Results on the existence of the Yamabe minimizer of $M^m \times \mathbb{R}^n$

Juan Miguel Ruiz *

Abstract

We let $(M^m, g)$ be a closed smooth Riemannian manifold $(m \geq 2)$ with positive scalar curvature $S_g$, and prove that the Yamabe constant of $(M \times \mathbb{R}^n, g + g_E)$ is achieved by a metric in the conformal class of $(g + g_E)$, where $g_E$ is the Euclidean metric. We also show that the Yamabe quotient of $(M \times \mathbb{R}^n, g + g_E)$ is improved by Steiner symmetrization with respect to $M$. It follows from this last assertion that the dependence on $\mathbb{R}^n$ of the Yamabe minimizer of $(M \times \mathbb{R}^n, g + g_E)$ is radial.

Keywords. Scalar curvature, non-compact Yamabe problem. Mathematics subject classification (2000). 53Cxx: 53C21.

1 Introduction

Let $(M^m, g_M)$ be a smooth closed Riemannian manifold (smooth compact manifold without boundary). Let $[g_M]$ denote the conformal class of the metric $g_M$. The Yamabe constant of the conformal class $(M^m, [g_M])$ is defined as the infimum of the normalized total scalar curvature restricted to $[g_M]$.

$$Y(M, [g_M]) = \inf_{h \in [g_M]} \frac{\int s_h dV_h}{\text{Vol}(M, h)^{\frac{m-2}{2}}},$$

where $S_h$ and $dV_h$ are the scalar curvature and the volume element of $h$. By writing $h = f^{\frac{4}{m-2}} g_M$ ($f$ positive and $C^\infty$), we can rewrite $Y(M, [g_M])$ in terms of functions of the sobolev space $L^2(M)$,

$$Y(M, [g_M]) = \inf_{f \in L^2(M), f \neq 0} Q_{g_M}(f)$$

$$:= \inf_{f \in L^2(M), f \neq 0} \frac{a \int_M |\nabla f|^2 dV_{g_M} + \int_M s_{g_M} f^2 dV_{g_M}}{\int_M f^p dV_{g_M}}^{\frac{p}{p-1}},$$

where $a = \frac{4(m-1)}{(m-2)}$, $p = \frac{2m}{m-2}$. $Q_{g}(f)$ is called the Yamabe quotient. It is a fundamental result, proven in many steps by H. Yamabe [21], N. Trudinger

*The author is supported by CONACYT
possibly the simplest example of several metrics of constant scalar curvature, and constant scalar curvature greater than $\kappa$. Examples include ($\kappa$ is positive, there may be several metrics of constant scalar curvature in the conformal class. On the other hand, when the Yamabe invariant $Y(M)$ of $M$ is defined as the supremum of the Yamabe constants over all the conformal classes (cf. in [10], [19]). Hence, it is an easy consequence that $Y(M) \leq Y(S^m) = Y(S^m, [g_0])$. In the following, we will denote $Y(S^m) = Y(S^m, [g_0])$ by $Y_m$.

When the Yamabe constant is non-positive, there is only one metric with constant scalar curvature in the conformal class. On the other hand, when the Yamabe constant is positive, there may be several metrics of constant scalar curvature. Examples include ($S^k \times M^m, g_0 + g_M$), where $M^m$ is a Riemannian manifold of constant scalar curvature $s_{g_M}$: it has been shown in [19] that in this case, the number of unit volume non-isometric metrics of constant scalar curvature in the conformal classes of $[g_0 + g_M]$ grows at least linearly with $\sqrt{s_{g_M}}$.

Possibly the simplest example of several metrics of constant scalar curvature, is the one exhibited in [1]: if $(M^m_1, g_1)$ and $(M^m_2, g_2)$ are Riemannian manifolds with constant scalar curvature, and $s_{g_1} > 0$, then $\delta^n g_1 + \delta^{-m} g_2$ has volume one and constant scalar curvature greater than $Y_{m+n}$.

Through the study of these cases, Akutagawa, Florit and Petean, found that if $(M^m_1, g_1)$ is a closed manifold ($m \geq 2$) of positive scalar curvature and $(M^m_2, g_2)$ any closed manifold, then

$$\lim_{r \to \infty} Y(M \times N, [g_1 + rg_2]) = Y(M \times \mathbb{R}^n, [g + g_E])$$

(3)

where $g_E$ is the Euclidean metric on $\mathbb{R}^n$ (Theorem 1.1 in [1]). Making thus the Yamabe constant $Y(M \times \mathbb{R}^n, [g + g_E])$ of high relevance in the study of the Yamabe constant of product manifolds, since, for instance, from (3) follows that the Yamabe invariant of $M \times N$ is bounded below,

$$Y(M \times \mathbb{R}^n, [g_1 + g_E]) \leq Y(M \times N).$$

As another example, the Yamabe constant of $Y(S^m \times \mathbb{R}^n, [g + g_E])$ is involved in a surgery formula for the Yamabe invariant of a compact manifold, as have shown recent results of B. Ammann, M. Dahl and E. Humbert [4].

Also, it was through the case where $n = 1$ that J. Petean found a lower bound to the Yamabe invariant of $M^m_1 \times S^1$ (when $M^m_1$ an Einstein manifold), among other interesting results involving $Y(M \times \mathbb{R}^n, [g + g_E])$. [14].

In this article we study the Yamabe constant $Y(M^m \times \mathbb{R}^n, [g + g_E])$, where $M^m$ ($m \geq 2$), as in (3), is a closed manifold with positive scalar curvature.

The Yamabe problem for non-compact manifolds has not been solved completely yet. Different counter-examples and conditions for existence and nonexistence of a constant scalar curvature in the conformal class of a metric, have
been published for non-compact manifolds (cf. in [22]). Results include, e.g.,
those of K. Akutagawa and B. Botvinnik in [2], where they study complete
manifolds with cylindrical ends and solve affirmatively the Yamabe problem on
cylindrical manifolds. Results include also some cases for noncompact complete
manifolds of positive scalar curvature. We cite here the work of S. Kim in [9],
where he introduces the notation
\[ Q(M) := \inf_{u \in C^\infty_0(M)} \frac{\int_M |\nabla f|^2 dV_g + 1/a \int_M s_g f^2 dV_g}{\int_M f^p dV_g}^{2/p}, \]
and
\[ \bar{Q}(M) := \inf_{u \in C^\infty_0(M \setminus B_r)} \frac{\int_M |\nabla f|^2 dV_g + 1/a \int_M s_g f^2 dV_g}{\int_M f^p dV_g}^{2/p}, \]
(where \( r \) is the distance from \( x \) to a fixed point \( x_0 \in M \), and \( B_r \) the ball of
radius \( r \) and centered at \( x_0 \)), and then proves the existence of a constant scalar
curvature in the conformal class of \((M, g_M)\) whenever \( Q(M) < \bar{Q}(M) \).

In our case, given some of the particularities of \((M^m \times \mathbb{R}^n, g_M + g_E)\), we
use a more direct approach to prove existence of a Yamabe minimizer. We
first show that the Steiner symmetrization of a function “improve” the Yamabe
quotient, making thus, the Steiner symmetrized functions, the best candidates
for the Yamabe minimizer. Then, along with this result, we use the fact that
\( Y((M^m \times \mathbb{R}^n, g_M + g_E) < Y_{m+n} \) (a known result of Akutagawa, Florit and
Petean (I)) to prove that the Yamabe minimizer exists and is positive and
\( C^\infty \).

The fact that that Steiner symmetrizations “improve” the Yamabe quotient
is a consequence of the following.

**Theorem 1.** Let \((N, g) = (M^m \times \mathbb{R}^n, g_M + g_E)\), and \( u \in L^1(N), \) \( 1 < s < \infty \). Let \( u^* \) be the Steiner symmetrization of \( u \), with respect to \( M \). Then
\( u^* \in L^1(N), \) and

\[ ||\nabla u^*||_s \leq ||\nabla u||_s. \]  

Indeed, using inequality \( (I) \) from the preceding theorem, and the fact that
the norm is preserved under Steiner symmetrizations \( (||u^*||_s = ||u||_s, \) for any \( s \),
the next corollary follows.

**Corollary 2.** Consider \((N, g) = (M^m \times \mathbb{R}^n, g_M + g_E)\), and the Yamabe quotient
for \( 2 \leq s \leq p \):
\[ Q_s(u) = \frac{a \int_N |\nabla u|^2 dV_g + \int_N s_g u^2 dV_g}{(\int_N u^p dV_g)^{2/p}}, \]
then \( Q_s(u^*) \leq Q_s(u) \).

The main result of this paper, the existence of the Yamabe minimizer of
\((N, g) = (M^m \times \mathbb{R}^n, g_M + g_E)\), is stated in the next Theorem.

3
Theorem 3. Let \((N, g) = (M^m \times \mathbb{R}^n, g_M + g_E)\), with \(m \geq 2\) and \(s_{g_M} > 0\). The Yamabe minimizer of \((N, g)\) exists, and is positive and \(C^\infty\).

The result we give is sharp, since counter-examples for manifolds of the type \((M^m \times \mathbb{R}^n, g_M + g_E)\), where \(M^m\) has non-positive scalar curvature or where \(m < 2\), and a positive Yamabe metric is not achieved are known to exist. An example of the former is \(N^9 = S^1 \times S^1 \times S^1 \times S^1 \times R^3\) (with the metric being the product of those usual ones on \(R^3\) and \(S^1\)), while an example of the latter is \(N^4 = S^1 \times R^3\) (with the metric being the product of the usual metrics), as is shown by Zhang in [22] and [23].

