A note on the boundedness of discrete commutators on Morrey spaces and their preduals

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Abstract

Dyadic fractional integral operators are shown to be bounded on Morrey spaces and their preduals. It seems that the proof of the boundedness by means of dyadic fractional integral operators is effective particularly on the preduals. In the present paper the commutators are proved to be bounded as well.

1 Introduction

In the present paper, we consider the dyadic analysis of Morrey spaces and their preduals. The Haar wavelet, which plays a central role in this field, is given as follows: First, we write

\[ h_{\varepsilon}^{\varepsilon}(t) := \chi_{(0,1)}(2t) + (-1)^{\varepsilon} \chi_{[1,2]}(2t) \quad (t \in \mathbb{R}) \]  

for \( \varepsilon \in \mathbb{Z}/2\mathbb{Z} \). Given \( \varepsilon \in E := (\mathbb{Z}/2\mathbb{Z})^{n} \setminus \{ (0, \ldots, 0) \} \), we define

\[ h_{\varepsilon} := h_{\varepsilon}^{1} \otimes h_{\varepsilon}^{2} \otimes \ldots \otimes h_{\varepsilon}^{n}, \quad \text{that is,} \quad h_{\varepsilon}(x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} h_{\varepsilon}^{i}(x_i). \]

By \( \mathcal{D} \) we mean the set of all dyadic cubes. If we write \( Q_{jm} := \prod_{\nu=1}^{n} \left[ \frac{m_{\nu}}{2^j}, \frac{m_{\nu} + 1}{2^j} \right] \) for \( j \in \mathbb{Z} \) and \( m \in \mathbb{Z}^{n} \), then we have \( \mathcal{D} = \{ Q_{jm} : j \in \mathbb{Z}, m \in \mathbb{Z}^{n} \} \). The set \( \mathcal{D}_j \) is the subset of \( \mathcal{D} \) made up of the cubes of volume \( 2^{-jn} \): \( \mathcal{D}_j = \{ Q_{jm} : m \in \mathbb{Z}^{n} \} \). Given a dyadic cube \( Q = Q_{jm} \quad (j \in \mathbb{Z}, m \in \mathbb{Z}^{n}) \), we define the corresponding Haar function by

\[ h_{Q}^{\varepsilon}(x) := 2^{jn/2} h^{\varepsilon}(2^{j} x - m). \]

The idea of discretizing \( I_{\alpha} \) dates back to Lacey’s 2007 paper [5].

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Now we will describe Morrey spaces, the function spaces considered in the present paper. Let $1 \leq q \leq p < \infty$. Then let us define the Morrey norm $\|f\|_{M^{p,q}_q}$ by

$$\|f\|_{M^{p,q}_q} := \sup_{Q \in D} |Q|^\frac{1}{q} \left( \int_Q |f(y)|^q \, dy \right)^{\frac{1}{q}},$$

where $f \in L^{q,\text{loc}}$. We will also use the dyadic BMO space. Given a cube $Q \in D$ and $f \in L^{1,\text{loc}}$, we can write $m_Q(f) := \frac{1}{|Q|} \int_Q f(x) \, dx$. The dyadic sharp maximal operator here is defined by

$$M^{\sharp,\text{dyadic}} f(x) := \sup_{x \in D} m_Q(|f - m_Q(f)|).$$

A function $a \in L^{1,\text{loc}}$ is said to belong to the dyadic BMO, which we will write as $BMO^{\text{dyadic}}$, if $M^{\sharp,\text{dyadic}} a \in L^{\infty}$. We define the dyadic BMO norm by $\|a\|_{BMO^{\text{dyadic}}} := \|M^{\sharp,\text{dyadic}} a\|_{\infty}$.

The present paper, based upon Theorem 1.1, considers the boundedness of commutators. Throughout the paper, for $A, B > 0$, we write $A \lessapprox B$ to indicate that there exists a constant $c > 1$ such that $A \leq c B$ and that this constant depends only on $p, q, s, t, \alpha$ which will appear in each theorem. We also use $A \approx B$ to denote $B \lessapprox A$ and $A \sim B$ to denote the two-sided inequality $A \lessapprox B \lessapprox A$.

**Theorem 1.1.** Let $1 < q \leq p < \infty$.

(i) Let $f \in \mathcal{M}^p_q$. Then we have equivalence

$$\|f\|_{\mathcal{M}^p_q} \sim \sum_{\varepsilon \in E} \left\| \left( \sum_{j=-\infty}^{\infty} \sum_{Q \in D_j} \langle f, h^\varepsilon_Q \rangle h^\varepsilon_Q \right)^2 \right\|_{\mathcal{M}^p_q}^{\frac{1}{2}}. \quad (6)$$

(ii) If a locally integrable function $f$ satisfies

$$\sum_{\varepsilon \in E} \left\| \left( \sum_{j=-\infty}^{\infty} \sum_{Q \in D_j} \langle f, h^\varepsilon_Q \rangle h^\varepsilon_Q \right)^2 \right\|_{\mathcal{M}^p_q}^{\frac{1}{2}} < \infty, \quad (7)$$

then the limit

$$g := \lim_{M \to \infty} \sum_{\varepsilon \in E} \sum_{j=-M}^{M} \sum_{Q \in D_j} \langle f, h^\varepsilon_Q \rangle h^\varepsilon_Q \quad (8)$$

exists in the topology of $L^{q,\text{loc}}$ and defines an $\mathcal{M}^p_q$-function. Furthermore,

$$\|g\|_{\mathcal{M}^p_q} \sim \sum_{\varepsilon \in E} \left\| \left( \sum_{j=-\infty}^{\infty} \sum_{Q \in D_j} \langle f, h^\varepsilon_Q \rangle h^\varepsilon_Q \right)^2 \right\|_{\mathcal{M}^p_q}^{\frac{1}{2}}. \quad (9)$$
The following paraproduct plays an important role in the proof of the boundedness of commutators. The next result follows.

**Theorem 1.2.** Let \( a \in \text{BMO}_{\text{dyadic}} \) and \( 1 < q \leq p < \infty \). Then we have

\[
\sum_{\varepsilon \in E} \sum_{j=-\infty}^{\infty} \left( \sum_{Q \in D_j} \langle f, \chi_Q \rangle \cdot \langle a, h_Q^\varepsilon \rangle \right) h_Q^\varepsilon \lesssim \|a\|_{\text{BMO}_{\text{dyadic}}} \|f\|_{\mathcal{M}_q^p}.
\]

One formally defines

\[
I_{\alpha, \text{dyadic}} f(x) := \sum_{\varepsilon \in E} \sum_{j=-\infty}^{\infty} \sum_{Q \in D_j} |Q|^{\frac{\alpha}{n}} \langle f, h_Q^\varepsilon \rangle h_Q^\varepsilon(x).
\]

We can justify the definition of \( I_{\alpha, \text{dyadic}} \). In particular, we can also justify the convergence of the sum (9) in the next theorem.

