AN EXPLICIT THEORY OF $\pi_{1}^{\text{un,crys}}(\mathbb{P}^1 - \{0, \mu_N, \infty\})$

I : Explicit computation of the Frobenius

I-3 : The number of iterations of the Frobenius viewed as a variable

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Abstract. Let $X_0$ be a curve $\mathbb{P}^1 - (\{0, \infty\} \cup \mu_N) / \mathbb{F}_q$, with $N \in \mathbb{N}^*$ and $\mathbb{F}_q$ of characteristic $p$ prime to $N$ and containing a primitive $N$-th root of unity. We establish an explicit theory of the crystalline pro-unipotent fundamental groupoid of $X_0$.

In this part I, we compute explicitly the Frobenius and in particular the periods associated with it, i.e. cyclotomic $p$-adic multiple zeta values.

In I-1 and I-2 the number of iterations of the Frobenius was a fixed integer $\alpha \in \mathbb{N}^*$. In this I-3, we study how the Frobenius depends on $\alpha$ viewed as a $p$-adic variable. One of the applications is to obtain a natural indirect computation of the variant of cyclotomic $p$-adic multiple zeta values that expresses the Frobenius-invariant path at $(\vec{1}_1, \vec{1}_0)$.

The results are expressed by defining a counterpart of the harmonic Ihara action which is different from the two ones defined in I-2.

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1. INTRODUCTION

1.1. Let $p$ be a prime number, $N \in \mathbb{N}^*$ an integer prime to $p$, and $R = W(F_q)$, the ring of Witt vectors of a finite field of characteristic $p$ which contains a primitive $N$-th root of unity. Let $\sigma$ be the order of $p$ in $(Z/NZ)^\times$, thus we have $q \in (p^\sigma)^\times$; in order to simplify certain statements, we will assume, without loss of generality, that $q = p^\sigma$. We denote by $\xi$ a primitive $N$-th root of unity in $R$, we denote also by $z_i = \xi^i$ for $i \in \{1, \ldots, N\}$, and by $K = \text{Frac}(R)$. Let $X$ be the curve $\mathbb{P}^1 - \{(0, \infty) \cup \mu_N \}/ R$, and $X_K$ be the base change of $X$ to $K$.

In this part I, we compute explicitly the Frobenius structure of the pro-unipotent De Rham fundamental groupoid $\pi_1^{\text{un,DR}}(X_K)$, in the sense of Deligne [D], §11 (or, alternatively, the version of this object defined by Chiarellotto and Le Stum [CL], or the one defined by Shiho [S1, S2]), in particular the periods associated with it, cyclotomic $p$-adic multiple zeta values.

1.2. The data of $\pi_1^{\text{un,DR}}(X_K)$ equipped with its Frobenius structure is the following, by [D], §11. Let us fix the tangential base-point $I_0$ as the origin of the paths of integration. Thus we consider the bundle of paths of $\pi_1^{\text{un,DR}}(X_K)$ starting at $I_0$; it is equipped with the universal unipotent connexion denoted by $\nabla_{KZ}$. Let $\sigma$ be the Frobenius of $R$, and $X^{(p)}$ the pull-back of $X$ by $\sigma$. Let $X_K$, resp. $X^{(p)}_K$ be the base change of $X$ resp. $X^{(p)}$ to Spec$(K)$. We have a pull-back by Frobenius $F^{\ast}_{X/K} \pi_1^{\text{un,DR}}(X_K^{(p)})$ of the de Rham fundamental groupoid of $X_K^{(p)}$, constructed analytically, and an isomorphism of groupoids
\[
(F_{X/K})_{\ast} : \pi_1^{\text{un,DR}}(X_K) \rightarrow F^{\ast}_{X/K} \pi_1^{\text{un,DR}}(X_K^{(p)})
\]
which is horizontal with respect to the connexions on the bundle of paths starting at the chosen base-point. The Frobenius $\phi$ of $\pi_1^{\text{un,DR}}(X_K)$ is defined as the inverse of $F_* = (F_{X/K})_{\ast}$ (D, §13.6, §13.13).

In I-1 and I-2, we have fixed an integer $\alpha \in \mathbb{N}^*$, and we have computed explicitly the Frobenius iterated $\alpha$ times, that is to say, depending on the choice of convention, either $(p^\alpha)^{\text{weight}}$, $\phi^\alpha$ or $F^{\alpha}$. Computing the Frobenius reduced to computing its restriction to canonical paths, that is to say a couple $(\mathbb{L}_1^{\prime}_{p,\pm,\alpha}, \zeta_{p,\pm,\alpha})$ (the objects indexed by $(p, +\alpha)$ are defined through $(p^\alpha)^{\text{weight}}, \phi^\alpha$, and the ones indexed by $(p, -\alpha)$ are defined through $F^{\alpha}$), where $\mathbb{L}_1^{\prime}_{p,\pm,\alpha}$ is an overconvergent function on the rigid analytic affinoid subspace $U_{\frac{\alpha}{\infty}} = ([\mathbb{P}^1_{\alpha} - \cup_{i=1}^N z_i]) / K$ of $\mathbb{P}^1_{\alpha} / K$, and $\zeta_{p,\pm,\alpha}$ denotes cyclotomic $p$-adic multiple zeta values.

The results of I-1 and I-2 were expressed in terms of weighted multiple harmonic sums, that are essentially the coefficients of the series expansion at 0 of the fundamental solution of $\nabla_{KZ}$ $(s_d, \ldots, s_1 \in \mathbb{N}^*)$:
\[
\text{hnr}_n \left( \frac{z_{id+1}}{s_d}, \ldots, \frac{z_1}{s_1} \right) = n^{s_d + \ldots + s_1} \sum_{0 < n_1 < \ldots < n_d < n} \left( \frac{z_{id+1}}{z_i} \right)^{n_1} \ldots \left( \frac{z_{id+1}}{z_{id}} \right)^{n_d} \left( \frac{1}{\sigma_{id+1}} \right)^n \in \mathbb{Q}[\xi]
\]

1.3. Our goal here is, on the contrary to I-1 and I-2 where $\alpha$ was fixed, to study the Frobenius as a function of $\alpha$, viewed as a variable $p$-adic integer. The concrete meaning of this natural question is the following.

Aside from $\zeta_{p,\pm,\alpha}$ introduced above, there exists another type of cyclotomic $p$-adic multiple zeta values, sometimes denoted by $\zeta_{p,\pm,\alpha}^{KZ}$; whereas $\zeta_{p,\pm,\alpha}$ expresses the action of Frobenius on the canonical De Rham path at $(\tilde{I}_1, \tilde{I}_0)$, $\zeta_{p,\pm,\alpha}^{KZ}$ expresses the unique Frobenius-invariant path at $(\tilde{I}_1, \tilde{I}_0)$; the two objects are related to each other by a simple formula which can be expressed in terms of the multiplication in a motivic Galois group, and are conjecturally arithmetically equivalent. The distinction between $\zeta_{p,\pm,\alpha}$ and $\zeta_{p,\pm,\alpha}^{KZ}$ is similar to the one between two types of hyperlogarithms which already appeared before: the overconvergent ones, $\mathbb{L}^{\prime}_{p,\alpha}$, which we studied in I-1 and in I-2, and $\mathbb{L}^{KZ}_{p,\alpha}$, characterized as the unique Coleman function solution to $\nabla_{KZ}$ satisfying certain asymptotic conditions, which appeared only as a
tool up to now. Here, the couple \((Li_p^{KZ}, \zeta_p^{KZ})\) will provide the natural way to express how the Frobenius varies in function of \(\alpha\).

1.4. We will proceed in the following way. In I-1, we solved directly the equation of horizontality of \(F\) with respect to \(\nabla_{KZ}\). In I-2, we wrote a kind of 'Frobenius for weighted multiple harmonic sums', and we expressed it in two ways: one by taking a certain limit of the Frobenius structure of \(s_\alpha^{un,\text{DR}}(X_K)\), and one in what we call the RT setting, in a way that involves exclusively multiple harmonic sums; the identification between the coefficients of the two computations provided an indirect computation of \((Li_{p,\alpha}^1, \zeta_{p,\alpha})\), identified as the coefficients of the Frobenius of weighted multiple harmonic sums. The notation RT applies to objects arising from computations on multiple harmonic sums, and is intended as a short for rational, Rham-Taylor and rigid-Taylor; indeed, multiple harmonic sums are rational numbers in the case of \(\mathbb{P}^1 - \{0, 1, \infty\}\) (and algebraic in the case of \(\mathbb{P}^1 - \{0, \mu_N, \infty\}\)), equal, up to a factor \(\frac{1}{\text{degree}}\), to Taylor coefficients of the fundamental solution to the connexion \(\nabla_{KZ}\) on the De Rham fundamental groupoid, and finally, certain equations that we write contain implicitly the Frobenius structure, which is defined in the rigid framework. In this paper, we will organize the proofs again according to a distinction between two types of computations and the comparison between them as in I-2.

In I-2, we viewed the De Rham fundamental groupoid of \(X_K\) not as a groupoid of affine schemes, but as a groupoid in the category of complete topological \(K\)-algebras. This will again be the case in this I-3, except that this topological setting will be here more important. Indeed, we will see that, in this context and using the Ihara multiplication, one can define a notion of contraction mapping, and that the Frobenius is a contraction in this sense; this will imply instantly certain properties of the Frobenius as a function of \(\alpha\) involving its fixed point. As in I-2, we will proceed in two steps and first restrict ourselves to tangential base points \((\overline{1}_z, \overline{0}_0)\), \(z \in \mathbb{Z} - \{0, \infty\}\), then place ourselves on the bundle of paths starting at \(\overline{0}_0\).

Again as in I-2, we will focus on a particular property of multiple harmonic sums, here, the study of prime weighted multiple harmonic sums \(\text{har}_{p,\alpha}\) as a function of \(\hat{\alpha}\) viewed as a \(p\)-adic integer; and we will identify to each other the coefficients of the De Rham and 'RT' computations. What the comparison will provide will be an indirect computation of \(\zeta_p^{KZ}\); although \(\zeta_p^{KZ}\) could be already computed via I-2 its relation to \(\zeta_{p,\alpha}\) for any \(\alpha \in \mathbb{N}^*\), we will obtain here a computation of \(\zeta_p^{KZ}\) which will be natural, and which will have more consequences.

The parameter of the computations will actually not be \(\hat{\alpha} \in \mathbb{N}^*\) but a couple \((\hat{\alpha}_0, \hat{\alpha}) \in (\mathbb{N}^*)^2\) where \(\hat{\alpha}_0 \hat{\alpha}\) : instead of considering that we iterate \(\hat{\alpha}\) times \(\phi^q\) we will consider that we iterate \(\hat{\alpha}_0\) times \(\phi^{\text{point}}\).

Aside from giving more general formulas, this also has specific applications. Indeed, for prime weighted multiple harmonic sums \(\text{har}_{p,\alpha}\), the nature of \((\hat{\alpha}_0, \hat{\alpha})\) is ambiguous: we can also view \(\hat{\alpha}_0\) (instead of \(\hat{\alpha}\)) as the variable power of Frobenius, and the variable \(q^\alpha\) as a restriction of the variable upper bound \(n \in \mathbb{N}\) of the underlying iterated sum. Thus, this degree of generality helps us to study \(\text{har}_{p,\alpha}\) as a function of \(n\) viewed as a \(p\)-adic integer. This question is only partially new, because in I-1, a central technical point was the study of the variant 'regularized by Frobenius' of multiple harmonic sums (the coefficients \(\text{har}_{n,\alpha}\) of the series expansion of \(Li_{p,\pm,0}^n\) at 0 of degree \(n \in \mathbb{N}\)) as a function of \(n\) viewed as a \(p\)-adic integer. This point of view on \((\hat{\alpha}_0, \hat{\alpha})\) will reappear in part V-1.

As in I-2, we proceed in two steps and consider first the Frobenius at tangential base-points \((\overline{1}_z, \overline{0}_0)\), \(z \in \mathbb{Z}\) (which gives Theorem I-3.a), and then the Frobenius on the bundle of paths starting at \(\overline{0}_0\) (which gives Theorem I-3.b).

1.5. The first step in the above sense was made in I-2 by defining two variants of the Ihara action, called the 'DR - RT harmonic Ihara action', \(\phi^{\text{DR-RT}}\), and the 'RT harmonic Ihara action', \(\phi^{\text{RT}}\); we used them
to express in two different ways a Frobenius for weighted multiple harmonic sums and we related them to each other.

Here, the central objects, that we are going to define, are the following: a "De Rham harmonic Ihara action" $\hat{c}^{\text{DR}}_{\text{har}}$ (Proposition-Definition 3.3.3), which can be easily compared to $\hat{c}^{\text{DR}-\text{RT}}_{\text{har}}$ (Proposition 3.4.3), and two "harmonic" variants of the operation of elevation to the $\hat{a}$-th power of the Ihara action weighted by $p^{o}$ (in the sense of Definition 2.3.1): a DR variant $e_{\hat{a}p^{o}}^{\text{DR}}$ (Definition 3.5.6) and a RT variant $e_{\hat{a}p^{o}}^{\text{RT}}$ (Definition 3.5.1).

For all the paper, it will be convenient to consider the Euclidean division of $\alpha$ by $\circ : \alpha = o\tilde{\alpha} + r$, to write $p^{r} = q^{a}p^{r}$ and to focus on the variable $\tilde{\alpha}$. By Theorem I-2.a (I-2, §1), we have, with the notations introduced in I-2,

\[
(h \hat{a} p^{r})_{\tilde{\alpha}} \in \mathbb{N}^{*} \text{ such that } \tilde{\alpha}_{0} | \tilde{\alpha}.
\]

Theorem I-3.a Let $\tilde{\alpha}, \tilde{\alpha} \in \mathbb{N}^{*}$ such that $\tilde{\alpha}_{0}|\tilde{\alpha}$.

a) (DR) There exists a continuous free group action of $(\text{Ad}\bar{\Pi}_{1,0}(K)_{\Sigma}(e_{1}), c^{\text{DR}}_{\text{Ad}})$, the "DR harmonic Ihara action" $c^{\text{DR}}_{\text{har}} : \text{Ad}\bar{\Pi}_{1,0}(K)_{\Sigma}(e_{1}) \times K(\langle e_{Z} \rangle) \rightarrow K(\langle e_{Z} \rangle)$

which extends to map $K(\langle e_{Z} \rangle)_{\Sigma} \times K(\langle e_{Z} \rangle)_{\text{har}} \rightarrow K(\langle e_{Z} \rangle)_{\text{har}}$

such that we have the analytic expansion (equality of functions of $\frac{\hat{a}}{\alpha_{0}} \in \mathbb{N}$):

\[
(1.5.1) \quad \hat{a}p^{r} = \sum_{z \in \mathbb{Z}^{\{0, \infty\}}} z^{-1}(x \mapsto zx) \left( \sum_{n \geq 0} \left( \text{pr}_{n+1}^{-1}(\hat{\Phi}_{q^{-\infty}}^{-1}c_{1}\hat{\Phi}_{q^{-\infty}}) c^{\text{DR}}_{\text{har}} \hat{\Sigma}_{\text{inv}}^{\text{DR}}(\hat{\Phi}_{q^{-\infty}}^{-1}c_{1}\hat{\Phi}_{q^{-\infty}}) \right)(q^{n})^{r} \right)
\]

i) (DR) There exists a map $e_{\hat{a}p^{o}}^{\text{DR}}_{\text{har}} : \text{Ad}\bar{\Pi}_{1,0}(K)_{\Sigma}(e_{1}) \rightarrow \mathbb{Q}_{p}[[\alpha]](\Lambda)(\langle e_{0}, e_{1} \rangle)_{\text{DR}}$, the (formal) "DR harmonic elevation to the $a$-th Ihara power weighted by $\Lambda$" such that, the map $e_{\hat{a}p^{o}}^{\text{DR}}_{\text{har}} : \bar{\Pi}_{1,0}(K)_{\Sigma} \rightarrow K(\langle e_{Z} \rangle)_{\text{har}}$ defined as the composition of $e_{\hat{a}p^{o}}^{\text{DR}}_{\text{har}}$ by the reduction modulo $(a - \frac{\hat{a}}{\alpha_{0}}, \Lambda - q^{ao})$ satisfies, at words $w$ such that $\frac{\hat{a}}{\alpha_{0}} > \text{depth}(w)$:

\[
(1.5.2) \quad \hat{a}p^{r} = \hat{c}^{\text{DR}}_{\text{har}} \hat{\Sigma}_{\text{inv}}^{\text{DR}}(\hat{\Phi}_{q^{-\infty}}^{-1}c_{1}\hat{\Phi}_{q^{-\infty}})
\]

ii) (RT) There exists a map $e_{\hat{a}p^{o}}^{\text{RT}}_{\text{har}} : (K(\langle e_{Z} \rangle)_{\text{RT}})_{\Sigma} \rightarrow \mathbb{Q}_{p}[[\alpha]](\Lambda)(\langle e_{Z} \rangle)_{\text{RT}}$, such that the map $e_{\hat{a}p^{o}}^{\text{RT}}_{\text{har}} : (K(\langle e_{Z} \rangle)_{\text{RT}})_{\Sigma} \rightarrow K(\langle e_{Z} \rangle)_{\text{DR}}$ defined as the composition of $e_{\hat{a}p^{o}}^{\text{RT}}_{\text{har}}$ by the reduction modulo $(a - \frac{\hat{a}}{\alpha_{0}}, \Lambda - q^{ao})$ satisfies,

\[
(1.5.3) \quad \hat{a}p^{r} = \hat{c}^{\text{RT}}_{\text{har}} \hat{\Sigma}_{\text{inv}}^{\text{RT}}(\hat{\Phi}_{q^{-\infty}}^{-1}c_{1}\hat{\Phi}_{q^{-\infty}})
\]

iii) (comparison) Let us fix $\tilde{\alpha}_{0}$, and view (1.5.1), (1.5.2), (1.5.3) as three expansions of a function $\mathbb{N}^{*} \rightarrow K(\langle e_{Z} \rangle)_{\text{har}}$ of $\frac{\hat{a}}{\alpha_{0}}$ in the ring $K(\langle p^{\alpha_{0}} \rangle)_{\text{har}}$. Then, the coefficients of these expansions are identical.

By iii) we get a natural indirect computation of the Frobenius-invariant path $\Phi_{q, \infty}$. 

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Example. In depth one, and for \( \mathbb{P}^1 - \{0, 1, \infty\} \) where \( \tilde{\alpha} = \alpha \) and \( \tilde{\alpha}_0 = \alpha_0 \), the o) and i) of the Theorem I-3.a give the same answer:

\[
\text{har}_{p^\alpha}(s) = \text{Ad}_{\Phi, \infty}(e_1) \left[ \frac{1}{1 - e_0} e_1 e_0^{-1} e_1 \right] + \sum_{b \geq 0} (p^\alpha)^{b+s} \text{Ad}_{\Phi, -\infty}(e_1) \left[ e_0^b e_1 e_0^{s-1} e_1 \right]
\]

whereas the ii) gives (with \( B_b^{L+b} = \frac{1}{L+b+1} B_{L+1} \)):

\[
\text{har}_{p^\alpha}(s) = \sum_{b \geq 1} \frac{p^\alpha(s+b) - 1}{p^\alpha(s+b) - 1} \sum_{L \geq -1} B_b^{L+b} \text{har}_{p^\alpha}(s + b + L)
\]

Thus the comparison gives (for all \( \alpha_0 \)):

\[
\text{Ad}_{\Phi, \infty}(e_1) \left[ e_0^b e_1 e_0^{s-1} e_1 \right] = \frac{1}{p^\alpha(s+b) - 1} \sum_{L \geq -1} B_b^{L+b} \text{har}_{p^\alpha}(s + b + L)
\]

It is in depth \( \geq 2 \) that the powers of \( \tilde{\alpha} \) appear in the formulas. One can introduce as in I-2 certain combinatorial tools to express the formulas for the two maps \( e_\text{har} \).

