ON THE QUASI-ORDINARY CUSPIDAL FOLIATIONS IN \((\mathbb{C}^3;0)\)

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1. Introduction and motivation

We would like to study the reduction of the singularities and the analytic classification, in some cases that we shall describe, of germs of singular holomorphic foliations in \((\mathbb{C}^3;0)\), with non-zero linear part. Consider, more generally, a germ in \((\mathbb{C}^n;0)\) of an integrable 1-form, and let

\[ ! = !_1 + !_2 + \]

be its decomposition in homogeneous forms \((!_1 = \sum_{j=1}^{n} A_{ij} dx_i; A_{ij} \text{ homogeneous polynomial of degree } i)\). Suppose, moreover, that \(!_1 \neq 0\). In general, we can write

\[ !_1 = c_{ij} x_j dx_i; \quad c_{ij} \in C. \]

The integrability condition \(! \wedge ! = 0\) implies that \(!_1 \wedge !_1 = 0\). Let \(C\) be the matrix \((c_{ij}) \in M_{n \times n}(\mathbb{C})\). Writing down explicitly the integrability condition, the coefficient of \(dx_i \wedge dx_j \wedge dx_k\) (\(i < j < k\)) in \(!_1 \wedge !_1\) is

\[ c_i (c_{kj} c_{ij} - c_{ki} c_{ij}) + c_j (c_{ki} c_{ij} - c_{kj} c_{ij}) + c_k (c_{ij} c_{ij} - c_{ij} c_{ij}); \]

where \(c_i = \sum_{j=1}^{n} c_{ij} x_j\). Two cases appear:

1. \(C\) is a symmetric matrix.
2. \(C\) is not symmetric. So, \(c_{ij} \neq c_{ji}\).

In the last case, the polynomials \(c_i, c_j, c_k\) are linearly dependent for every \(i, j, k\), and so \(rk(C) \leq 2\). Moreover, \(!_1(0) = !_1(0) \neq 0\), so we are in presence of a Kupka-type phenomenon and, in fact, it exists a biholomorphism \(f\) such that \(f^*! = 0\), where \(f\) is a form in 2-variables. For {	extsc{holomorphic}} phenomena, lots of work have been done.

We then focus on the symmetric case. A linear change of coordinates changes \(C\) in \(P^t C P, P\) invertible, so we can suppose \(C\) diagonal and moreover

\[ !_1 = x_i dx_i; \quad r \in \mathbb{N}; \]

If \(r = n, G. Reeb\), in his thesis, shows that there always exists a holomorphic first integral. The behavior of the foliation is then, the behavior of a function. Using \textsc{Malgrange’s} singular Frobenius theorem, we recover this result.

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If $r < n$, some work was done by R. M. Oussa under additional hypothesis. The fundamental paper of M. Atteia and M. Oussa completes the mentioned results. Let us recall in this case, briefly, the 2-dimensional situation. The foliations studied are defined by 1-form $sydy + x^p dx$. Following Takeuchi, such a foliation has a form at normal form

$$!_N = dy^2 + x^n + x^p U(x) dy$$

where $n \leq 3, p > 2, U(x) \in C[[x]], U(0) \neq 0$.

The generic case ($n = 3$) was studied by M. Oussa, and a generalization ($n = 2, 2p > n$) by Cerveau and M. Oussa. In both cases, the reduction of the singularities of $!$ and $!_N$ agrees with the reduction of the curve $y^2 + x^n = 0$. Projective holonom y classes and, generally, there is a rigidity phenomenon of all analytic. If $n$ is even and $2p = n$, it has been studied by M. Eizen and S. Sad under some restrictions on the values of $U(x)$. If $2n < n$ the study was done (not in full generality) by Berthier, M. Eizen, and S. S. We shall call "cuspidal" to these foliations.

The objective of the present work is to generalize this situation to dimension three. We want to study foliations whose linear part is given by $d(x^2 + y^2)$ or by $d(z^2)$. In this paper we shall focus in the case $d(z^2)$. A surface that controls the resolutions of the singularities, with an equation $z^2 + f = 0$ will appear in the considered cases.

Let us recall some results about reduction of the singularities of a complex surface, following Hironaka. A surface $X$ in $C^3$ has an equation

$$f = f + f_{+1} + \ldots = 0$$

where $f_i$ is homogeneous of degree $i$. For such a surface, define, at the origin:

1. The tangent cone, $C_X$, as the cone $f = 0$.
2. The Zariski tangent cone $T_X$, as Spec$(\mathcal{M} = M^2)$, $M$ being the maximal ideal corresponding to the origin of $C[[y; y; z]](f)$. This is the smallest linear space containing $C_X$.
3. The strict tangent cone $S_X$, as the largest linear subspace $T$ of $T_X$ such that $C_X = C_X + T$. The codimension of $S_X$ is the minimum number of variables required to write down the equations of $C_X$.

The resolution of singularities of an analytic surface $X$ is a problem that may stated as follows: to find a non-singular surface $\mathcal{X}$ and a birational morphism $\mathcal{X} \rightarrow X$ composed of quadratic (point blow-ups) and monoidal (curve blow-ups) transformations. These must be done in a precise order. The main case to consider is when the three tangent spaces defined above coincide, and the most difficult case is when, moreover, $\dim S_X = 2$. In this case, the tangent cone can be written as $z$. The resolution may be controlled by Hironaka's characteristic polyhedra of the singularities. The precise sequence of blow-ups needed can be read in the polyhedra.

