Gauge invariant two-point vertices of shadow fields, AdS/CFT, and conformal fields

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In the framework of gauge invariant Stueckelberg approach, totally symmetric arbitrary spin shadow fields in flat space-time of dimension greater than or equal to four are studied. Gauge invariant two-point vertices for such shadow fields are obtained. We demonstrate that, in Stueckelberg gauge frame, these gauge invariant vertices become the standard two-point vertices of CFT. Light-cone gauge two-point vertices of the shadow fields are also obtained. AdS/CFT correspondence for the shadow fields and the non-normalizable solutions of free massless totally symmetric arbitrary spin AdS fields is studied. AdS fields are considered in a modified de Donder gauge and this simplifies considerably the study of AdS/CFT correspondence. We demonstrate that the bulk action, when it is evaluated on solution of the Dirichlet problem, leads to the two-point gauge invariant vertex of shadow field. Also we shown that the bulk action evaluated on solution of the Dirichlet problem leads to new description of conformal fields. The new description involves Stueckelberg gauge symmetries and gives simple higher-derivative Lagrangian for the conformal arbitrary spin field. In the Stueckelberg gauge frame, our Lagrangian becomes the standard Lagrangian of conformal field. Light-cone gauge Lagrangian of the arbitrary spin conformal field is also obtained.

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1. INTRODUCTION

The present paper is a sequel to our paper [1] where gauge invariant approach to CFT was developed. Brief review of our results in Ref.[1] may be found in Sec. II D in this paper. In space-time of dimension $d \geq 4$, fields of CFT can be separated into two groups: conformal currents and shadow fields. This is to say that field having Lorentz algebra spin $s$ and conformal dimension $\Delta = s + d - 2$, is referred to as conformal current with canonical dimension, while field having Lorentz algebra spin $s$ and dual conformal dimension $\Delta = 2 - s$ is referred to as shadow field. We remind that in the framework of AdS/CFT correspondence [9], the conformal currents and shadow fields manifest themselves in two related ways at least. First, the conformal currents appear as boundary values of normalizable solutions of equations of motion for bulk fields of AdS supergravity theories, while the shadow fields appear as boundary values of non-normalizable solutions of equations of motion for bulk fields of AdS supergravity theories (see e.g. [10]-[14]). Second, the conformal currents, which are dual to string theory states, can be built in terms of fields of supersymmetric Yang-Mills (SYM) theory. In view of these relations to supergravity/superstring in AdS background and SYM theory we think that various alternative formulations of the conformal currents and shadow fields will be useful to understand string/gauge theory dualities better.

In our approach, starting with the field content of the standard formulation of currents (and shadow fields), we introduce additional field degrees of freedom (D.o.F), i.e., we extend space of fields entering the standard conformal field theory. We note that these additional field D.o.F are similar to the ones used in the gauge invariant Stueckelberg formulation of massive fields. Therefore such additional field D.o.F are referred to as Stueckelberg fields. As is well known, the Stueckelberg approach turned out be successful for study of theories involving massive fields. This is to say that all covariant formulations of string theories are realized by using Stueckelberg gauge symmetries. Therefore we expect that use of the Stueckelberg fields in CFT might be useful for the study of various aspects of AdS/CFT dualities.

In this paper we develop further our approach initiated in Ref.[1]. As in Ref.[1], we discuss bosonic arbitrary spin conformal currents and shadow fields in space-time of dimension $d \geq 4$. Our results in this paper can be summarized as follows. i) Using shadow field gauge symmetries found in Ref.[1], we obtain the two-point gauge invariant vertex for the arbi-

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2 We note that conformal currents with $s = 1, \Delta = d - 1$ and $s = 2, \Delta = d$, correspond to conserved vector current and conserved traceless rank-2 tensor field (energy-momentum tensor) respectively. Conserved conformal currents can be built from massless scalar, spinor and spin-1 fields (see e.g. [2]). Discussion of higher-spin conformal conserved charges bilinear in 4d massless fields of arbitrary spins may be found in [3].

3 In earlier literature, discussion of shadow field dualities may be found in [15, 16].
trary spin-$s$ shadow field. Imposing some gauge condition, which we refer to as Stueckelberg gauge, we demonstrate that our vertex is reduced to the two-point vertex appearing in the standard approach to CFT. Imposing light-cone gauge on the shadow field, we obtain light-cone gauge fixed two-point vertex. As usually, a kernel of our shadow field vertex gives a correlation function of the conformal current.

ii) We study $AdS/CFT$ correspondence for massless arbitrary spin-$s$ $AdS$ field and boundary spin-$s$ shadow field. Namely, using the modified de Donder gauge condition for $AdS$ field, we demonstrate that the two-point gauge invariant vertex of the shadow field does indeed emerge from massless $AdS$ field action when it is evaluated on solution of the Dirichlet problem. $AdS$ field action evaluated on solution of the Dirichlet problem will be referred to as effective action in this paper. We show that use of the modified de Donder gauge provides considerable simplification when computing the effective action.

iii) We show that the effective action of $AdS$ massless field leads to new interesting description of conformal field. As compared to the standard approaches to conformal fields [4, 5], our approach involves additional field D.o.F. and the respective additional gauge symmetries which are realized as the Stueckelberg gauge symmetries. We obtain very simple higher-derivative Lagrangian for conformal arbitrary spin field. Using the Stueckelberg gauge frame, we demonstrate that our Lagrangian is reduced to the standard Lagrangian of the conformal field. We also obtain light-cone gauge fixed Lagrangian of the conformal field.

The rest of the paper is organized as follows.

In Sec. III we summarize the notation used in this paper and review the standard approach to the conformal currents and shadow fields. Also we briefly review the gauge invariant approach developed in Ref. [1].

In Sec. III we start with the examples of low-spin, $s = 1, 2$, shadow fields. For these shadow fields, we obtain the gauge invariant two-point vertices. Using the Stueckelberg gauge frame, we show how our gauge invariant vertices are related to the vertices appearing in the standard approach to CFT. Light-cone gauge fixed vertices are also obtained.

Section IV is devoted to the study of the two-point vertex for the arbitrary spin-$s$ shadow field. In this section, we generalize results obtained in Sec. III to the case of the arbitrary spin field.

In Sec. V we study $AdS/CFT$ correspondence for low-spin $AdS$ massless fields and boundary shadow fields. One of remarkable features of the modified de Donder gauge is that the computation of the effective action for massless arbitrary spin-$s$, $s \geq 1$, $AdS$ field subject to the modified de Donder gauge is similar to the computation of the effective action for a massive scalar $AdS$ field. Therefore we begin with brief review of the computation of the effective action for the massive scalar field. After that we proceed with the discussion of the effective actions for the massless spin $s = 1, 2$, $AdS$ fields. We demonstrate that these effective actions coincide with the respective gauge invariant two-point vertices for the spin $s = 1, 2$ shadow fields.

Section VI is devoted to the study of $AdS/CFT$ correspondence for massless arbitrary spin-$s$ $AdS$ field and boundary arbitrary spin-$s$ shadow field. In this section we generalize results obtained in Sec. V to the case of arbitrary spin fields.

In Sec. VII we deal with conformal fields. We start with the examples of low-spin, $s = 1, 2$, conformal fields. For these fields, we discuss our new Lagrangian and show how this Lagrangian, taken in the Stueckelberg gauge frame, is reduced to the standard Lagrangian. Light-cone gauge fixed Lagrangian is also obtained. After that we discuss generalization of these results to the case of arbitrary spin-$s$ conformal field.

Section VIII summarizes our conclusions and suggests directions for future research.

We collect various technical details in five appendices. In Appendix A we study restrictions imposed on the shadow field two-point vertex by the Poincaré algebra symmetries, dilatation symmetry, and the shadow field gauge symmetries. We demonstrate that these restrictions allow us to determine the vertex uniquely. Invariance of the two-point gauge invariant vertex under the conformal boost transformations is demonstrated in Appendix B. In Appendix C we present details of the derivation of the effective action. In Appendix D we derive CFT adapted Lagrangian for massless spin-1 and spin-2 fields in $AdS_{d+1}$. In Appendix E we discuss some details of the derivation of normalization factor in the Dirichlet problem.

II. PRELIMINARIES

A. Notation

Our conventions are as follows. $x^a$ denotes coordinates in $d$-dimensional flat space-time, while $\partial_a$ denotes derivatives with respect to $x^a$, $\partial_a \equiv \partial/\partial x^a$. Vector indices of the Lorentz algebra $so(d-1,1)$ take the values $a,b,c,e = 0,1, \ldots, d-1$. We use mostly positive flat metric tensor $\eta^{ab}$. To simplify our expressions we drop $\eta_{ab}$ in scalar products, i.e., we use $X^aY^a \equiv \eta_{ab}X^aY^b$.

We use a set of the creation operators $\alpha^a$, $\alpha^a_\dagger$, and the re-
spective set of annihilation operators $\bar{\alpha}^a$, $\bar{\alpha}^z$,

$$[\bar{\alpha}^a, \bar{\alpha}^b] = \eta^{ab}, \quad [\bar{\alpha}^z, \bar{\alpha}^z] = 1, \quad (2.1)$$

$$\bar{\alpha}^a|0\rangle = 0, \quad \bar{\alpha}^z|0\rangle = 0, \quad (2.2)$$

$$\alpha^{\dagger} = \bar{\alpha}^a, \quad \alpha^{\dagger} = \bar{\alpha}^z. \quad (2.3)$$

These operators will often be referred to as oscillators in what follows.\(^4\) The oscillators $\alpha^a$, $\bar{\alpha}^a$ and $\alpha^z$, $\bar{\alpha}^z$, transform in the respective set of annihilation operators constructed out of the derivatives and the oscillators,

$$\Box = \partial^a \partial^a, \quad \alpha \partial = \alpha^a \partial^a, \quad \bar{\alpha} \partial = \bar{\alpha}^a \partial^a, \quad (2.4)$$

$$\alpha^2 = \alpha^a \alpha^a, \quad \bar{\alpha}^2 = \bar{\alpha}^a \bar{\alpha}^a, \quad (2.5)$$

$$N_\alpha \equiv \alpha^a \bar{\alpha}^a, \quad \bar{N}_\alpha \equiv \bar{\alpha}^a \alpha^a, \quad (2.6)$$

$$\Pi^{[a:b]} \equiv 1 - \alpha^2 \left( \frac{1}{2(\alpha^a \alpha^a)} \right) \bar{\alpha}^2. \quad (2.7)$$

B. Global conformal symmetries

In $d$-dimensional flat space-time, the conformal algebra $so(d, 2)$ consists of translation generators $P^a$, dilatation generator $D$, conformal boost generators $K^a$, and generators of the $so(d-1, 1)$ Lorentz algebra $J^{ab}$. We assume the following normalization for commutators of the conformal algebra:

$$[D, P^a] = -P^a, \quad [P^a, P^{bc}] = \eta^{ab} P^c - \eta^{ac} P^b, \quad (2.8)$$

$$[D, K^a] = K^a, \quad [K^a, J^{bc}] = \eta^{ab} K^c - \eta^{ac} K^b, \quad (2.9)$$

$$[P^a, K^b] = \eta^{ab} D - J^{ab}, \quad (2.10)$$

$$[J^{ab}, J^{cd}] = \eta^{bc} J^{ae} + 3 \text{ terms}. \quad (2.11)$$

Let $|\phi\rangle$ denotes conformal current (or shadow field) in flat space-time of dimension $d \geq 4$. Under conformal algebra symmetries the $|\phi\rangle$ transforms as

$$\delta_G |\phi\rangle = \tilde{G} |\phi\rangle, \quad (2.12)$$

where realization of the conformal algebra generators $\tilde{G}$ in terms of differential operators takes the form

$$P^a = \partial^a, \quad (2.13)$$

$$J^{ab} = x^a \partial^b - x^b \partial^a + M^{ab}, \quad (2.14)$$

$$D = x \partial + \Delta, \quad (2.15)$$

$$K^a = K^a_{\Delta, M} + R^a, \quad (2.16)$$

and we use the notation

$$K^a_{\Delta, M} \equiv -\frac{1}{2} x^2 \partial^a + x^a D + M^{ab} x^b. \quad (2.17)$$

$$x \partial \equiv x^a \partial^a, \quad x^2 = x^a x^a. \quad (2.18)$$

In (2.13)-(2.16), $\Delta$ is operator of conformal dimension, $M^{ab}$ is spin operator of the Lorentz algebra,

$$[M^{ab}, M^{ce}] = \eta^{bc} M^{ae} + 3 \text{ terms}. \quad (2.19)$$

For arbitrary spin conformal currents and shadow field studied in this paper, oscillator representation of the $M^{ab}$ takes the form

$$M^{ab} \equiv \alpha^a \bar{\alpha}^b - \alpha^b \bar{\alpha}^a. \quad (2.20)$$

$R^a$ is operator depending, in general, on derivatives with respect to space-time coordinates\(^5\) and not depending on space-time coordinates $x^a$, $[P^a, R^b] = 0$. In the standard formulation of the conformal currents and shadow fields, the operator $R^a$ is equal to zero, while, in the gauge invariant approach, the operator $R^a$ turns out be nontrivial\(^6\).

C. Standard approach to conformal currents and shadow fields

We begin with brief review of the standard approach to conformal currents and shadow fields. To keep our presentation as simple as possible we restrict our attention to the case of arbitrary spin totally symmetric conformal currents and shadow fields which have the appropriate canonical conformal dimensions given below. In this section we recall main facts of conformal field theory about these currents and shadow fields.

Conformal current with the canonical conformal dimension. Consider totally symmetric rank-$s$ tensor field $T^{a_1 \ldots a_s}$ of the Lorentz algebra $so(d-1, 1)$. The field is referred to as spin-$s$ conformal current with canonical dimension if $T^{a_1 \ldots a_s}$ satisfies the constraints

$$T^{a_1 a_2 \ldots a_s} = 0, \quad \partial^a T^{a_2 \ldots a_s} = 0 \quad (2.21)$$

and has the conformal dimension\(^6\)

$$\Delta = s + d - 2, \quad (2.22)$$

\(^4\) We use oscillator formulation to handle the many indices appearing for tensor fields (for recent discussion of oscillator formulation see \cite{17}). In a proper way, oscillators arise in the framework of world-line approach to higher-spin fields (see e.g. \cite{18,19}).

\(^5\) For the conformal currents and shadow fields studied in this paper, the operator $R^a$ does not depend on derivatives. Dependence on derivatives of $R^a$ appears e.g., in ordinary-derivative approach to conformal fields \cite{20}.

\(^6\) The fact that expression in r.h.s. of (2.22) is the lowest energy value of totally symmetric spin-$s$ massless fields propagating in $AdS_{d+1}$ space was demonstrated in Refs.\(^{21}\). Generalization of relation \(^{22}\) to mixed-symmetry fields in $AdS$ may be found in Ref.\(^{23}\).
which is referred to as the canonical conformal dimension of spin-\( s \) conformal current. Taking into account that the operator \( R^a \) of the conformal current \( T^{a_1 \ldots a_s} \) is equal to zero, using the well-known spin operator \( M_{ab} \) of the totally symmetric traceless current \( T^{a_1 \ldots a_s} \) and \( \Delta \) in (2.22), one can make sure that constraints (2.21) are invariant under conformal algebra transformations (2.12).

**Shadow field with the canonical conformal dimension.** Consider totally symmetric rank-\( s \) tensor field \( \Phi^{a_1 \ldots a_s} \) of the Lorentz algebra \( so(d - 1, 1) \). The field \( \Phi^{a_1 \ldots a_s} \) is referred to as shadow field if it meets the following requirements:

i) The field \( \Phi^{a_1 \ldots a_s} \) is traceless,
\[
\phi^{aaa_3 \ldots a_s} = 0. \tag{2.23}
\]

ii) The field \( \Phi^{a_1 \ldots a_s} \) transforms under the conformal algebra symmetries so that the following two point current-shadow field interaction vertex
\[
\mathcal{L} = \frac{1}{s!} \phi^{a_1 \ldots a_s} T^{a_1 \ldots a_s} \tag{2.24}
\]
is invariant (up to total derivative) under conformal algebra transformations.

We now note that:

i) Conformal dimension of the spin-\( s \) shadow field given by
\[
\Delta = 2 - s, \tag{2.25}
\]
is referred to as the canonical conformal dimension of spin-\( s \) shadow field. The operator \( R^a \) of the shadow field \( \Phi^{a_1 \ldots a_s} \) is equal to zero.

ii) Divergence-free constraint (2.21) and requirement for the vertex \( \mathcal{L} \) to be invariant imply that the shadow field is defined by module of gauge transformation
\[
\delta \phi^{a_1 \ldots a_s} = \Pi^{tr} \partial^{a_1} \xi^{a_2 \ldots a_s}, \tag{2.26}
\]
where \( \xi^{a_1 \ldots a_{s-1}} \) is traceless parameter of gauge transformation and the projector \( \Pi^{tr} \) is inserted to respect tracelessness constraint (2.23).

**D. Gauge invariant approach to conformal currents and shadow fields**

We now briefly review the gauge invariant approach to conformal currents and shadow fields developed in Ref.[1].

The gauge invariant approach to the conformal currents and shadow fields can be summarized as follows.

i) To discuss the arbitrary spin-\( s \) conformal current and spin-\( s \) shadow field we use the respective totally symmetric \( so(d - 1, 1) \) Lorentz algebra tensor fields \( \phi^{\text{cur}}_{a_1 \ldots a_s} \) and \( \phi^{s}_{sh} \), where \( s' = 0, 1, \ldots, s \). For \( s' \geq 4 \), these fields are restricted to be double-traceless, and the following two point current-shadow field interaction vertex
\[
\mathcal{L} = \frac{1}{s!} \phi^{\text{cur}}_{a_1 \ldots a_s} T^{a_1 \ldots a_s}, \tag{2.24}
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where \( \xi^{a_1 \ldots a_{s-1}} \) is traceless parameter of gauge transformation and the projector \( \Pi^{tr} \) is inserted to respect tracelessness constraint (2.23).

\[
\phi^{aaa_3 \ldots a_s} = 0, \quad \phi^{aaa_3 \ldots a_s} = 0. \tag{2.27}
\]

Conformal dimension of the field \( \phi^{a_1 \ldots a_s} \) is equal to \( s' + d - 2 \), while conformal dimension of the field \( \phi^{a_1 \ldots a_s} \) is equal to \( 2 - s' \).

ii) On space of the fields \( \phi^{a_1 \ldots a_s} \) (and separately on space of the fields \( \phi^{a_1 \ldots a_s} \)), we introduce new differential constraints, gauge transformations, and conformal algebra transformations.

iii) The new differential constraints are invariant under the gauge transformations and the conformal algebra transformations.

iv) The gauge symmetries and the new differential constraints make it possible to match our approach and the standard one, i.e., by appropriate gauge fixing of the Stueckelberg fields and by solving some differential constraints (Stueckelberg gauge frame) we obtain standard formulation of the conformal currents and shadow fields.

For the spin-\( s \) conformal current, use of the Stueckelberg gauge frame leads to \( \phi^{a_1 \ldots a_s} = 0 \) for \( s' = 0, 1, \ldots, s - 1 \) and divergence-free and tracelessness constraint for \( \phi^{a_1 \ldots a_s} \),
\[
\partial^{s} \phi^{a_2 \ldots a_s} = 0, \quad \phi^{s}_{sh} = 0. \tag{2.28}
\]

We see that, in the Stueckelberg gauge frame, our field \( \phi^{a_1 \ldots a_s} \) can be identified with the current \( T^{a_1 \ldots a_s} \) (2.21), i.e. our approach reduces to the standard one.

For the spin-\( s \) shadow field, use of the Stueckelberg gauge frame leads to tracelessness constraint for the field \( \phi^{a_1 \ldots a_s}_{sh} \),
\[
\phi^{aaa_3 \ldots a_s}_{sh} = 0, \tag{2.29}
\]
and this field is not subject to any differential constraint. Also, the Stueckelberg gauge frame makes it possible to express the fields \( \phi^{a_1 \ldots a_s}_{sh}, \quad s' = 0, 1, \ldots, s - 1 \), in terms of the field \( \phi^{a_1 \ldots a_s}_{sh} \), i.e., we are left with one traceless field \( \phi^{a_1 \ldots a_s}_{sh} \), which can be identified with the shadow field \( \Phi^{a_1 \ldots a_s} \) of the standard approach to \( CFT \).

