The Strong Approximation Conjecture holds for amenable groups

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Abstract. Let $G$ be a finitely generated group and $G 	riangleright G_1 	riangleright G_2 	riangleright \ldots$ be normal subgroups such that $\bigcap_{k=1}^{\infty} G_k = \{1\}$. Let $A \in \text{Mat}_{d \times d}(\mathbb{C}G)$ and $A_k \in \text{Mat}_{d \times d}(\mathbb{C}(G/G_k))$ be the images of $A$ under the maps induced by the epimorphisms $G \to G/G_k$. According to the strong form of the Approximation Conjecture of Lück $[4]$,

$$\dim_G(\ker A) = \lim_{k \to \infty} \dim_{G/G_k}(\ker A_k),$$

where $\dim_G$ denotes the von Neumann dimension. In $[2]$ Dodziuk, Linnell, Mathai, Schick and Yates proved the conjecture for torsion free elementary amenable groups. In this paper we extend their result for all amenable groups, using the quasi-tilings of Ornstein and Weiss $[6]$.

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1 Introduction

First, let us recall the approximation result of Dodziuk, Linnell, Mathai, Schick and Yates [2]. Let $G$ be a finitely generated group and let $A \in \text{Mat}_{d \times d}(\mathbb{C}G)$.

Let $l^2(G) = \{ f: G \to \mathbb{C} \mid \sum_{g \in G} |f(g)|^2 < \infty \}$. By left convolution, $A$ induces a bounded linear operator $A: (l^2(G))^d \to (l^2(G))^d$, which commutes with the right $G$-action. Let

$$\text{proj}_{\ker A} : (l^2(G))^d \to (l^2(G))^d$$

be the orthogonal projection onto $\ker A$. Then

$$\dim_G(\ker A) := \text{Tr}_G(\text{proj}_{\ker A}) := \sum_{i=1}^d \langle \text{proj}_{\ker A} 1_i, 1_i \rangle_{(l^2(G))^d}$$

where $1_i \in (l^2(G))^d$ is the function which takes the value $e_i$ on the unit element of $G$ and zero elsewhere ($\{e_1, e_2, \ldots, e_n\}$ is an orthonormal basis of $\mathbb{C}^d$). $\dim_G(\ker A)$ is called the von Neumann dimension of $\ker A$.

Now let $G \triangleright G_1 \triangleright G_2 \triangleright \ldots$ be normal subgroups such that $\cap_{k=1}^\infty G_k = \{1\}$.

Let $A_k \in \text{Mat}_{d \times d}(\mathbb{C}(G/G_k))$ be the images of $A$ under the maps induced by the epimorphisms $G \to G/G_k$. According to the strong form of the Approximation Conjecture of Lück [4]

$$\dim_G(\ker A) = \lim_{k \to \infty} \dim_{G/G_k}(\ker A_k).$$

In [2] the authors prove the conjecture above in the case when $G$ is a torsion-free elementary amenable group. The goal of this paper is to extend their result to arbitrary amenable groups. If $A \in \text{Mat}_{d \times d}(\mathbb{Z}(G))$ the problem is much easier to handle since one can use the method of Lück [5]. Then the conjecture holds for a large class of groups including amenable and residually finite groups. In the case of complex group algebra the situation seems much more complicated. Dodziuk et al. used noncommutative algebra to prove the conjecture, we shall use the quasi-tilings of Ornstein and Weiss.
2 Preliminaries

Let $G$ be a finitely generated amenable group with a finite symmetric set of generators $S$. Consider the Cayley-graph $C_G$, where $V(C_G) = G$ and

$$E(C_G) := \{(x, y) \in G \times G \mid y = sx, s \in S\}.$$  

Now we introduce some notation frequently used in the paper later on.

1. If $g \in G$, then its word-length $w(g)$ is defined as $d_{C_G}(g, 1)$, where $d_{C_G}$ is the shortest path distance on the Cayley-graph.

2. Let $F \subset G$ be a finite set, $k > 0$, then $B_k(F)$ denotes the $k$-neighborhood of $F$ in the $d_{C_G}$-metric.

3. We denote by $\Omega_k(F)$ the set of vertices $p$ in $F$, such that $d_{C_G}(p, F^c) > k$, where $F^c$ is the complement of $F$.

