Integrals of tau functions I: Round dance tau function and multi-matrix integrals

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Abstract

The simplest nontrivial tau functions of the Toda lattice and the N-component Toda lattice are compared in their applications to multimatrix integrals.

1 Introduction

I want to dedicate this article to Andrei Pogrebkov in connection with his 75th birthday. Andrei has made notable and varied contributions to the development of the theory of integrable systems - the vertex operator (the work by Pogrebkov and Sushko 1974, late published [1]), correctly defined Hamiltonian formalism, the inverse problem of scattering theory, a detailed study of a number of specific integrable equations, and a number of other often very original approaches to integrability theory. The text below represents a slightly different direction, but I am grateful for the always inspiring discussions with Andrey on this topic.

2 The simplest nontrivial tau functions of the multi-component TL

The well-known Cauchy-Littlewood relation can be written as

\[ e^{-\sum_{m>0} \frac{1}{m!} p_m \tilde{p}_m} = 1 + \sum_{\kappa>0} \sum_{\alpha_1 \geq \ldots \geq \alpha_\kappa} \sum_{\beta_1 \geq \ldots \geq \beta_\kappa} s_{(\alpha|\beta)}(p) s_{(\beta|\alpha)}(\tilde{p}) = \tau_o(p, \tilde{p}) \]  

(2.1)

Here \( p = (p_1, p_2, \ldots) \) and \( \tilde{p} = (\tilde{p}_1, \tilde{p}_2, \ldots) \) are two infinite sets of variables and \( s_{(\alpha|\beta)} \) denotes the Schur polynomial [2] in the power sum variables labeled by a partition \( (\alpha|\beta) \) written in the Frobenius coordinates \( \alpha = (\alpha_1, \ldots, \alpha_\kappa) \) and \( \beta = (\beta_1, \ldots, \beta_\kappa) \). This is also the simplest example of the Toda lattice (TL) [3] tau functions [4], [5] where the variables

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\( \mathbf{p} \) and \( \tilde{\mathbf{p}} \) play the role of the so-called higher times. \(^{*}\) It is written in form of series in the Schur functions as it was worked out in \([6]\), \([7]\). The straightforward generalization of this expression to the N-component Toda lattice (N-TL) yields

\[
\tau_\alpha(\mathbf{p}^1, \tilde{\mathbf{p}}^1, \ldots, \mathbf{p}^N, \tilde{\mathbf{p}}^N) = 1 + \sum_{\kappa > 0} \sum_{\mathbf{\alpha}_1 > \cdots > \mathbf{\alpha}_k} \cdots \sum_{\mathbf{\alpha}_N > \cdots > \mathbf{\alpha}_N} \prod_{1 \leq i \leq N} \mathcal{S}(\alpha^{i} | \beta^{i}) (\mathbf{p}^{i})^{\kappa \cdot \cdots \cdot \sum_{1 \leq i \leq N} \mathcal{S}(\beta^{i} | \alpha^{i+1}) (\tilde{\mathbf{p}}^{i}) \quad (2.2)
\]

where each set of the higher times is denoted by \( \mathbf{p}^i \) and \( \tilde{\mathbf{p}}^i \), \( i = 1, \ldots, N \). The symbol \( \circ \) above the symbol of the product means that we suppose the condition \( \alpha^{N+1} = \alpha^1 \), which closes the chain of Schur functions in a circle which explains the motivation of the name “round dance”. I like this expression and it somehow reminds me of Matisse’s painting “Dance”. There is a beautiful expression for series (2.2), similar to the left side in (2.1), see formula (38) in \([8]\).

