Analysis of Fractional-Order Nonlinear Dynamic Systems with General Analytic Kernels: Lyapunov Stability and Inequalities

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Abstract: In this paper, we study the recently proposed fractional-order operators with general analytic kernels. The kernel of these operators is a locally uniformly convergent power series that can be chosen adequately to obtain a family of fractional operators and, in particular, the main existing fractional derivatives. Based on the conditions for the Laplace transform of these operators, in this paper, some new results are obtained—for example, relationships between Riemann–Liouville and Caputo derivatives and inverse operators. Later, employing a representation for the product of two functions, we determine a form of calculating its fractional derivative; this result is essential due to its connection to the fractional derivative of Lyapunov functions. In addition, some other new results are developed, leading to Lyapunov-like theorems and a Lyapunov direct method that serves to prove asymptotic stability in the sense of the operators with general analytic kernels. The FOB-stability concept is introduced, which generalizes the classical Mittag–Leffler stability for a wide class of systems. Some inequalities are established for operators with general analytic kernels, which generalize others in the literature. Finally, some new stability results via convex Lyapunov functions are presented, whose importance lies in avoiding the calculation of fractional derivatives for the stability analysis of dynamical systems. Some illustrative examples are given.

Keywords: nonlinear fractional-order systems; general analytic kernels; FOB-stability; generalized lyapunov stability; inequalities

1. Introduction

Fractional calculus generalizes the well-known classical calculus to operators with non-integer orders. In this sense, this theory has had as a main purpose to extend classical mathematical results and develop a new formulation of calculus; this has been done along with the development of integer calculus. However, fractional calculus did not begin to have practical applications until the 1970s. The first theoretical and applied results in fractional calculus involve the Riemann–Liouville integral and the Riemann–Liouville and Caputo derivatives, which are still widely studied and used [1–10]; nevertheless, other definitions of non-integer operators have been developed. Some recent classifications of families of operators appear in [11–15].

Moreover, some fractional operators have been proposed, which seek to generalize several other existing operators. Some of these generalizations have been given in [16–21]. In this sense, Fernandez et al. introduced a family of operators that generalizes a wide variety of fractional operators [22], such as the classical Riemann–Liouville and Caputo operators, and the operators with a non-singular kernel; among these, the Caputo–Fabrizio [23]
and Atangana–Baleanu [24] derivatives are found between other definitions. An interesting peculiarity of the operators with non-singular kernel is that they are simple, yet they include many operators that are currently widely used; furthermore, they can be expressed in terms of the classical Riemann–Liouville operators by means of power series expansions. The fractional operators with a non-singular kernel have been widely studied recently [25–30] and used in multiple applications and modeling of systems and phenomena [31–39].

Thus, inspired by the general analytic kernel (GAK) fractional operators defined in [22], in this work, we extend several well-known results for classical operators (such as the Caputo and Riemann–Liouville derivatives) to these new operators, including the Leibniz rule and some widely used inequalities in Lyapunov-type stability. With these tools, we are able to extend classical asymptotic stability results for nonlinear dynamical systems modeled with the GAK operators of Riemann–Liouville (GAKRL) and Caputo (GAKC) type. In addition, a new stability definition is introduced, which generalizes previous definitions of stability for fractional systems; moreover, some generalizations of stability for GAK operators using convex Lyapunov functions are presented.

The rest of the paper is organized as follows: In Section 2, we introduce the mathematical background, some definitions concerning the GAK operators, and we show how other fractional operators are obtained as particular cases of them, along with some results necessary for the following sections. In Section 3, we establish the relationship between the GAKRL and GAKC fractional operators and present a novel form of the rule for the product of functions with these operators, avoiding the infinite series that appears in the Leibniz rule for classical fractional calculus. Section 4 addresses some results related to the Laplace transform of the GAKRL and GAKC operators, their relationship, and their application in the solution of fractional-order differential equations. Furthermore, some consequences for the generalized Lyapunov direct method with these operators are given through the Fernandez-Özarslan–Baleanu stability. The stability analysis task is more straightforward with adequate tools, and, in Section 5, we prove some functional inequalities for quadratic Lyapunov functions. Moreover, in Section 6, we use locally convex functions for generalizing stability results via convex Lyapunov functions, and, in Section 7, some representative examples are given. Finally, in Section 8, we present the conclusions of the work.

Throughout the paper, the following notation is used:

- \( I \) denotes fractional integrals. The left superscript of \( I \) denotes the type of integral, while the right superscript represents the order, and the right subscript indicates the lower limit of the integral:
  - \( RL^\alpha_a+ \) is the Riemann–Liouville fractional integral of order \( \alpha \).
  - \( AB^\alpha_a+ \) is the Atangana–Baleanu fractional integral of Riemann–Liouville type of order \( \alpha \).
  - \( A^\alpha_{\Delta a} \) is the fractional-order integral operator with general analytic kernel with orders \( \alpha \) and \( \beta \).

- \( D \) denotes fractional derivatives. The left superscript of \( D \) denotes the type of derivative, while the right superscript represents the order, and the right subscript indicates the lower limit of the set where the operator is being applied:
  - \( RL^\alpha_a+ \) is the Riemann–Liouville fractional differential operator of order \( \alpha \).
  - \( C^\alpha_a+ \) is the Caputo differential operator of order \( \alpha \).
  - \( CF^\alpha_a+ \) is the Caputo–Fabrizio differential operator of order \( \alpha \).
  - \( ABC^\alpha_a+ \) is the Atangana–Baleanu differential operator of Caputo type of order \( \alpha \).
  - \( ABR^\alpha_a+ \) is the Atangana–Baleanu differential operator of Riemann–Liouville type of order \( \alpha \).

In the case of the differential operators with general analytic kernel, both left superscript and subscripts are considered:
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- $\mathcal{D}_{a+}^{\alpha,\beta}$ is the Riemann–Liouville fractional differential operator with general analytic kernel with orders $\alpha$ and $\beta$.
- $\mathcal{D}_{C,a+}^{\alpha,\beta}$ is the Caputo fractional differential operator with general analytic kernel with orders $\alpha$ and $\beta$.

- For a candidate Lyapunov function $V(t, x(t))$, $\nabla V(t, x(t)) = \frac{\partial V}{\partial x}$ [40].

2. Preliminaries

As we mentioned, in this work, we propose new tools for the stability analysis of a class of fractional-order systems, where the main characteristic of those systems is the general analytic kernel in their derivative. This fractional derivative covers a wide variety of fractional operators, and, as particular cases, we find the classical Caputo, Riemann–Liouville, and Atangana–Baleanu derivatives. Therefore, we start this section with some basic definitions from the theory of fractional calculus and, subsequently, their generalization by using the general analytic kernel operators.

Definition 1. [41]. Let $\alpha \in \mathbb{R}^+$ and $\Gamma(\alpha) = \int_0^{\infty} t^{\alpha-1} e^{-t} dt$ be the Gamma function. The Riemann–Liouville fractional integral of order $\alpha$ is given by

$$\mathcal{I}^{\alpha}_{a+} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau) \, d\tau,$$

where $a \leq t \leq b$ and $\mathcal{I}^{\alpha}_{a+} f(t) \in L_1[a, b]$.

Definition 2. [41]. Let $\alpha \in \mathbb{R}^+$ and consider $n = \min\{k \in \mathbb{N} : k > \alpha\}$. Based on the Riemann–Liouville integral, the Riemann–Liouville fractional derivative of order $\alpha$ is defined by

$$\mathcal{D}^{\alpha}_{a+} f(t) = D^n \left[ \mathcal{I}^{n-\alpha}_{a+} f(t) \right] = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t (t-\tau)^{n-\alpha-1} f(\tau) \, d\tau,$$

where $D^n$ is the usual derivative of order $n \in \mathbb{Z}^+$.

Definition 3. [41]. Let $\alpha \in \mathbb{R}^+$ and consider $n = \min\{k \in \mathbb{N} : k > \alpha\}$. Whenever $D^n \in L_1[a, b]$, the Caputo derivative of fractional order $\alpha$ is given by

$$\mathcal{D}^{\alpha}_{C,a+} f(t) = \mathcal{I}^{n-\alpha}_{a+} [D^n f(t)] = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau) \, d\tau.$$

Definition 4. [23]. Let $0 < \alpha < 1$ and consider a function $f \in H^1(a, b)$ with $b > a$. The operator described by

$$\mathcal{D}^{\alpha}_{C,a+} f(t) = \frac{M(\alpha)}{1-\alpha} \int_a^t f'(\tau) \exp \left( -\frac{\alpha}{1-\alpha} (t-\tau) \right) d\tau$$

represents a fractional derivative with a nonsingular kernel called the Caputo–Fabrizio fractional derivative, where the normalization function $M(\alpha)$ satisfies $M(0) = M(1) = 1$.

Definition 5. [24]. The operator defined by

$$\mathcal{D}^{\alpha}_{ABR} f(t) = \frac{B(\alpha)}{1-\alpha} \frac{d}{dt} \int_a^t f(\tau) E_\alpha \left( \frac{-\alpha}{1-\alpha} (t-\tau)^\alpha \right) d\tau,$$

is called the Atangana–Baleanu Riemann–Liouville (ABR) fractional derivative for $f \in L^1(a, b)$, $0 < \alpha < 1$, $a < t < b$. 

...
Definition 6. [24]. Let \( f \) be a differentiable function on \([a, b] \) such that \( f' \in L^1(a, b) \). The operator defined by
\[
ABC D_{a+}^\alpha f(t) = \frac{B(\alpha)}{1 - \alpha} \int_a^t f'(\tau) E_a \left( -\frac{\alpha}{1 - \alpha} (t - \tau)^\alpha \right) d\tau,
\]
is called the Atangana–Baleanu–Caputo (ABC) fractional derivative of \( f \), where \( 0 < \alpha < 1 \) and \( a < t < b \).

Definition 7. [24]. Let \( \alpha \in \mathbb{R}^+ \). Based on the classical Riemann–Liouville fractional integral, the AB fractional integral of \( f(t) \) can be defined by
\[
AB I_{a+}^\alpha f(t) = \frac{1 - a}{B(\alpha)} f(t) + \frac{a}{B(\alpha)} RL I_{a+}^\alpha f(t),
\]
where \( a < t < b \) and \( f \in L^1(a, b) \).

Remark 1. The operators (5)–(7) use a real-valued normalization function \( B(\alpha) \) that satisfies \( B(\alpha) > 0 \) and \( B(0) = B(1) = 1 \). In addition, \( E_a(\cdot) \) represents the Mittag–Leffler function with one parameter \( \alpha > 0 \) defined by the convergent series [42]:
\[
E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(ak + 1)}.
\]

A generalization of function (8) is the Mittag–Leffler function with two parameters \( \alpha > 0, \beta > 0 \) defined by the convergent series [42]:
\[
E_{\alpha, \beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(ak + \beta)}.
\]

Now that the previous fractional derivatives and integrals have been defined, we may ask ourselves if there is a way to describe a family of derivatives to which operators (2) to (7) belong. In this sense, Fernández et al. proposed a simple integral model based on an analytical kernel and the Riemann–Liouville integral (1) that generalizes the already known operators [22]. Some useful definitions and results concerning these new operators are presented below.

Definition 8. [22]. Let \( [a, b] \subset \mathbb{R}, \alpha, \beta \in \mathbb{C} \) with \( \text{Re}(\alpha) > 0, \ \text{Re}(\beta) > 0 \), and \( R \in \mathbb{R}^+ \) such that \( R > (b - a)^{\text{Re}(\beta)} \). A general analytic kernel is a complex function analytic on the disc \( D(0, R) \) defined by the following locally uniformly convergent power series
\[
A(x) = \sum_{k=0}^{\infty} a_k x^k,
\]
where the coefficients \( a_k = a_k(\alpha, \beta) \) may have dependence on \( \alpha \) and \( \beta \).

Definition 9. [22]. Let \( A \) be a general analytic kernel satisfying Definition 8 and \( f \in L^1[a, b] \). The fractional-order integral operator with general analytic kernel (GAK) is given by:
\[
A I_{a+}^\alpha f(t) = \int_a^t (t - \tau)^{a-1} A \left( \text{Re}(\beta) \right) f(\tau) d\tau.
\]

Definition 10. [22]. Let \( A \) be a general analytic kernel satisfying Definition 8. The transformed function \( A_G \) is defined as follows:
\[
A_G(x) = \sum_{k=0}^{\infty} a_k \Gamma(\beta k + a) x^k.
\]
Theorem 1. [22]. Let $A$ be a general analytic kernel satisfying Definition 8. Then, for any function $f \in L^1[a, b]$, the integral (11) is equivalent to the locally uniformly convergent series on $[a, b]$:

$$A \mathcal{I}_a^\alpha f(t) = \sum_{k=0}^{\infty} a_k \Gamma(\beta k + \alpha) RL \mathcal{I}_a^{\alpha + k\beta} f(t),$$  \hspace{1cm} (13)

where $RL \mathcal{I}_a^{\alpha + k\beta}$ is the Riemann–Liouville integral.

