NOTES ON ATKIN–LEHNER THEORY FOR DRINFELD MODULAR FORMS

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Abstract

We settle a part of the conjecture by Bandini and Valentino [‘On the structure and slopes of Drinfeld cusp forms’, Exp. Math. 31(2) (2022), 637–651] for \(S_{k,l}(\Gamma_0(T))\) when \(\dim S_{k,l}(\text{GL}_2(A)) \leq 2\). We frame and check the conjecture for primes \(p\) and higher levels \(pm\), and show that a part of the conjecture for level \(pm\) does not hold if \(m \neq A\) and \((k,l) = (2,1)\).

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1. Introduction

The theory of oldforms and newforms is a well-understood area in the theory of classical modular forms. Certain properties of modular forms depend heavily on whether they are oldforms or newforms. For example, the space of newforms has a basis consisting of normalised eigenforms for all the Hecke operators and the Fourier coefficients of these normalised eigenforms generate a number field. The analogous theory of oldforms and newforms is much less developed for Drinfeld modular forms.

Bandini and Valentino [3–5, 19] defined \(p\)-oldforms and \(p\)-newforms, and studied some of their properties. In [3], they defined \(T\)-oldforms \(S_{k,l}^{T,\text{old}}(\Gamma_0(T))\) and \(T\)-newforms \(S_{k,l}^{T,\text{new}}(\Gamma_0(T))\) for \(p = (T)\). In [4], they made the following conjecture.

CONJECTURE 1.1 ([4, Conjecture 1.1] for \(\Gamma_0(T)\)).

(i) \(\ker(T_T) = 0\) where \(T_T\) is acting on \(S_{k,l}(\text{GL}_2(A))\);
(ii) \(S_{k,l}(\Gamma_0(T)) = S_{k,l}^{T,\text{old}}(\Gamma_0(T)) \oplus S_{k,l}^{T,\text{new}}(\Gamma_0(T))\);
(iii) for odd characteristic, \(U_T\) is diagonalisable on \(S_{k,l}(\Gamma_0(T))\).

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Throughout the article, we use the following notation. For an odd prime $p$, we consider the extent that this is valid for Drinfeld modular forms. By studying the action of the $T_p$-operators on the Fourier coefficients of Drinfeld modular forms, we prove the following result.

**Theorem 1.2 (Theorems 4.8 and 4.9).** If $\dim S_{k,l}(\Gamma_0(T)) \leq 1$, then Conjecture 1.1 is true for $S_{k,l}(\Gamma_0(T))$. Furthermore, if $\dim S_{k,l}(\Gamma_0(T)) = 2$, then $S_{k,l}(\Gamma_0(T)) = S_{k,l}^{T-\text{old}}(\Gamma_0(T)) \oplus S_{k,l}^{T-\text{new}}(\Gamma_0(T))$.

Our methods are completely different from those of [4, 6]. We are very optimistic that our methods can be used when $\dim S_{k,l}(\Gamma_0(T)) \geq 3$.

In [5], Bandini and Valentino extended the definition of $p$-oldforms and $p$-newforms from level $p$ to level $pm$ with $p \nmid m$. We frame Conjecture 1.1 for level $pm$ with $p \nmid m$ as a question (Question 4.3) and provide some evidence in favour of it.

First, we generalise the results of [4] for $p = (T)$ to an arbitrary prime ideal $p$ (Proposition 4.10). This implies that Question 4.3 has an affirmative answer in these cases. Then, we exhibit some cases where Question 4.3 for the level $pm$ is true (Proposition 4.11). If $m \neq A$, we show that the direct sum decomposition in Question 4.3(2) may fail when $l = 1$ (Proposition 4.13, Remark 4.14). More precisely, we exhibit nonzero Drinfeld cusp forms which are both $p$-oldforms and $p$-newforms. We believe that this is the only case where it may fail.

For classical modular forms, it is well known that the space of newforms can be characterised in terms of kernels of the Trace and twisted Trace operators [15, 18]. In the final section, we consider the extent that this is valid for Drinfeld modular forms.

**Notation.** Throughout the article, we use the following notation. For an odd prime number $p$ and $q = p^r$ for some $r \in \mathbb{N}$, $\mathbb{F}_q$ is the finite field of order $q$. For $k \in \mathbb{N}$ and $l \in \mathbb{Z}/(q - 1)\mathbb{Z}$ such that $k \equiv 2l \pmod{q - 1}$, let $l$ be a lift of $0 \leq l \leq q - 2$ be a lift of $l \in \mathbb{Z}/(q - 1)\mathbb{Z}$. By abuse of notation, we write $l$ for the integer as well as its class.

Set $A := \mathbb{F}_q[T]$, $K := \mathbb{F}_q(T)$. For $f \in A$, $g \in A \setminus \{0\}$, the norm of $f/g$ is defined as $|f/g| := q^{\deg(f) - \deg(g)}$ (with respect to the $1/T$-adic valuation).

Let $K_\infty = \mathbb{F}_q((1/T))$ be the completion of $K$ with respect to the infinite place $\infty$ (corresponding to the $1/T$-adic valuation) and $C := \overline{K_\infty}$, the completion of an algebraic closure of $K_\infty$. Let $p = (P)$ denote a prime ideal of $A$ with a monic irreducible polynomial $P$.

## 2. Basic theory of Drinfeld modular forms

In this section, we recall some basic theory of Drinfeld modular forms (see [10–13] for more details).

Let $L = \pi A \subseteq C$ be the $A$-lattice of rank 1 corresponding to the rank 1 Drinfeld module (also called a Carlitz module) $\rho_T = TX + X^q$, where $\pi \in K_\infty(\sqrt{-1})$ is defined up to a $(q - 1)$th root of unity. The Drinfeld upper half-plane $\Omega = C - K_\infty$, which is
analogous to the complex upper half-plane, has a rigid analytic structure. The group \( \Gamma_2(K_\infty) \) acts on \( \Omega \) via fractional linear transformations.

**Definition 2.1.** Let \( k \in \mathbb{N}, \ l \in \mathbb{Z}/(q-1)\mathbb{Z} \) and \( f : \Omega \to C \) be a rigid holomorphic function on \( \Omega \). For \( \gamma = (\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \in \Gamma_2(K_\infty) \), we define the slash operator \( f|_k, l \gamma \) on \( f \) by

\[
f|_k, l \gamma := (\det \gamma)^l (cz+d)^{-k} f(\gamma z).
\]

Define the congruence subgroup \( \Gamma_0(n) = \{(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \in \Gamma_2(A) : c \in n \} \), where \( n \subseteq A \) is an ideal. Next, we define a Drinfeld modular form of weight \( k \) and type \( l \) for \( \Gamma_0(n) \).

**Definition 2.2.** A rigid holomorphic function \( f : \Omega \to C \) is a Drinfeld modular form of weight \( k \), type \( l \) for \( \Gamma_0(n) \) if:

1. \( f|_k, l \gamma = f \) for all \( \gamma \in \Gamma_0(n) \);
2. \( f \) is holomorphic at the cusps of \( \Gamma_0(n) \).

The space of Drinfeld modular forms of weight \( k \), type \( l \) for \( \Gamma_0(n) \) is denoted by \( \mathcal{M}_{k,l}(\Gamma_0(n)) \). If \( f \) vanishes at the cusps of \( \Gamma_0(n) \), then we say \( f \) is a Drinfeld cusp form of weight \( k \), type \( l \) for \( \Gamma_0(n) \) and the space of such forms is denoted by \( \mathcal{S}_{k,l}(\Gamma_0(n)) \).

If \( k \not\equiv 2l \pmod{q-1} \), then \( \mathcal{M}_{k,l}(\Gamma_0(n)) = \{0\} \). So, without loss of generality, we can assume that \( k \equiv 2l \pmod{q-1} \). Let \( u(z) := 1/e_k(z) \), where \( e_k(z) := \prod_{0 \neq \lambda \in L} (1-z/\lambda) \) is the exponential function attached to the lattice \( L \). Then, each Drinfeld modular form \( f \in \mathcal{M}_{k,l}(\Gamma_0(n)) \) has a unique \( u \)-series expansion at \( \infty \) given by 

\[
f(z) = \sum_{i \geq 0} a_f(i) u^i.
\]

Since \( (\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}) \in \Gamma_0(n) \) for \( \zeta \in \mathbb{F}_q^\times \), Definition 2.2(1) implies \( a_f(i) = 0 \) if \( i \not\equiv l \pmod{q-1} \). Hence, the \( u \)-series expansion of \( f \) at \( \infty \) can be written as 

\[
\sum_{0 \leq i \leq l \pmod{q-1}} a_f(i) u^i.
\]

Any Drinfeld modular form of type \( l \neq 0 \) is a cusp form, that is, \( \mathcal{M}_{k,l}(\Gamma_0(n)) = \mathcal{S}_{k,l}(\Gamma_0(n)) \).

**2.1. Examples.** We now give some examples of Drinfeld modular forms.

**Example 2.3 [10, 12].** Let \( d \in \mathbb{N} \). For \( z \in \Omega \), the function

\[
g_d(z) := (-1)^{d+1} \tilde{\tau}^{1-q^d} L_d \sum_{a, b \in \mathbb{F}_q[T]} \frac{1}{(az+b)^{q^d-1}}
\]

is a Drinfeld modular form of weight \( q^d - 1 \), type \( 0 \) for \( \Gamma_2(A) \), where \( \tilde{\tau} \) is the Carlitz period and \( L_d := (T^{q^d} - T) \cdots (T^q - T) \) is the least common multiple of all monic polynomials of degree \( d \). We refer to \( g_d \) as an Eisenstein series and it does not vanish at \( \infty \).

