On a paucity result in Incidence Geometry *

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Annotation.

We obtain some asymptotic formulae (with power savings in their error terms) for the number of quadruples in the Cartesian product of an arbitrary set $A \subset \mathbb{R}$ and for the number of quintuplets in $A \times A$ for any subset $A$ of the prime field $\mathbb{F}_p$. Also, we obtain some applications of our results to incidence problems in $\mathbb{F}_p$.

1 Introduction

Incidence Geometry (see, e.g., [13, section 8]) deals with the incidences among different geometrical objects such as points, lines, curves, surfaces etc. A typical problem of this area is to estimate the quantity

$$I(\mathcal{P}, \mathcal{L}) := |\{(p, l) \in \mathcal{P} \times \mathcal{L} : p \in l\}|,$$

where the set of points $\mathcal{P}$ and the set of lines $\mathcal{L}$ belong to $\mathbb{F} \times \mathbb{F}$ with $\mathbb{F}$ be a field, say. In our paper the set of points $\mathcal{P}$ will be the Cartesian product $A \times B$ for some sets $A, B \subseteq \mathbb{F}$. This particular choice of $\mathcal{P}$ is very important for the applications see, e.g., [2], [5], [6], [9], [10], [13] because, basically, Cartesian products are naturally connected with arithmetic. In this note we study collinear tuples in $A \times B$. Namely, for any $k \geq 3$ we define $C_k(A, B)$ to be the number of collinear $k$–tuples in $A \times B$. Let $C_k(A) = C_k(A, A)$. A consequence of the famous Szemerédi–Trotter Theorem [12] gives us that for any $A \subset \mathbb{R}$ one has

$$C_3(A) \ll |A|^4 \log |A|$$

(a short proof of this result contains in [10]). Actually, for $\mathcal{P} = A \times A$ it is easy to see that bound (2) is equivalent to the Szemerédi–Trotter Theorem (up to logarithms). In our paper [2] we have obtained an asymptotic formula for the number of collinear quadruples in the case of the prime field $\mathbb{F} = \mathbb{F}_p$

$$C_4(A) = \frac{|A|^8}{p^2} + O(|A|^5 \log |A|).$$

The proof is based on Stevens–de Zeeuw’s Theorem [11]. Both bounds (2), (3) are known to be tight up to some powers of logarithms as the case $A = \{1, \ldots, n\}$ shows, namely, $C_3(A) \gg$

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\[ |A|^4 \log^c |A|, \ c > 0 \text{ for such } A \text{ (see discussion in \cite[Pages 603, 633]{2}). The problem of finding}\]

sharp asymptotic formula for the number of collinear triples in \( A \times A, A \subseteq \mathbb{F}_p \) is a well-known important open question and at the moment the best result is (see \cite{2}, \cite{6})

\[
C_3(A) = \frac{|A|^6}{p} + O(\min\{|A|^{9/2}, p^{1/2}|A|^{7/2}\}). \tag{4}
\]

As for \( C_k(A) \) with large \( k \), then one can easily obtain an analogues of formulae \( \cite{2}, \cite{3} \) but, generally speaking, the error term must be at least \( 2|A|^{k+1} \) (take horizontal/vertical lines), and hence the consideration of large \( k \) does not give anything new.

In this paper we make a further step and obtain a paucity result (see Theorems \cite{11}, \cite{12} below) for higher \( C_k(A) \). As we said before, in general, one cannot obtain non-trivial bounds for such quantities and thus we need some restrictions, which we formulate in terms of energies. For any two sets \( A, B \subseteq \mathbb{F} \) the additive energy of \( A \) and \( B \) is defined by

\[
E^+(A, B) = |\{(a_1, a_2, b_1, b_2) \in A \times A \times B \times B : a_1 - b_1 = a_2 - b_2\}|.
\]

If \( A = B \), then we simply write \( E^+(A) \) for \( E^+(A, A) \). Similarly, one can define the multiplicative energy of \( A \) and \( B \). Finally, put \( E^\times(A) = \max_{s \in \mathbb{F}} E^\times(A - s) \leq |A|^3 \).

**Theorem 1** Let \( A, B \subseteq \mathbb{R} \) be sets. Then

\[
C_4(A, B) - |B||A|^4 - |A||B|^4 \ll |B||A|^3 + |A||B|^3 +
\]

\[
(E^+(A)E^+(B))^{1/10}C_3^{2/5}(A, B)|A|^{8/5}|B|^{6/5} \log^{2/5} |A| + (E^\times(A)E^\times(B))^{1/4}|A|^2|B|^{3/2} \log^{3/2} |A|. \tag{5}
\]

Further let \( p \) be a prime number, \( A \subseteq \mathbb{F}_p \). Then

\[
C_5(A) - \frac{|A|^{10}}{p^3} - 2|A|^6 \leq \frac{|A|^7}{p} + \frac{|A|^4}{p^2}C_3(A) + (E^+(A))^{1/6}|A|^{11/2} + (E^\times(A))^{1/2}|A|^{9/2}. \tag{6}
\]

**Corollary 2** Let \( A, B \subseteq \mathbb{R}, \ A = B \) and \( E^+(A), E^\times(A) \leq |A|^{3-c} \), where \( c > 0 \). Substituting estimate \( \cite{2} \) to Theorem \( \cite{7} \) we get

\[
C_4(A) - 2|A|^5 \ll |A|^{5-c/5} \log^{4/5} |A|, \tag{7}
\]

and in the case of the prime field and, say, \( |A| \leq p^{2/3} \), we have

\[
C_5(A) - 2|A|^6 \leq |A|^{6-c/6} \log^{5/6} |A|. \tag{8}
\]

Once again the condition \( E^+(A) \leq |A|^{3-c} \) is required for \( \cite{7}, \cite{8} \). One can easily see that a random set \( A \) satisfies both assumptions \( E^+(A), E^\times(A) \leq |A|^{3-c} \), where \( c > 0 \). Also, there are some concrete constructions of such sets, see section \( \cite{9} \) namely, one can take \( A = B^{-1} + s \) for any \( s \in \mathbb{F} \) and \( B \subseteq \mathbb{F} \) such that \( |B + B| \ll |B| \) and then \( E^+(A), E^\times(A) \leq |A|^{3-c} \). It seems like that our Theorem \( \cite{11} \) is the first paucity result in the area. As for lower bounds on \( C_k(A) \) we
trivially have $\Omega(|A|^k)$ in the error term, counting $k$-tuples, which belong to the diagonal $(x,x)$, where $x$ runs over $A$.

