Bogolubov’s Recursion and Integrability of Effective Actions

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ABSTRACT

The Hopf algebra of Feynman diagrams, analyzed by A.Connes and D.Kreimer, is considered from the perspective of the theory of effective actions and generalized $\tau$-functions, which describes the action of diffeomorphism and shift groups in the moduli space of coupling constants. These considerations provide additional evidence of the hidden group (integrable) structure behind the standard formalism of quantum field theory.

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1 Introduction

Exponentiated effective action (partition function, statistical sum) is defined as a functional of the coupling constants $T$ and background fields $\varphi$ (the “vacuum configuration”), resulting from functional integration over quantum fields $\phi$:

$$Z\{T|\varphi\} = \int \exp (-S_T(\varphi + \phi)) \mathcal{D}\phi$$ (1.1)

When all possible coupling constants $T$ are taken into account (i.e. the theory is maximally deformed), $Z\{T\}$ becomes a generating function of all the correlation functions in entire family of models. Such $Z\{T\}$ possesses a hidden group-theoretical structure and – as a manifestation of this – satisfies bilinear (Hirota-like) and differential (Laplace-like) equations, i.e. belongs to the class of generalized $\tau$-functions. One can consider $Z\{T\}$ as a function (section) on the moduli space $\mathcal{M}$ of theories (parametrized by the coupling constants $T$).

There are two important groups, acting transitively on $\mathcal{M}$: the abelian group $\text{Shift}\mathcal{M}$ of shifts along $\mathcal{M}$ and non-abelian group $\text{Diff}\mathcal{M}$ of diffeomorphisms of $\mathcal{M}$. They act on partition functions in the same way:

$$Z\{T\} \rightarrow Z\{T + V(T)\},$$ (1.2)

but the composition rules are different:

$$Z\{T\} \rightarrow Z\{T + V_1(T) + V_2(T)\}$$ (1.3)

for $\text{Shift}\mathcal{M}$ and

$$Z\{T\} \rightarrow Z\{T + V_1(T) + V_2(T + V_1(T))\}$$ (1.4)

for $\text{Diff}\mathcal{M}$. In other words, the infinitesimal action of both groups is described by vector fields $\tilde{V}\{T\} = V(T)\partial/\partial T$, but the global action is by exponentiated vector field, $\exp \tilde{V}$ for $\text{Diff}\mathcal{M}$ and by the normal-ordered exponent $: \exp \tilde{V} :$ for abelian $\text{Shift}\mathcal{M}$. A map between $\text{Diff}\mathcal{M}$ and $\text{Shift}\mathcal{M}$ is provided by the relation

$$e^{\tilde{V}} = : e^{\tilde{V}} :$$ (1.5)

Looking from this perspective, one associates with every particular theory a group element $g_T$ with two basic properties

$$g_{T_1} = g_{T_2}g_{T_2}, \quad T_1 = T_{12} \circ T_2$$ (1.6)
and represents $Z\{T\}$ as a matrix element (generalized zonal-function or $\tau$-function): 

$$Z\{T\} = \langle 1|g_T|0 \rangle$$

between a Gaussian theory, labeled by $|0\rangle$, and some other state $\langle 1|$, depending on particular realization of $g_T$. The transitive action of the group basically puts all the points in the moduli space on equal footing (in particular the Gaussian point is not distinguished among the others), and this can explain the surprising power of the free-field formalism in quantum field theory. The composition rule $T_1 = T_{12} \circ T_2$ depends on which of the two groups, $\text{Shift}M$ or $\text{Diff}M$, we want $g_T$ to belong to. In the case of abelian $\text{Shift}M$ it is just an addition: $T_1 = T_{12} + T_2$ (if the space of coupling constants is big enough and appropriate choice of coordinates in the moduli space is made).

A more interesting non-abelian diffeomorphism group, and especially the stability subgroup $\text{Diff}\emptyset M$ of the Gaussian model in $M$, is relevant for description of one-parametric renormalization-group flows along $M$.

This old set of ideas [1, 2, 3, 4, 5] is supported by evidence in matrix models [6], Seiberg-Witten theory [7, 8, 9] and AdS/CFT correspondence [10, 11]. The purpose of this paper is to claim that additional evidence is provided by the recent studies of A.Connes and D.Kreimer [13], who actually define the action of the operators $g_T$ in the space of graphs (Feynman diagrams). In what follows we basically repeat their reasoning, making use of convenient quantum models and separating the algebraic structures, relevant for Hirota-like equations and Bogolubov’s $R$-operation, from peculiarities of particular models and prejudices of conventional local field theory. Some graph-theory and combinatorial routine is omitted.

### 2 The Basics of Bilinear Relations

The basic model for generic studies of integrable structures in quantum field theory is the one of arbitrarily many scalars $\phi^i$ in $0 + 0$ dimensions:

$$Z\{T\} = \int e^{\frac{1}{2}V_T(\phi)} \prod_i d\phi^i, \quad (2.1)$$

The AdS/CFT-correspondence [12] claims that certain Yang-Mills partition functions are represented by the boundary dependence of the bulk actions of certain classical gravities. Among other things, this implies that they satisfy bilinear Hamilton-Jacobi equations, which should be nothing but an avatar of bilinear Hirota and Laplace-like equations for the effective actions.

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1 The AdS/CFT-correspondence [12] claims that certain Yang-Mills partition functions are represented by the boundary dependence of the bulk actions of certain classical gravities. Among other things, this implies that they satisfy bilinear Hamilton-Jacobi equations, which should be nothing but an avatar of bilinear Hirota and Laplace-like equations for the effective actions.
\[ V_T(\phi) = \sum_n \frac{1}{n!} \left( \sum_{i_1, \ldots, i_n} T^{(n)}_{i_1 \ldots i_n} \phi^{i_1} \ldots \phi^{i_n} \right) \]  

(2.2)

Partition function \( Z\{T\} \) can be represented as a sum over all possible graphs without external legs (vacuum Feynman diagrams). One can exclude disconnected graphs by switching to \( \log Z\{T\} \) – in exchange one gets additional \( 1/n \) coefficients. These can be eliminated by consideration of correlation functions: graphs with external legs. This is the usual routine of diagram technique [14].

At this point it can make sense to comment on the choice of the model (2.1). Among its particular reductions (truncations) are: the single-scalar model,

\[ Z_{U(1)}\{T\} = \int \exp \left( \sum_{n=0}^{\infty} \frac{T_n}{n!} \phi^n \right) d\phi; \]  

(2.3)

the \( N \times N \) matrix model with \( N^2 \) scalars, assembled into a matrix \( \phi_{ab} \),

\[ Z_{U(N)}\{T\} = \int \exp \left( \sum_{n=0}^{\infty} \frac{T_n}{n!} \text{Tr} \phi^n \right) \prod_{a,b=1}^{N} d\phi_{ab}; \]  

(2.4)

the scalar field in \( d \) space-time dimensions, where indices \( i \) become continuous and \( T^{(2)}_{ij} \) is taken to be Laplace operator; etc. (In the last example, if the space-time is non-compact, it is unavoidable to introduce the background fields \( \varphi \), like it is done in (1.1), to label the boundary conditions and/or the asymptotics at infinities.) The most essential difference of all these popular models from the universal one in (2.1) is that the vertices in (2.1) are of the most general form, e.g. in \( d \)-dimensional theory the \( \phi^n \) coupling should allow any dependence of the coupling “constants” on all the \( n d \)-momenta (while in local field models they are indeed constants or at best polynomial in momenta). In such a large moduli space one can distinguish between any two Feynman diagrams, looking at their \( T \)-dependencies (while for the model (2.3) all the diagrams with the same number of propagators, \( l \), and vertices of valences \( k \), \( v_k \), give rise to the same expression, \( T_{2}^{-l} \prod_k T_{v_k}^{k} \); and switching to the matrix model (2.4) introduces nothing more than extra factor \( N^{-\chi} \), depending on Euler characteristics \( \chi \) of the corresponding fat graph, which is still not enough to distinguish between any two diagrams). Last, but not the least, such a large moduli space is preserved by renormalization group flow: effective actions at any stage of the flow remains in the class (2.1); also renormalizability is not a restriction on the form of the theory (once it is somehow regularized).

A further extension of (2.1), playing the same role of the universal model for fat graphs as the model (2.1) plays for the ordinary ones, is provided by the
matrix model with 2-index fields $\phi^{ij}$ and the action

$$V_T(\phi) = \sum_n \frac{1}{n!} \left( \sum_{i_1, \ldots, i_n} T^{(n)}_{i_1 \ldots i_n} \phi^{i_1 i_2} \phi^{i_2 i_3} \ldots \phi^{i_n i_1} \right)$$ (2.5)

Its principal difference from (2.1) is that the couplings $T^{(n)}_{i_1 \ldots i_n}$ are no longer symmetric under permutations of indices $i_1, \ldots, i_n$, only cyclic symmetric. This model remains beyond our consideration in the present paper.

The model (2.1) can be also regarded in a different way. Given any particular quantum theory one can switch to its $GL(N)$ or $GL(\infty)$ extension, just adding a vector index $i \in I$ to all the fields of the theory and ascribing the relevant tensor structure to all the coupling constants. For example, the $d$-dimensional $\phi^3$ theory can be substituted by

$$\int \prod_i D\phi^i(x) \exp \int d^d x \left( T^{(2)}_{ij} \left( (\nabla \phi^i)(\nabla \phi^j) - m^2 \phi^i \phi^j \right) - g_3 T^{(3)}_{ijk} \phi^i \phi^j \phi^k \right)$$ (2.6)

without requiring that $T$-variables are $x$-dependent. Then the partition function is $GL(\infty)$-invariant and can be expanded in a series over the basic $GL(\infty)$-invariant functions, provided by (2.1), however the expansion coefficients are now sophisticated functions not only of graphs, but also of many other parameters, including external momenta and the spins of particles. Still, some basic properties can be seen at the level of graph theory alone – and this is the subject of our further considerations.

We now return to the main line of discussion. The issue of our interest is the group (integrable) structure, hidden in partition functions $Z\{T\}$. This structure survives various reductions, e.g. the one to the matrix model (2.4), see [16], but many aspects are much more transparent in analysis of the universal model (2.1). Of course, the relation

$$V_{T+T'} = V_T + V_{T'}$$ (2.7)

for the potential under the integral does not imply that the average $Z\{T\}$ is a character of Shift,$\mathcal{M}$ group, $Z\{T + T'\} \neq Z\{T\}Z\{T'\}$, the true relation is between the group elements, $g_T g_{T'} = g_{T+T'}$, and

$$Z\{T + T'\} = \langle 1 | g_{T+T'} | 0 \rangle = \sum_{states} \langle 1 | g_T | states \rangle \langle states | g_{T'} | 0 \rangle$$ (2.8)

At the r.h.s. stands a non-trivial operator, acting on $T$ and $T'$ in the product $Z\{T\}Z\{T'\}$.

The question is, what are the relevant realizations of the Hilbert space of $|states\rangle$ and of the operators $g_T$ acting in it.
The simplest realization is implied by the functional integral and makes special use of the source-dependence of \( Z\{T\} \). Namely, the \( T^{(1)}\)-terms in (2.2) can be identified with the sources for the fields \( \phi^i \): \( \sum_j T_j^{(1)} \phi^i = i \sum_j J_j \phi^i \) and, as usual in the derivation of Hirota equations, one can use sources to construct a delta-function projector:

\[
\int e^{-i(\phi^i - \phi^i')} J'_j \prod_j dJ'_j \sim \prod_j \delta(\phi^i - \phi^i'),
\]

so that

\[
\int Z_{J - J'} \{T\} Z_{J'} \{T'\} dJ' = \int dJ' \left( \int d\phi e^{V_T(\phi)} e^{i(J - J')\phi} \int d\phi' e^{V_{T'}(\phi')} e^{i(J - J')\phi'} \right) = Z_J \{T + T'\} \tag{2.10}
\]

(we explicitly labeled the \( J\)- \( T^{(1)}\)-dependence of the action, suppressed all the indices \( i \) and made use of the addition formula (2.7)).