**Example 2.4 [10, 13].** For \( z \in \Omega \), the function

\[
\Delta(z) := (T - T^{q^d}) \tilde{\tau}^{1-q^d} E_{q^2-1} + (T^q - T)^q \tilde{\tau}^{1-q^d} (E_{q-1})^{q+1}
\]

is a Drinfeld cusp form of weight \( q^2 - 1 \), type \( 0 \) for \( \Gamma_2(A) \), where \( E_k(z) = \sum_{(0,0) \neq (a,b) \in A^2} (az+b)^{-k} \).
Example 2.5 [10]. For \( z \in \Omega \), the function

\[
h(z) := \sum_{\gamma = (a \ b \ c \ d) \in H / \text{GL}_2(A)} \frac{\det \gamma}{(cz + d)^{q+1}} u(\gamma z),
\]

where \( H = \{ (\alpha \ 0 \ 1) \in \text{GL}_2(A) \} \), is a Drinfeld cusp form of weight \( q + 1 \), type 1 for \( \text{GL}_2(A) \).

We end this section by introducing an important function \( E \), which is not modular. In [10], Gekeler defined the function \( E(z) := 1/\pi \sum_{a \in \mathbb{F}_q[T], a \text{ monic}} \sum_{b \in \mathbb{F}_q[T]} a/(az + b) \), which is analogous to the Eisenstein series of weight 2 over \( \mathbb{Q} \). For any prime ideal \( p = (P) \), using \( E \), we can construct the Drinfeld modular form

\[
E_p(z) := E(z) - PE(Pz) \in S_{2,1}(\Gamma_0(p))
\]

(see [9, Proposition 3.3] for a detailed discussion of \( E_p \)).

3. Certain important operators

3.1. Atkin–Lehner operators. Let \( r, n \) be two ideals of \( A \) generated by monic polynomials \( r, n \), respectively, with \( r \mid n \).

Definition 3.1 [17, page 331]. For \( \mathfrak{r} \mid \mathfrak{n} \) (that is, \( \mathfrak{r} \mid \mathfrak{n} \) with \( (\mathfrak{r}, \mathfrak{n} / \mathfrak{r}) = 1 \)), the (partial) Atkin–Lehner operator \( W_{\mathfrak{r}}^{(n)} \) is defined by the action of the matrix \( (ar \ b \ cn \ dr) \) on \( M_{k,l}(\Gamma_0(n)) \), where \( a, b, c, d \in A \) such that \( adr^2 - bcn = \zeta \cdot r \) for some \( \zeta \in \mathbb{F}_q^{\times} \).

By [9, Proposition 3.2], the action of \( W_{\mathfrak{r}}^{(n)} \) on \( M_{k,l}(\Gamma_0(n)) \) is well defined (here the action of the slash operator is different from the one in [9]). Assume that \( \mathfrak{p}^{\alpha} \mid \mathfrak{n} \) with \( \alpha \in \mathbb{N} \). Write \( n = Pm \) and \( m = (m) \). We now fix some representatives for the (partial) Atkin–Lehner operators \( W_{\mathfrak{r}}^{(n)} \) and \( W_{\mathfrak{r}}^{(m)} \).

Definition 3.2. For \( f \in S_{k,l}(\Gamma_0(n)) \), we write \( f|_{k,l} W_{\mathfrak{r}}^{(n)} := f|_{k,l}(\mathfrak{p}^{\alpha} b \mathfrak{p}^{\alpha} d) \), where \( b, d \in A \) such that \( P^{2\alpha}d - nb = P^\alpha \). Since \( (P^\alpha, n/P^\alpha) = 1 \), such \( b, d \in A \) exist.

When \( \alpha \geq 2 \), we can take \( (\mathfrak{p}^{\alpha-1} b \mathfrak{p}^{\alpha-1} d) \) as a representative for the (partial) Atkin–Lehner operator \( W_{\mathfrak{r}}^{(n)} \).

Lemma 3.3. The operator \( |_{k,l} W_{\mathfrak{r}}^{(n)} \) on \( S_{k,l}(\Gamma_0(n)) \) defines an endomorphism and for all \( f \in S_{k,l}(\Gamma_0(n)) \), we have \( (f|_{k,l} W_{\mathfrak{r}}^{(n)})|_{k,l} W_{\mathfrak{r}}^{(n)} = P^\alpha(2l-k) f \).

Proof. Since \( W_{\mathfrak{r}}^{(n)} : W_{\mathfrak{r}}^{(n)} = (\mathfrak{p}^{\alpha} b \mathfrak{p}^{\alpha} d) \gamma \) for some \( \gamma \in \Gamma_0(n) \), the lemma follows. \( \square \)

Lemma 3.4. For \( i = 1, 2 \), let \( \mathfrak{p}_i \) be two distinct prime ideals of \( A \) such that \( \mathfrak{p}_i^{\alpha_i} \mid \mathfrak{n} \) for some \( \alpha_i \in \mathbb{N} \). Then \( W_{\mathfrak{p}_1}^{(n)} W_{\mathfrak{p}_2}^{(n)} = W_{\mathfrak{p}_1}^{(n)} W_{\mathfrak{p}_2}^{(n)} \).
PROOF. The lemma follows from $W_{v_1'}^{(n)} W_{v_2'}^{(n)} = W_{v_1'}^{(n)} W_{v_2'}^{(n)} = W_{v_1'}^{(n)} W_{v_2'}^{(n)}$. □

3.2. Hecke operators. We now recall the definitions of $T_p$ and $U_p$-operators.

DEFINITION 3.5. For $f \in S_{k,l}(\Gamma_0(n))$, we define

$$T_p(f) := P^{k-l} \sum_{Q \in A, \deg Q < \deg P} f|_{k,l}(P^0_Q) + P^{k-l} f|_{k,l}(P^0_1) \quad \text{if } p \nmid n,$$

$$U_p(f) := P^{k-l} \sum_{Q \in A, \deg Q < \deg P} f|_{k,l}(P^0_Q) \quad \text{if } p \mid n.$$

The commutativity of the $T_p$ and $U_p$-operators follows from the next proposition.

PROPOSITION 3.6. Let $\mathfrak{n}$ be an ideal of $A$ and $\mathfrak{p}_1, \mathfrak{p}_2$ be two distinct prime ideals of $A$ generated by monic irreducible polynomials $P_1, P_2$, respectively. Suppose that $\mathfrak{p}_1 \mid \mathfrak{n}$. Then, $U_{\mathfrak{p}_1}$ commutes with $U_{\mathfrak{p}_2}$ if $\mathfrak{p}_2 \mid \mathfrak{n}$ and with $T_{\mathfrak{p}_2}$ if $\mathfrak{p}_2 \nmid \mathfrak{n}$, as operators on $S_{k,l}(\Gamma_0(n))$.

PROOF. Since $P_1$ and $P_2$ are distinct primes, for any $b \in A$ with $\deg b < \deg P_1$, there exists a unique $b' \in A$ with $\deg b' < \deg P_1$ such that $P_1 | (b-b'P_2)$. Thus, $(1_{0,0} - (b-b'P_2/P_1)) \in \Gamma_0(n)$ and $(1_{0,0} - (b-b'P_2/P_1)) = (1_{0,0} - (b-b'P_2/P_1))(P_2 0 1)(1_{0,0} - (b-b'P_2/P_1))$. Now the result follows from Definition 3.5 and the equality

$$\sum_{b \in A} \sum_{d \in A} (b+bP_1, P_2) = \sum_{e \in A} (1_{0,0} + eP_1) = \sum_{d \in A} \sum_{b' \in A} (d+bP_2, P_1).$$ □

3.3. The trace operators. We define the trace operators and mention their properties.

DEFINITION 3.7. For any ideal $\mathfrak{p} \mid n$, the trace operator $\Tr_{\mathfrak{p}}^n : M_{k,l}(\Gamma_0(n)) \to M_{k,l}(\Gamma_0(n))$ is defined by $\Tr_{\mathfrak{p}}^n(f) = \sum_{\gamma \in \Gamma_0(n) \backslash \Gamma_0(n)} f|_{k,l} \gamma$.

The next proposition gives the action of the trace operator in terms of the (partial) Atkin–Lehner operators and the Hecke operators.

PROPOSITION 3.8. Let $\mathfrak{m}, \mathfrak{n}$ be two ideals of $A$ generated by monic polynomials $m, n$, respectively, such that $n = Pm$. Let $\alpha \in \mathbb{N}$ such that $P^\alpha \mid n$. If $f \in S_{k,l}(\Gamma_0(n))$, then

$$\Tr_{\mathfrak{m}}^n(f) = \begin{cases} f + P^{-l} U_p(f)|_{\mathfrak{m}}^{(n)} & \text{if } \alpha = 1, \\ P^{-l-(\alpha-1)(2l-k)} U_p(f)|_{\mathfrak{m}}^{(n)} |_{\mathfrak{m}^{(n)}} & \text{if } \alpha \geq 2. \end{cases}$$

