Exact solutions for shells collapsing towards a pre-existing black hole

Yuan Liu\textsuperscript{a}, Shuang Nan Zhang\textsuperscript{a,b,c}

\textsuperscript{a}Physics Department and Center for Astrophysics, Tsinghua University, Beijing, 100084, China
\textsuperscript{b}Key Laboratory of Particle Astrophysics, Institute of High Energy Physics, Chinese Academy of Sciences, Beijing, China
\textsuperscript{c}Physics Department, University of Alabama in Huntsville, Huntsville, AL35899, USA

Abstract

The gravitational collapse of a star is an important issue both for general relativity and astrophysics, which is related to the well known “frozen star” paradox. This paradox has been discussed intensively and seems to have been solved in the comoving-like coordinates. However, to a real astrophysical observer within a finite time, this problem should be discussed in the point of view of the distant rest-observer, which is the main purpose of this paper. Following the seminal work of Oppenheimer and Snyder (1939), we present the exact solution for one or two dust shells collapsing towards a pre-existing black hole. We find that the metric of the inner region of the shell is time-dependent and the clock inside the shell becomes slower as the shell collapses towards the pre-existing black hole. This means the inner region of the shell is influenced by the property of the shell, which is contrary to the result in Newtonian theory. It does not contradict the Birkhoff’s theorem, since in our case we cannot arbitrarily select the clock inside the shell in order to ensure the continuity of the metric. This result in principle may be tested experimentally if a beam of light travels across the shell, which will take a longer time than without the shell. It can be considered as the generalized Shapiro effect, because this effect is due to the mass outside, but not inside as the case of the standard Shapiro effect. We also found that in real
astrophysical settings matter can indeed cross a black hole’s horizon according to the clock of an external observer and will not accumulate around the event horizon of a black hole, i.e., no “frozen star” is formed for an external observer as matter falls towards a black hole. Therefore, we predict that only gravitational wave radiation can be produced in the final stage of the merging process of two coalescing black holes. Our results also indicate that for the clock of an external observer, matter, after crossing the event horizon, will never arrive at the “singularity” (i.e. the exact center of the black hole), i.e., for all black holes with finite lifetimes their masses are distributed within their event horizons, rather than concentrated at their centers. We also present a worked-out example of the Hawking’s area theorem.

Key words: Black Holes, Classical Theories of Gravity, Spacetime Singularities

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1. Introduction

The “frozen star” is a well known novel phenomenon predicted by general relativity, i.e. a distant observer ($O$) sees a test particle falling towards a black hole moving slower and slower, becoming darker and darker, and is eventually frozen near the event horizon of the black hole. This process was vividly described and presented in many popular science writings [1, 2, 3] and textbooks [4, 5, 6, 7, 8, 9]. The time measured by an in-falling (with the test particle) observer ($O'$) is finite when the test particle reaches the event horizon. However, for $O$ it takes infinite time for the test particle to reach exactly the location of the event horizon. In other words, $O$ will never see the test particle falling into the event horizon of the black hole. It is therefore legitimate to ask the question whether in real astrophysical settings black holes can ever been formed and grow with time, since all real astrophysical black holes can only be formed and grown by matter collapsing into a “singularity”.

Two possible answers have been proposed so far. The first one is that since $O'$ indeed has observed the test particle falling through the event horizon, then in reality (for $O'$) matter indeed has fallen into the black hole. It seems that this paradox could be solved in the point of view of $O'$ or other comoving-like coordinates. However, since $O$ has no way to communicate with $O'$ once $O'$ crosses the event horizon, $O$ has no way to ‘know’ if the test particle
has fallen into the black hole. More importantly, in a real astrophysical observation within a finite time, it is more meaningful to discuss this problem in the coordinate of $O$, which is the main purpose of this paper. The second answer is to invoke quantum effects. It has been argued that quantum effects may eventually bring the matter into the black hole, as seen by $O$[10]. However, as pointed out by [11], even in that case the black hole will still take an infinite time to form and the pre-Hawking radiation will be generated by the accumulated matter just outside the event horizon. (However, as we shall show in this work, the in-falling matter will not accumulate around the event horizon.) It has also been realized that, even if matter did accumulate just outside the event horizon, the time scale involved for the pre-Hawking radiation to consume all accumulated matter (and thus make the black hole “black” again) is far beyond the Hubble time [12]. Thus this does not answer the question in the real world. Apparently $O$ cannot be satisfied with either answer.