A kind of surface singularities whose resolution is particularly simple, and combinatorial, are quasi-ordinary singularities. To define them, consider a finite projection $X ! C^2$ and let be discriminant locus of (i.e., the projection of the apparent contour). If has normal crossings the singularities of $X$ are called quasi-ordinary.
Quasi-ordinary singularities are studied not only because they are relatively simple, but because they arise in the Jungian approach to desingularization. First of all desingularize the discriminant locus in order to obtain quasi-ordinary singularities. Then, the problem (simply) is to reduce the singularities of a quasi-ordinary surface. Some good references of this are the articles of Giraudo and Cossart.

Quasi-ordinary singularities can be parametrized by fractional power series, as branches of curves:
\[
\begin{align*}
g &< x = x \\
y & = y \frac{1}{p} \\
z & = w_j c_j x^2 y^2 
\end{align*}
\]
By the condition of the discriminant, it can be seen that the set of points \( f(u; j) \) 2 \( \mathbb{R}^2 : c_{ij} \neq 0 \) is contained in a quadrant \( (a; b) + R_1 \), where \( c_{ij} \neq 0 \). Characteristic pairs may be needed for this parametrization, as is the case of curves, and they still determine the local topology of the singularity, while the converse is not known.

Coming back to foliations, this is related with the case we shall study. More precisely, we search a class of foliations in \( (\mathbb{C}; 0) \) whose reduction process can be read in a quasi-ordinary surface. For the case considered \( ! = d(z^2) \), by Weierstrass preparation theorem and Schichimhausen transformations we end that, in appropriate coordinates, the surface is \( z^2 + (x; y) = 0 \), that is not necessarily a separatrix.

The natural generalization of cuspidal foliations will be those with an equation:
\[
! = d(z^2 + (x; y)) + A(x; y)dz.
\]
In fact, in recent work, Frank Loray finds an analytic normal form as
\[
! = dF + zdG + zdz;
\]
where \( F; G \) 2 \( \mathbb{C} f(x; y) \), for integrable holomorphic foliations with linear part not tangent to the radial vector field. Note that a coordinate change of \( ! = G(x; y) \) in Loray’s form gives an equation like our expression for the foliations. This is integrable if and only if \( d^2 \cap dA = 0 \), i.e., if \( A \) are analytically dependent. As are shall restrict to the quasi-ordinary case, we have that \( (x; y) = x^2 y^2 U(x; y) \), with \( U \) a unit. A convenient change of variable in \( x, y \), allows us to suppose that \( (x; y) = x^2 y^2 \). Let \( d = gcd(p; g) \) \( p = dp^2 \) \( q = dq^2 \). The integrability condition \( d^2 \cap dA = 0 \) is then that \( A(x; y) = L(x^2 y^2) \) where \( L(u) \) 2 \( \mathbb{C} f(u) \).

The plan of this paper is as follows. In section 2 we shall review the notion of simple singularity of a foliation, in the sense defined by Cerveau and G/G/11, and its analytic classification according to Cerveau and M. Section 3 is devoted to describe the resolution of singularities of the quasi-ordinary foliations we are going to study, and the topology of the exceptional divisor. In section 4, we construct a Hopf branion associated to the quasi-ordinary foliations, making a reduction of the separatrix to a canonical form. Finally, section 5 is devoted to present the main result of the paper: In the considered cases, the holonomy of a certain component of the exceptional divisor class is analytically the foliation. The cases we study, as we shall see, are essentially the same as that are studied in dimension two.

Some notations used throughout the paper are presented here. \( D(\mathbb{C}; 0) \) will denote the group (under composition) of germ s of analytic diffeomorphism s of \( (\mathbb{C}; 0) \). If \( \mathbb{D} \) denotes a holomorphic integrable 1-form, defining a foliation, and \( D \) is a component of the divisor obtained after reduction of singularities, \( H : \mathbb{D} \rightarrow (\mathbb{D} n S) \)!
Di \((\mathbb{C}; 0)\) is the holonomy representation, defined over a transversal to \(D\) (omitted from the notation), where \(S\) is the singular set of the reduced foliation.

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2. Simple singularities of foliations and analytic classification

The process of reduction of singularities for a holomorphic foliation is well known in dimension two. After a finite number of point blow-ups performed in any order, a germ of analytic space and a foliation are obtained, and around the singular points, the foliation is generated by a one-form

\[ ! = (y + h_{\alpha}\cdot dx + (x + h_{\beta}\cdot dy; \]

with \(6_{0}, - E_{Q} < 0\).

The analytic classification is well studied in a wide variety of cases:

1. If \(-E > 0\), it is analytically linearizable, i.e., there exists an analytic diffeomorphism \(:(\mathbb{C}^{2}; 0)! \approx (\mathbb{C}^{2}; 0)\) such that

\[ ! ^{\dagger} = (ydx + xdy) = 0 \]

2. If \(-2 > 0\), but it is not \"well-approached" by rational numbers, it is also linearizable. If it is well-approached, we face a problem of small divisors, and the situation becomes more complicated.

3. If \(E = 0\) or \(-2 > 0\), Martinet and Ramis find a large moduli space formal/analytic. In this case the classification of the foliation agrees with the classification of the holonomy of a strong separatrix (i.e., a separatrix in the direction of a non-zero eigenvalue). Moreover in the resonant case \((-2 > 0\}) or in the saddle-node case \((E = 0)\) with analytic center manifold, the conjugation of the foliation is hered. This means the following: choose coordinates \(x, y\) such that the axis are the separatrices, \(y = 0\) being a strong one; the foliations are defined by 1-forms

\[ !_{i} = yA_{i}(x, y)dx + x(1 + B_{i}(x, y))dy; \]

with \(i = 1; 2\). Let \(h^{i}(x)\) be the holonomies of \(y = 0\), supposed conjugated. Then the foliations are conjugated by a diffeomorphism \((x, y) = (x; yg(x, y))\).

