Abstract

In [9] we presented a conjecture generalizing the Cauchy formula for Macdonald polynomials. This conjecture encodes the mixed Hodge polynomials of the character varieties of representations of the fundamental group of a punctured Riemann surface of genus $g$. We proved several results which support this conjecture. Here we announce new results which are consequences of those in [9].

1 Review of the results of [9]

1.1 Cauchy function

Fix integers $g \geq 0$ and $k > 0$. Let $x_1 = \{x_{1,1}, x_{1,2}, \ldots\}, \ldots, x_k = \{x_{k,1}, x_{k,2}, \ldots\}$ be $k$ sets of infinitely many independent variables and let $\Lambda$ be the ring of functions separately symmetric in each set of variables. Let $P$ be the set of partitions. For $\lambda \in P$, let $H_\lambda(z, w) \in \Lambda \otimes \mathbb{Q}(q, t)$ be the Macdonald symmetric function defined in [5, I.11].

Define the $k$-point genus $g$ Cauchy function

$$\Omega(z, w) = \sum_{\lambda \in P} H_\lambda(z, w) \prod_{i=1}^k H_\lambda(x_i; z^2, w^2).$$

where

$$H_\lambda(z, w) := \prod \frac{(z^{2a+1} - w^{2r+1})^{2g}}{(z^{2r+1} - w^{2a})(z^{2a} - w^{2r+2})}$$

is a $(z, w)$-deformation of the $(2g - 2)$-th power of the standard hook polynomial. Let Exp be the plethystic exponential and let Log be its inverse [9 2.3]. For $\mu = (\mu^1, \ldots, \mu^k) \in P^k$, let

$$H_\mu(z, w) := (z^2 - 1)(1 - w^2) \langle \text{Log}(\Omega(z, w), h_\mu) \rangle$$

where $h_\mu := h_{\mu^1}(x_1) \cdots h_{\mu^k}(x_k) \in \Lambda$ is the product of the complete symmetric functions and $\langle , \rangle$ is the extended Hall pairing.
1.2 Character and quiver varieties

Let $\mathcal{M}_\mu$ and $Q_\mu$ be generic character and quiver varieties corresponding to $\mu$ [9, 2.1.2.2]. Namely, we let $(C_1, \ldots, C_k)$ be a generic $k$-tuple of semisimple conjugacy classes of $GL_n(\mathbb{C})$ of type $\mu$, i.e., the coordinate $\mu^i$ of $\mu$ gives the multiplicities of the eigenvalues of $C_i$. Then $\mathcal{M}_\mu$ is the affine GIT quotient

$$\mathcal{M}_\mu := \{A_1, B_1, \ldots, A_g, B_g \in GL_n(\mathbb{C}), X_1 \in C_1, \ldots, X_k \in C_k \mid (A_1, B_1) \cdots (A_g, B_g) X_1 \cdots X_k = I_g\} // GL_n(\mathbb{C}),$$

where for two matrices $A, B$, we denote by $(A, B)$ the commutator $ABA^{-1}B^{-1}$. Let $(O_1, \ldots, O_k)$ be a generic $k$-tuple of semisimple adjoint orbits of $\mathfrak{gl}_n(\mathbb{C})$ of type $\mu$. The quiver variety $Q_\mu$ is defined as the affine GIT quotient

$$Q_\mu := \{A_1, B_1, \ldots, A_g, B_g \in \mathfrak{gl}_n(\mathbb{C}), C_1 \in O_1, \ldots, C_k \in O_k \mid [A_1, B_1] + \cdots + [A_g, B_g] + C_1 + \cdots + C_k = 0\} // GL_n(\mathbb{C}).$$

In [9], we proved that $\mathcal{M}_\mu$ and $Q_\mu$ are non-singular algebraic varieties of pure dimension $d_\mu = n^2(2g - 2 + k) - \sum_i (\mu^i)^2 + 2$.