Definition 11. [22]. Let $a, b, \alpha, \beta$, $A$ satisfying Definition 8, and let $f$ be a function $f \in L^1[a, b]$ with sufficient differentiability properties. The differential operators of Riemann–Liouville and Caputo type with general analytic kernel (GAKRL and GAKC, respectively) are given by

$$RL \mathcal{D}_a^{\alpha, \beta} f(t) = \frac{d^m}{dt^m} \left( A \mathcal{I}_a^{\alpha + \beta} f(t) \right),$$  \hspace{1cm} (14)

$$C \mathcal{D}_a^{\alpha, \beta} f(t) = A \mathcal{I}_a^{\alpha} \left( \frac{d^m}{dt^m} f(t) \right),$$  \hspace{1cm} (15)

respectively, where $m \in \mathbb{Z}^+$, and the orders $\alpha'$ and $\beta'$ depend on $\alpha$ and $\beta$.

Example 1. Let $x_0 \in \mathbb{R}$; then, by definition (15), one has

$$C \mathcal{D}_a^{\alpha, \beta} x_0 = 0.$$

On the other hand, by direct calculation employing Equation (14):

$$RL \mathcal{D}_a^{\alpha, \beta} x_0 = x_0 \sum_{k=0}^{\infty} a_k \frac{\Gamma(\alpha' + \beta' k)}{\Gamma(\alpha' + \beta' k + 1 - m)} (t - a)^{\alpha' + \beta' k - m}.$$

Remark 2. Some classical fractional operators may be obtained using different values of $\alpha$ and $\beta$ in definitions (11), (14), (15), as follows:

(a) If $\alpha' = \alpha$, $\beta' = 0$, $m = 1$ and $A \left( (t - \tau)^0 \right) = \frac{1}{\Gamma(\alpha)}$, then

$$A \mathcal{I}_a^{0, \alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t - \tau)^{\alpha - 1} f(\tau) \, d\tau = RL \mathcal{I}_a^{\alpha} f(t),$$  \hspace{1cm} (16)

represents the Riemann–Liouville integral operator of order $\alpha$.

(b) If $\alpha' = m - \alpha$, $\beta' = 0$, $m = 1$ and $A \left( (t - \tau)^0 \right) = \frac{1}{\Gamma(1 - \alpha)}$, then

$$RL \mathcal{D}_a^{0, \alpha} f(t) = \frac{d^m}{dt^m} \left( A \mathcal{I}_a^{0, \alpha} f(t) \right)$$

$$= \frac{1}{\Gamma(n - \alpha)} \frac{d^m}{dt^m} \int_a^t (t - \tau)^{m - \alpha - 1} f(\tau) \, d\tau = RL \mathcal{D}_a^{\alpha} f(t),$$  \hspace{1cm} (17)

$$C \mathcal{D}_a^{0, \alpha} f(t) = A \mathcal{I}_a^{0, \alpha} \left( \frac{d^m}{dt^m} f(t) \right)$$

$$= \frac{1}{\Gamma(m - \alpha)} \int_a^t (t - \tau)^{m - \alpha - 1} f^{(m)}(\tau) \, d\tau = C \mathcal{D}_a^{\alpha} f(t),$$  \hspace{1cm} (18)

represent the Riemann–Liouville fractional derivative and the Caputo fractional derivative, respectively.
(c) If \( \alpha' = 1, \beta' = 1, m = 1 \) and \( A((t - \tau)) = \frac{M(\alpha)}{1 - \alpha} \exp \left( -\frac{\alpha}{1 - \alpha} (t - \tau) \right) \), then

\[
\mathcal{C}D_{a+}^{\alpha, \beta} f(t) = \mathcal{C}I_{a+}^{1, 1} \left( \frac{d}{dt} f(t) \right)
\]

\[
= \frac{M(\alpha)}{1 - \alpha} \int_a^t f'(\tau) \exp \left( -\frac{\alpha}{1 - \alpha} (t - \tau) \right) d\tau = \mathcal{CF}D_{a+}^{\alpha} f(t),
\]

is the Caputo–Fabrizio derivative.

(d) If \( \alpha' = 1, \beta' = \alpha, m = 1 \) and \( A((t - \tau)\alpha) = \frac{B(\alpha)}{1 - \alpha} E_\alpha \left( -\frac{\alpha}{1 - \alpha} (t - \tau)^\alpha \right) \), then

\[
\mathcal{RL}D_{a+}^{\alpha, \beta} f(t) = \frac{d}{dt} \left( \mathcal{RL}I_{a+}^{1, \alpha} f(t) \right)
\]

\[
= \frac{B(\alpha)}{1 - \alpha} \frac{d}{dt} \int_a^t f(\tau) E_\alpha \left( -\frac{\alpha}{1 - \alpha} (t - \tau)^\alpha \right) d\tau = \mathcal{ABR}D_{a+}^{\alpha} f(t),
\]

\[
\mathcal{L}D_{a+}^{\alpha, \beta} f(t) = \mathcal{L}I_{a+}^{1, \alpha} \left( \frac{d}{dt} f(t) \right)
\]

\[
= \frac{B(\alpha)}{1 - \alpha} \int_a^t f'(\tau) E_\alpha \left( -\frac{\alpha}{1 - \alpha} (t - \tau)^\alpha \right) d\tau = \mathcal{ABC}D_{a+}^{\alpha} f(t),
\]

represent the ABR and ABC fractional derivatives, respectively.

3. Some Results with the General Analytic Kernel Operators

In this section, we relate the general analytic kernel differential operators \((14)\) and \((15)\) to each other, in a similar manner to the relationship between the classical Riemann–Liouville and Caputo derivatives. Moreover, the main result of the section consists of a new formula to calculate the fractional-order GAK derivative of the product of two functions.

**Theorem 2.** Let \( a, b, A \) satisfying Definition 8, \( \alpha, \beta \in \mathbb{C} \) with \( \text{Re}(\alpha) > 0, \text{Re}(\beta) > 0 \), and \( f \in L^1[a, b] \). Then

\[
\mathcal{RL}D_{a+}^{\alpha, \beta} f(t) = \mathcal{C}D_{a+}^{\alpha, \beta} f(t) + \sum_{j=0}^{m-1} \sum_{k=0}^{\infty} a_k \frac{\Gamma(\alpha' + \beta' k)}{\Gamma(j - m + \alpha' + k \beta' + 1)} (t - a)^j (t - a)^{-m + \alpha' + k \beta'} f^{(j)}(a).
\]

**Proof.** From Definition \((14)\) and the representation \((13)\) for the fractional integral operator, we have:

\[
\mathcal{RL}D_{a+}^{\alpha, \beta} f(t) = \frac{d^m}{dt^m} \left( \mathcal{RL}I_{a+}^{\alpha', \beta'} f(t) \right) = \frac{d^m}{dt^m} \left[ \sum_{k=0}^{\infty} a_k \Gamma(\beta'k + \alpha') \mathcal{RL}I_{a+}^{\alpha' + k \beta'} f(t) \right].
\]

The series for \( A \) is assumed to be locally uniformly convergent, so the order of summation can be swapped, to get:

\[
\mathcal{RL}D_{a+}^{\alpha, \beta} f(t) = \sum_{k=0}^{\infty} a_k \Gamma(\beta'k + \alpha') \frac{d^m}{dt^m} \left( \mathcal{RL}I_{a+}^{\alpha' + k \beta'} f(t) \right),
\]

i.e.,

\[
\mathcal{RL}D_{a+}^{\alpha, \beta} f(t) = \sum_{k=0}^{\infty} a_k \Gamma(\beta'k + \alpha') \mathcal{RL}I_{a+}^{\alpha' + k \beta' - m} f(t) = \sum_{k=0}^{\infty} a_k \Gamma(\beta'k + \alpha') \mathcal{RL}D_{a+}^{\alpha' + k \beta' - m} f(t).
\]
Moreover, considering that the classical Riemann–Liouville and Caputo operators are related to each other [3], then

\begin{align*}
A^\alpha D^\beta_{RL} f(t) &= \sum_{n=0}^{\infty} a_n \Gamma(\beta' k + \alpha') \left[ C D^m a_+^{m-a'-k\beta'} f(t) + \sum_{j=0}^{m-1} \frac{(t-a)^{j-m+a'+k\beta'}}{\Gamma(j-m+a'+k\beta'+1)} f^{(j)}(a) \right] \\
&= \sum_{n=0}^{\infty} a_n \Gamma(\beta' k + \alpha') RL D^{m-a'-k\beta'} f(t) + \sum_{j=0}^{m-1} \frac{\Gamma(\beta' k + \alpha')}{\Gamma(j-m+a'+k\beta'+1)} (t-a)^{j-m+a'+k\beta'} f^{(j)}(a) \\
&= \sum_{n=0}^{\infty} \frac{\Gamma(\beta' k + \alpha')}{\Gamma(j-m+a'+k\beta'+1)} (t-a)^{j-m+a'+k\beta'} f^{(j)}(a) \\
&= A^\alpha D^\beta_{RL} f(t) + \sum_{n=0}^{\infty} \frac{\Gamma(\beta' k + \alpha')}{\Gamma(j-m+a'+k\beta'+1)} (t-a)^{j-m+a'+k\beta'} f^{(j)}(a).
\end{align*}

\[\Box\]

**Theorem 3.** Let \(a, b, A\) satisfying Definition 8, \(\alpha, \beta \in \mathbb{C}\) with \(\text{Re}(\alpha) > 0, \text{Re}(\beta) > 0, f \in L^1[a, b]\) with sufficient differentiability properties such that \(f^{(j)}(a) \geq 0, 0 \leq j \leq m - 1, m \in \mathbb{Z}^+\). If \(a_k > 0\) and \(j - m + \alpha' + k\beta' + 1 > 0\) for all \(j, k\), then

\[A^\alpha D^\beta_{RL} f(t) \leq A^\alpha D^\beta_{RL} f(t)\]  

(23)

**Proof.** From the Theorem 2, we have

\[A^\alpha D^\beta_{RL} f(t) = A^\alpha D^\beta_{RL} f(t) - \sum_{j=0}^{m-1} \frac{\Gamma(\beta' k + \alpha')}{\Gamma(j-m+a'+k\beta'+1)} (t-a)^{j-m+a'+k\beta'} f^{(j)}(a).\]

Because \(f^{(j)}(a) \geq 0\), if \(a_k > 0\) and \(j - m + \alpha' + k\beta' + 1 > 0\), we get

\[A^\alpha D^\beta_{RL} f(t) \leq A^\alpha D^\beta_{RL} f(t).\]

This concludes the proof. \(\Box\)

