Quasiclassical expansion for $\text{Tr}\{(-1)^F e^{-\beta H}\}$.

A.V. Smilga

SUBATECH, Université de Nantes, 4 rue Alfred Kastler, BP 20 722, Nantes 44307, France.

Abstract

We start with some methodic remarks referring to purely bosonic quantum systems and then explain how corrections to the leading-order quasiclassical result for the fermion-graded partition function $\text{Tr}\{(-1)^F e^{-\beta H}\}$ can be calculated at small $\beta$. We perform such calculation for certain supersymmetric quantum mechanical systems where such corrections are expected to appear. We consider in particular supersymmetric Yang-Mills theory reduced to $(0+1)$ dimensions and were surprised to find that the correction $\propto \beta^2$ vanishes in this case. We discuss also a nonstandard $\mathcal{N}=2$ supersymmetric $\sigma$–model defined on $S^3$ and other 3–dimensional conformally flat manifolds and show that the quasiclassical expansion breaks down for this system.

1 Introduction

For many supersymmetric quantum systems, the fermion–graded partition function

\[ Z^{F–\text{grade}}(\beta) = \text{Tr}\{(-1)^F e^{-\beta \hat{H}}\} \]  

\[ (1.1) \]

\footnote{This is how the object (1.1) will be called in this paper (no established term for it exists in the literature). We admit that this name is awkward, but it is not misleading like “supersymmetric partition function” or “index” would be. (i) $Z^{F–\text{grade}}(\beta)$ is not the partition function $\text{Tr}\{e^{-\beta \hat{H}}\}$ of a supersymmetric system. Besides, though supersymmetric theories provide the main motivation and interest in studying the quantity (1.1), the latter can be defined for any system involving fermion degrees of freedom. (ii) If $Z^{F–\text{grade}}(\beta)$ depends on $\beta$, it does not coincide with the index (1.2).}
does not depend on $\beta$ and and defines the Witten index of the system

$$I_W = n_B^{(0)} - n_F^{(0)} = \lim_{\beta \to \infty} Z^{F-\text{grade}}(\beta).$$

(1.2)

Usually, $Z^{F-\text{grade}}(\beta)$ can be evaluated at small $\beta$ with quasi-classical methods. The leading order result is

$$Z^{F-\text{grade}}(\beta) = \int \prod_{i} \frac{dp_i dq_i}{2\pi} \prod_{a} d\bar{\psi}^a d\psi^a \exp \left\{ -\beta H^{\text{cl}}(p_i, q_i; \bar{\psi}^a, \psi^a) \right\},$$

(1.3)

where $(p_i, q_i)$ and $(\bar{\psi}^a, \psi^a)$ are bosonic and fermionic phase space variables, and $H^{\text{cl}}$ is the classical Hamiltonian function corresponding to the quantum Hamiltonian $\hat{H}$.

This philosophy does not always work, however. First of all, one can claim that $Z^{F-\text{grade}}(\beta)$ is $\beta$–independent only for the systems with discrete spectrum. Then the contributions of the degenerate boson and fermion states cancel in the trace and only the zero–energy states contribute. If the spectrum is continuous, it is not immediately clear what the trace or supertrace is. Some regularization is required or an additional definition for the quantity $Z^{F-\text{grade}}(\beta)$ should be given. After that $Z^{F-\text{grade}}(\beta)$ can well display a nontrivial dependence on $\beta$.

As an important example of supersymmetric quantum mechanics (SQM) with continuous spectrum, consider the super-Yang-Mills (SYM) quantum mechanics obtained by dimensional reduction of SYM theories. Consider first the system obtained from $\mathcal{N} = 1, d = 4$ SYM theory based on the gauge group $G$. The Hamiltonian has the form

$$\hat{H} = \frac{1}{2} \hat{P}_i^a \hat{P}_i^a + \frac{g^2}{4} f^{abc} f^{def} A_i^a A_j^b A_k^c A_l^d + ig f^{abc} \lambda_a \lambda_b \lambda_c,$$

$$i = 1, 2, 3, \quad \alpha = 1, 2 \quad a = 1, \ldots, \dim(G).$$

(1.4)

$A_i^a$ are the gauge potentials, $\hat{P}_i^a \equiv \hat{E}_i^a = -i\partial / \partial A_i^a$ are their canonical momenta operators, and $\lambda_a^\alpha$ and $\lambda_a^\lambda = \partial / \partial \lambda_a^\alpha$ are the fermionic gluino variables and their momenta. The Hilbert space includes only the physical states annihilated by the Gauss law constraints

$$\hat{G}^a \Psi = f^{abc} \left( \hat{P}_i^b A_i^c + i \lambda_a^{\alpha} \lambda_a^\lambda \right) \Psi = 0.$$

(1.5)
The system has two conserved complex supercharges

\[ \hat{Q}_\alpha = \frac{1}{\sqrt{2}} \lambda^{\alpha}_\beta \left[ \hat{E}^{\alpha}_i + \frac{i g}{2} \epsilon_{ijk} f^{abc} A^b_j A^c_k \right] \]  

(they formed a Weyl spinor before reduction) and, being restricted on the Hilbert space (1.5) enjoys the \( \mathcal{N} = 2 \) SQM algebra \( \{ \hat{Q}_\alpha^+, \hat{Q}_\beta \} = \delta^{\alpha}_\beta \hat{H} \).

The classical potential in Eq.(1.4) vanishes in the “vacuum valleys” with \( f^{abc} A^b_i A^c_j = 0 \). Due to supersymmetry, degeneracy along the valleys survives also after quantum corrections are taken into account. As a result, the system tends to escape along the valleys, the wave function of the low–energy states is delocalized, and the spectrum is continuous (this implies, incidentally, the continuity of the mass spectrum of supermembranes [2]). The calculation of \( Z^{F\text{–grade}}(\beta) \) for the system (1.4) in the quasiclassical aproximation gives a fractional number. For example, for the \( SU(2) \) gauge group [3, 4, 5]

\[ \lim_{\beta \to 0} Z^{F\text{–grade}}(\beta) = \frac{1}{4} . \]  

(1.7)

On the other hand, this system does not have normalized vacuum state with zero energy and

\[ I_W = \lim_{\beta \to \infty} Z^{F\text{–grade}}(\beta) = 0 . \]  

(1.8)

A similar mismatch between the values of \( Z^{F\text{–grade}}(\beta) \) in the two limits shows up in more complicated SYM QM systems. \( Z^{F\text{–grade}}(0) \) is always fractional [4, 6], while the Witten index vanishes for the \( \mathcal{N} = 2 \) and \( \mathcal{N} = 4 \) systems (the latter are obtained by dimensional reduction from 6–dimensional SYM theories) and acquires a nonzero integer value for the systems obtained by the reduction of 10–dimensional SYM theories and involving 8 complex supercharges [8, 9, 5].

In Sect. 5 we will deal with this system and calculate the 1–loop correction to the fractional leading order result for \( Z^{F\text{–grade}}(\beta) \) at small \( \beta \). Remarkably, the correction of order \( \beta^2 \) vanishes in all cases. We expect, however, nontrivial corrections to appear in higher loops.

For the systems with discrete spectrum, \( Z^{F\text{–grade}}(\beta) \) is \( \beta \)–independent, but, contrary to naive expectations, it does not always mean that it can be evaluated with the quasiclassical formula (1.3). In Sect. 6 we discuss a rather
nontrivial example of a nonstandard $\mathcal{N} = 2$ supersymmetric $\sigma$ model living on $S^3$, for which the phase space integral (1.3) does not give a correct result for the index.

The model belongs to a class of $\mathcal{N} = 2$ supersymmetric $\sigma$ models living on 3–dimensional conformally flat manifolds [10, 11]. The supercharges and the Hamiltonian of the model have the form

$$
\hat{Q}_\alpha = -i \sqrt{\frac{1}{2}} \left[ (\sigma_k)_\alpha^\beta \psi_\beta f(x) \hat{p}_k + i \partial_k f(x) \hat{\psi} \sigma_k \psi_\alpha \right],
$$

$$
\hat{Q}^\alpha = i \sqrt{\frac{1}{2}} \left[ \bar{\psi}^\beta (\sigma_k)^\beta_\alpha f(x) \hat{p}_k - i \partial_k f(x) \bar{\psi} \sigma_k \bar{\psi}^\alpha \right],
$$

$$
\hat{H} = \frac{1}{2} f(x) \hat{p}_k^2 f(x) - \epsilon_{jkp} \bar{\psi} \sigma_j \psi f(x) \partial_p f(x) \hat{p}_k - \frac{1}{2} f(x) \partial_k^2 f(x) (\hat{\bar{\psi}} \psi)^2,
$$

where the differential operators $\hat{p}_k = -i \partial/\partial x_k$ and $\hat{\bar{\psi}}_\alpha = \partial/\partial \psi_\alpha$ act on everything on the right they find. The algebra

$$
\{ \hat{Q}_\alpha, \hat{Q}_\beta \} = 0, \quad \{ \hat{Q}^\alpha, \hat{Q}^\beta \} = \delta^\alpha_\beta \hat{H}
$$

holds.

If the manifold is compact, the spectrum is discrete. For $S^3$ [with $f(x) = 1 + x^2/(4R^2)$], there are two bosonic states with zero energy, which gives $I_W = 2$. On the other hand, the integral (1.3) is not equal to 2. Moreover, its value depends on the way the ordering ambiguities are resolved and a classical counterpart of the quantum Hamiltonian in Eq.(1.9) is defined. Also, there are nonvanishing corrections, which are of the same order as the leading order result and as the two–loop and higher–loop corrections. The whole quasiclassical expansion for $Z^{F-\text{grade}}(\beta)$ breaks down.

To prepare ourselves for the discussion of these comparatively complicated problems, we start in the next section with recalling the technique of calculating the loop corrections to the ordinary partition function $Z = \text{Tr} \exp\{-\beta \hat{H}\}$ for purely bosonic systems. Gauge QM systems, where the calculation involves certain subtleties, are considered in Sect. 3. In Sect. 4 we generalize the analysis to the systems with fermion dynamic variables, being especially interested in supersymmetric systems. We consider the simplest SQM system due to Witten [12], where the corrections to $Z^{F-\text{grade}}(\beta)$ vanish even in the cases when the spectrum is continuous and one could expect a priori the
corrections to appear. Sect. 5 is devoted to SYM QM and Sect. 6 — to the $\mathcal{N} = 2$ supersymmetric $\sigma$–model on $S^3$. The last section is reserved, as usual, to recapitulating the results and to the discussion.

