The space of circular planar electrical networks

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Abstract

We discuss several parametrizations of the space of circular planar electrical networks. For any circular planar network we associate a canonical minimal network with the same response matrix, called a “standard” network. The conductances of edges in a standard network can be computed as a biratio of Pfaffians constructed from the response matrix. The conductances serve as coordinates that are compatible with the cell structure of circular planar networks in the sense that one conductance degenerates to 0 or ∞ when moving from a cell to a boundary cell.

We also show how to test if a network is well-connected by checking that \( \binom{n}{2} \) minors of the response matrix are positive; Colin de Verdière had previously shown that it was sufficient to check the positivity of exponentially many minors. For standard networks with \( m \) edges, positivity of the conductances can be tested by checking the positivity of \( m + 1 \) Pfaffians.

1 Introduction

A circular planar network (CPN, or simply network in this paper) is a finite graph \( G = (V, E) \) embedded in the plane with a distinguished set of vertices \( \mathcal{N} \subset V \), called nodes, on the outer face, and a positive real-valued function \( c : E \to \mathbb{R}_{>0} \) on the edges. The value \( c(e) \) is the conductance of the edge.

On a network, the Laplacian operator \( \Delta : \mathbb{R}^V \to \mathbb{R}^V \) is defined by \( \Delta(f)(v) = \sum_{v'} c_{v,v'}(f(v) - f(v')) \) where the sum is over neighbors \( v' \) of \( v \).

Given a network and a function \( u \) on \( \mathcal{N} \), let \( f \) be the unique harmonic extension to \( V \) of \( u \), and define \( L(u) = -\Delta f |_{\mathcal{N}} \). The linear function \( L : \mathbb{R}^\mathcal{N} \to \mathbb{R}^\mathcal{N} \) is the response matrix of the network. It is a symmetric, negative semidefinite matrix with (if \( G \) is connected) kernel consisting of the constant functions [CdV94].

Circular planar networks arise in a number of different situations: in electrical impedance tomography [BDGV08], in the study of the (positive part of the) orthogonal Grassmannian [LP11, ALT13a, ALT13b], in statistical mechanics [KW11a], in probability [Lyo14], and in string theory (see e.g., [HWX14, KL14]).

Their systematic study was first begun by Colin de Verdière [CdV94] and Curtis, Ingerman, Mooers, and Morrow [CMM94, CIM98], who studied the space \( \Omega_n \) of response matrices of all

\*Research partially supported by Microsoft Research and partially supported by NSF grant DMS-1208191.
networks with \( n \) nodes, proving that it is a semialgebraic set whose interior is homeomorphic to a ball of dimension \( n(n-1)/2 \). They showed that \( \Omega_n \) was defined by inequalities \( \det L_A^B \geq 0 \), where \( A, B \subset \mathcal{N} \) run over noninterlaced subsets of \( \mathcal{N} \), i.e., \( A \) and \( B \) are contained in disjoint intervals in the natural circular ordering of \( \mathcal{N} \), and \( L_A^B \) is the minor of \( L \) with rows \( A \) and columns \( B \).

For minimal, well-connected networks (see definitions below) it was shown that the edge conductances parametrize the interior \( \Omega_n^+ \) of \( \Omega_n \).

The boundary of \( \Omega_n \) has an interesting combinatorial structure. Lam and Pylyavskyy showed that \( \Omega_n \) has the structure of a cell complex \([LP11]\). The cells are parametrized by equivalence classes of minimal networks; two networks are equivalent if they can be obtained from one another by Y-\( \Delta \) moves. The dimension of the cell is equal to the number of edges of the minimal network. They conjectured that the cell structure is actually a regular CW complex, that is, the closure of each cell is homeomorphic to a closed ball, and the attaching maps from an \( m \)-cell to an \((m-1)\)-cell arise by sending an appropriate conductance to 0 or \( \infty \). Recently Lam showed that cell complex is Eulerian \([Lam14b]\).

For each cell of dimension \( m \), we define a canonical minimal network in its corresponding network equivalence class, called a standard network. This definition is closely related to the construction \([HW13]\), however we found our formulation better suited to our needs. We associate to a (standard or nonstandard) minimal network a set of \( B \) variables at each face and vertex. The conductances are biratios of the \( B \) variables at the adjacent vertices and faces. The \( B \) variables transform under Y-\( \Delta \) moves via the cube recurrence \([CS04, GK13]\). For standard minimal networks, the \( B \) variables (and a closely related set of variables, the tripod variables, see below) can be computed in terms of Pfaffians and determinants involving the response matrix. In particular, the reconstruction map from the response matrix to the set of conductances is given as an explicit rational function. We previously gave this explicit rational function in the case of well-connected networks \([KW09]\), and a recursive reconstruction procedure was previously given in \([CIM98]\) and studied further in \([CM02, Rus03]\) (see also \([Joh12]\)).

Another way to parametrize well-connected networks is through a collection of \( \binom{n}{2} \) central minors of the \( L \) matrix. Positivity of the central minors implies positivity of all non-interlaced minors. This result is analogous to testing whether an \( n \times n \) matrix is totally positivity by testing only \( n^2 \) minors (see \([FZ00]\)), but here there are \( n \) nodes each of which can index either a row or a column, while for the total positivity tests, there are \( n \) nodes that index rows and another \( n \) nodes that index columns.

Similar parametrizations (using minors of the \( L \) matrix) seem to hold for general minimal networks, but it is an open problem to find such parametrizations in general; see \([ALT13b]\) for some work in this direction.
2 Background

Background in this section comes from [CdV94, CM00, Ken12, LW08].