Summary of the gauge invariant approach to the low-spin, \( s = 1, 2 \) conformal currents and shadow fields is given in Table I.
Table I. In the Table, we present the field contents, conformal dimensions, differential constraints, gauge transformations and transformations under operator $R^a$ entering the gauge invariant approach of the low-spin, $s = 1, 2$ conformal currents and shadow fields. The operators $R^a$ enter conformal boost transformations given in (2.16).

| Field cont. | Conf. dim. | Differential constraint | Gauge transformation | Action of operator $R^a$ |
|-------------|------------|-------------------------|---------------------|-------------------------|
| $\phi^n_{\text{cur}}$ | $d - 1$ | $\partial^a \phi^n_{\text{cur}} + \Box \phi^n_{\text{cur}} = 0$ | $\delta \phi^n_{\text{cur}} = \partial^a \xi_{\text{cur}}$ | $R^a \phi^n_{\text{cur}} = (2 - d) \eta^{ab} \phi^n_{\text{cur}}$ |
| $\phi_{\text{cur}}$ | $d - 2$ | | $\delta \phi_{\text{cur}} = - \xi_{\text{cur}}$ | $R^a \phi_{\text{cur}} = 0$ |

**spin-1 shadow field**

| Field | Conformal dim. | Differential constraint | Gauge transformation | Action of operator $R^a$ |
|-------|----------------|-------------------------|---------------------|-------------------------|
| $\phi^0_{\text{sh}}$ | 1 | $\partial^a \phi^0_{\text{sh}} + \phi_{\text{sh}} = 0$ | $\delta \phi^0_{\text{sh}} = \partial^a \xi_{\text{sh}}$ | $R^a \phi^0_{\text{sh}} = 0$ |
| $\phi_{\text{sh}}$ | 2 | | $\delta \phi_{\text{sh}} = - \Box \xi_{\text{sh}}$ | $R^a \phi_{\text{sh}} = (d - 2) \phi^0_{\text{sh}}$ |

**spin-2 current**

| Field | Conformal dim. | Differential constraint | Gauge transformation | Action of operator $R^a$ |
|-------|----------------|-------------------------|---------------------|-------------------------|
| $\phi^{ab}_{\text{cur}}$ | $d$ | $\partial^b \phi^{ab}_{\text{cur}} - \frac{1}{2} \partial^a \phi^{bb}_{\text{cur}} + \Box \phi^n_{\text{cur}} = 0$ | $\delta \phi^{ab}_{\text{cur}} = \partial^a \xi_{\text{cur}} + \partial^b \xi_{\text{cur}} + \frac{2}{d-2} \eta^{ab} \Box \xi_{\text{cur}}$ | $R^a \phi^{bc}_{\text{cur}} = 2 \eta^{bc} \phi^{ab}_{\text{cur}}$ |
| $\phi^n_{\text{cur}}$ | $d - 1$ | $\partial^a \phi^n_{\text{cur}} + \frac{1}{2} \phi^{aa}_{\text{cur}} + u \Box \phi^n_{\text{cur}} = 0$ | $\delta \phi^n_{\text{cur}} = \partial^a \xi_{\text{cur}} - \xi_{\text{cur}}$ | $R^a \phi^n_{\text{cur}} = - \sqrt{2(d - 1)(d - 2)} \eta^{ab} \phi^n_{\text{cur}}$ |
| $\phi_{\text{cur}}$ | $d - 2$ | | $\delta \phi_{\text{cur}} = - u \xi_{\text{cur}}$ | $R^a \phi_{\text{cur}} = 0$ |

**spin-2 shadow field**

| Field | Conformal dim. | Differential constraint | Gauge transformation | Action of operator $R^a$ |
|-------|----------------|-------------------------|---------------------|-------------------------|
| $\phi^{ab}_{\text{sh}}$ | 0 | $\partial^b \phi^{ab}_{\text{sh}} - \frac{1}{2} \partial^a \phi^{bb}_{\text{sh}} + \phi^n_{\text{sh}} = 0$ | $\delta \phi^{ab}_{\text{sh}} = \partial^a \xi_{\text{sh}} + \partial^b \xi_{\text{sh}} + \frac{2}{d-2} \eta^{ab} \xi_{\text{sh}}$ | $R^a \phi^{bc}_{\text{sh}} = 0$ |
| $\phi^n_{\text{sh}}$ | 1 | $\partial^a \phi^n_{\text{sh}} + \frac{1}{2} \Box \phi^{aa}_{\text{sh}} + u \phi_{\text{sh}} = 0$ | $\delta \phi^n_{\text{sh}} = \partial^a \xi_{\text{sh}} - \xi_{\text{sh}}$ | $R^a \phi^n_{\text{sh}} = d \phi^{ab}_{\text{sh}} - \eta^{ab} \phi^{cc}_{\text{sh}}$ |
| $\phi_{\text{sh}}$ | 2 | $u \equiv \left( \frac{2d - 1}{4} \right)^{1/2}$ | $\delta \phi_{\text{sh}} = - u \xi_{\text{sh}}$ | $R^a \phi_{\text{sh}} = \sqrt{2(d - 1)(d - 2)} \phi_{\text{sh}}$ |
Table II. In the Table, we present the field contents, conformal dimensions, differential constraints, gauge transformations and the operators $R^a$ entering the gauge invariant approach of arbitrary spin-$s$ conformal currents and shadow fields. The operators $R^a$ enter conformal boost transformations given in (2.16).

| Field content | Conformal dimension | Differential constraint | Gauge transformation | Operator $R^a$ |
|---------------|---------------------|-------------------------|---------------------|----------------|
| $\phi_{\text{cur}}$ | $s + d$ | $\hat{C}_{\text{cur}}|\phi_{\text{cur}}\rangle = 0,$ | $\delta|\phi_{\text{cur}}\rangle = \left(\alpha\partial - e_{1,\text{cur}}\right)$ | $R^a = \tilde{r}\left(\alpha^a - \alpha^2\frac{1}{2N_s^a + d - 2\bar{\alpha}_a^2}\right),$ |
|               | $-2 - N_s$ | $\hat{C}_{\text{cur}} = \bar{\alpha}\partial - \frac{1}{2}\alpha\partial\bar{\alpha}^2,$ | $-2\int \frac{\alpha^2}{2s + d - 6 - 2N_s^a}\bar{e}_{1,\text{cur}}|\xi_{\text{cur}}\rangle,$ | $\tilde{r} = -\sqrt{2s + d - 4 - N_s} \times \sqrt{2s + d - 4 - 2N_s^a\bar{\alpha}_a^2}.$ |
| $\phi_{\text{sh}}$ | $2 - s + N_s$ | $\hat{C}_{\text{sh}}|\phi_{\text{sh}}\rangle = 0,$ | $\delta|\phi_{\text{sh}}\rangle = \left(\alpha\partial - e_{1,\text{sh}}\right)$ | $R^a = r\left(\alpha^a - \alpha^2\frac{1}{2N_s^a + d - 2\bar{\alpha}_a^2}\right),$ |
|               | | $\hat{C}_{\text{sh}} = \bar{\alpha}\partial - \frac{1}{2}\alpha\partial\bar{\alpha}^2,$ | $-2\int \frac{\alpha^2}{2s + d - 6 - 2N_s^a}\bar{e}_{1,\text{sh}}|\xi_{\text{sh}}\rangle,$ | $r = \alpha^{s}\sqrt{2s + d - 4 - N_s} \times \sqrt{2s + d - 4 - 2N_s^a\bar{\alpha}_a^2}.$ |
|               | | $\bar{e}_{1,\text{sh}} = \bar{\alpha}^s\bar{e}_1,$ | $e_{1,\text{sh}} = -\bar{e}_1\bar{\alpha}_s,$ | \( \bar{e}_1 \equiv \left(\frac{2s + d - 4 - N_s}{2s + d - 4 - 2N_s^a}\right)^{1/2} \) |

To simplify presentation of the gauge invariant approach to the arbitrary spin-$s$ conformal current and shadow field we use oscillators (2.1) and introduce the respective ket-vectors $|\phi_{\text{cur}}\rangle$ and $|\phi_{\text{sh}}\rangle$ defined by

$$|\phi_{\text{cur}}\rangle = \sum_{s' = 0}^{s} \alpha^{s-s'}_z |\phi_{\text{cur}, s'}\rangle,$$

$$|\phi_{\text{cur}, s'}\rangle \equiv \frac{\alpha^{s_1 \ldots a_{s'}}}{s'!\sqrt{(s - s')!}} \phi^{a_1 \ldots a_{s'}}_{\text{cur}}|0\rangle,$$  

(2.30)

$$|\phi_{\text{sh}}\rangle = \sum_{s' = 0}^{s} \alpha^{s-s'}_z |\phi_{\text{sh}, s'}\rangle,$$

$$|\phi_{\text{sh}, s'}\rangle \equiv \frac{\alpha^{s_1 \ldots a_{s'}}}{s'!\sqrt{(s - s')!}} \phi^{a_1 \ldots a_{s'}}_{\text{sh}}|0\rangle.$$  

(2.31)

To describe gauge symmetries of the arbitrary spin-$s$ conformal current and spin-$s$ shadow field we use gauge transformation parameters which are the respective totally symmetric $so(d-1, 1)$ Lorentz algebra tensor fields $\xi^{a_1 \ldots a_{s'}}_{\text{cur}}$ and $\xi^{a_1 \ldots a_{s'}}_{\text{sh}},$ where $s' = 0, 1, \ldots, s - 1$. For $s' \geq 2$, these gauge transformation parameters are restricted to be traceless,

$$\xi^{a_1 a_2 a_3 \ldots a_{s'} a'}_{\text{cur}} = 0, \quad \xi^{a_1 a_2 a_3 \ldots a_{s'} a'}_{\text{sh}} = 0. \quad (2.32)$$

Again, to simplify presentation of result, we collect the gauge transformation parameters in the respective ket-vectors $|\xi_{\text{cur}}\rangle$ and $|\xi_{\text{sh}}\rangle$ defined by

$$|\xi_{\text{cur}}\rangle \equiv \sum_{s' = 0}^{s-1} \alpha^{s-1-s'}_z |\xi_{\text{cur}, s'}\rangle,$$

$$|\xi_{\text{cur}, s'}\rangle \equiv \frac{\alpha^{a_1 \ldots a_{s'}}}{s'!\sqrt{(s - 1 - s')!}} \xi^{a_1 \ldots a_{s'}}_{\text{cur}}|0\rangle,$$  

(2.33)

$$|\xi_{\text{sh}}\rangle \equiv \sum_{s' = 0}^{s-1} \alpha^{s-1-s'}_z |\xi_{\text{sh}, s'}\rangle,$$

$$|\xi_{\text{sh}, s'}\rangle \equiv \frac{\alpha^{a_1 \ldots a_{s'}}}{s'!\sqrt{(s - 1 - s')!}} \xi^{a_1 \ldots a_{s'}}_{\text{sh}}|0\rangle.$$  

(2.34)

With these conventions for the ket-vectors, summary of our study of gauge invariant approach to the arbitrary spin-$s$ conformal current and shadow field is given in Table II.
III. GAUGE INVARIANT TWO-POINT VERTEX FOR LOW-SPIN SHADOW FIELDS

We now discuss shadow field two-point vertex. In the gauge invariant approach, the vertex is determined by requiring the vertex to be invariant under gauge transformations of the shadow field. Also, the vertex should be invariant under conformal algebra transformations. We consider the two-point vertices for spin-1, spin-2, and arbitrary spin-s shadow fields in turn.

A. Gauge invariant two-point vertex for spin-1 shadow field

We begin with the discussion of the two-point vertex for the spin-1 shadow field. This simplest example demonstrates all characteristic features of our approach. In the gauge invariant approach, the spin-1 shadow field is described by vector field \( \phi_{sh}^a \) and scalar field \( \phi_{sh} \) (see Table I). The gauge invariant two-point vertex we find takes the form

\[
\Gamma = \int d^d x_1 d^d x_2 \Gamma_{12},
\]

\[
\Gamma_{12} = \frac{\phi_{sh}^a(x_1)\phi_{sh}^a(x_2)}{2|x_{12}|^2(d-1)} + \frac{1}{4(d-2)^2} \frac{\phi_{sh}(x_1)\phi_{sh}(x_2)}{|x_{12}|^2(d-2)},
\]

\[|x_{12}|^2 \equiv x_{12}^a x_{12}^a, \quad x_{12}^a = x_1^a - x_2^a .
\]

One can check that this vertex is invariant under the gauge transformations of the spin-1 shadow field given in Table I. The vertex is obviously invariant with respect to the Poincaré algebra and dilatation symmetries. Also, using the operator \( R^a \) given in Table I, we check that the vertex is invariant under the conformal boost transformations.

The kernel of the vertex \( \Gamma \) is related to a two-point correlation function of the spin-1 conformal current. In our approach, the spin-1 conformal current is described by gauge fields \( \phi_{cur}^a \), \( \phi_{cur} \) (see Table I). Therefore, in order to discuss correlation functions of the fields \( \phi_{cur}^a \), \( \phi_{cur} \) in a proper way, we should impose gauge condition on the fields \( \phi_{cur}^a \), \( \phi_{cur} \). The two-point correlation functions of gauge fixed fields \( \phi_{cur}^a \), \( \phi_{cur} \) are obtained from the kernel of the two-point vertex \( \Gamma \) taken to be in appropriate gauge frame. To explain what has just been said we discuss two gauge conditions which can be used for studying the correlations functions - Stueckelberg gauge and light-cone gauge. We would like to discuss these gauges because of the following reasons.

i) As we have said, the Stueckelberg gauge reduces our approach to the standard formulation of CFT. Therefore the use of the Stueckelberg gauge allows us to demonstrate how the standard two-point correlation function of the spin-1 conformal current is obtained from the kernel of our gauge invariant two-point vertex \( \Gamma \).

ii) Motivation for considering the light-cone gauge vertex comes from conjectured duality of the free large \( N = 4 \) SYM theory and the theory of massless higher-spin AdS fields [23]. On the one hand, one expects that massless higher-spin AdS fields appear in the tensionless limit of AdS string. On the other hand, by analogy with flat space, we expect that a quantization of the Green-Schwarz AdS superstring [28] will be straightforward only in the light-cone gauge [29, 30]. As we shall demonstrate in Sec. [VI] correlation function of the arbitrary spin-s conformal current is obtained from the effective action of massless arbitrary spin-s AdS field. Therefore it seems that from the stringy perspective of AdS/CFT correspondence, light-cone approach to CFT is the fruitful direction to go.

We now discuss the Stueckelberg gauge and light-cone gauge in turn.

Stueckelberg gauge frame two-point vertex. We begin with the discussion of Stueckelberg gauge fixed two-point vertex of the spin-1 shadow field, i.e. we relate our vertex with the one in the standard approach to CFT. In general, Stueckelberg gauge frame is achieved through the use of the Stueckelberg gauge and differential constraints. The gauge invariant approach to the spin-1 shadow field does not involve the Stueckelberg gauge symmetries. This implies that we should just solve the differential constraint which is given in Table I, \( \partial^a \phi_{sh}^a + \phi_{sh} = 0 \). Solution to this constraint is obvious, \( \phi_{sh} = -\partial^a \phi_{sh}^a \). Plugging this \( \phi_{sh} \) into (3.2) and ignoring the total derivative, we find that two-point density \( \Gamma_{12} \) (3.2) takes the following form:

\[
\Gamma_{12}^{Stueck.g.fram} = k_1 \Gamma_{12}^{stand},
\]

\[
\Gamma_{12}^{stand} = \frac{\phi_{sh}^a(x_1)O_{12}^{ab}\phi_{sh}^b(x_2)}{|x_{12}|^2(d-1)},
\]

\[
O_{12}^{ab} \equiv \eta^{ab} - \frac{2x_a^{a}x_b^{b}}{|x_{12}|^2},
\]

\[
k_1 \equiv \frac{d-1}{2(d-2)},
\]

[7] We note that, in the gauge invariant approach, correlation functions of the conformal current can be studied without gauge fixing. To do that one needs to construct gauge invariant field strengths for the gauge potentials \( \phi_{cur}^a \), \( \phi_{cur} \). Study of field strengths for the conformal currents is beyond the scope of this paper.

[8] Discussion of this theme in the context of various limits in AdS superstring may be found in [24-27].
where $\Gamma_{12}^{\text{stand (3.5)}}$ stands for the two-point vertex of the spin-1 shadow field in the standard approach to $CFT$. From (3.4), we see that our gauge invariant vertex taken to be in the Stueckelberg gauge frame coincides, up to normalization factor $k_1$, with the two-point vertex in the standard approach to $CFT$. As is well known, $\Gamma_{12}^{\text{stand (3.5)}}$ is invariant under gauge transformation appearing in the standard approach to $CFT$,

$$\delta\phi_{\text{sh}}^a = \partial^a \xi_{\text{sh}}. \quad (3.8)$$

The kernel of vertex $\Gamma_{12}^{\text{stand (3.5)}}$ defines a correlation function of the spin-1 conformal current taken to be in the Stueckelberg gauge frame. This is to say that the gauge invariant approach to the spin-1 conformal current involves the Stueckelberg gauge frame. This is to say that, using the gauge symmetry associated with the gauge transformation parameter $\xi_{\text{cur}}$ (see Table I). This symmetry allows us to gauge away the scalar field $\phi_{\text{cur}}$, by imposing the Stueckelberg gauge $\phi_{\text{cur}} = 0$. Thus, in the Stueckelberg gauge frame, we are left with divergence-free vector field $\phi_{\text{cur}}^a$. Two-point correlation function of this vector field is given by the kernel of vertex $\Gamma_{12}^{\text{stand (3.5)}}$.

$$\langle \phi_{\text{cur}}^a(x_1), \phi_{\text{cur}}^b(x_2) \rangle = \frac{O_{12}^{ab}}{|x_{12}|^{2(d-1)}}. \quad (3.9)$$

**Light-cone gauge two-point vertex.** Light-cone gauge frame is achieved through the use of the light-cone gauge and differential constraints. Taking into account the gauge transformation of the field $\phi_{\text{sh}}^a$ (see Table I), we impose the light-cone gauge condition,$^9$

$$\phi_{\text{sh}}^+ = 0. \quad (3.10)$$

Plugging this gauge condition in the differential constraint for spin-1 shadow field (see Table I) we obtain solution for $\phi_{\text{sh}}^-$,

$$\phi_{\text{sh}}^- = -\frac{\partial^j}{\partial^+ \phi_{\text{sh}}^j} - \frac{1}{\partial^+} \phi_{\text{sh}}. \quad (3.11)$$

We see that we are left with vector field $\phi_{\text{sh}}^a$ and scalar field $\phi_{\text{sh}}$. These fields constitute the field content of the light-cone gauge frame. Note that, in contrast to the Stueckelberg gauge frame, the scalar field $\phi_{\text{sh}}$ becomes an independent field D.o.F in the light-cone gauge frame.

Using (3.10) in (3.2) leads to light-cone gauge fixed vertex

$$\Gamma_{12}^{(1, l.c.)} = \frac{\partial^j_{\text{sh}}(x_1) \partial^j_{\text{sh}}(x_2)}{2|x_{12}|^{2(d-1)}} + \frac{1}{4(d-2)^2} \frac{\phi_{\text{sh}}(x_1) \phi_{\text{sh}}(x_2)}{|x_{12}|^{2(d-2)}}. \quad (3.12)$$

We note that, as in the case of gauge invariant vertex (3.2), light-cone vertex (3.12) is diagonal with respect to the fields $\phi_{\text{sh}}^a$ and $\phi_{\text{sh}}$. Note however that, in contrast to the gauge invariant vertex, the light-cone vertex is constructed out of the fields which are not subject to any constraints.