4. For $A \in \text{Mat}_{d \times d}(\mathbb{C}G)$, its propagation $w(A)$ is just sup $w(g)$, where $g$ runs through the terms of non-zero coefficients in the entries of $A$. Observe that if $f \in (l^2(G))^d$ and supp$f = U \subset G$, then supp$A(f) \in B_{w(A)}(U)$, and if supp$f = \Omega_{w(A)}(U)$ then supp$A(f) \subseteq U$.

5. For a finite set $F \subset G$, $\partial F$ denotes the set of vertices in $F$ such that $d_{C_G}(p, F^c) = 1$.

6. Since $G$ is amenable, it has a Folner-exhaustion, that is a sequence of subsets $1 \in F_1 \subset F_2 \subset \ldots, \cup_{n=1}^{\infty} F_n = G$ such that $i(F_n) \to 0$.

Now we prove some approximation theorems for amenable groups. Let $1 \in F_1 \subseteq F_2 \subseteq \ldots$ be a Folner exhaustion of $G$ and $P_n : (l^2(G))^d \to (l^2(F_n))^d$ be the orthogonal projections. Then by Theorem 3.11 \cite{2} (or Proposition 1. \cite{3}) :

$$\dim_G(\ker A) = \lim_{n \to \infty} \frac{\dim_{C}(\ker P_n AP_n^*)}{|F_n|}.$$  

We define the following sequences of vector spaces:

$$Z_n := \{ f \in (l^2(G))^d \mid \text{supp} f \subseteq B_{w(A)}(F_n), A(f) \mid_{F_n} = 0 \}$$
\[ W_n := \{ f \in (l^2(G))^d \mid \text{supp} f \subseteq \Omega_{w(A)}(F_n), A(f) = 0 \} \]

\[ V_n := \ker(P_nAP_n^*) \]

**Proposition 2.1**

\[
\lim_{n \to \infty} \frac{\dim_C(Z_n)}{|F_n|} = \dim_G(\ker A), \quad \lim_{n \to \infty} \frac{\dim_C(W_n)}{|F_n|} = \dim_G(\ker A).
\]

**Proof.** It is enough to prove that

\[
\lim_{n \to \infty} \frac{\dim_C(Z_n)}{\dim_C(V_n)} = 1 \quad (1)
\]

and

\[
\lim_{n \to \infty} \frac{\dim_C(W_n)}{\dim_C(V_n)} = 1 \quad (2)
\]

Clearly, \( W_n = V_n \cap \{ f \in l^2(F_n)^d \mid \text{supp} f \subseteq \Omega_{w(A)}(F_n) \} \). Hence (2) follows from the fact that

\[
\lim_{n \to \infty} \frac{|\Omega_{w(A)}(F_n)|}{|F_n|} = 1.
\]

Also, \( W_n = Z_n \cap \{ f \in l^2(F_n)^d \mid \text{supp} f \subseteq \Omega_{w(A)}(F_n) \} \). Therefore (1) follows from the fact that

\[
\lim_{n \to \infty} \frac{|\Omega_{w(A)}(F_n)|}{|B_{w(A)}(F_n)|} = 1
\]

**Definition 2.1** Let \( F \subset G \) be a finite set and \( \delta, \epsilon > 0 \) be real numbers. We say that \( F \) has property \( A(\epsilon, \delta, -) \) if for any subset \( K \subseteq F \), \( \frac{|K|}{|F|} > 1 - \epsilon \), the following holds:

- If \( R = \{ f \in l^2(F_n)^d \mid \text{supp} f \subseteq \Omega_{w(A)}(K), A(f) = 0 \} \), then
  \( \dim_C(R) \geq (1 - \delta)(\dim_G(\ker A)) \).

Also, we say that \( F \) has property \( A(\delta, +) \) if the following holds:

- If \( Q \) is the restriction of the space

  \[ Z_F := \{ f \in (l^2(G))^d \mid \text{supp} f \subseteq B_{w(A)}(F), A(f) \mid_F = 0 \} \]
onto $F$, then
\[ \dim_C(Q) \leq (\dim_G(\ker A)) + \delta \]

Similarly to Proposition \[24\] one can easily prove the following proposition.

**Proposition 2.2** Let $1 \in F_1 \subseteq F_2 \subseteq \ldots$ be a Folner exhaustion of $G$ as above. Then for any pair of real numbers $\delta, \epsilon > 0$ there exists $n_{\delta, \epsilon}$ such that if $n \geq n_{\delta, \epsilon}$ then $F_n$ has both properties $A(\epsilon, \delta, -)$ and $A(\delta, +)$.

