ILL-POSEDNESS FOR THE 3D INHOMOGENEOUS NAVIER-STOKES EQUATIONS IN THE CRITICAL BESOV SPACE NEAR $L^6$ FRAMEWORK

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ABSTRACT. We prove the ill-posedness for the 3D incompressible inhomogeneous Navier-Stokes equations in critical Besov space. In particular, a norm inflation happens in finite time with the initial data satisfying

$$\|a_0\|_{B^\frac{2}{p}_p} + \|u_0\|_{B^{-\frac{2}{p}}_p} \leq \delta, \quad p > 6$$

or

$$\|a_0\|_{B^{\frac{2}{p}_p}} + \|u_0\|_{B^{\frac{2}{p}-1}_p} \leq \delta, \quad p > 6.$$  

To obtain the norm inflation, we construct a special class of initial data and introduce a modified pressure. Comparing with the classical Navier-Stokes equations in $L^\infty$ framework, we can obtain the ill-posedness for the inhomogeneous case in near $L^6$ framework.

1. INTRODUCTION

In this paper, we consider the cauchy problem for the 3D incompressible inhomogeneous Navier-Stokes equations:

$$\begin{cases}
\partial_t \rho + \text{div}(\rho u) = 0, \\
\rho \partial_t u + \rho u \cdot \nabla u - \mu \Delta u + \nabla P = 0, \\
\text{div} u = 0, \\
(\rho(0, x), u(0, x)) = (\rho_0(x), u_0(x)),
\end{cases} \quad (1.1)$$

where $(t, x) \in \mathbb{R}^+ \times \mathbb{R}^3$, $\rho \in \mathbb{R}$, $u = (u^1, u^2, u^3) \in \mathbb{R}^3$ stand for the density and the velocity field, respectively, $P$ represents the scalar pressure. The constant $\mu > 0$ is viscosity coefficient. $\rho_0$ and $u_0$ are the initial data satisfying $\text{div} u_0 = 0$. It is easy to check that the solution $(\rho, u)$ of (1.1) is scaling invariance under

$$\begin{pmatrix}
\rho_\lambda \\
u_\lambda
\end{pmatrix} = \begin{pmatrix}
\rho(\lambda^2 t, \lambda x) \\
\lambda u(\lambda^2 t, \lambda x)
\end{pmatrix}. \quad (1.2)$$

We say a function space is critical means the corresponding norm is invariant under (1.2).

Lions [25] showed (1.1) has a global weak solution $(\rho, u)$ with the following initial conditions:

$$0 \leq \rho_0 \in L^\infty, \quad \sqrt{\rho} u_0 \in L^2.$$  

2010 Mathematics Subject Classification. 35Q35, 35K55.

Key words and phrases. Navier-Stokes equations, ill-posedness, Besov space.
Then Ladyzenskaja and Solonnikov [24] obtained the local well-posedness for (1.1) with regular data. For the more results on the classical solution, one can see [5, 18, 25] and references therein.

Recently, many mathematicians have studied the well-posedness for the system (1.1) in the critical Besov space. Local well-posedness and small data global existence were obtained by Abidi [1] and Danchin [15], that is,

\[ \rho_0 - 1 \in \dot{B}^\frac{3}{p}_p, \quad u_0 \in \dot{B}^{\frac{3}{p} - 1}_p, \quad p < 6; \]  

(1.3)

global well-posedness:

\[ \|\rho_0 - 1\|_{\dot{B}^\frac{3}{p}_p} + \|u_0\|_{\dot{B}^{\frac{3}{p} - 1}_p} < \epsilon. \]  

(1.4)

Particularly, [1, 15] required the small condition of the initial density and the restriction of \( p \in [1, 3] \) for the uniqueness, which were removed by [4] and [16], respectively. In fact, all the above results are obtained around the premise that the initial density is near constant 1. It means that no vacuum is allowed. Let us introduce the new unknown \( a := \frac{1}{\rho} - 1 \), then (1.1) can be rewritten as follows:

\[
\begin{aligned}
\partial_t a + u \cdot \nabla a &= 0, \\
\partial_t u - \mu \Delta u &= -u \cdot \nabla u - (1 + a) \nabla P + \mu a \Delta u, \\
\text{div} u &= 0, \\
(a(0, x), u(0, x)) &= (a_0(x), u_0(x)).
\end{aligned}
\]  

(1.5)

Paicu and Zhang [26] proved the global well-posedness for (1.5) with large vertical velocity component (i.e., \( u^3 \)) with the initial data \((a_0, u_0) \in \dot{B}^{\frac{3}{q}, 1}_q \times \dot{B}^{\frac{3}{p} - 1}_p \) satisfying some restrictions on \((p, q)\), which was later improved by the authors in [31]. We refer to [2, 3, 10, 17, 19, 21, 27] for some other related results. Let us point out that it is not a trivial procedure to extend these results to \( L^p \) \((p \geq 6)\) framework, since there is no effective tool to deal with the nonlinear term \( \mu a \Delta u \).

When \( \rho = \text{constant} \), (1.1) reduces to the classical Navier-Stokes equations. Cannone [8] and Planchon [28] proved global solutions for small data in \( \dot{B}^{\frac{3}{p}, 1}_p \times \dot{B}^{\frac{3}{q}, 1}_q \) \((p < \infty, q \leq \infty)\). Bourgain and Pavlovic [7] obtained the ill-posedness in \( \dot{B}^{1, 1}_\infty \) by proving the solution map is discontinuous in \( \dot{B}^{-1, 1}_\infty \). And Germain [20] showed the solution map is not \( C^2 \) in \( \dot{B}^{1, 1}_\infty \) \((q > 2)\). Yoneda [30] showed the solution map is not continuous in \( \dot{B}^{1, 1}_\infty \) \((q > 2)\). Very recently, Wang [29] obtained the a new ill-posedness in \( \dot{B}^{1, 1}_\infty \) \((1 \leq q < 2)\). We refer to [13] for the ill-posedness in some Triebel-Lizorkin space and [14, 22] for other spaces. We point out that the norm inflation comes from the analysis of nonlinear term \( u \cdot \nabla u \).

Roughly speaking, (1.5) is locally well-posedness for the initial data \((a_0, u_0) \in \dot{B}^{\frac{3}{q}, 1}_q \times \dot{B}^{\frac{3}{p} - 1}_p \), \( p < 6 \). So a nature question is whether (1.5) is well-posedness in the critical Besov space with \( p \geq 6 \). To the best of our knowledge, similar question has been proposed for the compressible Navier-Stokes, see [12] for the details. Indeed, the authors [12] gave a negative answer to this question, that is, the solution of the compressible Navier-Stokes equations is ill-posedness with \( p > 6 \). Very recently, Chen and Wan [11] proved
ill-posedness with the initial velocity in \( L^6 \) framework by using a new approach to get a norm inflation which depends on a decomposition of the density. Motivated by the above analysis, we will show (1.5) is ill-posedness in the critical Besov space. Our main results read:

**Theorem 1.1.** Let \( p \in (6, \infty] \). For any \( \delta > 0 \), there exists initial data \((a_0, u_0)\) satisfying

\[
\|a_0\|_{B^\frac{3}{p},1} \leq \delta, \quad \|u_0\|_{\dot{B}^\frac{3}{p-1}_6,1} \leq \delta
\]

such that a solution \((a, u)\) to the system (1.5) satisfies

\[
\|u(t)\|_{\dot{B}^{\frac{3}{p-1}}_6,1} \geq \frac{1}{\delta^\alpha}
\]

for some \( 0 < t < \delta \) and \( \alpha > 0 \).

**Theorem 1.2.** Let \( p \in (6, \infty) \). For any \( \delta > 0 \), there exists initial data \((a_0, u_0)\) satisfying

\[
\|a_0\|_{B^{\frac{1}{2}}_6,1} \leq \delta, \quad \|u_0\|_{\dot{B}^{\frac{3}{p-1}}_p,1} \leq \delta
\]

such that a solution \((a, u)\) to the system (1.5) satisfies

\[
\|u(t)\|_{\dot{B}^{\frac{3}{p-1}}_p,1} \geq \frac{1}{\delta^\alpha}
\]

for some \( 0 < t < \delta \) and \( \alpha > 0 \).

**Remark 1.3.** The idea to the proof of Theorem 1.2 can not be applied directly to the case \( p = \infty \). For this case, we will give some comments on the barrier and provide a brief framework of the proof in the Appendix.

**Remark 1.4.** The norm inflation for the classical Navier-stokes equations coming from the nonlinear term \( u \cdot \nabla u \) is in \( L^\infty \) framework, while the norm inflation for (1.1) is in smaller space due to the appearance of \( \mu a \Delta u \). But we have no idea to extend the main results to \( L^6 \) framework.

**Remark 1.5.** In the proof, we only give a priori estimate, which is the key part. We give the structure of the existence in the Appendix.

Now, we give the idea of the proof and make some comments on the technics. Firstly, we present our idea. Like [11], the proof is based on a composition of the velocity and a new decomposition of the density (see Section 3.1), that is

\[
u = U_0 + U_1 + U_2, \quad a = a_0 + a_1.
\]

Then we obtain a norm inflation coming from the coupling term \( \mu a \Delta u \) yielding a norm inflation of \( U_1 \), while the corresponding norms of \( U_0 \) and \( U_2 \) are small. Secondly, let us show the technics.

1) We apply a small trick that a special class of initial velocity is constructed to obtain a large lower bound of the associated norm of \( U_1 \), see Remark 3.3.
2) Although we own the decomposition of the density, we will face the main difficulty coming from the estimate of gradient pressure (\(\nabla P\)). As a matter of fact, it seems hard to bound the nonlinear term \(\mu|D|^{-2}\nabla \text{div}(a\Delta u)\). To overcome this difficulty, we introduce a modified pressure \(\Pi\) satisfying

\[
\nabla \Pi := \cdots - \mu|D|^{-2}\nabla \text{div}\{a|D|^{-2}\nabla \text{div}(a\Delta u)\} + \cdots.
\]

Thanks to that this term \(-\mu|D|^{-2}\nabla \text{div}\{a|D|^{-2}\nabla \text{div}(a\Delta u)\}\) admits a good estimate, we can achieve this goal.

This paper is organized as follows:
In Section 2, we provide some lemmas and the definitions of some spaces. In Section 3, we prove Theorem 1.1, while Theorem 1.2 is proved in Section 4. We split each Section into several steps. In the Appendix, we will consider the case \(p = \infty\) in Theorem 1.2.

Let us complete this section by describing the notations we shall use in this paper.

**Notations**
In some places of this paper, we may use \(L^p\) and \(\dot{B}^{s}_{p,r}(\mathbb{R}^3)\) to stand for \(L^p(\mathbb{R}^3)\) and \(\dot{B}^{s}_{p,r}(\mathbb{R}^3)\), respectively. The uniform constant \(C\), which may be different on different lines, while the constant \(C(\cdot)\) means a constant depends on the element(s) in bracket.

\(a \approx b\) means \(N^{-1}a \leq b \leq N^1a\) for some constant \(N\), and \(a \gg b\) \((a \ll b)\) stands for \(a \geq Nb\) \((a \leq N^{-1}b)\), where \(N\) is a large enough constant.

2. Preliminaries

In this section, we give some necessary definitions, propositions and lemmas.