We obtain some applications of our paucity theorems in section 4. For example, let us formulate a consequence of our new incidence result in $\mathbb{F}_p$ concerning points/lines incidences. The basic regime here is the following: the sets $A$, $B$ are large and the sets $X, Y$ are small comparable to $p$.

**Theorem 3** Let $k \geq 2$ be an integer, $M \geq 1$ be a real number, $A, B, C, X, Y \subseteq \mathbb{F}_p$, $0 \notin X$ be sets, $|YC| \leq M|Y|$, and $|C| \geq k M^{2k+1}$, $|C|^{-1} \geq (p/|Y|)^2$. Suppose that

$$\max\{E_+^+(B), E_-^-(B)\} \leq |B|^{3-\delta},$$

where $\delta > 0$ is a constant. Then

$$|\{(a, b, x, y) \in A \times B \times X \times Y : y = bx + a\}| - |\frac{|A||B||X||Y|}{p}| \ll \sqrt{|A||B||X||Y|} \cdot |B|^{-\frac{\delta}{35.2^k}}. \quad (9)$$

There are two main advantages of Theorem 3. First of all, it is an asymptotic formula (and, again, the set $X \times Y$, which corresponds to the lines $L$ can be rather small, even of size $p^\varepsilon$) but not just upper bounds for $I^+(P, L)$ as in [11], say. Some asymptotic formulae for the quantity $I(P, L)$ were known before in the specific case of large sets (see [14]) and in the case of Cartesian products but with large sets of lines, see [6] and [11]. Such asymptotic formulae are important in the problem of estimating exponential sums (see, e.g., [6]) and in the questions on mixing times of some Markov chains [9], where one requires to have the set of points $A \times B$ to be really large.

The second advantage of Theorem 3 is that it works even for $|A|, |B| \gg p$ (for size of the set $B$ see the exact formulation of Theorem 13 below).

## 2 Definitions and preliminaries

By $G$ we denote an abelian group. Sometimes we underline the group operation writing $+$ or $\times$ in the considered quantities (as the energy, the representation function and so on, see below). Let $\mathbb{F}$ be the field $\mathbb{R}$ or $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ for a prime $p$. Let $\mathbb{F}^* = \mathbb{F} \setminus \{0\}$.

We use the same capital letter to denote set $A \subseteq \mathbb{F}$ and its characteristic function $A : \mathbb{F} \to \{0, 1\}$ and in the case of finite $\mathbb{F}$ we write $f_A(x) := A(x) - |A|/|\mathbb{F}|$ for the balanced function of $A$. Given two sets $A, B \subseteq G$, define the **sumset** of $A$ and $B$ as

$$A + B := \{a + b : a \in A, b \in B\}.$$  

In a similar way we define the **difference sets** and **higher sumsets**, e.g., $2A - A$ is $A + A - A$. We write $\perp$ for a direct sum, i.e., $|A \perp B| = |A||B|$. For an abelian group $G$ the Plünnecke–Ruzsa inequality (see, e.g., [13]) holds stating

$$|nA - mA| \leq \left(\frac{|A + A|}{|A|}\right)^{n+m} \cdot |A|, \quad (10)$$
where \( n, m \) are any positive integers. We use representation function notations like \( r_{A+B}(x) \) or \( r_{A-B}(x) \) and so on, which counts the number of ways \( x \in G \) can be expressed as a sum \( a+b \) or \( a-b \) with \( a \in A, b \in B \), respectively. For example, \( |A| = r_{A-A}(0) \).

For any two sets \( A, B \subseteq G \) the additive energy of \( A \) and \( B \) is defined by

\[
E(A, B) = E^+(A, B) = \{ (a_1, a_2, b_1, b_2) \in A \times A \times B \times B : a_1 - b_1 = a_2 - b_2 \}. 
\]

If \( A = B \), then we simply write \( E(A) \) for \( E(A, A) \). More generally, for sets (real functions) \( A_1, \ldots, A_{2k} (f_1, \ldots, f_{2k}) \) belonging to an arbitrary (noncommutative) group \( G \) and \( k \geq 2 \) define the energy \( T_k(A_1, \ldots, A_{2k}) \) as

\[
T_k(A_1, \ldots, A_{2k}) = |\{ (a_1, \ldots, a_{2k}) \in A_1 \times \cdots \times A_{2k} : a_1 a_2^{-1} \cdots a_{k-1} a_k^{-1} = a_{k+1} a_{k+2}^{-1} \cdots a_{2k-1} a_{2k}^{-1} \}|, 
\]

and

\[
T_k(f_1, \ldots, f_{2k}) = \sum_{a_1 a_2^{-1} \cdots a_{k-1} a_k^{-1} = a_{k+1} a_{k+2}^{-1} \cdots a_{2k-1} a_{2k}^{-1}} f_1(a_1) \cdots f_{2k}(a_{2k}). 
\]

In the abelian case put for \( k \geq 2 \)

\[
E^+_k(A) = \sum_x r^k_{A-A}(x) = \sum_{\alpha_1, \ldots, \alpha_{k-1}} |A \cap (A + \alpha_1) \cap \cdots \cap (A + \alpha_{k-1})|^2. 
\]

Clearly, \( |A|^k \leq E^+_k(A) \leq |A|^{k+1} \). Having a function \( f : F \to C \), we put \( E^x(f) = \max_{s \in F} E^x(f - s) \). In particular, \( E^x(A) \leq |A|^3 \) for any finite set \( A \subseteq F \).