2.1 Dual network

Given a circular planar network \( G \) with \( n \) nodes, the dual network is the network \( G^* \) with \( n \) nodes, with a node between every two adjacent nodes of \( G \), constructed from the dual graph of \( G \) embedded in the disk. (That is, the vertices of \( G^* \) are the regions of the disk which are bounded by edges of \( G \) or the boundary of the disk, see Figure 2.) The conductances of an edge and its dual edge are reciprocals.

2.2 Equivalence

Two networks are topologically equivalent if they have the same number of nodes and one can be obtained from the other by electrical transformations, see Figure 1, disregarding conductances.

![Figure 1: The electrical transformations: removing a dead branch (a degree-1 non-node vertex), removing a self-loop, combining edges in series (when the central vertex is not a node), combining edges in parallel, and a Y-∆ transformation (when the central vertex is not a node). This figure first appeared in [KW09].](image)

A network is minimal (also called reduced) if it has the smallest number of edges in its topological equivalence class. Two minimal equivalent networks can be obtained from one another using only Y-∆ moves.

Two networks are electrically equivalent if they have the same response matrix. In [CdVGV96] it is shown that networks are electrically equivalent if and only if they can be obtained from one another by electrical transformations.

2.3 Medial graph and strand matching

The medial graph of a network is a degree-4 graph with a vertex for every edge of \( G \), and an edge connecting two vertices if the corresponding edges of \( G \) are consecutive around a face of \( G \) (see Figure 2). For each node of \( G \) it is customary to break the edge of the medial graph separating it from \( \infty \) into two half-edges, called stubs. In this way the medial graph consists of \( n \) strands, which are paths in the medial graph which go straight (neither turning left nor right) through each vertex; strands begin and end at stubs.

A circular planar network is minimal if there are no closed strands, strands do not cross themselves, and two strands cross at most once [CdVGV96].
Figure 2: A strand diagram with 16 stubs. The strands divide the disk into cells, which are alternately colored blue or red. The blue cells are the vertices of the blue network, which has 8 nodes, and similarly the red cells define the red network, which is dual (within the disk) to the blue network. The vertices of the medial graph are where the blue and red network edges intersect, and the edges of the medial graph can be drawn where the strands are.

To each node $i$ of a minimal circular planar network there are two stubs of the medial graph, one just to the left of $i$ (in the circular order) and one just to the right of $i$. We label these two stubs $2i - 1$ and $2i$. There is an fixed-point free involution of the stubs defined by following the strands from one end to the other. This involution is called the **strand matching** $\pi = \pi(G)$ of the network.

Note that Y-∆ moves do not change the strand matching $\pi$. Thus $\pi$ is a function only of the topological equivalence class of the (minimal) network.

Every fixed-point free involution on $\{1, \ldots, 2n\}$ is the strand matching of some electrical network with $n$ nodes. This can be seen by taking $2n$ points in generic position on the circle; join them in pairs using chords according to the strand matching $\pi$. The chords form the medial graph of a network on $n$ nodes with strand matching $\pi$. For some strand matchings, some of the boundary nodes of the network will be glued together or in different components — the associated networks are called cactus networks in [Lam14a].
2.4 Groves and partitions

Given a network, a grove is a set of edges with the property that it contains no cycles and every component contains at least one node. A grove is similar in concept to an essential spanning forest on an infinite graph, which is a set of edges containing no cycles for which every connected component is infinite, in the sense that every tree reaches the “boundary”.

Each grove partitions the nodes according to its connected components. Associated to a partition \( \tau \) of the nodes is its partition sum

\[
Z_\tau = \sum_{T \in \tau} wt(T),
\]

where the sum is over groves having partition \( \tau \), and \( wt(T) \) is the product of the conductances of edges in \( T \).

In [KW11a] we showed how to compute \( Z_\tau \) (appropriately normalized) for any circular planar network, for any partition \( \tau \), as a polynomial of entries in \( L \). In [KW11c] we showed that \( Z_\tau \) (appropriately normalized) is a linear combination of minors of \( L \).

3 Standard networks

The standard well-connected network on \( n \) nodes (defined in [CdV94]) is the network illustrated in Figure 3. We construct a network of similar form for each strand matching \( \pi \); we call these standard networks.

![Figure 3: The standard well-connected network on 8 nodes.](image)

A Dyck path of order \( n \) is the graph of a simple random walk on \( \mathbb{Z} \) which starts at the origin, remains nonnegative, and returns to the origin after \( 2n \) steps. Given two Dyck paths \( \lambda \) and \( \mu \) of order \( n \), the domination partial order is defined by \( \lambda \preceq \mu \) if at each horizontal position, \( \mu \) is at least as high as \( \lambda \). If \( \lambda \preceq \mu \), then the region between them is denoted \( \lambda/\mu \) and is called a skew Young diagram, or simply a skew shape. A Dyck tiling (see [KW11b, SZJ12], where these were called cover-inclusive Dyck tilings) is a tiling of \( \lambda/\mu \) with Dyck tiles which are fattened Dyck paths, satisfying the following “cover-inclusive” constraint: if two tiles are vertically adjacent (in the sense that some square of one lies in the same column as a square of the other, and one position above it), then the horizontal extent of the upper tile is a subset of the horizontal extent of the lower tile. See Figure 4 for an example of a Dyck tiling. Dyck tilings have also been studied in [Kim12, KMPW14, KW11c, Fay13, FN12].
There is a bijection between Dyck tilings and perfect matchings of \(\{1, \ldots, 2n\}\) \cite{Kim12, KMPW14}. Here we need a slightly different bijection, which we now describe. To the Dyck tiling we associate a strand diagram as follows (see Figure 5). Along the upper Dyck path \(\mu\) of the tiling, there is a strand starting or ending in the center of every edge. Each Dyck tile contains two medial strands that cross once: the strand entering at the lower edge adjacent to the left-most point of the tile and exiting at the upper edge adjacent to the right-most point of the tile, and the strand entering at the upper edge adjacent to the left-most point of the tile and exiting at the lower edge adjacent to the right-most point of the tile. A tile may contain additional strands that pass through it horizontally, without crossing any other strands within the tile. From this strand diagram we can build the electrical network \(G\) and its dual network \(G^*\), as illustrated in Figure 5.

