Research Article

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An adaptive variational model for multireference alignment with mixed noise

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Abstract: Multireference alignment (MRA) problem is to estimate an underlying signal from a large number of noisy circularly-shifted observations. The existing methods are always proposed under the hypothesis of a single Gaussian noise. However, the hypothesis of a single-type noise is inefficient for solving practical problems like single particle cryo-EM. In this paper, we focus on the MRA problem under the assumption of Gaussian mixture noise. We derive an adaptive variational model by combining maximum a posteriori (MAP) estimation and soft-max method. There are two adaptive weights which are for detecting cyclical shifts and types of noise. Furthermore, we provide a statistical interpretation of our model by using expectation-maximization (EM) algorithm. The existence of a minimizer is mathematically proved. The numerical results show that the proposed model has a more impressive performance than the existing methods when one Gaussian noise is large and the other is small.

Keywords: Multireference alignment, Mixed noise, Adaptive variational model, Soft max, Expectation-maximization, Cryo-EM.

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1 Introduction

Multireference alignment (MRA) is the problem of estimating a true signal from a number of noisy and circularly-shifted observations. This mathematical model arises in many scientific and engineering fields, for instance, structural biology [5,8,26,30,37,38], single cell genomic sequencing [27], radar [17,28], robotics [9], crystalline simulations [6], image registration and super-resolution [10,13,22] and some algorithms and theoretical analysis [1,2,7,11–13,15,23,32,39]. There are some variants of MRA problem, such as heterogeneous MRA [25], super-resolution MRA [33].

The mathematical description of MRA problem is

\[ f_i = R_l u + v_i, \quad i = 1, 2, ..., M, \]  

where \( u = (u_1, u_2, ..., u_N) \), \( f_i = (f_{i,1}, f_{i,2}, ..., f_{i,N}) \), \( v_i = (v_{i,1}, v_{i,2}, ..., v_{i,N}) \) are the real signal, the \( i \)-th observation and the corresponding noise respectively. \( R_l \) is a circularly shifted operator, namely, \( R_l u[j] = u[j - l] \) in the sense of zero-based indexing and modulo \( N \), where \( l \in \{0, 1, 2, ..., N - 1\} \). Significantly, both the true signal \( x \) and the shifts \( r_i \) are unknown. The goal is to recover \( x \) from these observations \( f_i \).

As far as we know, the existing literatures for MRA problem have been always based on the hypothesis of a single Gaussian noise. The literature can roughly be divided into two patterns. One is first estimating shifts and then estimating the true signal [2], the other is estimating the true signal without seeking for shifts [15,32]. Here we are only concentrated on the latter. For the latter pattern, there are usually two general approaches. One is based on statistical knowledge like maximum likelihood estimation (MLE),

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maximum a posteriori (MAP) estimation, expectation-maximization (EM) algorithm\[6, 31\]. The other is based on shift variant features such as spectral method largest spectral gap\[15\], frequency marching (FM)\[32\], optimization on phase manifold\[32\] and optimization on phase synchronization\[32\]. The advantage of the former has higher accuracy, but the latter needs less time and computer resources.

MRA problem is a simplified mathematical model for single particle cryo-electron microscopy (cryo-EM)\[4, 31, 34\], which is a very popular technique for visualizing biological molecules\[21, 24, 35\]. Image denoising and image alignment are two tasks in single particle cryo-EM. The MRA problem is a simplified model for tackling these two tasks simultaneously. However, in practice, the noise in cryo-EM images\[36\] are always more complicated beyond a single Gaussian noise. Motivated by the above discussions, we extend the type of noise from a single Gaussian noise to Gaussian mixture noise to close the gap between the MRA model and the actual applications.

In this work, we derive an adaptive variational model for MRA problem with Gaussian mixture noise by combining MAP estimation and soft-max method. The big challenge is that both circularly-shifted translations and the types of noise are unknown. To solve this difficulty, the proposed model contains two weights, where one is for the circularly-shifted translation of each observation, the other is for the type of noise on each component of each observation. In addition, we provide a statistical explanation for our proposed model by using exception-maximization (EM) algorithm. Furthermore, we prove the existence of a minimizer in our proposed model with total-variation (TV) regularizer. We design an algorithm to by the alternating direction iterative method and the augmented Lagrange method. In addition, we provide some convergence analysis. In numerical experiments, our proposed model outperforms the existing algorithms when one Gaussian noise is large and the other is small.

Organization of this paper. We derive the proposed model in Section 2 including model hypothesis and modeling process. In Section 3 we provide a statistical interpretation of the proposed model. We prove the existence of a minimizer in Section 4. In Section 5 we design an algorithm by alternating direction iterative method and augmented Lagrange method. Section 6 provides some convergence analysis. Section 7 shows some numerical results. We summarize this paper in Section 8.

One can reproduce this work by the code in https://github.com/MIALAB-RUC/MRA-MGG-softmax.

2 The proposed model

In this section, we derive an adaptive variational model for multireference alignment (MRA) problem with mixed Gaussian-Gaussian (MGG) noise by combining maximum a posteriori (MAP) estimation and soft-max method.

We denote $\mathcal{U}$, $\mathcal{F}$, $\mathcal{V}$, $\mathcal{L}$ as random variables of pixels in the true signal, observation, noise, and circularly-shifted operator respectively. $f$, $u$, $v$, $l$ are corresponding sample values. $P_X(x)$ and $p_X(x)$ represent the cumulative distribution function and the probability density distribution function of random variable $X$ at point $x$ separately.

2.1 Model hypothesis

There are some basic hypotheses in the proposed model:

**A1** The true signal is corrupted by some mixed noise with mean 0. We denote $\alpha = (\alpha_1, \alpha_2, ..., \alpha_K)$ as the mixed ratio, where $\alpha_k$ is the ratio of the $k$-th kind of noise among the mixed noise. Moreover, $\sigma^2 = (\sigma_1^2, \sigma_2^2, ..., \sigma_K^2)$ denotes the mixed noise parameters, where $\sigma_k^2$ is the variance of the $k$-th kind of noise. Denote $\theta = (\alpha, \sigma^2)$ as all parameters of the mixed noise;

**A2** The value of noise is a realization of random variable $\mathcal{V}$. Each components of observations are mutually independently and identically distributed (i.i.d) with probability density function $p_V$;

**A3** The value of a circularly shifted operator is a realization of random variable $\mathcal{L}$. The circularly shifted
transmissions of different observations are mutually independently and identically distributed (i.i.d) with
discrete uniform distribution in \{0, 1, ..., N - 1\};

**A4** The true signal follows a Gibbs prior distribution.

By **A1**, the mixed noise can be expressed as

\[
\mathcal{V} = \begin{cases} 
\mathcal{V}_1, & \text{when event } A_1 \text{ occurs}, \\
\mathcal{V}_2, & \text{when event } A_2 \text{ occurs}, \\
\vdots & \\
\mathcal{V}_K, & \text{when event } A_K \text{ occurs}, 
\end{cases}
\tag{2}
\]

where \( P(A_k) = \alpha_k \), and \( \sum_{k=1}^{K} \alpha_k = 1 \). Moreover, we denote \( p_{\mathcal{V}_k}(v) \) as the probability density function of
random variable \( \mathcal{V}_k \) for any \( k = 1, 2 \).

By **A3**, we can get

\[
\mathcal{L} = \begin{cases} 
0, & \text{when event } B_0 \text{ occurs}, \\
1, & \text{when event } B_1 \text{ occurs}, \\
\vdots & \\
N - 1, & \text{when event } B_{N-1} \text{ occurs},
\end{cases}
\tag{3}
\]

where \( P(B_l) = \frac{1}{N} \) for any \( l = 0, 1, ..., N - 1 \). Furthermore, we can get

\[
P_{\mathcal{L}}(l) = \frac{1}{N}.
\tag{4}
\]

By **A4**, the probability density function of the true signal is

\[
p_{\mathcal{V}}(u) = \frac{1}{T} e^{-\gamma \phi(u)},
\tag{5}
\]

where \( T > 0 \) is a constant and \( \phi \) is a given function.

### 2.2 The modeling process

In this subsection, we apply MAP estimation and soft-max method to derive an adaptive variational model
for MRA problem under the assumption of mixture Gaussian noise.

#### 2.2.1 MAP estimation

The observations \( f_1, f_2, ..., f_M \) are some realizations of random variables \( \mathcal{F}_1, \mathcal{F}_2, ..., \mathcal{F}_M \) respectively. The \( i \)-th observation is \( f_i = (f_{i,1}, f_{i,2}, ..., f_{i,N}) \) and its corresponding random variable is
\( \mathcal{F}_i = (\mathcal{F}_{i,1}, \mathcal{F}_{i,2}, ..., \mathcal{F}_{i,N}) \) for any \( i=1,2,...,M \). Furthermore, we denote \( \mathcal{A} = (\mathcal{F}_1, \mathcal{F}_2, ..., \mathcal{F}_M) \). \( A = (f_1, f_2, ..., f_M) \) is denoted as a realization of \( \mathcal{A} \). The true signal \( u = (u_1, u_2, ..., u_N) \) is a realization of random vector \( \mathcal{U} = (\mathcal{U}_1, \mathcal{U}_2, ..., \mathcal{U}_N) \). The goal is to estimate the true signal \( u \) from all observations. We need to maximize \( P_{\mathcal{U} | \mathcal{A}}(u | A) \).