Let $H_c(\mathcal{M}_\mu; x, y, t) := \sum_{i,j,k} h^{ijk}_c(\mathcal{M}_\mu) x^iy^jy^k$ be the compactly supported mixed Hodge polynomial. It is a common deformation of the compactly supported Poincaré polynomial $P_c(\mathcal{M}_\mu; t) = H_c(\mathcal{M}_\mu; 1, 1, t)$ and the so-called $E$-polynomial $E(\mathcal{M}_\mu; x, y) = H_c(\mathcal{M}_\mu; x, y, -1)$. We have the following conjecture [9, Conjecture 1.1.1]:

**Conjecture 1.1.** The polynomial $H_c(\mathcal{M}_\mu; x, y, t)$ depends only on $xy$ and $t$. If we let $H_c(\mathcal{M}_\mu; q, t) = H_c(\mathcal{M}_\mu; \sqrt{q}, \sqrt{q}, t)$ then

$$H_c(\mathcal{M}_\mu; q, t) = (t \sqrt{q})^d_\mu E(\mathcal{M}_\mu; q, 1).$$

(1.1)

This conjecture implies the following one:

**Conjecture 1.2** (Curious Poincaré duality).

$$H_c\left( \frac{1}{q^{1/2}}, t \right) = (qt)^{-d_\mu} H_c(\mathcal{M}_\mu; q, t).$$

The two following theorems are proved in [9]:

**Theorem 1.3.** The $E$-polynomial $E(\mathcal{M}_\mu; x, y)$ depends only on $xy$ and if we let $E(\mathcal{M}_\mu; q) = E(\mathcal{M}_\mu; \sqrt{q}, \sqrt{q})$, we have

$$E(\mathcal{M}_\mu; q) = q^{d_\mu} E(\mathcal{M}_\mu; 1, 1) \sqrt{q}. $$

(1.2)

As a corollary we get a consequence of the curious Poincaré duality Conjecture 1.2:

**Corollary 1.4.** The $E$-polynomial is palindromic.

$$E(\mathcal{M}_\mu; q) = q^{d_\mu} E(\mathcal{M}_\mu; q^{-1}) = \sum_k \left( \sum_i (-1)^k h^{ijk}(\mathcal{M}_\mu) \right) q^k.$$

We say that $\mu$ is indivisible if the gcd of all the parts of the partitions $\mu^1, \ldots, \mu^k$ is equal to 1. It is possible to choose $k$ generic semisimple adjoint orbits of type $\mu$ if and only if $\mu$ is indivisible [9, Lemma 2.2.2].
Theorem 1.5. For \( \mu \) indivisible, the mixed Hodge structure on \( H^*_c(Q_{\mu}, \mathbb{C}) \) is pure. If we let \( E(Q_{\mu}; q) = E(Q_{\mu}; q^2) \), then
\[
P_c(Q_{\mu}; \sqrt{q}) = E(Q_{\mu}; q^2) = q^{1\frac{1}{2}}\mathbb{H}_p(0, \sqrt{q}).
\]

Note that Formula (1.2) is the specialization \( t \mapsto -1 \) of Formula (1.1). Assuming Conjecture 1.1 Formula (1.3) implies that the \( i \)-th Betti number of \( Q_{\mu} \) equals the dimension of the \( i \)-th piece of the pure part of the cohomology of \( M_{\mu} \), namely, \( \sum h^i_{-2i}(M_{\mu}) \). Furthermore, when \( g = 0 \), the first author conjectures [8] that there is an isomorphism between the pure part of \( H^*_c(M_{\mu}, \mathbb{C}) \) and \( H^*_c(Q_{\mu}, \mathbb{C}) \) induced by the Riemann-Hilbert monodromy map \( Q_{\mu} \rightarrow M_{\mu} \). This would give a geometric interpretation of Theorem 1.3 in this case.

1.3 Multiplicities in tensor products

Given \( \mu = (\mu^1, ..., \mu^k) \in \mathcal{P}^k \), we can choose a generic \( k \)-tuple \((R_1, ..., R_k)\) of semisimple irreducible complex characters of \( \text{GL}_n(\mathbb{F}_q) \) where \( \mathbb{F}_q \) is a finite field with \( q \) elements [9]. We also denote by \( \Lambda : \text{GL}_n(\mathbb{F}_q) \rightarrow \mathbb{C} \) the character \( h \mapsto q^{\dim Z(h)} \) where \( Z(h) \) is the centralizer of \( h \) in \( \text{GL}_n(\mathbb{F}_q) \). Then we have [9 6.1.1]:

Theorem 1.6.
\[
\left\langle \Lambda \otimes R_\mu, 1 \right\rangle = \mathbb{H}_p(0, \sqrt{q})
\]

where \( R_\mu = \bigotimes_{i=1}^{k} R_i \).