The fractional Laplacian operator \(|D|^\alpha = (-\Delta)^{\frac{\alpha}{2}}\) is defined through the Fourier transform, namely,

\[
|D|^\alpha f(\xi) := |\xi|^\alpha \hat{f}(\xi),
\]

where the Fourier transform is given by

\[
\hat{f}(\xi) := \int_{\mathbb{R}^3} e^{-ix\cdot\xi} f(x)dx, \text{ or } \mathcal{F}(f)(\xi) := \int_{\mathbb{R}^3} e^{-ix\cdot\xi} f(x)dx.
\]

Let \(\mathcal{C} = \{\xi \in \mathbb{R}^3, \frac{2}{3} \leq |\xi| \leq \frac{8}{3}\}\). Choose a nonnegative smooth radial function \(\varphi\) supported in \(\mathcal{C}\) such that

\[
\sum_{j \in \mathbb{Z}} \varphi(2^{-j}\xi) = 1, \quad \xi \in \mathbb{R}^3 \setminus \{0\}.
\]

We denote \(\varphi_j = \varphi(2^{-j}\xi), \ h = \mathfrak{F}^{-1}\varphi\), where \(\mathfrak{F}^{-1}\) stands for the inverse Fourier transform. Then the dyadic blocks \(\Delta_j\) and \(S_j\) can be defined as follows

\[
\Delta_j f = \varphi(2^{-j}D)f = 2^{3j} \int_{\mathbb{R}^3} h(2^jy)f(x-y)dy, \quad S_j f = \sum_{k \leq j-1} \Delta_k f.
\]

One easily verifies that with our choice of \(\varphi\)

\[
\Delta_j \Delta_k f = 0 \text{ if } |j - k| \geq 2 \quad \text{and} \quad \Delta_j(S_{k-1}f \Delta_k f) = 0 \text{ if } |j - k| \geq 5.
\]

Let us recall the definitions of the Besov space and Chemin-Lerner type space [9].
**Definition 2.1.** Let \( s \in \mathbb{R}, \ (p, q) \in [1, \infty]^2 \), the homogeneous Besov space \( \dot{B}^s_{p,q}(\mathbb{R}^3) \) is defined by

\[
\dot{B}^s_{p,q}(\mathbb{R}^3) = \{ f \in \mathcal{S}'(\mathbb{R}^3) : \| f \|_{\dot{B}^s_{p,q}(\mathbb{R}^3)} < \infty \},
\]

where

\[
\| f \|_{\dot{B}^s_{p,q}(\mathbb{R}^3)} = \begin{cases} 
\left( \sum_{j \in \mathbb{Z}} 2^{sqj} \| \Delta_j f \|^q_{L^p(\mathbb{R}^3)} \right)^{\frac{1}{q}}, & \text{for } 1 \leq q < \infty, \\
\sup_{j \in \mathbb{Z}} 2^{sj} \| \Delta_j f \|_{L^p(\mathbb{R}^3)}, & \text{for } q = \infty,
\end{cases}
\]

and \( \mathcal{S}'(\mathbb{R}^3) \) denotes the dual space of \( \mathcal{S}(\mathbb{R}^3) = \{ f \in C^\infty(\mathbb{R}^3) : \partial^n f(0) = 0; \ \forall \ \alpha \in \mathbb{N}^3 \} \) and can be identified by the quotient space of \( \mathcal{S}' / \mathcal{P} \) with the polynomials space \( \mathcal{P} \).

**Definition 2.2.** Let \( s \in \mathbb{R}, \ (p, q, r) \in [1, \infty]^3, \ 0 < T \leq \infty \). The Chemin-Lerner type space \( \dot{L}^r_T \dot{B}^s_{p,q}(\mathbb{R}^3) \) is defined by

\[
\dot{L}^r_T \dot{B}^s_{p,q}(\mathbb{R}^3) = \{ f \in \mathcal{S}'(\mathbb{R}^3) : \| f \|_{\dot{L}^r_T \dot{B}^s_{p,q}(\mathbb{R}^3)} < \infty \},
\]

where

\[
\| f \|_{\dot{L}^r_T \dot{B}^s_{p,q}(\mathbb{R}^3)} = \begin{cases} 
\left( \sum_{j \in \mathbb{Z}} 2^{sqj} \| \Delta_j f \|^q_{L^p(\mathbb{R}^3)} \right)^{\frac{1}{q}}, & \text{for } 1 \leq q < \infty, \\
\sup_{j \in \mathbb{Z}} 2^{sj} \| \Delta_j f \|_{L^p(\mathbb{R}^3)}, & \text{for } q = \infty.
\end{cases}
\]

It is clear that \( \dot{L}^r_T \dot{B}^s_{p,r} = \dot{L}^r_T \dot{B}^s_{p,r} \).

Let us introduce the homogeneous Bony’s decomposition.

\[
uv = T_u v + T_v u + R(u, v),
\]

where

\[
T_u v = \sum_{j \in \mathbb{Z}} S_{j-1} u \Delta_j v, \quad T_v u = \sum_{j \in \mathbb{Z}} \Delta_j u S_{j-1} v, \quad R(u, v) = \sum_{j \in \mathbb{Z}} \Delta_j u \tilde{\Delta}_j v,
\]

here \( \tilde{\Delta}_j = \Delta_{j-1} + \Delta_j + \Delta_{j+1} \).

The following proposition provides Bernstein type inequalities.

**Proposition 2.3.** Let \( 1 \leq p \leq q \leq \infty \). Then for any \( \beta, \gamma \in \mathbb{N}^3 \), there exists a constant \( C \) independent of \( f, j \) such that

1) If \( f \) satisfies

\[
supp \tilde{f} \subset \{ \xi \in \mathbb{R}^3 : \ |\xi| \leq K2^j \},
\]

then

\[
\| \partial^\beta f \|_{L^q(\mathbb{R}^3)} \leq C 2^{j|\gamma| + j(\frac{3}{p} - \frac{3}{q})} \| f \|_{L^p(\mathbb{R}^3)}.
\]

2) If \( f \) satisfies

\[
supp \tilde{f} \subset \{ \xi \in \mathbb{R}^3 : \ K_1 2^j \leq |\xi| \leq K_2 2^j \}
\]

then

\[
\| f \|_{L^p(\mathbb{R}^3)} \leq C 2^{-j|\gamma|} \sup_{|\beta| = |\gamma|} \| \partial^\beta f \|_{L^p(\mathbb{R}^3)}.
\]
The standard estimates of the heat equation and transport equation read in the following:

**Proposition 2.4.** Let $T > 0$, $s \in \mathbb{R}$ and $1 \leq r \leq \infty$. Assume that $u_0 \in \dot{B}^{s}_{r,1}$ and $f \in \dot{L}^p_T \dot{B}^{-2+\frac{2}{p}}_{r,1}$. If $u$ is the solution of the heat equation

\[
\begin{aligned}
\partial_t u - \mu \Delta u &= f, \\
\quad u(0, x) &= u_0(x),
\end{aligned}
\]

with $\mu > 0$, then $\forall \rho_1 \in [\rho, \infty]$, we have

\[
\mu^{\frac{1}{r_1}} \| u \|\dot{L}^{s_1}_{p_1} \dot{B}^{\frac{2}{p_1}}_{r,1} \leq C(\| u_0 \|\dot{B}^{s}_{r,1} + \| f \|\dot{L}^p_T \dot{B}^{-2+\frac{2}{p}}_{r,1}).
\]

**Proposition 2.5.** Let $T > 0$, $s \in (-3 \min(\frac{1}{r}, \frac{r-1}{r}), 1+\frac{3}{r}]$, and $1 \leq r \leq \infty$. Assume that $u$ is the solution of

\[
\begin{aligned}
\partial_t u + v \cdot \nabla u &= f, \\
\quad u(0, x) &= u_0(x),
\end{aligned}
\]

then we have $\forall \rho_1 \in [0, T]$

\[
\| u \|\dot{L}^{s}_{p,1} \dot{B}^{r}_{r,1} \lesssim (\| u_0 \|\dot{B}^{s}_{r,1} + \| f \|\dot{L}^1_T \dot{B}^{s}_{r,1}) \exp\{\| \nabla v \|\dot{L}^1_T \dot{B}^{s}_{r,1}\}.
\]

The Kato-Ponce estimate and some product estimates can be given by

**Lemma 2.6.** [23] Let $s > 0$, $1 \leq p, r \leq \infty$, then

\[
\| fg \|\dot{B}^{s}_{p,r} \leq C \left\{ \| f \|\dot{L}^p_1 \| g \|\dot{B}^{s}_{p,2} + \| g \|\dot{L}^1_T \| g \|\dot{B}^{s}_{r,2} \right\},
\]

where $1 \leq p, r \leq \infty$ such that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{r_1} + \frac{1}{r_2}$.

**Lemma 2.7.** [11] Let

\[
1 \leq s, s_1, s_2, s_{i1}, s_{i2} \leq \infty, 3 < r < \infty, 3 < q < 6, |\alpha| \geq 1, \quad \frac{1}{s} = \frac{1}{s_1} + \frac{1}{s_2}, \quad \frac{3}{p_0} + \frac{3}{r} > 1, \quad p_0 > 6.
\]

Then the following estimates hold:

(a)

\[
\| fg \|\dot{L}^{\frac{s}{p_0}}_T \dot{B}^{\frac{r}{p_0}}_{r,1} \leq C \left( \| g \|\dot{L}^{s_1}_{T} \| f \|\dot{L}^{s_2}_{T} \dot{B}^{\frac{r}{r_1}}_{r_1} + \| \partial^\alpha g \|\dot{L}^{s_1}_{T} \| f \|\dot{L}^{s_2}_{T} \dot{B}^{\frac{r}{r_1}}_{r_1} \right); \tag{2.2}
\]

(b)

\[
\| fg \|\dot{L}^{\frac{s}{p_0}}_T \dot{B}^{\frac{r}{p_0}}_{r,1} \leq C \left( \| f \|\dot{L}^{s_1}_{T} \| g \|\dot{L}^{s_2}_{T} \dot{B}^{\frac{r}{r_1}}_{r_1} + \| f \|\dot{L}^{s_1}_{T} \| g \|\dot{L}^{s_2}_{T} \dot{B}^{\frac{r}{r_1}}_{r_1} + \| g \|\dot{L}^{s_2}_{T} \dot{B}^{\frac{r}{r_1} - \frac{3}{2}} \right); \tag{2.3}
\]

(c)

\[
\| fg \|\dot{L}^{\frac{s}{p_0}}_T \dot{B}^{\frac{r}{p_0}}_{r,1} \leq C \| f \|\dot{L}^{s_1}_{T} \| g \|\dot{L}^{s_2}_{T} \dot{B}^{\frac{r}{r_1} - \frac{1}{2}}. \tag{2.4}
\]

For the readers’ convenience, we refer to [6] for more details about the Besov space.
3. Proof of Theorem 1.1

3.1. Reformulation of the equation. From (1.5), we have
\[
∇P = |D|^{-2} \nabla \text{div}(a \nabla P) + |D|^{-2} \nabla \text{div}(u \cdot \nabla u) - µ|D|^{-2} \nabla \text{div}(a \Delta u).
\]  
(3.1)

As the previous comments in Section 1, we require a modified pressure given by
\[
Π := P + µ|D|^{-2} \nabla \text{div}(a \Delta u).
\]

Then one gets
\[
∇Π = |D|^{-2} ∇ \text{div}(a \nabla Π) + µ|D|^{-2} \nabla \text{div}(a \Delta u),
\]
(3.2)

So we can write (1.5) as
\[
\begin{cases}
∂_t a + u \cdot ∇ a = 0, \quad \text{div} u = 0 \\
∂_t u - µ\Delta u = -u \cdot ∇ u - ∇Π - a∇P + µ|D|^{-2} \nabla \text{div}(a \Delta u), \\
(a(0, x), u(0, x)) = (a_0(x), u_0(x)).
\end{cases}
\]
(3.3)

Applying Duhamel principle to (3.3), we get
\[
u(t, x) = e^{µ\Delta t} u_0 + \int_0^t e^{µ\Delta(t-τ)} \{-u \cdot ∇ u - ∇Π - a∇P + µa\Delta u + µ|D|^{-2} \nabla \text{div}(a \Delta u)\} dτ.
\]

Denote
\[
U_0(t) = e^{µ\Delta t} u_0,
\]
\[
U_1(t) = µ \int_0^t e^{µ\Delta(t-τ)} \{a_0 \Delta U_0 + |D|^{-2} \nabla \text{div}(a_0 \Delta U_0)\} dτ,
\]
\[
U_2(t) = \int_0^t e^{µ\Delta(t-τ)} \{-u \cdot ∇ u - ∇Π - a∇P + F_1 + F_2\} dτ,
\]

where
\[
F_1 = µ(a_1 \Delta u + a_0 \Delta(U_1 + U_2)),
\]
\[
F_2 = µ|D|^{-2} \nabla \text{div}\{a_1 \Delta u + a_0 \Delta(U_1 + U_2)\}.
\]