**Lemma 1.** Let \(a, b, A\) satisfying Definition 8, \(\alpha, \beta \in \mathbb{C}\) with \(\text{Re}(\alpha) > 0, \text{Re}(\beta) > 0, f \in L^1[a, b]\) with sufficient differentiability properties, and the following relationship holds:

\[\begin{align*}
A^\alpha D^\beta_{RL} \left\{ RL D^m a_+^{m-a'} f(t) \right\} &= \sum_{n=0}^{\infty} a_n \Gamma(\alpha + \beta l) \frac{d}{dt} \left[ A^\alpha D^{m-a' k+\beta'}_{RL} f(t) - \sum_{j=1}^{m} \frac{d^{m-j}}{dt^{m-j}} A^\alpha D^{a' k+\beta'}_{RL} f(t) \right]_{t=a} \Gamma(2 + \alpha + \beta l - j) \\
&= \sum_{k=0}^{\infty} a_k \Gamma(\alpha + \beta k) RL D^m a_+^{m-a'} f(t) - \sum_{n=0}^{\infty} a_n \Gamma(\alpha' + \beta' n) RL D^{a' k+\beta' n-m}_{RL} f(t).
\end{align*}\n
**Proof.**

\[\begin{align*}
A^\alpha D^\beta_{RL} \left\{ RL D^m a_+^{m-a'} f(t) \right\} &= \sum_{k=0}^{\infty} a_k \Gamma(\alpha + \beta k) RL D^m a_+^{m-a'} f(t) - \sum_{n=0}^{\infty} a_n \Gamma(\alpha' + \beta' n) RL D^{a' k+\beta' n-m}_{RL} f(t).
\end{align*}\]
By hypothesis, the series is locally uniformly convergent, hence the order of integration and the summation can be changed. Considering the properties between the Riemann–Liouville fractional derivative and integral [1], one has that

\[
\frac{d^m}{dt^m} A^\alpha_{a+} J^\beta_{a+} f(t) = \frac{d^m}{dt^m} \left\{ A^\alpha_{a+} J^\beta_{a+} f(t) \right\} = \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} a_k a_n \Gamma(\alpha + \beta k) \Gamma(\alpha' + \beta' n) RL_D^{\alpha+\beta k}_{a+} \left( RL_D^{m-\alpha'-\beta' n}_{a+} f(t) \right)
\]

\[
= \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} a_k a_n \Gamma(\alpha + \beta k) \Gamma(\alpha' + \beta' n) \left\{ RL_D^{m-\alpha'-\beta' n}_{a+} f(t) \right\} - \sum_{j=1}^{q} \left( RL_D^{m-\alpha'-\beta' n-j}_{a+} f(t) \right) \bigg|_{t=a} \frac{(t-a)^{\alpha+\beta k-j}}{\Gamma(1 + \alpha + \beta k - j)} \bigg\}
\]

\[
= \frac{d^m}{dt^m} A^\alpha_{a+} J^\beta_{a+} f(t)
\]

\[
- \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} a_k a_n \Gamma(\alpha + \beta k) \Gamma(\alpha' + \beta' n) \sum_{j=1}^{q} \left( RL_D^{m-\alpha'-\beta' n-j}_{a+} f(t) \right) \bigg|_{t=a} \frac{(t-a)^{\alpha+\beta k-j}}{\Gamma(1 + \alpha + \beta k - j)} .
\]

It is well known that the fractional derivative of a product of functions obtained with the classical fractional operators is difficult to calculate. Therefore, we present a result that helps to solve this problem by using the GAKRL derivative. According to the integer-order definition, the first order derivative of the function \( f(t) \) is given by

\[
\frac{d}{dt} f(t) = \lim_{\varepsilon \to 0} \frac{f(t) - f(t - \varepsilon)}{\varepsilon} .
\]

Similarly,

\[
\frac{d^2}{dt^2} f(t) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2} [f(t) - 2f(t - \varepsilon) + f(t - 2\varepsilon)],
\]

\[
\frac{d^3}{dt^3} f(t) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^3} [f(t) - 3f(t - \varepsilon) + 3f(t - 2\varepsilon) - f(t - 3\varepsilon)].
\]

Iterating this process \( n \) times, we get

\[
\frac{d^n}{dt^n} f(t) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^n} \sum_{j=0}^{m} (-1)^j \binom{m}{j} f(t - j\varepsilon).
\]

On the other hand, observe that the product of two functions can be expressed as follows:

\[
u(\tau)v(\tau) = (u(\tau) - u(t))(v(\tau) - v(t)) + u(t)v(\tau) + u(\tau)v(t) - u(t)v(t).
\]

Based on the previous comments, we prove the following result.

**Lemma 2.** Let \( a, b, A \) satisfying Definition 8, \( a, \beta \in \mathbb{C} \) with \( \text{Re}(a) > 0, \text{Re}(\beta) > 0 \), \( u(t), v(t) \in L^1[a, b] \), \( u(\tau)v(t) \in L^1[a, b] \). If \( \alpha' + \beta' k - m > 0 \) for all \( j, k \), then

\[
\frac{A^\alpha_{a+} D^\beta_{a+} u(t)v(t)}{RL_D^{\alpha+\beta k}_{a+} u(t)} = u(t) \frac{A^\alpha_{a+} D^\beta_{a+} v(t) + v(t) A^\alpha_{a+} D^\beta_{a+} u(t)}{RL_D^{\alpha+\beta k}_{a+} u(t)} + \sum_{k=0}^{\infty} a_k \Gamma(\alpha + \beta k) \frac{\int_{a}^{t} (t - \tau)^{a' + \beta' k - 1 - m} [(u(\tau) - u(t))(v(\tau) - v(t))] d\tau}{\Gamma(\alpha' + \beta' k + 1 - m)}
\]

\[
- u(t)v(t) \sum_{k=0}^{\infty} a_k \frac{\Gamma(\alpha' + \beta' k)}{\Gamma(\alpha' + \beta' k + 1 - m)} (t - a)^{a' + \beta' k - m}. (29)
\]
Proof. Combining Definition (14) and expression (28), the derivative of the product between $u(t)$ and $v(t)$ can be expressed as follows:

$$kL^\alpha \mathcal{D}_a^\beta (u(t)v(t)) = \frac{d^m}{dt^m} \int_a^t (t-\tau)^{a'-1} A((t-\tau)^{\beta'})(u(\tau)v(\tau)) \, d\tau$$

$$= \sum_{k=0}^\infty \frac{d^m}{dt^m} \left\{ \int_a^t (t-\tau)^{a'+\beta k-1} u(\tau)v(\tau) \right\}$$

$$= I_1 + I_2 + I_3 - I_4,$$

where

$$I_1 = \sum_{k=0}^\infty \frac{d^m}{dt^m} \left\{ \int_a^t (t-\tau)^{a'+\beta k-1} (u(\tau) - u(t)) (v(\tau) - v(t)) \right\},$$

$$I_2 = \sum_{k=0}^\infty \frac{d^m}{dt^m} \left\{ \int_a^t (t-\tau)^{a'+\beta k-1} u(t)v(\tau) \right\},$$

$$I_3 = \sum_{k=0}^\infty \frac{d^m}{dt^m} \left\{ \int_a^t (t-\tau)^{a'+\beta k-1} u(\tau)v(t) \right\},$$

$$I_4 = \sum_{k=0}^\infty \frac{d^m}{dt^m} \left\{ \int_a^t (t-\tau)^{a'+\beta k-1} u(t)v(t) \right\}.$$ (30) (31) (32) (33)

By definition (14), it follows that

$$I_2 + I_3 = u(t) \mathcal{D}_a^\beta \mathcal{D}_a^\beta v(t) + v(t) \mathcal{D}_a^\beta \mathcal{D}_a^\beta u(t),$$

and the integral (30) can be expressed by using the form (27), i.e.,

$$I_1 = \sum_{k=0}^\infty \frac{d^m}{dt^m} \left\{ \int_a^t (t-\tau)^{a'+\beta k-1} ([u(\tau) - u(t)](v(\tau) - v(t))) \right\}.$$ (34)

By a simple calculation in the integral $I_1$, observe that $\lim_{\epsilon \to 0} \epsilon^m = 0$ and

$$\lim_{\epsilon \to 0} \sum_{j=0}^m (-1)^j \binom{m}{j} \int_a^t (t-j\epsilon - \tau)^{a'+\beta k-1} ([u(\tau) - u(t)](v(\tau) - v(t))) \, d\tau$$

$$= \sum_{j=0}^m (-1)^j \binom{m}{j} \int_a^t (t-\tau)^{a'+\beta k-1} ([u(\tau) - u(t)](v(\tau) - v(t))) \, d\tau$$

$$= (1 + (-1))^m \int_a^t (t-\tau)^{a'+\beta k-1} ([u(\tau) - u(t)](v(\tau) - v(t))) \, d\tau$$

$$= 0.$$
To solve the problem with the indeterminate form, by using the L’Hôpital’s rule:

\[
I_1 = \sum_{k=0}^{\infty} a_k \sum_{j=1}^{m} \frac{(-1)^j (-j)^m}{\Gamma(m+1)} \cdot \frac{m!}{\Gamma(m+j)} \cdot \frac{\Gamma(a' + \beta'k)}{\Gamma(a' + \beta'k - m)} \times \int_a^t (t - \tau)^{a' + \beta'k - 1 - m}(u(\tau) - u(t))(v(\tau) - v(t)) d\tau
\]

\[
= \sum_{k=0}^{\infty} a_k \frac{m!}{\Gamma(m+1)} \sum_{j=1}^{m} \frac{(-1)^j (-j)^m}{j!(m-j)!} \frac{\Gamma(a' + \beta'k)}{\Gamma(a' + \beta'k - m)} \times \int_a^t (t - \tau)^{a' + \beta'k - 1 - m}(u(\tau) - u(t))(v(\tau) - v(t)) d\tau
\]

\[
= \sum_{k=0}^{\infty} a_k \frac{\Gamma(a' + \beta'k)}{\Gamma(a' + \beta'k - m)} \int_a^t (t - \tau)^{a' + \beta'k - 1 - m}(u(\tau) - u(t))(v(\tau) - v(t)) d\tau,
\]

where we have used the fact that

\[
\sum_{j=1}^{m} \frac{(-1)^j (-j)^m}{j!(m-j)!} = 1.
\]

Finally, the integral (33) is solved by direct calculation:

\[
I_4 = u(t)v(t) \sum_{k=0}^{\infty} a_k \frac{d^m}{dt^m} \left[ \int_a^t (t - \tau)^{a' + \beta'k} d\tau \right]
\]

\[
= u(t)v(t) \sum_{k=0}^{\infty} a_k \frac{\Gamma(a' + \beta'k)}{\Gamma(a' + \beta'k + 1)} (t-a)^{a' + \beta'k - m}.
\]

Therefore,

\[
A_{RL}^{\alpha,\beta} D_{a+}^{\alpha,\beta}(u(t)v(t)) = u(t) A_{RL}^{\alpha,\beta} D_{a+}^{\alpha,\beta}v(t) + v(t) A_{RL}^{\alpha,\beta} D_{a+}^{\alpha,\beta}u(t)
\]

\[
+ \sum_{k=0}^{\infty} a_k \frac{\Gamma(a' + \beta'k)}{\Gamma(a' + \beta'k - m)} \int_a^t (t - \tau)^{a' + \beta'k - 1 - m}(u(\tau) - u(t))(v(\tau) - v(t)) d\tau
\]

\[
= u(t)v(t) \sum_{k=0}^{\infty} a_k \frac{\Gamma(a' + \beta'k)}{\Gamma(a' + \beta'k + 1 - m)} (t-a)^{a' + \beta'k - m}.
\]

\[
\square
\]

**Lemma 3.** If the assumptions in Lemma 2 are satisfied, but replacing \( A_{RL}^{\alpha,\beta} \) by \( C_{\alpha}^{\alpha,\beta} \), then

\[
A_{C}^{\alpha,\beta} D_{a+}^{\alpha,\beta}u(t)v(t) = u(t) A_{C}^{\alpha,\beta} D_{a+}^{\alpha,\beta}v(t) + v(t) A_{C}^{\alpha,\beta} D_{a+}^{\alpha,\beta}u(t)
\]

\[
+ u(t) \sum_{j=0}^{m-1} \sum_{k=0}^{\infty} a_k \frac{\Gamma(\beta'k + a')}{\Gamma(j - m + a' + k\beta' + 1)} (t-a)^{j-m+a'+k\beta'} v^{(j)}(a)
\]

\[
+ v(t) \sum_{j=0}^{m-1} \sum_{k=0}^{\infty} a_k \frac{\Gamma(\beta'k + a')}{\Gamma(j - m + a' + k\beta' + 1)} (t-a)^{j-m+a'+k\beta'} u^{(j)}(a)
\]

\[
+ \sum_{k=0}^{\infty} a_k \frac{\Gamma(a' + \beta'k)}{\Gamma(a' + \beta'k - m)} \int_a^t (t - \tau)^{a' + \beta'k - 1 - m}(u(\tau) - u(t))(v(\tau) - v(t)) d\tau
\]

\[
= u(t)v(t) \sum_{k=0}^{\infty} a_k \frac{\Gamma(a' + \beta'k)}{\Gamma(a' + \beta'k + 1 - m)} (t-a)^{a' + \beta'k - m}
\]

\[
- \sum_{j=0}^{m-1} \sum_{k=0}^{\infty} a_k \frac{\Gamma(\beta'k + a')}{\Gamma(j - m + a' + k\beta' + 1)} (t-a)^{j-m+a'+k\beta'} (uv)^{(j)}(a).
\]
we establish some results related to the Laplace transform of the differential operators $A$

**Definition 8.** The Laplace transform of the fractional integral $I^α(t)$ is given by:

$$\mathcal{L}\{I^α(t)\} = \frac{1}{s^{α+1}},$$

where the function $s^{α+1}$ satisfies Definition 10.

