SCATTERED REPRESENTATIONS OF COMPLEX CLASSICAL LIE GROUPS

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Abstract. As a continuation of [DW], this paper studies scattered representations of $G = SO(2n + 1, \mathbb{C})$, $Sp(2n, \mathbb{C})$ and $SO(2n, \mathbb{C})$, which lies in the ‘core’ of the unitary spectrum $G$ with nonzero Dirac cohomology. We describe the Zhelobenko parameters of these representations, count their cardinality, and determine their spin-lowest $K$-types. We also disprove a conjecture raised in 2015 asserting that the unitary dual can be obtained via parabolic induction from irreducible unitary representations with non-zero Dirac cohomology.

1. Introduction

Although many results quoted in this paper hold in a much wider setting, for convenience, we set $G$ as a complex connected classical Lie group. Fix a Cartan involution $\theta$ on $G$ such that its fixed points form a maximal compact subgroup $K$ of $G$. Then on the Lie algebra level, we have the Cartan decomposition

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{p}.$$ 

The subscripts will be dropped to stand for the complexified Lie algebras.

In the late 1990s, Vogan introduced the notion of Dirac cohomology [V], see (1). As a new invariant for admissible $(\mathfrak{g}, K)$-modules, Dirac cohomology has profound implications in various areas of representation theory and beyond.

One of the applications of Dirac cohomology is to gain a better understanding of unitary representations. More explicitly, it is known in [VZ] that all unitary $(\mathfrak{g}, K)$-modules with non-zero $(\mathfrak{g}, K)$ cohomology are $A_q(\lambda)$ modules. Moreover, in [S], it was shown that these modules characterize all unitary modules with real, integral, and strongly regular infinitesimal characters. On the other hand, it is shown in [HKP] that the set of unitary modules with non-zero Dirac cohomology $\hat{G}^d$ strictly contains all unitary modules with non-zero $(\mathfrak{g}, K)$ cohomology. Therefore, by understanding $\hat{G}^d$, one can have a better understanding of the unitary dual $\hat{G}$. 


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The paper [DD] reduced the study of $\hat{G}^d$ to a finite set called scattered representations $\hat{G}^{sc}$ (see Definition 2.1 for the precise meaning). Let us recall some of the recent progress in the study of $\hat{G}^{sc}$ in the case when $G$ is a complex simple Lie group:

- When $G = GL(n, \mathbb{C})$, [DW] gives a complete study of $\hat{G}^{sc}$, thus of $\hat{G}^d$ as well.
- When $G$ is exceptional of type $G_2$, $F_4$ and $E_6$, then [DD] and [D2] described the Zhelobenko parameters of all scattered representations. One can then use atlas [At] to verify that these modules satisfy Conjecture 1.1 of [BP1].

In this manuscript, we study scattered representations of $G$ other than type A. Our main results include Corollary 2.5 answering [DD, Conjecture C] for $G$ in the affirmative, Corollary 2.6 on the number of scattered representations, and Theorem 3.1 giving the form of spin-lowest $K$-types introduced in [D1] (see Section 1.2). At the end of the introduction, we will prove Conjecture 5.6 (ii)-(iv) of [DD]. In Section 4 we will disprove Conjecture 13.3 of [H], which asserts that any irreducible unitary representation can be parabolically induced from an irreducible unitary representation with non-zero Dirac cohomology. As we shall see, this conjecture captures some special feature of type A, which does not hold in general.

There is a couple of reasons why we are interested in studying $\hat{G}^{sc}$ for these groups even though $\hat{G}^d$ is already known. Firstly, when $\pi \in \hat{G}^d$, spin lowest $K$-types are precisely the $K$-types contributing to the Dirac cohomology of $\pi$ (see Section 1.2 below). The main result of [BDW] gives the Dirac cohomology of all $\pi \in \hat{G}^d$ for all complex classical groups, yet its spin lowest $K$-type is unknown. Therefore, it would still be advantageous to investigate the spin lowest $K$-types of $\hat{G}^{sc}$ even though $\hat{G}^d$ is known in such cases.

Secondly, the recent research announcement [BP2] suggests that Dirac cohomology could be used to construct interesting automorphic forms. The techniques involved in the study of $\hat{G}^{sc}$ in this manuscript should be applicable to (some) real reductive groups.

1.1. Preliminaries. Let $H$ be a Cartan subgroup of $G$. Let $\mathfrak{h}_0$ be the Lie algebra of $H$. Fix a positive root system $\Delta^+_G$ of $\Delta(\mathfrak{g}_0, \mathfrak{h}_0)$, and let $\rho$ be half the sum of all positive roots in $\Delta^+_G$.

We denote by $\langle \cdot, \cdot \rangle$ the Killing form on $\mathfrak{g}_0$. This form is negative definite on $\mathfrak{t}_0$ and positive definite on $\mathfrak{p}_0$. Moreover, $\mathfrak{t}_0$ and $\mathfrak{p}_0$ are orthogonal to each other under $\langle \cdot, \cdot \rangle$. We shall denote by $\| \cdot \|$ the norm corresponding to the Killing form. Let $Z_i$, $1 \leq i \leq m$, be an orthonormal basis of $\mathfrak{p}_0$ with respect to the norm $\| \cdot \|$. The Dirac operator

$$D := \sum_{i=1}^{m} Z_i \otimes Z_i$$

introduced by Parthasarathy [P1] lives in $U(\mathfrak{g}) \otimes C(\mathfrak{p})$, the tensor product of the universal enveloping algebra of $\mathfrak{g}$ and the Clifford algebra of $\mathfrak{p}$. Denote by $\text{Ad} : K \to \text{SO}(\mathfrak{p}_0)$ the adjoint map, and $p : \text{Spin}(\mathfrak{p}_0) \to \text{SO}(\mathfrak{p}_0)$ the double covering map. Let $\tilde{K}$ be the spin double covering group of $K$. That is,

$$\tilde{K} := \{ (k, s) \in K \times \text{Spin}(\mathfrak{p}_0) \mid \text{Ad}(k) = p(s) \}.$$
Let $S$ be a spin module of $C(p)$, and let $π$ be an admissible $(g, K)$-module. Then $D$ acts on $π \otimes S$, and the Dirac cohomology of $π$ is defined in [V] as

$$H_D(π) := \ker D/(\ker D \cap \text{im}D).$$

Since the Dirac operator $D$ is independent of the choice of the orthonormal basis $\{Z_i\}_{i=1}^m$, it is evident that as a $\tilde{K}$-module, the Dirac cohomology $H_D(π)$ is an invariant of $π$.