This paper is organized as follows. In section 2 we give the precise definition of the Steiner symmetrization of a function \(u\), \(u^*\), with respect to \(M\); we also give the definition of Polarizations and we introduce other preliminaries. We also give a proof of Theorem 1, many of the proofs and lemmas we give there are due to Brock and Solynin [7] and to Jean Van Schaftingen [17], with some minor modifications. Finally, in section 4, we give the proof of Theorem 3.

In this last section we follow the ideas of the classical proof of the Yamabe problem for compact manifolds (cf. in [11]), and we take into account the non-compactness of the situation through the techniques of the Compactness Concentration Principle of Lions [12], [13].

Acknowledgment. The author would like to thank his supervisor J. Petean for many useful observations and valuable conversations on the subject.

2 Proof of Theorem 1

In this section we state some preliminary definitions and results we will need for the proof of Theorem 1. We begin by stating the definitions in \((N, g) = (M^m \times \mathbb{R}^n, g_M + g_E)\) of a Steiner symmetrization with respect to \(M\), and of a Polarization by a polarizer in \(\mathbb{R}^n\). We then prove some properties of Polarizations, such as the fact that Polarizations preserve the \(s\) norm, for any \(s \leq 1\),

\[||\nabla u^H||_p = ||\nabla u||_p,\]  

(5) (lemma 5). At the end of this section we give a proof of Theorem 1 by showing that we can approximate any Steiner symmetrization by constructing a carefully chosen sequence of Polarizations, and then by verifying that a less or equal than relation in (6) between the gradient of \(u^*\), and the gradient of \(u\), is preserved in the limit of the sequence. These results are a more or less direct adaptation to our case of the work of Brock and Solynin [7] and of Jean Van Schaftingen [17].
$U$ be a measurable set in $(N, g)$, we define its Steiner symmetrization $U^*$ as follows.

For each $x_0 \in M$, if

$$Vol(U \cap \{x_0\} \times \mathbb{R}^n), g_E) > 0,$$

then

$$(U^* \cap \{x_0\} \times \mathbb{R}^n)) = \begin{cases} \{x_0\} \times B_\rho(0), & \text{if } U \text{ is open,} \\ \{x_0\} \times \overline{B}_\rho(0), & \text{if } U \text{ is compact,} \end{cases}$$

(6)

where $B_\rho(0)$ is an open ball in $\mathbb{R}^n$, of radius $\rho > 0$, centered at the origin, and $\rho$ is such that

$$Vol(U \cap \{x_0\} \times \mathbb{R}^n), g_E) = Vol(B_0(\rho), g_E).$$

In particular, $\rho$ depends on $x_0$.

On the other hand, if $U$ is measurable but neither open nor compact, then the sets $U^* \cap \{x_0\} \times \mathbb{R}^n$ are defined in almost everywhere sense by either one of (6). Finally, if $Vol(U \cap \{x_0\} \times \mathbb{R}^n), g_E) = 0$, then $U^* \cap \{x_0\} \times \mathbb{R}^n$ is either empty or the point $(x_0, 0)$, according to whether $U \cap \{x_0\} \times \mathbb{R}^n$ is empty or not.

It is not hard to see that for any sets $A, B \subset N$,

$$A \subset B \Rightarrow A^* \subset B^*$$

(7)

and that for measurable subsets $A \subset B \subset N$,

$$Vol_\phi(B^* \setminus A^*) \leq Vol_\phi(B \setminus A).$$

(8)

We now define Steiner symmetrizations for functions. Consider the measurable functions $u : N \to \mathbb{R}$ for which

$$Vol\{x \in N|u(x) > c\} < +\infty,$$

$\forall c > 0 \inf u$ (in the following, we will denote $\{x \in N|u(x) > c\}$ by $\{u > c\}$). We will call $Sym$ this class of functions. We note that $L^s(N)$, $L_2^s(N)$ and $C_0(N)$ are subspaces of $Sym$. The Steiner symmetrization of a measurable function $u : N \to \mathbb{R}^+$ in $Sym$ is defined as follows. Let $y \in N$, then

$$u^*(y) = \sup\{c \in \mathbb{R}|y \in \{u > c\}^*\}.$$ 

It follows that for any $c \in \mathbb{R}$,

$$\{u > c\}^* = \{u^* > c\}.$$

(9)

One important property of Steiner symmetrizations is that they are non-expansive.

5
Lemma 4. Given $1 \leq s < \infty$, we have

$$||u^* - v^*||_s \leq ||u - v||_s$$ (10)

Proof. Recall that

$$\int_N |u - v|^s d\nu = \int_{\{\sigma \leq \tau\}} Vol (\{v > \tau\} \setminus \{\nu u > \sigma\})$$

$$+ Vol (\{u > \tau\} \setminus \{v > \sigma\}) s (s - 1) |\sigma - \tau|^{s-2} d\sigma d\tau.$$

The result of the lemma then follows from equations (8) and (9).

\[ \square \]

2.2 Polarizations

Let $\Sigma$ be some $(n - 1)$ dimensional affine hyperplane in $\mathbb{R}^n$. Consider $M^m \times \Sigma$ and assume that $H$ is one of the open spaces into which $N = M^m \times \mathbb{R}^n$ is subdivided by $M^m \times \Sigma$. We will call $H$ a polarizer, and denote its complement in $N$ by $H^c$. Let $\bar{x}$ denote the reflection in $M^m \times \Sigma$ with respect to $H$. That is, for $x = (a, b) \in M^m \times \mathbb{R}^n$, with $a \in M^m$ and $b \in \mathbb{R}^n$,

$$\bar{x} = (a, b^{\Sigma}),$$

where $b^{\Sigma}$ denotes the reflection of $b \in \mathbb{R}^n$, through the hyperplane $\Sigma \subset \mathbb{R}^n$, which defines $H$.

If $u$ is measurable, we define its polarization with respect to a polarizer $H$, $u^H$, by

$$u^H(x) = \begin{cases} 
\max\{u(x), u(\bar{x})\} & \text{if } x \in H, \\
\min\{u(x), u(\bar{x})\} & \text{if } x \in H^c.
\end{cases}$$ (11)

One useful property of polarizations is that the s-norms of the gradient of a function $u \in L^s_1(M \times \mathbb{R}^n)$, do not change under polarizations, as it is shown in the next lemma.

Lemma 5. Let $u \in L^s_1(N)$, $(1 \leq s \leq \infty)$, and let $H$ be some polarizer. Then $u_H \in L^s_1(N)$, and $|\nabla u|$ and $|\nabla u^H|$ are rearrangements of each other. In particular, we have

$$||\nabla u^H||_s = ||\nabla u||_s$$ (12)

Proof. For the sake of simplicity, we first define the reflection of $u(x)$, and the reflection of the polarization of $u$ by $H$. That is, let

$$v(x) := u(\bar{x}),$$

$$w(x) := u^H(\bar{x}),$$

for $x \in H$.

Next, we note that

$$u^H(x) = \max\{u(x), v(x)\} = v(x) + (u(x) - v(x))_+,$$
and that
\[ w(x) = \min\{u(x), v(x)\} = u(x) - (u(x) - v(x))_+ , \]
for all \( x \in H \). Hence, we conclude that \( u^H, w \in L^1(N) \), and that
\[
\nabla u^H(x) = \begin{cases} 
\nabla u(x) & \text{a.e. on } \{ x \in N : u(x) > v(x) \} \cap H, \\
\nabla u(x) & \text{a.e. on } \{ x \in N : u(x) \leq v(x) \} \cap H,
\end{cases}
\]
\[
\nabla w(x) = \begin{cases} 
\nabla v(x) & \text{a.e. on } \{ x \in N : u(x) > v(x) \} \cap H, \\
\nabla u(x) & \text{a.e. on } \{ x \in N : u(x) \leq v(x) \} \cap H.
\end{cases}
\]

Now, to prove the assertions of the lemma, we define the following regions on \( N \),
\[
R_1 = \{ x \in N : u(x) > v(x) \} \cap H, \\
R_2 = \{ x \in N : u(x) \leq v(x) \} \cap H, \\
R_3 = \{ x \in N : u(x) > v(x) \} \cap H^c, \\
R_4 = \{ x \in N : u(x) \leq v(x) \} \cap H^c,
\]
and we observe that \( u^H = u \) in \( R_1 \) and \( R_4 \). Thus, we have
\[
\int_{R_1 \cup R_4} |\nabla u^H|^s dV_g = \int_{R_1 \cup R_4} |\nabla u|^s dV_g.
\]
We also note that \( u^H = v \) in \( R_2 \) and \( R_3 \), i.e., \( \int_{R_2} |\nabla u^H|^s dV_g = \int_{R_2} |\nabla u|^s dV_g \) and \( \int_{R_3} |\nabla u^H|^s dV_g = \int_{R_3} |\nabla u|^s dV_g \). And so, the assertion follows:
\[
\int_N |\nabla u^H|^s dV_g = \int_{R_1 \cup R_4} |\nabla u^H|^s dV_g + \int_{R_2} |\nabla u^H|^s dV_g + \int_{R_3} |\nabla u^H|^s dV_g
= \int_{R_1 \cup R_4} |\nabla u|^s dV_g + \int_{R_2} |\nabla u|^s dV_g + \int_{R_3} |\nabla u|^s dV_g = \int_N |\nabla u|^s dV_g.
\]

\[ \square \]

**Remark 6.** By following the scheme of the proof of Lemma 2, we may also note that \( ||u||_s = ||u^H||_s \), for any \( 1 \leq s \leq \infty \).