**Theorem 4.** Ref. [22] Let $f \in L^2[0, b], b > 0$ with Laplace transform $\mathcal{F}(s)$. Let $a, β, A$ satisfying Definition 8. The Laplace transform of the fractional integral $I^{α, β}_{0+} f(t)$ is given by:

$$\mathcal{L}\{I^{α, β}_{0+} f(t)\} = s^{-α} A_I(s^{-β}) \mathcal{F}(s),$$

(35)

where the function $A_I$ satisfies Definition 10.

4. Laplace Transform and Generalized Lyapunov Direct Method

As with their integer-order counterpart, Laplace transform has been one of the most useful tools for the solution of fractional differential equations. Thus, in this section, we establish some results related to the Laplace transform of the differential operators of Riemann–Liouville and Caputo type with general analytic kernels, the relationship between them, and their application to the analysis of existence and uniqueness of certain types of differential equations. In addition, some results of a generalized Lyapunov direct method for these operators are presented.

**Theorem 5** (Laplace transform for the GAKRL derivative). Let $f \in L^2[0, b], b > 0$ with Laplace transform $\mathcal{F}(s)$. Let $α', β', A$ satisfy Definition 8. The Laplace transform of the GAKRL derivative $A^{α, β}_{RL} D_{α, β} f(t)$ is given by:

$$\mathcal{L}\{A^{α, β}_{RL} D_{α, β} f(t)\} = A_I\left(s^{-β'}\right) s^{m-α'} \mathcal{F}(s).$$

(36)

**Proof.** From Definition (14) and its representation (13),

$$A^{α, β}_{RL} D_{α, β} f(t) = \sum_{k=0}^{∞} a_k \Gamma(β'k + α') R^α T^{α', kβ', m} f(t).$$

The series is locally uniformly convergent, due to $0 \leq |(t - τ)^β'| \leq b\text{Re}(β') < R$, and $A$ is locally uniformly convergent by hypothesis on $D(0, R)$. This allows for interchanging the order between the summation and the integration. Therefore, by using the Laplace transform for the classical Riemann–Liouville integral, we have that

$$\mathcal{L}\{A^{α, β}_{RL} D_{α, β} f(t)\} = \mathcal{L}\left\{\sum_{k=0}^{∞} a_k \Gamma(β'k + α') R^α T^{α', kβ', m} f(t)\right\} = \sum_{k=0}^{∞} a_k \Gamma(β'k + α') \mathcal{L}\left\{R^α T^{α', kβ', m} f(t)\right\} = \sum_{k=0}^{∞} a_k \Gamma(β'k + α') \frac{\mathcal{F}(s)}{s^{α'+kβ'-m}}.$$
Theorem 6 (Laplace transform for GAKC derivative). Let \( f \in L^2[0, b], \ b > 0 \) with Laplace transform \( F(s) \). Let \( a, \beta \) satisfying Definition 8. The Laplace transform of the GAKC derivative \( \mathcal{CD}_0^a f(t) \) is given by:

\[
\mathcal{L}\{\frac{\mathcal{C}D_0^a f(t)}{\mathcal{C}D_0^\beta f(t)}\} = A\Gamma\left(s^{-\beta}\right)s^{-a}\left[s^m F(s) - \sum_{j=0}^{m-1} s^{m-1-j} f(j)(0)\right]. \tag{37}
\]

Proof. The operator (15) can be rewritten in its integral form as follows:

\[
\mathcal{CD}_0^a f(t) = \int_0^t (t - \tau)^{a-1} A\left((t - \tau)^\beta\right) f^{(m)}(\tau) \, d\tau = \text{the convolution operator.}
\]

where * is the convolution operator. Applying the Laplace transform to the above equation, one has that

\[
\mathcal{L}\{\mathcal{CD}_0^a f(t)\} = \mathcal{L}\{t^{a-1} A\left(t^\beta\right)\} \cdot \mathcal{L}\{f^{(m)}(t)\} = \mathcal{L}\{t^{a-1} \sum_{k=0}^{\infty} a_k t^\beta \} \cdot \mathcal{L}\{f^{(m)}(t)\}.
\]

The series for \( A \) is assumed to be locally uniformly convergent, which allows for interchanging the order of the integration and the summation, i.e.,

\[
\mathcal{L}\{\mathcal{CD}_0^a f(t)\} = \sum_{k=0}^{\infty} a_k \mathcal{L}\{t^{a+\beta k-1}\} \cdot \mathcal{L}\{f^{(m)}(t)\}
\]

\[
= \sum_{k=0}^{\infty} a_k \Gamma(a' + \beta') \left[s^m F(s) - \sum_{j=0}^{m} f^{(m)}(0)\right]
\]

Finally, by Definition 10 and rewriting the summation in square brackets, the proof is completed. \( \square \)

Example 2. Let \( a' = 1, \beta' = \alpha, m = 1; \) then,

\[
\mathcal{CD}_0^{1+a} f(t) = \mathcal{CD}_0^{1}\mathcal{CD}_0^a f(t) = \mathcal{CD}_0^a f(t) = \int_0^t (t - \tau)^{a-1} B(\alpha) \left(1 - \frac{\alpha}{1 - \beta'} t^{-\beta'}\right) f'(\tau) \, d\tau,
\]

where \( \mathcal{CD}_0^a f(t) \) is the Atangana–Baleanu derivative \( [24] \) in the Caputo sense, with

\[
A((t - \tau)^a) = \sum_{k=0}^{\infty} a_k \Gamma(1 + ak) \left\{1 - \frac{\alpha}{1 - \beta'} \right\}^k ((t - \tau)^a)^k. \tag{40}
\]

On the other hand, from expression (37), with the parameters established previously,

\[
\mathcal{L}\{\mathcal{CD}_0^a f(t)\} = \sum_{k=0}^{\infty} a_k \Gamma(1 + ak) \left(s^{-\alpha}\right)^k s^{-1} [s^{-m} F(s) - f(0)].
\]

Now, by using \( a_k \) as in the Equation (40) and the geometric series, one has that

\[
\mathcal{L}\{\mathcal{CD}_0^a f(t)\} = \frac{B(\alpha)}{1 - \alpha s^\alpha} \frac{s^{\alpha-1}}{\alpha} [sF(s) - f(0)]. \tag{42}
\]
This result coincides with the Laplace transform for the ABC derivative. In a similar manner, we can obtain the Laplace transform pairs for other fractional operators.

In this paper, we consider fractional-order nonlinear systems with a general analytic kernel of the form

\[ A_D^{\alpha,\beta}x(t) = f(t, x(t)), \]

where the initial conditions have the form

\[ x^{(k)}(0) = \frac{d^k x(0)}{dt^k} = \gamma_k, \quad k = 0, 1, \ldots, m - 1. \]

The set of initial conditions is necessary to specify the unique solution to the system (43); moreover, \( f : [t_0, \infty) \times \Omega \to \mathbb{R}^n \) is piecewise continuous in \( t \) and locally Lipschitz in \( x \) on \([t_0, \infty) \times \Omega\), where \( \Omega \subset \mathbb{R}^n \) is a domain that contains the equilibrium point \( x = 0 \). The equilibrium point of (43) is defined as follows.

**Definition 12.** An equilibrium point of the fractional-order system (43) is a constant \( x_0 \) that satisfies \( f(t, x_0) = 0 \).

**Remark 3.** Without loss of generality, let the equilibrium point be \( x = 0 \). If the equilibrium point of system (43) is \( x \neq 0 \), let \( y(t) = x(t) - x \). Then,

\[ A_D^{\alpha,\beta}y(t) = A_D^{\alpha,\beta}(x(t) - x) = f(t, x(t)) = f(t, y(t) + x) = g(t, y(t)), \]

where \( g(t, 0) = 0 \). A direct implication of this analysis is that the system with the variable \( y(t) \) satisfies Definition 12 so that its equilibrium point lies at the origin.

Now, consider the following assumptions, and define a function that will be useful for the analysis of the solution \( x(t) \).

**Assumption 1.** (1) The series \( A \Gamma(s^{-\beta'}) \) is uniformly convergent and satisfies \( A \Gamma(s^{-\beta'})s^{-a'+m} \neq 0 \) for \( \text{Re}(s) > 0 \).

(2) \( f(t, x(t)) \in L^1[0, \infty) \).

**Definition 13.** Let \( \alpha, \beta \in \mathbb{C} \) with \( \text{Re}(\alpha) > 0, \text{Re}(\beta) > 0 \). Let \( \alpha', \beta' \) be parameters that depend on \( \alpha, \beta \), respectively, and \( m \in \mathbb{Z}^+ \). The function \( \mathcal{A}(t; \alpha', \beta', m) \) defined by

\[ \mathcal{A}_\lambda(t; \alpha', \beta', m) = \mathcal{L}^{-1}\left\{ \frac{1}{A \Gamma(s^{-\beta'})s^{-a'+m} + \lambda} \right\}, \]

whenever the inverse Laplace transform \( \mathcal{L}^{-1}\{ \cdot \} \) converges, is called the Fernandez–Özarslan–Baleanu function (FOB-function).

**Theorem 7.** If the conditions stated in Assumption 1 hold, then the solution of system (43) can be rewritten as

\[ x(t) = x(0) + \mathcal{A}_0(t; \alpha', \beta', m) * f(t, x(t)) + \sum_{j=1}^{m-1} \frac{x^{(j)}(0)}{\Gamma(j+1)} t^j. \]

**Proof.** Under Assumption 1, we apply the Laplace transform to (43):

\[ A \Gamma(s^{-\beta'})s^{-a'} \left[ s^m X(s) - \sum_{j=0}^{m-1} s^{m-1-j} x^{(j)}(0) \right] = F(s), \]
Then, the solution of

\[ C^{\alpha} \int_0^t (t - \tau)^{\alpha - 1} f(\tau, x(\tau)) \, d\tau = x(0) + \frac{\Gamma(\alpha)}{\Gamma(1 + \alpha)} * f(t, x(t)). \]

On the other hand, for the Atangana–Baleanu-Caputo derivative, we set \( m = 1, \alpha' = 1, \beta' = \alpha \); then,

\[ \mathcal{A}_0 t \mathcal{I}_{0+}^{1, \alpha} x(t) = \mathcal{A}_0 t \mathcal{I}_{0+}^{\alpha} f(t, x(t)) + x(0). \]

According to the convolution operation and by the Riemann–Liouville integral (1), we obtain the following Volterra integral equation:

\[ x(t) = x(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha - 1} f(\tau, x(\tau)) \, d\tau = x(0) + \frac{\Gamma(\alpha)}{\Gamma(1 + \alpha)} * f(t, x(t)). \]

Then, the solution of \( \mathcal{A}_0 t \mathcal{I}_{0+}^{1, \alpha} x(t) = f(t, x(t)) \) is the solution of the Volterra integral equation:

\[ x(t) = \mathcal{A}_0 t \mathcal{I}_{0+}^{1, \alpha} f(t, x(t)) + x(0). \]
By Definition 13 and Equation (48), one has that

\[
x(t) = \mathcal{L}^{-1} \left\{ \frac{1}{B(a) \Gamma(a)} \frac{s^{-1}}{1 - s^{\alpha} + \frac{\alpha}{1 - \alpha} s} f(t, x(t)) + x(0) \right\}.
\]

Applying the inverse Laplace transform to the previous expression, we have that

\[
x(t) = x(0) + \frac{\alpha}{B(a) \Gamma(a)} \int_0^t (t - \tau)^{\alpha-1} f(\tau, x(\tau)) d\tau + \frac{1 - \alpha}{B(a)} \delta(t) * f(t, x(t)),
\]

where \(\delta(t)\) is the Dirac delta function. Finally, considering the convolution operator and its properties, we obtain the following Volterra integral equation:

\[
x(t) = x(0) + \frac{\alpha}{B(a) \Gamma(a)} \int_0^t (t - \tau)^{\alpha-1} f(\tau, x(\tau)) d\tau + \frac{1 - \alpha}{B(a)} \int_0^t \delta(t - \tau) f(\tau, x(\tau)) d\tau,
\]

that is,

\[
x(t) = x(0) + \frac{\alpha}{B(a) \Gamma(a)} \int_0^t (t - \tau)^{\alpha-1} f(\tau, x(\tau)) d\tau + \frac{1 - \alpha}{B(a)} f(t, x(t)),
\]

and rearranging terms,

\[
x(t) = x(0) + A^B_{\alpha, \alpha} \mathcal{T}_{0+} f(t, x(t)),
\]

where \(A^B_{\alpha, \alpha} \mathcal{T}_{0+} f(t, x(t))\) represents the Atangana–Baleanu integral [24]. A similar analysis for different values of parameters \(\alpha', \beta'\) and \(m\) can be done to obtain the solutions of fractional differential equations that involve different fractional operators.

