Beltrami equations with coefficient
in the fractional Sobolev space $W^{\theta, \frac{2}{\theta}}$

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Abstract
In this paper, we look at quasiconformal solutions $\phi : \mathbb{C} \to \mathbb{C}$ of Beltrami equations
$$\partial \bar{z} \phi(z) = \mu(z) \partial_z \phi(z),$$
where $\mu \in L^\infty(\mathbb{C})$ is compactly supported on $\mathbb{D}$, $\|\mu\|_\infty < 1$ and belongs to the fractional Sobolev space $W^{\alpha, \frac{2}{\alpha}}(\mathbb{C})$. Our main result states that
$$\log \partial_z \phi \in W^{\alpha, \frac{2}{\alpha}}(\mathbb{C})$$
whenever $\alpha > \frac{1}{2}$. Our method relies on an $n$-dimensional result, which asserts the compactness of the commutator
$[b, (-\Delta)^{\beta}] : L^{n/(n-\beta)}(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$
between the fractional laplacian $(-\Delta)^{\beta}$ and any symbol $b \in W^{\beta, n}(\mathbb{R}^n)$, provided that $1 < p < \frac{n}{\beta}$.

1 Introduction

A Beltrami coefficient is a function $\mu \in L^\infty(\mathbb{C})$ with $\|\mu\|_\infty < 1$. By the well-known Measurable Riemann Mapping Theorem, to every compactly supported Beltrami coefficient $\mu$ one can associate a unique homeomorphism $\phi : \mathbb{C} \to \mathbb{C}$ in the local Sobolev class $W^{1,2}_{loc}$ such that the Beltrami equation
$$\partial \bar{z} \phi(z) = \mu(z) \partial_z \phi(z)$$
holds for almost every $z \in \mathbb{C}$, and at the same time, $|\phi(z) - z| \to 0$ as $|z| \to \infty$. One usually calls $\phi$ the principal solution, and it is known to be a $K$-quasiconformal map with $K = \frac{1 + \|\mu\|_\infty}{1 - \|\mu\|_\infty}$, since
$$|\partial \bar{z} \phi(z)| \leq \frac{K - 1}{K + 1} |\partial_z \phi(z)|$$
at almost every $z \in \mathbb{C}$.

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Recent works have shown an interest in describing the Sobolev smoothness of $\phi$ in terms of that of $\mu$. As noticed already at [5], remarkable differences are appreciated under the assumption $\mu \in W^{\alpha-p}$, depending on if $\alpha p < 2$, $\alpha p = 2$ or $\alpha p > 2$. In this paper, we focus our attention on the case $\alpha p = 2$.

It was proven at [5] that if $\mu \in W^{1,2}$ then $\phi$ belongs to the local Sobolev space $W^{2,2-\epsilon}_{loc}$ for each $\epsilon > 0$ (and further one cannot take $\epsilon = 0$ in general). The proof was based on the elementary fact that

$$\mu \in W^{1,2} \quad \Rightarrow \quad \log(\partial_z \phi) \in W^{1,2}. \quad (1)$$

In particular, $\log \partial_z \phi$ enjoys a slightly better degree of smoothness than $\partial_z \phi$ itself. It is a remarkable fact that this better regularity cannot be deduced only from the fact that $\partial_z \phi \in W^{1,2-\epsilon}_{loc}$ for every $\epsilon > 0$. Somehow, this means that $\log \partial_z \phi$ contains more information than $\partial_z \phi$.

Similar phenomenon had been observed much earlier in the work of Hamilton [6], where it is shown that

$$\mu \in VMO \quad \Rightarrow \quad \log(\partial_z \phi) \in VMO. \quad (2)$$

Again, the $VMO$ smoothness of $\log(\partial_z \phi)$ cannot be completely transferred to $\partial_z \phi$ itself. Indeed, the example $\phi(z) = z (\log |z| - 1)$, in a neighbourhood of the origin, has $VMO$ Beltrami coefficient (at least locally) but clearly $D\phi \notin VMO$.

The $VMO$ setting is interesting in our context since it can be seen as the limiting space of $W^{\alpha,2}$. Certainly, the complex method of interpolation shows that

$$[VMO, W^{1,2}]_\alpha = W^{\alpha, \frac{2}{\alpha}}, \quad 0 < \alpha < 1$$

(see for instance [12]). Thus, it is natural to ask if a counterpart to implication (1) holds in $W^{\alpha, \frac{2}{\alpha}}$. In the present paper, we prove the following theorem.

**Theorem 1.** Let $\alpha \in (\frac{1}{2}, 1)$. Let $\mu$ be a Beltrami coefficient with compact support and such that $\mu \in W^{\alpha, \frac{2}{\alpha}}(\mathbb{C})$. Let $\phi$ be the principal solution to the $\mathbb{C}$-linear Beltrami equation

$$\partial_{\overline{z}} \phi = \mu \partial_z \phi.$$ 

Then, $\log (\partial \phi) \in W^{\alpha, \frac{2}{\alpha}}(\mathbb{C})$.

