CYCLOTOMIC EXPANSIONS FOR $\mathfrak{gl}_N$ KNOT INVARIANTS
VIA INTERPOLATION MACDONALD POLYNOMIALS

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Abstract. In this paper we construct a new basis for the cyclotomic completion of the center of the quantum $\mathfrak{gl}_N$ in terms of the interpolation Macdonald polynomials. Then we use a result of Okounkov to provide a dual basis with respect to the quantum Killing form (or Hopf pairing). Two main applications are: 1) a cyclotomic expansion of the universal $\mathfrak{gl}_N$ knot invariant and 2) an explicit construction of the unified $\mathfrak{gl}_N$ invariants for $\mathbb{Z}$-homology 3-spheres obtained by knot surgeries. These results generalize those of Habiro for $\mathfrak{sl}_2$. In addition, we give a simple proof of the fact that the universal $\mathfrak{gl}_N$ invariant of any evenly framed link and the universal $\mathfrak{sl}_N$ invariant of any 0-framed algebraically split link are $\Gamma$-invariant, where $\Gamma = Y/2Y$ with the root lattice $Y$.

1. Introduction

In a series of papers [1] [2] [15] Habiro, the first author et al. defined unified invariants of homology 3-spheres that belong to the Habiro ring and dominate Witten–Reshetikhin–Turaev (WRT) invariants. Unified invariants provide an important tool to study structural properties of the WRT invariants. In [3] [5] they were used to prove integrality of the $\mathfrak{sl}_2$ WRT invariants for all 3-manifolds at all roots of unity.

The theory of unified invariants for $\mathfrak{sl}_2$ is based on cyclotomic expansions for the colored Jones polynomial and for the universal knot invariant constructed as follows. Given a framed oriented link $L$ in the 3-sphere, we open its components to obtain a bottom tangle $T$, presented by a diagram $D$. For a ribbon Hopf algebra $U_q\mathfrak{g}$, the universal link invariant $J_L(\mathfrak{g}; q)$ is obtained by splitting $D$ into elementary pieces: crossings, caps and cups and then by associating to them $R^{\pm 1}$-matrices, and pivotal elements, respectively.

For a knot $K$, $J_K(\mathfrak{g}; q)$ belongs to (some completion of) the center $Z(U_q\mathfrak{g})$. In the easiest case $\mathfrak{g} = \mathfrak{sl}_2$, the center is generated by the Casimir $C$. For a 0-framed knot $K$, the Habiro series [15] have the form

$$(1) \quad J_K(\mathfrak{sl}_2; q) = \sum_{m=0}^{\infty} a_m(K) \sigma_m \quad \text{with} \quad \sigma_m = \prod_{i=1}^{m} (C^2 - (q^i + q^{-i} + 2)),$$

where $a_m(K) \in \mathbb{Z}[q^{\pm 1}]$. 

\begin{figure}[h]
\centering
\includegraphics[width=0.2\textwidth]{clasp_tangle.png}
\caption{An example of the clasp bottom tangle}
\end{figure}
Replacing $C^2$ in (1) by its value $q^n + q^{-n} + 2$ on the $n$-dimensional representation $V_{n-1}$, we get the $n$-colored Jones polynomial of $K$ (normalized to 1 for the unknot)

\[ J_K(V_{n-1}, q) = \sum_{m=0}^{\infty} (-1)^m q^{-\frac{m(m+1)}{2}} a_m(K) (q^{1+n}; q)_m (q^{1-n}; q)_m \]

where $(a; q)_m = (1 - a)(1 - aq) \ldots (1 - aq^{m-1})$. Equation (2) is known as a cyclotomic expansion of the colored Jones polynomial. Thus, Habiro’s series (1) dominates all colored $J_n$ with highest weight $v$ algebraically split links for all quantum groups of type $\mathfrak{g}$ developed in [16].

To prove Theorem 1.1 we extend the Reshetikhin-Turaev invariants to tangles colored with $\mathfrak{g}$ for $\mathfrak{g} = \mathfrak{gl}_N$, $\Gamma = \mathbb{Z}_2^N$ and for $\mathfrak{g} = \mathfrak{sl}_N$, $\Gamma = \mathbb{Z}_2^{N-1}$.

**Theorem 1.1.** The universal $\mathfrak{gl}_N$ invariant of any evenly framed link is $\Gamma$-invariant. The universal $\mathfrak{sl}_N$ invariant of any 0-framed algebraically split link is $\Gamma$-invariant.

The quantum group $U_q \mathfrak{gl}_N$ admits a finite dimensional irreducible representation $V(\lambda)$ with highest weight $v^\lambda$ for any partition $\lambda = (\lambda_1 \geq \cdots \geq \lambda_N)$ with $N$ parts and $v^2 = q$. To prove Theorem 1.1 we extend the Reshetikhin-Turaev invariants to tangles colored with representations $L(\zeta) \otimes V(\lambda)$ where $L(\zeta)$ is a one-dimensional representation of $U_q \mathfrak{gl}_N$ for $\zeta \in \Gamma$. We prove that the $\mathfrak{gl}_N$ Reshetikhin-Turaev invariants of evenly framed bottom tangles colored with $L(\zeta) \otimes V(\lambda)$ and $V(\lambda)$ coincide. Note that for even $N$ we can drop the condition on framing (compare Theorem 4.6).

The next main result of the paper establishes an explicit basis in the $\Gamma$-invariant part of the center $\mathcal{Z}$ of $U_q \mathfrak{gl}_N$. It generalizes Habiro’s basis $\{\sigma_m | m \in \mathbb{N}\}$ for the even part of $\mathcal{Z}(U_q \mathfrak{sl}_2)$.

**Theorem 1.2.** There exists a family of central elements $\sigma_\lambda \in \mathcal{Z}$ labeled by partitions $\lambda$ with at most $N$ parts with the following properties:

(a) $\sigma_\lambda$ is $\Gamma$-invariant and annihilates $L(\zeta) \otimes V(\mu)$ for all $\zeta \in \Gamma$ and partitions $\mu$ with at most $N$ parts not containing $\lambda$;

(b) $\sigma_\lambda$ does not annihilate $V(\lambda)$ and acts on it by an explicit scalar (see Theorem 8.3).

The proof uses the theory of interpolation Macdonald polynomials developed in [22, 23, 28, 29, 30, 31, 33]. This theory allows one to reconstruct a symmetric function $f(x_1, \ldots, x_N)$ from its values at special points $x_i = q^{-\mu_i - N + i}$ where $\mu$ is an arbitrary partition with at most $N$ parts. The connection between the center of $U_q \mathfrak{gl}_N$ and symmetric functions goes through the quantum Harish-Chandra isomorphism, and we interpret $f(q^{-\mu_1 - N + 1}, \ldots, q^{-\mu_N})$ as the scalar by which the element of the center $f$ acts on the irreducible representation $V(\mu)$. Interpolation Macdonald polynomials then correspond to a natural basis in the center of $U_q \mathfrak{gl}_N$.

The polynomials $\sigma_\lambda$ yield a basis in the $\Gamma$-invariant parts of both the center $\mathcal{Z}$ and its completion (a function in the completion is determined by its values on all finite-dimensional representations). We use a formula of Okounkov [28] to give explicit expansion of a given central element $z$ in the basis $\sigma_\lambda$ in terms of the scalars by which $z$ acts on all finite-dimensional representations $V(\lambda)$. This leads to an explicit expansion of the universal knot invariant in the basis $\sigma_\lambda$, where the coefficients are related to Reshetikhin-Turaev invariants of the same knot colored by $V(\mu)$ via an explicit triangular matrix $(d_{\lambda, \mu})$ which does not depend on the knot.
Theorem 1.3. For any evenly framed knot $K$, there exist Laurent polynomials $a_\lambda(K) \in \mathbb{Z}[q, q^{-1}]$ such that the universal invariant of $K$ has the following expansion:

$$J_K(\mathfrak{gl}_N; q) = \sum_\lambda a_\lambda(K) \sigma_\lambda.$$  

Moreover, the coefficients $a_\lambda(K)$ can be computed in terms of the Reshetikhin-Turaev invariants as follows:

$$a_\lambda(K) = \sum_{\mu \subset \lambda} d_{\lambda, \mu}(q^{-1}) J_K(V(\mu), q),$$

where the coefficients $d_{\lambda, \mu}(q)$ are defined in Theorem 9.17.

We prove Theorem 1.3 as Proposition 8.7. We would like to emphasize that the fact that $a_\lambda(K)$ are Laurent polynomials in $q$ is highly nontrivial. Indeed, we have computed the tables of coefficients $d_{\lambda, \mu}(q)$ for $\mathfrak{gl}_2$ in Section 10.3 and these are complicated rational functions, so a priori $a_\lambda(K)$ are rational functions as well. Theorem 1.3 thus encodes certain divisibility properties for the linear combinations of colored invariants of $K$. We refer to Section 10.4 for the explicit computation of the coefficients $a_\lambda(K)$ for the figure eight knot.

We call (3) a cyclotomic expansion of the universal $\mathfrak{gl}_N$ knot invariant. The name cyclotomic is justified by the fact that (3) has well-defined evaluations at any root of unity by Lemma 9.29 below. Note that for $N = 2$ and a 0-framed knot, our expansion does not coincide with that of Habiro, simply because if an element $z \in U_q \mathfrak{gl}_2$ is central and $\Gamma$-invariant, it does not imply $z$ has a decomposition in even powers of the Casimir. Therefore, our cyclotomic expansion is rather a generalization of $F_\infty$ in [36] or [4, eq.(3.14)], both having interesting application in the theory of non semisimple invariants of links and 3-manifolds.

Theorem 1.3 plays a crucial role in our construction of the unified invariants for integral homology 3-spheres obtained by knot surgeries. Assume $M_{\pm}$ is a 3-manifold obtained by $(\pm 1)$-surgery on a knot $K$, then following [19] we define the unified invariant $I(M_{\pm})$ as

$$I(M_{\pm}) = \langle \tau^{\pm 1}, J_K(\mathfrak{gl}_N; q) \rangle$$

where $\tau$ is the ribbon element and $\langle \cdot, \cdot \rangle$ is the Hopf pairing. For simple Lie algebras Habiro–Lê proved [19] that this invariant belongs to a cyclotomic completion of the polynomial ring

$$\widehat{\mathbb{Z}[q]} := \lim_{\leftarrow n} \mathbb{Z}[q]/((q; q)_n).$$

Using interpolation, we extend this result to $\mathfrak{gl}_N$. In addition, we diagonalize the Hopf pairing, i.e. define a second basis $P_\mu$ such that $\langle P_\mu, \sigma_\lambda \rangle = \delta_{\mu, \lambda}$. This allows us to give explicit formulas for the universal Kirby colors $\omega_{\pm}$ (see (22)) in the basis $P_\mu$ and prove the following result.

Theorem 1.4. The unified invariant

$$I(M_{\pm}) = J_K(\omega_{\pm}, q) \in \widehat{\mathbb{Z}[v]} := \lim_{\leftarrow n} \mathbb{Z}[v]/((v; v)_n)$$

belongs to the Habiro ring and dominates $\mathfrak{gl}_N$ WRT invariants of $M_{\pm}$ at all roots of unity.

Note that an $\mathfrak{sl}_2$ analog of the basis $P_\mu$ was constructed by Habiro in [15]. Our proof of Theorem 1.4 uses some divisibility results for $\sigma_\lambda$ described in Section 8, and the Laplace transform method together with the Andrews’ generalization of the Rogers-Ramanujan identity from [3].
Finally, we would like to comment on potential ideas for categorification of these results. The ring of symmetric polynomials in \( N \) variables is naturally categorified by the category of annular \( \mathfrak{gl}_N \)-webs, with morphisms given by annular foams [6, 32, 33, 13, 11]. By the work of the second author and Wedrich [13], one can interpret it as a symmetric monoidal Karoubian category generated by one object \( E \) corresponding to a single essential circle. The symmetric polynomials are then categorified by the Schur functors of \( E \).

We expect the categorified interpolation polynomials to correspond to interpolation Macdonald polynomials where \( q \) plays the role of quantum grading and \( t \) of the homological grading (after some change of variables). We recall the general definitions and properties of these polynomials from [28] in Appendix. The key obstacle for categorification of interpolation polynomials is that they are not homogeneous. Therefore one needs to enrich the category and allow additional morphisms between \( E \) and identity.

On the other hand, the conjectures of the second author, Neguț and Rasmussen ([12], see [10, 11] for further discussions) relate a version of the annular category to the derived category of the Hilbert scheme of points on the plane. The interpolation Macdonald polynomials appear in that context as well [7].

The paper is organized as follows. After recalling the definitions, we compare the Reshetikhin–Turaev invariants of tangles colored by \( V(\lambda) \) and \( L(\zeta) \otimes V(\lambda) \) in Section 4. The next two sections summarize known results about the center of \( \mathcal{U}_q\mathfrak{gl}_N \) and Habiro’s basis for \( \mathbb{Z}(\mathcal{U}_q\mathfrak{sl}_2) \) and use them to prove Theorem 1.1. The remaining results are proven in Section 8 assuming some facts about interpolation. In the last sections we develop the theory of the interpolation Macdonald polynomials, starting from the one variable case. In Appendix A we collect some additional known facts about the interpolation Macdonald polynomials and the Habiro ring.

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2. Notations and conventions

2.1. \( q \)-binomial formulas. Throughout the paper we will use the following notations for the \( q \)-series. The \( q \)-Pochhammer symbols are defined as

\[
(a; q)_m = \prod_{i=0}^{m-1} (1 - aq^i), \quad (a; q)_\infty = \prod_{i=0}^{\infty} (1 - aq^i), \quad m \geq 0.
\]

It is easy to see that

\[
(a; q)_{m+k} = (a; q)_m (aq^m; q)_k, \quad (a; q)_m = \frac{(a; q)_\infty}{(aq^m; q)_\infty}.
\]

We will use two normalizations for \( q \)-binomial coefficients defined as follows:

\[
\{a\}_q = 1 - q^a, \quad [a]_q = \frac{\{a\}_q}{\{1\}_q}, \quad [a]_q! = [1]_q \cdots [a]_q, \quad \binom{a}{b}_q = \frac{[a]_q!}{[b]_q! [a-b]_q!}.
\]

Note that

\[
[a]_q = \frac{(q; q)_a}{(1 - q)^a}, \quad \binom{a}{b}_q = \frac{(q; q)_a}{(q; q)_b (q; q)_{a-b}}.
\]
Finally, the $q$-binomial formula gives
\[
(a; q)_m = \sum_{j=0}^{m} (-1)^j q^{\frac{j(j-1)}{2}} \binom{m}{j}_q a^j.
\]
Let us also define symmetric $q$-numbers. For this we chose $v$ such that $v^2 = q$ and set
\[
\{a\} = v^a - v^{-a}, \quad [a] := \frac{\{a\}}{\{1\}}, \quad \begin{bmatrix} a \\ b \end{bmatrix} := \frac{\{a\}!}{\{b\}! \{a-b\}!}.
\]
We will use all these formulas throughout the paper without a reference.

2.2. Partitions. We will work with partitions $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \lambda_N)$ which we will identify with the corresponding Young diagrams in French notation, where the rows have length $\lambda_i$. Transpose diagram to $\lambda'$ is denoted by $\lambda'$, and $|\lambda| = \sum \lambda_i$. Given a box in a Young diagram, we define its arm, co-arm, leg and co-leg as in Figure 2.

![Figure 2. Arm, co-arm, leg and co-leg](image)

We define the hook length as $h(\square) = a(\square) + l(\square) + 1$, and the content $c(\square) = a' - l'$. Let
\[
n(\lambda) = \sum (i-1)\lambda_i = \sum_{\square} t'(\square) = \sum_{\square} l(\square),
\]
then
\[
n(\lambda') = \sum \frac{\lambda_i(\lambda_i - 1)}{2} = \sum_{\square} a'(\square) = \sum_{\square} a(\square).
\]
The content of $\lambda$ is defined as
\[
c(\lambda) = \sum c(\square) = n(\lambda') - n(\lambda).
\]
Let $\tilde{\lambda}_i = \lambda_i + N - i$ for $1 \leq i \leq N$, then we have the following identity
\[
(4) \quad \prod_{\square \in \lambda} (1 - t^{h(\square)}) = \frac{\prod_{i \geq 1} \prod_{j=1}^{\tilde{\lambda}_i} (1 - t^j)}{\prod_{i<j} (1 - t^{\tilde{\lambda}_i - \tilde{\lambda}_j})}
\]
and we define
\[
(5) \quad D_N(\lambda) = \sum_{i=1}^{N} \frac{(\tilde{\lambda}_i)(\tilde{\lambda}_i - 1)}{2} = \sum_i \frac{\lambda_i(\lambda_i - 1)}{2} + \sum_i (N-i)\lambda_i + \sum_{i=1}^{N} \binom{N - i}{2}
\]
\[
(6) \quad = n(\lambda') + (N-1)|\lambda| - n(\lambda) + \binom{N}{3} = c(\lambda) + (N-1)|\lambda| + \binom{N}{3}.
\]
3. Quantum Groups

3.1. Quantum $\mathfrak{gl}_N$. The quantum group $U = U_q \mathfrak{gl}_N$ is a $\mathbb{C}(v)$-algebra generated by $E_1, \ldots, E_{N-1}, F_1, \ldots, F_{N-1}, K_i^\pm, K_N^\pm$ satisfying the following relations:

$$K_i E_i = v E_i K_i, \quad K_i F_i = v^{-1} F_i K_i, \quad K_{i+1} E_i = v^{-1} E_i K_{i+1}, \quad K_{i+1} F_i = v F_i K_{i+1}$$

$$[E_i, F_j] = \delta_{ij} \frac{K_i K_{i+1} - K_{i+1} K_i}{v - v^{-1}}, \quad [K_i, K_j] = 0,$$

$$E_i^2 - [2] E_i E_j E_i + E_j E_i^2 = 0 \text{ if } |i-j| = 1 \text{ and } [E_i, E_j] = 0 \text{ otherwise.}$$

and analogously for $F_i$, where $v^2 = q$. To simplify the notation we set $K_i := K_i K_{i+1}^{-1}$. Then the Hopf algebra structure on $U$ (i.e. coproduct, antipode and counit) can be defined as follows:

$$\Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i, \quad \Delta(F_i) = 1 \otimes F_i + F_i \otimes K_i^{-1}, \quad \Delta(K_i) = K_i \otimes K_i, \quad \epsilon(K_i) = 1, \quad \epsilon(E_i) = 0.$$

Usually $U$ is considered as a subalgebra of $U_h$ that is an $h$-adically complete $\mathbb{C}[[h]]$-algebra topologically generated by $E_i, F_i$ and $H_j$ for $1 \leq i \leq N - 1$ and $1 \leq j \leq N$ with

$$v = \exp h/2, \quad K_i = v^{H_i} = \exp h H_i/2$$

satisfying (8), (9) and

$$H_i E_i - E_i H_i = E_i, \quad H_i F_i - F_i H_i = -F_i, \quad H_{i+1} E_i - E_i H_{i+1} = -E_i, \quad H_{i+1} F_i - F_i H_{i+1} = F_i$$

replacing (7). Rewriting the defining relations in terms of the generators

$$e_i = E_i(v - v^{-1}), \quad F_i^{(n)} = \frac{F_i^n}{[n]!} \quad \text{and} \quad K_j \quad \text{for} \quad 1 \leq i \leq N - 1, \quad 1 \leq j \leq N$$

we obtain an integral version $U_z$ as a Hopf algebra over $\mathbb{Z}[v, v^{-1}] \subset \mathbb{C}(v) \subset \mathbb{C}[[h]]$.