### 3 Graphs and matrix models

Consider the set of complex \( GL_N \)-matrices \( Z_1, \ldots, Z_n \) and their Hermitian conjugates \( Z_{-1}, \ldots, Z_{-n} \), where \( Z_{-i} := Z_i^\dagger \). We multiply each \( Z_i \) by \( C_i \in GL_N \) on the right, \( Z_i \to (Z_i C_i) \), where \( i = \pm 1, \ldots, \pm n \). The matrices \( Z_i \) will be called random, and the matrices \( C_i \) will be called source ones. The elements of random matrices are distributed according to the Gauss law as follows:

\[
\langle (Z_a)_{ij} (Z_{a'})_{i'j'} \rangle = \frac{1}{N} \delta_{a+a',0} \delta_{ii'} \delta_{jj'} \quad (3.1)
\]

Such correlation functions are obtained for the so-called Ginibre complex multi-matrix ensembles \([9]\), \([11]\), \([10]\). To calculate various correlation functions, we use Wick’s rule. We will study special correlation functions, which are obtained as follows:

We denote the collection of random matrices by \( Z \). We consider the set of \( GL_N \) matrices

\[
W_1(Z), \ldots, W_v(Z)
\]

where each \( W_a(Z) \) is the product of \( Z_i C_i \) pairs. Each product \( W_a(Z) \) is defined up to a cyclic permutation. We set the condition: each pair enters one and only once to one of the members of the set \( \{ W_a(Z), a = 1, \ldots, v \} \). In this case, there is a two-dimensional orientable surface \( \Omega \) and an embedded graph \( \Gamma \) with \( n \) numbered edges and with \( \Gamma \) numbered vertices, drawn on \( \Omega \) as follows. Each edge labeled \( \mid i \rangle \ (\mid i \rangle = 1, \ldots, n) \) consists of two half-edges with numbers \( i \) and \( -i \) and associate the half-edge \( i \) with the matrix \( Z_i \). Each vertex labeled \( a \) is connected to \( W_a(Z) \) as follows: if we move clockwise, then the outgoing half-edges will have the same numbers as the random matrices that are included in the product

\(^{*}\)For higher times, other designations are sometimes used, for example, \( nx_n \) instead of \( p_n \), the sign can be also different. To fix the correspondence we write the Appendix.

\(^{†}\)For embedded graphs, there are other names: “thick graphs” and “ribbon graphs” in physical literature and “maps” in mathematical literature.
$W_a(Z)$, if we read it from left to right. Then we number the corners of the faces of the graph as follows: when traversing a vertex clockwise, each half-edge with the number $i$ ($i = \pm 1, \ldots, \pm n$) is followed by an angle with the same number. Then in corner $i$ we place $C_i$. We will use the notation $W_a(I)$ which denote $W_a(Z)$ where we put all matrices $Z_i$ to be the identity matrix. We see that $W_a(I)$ can be called the monodromy of the vertex $a$ and $W_a(Z)$ the monodromy dressed by random matrices. Suppose that the Euler characteristic of $\Omega$ is $\chi = f - n + v$, here $f$ is the number of faces of $\Gamma$. Now we will number all the faces of $\Gamma$. We also assign a monodromy to each face which is a matrix defined up to the cyclic order. Monodromy $W_a^*(I)$ of the face with number $b$ is introduced as the product of all source matrices $C_i$ at the corners of the face in the order in which we encounter them when going around the face counterclockwise. The dressed monodromy of the face $W_a(Z)$ is obtained from $W_a(I)$, when we replace each matrix $C_i$ that enters $W_a(I)$ by $Z_i C_i$.

It can be seen that the dressed monodromies of faces of $\Gamma$ is the dressed monodromy of the vertices of the dual graph $\Gamma^*$. This construction was analyzed in the works [13], [14], [15], [16].

**Integrals of products of the Schur functions** For any sets of partitions $\{\lambda^a, a = 1, \ldots, v\}$ and $\{\lambda^b, b = 1, \ldots, v\}$ we have

$$\langle \prod_{a=1}^{v} s_{\lambda^a}(W_a(Z)) \rangle = c \delta_{\lambda} \prod_{b=1}^{v} s_{\lambda}(W_b^*(I))$$

(3.2)

$$\langle \prod_{b=1}^{v} s_{\lambda^b}(W_b^*(Z)) \rangle = c \delta_{\lambda} \prod_{a=1}^{v} s_{\lambda}(W_a(I))$$