**Proposition 1.** \( \mathcal{V}, \mathcal{V}_1, \mathcal{V}_2, ..., \mathcal{V}_K \) are random variables that satisfy \( \mathcal{V}_k(v) \) is the probability density function of \( \mathcal{V}_k \) for any \( k = 1, 2, ..., K \); then

\[
p_{\mathcal{V}}(v) = \sum_{k=1}^{K} \alpha_k p_{\mathcal{V}_k}(v_k).
\tag{6}
\]
Proposition 2. Assume that $\mathcal{F} = R_l \mathcal{U} + \mathcal{V}$, where $l$ is a known constant. $\mathcal{U}$ and $\mathcal{V}$ are mutually independent. The probability density function of $\mathcal{V}$ is $p_{\mathcal{V}}(v)$; then

$$
p_{\mathcal{F}|R_l}\mathcal{U}(f|R_l u) = p_{\mathcal{V}}(f - R_l u).
$$

(7)

By Bayes’ law, we can get

$$
P_{\mathcal{U}|\mathcal{F}}(u|A) = \frac{P_{\mathcal{U}|\mathcal{F}}(A|u)P_{\mathcal{U}}(u)}{P_{\mathcal{F}}(A)}.
$$

(8)

Note that $P_{\mathcal{F}}(A)$ is a constant. By taking the logarithm of (8), the goal can be converted to the following problem

$$
\max_{u, \theta} \left\{ -\sum_{i=1}^{M} \log P_{\mathcal{F},i|R_l}\mathcal{U}(f_i|u) - \log P_{\mathcal{U}}(u) \right\}.
$$

(9)

Substitute (10) into (9), then the problem (9) equals to the following problem

$$
\min_{u, \theta} \left\{ -\sum_{i=1}^{M} \log P_{\mathcal{F},i|R_l}\mathcal{U}(f_i|u) - \log P_{\mathcal{U}}(u) \right\}.
$$

(11)

By the law of total probability and substituting (4) to (11), we can get

$$
P_{\mathcal{F},i|R_l}\mathcal{U}(f_i|u) = \frac{1}{N} \sum_{l=1}^{N} P_{\mathcal{F},l|R_l}\mathcal{U}(f_i|R_l u),
$$

(12)

for any $i = 1, 2, ..., M$. By Proposition 2, we can get

$$
P_{\mathcal{F},i|R_l}\mathcal{U}(f_i|u) = \frac{1}{N} \sum_{l=1}^{N} P_{\mathcal{V}}(f_i - R_l u).
$$

(13)

Substitute (13) into (11), then the problem can be converted to

$$
\min_{u, \theta} \left\{ \mathcal{L}(u, \theta) = -\sum_{i=1}^{M} \log \sum_{l=1}^{N} P_{\mathcal{V}}(f_i - R_l u) - \log P_{\mathcal{U}}(u) \right\}.
$$

(14)

We note that there is a logarithm of summation term, which is difficult to handle in energy minimization problem. Next, we will apply soft-max method to solve this difficulty.

2.2.2 Soft-max method

Definition 1 (Soft-max). \[29\] Given a vector $x = (x_1, x_2, ..., x_N)$; then for any fixed $\varepsilon > 0$, the soft-max operator is defined by

$$
\max_{\varepsilon}(x) := \varepsilon \log \sum_{l=1}^{N} e^{\frac{x_l}{\varepsilon}}.
$$

(15)

It is easy to check that $\lim_{\varepsilon \to 0} \max_{\varepsilon}(x) = \max(x)$.

Proposition 3. \[29\] Set $G_{\varepsilon}(x) = \max_{\varepsilon}(x)$; then for any fixed $\varepsilon > 0$, $G_{\varepsilon}(x)$ is convex with respect to $x$.

Definition 2 (Fenchel-Legendre transformation). \[29\] Denote $G^*$ the Fenchel-Legendre transformation of $G$, which is defined by

$$
G^*(w) := \max_{x} \{ < x, w > - G(x) \}.
$$

(16)
Proposition 4. \cite{16} A function $G : \mathbb{R}^M \to \mathbb{R} \cup \{+\infty\}$ is convex and lower semi-continuity if and only if $G = G^\ast$.

Proposition 5. \cite{16} For any fixed $\varepsilon > 0$, $G^\ast_\varepsilon$ is the Fenchel-Legendre transformation of the soft-max function $G_\varepsilon$, then

$$G^\ast_\varepsilon(w) = \max \left\{ \varepsilon \sum_{i=1}^{N} w_i \log w_i, \quad w = (w_1, w_2, ..., w_M) \in \Delta^+, \right\} \begin{cases} +\infty, & \text{else}, \end{cases}$$

where $\Delta^+ = \{w = (w_1, w_2, ..., w_M) | 0 \leq w_i \leq 1, \sum_{i=1}^{M} w_i = 1\}$, and thus

$$G_\varepsilon(x) = G^\ast_\varepsilon(x) = \max_{w \in \Delta^+} \left\{ \varepsilon \sum_{i=1}^{N} w_i \log w_i \right\},$$

where $G^\ast_\varepsilon$ is the Fenchel-Legendre transformation of $G^\ast_\varepsilon$.

Set $x_l = \varepsilon \log P_\gamma(f_i - R_i u_l)$, $l = 1, 2, ..., N$. For any fixed $i$, index, by Proposition 5, we can get

$$\log \sum_{l=1}^{N} P_\gamma(f_i - R_i u_l) = \frac{1}{\varepsilon} \cdot \varepsilon \log \sum_{l=1}^{N} \sum_{\theta \in \Theta} P_\gamma(f_i - R_i u_l)$$

$$= \frac{1}{\varepsilon} \max_{w_i \in \Delta^+} \left\{ \varepsilon \sum_{l=1}^{N} w_i \log w_i \right\}$$

$$= \max_{w_i \in \Delta^+} \left\{ \sum_{l=1}^{N} w_i \log P_\gamma(f_i - R_i u_l) - \sum_{l=1}^{N} w_i \log w_i \right\}.$$  

Substitute (19) into (14), then we can get

$$\min_{u, \theta} \min_{w_i \in \Delta^+} \left\{ \mathcal{H}(u, \theta, w) = - \sum_{i=1}^{M} \sum_{l=1}^{N} w_i \log P_\gamma(f_i - R_i u_l) + \sum_{i=1}^{M} \sum_{l=1}^{N} w_i \log w_i - \log P_\gamma(u) \right\}.$$  

In addition, we have

$$P_\gamma(f_i - R_i u_l) = \prod_{j=1}^{N} P_\gamma(f_i - R_i u_j), \quad P_\gamma(u) = \prod_{c \in C} P_\gamma(u_c),$$

where $C$ is the clicks in the graph representation of the prior. Substitute (5), (6), (21) into (20) successively, we can get

$$\min_{u, \theta} \min_{w_i \in \Delta^+} \left\{ \mathcal{H}(u, \theta, w) = - \sum_{i=1}^{M} \sum_{l=1}^{N} w_i \log \alpha_k P_\gamma(f_i - R_i u_j) + \sum_{i=1}^{M} \sum_{l=1}^{N} w_i \log w_i + \gamma \sum_{c \in C} \phi(u_c) \right\}.$$  

Because there is a log sum item in (22), we apply Proposition 5 again. Here let $N = K$, $x_k = \varepsilon \log [\alpha_k P_\gamma(f_i - R_i u_j)]$ for any $k = 1, 2, ..., K$. For fixed indexes $i$, $j$, $l$, we can get

$$\log \sum_{k=1}^{K} \alpha_k P_\gamma(f_i - R_i u_j)$$

$$= \max_{q_{i,j,l,k} \in Q^*} \left\{ \sum_{k=1}^{K} \left[ q_{i,j,l,k} \log \alpha_k + \log P_\gamma(f_i - R_i u_j) \right] - \sum_{k=1}^{K} q_{i,j,l,k} \log q_{i,j,l,k} \right\},$$

where $Q^*$ is the set of all possible $q_{i,j,l,k}$.
where \( Q^+ = \{ q = (q_1, q_2, ..., q_K) \mid 0 \leq q_k \leq 1, \sum_{k=1}^K q_k = 1 \} \). Then we substitute (23) to (22). By simple calculation, we can get

\[
\min_{u, \theta, w, q, \xi, j, l \in Q^+} \left\{ J(u, \theta, w, q) = -\sum_{i=1}^M \sum_{l=1}^N w_{i,l} \sum_{j=1}^N \sum_{k=1}^K [q_{i,j,l,k} (\log \alpha_k + \log p_{\xi_k} (f_{i,j} - R_l u_{j,k})] \\
+ \sum_{i=1}^M \sum_{l=1}^N w_{i,l} \sum_{j=1}^N q_{i,j,l,k} \log q_{i,j,l,k} + \sum_{i=1}^M \sum_{l=1}^N w_{i,l} \log w_{i,l} + \gamma \sum_{c \in \mathcal{C}} \phi(u_c) \right\},
\]

where \( \Delta^+ = \{ w = (w_1, w_2, ..., w_M) \mid 0 \leq w_i \leq 1, \sum_{i=1}^M w_i = 1 \} \) and \( Q^+ = \{ q = (q_1, q_2, ..., q_K) \mid 0 \leq q_k \leq 1, \sum_{k=1}^K q_k = 1 \} \).

### 2.2.3 Mixed Gaussian-Gaussian noise model

Here we set \( K = 2 \), \( \gamma_1 \), \( \gamma_2 \) follow Gaussian distributions with mean 0 and variances \( \sigma_1^2 \), \( \sigma_2^2 \) respectively. The probability density function of \( \gamma_k \) is

\[
p_{\gamma_k}(v) = \frac{1}{\sqrt{2\pi \sigma_k^2}} e^{-\frac{v^2}{2\sigma_k^2}},
\]

for any \( k = 1, 2 \).

Substitute (25) into (24). By simple calculation, we can get

\[
\min_{u, \theta, w, q, \xi, j, l \in \Delta^+, q_{i,j,l,k} \in Q^+} \left\{ J(u, \theta, w, q) = \sum_{i=1}^M \sum_{l=1}^N w_{i,l} \left[ \sum_{j=1}^N \sum_{k=1}^K \frac{(f_{i,j} - R_l u_{j,k})^2}{2\sigma_k^2} q_{i,j,l,k} \\
+ \sum_{j=1}^N q_{i,j,l,k} \log q_{i,j,l,k} \right] + \sum_{i=1}^M \sum_{l=1}^N w_{i,l} \log w_{i,l} + \gamma \sum_{c \in \mathcal{C}} \phi(u_c) \right\},
\]

where \( \Delta^+ = \{ w = (w_1, w_2, ..., w_M) \mid 0 \leq w_i \leq 1, \sum_{i=1}^M w_i = 1 \} \) and \( Q^+ = \{ q = (q_1, q_2) \mid 0 \leq q_k \leq 1, \sum_{k=1}^2 q_k = 1 \} \).