**Remark 7.** Polarizations are non-expansive (for \( u, v \in L^s(N) \), \( 1 \leq s \leq \infty \), \( ||u^H - v^H||_s \leq ||u - v||_s \)).

### 2.3 Approximation of Steiner symmetrizations by Polarizations

We will now show that any Steiner symmetrization \( u^* \) of a function \( u \), can be approximated by a sequence of polarizations of \( u \), \( \{u^H\} \). To do so, we will first show that sequences of iterated polarizations \( \{u^H\} \) are sequentially compact. Then, we will construct a sequence of polarizations, and establish some conditions for the convergence of the sequence to the Steiner symmetrization of the function.
We begin this section by joining together the concepts of Steiner symmetrizations we defined earlier, with the concepts of polarizations, to define a special set of halfspaces in $N = M \times \mathbb{R}^n$. Let $\Sigma$ be a halfspace of $\mathbb{R}^n$, we will denote by $H$ the set of all halfspaces $H$ of $N$ of the form $M \times \Sigma$, and by $H_0$ the set of all halfspaces $H \in H$, such that $M \times \{0\} \subset H$.

**Remark 8.** It follows from the definition of a polarization, from the definition of $H_0$, and from the symmetry of the Steiner symmetrization, that $(u^*)_H = u^*$, for any polarizer $H \in H_0$.

Another fact that makes $H_0$ a special set of halfspaces, is that there is always some polarizer $H \in H_0$, such that $u_H$ is strictly closer to $u^*$ than $u$.

**Lemma 9.** Let $u \in C_{0+}(N)$. If $u \neq u^*$, then there is some polarizer $H \in H_0$, such that for each $1 \leq s \leq \infty$,

$$||u^H - u^*||_s < ||u - u^*||_s,$$

for $1 \leq s \leq \infty$.

**Proof.** Since $u \neq u^*$, then there is some $c > 0$, such that $\{x \in N : u(x) > c\} \Delta \{x \in N : u^*(x) > c\} \neq \emptyset$. So, we choose some $y \in \{x \in N : u^*(x) > c\} \setminus \{x \in N : u(x) > c\}$.

There is a polarizer $H \in H^*$, such that $y^H \in \{x \in N : u(x) > c\} \setminus \{x \in N : u^*(x) > c\}$.

We now choose a sufficiently small neighborhood $W_0 \subset H$ of $y$, so that $W_0^H \subset \{x \in N : u(x) > c\} \setminus \{x \in N : u^*(x) > c\}$. We then have

$$u^H(x) = u(\bar{x}) > c \geq u^*(\bar{x})$$

and

$$u^*(x) > c \geq u(x) = u^H(\bar{x}),$$

and so, for $s \geq 1$,

$$|u(x) - u^*(x)|^s + |u(\bar{x}) - u^*(\bar{x})|^s > |u^H(x) - u^*(x)|^s + |u^H(x) - u^*(\bar{x})|^s. \quad (15)$$

If $x \in W_0$, the corresponding inequality is non-strict. The integral inequality is obtained by integration of (15) over $W_0$ and of the nonstrict inequality over $H \setminus W_0$.

We now prove that for a sequence of polarizations $u_m = u^{H_1, H_2, \ldots, H_m}$, it suffices that the polarizers satisfy $\{H_i\}_{i \leq m} \subset H_0$, for the existence of a function $f$, such that a subsequence of $\{u_m\}$ converges to $f$.

**Lemma 10.** Let $u \in C_{0+}(N)$. Let $\{u_m\}$ be a sequence of polarizations of $u$, with its respective sequence of polarizers $\{H_m\} \subset H_0$, ($u_m = u^{H_1, \ldots, H_m}$). Then there is a function $f \in C_{0+}(N)$, and an increasing subsequence $\{u_{m_k}\}$ of $\{u_m\}$, such that, for each $s$, $1 \leq s \leq \infty$, we have...
\[ \lim_{k \to \infty} \| f - u_{m_k} \|_s = 0. \]

**Proof.** This lemma follows from an application of the theorem of **Arzela-Ascoli** (cf. [16]). That is, to conclude that the sequence \( \{ u_m \} \) is compact, we need to prove that \( \{ u_m \} \) is equibounded, equicontinuous and that the supports are uniformly bounded.

1. Since \( \| u \|_s = \| u^H \|_s \), for any polarizer \( H \subset H_0 \) (remark[6]), it follows that \( \| u \|_s = \| u_m \|_s \) for \( m = 1, 2, \ldots \). Thus, the functions \( u_m \) are equibounded for all \( m \).

2. Let

\[ w_u(\delta) = \sup \{ u(x) - u(y) | d(x, y) \leq \delta \}, \]

be the modulus of continuity of a function \( u \). Let \( H \subset H_0 \) be any polarization. We proceed to analyze the different cases.

Let \( \delta > 0 \), and consider any ball \( B_\delta(p) \) in the domain of \( u \), such that

\[ w_u(\delta) = \sup_{B_\delta(p)} \{ u(x) - u(y) \}. \]

If either \( B_\delta(p) \subset H \) or \( B_\delta(p) \subset H^c \), we then have

\[ \sup \{ u^H(x) - u^H(y) | d(x, y) \leq \delta \} = \sup \{ u(x) - u(y) | d(x, y) \leq \delta \}. \]

If, on the other hand, \( B_\delta(p) \cap H \neq \phi \) and \( B_\delta(p) \cap H^c \neq \phi \), then we consider that

\[ \sup_{B_\delta(p)} \{ u^H(x) - u^H(y) \} = \sup_{B_\delta(p)} \cap (R_1 \cup R_4) \{ u(x) - u(y) \} \leq \sup_{B_\delta(p)} \{ u(x) - u(y) \}, \]

since \( (B_\delta(p) \cap (R_1 \cup R_4)) \subset (B_\delta(p)) \).

And so, we have that \( w_u(\delta) \leq w_u(\delta) \). Which yields, by induction, \( w_u \leq w_u \). Finally, since \( u \in C_0(N) \), \( u \) is uniformly continuous, and then the sequence \( \{ u_m \} \) is equicontinuous.

3. The fact that the supports are equibounded follows from the fact that polarizations are monotone: since \( u \in C_0(N) \), there is some \( R > 0 \), and some \( p \in N \), such that \( \text{Supp } u \subseteq B_R(p) \), and

\[ \text{Supp } u \subseteq B_R(p) \Rightarrow \text{Supp } u^H \subseteq B_R(p)^H = B_R(p), \]

since polarizations are monotone.

And then, by induction, \( \text{Supp } u_m \subseteq B_R(p) \).

We conclude by the **Arzela-Ascoli** theorem that there is some \( f \in C_{0+}(N) \), such that there is some subsequence \( \{ u_{m_k} \} \) of \( \{ u_m \} \), and that \( u_{m_k} \to f \). \( \square \)

We now construct a sequence of polarizations of \( u \in C_{0+}(N) \) that will converge to \( u^* \). We proceed inductively. As expected, we start with \( u_0 = u \). Then, to choose \( H_{m+1} \in H_0 \), so that \( u_{m+1} = u_{m+1}^{H_{m+1}} \), we look at

9
\[ \alpha_m = \sup_{H \in H_0} \{ \|u_m - u^*\|_1 - \|u^H_m - u^*\|_1 \}. \]

By lemma 9 we know that \( \alpha_m \) is always strictly positive. Now, for some fixed \( \kappa \) (0 < \( \kappa < 1 \)), taking \( \epsilon < \alpha_m (1 - \kappa) \) we note that we can always choose \( H_{m+1} \in H_0 \) so that,

\[ 0 < \alpha_m < \|u_m - u^*\|_1 - \|u^H_{m+1} - u^*\|_1 + \epsilon < \|u_m - u^*\|_1 - \|u^H_{m+1} - u^*\|_1 + \alpha_m (1 - \kappa). \]

Then, it follows that

\[ \kappa \sup_{H \in H_0} \{ \|u_m - u^*\|_1 - \|u^H_m - u^*\|_1 \} < \|u_m - u^*\|_1 - \|u^H_{m+1} - u^*\|_1. \] (16)

Next, we prove that the sequence of polarizations we have just constructed converges to \( u^* \).

**Lemma 11.** Let \( u \in C_0^+(N) \). Let \( \{u_m\} \) be a sequence of iterated polarizations of \( u \), with corresponding halfspaces \( \{H_m\} \subset H_0 \) (\( u_m = u^H_1 u^H_2 ... u^H_m \)), and suppose that the \( H_m \)'s are chosen so that equation (16) is satisfied. Then \( u_m \to u^* \) in any s-norm (\( 1 \leq s \leq \infty \)).