In the special case where the lower path \(\lambda\) is minimal (the zigzag path) and the upper path \(\mu\) is maximal, and every Dyck tile is a single box, then every pair of strands cross, and the resulting network is the standard well-connected network. The networks we obtain from other Dyck tilings are analogous to the standard networks in the well-connected case, so we call them standard networks.

For every crossing in the strand diagram, there is either a horizontal edge of \(G\) and vertical dual edge of \(G^*\), or else a vertical edge of \(G\) and a horizontal edge of \(G^*\). By the horizontal conductance of an edge (or crossing), we mean the conductance of either the edge or its dual, whichever one is horizontal, and similarly the vertical conductance is the conductance of whichever one is vertical (and is the reciprocal of the horizontal conductance).

We now describe, conversely, how to construct a Dyck tiling from a strand matching. This description is an inductive procedure, which is illustrated in Figure 6. The base case is \(n = 0\), in which the trivial strand matching, containing no strands or stubs, corresponds to the trivial Dyck tiling, which has no tiles, and whose upper and lower Dyck paths have length \(2n = 0\). We can build up any strand matching starting from the trivial matching by a sequence of two types of elementary moves, and while doing this, we build up the Dyck tiling and its associated rectilinear strand diagram starting from the trivial tiling. The two moves to build up a strand matching are (1) introduce into the matching a new strand with adjacent stubs, either at the beginning or end or between existing stubs, and (2) crossing a pair of adjacent uncrossed strands, i.e., replacing \((a, i)\) and \((i + 1, b)\) (where \(a < i < i + 1 < b\)) with \((a, i + 1)\) and \((i, b)\). The operation of crossing a pair of adjacent endpoints corresponds to adding a square Dyck tile. The operation of inserting a pair of paired endpoints corresponds to splitting a Dyck tiling along a given column, in which case any tile crossing that column gets split into two halves which are joined in a unique way (inserting a “tent”) to make a larger tile.
Figure 5: (a) The strand diagram associated to a Dyck tiling of Figure 4, (b) the strands in the region below the upper Dyck path (with tiles erased) together with the cells they bound colored red and blue, (c) the blue network, formed from the blue cells, (d) the red network formed from the red cells. Some of the vertices in the networks are drawn in an extended fashion. The blue and red networks are dual to one another. In this example, red node 11 is disconnected, and blue nodes 11 and 12 are glued together.
Figure 6: Going from a strand matching to a Dyck tiling with associated rectilinear strand diagram. This construction is a modification of the recursive procedure illustrated in [KMPW14, Fig. 7] (see also [Kim12, Fig. 18]). We start in the upper left with the matching represented as a Dyck path and V-shaped strands (these are up to reflection and rotation the “folded tableaux” in [HW13, Fig. 12]). This matching is deconstructed going down on the left by removing peaks in the Dyck path; each such peak corresponds to either matched neighbors \((i, i + 1)\), or crossing neighbors \((a, i + 1), (i, b)\) where \(a < i < i + 1 < b\). For each deconstruction move going down on the left there is a corresponding construction move going up on the right: going up, if two neighbors are made to cross, then a box is added, and if a matched pair of neighbors is added, then the tiles overlapping that column are enlarged.
3.1 Tripod partitions

Given a standard minimal network, consider the strand diagram coming from its associated Dyck tiling. For any crossing $\chi$ in this strand diagram, the tripod partition $\tau_\chi$ associated with $\chi$ is the partition of the boundary nodes defined in Figure 7, and $\tau_\chi^*$ is the dual partition in the dual network. There is one further pair of tripod partitions, the exterior tripod partition $\tau_-$ and its dual $\tau_-^*$, which consist in taking all the horizontal edges.

Let $\{Z_\tau\}$ be the collection of partition sums $Z_\tau$ as $\tau$ ranges over the exterior partition $\tau_-$ together with all the tripod partitions $\tau_\chi$ associated with crossings in the strand diagram. We refer to $\{Z_\tau\}$ as the tripod variables of the network. We similarly let $\{Z_{\tau^*}\}$ be the corresponding tripod variables in the dual network.

**Theorem 1.** The number of tripod variables is one more than the number of edges of $\mathcal{G}$. Each tripod variable is a product of conductances. The conductances are biratios of tripod variables: The vertical conductance of a crossing $e$ is

$$c_e = \frac{Z_{\tau_a}Z_{\tau_f}}{Z_{\tau_a}Z_{\tau_b}} = \frac{Z_{\tau_a^*}Z_{\tau_f^*}}{Z_{\tau_a^*}Z_{\tau_b^*}},$$

where $a$ is the first crossing on the strand going up-and-left from crossing $e$, $b$ is the first crossing on the strand going up-and-right from $e$, and $f$ is the first crossing on the strand going up-and-left from crossing $b$. (If one or more of the crossings $a$, $b$, or $f$ does not exist, then $\tau_a$, $\tau_b$, or $\tau_f$, respectively, is replaced with the exterior partition $\tau_-$.)