The singularities obtained after this reduction process are called simple or reduced. The class of simple singularities is stable under blow-ups. Let us observe that the notion of simple singularity is not only analytic, but formal if \!1, \!2 are analytic 1-forms, and \! is a local diffeomorphism such that \!1 \!1 \!2 = 0, then \!1 has a simple singularity if and only if \!2 has.

If the dimension of the ambient space is greater or equal than three, the notion of simple singularity has been developed in [3] and its analytic classification studied in [2]. The reduction of singularities is only achieved when the dimension of the ambient space is at most three, and in this case, simple singularities are the final ones obtained after the reduction process. Let us summarize here, for convenience of the reader, the main results in dimension three.
First of all, let us recall the notion of \( \text{dim ensionalt ype} \). A foliation has dimensionalt ype \( r \) if there exist analytic (resp. for mal) coordinates such that the foliation is defined by an integrable 1-form \( \omega \) that can be written in coordinates \( x_1; \ldots; x_r \), but not less. So, a three-dimensionalt ype of foliation has dimensiontal ypes 2 and 3. For instance, if we are in presence of a K upka phenomena, the dimensiontal type is 2. The notion of for mal dimensiontal type or analytic dimensiontal type are equivalent, as seen in \( [6] \). So, we have simple singularities of dimensiontal types 2 and 3. If the dimensional type is 2, simple singularities are defined by a simple 2-dimensiontal 1-form. They have 2 separatrices, of which at most one is for mal.

If the dimensiontal type is tree, simple singularities are the ones that admit one of the following form al 1-forms:

\[(1) \quad \omega = xyz \left( \frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} \right) ;\]

with \(-; -; - \in \mathbb{Q}\) (and \( \epsilon \neq 0, \) as the dimensiontal type is 3). This is the linearizable case. If, for instance, some of the quotients is not real, the linearization is analytic.

\[(2) \quad \omega = xyz \left( \frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} \right)
+ \frac{1}{(x^p y^q z^r)^s} \left( \frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} \right) ;\]

where \( p; q; r \in 2\mathbb{N}, qr \neq 0, s \in 2\mathbb{N}, \) constants, not both zero. This is the resonant case. Several things can be said about foliations that are for mal y equivalent to this form al 1-form:

1. If has three separatrices, of which at most one is for mal (which, in the preceding coordinates, would be \( x = 0 \)). This is a consequence of simple two-dimensiontal singularities defined along the axis. Saddle-nodes only appear if \( p = 0 \), and only in this case the existence of a for mal, non convergent separatrix is possible.

2. The holonom y group of \( z = 0 \) (strong separatrix) classifies analytically the foliation. Moreover, the conjugation is here if the three separatrices are convergent.

3. If \(- \in \mathbb{Q}\), there is a rigidity phenomenon: every such foliation is analytically equivalent to \( !_N \).

A typical case in which we are in presence of a simple singularity and that will appear in the sequel, is when the foliation is defined by a 1-form

\[(3) \quad \omega = xyz \left( p + A (x; y; z) \frac{dx}{x} + (q + B (x; y; z)) \frac{dy}{y} + (r + C (x; y; z)) \frac{dz}{z} \right) ;\]

with \( p; q; r \in 2\mathbb{N}, \) \( A; B; C > 0. \)

Moreover, the transformation that converts \( !_N \) in its for mal 1-form \( !_N \), even if it is not analytic, is transversally for mal and be red. This means in particular that such a can be found in the form

\[(k; y; z) = (k; y;’, (x; y; z));\]

The existence of local holom orphic first integrals, according to Mattei and Moussu, is equivalent to the periodicity of the holonomy group. Moreover, an integrable
1-form \( ! \), that generates a reduced foliation of dimension type three, has a holomorphic first integral if and only if there exists analytic coordinates \((\kappa; y; z)\) such that

\[ ! (pyzdx + qxzdy + rxydz) = 0; \]

where \( p; q; r \) \( \in \mathbb{N} \).

3. Reduction of singularities and topology of the divisor

In this paper, we shall study the analytic classification of quasi-ordinary cuspidal foliations in dimension three, i.e., foliations such that, in appropriate coordinates, can be defined by an integrable 1-form

\[ ! = d(x^p + y^q) + A(\kappa; y)dz; \]

The integrability condition here is equivalent to \( d(x^p y^q) \wedge dA = 0 \). So, let \( d = \gcd(p; q) \), \( p = dp_0 \), \( q = dq_0 \). Such a 1-form can be written as

\[ ! = d(x^p + y^q) + (x^p y^q)^k h(x^p y^q) dz; \]

where \( h(u) \in C \), \( h(0) \neq 0 \). Fixing \( p, q \), we shall call \( pq \) the set of holomorphic foliations that are analytically equivalent to the foliation defined by one of these 1-forms.

As it will become clear from the development of the paper, the separatrices of this foliation have the equation

\[ z^2 + x^p y^q + h yz = 0; \]

and Weierstrass preparation theorem and Tschirnhaus transformation show that this separatrix is analytically equivalent to \( z^2 + x^p y^q = 0 \).

The reduction of singularities for these foliations is quite simple, similar to plane curves, and it is the main objective of this section their detailed analysis. For convenience, we divide the problem in three cases:

Case 1. \( p, q \) even.
Case 2. \( p \) even, \( q \) odd.
Case 3. \( p, q \) odd.