The proof of Theorem 1 is based on two facts. The first one is the following a priori estimate for linear Beltrami equations with coefficients belonging to $W^{\alpha, \frac{2}{\alpha}}(\mathbb{C})$.
Theorem 2. Let $\alpha \in (0,1)$ and $1 < p < \frac{2}{\alpha}$. Let $\mu, \nu$ be a pair of Beltrami coefficients with compact support, such that $||\mu|| + ||\nu||_{\infty} \leq k < 1$ and $\mu, \nu \in W^{\alpha,2}(\mathbb{C})$. For every $g \in W^{\alpha,p}(\mathbb{C})$ the equation
\[ \partial_z f - \mu \partial_z f - \nu \overline{\partial_z f} = g \]
admits a solution $f$ with $Df \in W^{\alpha,p}(\mathbb{C})$, unique modulo constants, and such that the estimate
\[ ||Df||_{W^{\alpha,p}(\mathbb{C})} \leq C ||g||_{W^{\alpha,p}(\mathbb{C})} \]
holds for a constant $C$ depending only on $k, ||\mu||_{W^{\alpha,2}(\mathbb{C})}$ and $||\nu||_{W^{\alpha,2}(\mathbb{C})}$.

Theorem 2 is sharp, in the sense that one cannot take $p = \frac{2}{\alpha}$. Thus, Theorem 1 shows that $\log \partial_z \phi$ enjoys better regularity than $\partial_z \phi$ itself.

The study of logarithms of derivatives of quasiconformal maps goes back to the work of Reimann [11], where it was shown that the real-valued logarithm $\log |\partial_z \phi| \in BMO$ whenever $||\mu||_{\infty} < 1$. References involving the complex logarithm $\log \partial_z \phi$ also lead to [1]. More recently, in [3] the authors obtained sharp bounds for the $BMO$ norm of $\log \partial_z \phi$ also with the only assumption $||\mu||_{\infty} < 1$.

The second main ingredient in the proof of Theorem 1 is a compactness result for commutators of pointwise multipliers and the fractional laplacian, which holds in higher dimensions and has independent interest. In order to state it, given a measurable function $u : \mathbb{R}^n \to \mathbb{R}$ we denote
\[ D^\beta u(x) := \lim_{\epsilon \to 0} C_{n,\beta} \int_{|x-y|>\epsilon} \frac{u(x) - u(y)}{|x-y|^{n+\beta}} \, dy. \tag{3} \]
This is a principal value representation of the fractional laplacian $(-\Delta)^{\frac{\beta}{2}}$, whose symbol at the Fourier side is
\[ \widehat{D^\beta u}(\xi) = (-\Delta)^{\frac{\beta}{2}} \hat{u}(\xi) = |\xi|^{\beta} \hat{u}(\xi). \]
The operator $D^\beta$ can also be seen as the formal inverse of $I_\beta$, the classical Riesz potential of order $\beta$, which can be represented as
\[ \widehat{I_\beta u}(\xi) = |\xi|^{-\beta} \hat{u}(\xi). \]
With this notation, a function $u$ belongs to $W^{\beta,p}$, $1 < p < \infty$, if and only if $u$ and $D^\beta u$ belong to $L^p$, with the corresponding equivalent norm. Analogously, $u \in W^{\beta,p}$ if and only if $D^\beta u \in L^p$.

Let us remind that if $T$ and $S$ are two operators, one usually calls $[T,S] = T \circ S - S \circ T$ the commutator of $T$ and $S$. 

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Theorem 3. Let $\beta \in (0,1)$ and $b \in W^{\beta, \frac{2}{\beta}}(\mathbb{R}^n)$. Then, the commutator

$$[b, D^\beta] : L^{\frac{n}{n-\beta p}}(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$$

is bounded and compact whenever $1 < p < \frac{n}{\beta}$.

The boundedness of the commutator can be seen as a consequence of fractional versions of the Leibnitz rule. For the compactness, the Fréchet-Kolmogorov characterization of compact subsets of $L^p$ is combined with the cancellation properties of the kernel of the commutator. Also, in the proof of Theorem 1 one uses Theorem 3 with $\beta = 1 - \alpha$. This explains the restriction $\alpha > \frac{1}{2}$ in Theorem 1, as what one really uses is that $\mu \in W^{1-\alpha, \frac{2}{1-\alpha}}(\mathbb{C})$. Note that this space contains $W^{\alpha, \frac{2}{\alpha}}(\mathbb{C})$ if and only if $\alpha > \frac{1}{2}$.

A detailed proof of Theorem 3 is provided at Section 2. In Section 3, we find a priori estimates for generalized Beltrami equations with coefficients in $W^{\theta, \frac{2}{\theta}}$, and prove Theorem 1 and Theorem 2.

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2 Proof of Theorem 3

The proof of Theorem 3 we present here is based on classical ideas, see for instance [10]. We will need the following auxiliary result about the Leibnitz rule for fractional derivatives.

Proposition 4. (Kenig-Ponce-Vega’s Inequality [8])

Let $\beta \in (0,1)$ and $1 < p < \frac{n}{\beta}$. Then the inequality

$$\|D^\beta (f g) - f D^\beta g\|_p \leq C \|D^\beta f\|_{\frac{p}{\beta}} \|g\|_{\frac{n}{n-\beta p}}.$$

holds whenever $f, g \in C^\infty_c(\mathbb{R}^n)$.

