The causality-composition law for the non-Debye relaxations

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Abstract.

The causality-composition law is studied for the non-Debye relaxations, in particular for the stretched exponential, i.e. the Kohlrausch-Williams-Watts (KWW) function and the Cole-Cole model given by the Mittag-Leffler function with parameter \( \alpha \) such that \( 0 < \alpha < 1 \). Studying these two cases we show that the causality-composition law leads to the integral version of the evolution equations which are equivalent to the fractional differential equations used to describe just mentioned class of the relaxation phenomena.

1. Introduction

The most intuitive explanation of the notion of causality is that any result cannot precede its cause. This commonly used and quite sufficient for purposes of daily life statement has to be given much more precise meaning if one investigates the time evolution of a physical system and needs to take into account that the evolution is some time ordered series of events of which each may be identified as resulting from previous history of the system and simultaneously is the cause of events occurring later in time. In the language of the time evolution equations such a rule is usually formulated in terms of differential equations and means that we deal with the Cauchy problem, i.e. the mathematical task how to determine the state of the system at an instant of time \( t \) provided the knowledge of the state of the same system at an instant of time \( t_0 \) preceding \( t \). The choice of \( t_0 \) is arbitrary with the only restriction that it is causally related to \( t \). Obviously, solutions to the Cauchy problem depend on the properties of evolution equations used to describe the system under consideration but for any model which respects the principles of classical physics they have to share a composition rule which merges the evolution in subsequent time intervals. Namely, if the system evolves from \( t_0 \) to \( t \) such that \( t_0 < t \) and if we choose an intermediate instant of time \( t_{\text{int}} \), \( t_0 < t_{\text{int}} < t \), then the composition of evolution in the intervals \((t_0, t_{\text{int}})\) and \((t_{\text{int}}, t)\) must give the same result as the evolution in the interval \((t_0, t)\). For the dielectric relaxation it can be written as

\[
\frac{n(t - t_0)}{n(t_0)} = \frac{n(t - t_{\text{int}})}{n(t_{\text{int}} - t_0)} \odot \frac{n(t_{\text{int}} - t_0)}{n(t_0)},
\]

where \( n(t)/n(t_0) \) is the relaxation function and ‘\( \odot \)’ denotes some composition rule to be determined. Eq. (1) means that the state of the system at the moment \( t_{\text{int}} \), if taken as
initial condition for further going evolution, yields, independently from the choice \( t_{\text{int}} \) in the interval \( (t_0, t) \), the same result at the time \( t \). This property seems trivial if one restricts analysis to the Debye relaxation governed by the simple exponential law arising from standard differential equations where \( '\circ' \) is the ordinary multiplication. But for non-Debye relaxations phenomena, in particular those involving memory effects (which appears to be a typical case for dielectric relaxation and/or viscoelasticity) the situation becomes much more complicated - neither the composition rule is given by multiplication nor the evolution is described by standard differential equations. The composition rule which we shall call causality-composition law is directly related to the shape of the evolution equations. Investigations of existing models of relaxation phenomena strongly suggest that suitable evolution equations involve integro-differential operators (named fractional derivatives) and as such are called fractional differential equations.

In the paper we will concentrate ourselves on two kinds of non-Debye relaxation described by the Kohlrausch-Williams-Watts function (named also the KWW function and/or the stretched exponential) and the Cole-Cole relaxation. The general aim of the paper is to apply the causality-composition law for these two types of relaxation and derive the appropriated evolution equations in the integral and fractional differential versions.

The paper is organized as follows. Sec. 1 has an introductory character and contains the explanation of the causality-composition law. In Sec. 2 we briefly recall the basic facts about the relaxation theory putting the attention on the Cole-Cole and stretched exponential models. We propose that in the complex system the relaxation functions can be uniquely expressed as weighted sums of the Debye (exponential) relaxations, which, for both models under consideration, is guaranteed by the Bernstein theorem of the theory of completely monotone functions. The Cole-Cole relaxation is given by the Mittag-Leffler function whose properties are recalled in Sec. 3. In Secs. 4 and 5 we show that the causality-composition law is satisfied by the stretched exponential and the Cole-Cole models, respectively. The paper is concluded in Sec. 6.