### 3 Statistical interpretation of the proposed model

The observed data is \( A = (f_1, f_2, ..., f_M) \). Recall that in Subsection 2.2, by maximum a posteriori (MAP) estimation, the goal becomes

\[
\max_{u, \theta} \{ \mathcal{L}(u, \theta) \},
\]

where

\[
\mathcal{L}(u, \theta) = \log p(A|u, \theta) + \log p_{\mathcal{Y}}(u).
\]

Next, we will apply Expectation-Maximization (EM) algorithm to (27). Because \( A \) is an incomplete data, we introduce a hidden variable \( y = (y', x) \) to get the complete data \( Z = (A, y) \), where \( y' \) and \( x \)}
represent the random variables of circularly shifted operator and the type of noise separately. \( l \) and \( c \) are some realizations of the random variables \( \mathcal{L} \) and \( \mathcal{X} \) separately. By Bayes’ law, we get
\[
p(Z|u;\theta) = p(A,y|u;\theta) = p(y|(A,u);\theta)p(A|u;\theta).
\]
By taking the logarithm of (29), then
\[
\log p(A|u;\theta) = \log p(Z|u;\theta) - \log p(y|(A,u);\theta).
\]
Take the expectation of \( \log p(A|u;\theta) \) with respect of \( Z \) under the condition of \( \theta^\nu \), where \( \nu \) is the iteration step. We can get
\[
E_y[\log p(Z|u;\theta)|(A,u^\nu);\theta^\nu] = \sum_y p(y|(A,u^\nu);\theta^\nu) \log p(Z|u;\theta),
\]
which is called E-step. Then we maximize the above expectation to get \( u^{\nu+1},\theta^{\nu+1} \) as follows
\[
(u^{\nu+1},\theta^{\nu+1}) = \arg \max_{u,\theta} \{ E_y[\log p(Z|u;\theta)|(A,u^\nu);\theta^\nu] + \log p_y(u) \},
\]
which is called M-step.

Calculating expectations is a key step in EM algorithm. Next, we will derive the concrete expression of (31).

Firstly, we consider \( \log p(Z|u;\theta) \). For any \( y = (l,k) \),
\[
\log p(Z|u;\theta) = \log \prod_{i=1}^M p(f_i,y|u;\theta) = \sum_{i=1}^M \log p(f_i,y|u;\theta) = \sum_{i=1}^M \log \prod_{j=1}^N p(f_{i,j},y|u_j;\theta)
\]
\[
= \sum_{i=1}^M \sum_{j=1}^N \log p(f_{i,j},y|u_j;\theta) = \sum_{i=1}^M \sum_{j=1}^N \left[ \log p(y|u_j;\theta) + \log p(f_{i,j}|(y,u_j);\theta) \right]
\]
\[
= \sum_{i=1}^M \sum_{j=1}^N \left[ \log \left( \frac{1}{N} \alpha_k \right) + \log p_{\gamma_k}(f_{i,j} - R_l u_j;\theta) \right],
\]
where \( p((l,k)|u_j;\theta) = p(l,k) = p_{\mathcal{L}}(l) p_y(k) = \frac{1}{N} \alpha_k \) owing to the independence of \( \mathcal{L} \) and \( \mathcal{X} \), and \( p(f_{i,j}|(y,u_j);\theta) = p_{\gamma_k}(f_{i,j} - R_l u_j;\theta) \) for any \( y = (l,k) \).

Then we consider \( p(y|(A,u^\nu);\theta^\nu) \). By Bayes’ law, for any \( y = (l,k) \), we can get
\[
p(l,k|(A,u^\nu);\theta^\nu) = p(l|(A,u^\nu);\theta^\nu) p(k|l,A,u^\nu;\theta^\nu).
\]
Specifically,
\[
p(l|(f_i,u^\nu);\theta^\nu) = \frac{p(l,f_i,u^\nu;\theta^\nu)}{p(f_i,u^\nu;\theta^\nu)} = \frac{p(f_i,l|(u^\nu);\theta^\nu) p_{\mathcal{L}}(l)}{\sum_{l'=1}^N p(f_i,l'|(u^\nu);\theta^\nu) p_{\mathcal{L}}(l')}
\]
\[
= \frac{p_{\gamma}(f_i - R_l u^\nu_j;\theta^\nu)}{\sum_{l'=1}^N p_{\gamma}(f_i - R_l u^\nu_j;\theta^\nu)} = \frac{\prod_{j=1}^N p_{\gamma}(f_{i,j} - R_l u^\nu_j;\theta^\nu)}{\sum_{l'=1}^N \prod_{j=1}^N p_{\gamma}(f_{i,j} - R_l u^\nu_j;\theta^\nu) \equiv u^\nu_{i,l}}
\]
where \( p_{\mathcal{L}}(l) = \frac{1}{N} \) and
\[
p_{\gamma}(f_{i,j} - R_l u^\nu_j;\theta^\nu) = \sum_{k=1}^K \alpha_k p_{\gamma_k}(f_{i,j} - R_l u^\nu_j;\theta^\nu) = \sum_{k=1}^K \frac{\alpha_k}{\sqrt{2\pi}\sigma_k} \exp \left( \frac{(f_{i,j} - R_l u^\nu_j)^2}{2\sigma_k^2} \right).
\]
\[ p(k|l, f_{i,j}, u_j^\nu; \theta^\nu) = \frac{p(k, f_{i,j}, (l, u_j^\nu); \theta^\nu)}{p(f_{i,j}, (l, u_j^\nu); \theta^\nu)} = \frac{p(f_{i,j}|(l, u_j^\nu); \theta^\nu)p(k|l, u_j^\nu; \theta^\nu)}{\sum_{k'=1}^K p(f_{i,j}|(k', u_j^\nu); \theta^\nu)p(k'|(l, u_j^\nu); \theta^\nu)} \]

\[ = \frac{\alpha_k p_{y_k}(f_{i,j} - R_l u_j) + \sum_{k'=1}^K \alpha_{k'} p_{y_k'}(f_{i,j} - R_l u_j^*)}{\sum_{k'=1}^K \alpha_{k'} p_{y_k'}(f_{i,j} - R_l u_j^*)} \approx q_{i,j,l,k}^\nu \]

where \( p(f_{i,j}|(l, u_j^\nu); \theta^\nu) = p_{y_k}(f_{i,j} - R_l u_j) \) and \( p(k|l, u_j^\nu; \theta^\nu) = \alpha_k \). Substitute (35) and (37) into (34), then we can get

\[ p(l, k|f_{i,j}; \theta^\nu) = w_{i,j,k}^\nu \]

Substitute (33) and (38) into (31), the expectation (31) becomes

\[ E_{\mathbf{y}}[\log p(Z|u; \theta)|(A, u^\nu); \theta^\nu] = \sum_{i=1}^M \sum_{j=1}^N \sum_{l=1}^I \sum_{k=1}^K w_{i,j,l,k}^\nu q_{i,j,l,k}^\nu \log \left( \frac{1}{N} \alpha_k p_{y_k}(f_{i,j} - R_l u_j; \theta) \right). \]

Substitute \( K = 2 \) and

\[ p_{y_k}(f_{i,j} - R_l u_j; \theta) = \frac{1}{\sqrt{2\pi \sigma_k^2}} \exp \left( -\frac{\|f_{i,j} - R_l u_j\|^2}{2\sigma_k^2} \right) \]

into (39), then we can get the iterative formula as follows

\[ E_{\mathbf{y}}[\log p(Z|u; \theta)|(A, u^\nu); \theta^\nu] + \gamma \sum_{c \in C} \phi(u_c) \]

\[ = \sum_{i=1}^M \sum_{j=1}^N \sum_{l=1}^I \sum_{k=1}^K w_{i,j,l,k}^\nu q_{i,j,l,k}^\nu \left( \log \alpha_k - \frac{1}{2} \log \sigma_k^2 - \frac{\|f_{i,j} - R_l u_j\|^2}{2\sigma_k^2} \right) \]

\[ + \gamma \sum_{c \in C} \phi(u_c) \triangleq T(u, \theta|u^\nu; \theta^\nu). \]

Moreover, the M-step is

\[ (u^{\nu+1}, \theta^{\nu+1}) = \arg \max_{u, \theta} T(u, \theta|u^\nu; \theta^\nu). \]

### 4 The existence of a minimizer

In this section, we prove the existence of a minimizer for the proposed model with total variation (TV) regularizer.

For the convenience of discussion, we introduce the following notations. Denote \( \Omega \subset \mathbb{R} \) as a bound open sets. \( \mathbb{T}, \mathbb{X} \subset \mathbb{R}^+ \) are bounded close sets. \( BV(\Omega) \) is denoted as the space of bounded total variation function, i.e., \( BV(\Omega) = \{ u \in L^1(\Omega)|J(u) < +\infty \} \), where \( J(u) = \sup \{ \int_{\Omega} u(x) \text{div}(\phi(x)) dx | \phi \in C_0^\infty(\Omega, \mathbb{R}^2), \| \phi \|_{L^\infty(\Omega, \mathbb{R}^2)} \leq 1 \} \).

\( S(\Omega) = \{ u \in BV(\Omega)|u \geq 0 \} \).

\( \mathbb{K} = \{ \theta(t) = (\sigma^2(t), \alpha(t))|0 \leq \sigma_{\min}^2 \leq \sigma^2(t) \leq \sigma_{\max}^2, 0 \leq \alpha_{\min} \leq \alpha(t) < 1, \forall t \in \mathbb{T} \} \).