The kernel of the light-cone vertex gives two-point correlation function of spin-1 conformal current taken to be in the light-cone gauge. This is to say that, using the gauge symmetry of the spin-1 conformal current (see Table I), we impose light-cone gauge on the field $\phi_{\text{cur}}^a$,

$$\phi_{\text{cur}}^+ = 0. \quad (3.13)$$

Using this gauge condition in the differential constraint for the conformal spin-1 current (see Table I), we find

$$\phi_{\text{cur}}^- = -\frac{\partial^j}{\partial^+ \phi_{\text{cur}}^j} - \frac{\partial^a}{\partial^+ \phi_{\text{cur}}^a}. \quad (3.14)$$

We see that we are left with vector field $\phi_{\text{cur}}^a$ and scalar field $\phi_{\text{cur}}$. These fields constitute the field content of the light-cone gauge frame. Defining two-point correlation functions of the fields $\phi_{\text{cur}}^a$, $\phi_{\text{cur}}$ in a usual way,

$$\langle \phi_{\text{cur}}^a(x_1), \phi_{\text{cur}}^j(x_2) \rangle = \frac{\delta^{ij} \Gamma_{12}^{(1, l.c.)}}{\delta \phi_{\text{sh}}^a(x_1) \delta \phi_{\text{sh}}^j(x_2)}, \quad (3.15)$$

$$\langle \phi_{\text{cur}}(x_1), \phi_{\text{cur}}(x_2) \rangle = \frac{\delta^{ij} \Gamma_{12}^{(1, l.c.)}}{\delta \phi_{\text{sh}}(x_1) \delta \phi_{\text{sh}}(x_2)}, \quad (3.16)$$

and using (3.12), we obtain the two-point light-cone gauge correlation functions of the spin-1 conformal field,

$$\langle \phi_{\text{cur}}^a(x_1), \phi_{\text{cur}}^j(x_2) \rangle = \delta^{ij} |x_{12}|^{-2(d-1)}, \quad (3.17)$$

$$\langle \phi_{\text{cur}}(x_1), \phi_{\text{cur}}(x_2) \rangle = \frac{1}{2(d-2)^2} |x_{12}|^{-2(d-2)}. \quad (3.18)$$

**B. Gauge invariant two-point vertex for spin-2 shadow field**

We proceed with the discussion of two-point vertex for spin-2 shadow field. As compared to the spin-1 shadow field, this important case demonstrates some new features of our approach. In the gauge invariant approach, the spin-2 shadow field is described by rank-2 tensor field $\phi_{\text{sh}}^{ab}$, vector field $\phi_{\text{sh}}^a$, and scalar field $\phi_{\text{sh}}$. Note that the tensor field $\phi_{\text{sh}}^{ab}$ is not traceless. The gauge invariant two-point vertex we find takes the form given by (3.1), where the two-point density $\Gamma_{12}$ is given by

$$\Gamma_{12} = \frac{1}{4|x_{12}|^{2d}} \left( \phi_{\text{sh}}^{ab}(x_1) \phi_{\text{sh}}^{ab}(x_2) - \frac{1}{2} \delta^{ab} \phi_{\text{sh}}^{ab}(x_1) \phi_{\text{sh}}^{ab}(x_2) \right) + \frac{1}{4d(d-1)} \frac{\phi_{\text{sh}}^a(x_1) \phi_{\text{sh}}^a(x_2)}{|x_{12}|^{2(d-1)}} + \frac{1}{8d(d-1)(d-2)^2} \frac{\phi_{\text{sh}}(x_1) \phi_{\text{sh}}(x_2)}{|x_{12}|^{2(d-2)}}. \quad (3.19)$$
One can check that this vertex is invariant under the gauge transformations of the spin-2 shadow field given in Table I. Also, using the operator $R^a$ given in Table I, we check that the vertex is invariant under conformal boost transformations.

Remarkable feature of the vertex is its diagonal form with respect to the fields $\phi_{sh}^{ab}$, $\phi_{sh}^a$, and $\phi_{sh}$.

We now discuss Stueckelberg gauge and light-cone gauge fixed vertices in turn.

**Stueckelberg gauge frame two-point vertex.** As we have said, the standard approach to $CFT$ is obtained from our approach by using the Stueckelberg gauge frame. Therefore to illustrate our approach, we begin with the discussion of Stueckelberg gauge fixed two-point vertex of the spin-2 shadow field. The Stueckelberg gauge frame is achieved through the use of the Stueckelberg gauge and differential constraints. From the gauge transformations of the spin-2 shadow field given in Table I, we see that the trace of the rank-2 tensor field $\phi_{sh}^{ab}$ transforms as the Stueckelberg field, i.e., $\phi_{sh}^{aa}$ can be gauged away via the Stueckelberg gauge fixing.

\[
\phi_{sh}^{aa} = 0. \quad (3.20)
\]

Gauge condition (3.20) leads to traceless field $\phi_{sh}^{ab}$ which can be identified with the shadow field of the standard approach to $CFT$. Taking into account (3.20), we solve the differential constraints for the spin-2 shadow field (see Table I) to express the fields $\phi_{sh}^a$ and $\phi_{sh}$ in terms of the traceless tensor field $\phi_{sh}^{ab}$,

\[
\begin{align*}
\phi_{sh}^a &= -\partial^a \phi_{sh}^{ab}, \\
\phi_{sh} &= \frac{1}{u} \partial^a \partial^b \phi_{sh}^{ab},
\end{align*} \quad (3.21, 3.22)
\]

Relations (3.20)-(3.22) provide complete description of the Stueckelberg gauge frame for the spin-2 shadow field. Plugging (3.20)-(3.22) in (3.19) and ignoring the total derivative, we find that our two-point density $\Gamma_{12}^{\text{Stuck.g.fram}}$ (3.19) takes the following form:

\[
\Gamma_{12}^{\text{Stuck.g.fram}} = k_2 \Gamma_{12}^{\text{stand}}, \quad (3.24)
\]

\[
\Gamma_{12}^{\text{stand}} = \phi_{sh}^{a_1 a_2} (x_1) \frac{O^{a_1 b_1} O^{a_2 b_2}}{|x_1|^2} \phi_{sh}^{b_1 b_2} (x_2), \quad (3.25)
\]

\[
k_2 \equiv \frac{d + 1}{4(d - 1)}, \quad (3.26)
\]

where $O_{12}^{ab}$ is defined in (3.6), while $\Gamma_{12}^{\text{stand}}$ stands for the two-point vertex of the spin-2 shadow field in the standard approach to $CFT$. From (3.24), we see that our gauge invariant vertex taken to be in the Stueckelberg gauge frame coincides, up to normalization factor $k_2$, with the two-point vertex in the standard approach to $CFT$. We note that $\Gamma_{12}^{\text{stand}}$ (3.25) is invariant under gauge transformation appearing in the standard approach to $CFT$,

\[
\delta \phi_{sh}^{ab} = \partial^a \delta \phi_{sh}^{cb} + \partial^b \delta \phi_{sh}^{ca} - \frac{2}{d} \eta^{ab} \partial^c \phi_{sh}. \quad (3.27)
\]

The kernel of vertex $\Gamma_{12}^{\text{stand}}$ (3.25) defines two-point correlation function of the spin-2 conformal current taken to be in the Stueckelberg gauge frame. This is to say that the gauge invariant approach to the spin-2 conformal current involves the Stueckelberg gauge symmetries associated with the gauge transformation parameters $\xi^a_{\text{cur}}$, $\zeta_{\text{cur}}$ (see Table I). These symmetries allow us to gauge away the vector field $\phi_{cur}^a$ and the scalar field $\phi_{cur}$ by imposing the Stueckelberg gauge $\delta \phi_{cur}^a = 0$, $\phi_{cur} = 0$. After that, the differential constraints lead to traceless divergence-free tensor field $\phi_{ab}^{\text{cur}}$. Two-point correlation function of this tensor field is given by the kernel of vertex $\Gamma_{12}^{\text{stand}}$ (3.25).

**Light-cone gauge two-point vertex.** We now consider light-cone gauge fixed vertex. In our approach, the light-cone gauge frame is achieved through the use of the light-cone gauge and differential constraints. Taking into account the gauge transformations of the fields $\phi_{sh}^{ab}$, $\phi_{sh}^a$ (see Table I), we impose the light-cone gauge condition,

\[
\phi_{sh}^{+a} = 0, \quad \phi_{sh}^+ = 0. \quad (3.28)
\]

Plugging this gauge condition in the differential constraints for the spin-2 shadow field (see Table I) we find,

\[
\begin{align*}
\phi_{sh}^{ij} &= 0, \quad (3.29) \\
\phi_{sh}^{-i} &= -\partial^i \phi_{sh}^{ij} - \frac{1}{\partial^+} \phi_{sh}^i, \quad (3.30) \\
\phi_{sh}^{-a} &= -\partial^a \phi_{sh}^i + \frac{u}{\partial^+} \phi_{sh}, \quad (3.31) \\
\phi_{sh}^{-} &= \partial^i \partial^+ \phi_{sh}^{ij} + \frac{2\partial^i}{\partial^+ \partial^+} \phi_{sh}^i + \frac{u}{\partial^+ \partial^+} \phi_{sh}. \quad (3.32)
\end{align*}
\]

We see that we are left with the $so(d - 2)$ algebra traceless rank-2 tensor field $\phi_{sh}^{ij}$, vector field $\phi_{sh}^i$, and scalar field $\phi_{sh}$. These fields constitute the field content of the light-cone gauge frame. Note that, in contrast to the Stueckelberg gauge frame, the vector field $\phi_{sh}^i$, and the scalar field $\phi_{sh}$ become independent field D.o.F in the light-cone gauge frame.

Using (3.28), (3.29) in (3.19) leads to light-cone gauge fixed
We see that, as in the case of gauge invariant vertex (3.19), light-cone vertex (3.33) is diagonal with respect to the fields $\phi_{sh}^{ij}, \phi_{sh}^{i},$ and $\phi_{sh}$. Note however that, in contrast to the gauge invariant vertex, the light-cone vertex is constructed out of the fields which are not subject to any differential constraints.

The kernel of light-cone vertex (3.33) gives two-point correlation functions of the fields $\phi_{sh}^{ab}, \phi_{cur}$ (see Table I), we impose the light-cone gauge condition,

$$\phi_{cur}^{+a} = 0, \quad \phi_{cur}^{a} = 0.$$

(3.34)

Using this gauge condition in the differential constraints for the conformal spin-2 current (see Table I), we find

$$\phi_{cur}^{ij} = 0,$$

(3.35)

$$\phi_{cur}^{i} = -\frac{\partial^{i} \phi_{cur}}{\partial^{+} \phi_{cur}},$$

(3.36)

$$\phi_{cur}^{-i} = \frac{\partial^{i} \phi_{cur}}{\partial^{+} \phi_{cur}},$$

(3.37)

$$\phi_{cur}^{-} = \frac{\partial^{i} \partial^{j} \phi_{cur}}{\partial^{+} \phi_{cur}} + \frac{2 \partial^{i} \phi_{cur}}{\partial^{+} \phi_{cur}} + \frac{u \partial^{2} \phi_{cur}}{\partial^{+} \phi_{cur}}.$$

(3.38)

We see that we are left with traceless rank-2 tensor field $\phi_{cur}^{ij}$, vector field $\phi_{cur}^{i}$ and scalar field $\phi_{cur}$. These fields constitute the field content of the light-cone gauge frame.

Defining two-point correlation functions of the fields $\phi_{cur}^{ij}, \phi_{cur}^{i}, \phi_{cur}$ as the second functional derivative of $\Gamma$ with respect to the fields $\phi_{sh}^{ij}, \phi_{sh}^{i}, \phi_{sh}$ we obtain

$$\langle \phi_{cur}^{ij}(x_1), \phi_{cur}^{kl}(x_2) \rangle = \frac{1}{2} \langle x_{12} \rangle^{-2d} \Pi^{ijkl},$$

(3.39)

$$\langle \phi_{cur}^{i}(x_1), \phi_{cur}^{j}(x_2) \rangle = \frac{|x_{12}|^{-2d-1}}{2d} \delta^{ij},$$

(3.40)

$$\langle \phi_{cur}(x_1), \phi_{cur}(x_2) \rangle = \frac{|x_{12}|^{-2d-2}}{4d(d-1)(d-2)} \Pi^{ijkl},$$

(3.41)

where we use the notation

$$\Pi^{ijkl} = \frac{1}{2} \left( \delta^{ik} \delta^{jl} + \delta^{il} \delta^{jk} - \frac{2}{d-2} \delta^{ij} \delta^{kl} \right).$$

(3.42)
collected in the ket-vector $|\phi_{\text{cur}}\rangle$. Using the Stueckelberg gauge and light-cone gauge, we now demonstrate relation of the conformal current correlation function to the gauge fixed two-point vertex $\Gamma$.

**Stueckelberg gauge frame two-point vertex.** As we have said, standard approach to $CFT$ is obtained from our approach by using the Stueckelberg gauge frame. Therefore to illustrate our approach, we begin with the discussion of Stueckelberg gauge fixed two-point vertex of the spin-$s$ shadow field. The Stueckelberg gauge frame is achieved through the use of the Stueckelberg gauge and differential constraints. Using the gauge transformations and differential constraint of the spin-$s$ shadow field given in Table II, one can demonstrate that, in the Stueckelberg gauge frame, one has the following relations:

$$\hat{\alpha}^2|\phi_{sh}, s'\rangle = 0,$$  \hspace{1cm} (4.9)

$$|\phi_{sh}, s'\rangle = X_{s'}(\tilde{\alpha}\bar{\partial})^{s-s'}|\phi_{sh}, s\rangle,$$  \hspace{1cm} (4.10)

$$X_{s'} \equiv \frac{(-)^{s-s'}}{(s-s')!} \times \frac{2^{s-s'}\Gamma(s+s'+d-3)\Gamma(s+d\frac{d-2}{2})}{\Gamma(2s+d-3)\Gamma(s'+d\frac{d+2}{2})},$$  \hspace{1cm} (4.11)

$s' = 0,1,\ldots,s$. Relation (4.9) tells us that all fields $\phi_{sh}^{a_1\ldots a_s}$ are traceless, while from relation (4.10) we learn that the fields $\phi_{sh}^{a_1\ldots a_{s'}}$, with $s' = 0,1,\ldots,s-1$, can be expressed in terms of the rank-$s$ tensor field $\phi_{sh}^{a_1\ldots a_s}$. Plugging (4.10) in (4.1) and ignoring total derivative, we get

$$\Gamma_{12}^{\text{Stueck-g.fram}} = k_s\Gamma_{\text{stand}}^{12}$$ \hspace{1cm} (4.12)

$$\Gamma_{12}^{\text{stand}} = s!|\phi_{sh}, s(x_1)||O_{12}(\alpha, \bar{\alpha})|\phi_{sh}, s(x_2)||,$$ \hspace{1cm} (4.13)

$$O_{12}(\alpha, \bar{\alpha}) \equiv \sum_{k=0}^{s} \frac{(-)^{s-k}k!}{k!} \frac{(\alpha x_{12})^k(\bar{\alpha} x_{12})^k}{|x_{12}|^{2(s+d-2+k)}},$$ \hspace{1cm} (4.14)

$$k_s \equiv \frac{2s+d-3}{2s!(s+d-3)},$$ \hspace{1cm} (4.15)

where $\alpha x_{12} = \alpha^a x_{12}^a$, $\bar{\alpha} x_{12} = \bar{\alpha}^a x_{12}^a$ and $\Gamma_{12}^{\text{stand}}$ in r.h.s. of (4.12) stands for the two-point vertex of the spin-$s$ shadow field in the standard approach to $CFT$. Relation (4.13) provides oscillator representation for the $\Gamma_{12}^{\text{stand}}$. In terms of the tensor field $\phi_{sh}^{a_1\ldots a_s}$, vertex $\Gamma_{12}^{\text{stand}}$ (4.13) can be represented in the commonly used form,

$$\Gamma_{12}^{\text{stand}} = \phi_{sh}^{a_1\ldots a_s}(x_1)\frac{O_{12}^{a_1b_1}\cdots O_{12}^{a_rb_r}}{|x_{12}|^{2(s+d-2)}}\phi_{sh}^{b_1\ldots b_r}(x_2),$$ \hspace{1cm} (4.16)

where $O_{12}^{ab}$ is defined in (3.6). From (4.12), we see that our gauge invariant vertex $\Gamma_{12}$ taken to be in the Stueckelberg gauge frame coincides, up to normalization factor $k_s$, with the two-point vertex in the standard approach to $CFT$.

The kernel of vertex $\Gamma_{12}^{\text{stand}}$ (4.16) defines two-point correlation function of the spin-$s$ conformal current taken to be in the Stueckelberg gauge frame. This is to say that, in the Stueckelberg gauge frame, we obtain $\phi_{sh}^{a_{12}a_{s'}} = 0$ for $s' = 0,1,\ldots,s-1$ and we are left with the traceless divergence-free conformal current $\phi_{sh}^{a_{12}a_s}$. Two-point correlation function of this conformal current is given by the kernel of vertex $\Gamma_{12}^{\text{stand}}$ (4.16).

As a side of remark we note that $\Gamma_{12}^{\text{stand}}$ (4.16) is invariant under gauge transformation appearing in the standard approach to $CFT$,

$$\delta\phi_{sh}^{a_1\ldots a_s} = s!\Pi^{t\bar{r}}\phi^{(t\bar{r})(a_1\ldots a_s)}_{sh},$$ \hspace{1cm} (4.17)

where $\xi^{a_1\ldots a_{s-1}}$ is traceless parameter of gauge transformation and the projector $\Pi^{t\bar{r}}$ is inserted to respect tracelessness constraint for the field $\phi_{sh}^{a_1\ldots a_{s-1}}$.

**Light-cone gauge two-point vertex.** The light-cone gauge frame is achieved through the use of the light-cone gauge and differential constraints. We impose the conventional light-cone gauge,

$$\bar{\alpha}^+\Pi^{[1,2]}|\phi_{sh}\rangle = 0,$$ \hspace{1cm} (4.18)

where $\Pi^{[1,2]}$ is given in (2.7), and analyze the differential constraint for the spin-$s$ shadow field $|\phi_{sh}\rangle$ (see Table II). We find that solution to the differential constraint can be expressed in terms of light-cone ket-vector $|\phi_{sh}^{1c}\rangle$,

$$|\phi_{sh}\rangle = \exp\left(\frac{\alpha^+}{\partial^+}(\bar{\alpha}^i\partial^i - \bar{e}_{1sh})\right)|\phi_{sh}^{1c}\rangle,$$ \hspace{1cm} (4.19)

$$\tilde{\alpha}^i\alpha^i|\phi_{sh}^{1c}\rangle = 0,$$ \hspace{1cm} (4.20)

where the light-cone ket-vector $|\phi_{sh}^{1c}\rangle$ is obtained from $|\phi_{sh}\rangle$ (2.31) by equating $\alpha^+ = \alpha^- = 0$,

$$|\phi_{sh}^{1c}\rangle = |\phi_{sh}\rangle|_{\alpha^+ = \alpha^- = 0}.$$ \hspace{1cm} (4.21)

We see that we are left with fields $\phi_{sh}^{1\ldots i_{s'}}, s' = 0,1,\ldots,s$, which are traceless so $(d-2)$ algebra tensor fields, $\phi_{sh}^{i_{s+1}\ldots i_{s'}} = 0$. These fields constitute the field content of the light-cone gauge frame. Note that, in contrast to the Stueckelberg gauge frame, the fields $\phi_{sh}^{1\ldots i_{s'}}, s' = 0,1,\ldots,s-1$, become independent field D.o.F in the light-cone gauge frame. Also note that, in contrast to the gauge invariant approach, the fields $\phi_{sh}^{1\ldots i_{s'}}, s' = 0,1,\ldots,s$, are not subject to any differential constraints.
Using (4.19), (4.20) in (4.1) leads to light-cone gauge fixed vertex

\[ \Gamma^{i.c.}_{12} = \frac{1}{2} \langle \phi_{\text{sh}}^{i.c.}(x_1) \rangle \frac{f_{\nu}}{|x_{12}|^{2i_{\nu}}(\phi_{\text{sh}}^{i.c.}(x_2))}, \]  

(4.22)

where \( f_\nu \) is defined in (4.3).