The simplest example arises if all \( T^{(n)} = 0 \) for \( n > 2 \). Then

\[
Z_{Gauss}\{T\} \sim \frac{1}{\sqrt{\det T^{(2)}}} \exp \left( -\frac{1}{4} Tr T^{(1)} T^{(2)} \right), \tag{2.11}
\]

and the bilinear relation

\[
Z_{Gauss}\{T + T'\} = \int dT^{(1)} dT'^{(1)} \delta(T^{(1)} - T'^{(1)}) Z_{Gauss}\{T\} Z_{Gauss}\{T'\} \tag{2.12}
\]

is just the completeness formula for Gaussian propagators, which in the single-scalar case is widely-known in the form

\[
\int dx_2 \frac{e^{-x_{12}^2/4t_{12}}}{\sqrt{t_{12}}} \cdot \frac{e^{-x_{23}^2/4t_{23}}}{\sqrt{t_{23}}} \sim \frac{e^{-x_{13}^2/4t_{13}}}{\sqrt{t_{13}}} \tag{2.13}
\]

(in our case \( x_{12} = x_1 - x_2 = T^{(1)} \), \( x_{23} = T^{(1)} - T^{(2)} \), \( t_{12} = T^{(2)} \), \( t_{23} = T^{(2)} \)).

In a similar way one can obtain bilinear integral “summation formulae” for the Eiry functions (if all \( T^{(n)} = 0 \) for \( n > 3 \)) etc.

Instead of using the source-dependencies, one can exploit those on other coupling constants. All these dependencies are interrelated: in the universal model (2.1)
\[
\frac{\partial Z}{\partial T_{i_1 \ldots i_n}} = \frac{\partial^n Z}{\partial T_{i_1} \ldots \partial T_{i_n}},
\] (2.14)

in its various reductions one has more sophisticated Ward identities, like Virasoro and W-constraints in matrix models \[15, 16, 17\]. Also, Legendre transform relates source- and background-field dependencies of generic \(Z\{T|\phi\}\) in (1.1). For more discussion of interplay between the source-, coupling-constants and background-fields dependencies see \[6\], especially the example of generalized Kontsevich model \[18\]. The Ward identities like (2.14) play important role in building explicit maps like (1.5) between the \(\text{Shift}\mathcal{M}\) and \(\text{Diff}\mathcal{M}\) groups.

Drawback of such functional-integral approaches to bilinear identities is that they do not immediately provide representations in terms of conventional (perturbative) correlation functions: at least some operator, like the source term \(\sum_j J_j \phi^i\), should be exponentiated, i.e. one needs to consider a global deformation and the entire family of theories, not just infinitesimal vicinities of the given models \(g_T\) and \(g_T'\). Though there is nothing bad about this from the general perspective of string theory, such representations are not the best ones for the search of bilinear relations in conventional quantum field theories, where isolated points in the moduli spaces (i.e. isolated particular models) are usually analyzed. One possibility to obtain representations in terms of the ordinary Green functions is to use the Vermat-module realizations of \(\vert\text{states}\rangle\,\[4\]. Another – not unrelated – possibility is exploited by A.Connes and D.Kreimer (CK) in \[13\]: it is to look at the contributions of particular graphs (Feynman diagrams).

## 3 Hilbert Space of Graphs and Operators \(g_T\)

Operators, acting in the Hilbert space of Feynman diagrams naturally appear if the Gaussian measure is explicitly extracted from \(\exp V_T\) (as a starting point for perturbation expansion) and if the theory (2.1) is further "complexified":

\[
Z\{\tilde{T}, T\} = \int \left\{ \prod_{i, \tilde{i}} d\phi^{i \tilde{i}} \exp \left(-\frac{1}{2} \sum_{i, \tilde{i}, j, \tilde{j}} G_{ij} \tilde{G}_{\tilde{i}\tilde{j}} \phi^{i \tilde{i}} \phi^{j \tilde{j}} \right) \right\} e^{V_{\tilde{T}, T}(\phi)},
\] (3.1)

\[
V_{\tilde{T}, T}(\phi) = \sum_n \frac{1}{n!} \left( \sum_{i_1, \ldots, i_n, \tilde{i}_1, \ldots, \tilde{i}_n} \tilde{T}^{(n)}_{i_1, \ldots, i_n} T^{(n)}_{\tilde{i}_1, \ldots, \tilde{i}_n} \phi^{i_1 \tilde{i}_1} \ldots \phi^{i_n \tilde{i}_n} \right)
\] (3.2)

The fields \(\phi^{i \tilde{i}}\) are now labeled by a pair of indices, taking values in the sets \(I\) and \(\tilde{I}\), \(i \in I, \tilde{i} \in \tilde{I}\), and coupling constants are factorized (assumed to be the "squared
modules” of “holomorphic” \( T \)'s). In variance with ordinary complexification, we do not assume that the sets \( I \) and \( \tilde{I} \) are the same. In particular, we can return to the original model (2.1) by asking \( \tilde{I} \) to consist of a single element and putting all \( \tilde{T}_1^{(n)} = 1 \) and \( \tilde{G}_{11} = 1 \) (below, when necessary, we just write \( \tilde{T} = 1 \), implying that \( \tilde{I} = \{1\} \), and \( Z\{\tilde{T} = 1, T\} \)).

Expanding \( \exp V_{\tilde{T}, T}(\phi) \) in (3.1) into formal series and applying the Wick theorem to Gaussian integrals, one obtains the expansion over vacuum Feynman diagrams \( \Gamma^{(0)} \), which has specific structure, called “holomorphic factorization”:

\[
Z\{\tilde{T}, T\} = \sum_{\Gamma^{(0)}} \frac{Z_{\Gamma}(\tilde{T})Z_{\Gamma}(T)}{S_{\Gamma}} 
\]  

(3.3)

Here \( S_{\Gamma} \) is the “symmetry factor” of the graph \( \Gamma \), for connected graph it is the order of the discrete group which permutes links, while keeping their ends fixed. For vacuum diagrams (graphs without external legs) \( S_{\Gamma} \) contains an additional factor \( \text{Vert}(\Gamma) \) – the number of vertices in the graph \( \Gamma \). For disconnected graphs \( S(\prod_i \Gamma_i^{n_i}) = \prod_i n_i! S_{\Gamma_i}^{n_i} \). Expression \( Z_{\Gamma}\{T\}/S_{\Gamma} \) for the Feynman diagram \( \Gamma \) is a convolution of vertices and propagators, divided by \( S_{\Gamma} \). In particular, \( Z_{\Gamma}\{T = 1\} = 1 \), and for the model (2.1) eq.(3.3) gives:

\[
Z\{T\} = Z\{\tilde{T} = 1, T\} = \sum_{\Gamma^{(0)}} \frac{Z_{\Gamma}\{T\}}{S_{\Gamma}} 
\]  

(3.4)

In Feynman diagrams \( G_{ij} \) plays the role of inverse propagator. In what follows we lower and raise indices with the help of \( G_{ij} \) and its inverse \( G^{ij} \). In particular the switch from Green functions to the amputated correlators is nothing but lowering of indices on external legs. Coupling constants are defined to have lower indices.

Introduce now the Hilbert space \( \mathcal{H}^{(0)} \) of all possible graphs (with vertices of any valence, connected or disconnected) with no external legs (vacuum Feynman diagrams), i.e. with every graph \( \Gamma^{(0)} \) we associate a state \( |\Gamma^{(0)}\rangle \). The scalar product is

\[
(\Gamma^{(0)}|\Gamma'^{(0)}) = S_{\Gamma} \delta_{\Gamma, \Gamma'} 
\]  

(3.5)

Let us further define the coherent-like states

\[
|T\rangle = \sum_{\Gamma^{(0)}} \frac{Z_{\Gamma}\{T\}}{S_{\Gamma}} |\Gamma\rangle 
\]  

(3.6)

They are not orthonormal, instead (3.3) states that
\[ Z\{\tilde{T}, T\} = \langle \tilde{T}|T \rangle \quad (3.7) \]

Thus with every particular model (a point in the moduli space \( M \)) we associate a point \(|T\rangle\) in the Hilbert space of vacuum graphs, and partition function \( Z\{\tilde{T}, T\} \) \((3.1)\) is just a scalar product of associated states. In particular, the holomorphic partition function \( Z\{T\} \) in \((2.1)\) is a scalar product of \(|T\rangle\) with a special state \( \langle \tilde{T} = 1 | \) where \( \tilde{T} \) has single-valued indices and all \( \tilde{T} = 1 \).

Another special state is the Gaussian model, \(|Gauss\rangle = |0\rangle = |\emptyset\rangle\), associated with all \( T = 0 \) (with any set \( I \) and non-vanishing metric \( G_{ij} \)). At Gaussian point the only contribution to \((3.6)\) comes from the empty graph \( \Gamma = \emptyset \) (we assume the normalization \( Z\{T = 0\} = 1 \)). There is an even more distinguished “trivial” model with \( Z\{T\} = \exp T^{(0)} \), the transitive actions of \( \text{Shift}_M \) and \( \text{Diff}_M \) groups connect it to any other model, including the Gaussian one. However, the structures which are of interest for us are explicitly dependent on the metric \( G_{ij} \), and the minimal model which takes this into account is the Gaussian one: arbitrary diffeomorphisms and shifts can be expanded over basic functions, provided by Gaussian model, but not by the “trivial” one.

Let us now introduce an operator \( g_T \), acting in the Hilbert space of graphs, such that

\[ |T\rangle = g_T|Gauss\rangle = g_T|\emptyset\rangle \quad (3.8) \]

Then

\[ Z_{\Gamma}\{T\} = \langle \Gamma|T\rangle = \langle \Gamma|g_T|\emptyset\rangle \quad (3.9) \]

In sec.8 below we use these matrix elements to convert functions of coupling constants \( T \) into functions of graphs \( \Gamma \) and vice versa.

Eq.\((3.9)\) does not fully specify the operator \( g_T \). However, field theory implies a natural extension of \((3.9)\) to all the matrix elements

\[ \langle \Gamma^{(n)}_{i_1...i_n}|g_T|\gamma^{(m)}_{j_1...j_m}\rangle \quad (3.10) \]

in the enlarged Hilbert space \( \mathcal{H} = \oplus_n \mathcal{H}^{(n)} \) of all the graphs \( \Gamma^{(n)} \) with any number \( n \) of external legs with indices \( i \in I \) ascribed to every leg.

The space \( \mathcal{H}^{(n)} \) naturally appears if one considers the \( n \)-point correlation functions in the theory \((3.1)\). Such correlator carries extra \( 2n \) indices and is decomposed into a sum over all the graphs with \( n \) external legs:

\[ Z\{\tilde{T}, T\}_{\tilde{i}_1...\tilde{i}_n;i_1...i_n} = \sum_{\Gamma^{(n)}} \frac{Z_{\Gamma}\{\tilde{T}\}_{\tilde{i}_1...\tilde{i}_n}Z_{\Gamma}\{T\}_{i_1...i_n}}{S_{\Gamma}} \quad (3.11) \]
Similarly to the case of $\mathcal{H}^{(0)}$ one can now define a set of states $|\Gamma_{(n)}^{i_1\ldots i_n}\rangle$ with the scalar product

$$\langle \Gamma_{(n)}^{i_1\ldots i_n} | \Gamma'_{(m)}^{j_1\ldots j_m} \rangle = S_{\Gamma} G^{i_1j_1} \ldots G^{i_nj_n} \delta_{\Gamma,\Gamma'}$$

(3.12)

(Since scalar product is non-vanishing only for coincident graphs, the number of external legs are also the same, and $G^{ik}$ couples the indices ascribed to the same $k$-th leg.) One can amputate external legs by lowering the indices with the help of the metric $G_{ij}$.