\( \mathbb{W} = \{ w(x,y) \in L^\infty(\mathbb{X} \times \mathbb{X})|0 \leq w(x,y) dy \leq 1, \forall (x,y) \in \mathbb{X} \times \mathbb{X}; \int w(x,y) = 1, \forall x \in \mathbb{X} \} \).

\( \mathbb{Q} = \{ q(x,y,s,t)|0 \leq q(x,y,s,t) \leq 1, \forall (x,y,s,t) \in \mathbb{X} \times \mathbb{X} \times \mathbb{T}; \int q(x,y,s,t) = 1, \forall (x,y,s) \in \mathbb{X} \times \mathbb{X} \times \mathbb{T} \} \).

\( \Gamma = \{ (u, \theta, w, q)|u \in S(\Omega), \theta \in \mathbb{K}, w \in \mathbb{W}, q \in \mathbb{Q} \} \). Denote \( \hat{u} \) as an extension of \( u \), i.e., \( \hat{u}(z) = u(z) \), if \( z \in \Omega; \hat{u}(z) = u(z+N), \) if \( z \in \Omega - N = \{ x-N|x \in \Omega \} \). The continuous form of the proposed model can be written as
\[
\begin{align*}
\min_{(u, \theta, w, q)} \left\{ J(u, \theta, w, q) = \int_X \int_\Omega w(x, y) \left[ \int_T \int_T \frac{(f(x, s) - \tilde{u}(s - y))^2}{2\sigma(t)^2} q(x, y, s, t) dt ds + \int_T \int_T \left( \frac{1}{2} \log \sigma^2(t) - \log \alpha(t) \right) q(x, y, s, t) dt ds \right] 
+ \int_X \int \left( \frac{1}{2} \log \sigma^2(t) - \log \alpha(t) \right) q(x, y, s, t) dt ds + \int \int_T q(x, y, s, t) \log q(x, y, s, t) dt ds \right]\right\}.
\end{align*}
\]

**Theorem 1.** Assume \( f \in L^\infty(X \times \Omega) \), \( 0 < \inf_{X \times \Omega} f < \sup f < +\infty \). Set \( \phi(u) = J(u) \); then there exists at least a solution in \( \Gamma \) for the problem \[\text{(43)}\].

The proof is in Appendix A.

**5 Related algorithm**

In this section, we design an algorithm for problem \[\text{(26)}\]. Recall that problem \[\text{(26)}\] is written as

\[
\begin{align*}
\min_{u, \theta, w, q} \left\{ J(u, \theta, w, q) = \sum_{i=1}^M \sum_{l=1}^N w_{i,l} \left[ \sum_{j=1}^N \sum_{k=1}^2 \frac{(f_{i,j} - R_l u_{j,l,k})^2}{2\sigma_k^2} q_{i,j,l,k} \right. 
+ \sum_{j=1}^N \sum_{k=1}^2 \left( \frac{1}{2} \log \sigma^2_k - \log \alpha_k \right) q_{i,j,l,k} + \sum_{j=1}^N q_{i,j,l,k} \log q_{i,j,l,k} 
+ \sum_{j=1}^N q_{i,j,l,k} \log q_{i,j,l,k} \bigg] 
\bigg] + \sum_{i=1}^M \sum_{l=1}^N w_{i,l} \log w_{i,l} + \gamma \sum_{c \in C} \phi(u_c) \right\}.
\end{align*}
\]

Denote \( \nu \) as outer iteration steps. By the alternating direction iterative method, we can decompose the problem \[\text{(26)}\] into two subproblems as follows,

\[
\begin{align*}
(u^{\nu+1}, \theta^{\nu+1}) &= \arg \min_{u, \theta} J(u, \theta, w^{\nu}, q^{\nu}), \\
(w^{\nu+1}, q^{\nu+1}) &= \arg \min_{w, q_{i,j,l}} J(u^{\nu+1}, \theta^{\nu+1}, w, q).
\end{align*}
\]

We can get a close solution of \( w^{\nu+1} \), by taking the derivative of \( J(u^{\nu+1}, \theta^{\nu+1}, w, q) \) with respect to \( w_i \) as follows

\[
\begin{align*}
w^{\nu+1}_{i,l} = \frac{p_{\nu}(f_i - R_l w^{\nu+1})}{\sum_{v=1}^N p_{\nu}(f_i - R_l w^{\nu+1})}, \quad l = 1, 2, ..., N,
\end{align*}
\]

where

\[
\begin{align*}
p_{\nu}(f_i - R_l w^{\nu+1}) &= \prod_{j=1}^N p_{\nu}(f_i - R_l w^{\nu+1}) \\
&= \prod_{j=1}^N \alpha^{\nu+1}_k p_{\nu}(f_i - R_l w^{\nu+1}) \\
&= \prod_{j=1}^N \left[ \sum_{k=1}^2 \alpha^{\nu+1}_k \exp \left( \frac{(f_i - R_l w^{\nu+1})^2}{2\sigma_k^{2(\nu+1)}} \right) \right]^{\nu+1} \left[ \sum_{k=1}^2 \alpha^{\nu+1}_k \exp \left( \frac{(f_i - R_l w^{\nu+1})^2}{2\sigma_k^{2(\nu+1)}} \right) \right]^{-\nu+1} \\
&= \prod_{j=1}^N \left[ \sum_{k=1}^2 \alpha^{\nu+1}_k \exp \left( \frac{(f_i - R_l w^{\nu+1})^2}{2\sigma_k^{2(\nu+1)}} \right) \right]^{\nu+1} \left[ \sum_{k=1}^2 \alpha^{\nu+1}_k \exp \left( \frac{(f_i - R_l w^{\nu+1})^2}{2\sigma_k^{2(\nu+1)}} \right) \right]^{-\nu+1} \\
&= \prod_{j=1}^N \left[ \sum_{k=1}^2 \alpha^{\nu+1}_k \exp \left( \frac{(f_i - R_l w^{\nu+1})^2}{2\sigma_k^{2(\nu+1)}} \right) \right]^{\nu+1} \left[ \sum_{k=1}^2 \alpha^{\nu+1}_k \exp \left( \frac{(f_i - R_l w^{\nu+1})^2}{2\sigma_k^{2(\nu+1)}} \right) \right]^{-\nu+1}
\end{align*}
\]
We can get a close solution of \(q_{i,j,l,k}^{\nu+1}\) by taking the derivative of \(J(u^{\nu+1}, \theta^{\nu+1}, w, q)\) with the respect of \(q_{i,j,l,k}\) as follows,

\[
q_{i,j,l}^{\nu+1} = q_{i,j,l,1}^{\nu+1} = \frac{\alpha_k^{\nu+1} p_{y_k}(f_{i,j} - R_i u_j^{\nu+1})}{\sum_{k'=1}^{K} \alpha_{k'}^{\nu+1} p_{y_k}(f_{i,j} - R_i u_j^{\nu+1})} \exp \left( \frac{(f_{i,j} - R_i u_j^{\nu+1})^2}{2 \sigma_k^{2(\nu+1)}} \right)
\]

where \(i = 1, 2, ..., M; \ j, l = 1, 2, ..., N\).

It's easy to see that the first subproblem of (44) equals to the problem (42). In other wards, the iterative scheme (44) derived by soft-max method is same as the iterative scheme (42) derived by EM algorithm.

Now we come back to the first subproblem of (44). The discussion below is in the case of fixed outer iteration steps \(\nu\). By the alternating direction iterative method, we can get

\[
\begin{align*}
\{ \nu\} &= \arg \min_u J(u, \theta^\nu, w^\nu, q^\nu) , \\
\{ \theta^\nu+1\} &= \arg \min_u J(u^\nu+1, \theta^\nu, w^\nu, q^\nu).
\end{align*}
\]

We can get the following iterative formulas from the second subproblem of (48):

\[
\begin{align*}
\alpha_k^{\nu+1} &= \frac{\sum_{i=1}^{M} \sum_{l=1}^{N} u_{i,l}^{\nu+1} \sum_{j=1}^{N} q_{i,j,l,k}^{\nu+1}}{MN}, \\
\sigma_k^{2(\nu+1)} &= \frac{\sum_{i=1}^{M} \sum_{l=1}^{N} u_{i,l}^{\nu+1} \sum_{j=1}^{N} \left( f_{i,j} - R_i u_j^{\nu+1} \right)^2 q_{i,j,l,k}^{\nu+1}}{\sum_{i=1}^{M} \sum_{l=1}^{N} \sum_{j=1}^{N} \sum_{k'=1}^{2} q_{i,j,l,k'}^{\nu+1}}.
\end{align*}
\]

For the first subproblem of (48), we can rewrite it as

\[
u^{\nu+1} = \arg \min_u \left\{ \sum_{i=1}^{M} \sum_{l=1}^{N} w_{i,l}^{\nu} \sum_{j=1}^{N} \sum_{k=1}^{2} \frac{(f_{i,j} - R_i u_j^{\nu+1})^2}{2(\sigma_k^{2(\nu+1)})} q_{i,j,l,k}^{\nu+1} + \gamma \phi(u) \right\},
\]

where \(\phi(u)\) is a regularizer and \(\gamma > 0\) is a parameter. We apply the augmented Lagrange method to problem (51) to get

\[
\begin{align*}
\min_u \max_p & \left\{ \sum_{i=1}^{M} \sum_{l=1}^{N} w_{i,l}^{\nu} \sum_{j=1}^{N} \sum_{k=1}^{2} \frac{(f_{i,j} - R_i d_j)^2}{2(\sigma_k^{2(\nu)})} q_{i,j,l,k}^{\nu+1} \right. \\
& \left. + \frac{\gamma}{2} \| u - d \|_2^2 + \| p - d \|_2^2 + \lambda \phi(u) \right\}.
\end{align*}
\]