## 3 Graph convergence and dimension averaging

Let $C_G$ be the Cayley-graph of the previous section. Color the directed edge $(x \to y), x = sy$ by $s \in S$ (hence $(y \to x)$ shall be colored by $s^{-1}$). Thus we color all edges in both direction with the elements of the set $S$ such a way that for each $x \in G$ the edges outgoing from $x$ are colored in different ways. The following definition is a variation of the one on random weak convergence in [1].

Let $B_1, B_2, \ldots$ be an infinite sequence of finite graphs. Assume that for any $x \in V(B_n)$:
\[ \deg(x) \leq |S| \]

We also assume that the directed edges are colored by $S$ such a way that:

- the color of the edge $(x \to y)$ is the inverse of the color of $(y \to x)$.
- the outgoing edges from any vertex are colored differently.

We say that $p \in V(B_n)$ is $k$-similar to the identity of $G$, if its $k$-neighborhood in $B_n$ is edge-colored isomorphic to the $k$-neighborhood of the identity in $C_G$. Let $Q^B_k$ be the set of vertices in $B$ that are $k$-similar to the identity. Then we say that $\{B_n\}_{n=1}^\infty$ converge to $C_G$ if for any $\epsilon > 0$ and $k \in \mathbb{N}$ there exists $n_{\epsilon, k}$ such that if $n \geq n_{\epsilon, k}$ then
\[ Q^B_k > (1 - \epsilon)|V(B_n)|. \]

**Example 1.** Let $G$ be a finitely generated group and $\{B_n\}_{n=1}^\infty$ be a sequence of finite induced subgraphs forming a Folner-exhaustion. Then $\{B_n\}_{n=1}^\infty$ converge to $C_G$. 

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Example 2.: Let $G$ be a finitely generated residually finite group and $G \triangleright G_1 \triangleright G_2 \triangleright \ldots \ldots$ be a sequence of finite index normal subgroups such that $\cap_{n=1}^\infty G_n = \{1\}$. Let $C_n$ be the Cayley-graph of $G/G_n$. Then $\{C_n\}_{n=1}^\infty$ converge to $C_G$.

Now let $A \in \text{Mat}_{d \times d}(\mathbb{C}G)$. One can define the transformation kernel of $A$, $\tilde{A} : G \times G \to \text{Mat}_{d \times d}(\mathbb{C})$ the following way. First write $A$ in the form of $\sum_{\gamma \in G} A_{\gamma} \cdot \gamma$, where $A_{\gamma} \in \text{Mat}_{d \times d}(\mathbb{C})$. Then set $\tilde{A}(\gamma, \delta) := A_{\gamma\delta^{-1}}$. Thus if $f \in l^2(G)^d$, then

$$A(f)(\delta) = \sum_{\gamma \in G} \tilde{A}(\delta, \gamma) f(\gamma).$$

Now let $\{B_n\}_{n=1}^\infty$ be a sequence of graphs converging to $C_G$. Then we define the finite dimensional linear transformations $T_n : (l^2(V(B_n)))^d \to (l^2(V(B_n)))^d$ approximating $A$, the following way.

- If $x \in Q^{B_n}_{w(A)}$, $y \in V(B_n)$ and $d_{B_n}(y, x) \leq w(A)$, let $\tilde{T}_n(y, x) := A(\gamma, 1)$, where $\gamma$ is the element of $G$ satisfying $\phi^{x}_{w(A)}(\gamma) = y$. Here $\phi^{x}_{w(A)}$ is the unique colored isomorphism between the $w(A)$-neighborhood of 1 in $C_G$ and the $w(A)$-neighborhood of 1 in $B_n$.

- If $x \notin Q^{B_n}_{w(A)}$ or $d_{B_n}(y, x) > w(A)$, then let $\tilde{T}_n(x, y) := 0$.

Then if $f \in l^2(V(B_n))^d$ and $p \in V(B_n)$:

$$T_n(f)(p) = \sum_{q \in V(B_n)} \tilde{T}_n(p, q) f(q).$$

The main goal of our paper is to prove the following theorem.