**Theorem 1.3.** Let \( 0 < \alpha < n, 1 < q \leq p < \infty, 1 < t \leq s < \infty \). Assume

\[
\frac{1}{s} = \frac{1}{p} - \frac{\alpha}{n}, \quad \frac{t}{s} = \frac{q}{p}.
\]

Then, for every \( f \in \mathcal{M}_q^p \),

\[
I_{\alpha, \text{dyadic}} f(x) = \lim_{M \to \infty} \left( \sum_{\varepsilon \in E} \sum_{j=-M}^{M} \sum_{Q \in D_j} |Q|^{\frac{\alpha}{n}} \langle f, h_Q^\varepsilon \rangle h_Q^\varepsilon(x) \right)
\]

converges for almost every \( x \in \mathbb{R}^n \) and we have

\[
\|I_{\alpha, \text{dyadic}} f\|_{\mathcal{M}_s^t} \lesssim \|f\|_{\mathcal{M}_q^p}.
\]

**Theorem 1.4.** Let \( 0 < \alpha < n, 1 < q \leq p < \infty, 1 < t \leq s < \infty \) and \( a \in \text{BMO}_{\text{dyadic}} \). Assume

\[
\frac{1}{s} = \frac{1}{p} - \frac{\alpha}{n}, \quad \frac{t}{s} = \frac{q}{p}.
\]

Then, for every \( f \in \mathcal{M}_q^p \), the limit

\[
[a, I_{\alpha, \text{dyadic}}] f(x) = a(x) I_{\alpha, \text{dyadic}} f(x) - I_{\alpha, \text{dyadic}}[a \cdot f](x)
\]

\[
:= \lim_{M \to \infty} \sum_{\varepsilon \in E} \sum_{j=-M}^{M} \sum_{Q \in D_j} (a \cdot I_{\alpha, \text{dyadic}} f - I_{\alpha, \text{dyadic}}[a \cdot f], h_Q^\varepsilon) h_Q^\varepsilon(x)
\]

exists in the topology of \( L^{q,\text{loc}} \) and we have

\[
\|[a, I_{\alpha, \text{dyadic}}] f\|_{\mathcal{M}_s^t} \lesssim \|a\|_{\text{BMO}_{\text{dyadic}}} \|f\|_{\mathcal{M}_q^p}.
\]

Next, we prove that the operator norm is characterized by the dyadic BMO norm.
Theorem 1.5. Let \( a \in \text{BMO}_{\text{dyadic}} \). Suppose that we are given parameters \( p, q, s, t, \alpha \) satisfying
\[
1 < q \leq p < \infty, \quad 1 < t \leq s < \infty, \quad 0 < \alpha < n
\]
and
\[
\frac{p}{q} = \frac{t}{s}, \quad \frac{1}{s} = \frac{1}{p} - \frac{\alpha}{n}.
\]
Then we have
\[
\| [a, I_{\alpha, \text{dyadic}}] \|_{B(M^p_q, M^s_t)} \sim \| a \|_{\text{BMO}_{\text{dyadic}}}.
\]

Needless to say, it is significant to prove that
\[
\| [a, I_{\alpha, \text{dyadic}}] \|_{B(M^p_q, M^s_t)} \gtrsim \| a \|_{\text{BMO}_{\text{dyadic}}}
\]
in view of Theorem 1.4. In the usual setting of \( p = q \) and \( s = t \), Theorem 1.4 is known as the result due to S. Chanillo [1].

All the results above carry over to predual spaces. Recall that the predual space \( H^p_q \) of the Morrey space \( M^p_q \) is given as follows: Let \( 1 < p \leq q < \infty \).

(i) A function \( A \in L^q \) is said to be a \((p, q)\)-block, if there exists a dyadic cube \( Q \) such that \( \| A \|_{L^q} \leq |Q|^{\frac{1}{q} - \frac{1}{p}} \) and that \( A \) is supported on \( Q \).

(ii) The predual space \( H^p_q \) is given by
\[
H^p_q := \left\{ \sum_{j=1}^{\infty} \lambda_j a_j : \sum_{j=1}^{\infty} |\lambda_j| < \infty \text{ and each } a_j \text{ is a } (p, q)\text{-block} \right\}
\]
and the norm is given by
\[
\| f \|_{H^p_q} := \inf \left\{ \sum_{j=1}^{\infty} |\lambda_j| : f = \sum_{j=1}^{\infty} \lambda_j a_j \text{ and each } a_j \text{ is a } (p, q)\text{-block} \right\}
\]
for \( f \in H^p_q \).

A well-known fact is that the dual of \( H^p_q \) is \( M^p_q \) (see [13]). Therefore, it seems easy to prove this theorem by duality.

Theorem 1.6. Let \( 0 < \alpha < n \), \( 1 < r \leq r_0 < \infty \) and \( 1 < p \leq p_0 < \infty \). Assume in addition
\[
\frac{1}{r_0} = \frac{1}{p_0} - \frac{\alpha}{n}, \quad r = \frac{p}{p_0}.
\]

(i) The fractional integral operator \( I_{\alpha, \text{dyadic}} \), which is originally defined on \( L^{r_0} \), is bounded from \( H^{r_0}_p \) to \( H^{p_0}_r \). That is,
\[
\| I_{\alpha, \text{dyadic}} f \|_{H^{r_0}_p} \leq C \| f \|_{H^{r_0}_p}
\]
for all \( f \in H^{r_0}_p \).
The commutator \([a, I_{\alpha, \text{dyadic}}]\), which is originally defined on \(L^{r_0'}\), is bounded from \(\mathcal{H}^{r_0'}\) to \(\mathcal{H}^{r_0'}\).

Actually, we invoke duality to prove this theorem. However, we need to pay attention to perform duality argument. Here is a “wrong” proof for \(I_{\alpha, \text{dyadic}}\). The same can be said for \([a, I_{\alpha, \text{dyadic}}]\) or \(I_{\alpha}\).

Wrong proof of Theorem 1.6. By duality argument, we have

\[
\|I_{\alpha, \text{dyadic}}f\|_{\mathcal{H}^{r_0'}} = \sup \left\{ \left| \int_{\mathbb{R}^n} I_{\alpha, \text{dyadic}}f(x)h(x) \, dx \right| : h \in \mathcal{M}^{p_0}_p, \|h\|_{\mathcal{M}^{p_0}_p} = 1 \right\}.
\]

In view of the definition of \(I_{\alpha, \text{dyadic}}\), we have

\[
\int_{\mathbb{R}^n} I_{\alpha, \text{dyadic}}f(x)h(x) \, dx = \int_{\mathbb{R}^n} f(x)I_{\alpha, \text{dyadic}}h(x) \, dx.
\]

If we invoke the boundedness of \(I_{\alpha,\text{dyadic}}\) obtained in Theorem 1.3 and we denote by \(\|I_{\alpha,\text{dyadic}}\|_{B(\mathcal{M}^{p_0}_p, \mathcal{M}^{p_0}_p)}\) the operator norm, then we have

\[
\|I_{\alpha, \text{dyadic}}f\|_{\mathcal{H}^{r_0'}} \leq \|I_{\alpha, \text{dyadic}}\|_{B(\mathcal{M}^{p_0}_p, \mathcal{M}^{p_0}_p)} \|f\|_{\mathcal{H}^{r_0'}}.
\]

The proof is now complete.

Here is some gap in the proof: There is no guarantee for \(I_{\alpha, \text{dyadic}}f\) to be a member of \(\mathcal{H}^{r_0'}\). So to overcome this trouble, we need to take full advantage of the dyadic fractional integral operator \(I_{\alpha, \text{dyadic}}f = |R|h_{\mathcal{D}}^\alpha f\).

Next, we investigate the compactness of the commutator \([a, I_{\alpha, \text{dyadic}}]\). To this end we define VMO\(_{\text{dyadic}}\) as the closure of Span\(\{h_{Q}^\varepsilon \in E, Q \in \mathcal{D}\}\), where Span(\(A\)) denotes a linear subspace generated by a set \(A\).

**Theorem 1.7.** Let \(0 < \alpha < n\), \(1 < q \leq p < \infty\) and \(1 < t \leq s < \infty\). Assume

\[
\frac{1}{s} = \frac{1}{p} - \frac{\alpha}{n}, \quad \frac{t}{s} = \frac{q}{p}.
\]

Then \(a \in \text{BMO}_{\text{dyadic}}\) generates a compact commutator \([a, I_{\alpha, \text{dyadic}}] : \mathcal{M}^{p}_q \to \mathcal{M}^{s}_t\) if and only if \(a \in \text{VMO}_{\text{dyadic}}\).

We remark that the “if” part of Theorem 1.7 is investigated in [7].

All the theorems above are proved in Section 3 after collecting some auxiliary facts in Section 2.
2 Preliminaries

Here we collect some preliminary facts. For the proof of Proposition 2.1 we refer to [4, Chapter 2].

Proposition 2.1. Let $1 < q < \infty$.