Now we introduce in $\mathcal{H}^{(n)}$ the state

$$|T\rangle^{(n)} = \sum_{\Gamma(n); i_1,\ldots,i_n} \frac{Z_{\Gamma}}{S_{\Gamma}} |\Gamma^{(n)}_{i_1\ldots i_n}\rangle$$

(3.13)

(the indices are lowered with the help of the metric $G$) and finally the state

$$|T\rangle = \oplus_n |T\rangle^{(n)}$$

(3.14)

in entire $\mathcal{H}$.

In order to define the matrix elements of $g_T$ between any two states in $\mathcal{H}$ we need to introduce the notion of subgraph.

4 Subgraphs

There are two different notions of subgraph, relevant for our further discussion. Let $\Gamma^{(n)}$ be a graph (connected or disconnected, possibly one-particle reducible) with $n$ external legs. It has vertices of any valence (including one and two).

1) Vertex-subgraphs. Divide the set of vertices in two non-intersecting subsets and cut all the links, connecting vertices from different sets. If $m$ legs were cut, we decompose the original graph $\Gamma^{(n)}$ into two disconnected graphs $\gamma_1^{m+n_1}$ and $\gamma_2^{m+n_2}$, such that $n_1 + n_2 = n$. We call them vertex-subgraphs of $\Gamma^{(n)}$ and introduce a notation $\gamma_2 = \Gamma / \gamma_1$ (of course, also $\gamma_1 = \Gamma / \gamma_2$). The empty graph $\gamma^{(0)} = \emptyset$ and $\gamma^{(n)} = \Gamma^{(n)}$ are vertex-subgraphs of $\Gamma^{(n)}$. The number of vertices $Vert(\gamma) + Vert(\Gamma / \gamma) = Vert(\Gamma)$.

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2 When one cuts a link in a graph, two new external legs are formed at the place of a single propagator $G^{-1}$ and one glues them back with the help of the metric $G$, or, alternatively, amputate one leg in each pair. This can be done more symmetrically, if $G = D^2$: then one can associate with every external leg the matrix $D^{-1}$, instead of the usual rule, ascribing the propagator $G^{-1}$ to non-amputated leg and unity to the amputated one. In continuous case, when $G$ is Laplace operator, $D$ turns into a Dirac operator. Though the use of $D$ can make the bilinear relations below conceptually more symmetric, we ignore this possibility in the present text.
2) **Box-subgraphs.** Pick a non-empty\(^3\) subset of vertices and draw a box or a set of non-intersecting boxes around them. Boxes should not lie one inside another. Each box in the set should contain at least one vertex, and the subgraph inside the box should be connected. The sides of the box cut some links of original graph, in particular a link connecting two vertices from our subset can be cut (and these two vertices can belong to the same box or to two disconnected boxes). We call the subgraph \(\gamma\) lying in this system of boxes a box-subgraph of \(\Gamma^{(n)}\). Its complement is no longer a box-subgraph: it can contain just a remnant of a double-cut link with no vertices. Instead of a complement, for a box-subgraph \(\gamma^{(m)}\) one can always define a contraction \([\Gamma^{(n)}/\gamma^{(m)}]\) obtained when each connected component of a box, which cuts links at \(k\) places is substituted by a single valence-\(k\) vertex. The resulting graph \([\Gamma^{(n)}/\gamma^{(m)}]\) has \(n\) external legs, as the original \(\Gamma^{(n)}\) and the same number of connected components, \(\text{Con}(\Gamma/\gamma) = \text{Con}(\Gamma)\). According to this definition, the empty graph \(\emptyset\) is not a box-subgraph of \(\Gamma\), and there is no box-subgraph \(\gamma\), such that \([\Gamma/\gamma]\) = \(\emptyset\).

The same graph \(\gamma\) can happen to be a vertex-subgraph and a box-subgraph simultaneously, but the two sets \(V\Gamma\) and \(B\Gamma\) (of vertex- and box-subgraphs respectively) are different. \(V\Gamma\) is just a set-theory object: the set of all subsets of the set of vertices of \(\Gamma\), in particular there are always exactly \(2^{\text{Vert}(\Gamma)}\) vertex-subgraphs. As to \(B\Gamma\), this is a more sophisticated object, essentially depending on the graph structure (not just the set-theory one), in particular, the size of this set depends on the valences of vertices and on exact construction of the links.

The set \(V\Gamma\) is related to the abelian group \(\text{Shift}\mathcal{M}\) (and is relevant for description of bilinear identities), while \(B\Gamma\) is related to the non-abelian group \(\text{Diff}_0\mathcal{M}\), generated by vector fields on \(\mathcal{M}\) (and is relevant for description of Bogolubov’s recursion and renormalization flows).

### 4.1 Examples

It is now instructive to consider some examples. We mention three classes of simple graphs, useful for various illustrations.

**1) Single-vertex graph** \(\Gamma^{(p-2n)}_{p;n}\) has one valence-\(p\) vertex, \(n\) propagators and \(p - 2n\) external legs. \(\Gamma^{(p)}_{p;0}\) is the elementary (bare) vertex of valence \(p\).

\(\Gamma^{(p-2n)}_{p;n}\) has just two vertex-subgraphs: \(\gamma = \emptyset\) and \(\gamma = \Gamma^{(p-2n)}_{p;n}\) itself. The corresponding complements are \(\Gamma^{(p-2n)}_{p;n}/\emptyset = \Gamma^{(p-2n)}_{p;n}\) and \(\Gamma^{(p-2n)}_{p;n}/\Gamma^{(p-2n)}_{p;n} = \emptyset\).

At the same time, there are \(2^n\) box-subgraphs (of which \(n + 1\) are topologically different): \(\gamma = \frac{n!}{k!(n-k)!} \times \Gamma^{(p-2k)}_{p;k}, 0 \leq k \leq n\) (binomial coefficient

\(^3\) The would-be empty box is not well defined. If there are no vertices inside the box, it still can be not empty: contain fragments of some links. To avoid such ambiguities we exclude empty graphs from the set of box-subgraphs of \(\Gamma\). When necessary, their contributions will be explicitly added to sums over the set \(B\Gamma\) of box-subgraphs.
\( n!/k!(n-k)! \) denotes the multiplicity of the subgraph. The corresponding contractions \([\Gamma_{p:n}/\Gamma_{p:k}]\) are \( \frac{n!}{k!(n-k)!} \times \Gamma_{p-2k;n-k}^{(p-2n)} \).

2) Two-vertex graph \( \Gamma_{p+q-2n}^{(p+q-2n)} \) has one valence-\( p \) and one valence-\( q \) vertices, \( n \) propagators between them (i.e. \( 0 \leq n \leq p, q \)) and \( p+q-2n \) external legs.

It has 4 vertex-subgraphs, \( \gamma = \emptyset, \Gamma_{p:0}, \Gamma_{q:0}, \Gamma_{p,q:0} \), and \( 2^n + 2 \) box-subgraphs: \( \gamma = \Gamma_{p:0}, \Gamma_{q:0}, \frac{n!}{k!(n-k)!} \Gamma_{p:q:k}, 0 \leq k \leq n \). The corresponding \([\Gamma/\gamma] = \Gamma_{q:0}, \Gamma_{p:0}, \frac{n!}{k!(n-k)!} \times \Gamma_{p+q-2k;n-k} \).

3) Chain graph \( C_N \) has \( N \) valence-two vertices, connected chain-wise by \( N-1 \) propagators. It has 2 external legs. \( C_N \) has \( 2^N \) vertex-subgraphs and \( \beta_N \) box-subgraphs.

\( N = 1 \).

Vertex subgraphs:

\[
\gamma = \emptyset, \ C_1 \\
C_1/\gamma = C_1, \emptyset
\] (4.1)

Box-subgraphs (\( \beta_1 = 1 \)):

\[
\gamma = C_1 \\
[ C_1/\gamma ] = C_1
\] (4.2)

\( N = 2 \).

Vertex subgraphs:

\[
\gamma = \emptyset, \ 2 \times C_1, \ C_2 \\
C_2/\gamma = C_2, \ 2 \times C_1, \emptyset
\] (4.3)

Box-subgraphs (\( \beta_2 = 4 \)):

\[
\gamma = 2 \times C_1, \ C_2, \ C_1 \cdot C_1 \\
[ C_2/\gamma ] = 2 \times C_2, \ C_1, \ C_2
\] (4.4)

\( N = 3 \).

Vertex subgraphs:

\[
\gamma = \emptyset, \ 2 \times C_1, \ C_1, \ C_2, \ C_1 \cdot C_1, \ C_3 \\
C_3/\gamma = C_3, \ 2 \times C_2, \ C_1 \cdot C_1, \ 2 \times C_1, \ C_1, \emptyset
\] (4.5)

Box-subgraphs (\( \beta_3 = 12 \)):
\[ \gamma = 3 \times C_1, \quad 2 \times C_2, \]
\[ [C_3/\gamma] = 3 \times C_3, \quad 2 \times C_2, \quad C_1, \quad 2 \times (C_1 \cdot C_1), \quad C_1 \cdot C_1, \quad 2 \times (C_1 \cdot C_2), \quad C_1 \cdot C_1 \cdot C_1 \]
\[ C_2, \quad 2 \times C_2, \quad C_3, \quad 2 \times C_2, \quad C_3 \]  \hspace{1cm} (4.6)

Every box-subgraph of \( C_N \) is located in \( s \) non-intersecting boxes, with \( k \)-th box beginning at link \( i_k \) and ending at link \( j_k \). The total number

\[ \beta_N = \sum_{s=1}^{N} \beta(N; s) = \sum_{s=1}^{N} \left( \sum_{0 \leq i_1 < j_1 \leq i_2 < j_2 \leq \ldots \leq i_s < j_s \leq N} 1 \right) = \mu_{N-1} \mu_N \]  \hspace{1cm} (4.7)

The number of chain graphs with exactly \( s \) connected components is

\[ \beta(N; s) = \frac{(N + s)!}{(2s)!(N - s)!} \]  \hspace{1cm} (4.8)

The \( \mu_N \) are Fibonacci-like numbers, satisfying recurrent relations:

\[ \mu_{2k} = 4\mu_{2k-1} - \mu_{2k-3}, \quad \mu_{2k+1} = 3\mu_{2k-1} - \mu_{2k-3} \]  \hspace{1cm} (4.9)

and initial conditions \( \mu_0 = \mu_1 = 1 \). Consequently

\[ (\mu_0, \mu_1, \mu_2, \ldots) = (1, 1, 4, 3, 11, 8, 29, 21, 76, 55, \ldots) \]  \hspace{1cm} (4.10)

and

\[ (\beta_1, \beta_2, \ldots) = (1, 4, 12, 33, 88, 232, 609, 1596, \ldots) \]  \hspace{1cm} (4.11)