The signs \( \ll \) and \( \gg \) are the usual Vinogradov symbols. When the constants in the signs depend on a parameter \( M \), we write \( \ll_M \) and \( \gg_M \). All logarithms are to base 2. If we have a set \( A \), then we will write \( a \lesssim b \) or \( b \gtrsim a \) if \( a = O(b \cdot \log^c |A|) \), \( c > 0 \). Let us denote by \([n]\) the set \{1, 2, \ldots, n\}.

Given an arbitrary line \( l \) denote by \( i_{A,B}(l) = |l \cap (A \times B)| \) hence in particular, \( I(A \times B, L) = \sum_{l \in L} i_{A,B}(l) \). Clearly, for \( k \geq 3 \) one has

\[
C_k(A, B) = \sum_{l : i_{A,B}(l) > 1} i^k_{A,B}(l) + O_k(|A|^2|B|^2 \cdot (\min\{|A|, |B|\})^{k-3}. 
\]

One can think about the first term in \( \textbf{[13]} \) as another definition of the quantity \( C_k(A, B) \).

We need a result on the energy of an arbitrary set of affine transformations, see [3, Theorem 2.1]. We consider any set of affine transformations \( L \subseteq \text{Aff}(F) \) as a subset of \( F^* \times F \), i.e., we associate a map \( x \to ax + b \) with a point \((a, b) \in F^* \times F\).

\[\textbf{Theorem 4} \ Let \( L \subseteq \text{Aff}(F) \) be a set of lines such that no line in \( F^* \times F \) contains more than \( M(L) \) points of \( L \), and no vertical line in \( F^* \times F \) contains more than \( m(L) \) points of \( L \). Suppose that \( m(L)|L| \leq p^2 \). Then \( E(L) \ll m^{1/2}(L)|L|^{5/2} + M(L)|L|^2 \).\]
The following result is contained in [9, Proposition 7]. Here we have considered the case of the prime field (which is more difficult) and the case of the real numbers can be treated in analogous way.

**Proposition 5** Let \(A, B \subseteq \mathbb{F}\) be sets and \(\mathcal{L}\) be a set of affine transformations over \(\mathbb{F}\). Then for any positive integer \(k\) one has

\[
I(A \times B, \mathcal{L}) - \frac{|A||B||\mathcal{L}|}{|\mathbb{F}|} \ll \sqrt{|A||B||\mathcal{L}|} \cdot (T_{2^k}(\mathcal{L})|A| \log |A|)^{1/2k+2}.
\]

More precisely, if \(\mathbb{F} = \mathbb{R}\), then

\[
I(A \times B, \mathcal{L}) \ll \sqrt{|A||B||\mathcal{L}|}^{1/3} \cdot (T_{2^k}(\mathcal{L})|A| \log |A|)^{1/3k}.\]

We need the following structural result on the higher energies, see [8, Theorem 4].

**Theorem 6** Let \(G\) be an abelian group, \(A \subseteq G\) be a set, \(\delta, \varepsilon \in (0, 1]\) be parameters, \(\varepsilon \leq \delta\).

1) Then there is \(k = k(\delta, \varepsilon) = \exp(O(\varepsilon^{-1} \log(1/\delta)))\) such that either \(E_k(A) \leq |A|^{k+\delta}\) or there is \(H \subseteq G, |H| \gtrsim |A|^{\delta(1-\varepsilon)}\), \(|H + H| \ll |A|^\varepsilon |H|\) and there exists \(Z \subseteq G, |Z||H| \ll |A|^{1+\varepsilon}\) with

\[
|(H + Z) \cap A| \gg |A|^{1-\varepsilon}.
\]

2) Similarly, either there is a set \(A' \subseteq A, |A'| \gg |A|^{1-\varepsilon}\) and \(P \subseteq G, |P| \gtrsim |A|^{\delta}\) such that for all \(x \in A'\) one has \(r_{A-P}(x) \gg |P| |A|^{-\varepsilon}\) or \(E_k(A) \leq |A|^{k+\delta}\) with \(k \ll 1/\varepsilon\).

Thus either \(E_k(A) \leq |A|^{k+\delta}\), or \(A\) has some strong structural properties depending on the parameters \(\delta\) and \(\varepsilon\). In the latter case we say that \(A\) is \(E_k\)-exceptional with \(\delta, \varepsilon\).

We use a simplified version of [7, Theorem 5], as well as [9, Lemma 9] in our last section.

**Theorem 7** Let \(k \geq 2\) be an integer, \(M \geq 1\) be a real number, \(A, B \subseteq \mathbb{F}_p\) be sets, \(|AB| \leq M|A|\), and \(|B| \gtrsim_k M^{2k+1}\). Then

\[
T_{2^k}(A) \lesssim_k M^{2k+1} \left(\frac{|A|^{2k+1}}{p} + |A|^{2k+1} - |B|^{-k-1} + |B|^{-k-1/2}\right).
\]

**Lemma 8** Let \(A, B \subseteq \mathbb{F}_p^*\) be sets, and \(\mathcal{L} = \{(a, b) : a \in A, b \in B\} \subseteq \text{Aff}(\mathbb{F}_p)\). Then for any \(k \geq 2\) one has

\[
T_k(f_{\mathcal{L}}) \leq |A|^{2k-1}T_k^+(f_B).
\]
3 The proof of the main result

First of all, combining Theorem 4 and Proposition 5 we obtain a result on "rich" lines (i.e., lines having large intersections with \( A \times B \)) in terms of the quantities \( m(\mathcal{L}) \) and \( M(\mathcal{L}) \). We follow the method from [5] and [3]. Given a set \( \mathcal{L} \) of affine transformations over \( \mathbb{F} \) and a real number \( \tau \geq 2 \) put

\[
\mathcal{L}_\tau = \mathcal{L}_\tau(A) := \{ l \in \mathcal{L} : |l \cap (A \times B)| \geq \tau \}.
\]

Formulae (18), (19) below can be treated as an alternative (to the Szemerédi–Trotter type estimates) upper bound for size of \( |\mathcal{L}_\tau| \).