PROOF. If $\alpha = 1$, this proposition is [9, Proposition 3.6]. When $\mathfrak{n}$ is a prime ideal, it coincides with [20, Proposition 3.8] (but note that the action of the slash operator here is different). Now, we let $\alpha \geq 2$. By definition,
We now show that \( \{ (1-mQ, P^{p^{-1}}Q) : \text{deg} Q < \text{deg} P \} \) is a set of representatives for \( \Gamma_0(n) \backslash \Gamma_0(m) \). Let \( \left( \frac{s}{mx}, \frac{t}{my} \right) \in \Gamma_0(m) \), where \( s, t, x, y \in A \) satisfy \( sy - tmx = \zeta \in \mathbb{F}_q^{\times} \). Let \( \zeta^{-1}sx \equiv Q_1 \pmod{m} \), where \( Q_1 \in A \) is such that \( \text{deg} Q_1 < \text{deg} P \). Since \( P^{p-1}|m \), there exists a unique \( Q_2 \in A \) with \( \text{deg} Q_2 < \text{deg} P \) such that \( (m/P^{p-1})Q_2 \equiv 1 \pmod{P} \). Since \( P|m \), the choice of \( Q_1 \) and \( sy - tmx = \zeta \in \mathbb{F}_q^{\times} \) implies that \( x + yQ_1 \equiv 0 \pmod{P} \). Let \( Q \in A \) with \( \text{deg} Q < \text{deg} P \) such that \( Q_1Q_2 \equiv Q \pmod{P} \). Then \( x + (m/P^{p-1})Qy \equiv 0 \pmod{P} \). Hence, the equation

\[
\left( \frac{s}{mx}, \frac{t}{my} \right) = \left( \frac{s(m+Q^2)}{(mx+Q)(m+Q)} \right) \left( \frac{t(m+Q^2)}{(my+Q)(m+Q)} \right) \left( \frac{1-mQ}{1+mQ} \right) \left( \frac{P^{p-1}Q}{1+mQ} \right)
\]

shows that \( \{ (1-mQ, P^{p^{-1}}Q) : \text{deg} Q < \text{deg} P \} \) forms a complete set of representatives for \( \Gamma_0(n) \backslash \Gamma_0(m) \). Therefore,

\[
U_P(f\mid W_{pm}^{(m)}) = P^l \sum_{\text{deg} Q < \text{deg} P} \frac{1-mQ}{1+mQ} \frac{P^{p-1}Q}{1+mQ} \frac{P^{p-1}Q}{1+mQ} = P^l(\text{Tr}_m^n f) \mid_{k,l} W_{pm}^{(m)}.
\]

Apply the \( W_{pm}^{(m)} \) operator on both sides. The proposition follows from Lemma 3.3.

**Corollary 3.9.** If \( p, m \) satisfy \( (p, m) = 1 \) and \( f \in S_{k,l}(\Gamma_0(p)) \), then \( \text{Tr}_m^n(f) = \text{Tr}_1^n(f) \).

**Proof.** Since \( f\mid W_{pm}^{(m)} = f\mid W_{p}^{(m)} \), the result follows from Proposition 3.8.

## 4. \( \mathfrak{p} \)-oldforms and \( \mathfrak{p} \)-newforms for level \( \mathfrak{p}m \)

Let \( \mathfrak{p} \) be a prime ideal of \( A \). Throughout this section, we consider \( m \) an ideal of \( A \) generated by a monic polynomial \( m \) such that \( \mathfrak{p} \nmid m \). We first recall the definitions of \( \mathfrak{p} \)-oldforms and \( \mathfrak{p} \)-newforms (see [5, 19]). Consider the map

\[
(\delta_1, \delta_\mathfrak{p}): (S_{k,l}(\Gamma_0(m)))^2 \to S_{k,l}(\Gamma_0(pm)) \text{ defined by } (f, g) \mapsto \delta_1 f + \delta_\mathfrak{p} g,
\]

with \( \delta_1, \delta_\mathfrak{p}: S_{k,l}(\Gamma_0(m)) \to S_{k,l}(\Gamma_0(pm)) \) given by \( \delta_1(f) = f \) and \( \delta_\mathfrak{p}(f) = f\mid_{k,l}(\mathfrak{p} \ 0) \).

**Definition 4.1.** The space of \( \mathfrak{p} \)-oldforms \( S_{k,l}^{\mathfrak{p-old}}(\Gamma_0(p\mathfrak{m})) \) of level \( \mathfrak{p}m \) is the subspace of \( S_{k,l}(\Gamma_0(p\mathfrak{m})) \) generated by the image of \( (\delta_1, \delta_\mathfrak{p}) \).

**Definition 4.2.** The space of \( \mathfrak{p} \)-newforms \( S_{k,l}^{\mathfrak{p-new}}(\Gamma_0(p\mathfrak{m})) \) of level \( \mathfrak{p}m \) is

\[
S_{k,l}^{\mathfrak{p-new}}(\Gamma_0(p\mathfrak{m})) := \text{Ker}(\text{Tr}_m^{\mathfrak{p}\mathfrak{m}}) \cap \text{Ker}(\text{Tr}_m^{\mathfrak{p}^*\mathfrak{m}}) \text{ where } \text{Tr}_m^{\mathfrak{p}\mathfrak{m}} f := \text{Tr}_m^{\mathfrak{p}\mathfrak{m}}(f\mid W_{p}^{(m)}).
\]
We formulate Conjecture 1.1 for primes $p$ and higher levels $pm$ as the following question and provide some evidence in favour of it.

**QUESTION 4.3 (For level $pm$).** Suppose $m$ is an ideal of $A$ such that $p \nmid m$:

1. $\ker(T_p) = 0$, where $T_p \in \End(S_{k,l}(\Gamma_0(m)))$;
2. $S_{k,l}(\Gamma_0(pm)) = S_{k,l}^{\text{old}}(\Gamma_0(pm)) \oplus S_{k,l}^{\text{new}}(\Gamma_0(pm))$;
3. the $U_p$-operator is diagonalisable on $S_{k,l}(\Gamma_0(pm))$.

When we say that ‘Question 4.3 is true for level $pm$’, we mean all the statements of Question 4.3 are true. We first show that if $m = A$, $p = (P)$ with $\deg P = 1$, then Question 4.3 is true for level $p$ if $\dim S_{k,l}(\text{GL}_2(A)) \leq 1$. In particular, Conjecture 1.1 is true for $S_{k,l}(\Gamma_0(T))$ when $\dim S_{k,l}(\text{GL}_2(A)) \leq 1$. Furthermore, we show the direct sum decomposition in Question 4.3(2) holds for $S_{k,l}(\Gamma_0(p))$ if $\dim S_{k,l}(\text{GL}_2(A)) \leq 2$. Finally, we give some evidence to support Question 4.3 for level $pm$.

**4.1. Question 4.3 when $\dim S_{k,l}(\text{GL}_2(A)) \leq 2$.** If Question 4.3(2) is true, then the diagonalisability of the $U_p$-operator on $S_{k,l}(\Gamma_0(pm))$ depends on that of the $U_p$-operators on $S_{k,l}^{\text{old}}(\Gamma_0(pm))$, $S_{k,l}^{\text{new}}(\Gamma_0(pm))$. By [5, Remark 2.17], the $U_p$-operator is diagonalisable on $S_{k,l}(\Gamma_0(pm))$. However, the $U_p$-operator is diagonalisable on $S_{k,l}^{\text{old}}(\Gamma_0(pm))$ if and only if the $T_p$-operator is diagonalisable on $S_{k,l}(\Gamma_0(m))$ and is injective (see [5, Remark 2.4]). Therefore, if Questions 4.3(1) and 4.3(2) are true, then Question 4.3(3) is equivalent to checking the diagonalisability of the $T_p$-operator on $S_{k,l}(\Gamma_0(m))$.

**4.1.1. Reformulation of Question 4.3(2).** In [19], Valentino gave a necessary and sufficient condition for Question 4.3(2) to hold.

**THEOREM 4.4 [19, Theorem 3.15].** The map $\text{Id} - P^{k-2l}(\text{Tr}^\text{pm}_m)^2$ is bijective on $S_{k,l}(\Gamma_0(pm))$ if and only if Question 4.3(2) holds.

We now rephrase Theorem 4.4 in terms of the eigenvalues of the $T_p$-operator.

**PROPOSITION 4.5.** The $T_p$-operator has no eigenform on $S_{k,l}(\Gamma_0(m))$ with eigenvalues $\pm P^{k/2}$ if and only if Question 4.3(2) holds.

The proof of Proposition 4.5 depends on the following observations. For any $\varphi \in S_{k,l}(\Gamma_0(m))$,

$$\varphi|_{k,l}W_p^{(pm)} = \varphi|_{k,l}(\frac{1}{m \ dP}, (\frac{p}{0 \ 1})) = \varphi|_{k,l}(\frac{P}{0 \ 1}) = \delta_p \varphi,$$

$$(\delta_p \varphi)|_{k,l}W_p^{(pm)} = \varphi|_{k,l}(\frac{P}{0 \ 1})(\frac{P}{pm \ dP}) = P^{2l-k}\varphi. \tag{4.1}$$

Combining Proposition 3.8 with (4.1),

$$\text{Tr}^\text{pm}_m(\delta_1(\varphi)) = \varphi|_{k,l}W_p^{(pm)} + P^{l-k}(U_p(\delta_1(\varphi))) = \delta_p \varphi + P^{l-k}(U_p(\delta_1(\varphi))) = P^{l-k}T_p \varphi, \tag{4.2}$$

where $W_p^{(pm)} := (\frac{P}{pm \ dP})$ for some $b, d \in A$ with $dP^2 - bPm = P$. 

PROOF OF PROPOSITION 4.5. If \( f \in \ker(\Id - P^{k-2l}(\Tr^p_{m}'))^2 \), then \( f \in \Im(\delta_1) \) (from the proof of Theorem 4.4), so \( \ker(\Id - P^{k-2l}(\Tr^p_{m}'))^2 \subseteq S_{k,l}(\Gamma_0(\infty)) \). Therefore, \( \Id - P^{k-2l}(\Tr^p_{m}'))^2 \) is bijective on \( S_{k,l}(\Gamma_0(\infty)) \) if and only if it is bijective on \( S_{k,l}(\Gamma_0(\infty)) \).

For any \( f \in S_{k,l}(\Gamma_0(\infty)) \), \( (4.2) \) implies \( \Tr^p_{m}((\Tr^p_{m}(f)) = P^{l-k}(\Tr^p_{m}(T_p(f))) = P^{2l-2k}T_p(T_p(f)) \). Thus, \( \Id - P^{k-2l}(\Tr^p_{m}'))^2 = \Id - P^{k}T_p^2 \) on \( S_{k,l}(\Gamma_0(\infty)) \). The map \( \Id - P^{k-2l}(\Tr^p_{m}'))^2 \) is bijective on \( S_{k,l}(\Gamma_0(\infty)) \) if and only if the \( T_p \)-operator has no eigenform on \( S_{k,l}(\Gamma_0(\infty)) \) with eigenvalues \( \pm P^{k}/2 \). The result follows from Theorem 4.4. \( \square \)

We now discuss the validity of Question 4.3. We need a proposition, which is a generalisation of a result of Gekeler [10, Corollary 7.6], where \( T_p h = Ph \) for any prime ideal \( p = (P) \). We show that this continues to hold for \( f \in M_{k,1}(\Gamma_0(\infty)) \) with \( a_f(1) \neq 0 \).