(i) For $f \in L^q$, the following equivalence holds:

$$\|f\|_{L^q} \sim \sum_{\varepsilon \in E} \left( \sum_{j=-\infty}^{\infty} \left\| \left( \sum_{Q \in D_j} \langle f, h_{Q}^{\varepsilon} \rangle h_{Q}^{\varepsilon} \right)^2 \right\|_{L^q} \right)^{1/2}.$$

(ii) For $f \in L^q$ and $k \in \mathbb{Z}$, the following equivalence holds:

$$\|f\|_{L^q} \sim \sum_{\varepsilon \in E} \left( \sum_{j=k}^{\infty} \left\| \left( \sum_{Q \in D_j} \langle f, h_{Q}^{\varepsilon} \rangle h_{Q}^{\varepsilon} \right)^2 \right\|_{L^q} \right)^{1/2} + \sum_{Q \in D_k} \| \langle f, h_{Q}^{\varepsilon} \rangle \chi_{Q} \|_{L^q}.$$

(iii) For $f \in L^q, \text{loc}$ and $R \in D$, the following equivalence holds:

$$\|f - m_{R}(f)\|_{L^q(R)} \sim \sum_{\varepsilon \in E} \left( \sum_{j=-\infty}^{0} \sum_{Q \in D_j, Q \subset R} \langle f, h_{Q}^{\varepsilon} \rangle^2 \right)^{1/2}.$$

Here the implicit constant in (19) does not depend on $k$.

A counterpart of Proposition 2.1 for Herz spaces was proved in [6]. So, it seems possible to extend the results to these spaces.

This is the only proposition whose proof we omit in the present paper.

When $n = 1$, the next proposition is [5 Theorem 2.6].

Proposition 2.2. Let $a \in \text{BMO}_{\text{dyadic}}$ and $1 < q < \infty$. Then the following is an equivalent norm of $\|a\|_{\text{BMO}_{\text{dyadic}}}$:

$$\sup \left\{ \sum_{\varepsilon \in E} \left\| \sum_{j=-\infty}^{\infty} \left( \sum_{Q \in D_j} \langle f, \chi_{Q} \rangle \cdot \langle a, h_{Q}^{\varepsilon} \rangle \frac{h_{Q}^{\varepsilon}}{|Q|} \right)^2 \right\|_{L^q} : f \in L^q, \|f\|_{L^q} = 1 \right\}.$$

This theorem is motivated by the results due to Coifman and Meyer. (See [2, 3].)

Proof. This is somehow well known [5 Theorem 2.6]. The proof of

$$\sup \left\{ \sum_{\varepsilon \in E} \left\| \sum_{j=-\infty}^{\infty} \left( \sum_{Q \in D_j} \langle f, \chi_{Q} \rangle \cdot \langle a, h_{Q}^{\varepsilon} \rangle \frac{h_{Q}^{\varepsilon}}{|Q|} \right)^2 \right\|_{L^q} : f \in L^q, \|f\|_{L^q} = 1 \right\} \geq \|a\|_{\text{BMO}_{\text{dyadic}}}$$
can be proved by the cube testing and Proposition 2.1. For the reverse inequality, we use an argument of \( T_1 \)-type as well as the Carleson embedding theorem when \( p = 2 \). The situation resembles that in [12, p.302 (64)].

Here and below, for \( k \in \mathbb{N} \cup \{0\} \) and \( R \in \mathcal{D} \), we write \( R_{+k} \) for the unique dyadic cube containing \( R \) and of volume \( 2^{kn} |R| \).

Finally before we prove Theorems 1.1–1.7, we shall obtain a counterpart of Proposition 2.2 for Morrey spaces.

**Proposition 2.3.** Let \( 1 < q \leq p < \infty \). Then we have

\[
\sup_{f \in \mathcal{M}_q^p} \left\| \sum_{j = -\infty}^{\infty} \left( \sum_{Q \in \mathcal{D}_j} \langle f, \chi_Q \rangle \cdot \langle a, h_\varepsilon^Q \rangle \frac{h_\varepsilon^Q}{|Q|} \right) \right\|_{\mathcal{M}_q^p} \sim \|a\|_{\text{BMO}_{\text{dyadic}}},
\]

where the supremum is taken over all \( f \in \mathcal{M}_q^p \) such that \( \|f\|_{\mathcal{M}_q^p} = 1 \).

**Proof.** Observe that by Proposition 2.1 (see (20)) we have

\[
\|a\|_{\text{BMO}_{\text{dyadic}}} \sim \sup_{Q \in \mathcal{D}} \left( \frac{1}{|Q|} \int_{Q} \sum_{j = -\infty}^{\infty} \left| \sum_{R \in \mathcal{D}_j, R \subset Q} \langle a, h_\varepsilon^R \rangle h_\varepsilon^R(x) \right|^q \frac{dx}{|Q|} \right)^{\frac{1}{q}}.
\]

Consequently the inequality \( \gtrsim \) in (22) follows by considering \( f = |Q|^{-1/p-1/2} h_\varepsilon^Q \) for a cube \( Q \).

Let us prove the inequality \( \lesssim \) in (22). Let \( S \) be a fixed dyadic cube. Then we need to show that

\[
|S|^{\frac{1}{p} - \frac{1}{q}} \left( \int_{S} \left| \sum_{j = -\infty}^{\infty} \left( \sum_{Q \in \mathcal{D}_j} \langle \chi f, \chi_Q \rangle \cdot \langle a, h_\varepsilon^Q \rangle \frac{h_\varepsilon^Q(x)}{|Q|} \right) \right|^q dx \right)^{\frac{1}{q}} \lesssim \|a\|_{\text{BMO}_{\text{dyadic}}}
\]

for all \( f \in \mathcal{M}_q^p \) with norm 1.

By Proposition 2.2, we have

\[
|S|^{\frac{1}{p} - \frac{1}{q}} \left( \int_{S} \left| \sum_{j = -\infty}^{\infty} \left( \sum_{Q \in \mathcal{D}_j} \langle \chi f, \chi_Q \rangle \cdot \langle a, h_\varepsilon^Q \rangle \frac{h_\varepsilon^Q(x)}{|Q|} \right) \right|^q dx \right)^{\frac{1}{q}} \leq \|S\|^{\frac{1}{p} - \frac{1}{q}} \left( \int_{\mathbb{R}^n} \left| \sum_{j = -\infty}^{\infty} \left( \sum_{Q \in \mathcal{D}_j} \langle \chi f, \chi_Q \rangle \cdot \langle a, h_\varepsilon^Q \rangle \frac{h_\varepsilon^Q(x)}{|Q|} \right) \right|^q dx \right)^{\frac{1}{q}} \lesssim \|a\|_{\text{BMO}_{\text{dyadic}}} \|S\|^{\frac{1}{p} - \frac{1}{q}} \|\chi f\|_{L^q}.
\]
By the definition of the Morrey norm $\|f\|_{M^p_q}$ we have
\[
|S|^{\frac{1}{p} - \frac{1}{q}} \left( \int_S \left| \sum_{j=-\infty}^{\infty} \left( \sum_{Q \in D_j} \langle \chi_{S} f, \chi_Q \rangle : \langle a, h^e_Q \rangle \frac{h^e_Q(x)}{|Q|} \right)^q \right|^{\frac{1}{q}} \ dx \right)^{\frac{1}{q}} \lesssim \|a\|_{\text{BMO}_{\text{dyadic}}}.
\]

Meanwhile, a geometric observation shows that
\[
|S|^{\frac{1}{p} - \frac{1}{q}} \left( \int_S \left| \sum_{j=-\infty}^{\infty} \left( \sum_{Q \in D_j} \langle \chi_{R^n \setminus S} f, \chi_Q \rangle : \langle a, h^e_Q \rangle \frac{h^e_Q(x)}{|Q|} \right)^q \right|^{\frac{1}{q}} \ dx \right)^{\frac{1}{q}} \approx \|a\|_{\text{BMO}_{\text{dyadic}}}.
\]

The inequality \(\lesssim\) in (22) is proved and the proof is therefore complete. \(\square\)

3 Proof of Theorems

3.1 Proof of Theorem 1.1

We shall prove an auxiliary inequality which is interesting of its own right.