Now, we can decompose \(u(t, x)\) and \(a(t, x)\) as
\[
u(t, x) = U_0(t, x) + U_1(t, x) + U_2(t, x)
\]
and
\[
a(t, x) = a_0(x) + a_1(t, x)
\]
where \(a_1(t)\) satisfies the generalized transport equation given by
\[
\begin{cases}
∂_t a_1 + u \cdot ∇(a_0 + a_1) = 0, \\
a_1(0, x) = 0.
\end{cases}
\]
(3.4)
3.2. The choice of initial data. Due to \( \text{supp} \varphi(2^8 \xi) \subset \left\{ \frac{3}{4} \times 2^{-8} \leq |\xi| \leq \frac{5}{3} \times 2^{-8} \right\} \), we get there exists a positive constant \( A > 0 \) such that at least one of the following two inequalities holds:

\[
\varphi(2^8 \xi) \frac{\xi_1^2 + \xi_3^2}{|\xi|^2} \geq A \varphi(2^8 \xi), \tag{3.5}
\]

\[
\varphi(2^8 \xi) \frac{\xi_2^2 + \xi_3^2}{|\xi|^2} \geq A \varphi(2^8 \xi).
\]

Without loss of generality, we assume (3.5) holds in this present article. Let \( C(N) = 2^{m(\frac{p}{p} - \frac{3}{2} - 2\epsilon_1)} \) for some \( 0 < \epsilon_1 < \frac{1}{2} \left( \frac{1}{2} - \frac{3}{p} \right) \) and \( p > 6 \), where \( N > 0 \) determined later is a sufficiently large constant leading to \( \frac{N}{C(N)} \ll 1 \). We construct the initial data \((a_0, u_0)\) as follows:

\[
\hat{a}_0(\xi) = \frac{1}{C(N)} \sum_{k=100}^{N} 2^{-k^2} \left( \hat{\phi}(\xi - 2^k e_1) + \hat{\phi}(\xi + 2^k e_1) \right),
\]

\[
\hat{u}_0(\xi) = \frac{1}{C(N)} \sum_{k=100}^{N} 2^{k^2} \left( \frac{2^k}{|\xi|} \frac{\xi_2}{\xi_1} \right),
\]

where \( e_1 = (1, 0, 0) \) and \( \hat{\phi} \) is a smooth, radial and nonnegative function in \( \mathbb{R}^3 \) satisfying

\[
\hat{\phi} = \begin{cases} 
1 & \text{for } |\xi| \leq 1, \\
0 & \text{for } |\xi| \geq 2.
\end{cases}
\]

One can see \( a_0 \) is a real valued function, while \( u_0 \) is a real vector-valued function by observing

\[
u_0(x) = \frac{2}{C(N)} \sum_{k=100}^{N} 2^k \left\{ -\mathcal{R}_2(\phi(x) \sin(2^k x_1)), \mathcal{R}_1(\phi(x) \sin(2^k x_1)), 0 \right\},
\]

where \( \mathcal{R}_i \) is the Riesz transform defined by

\[
\mathcal{R}_i f(\xi) := \frac{-i \xi_i}{|\xi|} \hat{f}(\xi).
\]

One can also check the following estimates hold, i.e.,

\[
\|a_0\|_{L^\infty} \leq \frac{C}{C(N)}, \quad \|a_0\|_{\dot{B}^\frac{3}{p}_2} \leq \frac{CN}{C(N)}, \quad \|u_0\|_{\dot{B}^\frac{3}{2}_6} \leq \frac{CN}{C(N)} \tag{3.6}
\]

and

\[
\|a_0\|_{\dot{B}^{s_1}_{r_1}} \leq \frac{C'2^{N(s_1 - \frac{3}{p})}}{C(N)}, \quad \|u_0\|_{\dot{B}^{s_1}_{r_1}} \leq \frac{C'2^{N(s_1 + \frac{1}{2})}}{C(N)},
\]

with \((r, r_1) \in [1, \infty]^2 \) and \( s > \frac{3}{p}, s_1 > -\frac{1}{2} \).

Next, a lemma is given.
Lemma 3.1. Let \( p > 6 \). Then there exist some positive constants \( q, p_0, \epsilon, \epsilon_1 \) satisfying

\[
\begin{cases}
1 < \frac{3}{q} + \frac{3}{p_0} < \frac{5}{4} + \frac{3}{2p} - \frac{3}{p_0} + 2\epsilon - 3\epsilon_1, \\
\frac{3}{p_0} < \frac{1}{4} + \frac{3}{2p} - \epsilon_1, \quad p_0 \in (6, p), \\
\frac{3}{q} < 1 + 3\epsilon - 2\epsilon_1, \quad q \in (3, 6), \\
0 < 2\epsilon < 2\epsilon_1 < \frac{1}{2} - \frac{3}{p}.
\end{cases}
\]

(3.7)

Remark 3.2. We give the following explanations of the limitations in (3.7). Let \( T = 2^{-2(1+\epsilon)N} \), then we have

\[
\begin{cases}
\frac{2^N(\frac{3}{p_0} - \frac{3}{p})}{C(N)} \ll 1 \iff \frac{3}{p_0} < \frac{1}{4} + \frac{3}{2p} - \epsilon_1, \\
\frac{T^{2N}2^{N(\frac{3}{p_0} - \frac{3}{p} + \frac{3}{2p})}}{C(N)^2} \ll 1 \iff \frac{3}{q} < 1 + 3\epsilon - 2\epsilon_1, \\
\frac{T2^N(\frac{3}{p_0} - \frac{3}{p} + \frac{3}{2p})}{C(N)^3} \ll 1 \iff \frac{3}{q} + \frac{3}{p_0} < \frac{5}{4} + \frac{3}{2p} - \frac{3}{p_0} + 2\epsilon - 3\epsilon_1.
\end{cases}
\]

(3.8)

Actually, we assume the conditions on the right hand side of (3.8) to ensure the conditions on the left hand side which is required in our proof. Furthermore, we use \( \frac{3}{q} + \frac{3}{p_0} > 1 \) to ensure some product estimates like (2.4). The choice of \( C(N) \) needs \( 2\epsilon_1 < \frac{1}{2} - \frac{3}{p} \), while \( \epsilon < \epsilon_1 \) ensures the norm inflation of \( U_1 \) (see the end of subsection 3.3).

Lemma 3.1 can be proved easily, here we use the following example in this article:

\[
\epsilon_1 = \frac{1}{4}(\frac{1}{2} - \frac{3}{p}), \quad \epsilon = \frac{1}{5}(\frac{1}{2} - \frac{3}{p}),
\]

and

\[
\frac{3}{p_0} = \frac{1}{16} + \frac{21}{8p}, \quad \forall \frac{3}{q} \in (\frac{15}{16} - \frac{21}{8p}, \frac{19}{20} - \frac{27}{10p}).
\]

This gives that

\[
2(\epsilon_1 - \epsilon) = \frac{1}{10}(\frac{1}{2} - \frac{3}{p}), \quad \frac{1}{2}(\frac{1}{2} - \frac{3}{p} - 2\epsilon) = \frac{1}{4}(\frac{1}{2} - \frac{3}{p}).
\]

(3.9)

3.3. The analysis of \( U_1 \). Let \( V^j \) be the \( j \)-th component of the vector \( V \). Thanks to \( \dot{B}_{\infty,1}^{-\frac{1}{2}} \hookrightarrow \dot{B}_{\infty,\infty}^{-1} \), we have

\[
\|U_1\|_{\dot{B}_{\infty,1}^{-\frac{1}{2}}} \geq c\|U_1\|_{\dot{B}_{\infty,\infty}^{-1}} \geq c\int \varphi(2^8\xi)\widehat{U_1}(\xi)d\xi \geq c\int \varphi(2^8\xi)\widehat{U_1^2}(\xi)d\xi.
\]
Let us give the second component \( U_1^2 \) of \( U_1 \):

\[
U_1^2 = \mu \int_0^t e^{\mu \Delta (t-\tau)} \left\{ a_0 \Delta U_0^2 + |D|^{-2} \partial_2 \text{div}(a_0 \Delta U_0) \right\} d\tau \\
= \mu \int_0^t e^{\mu \Delta (t-\tau)} (1 + \partial_2^2 |D|^{-2})(a_0 \Delta U_0^2) d\tau \\
+ \mu \int_0^t e^{\mu \Delta (t-\tau)} |D|^{-2} \partial_1 \partial_2 (a_0 \Delta U_0^1) d\tau \\
=: U_{11}^2 + U_{12}^2.
\]

So

\[
\| U_1 \|_{\tilde{B}_{a,t}^{1,0}} \geq |B_1| - |B_2|,
\]

where

\[
B_1 = \int \varphi(2^8 \xi) \hat{U}_{11}^2(\xi) d\xi, \quad B_2 = \int \varphi(2^8 \xi) \hat{U}_{12}^2(\xi) d\xi.
\]

Now, we give the estimates of \( B_1 \) and \( B_2 \).

\textbf{The estimate of } \( B_2 \)

Using some facts of Fourier transform, we have

\[
B_2 = -\mu \int \varphi(2^8 \xi) \int_0^t e^{-\mu(t-\tau)|\xi|^2} \frac{\xi_1 \xi_2}{|\xi|^2} \mathcal{F}(a_0 \Delta U_0^1) d\tau d\xi.
\]

Thanks to the construction of initial data, we get

\[
\mathcal{F}(a_0 \Delta U_0^1) = -\int \hat{a}_0(\xi - \eta) |\eta|^2 \hat{U}_0^1(\eta) d\eta \\
= -\frac{C}{C(N)^2} \sum_{k=100}^N 2^{k(\frac{3}{2} - \frac{3}{p})} \int e^{-\mu|\eta|^2} \eta_2 |\eta| A(\xi, \eta, k) d\eta,
\]

where

\[
A(\xi, \eta, k) := -\hat{\phi}(\xi - \eta + 2^k e_1) \hat{\phi}(\eta - 2^k e_1) + \hat{\phi}(\xi - \eta - 2^k e_1) \hat{\phi}(\eta + 2^k e_1).
\]

Thus \( B_2 \) can be given by

\[
B_2 = \frac{C}{C(N)^2} \sum_{k=100}^N 2^{k(\frac{3}{2} - \frac{3}{p})} \int \int \varphi(2^8 \xi) \frac{\xi_1 \xi_2}{|\xi|^2} \eta_2 |\eta| A(\xi, \eta, k) \int_0^t e^{-\mu((t-\tau)|\xi|^2 + |\eta|^2)} d\tau d\xi d\eta.
\]

Due to \(|\xi| \approx 1, |\eta_2| \approx 1 \) and \(|\eta| \approx 2^k \), we get

\[
|B_2| \leq \frac{C t}{C(N)^2} \sum_{k=100}^N 2^{k(\frac{3}{2} - \frac{3}{p})} \int \int \varphi(2^8 \xi) |A(\xi, \eta, k)| d\xi d\eta \\
\leq \frac{C t}{C(N)^2} \sum_{k=100}^N 2^{k(\frac{3}{2} - \frac{3}{p})} \leq \frac{C t}{C(N)^2} 2^{N(\frac{3}{2} - \frac{3}{p})}.
\]

\textbf{The estimate of } \( B_1 \)
We will show the large lower bound of $B_1$ which yields the norm inflation of the solution. One can easily obtain

$$B_1 = \mu \int \varphi(2^8 \xi) \frac{\xi_1^2 + \xi_3^2}{|\xi|^2} \int_0^t e^{-\mu|\xi|^2(t-\tau)} \mathcal{F}(a_0 \Delta U_0^2) d\tau d\xi.$$  

By a similar way as before, we can obtain

$$\mathcal{F}(a_0 \Delta U_0^2) = - \int \hat{a}_0(\xi - \eta)|\eta|^2 \hat{U}_0^2(\eta) d\eta$$

= \frac{C}{C(N)^2} \sum_{k=100}^N 2^{k\left(\frac{2}{3} - \frac{2}{p}\right)} \int \varphi(2^8 \xi) \frac{\xi_1^2 + \xi_3^2}{|\xi|^2} \eta_1|\eta|A(\xi, \eta, k) \mathcal{A}(t, \xi, \eta) d\xi d\eta.$$

Hence

$$B_1 = - \frac{C}{C(N)^2} \sum_{k=100}^N 2^{k\left(\frac{2}{3} - \frac{2}{p}\right)} \int \varphi(2^8 \xi) \frac{\xi_1^2 + \xi_3^2}{|\xi|^2} \eta_1|\eta|A(\xi, \eta, k) \mathcal{A}(t, \xi, \eta) d\xi d\eta,$$

where

$$\mathcal{A}(t, \xi, \eta) := \int_0^t e^{-\mu(|\xi|^2(t-\tau)+|\eta|^2\tau)} d\tau.$$  

**Remark 3.3.** Due to the construction of initial data, we obtain two negative terms:

$$\eta_1 A(\xi, \eta, k) = - \eta_1 \hat{\phi}(\xi - \eta + 2^k e_1) \hat{\phi}(\eta - 2^k e_1) + \eta_1 \hat{\phi}(\xi - \eta - 2^k e_1) \hat{\phi}(\eta + 2^k e_1).$$

In the following proof, we only use one of them.