**Proof.** It follows by lemma 10 that there is some \( f \in C_0(N) \), and some subsequence \( \{u_{m_k}\} \) of \( \{u_m\} \), such that \( \{u_{m_k}\} \) converges to \( f \), for any \( L^p \) norm. Now, by the lower semi-continuity of the norm,

\[ \|u^* - f^*\|_1 = \lim_{k \to \infty} \|u^*_{m_k} - f^*\|_1, \]

and since the Steiner symmetrization is a non-expansive rearrangement, we have

\[ \|u^*_{m_k} - f^*\|_1 \leq \|u_{m_k} - f\|_1. \]

It follows that

\[ \|u^* - f^*\|_1 = \lim_{k \to \infty} \|u^*_{m_k} - f^*\|_1 \leq \lim_{k \to \infty} \|u_{m_k} - f\|_1 = 0, \]

that is, \( f^* = u^* \). Now, polarizations are also non-expansive, then, since \( m_{k+1} \geq m_k + 1 \), we have that

\[ \|u^*_{m_k+1} - u^*\|_1 \leq \|u_{m_k+1} - u^*\|_1, \]

on the other hand, by equation (16), for any polarizer \( H \in H_0 \) we have,

\[ \|u_{m_k+1} - u^*\|_1 \leq \|u_{m_k} - u^*\|_1 + \kappa(\|u^H_{m_k} - u^*\|_1 - \|u_{m_k} - u^*\|_1) \]

\[ = (1 - \kappa)\|u_{m_k} - u^*\|_1 + \kappa\|u^H_{m_k} - u^*\|_1 \leq \|u_{m_k} - u^*\|_1, \]
since $\kappa ||u^{H}_{m_k} - u^*||_1 - ||u_{m_k} - u^*||_1 \leq 0$.

Hence, making $m_k \to \infty$, we get,

$$||f - u^*||_1 \leq (1 - \kappa)||f - u^*||_1 + \kappa||f^H - u^*||_1 \leq ||f - u^*||_1,$$

that is

$$||f - u^*||_1 = ||f^H - u^*||_1. \quad (17)$$

Now, since $f^* = u^*$, then $||f - f^*||_1 = ||f^H - f^*||_1$.

So, we cannot have $f \neq u^* = f^*$, because then we would have $||f - f^*||_1 > ||f^H - f^*||_1$, for some $H_o$ by lemma 9 which would contradict equation (17).

Then, we can only have that $\{u_{m_k}\}$ converges to $u^*$ for any $L^s$ norm.

Finally, again by the non-expansiveness of polarizations, we note that, for any $s$,

$$\lim_{k\to\infty} ||u_k - u^*||_s \leq \lim_{k\to\infty} ||u_{m_k} - u^*||_s = 0,$$

as desired.

Finally, because $C_0^+(N)$ is dense in $L^s_+(N)$ ($1 \leq s \leq \infty$), we show that the same results of lemma 11 hold for functions in $L^s_+(N)$.

**Lemma 12.** Let $u \in L^s(N)$ ($1 \leq s < \infty$). For any steiner symmetrization, there is a sequence of polarizers $\{H_m\} \subset H_0$, such that the sequence $\{u_{m}\} = \{u^{H_1 \cdots H_m}\}$, converges to $u^*$ in $L^s(N)$.

**Proof.** First, we recall that there is a countable subset $V \subset C_0(N)$ that is dense in $L^s(N)$. Next, we choose a sequence $\{H_m\}$, for which (16) holds for all $f \in V$. Then, we take any $f \in V$, sufficiently close to $u$, $||u - f||_s < \epsilon/3$. By contraction we have,

$$||u_{m} - u^*||_s \leq ||u_{m} - f_m||_s + ||f_m - f^*||_s + ||f^* - u^*||_s.$$ 

It remains to show that the right hand side is bounded by $\epsilon$.

First, by non-expansiveness of the polarization, we have that $||u_{m} - f_m||_s \leq ||u - f||_s$. Second, by non-expansiveness of the Steiner symmetrization we have $||f^* - u^*||_s \leq ||f - u||_s$. Then, since $f \in V \subset C_0^+(N)$, choosing $m$ sufficiently large, we have $||f_m - f^*||_s < \epsilon/3$, and then

$$||u_{m} - u^*||_s \leq ||u_{m} - f_m||_s + ||f_m - f^*||_s + ||f^* - u^*||_s < \epsilon,$$

as desired.

**2.4 Proof of Theorem 1**

We are now in position to prove Theorem 1 and conclude that $||\nabla u^*||_s \leq ||\nabla u||_s$.
Proof. (of Theorem 1)

Let $u \in L^2_1(N)$, and consider the sequence $\{u_m\}$ of polarizations of $u$, given by lemma 12. Then, for $1 < s < \infty$,

$$\lim_{m \to \infty} ||u_m - u^*||_s = 0.$$ 

Also, $||\nabla u^H||_s = ||\nabla u||_s$, by lemma 5. Then there exists some function $f \in L^s_1(N)$, and a subsequence $\{u_{m_k}\}$ of $\{u_m\}$, such that $f$ is the weak limit of $u_{m_k}$ in $L^s_1(N)$.

That is, for any compactly supported function $\varphi \in C_0(N)$,

$$\lim_{k \to \infty} \int_N \varphi \text{ div } u_{m_k} dV_g = \int_N \varphi \text{ div } f dV_g,$$

and

$$\lim_{k \to \infty} \int_N \varphi \text{ div } u_{m_k} dV_g = - \lim_{k \to \infty} \int_N \text{ div } \varphi \text{ div } u_{m_k} dV_g = - \int_N \text{ div } \varphi \text{ div } u^* dV_g.$$

Of course, this means that $v = u^*$. Finally, we recall that for $1 < s < \infty$ the $s$-norm is weakly lower semicontinuous, that is, since $u_{m_k} \to u^*$ weakly in $L^s(N)$, then

$$||\nabla u^*||_s \leq \liminf_{k \to \infty} ||\nabla u_{m_k}||_s,$$

hence

$$||\nabla u^*||_s \leq ||\nabla u||_s,$$

since $||\nabla u^H||_s = ||\nabla u||_s$ for any $H$ (lemma 5).

\[ \square \]

3 Proof of Theorem 3

Let $(N, g) = (M^m \times \mathbb{R}^n, g_M + g_E)$, where $M^m$ is a closed manifold ($m \geq 2$) with positive scalar curvature, and $g_E$ is the Euclidean metric. In this section we will prove the existence of a Yamabe minimizer for $(N, g)$. The basic scheme of the proof we give is the following. We first note that the subcritical Yamabe equation for $(N, g)$,

$$a \Delta u + S_g u^2 = \lambda_s u^s,$$

where $S_g$ is the scalar curvature of $(N, g)$ and $a = \frac{4(n+m-1)}{n+m-2}$, can be solved for $s < p = \frac{2(n+m)}{n+m-2}$ by a positive $C^\infty$ function $u_s$. We achieve this by making use of the techniques of the Yamabe problem in the compact case (cf. in [11]), and those of the Concentration Compactness Principle of Lions, ([12], [13]). We then find a uniform bound in $L^r(N, g)$ (for some $r > p$) for the family of solution functions $\{u_s\}$, for $s$ sufficiently close to $p$. Then, using standard
regularity theory and the Sobolev Embedding Theorem, we note that the \( \{u_s\} \) are \( C^{2,\alpha} \) bounded in every compact subset \( K_R = M \times B_R \) of \((N, g)\), and thus that \( u_s \rightarrow u \) uniformly on every compact subset \( K_R \) of \( N \), by the Arzela-Ascoli Theorem. As a final step, we use again the techniques of the Concentration Compactness Principle to prove that \( u_s \rightarrow u \) uniformly on all of \( N \), where \( u \) is a positive and \( C^\infty \) function that solves the Yamabe equation.

### 3.1 The subcritical problem for \((N, g)\)

In this section we will prove that the equation

\[
a \Delta u + S_g u^2 = \lambda_s u^s,
\]

has a positive smooth solution, \( u_s \), for \( s < p \) and \( s \) sufficiently close to \( p \).

Let

\[
Q_s(\phi) = \frac{\int_{N} (a|\nabla \phi|^2 + S_g \phi^2)dV_g}{(\int_{N} \phi^s dV_g)^{2/s}},
\]

and

\[
\lambda_s = \inf \{Q_s(\phi)|\phi \in C_0^\infty(N, g)\}.
\]

Now, fix \( s < p \), and choose a minimizing sequence \( \{u_i\} \) of functions in \( C_0^\infty(N) \), such that \( Q_s(u_i) \rightarrow \lambda_s \), and such that \( ||u_i||_{s} = 1, \forall i \). We remark that, by Theorem 1, we can choose a minimizing sequence such that \( u_i = u_i^* \).

Next, we note that

\[
||u_i||_{1,2} \leq C_1,
\]

where \( C_1 \) is some constant, independent of \( i \) and \( s \). To prove (22), we start with the following.

**Lemma 13.** Consider the set \( \{\lambda_s\}, 2 \leq s \leq p \), with \( \lambda_s \) as defined by equation (21). Then, \( \lambda_s \) is upper semi-continuous at \( p \), as a function of \( s \) (for any \( \epsilon > 0 \), there is some \( \delta \) such that \( \lambda_s \leq \lambda_p + \epsilon , \forall s \in (p - \delta, p) \)).