If we use the GAKRL fractional derivative instead of the GAKC operator, the solution of

\[
A^r_{RL} \mathcal{D}_{0+}^{\alpha, \beta} x(t) = f(t, x(t)),
\]

takes the form (46), where all the initial conditions \(x^{(j)}(0), j = 0, 1, \ldots, m - 1\) are set to zero. An equilibrium point of a system (49) can be defined.

**Definition 14.** An equilibrium point of the fractional-order system (49) is a constant \(x_0\) that satisfies

\[
A^r_{RL} \mathcal{D}_{0+}^{\alpha, \beta} x_0 = f(t, x_0).
\]

**Remark 4.** Similarly to Remark 3, suppose the equilibrium point for (49) is \(\bar{x} \neq 0\) and consider the same change of variable \(y(t) = x(t) - \bar{x}\). Then,

\[
A^r_{RL} \mathcal{D}_{0+}^{\alpha, \beta} y(t) = A^r_{RL} \mathcal{D}_{0+}^{\alpha, \beta} (x(t) - \bar{x}) = f(t, y(t) + \bar{x}) - \bar{x} \sum_{k=0}^{\infty} a_k \Gamma(\alpha' + \beta' k)(t - t_0)^{\alpha' + \beta' - m} / \Gamma(\alpha' + \beta' k + 1 - m)
\]

\[
= g(t, y(t)),
\]

where \(g(t, 0) = 0\). In terms of the new variable, the system has an equilibrium at the origin.

**Lemma 4.** (Comparison Lemma) Let \(a, b, A\) satisfying Definition 8. Let \(x(0) = y(0), \ldots, x^{(m-1)}(0) = y^{(m-1)}(0)\) and \(A^C_{0+} \mathcal{D}_{0+}^{\alpha, \beta} x(t) \geq A^C_{0+} \mathcal{D}_{0+}^{\alpha, \beta} y(t)\), where \(m \in \mathbb{N}, \alpha, \beta \in \mathbb{C}\) with \(\text{Re}(\alpha) > 0, \text{Re}(\beta) > 0\). If the FOB-function \(\mathcal{A}_{0+}(t; \alpha', \beta', m)\) takes non-negative functions into non-negative functions, then \(x(t) \geq y(t)\).
Proof. From inequality $\frac{\Delta \mathcal{D}_{0}^{\alpha,\beta}}{\mathcal{D}_{0}^{\alpha,\beta}} x(t) \geq \frac{\Delta \mathcal{D}_{0}^{\alpha,\beta}}{\mathcal{D}_{0}^{\alpha,\beta}} y(t)$, it follows that there exists a nonnegative function $M(t)$ that satisfies the following equation:

$$\frac{\Delta \mathcal{D}_{0}^{\alpha,\beta}}{\mathcal{D}_{0}^{\alpha,\beta}} x(t) = \frac{\Delta \mathcal{D}_{0}^{\alpha,\beta}}{\mathcal{D}_{0}^{\alpha,\beta}} y(t) + M(t).$$

(51)

Applying the Laplace transform (37) to Equation (51), we have that

$$A_{\Gamma} \left( s^{-\beta'} \right) s^{-\alpha'} \left[ s^{m} X(s) - s^{m-1} x(0) - \ldots - x^{(m-1)}(0) \right] =$$

$$A_{\Gamma} \left( s^{-\beta'} \right) s^{-\alpha'} \left[ s^{m} Y(s) - s^{m-1} y(0) - \ldots - y^{(m-1)}(0) \right] + M(s).$$

(52)

By hypothesis, $x(0) = y(0), \ldots, x^{(m-1)}(0) = y^{(m-1)}(0)$, then equality (52) is reduced to:

$$A_{\Gamma} \left( s^{-\beta'} \right) s^{-\alpha'} \left[ s^{m} X(s) - s^{m-1} y(0) - \ldots - y^{(m-1)}(0) \right] + M(s).$$

(53)

Dividing by $A_{\Gamma} \left( s^{-\beta'} \right) s^{-\alpha'}$ produces

$$X(s) = Y(s) + \frac{M(s)}{A_{\Gamma} \left( s^{-\beta'} \right) s^{-\alpha'}}.$$  

(54)

Taking the inverse Laplace transform of (54) and using the Convolution Theorem, one has that

$$x(t) = y(t) + \omega_{0}(t; \alpha', \beta', m) + m(t).$$

(55)

The second term of the right-hand side of (55) is non-negative because $M(t)$ is non-negative. Based on this reasoning, the proof is completed and $x(t) \geq y(t)$. □

Definition 15. Let $\alpha, \beta \in \mathbb{C}$ with $\text{Re}(\alpha) > 0$, $\text{Re}(\beta) > 0$. Let $\alpha', \beta'$ be parameters that depend on $\alpha, \beta$, respectively, and $m \in \mathbb{Z}^{+}$, $\gamma' \in \mathbb{Z}^{+} \cup \{0\}$, $\lambda > 0$ and $\beta'$ such that $A_{\Gamma} \left( s^{-\beta'} \right)$ converges. The function defined by

$$\mathcal{G}_{X} \mathcal{I}_{\lambda}^{\gamma'}(t; \alpha', \beta', m) = \mathcal{L}^{-1} \left\{ \frac{A_{\Gamma} \left( s^{-\beta'} \right) s^{m-1-\alpha'-\gamma'}}{A_{\Gamma} \left( s^{-\beta'} \right) s^{-\alpha'+m+\lambda}} \right\},$$

(56)

whenever the inverse Laplace transform $\mathcal{L}^{-1}\{ \cdot \}$ converges, is called the generalized Fernandez–Özarslan–Baleanu function ($\mathcal{A}$–FOB-function).

Definition 16 (FOB-Stability). The solution of (43) is said to be Fernandez–Özarslan–Baleanu stable (FOB-stable) if

$$\| x(t) \| \leq \left[ \sum_{j=0}^{m-1} \nu_{j}(x(t)) \mathcal{G}_{X} \mathcal{I}_{\lambda}^{\gamma'}(t-t_{0}; \alpha', \beta', m) \right]^{b},$$

(57)

for all $t \geq t_{0}$, where $\nu(x) \geq 0$ is a locally Lipschitz continuous function on $x$, with $\nu(0) = 0$, $m - \alpha' > j > 0$, $m \in \mathbb{Z}^{+}$, $\lambda > 0$, $b > 0$ and $\beta'$ such that $A_{\Gamma} \left( s^{-\beta'} \right)$ converges.

Remark 5. The $\mathcal{A}$–FOB-function represents a family of Mittag–Leffler functions and their generalizations. Note that, for $\alpha' = 1 - \alpha$, $\beta' = 0$, $m = 1$, and $\gamma' = 0$

$$\mathcal{G}_{X} \mathcal{I}_{\lambda}^{0}(t; 1 - \alpha, 0, 1) = \mathcal{L}^{-1} \left\{ \frac{A_{\Gamma} \left( s^{0} \right) s^{\alpha-1}}{A_{\Gamma} \left( s^{0} \right) s^{\alpha} + \lambda} \right\} = \mathcal{L}^{-1} \left\{ \frac{s^{\alpha-1}}{s^{\alpha} + \lambda} \right\} = E_{\alpha}(-\lambda t^{\alpha}),$$

(58)
Theorem 8. Let \( x = 0 \) be an equilibrium point for system (43) and \( D \subset \mathbb{R}^n \) be a domain containing the origin. Let \( V(t, x(t)) : [0, \infty) \times D \to \mathbb{R} \) be a continuously differentiable function and locally Lipschitz with respect to \( x \) such that \( V^{(j)}(0, x(0)) > 0 \), \( 0 \leq j \leq m - 1 \), and

\[
\alpha_1 \|x\|^a \leq V(t, x(t)) \leq \alpha_2 \|x\|^b,
\]

\[
A_D^{a, b} V(t, x(t)) \leq -\alpha_3 \|x\|^b,
\]

where \( t \geq 0 \), \( x, \beta \in \mathbb{C} \) with \( \text{Re}(\alpha) > 0 \), \( \text{Re}(\beta) > 0 \), \( \alpha_1, \alpha_2, \alpha_3, a \) and \( b \) be arbitrary positive constants. Then, \( x = 0 \) is FOB-stable and asymptotically stable.

Proof. From inequalities (60) and (61), one has that

\[
A_D^{a, b} V(t, x(t)) \leq -\frac{\alpha_3}{\alpha_2} V(t, x(t)).
\]

Let \( M(t) \) be a non-negative function. Based on this function, the inequality (62) can be rewritten as

\[
A_D^{a, b} V(t, x(t)) + \lambda V(t, x(t)) + M(t) = 0,
\]

with \( \lambda = \alpha_3 \alpha_2^{-1} \). Applying the Laplace transform to Equation (63), we have that

\[
V(s) \left[ A_D^{a, b} \right] s^{-a} \left[ \sum_{j=0}^{m-1} s^{m-1-j} V^{(j)}(0, x(0)) \right] + \lambda V(s) + M(s) = 0.
\]

Rearranging this equation and solving for \( V(s) \), we have

\[
V(s) \left[ A_D^{a, b} \right] s^{-a' + m + \lambda} = A_D^{a, b} \sum_{j=0}^{m-1} s^{m-1-a'-j} V^{(j)}(0, x(0)) - \frac{M(s)}{A_D^{a, b} s^{-a' + m + \lambda}}.
\]

Therefore,

\[
V(s) = \sum_{j=0}^{m-1} \frac{A_D^{a, b} \left[ \sum_{j=0}^{m-1} s^{m-1-a'-j} V^{(j)}(0, x(0)) \right]}{A_D^{a, b} \left[ A_D^{a, b} \right] s^{-a' + m + \lambda}} - \frac{M(s)}{A_D^{a, b} s^{-a' + m + \lambda}}.
\]

Once the algebraic problem is solved, we apply the inverse Laplace transform of (64) to obtain

\[
V(t, x(t)) = \sum_{j=0}^{m-1} V^{(j)}(0, x(0))^{\gamma} \mathcal{G}_\lambda(t; \alpha', \beta', m) - \mathcal{G}_\lambda(t; \alpha', \beta', m) \ast M(t).
\]

Considering \( \mathcal{G}_\lambda(t; \alpha', \beta', m) \geq 0 \), \( \forall t \geq 0 \), then \( \mathcal{G}_\lambda(t; \alpha', \beta', m) \ast M(t) \geq 0 \) and thus

\[
V(t, x(t)) \leq \sum_{j=0}^{m-1} V^{(j)}(0, x(0))^{\gamma} \mathcal{G}_\lambda(t; \alpha', \beta', m).
\]
Combining (66) with condition stated in inequality (60) yields
\[
\|x(t)\| \leq \left[ \sum_{j=0}^{m-1} \frac{V^{(j)}(0, x(0))}{\alpha_1} G_{\alpha'}(t; \alpha', \beta', m) \right]^{1/\alpha},
\]
where \( \frac{V^{(j)}(0, x(0))}{\alpha_1} > 0 \) for \( x(0) \neq 0 \). Letting \( K_j = \frac{V^{(j)}(0, x(0))}{\alpha_1} \geq 0 \), then we have
\[
\|x(t)\| \leq \left[ \sum_{j=0}^{m-1} K_j G_{\alpha'}(t; \alpha', \beta', m) \right]^{1/\alpha},
\]
where \( K_j = 0 \) if and only if \( x(0) = 0 \). Because \( V(t, x(t)) \) is locally Lipschitz with respect to \( x \), its derivatives are bounded and \( V^{(j)}(0, x(0)) = 0 \) if and only if \( x(0) = 0 \), then it follows that \( K_j \) is also Lipschitz with respect to \( x(0) \) and \( K_j(0) = 0 \); this implies the FOB stability of system (43). Furthermore, by using the final value theorem on the right-hand side of (66) for \( m - \alpha' - j > 0 \), we get
\[
\lim_{t \to \infty} V(t, x(t)) \leq \lim_{s \to 0} \sum_{j=0}^{m-1} \frac{A_j \left( s^{-\beta'} \right)^{m - \alpha' - j}}{A_1 \left( s^{-\beta'} \right)^{s - \alpha' + m} + \lambda} V^{(j)}(0, x(0)) = 0. \tag{68}
\]
Combining inequalities (60), (68), and considering that \( V(t, x(t)) \geq 0 \) for all \( t \) yields
\[
\lim_{t \to \infty} a_1 \|x\|^a \leq \lim_{t \to \infty} V(t, x(t)) = 0. \tag{69}
\]
It follows from (69) that \( \lim_{t \to \infty} \alpha_1 \|x\|^a \leq 0 \). Finally, due to \( \alpha_1, a > 0 \), then
\[
\lim_{t \to \infty} \|x(t)\| = 0.
\]
This proves that the origin of system (43) is asymptotically stable. \( \Box \)

**Remark 6.** Note that FOB-stability implies asymptotic stability.

**Theorem 9.** If the assumptions in Theorem 8 are satisfied except replacing \( \tilde{\mathcal{D}}^{\alpha, \beta}_{0+} \) by \( RL\mathcal{D}^{\alpha, \beta}_{0+} \), then the origin \( x = 0 \) of system (49) is asymptotically stable.

