DESTABILIZATION THRESHOLD CURVES FOR DIFFUSION SYSTEMS WITH EQUAL DIFFUSIVITY UNDER NON-DIAGONAL FLUX BOUNDARY CONDITIONS

KUNIMOCIHI SAKAMOTO

Graduate School of Science, Department of Mathematical and Life Sciences Hiroshima University
1-3-1 Kagamiyama, Higashi-Hiroshima, 739-8526, Japan

In Memory of Professor Paul Waltman

ABSTRACT. This article deals with destabilizations of Turing type for diffusive systems with equal diffusivity under non-diagonal flux boundary conditions. Stability-instability threshold curves in the complex plane are described as the graph of a piecewise analytic function for simple $m$-dimensional domains ($m \geq 1$). Also analyzed are effects caused by imposing homogeneous boundary conditions of Dirichlet or Neumann type on appropriate portions of the domain boundary.

1. Introduction. Turing’s diffusion-induced-instability [7, 6] for reaction-diffusion systems requires the presence of a substantial difference in the diffusion rates between activator and inhibitor species. This “restriction” universally applies to reacting and diffusing systems in which no interactions are involved on the boundary of domain, i.e., under no flux boundary conditions. In many cell-biological systems, however, relevant reagents have nearly equal diffusivities, and hence, the traditional Turing-mechanism may not be applied to such systems in order to account for the emergence of spatial or temporal inhomogeneities out of uniform states.

Levine and Rappel may be the first to point out in [5] that a mechanism of diffusion induced destabilization is still operative in diffusive systems with equal diffusivity, provided that suitable interactions on the boundary (cell membrane) are taken into consideration. In [5] they proposed a hypothetical toy model of reaction-diffusion equations with boundary flux-interaction. Together with linear stability analysis, numerical simulations are performed on this model to display diffusion-induced-instabilities of steady and oscillatory types under the equal diffusivity of reagents.

This article tries to put the rather surprising phenomena found in [5] on a mathematical footing.

2010 Mathematics Subject Classification. Primary: 35K40, 35J25; Secondary: 35J15.

Key words and phrases. Stability, threshold, reaction-diffusion equations, non-diagonal boundary conditions.

The author is supported in part by Grant-in-Aid (Fundamental Research (C) 25400171) from Ministry of Education, Japan.
1.1. **Problem.** The problem to be investigated in this article is slightly different from, but related to, the model treated in [5].

Let \( \Omega \subset \mathbb{R}^m (m \geq 1) \) be a bounded domain with smooth boundary \( \partial \Omega \). We consider the following system of diffusion equations under the *Robin type boundary conditions*

\[
\partial_t u = D \Delta u \quad \text{in } \Omega, \quad D \partial_n u = Ju \quad \text{on } \partial \Omega,
\]

(1.1)

for the vector of \( N \)-species \( u(t,x) = (u_1(t,x), \ldots, u_N(t,x)) \), where

- \( \partial_t = \partial/\partial t \) is the partial derivative with respect to *time* \( t \);
- \( \Delta \) is the \( m \)-dimensional Laplace operator;
- \( D = \text{diag}(d_1, \ldots, d_N) \) is the diagonal diffusion matrix with \( d_j > 0 \);
- \( n \) is the unit outward normal vector field on \( \partial \Omega \);
- \( \partial_n = \partial/\partial n \) is the derivative along \( n \) on the boundary;
- \( D\partial_n u \) stands for the vector of fluxes projected onto \( n \) at the boundary, and
- \( J \) is a *non-diagonal* \( N \times N \) real matrix called the *mass transfer matrix*.

The system (1.1) is the linearization of the nonlinear problem

\[
\partial_t u = D \Delta u \quad \text{in } \Omega, \quad D \partial_n u = f(u) \quad \text{on } \partial \Omega,
\]

in which \( f : \mathbb{R}^N \to \mathbb{R}^N \) is a nonlinear mapping modeling the *mass transfer mechanisms* across the boundary. The initial value problem for the nonlinear system is known to be well-posed (see, [1, 3]). If \( u^* \in \mathbb{R}^N \) is a constant solution of the non-linear problem, i.e., \( f(u^*) = 0 \), then (1.1) is the linearization of the nonlinear problem around the constant solution, where \( J = \partial_u f(u^*) \) is the Jacobian matrix of \( f \) at \( u^* \).

We exclusively deal with the linear problem with emphasis on identifying the thresholds between stability and instability of the constant solution. Weakly nonlinear analyses, such as normal form and bifurcation analyses, will be presented elsewhere.

In order to determine whether the trivial solution \( u \equiv 0 \in \mathbb{R}^N \) of (1.1) is stable or unstable, the relevant eigenvalue problem is

\[
\lambda \phi = D \Delta \phi \quad \text{in } \Omega, \quad D \partial_n \phi = J \phi \quad \text{on } \partial \Omega.
\]

(1.2)

The complex number \( \lambda \in \mathbb{C} \) is called an *eigenvalue* when (1.2) has a nontrivial solution \( \phi \neq 0 \). The eigenvalue problem (1.2) is obtained from (1.1) by substituting the ansatz \( u(t,x) = e^{\lambda t} \phi(x) \). Therefore, if the eigenvalues of (1.2) have negative real part then the trivial solution of (1.1) is *asymptotically stable*. If, on the other hand, (1.2) has an eigenvalue with \( \text{Re } \lambda > 0 \), then the system (1.1) is *unstable*.

1.2. **Known results.** The problem (1.1) was studied in [2] from a general point of view. Several characteristic features of the results found in [2] are summarized as follows.

1) If the mass transfer matrix \( J \) is stable (respectively, unstable), then (1.1) *tends to be* stable (respectively, unstable). Here, the matrix \( J \) is stable if the eigenvalues of \( J \) have negative real part, and unstable if \( J \) has an eigenvalue with positive real part.

2) More precisely, for a *symmetric* matrix \( J \), (1.1) is stable if and only if \( J \) is stable. In this case, (1.1) is a gradient system and (1.2) is variational, implying that the eigenvalues are real numbers.

3) If \( J \) is stable (but not necessarily symmetric) and \( D \) is close to a scalar matrix (nearly equal diffusivities), then (1.1) is stable.
4) If $J$ is unstable (but not necessarily symmetric) with a positive eigenvalue and $D$ is close to a scalar matrix (nearly equal diffusivities), then (1.1) is unstable. In 3) and 4) above, the condition that $D$ must be close to a scalar matrix is in general indispensable. Counter examples for 3) and 4) without this condition are given in [2], in which these examples were interpreted as the manifestations of the mechanisms of Turing-type for 3) and of anti-Turing-type for 4).