The quantum group $\mathfrak{gl}_N$ has a fundamental representation $\mathbb{C}^N$ with basis $v_1, \ldots, v_N$ such that

$$K_i v_j = v^{\delta_{ij}} v_j, \quad E_i v_{i+1} = v_i, \quad F_i v_i = v_{i+1}.$$ 

It generates a braided monoidal category with simple objects $V(\lambda)$, where $\lambda$ is a partition with at most $N$ parts. These are highest weight modules where $K_i$ act on the highest weight vector by $v^\lambda$. The fundamental representation corresponds to $\lambda = (1)$. The representations $V(\lambda)$ have integral basis where $U_z$ acts by $\mathbb{Z}[v, v^{-1}]$-valued matrices.

3.2. Ribbon structure. The Hopf algebra $U_h$ admits a ribbon Hopf algebra structure (see e.g. [8 Cor. 8.3.16]). The universal $R$-matrix has the form $R = D \Theta$ where the diagonal part $D$ and the quasi-$R$-matrix are defined as follows

$$D = v^\sum_{i=1}^{N} H_i \otimes H_i \quad \text{and} \quad \Theta = \sum_{n \in N^{N-1}} F_n \otimes e_n$$

where for any sequence of non-negative integers $n = (n_1, \ldots, n_{N-1})$, the elements $e_n$ and $F_n$ are defined by equations (66) and (67) in [13] and form topological bases of the positive and negative parts in the triangular decomposition of $U_z$. The inverse matrix $R^{-1} = \iota(\Theta)D^{-1}$ is obtained by applying the involution $\iota : v \to v^{-1}$.

The ribbon element and its inverse have the form

$$r = \sum_n F_n K_n r_0 e_n \quad \text{and} \quad r^{-1} = \sum_n \iota(F_n) K^{-n} r_0^{-1} \iota(e_n)$$
where \( r_0 = K_{-2p} v^{-\sum_{i=1}^{N} H_i^2} \) and \( K_{-2p} = \prod_{i=1}^{N} K_i^{-2i-N-1} \) is the pivotal element. Here for any sequence of integers \( n \in \mathbb{Z}^{N-1} \) we set \( K_n = \prod_i K_i^{n_i} \), and denote by

\[
\rho = \left( \frac{N-1}{2}, \frac{N-3}{2}, \ldots, \frac{1-N}{2} \right) = \frac{1-N}{2}(1,\ldots,1) + (N-1, N-2, \ldots, 0). \]

Using the central element \( K = \prod_{i=1}^{N} K_i \), we can write the previous definitions as follows:

\[
r_0^{-1} = K^N \prod_{i=1}^{N} K_i^{-2i} v^{\sum_{i=1}^{N} H_i(H_i+1)}, \quad K_{-2\rho} = K^{-N-1} \prod_{i=1}^{N} K_i^{2i}.
\]

The central element \( r^{-1} \) acts on \( V(\lambda) \) by the multiplication with

\[
\theta_V(\lambda) = v^{(\lambda,\lambda+2\rho)} = v^{N|\lambda| q^{c(\lambda)}},
\]

where \( (\lambda, \mu) = \sum_{i=1}^{N} \lambda_i \mu_i \), \( c(\lambda) \) is the content of \( \lambda \) and \( v^2 = q \).

3.3. Even part of \( \mathcal{U} \). The algebra \( \mathcal{U} \) has a natural grading by \( \Gamma = \mathbb{Z}_2^N = \{ \pm 1 \}^N \) where \( \zeta = (\zeta_1, \ldots, \zeta_N) \in \Gamma \) acts on \( K_i \) by \( \zeta_i \), on \( E_i \) by 1 and on \( F_i \) by \( \zeta_i \zeta_i^{-1} \). It is easy to see that the defining relations are preserved under this action. Following [19], we call an element of \( \mathcal{U}_N \) even or \( \Gamma \)-invariant if it is preserved under the action of \( \Gamma \).

Let us denote by \( \mathcal{U}_\mathbb{Z}^{ev} \) a \( \mathbb{Z}[q, q^{-1}] \)-subalgebra of \( \mathcal{U} \) generated by \( e_i, F_i^{(n)} K_i \) and \( K_j^2 \) for \( 1 \leq i \leq N-1 \) and \( 1 \leq j \leq N \). It is easy to check that \( \mathcal{U}_\mathbb{Z}^{ev} \) is \( \Gamma \)-invariant.

The action of \( \Gamma \) descends on the category \( Rep(\mathcal{U}) \) of all finite-dimensional representations. Given \( \zeta = (\zeta_1, \ldots, \zeta_N) \in \Gamma \), we can define a one-dimensional representation \( L(\zeta) \) where \( E_i \) and \( F_i \) act by zero, and \( K_i \) act by \( \zeta_i \). We can also define representation \( V(\lambda) \otimes L(\zeta) \) where \( K_i \) act on the highest weight vector by \( \zeta_i v^\lambda \). Note that the action of \( \mathcal{U} \) on \( V(\lambda) \otimes L(\zeta) \) agrees with the \( \Gamma \)-twisted action of \( \mathcal{U} \) on \( V(\lambda) \).

3.4. The subalgebra \( U_q sl_N \). We define \( U_q sl_N \) as a subalgebra of \( \mathcal{U} \) generated by \( E_i, F_i \) and \( K_i^{\pm} := K_i^{1+} K_i^{-1} \) for \( 1 \leq i \leq N-1 \). The Hopf algebra \( U_q sl_N \) also admits an integral version \( U_{\mathbb{Z}} sl_N \) generated by

\[
e_i, \quad F_i^{(n)}, \quad K_i^{\pm}.
\]

over \( \mathbb{Z}[v, v^{-1}] \). The braiding \( D = D' \Theta \) with \( \Theta \) as for \( gl_N \), but different diagonal part

\[
D' = v^{\sum_{i=1}^{N-1} \frac{H_i H_{i+1}}{2}} \quad \text{where} \quad H_i = H_i - H_{i+1}.
\]

The ribbon element is defined by (10) with \( r_0 = \prod_{i=1}^{N-1} K_i^{N-i} q^{3i^2/2} \). The pivotal element is \( \prod_{i=1}^{N-1} K_i^{i-N} \).

Example 3.1. For \( N = 2 \) the product \( K_1 K_2 \) is central. By denoting \( K = K_1 K_2^{-1}, E = E_1, F = F_1 \) we get the standard presentation for \( U_q(sl_2) \):

\[
KE = v^{-2} EK, \quad KE = v^{-2} F K, \quad [E, F] = \frac{K - K^{-1}}{v - v^{-1}}.
\]

3.5. Universal invariant. Lawrence, Reshetikhin, Ohtsuki and Kauffman constructed quantum group valued universal link invariants. As it was already mentioned in the introduction, the universal invariant of a link is defined by splitting a diagram of its bottom tangle into elementary pieces and by associating \( R \)-matrices and pivotal elements to them. For more details and references we recommend to consult [16, Sec. 7.3]. However, we admit here the convention from [19, Sec. 2.7] and write the contributions from left to right along the orientation of each component.
Theorem 1.1 will follow.

The maps $\text{Rep}^V$ define a braiding on $\text{Rep}^V$. Indeed, let $\text{Rep}^V$ act as $\text{U}_L$, which acts by $\text{U}_L$, and is given by $\text{U}_L$ on the weight subspace of $V$ where $K_i$ acts as $v^{a_i}$.

Lemma 4.1. The maps
\[ c_{\zeta, V} := \text{swap} \circ (\text{Id} \otimes T_V(\zeta)) : L(\zeta) \otimes V \rightarrow V \otimes L(\zeta) \]
with inverses
\[ c_{V, \zeta} := \text{swap} \circ (T_V(\zeta) \otimes \text{Id}) : V \otimes L(\zeta) \rightarrow L(\zeta) \otimes V \]
define a braiding on $\text{Rep}(U)$.

Proof. First, let us check that $\text{swap} \circ (\text{Id} \otimes T_V(\zeta))$ intertwines the actions of $U$ on both sides. Indeed, let $v \in V$ be a vector with weight $(v^{a_1}, \ldots, v^{a_N})$, then $E_i v$ has weight $(v^{a_1}, \ldots, v^{a_i-1}, v^{a_i+1}, \ldots, v^{a_N})$ while $F_i v$ has weight $(v^{a_1}, \ldots, v^{a_i-1}, v^{a_i+1}, \ldots, v^{a_N})$.

Let $\bullet$ denote the basis vector in $L(\zeta)$, then
\[ c_{\zeta, V} E_i(\bullet \otimes v) = c_{\zeta, V}(\zeta_1 \cdot \cdot \cdot \zeta_{i-1} \cdot \otimes E_i(v)) = \zeta_1^{a_1} \cdot \cdot \cdot \zeta_{i-1}^{a_{i-1}} \zeta_i^{a_i} \cdot \cdot \cdot \zeta_N^{a_N} E_i(v) \otimes \bullet, \]
\[ c_{\zeta, V} F_i(\bullet \otimes v) = c_{\zeta, V}(\bullet \otimes F_i(v)) = \zeta_1^{a_1} \cdot \cdot \cdot \zeta_{i-1}^{a_{i-1}} \zeta_i^{a_i+1} \cdot \cdot \cdot \zeta_N^{a_N} F_i(v) \otimes \bullet, \]
\[ c_{\zeta, V} K_i(\bullet \otimes v) = c_{\zeta, V}(\zeta_i \bullet \otimes K_i(v)) = \zeta_1^{a_1} \cdot \cdot \cdot \zeta_{i-1}^{a_{i-1}} \zeta_i^{a_i} \cdot \cdot \cdot \zeta_N^{a_N} K_i(v) \otimes \bullet, \]
while
\[ E_i c_{\zeta, V}(\bullet \otimes v) = E_i(\zeta_1^{a_1} \cdot \cdot \cdot \zeta_N^{a_N} v \otimes \bullet) = \zeta_1^{a_1} \cdot \cdot \cdot \zeta_N^{a_N} E_i(v) \otimes \bullet, \]
\[ F_i c_{\zeta, V}(\bullet \otimes v) = F_i(\zeta_1^{a_1} \cdot \cdot \cdot \zeta_N^{a_N} v \otimes \bullet) = \zeta_1^{a_1} \cdot \cdot \cdot \zeta_i^{a_i-1} \cdot \cdot \cdot \zeta_N^{a_N} F_i(v) \otimes \bullet, \]
\[ K_i c_{\zeta, V}(\bullet \otimes v) = K_i(\zeta_1^{a_1} \cdot \cdot \cdot \zeta_N^{a_N} v \otimes \bullet) = \zeta_1^{a_1} \cdot \cdot \cdot \zeta_{i-1}^{a_{i-1}} \cdot \cdot \cdot \zeta_N^{a_N} K_i(v) \otimes \bullet. \]

Next, we observe that $T_V(\zeta) T_V(\zeta') = T_V(\zeta \zeta')$ and $T_{U \otimes V}(\zeta) = T_U(\zeta) \otimes T_V(\zeta)$, so $c_{\zeta, V}$ indeed defines a braiding. Even more concretely, we get the braiding as the composition
\begin{equation}
\tag{11}
c_{L(\zeta) \otimes V, L(\zeta') \otimes U} : L(\zeta) \otimes V \otimes L(\zeta') \otimes U \xrightarrow{cv} L(\zeta) \otimes L(\zeta') \otimes V \otimes U = L(\zeta') \otimes L(\zeta) \otimes V \otimes U \xrightarrow{cv} L(\zeta') \otimes L(\zeta) \otimes U \otimes V \xrightarrow{\zeta} L(\zeta') \otimes U \otimes L(\zeta) \otimes V.
\end{equation}

The representations $L(\zeta)$ are self-dual, and it is easy to see that the braiding $c_{\zeta, V}$ is compatible with changing $V$ to $V^*$. Therefore, $\text{Rep}(U)$ with objects $L(\zeta) \otimes V$ form a pivotal braided monoidal category.

The quantum dimension of $L(\zeta)$ equals to the trace of the action of the pivotal element, which is $(\prod_i \zeta_i)^{N+1}$. The twist coefficient $\theta_{L(\zeta)}$ is defined as the action of the ribbon element on $L(\zeta)$, and is given by $(\prod_i \zeta_i)^N$. 

Lemma 4.2. \( \text{Rep}(U) \) is a ribbon category with twist \( \theta_{L(\zeta) \otimes V} = \theta_{L(\zeta)} \theta_V \).  

Proof. By definition \( \theta_{L(\zeta) \otimes V} = c_{\zeta,V} \theta_{L(\zeta)} \theta_V c_{V,\zeta} = \theta_{L(\zeta)} \theta_V \).  

4.1. Braiding in \( \text{Rep}(U_q \mathfrak{sl}_N) \). In this section, we study the action of \( \Gamma \) and the corresponding braiding for \( U_q \mathfrak{sl}_N \), starting from \( N = 2 \). Similarly to the previous section, \( U_q \mathfrak{sl}_2 \) has a one dimensional representation \( L(-1) \) where \( E \) and \( F \) act by 0 and \( K \) acts by \(-1\). The action of \( U_q \mathfrak{sl}_2 \) on \( L(-1) \otimes V \) is equivalent to \( \mathbb{Z}_2 \)-twisted action on \( V \) where \( \mathbb{Z}_2 \) scales \( E \) by 1 and \( F, K \) by \(-1\).

One can attempt to define a braiding for \( U_q \mathfrak{sl}_2 \). Since \( E, F \) shift the weights by 2, it is easy to see that the analogue of \( T_V \) should act by \( (\sqrt{-1})^a \) on a subspace with weight \( v^a \), and it does not square to identity. Nevertheless, it squares to \( \pm \text{Id} \) on each irreducible representation. This means that braiding relations on \( \text{Rep}(U_q \mathfrak{sl}_2) \) hold up to sign.

To pin down this sign, we define the sign automorphism \( \Sigma_\zeta \) which acts by \((-1)^a\) on a subspace with weight \( v^a \). Since \( E, F \) shift the weight by \( \pm v^2 \), \( \Sigma_\zeta \) commutes with the action of \( U_q \mathfrak{sl}_2 \) on \( V \). The operator \( \Sigma_\zeta \) acts on the irreducible representation \( V(n) \) by a scalar \((-1)^a\). Also, it is easy to see that \( \Sigma_{V \otimes W} = \Sigma_V \otimes \Sigma_W \) and \( \Sigma_{V \oplus W} = \Sigma_V \otimes \Sigma_W \).

Lemma 4.3. The operators \( T_V \) and \( \Sigma_V \) satisfy the following properties: 

(a) We have 
\[
T_V^2 = \Sigma_V, \quad c_{L(-1),V} = c_{L(-1),V}^{-1}(1 \otimes \Sigma_V) = (\Sigma_V \otimes 1)c_{L(-1),V}^{-1}
\]

(b) Let \( c_{V,W} : V \otimes W \to W \otimes V \) be the braiding, then 
\[
c_{V,W}(\Sigma_V \otimes 1) = (1 \otimes \Sigma_V)c_{V,W}, \quad c_{V,W}(1 \otimes \Sigma_W) = (\Sigma_W \otimes 1)c_{V,W}
\]

(c) We have \( c_{L(-1),V \otimes W} = c_{L(-1),V} \circ c_{L(-1),W} \).

(d) The braiding with \( L(-1) \) satisfies Yang-Baxter equation, that is, the following diagram commutes: 
\[
\begin{array}{ccc}
L(-1) \otimes V \otimes W & c_{L(-1),V} & V \otimes L(-1) \otimes W & c_{L(-1),W} & W \otimes V \otimes L(-1) \\
| \downarrow c_{V,W} | & | \downarrow c_{V,W} | & | \downarrow c_{V,W} | & | \downarrow c_{V,W} | & \\
L(-1) \otimes W \otimes V & c_{L(-1),W} & W \otimes L(-1) \otimes V & c_{L(-1),V} & W \otimes V \otimes L(-1)
\end{array}
\]

Proof. Part (a) is clear. To prove (b), observe that the action of \( U_q \mathfrak{sl}_2 \otimes U_q \mathfrak{sl}_2 \) on \( V \otimes W \) commutes with both \( \Sigma_V \otimes 1 \) and \( 1 \otimes \Sigma_V \), and the \( R \)-matrix is an element of the completion of \( U_q \mathfrak{sl}_2 \otimes U_q \mathfrak{sl}_2 \).

Given a pair of vectors \( u \in V, w \in W \) such that \( K u = v^i u \) and \( K w = v^j w \), we get \( K(u \otimes w) = v^{i+j} u \otimes w \), so \( T_{V \otimes W} = T_V \otimes T_W \). Since \( c_{L(-1),V} = \text{swap} \circ (\text{Id} \otimes T_V) \), we get the desired relation. Finally, (d) follows from (c).