2 Quasiclassical expansion of the partition function.

Consider a purely bosonic QM system. To leading order, the partition function is given by the integral

$$Z = \text{Tr}\{e^{-\beta H}\} = \int \prod_i \frac{dp_i dq_i}{2\pi} \exp\left\{-\beta H^{\text{cl}}(p_i, q_i)\right\}. \quad (2.1)$$

Strictly speaking, the function $H^{\text{cl}}$ is not uniquely defined due to the ordering ambiguities, but the ambiguity in the prescription $\hat{H} \rightarrow H^{\text{cl}}(p_i, q_i)$ does not affect the results in the leading order in $\beta$. We will stick to the most convenient choice and assume that $H^{\text{cl}}(p_i, q_i)$ coincides with the Weyl symbol of the quantum Hamiltonian $\hat{H}$. For sure, the final results (when all corrections are taken into account) do not depend on convention.

The result (2.1) represents the leading term in the high-temperature (small $\beta$) expansion for $Z$. The next-to-leading term is suppressed compared to Eq.(2.1) as $\sim \beta^2 E_{\text{char}}^2$, where $E_{\text{char}}$ is the characteristic energy scale. For the Hamiltonian of the simplest type,

$$H(p_i, q_i) = \frac{p_i^2}{2} + V(q_i), \quad (2.2)$$

the partition function is given by the well-known expression [14]

$$Z = \int \prod_i \frac{dp_i dq_i}{2\pi} e^{-\beta H(p_i, q_i)} \left[1 - \frac{\beta^2}{24} \frac{\partial^2 V}{\partial q_i^2} + O(\beta^4)\right] \quad (2.3)$$

($H(p_i, q_i) \equiv H^{\text{cl}}(p_i, q_i)$). In a generic case, the result is [15]

$$Z = \int \prod_i \frac{dp_i dq_i}{2\pi} e^{-\beta H(p_i, q_i)} \left[1 - \frac{\beta^2}{24} \left(\frac{\partial^2 H}{\partial p_i \partial p_j} \frac{\partial^2 H}{\partial q_i \partial q_j} \right) + O(\beta^4)\right]. \quad (2.4)$$

\footnote{This is true for $Z$, but not always true for $Z^{F\text{-grade}}$. See Sect. 6.}
The result (2.4) can be derived in two ways. First, one can note that

\[ Z = \int \prod_i \frac{dp_i dq_i}{2\pi} \left[ e^{-\beta \hat{H}} \right]_W, \]  

(2.5)

where \([\hat{O}]_W\) is the Weyl symbol of the operator \(\hat{O}\). To leading order, the Weyl symbol of the exponential is given by the exponential of the Weyl symbol [this gives Eq.(2.1)], but there are corrections. In the modern language, they stem from the fact that the “star product” is not just a simple product. In the Appendix, we present, following Ref.[15], an accurate calculation of such corrections, which leads to Eq.(2.4).

Alternatively, one can use the functional integral representation for the partition function. To make the problem simpler, consider first the case with only one pair \((p,q)\) of phase space variables. The path integral for the partition function is

\[ Z = \int Dp Dq \exp \left\{ \int_0^\beta d\tau [ip\dot{q} - H(p, q)] \right\}, \]  

(2.6)

where \(\tau\) is the Euclidean time. Periodic boundary conditions

\[ q(\beta) = q(0), \quad p(\beta) = p(0) \]  

(2.7)

are imposed. For very small \(\beta\), we can ignore the \(\tau\)-dependence of our variables and set \(q(\tau) \approx \bar{q}\) and \(p(\tau) \approx \bar{p}\). The measure \(Dp Dq\) happens to go into \((dpdq)/(2\pi)\) and we are reproducing the result (2.1). To find the corrections, we write

\[ q(\tau) = \bar{q} + x(\tau), \quad p(\tau) = \bar{p} + s(\tau) \]  

(2.8)

with \(\int d\tau x(\tau) = \int d\tau s(\tau) = 0\) and assume \(x(\tau)\) and \(s(\tau)\) to be small. Expanding over \(x(\tau)\) and \(s(\tau)\) up to the second order (the linear terms vanish), we obtain

\[
Z \approx \int \frac{dp dq}{2\pi} e^{-\beta H(p, q)} \int D's D'x \\
\exp \left\{ \int_0^\beta d\tau \left[ is(\tau)\dot{x}(\tau) - \frac{A}{2}s^2(\tau) - \frac{C}{2}x^2(\tau) - Bs(\tau)x(\tau) \right] \right\}, \]

(2.9)
where
\[ A = \frac{\partial^2 H}{\partial \bar{p}^2}, \quad B = \frac{\partial^2 H}{\partial \bar{p} \partial \bar{q}}, \quad C = \frac{\partial^2 H}{\partial \bar{q}^2}. \] (2.10)

The prime in the measure \( \mathcal{D}' s \mathcal{D}' x \) means that the zero Fourier harmonics of \( s(\tau) \) and \( x(\tau) \) are constrained to be zero. Let us make now the canonical transformation
\[ S = A^{1/2} \left( s + \frac{B x}{A} \right), \quad X = A^{-1/2} x. \] (2.11)

It leaves invariant the measure and the Poisson bracket,
\[ \{S(\tau), X(\tau')\}_{P.B.} = \{s(\tau), x(\tau')\}_{P.B.} = \delta(\tau - \tau') - \frac{1}{\beta}. \] (2.12)

We obtain
\[
Z = \int \mathcal{D}' S \mathcal{D}' X \exp \left\{ \int_0^\beta d\tau \left[ iS(\tau) \dot{X}(\tau) - \frac{1}{2} S^2(\tau) - \frac{1}{2} \omega_{pq}^2 X^2(\tau) \right] \right\},
\] (2.13)
where
\[ \omega_{pq}^2 = \frac{\partial^2 H}{\partial \bar{p}^2} \frac{\partial^2 H}{\partial \bar{q}^2} - \left( \frac{\partial^2 H}{\partial \bar{p} \partial \bar{q}} \right)^2. \] (2.14)

If the integration over the zero harmonics of \( S(\tau) \) and \( X(\tau) \) were also included, the inner integral would determine the partition function of the harmonic oscillator. As it is not included, the integral represents the ratio of the full partition function of the oscillator
\[ Z_{osc} = \int \mathcal{D} S \mathcal{D} X \exp \left\{ \int_0^\beta d\tau \left[ iS(\tau) \dot{X}(\tau) - \frac{1}{2} S^2(\tau) - \frac{1}{2} \omega^2 X^2(\tau) \right] \right\} = \frac{1}{2 \sinh \left( \frac{\beta \omega}{2} \right)} \] (2.15)

and the integral
\[ \int \frac{d\bar{S} d\bar{X}}{2\pi} \exp \left\{ -\frac{\beta}{2} \left[ \bar{S}^2 + \omega^2 \bar{X}^2 \right] \right\} = \frac{1}{\beta \omega}. \] (2.16)
[Eq.(2.16) is none other than the partition function of the oscillator in the quasiclassical limit]. We finally obtain

\[ Z \approx \int \frac{d\bar{p}d\bar{q}}{2\pi} e^{-\beta H(\bar{p},\bar{q})} \frac{\beta \omega_{pq}}{2 \sinh \left( \frac{\beta \omega_{pq}}{2} \right)} \]

\[ \approx \int \frac{d\bar{p}d\bar{q}}{2\pi} e^{-\beta H(\bar{p},\bar{q})} \left[ 1 - \frac{\beta^2}{24} \omega_{pq}^2 + \cdots \right] \quad (2.17) \]

in accordance with Eq.(2.3). The higher–order terms of the expansion of \( \sinh^{-1}(\beta \omega/2) \) give the corrections \( \propto \beta^4 \) etc, but, to take the latter into account correctly, one has also to keep higher terms in the expansion of the Hamiltonian over \( s(\tau) \) and \( x(\tau) \).

In the general multidimensional case, the corrections may be found by solving the quantum mechanical problem with the Hamiltonian

\[ \tilde{H} = \frac{1}{2} A_{jk} s_j s_k + B_{jk} s_j x_k + \frac{1}{2} C_{jk} x_j x_k , \quad (2.18) \]

\[ A_{jk} = \frac{\partial^2 H}{\partial \bar{p}_j \partial \bar{p}_k} , \quad B_{jk} = \frac{\partial^2 H}{\partial \bar{p}_j \partial \bar{q}_k} , \quad C_{jk} = \frac{\partial^2 H}{\partial \bar{q}_j \partial \bar{q}_k} . \]

With a proper choice of variables, the Hamiltonian (2.18) describes the (multidimensional) motion of a scalar charged particle in a generic oscillator potential and in a constant magnetic field. The problem can be solved exactly. It is reduced to a set of oscillators whose frequencies can be found algebraically. The simplest way to do it is to calculate the Gaussian path integral for \( Z \). On the first step, we integrate over the momenta \( s_j(\tau) \) (with the zero Fourier modes subtracted) and express the multidimensional analog of the inner integral in Eq.(2.9) as

\[ I \propto \prod_j \int \mathcal{D}x_j \exp \left\{ - \int_0^\beta \mathcal{L}_E \right\} , \quad (2.19) \]

where

\[ \mathcal{L}_E = \frac{1}{2} (A^{-1})_{jk} \dot{x}_j \dot{x}_k - i (A^{-1}B)_{jk} \dot{x}_i x_k + \frac{1}{2} (C - B^T A^{-1} B)_{jk} x_j x_k . \quad (2.20) \]

Let us expand

\[ x_j(\tau) = \sum_{n=1}^\infty x_j^{(n)} e^{2\pi imn/\beta} + \text{complex conjugate} \quad (2.21) \]
and do the integral over
\[ \prod_j D' x_j \propto \prod_{j=1}^{\infty} dx_j^{(n)} d\bar{x}_j^{(n)}. \]

We obtain
\[ I = \prod_{n=1}^{\infty} \frac{\det \| A^{-1} \omega_n^2 \|}{\det \| A^{-1} \omega_n^2 + (A^{-1} B - B^T A^{-1}) \omega_n + C - B^T A^{-1} B \|} = \prod_{n=1}^{\infty} \frac{\omega^{2N}}{\det \| \omega_n^2 + (B - AB^T A^{-1}) \omega_n + AC - AB^T A^{-1} B \|}, \tag{2.22} \]

where
\[ \omega_n = \frac{2\pi n}{\beta} \tag{2.23} \]

and \( N \) is the number of degrees of freedom. The normalization factor in Eq.(2.22) is chosen such that \( I \) represents the ratio of the full partition function of the system (2.18) and this partition function in the quasiclassical limit. For the free Hamiltonian \( H = (1/2) A_{jk} p_j p_k \), \( I = 1 \).