PROPOSITION 4.6. Suppose the \( u \)-series expansion of \( f \in M_{k,1}(\Gamma_0(\infty)) \) at \( \infty \) is given by \( \sum_{j=0}^{\infty} a_f(j(q-1) + 1)u^{j(q-1)+1} \) with \( a_f(1) \neq 0 \). If \( T_p f = \lambda f \) for some \( \lambda \in C \), then \( \lambda = P \).

In particular, \( T_p f = P_{k/2} f \) can happen only when \( k = 2 \).

PROOF. Let \( G_{i,P}(X) \) denote the \( i \)th Goss polynomial corresponding to the lattice \( \Lambda_P = \ker(\rho_P) = \{ x \in C | \rho_P(x) = 0 \} \), where \( \rho_P \) is the Carlitz module with value at \( P \). By [1, Proposition 5.2] (the normalisation here is different),

\[
T_p f = P^k \sum_{j \geq 0} a_f(j(q-1) + 1)(u_P)^{j(q-1)+1} + \sum_{j \geq 0} a_f(j(q-1) + 1)G_{j(q-1)+1,P}(Pu),
\]

where \( u_P(z) = u(Pz) = u^q + \cdots \). To determine \( \lambda \), we compute the coefficient of \( u \) in the \( u \)-series expansion of \( T_p f \). In \( (4.3) \), the term involving \( u_P \) does not contribute to the coefficient of \( u \). By [10, Proposition 3.4(ii)],

\[
G_{i,P}(X) = X(G_{i-1,P}(X) + \alpha_1 G_{i-2,q,P}(X) + \cdots + \alpha_i G_{i-q',P}(X) + \cdots).
\]

In \( G_{i,P}(Pu) \), the coefficient of \( u \) in \( G_{j(q-1)+1,P}(Pu) \) is 0 for \( j > 0 \). Since \( G_{1,P}(X) = X \) (see [10, Proposition 3.4(v)]), \( T_p f = Pa_f(1)u + \text{higher terms} \). Comparing the coefficient of \( u \) on both sides gives \( \lambda = P \). \( \square \)

REMARK 4.7. The Goss polynomials, which occur as the coefficients of \( T_p f \), are very difficult to handle if \( l \neq 1 \) (see \( (4.3) \) and \([1, Proposition 5.2] \)). So, we have restricted ourselves to \( l = 1 \) in the last proposition.

We now prove that Conjecture 1.1 is true for \( S_{k,l}(\Gamma_0(T)) \) when \( \dim S_{k,l}(\Gamma_0(T)) \leq 1 \).

THEOREM 4.8. For \( m = A \), \( \deg P = 1 \), Question 4.3 is true for \( S_{k,l}(\Gamma_0(P)) \) when \( \dim S_{k,l}(\Gamma_0(T)) \leq 1 \). In particular, Conjecture 1.1 is true for \( S_{k,l}(\Gamma_0(T)) \) when \( \dim S_{k,l}(\Gamma_0(T)) \leq 1 \).

PROOF. Recall that \( \dim M_{k,l}(\Gamma_0(T)) \) is \( (k-l(q+1))/(q^2-1) + 1 \) (see [7, Proposition 4.3]). By [10, Theorem 5.13], the graded algebra \( \Delta_{k,l}M_{k,l}(\Gamma_0(T)) \) is generated by \( g_1, h \).

Suppose \( \dim S_{k,l}(\Gamma_0(T)) = 0 \). Then Question 4.3(1) is trivially true. Questions 4.3(2) and 4.3(3) are true by Proposition 4.5 and by the diagonalisability of the \( T_p \)-operator on \( S_{k,l}(\Gamma_0(T)) \).
Now, suppose \( \dim S_{k,l}(GL_2(A)) = 1 \). Clearly, the \( T_p \)-operator is diagonalisable on \( S_{k,l}(GL_2(A)) \). Therefore, combining Proposition 4.5 with the discussions in Section 4.1, Question 4.3 for the level \((P)\) is true if we show that \( \ker(T_p) = 0 \) and the \( T_p \)-operator has no eigenform on \( S_{k,l}(GL_2(A)) \) with eigenvalues \( \pm p^{k/2} \). Without loss of generality, we assume that \( p = (T) \).

We first consider the case \( l \neq 0 \). In this case, \( S_{k,l}(GL_2(A)) = \langle g_1^x h^y \rangle \) for some \( x \in \{0, \ldots, q\} \) such that \( k = x(q - 1) + l(q + 1) \). The \( u \)-series expansions of \( g_1, h \) are given by

\[
g_1 = 1 - (T^q - T)u^{q-1} - (T^q - T)u^{(q-1)(q^2-q+1)} + \cdots \in A[[u]],
\]

\[
h = -u - u^{1+(q-1)^2} + (T^q - T)u^{1+q(q-1)} - u^{1+(2q-2)(q-1)} + \cdots \in A[[u]].
\]

Therefore, \( g_1^x h^y = (-1)^j \sum_{i=0}^x (-1)^i (T^q - T)^i u^{(q-1)+l} + O(u^{(q-1)^2+l}) \in A[[u]] \). Let \( T_p(g_1^x h^y) = \sum_{j=0}^\infty a_{T_p(g_1^x h^y)}(j(q-1) + l)u^{(q-1)+l} \). By [10, Example 7.4],

\[
a_{T_p(g_1^x h^y)}(l) = \sum_{0 \leq j < l} (\frac{q-1}{l})^{T}a_{g_1^x h^y}(j(q-1) + l) \in A.
\]

Define \( x_0 := \min\{x, l-1\} \). Then, \( a_{T_p(g_1^x h^y)}(l) = \sum_{0 \leq j \leq x_0} (\frac{q-1}{l})^{T}a_{g_1^x h^y}(j(q-1) + l) \).

Clearly, the set \( \{0 \leq j \leq x_0 \mid (\frac{q-1}{l})^{T}a_{g_1^x h^y}(j(q-1) + l) \neq 0\} \) is nonempty; let \( j_{\max} \) be its maximum. Since \( \deg(T\sum_{j \neq j_{\max}} (\frac{q-1}{l})^{T}a_{g_1^x h^y}(j(q-1) + l)) < \deg(T\sum_{j \neq j_{\max}} (\frac{q-1}{l})^{T}a_{g_1^x h^y}(j(q-1) + l))\),

\[
0 < \deg(a_{T_p(g_1^x h^y)}(l)) = l + j_{\max}(q - 1) \leq l + x_0(q - 1) < \frac{x(q - 1) + l(q + 1)}{2}.
\]

Hence,

\[
1 < |a_{T_p(g_1^x h^y)}(l)| < q^{(x(q-1)+l(q+1))}/2. \tag{4.4}
\]

The first inequality in (4.4) shows that \( \ker(T_p) = 0 \). From (4.4), \( T_p(g_1^x h^y) \) cannot be equal to \( \pm T^{(x(q-1)+l(q+1))/2} g_1^x h^y \). In particular, the \( T_p \)-operator has no eigenform on \( S_{k,l}(GL_2(A)) \) with eigenvalues \( \pm p^{k/2} \).

We now consider the case \( l = 0 \). In this case, \( S_{k,0}(GL_2(A)) = \langle g_1^x \Delta \rangle \) for some \( x \in \{0, \ldots, q\} \) such that \( k = x(q - 1) + (q^2 - 1) \). Since \( \Delta = h^{q-1} \), we can argue as before replacing \( l \) with \( q - 1 \). We briefly sketch the proof. Recall that

\[
g_1^x \Delta = \sum_{i=0}^x (\frac{q-1}{i})^{T}(-1)^i u^{(i+1)(q-1)+l} + O(u^{(q-1)^2+l}) \in A[[u]].
\]

Since \( a_{g_1^x \Delta}(0) = 0 \), we have \( a_{T_p(g_1^x \Delta)}(q - 1) = \sum_{0 \leq j \leq q-1} (\frac{q-2}{j})^{T}a_{g_1^x \Delta}(j(q-1) + l) \) (see [10, Example 7.4]). Set \( y_0 := \min\{x, q-2\} \). Then,

\[
a_{T_p(g_1^x \Delta)}(q - 1) = \sum_{0 \leq j \leq y_0} (\frac{q-2}{j})^{T}a_{g_1^x \Delta}(j(q-1) + l) \sum_{0 \leq j \leq y_0} \Delta(\frac{q-2}{j})^{T}a_{g_1^x \Delta}(j(q-1) + l).
\]
As in the previous case, \( 1 < |a_{T_{\frac{1}{2}}(\Gamma_0)}(q - 1)| < q^{(x(q-1)+(q^2-1))/2} \), which shows that \( \ker(T_\nu) = 0 \) and the \( T_\nu \)-operator has no eigenform on \( S_{k,l}(\text{GL}_2(A)) \) with eigenvalues \( \pm T^{k/2} \). This completes the proof of the proposition.

We show part of Conjecture 1.1 is true for \( S_{k,l}(\Gamma_0(T)) \) when \( \dim S_{k,l}(\text{GL}_2(A)) = 2 \).

**Theorem 4.9.** Let \( m = A \) and \( \deg P = 1 \). If \( \dim S_{k,l}(\text{GL}_2(A)) = 2 \), then the direct sum decomposition in Question 4.3(2) is true for \( S_{k,l}(\Gamma_0(p)) \).