We can generalize the above results to representations of \( U_q \mathfrak{sl}_N \) as follows. For \( \zeta \in \mathbb{Z}_2^{N-1} \) there is a one-dimensional representation \( L(\zeta) \) of \( U_q(\mathfrak{sl}_N) \) where \( E_i, F_i \) act by 0 and \( K_i = K_i K_{i+1}^{-1} \) act by \( \zeta_i \) (\( 1 \leq i \leq N - 1 \)). Given a representation \( V \) where all weights of \( K_i \) are integral powers of \( v \), we can define an operator \( T_{\zeta,V} : V \to V \) which acts by \( \zeta^{A+1} \) on a subspace where \( K_i \) acts by \( v^{a_i} \). Here \( A \) is the Cartan matrix for \( \mathfrak{sl}_N \) given by

\[
(12) \quad A = \begin{pmatrix}
2 & -1 & 0 & \ldots & 0 \\
-1 & 2 & -1 & \ldots & 0 \\
0 & -1 & 2 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & 2
\end{pmatrix}
\]
and \(a = (a_1, \ldots, a_{N-1})\). Note that \(\det(A) = N\), so \(A^{-1}\) has rational entries with denominator \(N\) and one needs to choose an \(N\)-th root of \((-1)\) to define \(\zeta^{A^{-1}a}\). Define \(\Sigma_{\zeta,V} = T^2_{\zeta,V}\).

**Lemma 4.4.** The operators \(T_{\zeta,V}\) and \(\Sigma_{\zeta,V}\) satisfy the following properties:

(a) \(T_{\zeta,V}E_i = \zeta_i E_i T_{\zeta,V}, T_{\zeta,V}F_i = \zeta_i F_i T_{\zeta,V}\)

(b) \(\Sigma_{\zeta,V}\) commutes with the action of \(U_q\mathfrak{sl}_N\) on \(V\)

(c) The map \(c_{L(\zeta),V} = \text{swap} \circ (\text{Id} \otimes T_{\zeta,V}) : L(\zeta) \otimes V \to V \otimes L(\zeta)\) is a morphism of \(U_q\mathfrak{sl}_N\)-representations

(d) The maps \(T_{\zeta,V}\) and \(\Sigma_{\zeta,V}\) satisfy all equations in Lemma 4.3 with \((-1)\) changed to \(L(\zeta)\).

**Proof.** (a) The operator \(F_i\) changes the weight \(a = (a_1, \ldots, a_{N-1})\) by \(Ae_i\), so if \(K_i v = v^{a_i} v\) then

\[
T_{\zeta,V}F_i(v) = \zeta^{A^{-1}(a+ Ae_i)} F_i v = \zeta^{A^{-1}a + e_i} F_i v = \zeta_i F_i T_{\zeta,V}(v).
\]

The proof for \(E_i\) is similar. Part (b) immediately follows from (a).

For (c), we observe that the action of \(E_i\) on \(L(\zeta) \otimes V\) is the same as the action on \(V\), while the actions of \(F_i, K_i\) are twisted by \(\zeta_i\). On the other hand, the action of \(F_i\) on \(V \otimes L(\zeta)\) is the same as the action on \(V\), while the actions of \(E_i, K_i\) are twisted by \(\zeta_i\). Therefore by (a) the operator \(c_{L(\zeta),V}\) intertwines the actions of \(U_q\mathfrak{sl}_N\) on \(L(\zeta) \otimes V\) and \(V \otimes L(\zeta)\).

Finally, the proof of the rest of Lemma 4.3 extends to \(U_q\mathfrak{sl}_N\) verbatim. \(\square\)

**Remark 4.5.** The above construction of \(T_{\zeta,V}\) and \(\Sigma_{\zeta,V}\) can be extended to an arbitrary semisimple Lie algebra with Cartan matrix \(A\). The action of \(\Sigma_{\zeta,V}\) can be interpreted in terms of projection of the weight lattice to its quotient by the root lattice.

We draw a tangle colored by a representation \(V = V(\lambda)\) using solid lines, and a tangle colored by \(L(\zeta)\) by dotted lines. If a component is colored by \(L(\zeta) \otimes V\), we draw a dotted line on the left of a solid line and parallel to it. The crossings between solid and dotted lines correspond to \(c_{L(\zeta),V}^\pm\) depicted in Figure 3. Note that unlike \(\mathfrak{gl}_N\) case, \(c_{L(\zeta),V}\) does not square to identity and we have to distinguish under- and over-crossings between solid and dotted lines. This allows us to define Reshetikhin-Turaev invariants for framed tangles colored by representations of \(U_q\mathfrak{sl}_N\) of the form \(L(\zeta) \otimes V(\lambda)\).

Using the notations as in Figure 3, we can visualize the statement of Lemma 4.3 in Figure 4.

**Figure 3.** The operators \(c_{L(\zeta),V}, c_{L(\zeta),V}^{-1}\) and \(\Sigma_{\zeta}\).

**Theorem 4.6.** (a) Let \(T\) be an algebraically split 0-framed bottom tangle. Then the Reshetikhin–Turaev invariants of \(T\) colored with the \(U_q\mathfrak{sl}_N\) modules \(V(\lambda)\) and \(L(\zeta) \otimes V(\lambda)\) coincide.

(b) Let \(T\) be an arbitrary bottom tangle with evenly framed components, if \(N\) is odd. Then the Reshetikhin-Turaev invariants of \(T\) corresponding to the \(U_q\mathfrak{gl}_N\) representations \(V(\lambda)\) and \(L(\zeta) \otimes V(\lambda)\) coincide.

**Proof.** (a) We use the results of Lemmas 4.3 and 4.4 and the above diagrammatic notation. By Lemma 4.3(a), we can change crossings between dotted and solid lines at a cost of
 placing $\Sigma_\zeta$ on solid lines. By doing this iteratively, we can make all dotted lines to be above solid lines. At this stage, each solid component of $L$ acquires several copies of $\Sigma_\zeta$ and $\Sigma_{-\zeta}$ at various places of the link diagram. The number of these copies (with signs) equals the linking number between this component and the dotted part which is even by our assumption. By Lemma 4.3(b) we can combine all these copies of $\Sigma_\zeta$ together and cancel out. Since the contributions from the pivotal elements, crossings and framings on the dotted part depend on the linking matrix only, they have to cancel as well.

The proof of (b) is similar, except that $\Sigma_V$ is trivial for all $V$. To compute the contribution from self-crossings (or framing) and the pivotal elements we argue as follows.

Choose a braid diagram $D$ such that the closures of $D$ and $T$ coincide. Note that the power of the pivotal element in $T$ (that is, the number of minima in $T$) has the same parity as the number of crossings. Indeed, for a braid on $n$ strands which closes up to a link with $r$ components, the number of minima is $n - r$ (we close all but one strand for each component). On the other hand, the corresponding permutation in $S_n$ has $r$ cycles, hence its sign equals $(-1)^{n-r}$ and the number of crossings equals $n - r$ modulo 2.

Since, each self-crossing together with a pivotal element acts on $L(\zeta)$ in the same way as a ribbon element $\theta_{L(\zeta)} = (\prod_i \zeta_i)^N$, we have the result.

5. CENTER OF $\mathcal{U}$

Let $Z$ be the center of $\mathcal{U}_Z$. In this section we recall the main known facts about $Z$.

5.1. Harish-Chandra isomorphism. Let $(\mathcal{U}_Z^0)^{S_N} := \mathbb{Z}[v, v^{-1}][K_1^{\pm 1}, \ldots, K_N^{\pm 1}]^{S_N}$ be the Cartan part of $\mathcal{U}_Z$ invariant under the Weyl group action. After a multiplication by an appropriate power of the central element $K := \prod_{i=1}^N K_i$, each element of $(\mathcal{U}_Z^0)^{S_N}$ can be viewed as a symmetric function in $N$ variables. This allows to identify $(\mathcal{U}_Z^0)^{S_N}$ with the ring of symmetric functions divided by powers of the elementary symmetric polynomial $e_N = K$. In the classical case, this ring can be identified with the center using the Harish-Chandra isomorphism. After quantization, the image of the Harish-Chandra homomorphism belongs to

$$Sym = \mathbb{Z}[v^{\pm 1}, e_N^{-1}][x_1, \ldots, x_N]^{S_N}$$
where \( x_i = K_i^2 \) (compare e.g. [20, Ch. 6]). In this section we will furthermore identify \( \text{Sym} \) with the Grothendieck ring \( R \) of \( \text{Rep}(U) \).

First, the character map
\[
ch : R \rightarrow \text{Sym}
\]
sends a representation \( U \) to its character \( ch(U) \). Clearly, \( ch(U \oplus V) = ch(U) + ch(V) \) and \( ch(U \otimes V) = ch(U)ch(V) \), so \( ch \) is a ring homomorphism. The character of \( V(\lambda) \) equals the Schur function \( s_\lambda(x_1, \ldots, x_N) \), while the character of \( L(\zeta) \) equals \( \zeta_1 \cdots \zeta_N \).

The Harish-Chandra map
\[
hc : Z \rightarrow \text{Sym}
\]
is defined as follows. Let \( \phi \) be a central element in \( U \), it acts in the Verma module \( \Delta(\lambda) \) by some scalar \( \phi|_{\Delta(\lambda)} \). We define \( hc(\phi) \) to be the polynomial in \( \text{Sym} \) defined by the condition
\[
hc(\phi)(q^{\rho+\lambda}) = \phi|_{\Delta(\lambda)} \quad \text{for all } \lambda
\]
where \( \rho = \left( \frac{N-1}{2}, \frac{N-3}{2}, \ldots, \frac{1-N}{2} \right) \). Note that the product \( \phi \phi' \) acts on \( \Delta(\lambda) \) by the product of the corresponding scalars, so \( hc \) is also a ring homomorphism. It is known to be an isomorphism (see e.g. [20, Ch. 6]).

Finally, the map \( \xi : R \rightarrow Z \) is defined by \( \xi = hc^{-1} \circ ch \). It is a composition of two ring homomorphisms and hence a ring homomorphism too. Hence, we get the commutative diagram:
\[
\begin{array}{ccc}
R & \xrightarrow{ch} & Z \\
\downarrow{\xi} & & \downarrow{hc} \\
\text{Sym} & & 
\end{array}
\]

In Lemma \( 5.3 \) we will show that \( \xi \) actually coincides with the Drinfeld map.

**Example 5.1.** The central element \( K = K_1 \cdots K_N \) acts on \( V(\lambda) \) by a scalar \( v^{\sum \lambda_i} \). Since \( \sum \rho_i = 0 \), we get \( hc(K_1 \cdots K_N) = y_1 \cdots y_N \).

**Example 5.2.** The center of \( \mathfrak{u}_q \mathfrak{sl}_2 \) is generated by the Casimir element:
\[
C = (v - v^{-1})^2FE + vK + v^{-1}K^{-1}
\]
It acts on a representation \( V_m \) by \( v^{m+1} + v^{-m-1} \), so \( hc(C) = y + y^{-1} \) (note that \( v^\rho = v \) in this case). On the other hand, \( ch(V_1) = y + y^{-1} \), so \( \xi(V_1) = C \), where \( V_1 \) the 2-dimensional representation.

Similarly, we can consider the corresponding central element in \( \mathfrak{u}_q \mathfrak{b}_2 \) defined by
\[
C_{\mathfrak{gl}_2} = (v - v^{-1})^2FE + vK_1K_2^{-1} + v^{-1}K_1^{-1}K_2.
\]
It acts on a representation \( V(\lambda) \) by a scalar
\[
v^{1+\lambda_1-\lambda_2} + v^{-1-\lambda_1+\lambda_2} = \frac{y_1}{y_2} + \frac{y_2}{y_1}, \quad y_1 = v^{1/2+\lambda_1}, y_2 = v^{-1/2+\lambda_2},
\]
so \( hc(C_{\mathfrak{gl}_2}) = \frac{y_1}{y_2} + \frac{y_2}{y_1} = \frac{y_1^2 + y_2^2}{y_1y_2} = e_2^{-1}(y_1, y_2)(x_1 + x_2) \).

5.2. **Hopf pairing.** The Hopf pairing \( \langle U, V \rangle \) of two representations \( U, V \in R \) is defined as the Reshetikhin-Turaev invariant of the Hopf link with components labeled by \( U \) and \( V \). This is a symmetric bilinear pairing on \( R \). The map \( \xi \) is related to the Hopf pairing as follows:

**Lemma 5.3.** The Hopf pairing on \( R \) can be computed as
\[
\langle U, V \rangle = Tr_q(U)(\xi(V)).
\]
Lemma 6.1. We have
\[(13) \quad \text{two-sided ideal generated by the products}\]

Proof. Consider the Drinfeld map \( D \) \[9\] which sends a representation \( V \) to a central element corresponding to the universal invariant of the following tangle:

\[
D(V) := \begin{tikzpicture}
\draw[thick] (0,0) -- (1,1);
\draw[thick] (1,0) -- (0,1);
\end{tikzpicture}
\]

By e.g. \[14\] eq. (20)] (see also \[19\] Proposition 8.19] and references therein] the eigenvalue of \( D(V) \) on the irreducible representation \( V(\lambda) \) equals \( ch(q^{\lambda + \rho}) \) where \( ch \) is the character of \( V \). By the definition of the Harish-Chandra map, this means that \( hc(D(V)) = ch(V) \), and

\[
D(V) = hc^{-1}(ch(V)) = \xi(V),
\]
so \( \xi \) agrees with the Drinfeld map. Now \( \langle U, V \rangle = Tr_U(D(V)) = Tr_U(\xi(V)) \).

Using the Drinfeld isomorphism \( \xi \) we can extend the Hopf pairing to the center by setting

\[
\langle z_1, z_2 \rangle := \langle \xi^{-1}(z_1), \xi^{-1}(z_2) \rangle \quad \text{for any} \quad z_1, z_2 \in \mathcal{Z}.
\]

6. Cyclotomic completion and the universal invariant

The universal invariant of a link (or bottom tangle) is defined in full details in \[16\] Sec. 7.3. A priori it belongs to a (completed) tensor product of copies of \( \mathcal{U}_h \), rather than \( \mathcal{U} \), due to the diagonal part of the \( R \)-matrix.

The aim of this section is to define a certain completion of \( \mathcal{U} \) and its tensor powers, such that the universal \( \mathfrak{gl}_N \) invariant of evenly framed links belongs to it. Since the action of \( \Gamma \) extends to the completion, this will allow us to speak about \( \Gamma \)-invariance of \( J_L(\mathfrak{gl}_N; q) \).

6.1. Cyclotomic completion of \( \mathcal{U} \). Given \( n \in \mathbb{N} \), we define a family of two-sided ideals \( \mathcal{U}_n^{(n)} \) as the minimal filtration such that \( \mathcal{U}_n^{(n)} \mathcal{U}_m^{(m)} \subset \mathcal{U}_{n+m}^{(n+m)} \) and

\[
(q; q)_n, \quad e_i, \quad f_n(K_j^2) \in \mathcal{U}_n^{(n)}
\]
for any \( 1 \leq i \leq N - 1 \) and \( 1 \leq j \leq N \) where \( f_n(x) = (x; q)_n \). In other words, \( \mathcal{U}_n^{(n)} \) is the two-sided ideal generated by the products

\[
(q; q)_a e_1^{b_1} \cdots e_{N-1}^{b_{N-1}} f_{c_1}(K_1^2) \cdots f_{c_N}(K_N^2), \quad \text{with} \quad a + \sum_i b_i + \sum_i c_i = n.
\]

Lemma 6.1. We have

\[
\Delta(f_n(K_i)) = \sum_{a=0}^{n} \binom{n}{a} f_a(K_i) \otimes K_i^{(n-a)} f_{n-a}(K_i).
\]

Proof. We prove Lemma by induction in \( n \). For \( n = 0 \) it is clear. The induction step follows from the identities

\[
f_{n+1}(K_i) = f_n(K_i)(1 - q^n K_i)
\]
and

\[
\Delta(1 - q^n K_i) = 1 \otimes 1 - q^n K_i \otimes K_i = (1 - q^a K_i) \otimes q^{-a} K_i + 1 \otimes (1 - q^{n-a} K_i).
\]

Observe that Lemma 6.1 holds after replacing \( K_i \) with \( K_i^2 \).
Proposition 6.2. a) $\mathcal{U}^{(n)}_c$ is the left ideal generated by $13$.

b) $\mathcal{U}^{(n)}_c$ form a Hopf algebra filtration, that is $\Delta \mathcal{U}^{(n)}_c \subset \sum_{i+j=n} \mathcal{U}^{(i)}_c \otimes \mathcal{U}^{(j)}_c$.

c) Assume that $\lambda_i \leq k$ for all $i$. Given arbitrary $m$, there exists $n = n(k, m)$ such that the elements of $\mathcal{U}^{(n)}_c$ act on the integral basis of $V(\lambda)$ by matrices divisible by $(q; q)_m$.

Proof. a) Observe that by Lemma 9.5 we get $f_n(q^s K_i) \in \mathcal{U}^{(n)}_c$ for all integer $s$. Now the statement follows from the identities $f_n(K^2_i)F_i(q) = f_n(q^{-a_i^s} K^2_i)F_i(q)$ and $f_n(K^2_i) = e^s f_n(q^{a_i^s} K^2_i)$. Recall that $A = (a_{ij})$ is the Cartan matrix given in (12).

b) Follows from the identity $\Delta(e^n) = \sum_{i=0}^{m} \binom{m}{i} K^i e^{m-i} \otimes e^i$ and Lemma 6.1.

c) By (a), it is sufficient to check the statement for $n > k e^n$ annihilates $V(\lambda)$, while $f_n(K^2_i)$ acts on a vector with weight $(v_{\lambda_1}, \ldots, v_{\lambda_N})$ by $f_n(q^{\lambda_i}) = (q^{\lambda_i}; q)_n$ which is divisible by $(q; q)_n$.