5 Vertex-subgraphs, action of \( g_T \) in \( \mathcal{H} \) and bilinear relations

We are now ready to define the matrix elements of \( g_T \). They are different from zero only for \( \gamma \) which is a vertex-subgraph of \( \Gamma \) (consequently, \( g_T \) is triangular and can not be Hermitean operator – this is natural, since it is an element of a group, not algebra,– moreover, triangularity implies that \( g_T^\dagger \neq g_{-T} \)). For
simplicity we first assume that no external legs of $\Gamma^{(n)}$ were cut to make the subgraph $\gamma^{(m)}$. Then

$$\langle \Gamma^{(n)}_{i_1\ldots i_n} | g_T | \gamma^{(m)}_{j_1\ldots j_m} \rangle = Z_{\Gamma/\gamma}^{i_1\ldots i_n j_1\ldots j_m} \{ T \}$$  \hspace{1cm} (5.1)

In other words, the matrix element is given by the expression for Feynman diagram $\Gamma/\gamma$ in the theory $g_T$ without the usual symmetry factor $1/S_T/\gamma$. This means that every link in $\Gamma/\gamma$ carries a propagator $G_{ij}^{(k)}$ and every vertex of valence $k$ in $\Gamma/\gamma$ contributes $T_{ii_1\ldots i_k}^{(k)}$. The indices are contracted and summed over. Since $\Gamma^{(n)}/\gamma^{(m)}$ has $n$ original and $m$ new-formed external legs, the whole matrix element has $n + m$ free indices. If some $p$ of external legs of $\gamma$ coincide with external legs of $\Gamma$, the corresponding indices appear as $\delta^i_j$ (or $G_{ij}^{(2)}$) factors.

$$\langle \Gamma^{(n)}_{i_1\ldots i_n-p1\ldots p} | g_T | \gamma^{(m)}_{j_1\ldots j_m-p} \rangle_{k_1\ldots k_p} = Z_{\Gamma/\gamma}^{i_1\ldots i_n-p j_1\ldots j_m-p} \{ T \} \delta^i_{k_1} \ldots \delta^p_{k_p}$$  \hspace{1cm} (5.2)

According to our definition, if $\Gamma^{(n)}$ consists of two disconnected components $\Gamma^{(n_1)}$ and $\Gamma^{(n_2)}$, $n = n_1 + n_2$, then the same is true about $\gamma^{(m)}$, it also consists of disconnected $\gamma^{(m_1)}$ and $\gamma^{(m_2)}$ (both can still be disconnected), $m = m_1 + m_2$, and

$$\langle \Gamma^{(n)} | g_T | \gamma^{(m)} \rangle = \langle \Gamma^{(n_1)} | g_T | \gamma^{(m_1)} \rangle \langle \Gamma^{(n_2)} | g_T | \gamma^{(m_2)} \rangle$$  \hspace{1cm} (5.3)

It is natural to introduce the product of disconnected graphs as their unification and then interpret $\langle 3 \rangle$ as the group-element property $\langle 1 \rangle$ of $g_T$.

From the definition it immediately follows that

$$\sum_j \langle \Gamma^{(n)}_{i_1\ldots i_n} | g_T | \gamma^{(m)}_{j_1\ldots j_m} \rangle \langle \gamma^{(m)}_{j_1\ldots j_m} | g_T | \gamma^{(k)}_{i_1\ldots i_k} \rangle = \langle \Gamma^{(n)}_{i_1\ldots i_n} | g_T | \gamma^{(k)}_{i_1\ldots i_k} \rangle$$  \hspace{1cm} (5.4)

for any fixed triple of vertex-subgraphs $\gamma \subset \gamma \subset \Gamma$ and given $g_T$.

The basic relation $\langle 3 \rangle$ now acquires the form:

$$\sum_{\gamma \in \gamma \subset \gamma \subset \Gamma} \langle \Gamma | g_T | \gamma \rangle \langle \gamma | g_T \cdot \gamma \rangle = \langle \Gamma | g_T + T | \gamma \rangle$$  \hspace{1cm} (5.5)

for any fixed $\tilde{\gamma} \in \gamma \Gamma$ and any two $g_T$ and $g_T \cdot \gamma$. In more detail, the multiplication relation states:

$$\sum_m \left( \sum_{\gamma \in \gamma \subset \gamma \subset \Gamma} \left( \sum_j \langle \Gamma^{(n)}_{i_1\ldots i_n} | g_T | \gamma^{(m)}_{j_1\ldots j_m} \rangle \langle \gamma^{(m)}_{j_1\ldots j_m} | g_T \cdot \gamma^{(k)}_{i_1\ldots i_k} \rangle \right) \right) = \langle \Gamma^{(n)}_{i_1\ldots i_n} | g_T + T | \gamma^{(k)}_{i_1\ldots i_k} \rangle$$  \hspace{1cm} (5.6)

Let us illustrate the relation $\langle 3 \rangle$ by a couple of examples.
5.1 Examples

1) Let \( \tilde{\gamma} = \emptyset \) and take a double-vertex graph \( \Gamma_{(3)}^{(3)} \) for \( \Gamma \). Then

\[
\langle \Gamma_{i_0;i_1 i_2} (3) | g_T | \emptyset \rangle = \sum_{m,n,\tilde{m},\tilde{n}} T_i^{(3)} G_{i_0 m n} G_{i_1 i_2 m \tilde{n}} T^{(3)}_{i_1 i_2 m \tilde{n}} = \sum_{m,n} T_i^{(3)} T_{i_1 i_2 m n} \tag{5.7}
\]

For the sake of brevity we omitted the labels (3) and (4) in coupling constants.

Four different vertex-subgraphs \( \gamma \) contribute to the sum in (5.6):

\[
\gamma^{(0)} = \emptyset; \quad \gamma^{(3)} = \Gamma_{(3)}^{(3)}; \quad \gamma^{(4)} = \Gamma_{(4)}^{(4)} \quad \text{and} \quad \gamma^{(4)} = \Gamma_{(3)}^{(3)}_{(4)} \tag{5.8}
\]

The corresponding \( \Gamma_{(3)} / \gamma^{(0)} = \Gamma_{(3)}^{(3)}; \quad \Gamma_{(3)} / \gamma^{(3)} = \Gamma_{(4)}^{(4)}; \quad \Gamma_{(3)} / \gamma^{(4)} = \Gamma_{(3)}^{(3)}_{(4)}; \quad \Gamma_{(3)} / \gamma^{(4)} = \emptyset \tag{5.9} \)

Eq.(5.6) states that

\[
\sum_{m,n} (T_{i_0}^{m n} T_{i_1 i_2 m n} \cdot 1 + T_{i_0}^{m n} \cdot T_{i_1 i_2 m n} + T_{i_0}^{m n} \cdot T_{i_1 i_2 m n} + \quad \tag{5.10}
\]

what is indeed true. Note that in this check it is important that the metric \( G_{i j} \) is the same for all the three theories \( g_T, g_T' \) and \( g_T + g_T' \).

2) Let \( \Gamma \) be a chain graph

\[
\Gamma_{(N)}^{(2)} = C_N \tag{5.11}
\]

with \( N \) vertices of valence two, connected by \( N - 1 \) propagators. Then

\[
\langle \Gamma_{(N)}^{(2)} | g_T | \emptyset \rangle_{i j} = \sum_{i_1 \ldots i_{N-1}} T_{i j_1 j_2}^{(2)} G_{i j_1}^{(2)} T_{i j_2}^{(2)} G_{i j_2}^{(2)} \ldots G_{i j_{N-1}}^{(2)} T_{i j_{N-1}}^{(2)} \tag{5.12}
\]

In this case \( \gamma \) and \( \tilde{\gamma} \) in (5.6) can be any collections of disconnected chains of the same type with the total length of no more than \( N \). If \( \tilde{\gamma} = \emptyset \), there are as many as \( 2^N \) possible choices of vertex-subgraphs \( \gamma \) in (5.6), specified by all possible subsets of \( N \) crosses in (5.11). In particular, there are \( \frac{N!}{k!(N-k)!} \) vertex-subgraphs with \( k \) vertices (connected and disconnected), and in the single-scalar case the identity (5.6) is just the binomial formula:

\[
\frac{1}{G^{N-1}} \sum_{k=0}^{N} \frac{N!}{k!(N-k)!} T_{(2)}^{k} (T_{(2)}')^{N-k} = \frac{(T_{(2)} + T_{(2)}')^N}{G^{N-1}} \tag{5.13}
\]

One can easily restore the indices \( i \) and also consider non-trivial subchains \( \tilde{\gamma} \).
Two Hopf algebras of graphs

The universal set-theoretical Hopf algebra defines a product of two graphs $\Gamma_1$ and $\Gamma_2$ to be a disconnected graph with components $\Gamma_1$ and $\Gamma_2$,

$$\Gamma_1 \cdot \Gamma_2 = \Gamma_1 \cup \Gamma_2 \quad \text{for} \quad \Gamma_1 \cap \Gamma_2 = \emptyset$$

(6.1)

(the role of unity is played by the empty graph $\emptyset$), and the coproduct

$$\Delta_{ST}(\Gamma) = \sum_{\gamma \in V(\Gamma)} \gamma \otimes \Gamma/\gamma$$

(6.2)

This Hopf algebra is both commutative and cocommutative, associative and coassociative. Because of its cocommutativity, it is not associated with any non-trivial Lie algebra (the dual algebra $\text{Shift}_M$, introduced in (5.6), is obviously commutative: $g_T g_T' = g_{T+T'} = g_{T} g_{T'}$). One can define such a Hopf algebra not only on graphs, but on any set and its subsets and we call it the “set-theory” (ST) Hopf algebra.

For functions on graphs, taking values in some commutative associative ring $\mathcal{K}$, one can define ST multiplication:

$$(F \circ_{ST} G)(\Gamma) = m((F \otimes G)(\Delta_{ST}(\Gamma))) = \sum_{\gamma \in V(\Gamma)} F(\gamma) G(\Gamma/\gamma),$$

(6.3)

(operation $m$ multiplies two components of the tensor product: $m((F(\gamma_1) \otimes G(\gamma_2)) = F(\gamma_1) G(\gamma_2)$).

Using the specifics of graphs, one can substitute vertex-subgraphs in (6.2) by box-subgraphs and construct a non-cocommutative comultiplication [13]. First of all, the matrix element of $g_T$ for contracted graph $[\Gamma/\gamma]$ obtained by contraction of a box-subgraph $\gamma^{(m)}$ in $\Gamma^{(n)}$, is given by

$$\langle [\Gamma/\gamma]_{i_1 \ldots i_n} | g_T | \emptyset \rangle = \sum_j \langle \Gamma^{(n)}_{i_1 \ldots i_n} | g_T | \gamma^{(m)}_{j_1 \ldots j_m} \rangle \Gamma^{(m)}_{j_1 \ldots j_m}$$

(6.4)

for connected $\gamma^{(m)}$,

$$\langle [\Gamma/(\gamma_1 \cdot \gamma_2)]_{i_1 \ldots i_n} | g_T | \emptyset \rangle = \sum_{j,k} \langle \Gamma^{(n)}_{i_1 \ldots i_n} | g_T | \gamma^{(m_1)}_{j_1 \ldots j_{m_1}} \cdot \gamma^{(m_2)}_{k_1 \ldots k_{m_2}} \rangle T^{(m_1)}_{j_1 \ldots j_{m_1}} T^{(m_2)}_{k_1 \ldots k_{m_2}}$$