When the diffusion matrix $D$ is scalar, i.e., $D$ is a positive constant multiple of the $N \times N$ identity matrix, the following theorem was established in [2].

**Theorem 1.1 (THEOREM 1.3 (iii) in [2]).** There exists an open set $U \subset \mathbb{C}$ with the following properties.

(i) The set $U$ has the reflection symmetry with respect to the real axis, and satisfies $U \subset \{z \in \mathbb{C} | \text{Re} \, z > 0\}$.

(ii) The boundary $\partial U := \mathbb{C}$ touches the imaginary axis only at the origin $0 \in \mathbb{C}$.

For the scalar diffusion matrix $D = \text{diag}(d, \ldots, d)$, $d > 0$, the stability criteria for (1.1) are as follows.

(iii) (1.1) is stable, if and only if $\alpha/d \in S := \mathbb{C} \setminus U$ for all eigenvalues $\alpha$ of $J$.

(iv) (1.1) is unstable, if and only if $\alpha/d \in U$ for some eigenvalue $\alpha$ of $J$.

(v) If $\alpha/d \in C$ for some eigenvalue $\alpha$ of $J$, then (1.2) has a purely imaginary eigenvalue $\lambda$.

The threshold set $C$ plays the role of the imaginary axis which is the stability/instability threshold in the usual context of reaction-diffusion systems under no flux boundary conditions, while $S$ (or, $U$) corresponds to the left (or, right) half plane. The main focus of this article is to give a concrete representation of $C \subset \mathbb{C}$ in terms of a piecewise smooth function, mapping from the imaginary axis to the real axis, when the domain $\Omega$ has a simple geometry.

2. **Main results.** As mentioned in INTRODUCTION, our interest is to give a concrete description of the critical set $C$ for $m$-dimensional ($m \geq 1$) domains $\Omega$ with special geometries. It turns out that $C$ is either analytic or piecewise analytic for such domains.

2.1. **When $\partial \Omega$ has two connected components.** We first deal with the situation in which $\partial \Omega$ consists of two connected components.

**Theorem 2.1.** Let $\Omega$ be either

- the interval $\Omega = (-1, +1) \subset \mathbb{R}$, or
- $\Omega = (-1, +1) \times M$ in general, where $M$ is a smooth closed manifold with $\dim M \geq 1$.

Then, the threshold set $C$ in Theorem 1.1 is the graph of a piecewise analytic function $g : \mathbb{R} \to \mathbb{R}$, namely,

$$C = \{z \in \mathbb{C} | \text{Re} \, z = g(\text{Im} \, z)\} \quad (\text{and } U := \{z \in \mathbb{C} | \text{Re} \, z > g(\text{Im} \, z)\}).$$

The function $g$ is even ($g(-s) = g(s)$, $s \in \mathbb{R}$) and satisfies

$$g(0) = 0, \quad g'(0) = 0, \quad g''(0) > 0 \quad \text{and} \quad g(s) > 0 \quad (s \in \mathbb{R} \setminus \{0\}).$$

Moreover, for the scalar diffusion matrix $D = \text{diag}(d, \ldots, d)$, an eigenvalue $\lambda$ of (1.2) transversely crosses the imaginary axis as $\alpha/d$ transversely crosses the curve $C$, where $\alpha$ is an eigenvalue of $J$. 

Let us give a more concrete form to the function $g$. We use two functions $g_0(s)$ and $g_1(s)$ to define $g(s) := \min[g_0(s), g_1(s)]$ for $s \geq 0$. Then for $s \leq 0$, $g(s)$ is simply defined by reflection, i.e., $g(s) = g(-s)$ for $s \leq 0$. The functions $g_k(s)$ ($k = 0, 1$) for $s \geq 0$ are implicitly defined by

\[ s = \frac{\sqrt{2\tau}}{2} \sinh \frac{\sqrt{2\tau}}{2} + \sin \frac{\sqrt{2\tau}}{2} \cos \frac{\sqrt{2\tau}}{2} \quad (\tau \geq 0) \]  

(2.1-Im)

\[ g_0(s) = \frac{\sqrt{2\tau}}{2} \sinh \frac{\sqrt{2\tau}}{2} - \sin \frac{\sqrt{2\tau}}{2} \cos \frac{\sqrt{2\tau}}{2} \quad (\tau \geq 0) \]  

(2.1-Re)

and

\[ s = \frac{\sqrt{2\tau}}{2} \sinh \frac{\sqrt{2\tau}}{2} - \sin \frac{\sqrt{2\tau}}{2} \cos \frac{\sqrt{2\tau}}{2} \quad (\tau \geq 0) \]  

(2.2-Im)

\[ g_1(s) = \frac{\sqrt{2\tau}}{2} \sinh \frac{\sqrt{2\tau}}{2} + \sin \frac{\sqrt{2\tau}}{2} \cos \frac{\sqrt{2\tau}}{2} \quad (\tau \geq 0) \]  

(2.2-Re)

The right hand sides in (2.1-Im) - (2.2-Re) appear to be the functions of $\sqrt{2\tau}$, but actually they are analytic functions of $\tau$. The right hand sides of (2.1-Im) and (2.2-Im) are strictly increasing functions of $\tau$, taking all values $s \geq 0$. Solving these equations in $\tau$ as $\tau = \tau_0(s)$ and $\tau = \tau_1(s)$, and substituting these into (2.1-Re) and (2.2-Re), respectively, we arrive at the definitions of $g_0(s)$ and $g_1(s)$. It is elementary to show that

\[ g_0(s) = \frac{1}{3} s^2 + O(s^4) \text{ and } g_1(s) = 1 + \frac{1}{5} s^2 + O(s^4) \text{ as } |s| \to 0, \]

\[ \lim_{s \to \pm \infty} \frac{g_0(s)}{s} = \pm 1 = \lim_{s \to \pm \infty} \frac{g_1(s)}{s} \]

and that $\lim_{s \to \pm \infty} |g_0(s) - s| = 0 = \lim_{s \to \pm \infty} |g_1(s) - s|$.

By plotting the graphs of $\text{Re } z = g_k(\text{Im } z)$ for $k = 0, 1$ as in Figure 1, we observe that these two curves intersect many times (and actually, infinitely many times). This is the reason why the critical curve $C$ in Theorem 2.1 must be piecewise analytic, but not analytic. From the discussion so far, an interesting corollary follows.