Applying the Taylor expansion $e^x = \sum_{r \geq 0} \frac{x^r}{r!}$, $|\xi| \approx 1$ and $|\eta| \approx 2^k$, we get

$$\mathcal{A}(t, \xi, \eta) = \frac{e^{-\mu|\eta|^2} - e^{-\mu|\xi|^2}}{\mu(|\xi|^2 - |\eta|^2)} = t + \mathcal{O}(t^2 |\eta|^2)$$  

(3.11)

when $t2^{2N} < 1$. Thanks to (3.11) and (3.12), one has

$$|B_1| \geq \frac{C}{C(N)^2} \sum_{k=100}^N 2^{k\left(\frac{2}{3} - \frac{2}{p}\right)} \int \varphi(2^8 \xi) \frac{\xi_1^2 + \xi_3^2}{|\xi|^2} \eta_1|\eta| \hat{\phi}(\xi - \eta + 2^k e_1) \hat{\phi}(\eta - 2^k e_1) \{t + \mathcal{O}(t^2 |\eta|^2)\} d\xi d\eta$$

$$\geq \frac{C}{C(N)^2} \sum_{k=100}^N 2^{k\left(\frac{2}{3} - \frac{2}{p}\right)} (t2^{2k} - \mathcal{O}(t^2 2^{4k}))$$

$$\geq \frac{ct}{C(N)^2} \sum_{k=100}^N 2^{k\left(\frac{2}{3} - \frac{2}{p}\right)} - \frac{Ct^2}{C(N)^2} \sum_{k=100}^N 2^{k\left(\frac{2}{3} - \frac{2}{p}\right)}$$

$$= \frac{ct2^{N\left(\frac{2}{3} - \frac{2}{p}\right)}}{C(N)^2} - \frac{Ct^2 2^{N\left(\frac{2}{3} - \frac{2}{p}\right)}}{C(N)^2}. $$
Choosing $t = T_0 := 2^{-(1+\epsilon)N}$, $0 < \epsilon < \epsilon_1$, which ensures $t2^{2N} < 1$, and combining with (3.10), (3.12) yields
\[
\|U_1(t)\|_{L^{\frac{q}{2}}_t B^{\frac{q}{4}}_{q,1}} \geq |B_1| - |B_2| \\
\geq \frac{c t2^{N\left(\frac{q}{4} - \frac{q}{p}\right)}}{C(N)^2} - \frac{C t2^{N\left(\frac{q}{4} - \frac{q}{p}\right)}}{C(N)^2} - \frac{C t2^{N\left(\frac{q}{4} - \frac{q}{p}\right)}}{C(N)^2} \\
\geq \frac{c t2^{N\left(\frac{q}{4} - \frac{q}{p}\right)}}{2C(N)^2} - \frac{C t2^{N\left(\frac{q}{4} - \frac{q}{p}\right)}}{C(N)^2} \geq \frac{c}{4} 2^{2(\epsilon_1 - \epsilon)N}.
\] (3.13)

3.4. **The analysis of $U_2$.** Let $(p, p_0, q)$ be given as in Lemma 3.1. Let $0 \leq T \leq T_0$. We split the analysis into five steps.

**Step 1. Some estimates of $U_1$** We provide some estimates of $U_1$ which will be used in the following proof.
\[
\|U_1\|_{L^p T^1 L^\infty} \leq T^{\frac{1}{2}} \int_0^T \|a_0\|_{L^\infty} \||\xi|^{2} \hat{U}_0\|_{L^1} \, d\tau \leq \frac{CT^{1+\frac{1}{4}} 2^{\frac{3}{4}N}}{C(N)^2},
\]
and
\[
\|U_1\|_{L^p_t B^{\frac{q}{4}}_{q,1}} + \|U_1\|_{\tilde{L}^p_t B^{\frac{q}{4}}_{q,1}} \leq T^{\frac{1}{2}} (\|U_1\|_{\tilde{L}^p_t B^{\frac{q}{4}}_{q,1}} + \|U_1\|_{\tilde{L}^p_t B^{\frac{q}{4}}_{q,1}}) \\
\leq CT^{\frac{1}{2}} \int_0^T \|a_0 \Delta U_0\|_{\tilde{B}^{\frac{q}{4}}_{q,1}} \, d\tau \\
\leq CT^{\frac{1}{2}} (\|a_0\|_{L^\infty} \|\Delta U_0\|_{\tilde{L}^p_t B^{\frac{q}{4}}_{q,1}} + \|a_0\|_{B^{\frac{q}{4}}_{q,1}} \|\|\xi|^{2} \hat{U}_0\|_{\tilde{L}^p L^1}) \\
\leq \frac{CT^{\frac{3}{4}} 2^{N\left(\frac{q}{4} - \frac{q}{p}\right)}}{C(N)^2},
\]
where $q_1 = q$ or $p_0$.

**Step 2. The estimate of $\|u \cdot \nabla u\|_{L^p_t B^{rac{3}{4}}_{q,1}}$** Thanks to $\text{div} u = 0$ and Bernstein inequality, it suffices to bound $\|u \otimes u\|_{L^p_t B^{rac{3}{4}}_{q,1}}$. Using the decomposition $u = U_0 + U_1 + U_2$, we can split this estimate into six parts. Applying (2.1), one has
\[
\|U_0 \otimes U_0\|_{L^p_t B^{rac{3}{4}}_{q,1}} \leq C \|U_0\|_{L^p L^\infty} \|U_0\|_{\tilde{L}^p B^\frac{3}{4}} \leq \frac{CT^{2N\left(\frac{q}{4} + 1\right)}}{C(N)^2},
\]
\[
\|U_0 \otimes U_1\|_{L^p_t B^\frac{3}{4}} \leq CT^{\frac{3}{4}} (\|U_0\|_{L^p L^\infty} \|U_1\|_{L^p_t B^\frac{3}{4}} + \|U_1\|_{L^p L^\infty} \|U_0\|_{L^p_t B^\frac{3}{4}}) \leq \frac{CT^{2N\left(\frac{q}{4} + 3\right)}}{C(N)^3},
\]
\[
\|U_1 \otimes U_1\|_{L^p_t B^\frac{3}{4}} \leq C \|U_1\|_{L^p L^\infty} \|U_1\|_{L^p_t B^\frac{3}{4}} \leq C \frac{CT^{3N\left(\frac{q}{4} + 5\right)}}{C(N)^4}.
\]
Using (2.2),
\[
\|U_0 \otimes U_2\|_{L^1_t B^{\frac{3}{q},1}_{q,1}} \leq C(\|U_0\|_{L^2_t L^\infty} \|U_2\|_{L^2_t B^{\frac{3}{q},1}_{q,1}} + \|\nabla U_0\|_{L^1_t L^\infty} \|U_2\|_{L^\infty_t B^{\frac{3}{q},1}_{q,1}})
\]
\[
\leq \frac{CT^2 \sqrt{2}N}{C(N)} (\|U_0\|_{L^\infty_t B^{\frac{3}{q},1}_{q,1}} + \|U_2\|_{L^3_t B^{\frac{3}{q},1}_{q,1}}),
\]
\[
\|U_1 \otimes U_2\|_{L^1_t B^{\frac{3}{q},1}_{q,1}} \leq C(\|U_1\|_{L^2_t L^\infty} \|U_2\|_{L^2_t B^{\frac{3}{q},1}_{q,1}} + \|\nabla U_1\|_{L^1_t L^\infty} \|U_2\|_{L^\infty_t B^{\frac{3}{q},1}_{q,1}})
\]
\[
\leq \frac{CT^2 \sqrt{2}N}{C(N)^2} (\|U_2\|_{L^\infty_t B^{\frac{3}{q},1}_{q,1}} + \|U_2\|_{L^3_t B^{\frac{3}{q},1}_{q,1}}).
\]

In reality, we have applied Proposition (2.4) with \(f = 0\) to some estimates of \(U_0\). Using (2.1) again, with \(B^{\frac{3}{q},1} \hookrightarrow L^\infty\), we have
\[
\|U_2 \otimes U_2\|_{L^1_t B^{\frac{3}{q},1}_{q,1}} \leq C \|U_2\|_{L^2_t B^{\frac{3}{q},1}_{q,1}}^2.
\]

Thus we get
\[
\|u \cdot \nabla u\|_{L^1_t B^{\frac{3}{q},1}_{q,1}} \leq \frac{CT^2 \sqrt{2}N^{\frac{3}{q}+1}}{C(N)^2} + \frac{C}{C(N)} (\|U_2\|_{L^\infty_t B^{\frac{3}{q},1}_{q,1}} + \|U_2\|_{L^\infty_t B^{\frac{3}{q},1}_{q,1}}) + C \|U_2\|_{L^\infty_t B^{\frac{3}{q},1}_{q,1}}^2.
\]

Step 3. The estimate of \(\|F_1\|_{L^1_t B^{\frac{3}{q},1}_{q,1}}\) and \(\|F_2\|_{L^1_t B^{\frac{3}{q},1}_{q,1}}\) Thanks to the product estimate (2.4) with \(\frac{3}{q} + \frac{3}{p_0} > 1\), we can deduce that
\[
\|a_1 \Delta U_0\|_{L^1_t B^{\frac{3}{q},1}_{q,1}} \leq C \|a_1\|_{L^\infty_t B^{\frac{3}{q},1}_{q,1}} \|U_0\|_{L^1_t B^{\frac{3}{q},1}_{q,1}} \leq \frac{CT^2 \sqrt{2}N^{\frac{3}{q}+\frac{3}{p_0}}}{C(N)} \|a_1\|_{L^\infty_t B^{\frac{3}{q},1}_{q,1}},
\]
\[
\|a_1 \Delta U_1\|_{L^1_t B^{\frac{3}{q},1}_{q,1}} \leq C \|a_1\|_{L^\infty_t B^{\frac{3}{q},1}_{q,1}} \|U_1\|_{L^1_t B^{\frac{3}{q},1}_{q,1}} \leq \frac{CT^2 \sqrt{2}N^{\frac{3}{q}+\frac{3}{p_0}}}{C(N)} \|a_1\|_{L^\infty_t B^{\frac{3}{q},1}_{q,1}},
\]
\[
\|a_1 \Delta U_2\|_{L^1_t B^{\frac{3}{q},1}_{q,1}} \leq C \|a_1\|_{L^\infty_t B^{\frac{3}{q},1}_{q,1}} \|U_2\|_{L^1_t B^{\frac{3}{q},1}_{q,1}} \leq \frac{CT^2 \sqrt{2}N^{\frac{3}{q}+\frac{3}{p_0}}}{C(N)} \|a_1\|_{L^\infty_t B^{\frac{3}{q},1}_{q,1}},
\]
\[
\|a_0 \Delta U_1\|_{L^1_t B^{\frac{3}{q},1}_{q,1}} \leq \|a_0\|_{L^\infty_t B^{\frac{3}{q},1}_{q,1}} \|U_1\|_{L^1_t B^{\frac{3}{q},1}_{q,1}} \leq \frac{CT^2 \sqrt{2}N^{\frac{3}{q}+\frac{3}{p_0}}}{C(N)^2} \|a_0\|_{L^\infty_t B^{\frac{3}{q},1}_{q,1}}.
\]

