Forests and the $W$ construction

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May 3, 2014

Abstract

A modified definition of the category of forests found in Costello [Cos04] and Getzler [Get09] is given. This language is used to prove some cofibration properties of the $W$ construction for colored operads. An application is given to the swiss cheese operad. The language of forests is convenient for showing that the swiss cheese operad is equivalent to its free degree 0 and 1 pieces.

1 The category of Forests

Definition 1. Let Fin be the symmetric monoidal category of finite sets, where the monoidal structure is disjoint union. Let $K$ be any set, the over category $\text{Fin}_K$ has objects given by pairs $(I, \text{col}_I)$ where $I$ is a finite set and $\text{col}_I: I \to K$ is any function. A map $f: (I, \text{col}_I) \to (J, \text{col}_J)$ is a map of sets $f: I \to J$ such that $\text{col}_J f = \text{col}_I$. We call $K$ the set of colors, and we call $\text{col}_I$ the coloring of $I$. We usually leave the coloring implicit and simply refer to $I$ as a $K$-colored set.

A map of $K$-colored sets is called a color-preserving map or colored map. If we do not require $f$ to preserve the coloring, we call it an uncolored map. Disjoint union of sets over $K$ defines on $\text{Fin}_K$ the structure of a symmetric monoidal category.

The following definition is an amalgamation of those found in [KS00], [Cos04] and [Get09].

Definition 2. A $K$-colored young forest is an uncolored map of finite $K$-colored sets $x: I_x \to J_x$. A $K$-colored forest $f: x \to y$ is a color-preserving isomorphism $f: I_y \sqcup J_x \to J_y \sqcup I_x$ such that $y$ is realized by a colimit of $f$ and $x$ as in diagram

$$
\begin{array}{ccc}
I_y & \xrightarrow{f} & I_x \\
\downarrow & \colim & \downarrow \\
J_y & \xleftarrow{y} & J_x \\
\end{array}
$$

(3)

More specifically we require the map from $J_y$ to be an isomorphism and the corresponding composite with the inverse to be $y: I_y \to J_y$. Now if $g: y \to z$ is
another forest, then \( gf: I_z \sqcup J_x \to J_z \sqcup I_x \) is defined by diagram \(4\)

\[
\begin{align*}
I_z & \xrightarrow{g} J_z \sqcup I_y \xrightarrow{f} J_y \sqcup I_x \xrightarrow{f} J_x \\
J_z & \xrightarrow{g} J_z \sqcup I_y \xrightarrow{f} J_y \sqcup I_x \xrightarrow{f} J_x
\end{align*}
\]

By this we mean take the colimit of the square in the diagram. The inclusions from \(J_z\) and \(I_x\) will induce an isomorphism with this colimit. Indeed, if \(i \in I_y\) and \(f(i) \in J_y\), then diagram \(3\) implies that \(f(i) = y(i)\). Moreover, the condition on \(g\) and \(y\) implies that a sufficiently large power of the endomorphism \((\text{id}_{J_z}, gy)\) of \(J_z \sqcup I_y\) has image contained in \(J_z\). The condition on \(gf\) can be checked by inserting the map \(x\) into the right hand side of diagram \(4\), then taking colimits. We denote the natural maps given by the colimit in \(3\) as \([f|x]: I_y \sqcup J_y \sqcup I_x \sqcup J_x \to J_y\).

This composition law defines a category called \(\text{For}_K\) whose objects are young forests and whose morphisms are forests. Under disjoint union of sets \(\text{For}_K\) becomes a symmetric monoidal category.

Given a \(K\)-colored forest \(f: x \to y\) we call \(V(f) := J_x\) the set of internal vertices of \(f\). We call \(\text{m}(f) := I_y\) the set of input vertices of \(f\) and \(\text{rt}(f) := J_y\) the set of root vertices of \(f\).

Given a forest \(f: x \to y\) we define \(\text{Ex}(f) := J_y \sqcup I_x\) to be the set of extended edges of \(f\). The isomorphism \(f: I_y \sqcup J_x \to \text{Ex}(f)\) gives a decomposition

\[
\text{Ex}(f) = (I_y \times_f J_y) \sqcup (I_y \times_f I_x) \sqcup (J_x \times_f J_y) \sqcup (J_x \times_f I_x).
\]

We call \(\text{un}(f) := I_y \times_f J_y\) the set of trivial or unit edges; \(\text{leaf}(f) := I_y \times_f I_x\) the set of leaf edges or the set of leaves; \(\text{rt}(f) := J_x \times_f J_y\) the set of root or output edges; and \(E(f) := J_x \times_f I_x\) the set of internal edges or simply edges. The internal edges have the most compact notation because they will be referred to the most often. The inclusion of edges into extended edges will be denoted \(e(f): E(f) \to \text{Ex}(f)\). The extended edges can be remembered by taking four pullbacks in the standard diagram representation of \(f\) as in diagram \(5\)

\[
\begin{align*}
\text{un}(f) & \xleftarrow{f} I_y \xleftarrow{f} \text{Ex}(f) \xleftarrow{f} J_y \xleftarrow{f} E(f) \\
\text{rt}(f) & \xrightarrow{f} J_x \xrightarrow{f} \text{Ex}(f) \xrightarrow{f} J_y \xrightarrow{f} E(f)
\end{align*}
\]

Remark 6. Throughout this paper \(K\) denotes a finite set of colors. We often drop it from the notation. Forests and young forests are always \(K\)-colored and \(\text{For}_K\) will be abbreviated \(\text{For}\).

2 The \(W\) construction

Let \((\text{Top}, \times)\) be the symmetric monoidal category of compactly generated topological spaces with the Cartesian product. We show how the \(W\) construction of Boardman and Vogt \([BV]\) can be realized as a coend construction using \(\text{For}\). In
nice situations the $W$ construction gives a cofibrant replacement for an operad, as shown in [BM06]. We use $[0, \infty]$ as our edge labels as in [Kon99].

Given any young forest $z$ there is a contravariant functor $W: \text{For}_z \to \text{Top}$ from the over category of $z$ to topological spaces. For any object $g: y \to z$ of this over category, set $W(g) = \text{map}(E(g), [0, \infty])$. If $f: x \to y$ is a forest, there is a co-correspondence

$$E(g) \hookrightarrow I_y \xrightarrow{[g,f]} J_z \sqcup I_x \hookleftarrow I_x \leftarrow E(gf),$$

where the map $[g,f]: I_z \sqcup J_z \sqcup I_y \sqcup J_y \sqcup I_x \sqcup J_x \to J_z \sqcup I_z$ is given by including into the square in diagram (7) then passing to the colimit. By pushing forward and pulling back functions, we get a map $W_\Sigma(f): W(g) \to W(gf)$. This process uses the sum operation on $[0, \infty]$. This is an extension of $+$ on $[0, \infty)$ such that $t + \infty = \infty = \infty + t$ for all values of $t$. Concretely, any morphism $gf \to g$ in the over category collapses some internal edges and deletes some unary vertices. Any collapsed edge in $gf$ is labeled with 0 in $W(gf)$. Any vertex of $gf$ which is deleted in $g$ causes two edges of $gf$ to have the same image under $[g,f]$. This merged edges in $gf$ is labeled with the sum of the two edges in its pre image under $[g,f]$.