**Corollary 1.** Let $\Omega$ be as in Theorem 2.1.

(i) If the eigenvalues of $J$ satisfy $\text{Re } \alpha \leq 0$, then the system (1.1) is asymptotically stable for all positive scalar diffusion matrix $D = \text{diag}(d, \ldots, d)$.

(ii) If the eigenvalues of $J$ satisfy $\text{Re } \alpha < |\text{Im } \alpha|$ and there exists at least one eigenvalue with $\text{Re } \alpha > 0$, then (1.1) with the scalar diffusion matrix $D = \text{diag}(d, \ldots, d)$ is stable for sufficiently small $d > 0$ and unstable for sufficiently large $d > 0$.

(iii) If $J$ has an eigenvalue with $\text{Re } \alpha > |\text{Im } \alpha|$, then (1.1) with the scalar diffusion matrix $D = \text{diag}(d, \ldots, d)$ is unstable for all $d > 0$.

(iv) If $J$ has an eigenvalue with $\text{Re } \alpha = |\text{Im } \alpha|$, then some eigenvalues $\lambda$ of (1.2) with the scalar diffusion matrix $D = \text{diag}(d, \ldots, d)$ transversely cross the imaginary axis $\mathbb{R}\backslash\{0\}$ back and forth infinitely many times as $d$ varies over all positive real numbers.
2.2. When $\partial \Omega$ is connected. A typical example of $\Omega$ with $\partial \Omega$ being connected is the flat Möbius band

$$B = \{(x, \theta) \in (-1, +1) \times \mathbb{R}\}/ \sim \text{ where } (x, \theta) \sim (-x, \theta + 2\pi).$$

**Theorem 2.2.** Let $\Omega$ be either

- the flat Möbius band $\Omega = B$,
- $\Omega = B \times M$, where $M$ is a smooth closed manifold with $\dim M \geq 1$.

Then, the threshold set $C$ in Theorem 1.1 is the graph

$$C = \{z \in \mathbb{C} \mid \Re z = g_0(\Im z)\}$$

of the analytic function $g_0$ defined in (2.1).

For the scalar diffusion matrices $D = \text{diag}(d, \ldots, d)$, the stability properties of (1.1) are characterized as follows.

(i) If the eigenvalues of $J$ satisfy $\Re \alpha \leq 0$, then the system (1.1) is asymptotically stable for all positive scalar diffusion matrix $D = \text{diag}(d, \ldots, d)$.

(ii) If the eigenvalues of $J$ satisfy $\Re \alpha < |\Im \alpha|$ and there exists at least one eigenvalue with $\Re \alpha > 0$, then (1.1) with the scalar diffusion matrix $D = \text{diag}(d, \ldots, d)$ is stable for sufficiently small $d > 0$ and unstable for sufficiently large $d > 0$.

(iii) If $J$ has an eigenvalue with $\Re \alpha > |\Im \alpha|$, then (1.1) with the scalar diffusion matrix $D = \text{diag}(d, \ldots, d)$ is unstable for all sufficiently large or small $d > 0$.

(iv) If $J$ has an eigenvalue with $\Re \alpha = |\Im \alpha|$, then some eigenvalues $\lambda$ of (1.2) with the scalar diffusion matrix $D = \text{diag}(d, \ldots, d)$ transversely cross the imaginary axis $i\mathbb{R}\setminus\{0\}$ back and forth infinitely many times as $d$ varies over all positive real numbers.

2.3. Other $\Omega$ with connected $\partial \Omega$. We denote by $B^2$ the two dimensional unit open disk with center at the origin.
Theorem 2.3. Let $\Omega$ be either
- $\Omega = B^2 \subset \mathbb{R}^2$, or
- $\Omega = B^2 \times M$, where $M$ is a smooth closed manifold with $\dim M \geq 1$.

Then, the threshold set $C$ in Theorem 1.1 is the graph of a piecewise-analytic function $\hat{g} : \mathbb{R} \to \mathbb{R}$. The function $\hat{g}$ is even and satisfies
$$\hat{g}(0) = 0, \quad \hat{g}'(0) = 0, \quad \hat{g}''(0) > 0 \quad \text{and} \quad \hat{g}(s) > 0 \quad (s \in \mathbb{R}\setminus\{0\}).$$

Moreover, for the scalar diffusion matrix $D = \text{diag}(d, \ldots, d)$, an eigenvalue $\lambda$ of (1.2) transversely crosses the imaginary axis as $\alpha/d$ transversely crosses the curve $C$, where $\alpha$ is an eigenvalue of $J$.

To define the function $\hat{g}$, we first define a family of functions $\{\hat{g}_n\}_{n=0}^\infty$. The graph of the function $\hat{g}_0$, $\{z \in C \mid \Re z = \hat{g}_0(\Im z)\}$, is the image of the imaginary axis under the mapping
$$\mathbb{C} \ni \lambda \mapsto \frac{\lambda}{2} \left( \sum_{j=0}^\infty \frac{(\lambda/4)^j}{j!(j+1)!} \right) \left( \sum_{j=0}^\infty \frac{(\lambda/4)^j}{(j!)^2} \right)^{-1} \in \mathbb{C},$$
while the graph of the function $\hat{g}_n$ ($n \geq 1$) is the image of imaginary axis under the mapping
$$\mathbb{C} \ni \lambda \mapsto \left( \sum_{j=0}^\infty \frac{(2j+n)(\lambda/4)^j}{j!(j+n)!} \right) \left( \sum_{j=0}^\infty \frac{(\lambda/4)^j}{j!(j+n)!} \right)^{-1} \in \mathbb{C}.$$

We then define $\hat{g}(s) := \inf \{\hat{g}_n(s) \mid n = 0, 1, \ldots\}$. Numerical evaluations suggest that $\hat{g}(s) = \hat{g}_0(s)$, in which case $\hat{g}(s)$ is analytic, instead of being piecewise-analytic. To rigorously prove this would require delicate analyses involving modified Bessel functions of non-negative integer orders. The graph of $\hat{g}$ is numerically depicted in

![Graph of Re z = g(Im z) near the origin.](image)

Figures 2 and 3, in which the graph of $\Re z = |\Im z|$ is also shown for comparison.
To draw Figures 2 and 3, approximations are made by replacing the upper limit \( \infty \) of the summations by 25, and \( \lambda = i\tau \) \((\tau \in \mathbb{R})\) is substituted with \(|\tau| \leq 50\) for Figure 2 and \(|\tau| \leq 500\) for Figure 3.