Fix the index \(\nu\). Denote \(\iota\) the inner iteration steps of subproblem (52). The problem (52) can be decomposed into

\[
\begin{align*}
(u^{\iota+1}, d^{\iota+1}) &= \arg \min_u \max_d \left\{ \sum_{i=1}^{M} \sum_{l=1}^{N} w_{i,l}^{\nu} \sum_{j=1}^{N} \sum_{k=1}^{2} \frac{(f_{i,j} - R_i d_j)^2}{2(\sigma_k^{2(\nu)})} q_{i,j,l,k}^{\nu+1} \right. \\
& \left. + \frac{\gamma}{2} \| u - d \|_2^2 + \frac{r}{2} \left\| p - d \right\|_2^2 + \lambda \phi(u) \right\}, \\
\end{align*}
\]

\[
p^{\iota+1} = p^{\iota} + \tau(u^{\iota} - d^{\iota}).
\]
Furthermore, the problem (53) can be decomposed once again and we can get

\[
\begin{align*}
\begin{cases}
u^{+1} &= \arg\min_u \left\{ \frac{r}{2} \left\| u - d^t + P^t \right\|_2^2 + \lambda \phi(u) \right\}, \\
d^{+1} &= \arg\min_d \left\{ \sum_{i=1}^M \sum_{l=1}^N w^t_{i,l} \left( \sum_{j=1}^N (f_{i,j} - R_d d_j)^2 \right) \right. \\
&\left. + \frac{r}{2} \left\| d - u^{+1} - P^t \right\|_2^2 \right\}, \\
p^{+1} &= P^t + \tau (u^t - d^t).
\end{cases}
\end{align*}
\] (54)

The first subproblem of (54) is a Gaussian denoiser. For the second subproblem of (54), we take the derivative with respect of \( d \). We can get

\[
d^{+1}_j = \sum_{i=1}^M \sum_{l=1}^N w^t_{i,l} t_1 + \sigma^2_2(\nu) \sigma^2_1(\nu) \left( ru^{+1} + p^t_j \right)
\]

\[
= \sum_{i=1}^M \sum_{l=1}^N w^t_{i,l} t_2 + r \sigma^2_2(\nu) \sigma^2_1(\nu)
\]

where

\[
t_1 = \left[ q^t_{i,j,l} \sigma^2_2(\nu) + (1 - q^t_{i,j,l}) \sigma^2_1(\nu) \right] R^{-1} f_{i,j},
\]

\[
t_2 = q^t_{i,j,l} \sigma^2_2(\nu) + (1 - q^t_{i,j,l}) \sigma^2_1(\nu).
\]

According to the above discussion, We propose the following algorithm.

Algorithm 1 (MGG SoftMax)

1: Initialization. Let \( \nu = 0 \). Set \( u^0 = f^t, \theta^0 \). Then calculate \( w^0 \) by (45) and calculate \( q^0 \) by (47).
2: Smoothness. Set the inner iteration \( \nu = 0 \), \( u^{\nu+1, 0} = u^\nu \). Update \( u^{\nu+1, \nu+1} \) by (54) until convergence.
3: Parameter estimation. Update \( \alpha^{\nu+1} \) and \( \sigma^{(\nu+1)}_k \) by (49) and (50) separately.
4: Noise classification. Update \( q^{\nu+1} \) by calculating (47).
5: Circularly-shifted classification. Update \( w^{\nu+1} \) by calculating (45).
6: Convergence condition. If \( \|u^{\nu+1} - u^\nu\|_2 < \epsilon \), stop iterating the algorithm; Else, go to the step 2.

6 Convergence analysis

In this section, we will show some convergence analysis for the proposed algorithm.

Theorem 2. \[ \mathcal{L}, \mathcal{H} \text{ and } \mathcal{J} \text{ have the same minimizer } (u^*, \theta^*). \]

Theorem 3. \[ \mathcal{L}, \mathcal{H} \text{ (Energy Descent)} \text{ If the sequence } (u^\nu, \theta^\nu) \text{ satisfies } \mathcal{J}(u^{\nu+1}, \theta^{\nu+1}) \leq \mathcal{J}(u^\nu, \theta^\nu); \text{ then we can get } \mathcal{L}(u^{\nu+1}, \theta^{\nu+1}) \leq \mathcal{L}(u^\nu, \theta^\nu). \] (58)

Theorem 4. \[ \text{Fixed the iteration steps } \nu. \text{ Assume } \|f\|_\infty < +\infty. \text{ Set } u^* \text{ be the minimizer of the problem } (51). \forall 0 < \tau < 2r, \text{ then the sequence } u^t \text{ generated by the iteration scheme converges to } u^*, \text{ i.e. } \lim_{t \to +\infty} u^t = u^*.$
Theorem 5. [19] Fixed the iteration steps $\nu$. Assume $\|f\|_\infty < +\infty$. Set $u^*$ be the minimizer of the problem [31]. Let $\tau = r$, then the sequence $u^t$ generated by the iteration scheme [54] converges to $u^*$, i.e., $\lim_{t \to +\infty} u^t = u^*$.

7 Numerical Experiments and Results

This section is focused on numerical experiments. We consider the case of mixed Gaussian-Gaussian (MGG) noise. The true signal $u$ is a 1-dimensional signal of length $N$. We can generate $M$ noisy and circularly-shifted observations by the formula (1). The goal is to recover the true signal $u$ from $M$ noisy circularly-shifted observations. All methods are evaluated by relative recovery error defined as

$$\text{Relative Error}(\tilde{u}, u) = \frac{\|R(\tilde{u} - u)\|_2}{\|u\|_2}$$

where $\tilde{u}$ is an estimation of the true signal $u$.

To show the validity of the proposed method, we make a number of comparisons between the proposed model and some existing algorithms like Expectation Maximization (EM) [3], spectral method largest spectral gap [15], frequency marching (FM) [32], optimization on phase manifold [32] and optimization on phase synchronization [32]. The proposed model is abbreviated by MGG SoftMax. The experiments are run on a computer with 8 Inter Core i7-8550U CPUs. These CPUs are used to compute thousands of FFTs for the proposed method and EM algorithm, while they are also used to compute the invariants in parallels.

Fig. 1: Relative error as a function of the number of observations $M$. The observations are obtained through corrupting a real signal of length $N = 41$ by a mixed Gaussian-Gaussian noise with fixed parameters $\alpha = 0.2, \sigma_1 = 10, \sigma_2 = 0.1$. Note that curves corresponding to the optim. phase manifold and the iter. phase synch almost overlap.

Fig. 2: Average computation time over 10 repetitions corresponding to Fig. 1. Note that the curves corresponding to the Spectral M. largest spectral gap and FM almost overlap.

There are two groups of experiments with different true signals. The experiments are under the number of observations $M = 10^4$. One of the true signal $u$ of length $N = 41$ is a 1-dimensional standard Gaussian random signal. The experiments are conducted under different noise levels where the noise ratio $\alpha$ varies over $\{0, 0.2, 0.4, 0.6, 0.8, 1\}$, the standard deviation of the first Gaussian noise $\sigma_1$ varies over $\{10, 5\}$ and the standard deviation of the first Gaussian noise $\sigma_1$ varies over $\{0.01, 0.1, 0.5\}$. The corresponding relative errors are shown in Tab. 1. The other of true signal $u$ of length $N = 101$ is a 1-dimensional piecewise
constant signal. Specifically, the components of \( u \) from 30th to 60th equal to 1, and others equal to 0. We design the latter true signal to further show the effect of the regularization term. The corresponding relative errors are shown in Tab. 2. We can easily see that the relative errors in Tab. 2 are better than those in

| \( \alpha \) | \( \sigma_1 \) | \( \sigma_2 \) | Existing Methods | Proposed Method |
|----------|------------|------------|-----------------|-----------------|
| 0        | 10         | 0.01       | 0.0002082       | 0.0002078       |
| 0.2      | 10         | 0.01       | 0.7445          | 0.6365          |
| 0.4      | 10         | 0.01       | 0.7450          | 0.7144          |
| 0.6      | 10         | 0.01       | 0.7448          | 0.7185          |
| 0.8      | 10         | 0.01       | 0.7453          | 0.7422          |
| 1        | 10         | 0.01       | 0.7456          | 0.7318          |

| \( \alpha \) | \( \sigma_1 \) | \( \sigma_2 \) | Existing Methods | Proposed Method |
|----------|------------|------------|-----------------|-----------------|
| 0        | 10         | 0.1        | 0.0002082       | 0.0002078       |
| 0.2      | 10         | 0.1        | 0.7445          | 0.6438          |
| 0.4      | 10         | 0.1        | 0.7450          | 0.7136          |
| 0.6      | 10         | 0.1        | 0.7448          | 0.7186          |
| 0.8      | 10         | 0.1        | 0.7453          | 0.7422          |
| 1        | 10         | 0.1        | 0.7456          | 0.7318          |

| \( \alpha \) | \( \sigma_1 \) | \( \sigma_2 \) | Existing Methods | Proposed Method |
|----------|------------|------------|-----------------|-----------------|
| 0        | 10         | 0.5        | 0.01046         | 0.1356          |
| 0.2      | 10         | 0.5        | 0.7446          | 0.6496          |
| 0.4      | 10         | 0.5        | 0.7450          | 0.6943          |
| 0.6      | 10         | 0.5        | 0.7448          | 0.7219          |
| 0.8      | 10         | 0.5        | 0.7453          | 0.7455          |
| 1        | 10         | 0.5        | 0.7456          | 0.7318          |

| \( \alpha \) | \( \sigma_1 \) | \( \sigma_2 \) | Existing Methods | Proposed Method |
|----------|------------|------------|-----------------|-----------------|
| 0        | 5          | 0.01       | 0.0002082       | 0.0002078       |
| 0.2      | 5          | 0.01       | 0.7444          | 0.6200          |
| 0.4      | 5          | 0.01       | 0.7445          | 0.5791          |
| 0.6      | 5          | 0.01       | 0.7445          | 0.6759          |
| 0.8      | 5          | 0.01       | 0.7446          | 0.6576          |
| 1        | 5          | 0.01       | 0.7447          | 0.6415          |