In part I-2, the harmonic Ihara actions \( \sigma_{\text{har}}^{\text{DR} - \text{RT}} \) and \( \sigma_{\text{har}}^{\text{RT}} \) were related to each other by comparison maps \( \Sigma_{\text{RT}} \) and \( \Sigma_{\text{inv}}^{\text{DR}} \), satisfying \( \Phi_{p, \alpha} = \Sigma_{\text{RT}} \text{har}_{p^\alpha} \), \( \text{har}_{p^\alpha} = \Sigma_{\text{inv}}^{\text{DR}} \Phi_{p, \alpha} \), and \( \Sigma_{\text{inv}}^{\text{DR} \circ \text{RT}} = \text{id} \). In the final version of this work we will show that the two maps \( e_\text{har} \) can be related to each other via the map \( \Sigma_{\text{RT}} \).

From the right-hand side of the formula (1.5.1), one can define a map \( \Sigma_{\text{inv}}^{\text{DR}, \text{iter}} \), such that (1.5.1) is reformulated as \( \text{har}_{p^\alpha} = \Sigma_{\text{inv}}^{\text{DR}, \text{iter}} \Phi_{p, \alpha} \). It is the variant of \( \Sigma_{\text{inv}}^{\text{DR}} \) which takes into account all the values of \( \tilde{\alpha} \) at the same time, and the information of how the Frobenius depends on its number of iterations. The following statement is subjacent to the computation of \( \Phi_{p, \alpha} \) through Theorem I-3-a, and can be viewed as an advantage of \( \Sigma_{\text{inv}}^{\text{DR}, \text{iter}} \) with respect to \( \Sigma_{\text{inv}}^{\text{DR}} \):

**Appendix to Theorem I-3.a** \( \Sigma_{\text{inv}}^{\text{DR}, \text{iter}} \) is injective.

The second step of our study is to consider the Frobenius on the bundle of paths starting at \( \tilde{I}_0 \). The result will follow easily from the Theorem I-3-a and from I-1. Let \( w \) be a word on the alphabet \( e_Z = \{ e_0, e_1, \ldots, e_N \} \). We recall from I-1 that the map \( n \in \mathbb{N} \mapsto \text{Li}_n^{\tilde{\alpha}}[w][z^n] \), where \( [z^n] \) means the coefficient of degree \( n \) in the power series expansion at \( 0 \), extends to a map \( \lim \mathbb{K}(U^{\text{an}}_{\text{inv}}) (\text{Li}_n^{\tilde{\alpha}, \tilde{\alpha}}[w]) : \lim \mathbb{Z}/p^n \mathbb{N} \mathbb{Z} \rightarrow \mathbb{K} \) which is locally analytic (see §3.5.2 for a more precise description of its features, in particular in terms of multiple harmonic sums).

**Theorem I-3.b** (see Corollary 3.5.2 for a precise statement) The map \( \text{Li}_n^{\tilde{\alpha}, \tilde{\alpha}} \) admits a power series expansion of the form \( \sum_{i \geq 0} C_i^{\tilde{\alpha}}(q^\alpha)^i \), where for each word \( w \), \( C_i^{\tilde{\alpha}}[w] \in \mathbb{K}(U^{\text{an}}_{\text{inv}}) \) and the maps \( n \in \mathbb{N}^* \mapsto C_i^{\tilde{\alpha}}[w][z^n] \) extend to locally analytic maps on \( \lim \mathbb{Z}/p^n \mathbb{N} \mathbb{Z} \), of the same type with the maps \( n \in \mathbb{N}^* \mapsto \text{Li}_n^{\tilde{\alpha}, \tilde{\alpha}}[w][z^n] \), with radius of local analyticity depending on \( \tilde{\alpha} \) but with coefficients independent of \( \tilde{\alpha} \).

**Outline.** The plan is similar to the one of I-2. The computations in the framework of the pro-unipotent fundamental groupoid are made in §2 and §3; the ones involving exclusively multiple harmonic sums in §4 and §5; we compare the results of the two types of computations in §6. The §2 and §4 are preliminaries; the cores of the results are obtained in §3 and §5.

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2. Properties of the Ihara action and of the Frobenius

We recall some definitions and facts about the fundamental groupoid, including the fixed points of Frobenius (§2.1 and §2.2), then we define the setting for studying in §3 the power of the Frobenius as a variable (§2.3 and §2.4).

2.1. Preliminaries.

2.1.1. Notations. Let $Z = \{(0, \infty) \cup \mu_N\}(K)$

**Notation 2.1.1.** For all $z \in Z$, we denote by $\Pi_{z,0} = \pi_1^{un,DR}(X_K, \tilde{\imath}_z, \tilde{\imath}_0)$.

Other notations: the canonical De Rham base-point, on $X_K$ and $X_K^{(\rho^*)}$, is denoted by $\omega_{DR}$; the shuffle Hopf algebra over the alphabet $e_Z = \{e_0, e_{z_1}, \ldots, e_{z_N}\}$, resp. $e_{Z(\rho^*)} = \{e_0, e_{z_1^{(\rho^*)}}, \ldots, e_{z_N^{(\rho^*)}}\}$ is denoted by $\mathcal{O}^{m.e_z}$, resp. $\mathcal{O}^{m.e_Z(\rho^*)}$. We have $\pi_1^{un,DR}(X_K, \omega_{DR}) = \text{Spec}(\mathcal{O}^{m.e_z})$, and $\pi_1^{un,DR}(X_K^{(\rho^*)}, \omega_{DR}) = \text{Spec}(\mathcal{O}^{m.e_Z(\rho^*)})$. We read the groupoid structure from the right to the left: $\pi_1^{un,DR}(X_K, y, x)$ is the scheme of paths $\text{from } x \text{ to } y$. The canonical De Rham path from $x$ to $y$ is denoted by $\gamma_1$.

The groupoid multiplication is a morphism as follows:

$$\pi_1^{un,DR}(X_K, z, y) \times \pi_1^{un,DR}(X_K, y, x) \rightarrow \pi_1^{un,DR}(X_K, z, x).$$

We denote by $\text{Ad}(\epsilon_z) : f \in \Pi_{z,0}(A) \rightarrow f^{-1}e_zf \in \mathcal{A}(\langle e_Z \rangle)$ (unlike in the usual convention $f^{-1}$ is on the left). For each $\lambda \in K^*$, the map $\tau(\lambda) : \mathcal{A}(\langle e_Z \rangle) \rightarrow \mathcal{A}(\langle e_Z \rangle)$ maps $f = \sum_w \text{word on } e_z \hat{f}[w]w$ to $\sum_w \text{word on } e_z \lambda^{\text{weight}(w)}f[w]w$.

For each $z \in Z \setminus \{0, \infty\}$, $\Pi_{z,0}$ is the subgroup scheme of $\Pi_{z,0}$ defined by the additional equations $f[e_z] = f[e_0] = 0$; the adjoint action $\text{Ad}(\epsilon_z)$ written above restricts to an injective map on each $\Pi_{z,0}(A)$.

2.1.2. Ihara action. For all $z \in Z \setminus \{0, \infty\}$, the Ihara action on $\Pi_{z,0}(A)$ is the map

$$\Pi_{z,0}(A) \times \Pi_{z,0}(A) \rightarrow \Pi_{z,0}(A)$$

$$(g_z, f_z) \mapsto g_z.f_z(e_0, g_z^{-1}e_1, \ldots, g_z^{-1}e_{z_N}g_z)$$

where $g_z = (z \mapsto \epsilon^*(z), (g_{z_i}))$ for all $i \in \{1, \ldots, N\}$. It extends to a map

$$\Pi_{z,0}(A) \times \mathcal{A}(\langle e_Z \rangle) \rightarrow \mathcal{A}(\langle e_Z \rangle).$$

The dual of the Ihara product $\circ^{DR}$ on $\Pi_{z,0}(A)$ is a map $\mathcal{O}^{m.e_Z} \rightarrow \mathcal{O}^{m.e_Z} \otimes \mathcal{O}^{m.e_Z}$, and one can deduce from it, by dualizing again, a map

$$\mathcal{A}(\langle e_Z \rangle) \times \mathcal{A}(\langle e_Z \rangle) \rightarrow \mathcal{A}(\langle e_Z \rangle)$$

In particular it applies to series that are not necessarily grouplike. We will denote again by $\circ^{DR}$ this extension. We defined in I-2 the adjoint Ihara action on $\Pi_{z,0}(A)$ : it is the unique map

$$\text{Ad}^{DR}_{\text{Ad}} : \text{Ad}_{\Pi_{z,0}(A)}(e_z) \times \text{Ad}_{\Pi_{z,0}(A)}(e_z) \rightarrow \text{Ad}_{\Pi_{z,0}(A)}(e_z)$$

satisfying, for all $g_z, f_z$

$$\text{Ad}_{g_z}(e_z) \circ^{DR}_{\text{Ad}} \text{Ad}_{f_z}(e_z) = \text{Ad}_{g_z \circ^{DR} f_z}(e_z)$$

It is given by

$$h_z \circ^{DR}_{\text{Ad}} f_z = f_z(e_0, h_{z_1}, \ldots, h_{z_N})$$

where $h_{z_i} = (z \mapsto \epsilon^*(z), (h_{z_i}))$ for all $i \in \{1, \ldots, N\}$.

Let us now take $U_{0,0}^{an}$ the rigid analytic affinoid space $(\mathbb{P}^{1,an} \setminus \cup_{i=1}^N \{z_i\})/K$, and let us consider the bundle of paths of $\pi_1^{un,DR}(X_K)$ starting at $\tilde{\imath}_0$, viewed as trivialized at $\tilde{\imath}_0$. The group of its rigid analytic sections on $U_{0,0}^{an}$ can be viewed as $\Pi_{0,0}(\mathfrak{A}(U_{0,0}^{an}))$. It is equipped with the generalized Ihara action

$$(\Pi_{0,0}(\mathfrak{A}(U_{0,0}^{an})) \times \Pi_{z,0}) \times (\Pi_{0,0}(\mathfrak{A}(U_{0,0}^{an})) \times \Pi_{z,0}) \rightarrow (\Pi_{0,0}(\mathfrak{A}(U_{0,0}^{an})) \times \Pi_{z,0})$$

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Proposition 2.2.1. ii) \(\Pi\) 

\[\langle\langle \xi\rangle\rangle, (g_{\alpha}) \mapsto (g_{\epsilon_0, g_{\epsilon_1}^{-1} g_{\epsilon_2}, \ldots, g_{\epsilon_N}^{-1} g_{\epsilon_{N^2}}), g_{\epsilon_1} f_{\epsilon_0, g_{\epsilon_1}^{-1} g_{\epsilon_2}, \ldots, g_{\epsilon_N}^{-1} g_{\epsilon_{N^2}}})\]

where again \(g_{\epsilon_i} = (z \mapsto \xi^i z), (g_{\epsilon_{N^2}})\) for all \(i \in \{1, \ldots, N\} \).

2.1.3. Topological framework. Most of the next following definitions and facts have been established in I-2, §2. Let \(A\) be a complete topological \(K\)-algebra. In practice, \(A\) can be equal to \(K\), to a ring of polynomials over \(K\), or to \(\mathfrak{A}(U_{\mu_{\infty}})\) equipped with the supremum norm attached to the sequences of coefficients of the series expansion at 0 of its elements. For \(n, d \in \mathbb{N}^*\), let \(W_{n, d}(e_{\mathbb{Z}})\), resp. \(W_{n, d}(e_{\mathbb{Z}})\) the set of words on \(e_{\mathbb{Z}}\) that are of depth \(d\), resp. of weight \(n\) and depth \(d\).

Definition 2.1.2. i) Let \(A(\langle\langle e_{\mathbb{Z}}\rangle\rangle)_b \subset A(\langle\langle e_{\mathbb{Z}}\rangle\rangle)_b\) be the subset of the elements \(f\) such that, for each \(d \in \mathbb{N}^*\), we have \(\sup_{w \in W_{n, d}(e_{\mathbb{Z}})} |f[w]|_p < +\infty\).

ii) Let \(A(\langle\langle e_{\mathbb{Z}}\rangle\rangle)_\Sigma \subset A(\langle\langle e_{\mathbb{Z}}\rangle\rangle)_\Sigma\) be the subset of the elements \(f\) such that, for each \(d \in \mathbb{N}^*\), we have \(\sup_{w \in W_{n, d}(e_{\mathbb{Z}})} |f[w]|_{p, n} \to +\infty\).

Definition 2.1.3. i) Let \(N_{\Lambda, D} : A(\langle\langle e_{\mathbb{Z}}\rangle\rangle) \to \mathbb{R}_+[\Lambda, D]\) be the map 

\[f \mapsto N_{\Lambda, D}(f) = \sum_{(n, d) \in \mathbb{N}^2} \max_{w \in W_{n, d}(e_{\mathbb{Z}})} |f[w]|_p A^n D^d\]

ii) Let \(N_D : A(\langle\langle e_{\mathbb{Z}}\rangle\rangle)_b \to \mathbb{R}_+[\|D\|]\) be the map defined by:

\[N_D(f) = \sup_{d \in \mathbb{N}} \left( \left( \max_{w \in W_{n, d}(e_{\mathbb{Z}})} |f[w]|_p \right) D^d \right)\]

We proved in I-2, §2 the following facts. For all \(f \in \Pi_{z, 0}(A), \)

\[N_{\Lambda, D}(\text{Ad}_f(e_{\mathbb{Z}})) \leq \Lambda D N_{\Lambda, D}(f)\]

We have, for all \(f, g \in \Pi_{z, 0}(A)\):

\[N_{\Lambda, D}(g \circ_{\text{DR}} f) \leq N_{\Lambda, D}(g) \times N_{\Lambda, D}(f)\]

\[N_{\Lambda, D}(\tau(\lambda)(f)) (\Lambda, D) = N_{\Lambda, D}(f)(\Lambda, D)\]

These facts stay true for \(N_D\). Moreover, the Ihara product \(\circ_{\text{DR}} : \Pi_{z, 0}(A) \times \Pi_{z, 0}(A) \to \Pi_{z, 0}(A)\) and \(\tau : A^* \times \Pi_{z, 0}(A) \to \Pi_{z, 0}(A)\) are continuous relatively to the \(N_{\Lambda, D}\)-topology on \(\Pi_{z, 0}(A)\) and the \(p\)-adic topology on \(A\). Finally, \(A(\langle\langle e_{\mathbb{Z}}\rangle\rangle)_\Sigma\) equipped with \(N_D\) is a complete normed \(K\)-algebra. Let \(\Pi_{z, 0}(A)_\Sigma = \Pi_{z, 0}(A) \cap A(\langle\langle e_{\mathbb{Z}}\rangle\rangle)_\Sigma\); we equip it and its image by \(\text{Ad}(e_{\mathbb{Z}})\) with the topology induced \(N_D\).

Then:

i) \(\Pi_{z, 0}(A)_\Sigma\) is stable by the usual product of grouplike series, and by the Ihara group law.

ii) \(\Pi_{z, 0}(A)_\Sigma\) and its image by \(\text{Ad}(e_{\mathbb{Z}})\) are closed inside, respectively, \(\Pi_{z, 0}(A)\) and \(\text{Ad}_{\Pi_{z, 0}(A)}(e_{\mathbb{Z}})\).

In view of some next computations we also want to stress the following fact which is a consequence of the definitions, and was implicit in I-2:

**Fact 2.1.4.** For each sequence \((w_l)_{l \in \mathbb{N}}\) of \(O_{\mu_{\infty}}\) such that \(\lim_{l \to +\infty} \sup_{m \geq 0} f[w_m] < +\infty\) and \(\text{weight}(w_l) \to_{l \to +\infty} +\infty\), the map "sum of series" \(\Pi_{z, 0}(A)_\Sigma \to A\) resp. \(\text{Ad}_{\Pi_{z, 0}(A)}(e_{\mathbb{Z}}) \to A\), defined by \(f \mapsto \sum_{m \geq 0} f[w_m]\) is continuous.

2.2. The Frobenius and its fixed point.

2.2.1. At tangential base-points \((\overline{\imath}_z, \overline{\imath}_0)\). We recall the formula for the Frobenius at the base-points \((\overline{\imath}_z, \overline{\imath}_0), z \in \mathbb{Z} - \{0, \infty\}\):

**Proposition 2.2.1.** For all \(\alpha \in \mathbb{N}^*\), we have:

\[F_\alpha : \pi_{\overline{\imath}_z, 0}^{\text{un, DR}}(X_K, \overline{\imath}_z, \overline{\imath}_0)(K) \to \pi_{\overline{\imath}_z, 0}^{\text{un, DR}}(X_K^{(\alpha)}, \overline{\imath}_{z^\alpha}, \overline{\imath}_0)(K)\]

\[f \mapsto \varphi_{p, \text{un}}(\alpha, \alpha) \circ_{\text{DR}} \tau(\alpha)(f)\]
\[
\tau(p^\alpha)\phi^\alpha : \pi^\text{un,DR}_1(X_K, \bar{\Gamma}_z^\alpha, \bar{\Gamma}_0)(K) \to \pi^\text{un,DR}_1(X_K, \bar{\Gamma}_z^\alpha, \bar{\Gamma}_0)(K)
\]

where \(\Phi^{(z_i^\alpha)}_{p,-\alpha} = F_\alpha^{(z_i)}(1, \bar{\Gamma}_0)\) and \(\Phi^{(z_i)}_{p,\alpha} = \tau(p^\alpha)\phi^\alpha(1, \bar{\Gamma}_0)\).

We have \(\phi^\alpha \circ F_\alpha = \text{id}\). The formulas imply that for all \(\alpha \in \mathbb{N}^+\), we have \(\Phi_{p,\alpha} \circ \text{DR} \Phi_{p,-\alpha} = 1\). It implies also a formula for \(\Phi_{p,\epsilon}\) in terms of \(\Phi_{p,\epsilon}(\epsilon \in \{1, -1\})\), (which we will write in §2.3.2 with the notations introduced there).

Cyclotomic \(p\)-adic multiple zeta values (when \(N = 1\), \(p\)-adic multiple zeta values) are the numbers \(\zeta_{p,\alpha}(w) = \Phi_{p,\alpha}(w)\) for all words \(w\).