![Figure 7](image_url)

**Figure 7:** In panel (a) is the rectilinear strand diagram coming from the Dyck tiling associated with a standard network. Also shown in panel (a) is a particular crossing of the strand diagram marked in green, together with the “comb” based at that crossing. The spine of the comb is formed by following the up-left strand segment from the green crossing. The teeth of the comb are formed by following the up-right strand segments from each crossing in the spine. In panel (b) is the associated grove and dual grove. For each strand crossing not in the comb, the horizontal edge or dual edge is selected; for each crossing in the comb, the vertical edge or dual edge is selected. In this example, the blue grove is a tripod, and the red grove is a dual tripod.
Proof. The first statement follows since each edge of $G$ is associated with a crossing in the strand diagram; thus there is one tripod variable for each edge, as well as the exterior tripod variable.

By construction, for each tripod partition $\tau$, there is only one grove of partition type $\tau$. Hence $Z_\tau$ is the product of the conductances of the edges in this grove.

Let $e$ be a crossing of the strand diagram, and let $a$ be the first crossing up and left from $e$. Comparing the combs based at $e$ and $a$, the comb based at $e$ has one more tooth, which starts at $e$. Thus $Z_{\tau_e}/Z_{\tau_a}$ is the product of the vertical conductances of all crossings along the up-and-right strand segment starting at $e$. (If there is no such crossing $a$, then the same statement holds when we replace $\tau_a$ with $\tau_- \tau_a$.) The ratio $Z_{\tau_b}/Z_{\tau_f}$ has a similar interpretation, except starting at the crossing $b$ (if crossing $b$ exists). Since crossing $b$ is the first crossing on the up-and-right strand segment from crossing $e$, the ratio $(Z_{\tau_e}/Z_{\tau_a})/(Z_{\tau_b}/Z_{\tau_f})$ is just the vertical conductance at $e$ (with $\tau_x$ replaced by $\tau_- \tau_x$ if crossing $x$ does not exist).

The formula in terms of the dual groves follows because $Z^*_\tau = Z_\tau/(\text{product of all edge weights})$ for each $\tau$.

There is one monomial relation between either the tripod variables or the dual tripod variables: Consider any lower-most crossing $\chi$ of the strand diagram. The edge associated with $\chi$ is horizontal for either the network or the dual network; for convenience say that it is horizontal for the network. The conductance of the associated edge does not appear as a factor in the tripod variable $Z_{\tau_\chi}$. Since each edge weight is the biratio of tripod variables, each tripod variable is a Laurent monomial in the set of tripod variables. Because $\chi$ is a lowermost crossing, its tripod variable appears in the biratio of only the associated edge. Thus $Z_{\tau_\chi}$ is a Laurent monomial in the other tripod variables, giving a nontrivial monomial relation.

3.2 Tripods and determinants

In this section we show how to write each tripod variable (up to a normalization factor) as a product of a Pfaffian (or Pfaffianoid), and a determinant, of matrices derived from the response matrix $L$. Combined with Theorem 1 this gives us an explicit expression for each conductance as a rational function of $L$.

Let $\tau$ be a tripod partition. We add edges of weight 1 to $G$ connecting parts of $\tau$ and internalizing the nodes (to which we added edges) so that the new partition $\tilde{\tau}$ is a pure tripod partition, that is, has the form of one triple, and the rest doubleton parts, ordered in parallel in the three regions complementary to the triple part, as on the right in Figure 9. Algorithmically this can be accomplished from the Dyck tiling as illustrated in Figure 8.

1The Pfaffianoid is an analog of the Pfaffian for odd-order antisymmetric matrices [KW09]: one formula for it is

\[
Pfd(M) = \sum_{1 \leq a < b < c \leq 2n+1} (-1)^{a+b+c} (M_{a,b}M_{b,c} + M_{b,c}M_{c,a} + M_{c,a}M_{a,b}) \text{Pf}(M \setminus \{a,b,c\})
\]

where $M$ is an $(2n+1) \times (2n+1)$ antisymmetric matrix, and $M \setminus \{a,b,c\}$ is the matrix $M$ with rows and columns $a,b,c$ deleted. The Pfaffianoid corresponds to a Pfaffian on the dual network.

2Internalizing means demoting a node to a regular (“internal”) vertex.
Figure 8: To make a pure tripod and dual tripod partition, we extend the strand diagram and the comb as indicated. Edges or dual edges in the extended network are given infinite conductance (i.e., an edge has zero conductance if the dual edge is in the extended network). We then contract the infinite-conductance edges or dual edges, as indicated in Figure 9.
Lemma 2. For the pure tripod partition $\tilde{\tau}$ defined in Figures 8 and 9, there is a unique grove in $\tilde{\mathcal{G}}$ whose partition is $\tilde{\tau}$.

Proof. The tripod partition $\tau$ is realized by a unique crossing of $\mathcal{G}$. The only way to make $\tilde{\tau}$ in $\tilde{\mathcal{G}}$ is to use that crossing of $\mathcal{G}$ and to add in the extra edges in $\tilde{\mathcal{G}} \setminus \mathcal{G}$.

Since the new partition $\tilde{\tau}$ is a pure tripod, from [KW09] we have

$$\frac{\tilde{Z}_{\tilde{\tau}}}{\tilde{Z}_{\text{unc}}} = \text{Pfd } \tilde{L}_{A,B,C},$$

where $\tilde{Z}_{\tilde{\tau}}$ is the weighted sum of pairings of $\tilde{\mathcal{G}}$ type $\tilde{\tau}$, $\tilde{Z}_{\text{unc}}$ is the weighted sum of uncrossings of $\tilde{\mathcal{G}}$ (groves with a unique node per component), and $\tilde{L}_{A,B,C}$ is the antisymmetric matrix obtained from $\tilde{L}$ by negating all entries on or below the diagonal, and zeroing entries both of whose index is in the same part $A, B$ or $C$ of the nodes of $\tilde{\mathcal{G}}$.

