Superscattering for non-spherical objects

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Abstract. In this work we generalize the notion of superscattering and associate it with a symmetry group of a scattering object. Using the group theory approach we describe a way to spectrally overlap several eigenmodes of a resonator in order to achieve scattering enhancement. Importantly, this can be done by simple variation of geometric parameters of the system, implying that the symmetry is preserved. We also demonstrate that a scattering cross-section limit of a spherical object is not valid for the case of non-spherical geometries. As an example, we use finite-size ceramic cylinder and demonstrate that a dipolar scattering cross-section limit of a spherical object can be exceeded by more than 3 times. The obtained results may be promising for design of antennas and radio frequency identification systems.

1. Introduction
Electromagnetic scattering is a phenomena which plays a crucial role in a wide range of practical applications. Antenna devices, radars, and radio frequency identification (RFID) technologies are based on efficient manipulation of scattering. One of the challenging problems is the miniaturization of resonant elements, since the scattering peak bandwidth decreases with the size of the scatterer (Chu-Harrington limit \cite{1, 2}). Possibly the only way to enhance scattering cross-section (SCS) of subwavelength particles is to spectrally overlap several resonances. The scattered field can be expanded into a series of vector spherical harmonics (multipoles), associated with the far-field of a point multipole (dipole, quadrupole, octupole, etc.) \cite{3}. Multipoles are orthogonal and therefore represent independent scattering channels and the total SCS ($C_{\text{SCS}}$) can be considered as the sum of partial cross-sections. For objects with spherical symmetry partial cross-sections are limited by the value \cite{4, 5}

\[ C_{\ell}^{\max} = \frac{2\ell + 1}{2\pi} \lambda^2, \]

where $\ell$ is the orbital angular momentum and $\lambda$ is the wavelength. The scattering cross-section reaches this limit at the Mie resonance condition. But it is possible to overcome the limit by spectral overlapping of several resonances. In this case the scattering structure is usually called a superscatterer \cite{6, 7}. It should be highlighted that the notation of superscattering is...
applicable only if scattering channels can be identified. For that, there has to be a one-to-one correspondence between a resonance of a structure and a single scattering channel. This is possible only for high-symmetry objects, such as spheres and two-dimensional cylinders [6, 8, 9]. In other cases, the single-channel limit Eq. 1 defined for spherical objects is not valid, since in a non-spherical case there is a rescattering between the channels.

In this work we provide a generalization of superscattering notion for objects of non-spherical symmetry. As an illustrative example we use finite-size cylindrical resonator and demonstrate that collocation of several resonances of different symmetry can significantly enhance the SCS.

2. Generalization of superscattering for non-spherical objects

Electromagnetic scattering can be analyzed via T-matrix [10] which links incident and scattered fields via the relation

$$b = \hat{T}a,$$

where $a$ and $b$ are the vectors of complex amplitudes, corresponding to the incident and scattered fields, respectively. For the case of a spherical resonator $\hat{T}$ is diagonal, which means that there is no coupling between different multipoles and the field can only be scattered into specific channels. For example, a dipole harmonic of the incident field can only be scattered into the dipole channel.

The situation is different for non-spherical objects since in this case the multipoles of the scattered field are partially mixed. However, they still form independent sets linked to eigenmodes of a resonator, which can be classified with different symmetry groups. Therefore, for the case of non-spherical resonators T-matrix is block-diagonal [11]:

$$\hat{T} = \text{diag}\{\hat{T}_1, \hat{T}_2, \ldots\},$$

where the blocks $\hat{T}_s$ correspond to the modes of different symmetry. If modes are transformed differently under the operations of the resonator’s symmetry group, they are affiliated with two different blocks. Similarly, transformation properties of the multipoles defines a block into which they are contributing. Therefore, mixing rules can be formulated using a group theory approach.

The multipoles contributing to the same block of $\hat{T}$ form an orthogonal basis of the irreducible representations (irreps) of the resonator’s symmetry group [11]. The number of blocks is equal to the number of its irreducible representations [12]. Importantly, different types of modes and irreducible representations are related through a one-to-one correspondence (the Wigner theorem) [13] and therefore, the independent scattering channels are associated with irreducible representation but not with particular multipoles. This is a crucial difference between spherical and non-spherical resonators.

