Time-evolution of the Rule 150 cellular automaton activity from a Fibonacci iteration

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The total activity of the single seeded cellular automaton does not follow a one-step iteration like other elementary cellular automata, but can be solved as a two-step vectorial, or string, iteration, which can be viewed as a generalization of Fibonacci iteration generating the time series from a sequence of vectors of increasing length. This allows to compute the total activity time series more efficiently than by simulating the whole spatio-temporal process, or even by using the closed expression.

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I. INTRODUCTION

Since the coining paper of Bak, Tang, and Wiesenfeld [1], there has been considerable interest in the long-time behavior of cellular automata, especially for occurrence of long range correlations, and correspondingly for power spectra exhibiting a power law decay, which have become a paradigm for complex dynamical systems in general[2]. Cellular automata[3,4] are a widely studied class of discrete dynamical systems showing emergence of complex spatio-temporal patterns from a simple dynamical rule.

ECA and sum signals — A cellular automaton consists of an infinite lattice of cells of e.g. two possible states (0,1), and a local deterministic update rule. At each discrete time step, a cell is updated $x_n \rightarrow x_{n+1}$ according to the state within a local neighborhood. For Conway's Game of Life[5] the 3 $\times$ 3 Moore neighborhood on a two-dimensional lattice is used. A simpler, yet complex, class are the elementary cellular automata (ECA) [3,4], defined on a one-dimensional lattice, and the update rule depends on the next-neighbor sites and its own state one time step before:

$$x_{n+1}^{t+1} = f(x_n^{t+1}, x_n^t, x_{n-1}^t)$$

where $f$ (the rule) is determined by 8 bits being the output of the possible input bits 000, 001, ..., 111; this 8-bit number forms the “rule” number which enumerates the 256 possible ECA rules. The power spectra of rule 90 [6] and some of the other ECA rules [7] exhibit a $1/f^\alpha$ decay. Rule 90 and rule 150 can be expressed also as

$$x_{n+1}^t = [x_{n-1}^t + rx_n^t + x_{n+1}^t] \mod 2$$

where $r = 0$ defines rule 90, and $r = 1$ rule 150, respectively. In the context of catalytic processes, both can be interpreted as local self-limiting reaction processes [3,4]. As models for chemical turbulence, similar dynamics with a continuous phase variable have been discussed in [8] and [9], including solitonic behavior, periodic, and turbulent states.

Total activity. — In the chemical picture, the total reaction rate at a given time $t$ corresponds to the total number of sites with $x_n = 1$, described by the sum signal

$$X(t) = \sum_n x_n.$$  

While for rule 150 there is no convenient solution of $X(t)$ except a formal one [10], and a fairly complicated expression (see Sec. II), it is computationally quite costly to perform the full spatiotemporal dynamical simulation, even if one is interested only in the time series. This paper gives an iterative solution of $X(t)$ from a geometrical iteration and investigates the relationship to the Fibonacci iteration. The block sums over $0 \leq t \leq N-1$ can even be expressed directly via Fibonacci numbers.

Throughout this paper the pure pattern generated by a single 1 are considered on an infinite lattice.

II. EXACT SOLUTION

While in the Sierpinski (rule 90) case $X(t)$ factorizes in a product of $X_i(\sigma_i)$ for all “time spins” [6], for rule 150 it does not. But again, a “spin decomposition” of time $t = \sum_{j=0}^{N-1} \sigma_j 2^j$ with $\sigma_j \in \{0,1\}$ can be utilized as an efficient coordinate system for the time axis.

Before turning to the geometric iteration, it should be mentioned that a closed expression in fact can be written down as follows. As pointed out by Wolfram [6], for rule 150 the “correlation” of the time spins comes into play, i.e. $X(t)$ is exactly multiplicative for blocks of spins of value 1 which are separated by one or more zeroes. Then $X(t) = \prod_{n=1}^{N} \chi(n)^{c_n}$, where $c_n$ is the multiplicity of blocks of length $n$. The series $\chi(n)$ should correctly read [10] to the iteration $\chi(n) = 2\chi(n-1) - (-1)^n$ for $n \leq 1$ and $\chi(0) = 1$. Obviously $\chi(n) = X(2^N - 1)$ holds. The $\chi(n)$ on the other hand turn out as the most decaying frequencies in the spectrum of rule 90 (see Fig. 2 in Ref. [6]), and can be expressed in closed form by

$$\chi(n) = \left[\frac{2^n+2}{3}\right],$$

where $[ \cdot ]$ is the floor function. Defining $\sigma_{-1} := 0$ and $\sigma_{N} := 0$, one can formalize the spin-block counting as

$$X(t) = \prod_{n=1}^{N} \chi(n) \sum_{i=0}^{N-n} (1-\sigma_{i-1})(1-\sigma_{i+n}) \prod_{l=0}^{n-1} \sigma_{i+l}.$$  

With our expression [6], this is a closed solution, and corresponds to $X_{150}(t) = \prod_{j=0}^{N-1} 2^{\sigma_j}$ in the Sierpinski case.
Due to the complicated time spin correlations, it however looks quite unwieldy for analytical use, and even is numerically unfeasible \cite{17}.