Now let us come to the representations of $G$. Let $H = TA$ be the Cartan decomposition of $H$, with $h_0 = t_0 + a_0$. We make the following identifications:

$$h \cong h_0 \times h_0, \quad t = \{(x, -x) : x \in h_0\} \cong h_0, \quad a \cong \{(x, x) : x \in h_0\} \cong h_0.$$

Take an arbitrary pair $(λ_L, λ_R) \in h_0^* \times h_0^*$ such that $η := λ_L - λ_R$ is integral. Denote by $\{η\}$ the unique dominant weight to which $G$ is conjugate under the action of the Weyl group $W := W(g_0, h_0)$. Write $ν := λ_L + λ_R$. We can view $η$ as a weight of $T$ and $ν$ a character of $A$. Put

$$I(λ_L, λ_R) := \text{Ind}_B^G(C_η \otimes C_ν \otimes \text{triv})_{K\text{-finite}},$$

where $B \supset H$ is the Borel subgroup of $G$ determined by $Δ^+$. Then the $K$-type with highest weight $\{η\}$, denoted by $V^K_{\{η\}}$, occurs exactly once in $I(λ_L, λ_R)$. Let $J(λ_L, λ_R)$ be the unique irreducible subquotient of $I(λ_L, λ_R)$. By [Zh], every irreducible admissible $(g, K)$-module has the form $J(λ_L, λ_R)$. Indeed, up to equivalence, $J(λ_L, λ_R)$ is the unique irreducible admissible $(g, K)$-module with infinitesimal character the $W(g, h) = W \times W$ orbit of $(λ_L, λ_R)$, and lowest $K$-type $V^K_{\{λ_L - λ_R\}}$ (note that under the identification $t \cong h_0$ above, the compact Weyl group $W(\frak{t}, \frak{t})$ is isomorphic to $W$). We will refer to the pair $(λ_L, λ_R)$ as the Zhelobenko parameter for the module $J(λ_L, λ_R)$.

For $J(λ_L, λ_R)$ to live in $\hat{G}^d$, it should admit a nondegenerate Hermitian form in the first place. Moreover, it should satisfy the Vogan conjecture proved by Huang and Pandžić [HP1]. Then one deuces that, as carried out on page 5 of [BP1], $J(λ_L, λ_R)$ must have the form

$$J(λ, -sλ), \quad 2λ \text{ is dominant integral regular and } s \in W \text{ is an involution}.$$

1.2. Spin-lowest $K$-type. From now on, let $π = J(λ, -sλ)$ be an irreducible unitary $(g, K)$-module bearing the form (3). To achieve the classification of $\hat{G}^d$, the first-named author introduced the notion of spin-lowest $K$-type [D1]: Given an arbitrary $K$-type $V^K_δ$, its spin norm is defined as

$$||δ||_{\text{spin}} := ||\{δ - ρ\} + ρ||.$$

Then a $K$-type $V^K_δ$ occurring in $π$ is called a spin-lowest $K$-type of $π$ if it attains the minimum spin norm among all the $K$-types showing up in $π$.

Using the definition of spin norm, Parthasarathy’s Dirac operator inequality [P2] can be rephrased as:

$$||δ||_{\text{spin}} \geq ||2λ|| \quad \text{for all } V^K_δ \text{ appearing in } π.$$
Moreover, one can deduce from [HP2, Theorem 3.5.3] that \( \pi \in \hat{G}^d \) if and only if

\[
\{ \mu - \rho \} + \rho = 2\lambda
\]

for some \( K \)-type \( V^K_\mu \) appearing in \( \pi \). In other words, \( \pi \in \hat{G}^d \) if and only if its spin lowest \( K \)-types make (\ref{6}) an equality. Moreover, in such cases, the Dirac cohomology of \( \pi \) consists only of certain copies of the \( \tilde{K} \)-module \( V^K_{2\lambda - \rho} \). Put it in a different way, if \( \pi \in \hat{G}^d \), then the spin-lowest \( K \)-types of \( \pi \) are exactly the \( \tilde{K} \)-types contributing to its Dirac cohomology (see Proposition 3.3 of [DL] for more details).

1.3. The classification of \( \hat{G}^d \). Let us recall the classification of \( \hat{G}^d \) in [BDW].

**Theorem 1.1.** Let \( G \) be a connected complex classical Lie group, and \( \pi = J(\lambda, -s\lambda) \) be an irreducible \((g, K)\)-module as in [3]. Then \( \pi \in \hat{G}^d \) if and only if it has the form

\[
\pi = \text{Ind}_P^G \left( \bigotimes_{i=1}^l \mathbb{C}_{\chi_i} \otimes \pi_O \right),
\]

where

- \( P = LU \) is the parabolic subgroup with \( L = \prod_{i=1}^l GL(a_i) \times G' \), with \( G' \) being the same type as \( G \) of smaller rank (we also allow \( G' = G \), or \( G' \) to be absent in \( L \));
- \( \mathbb{C}_{\chi_i} \) is a unitary character of \( GL(a_i) \); and
- \( \pi_O \) is equal to one of the following unipotent representations of \( G' \):

**Type B:** \( \pi_{[2k+1,2n]} := J \left( \begin{array}{c} 1, 1/2, \ldots, 1/2, n \end{array} \right), \ k \geq n \geq 0 \)

\[
\pi_{[2n]} := J \left( \begin{array}{c} 1, 2, \ldots, n \end{array} \right), \ n > 0
\]

**Type C:** \( \pi_{[2n-1,1]} := J \left( \begin{array}{c} 1/2, \ldots, 1/2, 1, 2, \ldots, n \end{array} \right), \ n > 0 \)

**Type D:** \( \pi_{[2n,2k-1,1]} := J \left( \begin{array}{c} 0, \ldots, n-1, \frac{1}{2}, \ldots, k - \frac{1}{2} \end{array} \right), \ n \geq k > 1 \)

The subscripts of the representations above give the column sizes of the Young diagram of the nilpotent orbit \( O \subset g' \).

**Remark 1.2.** (a) In Type D of the above theorem, we excluded the unipotent representations \( \pi_{[2n,1,1]} = J \left( \begin{array}{c} 0, \ldots, n-1, \frac{1}{2}, \ldots, k - \frac{1}{2} \end{array} \right) \) which appears in [BP1]. In fact, \( \pi_{[2n,1,1]} = \text{Ind}_P^G(\mathbb{C}_{(1)} \otimes \pi_{[2n])} \), where the Levi subgroup \( L \) of \( P \) is equal to \( GL(1) \times SO(2n) \).
(b) Note that Conjecture 5.6 (ii)-(iv) of [DD] follows directly from Theorem 1.1. More precisely, any spherical, non-trivial representation in Theorem 1.1 must be a unipotent representation $\pi_O$ with Zhelobenko parameters $J\left(\lambda, -s\lambda\right)$. This includes exactly the following representations:

Type B: $\pi_{[2k+1,2n]}$ for $k \geq n \geq 0$, with $2\lambda = \left(2k - 1, 2k - 3, \ldots, 2n + 1, 2n, \ldots, 2, 1\right)$. In terms of fundamental weights, it is equal to

$$a \left[\begin{array}{c} 2, \ldots, 2, 1, \ldots, 1, 2 \end{array}\right]$$

where $a := \max\{k - n - 1, 0\}$.

Type C: The spherical metaplectic (or minimal) representation $\pi_{[2n-1,1]}$ for $n$ even, with $2\lambda = \left(2n - 1, 2n - 3, \ldots, 1\right)$. In terms of fundamental weights, it is equal to

$$\left[\begin{array}{c} 2, \ldots, 2, 1 \end{array}\right].$$

Type D: $\pi_{[2n,4r-1,1]}$ with $n \geq 2r > 1$, with $2\lambda = \left(2n - 2, 2n - 4, \ldots, 4r, 4r - 1, \ldots, 1, 0\right)$. In terms of fundamental weights, it is equal to

$$a \left[\begin{array}{c} 2, \ldots, 2, 1, \ldots, 1 \end{array}\right],$$

where $a := \max\{n - 2r - 1, 0\}$.

2. Scattered Representations

In this section, we investigate which representations in Theorem 1.1 are scattered. Firstly, we recall the definition of scattered representations.

Definition 2.1. Let $\pi = J(\lambda, -s\lambda) \in \hat{G}^d$. Then $\pi$ is a scattered representation of $G$ if all the simple reflections of $W$ occur in any reduced expression of $s$. In such a case, we write that $\pi \in \hat{G}^{sc}$.

