Decay constant of the $\eta$-meson from QCD sum rule

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Abstract

Decay constant of the $\eta$-meson $f_\eta$ is calculated on the basis of the QCD sum rule method. An instanton contribution is taken into account. The result is $f_\eta = 0.17 \pm 0.01 \, \text{GeV}$.

The quantum chromodynamics (QCD) sum rule technique was originally suggested by M. Shifman, A. Vainshtein and V. Zakharov [1]. According to this technique the operator product expansion of the polarization operator for various currents is considered. The terms of this expansion are the vacuum expectation values of different operators, such as quark condensate $\langle 0 | \bar{q} q | 0 \rangle$, gluon condensate $\langle 0 | G^a_{\mu\nu} G^a_{\mu\nu} | 0 \rangle$ and other operators with higher dimensions, multiplied by calculated in QCD factors. Such way the non-perturbative corrections in QCD are taken into account. Perturbative contribution is also included. On the other hand, polarization operator is expressed phenomenologically via the characteristics of physical states. In order to obtain sum rule, both representations are equated to each other after the Borel transformation $B_{M^2}$:

$$B_{M^2} f(Q^2) = \lim_{n, Q^2 \to \infty, Q^2/n=M^2} \left( \frac{(Q^2)^{n+1}}{n!} \left( -\frac{d}{dQ^2} \right)^n f(Q^2) \right).$$  \hspace{1cm} (1)
where $M^2$ is the Borel parameter.

Decay constant of the $\eta$-meson $f_\eta$ is defined in the following way:

$$\langle 0|j_{\mu 5}|\eta(p)\rangle = i f_\eta p_\mu,$$

where $j_{\mu 5}$ is the axial vector current with $\eta$-meson quantum numbers:

$$j_{\mu 5} = \frac{1}{\sqrt{6}}(\bar{u}\gamma_\mu \gamma_5 u + \bar{d}\gamma_\mu \gamma_5 d - 2\bar{s}\gamma_\mu \gamma_5 s). \quad (2)$$

In order to calculate $f_\eta$ on the basis of QCD sum rule we consider polarization operator $\Pi_{\mu\nu}(q)$:

$$\Pi_{\mu\nu}(q) = i \int d^4 x e^{iqx} \langle 0|T(j_{\mu 5}(x) j_{\nu 5}(0))|0\rangle,$$  

where $j_{\mu 5}$ is defined in (2). Polarization operator has the following structure:

$$\Pi_{\mu\nu}(q) = g_{\mu\nu}\Pi_L(q^2) + (q_\mu q_\nu - q^2 g_{\mu\nu})\Pi_T(q^2).$$

Then:

$$\Pi_{\mu\nu}(q)q_\mu q_\nu = q^2\Pi_L(q^2).$$

We assume that the masses of $u, d$ quarks are zero, but the mass of $s$ quark has the non-zero value, $m_s(1 GeV) = 0.15 GeV$.

We consider negative $q^2$: $q^2 < 0$, $-q^2 >> R_c^{-2}$, where $R_c$ is the confinement radius.

For polarization operator (3) with current (2) one can obtain:

$$\Pi_{\mu\nu}q_\mu q_\nu = \frac{16}{3} m_s \langle 0|\bar{s}s|0\rangle + i \int d^4 x e^{iqx} \langle 0|T(\partial j_5(x) \partial j_5(0))|0\rangle.$$ 

In this formula $\partial j_5 = -4/\sqrt{6}im_s\bar{s}\gamma_5 s$ is the pseudoscalar current. The polarization operator for the pseudoscalar current was obtained in [1]. For our choice of $j_{\mu 5}$ we have:

$$\Pi_L = -\frac{16}{3} \frac{m_s \langle 0|\bar{s}s|0\rangle}{Q^2} + \frac{m_s^2}{\pi^2} \ln \frac{Q^2}{\mu^2} +$$

$$+ \frac{8}{3} m_s^2 \left[ -\frac{m_s \langle 0|\bar{s}s|0\rangle}{Q^4} + \frac{\alpha_s \langle 0|G^a_{\mu\nu} G^a_{\mu\nu}|0\rangle}{8\pi Q^4} + \frac{112}{27} \frac{\pi \alpha_s \langle 0|\bar{s}s|0\rangle}{Q^6} + I(Q^2) \right]. \quad (4)$$
where $Q^2 = -q^2$, $\mu$ is the normalization scale and $I(Q^2)$ is the instanton term. In the pseudoscalar channel the direct instanton contribution is not small. It was shown by B.V. Geshkenbein and B.L. Ioffe [2], also see [3,4].