With this result at hand, we immediately get that the commutator

$$[b, D^\beta] : L^{\frac{n}{n-\beta p}}(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$$

admits a unique bounded extension. Remarkably,

$$\|[b, D^\beta]\|_{L^{\frac{n}{n-\beta p}}(\mathbb{R}^n) \to L^p(\mathbb{R}^n)} \leq C \|b\|_{W^{\beta, \frac{2}{\beta}}(\mathbb{R}^n)}.$$
As a consequence, if \( b_n \in C_c^\infty(\mathbb{R}^n) \) is such that
\[
\lim_{n \to \infty} \| b_n - b \|_{W^{\beta, \frac{n}{\alpha}}(\mathbb{R}^n)} = 0
\]
then
\[
\lim_{n \to \infty} \| [b_n, D^\beta] - [b, D^\beta] \|_{L^{\frac{n}{\alpha-p}}(\mathbb{R}^n) \to L^p(\mathbb{R}^n)} = 0
\]
Thus, we are reduced to prove Theorem 3 with the extra assumption \( b \in C_c^\infty(\mathbb{R}^n) \). To this end, we observe that the commutator \( C_b = [b, D^\beta] \) can be represented as an integral operator
\[
C_b f(x) = b(x) \text{ P.V.} \int K(x, y) (f(x) - f(y)) \, dy - \text{P.V.} \int K(x, y) (f(x) b(x) - b(y) f(y)) \, dy
= \text{P.V.} \int K(x, y) (b(y) - b(x)) f(y) \, dy
= \int K(x, y) f(y) \, dy
\]
where
\[
K(x, y) = C_{n, \beta} \frac{(b(y) - b(x))}{|y - x|^{n+\beta}}
\]
and the principal value has been removed from the last integral because the smoothness of \( b \) ensures that \( x \mapsto K(x, y) \) is integrable. For \( C_b \) to be compact, we need to prove that the image under \( C_b \) of the unit ball of \( L^{\frac{n}{\alpha-p}}(\mathbb{R}^n) \) is compact in \( L^p(\mathbb{R}^n) \). To this end, we denote
\[
\mathcal{F} = \{ C_b f : \| f \|_{L^{\frac{n}{\alpha-p}}(\mathbb{R}^n)} \leq 1 \}.
\]
The classical Fréchet-Kolmogorov’s Theorem asserts that \( \mathcal{F} \) is relatively compact if and only if the following conditions hold:

(i) \( \mathcal{F} \) is uniformly bounded, i.e. \( \sup_{\psi \in \mathcal{F}} \| \psi \|_{L^p(\mathbb{R}^n)} < \infty \).

(ii) \( \mathcal{F} \) vanishes uniformly at \( \infty \), i.e. \( \sup_{\psi \in \mathcal{F}} \| \psi \chi_{|x| > R} \|_{L^p(\mathbb{R}^n)} \to 0 \) as \( R \to \infty \).

(iii) \( \mathcal{F} \) is uniformly equicontinuous, i.e. \( \sup_{\psi \in \mathcal{F}} \| \psi(\cdot + h) - \psi(\cdot) \|_{L^p(\mathbb{R}^n)} \to 0 \) as \( |h| \to 0 \).

In our particular case, every element \( \psi \in \mathcal{F} \) has the form \( \psi = C_b f \) with \( \| f \|_{L^{\frac{n}{\alpha-p}}(\mathbb{R}^n)} \leq 1 \). Thus (i) follows automatically from the boundedness of \( [b, D^\beta] : L^{\frac{n}{\alpha-p}}(\mathbb{R}^n) \to L^p(\mathbb{R}^n) \).

To prove (ii), let \( R_0 > 0 \) be such that \( \text{supp}(b) \subset B(0, R_0) \). At points \( x \) with \( |x| > 3R_0 \) we have
\[
|C_b f(x)| \leq \int \frac{|f(y) b(y)|}{|x - y|^{n+\beta}} \, dy \leq C \| b \|_{\infty} \int_{B(0, R_0)} \frac{|f(y)|}{|x|^{n+\beta}} \, dy \leq C \| b \|_{\infty} \frac{|x|^{\frac{n+\beta}{n}}}{|x|^{n+\beta}} \| f \|_q R_0^{\frac{n+\beta}{n}}.
\]
Thus, if $R > 3R_0$ then
\[
\int_{|x| > R} |C_b f(x)|^p \, dx \leq C_R \|b\|_{\mathcal{P}_n} \|f\|_{\mathcal{P}_{\frac{n-p}{n-p}}} \int_{|x| > R} |x|^{-p(n+\beta)} \, dx \to 0 \quad \text{as } R \to \infty
\]
as needed.

For the proof of (iii), we could proceed as usually, which means to regularize the kernel $K$ in the diagonal \( \{x = y\} \). Then we would prove the compactness of this regularization and finally the limit of compact operators would give us the result. However, a more direct approach is available, since $\|K(x, \cdot)\|_{L^1(\mathbb{R}^n)}$ is uniformly bounded.