(3.3)

$$c = \left( \frac{\text{dim } \lambda}{|\lambda|!} \right)^{-n} N^{-n|\lambda|}$$

(3.4)

where $\text{dim } \lambda$ is the dimension of the irreducible representation of the symmetric group $S_d$ ($d = |\lambda|!$) labeled by $\lambda$. The symbol $\delta_{\lambda}$ is equal to 1 if all partitions are equal to each other: $\lambda^1 = \lambda^2 = \cdots = \lambda$, otherwise $\delta_{\lambda} = 0$. Notice the diagonalization property for the products of the Schur function when we evaluate the expectation $\langle \rangle$.

**Remark 3.1.** These nice relations can be compared with the relations for embedded graphs [12]

$$\sigma \alpha = \varphi$$

(3.5)

$$\sigma \varphi = \alpha$$

(3.6)

where, $\sigma, \alpha, \varphi \in S_{2n}$ and $\sigma$ is the involution without fix points, $\alpha$ is the product of cycles related to vertices and $\varphi$ is the product of cycles.

For a given matrix $X$ and a partition $\mu = (\mu_1, \ldots, \mu_\ell)$ we introduce the notation

$$p_\mu(X) = \text{tr}(X)^{\mu_1} \cdots \text{tr}(X)^{\mu_\ell}$$

(3.7)
One can associate $p_\mu(X)$ with a set of polygons consisting of $\mu_1$-gons,...,$\mu_\ell$-gons. Suppose that an $i$-gon is included in this set $m_i$ times. Introduce number $\text{Aut}(\mu) := \prod_{i>0} m_i!^{m_i}$, which can be called the number of automorphisms of this collection: number of permutations polygons with the same number of edges multiplied by the number of rotations of each polygon.

One can derive (3.2)-(3.3) in different ways, in particular [13], starting from the relation

$$\langle \prod_{b=1}^{F} \frac{p_{\mu_b}(W_b^*Z)}{|\text{Aut}(\mu_b)|} \rangle = \sum_{\nu_1,\ldots,\nu_\nu} \mathcal{H}_\Omega(\mu^1,\ldots,\mu^F,\nu^1,\ldots,\nu^\nu) \prod_{a=1}^{v} p_{\nu_a}(W_a(I))$$  \hspace{1cm} (3.8)

where all partitions $\mu^1,\ldots,\mu^F,\nu^1,\ldots,\nu^\nu$ are of the same weight, say $d$, $d \leq N$. Here $\mathcal{H}_\Omega$ is the Hurwitz number, $\Omega$ the base surface and partitions $\mu^1,\ldots,\mu^F,\nu^1,\ldots,\nu^\nu$ are the ramification profiles.

The meaning of the formula (3.8) is as follows. For all partitions $\mu^1,\nu^1,\ldots$ equal to the partition (1), this formula describes the base surface, glued from polygons associated with $\{\text{tr}(W_b^*Z), b = 1,\ldots,F\}$, see (3.6). Expression for $p_\mu$ corresponds to the set of polygons on the covering surface, which correspond to the set of profiles $\mu^b$ in the center (in the ”capital”) of the polygon $b$. Wick’s rule gives all possible gluings of the collection of polygons given by $p_\nu$. Each way of gluing leads to a set $\{\nu^a, a = 1,\ldots,v\}$ of branching profiles at the vertices. The number of ways to glue polygons (up to the automorphisms) for given sets of all ramification profiles $\mu^1,\ldots,\mu^F,\nu^1,\ldots,\nu^\nu$ is the Hurwitz number in its geometric meaning.

Then from the Frobenius formula for the Hurwitz numbers and from the orthogonality relations of the characters symmetric group, we get (3.3) and (3.2). Another way to get (3.3) and (3.2) - consistent use of well-known formulas for integrating Schur functions (using analytic continuation of parameters), see for instance [15].

If instead of the Ginibra ensemble we consider ensembles of unitary matrices (circular ensembles), then the relations (3.2) and (3.3) will change slightly, in particular, the dimension of the representation of the symmetric group will have to be replaced by the dimension of the representation of the linear group. On the related topics see also [17] and [18].