**Theorem 1** If $G$ is a finitely generated amenable group and $\{B_n\}_{n=1}^\infty$, $\{T_n\}_{n=1}^\infty$ are as above, then

$$\lim_{n \to \infty} \frac{\dim_{\mathbb{C}} \ker T_n}{|V(B_n)|} = \dim_G(\ker A).$$

The Strong Approximation Conjecture for amenable groups follows from the theorem:

**Corollary 3.1** If $G$ is a finitely generated amenable group and $G \triangleright G_1 \triangleright G_2 \ldots \ldots \cap_{n=1}^\infty G_n = \{1\}$ are normal subgroups, then

$$\lim_{n \to \infty} \dim_{G/G_n}(\ker A_n) = \dim_G(\ker A),$$

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where \( A \in \text{Mat}_{d \times d}(\mathbb{C}G) \) and \( A_n \in \text{Mat}_{d \times d}(\mathbb{C}(G/G_n)) \) are the images of \( A \) under the maps induced by the epimorphisms \( G \rightarrow G/G_n \).

Proof. (of the Corollary)

Case 1. Suppose that all \( G_n \) has finite index. Note that in this case \( T_n = A_n \) if \( n \) is large enough, hence the corollary immediately follows.

Case 2. Assume that for large enough \( n \), the amenable group \( G/G_n \) is infinite. Let \( 1 \in F_1^n \subset F_2^n \subset \ldots \) be a Folner-exhaustion of the Cayley graph \( C_{G/G_n} \) (using the image of the generator system \( S \)). Then

\[
\dim_{G/G_n}(A_n) = \lim_{k \to \infty} \frac{\dim_{\mathbb{C}}(\ker P^n_k A_n (P^n_k)^*)}{|F^n_k|},
\]

where \( P^n_k : (l^2(G/G_n))^d \rightarrow (l^2(F^n_k))^d \) is the orthogonal projection.

Pick a sequence \( F^1_{m_1}, F^2_{m_2}, \ldots \) such that

\[
\begin{align*}
&\bullet i(F^j_{m_j}) \to 0. \\
&\bullet (\dim_{G/G_n}(A_n) - \frac{\dim_{\mathbb{C}}(\ker P^n_{m_n} A_n (P^n_{m_n})^*)}{|F^n_{m_n}|}) \to 0.
\end{align*}
\]

Now let \( B^n_{m_n} \) be the graph induced by \( F^n_{m_n} \).

Lemma 3.1 \( \{B^n_{m_n}\}_{n=1}^{\infty} \) converge to \( C_G \).

Proof. Since \( \cap_{k=1}^{\infty} G_k = \{1\} \), for any \( d \in \mathbb{N} \) there exists \( n_d > 0 \) such that if \( n \geq n_d \) then the \( d \)-balls in \( C_{G/G_n} \) are colored-isomorphic to the \( d \)-ball of \( C_G \). Let \( H^n_{m_n} = \Omega_d(F^n_{m_n}) \). Clearly \( H^n_{m_n} \subseteq Q^n_{m_n} \). Since the vertex degrees of \( G/G_n \) are at most \( S \), \( |H^n_{m_n}| \geq |F^n_{m_n}| - |S|^d |\partial F^n_{m_n}| \).

Now our lemma easily follows.

Lemma 3.2

\[
\lim_{n \to \infty} \frac{\dim_{\mathbb{C}}(\ker P^n_k A_n (P^n_k)^*)}{\dim_{\mathbb{C}}(\ker T_n) \dim_{\mathbb{C}}(\ker T_n)} = 0.
\]

Here \( T_n \) is the linear operator associated to \( B^n_{m_n} \).

If \( \text{supp} f \subset F^n_{m_n} \setminus B_w(A)(\partial F^n_{m_n}) \) then \( T_n(f) = P^n_k A_n (P^n_k)^* (f) \). Since

\[
\frac{|F^n_{m_n} \setminus B_w(A)(\partial F^n_{m_n})|}{|F^n_{m_n}|} \to 1
\]

our lemma follows.

Obviously, Lemma 3.1 and Lemma 3.2 imply the corollary.
4 Quasi-tilings

Let us recall the notion of quasi-tilings from [6]. Let $X$ be a finite set and $\{A_i\}_{i=1}^n$ be finite subsets of $X$. Then we say that $\{A_1, A_2, \ldots, A_n\}$ are $\epsilon$-disjoint if there exist subsets $\overline{A_i} \subseteq A_i$ such that

- For any $1 \leq i \leq n$, $\frac{\overline{A_i}}{|A_i|} \geq 1 - \epsilon$.
- If $i \neq j$ then $\overline{A_i} \cap \overline{A_j} = \emptyset$.