By Theorem A of [DD], $\hat{G}^{sc}$ is a finite set. Moreover, the set $\hat{G}^d$ can be completely recovered from the scattered representations of $[L, L]$, where $L$ runs over the Levi factors of all the $\theta$-stable parabolic subgroups of $G$.

As in the case of $GL(n, \mathbb{C})$ in [DW], we use chains to study the representations in $\hat{G}^{sc}$.

Definition 2.2. The chains of each classical type is defined as follows:

(i) Type A: Let $T \geq t$ be positive integers of the same parity, define

$$A_{T,t} := \{T, T - 2, \ldots, t + 2, t\} := \left(\begin{array}{cccc} T/2 & T/2 & \ldots & T/2 \\ -T/2 & -T/2 & \ldots & -T/2 \\ \ldots & \ldots & \ldots & \ldots \\ -t/2 & -t/2 & \ldots & -t/2 \end{array}\right).$$
(ii) Type B: Let $k \geq n \geq 0$ be integers, define
\[
B_{[2k+1,2n]} := \{2k-1, 2k-3, 2n+1, 2n, 2n-1, \ldots, 1\}_B := \left(\frac{1}{2}, \frac{3}{2}, \ldots, \frac{2k-1}{2}; 1, 2, \ldots, n\right)
\]
(iii) Type C: Let $n$ be a positive integer, define
\[
C_{[2n]} := \{2n, 2n-2, \ldots, 2\}_C := \left(1, \ldots, n\right);
C_{[2n-1,1]} := \{2n-1, 2n-3, \ldots, 1\}_C := \left(\frac{1}{2}, \ldots, \frac{2n-1}{2}\right)
\]
(iv) Type D: Let $n > 0$ be an integer, define
\[
D_{[2n]} := \{2n-2, 2n-4, \ldots, 0\}_D := \left(0, \ldots, n-1\right);
D_{[2n,2k-1,1]} := \{2n-2, 2n-4, \ldots, 2k, 2k-1, \ldots, 1, 0\}_D := \left(0, \ldots, n-1; \frac{2k-1}{2}; \frac{(-1)^k}{2}, \ldots, \frac{(-1)^{2k-1}}{2}\right)
\]

If the subscript of a chain is omitted, we assume that the chain is of Type A.

Using Theorem 1.1 and the definition above, all $J(\lambda, -s\lambda) \in \hat{G}^d$ have Zhelobenko parameters of the form
\[
(\lambda, -s\lambda) = \chi_\mathcal{O} \text{ or } \bigcup_{i=1}^l \mathcal{A}_{T_i, t_i} \text{ or } \bigcup_{i=1}^l \mathcal{A}_{T_i, t_i} \cup \chi_\mathcal{O},
\]
where $\mathcal{X} = \mathcal{B}, \mathcal{C}$ or $\mathcal{D}$, and the coordinates of all chains constitute $2\lambda$. In particular, the coordinates of the union of chains must be distinct.

In order to show which $(\lambda, -s\lambda)$ gives rise to scattered representations, we extend the definition of interlaced chains in [DW] as follows:

**Definition 2.3.**

(a) Two chains $\mathcal{X}_1 = \{M, \ldots, m\}_X$, $\mathcal{X}_2 = \{N, \ldots, n\}_X$, where $X = A, B, C$ or $D$, are linked if the entries of $\mathcal{X}_1$ and $\mathcal{X}_2$ are disjoint, and either one of the following holds:
- $M > N > m$; or
- $N > M > n$; or
- $\{\mathcal{X}_1, \mathcal{X}_2\} = \{\mathcal{C}_{[2n]}, \mathcal{A}_{(1,1)}\}$.

(b) We say a union of chains $\bigcup_{i \in I} \mathcal{X}_i$ is interlaced if for all $i \neq j$ in $I$, there exist indices $i = i_0, i_1, \ldots, i_k = j$ in $I$ such that $\mathcal{X}_{i_{l-1}}$ and $\mathcal{X}_{i_l}$ are linked for all $1 \leq l \leq k$. (By convention, we also let the single chain $\mathcal{X}$ to be interlaced).

Among all $\pi \in \hat{G}^d$ given in [BDW], the following theorem allows us to pick out those appearing in $\hat{G}^{sc}$.
Theorem 2.4. Take any \( \pi \in \hat{G}^d \) as described in Theorem 1.1. Then \( \pi \) is scattered if and only if its Zhelobenko parameters \( \bigcup_{i=1}^{t_i} A_{T_i,t_i} \cup \mathcal{X}_Q \) are interlaced.

Proof. There must be a chain \( \mathcal{X}_Q \) of Type \( X \), or else the reduced expressions of \( s \in W \) do not contain the short/long root in Type B/C, or one of the roots at the fork in Type D. The fact that they are all interlaced follows directly from the arguments in [DW]. \( \square \)

Now let us give two applications of Theorem 2.4.

2.1. The spin-lowest \( K \)-type is unitarily small. The notion of unitarily small \( K \)-types was introduced by Salamanca-Riba and Vogan [SV] in their uniform conjecture of the unitary dual of real reductive Lie groups. In our setting, a \( K \)-type \( V^K_\mu \) is unitarily small if and only if its highest weight \( \mu \) lies in the convex hull generated by the points \( \{ w \rho \mid w \in W \} \).

Corollary 2.5. The spin-lowest \( K \)-type of any scattered representation is unitarily small.

Proof. Note that by our construction of interlaced chains, the adjacent coordinates of \( 2\lambda \) differ by at most one. The corollary therefore follows directly from Lemma 3.4 of [DW]. \( \square \)

2.2. Number of scattered representations. Now let us count the number of scattered representations as in Section 3 of [DW].

Corollary 2.6. Let \( b_n, c_n, d_n \) be the number of scattered representations of Type \( B \), \( C \), \( D \) of rank \( n \) respectively. Then we have the following recursive formulas:

- **Type B**: \( b_2 = 2 \)
  \[
  b_{n+1} = \begin{cases} 
  2b_n - 1 & \text{if } n \text{ is even} \\
  2b_n & \text{if } n \text{ is odd}
  \end{cases} \text{ for } n \geq 2.
  \]
- **Type C**: \( c_2 = 3 \), \( c_{n+1} = 2c_n \) for \( n \geq 2 \).
- **Type D**: \( d_3 = 2 \), \( d_4 = 5 \)
  \[
  d_{n+1} = \begin{cases} 
  2d_n - 1 & \text{if } n \text{ is even} \\
  2d_n & \text{if } n \text{ is odd}
  \end{cases} \text{ for } n \geq 4.
  \]

In particular, the number of scattered representations of Type \( C_n \) is given by \( c_n = 3 \cdot 2^{n-2} \) for all \( n \geq 2 \).