Besides \(\phi^\alpha\), the other natural isomorphism between \(\mathcal{O}^{m,*z}\) and \(\mathcal{O}^{m,*z}(p^\alpha)\) is the one that sends \(e_{z_i^{(p^\alpha)}} \mapsto e_{z_i^\alpha}\) for all \(i\). We recall that \(\sigma \in \mathbb{N}^+\) is the order of \(p\) in \((\mathbb{Z}/\mathbb{N}\mathbb{Z})^\times\), and that we have assumed \(q = p^\sigma\); thus, assume now that \(\alpha \in \mathbb{Q}^\times : p^\sigma = q^\alpha\) where \(\alpha = \sigma \bar{\alpha}\) : we have \(z_i^{p^\alpha} = z_i^\alpha\) for all \(i \in \{1, \ldots, N\}\), and the isomorphism above sends \(e_{z_i^{(p^\alpha)}} \mapsto e_{z_i^\alpha}\), \(i \in \{1, \ldots, N\}\).

We can now write the second notion of cyclotomic \(p\)-adic multiple zeta values; this definition is due to Furusho in [FII] when \(N = 1\), and to Yamashita in [Y] for any \(N\). Note that, in [Y], cyclotomic \(p\)-adic multiple zeta values are called \(p\)-adic multiple \(L\)-values; we will not use this terminology in order not to confuse this object with the values at tuples of positive integers of the newly introduced \(p\)-adic multiple \(L\)-functions of [FKMT]. We fix again \(z \in Z - \{0, \infty\}\).

**Definition 2.2.2.** Let \((\Phi_{q}^{KZ}(z)) \in \Pi_{z,0}(K)\) be the unique element of \(\Pi_{z,0}(K)\) such that \(F_\alpha^{(z)}(\Phi_{q}^{KZ}(z)) = (\phi^\alpha)(\Phi_{q}^{KZ}(z))\).

For all words \(w\) on \(e_{z_i}\), the coefficients \(\Phi_{q}^{KZ}(w) = (\Phi_{q}^{KZ}(z))_i(w) \in K\), are called cyclotomic \(p\)-adic multiple zeta values (when \(N = 1\), \(p\)-adic multiple zeta values).

Conjecturally, for any \(\epsilon \in \{-1, 1\}\) and \(\alpha \in \mathbb{N}^+\), the \(\mathbb{Q}\)-algebra generated by the numbers \(\zeta_{p,\alpha}(w)\) is isomorphic to the \(\mathbb{Q}\)-algebra generated by the numbers \((\phi_{q}^{KZ}(z))_i(w)\), by the isomorphism that sends \(\zeta_{p,\alpha}(w) \mapsto (\phi_{q}^{KZ}(z))_i(w)\).

By functoriality of the Frobenius we have, as for \(\Phi_{p,\alpha}\) :

\(\Phi_{q}^{KZ(z)}(z) = (z \mapsto zx)_*(\Phi_{q}^{KZ(z)})^{(1)}\)

**Notation 2.2.3.** For all \(\bar{\alpha} \in \mathbb{N}^+\), \(z \in Z - \{0, \infty\}\), let \(\Phi_{q,\bar{\alpha}} = \Phi_{p,\sigma \bar{\alpha}}\), and \(\Phi_{q,\bar{\alpha}} = \Phi_{p,\sigma \bar{\alpha}}\).

For all words \(w\), let \(\zeta_{q,\bar{\alpha}}(w) = \Phi_{q,\bar{\alpha}}(w)\) and \(\zeta_{q,\bar{\alpha}}(w) = \Phi_{q,\bar{\alpha}}(w)\).

**2.2.2. At all base-points.** We consider again the bundle of paths of \(\pi_{1,\text{un,DR}}(X_K)\) starting at \(\bar{\Gamma}_0\), trivialized at \(\bar{\Gamma}_0\), the affinoid space \(U_{\infty}^\text{an} = (\mathcal{P}^{\text{an}} - \cup_{i=1}^N \{z_i\})/K\) equipped with the canonical lift of Frobenius \(z \mapsto z^p\), and the group of rigid analytic sections on \(U_{\infty}^\text{an}\) of this bundle, which can be viewed as \(\Pi_{0,0}(\mathfrak{A}(U_{\infty}^\text{an}))\).

The Frobenius on it is very close to the Ihara action written in §2.1.2 : a formula for \(\phi^\alpha\) is

\[\Pi_{0,0}(\mathfrak{A}(U_{\infty}^\text{an})) \to \Pi_{0,0}(\mathfrak{A}(U_{\infty}^\text{an}))\]

\[f \mapsto \text{L}_{f}^{(z_i^\alpha)}(z)(f(z^p))(e_{0, \Phi_{p,\alpha}(z_1)}^{-1} e_{z_1} e_{\Phi_{p,\alpha}(z_1)^{-1}} \ldots e_{z_N} e_{\Phi_{p,\alpha}(z_N)^{-1}})\]

and we have a similar formula for \(F_{\alpha}^{(z)}\).

The most standard definition of \(L_{f}^{(z_i^\alpha)}\) in terms of the Frobenius-invariant paths takes place in the framework of the rigid pro-unipotent fundamental groupoid of the reduction modulo \(p\) of \(X\); see for example [F2], Theorem 2.3 for \(N = 1\). For our purposes, a convenient variant is the following. Let \(A_{\text{Col}}(U_{\infty}^\text{an})\) the algebra of Coleman functions over \(U_{\infty}^\text{an}\); it contains \(\mathfrak{A}(U_{\infty}^\text{an})\) and is a \(\mathfrak{A}(U_{\infty}^\text{an})\)-algebra. The Frobenius \(\phi^\alpha\) defines a map

\[\Pi_{0,0}(A_{\text{Col}}(U_{\infty}^\text{an})) \to \Pi_{0,0}(A_{\text{Col}}(U_{\infty}^\text{an}))\]
\[ f \mapsto \text{Li}_{p,\alpha}^i (z) f (z^{p^\alpha} (c_0, \Phi(z_1), \ldots, \Phi(z_N))^{-1} e_{z_1} \Phi(z_1), \ldots, e_{z_N} \Phi(z_N)) \]

Under this point of view, the equation of horizontality of Frobenius, which was at the center of I-1 and I-2, and which can be written as

(2.2.1) \[ \text{Li}_{p,\alpha}^i (z) = \text{Li}_{p,\alpha}^{KZ} (z) (p^{\alpha} c_0, p^{\alpha} e_{z_1}, \ldots, p^{\alpha} e_{z_N}) \]

(2.2.2) \[ \text{Li}_{p,\alpha}^i (z) = \text{Li}_{p,\alpha}^{KZ} (z) (p^{p^\alpha} c_0, p^{p^\alpha} e_{z_1}, \ldots, p^{p^\alpha} e_{z_N}) \]

is the natural lift of the definition of \( \text{Li}_{p,\alpha}^{KZ} \) in terms of the unique invariant paths.

2.3. \textbf{Ihara action weighted by a small number} \( \lambda \in K^* \) \textbf{and its iterates}. We state now some preliminary facts on the Ihara action pre-composed with \( \tau(\lambda) \), where \( \lambda \in K^* \), in view of considering in §3.1 the family of the \( a \)-th iterates of this map \( (a \in \mathbb{N}) \) as a function of \( \lambda^a \). Let \( z \in \mathbb{Z} - \{0, \infty\} \), and let \( A \) a complete topological \( K \)-algebra.

2.3.1. \textbf{Definition}.

\textbf{Definition 2.3.1.} Let \( (\lambda, g) \in \mathbb{G}_m(K) \times \Pi_{z,0}(A) \). We call Ihara action by \( g \) weighted by \( \lambda \), and denote by \( (\lambda, g) \circ \overline{\text{DR}} \), the map

\[ \Pi_{z,0}(A) \to \Pi_{z,0}(A) \]

\[ f \mapsto (\lambda, g) \circ \overline{\text{DR}} f = g \circ \overline{\text{DR}} \tau(\lambda)(f) \]

\textbf{Lemma 2.3.2.} For all \( (\lambda, g) \in \mathbb{G}_m(K) \times \Pi_{z,0}(A) \), the Ihara action of \( g \) weighted by \( \lambda \) is an automorphism of the scheme \( \Pi_{z,0} \times_{\text{Spec} \mathbb{Q}} \text{Spec} A \), whose inverse is

\[ f \mapsto \tau(\lambda^{-1}) (g^{-1} \circ \overline{\text{DR}} f) \]

\textbf{Definition 2.3.3.} Let \( a \in \mathbb{N}^* \). Let the map of elevation to the \( a \)-th Ihara power weighted by \( \lambda \),

\[ e_{\alpha,\overline{\text{DR}},\lambda} : \Pi_{z,0}(A) \to \Pi_{z,0}(A) \]

be defined by

\[ g^{\alpha,\overline{\text{DR}},\lambda} = (\lambda, g) \circ \overline{\text{DR}} \circ \overline{\text{DR}} \circ \overline{\text{DR}} \circ \ldots \circ \overline{\text{DR}} \circ \overline{\text{DR}} \circ 1_{\Pi_{z,0}(A)} = g \circ \overline{\text{DR}} \tau(\lambda) (g) \circ \overline{\text{DR}} \circ \overline{\text{DR}} \circ \tau(\lambda^{-1})(g) \]

i.e. \( g^{\alpha,\overline{\text{DR}},\lambda} \) is the unique element of \( \Pi_{z,0}(A) \) such that we have, for all \( f \in \Pi_{z,0}(A) \),

\[ (\lambda, g) \circ \overline{\text{DR}} \circ \overline{\text{DR}} \circ \overline{\text{DR}} \circ \overline{\text{DR}} \circ \overline{\text{DR}} \circ \overline{\text{DR}} \circ \overline{\text{DR}} \circ 1_{\Pi_{z,0}(A)} = (\lambda^a, g^{\alpha,\overline{\text{DR}},\lambda}) \circ \overline{\text{DR}} f \]

2.3.2. \textbf{Application to the Frobenius}. As one can see by §2.2, when \( p^a \) is a power of \( q \), the \( q \)-th power of Frobenius at tangential base-points can be identified with the map

\[ F_{q,\alpha} : \Pi_{z,0}(A) \to \Pi_{z,0}(A) \]

\[ f \mapsto (q^a, \Phi_{q,\alpha}^\circ \overline{\text{DR}} f) \]

and we have

\[ \Phi^\circ_{q,\alpha} = \Phi^\circ_{q,\overline{\text{DR}},\alpha} \]

\[ \Phi^\circ_{q,\alpha} \circ \overline{\text{DR}} (q^\circ) (\Phi^\circ_{q,KZ}^\circ) (z) = (\Phi^\circ_{q,KZ}^\circ) (z) \]

\[ \Phi_{q,\alpha}^\circ (\Phi^\circ_{q,KZ}^\circ) (z) = \tau(q^\circ) (\Phi^\circ_{q,KZ}^\circ) (z) \]
When $N = 1$, and $\alpha = 1$, this last equality has been written by Furusho (with different conventions and notations) in [F2], Theorem 2.8.

2.4. Projection onto the parts of given weight and depth in $A\langle \langle e_Z \rangle \rangle$. The following definitions and facts are stated in view of §3.1. We take again $A$ a complete topological $K$-algebra.

2.4.1. Definition.

**Definition 2.4.1.** i) For $s \in \mathbb{N}^*$, let

$$\text{pr}_s : A\langle \langle e_Z \rangle \rangle \rightarrow A\langle e_Z \rangle$$

of weight $s$

$$f \mapsto \sum_{w \in W} (e_Z)^w f[w] w$$

Equivalently, $(\text{pr}_s)_{s \in \mathbb{N}}$ is the sequence of coefficients of $\tau :$ the maps $A\langle \langle e_Z \rangle \rangle \rightarrow A\langle e_Z \rangle$ characterized by,

$$\tau(\lambda) = \sum_{s \in \mathbb{N}} \lambda^s \text{pr}_s : A\langle \langle e_Z \rangle \rangle \rightarrow A\langle e_Z \rangle$$

ii) For $d \in \mathbb{N}^*$, let

$$\text{pr}_{s,d} : A\langle \langle e_Z \rangle \rangle \rightarrow A\langle e_Z \rangle$$

of depth $d$

$$f \mapsto \sum_{w \in W} (e_Z)^w f[w] w$$

iii) For $s,d \in \mathbb{N}^*$, let

$$\text{pr}_{s,d} = \text{pr}_s \circ \text{pr}_{s,d} = \text{pr}_{s,d} \circ \text{pr}_s$$

2.4.2. Almost commutation of $\text{pr}_s$ with the usual algebraic operations. Let $z \in \mathbb{Z} - \{0, \infty\}$. It follows from Definition 2.4.1 that:

**Lemma 2.4.2.** i) For all $f,g \in A\langle \langle e_Z \rangle \rangle$, we have:

$$(2.4.1) g \circ_{\text{DR}} (\tau(\lambda)f) = \sum_{s \in \mathbb{N}} \lambda^s g \circ_{\text{DR}} (\text{pr}_s f)$$

ii) For all $s \in \mathbb{N}^*$, we have

$$\text{pr}_{s+1} \circ \text{Ad}(e_Z) = \text{Ad}(e_Z) \circ \text{pr}_s$$

By these two facts and by §2.1.2, we deduce:

**Corollary 2.4.3.** For all $f,g \in \Pi_{z,0}(A)$, for all $\lambda \in \mathbb{A}^*$:

$$\text{Ad}_g(e_Z) \circ_{\lambda} \text{Ad}(e_Z) (f) = \sum_{s \geq 0} \lambda^s \text{Ad}_g(e_Z) \circ_{\lambda} \text{pr}_{s+1} \text{Ad}_f(e_Z)$$

2.4.3. Relation with the topological setting. Clearly we have:

**Lemma 2.4.4.** For all $s \in \mathbb{N}^*$, for all $f \in A\langle \langle e_Z \rangle \rangle$,

$$\mathcal{N}_{\lambda,d}(\text{pr}_s f) \leq \mathcal{N}_{\lambda,d}(f)$$

and in particular $\text{pr}_s$ is a continuous linear map for the $\mathcal{N}_{\lambda,d}$-topology ; moreover, $A\langle \langle e_Z \rangle \rangle_b$ and $A\langle \langle e_Z \rangle \rangle_\Sigma$ are stable by $\text{pr}_s$.

This lemma and §2.1.2 imply the following last fact:

**Corollary 2.4.5.** For all $f,g \in \Pi_{z,0}(A)$ and $s \in \mathbb{N}$ we have:

$$\mathcal{N}_{\lambda,d}(\text{Ad}_g \circ_{\text{DR}} \text{pr}_s f)(e_z)) \leq \Lambda_d \mathcal{N}_{\lambda,d}(g) \mathcal{N}_{\lambda,d}(f)$$

We single out this particular consequence because we will specifically need it in §3.
3. Proofs within the fundamental groupoid

We study the Ihara action weighted by $\lambda \in K^\times$ iterated $a$ times, as a function of $\lambda^a$, where $a$ varies in $\mathbb{N}^*$ ($\S 3.1$, $\S 3.2$, $\S 3.3$) with a particular treatment of the case where $|\lambda|^p < 1$ ($\S 3.2$); we define the 'De Rham' counterpart of the harmonic Ihara action of I-2 ($\S 3.4$); we finally apply these results to the Frobenius ($\S 3.5$-$\S 3.6$). In this paragraph, $A$ is any complete topological $K$-algebra.

3.1. Expansion of the iterates of the weighted Ihara action.

3.1.1. Preliminary. Let us write $e_{\lambda,DR}^a$ in terms, not of the Ihara multiplication $\circ^DR$, but the usual De Rham multiplication $\times = \times^{DR}$ (the multiplication of formal power series of $A([e_Z])$).

Lemma 3.1.1. For all $g_z \in \Pi_{z,0}(A)$, for all $a \in \mathbb{N}^*$, and $\lambda \in A$ which is not a root of unity:

(3.1.1) $e_{\lambda,DR}^a(g_z) = g_z(e_{0}, e_{z_1}, \ldots, e_{z_N}) \times g_z(\lambda e_0, \lambda \Ad_{g_{z_1}}(e_{z_1}), \ldots, \lambda \Ad_{g_{z_N}}(e_{z_N})) \times \ldots \times$

\[ g_z(\lambda^{a-1} e_0, \lambda^{a-1} \Ad_{g_{z_1}}(e_{z_1}), \ldots, \lambda^{a-1} \Ad_{g_{z_N}}(e_{z_N})) \]

This follows from the formula for the weighted Ihara action (by induction on $a$).

3.1.2. Main result. We can now write how $e_{\lambda,DR}^a$ depends on $a$.

Proposition 3.1.2. Let $\Lambda, \Lambda^a, a$ be three (independent, despite the notation) formal variables. There exists a map

\[ e_{\lambda,DR}^a : \Pi_{z,0}(A) \to A ([e_Z])[\Lambda^a, a](\Lambda) \]

such that, for all $f \in \Pi_{z,0}(A)$, for all word $w$, for all $a \in \mathbb{N}^*$ such that $a > \text{depth}(w)$ and for all $\lambda \in A - \{0\}$ which is not a root of unity, we have:

\[ (e_{\lambda,DR}^a(f))[w] = (e_{\lambda,DR}^{a,\text{weighted}}(f)[w])(\Lambda^a, a)(\Lambda) \]

Proof. Knowing that $g_z[0] = 1$, dualizing the multiplication of the $a$ factors of (3.1.1) gives

(3.1.2) $e_{\lambda,DR}^a(f)[w] = \sum_{0 \leq d' \leq d} \sum_{0 \leq i_1 < \ldots < i_{d'} \leq a-1} \sum_{w_{i_1} \ldots w_{i_{d'}} = w} g_z(\lambda^{i_{d'}-1} e_0, \lambda^{i_{d'}-1} \Ad_{g_{z_1}}(e_{z_1}), \ldots, \lambda^{i_{d'}-1} \Ad_{g_{z_N}}(e_{z_N})[w_{i_{d'}}]) \times \ldots \times$

\[ g_z(\lambda^{w-1} e_0, \lambda^{w-1} \Ad_{g_{z_1}}(e_{z_1}), \ldots, \lambda^{w-1} \Ad_{g_{z_N}}(e_{z_N})[w_{i_{d'}}]) \]

We have assumed that $a > d$; let us thus separate the indices $i_j \leq d$ and $i_j > d$:

\[ \sum_{0 \leq d' \leq d} \sum_{0 \leq i_1 < \ldots < i_{d'} \leq a-1} \sum_{w_{i_1} \ldots w_{i_{d'}} = w} \sum_{0 \leq d'' \leq d'} \sum_{0 \leq i_1 < \ldots < i_{d''} \leq a} \sum_{d < i_{d'+1} < \ldots < i_{d''} \leq a-1} \]

This yields an expression of $\lambda$, as a $K$-linear combination indexed by $\{(d'',d') \mid 0 \leq d'' \leq d' \leq d\} \times \{$deconcatenations of $w$ in $d''$-non-empty subwords$\}$ which is independent of $a$ but depends polynomially of $\lambda$, and with coefficients as well independent of $a$ and polynomial functions of $\lambda$, of the numbers