On the other hand, if $\{H_j\}_{j=1}^m$ are finite subsets of $X$, then we say that they $\alpha$-cover $X$ if

$$\frac{|X \cap (\bigcup_{j=1}^m H_j)|}{|X|} \geq \alpha.$$ 

Finally, we say that the collection $\{H_1, H_2, \ldots, H_m\}$ $\delta$-evenly covers $X$ if there exists some $M \in \mathbb{R}^+$ such that

- For any $p \in X$, $\sum_{j:p \in H_j} 1 \leq M$.
- $\sum_{j=1}^m |H_j| \geq (1 - \delta)M|X|$.

According to Lemma 4. [6], if $\{H_1, H_2, \ldots, H_m\}$ form a $\delta$-even covering of $X$, then for any $0 < \epsilon < 1$ there exists an $\epsilon$-disjoint subcollection of the $H_j$'s that $\epsilon(1 - \delta)$-covers $X$.

Now we define tiles for our $S$-edge colored graphs. Let $G$ be a finitely generated group with a symmetric generator set $S$ and let $1 \in F_1 \subseteq F_2 \subseteq \ldots \cup_{n=1}^\infty F_n = G$ be a Folner-exhaustion. Let $B$ be a finite graph as in the previous section with edge-colorings by the elements of $S$. Also, let $L$ be a natural number. Let $\{F_{\alpha_1}, F_{\alpha_2}, \ldots, F_{\alpha_n}\}$ be a finite collection of the Folner sets above such that for any $1 \leq i \leq n$, $F_{\alpha_i} \subset B_{\frac{1}{2L}}(1)$. Then for any $x \in Q^B_L$ and $1 \leq i \leq n$, $T_x(F_{\alpha_i})$ is the image of $F_{\alpha_i}$ under the unique colored isomorphism $\phi^x_L : B_L(1) \rightarrow B_L(x)$ mapping 1 to $x$. We call such a subset a tile of type $F_{\alpha_i}$ and say that $x$ is the center of $T_x(F_{\alpha_i})$. A system of tiles $\epsilon$-quasi tile $V(B)$ if they are $\epsilon$-disjoint and form an $(1 - \epsilon)$-cover.

The following theorem is a version of Theorem 6. in [6].

**Theorem 2** For any $\epsilon > 0$, $n > 0$, there exist $L > 0$, $\delta > 0$ and a finite collection $\{F_{n_1}, F_{n_2}, \ldots, F_{n_n}\} \subset B_L(1)$ of the Folner sets, such that $n_i > n$ and if

$$\frac{|Q^B_L|}{|V(B)|} > 1 - \delta$$
then $V(B)$ can be $\epsilon$-quasi-tiled by tiles of the form $T_x(F_{n_i})$, $x \in Q^B_L$, $1 \leq i \leq s$.

5 The inducational step

First of all fix a constant $\epsilon_1 < \frac{\epsilon}{100}$. Let us call a finite set $H \subset G$ a set of type $(K, \alpha)$, $K \in \mathbb{N}$, $\alpha \geq 0$ if

$$\frac{|B_K(H)|}{|H|} < 1 + \alpha. \quad (3)$$

Now let $B$ be our $S$-edge colored finite graph and suppose that

$$\frac{|Q^B_L|}{|V(B)|} > 1 - \beta. \quad (4)$$

The exact values of $\beta$ and $L$ shall be given later. Assume that $H$ is of type $(K, \alpha)$, where

$$H \subset B^{1+L}(1) \quad \text{and} \quad K < \frac{L}{10}. \quad (5)$$

Now consider all tiles in $B$ in the form $T_x(H)$, where $x \in Q^B_{\frac{L}{2}}$. Note that no vertices of $B$ is covered by more than $|H|$ tiles. Indeed, if $z$ is covered, then the $\frac{L}{2}$-neighborhood of $z$ in $B$ is colored isomorphic to the $\frac{L}{2}$-neighborhood of 1 in $G$. Hence if $z \in T_x(H)$, then $z \in Q^B_{\frac{L}{2}}$ and $x \in T_z(H^{-1})$. Summarizing these:

- For any $y \in V(B)$, $\sum_{x:y \in T_x(H)} 1 \leq |H|$.
- $\sum_{x \in Q^B_L} |T_x(H)| = |Q^B_L||H| \geq (1 - \beta)|V(B)||H|$.