The total SCS can be represented as a sum of partial SCSs corresponding to the different irreducible representations:

$$C_{\text{scat}} = \sum_{s=\text{irreps}} \sum_{\ell,m,\sigma} C_{\ell\ell\mu}^{\sigma s},$$

where $\sigma$ is the polarization and $\ell$ is the angular momentum. Therefore, scattering enhancement or superscattering for the non-spherical resonator can be achieved via spectral overlapping of several modes from different irreducible representations or, in other words, from different blocks of T-matrix.

3. Maximization of scattering cross-section for cylindrical resonator

As an illustrative example, we would like to consider a finite-size cylindrical resonator made of ceramics with permittivity $\varepsilon_{\text{Cyl}} = 44.8$ and and loss tangent $\tan\delta = 10^{-4}$. The cylinder is illuminated by the TE-polarized incident wave [see Fig. 1(a)]. The color map in Fig. 1(b)
Figure 1. (a) Schematic picture of the considered system. The cylinder with height \( h \), radius \( r \) and permittivity \( \varepsilon_{\text{cyl}} = 44.8 \) is illuminated by a TE-polarized plane wave. Colormap (b) illustrates the numerically calculated SCS as the function of the normalized frequency \( \omega r/c \) and cylinder’s aspect ratio \( r/h \). Here, values of SCS are normalized to the corresponding dipole single-channel limit [Eq. (1)]. The blue dot indicates one of the most pronounced peaks of SCS and the white dashed rectangle indicates the area around it. (c) Example of ceramic cylinders used for the experimental measurements. Each cylinder has different height but the same radius \( r = 15.7 \) mm. Dielectric permittivity of the sample is \( \varepsilon_{\text{cyl}} = 44.8 \) and loss tangent \( \tan \delta = 10^{-4} \). (d) The measured map of the SCS spectra.

demonstrates the evolution of modes as a function of parameters of the system, namely, the aspect ratio and the frequency. The calculated values of \( C_{\text{sc}} \) are normalized to the single-channel dipolar limit for spherical objects [Eq. (1), \( \ell = 1 \)] to highlight the relative enhancement. It can be seen that SCS prevails the dipolar limit by a factor of 3-4 in the vicinity of the points where several modes of the cylinder are overlapped. Importantly, this enhancement can be achieved by tuning of only one geometric parameter of the resonator, contrary to the cases where additional coating layers or surface structuring is needed, as for example in Refs. [8, 9]. Therefore, the proposed design is very simple and can be easily implemented in practice.

The same result also was demonstrated experimentally. For that the manufactured ceramic rod with radius 16.2 mm was sliced into several sections of different heights, which allow to set the height of the resonator within the range from 0.25 mm to 15 mm with the step 0.2 mm. Figure 1(c) shows an example of several slices. The rod is manufactured by sintering ceramic powder of calcium titanate-lanthanum aluminate (LaAlO\(_3\)-CaTiO\(_3\)) into a solid. The permittivity of the rod and loss tangent are the same as those used for numerical calculations, \( \varepsilon_{\text{cyl}} = 44.8 \) and \( \tan \delta = 10^{-4} \) [14]. The rod was placed on a foam holder made of Penoplex, which is transparent in GHz region. Figure 1(d) demonstrates the measured map of the SCS.
The results are in a good agreement with the numerical calculations and the provided map also demonstrates that maximal SCS is achieved at the points where modes are overlapped.

4. Modal and multipole description of scattering enhancement

Figure 2. (a) Eigenmodes and irreducible representations of the resonator calculated in the vicinity of the SCS peak. (b) Multipole decomposition of the total SCS at $r/h = 0.48$. Solid black line corresponds to the result of summation of the partial SCSs (ED, MD, EQ, and MQ) shown by colored lines and dashed black line corresponds to full-wave numerical calculations. (c) Characteristic field distributions of eigenmodes corresponding to the irreducible representations $E_{1u}$, $A_{2g}$ and $E_{2g}$, and their multipole far-field decompositions.

