Novel Lagrangian hierarchies, generalized variational ODE’s, and families of regular and embedded solitary waves

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Abstract

Hierarchies of Lagrangians of degree two, each only partly determined by the choice of leading terms and with some coefficients remaining free, are considered. The free coefficients they contain satisfy the most general differential geometric criterion currently known for the existence of a Lagrangian and variational formulation, and derived by solution of the full inverse problem of the calculus of variations for scalar fourth-order ordinary differential equations (ODEs) respectively. However, our Lagrangians have significantly greater freedom since our existence conditions are for individual coefficients in the Lagrangian. In particular, the classes of Lagrangians derived here have four arbitrary or free functions, including allowing the leading coefficient in the resulting variational ODEs to be arbitrary, and with models based on the earlier general criteria for a variational representation being special cases. For different choices of leading coefficients, the resulting variational equations could also represent traveling waves of various nonlinear evolution equations, some of which recover known physical models. Families of regular and embedded solitary waves are derived for some of these generalized variational ODEs in appropriate parameter regimes, with the embedded solitons occurring only on isolated curves in the part of parameter space where they exist. Future work will involve higher order Lagrangians, the resulting equations of motion, and their solitary wave solutions.

Keywords: novel Lagrangian families, generalized variational ODEs, regular and embedded solitary waves

(Some figures may appear in colour only in the online journal)

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1. Introduction

Derivations and use of Lagrangians for higher-order differential equations have seen renewed interest in recent years. This is due to many recent applications including, but not limited to, higher-order field-theoretic models [1, 2], investigations of isochronous behaviors in a variety of nonlinear models [3], treatments of higher-order Painlevé equations [4], and variational treatments of embedded solitary waves of a variety of nonlinear wave equations [5].

The problem of finding a Lagrangian for a given ordinary differential equation (ODE) is usually referred to as the inverse problem of the calculus of variations. The classical problem, restricting the Lagrangian to a function of the first derivatives of the dependent variable $u$ on $z$, has the form $u'' = F(z, u, u')$. The necessary and sufficient conditions for such an equation to be derivable from the Euler–Lagrange equation

$$\frac{\partial_{u}L - d(\partial_{u'}L)}{dz} = 0 \quad (1)$$

are classical, and were first derived by Helmholtz [6, 7], and also investigated by Darboux [8]. The inverse problem for a fourth-order ODE was treated by Fels [9]. We shall consider his main result later, and compare it to those derived in this paper.

In this paper, we approach this inverse problem differently. By treating the coefficients of the highest derivative in the ODE as arbitrary functions of the dependent variable and its derivatives, we are able to derive Lagrangians for entire classes of differential equations. The resulting generalized families of Lagrangians may then be employed to investigate solutions of various members in the corresponding families of variational ODEs. In this paper, we employ them to derive regular and embedded solitary wave solutions of member equations of the ODE family.

The remainder of this paper is organized as follows. Section 2 considers a preliminary example where the Lagrangian follows easily by matching the terms in the ODE to those in the Euler–Lagrange equation. Less trivially, we then illustrate the procedure for constructing regular and embedded solitary wave solutions of the original family of ODEs using this Lagrangian.

Section 3 develops our new approach to the inverse problem for generalized classes of nonlinear fourth-order ODEs with leading-order coefficients that may be dependent on up to the second derivative of the dependent variable. By matching the ODE to the Euler–Lagrange variational equations, the general form of Lagrangian for all possible variational ODEs of this class is derived, together with the conditions on the ODE coefficients. Section 4 then employs the resulting Lagrangian to construct both regular and embedded solitary waves for a representative member of this most general variational ODE of this family. Section 5 shows that our conditions on the variational ODE coefficients satisfy the most general conditions currently known [9] for variational fourth-order ODEs. However, our ODEs are in fact significantly more general in that they leave several individual coefficients in the Lagrangian free, as well as allowing the resulting variational ODE to have arbitrary leading-order coefficients. Section 6 briefly reviews the results and conclusions of the paper.