**Proof.** Let \( \phi \in L_2^p(N) \). Given \( s' \), \( s \leq p \), since

\[
Q_s(\phi) = a \int_{N} |\nabla \phi|^2 dV_g + \int_{M} S_g \phi^2 dV_g
\]

then

\[
Q_s(\phi) = Q_{s'}(\phi) \|\phi\|_2^2.
\]

Now, since \( \lambda_p \) is an infimum, given \( \epsilon > 0 \) we may choose \( \phi_0 \) such that

\[
\lambda_p + \epsilon > Q_p(\phi_0).
\]

On the other hand, by continuity of the norm, we have, for some \( \delta > 0 \),
\[ 1 - \epsilon \leq \frac{||\varphi_0||^2}{||\varphi_0||^2} \leq 1 + \epsilon, \]

for all \( s \in (p - \delta, p + \delta) \). Hence,

\[ \frac{||\varphi_0||^2}{||\varphi_0||^2} Q_p(\varphi_0) \leq Q_p(\varphi_0)(1 + \epsilon), \]

for all \( s \in (p - \delta, p) \). Then, taking into account equation (23), we have

\[ Q_s(\varphi_0) \leq Q_p(\varphi_0)(1 + \epsilon), \]

and then, by (24)

\[ Q_s(\varphi_0) \leq Q_p(\varphi_0)(1 + \epsilon) \leq (\lambda_p + \epsilon)(1 + \epsilon). \]

Finally, since \( \lambda_s < Q_s(\varphi_0) \), we have

\[ \lambda_s < \lambda_p + C\epsilon + \epsilon^2. \]

for all \( s \in (p - \delta, p + \delta) \), with \( C = \lambda_p + 1 \).

\[ \square \]

**Remark 14.** It is a recent result of Akutagawa, Florit and Petean (Theorem 1.3 in [1]) that \( \lambda_p = Y(M \times \mathbb{R}^n, g_M + g_E) < Y(S^{n+m}, g_0) = Y_{n+m} \), when \( M \) is closed, of positive scalar curvature and \( m \geq 2 \). Since the inequality is strict, we may choose \( \epsilon > 0 \) small enough so that \( \lambda_p + \epsilon < c < Y_{n+m} \), for some \( c \in \mathbb{R} \). It then follows from lemma (13), that for some \( \epsilon > 0 \) small enough, there is some \( \delta \), such that

\[ \lambda_s < \lambda_p + \epsilon < Y_{n+m}, \]

for every \( s \in (p - \delta, p) \). That is

\[ \frac{\lambda_s}{Y_{n+m}} < 1, \quad (25) \]

for \( s \) close enough to \( p \).

We now go back to prove (22). We note that

\[ ||u_i||_{1,2} = \int_N |\nabla u_i|^2 + \int_N u_i^2 \leq \frac{\lambda_s + 1}{a} + \frac{\lambda_s + 1}{\min_M \{S_g\}} \leq \frac{Y_{n+m} + 1}{a} + \frac{Y_{n+m} + 1}{\min_M \{S_g\}} = C_1, \]

for \( s \) close enough to \( p \), by (26). That is \( \{u_i\} \) is \( L^2 \) bounded independently of \( i \) and of \( s \).

It then follows from the Rellich-Kondrakov theorem (cf. in [1]) that for every compact \( K \subset N \), there is some subsequence \( \{u_{i_k}\} \subset \{u_i\} \) that converges
weakly in $L^2(K)$ and strongly in $L^s(K)$ to a function that we will denote by $u_s|_K$.

Consider now the compact subsets $K_R = M \times B_R \subset N$, and note that since $K_R \subset K_{R'}$, for $R < R'$, then we have uniqueness of limits on each compact (because the convergence on $L^s(K_R)$ is strong for each $R$). Also, note that $N = \bigcup_{i}^\infty K_i$. Then, we have our limit function $u_s$, as a well defined function on all of $N$ by taking $u_s = \lim_{R \to \infty} u_s|_{K_R}$.

Furthermore, on each $K_R$, by the weak convergence on $L^2_1(K)$, we have

$$||\nabla u_s|_{K_R}||_2^2 \leq \lim_{k \to \infty} \int_{K_R} \langle \nabla u_s|_{K_R}, \nabla u_{i_k}\rangle dV_g,$$

and this implies that

$$||\nabla u_s|_{K_R}||_2^2 \leq \limsup_{k \to \infty} ||(\nabla u_{i_k})|_{K_R}||_2^2.$$

On the other hand, by the strong convergence of $u_{i_k}$ to $u_s|_{K_R}$ in $L^s(K_R)$, and by Hölder’s inequality, we have

$$\int_{K_R} u_{i_k}^2 dV_g = \lim_{k \to \infty} \int_{K_R} u_{i_k}^2 dV_g,$$

and so, it follows that

$$\int_{K_R} (a|\nabla u_s|_{K_R}|^2 + S_g u_s|_{K_R}^2) dV_g \leq \limsup_{k \to \infty} \int_{K_R} (a|\nabla u_{i_k}|^2 + S_g u_{i_k}^2) dV_g. \quad (26)$$

Hence, to prove that $u_s$ in fact minimizes $Q_s$ on $N$, it remains to show that $||u_s||_a = 1$. To this purpose, we introduce in the following lemmas the techniques of the Concentration Compactness Principle, due to Lions [12, 13].

**Lemma 15.** Consider a sequence $\{\rho_k\}$ of $C^\infty$, non-negative functions, such that $\rho_k = \rho_k^*$, and

$$\int_{N} \rho_k dV_g = 1.$$  

Then, there exists a subsequence $\{\rho_{k_j}\} \subset \{\rho_k\}$, and some $\alpha$ ($0 \leq \alpha \leq 1$), such that the following is satisfied: for all $\epsilon > 0$, there exists some $R_\epsilon$ ($0 < R_\epsilon < \infty$), and some $j_0 > 1$ such that

$$\int_{M \times B_{R_\epsilon}} \rho_{k_j} dV_g \geq \alpha - \epsilon,$$

$\forall j > j_0$.

Furthermore, for each $R > 0$, given $\epsilon > 0$, there is some $j_1 > 1$ such that

$$\int_{M \times B_R} \rho_{k_j} dV_g \leq \alpha + \epsilon,$$

$\forall j > j_1$.  

15
Proof. First note that since $\rho_k = \rho_k^*$ for each $k > 1$, then, for each $R$ we have

$$\sup_{y \in \mathbb{R}^n} \int_{M \times \{y + B_R\}} \rho_k dV_g = \int_{M \times B_R} \rho_k dV_g,$$

where $B_R$ is the ball of radius $R$ centered at 0, and $y + B_R$ the ball of radius $R$ centered at $y$. Now, consider the functions

$$Q_k(t) = \int_{M \times B_t} \rho_k dV_g.$$ 

It follows that for each $k$, $0 \leq Q_k(t) \leq 1$. Thus, the functions $Q_k(t)$ are non-negative and uniformly bounded in $\mathbb{R}^+$. Furthermore, since the $\rho_k$ are non-negative, the functions $Q_k(t)$ are non-decreasing as functions of $t$.

It follows then, from the Heine-Borel theorem, that there is a subsequence ${Q_{k_j}} \subset {Q_k}$, and a non-negative function $Q(t)$, such that

$$\lim_{j \to \infty} Q_{k_j} = Q(t),$$

for each $t \geq 0$.

Now, let $\lim_{t \to \infty} Q(t) = \alpha$. We note that, $0 \leq \alpha \leq 1$. Also, since $Q(t)$ is non-decreasing, and $\lim_{t \to \infty} Q(t) = \alpha$, then, given $\epsilon > 0$, we may choose some $t_\epsilon$ such that $Q(t_\epsilon) > \alpha - \epsilon$. Of course this implies that

$$\int_{M \times B_{t_\epsilon}} \rho_k dV_g \geq \alpha - \epsilon,$$  \hspace{1cm} (27)$$

for all $j > j_0$, for $j_0$ large enough. Moreover, since $Q(t)$ is non-decreasing, for all $t > 0$ we have $Q(t) \leq \alpha$. This implies that

$$\int_{M \times B_t} \rho_k dV_g \leq \alpha + \epsilon,$$  \hspace{1cm} (28)$$

for all $j > j_1$, for $j_1$ large enough.

We now show that given $\beta \in (2, p)$ the $u_k^\beta$ “concentrate” in a compact set.

**Lemma 16.** Consider a sequence $\{u_k^b\}$ of $C^\infty$, non-negative functions ($b_k > 2, \forall k$), such that $u_k = u_k^b$, and

$$\int_N u_k^b dV_g = 1,$$

for each $k$. Assume also that the sequence $\{u_k\}$ is bounded in $L_2^1(N)$.

Then, there exists a subsequence $\{u_{k_j}\} \subset \{u_k\}$, such that for each $\beta$ ($\beta \in (2, p)$), we have that given $\epsilon > 0$, there exists some $R_\epsilon$ ($0 < R_\epsilon < \infty$), such that

$$\int_{N \setminus (M \times B_{R_\epsilon})} u_{k_j}^\beta dV_g \leq \epsilon,$$

for all $j > j_0$, for some $j_0 > 1$.  

16
Proof. Take $\rho_k = u_k^{b_k}$. Then, by Lemma 15 we have a subsequence $\{u_{k_j}^{b_j}\}$ of $\{u_k^b\}$, and an $\alpha$, $0 \leq \alpha \leq 1$, such that, for every $\epsilon/2 > 0$, there is some $R_{\epsilon/2}$, such that
\[
\alpha - \epsilon/2 < \int_{M \times B_{R_{\epsilon/2}}} u_j^{b_j} dV_g < \alpha + \epsilon/2,
\] (29)
for all $j > j_0$, for some $j_0$ (for simplicity, we will denote $u_{k_j}^{b_j}$ by $u_j^{b_j}$).

Also, since for every $R > 0$ we have $\int_{M \times B_R} \rho_k dV_g \leq \alpha + \epsilon/2$ (for $j > j_1$, $j_1$ large enough), then, it follows from (29) that for every for every compact $K$, $K \subset N \setminus M \times B_R$, we have
\[
\int_K u_j^{b_j} dV_g < \epsilon,
\] for all $j > j_1$.