The modes of a finite cylinder can be characterized by azimuthal number $m = 0, 1, 2, ...$ and parity (odd or even) of the electric field distribution with respect to the mirror reflection $z \rightarrow -z$. However, contrary to spherical resonators, an orbital quantum number $\ell$ is not a good quantum number for a non-spherical resonator. To explicitly demonstrate the relation between the scattering enhancement and spectral overlapping of the modes associated to different irreducible representations, we calculate the eigenmode spectrum as a function of aspect ratio for the area marked in Fig. 1(b) by the white dashed rectangle. The spectrum obtained with COMSOL Multiphysics® software is demonstrated in Fig. 2(a). This result clearly shows that only the modes from different irreducible representation can intersect while modes from the same irreducible representations form avoid crossings. Associated examples of the electric field distributions and the multipole content for several eigenmodes from different irreducible representations are shown in Fig. 2(c).

Let us consider one of the points where SCS reaches a maximal value. In particular, when $r/h = 0.48$ [see blue dot in Fig. 1(b)] SCS prevails the dipolar single-channel limit ($\ell = 1$) for a spherical object by more than three times. This enhancement is a result of spectral overlapping of two modes from different irreducible representations, $A_{2g}$ and $E_{1u}$ (we use the standard
notations [15]). The dominant contribution to SCSs for these modes is given by the electric and magnetic dipoles, as Fig. 2(b) demonstrates. It should be highlighted that the partial SCSs corresponding to the electric and magnetic dipole moments exceed the single-channel limit for a sphere due to rescattering between the channels corresponding to different multipoles. Therefore, the single-channel limit for a spherical resonator is not applicable for a non-spherical one. However, the question of whether the limit for low-symmetry objects exists or not, remains open and it will be the subject of further research.

5. Conclusion
In this work the concept of superscattering is generalized for the case of non-spherical resonators. The enhancement of scattering cross-section can be achieved via spectral overlapping between eigenmodes of different symmetry, which was demonstrated for cylindrical resonator with finite size. Importantly, overlapping can be achieved by simple tuning of geometrical parameters of the system, which for the case of cylindrical resonator is its aspect ratio. It is also demonstrated that the partial SCS of a non-spherical object may exceed the corresponding single-channel limit for a sphere, hence, the limit for non-spherical objects has to be introduced, which is an open question yet.

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References
[1] Harrington R F 1960 J. Res. Nat. Bur. Stand 64 1–12  
[2] Chu L J 1948 J. Appl. Phys. 19 1163–1175  
[3] Evlyukhin A B, Fischer T, Reinhardt C and Chichkov B N 2016 Phys. Rev. B 94 205434  
[4] Foot C J et al. 2005 Atomic Physics vol 7 (Oxford University Press)  
[5] Tribel’skii M I 1984 JETP 86 915–919  
[6] Ruan Z and Fan S 2010 Phys. Rev. Lett. 105 013901  
[7] Ruan Z and Fan S 2011 Appl. Phys. Lett. 98 043101 ISSN 0003-6951  
[8] Qian C, Lin X, Yang Y, Xiong X, Wang H, Li E, Kaminer I, Zhang B and Chen H 2019 Phys. Rev. Lett. 122 063901  
[9] Mirzaei A, Miroshnichenko A E, Shadrivov I V and Kivshar Y S 2014 Appl. Phys. Lett. 105 011109  
[10] Mishchenko M I, Travis L D and Mackowski D W 1996 J. Quant. Spectrosc. Radiat. Transfer 55 535–575  
[11] Kahnert M 2005 JOSA A 22 1187–1199  
[12] Tinkham M 2003 Group Theory and Quantum Mechanics (Mineola, N.Y: Dover Publications) ISBN 978-0-486-43247-2  
[13] Wigner E P 1959 Group Theory and Its Application to the Quantum Mechanics of Atomic Spectra (Pure and Applied Physics no 5) (New York, NY: Academic Press) ISBN 978-0-323-15278-5  
[14] Nenasheva Y A 1992 MRS Online Proceedings Library Archive 269 ISSN 0272-9172, 1946-4274  
[15] Ivchenko E L and Pikus G 1995 Superlattices and Other Heterostructures: Symmetry and Optical Phenomena Springer Series in Solid-State Sciences (Berlin Heidelberg: Springer-Verlag) ISBN 978-3-642-97589-9