If $h: z \to w$ is a forest there is a morphism $W_\infty(h): W(g) \to W(hg)$ which uses the maps $E(g) \hookrightarrow I_y \xrightarrow{[g,f]} E(hg)$. In this case we do not push forward and pull back functions. Rather we extend a function $t: E(g) \to [0, \infty]$ to a function on $E(hg)$ by setting $t(\epsilon) = \infty$ if $\epsilon \notin E(g)$. This defines a natural transformation $W_\infty(h): W \to Wh_*$ where $h_*: \text{For}_z^{\text{op}} \to \text{For}_w^{\text{op}}$ is induced by $h: z \to w$. In other words, diagram (7) commutes for all composable triples $h, g, w$.

$$\begin{array}{ccc}
W(g) & \xrightarrow{W_\Sigma(f)} & W(gf) \\
\downarrow W_\infty(h) & & \downarrow W_\infty(h) \\
W(hg) & \xrightarrow{W_\Sigma(f)} & W(hgf)
\end{array} \quad (7)$$

**Definition 8.** A $K$-colored operad $\mathcal{O}$ is a symmetric monoidal functor $(\text{For}, \sqcup) \to (\text{Top}, \times)$.

Consider an operad $\mathcal{O}$ as a collection of functors $\mathcal{O}_z: \text{For}_z \to \text{Top}$ by setting $\mathcal{O}_z(y) := \mathcal{O}(y)$ for $g: y \to z$. For a young forest $z$ the topological space $WO(z)$ is the coend

$$WO(z) = W \otimes_{\text{For}_z} \mathcal{O}_z = \left( \coprod_{g: y \to z} W(g) \times \mathcal{O}(y) \right) / \sim \quad (9)$$

where $(W_\Sigma(f)t, \alpha) \sim (t, \mathcal{O}(f)\alpha)$ for every $f: x \to y$, $t \in W(g)$ and $\alpha \in \mathcal{O}(x)$. Now a forest $h: z \to w$ with the natural transformations above gives us a map

$$WO(z) = W \otimes_{\text{For}_z} \mathcal{O}_z \xrightarrow{W_\infty(h) \otimes \text{id}} Wh_* \otimes_{\text{For}_w} \mathcal{O}_w \xrightarrow{W \otimes_{\text{For}_w} \mathcal{O}_w} WO(w). \quad (10)$$

This defines $WO$ as a functor $\text{For} \to \text{Top}$. This functor is symmetric monoidal, so $WO$ is a $K$-colored operad.
In the sequel, we will define several variants on the category \( \text{For} \). Each of these variants admits a functor to \( \text{For} \) and we want a corresponding \( W \) construction for each.

**Definition 11.** If \( \text{For}^{\text{var}} \) is any symmetric monoidal category equipped with a functor \( \text{For}^{\text{var}} \to \text{For} \), let \( W_{\text{var}} \text{O} : \text{For}^{\text{var}} \to \text{Top} \) be defined by \(^9\) and \(^{10}\) where use the given functor to replace \( \text{For}/_v \) by \( \text{For}^{\text{var}} \) for an object \( v \in \text{For}^{\text{var}} \).

### 2.1 \( W\text{O} \) is cofibrant

**Definition 12.** Let \( f : x \to y \) be a forest. For \( I \subseteq I_y \sqcup I_x \sqcup J_x \) and \( j \in J_y \) let \( I(j) \) denote \( I \cap [f|x]^{-1}(j) \), the set of elements of \( I \) living over \( j \). A **weighted young forest** is a pair \( (x, \omega_x) \) where \( x \) is a young forest and \( \omega_x : J_x \to \mathbb{Z}_{\geq 0} \) is any function, called the **weight** of \( x \). A weighted forest \( f : (x, \omega_x) \to (y, \omega_y) \) is a forest \( f : x \to y \) such that for all \( j \in J_y \),

\[
\omega_y(j) \geq \#E(f) + \sum_{i \in J_y(j)} \omega_x(i). \tag{13}
\]

If \( g : (y, \omega_y) \to (z, \omega_z) \) is a weighted forest. Then one can show that for each \( j \in J_z \), we have

\[
\#E(gf)(j) = \#E(g)(j) + \sum_{i \in J_y(j)} \#E(f)(i) - \#\text{un}(f)(i) \tag{14}
\]

\[
= \#E(g)(j) + \#E(f)(j) - \#\text{un}(f)(j).
\]

This implies \( gf : x \to z \) defines a weighted forest \( gf : (x, \omega_x) \to (z, \omega_z) \).

**Remark 15.** If \( f : (x, \omega_x) \to (y, \omega_y) \) is a weighted forest and \( \omega_y \leq k \) then certainly \( \#E(f) \leq k \). Furthermore, if \( g : (y, \omega_y) \to (z, \omega_z) \) is a weighted forest where \( \omega_z \leq k + 1 \) and \( \#E(gf)(j) = k + 1 \) for some \( j \in J_z \), then equations \(^{13}\) and \(^{14}\) imply that \( \omega_x(i) = 0 \) for all \( i \in J_y(j) \), \( \#\text{un}(f)(j) = 0 \), and \( \#E(g) \geq 1 \).

**Definition 16.** Disjoint union of forests extends to disjoint union of weighted forests. Let \( \text{For}^{\omega} \) denote the symmetric monoidal category of weighted forests. For each \( k \geq 0 \), let \( \text{For}_k \) denote the full subcategory of \( \text{For}^{\omega} \) generated by objects of the form \( (x, \omega_x) \) such that \( \omega_x(j) \leq k \) for every \( j \in J_x \). Let \( \text{Op}_k \) denote the category of **weignt \( k \) operads**, which are symmetric monoidal functors \( \text{For}_k \to \text{Top} \). The category \( \text{Op}_k \) is often denoted \( \text{Coll} \) in the literature and is called the category of **\( K \)-colored pointed collections** in \( \text{Top} \). All of these categories are endowed with the projective model structure via the forgetful functor \( \text{For}_k \to \text{Coll} \). In particular, left adjoints preserve cofibrations.

We will often refer to the weighted forest \( (x, \omega_x) \) simply as \( x \) when the weight is understood. If \( \omega_x(j) \leq k \) for all \( j \in J_x \), we write \( \omega_x \leq k \) or say \( x \) has **weight** \( \leq k \). Note that if \( f : x \to y \) is a weighted forest, then \( x \) has weight \( \leq k \) if \( y \) has weight \( \leq k \).
Remark 17. The only morphisms in the category \( \text{For}_0 \) are those \( f: x \rightarrow y \) which satisfy \( f(J_x) \subseteq J_y \). In other words, \( f \) has no internal edges. However \( f \) may still have some unit edges. Every young forest \( x \) is a disjoint union of young forests of the form \( I \to \{c\} \), i.e. young trees. Denote this young tree by \( (I; c) \). A pointed collection \( O: \text{For}_0 \to \text{Top} \) consists of spaces \( O(I; c) \) for every young tree \( (I; c) \). In addition the trees \( f: (I; c) \to (I; c) \) with no unit edges define an action of \( \text{aut}(I) \) on \( O(I; c) \). Each unit tree \( (\emptyset; \emptyset) \to (c; c) \) defines a map \( * \to O(c; c) \) which, if \( O \) is an operad, is the \( c \)-colored unit of \( O \).