2. An example Lagrangian

To introduce some necessary techniques, let us first consider the traveling-wave equation given by

$$u^{(4)} \left( d_{2} - d_{3} u' \right) - 2 d_{3} u'' u''' + u'' \left( d_{1} - 2 \mu u \right) - \mu u'^{2} = 0. \quad (2)$$
This equation is related to that obtained from equation (1) of Ilison et al [10]. Matching derivatives of various orders to the Euler–Lagrangian equation of form (1), gives us the Lagrangian

$$L = - \frac{d_1 z^2}{2} u_z + \mu \frac{1}{2} u_z u_{zz} + \frac{1}{2} (d_2 - 3) u_z^2. \quad (3)$$

In the next two sub-sections we introduce the techniques for finding both regular and embedded solitons for the equation (2) using its Lagrangian (3) [11–13].

2.1. Regular solitons

In this section, we shall use the well-known Rayleigh–Ritz method for constructing regular solitary waves with exponentially decaying tails of equation (2), as widely employed in many areas of applications. The extension to embedded solitary waves in the next sub-section is much more recent, and also less widely known.

As usual, the localized regular solitary wave solutions will be found using a Gaussian trial function (4). This is standard even for simpler nonlinear partial differential equations (PDEs) where exact solutions may be known and have the usual sech or sech$^2$ functional forms. We use the form:

$$u = A \exp \left( - \frac{z^2}{\rho^2} \right). \quad (4)$$

Then, after substituting the trial function into the Lagrangian and integrating over the entire $z$ axis one gets the ‘averaged action’ (5):

$$S = \frac{A^2 \left( 27 \sqrt{2} d_2 + \left( -9 \sqrt{2} d_1 + 8 \sqrt{3} \mu A \right) \right)}{36 \rho^3} \sqrt{\frac{\pi}{2}}. \quad (5)$$

At this point the goal is to optimize the trial function by varying the action with respect to the amplitude, $A$, and the width, $\rho$. This determines the optimal parameters for the trial function or solitary wave solution, but within the particular functional form chosen for the ansatz. The
resulting variational Euler–Lagrange equations, by varying $A$ and $\rho$ respectively, are the system of algebraic equations:

\[
\begin{align*}
\frac{2\sqrt{\pi}A^2\mu}{3\rho} + \sqrt{\pi}A \left( \rho^2 \left( 8\sqrt{3}A\mu - 9\sqrt{2d_1} \right) + 27\sqrt{2d_2} \right) &= 0, \\
\frac{\sqrt{\pi}A^2 \left( 8\sqrt{3}A\mu - 9\sqrt{2d_1} \right)}{18\rho^3} - \frac{\sqrt{\pi}A^2 \left( \rho^2 \left( 8\sqrt{3}A\mu - 9\sqrt{2d_1} \right) + 27\sqrt{2d_2} \right)}{12\rho^4} &= 0.
\end{align*}
\] (6a, 6b)

The nontrivial solution to these equations is

\[
\begin{align*}
A &= \frac{3\sqrt{2}d_1}{7\mu}, \\
\rho &= \frac{21d_2}{d_1}. \quad \text{(7a, 7b)}
\end{align*}
\]

The optimized variational soliton for the regular solitary waves of the traveling-wave equation (2) is given by the trial function (4) with the above $A$ and $\rho$. Figures 1(a) and (b) respectively show the resulting regular solitary wave solution for $\mu = d_3 = 1$, and various positive values of either $d_2$ or $d_1$ (with the other one set to the value one).

Figures 2 and 3 show a direct analysis of the accuracy of the variational regular solitary waves obtained above. It is possible to do this direct accuracy analysis since our variational solution for the regular solitary waves given by (4), (7a), and (7b) is analytical. Substituting this variational solution (4) (with (7a) and (7b)) into the traveling-wave (2), the deviation of the left-hand side of (2) from zero gives a direct measure of the goodness of the variational solution.