**Proof.** By Proposition 4.5, it is enough to show the \( T_\nu \)-operator has no eigenform on \( S_{k,l}(\text{GL}_2(A)) \) with eigenvalues \( \pm T^{k/2} \). Without loss of generality, we assume that \( \nu = (T) \). We give a complete proof only for \( l \neq 0 \). The proof is similar when \( l = 0 \).

Assume \( l \neq 0 \). Since \( \dim S_{k,l}(\text{GL}_2(A)) = 2 \), \( S_{k,l}(\text{GL}_2(A)) = \langle g_1^h, g_1^\Delta h \rangle \) for some \( y \in \{q + 1, \ldots, 2q + 1\} \) such that \( k = y(q - 1) + l(q + 1) \) and where \( x := y - (q + 1) \).

There are three cases to be considered. We first assume that \( l \neq 1 \). Recall the following \( u \)-expansions:

\[
\begin{align*}
\delta_1^y & = \begin{cases} 
\sum_{i=0}^{y} (\frac{y}{i}) (-1)^i (T^q - T)^i u^{i(q-1)} + O(u^{(l+q)(q-1)}) & \text{if } y < l + (q - 1), \\
\sum_{i=0}^{l+q-1} (\frac{y}{i}) (-1)^i (T^q - T)^i u^{i(q-1)} + O(u^{(l+q)(q-1)}) & \text{if } y \geq l + (q - 1),
\end{cases} \\
\delta_1^x & = \sum_{i=0}^{x} (\frac{x}{i}) (-1)^i (T^q - T)^i u^{i(q-1)} + O(u^{(q-1)(q^2-q+1)}),
\end{align*}
\]

\[
\begin{align*}
h' & = (-1)^l u^l + (-1)^l T_{\nu}^{-1} u^{(q-1)^2+l} + (-1)^l T_{\nu}^{-1} (T^q - T) u^{(q-1)^2+l} + O(u^{(l+q)(q-1)+l}), \\
\Delta & = -u^{q-1} + u^{(q-1)} - (T^q - T) u^{(q+1)(q-1)} + O(u^{(q^2-q+1)(q-1)}), \\
\Delta h' & = (-1)^l u^{l+1} + (-1)^l (1 - l) u^{(q-1)^2+l} + (-1)^l (l - 1) (T^q - T) u^{(q-1)^2+l} + O(u^{(l+q)(q-1)+l}),
\end{align*}
\]

from which we obtain

\[
\begin{align*}
\delta_1^x \Delta h' & = \begin{cases} 
(-1)^l \sum_{i=1}^{x+1} \frac{x}{i-1} (1) (-1)^{i-1} (T^q - T)^{i-1} u^{i(q-1)+l} + O(u^{(l+q)(q-1)+l}) & \text{if } x + 1 < l, \\
(-1)^l \sum_{i=1}^{x+1} \frac{x}{i-1} (1) (-1)^{i-1} (T^q - T)^{i-1} u^{i(q-1)+l} + O(u^{(l+q)(q-1)+l}) & \text{if } x + 1 \geq l,
\end{cases} \\
\delta_1^y h' & = (-1)^l \sum_{i=0}^{l-1} \frac{y}{i} (-1)^i (T^q - T)^i u^{i(q-1)+l} + O(u^{l(q-1)+l}).
\end{align*}
\]
We first show that \( T_x(g_x^1\Delta h^1) \neq \pm T^{(x(q-1)+(q^2-1)+l(q+1))/2}g_x^1\Delta h^1 \). From [10, Example 7.4], the \((l + (q - 1))\)th coefficient of \( T_x(g_x^1\Delta h^1) \) is given by

\[
a_{T_x(g_x^1\Delta h^1)}(l + (q - 1)) = \sum_{0 \leq i \leq l+q-1} (i+q-2)^{l+q-1-i} a_{g_x^1\Delta h^1}(l + (i + 1)(q - 1)). \tag{4.8}
\]

Taking the norm,

\[
|a_{T_x(g_x^1\Delta h^1)}(l + (q - 1))| \leq \max_{1 \leq i \leq l+q-1} \left( \left| T^{l+q-i} a_{g_x^1\Delta h^1}(l + i(q - 1)) \right| \right)
= \max_{1 \leq i \leq l+q-1} \left\{ T^{l+q-i} \sum_{\alpha \in \mathbb{N} \cup \{0\}, \beta \in \mathbb{N}} a_{g_x^1}(\alpha(q - 1)) \cdot a_{\Delta h^1}(\beta(q - 1) + l) \right\}.
\]

By (4.5) and (4.7), \( a_{g_x^1}(i(q - 1)) = 0 \) for \( x < i \leq l + q - 1 \) and \( a_{\Delta h^1}(\beta(q - 1) + l) = 0 \) for \( 1 \leq \beta \leq l + q - 1 \) with \( \beta \notin \{1, q, q + 1\} \). Therefore, the maximum above is

\[
\leq \max_{\beta \in \{1, q, q + 1\}} \left\{ \max_{0 \leq i \leq l+q-1} \left| T^{l+q-i} a_{g_x^1}((i - \beta)(q - 1)) a_{\Delta h^1}(\beta(q - 1) + l) \right| \right\}
= \max \left\{ \max_{1 \leq i \leq l+q-1} \left| q^{|i\beta-q+1|} \right|, \max_{1 \leq i \leq l+q-1} \left| q^{|i\beta-(q-1)+1|} \right|, \max_{1 \leq i \leq l+q-1} \left| q^{|i\beta-(q+1)+1|} \right| \right\}
\leq \max \left\{ q^{(x+1)(q-1)+l}, q^{(x-1)(q-1)+l}, q^{(x+1)(q-1)+l} \right\} = q^{(x+1)(q-1)+l}.
\]

Hence,

\[
|a_{T_x(g_x^1\Delta h^1)}(l + (q - 1))| \leq q^{(x+1)(q-1)+l}. \tag{4.9}
\]

The assumption \( l \neq 1 \) implies \((x + 1)(q - 1) + l < (x(q - 1) + (q^2 - 1) + l(q + 1))/2\). Since \( a_{g_x^1\Delta h^1}(l + (q - 1)) = (-1)^{l+1} \), combining the last inequality with (4.9), we get

\[
T_x(g_x^1\Delta h^1) \neq \pm T^{(x(q-1)+(q^2-1)+l(q+1))/2}g_x^1\Delta h^1.
\]

By the same technique, we give an upper bound on the coefficient \( a_{T_x(g_x^1\Delta h^1)}(l + (q - 1)) \). Recall that

\[
a_{T_x(g_x^1\Delta h^1)}(l + (q - 1)) = \sum_{0 \leq i \leq l+q-1} (i+q-2)^{l+q-1-i} a_{g_x^1\Delta h^1}(l + (i + 1)(q - 1)). \tag{4.10}
\]

Taking the norm,

\[
|a_{T_x(g_x^1\Delta h^1)}(l + (q - 1))| \leq \max_{1 \leq i \leq l+q-1} \left( \left| T^{l+q-i} a_{g_x^1\Delta h^1}(l + i(q - 1)) \right| \right)
= \max_{1 \leq i \leq l+q-1} \left\{ T^{l+q-i} \sum_{\alpha \in \mathbb{N} \cup \{0\}, \beta \in \mathbb{N}} a_{g_x^1}(\alpha(q - 1)) \cdot a_{\Delta h^1}(\beta(q - 1) + l) \right\}.
\]

By (4.6), \( a_{\Delta h^1}(\beta(q - 1) + l) = 0 \) for \( 0 \leq \beta \leq l + q - 1 \) with \( \beta \notin \{0, q, q + 1\} \). When \( y < l + q - 1 \), we have \( a_{g_x^1}(i(q - 1)) = 0 \) for \( y < i \leq l + q - 1 \), and when \( y \geq l + (q - 1) \), we have \( a_{g_x^1}(i(q - 1)) = 0 \) for \( l + q - 1 \leq i \leq y \). Computing the maximum in each case as before,
\[ |a_{T_p(g_1^i h^l)}(l + (q - 1))| \leq \begin{cases} q^{y(q-1)+l+q} & \text{if } y < l + (q - 1), \\ q^{(l+(q-1))(q-1)+l+q} & \text{if } y \geq l + (q - 1). \end{cases} \tag{4.11} \]

Since \( T_p(g_1^i \Delta h^l) \neq \pm T(x(q-1)+(q^2-1)+k(q+1))/2 g_1^i \Delta h^l \), it is now enough to show that there does not exist any \( c \in C \) such that

\[ T_p(g_1^i h^l + cg_1^i \Delta h^l) = \pm T^{y(q-1)+k(q+1))/2} (g_1^i h^l + cg_1^i \Delta h^l). \tag{4.12} \]

In contrast, suppose there is an element \( c \in C \) such that (4.12) holds with the ‘+’ sign. A similar argument works with ‘−’. The \( l \)th coefficients of \( T_p(g_1^i h^l) \) and \( T_p(g_1^i \Delta h^l) \) are given by (see [10, Example 7.4])

\[
a_{T_p(g_1^i h^l)}(I) = (-1)^l \sum_{0 \leq j < l} \binom{l}{j} T^{l-j}(\sum_{j_1}(-1)^j(T^q - T)^j, \\
a_{T_p(g_1^i \Delta h^l)}(I) = \begin{cases} (-1)^{l+1} \sum_{j=1}^{x_1} \binom{l}{j} T^{l-j}(\sum_{j_1}(-1)^j(T^q - T)^j) & \text{if } x_1 < l, \\
(1)^{l+1} \sum_{j=1}^{x_1} \binom{l}{j} T^{l-j}(\sum_{j_1}(-1)^j(T^q - T)^j) & \text{if } x_1 \geq l. \end{cases} \]