Now we represent polarization operator phenomenologically as a sum of $\eta$-meson and continuum contributions:

$$
\Pi_L(q^2) = f_{\eta}^2 q^2 + \frac{1}{\pi} \int_{s_0}^\infty \frac{c(t) dt}{t - q^2}.
$$

In this formula $m_\eta$ is the mass of $\eta$-meson, $s_0$ is the continuum threshold. Such representation is typical for polarization operator in the sum rule method. It incorporates its main features: existence of some lowest resonance and approach to the perturbative result at large $Q^2$, as demanded by asymptotic freedom.

After the substitution (5) into equation (4) and Borel transformation (1) we obtain:

$$
f_\eta = \sqrt{8} \frac{m_\eta^2}{\sqrt{3m_\eta}} \exp\left(\frac{m_\eta^2}{2M^2}\right) \times
$$

$$
\times \left[2a + m_s^2(\mu^2)L^{-\frac{8}{9}}\left(\frac{b_1}{M^2} + \frac{b_2}{M^4} - \frac{3}{8\pi^2}M^2(1 - \exp(-\frac{s_0}{M^2})) + \tilde{I}(M^2)\right)\right]^{1/2},
$$

where

$$
a = -m_s \langle 0|\bar{s}s|0 \rangle, \quad b_1 = a + \frac{\alpha_s \langle 0|G_{\mu\nu}G_{\mu\nu}^a|0 \rangle}{8\pi}, \quad b_2 = \frac{56}{27\pi}\alpha_s \langle 0|\bar{s}s|0 \rangle^2,
$$

$$
L = \ln \frac{M}{\Lambda}/\ln \frac{\mu}{\Lambda}.
$$

$\tilde{I}(M^2)$ is the instanton contribution. It calculated in the single instanton approximation [4,5,6].

$$
\tilde{I}(M^2) = \frac{3\sqrt{2}}{8\pi^2} \xi_\eta \rho_c (M^2)^{3/2} \int_0^\infty \int_0^\infty dxdy w \exp\left(-\frac{1}{4}\rho_c^2 w^2 M^2\right),
$$

where

$$
w = \sqrt{1 + x^2} + \sqrt{1 + y^2} \quad \text{and} \quad \xi_\eta = 1 - \frac{4}{3} \frac{1}{1 + \frac{2\pi \rho_c \sqrt{\rho_c}}{\sqrt{3m_\eta}}}.
$$
Instanton density has the form [4]:

\[ \frac{dn}{d\rho} = n_c \delta(\rho - \rho_c), \]

where \( \rho_c \) and \( n_c \) are two parameters, instanton radius and total instanton density. Instanton radius \( \rho_c \approx 0.3 \text{ fm} \) is assumed relatively small compared average instanton separation, which is approximately equal to 1 \text{ fm}.

The parameters in (6) have the following values:

\[ \mu = 1 \text{ GeV}, \]
\[ \Lambda = 0.2 \text{ GeV}, \]
\[ m_\eta = 0.55 \text{ GeV}, \]
\[ m_s(1 \text{ GeV}) = 0.15 \text{ GeV}, \]
\[ \langle 0|\bar{s}s|0 \rangle = 0.7 \langle 0|\bar{q}q|0 \rangle = 0.97 \cdot 10^{-2} \text{ GeV}^3 \] (This value is considered in [7,4]),
\[ \frac{\alpha_s}{\pi} \langle 0|G_{\mu\nu}^a G_{\mu\nu}^a|0 \rangle = 0.012 \text{ GeV}^4, \]
\[ s_0 = 2.5 \text{ GeV}^2, \]
\[ \rho_c = 1.67 \text{ GeV}^{-1}, \]
\[ n_c = 0.001 \text{ GeV}^4. \]

The values of \( \rho_c \) and \( n_c \) are discussed in [6].

Substituting these values into equation (6), we obtain \( f_\eta \) as the function of \( M^2 \). This function is shown in fig.1. The range of \( M^2 \) is limited by two conditions: the continuum contribution does not exceed 40-50\% and the power corrections don’t exceed 10-15\%. In our case these conditions give too wide interval of \( M^2 \). We use the typical values: \( 0.9 \text{ GeV}^2 < M^2 < 1.3 \text{ GeV}^2 \). In this interval we obtain:

\[ f_\eta = 0.17 \pm 0.01 \text{ GeV}. \]

This result is in a good agreement with the values of \( f_\eta \) from number of papers [8,9,10].

It is worth noting, however, that Particle Data Group [11] gives the different value of \( f_\eta \): \( f_\eta^{PDG} = 0.13 \pm 0.01 \text{ GeV}. \)

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Fig. 1. Function $f_\eta = f_\eta(M^2)$. 