**Figure 3.** The graph of \( \text{Re } z = \hat{g}(\text{Im } z) \); a global view.

It is probable that the threshold curve \( \mathcal{C} \) in Theorem 1.1 is represented by an analytic function when the boundary of \( \Omega \) consists of one connected component.

3. **Proof of main results.** We investigate how the eigenvalues of (1.2) depends on the eigenvalues of the mass transfer matrix \( J \). For scalar diffusion matrices \( D = \text{diag}(d, \ldots, d) \), it is shown in [2, §2.3] that \( \lambda \) is an eigenvalue of (1.2) if and only if the scalar problem

\[
\lambda \phi = d \Delta \phi \text{ in } \Omega, \quad d \partial_n \phi = \alpha \phi
\]

has a nontrivial solution \( \phi \neq 0 \), where \( \alpha \) ranges over the eigenvalues of \( J \). This means that one can enumerate the eigenvalues of (1.2) as the family

\[
\{ \lambda_j(\alpha) \mid j \in \mathbb{N}, \alpha \in \sigma(J) \}
\]

of eigenvalues \( \lambda_j(\alpha) \) of (3.1), parametrized by the eigenvalues of \( J \). Since \( J \) is a real matrix, its non real eigenvalues appear in complex conjugate pairs \( \alpha, \bar{\alpha} \). Therefore, if the triplet \((\lambda, \phi, \alpha)\) is a nontrivial solution of (3.1), then so is its complex conjugate \((\bar{\lambda}, \bar{\phi}, \bar{\alpha})\). This observation will imply that the critical set \( \mathcal{C} \), to be defined below, is symmetric with respect to the real axis.

The scalar problem (3.1) is now reformulated in terms of the so called *Dirichlet-to-Neumann map*. Let us begin with recalling the definition and several properties of this map (see [4]).

Consider Dirichlet boundary value problem for the \( \mathbb{C} \)-valued function \( v(x) \):

\[
\lambda v(x) = \Delta v(x) \quad x \in \Omega, \quad v(x) = p(x) \quad x \in \partial \Omega,
\]

(3.2)
where $p \in H^{3/2}(\partial \Omega)$ is a given function (Dirichlet data) and $\lambda \in \mathbb{C}$ plays the role of a parameter. The Dirichlet-to-Neumann map is defined as follows.

- The problem (3.2) has a unique solution $v(\cdot; \lambda, p) \in H^2(\Omega)$ for each $p \in H^{3/2}(\partial \Omega)$ if $\lambda \in \mathbb{C} \setminus \sigma(\Delta_{\text{Dir}})$, where $\sigma(\Delta_{\text{Dir}})$ stands for the spectrum of the Laplacian under the homogeneous Dirichlet boundary condition on $\partial \Omega$.
- The Dirichlet-to-Neumann map $T(\lambda) : H^{3/2}(\partial \Omega) \to H^{1/2}(\partial \Omega)$ is then defined by assigning to $p$ the Neumann data of $v(\cdot; \lambda, p)$ on $\partial \Omega$;

$$\left( T(\lambda) p \right)(x) := \partial_n v(x; \lambda, p) \quad x \in \partial \Omega.$$ 

The eigenvalues of $\mathcal{T}(0)$, called the Steklov eigenvalues, play a pivotal role in what follows. It is known [4] that the Steklov eigenvalues are infinite nonnegative numbers $\nu_0 = 0 < \nu_1 \leq \nu_2 \leq \ldots \leq \nu_j \to \infty$ for smooth bounded domains of dimension $m \geq 2$. For the one dimensional interval $\Omega = (-1, 1)$, there are only two Steklov eigenvalues $\nu_0 = 0$, $\nu_1 = 1 = 2/|\Omega|$.

The definition of $T(\lambda)$ implies that (3.1) is equivalent to

$$T \left( \frac{\lambda}{d} \right) \phi = \frac{\alpha}{d} \phi. \quad (3.3)$$

This allows us to approach the scalar problem (3.1) from another viewpoint. Namely, we now ask how the eigenvalues of $T(\lambda)$ depend on $\lambda \in \mathbb{C} \setminus \sigma(\Delta_{\text{Dir}})$.

Let us denote by $E(\lambda)$ an eigenvalue of $T(\lambda)$ for $\lambda \in \mathbb{C} \setminus \sigma(\Delta_{\text{Dir}})$. From the observation made above in the paragraph following (3.1), we find that $E(\lambda)$ satisfies $E(\lambda) = E(\overline{\lambda})$. It is expected that $E(\lambda)$ depends analytically in $\lambda$. In fact, for the domain $\Omega$ under consideration in Theorems 2.1, 2.2 and 2.3 (as well as Theorems 4.1 and 4.2, in §4), we will show that the eigenvalues $E(\lambda)$ are analytic in $\lambda$. This will be done in general context as follows. First of all, note that $E(0)$ must be one of the Steklov eigenvalues $\{\nu_j\}_{j=0}^{\infty}$. We then analytically continue $E(0) = \nu_j$ for all possible values $\lambda \in \mathbb{C}$, and denote the continuation by $E_j(\lambda)$. It is shown in [4] that such analytic continuations are possible for all $\nu_j$ that is a simple eigenvalue of $\mathcal{T}(0)$. In the situations of Theorems 2.1, 2.2 and 2.3, however, we can bypass these steps, since the Fourier expansions of $H^2(\Omega)$-functions allow us to directly obtain the analyticity of $E(\lambda)$, even when some of the Steklov eigenvalues are not simple.