To illustrate the structure of vertex \( \Gamma^{i.c.}_{12} \) (4.22) we note that, in terms of the fields \( \phi_{\text{sh}}^{i_1\ldots i_s'} \), the vertex can be represented as

\[ \Gamma^{i.c.}_{12} = \sum_{s'=0}^s \Gamma^{i.c.}_{12}^{(s')} \]  

(4.23)

where \( \Gamma^{i.c.}_{12}^{(s')} \) is given in (4.8). We see that, as in the case of gauge invariant vertex, light-cone vertex (4.22) is diagonal with respect to the light-cone fields \( \phi_{\text{sh}}^{i_1\ldots i_s'} \), \( s' = 0, 1, \ldots, s \).

Note however that, in contrast to the gauge invariant vertex, the light-cone vertex is constructed out of the light-cone fields which are not subject to any differential constraints.

As usually, the kernel of light-cone vertex (4.22) gives two-point correlation function of the spin-\( s \) conformal current (see Table I) we impose light-cone gauge condition on the ket-vector

\[ \phi_{\text{cur}} | \alpha \rangle, \]  

(4.25)

where \( \alpha \) is given in (2.7). Using this gauge condition in the differential constraint for the conformal spin-\( s \) current (see Table I), we find

\[ |\phi_{\text{cur}}\rangle = \exp \left( -\frac{\alpha^+}{\partial^+} (\bar{\alpha}^i \partial_i - \bar{e}_1 \text{cur} \mp) \right) |\phi_{\text{cur}}^{i.c.}\rangle, \]  

(4.26)

where a light-cone ket-vector \( |\phi_{\text{cur}}^{i.c.}\rangle \) is obtained from \( |\phi_{\text{cur}}\rangle \) (2.30) by equating \( \alpha^+ = \alpha^- = 0 \),

\[ |\phi_{\text{cur}}^{i.c.}\rangle = |\phi_{\text{cur}}\rangle \bigg|_{\alpha^+ = \alpha^- = 0}. \]  

(4.27)

We see that we are left with light-cone fields \( \phi_{\text{cur}}^{i_1\ldots i_s'} \), \( s' = 0, 1, \ldots, s \), which are traceless tensor fields of \( \text{so}(d-2) \) algebra, \( \phi_{\text{cur}}^{i_1\ldots i_1} = 0 \). These fields constitute the field content of the light-cone gauge frame. Note that, in contrast to the Stueckelberg gauge frame, the fields \( \phi_{\text{cur}}^{i_1\ldots i_s'} \), with \( s' = 1, \ldots, s \), are not equal to zero. Also note that, in contrast to the gauge invariant approach, the fields \( \phi_{\text{cur}}^{i_1\ldots i_s'} \), \( s' = 0, 1, \ldots, s \), are not subject to any differential constraints.

Defining two-point correlation functions of the fields \( \phi_{\text{cur}}^{i_1\ldots i_s'} \), as the second functional derivative of \( \Gamma \) with the respect to the shadow fields \( \phi_{\text{sh}}^{i_1\ldots i_s'} \), we obtain the following correlation functions:

\[ \langle \phi_{\text{cur}}^{i_1\ldots i_s'}(x_1), \phi_{\text{cur}}^{i_1\ldots i_s'}(x_2) \rangle = \frac{w_{s'}}{|x_{12}|^{2(s'+d-2)} \Pi^{i_1\ldots i_s' j_1\ldots j_s'}}, \]  

(4.29)

\( s' = 0, 1, \ldots, s \), where \( w_{s'} \) is defined in (4.8) and \( \Pi^{i_1\ldots i_s' j_1\ldots j_s'} \) stands for the projector on traceless spin-\( s' \) tensor field of the \( \text{so}(d-2) \) algebra. Explicit form of the projector may be found e.g. in Ref. [31].

V. ADS/CFT CORRESPONDENCE

We now apply our results to the study of AdS/CFT correspondence for free massless arbitrary spin AdS fields and boundary shadow fields. To this end we use the gauge invariant CFT adapted description of AdS massless fields and modified (Lorentz) de Donder gauge found in Ref. [32]. Our massless fields are obtained from Fronsdal fields by the invertible transformation which is described in Sec V in Ref. [32]. It is the use of our fields and the modified (Lorentz) de Donder gauge that leads to decoupled form of gauge fixed equations of motion and surprisingly simple Lagrangian. Owing these properties of our fields and the modified (Lorentz) de Donder gauge, it is possible to significantly simplify the computation of the effective action. We remind that the bulk action evaluated on solution of the Dirichlet problem is referred to as effective action in this paper. Note also that, from the very beginning, our CFT adapted gauge invariant Lagrangian is formulated in the Poincaré parametrization of AdS space,

\[ ds^2 = \frac{1}{2} (dx^a dx^a + dz dz). \]  

(5.1)

Therefore our Lagrangian is explicitly invariant with respect to boundary Poincaré symmetries, i.e., manifest symmetries of our Lagrangian are adapted to manifest symmetries of boundary CFT.

In this Section, using the modified (Lorentz) de Donder gauge, we are going to demonstrate that action of massless spin-\( s \) AdS field, when it is evaluated on solution of equations of motion with the Dirichlet problem corresponding to the boundary shadow field, is equal, up to normalization factor, to the gauge invariant two-point vertex of spin-\( s' \) shadow field which was obtained in Secs. III and IV. Also we find the normalization factor.

Remarkable feature of our approach is that it can be generalized to the case of massive fields in a relatively straightforward way. This can be done by using CFT adapted approach to massive AdS fields developed in Ref. [33].
In the standard approach to the on-shell leftover gauge symmetries of bulk $AdS$ fields. Note however that, in our approach, we have gauge symmetries not only at $AdS$ side, but also at the boundary $CFT$.

These gauge symmetries are also related via $AdS/CFT$ correspondence. Namely, in Ref. [1], we demonstrated that the on-shell leftover gauge symmetries of non-normalizable solutions of bulk $AdS$ fields match with the gauge symmetries of shadow fields. It is this matching of on-shell leftover gauge symmetries of non-normalizable solutions and the gauge symmetries of shadow fields that explains why the effective action coincides with the gauge invariant two-point vertex of shadow field.

A. AdS/CFT correspondence for scalar field

Action of arbitrary spin $AdS$ field taken to be in the modified (Lorentz) de Donder gauge is similar to the action for a massive scalar $AdS$ field. In fact, this is main advantage of using the modified (Lorentz) de Donder gauge condition. Therefore we begin with brief review of the computation of the effective action for massive scalar field.

Action and Lagrangian for the massive scalar field in $AdS_{d+1}$ background take the form

$$S = \int d^dxdz \mathcal{L}, \quad (5.2)$$

$$\mathcal{L} = \frac{1}{2} \sqrt{|g|} \left( g^{\mu \nu} \partial_\mu \Phi \partial_\nu \Phi + m^2 \Phi^2 \right). \quad (5.3)$$

In terms of the canonical normalized field $\phi$ defined by

$$\Phi = z^{\frac{\Delta}{2}} \phi, \quad (5.4)$$

the Lagrangian takes the form (up to total derivative)

$$\mathcal{L} = \frac{1}{2} (d\phi)^2 + \frac{1}{2} |T - \frac{\nu}{z}|^2, \quad (5.5)$$

$|d\phi|^2 \equiv \partial^\alpha \phi \partial^\alpha \phi$, where we use the notation

$$T_\nu \equiv \partial_z + \frac{\nu}{z}, \quad (5.6)$$

$$\nu = \sqrt{m^2 + \frac{d^2}{4}}. \quad (5.7)$$

We note that $\nu$ is related with the conformal dimension of boundary conformal spin-0 current, $\phi_{\text{cur}}$ as

$$\nu = \Delta - \frac{d}{2}. \quad (5.8)$$

We assume that $\nu > 0$. Also note that, for massless scalar field, $m^2 = (1 - d^2)/4$.

Equations of motion obtained from Lagrangian (5.5) take the form

$$\Box \phi = 0, \quad (5.9)$$

$$\Box \nu \equiv \Box + \partial_\nu^2 - \frac{1}{z^2} (\nu^2 - \frac{1}{4}). \quad (5.10)$$

It is easy to see that by using equations of motion (5.9) in bulk action (5.2) with Lagrangian (5.5) we obtain the effective action given by

$$S_{\text{eff}} = - \int d^d x \mathcal{L}_{\text{eff}} \bigg|_{z \to 0}, \quad (5.11)$$

$$L_{\text{eff}} = \frac{1}{2} \phi T _{\nu} \phi. \quad (5.12)$$

Following the procedure in [34], we note that solution of equations (5.9) with the Dirichlet problem corresponding to boundary shadow scalar field $\phi_{\text{sh}}$ takes the form

$$\phi(x, z) = \sigma \int d^d y G_\nu(x - y, z) \phi_{\text{sh}}(y), \quad (5.13)$$

$$G_\nu(x, z) = \frac{c_\nu z^{\nu + \frac{d}{2}}}{(z^2 + |x|^2)^{\nu + \frac{d}{2}}}, \quad (5.14)$$

$$c_\nu = \frac{\Gamma(\nu + \frac{d}{2})}{\pi^{d/2} \Gamma(\nu)}. \quad (5.15)$$

To be flexible, we use normalization factor $\sigma$ in (5.13). For the case of scalar field, commonly used normalization in (5.13) is achieved by setting $\sigma = 1$.

Using asymptotic behavior of the Green function

$$G_\nu(x, z) \xrightarrow{z \to \infty} z^{-\nu + \frac{d}{2}} \delta^d(x), \quad (5.16)$$

we find the asymptotic behavior of our solution

$$\phi(x, z) \xrightarrow{z \to \infty} z^{-\nu + \frac{d}{2}} \sigma \phi_{\text{sh}}(x). \quad (5.17)$$

From this expression, we see that our solution has indeed asymptotic behavior corresponding to the shadow scalar field.

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\[11\] In the standard approach to $CFT$, only the shadow fields are transformed under gauge transformations, while in our gauge invariant approach both the currents and shadow fields are transformed under gauge transformations, i.e., our approach allows us to study the currents and shadow fields on an equal footing.

\[12\] In Secs. V and VI we use the Euclidian signature.
Plugging solution of the Dirichlet problem (5.13) into (5.11), (5.12), we obtain the effective action

\[-S_{\text{eff}} = \nu c_{\nu} \sigma^{2} \int d^{d}x_{1}d^{d}x_{2} \frac{\phi_{\text{sh}}(x_{1})\phi_{\text{sh}}(x_{2})}{|x_{12}|^{2d+2}}.\] (5.18)

Plugging the commonly used value of $\sigma$, $\sigma = 1$, in (5.18), we obtain the properly normalized effective action found in Refs. [35, 36]. Note however that our Lagrangian (5.5) differs from the one in Refs. [35, 36] by total derivative with respect to the radial coordinate $z$, which gives nontrivial contribution to the effective action. Coincidence of our result and the one in Refs. [35, 36] is related to the fact that we fix boundary value from the one in Refs. [35, 36] by total derivative with respect to the radial coordinate $z$, which gives nontrivial contribution to the effective action. Details of our computation may be found in Appendix C.

B. AdS/CFT correspondence for spin-1 field

We now discuss AdS/CFT correspondence for bulk massless spin-1 AdS field and boundary spin-1 shadow field. To this end we use CFT adapted gauge invariant Lagrangian and the modified Lorentz gauge condition [12, 32, 33] (see Appendix D).

In AdS$_{d+1}$ space, the massless spin-1 field is described by fields $\phi^{\mu}(x, z)$ and $\phi(x, z)$ which are the respective vector and scalar fields of the $so(d)$ algebra. CFT adapted gauge invariant action and Lagrangian for these fields take the form,

\[S = \int d^{d}xdz \mathcal{L},\] (5.19)

\[\mathcal{L} = \frac{1}{2}|d\phi^{\mu}|^{2} + \frac{1}{2}|d\phi|^{2} + \frac{1}{2}|\mathcal{T}_{\nu_{1} - \frac{1}{2}} \phi|^{2} + \frac{1}{2}|\mathcal{T}_{\nu_{0} - \frac{1}{2}} \phi|^{2} - \frac{1}{2} C^{2},\] (5.20)

where we use the notation

\[C \equiv \partial^{\mu} \phi^{\mu} + \mathcal{T}_{\nu_{1} - \frac{1}{2}} \phi,\] (5.21)

\[\nu_{1} = \frac{d - 2}{2}, \quad \nu_{0} = \frac{d - 4}{2},\] (5.22)

and $\mathcal{T}_{\nu}$ is given in (5.6). Lagrangian (5.20) is invariant under gauge transformations

\[\delta \phi^{\mu} = \partial^{\mu} \xi,\] (5.23)

\[\delta \phi = \mathcal{T}_{\nu_{1} - \frac{1}{2}} \xi,\] (5.24)

where $\xi$ is a gauge transformation parameter.

Gauge invariant equations of motion obtained from Lagrangian (5.20) take the form

\[\Box_{\nu_{1}} \phi^{\mu} - \partial^{\mu} C = 0,\] (5.25)

\[\Box_{\nu_{0}} \phi - \mathcal{T}_{\nu_{1} - \frac{1}{2}} C = 0,\] (5.26)

where the operator $\Box_{\nu}$ and $\nu$’s are given in (5.10) and (5.22) respectively. We see that gauge invariant equations (5.25), (5.26) are coupled. Using equations of motion (5.25), (5.26) in bulk action (5.19), we obtain the following boundary effective action:

\[S_{\text{eff}} = - \int d^{d}x L_{\text{eff}} \bigg|_{z \to 0},\] (5.27)

\[\mathcal{L}_{\text{eff}} = \frac{1}{2} \phi^{\mu} \mathcal{T}_{\nu_{1} - \frac{1}{2}} \phi^{\mu} + \frac{1}{2} \phi \mathcal{T}_{\nu_{0} - \frac{1}{2}} \phi - \frac{1}{2} C^{2}.\] (5.28)

Now we would like to demonstrate how use of the modified Lorentz gauge condition provides considerable simplification in solving the equations of motion and computing effective action (5.27). To this end we note that it is the quantity $C$ given in (5.21) that defines the modified Lorentz gauge condition,

\[C = 0, \quad \text{modified Lorentz gauge}.\] (5.29)

Using this gauge condition in equations of motion (5.25), (5.26) gives simple gauge fixed equations of motion,

\[\Box_{\nu_{1}} \phi^{\mu} = 0,\] (5.30)

\[\Box_{\nu_{0}} \phi = 0.\] (5.31)

Thus, we see that the gauge fixed equations of motions are decoupled. Using modified Lorentz gauge (5.29) in (5.28), we obtain

\[\mathcal{L}_{\text{eff}} \bigg|_{C = 0} = \frac{1}{2} \phi^{\mu} \mathcal{T}_{\nu_{1} - \frac{1}{2}} \phi^{\mu} + \frac{1}{2} \phi \mathcal{T}_{\nu_{0} - \frac{1}{2}} \phi,\] (5.32)

i.e. we see that $\mathcal{L}_{\text{eff}}$ is also simplified.

In order to find $S_{\text{eff}}$ we should solve equations of motion (5.30), (5.31) with the Dirichlet problem corresponding to the boundary shadow field and plug the solution into (5.27), (5.28). We now discuss solution of equations of motion (5.30), (5.31). Because equations of motion (5.30), (5.31) are similar to the ones for scalar AdS field (5.9) we can simply
apply result in Sec. VA This is to say that solution of equations (5.30), (5.31) with the Dirichlet problem corresponding to the spin-1 shadow field takes the form

\[ \phi^a(x, z) = \sigma_{1, \nu_1} \int d^d y G_{\nu_1}(x - y, z) \phi_{sh}^a(y), \quad (5.33) \]

\[ \phi(x, z) = \sigma_{1, \nu_0} \int d^d y G_{\nu_0}(x - y, z) \phi_{sh}(y), \quad (5.34) \]

\[ \sigma_{1, \nu_1} = 1, \quad (5.35) \]

\[ \sigma_{1, \nu_0} = -\frac{1}{d - 4}, \quad (5.36) \]

where the Green function is given in (5.14). Note that coefficient \( \sigma_{1, \nu_0} \) is singular when \( d = 4 \). In addition to this singularity, there are other singularities when \( d \) is even integer (see Appendix C). Therefore to keep the discussion from being singularity, there are other singularities when simply use the dimensional regularization in the intermediate

ducient

tation (5.14). Note that coefficients (5.30), (5.31) with the Dirichlet problem corresponding to the spin-1 shadow field takes the form

\[ \phi^a(x, z) \rightarrow 0 \rightarrow z^{-\nu_1 + \frac{1}{2}} \phi_{sh}^a(x), \quad (5.37) \]

\[ \phi(x, z) \rightarrow 0 \rightarrow -\frac{z^{-\nu_0 + \frac{1}{2}}}{d - 4} \phi_{sh}(x). \quad (5.38) \]

From these expressions, we see that our solution has indeed asymptotic behavior corresponding to the spin-1 shadow field. Note that because the solution has non-integrable asymptotic behavior (5.37), (5.38), such solution is referred to as the non-normalizable solution in the literature.

We now explain the choice of the normalization factors \( \sigma_{1, \nu_1}, \sigma_{1, \nu_0} \) in (5.33), (5.36). The choice of \( \sigma_{1, \nu_1} \) is a matter of convention. Following commonly used convention, we set this coefficient to be equal to 1. The remaining normalization factor \( \sigma_{1, \nu_0} \) is then determined uniquely by requiring that the modified Lorentz gauge condition for the spin-1 AdS field (5.29) be amout the differential constraint for the spin-1 shadow field (see Table I). With the choice made in (5.33)-(5.36) we find the relations

\[ \delta^a \phi^a = \int d^d y G_{\nu_1}(x - y, z) \delta^a \phi_{sh}^a(y), \quad (5.39) \]

\[ \delta \phi = \int d^d y G_{\nu_1}(x - y, z) \delta \phi_{sh}(y). \quad (5.40) \]

From these relations and (5.21), we see that our choice of \( \sigma_{1, \nu_1}, \sigma_{1, \nu_0} \) allows us to match modified Lorentz gauge for the spin-1 AdS field (5.29) and differential constraint for the spin-1 shadow field given in Table I.

All that remains to obtain \( S_{\text{eff}} \) is to plug solution of the Dirichlet problem for AdS fields, (5.33), (5.34) into (5.27), (5.28). Using general formula given in (5.18), we obtain

\[ -S_{\text{eff}} = (d - 2)c_{\nu_1} \Gamma, \quad (5.41) \]

\[ c_{\nu_1} = \frac{\Gamma(d - 1)}{\pi^{d/2} \Gamma(d - 2)}, \quad (5.42) \]

where \( \Gamma \) is gauge invariant two-point vertex of the spin-1 shadow field given in (3.1), (3.2).

Thus, we see that imposing the modified Lorentz gauge on the massless spin-1 AdS field and computing the bulk action on the solution of equations of motion with the Dirichlet problem corresponding to the boundary shadow field we obtain the gauge invariant two-point vertex of the spin-1 shadow field.

Because in the literature \( S_{\text{eff}} \) is expressed in terms of two-point vertex taken in the Stueckelberg gauge frame, \( \Gamma_{\text{stand}} \), we use (3.4) and represent our result (5.41) as

\[ -S_{\text{eff}} = \frac{1}{2} (d - 1)c_{\nu_1} \Gamma_{\text{stand}}. \quad (5.43) \]

This relation, by using the radial gauge for AdS fields, was obtained in Ref. [34]. The normalization factor in r.h.s. of (5.43) was found in Ref. [36]. Note that we have obtained more general relation given in (5.41), while relation (5.43) is obtained from (5.41) by using the Stueckelberg gauge frame. The fact that \( S_{\text{eff}} \) is related to \( \Gamma_{\text{stand}} \) is expected because of conformal symmetry. What is important for the systematic study of AdS/CFT correspondence is the computation of the normalization factor in front of \( \Gamma_{\text{stand}} \).