In desperation, $O$ may take the attitude of “who cares?” When the test particle is sufficiently close to the event horizon, the redshift is so large that practically no signals from the test particle can be seen by $O$ and apparently the test particle has no way of turning back, therefore the “frozen star” does appear “black” and is an infinitely deep “hole”. For practical purposes $O$ may still call it a “black hole”, whose total mass is also increased by the in-falling matter. Apparently this is the view taken by most people in the astrophysical community and general public, as demonstrated in many well known textbooks $[4, 5, 6, 7, 8, 9, 13]$ and popular science writings $[1, 2, 3]$. However when two such “frozen stars” merge together, strong electromagnetic radiations will be released, in sharp contrast to the merging of two genuine black holes (i.e., Schwarzschild singularities); the latter can only produce gravitational wave radiation $[12]$. Thus this also does not answer the question in the real world. As we shall show in this work this view should be changed, because in real astrophysical settings $O$ (in the Schwarzschild coordinates) does observe matter falling into the black hole.

This problem is closely linked to the problem of collapse. Great interests have been paid on the problem of gravitational collapse. Since the seminal work of Oppenheimer and Snyder (1939) $[14]$, numerous works have been done on this issue. For example, the result of $[14]$ has been extended to more realistic and complicated cases, e.g. considering pressure, inhomogeneous, the effect of radiation and rotation, and so on (e.g. $[15, 16, 17, 18, 19]$). There are also a lot of works concerning collapsing shells. The paper of Isarel
found the equation of motion for one thin (without thickness) shell in the vacuum. Khakshournia and Mansouri \cite{21} obtained the equation of motion for one shell in vacuum with thickness up to the first order. Gonçalves \cite{22} discussed the solution for two thin shells in vacuum. However, in many real astrophysical settings, the matter shell collapsing towards a black hole should have a finite thickness and is much less massive than the black hole. In this paper we investigate such more realistic cases, and in particular to find the differences between these solutions and that in \cite{14}, and the implication for the “frozen star” paradox. In Sec. \ref{sec:2} we present the dynamical solutions for one and two shells, in which the shell is spherically symmetrical and pressureless. We also show the evolution of event horizon (the boundary of the region from where the photon cannot escape to space-like infinity) and apparent horizon (the boundary of the trapped region, see the detail of the definition of event and apparent horizon in \cite{13} and \cite{23}) in the one shell case. In Sec. \ref{sec:3} we discuss the implications of these solutions and make our conclusions. We adopt $G = c = 1$ throughout this paper.

2. The dynamical solution

2.1. The solution for one shell

Fig. \ref{fig:1} is the configuration of the one shell case. To investigate the dynamical solution, we follow the method in \cite{14}, i.e. obtaining the solution

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{The configuration of the dynamical case for one shell in the comoving coordinates. $m$ and $m_s$ are the total gravitating masses of the black hole and the shell, respectively. $a'$ and $a$ are the radii of the inner and outer boundaries of the shell in the comoving coordinates, respectively.}
\end{figure}
in the comoving coordinates first. The general form of the metric is

\[ ds^2 = d\tau^2 - e^{\bar{\omega}} dR^2 - e^{\omega} (d\theta^2 + \sin^2 \theta d\phi^2), \]

where \( \bar{\omega}(R, \tau) \) and \( \omega(R, \tau) \) are the unknown functions. In the comoving coordinates, the only non-zero component of the energy momentum tensor is \( T^4_{4} = \rho \). As the results in [14], we can obtain the following equations from the field equation

\[ e^{\omega} = (F \tau + G)^{4/3}, \tag{2} \]
\[ e^{\bar{\omega}} = e^{\omega} \omega'^2 / 4, \tag{3} \]
\[ 8\pi \rho = \frac{4}{3} (\tau + G/F)^{-1} (\tau + G'/F')^{-1}, \tag{4} \]

where \( F \) and \( G \) are arbitrary functions of \( R \). Dot and prime are the partial differentiation with respect to \( \tau \) and \( R \), respectively. At \( \tau = 0 \), we obtain

\[ F = -\sqrt{6\pi \rho_0 (G^2 + C_1)}. \tag{5} \]

As in [14], we choose \( G = R^{3/2} \).