Proof. Firstly, we list the scattered representations for each type of small rank:

- **Type B2**: \( \{ 3 \ 1 \}_B, \{ 2 \ 1 \}_B \).
- **Type B3**: \( \{ 5 \ 3 \ 1 \}_B, \{ 2 \} \cup \{ 3 \ 1 \}_B, \{ 3 \ 2 \ 1 \}_B \).
- **Type C2**: \( \{ 3 \}_C, \{ 2 \}_C \cup \{ 1 \}, \{ 4 \ 2 \}_C \).
- **Type C3**: \( \{ 5 \ 3 \ 1 \}_C, \{ 2 \} \cup \{ 3 \ 1 \}_C, \{ 1 \} \cup \{ 4 \ 2 \}_C, \{ 3 \ 1 \} \cup \{ 2 \}_C, \{ 6 \ 4 \ 2 \}_C, \{ 3 \} \cup \{ 4 \ 2 \}_C \).
- **Type D3**: \( \{ 4 \ 2 \ 0 \}_D, \{ 1 \} \cup \{ 2 \ 0 \}_D \).
- **Type D4**: \( \{ 6 \ 4 \ 2 \ 0 \}_D, \{ 3 \} \cup \{ 4 \ 2 \ 0 \}_D, \{ 1 \} \cup \{ 4 \ 2 \ 0 \}_D, \{ 3 \ 1 \} \cup \{ 2 \ 0 \}_D, \{ 3 \ 2 \ 1 \ 0 \}_D \).
This verifies the corollary for small ranks. For the recursive formula, we can apply an analog of Algorithm 3.6 of [DW] to construct new scattered representations of Type $X_{n+1}$ ($X = B, C, D$) from those of Type $X_n$ (see the example below). Then the result follows.

Example 2.7. We begin with the recursion of Type $C$, which is exactly the same as that in [DW]: For example, the $c_4 = 12$ scattered representations of Type $C_4$ are obtained from the $c_3 = 6$ scattered representations of Type $C_3$ by:

\[
\begin{align*}
\{5 \ 3 \ 1\}_C & \mapsto \{7 \ 5 \ 3 \ 1\}_C, & \{4\} \cup \{5 \ 3 \ 1\}_C \\
\{2\} \cup \{3 \ 1\}_C & \mapsto \{2\} \cup \{5 \ 3 \ 1\}_C, & \{4 \ 2\} \cup \{3 \ 1\}_C \\
\{1\} \cup \{4 \ 2\}_C & \mapsto \{3\} \cup \{1\} \cup \{4 \ 2\}_C, & \{1\} \cup \{6 \ 4 \ 2\}_C \\
\{3 \ 1\} \cup \{2\}_C & \mapsto \{5 \ 3 \ 1\} \cup \{2\}_C, & \{3 \ 1\} \cup \{4 \ 2\}_C \\
\{6 \ 4 \ 2\}_C & \mapsto \{8 \ 6 \ 4 \ 2\}_C, & \{5\} \cup \{6 \ 4 \ 2\}_C \\
\{3\} \cup \{4 \ 2\}_C & \mapsto \{5 \ 3\} \cup \{4 \ 2\}_C, & \{3\} \cup \{6 \ 4 \ 2\}_C.
\end{align*}
\]

As for special orthogonal groups, Algorithm 3.6 of [DW] applies as in the case of symplectic groups with the following exceptions:

- In Type $B_{2n}$, we only have

\[
\{2n \ 2n-1 \ldots \ 2 \ 1\}_B \mapsto \{2n+1 \ 2n \ 2n-1 \ldots \ 2 \ 1\}_B.
\]

Namely, there is no $\{2n+2 \ 2n \ 2n-1 \ldots \ 2 \ 1\}_B$.

- In Type $D_{2n}$, we only have

\[
\{2n-1 \ 2n-2 \ldots \ 1 \ 0\}_D \mapsto \{2n \ 2n-1 \ 2n-2 \ldots \ 1 \ 0\}_D.
\]

Namely, there is no $\{2n+1 \ 2n-1 \ 2n-2 \ldots \ 1 \ 0\}_D$.

In both cases, the latter parameter does not give a unipotent representation in Theorem 1.1. This explains the discrepancy between the even and odd $b_n$ and $d_n$ in the formulas of Corollary 2.6.

3. Spin Lowest $K$-type

We now investigate the spin lowest $K$-type of all the scattered representations, which must be of the form

\[
\text{Ind}^G_P(\pi_L) = \text{Ind}^G_{\prod_i GL(a_i) \times G'} \left( \bigotimes_i \mathbb{C}_{\chi_i} \otimes \pi_O \right) \in \hat{G}^{sc},
\]

according to Theorem 1.1. Here each $\mathbb{C}_{\chi_i}$ has Zhelobenko parameter $A_i := A_{T_i, t_i}$. By switching the order of the Levi factors if necessary, we assume that $T_i + t_i \geq T_j + t_j$ for all $i \leq j$.

By induction in stages, consider

\[
\pi_A := \text{Ind}^{GL(\bigcup a_i)}_{\prod_i GL(a_i)} \left( \bigotimes_i \mathbb{C}_{\chi_i} \right).
\]
The spin lowest $K$-type of $\pi_A$ is equal to $V^{A}_{(\theta_1; \ldots; \theta_l)}$, where $(\theta_1; \ldots; \theta_l)$ is given in Section 2 of [DW] (our choice of ordering in the previous paragraph guarantees that it is a dominant weight). Also, the spin lowest $K$-type of $\pi_O$ is given in Sections 5.4 – 5.6 of [BP1]. More precisely, it is equal to $V^{K'}_{\theta_O}$, where

$$\theta_O = \begin{cases} 
((k + n - 1)^2(k + n - 3)^2 \ldots (k - n + 1)^20^{k-n}) & \text{if } O = [2k + 1, 2n] \text{ in Type B} \\
(0^2) & \text{if } O = [2n] \text{ in Type C} \\
(n^10^{n-1}) & \text{if } O = [2n - 1, 1] \text{ in Type C} \\
(n^20^{n-2}) & \text{if } O = [2n] \text{ in Type D} \\
((n + k - 1)^1(n + k - 2)^1 \ldots (n - k)^10^{n-k}) & \text{if } O = [2n, 2k - 1, 1] \text{ in Type D}
\end{cases}$$

where $(n_1^1, n_2^2, \ldots)$ is the shorthand of $(n_1, n_1, n_2, \ldots, n_2, \ldots)$. In all cases, we separate the non-zero and zero coordinates of $\theta_O$ by writing $\theta_O = (\theta_O^+, 0, \ldots, 0)$.

**Theorem 3.1.** Let $\pi \in \tilde{G}^{sc}$ be of the form of (7). Then the spin lowest $K$-type of $\pi$ is obtained as follows:

(i) Take $V^{A}_{(\theta_1; \ldots; \theta_l)}$ and $V^{K'}_{\theta_O}$ as given in (7).

(ii) For each $1 \leq i \leq l$, construct $\mu_i$ from $\theta_i$ by the following:

- If $A_i$ is linked with $X_O$, we have two possibilities:
  - Suppose $A_i$ and $X_O$ are linked by
    $$A_i = \{A_1, \ldots, A_{q-p}, A_{q-p+1}, \ldots, A_q\}$$
    $$\{X_1, \ldots, X_{p}, X_{p+1}, \ldots, X_r\} = X_O,$$
    take $\nu_i := (p, p-1, \ldots, 1)$, and $\mu_i$ is obtained from $\theta_i$ by adding $\nu_i$ on the $p$ coordinates $((A_1 + A_q)/2, \ldots, (A_1 + A_q)/2, \ldots)$ of $\theta_i$.
  - Suppose $A_i$ and $X_O$ are linked by
    $$A_i = \{A_1, \ldots, A_q\}$$
    $$\{X_1, \ldots, X_{p}, X_{p+1}, \ldots, X_r\} = X_O,$$
    (this includes the case $\{1\}$), take $\nu_i := (p, p-1, \ldots, p - q + 1)$ and $\mu_i$ is obtained from $\theta_i$ by adding $\nu_i$ on $\theta_i = ((A_1 + A_q)/2, \ldots, (A_1 + A_q)/2)$.