As a side of a remark we note that the modified Lorentz gauge and gauge-fixed equations have left-over on-shell gauge symmetry. Namely, modified Lorentz gauge (5.29) and gauge-fixed equations (5.30), (5.31) are invariant under gauge transformations given in (5.23), (5.24). Provided the gauge transformation parameter satisfies the equation

\[ \Box \phi \xi = 0. \quad (5.44) \]

Solution to this equation is given by

\[ \xi(x, z) = \int d^d y G_{\nu_1}(x - y, z) \xi_{sh}(y). \quad (5.45) \]

Plugging this solution in (5.23), (5.24) we represent the gauge transformations of \( \phi^a(x, z) \) and \( \phi(x, z) \) as

\[ \delta \phi^a = \int d^d y G_{\nu_1}(x - y, z) \delta^a \xi_{sh}(y), \quad (5.46) \]

\[ \delta \phi = \int \frac{1}{d - 4} d^d y G_{\nu_0}(x - y, z) \Box \xi_{sh}(y). \quad (5.47) \]
Comparing (5.46), (5.47) with (5.33), (5.34), we see that the on-shell left-over gauge symmetries of solution of the Dirichlet problem for AdS spin-1 field amount to the gauge symmetries of the spin-1 shadow field (see Table I). It is this matching of the on-shell leftover gauge symmetries of solutions of the Dirichlet problem and the gauge symmetries of the shadow field that explains why the effective action coincides with the gauge invariant two-point vertex for the boundary shadow field.

C. AdS/CFT correspondence for spin-2 field

We now proceed with the discussion of AdS/CFT correspondence for bulk massless spin-2 AdS field and boundary spin-2 shadow field. To this end we use CFT adapted gauge invariant Lagrangian and modified de Donder gauge condition for the massless spin-2 AdS field found in Ref. [32]. We begin therefore with the presentation of our result in Ref. [32]. Some helpful details of the derivation of the CFT adapted Lagrangian for the massless spin-2 AdS field may be found in Appendix D.

In $AdS_{d+1}$ space, the massless spin-2 field is described by fields $\phi^{ab}(x,z), \phi^a(x,z), \phi(x,z)$. The field $\phi^{ab}$ is rank-2 tensor field of the $so(d)$ algebra, while $\phi^a$ and $\phi$ are the respective vector and scalar fields of the $so(d)$ algebra. The CFT adapted gauge invariant Lagrangian for these fields takes the form

$$\mathcal{L} = \frac{1}{4} |d\phi^{ab}|^2 - \frac{1}{8} |d\phi^a|^2 + \frac{1}{2} |d\phi|^2 + \frac{1}{2} |T_{\nu_2 - \frac{1}{2} \phi^a}|^2 + \frac{1}{2} |T_{\nu_2 - \frac{1}{2} \phi}|^2 - \frac{1}{2} \nu_2 C^a C_a - \frac{1}{2} C^2 ,$$

(5.48)

where $T_{\nu}$ is defined in (5.6) and we use the notation

$$C^a = \partial^a \phi^{ab} - \frac{1}{2} \partial^a \phi^{bb} + T_{\nu_2 - \frac{1}{2} \phi^a} ,$$

(5.49)

$$C = \partial^a \phi^a - \frac{1}{2} T_{\nu_2 - \frac{1}{2} \phi^a} + u T_{\nu_2 - \frac{1}{2} \phi} ,$$

(5.50)

$$\nu_2 = \frac{d}{2} , \quad \nu_1 = \frac{d - 2}{2} , \quad \nu_0 = \frac{d - 4}{2} ,$$

(5.51)

$$u \equiv \left( \frac{2}{d - 1} \frac{d - 2}{2} \right)^{1/2} .$$

(5.52)

Lagrangian (5.48) is invariant under the gauge transformations

$$\delta \phi^{ab} = \partial^a \xi^b + \partial^b \xi^a + \frac{2}{d - 2} \eta^{ab} T_{\nu_2 - \frac{1}{2} \phi} \xi ,$$

(5.53)

$$\delta \phi^a = \partial^a \xi + T_{\nu_2 - \frac{1}{2} \phi^a} \xi ,$$

(5.54)

$$\delta \phi = u T_{\nu_1 - \frac{1}{2} \phi} \xi ,$$

(5.55)

where $\xi^a, \xi$ are gauge transformation parameters.

Gauge invariant equations of motion obtained from Lagrangian (5.48) take the form

$$\Box_{\nu_2} \phi^{ab} - \partial^a C^b - \partial^b C^a - \frac{2\nu^{ab}}{d - 2} T_{\nu_2 - \frac{1}{2} \phi} \phi^{ab} = 0 ,$$

(5.56)

$$\Box_{\nu_1} \phi^a - \partial^a C - T_{\nu_2 - \frac{1}{2} \phi^a} = 0 ,$$

(5.57)

$$\Box_{\nu_0} \phi - u T_{\nu_1 - \frac{1}{2} \phi} = 0 ,$$

(5.58)

where $\Box_{\nu}$ and $\nu$’s are defined in (5.10) and (5.51) respectively. We see that the gauge invariant equations of motion are coupled.

Using equations of motion (5.56)-(5.58) in bulk action (5.19) with Lagrangian (5.48), we obtain boundary effective action (5.27) with $\mathcal{L}_{\text{eff}}$ given by

$$\mathcal{L}_{\text{eff}} = \frac{1}{4} \phi^{ab} T_{\nu_2 - \frac{1}{2} \phi^{ab}} - \frac{1}{8} \phi^{aa} T_{\nu_2 - \frac{1}{2} \phi^{bb}} + \frac{1}{2} \phi^{a} T_{\nu_2 - \frac{1}{2} \phi^{a}} + \frac{1}{2} \phi T_{\nu_2 - \frac{1}{2} \phi} - \frac{1}{2} C^a C_a + \left( \frac{1}{4} \phi^{aa} - \frac{u}{2} \phi \right) C .$$

(5.59)

Now we would like to demonstrate how use of the modified de Donder gauge condition provides considerable simplification in solving the equations of motion and computing effective action (5.27). To this end we note that it is the quantities $C^a, C$ given in (5.59), (5.50) that define the modified de Donder gauge condition,

$$C^a = 0 , \quad C = 0 \quad \text{modified de Donder gauge} .$$

(5.60)

Using this gauge condition in equations of motion (5.56)-(5.58) gives the following surprisingly simple gauge fixed equations of motion:

$$\Box_{\nu_2} \phi^{ab} = 0 ,$$

(5.61)

$$\Box_{\nu_1} \phi^a = 0 ,$$

(5.62)

$$\Box_{\nu_0} \phi = 0 .$$

(5.63)

Thus, we see that the gauge fixed equations of motions are decoupled. Using modified de Donder gauge (5.60) in (5.59),
we obtain

\[ L_{\text{eff}} \bigg|_{c_{s}=0} = \frac{1}{4} \phi^{ab} \mathcal{T}_{ab} - \frac{1}{8} \phi_{ab} \mathcal{T}^{ab} - \phi^{ab} \mathcal{T}_{ab} - \frac{1}{2} \phi^{a} \mathcal{T}_{a} - \frac{1}{8} \phi^{bb} + \frac{1}{2} \phi^{a} \mathcal{T}_{a} - \frac{1}{2} \phi^{bb} \],

(5.64)

i.e. we see that \( L_{\text{eff}} \) is also simplified.

In order to find \( S_{\text{eff}} \) we should solve equations of motion (5.61)-(5.63) with the Dirichlet problem corresponding to the boundary shadow field and plug the solution into \( L_{\text{eff}} \). To this end we discuss solution of equations of motion (5.61)-(5.63).

Because our equations of motion take decoupled form and similar to the equations of motion for the massive scalar \( AdS \) field we can apply the procedure described in Sec. V A. Doing so, we obtain solution of equation (5.61)-(5.63) with the Dirichlet problem corresponding to the spin-2 shadow field,

\[ \phi^{ab}(x, z) = \sigma_{2, \nu_{2}} \int d^{d}y \ G_{\nu_{2}}(x - y, z) \phi_{ab}^{\nu_{2}}(y), \] \hspace{1cm} (5.65)

\[ \phi^{a}(x, z) = \sigma_{2, \nu_{1}} \int d^{d}y \ G_{\nu_{1}}(x - y, z) \phi_{a}^{\nu_{1}}(y), \] \hspace{1cm} (5.66)

\[ \phi(x, z) = \sigma_{2, \nu_{0}} \int d^{d}y \ G_{\nu_{0}}(x - y, z) \phi_{sh}^{\nu_{0}}(y), \] \hspace{1cm} (5.67)

\[ \sigma_{2, \nu_{2}} = 1, \] \hspace{1cm} (5.68)

\[ \sigma_{2, \nu_{1}} = - \frac{1}{d - 2}, \] \hspace{1cm} (5.69)

\[ \sigma_{2, \nu_{0}} = \frac{1}{(d - 2)(d - 4)}, \] \hspace{1cm} (5.70)

where the Green function \( G_{\nu} \) is given in (5.14), while \( \nu_{s} \)’s are defined in (5.51). Choice of normalization factor \( \sigma_{2, \nu_{2}} \) is a matter of convention. The remaining normalization factors \( \sigma_{2, \nu_{1}}, \sigma_{2, \nu_{0}} \) are uniquely determined by requiring that modified de Donder gauge conditions (5.60) be amount to the differential constraints for the spin-2 shadow field (see Table I). The derivation of the normalization factor \( \sigma_{\nu, \nu} \) for arbitrary spin-\( s \) \( AdS \) field may be found in Appendix E.

Using asymptotic behavior of the Green function given in \( G_{\nu} \) (5.16), we find the asymptotic behavior of our solution

\[ \phi^{ab}(x, z) \xrightarrow{z \rightarrow 0} z^{-\nu_{2} + \frac{1}{2}} \phi_{ab}^{\nu_{2}}(x), \] \hspace{1cm} (5.71)

\[ \phi^{a}(x, z) \xrightarrow{z \rightarrow 0} \frac{z^{-\nu_{1} + \frac{1}{2}}}{d - 2} \phi_{a}^{\nu_{1}}(x), \] \hspace{1cm} (5.72)

\[ \phi(x, z) \xrightarrow{z \rightarrow 0} \frac{z^{-\nu_{0} + \frac{1}{2}}}{(d - 2)(d - 4)} \phi_{sh}(x). \] \hspace{1cm} (5.73)

From these expressions, we see that our solution (5.65)-(5.67) has indeed asymptotic behavior corresponding to the spin-2 shadow field.

Finally, to obtain the effective action we plug solution of the Dirichlet problem for \( AdS \) fields, (5.65)-(5.67) into (5.27), (5.64). Using general formula given in (5.18), we obtain

\[ -S_{\text{eff}} = dc_{\nu_{2}} \Gamma, \] \hspace{1cm} (5.74)

\[ c_{\nu_{2}} = \frac{\Gamma(d)}{\pi^{d/2} \Gamma(\frac{d}{2})}, \] \hspace{1cm} (5.75)

where \( \Gamma \) is gauge invariant two-point vertex of the spin-2 shadow field given in (5.11), (5.19).

To summarize, using the modified de Donder gauge for the massless spin-2 \( AdS \) field and computing the bulk action on solution of equations of motion with the Dirichlet problem corresponding to the boundary shadow field we obtain the gauge invariant two-point vertex of the spin-2 shadow field.

Because in the literature \( S_{\text{eff}} \) is expressed in terms of two-point vertex taken in the Stueckelberg gauge frame, \( \Gamma_{\text{stand}}^{ \nu_{2}} \), we use (5.24), (5.26) to represent our result (5.74) as

\[ -S_{\text{eff}} = \frac{d(d + 1)}{4(d - 1)} c_{\nu_{2}} \Gamma_{\text{stand}}. \] \hspace{1cm} (5.76)

This relation, by using the radial gauge for \( AdS \) fields, was obtained in Ref.[38]. Note that we have obtained more general relation given in (5.74), while relation (5.76) is obtained from (5.74) by using the Stueckelberg gauge frame. The fact that \( S_{\text{eff}} \) is related to \( \Gamma_{\text{stand}} \) is expected because of the conformal symmetry. What is important for the systematical study of \( AdS/CFT \) correspondence is the computation of the normalization factor in front of \( \Gamma_{\text{stand}} \) (5.76). We note that our normalization factor in (5.76) coincides with the one found in Ref.[38].

VI. \( AdS/CFT \) CORRESPONDENCE FOR ARBITRARY SPIN FIELD

We proceed with the discussion of \( AdS/CFT \) correspondence for bulk massless arbitrary spin-\( s \) \( AdS \) field and boundary spin-\( s \) shadow field. To discuss the correspondence we use the \( CFT \) adapted gauge invariant Lagrangian and modified de Donder gauge condition for the massless arbitrary spin \( AdS \) field found in Ref.[32]16. We begin therefore with the presentation of our result in Ref.[32]17.

16 In light-cone gauge, \( AdS/CFT \) correspondence for arbitrary spin massless \( AdS_{d+1} \) fields was studied in Ref.[12, 34]. Recent interesting applications of the standard de Donder-Feynman gauge to the various problems of higher-spin fields may be found in Refs.[42-44]. We believe that our modified de Donder gauge will also be useful for better understanding of various aspects of \( AdS/QCD \) correspondence which are discussed e.g. in [45-47].

17 Representation for the Lagrangian, which we use in this paper, is different from the one given in Ref.[34]. Namely, in this paper, we use \( CFT \)
In $AdS_{d+1}$ space, massless spin-$s$ field is described by the following scalar, vector, and totally symmetric tensor fields of the $so(d)$ algebra:

$$\phi^{a_1\ldots a_s}, \quad s' = 0, 1, \ldots, s.$$  \hfill (6.1)

The fields $\phi^{a_1\ldots a_{s'}}$ with $s' \geq 4$ are double-traceless,\footnote{In this paper, we adopt the formulation in terms of the double traceless gauge fields [58]. Discussion of various formulations in terms of unconstrained gauge fields may be found in [49]-[53]. For recent review, see [54]. Discussion of other formulations which seem to be most suitable for the theory of interacting fields may be found e.g. in [55].}

$$\phi^{aabb\ldots a_{s'}} = 0, \quad s' = 4, 5, \ldots, s.$$  \hfill (6.2)

In order to obtain the gauge invariant description in an easy-to-use form we use the oscillators and introduce a ket-vector $|\phi\rangle$ defined by

$$|\phi\rangle \equiv \sum_{s'=0}^{s} \alpha_{s}^{s-s'} |\phi_{s'}\rangle,$$

$$|\phi_{s'}\rangle \equiv \frac{\alpha_{a_1}\ldots\alpha_{a_{s'}}}{s'!\sqrt{(s-s')!}} \phi^{a_1\ldots a_{s'}}|0\rangle.$$  \hfill (6.3)

From (6.3), we see that the ket-vector $|\phi\rangle$ is degree-$s$ homogeneous polynomial in the oscillators $\alpha^{a}$, $\alpha^{\bar{a}}$, while the ket-vector $|\phi_{s'}\rangle$ is degree-$s'$ homogeneous polynomial in the oscillators $\alpha^{a}$. In terms of the ket-vector $|\phi\rangle$, double-tracelessness constraint (6.2) takes the form

$$\langle \tilde{\alpha}^{2}\rangle^{2}|\phi\rangle = 0.$$  \hfill (6.4)

The $CFT$ adapted gauge invariant Lagrangian is given by

$$\mathcal{L} = \frac{1}{2}(\partial^{\mu}\phi)(\partial^{\nu}\phi) + \frac{1}{2}(\mathcal{T}_{\nu-\frac{1}{2}}\phi_{s}^{\nu}|T_{\nu-\frac{1}{2}}\phi_{s}^{\nu}) - \frac{1}{2}(\mathcal{C}\phi)|\mathcal{C}\phi\rangle,$$  \hfill (6.5)

where $\mathcal{T}_{\nu}$ is defined in (5.6) and we use the notation

$$\bar{C} \equiv \delta \partial - \frac{1}{2}\alpha \partial \tilde{\alpha}^{2} - \bar{e}_{1} \Pi^{[1,2]} + \frac{1}{2}\bar{e}_{1} \bar{\alpha}^{2},$$ \hfill (6.6)

$$e_{1} = e_{1,1} \mathcal{T}_{\nu-\frac{1}{2}}, \quad \bar{e}_{1} = \mathcal{T}_{\nu-\frac{1}{2}} \bar{e}_{1,1},$$ \hfill (6.7)

$$e_{1,1} = -\alpha^{z} \bar{e}_{1}, \quad \bar{e}_{1,1} = -\bar{e}_{1} \alpha^{z},$$ \hfill (6.8)

$$\bar{e}_{1} = \left(\frac{2s + d - 4 - N_{\phi}}{2s + d - 4 - 2N_{\phi}}\right)^{1/2},$$ \hfill (6.9)

$$\mathbf{\mu} \equiv 1 - \frac{1}{4} \alpha^{2} \bar{\alpha}^{2},$$ \hfill (6.10)

$$\nu \equiv s + \frac{d - 4}{2} - N_{z},$$  \hfill (6.11)

where $\Pi^{[1,2]}$ is given in (2.7). Lagrangian (6.5) is invariant under the gauge transformation

$$\delta|\phi\rangle = (\alpha \partial - e_{1} - \alpha^{2} \frac{1}{2N_{\phi} + d - 2}\bar{e}_{1})(|\xi\rangle).$$  \hfill (6.12)

In terms of the $so(d)$ algebra tensor fields, the ket-vector $|\xi\rangle$ is represented as

$$|\xi\rangle \equiv \sum_{s'=0}^{s-1} \alpha_{s}^{s-1-s'} |\xi_{s'}\rangle,$$

$$|\xi_{s'}\rangle \equiv \frac{\alpha_{a_1}\ldots\alpha_{a_{s'}}}{s'!\sqrt{(s-1-s')!}} \xi^{a_1\ldots a_{s'}}|0\rangle,$$ \hfill (6.13)

where gauge transformation parameters are traceless, $\xi^{aabb\ldots a_{s'}} = 0$.

Gauge invariant equations of motion obtained from Lagrangian (6.5) take the form

$$\mathbf{\mu}\Box_{\nu}|\phi\rangle - C \bar{C}|\phi\rangle = 0,$$ \hfill (6.15)

where $C$ is given by

$$C \equiv \alpha \partial - \frac{1}{2} \alpha^{2} \partial \bar{\alpha} - e_{1} \Pi^{[1,2]} + \frac{1}{2} \bar{e}_{1} \bar{\alpha}^{2}.$$  \hfill (6.16)

Note that for the derivation of equations of motion (6.15) we use the relations $C^{\dagger} = -\bar{C}$ and

$$\mathcal{T}_{\nu-\frac{1}{2}}^{\dagger} \mathcal{T}_{\nu-\frac{1}{2}} = -\partial^{2} + \frac{1}{2^{2}}(\nu^{2} - \frac{1}{4}).$$  \hfill (6.17)

Using equations of motion (6.15) in bulk action (5.19) with Lagrangian (6.5), we obtain boundary effective action (5.27) with the following $\mathcal{L}_{\text{eff}}$:

$$\mathcal{L}_{\text{eff}} = \frac{1}{2}(\phi)(\mathcal{T}_{\nu-\frac{1}{2}}\phi) + \langle (\frac{1}{4} \bar{e}_{1,1} - \frac{1}{2} e_{1,1} \bar{\alpha}^{2}) \phi |\mathcal{C}\phi\rangle.$$  \hfill (6.18)

We now, as before, demonstrate how use of the modified de Donder gauge condition provides considerable simplification in solving equations of motion (6.15) and computing effective action (5.27).\footnote{Powerful methods of solving $AdS$ field equations of motion based on star algebra products in auxiliary spinor variables are discussed in Refs [54]-[57]. One of interesting features of these methods is that they do not use any gauge conditions when solving the equations of motion.}

The adapted Lagrangian represented in terms of operator $\mathcal{T}_{\nu}$. This operator was introduced in Ref.[35].

$$\mathcal{C}|\phi\rangle = 0, \quad \text{modified de Donder gauge}.$$  \hfill (6.19)
Using this gauge condition in (6.15) gives the following surprisingly simple gauge fixed equations of motion:

\[ \Box_{\nu} |\phi\rangle = 0, \quad (6.20) \]

where \( \Box_{\nu} \) and \( \nu \) are given in (5.10) and (6.11) respectively. Note that for the derivation of Eq. (6.20) we use the fact that kernel of operator \( \mu (6.10) \) is trivial on space of the double-traceless ket-vectors. Thus, we see that the modified de Donder gauge leads to the decoupled equations of motion.