The categories \( \text{For}_k \) define a functor from the poset \( \mathbb{Z}_{\geq 0} \) to the category of symmetric monoidal categories. That is, for each \( k \leq \ell \) there are inclusions \( \text{For}_k \to \text{For}_\ell \), and these inclusions are all compatible with one another. The colimit of this these categories is \( \text{For}_\omega \). There are forgetful functors \( \text{For}_k \to \text{For} \), compatible with the inclusions \( \text{For}_k \to \text{For}_\ell \), and the induced map from the colimit is the canonical forgetful functor \( \text{For}_\omega \to \text{For} \). For \( k \leq \ell \) let \( U_k \) denote the induced forgetful functor \( \text{Op}_ \to \text{Op}_\ell \), and let \( U_k: \text{Op} \to \text{Op}_k \) be the functor sending an operad \( O \) to underlying pointed collection of \( O \). Let \( F_k \) and \( F^k \) denote the left adjoints of \( U_k \) and \( U_k \) respectively.

Let \( O \) be an operad. Since each \( k \geq 0 \) defines a symmetric monoidal category \( \text{For}_k \) equipped with a symmetric monoidal functor \( \text{For}_k \to \text{For} \) definition[11] to define \( \text{For}_k: \text{For}_k \to \text{Top} \). Precisely, this is the weight \( k \) operad which sends \((z, \omega_z)\) to

\[
W_k O(z, \omega_z) := W \otimes_{\text{For}_k} O_z = \left( \prod_{g: (y, \omega_y) \rightarrow (z, \omega_z)} W(g) \times O(y) \right)/\sim, \tag{18}
\]

where \((W_z(f) t, \alpha) \sim (t, O(f) \alpha)\) for \( f: (x, \omega_x) \rightarrow (y, \omega_y), t \in W(g), \) and \( \alpha \in O(x) \). Note that \( x \) and \( y \) necessarily have weight \( \leq k \) since \( z \) does. If \( w \) is a young forest of weight \( \leq k \) and \( h: (z, \omega_z) \rightarrow (w, \omega_w) \) is a weighted forest, then \( W_k O(h): W_k O(z) \rightarrow W_k O(w) \) is defined exactly as in[10].

The inclusion \( \text{For}_k \to \text{For}_{k+1} \) induces a map of operads \( W_k O \to U_{k+1} W_{k+1} O \). To describe the adjoint of this map, note that

\[
F^k_{k+1} W_k O(z, \omega_z) = \left( \prod_{g: (x, \omega_x) \rightarrow (y, \omega_y)} W(f) \times O(x) \right)/\sim, \tag{19}
\]

where \( \omega_x, \omega_y \leq k \) and \( \omega_z \leq k+1 \). To describe the relation \( \sim \) we describe maps \( \mu: F^k_{k+1} W_k O(z) \rightarrow X \) for an arbitrary space \( X \). Such a map is given by a collection \( \{ \mu(g, f): W(f) \times O(y) \rightarrow X \} \) where \( g: y \rightarrow z \) and \( f: x \rightarrow y \) are weighted forests, and \( \omega_x, \omega_y \leq k \). These maps must make the diagrams in[20] commute for all \( q: x' \rightarrow x, p: y' \rightarrow y, \) and \( f': x' \rightarrow y' \) in \( \text{For}_k \).

\[
\begin{align*}
W(f) \times O(x') &\xrightarrow{W(g) \times 1} W(f g) \times O(x') \\
W(f) \times O(x) &\xrightarrow{1 \times O(q)} W(f) \times O(x) \xrightarrow{\mu(f, g)} X \xrightarrow{\mu(g, f')} W(p f') \times O(x') \\
W(f) \times O(x) &\xrightarrow{\mu(f, g)} X \xrightarrow{\mu(g, f')}
\end{align*}
\tag{20}
\]
Given $g$ and $f$ as above, let $\iota(g,f): W(f) \times \mathcal{O}(x) \to W(gf) \times \mathcal{O}(x)$ be $W_\infty(g) \times 1$. If $((g,f),t,\alpha)$ represents a point $\beta$ of $F_{k+1}W_k\mathcal{O}(z,\omega_z)$ then its image in $W_{k+1}\mathcal{O}(z,\omega_z)$ is represented by $\iota(g,f)((g,f),t,\alpha) = (gf, W_\infty(g)t, \alpha)$.

A map $W_{k+1}\mathcal{O}(z) \to X$, where $\omega_z \leq k + 1$, consists of a collection maps $\{\eta(g): W(g) \times \mathcal{O}(y) \to X\}$ indexed by the set of all weighted forests $g: y \to z$ in For$_{k+1}$. These maps must make diagram (21) commute for every weighted forest $g: y' \to y$ in For$_{k+1}$.

\[
\begin{array}{ccc}
W(g) \times \mathcal{O}(y') & \xrightarrow{W_\mathcal{O}(g) \times 1} & W(gq) \times \mathcal{O}(y') \\
1 \times \mathcal{O}(y') & \downarrow & 1 \\
W(g) \times \mathcal{O}(y) & \xrightarrow{\eta(g)} & X
\end{array}
\] (21)

For $g: y \to z$ a forest in For$_{k+1}$, let $(W \times \mathcal{O})_k^+ (g)$ be $W(g) \times \mathcal{O}(y)$ if $g$ has $\leq k$ internal edges. Otherwise let $(W \times \mathcal{O})_k^+ (g) \subset W(g) \times \mathcal{O}(y)$ be the set of $(t, \alpha)$ such that $t(e) = 0$ or $t(e) = \infty$ or $\alpha(j) = id$ for some $e \in E(g)$ or $j \in V(g)$.

Define the map $(W \times \mathcal{O})_k^+ (g) \to F_{k+1}W_k\mathcal{O}(z)$ by collapsing any edge labeled 0 and deleting any vertex labeled with the identity. The square in diagram (22) is a pushout. Indeed, if we are given $\eta_T([g]): (W(g) \times \mathcal{O}(\text{dom } g))_{\text{aut}(g)} \to T$ for every $[g] \in \pi_0\text{For}_{k+1/z}$ and $\mu_T: F_{k+1}W_k\mathcal{O}(z) \to T$, such that

\[
\begin{array}{ccc}
\bigsqcup_{[g] \in \pi_0\text{For}_{k+1/z}} (W \times \mathcal{O})_k^+ (g)_{\text{aut}(g)} & \xrightarrow{\bigsqcup_{[g] \in \pi_0\text{For}_{k+1/z}} F_{k+1}W_k\mathcal{O}(z)} & F_{k+1}W_k\mathcal{O}(z) \\
\downarrow & & \downarrow \\
\bigsqcup_{[g] \in \pi_0\text{For}_{k+1/z}} (W \times \mathcal{O})(\text{dom } g)_{\text{aut}(g)} & \xrightarrow{\bigsqcup_{[g] \in \pi_0\text{For}_{k+1/z}} W_{k+1}\mathcal{O}(z)} & W_{k+1}\mathcal{O}(z)
\end{array}
\] (22)

Using the techniques of Berger and Moerdijk [ , section 2] one can also show that if $U_0\mathcal{O}$ is a cofibrant pointed collection then $(W \times \mathcal{O})_k^+ (g) \to W(g) \times \mathcal{O}(\text{dom } g)$ is an aut($g$)-cofibration, where aut($g$) is the automorphism group of $g$ as an object of the category For$_{k+1/z}$. This implies the map on the left in (22) is an aut($z$)-cofibration. Thus $U_0F_{k+1}W_k\mathcal{O} \to U_0W_{k+1}\mathcal{O}$ is a cofibration of pointed collections.

**Lemma 23.** The natural map $\iota_k: F_{k+1}W_k\mathcal{O} \to W_{k+1}\mathcal{O}$ is a cofibration of weight $k + 1$ operads.