Consequently, the tiles $\{T_x(H)\}_{x \in Q^B_L}$ form a $\delta$-even covering of $V(B)$, where $\delta = 1 - \beta$. Then by Lemma 4. of [3], there exists an $\epsilon_1$-disjoint subcollection of tiles, $\cup_{x \in I} T_x(H)$ such that they form a $\epsilon_1(1 - \beta)$-covering of $V(B)$.

Now suppose that the number of vertices in $V(B)$ not covered by this subcollection above is greater than $\frac{\epsilon}{2}|V(B)|$. Let $B_1$ be the graph induced by the uncovered vertices. We would like to estimate the quotient: $\frac{|Q^B_L|}{|V(B)|}$. Note that if $y \in V(B_1)$ and

$$y \notin \cup_{x \in I}(T_x(B_K(H)) \setminus T_x(H))$$
then \( y \in Q^B_K \). Hence by \( \epsilon_1 \)-disjointness,

\[
| \bigcup_{x \in I} (T_x(B_K(H)) \setminus T_x(H)) | \leq \alpha \sum_{x \in I} |H| \leq \alpha (1 - \epsilon_1)^{-1} |V(B)| .
\]

Hence

\[
|Q^B_K| \geq |V(B_1)| - \alpha (1 - \epsilon_1)^{-1} |V(B)| ,
\]

that is

\[
\frac{|Q^B_K|}{|V(B)|} \geq 1 - \beta_1 ,
\]

where \( \beta_1 = \alpha (1 - \epsilon_1)^{-1} \frac{1}{\epsilon} \). Also note that

\[
\frac{\epsilon}{2} |V(B)| \leq |V(B_1)| \leq (1 - \epsilon_1(1 - \beta)) |V(B)| .
\]

### 6 The proof of Theorem 2

Let \( \{\alpha_k\}_{k=1}^\infty \) be a sequence of real numbers tending to zero and let \( \{s_k\}_{k=1}^\infty \) be a sequence of real numbers tending to infinity, satisfying the following inequalities:

\[
s_k \geq 1, s_{k+1} \geq 10s_k .
\]

We call a subsequence of the Folner exhaustion \( \{F_n\}_{n=1}^\infty \) an \((\alpha, s)\)-good subsequence if it satisfies the following conditions:

- \( 1 \in F_{n_1} \subset B_{s_1}(1) \subset F_{n_2} \subset B_{s_2}(1) \subset F_{n_3} \subset \ldots \)
- \( F_{n_{i+1}} \) is of type \((100s_i, \alpha_i)\).

Obviously one can choose \( \{s_k\}_{k=1}^\infty \) for any fixed \( \{\alpha_k\}_{k=1}^\infty \) to have such \((\alpha, s)\)-good subsequences. Now let \( M \) be an integer such that

\[
(1 - \frac{\epsilon_1}{2})^M < \frac{\epsilon}{100} .
\]

Also, pick \( \beta > 0 \) so that

\[
\beta M < \frac{\epsilon}{100} .
\]
And finally fix a sequence \( \{\alpha_k\}_{k=1}^{\infty} \) such that
\[
\alpha_i (1 - \epsilon_1)^{-2} < \beta . \tag{9}
\]
Now let \( B \) a finite \( S \)-colored graph such that
\[
\frac{Q_B^{B_1}}{|V(B)|} > 1 - \beta ,
\]
where \( \beta, M \) are as above. Then by the argument of the previous section we can \( \epsilon_1 (1 - \beta) \)-cover the vertices of \( B \) by \( \epsilon_1 \)-disjoint tiles of type \( F_{nM} \). If \( B_1 \) is the graph induced by the uncovered vertices of \( V(B) \), by (8):
\[
\frac{|Q_{sB_1}^{B_1}|}{|V(B_1)|} > 1 - \beta_1 ,
\]
where \( \beta_1 = \alpha_M (1 - \epsilon_1)^{-2} \). Now we can \( \epsilon_1 (1 - \beta_1) \)-cover \( V(B_1) \) by tiles of type \( F_{nM-1} \). If \( B_2 \) denotes the graph induced by the uncovered part of \( V(B_1) \) then
\[
\frac{|Q_{sB_2}^{B_2}|}{|V(B_2)|} > 1 - \beta_2 ,
\]
where \( \beta_2 = \alpha_{M-1} (1 - \epsilon_1)^{-2} \).