III. ITERATIVE SOLUTION BY GENERALIZED HYPER-FIBONACCI SERIES

In contrast to the rule 90 (Sierpinski) case, for rule 150 the time evolution does not follow the same type of initiator-generator mechanism as it is well known for fractal sets. However, it is possible to define a geometric or measure-theoretic \cite{10} iteration based on the last and the last but one iterate, see Fig. 1. This corresponds to a difference equation with the r.h.s. depending on the last two time steps, and in fact, for the total activity within $2^N$ time steps we will derive a difference equation later.

![FIG. 1: Left: Time-evolution of rule 150 for the first 64 time steps, started with a single seed. Right: Illustration of the replication rule. The whole system is symmetric with respect to the vertical axis. The whole triangle above is replicated once (left part of the triangle). The upper part is reproduced quadruplicate (right part of the triangle).](image)

According to the replication law (see Fig. 1), the time series of the total activity $X(t)$ follows the two-step iteration

$$X(1, 1, \sigma_{n-2} \cdots \sigma_0) = X(0, 1, \sigma_{n-2} \cdots \sigma_0) + 2X(0, 0, \sigma_{n-2} \cdots \sigma_0)$$
$$X(1, 0, \sigma_{n-2} \cdots \sigma_0) = 3X(0, 0, \sigma_{n-2} \cdots \sigma_0).$$

If we define

$$Y(\bullet, - - -) := X(\bullet, 0, - - -)$$
$$Z(\bullet, - - -) := X(\bullet, 1, - - -),$$

the iteration reads

$$Z(1, \sigma_{n-2} \cdots \sigma_0) = Z(1, \sigma_{n-2} \cdots \sigma_0) + 2Y(1, \sigma_{n-2} \cdots \sigma_0)$$
$$Y(1, \sigma_{n-2} \cdots \sigma_0) = 3Y(1, \sigma_{n-2} \cdots \sigma_0),$$

or short

$$Z_n^0 = 3Y_{n-1},$$
$$Z_n^1 = 2Y_{n-1} + Z_{n-1},$$

and the concatenations

$$Y_n = (Y_{n-1}, Z_{n-1})^T$$
$$Z_n = (Z_n^0, Z_n^1)^T$$

complete the iteration

$$\begin{pmatrix} Y_n \\ Z_n \end{pmatrix} = \begin{pmatrix} Y_{n-1} \\ Z_{n-1} \end{pmatrix},$$
$$\begin{pmatrix} Y_n \\ Z_n \end{pmatrix} = \begin{pmatrix} 3 \cdot 1 \cdot 0 \\ 2 \cdot 1 \cdot 1 \end{pmatrix} \begin{pmatrix} Y_{n-1} \\ Z_{n-1} \end{pmatrix}.$$

The initial vector is given by $\begin{pmatrix} Y_0 \\ Z_0 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$. Eqs. \cite{12, 13} can be collected together to the iteration

$$\begin{pmatrix} Y_n \\ Z_n \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 3 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} Y_{n-1} \\ Z_{n-1} \end{pmatrix}$$

where the dimension of the vectors $Y_n, Z_n$ is $2^n$, growing in the same way as for the Sierpinski iteration (see Fig. 2) $\begin{pmatrix} Z_n \\ Z_n \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \cdot 1 \end{pmatrix} = \begin{pmatrix} Y_{n-1} \\ Z_{n-1} \end{pmatrix}$, or the Thue-Morse iteration $\begin{pmatrix} Z_n \\ Z_n \end{pmatrix} = \begin{pmatrix} 1 \\ (-1) \cdot 1 \end{pmatrix} \begin{pmatrix} Y_{n-1} \\ Z_{n-1} \end{pmatrix}$ (both use \cite{12}}
and start with $Z_0 = (1)$. 

![Diagram](https://example.com/diagram.png)

**FIG. 2:** Illustration of the replication rule for rule 90.

These iterations look formally similar to the Fibonacci or Lucas iteration $(F_n \ F_{n-1})^T = (1 \ 1) \ (F_{n-1} \ F_{n-2})$. Here $(F_1, F_0)^T = (1, 0)^T$ defines the Fibonacci series and $(F_1, F_0)^T = (1, 2)^T$ is the initial condition of the Lucas series. For the latter two, the length of the iterates is not growing.

Eqns. 11 or 14, equivalently, together with the initial condition $(1, 3)^T$ generate $X(t)$ iteratively for all $t$. Formally this iteration is analogous to the Fibonacci iteration, but acts on vectors of growing length within an infinite-dimensional vector space indexed by nonnegative integer values.