Case 1. Suppose \( p, q \) are even, and \( d = 2d_0 \). If \( k > d_0 \), the reduction of the singularities is obtained after \( \frac{p + q}{2} \) blow-ups:

(a) First of all, blow up \( \frac{p}{2} \) times the y-axis. We obtain a sequence of divisors \( D_1; \ldots; D_{p/2} \), topologically germs \( (\mathbb{C}^1, C; \mathbb{P}^1_\mathbb{C}) \). The intersection of two consecutive components is a germ of a line \( (C; 0), \)

\[ L_i = D_i \setminus D_{i+1}, i < \frac{p}{2}. \]

In the appropriate chart, these blow-ups have the equations

\[ \begin{align*}
&< x = x \\
&\quad y = Y \\
&\quad t_{i+1} = x \neq t_i,
\end{align*} \]

where \( t_0 = z, 1 \leq i < \frac{p}{2} \).

(b) Then blow-up \( \frac{q}{2} \) times the x-axis, obtaining again a sequence of divisors \( D_{p+1}; \ldots; D_{p+q/2} \), topologically equal to \( (\mathbb{C}^1, C; \mathbb{P}^1_\mathbb{C}) \). Again, the intersection between two consecutive components is a line \( L_1 = \)
\[ D_i \setminus D_{i+1}, \frac{P}{2} + 1 \quad \text{for} \quad i < \frac{P + q}{2}. \]

Now, the coordinates of the blow-ups are
\[ g < x = x \]
\[ y = y \]
\[ t_{i+1} = y \]

where \( \frac{P}{2} < i < \frac{P + q}{2}. \)

The result of the composition of all the blow-ups in the preceding charts is the map \((x; y; t_{i+1}) = (x; y; x^\frac{P}{q} + D_{i+1}).\) The pullback of the foliation is given by
\[ ! = x^P y^q 1 \]
\[ 2xyt^d + (t^2 + 1)xy \quad \frac{dx}{x} + \frac{dy}{y} + \]
\[ + (x^P y^q) dx + (x^P y^q) y \quad \frac{dx}{2} + \frac{dy}{2} + \frac{dt}{t} \]

(here \( t = t_{i+1} \).

The foliation, now, is reduced. Let \( S \) be the singular locus of this reduced foliation. \( S \) is an analytic, normal crossing space of dimension one, composed of:

(i) The lines \( L_i \) of intersection of the divisors. These are resonant singular points of dimension type two.

(ii) The lines \( L_i, L^0_\infty \) in \( D_{i+1} \) of equations \((y = 0; t = 1), (y = 0; t = i),\) and also the lines \( M^0, M^q \) in \( D_{i+1} \) of equations \((x = 0; t = i), (x = 0; t = i)\) (in the last chart). These lines are the intersections of the two separatrices \( S^0, S^q \) with the divisors.

(iii) The intersection \( P_i = D_{i+1} \setminus D_i, \frac{P}{2} < i < \frac{P + q}{2} \) is a projective line composed of points of dimension type two, except at the corners:

(A) \( m_i = P_i \setminus L_i = D_{i+1} \setminus D_i \setminus D_{i+1}, \frac{P}{2} < i < \frac{P + q}{2}. \) These are the resonant singular points of dimension type three, having \( D_{i+1}, D_i, D_{i+1} \) as separatrices.

(B)\[ m^0 = D_{i+1} \setminus D_{i+1} \setminus S^0 = L^0_\infty \setminus M^0 \setminus P_{i+1} ; \text{ and} \]
\[ m^{00} = D_{i+1} \setminus D_{i+1} \setminus S^{00} = L_\infty \setminus M^{00} \setminus P_{i+1} ; \]

These are the resonant singular points of dimension type three corresponding to the separatrices of the foliations. According to the preceding description of the resolution of the singularities, we have all the information about the topology of \( D_i n S, \) and more precisely about the fundamental group of these components. We have:

\( D_i n S \) is topologically \( C \times C, \) so simply connected.

\( D_i n S \) \((1 < i < \frac{P}{2})\) is topologically \( C \times C. \) The generator of the fundamental group is a loop \( _i \) that turns around \( L_i \) (or \( _i \) around \( L_{i+1} \)).

\( D_{i+1} n S = C \times C. \) The fundamental group is generated by a loop \( _i \) around \( P_{i+1} \).
\[ D_1 \cap S = C \quad (\frac{p}{2} + 1 < i < \frac{p+q}{2}). \]

The fundamental group has generators \( \gamma \) around \( L_1 \) and \( \gamma \) around \( P_1 \), that commute.

\[ D_{\frac{p}{2}+1} \cap S = (C \cap \mathcal{F} \cap m_0 \cap m_0) \quad C. \]

We now have one loop \( \frac{p}{2}+1 \) around \( P_{\frac{p}{2}+1} \) and loops \( \gamma^i, \gamma^j \) around the separatrices (i.e., around \( m_0, m_0 \)).

\[ D_{\frac{p}{2}} \cap S = C^{ \frac{p}{2} } \cap C, \]

where \( C \) is the curve with coordinates \( \gamma^2 + y^3 = 0 \), composed of two smooth branches that meet tangentially at the origin. In this case (see below), \( (C^{\frac{p}{2}} \cap C) \) is the group, written in terms of generators and relations as

\[ \sqrt{i} (C^{\frac{p}{2}} \cap C) = h; \quad \frac{d}{t} = \frac{i}{d}i. \]

These loops go as follows. Consider the curve \( t^2 + y^3 = 0 \) on \( C^2 \), and cut by \( y = 1 \). You obtain \( C \cap \mathcal{F} \cap m_0 \cap m_0 \); then \( \gamma^i \) is a loop in \( y = 1 \) that turns around these two points \( m_0, m_0 \) and \( \gamma^j \) is a loop in \( t^2 = 0 \) that turns around the origin. At the end of the reduction process, is going to be a loop in \( D_{\frac{p}{2}} \cap S \) around the two separatrices, and a loop around \( P_{\frac{p}{2}+1} \) between \( S^0 \) and \( S^{2m} \).