Figures 2(a) and (b) show the absolute value of this left-hand side for $\mu = 1$, $d_3 = 1$ and $d_2 = 5$ and $d_2 = 1$ respectively for $z$ from $-3$ to $3$ and $d_1$ from $0$ to $10$. It is clear that, for these fixed values, our variational solution is increasingly accurate when $d_1$ values are small.

Figure 3 displays the absolute value of this left-hand side of (2) in a somewhat different fashion, in this case for $d_1 = 1$. Over the entire ranges of $z$ from $-3$ to $3$ and $d_2$ from $0$ to $10,$
it remains small, thus showing that our variational solution is accurate for all $z$ with small $d_2$ values. At larger $d_2$ values, the deviation from zero becomes even smaller, and the variational solution even more accurate.

2.2. Embedded solitons

In the variational approach to embedded solitary waves, the tail of a delocalized soliton is modeled by

$$u_{\text{tail}} = \alpha \cos(\kappa(c)z), \quad (8)$$

where the cosine ensures an even solution, and the arbitrary function $\kappa(c)$ will be sought so it ensures the integrability of the action [13].

Since embedded solitary waves are not as well-known as regular solitons, we provide some background detail about them here. In particular, they occur in parameter regimes where the (exponentially small) tails of the solitary waves are oscillatory. These regimes are straightforwardly derived [12, 13] by substituting an exponential solution in the linearized version of the ODE, since the tail solutions are small and nonlinear terms may thus be neglected. In the parameter regimes where the arguments of this exponential ansatz are complex, the tail solutions are oscillatory. As detailed investigations have revealed, the resulting solitary waves, now called embedded solitons, are embedded in two different ways. The original name ‘embedded’ came from the fact [14, 15] that the frequency spectrum of all such solutions turned out to be embedded in the continuous spectrum of the linearized operator corresponding of the ODE whose solitary waves are being considered. However, besides this embedding or resonance with the linear spectrum in the frequency domain, they are also discrete structures embedded within continuous seas of ‘delocalized’ solitary waves with exponentially small tails in the parameter regimes where the waves may have oscillatory tails. On isolated curves in those parameter regimes, the tail amplitudes become strictly zero [16, 17], and we thus get a genuine soliton embedded within a sea of delocalized ones surrounding it.
Our ansatz for the embedded soliton uses a second order exponential core model plus the above tail model, i.e.,

$$u = A \exp \left( -\frac{z^2}{\rho^2} \right) + u_{\text{tail}}. \tag{9}$$

Substituting this trial function into the Lagrangian (3) and reducing the higher powers of the trigonometric functions yields an equation with trigonometric functions of the double and triple angles, as well as terms linear in $z$. The former would make averaging of the Lagrangian divergent. However, it is possibly to rigorously establish that such terms may be averaged out, so we shall set them to zero \textit{a priori} [12, 13].

The terms linear in $z$ would also cause the Lagrangian to be non-integrable as investigated in detail in [13]. Hence, we must set $\kappa(c) = \pm \sqrt{\frac{4}{d_2}}$, to force these linear terms to equal zero [13]. This wavenumber of the tail oscillation (8) may also be directly derived by substituting (8) into the linearized version of (2) (for the small amplitude tail oscillations).

These two steps are not part of the traditional Rayleigh–Ritz method used for the construction of regular solitary waves and are used only for the variational approximation of embedded solitary waves.

Next, we can integrate the rest of the equation to obtain the action

$$0 = \frac{\sqrt{\pi} A e^{\frac{d_1^2}{d_2}}}{72 d_2 \rho^3} \left( 9 \sqrt{2} \alpha \mu \rho^2 \left( 3 d_1 \rho^2 + 4 d_2 \right) \right. \right.$$

$$\left. + 2 e^{\frac{d_1^2}{d_2}} \left( 8 \sqrt{3} A^2 d_2 \mu \rho^6 + 9 \sqrt{2} A d_2 \left( 3 d_2 - d_1 \rho^2 \right) + 18 \alpha^2 d_1 \mu \rho^4 \right) \right) = 0 \tag{10}$$