Lemma 3.1. Let $1 \leq q \leq p < \infty$. Let $f \in M^p_q$. Then we have equivalence
\[
\|f\|_{M^p_q} \sim \sum_{\varepsilon \in E} \left\| \left( \sum_{j=-\infty}^{\infty} \left| \sum_{Q \in D_j} \langle f, h^e_Q \rangle h^e_Q \right|^2 \right)^{\frac{1}{2}} \right\|_{M^p_q} + \|f\|_{M^p_1}.
\]

Proof. Let $R \in D$ be fixed throughout the proof.

By virtue of a crude estimate and the Hölder inequality
\[
|\langle f, h^e_{R+m} \rangle h^e_{R+m}| \leq |R+m|^\frac{1}{p} - 1 \|f\|_{M^p_1}
\]

(25)
we obtain
\[
|R|^{\frac{1}{\nu} - \frac{1}{q}} \left\{ \int_R \left( \sum_{j=-\infty}^\infty \left| \sum_{Q \in D_j, Q \supset R} \langle f, h^\nu_Q \rangle h^{\nu}_Q \right|^2 \right)^{\frac{q}{2}} \right\} \lesssim \|f\|_{M^\nu_q} \leq \|f\|_{M^p_\nu}, \quad (26)
\]

Keeping in mind (26), let us first prove that
\[
\|f\|_{M^\nu_q} \gtrsim \sum_{\varepsilon \in E} \left\| \left( \sum_{j=-\infty}^\infty \left| \sum_{Q \in D_j} \langle f, h^\nu_Q \rangle h^{\nu}_Q \right|^2 \right)^{\frac{1}{2}} \right\|_{M^p_\nu} + \|f\|_{M^p_\nu}. \quad (27)
\]

By Proposition 2.1 (i) and (26) we have
\[
|R|^{\frac{1}{\nu} - \frac{1}{q}} \left\{ \int_R \left( \sum_{j=-\infty}^\infty \left| \sum_{Q \in D_j} \langle f, h^\nu_Q \rangle h^{\nu}_Q \right|^2 \right)^{\frac{q}{2}} \right\}^{\frac{1}{q}} \quad (26)
\]
\[
\lesssim |R|^{\frac{1}{\nu} - \frac{1}{q}} \left( \int_R |f(x)|^q \, dx \right)^{\frac{1}{q}} + |R|^{\frac{1}{\nu} - \frac{1}{q}} \left\{ \int_R \left( \sum_{j=-\infty}^\infty \left| \sum_{\chi_{R^\nu \setminus R} f, h^\nu_Q h^{\nu}_Q \right|^2 \right)^{\frac{1}{2}} \right\} \quad (27)
\]
\[
\lesssim \|f\|_{M^\nu_q}.
\]
Thus, (27) is established.

Now let us prove the converse inequality of (27). First by the triangle inequality and the definition of the Morrey norm (41), we have
\[
|R|^{\frac{1}{\nu} - \frac{1}{q}} \left( \int_R |f(x)|^q \, dx \right)^{\frac{1}{q}} \leq |R|^{\frac{1}{\nu} - \frac{1}{q}} \left( \int_R |f(x) - m_R(f)|^q \, dx \right)^{\frac{1}{q}} + \|f\|_{M^p_\nu}.
\]

By Proposition 2.1 (iii) we obtain
\[
|R|^{\frac{1}{\nu} - \frac{1}{q}} \left( \int_R |f(x)|^q \, dx \right)^{\frac{1}{q}} \quad (27)
\]
\[
\lesssim |R|^{\frac{1}{\nu} - \frac{1}{q}} \left( \int_R \left( \sum_{j=-\infty}^\infty \left| \sum_{Q \in D_j} \langle h^\nu_Q f \rangle h^{\nu}_Q(x) - m_R \left[ \sum_{Q \in D_j} \langle h^\nu_Q f \rangle h^{\nu}_Q \right] \right|^2 \right)^{\frac{1}{2}} \, dx \right)^{\frac{1}{q}} + \|f\|_{M^p_\nu}
\]
\[
= |R|^{\frac{1}{\nu} - \frac{1}{q}} \left( \int_R \left( \sum_{j=-\infty}^\infty \left| \sum_{Q \in D_j, \Omega \supset R} \langle h^\nu_Q f \rangle h^{\nu}_Q(x) \right|^2 \right)^{\frac{1}{2}} \, dx \right)^{\frac{1}{q}} + \|f\|_{M^p_\nu}.
\]
If we use (26) again, then we have
\[
|R|^k \int R \left( \sum_{j=-\infty}^{\infty} \sum_{Q \in D_j, Q \subset R} \langle h_Q^k f \rangle h_Q \right)^q \left| f(x) \right| dx
\]
\[
\lesssim |R|^{\frac{1}{p} - \frac{k}{q}} \int R \left( \sum_{j=-\infty}^{\infty} \sum_{Q \in D_j} \langle h_Q^k f \rangle h_Q \right)^q \left| f(x) \right| dx + \|f\|_{\mathcal{M}^q_{q/2}}
\]
Thus, the proof of Lemma 3.1 is complete.

Let us now prove Theorem 1.1. In view of Lemma 3.1 for the proof of (i) it suffices to establish
\[
|R|^\frac{1}{p} m_R(|f|) = |R|^\frac{1}{p} - 1 \int R \left| f(x) \right| dx \lesssim \sum_{\varepsilon \in E} \left\| \left( \sum_{j=-\infty}^{\infty} \sum_{Q \in D_j} \langle f, h_Q \rangle h_Q^\varepsilon \right)^2 \right\|_{\mathcal{M}^q_{q/2}}.
\]  
By the triangle inequality, we have
\[
|R|^\frac{1}{p} m_R(|f|) = \lim_{k \to \infty} |R|^\frac{1}{p} - 1 \int R \left| f(x) - m_{R+k}(f) \right| dx
\]
\[
\lesssim \sum_{k=0}^{\infty} 2^{-k} \left(\frac{1}{p} - 1\right) \int R_{+k} \left| f(x) - m_{R+k}(f) \right| dx
\]
We calculate, by using Proposition 2.1 and the fact that \( p > 1 \),
\[
|R|^\frac{1}{p} m_R(|f|) \lesssim \sum_{k=0}^{\infty} \sum_{\varepsilon \in E} 2^{-k} \left(\frac{1}{p} - 1\right) \left( \int R_{+k} \left( \sum_{j=-\infty}^{\infty} \sum_{Q \in D_j} \langle h_Q^k f \rangle h_Q \right)^q \right)^{\frac{1}{q}}
\]
\[
\lesssim \sum_{\varepsilon \in E} \left\| \left( \sum_{j=-\infty}^{\infty} \sum_{Q \in D_j} \langle f, h_Q \rangle h_Q^\varepsilon \right)^2 \right\|_{\mathcal{M}^q_{q/2}}.
\]
As a consequence (28) is proved.

Therefore, the proof of (i) is complete.

For the proof of (ii) we fix a compact set \( K \subset \mathbb{R}^n \) and prove that
\[
\lim_{M \to \infty} \sum_{\varepsilon \in E} \left| \sum_{j=-\infty}^{\infty} \sum_{Q \in D_j} \langle f, h_Q \rangle h_Q^\varepsilon - \sum_{j=-M}^{M} \sum_{Q \in D_j} \langle f, h_Q \rangle h_Q \right|_{L^q(K)} = 0.
\]  
However, since \( K \) can be covered by \( 3^n \) dyadic cubes of the same size, we have only to prove (29) with \( K \) replaced by a dyadic cube \( R \in D_k \), where \( k \in \mathbb{Z} \) is a fixed integer. Let us denote
\[
f_k^j := \sum_{Q \in D_j} \langle f, h_Q \rangle h_Q^j
\]  
(30)
A geometric observation shows that for \( \epsilon \in E \) and \( j \in \mathbb{Z} \). If \( x \in \mathbb{R} \) and \( M \geq 1 + |k| \), then we have

\[
\sum_{\epsilon \in E} \left| \sum_{j=-\infty}^{\infty} f_j^\epsilon(x) - \sum_{j=-M}^{M} f_j^\epsilon(x) \right| = \sum_{\epsilon \in E} \left| \sum_{j=M+1}^{\infty} f_j^\epsilon(x) + \sum_{j=-\infty}^{-M-1} f_j^\epsilon(x) \right|
\]

\[
\leq \sum_{\epsilon \in E} \left| \sum_{j=M+1}^{\infty} \sum_{Q \in D_j, Q \subset R} (f, h_Q^\epsilon)h_Q^\epsilon(x) \right| + \sum_{\epsilon \in E} \left| \sum_{j=-\infty}^{-M} f_j^\epsilon(x) \right|
\]