By Proposition 6.2(b), the filtration

$$\mathcal{U}_c = \mathcal{U}^{(0)}_c \supset \mathcal{U}^{(1)}_c \supset \ldots \mathcal{U}^{(n)}_c \supset \ldots$$

is a Hopf algebra filtration of $\mathcal{U}_c$ with respect to a descending filtration of ideals $I_n = ((q; q)_n)$ in $\mathbb{Z}[v, v^{-1}]$ in the sense of [18, Sec. 4]. Hence, the completion

$$\mathcal{U} := \lim_{\leftarrow n} \mathcal{U}^{(n)}_c$$

is a complete Hopf algebra over the Habiro ring

$$\mathbb{Z}[v] := \lim_{\leftarrow n} \mathbb{Z}[v] / ((v; v)_n).$$

We refer to [18, Sec. 4] for details. Analogously, we define

$$\mathcal{U}^{\otimes} := \lim_{\leftarrow n} \mathcal{U}^{(n)}_c$$

as a complete Hopf algebra over $\mathbb{Z}[q]$. Let us now extend the completion to the tensor powers of $\mathcal{U}_c$. For this we define the filtration for $\mathcal{U}^{\otimes}_c$ for $l \geq 1$ as follows

$$F_n(\mathcal{U}^{\otimes}_c) = \sum_{i=1}^{l} \mathcal{U}^{\otimes l-1} \otimes \mathcal{U}^{(n)}_c \otimes \mathcal{U}^{\otimes l-i}$$

and the completed tensor product $\mathcal{U}^{\otimes}_c$ with respect to this filtration will be the image of the homomorphism

$$\lim_{\leftarrow n} \frac{\mathcal{U}^{\otimes l}_c}{F_n(\mathcal{U}^{\otimes l}_c)} \to \mathcal{U}^{\otimes l}_h$$

where on the right hand side we use the $h$-adically completed tensor product.

6.2. Universal invariants. Let us denote by $c \in \mathcal{U}_c \otimes \mathcal{U}_h$ the double braiding or the universal invariant of the clasp tangle in Figure 1 given by

$$c = (S \otimes \id) \mathcal{R}_2 \mathcal{R}.$$

The main point about this element is that it is dual to the quantum Killing form (compare [19, Sec. 4]). Hence, we can rewrite $c$ as

$$c = \sum_i c(i) \otimes c'(i) \quad \text{with} \quad \langle c(i), c'(j) \rangle = \delta_{ij}.$$
where \( c(i) \) and \( c'(i) \) are dual \( \mathbb{Z}[u^\pm 1] \)-bases of \( \widehat{U}_q \) with respect to the Hopf pairing or the quantum Killing form. Restricting to the Cartan part this gives us (compare [19, Lemma 3.12])

\[
D^{-2} = \prod_{i=1}^{N} q^{-H_i \otimes H_i} = \prod_{i=1}^{N} \sum_{n_i} (-1)^{n_i} \frac{h_i^{n_i}}{n_i!} H_i^{n_i} \otimes H_i^{n_i}
\]

and hence, \( \langle H_i^n, H_j^m \rangle = \delta_{ij} \delta_{nm} (-1)^n \frac{m!}{n!} \). We deduce that \( \langle K_i^2, K_j^2 \rangle = q^{-1} \) or, more generally,

\[
\langle K_i^{2a}, K_j^{2b} \rangle = \delta_{ij} q^{-ab}
\]
defines the Hopf pairing on the \( \Gamma \)-invariant part of the Cartan. In Section 9 we construct another basis for the Cartan given by \( \prod_{i=1}^{N} f_n(K_i^2) \) such that \( \langle f_n, f_m \rangle = \delta_{nm}(-1)^n q^{-n}(q; q)_n \).

In this new basis, we can rewrite the Cartan part of the clasp element as follows:

\[
D^{-2} = \sum_{n \in \mathbb{N}^N} \prod_{i=1}^{N} (-1)^n q^{n_i} \frac{f_{n_i}(K_i^2)}{(q; q)_n_i} \otimes f_{n_i}(K_i^2)
\]

For \( \mathfrak{sl}_N \) similar computations will give

\[
(D')^{-2} = \sum_{n \in \mathbb{N}^{N-1}} \prod_{i=1}^{N-1} (-1)^n q^{n_i} \frac{f_{n_i}(K_i)}{(q; q)_n_i} \otimes f_{n_i}(K_i^2)
\]

(compare Section B.1 in [19]). As a corollary, we get the following:

**Proposition 6.3.** Given an \( l \)-component evenly framed link \( L \), the universal invariant \( J_L(\mathfrak{gl}_N; q) \) is a well defined element of \( \widehat{U}^{\otimes l} \).

**Proof.** By definition, \( J_K \) is obtained by multiplying together elementary pieces, such as \( F_n, e_n, K_i^{\pm 1}, D^{\pm 1} \), and then taking a sum over all indices. The linking between different components and framing will make appear powers of \( D^{\pm 2} \) that we can decompose using the basis elements \( f_n(K_i^2) \) of the completion by (15). Note that we can collect all diagonal contributions of each component by using formulas like

\[
D(E_i \otimes 1)D^{-1} = E_i \otimes K_i^2 \quad \text{and} \quad D(1 \otimes F_j)D^{-1} = K_j^{-2} \otimes F_j.
\]

Since framing is assumed to be even, we will have an even number of \( D \)-parts and pivotal elements (by the same argument as in Theorem 4.6(b)). Hence using (15) and the explicit form of the quasi \( R \)-matrix \( \Theta \), we get the claim. \( \square \)

**Remark 6.4.** For \( \mathfrak{sl}_N \) we can build the same completion after replacing \( K_i \) with \( \hat{K}_i \). Then the arguments in the proof of Proposition 6.3 will show us that for any algebraically split link the universal invariants belongs to this completion.

**Proof of Theorem 4.11.** Using Proposition 6.3 and remark above, we can define the action of \( \Gamma \) on both invariants. Now Theorem 4.11 follows from Theorem 4.6 if we observe that the action of the universal invariant \( J_L(\mathfrak{gl}_N; q) \) on \( V(\lambda) \) is \( J_L(V(\lambda), q) \), however the action of \( \zeta \cdot J_L(\mathfrak{gl}_N; q) \) on \( V(\lambda) \) is \( J_L(L(\zeta) \otimes V(\lambda), q) \). \( \square \)

**Corollary 6.5.** For any evenly framed knot \( K \), \( J_K(\mathfrak{gl}_N; q) \in \widehat{U}^{\otimes n} \).
6.3. **Twist forms.** Let us denote by $\hat{Z}$ the center of $\hat{U}$. In what follows, we will be particularly interested in the following twist forms

$$ T_\pm : \hat{Z} \to \hat{Z} \quad \text{given by} \quad T_\pm(z) := (r^\pm 1, z) $$

the Hopf pairing with the ribbon element. On the $\Gamma$-invariant Cartan part they are easy to compute and are given by

$$ T_\pm(K_{2m}) = v^{\pm(m, 2\rho - m)} \in \mathbb{Z}[v, v^{\pm 1}] $$

for any $m \in \mathbb{Z}^N$.

7. **Habiro’s basis for $\mathcal{Z}(U_q\mathfrak{sl}_2)$**

In this section we summarize Habiro’s results for $\mathfrak{sl}_2$ in the way suitable for our generalization.

Habiro [15] defined a remarkable family of central elements in $\mathcal{Z}(U_q\mathfrak{sl}_2)$:

$$ \sigma_m := \prod_{i=1}^m (C^2 - (v^i + v^{-i})^2) = \prod_{i=1}^m (C - v^i - v^{-i})(C + v^i + v^{-i}). $$

Since $C$ acts on the $(j + 1)$-dimensional representation $V_j$ by a scalar $v^{j+1} + v^{-j-1}$, the polynomial $\sigma_m$ is completely characterized by the following properties:

(a) **(Parity)** $\sigma_m$ is $\Gamma = \mathbb{Z}_2$-invariant.

(b) **(Vanishing)** $\sigma_m$ annihilates the representations $V_j$ for $j < m$.

(c) **(Normalization)** $\sigma_m$ acts on the representation $V_m$ by a scalar

$$ \prod_{i=1}^m ((v^{m+1} + v^{-m-1})^2 - (v^i + v^{-i})^2). $$

Note that parity implies that $\sigma_m$ also annihilates the representations $L(-1) \otimes V_j$ for $j < m$.

By using the Harish-Chandra isomorphism, we can alternatively consider the polynomials

$$ T_m(y) := hc(\sigma_m) := \prod_{i=1}^m (yv^i - y^{-1}v^{-i})(yv^{-i} - y^{-1}v^i) = (-1)^m \prod_{i=1}^m q^{-i}(1 - y^2 q^i)(1 - y^{-2} q^i) $$

which are characterized by the following properties:

(a) **(Parity)** $T_m$ is $\mathbb{Z}_2$-invariant, that is, $T_m(-y) = T_m(y)$

(b) **(Vanishing)** $T_m(\pm v^{j+1}) = 0$ for $j < m$

(c) **(Normalization)** $T_m(v^{m+1})$ is given in (18).

Habiro proved that $\{\sigma_m\}_{m \geq 0}$ form a basis in (a certain completion of) the $\Gamma$-invariant part of the center. Hence, the elements $S_m = \xi^{-1}(\sigma_m)$, given by

$$ S_m := \prod_{i=1}^m (V_1 - v^i - v^{-i})(V_1 + v^i + v^{-i}) $$

form a basis of $\mathcal{R}$. We will show that

$$ P_n = \prod_{i=0}^{n-1} (V_1 - v^{2i+1} - v^{-2i-1}) \in \mathcal{R} $$

is a dual basis to $\{S_m\}_{m \geq 0}$ with respect to the Hopf pairing. The following is a slight reformulation of [15] Prop. 6.3.

**Lemma 7.1.** We have

$$ \langle P_n, S_m \rangle = \frac{(2n + 1)!}{\{1\}} \delta_{n,m}. $$
Proof. Clearly, one has

$$\xi(P_n) = \prod_{i=0}^{n-1} (C - v^{2i+1} - v^{-2i-1})$$

which annihilates $V_{2i}$ for $i < n$. We have the following cases:

1) For $n < m$ we have $\langle P_n, S_m \rangle = \text{Tr}_q^a(\sigma_m)$. Since $P_n$ is in span of $V_i$ for $i \leq n$ and $\sigma_m$ annihilates all these, we get $\langle P_n, S_m \rangle = 0$.

2) For $m < n$ we have $\langle P_n, S_m \rangle = \text{Tr}_q^m(\xi(P_n))$. Since $S_m$ is in span of $V_{2i}$ for $i \leq n$ and

$$\langle P_n, V_{2i} \rangle = \{i + n\} \ldots \{i - n + 1\}[2i + 1].$$

Hence $P_n$ annihilates all these, we get $\langle P_n, S_m \rangle = 0$.

3) Finally, for $n = m$ we observe that $P_n$ has a unique copy of $V_n$ and

$$\langle P_n, S_n \rangle = \langle V_n, S_n \rangle = \text{Tr}_q^V(\sigma_n)$$

which is easy to compute. \hfill \Box

We can use the above results to compute the coefficients in the decomposition of any central element into $\{\sigma_m\}_{m \geq 0}$.

Lemma 7.2. Let $\phi$ be a $\mathbb{Z}_2$-invariant element in $\mathcal{Z}(U_q\mathfrak{sl}_2)$ which acts on $V_j$ by a scalar $\phi_j$. Then

$$\phi = \sum a_n \sigma_n, \text{ where } a_n = \sum_{i=0}^n (-1)^{n-i} \frac{2i + 2}{n + i + 2} \frac{i + 1}{\{n + 2\}} \frac{1}{\{n - i\}} \phi_i.$$

Proof. We have ([15 Lemma 6.1])

$$P_n = \sum_{i=0}^n (-1)^{n-i} \frac{2i + 2}{n + i + 2} \left[ \frac{2n + 1}{n + 1 + i} \right] V_i.$$

If $\phi = \sum a_m \sigma_m$ then

$$a_n = \frac{\{1\}}{\{2n + 1\}} \text{Tr}_q^P = \frac{\{1\}}{\{2n + 1\}} \sum_{i=0}^n (-1)^{n-i} \frac{2i + 2}{n + i + 2} \left[ \frac{2n + 1}{n + 1 + i} \right] \text{Tr}_q^V =$$

$$\sum_{i=0}^n (-1)^{n-i} \frac{2i + 2}{n + i + 2} \frac{\{1\}}{\{n - i\}} \dim_q(V_i) \phi_i.$$

Using $\dim_q(V_i) = \{i + 1\}$ we obtain the result. \hfill \Box

Habiro proved that for any 0-framed knot $K$, there exist $a_n(K) \in \mathbb{Z}[q, q^{-1}]$ such that

$$J_K(\mathfrak{sl}_2; q) = \sum_{n \geq 0} a_n(K) \sigma_n$$

known as a cyclotomic expansion of the colored Jones polynomial.

8. NEW BASIS FOR THE CENTER OF $\hat{U}$

Recall that $\hat{Z}$ is the center of the completion $\hat{U}$. In this section we construct the basis $\{\sigma_\lambda\}_\lambda$ of the $\Gamma$-invariant part of $\hat{Z}$. Furthermore, we explicitly define its dual $\{P_\lambda\}_\lambda$ with respect to the Hopf pairing. This allows us to construct the cyclotomic expansion of $J_K(\mathfrak{g}\mathfrak{l}_N; q)$ for any 0-framed knot $K$.

The proof uses the existence and properties of interpolation Macdonald polynomials [25] which are summarized in the following theorem.

Theorem 8.1. There is a family of symmetric polynomials $F_\lambda(x_1, \ldots, x_N; q)$ such that:
(a) \( F_\lambda \) is in the span of Schur functions \( s_\mu \) for \( \mu \leq \lambda \) with the leading term
\[
F_\lambda = (-1)^{|\lambda| + \binom{\lambda}{2}} q^{D_N(\lambda)} s_\lambda + \ldots .
\]

(b) \( F_\lambda(q^{-\mu_1-N+1}, \ldots, q^{-\mu_N}) = 0 \) unless \( \mu \) contains \( \lambda \).

(c) \( F_\lambda(q^{-\lambda_1-N+1}, \ldots, q^{-\lambda_N}) = (-1)^{\binom{\lambda}{2}} q^{\nu(\lambda) + \binom{\lambda}{3}} \prod_{i \in \lambda}(1 - q^{-h(i)}) \).

(d) Any function \( F \) in the completion can be written as
\[
F(x_1, \ldots, x_N) = \sum_{\lambda, \mu \subseteq \lambda} d_{\lambda, \mu}(q) F(q^{-\mu_1-N+1}, \ldots, q^{-\mu_N}) F_\lambda(x_1, \ldots, x_N; q)
\]
where \( d_{\lambda, \mu} \) are explicit coefficients prescribed by Theorem 9.17.

We discuss the definition and give more details on interpolation Macdonald polynomials in Section 9.

**Theorem 8.2.** There exists a family of central elements \( \sigma_\lambda \in \mathcal{Z} \) with the following properties:
(a) \( \sigma_\lambda \) is \( \Gamma \)-invariant and annihilates \( L(\zeta) \otimes V(\mu) \) for all \( \mu \) not containing \( \lambda \) and \( \zeta \in \Gamma \).
(b) \( hc(\sigma_\lambda) \) is in the span of \( s_\mu(x_1, \ldots, x_N) \) for \( \mu \leq \lambda \), with the leading term
\[
hc(\sigma_\lambda) = (-1)^{|\lambda| + \binom{\lambda}{2}} q^{(N-1)|\lambda|} q^{D_N(\lambda)} s_\lambda + \ldots .
\]
(c) \( \sigma_\lambda \) acts on \( V(\lambda) \) by a scalar
\[
\sigma_\lambda|_{V(\lambda)} = (-1)^{\binom{\lambda}{2}} q^{-\nu(\lambda) - \binom{\lambda}{3}} \prod_{i \in \lambda}(1 - q^{-h(i)}) .
\]

Proof. Define \( \sigma_\lambda = hc^{-1}(g_\lambda) \), where \( g_\lambda(x_1, \ldots, x_N) = F_\lambda(v^{N-1}x_1, \ldots, v^{N-1}x_N; q^{-1}) \). Then \( \sigma_\lambda \) is clearly \( \Gamma \)-invariant and
\[
\sigma_\lambda|_{L(\zeta) \otimes V(\mu)} = g_\lambda(\zeta \cdot v^{\mu_1+\rho_i}) = F_\lambda(q^{(\mu_1+N-1)}, \ldots, q^{\mu_N}; q^{-1}).
\]

Indeed, if \( y_i = \zeta \cdot v^{\mu_1+\rho_i} = \zeta v^{(\mu_1-N+1)+N-i} \) then \( v^{N-1}y_i^2 = q^{(\mu_1+N+i)} \).

Now \( F_\lambda(q^{(\mu_1+N-1)}, \ldots, q^{\mu_N}; q^{-1}) \) vanishes unless \( \mu \) contains \( \lambda \), and has the nonzero value prescribed by the previous theorem for \( \mu = \lambda \).

**Theorem 8.3.** Define the following formal elements of \( \mathcal{R} \)
\[
P_\lambda = \sum_{\mu \subseteq \lambda} \frac{d_{\lambda, \mu}(q^{-1})}{\dim_q(V(\mu))} V(\mu) \in \mathcal{R},
\]
then one has
\[
\langle P_\lambda, \sigma_\nu \rangle := Tr_q F_\lambda(\sigma_\nu) = \delta_{\lambda, \nu} .
\]

Proof. First, let us write the interpolation formula (19) for \( F = F_\nu \):
\[
F_\nu(x_1, \ldots, x_N; q) = \sum_{\mu \subseteq \lambda} d_{\lambda, \mu}(q) F_\nu(q^{-\mu_1-N+1}, \ldots, q^{-\mu_N}; q) F_\lambda(x_1, \ldots, x_N; q),
\]
so
\[
\sum_{\mu \subseteq \lambda} d_{\lambda, \mu}(q) F_\nu(q^{-\mu_1-N+1}, \ldots, q^{-\mu_N}; q) = \delta_{\lambda, \nu}.
\]

By changing \( q \) to \( q^{-1} \) we get
\[
\sum_{\mu \subseteq \lambda} d_{\lambda, \mu}(q^{-1}) F_\nu(q^{\mu_1+N-1}, \ldots, q^{\mu_N}; q^{-1}) = \delta_{\lambda, \nu}.
\]
Now $\text{Tr}_q^{V(\mu)}(\sigma_\nu) = \dim_q(V(\mu)) g_\nu(q^{\mu_1+N-1}, \ldots, q^{\mu_N})$, hence

$$\text{Tr}_q^{P_\lambda}(\sigma_\nu) = \sum_{\mu \subseteq \lambda} \frac{d_{\lambda,\mu}(q^{-1})}{\dim_q(V(\mu))} \text{Tr}_q^{V(\mu)}(\sigma_\nu) = \delta_{\lambda,\nu}. \tag*{□}$$

Next, we would like to study the integrality properties of the universal knot invariant.