**Corollary 9** Let \( A, B \subseteq \mathbb{F} \) be sets and \( \mathcal{L} \) be a set of affine transformations over \( \mathbb{F} \). Also, let \( \tau \geq 1 \) be a real number, \( \frac{2|A||B|}{|F|^2} \leq \tau \) and \( m(\mathcal{L}_\tau)|\mathcal{L}_\tau| \leq |F|^2 \). Then

\[
|\mathcal{L}_\tau| \ll \tau^{-16/3}m^{1/3}(\mathcal{L}_\tau)|A|^{10/3}|B|^{8/3}\log^{2/3}|A| + \tau^{-4}M^{1/2}(\mathcal{L}_\tau)|A|^{5/2}|B|^{2}\log^{1/2}|A|.
\]

More precisely, if \( \mathbb{F} = \mathbb{R} \), then

\[
|\mathcal{L}_\tau| \ll \tau^{-4}(m(\mathcal{L}_\tau)|A|^8|B|^{2}\log^{2}|A|^{1/3} + \tau^{-3}M^{1/2}(\mathcal{L}_\tau)|A|^2|B|^{3/2}\log^{1/2}|A|.
\]

**Proof.** Let \( m = m(\mathcal{L}_\tau) \) and \( M = M(\mathcal{L}_\tau) \). By Proposition 5 with \( k = 1 \), the definition of the set \( \mathcal{L}_\tau \) and our condition \( \frac{2|A||B|}{|F|^2} \leq \tau \), we get

\[
\tau|\mathcal{L}_\tau| \ll \sqrt{|A||B||\mathcal{L}_\tau|} \cdot (E(\mathcal{L}_\tau)|A| \log |A|)^{1/8}.
\]

We now apply Theorem 4 and obtain

\[
\tau|\mathcal{L}_\tau| \ll \sqrt{|A||B||\mathcal{L}_\tau|(|A| \log |A|)^{1/8} \cdot (m^{1/2}|\mathcal{L}_\tau|^{5/2} + M|\mathcal{L}_\tau|^2)^{1/8}.
\]

The last inequality is equivalent to (18). To obtain estimate (19) we just use formula (15) instead of (14). This completes the proof. \( \square \)

Now let us give simple upper bounds for the quantities \( m(\mathcal{L}_\tau), M(\mathcal{L}_\tau) \).

**Lemma 10** Let \( A, B \subseteq \mathbb{F} \) be sets, \( \tau \geq 2 \) and \( \mathcal{L} \) be a set of affine transformations over \( \mathbb{F} \). Then for any integer \( k \geq 2 \) one has

\[
m(\mathcal{L}_\tau) \leq \tau^{-k}(E_k^+(A)E_k^+(B))^{1/2} \quad \text{and} \quad M(\mathcal{L}_\tau) \leq \tau^{-k}(E_k^x(A)E_k^x(B))^{1/2}.
\]

**Proof.** Let \( m = m(\mathcal{L}_\tau) \) and \( M = M(\mathcal{L}_\tau) \). By the definition of the quantity \( m \) we can find \( m \) lines in \( \mathcal{L}_\tau \) having the form \( \alpha x + \beta \), where \( \alpha \neq 0 \) is a fixed number and \( \beta \) runs over a set \( \mathcal{B} \) of cardinality \( m \). Using the definition of the set \( \mathcal{L}_\tau \) and the Hölder inequality, we obtain

\[
\tau m \leq \sum_{\beta \in \mathcal{B}} \sum_{x \in \mathbb{B}} A(\alpha x + \beta) = \sum_{\beta \in \mathcal{B}} r_{\alpha B - A}(\beta) \leq m^{1-1/k}(E_k^+(\alpha B, A))^{1/k} \leq m^{1-1/k}(E_k^+(A)E_k^+(B))^{1/2k}.
\]
Similarly, by the definition of the quantity $M$ we find $M$ lines concurrent at a certain point $(x_0, y_0)$. We parameterise them by their slopes: $\beta = \frac{y - y_0}{x - x_0}$, $\beta \in B$, $|B| = M$. Hence as above

$$\tau M \leq \sum_{\beta \in B} r_{(A-y_0)/(B-x_0)}(\beta) \leq M^{1-1/k}(E^\times(A - y_0, B - x_0))^{1/2} \leq M^{1-1/k}(\overline{E}^\times_k(A)\overline{E}^\times_k(B))^{1/2k}.$$ 

This completes the proof. □

Now we are ready to obtain our main result in the case of the real field and a result on $C_5(A, B)$ for small $A$ and $B$ in the case of the prime field.

**Theorem 11** Let $A, B \subset \mathbb{R}$ be sets. Then

$$C_4(A, B) - |B||A|^4 - |A||B|^4 \ll |B||A|^3 + |A||B|^3 + \quad (A, B) \log |A| \leq p|A|^5|B|^4 \quad \text{and} \quad |A|^5|B|^6 \leq p^9,$$ 

If $A, B \subseteq \mathbb{F}_p$ such that $C_5(A, B) \log |A| \leq p|A|^5|B|^4$ and $|A|^5|B|^6 \leq p^9$, then

$$C_5'(A, B) = |B||A|^5 - |A||B|^5 \ll |B||A|^4 + |A||B|^4 + C_5(A, B),$$

where the error term $C_5'(A, B)$ is at most

$$C_5(A, B) \log |A| \ll |B||A|^5 - |A||B|^5 \ll |B||A|^4 + |A||B|^4 + C_5'(A, B) \log |A|.$$ 

Proof. We start with (21). The term $|B||A|^4 + |A||B|^4$ corresponds to vertical/horizontal lines plus an error, which is $O(|B||A|^3 + |A||B|^3)$. Let $\mathcal{L}$ be the set of non-vertical and non-horizontal lines, having at least two points in $A \times B$. Denote by $\mathcal{E}$ the error term in (22). Let $\Delta > 0$ be a parameter, which we will choose later. Having the definition of the quantity $C_3(A, B)$ and using the diadic Dirichlet principle, we get

$$\mathcal{E} \ll \Delta C_3(A, B) + \sum_{j : 2^j \geq \Delta} 2^{4j}|L_{2^j}| = \Delta C_3(A, B) + \mathcal{E}' + \sum_{j : 2^j \geq \Delta} 2^{4j}|L_{2^j}|.$$