The case \( k = d^0 \) is almost identical, except for some values of the coefficient \( h(0) \). More precisely, after \( \frac{p+q}{2} \) blow-ups, in order to obtain the complete reduction of singularities (i.e., simple singular points) it is necessary and sufficient that

\[ h(0)^2 \equiv \frac{(16 + r)^2}{16 + 2r}; \quad 8r > 0 \]

Moreover, if in the preceding expression we put \( r = 0 \), we have then \( h(0) = 4 \). In this case, only one separatrix is obtained, but it is a three-dimensional saddle-node, the divisor being the weak separatrix (then convergent). We shall assume that this is not the case, i.e., if \( k = d^0 \) we shall assume that

\[ h(0)^2 \equiv \frac{(16 + r)^2}{16 + 2r}; \quad 8r > 0 \]

The reader may verify that this condition is equivalent to \( P_2 \) property in \( \mathbb{Q} \) (i.e., \( h(0) \leq 2 \frac{p+q}{2} + 1, 8r > 0 \) \( \{0;1\} \cap \mathbb{Q} \)).

Suppose now that \( k < d^0 \). In this case, the reduction of singularities is achieved blowing-up \( kq \) times the \( x \)-axis and \( kq \) times the \( y \)-axis. A finer these, in the last chart we obtain as singularities the sets \( L_0 = (x = t = 0) \), \( M_0 = (y = t = 0) \), \( L_0 = (x = 0; t = 1) \), and \( M_0 = (y = 0; t = 1) \). These are also two singular points of dimension three, namely \( m_0 = L_0 \cap M_0 \cap P_{k(q+1)} \), \( m_0 = L_0 \cap M_0 \cap P_{k(q+1)} \) with analogous notations as before, corresponding respectively to the points \( (0;0;0) \) and \( (0;0;1) \). But now \( m_0 \) is a saddle-node, so the separatrix \( S_0 \) is maybe formal. In this paper, we shall assume that always \( S_0 \) is convergent, i.e., there is a center manifold.

**Case 2.** Suppose \( p \) even, \( q \) odd. If \( k > d \), the reduction of singularities is obtained after the following sequence of blow-ups.
(a) First, blow-up $p=2$ times the $y$-axis, obtaining divisors $D_1;:::;D_8$ linked by lines $L_1;:::;L_8$. The equations of these blow-ups are

$$x = x$$

$$y = y$$

$$t_i = x^{1/2}$$

where $t_0 = z, i < \frac{p}{2}$.

(b) Blow-up $q \leq \frac{p}{2}$ times the $x$-axis, obtaining $D_{\frac{p}{2}+1};:::;D_8$ joined by lines $L_1 = D_1 \setminus D_{\frac{p}{2}+1}$, and $D_1$ joined to $D_{p=2}$ by a projective $P_1$. The equations are

$$x = x$$

$$y = y$$

$$t_i = y^{1/2}$$

$$\frac{p}{2} < i < \frac{p+q}{2}$$

(c) It appears a tangency in the singular locus. In order to break $z$, blow-up again the $x$-axis and take a chart centered in the point corresponding to $t_{\frac{p}{2}+1}$. The equations are now

$$x = x$$

$$y = y$$

$$t_{\frac{p}{2}+1} = t_{\frac{p}{2}+1}$$

and we obtain a new component $D^0$ such that $D^0 \setminus D_{\frac{p}{2}+1} = L_{\frac{p}{2}+1}$, $D^0 \setminus D_{\frac{p}{2}} = P^0$.

(d) Finally, blow-up again the $x$-axis, in order to obtain normal crossings. We obtain a new component $D^0$ and the only separatrix $S$ of the foliation cuts $D^0$ transversely in a line $L$ (and $D_{p=2}$ in a line $M$). We have $L^0 = D^0 \setminus D_{\frac{p}{2}}$ and $D^0 = D^0 \setminus D_{p=2}$.

The singular points of dimension type three are $m_1 = D_{p=2} \setminus D_1 \setminus D_{\frac{p}{2}+1}$. The equations are $m_{\frac{p}{2}+1} = D_{p=2} \setminus D_{\frac{p}{2}+1} \setminus D^0, m^0 = D_{p=2} \setminus D_{p=2} \setminus D^0$, and $m = D_{p=2} \setminus D^0 \setminus S$.

The topology of the components is as in Case 1. If $S$ is the singular locus, $D_1 n S = C^2$ is simply connected, $D_1 n S = C^2$ if $1 < i < \frac{P}{2}$, $D_{\frac{p}{2}+1} n S = C^2, D_1 n S = C^2$ if $\frac{P}{2} + 1 < i < \frac{P+q}{2}$, $D^0 n S = (C^2 n C) \setminus (C n n^0)$, $D^0 n S = C^2$. Finally, $D_{p=2} n S = C^2 n C$, where $C$ is the curve with coordinates $t_{p=2}^2 + y^q = 0$. As before,

$$1(C^2 n C) = h; ; q = 2i$$

When $k < d$, as in dimension two, the situation is as in Case 1, with $k < d^2$.