Now, we write
\[ \det \| \omega_n^2 + (B - AB^T A^{-1}) \omega_n + AC - AB^T A^{-1} B \| = \prod_{j=1}^{N} (\omega_n^2 + \Omega_j^2), \tag{2.24} \]

where \(-\Omega_j^2\) are the roots of the corresponding polynomial.

The ratio (2.22) acquires the form
\[ I = \prod_{j=1}^{N} \left( \prod_{n=1}^{\infty} \frac{\omega_n^2}{\omega_n^2 + \Omega_j^2} \right) = \prod_{j=1}^{N} \frac{\beta \Omega_j}{2 \sin \frac{\beta \Omega_j}{2}}. \tag{2.25} \]

The full partition function
\[ Z = \prod_{j=1}^{N} \frac{1}{2 \sin \frac{\beta \Omega_j}{2}} \tag{2.26} \]

To prove that the left side of Eq.(2.24) is, indeed, a polynomial in \( \omega_n^2 \), let us choose a basis where \( A = 1 \). We are left with the expression \( \det \| \omega_n^2 + \omega_n F + S \| \), where \( S \) and \( F \) are symmetric and antisymmetric real matrices. It is not difficult to see now that the odd powers of \( \omega_n \) in the expansion of the determinant cancel out.
represents a product of the partition functions of the harmonic oscillators with frequencies $\Omega_j$. Generically, $\Omega_j$ appearing on the right side of Eq.(2.24) are complex. For example, it is so for a not positive definite and hence non-Hermitian Hamiltonian (2.18) with $A = -C = 1$, $B = 0$. If the Hamiltonian is Hermitian, all $\Omega_j$ are real. The spectrum has the form

$$E_{\{n_j\}} = \sum_{j=1}^{N} \left( \frac{1}{2} + n_j \Omega_j \right). \quad (2.27)$$

As a simplest nontrivial example, consider the 2-dimensional Hamiltonian

$$H = \frac{1}{2} \left( p_x - \frac{Hy}{2} \right)^2 + \frac{1}{2} \left( p_y + \frac{Hx}{2} \right)^2 + \frac{1}{2} \left( \omega_1^2 x^2 + \omega_2^2 y^2 \right). \quad (2.28)$$

Eq.(2.24) acquires the form

$$\det \begin{vmatrix} \omega_n^2 + \omega_1^2 & \omega_n H \\ -\omega_n H & \omega_n^2 + \omega_2^2 \end{vmatrix} = (\omega_n^2 + \Omega_1^2)(\omega_n^2 + \Omega_2^2), \quad (2.29)$$

which gives the frequencies

$$\Omega_{1,2} = \frac{1}{2} \left[ \sqrt{H^2 + (\omega_1 + \omega_2)^2} \pm \sqrt{H^2 + (\omega_1 - \omega_2)^2} \right]. \quad (2.30)$$

The result (2.30) was obtained earlier by a different method [16].

In this paper, our primary concern are the corrections $\propto \beta^2$ in the partition function. To find them, we do not need to determine all eigenfrequencies $\Omega_j$. It suffices to expand the right side of Eq.(2.22) in $\beta$ using the identity

$$\det \| 1 + \alpha \| = 1 + \text{Tr} \alpha + \frac{1}{2} \left[ (\text{Tr} \alpha)^2 - \text{Tr} \alpha^2 \right] + o(\alpha^2).$$

We obtain

$$I \approx \prod_{n=1}^{\infty} \frac{\omega_n^2}{\omega_n^2 + \text{Tr}(AC - B^2)} \approx 1 - \frac{\beta^2}{24} \text{Tr}(AC - B^2), \quad (2.31)$$

where the relation $\sum_{n=1}^{\infty} (1/n^2) = \pi^2/6$ was used. The result (2.31) coincides with the square bracket in Eq.(2.4).
3 Gauge quantum mechanics.

In order to prepare ourselves for the discussion of the SYM quantum mechanics, we are going to be as instructive as possible and consider two simple quantum mechanical models involving gauge symmetry.

3.1 $SO(2)$ gauge oscillator.

The simplest gauge QM system is the constrained two–dimensional oscillator (see e.g. Ref.[17, 18]). It is the system described by the Hamiltonian

$$\hat{H} = \frac{1}{2} \hat{p}_i^2 + \frac{1}{2} \omega^2 x_i^2$$

with the constraint

$$\hat{p}_\phi \Psi^{\text{phys}} = \epsilon_{jk} x_j \hat{p}_k \Psi^{\text{phys}} = 0 .$$

The spectrum of the system involves all rotationally invariant states $\Psi^{\text{phys}}_n$, $\epsilon_n = \omega(1 + 2n)$, and the partition function can be found straightforwardly

$$Z^{O(2)}(\beta) = \sum_n e^{-\beta \epsilon_n} = \frac{1}{2 \sinh(\beta \omega)} .$$

The same result is obtained by using functional methods. We start from the representation

$$Z^{O(2)}(\beta) = \int_0^{2\pi} d\phi \int dx \ K(x^\phi, x; -i\beta) ,$$

where $K(x^\phi, x; -i\beta)$ is the Euclidean evolution operator of the unconstrained system ($\beta$, $\infty$) and $x^\phi_i = O_{ij}(\phi)x_j$, $O_{ij}$ being an $SO(2)$ matrix. The integral over $d\phi$ kills all rotationally noninvariant states in the spectral decomposition of $K$. Eq.($\beta$) can be expressed into a path integral,

$$Z^{O(2)}(\beta) = \int_0^{2\pi} d\phi \int \prod d\chi(\tau) \exp \left\{ - \int_0^\beta d\tau L_E[\chi(\tau)] \right\} ,$$

where

$$L_E = \frac{1}{2} \dot{x}_i^2 + \frac{1}{2} \omega^2 x_i^2 ;$$

$$\chi(\beta) = x^\phi(0) .$$
The boundary conditions can be rendered periodic by changing the variables
\[ x(\tau) = y^{\phi/\beta}(\tau). \]  

We arrive at
\[ Z^{O(2)}(\beta) = \int_0^{2\pi} \frac{d\phi}{2\pi} \int dy(\tau) \exp \left\{ - \int_0^\beta \frac{d\tau}{\beta} \mathcal{L}_E[y(\tau)] \right\}, \tag{3.8} \]

where
\[ \mathcal{L}_E = \frac{1}{2} \left( \dot{y}_i + \frac{\phi}{\beta} \epsilon_{ij} y_j \right)^2 + \frac{1}{2} \omega^2 y_i^2 \tag{3.9} \]

and \( y(\beta) = y(0) \). If one wishes, one can upgrade the integral over \( d\phi \) to the path integral over the periodic functions \( \phi(\tau) \). Then the Lagrangian \( \mathcal{L}_E \) with time-dependent \( \phi(\tau) \) is invariant under the gauge transformations
\[ y_i = O_{ij}(\chi) y'_i, \quad \phi = \phi' - \beta \dot{\chi}. \tag{3.10} \]

The form \( (3.8) \) is more convenient, however. Substituting there the Fourier expansion for \( y(\tau) \) and doing the Gaussian integrals, we obtain
\[ Z^{O(2)}(\beta) = \int_0^{2\pi} \frac{d\phi}{2\pi} \left[ \frac{1}{\cosh(\beta \omega)} - \cos \phi \right] = \frac{1}{2} \frac{\sinh(\beta \omega)}{\sinh(\beta \omega)} \tag{3.12} \]

Let us now study the quasiclassical expansion of \( Z \). To leading order, the product over nonzero modes in Eq.\( (3.11) \) can be ignored, and we have
\[ [Z^{O(2)}(\beta)]_{\text{quasicl}} \approx \int_{-\pi}^{\pi} \frac{d\phi}{2\pi} \frac{1}{(\beta \omega)^2 + \phi^2} \approx \int_{-\infty}^{\infty} \frac{d\phi}{2\pi} \frac{1}{(\beta \omega)^2 + \phi^2} = \frac{1}{2\beta \omega}. \tag{3.13} \]
The same result is obtained in the Hamiltonian approach

\[ \left[ Z^{O(2)}(\beta) \right]^{\text{quasicl}} = \int \prod_{i=1}^{2} \frac{dp_i dx_i}{2\pi} \delta(p_\phi) \exp\{-\beta H\} \]

\[ = \int_\infty^{-\infty} d\phi \int \prod_{i=1}^{2} \frac{dp_i dx_i}{2\pi} \exp\{-\beta \tilde{H}\} , \]

(3.14)

where

\[ \tilde{H} = H + \frac{i\phi}{\beta} p_\phi . \]

(3.15)

To find the higher order terms in the quasiclassic expansion, let us expand the integrand in Eq.(3.11) [or in Eq.(3.12), which is easier] over \( \phi \) and \( \beta \omega \). Taking into account the leading and the next–to–leading term, we obtain

\[ Z \approx \int d\phi \frac{1}{2\pi (\beta\omega)^2 + \phi^2} \left[ 1 - \frac{(\beta\omega)^2 - \phi^2}{12} + \ldots \right] . \]

(3.16)

We seem to be in hot water now: the integral over \( \phi \) diverges linearly at large \( \phi \) or, if we keep the integration limits finite, it is determined by the region of large \( \phi \), where the expansion is not valid. Still, the correction can be calculated with the following recipe:

- Extend the limits of integration over \( \phi \) to \( \pm \infty \)
- Forget about the divergence and calculate the correction as the residue of the integrand at the pole at \( \phi = i\beta\omega \).