Recall that \( R \in D_k \). Consequently, we can write

\[
f_j^\epsilon(x) = \sum_{Q \in D_j} (f, h_Q^\epsilon)h_Q^\epsilon(x) = (f, h_{R_{+j+k}}^\epsilon)h_{R_{+j+k}}^\epsilon(x) \quad (x \in R_k), \quad (31)
\]

if \( j \) is negative enough, that is, \( j \leq -M - 1 < -|k| \). If we use (31), then we obtain

\[
\sum_{\epsilon \in E} \left| \sum_{j=-\infty}^{\infty} f_j^\epsilon(x) - \sum_{j=-M}^{M} f_j^\epsilon(x) \right|
\]

\[
\leq \sum_{\epsilon \in E} \left| \sum_{j=M+1}^{\infty} \sum_{Q \in D_j, Q \subset R} (f, h_Q^\epsilon)h_Q^\epsilon(x) \right| + \sum_{\epsilon \in E} \left| \sum_{m=M+k+1}^\infty (f, h_{R_{+m}}^\epsilon)h_{R_{+m}}^\epsilon(x) \right|
\]

Thus, by the triangle inequality, we have

\[
\left\| \sum_{j=-\infty}^{\infty} f_j^\epsilon - \sum_{j=-M}^{M} f_j^\epsilon \right\|_{L^q(R)}
\]

\[
\leq \sum_{j=M+1}^{\infty} \sum_{Q \in D_j, Q \subset R} (f, h_Q^\epsilon)h_Q^\epsilon_{L^q(R)} \right| + \sum_{m=M+k+1}^{\infty} \| (f, h_{R_{+m}}^\epsilon)h_{R_{+m}}^\epsilon \|_{L^q(R)}.
\]

A geometric observation shows that

\[
\left\| (f, h_{R_{+m}}^\epsilon)h_{R_{+m}}^\epsilon \right\|_{L^q(R)}
\]

\[
= 2^{-mn/q} \left\| (f, h_{R_{+m}}^\epsilon)h_{R_{+m}}^\epsilon \right\|_{L^q(R_{+m})}
\]

\[
= 2^{-mn/p+kn(1/p-1/q)} \left\| (f, h_{R_{+m}}^\epsilon)h_{R_{+m}}^\epsilon \right\|_{L^q(R_{+m})}.
\]

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If we use this equality, then we have
\[
\left\| \sum_{j=-\infty}^{\infty} f_j^\varepsilon - \sum_{j=-M}^{M} f_j^\varepsilon \right\|_{L^q(R)} = \left\| \sum_{j=M+1}^{\infty} \sum_{Q \in \mathcal{D}_j, Q \subset R} \langle f, \chi_Q \rangle \cdot \langle a, h_Q^\varepsilon \rangle h_Q^\varepsilon(x) \cdot \langle a, h_Q^\varepsilon \rangle h_Q^\varepsilon(x) \right\|_{L^q(R)} + \sum_{m=M+k+1}^{\infty} 2^{-m/n/q} \left\| \langle f, h^\varepsilon_{R+m} \rangle h^\varepsilon_{R+m} \right\|_{L^q(R)}
\]
\[
\sim \left( \sum_{j=M+1}^{\infty} \left| \langle f_j^\varepsilon \rangle \right|^2 \right)^{\frac{1}{2}}_{L^q(R)} + 2^{-nM/p-nk/q} \left\| \sum_{j=-\infty}^{\infty} \left| f_j^\varepsilon \right| \right\|_{\mathcal{M}^p_q}.
\]
Thus, we obtain (29), which shows that (8) holds in the topology of \(L^q_{\text{loc}}\). As a consequence Theorem 1.1 is proved completely.

**Remark 3.2.** It may be interesting to compare Theorem 1.1 and Lemma 3.1 with the result in [13] Theorem 1.3. In [13] Theorem 1.3], we have proved that
\[
\|f\|_{\mathcal{M}^p_q} \sim \|M^f f\|_{\mathcal{M}^p_q} + \|f\|_{\mathcal{M}^p_q} (1 < q \leq p < \infty).
\]
Here \(M^f\) denotes the sharp maximal operator due to Fefferman, Stein and Stromberg.

### 3.2 Proof of Theorem 1.2

Let \(\varepsilon \in E\) be fixed. We also take a dyadic cube \(R\). Then it suffices from Theorem 1.1
\[
I := |R|^{\frac{1}{p} - \frac{1}{q}} \left\{ \int_R \left( \sum_{j=-\log_2 \ell(R)}^{\infty} \sum_{Q \in \mathcal{D}_j, Q \subset R} \langle f, \chi_Q \rangle \cdot \langle a, h_Q^\varepsilon \rangle h_Q^\varepsilon(x) \right)^{2} \right\}^{\frac{q}{2}} dx
\]
by \(C\|a\|_{\text{BMO}_{\text{dyadic}}} \|f\|_{\mathcal{M}^p_q}\) with constants independent of \(R, a\) and \(f\). By using Proposition 2.2 we obtain
\[
I = |R|^{\frac{1}{p} - \frac{1}{q}} \left\{ \int_R \left( \sum_{j=-\log_2 \ell(R)}^{\infty} \sum_{Q \in \mathcal{D}_j, Q \subset R} \langle \chi_R f, \chi_Q \rangle \cdot \langle a, h_Q^\varepsilon \rangle h_Q^\varepsilon(x) \right)^{2} \right\}^{\frac{q}{2}} dx
\]
Note that \(\{Q : Q \in \mathcal{D}_j\}\) partitions \(\mathbb{R}^n\). Consequently, we have
\[
I \lesssim \|a\|_{\text{BMO}_{\text{dyadic}}} |R|^{\frac{1}{p} - \frac{1}{q}} \left\{ \int_R \left( \sum_{j=-\log_2 \ell(R)}^{\infty} \sum_{Q \in \mathcal{D}_j, Q \subset R} \langle \chi_R f, \chi_Q \rangle |Q|^{\frac{1}{q}} h_Q^\varepsilon(x) \right)^{2} \right\}^{\frac{q}{2}} dx
\]
from the definition of \(\|a\|_{\text{BMO}_{\text{dyadic}}}\). If we use the definition of the Morrey norm [4] crudely, then we have
\[
I \lesssim \|a\|_{\text{BMO}_{\text{dyadic}}} |R|^{\frac{1}{p} - \frac{1}{q}} \left( \int_R |f(x)|^q dx \right)^{\frac{1}{q}} \lesssim \|a\|_{\text{BMO}_{\text{dyadic}}} \|f\|_{\mathcal{M}^p_q}.
\]
Therefore, since \(R\) is arbitrary, the proof of Theorem 1.2 is complete.
3.3 Proof of Theorem 1.3

We shall make use of the following estimate in the proof of Theorem 1.4 as well as Theorem 1.3. Actually Proposition 3.3 is a little stronger than Theorem 1.3.