In region I and III, \( \rho_0 = 0 \), therefore, \( F = \text{constant} \). We will determine this constant and \( C_1 \) by the matching condition in the next paragraph. As obtained in [14], in region III \( F_{\text{III}} = -\frac{3}{2} \sqrt{r_0} \), where \( r_0 = 2M \). According to the well-known Birkhoff’s theorem for spherically symmetric mass distribution, \( M \) here is the total gravitating mass of the system, i.e., including rest mass and gravitational energy.

Throughout this paper, our matching conditions are chosen such that the metric is continuous across the boundaries between the different regions. In Sec. 3 we will verify that in our solutions found this way the extrinsic curvature is also continuous at the boundaries. Since \( F_{\text{III}} = -\frac{3}{2} \sqrt{r_0} \), to assure the metric is continuous at the boundary between region III and II, i.e. \( F_{\text{II}}(R = a) = F_{\text{III}}(R = a) \), according to Eq. (5), we obtain the value of \( C_1 \) as \( C_1 = \frac{3}{4\pi \rho_0} M - a^3 \). Therefore, to assure the metric is continuous at the boundary between region II and I, i.e. \( F_{\text{II}}(R = a') = F_{\text{I}}(R = a') \), the value of \( F \) in region I should be \( F_{\text{I}} = -\frac{3}{2} \sqrt{2[M - \frac{4}{3} \pi \rho_0 (a^3 - a'^3)]} \equiv -\frac{3}{2} \sqrt{2m} \equiv -\frac{3}{2} \sqrt{r_0} \) (we also denote \( m_s \equiv \frac{4}{3} \pi \rho_0 (a^3 - a'^3) \)). In Sec. 2.1.2 we will show that \( m \) and \( m_s \) are the total gravitating masses of the central black hole and the shell, respectively.

For the spherically symmetric ball case (just as the solution in [14]), \( C_1 \) must be 0. However, in the case that a shell collapses towards a pre-existing
black hole, $C_1$ can be any value. For example, $C_1 < 0$ means that the mass of the central black hole is smaller than the mass of the inner region interior to the shell in the spherically symmetric ball case. Thus, the gravitational force on the inner layers of the shell will become so weak that the inner layer will take more time than the outer layers (it can be seen from Eq. 7) to arrive at $r = 0$ [17]. Actually, it means the crossing of the layers will happen. We will not discuss the crossing case in this paper, since no comoving coordinates exist in that case. This means that we will only study the cases when $C_1 \geq 0$. Therefore, our result cannot be compared directly with the result obtained for a collapsing shell (without a black hole in the center, i.e., $C_1 < 0$) either without any thickness [20] or only with first-order thickness [21].

Up to now we have obtained the solution in the comoving coordinates. Then we try to transform the solution into the ordinary coordinates, in which the metric has the form as

$$ds^2 = B(r, t)dt^2 - A(r, t)dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2).$$

(6)

The transformation of $r$ is obvious. We must choose

$$r = e^{\omega/2} = (F\tau + G)^{2/3}.$$  

(7)

Using the contravariant form of the metric and requiring that the $g^{tr}$ term vanishes, we have

$$t'/\dot{t} = \dot{r}r'.$$  

(8)

Substituting the expressions of $r$ in three regions into Eq. (8), we can obtain three partial differential equations of $t(R, \tau)$.