We obtain, indeed

\[ Z^{O(2)}(\beta) \approx \frac{1}{2\beta\omega} \left[ 1 - \frac{(\beta\omega)^2}{6} + \ldots \right] , \]

(3.17)

which coincides with the expansion of Eq.(3.3). Also next–to–next–to–leading and all other corrections can be obtained in this way.
This recipe seems to be rather wild, but it is not difficult to justify it quite rigourously. This is done with the following chain of relations

\[
Z^{O(2)}(\beta) = \int_{-\pi}^{\pi} \frac{d\phi}{4\pi} \frac{1}{\cosh(\beta \omega) - \cos \phi}
\]

\[
= \lim_{N \to \infty} \frac{1}{1 + 2N} \int_{-\pi(2N+1)}^{\pi(2N+1)} \frac{d\phi}{4\pi} \frac{1}{\cosh(\beta \omega) - \cos \phi}
\]

\[
\approx \lim_{N \to \infty} \frac{1}{1 + 2N} \sum_{k=-N}^{N} \frac{2\pi i}{4\pi} \text{res}_{\phi=i\beta \omega+2k\pi} \frac{1}{\cosh(\beta \omega) - \cos \phi} = 
\]

\[
i \text{res}_{\phi=i\beta \omega} \left\{ \frac{1}{(\beta \omega)^2 + \phi^2} \left[ 1 - \frac{(\beta \omega)^2 - \phi^2}{12} + \ldots \right] \right\} .
\]

We want to note here that essentially the same procedure of replacing divergent integrals by the residue contributions was used in Refs.\cite{6, 7} when calculating the fermion–graded partition function for SYM QM in the leading quasiclassical approximation. No justification of this procedure was given there, but we believe that it can be eventually found along the same lines as in the trivial example discussed above.

The final remark is that the corrections \( \propto (\beta \omega)^2 \) and \( \propto \phi^2 \) in the expansion (3.16) can, in the full analogy with Eq.(2.4), be cast in the form

\[
-\frac{\beta^2}{24} \left( \frac{\partial^2 \tilde{H}}{\partial p_i \partial p_j} \frac{\partial^2 \tilde{H}}{\partial q_i \partial q_j} - \frac{\partial^2 \tilde{H}}{\partial p_i \partial q_j} \frac{\partial^2 \tilde{H}}{\partial p_j \partial q_i} \right)
\]

(3.19)

with \( \tilde{H} \) given by Eq.(3.15).

### 3.2 SO(3) gauge oscillator.

The next in complexity example is the SO(3) gauge oscillator\cite{17} with the Hamiltonian

\[
\hat{H} = \frac{\hat{P}^a_i \hat{P}^a_i}{2} + \frac{1}{2} \omega^2 A^a_i A^a_i, \quad i, a = 1, 2, 3
\]

(3.20)

and the constraints

\[
\tilde{G}^a_i = \epsilon^{abc} \hat{P}^b_i A^c_i = 0 .
\]

(3.21)
\( \hat{G}^a \) can be interpreted as generators of isotopic gauge rotations. Only the isosinglet states are present in the physical spectrum.

Proceeding in the same way as above, we can represent the partition function of this system as the following path integral

\[
Z^{O(3)}(\beta) = \int \mathcal{D}\Omega(\phi) \int \prod_{i=1}^{6} dA^a_i(\tau) \exp \left\{ - \int_0^\beta d\tau \tilde{L}_E[A^a_i(\tau)] \right\},
\]

where

\[
\tilde{L}_E[A^a_i(\tau)] = \frac{1}{2} \left( \dot{A}^a_i + \epsilon^{abc} \frac{\phi^b}{\beta} A^c_i \right)^2 + \frac{1}{2} \omega^2 A^a_i A^a_i.
\]

Periodic boundary conditions for \( A^a_i(\tau) \) are imposed. Now, \( \mathcal{D}\Omega(\phi) \) is the Haar measure on the \( SO(3) \) group normalized to \( \int \mathcal{D}\Omega(\phi) = 1 \). Explicitly,

\[
\mathcal{D}\Omega(\phi) = \frac{d\phi(1 - \cos |\phi|)}{4\pi^2 \phi^2}, \quad 0 \leq |\phi| \leq \pi.
\]

To find the inner integral in Eq. (3.22), we can set \( \phi^a = (0, 0, \phi) \). Then the variables \( A^3_i \) are “neutral” with respect to \( \phi^a \) and 6 remaining “charged” variables are decomposed into three pairs, which are coupled to \( \phi \) in the same way as the pair \( x_i \) was coupled to \( \phi \) in the \( SO(2) \) case.

The calculation of the functional integral for the partition function involves the products over the modes. Each pair of charged variables brings about the factor

\[
\frac{1}{(\beta \omega)^2 + \phi^2} \prod_n \frac{\omega_n^2}{(\omega_n + \phi\beta)^2 + \omega^2} = \frac{1}{2 \cosh(\beta \omega) - \cos \phi}
\]

and each neutral variable provides a \( \phi \)-independent factor

\[
\frac{1}{\beta \omega} \prod_{n=1}^{\infty} \frac{\omega_n^2}{\omega_n^2 + \omega^2} = \frac{1}{2 \sinh \left( \frac{\beta \omega}{2} \right)}.
\]

Integrating it over \( \phi \) with the measure (3.24), we obtain the result

\[
Z^{O(3)}(\beta) = \frac{1}{8 \sinh^3 \left( \frac{\beta \omega}{2} \right)} \int_0^{\pi} \frac{d\phi}{\pi} (1 - \cos \phi) \left\{ \frac{1}{2 \cosh(\beta \omega) - \cos \phi} \right\}^3
\]

\[
= \frac{1}{64 \sinh^3 \left( \frac{\beta \omega}{2} \right)} \frac{2 \cosh^2(\beta \omega) + 1 - 3 \cosh(\beta \omega)}{2 \sinh^5(\beta \omega)}. \quad (3.27)
\]
At large $\beta$, 

$$Z^{O(3)}(\beta \omega \gg 1) \approx e^{-9\beta \omega/2} \left[ 1 + 6e^{-2\beta \omega} + e^{-3\beta \omega} + \ldots \right]. \quad (3.28)$$

This asymptotic expansion corresponds to the presence of the ground state with energy $9\omega/2$, which is an isosinglet; the absence of isosinglets with energy $9\omega/2 + \omega$; the presence of 6 isosinglets with energy $9\omega/2 + 2\omega$ (their wave functions have the form $\Psi_{ij} \propto A^a_i A^a_j \exp\{-\omega(A^b_k)^2/2\}$; the presence of one isosinglet with energy $9\omega/2 + 3\omega$ (its wave function involves the factor $\epsilon_{ijk} \epsilon^{a,b,c} A^a_i A^b_j A^c_k$); etc.

At small $\beta$, 

$$Z^{O(3)}(\beta \omega \ll 1) \approx \frac{1}{32(\beta \omega)^6} \left[ 1 + \frac{(\beta \omega)^2}{8} + \ldots \right]. \quad (3.29)$$

This result can be reproduced by quasiclassical expansion of the integrand in the path integral. To leading order, the partition function is given by the integral

$$Z_{\text{quasicl}}^{O(3)} = \int \frac{d\phi}{8\pi^2} \int \prod_{ia} \frac{dP^a_i dA^a_i}{2\pi} e^{-\beta \tilde{H}} \quad (3.30)$$

with

$$\tilde{H} = H + \frac{i\phi^a}{\beta} G^a. \quad (3.31)$$

If corrections in quasiclassical expansion are taken into account, the integral for the partition function reads

$$Z^{O(3)}(\beta \omega \gg 1) = \int \frac{d\phi}{8\pi^2} \left( 1 - \frac{\phi^2}{12} + \ldots \right) \int \prod_{ia} \frac{dP^a_i dA^a_i}{2\pi} e^{-\beta \tilde{H}}$$

$$\left[ 1 - \frac{\beta^2}{24} \left( \frac{\partial^2 \tilde{H}}{\partial P^a_i \partial P^b_j} \frac{\partial^2 \tilde{H}}{\partial A^a_i \partial A^b_j} - \frac{\partial^2 \tilde{H}}{\partial P^a_i \partial A^b_j} \frac{\partial^2 \tilde{H}}{\partial P^b_j \partial A^a_i} \right) + \ldots \right], \quad (3.32)$$

where the factor $1 - \phi^2/12$ comes from the expansion of the factor $2(1 - \cos \phi)/\phi^2$ in the measure (3.24) and the correction $\propto \beta^2$ in the inner integral has the same origin as before. We emphasize that the integrals in Eqs. (3.31)
are done over the whole range of $\phi$. Calculating the derivatives and performing the integral over $\prod_\alpha dP^\alpha dA^\alpha$, we obtain
\[
\frac{1}{(\beta \omega)^3} \int_0^\infty \frac{\phi^2 d\phi}{2\pi} \left( 1 - \frac{\phi^2}{12} \right) \frac{1}{[(\beta \omega)^2 + \phi^2]^3} \left[ 1 + \frac{2\phi^2 - 3(\beta \omega)^2}{8} \right],
\]
which coincides with the expansion of the integral in Eq. (3.27) where the upper limit is set to infinity. The integral (3.33) converges, but the expansion up to the terms $\propto \phi^4$, $\propto \phi^6$, etc results in divergent integrals and only the contribution of the residues at $\phi = i \omega \beta$ should be taken into account.

The recipe given after Eq. (3.16) works again. It can be justified in the same way as above by using the periodicity of the integrand in the exact expression for $Z^{O(3)}(\beta)$ in Eq. (3.27).