### 3.1. Proof of Theorem 2.1.

We now compute the eigenvalues $E_0(\lambda)$ and $E_1(\lambda)$ for $\Omega = (-1, 1) \subset \mathbb{R}$. In this case, it is easy to show that the Dirichlet-to-Neumann map $T(\lambda)$ is represented by the $2 \times 2$ matrix

$$\begin{pmatrix}
  u(-1) \\
  u(+1)
\end{pmatrix} \mapsto \begin{pmatrix}
  -u'(-1) \\
  +u'(1)
\end{pmatrix} = \frac{\sqrt{\lambda}}{\sinh 2\sqrt{\lambda}} \begin{pmatrix}
  \cosh 2\sqrt{\lambda} & -1 \\
  -1 & \cosh 2\sqrt{\lambda}
\end{pmatrix} \begin{pmatrix}
  u(-1) \\
  u(+1)
\end{pmatrix}.$$

The eigenvalues $E_0(\lambda)$ and $E_1(\lambda)$ of the matrix are given by

$$E_0(\lambda) = \sqrt{\lambda} \frac{\sinh \sqrt{\lambda}}{\cosh \sqrt{\lambda}} = \lambda \sum_{n=0}^{\infty} \frac{\lambda^n}{(2n+1)!};$$

$$E_1(\lambda) = \sqrt{\lambda} \frac{\cosh \sqrt{\lambda}}{\sinh \sqrt{\lambda}} = \sum_{n=0}^{\infty} \frac{\lambda^n}{(2n)!}. $$

By expanding the numerator and denominator, one finds that these eigenvalues are analytic functions of $\lambda$. The domains of analyticity are, respectively,

$\mathbb{C} \setminus \{\lambda \mid \cosh \sqrt{\lambda} = 0\}$ for $E_0(\lambda)$ and $\mathbb{C} \setminus \{\lambda \neq 0 \mid \sinh \sqrt{\lambda} = 0\}$ for $E_1(\lambda)$.
Moreover, the excluded sets together constitute \( \sigma(\triangle_{\text{Dir}}) \):

\[
\sigma(\triangle_{\text{Dir}}) = \{ \lambda \mid \cosh \sqrt{\lambda} = 0 \} \cup \{ \lambda \neq 0 \mid \sinh \sqrt{\lambda} = 0 \}
\]

\[
= \{-\pi^2(n + \frac{1}{2})^2 \mid n \in \mathbb{N} \} \cup \{-\pi^2n^2 \mid n \in \mathbb{N} \} = \{-\pi^2n^2/4 \mid n \in \mathbb{N} \}.
\]

Let us denote by \( C_k \) the image of \( E_k(\lambda) \) \((k = 0, 1)\) as \( \lambda \) ranges over the imaginary axis \( \{ \lambda = i\tau \mid \tau \in \mathbb{R} \} \). The parametric representations of these curves are respectively given by (2.1) and (2.2), or equivalently, \( \Re z = g_0(\Im z) \) for \( C_0 \) and \( \Re z = g_1(\Im z) \) for \( C_1 \).

It is elementary to show that \( \frac{d}{d\lambda}E(\lambda)|_{\lambda=i\tau} \neq 0 \) for all \( \tau \in \mathbb{R} \). To prove the transverse crossing of eigenvalues, we use the Cauchy-Riemann relation

\[
\frac{\partial E_k(\lambda)}{\partial \zeta} = -i\frac{\partial E_k(\lambda)}{\partial \tau} \quad \text{for} \quad \lambda = \zeta + i\tau \quad \text{at} \quad \zeta = 0 \tag{3.4}
\]

for \( k = 0, 1 \). Note that \( \frac{\partial E_k(\lambda)}{\partial \tau} \) is the tangent vector along \( C_k \), while \( \frac{\partial}{\partial \zeta} \) means that \( \lambda \) transversely crosses the imaginary axis from left to right in the right angle \( \pi/2 \).

Then the relation in (3.4) says that \( E_k(\lambda) \) transversely crosses \( C_k \) from left to right, \( \lambda \) transversely crosses the imaginary axis from left to right. Moreover, the left half plane minus \( \sigma(\triangle_{\text{Dir}}) \) is mapped into the domain left to the curve \( C_k \), while the right half plane is mapped into the domain right to the curve.

To show that the curves \( C_0 \) and \( C_1 \) intersect infinitely many times, let \( \tau > 0 \) be such that \( \sqrt{2\tau} = \left( n + \frac{1}{2} \right)\pi \) for an even integer \( n > 0 \). Then, the corresponding values of \( s, g_k(s) \) for (2.1) and (2.2) satisfy

\[
g_0(s_0^{(n+1/2)}) = s_1^{(n+1/2)} < s_0^{(n+1/2)} = g_1(s_1^{(n+1/2)}).
\]

For \( n + 1 \) (odd integer) the relations are reversed as follows.

\[
g_0(s_0^{(n+3/2)}) = s_1^{(n+3/2)} > s_0^{(n+3/2)} = g_1(s_1^{(n+3/2)}).
\]

If \( \tau \) is such that \( \sqrt{2\tau} = n\pi \) \((n \in \mathbb{N})\), then \( s_0^{(n)} = g_0(s_0^{(n)}), s_1^{(n)} = g_1(s_1^{(n)}) \), i.e., \( C_k \) intersect the diagonal line \( \Re z = \Im z \) infinitely many times. Therefore, the curves \( C_k \) \((k = 0, 1)\) interlace with each other around the diagonal \( \Re z = \Im z \), and we conclude that they intersect infinitely many times. The proof of Theorem 2.1 is complete for \( \Omega = (-1, +1) \subset \mathbb{R} \).

To prove the theorem for \( \Omega = (-1, +1) \times M \), we only need to describe the eigenvalues of \( T(\lambda) \). Applying the variable separation, we find that the eigenvalues are given by \( E_0(\lambda + \kappa_j) \) and \( E_1(\lambda + \kappa_j) \) with \( j \in \{0\} \cup \mathbb{N} \), where \( \{-\kappa_j\}_{j=0}^{\infty} \) is the set of the eigenvalues of \( \triangle_M \), the Laplacian on \( M \). These eigenvalues satisfy

\[
\kappa_0 = 0 < \kappa_1 \leq \ldots \leq \kappa_j \to \infty \quad \text{as} \quad j \to \infty.
\]

Thanks to the arguments above, it is easy to see that the curves generated by \( E_k(i\tau + \kappa_j) \) for \( j \in \mathbb{N}, k = 0, 1 \) are shifted to the right by the amount corresponding to \( \kappa_j > 0 \). Therefore, the threshold curves are determined by \( E_k(i\tau + \kappa_0) = E_k(i\tau) \) which are the same as the critical eigenvalue for the interval \( \Omega = (-1, +1) \). This completes the proof of Theorem 2.1.
3.2. Proof of Theorem 2.2. On the flat Möbius band

\[ B = \{(x, \theta) \in (-1, +1) \times \mathbb{R}\} / \sim \text{ with } (x, \theta) \sim (-x, \theta + 2\pi), \]

our eigenvalue problem is \( \lambda u = u_{xx} + u_{\theta\theta} \) for \( (x, \theta) \in (-1, +1) \times (0, 2\pi) \), where \( u \) must satisfy \( u(x, 0) = u(-x, 2\pi) \). Therefore, in the variable separation \( u(x, \theta) = A(x)B(\theta) \), \( A \) is even, \( A(x) = A(-x) \), and \( B(\theta) \) must be \( 2\pi \)-periodic in \( \theta \). Substituting the ansatz, and using the \( 2\pi \)-periodicity, we have a family of problems

\[ (\lambda + k^2)A(x) = A''(x) \text{ for } x \in (-1, +1) \text{ and } A(x) = A(-x). \]

parametrized by non-negative integers \( k \). The solutions are exhausted by the constant multiples of \( e^{\sqrt{\lambda + k^2}x} + e^{-\sqrt{\lambda + k^2}x} \). Therefore, the eigenvalues of the Dirichlet-to-Neumann map in this case are given by

\[ E(\lambda, k) = \frac{A'(1)}{A(1)} = \sqrt{\lambda + k^2} \tanh \sqrt{\lambda + k^2}. \]