As for the regular solitary waves, the action is now varied with respect to the amplitude ($A$), the width ($\rho$), and this time also the small amplitude ($\alpha$) of the oscillating tail to give the following system of equations:

$$0 = 8 \sqrt{3} A d_2 \mu \rho^3 + 6 \alpha^2 d_1 \mu \rho^4 + 3 \sqrt{2} \left( 2 d_1 \rho^2 \left( 2 \alpha \mu e^{\frac{d_1^2}{d_2}} - d_1 \right) \right. \right.$$

$$\left. + 3 \alpha d_1 \mu \rho^4 e^{\frac{d_1^2}{d_2}} + 6 d_2^2 \right) \tag{11a}$$

$$0 = 64 \sqrt{3} A^2 d_1^2 \mu \rho^5 e^{\frac{d_1^2}{d_2}} - 144 \alpha^2 d_1 d_2 \mu \rho^6 e^{\frac{d_1^2}{d_2}} + 9 \sqrt{2} A \right.$$

$$\times \left( 3 \alpha d_1^2 \mu \rho^5 - 8 d_1^2 \rho^2 \left( d_1 e^{\frac{d_1^2}{d_2}} - 2 \alpha \mu \right) - 8 \alpha d_1 d_2 \mu \rho^4 + 72 d_2^2 e^{\frac{d_1^2}{d_2}} \right) \tag{11b}$$

$$0 = \sqrt{2} A \left( 3 d_1 \rho^2 + 4 d_1 \right) + 8 \alpha d_1 \mu \rho^2 e^{\frac{d_1^2}{d_2}}. \tag{11c}$$

For embedded solitary waves, which live in a sea of delocalized solitary waves with small tails, the amplitude of the tail is strictly zero [11, 13]. Hence, we set $\alpha = 0$ in the above
In order to recover such embedded solitary waves, yielding

\[ 8 \sqrt{3} \mu A d_2 \rho^2 + 3 \sqrt{2} (6d_1^2 - 2d_1 d_2 \rho^2) = 0 \]  
\[ 64 \sqrt{3} \mu A d_2 \rho^2 + 9 \sqrt{2} (72d_1^3 - 8d_1 d_2^2 \rho^2) = 0 \]  
\[ \sqrt{2} A (3d_1 \rho^2 + 4d_2^3) = 0. \]

A nontrivial analytical solution to these equations is found to be

\[ A = \frac{13 \sqrt{2} d_1}{8 \mu} \]  
\[ \rho^2 = -\frac{4d_2}{3d_1}. \]

Note that using the model \( \exp(-z^2/\rho^2) \) for the core of the embedded solitary wave, with the tail \( (8) \) added on results in an imaginary \( \rho \) unless \( d_1 \) and \( d_2 \) have opposite signs, which would have made the argument to the exponential positive, and there would have been no decaying wave corresponding to a genuine embedded solitary wave.

In figures 4(a) and (b) the embedded soliton \( (9) \), with \( (13a) \) and \( (13b) \), is presented for \( d_1 = -1, \mu = 1 \) and \( \alpha = 0 \) and \( \alpha = 1 \) respectively, with \( z \) varying from \(-3\) to \(3\) and \( d_2 \) from \(0.5\) to \(5\).

In figures 5 and 6 a direct analysis of the accuracy is carried out for \( \mu = 1 \) and \( z \) varying from \(-3\) to \(3\) while \( d_2 \) goes from \(0\) to \(10\). Figures 5(a), (b) and (c) represent the results when \( \alpha = 1 \) and \( d_1 = -3, -1 \) and \(-1/3\) respectively. Figure 6 presents the results when \( \alpha = 0 \) and \( d_1 = -1 \).