2. The relaxation phenomena

In physics, as well as in other natural sciences, relaxation phenomena are usually understood as processes describing the return of a system perturbed by the action of an external field to the equilibrium. Switching off the external filed is immediate whereas the system needs some time to relax. This time is called the relaxation time \( \tau \). If all elements of the system relax with the same relaxation time we have the simplest situation described by the function \( \exp(-t/\tau) \) i.e. the Debye or the exponential law. However, relaxations observed in complex systems rarely follow this law. Observed relaxation patterns are usually effects of cumulation of numerous processes, of which each is characterized by its own relaxation time. These processes influence each other and effectively lead to the quite complicated relaxation behaviour which reflects so-called "memory effects". As an illustrative example we can consider the relaxation (depolarisation, or discharging) process of a capacitor whose plates are separated by a dielectric medium. The external electric field \( E \) acting on the capacitor induces charges on its plates and the field inside the capacitor, namely the electric induction \( D \), differs from the external field by some quantity (polarization) \( P = D - E \) which depends on the properties of material filling the capacitor. If the external field \( E \) does not depend on time, i.e. in the static case, the relation between \( E \) and \( D \), named the constitutive relation, is simple proportionality \( D = \varepsilon_0 \varepsilon_r E \) where the vacuum permittivity (vacuum dielectric constant) \( \varepsilon_0 \) denotes an universal quantity and the relative permittivity \( \varepsilon_r \) characterizes the medium.

This picture changes if the external electric field begins to depend on time - polarization effects need time to follow its changes which leads to the constitutive relations traditionally
expressed by the formula $[1, 2, 3]$

$$D(t) = \varepsilon_\infty E(t) + \int_{t_0}^{t} \kappa(t - t') E(t') \, dt'.$$

(2)

In the above the dielectric constant of the induced polarisation $\varepsilon_\infty$ is positive constant defined below Eq. (3) while the integral kernel $\kappa(t-t')$ contains the information on depolarization effects characteristic for the specific material, i.e. informs how the "memory effects" influence processes occurring in the sample. The time behaviour of these phenomena is usually described by two functions, namely the response function $\phi(t)$ and the relaxation function $n(t)/n_0$, closely related to the previous one. In dielectric materials the response function is depolarization current and informs us about the current flowing through the experimental setup under the action of the step function-like electric field. Transformed into the frequency domain it is called the spectral function $\tilde{\phi}(i\omega)$ related to the ratio of permittivities

$$\tilde{\phi}(i\omega) = L[\phi(t); i\omega] = \frac{\tilde{\varepsilon}(\omega) - \varepsilon_\infty}{\varepsilon_{\text{static}} - \varepsilon_\infty},$$

(3)

where $\varepsilon_\infty \hat{=} \lim_{\omega \to \infty} \tilde{\varepsilon}(\omega)$ and $\varepsilon_{\text{static}} \hat{=} \lim_{\omega \to 0} \tilde{\varepsilon}(\omega)$. In the time domain the response function $\phi(t)$ and the relaxation function $n(t)/n_0$ are related by

$$\phi(t) = -\frac{d}{dt} \left\{ \frac{n(t)}{n_0} \right\},$$

(4)

where $[n(t)/n_0]$ is treated as one symbol providing us with the ratio of the number $n(t)$ of some objects counted at an instant of time $t$ and the number $n(t_0) = n_0$ of the same kind of objects counted at $t_0 \leq t$. Combining Eq. (3) with Eq. (4) we get the equation which enables us to find the relaxation function from the knowledge of spectral function:

$$\frac{n(t)}{n_0} = L^{-1} \left\{ [1 - \tilde{\phi}(i\omega)]/(i\omega); t \right\},$$