(3.1.3) $\sum_{d < i_{d'+1} < \ldots < i_{d''} \leq a-1} g_z(\lambda^{i_{d''}-1} e_0, \lambda^{i_{d''}-1} \Ad_{g_{z_1}}(e_{z_1}), \ldots, \lambda^{i_{d''}-1} \Ad_{g_{z_N}}(e_{z_N})[w_{i_{d''}}]) \times \ldots \times$

\[ g_z(\lambda^{w-1} e_0, \lambda^{w-1} \Ad_{g_{z_1}}(e_{z_1}), \ldots, \lambda^{w-1} \Ad_{g_{z_N}}(e_{z_N})[w_{i_{d''}}]) \]
Let \( \epsilon_z = g_z - 1 \), and \( \tilde{\epsilon}_z = g_z^{-1} - 1 \). For \( 0 \leq i_j \leq a - 1 \), we have (where \( \text{pr}_{*, \leq d} \), defined in §2.4.1, is the projection onto the part of depth \( \leq d \)):

\[
\text{pr}_{*, \leq d}(\text{Ad}_{g_{i_j}^{-1}}(\epsilon_z)) = \sum_{m_j, m'_j \in \mathbb{N}, m_j \leq i_j, m'_j \leq i_j, m_j + m'_j + 1 \leq d} \left( \frac{i_j}{m_j} \right) \left( \frac{i_j}{m'_j} \right) \epsilon_{z,j} e_{z,j} m_j m'_j
\]

When \( i_j > d \), the collection of conditions \( \{m_j, m'_j \in \mathbb{N}, m_j \leq i_j, m'_j \leq i_j, m_j + m'_j + 1 \leq d \} \) is equivalent to \( \{m_j, m'_j \in \mathbb{N}, m_j + m'_j + 1 \leq d \} \); thus, dualizing in (3.1.3) each factor \( g_z(\lambda^{i-1} \epsilon_0, \lambda^{i-1} \text{Ad}_{g_{i_{j-1}}^{-1}}(\epsilon_{z,j})), \ldots, \lambda^{i-1} \text{Ad}_{g_{i_{j-1}}^{-1}}(\epsilon_{z,\bar{n}})) \) gives that (3.1.3) is a linear combination, independent of \( a \) and \( \lambda \), of sums:

\[
\sum_{d < i_{j', \ldots, j' \leq a-1}} \prod_{j=1}^{d} P_j(I_j) \lambda^{i_j C_j}
\]

where \( \text{weight}_j \in \mathbb{N}^* \) arises as the weight of a certain quotient sequence of \( w_{i_j} \), and \( (\frac{i_j}{m_j}) (\frac{i_j}{m'_j}) \) are polynomials of \( i_j \). Finally, any function of \( a \) of the form

\[
\sum_{L \leq I_1 < \ldots < I_d \leq a-1} \prod_{j=1}^{d} P_j(I_j) \lambda^{i_j C_j}
\]

with \( L, \delta \in \mathbb{N}^*, C_1, \ldots, C_\delta \in \mathbb{N}^* \) and \( P_1, \ldots, P_\delta \in \mathbb{K}[T] \) polynomials, depend on \( a \) as a polynomial function of \( (a, \lambda^\delta) \): one can reduce this statement to \( L = 0 \) by splitting an iterated sum over \( 0 \leq I_1 < \ldots < I_\delta \leq a-1 \) at \( L \) and by induction on \( \delta \), then use, again by induction on \( \delta \) that, for all \( \text{deg}_j \in \mathbb{N}^* \), we have \( \sum_{I_j=0}^{\text{deg}_j} f_{i_j}^j \lambda^{C_j I_j} = (\lambda^{C_j} \frac{d}{d\lambda^{C_j}})^j (\frac{1}{\lambda^{C_j} - 1}) \).

3.2. Contraction property and expansion of the iterates of the Ihara action weighted by a small \( \lambda \in \mathbb{K}^* \).

3.2.1. Contractance. Because of the sub-multiplicativity of \( N_{\lambda,D} \) with respect to the Ihara product (§2.1.3), and because of the completeness of \( A(\epsilon_z) \) with respect to \( N_{\lambda,D} \), one can define through these objects a notion of contraction mapping; namely, the contraction mappings in the sense below satisfy the usual properties of contraction mappings regarding fixed points, as we write below:

**Definition 3.2.1.** Let \( \kappa \in \mathbb{K}^* \) with \( |\kappa|_p < 1 \). A map

\[
\varphi : \Pi_{z,0}(A) \rightarrow \Pi_{z,0}(A)
\]

is said to be a \( \kappa \)-contraction (with respect to \( N_{\lambda,D} \) and \( \diamond \text{DR} \)) if, for all \( f, f' \in \Pi_{z,0}(A) \), we have:

\[
N_{\lambda,D}(\varphi(f')^{-1} \diamond \text{DR} \varphi(f))(\Lambda, D) \leq N_{\lambda,D}(f'^{-1} \diamond \text{DR} f)(\kappa \Lambda, D)
\]

The example which we are interested in is the following:

**Lemma 3.2.2.** i) Let \( (\lambda, g) \in \mathbb{G}_m(K) \times \Pi_{z,0}(A) \). Then, if \( |\lambda|_p < 1 \), the Ihara action of \( g \) weighted by \( \lambda \) is a \( \lambda \)-contraction. More precisely we have, for all \( f, f' \in \Pi_{z,0}(A) \),

\[
N_{\lambda,D}((\lambda, g) \diamond \text{DR} f'^{-1} \diamond \text{DR} ((\lambda, g) \diamond \text{DR} f))(\Lambda, D) = N_{\lambda,D}(f'^{-1} \diamond \text{DR} f)(\lambda \Lambda, D)
\]

This statement follows from §2.1.3. On this example, some usual properties of contractions regarding fixed points are the following:

**Corollary 3.2.3.** i) Let \( (\lambda, g) \in \mathbb{G}_m(K) \times \Pi_{z,0}(A) \) such that \( |\lambda|_p < 1 \). Then, the Ihara action of \( g \) weighted by \( \lambda \) is continuous and has a unique fixed point, which is equal to:

\[
\text{fix}_{\lambda,g} = \lim_{a \to \infty} (\lambda, g) a^{-1} \diamond \text{DR} \ldots a^{-1} \diamond \text{DR} (\lambda, g) \diamond \text{DR} f = \lim_{a \to \infty} g^{a^{-1} \diamond \lambda} \diamond \text{DR} \tau(\lambda^a)(f)
\]
for all \( f \in \Pi_{\geq 0}(A) \). In particular, for all \( a \in \mathbb{N}^* \):
\[
\text{fix}_{\lambda,g} = \text{fix}_{\lambda^a,g^a \cdot \text{pr}_{\lambda},\lambda} = \lim_{a' \to \infty} g^{a'\text{DR},\lambda}
\]

ii) Let \( \lambda \in \mathbb{G}_m(K) \) such that \( |\lambda|_p < 1 \). The map \( g \mapsto \text{fix}_{\lambda,g} \) is an automorphism of the scheme \( \Pi_{\geq 0} \times_{\Spec A, \lambda} \Spec A \), whose inverse is
\[
\text{fix}_{\lambda}^{-1} : f \mapsto f \circ_{\text{DR}} \tau(\lambda)(f)^{-1}\text{DR}
\]

**Proof.** The part i) follows from a standard formal argument on contraction mappings, knowing that \( \Pi_{\geq 0}(A) \subset A(\langle e_\Lambda \rangle) \) equipped with the \( \mathcal{N}_{\Lambda,p} \)-topology is closed, thus is complete. The part ii) is essentially immediate, given the equation \( g \circ_{\text{DR}} \tau(\lambda)(\text{fix}_{\lambda,g}) = \text{fix}_{\lambda,g} \).

\[\Box\]

### 3.2.4. Proposition 3.2.4

Let \( \Lambda \) a formal variable.

i) There is a unique formal power series, namely
\[
S_{\lambda,g}(A) = \sum_{a \in \mathbb{N}} A^a \text{fix}_{\lambda,g} \circ_{\text{DR}} \text{pr}_a((\text{fix}_{\lambda,g})^{-1}\text{DR}) \in A(\langle e_\Lambda \rangle)[[A]]
\]
satisfying for all \( a \in \mathbb{N}^* \),
\[
g^{a\cdot \text{DR},\lambda} = S_{\lambda,g}(\lambda^a)
\]

ii) For all \( f \in \Pi_{\geq 0}(A) \), there is a unique formal power series, namely
\[
S_{\lambda,g,f}(A) = \sum_{t \in \mathbb{N}} A^t \left( \sum_{s+t'=t} \text{fix}_{\lambda,g} \circ_{\text{DR}} \left( \text{pr}_a((\text{fix}_{\lambda,g})^{-1}\text{DR}) \circ_{\text{DR}} \text{pr}_{s'}(f) \right) \right) \in A(\langle e_\Lambda \rangle)[[A]]
\]
satisfying for all \( a \in \mathbb{N}^* \),
\[
(\lambda^a, g^{a \cdot \text{DR},\lambda}) \circ_{\text{DR}} f = S_{\lambda,g,f}(\lambda^a)
\]

**Proof.** By Corollary 3.2.3 we have
\[
(\lambda^a, g^{a \cdot \text{DR},\lambda}) \circ_{\text{DR}} f = \text{fix}_{\lambda,g} \circ_{\text{DR}} \tau(\lambda^a)((\text{fix}_{\lambda,g})^{-1}\text{DR}) \circ_{\text{DR}} \tau(\lambda^a)(f) = \text{fix}_{\lambda,g} \circ_{\text{DR}} \tau(\lambda^a)((\text{fix}_{\lambda,g})^{-1}\text{DR} \circ_{\text{DR}} f)
\]
and the rest follows from §2.4.2. \[\Box\]

### 3.3. Properties of the maps subjacent to the expansions of iterates of the Ihara action.

In view of studying the relationship between all the variants of cyclotomic \( p \)-adic multiple zeta values (§3.5), we are interested in describing the duals of the map \( \text{fix}_\lambda \) (of §3.1.1), when \( |\lambda|_p < 1 \), the map \( e_{\lambda,\text{DR},\lambda} : g \mapsto g^{\lambda \cdot \text{DR},\lambda} \) (of §2.3.1) for which we will assume that \( \lambda \in K^* \) is not a root of unity, and also the inversion for the Ihara product, \( e_{-1,\text{DR}} : g \mapsto g^{-1\cdot \text{DR}} \).

#### 3.3.1. Algebraic properties.

We must first write the maps above in terms, not of the Ihara product \( \circ_{\text{DR}} \), but the usual multiplication \( \times = \times_{\text{DR}} \) of the group \( \pi^\text{an}_{\text{DR}}(X_K, \omega_{\text{DR}}) \).

**Lemma 3.3.1.** i) The map \( \text{fix}_\lambda \) on \( \Pi_{\geq 0}(A) \) is characterized by the equation :
\[
g_z(e_0, e_1, \ldots, e_{zN}) \cdot \text{fix}_{\lambda,g_z}(\lambda e_0, \lambda g_z^{-1} e_1, g_{z_1}, \ldots, g_{z_1}^{-1} e_{zN} g_{zN}) = \text{fix}_{\lambda,g_z}(e_0, e_1, \ldots, e_{zN})
\]
where \( g_{z_i} = (x \mapsto z_i x)_* (g_1) \) for all \( i \in \{1, \ldots, N\} \) (this determines \( \text{fix}_{\lambda,g_z}[w] \) for every word \( w \) by induction on the weight.)

ii) The map \( e_{-1,\text{DR}} : g_z \mapsto g_z^{-1\cdot \text{DR}} \) on \( \Pi_{\geq 0}(A) \) is characterized by
\[
g_z(e_0, e_1, \ldots, e_{zN}) \cdot g_z^{-1\cdot \text{DR}}(e_0, g_z^{-1} e_1, g_{z_1}, \ldots, g_{z_1}^{-1} e_{zN} g_{zN}) = 1
\]
where \( g_{z_i} = (x \mapsto z_i x)_* (g_{zN}) \) for all \( i \in \{1, \ldots, N\} \).

From these expressions one can read a formula for the dual of these maps, in terms of the operations of taking subwords and quotients words of words on \( e_\Lambda \), and which it is not necessary to write explicitly; we only want to point out certain features of them.
Notation 3.3.2. For any \( \mathbb{Q} \)-algebra \( R \), and for each \( s, d \in \mathbb{N}^* \), let \( R^{o_{s,d}^{\epsilon_2}} \), \( R^{o_{s,d}^{\epsilon_2}} \), \( R^{o_{s,d}^{\epsilon_2}} \) be the sub-\( R \)-module of \( O^{m,\epsilon_2} \otimes R \) generated by words on \( e_Z \) of weight \( s \), resp. depth \( \leq d \), resp. weight \( s \) and depth \( \leq d \).

The dual of the maps \( \text{fix}_\lambda, \text{el}_{s,\text{DR},\lambda} \) and \( \text{el}_{-1,\text{DR}} \) are compatible with the depth filtration:

Lemma 3.3.3. Let \( s, d \in \mathbb{N}^* \).

i) Let \( \lambda \in K \) such that \( |\lambda|_K < 1 \).

\[
\text{fix}_\lambda^\vee \left( z(\lambda) O^{m,Z}_{s,\leq d} \right) = z(\lambda) O^{m,Z}_{s,\leq d}
\]

Let \( \lambda \in K \) which is not a root of unity and \( a \in \mathbb{N}^* \); then

\[
\text{el}_{s,\text{DR},\lambda}^\vee \left( z(\lambda) O^{m,Z}_{s,\leq d} \right) = z(\lambda) O^{m,Z}_{s,\leq d}
\]

and finally

\[
\text{el}_{-1,\text{DR}}^\vee \left( z(\lambda) O^{m,Z}_{s,\leq d} \right) = z(\lambda) O^{m,Z}_{s,\leq d}
\]

ii) More precisely, for each \( w \) word on \( e_Z \), of weight \( s \) and depth \( d \), for each \( a \in \mathbb{N}^* \), and with the above respective hypothesis for \( \lambda \), we have the three following congruences modulo shuffle products of words of lesser depth, i.e. modulo \( \sum_{\substack{r \geq 2 \\ s_i + \cdots + s_r = s \\ d_1 + \cdots + d_r \leq d}} O^{m,\epsilon_2}_{s_1,d_1} \circ \cdots \circ O^{m,\epsilon_2}_{s_r,d_r} : \)

\[
(1 - \lambda^s) \text{fix}_\lambda^\vee (w) \equiv w
\]

\[
\text{el}_{s,\text{DR},\lambda}^\vee (w) \equiv \frac{1 - \lambda^a}{1 - \lambda^s} w
\]

\[
\text{el}_{-1,\text{DR}}^\vee (w) \equiv -w
\]

Proof. This is a direct consequence of the previous lemmas and the formula for \( c_{\text{DR}} \) (recalled in §2.1.2). \( \Box \)

Remark 3.3.4. We note in particular the compatibility between these three maps and the depth filtration; we will compute all variants of cyclotomic \( p \)-adic multiple zeta values; since our computations of cyclotomic \( p \)-adic multiple zeta values are either inductive on the depth or done at each depth separately, this means that we can bring together all the explicit computations.

3.3.2. Effect on valuations. Let us now bring together these maps and the topological setting:

Lemma 3.3.5. i) Let \( \lambda \in K^\times \) such that \( |\lambda|_p < 1 \) and \( a \in \mathbb{N}^* \). The maps \( \text{fix}_\lambda, \text{el}_{s,\text{DR},\lambda}^\vee, \text{el}_{-1,\text{DR},\lambda}^\vee : \Pi_{s,0}(A) \to \Pi_{s,0}(A) \) are continuous for the \( N_{\Lambda,D} \)-topology; for \( \text{el}_{s,\text{DR},\lambda} ^\vee \), this remains true for any \( \lambda \in K^\times \) which is not a root of unity.

ii) Let \( \lambda \in K^\times \) such that \( |\lambda|_p < 1 \) and \( a \in \mathbb{N}^* \); for all \( g \in \Pi_{s,0}(K) \), we have

\[
N_{\Lambda,D}(\text{fix}_{\lambda,g}) = N_{\Lambda,D}(g_{\text{DR},\lambda}^\vee) = N_{\Lambda,D}(g^{-1,\text{DR}}) = N_{\Lambda,D}(g)
\]

This follows from the previous lemmas and the definitions.

3.3.3. Functoriality.

Lemma 3.3.6. For every automorphism \( \sigma \) of \( X_K \), we have a commutative diagram

\[
\begin{array}{ccc}
\Pi_{s,0}(A) & \xrightarrow{\sigma} & \Pi_{\sigma(z),\sigma(0)}(A) \\
\downarrow & & \downarrow \\
\Pi_{s,0}(A) & \xrightarrow{\sigma} & \Pi_{\sigma(z),\sigma(0)}(A)
\end{array}
\]

where the couple of vertical arrows can be a couple \( (\text{fix}_\lambda, \text{fix}_\lambda^\vee) \), or \( (\text{el}_{s,\text{DR},\lambda}^\vee, \text{el}_{s,\text{DR},\lambda}^\vee) \), or \( (\text{el}_{-1,\text{DR},\lambda}^\vee, \text{el}_{-1,\text{DR},\lambda}^\vee) \), where the variants with the exponent \( \sigma \) are defined through the Ihara action on \( \Pi_{\sigma(z),\sigma(0)}(A) \) (we leave to the reader the precise definition).

Proof. Clear. \( \Box \)
3.4. The De Rham harmonic Ihara action. We introduced in I-2 a DR-RT harmonic Ihara action $\sigma_{\text{har}}$, and a RT harmonic Ihara action $\sigma_{\text{RT}}$; in view of expressing the results of this part we introduce now a ‘De Rham’ harmonic Ihara action $\sigma_{\text{DR}}$.

3.4.1. Definition. We recall from I-2 the following definition. Let $A(\langle e_z \rangle) \subset A(\langle e_z \rangle)$ be the submodule of elements $h$ such that, for all words $w$, the series $\sum_{l \in \mathbb{N}} h[w^l w]$ is absolutely convergent. It contains in particular $\Pi, 0(A)_\Sigma$ and its image by $\text{Ad}(e_z)$ for all $z \in \mathbb{Z} - \{0, \infty\}$.

Definition 3.4.1. Let $\Sigma_{\text{DR}}$ making the following diagram commutative:

There exists a unique map $\Pi_{\text{DR}} : A(\langle e_z \rangle) \to A(\langle e_z \rangle)$ where $\cdot w$ is omitted when $s_0 = 0$. By considering an infinite countable number of copies of it, and pre-composing the copy indexed by each $n \in \mathbb{N}$ by $\tau(n) \times id$, we obtain a map

$\Pi_{\text{DR}} : A(\langle e_z \rangle) \to A(\langle e_z \rangle)$

which we also denote by $\sigma_{\text{DR}}$ and also call the De Rham harmonic Ihara action.

Proof. Let us consider the coefficient $\Sigma_{\text{inv}}(\text{Ad}(e_z) \circ \sigma_{\text{DR}} \circ \text{Ad}(e_z)[w])$ at any word $w$, viewed as a function of $g$ and $f$. By the formula for the dual of the adjoint Ihara action, one can see that its dependence on $f$ factorizes in a natural way through $\Sigma_{\text{inv}}(f)$. This defines $\sigma_{\text{DR}}$ and one can check that this map is unique.