3. A general class of Lagrangians and the associated variational ODEs

Now, let us generalize the previous ideas to a class of Lagrangians. To do so, we consider the leading-order fourth derivative term of a variational ODE to be of the most general form.
which one may get in the fourth-order Euler–Lagrange equation of a Lagrangian of the form
\( L(u, u', u'') \). Hence, we consider
\[
c_1(u, u')c_2(u')u^{(4)},
\]
and we want to match it to the Euler–Lagrange equation
\[
u^{(3)}L_{\nu''\nu''} + (u^{(3)})^2 L_{\nu''\nu''\nu''} + 2u^{(3)}L_{\nu''\nu''} - u''(L_{\nu''} - L_{\nu''})
\]
\[
+ (u')^2 L_{\nu''\nu''} + 2u'u^{(3)}L_{\nu''\nu''} - u'L_{\nu''} + 2u'u'L_{\nu''\nu''} + (u')^2 L_{\nu''\nu''} + L_{\nu''} = 0.
\]

Note that the form (14) assumed above is more general than that employed in [9], and this will lead to more general variational equations than derived in that paper.

Thus, at order \( O(u^{(4)}) \): we get
\[
L_{\nu''\nu''} = c_1(u, u')c_2(u'),
\]
or, on integrating,

\[ L_{u''} = c_1(u, u') \int c_2(u'') du'' + c_3(u, u'). \]

A second integration now yields:

\[ L = c_1(u, u') c_4(u'') + c_3(u, u') u'' + c_5(u, u'), \]

where

\[ c_4(u'') = \int \int c_2(u'', u'') du'' du'. \]

Hence, we must have

\[ L_{u''u''u''} = c_1(u, u') \frac{\partial c_2(u'')}{\partial u''}, \]

\[ L_{u'u'u''} = \frac{\partial c_1(u, u')}{\partial u} c_2(u''), \]

\[ L_{uu'u''} = \frac{\partial c_1(u, u')}{\partial u} c_4(u''), \]

\[ L_{u} = \frac{\partial c_1(u, u')}{\partial u} c_4(u'') + \frac{\partial c_3(u, u')}{\partial u} u'' + \frac{\partial c_5(u, u')}{\partial u}, \]

\[ L_{u'} = \frac{\partial c_1(u, u')}{\partial u'} c_4(u'') + \frac{\partial c_3(u, u')}{\partial u'} u'' + \frac{\partial c_5(u, u')}{\partial u'}, \]

\[ L_{u'u'} = \frac{\partial^2 c_1(u, u')}{\partial u'^2} c_4(u'') + \frac{\partial^2 c_3(u, u')}{\partial u'^2} u'' + \frac{\partial^2 c_5(u, u')}{\partial u'^2}. \]
Therefore, equation (15) defines the most general variational fourth order ODE (which may often be the traveling wave equation of a particular nonlinear PDE of interest [12, 13]) with coefficients given by (16)–(19), and can now be written, for completely general variable dependence of each function, as

\[
\begin{align*}
L_{uu''} & = \frac{\partial c_1(u, u')}{\partial u} \frac{dc_4(u'')}{du''} + \frac{\partial c_3(u, u')}{\partial u}, \\
L_{u'u''} & = \frac{\partial^2 c_1(u, u')}{\partial u^2} \frac{dc_4(u'')}{du''} + \frac{\partial^2 c_3(u, u')}{\partial u^2}, \\
L_{uu''} & = \frac{\partial^2 c_1(u, u')}{\partial uu''} \frac{dc_4(u'')}{du''} + \frac{\partial^2 c_3(u, u')}{\partial uu''}, \\
L_{u'u''} & = \frac{\partial^2 c_1(u, u')}{\partial uu''} \frac{dc_4(u'')}{du''} + \frac{\partial^2 c_3(u, u')}{\partial uu''}, \\
L_{u''u''} & = \frac{\partial^2 c_1(u, u')}{\partial u^2 u''} \frac{dc_4(u'')}{du''} + \frac{\partial^2 c_3(u, u')}{\partial u^2 u''}.
\end{align*}
\]

(19g–19k)

In section 5, we shall prove that (20) generalizes the most general currently discussed variational fourth-order variational ODEs, as derived by Fels [9], and which in fact turn out to be special cases of (20).