4 Supersymmetric quantum mechanics.

In this section, we will study the fermion–graded partition function (1.1) in the simplest supersymmetric quantum mechanical system[12]. The (classical) Hamiltonian of the model is
\[
H = \frac{p^2}{2} + \frac{1}{2} [V'(x)]^2 + V''(x) \bar{\psi} \psi,
\]
where $\psi, \bar{\psi}$ are holomorphic fermion Grassmann variables. The function $V(x)$ is called superpotential. The fermion–graded partition function is given by the Euclidean path integral where both bosonic and fermionic variables are periodic in Euclidean time $\tau$. To leading order, one can assume $x, p, \psi, \bar{\psi}$ to be constant, and we arrive at the result (1.3). Performing the integrals over $d\bar{\psi}d\psi dp$, we obtain
\[
Z^{F-\text{grade}} = \sqrt{\frac{\beta}{2\pi}} \int_{-\infty}^\infty dx V''(x) \exp \left\{ -\frac{\beta}{2} [V'(x)]^2 \right\}.
\]
If the potential $[V'(x)]^2/2$ grows at large $x$, the spectrum is discrete and $Z^{F-\text{grade}} = \pm 1$ or 0, depending on the asymptotics of $V(x)$. $Z^{F-\text{grade}}$ defines

\footnote{Recall that the path integral for the ordinary partition function $\text{Tr} \{ e^{-\beta H} \}$ involves antiperiodic boundary conditions for fermionic variables.}
In this case the Witten index of the system. In the next-to-leading order, $Z^F_{\text{grade}}$ is given by the integral

$$Z^F_{\text{grade}}(\beta) = \int \prod_i \frac{dp_i dq_i}{2\pi} \prod_a d\bar{\psi}_a d\psi_a \exp \{-\beta H\} (1 + \delta), \quad (4.3)$$

where

$$\delta(p_i, q_i; \psi_a, \bar{\psi}_a) = \frac{\beta^2}{48} \left[ \frac{\partial^2}{\partial \psi_a \partial \psi_a} - \frac{\partial^2}{\partial \bar{\psi}_a \partial \bar{\psi}_a} + i \left( \frac{\partial^2}{\partial q_i \partial p_i} \right) \right] H(p_i, q_i; \psi_a, \bar{\psi}_a) H(P_i, Q_i; \Psi_a, \bar{\Psi}_a) \bigg|_{P=p, Q=q, \Psi=\psi, \bar{\Psi}=\bar{\psi}}. \quad (4.4)$$

The formulae (4.3), (4.4) represent a rather straightforward generalisation of Eq.(2.4) and can be derived either using the methods of Appendix [the right side of Eq.(4.4) is expressed via the star product $H \ast H$ in the case when $H$ depends on both bosonic and fermionic variables] or with the functional methods of Sect.2. We leave it to the reader as an exercise.

For the Hamiltonian (4.1) under consideration,

$$\delta = -\frac{\beta^2}{24} [V'(x) V^{(3)}(x) + V^{(4)}(x) \bar{\psi} \psi]. \quad (4.5)$$

Substituting it in Eq.(4.3) and doing the integral over $d\bar{\psi}d\psi dp$, we obtain

$$\Delta Z^F_{\text{grade}}(\beta) = \frac{\beta^{3/2}}{24 \sqrt{2\pi}} \int_{-\infty}^{\infty} dx [V^{(4)} - \beta V' V'' V^{(3)}] \exp \left\{ -\frac{\beta [V']^2}{2} \right\}$$

$$= \frac{\beta^{3/2}}{24 \sqrt{2\pi}} \int_{-\infty}^{\infty} dx \frac{\partial}{\partial x} \left[ V^{(3)} \exp \left\{ -\frac{\beta [V']^2}{2} \right\} \right] = 0. \quad (4.6)$$

In other words, the corrections to the leading order result for $Z^F_{\text{grade}}(\beta)$ vanish in this case, as should have been expected in advance.

Actually, the correction (4.6) always vanishes, even for the systems with continuum spectrum, where $\text{Tr}\{(-1)^F e^{-\beta H}\}$ may depend on $\beta$. As an example, consider the Hamiltonian (4.1) with

$$V(x) = \ln \left[ \cosh \left( \frac{x}{a} \right) \right]. \quad (4.7)$$
In this case, \( V'(x) = (1/a) \tanh(x/a) \) tends to a constant at large \( x \) and the spectrum is continuous. The fermion–graded partition function depends on \( \beta \) and it is seen already in the leading quasiclassical order,

\[
Z^{F-\text{grade}}(\beta) = \Phi \left( \frac{\sqrt{\beta}}{a} \right),
\]

where \( \Phi(x) \) is the probability integral. This expression alone provides the correct asymptotics \( \lim_{\beta \to \infty} Z^{F-\text{grade}}(\beta) = 1 \) corresponding to the presence of the normalized vacuum state with

\[
\Psi(x) \propto e^{-V(x)} = \frac{1}{\cosh \left( \frac{x}{a} \right)}.
\]

And the integral (4.6) and probably also all higher–order terms in the quasiclassical expansion vanish.

As another example with continuous spectrum and mismatch between \( \text{Tr}\{(-1)^F e^{-\beta H}\} \) and the Witten index, consider the superconformal SQM with \( V(x) = \lambda \ln(x/a) \) [19]. If \( \lambda > 1 \), we have the singular repulsive potential \( \propto 1/x^2 \) in both the bosonic and fermion sectors so that the motion is restricted to the half–line \( x \in (0, \infty) \). Hence, the integral (4.2) is done within the limits \( 0 \leq x < \infty \), and we obtain \( Z^{F-\text{grade}}(\beta) = 1/2 \) in the leading order. The integral (4.6) and all higher–loop corrections vanish, however. In this case, it is not so surprising. The Hamiltonian (1.1) with \( V(x) \propto \ln x \) does not involve a dimensionful parameter and neither \( E_{\text{char}} \) nor the dimensionless parameter of the quasiclassical expansion \( \beta E_{\text{char}} \) can be defined.

5 SYM quantum mechanics.

There are systems, however, for which the quasiclassical series for \( Z^{F-\text{grade}}(\beta) \) is expected to be “alive”. In this section we consider the gauge supersymmetric quantum mechanical systems obtained by dimensional reduction from \( \mathcal{N} = 1 \) SYM field theories in 4, 6, and 10 dimensions. We will refer to them as \( \mathcal{N} = 2 \), \( \mathcal{N} = 4 \), and \( \mathcal{N} = 8 \) SYM QM systems, indicating the number of different complex supercharges \( Q_\alpha \). Consider first the \( \mathcal{N} = 2 \) system described by the Hamiltonian (1.4) and the constraints (1.3). To leading order,
the partition function is given by the integral

\[
Z_{\text{SYM}}(\text{small } \beta) = \frac{1}{V_G} \int \prod_a d(\beta g A^a_0) \int \prod_{ai} \frac{dP^a_idA^a_i}{2\pi} \prod_{a\alpha} d\bar{\lambda}^\alpha_a d\lambda^a_\alpha \exp\{-\beta[H + igA^a_0G]\},
\]

(5.1)

where \( \beta g A^a_0 \equiv \phi^a \) are gauge parameters and \( V_G \) is the volume of the gauge group. The notation \( A^a_0G \equiv A^a_0G^a \) is used.

As was mentioned in the introduction, the calculation of the integral (5.1) gives a fractional number, while the Witten index \( Z_{\text{SYM}}(\beta = \infty) \) is integer (it is zero for the \( \mathcal{N} = 2 \) and \( \mathcal{N} = 4 \) systems). Also, the Hamiltonian (1.4), in contrast to the Hamiltonian of the superconformal quantum mechanics considered at the end of the previous section, involves a dimensional parameter \( g \) [remember, we are not in (3+1), but in (0+1) dimensions!]. The characteristic energy scale is \( E_{\text{char}} \sim g^{2/3} \) (this estimate is easily obtained by equating the characteristic kinetic and potential energies, \( 1/A^2 \sim g^2 A^4 \)), and we expect \( Z_{\text{SYM}}(\beta) \) to be a nontrivial function of \( \beta \) tending to the limit (5.1) when \( \beta g^{2/3} \ll 1 \) and to the integer Witten index when \( \beta g^{2/3} \gg 1 \).

We want to calculate the correction \( \propto \beta^2 \) to the result (5.1). Using the experience acquired when studying toy models in the previous sections, we understand that we have to (i) take into account the expansion of the Haar measure on \( G \) and (ii) calculate the correction \( 1 + \delta \) in the integrand, where \( \delta(P^a_1, A^a_i; \lambda^a_\alpha, \bar{\lambda}^\alpha_a) \) is evaluated using the rule (4.4) with \( H \rightarrow \tilde{H} = H + igA^a_0G \).

Let us first evaluate the contribution of the bosonic derivatives in Eq.(4.4). The direct calculation gives

\[
\delta^{N=2}_{\text{bos}} = -\frac{\beta^2 g^2 c_V A^2_i}{12} + \frac{\beta^2 g^2 c_V A^2_0}{8},
\]

(5.3)

where \( c_V \) is the adjoint Casimir eigenvalue. The first term in the right side of Eq.(5.3) comes from differentiation of \( H \) and the second term from differentiation of \( igA^a_0G \) in Eq.(5.2).

One can also easily perform the calculation for the \( \mathcal{N} = 4 \) and \( \mathcal{N} = 8 \) systems. The bosonic part of the Hamiltonian and constraints have the same form as in Eqs.(1.4), (1.5), only the spatial index runs now from 1 to
\( D - 1 = 5, 9 \). We obtain

\[
\delta_{\text{bos}} = \frac{\beta^2 g^2 c V}{24} \left[ -(D - 2)A_i^2 + (D - 1)A_0^2 \right].
\] (5.4)

In the \( \mathcal{N} = 2 \) case the contribution of the fermion derivatives in Eq. (4.4) is

\[
\delta_{\text{ferm}}^{\mathcal{N}=2} = \frac{\beta^2 g^2 c V}{12} (A_i^2 - A_0^2).
\] (5.5)

Consider now \( \mathcal{N} = 4 \) theory. The fermion terms in the Hamiltonian and
constraints have, again, the same form as Eqs. (1.4), (1.5), but \( \sigma_i \) are replaced
by 5–dimensional Euclidean \( \gamma \) matrices,

\[
\{ \gamma_j, \gamma_k \} = 2 \delta_{jk},
\] (5.6)

and the spinorial index \( \alpha \) runs now from 1 to 4 (the variables \( \lambda^a_{\alpha} \) are a result of reduction of 6–dimensional Weyl spinors). Twice as much spinorial
components bring about the factor 2 in Eq. (5.5).