Applying (2.3), we can get
\[
\|a_0 \Delta U_2\|_{L^1_t B^{\frac{3}{q},1}_{q,1}} \leq C(\|a_0\|_{L^\infty_t B^{\frac{3}{q},1}_{q,1}} \|U_2\|_{L^1_t B^{\frac{3}{q},1}_{q,1}} + T \|a_0\|_{L^\infty_t B^{\frac{3}{q},1}_{q,1}} \|U_2\|_{L^\infty_t B^{\frac{3}{q},1}_{q,1}})
\]
\[
\leq \frac{CT^2 N^{\frac{3}{q}+\frac{3}{p_0}+2}}{C(N)} + \frac{1}{C(N)} (\|U_2\|_{L^\infty_t B^{\frac{3}{q},1}_{q,1}} + \|U_2\|_{L^\infty_t B^{\frac{3}{q},1}_{q,1}}).
\]
Collecting the above estimates leads to

\[ \| F_1 \|_{L^\infty_t B^\frac{3}{4},_q^1} + \| F_2 \|_{L^\infty_t B^\frac{3}{4},_q^1} \leq \frac{CT^2 2^N (\frac{2}{3} + \frac{1}{p_0} - \frac{2}{p} + \frac{2}{3})}{C(N)^3} \| a_1 \|_{L^\infty_t B^\frac{3}{2},_q^0} + \frac{CT^2 2^N (\frac{2}{3} + \frac{6}{p_0} - \frac{2}{p} + \frac{2}{3})}{C(N)^3} \| a_1 \|_{L^\infty_t B^\frac{3}{2},_q^0} + \| U_2 \|_{L^\infty_t B^\frac{3}{4},_q^1} + \frac{CT^2 2^N (\frac{2}{3} + \frac{5}{p_0} - \frac{2}{p} + \frac{2}{3})}{C(N)^3} \| a_1 \|_{L^\infty_t B^\frac{3}{2},_q^0} + \frac{CT^2 2^N (\frac{2}{3} + \frac{5}{p_0} - \frac{2}{p} + \frac{2}{3})}{C(N)^3} \| a_1 \|_{L^\infty_t B^\frac{3}{2},_q^0} + \| U_2 \|_{L^\infty_t B^\frac{3}{4},_q^1} \]

\[ (3.15) \]

Step 4. Some estimates of the pressure \( P \) and the modified pressure \( \Pi \) By using (3.11), (3.14), (3.15) and

\[ \| a_0 \Delta U_0 \|_{L^\infty_t B^\frac{3}{4},_q^1} \leq C \| \nabla u \|_{L^\infty_t B^\frac{3}{4},_q^1},\]

we obtain

\[ \| \nabla P \|_{L^\infty_t B^\frac{3}{4},_q^1} \leq C \| a_0 \|_{L^\infty_t B^\frac{3}{4},_q^1} + C \| \nabla u \|_{L^\infty_t B^\frac{3}{4},_q^1} + C \| a \|_{L^\infty_t B^\frac{3}{4},_q^1} + \frac{CT^2 2^N (\frac{2}{3} + \frac{5}{p_0} - \frac{2}{p} + \frac{2}{3})}{C(N)^2} \| U_2 \|_{L^\infty_t B^\frac{3}{4},_q^1}, \]

\[ (3.16) \]

Similarly, using (3.2) yields that

\[ \| \nabla \Pi \|_{L^\infty_t B^\frac{3}{4},_q^1} \leq C \| a \|_{L^\infty_t B^\frac{3}{4},_q^1} + C \| \nabla u \|_{L^\infty_t B^\frac{3}{4},_q^1} + C \| \nabla \Pi \|_{L^\infty_t B^\frac{3}{4},_q^1} + \frac{CT^2 2^N (\frac{2}{3} + \frac{5}{p_0} - \frac{2}{p} + \frac{2}{3})}{C(N)^2} \| U_2 \|_{L^\infty_t B^\frac{3}{4},_q^1}, \]

\[ (3.17) \]

Step 5. The estimate of \( \| a_1 \|_{L^\infty_t B^\frac{3}{4},_q^0} \) Applying Proposition 2.5 to the transport equation (3.4), one deduces

\[ \| a_1 \|_{L^\infty_t B^\frac{3}{4},_q^0} \leq \| \nabla u \|_{L^\infty_t B^\frac{3}{4},_q^0} \exp \{ C \| \nabla u \|_{L^\infty_t B^\frac{3}{4},_q^0} \}. \]
It follows from using (2.2) that

$$\|U_0 \cdot \nabla a_0\|_{L_T^1 B_{p,0}^{\frac{3}{p}}} \leq C(\|U_0\|_{L_T^1 L^\infty} \|\nabla a_0\|_{B_{p,0}^1} + \|U_0\|_{L_T^1 B_{p,0}^{\frac{1}{p}} \|\nabla a_0\|_{L^\infty}} \leq \frac{CT2^{N(\frac{3}{p} - \frac{3}{p} + \frac{3}{p})}}{C(N)^2},$$

$$\|U_1 \cdot \nabla a_0\|_{L_T^1 B_{p,0}^{\frac{3}{p}}} \leq C(\|U_1\|_{L_T^1 L^\infty} \|\nabla a_0\|_{B_{p,0}^1} + \|U_1\|_{L_T^1 B_{p,0}^{\frac{1}{p}} \|\nabla a_0\|_{L^\infty}} \leq \frac{CT2^{N(\frac{3}{p} - \frac{3}{p} + \frac{3}{p})}}{C(N)^3}.$$

Using (2.2), we have

$$\|U_2 \cdot \nabla a_0\|_{L_T^1 B_{p,0}^{\frac{3}{p}}} \leq C(T^\frac{1}{2} \|\nabla a_0\|_{L^\infty} \|U_2\|_{L_T^2 B_{p,0}^{\frac{3}{2}}} + T \|\nabla^2 a_0\|_{L^\infty} \|U_2\|_{L_T^2 B_{p,0}^{\frac{3}{2}}})$$

$$\leq \frac{CT2^{N(1 - \frac{3}{p})}}{C(N)}(\|U_2\|_{L_T^\infty B_{p,0}^{\frac{3}{2}}} + \|U_2\|_{L_T^2 B_{p,0}^{\frac{3}{2}}}).$$

Using $\dot{B}^s_{q,1} \hookrightarrow \dot{B}^\frac{s+3}{p}_{q,1}$, thus we get

$$\|a_1\|_{L_T^\infty B_{p,1}^{\frac{3}{p}}} \leq \frac{CT2^{N(\frac{3}{p} - \frac{3}{p} + \frac{3}{p})}}{C(N)^2} + \frac{CT2^{N(1 - \frac{3}{p})}}{C(N)}(\|U_2\|_{L_T^\infty B_{q,1}^{\frac{3}{2}}} + \|U_2\|_{L_T^2 B_{q,1}^{\frac{3}{2}}})$$

$$\exp\left\{\frac{CT2^{N(\frac{3}{p} - \frac{3}{p} + \frac{3}{p})}}{C(N)} + C\|U_2\|_{L_T^\infty B_{p,1}^{\frac{3}{2}}} \right\}. \tag{3.18}$$

3.5. **Proof of Theorem 1.1**

Denote

$$X_T := \|a_1\|_{L_T^\infty B_{p,1}^{\frac{3}{p}}} , \ Y_T := \|U_2\|_{L_T^\infty B_{q,1}^{\frac{3}{2}}} + \|U_2\|_{L_T^2 B_{q,1}^{\frac{3}{2}}},$$

and

$$\bar{T} = \sup \left\{ t \in (0, T) : Y_T \leq M_1 \left(\frac{T2^{N(\frac{3}{p} - \frac{3}{p} + \frac{3}{p})}}{C(N)^3} + \frac{1}{C(N)^2}\right), \ X_T \leq M_2 \frac{2^{N(\frac{3}{p} - \frac{3}{p})}}{C(N)^2} \right\},$$

where $M_i, i = 1, 2$ are large enough constants, which will be determined later on. Assume $\bar{T} < T$. Choosing $N$ such that

$$\frac{2^{N(\frac{3}{p} - \frac{3}{p})}}{C(N)} \ll 1, \ M_2 \frac{2^{N(\frac{3}{p} - \frac{3}{p})}}{C(N)^2} \ll 1,$$

thanks to (3.17) and (3.16), we get

$$\|\nabla P\|_{L_T^\infty B_{q,1}^{\frac{3}{2}}} \leq \frac{CT2^{N(\frac{3}{p} + \frac{3}{p} - \frac{3}{p} + \frac{3}{p})}}{C(N)^2} + \frac{CT2^{N(\frac{3}{2} + \frac{3}{2})}}{C(N)} X_T^2$$

$$+ \frac{CT2^{N(\frac{3}{p} - \frac{3}{p})}}{C(N)^2} Y_T + C \bar{T} X_T Y_T + CY_T^2,$$

and

$$\|\nabla P\|_{L_T^\infty B_{q,1}^{\frac{3}{2}}} \leq \frac{CT2^{N(\frac{3}{p} + \frac{3}{p} - \frac{3}{p} + \frac{3}{p})}}{C(N)^2} + \frac{CT2^{N(\frac{3}{2} + \frac{3}{2})}}{C(N)} X_T$$

$$+ \frac{CT2^{N(\frac{3}{p} - \frac{3}{p} + 2)}}{C(N)} + 1 Y_T + CY_T(X_T + Y_T), \tag{3.19}$$
we can obtain 

\[
\|a\nabla P\|_{L^1_tB^\frac{3}{2}} \leq C\left(\frac{2^{N\left(\frac{6}{p} - \frac{2}{p} + \frac{3}{2}\right)}}{C(N)} + X_T\right)\|\nabla P\|_{L^1_tB^\frac{3}{2}}
\]

\[
\leq \frac{T2^{N\left(\frac{6}{p} - \frac{2}{p} + \frac{3}{2}\right)}}{C(N)^3} + \frac{CT2^{N\left(\frac{6}{p} - \frac{2}{p} + \frac{3}{2}\right)}}{C(N)^2}X_T
\]

\[
+ \frac{C2^{N\left(\frac{6}{p} - \frac{2}{p} + \frac{3}{2}\right)}}{C(N)^2}Y_T + \frac{C2^{N\left(\frac{6}{p} - \frac{2}{p} + \frac{3}{2}\right)}}{C(N)}Y_T(X_T + Y_T)
\]

\[
+ \frac{CT2^{N\left(\frac{6}{p} - \frac{2}{p} + \frac{3}{2}\right)}}{C(N)}X_T^2 + CX_TY_T(X_T + Y_T).
\]

Setting \(N\) such that

\[
M_1 \frac{T^{\frac{3}{2}}2^{N\left(\frac{6}{p} - \frac{2}{p} + \frac{3}{2}\right)}}{C(N)^2} \ll 1,
\]

and using (3.18), we have

\[
X_T \leq \frac{CT2^{N\left(\frac{6}{p} - \frac{2}{p} + \frac{3}{2}\right)}}{C(N)^2} + \frac{CM_1T^{\frac{3}{2}}2^{N\left(\frac{6}{p} - \frac{2}{p} + \frac{3}{2}\right)}}{C(N)^4} + \frac{CM_1T^{\frac{3}{2}}2^{N\left(1 - \frac{2}{p}\right)}}{C(N)^3} \leq \frac{C_12^{N\left(\frac{6}{p} - \frac{2}{p} + \frac{3}{2}\right)}}{C(N)^2}.
\]

Thanks to the above estimates, choosing \(N\) such that

\[
\frac{T2^{N\left(\frac{6}{p} - \frac{2}{p} + \frac{3}{2}\right)}}{C(N)^3} \ll 1,
\]

we can obtain

\[
Y_T \leq C\|u \cdot \nabla u\|_{L_t^1B^\frac{3}{2}} + C\sum_{i=1,2} \|F_i\|_{L_t^1B^\frac{3}{2}_q} + C\|a\nabla P\|_{L_t^1B^\frac{3}{2}} + C\|\nabla \Pi\|_{L_t^1B^\frac{3}{2}}
\]

\[
\leq C\left(\frac{T2^{N\left(\frac{6}{p} - \frac{2}{p} + \frac{3}{2}\right)}}{C(N)^3} + \frac{1}{C(N)^2} + \frac{CT2^{N\left(\frac{6}{p} + \frac{3}{2}\right)}}{C(N)}X_T
\]

\[
+ C\left(\frac{1 + T2^{N\left(\frac{6}{p} - \frac{2}{p} + 2\right)}}{C(N)} + \frac{2^{N\left(\frac{6}{p} - \frac{2}{p}\right)}}{C(N)^2}Y_T + CY_T(X_T + Y_T)
\]

\[
\leq C_2\left(\frac{T2^{N\left(\frac{6}{p} - \frac{2}{p} + \frac{3}{2}\right)}}{C(N)^3} + \frac{1}{C(N)^2} + \frac{1}{2}Y_T
\]

which follows

\[
Y_T \leq 2C_2\left(\frac{T2^{N\left(\frac{6}{p} + \frac{3}{2}\right)}}{C(N)^3} + \frac{1}{C(N)^2}\right).
\]

We can see from the Remark 3.2 that the conditions in Lemma 3.1 can ensure the above requirements of \(N\). If we set \(M_2 = 4C_1\) and \(M_1 = 4C_2\), then a contradiction is obtained. Therefore, we have \(T = T_0\), and

\[
Y_T \leq 4C_2\left(\frac{T2^{N\left(\frac{6}{p} - \frac{2}{p} + \frac{3}{2}\right)}}{C(N)^3} + \frac{1}{C(N)^2}\right), \quad X_T \leq 4C_12^{N\left(\frac{6}{p} - \frac{2}{p}\right)}C(N)^2, \quad \forall T \leq T_0.
\]

(3.20)
Combining with (3.13) and (3.20), we get
\[ \|u(T_0)\|_{\dot{B}^{-\frac{1}{2}}_{6,1}} \geq \|U_1(T_0)\|_{\dot{B}^{-\frac{1}{2}}_{6,1}} - (\|U_0(T_0)\|_{\dot{B}^{-\frac{1}{2}}_{6,1}} + Y_{T_0}) \geq \frac{C}{8} 2^{2(\epsilon_1 - \epsilon)N}. \]

Thanks to (4.1) and (3.9), then we conclude the proof of Theorem 1.1.