2 Absolutely indecomposable representations

Let \( s = (s_1, ..., s_k) \in (\mathbb{Z}_{\geq 0})^k \). Let \( \Gamma \) be the comet-shaped quiver with \( g \) loops on the central vertex represented as below:

\[
\begin{array}{c}
  \vdots \\
  \vdots \\
  \vdots \\
  [k, 1] \quad [k, 2] \\
  \vdots \\
  \vdots \\
  [1, 1] \quad [1, 2] \rightarrow \cdots \rightarrow [1, s_1] \\
  \vdots \quad \vdots \\
  0 \\
  \vdots \\
  [2, 1] \quad [2, 2] \\
  \vdots \quad \vdots \\
  \vdots \\
  \vdots \quad \vdots \\
  [1, 1] \quad [1, 2] \rightarrow \cdots \rightarrow [1, s_1] \\
  \vdots \\
  \vdots \\
  \vdots \\
  \vdots \\
  [k, 1] \quad [k, 2] \\
  \vdots \\
  \vdots \\
  \vdots \\
\end{array}
\]

Let \( I = \{0\} \cup \{[i, j]\}_{1 \leq i < k, 1 \leq j \leq s_i} \) denote the set of vertices and let \( \Omega \) be the set of arrows. For \( \gamma \in \Omega \), we denote by \( h(\gamma) \in I \) the head of \( \gamma \) and \( t(\gamma) \in I \) the tail of \( \gamma \). A dimension vector for \( \Gamma \) is a collection of non-negative integers \( \nu = (\nu_i)_{i \in I} \) and a representation \( \varphi \) of \( \Gamma \) of dimension \( \nu \) over a field \( \mathbb{K} \) is a collection of \( \mathbb{K} \)-vector spaces \( \{V_i\}_{i \in I} \) with \( \dim V_i = \nu_i \) together with a collection of \( \mathbb{K} \)-linear maps \( \{\varphi_\gamma : V_{t(\gamma)} \rightarrow V_{h(\gamma)}\}_{\gamma \in \Omega} \). We denote by \( \text{Rep}_{\mathbb{K}}(\Gamma, \nu) \) the \( \mathbb{K} \)-vector space of all representations of \( \Gamma \) over \( \mathbb{K} \) of dimension vector \( \nu \). We also denote by \( \text{Rep}_{\mathbb{K}}^*(\Gamma, \nu) \) the subset of representations \( \varphi \in \text{Rep}_{\mathbb{K}}(\Gamma, \nu) \) such that \( \varphi_\gamma \) is injective for all \( \gamma \in \Omega \) such that \( r(\gamma) \) is not the central vertex 0.

Assume from now that \( \mathbb{K} \) is a finite field \( \mathbb{F}_q \). We denote by \( \text{Rep}_{\mathbb{K}}^{(i)}(\Gamma, \nu) \) the set of absolutely indecomposable representations in \( \text{Rep}_{\mathbb{K}}(\Gamma, \nu) \). We also assume that \( \nu_0 \neq 0 \). Under this assumption, note that \( \text{Rep}_{\mathbb{K}}^{(i)}(\Gamma, \nu) \subset \text{Rep}_{\mathbb{K}}^*(\Gamma, \nu) \). We may assume that \( \nu_0 \geq \nu_{[i,1]} \geq \cdots \geq \nu_{[i,k]} \) for all \( i \in \{1, ..., k\} \) since otherwise \( \text{Rep}_{\mathbb{K}}^{(i)}(\Gamma, \nu) \) is empty. For each \( i \), take the strictly decreasing subsequence \( \nu_0 > \nu_{n_i} > \cdots > \nu_{n_i} \) of \( \nu_0 \geq \nu_{[i,1]} \geq \cdots \geq \nu_{[i,k]} \) of maximal length. This defines a partition \( \mu_i' : = \mu_i' + \cdots + \mu_{i+1}' \) of \( \nu_0 \) as follows: \( \mu_i' = \nu_0 - \nu_i, \mu_i'' = \nu_i - \nu_i, \ldots, \mu_{i+1}'' = \nu_i \). The dimension vector \( \nu \) defines thus a unique multipartition
\( \mu = (\mu^1, ..., \mu^k) \in \mathcal{P}^k \). The number \( A_\mu(q) \) of isomorphism classes in \( \text{Rep}_{\mathbb{C}}^G(\Gamma, v) \) depends only on \( \mu \) and not on the choice of \( v \).