**Proof.** Consider a commutative diagram of weight $k$ operads where $\pi(z): P(z) \to Q(z)$ is a fibration for every $z \in \text{For}_{k+1}$,

\[
\begin{array}{ccc}
F_{k+1}W_k\mathcal{O} & \xrightarrow{\eta} & P \\
\downarrow & & \downarrow \\
W_{k+1}\mathcal{O} & \xrightarrow{\mu} & Q
\end{array}
\] (24)

By the discussion below diagram (22) there is a lift on the level of pointed collections, $\nu: U_0W_{k+1}\mathcal{O} \to U_0P$, $\nu \iota = \eta$, $\pi \nu = \mu$. We will show that this is automatically a lift on the level of weight $k + 1$ operads. The condition that
ν is a morphism of pointed collections means that the square on the right in diagram 25 commutes for all h: z → z′ with no internal edges, where z, z′ are objects of For_{k+1}. We want to show that the square on the right in 25 commutes for all h: z → z′ in For_{k+1}. By assumption, the square on the left, and the top and bottom triangles commute.

$$\eta(z) \quad \nu(z) \quad \nu'(z) \quad \nu'(z) \quad \nu(z') \quad \nu(z') \quad \nu(z) \quad \nu(z')$$

If h has at least one internal edge, then z has weight at most k. This implies ε(z) is an isomorphism. Thus,

$$\nu(z')W_{k+1}O(h) = \nu(z')\epsilon(z)F_{k+1}^kW_kO(h)(\epsilon(z))^{-1}$$

$$= \eta(z')F_{k+1}^kW_kO(h)(\epsilon(z))^{-1}$$

$$= \mathcal{P}(h)\eta(z)(\epsilon(z))^{-1}$$

$$= \mathcal{P}(h)\nu(z).$$

**Definition 26.** Let W_ωO denote the weight ω operad colim_k F^k_ωW_kO. Let W_O = F^ωW_ωO ∈ Op. Let ε: W_O → O denote the adjoint of the natural map W_ωO → U_ωO.

**Proposition 27.** If U_0O is a cofibrant collection, then the natural map ε: W_O → O is a weak equivalence and F^0U_0O → W_O is a cofibration. In particular, W_O is a cofibrant operad.

**Proof.** By induction, F^k_0U_0O → W_kO is a cofibration in Op_k. Indeed, U_0O = W_0O and if F^0_{k-1}U_0O → W_{k-1}O is a cofibration in Op_{k-1}, then applying F^k_{k-1} we get a cofibration in Op_k, F^k_0U_0O → F^k_{k-1}W_{k-1}. Then lemma ?? shows that F^k_{k-1}W_{k-1}O → W_kO is a cofibration in Op_k. Taking colim_k, we get F_0^kU_0O → W_ωO a cofibration in Op_ω. Apply F_ω, then F^0U_0O → W_O is a cofibration in Op. Since F^0 preserves cofibrant objects, F^0U_0O is cofibrant. Thus W_O is cofibrant. Finally, the map U_0W_O → U_0O has homotopy inverse given by the adjoint of the inclusion of the free operad. □

### 3 The swiss cheese operad

Throughout the remainder of this paper, let K = {f, h} be the set of colors. Let C_h → For be the full subcategory of For spanned by young forests x with J_x.
of color h. Let \( C^h \) denote the full subcategory of \( C \) spanned by young forests \( x \) such that \( \#_c x^{-1}(j) \leq 1 \) for all \( j \in J_x \), where \( \#_c I \) denotes the number of \( c \)-colored elements of a colored set \( I \). Using the construction in definition \ref{colored_operad}
we can define the operad \( W_h \): \( C \to \text{Top} \) using the functor \( C \to \text{For} \). In the
same way, define \( W_{h}^{\text{op}} \): \( C \to \text{Top} \) using the functor \( C \to \text{For} \).

For \( k \in \mathbb{Z}_{\geq 1} \cup \{ \omega \} \) define \( C_k \) to be the full subcategory of weighted young forests \( (x, \omega_x) \), where \( x \in C \), and \( \omega_x(j) \leq k \) if \( j \in J_x \) satisfies \( \#_c x^{-1}(j) \geq 2 \).
For \( f: x \to y \), we require condition \ref{condition} For each \( k \geq -1 \) define \( W_k \): \( C \to \text{Top} \) using the functor \( C \to \text{For} \) and the construction in definition \ref{colored_operad}.

The functor categories \( \text{Op}_k \), \( \text{Op}_h \), \( \text{Op}^{\text{op}}_k \) are defined in the obvious way. Adjunctions are
denoted
\[
F_{h,c}^k: \text{Op}_k \rightleftharpoons \text{Op}_h: (h,c)
\]

Lemma 30. The natural map \( \iota: F_{0}^{-1}W_{C}^{\text{op}} \to W_{0}^{\text{op}} \) is a cofibration in \( \text{Op}^{\text{op}}_0 \).

Proof. If \( y \in C_{-1} \), then \( F_{0}^{-1}W_{C}^{\text{op}}(y) \to W_{0}^{\text{op}}(y) \) is an isomorphism. If \( y \in C_{0} - C_{-1} \), there is no weighted forest \( f: x \to y \) with \( x \in C_{-1} \). Thus \( F_{0}^{-1}W_{C}^{\text{op}}(y) = \emptyset \) and \( W_{0}^{\text{op}}(y) = \emptyset \).
Thus \( \iota \) is certainly a cofibration of collections. If \( f: x \to y \) has at least one internal edge then \( x \in C_{0} \), so \( \iota(x) \) is an
isomorphism. Thus \( \iota \) is a cofibration of operads in \( \text{Op}^{\text{op}}_0 \).

Lemma 31. For any \( k \geq 0 \) the natural map \( F_{k+1} W_{C}^{\text{op}} \to W_{k+1}^{\text{op}} \) is a cofibration in \( \text{Op}^{\text{op}}_{k+1} \).

Proof. The argument from lemma \ref{cofibration} works in this case.

Corollary 32. The natural map \( F_{1} W_{C}^{\text{op}} \to W_{h}^{\text{op}} \) is a cofibration in \( \text{Op}_h \).

Proof. We have \( W_{h}^{\text{op}} = F_{h,c}^{-1} W_{C}^{\text{op}} \) and \( F_{h,c} W_{C}^{\text{op}} = W_{h}^{\text{op}} \).
By lemmas \ref{cofibration} and \ref{cofibration}, \( F_{h}^{-1} W_{C}^{\text{op}} \to W_{C}^{\text{op}} \) is a cofibration in \( C \).
Apply \( F_{h,c} \) and use the fact \( F_{h,c} F_{h}^{-1} = F_{1} F_{h,c}^{-1} \) to get \( F_{1} W_{C}^{\text{op}} \to W_{h}^{\text{op}} \).

The main example of an \{f, h\}-colored operad is \( SC_d \), where \( f \) stands for full disc and \( h \) stands for half disc. Fix a dimension \( d \geq 1 \). Let \( D_t \) denote the closed
unit disc inside \( \mathbb{R}^d \), and let \( D_h \) denote the closed unit half-disc \( \{ p \in D_t \mid p_d \geq 0 \} \).
For any \( K \)-colored set \( (I, \text{col}_I) \) and any \( i \in I \) let \( D_i = D_{\text{col}_I(i)} \). Finally, put
\( D(I) = \prod_{i \in I} D_i \).