For the \( \mathcal{N} = 8 \) system obtained by dimensional reduction from 10–
dimensional SYM theory, we meet a difficulty: the \( SO(9) \) group admits only
real spinor representations and, though one can still introduce complex holo-
morphic fermion variables, the Hamiltonian expressed in these terms does
not have a natural structure, the fermion number is not conserved, etc. It is
more convenient to write the fermionic term in \( \tilde{H} \) as

\[
\tilde{H}_{\text{ferm}}^{\mathcal{N}=8} = \frac{ig}{2} f^{abc} \lambda^a_{\alpha} (\Gamma_\mu)_{\alpha\beta} \lambda^b_{\beta} A^c_\mu,
\] (5.6)

where \( \lambda^a_{\alpha} \) are now real fermion variables; \( \alpha, \beta = 1, \ldots, 16 \) and \( \mu = 0, 1 \ldots, 9 \).
When \( \mu = j \) is spatial, \( \Gamma_j \) are \( 16 \times 16 \) real and symmetric 9–dimensional \( \Gamma \)
matrices, \( \{ \Gamma_j, \Gamma_k \} = 2 \delta_{jk}. \) Also, \( \Gamma_0 = i. \) Eight holomorphic variables in
terms of which Eq. (4.4) is written are expressed via \( \lambda^a_{\alpha} \) as

\[
\mu^a_1 = \frac{1}{\sqrt{2}} (\lambda^a_1 + i\lambda^a_9), \ldots, \mu^a_8 = \frac{1}{\sqrt{2}} (\lambda^a_8 + i\lambda^a_{16}).
\] (5.7)

If using the variables \( \lambda \) instead of \( \mu \), the fermionic contribution to \( \delta \) is cast
in the form

\[
\delta_{\text{ferm}}^{\mathcal{N}=8} = -\frac{\beta^2 g^2}{192} f^{abc} f^{def} A^a_\mu A^d_\nu \left( \frac{\partial^2}{\partial \lambda^a_{\alpha} \partial \lambda^b_{\beta}} \right)^2 \lambda^b \Gamma_\mu \lambda^c \Lambda^\epsilon \Gamma_\nu \Lambda^\delta \bigg|_{\Lambda=\lambda}
\]

\[
= \frac{\beta^2 g^2 c V}{3} (A_i^2 - A_0^2).
\] (5.8)

21
Combining this with Eq.(5.5) and with the twice as large value of $\delta_{\text{ferm}}$ for $N = 4$, we can write

$$
\delta_{\text{ferm}} = \frac{\beta^2 g^2 c_V (D-2)}{24} (A_i^2 - A_0^2). \tag{5.9}
$$

Adding Eqs.(5.9) and (5.4) (the mixed contributions involving both bosonic and fermionic derivatives vanish in this case), we obtain

$$
\delta = \frac{\beta^2 g^2 c_V}{24} A_0^2. \tag{5.10}
$$

This is not yet the full story. The total correction is obtained if adding to Eq.(5.10) the correction coming from the expansion of the measure. We have already calculated this correction for the $SU(2)$ group. The corresponding factor in the measure is

$$
\frac{2[1 - \cos(\beta g |A_0|)]}{\beta^2 g^2 A_0^2} = 1 - \frac{\beta^2 g^2 A_0^2}{12} + \ldots \tag{5.11}
$$

We see that the correction coming from the measure cancels exactly the correction (5.10) and the total correction $\propto \beta^2$ to the integrand in Eq.(5.1) vanishes!

One can show that this cancellation works not only for $SU(2)$, but for any gauge group $G$. The measure on an arbitrary Lie group is given by the Weyl formula. To write it, represent an element $g$ of the group $G$ as

$$
g = \Omega^{-1} e^{ih}\Omega,
$$

where $h$ belongs to the Cartan subalgebra $\mathfrak{h}$ of the corresponding Lie algebra $\mathfrak{g}$ and $\Omega$ belongs to the coset $G/T$, where $T = [U(1)]^r$ is the maximal torus in $G$, $r$ is the rank of the group. After integrating over angular variables (residing in $\Omega$), we obtain (see e.g. [20])

$$
d\mu_G \propto \prod_i \sin^2 \left( \frac{\alpha_i(h)}{2} \right), \tag{5.12}
$$

where the product runs over all positive roots $\alpha_i$ of $\mathfrak{g}$.

We remind that the roots are certain linear forms on $\mathfrak{h}$. Each positive root $\alpha$ correspond to a pair of root vectors $e_\alpha, e_{-\alpha}$ such that

$$
[h, e_{\pm\alpha}] = \pm \alpha(h) e_{\pm\alpha}
$$
for any $h \in \mathfrak{h}$. The expansion of the measure (5.12) at small $h$ gives
\[
d\mu_G \propto \prod_i \alpha_i^2(h) \left[ 1 - \frac{1}{12} \sum_i \alpha_i^2(h) \right]. \tag{5.13}
\]
Using the identity\[20\]
\[
\sum_i \alpha_i^2(h) = c_V \text{Tr}\{h^2\}, \tag{5.14}
\]
where $h$ are the matrices in the fundamental representation, substituting $h = \beta g A_0^a t^a$, and restoring the angular variables, we obtain
\[
d\mu_G \propto \prod_{a=1}^{\text{dim}(G)} dA_0^a \left( 1 - \frac{\beta^2 g^2 c_V}{24} A_0^a \right), \tag{5.15}
\]
which cancels with Eq.(5.10).

Let us illustrate this general result by calculating the correction to the measure for the group $SU(3)$. After conjugating $A_0^a t^a$ to the Cartan subalgebra, we can write
\[
h = \frac{\beta g}{2} \text{diag} \left( A_0^3 + \frac{A_0^8}{\sqrt{3}}, -A_0^3 + \frac{A_0^8}{\sqrt{3}}, -2 \frac{A_0^8}{\sqrt{3}} \right). \tag{5.16}
\]
There are three positive roots:
\[
\alpha_1 = (1, -1, 0), \quad \alpha_2 = (0, 1, -1) \quad \text{and} \quad \alpha_3 = \alpha_1 + \alpha_2 = (1, 0, -1).
\]
The measure (5.12) has the form
\[
\sin^2 \left( \frac{\beta g A_0^3}{2} \right) \sin^2 \left( \frac{\beta g (-A_0^3 + A_0^8 \sqrt{3})}{4} \right) \sin^2 \left( \frac{\beta g (A_0^3 + A_0^8 \sqrt{3})}{4} \right). \tag{5.16}
\]
Expanding this, we obtain the factor
\[
1 - \frac{\beta^2}{8} \left[ (A_0^3)^2 + (A_0^8)^2 \right]
\]
in agreement with (5.13).

Thus, we arrive at a rather unexpected and remarkable result: the total correction $\propto \beta^2$ to the fermion-graded partition function of the SYM quantum mechanics vanishes for all groups and all $\mathcal{N}$. 

23
6 $\mathcal{N} = 2$ SUSY $\sigma$ model on $S^3$.

Consider the system (1.9) with

$$ f(x) = 1 + \frac{x^2}{4R^2} . \quad (6.1) $$

The bosonic part of the Hamiltonian coincides up to a constant shift with the Laplacian on $S^3$,

$$ \hat{H}_{\text{bos}} = -\frac{1}{2}\Delta_{S^3} + \frac{3}{8R^2} , \quad (6.2) $$

where the metric is written in the stereographic coordinates,

$$ ds^2 = \frac{dx^2}{f^2} = \frac{dx^2}{\left(1 + \frac{x^2}{4R^2}\right)^2} . \quad (6.3) $$

The relation

$$ r = |x| = 2R \tan \frac{\Theta}{2} $$

holds, where $\Theta$ is the polar angle on $S^3$. The constant $R$ is the radius of the sphere. The Hamiltonian (6.2) acts upon the wave functions with canonical normalization

$$ \int |\Psi|^2 d\mathbf{x} = 1 . \quad (6.4) $$

Alternatively, one can perform a unitary transformation and define $\hat{H}_{\text{cov}} = f^{3/2}\hat{H} f^{-3/2}$, which acts on the covariantly normalized wave functions $\Psi_{\text{cov}} = f^{3/2}\Psi$,

$$ \int |\Psi_{\text{cov}}|^2 \sqrt{g} d\mathbf{x} = \int |\Psi_{\text{cov}}|^2 \frac{dx}{f^3} = 1 . \quad (6.5) $$

Now, $S^3$ is a compact manifold, the motion is finite, and the spectrum is discrete. There are two bosonic supersymmetric vacua, which are annihilated upon the action of the Hamiltonian and supercharges (1.9). Their wave functions are

$$ \Psi^{(1)}_{\text{vac}} \propto f^{-1}(x), \quad \Psi^{(2)}_{\text{vac}} \propto \psi^2 f^{-1}(x) . \quad (6.6) $$
The functions (6.6) correspond to the covariant wave functions

$$\Psi_{\text{cov}} \propto \sqrt{f} = \sqrt{1 + \frac{x^2}{4R^2}} = \frac{1}{\cos \frac{\Theta}{2}}.$$  \hspace{1cm} (6.7)

The wave function (6.7) is singular at the north pole of the sphere, but this singularity is of a benign, normalizable kind.

Let us discuss this important point in some details. When one considers a purely mathematical problem of the spectrum of the Laplacian on a sphere, only nonsingular eigenfunctions are usually considered. One of the reasons for that is that the function (6.7) is not strictly speaking an eigenfunction of the Laplacian, the action of $\Delta_{\text{cov}}$ on (6.7) gives besides $(-3/8R^2)\Psi_{\text{cov}}$ also a $\delta$ function singularity at $\Theta = \pi$. However, if one considers $S^3$ with its north pole removed, nothing prevents us to bring into consideration singular normalizable wave functions of the type (6.7).

Consider now the standard $\mathcal{N} = 1$ supersymmetric quantum mechanics on $S^3$ [21]. The bosonic part of the Hamiltonian is, again, the Laplacian, but in this case the proper Hilbert space does not include singular wave functions. The matter is that, though the function (6.7) is normalizable, the function obtained from it by the action of the $\mathcal{N} = 1$ supercharge $\hat{Q}$ (we remind that $\hat{Q}^\mathcal{N}=1$ is the operator of external differentiation acting on the forms) is not. Indeed, $\hat{Q}^\mathcal{N}=1 f^{1/2} \sim e^i_a \psi^a \partial_i f^{1/2} \propto f \partial_i f^{1/2} \propto r^2$ (the vielbein $e^i_a$ was chosen in the form $e^i_a = f \delta_{ai}$), and the normalization integral (6.3) diverges linearly at large distances. Therefore, the function (6.7) does not have a normalizable superpartner and is not admissible by that reason [22].

In our case, however, the wave functions (6.6) are annihilated by the supercharges $\hat{Q}_\alpha$, $\hat{Q}^\alpha$, and there is no reason whatsoever to ignore them. In the sector with $F = 1$, $\Psi(x, \psi_\alpha) = P^\alpha(x) \psi_\alpha$, normalizable solutions to the equations $\hat{Q}_\alpha \Psi = \hat{Q}^\alpha \Psi = 0$ are absent. Thus, there are no normalizable fermionic vacuum states and the Witten index is equal to $2 - 0 = 2$.