**Lemma 5.** One has
\[
\lim_{h \to 0} \sup_{f \neq 0} \frac{\|C_b f(\cdot + h) - C_b f(\cdot)\|_{L^q(\mathbb{R}^n)}}{\|f\|_{L^q(\mathbb{R}^n)}} = 0
\]
whenever $1 \leq q \leq \infty$.

**Proof.** We start by observing that
\[
\|K(x, \cdot)\|_{L^1(\mathbb{R}^n)} = \int_{|x-y| \leq 1} |K(x, y)| \, dy + \int_{|x-y| > 1} |K(x, y)| \, dy
\leq C \|\nabla b\|_\infty \int_{|x-y| \leq 1} |x-y|^{-n-\beta+1} \, dy + C \|b\|_\infty \int_{|x-y| > 1} |x-y|^{-n-\beta} \, dy
\leq C \left( \frac{\|\nabla b\|_\infty}{1 - \beta} + \frac{\|b\|_\infty}{\beta} \right) := A
\]
As a consequence, the behavior of $C_b f$ is like the convolution of the function $f$ with a $L^1$-kernel. In particular, by Jensen’s inequality one gets
\[
\|C_b f\|_q \leq A \|f\|_q, \quad 1 \leq q \leq \infty,
\]
so that $C_b : L^q(\mathbb{R}^n) \to L^q(\mathbb{R}^n), 1 \leq q \leq \infty$.

Towards (5), we need to estimate the translates of $C_b$. Clearly,
\[
\|C_b f(\cdot + h) - C_b f(\cdot)\|_q \leq \int \left( \int |f(y)(K(x + h, y) - K(x, y))\, dy \right)^q \, dx
\leq \left( \int |f(y)|^q |K(x + h, y) - K(x, y)| \, dy \right)^{\frac{q}{q-1}} \left( \int |K(x + h, y) - K(x, y)| \, dy \right)^{\frac{q-1}{q}} \, dx
\leq (2A)^{q-1} \left( \int |K(x + h, y) - K(x, y)| \, dx \right) \|f(y)\|^q \, dy
\leq (2A)^{q-1} B(h) \int |f(y)|^q \, dy
\]
where \( B(h) = \sup_y \| \mathcal{K}(\cdot, h, y) - \mathcal{K}(\cdot, y) \|_{L^1(\mathbb{R}^n)} \). In order to find estimates for \( B(h) \), we choose an arbitrary \( \rho > 0 \) and write

\[
\int |\mathcal{K}(x+h, y) - \mathcal{K}(x, y)| \, dx = \int_{|x-y| \leq \rho} \cdots + \int_{|x-y| > \rho} \cdots =: I + II.
\]

The integrability of \( \mathcal{K} \) gives that \( I \) is small if \( \rho \) is small enough. Indeed,

\[
\int_{|x-y| \leq \rho} |\mathcal{K}(x, y)| \, dx \leq \| \nabla b \|_{\infty} \int_{|x-y| \leq \rho} |x-y|^{-n-\beta+1} \, dx = C \frac{\| \nabla b \|_{\infty}}{1-\beta} \rho^{1-\beta}.
\]

Moreover, if \( x \in B(y, \rho) \) then \( x+h \in B(y, \rho + |h|) \) so that

\[
\int_{|x-y| \leq \rho} |\mathcal{K}(x+h, y) - \mathcal{K}(x, y)| \, dx \leq \int_{|x-(y-h)| \leq 2\rho} |\mathcal{K}(x+h, y)| \, dx \leq C \frac{\| \nabla b \|_{\infty}}{1-\beta} (\rho + |h|)^{1-\beta}.
\]

Therefore, there exists \( \rho_0 > 0 \) such that if \( \rho < \rho_0 \) and \( |h| < \rho_0/2 \) then \( I \leq \varepsilon/((2A)^{\eta-1}) \). Let us then fix \( \rho = \rho_0/2 \), and take care of \( II \). Note that, since \( |h| < \rho_0/2 \) and \( |x-y| > \rho \), we have

\[
|\mathcal{K}(x+h, y) - \mathcal{K}(x, y)| = \left( b(y) - b(x+h) \right) \left( \frac{1}{|x+h-y|^{n+\beta}} + \frac{1}{|x-y|^{n+\beta}} \right) \beta/\rho_0 + \| \nabla b \|_{\infty} |h|/|x-y|^{n+\beta}.
\]

Then, since we fixed \( \rho = \rho_0/2 \),

\[
II \leq C \frac{\| \nabla b \|_{\infty}}{\beta} \int_{|x-y| > \rho} \frac{dx}{|x-y|^{n+1+\beta}} + C \frac{\| \nabla b \|_{\infty}}{\beta} \int_{|x-y| > \rho} \frac{dx}{|x-y|^{n+\beta}}
\]

\[
\leq C \frac{\| \nabla b \|_{\infty}}{\beta} \left( \frac{\| b \|_{\infty}}{\rho_0^{1+\beta}} + \frac{\| \nabla b \|_{\infty}}{\rho_0^\beta} \right).
\]

Thus, by taking \( |h| \) sufficiently small, we see that \( II \leq \varepsilon/((2A)^{\eta-1}) \). Hence \( B(h) \to 0 \) as \( |h| \to 0 \), and thus (5) follows.