4. Proof of Theorem 1.2

The proof is very similar to the proof of Theorem 1.1. Let us keep the process in Section 3.1 in mind. Now we begin with the choice of initial data.

4.1. The choice of initial data. As Section 3.2, we assume (3.5) and make the same assumptions of \( C(N), \epsilon_1, \hat{\phi} \). Let us construct the initial data as follows:

\[
\hat{a}_0(\xi) = \frac{1}{C(N)} \sum_{k=100}^{N} 2^{-\frac{k}{2}} \left( \hat{\phi}(\xi - 2^k e_1) + \hat{\phi}(\xi + 2^k e_1) \right),
\]

\[
\hat{u}_0(\xi) = \frac{1}{C(N)} \sum_{k=100}^{N} 2^{k(1-\frac{3}{p})} \left( \hat{\phi}(\xi + 2^k e_1) - \hat{\phi}(\xi - 2^k e_1) \right) \frac{1}{|\xi|} \left( \begin{array}{c} \xi_2 \\ -\xi_1 \end{array} \right).
\]

One can see \( a_0 \) and \( u_0 \) are real valued function and real vector-valued function, respectively. One can also check the following estimates hold, i.e.,

\[ \|a_0\|_{L^\infty} \leq \frac{C}{C(N)} , \quad \|a_0\|_{\dot{B}^{s}_{6,1}} \leq \frac{CN}{C(N)}, \quad \|u_0\|_{\dot{B}^{r,s}_{p,1}} \leq \frac{C N}{C(N)} \]  

and

\[ \|a_0\|_{\dot{B}^s_{r,1}} \leq \frac{C 2^{N(s-\frac{3}{p})}}{C(N)}, \quad \|u_0\|_{\dot{B}^{r,s}_{p,1}} \leq \frac{C 2^{N(s_1+1-\frac{3}{p})}}{C(N)}, \]

with \((r, s_1) \in [1, \infty]^2\) and \(s > \frac{3}{2}, s_1 > \frac{3}{3}p - 1\).

Now, we give a lemma.

Lemma 4.1. Let \( p \in (6, \infty) \). Then there exist some positive constants \( q, \epsilon, \epsilon_1 \) satisfying

\[
\begin{align*}
3 \frac{q}{p} < & \min \{1 + 2\epsilon - 2\epsilon_1, \frac{3}{4} + \frac{9}{2p} + 2\epsilon - \epsilon_1, \\
& \frac{3}{4} - \frac{3}{2p} + 2\epsilon - 3\epsilon_1, \frac{1}{2} + \frac{3}{p} + 2\epsilon - 2\epsilon_1 \} \\
q & \in (3, 6), \quad 0 < 2\epsilon < 2\epsilon_1 < \frac{1}{2} - \frac{3}{p}.
\end{align*}
\]

Remark 4.2. Although one can easily find that \( 1 + 2\epsilon - 2\epsilon_1 \) and \( \frac{3}{4} + \frac{9}{2p} + 2\epsilon - \epsilon_1 \) in (4.2) can be dropped, here we keep it in the bracket in order to giving a detailed analysis of the conditions in (4.3). Now, let us give the following explanations of the limitations in...
Let \( T = 2^{-2(1+\epsilon)N} \), then we have

\[
\begin{cases}
T_2 N \left( \frac{3}{q} - \frac{2}{2p} + \frac{3}{2} \right) < C(N) & \iff \frac{3}{q} < 1 + 2\epsilon - 2\epsilon_1, \\
T_2 N \left( \frac{3}{q} - \frac{2}{2p} + \frac{1}{2} \right) < C(N) & \iff \frac{3}{q} < \frac{3}{4} + \frac{9}{2p} + 2\epsilon - \epsilon_1, \\
T_2 N \left( \frac{2}{q} - \frac{2}{p} + \frac{3}{2} \right) < C(N)^3 & \iff \frac{3}{q} < \frac{3}{4} - \frac{3}{2p} + 2\epsilon - 3\epsilon_1, \\
T_2 N \left( \frac{2}{q} - \frac{2}{p} + \frac{1}{2} \right) < C(N)^2 & \iff \frac{3}{q} < \frac{1}{2} + \frac{3}{p} + 2\epsilon - 2\epsilon_1,
\end{cases}
\]

Actually, we assume the conditions on the right hand side of (4.3) to ensure the conditions on the left hand side which is required in our proof. Furthermore, we use \( q \in (3, 6) \) to ensure some product estimates like (2.4). The choice of \( C(N) \) needs \( 2\epsilon_1 < \frac{1}{2} - \frac{2}{p} \), while \( \epsilon < \epsilon_1 \) ensures the norm inflation of \( U_1 \) (see Subsection 4.2).

Lemma 4.1 can be proved easily, here we use the following example in this article.

If \( p \in (6, 18] \), the first inequality in (4.2) reduces to \( \frac{3}{q} < \frac{3}{4} - \frac{3}{2p} + 2\epsilon - 3\epsilon_1 \), then we can choose

\[
\epsilon_1 = \frac{1}{4} \left( \frac{1}{2} - \frac{3}{p} \right), \quad \epsilon = \frac{1}{5} \left( \frac{1}{2} - \frac{3}{p} \right)
\]

and

\[
\forall \frac{3}{q} \in \left( \frac{1}{2}, \frac{23}{40} - \frac{9}{20p} \right),
\]

leading

\[
2(\epsilon_1 - \epsilon) = \frac{1}{10} \left( \frac{1}{2} - \frac{3}{p} \right), \quad \frac{1}{2} \left( \frac{1}{2} - \frac{3}{p} - 2\epsilon_1 \right) = \frac{1}{4} \left( \frac{1}{2} - \frac{3}{p} \right);
\]

If \( p \in (18, \infty) \), the first inequality in (4.2) can be ensured by \( \frac{3}{q} < \frac{1}{2} + \frac{2}{p} + 2\epsilon - 3\epsilon_1 \), then we choose

\[
\epsilon_1 = \frac{2.1}{p}, \quad \epsilon = \frac{2}{p}, \quad \forall \frac{3}{q} \in \left( \frac{1}{2}, \frac{1}{2} + \frac{0.7}{p} \right)
\]

leading

\[
2(\epsilon_1 - \epsilon) = \frac{0.2}{p}, \quad \frac{1}{2} \left( \frac{1}{2} - \frac{3}{p} - 2\epsilon_1 \right) = \frac{1}{4} - \frac{3.6}{p} \geq \frac{0.2}{p}.
\]

4.2. The lower bound of \( U_1 \). Following the same idea and process as Section 3.3, one can easily get the finial lower bound of \( \| U_1(t) \|_{B_{\gamma_1}} \), that is,

\[
\| U_1(t) \|_{B_{\gamma_1}} \geq \frac{c}{4} 2^{2(\epsilon_1 - \epsilon)N},
\]

where \( t = T_0 := 2^{-2(1+\epsilon)N} \). We omit the details to avoid the repetition.
4.3. The analysis of $U_2$. Let $0 \leq T \leq T_0$, we split this subsection into several steps.

**Step 1** The estimate of $U_1$. Like the previous section, we list first some estimates of $U_1$:

$$\|U_1\|_{L_T^q L^\infty} \leq C T^{1+\frac{2}{3} N(3-\frac{2}{p})} C(N)^2,$$

$$\|U_1\|_{L_T^q B_{q,1}^{\frac{3}{p}+1}} + \|U_1\|_{L_T^{2} B_{q,1}^{\frac{3}{p}+3}} \leq C T^{\frac{2}{3} N(\frac{3}{q} - \frac{3}{p} + 3)} C(N)^2, \quad (q_2 = 6 \text{ or } q).$$

**Step 2** The estimate of $\|u \cdot \nabla u\|_{L_T^{\frac{3}{q},1} B_{q,1}^{\frac{3}{q}-1}}$. By using (2.1),

$$\|u \cdot \nabla u\|_{L_T^{\frac{3}{q},1} B_{q,1}^{\frac{3}{q}-1}} \leq C \|u\|_{L_T^{\infty} L^\infty} \|u\|_{L_T^{\infty} B_{q,1}^{\frac{3}{q}-1}} \leq C T^{2 N(\frac{2}{q} - \frac{6}{p} + 2)} C(N)^2,$$

$$\|u \cdot \nabla u\|_{L_T^{\infty} L^\infty} \|u\|_{L_T^{\infty} B_{q,1}^{\frac{3}{q}-1}} \leq C \|u\|_{L_T^{\infty} L^\infty} \|u\|_{L_T^{\infty} B_{q,1}^{\frac{3}{q}-1}} \leq C T^{2 N(\frac{2}{q} - \frac{6}{p} + 4)} C(N)^3,$$

$$\|u \cdot \nabla u\|_{L_T^{\infty} L^\infty} \|u\|_{L_T^{\infty} B_{q,1}^{\frac{3}{q}-1}} \leq C \|u\|_{L_T^{\infty} B_{q,1}^{\frac{3}{q}-1}} \leq C T^{2 N(\frac{2}{q} - \frac{6}{p} + 6)} C(N)^4.$$

Thanks to (2.2),

$$\|u \cdot \nabla u\|_{L_T^{\infty} L^\infty} \|u\|_{L_T^{\infty} B_{q,1}^{\frac{3}{q}-1}} \leq C \|u\|_{L_T^{\infty} L^\infty} \|u\|_{L_T^{\infty} B_{q,1}^{\frac{3}{q}-1}} \leq C T^{2 N(\frac{2}{q} - \frac{6}{p} + 4)} C(N)^3.$$