Let us calculate now the functional integral for the fermion–graded partition function. As the spectrum is discrete, one could expect a priori that, as it was the case for the simple models of Sect. 4, the leading order calculation gives the correct result $\text{Tr}\{( -1)^F e^{-\beta H}\} = 2$, and the higher–order corrections vanish. The actual situation is much more interesting.
To leading order, the fermion–graded partition function is given by the integral (1.3). A novelty is that the value of this integral depends in essential way on how the ordering ambiguities are resolved and the classical Hamiltonian is chosen. One of the choices is to define $H^{\mathrm{cl}}$ as the Poisson bracket of two classical supercharges, which are defined in turn as the Weyl symbols of the quantum ones \cite{23} and have the same functional form as the expressions (1.9) up to the change $\hat{p}_k \to p_k$, $\bar{\psi}^\alpha \to \bar{\psi}^\alpha$. In this case,

$$H^{\mathrm{cl}} = \frac{1}{2}\{Q^\alpha, \bar{Q}^\alpha\}_{\mathrm{P.B.}} =$$

$$\frac{1}{2}f^2(x)p_k^2 - \epsilon_{jkp}\bar{\psi}_j\psi f(x)\partial_p f(x)p_k - \frac{1}{2}f(x)\partial^2_k f(x)(\bar{\psi}\psi)^2 . \quad (6.8)$$

Substituting this in Eq. (1.3) and performing the integral over momenta and fermion variables, we obtain

$$Z^{\mathrm{F–grade}} = \frac{1}{\sqrt{\beta}(2\pi)^3/2} \int d\mathbf{x} \partial_k \left( \frac{\partial_k f}{f^2} \right) = 0 . \quad (6.9)$$

Note the presence of the large factor $\propto 1/\sqrt{\beta}$ in front of the integral. The latter vanishes, however.

Another option is to use $\tilde{H}^{\mathrm{cl}}$ defined as the Weyl symbol of the quantum Hamiltonian in Eq. (1.9). Weyl symbol of an anticommutator $\{\hat{Q}^\alpha, \hat{Q}^\alpha\}_+$ is not given by the Poisson bracket of the Weyl symbols $Q^{\mathrm{cl}}_\alpha, \bar{Q}^{\mathrm{cl}}_\alpha$, but rather by their Moyal bracket. \footnote{We remind its definition in the Appendix.} We derive

$$\tilde{H}^{\mathrm{cl}} = H^{\mathrm{cl}} + \frac{1}{4} [\partial_k f(x)]^2 . \quad (6.10)$$

The fermion–graded partition function is

$$Z^{\mathrm{F–grade}} = \frac{1}{\sqrt{\beta}(2\pi)^3/2} \int d\mathbf{x} \partial_k \left( \frac{\partial_k f}{f^2} \right) \exp \left\{ -\frac{\beta}{4} [\partial_k f(x)]^2 \right\} = 2\sqrt{2} + o(\beta) \quad (6.11)$$

for $f(x) = 1 + x^2/(4R^2)$. Neither 0 nor $2\sqrt{2}$ is the correct result for the index, which means that, contrary to naive expectations, higher–order terms in the quasiclassical expansion must be relevant in this case.
And they are. First, note that, for small $\beta$, the characteristic values of $r = |x|$ are large: $r_{\text{char}} \sim R^2/\sqrt{\beta}$. Also, one can estimate

$$\beta H^\text{cl} \sim 1 \rightarrow \beta \frac{p^4}{R^4} \sim 1 \rightarrow p_{\text{char}} \sim \frac{R^2}{r_{\text{char}}^2 \sqrt{\beta}} \sim \frac{\sqrt{\beta}}{R^2}.$$  

It follows that $p_{\text{char}} r_{\text{char}} \sim 1$ and the correction $\delta$ calculated according to the rule (4.4) is estimated to be of order 1. Thus, the “correction $\propto \beta^2$ ” does not depend on $\beta$ at all in this case, but is simply a number! The same concerns the higher-loop corrections: they are all equal to some numbers, and there is no expansion parameter.

This remarkable result can be given the following explanation. First, for a supersymmetric system with discrete spectrum, $\text{Tr}\{(−1)^F e^{-\beta H}\}$ just cannot depend on $\beta$ and that refers also to the individual terms in the quasiclassical expansion. If higher-loop corrections appear, they have to be $\beta$–independent numbers. Second, as was discussed before, the proper quasiclassical expansion parameter is $\beta E_{\text{char}}$. For the system under consideration, the characteristic energy is determined by the radius of the sphere: $E_{\text{char}} \sim 1/R^2$. On the other hand, $r_{\text{char}} \propto \beta^{-1/2}$ are large, which means that the integral is saturated by the region at the vicinity of the north pole, where the metric is almost flat. In other words, our integral does not “know” about the existence of the sphere and about the value of $R$. It does not depend on $E_{\text{char}}$ and cannot thereby depend on $\beta$. The situation is similar to the situation for the superconformal quantum mechanics, discussed at the end of Sect. 4. The difference is that in the case of superconformal quantum mechanics higher-order corrections vanish. Here, all such corrections have the same order and the quasiclassical expansion breaks down.

7 Discussion.

The main subject of this paper was the studying of the quasiclassical expansion of $Z^{F\text{-grade}}(\beta)$ for supersymmetric systems. But the results obtained in Sect. 2,3 for purely bosonic systems also present a certain methodical interest.

To derive the expression (2.4) by functional methods, we had to solve the spectral problem with generic quadratic Hamiltonian (2.18). The explicit
solution is given in Eqs. (2.27), (2.24).

An important particular case is the Hamiltonian describing multidimensional motion in a constant magnetic field $F_{ij}$ with generic oscillatory potential,

$$\hat{H} = \frac{1}{2} \left( \hat{p}_i - \frac{1}{2} F_{ij} \hat{x}_j \right)^2 + \frac{1}{2} S_{ij} \hat{x}_i \hat{x}_j.$$  \hspace{1cm} (7.1)

The eigenvalues $\Omega_j$ are determined by the roots $\lambda^{(j)}$ of the polynomial

$$\det \| \lambda + \sqrt{\lambda} F + S \|,$$  \hspace{1cm} (7.2)

$\lambda^{(j)} = -\Omega_j^2$. If all eigenvalues of $S$ are positive, the quadratic form $S_{ij} \hat{x}_i \hat{x}_j$ is positive definite and the whole differential operator (7.1) is positive definite. Hence, it has a real positive spectrum (2.27) and hence all $\Omega_j$ and $\Omega_j^2$ are real and positive.

We have derived a purely mathematical result: for a positive definite $S$ and antisymmetric $F$, all roots of the polynomial (7.2) are real and negative. This simple but amusing fact can also be derived by purely algebraic means [24]. As $A = \lambda + \sqrt{\lambda} F + S$ has zero determinant, it has an eigenvector $v$ with zero eigenvalue, $Av = 0$ and hence $\langle v, Av \rangle = 0$. We obtain $\lambda \langle v, v \rangle + \langle v, Sv \rangle = 0$ ($\langle v, Fv \rangle = 0$ due to antisymmetry of $F$). As $S$ is positive definite, $\lambda$ must be real and negative.

In Sect. 3 we studied the quasiclassical expansion of $Z$ for gauge QM systems. We have learned that, when calculating the corrections, (i) We have to extend the limits of integration over the gauge parameters to infinity even if the gauge group is compact. (ii) If the integral thus obtained diverges, the correction is still finite and is given by the residue of the integrand at the same pole which shows up in the integral to leading order.

We believe that this lesson might help to justify the calculation of the leading quasiclassical contribution to $Z^{F-\text{grade}}(\beta)$ in SYM quantum mechanics, performed in Refs. [6, 7]. (To calculate the integral, the authors of Refs. [6, 7] first deformed the system by introducing mass parameters and then replaced the divergent integrals by the pole contributions as if the integrals were convergent.)

Our initial guesses when studying supersymmetric QM systems were that

\footnote{We were not able to find these results in the literature.}
1. For the systems with discrete spectrum, \( Z^{F-\text{grade}} \), which does not depend on \( \beta \), is calculated correctly in the leading quasiclassical approximation and all corrections vanish.

2. For the systems with continuous spectrum, the quasiclassical series comes to life. The sum of the series gives a nontrivial function \( Z^{F-\text{grade}}(\beta) \) determining the index (1.2) in the limit \( \beta \to \infty \).

Unexpectedly, we have discovered a lot of other scenarios.

1. For 1-dimensional SQM systems with continuous spectrum, the function is determined by the leading quasiclassical contribution as it is the case when the spectrum is discrete. (The difference is that, for the system with continuum spectrum, \( Z^{F-\text{grade}} \) may depend on \( \beta \).) The loop corrections vanish.

2. The 1-loop corrections vanish also for the SYM systems for all \( N = 2, 4, 8 \) and for all groups. (The cancellation occurs if all the effects of order \( \beta^2 \), including the expansion of the Haar measure, are taken into account.)

Actually, this result may be not so surprising. Indeed, guess 2 of the previous list was based mainly on the known calculation for the index where the trace was regularized by putting a boundary in the field space, which makes the spectrum discrete. The associated boundary conditions break supersymmetry and result in \( \beta \)-dependence of the fermion-graded partition function. In this paper, we adopted a different philosophy. We defined \( Z^{F-\text{grade}}(\beta) \) via the corresponding path integral and studied the quasiclassical expansion for the latter. It is reasonable to expect that the path integral still does not depend on \( \beta \) if the characteristic values of the fields contributing to the integral are not large so that, even if the boundaries are set, their effect [breaking of supersymmetry and \( \beta \)-dependence of \( Z^{F-\text{grade}}(\beta) \)] is not felt yet. This is so

\footnote{We have checked it only at the 1-loop level, but our conjecture is that the corrections vanish also for higher loops. Indeed, higher-loop corrections must vanish for the systems (1.1) with discrete spectrum. This implies that the corresponding contribution is reduced to the integral of total derivative, as it was the case for the 1-loop corrections. But then the integral must vanish for any superpotential \( V(x) \).}

\footnote{The reasoning below belongs to A. Vainshtein.}
for the leading order integral (5.1): it converges with $A_{\text{char}} \sim (\beta g^2)^{-1/4}$. The integrals with account of “individual” corrections of Eq.(5.3) etc converge in a similar way for $\mathcal{N} = 4$ and $\mathcal{N} = 8$ theories. There is a potential logarithmic divergence in the $\mathcal{N} = 2$ case, but seemingly this divergence is not strong enough to make the fermion-graded partition function $\beta$-dependent.