(5)

It depends on the experimental setup which function, i.e either the response function $\phi(t)$ or the spectral function $\tilde{\phi}(i\omega)$, is measured. Both approaches permit to determine the relaxation function $n(t)/n_0$ which traditionally is fitted to the experimental data using the (standard) non-Debye relaxations like the Cole-Cole [3, 5, 6], the Cole-Devidson [3, 5, 6], the Havriliak-Negami [3, 5, 6] relaxations and the Kohlrausch-Williams-Watts function [3, 7]. In this paper we will restrict ourselves to the cases of the stretched exponential

$$[n(t)/n_0]_{\alpha;KW} = \exp[-(t/\tau)^\alpha], \quad 0 < \alpha < 1,$$

(6)

whose response function $\phi(t)$ is given by the Weibull distribution, and of the Cole-Cole relaxation

$$[n(t)/n_0]_{\alpha;CC} = E_\alpha[-(t/\tau)^\alpha], \quad 0 < \alpha < 1,$$

(7)

whose spectral function $\tilde{\phi}(i\omega)$ is equal to $[1 + (i\omega t)^\alpha]^{-1}$. In the Eq. (7) $E_\alpha(\sigma)$ denotes the one-parameter Mittag-Leffler (ML) for $0 < \alpha < 1$ which main properties will be introduced and briefly discussed in the next section.

1 The Laplace transform denoted as $L[f(t); p]$ [4] is defined as

$$L[f(t); p] = \int_0^\infty e^{-pt} f(t) \, dt = \hat{f}(p)$$

with $f(t)$ given by the inverse Laplace transform defined on the Bromwich contour $L$ as follows

$$f(t) = L^{-1}[f(p); t] = \frac{1}{2\pi i} \int_L e^{pt} \hat{f}(p) \, dp.$$
2.1. Relaxation in the complex systems

Let us consider the system which contains \( N \) relaxing centers and suppose that each of these centers relax according to the Debye law with a different characteristic relaxation time \( \tau_k = \tau/u_k \), \( k = 1, 2, \ldots, N \). Then the relaxation of the whole sample can be presented as a weighted sum of the Debye relaxations [8, 9, 10], namely

\[
n(t)/n_0 = \sum_k e^{-t u_k/\tau} g(u_k) \Delta u_k, \tag{8}
\]

where the weight function \( g(u_k) \) is the probability density distribution i.e. it is positively defined and normalised \( \sum_k g(u_k) \Delta u_k = 1 \). Going with \( N \) to infinity (or, equivalently, taking infinitesimally small \( \Delta u_k \)) we get the Laplace representation of the relaxation function:

\[
n(t)/n_0 = \int_0^\infty e^{-t u/\tau} g(u) \, du = \int_0^\infty e^{-v} g(\tau v/t) \frac{\tau}{t} \, dv. \tag{9}
\]

According to the Bernstein theorem [4, 11] Eq. (9) says that \( n(t)/n_0 \) is completely monotone, i.e. \((-1)^n d^n[n(t)/n_0]/dt^n \leq 0\) and we have the unique Laplace representation of \( n(t)/n_0 \) by the nonnegative function \( g(u) \). The completely monotonic character of the (standard) non-Debye dielectric relaxations has been shown in [6, 12, 13, 14, 15, 16]. For instance, the completely nonnegative function \( g(u) \) centers relaxing according to the Debye law with a different characteristic relaxation time \( \tau_0 \). Let us consider the system which contains \( N \) centers relaxing according to the Debye law with different characteristic relaxation times, namely

\[
E = \frac{1}{N} \sum_k \frac{e^{-t \tau_0}}{\tau_0} g(u_k) \Delta u_k,
\]

\( k = 1, 2, \ldots, N \), of the Debye relaxations [8, 9, 10], namely

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treated as the generalization of the exponential function to which it goes for $\alpha = 1$. In this case the composition rule of Eq. (1) is multiplication and we have that the product of exponential functions gives another exponential function. We point out that this condition is not satisfied for $\alpha \neq 1$.