Put $m_* = (E^\times(A)E^\times(B))^{1/2}$, $M_* = (\overline{E}^\times(A)\overline{E}^\times(B))^{1/2}$. Applying bound (19) of Corollary 9, combining with Lemma 10 (with the parameter $k = 2$), we obtain

$$\mathcal{E}' \ll \sum_{j : 2^j \geq \Delta} 2^{4j} \cdot \left(2^{-14j/3}(m_*|A|^8|B|^6 \log^2 |A|)^{1/3} + 2^{-4j}M_*^{1/2}|A|^2|B|^{3/2} \log^{1/2} |A| \right)$$

$$\ll \Delta^{-2/3}(m_*|A|^8|B|^6 \log^2 |A|)^{1/3} + M_*^{1/2}|A|^2|B|^{3/2} \log^{3/2} |A|.$$ 

Now choosing $\Delta^{5/3} = (m_*|A|^8|B|^6 \log^2 |A|)^{1/3}C_3^{-1}(A, B)$ (one can check that $\Delta \geq 2$ thanks to formula (13)), say, we get

$$\mathcal{E} \ll C_3^{2/5}(A, B)m_*^{1/5}|A|^{8/5}|B|^{6/5} \log^{2/5} |A| + M_*^{1/2}|A|^2|B|^{3/2} \log^{3/2} |A|.$$
as required. To obtain (22) we use the same argument, namely,
\[ \mathcal{E} \ll \Delta \mathcal{C}_4(A, B) + \sum_{j : 2^j \geq \Delta} 2^{5j} |L_{2j}| = \Delta \mathcal{C}_4(A, B) + \mathcal{E}', \]
and
\[ \mathcal{E}' \ll \sum_{j : 2^j \geq \Delta} 2^{5j} \left( 2^{-6j} m_{s_1}^{1/3} |A|^{10/3} |B|^{8/3} \log^{2/3} |A| + 2^{-5j} M_{s_2}^{1/2} |A|^{5/2} |B|^2 \log^{1/2} |A| \right) \]
\[ \ll \Delta^{-1} m_{s_1}^{1/3} |A|^{10/3} |B|^{8/3} \log^{2/3} |A| + M_{s_2}^{1/2} |A|^{5/2} |B|^2 \log^{3/2} |A|. \]

Now the optimal choice of \( \Delta \) is \( \Delta^2 = m_{s_1}^{1/3} |A|^{10/3} |B|^{8/3} \log^{2/3} |A| \cdot C_4^{-1}(A, B) \) (again \( \Delta \geq 2 \) thanks to formula (13), say) and hence
\[ \mathcal{E} \ll C_4^{1/2}(A, B) m_{s_1}^{1/6} |A|^{5/3} |B|^{4/3} \log^{1/3} |A| + M_{s_2}^{1/2} |A|^{5/2} |B|^2 \log^{3/2} |A|. \]

It remains to check that \( \Delta \geq 2 |A| |B|/p \) and \( m(\mathcal{L}_{\Delta}) \mathcal{L}_{\Delta} | \leq p^2. \) Since by our condition \( C_4^2(A, B) \log |A| \leq p |A| |B|^4, \) it follows that in view of trivial lower bounds \( \mathcal{E}^+(A) \geq |A|^2, \)
\( \mathcal{E}^+(B) \geq |B|^2, \) we get
\[ \Delta^2 \gg m_{s_1}^{1/3} |A|^{5/6} |B|^{2/3} p^{-1/2} \gg |A|^{7/6} |B| p^{-1/2}. \]

It is easy to see that the last quantity is greater than \( (2 |A| |B|/p)^2 \times 2 \) thanks to \( |A|^{5/6} |B|^6 \leq p^9. \) Finally, using \( |\mathcal{L}_{\Delta}| \leq C_4(A, B) \Delta^{-4}, \) Lemma 10 with \( k = 2, \) the definition of the quantity \( \Delta \) and our assumption, we obtain
\[ m(\mathcal{L}_{\Delta}) |\mathcal{L}_{\Delta}| \leq m_s C_4(A, B) \Delta^{-6} \leq C_4^3(A, B) |A|^{-10} |B|^{-8} \log^2 |A| \leq p^2. \]

This completes the proof. \( \square \)

Of course one can rewrite the condition \( C_4^2(A, B) \log |A| \leq p |A| |B|^4 \) in terms of cardinalities of \( A \) and \( B, \) using (3) but we leave this a little bit more precise condition. Now let us consider a constructive family of sets satisfying the assumptions of Theorem 11.

**Example.** Let \( A \subseteq \mathbb{F}, |A| < \sqrt{|\mathbb{F}|}, \) say, \( |A + A| \leq K |A| \) and consider \( X := A^{-1}. \) Then one can quickly show that \( \mathcal{E}^+(X) \ll_K |A|^{3 - 1/4} \) (see [14, Lemma 14]) and \( \mathcal{E}^+(X + s) \ll_K |A|^{3 - c} \) for a certain \( c > 0 \) and any \( s \in \mathbb{F}. \) We give a sketch of the proof of the last estimate. Indeed, for \( s = 0 \) it immediately follows from Rudnev’s Theorem 11 (with \( c = 1/2 \)) and for \( s \neq 0 \) (dividing we can suppose that \( s = 1 \)) we see that it is enough to solve the equation
\[ \left( \frac{1}{a + b} + 1 \right) \left( \frac{1}{c + d} + 1 \right) = \lambda, \]
where \( \lambda \neq 1 \) is a fixed number. If we consider \( a, c \) as variables and \( b, d \) as coefficients, then, clearly, (27) determines a family of conics and the number of the solutions to the equation can be estimated via the main result of [1], say.