We have the following theorem \([10]\):

**Theorem 2.1.** For any \( \mu \in \mathcal{P}^k \)

\[
A_\mu(q) = \mathbb{H}_\mu(0, \sqrt{q}).
\]

We know by a theorem of V. Kac that \( A_\mu(q) \in \mathbb{Z}[q] \), see \([12]\). It is also conjectured in \([12]\) that the coefficients of \( A_\mu(q) \) are non-negative. Assuming Conjecture \([12]\), Theorem 2.1 gives a cohomological interpretation of \( A_\mu(q) \); indeed, it implies that \( A_\mu(q) \) is the Poincaré polynomial of the pure part of the cohomology of \( M_\mu \), thus implying Kac’s conjecture for comet-shaped quivers. In particular, combining Conjectures \([1.1]\) and \([1.2]\) and Theorem 2.1, we obtain the conjectural equality of the middle Betti number of \( M_\mu \) and \( A_\mu(1) \). These remarks can be compared to the fact that, when \( \mu \) is indivisible, \( r^h A_\mu(t^2) \) is \([3]\) the compactly supported Poincaré polynomial of \( Q_0 \) and thus the middle Betti number of \( Q_0 \).

Also, Theorems 1.6 and 2.1 imply that \( \langle A \otimes R_\mu, 1 \rangle = A_\mu(q) \). This gives an unexpected connection between the representation theory of \( \text{GL}_n(\mathbb{C}) \) and that of comet-shaped, typically wild, quivers.

### 3 Connectedness of character varieties

The quiver variety \( Q_\mu \) is known to be connected \([2]\). Here we use Theorem 1.3 to prove the following theorem \([10]\):

**Theorem 3.1.** The character variety \( M_\mu \) is connected.

Since the character variety \( M_\mu \) is non-singular, the mixed Hodge numbers \( h^{i-j,k}(M_\mu) \) equal zero if \((i, j, k) \notin \{(i, j, k) | i \leq k, j \leq k, i + j \} \), see \([4]\). The number of connected components of \( M_\mu \) is equal to \( h^{0,0}(M_\mu) \) and \( h^{0,0}(M_\mu) = 0 \) if \( k > 0 \). Hence by Corollary \([1.4]\) we see that the number of connected components of \( M_\mu \) equals the constant term of the \( E \)-polynomial \( E(M_\mu; q) \). To prove the theorem, we are thus reduced to prove that the coefficient of the lowest power \( q^{\frac{n_i}{2}} \) of \( q \) in \( \mathbb{H}_\mu(\sqrt{q}, 1/\sqrt{q}) \) is 1.

We use the following expansion \([9]\) Lemma 5.1.5):

\[
\sum_{\mu \in \mathcal{P}^k} \frac{q^{\mathbb{H}_\mu(\sqrt{q}, 1/\sqrt{q})}}{(q - 1)^2} m_\mu = \text{Log} \left( \sum_{\lambda \in \mathcal{P}} q^{\mathbb{H}_\lambda(\sqrt{q}, 1/\sqrt{q})} (q^{-n(\lambda)} H_\lambda(q))^k \prod_{j=1}^k s_j(x,y) \right)
\]

where \( y = \{1, 1, q^2, ..., \} \), \( H_\lambda(q) \) is the hook polynomial and \( s_j \) is the Schur symmetric function. The key-point in the proof of Theorem 3.1 for \( g > 0 \) is the following result \([10]\):

**Theorem 3.2.** Given a partition \( \lambda \in \mathcal{P} \), let \( v(\lambda) \) be the lowest power of \( q \) in

\[
\mathcal{A}_\lambda(q) := \mathbb{H}_\lambda(\sqrt{q}, 1/\sqrt{q})(q^{-n(\lambda)} H_\lambda(q))^k \prod_{j=1}^k \langle h_\mu(x), s_j(x,y) \rangle.
\]

If \( g > 0 \), then the minimum of the \( v(\lambda) \) where \( \lambda \) runs over the partitions of a given size \( n \), occurs only at \( \lambda = (1^n) \). Moreover \( v(\lambda) = -\frac{1}{2} d_\mu + 1 \) and the coefficient of \( q^{-\frac{1}{2} d_\mu + 1} \) in \( \mathcal{A}_\lambda(q) \) is 1.

When \( g = 0 \), Theorem 3.2 is known to fail in some cases. Instead we proceed with a proof which combines the use of Weyl symmetry or Katz convolution at the middle vertex and an analogue of Theorem 3.2. Here the partition \( \lambda = (1^n) \) may be not the only one for which \( v(\lambda) \) is minimal. However, we show that an appropriate cancellation occurs after taking the Log.
4 Relation with Hilbert schemes on $\mathbb{C}^* \times \mathbb{C}^*$

Put $X := \mathbb{C}^* \times \mathbb{C}^*$ and denote by $X^{[n]}$ the Hilbert scheme of $n$ points on $X$. We have [10]:

**Theorem 4.1.** Assume that $g = 1$ and $\mu$ is the single partition $\mu = (n - 1, 1)$. Then $X^{[n]}$ and $M_{\mu}$ have the same mixed Hodge polynomial.