Definition 33. Given a \( K \)-colored young forest \( x \), let \( SC_d(x) \) denote the set of maps \( \alpha: D(I_x) \to D(J_x) \) such that

- For each \( i \in I_x \) the restriction of \( \alpha \) to \( D_t \) lands in \( D_x(i) \).
- For each \( i \in I_x \) of color \( f \) there is an \( r(i) > 0 \) and \( c(i) \in \mathbb{R}^d \) such that
  \( \alpha(p) = r(i)p + c(i) \) for all \( p \in D(i) \).
• For each \( i \in I_x \) of color \( h \), \( x(i) \) has color \( h \) and there is an \( r(i) > 0 \) and \( c(i) \in \mathbb{R}^{d-1} \times \{0\} \) such that \( \alpha(p) = r(i)p + c(i) \) for all \( p \in D(i) \).

• \( \alpha \) is an embedding of \( D(I_x) \) into \( D(J_x) \).

Let \( (n, m) \) denote the \( K \)-colored set \( \{1, \ldots, n + m\} \) where \( 1 \leq i \leq n \) has color \( f \) and \( n + 1 \leq i \leq n + m \) has color \( h \). Suppose \( J_x = \{j\} \), then an isomorphism \( I_x \rightarrow (n, m) \) defines an embedding of \( SC_d(x) \) into \( \mathbb{R}^N \), where \( N = n + nd + m + m(d - 1) \). We endow \( SC_d(x) \) with the topology induced from such an embedding. If \( J_x \) is not a singleton then \( x = \bigcup_j x^{-1}(j) \) and \( SC_d(x) = \prod_{j \in J_x} SC_d(x|x^{-1}(j)) \), and we take the product topology on \( SC_d(x) \).

If \( f: x \rightarrow y \) is a \( K \)-colored forest, the map \( SC_d(f): SC_d(x) \rightarrow SC_d(y) \) is defined to be

\[
SC_d(f)(\alpha)(p) = (D(f)\alpha)^k D(f)(p)
\]

where \( \alpha \in SC_d(x) \), \( p \in D_i \), \( i \in I_y \), and \( [f|x](i) = (fx)^k f(i) \). The isomorphism \( f: I_y \sqcup J_x \rightarrow J_y \sqcup I_x \) induces the isomorphism \( D(f): D(I_y) \sqcup D(J_x) \rightarrow D(J_y) \sqcup D(I_x) \).

In [Tho12], \( SC_d^{h,\infty} \) denotes \( F_{\text{th}} W_1^{\text{th}} SC_d \) and \( SC_d^h \) denotes \( W_1^{k} SC_d \). In this special case corollary 32 gives theorem 34, which is an essential technical result of that paper. Informally, theorem 34 combined with theorem 33 means that the swiss cheese operad is generated in degrees 0 and 1.

**Theorem 34.** The natural map \( SC_d^{h,\infty} \rightarrow SC_d^h \) is a cofibration in Op\(_h\).

### 3.1 Weak equivalence proof

This section contains a proof of

**Theorem 35.** The natural map \( SC_d^{h,\infty} \rightarrow SC_d^h \) is a weak equivalence of operads in Op\(_h\).

The idea of the proof is to consider the maps \( p_1: SC_d^{h,\infty}(n, m) \rightarrow SC_d^{h,\infty}(n-1, m) \) and \( p: SC_d^h(n, m) \rightarrow SC_d^h(n-1, m) \) given by forgetting the \( n \)th disc. By induction, we can suppose \( SC_d^{h,\infty}(n-1, m) \rightarrow SC_d^h(n-1, m) \) is a weak equivalence. We continue the induction by showing that \( p_1^{-1}(\alpha) \rightarrow p^{-1}(\alpha) \) is a weak equivalence for every \( \alpha \in SC_d^{h,\infty}(n-1, m) \).

To make the computation of \( p_1^{-1}(\alpha) \) and \( p^{-1}(\alpha) \) accessible, we will collapse the \( n \)th disc of \( \alpha \in SC_d^h(n, m) \) to a point. Our goal in the next section is to make this precise.

#### 3.1.1 Defining \( SC_d, (k, l|n, m) \)

When we collapse the \( n \)th disc of \( \alpha \in SC_d^h(n, m) \) to its center, we think of the result \( \alpha \) as living in a four-colored operad which we denote by \( SC_d, (k, l|n, m) \). We add the colors \( f \) and \( h \). Let \( K_\bullet = \{f_\bullet, h_\bullet, f, h\} \) be the set of colors for this new operad. The color \( f_\bullet \) stands for collapsed full disc. It is convenient to also allow a collapsed half disc, which we color with \( h_\bullet \). Let \( (k, l|n, m) \) denote the \( K_\bullet \)-colored
finite set with \((k, l, n, m)\) elements of color \((f_\bullet, h_\bullet, f, h)\). Let \(\text{For}^{K_\bullet}_{\text{h}}\) denote the full sub category of \(\text{For}^{K_\bullet}\) with objects isomorphic to disjoint unions of the young forests

\[
(0,0|n,m) \to \{h\} \quad (1,0|n,m) \to \{h\} \quad (0,1|n,m) \to \{h\} \quad (1,0|0,0) \to \{h_\bullet\}
\]

(36)

To define \(SC_{d, \bullet}: \text{For}^{K_\bullet}_{\text{h}} \to \text{Top}\) we need the notion of the geometric realization of \(\beta \in SC_d^{h}(n,m)\).

**Definition 37.** Given \(\beta \in SC_d^h(n,m)\), let \(\lvert \beta \rvert\) be its geometric realization. This is the subset of \(\mathbb{R}^d\) given by deleting the open discs and half-discs of \(\beta\) from the closed unit half-disc. More precisely, if \(D^+_d\) is the closed unit half-disc in \(\mathbb{R}^d\), \(\{(D^+_d)^i_{j=1}\}^n_{i=1}\) are the open discs of \(\beta\), and \(\{(D^+_d)^{i}_{j=1}\}^n_{i=1}\) are the open half-discs of \(\beta\) considered as open discs in \(\mathbb{R}^d\) whose center lies in \(\mathbb{R}^{d-1}\), then

\[
\lvert \beta \rvert = D^+_d - \left( \bigcup_{i=1}^{n} (D^+_d)^i \cup \bigcup_{j=1}^{m} (D^+_d)^j \right).
\]

Let \(\partial_h \lvert \beta \rvert := \partial (D^+_d - (\bigcup_i (D^+_d)^i))\) be the \(h\)-colored boundary of \(\lvert \beta \rvert\). Let \(\partial_t \lvert \beta \rvert\) be the upper hemisphere \(S^d_{d-1} \subset \partial D^+_d\) and let \(\partial_i \lvert \beta \rvert\) be the upper hemisphere of \(\partial (D^+_d)^i\) for \(1 \leq i \leq n\).

Now we can set

\[
\begin{align*}
SC_{d, \bullet}^{h}(0,0|n,m) & = SC_{d}^{h}(n,m) \\
SC_{d, \bullet}^{h}(1,0|n,m) & = \{ (\alpha, q) \mid \alpha \in SC_{d}^{h}(n,m), q \in \lvert \alpha \rvert \} \\
SC_{d, \bullet}^{h}(0,1|n,m) & = \{ (\alpha, q) \mid \alpha \in SC_{d}^{h}(n,m), q \in \lvert \alpha \rvert \cap \mathbb{R}^{d-1} \} \\
SC_{d, \bullet}^{h}(k, |n,m) & = \ast
\end{align*}
\]

We think of the point \(q \in \lvert \alpha \rvert\) as a collapsed disc and the point \(q \in \lvert \alpha \rvert \cap \mathbb{R}^{d-1}\) as a collapsed half-disc. Composition in \(SC_{d, \bullet}\) takes place in the half-discs and collapsed half-discs only. The discs play no part in composition. However the collapsed half-discs and collapsed discs only play a part in composition when we plug a collapsed disc into a collapsed half-disc. The result is a collapsed disc which happens to live on the boundary of the geometric realization.