Case 3. $p, q$ odd. Now, the resolution is something different than before. First, blow-up $p \geq 1$ times the $y$-axis and $q \geq 1$ times the $x$-axis obtaining...
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$D_1; \ldots ; D_{n-1} \subset \{ z^2 + x^2y^3 = 0 \}$. In the new coordinates $\{ x; y; t = t_{n-1} \}$ the singular locus is given by the three coordinate axes, that corresponds to the intersection of the divisors and the intersection of the cone $t^2 + xy = 0$ with the divisors.

Now, blow-up the origin, obtaining $\mathbb{P}$, a projective $\mathbb{P}^2$. The three coordinate axis, now transverse to $\mathbb{P}$, continue being singular. Over $\mathbb{P}$, the singular locus is composed by two projective lines and a conic tangent to both lines. In order to finish, blow-up twice each of the axis $x$ and $y$ transverse to $\mathbb{P}$, obtaining $D^0_{(1)}, D^0_{(2)}, D^0_{(3)}$.

With respect to the topology of the divisors, the only interesting case (i.e., not similar to the preceding ones) to comment is $\mathbb{P} \setminus S$. As we said before, $\mathbb{P} \setminus S$ is composed by two lines and a regular conic, so

$$1 \cap (\mathbb{P} \setminus S) = h; \quad 2 = 2i,$$

4. Reduction of the separatrix to a canonical form

Let $F$ be a germ of a singular foliation defined on $\mathbb{C}^3; 0$, and let $\phi : (M; D) ! (\mathbb{C}^3; 0)$ be the minimal reduction of the singularities of $F$ in Cano-Cerveau sense, as described above. Let $F$ be the strict transform of the foliation $F$ by $\phi$ and let $D_i$ be a component of the exceptional divisor $D$.

We recall that a Hopf bration $H_F$ adapted to $\mathbb{P}^2_{pq}$ is a holomorphic transversal bration $f : M ! D$ to the foliation $F$, i.e:

1. $f$ is a retraction, more precisely, $f$ is a submersion and $f|_{D_i} = Id_{D_i}$.
2. The bres $f^{-1}(p)$ of $H_F$ are contained in the separatrices of $F$, for all $p \in D_i \setminus \text{Sing}(F^-)$.
3. The bres $f^{-1}(p)$ of $H_F$ are transversal to the foliation $F$, for all $p \in D_i \setminus \text{Sing}(F^-)$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{The reduction of the surface $x^2 + x^2y^3 = 0$.}
\end{figure}
We shall be interested in nding a Hopf bration adapted to the foliation, relative to a particular component of the exceptional divisor. For, if \( p \) is even, call \( D = D_{p-2} \), i.e., the last component obtained after the first sequence of line blow-ups. If \( p \) and \( q \) are odd, \( D = F \), i.e., the projective obtained after the (only) point blow-up.

The task of nding a Hopf bration associated to the foliation \( F \) is not easy in the actual coordinates \((x; y; z)\). As it is done in the two-dimensional case, to overcome this obstacle, we analyze the desingularization of \( F \) in order to obtain a simple equation for the separatrices.

From Section 1, we know that the foliation \( F \) is defined by the one-form

\[
d(z^2 + (x^p y^q)^2) + \beta (x^p y^q)^k h (x^p y^q) dz;
\]

has a separatrix analytically equivalent to \( S: z^2 + (x^p y^q)^2 = 0 \) for some \( 2 \leq N \).

In order to nd a Hopf bration \( H_F \) of the foliation \( F \), we need to normalize the one-form such that the foliation defined by this normal form has exactly \( S: z^2 + (x^p y^q)^2 = 0 \) as separatrix, for certain \( r \). So, the standard transformed of \( S \) by the desingularization is an hyperplane in these coordinates and invariant by Hopf bration.

**Proposition 1.** The foliation \( F \) is analytically equivalent to a foliation defined by the one-form

\[
d(z^2 + (x^p y^q)^2) + g (x^p y^q)^k h (x^p y^q) dz; \]

where \( r = d \) if \( 2k \leq d \) and \( r = 2k - d \) if \( 2k < d \). In particular, the separatrix of the foliation \( F \) is analytically equivalent to \( S: z^2 + (x^p y^q)^2 = 0 \).

**Proof:** The foliation \( F \) is defined by the 1-form

\[
d(z^2 + (x^p y^q)^2 + \beta (x^p y^q)^k h (x^p y^q)) dz;
\]

where \((p; q) = d, p = p; d, q = q; d\). That is, is the pull-back of the 1-form

\[
o = d(z^2 + u^d) + u^d h(u) dz\]

by the ramified bration

\[
(C^3; 0) ! (C^2; 0) \quad (x; y; z) ! (x^p y^q; z) = (u; z);
\]

The equation of the separatrices of \( F \) is of the form \( z^2 + u^d + h(z) = 0 \) (if \( d \) is even, this is a joint equation, i.e., the product of the two separatrices).

Using Weierstrass' preparation theorem, we can assume that the local equation of the separatrix is a polynomial in \( z; z^2 + a(u)z + b(u) = 0 \), with \( a(0) = b(0) = 0 \).

If \( 1 (u; z) = (u; z) \frac{a(u)}{2} \) is the Tsushima-Hausdorff transformation, then the pullback

\[
1 \times 0 has z^2 + c(u) = 0 as separatrix, with c(u) = b(u) \frac{a(u)}{2} = u^d f(u), f(0) \neq 0.
\]

If \( d > 2 \) (cuspical case), we have that \((a) > \frac{d}{2} > 3\), and then \( r > 2 \). In fact, \( r = d \) when \( 2k > d \), or \( r = 2k \) when \( 2k < d \) (see Section 2). Similar computations are valid when \( d = 1 \) or \( d = 2 \) (in these cases, \( 1 = d \)).