Combining with the above six estimates, we have

$$\|u \cdot \nabla u\|_{L_T^{\frac{3}{q},1} B_{q,1}^{\frac{3}{q}-1}} \leq \frac{CT^{2 N(\frac{2}{q} - \frac{6}{p} + 2)} C(N)^2}{C(N)} + \frac{CT^{1+\frac{2}{3} N(1-\frac{2}{p})} C(N)}{C(N)} \times (\|U_2\|_{L_T^{\infty} B_{q,1}^{\frac{3}{q}-1}} + \|U_2\|_{L_T^{\infty} B_{q,1}^{\frac{3}{q}-1}} + C \|U_2\|_{L_T^{\infty} B_{q,1}^{\frac{3}{q}-1}}), \quad (4.7)$$
Step 3. The estimate of $\|F_1\|_{L^1_T B^{\frac{3}{q}, 1}_{q, 1}}$ and $\|F_2\|_{L^1_T B^{\frac{3}{q}, 1}_{q, 1}}$ Using the product estimate (2.4) and $B^{\frac{5}{6}, 1}_r \hookrightarrow \tilde{B}^{\frac{3}{p_0}, 1}_r$, one gets

$$
\begin{align*}
\|a_1 \Delta U_0\|_{L^1_T B^{\frac{3}{q}, 1}_{q, 1}} & \leq C \|a_1\|_{L^\infty_T B^{\frac{1}{2}, 1}_{q, 1}} \|U_0\|_{L^1_T B^{\frac{3}{q}, 1}_{q, 1}} \leq \frac{CT2^{\frac{N(q-\frac{3}{q}+2}{C(N)}}}{C(N)} \|a_1\|_{L^\infty_T B^{\frac{1}{2}, 1}_{q, 1}}, \\
\|a_1 \Delta U_1\|_{L^1_T B^{\frac{3}{q}, 1}_{q, 1}} & \leq C \|a_1\|_{L^\infty_T B^{\frac{1}{2}, 1}_{q, 1}} \|U_1\|_{L^1_T B^{\frac{3}{q}, 1}_{q, 1}} \leq \frac{CT^{\frac{N(q-\frac{3}{q}+3)}{C(N)}}}{C(N)^2} \|a_1\|_{L^\infty_T B^{\frac{1}{2}, 1}_{q, 1}}, \\
\|a_1 \Delta U_2\|_{L^1_T B^{\frac{3}{q}, 1}_{q, 1}} & \leq C \|a_1\|_{L^\infty_T B^{\frac{1}{2}, 1}_{q, 1}} \|U_2\|_{L^1_T B^{\frac{3}{q}, 1}_{q, 1}} \leq \frac{C}{C(N)} \|a_1\|_{L^\infty_T B^{\frac{1}{2}, 1}_{q, 1}} \|U_2\|_{L^1_T B^{\frac{3}{q}, 1}_{q, 1}}.
\end{align*}
$$

Combining with the above estimates, then we get

$$
\begin{align*}
\|F_1\|_{L^1_T B^{\frac{3}{q}, 1}_{q, 1}} + \|F_2\|_{L^1_T B^{\frac{3}{q}, 1}_{q, 1}} & \leq \frac{CT2^{\frac{N(q-\frac{3}{q}+3)}{C(N)}}}{C(N)^3} + \frac{CT2^{\frac{N(q-\frac{3}{q}+2)}{C(N)}}}{C(N)^2} \|a_1\|_{L^\infty_T B^{\frac{1}{2}, 1}_{q, 1}} \\
& + \frac{C}{C(N)} \|U_2\|_{L^1_T B^{\frac{3}{q}, 1}_{q, 1}} + \frac{C}{C(N)} \|a_1\|_{L^\infty_T B^{\frac{1}{2}, 1}_{q, 1}} \|U_2\|_{L^1_T B^{\frac{3}{q}, 1}_{q, 1}}.
\end{align*}
$$

Step 4. The estimates of the pressure $P$ and the modified pressure $\Pi$ Using the product estimate (2.4) again, we have

$$
\|a_0 \Delta U_0\|_{L^1_T B^{\frac{3}{q}, 1}_{q, 1}} \leq C \|a_0\|_{L^\infty_T B^{\frac{1}{2}, 1}_{q, 1}} \|U_0\|_{L^1_T B^{\frac{3}{q}, 1}_{q, 1}} \leq \frac{CT2^{\frac{N(q-\frac{3}{q}+2)}{C(N)}}}{C(N)^2}.
$$

Using (3.1), (2.4), (4.7), (4.8) and (4.9), one obtains

$$
\begin{align*}
\|\nabla P\|_{L^1_T B^{\frac{3}{q}, 1}_{q, 1}} & \leq C \|a\|_{L^\infty_T B^{\frac{1}{2}, 1}_{q, 1}} \|\nabla P\|_{L^1_T B^{\frac{3}{q}, 1}_{q, 1}} + C \|u \cdot \nabla u\|_{L^1_T B^{\frac{3}{q}, 1}_{q, 1}} + C \|a \Delta u\|_{L^1_T B^{\frac{3}{q}, 1}_{q, 1}} \\
& \leq C\left(\frac{C}{C(N)} + \|a_1\|_{L^\infty_T B^{\frac{1}{2}, 1}_{q, 1}}\right) \|\nabla P\|_{L^1_T B^{\frac{3}{q}, 1}_{q, 1}} \\
& + \frac{CT2^{\frac{N(q-\frac{3}{q}+2)}{C(N)}}}{C(N)^2} + \frac{CT2^{\frac{N(q-\frac{3}{q}+2)}{C(N)}}}{C(N)} \|a_1\|_{L^\infty_T B^{\frac{1}{2}, 1}_{q, 1}} \\
& + \frac{CT2^{\frac{3\frac{N(q-\frac{3}{q})+4}{C(N)}}}{C(N)^2} + C(N)}{C(N)^2} \|U_2\|_{L^1_T B^{\frac{3}{q}, 1}_{q, 1}} + \|U_2\|_{L^1_T B^{\frac{3}{q}, 1}_{q, 1}} \\
& + C\left(\|a_1\|_{L^\infty_T B^{\frac{1}{2}, 1}_{q, 1}} + \|U_2\|_{L^1_T B^{\frac{3}{q}, 1}_{q, 1}}\right) \|U_2\|_{L^1_T B^{\frac{3}{q}, 1}_{q, 1}}.
\end{align*}
$$
And using (3.2), (4.7) and (4.8), we have
\[
\|\nabla u\|_{L^\frac{4}{3}B_2^{\frac{2}{3}},1} \leq C \|a\|_{L^\infty L^\frac{4}{3}} \|\nabla u\|_{L^\frac{4}{3}B_2^{\frac{2}{3}},1} + C \|u\cdot \nabla u\|_{L^\frac{4}{3}B_2^{\frac{2}{3}},1} + C \|u\|_{L^\frac{4}{3}B_2^{\frac{2}{3}},1} \|\nabla u\|_{L^\frac{4}{3}B_2^{\frac{2}{3}},1}
\]
\[
\leq C \left( \frac{1}{C(N)} + \|a_1\|_{L^\infty L^\frac{4}{3}} \|\nabla u\|_{L^\frac{4}{3}B_2^{\frac{2}{3}},1} \right) + C \|u\|_{L^\frac{4}{3}B_2^{\frac{2}{3}},1} \|\nabla u\|_{L^\frac{4}{3}B_2^{\frac{2}{3}},1}
\]
\[
+ C \left( \frac{1}{C(N)^2} + \|a_1\|_{L^\infty L^\frac{4}{3}} \right) \left( \frac{T^{2N} \left( 3- \frac{N}{p} \right)}{C(N)} \right) + C \|u\|_{L^\frac{4}{3}B_2^{\frac{2}{3}},1} \|\nabla u\|_{L^\frac{4}{3}B_2^{\frac{2}{3}},1} \|U_2\|_{L^\frac{4}{3}B_2^{\frac{2}{3}},1} \|\nabla u\|_{L^\frac{4}{3}B_2^{\frac{2}{3}},1}
\]
(4.11)
\[
\leq C \left( \frac{1}{C(N)} + \|a_1\|_{L^\infty L^\frac{4}{3}} \right) \left( \frac{T^{2N} \left( 3- \frac{N}{p} \right)}{C(N)} \right) + C \|U_2\|_{L^\frac{4}{3}B_2^{\frac{2}{3}},1} \|\nabla u\|_{L^\frac{4}{3}B_2^{\frac{2}{3}},1} \|U_2\|_{L^\frac{4}{3}B_2^{\frac{2}{3}},1} \|\nabla u\|_{L^\frac{4}{3}B_2^{\frac{2}{3}},1}
\]
Step 5. The estimate of \( \|a_1\|_{L^\infty L^\frac{4}{3}} \). Thanks to Proposition 2.5, we have
\[
\|a_1\|_{L^\infty L^\frac{4}{3}} \leq \|\nabla u\|_{L^\frac{4}{3}B_2^{\frac{2}{3}},1} \exp \{ C \|\nabla u\|_{L^\frac{4}{3}B_2^{\frac{2}{3}},1} \}.
\]
And by (2.1), one gets
\[
\|u \cdot \nabla a_0\|_{L^\frac{4}{3}B_2^{\frac{2}{3}},1} \leq \|u\|_{L^\frac{1}{2}L^\infty} \|\nabla a_0\|_{L^\frac{1}{2}B_2^{\frac{2}{3}},1} \|\nabla u\|_{L^\frac{1}{2}L^\infty} \leq \frac{CT^{2N(2- \frac{N}{p})}}{C(N)^2},
\]
\[
\|U_1 \cdot \nabla a_0\|_{L^\frac{4}{3}B_2^{\frac{2}{3}},1} \leq C \|U_1\|_{L^\frac{2}{3}L^\infty} \|\nabla a_0\|_{L^\frac{2}{3}B_2^{\frac{2}{3}},1} \|\nabla u\|_{L^\frac{2}{3}L^\infty} \leq \frac{CT^2 2^N \left( 4- \frac{N}{p} \right)}{C(N)^3}.
\]
Applying (2.2), we can obtain
\[
\|U_2 \cdot \nabla a_0\|_{L^\frac{4}{3}B_2^{\frac{2}{3}},1} \leq C (T^{\frac{2N}{3}} \|\nabla a_0\|_{L^\infty} \|U_2\|_{L^\frac{2}{3}B_2^{\frac{2}{3}},1} + T \|\nabla^2 a_0\|_{L^\infty} \|U_2\|_{L^\frac{2}{3}B_2^{\frac{2}{3},1}})
\]
\[
\leq \frac{CT^{\frac{2N}{3}}}{C(N)} \left( \|U_2\|_{L^\frac{2}{3}B_2^{\frac{2}{3},1}} + \|U_2\|_{L^\frac{2}{3}B_2^{\frac{2}{3},1}} \right).
\]
Thus we get
\[
\|a_1\|_{L^\infty L^\frac{4}{3}} \leq C \left( \frac{T^{2N(2- \frac{N}{p})}}{C(N)^2} + \frac{T^{\frac{2N}{3}}}{C(N)} \left( \|U_2\|_{L^\frac{2}{3}B_2^{\frac{2}{3},1}} + \|U_2\|_{L^\frac{2}{3}B_2^{\frac{2}{3},1}} \right) \right)
\]
\[
\times \exp \{ C \|\nabla U_2\|_{L^\frac{4}{3}B_2^{\frac{2}{3},1}} \}.
\]
(4.12)
4.4. Proof of Theorem 1.2. Denote
\[
X_T := \|a_1\|_{L^\infty L^\frac{4}{3}}, \quad Y_T := \|U_2\|_{L^\infty L^\frac{4}{3}} + \|U_2\|_{L^\frac{4}{3}B_2^{\frac{2}{3}},1},
\]
and
\[
\bar{T} := \sup \{ t \in (0, T_0) : Y_T \leq \frac{2^{N(2- \frac{N}{p})}}{C(N)^3} + \frac{2^{N(2- \frac{N}{p}+2)}}{C(N)^2}, \quad X_T \leq M_2 \frac{T^{\frac{2N}{3}}}{C(N)} \}
\]
where \( M_3 \) and \( M_4 \) will be fixed later. Using (4.12), choosing \( N \) such that
\[
M_3 T \left\{ \frac{2^{N(2- \frac{N}{p})}}{C(N)^3} + \frac{2^{N(2- \frac{N}{p}+2)}}{C(N)^2} \right\} \ll 1,
\]
we have

\[ X_T \leq \frac{CT^{1/2}2^N}{C(N)^2} + \frac{CT^{1/2}2^N}{C(N)^2}\{M_3T\left(\frac{2^{N(\frac{1}{2} - \frac{2}{p} + \frac{3}{2})}}{C(N)^2} + \frac{2^{N(\frac{1}{2} - \frac{2}{p} + \frac{3}{2})}}{C(N)^2}\right)\} \leq \frac{C_1T^{1/2}2^N}{C(N)^2}. \]  

(4.13)