Comparing the \( l \)th coefficients on both sides of (4.12) gives

\[
\sum_{0 \leq j < l} \binom{l}{j} T^{l-j}(\sum_{j_1}(-1)^j(T^q - T)^j) - c \sum_{j=1}^{x_1} \binom{l}{j} T^{l-j}(\sum_{j_1}(-1)^j(T^q - T)^j) = T^{y(q-1)+l(q+1))/2}, \tag{4.13} \]

where \( x_0 := \min\{x, l - 2\} + 1 \). Here, \( c \sum_{j=1}^{x_0} \binom{l}{j} T^{l-j}(\sum_{j_1}(-1)^j(T^q - T)^j) \neq 0 \). Otherwise, the inequality \( lq < (y(q-1) + l(q+1))/2 \) would imply that the two sides of (4.13) have different degrees. Let \( j_{\max} := \max\{1 \leq j \leq x_0 \mid \binom{l}{j} \neq 0 \} \). Then, \( \sum_{j=1}^{x_0} \binom{l}{j} T^{l-j}(\sum_{j_1}(-1)^j(T^q - T)^j) \neq 0 \). Again, since \( lq < (y(q-1) + l(q+1))/2 \), it follows that

\[
|T^{y(q-1)+l(q+1))/2} - \sum_{0 \leq j < l} \binom{l}{j} T^{l-j}(\sum_{j_1}(-1)^j(T^q - T)^j)| = q^{y(q-1)+l(q+1))/2}. \]

Therefore, (4.13) gives

\[
c = \frac{T^{y(q-1)+l(q+1))/2} - \sum_{0 \leq j < l} \binom{l}{j} T^{l-j}(\sum_{j_1}(-1)^j(T^q - T)^j)}{\sum_{j=1}^{j_{\max}} \binom{l}{j} T^{l-j}(\sum_{j_1}(-1)^j(T^q - T)^j)} \in K \tag{4.14} \]

and \( |c| = q^{y(q-1)+l(q+1))/2} - (j_{\max}(q+1)+l-q) \). Since \( lq < (y(q-1) + l(q+1))/2 \), from (4.14),

\[
|a_{T_p(g_1^i h^l + cg_1^i \Delta h^l)}((q - 1) + l)| = \left| (-1)^l y(T^q - T) + (-1)^{l+1} c \right| = q^{y(q-1)+l(q+1))/2} - (j_{\max}(q-1)+l-q). \tag{4.15} \]
Comparing the \((q - 1) + l\)th coefficients on both sides of (4.12),
\[
|a_{T_p(g_1^\infty h + c g_1^\Delta h)}((q - 1) + l)| = q^{y(q-1)+(q+1)-(j_{\max}(q-1)+l-q)}, \tag{4.16}
\]
However, from (4.11),
\[
|a_{T_p(g_1^\infty h + c g_1^\Delta h)}((q - 1) + l)| \leq \max\{|a_{T_p(g_1^\Delta h)}((q - 1) + l)|, |c||a_{T_p(g_1^\Delta h)}((q - 1) + l)|\} \\
\leq \max\{q^{\gamma_0(q-1)+l+q}, q^{\gamma_0(q-1)+l(q+1)/2-(j_{\max}(q-1)+l-q)}q^{(x+1)(q-1)+l}\},
\]
where \(\gamma_0 := \min[y, l + (q - 1)].\) Since \(0 \leq j_{\max} < l,\) an easy verification shows that
\[
q^{\gamma_0(q-1)+l+q} < q^{\gamma_0(q-1)+l(q+1)/2-(j_{\max}(q-1)+l-q)}.\]
Moreover, \((x+1)(q-1)+l < (x(q-1) + (q^2 - 1) + l(q+1))/2\) implies
\[
q^{\gamma_0(q-1)+l(q+1)/2-(j_{\max}(q-1)+l-q)}q^{(x+1)(q-1)+l} < q^{(x(q-1) + l(q+1)-(j_{\max}(q-1)+l-q)}.
\]
Therefore, we can conclude
\[
|a_{T_p(g_1^\infty h + c g_1^\Delta h)}((q - 1) + l)| < q^{(x(q-1) + l(q+1)-(j_{\max}(q-1)+l-q)}
\]
which contradicts (4.16). Hence, the \(T_p\)-operator has no eigenform on \(S_{k,l}(\text{GL}_2(A))\) with eigenvalue \(\pm T^{k/2}\), and the result now follows from Proposition 4.5.

We now consider the case when \(l = 1\) and \(y \neq 2q + 1.\) Since \(a_{g_1^\Delta h}(1) = 0\) and \((x+1)(q-1) + 1 < (x(q-1) + (q^2 - 1) + (q + 1))/2,\) a similar argument shows that
\[
T_p(g_1^\Delta h) \neq \pm T^{(x(q-1)+(q^2-1)+(q+1))}/g_1^\Delta h
\]
(see (4.9)). However, since \(a_{c_1 g_1^\Delta h + c_2 g_1^\Delta h}(1) \neq 0\) for \(c_2 \in C \setminus \{0\}\) and \((x(q-1) + (q^2 - 1) + (q + 1)) > 2,\) Proposition 4.6 implies that
\[
T_p(c_1 g_1^\Delta h + c_2 g_1^\Delta h) \neq \pm T^{(x(q-1)+(q^2-1)+(q+1))}/(c_1 g_1^\Delta h + c_2 g_1^\Delta h).
\]
We now consider the case \((y, l) = (2q + 1, 1).\) By the \(u\)-series expansion, \(a_{T_p(g_1^{2q+1} h)}(1) = -T, a_{T_p(g_1^\infty h)}(1) = 0\) and \(a_{T_p(g_1^\Delta h)}(q) = T^q.\) This implies that for any \((c_1, c_2) \in C^2 \setminus \{(0, 0)\},\)
\[
T_p(c_1 g_1^{2q+1} h + c_2 g_1^\infty h) \neq T^q (c_1 g_1^{2q+1} h + c_2 g_1^\infty h).
\]
This can be checked by comparing the \(q\)th coefficients if \(c_1 = 0\) and by Proposition 4.6 if \(c_1 \neq 0.\) Now, we are done by Proposition 4.5.

We now consider the last case \(l = 0.\) Here, the proof is similar to \(l \neq 0,\) except that we need to consider the \((q - 1)\)th, \(2(q - 1)\)th coefficients and use the inequality \((x + 2)(q - 1) < (x(q-1) + 2(q^2 - 1))/2.\)

\section{4.2. Evidence for Question 4.3 for prime ideals \(p.\)}

We now give some instances where Question 4.3 for prime ideals \(p\) has an affirmative answer.

\begin{proposition}
For any prime ideal \(p,\) Question 4.3 is true for level \(p\) in the following cases:
\begin{enumerate}
\item \((a)\) \(1 \leq l \leq q - 2\) and \(k = 2l + \alpha(q - 1),\) where \(\alpha \in \{0, \ldots, l\};\)
\item \((b)\) \(l = 0\) and \(k = \beta(q - 1),\) where \(\beta \in \{1, \ldots, q + 1\};\)
\item \((c)\) \(l = 1\) and \(k = \alpha(q - 1) + (q + 1),\) where \(\alpha \in \{0, \ldots, q\};\)
\item \((2)\) \(k \leq 3q.\)
\end{enumerate}
\end{proposition}

\begin{proof}
In all of these cases, \(\dim S_{k,l}(\text{GL}_2(A)) \leq 1.\) Hence, the \(T_p\)-operator is diagonalisable on \(S_{k,l}(\text{GL}_2(A)).\) As in our earlier discussion, Question 4.3 has an affirmative answer for \(p\) if we show that \(\ker(T_p) = 0\) and the \(T_p\)-operator has no eigenform on \(S_{k,l}(\text{GL}_2(A))\) with eigenvalues \(\pm T^{k/2}.\) We prove these statements in all cases.
\end{proof}
PROOF. We may assume that $l > 0$, $M_{k,l}(\mathbb{G}_2) = S_{k,l}(\mathbb{G}_2)$. If $\alpha \in \{0, \ldots, l - 1\}$, then $S_{2l+\alpha(q-1),l}(\mathbb{G}_2) = 0$ and the result follows trivially. If $\alpha = l$, then $S_{2l+\alpha(q-1),l}(\mathbb{G}_2) = 1$ and $S_{2l+\alpha(q-1),l}(\mathbb{G}_2) = \langle h^l \rangle$. By [14, (9)] (or by [16, Theorem 3.17]), the $T_q$-operator acts on $h^l$ by $P^l$ for $1 \leq i \leq q - 2$. Since $P^l \neq \pm P^{(q+1)/2}$ for $1 \leq l \leq q - 2$, the result follows.

(1b). When $l = 0$, we prove the required claim in two steps. For $\beta \in \{1, \ldots, q\}$, $M_{\beta(q-1),0}(\mathbb{G}_2) = \langle g_1^{\beta} \rangle$. Therefore, $S_{\beta(q-1),0}(\mathbb{G}_2) = \{0\}$ and the result follows. If $\beta = q + 1$, $S_{q^2-1,0}(\mathbb{G}_2) = \langle \Delta \rangle$. By [10, Corollary 7.5], $T_\psi(\Delta) = P^{q-1}\Delta$. Since $P^{q-1} \neq \pm P^{q^2-1/2}$, the result follows.

(1c). If $\alpha \in \{0, \ldots, q\}$, $S_{k,1}(\mathbb{G}_2) = \langle g_1^\alpha h \rangle$. Since $a_{g_1^\alpha h}(1) \neq 0$, by Proposition 4.6, $T_\psi(g_1^\alpha h) = Pg_1^\alpha h$ and the result follows.

(2). This part can be deduced from the earlier known cases as follows. Let $0 \leq l \leq q - 2$. If $k \neq 2l$ (mod $q - 1$), then $M_{k,0}(\mathbb{G}_2) = \{0\}$ and Question 4.3 is trivially true. So, we only consider the cases $k \equiv 2l$ (mod $q - 1$), that is, $k = 2l + x(q - 1)$ for some $x \in \mathbb{N} \cup \{0\}$. The condition $k \leq 3q$ implies $x \leq 4$.