| \( \alpha \) | \( \sigma_1 \) | \( \sigma_2 \) | Existing Methods | Proposed Method |
|----------|------------|------------|-----------------|-----------------|
| 0        | 5          | 0.1        | 0.0002082       | 0.0002078       |
| 0.2      | 5          | 0.1        | 0.7444          | 0.6267          |
| 0.4      | 5          | 0.1        | 0.7445          | 0.6581          |
| 0.6      | 5          | 0.1        | 0.7445          | 0.6795          |
| 0.8      | 5          | 0.1        | 0.7446          | 0.6564          |
| 1        | 5          | 0.1        | 0.7447          | 0.6415          |

| \( \alpha \) | \( \sigma_1 \) | \( \sigma_2 \) | Existing Methods | Proposed Method |
|----------|------------|------------|-----------------|-----------------|
| 0        | 5          | 0.5        | 0.01046         | 0.1356          |
| 0.2      | 5          | 0.5        | 0.7444          | 0.6746          |
| 0.4      | 5          | 0.5        | 0.7445          | 0.6525          |
| 0.6      | 5          | 0.5        | 0.7445          | 0.6514          |
| 0.8      | 5          | 0.5        | 0.7446          | 0.6711          |
| 1        | 5          | 0.5        | 0.7447          | 0.6415          |
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Fig. 3: Relative error as a function of the noise ratio $\alpha$ with fixed noise parameters $\sigma_1 = 10, \sigma_2 = 0.1$ on a real signal of length $N = 41$. The number of observations is fixed at $M = 10^4$. Note that curves corresponding to the optim. phase manifold and the iter. phase synch almost overlap.

Fig. 4: Average computation time over 10 repetitions corresponding to Fig.3. Note that the curves corresponding to the Spectral M. largest spectral gap and FM almost overlap.

Fig. 5: Relative error as a function of the second noise standard deviation $\sigma_2$ with fixed noise parameters $\alpha = 0.2, \sigma_1 = 5$ on a real signal of length $N = 41$. The number of observations is fixed at $M = 10^4$. Note that curves corresponding to the optim. phase manifold and the iter. phase synch almost overlap.

Fig. 6: Average computation time over 10 repetitions corresponding to Fig.5. Note that the curves corresponding to the Spectral M. largest spectral gap and FM almost overlap.

Tab. 1. As both Tab. 1 and Tab. 2 show, it’s clear to see that the proposed method named MGG SoftMax outperforms the other existing algorithms mentioned above except for the case of $\alpha = 0, 1, 1$. However, notice that there is usually a tiny difference between the proposed method and the best results in the case of $\alpha = 0, 1$, which can be usually negligible.

To intuitively see that the effect of parameters like $M, \alpha, \sigma_2$ on the relative error and computation time, we show Fig. 1, Fig. 2 based on the true random signal of length $N = 41$. Specific details are as follows.

C1 Effect of the number of observations ($M$) on the relative error and computation time.

Fig. 1 and Fig. 2 show the relative error and computation time of all methods mentioned above as a function of the number of observations $M$ with fixed noise level $\alpha = 0.2, \sigma_1 = 10, \sigma_2 = 0.1$. 
Tab. 2: Comparison of relative recovery error values for a fixed real piecewise constant signal $u$ of length $N = 101$ under different noise levels for EM [3], spectral M. largest spectral gap [15], optim. phase manifold [32], FM [32], iter. phase synch [32] and MGG SoftMax model with setting different initial values. (The largest relative errors are shown in bold fonts.)

| $\alpha$ | $\sigma_1$ | $\sigma_2$ | Existing Methods | Proposed Method |
|----------|-------------|-------------|------------------|-----------------|
| 0        | 10          | 0.001       | 0.0001128        | 0.001273        |
| 0.2      | 10          | 0.001       | 0.3907           | 0.4563          |
| 0.4      | 10          | 0.001       | 0.4198           | 0.6123          |
| 0.6      | 10          | 0.001       | 0.4250           | 0.6474          |
| 0.8      | 10          | 0.001       | 0.4672           | 0.5103          |
| 1        | 10          | 0.001       | **0.4064**       | 0.4637          |
| 0        | 10          | 0.1         | 0.01128          | 0.01273         |
| 0.2      | 10          | 0.1         | 0.4520           | 0.5463          |
| 0.4      | 10          | 0.1         | 0.4248           | 0.6085          |
| 0.6      | 10          | 0.1         | 0.4665           | 0.5126          |
| 0.8      | 10          | 0.1         | **0.4064**       | 0.4637          |
| 1        | 10          | 0.1         | **0.4064**       | 0.4637          |
| 0        | 10          | 0.5         | 0.006857         | 0.3769          |
| 0.2      | 10          | 0.5         | 0.5142           | 0.5676          |
| 0.4      | 10          | 0.5         | 0.5346           | 0.5705          |
| 0.6      | 10          | 0.5         | 0.4163           | 0.6563          |
| 0.8      | 10          | 0.5         | 0.4596           | 0.5223          |
| 1        | 10          | 0.5         | **0.4064**       | 0.4637          |
| 0        | 5           | 0.001       | 0.001218         | 0.0117         |
| 0.2      | 5           | 0.001       | 0.2920           | 0.4829          |
| 0.4      | 5           | 0.001       | 0.3712           | 0.5980          |
| 0.6      | 5           | 0.001       | 0.3213           | 0.6738          |
| 0.8      | 5           | 0.001       | 0.3371           | 0.4450          |
| 1        | 5           | 0.001       | **0.328**        | 0.3465          |
| 0        | 5           | 0.1         | 0.001218         | 0.1044          |
| 0.2      | 5           | 0.1         | 0.2925           | 0.4906          |
| 0.4      | 5           | 0.1         | 0.3707           | 0.6011          |
| 0.6      | 5           | 0.1         | 0.3204           | 0.6792          |
| 0.8      | 5           | 0.1         | 0.3791           | 0.4813          |
| 1        | 5           | 0.1         | **0.328**        | 0.3465          |
| 0        | 5           | 0.5         | 0.006857         | 0.3769          |
| 0.2      | 5           | 0.5         | 0.2812           | 0.5225          |
| 0.4      | 5           | 0.5         | 0.3632           | 0.6259          |
| 0.6      | 5           | 0.5         | 0.3162           | 0.6857          |
| 0.8      | 5           | 0.5         | 0.3343           | 0.4827          |
| 1        | 5           | 0.5         | **0.3280**       | 0.3465          |

As Fig. 1 shows, the proposed method named MGG SoftMax outperforms the other algorithms for $M = 10, 10^2, 10^3, 10^4$. Furthermore, with the increase of the observations, the relative error of the
proposed method decreases obviously. In particular, when $M$ equals to $10^4$, the relative error of proposed method named MGG SoftMax is less than the best approaches by a factor of 65. However, the computation time of the proposed method named MGG SoftMax is more than the best approaches by a factor of 10. As Fig. 2 shows.

C2 Effect of the noise ratio ($\alpha$) on the relative error and computation time.
We fix the number of observations $M = 10^4$ and the two standard deviations $\sigma_1 = 10, \sigma_2 = 0.1$ and vary the mixture noise ratio $\alpha$ from 0 to 1 at intervals of 0.2. In Fig. 3, the proposed method named MGG SoftMax outperforms than the other existing algorithms. The corresponding computation time is shown in Fig. 4.

C3 Effect of the one of the noise standard deviation($\sigma_2$) on the relative error and computation time.
With fixing the number of observations $M = 10^4$ and noise parameters $\alpha = 0.2, \sigma_1 = 5$, we vary the standard deviation of the second Gaussian noise $\sigma_2$ among $\{0.01, 0.1, 0.5\}$. As Fig. 5 shows, it’s clear that the proposed method MGG softMax has a better performance with the relative error than the other existing algorithms. Fig. 6 shows the corresponding computation time.

8 Conclusion
We derive a general adaptive variational model for MRA problem with Gaussian mixture noise. Compared with the existing methods, the difficulty is that besides shifts are unknown, the type of noise at each point is unknown. To overcome it, the proposed model contains two weights to determine unknown shifts and unknown parameters of the mixed noise. We prove the existence of a minimizer. Furthermore, we design an algorithm to calculate the two weights by alternate iterations separately. We provide some convergence analysis. Numerical experimental results illustrate that our model provides an impressive performance.

Note that accurate estimation of two weights by updating alternately is a difficult problem in itself. In this model, the error of one step does not spread as the calculation step goes further. But the weakness is that the numerical performance of our model partly depends on initial values of mixed noise parameters. It may be useful to import other algorithms to reduce the dependence of the model on initial values.

Furthermore, one can continue to apply the proposed model to some practical problems, like single particle cryo-EM problem or other scientific fields. The model of denoising and alignment of 2D images in single particle cryo-EM problem is more complicated than that of the proposed model for 1D signal.

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Therefore, \( \mathcal{J} \)

Theorem 1 is shown as follows.

That is to say, \( \exists \) \( \langle 38 \rangle \) W. Park and C. R. Midgett and D. R. Madden and G. S. Chirikjian, A stochastic kinematic model of class averaging

Set \( \text{Proof.} \)

\[
\text{for any } x, y, s, t, \in X \times \Omega, \text{ we have } J_1(u, \theta, w, q) = \int \int w(x, y)q(x, y, s, t)\,dy\,dx, \quad (A.1)
\]

where

\[
h_1(x, y) = \int \int \frac{(f(x, s) - \bar{u}(s - y))^2}{2\sigma^2(t)}q(x, y, s, t)\,dt\,ds, \quad (A.2)
\]

\[
h_2(x, y) = \int \int \left( \frac{1}{2} \log \sigma^2(t) - \log \alpha(t) \right)q(x, y, s, t)\,ds\,dt. \quad (A.3)
\]

It's easy to verify that \( h_1(x, y) \geq 0 \), for any \( (x, y) \in X \times \Omega \). For \( h_2(x, y) \), we have \( h_2(x, y) \geq \int_X |\Omega| \left( \frac{1}{2} \log \sigma^2(t) - \log \alpha(t) \right) dt \geq |\Omega| \left( \frac{1}{2} \log \sigma_{\max}^2 - \log \alpha_{\min} \right) \) for any \( (x, y) \in X \times \Omega \). Combing the condition \( 0 \leq w(x, y) \leq 1 \) for any \( (x, y) \in X \times \Omega \), \( J_1(u, \theta, w, q) \) is lower bounded. The function \( z \log z \) is continuous and convex. \( z \log z \) reaches its maximum value \(-1/e\) at the point \( z = e^{-1} \). So \( J_2(w, q) \geq -\frac{1}{e} |\Omega| \int_X w(x, y)\,dy \geq -\frac{1}{e} |\Omega| |X| \), the last inequality is based on \( \int_\Omega w(x, y)\,dy = 1 \) for any \( x \in X \).