**Lemma 8.4.** (a) Let $\sigma \in \mathcal{U}_Z^{ev}$. Then $\sigma = (K_1 \cdots K_N)^{-2s} \sum a_{\lambda,\sigma} \lambda$ with $a_\lambda \in \mathbb{Z}[q, q^{-1}]$.

(b) Given $k$ and $m$, there exists $n = n(k, m)$ such that for all $\Gamma$-invariant central elements $\sigma$ in the ideal $\mathcal{U}_Z^{(n)}$ the coefficients $a_\lambda$ are divisible by $(q;q)_m$ for $|\lambda| \leq k$.

**Proof.** (a) Recall that Harish-Chandra transform $hc$ identifies the $\Gamma$-invariant part of the center of $\mathcal{U}_Z$ with the space of symmetric functions in $x_1, \ldots, x_N$ with coefficients in $\mathbb{Z}[q, q^{-1}]$. Since $F_\lambda$ is a polynomial with top degree part equal to the Schur polynomial, up to a monomial in $q$, we can write $(x_1 \cdots x_N)^s f(x_1, \ldots, x_N) = \sum_\lambda a_\lambda F_\lambda(x_1, \ldots, x_N; q^{-1})$ and the result follows.

(b) If $\sigma$ is in the ideal $\mathcal{U}_Z^{(n)}$ for sufficiently large $n$, then by Proposition 6.2 its matrix elements in the integral basis of $V(\lambda)$ are divisible by $(q;q)_m$. By definition of Harish-Chandra transform, this implies that the values $f(q^{-\lambda_1-N+1}, \ldots, q^{-\lambda_N})$ are divisible by $(q;q)_m$ and hence by the interpolation formula (19) the coefficients $a_\lambda$ are divisible by $(q;q)_m$ as well. \tag*{□}

**Corollary 8.5.** The center of the completion $\hat{\mathcal{U}}$ is isomorphic to the completion of the space of symmetric polynomials with coefficients in $\mathbb{Z}[v]$ with respect to the basis $F_\lambda$.

**Proof.** By Lemma 8.4 any element of the center of $\hat{\mathcal{U}}$ can be written as an infinite series $\sum a_\lambda F_\lambda$ with coefficients in $\mathbb{Z}[v]$, up to a factor $(x_1 \cdots x_N)^{-s}$. By Corollary 10.7 the multiplication by $(x_1 \cdots x_N)^{-s}$ preserves the space of such series. \tag*{□}

**Corollary 8.6.** Any $\sigma \in \mathcal{U}_Z^{ev}$ can be written as an infinite sum $\sigma = \sum a_\lambda \sigma_\lambda$ with coefficients $a_\lambda = \text{Tr}_{q}^{P_\lambda}(\sigma) \in \mathbb{Z}[q]$.

**Proposition 8.7.** The universal knot invariant admits an expansion

$$J_K(\mathfrak{gl}_N; q) = \sum_\lambda a_\lambda(K) \sigma_\lambda \quad \text{with} \quad a_\lambda(K) = \sum_{\mu \subseteq \lambda} d_{\lambda,\mu}(q^{-1}) J_K(V(\mu), q) \in \mathbb{Z}[q, q^{-1}]$$

called a cyclotomic expansion of the universal $\mathfrak{gl}_N$ knot invariant.

**Proposition 8.7** implies Theorem 1.3 in Introduction.

**Proof.** By Corollary 6.5 $J_K(\mathfrak{gl}_N; q)$ is a central element in $\mathcal{U}_Z^{ev}$, so it can be written as $\sigma = \sum a_\lambda \sigma_\lambda$ with coefficients $a_\lambda \in \mathbb{Z}[q]$. On the other hand, the value of $J_K$ on any representation $V_\lambda$ is in $\mathbb{Z}[q, q^{-1}]$, so by the interpolation formula (19) the coefficients $a_\lambda$ can be written as rational functions with numerators in $\mathbb{Z}[q, q^{-1}]$ and cyclotomic denominators. By Proposition 11.1 this implies that $a_\lambda \in \mathbb{Z}[q, q^{-1}]$. The explicit formula for $a_\lambda$ is obtained by taking Hopf pairing with $P_\mu$ and observing that $\text{Tr}_q^{V(\mu)} J_K = \dim_q(V(\mu)) J_K(V(\mu), q)$ according to our normalization. \tag*{□}

Let us denote by

$$P_\lambda' = \sum_{\mu \subseteq \lambda} d_{\lambda,\mu}(q^{-1}) V(\mu)$$

The last result shows that $a_\lambda(K) = J_K(P_\lambda', q) \in \mathbb{Z}[q, q^{-1}]$, even through the coefficients $d_{\lambda,\mu}(q)$ are rational functions in $q$ (compare Example 9.23).
8.1. Applications for 3-manifold invariants. Here we use the cyclotomic expansion
defined above to give explicit formulas for the unified invariants of 3-manifolds ob-
tained by \((\pm 1)\)-surgeries on a knot.

Assume \(M_\pm\) is a 3-manifold obtained by \((\pm 1)\)-surgery on a knot \(K\), then the unified
invariant \(I(M_\pm)\) is defined as follows [19]:

\[ I(M_\pm) = T_\pm(J_K(\mathfrak{gl}_N; q)) = \langle r^{\pm 1}, J_K(\mathfrak{gl}_N; q) \rangle \]

where \(r\) is the ribbon element and \(J_K(\mathfrak{gl}_N; q)\) is the universal invariant of the 0-framed
knot \(K\). One of the main applications of the cyclotomic expansion is the following result.

Lemma 8.8. Let us define

\[ \omega_{\pm} = \sum_{\lambda} (-1)^{|\lambda|+(\frac{N}{2})} v^{\tau(\lambda,\lambda+2\rho)} v^{w(\lambda)} P_\lambda^r \in \hat{\mathcal{R}} \quad \text{with} \quad w(\lambda) = 2D_N(\lambda) + (N-1)|\lambda| \]

then for any \(x \in \mathcal{R}\), we have

\[ \langle \omega_{\pm}, x \rangle = \tilde{J}_{U_{\pm}}(x) = \langle r^{\pm 1}, \xi(x) \rangle \]

where \(\tilde{J}_{U_{\pm}}(x)\) is the invariant of the \((\pm 1)\)-framed unknot colored by \(x\) and normalized
\(\tilde{J}_U(V(\lambda)) = \dim_q(V(\lambda))\). Hence, \(\omega_{\pm}\) is the universal Kirby color for \((\pm 1)\)-surgery.

Proof. It is enough to check (23) for the basis elements \(x = V(\nu)\). We compute

\[ \langle P_\lambda, V(\nu) \rangle = \sum_{\mu} d_{\lambda,\mu}(q^{-1}) \langle V(\mu), V(\nu) \rangle = \sum_{\mu} d_{\lambda,\mu}(q^{-1}) s_\nu(q^{\mu+N-1})s_\nu(q^\nu) = C_{\lambda} \delta_{\lambda,\nu} s_\nu(q^\nu) \]

where we used Lemma 5.3 (21) and the expansion \(s_\nu = (-1)^{|\lambda|+(\frac{N}{2})} v^{-w(\lambda)} F_\nu + \text{lower terms}\) and hence,

\[ C_\lambda = (-1)^{|\lambda|+(\frac{N}{2})} v^{-w(\lambda)} . \]

Using this computation it is easy to check that

\[ \langle \omega_{\pm}, V(\nu) \rangle = v^{\tau(\nu,\nu+2\rho)} \dim_q(V(\nu)) = \text{Tr}^{V(\nu)}(r^{\pm 1}) \]

is equal to \(\tilde{J}_{U_{\pm}}(V(\nu))\). \(\square\)

We see that \(\omega_+\) and \(\omega_-\) are inverse to each other in the algebra \(\hat{\mathcal{R}}\) isomorphic to \(\hat{\mathcal{Z}}\).

Theorem 8.9. The unified invariant \(I(M_\pm) \in \mathbb{Z}[v]\) and its evaluation of any root of unity
coincides with the Witten–Reshetikhin–Turaev invariant of \(M_\pm\).

Proof. By Theorem 1.3 we know

\[ J_K(\mathfrak{gl}_N; q) = \sum_{\mu} a_\mu(K) \sigma_\mu \quad \text{with} \quad a_\mu(K) \in \mathbb{Z}[q, q^{-1}] \]

The fact that \(I(M_\pm)\) belongs to the Habiro ring easily follows from the claim that \(T_\pm(\sigma_\mu)\)
is divisible by \((q; q)_m\) for some \(m\) depending on \(\mu\) and \(N\). Let us prove the claim. By
10 the Hopf pairing with \(r^{\pm 1}\) replaces each \(x_i^{k_i}\) with \(v^{Q_\pm(k_i)}\) where \(Q_\pm\) is a quadratic form.
By Lemma 9.27 we can rewrite \(\sigma_\mu\) as a linear combination of \(\prod_{i=1}^d f_{n_i}(q^m x_i)\) such that
\(\sum_i n_i = |\mu|, d = N(N+1)/2\) and \(s_i \in \mathbb{Z}\). Moreover, each \(f_{n_i}(q^{m} x_i)\) is divisible by
\(f_n(v^{m} y_i)\) where \(y_i^\pm = x_i\) and hence belongs to the ideal \(I_n\) of \(\mathbb{Z}[v^{\pm 1}, y_i^{\pm 1}]\) characterized in
Proposition 2.1 of [3]. The result follows now from [3, Theorem 2.2]. The number \(m\) we
are looking for is \(\left\lfloor \frac{|\mu|}{N(N+1)} \right\rfloor \).
To check that second claim let us recall that the WRT invariant is obtained from $J_K(\mathfrak{gl}_N; q)$ by taking trace along the Kirby color

$$\Omega_{\pm} = \sum_{\lambda} \sum_{x} \dim_q(V(\lambda)) x^{\pm(\lambda, \lambda+2\mu)} d_{\lambda, \mu}(q^{-1}) \tilde{J}_K(V(\mu))$$

where the sum is taking over all $\lambda \in \mathcal{R}^\infty = \{ \lambda | \dim_q(V(\lambda)) \neq 0 \}$ and $v$ is a root of unity. Since $\Omega_{\pm}$ satisfies Kirby equations, we have

$$\langle \Omega_{\pm}, V(\nu) \rangle = v^{\mp(\nu, \nu+2\rho)} \dim_q(V(\nu))$$

where we interpret the left hand side as a Hopf link with components colored by $\Omega_{\pm}$ and $V(\nu)$, and the right hand side is the result of the sliding. Comparing this computation with (24), we deduce that at roots of unity the actions of $\Omega_{\pm}$ and $\omega_{\pm}$ on $V(\nu)$ from $\mathcal{R}^\infty$ do coincide. This implies the claim since $V(\nu)$ is a basis for $\mathcal{R}^\infty$. \hfill $\square$

Combining the previous results we obtain an explicit expression for the unified invariant using the universal Kirby color

$$I_{M_{\pm}} = J_K(\omega_{\pm}, q) = \sum_{\lambda, \mu \subset \lambda} C_{\lambda} v^{\mp(\lambda, \lambda+2\mu)} d_{\lambda, \mu}(q^{-1}) \tilde{J}_K(V(\mu)).$$

This implies Theorem [14] from Introduction.

**Corollary 8.10.** For all but finitely many partitions $\lambda$, there exist positive integers $n = n(\lambda, N)$, such that the coefficients

$$\sum_{\mu \subset \lambda} d_{\lambda, \mu}(q^{-1}) \tilde{J}_K(V(\mu)) \in (v; v)_n \mathbb{Z}[v, v^{-1}].$$

This is a generalization of the famous integrability theorem in [15, Thm. 8.2] for knots.

### 9. Interpolation Polynomials

In this section we summarize the theory of interpolation Macdonald polynomials.

**9.1. One variable case.** Consider the space of polynomials in one variable $x$ over $\mathbb{C}(q)$ with the following bilinear form

$$(x^k, x^m) = q^{-km}.$$

Let us define polynomials $f_m(x)$, $m = 0, 1, \ldots$ by the equation $f_0(x) = 1$ and

$$f_m(x) = (x; q)_m = (1 - x) \cdots (1 - xq^{m-1}) \quad \text{for} \quad m \geq 1. \tag{26}$$

Clearly, $f_m(x)$ is a degree $m$ polynomial with leading term $(-1)^m q^{\frac{m(m-1)}{2}} x^m$, so $\{f_m\}_{m \geq 0}$ form a basis in $\mathbb{Z}_{q^{-1}}[x]$. Our next aim is to show that this basis is orthogonal. Observe that $f_m(q^{-k}) = 0$ for $k < m$.

**Lemma 9.1.** We have $(f_m(x), f_k(x)) = \delta_{km}q^{-m}(q; q)_m$

**Proof.** First, observe that $(g(x), x^k) = g(q^{-k})$ for any polynomial $g(x)$. Therefore for $m > k$ we have $(f_m(x), x^k) = f_m(q^{-k}) = 0$, so $(f_m(x), g(x)) = 0$ for any polynomial $g(x)$ of degree strictly less than $m$. In particular, $(f_m(x), f_k(x)) = 0$ for $m > k$ and

$$(f_m(x), f_m(x)) = (-1)^m q^{\frac{m(m-1)}{2}} f_m(x, x^m) = (-1)^m q^{\frac{m(m-1)}{2}} f_m(q^{-m}) = (-1)^m q^{-\frac{m(m-1)}{2}} (1 - q^{-m}) \cdots (1 - q^{-1}) = q^{-m}(1 - q^m) \cdots (1 - q).$$

$\square$
Lemma 9.2. The transition matrix between the monomial basis $x^a$ and the basis $f_b(x)$ has the following form:

\begin{equation}
(27) \quad x^a = \sum_{b \leq a} k_{a,b} f_b(x), \quad k_{a,b} = (-1)^b q^{-ab + \frac{b(b-1)}{2}} \binom{a}{b}_q.
\end{equation}

**Proof.** To find the coefficients we compute the pairing $(f_b(x), x^a)$, then using orthogonality we obtain

$$k_{a,b} = \frac{(f_b(x), x^a)}{(f_b(x), f_b(x))} = \frac{f_b(q^{-a})}{(f_b(x), f_b(x))},$$

For $a \geq b$ from Lemma 9.1 we get

$$(f_b(x), f_b(x)) = q^{-b}(q; q)_b,$$

while

$$f_b(q^{-a}) = (1 - q^{-a}) \cdots (1 - q^{-a+b-1}) = (-1)^b q^{-ab + \frac{b(b-1)}{2}}(1 - q^a) \cdots (1 - q^{a-b+1})$$

$$= (-1)^b q^{-ab + \frac{b(b-1)}{2}} \binom{q^{-a}}{a}_q \binom{q^{-a-b}}{q^{-a-b}}(q; q)_a^{-b},$$

and the equation follows. \(\square\)

Our next goal is to expand arbitrary polynomial $f(x)$ in the basis $f_m(x)$. This can be done in two different ways. First, we can expand $f(x)$ in the monomial basis and apply (27). Alternatively, we can apply Newton interpolation method: if $f(x) = \sum a_m f_m(x)$ then

$$f(q^{-j}) = \sum_{m \geq j} a_m f_m(q^{-j}),$$

which is a triangular system of equations for the unknown coefficients $a_m$. Thus knowing $f(q^{-j})$ one can at least theoretically reconstruct the coefficients $a_m$. This can be made explicit by the following:

Lemma 9.3. We have

\begin{equation}
(28) \quad f(x) = \sum_{m=0}^{\infty} a_m f_m(x), \quad a_m = \frac{1}{(f_m, f_m)} \sum_{j=0}^{m} (-1)^j q^{\frac{j(j-1)}{2}} \binom{m}{j}_q f(q^{-j}).
\end{equation}

**Proof.** By $q$-binomial theorem we have

\begin{equation}
(29) \quad f_m(x) = \sum_{j=0}^{m} (-1)^j q^{\frac{j(j-1)}{2}} \binom{m}{j}_q x^j.
\end{equation}

Now

$$a_m = \frac{(f, f_m)}{(f_m, f_m)} = \frac{1}{(f_m, f_m)} \sum_{j=0}^{m} (-1)^j q^{\frac{j(j-1)}{2}} \binom{m}{j}_q (f, x^j).$$

Finally, $(f, x^j) = f(q^{-j})$. \(\square\)

Remark 9.4. Equation (29) can be interpreted as an explicit inverse of the matrix in (27).

One can consider completion $\widehat{\mathbb{Z}_q[x]}$ of the space of polynomials with respect to the basis $f_m(x)$. In this completion, infinite sums $\sum_{m=0}^{\infty} a_m f_m(x)$ are allowed. Newton interpolation method and (28) identify this completion with the space of distributions on the interpolation nodes $1, q^{-1}, \ldots$.