From Lemma 2 we have $\tilde{Z}_{\tilde{\tau}} = Z_{\tau}$, since the added edges have weight 1.

The last step is to relate $\tilde{Z}_{\text{unc}}$ to $Z_{\text{unc}}$, the weighted sum of uncrossings of $\mathcal{G}$. Let $\Delta$ be the Laplacian of $\mathcal{G}$ written in block form, where the first block consists of the nodes of $\tilde{\mathcal{G}}$, the second block consists of the nodes of $\mathcal{G}$ which are internalized in $\tilde{\mathcal{G}}$, and the last block consists of the internal vertices of $\mathcal{G}$. The edges added when going from $\mathcal{G}$ to $\tilde{\mathcal{G}}$ change $\Delta$ by adding a matrix $N$ to $D$ only: adding a single edge from node $i$ to node $j$ corresponds to adding a matrix $E_{ii} + E_{jj} - E_{ij} - E_{ji}$ where here $E_{**}$ represents an elementary matrix; $N = \tilde{\Delta} - \Delta$ is a sum of these.

Lemma 3. We have $\tilde{Z}_{\text{unc}} = \det(L_M^M + (\tilde{\Delta} - \Delta)_M M)$, where $M$ is the set of nodes of $\mathcal{G}$ that are internalized in $\tilde{\mathcal{G}}$.

Proof. We have $Z_{\text{unc}} = \det F$, and

$$\tilde{Z}_{\text{unc}} = \det \begin{pmatrix} D + N & E^t \\ E & F \end{pmatrix}.$$
The ratio is the determinant of the Schur reduction to the internalized nodes:

\[
\frac{\tilde{Z}_{\text{unc}}}{Z_{\text{unc}}} = \det(D + N - EF^{-1}E^t).
\]

The response matrix \( L \) of \( G \) is

\[
L = \begin{pmatrix} A & B \\ B^t & D \end{pmatrix} - \begin{pmatrix} C \\ E \end{pmatrix} F^{-1} \begin{pmatrix} C & E \end{pmatrix},
\]

so that

\[
L^M_M = D - EF^{-1}E^t.
\]

Finally we have

\[
\frac{Z_\tau}{Z_{\text{unc}}} = \frac{\tilde{Z}_\tau}{Z_{\text{unc}}} \frac{\tilde{Z}_{\text{unc}}}{Z_{\text{unc}}} = \text{Pfd} \tilde{L}_{A,B,C} \det(L^M_M + N).
\]

The situation when \( \tau \) is a tripod centered on a dual vertex is similar, and leads to the same formula with the Pfaffianoid replaced by a Pfaffian.

3.3 B variables

Above we defined the tripod variables of a standard network. Here we associate a different set of variables, called B variables, to the vertices and faces of an arbitrary minimal network, or equivalently, to the cells of its associated strand diagram. The defining characteristic of B variables is that each conductance \( c = c(e) \) satisfies

\[
c = \frac{B_v B_{v'}}{B_f B_{f'}}, \tag{2}
\]

where \( v \) and \( v' \) are the endpoints of edge \( e \) and \( f \) and \( f' \) are the adjacent faces. This biratio formula is very similar to the biratio formula (1) for the tripod variables, but the tripod variables do not comprise a set of B variables because there are different tripod variables along different portions of the outer face, and there can only be one B variable per face. The B variables are not uniquely defined by (2), since we can multiply the B variables on the cells to one side of strand without affecting (2), but these are the only degrees of freedom.

We construct a valid set of B variables as described in Figure 11. We can draw the strand diagram so that each strand monotonically goes from left to right. For each cell of the diagram (face or vertex of the network) there is a set of crossings to its right, for which the cell is in the left shadow. The product of horizontal conductances on these crossings is the B variable.

**Proposition 4.** With the construction in Figure 11, the B variables satisfy (2).

**Proof.** Consider the contribution of a particular crossing \( e \) to the biratio of B variables at any other crossing \( e' \). If \( e' \) is in the interior of the shadow of \( e \), then all four B variables have a factor of \( c_e \). If \( e' \) is in the interior of the complement of the shadow, none of the four B variables have a factor of \( c_e \). If \( e' \) is on the boundary of the shadow but not at \( e \), then two adjacent B variables have the weight \( c_e \) and this gets divided out in the biratio. The only net contribution comes when \( e' = e \).
One advantageous property of the $B$ variables is that they transform in a simple manner under Y-Δ moves: via the cube recurrence \cite{CS04,GK13}, see Figure 10.

![Figure 10: Transformation of $B$ variables under a Y-Δ transformation. We have $B_0B_7 = B_1B_4 + B_2B_5 + B_3B_6$.](image)

### 4 Well-connected networks

The top-dimensional cell of $\Omega_n$ consists of networks which are called well-connected. There are a number of equivalent definitions of well-connected networks, due to \cite{CdV94}. A network $\mathcal{G}$ is well-connected if for any pair of non-interlaced subsets $A, B \subset \mathcal{N}$ of the same cardinality $k$, there is a pairwise vertex-disjoint set of $k$ paths in $\mathcal{G}$ connecting in pairs the nodes in $A$ to those in $B$. A minimal well-connected network has exactly $n(n-1)/2$ edges; it has strand matching $\pi = \{\{1, n+1\}, \{2, n+2\}, \ldots, \{n, 2n\}\}$. A network is well-connected if and only if all non-interlaced minors of $L$ are strictly positive.

For well-connected networks, we show here that one can test positivity using $\binom{n}{2}$ minors of $L$, rather than Pfaffians. We conjecture that an analogous statement holds for general minimal networks.