Now, we choose $R_0 > 0$ such that $\text{Vol}(M \times B_{R_0}) \leq 1$. Then, by H"older’s inequality, for any $y \in B_{R_0}$,
\[
\int_{M \times \{y + B_{R_0}\}} u_j^2 dV_g \leq \int_{M \times \{y + B_{R_0}\}} u_j^{b_j} dV_g < \epsilon.
\]
Now, let $R_1 = R_0 + 2R_0$, then,
\[
\sup_{y \in B_{R_1}} \int_{M \times \{y + B_{R_0}\}} u_j^2 dV_g < \epsilon. \tag{30}
\]
Of course, we can make $\epsilon \to 0$ by making $R_1$ (and thus $R_0$) go to infinity.

We next divide the proof in cases.

**Case 1.** The sequence $\{u_k\}$ is bounded in $L^\infty(N)$.

Let $\|u_k\|_\infty < A_\infty$. Also, since $u_k$ is bounded in $L^2(N)$, let $A_{1,2}$ ($1 < A_{1,2} < \infty$) be such that $\|u_k\|_{1,2} < A_{1,2}$. Then, we have, for all $\beta > 0$, given any $y \in B_{R_1}$,
\[
\int_{M \times \{y + B_{R_0}\}} u_j^{\beta_0} dV_g \leq A_{\infty}^{\beta_0 - 2} \int_{M \times \{y + B_{R_0}\}} u_j^2 dV_g < A_{\infty}^{\beta_0 - 2} \epsilon. \tag{31}
\]
by (30). Now, take $\tilde{\beta}$, such that $\tilde{\beta} > 2$. Of course, $2 < 2(\tilde{\beta} - 1) < \infty$. Then, by H"older’s inequality, for any given $y \in B_{R_1}$,
\[
\int_{M \times \{y + B_{R_0}\}} |u_j|^{|\tilde{\beta} - 1|} |\nabla u| dV_g
\] 
\[
\leq \left( \int_{M \times \{y + B_{R_0}\}} |u_j|^{2(\tilde{\beta} - 1)} dV_g \right)^{\frac{1}{2}} \left( \int_{M \times \{y + B_{R_0}\}} |\nabla u|^2 dV_g \right)^{\frac{1}{2}} \leq (A_{\infty}^{\beta_0 - 2})^{1/2} (A_{1,2}), \tag{32}
\]
where the last inequality follows from (31) and the fact that \( \int_N |\nabla u|^2 dV_g \) is uniformly bounded by \( A_{1,2} \). Then, by the Sobolev imbedding, for any \( \gamma \in (1, \frac{m}{m+n-1}) \), there is a constant \( c_0 \), independent of \( y \), such that
\[
\left( \int_{M \times \{ y + B_{R_0} \}} u_j^{\beta \gamma} dV_g \right)^{1/\gamma} \leq c_0 \int_{M \times \{ y + B_{R_0} \}} u_j^\beta + |\nabla u_j|^{\beta} dV_g
\]
\[
\leq c_0 \int_{M \times \{ y + B_{R_0} \}} u_j^\beta + \beta(u_j)^{\beta - 1} |\nabla u_j| dV_g,
\]
for any \( y \in B_{R_0}^c \). That is,
\[
\int_{M \times \{ y + B_{R_0} \}} u_j^{\beta \gamma} dV_g \leq C_0 \left( \int_{M \times \{ y + B_{R_0} \}} u_j^\beta + \beta(u_j)^{\beta - 1} |\nabla u_j| dV_g \right)^\gamma
\]
\[
\leq C_0 \left( A_{\infty}^{\beta - 2} + \beta A_{1,2} (A_{\infty}^{\beta - 2} \epsilon^{1/2}) \right)^{\gamma - 1} \left( \int_{M \times \{ y + B_{R_0} \}} u_j^\beta + \beta(u_j)^{\beta - 1} |\nabla u_j| dV_g \right)
\]
\[
\leq C_1 \epsilon^{(\gamma - 1)/2} \left( \int_{M \times \{ y + B_{R_0} \}} u_j^\beta + \beta(u_j)^{\beta - 1} |\nabla u_j| dV_g \right),
\]
with \( C_1 = C_0 A_{\infty}^{\beta - 2} (\beta A_{1,2})^{\gamma - 1} \), and \( C_0 = c_0^2 \).

We then cover \( \mathbb{R}^n \setminus B_{R_1} \) with balls of radius \( R_0 \) in some way that any point \( y \in (\mathbb{R}^n \setminus B_{R_1}) \) is not covered by more than \( m \) balls (\( m \) a prescribed integer).

It follows that
\[
\int_{N \setminus M \times B_{R_1}} u_j^{\beta \gamma} dV_g \leq m C_1 \epsilon^{(\gamma - 1)/2} \int_{N \setminus (M \times B_{R_1})} u_j^\beta + \beta(u_j)^{\beta - 1} |\nabla u_j| dV_g
\]
\[
\leq m C_1 \epsilon^{(\gamma - 1)/2} (A_{\infty}^{\beta - 2} A_{1,2} + \beta A_{1,2} A_{\infty}^{\beta - 2} A_{1,2}^{1/2})
\]
\[
\leq (m C_0 p A_{1,2}^2 A_{\infty}^2) \epsilon^{(\gamma - 1)/2}
\]
\[
\leq C_2 \epsilon^{(\gamma - 1)/2}.
\]

Finally, by noting that \( C_2 \) does not depend on \( y \in \mathbb{R}^n \), we can make \( \epsilon \to 0 \), by making \( R_\epsilon \) (and thus \( R_1 \)) go to infinity. That is, given \( \beta \in (2, p) \), for every \( \delta > 0 \) we may find \( R_\delta \) such that
\[
\int_{N \setminus M \times B_{R_\delta}} u_j^\beta dV_g < \delta.
\]

This finishes the proof of case 1. We now remove the assumption that \( u_j \) is bounded in \( L^\infty(N) \).

**Case 2.** The sequence \( \{ u_j \} \) is not bounded in \( L^\infty(N) \).
We note that for any $A > 1$, the function $v_j = \min\{u_j, A\}$ is bounded in $L^\infty(N)$, and still satisfies the conditions needed for the previous proof, so that for any $2 < \beta_1 < p$, given $\epsilon > 0$, we have, by equation (33), some $R_1 > 0$ such that
\[
\int_{N \setminus M \times B_{R_1}} v_j^{\beta_1} < C_3 A^2 \epsilon. \tag{34}
\]
where $C_3$ is a constant that does not depend on $A$. We also have
\[
\int_{N \setminus M \times B_{R_1}} u_j^{\beta_1} dV_g \leq \int_{N \setminus M \times B_{R_1}} v_j^{\beta_1} dV_g + \int_{N \setminus M \times B_{R_1}} (u_j|\{u_j > A\})^{\beta_1} dV_g. \tag{35}
\]
We next choose $\beta_2 \in (2, p)$, such that $\beta_2 > \beta_1$. And then,
\[
A^{\beta_2 - \beta_1} \int_{N \setminus M \times B_{R_1}} (u_j|\{u_j > A\})^{\beta_2} dV_g \leq \int_{N \setminus M \times B_{R_1}} (u_j|\{u_j > A\})^{\beta_1} dV_g,
\]
it follows that
\[
\int_{N \setminus M \times B_{R_1}} (u_j|\{u_j > A\})^{\beta_1} dV_g \leq \frac{K}{A^{\beta_2 - \beta_1}}, \tag{36}
\]
since $\int_N u_j^{\beta_1} dV_g < K$, for some $K > 0$, because $\beta_1 < p$. Hence, from (34), (35) and (36), we have
\[
\int_{N \setminus M \times B_{R_1}} u_j^{\beta_1} < C_3 A^2 \epsilon + \frac{K}{A^{\beta_2 - \beta_1}}. \tag{37}
\]
Then, given $\delta > 0$, we may first choose $A$ such that $\frac{K}{A^{\beta_2 - \beta_1}} < \frac{\delta}{2}$, and then we choose $\epsilon > 0$, such that $C_3 A^2 \epsilon < \frac{\delta}{2}$. Of course, for this $\epsilon$ there is some $R_1$ such that $\int_{N \setminus M \times B_{R_1}} u_j^{\beta_1} < C_3 A^2 \epsilon$, and then by equation (37) we have
\[
\int_{N \setminus M \times B_{R_1}} u_j^{\beta_1} < \delta. \tag{38}
\]
The conclusion of the lemma follows.

We now go back to prove that $||u_s||_s = 1$. By taking $b_k = s$, we note that the minimizing sequence $\{u_k\}$ satisfies the hypothesis of lemma 16 since in its construction we assumed that the minimizing sequence was symmetrized ($u_k = u_k^*$), and that $||u_k||_s = 1$, for all $k > 1$. On the other hand, equation (22) showed that $\{u_k\}$ was uniformly bounded in $L^2_1(N)$. Then, by taking $\beta = s < p$ in lemma 16 we have that for every $\delta > 0$ there is some $R_1$ such that
\[
\int_{N \setminus M \times B_{R_1}} u_j^s dV_g < \delta.
\]
Of course, this implies that $\alpha = 1$. That is, $||u_s||_s = 1$. Then, $u_s$ is a weak solution to equation (19). It follows from a result of N. Trudinger (Theorem 3 in [20]) that $u_s$ is smooth, since it is a weak solution of (18), and from the maximum principle (cf. in [11]) that $u_s$ is positive, since $S_g$ is positive.

We resume in the next lemma what we have just proved.

**Lemma 17.** For $s > 2$ and close enough to $p$ (close enough so that equation (25) is satisfied), equation (19) has a solution $u_s$, such that $Q_s(u_s) = \lambda_s$, and $||u_s||_s = 1$.