Similarly, we obtain an analogue of estimate (22) without any restrictions on size of \( A. \)
Theorem 12 Let $A \subseteq \mathbb{F}_p$ be a set and $f_A(x) = A(x) - |A|/p$. Then

$$\sum_{l \in \text{Aff}(\mathbb{F}_p)} \left| \sum_x f_A(x) f_A(lx) \right|^5 \lesssim \left( \mathbb{E}^+(f_A) \right)^{1/6} |A|^{11/2} + \left( \mathbb{E}^x(f_A) \right)^{1/2} |A|^{9/2},$$

(28)

where the summation over $l$ in the last formula is taken over all affine transformations having at least two points in $A \times A$. In particular,

$$C_5(A) - \frac{|A|^{10}}{p^3} - 2|A|^6 \lesssim \frac{|A|^7}{p} + \frac{|A|^4}{p^2} C_3(f_A) + \left( \mathbb{E}^+(f_A) \right)^{1/6} |A|^{11/2} + \left( \mathbb{E}^x(f_A) \right)^{1/2} |A|^{9/2}. \quad (29)$$

Proof. Let $\mathcal{E}$ be the left–hand side of (28). First of all, we redefine the set $\mathcal{L}_\tau$ as

$$\mathcal{L}_\tau = \mathcal{L}_\tau(f_A) := \left\{ l \in \mathcal{L} : \left| \sum_x f_A(x) f_A(lx) \right| \geq \tau \right\}.$$

Then estimate (14) of Proposition 5 with $A = B$ gives us

$$\tau |\mathcal{L}_\tau| \ll \sqrt{|A||B|L} \cdot (E(\mathcal{L}_\tau)|A|^3 \log |A|^{1/8}.$$

(30)

Indeed,

$$\tau |\mathcal{L}_\tau| \leq \sum_{l \in \mathcal{L}_\tau} \left| \sum_x f_A(x) f_A(lx) \right| = \sum_{l \in \mathcal{L}_\tau} \varepsilon(l) \sum_x f_A(x) f_A(lx),$$

where by $\varepsilon(l)$ we have denoted the sign of $\left| \sum_x f_A(x) f_A(lx) \right|$. In other words, we consider a new function $\mathcal{L}_\tau^\varepsilon(l) := \mathcal{L}_\tau(l) \varepsilon(l)$. Clearly, $\|\mathcal{L}_\tau^\varepsilon\|_1 = |\mathcal{L}_\tau|$ and for any integer $l \geq 2$ one has $T_l(\mathcal{L}_\tau^\varepsilon) \leq T_l(\mathcal{L}_\tau)$. Thus using the Cauchy–Schwarz inequality as in the proof of [9, Proposition 7] (alternatively see the proof of Theorem 13 below), we obtain (30) (without any condition on $\tau$). Thus Proposition 5 (estimate (30)) holds for such defined $\mathcal{L}_\tau$ and hence Corollary 5 takes place as well. As for an analogue of Lemma 10, we have for $m = m(\mathcal{L}_\tau)$ and an arbitrary even $k$ that

$$\tau m \leq \sum_{\beta \in \mathcal{B}} \left| \sum_{x \in \mathcal{B}} f_A(\alpha x + \beta) \right| = \sum_{\beta \in \mathcal{B}} |r_{\alpha B - f_A}(\beta)| \leq m^{1-1/k} (\mathbb{E}^+(\alpha f_B, f_A))^{1/k} \leq m^{1/2} (\mathbb{E}^+(f_A) E^+(f_B))^{1/4}.$$

and similarly for $M(\mathcal{L}_\tau)$. Finally, in [2] it was proved that

$$\sigma = \sum_{l \in \text{Aff}(\mathbb{F}_p)} \left| \sum_x f_A(x) f_A(lx) \right|^4 \ll |A|^5 \log |A|,$$

(31)

where again the summation over $l$ in the last formula is taken over all affine transformations having at least two points in $A \times A$. Actually, (31) is equivalent to asymptotic formula (3). Thus we can repeat the calculations in (23)–(25) and obtain

$$\mathcal{E} \ll \sigma^{1/2} m^{1/6} |A|^3 \log^{1/3} |A| + M^{1/2} |A|^{9/2} \log^{3/2} |A| \ll$$
where \( m_\ast = \mathbb{E}^\times(f_A), M_\ast = \overline{\mathbb{E}}^\times(f_A) \). It remains to check the condition \( m(\mathcal{L}_\tau) | \mathcal{L}_\tau | \leq p^2 \) as in (26). Using these calculations, as well as (31), we see that the condition \( |A| \ll p/ \log |A| \) is enough. Splitting our set \( A \) if its needed and loosing some logarithms, we arrive to (28) for all \( A \).

Now let us obtain (29). The term \( 2 |A|^6 \) corresponds to vertical/horizontal lines and since \( \mathbb{E}^+(f_A), \overline{\mathbb{E}}^\times(f_A) \gg |A|^2 \), it follows that all appearing terms, which less than \( |A|^{35/6} \) are negligible. Let \( i(l) = |l \cap (A \times A)| \) and below we consider just \( l \) with \( i(l) > 1 \). Then \( \sum_i i^2(l) = |A|^4 - |A|^2 \) and

\[
\sum_i i^5(l) = \sum_i i^2(l) \left( i(l) - \frac{|A|^2}{p} + \frac{|A|^2}{p} \right)^3 = \sum_i i^2(l) \left( i(l) - \frac{|A|^2}{p} \right)^3 + \frac{3 |A|^2}{p} \sum_i i^2(l) \left( i(l) - \frac{|A|^2}{p} \right)^2 + \frac{3 |A|^4}{p^2} \left( \sum_i i^3(l) - \frac{|A|^2}{p} \sum_i i^2(l) \right) + \frac{|A|^{10}}{p^3} - \frac{|A|^8}{p^3}.
\]

It is easy to check that

\[
\sum_i i^3(l) - \frac{|A|^2}{p} \sum_i i^2(l) \leq C_3(f_A).
\]