In brief, the De Rham harmonic Ihara action is thus $\Sigma_{\text{inv}}$ of $\text{Ad}(e_z)$, of the Ihara action (where the push-forward refers to the set which is acted upon), characterized by the equation:

$\Sigma_{\text{inv}}(\text{Ad}(e_z) \circ \sigma_{\text{DR}} \circ \text{Ad}(e_z) \circ \text{Ad}(e_z)) = g \circ \Sigma_{\text{inv}}(\text{Ad}(e_z))$

We authorize ourselves to use the same notation and terminology for the two versions of this group action, the one on $A(\langle e_z \rangle)$ and the one on $\text{Map}(\mathbb{N}, A(\langle e_z \rangle))$, because it will be clear by the context which one (or both) we are referring to.

Example 3.4.4. i) In depth one and for $\mathbb{P}^1 - \{0, 1, \infty\}$, we have:

$$(g \circ \Sigma_{\text{inv}} f)(s_1) = \Sigma_{\text{inv}} f(s_1) + g[\frac{1}{1 - e_0} e_1 s_0^{-1} e_1]$$
ii) In depth two and for $F$, we have:

$$(g \circ \text{DR}_{\text{inv}}(f))(s_2, s_1) = (\Sigma \text{inv}_f)(s_2, s_1) + g\left[\frac{1}{1 - e_0}e_1e_0^{s_2-1}e_1e_0^{s_1-1}e_1\right]
+ \sum_{r=0}^{s_1-1} g\left[\frac{1}{1 - e_0}e_1e_0^{s_2-1}e_1e_0^r\right] (\Sigma \text{inv}_f)(s_1 - r) + \sum_{r=0}^{s_2-1} g[e_0^r e_1 e_0^{s_1-1} e_1] (\Sigma \text{inv}_f)(s_2 - r)$$

**Definition 3.4.5.** For any $\alpha \in \mathbb{N}^+$, let the (De Rham) harmonic Frobenius map at $(\Pi_1, \Gamma_0)$ of $\pi_{1}^{\text{in,DR}}(X_K)$ be the map $\phi_{\text{har}}$ defined by

$$\tau(q^2)\phi_{\text{har}}: \text{Map}(\mathbb{N}, A(\langle e_z \rangle))_{\text{har}} \rightarrow \text{Map}(\mathbb{N}, A(\langle e_z \rangle))_{\text{har}}$$

**Proposition 3.4.6.** $\phi_{\text{har}}$ is a group action of $\Pi_1(0)(A)\Sigma$: on $A(\langle e_z \rangle)$.

**Proof.** This follows from a formal argument. Take $g_1, g_2 \in \text{Ad}_{\Pi_1(0)(A)\Sigma}(e_1)$ and $f \in A(\langle e_z \rangle)$. We have $g_2 \circ \text{DR}_{\text{Ad}}(g_1 \circ \text{DR}_{\text{Ad}} f) = (g_2 \circ \text{DR}_{\text{Ad}} g_1) \circ \text{DR}_{\text{Ad}} f$; applying the map $\Sigma \text{inv}$ and the definition of $\phi_{\text{har}}$ gives:

$$g_2 \circ \phi_{\text{har}}(\Sigma \text{inv}(g_1 \circ \text{DR}_{\text{Ad}} f)) = (g_2 \circ \text{DR}_{\text{Ad}} g_1) \circ \phi_{\text{har}}(\Sigma \text{inv}(f)),$$

and applying again the definition of $\phi_{\text{har}}$ we deduce:

$$g_2 \circ \phi_{\text{har}}(g_1 \circ \text{DR}_{\text{Ad}} f) = (g_2 \circ \text{DR}_{\text{Ad}} g_1) \circ \phi_{\text{har}}(f).$$

**Proposition 3.4.7.** $\phi_{\text{har}}$ is continuous for the $N_D$-topology on $\Pi_1(0)(A)\Sigma$ and on $A(\langle e_z \rangle)$. The product indexed by $\mathbb{N}$ of the $N_D$-topologies on $A(\langle e_z \rangle)$.

**Proof.** This is clear, in particular using Lemma 2.1.3.

In I-2, §3.3, we have defined $\phi^\text{DR-RT}_{\text{har}}$ as a map

$$\Pi_1(0)(A)\Sigma \times A(\langle e_z \rangle)_{\text{har}} \rightarrow A(\langle e_z \rangle)_{\text{har}}$$

where $A(\langle e_z \rangle)_{\text{har}} \subset A(\langle e_z \rangle)$ is the subspace of elements $f$ such that, for all words $w$ on $e_z$, the sequence $(f[e_0^w])_{w \in \mathbb{N}}$ is constant. The formula $f \mapsto \sum_w \text{word} f[w] \sum_{w \in \mathbb{N}} e_0^w$ defines a natural isomorphism of free $A$-modules

$$\text{comp}_{A(\langle e_z \rangle)_{\text{har}}}: A(\langle e_z \rangle)_{\text{har}} \rightarrow A(\langle e_z \rangle)_{\text{har}}$$

In I-2 we have related $\phi^\text{DR-RT}_{\text{har}}$ and $\phi_{\text{har}}$ to each other, and it was a central aspect of I-2 (see I-2, §3.3 for the definition of $\phi^\text{DR-RT}_{\text{har}}$). Let us now compare $\phi^\text{DR}_{\text{har}}$ and $\phi^\text{DR-RT}_{\text{har}}$:

**Proposition 3.4.8.** The isomorphism of equation 3.4.2 induces an isomorphism of sets with group actions of $\Pi_{1,0}(A)\Sigma$, $\phi_{\text{DR}}$:

$$\text{comp}_{A(\langle e_z \rangle)_{\text{har}}}(\phi^\text{DR}_{\text{har}}): (\text{Map}(\mathbb{N}, A(\langle e_z \rangle)_{\text{har}}), \phi^\text{DR}_{\text{har}}) \simeq (\text{Map}(\mathbb{N}, A(\langle e_z \rangle)_{\text{har}}), \phi^\text{DR-RT}_{\text{har}})$$

**Proof.** More precisely, we have the following property of $\phi_{\text{DR}}$: for all $h \in \text{Ad}_{\Pi_{1,0}(A)\Sigma}(e_z)$ and $g \in \Pi_{1,0}(A)\Sigma$,

$$(Ad_g(e_z) \circ \phi_{\text{Ad}} h)[\frac{1}{1 - e_0} w] = \lim_{t \rightarrow \infty} (Ad_g(e_z) \circ \phi_{\text{Ad}} \text{comp}_{A(\langle e_z \rangle)_{\text{har}}} \Sigma \text{inv} h)[e_0^t w]$$

which follows from the formula for the dual of $\phi_{\text{Ad}}$.

**3.5. Application to the Frobenius of $\Pi_{1,0}$, $z \in Z - \{0, \infty\}$.**

**3.5.1. Notation and main statement.** Because of the previous results, and their applications to the Frobenius which we shall state below, we will use, from now on until the end of this I-3, a different notation for the Frobenius-invariant paths:

**Notation 3.5.1.** We fix $z \in Z - \{0, \infty\}$. Let $\Phi_{q, \infty}^{(z)} = \Phi_{q, \infty}^{KZ}$, and $\Phi_{q, \infty}^{(z)} = (\Phi_{q, \infty}^{(z)})^{-1}$.

For all words $w$, and $\epsilon \in \{-1, 1\}$, let $\Phi_{q, \epsilon}^{(z)}(w) = \Phi_{q, \epsilon}^{(z)}[w]$. As usual, the exponent $(z)$ can be omitted in the notation when it is equal to $(1)$. 

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We can apply §3.1 to the Frobenius by §2.3.2; this gives the following result and justifies Notation 3.5.1:

**Corollary 3.5.2.** i) For all $\tilde{\alpha} \in \mathbb{N}^*$, we have the expansion:

$$\Phi_{q,-\tilde{\alpha}} = \sum_{s \geq 0} \Phi_{q,\infty} \circ^{\text{DR}} \left( \text{pr}_s \Phi_{q,-\infty} \right) \left( q^{\tilde{\alpha}} \right)^s$$

$$\Phi_{q,\tilde{\alpha}} = \sum_{s \geq 0} \left( \text{pr}_s \Phi_{q,\infty} \right) \circ^{\text{DR}} \Phi_{q,-\infty} \left( q^{\tilde{\alpha}} \right)^s$$

For the $\mathcal{N}_D$-topology on $\Pi_{z,0}(K)$ we have:

$$\Phi_{q,-\tilde{\alpha}} \rightarrow_{\tilde{\alpha} \rightarrow \infty} \Phi_{q,\infty} \circ^{\text{DR}} \left( \Phi_{q,-\infty} \circ^{\text{DR}} \Phi_{q,-\tilde{\alpha}} \right)$$

iii) For $\tilde{\alpha} \in \mathbb{N}^*$, the Frobenius $F_{\tilde{\alpha}}^{\circ} \Phi$ has an expansion:

$$F_{\tilde{\alpha}}^{\circ} \left( f \right) = \sum_{s \geq 0} \Phi_{q,\infty} \circ^{\text{DR}} \left( \left( \Phi_{q,-\infty} \circ f \right) \text{pr}_s \right) . \left( q^{\tilde{\alpha}} \right)^s$$

For $\tilde{\alpha} \in \mathbb{N}^*$, the Frobenius $\phi^{\tilde{\alpha}} \Phi$ has an expansion:

$$\phi^{\tilde{\alpha}} \left( f \right) = \sum_{s \geq 0} \Phi_{q,\infty} \circ^{\text{DR}} \left( \left( \Phi_{q,-\infty} \circ f \right) \text{pr}_s \right) . \left( q^{\tilde{\alpha}} \right)^s$$

**Proof.** The statements concerning $\Phi_{q,-\tilde{\alpha}}$ and $F_{\tilde{\alpha}}^{\circ} \Phi$ are, knowing §2.3.2, the consequence of Corollary 3.2.3 and Proposition 3.2.4. The rest follows from that $\Phi_{q,\tilde{\alpha}} = \Phi_{q,-\tilde{\alpha}}^{-1}$, from the compatibility of $\Phi_{q,\tilde{\alpha}}$ and $\circ^{\text{DR}}$ (§2.4.2) and from the continuity of $g \mapsto g^{-1}$ for the $\mathcal{N}_D$-topology. □

Let us note that these facts reprove, in our extremely particular case, the existence and uniqueness of a Frobenius-invariant path which are known by the general theory of Coleman integration. When $N = 1$, a statement equivalent to the information concerning the limit of $\Phi_{p,\alpha}$ when $\alpha \rightarrow \infty$ for the $\mathcal{N}_L \Lambda^D$-topology appears in [P2], proof of Proposition 3.1.3.

**3.5.2. Application to $\zeta_{q,\alpha}$.** The considerations of §3.1.3 can be applied to the Frobenius via the three following formulas, valid for all $\tilde{\alpha}, \tilde{\alpha}_0 \in \mathbb{N}^*$ such that $\tilde{\alpha}_0|\tilde{\alpha}$:

$$\Phi_{q,\infty}^{(z)} = \text{fix}_{\tilde{\alpha}_0} \left( \Phi_{q,\tilde{\alpha}}^{(z)} \right); \quad \Phi_{q,\alpha}^{(z)} = \text{el}_{\tilde{\alpha}_0} \Phi_{q,\tilde{\alpha}}^{(z)} \left( \Phi_{q,\tilde{\alpha}} \right); \quad \Phi_{q,\tilde{\alpha}}^{(z)} = \text{el}_{-1} \Phi_{q,\tilde{\alpha}}^{(z)}$$

This gives:

**Corollary 3.5.3.** i) Let $s \in \mathbb{N}^*$. The $\mathbb{Q}$-vector spaces $Z_s^{(z)}$ generated respectively by cyclotomic $p$-adic multiple zeta values $\zeta_{p,\alpha}^{(z)}$ of weight $s$, associated with $\alpha \in \mathbb{Z} \cup \{\pm \infty\} \setminus \{0\}$, $\epsilon \in \{-1,1\}$ and $z \in \{z_1, \ldots, z_N\}$, are all equal to each other.

ii) $\mathcal{N}_L \Lambda^D \left( \zeta_{p,\alpha}^{(z)} \right)$ is independent of $\alpha \in \mathbb{Z} \cup \{\pm \infty\} \setminus \{0\}$, $\epsilon \in \{-1,1\}$ and $z \in \{z_1, \ldots, z_N\}$.

The i) when $N = 1$ and $\alpha \in \{1,-\infty\}$, is almost implicit in [P2], statement before Example 2.10; the Example 2.10 of [P2] consists of the relation between $\Phi_{p,1}$ and $\Phi_{p,-\infty}$ in depth one and two, when $N = 1$.

Let us write more generally the first examples of this corollary. The notation $\zeta^{\Lambda^D}$ refers to adjoint multiple zeta values, defined in the Appendix of I-2.

**Example 3.5.4.** i) In depth one and for $\mathbb{P}^1 - \{0,1,\infty\}$, we have: for all $\alpha_0, \alpha \in \mathbb{N}^*$ such that $\alpha_0|\alpha$:

$$\zeta_{p,\alpha}^{(z)} = \zeta_{p,\infty}^{(z)} + \left( p^\alpha \right)^s \zeta_{p,\infty}^{(z)}(s) = (1 - (p^\alpha)^s) \zeta_{p,\infty}^{(z)}(s)$$

$$\zeta_{p,\alpha}^{(z)} = \frac{\left( p^\alpha \right)^s - 1}{17} \zeta_{p,\alpha_0}^{(z)}(s)$$
and for all $\alpha \in \mathbb{N}^* \cup \{\infty\}$:

$$\zeta_{p,\alpha}(s) = -\zeta_{p,-\alpha}(s)$$

ii) In depth two and for $\mathbb{P}^1 - \{0, 1, \infty\}$, we have, for all $\alpha \in \mathbb{N}^*$:

$$
\zeta_{p,\alpha}(s_2, s_1) = \left(\frac{p^{\alpha}}{p^{\alpha_0}}\right)_{s_2 + s_1 - 1} \zeta_{p,\alpha_0}(s_2, s_1) + \sum_{r_2 = 0}^{s_2-1} \left(\frac{p^{\alpha}}{p^{\alpha_0}}\right)_{s_2 + s_1 - 1} \zeta_{p,\alpha_0}(s_2 - r_2) \zeta_{p,\alpha_0}^A(r; s_1) \\
+ \sum_{r_2 = 0}^{s_2-1} \frac{1}{p^{r_2+s_1+1}} \left(\frac{p^{\alpha}}{p^{\alpha_0}}\right)_{s_2 + s_1 - 1} \zeta_{p,\alpha_0}(s_2 + r) \zeta_{p,\alpha_0}(s_1 - r)
$$

$$
\zeta_{p,\alpha}(s_2, s_1) = \zeta_{p,\alpha}(s_2 + s_1) + \sum_{r_1 = 0}^{s_2-1} \left(\frac{-s_2}{r_1}\right) \zeta_{p,-\alpha}(s_2 + r_1) \zeta_{p,\alpha}(s_1 - r_1) + \sum_{r_2 = 0}^{s_2-1} \zeta_{p,\alpha_0}^A(r_2; s_1) \zeta_{p,\alpha_0}(s_2 - r_2)
$$

And, for all $\alpha \in \mathbb{N}^* \cup \{\infty\}$:

$$
\zeta_{p,\alpha}(s_2, s_1) = -\zeta_{p,-\alpha}(s_2, s_1) + \sum_{r_1 = 0}^{s_2-1} \left(\frac{-s_2}{r_1}\right) \zeta_{p,-\alpha}(s_2 + r_1) \zeta_{p,\alpha}(s_1 - r_1) \\
+ \sum_{r_2 = 0}^{s_2-1} \zeta_{p,\alpha_0}^A(r_2; s_1) \zeta_{p,-\alpha}(s_2 - r_2)
$$

3.5.3. Application to $\text{har}_{q^\varphi}$. Let us recall the expression of prime weighted multiple harmonic sums as infinite sums of cyclotomic $p$-adic multiple zeta values (I-2, §1, Corollary I-2.a) : for all indices $(w, \tilde{w}) = (e_{z_{d+1}} e_{0}^{d-1} e_{z_{d}} \ldots e_{1}^{d-1} e_{z_{1}}\ldots, z_{d}, \ldots, z_{1})$, we have:

$$
\text{har}_{q^\varphi}(\tilde{w}) = (-1)^d \sum_{z \in \mathbb{Z} - \{0, \infty\}} z^{-p^\varphi} \text{Ad}_{q^\varphi,\alpha}(e_z) \left[ \frac{1}{1 - e_0^{-1} w} \right]
$$

We have defined (I-2, §3.1) a notion of weighted multiple harmonic sums for words that are not "convergent at 0", i.e. words of the form $w e_l^r$, $r > 0$. They are elements of $K[l]/I$ with $I$ a formal variable (which does not represent a branch of the $p$-adic logarithm, but a power of $e_0$ in a word $e_0^l w$, $l \in \mathbb{N}$). It is convenient to adopt the following notation:

**Notation 3.5.5.** i) For all $(w, \tilde{w})$ as above, let

$$
\text{har}_{q^\varphi}(\tilde{w}) = (-1)^d \sum_{z \in \mathbb{Z} - \{0, \infty\}} z^{-1} \text{Ad}_{q^\varphi,\alpha}(e_z) \left[ \frac{1}{1 - e_0^{-1} w} \right]
$$

ii) Let $\text{har}_{q^\varphi} = (\text{har}_{q^\varphi}(\tilde{w}))_{(\tilde{w} \text{ word})}$ and $\text{har}^\infty_{q^\varphi} = (\text{har}_{q^\varphi}(\tilde{w}))_{(\tilde{w} \text{ word})}$; they are both elements of $K[l]/\langle e_Z \rangle_{\text{har}}^\mathbb{R}$.

**Definition 3.5.6.** Let $\tilde{e}_{\text{har}, \text{DR}} = \sum_{\in \mathbb{R}} \tilde{e}_l^{\text{DR, \text{weighted}}}$, where $\tilde{e}_l^{\text{DR, \text{weighted}}}$ is defined in Proposition 3.1.2.