- If $A_i$ is not linked with $X_O$, take $\mu_i = \theta_i$.

(iii) Suppose $A_{i_1}, \ldots, A_{i_j}$ are the chains that are linked to $X_O$ such that $i_1 < i_2 < \cdots < i_j$.

Then the spin LKT of $\pi$ is given by $V^K_{\mu}$, where $\mu := (\mu_1, \ldots, \mu_i; \mu_O)$.

**Example 3.2.** Consider the scattered representation

$$\text{Ind}_{GL(4) \times GL(1) \times GL(2) \times SO(17)}^{G} (C_{(15, 15, 15)} \otimes C_{(8)} \otimes C_{(5, 5)} \otimes \pi_{[15, 2]})$$
with Zhelobenko parameter:
\[
\{18 16 14 12\} \quad \{8\} \quad \{6 4\} \\
\{13 11 9 7\} \quad 5 \quad 3 \quad 2 \quad 1\}
\]

By our ordering of \(A_i\), we label the chains corresponding to \(GL(4)\), \(GL(1)\) and \(GL(2)\) by \(A_1, A_2\) and \(A_3\), respectively. By [DW], the spin lowest \(K\)-type of \(GL\) part is equal to its lowest \(K\)-type, which is \(V^A_{(\theta_1,\theta_2,\theta_3)} = V^A_{(15,15,15;5,5)}\), and the unipotent representation \(\pi_{[15,2]}\) has spin lowest \(K\)-type \(V^K_{(7,7;0,0,0,0,0)}\). Then \(A_1, A_2\) and \(A_3\) are all linked to \(X_O\), with \(\nu_1 = (1)\) on the \(GL(4)\) coordinates, \(\nu_2 = (3)\) on the \(GL(1)\) coordinates, and \(\nu_3 = (5,4)\) on the \(GL(2)\) coordinates. So \(\mu = (16,15,15;11;10,9;7,7,5,4,3,1,0,0)\).

Note that \(\{\mu - \rho\} = (2,3,2,\frac{3}{2},\frac{3}{2},\frac{3}{2},\frac{3}{2},\frac{3}{2},\frac{3}{2},\frac{3}{2},\frac{3}{2},\frac{3}{2},\frac{3}{2},\frac{3}{2},\frac{3}{2},\frac{3}{2})\) and hence
\[
\{\mu - \rho\} + \rho = \{\mu - \rho\} + \left(\frac{29}{2},\frac{27}{2},\ldots,\frac{3}{2},\frac{1}{2}\right) = (18,16,14,13,12,11,9,8,7,6,5,4,3,2,1) = 2\lambda
\]
satisfies (6).

**Example 3.3.** Let \(G = Sp(6,\mathbb{C})\), then the six scattered representations in Example 2.7 along with their spin lowest \(K\)-types are given by

| Parameters | Scattered Representations | LKT | Spin LKT |
|------------|--------------------------|-----|----------|
| \{5 3 1\}_C | \(\pi_{[5,1]}\) | (1,0,0) | (3,0,0) |
| \{6 4 2\}_C | \(\pi_{[6]}\) | (0,0,0) | (0,0,0) |
| \{1\} \cup \{4 2\}_C | \text{Ind}_{GL(1) \times Sp(4)}^G(C_{(1)} \otimes \pi_{[4]}) | (1,0,0) | (3,2,0) |
| \{2\} \cup \{3 1\}_C | \text{Ind}_{GL(1) \times Sp(4)}^G(C_{(2)} \otimes \pi_{[3,1]}) | (2,0,0) | (3,2,1) |
| \{3\} \cup \{4 2\}_C | \text{Ind}_{GL(1) \times Sp(4)}^G(C_{(3)} \otimes \pi_{[4]}) | (3,0,0) | (4,1,0) |
| \{3 1\} \cup \{2\}_C | \text{Ind}_{GL(2) \times Sp(2)}^G(C_{(2,2)} \otimes \pi_{[2]}) | (2,2,0) | (3,2,1) |

For instance, the scattered representation with parameter \(\{2\} \cup \{3 1\}_C\) has \(\theta_1 = (2)\) and \(\theta_O = (2,0)\). Here \(\nu_1 = (1)\), so \(\mu = (2 + 1;2,0 + 1) = (3,2,1)\).

In order to prove Theorem 3.1 it suffices to show that \(V^K_{\mu}\) appears in \(\pi\), and that \(\mu\) satisfies (6). Indeed, by the main result of [BDW], then \(V^K_{\mu}\) would be the unique spin-lowest \(K\)-type in \(\pi\) appearing with multiplicity one. The rest of the manuscript is devoted to proving these two results.

**Proposition 3.4.** Let \(\pi\) be a scattered representation of the form given in (7). Then \([\pi |_{K} : V^K_{\mu}] > 0\).

**Proof.** Note that we have an inclusion of \(M \cap K\)-types
\[
V^A_{(\theta_1,\ldots,\theta_t)} \otimes V^K_{\theta_O} \subseteq (\pi_A \otimes \pi_O)|_{M \cap K},
\]
where \(M = GL(\sum_i a_i) \times G^t\) is a maximal Levi subgroup of \(G\) containing the maximal compact torus \(T \leq K\), and \(\pi_A\) is as defined in (8). Therefore we have the inclusion of \(K\)-types
\[
\text{Ind}_{Q \cap K}^K(V^A_{(\theta_1,\ldots,\theta_t)} \otimes V^K_{\theta_O}) \subseteq \text{Ind}_{Q \cap K}^K((\pi_A \otimes \pi_O)|_{M \cap K}) \cong \pi |_{K},
\]
where \( Q = MN \) is the parabolic subgroup such that the roots of \( n \) are all contained in \( \Delta_G^+ \).

So to prove the proposition, it suffices to check that

\[
[\text{Ind}_{Q \cap K}^K(V^A_{\theta_1, \ldots, \theta_l} \boxtimes V^K_{\theta_\emptyset}) : V^K_\mu] > 0.
\]

This would follow immediately from (9), Lemmas 3.5 and 3.9.

For induced representations, we have a Blattner-type formula

(9)

\[
[\text{Ind}_{Q \cap K}^K(V^A_{\theta_1, \ldots, \theta_l} \boxtimes V^K_{\theta_\emptyset}) : V^K_\mu] = \sum_{m \in \mathbb{N}} \sum_i (-1)^i [(V^A_{\theta_1, \ldots, \theta_l}) \boxtimes V^K_{\theta_\emptyset}] \otimes S^m(n \cap \mathfrak{t}) : H^i(n \cap \mathfrak{t}, V^K_\mu),
\]

where

(10)

\[
H^i(n \cap \mathfrak{t}, V^K_\mu) = \bigoplus_{\{w \in W | \ell(w) = i, \langle w(\mu + \rho), \alpha^\vee \rangle > 0, \forall \alpha \in \Delta^+_M \}} V^{W \cap K}_{w(\mu + \rho) - \rho}.
\]

The Lie algebra cohomology formula (10) was due to Kostant [K]. The formula (9) can be seen by looking at the Weyl character formula of the restricted representation \( V^K_{\mu} \mid_{M \cap K} \).