We will need the following lemma.
Lemma 9.5. We have

\[(x - q^s)(x - q^{s+1}) \cdots (x - q^{s+m-1}) = \sum_{j=0}^{m} (-1)^j q^{-jm+\binom{j+1}{2}} \binom{m}{j}_q (1 - q^{s+j}) \cdots (1 - q^{s+m-1}) f_j(x).\]

Proof. We prove it by induction in \(m\). For \(m = 1\) we get

\[x - q^s = -(1 - x) + (1 - q^s) = -f_1 + (1 - q^s)f_0.\]

For the step of induction we observe

\[(x - q^{s+m}) f_j(x) = -q^{-j} (1 - q^j x) f_j(x) + (q^{-j} - q^{s+m}) f_j(x)\]

\[= -q^{-j} f_{j+1}(x) + q^{-j} (1 - q^{s+m+j}) f_j(x).\]

Using (30), it is easy to identify the coefficient at \(f_j(x)\) in

\[(x - q^{s+m}) \sum (\cdots) (1 - q^{s+j}) \binom{m}{j}_q (1 - q^{s+m-1}) f_j(x)\]

as

\[-q^{-j+1} (1 - q^{s+j+1}) \binom{m}{j-1}_q (1 - q^{s+j-1}) \cdots (1 - q^{s+m-1})\]

\[+ q^{-j} q^{-jm+\binom{j+1}{2}} \binom{m}{j}_q (1 - q^{s+j}) \cdots (1 - q^{s+m-1})(1 - q^{s+m+j})\]

\[= -q^{-j(m+1)+\binom{j+1}{2}} (1 - q^{s+j}) \cdots (1 - q^{s+m-1})\]

\[\times \left[ q^{m-j+1} \binom{m}{j-1}_q (1 - q^{s+j-1}) + \binom{m}{j}_q (1 - q^{s+m+j}) \right].\]

It remains to notice that

\[q^{m-j+1} \binom{m}{j-1}_q (1 - q^{s+j-1}) + \binom{m}{j}_q (1 - q^{s+m+j})\]

\[= \left[ q^{m-j+1} \binom{m}{j-1}_q + \binom{m}{j}_q \right] - q^{s+m} \left[ \binom{m}{j-1}_q + q^j \binom{m}{j}_q \right] \]

\[= \binom{m+1}{j}_q - q^{s+m} \binom{m+1}{j}_q = (1 - q^{s+m}) \binom{m+1}{j}_q.\]

\[\square\]

Remark 9.6. If we set a formal variable \(y = q^s\) in Lemma 9.5, then we get the identity

\[(x - y)(x - qy) \cdots (x - yq^{m-1}) = \sum_{j=0}^{m} (-1)^j q^{-jm+\binom{j+1}{2}} \binom{m}{j}_q f_{m-j}(yq^j) f_j(x).\]

This is a \(q\)-analogue of the binomial identity

\[(x - y)^m = \sum_{j=0}^{m} (-1)^j \binom{m}{j} (1 - y)^{m-j} (1 - x)^j.\]
9.2. Multi-variable case: polynomials. Let us generalize the above results to the case of $N$ variables. The pairing has the form
\[(x_1^{a_1} \cdots x_N^{a_N}, x_1^{b_1} \cdots x_N^{b_N}) = q^{-\sum a_i b_i} (x_1^{a_1} x_1^{b_1}) \cdots (x_N^{a_N} x_N^{b_N}).\]
Note that for $x = (x_1, \ldots, x_N)$
\[(g(x), x_1^{b_1} \cdots x_N^{b_N}) = g(q^{-b_1}, \ldots, q^{-b_N}).\]
Consider the products
\[f_{k_1, \ldots, k_N}(x) = f_{k_1}(x_1) \cdots f_{k_N}(x_N).\]
Since $f_k(x)$ give a basis in $\mathbb{C}(q)[x]$, the polynomials $f_{k_1, \ldots, k_N}$ give a basis in $\mathbb{C}(q)[x_1, \ldots, x_N]$. Clearly,
\[(f_{k_1, \ldots, k_N}, x_1^{b_1} \cdots x_N^{b_N}) = 0 \text{ unless } b_i \geq k_i \text{ for all } i.\]

**Lemma 9.7.** We have $(f_{k_1, \ldots, k_N}, f_{m_1, \ldots, m_N}) = 0$ unless $k_i = m_i$ for all $i$.

**Proof.** Suppose that $k_i > m_i$ for some $i$. Since $f_{m_1, \ldots, m_N}$ contains only monomials of the form $x_1^{b_1} \cdots x_N^{b_N}$ with $b_i \leq m_i$, we have $(f_{k_1, \ldots, k_N}, x_1^{b_1} \cdots x_N^{b_N}) = 0$ for all such monomials and hence $(f_{k_1, \ldots, k_N}, f_{m_1, \ldots, m_N}) = 0$. \(\square\)

Next, we would like to describe the basis in symmetric polynomials. It will be labeled by partitions $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_N)$ with at most $N$ parts. We define
\[(31) \quad F_\lambda(x) = \det(f_{\lambda_i+N-i}(x_j)) / \prod_{i<j} (x_i - x_j).\]
Clearly, the numerator in $(31)$ is antisymmetric in $x_i$, so it is divisible by $\prod_{i<j} (x_i - x_j)$ and the ratio is a symmetric function. It is easy to see that $F_\lambda(x)$ is a non-homogeneous polynomial of degree $|\lambda|$, and the top degree component equals $(-1)^{|\lambda|+\binom{N}{2}} q^{D_N(\lambda)} s_\lambda$ where $s_\lambda$ is the Schur function and $D_N(\lambda)$ is defined by (5). The function $F_\lambda(x)$ is known as a special case of a factorial Schur function $[25]$, it is also a specialization of nonsymmetric Macdonald polynomials described below.

**Lemma 9.8.** Suppose that $b_1 > \ldots > b_N$. Then $F_\lambda(q^{-b_1}, \ldots, q^{-b_N}) = 0$ unless $b_i \geq \lambda_i + N - i$ for all $i$.

**Proof.** Suppose that $b_j < \lambda_j + N - j$ for some $j$, then for all $i \leq j$ and $\ell > j$ one has $\lambda_i + N-i \geq \lambda_j + N-j > b_j \geq b_i$, so $f_{\lambda_i+N-i}(q^{-b_i}) = 0$. This implies $\det(f_{\lambda_i+N-i}(q^{-b_i}))_{i,j=1}^N = 0$. On the other hand, since $b_j \neq b_i$ the denominator $\prod_{i<j} (q^{-b_i} - q^{-b_j})$ does not vanish. \(\square\)

**Corollary 9.9.** If $\mu$ is another partition then we can define $b_i = \mu_i + N - i$, and conclude that $F_\lambda(q^{-\mu_i-N+i}) = 0$ unless $\mu_i \geq \lambda_i$ for all $i$, that is, partition $\mu$ contains $\lambda$.

**Example 9.10.** Suppose that $\lambda = (1)$, then $F_{(1)}$ is a symmetric function of degree 1 with leading term $(-1)^1 \binom{N}{2} q^{D_N(1)} s_{(1)} = q^{D_N(1)} \sum x_i$. We have $D_N(1) = N-1 + \binom{N}{2}$, so $F_{(1)}(x_1, \ldots, x_N) = (-1)^1 \binom{N}{2} q^{N-1+\binom{N}{2}} \sum x_i + c$. To find the constant $c$, we observe that by Corollary 9.9 we get $F_{(1)}(q^{-N+1}, q^{-N+2}, \ldots, 1) = 0$, so
\[c = (-1)^1 \binom{N}{2} q^{N-1+\binom{N}{2}} (q^{-N+1} + q^{-N+2} + \ldots + 1) = (-1)^1 \binom{N}{2} q^{\binom{N}{2}} [N].\]

**Lemma 9.11.** We have
\[F_\lambda(q^{-\lambda_i-N+i}) = (-1)^{\binom{N}{2}} q^{s_\lambda+\binom{N}{2}} \prod_{\square \in \lambda} (1 - q^{-h(\square)}),\]
where $h(\square)$ is the hook length of a box $\square$ in the Young diagram corresponding to $\lambda$. 
Proof. Since the sequence $\lambda_i + N - i$ is strictly decreasing, we have $f_{\lambda_i+N-j}(q^{-\lambda_i-N+i}) = 0$ for $j > i$ and

$$f_{\lambda_i+N-i}(q^{-\lambda_i-N+i}) = \{\lambda_i + N - i\}_{q^{-1}}!$$

and

$$\det(f_{\lambda_j+N-j}(q^{-\lambda_j-N+i})) = \prod_i \{\lambda_i + N - i\}_{q^{-1}}!.$$  

On the other hand,

$$\prod_{i<j}(q^{-\lambda_i-N+i} - q^{-\lambda_j-N+j}) = (-1)^{\binom{N}{2}}q^{-\sum(\lambda_j+N-j)(j-1)}\prod_{i<j}(1 - q^{-\lambda_i+N+j}).$$

and the statement follows now from formula (4) and the identity

$$\sum(\lambda_j + N - j)(j-1) = n(\lambda) + \binom{N}{3}.$$ 

\[\square\]

Example 9.12. For arbitrary $N$ and $\lambda = (1)$ we computed in Example 9.10

$$F_{(1)} = (-1)^{1+\binom{N}{2}}q^{\binom{N}{3}}(q^{N-1}(x_1 + \ldots + x_N) - [N]_q).$$

Hence,

$$F_{(1)}(q^{-N}; q^{-N+2}, \ldots, 1) = q^{\binom{N}{3}}(q^{N-1}(q^{-N} + q^{-N+2} + \ldots + 1) - [N]_q)$$

$$= (-1)^{1+\binom{N}{2}}q^{\binom{N}{3}}(q^{-1} - 1) = (-1)^{\binom{N}{2}}q^{\binom{N}{3}}(1 - q^{-1}).$$

We summarize the above results in the following proposition:

Proposition 9.13. \[29]\] There exists a unique collection of nonhomogeneous symmetric polynomials $F_\lambda(x_1, \ldots, x_N)$ with the following properties:

- $F_\lambda(x_1, \ldots, x_N)$ has degree $|\lambda|$.
- $F_\lambda(q^{-\mu-N+i}) = 0$ for all partitions $\mu$ not containing $\lambda$.
- $F_\lambda(q^{-\lambda-N+i}) = (-1)^{\binom{N}{2}}q^{n(\lambda)+\binom{N}{3}}\prod_{i<j}(1 - q^{-h(i:j)})$.

We will denote the value $F_\lambda(q^{-\lambda-N+i}) = (-1)^{\binom{N}{2}}q^{n(\lambda)+\binom{N}{3}}\prod_{i<j}(1 - q^{-h(i:j)})$ by $c_{\lambda,\lambda}$.

Lemma 9.14. Suppose that $q$ is a root of unity. Then $c_{\lambda,\lambda}$ vanishes for all but finitely many partitions $\lambda$.

Proof. Observe that $\prod_{i<j}(1 - q^{-h(i:j)})$ is divisible by $\prod_{i}(|\lambda_i - \lambda_{i+1}|_q)!$ and

$$\sum_{i=1}^N i(\lambda_i - \lambda_{i+1}) = |\lambda|.$$ 

This means that for some $i$ we must have

$$i(\lambda_i - \lambda_{i+1}) \geq \frac{|\lambda|}{N}, \quad \lambda_i - \lambda_{i+1} \geq \frac{|\lambda|}{iN} \geq \frac{|\lambda|}{N^2},$$

and $c_{\lambda,\lambda}$ is divisible by $(1 - q) \cdots (1 - q^{-\frac{|\lambda|}{N^2}})$. If $q^s = 1$ then it vanishes for $|\lambda| \geq sN^2$. \[\square\]

Remark 9.15. A partition is called an $s$-core if none of its hook lengths is divisible by $s$. The $s$-core partitions play an important role in representation theory of symmetric groups in finite characteristic, and of Hecke algebras at roots of unity \[21\]. If $q^s = 1$ then clearly $c_{\lambda,\lambda}(q) \neq 0$ if and only if $\lambda$ is an $s$-core. Although there are infinitely many $s$-cores, Lemma 9.14 shows that there are finitely many $s$-cores with at most $N$ rows.

For example, for $s = 2$ the 2-cores are “staircase partitions” $\lambda = (k, k-1, \ldots, 1)$, and the maximal 2-core with at most $N$ rows has size $N + (N - 1) + \ldots + 1 = \binom{N+1}{2}$. 
Problem 9.16. Find a symmetric function $f = \sum a_\lambda F_\lambda$ given its values $f(q^{-\mu_i-N+i})$ for all $\mu$.

We have

$$f(q^{-\mu_i-N+i}) = \sum a_\lambda F_\lambda(q^{-\mu_i-N+i})$$

This is a linear system on $a_\lambda$ with the triangular matrix

$$(32) \quad C = (c_{\lambda,\mu}), \quad c_{\lambda,\mu}(q) := [F_\lambda(q^{-\mu_i-N+i})]_{\lambda,\mu}$$

It is clear from Proposition 9.13 that to find $a_\lambda$ for a given $\lambda$ it is sufficient to know all coefficients $c_{\mu,\nu}$ for $\mu \subset \nu \subset \lambda$.

In [28] Okounkov computed the inverse matrix $D = C^{-1}$ which allows one to explicitly compute the coefficients $a_\lambda$.

Theorem 9.17. [28] Define $c_{\lambda,\mu}^*(q) = c_{\lambda,\mu}(q^{-1})$ and $\text{cont}(\lambda) = n(\lambda) - n(\lambda')$. Then

$$D = (d_{\lambda,\mu}), \quad d_{\lambda,\mu} = (-1)^{|\mu|-|\lambda|} q^{\text{cont}(\lambda) - \text{cont}(\mu)} \frac{c_{\lambda,\mu}^*}{c_{\mu,\nu} c_{\lambda,\lambda}}$$

and

$$a_\mu = \sum_{\lambda \subset \mu} d_{\lambda,\mu} f(q^{-\lambda_i-N+i}) = \frac{1}{c_{\mu,\nu}} \sum_{\lambda \subset \mu} (-1)^{|\mu|-|\lambda|} q^{\text{cont}(\lambda) - \text{cont}(\mu)} \frac{c_{\lambda,\mu}^*}{c_{\lambda,\lambda}} f(q^{-\lambda_i-N+i}).$$

Example 9.18. If $\lambda = \mu$ then clearly $d_{\lambda,\mu} = \frac{1}{c_{\lambda,\lambda}}$.

Example 9.19. We have $F_{(\emptyset)} = (-1)^{\binom{N}{2}} q^{\binom{N}{3}}$, so

$$c_{(\emptyset),(\emptyset)} = c_{(\emptyset),(1)} = (-1)^{\binom{2}{2}} q^{\binom{3}{3}}, \quad c_{(\emptyset),(\emptyset)}^* = c_{(\emptyset),(1)}^* = (-1)^{\binom{2}{2}} q^{-\binom{3}{3}}.$$ Since

$$c_{(1),(1)} = (-1)^{\binom{2}{2}} q^{\binom{3}{3}} (1-q^{-1}) = (-1)^{\binom{2}{2}+1} q^{\binom{3}{3}-1} (1-q),$$

we get

$$d_{(\emptyset),(1)} = \frac{(-1)^{\binom{2}{2}} q^{-\binom{3}{3}+1}}{(1-q)}.$$ So the first two terms of interpolation series have the following form:

$$f(x_1, \ldots, x_N) = (-1)^{\binom{N}{2}} q^{-\binom{N}{3}} f(q^1, q^{2-N}, \ldots, 1) F_{(\emptyset)}(x) +$$

$$\frac{(-1)^{\binom{2}{2}+1} q^{-\binom{3}{3}+1}}{1-q} \left[ -f(q^1, q^{2-N}, \ldots, 1) + f(q^{-N}, q^{2-N}, \ldots, 1) \right] F_{(1)}(x) + \ldots$$

Example 9.20. For $N = 1$ and $a \geq b$ we have

$$c_{(b),(a)} = f_b(q^{-a}) = (1-q^{-a}) \cdots (1-q^{-a+b-1})$$

hence

$$c_{(b),(a)}^* = (1-q^a) \cdots (1-q^{a+b+1})$$

Now

$$c_{(b),(b)}^* = \frac{(1-q^a) \cdots (1-q^{a+b+1})}{(1-q^b) \cdots (1-q)} = \binom{a}{b},$$

and

$$d_{(b),(a)} = (-1)^{a-b} q^{\frac{b(b-1)}{2} + \frac{a(a-1)}{2}} c_{(b),(a)}^* c_{(a),(b)}^* = \frac{(-1)^{a-b}}{(1-q) \cdots (1-q)} q^{\frac{b(b-1)}{2} + \frac{a(a-1)}{2}} \binom{a}{b}.$$
which matches [28].

Example 9.21. Let \( N = 2, \lambda = (1) \) and \( \mu = (3, 2) \). We have \( F_\lambda = q(x_1 + x_2) - (1 + q) \), so

\[
c_{\lambda, \mu} = F_\lambda(q^{-4}, q^{-2}) = (-q - 1 + q^{-1} + q^{-3}), \quad c_{\lambda, \mu}^* = q^3 + q - 1 - q^{-1},
\]

and using Lemma [9.17]

\[
c_{\lambda, \lambda} = -(1 - q^{-1}), \quad c_{\lambda, \lambda}^* = -(1 - q),
\]

\[
c_{\mu, \mu} = -q^2(1 - q^{-1})^2(1 - q^{-2})(1 - q^{-3}) = q^{-9}(1 - q)^2(1 - q^2)(1 - q^3)(1 - q^4).
\]

Now

\[
d_{\lambda, \mu} = q^{-2} \frac{c_{\lambda, \mu} c_{\lambda, \lambda}^*}{c_{\mu, \mu} c_{\lambda, \lambda}^*} = -q^6 \frac{q^4 + q^2 - q - 1}{(1 - q)^3(1 - q^2)(1 - q^3)(1 - q^4)}.
\]

9.4. Hopf pairing. We have a symmetric bilinear form \( \langle \cdot \cdot \rangle \) on \( \mathbb{Z}[x_1, \ldots, x_N]^{S_N} \) defined by its values on Schur polynomials

\[
(s_\lambda, s_\mu) = s_\lambda(q^{-\mu_1 - N + 1}, \ldots, q^{-\mu_N}) s_\mu(q^{-N + 1}, \ldots, 1).
\]

It is closely related to the Hopf pairing \( \langle \cdot \cdot \rangle \) for \( \mathcal{R} = \text{Rep}(\mathcal{U}) \) defined in Section 5.2. Note that

\[
(f, s_\mu) = f(q^{-\mu_1 - N + 1}, \ldots, q^{-\mu_N}) s_\mu(q^{-N + 1}, \ldots, 1).
\]

for any symmetric function \( f \).

Proposition 9.22. We have

\[
(F_\lambda, F_\nu) = \delta_{\lambda, \nu} q^{-|\lambda| + 2(\frac{N}{2})} \prod_{\square \in \lambda} (1 - q^{N + c(\square)}),
\]

so the Hopf pairing is diagonal in the basis \( \{F_\lambda\}_\lambda \).

Proof. We have

\[
(F_\lambda, s_\mu) = F_\lambda(q^{-\mu_1 - N + 1}, \ldots, q^{-\mu_N}) s_\mu(q^{-N + 1}, \ldots, 1) = 0
\]

unless \( \lambda \subset \mu \). On the other hand, \( F_\nu \) can be expanded in \( s_\mu \) for \( \mu \leq \nu \), so \( (F_\lambda, F_\nu) \) vanishes unless there exists \( \mu \leq \nu \) such that \( \lambda \subset \mu \), in particular, \( \lambda \leq \nu \).