### 4.1 Contiguous and central minors

We define a contiguous minor of an $n \times n$ matrix $M$ to be a minor of the form

$$CM_{a,b,y}(M) = \det M_{a,a+1,\ldots,a+y-1}^{b+y-1,\ldots,b+1}$$

where the indices are interpreted modulo $n$. In other words, the row and column indices are both contiguous modulo $n$. We define the central minor $CM_{x,y}(M)$ to be the contiguous minor $CM_{a,b,y}(M)$ with

$$a = \left\lfloor \frac{x-y}{2} \right\rfloor \quad b = \left\lfloor \frac{x-y+n-(n-1 \mod 2)}{2} \right\rfloor .$$

(Central minors were implicitly defined within a proof in [CM00 Chapt. 5.3].) In other words, the central minor $CM_{x,y}$ is a $y \times y$ minor of $M$ whose row and column indices are (cyclically) contiguous, and when the matrix indices are arranged on a circle, the chords connecting the row and column indices are generally as central as possible, modulo some details about rounding. Increasing $x$ by 2 is equivalent to cyclically shifting the indices, so $x$ is naturally interpreted modulo $2n$. The parameter $y$ naturally ranges from 0 to $n$. 

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Figure 11: (a) The minimal strand diagram from Fig. 2, (b) the strand diagram drawn rectilinearly with each strand going from left to right monotonically, (c) the region to the left of a crossing, (d) a cell together with the crossings to which it is to the left.

If the matrix $M$ is symmetric, then $CM_{x+n,y} = CM_{x,y}$ when $n$ is odd, and also when $n$ is even and $x+y$ is odd. If $n$ is even and $x+y$ is even, then $CM_{x,y}$ is off-center by one and $CM_{x+n,y}$ is off-center by one in the other direction, so they are different, but for some purposes they will turn out to be essentially interchangeable.

The row and column indices of $CM_{x,y}(M)$ are disjoint when either $y < n/2$ or $y = n/2$ and $x+y$ is odd.
Figure 12: The 15 small central minors of a $6 \times 6$ matrix, and the 21 small central minors of a $7 \times 7$ matrix. The row indices are shown in green, the column indices are shown in white.

We define the small central minors of a symmetric matrix $M$ to be

$$\left\{ \text{CM}_{x,y}(M) \right\}_{1 \leq x \leq n}^{1 \leq y < n/2 \text{ or } y=n/2 \text{ and } x+y \text{ odd}}.$$  

See Figure 12 for illustrations of the small central minors when $n = 6$ and $n = 7$.

There are \( \binom{n}{2} \) small central minors, whether $n$ is even or odd. The number of edges in a minimal well-connected network is also \( \binom{n}{2} \). We shall see that the small central minors of the response matrix are all positive precisely when the network is well connected, and can be used for reconstruction of the conductances.

We will show that a general contiguous minor $\text{CM}_{a,b,y}(M)$ can be expressed as a Laurent polynomial in the central minors, where the terms in this Laurent polynomial are in bijective correspondence with domino tilings of a certain region.

To define the Laurent polynomial, for each $x, y \in \mathbb{Z}$ let $v_{x,y}$ be a variable. We let $\text{AD}_{x_0,y_0,\ell}$ denote the "Aztec diamond" region of order $\ell$ centered at $(x_0, y_0)$, which consists of those squares of the square lattice centered at points $(x, y)$ for which $x, y \in \mathbb{Z} + \frac{1}{2}$ and $|x - x_0| + |y - y_0| \leq \ell$. There are some examples of Aztec diamonds illustrated in the next
few pages. As is well-known, there are $2^{\ell(\ell+1)/2}$ ways to tile an order-$\ell$ Aztec diamond by $2 \times 1$ dominoes [EKLP92].

We give weights to the dominoes: a horizontal domino covering the squares centered at $(x + \frac{1}{2}, y + \frac{1}{2})$ and $(x - \frac{1}{2}, y + \frac{1}{2})$ has weight $1/(v_{x,y}v_{x,y+1})$, and similarly, a vertical domino covering the squares $(x + \frac{1}{2}, y + \frac{1}{2})$ and $(x + \frac{1}{2}, y - \frac{1}{2})$ has weight $1/(v_{x,y}v_{x+1,y})$. The weight of a domino tiling is the product of the weights of its dominoes. Given a region $R$ tileable by dominoes, we define its weight $W(R)$ to be the sum of the weights of its domino tilings.

To the Aztec diamond $AD_{x_0,y_0,\ell}$, we associate the Laurent polynomial which is the Aztec diamond’s weight times the monomial factor

$$\prod_{x,y \in \mathbb{Z}} v_{x,y},$$

$$|x-x_0|+|y-y_0| \leq \ell$$

For other regions $R$, we similarly define the Laurent polynomial $P(R)$ of $R$ to be $W(R)$ times the monomial which is (roughly) the product of the variables on which the domino weights depend. (For the Aztec diamond $AD_{x_0,y_0,0}$, the monomial factor is $v_{x_0,y_0}$ even though the weight $W(AD_{x_0,y_0,0}) = 1$ does not actually depend on $v_{x_0,y_0}$. More precisely, the monomial factor is a product of variables that include the variables on which $W(R)$ depends, and which will be clear from context.) For each monomial in the Laurent polynomial $P(R)$ of the region, the degree of any variable $v_{x,y}$ is one of $0, -1, +1$.

Let the truncated Aztec diamond $TAD_{x_0,y_0,\ell,n}$ be Aztec diamond $AD_{x_0,y_0,\ell}$ truncated to contain only squares centered at points with $y$-coordinate between 0 and $n$, and depend only on variables $v_{x,y}$ for which $|x-x_0|+|y-y_0| \leq \ell$ and $0 \leq y \leq n$.