With the above Lemma, the proof of (iii) is almost immediate. Indeed, by (4) we see that

\[
\| C_b f(\cdot + h) - C_b f(\cdot) \|_p^p = \int_{|x| \leq R} |C_b f(x+h) - C_b f(x)|^p \, dx
\]

\[
\quad + \int_{|x| > R} |C_b f(x+h) - C_b f(x)|^p \, dx
\]

\[
\leq \| C_b f(\cdot + h) - C_b f(\cdot) \|_p^p \frac{1}{R^{\beta p}} \frac{1}{R^{\beta p}}
\]

\[
\quad + C_R \frac{\| b \|_{t, \infty} \| f \|_p^p \frac{1}{R^{\beta p}}}{R^{\beta p}} \int_{|x| > R} |x|^{-p(n+\beta)} \, dx.
\]

at least for \( R > 3R_0 \). In particular, the last term is small if \( R \) is large enough. But for this particular \( R \), and using (5), the penultimate term is also small if \( |h| \) is small. Therefore (iii) follows. Theorem 3 is proved.
3 Beltrami operators in fractional Sobolev spaces

The regularity theory for Beltrami equations relies on the behavior of the Beurling operator, which is formally defined as a principal value operator,

$$\mathcal{B}f(z) = -\frac{1}{\pi} \text{p.v.} \int_C f(z-w) \frac{1}{w^2} dA(w).$$

This operator intertwines the $\partial_z$ and $\partial_{\overline{z}}$ derivatives. More precisely, its Fourier representation

$$\hat{\mathcal{B}}f(\xi) = \frac{\overline{\xi}}{\xi} \hat{f}(\xi).$$

makes it clear that $\mathcal{B}(\partial_{\overline{z}} f) = \partial_z f$, at least when $f$ is smooth and compactly supported.

Furthermore, $\mathcal{B}$ is an isometry on $L^2(\mathbb{C})$, and as a Calderón-Zygmund operator, it can be boundedly extended to $L^p(\mathbb{C})$ whenever $1 < p < \infty$.

Before proving Theorem 1, we first state and prove the following fact about generalized Beltrami equations. Let us recall that $\overline{\mathcal{B}}$ denotes the composition of $\mathcal{B}$ with the complex conjugation operator, that is, $\overline{\mathcal{B}}(f) = \overline{\mathcal{B}(f)}$.

**Proposition 6.** Let $\alpha \in (0, 1)$. Let $\mu, \nu \in W^{\alpha, 2}(\mathbb{C})$ be compactly supported Beltrami coefficients, with $\|\mu\| + \|\nu\|_{\infty} \leq k < 1$. Then the generalized Beltrami operators

$$\text{Id} - \mu \mathcal{B} - \nu \overline{\mathcal{B}} : \dot{W}^{\alpha,p}(\mathbb{C}) \to \dot{W}^{\alpha,p}(\mathbb{C})$$

are bounded and boundedly invertible if $1 < p < \frac{2}{\alpha}$.

**Proof.** The operators $\text{Id} - \mu \mathcal{B} - \nu \overline{\mathcal{B}}$ are clearly bounded in $\dot{W}^{\alpha,p}(\mathbb{C})$, since $\mathcal{B}$ preserves $\dot{W}^{\alpha,p}(\mathbb{C})$ (recall that we are assuming $1 < p < \frac{2}{\alpha}$) and also because if $\mu \in \mathcal{L}^\infty(\mathbb{C}) \cap W^{\alpha, 2}(\mathbb{C})$ then $\mu$ is a pointwise multiplier of $\dot{W}^{\alpha,p}(\mathbb{C})$ (similarly for $\nu$). This fact follows directly working on the expression (3) for $D^\alpha$ or see [13, p. 250]. Also, the operator $\text{Id} - \mu \mathcal{B} - \nu \overline{\mathcal{B}}$ is clearly injective in $\dot{W}^{\alpha,p}(\mathbb{C})$, as its kernel is a subset of $L^{2^{\alpha, 2}}(\mathbb{C})$ were we already know it is injective (see [7] for a proof in the $\mathbb{C}$-linear setting, and [9] or also [4] for a proof in the general case). Thus, in order to get the surjectivity (and finish the proof by the Open Mapping Theorem) we will prove that $\text{Id} - \mu \mathcal{B} - \nu \overline{\mathcal{B}}$ is a Fredholm operator on $\dot{W}^{\alpha,p}(\mathbb{C})$ with index 0. To do this, it is sufficient if we prove that

$$D^\alpha(\text{Id} - \mu \mathcal{B} - \nu \overline{\mathcal{B}})I_\alpha : L^p(\mathbb{C}) \to L^p(\mathbb{C})$$

is a Fredholm operator of index 0, since both properties stay invariant under the topological isomorphisms

$$D^\alpha : \dot{W}^{\alpha,p}(\mathbb{C}) \to L^p(\mathbb{C}),$$

$$I_\alpha : L^p(\mathbb{C}) \to \dot{W}^{\alpha,p}(\mathbb{C}).$$
But this follows easily. Indeed,
\[
D^\alpha (\text{Id} - \mu \mathcal{B} - \nu \overline{\mathcal{B}}) I_\alpha = \text{Id} - D^\alpha (\mu \mathcal{B} + \nu \overline{\mathcal{B}}) I_\alpha \\
= \text{Id} - \mu \mathcal{B} - \nu \overline{\mathcal{B}} - [D^\alpha, \mu] \mathcal{B} I_\alpha - [D^\alpha, \nu] \overline{\mathcal{B}} I_\alpha
\]