(6.5)

for $\gamma^{(m)}$ consisting of two connected parts, and so on.
The Connes-Kreimer (CK) comultiplication

\[ \Delta_{CK} \Gamma = \emptyset \otimes \Gamma + \Gamma \otimes \emptyset + \sum_{\gamma \in B \Gamma} \gamma \otimes [\Gamma / \gamma], \quad \text{Con}(\Gamma) = 1, \]

and the CK product

\[ (F \circ_{CK} G)(\Gamma) = \text{Con}(\Gamma)=1 = F(\emptyset)G(\Gamma) + F(\Gamma)G(\emptyset) + \sum_{\gamma \in B \Gamma} F(\gamma)G([\Gamma / \gamma]) \] (6.7)

are no longer cocommutative. Thus the dual algebra is the universal enveloping of non-trivial Lie algebra. This Lie algebra, \( \text{diff}_0 \mathcal{M} \), is straightforwardly realized by vector fields on the moduli space \( \mathcal{M} \) of coupling constants. The universal model (2.1) provides a basis in \( \text{diff}_0 \mathcal{M} \), labeled by connected graphs: for any connected \( \Gamma^{(n)} \) one explicitly defines \( \hat{Z}_\Gamma \in T \mathcal{M} \) as

\[ \hat{Z}_\Gamma = \sum_i \langle T^{(n)}_{i_1...i_n} | g_T | \emptyset \rangle \frac{\partial}{\partial T^{(n)}_{i_1...i_n}} = \sum_i Z^{(n)}_{i_1...i_n} \{ T \} \frac{\partial}{\partial T^{(n)}_{i_1...i_n}} \] (6.8)

In what follows we denote by hats the vector fields and other elements of the universal enveloping \( U(\text{diff}_0 \mathcal{M}) \) to distinguish them from scalars and other elements of modules (representations) of \( \text{diff}_0 \mathcal{M} \). The commutator

\[ [\hat{Z}_{\Gamma_1}, \hat{Z}_{\Gamma_2}] = \hat{Z}_{[\Gamma_1, \Gamma_2]}, \] (6.9)

where commutator \([\Gamma_1, \Gamma_2]\) is a linear combination of all graphs \( \Gamma \), such that \([\Gamma / \Gamma_1] = \Gamma_2 \) – these enter with the coefficient +1, – or \([\Gamma / \Gamma_2] = \Gamma_1 \) – these enter with −1. (i.e. one blows any valence-\( m \) vertex in \( \Gamma_2 \) by insertion of \( \Gamma_1^{(m)} \) and any valence-\( n \) vertex in \( \Gamma_1 \) by gluing in \( \Gamma_2^{(n)} \) and takes an algebraic sum over such graphs with insertions.)

Disconnected graphs are associated with higher-order differential operators, e.g.

\[ \hat{Z}_{\Gamma^{(n_1)}, \Gamma^{(n_2)}} = \sum_{i,j} \langle T^{(n_1)}_{i_1...i_{n_1}} \Gamma^{(n_2)}_{j_1...j_{n_2}} | g_T | \emptyset \rangle \frac{\partial^2}{\partial T^{(n_1)}_{i_1...i_{n_1}} \partial T^{(n_2)}_{j_1...j_{n_2}}} = \]

\[ = \hat{Z}_{\Gamma^{(n_1)}} \hat{Z}_{\Gamma^{(n_2)}} : \neq \hat{Z}_{\Gamma^{(n_1)}} \hat{Z}_{\Gamma^{(n_2)}} \] (6.10)

In other words, we associate with disconnected graphs the normal ordered products of vector fields, corresponding to each connected component. This provides
a differential operator of certain order, equal to the number of connected components.

The vacuum graphs with no external legs define vector fields in the $T^{(0)}$ direction:

$$\hat{Z}_{\Gamma}^{(0)} = Z_{\Gamma}^{(0)} \left\{T\right\} \frac{\partial}{\partial T^{(0)}}$$ (6.11)

The matrix elements $\langle \Gamma | g_T | \gamma \rangle$ can be associated either with Beltrami differentials:

$$\mu_{\Gamma/\gamma} = \sum_{i,j} \langle \Gamma_i \cdots i_n | g_T | \gamma_j^{j_1 \cdots j_m} \rangle dT^{(m)}_{j_1 \cdots j_m} \frac{\partial}{\partial T^{(n)}_{i_1 \cdots i_n}}$$ (6.12)

(for connected $\Gamma$ and $\gamma$),

$$\mu_{\Gamma/(\gamma_1, \gamma_2)} = \sum_{i,j,k} \langle \Gamma_i \cdots i_n | g_T | \gamma_1^{j_1 \cdots j_{m_1}} \gamma_2^{k_1 \cdots k_{m_2}} \rangle dT^{(m_1)}_{j_1 \cdots j_m} dT^{(m_2)}_{k_1 \cdots k_{m_2}} \frac{\partial}{\partial T^{(n)}_{i_1 \cdots i_n}}$$ (6.13)

(for connected $\Gamma$, $\gamma_1$ and $\gamma_2$) etc; or with the vector fields

$$\hat{Z}_{\Gamma/\gamma} = \sum_{i,j} \langle \Gamma_i \cdots i_n | g_T | \gamma_j^{j_1 \cdots j_m} \rangle T^{(m)}_{j_1 \cdots j_m} \frac{\partial}{\partial T^{(n)}_{i_1 \cdots i_n}}$$ (6.14)

(for connected $\Gamma$ and $\gamma$). Note that the only difference between (6.12) and (6.14) is in the letter $d$ in front of $T^{(m)}$, but it makes a lot of difference.

Operators $g_T$ form a subgroup in abelian group $\text{Shift}_M$, which acts transitively on the moduli space $M$. They are complemented by the non-abelian subgroup $\text{Diff}_\emptyset(M)$ of $\text{Diff}_M$, which is generated by the vector fields $\hat{Z}_\Gamma$, defined in (6.8), and is the stability subgroup of the Gaussian point $T^{(n)} = 0$ (since $\langle \Gamma | g_T = 0 | \emptyset \rangle = \delta_{\Gamma, \emptyset}$ and all $\hat{Z}_\Gamma(T = 0) = 0$). The moduli space itself can be represented as a homogeneous factor-space $M = \text{Diff}_\emptyset(M)/\text{Diff}_\emptyset(M) = \text{Shift}(M)/\text{Shift}_\emptyset(M)$. The action of $\text{Diff}_\emptyset(M)$ on non-Gaussian models is relevant for description of renormalization group flows in $M$.

The Lie algebra $\text{diff}_\emptyset M$ of vector fields $\hat{Z}_\Gamma$ on entire $M$ has a variety of reductions to smaller Lie algebras on subspaces $M_{red} \subset M$, i.e. there are Lie algebras associated with smaller families of models than the universal (2.1). For example, one can consider only interactions of a given valence, i.e. all $T^{(n)} = 0$ for $n \neq k$, then vector fields (6.8) associated with connected graphs with exactly $k$ external legs form a closed Lie subalgebra. Alternative reduction is to the tree graphs ($\hat{Z}_\Gamma = 0$ for any $\Gamma$ with loops).
One can also consider the finite sets of indices $I = \{1, \ldots, N\}$ and accordingly reduced moduli spaces $\mathcal{M}^{(N)}$, in this case

$$Vect(\mathcal{M}^{(N)}) = Vect(\mathcal{M})|_{\mathcal{M}^{(N)}}$$

(6.15)

7 Inverse operators, projectors and R-Operation

Due to (5.3) partition functions are characters of the graph multiplication:

$$\langle \Gamma \cdot \Gamma_2 | g_T | \emptyset \rangle = \langle \Gamma_1 | g_T | \emptyset \rangle \langle \Gamma_2 | g_T | \emptyset \rangle, \langle \emptyset | g_T | \emptyset \rangle = 1.$$  

Characters take values in some commutative associative ring $\mathcal{K}$ and satisfy:

$$F(\emptyset) = 1,$$

$$F(\Gamma \cdot \Gamma_2) = F(\Gamma_1)F(\Gamma_2),$$

$$(F \circ_{ST} G)(\Gamma) = m((F \otimes G)(\Delta_{ST}(\Gamma))) = \sum_{\gamma \in \mathcal{V}_{\Gamma}} F(\gamma)G(\Gamma/\gamma),$$

$$(F \circ_{CK} G)(\Gamma) = m((F \otimes G)(\Delta_{CK}(\Gamma))) = \sum_{\gamma \in \mathcal{B}_{\Gamma}} F(\gamma)G([\Gamma/\gamma]).$$

(7.1)

(operation $m$ multiplies two components of the tensor product: $m((F(\gamma_1) \otimes G(\gamma_2))) = F(\gamma_1)G(\gamma_2)$).

One can define the inverses (antipodes) of a character $F$, $F_{ST}^{-1}$, $F_{CK}^{-1}$, which satisfy $F_{ST}^{-1} \circ_{ST} F(\Gamma) = \delta_{\Gamma, \emptyset}$, $F_{CK}^{-1} \circ_{CK} F(\Gamma) = \delta_{\Gamma, \emptyset}$, by recursive formulas:

$$F_{ST}^{-1}(\emptyset) = F_{CK}^{-1}(\emptyset) = 1,$$

$$F_{ST}^{-1}(\Gamma) = -F(\Gamma) - \sum_{\gamma \in \mathcal{V}_{\Gamma}, \gamma \neq \emptyset, \Gamma} F_{ST}^{-1}(\gamma)F(\Gamma/\gamma),$$

$$F_{CK}^{-1}(\Gamma) = \sum_{\gamma \in \mathcal{B}_{\Gamma}} F_{CK}^{-1}(\gamma)F([\Gamma/\gamma]).$$

(7.2)

According to (6.6), if $F(\Gamma) = \langle \Gamma | g_T | \emptyset \rangle$, then $F_{ST}^{-1}(\Gamma) = \langle \Gamma | g_{-T} | \emptyset \rangle$, but $F_{CK}^{-1}(\Gamma)$ is given by a more sophisticated expression. In fact, eq. (7.2) for $F_{CK}^{-1}(\Gamma)$ is closely associated with Bogolubov’s recursive formula, defining the $R$-operation [19].

Assume that the ring $\mathcal{K}$ as a linear space can be decomposed into two components, $\mathcal{K} = \mathcal{K}_1 \oplus \mathcal{K}_2$, with the help of projectors $\mathcal{P}_\pm$, $\mathcal{P}_\pm^2 = \mathcal{P}_\pm$, $\mathcal{P}_- = I - \mathcal{P}_+$; $\mathcal{K}_\pm = \mathcal{P}_\pm \mathcal{K}$. These projectors can be used to define the “$\mathcal{P}$-inverse” ($\mathcal{P}$-antipode) $PF^{-1}$ of $F(\Gamma)$ [3].
\[ \mathcal{P}_- \left( (PF^{-1}_\odot \odot F)(\Gamma) - \delta_{\Gamma,\emptyset} \right) = 0, \]
\[ \mathcal{P}_+ \left( PF^{-1}_\odot(\Gamma) \right) = 0 \quad (7.3) \]

The second condition makes the definition of \( \mathcal{P} \)-antipode unambiguous. The ordinary inverse \( F^{-1}_\odot \) is associated with the trivial projector \( \mathcal{P}_+ = 0 \). One can easily write down recursive formulae for the \( \mathcal{P} \)-antipodes for comultiplications \( \odot_{ST} \) and \( \odot_{CK} \) by applying \( \mathcal{P}_- \) to the r.h.s. of (7.2):

\[ PF^{-1}_{ST}(\Gamma) = -\mathcal{P}_- \left( F(\Gamma) + \sum_{\gamma \in \Gamma; \gamma \neq \emptyset, \Gamma} PF^{-1}_{ST}(\gamma)F(\Gamma/\gamma) \right), \]
\[ PF^{-1}_{CK}(\Gamma) \overset{\text{Con}(\Gamma)=1}{=} -\mathcal{P}_- \left( F(\Gamma) + \sum_{\gamma \in BF} PF^{-1}_{CK}(\gamma)F([\Gamma]/\gamma) \right) \quad (7.4) \]