In region III, the solution of Eq. (8) is

$$t = L(x), \quad x = \frac{2}{3\sqrt{r_0}}(R^{3/2} - r^{3/2}) - 2\sqrt{r_0} + r_0 \ln \frac{\sqrt{r} + \sqrt{r_0}}{\sqrt{r} - \sqrt{r_0}}.$$  

(9)

In region II, we note the solution of

$$t'/\dot{t} = -\frac{2}{3}\sqrt{R} \left[ R^{3/2} - \tau \sqrt{6\pi \rho_0 (R^3 + C_1^1)} \right]^{-\frac{2}{3}} \left[ \sqrt{6\pi \rho_0 (R^3 + C_1)} - 6\pi \rho_0 R^{3/2} \right]$$

as $t = M(y)$, where $y$ is a function of $R$ and $\tau$.

The characteristic equation of Eq. (10) is
\[
\frac{d\tau}{dR} = \frac{2}{3}\sqrt{R} \left[R^{3/2} - \tau \sqrt{6\pi \rho_0 (R^3 + C)}\right]^{-\frac{2}{3}} \left[\sqrt{6\pi \rho_0 (R^3 + C)} - 6\pi \rho_0 R^{3/2}\tau}\right].
\]  
(11)

In region I, the solution of Eq. (8) is
\[
t = N(x'), \quad x' = \frac{2}{3\sqrt{r_0}}(R^{3/2} - r^{3/2}) - 2\sqrt{rr_0'} + r_0' \ln \frac{\sqrt{r} + \sqrt{r_0}}{\sqrt{r} - \sqrt{r_0}}
\]  
(12)

where \(L, M, N\) are arbitrary functions of their arguments. Since the metric in the ordinary coordinates in region III is the Schwarzschild metric, \(L(x)\) should be \(x\).

2.1.1. Analytic solution when \(C_1 = 0\)

Eq. (10) can be solved analytically only when \(C_1 = 0\) and the results are
\[
t = M(y), \quad y = \frac{1}{2}(R^2 - a^2) + \frac{1}{k}(1 - \frac{3}{2}\sqrt{k\tau})^{\frac{3}{2}},
\]  
(13)

where \(k = \frac{8\pi \rho_0}{3}\).

Then if we require \(L = M\) at \(R = a\) for any \(\tau\), the form of \(M\) must be
\[
M(y) = \frac{2}{3\sqrt{r_0}}[a^{\frac{3}{2}} - (ka)^{3/2}] - 2\sqrt{ka}r_0 + r_0' \ln \frac{\sqrt{ka} + \sqrt{r_0}}{\sqrt{ka} - \sqrt{r_0}},
\]  
(14)

where \(y = f(x')\) is determined by the following equation,
\[
x' = \frac{2}{3\sqrt{r_0}}[a^{\frac{3}{2}} - (ka')^{3/2}(y - \frac{1}{2}(a^2 + a'^2)^{3/2}) - 2\sqrt{r_0'}ka' [y - \frac{1}{2}(a'^2 - a^2)]
\]
\[
+ r_0' \ln \frac{\sqrt{ka'} [y - \frac{1}{2}(a'^2 - a^2)] + \sqrt{r_0}}{\sqrt{ka'} [y - \frac{1}{2}(a'^2 - a^2)] - \sqrt{r_0}}
\]  
(15)

Using the relations
\[
g^{rr} = -(1 - r^2),
\]  
(16)

and
\[
g^{tt} = t^2(1 - t^2),
\]  
(17)
we can obtain the metric in the ordinary coordinates

In region III,
\[ g_{tt} = 1 - \frac{r_0}{r}, \]
\[ g_{rr} = -(1 - \frac{r_0}{r})^{-1}. \]  
\[(18)\]  
\[(19)\]

In region II,
\[ g_{rr} = -(1 - \frac{kR^3}{r})^{-1}. \]  
\[(20)\]

Since
\[ \dot{t} = \frac{a^{5/2}k^2y^2}{\sqrt{r_0(aky - r_0)(1 - \frac{3}{2}\sqrt{k\tau})^{3/2}}}, \]
\[ (21)\]

therefore,
\[ g_{tt} = \frac{r_0(aky - r_0)^2(1 - \frac{3}{2}\sqrt{k\tau})^{3/2}}{a^5k^4y^3}(1 - \frac{kR^3}{r})^{-1}, \]
\[ (22)\]

where \( k \equiv \frac{8}{3}\pi\rho_0 \).