Accordingly, using the modified de Donder gauge in (6.18) leads to the simplified expression for \( \mathcal{L}_{\text{eff}} \),

\[ \mathcal{L}_{\text{eff}} \big|_{\Box_{\nu} |\phi\rangle = 0} = \frac{1}{2} \langle \phi | \mu | \mathcal{T}_{\nu} - \frac{1}{2} \phi \rangle. \quad (6.21) \]

In order to find \( S_{\text{eff}} \) we should solve equations of motion (6.20) with the Dirichlet problem corresponding to the boundary shadow field and plug the solution into \( \mathcal{L}_{\text{eff}} \). To this end we discuss solution of equations of motion (6.20). Solution of equation (6.20) with the Dirichlet problem corresponding to the boundary shadow field is given by

\[ |\phi(x,z)\rangle = \sigma_{s,\nu} \int d^d y G_{\nu}(x-y,z) |\phi_{\text{sh}}(y)\rangle, \quad (6.22) \]

\[ \sigma_{s,\nu} = \frac{(-)^{\nu_s-\nu} \Gamma(\nu)}{2^{\nu_s-\nu} \Gamma(\nu_s)}, \quad (6.23) \]

\[ \nu_s = s + \frac{d-4}{2}, \quad (6.24) \]

where the Green function \( G_{\nu} \) and \( \nu \) are given in (5.14) and (6.11) respectively. The derivation of normalization factor \( \sigma_{s,\nu} (6.23) \) may be found in Appendix E.

Using asymptotic behavior of the Green function given in \( G_{\nu} (5.16) \), we find the asymptotic behavior of our solution

\[ |\phi(x,z)\rangle \underset{z \rightarrow 0}{\longrightarrow} \sigma_{s,\nu} z^{-\nu+\frac{1}{2}} |\phi_{\text{sh}}(x)\rangle. \quad (6.25) \]

From this expression, we see that our solution (6.22) has indeed asymptotic behavior corresponding to the spin-\( s \) shadow field. Finally, plugging our solution (6.22) into (6.21) and using general formula given in (5.18), we obtain the effective action

\[ -S_{\text{eff}} = (2s + d - 4) c_{\nu_s} \Gamma, \quad (6.26) \]

\[ c_{\nu_s} = \frac{\Gamma(s + d - 2)}{\pi^{d/2} \Gamma(s + \frac{d-4}{2})}, \quad (6.27) \]

where \( \Gamma \) stands for two-point gauge invariant vertex (4.1) of the spin-\( s \) shadow field.

To summarize, imposing the modified de Donder gauge on the massless spin-\( s \) AdS field and computing the bulk action on solution of equations of motion with the Dirichlet problem corresponding to the boundary shadow field we obtain the gauge invariant two-point vertex of the spin-\( s \) shadow field.

Because in the literature \( S_{\text{eff}} \) is expressed in terms of two-point vertex taken in the Stueckelberg gauge frame, \( \Gamma^{\text{stand}} (4.16) \), we use (4.12) and represent our result (6.26) as

\[ -S_{\text{eff}} = \frac{(2s + d - 3)(2s + d - 4)}{2s!(s + d - 3)} c_{\nu_s} \Gamma^{\text{stand}}. \quad (6.28) \]

For the particular values \( s = 1 \) and \( s = 2 \), our normalization factor in front of \( \Gamma^{\text{stand}} (6.28) \) coincides with the respective normalization factors given in (5.44), (6.26). Thus, our result agrees with the previously reported results for the particular values \( s = 1, 2 \) (see Refs. [36, 38, 40]) and gives the normalization factor for arbitrary values of \( s \) and \( d \).

Knowledge of the normalization factor for arbitrary values of \( s \) is important for the systematical study of \( AdS/CFT \) correspondence because higher-spin gauge field theories [37, 59] involve infinite tower of \( AdS \) fields with all values of \( s, s = 0, 1, \ldots, \infty \).

As a side of remark we note that the modified de Donder gauge and the gauge-fixed equations of motion have the on-shell left-over gauge symmetry. Namely, modified de Donder gauge (6.19) and gauge-fixed equations of motion (6.20) are invariant under gauge transformation (6.12) provided the gauge transformation parameter satisfies the equation

\[ \Box_{\nu} |\xi\rangle = 0, \quad (6.29) \]

where \( \Box_{\nu} \) and \( \nu \) are given in (5.10) and (6.11) respectively. Solution to Eq. (6.29) is given by

\[ |\xi(x,z)\rangle = \sigma_{s,\nu} \int d^d y G_{\nu}(x-y,z) |\phi_{\text{sh}}(y)\rangle. \quad (6.30) \]

Plugging this solution in (6.12) we find the following expression for the on-shell left-over gauge transformation of solution of the Dirichlet problem:

\[ \delta|\phi\rangle = \sigma_{s,\nu} \int d^d y G_{\nu}(x-y,z) \delta|\phi_{\text{sh}}(y)\rangle, \quad (6.31) \]

where \( \delta|\phi_{\text{sh}}\rangle \) is the gauge transformation of the spin-\( s \) shadow field (see Table II). From (6.31), we see that the on-shell left-over gauge symmetry of solution of the Dirichlet problem for spin-\( s \) AdS field amounts to the gauge symmetry of the spin-\( s \) shadow field (see Table II). It is matching of the on-shell leftover gauge symmetry of solution of the Dirichlet problem for AdS field and the gauge symmetry of the shadow field that explains why the effective action coincides with the gauge invariant two-point vertex for the boundary shadow field.

\[ \text{20} \] For massless arbitrary spin \( AdS_5 \) fields, the effective action, by using radial gauge, was studied in [58]. Normalization factor in Ref. [58] disagrees with our result and results reported in Refs. [34, 35, 40].
VII. CONFORMAL FIELDS

The kernel of two-point vertex (4.1) is not well-defined when \( d \) is even integer and \( \nu \) takes integer values (see e.g. [60]). However this kernel can be regularized and after that it turns out that the leading logarithmic divergence of the two-point vertex \( \Gamma \) leads to Lagrangian of conformal fields. To explain what has just been said we note that the kernel of \( \Gamma \) can be regularized by using dimensional regularization. This is to say that using the dimensional regularization and denoting the integer part of \( d \) by \([d]\), we introduce the regularization parameter \( \epsilon \) as

\[
d - [d] = -2 \epsilon, \quad [d] - \text{even integer}.
\]

With this notation we have the following behavior of the regularized expression for the kernel in (4.1):

\[
\frac{1}{|x|^{2\nu+d}} \overset{\epsilon \to 0}{\sim} \frac{1}{\epsilon} \partial_\nu \Gamma^{(\nu)} \delta(x),
\]

\[
\partial_\nu = \frac{\pi d/2}{4\nu\Gamma(\nu+1)\Gamma(\nu+\frac{d}{2})},
\]

when \( \nu \) is integer. Note that, in view of (6.11), \( \nu \) takes integer values when \( d \) is even integer. Using (7.2) in (3.1), (4.1), we obtain

\[
\Gamma \overset{\epsilon \to 0}{\sim} \frac{1}{\epsilon} \delta_{\nu,\nu} \int d^3x \mathcal{L},
\]

where \( \nu, \) is defined in (6.2) and \( \mathcal{L} \) is a higher-derivative Lagrangian for conformal spin-s field. We now discuss the Lagrangian for spin \( s = 1, 2 \) and arbitrary spin-s conformal fields in turn.

A. Spin-1 conformal filed

Using two-point vertex for spin-1 shadow field (3.2) and adopting relation (7.2) to the case of \( s = 1 \), we obtain the following gauge invariant Lagrangian for the spin-1 conformal field:

\[
\mathcal{L} = \frac{1}{2} \partial^a \Box^{k+1} \partial_a + \frac{1}{2} \partial^a \phi \phi_a , \quad k = \frac{d-4}{2},
\]

where we have made the identification

\[
\phi^a = \phi^{a}_{sh}, \quad \phi = \phi_{sh}.
\]

Using this identification, we note that the differential constraint for spin-1 shadow field (see Table I) implies the same differential constraint for \( \phi^a \) and \( \phi \),

\[
\partial^a \phi^a + \phi = 0.
\]

Lagrangian (7.5) and constraint (7.7) are invariant under gauge transformations

\[
\delta \phi^a = \partial^a \xi, \quad \delta \phi = -\Box \xi.
\]

To check gauge invariance of the Lagrangian we use the notation \( C \) for the left hand side of (7.7) and note that gauge variation of Lagrangian (7.5) takes the form

\[
\delta \mathcal{L} = -\xi \Box^{k+1} C,
\]

i.e. we see that Lagrangian (7.5) is indeed invariant when \( C = 0 \).

Interrelation between our approach and standard approach. Standard formulation of the spin-1 conformal field is obtained from our approach by solving the differential constraint. This is to say that using differential constraint (7.7) we express the scalar field in terms of the vector field,

\[
\phi = -\partial^a \phi^a,
\]

and plug \( \phi \) in Lagrangian (7.5). By doing so, we obtain the standard Lagrangian for the spin-1 conformal field,

\[
\mathcal{L} = -\frac{1}{4} F^{ab} \Box^k F_{ab}, \quad F^{ab} = \partial^a \phi^b - \partial^b \phi^a.
\]

Light-cone gauge Lagrangian. We now consider light-cone gauge Lagrangian for the spin-1 conformal field. As usually, the light-cone gauge frame is achieved through the use of the light-cone gauge and differential constraints. Gauge transformations (7.8), (7.9) and differential constraint (7.7) of the spin-1 conformal field take the same form as the ones for spin-1 shadow field (see Table I). Therefore to discuss the light-cone gauge for the spin-1 conformal field we can use results obtained for the spin-1 shadow field in Sec. III.A. This is to say that light-cone gauge and solution for differential constraint for the spin-1 conformal field are obtained from the respective expressions (3.10) and (3.11) by using identification (7.6). Doing so, we are left with so(\( d - 2 \)) algebra vector field \( \phi^i \) and scalar field \( \phi \). These fields constitute the field content of the light-cone gauge frame. Note that, in contrast to the standard approach, the scalar field \( \phi \) becomes an independent field D.o.F in the light-cone gauge frame. Making use of the light-cone gauge in gauge invariant Lagrangian (7.5), we obtain the light-cone gauge Lagrangian

\[
\mathcal{L}^{l.c.} = \frac{1}{2} \phi^{i} \Box^{k+1} \phi^{i} + \frac{1}{2} \phi \partial^k \phi , \quad k = \frac{d-4}{2}.
\]

B. Spin-2 conformal filed

We proceed with the discussion of spin-2 conformal field. Using two-point vertex for the spin-2 shadow field (3.19) and
adopting relation (7.2) to the case of $s = 2$, we obtain the gauge invariant Lagrangian for the spin-2 conformal field,

\[
\mathcal{L} = \frac{1}{4} \phi^{ab} \Box^{k+1} \phi^{ab} - \frac{1}{8} \phi^{aa} \Box^{k+1} \phi^{bb} + \frac{1}{2} \phi^{a} \Box^{k-1} \phi, \quad k = \frac{d - 2}{2}, \tag{7.14}
\]

where we have made the identification

\[
\phi^{ab} = \phi^{ab}_{sh}, \quad \phi^{a} = \phi^{a}_{sh}, \quad \phi = \phi_{sh}. \tag{7.15}
\]

Using this identification, we note that the differential constraints for the spin-2 shadow field (see Table I) imply the same differential constraints for the fields $\phi^{ab}$, $\phi^{a}$, and $\phi$,

\[
\partial^{b} \phi^{ab} - \frac{1}{2} \partial^{a} \phi^{bb} + \phi^{a} = 0, \tag{7.16}
\]

\[
\partial^{a} \phi^{a} + \frac{1}{2} \Box \phi^{aa} + u \phi = 0, \tag{7.17}
\]

\[
u = \left( \frac{2}{d - 2} \right)^{1/2}. \tag{7.18}
\]

The Lagrangian and the constraints are invariant under the gauge transformations

\[
\delta \phi^{ab} = \partial^{a} \xi^{b} + \partial^{b} \xi^{a} + \frac{2}{d - 2} \eta^{ab} \xi, \tag{7.19}
\]

\[
\delta \phi^{a} = \partial^{a} \xi - \Box \xi^{a}, \tag{7.20}
\]

\[
\delta \phi = -u \Box \xi. \tag{7.21}
\]

To demonstrate gauge invariance of the Lagrangian we use the notation $C^{a}$ and $C$ for the respective left hand sides of (7.16) and (7.17) and find that gauge variation of Lagrangian (7.14) takes the form

\[
\delta \mathcal{L} = -\xi^{a} \Box^{k+1} C^{a} - \xi \Box^{k} C, \tag{7.22}
\]

i.e. we see that Lagrangian (7.14) is indeed invariant when $C^{a} = 0, C = 0$.

**Interrelation between our approach and standard approach.** Standard formulation of the spin-2 conformal field is obtained from our approach as follows. First, we use differential constraints (7.16), (7.17) and express the vector field and scalar field in terms of the field $\phi^{ab}$,

\[
\phi^{a} = -\partial^{b} \phi^{ab} + \frac{1}{2} \partial^{a} \phi^{bb}, \tag{7.23}
\]

\[
\phi = \frac{1}{u} (\partial^{b} \partial^{b} \phi^{ab} - \Box \phi^{aa}). \tag{7.24}
\]

Second, we plug $\phi^{a}$, $\phi$ (7.23), (7.24) in Lagrangian (7.14) and obtain the standard Lagrangian for the spin-2 conformal field,

\[
\mathcal{L} = \frac{1}{4} \phi^{ab} \Box^{k+1} \phi^{ab} - \frac{1}{4(d - 1)} \phi^{aa} \Box^{k+1} \phi^{bb} + \frac{1}{2} \phi^{a} \Box^{k-1} \phi, \quad k = \frac{d - 2}{2}, \tag{7.25}
\]

\[
\frac{d - 2}{4(d - 1)} \Box \phi^{a} \Box^{k-1} \phi, \tag{7.26}
\]

\[
(\partial \phi)^{a} \equiv \partial^{b} \phi^{ab}, \quad (\partial \phi) = \partial^{a} \partial^{b} \phi^{ab}. \tag{7.27}
\]

Lagrangian (7.25) is invariant under gauge transformation of $\phi^{ab}$ given in (7.19).

As is well known, Lagrangian (7.25) can be represented in terms of the linearized Ricci tensor and Ricci scalar

\[
\mathcal{L} = R^{ab} \Box^{k-1} R^{ab} - \frac{d}{4(d - 1)} R \Box^{k-1} R, \tag{7.28}
\]

or equivalently in terms of the Weyl tensor

\[
\mathcal{L} = \frac{1}{q^{2}} C^{abce} \Box^{k-1} C^{abce}, \quad q^{2} \equiv \frac{4}{d - 2}. \tag{7.29}
\]

For the derivation of relation (7.28), we use the following expressions for the linearized Ricci tensor and Ricci scalar:

\[
R^{ab} = \frac{1}{2} (\Box \phi^{ab} + \partial^{b} (\partial \phi)^{a} + \partial^{a} (\partial \phi)^{b} - \partial^{a} \partial^{b} \phi^{cc}), \tag{7.30}
\]

\[
R = (\partial \phi)^{a} \Box \phi^{aa}, \tag{7.31}
\]

while for the derivation of relation (7.29) we use the fact that the Gauss-Bonnet combination taken at second order in the field $\phi^{ab}$ is a total derivative,

\[
R^{ab} \Box^{d} R^{d} R^{ab} - 4 R^{ab} R^{ab} + R^{2} = 0 \quad \text{up to total deriv.}, \tag{7.32}
\]

**Light-cone gauge Lagrangian.** As before, the light-cone gauge frame is achieved through the use of light-cone gauge and differential constraints. Gauge transformations (7.19)-(7.21) and differential constraints (7.16), (7.17) for the spin-2 conformal field take the same form as the ones for the spin-2 shadow field (see Table I). Therefore to discuss the light-cone gauge for the spin-2 conformal field we can use results obtained for the spin-2 shadow field in Sec. III B. This is to say that the light-cone gauge fixing and solution for differential constraints for the spin-2 conformal field are obtained from the respective expressions (3.28) and (3.29)-(3.32) by using identification (7.15). Doing so, we are left with traceless tensor field $\phi^{ij}$, vector field $\phi^{i}$ and scalar field $\phi_{sh}$. These fields
constitute the field content of the light-cone gauge frame. Note that, in contrast to the standard approach, the vector field \( \phi^i \) and scalar field \( \phi \) become independent field D.o.F in the light-cone gauge frame. Making use of the light-cone gauge in light-cone gauge Lagrangian \((7.14)\), we obtain the light-cone gauge Lagrangian for the spin-2 conformal field, \( L \) and constraint \((7.38)\), we find that variation of the Lagrangian under gauge transformation \((7.44)\) takes the form
\[
\delta L = -\langle \xi | \square' \bar{C} | \phi \rangle ,
\] (7.45)
i.e. we see that \( L \) is indeed gauge invariant provided the ket-vector |\( \phi \rangle \) satisfies differential constraint \((7.38)\).