If $x < l$, then dim $M_{k,0}(\mathbb{G}_2) = 0$ and the result follows. If $x = l$, then $k = l(q + 1)$. If $l \neq 0$, then $S_{k,0}(\mathbb{G}_2) = \langle h^l \rangle$. So, we are back to case 1(a). If $l = 0$, then $S_{k,0}(\mathbb{G}_2) = \{0\}$ and the result follows. Therefore, the remaining cases of interest are $l < x \leq 4$. If $l \geq 2$, the inequality $k \leq 3q$ forces $x \leq 2$ and we are back to the case $x \leq l$. So, it is enough to consider $l \in \{0, 1\}$ with $l < x \leq 4$.

For $l = 0$: if $(q, x) \neq (3, 4)$, then $M_{3q-1,0}(\mathbb{G}_2) = \langle g_1^3 \rangle$, $S_{3q-1,0}(\mathbb{G}_2) = \{0\}$ and the result follows; if $(q, x) = (3, 4)$, then $k = (q + 1)(q - 1)$ and we are back to case 1(b). For $l = 1$, we have $k = (x - 1)(q - 1) + (q + 1)$ where $1 < x \leq 3$. Since $q \geq 3$, we are back to case 1(c). This completes the proof of the proposition. □

We remark that our Proposition 4.10 is similar to [4, Theorem 5.8, Corollary 5.11 and Theorem 5.14] for the $\psi = (T)$-case. In contrast to Proposition 4.10, we have the following proposition.

**Proposition 4.11.** For $\deg m = 1$ and $\psi \neq m$, Question 4.3 is true for level $\psi m$ when:

(i) $l > (q - 1)/2$ and $k = 2l - (q - 1)$; or

(ii) $l = 1$ and $k = q + 1$.

**Proof.** We may assume that $m = (T)$, since a similar calculation works for any ideal $m$ with $\deg m = 1$. We now follow the strategy in the proof of the Proposition 4.10.

(i). In this case, $S_{k,l}(\Gamma_0(T)) = \{0\}$ by [8, Proposition 4.1] and the result follows trivially.

(ii) First, we show that the operator $T_\psi - P$ is zero on $S_{q+1,1}(\Gamma_0(T))$. Recall that $\Delta_T(z) := (g_1(Tz) - g_1(z))/(T - T), \Delta_W(z) := (Tg_1(Tz) - Tg_1(z))/(T - T) \in M_{q+1,0}(\Gamma_0(T))$.

By [8, Proposition 4.3(3)], $\dim C_S_{q+1,1}(\Gamma_0(T)) = 2$ with basis $\{\Delta_T E_T, \Delta_W E_T\}$. By [8, Proposition 4.3(8)], $h = -\Delta_W E_T$. Since $T_q h = Ph$, we obtain $T_\psi(\Delta_W E_T) = P\Delta_W E_T$. Note that $\Delta_T = -T^{-1}\Delta_W W_{q-1,0} W_T$ and $T_\psi W_T = W_T^{(T)} T_\psi$ by [19, Theorem 1.1]. Using $E_T|_{2,1} W_T^{(T)} = -E_T$ [9, Proposition 3.3],
PROOF. Let 

\[ T_\wp(\Delta_T E_T) = T_\wp((T^{-1} \Delta w E_T)_{|q+1,1} W_T^{(T)}) = (T_\wp(T^{-1} \Delta w E_T))_{|q+1,1} W_T^{(T)} \]

\[ = T^{-1}(P \Delta w E_T)_{|q+1,1} W_T^{(T)} = P \cdot T^{-1} \Delta w_{|q-1,0} W_T^{(T)} \cdot E_T|_{2,1} W_T^{(T)} = P \Delta_T E_T. \]

Thus, \( T_\wp \equiv P \) on \( S_{q+1,1}(\Gamma_0(T)) \). So, the \( T_\wp \)-operator is injective and diagonalisable on \( S_{q+1,1}(\Gamma_0(T)) \), which proves Question 4.3(1). Question 4.3(2) follows from Proposition 4.5. Finally, Question 4.3(3) follows from the diagonalisability of the \( T_\wp \)-operator on \( S_{q+1,1}(\Gamma_0(T)) \). □

4.3. Counterexample to question in Question 4.3(2). In this section, we show that the direct sum decomposition in Question 4.3(2) does not hold if \( m \neq A \) and \( (k, l) = (2, 1) \) because there are nonzero Drinfeld cusp forms which are both \( \wp \)-oldforms and \( \wp \)-newforms. We first prove a result which is of independent interest.

LEMMA 4.12. Let \( \wp_1, \wp_2 \) be two distinct prime ideals of \( A \) generated by monic irreducible polynomials \( P_1, P_2 \), respectively. Then, \( T_{\wp_1} E_{\wp_2} = P_1 E_{\wp_2} \).

PROOF. By [10, (8.2)], the function \( E(z) = \sum_{a \in A} \wp(a z) \), where \( A \) denotes the set of all monic polynomials in \( A \). Hence, \( E_{\wp_1}(z) = \sum_{a \in A,} \wp(a z) = P_2 \sum_{a \in A,} \wp(P_2 a) \). We now use an argument in the proof of [16, Theorem 2.3] to get

\[ T_{\wp_1} E_{\wp_2} = \sum_{Q \in A, \deg Q < \deg P_1} E_{\wp_2}\left(\frac{z + Q}{P_1}\right) + P_1^2 E_{\wp_2}(P_1 z) \]

\[ = \sum_{Q \in A, \deg Q < \deg P_1} \sum_{a \in A,} \wp(a \frac{z + Q}{P_1}) + P_1^2 \sum_{a \in A,} \wp(P_1 a)z \]

\[ = \frac{1}{\wp} \sum_{Q \in A, \deg Q < \deg P_1} \sum_{a \in A,} \wp(a z + aQ + P_1 b) + P_1 \sum_{a \in A,} \wp(P_1 a)z \]

\[ = \frac{1}{\wp} \sum_{a \in A,} \wp(a z + aQ + P_1 b) + P_1 \sum_{a \in A,} \wp(P_1 a)z \]

\[ = P_1 \sum_{a \in A,} \wp(a z) + P_1 \sum_{a \in A,} \wp(P_1 a)z = P_1 \sum_{a \in A,} \wp(a z) = P_1 E_{\wp_2}. \]

This completes the proof of the Lemma. □

PROPOSITION 4.13. Suppose \( m \neq A \). For any prime ideal \( \wp \nmid m \),

\[ S_{2,1}^{\text{old}}(\Gamma_0(pm)) \cap S_{2,1}^{\text{new}}(\Gamma_0(pm)) \neq \{0\}. \]

PROOF. Let \( \wp \) be a prime divisor of \( m \) generated by a monic irreducible polynomial \( P_2 \). Clearly, \( 0 \neq E_{P_2} - \delta_{P} E_{P_2} \in S_{2,1}(\Gamma_0(pm)) \).
We show that $P_2 - \delta_P P_2 \in S_{p_{old}}(\Gamma_0(pm)) \cap S_{p_{new}}(\Gamma_0(pm))$. By definition, $P_2 - \delta_P P_2 \in S_{p_{old}}(\Gamma_0(pm))$. Combining (4.2), (4.1) and Lemma 4.12,

$$Tr_m^p(P_2 - \delta_P P_2) = E_{P_2} - P^{-1}T_p(E_{P_2}) = E_{P_2} - P_{P_2} = 0. \quad (4.17)$$

By (4.1),

$$Tr_m^p((P_2 - \delta_P P_2)|W_{p}(pm)) = Tr_m^p(E_{P_2}|W_{p}(pm) - (\delta_P P_2)|W_{p}(pm))$$

$$= Tr_m^p(\delta_P P_2 - P_{P_2}) = 0.$$

This proves that $P_2 - \delta_P P_2 \in S_{p_{new}}(\Gamma_0(pm))$. The result follows. \qed

**Remark 4.14.** For $f \in S_k(\Gamma_0(n))$, $T_p(f^{q^n}) = (T_p(f))^{q^n}$ for any $n \in \mathbb{N}$. An argument similar to Proposition 4.13 gives

$$0 \neq P_{P_2} - P^{q^n-1}\delta_P P^{q^n} \in S_{p_{old}}(\Gamma_0(pm)) \cap S_{p_{new}}(\Gamma_0(pm)). \quad (4.18)$$

Since $E$ behaves like a classical weight 2 Eisenstein series, we believe that the phenomenon in (4.18) may not happen for $l \neq 1$.

Proposition 4.13 and Remark 4.14 imply that either one needs to reformulate the definition of $p$-newforms for level $pm$ or exclude the cases above in formulating Question 4.3 for level $pm$.

### 5. Oldforms and newforms for square-free level $n$

For classical modular forms, it is well known that the space of newforms can be characterised in terms of kernels of the Trace and twisted Trace operators [15, 18]. In this section, we examine the extent that this is valid for Drinfeld modular forms.

Throughout this section, we assume that $n$ is a square-free ideal of $A$ generated by a (square-free) monic polynomial $n \in A$. Let $p, p_1$ be two prime ideals of $A$ generated by monic irreducible polynomials $P, P_1 \in A$, respectively.

**Definition 5.1 (Oldforms).** The space of oldforms of weight $k$, type $l$ and square-free level $n$ is defined as

$$S_{k,l}^{old}(\Gamma_0(n)) := \sum_{p | n} (\delta_1, \delta_P)((S_{k,l}(\Gamma_0(n/p)))^2).$$

**Definition 5.2 (Newforms).** The space of newforms of weight $k$, type $l$ and square-free level $n$ is defined as

$$S_{k,l}^{new}(\Gamma_0(n)) := \bigcap_{p | n} (\ker(Tr_{n/p}^n) \cap \ker(Tr_{n/p}^n)), \quad \text{where} \ \ Tr_{n/p}^n f = Tr_{n/p}^n(f|_kW_{p}(n)).$$

The action of Hecke operators on $S_{k,l}(\Gamma_0(n))$, $S_{k,l}^{new}(\Gamma_0(n))$ depends on the commutativity of the (partial) Atkin–Lehner operators with the $T_p$ and $U_p$-operators.
**Theorem 5.3** [19, Theorem 1.1]. Let \( n, p \subseteq A \) be ideals such that \( p \nmid n \) and \( p \) is prime. For any ideal \( \mathfrak{d} \) of \( A \) such that \( \mathfrak{d} \mid n \), the actions of \( T_p W_{\mathfrak{d}}^{(n)} \) and \( W_{\mathfrak{d}}^{(n)} T_p \) on \( S_{k,l}(\Gamma_0(n)) \) are equal.