Because we will use the $n$th derivative of the three-parameters ML function given by [23, Eq. (11.9)], its Laplace transform given by [21, Eq. (2.5)], and Theorem 11.2 of [23] we quote the needed expressions

$$\frac{d^n}{dt^n} [t^{\beta-1} E_{\alpha,\beta}^{\gamma}(\lambda t^\alpha)] = t^{\beta-n-1} E_{\alpha,\beta-n}^{\gamma}(\lambda t^\alpha),$$  \hfill (14)

$$L[t^{\beta-1} E_{\alpha,\beta}^{\gamma}(\lambda t^\alpha); p] = p^{-\beta}(1 - \lambda p^{-\alpha})^{-\gamma},$$ \hfill (15)

for $\Re(\beta), \Re(p) > 0$, $|p| > \lambda^{1/\Re(\alpha)}$, and

$$\int_0^x (x-t)^{\beta-1} E_{\alpha,\beta}^{\gamma}[a(x-t)^\alpha] t^{\nu-1} E_{\alpha,\nu}^{\sigma}(at^\alpha) \, dt = x^{\beta+\nu-1} E_{\alpha,\beta+\nu}^{\gamma+\sigma}(ax^\alpha)$$ \hfill (16)

satisfied by complex $\alpha, \beta, \gamma, \alpha, \nu, \sigma$ such that $\Re(\alpha), \Re(\beta), \Re(\gamma), \Re(\nu), \Re(\sigma) > 0$, respectively.

3.1. The Cole-Cole relaxation as a special case of Havriliak-Negami relaxation

Eq. (15) enable us to connect the Havriliak-Negami relaxation given by the spectral function

$$[\tilde{\phi}(\omega)]_{\alpha,\beta;HN} = [1 + (\omega t)^\alpha]^{-\beta}$$ \hfill (17)

with the time domain response function $\phi(t)$ defined through the three-parameters ML function [5, 14, 24] as $[\phi(T)]_{HN} = T^{\alpha+1} E_{\alpha,1+\alpha}^{\beta}(T^\alpha)/\tau$ for $T = t/\tau$. Due to Eq. (4) the relaxation function $[n(t)/n_0]_{HN}$ can be calculated taking the first derivative of $[\phi(T)]_{HN}$. Applying Eq. (14) it equals to [5, Eq. (24)]:

$$[n(t)/n_0]_{\alpha,\beta;HN} = 1 - T^{\alpha+1} E_{\alpha,1+\alpha}^{\beta}(T^\alpha), \quad T = t/\tau.$$ \hfill (18)

The representation of Eq. (18) in terms of series and the high transcendental function called the Fox $H$ function can be found in [25, 26]. Eqs. (17) and (18) reduce to the spectral function of the Debye relaxation for $\alpha = \beta = 1$, the Cole-Cole relaxation for $\beta = 1$, and for the Cole-Davidson relaxation for $\alpha = 1$, for example Eq. (18) for $\beta = 1$ yields

$$[n(t)/n_0]_{\alpha,1;HN} = 1 - T^{\alpha} E_{\alpha,1+\alpha}^{\beta}(T^\alpha) = E_{\alpha}(T^\alpha) = [n(t)/n_0]_{\alpha;CC}$$ \hfill (19)

with the second equality in Eq. (19) derived using the series form of the one-parameter ML function given by Eq. (13) for $\gamma = \beta = 1$.