**Proof.** By using the inequality of Theorem 3 and \( V(t, x(t)) \geq 0 \), we obtain
\[
\tilde{\mathcal{D}}^{\alpha, \beta}_{0+} V(t, x(t)) \leq RL\mathcal{D}^{\alpha, \beta}_{0+} V(t, x(t)) \leq -\alpha_3 \|x\|^a.
\]
Following the proof of Theorem 8, the proof is completed. \( \Box \)

5. Useful Inequalities for Lyapunov Stability Analysis

In this section, some inequalities are established. These results help develop tools for the stability analysis of fractional-order nonlinear systems, employing the generalized Lyapunov direct method shown in the previous section. We start with some significant lemmas that will help us prove the main results.

**Lemma 5.** \([44]\). Let \( u(t) \) be a continuous and differentiable real-valued function. Then, for any time instant \( t \geq t_0 \) and for all \( 0 < \alpha < 1 \):
\[
RL\mathcal{D}^\alpha_{t_0+} u^2(t) \leq 2u(t) RL\mathcal{D}^\alpha_{t_0+} u(t).
\]
Lemma 6. [45]. Let \( u(t) \) be a continuous and differentiable real-valued function. Then, for any time instant \( t \geq t_0 \) and for all \( 0 < \alpha < 1 \):

\[
\mathcal{D}_t^\alpha u^2(t) \leq 2u(t) \mathcal{D}_t^\alpha u(t).
\]

Lemma 7. Let \( u(t) \in \mathbb{R} \) be a continuously differentiable and monotonically increasing function and \( u(t) \geq 0 \). Then, for all \( t \geq t_0 \), \( \alpha' + \beta'k > m, m \in \mathbb{Z}^+ \), the following inequality holds:

\[
\mathcal{I}_t^\alpha f + \beta'k - m u^2(t) \leq 2u(t) \mathcal{I}_t^\alpha f + \beta'k - m u(t).
\] (70)

Proof. Rewriting the inequality (70), one has that

\[
\mathcal{I}_t^\alpha f + \beta'k - m u^2(t) \leq 2u(t) \mathcal{I}_t^\alpha f + \beta'k - m u(t).
\] (71)

Differentiating inequality (71) with respect to time, we have

\[
\mathcal{I}_t^\alpha f + \beta'k - m u^2(t) - 2u(t) \mathcal{I}_t^\alpha f + \beta'k - m u(t) \leq 0.
\] (72)

By the hypotheses on function \( u \), and using the fact that \( \alpha' + \beta'k > m \),

\[
\mathcal{I}_t^\alpha f + \beta'k - m u(t) \geq 0.
\] (73)

Then, expression (72) is clearly true due to Lemma 5. This concludes the proof. \( \square \)

Remark 7. If we set \( a_k > 0 \) for all \( k \geq 0 \), \( \alpha' > 0 \), \( \beta' > 0 \); then, from the previous Lemma,

\[
\begin{align*}
& a_0 \Gamma(\alpha') \mathcal{I}_t^\alpha f + \beta'k - m u^2(t) \leq 2a_0 \Gamma(\alpha') u(t) \mathcal{I}_t^\alpha f + \beta'k - m u(t), \\
& a_1 \Gamma(\alpha' + \beta') \mathcal{I}_t^\alpha f + \beta'k - m u^2(t) \leq 2a_1 \Gamma(\alpha' + \beta') u(t) \mathcal{I}_t^\alpha f + \beta'k - m u(t), \\
& a_2 \Gamma(\alpha' + 2\beta') \mathcal{I}_t^\alpha f + \beta'k - m u^2(t) \leq 2a_2 \Gamma(\alpha' + 2\beta') u(t) \mathcal{I}_t^\alpha f + \beta'k - m u(t), \\
& \vdots \\
& a_j \Gamma(\alpha' + j\beta') \mathcal{I}_t^\alpha f + \beta'k - m u^2(t) \leq 2a_j \Gamma(\alpha' + j\beta') u(t) \mathcal{I}_t^\alpha f + \beta'k - m u(t).
\end{align*}
\]

Then, under the conditions that permit local uniform convergence on a disc \( D(0, R) \) such that \( R > (b - a) \text{Re}(\beta) \), it follows that

\[
\lim_{j \to \infty} \sum_{k=0}^{j} a_k \Gamma(\alpha' + \beta'k) \mathcal{I}_t^\alpha f + \beta'k - m u^2(t) \leq \lim_{j \to \infty} \sum_{k=0}^{j} a_k \Gamma(\alpha' + \beta'k) 2u(t) \mathcal{I}_t^\alpha f + \beta'k - m u(t).
\]

This idea can be summarized in the following corollary.

Corollary 1. Let \( a, b, A \) satisfying Definition 8, \( \alpha', \beta' \in \mathbb{C} \) with \( \text{Re}(\alpha') > 0, \text{Re}(\beta') > 0 \). If \( a_k > 0 \) for all \( k \geq 0 \), \( \alpha' + \beta'k > m, m \in \mathbb{Z}^+ \). Then,

\[
\sum_{k=0}^{\infty} a_k \Gamma(\alpha' + \beta'k) \mathcal{I}_t^\alpha f + \beta'k - m u^2(t) \leq \sum_{k=0}^{\infty} a_k \Gamma(\alpha' + \beta'k) 2u(t) \mathcal{I}_t^\alpha f + \beta'k - m u(t).
\] (74)

Theorem 10. Let \( u \in L^1[a, b] \). If \( a_k > 0 \) and \( \alpha' + \beta'k - m > 0 \) for all \( k \), then, for any time instant \( t \geq t_0 \), the following inequality holds:

\[
\mathcal{A}_t^{\lambda, \alpha} u^2(t) \leq 2u(t) \mathcal{A}_t^{\lambda, \alpha} u(t).
\]
Proof. From Lemma 2 with \( u(t) = v(t) \), we have

\[
\begin{align*}
\mathcal{A}_{RL}^\alpha D_0^{\alpha + \beta} u^2(t) &= 2u(t) \mathcal{A}_{RL}^\alpha D_0^{\alpha + \beta} u(t) \\
& \quad + \sum_{k=0}^{\infty} a_k \frac{\Gamma(\alpha' + \beta'k)}{\Gamma(\alpha' + \beta'k - m)} \int_{t_0}^t (t - \tau)^{\alpha' + \beta'k - m - 1} \left[ u^2(\tau) - 2u(\tau)u(t) \right] d\tau \\
& \quad + \Gamma(\alpha' + \beta'k) \int_{t_0}^t (t - \tau)^{\alpha' + \beta'k - m} d\tau \\
& \quad - \frac{u^2(t)}{\Gamma(\alpha' + \beta'k - m)} \int_{t_0}^t (t - \tau)^{\alpha' + \beta'k - m - 1} d\tau. \\
&= 2u(t) \mathcal{A}_{RL}^\alpha D_0^{\alpha + \beta} u(t)
\end{align*}
\]

Since

\[
\int_{t_0}^t (t - \tau)^{\alpha' + \beta'k - m - 1} d\tau = (t - t_0)^{\alpha' + \beta'k - m} \frac{\Gamma(\alpha' + \beta'k - m)}{\Gamma(\alpha' + \beta'k - m + 1)},
\]

and by using the classical Riemann–Liouville integral, one has that

\[
\begin{align*}
\mathcal{A}_{RL}^\alpha D_0^{\alpha + \beta} u^2(t) &= 2u(t) \mathcal{A}_{RL}^\alpha D_0^{\alpha + \beta} u(t) \\
& \quad + \sum_{k=0}^{\infty} a_k \frac{\Gamma(\alpha' + \beta'k)}{\Gamma(\alpha' + \beta'k - m)} \int_{t_0}^t (t - \tau)^{\alpha' + \beta'k - m} u^2(\tau) - 2u(\tau)u(t) \int_{t_0}^t (t - \tau)^{\alpha' + \beta'k - m} d\tau. \\
&= 2u(t) \mathcal{A}_{RL}^\alpha D_0^{\alpha + \beta} u(t),
\end{align*}
\]

Since the integral converges for all \( k \), then, from Lemma 7 by setting \( a_k > 0 \) for \( \alpha' + \beta'k - m > 0 \), the series is convergent and satisfies (74). Then, it follows that

\[
\begin{align*}
\mathcal{A}_{RL}^\alpha D_0^{\alpha + \beta} u^2(t) &\leq 2u(t) \mathcal{A}_{RL}^\alpha D_0^{\alpha + \beta} u(t),
\end{align*}
\]

and the proof is completed. \( \square \)

Theorem 11. Let \( u \in L^1[a, b] \). If \( a_k > 0 \) and \( \alpha' + \beta'k - m > 0 \) for all \( k \), then, for any time instant \( t \geq t_0 \), the following inequality holds:

\[
\mathcal{A}_{RL}^\alpha D_0^{\alpha + \beta} u^2(t) \leq 2u(t) \mathcal{A}_{RL}^\alpha D_0^{\alpha + \beta} u(t). \tag{75}
\]

Proof. Proving that inequality (75) is true, is equivalent to prove that

\[
2u(t) \mathcal{A}_{RL}^\alpha D_0^{\alpha + \beta} u(t) - \mathcal{A}_{RL}^\alpha D_0^{\alpha + \beta} u^2(t) \geq 0. \tag{76}
\]

Using Definition (15) and representation (13), the expression (76) can be written as

\[
\begin{align*}
\sum_{k=0}^{\infty} a_k \Gamma(\beta'k + \alpha') \left[ 2u(t) \mathcal{A}_{RL}^{\alpha' + \beta'k} \frac{d}{dm} u(t) - \mathcal{A}_{RL}^{\alpha' + \beta'k} \frac{d}{dm} u^2(t) \right] &\geq 0. \tag{77}
\end{align*}
\]

Furthermore, from the classical Caputo derivative (Definition 3), the inequality takes the following form:

\[
\begin{align*}
\sum_{k=0}^{\infty} a_k \Gamma(\beta'k + \alpha') \left[ 2u(t) \mathcal{C} D_0^{m - \alpha' - \beta'k} u(t) - \mathcal{C} D_0^{m - \alpha' - \beta'k} u^2(t) \right] &\geq 0. \tag{78}
\end{align*}
\]

Since the Riemann–Liouville integral converges for all \( k, \alpha' + \beta'k > 0 \); then, from Lemma 6, by setting \( a_k > 0 \), the series is convergent, and this concludes the proof. \( \square \)

Corollary 2. Assume the conditions of Theorems 10 and 11.
(a) If \( \alpha' = 1 - \alpha, \beta' = 0, m = 1 \) and \( A((t - \tau)^{\alpha}) = \frac{1}{\Gamma(1 - \alpha)}, \) then for all \( t \geq t_0 \)

\[
\begin{align*}
    RL^\alpha_{t_0+}u^2(t) &\leq 2u(t) RL^\alpha_{t_0+}u(t), \\
    CD^\alpha_{t_0+}u^2(t) &\leq 2u(t) CD^\alpha_{t_0+}u(t).
\end{align*}
\]

(b) If \( \alpha' = 1, \beta' = 1, m = 1 \) and \( A((t - \tau)^{\alpha}) = \frac{M(\alpha)}{1 - \alpha} \exp\left(-\frac{\alpha}{1 - \alpha}(t - \tau)\right), \) then for all \( t \geq t_0 \)

\[
\begin{align*}
    CF^\alpha_{t_0+}u^2(t) &\leq 2u(t) CF^\alpha_{t_0+}u(t).
\end{align*}
\]

(c) If \( \alpha' = 1, \beta' = \alpha, m = 1 \) and \( A((t - \tau)^{\alpha}) = \frac{B(\alpha)}{1 - \alpha} E_\alpha\left(-\frac{\alpha}{1 - \alpha}(t - \tau)^\alpha\right), \) then for all \( t \geq t_0 \)

\[
\begin{align*}
    ABR^\alpha_{t_0+}u^2(t) &\leq 2u(t) ABR^\alpha_{t_0+}u(t), \\
    ABC^\alpha_{t_0+}u^2(t) &\leq 2u(t) ABC^\alpha_{t_0+}u(t).
\end{align*}
\]

**Proof.** The proof is straightforward by a simple substitution of \( \alpha', \beta' \) and \( A(\cdot) \) of Definitions (14) and (15) in Theorems 10 and 11. \( \square \)

**Remark 8.** We presented two general cases for inequalities related to quadratic forms. Particular cases for classical operators are treated in Corollary 2. All these inequalities were previously treated as the main results in different papers [35,44–46].