As \( \text{GL}(\sum_{i} a_i \otimes \mathbb{C}^x) \) contains a copy of \( \mathbb{C} \), for each classical Type with

\[
x = \begin{cases} 
2 \cdot \text{rank}(G') + 1 & \text{for Type B} \\
2 \cdot \text{rank}(G') & \text{for Type C, D}.
\end{cases}
\]

As \( \text{GL}(\sum_{i} a_i) \times \text{GL}(x) \)-modules, \( S^k(\mathbb{C} \sum_{i} a_i \otimes \mathbb{C}^x) \) contains a copy of

\[
V^A_{(k_1, k_2, \ldots, k_j, 0, \ldots, 0)} \boxtimes V^{U(x)}_{(k_1, k_2, \ldots, k_j, 0, \ldots, 0)},
\]

where \( k_1 \geq \cdots \geq k_j \geq 0 \) and \( k = \sum_i k_i \).
Suppose \( j \leq \min\{\sum_i a_i, \text{rank}(G')\} \). By restricting the \( U(x) \)-module \( V_{(k_1, \ldots, k_j, 0, \ldots, 0)}^{U(x)} \) to \( K' \), it must have a copy of \( V_{(k_1, \ldots, k_j, 0, \ldots, 0)}^{K'} \) with multiplicity one.

For any \( v \in \mathbb{R}^n \), let \( |v| \) be the sum of the coordinates of \( v \). By taking \( k = |\nu_j| + \cdots + |\nu_1| \) with each \( \nu_t \) being given by Theorem 3.1, we conclude that \( S^k(n \cap \mathfrak{k}) \) contains a copy of \( V_{(\nu_j, \ldots, \nu_1, 0, \ldots, 0)}^A \otimes V_{(\nu_j, \ldots, \nu_1, 0, \ldots, 0)}^K \).

On the \( GL(\sum_i a_i) \) factor, we will show in Lemma 3.7 that
\[
[V_{A_{\theta_1; \ldots; \theta_l}}^A \otimes V_{A_{\nu_j; \ldots; \nu_1, 0, \ldots, 0}}^A : V_{\mu_{\gamma_1; \ldots; \gamma_l}}^A] \geq 1.
\]
On the \( G' \) factor, by using Section 2.1 of [HTW], we have that
\[
[V_{\theta_1; \ldots; \theta_l}^{K'} \otimes V_{\nu_j; \ldots; \nu_1, 0, \ldots, 0}^{K'} : V_{\mu_{\gamma_1; \ldots; \gamma_l}}^{K'}] \geq 1.
\]
Indeed, the number of non-zero entries of \( \mu_{\gamma} = (\theta_{\gamma, \nu}; \nu_1; \ldots; \nu_1; 0; \ldots, 0) \) is upper bounded by \( \text{rank}(G') \). Thus Sections 2.1.2 and 2.1.3 of [HTW] apply, and give the following lower bound for the left hand side of (13):
\[
\epsilon_{\theta_{\gamma, \nu}; (\nu_1; \ldots; \nu_1; 0; \ldots, 0)}^{\mu_{\gamma}}.
\]
One sees that this Littlewood-Richardson coefficient equals to one, and (13) follows.

To summarize, we have
\[
\begin{align*}
&\left[ (V_{A_{\theta_1; \ldots; \theta_l}}^A \otimes V_{A_{\nu_j; \ldots; \nu_1, 0, \ldots, 0}}^A) \otimes S^k(n \cap \mathfrak{k}) : H^0(n \cap \mathfrak{k}, V_{\mu}^{K}) \right] \\
\geq &\left[ (V_{A_{\theta_1; \ldots; \theta_l}}^A \otimes V_{A_{\nu_j; \ldots; \nu_1, 0, \ldots, 0}}^A) \otimes (V_{A_{\nu_j; \ldots, \nu_1, 0, \ldots, 0}}^A \otimes V_{A_{(\nu_j, \ldots, \nu_1, 0, \ldots, 0)}}^K) : V_{\mu}^{M \cap K} \right] \\
\geq &\left[ V_{A_{\mu_1; \ldots; \mu_l}}^A \otimes (V_{\theta_{\gamma, \nu}; \mu_{\gamma}}^A \otimes V_{A_{\nu_j; \ldots; \nu_1, 0, \ldots, 0}}^K) : V_{\mu}^{M \cap K} \right] \\
\geq &\left[ V_{A_{\mu_1; \ldots; \mu_l}}^A \otimes V_{\mu_{\gamma}}^{K'} : V_{\mu}^{M \cap K} \right] = 1,
\end{align*}
\]
where the second inequality uses (12), while the third inequality uses (13). Hence the result follows.

**Remark 3.6.** From the proof of the above lemma, one can also check that any \( V_{A_{\gamma_1}}^A \otimes V_{K'}^{A_{\gamma_2}} \) appearing in \( S^m(n \cap \mathfrak{k}) \) must have \( |\gamma_1| \geq |\gamma_2| \) for all \( m \geq 0 \).

**Lemma 3.7.** In the setting of Lemma 3.7, the inequality (12) holds.

**Proof.** By the Littlewood-Richardson Rule as stated on page 420 of [GW], it suffices to find one \( L-R \) skew tableaux of shape \((\mu_1; \ldots; \mu_l)/(\theta_1; \ldots; \theta_l)\) and weight \((\nu_j, \ldots, \nu_1, 0, \ldots, 0)\) in the sense of Definition 9.3.17 of [GW].

Construct the Ferrers diagram \((\mu_1; \ldots; \mu_l)/(\theta_1; \ldots; \theta_l)\). Counting from top to bottom, its row sizes are equal to \((\nu_1, \nu_2, \ldots, \nu_j)\).

We now fill each partition \( \nu_t \) in \((\mu_1; \ldots; \mu_l)/(\theta_1; \ldots; \theta_l)\) for each \( 1 \leq t \leq j \) as follows: Let \( T \) be the semi-standard Young tableau whose shape is given by the partition \((\nu_j, \ldots, \nu_2, \nu_1)\), and the entries of \( k \)-th row of \( T \) are all equal to \( k \). Take a sequence of sub-tableaux of \( T \)
\[
T_1 \subset T_2 \subset \cdots \subset T_j = T
\]
such that $T_t$ has the same shape as $(\nu_{t_1}, \nu_{t_2}, \ldots, \nu_{t_t})$ for all $1 \leq t \leq j$. We now look at the skew-tableau $T_t/T_{t-1}$ (where $T_0$ is the empty tableau). By construction, the column sizes of $T_t/T_{t-1}$ is the same as that of $\nu_{t_i}$.

Fill the $k$-th row of the partition $\nu_{t_i}$ in $(\mu_1; \ldots; \mu_l)/(\theta_1; \ldots; \theta_l)$ by the $k$-th entries on the columns of $T_t/T_{t-1}$ counting from the top in ascending order. Due to the construction in Theorem 3.1, this will give us a semi-standard skew tableau of shape $(\mu_1; \ldots; \mu_l)/(\theta_1; \ldots; \theta_l)$ and weight $(\nu_{t_1}, \ldots, \nu_{t_i}, 0, \ldots, 0)$ (see Definition 9.3.16 of [GW]), which is a reverse lattice word by Definition 9.3.17 of [GW]. Therefore, it is a desired L-R skew tableaux and the proof finishes. □

**Example 3.8.** Let us come back to Example 3.2 where $l = 3$ and

$$(\mu_1; \mu_2; \mu_3) = (16, 15, 15, 11, 10, 9), \quad (\theta_1; \theta_2; \theta_3) = (15, 15, 15, 8, 5, 5).$$

Recall that $\nu_{t_1} = (1), \nu_{t_2} = (3), \nu_{t_3} = (5, 4)$. So the skew Ferrers diagram $(\mu_1; \mu_2; \mu_3)/(\theta_1; \theta_2; \theta_3)$ looks like:

```