At the two-loop and higher level, the integrals start to diverge as a power. Though the answer obtained using the recipe of Sect. 3 should be finite, we do not expect the coefficient of $\beta^4$ in the quasiclassical expansion of $Z_{F-\text{grade}}(\beta)$ to vanish. An explicit calculation of such corrections is an interesting though not so simple problem.

3. On the other hand, we have found a system with discrete spectrum, described in Eq.(1.9), where the corrections to the leading quasiclassical result do not vanish, though they do not depend on $\beta$ and are all of the same order.

A following interpretation of this fact can be suggested. In the standard supersymmetric $\sigma$–model on a compact manifold, the Witten index coincides with the Euler characteristics $\chi$ of the manifold. The leading term in the quasiclassical expansion for $Z_{F-\text{grade}}$ is none other than the known integral representation for $\chi$. In the nonstandard model (1.9), the Witten index is equal to 2 for $S^3$, while $\chi(S^3) = 0$. One can represent $2 = \beta_0 + \beta_3$, where $\beta_i$ are the Betti numbers, but no integral representation for this quantity is known. And that is why a “normal” scenario for a system with discrete spectrum — $Z_{F-\text{grade}}$ is determined by the leading–order formula and the corrections vanish — is not realized in this case.

4. We did not find a system where the 1–loop correction to $Z_{F-\text{grade}}(\beta)$ would have a normal order $\sim (\beta E_{\text{char}})^2$ with a nonvanishing coefficient. It would be interesting to find one.

I am indebted to K.-H. Rehren, D. Robert and A. Vainshtein for illuminating discussions.
Appendix: Quasiclassical expansion and star product.

We outline here how the result (2.4) for the quasiclassical correction to the partition function of a quantum system is derived using operator methods. Bearing in mind the representation (2.5), the problem is reduced to evaluating the Weyl symbol of the exponential \( \exp\{-\beta \hat{H}\} \). The Weyl symbol of the product of two operators is given by the expression [13]

\[
[\hat{A}\hat{B}]_W = \exp\left\{ \frac{i\hbar}{2} \left( \frac{\partial^2}{\partial q_i \partial p_i} - \frac{\partial^2}{\partial p_i \partial q_i} \right) \right\} A(p_i, q_i) B(P_i, Q_i) \bigg|_{P=p, Q=q}, \tag{A.1}
\]

where \( A(p_i, q_i) \) and \( B(P_i, Q_i) \) are Weyl symbols of the operators \( \hat{A}, \hat{B} \). Now, \( \hbar \) is the Planck’s constant, which we preferred to retain here to facilitate bookkeeping.

In the modern language, the right side of Eq.(A.1) is called the star product \( \hat{A} \ast \hat{B} \) of the functions \( A(p_i, q_i) \) and \( B(p_i, q_i) \) [25]. The star product is not commutative. The star commutator \( \hat{A} \ast \hat{B} - \hat{B} \ast \hat{A} \) is called the Moyal bracket of the functions \( A, B \). In the leading order in \( \hbar \), the Moyal bracket is reduced to the Poisson bracket. The star product is associative, however:

\[
(A \ast B) \ast C = A \ast (B \ast C). \tag{A.2}
\]

We need to determine

\[
\left[ e^{-\beta \hat{H}} \right]_W = 1 - \beta H + \frac{\beta^2}{2} H \ast H - \frac{\beta^3}{6} H \ast H \ast H + \ldots , \tag{A.2}
\]

where \( H \) is the Weyl symbol of the Hamiltonian. Keeping only the terms \( \propto 1 \) and \( \propto \hbar^2 \), we obtain

\[
H \ast H = H^2 - \frac{\hbar^2}{4} \left( \frac{\partial^2 H}{\partial p_i \partial p_j} \frac{\partial^2 H}{\partial q_i \partial q_j} - \frac{\partial^2 H}{\partial p_i \partial q_j} \frac{\partial^2 H}{\partial p_j \partial q_i} \right) \tag{A.3}
\]

where \( \Delta \) is defined as

\[
\Delta = \frac{\partial^2}{\partial p_i \partial q_i} - 2 \frac{\partial^2}{\partial p_i \partial q_j} \frac{\partial}{\partial q_j} \frac{\partial}{\partial q_i} - \frac{\partial^2}{\partial p_i \partial q_j} \frac{\partial}{\partial q_j} \frac{\partial}{\partial p_i} = \frac{\partial^2}{\partial p_i \partial q_i} - \frac{\partial^2}{\partial p_i \partial q_j} \frac{\partial}{\partial q_j} \frac{\partial}{\partial p_i}.
\]

Further,

\[
H \ast H \ast H = H^3 + 3\hbar^2 \Delta H - \frac{\hbar^2}{4} \left( \frac{\partial^2 H}{\partial q_i \partial q_j} \frac{\partial H}{\partial p_i} \frac{\partial H}{\partial p_j} - 2 \frac{\partial^2 H}{\partial p_i \partial q_j} \frac{\partial H}{\partial p_j} \frac{\partial H}{\partial q_i} \right) + \frac{\partial^2 H}{\partial p_i \partial p_j} \frac{\partial H}{\partial q_i} \frac{\partial H}{\partial q_j} + o(\hbar^2) \tag{A.4}
\]

\[
\equiv H^3 + 3\hbar^2 \Delta H + \hbar^2 Q + o(\hbar^2). \tag{A.4}
\]
To order $\hbar^2$, the products $H \ast H \ast H \ast H$ etc. are all expressed via $\Delta$ and $Q$,

$$H \ast \cdots \ast H = H^n + \hbar^2 \frac{n(n-1)}{2} \Delta H^{n-2} + \hbar^2 \frac{n(n-1)(n-2)}{6} Q H^{n-3} + o(h^2). \quad (A.5)$$

Substituting (A.5) into (A.2), we obtain

$$\left[ e^{-\beta H} \right]_W = e^{-\beta H} \left[ 1 + \frac{\beta^2 \hbar^2}{2} \Delta - \frac{\beta^3 \hbar^2}{6} Q + o(h^2) \right]. \quad (A.6)$$

Integrating by parts, one can derive

$$\int \beta Q e^{-\beta H} \prod_i \frac{dp_i dq_i}{2\pi} = 2 \int \Delta e^{-\beta H} \prod_i \frac{dp_i dq_i}{2\pi}, \quad (A.7)$$

which leads to the result (2.4).

It is not difficult to generalize all this to the systems involving fermion variables. We have

$$\text{Tr}\{(-1)^F e^{-\beta H}\} = \int \prod_i \frac{dp_i dq_i}{2\pi} \prod_a d\bar{\psi}_a d\psi_a \left[ e^{-\beta H} \right]_W. \quad (A.8)$$

The star product of two functions on the phase space $(p_i, q_i; \bar{\psi}_a, \psi_a)$ is given, again, by the expression (A.1), only we have to write the differential operator

$$\frac{\partial^2}{\partial \Psi_a \partial \bar{\psi}_a} - \frac{\partial^2}{\partial \psi_a \partial \Psi_a} + i \left( \frac{\partial^2}{\partial q_i \partial P_i} - \frac{\partial^2}{\partial Q_i \partial p_i} \right) \quad (A.9)$$

in the exponent. Repeating all the steps of the derivation above, we are led to the result (4.3), (4.4).

References

[1] S. Cecotti and L. Girardello, Phys. Lett. B110 (1982) 39; L. Girardello et al, Phys. Lett. B132 (1983) 69.

[2] A.V. Smilga, in: Proc. Int. Workshop on Supermembranes and Physics in 2+1 dimensions (Trieste, July, 1989), eds. M.J. Duff, C.N. Pope, E. Sezgin (Worlds Scientific, Singapore, 1987); B. de Witt, M. Lüscher, and H. Nicolai, Nucl. Phys. B320 (1989) 135.
[3] A.V. Smilga, Nucl. Phys. B266 (1986) 45; Yad. Fiz. 43 (1986) 45.

[4] P. Yi, Nucl. Phys. B505 (1997) 307; S. Sethi and M. Stern, Comm. Math. Phys. 194 (1998) 675.

[5] V.G. Kac and A.V. Smilga, Nucl. Phys. B571[PM] (2000) 515.

[6] G. Moore, N. Nekrasov, and S. Shatashvili, Comm. Math. Phys. 209 (2000) 77.

[7] W. Krauth and M. Staudacher, Nucl. Phys. B584 (2000) 641.

[8] M.B. Halpern and C. Schwarz, Int. J. Mod. Phys. A13 (1998) 4367.

[9] G.M. Graf et al, Nucl. Phys. B567 (2000) 231.

[10] A.V. Smilga, Nucl. Phys. B291 (1987) 241.

[11] E.A. Ivanov and A.V. Smilga, Phys. Lett. B257 (1991) 79.

[12] E. Witten, Nucl. Phys. B188 (1981) 513.

[13] H. Weyl, The Theory of Groups and Quantum Mechanics, Dover, 1931; I.E. Moyal, Proc. Cambr. Phil. Soc. 45 (1949) 99. For an excellent review, see F.A. Berezin, Usp. Fiz. Nauk 132 (1980) 497.

[14] R.P. Feynman and A.R. Hibbs, Quantum Mechanics and Path Integrals, McGraw-Hill, 1965.

[15] B. Helffer and D. Robert, Asympt. Anal. 3 (1990) 91.

[16] H. Matsumoto, J. Funct. Anal. 129 (1995) 168.

[17] A.V. Smilga, Ann. Phys. 234 (1994) 1, Sect.2.

[18] A.V. Smilga, Lectures on Quantum Chromodynamics (World Scientific, to appear), Lecture 4.

[19] S. Fubini and E. Rabinovici, Nucl. Phys. B245 (1984) 17.

[20] D.P. Želobenko, Compact Lie groups and their representations, American Mathematical Society, Providence, 1973.
[21] E. Witten, Nucl. Phys. B202 (1982) 253.

[22] M. Shifman, A. Smilga, and A. Vainshtein, Nucl. Phys. B299 (1988) 79.

[23] A. Smilga, Nucl. Phys. B292 (1987) 363.

[24] K.-H. Rehren, private communication.

[25] N. Seiberg and E. Witten, JHEP 9909:032, 1999.