In region I, in the ordinary coordinates, we can obtain
\[ g_{tt} = h(t)(1 + C_4/r), \]  
\[ (23)\]
\[ g_{rr} = -(1 + C_4/r)^{-1}. \]  
\[ (24)\]

If we require \( g_{rr} \) and \( g_{tt} \) are continuous at the inner boundary of the shell with Eq. (20) and (22), respectively, we have
\[ C_4 = -r'_0 \]  
\[ (25)\]

and
\[ h(t) = \frac{(1 - \frac{3}{2}\sqrt{k\tau})^2}{[(1 - \frac{3}{2}\sqrt{k\tau})^2 - r'_0/a']^2} \left\{ \frac{1}{2}(a'^2 - a_0^2) + \frac{1}{2k}(1 - \frac{3}{2}\sqrt{k\tau})^2 - a_0^2 \right\}^2 \left\{ k[(\frac{1}{2}a'^2 - a_0^2) + \frac{1}{2k}(1 - \frac{3}{2}\sqrt{k\tau})^2]^3 \right\}. \]
\[ (26)\]

where \( \tau = \tau(t) \) is determined by the following equation
\[ t = \frac{2}{3\sqrt{r_0}}\left[a_0^2 - (kay(\tau))^{3/2}\right] - 2\sqrt{kay(\tau)r_0} + r_0\ln\frac{\sqrt{kay(\tau)} + \sqrt{r_0}}{\sqrt{kay(\tau)} - \sqrt{r_0}}. \]
\[ (27)\]
2.1.2. Numerical solution when $C_1 > 0$

In general cases, Eq. (10) can be solved numerically only. From Eq. (9), the value of $t$ at the outer boundary of the shell can be determined. It can be easily shown that the value of $t$ along every curve determined by Eq. (11), i.e. $\tau = \tau(R)$, is constant. Therefore, using the value of $t$ from Eq. (9) as the boundary value, we integrate Eq. (11) along the characteristic lines and obtain how the shell evolves with coordinate time $t$. In Fig. 2, we show the evolution curves for the cases $C_1 = 0$ and $C_1 > 0$. As it can be seen, the inner boundary of the shell crosses the Schwarzschild radius with finite time and approaches to a position $r^* = \frac{3}{2} \frac{r_0^2}{r_0^4} - \frac{1}{2} r_0'$ (This is obtained from the transformations Eq. [7] and Eq. [14]), whereas the outer boundary approaches to the Schwarzschild radius. (However, if there is another shell outside this shell, the outer boundary of this shell will also cross the Schwarzschild radius with finite time, as we will show in Sec. 2.2)

As shown in Fig. 2 the event horizon of the system increases just from the location $r = 2m \equiv r_0'$. Therefore, $r = r_0'$ corresponds to the event horizon of the central black hole and $m$ is the total gravitating mass of the central black hole. As a result, $m_s$ can be recognized as the total gravitating mass of the shell.

We find that the metric in region I is not the standard Schwarzschild metric; actually, it should be a function of time in the dynamical case. The evolution curve of $h(t)$ is shown in Fig. 3. When the shell is far from the central black hole, the value of $h(t)$ is almost equal to 1, i.e. the metric is approximately the standard Schwarzschild solution. When the shell is near the horizon, the value of $h(t)$ decreases rapidly and approaches to 0, i.e. the clock in region I becomes slower and slower. It does not contradict the Birkhoff’s theorem, since in the proof of the Birkhoff’s theorem the region outside the central mass is vacuum [4]. Therefore, in this case we could adjust the clock to ensure that metric asymptotically approaches to the Minkowski metric at infinity. However, in our problem there is a shell, but not the vacuum outside the region I. Thus we should ensure that the metric is continuous at the boundary between region I and the shell and therefore cannot arbitrarily select the clock inside the shell, i.e. region I.