```

To fill in the entries of the above diagram, consider

$${T_1 = \begin{array}{c} 1 \\ 2 \end{array} \subseteq T_2 = \begin{array}{ccc} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{array} \subseteq T_3 = T,}$$

where the highlighted blocks are $T_t/T_{t-1}$ for $t = 1, 2, 3$. Note that the column sizes of $T_t/T_{t-1}$ are the same as that of $\nu_{t_i}$. This leads us to the following tableau:

```

```

Note that the row word of $T$ is 2334112231121, which is a reverse lattice word. Thus $T$ is an L-R skew tableau of shape $(\mu_1; \mu_2; \mu_3)/(\theta_1; \theta_2; \theta_3)$ and weight $(5, 4, 3, 1, 0, 0, 0, 0)$. Now the Littlewood-Richardson Rule guarantees that

$$[V^{A}_{(15,15,15;8,5,5)} \otimes V^{A}_{(5,4,3,1,0,0,0)} : V^{A}_{(16,15,15,11,10,9)}] \geq 1.$$
Lemma 3.9. For all $i > 0$ and $m \geq 0$, we have

\[(V_{\theta_i:;\theta_i}^A \boxtimes V_{\theta_\infty}^{K'}) \otimes S^m(n \cap \mathfrak{t}) : H^i(n \cap \mathfrak{t}, V^K_\mu) = 0.\]

Proof. Let $\theta_A := (\theta_1; \ldots; \theta_i)$. By Remark 3.6, the $M \cap K$-types appearing on the left hand side of (14) must be of the form

\[V^A_{\theta_A + \gamma_1} \boxtimes V^{K'}_{\theta_\infty + \gamma_2}, \quad |\gamma_1| \geq |\gamma_2|,\]

where $\gamma_1$ and $\gamma_2$ consists solely of non-negative integers. We claim that for all $i > 0$, if $H^i(n \cap \mathfrak{t}, V^K_\mu)$ consists of $M \cap K$-types of the form

\[V^A_{\theta_A + \delta_1} \boxtimes V^{K'}_{\theta_\infty + \delta_2}\]

where $\delta_1, \delta_2$ consists only of non-negative integers, then $|\delta_1| < |\delta_2|$.

Indeed, recall from the construction of $\mu$ in Theorem 3.1 that

\[\mu = (\theta_A + \nu_1; \theta_\infty + \nu_2), \quad |\nu_1| = |\nu_2|\]

If $w \in W$ is such that $w(\mu + \rho) = (\beta_1; \beta_2)$ is regular on $\Delta^+_\mu$, $\beta_1, \beta_2$ are obtained by the following:

(a) Take any sub-collection of $q$ entries inside $\mu + \rho$, and assign either $+$ or $-$ to each coordinate;

(b) $\beta_1$ is obtained by rearranging the $q$ coordinates chosen in (a) in descending order;

(c) For the remaining $r$ entries of $\mu + \rho$, rearrange in descending order and get $\beta_2$.

It is obvious that if $w(\mu + \rho) - \rho = (\beta_1; \beta_2) - \rho$ contributes to any multiplicities in (14), the entries of $\beta_1$ must be all positive, i.e., we always assign $+$ in Step (a) above. So we focus on $w \in W$ consisting of transpositions only, which implies that $|w(\mu + \rho) - \rho| = |\mu|$.

Therefore, if $w$ is not identity, then the sum of the first $q$ coordinates of $w(\mu + \rho) - \rho$ must be strictly less than $\theta_A + \nu_1$, and the sum of the last $r$ coordinates of $w(\mu + \rho) - \rho$ must be strictly greater than that of $\theta_\infty + \nu_2$. This proves our claim, and the lemma follows immediately.

The proof of Theorem 3.1 ends with:

Proposition 3.10. Equation (15) holds for $V^K_\mu$, i.e., $\{\mu - \rho\} = 2\lambda - \rho$.

Proof. In our construction of $V^K_\mu = V^K_{(\mu_1; \ldots; \mu_i; \mu_\infty)}$, the coordinates of Type A chains $\mu_i$ are determined in exactly the same way as in Algorithm 2.2 of [DW] (this is true also when it is linked to $\mathcal{X}_\infty$). Hence the proof in the [DW] applies to all $\mu_i$ appearing in $\mu$.

We now focus on studying the coordinates corresponding to $\theta_\infty \Rightarrow \mu_\infty$. For convenience, we reorder $\mathcal{A}_i$ (if necessary) such that

(15)

\[
\{A_1\} \quad \{A_2\} \quad \ldots \ldots \quad \{A_k\} \quad \{X_1, \ldots, X_p, \ldots, \ldots, X_r\} = \mathcal{X}_\infty.
\]
In particular, we have \( r = \text{rank}(G') \).

Let \( \lambda_\mathcal{O} \) be such that \( 2\lambda_\mathcal{O} \) is equal to the coordinates of \( X_\mathcal{O} \). By Sections 5.4 – 5.6 of [BPI],

\[
2\lambda_\mathcal{O} - \rho_r = \{ \theta_\mathcal{O} - \rho_r \},
\]

where \( \rho_k \) is half sum of the positive roots in the Dynkin diagram of Lie type \( X_k \) \((X = B, C, D)\).

We keep track of the difference between \( 2\lambda_\mathcal{O} - \rho_r \) and the last \( r \) coordinates in \( 2\lambda - \rho \), along with the difference between \( \{ \theta_\mathcal{O} - \rho_r \} \) and the last \( r \) coordinates of \( \{ \mu - \rho \} \). If their differences are equal, then the proposition follows from (16).

By subtracting \( 2\lambda \) given in (15) with \( \rho \), to see that the difference between \( 2\lambda_\mathcal{O} - \rho_r \) and the last \( r \) coordinates in \( 2\lambda - \rho \) is equal to

\[
((\nu_k; \ldots; \nu_1; 0, \ldots, 0)^t; 0, \ldots, 0),
\]

where \( \nu_i \) are determined in Theorem 3.1, and \( \mathbf{p}^t \) is the transpose of the partition \( \mathbf{p} \) by switching the rows of \( \mathbf{p} \) into columns. In other words, if \( \mathbf{p} = (\alpha_1, \ldots, \alpha_{r-Z}) \), then

\[
\mathbf{p}^t := (\beta_1, \ldots, \beta_{r-Z}), \quad \text{where } \beta_i = \# \{ j \mid \alpha_j \geq i \} \quad \forall i \geq 1.
\]

On the other hand, note that \( 0 \leq s := \text{number of coordinates of } \theta_{\mathcal{O},+} \leq Z \leq r \), and

\[
\{ \theta_\mathcal{O} - \rho_r \}
= \{ (\theta_{\mathcal{O},+}; 0, \ldots, 0) - (p_1, \ldots, p_r) \}
= \{ (\theta_{\mathcal{O},+} - (p_1, \ldots, p_s); (0, \ldots, 0) - (p_{s+1}, \ldots, p_{s+r-Z}); (0, \ldots, 0) - (p_{r-(Z-s)+1}, \ldots, p_r)) \}
= \left( \begin{array}{c}
\underbrace{p_{s+1}, \ldots, p_{s+r-Z}}_{r-Z} ; \underbrace{p_{r-(Z-s)+1}, \ldots, p_r}_{Z} ; (\theta_{\mathcal{O},+} - (p_1, \ldots, p_s))\end{array} \right),
\]

where \( \rho_r = (p_1, p_2, \ldots, p_r) \). The coordinates of \( \{ \theta_{\mathcal{O},+} - (p_1, \ldots, p_s) \} \) appear in the last coordinates in the dominant form above, since its coordinates are equal to either \( \frac{1}{2} \) or 0 by the definition of \( \theta_{\mathcal{O}} \) above and direct calculation.