That is to say, \( J_2(w, q) \) is lower bounded. At the same way, \( J_3(w) \geq -\frac{1}{e} |\Omega| |X| \), \( J_4(u) = \lambda J(u)\,dy \geq 0 \). Therefore, \( J(u, \theta, w, q) \) is lower bounded. Then there exists a sequence \( \{ (u_n, \theta_n, w_n, q_n) \} \subseteq \Gamma \) such that

\[
J(u_n, \theta_n, w_n, q_n) \rightarrow \inf_{(u, \theta, w, q) \in \Gamma} J(u, \theta, w, q), \quad n \rightarrow +\infty. \quad (A.6)
\]

For any \( t \in \mathbb{T} \), we have \( \theta_n(t) = (\sigma_n^2(t), \alpha_n(t)) \in K \). Moreover, \( K \) is a bounded close set in \( \mathbb{R}^2 \). So there exists a subsequence of \( \theta_n(t) \) (still denote label as \( n \)) and \( \theta(t) = (\sigma^2(t), \alpha(t)) \in K \) such that

\[
\theta_n(t) \rightarrow \theta(t) \quad \text{as } n \rightarrow +\infty. \quad (A.7)
\]

That is to say, \( \theta_n(t) \rightarrow \theta(t)(n \rightarrow +\infty) \) is pointwise convergence with respect of \( t \).

Recall that \( w_n \in L^\infty(X \times \Omega) \). Since \( L^\infty \) is the dual space of separable linear normed space \( L^1 \), by Banach-alagough theorem, we can get that there exists a weak * subsequence of \( w_n \) (still denote label as \( n \)) and weak * limitation \( w \in L^\infty(X \times \Omega) \) such that \( w_n \rightharpoonup w(n \rightarrow +\infty) \) in \( L^\infty(X \times \Omega) \). That is to say, for any \( \varphi \in L^1(X \times \Omega) \), we have

\[
\int_X \int_\Omega w_n(x, y)\varphi(x, y)\,dy\,dx \rightarrow \int_X \int_\Omega w(x, y)\varphi(x, y)\,dy\,dx, \quad n \rightarrow +\infty. \quad (A.8)
\]
Next, we will show that \( w \in \mathcal{W} \) holds, i.e., \( 0 \leq w(x, y) \leq 1 \), a.e. in \( \mathbb{K} \times \Omega \) and \( \int_{\Omega} w(x, y)dy = 1 \) for any \( x \in \mathbb{K} \). At first, we will show that \( 0 \leq w(x, y) \leq 1 \) a.e. in \( \mathbb{K} \times \Omega \). Denote \( A_1 = \{(x, y) \in (\mathbb{K}, \Omega) | w(x, y) > 1\} \), \( A_2 = \{(x, y) \in (\mathbb{K}, \Omega) | w(x, y) < 0\} \). Set \( \varphi_1(x, y) = \chi_{A_1}(x, y) \), \( \varphi_2(x, y) = \chi_{A_2}(x, y) \), then \( \varphi_1, \varphi_2 \in \mathcal{L}^1(\mathbb{K} \times \Omega) \). Substitute \( \varphi_1 \) into \([A, \delta]\), then we can get
\[
\int_{A_1} \int_{\Omega} w_n(x, y)dy dx \to \int_{A_1} \int_{\Omega} w(x, y)dy dx, \quad n \to +\infty. \tag{A.9}
\]
If \(|A_1| \neq 0\), the right side of \([A.9]\) is greater than \(|A_1|\) by the sign-preserving property of limitation, there exists a large integer \( N > 0 \) such that for any \( n \geq N \), \( \int_{A_1} w_n(x, y)dy dx > |A_1| \) holds. However, owing to \( 0 \leq w_n(x, y) \leq 1 \) a.e. \((x, y) \in \mathbb{K} \times \Omega\), the left side of \([A.9]\) equals to or less than \(|A_1|\). It’s a contradiction. So we can get \(|A_1| = 0\). At the same way, we can get \(|A_2| = 0\). That is to say, \( 0 \leq w(x, y) \leq 1 \) holds a.e. in \( \mathbb{K} \times \Omega \). Next we will show that \( \int_{\Omega} w(x, y)dy = 1 \) holds for any \( x \in \mathbb{K} \). Set \( h(x) = \int_{\Omega} w(x, y)dy \), \( \psi(x, y) = \psi(x) = \text{sign}(h(x) - 1) \), then \( \psi \in \mathcal{L}^1(\mathbb{K} \times \Omega) \) and
\[
\int_{\mathbb{K}} \int_{\Omega} w_n(x, y)\psi(x, y)dy dx \to \int_{\mathbb{K}} \int_{\Omega} w(x, y)\psi(x, y)dy dx, \quad n \to +\infty, \tag{A.10}
\]
which can be rewritten as
\[
\int_{\mathbb{K}} \hat{\psi}(x)(h_n(x) - 1)dx \to \int_{\mathbb{K}} \hat{\psi}(x)(h(x) - 1)dx, \quad n \to +\infty. \tag{A.11}
\]
Note that the left of \([A.11]\) equals to 0 for any \( n \), so we can get
\[
\int_{\mathbb{K}} |g(x) - 1|dx = 0, \tag{A.12}
\]
which implies \( g(x) = \int_{\mathbb{K}} w(x, y)dy = 1 \), a.e. \( x \in \mathbb{K} \).

Since \( \log z \) is continuous and convex, \( \mathcal{J}_3(w) \) is quasiconvex and weak * lower semicontinuous, i.e.
\[
\lim_{n \to +\infty} \inf \int_{\mathbb{K}} \int_{\Omega} w_n(x, y) \log w_n(x, y)dy dx \geq \int_{\mathbb{K}} \int_{\Omega} w(x, y) \log w(x, y)dy dx, \tag{A.13}
\]
that is to say,
\[
\lim_{n \to +\infty} \inf \mathcal{J}_3(w_n) \geq \mathcal{J}_3(w). \tag{A.14}
\]
By the same way, we can also get that there exists a weak * subsequence of \( \{q_n(x, y, s, t)\} \) (still denote label as \( n \)) and weak * limitation \( q(x, y, s, t) \in \mathcal{Q} \) such that \( q_n(x, y, s, t) \to q(x, y, s, t) \) \( (n \to +\infty) \), i.e., for any \( \kappa \in \mathcal{L}^1(\mathbb{T} \times \Omega) \), we have
\[
\lim_{n \to +\infty} \int_{\mathbb{T}} \int_{\Omega} q_n(x, y, s, t)\kappa(s, t)ds dt \to \int_{\mathbb{T}} \int_{\Omega} q(x, y, s, t)\kappa(s, t)ds dt, \quad n \to +\infty. \tag{A.15}
\]
What’s more, we can get that for any \((x, y) \in \mathbb{K} \times \Omega, \)
\[
\lim_{n \to +\infty} \inf \int_{\mathbb{T}} \int_{\Omega} q_n(x, y, s, t) \log q_n(x, y, s, t)ds dt \geq \int_{\mathbb{T}} \int_{\Omega} q(x, y, s, t) \log q(x, y, s, t)ds dt. \tag{A.16}
\]
Denote \( d_n(x, y) = \int_{\mathbb{T}} \int_{\Omega} q_n(x, y, s, t) \log q_n(x, y, s, t)ds dt \), \( d(x, y) = \int_{\mathbb{T}} \int_{\Omega} q(x, y, s, t) \log q(x, y, s, t)ds dt \). Then by the sign-preserving of limitation we can get for the subsequence satisfying \([A.16]\) (still denote label as \( n \)), there exists a large integer \( N_0 \) such that for any \( n \geq N_0 \), \( d_n(x, y) \geq d(x, y) \); moreover, Combining \( 0 \leq w_n(x, y) \leq 1 \) a.e., for the same \( n, w_n(x, y)(b_n(x, y) - b(x, y)) \geq 0 \). What’s more, \( d_n, d \in \mathcal{L}^1(\mathbb{K} \times \Omega) \) and
\[
\mathcal{J}_2(w_n, q_n) - \mathcal{J}_2(w, q) = \int_{\mathbb{K}} \int_{\Omega} w_n(x, y)d_n(x, y)dy dx - \int_{\mathbb{K}} \int_{\Omega} w(x, y)d(x, y)dy dx
\]
\[
= \int_{\mathbb{K}} \int_{\Omega} w_n(x, y)(d_n(x, y) - d(x, y))dy dx + \int_{\mathbb{K}} \int_{\Omega} (w_n(x, y) - w(x, y))d(x, y)dy dx. \tag{A.17}
\]
By Fatou’s lemma, we can get
\[
\lim_{n \to +\infty} \inf_{x} \int_{\Omega} w_n(x, y) (d_n(x, y) - d(x, y)) dy dx \\
\geq \int_{\Omega} \lim_{n \to +\infty} \inf_{x} w_n(x, y) (d_n(x, y) - d(x, y)) dy dx \\
\geq \int_{\Omega} \lim_{n \to +\infty} \inf_{x} w_n(x, y) \lim_{n \to +\infty} \inf_{x} (d_n(x, y) - d(x, y)) dy dx \\
\geq 0.
\]