The compactly supported mixed Hodge polynomial of $X^{[n]}$ is given by the following generating function [7]:

$$1 + \sum_{n \geq 1} H_c(X^{[n]}, q, t) T^n = \prod_{n \geq 1} \frac{(1 + T^{n+1})^3}{(1 - q^n T^n)(1 - t^{2n+1}q^n T^n)},$$

(4.1)

The identity (4.1) combined with the case $g = 1$ and $\mu = (n - 1, 1)$ of our Conjecture 1.1 becomes the following purely combinatorial conjectural identity:

**Conjecture 4.2.**

$$1 + \sum_{n \geq 1} H(\mu; z, w) T^n = \prod_{n \geq 1} \frac{(1 - zwT^n)^2}{(1 - z^2 T^n)(1 - w^2 T^n)},$$

(4.2)

where $\phi(0) := 0$ and if $\lambda$ is a non-zero partition

$$\phi_{\lambda}(z, w) := \sum_{(i, j) \in \lambda} z^i w^j,$$

where the sum runs over the boxes of $\lambda$.

**Theorem 4.3.** Equation (4.2) is true in the specialization $(z, w) \mapsto (1/\sqrt{q}, \sqrt{q})$.

This theorem is a consequence of (4.1), Theorems 1.3 and 4.1 in [10] we give an alternative purely combinatorial proof. Putting $q = e^u$ yields the following

**Corollary 4.4.**

$$1 + \sum_{n \geq 1} H(\mu, e^{u/2}, e^{-u/2}) T^n = \frac{1}{u} \left(e^{u/2} - e^{-u/2}\right) \exp \left(2 \sum_{k \geq 2} G_k \frac{u^k}{k!}\right)$$

where $G_k, k \geq 2$ are the standard Eisenstein series. In particular, the coefficient of any power of $u$ of the left hand side is in the ring of quasi-modular forms, generated by the $G_k, k \geq 2$ over $\mathbb{Q}$.

The fact that modular forms might be involved in this situation was pointed out in [13], see also [6] and [1].

**References**

[1] S. Bloch and A. Okounkov The character of the infinite wedge representation. Adv. Math. 149 (2000), no. 1, 1–60

[2] W. Crawley-Boevey: Geometry of the moment map for representations of quivers. Comp. Math. 126 (2001), 257–293.

[3] Crawley-Boevey, W. and Van den Bergh, M.: Absolutely indecomposable representations and Kac-Moody Lie algebras. With an appendix by Hiraku Nakajima. Invent. Math. 155 (2004), no. 3, 537–559.

[4] P. Deligne: Théorie de Hodge II. Inst. hautes Etudes Sci. Publ. Math. 40 (1971), 5–47.
[5] **Garsia, A.M. and Haiman, M.**: A remarkable q,t-Catalan sequence and q-Lagrange inversion, *J. Algebraic Combin.* 5 (1996) no. 3, 191–244.

[6] **L. Görtschče**: Theta functions and Hodge numbers of moduli spaces of sheaves on rational surfaces. *Comm. Math. Phys.* 206, (1999), 105–136.

[7] **L. Görtsche and W. Soergel**: Perverse sheaves and the cohomology of Hilbert schemes of smooth algebraic surfaces. *Math. Ann.* 296 (1993), 235–245.

[8] **T. Hausel**: Arithmetic harmonic analysis, Macdonald polynomials and the topology of the Riemann-Hilbert monodromy map (with an Appendix by E. Letellier) (in preparation)

[9] **T. Hausel E. Letellier and F. Rodriguez-Villegas**, Arithmetic harmonic analysis on character and quiver varieties. [arXiv:0810.2076](https://arxiv.org/abs/0810.2076)

[10] **T. Hausel E. Letellier and F. Rodriguez-Villegas**, Arithmetic harmonic analysis on character and quiver varieties II. In preparation.

[11] **J. Hua**: Counting representations of quivers over finite fields. *J. Algebra* 226, (2000) 1011–1033

[12] **V. Kac**: Root systems, representations of quivers and invariant theory. *Lecture Notes in Mathematics*, *vol. 996*, Springer-Verlag (1982), 74–108.

[13] **C. Vafa and E. Witten**: A strong coupling test of S-duality. *Nuclear Phys. B* 431 (1994), no. 1-2, 3–77.