**Proposition 3.3.** Let \( 0 < \alpha < n \), \( 1 < q \leq p < \infty \) and \( 1 < t \leq s < \infty \). Assume

\[
\frac{1}{s} = \frac{1}{p} - \frac{\alpha}{n}, \quad \frac{t}{s} = \frac{q}{p}.
\]

Then

\[
\| I_{\alpha, \text{dyadic}} f \|_{M^q_s} \lesssim \left\| \sum_{j=-\infty}^{\infty} \sum_{Q \in \mathcal{D}_j} |Q|^\frac{\alpha}{n} \langle f, h_{\xi_Q}^\varepsilon \rangle h_{\xi_Q}^\varepsilon \right\|_{M^q_s} \lesssim \| f \|_{M^p_q}.
\]

**(32)**

**Proof.** The proof is simple: Let \( x \in \mathbb{R}^n \) and \( j \in \mathbb{Z} \) be fixed and choose \( Q_0 \in \mathcal{D}_j \) so that \( x \in Q_0 \). If we use a simple inequality

\[
\left| \sum_{Q \in \mathcal{D}_j} |Q|^\frac{\alpha}{n} \langle f, h_{\xi_Q}^\varepsilon \rangle h_{\xi_Q}^\varepsilon (x) \right| \leq \ell(Q_0)^\alpha m_{Q_0}(|f|) \leq \ell(Q_0)^\alpha - \frac{\alpha}{p} \parallel f \parallel_{M^q_s} = 2^{-j(\alpha - \frac{\alpha}{p})} \parallel f \parallel_{M^q_s},
\]

we have

\[
\sum_{j=-\infty}^{\infty} \sum_{Q \in \mathcal{D}_j} |Q|^\frac{\alpha}{n} \langle f, h_{\xi_Q}^\varepsilon \rangle h_{\xi_Q}^\varepsilon \leq \sum_{j=-\infty}^{\infty} 2^{-j\alpha} \min \left( \sum_{Q \in \mathcal{D}_j} \langle f, h_{\xi_Q}^\varepsilon \rangle h_{\xi_Q}^\varepsilon, 2^{jn} \parallel f \parallel_{M^q_s} \right)
\]

\[
\leq \sum_{j=-\infty}^{\infty} 2^{-j\alpha} \min \left( \sup_{l \in \mathbb{Z}} \left| \sum_{Q \in \mathcal{D}_j} \langle f, h_{\xi_Q}^\varepsilon \rangle h_{\xi_Q}^\varepsilon \right|, 2^{jn} \parallel f \parallel_{M^q_s} \right)
\]

\[
\lesssim \| f \|_{M^q_s} \sup_{l \in \mathbb{Z}} \left| \sum_{Q \in \mathcal{D}_j} \langle f, h_{\xi_Q}^\varepsilon \rangle h_{\xi_Q}^\varepsilon \right|^{\frac{p}{p}}.
\]

If we use this pointwise estimate, then we obtain the desired estimate. \( \square \)

**Remark 3.4.** It may be interesting compare Proposition 3.3 with the following result. Let \( \varphi \in \mathcal{S} \) be chosen so that \( \varphi(\xi) = 1 \) if \( 2 \leq |\xi| \leq 4 \) and that \( \varphi(\xi) = 0 \) if \( |\xi| \leq 1 \) or if \( |\xi| \geq 8 \). Then in [S], we have established

\[
\left\| \sum_{j=-\infty}^{\infty} |2^{jn} F^{-1}[\varphi(2^{-j} \cdot) F f]| \right\|_{M^q_s} \lesssim \| f \|_{M^p_q},
\]

**Estimate** (35) admits an extension to Triebel-Lizorkin-Morrey spaces as we did in [S].
3.4 Proof of Theorem 1.4

We freeze $\varepsilon \in E$ for a while. By definition of $I_{\alpha,\text{dyadic}} f$, we obtain

$$\sum_{j=-\infty}^{\infty} \sum_{Q \in D_j} \langle a, h_Q^\varepsilon \rangle \{h_Q^\varepsilon(x)I_{\alpha,\text{dyadic}} f(x) - I_{\alpha,\text{dyadic}} [h_Q^\varepsilon f](x)\}$$

$$= \sum_{j=-\infty}^{\infty} \sum_{Q \in D_j} \sum_{\varepsilon' \in E} \sum_{l=-\infty}^{\infty} \sum_{R \in D_l} \langle a, h_Q^\varepsilon \rangle \langle f, h_R^\varepsilon \rangle \{h_Q^\varepsilon(x)I_{\alpha,\text{dyadic}} h_R^\varepsilon(x) - I_{\alpha,\text{dyadic}} [h_Q^\varepsilon h_R^\varepsilon](x)\}.$$ 

Observe that, if $Q \supseteq R$, then $h_Q^\varepsilon h_R^\varepsilon = m_R(h_Q^\varepsilon)h_R^\varepsilon$ and hence

$$h_Q^\varepsilon(x)I_{\alpha,\text{dyadic}} h_R^\varepsilon(x) = I_{\alpha,\text{dyadic}} [h_Q^\varepsilon h_R^\varepsilon](x). \quad (36)$$

Therefore, we have

$$\sum_{j=-\infty}^{\infty} \sum_{Q \in D_j} \langle a, h_Q^\varepsilon \rangle \{h_Q^\varepsilon(x)I_{\alpha,\text{dyadic}} f(x) - I_{\alpha,\text{dyadic}} [h_Q^\varepsilon f](x)\}$$

$$= \sum_{j=-\infty}^{\infty} \sum_{Q \in D_j} \sum_{\varepsilon' \in E} \sum_{l=-\infty}^{\infty} \sum_{R \in D_l} \langle a, h_Q^\varepsilon \rangle \langle f, h_R^\varepsilon \rangle \{h_Q^\varepsilon(x)I_{\alpha,\text{dyadic}} h_R^\varepsilon(x) - I_{\alpha,\text{dyadic}} [h_Q^\varepsilon h_R^\varepsilon](x)\}.$$ 

If $Q = R$ and $\varepsilon \neq \varepsilon'$, then we obtain

$$h_Q^\varepsilon(x)I_{\alpha,\text{dyadic}} h_R^\varepsilon(x) = I_{\alpha,\text{dyadic}} [h_Q^\varepsilon h_R^\varepsilon](x). \quad (37)$$

Let us write

$$I_1(x) := \sum_{j=-\infty}^{\infty} \sum_{Q \in D_j} \sum_{\varepsilon' \in E} \sum_{l=-\infty}^{\infty} \sum_{R \in D_l} |R|^\frac{n}{n} \langle a, h_Q^\varepsilon \rangle \langle f, h_R^\varepsilon \rangle h_Q^\varepsilon(x) h_R^\varepsilon(x),$$

$$I_2(x) := \sum_{j=-\infty}^{\infty} \sum_{Q \in D_j} \sum_{\varepsilon' \in E} \sum_{l=-\infty}^{\infty} \sum_{R \in D_l} |Q|^\frac{n}{n} \langle a, h_Q^\varepsilon \rangle \langle f, h_R^\varepsilon \rangle h_Q^\varepsilon(x) h_R^\varepsilon(x),$$

$$II(x) := \sum_{j=-\infty}^{\infty} \sum_{Q \in D_j} \langle a, h_Q^\varepsilon \rangle \langle f, h_Q^\varepsilon \rangle |Q|^\frac{n}{n} |h_Q^\varepsilon(x)|^2,$$

$$III(x) := \sum_{j=-\infty}^{\infty} \sum_{Q \in D_j} \langle a, h_Q^\varepsilon \rangle \langle f, h_Q^\varepsilon \rangle I_{\alpha,\text{dyadic}}[|h_Q^\varepsilon|^2](x).$$

Note that both $I_1$ and $III$ have another expression:

$$I_1(x) = \sum_{j=-\infty}^{\infty} \left( \sum_{Q \in D_j} \langle I_{\alpha,\text{dyadic}} f, \chi_Q \rangle \cdot \langle a, h_Q^\varepsilon \rangle \frac{h_Q^\varepsilon}{|Q|} \right),$$

$$III(x) = I_{\alpha,\text{dyadic}} \left[ \sum_{j=-\infty}^{\infty} \sum_{Q \in D_j} \langle a, h_Q^\varepsilon \rangle \langle f, h_Q^\varepsilon \rangle |h_Q^\varepsilon|^2 \right](x).$$
Hence, we have
\[
\begin{align*}
\sum_{j=-\infty}^{\infty} \sum_{Q \in \mathcal{D}_j} \langle a, h_Q^j \rangle (h_Q^j(x)I_{\alpha, \text{dyadic}}f(x) - I_{\alpha, \text{dyadic}}[h_Q^j f](x)) \\
= \sum_{j=-\infty}^{\infty} \sum_{Q \in \mathcal{D}_j} \sum_{j'=1}^{j-1} \sum_{\epsilon' = 0}^{\epsilon} \sum_{l = -\infty}^{l} \sum_{R \in \mathcal{D}_l} \langle a, h_Q^j \rangle \langle f, h_R^\epsilon \rangle \{h_Q^j(x)I_{\alpha, \text{dyadic}}h_R^\epsilon(x) - I_{\alpha, \text{dyadic}}[h_Q^j h_R^\epsilon](x)\} \\
+ \sum_{j=-\infty}^{\infty} \sum_{Q \in \mathcal{D}_j} \langle a, h_Q^j \rangle \{h_Q^j(x)I_{\alpha, \text{dyadic}}h_Q^j(x) - I_{\alpha, \text{dyadic}}[h_Q^j h_Q^j](x)\} \\
= I_1(x) - I_2(x) + II(x) - III(x).
\end{align*}
\]