### 3.2 The limit as $s \to p$

We now investigate the limit of the functions $u_s$, as $s \to p$. We will show that the functions $u_s$ converge to a function $u$, which in turn will be the Yamabe minimizer for $(N,g)$. We will also show that $u$ is positive and $C^\infty$.

By lemma 17 we have a family $\{u_s\}$ of functions that solve equation (19) and such that $||u_s||_s = 1$. Next, we will prove that this family is uniformly $C^{2,\alpha}$ bounded in each compact set $M \times B_R \subset N$. We will achieve this by finding first a uniform bound for $||u_s||_r$, (for some $r > p$) and then, using standard elliptic regularity theory, and the Sobolev Embedding Theorem, we will find our $C^{2,\alpha}$ bound. We follow the techniques of Parker and Lee, [11].

We begin by proving that the functions $||u_s||$ are uniformly bounded in $L^r(N)$, for some $r > p$, as $s \to p$.

**Proposition 18.** Given the collection of functions of lemma 17, $\{u_s\}$, there are some constants $s_0 < p$, $r > p$, and $C > 0$, such that

$$||u_s||_r \leq C,$$

for all $s > s_0$.

**Proof.** Consider the Yamabe subcritical equation (19). Let $\delta > 0$ and multiply (19) by $u_s^{1+2\delta}$. Then, integrating over $N$, we have

$$a \int_N u_s^{1+2\delta} \Delta u_s dV_g + \int_N S_g u_s^{2+2\delta} dV_g = \int_N \lambda_s u_s^{s+2\delta} dV_g. \quad (39)$$

Next, by setting $w = u_s^{1+\delta}$, we get $dw = (1+\delta)u_s^\delta du_s$. And so, multiplying both sides of (39), by $(1+\delta)^2$, it simplifies to

$$\frac{1+2\delta}{(1+2\delta)^2} a \int_N |dw|^2 dV_g + = \lambda_s \int_N u_s^{s-2} w^2 dV_g - \int_N S_g w^2 dV_g.$$

Then by using the “integration by parts” formula,

$$\int_N \langle \nabla \varphi, \nabla \psi \rangle dV_g = \int_N \varphi \Delta \psi dV_g,$$

(cf. in [11], page 42), we have
\[
\int_N |dw|^2 dV_g \leq \frac{(1 + \delta)^2 \lambda_s}{1 + 2\delta} \frac{a}{m + n} \lambda_s \int_N u_s^{s-2} w^2 dV_g.
\] (40)

Now, since \((N, g)\) is a complete manifold, it has bounded sectional curvature, and strictly positive injective radius, then the Sobolev Embedding Theorem holds (cf. in [6]), that is, for any \(\epsilon > 0\), there is some \(C_\epsilon\) such that
\[
||w||^2_p \leq (1 + \epsilon) \frac{\lambda_s}{m + n} \int_N |dw|^2 dV_g + C_\epsilon \int_N w^2 dV_g,
\]
hence, by equation (40),
\[
||w||^2_p \leq (1 + \epsilon) \frac{(1 + \delta)^2 \lambda_s}{1 + 2\delta} \frac{a}{m + n} \lambda_s \int_N u_s^{s-2} w^2 dV_g + C_\epsilon \int_N w^2 dV_g,
\]
and so, by H"{o}lder’s inequality,
\[
||w||^2_p \leq (1 + \epsilon) \frac{(1 + \delta)^2 \lambda_s}{1 + 2\delta} \frac{a}{m + n} \lambda_s \int_N u_s^{s-2} w^2 dV_g + C_\epsilon ||w||^2_p + C_\epsilon^\prime ||w||^2.
\] (41)

Now we recall that by remark 14, there is some \(\delta_1 > 0\) such that
\[
\lambda_s \frac{1}{m + n} < 1,
\]
for all \(s, p - \delta_1 \leq s \leq p\).

On the other hand, we note that if \(p - \delta \leq s \leq p\), then
\[
0 \leq s - \left(\frac{(s - 2)(n + m)}{2}\right) \leq \delta \left(\frac{n + m}{2}\right).
\]
Meanwhile, by continuity of the norm, given \(\epsilon > 0\), there is some \(\delta_\epsilon > 0\) such that
\[
||u_s||_s \leq ||u_s||_s + \epsilon,
\]
if \(|s - s'| \leq \delta_\epsilon\). Then, by taking \(\delta_\epsilon = \delta_\epsilon \left(\frac{2}{n + m}\right)\), we have that for \(s \in (p - \delta_\epsilon, p)\),
\[
||u_s||_{(s-2)(n+m)/2} \leq ||u_s||_s + \epsilon = 1 + \epsilon,
\] (42)
since \(0 \leq s - ((s - 2)(n + m)/2) \leq \delta_\epsilon\).

Thus, in (41), we can choose \(\delta\) and \(\epsilon\) small enough so that the coefficient of the first term is less than 1 and hence, can be absorbed by the left-hand side. We note that we need \(s\) to be close enough to \(p\) so that both (42) and (25) are satisfied.

We then have from (41)
\[
||w||^2_p \leq C||w||^2_2.
\] (43)
Hence, to finish the proof, we only need to show that
\[ ||w||_2^2 = ||u_s||_{2(1+\delta)}^{1+\delta}, \]
is bounded independently of \( s \). We proceed as follows. First, we divide the support of \( u_s \) in \( \Omega_s = u_s^{-1}(1, \infty) \) and \( \Omega_s^c \). Then we note that since \( ||u_s||_s = 1 \), then Vol(\( \Omega_s \)) \leq 1, independently of \( s \), and hence, by Hölder’s inequality
\[
\left( \int_{\Omega_s} u_s^{2(1+\delta)} \right)^{\frac{1+\delta}{2(1+\delta)}} \leq ||u_s||_{2(1+\delta)}^{1+\delta} < ||u_s||_{s}^{1+\delta} = 1. \tag{44}
\]

Meanwhile, outside \( \Omega_s \), since \( u_s < 1 \), then
\[ u_s^{2(1+\delta)} < u_s^2, \]
and then
\[
\int_{\Omega_s^c} u_s^{2(1+\delta)} < \int_N u_s^2 < C_1, \tag{45}
\]
where \( C_1 \) is independent of \( s \), by (22). It follows from (44) and (45) that
\[ ||w||_p = ||u_s||_{p(1+\delta)}^{1+\delta} \]
is bounded independently of \( s \).

It follows from this \( L^r \) bound that we may find a \( C^{2,\alpha} \) bound for the family \( \{u_s\} \) on each compact subset of \( N \).

**Lemma 19.** For the family of solutions \( \{u_s\} \) in lemma 18 that are bounded uniformly in \( L^r(N) \), there is a \( C^{2,\alpha} \) bound on each compact \( M \times B_R \subset N \).

**Proof.** Consider any compact subset \( M \times B_R \subset N \), and take \( R_0, R_1, R_2, (R < R_0 < R_1 < R_2) \) large enough. Of course, for any \( r > 0 \), \( Y(M \times B_r, g_M + g_E) \leq Y(M \times \mathbb{R}^n, g_M + g_E) < Y_{m+n} \). Now, since \( u_s \in L^r(N) \) (lemma 18), then by (18),
\[ |\Delta u_s| = |\lambda_s u_s^{s-1} - \frac{S_a}{a} u_s| \in L^q(M \times B_{R_2}), \]
with \( q = \frac{r}{r-1} \). Then, by standard elliptic regularity theory (for example, Gilbarg and Trudinger, [3]), we have \( u_s \in L^q(M \times B_{R_1}) \). And then, from the Sobolev Embedding Theorem, \( u_s \in L^{r'}(M \times B_{R_1}) \), with \( r' = \frac{2}{(n+m)(s-(n+m)-2r)} \). Of course, \( r > r' \), since \( r > p = \frac{(n+m)(p-2)}{2} > \frac{(n+m)(s-2)}{2} \). By iterating this procedure we get \( u_s \in L^q \) for all \( q > 1 \).

Then, again by the Sobolev Embedding Theorem, we have \( u_s \in C^{\alpha}(M \times B_R) \) for some \( \alpha > 0 \). Thus, using standard elliptic regularity theory one more time, we conclude that \( u_s \in C^{2,\alpha}(M \times B_R) \).

This implies that we have a uniform \( C^{2,\alpha} \) bound on each compact subset \( M \times B_R \subset N \).
It follows now from the Arzela-Ascoli Theorem that we can find a subsequence \( \{u_{s_k}\} \subset \{u_s\} \) which converges to its limit \( u \) on each compact subset of \((N, g)\). From this, we can construct the limit function \( u \) such that \( u_{s_k} \) converges to \( u \) on all of \( N \). Then, using Lemma 16 we will prove that \( \lim_{k \to \infty} || u_{s_k} ||_p = 1 \). Naturally, the limit function \( u \) would be a solution to the Yamabe equation, completing thus the proof of Theorem 3.