2.2. The solution for two shells

In this section, we investigate the solution for two shells. The configuration of this problem is shown in Fig. 4. The process to obtain the solution
The event horizon
The apparent horizon

Figure 2: The dynamical solutions for one shell case. (a) and (b) are evolution curves for $a = 5r_0$, $a' = 2.5r_0$, and $r'_0 = 1/8r_0$ (i.e. $C_1 = 0$) with comoving time and coordinate time, respectively. The evolution of the event and apparent horizons are also shown. (c) and (d) for $a = 5r_0$, $a' = 2.5r_0$, and $r'_0 = 0.5r_0$ (i.e. $C_1 > 0$) with comoving time and coordinate time, respectively.
Figure 3: The evolution of $h(t)$ with the position of the outer boundary of the shell for the case $a = 5r_0$, $a' = 2.5r_0$, and $r'_0 = 1/8r_0$.

Figure 4: The configuration of the dynamical case for two shells in comoving coordinates. $m$, $m_1$ and $m_2$ are the total gravitating masses of the black hole, the shell 1 and 2, respectively. $a'_1$ and $a_1$ are the radii of the inner and outer boundaries of shell 1, respectively. $a'_2$ and $a_2$ are the radii of the inner and outer boundaries of shell 2, respectively.
is similar to the one shell case, however, there are five regions. The result of $F$ is

$$
F = \begin{cases} 
\frac{3}{2} \sqrt{r_0} : R < a_1' \\
\frac{3}{2} \sqrt{\frac{8\pi \rho_1}{3}(R^3 - a_1^3)} + r_0'' : a_1' < R < a_1 \\
\frac{3}{2} \sqrt{r_0''} : a_1 < R < a_2' \\
\frac{3}{2} \sqrt{\frac{8\pi \rho_2}{3}(R^3 - a_2^3)} + r_0 : a_2' < R < a_2 \\
\frac{3}{2} \sqrt{r_0} : R > a_2
\end{cases},
$$

(28)

where $r_0 = 2M$, $r_0' = 2m$, $r_0'' = 2(m + m_1)$.

The form of the transformation in region V is the same as that in region III of the one shell case. The transformations for other regions are shown below.

In region IV,

$$
t'/t = -\frac{2}{3} \sqrt{R} \left[R^{3/2} - \tau \sqrt{6\pi \rho_2(R^3 + C_2)}\right]^{-\frac{2}{3}} \left[\sqrt{6\pi \rho_2(R^3 + C_2)} - 6\pi \rho_2 R^{3/2} \tau\right].
$$

(29)

The characteristic equation of Eq. (29) is

$$
\frac{d\tau}{dR} = \frac{2}{3} \sqrt{R} \left[R^{3/2} - \tau \sqrt{6\pi \rho_2(R^3 + C_2)}\right]^{-\frac{2}{3}} \left[\sqrt{6\pi \rho_2(R^3 + C_2)} - 6\pi \rho_2 R^{3/2} \tau\right],
$$

(30)

where $C_2 = \frac{3}{4\pi \rho_2}M - a_2^3$.

In region III,

$$
t = O(x''), \quad x'' = \frac{2}{3} \sqrt{r_0''} \left[R^{3/2} - r^{3/2}\right] - 2\sqrt{r_0''} + r_0'' \ln \frac{\sqrt{r} + \sqrt{r_0''}}{\sqrt{r} - \sqrt{r_0''}},
$$

(31)

where $O(x'')$ is an arbitrary function of $x''$.

In region II,

$$
t'/t = -\frac{2}{3} \sqrt{R} \left[R^{3/2} - \tau \sqrt{6\pi \rho_1(R^3 + C_1)}\right]^{-\frac{2}{3}} \left[\sqrt{6\pi \rho_1(R^3 + C_1)} - 6\pi \rho_1 R^{3/2} \tau\right].
$$

(32)

The characteristic equation of Eq. (32) is

$$
\frac{d\tau}{dR} = \frac{2}{3} \sqrt{R} \left[R^{3/2} - \tau \sqrt{6\pi \rho_1(R^3 + C_1)}\right]^{-\frac{2}{3}} \left[\sqrt{6\pi \rho_1(R^3 + C_1)} - 6\pi \rho_1 R^{3/2} \tau\right],
$$

(33)

where $C_1 = \frac{3}{4\pi \rho_1}(m + m_1) - a_1^3$. 