The following result can be thought of as a generalisation of Theorem 5.3 to the \( U_p \)-operator. Note that Theorem 5.4 holds for any integral ideal \( n \).

**Theorem 5.4.** Assume that \( n^\alpha \mid n \) for some \( \alpha \in \mathbb{N} \). For all prime divisors \( p_1 \) of \( n \) with \( p_1 \neq p \), the actions of \( U_{p_1} W_{p_1}^{(n)} \) and \( W_{p_1}^{(n)} U_{p_1} \) on \( S_{k,l}(\Gamma_0(n)) \) are equal.

**Proof.** By definition,

\[
P_{1-k}^{P} U_{p_1} W_{p_1}^{(n)} = \sum_{Q \subseteq A} \left( \sum_{\deg Q < \deg P_1} \left( \frac{P}{n} P_1 \right) \left( \frac{Q}{nP_1} P_1 \right) \right) \sum_{Q \subseteq A} \left( \sum_{\deg Q < \deg P_1} \left( \frac{P}{n} P_1 \right) \left( \frac{Q}{nP_1} P_1 \right) \right),
\]

which can be computed as

\[
P_{1-k}^{P} W_{p_1}^{(n)} U_{p_1} = \sum_{Q \subseteq A} \left( \sum_{\deg Q < \deg P_1} \left( \frac{Q}{nP_1} P_1 \right) \left( \frac{P}{n} P_1 \right) \right) \sum_{Q \subseteq A} \left( \sum_{\deg Q < \deg P_1} \left( \frac{Q}{nP_1} P_1 \right) \left( \frac{P}{n} P_1 \right) \right).
\]

To prove the proposition, it suffices to show that for any \( Q \in A \) with \( \deg Q < \deg P_1 \), there exists a unique \( Q' \in A \) with \( \deg Q' < \deg P_1 \) such that

\[
\left( \frac{P}{n} P_1 \right) \left( \frac{Q}{nP_1} P_1 \right) = \left( \frac{x}{w} \right) \left( \frac{y}{z} \right) \left( \frac{P}{n} P_1 \right) \left( \frac{Q}{nP_1} P_1 \right),
\]

for some \( (x, y, z) \in \Gamma_0(n) \). For any \( Q, Q' \in A \), (5.1) implies \( x, w \in A, z \in n \) and

\[
-P_1y = P^n Q' - P^n Qd - b + \left( nQQ' + bP_1 + \frac{n}{P} bP_1 Q \right).
\]

Thus, we are reduced to showing that for any \( Q \in A \) with \( \deg Q < \deg P_1 \), there exists a unique \( Q' \in A \) with \( \deg Q' < \deg P_1 \) such that \( y \in A \).

Since \( P_1 \mid n \), we have \( P_1 \mid nQQ' + bP_1 + (P^n/P^n)bP_1 Q \). Now it is enough to show that there exists a unique \( Q' \in A \) such that \( P_1 \mid P^n Q' - Qd - b \).

Recall that \( P^n d - bn/P^n = 1 \). Since \( P_1 \) divides \( n/P^n \), we get \( QP^n d \equiv Q \) (mod \( P_1 \)) for any \( Q \in A \). So, it is enough to show that with a unique \( Q' \in A \) such that \( P_1 \mid P^n Q' - (Q + b) \). Since \( (P^n, P_1) = 1 \), the congruence \( P^n f(X) \equiv (Q + b) \) (mod \( P_1 \)) has a unique solution in \( A \) with \( \deg(f(X)) < \deg P_1 \).

**Theorem 5.5.** The spaces \( S_{k,l}^{\text{old}}(\Gamma_0(n)) \), \( S_{k,l}^{\text{new}}(\Gamma_0(n)) \) are invariant under the action of the Hecke operators \( T_p \), for \( p \nmid n \) and \( U_p \) for \( p \mid n \).

**Proof.** Let \( p \) be a prime ideal of \( A \) such that \( p \mid n \). We first show that the space \( S_{k,l}^{\text{new}}(\Gamma_0(n)) \) is stable under the \( U_p \)-operator. Let \( p_1 \neq p \) be a prime divisor of \( n \) and \( f \in S_{k,l}^{\text{new}}(\Gamma_0(n)) \). By Theorem 5.4 and Proposition 3.6, the \( U_p \)-operator commutes with the \( W_{p_1}^{(n)} \)-operator and the \( U_{p_1} \)-operator. Since \( f \in S_{k,l}^{\text{new}}(\Gamma_0(n)) \), from Proposition 3.8,

\[
\text{Tr}_{n/p_1}^n (U_p(f)) = U_p(f) + P_{1-k}^{P} U_{p_1} (U_p(f) W_{p_1}^{(n)}) = U_p(\text{Tr}_{n/p_1}^n (f)) = 0.
\]
A similar argument shows that \( \operatorname{Tr}_{n/p_1}^n U_p(f) = 0 \). Thus, \( S_{k,l}^{\text{new}}(\Gamma_0(\mathfrak{n})) \) is stable under the \( U_p \)-operator. Since the space \( S_{k,l}^{\text{old}}(\Gamma_0(\mathfrak{n})) \) is stable under the action of the \( U_p \)-operator by [5, Proposition 2.15], the space \( S_{k,l}^{\text{new}}(\Gamma_0(\mathfrak{n})) \) is stable under the action of the \( U_p \)-operator.

Next, we show that the space \( S_{k,l}^{\text{old}}(\Gamma_0(\mathfrak{n})) \) is stable under the action of the \( U_p \)-operator. Let \( p_1 \neq p \) be a prime divisor of \( \mathfrak{n} \). Let \( \psi, \varphi \in S_{k,l}(\Gamma_0(\mathfrak{n}/p_1)) \). Since \( p \mid (\mathfrak{n}/p_1) \), we have \( U_p(\psi), U_p(\varphi) \in S_{k,l}(\Gamma_0(\mathfrak{n}/p_1)) \). Moreover, (4.1) and Theorem 5.4 yield

\[
U_p(\delta_p \varphi) = U_p(\varphi|W_p^{(n)}) = (U_p(\varphi)|W_p^{(n)} = \delta_p(U_p(\varphi)).
\]

Hence, for all \( p_1 \mid \mathfrak{n} \) with \( p_1 \neq p \), we have \( U_p(\psi + \delta_p \varphi) = U_p(\psi) + \delta_p U_p(\varphi) \) with \( U_p(\psi), U_p(\varphi) \in S_{k,l}(\Gamma_0(\mathfrak{n}/p_1)) \). Since the space \( S_{k,l}^{\text{old}}(\Gamma_0(\mathfrak{n})) \) is stable under the action of the \( U_p \)-operator by [5, Proposition 2.15], the space \( S_{k,l}^{\text{old}}(\Gamma_0(\mathfrak{n})) \) is stable under the action of the \( U_p \)-operator.

By a similar argument, the spaces \( S_{k,l}^{\text{new}}(\Gamma_0(\mathfrak{n})) \) and \( S_{k,l}^{\text{old}}(\Gamma_0(\mathfrak{n})) \) are stable under the \( T_p \)-operator for \( p \nmid \mathfrak{n} \).

\[\square\]

**Corollary 5.6.** The set of \( U_p \)-operators (for \( p \mid \mathfrak{n} \)) are simultaneously diagonalisable on \( S_{k,l}^{\text{new}}(\Gamma_0(\mathfrak{n})) \).

**Proof.** Let \( \mathfrak{p} \) be a prime ideal of \( A \) such that \( p \nmid \mathfrak{p} \). By [5, Remark 2.17], the \( U_p \)-operator is diagonalisable on \( S_{k,l}^{\text{new}}(\Gamma_0(\mathfrak{n})) \). By Theorem 5.5, the space \( S_{k,l}^{\text{new}}(\Gamma_0(\mathfrak{n})) \) is a \( U_p \)-invariant subspace of \( S_{k,l}^{\text{new}}(\Gamma_0(\mathfrak{n})) \); hence, the \( U_p \)-operator is also diagonalisable on \( S_{k,l}^{\text{new}}(\Gamma_0(\mathfrak{n})) \). Now, the corollary follows from Proposition 3.6 and the fact that a commuting set of diagonalisable operators on a finite dimensional vector space are simultaneously diagonalisable.

Finally, we remark that \( S_{k,l}^{\text{old}}(\Gamma_0(\mathfrak{n})) \cap S_{k,l}^{\text{new}}(\Gamma_0(\mathfrak{n})) = \{0\} \) may happen only for \( l \neq 1 \).

**Lemma 5.7.** For any two distinct prime ideals \( p, q \) generated by monic irreducible polynomials \( P, Q \), respectively, the intersection \( S_{2q^r,1}^{\text{old}}(\Gamma_0(pq)) \cap S_{2q^r,1}^{\text{new}}(\Gamma_0(pq)) \neq \{0\} \) for any \( n \in \mathbb{N} \).

**Proof.** By an argument similar to that in the proof of Proposition 4.13, it follows that \( 0 \neq E_Q^{(nP)} - P_q r^{-1} \delta P E_Q^{(nP)} \in S_{2q^r,1}^{\text{old}}(\Gamma_0(pq)) \cap S_{2q^r,1}^{\text{new}}(\Gamma_0(pq)) \).

\[\square\]

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