Let us write this reduced equation of the separatrices as

\[
\frac{z^2}{f(u)} + u^d = 0;\]
and let $f(u)^{1/2}$ be a square root of the unit $f(u)$. If $f(u; z) = (u; z f(u)^{1/2})$, and $= 1_2$, then $0$ has $z^2 + u = 0$ as separatrix. This map has the form

$$(u; z) = u; z f(u)^{1/2}, \quad \frac{a(u)}{2} :$$

Consider the diagram

$$\begin{array}{ccc}
C^3 & \rightarrow & C^2 \\
\uparrow & & \uparrow \\
? & \rightarrow & ? \\
\downarrow & & \downarrow \\
\mathcal{C}^3 & \rightarrow & C^2
\end{array}$$

We want to find a di eomorphism $F = (F_1; F_2; F_3)$ that makes commutative the diagram, i.e., that

$$(F_1^0 F_2^0; F_3^0) = \frac{a(k^0 y^0)}{2} :$$

For, we may choose $F_1 = x, F_2 = y, F_3 = z f(k^0 y^0)^{1/2} = \frac{a(k^0 y^0)}{2}$. The form $0$, having $z^2 + u = 0$ as a separatrix is, up to a unit, $d(z^2 + u^2) + g(u; z) \zeta dz du$, so $F_0$ determines the same foliation that

$$(z^2 + (k^0 y^0)^2) + g(k^0 y^0; z) k^0 y^0 z \frac{2dz}{z} - p \frac{dy}{z} = 0 :$$

We reproduce part of the proof presented in order to find the transformation $F$ needed.

As a consequence of this normal form for $F$, there exists coordinates $(x; y; z)$, such that the separatrix $S$ of the normal form is given by the equation: $z^2 + (k^0 y^0)^2 = 0$, where $r$ is as in the Proposition and not only analytically equivalent to $0$. Now, we can find a Hopf bration, from a holomorphic vector field $X$, for which $S$ is an invariant set, that is

$$\begin{cases}
\frac{\partial}{\partial x} = x \frac{\partial}{\partial x} + p \frac{\partial}{\partial y}; & \text{p is even} \\
\frac{\partial}{\partial y} = y \frac{\partial}{\partial y} + \frac{p + q}{2} \frac{\partial}{\partial z}; & \text{p and q are odd}.
\end{cases}$$

So, we have that the Hopf bration $H \phi_{(F)}$ adapted to the foliation defined by the one-form $\omega = p \frac{dy}{z} + q \frac{dz}{x}$, will be determined (not uniquely) by a linearizable singularity of a holomorphic vector field $X = X_1 + X_2 + \ldots$.

Having defined a Hopf bration adapted to $F$, we can define the holonomy of the leaf $\mathcal{U} \cap \operatorname{Sing}(F)$ respect to this bration. In order to determine $x$, we x a point $p_0 \in \mathcal{U} \cap \operatorname{Sing}(F)$, over this point we have a transformation $p_1(p_0)$ and by path lifting construction, a representation of the fundamental group of $\mathcal{U} \cap \operatorname{Sing}(F)$ in $\operatorname{Diff}(C; 0)$ is determined, denoted by $H_{p_1}$

$$H_{p_1} : \mathcal{U} \cap \operatorname{Sing}(F); p_0) \to \operatorname{Diff}(C; 0);$$

This representation is independent of $p_0$ modulo conjugacy and its image will be called the exceptional holonomy and denoted $H_{p_1}$.
5. Classification of the singularities

From section ..., we know that the homotopy group \( \pi_1(\mathbb{C}^2;0) \) can be generated by two elements \( \alpha, \beta \) in all the cases considered, with different relations in each case:

1. If \( p = 2 \) and \( q = 2 \),
2. If \( p = 2 \) and \( q = 2 \),
3. If \( p = 2 \) and \( q = 2 \).

If \( \alpha \) is an element of the homotopy group, let us denote \( \alpha \)'s image by the map \( H_{\alpha} \) in the exceptional holonomy. This holonomy can be generated by \( h, h' \), which at least satisfy the same relations than \( \alpha \). But in some cases, these relations may be improved. The following proposition collects some of these in proven facts:

**Proposition 2.**
1. If \( p \) is even, \( h^{\pi} = \text{id} \).
2. If \( q \) is odd, \( h^{\pi} = h^{\delta} = \text{id} \).

**Proof.** Consider \( h \). A first blow-up, the strict transform of the separatrix \( S \) is given by a surface analytically equivalent to \( t_{p+1}^{2} + y^2 = 0 \). This singular surface is a cylinder over a curve, that is either a cuspidal curve of characteristic pair \( (2,g) \) or a couple of regular curves tangent at the origin at order \( \frac{3}{2} \). Applying Picard-Lefschetz techniques, it can be seen that the loop \( \alpha \) is a simple curve contained in the plane \( y = 0 \), with \( f \)'s small enough, that turns around the points \( (t,y) = (1, \pm 2) \) in the plane \( y = 0 \).

Thus, the holonomy \( h \) is completely determined by the holonomy of a loop that turns around the line \( D_{p+1} \setminus D_{p+2} \). A long this line, the foliation is a reduced foliation of dimension \( \leq 2 \) (in fact, we are in presence of a Kupka phenomenon), and its analytic type is determined by a two-dimensional section transversal to the \( y \)-axis. This foliation has a linearizable, periodic holonomy, and \( h^{0}(0) = e^{2 \pi i \beta} \).