4. The causality-composition law for the stretched exponential

The causality-composition law applied to the stretched exponential illustrates the stable law, i.e. for all positive constants $c_1$ and $c_2$ the sum $c_1 X_1 + c_2 X_2$ of two independent random variables $X_1$ and $X_2$ with the distribution $g$ again has the distribution $g$ [27, 28]. In the language of the inverse Laplace transform this stable law can be written as [8, 9]

$$\int_0^x g_\alpha([T_2 - T_1]^\alpha, y) g_\alpha([T_1 - T_0]^\alpha, x - y) \, dy = g_\alpha([T_2 - T_0]^\alpha, x)$$ \hfill (20)

with $g_\alpha(T^\alpha, x)$ given by Eq. (11) and $T_i = t_i/\tau$, $i = 0, 1, 2$. Note that substituting Eq. (10) in the causality-composition law for the stretched exponential function we get the integral version of the evolution equation of the Lévy stable distribution, i.e. Eq. (20).
The differential equation relevant to Eq. (20) can be obtained by renaming $T_2 - T_1$ and $T_1 - T_0$ as $\bar{T}$ and $\Delta \bar{T}$, respectively, and considering the infinitesimally small $\Delta \bar{T}$. We start with finding the asymptotics of the right-hand side of Eq. (20) written down for $T_2 - T_1 = \bar{T}$ and infinitesimal $T_1 - T_0 = \Delta \bar{T}$. First, we employ the Taylor expansion restricted to the first two terms and get

$$g_a[(\bar{T} + \Delta \bar{T})^\alpha, x] \simeq g_a(\bar{T}^\alpha, x) + (\Delta \bar{T})^\alpha \partial_{\bar{T}} g_a(\bar{T}^\alpha, x). \quad (21)$$

Next, we calculate the asymptotic behaviour $g_a(\Delta \bar{T}^\alpha, x - y)$ for $\Delta \bar{T} \ll 1$. This asymptotics can be found using Eq. (11) and the integral representation of the Lévy stable distribution given by unnumbered formula on the last page in [16], namely

$$g_a[(\Delta \bar{T})^\alpha, x - y] = \frac{1}{\pi} \text{Im} \left\{ \int_0^\infty e^{-(x-y)\xi} e^{-(e^{-i\pi} \Delta \bar{T}^\alpha)\xi} d\xi \right\}.$$

(22)

Taking the first two terms in the Taylor series of $\exp[-(e^{-i\pi} \Delta \bar{T}^\alpha)\xi]$, i.e. $1 - (\Delta \bar{T}^\alpha)^\alpha e^{-i\alpha \xi}$, we can represent Eq. (22) as

$$g_a[(\Delta \bar{T})^\alpha, x - y] \simeq \text{Im} \left\{ \int_0^\infty e^{-(x-y)\xi} \frac{1}{\pi} \right\}
\times \left( - (\Delta \bar{T}^\alpha)^\alpha e^{-i\alpha \xi} \frac{\Gamma(1 + \alpha)}{\pi(x-y)^{1+\alpha}} \right)
\times \left( \frac{1}{\pi(x-y)} - (\Delta \bar{T}^\alpha)^\alpha \frac{\Gamma(\alpha)}{e^{i\pi\alpha} \frac{d}{dy} (x-y)^{-\alpha}} \right).$$

(23)

The substitution Eq. (23) to the left-hand side of Eq. (20), where $T_2 - T_1 = \bar{T}$ and $T_1 - T_0 = \Delta \bar{T}$, and subsequent integration by parts enable us to obtain

$$\int_0^x g_a(\bar{T}^\alpha, y) g_a[(\Delta \bar{T})^\alpha, x - y] dy \simeq \text{Im} \left[ \int_0^x \frac{g_a(\bar{T}^\alpha, y) dy}{x-y} \right]
\times \left[ 1 + \Gamma(\alpha)(\Delta \bar{T}^\alpha)^\alpha \text{Im} \left\{ \lim_{y \to x} \frac{g_a(\bar{T}^\alpha, y)}{y-x} \right\} - (\Delta \bar{T}^\alpha)^\alpha \text{Im} \left\{ \frac{\Gamma(\alpha)}{e^{i\pi\alpha}} \int_0^x \frac{g_a(\bar{T}^\alpha, y) dy}{(x-y)^{\alpha}} \right\} \right].$$

(24)