12
In region I,

\[ t = P(x'), \quad x' = \frac{2}{3\sqrt{r_0'}} (R^{3/2} - r^{3/2}) - 2\sqrt{rr_0'} + r_0' \ln \frac{\sqrt{r} + \sqrt{r_0'}}{\sqrt{r} - \sqrt{r_0'}}, \]  

(34)

where \( P(x') \) is an arbitrary function of \( x' \).

To obtain the evolution curves of the shells, we solve the above equations numerically with the same method in Sec. 2.1.2. Fig. 5 shows the evolution curves of two shells. The outer boundary of shell 1 crosses the Schwarzschild radius corresponding to the total gravitating mass within finite coordinate time. In Fig. 6, we show the evolution curves of the outer boundary of shell 1 in the case with or without shell 2. As we have pointed out in Sec. 2.1, the clock inside the shell is slower, therefore the outer boundary of shell 1 will take more time to approach the asymptotic line in the case with shell 2.

3. Discussions and Conclusions

In this work, our matching conditions selection requires that the metric is continuous at the boundaries. Here we verify that in our solutions found this way the extrinsic curvature is also continuous at each boundary (see the detailed discussion about the extrinsic curvature in [23]). Since the boundary is defined as \( R = \text{constant} \), the normalized normal vector is
Figure 6: The comparison of the evolution of the outer boundary of shell 1 between the case with (solid) or without (dashed) shell 2. The parameters of the shells are $a_2 = 10r_0$, $a'_2 = 6r_0$, $a'_1 = 2.5r_0$, $a_1 = 5r_0$, $r'_0 = 1/5r_0$, and $r''_0 = 2/5r_0$, where $r_0$ is the Schwarzschild radius corresponding to the total gravitating mass of the system, i.e., the sum of the gravitating masses of the pre-existing black hole, shell 1 and shell 2. The inset is the ratio of the solid line to the dashed line. It is seen clearly that outside matter, though spherically symmetric, does influence the motion of matter inside.

The definition of extrinsic curvature is $K_{ab} = n_{\alpha} e^a_{\alpha} e^b_{\beta}$. We choose $y^a = (\tau, \theta, \varphi)$ as the coordinates on the boundary. Therefore, the non-vanishing components of $K_{ab}$ are

$$n_\alpha = (0, \frac{1}{|g_{RR|^{1/2}}}, 0, 0).$$

(35)

$$K_{\tau \tau} = n_{\tau;\tau} = -\Gamma^R_{\tau \tau} n_R = \frac{1}{2} n_R g^{RR} g_{\tau \tau, R} = 0,$$

(36)

$$K_{\theta \theta} = n_{\theta;\theta} = -\Gamma^R_{\theta \theta} n_R = \frac{1}{2} g^{RR} g_{\theta \theta, R} n_R = (F + G)^{2/3},$$

(37)

$$K_{\varphi \varphi} = n_{\varphi;\varphi} = -\Gamma^R_{\varphi \varphi} n_R = \frac{1}{2} g^{RR} g_{\varphi \varphi, R} n_R = (F + G)^{2/3} \sin^2 \theta.$$ 

(38)

Obviously, they are all continuous at all boundaries.