Above, \( \text{Id} - \mu \mathcal{B} - \nu \overline{\mathcal{B}} \) is invertible in \( L^p(\mathbb{C}) \) by [7]. Also, \([D^\alpha, \mu] \mathcal{B} I_\alpha \) is the composition of the bounded operators \( I_\alpha : L^p(\mathbb{C}) \to L^{\frac{2p}{\alpha-p}}(\mathbb{C}) \) and \( \mathcal{B} : L^{\frac{2p}{\alpha-p}}(\mathbb{C}) \to L^{\frac{2p}{\alpha-p}}(\mathbb{C}) \) with the operator \([D^\alpha, \mu] : L^{\frac{2p}{\alpha-p}}(\mathbb{C}) \to L^p(\mathbb{C}) \), which is compact by Theorem 3. Hence \([D^\alpha, \mu] \mathcal{B} I_\alpha : L^p(\mathbb{C}) \to L^p(\mathbb{C}) \) is compact, and the same happens to \([D^\alpha, \nu] \overline{\mathcal{B}} I_\alpha \). Thus the term on the right hand side is the sum of an invertible operator with two compact operators. Hence it is a Fredholm operator. The claim follows. \( \square \)

We are now ready to prove Theorem 2.

**Proof of Theorem 2.** By simplicity, we assume that \( \nu = 0 \). Otherwise, the proof follows similarly. First of all, let us observe that if \( g \in \dot{W}^{\alpha,p}(\mathbb{C}) \) and \( \alpha p < 2 \) then automatically \( g \in L^{\frac{2p}{\alpha-p}}(\mathbb{C}) \) by the Sobolev embedding. On the other hand, and since \( W^{\alpha,\frac{2}{p}}(\mathbb{C}) \subset VMO \), we know from [7] that a solution \( f \in \dot{W}^{1,\frac{2p}{\alpha-p}}(\mathbb{C}) \) exists, and moreover
\[
\|Df\|_{L^{\frac{2p}{\alpha-p}}(\mathbb{C})} \leq C \|g\|_{L^{\frac{2p}{\alpha-p}}(\mathbb{C})} \leq C \|g\|_{\dot{W}^{\alpha,p}(\mathbb{C})}.
\]

Our goal consists of replacing the term on the left hand side by \( \|Df\|_{\dot{W}^{\alpha,p}(\mathbb{C})} \).

To do this, we first note that \( \partial_z f = \mathcal{B}(\partial_z f) \), since \( f \in \dot{W}^{1,\frac{2p}{\alpha-p}} \). Thus (??) is equivalent to
\[
(\text{Id} - \mu \mathcal{B})(\partial_z f) = g
\]

Now, from Proposition 6 and our assumption \( g \in \dot{W}^{\alpha,p}(\mathbb{C}) \), we also know that there is a unique \( F \in \dot{W}^{\alpha,p}(\mathbb{C}) \) such that
\[
(\text{Id} - \mu \mathcal{B})F = g \tag{7}
\]

for which we know the estimate \( \|F\|_{\dot{W}^{\alpha,p}(\mathbb{C})} \leq C \|g\|_{\dot{W}^{\alpha,p}(\mathbb{C})} \) holds. Of course, by the Sobolev embedding, \( F \in L^{\frac{2p}{\alpha-p}}(\mathbb{C}) \). From the invertibility of \( \text{Id} - \mu \mathcal{B} \) on \( L^{\frac{2p}{\alpha-p}}(\mathbb{C}) \), we immediately get that \( F = \partial_z f \) almost everywhere, and therefore \( \partial_z f \in \dot{W}^{\alpha,p}(\mathbb{C}) \). Proving that \( \partial_z f \in \dot{W}^{\alpha,p}(\mathbb{C}) \) is very easy, as we already knew that \( f \in \dot{W}^{1,\frac{2p}{\alpha-p}}(\mathbb{C}) \) and so we can be sure that \( \partial_z f = \mathcal{B}(\partial_z f) \). Thus, \( Df \in \dot{W}^{\alpha,p}(\mathbb{C}) \) and certainly
\[
\|Df\|_{\dot{W}^{\alpha,p}(\mathbb{C})} \leq C \|F\|_{\dot{W}^{\alpha,p}(\mathbb{C})} \leq C \|g\|_{\dot{W}^{\alpha,p}(\mathbb{C})}
\]
as desired. \( \square \)
Towards the proof of Theorem 1, we denote by $C(h)$ the solid Cauchy transform,
\[ C(h)(z) = \frac{1}{\pi} \int_{\mathbb{C}} h(z - w) \frac{1}{w} \, dA(w). \] (8)
This operator appears naturally as a formal inverse to the $\partial_z$ derivative, that is, the formula
\[ \partial_z C(h) = h \] holds if $h \in L^p(\mathbb{C})$ and $1 < p < \infty$. Another important feature about the Cauchy transform is that $\partial C = B$. The Cauchy and Beurling transforms allow for a nice representation of the principal solution $\phi$ of the Beltrami equation $\partial_z \phi = \mu \partial_z \phi$,
\[ \phi(z) = z + C(h)(z), \]
see for instance [2, p. 165]. In this representation, $h$ is a solution to the integral equation
\[ (\text{Id} - \mu B)(h) = \mu. \]
As a consequence, the invertibility of the Beltrami operators $\text{Id} - \mu B$ also plays a central role in determining the smoothness of $\phi$. In particular, by applying Proposition 6 with $\mu \in W^{\alpha, \frac{2}{\alpha}}$, we see that $Dh \in W^{\alpha, p}$ provided that $p < \frac{2}{\alpha}$, whence $D\phi \in W^{\alpha, p}_{\text{loc}}$. As a consequence, by Stoilow’s Factorization Theorem (e.g., [2, section 5.5]), the same conclusion holds for any quasiregular solution $f$ of $\partial_z f - \mu \partial_z f = 0$. However, this is not enough for Theorem 1, which we prove now.