Using the integral representation of the one-sided Lévy stable distribution given in [16] we get that the first integral in Eq. (24) is equal to $g_a(\bar{T}^\alpha, x)$. The limit present in Eq. (24) is real such that its imaginary part vanishes (detailed calculations of the first two terms in Eq. (24) can be found in [9]). The imaginary part of the third term in Eq. (24) is the $\alpha$-fractional derivative taken in the Caputo sense $CD_{x}^{\alpha}$, $0 < \alpha < 1$, and because of that

$$\int_0^x g_a(\bar{T}^\alpha, y) g_a[(\Delta \bar{T})^\alpha, x - y] dy \simeq g_a(\bar{T}^\alpha, x) - (\Delta \bar{T})^\alpha CD_{x}^{\alpha} g_a(\bar{T}^\alpha, x),$$

(25)

where

$$CD_{x}^{\alpha} f(x) = \frac{1}{\Gamma(1 - \alpha)} \int_0^x \frac{f'(y)}{(x-y)^\alpha} dy, \quad 0 < \alpha < 1.$$  

(26)

Eq. (20) for $T_2 - T_1 = \bar{T}$ and $T_1 - T_0 = \Delta \bar{T}$ in the limit of small $\Delta \bar{T}$ can be obtained comparing Eq. (25) with Eq. (21). As the result we get the fractional differential equation similar to the anomalous diffusion equation

$$\partial_{\bar{T}} g_a(\bar{T}^\alpha, x) = -CD_{x}^{\alpha} g_a(\bar{T}^\alpha, x),$$

(27)

in which $\bar{T}$ is related to time and $x$ corresponds to the variable from the space dual to time in the sense of the Laplace transform.
5. The causality-composition law for the Cole-Cole relaxation

The causality-composition law for the Cole-Cole relaxation is given by Eq. (1) or Eq. (7) of [29] which is also the composition law of the one-parameter ML function. We remind once more that only for $\alpha = 1$ the composition ‘$\circ$’ is multiplication. In another cases it is not fulfilled [30, 31]. In this section we will find how the ‘$\circ$’ should be constructed and understand.

We begin with rewriting the composition law using the series representation of the one-parameter ML function defined by Eq. (13) for $\beta = \gamma = 1$. Thus, we have

$$
\sum_{n=0}^{\infty} (-1)^n \sum_{r=0}^{n} \frac{(T_2 - T_1)^{\alpha r} \circ (T_1 - T_0)^{\alpha(n-r)}}{\Gamma(1 + \alpha r)\Gamma[1 + \alpha(n-r)]} = \sum_{n=0}^{\infty} \frac{(-1)^n(T_2 - T_0)^{\alpha n}}{\Gamma(1 + \alpha n)},
$$

where $T_i = t_i/\tau$, $i = 0, 1, 2$. Eq. (28) will be satisfied if $(T_2 - T_1)^{\alpha r} \circ (T_1 - T_0)^{\alpha(n-r)}$ is defined as $\Gamma(1 + \alpha r)\Gamma[1 + \alpha(n-r)](T_2 - T_0)^{\alpha n}/[(n + 1)\Gamma(1 + \alpha n)]$. Consequently, ‘$\circ$’ may be defined by the integral

$$
(T_2 - T_1)^{\alpha r} \circ (T_1 - T_0)^{\alpha(n-r)} \overset{df}{=} \frac{1 + \alpha n}{1 + n} \int_{T_0}^{T_1} (T_2 - T_1)^{\alpha r}(T_1 - T_0)^{\alpha(n-r)} \frac{dT_1}{T_2 - T_0}.
$$