From the estimate (4.10), one has

\[ \|\nabla P\|_{L^2_t L^{\frac{3}{2},1}_{x}} \leq \frac{1}{2} \|\nabla P\|_{L^2_t L^{\frac{3}{2},1}_{x}} + \frac{CT^22^{N(\frac{3}{2} - \frac{1}{p} + \frac{1}{2})}}{C(N)^2} + \frac{C(N)}{C(N)^2}YT \]

\[ + \frac{CT^22^{N(\frac{3}{2} - \frac{1}{p} + \frac{1}{2})}}{C(N)^2}X_T + CY_T(X_T + Y_T), \]

which leads to

\[ \|\nabla P\|_{L^2_t L^{\frac{3}{2},1}_{x}} \leq \frac{CT^22^{N(\frac{3}{2} - \frac{1}{p} + \frac{1}{2})}}{C(N)^2} + \frac{C(N)}{C(N)^2}YT \]

\[ + \frac{CT^22^{N(\frac{3}{2} - \frac{1}{p} + \frac{1}{2})}}{C(N)^2}X_T + CY_T(X_T + Y_T), \]

Applying product estimate again, and using \( X_T \ll 1 \), we can deduce that

\[ \|a \nabla P\|_{L^2_t L^{\frac{3}{2},1}_{x}} \leq \frac{CT^22^{N(\frac{3}{2} - \frac{1}{p} + \frac{1}{2})}}{C(N)^3} + \frac{CT^22^{N(\frac{3}{2} - \frac{1}{p} + \frac{1}{2})}}{C(N)^2}YT \]

\[ + \frac{CT^22^{N(\frac{3}{2} - \frac{1}{p} + \frac{1}{2})}}{C(N)^2}X_T + CY_T(X_T + Y_T). \]  

(4.14)

Similarly, thanks to (4.11), we have

\[ \|\nabla II\|_{L^2_t L^{\frac{3}{2},1}_{x}} \leq CT\left(\frac{2^{N(\frac{1}{2} - \frac{2}{p} + \frac{1}{2})}}{C(N)^3} + \frac{2^{N(\frac{1}{2} - \frac{2}{p} + \frac{1}{2})}}{C(N)^2}\right) \]

\[ + \frac{CT^22^{N(\frac{3}{2} - \frac{1}{p} + \frac{1}{2})}}{C(N)^2}X_T^2 + \left(\frac{C}{C(N)^2}\right)YT + CX_T^2YT. \]  

(4.15)

Collecting the above estimates (4.11, 4.12, 4.13, 4.14) and (4.15), and setting \( N \) such that

\[ M_3T\left(\frac{2^{N(\frac{1}{2} - \frac{2}{p} + \frac{1}{2})}}{C(N)^3} + \frac{2^{N(\frac{1}{2} - \frac{2}{p} + \frac{1}{2})}}{C(N)^2}\right) \ll 1, \]

we have

\[ Y_T \leq C\left(\|u \cdot \nabla u\|_{L^2_t L^{\frac{3}{2},1}_{x}} + \sum_{i=1,2} \|F_i\|_{L^2_t L^{\frac{3}{2},1}_{x}} + \|a \nabla P\|_{L^2_t L^{\frac{3}{2},1}_{x}} + \|\nabla II\|_{L^2_t L^{\frac{3}{2},1}_{x}}\right) \]

\[ \leq C_2T\left(\frac{2^{N(\frac{1}{2} - \frac{2}{p} + \frac{1}{2})}}{C(N)^3} + \frac{2^{N(\frac{1}{2} - \frac{2}{p} + \frac{1}{2})}}{C(N)^2}\right) + \frac{1}{2}YT. \]

This yields

\[ Y_T \leq 2C_2T\left(\frac{2^{N(\frac{1}{2} - \frac{2}{p} + \frac{1}{2})}}{C(N)^3} + \frac{2^{N(\frac{1}{2} - \frac{2}{p} + \frac{1}{2})}}{C(N)^2}\right) \]
One can see from the Remark 4.2 that the conditions in Lemma 4.1 can ensure the above requirements of \(N\). Choosing \(M_3 = 4C_1\) and \(M_4 = 4C_2\), we can get a contradiction by using the continuation argument. Therefore, we have \(\tilde{T} = T_0\), and

\[
Y_T \leq 4C_2 T \left( \frac{2^{N(\frac{3}{4} - \frac{3}{p} + 2)}}{C(N)^3} + \frac{2^{N(\frac{3}{4} - \frac{3}{p} + 2)}}{C(N)^2} \right), \quad X_t \leq 4C_1 T^{\frac{3}{2}} 2^{N} / C(N)^2, \quad \forall \ T \leq T_0. \tag{4.16}
\]

Combining with (4.6) and (4.16), we get

\[
\|u(T_0)\|_{B_{p,1}^{\frac{3}{4} - 1}} \geq \|U_1(T_0)\|_{B_{p,1}^{\frac{3}{4} - 1}} - (\|U_0(T_0)\|_{B_{p,1}^{\frac{3}{4} - 1}} + Y_T) \geq \frac{c}{2^{(\epsilon_1 - \epsilon)}}. \tag{4.11}
\]

Thanks to (4.11), (4.14) and (4.3), we can complete the proof of Theorem 1.2.

**APPENDIX A. THE ENDPOINT CASE FOR THEOREM 1.2: \(p = \infty\)**

In this section, we give some comments on the endpoint case for Theorem 1.2, that is, the case \(p = \infty\). The previous proof in Section 4 is not suitable for this case, since some difficulties occur when we bound \(\|U_0 \cdot \nabla U_0\|_{L^{\frac{3}{2},B_{q,1}^{\frac{3}{4}}}}\) in the estimate of \(Y_T\). More precisely, we have

\[
\|U_0 \cdot \nabla U_0\|_{L^{\frac{3}{2},B_{q,1}^{\frac{3}{4}}}} \leq \frac{CT 2^{N(\frac{3}{4} + 2)}}{C(N)^2},
\]

combined with (4.2) and (4.3) yielding

\[
\frac{3}{q} < \frac{1}{2}, \quad 2\epsilon - 2\epsilon_1 < \frac{1}{2}.
\]

This is a contradiction with \(q \in (3, 6)\). However, we can break this barrier by introducing another new modified pressure

\[
\Pi_1 := P + \mu|D|^{-2} \text{div}(a\Delta u) - |D|^{-2} \text{div}(u \cdot \nabla u)
\]

leading

\[
\nabla \Pi_1 = |D|^{-2} \nabla \text{div}(a\nabla \Pi_1) - \mu|D|^{-2} \nabla \text{div}(a|D|^{-2} \nabla \text{div}(a\Delta u)) + \mu|D|^{-2} \nabla \text{div}(a|D|^{-2} \nabla \text{div}(u \cdot \nabla u)).
\]

Then we consider

\[
\begin{aligned}
\partial_t a + u \cdot \nabla a &= 0, \quad \text{div} u = 0 \\
\partial_t u - \mu \Delta u &= -u \cdot \nabla u - \nabla \Pi_1 - a \nabla P \\
&\quad + \mu a \Delta u + \mu|D|^{-2} \nabla \text{div}(a\Delta u) - \nabla |D|^{-2} \text{div}(u \cdot \nabla u), \\
(a(0, x), u(0, x)) &= (a_0(x), u_0(x)), \tag{A.1}
\end{aligned}
\]

and have

\[
u(t, x) = e^{\mu \Delta t} u_0 + \int_0^t e^{\mu \Delta (t - \tau)} \left(-u \cdot \nabla u - \nabla \Pi_1 - a \nabla P \\
+ \mu a \Delta u + \mu|D|^{-2} \nabla \text{div}(a\Delta u) - \nabla |D|^{-2} \text{div}(u \cdot \nabla u)\right) d\tau.
\]
Denote
\[ U_0(t) = e^{\mu\Delta t} u_0, \]
\[ U_1(t) = \mu \int_0^t e^{\mu\Delta(t-\tau)} \left\{ a_0 \Delta U_0 + |D|^{-2} \nabla \text{div}(a_0 \Delta U_0) - U_0 \cdot \nabla U_0 - \nabla |D|^{-2} \text{div}(U_0 \cdot \nabla U_0) \right\} d\tau, \]
\[ U_2(t) = \int_0^t e^{\mu\Delta(t-\tau)} \left\{ K_1 + K_2 - u \cdot \nabla u - \nabla \Pi_1 - a \nabla P + F_1 + F_2 \right\} d\tau, \]
where
\[ K_1 = (U_1 + U_2) \cdot \nabla (U_0 + U_1 + U_2) + U_0 \cdot \nabla (U_1 + U_2), \]
\[ K_2 = \nabla |D|^{-2} \text{div} \{(U_1 + U_2) \cdot \nabla (U_0 + U_1 + U_2) + U_0 \cdot \nabla (U_1 + U_2)\}, \]
\[ F_1 = \mu (a_1 \Delta u + a_0 \Delta (U_1 + U_2)), \quad F_2 = \mu |D|^{-2} \text{div} \{a_1 \Delta u + a_0 \Delta (U_1 + U_2)\}. \]

Thus, we have the decomposition of \( u \). We also use the previous decomposition of \( a \) in the following. We choose the initial data as the Section 4 by setting \( p = \infty \). The proof is very similar, here we only show the framework.

Firstly, one can get the large lower bound of \( U_1 \), which can be obtained from the estimate of \( \mu \int_0^t e^{\mu\Delta(t-\tau)} \| \nabla U_0 \|_{L^4 B^{\frac{4}{3}}_{2,1}}^4 \) and get the small bound of \( X_T \) and \( Y_T \) by following the procedure as Section 4.

Secondly, thanks to the modified pressure \( \Pi_1 \), we can avoid the estimate of \( \| U_0 \cdot \nabla U_0 \|_{L^4 B^{\frac{4}{3}}_{2,1}} \) and get the small bound of \( X_T \) and \( Y_T \) by following the procedure as Section 4.

At last, combining with the above arguments yields the desired result by following the same procedure as Section 4.

**Appendix B. Proof of the existence of the solution**

In this section, we give a brief structure to show the existence of the solution. In fact, \( K \) is equal to
\[
\begin{cases}
\partial_t a + u \cdot \nabla a = 0, & \text{div} u = 0 \\
u(t,x) = e^{\mu\Delta t} u_0 + \int_0^t e^{\mu\Delta(t-\tau)} \left\{ -u \cdot \nabla u - \nabla \Pi - a \nabla P \right\} d\tau, \\
+ \mu a \Delta u + \mu |D|^{-2} \text{div}(a \Delta u) \right\} d\tau, \\
(a(0,x),u(0,x)) = (a_0(x),u_0(x)).
\end{cases}
\] (B.1)

From the previous parts, one can see \( U_0 \) and \( U_1 \) are chose as follows:
\[ U_0(t) = e^{\mu\Delta t} u_0, \]
\[ U_1(t) = \mu \int_0^t e^{\mu\Delta(t-\tau)} \{ a_0 \Delta U_0 + |D|^{-2} \text{div}(a_0 \Delta U_0) \} d\tau. \]
In addition, we can get from section 3.5 that \((a_1, U_2)\) is the solution to the system below
\[
\begin{aligned}
\partial_t a_1 + (U_0 + U_1 + U_2) \cdot \nabla (a_0 + a_1) &= 0, \quad \text{div} u = 0 \\
U_2(t) &= \int_0^t e^{\mu \Delta (t-t')} \\
\{(U_0 + U_1 + U_2) \cdot \nabla (U_0 + U_1 + U_2) - \nabla \Pi - (a_0 + a_1) \nabla P + F_1 + F_2\} d\tau, \\
(a(0, x), U_2(0, x)) &= (a_0(x), 0),
\end{aligned}
\]
where
\[
F_1 = \mu (a_1 \Delta u + a_0 \Delta (U_1 + U_2)),
\]
\[
F_2 = \mu |D|^{-2} \nabla \text{div} \{a_1 \Delta (U_0 + U_1 + U_2) + a_0 \Delta (U_1 + U_2)\},
\]
and
\[
\nabla \Pi = |D|^{-2} \nabla \text{div} ((a_0 + a_1) \nabla \Pi) - \mu |D|^{-2} \nabla \text{div} \{(a_0 + a_1) |D|^{-2} \nabla \text{div} ((a_0 + a_1) \Delta (U_0 + U_1 + U_2))
\]
\[
+ |D|^{-2} \nabla \text{div} ((U_0 + U_1 + U_2) \cdot \nabla (U_0 + U_1 + U_2)),
\]
although section 3.5 only gives a priori estimate. The strict proof can follow the Chapter 10 in [6], which is very standard, so we omit the details. Therefore, one can see \(a := a_0 + a_1\) and \(u := U_0 + U_1 + U_2\) is a solution to (B.1).

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