This type of series should be distinguished from the $(r$th$)$ hyper-Fibonacci series $12$, where $f_l = 2^{l-1}$ for $l = 1, \ldots, r+1$ and $f_l = f_{l-1} + \cdots + f_{l-r+1}$ for $l > r+1$. On the other hand, the terminus generalized Fibonacci series is widely used for the ordinary Fibonacci or Lucas iteration with two arbitrary start values $f_0$ and $f_1$, where $f_0 = 0$, $f_1 = 1$ defines the Fibonacci series and $f_0 = 2$, $f_1 = 1$ defines the Lucas series; and in fact both can be used as (nonorthogonal) basis vectors of the linear space of generalized Fibonacci series. The $r = 1$ hyper-Fibonacci series corresponds to a generalized Fibonacci series with $f_0 = 1$, $f_1 = 2$. Consequently, an iteration of the algebraic structure of Eq. 14 could be denoted as a generalized hyper-Fibonacci series.

Another observation is the partial self-similarity relation

$$X(\sigma_1 \cdots \sigma_3, 0, \sigma_1, \sigma_0) = X(\sigma_1, \sigma_0) \cdot X(\sigma_1 \cdots \sigma_3) \quad (15)$$

(leading zeroes omitted in notation), i.e. the sequence generated by every second four-block $(\sigma_2 = 0)$ factorizes into the first block $(1, 3, 3, 5)$ and the whole sequence itself. A closed expression for $X(\sigma_4 \cdots \sigma_3, 1, \sigma_1, \sigma_0$) is however not known yet.

The first values of $X(t) = X(t_1 + t_2)$ are listed in Tab. 12 (see Fig. 3).

| $t_2$ | 0 | 16 | 32 | 48 | 64 | 80 | 96 | 112 | 128 | 144 | 160 | 176 | 192 | 208 | 224 | 240 |
|------|---|----|----|----|----|----|----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| $t_1$ | 0 | 1 | 3 | 9 | 15 | 21 | 33 | 55 | 85 | 129 | 194 | 323 | 527 | 850 | 1377 | 2227 |
| 11 | 33 | 55 | 85 | 129 | 194 | 323 | 527 | 850 | 1377 | 2227 |
| 13 | 33 | 55 | 85 | 129 | 194 | 323 | 527 | 850 | 1377 | 2227 |
| 15 | 33 | 55 | 85 | 129 | 194 | 323 | 527 | 850 | 1377 | 2227 |

**TABLE I:** The total activity $X(t)$ for the first 256 time steps.

![Diagram](https://example.com/diagram.png)

**FIG. 3:** Rule 150: Plot of $X(t)$ for the first 256 time steps.

$$\text{len}(a) = \text{len}(b) \quad (16)$$

This is equivalent to Eq. 14.

V. BLOCK-SUMS AND THE FIBONACCI SERIES

Following the same geometrical argument as for the row sums, the sum $S_n = \sum_{i=0}^{2^n-1} X(i)$ is given by the iteration $S_n - S_{n-1} = S_{n-1} + 4S_{n-2}$, or

$$S_n = 2S_{n-1} + 4S_{n-2} \quad (17)$$

and the first elements of the series are listed in Tab. 13.

The matrix of the iteration in time-delayed coordinates

$$
\begin{pmatrix}
S_n \\
S_{n-1}
\end{pmatrix} =
\begin{pmatrix}
2 & 4 \\
1 & 0
\end{pmatrix}
\begin{pmatrix}
S_{n-1} \\
S_{n-2}
\end{pmatrix}
$$

(18)

has the eigenvalues $\lambda_{1,2} = 1 \pm \sqrt{5}$, indicating that it differs from the Fibonacci iteration matrix by an additional expansion factor of 2, i.e. by a suitable transformation

$$
\begin{pmatrix}
S_n \\
2S_{n-1}
\end{pmatrix} =
\begin{pmatrix}
1 & 1 \\
1 & 0
\end{pmatrix}
\begin{pmatrix}
2S_{n-1} \\
2^2S_{n-2}
\end{pmatrix}
$$

(19)
TABLE II: The block sums $S_n$ for the first 18 time steps.

\[
\begin{array}{cccccc}
 n & S_n & n & S_n & n & S_n \\
 0 & 1 & 6 & 1344 & 12 & 1544192 \\
 1 & 4 & 7 & 4352 & 13 & 4997120 \\
 2 & 12 & 8 & 14080 & 14 & 16171008 \\
 3 & 40 & 9 & 45568 & 15 & 52330496 \\
 4 & 128 & 10 & 147456 & 16 & 160835024 \\
 5 & 416 & 11 & 477184 & 17 & 548012032 \\
\end{array}
\]

Thus, if the time-doubling iteration is interpreted as generation rule of the resulting self-similar fractal (rescaled to the unit interval), its Hausdorff-Besicovic dimension is given by $(1 + \sqrt{5})/2$.

VI. CONCLUSIONS

The self-similarity structure of the rule 150 elementary cellular automaton generated space-time fractal is qualitatively different from the Sierpinski triangle generated by rule 90. While the iteration itself generalizes the concept of a Fibonacci iteration to vectors of growing dimension, the blockwise sum exactly is given by the Fibonacci series multiplied by a scaling factor $2^n$. The iteration rule for the total activity derived here allows to compute the total activity without simulating the spatial dynamics, thus considerably eases the numerical computation.

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