This is nothing but \( E_0(\lambda + k^2) \), where \( E_0 \) is the one that has appeared in §3.1. Therefore, the remaining part of the proof has been already supplied. This completes the proof of Theorem 2.2.

3.3. Proof of Theorem 2.3. Thanks to the line of arguments in §3.1, it suffices to compute the eigenvalues of \( T(\lambda) \) for \( \Omega = B^2 \), the unit open disk in \( \mathbb{R}^2 \) with center at the origin.

We represent \( z \in B^2 \subset \mathbb{C} \) by \( z = re^{i\theta} \) with \( 0 \leq r < 1 \), \( \theta \in [0, 2\pi) \). To solve the eigenvalue problem

\[ \lambda \phi = \Delta \phi \text{ in } B^2, \quad \partial_n \phi = E(\lambda)\phi \text{ on } \partial B^2, \]

we use the variable separation \( \phi = u(r)e^{in\theta} \), imposing the Dirichlet boundary condition \( \phi(r = 1) = e^{in\theta} \) for integers \( n \geq 0 \), and hence \( u(1) = 1 \). The equation for \( u(r) \) is the following.

\[ u_{rr} + \frac{1}{r} u_r - \left( \lambda + \frac{n^2}{r^2} \right) u = 0 \quad (0 \leq r < 1), \quad u(1) = 1. \]

In terms of the modified Bessel function [8] of the first kind of order \( n \), \( I_n(z) \), the solution is expressed as

\[ u(r) = \frac{I_n(\sqrt{\lambda}r)}{I_n(\sqrt{\lambda})}, \]

and hence the corresponding eigenvalue \( E_n(\lambda) \) of \( T(\lambda) \) is

\[ E_n(\lambda) = \frac{d}{dr}u(r)|_{r=1} = \sqrt{\lambda} \frac{I'_n(\sqrt{\lambda})}{I_n(\sqrt{\lambda})}, \quad \text{where } I'_n(z) = \frac{d}{dz}I_n(z). \quad (3.5) \]

From the series expansion

\[ I_n(z) = \sum_{j=0}^{\infty} \frac{1}{j!(j+n)!} \left( \frac{z}{2} \right)^{2j+n} \]
it follows that $E_n(\lambda)$ is analytic in $\lambda$:

$$E_0(\lambda) = \frac{\lambda}{2} \left( \sum_{j=0}^{\infty} \frac{(\lambda/4)^j}{j!(j+1)!} \right) \left( \sum_{j=0}^{\infty} \frac{(\lambda/4)^j}{(j!)^2} \right)^{-1},$$

$$E_n(\lambda) = \left( \sum_{j=0}^{\infty} \frac{(2j+n)(\lambda/4)^j}{j!(j+n)!} \right) \left( \sum_{j=0}^{\infty} \frac{(\lambda/4)^j}{j!(j+n)!} \right)^{-1}, \quad n \geq 1.$$

Let $\{j_{n,k} \mid k \in \mathbb{N}\}$ be the set of the positive zeros of the Bessel function $J_n(z)$. Since, by definition, $I_n(z) = e^{-in\pi/2}J_n(iz)$, we find from (3.5) that $E_n(\lambda)$ is defined on $\mathbb{C} \setminus \{-(j_{n,k})^2 \mid k \in \mathbb{N}\}$. The excluded sets together constitute the Dirichlet-eigenvalues of the Laplacian on $B^2$: $\sigma(\Delta_{\text{Dir}}) = \bigcup_{n=0}^{\infty} \{-(j_{n,k})^2 \mid k \in \mathbb{N}\}$. Clearly, the Steklov eigenvalues are given by $\{E_n(0) = n\}_{n=0}^{\infty}$. For each non-negative integer $n$, let us define the curve $C_n := \{E_n(\tau i) \mid \tau \in \mathbb{R}\}$, the image of the imaginary axis under the analytic map $E_n(\cdot)$. Evidently, $C_n$ intersects the real axis at $n$, one of the Steklov eigenvalues.

We now define the function $\hat{g}_n$ by $C_n := \{z \in \mathbb{C} \mid \text{Re } z = \hat{g}_n(\text{Im } z)\}$ for each $n \geq 0$, and define $\hat{g}(s) := \inf \{\hat{g}_n(s) \mid n = 0, 1, 2, \ldots\}$. With this $\hat{g}$, the statements in §2.3 easily follow by elementary arguments. This completes the proof of Theorem 2.3.

4. Fixed boundary conditions on part of the boundary. It is technically difficult to study (1.2) for rectangular domains $\Omega$. This is partly due to the fact that the boundary of such a domain is piecewise smooth but not smooth. For such domains, the Dirichlet-to-Neumann map tends to be singular. Moreover, the procedure of separating variables do not work for rectangular domains. However, by imposing fixed boundary conditions on appropriate portions of the boundary, and thus making smooth the free remaining boundary portions on which the Dirichlet-to-Neumann map is defined, our analysis easily extends to piecewise smooth domains.

Let us, therefore, consider the following variant of (1.1).

$$\begin{cases}
\partial_t u = D\Delta u & \text{in } \Omega, \\
D\partial_n u = J u & \text{on } \partial_0 \Omega \text{ and } B u = 0 \text{ on } \partial_0 \Omega,
\end{cases} \quad (4.1)$$

where $\partial \Omega = \overline{\partial_0 \Omega}$, $\partial_0 \Omega$ is a bounded piecewise smooth domain with $\partial_0 \Omega$ being smooth ($\partial_0 \Omega = \partial \Omega \setminus \overline{\partial_0 \Omega}$), while $Bu = 0$ is a set of fixed diagonal boundary conditions such as $\partial_n u = 0$ or $u = 0$.