Let us start with dealing with $I_1$. If we invoke again Theorem 1.2 then we have
\[
\|I_1\|_{\mathcal{M}_t^p} \lesssim \|a\|_{\text{BMO}_{\text{dyadic}}} \|I_{\alpha, \text{dyadic}} f\|_{\mathcal{M}_t^p} \lesssim \|a\|_{\text{BMO}_{\text{dyadic}}} \|f\|_{\mathcal{M}_t^p}.
\]
Since
\[
I_2(x) = \sum_{\epsilon' \in E} \sum_{j=-\infty}^{\infty} \sum_{l = -\infty}^{l} \sum_{R \in \mathcal{D}_l} \langle R f, a, h_{Q}^{\epsilon'} \rangle \langle f, h_{R}^{\epsilon} \rangle h_Q^j(x)h_R^\epsilon(x)
\]
we have by Theorem 1.2
\[
\|I_2\|_{\mathcal{M}_t^p} = \left( \sum_{j=-\infty}^{\infty} \sum_{Q \in \mathcal{D}_j} \langle a, h_Q^j \rangle \langle f, h_Q^j \rangle h_Q^j(x) \right) \|I_{\alpha, \text{dyadic}} f\|_{\mathcal{M}_t^p} \lesssim \|a\|_{\text{BMO}_{\text{dyadic}}} \|I_{\alpha, \text{dyadic}} f\|_{\mathcal{M}_t^p}.
\]
If we invoke Theorem 1.3 then we have
\[
\|I_2\|_{\mathcal{M}_t^p} \lesssim \|a\|_{\text{BMO}_{\text{dyadic}}} \|f\|_{\mathcal{M}_t^p}.
\]
By Proposition 3.3 we obtain
\[
\|II\|_{\mathcal{M}_t^p} = \left( \sum_{j=-\infty}^{\infty} \sum_{Q \in \mathcal{D}_j} |Q|^\alpha \langle a, h_Q^j \rangle \langle f, h_Q^j \rangle h_Q^j(x) \right) \|I_{\alpha, \text{dyadic}} f\|_{\mathcal{M}_t^p} \lesssim \|a\|_{\text{BMO}_{\text{dyadic}}} \|f\|_{\mathcal{M}_t^p}.
\]
Next, by Proposition 3.3 and equality $I_{\alpha, \text{dyadic}}[h_Q^j h_Q^j](x) = c_a |Q|^\alpha \chi_Q(x)$, we have
\[
\|III\|_{\mathcal{M}_t^p} \lesssim \left( \sum_{j=-\infty}^{\infty} \sum_{Q \in \mathcal{D}_j} \langle a, h_Q^j \rangle \langle f, h_Q^j \rangle |Q|^\alpha \chi_Q \right) \|I_{\alpha, \text{dyadic}} f\|_{\mathcal{M}_t^p} \lesssim \|a\|_{\text{BMO}_{\text{dyadic}}} \|f\|_{\mathcal{M}_t^p}.
\]
Thus, the proof of Theorem 1.4 is complete.
3.5 Proof of Theorem 1.5

Let \( U \in \mathcal{D} \) be fixed. Then we have

\[
[a, I_{a, \text{dyadic}}] h^{\varepsilon''}_U(x) = \sum_{\varepsilon \in E} \sum_{j = -\infty}^{\infty} \sum_{Q \subseteq U} \langle |Q|^\frac{2}{p} - |U|^\frac{2}{p} \rangle \langle a, h^{\varepsilon}_Q \rangle h^{\varepsilon}_Q(x) h^{\varepsilon''}_U(x) + \langle a, h^{\varepsilon''}_U \rangle \langle f, h^{\varepsilon''}_U \rangle \langle |U|^\frac{2}{p} h^{\varepsilon''}_U(x) h^{\varepsilon''}_U(x) - I_{a, \text{dyadic}}[h^{\varepsilon''}_Q h^{\varepsilon''}_U](x) \rangle.
\]

By virtue of the non-homogeneous wavelet expansion (see (19)), we obtain

\[
\| [a, I_{a, \text{dyadic}}] \|_{B(\mathcal{M}^p_t, \mathcal{M}^q_t)} \geq \| [a, I_{a, \text{dyadic}}] h^{\varepsilon''}_U \|_{\mathcal{M}^q_t} |U|^{-\frac{1}{p} + \frac{1}{2}} \cdot \left( \sum_{j = -\infty}^{\infty} \sum_{Q \subseteq U} \langle |Q|^\frac{2}{p} - |U|^\frac{2}{p} \rangle \langle a, h^{\varepsilon}_Q \rangle h^{\varepsilon}_Q \right) \|_{\mathcal{M}^q_t}.
\]

By Theorem 1.1, we have

\[
\| [a, I_{a, \text{dyadic}}] \|_{B(\mathcal{M}^p_t, \mathcal{M}^q_t)} \geq \| [a, I_{a, \text{dyadic}}] h^{\varepsilon''}_U \|_{\mathcal{M}^q_t} |U|^{-\frac{1}{p} + \frac{1}{2}} \cdot \left( \sum_{j = -\infty}^{\infty} \sum_{Q \subseteq U} \langle a, h^{\varepsilon}_Q \rangle h^{\varepsilon}_Q \right) \|_{\mathcal{M}^q_t}.
\]

By the Hölder inequality, we have

\[
\frac{1}{|U|} \int_U \left( \sum_{j = -\infty}^{\infty} \sum_{Q \subseteq U} \langle a, h^{\varepsilon}_Q \rangle h^{\varepsilon}_Q(x) \right) \, dx \lesssim \| [a, I_{a, \text{dyadic}}] \|_{B(\mathcal{M}^p_t, \mathcal{M}^q_t)}.
\]

Therefore, we obtain

\[
m_U \left( \left( \sum_{j = -\infty}^{\infty} \sum_{Q \subseteq U} \langle a, h^{\varepsilon}_Q \rangle h^{\varepsilon}_Q \right) \right) \lesssim \frac{\| [a, I_{a, \text{dyadic}}] h^{\varepsilon''}_U \|_{\mathcal{M}^q_t}}{\| h^{\varepsilon''}_U \|_{\mathcal{M}^q_t}}.
\]

(38)

for all \( U \in \mathcal{D} \). Denote by \( U^* \) the dyadic parent of \( U \), that is, the smallest dyadic cube engulfing \( U \). With \( U \) replaced by \( U^* \) above, we obtain

\[
m_U \left( \left( \sum_{j = -\infty}^{\infty} \sum_{Q \subseteq U} \langle a, h^{\varepsilon}_Q \rangle h^{\varepsilon}_Q \right) \right) \lesssim \frac{\| [a, I_{a, \text{dyadic}}] h^{\varepsilon''}_U \|_{\mathcal{M}^q_t}}{\| h^{\varepsilon''}_U \|_{\mathcal{M}^q_t}}.
\]

(39)

It follows from the definition of the operator norm \( \| [a, I_{a, \text{dyadic}}] \|_{\mathcal{M}^p_t \rightarrow \mathcal{M}^q_t} \) that

\[
m_U \left( \left( \sum_{j = -\infty}^{\infty} \sum_{Q \subseteq U} \langle a, h^{\varepsilon}_Q \rangle h^{\varepsilon}_Q \right) \right) \lesssim \| [a, I_{a, \text{dyadic}}] \|_{\mathcal{M}^p_t \rightarrow \mathcal{M}^q_t}.
\]

(40)
If we take the supremum over $U \in \mathcal{D}$ in (40), then we obtain

$$\|a\|_{\text{BMO}_{\text{dyadic}}} \lesssim \|[a, I_{\alpha, \text{dyadic}}]\|_{B(M^*_p, M^0_p)}.$$  

Thus, the proof is complete.