**Definition 38.** Let \(\text{For}^{K_\bullet}_{\leq 1,h}\) denote the full sub category of \(\text{For}^{K_\bullet}_{\text{h}}\) given by disjoint unions of the young forests from \(\text{[36]}\) with \(n \leq 1\). In the same way, let \(\text{For}^{K_{\leq 1,h}}\) denote the full sub category of \(\text{For}^{h}\) given by disjoint unions of forests from \(\text{[36]}\) with \(k = l = 0\) and \(n \leq 1\). We write \(x \leq 1\) if \(x\) is a \(K_\bullet\)-colored young forest in \(\text{For}^{K_{\leq 1,h}}\) and we also write \(x \leq 1\) if \(x\) is a \(K\)-colored young forest in \(\text{For}^{K}_{\leq 1,h}\). Let \(SC_{d, \bullet}^{h_{1}}\) denote the restriction of \(SC_{d, \bullet}\) to \(\text{For}^{K_{\leq 1,h}}\).

Let \(SC_{d, \bullet}^{h_{1}}\) and \(SC_{d, \bullet}^{h_{1}}\) denote the \(W\) construction applied to the four-colored operads \(SC_{d, \bullet}\) and \(SC_{d, \bullet}^{h_{1}}\). Let \(F\) denote Kan extension along \(\text{For}^{K_{\leq 1,h}} \to \text{For}^{K_{\bullet}}_{\text{h}}\).
Figure 1: The collapsed discs are denoted by dots and the collapsed half-discs by tick marks. Collapsed discs are color $f_\bullet$ input edges and collapsed half-discs are color $h_\bullet$ input edges. To keep the collapsed discs and half-discs from coinciding, we only allow one or the other in any composition. Composition in $SC_{d,\bullet}$ takes place only in the half-discs and collapsed half-discs. The only composition we can do in a collapsed half-disc is given by plugging in a collapsed disc. The result is a collapsed disc replacing the collapsed half-disc.

Consider the commutative diagram of topological spaces where the horizontal arrows do not assemble to operad maps,

$$
\begin{array}{c}
SC_{d,\bullet}^{h}(n+1,m) & \xrightarrow{\sim} & F(SC_{d,\bullet}^{h})(1,0|n,m) & \xrightarrow{p_1} & SC_{d}^{h\infty}(n,m) \\
\downarrow & & \downarrow & & \downarrow \\
SC_{d}^{h}(n+1,m) & \xrightarrow{\sim} & SC_{d,\bullet}^{h}(1,0|n,m) & \xrightarrow{p} & SC_{d}^{h}(n,m).
\end{array}
$$

The maps $p_1$ and $p$ delete the collapsed disc and, if necessary, a left over collapsed half-disc. By induction on $n$ we assume the right vertical arrow is an equivalence. We will show that for each $\alpha \in SC_{d}^{h\infty}(n,m)$ the inclusion $p_1^{-1}(\alpha) \rightarrow p^{-1}(\alpha)$ is an equivalence. Then by the long exact sequence of homotopy groups we conclude that the middle vertical arrow is an equivalence. The top left and bottom right maps collapse the $n^{th}$ full disc. One can show that these are equivalences. We conclude that the left vertical arrow is also an equivalence. This will prove theorem 35.

3.1.2 Computing $p^{-1}(\alpha)$ and $p_1^{-1}(\alpha)$.

We have shown that the proof rests on the following proposition 39. This section is dedicated to the proof of this proposition.

**Proposition 39.** Fix $\alpha \in SC_{d}^{h\infty}(n,m)$. The inclusion of the fiber $p_1^{-1}(\alpha)$ into the fiber $p^{-1}(\alpha)$ is a weak equivalence.
Definition 40. Define $p : \text{For}^{K\bullet}_h \to \text{For}^K_h$ by sending the $K\bullet$-colored young forest $x : I_x \to J_x$ to the $K$-colored forest $px$ with

$$I_{px} = I_x \setminus (I_x)_{h\bullet}, \quad J_{px} = J_x \setminus (J_x)_{h\bullet},$$

where $I_{h\bullet}$ is the $f_{h\bullet}$ and $h_{\bullet}$-colored portion of the $K_h\bullet$-colored set $I$. In \cite{36} we see that we must have $x(I_{px}) \subset J_{px}$ so that we can define $px$ as the restriction of $x$ to $I_{px}$. Observe that $p(1,0|n,m) = (n,m)$. If $f : y \to x$ is a forest, then $pf : py \to px$ is defined using $f$. Since $f$ preserves the colorings $pf$ is indeed a forest from $py$ to $px$. If $f$ is in $\text{For}^{K\bullet}_h$, then $pf$ is a morphism in $\text{For}^K_h$.

If $\beta \in \text{SC}_{d\bullet}(z)$ for a $K\bullet$-colored young forest $z$, then we get $p\beta \in \text{SC}_{d}(pz)$. To define $p\beta$ write $\beta = (\beta_j)_{j \in J_\alpha}$ where $\beta_j \in \text{SC}_{d\bullet}(z^{-1}(j))$. Each $\beta_j$ is of the form $(\gamma_j, q_j)$ with $q_j \in |\gamma_j|$ or of the form $\beta_j \in \text{SC}_{d}(z^{-1}(j))$. Set $p\beta = (\gamma_j)_{j \in J_\alpha}$.

If $t \in W(f)$ and $f \in \text{For}^{K\bullet}_h$, then $E(pf) \subset E(f)$ and $pt \in W(pf)$ is defined to be the pullback of $t$ : $E(f) \to [0,\infty]$.

Combining the colimits defining the $W$ construction and the left adjoint $\text{Op}(\text{Coll}^{\leq 1}) \to \text{Op}(\text{Coll})$ we get

$$\text{SC}^{h\infty}_d(n,m) = \left( \coprod_{g : z \to y} W(g) \times \text{SC}_d(z) \right)/\sim,$$

where $y \leq 1$ (definition \cite{38}) and the same relations hold as in equation \cite{19}. If $\alpha \in \text{SC}^{h\infty}_d(n,m)$ is represented by $(f,g,t,\bar{\alpha})$ where $f : y \to (n,m)$, $g : z_\alpha \to y, t \in W(g)$, and $\alpha \in \text{SC}_d(z)$, then $\alpha \in \text{SC}^{h\infty}_d(n,m)$ is represented by $(f,g,W_\infty(f)t,\bar{\alpha})$. Let $T_\alpha = fg : z_\alpha \to (n,m)$ and $t_\alpha = W_\infty(f)t$. Without loss of generality, we may assume $t_\alpha(i) > 0$ for every $i \in E(T_\alpha)$ and that $\bar{\alpha}(j) \neq \text{id}_{\text{SC}_d}$ for any $j \in J_\alpha$.

Definition 41. Let $\text{Trees}(1,0|n,m)$ denote the over category $\text{For}^{K\bullet}_{h/(1,0|n,m)}$. Let $\text{Trees}(n,m) = \text{For}^{K\bullet}_{\text{h}/(n,m)}$. Let $p : \text{Trees}(1,0|n,m) \to \text{Trees}(n,m)$ denote the functor induced by $p$ from definition \cite{40}.