**Theorem 5.** A general contiguous minor $CM_{a,b,y}(M)$ of an $n \times n$ matrix $M$ can be expressed as a Laurent polynomial in the central minors of $M$ where all the coefficients are 1. Specifically, let $\ell$ be any integer for which $CM_{a,b+\ell,y}$ is a central minor $CM_{x,y}$. Let $P$ be the Laurent polynomial $P(TAD_{x-\ell,y,|\ell|,n})$. We can evaluate $P$ with each $v_{x',y'} = CM_{x',y'}(M)$ to obtain $CM_{a,b,y}(M)$.

To illustrate the theorem, in the case of 7 nodes, the simplest nontrivial example (with $\ell = 1$) is

$$\det [M_{1,3}] = \begin{array}{c}
\text{rectangle} \quad \text{+} \quad \text{rectangle} \\
\end{array} = \begin{array}{c}
\text{rectangle} \quad \text{+} \quad \text{rectangle} \\
\end{array}
$$

$$= \det [M_{7,3}] \det [M_{1,4}] + \det \begin{bmatrix} M_{7,4} & M_{7,3} \\ M_{1,4} & M_{1,3} \end{bmatrix} = \frac{\det [M_{7,4}] \det [M_{1,3}]}{\det [M_{7,4}]}.$$
An example illustrating the truncation is the following (with $\ell = 2$)

\[
\begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\
3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8
\end{array}
\]

which represents $M_{6,5} = \det[M_{6,5}]$ as a sum of 6 Laurent monomials in the central minors.

A larger example with $\ell = 3$ makes the Aztec diamond structure of the region apparent:

\[
\begin{array}{cccccccccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\
4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 \\
3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 \\
2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18
\end{array}
\]

**Proof of Theorem** We use the graphical condensation of Kuo [Kuo04, Thm. 5.5] together with the Desnanot-Jacobi identity, which are both related to Dodgson condensation.

For a matrix $M$, let $M^{c_1,\ldots,c_k}_{r_1,\ldots,r_k}$ denote the submatrix obtained by deleting rows $r_1, \ldots, r_k$ and columns $c_1, \ldots, c_k$. The Desnanot-Jacobi identity is

\[
\det M_a^c \det M_b^d = \det M \det M_{a,b}^{c,d} + \det M_a^c \det M_b^d,
\]

where row indices $a$ and $b$ are in sorted order, and column indices $c$ and $d$ are in sorted order. For example,

\[
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
2 & 3 & 4 & 5 \\
3 & 4 & 5 & 6 \\
4 & 5 & 6 & 7 \\
5 & 6 & 7 & 8
\end{array}
= \begin{array}{cccc}
1 & 2 & 3 & 4 \\
3 & 4 & 5 & 6 \\
5 & 6 & 7 & 8 \\
7 & 8 & 9 & 10 \\
9 & 10 & 11 & 12
\end{array} + \begin{array}{cccc}
2 & 3 & 4 & 5 \\
4 & 5 & 6 & 7 \\
6 & 7 & 8 & 9 \\
8 & 9 & 10 & 11 \\
10 & 11 & 12 & 13
\end{array}
\]

Kuo’s graphical condensation relates the weighted sum of domino tilings of weighted Aztec diamonds. Kuo’s formula is

\[
W(\text{AD}_{x,y,\ell})W(\text{AD}_{x,y,\ell-2}) = w[(x-\ell+\frac{1}{2}, y-\frac{1}{2}), (x-\ell+\frac{1}{2}, y+\frac{1}{2})] w[(x+\ell+\frac{1}{2}, y-\frac{1}{2}), (x+\ell+\frac{1}{2}, y+\frac{1}{2})] W(\text{AD}_{x,y+1,\ell-1})W(\text{AD}_{x,y-1,\ell-1})
\]

\[
+ w[(x-\frac{1}{2}, y-\ell+\frac{1}{2}), (x+\frac{1}{2}, y-\ell+\frac{1}{2})] w[(x-\frac{1}{2}, y+\ell-\frac{1}{2}), (x+\frac{1}{2}, y+\ell-\frac{1}{2})] W(\text{AD}_{x-1,y,\ell-1})W(\text{AD}_{x+1,y,\ell-1})
\]

\]
for any domino weights, not just the ones defined above. Here \( w \) refers to the weights of individual dominos. Notice that this formula extends to truncated Aztec diamonds, simply by setting the vertical domino weights to 0 and horizontal domino weights to 1 for dominos whose \( y \)-coordinate is too low or too high. When we divide both sides by the monomial factors of \( TAD_{x,y,\ell,n} \) and \( TAD_{x,y,\ell-2,n} \), the individual domino weights drop out, and we obtain

\[
P(TAD_{x,y,\ell,n})P(TAD_{x,y,\ell-2,n}) = P(TAD_{x,y+1,\ell-1,n})P(TAD_{x,y,\ell-1,n})
+ P(TAD_{x-1,y,\ell-1})P(TAD_{x+1,y,\ell-1}).
\]

Here is an example:

\[
\begin{array}{ccc}
\begin{array}{ccc}
\times & \times & \\
\end{array}
\end{array}
= \begin{array}{ccc}
\begin{array}{ccc}
\times & \times & \\
\end{array}
\end{array} + \begin{array}{ccc}
\begin{array}{ccc}
\times & \times & \\
\end{array}
\end{array}
\]

We prove the theorem by induction on \(|\ell|\). The case \( \ell = 0 \) is a tautology. The case \(|\ell| = 1\) is straightforward to verify; this was the first example given after the theorem statement. For \(|\ell| \geq 2\) we observe that the contiguous minors and the truncated Aztec diamond Laurent polynomials satisfy the same recurrence.