Actually we need only formula (3.2) for the next part.

## 4 Round dance tay function and multimatrix Ginibre ensemble

Now consider the construction of the previous section and in formula (2.2) (where we replace $N$ by $v$ to avoid a mess in the usage of the capital $N$) we choose the times $\tilde{p}_a$ as follows

$$\tilde{p}_k^{(a)} = \text{tr}(W_a(Z))^k, \quad a = 1,\ldots,v$$  \hspace{1cm} (4.1)
We get
\[
\langle \tau(p^1, \tilde{p}^1), \ldots, p^v, \tilde{p}^v) \rangle = \sum_{\lambda} \left( \frac{\dim \lambda}{|\lambda|!} \right)^{-n} N^{-n|\lambda|} \prod_{a=1}^{v} s_{\lambda}(p^a) \prod_{b=1}^{F} s_{\lambda}(W^*_b(I))
\] (4.2)

The series (4.2) appeared in the works [19], [20], [21], [22], [23], [24] as a generating function for Hurwitz numbers. To achieve these series we chose the condition (4.1).

Note that, apparently, the right-hand side can only be a tau function of the hypergeometric type of the TL hierarchy [25], [26], [27], [28]. It’s not an exact statement, but I don’t see any other possibilities at the moment. That is, there may be a tau function of the form
\[
\sum_{\lambda} s_{\lambda}(p^1)s_{\lambda}(p^2) \prod_{(i,j) \in \lambda} r(j - i)
\] (4.3)

with some function $r$.

What are our chances of getting it. First, the Euler characteristic of the base surface $\Omega$ must be equal to two, therefore we have a graph $\Gamma$ on a Riemann sphere. Then only two sets of free parameters should remain. Let’s look at all the possibilities, there are three of them:

1) two sets among $p^a$ are free, all others have a form
\[
p^i = p(a_i) := (a_i, a_i, \ldots), \quad i = 1, \ldots, k
\] (4.4)

(in the considered case $k \leq v$) or coincide with
\[
p^i = p_\infty := (1, 0, 0, \ldots), \quad i = 1, \ldots, v - k
\] (4.5)

where $\alpha$ is a complex number. In this case the spectrum of matrices $W^*_b$, $b = 1, \ldots, F$ should be
\[
\text{Spectr} [W^*_b(I)] = (1, \ldots, 1, 0, \ldots, 0)
\] with some $N_b$. In this case the right hand side of (4.2) is as follows
\[
\sum_{\lambda} N^{-n|\lambda|} s_{\lambda}(p^1)s_{\lambda}(p^2) \prod_{(i,j) \in \lambda} \prod_{c=1}^{k} (a_c + j - i) \prod_{b=1}^{v} (N_b + j - i)
\] (4.7)

About specializations (4.4), (4.5) see [27], [17], and see [15], [16] on (4.6).

Among these tau functions, one can be found, which was studied in [29]. The authors use rectangular matrices $Z$, in our approach this is the choice of source matrices which may degenerated. The graph $\Gamma^*$ in this case is just an open chain.

Tau functions of type (4.7) for the first time were used in [30], then in [31] in the context of combinatorial problems and also in [32], [33], [34] as the generating functions of Hurwitz numbers of certain type.
We have one free set, say, $p^1$, all others have forms (4.5) and (4.4). We have a single free monodromy, say, $W^*_1$, all others have their spectrum as in (4.6). In this case the right hand side of (4.2) is

$$
\sum_{\lambda} N^{-n|\lambda|} s_{\lambda}(p^1) s_{\lambda}(p^2(W^*_1(I))) \prod_{(i,j) \in \lambda} \prod_{c=1}^{k+1} (a_c + j - i) \prod_{b=2}^{v} (N_b + j - i) \quad (4.8)
$$

It is written as a function of the set of the higher times $p^1$ and as a function in the so-called Miwa variables given by the spectrum of $W^*_1(I)$