4.1. Neumann boundary conditions on $\overline{\partial_0 \Omega}$.

**Theorem 4.1.** For scalar diffusion matrix $D = \text{diag}(d, \ldots, d)$, consider the system (4.1) under the Neumann boundary conditions $Bu = \partial_n u$ on $\overline{\partial_0 \Omega}$.

(i) For the rectangle $\Omega = (-1, +1) \times (-\pi, +\pi)$ with $\partial_0 \Omega = (-1, +1) \times \{-\pi, +\pi\}$, $\partial_1 \Omega = (-1, +1) \times (-\pi, +\pi)$, the threshold curve is given by $C = \{z \in \mathbb{C} \mid \text{Re } z = g(\text{Im } z)\}$, where $g$ is the same function as appeared in §2.1.
Theorem 4.2. Dirichlet boundary conditions on connected component.

(i) For the rectangle \( \Omega = (-1,+1) \times (-\pi,+\pi) \) with
\[
\partial_\Omega = (-1,+1) \times \{-\pi,+\pi\} \cup \{-1\} \times (-\pi,+\pi),
\]
the threshold curve is given by \( C = \{ z \in \mathbb{C} \mid \text{Re } z = g_0(\text{Im } z) \} \), where \( g_0 \) is the same function as appeared in §2.1.

(ii) For the rectangle \( \Omega = (-1,+1) \times (-\pi,+\pi) \) with
\[
\partial_\Omega = (-1,+1) \times \{-\pi,+\pi\} \cup \{-1\} \times (-\pi,+\pi),
\]
the threshold curve is given by \( C = \{ z \in \mathbb{C} \mid \text{Re } z = g(\text{Im } z) \} \), where \( g \) is the same function as appeared in §2.1.

The proof is completed by following the same line of arguments in §2.1.

For the solid cylinder \( \Omega = (-1,+1) \times B^2 \) with
\[
\partial_\Omega = (-1,+1) \times \partial B^2, \quad \partial_\Omega = \{-1,+1\} \times B^2,
\]
the threshold curve is given by \( C = \{ z \in \mathbb{C} \mid \text{Re } z = \hat{g}(\text{Im } z) \} \), where \( \hat{g} \) is the same function as appeared in §2.3.

(iv) For the solid cylinder \( \Omega = (-1,+1) \times B^2 \) with
\[
\partial_\Omega = (-1,+1) \times \partial B^2 \cup \{-1\} \times B^2, \quad \partial_\Omega = \{+1\} \times B^2,
\]
the threshold curve is given by \( C = \{ z \in \mathbb{C} \mid \text{Re } z = g_0(\text{Im } z) \} \), where \( g_0 \) is the same function as appeared in §2.1.

The proofs of (iii), (iv) and (v) are similar.

Proof. (i) The eigenvalue problem to be investigated is
\[
\lambda u = u_{xx} + u_{yy} \text{ for } (x,y) \in (-1,+1) \times (-\pi,+\pi)
\]
\[
u_y(x,\pm\pi) = 0 \text{ for } x \in (-1,+1),
\]
\[
\pm u_x(\pm1,y) = E(\lambda)u(\pm1,y) \text{ for } y \in (-\pi,+\pi).
\]
We now apply the variable separation \( u(x,y) = A(x)B(y) \), and obtain the one-dimensional problem
\[
\lambda A(x) = A''(x) - k^2 A(x) \text{ for } x \in (-1,+1), \quad \pm A'(\pm1) = E(\lambda)A(\pm1).
\]
for each non-negative integer \( k \). The proof now follows from that of Theorem 2.1.

(ii) Arguing as in the proof for (i) above, we arrive at
\[
\lambda A(x) = A''(x) - k^2 A(x) \text{ for } x \in (-1,+1),
\]
\[
A'(-1) = 0, \quad A'(1) = E(\lambda)A(+1).
\]
The proof is completed by following the same line of arguments in §3.

The proofs of (iii), (iv) and (v) are similar. \( \square \)

In Theorem 4.1 (ii) and (v), and possibly in Theorem 4.1 (iv), note that the threshold curve \( C \) is analytic. These examples as well as Theorem 2.2 suggest that the threshold curve may be analytic for domains whose boundary consists of one connected component.

4.2. Dirichlet boundary conditions on \( \partial_0 \Omega \). If the fixed boundary conditions on \( \partial_0 \Omega \) are replaced by the Dirichlet boundary conditions, then the threshold curve shifts to the right.