The sum \( \sum_i i^2(l) \left( i(l) - \frac{|A|^2}{p} \right)^2 \) was estimated in [23] page 606, namely, splitting the summation over \( i(l) < 2 |A|^2/p \) and \( i(l) \geq 2 |A|^2/p \), as well as using (3), we get

\[
\sum_i i^2(l) \left( i(l) - \frac{|A|^2}{p} \right)^2 \ll \left( |A|^2 \right)^2 p |A|^2 + |A|^5 \log |A| \ll |A|^5 \log |A|
\]

and hence we have the remaining term \( O(p^{-1} |A|^7 \log |A|) \). Finally,

\[
\sum_i \left( i(l) - \frac{|A|^2}{p} + \frac{|A|^2}{p} \right)^2 \left( i(l) - \frac{|A|^2}{p} \right)^3 = \sum_i \left( i(l) - \frac{|A|^2}{p} \right)^5 + \frac{2 |A|^2}{p} \sum_i \left( i(l) - \frac{|A|^2}{p} \right)^4 + \frac{4 |A|^4}{p^2} \sum_i \left( i(l) - \frac{|A|^2}{p} \right)^3 = \mathcal{E} + O(p^{-1} |A|^7 \log |A| + \frac{|A|^4}{p^2} C_3(f_A)).
\]

Combining all bounds and estimate (28), we obtain (29). This completes the proof. \( \square \)

4 Some applications

The power saving in (28) (for sets with small additive/multiplicative energies of its shifts) allows us to obtain our new incidence result Theorem 3 from the introduction. We follow the method of the proof from [9 Proposition 7].

**Theorem 13** Let \( k \geq 2 \) be an integer, \( M \geq 1 \) be a real number, \( A, B, C, X, Y \subseteq \mathbb{F}_p, 0 \notin X \) be sets, \( |Y| \subseteq M |Y| \), and \( |C| \gg M^{2k+1}, |C|^{k-1} \gg (p/|Y|)^2 \). Suppose that \( \max \{ \mathbb{E}^+(f_B), \overline{\mathbb{E}}^\times(f_B) \} \leq |B|^{3-\delta} \), where \( \delta > 0 \) is a constant. Then

\[
| \{(a, b, x, y) \in A \times B \times X \times Y : y = bx + a \} | - \frac{|A||B||X||Y|}{p} \ll \sqrt{|A||B||X||Y|} \cdot |B|^{- \frac{\delta}{35 + 2\delta}} \cdot (32)
\]
Proof. At the beginning we repeat the arguments from the proofs of [4] Proposition 8 and Proposition 5. Let $\mathcal{L}$ be the set of the lines $\{x,y\}$, $x \in X$, $y \in Y$ defined as $y = bx + a$, so $a, b$ are variables. Thus the equation from the left–hand side of (32) can be treated as a question about incidences between points $P = A \times B$ and lines $\mathcal{L}$. We have

$$I(A \times B, \mathcal{L}) = \frac{|A||B||\mathcal{L}|}{p} + \sum_{x \in A} \sum_{l \in \mathcal{L}} f_B(lx) = \frac{|A||B||\mathcal{L}|}{p} + \sigma.$$  \hspace{1cm} (33)

Using the Hölder inequality several times, we get

$$\sigma_2 \lesssim |A|^{2k-1} |B|^{2k-1} - \sum_{h \in \mathcal{L}} \sum_{x} f_B(x) f_B(hx). \hspace{1cm} (34)$$

Assume that the summation in the last formula is taken over lines $h$, having at least two points in $B \times B$, denote the rest as $\sigma_*$ and suppose that $\sigma_* \leq 2^{-1} \sigma$, say. Then applying the Hölder inequality one more time, combining with formula (28) of Theorem 12 we obtain

$$\sigma_5 \lesssim |A|^{5k-1} |B|^{5k-1} - \sum_{h \in \mathcal{L}} \sum_{x} f_B(x) f_B(hx) \lesssim |A|^{5k-1} |B|^{5k-1} - \sum_{h \in \mathcal{L}} \sum_{x} f_B(x) f_B(hx).$$

By Lemma 8, we know that $T_{2k}(\mathcal{L}) \leq |X|^{2k-1} - \sum_{Y} T_{2k}(Y)$. Using our conditions $|C| \gtrsim_k M^{2k+1}$, $|C|^{k-1} \gtrsim (p/|Y|)$, as well as Theorem 7 for $A = Y$ and $B = C$ to estimate the quantity $T_{2k}(Y)$, we obtain

$$\sigma \ll \sqrt{|A||B||X||Y|(|B|^{1-\delta/7} / (|X|p))^{1/5:k}} \lesssim \sqrt{|A||B||X||Y|}. \hspace{1cm} (35)$$

It remains to estimate $\sigma_*$. Returning to (34), we see that

$$\sum_{x} f_B(x) f_B(hx) = \sum_{x} B(x)B(hx) - \frac{|B|^2}{p}$$

and hence

$$\sigma_* \lesssim |A|^{5k-1} |B|^{5k-1} - |\mathcal{L}|^{5k-2k}.$$ and this is better than (35). This completes the proof. $\square$

We now obtain another application, using our structural Theorem 6 (also, see the discussion and the definitions after this result). We show that for all sets having no special form one can obtain a good power saving for $C_n(f_A)$. It allows us to estimate nontrivially (in a rather strong sense) some mixed energies of shifts of $A$, see Theorem 15 below.
Theorem 14 Let $A \subseteq \mathbb{F}$ be a set and $f_A(x) = A(x) - |A|/|\mathbb{F}|$. Suppose that $A$ is not $E^+$-exceptional and for any $s \in \mathbb{F}$ the set $A - s$ is not $E^+$-exceptional with $\delta, \varepsilon$. Then for any number $n \geq \exp(C\varepsilon^{-1}\log(1/\delta))$, where $C > 0$ is an absolute constant one has

$$\sum_{l \in \text{Aff}(\mathbb{F})} \left| \sum_x f_A(x)f_A(lx) \right|^n \ll |A|^{n+2/3+\delta}.$$  \hfill (36)

Proof. Suppose that $n \geq 6$ is an even number and let $\mathcal{E}_n$ be the left-hand side of (36), i.e. the $n$th moment of the function $\sum_x f_A(x)f_A(lx)$. We mimic the calculations in (23)–(25) and obtain

$$\mathcal{E}_n \ll \sum_j 2^{nj}|\mathcal{L}_2|.$$  \hfill (37)