3.1. A case worth mentioning

Notice that for the very general case

\[
c_1(u, u') = \left( \sum_{k=0}^{n} a_k u^k \right) \left( \sum_{l=0}^{n} a_l (u')^l \right),
\]

one may directly obtain the variational ODE (20). We consider a specific example of this class of Lagrangians in detail in the following section, before returning to the general case and comparing our results to the most general differential geometric criteria obtained earlier for the existence of Lagrangians of fourth-order ODEs.

4. A member of the family of generalized Lagrangians

If in the previous section we set

\[
\begin{align*}
c_1(u, u') & = -d_1 + d_2 u + d_3 u', \\
c_3(u, u') & = u^2,
\end{align*}
\]

(21)

then the Lagrangian reads

\[
L = (-d_1 + d_2 u + d_3 u') u_z^2 + u^2 u_{zz} + u_z^2.
\]

(23)
Figure 7. The regular soliton plotted for $d_2 = 1$.

(a) The embedded soliton plotted for $\alpha = 5$. (b) The embedded soliton plotted for $d_1 = 5$.

Figure 8. Embedded soliton for $z$ and different values of the parameters.

and its Euler–Lagrange equation is

$$2u'^2 + 2(-1 + 2u)u'' + \frac{3d_2u'^2}{2} + 2d_2u' + u''' - d_1u^{(4)} + d_2uu^{(4)} = 0. \quad (24)$$

In the next subsections we will apply the variational approaches of section 2 to obtain

solitary wave solutions for the variational ODEs of this general class of Lagrangians.

4.1. Regular solitons

Using the same ansatz (4) and following the process outlined in subsection 2.1 one gets the action

$$S = \frac{A^2 \left( 27\sqrt{2}d_1 - 18\sqrt{2}\rho^2 + 16\sqrt{3}A (d_2 + \rho^2) \right)}{36\rho^3} \sqrt{\pi}. \quad (25)$$
By solving the system of equations generated by differentiating the action with respect to the parameters one gets the non-trivial solution

\[ A = \frac{3\sqrt{6}}{64d_2^2} \nu \quad \text{and} \quad \rho^2 = \frac{3}{4} \nu, \]

(26)

where

\[ \nu = 7d_1 - 2d_2 + \sqrt{49d_1^2 - 36d_1d_2 + 4d_2^2} \quad \text{and} \quad \tilde{\nu} = 7d_1 + 2d_2 - \sqrt{49d_1^2 - 36d_1d_2 + 4d_2^2}. \]

In figure 7 the regular soliton for these values of \( \rho \) and \( A \) is presented, using \( d_2 = 1 \) and varying \( z \) from \(-5\) to \(5\) and \( d_1 \) from \(0.5\) to \(5\).

### 4.2. Embedded solitons

Next, we follow the procedure presented in subsection 2.2, we start by considering an ansatz like (9), which yields the averaged action

\[
\sqrt{\pi} A e^{-\rho^2/4d_1^2} \left( e^{-\rho^2} \left( -144\alpha^2 (2d_1 - d_2) + 64\sqrt{3}A^2 (d_2 - \rho^2) \right) - 9\sqrt{2}A + \alpha \left( -9d_2 \rho^4 - 4d_1^2 (3d_2 - 4\rho^2) - 12d_1 \rho^2 (d_2 - 2\rho^2) + 4d_1^2 e^{\rho^2/2} (3d_1 - 2\rho^2) \right) \right) = S,
\]

(27)

when

\[ \kappa(c) = \frac{\sqrt{2}}{\sqrt{d_1}}. \]

(28)

These yield the non-trivial solutions for \( A \) and \( \rho^2 \):

\[ A = \frac{\sqrt{2}}{4} \left( 3d_1 - \frac{4}{3}\xi \right) \quad \text{and} \quad \rho^2 = \frac{2}{3} \xi, \]

where

\[ \xi = \frac{3d_1d_2 - 4d_1^2 - \sqrt{3}d_1^2 (8d_1^2 + 24d_1d_2 - 9d_2^2)}{8d_1 - 3d_2}. \]

In figure 8 the embedded soliton is presented. In it \( z \) varies from \(-3\) to \(3\), with \( d_2 = 2 \) and \( d_1 = 5 \) respectively. In figure 8(a) \( \alpha = 5 \) while \( d_1 \) varies between \(0.5\) to \(5\). In figure 8(b) \( d_1 = 5 \) while \( \alpha \) varies from \(0\) to \(5\).