To illustrate the structure of the Lagrangian we note that, in terms of tensor fields \( \phi^{a_1 \ldots a_s} \) defined as
\[
|\phi \rangle \equiv \sum_{s'=0}^{s} \frac{\alpha_{a_1 \ldots a_{s'}}}{s!} \! \phi^{a_1 \ldots a_{s'}} |0\rangle ,
\] (7.46)

Lagrangian \((7.34)\) takes the form
\[
L = \sum_{s'=0}^{s} L_{s'} ,
\] (7.47)
\[
L_{s'} = \frac{1}{2s!} \left( \phi^{a_1 \ldots a_{s'}} \square^{a_1 \ldots a_{s'}} \phi^{a_1 \ldots a_{s'}} - \frac{s'(s' - 1)}{4} \phi^{a_1 a_2 a_3 \ldots a_{s'} \phi^{b_1 b_2 b_3 \ldots a_{s'}}} \right) ,
\] (7.48)
\[
\nu_{s'} = s' + \frac{d - 4}{2} .
\] (7.49)

**Stueckelberg gauge frame.** Standard formulation of the conformal field is obtained from our approach by using the Stueckelberg gauge frame. Therefore to illustrate our approach, we now present Stueckelberg gauge fixed Lagrangian of the spin-s conformal field. The Stueckelberg gauge frame is achieved through the use of the Stueckelberg gauge and differential constraints. Gauge transformation \((7.44)\) and differential constraint \((7.38)\) for the spin-s conformal field take the same form as the ones for the spin-s shadow field (see Table II). Therefore to discuss the Stueckelberg gauge frame for the spin-s conformal field we can use results obtained for the spin-s shadow field (see Table II). This is to say that, by imposing the Stueckelberg gauge, solution to differential constraint for the spin-s conformal field is obtained from \((4.9)-(4.11)\) by using identification \((7.37)\) and \( |\phi_{sh} \rangle = |\phi \rangle , \ s' = 0, 1, \ldots, s \). After that, plugging \( |\phi \rangle \) in \((7.34)\) we obtain the Stueckelberg
gauge frame Lagrangian,\textsuperscript{21}
\[
\mathcal{L} = \frac{1}{2} \sum_{s'=0}^{s} \frac{2^{s-s'}(s'+d-1)}{(s-s')!(s+s'+d-3)s-s'} \times \left( (\tilde{\alpha} \partial)_{s-s'} \phi_{s'} \right) \left( (\tilde{\alpha} \partial)_{s-s'} \phi_{s} \right),
\]
where \(\nu\) is defined in (7.49) and we use the notation \((p)_{q}\) to indicate the Pochhammer symbol, \((p)_{q} \equiv \Gamma(p+q)_{n-p}/n!\). To illustrate the structure of Lagrangian (7.50) we note that, in terms of tensor fields \(\phi_{a_1 \ldots a_{s'}}\) defined in (7.46), Lagrangian (7.50) takes the form
\[
\mathcal{L} = \frac{1}{2} \sum_{s'=0}^{s} \frac{2^{s-s'}(s'+d-1)}{(s-s')!(s+s'+d-3)s-s'} \times \left( (\tilde{\alpha} \partial)_{s-s'} \phi_{a_1 \ldots a_{s'}} \right) \left( (\tilde{\alpha} \partial)_{s-s'} \phi_{a_1 \ldots a_{s'}} \right),
\]
(7.51)
For the readers convenience, we write down leading terms in Lagrangian (7.51).
\[
\mathcal{L} = \frac{1}{2s!} \phi_{a_1 \ldots a_s} \tilde{\alpha}_{\nu\rho} \partial_{\nu} \phi_{a_1 \ldots a_s}
\]
+ \[
\frac{1}{2(s-1)!} (\partial_{\nu})_{a_1 \ldots a_{s-1}} \tilde{\alpha}_{\nu\rho} \phi_{a_1 \ldots a_{s-1}}
\]
+ \[
\frac{1}{4(s-2)!} \frac{2s+d-6}{2s+d-5} \times \left( (\partial^2)_{\nu_1 \nu_2} a_1 \ldots a_{s-2} \phi_{a_1 \ldots a_{s-2}} + \ldots \right).
\]
\textbf{Light-cone gauge Lagrangian.} The light-cone gauge frame is achieved through the use of the light-cone gauge and differential constraints. Because the gauge transformation and differential constraint of the spin-\(d\) conformal field take the same form as the ones for the spin-\(d\) shadow field we can use the results obtained for the spin-\(d\) shadow field. This is to say that the light-cone gauge condition and solution for the differential constraint for the spin-\(s\) conformal field are obtained from the respective expressions (7.18) and (7.19), (7.20) by using identification (7.37). Doing so, we are left with traceless \(so(d-2)\) algebra fields \(\phi_{i_1 \ldots i_{s'}}\), \(s'=0,1,\ldots,s\). These fields constitute the field content of light-cone gauge frame. Note that, in contrast to the standard approach, the fields \(\phi_{i_1 \ldots i_{s'}}\), with \(s'=0,1,\ldots,s-1\), become independent field D.o.F in the light-cone gauge frame. Making use of the light-cone gauge in gauge invariant Lagrangian (7.54), we obtain the light-cone gauge Lagrangian for the spin-\(s\) conformal field
\[
\mathcal{L}^{l.c.} = \frac{1}{2} \left( \phi_{i_1 \ldots i_{s'}} \right) \left( \tilde{\alpha} \partial_{\nu} \phi_{i_1 \ldots i_{s'}} \right),
\]
(7.54)
To illustrate structure of the light-cone gauge Lagrangian we note that, in terms of the tensor fields \(\phi_{i_1 \ldots i_{s'}}\), Lagrangian (7.54) takes the form
\[
\mathcal{L}^{l.c.} = \sum_{s'=0}^{s} \frac{1}{2s!} \phi_{i_1 \ldots i_{s'}} \tilde{\alpha}_{\nu\rho} \phi_{i_1 \ldots i_{s'}},
\]
(7.55)
where \(\nu\) is given in (7.49).

\textbf{VIII. CONCLUSIONS}

In this paper, we have further developed the gauge invariant Stueckelberg approach to \textit{CFT} initiated in Ref.\textsuperscript{[1]}. The Stueckelberg approach turned out to be efficient for the study of massive fields and therefore we believe that this approach might also be useful for the study of \textit{CFT}. In this paper, we studied the two-point gauge invariant vertices of the shadow fields and applied our approach to the discussion of \textit{AdS/CFT} correspondence. In our opinion, use of the Stueckelberg approach to conformal currents and shadow fields turns out to be efficient for the study of \textit{AdS/CFT} correspondence and therefore this approach seems to be very promising. The results obtained should have a number of the following interesting applications and generalizations.

(i) In this paper we considered the gauge invariant approach for the conformal currents and shadow fields which, in the framework of \textit{AdS/CFT} correspondence, are related to the respective normalizable and non-normalizable solutions of massless \textit{AdS} fields. It would be interesting to generalize our approach to the case of anomalous conformal currents and shadow fields. In the framework of \textit{AdS/CFT} correspondence, the anomalous conformal currents and shadow fields are related to solution of equations of motion for massive \textit{AdS} fields. Therefore such generalization might be interesting for the study of \textit{AdS/CFT} duality between string massive states and the boundary conformal currents and shadow fields.

(ii) We studied the bosonic conformal currents and shadow fields. Generalization of our approach to the case of fermionic conformal currents and shadow fields will make it possible to involve the supersymmetry and apply our approach to the type IIB supergravity in \textit{AdS} background and then to the string in this background.

(iii) This paper was devoted to the study of the two-point gauge invariant vertices. Generalization of our approach to

\textsuperscript{21} In the Stueckelberg gauge frame, Lagrangian of the arbitrary spin conformal field for \(d = 4\) and \(d \geq 4\) was discussed in Ref.\textsuperscript{[4]} and Ref.\textsuperscript{[5]} respectively. Our expression (7.50) provides the explicit representation for Lagrangian of the conformal arbitrary spin field. Note that Lagrangian in Ref.\textsuperscript{[5]} and the one given in (7.50) involve the higher derivatives. Discussion of ordinary-derivative Lagrangian of the conformal arbitrary spin field may be found in Ref.\textsuperscript{[20]}.\textsuperscript{22}
the case of 3-point and 4-point gauge invariant vertices will give us the possibility to the study of various applications of our approach along the lines of Refs. [61,62].

(iv) In recent years, mixed symmetry fields have attracted a considerable interest (see e.g. Refs. [63]-[68]). We think that generalization of our approach to the case of mixed symmetry fields appears in string theory and higher-spin gauge fields theory.

(v) In this paper, we have discussed $AdS/CFT$ correspondence between the massless arbitrary spin $AdS$ fields and the boundary shadow fields. By now it is known that to construct self-consistent interaction of massless higher spin fields it is necessary to introduce, among other things, a infinite chain of massless $AdS$ fields which consists of every spin just once [37, 59]. This implies that to maintain $AdS/CFT$ correspondence for such interaction equations of motion we should also introduce a infinite chain of the boundary shadow fields. We have demonstrated that use of the modified de Donder gauge provides considerably simplifications in the analysis of free equations of motion of $AdS$ fields. In this respect it would be interesting to apply the modified de Donder gauge to the study of the consistent equations for interacting gauge fields of all spins [59] and extend the analysis of this paper to the case of infinite chain of interacting massless fields and the corresponding boundary shadow fields.

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Appendix A: Derivation of two-point vertex

In this Appendix, we demonstrate that two-point vertex (4.1) is uniquely determined by requiring the vertex to be invariant under the shadow field gauge symmetries, Poincaré algebra symmetries, and dilatation symmetry. We proceed in the following way.

i). Taking into account double-tracelessness constraint, the shadow field differential constraint, and Poincaré algebra symmetries, we note that the general form of two-point vertex is given by

$$\Gamma = \int d^d x_1 d^d x_2 \Gamma_{12}, \quad (A1)$$

$$\Gamma_{12} = \frac{1}{2} \langle \phi_1 | H_{12} | \phi_2 \rangle, \quad (A2)$$

$$H_{12} = h_1 + \alpha^2 h_2 + h_3 \tilde{\alpha}^2 + \alpha^2 h_4 \tilde{\alpha}^2, \quad (A3)$$

$$h_1 = h_1 (|x_{12}|, N_\tilde{\alpha}), \quad (A4)$$

$$h_2 = h_2 (|x_{12}|, \alpha x_{12}, \alpha^2, \tilde{\alpha}^2), \quad (A5)$$

$$h_3 = h_3 (|x_{12}|, \tilde{\alpha} x_{12}, \alpha^2, \tilde{\alpha}^2), \quad (A6)$$

$$h_4 = h_4 (|x_{12}|, \alpha x_{12}, \alpha x_{12}, \alpha^2, \tilde{\alpha}^2), \quad (A7)$$

where $N_\tilde{\alpha} = \alpha^2 \tilde{\alpha}^2$, $\alpha x_{12} = \alpha^a x_{12}^a$, $\tilde{\alpha} x_{12} = \tilde{\alpha}^a x_{12}^a$ and we use the shortcuts $\langle \phi_1 |$ and $| \phi_2 \rangle$ for the respective ket-vectors $|\phi_1 (x_1)\rangle$ and $|\phi_2 (x_2)\rangle$. All that is required is to find the functions $h_a$, $a = 1, 2, 3, 4$. To this end we note that variation of $\Gamma_{12}$ under gauge transformation

$$\delta |\phi_1 \rangle = \alpha \partial |\xi_2 \rangle + \ldots , \quad (A8)$$

takes the form (up to total derivative)

$$\delta \Gamma_{12} = \frac{1}{2} \langle \phi_1 | h_1 \alpha^2 \tilde{\alpha} \partial |\xi_2 \rangle + \langle \phi_1 | h_2 \alpha^2 \alpha \partial |\xi_2 \rangle + \langle \phi_1 | h_3 \tilde{\alpha} \partial |\xi_2 \rangle + 2 \langle \phi_1 | h_4 \alpha^2 \tilde{\alpha} \partial |\xi_2 \rangle + \ldots \quad (A9)$$

where dots in (A8) and (A9) stand for the contributions that are independent of the derivative $\partial^a$. From (A9), we see that requiring the variation $\delta \Gamma_{12}$ to vanish gives constraints

$$h_4 = -\frac{1}{4} \alpha^2 \tilde{\alpha}^2 h_1, \quad h_2 = h_3 = 0. \quad (A11)$$

Plugging (A11) in (A3) we obtain

$$H_{12} = \mu h_1, \quad (A12)$$

where $\mu$ is given in (A2). We now note that requiring invariance of the vertex under shadow field dilatation symmetry (see Table II) we are led to the following solution to $h_1$:

$$h_1 = f_\nu \rho, \quad \rho \equiv |x_{12}|^{-(2\nu + d)}, \quad (A13)$$

where $f_\nu$ depends only on the operator $N_\tilde{\alpha}$. Taking into account (6.11) we consider $f_\nu$ as function of $\nu$. Thus, restrictions imposed by Poincaré symmetries, dilatation symmetry
and some restrictions imposed by gauge symmetries lead to the following expression for \( H_{12} \):

\[
H_{12} = \mu f_\nu \rho .
\]  

(A14)

ii) We now consider all restrictions on \( f_\nu \) imposed by gauge symmetries. In other words, taking into account the gauge transformation of the shadow field,

\[
\delta \phi_{sh} = G_{sh} |\xi_{sh}\rangle ,
\]  

(A15)

we consider restrictions imposed on \( f_\nu \) by equation (up to total derivative)

\[
\langle G_{sh} \xi_1 | \mu f_\nu | \phi_2 \rangle = 0 .
\]  

(A17)

To this end we note the following helpful relation (up to total derivative):

\[
\langle G_{sh} \xi_1 | \mu f_\nu | \phi_2 \rangle = \langle (\xi_1 C_{\perp} - \bar{e}_{\perp 1} \alpha^2) f_\nu \rho | \phi_2 \rangle .
\]  

(A18)

Using differential constraint for the shadow field (see Table II), we transform \( C_{\perp} \)-term in (A18) as

\[
\langle \xi_1 | C_{\perp} f_\nu \rho | \phi_2 \rangle = \langle \xi_1 | f_\nu \rho C_{\perp} | \phi_2 \rangle = \langle \xi_1 | f_\nu \rho e_{\perp 1} \alpha^2 | \phi_2 \rangle = \langle \xi_1 | f_\nu \rho e_{\perp 1} | \phi_2 \rangle - \langle \xi_1 | (\nu + 1)(2\nu + d) f_\nu \rho | \phi_2 \rangle .
\]  

(A19)

The \( e_{\perp 1} \)- and \( e_{\perp 1} \)-terms in (A18) can be transformed as

\[
\langle \xi_1 | (e_{\perp 1} \alpha^2) f_\nu \rho | \phi_2 \rangle = \langle \xi_1 | f_\nu \rho e_{\perp 1} \alpha^2 | \phi_2 \rangle = \langle \xi_1 | f_\nu \rho e_{\perp 1} \alpha^2 | \phi_2 \rangle - \langle \xi_1 | (\nu + 1)(2\nu + d) f_\nu \rho | \phi_2 \rangle .
\]  

(A20)

Taking into account (A19), (A20) we see that requiring the contribution of \( e_{\perp 1} \)-terms in (A18) to vanish we find the following equation for \( f_\nu \):

\[
\frac{f_\nu}{f_{\nu - 1}} = 2\nu(2\nu + d - 2) .
\]  

(A21)

Using (A19), (A20), we see that that requiring the contribution of \( e_{\perp 1} \)-terms in (A18) to vanish also leads to (A21). Thus, equations (A21) amount to requiring that gauge variation of two-point vertex vanishes (A17). Solution to Eq. (A21) with initial condition \( f_{\nu_0} = 1 \) is given in (A23).

Appendix B: Invariance of two-point vertex under conformal boost transformations

We now demonstrate invariance of the shadow field two-point vertex \((\xi_1, \xi_2)\) under conformal boost transformations \((K^a)\). We start with expression for two-point vertex \((\xi_1, \xi_2)\) with \(\Gamma_{12}\) given by

\[
\Gamma_{12} = \frac{1}{2} \langle \phi_1 H_{12} | \phi_2 \rangle ,
\]  

\[
H_{12} = \mu f_\nu \rho ,
\]  

(B1)

where \(\mu, f_\nu \) and \(\rho \) are defined in (4.2), (4.3) and (A13) respectively. In (B1), we use the shortcuts \(\langle \phi_1 \rangle \) and \(\langle \phi_2 \rangle \) for the respective shadow field ket-vectors \(\langle \phi_{sh}(x_1) \rangle \) and \(\langle \phi_{sh}(x_2) \rangle \). We proceed in the following way.

i) Before analyzing restrictions imposed on the two-point vertex by the conformal boost symmetries we find explicit form of the restrictions imposed on the two-point vertex by the Poincaré algebra and dilatation symmetries. Invariance with respect to the Poincaré translations implies that \(H_{12}\) depends on \(x_{12}^a \equiv x_1^a - x_2^a\). Requiring the vertex \(\Gamma\) to be invariant under dilatation and Lorentz algebra symmetries

\[
\delta_D \Gamma = 0 ,
\]  

\[
\delta_P \Gamma = 0 ,
\]  

(B2)

amounts to the following respective equations for \(H_{12}\):

\[
\langle x_{12}^a \partial_a x_{12} + 2d - \Delta_{sh} \rangle H_{12} - H_{12} \Delta_{sh} = 0 ,
\]  

(B3)

\[
\langle l_{12}^{ab} + M^{ab} \rangle H_{12} - H_{12} M^{ab} = 0 ,
\]  

(B4)

\[
l_{12}^{ab} \equiv x_{12}^a \partial_b - x_{12}^b \partial_a ,
\]  

(B5)

\[
\Delta_{sh} \equiv 2 - s + N_z ,
\]  

(B6)

where \(M^{ab}\) is given in (2.20). It easy to see that \(H_{12}\) given in (A14) satisfies Eqs. (B3), (B4).

ii) We now prove invariance of \(\Gamma\) under the \(K^a\) transformations. To this end we analyze variation of the \(\Gamma\) under the \(K^a\) transformations. Taking into account (2.16), we see that variation of \(\Gamma\) can be represented as

\[
\delta_{K^a} \Gamma = \delta_{K^a, M} \Gamma + \delta_{R^a} \Gamma .
\]  

(B7)

Taking into account Eqs. (B3), (B4), one can make sure that the variation of \(\Gamma_{12}\) under \(K^a, M\) transformations takes the form (up to total derivative)

\[
\delta_{K^a, M} \Gamma_{12} = \frac{1}{2} \langle \phi_1 \langle \frac{1}{2} \delta x_{12}^a \partial_x \left| P_x^a - M^{ab} x_{12}^b \right| H_{12} \rangle ,
\]  

(B8)

\[
\delta_{R^a} \Gamma_{12} = \frac{1}{2} x_{12}^a \partial_x \left| \frac{1}{2} x_{12}^a \partial_x \right| H_{12} ,
\]  

(B9)
Using the notation $R_{sh}^{ab}$ for the shadow field operator $R^a$ given in Table II, it is easy to see that variation of the vertex $\Gamma$ under action of the operator $R^a$ is given by

$$\delta_{R^a} \Gamma = \int d^4x_1 d^4x_2 \langle R_{sh}^{ab} \phi_1 | H_{12} | \phi_2 \rangle .$$  \hfill (B10)

We note that relations (B8) and (B10) are valid for arbitrary $H_{12}$ which satisfies Eqs. (B3), (B4). Making use of expression for $H_{12}$ in (B1), it is easy to see that requiring the vertex $\Gamma$ to be invariant under $K^a$ transformations,

$$\delta_{K^a} \Gamma = 0 ,$$  \hfill (B11)

amounts to the following equation (up to total derivative):

$$- \frac{1}{2} \langle \phi_1 | M^{ab} x_{12}^b H_{12} | \phi_2 \rangle + \langle R_{sh}^{ab} \phi_1 | H_{12} | \phi_2 \rangle = 0 .$$  \hfill (B12)

Note that for the derivation of this equation we use the relations $[P_a^2, x] = 0$ and $[H_{12}, \Delta_{sh}] = 0$.

Thus, all that remains to be done is to prove Eq. (B12) with $H_{12}$ given in (B1). To this end we note the relations

$$x_{12}^a H_{12} = - \frac{1}{2 \nu + d - 2} \partial_{x_{12}}^a (|x_{12}|^2 H_{12}) ,$$  \hfill (B13)

$$M^{ab} \partial^b + G_{sh} \bar{C}^a \equiv \alpha^a \bar{C}_{sh} - e_{1sh} \alpha^a + \bar{T}^a e_{1sh} ,$$  \hfill (B14)

$$\bar{T}^a \equiv \alpha^a - \alpha^2 \frac{1}{2 N_{\alpha} + d - 2} \bar{\alpha}^a ,$$  \hfill (B15)

$$\bar{C}^a \equiv \alpha^a - \frac{1}{2 \alpha^a} \bar{\alpha}^a ,$$  \hfill (B16)

where $\nu$ and $G_{sh}$ are given in (4.4) and (A16) respectively, while $\bar{C}_{sh}, e_{1sh}, \bar{e}_{1sh}$ are given in Table II. Using (B13), (B14), the constraint $\bar{C}_{sh} | \phi_{sh} \rangle = 0$, and the relation $M^{ab} = - M^{ab}$, we obtain (up to total derivative)

$$- \langle \phi_1 | M^{ab} x_{12}^b H_{12} | \phi_2 \rangle = \langle \frac{1}{2 \nu + d - 2} M^{ab} \partial^b \phi_1 | x_{12}^2 H_{12} | \phi_2 \rangle$$

$$= \langle \left( -G_{sh} \bar{C}^a_{sh} - e_{1sh} \alpha^a + \bar{T}^a e_{1sh} \right) \phi_1 |$$

$$\times |x_{12}|^2 H_{12} | \phi_2 \rangle .$$  \hfill (B17)

We now consider expressions appearing in (B17) in turn. $G_{sh}$-term can be transformed as (up to total derivative)

$$- \langle G_{sh} \bar{C}^a \phi_1 | x_{12}^2 H_{12} | \phi_2 \rangle$$

$$= \langle (\alpha^a \bar{C}_{sh} - e_{1sh} \alpha^a + \bar{T}^a e_{1sh} \phi_1 |$$

$$\times |x_{12}|^2 H_{12} | \phi_2 \rangle .$$  \hfill (B18)

where the operator $\bar{C}_{cur}$ is given in Table II. Also, note that we use the differential constraint $\bar{C}_{sh} | \phi_{sh} \rangle = 0$. The $e_{1sh}$- and $\bar{e}_{1sh}$- terms in (B17) can be transformed as

$$- \langle e_{1sh} \alpha^a \phi_1 | x_{12}^2 H_{12} | \phi_2 \rangle$$

$$= - \langle 2\nu e_{1sh} \alpha^a \phi_1 | H_{12} | \phi_2 \rangle ,$$  \hfill (B19)

$$\langle \bar{F}^a e_{1sh} \phi_1 | x_{12}^2 H_{12} | \phi_2 \rangle$$

$$= - \langle 2(\nu + 1) e_{1sh} \alpha^a \phi_2 | H_{12} | \phi_1 \rangle .$$  \hfill (B20)

Plugging (B18), (B19) and (B20) in (B17) and taking into account $R_{sh}^{ab}$ given in Table II we make sure that relation (B12) holds true.