With the dynamical solutions presented in this work, we conclude that
(a) In general relativity, the outer mass influences the inner space-time even if the mass distribution is spherically symmetric;

(b) The metric inside the shell can not be the standard Schwarzschild form and is time dependent in the dynamical case;

(c) A dust shell of finite thickness can indeed cross the black hole’s event horizon, as observed by an external observer, except for its outer surface. This means that asymptotically only an infinitesimal amount of matter remains just outside the event horizon. Therefore such a “star” is qualitatively different from the so-called the “frozen star”, which supposedly holds all in-falling matter outside its event horizon;

(d) In the two-shell case, even the outer boundary of the inner shell can be observed by an external observer to collapse into the Schwarzschild radius, thus no matter can be accumulated outside a black hole’s event horizon. Since we can mimic the outer shell as all matter between the observer and the in-falling shell being observed, we can conclude that the outside observer indeed “sees” the inner shell falls into the black hole completely. We note that the outer layer of matter does not necessarily block the view of the outside observer to the inner layer, since the outer layer can be optically thin to at least a certain wavelength of electromagnetic radiation, or is made of dark matter and is thus completely transparent to electromagnetic radiation. Even if the outer layer is optically thick completely, such as the case when the fall-back matter approaches to the black hole after the core collapse of a massive star, neutrinos can still find their way out and photons may eventually escape after a lengthy radiative transfer process.

In our solution, the metric between the shell and the black hole should not be the standard Schwarzschild form, and there should be an additional time-dependent factor $h(t)$ in $g_{tt}$ (please refer to equations Eq. 23 and 26), Fig. 3. This factor $h(t)$ decreases from 1 to 0 when the shell collapses towards the Schwarzschild radius. This means that the clock in the inner region of the shell is slower compared with the case without the shell, and becoming even slower when the shell is falling in. This effect may be testable experimentally in principle, e.g., if a beam of light travels across the shell and a parallel beam of light passes outside the shell (far enough from the shell), then the two beams of light will travel at different velocities with respect to the observers outside the shell. This previously unrecognized effect is caused by the outer mass’s influence to the inner space-time; such effect does not exist in the Newtonian Gravity. We could recognize it as the generalized Shapiro effect, because this effect is due to the mass outside, but not inside
as the case of the standard Shapiro effect. We comment in passing that in [22], the author directly identified the metric between two thin shells as Schwarzschild metric and therefore his result can recover to our one shell solution only when the mass of the outer shell vanishes.

In the one shell case, the shell could almost cross the Schwarzschild radius except the outer boundary; such a star is already qualitatively different from a “frozen star”. Nevertheless, a shell can indeed cross the Schwarzschild radius in the two shell case, and the external observer can indeed observe a complete shell falling into a black hole, since the outer shell in general cannot block the view to the inner shell completely, as discussed above. In this sense we could observe the matter falls into a black hole and the “frozen star” paradox discussed in [10] is naturally solved. As pointed out previously by us, the origin of the “frozen star” is the “test particle” assumption for the in-falling matter, which neglects the influence to the metric by the in-falling matter itself [24]. In fact, for some practical astrophysical settings, the time taken for matter falling into a black hole is quite short, even for the external observer [24]. Therefore with full and self-consistent general relativity calculation, we find that the in-falling matter will not accumulate outside the event horizon, and thus the quantum radiation and Gamma Ray bursts predicted in [11] and [12] are not likely to be generated. We predict that only gravitational wave radiation can be produced in the final stage of the merging process of two coalescing black holes. Future simultaneous observations by X-ray telescopes and gravitational wave telescopes shall be able to verify our prediction.

It is also interesting to note that, as can be seen from Fig. 2b, 2d and 5b, in ordinary coordinates, the matter will not collapse to the singularity \( r = 0 \) even with infinite coordinate time (if \( r_0' = 0 \), the inner boundary will take infinite time to arrive at \( r = 0 \)). It means that in real astrophysics sense, matter can never arrive at the singularity (i.e., the exact center of the black hole) with respect to the clock of the external observer. Therefore, no gravitational singularity exists physically, even within the framework of the classical general relativity.

As predicted by Hawking’s area theorem, the event horizon grows monotonously and continuously from \( r_0' \) to \( r_0 \) when the shell collapses towards the black hole, whereas the apparent horizon jumps to the Schwarzschild radius when all the mass of the shell crosses into the Schwarzschild radius (see Fig. 2a and 4b). Therefore, we have presented a worked-out example of the area theorem.
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