### 5. Generalizations of previous existence conditions for Lagrangians

As mentioned in section 1, the inverse problem for a fourth-order ODE was treated by Fels [9]. We now proceed to compare our generalized family of Lagrangians (17) and (18) for variational fourth-order ODEs against the criteria obtained by Fels using differential geometric approaches, as well as to various families of variational ODEs based on Fels’ criteria [4, 18].
To this end, we state Fels’ principal result here:

Theorem: a fourth-order ODE

\[ u^{(4)} = F(z, u, u', u'', u''') \]  

admits a variational representation with a non-degenerate second order Lagrangian if and only if the following two conditions are satisfied

\[ F_{u'''}u'''u''' = 0 \]  

and

\[ F_{u'} + F_{u''}z/2 - F_{u''}z^2/4 + F_{u''}F_{u''}'/2 + F_{u''}^3/8 = 0. \]

Clearly, our general variational equation (20) corresponds to

\[ F(z, u, u', u'') = -\left( (u'')^2 c_1 e_2 + 2u''u(3)c_1 u' e_2 + 2u'u(2)c_1 u' c_4 + c_3 u'' u u' + c_4 u'' u u' + c_5 u'' u u' \right) / (e_2 c_3). \]

It is now straightforward to check that this general \( F(z, u, u', u'') \) for our generalized family of Lagrangians (which we copy here for ease of understanding of the following discussion)

\[ L = c_1(u, u')c_4(u'') + c_3(u, u')u'' + c_5(u, u'), \]

where

\[ c_4(u'') = \int\int c_2(u'') du'' du'', \]

satisfies the first condition (30) above for arbitrary functions \( c_j \) in the Lagrangian (33).

However, the second condition (31) above is not satisfied for arbitrary functions \( c_i \) in the Lagrangian (33). It is however satisfied for the special case

\[ c_1(u, u') = 1, c_2(u'') = 1, \]

considered in Fels’ [9] differential geometric derivation of this criterion, as well as in the treatments of various models based on this condition (31) [4, 18].

This leads us to the following two conclusions about the two ways that our generalized family of Lagrangians (33) and (34) and the associated variational ODEs (20) are more general than the criteria developed in Fels [9]:

(a) Our Lagrangian has four arbitrary or free functions \( c_1, c_3, c_4, \) and \( c_5 \), in place of Fels’ single function \( F \), which, in our case is given by (32)

and

(b) The leading coefficient \( c_1(u, u')c_4(u'') \) in our variational ODE (20) may be far more general than for Fels or other models based on his criteria for a variational representation.
6. Conclusions

In conclusion, we have derived a novel, significantly generalized hierarchy of Lagrangians for fourth-order nonlinear ODEs, and employed it to construct families of regular and embedded solitary waves of some member equations.

The method of derivation of the family of Lagrangians is both straightforward and novel. In particular, our family of Lagrangians and the associated variational ODEs are more general than the most general variational formulations derived earlier for fourth-order ODEs [9] in two significant ways:

(a) Our Lagrangian has four arbitrary or free functions $c_1, c_3, c_4,$ and $c_5$, in place of the single function $F$ in the earlier general variational criteria

and

(b) The leading coefficient $c_1(u,u')c_4(u'')$ in our variational ODEs (20) may be far more general than in earlier work, or in other models based on the earlier general criteria for a variational representation. Future work on extending the approach developed here to sixth and eighth-order variational ODE models is ongoing.

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