Note that $T_\alpha \in \text{Trees}(n,m)$. Let $(S,\nu) \in \text{Trees}(1,0|n,m)_{T_\alpha}$ where $S : x \to (1,0|n,m)$ is any $K\bullet$-colored tree and $\nu : z_\alpha \to px$ is a forest such that $(pS)\nu = T_\alpha$.

Define functors $W_\alpha : \text{Trees}(1,0|n,m)_{P_{T_\alpha}} \to \text{Top}$ and $S\alpha : \text{Trees}(1,0|n,m)_{T_\alpha} \to \text{Top}$ via the pullbacks

$$\text{SC}_\alpha(S) \quad \text{SC}_{d\bullet}(x) \quad W_\alpha(S) \to W(S) \to (42)$$

$$\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow$$

$$\bar{\alpha} \quad S\alpha(z) \quad SC_d(pz) \quad W(pS) \quad \text{Op}(\text{Coll}^{\leq 1}) \quad \text{Op}(\text{Coll}) \quad \text{Op}(\text{Coll})$$

We want to replace $\text{Trees}(1,0|n,m)$ by a much smaller category. First we need the wedge operation on forests.

Definition 43. Let $f : x \to y$ be a $K_f$-colored forest and let $g : z \to w$ be an $K_g$-colored forest for some finite sets $K_f, K_g$. Let $\tau : J_w \to J_x$ be any map. Define $x \lor_{\tau} z$ to be the young $K_f \sqcup K_g$-colored forest $(x,\tau,z) : I_x \sqcup J_w \sqcup I_z \to J_x \sqcup J_z$. 

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Thus we get the commutative diagram on the left. The image of this diagram

\[ I_y \sqcup I_w \to J_y. \]

Finally, set \( f \circ \tau \circ g : x \circ \tau \circ z \to y \circ \tau \circ w \) to be the forest \( (f, g, f, \tau) : I_y \sqcup I_w \sqcup J_x \sqcup J_z \to I_x \sqcup J_w \sqcup I_z \sqcup J_y. \)

**Definition 44.** Let \( \Gamma_0 \) be the tree with with no internal vertices and a single input vertex of color \( f_\bullet \). Let \( \Gamma_1 \) be the tree with a single internal vertex of color \( h_\bullet \) and a single input vertex of color \( f_\bullet \).

For any edge \( i \in \text{Edge}(T_\alpha)_h \) define \( \nu(i) : T_\alpha \to T_\alpha(i) \) to be the morphism in \( \text{Trees}(n, m) \) which inserts a unary vertex along \( i \). Call this new vertex \( \mathcal{X}_i \). Let \( \mathcal{X}_{i,k} = T_\alpha(i) \vee_{\mathcal{X}_i} \Gamma_k \). For any internal vertex \( j \in J_{\alpha} \) let \( S_{j,k} = T_\alpha \vee_j \Gamma_k \). Note that \( p_{\mathcal{X}_{i,k}} = T_{\alpha}(i) \) and \( p_{S_{j,k}} = T_{\alpha} \).

Let \( \text{Trees}_\alpha \) be the full subcategory of \( \text{Trees}(1, 0|n, m)/T_\alpha \) given by the objects \( S_{i,k} = (\mathcal{X}_{i,k}, \nu(i)) \) and \( S_{j,k} = (S_{j,k}, \text{id}_{T_\alpha}) \) where \( i \in (I_{\alpha})_h \sqcup \{rt\} \), \( j \in J_{\alpha} \) and \( k \in \{0, 1\} \).

**Remark 45.** The advantage of \( T_\alpha \) is that it is easy to understand and computes the space \( p^{-1}(\alpha) \) (Lemma 46). There is a unique morphism \( \mathbb{S}_{\ell, \alpha} \to \mathbb{S}_{0, \alpha} \) for every \( \ell \) and unique morphisms \( S_{i,k} \to S_{T_\alpha^{-1}(i), k} \) and \( S_{i,k} \to S_{\alpha(i), k} \). See figure 3.1.2 for an illustration.

**Lemma 46.** The fiber \( p^{-1}(\alpha) \) is given by the coend

\[ W_\alpha \otimes_{\text{Trees}_\alpha} \text{SC}_\alpha. \]

**Proof.** Let \( \gamma = [S, s, \gamma] \in \text{SC}_d^{h, \bullet}(1, 0|n, m) \) where \( S : x \to (1, 0|n, m) \) is a forest in \( \text{For}^{h, \bullet} \), \( s \in W(S) \), and \( \gamma \in \text{SC}_d^{h, \bullet}(x) \). Let us assume that \( \gamma(j) \neq \text{id} \) for all \( j \in J_x \) and \( s(i) > 0 \) for all \( i \in E(S) \). Observe that \( p_{\gamma} \in \text{SC}_d(n, m) \) is given by \( [pS, ps, p\gamma] \). If \( p_{\gamma} = \alpha \) there must be some \( \nu : T_\alpha \to pS \) in \( \text{Trees}(n, m) \) such that \( SC_d(\nu)\alpha = p\gamma \) and \( W(\nu)ps = t_\alpha \). The condition \( t_\alpha(i) > 0 \) for all \( i \in E(T_\alpha) \) implies that \( t_\alpha \neq W_\gamma(\nu)(t') \) for any \( t' \) and any \( \nu \) which collapses any edges. Moreover the condition \( \gamma(j) \neq \text{id} \) for all \( j \) implies that \( p_{\gamma}(j) \neq \text{id} \) for all \( j \in J_x \) such that \( x^{-1}(j)_{\alpha, h_\bullet} \) is not empty. We conclude that either \( \nu = \text{id} \) or \( \nu \) is the insertion of the unique unary (in \( pS \), not in \( S \)) vertex \( j \) such that \( x^{-1}(j)_{\alpha, h_\bullet} \) is not empty. In the former case we must have \( S = S_{j, k} \) for some vertex \( j \in J_x \) and some \( k \in \{0, 1\} \). In the latter case we have \( S = S_{i, k} \) for some edge \( i \) of \( T_\alpha \) and some \( k \). This defines the map \( p^{-1}(\alpha) \to W_\alpha \otimes_{\text{Trees}_\alpha} \text{SC}_\alpha \). The map in the other direction is clear and the verification that they are inverses is left to the reader.

In diagram 47 we have \( h \)-colored edges \( i_1, i_2 \) of \( T_\alpha \) with \( z_\alpha(i_1) = j = T_\alpha^{-1}(i_2) \). Thus we get the commutative diagram on the left. The image of this diagram under \( \text{SC}_\alpha \) is shown on the right.

\[
\begin{array}{ccc}
S_{i_1,1} & \to & S_{j,1} \\
\downarrow & & \downarrow \\
S_{i_0,1} & \leftrightarrow & S_{i_2,1} \\
\downarrow & & \downarrow \\
S_{i_1,0} & \to & S_{j,0} \\
\downarrow & & \downarrow \\
S_{i_2,0} & \leftrightarrow & S_{i_0,0} \\
\end{array}
\]

\[
\begin{array}{cccc}
|id_h| \cap \mathbb{R}^{d-1} & \to & |\alpha(j)| \cap \mathbb{R}^{d-1} & \leftarrow |id_h| \cap \mathbb{R}^{d-1} \\
\downarrow & & \downarrow & \downarrow \\
|id_h| & \leftrightarrow & |\alpha(j)| & \leftarrow |id_h| \\
\end{array}
\]

(47)
Figure 2: The edge $i$ and vertex $j$ of $T_\alpha$ give a commutative square in $\text{Trees}_\alpha$. The input vertices are circles. The output vertex ends in an $\times$. The internal vertices are filled dots. The input and internal vertices of $\Gamma_0$ and $\Gamma_1$ are labeled with their colors. In addition, the image of $S_{i,1}$ under the functor $p$ is shown. This makes it clear that the map $T_\alpha \to pS_{i,1}$ is given by inserting a single vertex.