We proceed inductively. In each step the new tiles are disjoint from all previous ones. Also,
\[
V(B_i) \leq V(B_{i-1})(1 - \frac{\epsilon_1}{2}) .
\]

Hence by our conditions, in at most \( M \) steps we obtain an \( \epsilon \)-disjoint \( (1 - \epsilon) \)-covering of \( V(B) \).

7 The proof of Theorem

Let \( G \) be a finitely generated amenable group, \( A \in \text{Mat}_{d \times d}(\mathbb{C}G) \) and \( \{B_n\}_{n=1}^{\infty} \) be a sequence converging to \( C_G \). Let \( \{T_n\}_{n=1}^{\infty} \) be the sequence of approximating operators as in Section 3.

Proposition 7.1 For any pair \( \delta, \epsilon > 0 \) there exists \( k_{\delta, \epsilon} > 0 \) such that if \( k \geq k_{\delta, \epsilon} \) then
\[
\frac{\dim_{\mathbb{C}}(\ker T_k)}{|V(B_k)|} \geq (\dim_G(\ker A) - \delta) (1 - \epsilon) .
\]
Proof. Let $1 \in F_1 \subseteq F_2 \subseteq \ldots$ be a Folner exhaustion of $G$, such that all the $F_n$’s have property $A(\epsilon, \delta, -)$ (see Lemma 2.1). Let $\{H_1, H_2, \ldots H_s\}$ be an $\epsilon$-quasi-tiling of $B_k$ by tiles from this Folner sequence. Such $\epsilon$-quasi-tiling exists by Theorem 2 if $k$ is large enough. For $1 \leq i \leq s$ let $K_i \subset H_i$ be a subset such that

- $\frac{|K_i|}{|H_i|} > 1 - \epsilon$.
- $K_i \cap K_j = \emptyset$ if $i \neq j$.

Since the $F_n$’s have property $A(\epsilon, \delta, -)$ there exist subspaces $V_i \subset (l^2(B_n))^d$ such that

- If $f \in V_i$ then supp $f \subseteq K_i$.
- $f \in \ker T_k$.
- $\frac{\dim_{V_i}}{|H_i|} \geq \dim_G(\ker A) - \delta$.

Now consider the subspace $\oplus_{i=1}^s V_i \subseteq \ker T_k$. Then

$$\dim_C(\oplus_{i=1}^s V_i) \geq \left( \sum_{i=1}^s |H_i| \right) \left( \dim_G(\ker A) - \delta \right) \geq (1 - \epsilon)|V(B_k)| \left( \dim_G(\ker A) - \delta \right).$$

That is

$$\frac{\dim_C(\ker T_k)}{|V(B_k)|} \geq (\dim_G(\ker A) - \delta)(1 - \epsilon).$$

Proposition 7.2 For any pair $\delta, \epsilon > 0$ there exists $m_{\delta, \epsilon} > 0$ such that if $k \geq m_{\delta, \epsilon}$ then

$$\frac{\dim_C(\ker T_k)}{|V(B_k)|} \leq (1 - \epsilon)^{-1}(\dim_G(\ker A) + \delta) + \epsilon.$$  

Proof. Again let $1 \in F_1 \subseteq F_2 \subseteq \ldots$ be a Folner exhaustion of $G$, such that all the $F_n$’s have property $A(\delta, +)$ (see Lemma 2.1). Consider the $\epsilon$-quasi-tilings of the previous proposition. Now let $W_i \subset l^2(H_i)$ be the restriction of $\ker T_k$ onto $H_i$. By our assumption,

$$\dim_G(\ker T_k) \leq |H_i| \left( \dim_G(\ker A) + \delta \right).$$

Since $\{H_i\}_{i=1}^s$ form an $\epsilon$-covering

$$\dim_C(\ker T_k) \leq \epsilon |V(B_k)| + \sum_{i=1}^s |H_i| \left( \dim_G(\ker A) + \delta \right).$$
Note that by $\epsilon$-disjointness
\[
\sum_{i=1}^{s} |H_i| \leq (1 - \epsilon)^{-1}|V(B_k)|.
\]
Thus
\[
\frac{\dim C(\ker T_k)}{|V(B_k)|} \leq (1 - \epsilon)^{-1}(\dim G(\ker A) + \delta) + \epsilon.
\]
Clearly, Propositions 7.1 and 7.2 imply Theorem 1.

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