Theorem 4.2. For scalar diffusion matrix \( D = \text{diag}(d,d,\ldots,d) \), consider the system
\[
(4.1) \text{ under the Dirichlet boundary conditions } Bu = u \text{ on } \partial_0 \Omega.
\]
(i) For the rectangle \( \Omega = (-1, +1) \times (-\pi, +\pi) \) with
\[
\partial_0 \Omega = (-1, +1) \times \{-\pi, +\pi\}, \\
\partial_1 \Omega = (-1, +1) \times (-\pi, +\pi).
\]
Then the threshold curve \( C \) is the graph of the piecewise analytic function \( h(s) \);
\( C = \{ z \in \mathbb{C} \mid \text{Re} \, z = h(\text{Im} \, z) \} \), with
\[
h(0) = \frac{e - 1}{2(e + 1)} \leq h(s) \quad (s \in \mathbb{R}), \quad h'(0) = 0, \quad h''(0) > 0,
\]
where \( h(s) \) is defined by \( h(s) := \min \{h_0(s), h_1(s)\} \). The analytic functions \( h_0(s) \) and \( h_1(s) \) are implicitly defined, respectively, by
\[
s = \pm \frac{b(\tau) \sin 2a(\tau) + a(\tau) \sin 2b(\tau)}{\cosh 2a(\tau) + \cos 2b(\tau)} \quad \text{(for} \, \pm \tau \geq 0), \\
h_0(s) = \frac{a(\tau) \sin 2a(\tau) - b(\tau) \sin 2b(\tau)}{\cosh 2a(\tau) + \cos 2b(\tau)}, \tag{4.2}
\]
and
\[
s = \pm \frac{b(\tau) \sin 2a(\tau) - a(\tau) \sin 2b(\tau)}{\cosh 2a(\tau) - \cos 2b(\tau)} \quad \text{(for} \, \pm \tau \geq 0), \\
h_1(s) = \frac{a(\tau) \sin 2a(\tau) + b(\tau) \sin 2b(\tau)}{\cosh 2a(\tau) - \cos 2b(\tau)}, \tag{4.3}
\]
where
\[
a(\tau) = \sqrt{-\frac{1}{8} + \frac{1}{8} \sqrt{1 + 16\tau^2}}, \quad b(\tau) = \sqrt{-\frac{1}{8} + \frac{1}{8} \sqrt{1 + 16\tau^2}} \quad \tau \in \mathbb{R}.
\]
(ii) For the solid cylinder \( \Omega = (-1, +1) \times B^2 \) with
\[
\partial_0 \Omega = (-1, +1) \times \partial B^2, \quad \partial_1 \Omega = (-1, +1) \times B^2,
\]
the statements in (i) above are valid with \( h(0) = \sqrt{2k_1} \tanh \sqrt{2k_1} \) and \( a(\tau), b(\tau) \) in (4.2), (4.3) being replaced by
\[
a(\tau) = \sqrt{\frac{k_1}{2} + \frac{k_1}{2} \sqrt{1 + (\tau/k_1)^2}}, \quad b(\tau) = \sqrt{-\frac{k_1}{2} + \frac{k_1}{2} \sqrt{1 + (\tau/k_1)^2}}, \tag{4.4}
\]
where \( k_1 > 0 \) is the first Dirichlet eigenvalue of the Laplacian on \( B^2 \).
(iii) For the solid cylinder \( \Omega = (-1, +1) \times B^2 \) with
\[
\partial_0 \Omega = (-1, +1) \times B^2, \quad \partial_1 \Omega = (-1, +1) \times \partial B^2,
\]
the threshold curve \( C \) is the graph of the piecewise analytic function \( \tilde{h}(s) \);
\( C = \{ z \in \mathbb{C} \mid \text{Re} \, z = \tilde{h}(\text{Im} \, z) \} \), where \( \tilde{h}(s) := \inf \{ h_n(s) \mid n = 0, 1, 2, \ldots \} \) and each \( h_n \) is implicitly defined by
\[
s = \text{Im} \sqrt{1 + \pi^2/4} I_n \left( \sqrt{1 + \pi^2/4} \right) \left/ I_n \left( \sqrt{1 + \pi^2/4} \right) \right. \\
\tilde{h}(s) = \text{Re} \sqrt{1 + \pi^2/4} I_n \left( \sqrt{1 + \pi^2/4} \right) \left/ I_n \left( \sqrt{1 + \pi^2/4} \right) \right.
\]
(iv) For the solid cylinder \( \Omega = (-1, +1) \times B^2 \) with
\[
\partial_0 \Omega = (-1, +1) \times \partial B^2 \cup \{-1\} \times B^2, \quad \partial_1 \Omega = \{+1\} \times B^2,
\]
the threshold curve given by \( C = \{ z \in \mathbb{C} \mid \text{Re} \, z = h_1(\text{Im} \, z) \} \), where \( h_1(s) \) is defined by (4.3) with \( a(\tau), b(\tau) \) given by (4.4).
Proof. (i) After separating variables, we find that the relevant eigenvalue problem reduces to

\[ \lambda A(x) = A''(x) - \frac{1}{4} A(x) \quad \text{for} \quad x \in (-1, +1), \]
\[ \pm A'(\pm 1) = E(\lambda) A(\pm 1), \]

where \( \mu = -1/4 \) is the principal eigenvalue of the Dirichlet problem:

\[ B''(y) = -\mu B(y) \quad \text{for} \quad y \in (-\pi, +\pi), \quad B(\pm \pi) = 0. \]

Now, arguing as in the proof of Theorem 2.1 (i), we find that the eigenvalues \( E(\lambda) \) are given by

\[ E_0(\lambda) = \sqrt{\lambda + \frac{1}{4}} \tanh \sqrt{\lambda + \frac{1}{4}}, \quad E_1(\lambda) = \sqrt{\lambda + \frac{1}{4}} \cosh \sqrt{\lambda + \frac{1}{4}}. \]

By using the identity \( \sqrt{i \tau + 1/4} = a(\tau) + ib(\tau) \), we find that the graph of \( h_0 \) is the image \( \{ E_0(i \tau) \mid \tau \in \mathbb{R} \} \) and the graph of \( h_1 \) is the image \( \{ E_1(i \tau) \mid \tau \in \mathbb{R} \} \). Then the remaining part of the proof is similar to that of Theorem 2.1.

(ii) The proof is almost identical to that of (i) above.

(iii) The proof is similar to the proof of Theorem 2.3, and hence omitted.

(iv) The problem reduces to

\[ \lambda A(x) = A''(x) - k_1 A(x) \quad \text{for} \quad x \in (-1, +1), \]
\[ A(-1) = 0, \quad A'(1) = E(\lambda) A(1). \]

Then the eigenvalue is \( E(\lambda) = \sqrt{\lambda + k_1} \cosh \sqrt{\lambda + k_1} \). The remaining part is similar to the foregoing discussions.

Acknowledgments. I am thankful to Professors Hal Smith and Sze-Bi Hsu for their support and encouragement during the preparation of this article.

REFERENCES

[1] H. Amann, Parabolic evolution equations and nonlinear boundary conditions, *J. Differential Equations*, **72** (1988), 201–269.

[2] A. Anma and K. Sakamoto, Turing type mechanisms for linear diffusion systems under non-diagonal Robin boundary conditions, *SIAM Journal on Mathematical Analysis*, **45** (2013), 3611–3628.

[3] J. M. Arrieta, A. N. Carvalho and A. Rodríguez-Bernal, Upper semicontinuity for attractors of parabolic problems with localized large diffusion and nonlinear boundary conditions, *J. Differential Equations*, **168** (2000), 33–59.

[4] G. Auchmuty, Steklov eigenproblems and the representation of solutions of elliptic boundary value problems, *Numerical Funct. Anal. Optim.*, **25** (2004), 321–348.

[5] H. Levine and W.-J. Rappel, Membrane-bound Turing patterns, *Physical Review E*, **72** (2005), 061912, 5pp.

[6] J. D. Murray, *Mathematical Biology*, Biomathematics Texts, Springer-Verlag Berlin Heidelberg, 1989.

[7] Alan M. Turing, The chemical basis for morphogenesis, *Phil. Trans. R. Soc. London*, **B 273** (1952), 37–72.

[8] G. N. Watson, *A Treatise on the Theory of Bessel Functions*, Cambridge Mathematical Library. Cambridge University Press, Cambridge, 1995.

Received December 2014; revised September 2015.

E-mail address: kuni@math.sci.hiroshima-u.ac.jp