The causality-composition law for the Cole-Cole relaxation (the ML function) can be found by substituting the above definition into Eq. (28), changing the order of summation and integration and replacing $(1 + n)^{-1}$ by $\int_0^1 u^n \, du$. Moreover, instead of $(1 + \alpha n)(T_2 - T_1)^{\alpha r}(T_1 - T_0)^{\alpha(n-r)}$ we write

$$
(T_1 - T_0)^{\alpha(n-r)}(T_2 - T_1)^{\alpha r} + (T_2 - T_1)^{\alpha r}(T_1 - T_0) d(T_1 - T_0)(T_1 - T_0)^{\alpha(n-r)} + (T_1 - T_0)^{\alpha(n-r)}(T_2 - T_1) d(T_2 - T_1)(T_2 - T_1)^{\alpha r},
$$

where the abbreviation $d_a f(a)$ is used for $d f(a)/da$. Laborious calculations end up with the evolution equation in the integral form:

$$
\frac{d}{dT_2} \int_0^1 du \int_{T_0}^{T_2} dT_1 E_\alpha[-u(T_2 - T_1)^\alpha] E_\alpha[-u(T_1 - T_0)^\alpha] = E_\alpha[-(T_2 - T_0)^\alpha].
$$

Validity of Eq. (31) can be checked for the Debye and the Cole-Cole relaxations. In order to proceed we change in the Eq. (31) the variable $T_1 = s + T_0$ and insert $T_2 - T_0 = S$ such that $s \in [0, S]$. The causality-composition law for the Debye relaxation, namely Eq. (31) with $\alpha = 1$ (the ML function becomes the exponential function) leads to

$$
\frac{d}{dS} \int_0^S ds \int_0^1 du \, e^{-uS} \frac{d}{dS} = \int_0^S \frac{1 - \exp(-S)}{S} \, ds = e^{-S}.
$$

For the Cole-Cole relaxation for which $0 < \alpha < 1$ we use the Eq. (16) taken for $\beta = \gamma = \nu = \sigma = 1$. That gives

$$
\int_0^S ds E_\alpha[-u(S - s)^\alpha]E_\alpha[-uS^\alpha] = SE_{\alpha,2}^2(-uS^\alpha),
$$

where $E_{\alpha,\beta}(\sigma)$ is the three-parameter Mittag-Leffler function introduced by Eq. (13). Furthermore, the Eq. (14) for $\beta = \gamma = 2$ yields to

$$
\frac{d}{dS} [SE_{\alpha,2}^2(-uS^\alpha)] = E_{\alpha,1}^2(-uS^\alpha).
$$
The last operation which has to be done is to integrate $E_{\alpha,1}^2(-uS^\alpha)$ over $u \in [0, 1]$. To perform this task we employ the series form in Eq. (13). That gives $E_\alpha(-S^\alpha)$, i.e. the right hand side of Eq. (31), which completes the proof.

Eq. (31) is the integral counterpart of the fractional differential equation

$$C D^\alpha_T E_\alpha(-T^\alpha) = -E_\alpha(T^\alpha),$$

(35)

for $0 < \alpha < 1$ where the fractional derivative in Caputo the sense is defined in Eq. (26). The evolution equation 35 is the Fokker-Planck equation with the constant operator equal to $-1$ [9, 18].

6. CoInclusion

In the paper we studied the causality-composition law for two examples of the non-Debye relaxations: the stretched exponential and the Cole-Cole models. Due to this causality-composition law the evolution of the system described by a function at the time interval $t_2 - t_0$ is equal to the composition of the same function in the intervals $t_2 - t_1$ and $t_1 - t_0$ where $t_1$ splits the interval $t_2 \geq t_1 \geq t_0$. The causality-composition law, written in the form $f(t_2 - t_0) = f(t_2 - t_1) \circ f(t_1 - t_0)$, leads to the evolution equation in the integral form. For the case of the stretched exponential model we obtain the Laplace convolution of the one-sided Lévy stable distributions depending on two variables. The differential version of this equation has also been found and it contains the fractional derivative in the Caputo sense. Notice that the obtained differential equation is similar to the anomalous diffusion. The integral form of the evolution equation for the Cole-Cole relaxation contains the derivative and integrals. It is equivalent to the fractional differential equation where the time derivative is taken in the Caputo sense.

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