Appendix C: Derivation of effective action

We now discuss details of the derivation of effective action (5.18). We are going to prove the following relations:

$$\int d^4 x \phi(x, z) \partial_z \phi(x, z) \rightarrow \frac{\nu^2}{c_{\nu}} \int d^4 x_1 d^4 x_2 \frac{\phi_1 \phi_2}{|x_{12}|^{2\nu d}} ,$$  \hfill (C1)

$$\int d^4 x \frac{1}{z} \phi(x, z) \phi(x, z) \rightarrow \frac{\nu^2}{c_{\nu}} \int d^4 x_1 d^4 x_2 \frac{\phi_1 \phi_2}{|x_{12}|^{2\nu d}} ,$$  \hfill (C2)

$$\bar{c}_{\nu} \equiv c_{\nu} \nu ,$$  \hfill (C3)

where $\phi(x, z)$ is solution of the Dirichlet problem given in (5.13) and we use the shortcuts $\phi_1$ and $\phi_2$ for the respective boundary shadow fields $\phi_{sh}(x_1)$ and $\phi_{sh}(x_2)$. The $|x_{12}|$ is given in (3.3). Taking into account expressions for the effective action given in (5.11), (5.12) it is easy to see that relations (C1), (C2) do indeed lead to effective action (5.18). Note that in r.h.s. of (C1), (C2) we keep, as usually, only non-local contributions.
We now prove relations (C1), (C2). To this end we use expression for solution \( \phi(x, z) \) in (5.13) to find the relations

\[
\int d^d x \phi(x, z) \partial_z \phi(x, z) = c^2_\nu (\nu + 1) X_1 - (2\nu + d) X_2, \\
\int d^d x \frac{1}{z} \phi(x, z) \phi(x, z) = c^2_\nu X_1,
\]

where we use the notation

\[
X_1 \equiv \int d^d x_1 d^d x_2 d^d x_3 \phi_1 \phi_2 \frac{z^{2\nu}}{f_{13}^{\nu + 2} f_{23}^{\nu + 4}}, \\
X_2 \equiv \int d^d x_1 d^d x_2 d^d x_3 \phi_1 \phi_2 \frac{z^{2\nu + 2}}{f_{13}^{\nu + 2} f_{23}^{\nu + 4 + 1}}, \\
f_{mn} \equiv z^2 + |x_{mn}|^2, \quad x_{mn}^n \equiv x_m^n - x_n^n.
\]

Using the Fourier transform of the kernels in (C6), (C7)

\[
\frac{z^\nu}{(z^2 + |x|^2)^{\nu + 4}} = \omega_\nu \int d^d k e^{i k \cdot x} K_\nu(k z), \\
\omega_\nu^{-1} \equiv \pi^{d/2} \omega^2 \Gamma(\nu + \frac{d}{2}),
\]

where \( K_\nu \) is the modified Bessel, and integrating over \( x_3 \), we cast \( X_1 \) and \( X_2 \) into the form

\[
X_1 = (2\pi)^d \int d^d x_1 d^d x_2 d^d k \phi_1 \phi_2 e^{ik \cdot x_{12}} \\
\times \omega_\nu^2 K_\nu^2(K_\nu(z)) \Gamma^2(\nu + \frac{d}{2}); \\
X_2 = (2\pi)^d \int d^d x_1 d^d x_2 d^d k \phi_1 \phi_2 e^{ik \cdot x_{12}} \\
\times \omega_\nu \omega_{\nu + 1} k^{2\nu + 1} K_\nu(K_\nu(z)) K_{\nu + 1}(K_\nu(z)).
\]

We now consider the asymptotic behavior, as \( z \to 0 \), of \( X_1 \) and \( X_2 \). As usually, we are interested in non-local contributions to (C1), (C2). To this end we use the definition of \( K_\nu \),

\[
K_\nu(z) \to \frac{\pi}{2 \sin \pi \nu} (I_{-\nu}(z) - I_\nu(z)), \\
I_\nu(z) = \sum_{k=0} \frac{1}{k! \Gamma(k + \nu + 1)} \left( \frac{z}{2} \right)^{\nu + 2k},
\]

to obtain the following well-known formula:

\[
K_\nu(z) \to 2^{\nu + 1} \Gamma(\nu) \frac{(kz)^\nu \Gamma(-\nu)}{2^{\nu + 1}} + \ldots,
\]

where dots stand for the terms which are not relevant for the analysis of non-local contributions to (C1), (C2). Making use of (C15), we obtain

\[
X_1 \to (2\pi)^d \int d^d x_1 d^d x_2 d^d k \phi_1 \phi_2 k^{2\nu} e^{ik \cdot x_{12}} \\
\times \frac{1}{2} \omega_\nu^2 \Gamma(\nu) \Gamma(-\nu),
\]

\[
= \frac{2}{\omega_\nu} \int d^d x_1 d^d x_2 \phi_1 \phi_2 |x_{12}|^{2\nu + d},
\]

\[
X_2 \to (2\pi)^d \int d^d x_1 d^d x_2 d^d k \phi_1 \phi_2 k^{2\nu} e^{ik \cdot x_{12}} \\
\times \frac{1}{2} \omega_\nu \omega_{\nu + 1} \Gamma(-\nu) \Gamma(\nu + 1),
\]

\[
= \frac{2\nu}{\omega_{\nu + 1}(2\nu + d)} \int d^d x_1 d^d x_2 \phi_1 \phi_2 |x_{12}|^{2\nu + d}.
\]

For the derivation of relations (C16), (C17), we use the formulas

\[
\int d^d k k^{2\nu} e^{ik \cdot x} = \frac{2^{2\nu + d} \omega^d 2\Gamma(\nu + \frac{d}{2})}{\Gamma(-\nu) |x|^{2\nu + d}},
\]

\[
\frac{\omega_{\nu + 1}}{\omega_\nu} = \frac{1}{2\nu + d},
\]

where (C19) is simply obtained by using (C10). Making use of (C16), (C17) in (C4), (C5), we arrive at desired relations (C1), (C2).

**Appendix D: CFT adapted Lagrangian for massless spin-1 and spin-2 fields in \( AdS_{d+1} \)**

In this Appendix, we explain some details of the derivation of the CFT adapted gauge invariant Lagrangian for massless spin-1 and spin-2 fields given in (5.20) and (5.48). Presentation in this Appendix is given by using Lorentzian signature. Euclidean signature Lagrangian in Sec. VII is obtained from the Lorentzian signature Lagrangian by simple substitution \( \mathcal{L} \to -\mathcal{L} \).

**Spin-1 massless field.** We use field \( \Phi^4 \) carrying flat Lorentz algebra \( so(d, 1) \) vector indices \( A, B = 0, 1, \ldots, d - 1, d \). The field \( \Phi^4 \) is related with field carrying the base manifold indices \( \Phi^\mu, \mu = 0, 1, \ldots, d, \) in a standard way \( \Phi^4 = e^A \Phi^\mu \), where \( e^A \) is vielbein of \( AdS_{d+1} \) space. For the Poincaré parametrization of \( AdS_{d+1} \) space (5.1), vielbein \( e^A = e^A dx^\mu \) and Lorentz connection, \( de^A + \omega^{AB} \wedge e^B = 0 \), are given by

\[
e^A = \frac{1}{z} \delta^A \mu, \quad \omega^{AB} = \frac{1}{z} (\delta^A_\mu \delta^B_\nu - \delta^B_\mu \delta^A_\nu),
\]
where $\delta^A_\mu$ is Kronecker delta symbol. We use a covariant derivative with the flat indices $\mathcal{D}^A$,

$$\mathcal{D}_A \equiv e^\mu_A \partial_\mu, \quad \mathcal{D}^A = \eta^{AB} \mathcal{D}_B,$$

(D2)

where $e^\mu_A$ is inverse of AdS vielbein, $e^\mu_A e^\nu_B = \delta^A_B$ and $\eta^{AB}$ is flat metric tensor. With choice made in (D1), the covariant derivative takes the form

$$\mathcal{D}^A \Phi^B = \partial^A \Phi^B + \delta^B_\mu \Phi^A - \eta^{AB} \Phi^z, \quad \partial^A \equiv z \partial^A,$$

(D3)

where we adopt the following conventions for the derivatives and coordinates: $\partial^A = \eta^{AB} \partial_B$, $\partial_A = \partial / \partial x^A$, $x^A \equiv \delta^A_\mu x^\mu$, $x^A = x^a, x^d, x^d = z$.

In arbitrary parametrization of AdS, Lagrangian of the massless spin-1 field takes the standard form

$$e^{-1} \mathcal{L} = \frac{1}{4} F^{AB} F^{AB}, \quad F^{AB} = \mathcal{D}^A \Phi^B - \mathcal{D}^B \Phi^A,$$

(D4)

where $e \equiv \det e^\mu_A$. Using the notation

$$C_{st} = \mathcal{D}^A \Phi^A,$$

(D5)

we note that it is the relation $C_{st} = 0$ that defines the standard Lorentz gauge. Lagrangian (D4) can be represented as

$$e^{-1} \mathcal{L} = \frac{1}{2} \Phi^A (D^2 + d) \Phi^A + \frac{1}{2} C_{st}^2.$$

(D6)

We now use the Poincaré parametrization of AdS and introduce the following quantity:

$$C = \mathcal{D}^A \Phi^A + 2 \Phi^z.$$

(D7)

We note that it is the relation $C = 0$ that defines the modified Lorentz gauge. Using the relations (up to total derivative)

$$e \Phi^A D^2 \Phi^A = e \left( \Phi^A (\square_{AdS} - 1) \Phi^A \right)$$

$$+ 4 \Phi^z C + (d - 7) \Phi^z \Phi^z,$$

(D8)

$$C_{st}^2 = C^2 - 4 \Phi^z C + 4 \Phi^z \Phi^z,$$

(D9)

$$\square_{AdS} \equiv z^2 (\square + \partial_z^2) + (1 - d) z \partial_z,$$

(D10)

we represent Lagrangian (D6) and $C$ (D7) as

$$e^{-1} \mathcal{L} = \frac{1}{2} \Phi^A (\square_{AdS} + d - 1) \Phi^A$$

$$+ \frac{d - 3}{2} \Phi^z \Phi^z + \frac{1}{2} C^2,$$

(D11)

$$C = \partial^A \Phi^A + (2 - d) \Phi^z.$$

(D12)

In terms of $so(d - 1, 1)$ tensorial components of the field $\Phi^A$ given by $\Phi^a, \Phi^z$, Lagrangian (D11) takes the form

$$e^{-1} \mathcal{L} = \frac{1}{2} \Phi^a (\square_{AdS} + d - 1) \Phi^a$$

$$+ \frac{1}{2} \Phi^z (\square_{AdS} + 2d - 4) \Phi^z + \frac{1}{2} C^2,$$

(D13)

$$= z \partial^a \Phi^a + z T_{2d} \Phi^z.$$

(D14)

Introducing the canonically normalized field $\phi^A$,

$$\phi^A = \frac{z^{d - 1}}{z} \phi^A,$$

(D15)

and using the identification $\phi^z = \phi$ we make sure that Lagrangian (D13) takes the form

$$\mathcal{L} = \frac{1}{2} \phi^a \square_{\nu} \phi^a + \frac{1}{2} \phi \square_{\nu} \phi + \frac{1}{2} C^2,$$

(D16)

where $\square_{\nu}, C$, and $\nu$’s are defined in (5.10), (5.21), and (5.22) respectively. Note that $C = z^{(d+1)/2} \mathcal{E}$. Finally, taking into account expression for operator $\square_{\nu}$ (5.10) and relation (6.17), we see that Lagrangian (D16) is equal, up to total derivative and overall sign, to the one given in (5.20).

Lagrangian (D4) is invariant under the gauge transformations $\delta \Phi^A = \partial^A \Xi$. Making the rescaling $\Xi = z^{(d-3)/2} \mathcal{E}$, we check that these gauge transformations lead to the ones given in (5.23), (5.24).

Spin-2 massless field. We use a tensor field $\Phi^{AB}$ with flat indices. This field is related with the tensor field carrying the base manifold indices in a standard way $\Phi^{AB} = e^\mu_A e^\nu_B \Phi^{\mu\nu}$. In arbitrary parametrization of AdS, Lagrangian of massless spin-2 field takes the standard form

$$e^{-1} \mathcal{L} = \frac{1}{4} \Phi^{AB} (E_{EH} \Phi)^{AB} + \frac{1}{2} \Phi^{AB} \Phi^{AB}$$

$$+ \frac{d - 2}{4} \Phi^z,$$

(D17)

$$(E_{EH} \Phi)^{AB} = \mathcal{D}^2 \Phi^{AB} - \mathcal{D}^A (\mathcal{D}^B \Phi) - \mathcal{D}^B (\mathcal{D}^A \Phi)^A$$

$$+ \mathcal{D}^A \mathcal{D}^B \Phi + \eta^{AB} (\mathcal{D}^C \mathcal{D}^E \Phi^{CE} - \mathcal{D}^2 \Phi),$$

$$\Phi \equiv \Phi^{AA}, \quad (\mathcal{D}^A \Phi)^A \equiv \mathcal{D}^B \Phi^{AB}.$$

(D18)

Using the notation

$$C_{st}^A \equiv \mathcal{D}^B \Phi^{AB} - \frac{1}{2} \mathcal{D}^A \Phi,$$

(D19)

we note that it is the relation $C_{st}^A = 0$ that defines the standard de Donder gauge condition. Using $C_{st}^A$, Lagrangian (D17) can be represented as

$$e^{-1} \mathcal{L} = \frac{1}{4} \Phi^{AB} (D^2 + 2) \Phi^{AB}$$

$$- \frac{1}{8} \Phi (D^2 - 2d + 4) \Phi + \frac{1}{2} C_{st}^A C_{st}^A.$$

(D20)
We note the relation $C^A = 0$ that defines the modified de Donder gauge condition. Using the relations (up to total derivative)

$$
\frac{1}{4} e \Phi^{AB} D^2 \Phi^{AB} = e \left( \frac{1}{4} \Phi^{AB}(\Box_0 \text{AdS} - 2)\Phi^{AB} + \frac{d - 5}{2} \Phi^{zA} \Phi^{zA} + 2\Phi^{zA} \Phi - \frac{d}{4} \Phi^2 \right)
$$

$$
+ 2\Phi^{zA} C^A - \Phi C^z ,
$$

(D22)

we represent Lagrangian (D20) and $C^A$ as

$$
e^{-1} \mathcal{L} = \frac{1}{4} \Phi^{AB} \Box_0 \text{AdS} \Phi^{AB} - \frac{1}{8} \Phi^{AB} \Box_0 \text{AdS} \Phi + \frac{d - 1}{2} \Phi^{zA} \Phi^{zA} + \frac{1}{2} C^A C^A ,
$$

(D24)

$$
C^A = \partial^B \Phi^{AB} - \frac{1}{2} \partial^A \Phi + (1 - d) \Phi^{zA} ,
$$

(D25)

where $\Box_0 \text{AdS}$ is given in (D10). In terms of canonically normalized fields $\Phi^{AB}$, defined by

$$
\Phi^{AB} = z^{-\frac{d+1}{2}} \Phi^{AB} ,
$$

(Lagrange (D24) takes the form

$$
\mathcal{L} = \frac{1}{4} \Phi^{AB} \Box_0 \nu_\theta \Phi^{AB} - \frac{1}{8} \Phi^{AB} \Box_0 \nu_\theta \Phi^{BB} + \frac{d - 1}{2} \Phi^{zA} \Phi^{zA} + \frac{1}{2} C^A \Phi^{\bar{A}} + \frac{1}{2} \bar{C}^z \bar{C}^z ,
$$

(D27)

where $\Box$, $\nu$’s are defined in (6.15), i.e. we see that if we find $p_\nu$ then we fix the coefficient $\sigma_{s,\nu}$. The coefficient $p_\nu$ is uniquely determined by the following two requirements:

i) Modified de Donder gauge condition for AdS field $|\phi|$ should lead to the differential constraint for the shadow field $|\phi_{sh}|$ (see Table II).

ii) For $\nu = \nu_\theta$ (see (4.5)), the $p_{\nu_\theta}$ is normalized to be

$$
p_{\nu_\theta} = c_{\nu_\theta} \nu_\theta ,
$$

(E4)

where $c_{\nu_\theta}$ is given in (6.27).

We note that the choice of normalization condition (E2) is a matter of convenience. This condition implies that solution (E1) leads to the following asymptotic behavior for the leading rank-$s$ tensor field $\phi^{a_1\ldots a_s}$ (see (6.3)):

$$
\phi^{a_1\ldots a_s}(x, z) \sim z^{-\frac{d}{2}} z^{-\nu_s + \frac{1}{p}} \phi^{a_1\ldots a_s}_{sh}(x) ,
$$

(E5)
where $\phi_{sh}^{a_1...a_n}$ is the leading rank-$s$ tensor field in $|\phi_{sh}\rangle$ (see (2.31)).

We now analyze restrictions imposed by the first requirement. To this end we note the relation

$$C|\phi(x,z)\rangle = \int d^d y p_\nu F(x-y)\phi_{sh}(y), \quad (E6)$$

where $C$ is modified de Donder operator (6.6). Matching of modified de Donder gauge condition for $AdS$ field $|\phi\rangle$ and the differential constraint for the shadow field $|\phi_{sh}\rangle$ implies the relation

$$\bar{C}|\phi(x,z)\rangle = \int d^d y p_\nu F(x-y)\bar{C}_{sh}|\phi_{sh}(y)\rangle. \quad (E8)$$

Comparison of (E6) and (E8) gives the equation

$$\bar{C}_{sh} = W. \quad (E9)$$

Comparing $\bar{C}_{sh}$ given in Table II and $W$ given in (E7) we see that Eq. (E9) amounts to the following equation for $p_\nu$:

$$p_{\nu-1}(2\nu + d - 2) = -p_\nu. \quad (E10)$$

Solution to this equation is given by

$$p_\nu = (-2)^{\nu}\Gamma(\nu + \frac{d}{2})p_0, \quad (E11)$$

where $p_0$ does not depend on $\nu$. Requiring normalization condition (2.4), we find

$$p_0 = \frac{c_{\nu s}}{(-2)^{\nu}\Gamma(\nu + \frac{d}{2})}. \quad (E12)$$

Plugging this $p_0$ in (E11) we get

$$p_\nu = (-)^{\nu s}\Gamma(\nu + \frac{d}{2})c_{\nu s}. \quad (E13)$$

Taking into account (E3), (E13) and $c_{\nu s}$ (6.15), we obtain solution for $\sigma_{s,\nu}$ given in (6.23).

For the readers convenience, we note the formulas which are helpful for the derivation of relation (E6).

$$e_1|\phi(x,z)\rangle = -\int d^d y F(x-y)\frac{p_{\nu+1}}{2\nu + d}e_{1sh}|\phi_{sh}(y)\rangle, \quad (E14)$$

$$\bar{e}_1|\phi(x,z)\rangle = -\int d^d y F(x-y)

\times p_{\nu-1}(2\nu + d - 2)e_{1sh}|\phi_{sh}(y)\rangle, \quad (E15)$$

$$e_1F(x) = \frac{1}{2\nu + d}\square F(x)e_{1sh}, \quad (E16)$$

$$\bar{e}_1F(x) = -F(x)(2\nu + d - 2)e_{1sh}, \quad (E17)$$

where the operators $e_1$ and $\bar{e}_1$ are defined in (6.7).

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