If \( m \) is even, \( q \) is odd, the periodicity of \( h \) implies the periodicity of \( h' \), and so, \( h' \) is linearizable, \( h'(0) = e^{2 \pi i \beta} \). Nevertheless, it does not mean necessary that the holonomy group \( H_{\alpha} \) is linearizable, as in particular we don't know if it is abelian or not.

The following theorem contains the main result of the paper. In the proof, several techniques from \( \pi_1 \) are used, and we shall not enter in details about them:

**Theorem 1.** Let \( \gamma, \delta \) be elements of \( \pi_{1} \). Consider the foliations \( F_{\gamma}, F_{\delta} \), and their exceptional holonomies \( H_{\gamma}, H_{\delta} \), defined as before.

Then, the foliations are algebraically conjugated if and only if the couples \( (h^{i};h^{i}) \) are also linearly conjugated, i.e., if and only if there exists \( 2 \) \( \mathbb{C} \) such that \( h^{i} = h^{i} \), where \( \mathbb{C} \) is the singular points. These singular points are the intersections of \( \mathbb{C} \) with the other components of the divisor, and with the separatrix (the separatrices in the even-even case). All these points are singular points of
dimensions types two or three, and for all of them, $D^*$ is a strong separatrix. In this situation, the conjugation of the holonomy is of $D^*$ implies conjugation of the reduced foliations in a neighborhood of the singular points.

So, we have that $F^-$, $F_3$ are conjugated in a neighborhood of $D^*$. Suppose now that $p$ is even. We need to conjugate the foliations also in a neighborhood of $D_{2}$; $D_{0}=D_{2}$ as $D_{2}$ is simply connected, its holonomy is trivial. So, the holonomy of $D_{2}$, generated by one loop around $L_{1}=D_{1}\setminus D_{2}$ is periodic (the argument is the same as in $D^*$). The same argument shows that $D_{1}$ has a periodic holonomy, 1, $i<\frac{D}{2}$, and so, the foliations have real integrals in a neighborhood of each $L_{i}$, $1<i<\frac{D}{2}$. These are points of dimension two type. By analogous reasons as in the two-dimensional case, can be extended to a neighborhood of the exceptional divisor, so, $F^-$, $F_3$ are conjugated outside the singular locus, which has codimension two. We conclude using Hartogs' theorem to extend the conjugation to a neighborhood of the origin.

Suppose now that $p, q$ are odd. $F^-$, $F_3$ are conjugated in a neighborhood of $D^*$, that is a projective $P^2$ in this case. The fundamental group of $D_{2}$ is generated by only one loop, that, after the resolution, can be seen as a loop around $D_{2} \cup D_{0}=D_{2}$; $D_{0}$ locally at the reduced singular points, and following similar arguments as in the preceding cases, and as the two-dimensional case, the foliation is linearizable around these points. Let us detail, in this case, how the use of real integrals allows the extension of the conjugation.

Consider, for instance, the singular point $D_{2} \cup D_{2}$; $D_{2} \cup D_{2}$; $D_{0}$, with coordinates $(x_{0}^{0}; s_{0}^{0}; t_{0}^{0})$ as in picture.

We have a conjugation between the foliations $F^-$, $F_3$ defined over an annulus

$$f_{0}^{0}j< *g \quad f_{0}^{0}j< *g \quad f_{0}^{0}j< *g$$

that respects the branched. In these coordinates, the foliation is given by $x^{0}t^{0}=cst; s_{0}^{0}t^{0}=cst; and the real integral of $F^-$ is $x_{0}^{0}y^{0}t^{0}u^{0}v^{0}$, $U_{j}(0)=1$. This real integral may be extended to

$$f_{0}^{0}j< *g \quad f_{0}^{0}j< *g \quad f_{0}^{0}j< *g;$$

where $c_{i}<c$, eventually making $\mu$ small enough. We look for a diemorphism $j$ that transforms this real integral into $x_{0}^{0}y^{0}t^{0}u^{0}v^{0}$, respecting the branching. This diemorphism is

$$j(x_{0}^{0}; s_{0}^{0}; t_{0}^{0}) = (x_{0}^{0}V_{j}; s_{0}^{0}V_{j}; t_{0}^{0}V_{j});$$

and the conditions mean that

$$V_{j}^{0}x^{0}y^{0}t^{0}= U_{j}$$

$$V_{1}y^{0}= 1$$

$$V_{2}y^{0}= 1.$$
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Figure 2. Coordinates $(x_0, y_0, t_0)$ the singular point $D = \mathbb{D}_{p+q} \cap \mathbb{D}_{p+q} \cap \mathbb{D}_{p+q}$. Mean that

$$
\begin{align*}
1 & = x_0^0 \\
2 & = s_0^q \\
3 & = t_0^{p+q}
\end{align*}
$$

with $g(0) = 0$. As before, we have that $1 = x^0 g(p+q)$; $2 = s^0 g^2(p+q)$; $3 = t^0 g^q$. This is a map defined, in the considered chart, over a set of the type $f: \mathbb{D}_{p+q} \cap \mathbb{D}_{p+q} \cap \mathbb{D}_{p+q}$.

Repeating the argument, we extend the conjugation to a neighbourhood of $D = \mathbb{D}_{p+q} \cap \mathbb{D}_{p+q} \cap \mathbb{D}_{p+q}$. Now, similar arguments as in the preceding situations, and as in the two-dimensional case, allow us to extend to a neighbourhood of the exceptional divisor, and again Hartogs' theorem completes the result.

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