For projector \( \mathcal{P}_+ \), possessing additional triangular property w.r.t. multiplication in \( K \), namely

\[ K_+ \cdot K_+ \subset K_+, \quad K_- \cdot K_- \subset K_- \quad (7.5) \]

(i.e. the product of any two elements from \( K_+ \) lies again in \( K_+ \) and similarly for \( K_- \)), the ST \( \mathcal{P} \)-inverse of \( F(\Gamma) \) is a character whenever \( F(\Gamma) \) is a character:

\[ PF^{-1}_{ST}(\Gamma_1 \cdot \Gamma_2) = PF^{-1}_{ST}(\Gamma_1)PF^{-1}_{ST}(\Gamma_2) \quad (7.6) \]

if

\[ F(\Gamma_1 \cdot \Gamma_2) = F(\Gamma_1)F(\Gamma_2) \quad \forall \Gamma_1, \Gamma_2 \quad (7.7) \]

Indeed, assume that this is true for all smaller vertex-subgraphs of \( \Gamma_1 \) and \( \Gamma_2 \). Then

\[ PF^{-1}_{ST}(\Gamma_1 \cdot \Gamma_2) = -\mathcal{P}_- \left( F(\Gamma_1 \cdot \Gamma_2) + \right. \]
\[ + \sum_{\gamma_1 \in BF_1} \sum_{\gamma_2 \in BF_2} PF^{-1}_{ST}(\gamma_1)PF^{-1}_{ST}(\gamma_2)F(\Gamma_1/\gamma_1)F(\Gamma_2/\gamma_2) - \]
\[ - F(\Gamma_1)F(\Gamma_2) - PF^{-1}_{ST}(\Gamma_1)PF^{-1}_{ST}(\Gamma_2) \right) \quad (7.8) \]

20
The last two items at the r.h.s. subtract the contributions from $\gamma_1 \cdot \gamma_2 = \emptyset$ and $\gamma_1 \cdot \gamma_2 = \Gamma_1 \cdot \Gamma_2$. The double sum in (7.8) is equal to the product of two sums, defining the ST $\mathcal{P}$-inverses of $\Gamma_1$ and $\Gamma_2$, which (the sums) are both $\mathcal{P}$-positive. Due to triangularity the product is also $\mathcal{P}$-positive and is eliminated by $\mathcal{P}^-$. Therefore (7.8) states that

$$PF_{ST}^{-1}(\Gamma_1 \cdot \Gamma_2) = \mathcal{P}_-(PF_{ST}^{-1}(\Gamma_1)PF_{ST}^{-1}(\Gamma_2)) = PF_{ST}^{-1}(\Gamma_1)PF_{ST}^{-1}(\Gamma_2) \quad (7.9)$$

The last equality is again implied by triangularity, since both $\mathcal{P}$-inverses are $\mathcal{P}$-negative.

Not every projector is triangular, for example projection on positive numbers in the ring of reals is not triangular: the product of two negatives is no longer negative. A natural triangular projector exists in a ring of Laurent series $\{A = \sum_{k=-N}^{\infty} a_k z^k\}$: $\mathcal{P}_+ A = \sum_{k=0}^{\infty} a_k z^k$. The difference $\mathcal{R} = \mathcal{P}_+ - \mathcal{P}_-$ is the $r$-matrix, widely used in the theory of integrable systems and its applications (see, for example, [20] and also [1, 21]). To get a field-theory model with such $K$ one can, for example, consider the $z$-dependent couplings $T^{(n)} = \sum_{k=-N}^{\infty} T^{(n)}_k z^k$ in (2.1). In the study of continuous field theory $z$ rather enters through regularization of infinite sums (integrals) over indices $i$ in (2.1): it can be identified with $d - d_{crit}$ for dimensional regularization [13] or with $1/M$ for Pauli-Villars regularization etc.

According to (7.3), the $R$-operation

$$F(\Gamma) \longrightarrow R\odot F(\Gamma) = (PF_{\odot}^{-1} \odot F)(\Gamma),$$

$$\mathcal{P}_-(R\odot F(\Gamma)) = 0, \quad (7.10)$$

acting on the space of functions of graphs, converts any function into a $\mathcal{P}$-positive (“finite”) one. Moreover, since $\odot$-product of characters is again a character, it converts characters into characters. The main claim of [13] is that eq.(7.10) for $\odot_{CK}$ can be considered as group-theory interpretation of Bogolubov’s recursion formula [19]. In sec.8 we shall see that more relevant in generic case is the corepresentation $\odot_{CK}$.

Of course, from algebraic perspective there is nothing special about continuous theory, divergencies and dimensional regularization: the only things that matter are algebraic structures and triangular projectors.

8 Representations of $\text{diff}_\emptyset \mathcal{M}$ and $\mathcal{U}(\text{diff}_\emptyset \mathcal{M})$ in differential operators on $\mathcal{M}$

Returning to the beginning of sec.7, model (2.1) provides $Z_T\{T\}$ for the role of characters, if these quantities are considered as functions of $\Gamma$, and $T$-dependence is not taken into account. Similar treatment can be given to vacuum diagrams
in generic (2.1). However, it is more adequate to treat \( Z_\Gamma \{ T \} = \langle \Gamma | T \rangle \) as describing a transformation between the functions of \( T \) and functions of \( \Gamma \) (not obligatory characters), as suggested in sec.3 above.

Consider the action of vector fields on linear modules over \( \mathcal{M} \). Namely, a vector field

\[
\hat{V} = \sum_n \left( \sum_{i_1, \ldots, i_n} V_{i_1 \ldots i_n}^{(n)} (T) \frac{\partial}{\partial T_{i_1 \ldots i_n}} \right)
\]

(8.1)
can act on a function of \( T \)-variables \( F \{ T \} \) with or without free indices:

\[
F \{ T \} \rightarrow \hat{V} \{ T \} F \{ T \}
\]

(8.2)

Now we can exploit the power given by the use of the universal model (2.1). It provides a large enough set of functions on \( \mathcal{M} \) to establish the one-to-one correspondence between linear combinations of graphs and invariant functions of coupling constants (while for smaller models the set of such functions is much smaller: graphs label different types of contracting indices, and there should be many enough indices to distinguish between different contractions). Because of this, every invariant (i.e. with all indices \( i \) contracted with the help of the metric \( G_{ij} \) ) function on \( \mathcal{M} \) can be uniquely decomposed into a sum over graphs of the basic functions \( Z_\Gamma \{ T \} \), introduced in sec.2 (of course, such expansions survive certain reductions of \( \mathcal{M} \), but this is a separate story). Actually, functions are decomposed into sums over vacuum graphs \( \Gamma^{(0)} \) without external legs,

\[
F \{ T \} = \sum_{\Gamma^{(0)}} F(\Gamma) Z_\Gamma \{ T \}
\]

(8.3)

with \( T \)-independent coefficients \( F_\Gamma \); vector fields – over connected graphs with any number of external legs,

\[
\hat{V} \{ T \} = \sum_{n \geq 1} \left( \sum_{\text{connected } \Gamma^{(n)}} V(\Gamma) \hat{Z}_\Gamma \{ T \} \right)
\]

(8.4)

the \( k \)-differentials – over graphs with \( k \) connected components and non-vanishing number of external legs in each component,

\[
\hat{W}_k \{ T \} := \sum_{n_1, \ldots, n_k \geq 1} \left( \sum_{\text{connected } \Gamma_1^{(n_1)} \ldots \Gamma_k^{(n_k)}} W(\Gamma_1 \cdot \ldots \cdot \Gamma_k) \hat{Z}_{\Gamma_1} \{ T \} \cdot \ldots \cdot \hat{Z}_{\Gamma_k} \{ T \} \right)
\]

(8.5)
generic elements of the universal module (generic differential operators on \(\mathcal{M}\)) over all possible graphs (with any number of connected components and external legs). In what follows \(F\{T\}\) can be arbitrary element of the universal module. Also, we assume that for a vector field the coefficients \(V(\Gamma)\) are defined for all graphs \(\Gamma\), just \(V(\Gamma) = 0\) if \(\Gamma\) is not connected (of course, \(V(\Gamma)\) is not a character, characters are associated with group elements \(\hat{G} = e^V\), not vector fields themselves).

The result of the action of \(\hat{V}\) on \(F\) can also be decomposed in the basis \(Z_{\Gamma}\{T\}\),

\[
\hat{V}\{T\}F\{T\} = \sum_{\Gamma} (\hat{V}F)(\Gamma)Z_{\Gamma}\{T\}
\]

and one obtains a relation between the coefficients \((\hat{V}F)(\Gamma), F(\Gamma)\) and \(V(\Gamma)\):

\[
\sum_{\Gamma}(\hat{V}F)(\Gamma)Z_{\Gamma}\{T\} = \sum_{\Gamma'} \sum_{\gamma} V(\gamma)F(\Gamma') \left(\hat{Z}_{\gamma}(T)Z_{\Gamma'}\{T\}\right),
\]

and

\[
\hat{Z}_{\gamma}(T)Z_{\Gamma'}\{T\} = \sum_{\Gamma: \Gamma' = [\Gamma/\gamma]} Z_{\Gamma}\{T\}
\]

we get a convolution formula

\[
(\hat{V}F)(\Gamma) = \sum_{\gamma \in B_{\Gamma}} V(\gamma)F([\Gamma/\gamma]) = (V \hat{\circ}_{CK} F)(\Gamma)
\]

Operation \(\hat{\circ}_{CK}\),

\[
(W \hat{\circ}_{CK} F)(\Gamma) = m((W \otimes F)(\hat{\Delta}_{CK} \Gamma)) = \Con(\Gamma) = 1
\]

\[
W(\emptyset)F(\Gamma) + \sum_{\gamma \in B_{\Gamma}} W(\gamma)F([\Gamma/\gamma]),
\]

is expressed in terms of the corepresentation of the CK Hopf algebra of graphs,

\[
\hat{\Delta}_{CK} \Gamma = \emptyset \otimes \Gamma + \sum_{\gamma \in B_{\Gamma}} \gamma \otimes [\Gamma/\gamma], \quad \Con(\Gamma) = 1,
\]

the same way as \(\circ_{CK}\), eq. (6.7), is expressed through the comultiplication.
\[
\Delta_{CK} \Gamma = \emptyset \otimes \Gamma + \Gamma \otimes \emptyset + \sum_{\gamma \in B^\Gamma} \gamma \otimes [\Gamma/\gamma], \quad \text{Con}(\Gamma) = 1 \quad (8.12)
\]

In (8.9) \( \hat{V} \) is a vector field, therefore \( V(\emptyset) = 0 \). The difference between comultiplication \( \Delta \) and corepresentation \( \hat{\Delta} \) is in associativity conditions:

\[
(\Delta \otimes \text{id})\Delta = (\text{id} \otimes \Delta)\Delta \quad (8.13)
\]

for \( \Delta \) and

\[
(\Delta \otimes \text{id})\hat{\Delta} = (\text{id} \otimes \hat{\Delta})\hat{\Delta} \quad (8.14)
\]

for \( \hat{\Delta} \).