**Lemma 20.** Let \( \{u_s\} \) be the sequence of functions given by Lemma 17, then, as \( s \to p \) there is a subsequence \( \{u_{s_k}\} \subset \{u_s\} \) such that it converges to a positive, \( C^\infty \) solution, of

\[
a \Delta u + S_g u = \lambda u^{p-1},
\]

with

\[
||u||_p = 1
\]

and

\[
Q_p(u) = Y(N, [g]) = \lambda.
\]

**Proof.** By Lemma 19 we have that the sequence \( \{u_s\} \) is \( C^{2, \alpha} \) uniformly bounded on each compact \( M \times B_R \subset N \). Then, by the Arzela-Ascoli theorem (cf. in [16]), this implies that for each compact \( K_R = M \times B_R \subset N \), there is a subsequence \( \{u_{s_k}\} \subset \{u_s\} \) such that it converges in \( C^2(K_R) \) norm to a function in \( C^2(K_R) \)

that we will denote by \( u|_{K_R} \). Then, since \( K_R \subset K_R' \) for \( R < R' \), we have uniqueness of limits on each compact (because of the \( C^2(K_R) \) convergence for each \( R \)). Also, since \( N = \bigcup_i K_i \), then we have our limit function \( u \) as a well defined function on all of \( N \) by taking \( u = \lim_{R \to \infty} u|_{K_R} \).

We now prove that \( \lim_{k \to \infty} || u_{s_k} ||_q = ||u||_p = 1 \). We use Lemma 16. First, we note that the hypothesis are satisfied by \( \{u_{s_k}\} \). We already know that \( u_{s_k} = u_{s_k}^* \) and that \( || u_{s_k} ||_q = 1 \), for each \( k > 1 \). On the other hand, equation 22 shows that the \( u_{s_k} \) are uniformly bounded in \( L^2(N) \). To prove that the \( u_{s_k} \) are uniformly bounded in \( L^\infty(N) \), consider the compact set \( K_1 = (M \times \bar{B}_1) \).

We recall that \( u_{s_k} \to u|_{K_1} \) on \( K_1 \), in \( C^2 \) norm. Hence, for all \( k > k_1 \), \( k_1 \) large enough,

\[
\sup_{K_1} u_{s_k} \leq (\sup_{K_1} u|_{K_1}) + 1,
\]

Then, since \( u_{s_k} = u_{s_k}^* \) for all \( k > 1 \), we know that

\[
\sup_N u_{s_k} \leq \sup_{K_1} u_{s_k}.
\]

Of course this implies that \( (\sup_{K_1} u_{s_k}) \leq (\sup_{K_1} u|_{K_1}) + 1 \), and then the \( u_{s_k} \) are uniformly bounded in \( L^\infty(N) \) for all \( k > k_1 \).

Now, let \( \beta \in (2, p) \). Let \( \epsilon > 0 \), then, by Lemma 16 there is some \( R_\epsilon > 0 \) and some \( k_2 > 1 \) such that

\[
\int_{N \setminus (M \times B_{R_\epsilon})} u_{s_k}^\beta dV_g < \epsilon
\]

for all \( k > k_2 \).
On the other hand, since \( u_k \) is bounded uniformly in \( L^\infty(N) \), say \( u_k \leq A_\infty \) (for all \( k > k_3 \), for some \( k_3 > 1 \)) we have

\[
\int_{N \setminus (M \times B_{R_\epsilon})} u_k^{s_k} dV_g \leq A_\infty^{s_k - \beta} \int_{N \setminus (M \times B_{R_\epsilon})} u_k^\beta dV_g \leq C_A \int_{N \setminus (M \times B_{R_\epsilon})} u_k^\beta dV_g \leq C_A \epsilon \quad (47)
\]

where \( C_A \) is a constant such that \( C_A = \max\{1, A_\infty\} \) (and of course, we have chosen \( k_4 \) large enough so that \( s_k - \beta > 0 \), for all \( k > k_4 \)). The last inequality of (47) is an application of (46). It follows from (47) that

\[
\lim_{k \to \infty} ||u_k||_{s_k} = \alpha = 1.
\]

Hence \( ||u||_p = 1 \). Of course, this implies that there is a subsequence \( \{u_{s_k}\} \subset \{u_s\} \) such that it converges in \( C^2 \) norm to a solution \( u \in C^2(N) \) that satisfies

\[
a \Delta u + S_g u = \lambda u^{p-1},
\]

with

\[
Q_p(u) = \lambda,
\]

where \( \lambda = \lim_{s \to p} \lambda_s \). The following continuity lemma implies that \( \lambda = \lambda_p = Y(N, [g]) \).

**Lemma 21.** Consider the set \( \{\lambda_s\} \) as defined by equation (21), then \( \lambda_s \to \lambda_p \) as \( s \to p \).

**Proof.** Since \( \lim_{k \to \infty} ||u_k||_p = 1 \), recalling that \( ||u_k||_{s_k} = 1 \) and \( Q_{s_k}(u_k) = \lambda_{s_k} \), we have by (23),

\[
Q_p(u_k) = \frac{\lambda_{s_k}}{||u_k||_p}.
\]

Then, for \( s_k \) close enough to \( p \),

\[
\lambda_p \leq Q_p(u_k) = \frac{\lambda_{s_k}}{||u_k||_p} \leq \lambda_{s_k}(1 + \epsilon) \leq \lambda_{s_k} + \epsilon Y_{n+m},
\]

since \( \lambda_{s_k} < Y_{n+m} \), for all \( s_k \leq p \), by (24).

We conclude, using lemma 13 that \( \lambda_s \to \lambda_p \) as \( s \to p \). \( \square \)

Finally, the regularity of \( u \) follows from a result of N. Trudinger (Theorem 3 in [20]), since \( u \) is an \( L^2(N, g) \) solution of the Yamabe equation.

On the other hand, since \( S_g > 0 \) and \( u \) is smooth, it follows from the maximum principle (cf. in [11]) that \( u \) is positive.

\( \square \)

Of course, from lemma 20, Theorem 3 follows.
References

[1] K. Akutagawa, L. Florit, and J. Petean, On Yamabe constants of Riemannian products Comm. Anal. Geom. 15 (2008), 947-969

[2] K. Akutagawa, B. Botvinnik, Yamabe metrics on cylindrical manifolds Geometric and Functional Analysis. Vol. 13, No. 2 (2003), 259-333.

[3] B. Amman, M. Dahl and E. Humbert, Smooth Yamabe invariant and surgery arXiv:0804.1418v3 [math.DG] (2008)

[4] T. Aubin, Some non-linear problems in Riemannian geometry Springer monographs in mathematics. Springer-Verlag Berlin Heidelberg 1998.

[5] T. Aubin, Equations différentielles non linéaires et problème de Yamabe concernant la courbure scalaire, J. Math. Pures Appl. 55 (1976), 269-296

[6] T. Aubin, Espaces de Sobolev sur les variétés Riemanniennes, Bull. Sci. Math. 100 149-173, (1976).

[7] F. Brock and A. Yu. Solynin, An approach to symmetrization via polarization. Transactions of the American Mathematical Society, 352 (2000), no. 4, 1759-1796. MR1695019 (2001a:26014).

[8] Gilbarg, TrudingerElliptic partial differential equations of second order. Springer, New York, 2001.

[9] S. Kim, Scalar curvature on non-compact complete Riemannian manifolds. Nonlinear Analysis T.M. & A. Vol 26. No. 12 (1996), 1985-1993.

[10] O. Kobayashi, On the large scalar curvature, Research Report 11, Dept. Math. Keio Univ.,1985.

[11] J.M. Lee and T. Parker The Yamabe Problem. Bulletin (New Series) of the AMS Vol. 17 number 1,(1987). 37-91.

[12] P.L. Lions The concentration compactness principle in the calculus of variations. The locally compact case, part 1. Ann. Inst. Henri Poincaré Anal. Nonlin., Vol. 1 no. 2, (1984) p. 109-145.

[13] P.L. Lions The concentration compactness principle in the calculus of variations. The locally compact case, part 2. Ann. Inst. Henri Poincaré Anal. Nonlin., Vol. 1 no. 4, (1984) p. 223-283.

[14] J. Petean, Isoperimetric regions in spherical cones and Yamabe constants of $M \times S^1$ Geometriae Dedicata, (2009). arXiv:0710.2536v2 [math.DG]
[15] J. Petean, *Metrics of constant scalar curvature conformal to a Riemannian product with a sphere*, [arXiv:0812.4328v1 [math.DG]](http://arxiv.org/abs/0812.4328) (2008)

[16] P. Petersen, *Riemannian Geometry*. Second edition. Springer, New York, 2006.

[17] J. Van Schaftingen, *Universal approximation of symmetrizations by polarizations*. Proceedings of the American Mathematical Society, Volume 134, Number 1, pages 177-186.

[18] R. Schoen, *Conformal deformation of a Riemannian metric to constant scalar curvature*, J. Differential Geometry 20 (1984), 479-495.

[19] R. Schoen, *Variational Theory for the Total Scalar Curvature Functional for Riemannian Metrics and Related Topics*, Lecture Notes in Math. 1365, Springer-Verlag, Berlin, (1987) 120-154.

[20] N. Trudinger, *Remarks concerning the conformal deformation of Riemannian structures on compact manifolds*, Ann. Scuola Norm. Sup. Pisa 22 (1968), 265-274.

[21] H. Yamabe, *On a deformation of Riemannian structures on compact manifolds*, Osaka Math. J, 12 (1960), 21-37

[22] Zhang, *Nonlinear parabolic problems on manifolds, and a nonexistence result for the noncompact Yamabe problem*, Electronic Res. Announcements of the AMS, Vol. 3 (1997), 45-51

[23] Zhang, *An optimal parabolic estimate and its applications in prescribing scalar curvature on some open manifolds with Ricci ≥ 0*, Math. Ann, 316, (2000), 703-731

CIMAT, Jalisco S/N, Col. Valenciana, CP 36240 Guanajuato
Guanajuato, Mexico
E-mail: miguel@cimat.mx