**Proposition 3.5.7.** Let $(w, \tilde{w})$ as above. For each $\alpha \in \mathbb{N}^*$, we have the expansion :

$$
\text{har}_{q^\varphi} = \tau(q^\varphi) \Phi_{q^\varphi,\alpha} \text{har}_{q^\varphi}^\infty
$$

In other words, the map $q^\varphi \mapsto \text{har}_{q^\varphi}(w)$ is the analytic function characterized by :

$$
\text{har}_{q^\varphi}(\tilde{w}) = \sum_{z \in \mathbb{Z} - \{0, \infty\}} z^{-1} \sum_{\alpha \geq 0} (q^\alpha) (\text{Ad}_{q^\varphi,\alpha}(e_z)) \text{Ad}_{q^\varphi,\alpha}(e_z) \left[ \frac{1}{1 - e_0^{-1} w} \right]
$$

In particular, we have

$$
\text{har}_{q^\varphi}(\tilde{w}) \underset{\alpha \to \infty}{\to} \text{har}_{q^\varphi}^\infty
$$

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Proof. We have, by definition, \( \Phi_{q,-\infty}^{(z)} \circ DR \tau(q^{\bar{z}})(\Phi_{q,-\infty}^{(z)}) = \Phi_{p,-\infty}^{(z)} \), whence \( \Phi_{q,\bar{z}}^{(z)} = \tau(q^{\bar{z}})(\Phi_{q,-\infty}^{(z)}) \circ DR \Phi_{q,-\infty}^{(z)} \); whence by Lemma 2.4.2

\[
\text{Ad}_{q,z}^{(z)}(e_{z}) = \sum_{s \geq 0} (q^{\bar{z}})^{s}(pr_{s} + 1) \text{Ad}_{q,z}^{(z)}(e_{z}) \circ \text{Ad}_{q,z}^{(z)}(e_{z})
\]

whence the result by equation (3.5.3) and §3.1.1, inverting an absolutely convergent double sum and using that \( z^{-\bar{z}} = z^{-1} \) for all \( \bar{z} \in \mathbb{N}^* \).

\[\Box\]

Remark 3.5.8. The previous Proposition indicates that \( e_{\text{har}}^{(z)} \) arises in a natural way from a map:

\[
\text{Map}(\mathbb{N}, \tilde{\Pi}_{1,0}(A)) \times A(\langle e_{Z} \rangle)_{\text{har}} \rightarrow \text{Map}(\mathbb{N}, A(\langle e_{Z} \rangle)_{\text{har}})
\]

which can be thought of as another form of \( e_{\text{har}}^{(z)} \), and which is not a group action.

Remark 3.5.9. Let a word \( \tilde{w} = (z_{d+1}, \ldots, z_{1}) \). The coefficients of the analytic expansion of \( \text{har}_{q^{\bar{z}}}(\tilde{w}) \) in terms of \( q^{\bar{z}} \) with respect to degrees in \( \{1, \ldots, \min s_{1} = 1 \} \) vanish.

This follows from that \( \Phi_{q,-\infty}^{(z)}[e_{0}] = 0 \) which implies \( \text{Ad}_{q,z}^{(z)}(e_{z}) = e_{z} + \text{terms of depth } \geq 2 \).

Example 3.5.10. i) In depth one and for \( \mathbb{P}^{1} - \{0, 1, \infty \} \), we have, for all \( \alpha \in \mathbb{N}^* \):

\[
\text{har}_{q^{\bar{z}}}(s) = \text{har}_{q^{\bar{z}}}(s_{1}) + \sum_{s_{2} \leq s_{1}} (q^{\bar{z}})^{s_{2}} \text{Ad}_{q,-\infty}^{(z)}(e_{1})[e_{0}^{s_{1} - s_{2}} e_{1} e_{0}^{s_{2} - 1} e_{1}]
\]

ii) In depth two and for \( \mathbb{P}^{1} - \{0, 1, \infty \} \), we have, for all \( \alpha \in \mathbb{N}^* \):

\[
\text{har}_{q^{\bar{z}}}(s_{2}, s_{1}) = \text{har}_{q^{\bar{z}}}(s_{2}, s_{1}) + \sum_{s_{2} \leq s_{1}} (q^{\bar{z}})^{s_{2}} \text{Ad}_{q,-\infty}^{(z)}(e_{1})[e_{0}^{s_{1} - s_{2}} e_{1} e_{0}^{s_{2} - 1} e_{1} e_{0} e_{1}^{r_{1}}]
\]

\[
+ \sum_{r_{1} = 0}^{s_{2} - 1} \sum_{s_{1} \geq s_{2} + r_{1}} (q^{\bar{z}})^{s_{1} + r_{1}} \text{Ad}_{q,-\infty}^{(z)}(e_{1})[e_{0}^{s_{1} - r_{1}} e_{1} e_{0}^{s_{2} - 1} e_{1} e_{0}^{r_{1}}]
\]

3.6. The power of the Frobenius as a variable on the bundle of paths starting at \( \tilde{1}_{0} \). Since the Frobenius action on the fundamental bundle of paths is entirely characterized by \( \zeta_{q,\alpha} \) and \( \text{Li}_{q,\alpha}^{1} \), by §3.5 it remains to study how \( \text{Li}_{q,\alpha}^{1} \) depends on \( \alpha \). For simplicity, we restrict to \( \mathbb{P}^{1} \), knowing that \( \text{Li}_{q,\alpha}^{1} \) is characterized as an element of \( \frak{A}(U_{0}^{\mathbb{N}}) \) by its restriction to \( \mathbb{P}^{1} \).

3.6.1. \( \text{Li}_{q,\alpha}^{1} \) as a function of \( \alpha \) on \( \mathbb{P}^{1} \). The formula for the equations (2.2.) and (2.2.) of horizontality, and the formula for the analytic expansion of \( \text{Li}_{p}^{1} \) on \( \mathbb{P}^{1} \) (I-2, §3.1, Lemma 3.1.1 and Lemma 3.1.3) give the dependence of \( \text{Li}_{q,\alpha}^{1} \) as a function of \( \alpha \) on \( \mathbb{P}^{1} \). Let us denote by \( \sigma \) the non-weighted multiple harmonic sums:

\[
\sigma_{n}(z_{d+1}, \ldots, z_{1}) = \sum_{0 < n_{1} < \ldots < n_{d} < n} \frac{(z_{d+1})^{n_{1}} \ldots (z_{d+1})^{n_{2}} \ldots (z_{d})^{n_{d-1}}}{n_{1}^{d} \ldots n_{d}^{n}}
\]

Proposition 3.6.1. (rough version) For \( z \in \mathbb{P}^{1} \), we have

\[
\text{Li}_{p,\alpha}^{1}(w) = \sum_{n_{1}, n_{2} \in \mathbb{N}} \sum_{n_{1} + p^{n} n_{2} = n} c_{n_{1}, n_{2}}(w) z^{p^{n} n_{1} + n_{2}}
\]

with \( c_{n_{1}, n_{2}}(w) \) of the form

\[
c_{n_{1}, n_{2}}(w) = \sum_{\text{weight}(w_{1}) = n_{1}} (p^{n})^{\text{weight}(w_{1})} \frac{\sigma_{n_{1}}(w_{1}) \sigma_{n_{2}}(w_{2})}{n_{1}^{p^{n} n_{2}} L_{1}^{p^{n} n_{2}} \tau_{p,\alpha}(w_{3})}
\]
We leave to the reader the explicit formulation of the conditions (***) and (**'), which can be written by using the formulas for the dual of the composition of series and adjoint action of $\varepsilon$, which appear in I-1, §2.3.1. A different way to express this formula, which takes into account the analytic nature of $L^{{\dagger}}_{p,\alpha}$ and thus is more precise, is the following.

3.6.2. $\text{comp}_{\mathcal{A}(U_{0,\infty}^n)} L^{{\dagger}}_{q,\epsilon\alpha}$ as a function of $\alpha$ on $[0]$ . In I-1, §4.1, we defined an isomorphism of normed $K$-vector spaces $\text{comp}_{\mathcal{A}(U_{0,\infty}^n)}$ from the $K$-vector space $\mathcal{A}(U_{0,\infty}^n)$ of rigid analytic functions over $U_{0,\infty}^n$ to a certain explicit subspace of the maps $\lim Z/Np^\infty Z \to K$ (where $\lim Z/Np^\infty Z$ is isomorphic to a topological space to the disjoint union of $N$ copies of $Z_p$, and we have a natural inclusion $\mathbb{N}^* \subset \lim Z/Np^\infty Z$ defined by partitioning $\mathbb{N}^*$ into the classes of congruence modulo $N$ and taking $p$-adic completions). The restriction of $\text{comp}_{\mathcal{A}(U_{0,\infty}^n)}$ to the subspace of codimension 1 of $\mathcal{A}(U_{0,\infty}^n)$ of functions vanishing at 0 was defined through

$$\left( f : z \mapsto \sum_{n>0} c_n z^n \right) \mapsto \left( c : n \in \mathbb{N}^* \mapsto c_n \in K \right)$$

where the right-hand side above extended in a canonical way to a function of $\lim Z/Np^\infty Z$ by a continuity property. Thus, for every word $w$, we call $\text{comp}_{\mathcal{A}(U_{0,\infty}^n)} L^{{\dagger}}_{p,\epsilon\alpha}[w]$ the map $n \in \mathbb{N}^* \mapsto L^{{\dagger}}_{p,\epsilon\alpha}[w][z^n] \in K$ where $[z^n]$ refers to the coefficient of degree $n$ of the series expansion at 0.

We have proved in I-1, §6 that the map $\text{comp}_{\mathcal{A}(U_{0,\infty}^n)} L^{{\dagger}}_{p,\epsilon\alpha}[w]$ satisfies the following: for all $n_0 \in Z_p$ and $i \in \{1, \ldots, N\}$, there exist sequences $(\text{comp}_{\mathcal{A}(U_{0,\infty}^n)} L^{{\dagger}}_{p,\epsilon\alpha}[w])_{i \in \mathbb{N}} \subset K_{\mathbb{N}}$, such that, for $n \in \mathbb{N}^*$ such that $|n - n_0|_p \leq p^{-\alpha}$ we have the absolutely convergent series expansion:

$$\sum_{i=1}^{N} \text{comp}_{\mathcal{A}(U_{0,\infty}^n)} L^{{\dagger}}_{p,\epsilon\alpha}[w](n) = \sum_{i=1}^{N} \text{comp}_{\mathcal{A}(U_{0,\infty}^n)} L^{{\dagger}}_{p,\epsilon\alpha}[w](n_0) \xi^{-in} (n - n_0)^i$$

For $r \in \mathbb{N}^*$ let

$$\mathbf{Har}(p^r) = \text{Vect}_{\mathbb{Q}} \{ \left( \sum_{L \in \mathbb{N}} F_L(\xi, \ldots, \xi^{N-1}) \text{har}_{r}(w_L) \prod_{\eta=1}^{\eta_0} \text{har}_{p^\infty}(w_{L,\eta}) \right)_{(p,\alpha)} \mid (\ast) \} \subset \prod_{(p,\alpha) \in \mathcal{P}_N \times \mathbb{N}^*} \mathbb{Q}_p(\mu_N)$$

and let

$$\mathbf{Har}_{p^r} = \text{Vect}_{\mathbb{Q}} \{ \left( \sum_{L \in \mathbb{N}} F_L(\xi, \ldots, \xi^{N-1}) \prod_{\eta=1}^{\eta_0} \text{har}_{p^\infty}(w_{L,\eta}) \right)_{(p,\alpha)} \mid (\ast') \} \subset \prod_{(p,\alpha) \in \mathcal{P}_N \times \mathbb{N}^*} \mathbb{Q}_p(\mu_N)$$

where $(\ast)$ resp. $(\ast')$ means that $(w_{L,1})_{L \in \mathbb{N}}, \ldots, (w_{L,\eta})_{L \in \mathbb{N}} (\eta_0 \in \mathbb{N}^*)$ (resp. and $(\tilde{w}_L)_{L \in \mathbb{N}}$) are sequences of words on $\epsilon^2$ satisfying $\sum_{\eta=1}^{\eta_0} \text{weight}(w_{L,\eta}) \to_{i\to\infty} \infty$ and $\limsup_{L \to \infty} \sum_{\eta=1}^{\eta_0} \text{depth}(w_{L,\eta}) < \infty$, (resp. weight($\tilde{w}_L$) + $\sum_{\eta=1}^{\eta_0} \text{weight}(w_{L,\eta}) \to_{L \to \infty} \infty$ and $\limsup_{L \to \infty} \text{depth}(\tilde{w}_L) + \sum_{\eta=1}^{\eta_0} \text{depth}(w_{L,\eta}) < \infty$) and that $(F_L)_{L \in \mathbb{N}}$ is a sequence of elements of $\mathbb{Q}$ if $N = 1$, resp. of $\mathbb{Q}[T_1, \ldots, T_{N-1}, T_{N-1}^{-1}, \ldots, T_1^{-1}, \ldots, T_N^{-1}]$ if $N \neq 1$. We have proved in I-1 that, when $\alpha$ and $p$ vary, $\text{comp}_{\mathcal{A}(U_{0,\infty}^n)} L^{{\dagger}}_{p,\epsilon\alpha}[w](\epsilon, S)(n_0)$ define elements of $\mathbf{Har}_{p^r}$ when $n_0 = 0$ and elements of $\mathbf{Har}_{p^r}$ when $n_0 \in \{1, \ldots, q^d - 1\}$ (I-1,§4.5). We deduce now:

**Corollary 3.6.2.** For each $n_0 \in \{0, \ldots, p^{-\alpha} - 1\}, \epsilon \in (-1, 1), i \in \{1, \ldots, N\}$, there exists a sequence $(\kappa_{q,\epsilon}[w]\epsilon,S(n_0))_{S \subseteq \mathbb{N}^*}$ (it does not depend on $\alpha$) such that, for $n_0 \in \mathbb{N}^*$ satisfying $|n - n_0|_p \leq p^{-\alpha}$, we have

$$\text{comp}_{\mathcal{A}(U_{0,\infty}^n)} L^{{\dagger}}_{q,\epsilon\alpha}(n) = \sum_{S \geq 0} (q^d)^S \left( \sum_{L \in \mathbb{N}} \kappa_{q,\epsilon}[w]\epsilon,S(n_0) \xi^{-in} (n - n_0)^i \right)$$
This formula extends to $n \in \lim_{\delta \to \infty} \mathbb{Z}/N^p\mathbb{Z}$ by continuity.

**Proposition 3.6.3.** i) For all $z \in [0,1]$, we have, for the $N_{\lambda,D}$-topology:

$$
\tau(q^{-\tilde{\alpha}}) \text{Li}_{q,\tilde{\alpha}}^1(z) \to_{\tilde{\alpha} \to \infty} \text{Li}_{\tilde{q}}^{KZ}(z)(e_0,e_{z_1},\ldots,e_{z_N})
$$

$$
\tau(q^{-\tilde{\alpha}}) \text{Li}_{q,\tilde{\alpha}}^{KZ}(z) \to_{\tilde{\alpha} \to \infty} \text{Li}_{\tilde{q}}^{KZ}(z)(e_0,\text{Ad}_{\phi_{\tilde{q},1}}(e_{z_1}),\ldots,\text{Ad}_{\phi_{\tilde{q},N}}(e_{z_N}))
$$

ii) Moreover, this convergence is uniform on all the closed disks of center 0 and radius $\rho < 1$.

**Proof.** $\tau(q^{-\tilde{\alpha}}) \text{Li}_{q,\tilde{\alpha}}^1(z)$ is the product of $\text{Li}_{\tilde{q}}^{KZ}(z)(e_0,e_{z_1},\ldots,e_{z_N})$ by

$$
\tau(q^{-\tilde{\alpha}}) \text{Li}_{p,\tilde{q},z_q^{\tilde{\alpha}}}(z^{\tilde{\alpha}})(e_0,\text{Ad}_{\phi_{\tilde{q},1}}(e_{z_1}),\ldots,\text{Ad}_{\phi_{\tilde{q},N}}(e_{z_N}))^{-1}
$$

Each coefficient of $(3.6.4)$ relative to a word $w$ is a sum, over certain couples of words $(w_1,w_2)$, of terms of the following form, with $w_1,w_2$ words, $L \in \mathbb{N}^+$:

$$
q^{-\tilde{\alpha}} \sum_{n \geq 0} \sigma_{q,n}(w_1)z^{\tilde{\alpha}n}(q^{\tilde{\alpha}})^{\tilde{\alpha}} \cdot \tilde{\alpha} \log(q) + \log(n)
$$

For all $m \in \mathbb{N}^+$ we have $v_p(m) \geq \frac{\log(n)}{\log(p)}$, thus

$$
v_p(\text{har}_{q,n}(w)) \geq -\text{weight}(w) \cdot \frac{\tilde{\alpha} \log(q) + \log(n)}{\log(p)}
$$

For all $C,C' \in \mathbb{R}^+$, and $z \in [0,1]$, we have

$$
q^{\tilde{\alpha}} n v_p(z) - C\tilde{\alpha} - C' \log(n) \to_{\tilde{\alpha} \to \infty} +\infty
$$

and this convergence is uniform with respect to $n$. Indeed, let $n_0$ such that for all $n \geq n_0$, we have $C' \log(n) \leq \frac{\log(n)}{\log(p)} v_p(z)$; then $n_0$ is independent of $\tilde{\alpha}$ and we have, for all $n \geq n_0$, $q^{\tilde{\alpha}} n v_p(z) - C\tilde{\alpha} - C' \log(n) \geq q^{\tilde{\alpha}} n v_p(z) - C\tilde{\alpha}$. Because of the bounds of valuations on cyclotomic $p$-adic multiple zeta values (I-1, §1, appendix to Theorem I-1, or alternatively, their consequences stated in §3.3 of this paper) the sequence $(N_{\lambda,D}(\Phi_{q,-\alpha}))_{\tilde{\alpha} \in \mathbb{N}^+}$ is bounded. Thus, we can deduce i) for $\tau(q^{-\tilde{\alpha}}) \text{Li}_{q,\tilde{\alpha}}^1(z)$ by an application of the dominated convergence theorem. The ii) for $\tau(q^{-\tilde{\alpha}}) \text{Li}_{q,\tilde{\alpha}}^1(z)$ comes from that $n_0$ can be chosen the same for each $z$ in a closed disk of center 0 and radius $\rho < 1$.

The proofs concerning $\tau(q^{-\tilde{\alpha}}) \text{Li}_{q,\tilde{\alpha}}^1(z)$ are similar.

**Remark 3.6.4.** We placed ourselves only on $[0,1]$. The rest of $U_{an}^{\mathbb{N}}(\mathbb{K})$ can be treated by using what follows.

The set $P^1(K)$ is the disjoint union of the tubes $[0,\infty]$ and the $q-1$ tubes of the roots of unity of order $q-1$; then, $U_{an}^{\mathbb{N}}(\mathbb{K}) = P^1(K) - \cup_{n \in \mathbb{N}^+}[z_n](K)$ is the union of $[0,\infty]$ and certain of the tubes of roots of unity (in certain cases, none of them); when $N = 1$, they are given by $U_{an}^{\mathbb{N}}(\mathbb{Q}_p) = [0,\infty] \cup [2,\ldots]\cup [p-1]$. We have of course $z^n \to_{n \in \mathbb{N}} \infty$ if $z \in ]\infty[,]$ and if $|z|_p = 1$, we have $\lim_{a \to \infty} z^{p^a} = \omega(z)$ where $\omega : \mathbb{C}_p^\times \to$ { roots of 1 in $\mathbb{C}_p^\times$ of order prime to p } is the Teichmüller character : for any $z \in \mathbb{C}_p^\times$ such that $|z|_p = 1$, $\omega(z)$ is the unique root of unity of order prime to $p$ such that $|z - \omega(z)|_p < 1$.