Repeated application of formula (8.9) defines the action of products and normal ordered products of vector fields on \( F \). For two vectors, since

\[
\hat{Z}_{\gamma_1} \left( \hat{Z}_{\gamma_2} Z_{\Gamma''} \right) = \sum_{\Gamma \colon [\Gamma/\gamma_1] = \Gamma''} \left( \sum_{\Gamma' \colon [\Gamma'/\gamma_2] = \Gamma''} Z_{\Gamma'} \right) \quad (8.15)
\]

and

\[
: \hat{Z}_{\gamma_1} \hat{Z}_{\gamma_2} : Z_{\Gamma'} = \sum_{\Gamma \colon [\Gamma/(\gamma_1 \cdot \gamma_2)] = \Gamma''} Z_{\Gamma'} \quad (8.16)
\]

we have:

\[
(\hat{V}_1 \hat{V}_2 F)(\Gamma) = \sum_{\gamma_2 \in B^\Gamma} V_2(\gamma_2) \left( \sum_{\gamma_1 \in B^{[\Gamma/\gamma_2]}} V_1(\gamma_1) F([\Gamma/\gamma_2]/\gamma_1) \right) \quad (8.17)
\]

and

\[
(\hat{V}_1 \hat{V}_2 : F)(\Gamma) = \sum_{\gamma_1 \cdot \gamma_2 \in B^\Gamma} V_1(\gamma_1) V_2(\gamma_2) F([\Gamma/(\gamma_1 \cdot \gamma_2)]) = (\hat{V}_1 V_2 : \hat{\circ}_{CK} F)(\Gamma) \quad (8.18)
\]

Note that in these formulas \([\Gamma/\gamma_2]/\gamma_1 \neq [\Gamma/(\gamma_1 \cdot \gamma_2)]\): a graph \( \gamma_1 \) can appear after contraction \([\Gamma/\gamma_2]\) is made. Relatively simple formulae in terms of
the corepresentation $\hat{\circ}_{CK}$ exist only for the normal ordered elements: $\hat{W} \in \mathcal{U}(\text{diff}_\emptyset \mathcal{M})$:

$$\langle \hat{W} : F \rangle(\Gamma) \overset{\text{Con}(\Gamma)=1}{=} W : (\emptyset)F(\Gamma) + \sum_{\gamma \in \Gamma} : W : (\gamma)F([\Gamma/\gamma]) = \langle \hat{W} : \hat{\circ}_{CK}F \rangle(\Gamma)$$  (8.19)

Of special interest for us are specific elements of $\mathcal{U}(\text{diff}_\emptyset \mathcal{M})$, which are the group elements, and form the diffeomorphism group $\text{Diff}_\emptyset \mathcal{M}$.

Given a vector field $\hat{V} = V^\alpha \partial_\alpha$ ($\alpha$ is a multiindex, labeling connected graph with indices or any linear combinations of such graphs), one can make an element of $\text{Diff}_\emptyset (\mathcal{M})$ by exponentiation:

$$G_{\hat{V}} = e^{\hat{V}} = \sum_{n=0}^{\infty} \frac{\hat{V}^n}{n!}$$  (8.20)

However, it is not normal ordered, and the action of $G_{\hat{V}}$ on $F(\Gamma)$ is described by sophisticated expression:

$$(G_{\hat{V}}F)(\Gamma) = \sum_{n=0}^{\infty} \frac{1}{n!} \left( \sum_{\{\gamma\}_n} V(\gamma_1) \ldots V(\gamma_n)F([\Gamma/\{\gamma\}_n]) \right)$$  (8.21)

where $\{\gamma\}_n$ denotes a hierarchy of subgraphs $\gamma_n \in \mathcal{B}_\Gamma$, $\gamma_{n-1} \in \mathcal{B}[\Gamma/\gamma_n]$, $\ldots$, $\gamma_1 \in \mathcal{B}[[\ldots[[\Gamma/\gamma_n]/\gamma_{n-1}] / \ldots]/\gamma_2]$.

One can instead expand $G_{\hat{V}}$ in normal order constituents with the help of a forest formula:

$$G_{\hat{V}} = e^{\hat{V}} = \sum_{n=0}^{\infty} \frac{\hat{V}^n}{n!} = 1 + V^\alpha \partial_\alpha + \frac{1}{2} V^\gamma \partial_\gamma V^\alpha \partial_\alpha + \frac{1}{6} V^\gamma \partial_\gamma V^\beta \partial_\beta V^\alpha \partial_\alpha + \ldots =$$

$$= 1 + \left( V^\alpha + \frac{1}{2} V^\gamma (\partial_\gamma V^\alpha) + \frac{1}{6} V^\gamma (\partial_\gamma V^\beta)(\partial_\beta V^\alpha) + \ldots \right)$$

Note that the only component of $\mathcal{U}(\text{diff}_\emptyset \mathcal{M})$ which has $W(\emptyset) \neq 0$ is a counity, i.e. $W(T) = \text{const}$. Non-trivial functions $\hat{f}(T) \notin \mathcal{U}(\text{diff}_\emptyset \mathcal{M})$, and ordinary product of functions is not expressible in terms of coproduct $\hat{\circ}_{CK}$. Instead the scalar $f(T) = \sum_{T^{(0)}} f(T) Z_T \{T\}$ acts on $F\{T\}$ as

$$\langle \hat{f} \rangle F(\Gamma) = \sum_{\Gamma_1, \Gamma_2} f(\Gamma_1)F(\Gamma_2)$$

In particular, for connected $\Gamma$, $\langle \hat{f} \rangle F(\Gamma) = f(\Gamma) + F(\Gamma)$. 25
\[ + \frac{1}{2} V^\gamma (\partial_\gamma V^\alpha) + \ldots \] \[ + \frac{1}{6} V^\beta V^\gamma (\partial_\beta \partial_\gamma V^\alpha) + \ldots \] \[ + \frac{1}{2} \left( V^\alpha + \frac{1}{2} V^\gamma (\partial_\gamma V^\alpha) + \ldots \right) \left( V^\beta + \frac{1}{2} V^\gamma (\partial_\gamma V^\beta) + \ldots \right) \partial_\alpha \partial_\beta + \]
\[ + \frac{1}{6} (V^\alpha + \ldots) (V^\beta + \ldots) (V^\gamma + \ldots) \partial_\alpha \partial_\beta \partial_\gamma + \ldots = \]
\[ = 1 + \sum_{\mathcal{F} : \text{Tree}(\mathcal{F})!} \frac{1}{\sigma_\mathcal{T}!} \prod_{\mathcal{T} \in \mathcal{F}} \hat{V}_\mathcal{T} : \quad (8.22) \]

The forest \( \mathcal{F} \) is an ordered set of rooted trees. Rooted tree has a single external leg (root), all other external legs end at the valence-one vertices. \( \text{Tree}(\mathcal{F}) \) is the number of trees in the forest, and \( \text{Vert}(\mathcal{T}) \) is the number of vertices in the tree \( \mathcal{T} \). For every rooted tree \( \sigma_\mathcal{T} \) is the symmetry factor (the order of the discrete group which interchanges subtrees, leaving the tree intact), while the tree-factorial \( 22 \) is defined iteratively: \( \mathcal{T}! = \text{Vert}(\mathcal{T}) ! \prod_\mathcal{T}_a \sigma_\mathcal{T}_a ! \), where \( \mathcal{T}_a \) are the root subtrees formed after the root is cut away. In every vertex of a tree stands the vector field \( \hat{V} \), acting on the neighbor vertex downwards (in the direction to the root), and not further. Then with every tree we associate a vector field \( \hat{V}_\mathcal{T} \), which contains the \( \text{Vert}(\mathcal{T}) \)’s power of \( \hat{V} \) and \( \text{Link}(\mathcal{T}) \) derivatives (\( \text{Link}(\mathcal{T}) \) is the number of links in the tree). For example, for the 1-vertex (\( \mathcal{T}_1 \)), 2-vertex (\( \mathcal{T}_2 \)) and 3-vertex/2-branch (\( \mathcal{T}_Y \)) trees:

\[ V(\mathcal{T}_1) = V^\alpha \partial_\alpha, \]
\[ V(\mathcal{T}_2) = V^\alpha (\partial_\alpha V^\beta) \partial_\beta, \]
\[ V(\mathcal{T}_Y) = V^\alpha V^\beta (\partial_\alpha \partial_\beta V^\gamma) \partial_\gamma, \]
\[ \quad (8.23) \]

etc. With a forest we associate a differential operator, which is a normal-ordered product of vector fields \( \hat{V}_\mathcal{T} \) over the trees (as usual, normal ordering means, that all derivatives are written to the right of \( V^\alpha \)’s, this is a coordinate-dependent operation, e.g.: \( \hat{V}^3 := V^\alpha V^\beta V^\gamma \partial_\alpha \partial_\beta \partial_\gamma \)).

One can apply (8.19) to obtain an alternative expression to (8.21) in terms of the corepresentation \( \hat{\circ}_{CK} \). Complexity of the formula is now encoded in the sum over forests. One can efficiently handle this complexity by the following trick. Since \( G_{\hat{V}} = e^{\hat{V}} \) is a diffeomorphism of \( \mathcal{M} \), for any \( F\{T\} \) we have:

\[ (G_{\hat{V}} F)\{T\} = F\{T + \hat{V}(T)\} \]
\[ \quad (8.24) \]

with

\[ \hat{V}^{(n)}_{i_1 \ldots i_n} = \left( e^{\hat{V}} - 1 \right) T^{(n)}_{i_1 \ldots i_n} = \left( \sum_\mathcal{T} \frac{\hat{V}_{\mathcal{T}}}{\sigma_\mathcal{T}!} \right) T^{(n)}_{i_1 \ldots i_n} \]
\[ \quad (8.25) \]
The first equality in (8.24) is obtained by substitution of \( T(n)_{i_1...i_n} \) instead of \( F\{T\} \) in (8.23), the second equality is implied by the forest formula (8.22), because a normal product of two or more vector fields annihilates \( T(n)_{i_1...i_n} \). Now introduce a new vector field

\[
\hat{V}\{T\} = \sum_n \left( \sum_{i_1...i_n} \hat{V}(n)_{i_1...i_n} \frac{\partial}{\partial T(n)_{i_1...i_n}} \right) = \sum_{\text{connected } \Gamma} \hat{V}(\Gamma) \hat{Z}_\Gamma\{T\}, \quad (8.26)
\]

such that the shift operator

\[
G_{\hat{V}} = e^{\hat{V}} = : e^{\hat{V}} : \quad (8.27)
\]

Equality (8.27) is implied by (8.24) and by Taylor expansion

\[
F\{T + \hat{V}(T)\} = : e^{\hat{V}} : F\{T\} \quad (8.28)
\]

Now we can make use of (8.19) to obtain a simple substitute for (8.21):

\[
(G_{\hat{V}}F)(\Gamma) = \sum_{\text{non-intersecting } \gamma_1...\gamma_n \in B\Gamma} \hat{V}(\gamma_1)...\hat{V}(\gamma_n)F([\Gamma/(\gamma_1\cdot\cdot\cdot\gamma_n)])
\]

\[
= \sum_{n=0}^{\infty} \sum_{\text{non-intersecting } \gamma_1...\gamma_n \in B\Gamma} \hat{V}(\gamma_1)...\hat{V}(\gamma_n)F([\Gamma/(\gamma_1\cdot\cdot\cdot\gamma_n)]) \quad (8.29)
\]

### 9 Bogolubov’s recursion and renormalized Lagrangian

One can apply diffeomorphisms in moduli space to “improve” partition functions. This is important if one wants to eliminate undesired dependence on one or another parameter of the theory, like ultraviolet cut-off in continuous local field models. Basically, one needs to project the entire moduli space \( \mathcal{M} \) onto certain subspace \( \mathcal{M}_{\text{ren}} \) of “renormalized models”. The problem is that parameter-dependence arises in partition functions, and arbitrary elimination of unwanted parameters from particular correlators can break the relation to Lagrangian formalism and moduli space. It is exactly the problem, which is resolved by Bogolubov’s \( R \)-operation [19], and which can be most straightforwardly described in terms of diffeomorphisms of \( \mathcal{M} \).