### 6. Convex Lyapunov Functions and Stability

In this section, the stability results presented formerly are extended considering convex Lyapunov functions. Previously, in [47,48], the convex analysis was treated for the classic Caputo and Riemann–Liouville derivatives. Now, we extend the results considering general analytic kernel operators, which represent a generalization of earlier studies. In this section, we will assume that \( \Omega \) is a convex and compact set [49].

**Definition 17.** Let \( f(x) \) be a continuously differentiable function. The function \( f(x) \) is said to be convex in a convex domain if it satisfies

\[
f(y) \geq f(x) + \langle f'(x), y - x \rangle
\]

for all \( x, y \) in its domain.

**Theorem 12.** Let \( V(t, x(t)) : \Omega \rightarrow \mathbb{R} \), where \( \Omega = [0, \infty) \times D, D \subset \mathbb{R}^n, \) and \( x(t) : [t_0, \infty) \rightarrow D \) be two continuous and differential functions. Suppose that \( V(t, x(t)) \) is convex over \( \Omega \), with \( V(t_0, x(t_0)) = 0, a_k > 0 \) and \( m - \alpha' - \beta'k > 0 \) for all \( k \). Then, for all \( t \geq t_0 \)

\[
\begin{align*}
    ARL^\alpha_{t_0+}V(t, x(t)) &\leq \nabla^T V(t, x(t)) ARL^\alpha_{t_0+}x(t).
\end{align*}
\]

**Proof.** The function \( V(t, x(t)) \) can be expressed as

\[
V(t, x(t)) = V(t_0, x(t_0)) + \int_{t_0}^{t} \nabla V(\tau, x(\tau)) d\tau = V(t_0, x(t_0)) + RL^1_{t_0+} \dot{V}(t, x(t)).
\]

Taking the GAKRL (14) in both sides of (86), then

\[
\begin{align*}
    ARL^\alpha_{t_0+}V(t, x(t)) &= V(t_0, x(t_0)) \sum_{k=0}^{\infty} a_k \frac{\Gamma(\alpha' + \beta'k)}{\Gamma(\alpha' + \beta'k + 1 - m)} (t - t_0)^{\alpha' + \beta'k - m} \\
    &+ \sum_{k=0}^{\infty} a_k \Gamma(\beta'k + \alpha') RL^\alpha_{t_0+}k^{\beta'k + 1 - m} V(t, x(t)).
\end{align*}
\]
Following the same idea for the function \( x(t) \), we have that
\[
 x(t) = x(t_0) \sum_{k=0}^{\infty} a_k \frac{\Gamma(a' + \beta' k)}{\Gamma(a' + \beta' k + 1 - m)} (t - t_0)^{a' + \beta' k - m} + \sum_{k=0}^{\infty} a_k \Gamma(\beta' k + \alpha') RL_{t_0}^{a' + \beta' k + 1 - m} x(t).
\]

(88)

Thus, to justify the proposed inequality, we can rewrite inequality (85) as follows:
\[
(\nabla^T V(t, x(t))x(t_0) - V(t_0, x(t_0))) \sum_{k=0}^{\infty} a_k \frac{\Gamma(a' + \beta' k)}{\Gamma(a' + \beta' k + 1 - m)} (t - t_0)^{a' + \beta' k - m} + \sum_{k=0}^{\infty} a_k \frac{\Gamma(a' + \beta' k)}{\Gamma(a' + \beta' k + 1 - m)} \left[ \lim_{\tau \to t} (t - \tau)^{m - a' - \beta' k} - \frac{\phi(t_0)}{(t - t_0)^{m - a' - \beta' k}} \right]
\]
\[
+ (m - a' - \beta' k) \int_{t_0}^{t} -\phi'(\tau) (t - \tau)^{a' + \beta' k - m - 1} d\tau \geq 0.
\]

(90)

Considering that \( a' + \beta' k + 1 - m > 0 \), then the limit is calculated by applying the L'Hôpital's rule, i.e.,
\[
\lim_{\tau \to t} (t - \tau)^{m - a' - \beta' k} = \lim_{\tau \to t} \frac{\nabla^T V(t, x(t)) - \nabla^T V(t, x(\tau))}{(m - a' - \beta' k)(t - \tau)^{m - a' - \beta' k - 1}} = 0.
\]

Evaluating \( \phi(t_0) \) and considering that \( V(t_0, x(t_0)) = 0 \), we have
\[
\sum_{k=0}^{\infty} a_k \frac{\Gamma(a' + \beta' k)}{\Gamma(a' + \beta' k + 1 - m)} \left[ (t - t_0)^{a' + \beta' k - m} \nabla^T V(t, x(t)) x(t) - V(t, x(t)) \right]
\]
\[
+ (m - a' - \beta' k) \int_{t_0}^{t} -\phi'(\tau) (t - \tau)^{a' + \beta' k - m - 1} d\tau \geq 0.
\]

(91)

Finally, due to the convexity of \( V(t, x(t)) \), \( \nabla^T V(t, x(t)) x(t) - V(t, x(t)) \geq 0 \). Furthermore, \( \phi(t) \leq 0 \) for all \( t \) such that the series converges. Then, the inequality (91) is true, and the proof is completed. \( \square \)

**Corollary 3.** Assume the conditions of Theorem 12:

(a) If \( a' = 1 - \alpha, \beta' = 0, m = 1 \) and \( A((t - \tau)^0) = \frac{1}{\Gamma(1 - \alpha)} \), then for all \( t \geq t_0 
\]
\[
RL_{t_0}^{\alpha} V(t, x(t)) = \nabla^T V(t, x(t)) RL_{t_0}^{\alpha} x(t).
\]

(b) If \( a' = 1, \beta' = \alpha, m = 1 \) and \( A((t - \tau)\alpha) = \frac{B(\alpha)}{1 - \alpha} E_{\alpha} \left(-\frac{\alpha}{1 - \alpha}(t - \tau)^{\alpha}\right) \), then for all \( t \geq t_0 
\]
\[
ABR_{t_0}^{\alpha} V(t, x(t)) \leq \nabla^T V(t, x(t)) ABR_{t_0}^{\alpha} x(t).
\]

(93)
Proof. The proof is immediate and is omitted. □

Theorem 13. Let \( V(t, x(t)) : \Omega \to \mathbb{R}, \) where \( \Omega = [0, \infty) \times D, D \subseteq \mathbb{R}^n, \) and \( x(t) : [t_0, \infty) \to D \) be two continuous and differential functions. Suppose that \( V(t, x(t)) \) is convex over \( \Omega, \) with \( V^{(j)}(t_0, x(t_0)) = 0 \) for all \( j > 0, a_k > 0 \) and \( m - \alpha' - \beta'k > 0 \) for all \( k. \) Then, for all \( t \geq t_0, \)
\[
{\mathcal{A}}D_\Gamma ^{\alpha, \beta} V(t, x(t)) \leq \nabla^TV(t, x(t)) {\mathcal{A}}D_\Gamma ^{\alpha, \beta} x(t). \tag{94}
\]

Proof. The proof is obtained directly by combining Theorem 2 and Theorem 12, considering that \( V^{(j)}(t_0, x(t_0)) = 0 \) for all \( j. \) □

Corollary 4. Assume the conditions of Theorem 13.

(a) If \( \alpha' = 1 - \alpha, \beta' = 0, m = 1, \) and \( A\left( t - \tau \right) = \frac{1}{1 - \alpha} \), then, for all \( t \geq t_0, \)
\[
{\mathcal{C}}D_\Gamma ^{\alpha} V(t, x(t)) \leq \nabla^TV(t, x(t)) {\mathcal{C}}D_\Gamma ^{\alpha} x(t). \tag{95}
\]

(b) If \( \alpha' = 1, \beta' = 1, m = 1 \) and \( A\left( t - \tau \right) = \frac{M(\alpha)}{1 - \alpha} \exp \left( -\frac{\alpha}{1 - \alpha} (t - \tau) \right), \) then, for all \( t \geq t_0, \)
\[
{\mathcal{C}}D_\Gamma ^{\alpha} V(t, x(t)) \leq \nabla^TV(t, x(t)) {\mathcal{C}}D_\Gamma ^{\alpha} x(t). \tag{96}
\]

(c) If \( \alpha' = 1, \beta' = \alpha, m = 1 \) and \( A\left( t - \tau \right) = \frac{B(\alpha)}{1 - \alpha} \left( \frac{\alpha}{1 - \alpha} (t - \tau) \right), \) then, for all \( t \geq t_0, \)
\[
{\mathcal{A}}BC_\Gamma ^{\alpha} V(t, x(t)) \leq \nabla^TV(t, x(t)) {\mathcal{A}}BC_\Gamma ^{\alpha} x(t). \tag{97}
\]

Proof. The proof is straightforward by a simple substitution of \( \alpha', \beta' \) and \( A(\cdot) \) of Definitions (14) and (15) in Theorem 13. □

Theorem 14. Let \( x = 0 \) be an equilibrium point for the fractional-order system (43). Let \( V(t, x(t)) \) such that it satisfies Theorem 13, locally Lipschitz with respect to \( x \) such that the following inequalities hold:
\[
\alpha_1 \|x\|^\alpha \leq V(t, x(t)) \leq \alpha_2 \|x\|^{\alpha b}, \tag{98}
\]
\[
\nabla^TV(t, x(t)) f(t, x(t)) \leq -\alpha_3 \|x\|^{ab}, \tag{99}
\]
where \( \alpha_1, \alpha_2, \alpha_3, a, \) and \( b \) are arbitrary positive constants. Then, \( x = 0 \) is asymptotically stable.

Proof. From Theorem 13 and inequality (99), the fractional-order derivative of \( V(t, x(t)) \) satisfies
\[
{\mathcal{A}}D_\Gamma ^{\alpha, \beta} V(t, x(t)) \leq \nabla^TV(t, x(t)) {\mathcal{A}}D_\Gamma ^{\alpha, \beta} x(t) = \nabla^TV(t, x(t)) f(t, x(t)) \leq -\alpha_3 \|x\|^{ab}.
\]
From this analysis and due to \( V(t, x(t)) \) satisfying (98), it follows that conditions of Theorem 8 are satisfied. Therefore, the origin of system (43) is asymptotically stable. □

Theorem 15. Let \( x = 0 \) be an equilibrium point for the fractional-order system (49). Let \( V(t, x(t)) \) be as in Theorem 12, locally Lipschitz with respect to \( x \) such that
\[
\alpha_1 \|x\|^\alpha \leq V(t, x(t)) \leq \alpha_2 \|x\|^{\alpha b}, \tag{100}
\]
\[
\nabla^TV(t, x(t)) f(t, x(t)) \leq -\alpha_3 \|x\|^{ab}, \tag{101}
\]
where \( \alpha_1, \alpha_2, \alpha_3, a, \) and \( b \) are arbitrary positive constants. Then, \( x = 0 \) is asymptotically stable.

Proof. The proof follows the same outline as the one from Theorem 14 and is omitted. □
Remark 9. If we set different values of $\alpha'$ and $\beta'$, the conclusions of Theorems (14) and (15) hold for different fractional-order derivatives. For example, for $\alpha' = \beta' = 1$, $m = 1$ and $A((t - \tau)) = M(\alpha) \exp \left(-\frac{\alpha}{1 - \alpha} (t - \tau)\right)$, then, for all $t \geq t_0$, we obtain the conclusion for systems with a Caputo–Fabrizio derivative [50].