Since the Hopf pairing is symmetric, \( (F_\lambda, F_\nu) \) vanishes unless \( \lambda \leq \nu \) and \( \nu \leq \lambda \), so \( \lambda = \nu \). Finally,

\[
(F_\lambda, F_\lambda) = (-1)^{|\lambda| + (\frac{N}{2})} q^{D_N(\lambda)}(F_\lambda, s_\lambda) = (-1)^{|\lambda| + (\frac{N}{2})} q^{D_N(\lambda)} F_\lambda(q^{-\lambda_1 - N + 1}, \ldots, q^{-\lambda_N}) s_\lambda(q^{-N + 1}, \ldots, 1).
\]

Now

\[
F_\lambda(q^{-\lambda_1 - N + 1}, \ldots, q^{-\lambda_N}) = (-1)^{(\frac{N}{2})} q^{n(\lambda) + (\frac{N}{2})} \prod_{\square \in \lambda} (1 - q^{-h(\square)})
\]

while

\[
s_\lambda(q^{-N + 1}, \ldots, 1) = q^{-n(\lambda)} \prod_{\square \in \lambda} \frac{1 - q^{-N + c(\square)}}{1 - q^{-h(\square)}}.
\]

hence

\[
F_\lambda(q^{-\lambda_1 - N + 1}, \ldots, q^{-\lambda_N}) s_\lambda(q^{-N + 1}, \ldots, 1) = (-1)^{(\frac{N}{2})} q^{(\frac{N}{2})} \prod_{\square \in \lambda} (1 - q^{-N + c(\square)}) =
\]

\[
(-1)^{|\lambda| + (\frac{N}{2})} q^{-N|\lambda| - c(\lambda) + (\frac{N}{2})}(1 - q^{N + c(\square)}).
\]

On the other hand, \( D_N(\lambda) = c(\lambda) + (N - 1)|\lambda| + (\frac{N}{2}) \). □
This provides us with a different perspective for the interpolation problem. Suppose that we have a Schur expansion for $F_\lambda$:

$$F_\lambda = \sum_{\mu \leq \lambda} b_{\lambda,\mu} s_\mu.$$ 

Then for an arbitrary symmetric function $f(x_1, \ldots, x_N)$ we can write

$$f = \sum_{\lambda} \frac{(f, F_\lambda)}{(F_\lambda, F_\lambda)} F_\lambda = \sum_{\lambda} \sum_{\mu \leq \lambda} b_{\lambda,\mu} \frac{(f, s_\mu)}{(F_\lambda, F_\lambda)} F_\lambda = \sum_{\lambda} \sum_{\mu \leq \lambda} b_{\lambda,\mu} s_\mu \frac{(q^{-N-i})}{(F_\lambda, F_\lambda)} f(q^{-\mu,-N+i}) F_\lambda,$$

and the interpolation coefficient is equal to

$$d_{\lambda,\mu} = \frac{b_{\lambda,\mu} s_\mu (q^{-N+i})}{(F_\lambda, F_\lambda)}.$$

**Example 9.23.** For $N = 2$ and $\lambda = (3, 2)$ we have

$$F_{(3,2)} = q^2(1 - x_1)(1 - qx_1)(1 - x_2)(1 - qx_2)(q^3(x_1 + x_2) - (1 + q)) = q^7 s_{3,2} - q^6(1 + q)s_{3,1} - q^4(1 + q + q^2 + q^3)s_{2,2} + q^6 s_{3,0} + q^3(1 + q + q^2 + q^3)(1 + q)s_{2,1} - q^3(1 + q + q^2 + q^3)s_{2,0} - q^2(1 + q + q^2 + q^3)(1 + q)s_{1,1} + (q^5 + q^4 + 2q^3 + q^2)s_{1,0} - (q^3 + q^2).$$

Also

$$(F_{3,2}, F_{3,2}) = -q^{-5}(1 - q^4)(1 - q^3)(1 - q^2)^2(1 - q)$$

Therefore the interpolation coefficient for $\lambda = (3, 2)$ and $\mu = (1, 0)$ equals

$$d_{(3,2),(1,0)} = (q^5 + q^4 + 2q^3 + q^2)s_{1,0}(q^{-1}, 1) =$$

$$-\frac{(q^5 + q^4 + 2q^3 + q^2)(1 + q^{-1})}{q^{-5}(1 - q^4)(1 - q^3)(1 - q^2)^2(1 - q)} = -\frac{q^6(q^4 + q^3 + q^2 - q - 1)}{(1 - q^4)(1 - q^3)(1 - q^2)(1 - q)^3}. $$

This agrees with Example [9.21]

9.5. **Divisibility.** Given a polynomial $f(x)$, define

$$\partial_{xy}(f) := \frac{f(x) - f(y)}{x - y}.$$ 

Observe that

$$\partial_{xy}(fg) = \frac{f(x) - f(y)}{x - y} g(x) + f(y) \frac{g(x) - g(y)}{x - y} = \partial_{xy} f \cdot g(x) + f(y) \cdot \partial_{xy} (g).$$

More generally, we have

$$\partial_{xy}(f_1 \cdots f_k) = \partial_{xy}(f_1) f_2(x) \cdots f_k(x) + f_1(y) \partial_{xy}(f_2) f_3(x) \cdots f_k(x) + \cdots + f_1(y) f_2(y) \cdots \partial_{xy}(f_k).$$

**Example 9.24.** For $f_n(x) = (1 - x) \cdots (1 - q^{n-1} x)$, note that $\partial_{xy}(1 - q^ix) = -q^i$, so we get

$$F_{n,0}(x, y) = \partial_{xy} f_{n+1}(x) = \sum_{i=0}^{n} (1 - y) \cdots (1 - q^{i-1} y) [\partial_{x,y}(1 - q^i x)] (1 - q^{i+1} x) \cdots (1 - q^n x)$$

$$= \sum_{i=0}^{n} f_i(y) \cdot (-q^i) f_{n-i}(q^{i+1} x).$$
Example 9.25. For example,
\[ F_{1,0}(x, y) = q(x + y) - (1 + q) = q(y - 1) + (qx - 1) = -[qf_1(y) + f_1(qx)]. \]
Similarly,
\[ F_{2,0}(x, y) = -q^2(x^2 + xy + y^2) + (q + q^2 + q^3)(x + y) - (1 + q + q^2) \]
\[ = -[(1 - qx)(1 - q^2x) + q(1 - q^2x)(1 - y) + q^2(1 - y)(1 - qy)] \]
\[ = -[f_2(qx) + qf_1(x)f_2(y) + q^2f_2(y)]. \]

Corollary 9.26. For all integers \(a\) and \(b\) the value \(F_{n,0}(q^a, q^b)\) is divisible by \((\left\lfloor \frac{n}{2} \right\rfloor)_q!\)

Proof. Let \(k = \left\lfloor \frac{n}{2} \right\rfloor\). In the above equation either \(i \geq k\) or \(n - i \geq k\), so each term in the sum is either divisible by \(f_k(q^{i+1}a)\) or by \(f_k(q^b)\), so by \(q\)-binomial theorem it is divisible by \((k)_q!\).

More generally, let \(\partial_i = \partial_{x_i, x_{i+1}}\) then it is well known that \(\partial_i\) satisfy braid relations, so one can define \(\partial_w\) for any permutation \(w\). Furthermore,
\[ F_\lambda(x_1, \ldots, x_N) = \partial_{w_0}[f_{\lambda_1+N-1}(x_1) \cdots f_{\lambda_N}(x_N)], \]
where \(w_0 = (N N - 1 \ldots 1)\) is the longest element in \(S_N\).

Lemma 9.27. For all \(\lambda\) one can write \(F_\lambda(x_1, \ldots, x_N)\) as the sum where each term has the form
\[ f_{j_1}(q^{s_1}x_{m_1}) \cdots f_{j_d}(q^{s_d}x_{m_d}), \]
where \(j_1 + \ldots + j_d = |\lambda|\) and \(d = \binom{N + 1}{2}\).

Here the indices \(m_i\) might repeat arbitrarily.

Proof. From (35) and Example 9.24 it is clear that \(\partial_i\) applied to a product (36) with \(\ell\) factors produces a sum of similar products with \(\ell + 1\) factors. We start from a product of \(N\) factors, and \(\partial_w\) is a composition of \(\binom{N}{2}\) operators \(\partial_i\), so the terms in the resulting sum have \(N + \binom{N}{2} = \binom{N + 1}{2}\) factors. Also, each \(\partial_i\) decreases the degree by 1, so
\[ j_1 + \ldots + j_d = \sum(\lambda_i + N - i) - \binom{N}{2} = |\lambda|. \]

Remark 9.28. A more careful analysis of this proof leads to a combinatorial formula for \(F_\lambda\) where the terms are labeled by semistandard tableaux, but we do not need it here. This is a \(q\)-analogue of the expansion of a Schur function in the monomial basis.

Lemma 9.29. For any sequence of integers \(a_1, \ldots, a_N\) the value \(F_\lambda(q^{a_1}, \ldots, q^{a_N})\) is divisible by \((k)_q!\), where \(k = \left\lfloor \frac{|\lambda|}{\binom{N}{2}} \right\rfloor\).

Proof. In each term (36) there are \(d = \binom{N + 1}{2}\) indices \(j_1, \ldots, j_d\) which add up to \(|\lambda|\), so at least one of these indices is greater than \(|\lambda|/d\). It remains to notice that \(f_j(q^a)\) is divisible by \((q)_j!\) for all integers \(a\).

The following lemma gives a rough description of the expansion
\[ F_\lambda(x_1, \ldots, x_N) = \sum_{m_1, \ldots, m_N} b_{m_1, \ldots, m_N} f_{m_1}(x_1) \cdots f_{m_N}(x_N). \]
of the symmetric interpolation polynomial \(F_\lambda\) in terms of nonsymmetric ones.
Lemma 9.30. Given $k$, for sufficiently large $|\lambda|$ for all terms of the expansion (37) either the coefficient $b_{m_1,\ldots,m_k}$ is divisible by $(k)_q^i$ or there exists $m_i \geq k$ for some $1 \leq i \leq N$.

Proof. We follow the same logic as in Lemma 9.29. For $|\lambda| > 2k(N+1)$ every term (37) is divisible by $f_{2k}(q^ix_i)$ for some $s$ and $i$. By Lemma 10.1 this can be further decomposed into terms which are divisible by $(j)_q^i f_{2k-j}(x_i)$, and either $j$ or $2k-j$ is greater than or equal to $k$. Overall, we presented

$$F_\lambda(x_1, \ldots, x_N) = A(k)_q^N + \sum B_i f_k(x_i)$$

for some polynomials $A$ and $B_i$. It remains to notice that the polynomial $B_i f_k(x_i)$ can be presented as the sum of $f_{m_1}(x_1) \cdots f_{m_N}(x_N)$ where $m_i \geq k$. $\square$

10. Stability of interpolation and the case $N = 2$

In this section study the dependence of the interpolation polynomials on $N$.

As above, if partition $\lambda$ has less than $N$ parts we can complete it with zeroes. We denote by $F_{\lambda,N}(x_1, \ldots, x_N)$ the corresponding polynomial in $N$ variables.

Lemma 10.1. Let $\lambda$ be a partition with at most $N$ parts. Then

$$F_{\lambda,N}(x_1, \ldots, x_{N-1}, 1) = \begin{cases} (-1)^{N-1}q^{(N-2)}F_{\lambda,N-1}(q x_1, \ldots, q x_{N-1}) & \text{if } \lambda_N = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let $\mu$ be a partition with at most $N-1$ parts. Then by Proposition 9.13

$$F_{\lambda,N}(q^{-\mu_1-N+1}, \ldots, q^{-\mu_{N-1}-1}, 1) = 0$$

unless $\mu$ contains $\lambda$. If $\lambda_N > 0$ then this never happens and $F_{\lambda,N}(x_1, \ldots, x_{N-1}, 1) = 0$. If $\lambda_N = 0$ we write $L(x_1, \ldots, x_{N-1}) = F_{\lambda,N-1}(q x_1, \ldots, q x_{N-1})$. We have

$$L(q^{-\mu_1-N+1}, \ldots, q^{-\mu_{N-1}-1}) = F_{\lambda,N-1}(q^{-\mu_1-(N-1)+1}, \ldots, q^{-\mu_{N-1}})$$

which vanishes unless $\mu$ contains $\lambda$, so by Proposition 9.13 $F_{\lambda,N}(x_1, \ldots, x_{N-1}, 1)$ is proportional to $L(x_1, \ldots, x_{N-1})$. Finally, at $\mu = \lambda$ we can use Lemma 9.11 to determine the coefficient. $\square$

Remark 10.2. We can also prove the lemma using the explicit determinantal formula. Indeed, $f_{\lambda+N-i}(1) = 0$ unless $f_{\lambda+N-i} = 0$ which is equivalent to $i = N$ and $\lambda_N = 0$. Therefore for $\lambda_N \neq 0$ the last row in the matrix $f_{\lambda+N-i}(x_j)$ vanishes (where $x_N = 1$), and $F_{\lambda,N}(x_1, \ldots, x_{N-1}, 1) = 0$. For $\lambda_N = 0$ we have

$$F_{\lambda,N}(x_1, \ldots, x_{N-1}, 1) = \frac{\det [f_{\lambda+N-i}(x_j)]_{i,j=1}^{N-1}}{\prod_{i<j \leq N-1}(x_i - x_j) \prod_{i \leq N-1}(x_i - 1)}.$$

Note that $f_{k+1}(x) = (1-x)f_k(qx)$, so

$$f_{\lambda+N-i}(x_j) = (1-x_j)f_{\lambda+(N-1)-i}(qx_j)$$

Therefore

$$F_{\lambda,N}(x_1, \ldots, x_{N-1}, 1) = \frac{\prod_{i=1}^n(1-x_i) \det [f_{\lambda+(N-1)-i}(qx_j)]_{i,j=1}^{N-1}}{\prod_{i<j \leq N-1}(x_i - x_j) \prod_{i \leq N-1}(x_i - 1)} = (-1)^{N-1}q^{(N-2)}F_{\lambda,N-1}(qx_1, \ldots, qx_{N-1})$$.
Corollary 10.3. Let \( c_{\lambda, \mu}^{(N)} \) be the coefficient defined in previous section for symmetric functions in \( N \) variables. Then the expressions

\[
(-1)^{\lambda_2} q^{-\lambda_2} c_{\lambda, \mu}^{(N)}, (-1)^{\lambda_2} q^{\lambda_2} d_{\lambda, \mu}^{(N)}
\]

are independent of \( N \) (provided that \( \lambda \) and \( \mu \) have at most \( N \) parts).

Example 10.4. For one-row partitions \( \lambda = (b) \) and \( \mu = (a) \) the interpolation coefficients are given by the formulas in Example 10.20 up to a monomial factor.

10.1. Adding a column. It is well known that in symmetric functions in \( N \) variables one has the identity

\[
s_{\lambda + 1} = x_1 \cdots x_N \cdot s_{\lambda}.
\]

Here \( \lambda + 1^N = (\lambda_1 + 1, \ldots, \lambda_N + 1) \) and the corresponding Young diagram is obtained from the Young diagram for \( \lambda \) by adding a vertical column.

For interpolation polynomials we have two different generalizations of this identity: the first relates \( F_{\lambda+1} \) to \( F_{\lambda} \) and the second describe the action of the multiplication by \( x_1 \cdots x_N \).

Proposition 10.5. We have \( F_{\lambda+1}(x_1, \ldots, x_N) = q^{\lambda_2} \prod_{i=1}^{N} (1 - x_i) F_{\lambda}(qx_1, \ldots, qx_N) \). More generally,

\[
F_{\lambda+k}(x_1, \ldots, x_N) = q^{\lambda_2} \prod_{i=1}^{N} f_k(x_i) F_{\lambda}(q^k x_1, \ldots, q^k x_N).
\]

Proof. We have \( f_{m+1}(x) = (1 - x) f_m(x) \), therefore

\[
\det [f_{\lambda+i}(x_j)] = \det [(1 - x_j) f_{\lambda+i}(qx_j)] = \prod_{j=1}^{N} (1 - x_j) \det [f_{\lambda+i}(x_j)].
\]

Since each factor \((x_i - x_j)\) in the denominator gets multiplied by \( q \) after changing \( x_i \rightarrow qx_i \), this implies the first equation. Now (38) can be obtained by applying it \( k \) times.

Let \( e_i \) denote the \( i \)-th basic vector in \( \mathbb{Z}^N \) with 1 at \( i \)-th position and 0 at other positions. Given \( I \subset \{1, \ldots, n\} \), we define \( e_I = \sum_{i \in I} e_i \).

Proposition 10.6. We have

\[
x_1 \cdots x_N F_{\lambda}(x_1, \ldots, x_N) = q^{-|\lambda| - \binom{|I|}{2}} \sum_{I \subset \{1, \ldots, n\}} (-1)^{|I|} F_{\lambda+e_I}(x_1, \ldots, x_N).
\]

Here we use the convention that \( F_{\lambda+e_I} = 0 \) unless the entries of \( \lambda + e_I \) are non-increasing (that is, \( \lambda + e_I \) is a partition).