The \( R \)-operation can be formulated as follows:
Given (triangular) projectors $P_\pm$ in the ring $K$ (where matrix elements and partition functions are taking values) and a function $F\{T\}$ (with or without free indices, i.e., any element of the universal module over $M$), one finds a specific diffeomorphism $G_\rho \in \text{Diff}_p M$ which makes $F\{T\}$ $P$-positive:

$$P_- \left( F\{T + \hat{P}(T)\} \right) = 0, \quad (9.1)$$

$$F\{T + \hat{P}(T)\} = e^\rho F\{T\} = (G_\rho F\{T\}) \quad (9.2)$$

To define such diffeomorphism unambiguously, one imposes additional constraint on $\hat{P}\{T\}$, for example,

$$P_+ \left( \hat{P}^{(n)}_{i_1 \ldots i_n} \right) = 0 \quad \forall \ n; \ i_1, \ldots, i_n \quad (9.3)$$

(see eq. (9.6) below for a more adequate constraint). Some constraint of this type is needed to distinguish between “renormalizations”, needed to eliminate $P$-negative contributions to the correlation functions from arbitrary diffeomorphisms of $M$, which can map $P$-positive models into other $P$-positive ones.

Eqs. (9.1-9.3) define the Bogolubov’s $R$-operation for any projector $P_+$. One can apply the machinery of the previous sections to rewrite (9.2) either in terms of Gauss-Birkhoff decomposition of the shift operator,

$$g_{T+\hat{P}(T)} = g_T g_{\hat{P}(T)} \quad (9.4)$$

(decomposing $P$-positive renormalized model into the bare one and $P$-negative counterterm model), or in terms of CK algebra of functions on graphs. In the last case one can use any of the three representations (8.21), (8.22) or (8.29). The most convenient is the third choice, and it is exactly the one providing the Bogolubov’s recursion formula. Eq. (8.29) can indeed be rewritten in the form of a recurrent relation for $\hat{P}(\Gamma)$, expressing it through $\hat{P}(\gamma)$ for smaller box-subgraphs $\gamma$ (with less vertices), provided $F(\Gamma)$ does not vanish on elementary vertices $[\Gamma/\Gamma]$. Indeed, one can extract from the r.h.s. of (8.29) two items: one with $n = 0$ and another with $n = 1$ and $\gamma = \Gamma$. Then we obtain:

$$\hat{P}(\Gamma) F([\Gamma/\Gamma]) \overset{\text{Con}(\Gamma) = 1}{=} -P_- \left( F(\Gamma) + \sum_{\gamma \in S^\Gamma; \gamma \neq \Gamma} G_\rho(\gamma) F([\Gamma/\gamma]) \right) =$$

$$= -P_- \left( F(\Gamma) + \sum_{n = 1} F(\Gamma) + \sum_{\gamma_1, \ldots, \gamma_n \in S^\Gamma} \hat{P}(\gamma_1) \ldots \hat{P}(\gamma_n) F([\Gamma/(\gamma_1 \cdot \ldots \cdot \gamma_n)]) \right) \quad (9.5)$$

\(^5\) In notation of [8.3] $G_\rho(\Gamma) = C(\Gamma).$
Here $\gamma \in \mathcal{B}^\Gamma$ means that the sum goes over non-intersecting box-subgraphs $\gamma_1, \ldots, \gamma_n \neq \Gamma$. We also assumed that (9.3) is in fact substituted by a more sophisticated constraint

$$\mathcal{P}_+ \left( \tilde{P}(\Gamma) F([\Gamma/\Gamma]) \right) = 0$$

Then we can omit $\mathcal{P}_-$ acting on the l.h.s. of (9.3). Eq.(9.3) provides a recursion formula for $\tilde{P}(\Gamma)$ if $F([\Gamma/\Gamma]) \neq 0$ whenever the r.h.s. of (9.3) is non-vanishing.

In fact, this is a necessary requirement for renormalizability of the theory (of a particular reduction of the universal model (2.1)): all the elementary vertices $[\Gamma/\Gamma]$ should be included into the bare Lagrangian, if they have the structure which can be generated in perturbation theory with $\mathcal{P}$-negative coefficients (in traditional language of quantum field theory: if there are divergent diagrams with a given number of external legs and external-momenta dependence, an elementary vertex with such valence and momentum dependence should be included into the bare Lagrangian).

Recursive formula (9.5) has a formal solution in terms of its own forest formula, involving decorated rooted trees. For connected graph $\Gamma$ consider a sequence of embedded box-subgraphs, complementary to $\{\gamma\}_n$ in (8.21), $\{\{\gamma\}\}_n$:

$$\gamma_0 = \Gamma, \gamma_1 \in \mathcal{B}^\Gamma, \gamma_2 \in \mathcal{B}^{\gamma_1} \subset \mathcal{B}^\Gamma, \ldots, \gamma_n \in \mathcal{B}^{\gamma_{n-1}} \subset \ldots \subset \mathcal{B}^{\gamma_1} \subset \mathcal{B}^\Gamma.$$ It corresponds to a collection of non-intersecting boxes, which can now (in variance with the set, used in the definition of particular box-subgraph) lie on inside another. Such collection allows one to build a decorated rooted tree $\mathcal{T}$ [13]. If $\Gamma$ is disconnected, there will be trees, associated with every connected component. With lower site of each box one associates a vertex of the tree, two vertices are connected by a link if one of the corresponding boxes lies immediately inside another (i.e. there are no boxes in between the two). The root link ends at a vertex, associated with $\gamma_0 = \Gamma$. According to this construction, every vertex of the tree is associated with connected box-subgraph $\hat{\gamma}_k \subset \gamma_k$ ($\gamma_k$ need not be connected), and there is exactly one link, going downwards (towards the root, i.e. associated with the neighbor bigger box) and connecting $\hat{\gamma}_k$ to some $\hat{\gamma}_{k-1}$, and unrestricted number of links, going upwards and connecting $\hat{\gamma}_k$ to some collection $\hat{\gamma}_{k+1}, \ldots, \hat{\gamma}_{k+1} \subset \gamma_{k+1}$. The solution to (9.5) associates with every vertex $\hat{\gamma}_k$ an operator $(F([\hat{\gamma}_k/\hat{\gamma}_k]))^{-1} (-\mathcal{P}_-) F([\hat{\gamma}_k/\hat{\gamma}_{k+1} \cdot \ldots \cdot \hat{\gamma}_{k+1}])$, where projector $\mathcal{P}_-$ acts upwards along the branches of the tree. The root vertex $\hat{\gamma}_0$ (i.e. a connected component of $\Gamma$) contributes just $F([\gamma_0/(\hat{\gamma}_1 \cdot \ldots \cdot \hat{\gamma}_1^{(0)})])$. In these terms the result of $R$-operation can be written as follows [13]:

$$\langle G_{\hat{\rho}} F \rangle(\Gamma) = \sum_{\mathcal{F}_T} \prod_{T \in \mathcal{F}_T} \prod_{\text{vertices of } T} \frac{1}{F([\hat{\gamma}_k/\hat{\gamma}_k])} (-\mathcal{P}_-) F([\hat{\gamma}_k/\hat{\gamma}_{k+1} \cdot \ldots \cdot \hat{\gamma}_{k+1}])$$

(9.7)
Arrow over the product sign means that the product is ordered along the branches.

Importance of Bogolubov’s recursion in the space of function $F\{T\}$ is that it converts partition functions ($\tau$-functions) into partition functions, while arbitrary subtraction procedure, like the naive $F_T\{T\} \rightarrow \mathcal{P}_-(F_T\{T\})$, does not have this property: it may not be represented as an action of $\text{Diff} \mathcal{M}$ and no operator $g_{T+\hat{P}(T)}$ results from such a subtraction.

It deserves noting that $\left(F\{T + \hat{P}(T)\}\right)_\Gamma \neq F_T\{T + \hat{P}(T)\}$. For example, for the simplest chain graph $C_1$ (one valence-two vertex)

$$Z^{ij}_{C_1}\{T + \hat{P}(T)\} = T_{(2)}^{ij} + \hat{P}_{(2)}^{ij}\{T\} =$$

$$= Z^{ij}_{C_1}\{T\} + \hat{P}^{ij}_{C_1}\{T\} + \hat{P}^{ij}_{C_2}\{T\} + \ldots, \quad (9.8)$$

while

$$\left(Z^{ij}\{T + \hat{P}(T)\}\right)_{C_1} = Z^{ij}_{C_1}\{T\} + \hat{P}^{ij}_{C_1}\{T\} \quad (9.9)$$

Because of this difference one sometime says that renormalization of Lagrangian does not make contribution of each individual graph $\mathcal{P}$-positive (in the sense that sometime $\mathcal{P}_-(Z_T\{T + \hat{P}(T)\}) \neq 0$), while $R$-operation does (in the sense that always $\mathcal{P}_-(Z\{T + \hat{P}(T)\}) \Gamma = 0$). However, as we just explained, if interpreted properly, renormalization of Lagrangian and $R$-operation are just the same.

From here on – if one wants to continue – one needs to split the universal model (2.1) into smaller universality classes, which differ by the choice and properties of the sets $I$ (where indices $i$ in (2.1) take values), especially by the ways the possibly-divergent sums over indices (e.g. integrals over momenta) are regularized (it still makes sense to keep the full set of coupling constants $T_{i_1,\ldots,i_n}^{(n)}$). The most interesting projectors exploit particular properties of particular $I$’s. They can act non-trivially on the basic functions $Z_I\{T\}$, not only on the coefficient functions $F(\Gamma)$ (this actually happens in the case of regularized continuous field models, at least in the naive approach). For particular projectors the counter-terms $\hat{P}(\Gamma)$ can vanish for certain classes of graphs (for divergency-eliminating projectors in renormalizable field models contributing are only graphs with loops and restricted number of external legs). Given $I$ and $\mathcal{P}_\pm$, one can say that the $R$-operation (9.2) provides a full set of $\mathcal{P}$-positive functions on $\mathcal{M}(I)$: a linear basis is provided by the set of $Z_I\{T + \hat{P}(T)\}$ (generically this space is smaller than the one with the basis $Z_T\{T\}$).
10 Conclusion

We described the relation between the algebraic structures, introduced by A.Connes and D.Kreimer, and the generic bilinear relations (Hirota equations) for effective actions in quantum field theory. We discussed two groups acting on the moduli space $\mathcal{M}$ of theories: one, essentially commutative $\text{Shift}\mathcal{M}$, acting transitively on $\mathcal{M}$ and responsible for bilinear relations; another, the non-commutative stability subgroup of the Gaussian point $\text{Diff}_0\mathcal{M}$ in the diffeomorphism group $\text{Diff}\mathcal{M}$, is associated with Lie algebra of vector fields on $\mathcal{M}$, it is related to the CK Hopf algebra of graphs, to Bogolubov’s $R$-operation and to renormalization group flows. Bogolubov’s $R$-operation is defined in terms of projector operators and can be expressed as renormalization of the action ($T$-dependent shift of the coupling constants $T$). This study provides a long awaited support to the idea of hidden integrability of non-perturbative quantum phenomena from the field of conventional field theory (Feynman diagram technique).

It also opens a way for the study of analogous phenomena in perturbative string theory, where graphs are substituted by open Riemann surfaces and CK Hopf algebra has interesting generalizations (an infinitesimal deformation in that direction is to the Hopf algebra of fat graphs, associated with the universal matrix model (2.5).

The old belief that the moduli space $\mathcal{M}$ of theories and diffeomorphism group $\text{Diff}\mathcal{M}$ are indeed very similar to conventional simple moduli spaces, studied in mathematics and elementary string theory, gains a new support from the observations in earlier papers of D.Kreimer [23]. However, this subject is beyond the scope of the present paper.

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