7. Some Representative Examples

7.1. Scalar Systems

In this first example, we study scalar systems. Consider the system given by [51]:

$$A_C^{0,0} \frac{D_t^\alpha x}{C} = ax^p + g(x),$$

(102)

where the Caputo derivative of general analytic kernel (15) is considered. In system (102), the parameter $p > 0$ is odd, $a < \mathbb{R}^-$ and $g(x)$ is bounded as follows:

$$|g(x)| \leq k|x|^{p+1},$$

(103)

where the inequality (103) is valid in some neighborhood of the origin $x = 0$. To carry out the stability analysis and show the validity, in practical cases, of the inequalities demonstrated in previous sections, let us consider the Lyapunov candidate function $V(x) = \frac{1}{2} x^2$. Then, applying Theorem 11 and considering system (102), one has that

$$A_C^{0,0} \frac{D_t^{\alpha,\beta} V}{C} \leq A_C^{0,0} \frac{D_t^{\alpha,\beta} x}{C} = x[a x^p + g(x)] \leq ax^{p+1} + |x||g(x)|.$$

Hence, due to the bound (103), the fractional-order derivative of $V(x)$ is bounded as follows:

$$A_C^{0,0} \frac{D_t^{\alpha,\beta} V}{C} \leq ax^{p+1} + k|x|^{p+2}. $$

Near the origin, the term $ax^{p+1}$ is dominant, and then $A_C^{0,0} \frac{D_t^{\alpha,\beta} V}{C} \leq k|x|^{p+2}$. This implies that the origin is Fernandez–Özarslan–Baleanu stable and therefore asymptotically stable.

7.2. Second Order Systems

There is a wide class of systems of second order. In particular, consider the following system modeled using the Caputo derivative of general analytic kernel (15):

$$\begin{align*}
A_C^{0,0} \frac{D_t^{\alpha,\beta} x_1}{C} &= -x_1 + x_1 x_2, \\
A_C^{0,0} \frac{D_t^{\alpha,\beta} x_2}{C} &= -x_2.
\end{align*}$$

(104)

It is not difficult to show that the equilibrium point of system (104) is $(0,0)$. To analyze stability, consider $V(x) = \frac{1}{2} (\bar{x}_1^2 + \bar{x}_2^2)$. Applying the inequality given in Theorem 11 and considering system (104), one has that

$$A_C^{0,0} \frac{D_t^{\alpha,\beta} V}{C} \leq A_C^{0,0} \frac{D_t^{\alpha,\beta} x_1}{C} + A_C^{0,0} \frac{D_t^{\alpha,\beta} x_2}{C} = x_1(-x_1 + x_1 x_2) - \bar{x}_2^2.$$

To analyze the last inequality, consider the ball $\{||x||^2 \leq r^2\}$ where $|x_1| \leq r$. If we restrict the analysis to this set, then

$$A_C^{0,0} \frac{D_t^{\alpha,\beta} V}{C} \leq -x_1^2 - \bar{x}_2^2 + r|x_1||x_2| = -\left(|x_1|^2 - 2 \cdot \frac{r}{2} |x_1||x_2| + |x_2|^2\right).$$
The right-hand side of the above inequality can be rewritten by using a matrix form, so that
\[
\frac{\text{d}}{\text{d}t} D^{\alpha,\beta}_{0+} V \leq - \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T \begin{bmatrix} 1 \\ -r/2 \\ 1 \\ -r/2 \end{bmatrix} \begin{bmatrix} |x_1| \\ |x_2| \end{bmatrix}.
\]

By a simple calculation, it is not difficult to show that the associated eigenvalues of \( M \) are \( \lambda_1 = \frac{r + \frac{3}{2}}{2}, \lambda_2 = \frac{-r + \frac{3}{2}}{2} \); then, if we choose \( r < 2 \) and applying Theorem 8, it follows that the origin is asymptotically stable.

### 7.3. A Spacecraft Modeled by Generalized Dynamics

A rotating rigid spacecraft is studied in [52] by using the well-known Euler equations. The differential equations can be generalized considering the Caputo (or Riemann–Liouville (14)) derivative of general analytic kernel (15). This allows for considering different kinds of analysis in engineering applications. For this case, consider the following set of differential equations:

\[
\begin{align*}
\frac{\text{d}}{\text{d}t} D^{\alpha,\beta}_{0+} \omega_1 &= J_1^{-1}[J_2 - J_3] \omega_2 \omega_3 + u_1, \\
\frac{\text{d}}{\text{d}t} D^{\alpha,\beta}_{0+} \omega_2 &= J_2^{-1}[J_3 - J_1] \omega_3 \omega_1 + u_2, \\
\frac{\text{d}}{\text{d}t} D^{\alpha,\beta}_{0+} \omega_3 &= J_3^{-1}[J_1 - J_2] \omega_1 \omega_2 + u_3,
\end{align*}
\]

(105)

where the scalar components of the vector \( \omega \) are, respectively, \( \omega_1, \omega_2 \) and \( \omega_3 \). On the other hand, some torque inputs, denoted by \( u_1, u_2 \) and \( u_3 \), are considered and applied about the principal axes. In addition to the dynamical analysis, the components \( J_1, J_2 \) and \( J_3 \) represent the moments of inertia. The stability analysis will be done by considering the controlled system (105). Suppose that the torque inputs apply the feedback control \( u_i = -k_i \omega_i \), where \( k_i > 0 \) and \( 1 \leq i \leq 3 \). Then, the close-loop system is given by

\[
\begin{align*}
\frac{\text{d}}{\text{d}t} D^{\alpha,\beta}_{0+} \omega_1 &= J_1^{-1}[J_2 - J_3] \omega_2 \omega_3 - k_1 \omega_1, \\
\frac{\text{d}}{\text{d}t} D^{\alpha,\beta}_{0+} \omega_2 &= J_2^{-1}[J_3 - J_1] \omega_3 \omega_1 - k_2 \omega_2, \\
\frac{\text{d}}{\text{d}t} D^{\alpha,\beta}_{0+} \omega_3 &= J_3^{-1}[J_1 - J_2] \omega_1 \omega_2 - k_3 \omega_3,
\end{align*}
\]

(106)

Now, taking \( V(\omega) = \frac{1}{2} \left( J_1 \omega_1^2 + J_2 \omega_2^2 + J_3 \omega_3^2 \right) \) as a Lyapunov function candidate and using the inequality (75), it is not difficult to obtain

\[
\frac{\text{d}}{\text{d}t} D^{\alpha,\beta}_{0+} \leq -k_1 \omega_1^2 - k_2 \omega_2^2 - k_3 \omega_3^2.
\]

Thus, by applying Theorem 8, the origin is Fernandez–Özarslan–Baleanu stable and therefore asymptotically stable.

### 7.4. Financial Analysis

Financial systems are employed to analyze different situations in society, and many points of view have been proposed to model financial scenarios [53]. Consider, for example, the following fractional-order system modeled by using the Caputo derivative with general analytic kernel (15):

\[
\begin{align*}
\frac{\text{d}}{\text{d}t} D^{\alpha,\beta}_{0+} x_1(t) &= x_3(t) + (x_2(t) - a)x_1(t) + u_1(t), \\
\frac{\text{d}}{\text{d}t} D^{\alpha,\beta}_{0+} x_2(t) &= 1 - bx_2(t) - x_1^2(t) + u_2(t), \\
\frac{\text{d}}{\text{d}t} D^{\alpha,\beta}_{0+} x_3(t) &= -x_1(t) - cx_3(t) + u_3(t),
\end{align*}
\]

(107)
where $x_1(t)$ represents the interest rate, $x_2(t)$ the investment demand, and $x_3(t)$ the price index, and the constants $a = 1$ (the saving amount), $b = 0.1$ (the cost per investment), and $c = 1$ (the elasticity of demand of commercial markets). The stability analysis considers the equilibrium point of the closed-loop system considering the control laws given by

$$
\begin{align*}
    u_1(t) &= x_1^{2/3}(t)x_3^{1/3}(t) + x_1^{5/3}(t)x_2^{1/3}(t), \\
    u_2(t) &= -1 - x_2(t) - x_1^{4/3}(t)x_2^{2/3}(t), \\
    u_3(t) &= -x_1^{1/3}(t)x_2^{2/3}(t).
\end{align*}
$$

(108)

On the other hand, let $V(x(t))$ be the candidate Lyapunov function described by

$$
V(x(t)) = x_1^{4/3}(t) + x_2^{4/3}(t) + x_3^{4/3}(t).
$$

(109)

It is not difficult to show, by using the Hessian matrix, that the function (109) is convex. This function can be rewritten as follows:

$$
V(x(t)) = (x_1^2(t))^{2/3} + (x_2^2(t))^{2/3} + (x_3^2(t))^{2/3},
$$

and, considering a well known inequality in control systems [54], the function $V(x(t))$ is bounded as follows:

$$
\left( x_1^2(t) + x_2^2(t) + x_3^2(t) \right)^{2/3} \leq V(x(t)) \leq 3^{1/3} \left( x_1^2(t) + x_2^2(t) + x_3^2(t) \right)^{2/3}
$$

$$
\|x\|^{4/3} \leq V(x(t)) \leq 3^{1/3} \|x\|^{4/3}.
$$

(110)

The advantage of Theorem 14 is that we do not need to calculate fractional derivatives of the Lyapunov function, and we only need to restrict the calculation to determine the gradient. Then, one has that

$$
\begin{align*}
    \left( \frac{\partial V}{\partial x} \right)^T f(x,t) &= \frac{4}{3} \left[ x_1^{1/3}(t) \left( x_3(t) + (x_2(t) - a)x_1(t) + x_1^{2/3}(t)x_3^{1/3}(t) + x_1^{5/3}(t)x_2^{1/3}(t) \right) \\
    &\quad + \frac{4}{3} x_2^{1/3}(t) \left( 1 - bx_2(t) - x_2(t) - x_1^{4/3}(t)x_2^{2/3}(t) \right) \\
    &\quad + \frac{4}{3} x_3^{1/3}(t) \left( -x_1(t) - cx_3(t) - x_1^{1/3}(t)x_3^{2/3}(t) \right) \right].
\end{align*}
$$

(111)

Finally, after some algebraic reductions and from (110), we have

$$
\begin{align*}
    \left( \frac{\partial V}{\partial x} \right)^T f(x,t) &= -\frac{4}{3} \left( ax_1^{1/3}(t) + (1 + b)x_2^{1/3}(t) + cx_3^{1/3}(t) \right) \\
    &\leq -\frac{4}{3} \left( x_1^{1/3}(t) + x_2^{1/3}(t) + x_3^{1/3}(t) \right) \\
    &\leq -\|x\|^{4/3}.
\end{align*}
$$

(112)

Finally, according to Theorem 14, it is concluded that the origin of system (107) with the control law (108) is Fernandez–Özarslan–Baleanu stable and asymptotically stable.

**Remark 10.** As proposed throughout the text, the stability results shown generalize the results previously established in the literature. In the presented examples, when considering particular values for $a$ and $b$, different dynamics can be obtained, which allows for comparing and analyzing dynamical behaviors that permit to choose one fractional derivative or propose one that adapts to the objectives of the research.
8. Conclusions

In this paper, we studied some properties of fractional-order derivatives and integrals with general analytic kernels. Some results concerning the Laplace transform were proved and used to establish some remarks on the solutions of fractional-order differential equations that involve these fractional operators, along with a generalized comparison principle. We proposed the FOB-function and the concept of FOB stability, which generalizes Mittag–Leffler stability for a comprehensive family of systems with different fractional-order derivatives.

Moreover, one of the main results presented consists of the generalization of the Lyapunov direct method, which is directly related to the FOB-stability and the boundedness of solutions. In addition, some inequalities for quadratic forms have been proposed. These results allow using the Lyapunov direct method for the stability analysis of fractional-order systems with the operators considered.

Furthermore, since the inequalities are established for operators with general analytic kernels, we have shown that the inequalities previously presented in the literature emerge from our work as particular cases. In addition, as an extension of the stability analysis, we have treated the stability problem and its solution through convex Lyapunov functions; some theorems are obtained directly from this part. Finally, we provided some illustrative examples to demonstrate the applicability of the proposed approach.

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