The value of $\text{Li}_{q,\tilde{\alpha}}$ at any point is expressed by the formula for $\text{comp}_{2}(U_{an}^{\mathbb{N}})$ given in I-1, §4.1.2. One can express the series expansion of $\text{Li}_{q,\tilde{\alpha}}$ at any point by using the formula of composition of paths for iterated integrals. The isomorphism $z \mapsto \frac{1}{2}$ of $X_K$ exchanges $[0]$ and $]\infty[$.
**Remark 3.6.5.** One cannot hope that the convergence of Proposition extends to $U_{\infty}^{an}$ while being uniform on $U_{\infty}^{an}$. Indeed, otherwise the map $L_{p}^{KZ}$ would be rigid analytic on $U_{\infty}^{an}$; by the characterization of the algebra of rigid analytic functions $U_{\infty}^{an}$ from I-1 §4.1, this would imply that, for any word $w$ on $e_{p}$, the multiple harmonic sums functions $n \mapsto \text{har}_{n}(w)$ restricted to classes of congruences modulo $N$ should be continuous. This contradicts the computations of the next paragraph (§4).

### 4. Properties of multiple harmonic sums

We review from I-2, §4, some definitions and properties concerning multiple harmonic sums. We deduce from it some information on how $\text{har}_{n}$ depends on $n$ viewed as a $p$-adic variable.

#### 4.1. Preliminaries

In I-2, we introduced the following generalization of multiple harmonic sums. Let $G_{\alpha}$ be the subgroup of $\text{Hom}_{\text{gp}}(K^{*}, K^{*})$ made of elements $\chi$ such that, for all $\epsilon \in K$ satisfying $|\epsilon|_{p} \leq \frac{1}{p^{r}}$, we have an absolutely convergent expansion $\chi(1 + \epsilon) = \sum_{i \geq 0} \chi^{(i)}(1) \epsilon^{i}$ with $(\chi^{(i)}(1))_{i \in \mathbb{N}} \in K^{\mathbb{N}}$. Let $\text{wt} : (G_{\alpha}, \times) \to (K, +)$, the weight map, be the morphism of groups that sends $\chi \mapsto -\frac{\log(\chi(p))}{\log_{p}(p)}$. Let $W'(e_{Z})$ be the set of sequences $(z_{i_{d+1}}, \ldots, z_{i_{1}})$ with $d \in \mathbb{N}^{*}$, where $i_{1}, \ldots, i_{d+1} \in \{1, \ldots, N\}$, and $\chi_{d}, \ldots, \chi_{1} \in \text{Vect}_{K}(G_{\alpha}); d$ is called the depth of such a sequence. For all $d \in \mathbb{N}^{*}$, let $W'_{d}(e_{Z}) \subset W_{d}(e_{Z})$ be the subset of words of depth $d$.

To each element $w = (z_{i_{d+1}}, \ldots, z_{i_{1}})$ of $W'(e_{Z})$ and $(n_{0}, n) \in \mathbb{Z}^{2}$ such that $0 \leq n_{0} < n$ or $n_{0} < n \leq 0$, we associate the multiple harmonic sum:

$$\sigma_{n_{0}, n}(w) = z_{i_{1}}^{n_{0}} \left( \sum_{n_{0} < n_{1} < \ldots < n_{d} < n} \prod_{j=1}^{d} \left( \frac{z_{i_{j+1}}}{z_{i_{j}}} \right)^{n_{j}} \chi_{j}(n_{j}) \right) \frac{1}{z_{i_{d+1}}}$$

and the weighted multiple harmonic sum

$$\text{har}_{n_{0}, n}(w) = (n - n_{0})^{	ext{wt}(\chi_{d} \ldots \chi_{1})} \sigma_{n_{0}, n}(w)$$

We omit $n_{0}$ in the notations $\text{har}_{n_{0}, n}$ and $\sigma_{n_{0}, n}$ when $n_{0} = 0$.

The results of the next paragraphs apply to this more general notion of multiple harmonic sums, although we will restrict, for the simplicity of the statements, to the usual values of $\chi_{d}, \ldots, \chi_{1}$.

#### 4.2. Multiple harmonic sums and the $p$-adic expansion of their upper bound

The question of studying $\tilde{n}$ as a variable applied to prime weighted multiple harmonic sums $\text{har}_{q^{n}}$ is a particular case of the more general and natural question of studying $\text{har}_{n}$, or equivalently $\sigma_{n}$, as a function of $n$ viewed as a $p$-adic variable.

In I-1, one of the main technical points was to show that the variant $\text{har}_{n}^{1}$ of multiple harmonic sums defined through the series expansion of $L_{p}^{K_{p}}$ extended into a certain type of locally analytic function of $n$ on $\mathbb{Z}_{p}^{(N)} = \lim_{\leftarrow} \mathbb{Z}/Np^{\nu}Z$, as recalled in §3.4.2.

We did not treat in I-1 the question of studying $\text{har}_{n}$ itself as a function of $n$ viewed as a $p$-adic variable. Let us state one feature of this problem. Other features of this problem will appear in the final version of this work. The next proposition says, roughly speaking, that $\text{har}_{n}$ is a rational function of the coefficients of the decomposition of $n$ in base $q$, whose coefficients are $\text{har}_{q^{\nu}}$ such that $q^{\nu}$ occurs in the decomposition of base $q$ of $n$ (it remains of course true when we replace $q$ by $p$).

**Proposition 4.2.1.** (see below for the definition of the condition $(\ast)$)

1. Let $n \in \mathbb{N}^{*}$, and let its decomposition in base $q$: $n = a_{y_{d'}}q^{y_{d'}} + a_{y_{d' - 1}}q^{y_{d' - 1}} + \ldots + a_{y_{1}}q^{y_{1}}$, with $y_{d'} > \ldots > y_{1}$, and $a_{y_{d'}}, \ldots, a_{y_{1}} \in \{1, \ldots, q - 1\}$. Let $\nu_{j} = a_{y_{j}}q^{y_{j}} + \ldots + a_{y_{j - j'}}q^{y_{j - j'} + 1}$ for
\[j' \in \{1, \ldots, d'\}. \text{ We have} \]

\[
\sigma_n(w) = \sum_{n=\nu'_1, \ldots, \nu'_d} \prod_{j'=1}^{d'} \frac{1}{\nu_{j'}^{\nu_j}} \prod_{j'=0}^{d'} \sum_{l_{j'} \geq 0} \frac{1}{\nu_{j'}^{l_{j'}}} \sum_{\nu_{j'} \in \{1, \ldots, d'\} - \{j(n)\}, \text{satisfying } (\ast)} \left( \prod_{u=\nu_{j_{\min}}^{\nu_{j_{\max}}}} \left(-s_u\right) \right) \left( \sum_{l_u} a_{y_u} q_{y_u} \right) \sigma_{\nu_{j_{\prime}}^{\nu_{j_{\prime}+1}}(w) \mid j_{\prime}}.
\]

\[
\text{ii) Let } n \in \mathbb{N}^*, \text{ whose decomposition in base } q \text{ is of the form } aq^y, \text{ with } a \in \{1, \ldots, q-1\} \text{ and } y \in \mathbb{N}^*. \text{ Let } \nu_{j'} = j'q^y \text{ for } j' \in \{1, \ldots, a-1\}. \text{ We have} \]

\[
\sigma_{aq^n}(w) = \sum_{n=\nu'_1, \ldots, \nu'_d} \prod_{j'=1}^{d'} \frac{1}{\nu_{j'}^{\nu_j}} \prod_{j'=0}^{d'} \sum_{l_{j'} \geq 0} \frac{1}{\nu_{j'}^{l_{j'}}} \sum_{\nu_{j'} \in \{1, \ldots, d'\} - \{j(n)\}, \text{satisfying } (\ast)} \left( \prod_{u=\nu_{j_{\min}}^{\nu_{j_{\max}}}} \left(-s_u\right) \right) \left( \sum_{l_u} a_{y_u} q_{y_u} \right) \sigma_{\nu_{j_{\prime}}^{\nu_{j_{\prime}+1}}(w) \mid j_{\prime}}.
\]

where, in i) and ii), the condition \((\ast)\) is that \(J_0 \supseteq \ldots \supseteq J_{d'} \) is an increasing connected partition of \(\{1, \ldots, d\} - \{j(n)\}\), such that each \(J_{\nu'_{j'+1}}(w) \mid j' \) is an increasing connected partition of \((\{1, \ldots, d\} - \{j(n)\}) \cap [j(\nu'_j), j(\nu'_{j'+1})]$. In this statement, increasing connected partition means partition into sets of consecutive integers, such that each element of \(J_i\) inferior to each element of \(J_i'\) if \(i < i'\).

**Proof.** (for i) and ii)) we apply the formula of splitting of multiple harmonic sums (1-2, §6.4.2) at \(\nu_1, \ldots, \nu_r\); this gives

\[\sigma_n(w) = \sum_{n=\nu'_1, \ldots, \nu'_d} \prod_{j'=1}^{d'} \frac{1}{\nu_{j'}^{\nu_j}} \prod_{j'=0}^{d'} \sum_{l_{j'} \geq 0} \frac{1}{\nu_{j'}^{l_{j'}}} \sum_{\nu_{j'} \in \{1, \ldots, d'\} - \{j(n)\}, \text{satisfying } (\ast)} \left( \prod_{u=\nu_{j_{\min}}^{\nu_{j_{\max}}}} \left(-s_u\right) \right) \left( \sum_{l_u} a_{y_u} q_{y_u} \right) \sigma_{\nu_{j_{\prime}}^{\nu_{j_{\prime}+1}}(w) \mid j_{\prime}}.
\]

and we express each factor \(\sigma_{\nu_{j_{\prime}}^{\nu_{j_{\prime}+1}}(w) \mid j_{\prime}}\) in terms of \(\sigma_{\nu_{j_{\prime}}^{\nu_{j_{\prime}+1}}(w)}\) by the p-adic formula of shifting of multiple harmonic sums (1-2, §4.2), writing \(J' = [j_{\min'}^{j_{\max'}-1}]\). \(\square\)

**Example 4.2.2.** In depth one for \(\mathbb{P}^1 - \{0, 1, \infty\}\), the Proposition 4.2.1 with the same notations for the decomposition of \(n \in \text{base } q\), can be written as follows: for all \(s \in \mathbb{N}^*:\)

\[
(4.2.3) \quad \sigma_n(s) = \frac{\sigma_{n_{w'_{+1}}}^s(s)}{(p^{w'_{+1}})^s} + \sum_{i=1}^{d'-1} \sum_{l \geq 0} \sigma_{n_{w'_{+1}}}^s(s + l) \left( \frac{-s}{l} \right) \left( a_{w'_{+1}} p^{w'_{+1}} + \ldots + a_{w'_{i}} p^{w'_{i+1}} \right)^l
\]

\[+ \sum_{0 \leq s_j' \leq a_{y_j} - 1} \sum_{l \geq 0} \left( \frac{-s}{l} \right) \sigma_{y_j}^s(s + l) \left( \sum_{m=j+1}^{d'} a_{y_m} p^{y_m} + a_{y_j} p^{y_j} \right)^l\]

5. **Proofs involving only multiple harmonic sums**

As suggested by §3 and §4, we study in an elementary way how the prime weighted multiple harmonic sums \(\text{har}_{\alpha} \) depend on \(\alpha\) viewed as a p-adic integer, and we prove the part ii) of Theorem I-3.a.
5.1. A $p$-adic expansion of $\text{har}_{q^a}$. We fix $(\tilde{a}_0, \tilde{a}) \in \mathbb{N}^*$ such that $\tilde{a}_0 | \tilde{a}$. The first lemma is a $p$-adic expression of $\text{har}_{q^a}$, obtained by regrouping the indices $(n_1, \ldots, n_d)$ of the iterated sums underlying $\text{har}_{q^a}$ according to certain features of their $q^{n_0}$-adic expansion.

**Lemma 5.1.1.** We have, for all words:

\[
(\text{har}_{q^a})_{\tilde{z}_{i+1}^d, \ldots, \tilde{z}_1^s} = \sum_{(v_1, \ldots, v_d) \in \mathbb{N}^d} \left( \frac{1}{z_{i+1}} \right)^{q^a (\prod_{j=1}^d (\tilde{z}_{i,j} + r_j) (\tilde{r}_{i,j} + 1))} \right)
\]

**Proof.** We view each $n_j$ as an element of $\mathbb{Z}_p$ viewed as $\lim_{l \to \infty} \mathbb{Z}/(q^{n_0})^l \mathbb{Z}$. There is a unique way to write $n_j = (q^{n_0})^v (q^{n_0} u_j + r_j)$ with $v_j \in \mathbb{N}$, $u_j \in \mathbb{N}$, $r_j \in \{1, \ldots, q^{n_0} - 1\}$. Moreover, for each $z \in \mathbb{Z} - \{0, \infty\}$, since $q^{n_0}$ is a multiple of $q$ we have $z(q^{n_0} u_j + r_j) = z^{q^{n_0} u_j + r_j} = z^{u_j + r_j}$. Finally, we write $(q^{n_0} u_j + r_j)^{-s_j} = r_j^{-s_j} (q^{n_0} u_j + r_j)^{1-s_j} = \prod_{i \geq 0} (1 - s_j)^i (q^{n_0} u_j)^{s_j - r_j}$ for each $j$. \[\square\]

5.2. Elimination of the additional parameters in the $p$-adic expansion of $\text{har}_{q^a}$. We want to reindex (5.1.1) as an absolutely convergent sum indexed by powers of $\tilde{a}$ and $q^{\tilde{a}}$, and this requires to eliminate the parameters $v_i$, $u_i$, $r_i$. We must first express more conveniently the dependance of multiple harmonic sums with respect to these parameters.

5.2.1. Indexation of multiple harmonic sums in terms of the additional parameters. The condition $0 < q^{n_0} v_j (q^{n_0} u_j + r_j) < q^a$ appearing in (5.1.1) can be restated as follows.

**Lemma 5.2.1.** Let $(v_1, \ldots, v_d) \in \{0, \ldots, \frac{1}{q^{n_0}} - 1\}^d$, $(u_1, \ldots, u_d) \in \{0, \ldots, q^{n_0} - 1\}^d$, $(r_1, \ldots, r_d) \in \{1, \ldots, q^{n_0} - 1\}^d$ such that, for all $j \in \{1, \ldots, d\}$ we have $0 < q^{n_0} v_j (q^{n_0} u_j + r_j) < q^a$. Then, for all $j \in \{1, \ldots, d - 1\}$, we have:

\[
q^{n_0} v_j (q^{n_0} u_j + r_j) < q^{n_0} v_{j+1} (q^{n_0} u_{j+1} + r_{j+1}) \Leftrightarrow \begin{cases} u_j \leq q^{n_0} (v_j + v_{j+1} - 1) (q^{n_0} u_{j+1} + r_{j+1} - 1) - 1 \\
\text{if } v_j < v_{j+1} \\
u_j \leq u_{j+1} - 1 \\
\text{if } v_j = v_{j+1} \text{ and } r_j < r_{j+1} \\
u_j \leq u_{j+1} - 1 \\
\text{if } v_j = v_{j+1} \text{ and } r_j \geq r_{j+1} \\
q^{n_0} (v_j - v_{j+1} - 1) (q^{n_0} u_j + r_j) \leq u_{j+1} - 1 \\
\text{if } v_j > v_{j+1} \end{cases}
\]

**Proof.** Clear. \[\square\]

5.2.2. Suppression of the parameters $(u_1, \ldots, u_d)$ and $(r_1, \ldots, r_d)$. We sum over the parameters $u_1, \ldots, u_d$ and $r_1, \ldots, r_d$, representing the Euclidean division by $q^{n_0}$ of the indices $n_1, \ldots, n_d$ of the iterated sum $\text{har}_{q^a}$. We recall that $z_i$ is a primitive $N$-th root of unity in $K$, and $z_i = z_i^1$ for all $i \in \{1, \ldots, N\}$.

Let us recall a few notations and facts from I-1, §4.

1) For $l, m \in \mathbb{N}^2$ such that $1 \leq m \leq l + 1$, we denote by $\mathcal{B}_l^m = \mathcal{B}_l^m(1) = \frac{1}{l+1} (l+1) B_{l+1-m}$, where $B$ denotes Bernoulli numbers; these are the rational numbers characterized by: for all $n, l \in \mathbb{N}^*$, $\sum_{n=0}^{l-1} n^l = \frac{1}{l+1} \sum_{m=1}^{l+1} \mathcal{B}_l^m n^m$. For any $n \in \mathbb{N}^*$, it follows from $p^{v_p(n)} \leq n$ that $v_p(\frac{1}{n}) \geq - \frac{\log_2(n)}{\log(p)}$; moreover, for any $n \in \mathbb{N}^*$, we have $v_p(B_n) \geq -1$ (this is part of Von-Staudt Clausen’s theorem); thus

**Fact 5.2.2.** For all $l, m$, we have $v_p(B_l^m) \geq -1 - \frac{\log(l+1)}{24 \log(p)}$. \[\square\]
2) For \( i \in \{1, \ldots, N-1\} \) (thus \( z_i \neq 1 \)), we denote by \( B_{m}^i(z_i) \in \mathbb{Z}[z_i, \frac{1}{z_i}, \frac{1}{z_i-1}] \) the numbers characterized by, for all \( n, l \in \mathbb{N}^* \), \( \sum_{n_{i=0}^{n-1}} z_i^{n_{i=0}^{l-1}}n_{i=0}^{l} = z_i^{n_{i=0}^{l-1}} \sum_{n=0}^{l} B_{m}^i(z_i) n^m \); they are defined by applying \( (T_{d}^T)^l \) to the equality \( \frac{T_{d}^n-1}{T_{d}^n-1} = \sum_{n_{i=0}^{n-1}} T_{d}^n \). Since \( |z_i|_p = |z_i-1|_p = 1 \), we have:

**Fact 5.2.3.** For all \( l, m \), and \( i \neq N \), we have \( v_p(B_{m}^i(z_i)) \geq 0 \).

**Lemma 5.2.4.** Let a word \( w = \left( \frac{z_i}{z_i}, \ldots, \frac{z_1}{z_1} \right) \); \( (l_1, \ldots, l_d) \in \mathbb{N}^d \), and \( v_1, \ldots, v_d \in \{0, \ldots, \frac{d}{m} - 1\} \). For every word \( w' \) over \( e_{Z} \), there exists a polynomial

\[
P_{w, w', (l_1, \ldots, l_d), (v_1, \ldots, v_d)} \in \begin{cases} \mathbb{Z}[z_1, \frac{1}{z_1}, \ldots, \frac{1}{z_1}] & \text{if } N \neq 1 \\ \mathbb{Z}[(Q_{j, j+1})_{1 \leq i \leq l-1}, (B_{m, l, z_i})_{1 \leq i \leq l_1 + \ldots + l_d + d}] & \text{if } N = 1 \end{cases} \]

with degree at most \( l_1 + \ldots + l_d + d \) in the variables \( Q_{j, j+1} \), and with total degree at most \( d \) in the variables \( B_{m, l, z_i} \), such that we have

(5.2.1)

\[
\sum_{(u_1, \ldots, u_d) \in \mathbb{N}^d} \left( \prod_{j=1}^{d} \left( \frac{z_{j+1}}{z_j} \right)^{u_j + r_j \left( q^{o_0}u_j \right)} \right) \left( \frac{q^{o_0}(v_{j+1}-v_j-1)}{r_j^{l_j}} \right) = \sum_{w' \text{ word on } e_{Z}} P_{w, w', (l_1, \ldots, l_d), (v_1, \ldots, v_d)} \left( (q^{o_0}(v_{j+1}-v_j-1))_{1 \leq j \leq d}, (B_{m}^i(z_i))_{1 \leq i \leq l_1 + \ldots + l_d + d}] \right) \]

**Proof.** Induction on \( d \). For \( d = 1 \) we can apply the definition of the numbers \( B_{m}^i(z_i) \), \( i \in \{1, \ldots, N\} \).