### 3.6 Proof of Theorem 1.6

By Theorem 1.1 (iii), we see that $\{h^\varepsilon_Q\}_{Q \in \mathcal{D}, \varepsilon \in \mathcal{E}}$ is dense in $H^r_{p', 0}$. Thus, to check (i) and (ii), we need only to prove

$$\|I_{\alpha, \text{dyadic}}f\|_{H^r_{p', 0}} + \|[a, I_{\alpha, \text{dyadic}}]f\|_{H^r_{p', 0}} \lesssim \|f\|_{H^r_{p', 0}}$$

for all $f \in \text{Span}(\{h^\varepsilon_Q\}_{Q \in \mathcal{D}, \varepsilon \in \mathcal{E}})$. If we assume $f \in \text{Span}(\{h^\varepsilon_Q\}_{Q \in \mathcal{D}, \varepsilon \in \mathcal{E}})$, then from the definition we have

$$I_{\alpha, \text{dyadic}}f, [a, I_{\alpha, \text{dyadic}}]f \in H^r_{p'}.$$  

Observe that (41) counts in that we can obtain (41) only by using the discrete fractional integral operators. Consequently, if we invoke Theorem 1.1, we obtain

$$\|I_{\alpha, \text{dyadic}}f\|_{H^r_{p', 0}} + \|[a, I_{\alpha, \text{dyadic}}]f\|_{H^r_{p', 0}} \lesssim \|f\|_{H^r_{p', 0}}.$$  

Thus, the proof is now complete.

**Remark 3.5.** A usual averaging procedure yields the following corollaries. For this technique we refer to [5].

**Corollary 3.6.** Maintain the same conditions on the parameters $p, p_0, r, r_0, \alpha$. If a function $a$ belongs to $\text{BMO}$, then the following boundedness is true:

$$\|I_{\alpha}f\|_{H^r_{p', 0}} + \|[a, I_{\alpha}]f\|_{H^r_{p', 0}} \lesssim \|f\|_{H^r_{p', 0}},$$

where $I_{\alpha}$ is given by

$$I_{\alpha}f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x - y|^{n-\alpha}} \, dy.$$  

As for $I_{\alpha}$ we made an alternative approach in [9, Theorem 3.1] and [10, Theorem 3.1].
3.7 Proof of Theorem 1.7

"If part" is a direct consequence of Theorem 1.5. Let us prove the converse. To this end, we need the following fundamental lemma.

**Lemma 3.7.** Let X and Y be Banach spaces. Suppose that we are given a compact linear operator $T : X \to Y$. If $\{f_j\}_{j \in \mathbb{N}}$ is a sequence in $X^*$ that is weak-* convergent to 0. Then $\{T^* f_j\}_{j \in \mathbb{N}}$ is norm-convergent to 0.

Now let us prove $a \in \text{VMO}_{\text{dyadic}}$ assuming that $[a, I_{\text{dyadic}}]$ is compact.

Let us set

$$[a, I_{\text{dyadic}}] \geq L := \lim_{M \to \infty} \sum_{Q \in D_j} \langle a \cdot I_{\text{dyadic}} f - I_{\text{dyadic}}[a \cdot f], h_Q^\varepsilon \rangle h_Q^\varepsilon. \quad (42)$$

Then we have

$$\| [a, I_{\text{dyadic}}] \|_{\mathcal{M}_p^\varepsilon \to \mathcal{M}_p^\varepsilon} \lesssim \sum_{U \in D} \sup_{U \in \mathcal{D}_0} \left( \sum_{j=L}^{\infty} \sum_{Q \subset U} \langle a, h_Q^\varepsilon h_Q^\varepsilon \rangle \right). \quad (43)$$

by virtue of Theorem 1.4. The triangle inequality yields

$$\sup_{U \in \mathcal{D}_0} \left( \sum_{j=L}^{\infty} \sum_{Q \subset U} \langle a, h_Q^\varepsilon h_Q^\varepsilon \rangle \right) = \sup_{U \in \mathcal{D}_0} \left( \sum_{j=L}^{\infty} \sum_{Q \subset U} \langle a, h_Q^\varepsilon h_Q^\varepsilon \rangle \right).$$

According to (39) and Lemma 3.7 we have

$$\lim_{L \to \infty} \sup_{U \in \mathcal{D}} \left( \sum_{j=L}^{\infty} \sum_{Q \subset U} \langle a, h_Q^\varepsilon h_Q^\varepsilon \rangle \right) = 0.$$ \quad (44)

Thus, assuming that $[a, I_{\text{dyadic}}]$ is compact, we have

$$\lim_{L \to \infty} [a, I_{\text{dyadic}}] \geq L = 0 \quad (44)$$

in the operator topology. Also, we set

$$[a, I_{\text{dyadic}}] \leq -L := \lim_{M \to \infty} \sum_{Q \in D_j} \langle a \cdot I_{\text{dyadic}} f - I_{\text{dyadic}}[a \cdot f], h_Q^\varepsilon h_Q^\varepsilon \rangle. \quad (45)$$

Then we have

$$\| [a, I_{\text{dyadic}}] \leq -L \|_{\mathcal{M}_p^\varepsilon \to \mathcal{M}_p^\varepsilon} \lesssim \sum_{U \in \mathcal{D}} \sup_{U \in \mathcal{D}_0} \left( \sum_{j=-L}^{-\infty} \sum_{Q \subset U} \langle a, h_Q^\varepsilon h_Q^\varepsilon \rangle \right). \quad (46)$$
by virtue of Theorem 1.4. Note that
\[
\sup_{U \in \mathcal{D}} m_U \left( \left| \sum_{j=-\infty}^{-L} \sum_{Q \in \mathcal{D}_j, Q \subset U} \langle a, h^\xi_Q \rangle h^\xi_Q \right| \right) \\
= \sup \left\{ m_U \left( \left| \sum_{j=-\infty}^{-L} \sum_{Q \in \mathcal{D}_j, Q \subset U} \langle a, h^\xi_Q \rangle h^\xi_Q \right| \right) : U \in \bigcup_{\nu=-\infty}^{-L} \mathcal{D}_\nu \right\}
\]
in order that \( Q \subset U \) actually happens. Thus, assuming that \([a, \mathcal{I}_\alpha, \text{dyadic}]\) is compact, we have
\[
\lim_{L \to \infty} [a, \mathcal{I}_\alpha, \text{dyadic}] \leq -L = 0 \tag{47}
\]
again in the operator topology. From (44) and (47) we have
\[
\lim_{L \to \infty} \|a - a_{(L)}\|_{\text{BMO}_\text{dyadic}} = 0, \tag{48}
\]
if we write
\[
a_{(L)} := \sum_{\epsilon \in E} \sum_{-L}^{L} \sum_{Q \in \mathcal{D}_j} \langle a, h^\xi_Q \rangle h^\xi_Q.
\]
It is not so hard to see that
\[
\lim_{R \to \infty} \sup \{ |\langle a, h^\xi_{Q_{\nu m}} \rangle| : m \in \mathbb{Z}^n, |m| \geq R \} = 0 \tag{49}
\]
for all \( \nu \in \mathbb{Z} \) if \([a, \mathcal{I}_\alpha, \text{dyadic}]\) is compact. Thus, it follows that
\[
\lim_{R \to \infty} \left\| \sum_{m \in \mathbb{Z}^n, |m| > R} \langle a, h^\xi_{Q_{jm}} \rangle h^\xi_{Q_{jm}} \right\|_{\text{BMO}_\text{dyadic}} = 0 \tag{50}
\]
for all \( j \in \mathbb{Z} \).

From (50) we learn that \( a_{(L)} \in \text{VMO}_\text{dyadic} \), which in turn yields \( a \in \text{VMO}_\text{dyadic} \) by virtue of (48).

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