Proof of Theorem 1. We will first prove that if $\mu \in W^{\alpha, \frac{2}{\alpha}}(\mathbb{C})$ is a compactly supported Beltrami coefficient and $\alpha > \frac{1}{2}$ (this is the point where we use that restriction) the operator
\[ T_\mu := I_{1-\alpha}(\text{Id} - \mu B)D^{1-\alpha} : L^2(\mathbb{C}) \to L^2(\mathbb{C}) \]
is continuously invertible, with lower bounds depending only on $\|\mu\|_{L^\infty(\mathbb{C})}$ and $\|\mu\|_{W^{\alpha, \frac{2}{\alpha}}(\mathbb{C})}$. To do this, we proceed as usually,
\[ T_\mu = I_{1-\alpha}(\text{Id} - \mu B)D^{1-\alpha} = \text{Id} - I_{1-\alpha}\mu BD^{1-\alpha} = \text{Id} - \mu B + I_{1-\alpha}[D^{1-\alpha}, \mu]B. \]
Here, the term $\text{Id} - \mu B$ is bounded and continuously invertible in $L^{\frac{2}{\alpha}}(\mathbb{C})$ by [7]. Concerning the second term on the right hand side, from $\mu \in W^{\alpha, \frac{2}{\alpha}}(\mathbb{C}) \cap L^\infty(\mathbb{C})$ and $\frac{1}{2} < \alpha$ we easily get that $\mu \in W^{1-\alpha, \frac{2}{\alpha}}(\mathbb{C})$. Thus we are legitimate to use Theorem 3 with $\beta = 1 - \alpha$ and $p = \frac{2}{\alpha}$ and get that $[\mu, D^{1-\alpha}]$ is a compact operator from $L^{\frac{2}{\alpha}}(\mathbb{C})$ into $L^2(\mathbb{C})$. As a consequence, we obtain that $T_\mu$ is a Fredholm operator from $L^{\frac{2}{\alpha}}(\mathbb{C})$ into itself, which clearly has index 0. So the desired lower bounds will be automatic if we see that it is injective.

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Let $F \in L^\frac{2}{\alpha}$ such that $T_\mu(F) = 0$. We want to show that $F = 0$. First, if $F \in \dot{W}^{1-\alpha,2}(\mathbb{C})$ then the result follows easily. Indeed, we can then write $F := I_{1-\alpha}f$ for some $f \in L^2$ and write the equation in terms of $f$. We get $I_{1-\alpha}(\mathrm{Id} - \mu \mathcal{B})f = 0$. From the classical $L^2$ theory, we have that $f = 0$ and hence $F = 0$. For a general $F \in L^\frac{2}{\alpha}$ satisfying $T_\mu(F) = 0$ we will prove that necessarily $F \in \dot{W}^{1-\alpha,2}(\mathbb{C})$, and therefore $F = 0$. To do this, again we decompose $T_\mu$ in terms of the commutator,

$$(\mathrm{Id} - \mu \mathcal{B})F = I_{1-\alpha}[\mu, D^{1-\alpha}]BF.$$ 

Then by Theorem 3 the term on the right hand side above belongs to $\dot{W}^{1-\alpha,2}(\mathbb{C})$, because $F \in L^\frac{2}{\alpha}(\mathbb{C})$. Using again that $\alpha > \frac{1}{2}$ one has $\mu \in W^{1-\alpha,\frac{2}{\alpha}}(\mathbb{C})$, and therefore we can use Proposition 6 to get that $\mathrm{Id} - \mu \mathcal{B} : \dot{W}^{1-\alpha,2}(\mathbb{C}) \to \dot{W}^{1-\alpha,2}(\mathbb{C})$ is continuously invertible. Hence

$$F = (\mathrm{Id} - \mu \mathcal{B})^{-1}I_{1-\alpha}[\mu, D^{1-\alpha}]BF$$

belongs to $\dot{W}^{1-\alpha,2}(\mathbb{C})$. The claim follows.