**Corollary 6.** Any minor of a matrix can be expressed as a Laurent polynomial in the small central minors.

**Proof.** By Theorem 5, \( M_{i,j} \) is a Laurent polynomial in the small central minors. Since any minor is a polynomial of the \( M_{i,j} \)'s, this implies that any minor \( \det M_{A}^{B} \) is a Laurent polynomial in the small central minors.

### 4.2 Positivity

A non-interlaced minor of a matrix is a Laurent polynomial in the central minors (by Corollary 6), but we do not know whether or not the coefficients are always positive. But as the next theorem shows, non-interlaced minors are positive rational functions of the central minors. Our positivity results in this section were discussed in the survey [Ken12].

**Theorem 7.** Any non-interlaced minor of a matrix is a positive rational function of its central minors.

**Proof.** Let \( A \) be a matrix, and let \( a, b, c \) index some of its columns, and \( z, d \) index some of its rows. It is elementary that

\[
0 = \begin{vmatrix} A_{z,a} & A_{z,b} & A_{z,c} \\ A_{z,a} & A_{z,b} & A_{z,c} \\ A_{d,a} & A_{d,b} & A_{d,c} \end{vmatrix} = A_{z,a} \begin{vmatrix} A_{z,b} & A_{z,c} \\ A_{d,a} & A_{d,c} \end{vmatrix} - A_{z,b} \begin{vmatrix} A_{z,a} & A_{z,c} \\ A_{d,a} & A_{d,c} \end{vmatrix} + A_{z,c} \begin{vmatrix} A_{z,a} & A_{z,b} \\ A_{d,a} & A_{d,b} \end{vmatrix}.
\]
Suppose \( A \) is invertible, and let \( M \) denote its inverse; \( a, b, c \) index rows of \( M \) and \( z, d \) index columns of \( M \). Dividing through by \((\det A)^2\) and using Jacobi’s identity, we obtain

\[
0 = \det M_{\hat{a}}^\hat{d} \det M_{\hat{b}, \hat{c}}^{\hat{d}, \hat{a}} - \det M_{\hat{b}}^\hat{c} \det M_{\hat{a}, \hat{c}}^{\hat{d}, \hat{b}} + \det M_{\hat{c}}^\hat{d} \det M_{\hat{a}, \hat{d}}^{\hat{b}, \hat{c}}. 
\]

Since this is a polynomial equation that holds generically, it must always hold. As column \( z \) is always excluded, it need not index an actual column of \( M \). Dropping \( z \) and rearranging terms we obtain

\[
\det M_{\hat{b}} \det M_{\hat{a}, \hat{c}}^{\hat{d}, \hat{b}} = \det M_{\hat{a}} \det M_{\hat{b}, \hat{c}}^{\hat{d}, \hat{a}} + \det M_{\hat{c}} \det M_{\hat{a}, \hat{d}}^{\hat{b}, \hat{c}}.
\]

For example, if \( M = M_{1,2,3,4}^{9,8,7,6,5} \) and \( a, b, c, d = 9, 8, 5, 4 \), we have

\[
\begin{align*}
\text{Diagram 1} & = \text{Diagram 2} + \text{Diagram 3}.
\end{align*}
\]

We call transformations of this type the “jaw move”. For a given non-interleaved determinant interspersed with at least one isolated node, we can take \( b \) to be one of the interspersed isolated nodes, and \( a \) and \( c \) to be the first and last of the nodes on the same side as \( b \), and \( d \) to be either first or last node on the other side as \( b \). With this choice of \( a, b, c, d \), the jaw move expresses the original determinant as a positive rational function of “simpler” non-interleaved determinants, where a determinant is simpler if it has fewer strands, or else the same number of strands but fewer interspersed isolated nodes.

By repeated application of the jaw move, any non-interleaved determinant can be expressed as a positive rational function of non-interleaved contiguous determinants. Then we can use Theorem 5 to express the non-interleaved contiguous minors in terms of the small central minors.

The remaining case to check is for \( n \) even, with the (minimally) off-center contiguous crossings come in pairs, of which only one is included in the base cluster. Using a condensation move we can express a minimally off-center contiguous crossing in terms of its opposite minimally off-center contiguous crossing and central contiguous crossings as shown below:

\[
\begin{align*}
\text{Diagram 4} & = \text{Diagram 5} + \text{Diagram 6}.
\end{align*}
\]

Corollary 8. If each of the \( \binom{n}{2} \) small central minors of the response matrix of a network is positive then the network is well-connected.

4.3 Domino regions for semicontiguous minors

We believe that Theorem 5 can be extended to semicontiguous minors, i.e., minors \( M_{A}^{B} \) where only one of \( A \) or \( B \) is contiguous. We include here some example formulas, each expressing
a semicontiguous minor in terms of domino tilings of a region, where each domino tiling corresponds to a Laurent monomial in the central minors according to the same rule as in Theorem 5. We believe that every semicontiguous minor has an associated domino tiling region, and think it would be interesting to work out what the regions are in general.
5 Questions

Is there a decision-procedure which on any response matrix makes at most \( \binom{n}{2} \) positivity tests and determines the strand matching for which all the conductances are positive?

Rather than test the positivity of Pfaffians, for well-connected networks we have seen that we can instead test the positivity of minors of the response matrix (the central minors), where the number of tests equals the number of edges. For networks that are not well connected, is it possible to test the positivity of the conductances by testing the positivity of a number of minors equal to the number of edges? The answer appears to be yes, but we do not know a proof for general non-well-connected networks.

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