Substitute $\varphi(x, y) = d(x, y)$ into (A.8), we can get
\[
\lim_{n \to +\infty} \int_{\Omega} (w_n(x, y) - w(x, y)) dy dx = 0. \tag{A.19}
\]

Combing (A.17), (A.18) and (A.19), then
\[
\lim_{n \to +\infty} \inf_{n} J_2(w_n, q_n) \geq J_2(w, q). \tag{A.20}
\]

Since $\lim_{n \to +\infty} J(u, \theta, w, q) = +\infty$ and the sequence $\{u_n, \theta_n, w_n, q_n\} \in \Gamma$ is a minimizing sequence of $J(u, \theta, w, q)$, we can get the sequence $\{u_n\}$ is uniformly bounded with respect to $n$ and $x$ and the boundary is denoted as $M_n$. By the definition of the sequence $\{(u_n, \theta_n, w_n, q_n)\}$, then when $n$ is large enough, $\mathcal{J}(u_n, \theta_n, w_n, q_n)$ is bounded, which denoted by $C$, i.e., $\mathcal{J}(u_n, \theta_n, w_n, q_n) \leq C$. Recall that $J_1, J_2, J_3$ is lower bounded, then $J_3(u_n)$ is upper bounded. Recall $\|u_n\|_{BV(\Omega)} = \|u_n\|_{L^1(\Omega)} + J(u_n)$, then $\{u_n\}$ is bounded in $BV(\Omega)$. So there exists a subsequence label still denoted as $n$ and $u \in BV(\Omega)$ such that $u_n \rightharpoonup u$ strongly in $BV(\Omega)$ and $Du_n \rightharpoonup Du$ in the sense of distribution, i.e., $\langle Du_n, \psi \rangle \to \langle Du, \psi \rangle$, $(n \to +\infty)$ for all $\psi \in (C_0^\infty(\Omega))^2$. By the lower semicontinuity of the total variation and Fatou’s lemma, we can get
\[
\lim_{n \to +\infty} \inf_{n} J_2(u_n) \geq J_2(u). \tag{A.21}
\]

Due to $u_n(y) \geq 0$ a.e. and $u_n \to u$ in $L^1(\Omega)$ strongly, we can get $u \geq 0$ a.e. So we can get $u \in S(\Omega)$.

Now let’s consider the first term $J_1(u, \theta, w, q)$. Recall that
\[
h_1(x, y) = \int_{\Omega} \int_{\mathbb{T}} \frac{(f(x, s) - \hat{u}(s - y))^2}{2\sigma^2(t)} q(x, y, s, t) dt ds,
\]
then
\[
h_{1,n}(x, y) - h_1(x, y) = \int_{\Omega} \int_{\mathbb{T}} \left( \frac{(f(x, s) - \hat{u}_n(s - y))^2}{2\sigma_n^2(t)} - \frac{(f(x, s) - \hat{u}(s - y))^2}{2\sigma^2(t)} \right) q(x, y, s, t) dt ds \\
\geq \int_{\Omega} \int_{\mathbb{T}} \left( \frac{(f(x, s) - \hat{u}_n(s - y))^2}{2\sigma_n^2(t)} - \frac{(f(x, s) - \hat{u}(s - y))^2}{2\sigma^2(t)} \right) q_n(x, y, s, t) dt ds \\
+ \int_{\Omega} \int_{\mathbb{T}} \frac{(f(x, s) - \hat{u}_n(s - y))^2}{2\sigma^2(t)} (q_n(x, y, s, t) - q(x, y, s, t)) dt ds. \tag{A.22}
\]

For any fixed point $(s, t) \in \mathbb{T} \times \Omega$, substitute $\kappa(s, t) = \frac{(f(x, s) - \hat{u}(s - y))^2}{2\sigma^2(t)} \in L^1(\mathbb{T} \times \Omega)$ into (A.15), then we can get
\[
\lim_{n \to +\infty} \int_{\Omega} \int_{\mathbb{T}} \frac{(f(x, s) - \hat{u}(s - y))^2}{2\sigma^2(t)} (q_n(x, y, s, t) - q(x, y, s, t)) dt ds = 0. \tag{A.23}
\]
\[
\begin{align*}
\int \int \int & (f(x, s) - \hat{u}_n(s,y))^2 \left( \frac{1}{2\sigma_n^2(t)} - \frac{1}{2\sigma^2(t)} \right) q_n(x, y, s, t) dt ds \\
& = \int \int \int (f(x, s) - \hat{u}_n(s,y))^2 \left( \frac{1}{2\sigma_n^2(t)} - \frac{1}{2\sigma^2(t)} \right) q_n(x, y, s, t) dt ds \\
& + \int \int \int (f(x, s) - \hat{u}_n(s,y))^2 \left( \frac{1}{2\sigma_n^2(t)} - \frac{1}{2\sigma^2(t)} \right) q_n(x, y, s, t) dt ds \\
& \quad \text{for any } x, y \in \mathbb{R}^2, \quad \text{for any fixed } t.
\end{align*}
\]

Since \( |f(x, s) - \hat{u}(s,y)|^2 \left( \frac{1}{2\sigma_n^2(t)} - \frac{1}{2\sigma^2(t)} \right) q_n(x, y, s, t) \), it follows from \( L^1(\mathbb{T} \times \Omega) \), by Lebesgue Control Convergent Theorem, we can get

\[
\begin{align*}
\lim_{n \to +\infty} \int \int (f(x, s) - \hat{u}(s,y))^2 \left( \frac{1}{2\sigma_n^2(t)} - \frac{1}{2\sigma^2(t)} \right) q_n(x, y, s, t) dt ds \\
& = \int \int (f(x, s) - \hat{u}(s,y))^2 \left( \frac{1}{2\sigma_n^2(t)} - \frac{1}{2\sigma^2(t)} \right) q_n(x, y, s, t) dt ds \\
& = 0,
\end{align*}
\]

where the last equation is based on the facts \( \lim_{n \to +\infty} \left( \frac{1}{2\sigma_n^2(t)} - \frac{1}{2\sigma^2(t)} \right) = 0 \) and \( 0 \leq q_n(x, y, s, t) \leq 1 \) is bounded in \( \Gamma \) for any \( n \).

\[
\begin{align*}
\lim_{n \to +\infty} \int \int (f(x, s) - \hat{u}_n(s,y))^2 \left( \frac{1}{2\sigma_n^2(t)} - \frac{1}{2\sigma^2(t)} \right) q_n(x, y, s, t) dt ds \\
\leq & \lim_{n \to +\infty} \int \int \left( \frac{2f(x, s) - \hat{u}_n(s,y) - \hat{u}(s,y)}{2\sigma_n^2(t)} \right) q_n(x, y, s, t) dt ds \\
\leq & \frac{f + M_n}{\sigma^2_{\min}} \lim_{n \to +\infty} \int \int |\hat{u}(s,y) - \hat{u}_n(s,y)|^2 ds dt \\
= & 0.
\end{align*}
\]

where the last equation is based on \( u_n \to u \) (\( n \to +\infty \)) strongly in \( L^1(\mathbb{T} \times \Omega) \). Combing (A.22), (A.23), (A.24), (A.25) and (A.26), then for any \( \Omega \subseteq \mathbb{R} \times \Omega \),

\[
\lim_{n \to +\infty} h_{1,n}(x, y) = h_1(x, y).
\]

By the similar way, we can easily get

\[
\lim_{n \to +\infty} h_{2,n}(x, y) = h_2(x, y).
\]

Furthermore, for any fixed \( x, y \in \mathbb{R} \times \Omega \), we have

\[
\lim_{n \to +\infty} h_n(x, y) = \lim_{n \to +\infty} h_{1,n}(x, y) + \lim_{n \to +\infty} h_{2,n}(x, y) = h(x, y).
\]

Next we consider

\[
\begin{align*}
\mathcal{J}_1(u_n, \theta_n, w_n, q_n) - \mathcal{J}_1(u, \theta, w, q) \\
= & \int_{\mathbb{X}} \int_{\mathbb{Y}} (w_n(x, y)h_n(x, y) - w(x, y)h(x, y)) dy dx \\
= & \int_{\mathbb{X}} \int_{\mathbb{Y}} w_n(x, y)(h_n(x, y) - h(x, y)) dy dx + \int_{\mathbb{X}} \int_{\mathbb{Y}} (w_n(x, y) - w(x, y)) h(x, y) dy dx.
\end{align*}
\]

Substitute \( \varphi(x, y) = h(x, y) \in L^1(\mathbb{X} \times \Omega) \) into (A.8), we can get

\[
\int_{\mathbb{X}} \int_{\mathbb{Y}} (w_n(x, y) - w(x, y)) h(x, y) dy dx = 0.
\]
By Lebesgue Control Convergence Theorem, we can get

\[
\lim_{n \to +\infty} \int_X \int_\Omega w_n(x, y)(h_n(x, y) - h(x, y)) dy dx \\
\leq \lim_{n \to +\infty} \left\| w_n \right\|_{L^\infty(X \times \Omega)} \int_X \int_\Omega (h_n(x, y) - h(x, y)) dy dx \\
\leq \lim_{n \to +\infty} \int_X \int_\Omega (h_n(x, y) - h(x, y)) dy dx
\]

(A.32)

The last equality sign is based on (A.29). So

\[
\lim_{n \to +\infty} \int_X \int_\Omega w_n(x, y)(h_n(x, y) - h(x, y)) dy dx = 0.
\]

(A.33)

Substitute (A.31) and (A.33) into (A.30), we can get

\[
\lim_{n \to +\infty} \mathcal{J}_1(u_n, \theta_n, w_n, q_n) = \mathcal{J}_1(u, \theta, w, q).
\]

(A.34)

Above all, we can get

\[
\lim_{n \to +\infty} \mathcal{J}(u_n, \theta_n, w_n, q_n) \geq \mathcal{J}(u, \theta, w, q),
\]

(A.35)

which implies \((u, \theta, w, q)\) is a minimizer. \qed