We now finish the proof. Given $\mu \in W^{\alpha,\frac{2}{\alpha}}(\mathbb{C})$, we approximate it by $\mu_n \in C^\infty_c(\mathbb{C})$ in the $W^{\alpha,\frac{2}{\alpha}}(\mathbb{C})$ topology, in such a way that $\|\mu_n\|_{L^\infty(\mathbb{C})} \leq \|\mu\|_{L^\infty(\mathbb{C})}$. Then every $\mu_n$ admits a principal quasiconformal map $\phi_n$, for which the function $g_n = \log \partial_z \phi_n$ is well defined and solves

$$\partial_\bar{z} g_n - \mu_n \partial_z g_n = \partial_z \mu_n.$$ 

Therefore

$$(\mathrm{Id} - \mu_n \mathcal{B})\partial_\bar{z} g_n = \partial_z \mu_n.$$ 

We use the Fourier representation of the classical Riesz transforms in $\mathbb{R}^2$,

$$\widehat{\mathcal{R}_j u}(\xi) = -i \frac{\xi_j}{|\xi|} \hat{u}(\xi) \quad j = 1, 2$$

to represent

$$\partial_\bar{z} g = -\pi D^{1-\alpha}(\mathcal{R}_1 + i\mathcal{R}_2)(D^\alpha g)$$
$$\partial_z g = -\pi D^{1-\alpha}(\mathcal{R}_1 - i\mathcal{R}_2)(D^\alpha g).$$

As a consequence, we obtain

$$(\mathrm{Id} - \mu_n \mathcal{B})D^{1-\alpha}(\mathcal{R}_1 + i\mathcal{R}_2)(D^\alpha g_n) = D^{1-\alpha}(\mathcal{R}_1 - i\mathcal{R}_2)(D^\alpha \mu_n),$$

and therefore

$$T_{\mu_n}(\mathcal{R}_1 + i\mathcal{R}_2)(D^\alpha g_n) = (\mathcal{R}_1 - i\mathcal{R}_2)(D^\alpha \mu_n).$$

We recall that both $\mathcal{R}_1 + i\mathcal{R}_2$ and $\mathcal{R}_1 - i\mathcal{R}_2$ are bounded and continuously invertible operators in $L^p(\mathbb{C})$, $1 < p < \infty$. Moreover, we have just seen that $T_{\mu_n}$ is boundedly invertible in $L^\frac{2}{\alpha}(\mathbb{C})$. 

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with bounds depending only on $\|\mu_n\|_{L^\infty(C)}$ and $\|\mu_n\|_{W^{\alpha, 2}(C)}$. However, each $\|\mu_n\|_\infty$ (and respectively $\|\mu_n\|_{W^{\alpha, 2}(C)}$) is bounded by a constant multiple of $\|\mu\|_\infty$ (respectively $\|\mu\|_{W^{\alpha, 2}(C)}$).

Hence
\[
\|g_n\|_{\dot{W}^{\alpha, 2}(C)} = \|D^\alpha g_n\|_{L^2(C)} \leq C(\alpha) \|(R_1 + iR_2)D^\alpha g_n\|_{L^2(C)} \\
\leq C \left( \alpha, \|\mu\|_{L^\infty(C)}, \|\mu\|_{W^{\alpha, 2}(C)} \right) \|T_{\mu_n}(R_1 + iR_2)(D^\alpha g_n)\|_{L^2(C)} \\
\leq C \left( \alpha, \|\mu\|_{L^\infty(C)}, \|\mu\|_{W^{\alpha, 2}(C)} \right) \|(R_1 - iR_2)D^\alpha \mu_n\|_{L^2(C)} \\
\leq C \left( \alpha, \|\mu\|_{L^\infty(C)}, \|\mu\|_{W^{\alpha, 2}(C)} \right).
\]

It then follows that $g_n$ is a bounded sequence in $\dot{W}^{\alpha, 2}(C)$. By the Banach-Alaoglu theorem there exists $h \in \dot{W}^{\alpha, 2}(C)$ such that
\[
\lim_{n \to \infty} \langle g_n, \varphi \rangle = \langle h, \varphi \rangle
\]
for each $\varphi \in W^{-\alpha, \frac{2}{2-\alpha}}(C)$. Remarkably, by the weak lower semicontinuity of the norm,
\[
\|h\|_{\dot{W}^{\alpha, 2}(C)} = \|D^\alpha h\|_{L^2(C)} \leq \liminf_{n \to \infty} \|D^\alpha g_n\|_{L^2(C)} \leq C \left( \alpha, \|\mu\|_{L^\infty(C)}, \|\mu\|_{W^{\alpha, 2}(C)} \right).
\]

Incidentally, we already knew from the classical theory that $\phi_n$ converges in $W^{1,p}_{loc}(C)$ to the principal quasiconformal map $\phi$ associated to $\mu$. In particular, modulo subsequences, $\partial_z \phi_n$ converges to $\partial_z \phi$ almost everywhere. But then $g_n$ converges almost everywhere to $\log(\partial_z \phi)$. It then follows that $\log(\partial_z \phi) = h$ and so we deduce that $\log(\partial_z \phi)$ belongs to $\dot{W}^{\alpha, 2}(C)$, with the same bound than $h$. The theorem follows.

\[
\square
\]

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