(3) We two free monodromies, say, these are $W^*_1(I)$ and $W^*_2(I)$ while the spectrum of all others are given by (4.6). Each $p^a$ is either (4.4) or (4.5). In this case we have the right hand side of (4.2) written in Miwa variables as

$$
\sum_{\lambda} N^{-n|\lambda|} s_{\lambda}(W^*_1(I)) s_{\lambda}(p^2(W^*_2(I))) \prod_{(i,j) \in \lambda} \prod_{c=1}^{k+2} (a_c + j - i) \prod_{b=3}^{v} (N_b + j - i) \quad (4.9)
$$

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A Partitions and Schur functions

The partition is a set of integers $\lambda = (\lambda_1, \ldots, \lambda_\ell)$, where $\lambda_1 \geq \cdots \geq \lambda_\ell > 0$ and $\lambda_i$ are called parts of $\lambda$. The sum of the parts is called the weight of $\lambda$ and is usually denoted as $|\lambda|$. There is a well-known notion of Young diagrams $\lambda$ whose row lengths are equal to parts of $\lambda$. We accept the agreement as in [2]. Let us denote the horizontal lengths of the part of the strings that starts at the $i$th node (and does not include it) and goes to the right as $\alpha_i$, and the vertical part of the column that starts at (not including) the same node and goes down we denote $\beta_i$. In Frobenius coordinates, $\lambda$ is written as $(\alpha|\beta) = (\alpha_1, \ldots, \alpha_k|\beta_1, \ldots, \beta_k)$, where $k$ is the number of nodes on the main diagonal of the Young diagram, see [2] for details.
The following relations define Schur polynomials (Schur functions) as a function of power sum variables \( \mathbf{p} = (p_1, p_2, \ldots) \):

\[
e^{\sum_{k>0} \frac{1}{k} p_k x^k} = \sum_{k \geq 0} s_{(k)}(\mathbf{p})
\]

\[
s_\lambda(\mathbf{p}) = \det \left[ s_{(\lambda_i - i + j)}(\mathbf{p}) \right]_{i,j}
\]

In case \( p_m = \text{tr}(X)^m \) one can write \( s_\lambda(\mathbf{p}(X)) =: s_\lambda(X) \)

We have

\[
s_\lambda(\mathbf{p}) = \frac{\dim \lambda}{|\lambda|!} \sum_\mu \varphi_\lambda(\mu) \mathbf{p}_\mu, \quad \dim \lambda = \prod_{i<j} (\lambda_i - \lambda_j - i + j) \prod_i (\lambda_i - i + \ell)!.
\]

The known orthogonality relations for the factors \( \varphi_\lambda \) (which are the normalized characters of the symmetric group in representation \( \lambda ) [2] are as follows:

\[
\zeta_\Delta \sum_\lambda \left( \frac{\dim \lambda}{d!} \right)^2 \varphi_\lambda(\mu) \varphi_\lambda(\Delta) = \delta_{\Delta,\mu} \tag{A.1}
\]

and

\[
\left( \frac{\dim \lambda}{d!} \right)^2 \sum_\Delta \zeta_\Delta \varphi_\lambda(\Delta) \varphi_\mu(\Delta) = \delta_{\lambda,\mu} \tag{A.2}
\]

where \( d = |\Delta| = |\lambda| \) and if each part \( i \) enters \( m_i \) times the partition \( \Delta \) we define

\[
\zeta_\Delta = \prod_i m_i! m_i! \tag{A.3}
\]

(The same number was denoted \( |\text{Aut}(\Delta)| \) in the paragraph "Integrals of products of the Schur functions"). All details can be found in [2].

The Frobenius formula for Hurwitz numbers mentions in Section 3 is

\[
\mathcal{H}_\Omega(\Delta^1, \ldots, \Delta^m) = \sum_\lambda \left( \frac{\dim \lambda}{|\lambda|!} \right)^E \varphi_\lambda(\Delta^1) \cdots \varphi_\lambda(\Delta^m)
\]

where \( E \) is the Euler characteristic of the base surface \( \Omega \), the partitions \( \Delta^1, \ldots, \Delta^m \) is the set of the ramification profiles in branch points, see the textbook [12] for the details.