For \( d > 1 \), let \( j \in \{1, \ldots, d\} \) such that \( v_j = \min(v_1, \ldots, v_d) \); the sum over \( u_j \) and \( r_j \) can be expressed through Lemma 5.2.2.2 and eliminated using the formula in depth one. By Lemma 5.2.2.1 the dependence on \( r_i \) is only via inequalities of the form \( r_i > r_{i+1} \) or \( r_i \leq r_{i+1} \), and the computation gives the generalization of multiple harmonic sums \( \text{har}_{q^{o_0}} \) involving positive and negative powers of the \( r_i \)'s, to which we can apply to them the map of elimination of positive powers of I-2, §5. 

5.2.3. Suppression of the parameter \( (v_1, \ldots, v_d) \). Starting from the formula of Lemma 5.2.4 we eliminate the parameter \( (v_1, \ldots, v_d) \) representing the \( q^{o_0} \)-adic valuation of the indices \( n_1, \ldots, n_d \) of the iterated sum \( \text{har}_{q^{o_0}} \).

**Lemma 5.2.5.** Let a word \( w = \left( \frac{z_i}{z_i}, \ldots, \frac{z_1}{z_1} \right) \); \( (l_1, \ldots, l_d) \in \mathbb{N}^d \).

For every word \( w' \) over \( e_{Z} \), there exists a polynomial

\[
P_{w, w', (l_1, \ldots, l_d)} \in \begin{cases} \mathbb{Z}[z_1, \frac{1}{z_1}, \ldots, \frac{1}{z_1}] & \text{if } N \neq 1 \\ \mathbb{Z}[\bar{Q}, (B_{m, l, z_i})_{1 \leq i \leq l_1 + \ldots + l_d + d}] & \text{if } N = 1 \end{cases} \]
with degree at most $l_1 + \ldots + l_d + d$ in $\hat{Q}$, and with total degree at most $d$ in the variables $B_{m,l,z}$, such that we have

\begin{equation}
(5.2.2) \quad \sum_{(v_1, \ldots, v_d) \in \{0, \ldots, \frac{\alpha}{d} - 1\}} \left( \prod_{j=1}^{d} \frac{z_{j+1}}{z_j} \right)^{u_j+r_j} \left( \frac{q^{\alpha} u_j}{j^{l_j+x_j}} \right)^{t_j} = \sum_{\text{w word on } \epsilon \hat{z}} \sum_{w, w', (l_1, \ldots, l_d)} P_{w,w', (l_1, \ldots, l_d)}(q^\alpha) \text{ har}_{q^{\alpha}}(w)
\end{equation}

**Proof.** The set $\{0, \ldots, \frac{\alpha}{d} - 1\}^d$ admits a partition indexed by the set of couples $(P, \sigma)$, where $P$ is a partition of $\{1, \ldots, d\}$ and $\sigma$ is a permutation of $\{1, \ldots, d\}$ defined as follows: for each $(v_1, \ldots, v_d) \in [0, k - 1]^d$, and each such $(P, \sigma)$, we say that $(v_1, \ldots, v_d) \in (P, \sigma)$ if and only if, for all, $i, i', a$ : 

\[
\begin{cases}
 v_i = v_{i'} \text{ for } i, i' \in P_{\sigma}(a) \\
 v_i < v_{i'} \text{ for } i, i' \in P_{\sigma}(a) \setminus P_{\sigma}(a+1)
\end{cases}
\]

We expressed the left hand side of (5.2.2) as the sum over values of $(v_1, \ldots, v_d)$ of the right-hand side of (5.2.1) and we split it according to the terms of this partition. We note that, since $z_i^q = z_i$ for all $i \in \{1, \ldots, N\}$, we have $z_i^{q^{\alpha}v_i} = z_i$ for all $v \in \mathbb{N}^*$, and the coefficients of $P_{w,w', (l_1, \ldots, l_d)}(v_1, \ldots, v_d)$ of Lemma 5.2.4 are independent of $(v_1, \ldots, v_d)$. We obtain an expression for the left hand side of (5.2.2) in terms of functions of the following type, where $d', k \in \mathbb{N}^*$, $I = \{i_1, \ldots, i_r\} \subset \{1, \ldots, d\}$ with $i_1 < \ldots < i_r$, and $T_{i_1, \ldots, i_r}$ are formal variables:

\begin{equation}
(5.2.3) \quad \sum_{0 \leq w_1 \ldots w_{d'} \leq k-1} T_{w_1}^{w_{d'}} = \sum_{0 \leq w_0 \leq w_1 \ldots w_{d} \leq k-1} T_{w_1}^{w_{d}} \sigma_{w_1} \left( \sigma_{w_1}, 0, \ldots, 0 \right) \prod_{j=1}^{r-1} \sigma_{w_{j+1}}(0, \ldots, 0) \sigma_{w_{j}} \leq k-1 \left( 0, \ldots, 0 \right)_{k-i_r-1} \sigma_{w_{j+1}}(0, \ldots, 0)
\end{equation}

where $\sigma$ is the variant of multiple harmonic sums defined through an iterated sum over $0 \leq n_1 < \ldots < n_d$ instead of $0 < n_1 < \ldots < n_d$. By induction on the depth, the functions $\sigma$ and $\tilde{\sigma}$ are polynomial functions of $w_1, \ldots, w_d$, whose coefficients are polynomials of the numbers $B_{m,l}$. By the next sub-lemma, one can eliminate the indices $w_1, \ldots, w_{d'}$ in the functions of the type $\tilde{\sigma}(w)$ and obtain the result. \hfill \Box

**Sub-lemma 5.2.6.** Let $d, k \in \mathbb{N}^*$; let $A_1, \ldots, A_d \in \mathbb{Q}[T]$ be polynomials, $T_1, \ldots, T_d$ formal variables. There exists $\mathbb{C}_{l_1, \ldots, l_d} \in \mathbb{Z}[\text{coefficients of } A_1, \ldots, A_d][T]$ such that we have

\[\sum_{0 \leq w_1 \ldots w_d \leq k-1} \prod_{i=1}^{d} T_i^{w_i} A_i(w_i) = \sum \mathbb{C}_{l_1, \ldots, l_d}(k) \prod_{i=1}^{d} U_i^{l_i} \left( U_i - 1 \right)^{l_i+1} \]

**Proof.** Let us denote by $F_k(A_1, \ldots, A_d)(T_1, \ldots, T_d) = \sum_{0 \leq w_1 \ldots w_d \leq k-1} \prod_{i=1}^{d} T_i^{w_i} A_i(w_i)$

1) In depth one we prove

\[\sum_{0 \leq w \leq W-1} T^w w^\alpha = \sum_{\beta} (-1)^{\beta} T^{\beta} \left( T^w - 1 \right)^{\beta} \sum_{\beta_0 + \ldots + \beta_i = \alpha - l} w^{\beta} (w + 1)^{\beta_1} \ldots (w + l)^{\beta_l}
\]

Indeed, we have $\sum_{0 \leq w \leq W-1} T^w w^\alpha = \left( T \frac{\partial}{\partial T} \right)^\alpha \left( \sum_{0 \leq w \leq W-1} T^w \right) = (T \frac{\partial}{\partial T})^\alpha (T^{W-1})$. On the other hand, for all $\alpha \in \mathbb{N}^*$, 

\[\left( T \frac{\partial}{\partial T} \right)^\alpha (T^{W-1}) = \sum_{\beta} (-n)^{\alpha - l} \left( -n - 1 \right) \ldots \left( -n - l + 1 \right) \frac{T^{W-1}}{(T-1)^{\alpha}}
\]

Whence the result by linearity with respect to $A_1$. \hfill 26
2) Now we prove the result by induction on $d$ : assume that $A_1 = \sum_{\alpha_1=0}^{\deg A_1} u_{\alpha_1} T^{\alpha_1}$ with $u_{\alpha_1} \in \mathbb{Q}$. We have, for all $\alpha_1 \in \{0, \ldots, \deg A_1\}$ :

$$F_k(T^{\alpha_1}, A_2, \ldots, A_d) = \sum_{l=0}^{\alpha_1} (-1)^l \frac{T^l}{(T_1-1)^{l+1}} \left[ F_k(B_{2,l}, A_3, \ldots, A_d) - F_k(B_{2,l}, A_3, \ldots, A_d)(T_2, \ldots, T_d) \right]$$

with $B_{2,l} = (\sum_{\beta_0 + \beta_1 = \alpha_1} T^{\beta_0} \ldots (T + l)^{\beta_1}) A_2$. Whence the result by induction and by linearity. $\square$

5.3. End of the proof. We now finish the proof of ii) of Theorem I-3.a.

**Proposition-Definition 5.3.1.** The numbers $\text{har}_{p}^{\alpha}$ have an expression valid for all $\bar{\alpha}$ as elements of the ring of absolutely convergent power series $K[[q^\alpha]](\frac{\alpha}{\bar{\alpha}})$, arising from the previous lemmas. Let the map of elevation to the $\bar{\alpha}$-th power as the map formalizing this series expansion above, with $q^\alpha$ replaced by $\Lambda$, and $\frac{\alpha}{\bar{\alpha}}$ replaced by $A$.

**Proof.** We inject the formula of the Lemma 5.2.5 in the one of Lemma 5.1.1, which amounts to summing over the tuples $(l_1, \ldots, l_d)$. We show that that the infinite sum which appears is absolutely convergent, and we invert it. This follows from the following facts :

i) We have bounds of valuations of the coefficients of the polynomials $P_{w,w'/(l_1, \ldots, l_d)}$, arising from bounds of valuations of the coefficients $B^l_{m}(z_i)$ (Fact 5.2.2) and Fact 5.2.3) and the bound of the degree of $P_{w,w'/(l_1, \ldots, l_d)}$ with respect to the variables $B_{m,l,z}$,

ii) If $T_1, T_2$ are formal variables and $m \in \mathbb{N}^*$, we have $\frac{T_1^{m+1}}{T_2^{m+1}} - \frac{T_1^{m+1} - T_1^{m}}{T_2^{m+1} - T_2^{m}} = \frac{T_1^{m+1} - T_1^{m}}{T_2^{m+1}} - \frac{T_2^{m+1} - T_2^{m}}{T_2^{m+1}}$.

Finally, let $z \in K$ with $v_p(z) \neq 0$. Then we have $v_p(\sum_{l=0}^{\alpha_1} (-1)^l \frac{T^l}{(T_1-1)^{l+1}} (z^{l+1}) > 0$ if $v_p(z) > 0$, and $v_p(\frac{1}{z^{l+1}}) > v_p(z^{-1})$ if $v_p(z) < 0$.

This finishes the proof of ii) of Theorem I-3.a. In the final version of this work, we will introduce combinatorial tools to define a simpler way to express the formula for this map. One can for example write a formula by regrouping the sequences of consecutive indices $n_i$ on which $v_p$ is constant, considering the sequence associated with the lowest value of $v_p$, eliminating the variables $r_i, u_i, v_i$ corresponding to this sequence, and so on. We will also relate this formula to the map $\Sigma^{RT}$.

**Remark 5.3.2.** The ii) of Theorem I-3.a can be seen as a $p$-adic analogue of the complex asymptotic expansion of multiple harmonic sums $\sigma_n$ when $n \to \infty$, in the ring $\mathbb{Q}[[\text{multiple zeta values}]](\frac{\gamma}{\bar{\gamma}})[\log(n)]$, where $\gamma$ is Euler’s constant.

**Remark 5.3.3.** The formula $\text{har}_{p}^{\alpha} = \text{har}_{p}^{\alpha'} \text{har}_{p}^{\alpha''}$, when restricted to $n \in \mathbb{N}$ equal to powers of $p$, can be seen as a functional equation of the map $\alpha \mapsto \text{har}_{p}^{\alpha}$, expressing $\text{har}_{p^{\alpha+s}}$ in terms of $\text{har}_{p}^{\alpha}$ and $\text{har}_{p^{s}}$.

5.4. Examples. For $\mathbb{P}^1 = \{0, 1, \infty\}$, we have $q = p \bar{\alpha} = \alpha$, $\alpha_0 = \alpha_0$. In depth one and two, the computation of $\S 5.1$-$\S 5.2$ gives respectively :

**Example 5.4.1.** For all $s \in \mathbb{N}^*$,

$$(5.4.1) \quad \text{har}_{p}^{\alpha}(s) = - \sum_{l \geq 0} \binom{-s}{l} \text{har}_{p^{\alpha_0}}(l + s) \sum_{u=1}^{l+1} \frac{B^{l}_{u}}{1 - p^{\alpha_0(u+s)}} - \sum_{u \geq s+1} \frac{p^{\alpha_u}}{1 - p^{\alpha_u}} \sum_{l \geq u-s-1} \binom{-s}{l} \text{har}_{p^{\alpha_0}}(l + s)B^{l}_{u-s}$$

**Example 5.4.2.** For all $s_1, s_2 \in \mathbb{N}^*$, $\text{har}_{p}^{\alpha}(s_2, s_1)$ is the sum of the following subsums, where the variables $v_1, v_2$ are those defined in Lemma 5.1.1.
• its restriction to $v_1 = v_2$, equal to:

$$\sum_{u \geq 1} \frac{p_0^u} {\rho_0(p_0 + s_2 - u)} \prod_{i=1}^{\frac{2}{l_i}} (-s_i) \left( B_u \prod_{i=1}^{l_i} \varphi_i = \rho_0(s_1 + l_i) + \sum_{i=1}^{l_i+2} \varphi_i \rho_0(s_2 + l_2, s_1 + l_1) \right)$$

• its restriction to $v_1 < v_2$, equal to:

$$\sum_{u \geq 1} \frac{p_0^u} {\rho_0(p_0 + s_2 - u)} \prod_{i=1}^{\frac{2}{l_i}} (-s_i) \left( B_u \prod_{i=1}^{l_i} \varphi_i = \rho_0(s_1 + l_i) + \sum_{i=1}^{l_i+2} \varphi_i \rho_0(s_2 + l_2, s_1 + l_1) \right)$$

• its restriction to $v_1 > v_2$, equal to a expression similar to the previous one, and also equal, by using a change of variable $(n_1, n_2) \mapsto (p^n - n_1, p^n - n_2)$, to:

$$\sum_{0 < n_1 < n_2 < p^n \varepsilon, v_p(n_1) > v_p(n_2)} \sum_{l_1, l_2 \geq 0} \frac{(p^n)^{s_1 + s_2}} {n_1^n n_2^n} \sum_{l_1, l_2 \geq 0} (-s_1) (-s_2) \frac{(p^n)^{l_1 + l_2 + s_1 + s_2}} {n_1^n n_2^n}$$

6. Comparisons

6.1. The map $\Sigma_{\text{inv}}^{\text{DR}}$. Let again $A$ be a complete topological $K$-algebra. We define the variant of $\Sigma_{\text{inv}}^{\text{DR}}$ which takes into account all numbers of iterations of Frobenius at the same time and how the Frobenius varies in function of the number of iterations.

Definition 6.1.1. Let the map

$$\Sigma_{\text{inv}}^{\text{DR, iter}} : \Pi_{1.0}(A) \Sigma \to \text{Map}(q^{n^*}, A(\{e^z\})^{\text{DR}})$$

defined by

$$f \mapsto \left( a \in \mathbb{N}^* \mapsto \sum_{z \in Z - \{0, \infty\}} z^{-1}(x \mapsto z^2) \left( \sum_{s \geq 0} (pr_{s+1}((f^{-1}e_f)^{\text{DR}})^{-1} \varphi_S) \right) \right)$$

Proposition 6.1.2. The map $\Sigma_{\text{inv}}^{\text{DR, iter}}$ is injective.

Proof. Let a word $w = e_{z_1} e_{z_2} \ldots e_{z_{l+1}} s_0^{d-1} e_{z_{l+2}} \ldots e_0^{s_1} e_{z_1}$. For $s \geq \text{weight}(w)$, let us consider the coefficient of $(q^a)^s$ in $\Sigma_{\text{inv}}^{\text{DR, iter}}(f)[w]$:

$$\left( \sum_{z \in Z - \{0, \infty\}} z^{-1}(x \mapsto z^2) \left( \sum_{s \geq 0} (pr_{s+1}((f^{-1}e_f)^{\text{DR}})^{-1} \varphi_S) \right) \right) \varphi_S \left( \sum_{s \geq 0} (f^{-1}e_f)^{\text{DR}} \right)$$

It is equal to $\sum_{z \in Z - \{0, \infty\}} z^{-1}(x \mapsto z^2) \left( f^{-1}e_f \right)$ terms of lower depth, where the depth is the one of coefficients of $f^{-1}e_f$ and one can express $(f^{-1}e_f)^{-1} \varphi_S$ in terms of $f^{-1}e_f$. This implies by induction on $d$ that we can retrieve the coefficients of $f^{-1}e_f$ of depth up to $d$ from $\Sigma_{\text{inv}}^{\text{DR, iter}}(f)$. □

6.2. Comparison for $\varphi_S$ viewed as a function of $\alpha$ and the fixed point.

Lemma 6.2.1. Let $\delta \in \mathbb{N}^*$, and a map $C : \mathbb{N}^* \cap [\delta, +\infty] \to K$ such that there exist $M \in \mathbb{N}^*$, and elements $c_{l, m} \in K$, for $l \in \mathbb{N}$, $m \in [0, M]$, such that, for all $a \in \mathbb{N}^* \cap [\delta, +\infty]$, $C(a)$ is equal to the absolutely
convergent sum $\sum_{l \in \mathbb{N}} \sum_{m=0}^{M} c_{l,m} (q^a)^l a^m$. Then, if $C(a) = 0$ for all $a$, then we have $c_{l,m} = 0$ for all $l, m$. In particular, the coefficients of (1.5.1), (1.5.2), (1.5.3) are the same.

Proof. Let $a_0 \in \mathbb{N}^* \cap [\delta, +\infty[$ and $u \in \mathbb{N}$. By taking $a = a_0 + pu$ in the equation $\sum_{l \in \mathbb{N}} \sum_{m=0}^{M} c_{l,m} (q^a)^l a^m = 0$ and by taking the limit $u \to \infty$, we get $\sum_{m=0}^{M} c_{0,m} a_0^m = 0$. Since this is true for infinitely many $a_0$, we get $c_{0,m} = 0$ for all $m$. This implies, for all $a$, $\sum_{l \geq 1} \sum_{m=0}^{M} c_{l,m} (q^a)^l a^m = (q^a) \left( \sum_{l \geq 1} \sum_{m=0}^{M} c_{l,m} (q^a)^{l-1} a^m \right) = 0$, thus, by simplifying by $q^a$, $\sum_{l \in \mathbb{N}} \sum_{m=0}^{M} c_{l+1,m} (q^a)^l a^m = 0$. Thus, if we have $c_{l,m} \neq 0$ for a certain $(l, m)$, we get a contradiction by considering the minimal value of $l$ among such $(l, m)$.

\[\square\]
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