Meanwhile, the last \( r \) coordinates of \( \{ \mu - \rho \} \) are equal to:

\[
((\nu_k; \ldots; \nu_1; 0, \ldots, 0) - (p_{s+1}, \ldots, p_{s+r-Z}); p_{r-(Z-s)+1}, \ldots, p_r; (\theta_{\mathcal{O},+} - (p_1, \ldots, p_s)))
\]

Writing \( \ell := (p_{s+1}, \ldots, p_{s+r-Z}) \), the difference between \( \{ \theta_\mathcal{O} - \rho_r \} \) and the last \( r \) coordinates of \( \{ \mu - \rho \} \) is equal to

\[
(\ell - ((\nu_k; \ldots; \nu_1; 0, \ldots, 0) - \ell); 0, \ldots, 0)
\]

Therefore, the proposition follows if one can show that

\[
\ell - ((\nu_k; \ldots; \nu_1; 0, \ldots, 0) - \ell) = (\nu_k; \ldots; \nu_1; 0, \ldots, 0)^t
\]
or equivalently

\[(17) \quad \{\ell - (\nu_k; \ldots; \nu_1; 0, \ldots, 0)\} = \ell - (\nu_k; \ldots; \nu_1; 0, \ldots, 0)^t.\]

The coordinates of \(\ell\) in \((17)\) can be translated by any fixed integer as long as the coordinates inside the bracket on the left remain non-negative. In particular, we can prove \((17)\) holds by replacing \(\ell\) with \(\rho_{r-Z}\).

For simplicity, we only prove \((17)\) in Type \(C\) with \(\ell = \rho_{r-Z} = (r-Z, \ldots, 2, 1)\). We claim that for all partitions \(p = (p_1, \ldots, p_m)\) such that \(p_1 \leq m\) and all positive entries of \(p\) are distinct (for example, \(p = (\nu_k; \ldots; \nu_1; 0, \ldots, 0)\)), then

\[\{\ell - p\} = \ell - p^t.\]

Indeed, \(\ell/p\) defines a skew partition by hypothesis, and the row sizes and and column sizes give the sizes of \(\{\ell - p\}\) and \(\ell - p^t\) respectively. So we need to show that \(\ell/p\) have the same row and column sizes.

Mark the \((i, j)\)-block of \(\ell/p\) by \((m+2) - (i + j)\), so that the leftmost entry of each row of \(\ell/p\) gives the size of the row, and the topmost entry of each column of \(\ell/p\) gives the size of the column. We now define a one-to-one map between these two sets as follows: If the leftmost block of a row of \(\ell/p\) is also the topmost block of a column, then we have a natural one-to-one map between the entries corresponding to these two blocks.

Suppose that the leftmost block of a row of \(\ell/p\) is not the topmost block of any column, then (since \(p\) is a strictly decreasing partition) it must occur at the first column of \(p\). More precisely, if \(p = (p_1 > \cdots > p_t > p_{t+1} = 0 \ldots p_m = 0)\), then the entry of \((t+1,1)\)-block of \(\ell/p\) is \(m - t\), and the blocks below it cannot be the topmost block of a column, which have entries

\[\{1, 2, \ldots, m - t - 1\}.\]

On the other hand, we study which topmost blocks of a column are not the leftmost blocks of any row. Namely, suppose the entries of the \(t\)th-row of \(\ell - p\) are \(v_i, v_i - 1, \ldots, v_i-1\), then the blocks corresponding to \(v_i - 1, v_i - 2, \ldots, v_i-1\) are the topmost blocks of some columns that are not the leftmost blocks of row \(i\). Collecting such entries, we have

\[(19) \quad \bigcup_{i=1}^{t+1}\{v_i - 1, v_i - 2, \ldots, v_i-1\} = \{v_{t+1} - 1, v_{t+1} - 2, \ldots, 2, 1\}.\]

Note that the entry \(v_{t+1}\) appears at the \((t+1,1)\)-block, so \(v_{t+1} = m - t\) as in the previous paragraph, and consequently we have a natural one-to-one correspondence between \((18)\) and \((19)\), and the result follows.

**Example 3.11.** Here is an example of the claim above with \(\ell = \rho_{10}\) and \(p = (10, 7, 5, 4, 1)\). The skew tableau \(\ell/p\) is given by the numbered blocks below:
The blocks with circled entries are both the leftmost block of a row and topmost block of a column of $\ell/p$. Also, the blocks with bold entries are the leftmost block of a row but not the topmost block of a column (and vice versa for the entries with an asterisk). Then there is an obvious bijection between the bold entries and the entries with an asterisk. This implies that the rows sizes of $\ell/p$ is equal to the columns sizes of $\ell/p$, which are both \{5, 5, 4, 3, 3, 3, 2, 2, 1\}, and hence we have $\ell - p = \ell - p'$.

4. ON A CONJECTURE OF HUANG

Let us investigate Conjecture 13.3 of [H] raised by Huang in 2015.

**Conjecture 4.1.** ([H]) A unitary representation either has nonzero Dirac cohomology or is induced from a unitary representation with nonzero Dirac cohomology by parabolic induction.

**Example 4.2.** Let $G = \text{Sp}(6, \mathbb{C})$. Fix a positive root system $\Delta^+_G$ so that it has simple roots \{\(e_1 - e_2, e_2 - e_3, 2e_3\}\). Consider the spherical irreducible unitary representation $J(\lambda, \lambda)$ with $\lambda = (\frac{5}{2}, \frac{3}{2}, \frac{1}{2})$. This is the metaplectic representation $\pi_{\text{even}}$ described in Section 5.5 of [BP1]. As computed there, $H_D(\pi_{\text{even}}) = 0$.

We claim that $\pi_{\text{even}}$ cannot be parabolically induced from any unitary representation with non-zero Dirac cohomology. Indeed, if there exists such a representation, say $\pi_L$, then $\pi_{\text{even}} = \text{Ind}_{G}^{G}(\pi_L)$.

Since the infinitesimal character $2\lambda = (5, 3, 1)$ of $\pi_{\text{even}}$ is dominant, integral and regular for $\Delta^+_G$, one would conclude from Theorem 2.4 of [BP1] that $H_D(\pi_{\text{even}}) \neq 0$, contradiction. Thus the claim holds, and $\pi_{\text{even}}$ violates Conjecture 4.1.

More generally, there are other unipotent representations in $G = \text{Sp}(2n, \mathbb{C})$ violating the conjecture. Consider the unipotent representations corresponding to a cuspidal special nilpotent orbit $O \subset \mathfrak{g}$ not equal to the trivial orbit. For example, one can take $O = [4m_1, 4m_2, \ldots, 4m_k]$ with integers $m_1 > m_2 > \cdots > m_k > 0$. Then its Dirac cohomology is zero since $h^\vee = 2\lambda$ in Equation (6) is singular (here $h^\vee$ is the semisimple element of a Jacobson-Morozov triple of the Lusztig-Spaltenstein dual $O^\vee \subset \mathfrak{g}^\vee$). On the other hand, it is not parabolically induced from any representations $\pi$ given in Theorem 1.1 tensored with a unitary character of Type A, or else its associated variety (in this case) will be a non-cuspidal nilpotent orbit. □
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