## B Fermions. Multicomponent tau function

I will only briefly give the notation and some formulas. Fermion operators \( \psi_n^{(a)}, \psi_n^{(d)} \) where \( a = 1, \ldots, N, n \in \mathbb{Z} \), satisfy the relations

\[
[\psi_n^{(a)}, \psi_m^{(b)}]_+ = 0 = [\psi_n^{(a)}\dagger, \psi_m^{(b)}\dagger], \quad [\psi_n^{(a)}\dagger, \psi_m^{(b)}]_+ = \delta_{a,b}\delta_{n,m}
\]

and

\[
\langle 0 | \psi_n^{(a)} = \langle 0 | \psi_{-n-1}^{(a)} = 0 = \psi_{-n-1}^{(a)} | 0 \rangle = \psi_n^{(a)} | 0 \rangle, \quad n > 0
\]
The wonderful observation of Kyoto school (see for instance [4]) is the following relation
\[ s_{(\alpha|\beta)}(\mathbf{p}^a) = (-1)^{\sum_{i=1}^{k} \alpha_i} \langle 0 | e^{\sum_{k>0} \frac{1}{k} p_k^{(a)} J_k^{(a)}} \sum_{i=1}^{k} \psi_{\alpha_i}^{(a)} \psi_{\beta_i - 1}^{\dagger (a)} | 0 \rangle, \quad (B.1) \]
which is also known as bosonization formula (a version of this formula). Here
\[ J_n^{(a)} = \sum_{i \in \mathbb{Z}} \psi_i^{(a)} \psi_{n+i}^{\dagger (a)} \]
. We also need the property \( s_{(\alpha|\beta)}(\mathbf{p}) = (-1)^{|(\alpha|\beta)|} s_{(\beta|\alpha)}(-\mathbf{p}) \).
Following [4] we introduce 2N-component KP (N-component Toda) tau function in form of a fermionic expectation value which we choose as follows:
\[ \langle 0 | e^{\sum_{a} \sum_{k>0} \frac{1}{k} p_k^{(a)} J_k^{(a)}} e^{\sum_{1 \leq a \leq 2N} \sum_{i=1}^{2N} \psi_{i-1}^{(a)} \psi_{i}^{(a+1)} \psi_{i}^{(a+1)} \psi_{i-1}^{(a)} | 0 \rangle =: \tau_{\sigma}(\mathbf{p}^1, \ldots, \mathbf{p}^N) \quad (B.2) \]
where \( \circ \) means that \( \psi_{(2N+1)}^{(1)} := \psi_{(1)}^{(1)} \). We call it round dance tau function. Expanding the exponential in the last expression in Taylor series and using the relations above, we get (2.2).
In case \( N = 1 \) we get the right hand side of (2.1).

C Complex Ginibre ensemble

On this subject there is an extensive literature, for instance see lists of references in [10][11][13][14][16][18][29].
Let us consider integrals over \( N \times N \) complex matrices \( Z_1, \ldots, Z_n \) where the measure is defined as
\[ d\Omega(Z_1, \ldots, Z_n) = c_N^n \prod_{i=1}^{N} \prod_{a,b=1}^{N} \mathcal{d}\mathbb{R}(Z_i)_{ab} \mathcal{d}\mathbb{I}(Z_i)_{ab} e^{-N |(Z_i)_{ab}|^2} \quad (C.1) \]
where the integration domain is \( \mathbb{C}^{N^2} \times \cdots \times \mathbb{C}^{N^2} \) and where \( c_N^n \) is the normalization constant defined via \( \int d\Omega(Z_1, \ldots, Z_n) = 1 \). We get (3.1) for the correlation functions of the entries of the matrices \( Z_1, \ldots, Z_n \).

The set of \( n \ N \times N \) complex matrices with measure (C.1) is called the set of \( n \) independent complex Ginibre ensembles. Such ensembles have wide applications in physics and in information transfer theory.

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