Now to estimate $|\mathcal{L}_2|$ we use formula (18) of Corollary 3 and Lemma 10 with $k_1 = 3n - 16 \geq 2$, $k_2 = 2n - 8 \geq 4$ (hence $k_1, k_2$ are even numbers automatically) to get

$$|\mathcal{L}_2| \ll \tau^{-(k_1+16)/3}(\mathcal{E}_{k_1}^+(f_A))^{1/3}|A|^6 \log^{2/3}|A| + \tau^{-(k_2+8)/2}(\mathcal{E}_{k_2}^+(f_A))^{1/2}|A|^{9/2} \log^{1/2}|A|.$$  \hfill (38)

Hence summing the last estimate over $j$ and using our basic bound (37), we get

$$\mathcal{E}_n \ll \sum_j 2^{nj}j^{(k_1+16)/3}(\mathcal{E}_{k_1}^+(f_A))^{1/3}|A|^6 \log^{2/3}|A| + \sum_j 2^{nj-j(k_2+8)/2}(\mathcal{E}_{k_2}^+(f_A))^{1/2}|A|^{9/2} \log^{1/2}|A|$$

$$\ll (\mathcal{E}_{3n-16}^+(f_A))^{1/3}|A|^6 \log^{5/3}|A| + (\mathcal{E}_{2n-8}^+(f_A))^{1/2}|A|^{9/2} \log^{3/2}|A|.$$  \hfill (39)

Now applying Theorem 6 and the condition that $A - s$ is not exceptional for both energies $E^+, E^x$ with $\delta, \varepsilon$, we derive

$$\mathcal{E}_n \ll |A|^{n+2/3+\delta/3} \log^{5/3}|A| \ll |A|^{n+2/3+\delta/10}$$

as required. \hfill \Box

Of course one can obtain an analogue of asymptotic formula (29) of Theorem 12 for larger $k$ but it requires some calculations and we leave it for the interested reader.

Having sets $A, B \subseteq \mathbb{F}$ and an integer $k \geq 2$ let $\tilde{C}_k(A, B)$ be $\mathcal{C}_k(A, B) - |A||B|^k - |B||A|^k$, that is, we have deleted all horizontal/vertical lines from our consideration. Also, we put

$$\tilde{E}_k^x(A, B) = \left\{ (a_1, \ldots, a_k, b_1, \ldots, b_k) \in (A \setminus \{0\})^k \times (B \setminus \{0\})^k : \frac{a_1}{b_1} = \cdots = \frac{a_k}{b_k} \right\}.$$  \hfill (40)

Then from the equation of a line $y = \lambda x + \mu$, intersecting $A \times B$, we derive

$$\tilde{C}_k(A, B) + O_k(|A||B|^2 \cdot (\min\{|A|, |B|\})^{k-3}) = \sum_{\lambda \neq 0} \mathcal{E}_k^+(B, \lambda A) = \sum_{\mu} \tilde{E}_k^x(B - \mu, A)$$  \hfill (41)

similar to formula (13). As we will see from the proof of Theorem 15 there are other expressions for $\mathcal{C}_k(A, B)$ in terms of higher energies $E_k$. Notice that in the symmetric case $A = B$ we always have $\tilde{E}_k^x(A, A) \gg |A|^k$. In contrast, using our paucity result, we obtain
Theorem 15 Let $A, B \subset \mathbb{F}_p$ be sets, $|A| = |B| \leq p^{2/3}$, and $E^+(A)$, $E^\times(A) \leq |A|^{3-c}$, where $c \in [0, 1)$. Then there are $b_1, b_2 \in B$ such that

$$\tilde{E}_n^\times(A - b_1, A - b_2) \ll |A|^{4-\frac{16}{15}} \log |A|.$$  \hspace{1cm} (39)

Now suppose that our set $A$ is not $E^+$–exceptional and for any $s \in \mathbb{F}$ the set $A - s$ is not $E^\times$–exceptional with $\delta, \varepsilon$. Then for any number $n \geq \exp(C\varepsilon^{-1} \log(1/\delta))$, where $C > 0$ is an absolute constant one can find a pair $b_1, b_2 \in B$ with

$$\tilde{E}_n^\times(f_A - b_1, f_A - b_2) \ll |A|^{n-1/3+\delta}.$$  \hspace{1cm} (40)

Proof. Let $L = \log |A|$. The number of collinear quintuplets $(b_1, b_2), (a_1, a'_1), \ldots, (a_4, a'_4) \in (B \times B) \times (A \times A)^4$ equals the number $\sigma$ of the solutions to the system

$$\frac{a_1 - b_1}{a'_1 - b_2} = \cdots = \frac{a_4 - b_1}{a'_4 - b_2}$$

with non–zero numerators and denominators plus quintuplets, which correspond to vertical/horizontal lines. Thus the sum $\sigma := \sum_{b_1, b_2 \in B} \tilde{E}_n^\times(A - b_1, A - b_2)$ can be estimated via formula (22) of Theorem 11. We cannot write $\sigma \leq {\cal C}_4(A, A)^{4/5} {\cal C}_5(B, B)^{1/5}$ using the Hölder inequality because one can have $i_{A, A}(l) = 1$, say. Nevertheless, the contribution of these terms in view of formula (3) is at most ${\cal C}_4(A \cup B) \ll |A|^5 L$ and hence it is negligible. Thus by estimate (22) and our restrictions $|A| = |B| \leq p^{2/3}$, we obtain

$$\sum_{b_1, b_2 \in B} \tilde{E}_n^\times(A - b_1, A - b_2) \ll (|A|^{6-c/6}L^{5/6} + |A|^5L)^{4/5}(|A|^{6}L + |A|^5L)^{1/5} + |A|^5L \ll |A|^{6-2c/15} L$$

and we have proved bound (39). To obtain (40) we just apply Theorem 14 instead of Theorem 11. This completes the proof. \hspace{1cm} $\square$

Of course, in a similar way, using formula (38) and our bounds for $\tilde{C}_k(A, B)$, one can derive some upper bounds for $E_k^+(B, \lambda A)$, $E_k^\times(B - \mu, A)$ for typical shifts $\lambda, \mu$.

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