Proof. We have \( f_{m+1}(x) = f_m(x)(1 - q^m x) \), so

\[
x f_m(x) = q^{-m} (f_m(x) - f_{m+1}(x)).
\]

Therefore

\[
x_1 \cdots x_N \det [f_{\lambda+i}(x_j)] = \det [x_j f_{\lambda+i}(x_j)] = \det [q^{-\lambda_i-N+i}(f_{\lambda+i}(x_j) - f_{\lambda+i+1}(x_j))].
\]
Corollary 10.7. Consider the completion of the space of symmetric functions with coefficients in \( \mathbb{Z}[q, q^{-1}] \) with respect to the basis \( F_{\lambda} \). Then the operator of multiplication by \( x_1 \cdots x_N \) is invertible in this completion and its inverse is given by the equation

\[
(x_1 \cdots x_N)^{-1} F_{\lambda}(x_1, \ldots, x_N) = q^{|\lambda|+v} F_{\lambda+e}(x_1, \ldots, x_N).
\]

Proof. Define the operators \( A_i \) by \( A_i(F_{\lambda}) = F_{\lambda+e_i} \), and \( p_i(F_{\lambda}) = q^{\lambda_i} F_{\lambda} \) for \( i = 1, \ldots, N \). Clearly, \([A_i, A_j] = [p_i, p_j] = [A_i, p_j] \) for \( i \neq j \) and by Proposition [10.6] we have

\[
x_1 \cdots x_N = q^{|\lambda|} \prod_i (1 - A_i)p_i^{-1},
\]

hence

\[
(x_1 \cdots x_N)^{-1} = q^{|\lambda|} \prod_i p_i(1 + A_i + A_i^2 + \ldots).
\]

\( \square \)

Example 10.8. For \( N = 1 \) and \( \lambda = (0) \) we get a curious identity

\[
x^{-1} = \sum_{m=0}^{\infty} f_m(x)q^m
\]

We can check this identity directly, by computing the values of both sides at \( q^{-j} \) for all \( j \). Denote

\[
u_j = \sum_{m=0}^{\infty} f_m(q^{-j})q^m = \sum_{m=0}^{j} f_m(q^{-j})q^m.
\]

Then \( u_{j+1} = 1 + q(1 - q^{-j-1})u_j \) and \( u_0 = 1 \), so it is easy to see that \( u_j = q^j \).

10.2. Interpolation polynomials for \( \mathfrak{g}l_2 \). In this subsection we describe the interpolation polynomials for \( \mathfrak{g}l_2 \) explicitly. By definition, we have polynomials \( F_{\lambda}(x_1, x_2) \) where \( \lambda_1 \geq \lambda_2 \):

\[
F_{\lambda_1, \lambda_2}(x_1, x_2) = \frac{1}{x_1 - x_2} \begin{vmatrix} f_{\lambda_1+1}(x_1) & f_{\lambda_1+1}(x_2) \\ f_{\lambda_2}(x_1) & f_{\lambda_2}(x_2) \end{vmatrix}
\]

Let us consider the case \( \lambda_2 = 0 \) first, and write \( \lambda_1 = k \). Then

\[
F_{k,0}(x_1, x_2) = \frac{1}{x_1 - x_2} \det \begin{vmatrix} f_{k+1}(x_1) & f_{k+1}(x_2) \\ 1 & 1 \end{vmatrix} = \frac{f_{k+1}(x_1) - f_{k+1}(x_2)}{x_1 - x_2}.
\]

Let

\[
h_k(x_1, x_2) = \frac{x_1^{k+1} - x_2^{k+1}}{x_1 - x_2}.
\]

Recall that \( f_{k+1}(x) = \sum_{j=0}^{k+1} (-1)^j q^{\frac{j(j+1)}{2}} \binom{k+1}{j} x^j \), so

\[
F_{k,0}(x_1, x_2) = \sum_{j=0}^{k+1} (-1)^j q^{\frac{j(j-1)}{2}} \binom{k+1}{j} h_{j-1}(x_1, x_2).
\]

We just replace each \( x^j \) in the expression for \( f_{k+1}(x) \) by \( h_{j-1}(x_1, x_2) \).

Example 10.9. We have

\[
f_1(x) = 1 - x, \quad f_2(x) = (1 - x)(1 - qx) = 1 - (1 + q)x + qx^2,
\]

\[
f_3(x) = (1 - x)(1 - qx)(1 - q^2x) = 1 - (1 + q + q^2)x + (q + q^2 + q^3)x^2 - q^3x^3
\]

so

\[
F_{0,0}(x_1, x_2) = -1, \quad F_{1,0}(x_1, x_2) = q(x_1 + x_2) - (1 + q),
\]
\[ F_{2,0} = -q^2(x_1^2 + x_1x_2 + x_2^2) + (q + q^2 + q^3)(x_1 + x_2) - (1 + q + q^2). \]

By Proposition 10.5 we have
\[ F_{\lambda_1,\lambda_2}(x_1, x_2) = q^{\lambda_2} f_{\lambda_2}(x_1) f_{\lambda_2}(x_2) F_{\lambda_1 - \lambda_2,0}(q^{\lambda_2} x_1, q^{\lambda_2} x_2). \]

In particular, for \((\lambda_1, \lambda_2) = (k, k)\) we have
\[ F_{k,k}(x_1, x_2) = q^k f_k(x_1) f_k(x_2). \]

Also, by Lemma 10.1 we get
\[ F_{\lambda_1,\lambda_2}(x_1, 1) = \begin{cases} -f_{\lambda_1}(q x_1) & \text{if } \lambda_2 = 0 \\ 0 & \text{otherwise.} \end{cases} \]

10.3. **Interpolation tables for \( \mathfrak{gl}_2 \).** For the reader’s convenience, we have computed the polynomials \( F_\lambda(x_1, x_2) \) and the corresponding interpolation matrices using **Sage** [34]. First, we present \( F_\lambda \) in Schur basis:

\[
\begin{align*}
F_0 &= -1, & F_1 &= q s_1 - (q + 1), & F_2 &= -q^3 s_2 + (q^3 + q^2 + q) s_1 - (q^2 + q + 1), \\
F_{1,1} &= -q s_{1,1} + q s_1 - q = -q(1 - x_1)(1 - x_2) \\
F_3 &= q^6 s_3 - (q^6 + q^5 + q^4 + q^3) s_2 + (q^5 + q^4 + 2q^3 + q^2 + q) s_1 - (q^3 + q^2 + q + 1), \\
F_{2,1} &= -q^3 s_{2,1} - q^3 s_2 - (q^3 + q^2 + q) s_{1,1} + (q^3 + q^2 + q) s_1 - (q^2 + q) \\
F_{3,1} &= -q^6 s_{3,1} + q^6 s_1 + (q^6 + q^5 + q^4 + q^3) s_{2,1} - (q^6 + q^5 + q^4 + q^3) s_2 - \\
& \quad (q^5 + q^4 + 2q^3 + q^2 + q) s_{1,1} + (q^5 + q^4 + 2q^3 + q^2 + q) s_1 - (q^3 + q^2 + q) \\
F_{2,2} &= -q^4 s_{2,2} + (q^4 + q^3) s_{2,1} - q^3 s_2 - (q^4 + q^3 + q^2) s_{1,1} + (q^3 + q^2) s_1 - q^2. \\
F_{3,3} &= q^7 s_{3,2} - (q^7 + q^6) s_{3,1} - (q^6 + q^5 + q^4) s_{2,2} + (q^6 + q^5 + q^4 + q^3) s_{2,1} + \\
& \quad (q^6 + q^5 + q^4 + q^3) s_2 - (q^6 + 2q^5 + 2q^4 + 2q^3 + q^2) s_{1,1} + (q^5 + q^4 + 2q^3 + q^2) s_1 - (q^3 + q^2) \\
& \quad (q^6 + q^5 + q^4 + q^3) s_2 - (q^6 + 2q^5 + 2q^4 + 2q^3 + q^2) s_{1,1} + (q^5 + q^4 + 2q^3 + q^2) s_1 - (q^3 + q^2). \\
F_{3,3} &= -q^9 s_{3,3} + (q^9 + q^8 + q^7) s_{3,2} - (q^8 + q^7 + q^6) s_{3,1} - (q^9 + q^8 + 2q^7 + q^6 + q^5) s_{2,2} + (q^9 + q^8 + 2q^7 + q^6 + q^5 + q^4 + q^3) s_{1,1} + (q^9 + q^8 + 2q^7 + q^6 + q^5 + q^4 + q^3) s_1 - q^3. 
\end{align*}
\]

Next, we list the values of the evaluations \( c_{\lambda,\mu} = F_\lambda(q^{-\mu-1}, q^{-\mu_2}) \) for various \( \lambda \) and \( \mu \) in Tables [123] below. The resulting matrix \( C = (c_{\lambda,\mu}) \) is upper-triangular, with diagonal entries prescribed by Lemma 9.11. Zero entries correspond to pairs \((\lambda, \mu)\) where \( \mu \) does not contain \( \lambda \). The entry corresponding to \((\lambda, \mu) = ((1), (3, 2))\) is marked in bold, it is divisible by \( 1 - q \) but does not factor any further.

Using either Theorem 9.11 or equation (34), one can easily reconstruct the inverse matrix \( D = C^{-1} \), and we list part of it in Table [4] (see Examples [9.21] and [9.23] for more computations).

Note that by Corollary 10.3 this determines the coefficients \( c_{\lambda,\mu} \) and \( d_{\lambda,\mu} \) for \( \lambda \subset \mu \subset (3, 3) \) and arbitrary \( N \).

10.4. **Link invariants for \( \mathfrak{gl}_2 \).** We can use the interpolation tables to expand the invariants of simple knots in the basis \( F_\lambda \). Indeed, the colored \( \mathfrak{gl}_2 \) invariants are determined by the colored \( \mathfrak{gl}_2 \) invariants (that is, colored Jones polynomial) by the formula
\[ J_K(V(\lambda_1, \lambda_2), q) = J_K(V_{\lambda_1 - \lambda_2}, q). \]

The coefficients \( a_\lambda(K) \) are then determined by Theorem 10.3
\[ a_\lambda(K) = \sum_{\mu \subset \lambda} d_{\lambda,\mu} (q^{-1}) J_K(V(\mu), q). \]
For example, for the figure eight knot we have the following values of the colored Jones polynomial:

\[ J_K(V_0, q) = 1 = J_K(V(1, 1), q), \quad J_K(V_1, q) = J_K(V(2, 1), q) = 1 + q^2 + q^{-2} - q - q^{-1}, \]
\[ J_K(V_2, q) = 1 + q^3 + q^{-3} - q - q^{-1} + (q^3 + q^{-3} - q - q^{-1})(q^3 + q^{-3} - q^{-2}). \]

Using the values of \( d_{\lambda, \mu} \) from Table 3 (and changing \( q \) to \( q^{-1} \)) we obtain

\[ a_0(K) = -J_K(V_0, q) = -1, \quad a_1(K) = -\frac{q^{-1}}{1-q^{-1}}J_K(V_0, q) + \frac{q^{-1}}{1-q^{-1}}J_K(V_1, q) = q^{-2}(q^3 - 1), \]
\[ a_2(K) = -\frac{q^{-2}}{(1-q^{-1})(1-q^{-2})}J_K(V_0, q) + \frac{q^{-2}}{(1-q^{-1})^2}J_K(V_1, q) - \frac{q^{-3}}{(1-q^{-1})(1-q^{-2})}J_K(V_2, q) = q^{-6}(-q^9 + q^5 + q^4 - q^3 - 1), \]
\[ a_{1,1}(K) = -\frac{q^{-3}}{(1-q^{-1})(1-q^{-2})}J_K(V_0, q) + \frac{q^{-2}}{(1-q^{-1})^2}J_K(V_1, q) - \frac{q^{-2}}{(1-q^{-1})(1-q^{-2})}J_K(V(1, 1), q) = q^{-2}(q^2 + q + 1), \]
\[ a_{2,1}(K) = -\frac{q^{-4}}{(1-q^{-1})^2(1-q^{-3})}J_K(V_0, q) + \frac{q^{-3}}{(1-q^{-1})^3}J_K(V_1, q) - \frac{q^{-4}}{(1-q^{-1})^2(1-q^{-2})}J_K(V_2, q) - \frac{q^{-3}}{(1-q^{-1})^2(1-q^{-2})}J_K(V(1, 1), q) + \frac{q^{-4}}{(1-q^{-1})^2(1-q^{-3})}J_K(V(2, 1), q) = q^{-6}(-q^8 - q^7 - q^6 - q^5 + q^4 + 2q^3 + q^2 + q + 1). \]

Using Tables 1, 2, 3 one can similarly compute the values of \( a_\lambda(K) \) for all \( \lambda \subset (3, 3) \) and verify that these are indeed Laurent polynomials in \( q \).

| \( \lambda \backslash \mu \) | (0) | (1) | (2) | (3) |
|----------------|-----|-----|-----|-----|
| (0)  | -1  | -1  | -1  | -1  |
| (1)  | 0   | \( q^{-1}(1-q) \) | \( q^{-2}(1-q^2) \) | \( q^{-3}(1-q^3) \) |
| (2)  | 0   | 0   | \( -q^{-3}(1-q)(1-q^2) \) | 0   |
| (3)  | 0   | 0   | 0   | 0   |
| (1,1) | 0   | 0   | \( -q^{-2}(1-q)(1-q^2) \) | 0   |
| (2,1) | 0   | 0   | 0   | 0   |
| (3,1) | 0   | 0   | 0   | 0   |
| (2,2) | 0   | 0   | 0   | 0   |
| (3,2) | 0   | 0   | 0   | 0   |
| (3,3) | 0   | 0   | 0   | 0   |

**Table 1. Evaluations of interpolation polynomials: matrix \( C = (c_{\lambda, \mu}) \)**

11. **Appendix**

Here we collect some useful definitions and facts about Habiro’s ring and interpolation Macdonald polynomials.
The Habiro ring \([17]\) is defined as

\[
\hat{\mathbb{Z}}_q := \lim_{n \to \infty} \frac{\mathbb{Z}[q]}{((q; q)_n)}
\]

Any element of \(\hat{\mathbb{Z}}_q\) can be presented (not uniquely) as infinite series

\[
f(q) = \sum_{n=0}^{\infty} f_n (q; q)_n, \quad f_n \in \mathbb{Z}[q].
\]

Evaluations of such \(f(q)\) at all roots of unity are well defined, since if \(q^r = 1\) one has \(f(q) = \sum_{n=0}^{r-1} f_n(q)_n\). It is easy to expand every \(f(q) \in \hat{\mathbb{Z}}_q\) into formal power series in \((q - 1)\), denoted by \(T(f)\) and called the Taylor series of \(f(q)\) at \(q = 1\). One important property of the Habiro ring is that any \(f \in \hat{\mathbb{Z}}_q\) is uniquely determined by its Taylor series. In other words, the map \(T : \hat{\mathbb{Z}}_q \to \mathbb{Z}[[q - 1]]\) is injective \([17]\) Thm 5.4.
particular, \(\widehat{\mathbb{Z}[q]}\) is an integral domain. Moreover, every \(f \in \widehat{\mathbb{Z}[q]}\) is determined by the values of \(f\) at any infinite set of roots of unity of prime power order. Because of these properties, Habiro ring is also known as a ring of analytic functions at roots of unity.

Since \(\cap_{n \geq 0} I_n = 0\) with \(I_n = (q; q)_n \mathbb{Z}[q]\), the natural map \(\mathbb{Z}[q] \to \widehat{\mathbb{Z}[q]}\) is injective. The image of \(q\) under this map is invertible, and the inverse is given by

\[
q^{-1} = \sum_{n=1}^{\infty} q^n (q; q)_n,
\]

compare with Example 10.8. This implies that there is an injective map \(\mathbb{Z}[q, q^{-1}] \to \widehat{\mathbb{Z}[q]}\). The following result is proved in \[17\] Proposition 7.5, but we give a slightly different proof here for the reader’s convenience. We will denote by \(\Phi_n(q)\) the \(n\)th cyclotomic polynomial \(\Phi_n(q) = \prod_{(a, n) = 1} (q - \zeta_n^a)\) where \(\zeta_n\) is any primitive \(n\)th root of unity.

**Proposition 11.1.** Suppose that \(f(q) \in \widehat{\mathbb{Z}[q]}\) and \(f(q)h(q) \in \mathbb{Z}[q, q^{-1}]\) for some product of cyclotomic polynomials \(h(q) = \Phi_{n_1}(q) \cdots \Phi_{n_r}(q)\). Then \(f(q) \in \mathbb{Z}[q, q^{-1}]\).

**Proof.** Let us denote \(g(q) = f(q)h(q) \in \mathbb{Z}[q, q^{-1}]\), we prove the statement by induction in \(r\). For \(r = 1\) we get \(h(q) = \Phi_n(q)\) and \(g(q) = f(q)\Phi_n(q)\), so for any primitive \(n\)th root of unity \(\zeta_n\) we have \(g(\zeta_n) = f(\zeta_n)\Phi_n(\zeta_n) = 0\), so \(g(q) = \alpha(q)\Phi_n(q)\) for some \(\alpha \in \mathbb{Z}[q, q^{-1}]\). This implies \((f(q) - \alpha(q))\Phi_n(q) = 0\), and since \(\mathbb{Z}[q]\) is an integral domain we get \(f(q) = \alpha(q)\).

For \(r > 1\) we get

\[
f(q)\Phi_{n_1}(q) \cdots \Phi_{n_r}(q) \in \mathbb{Z}[q, q^{-1}],
\]

so by the above

\[
f(q)\Phi_{n_1}(q) \cdots \Phi_{n_{r-1}}(q) \in \mathbb{Z}[q, q^{-1}],
\]

and by the assumption of induction \(f(q) \in \mathbb{Z}[q, q^{-1}]\). \(\square\)

11.2. **Interpolation Macdonald polynomials.** We consider partitions with at most \(N\) parts.

**Theorem 11.2.** \([22, 23, 28, 29, 30, 31, 35]\) There exists unique up to scalar factors family of symmetric polynomials \(I_\lambda(x_1, \ldots, x_N; q, t)\) with the following properties:

(a) \(I_\lambda(q^{-\mu}t^{N-1}) = 0\) unless \(\mu\) contains \(\lambda\)

(b) \(I_\lambda(q^{-\lambda}t^{N-1}) \neq 0\)

(c) \(I_\lambda\) is a nonhomogeneous polynomial of degree \(|\lambda|\), and its degree \(|\lambda|\) part is proportional to the Macdonald polynomial \(F_\lambda(x_1, \ldots, x_N; q, t)\).

The polynomials \(I_\lambda\) are called interpolation Macdonald polynomials. In fact, the properties (a) and (b) already uniquely determine \(I_\lambda\) (up to a scalar), and their existence follows from the fact that \(q^{-\lambda}t^{N-1}\) for a nondegenerate grid in the sense of \[30\]. Part (c) is then a deep property of these polynomials.

It is easy to see that at \(q = t\) interpolation Macdonald polynomials \(I_\lambda\) specialize to \(F_\lambda\). Unlike \(F_\lambda\), there is no determinant formula for \(I_\lambda\) but there is a different combinatorial formula \[29\].

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