The geometric realization of the identity $id_\alpha$ is just $S_d^{d-1}$, the top half of the $(d-1)$-sphere. The input of $\tilde{\alpha}(j)$ corresponding to $i_{1}$ is a half disc and the map $|id_\alpha| \to |\tilde{\alpha}(j)|$ corresponding to $S_{i_{1},0} \to S_{j,0}$ is just $\partial_1 |\tilde{\alpha}(j)| \to |\tilde{\alpha}(j)|$ (see definition 37). On the other hand the image of $S_{i_{2},0} \to S_{j,0}$ is the inclusion of the output boundary $\partial rt|\tilde{\alpha}(j)| \to \tilde{\alpha}(j) |\tilde{\alpha}(j)|$.

**Definition 48.** Let $\epsilon_\bullet \in E(S_{i,1})$ be the unique internal edge of color $h_\bullet$. If $i \in \text{Edge}(T_\alpha)$, let $i_v$ denote the vertex inserted by $\nu: T_\alpha \to pS_{i,k}$. Let $i_{in}$ and $i_{out}$ respectively denote the incoming and outgoing edges of $i_v$ considered as internal edges of $S_{i,k}$. For any object $S_{i,k}$ of $\text{Trees}_\alpha$, let $E_\alpha(S_{i,k}) = \{\epsilon_\bullet\}^k \sqcup (\{i_{in}, i_{out}\} \cap E(S_{i,k}))$. This defines a functor $E_\alpha: \text{Trees}_\alpha^{op} \to \text{Set}$.

The image under $W_\alpha$ of the square in diagram 47 is in diagram 49.

\[
\begin{align*}
[0, \infty]^2 & \xrightarrow{(id,0)} [0, \infty] \xleftarrow{0} [0, \infty]^2 \\
0 & \xrightarrow{(0, id)} [0, \infty] \xrightarrow{(id,0)} [0, \infty]^2
\end{align*}
\]

More precisely,

\[
W_\alpha(S) = \{s: E_\alpha(S) \to [0, \infty] \mid s(i_{in}) + s(i_{out}) = t_\alpha(i)\},
\]

and $W_\alpha(S) \to W_\alpha(S')$ for a map $S' \to S$ in $\text{Trees}_\alpha$ is given by push forward of functions along the map of finite sets $E_\alpha(S) \to E_\alpha(S')$. There is no condition on $s(\epsilon_\bullet)$, the length of the edge of color $h_\bullet$. The isomorphism $W_\alpha(S_{i,1}) \to [0, \infty]^2$ sends $s$ to $(s(\epsilon_\bullet), r(s(i_{out}), s(i_{in})))$ where

\[
r(s_0, s_1) = \frac{1 - e^{-s_0}}{1 - e^{-s_1}},
\]

14
which lands in $[0, \infty]$ because $s_0 + s_i = t_\alpha > 0$. Note that $s_\alpha = 0$ if and only if $r(s_0, s_i) = 0$ and $s_\alpha = t_\alpha$ if and only if $r(s_\alpha, s_i) = \infty$. Since the morphism $S_{i,1} \rightarrow S_{j,1}$ from diagram \[47\] collapses the edge $(i_1)_{\text{out}}$ we get $W_\alpha(S_{j,1}) \cong \{(r_*, r) \in W_\alpha(S_{i,1}) \mid r = 0\}$. In the same diagram, the morphism $S_{i,0} \rightarrow S_{j,1}$ collapses the edge $(i_2)_{\text{in}}$ so we have $W_\alpha(S_{j,1}) \cong \{(r_*, r) \in W_\alpha(S_{i,1}) \mid r = \infty\}$.

The unique morphism $S_{1,1} \rightarrow S_{1,0}$ collapses the edge $i_0$ so that $W_\alpha(S_{1,0}) \cong \{(r_*, r) \in W_\alpha(S_{1,1}) \mid r_* = 0\}$. The rest can be deduced from these cases.

**Lemma 51.** For any functor $F$: Trees$_\alpha \rightarrow \text{Top}$ the coend $W_\alpha \otimes_{\text{Trees}_\alpha} F$ is the homotopy colimit of $F$ over Trees$_\alpha$.

**Proof.** It is clear from diagrams \[49\] and \[47\] that $W_\alpha(S)$ is the geometric realization of the nerve of the under category of $S$ for each object $S \in \text{Trees}_\alpha$. In addition the maps $W_\alpha(S) \rightarrow W_\alpha(S')$ for $S' \rightarrow S$ agree with the maps obtained from the nerves of under categories. □

**Lemma 52.** We can explicitly compute $p^{-1}(\iota \alpha)$ as

$$p^{-1}(\iota \alpha) \simeq |SC_d(T_\alpha)\tilde{\alpha}| \simeq (S^{d-1})^\vee n,$$

where $SC_d(T_\alpha)\tilde{\alpha}$ is the composition of all vertex labels from $\iota \alpha$.

**Proof.** Let Trees$_{s,0}$ denote the full subcategory of Trees$_\alpha$ consisting of objects $S_{j,0}$ and $S_{i,0}$ for internal vertices $j$ and internal edges $i$. This category is homotopy terminal, so by lemma \[51\] and lemma \[46\] we have $p^{-1}(\iota \alpha) = \text{hocolim}_{\text{Trees}_{s,0}} SC_\alpha$. This is the same as the homotopy colimit of the coequalizer diagram

$$\bigoplus_{i \in E(T_\alpha)} |dh| \Rightarrow \bigoplus_{j \in V(T_\alpha)} |\tilde{\alpha}(j)|,$$

where one arrow is given by including into output parts of the boundaries of $|\tilde{\alpha}(j)|'$s, and the other arrow is given by including into input boundaries. These maps are cofibrations with disjoint images. Each space in the coequalizer diagram is cofibrant. Thus the coequalizer diagram is already cofibrant as a functor $(\cdot \Rightarrow \cdot) \rightarrow \text{Top}$. Thus we can compute the normal colimit. It is clear that this is the same as composing the $\tilde{\alpha}(j)$'s via $T_\alpha$ then taking the realization of the result. In addition $|\beta|$ is equivalent to a wedge of $n$ spheres of dimension $d - 1$ if $\beta \in SC_d(n, m)$. □

**Definition 53.** Let Trees$_{s,1}$ denote the full subcategory of Trees$_\alpha$ where we discard the objects $S_{j,0}$ and $S_{i,0}$ for $j \in J_{z_\alpha}$ and $i \in E(T_\alpha)$. Define a functor $W_{\alpha,1}$: Trees$_{s,1} \rightarrow \text{Top}$ by setting

$$W_{\alpha,1}(S) = \{s: E_\alpha(S) \rightarrow [0, \infty] \mid \sum_{i \in E_\alpha(S)} s(i) = \infty\}.$$ 

**Lemma 54.** Suppose $t_\alpha < \infty$, and $n = 1$, then $p^{-1}_1(\alpha)$ is given by the coend

$$W_\alpha \otimes_{\text{Trees}_{s,1}} SC_\alpha,$$

where $SC_\alpha$ is the functor in definition \[53\] restricted to Trees$_{s,1}$ and $W_{\alpha,1}$ is defined in \[53\].
Proof. Let $\gamma \in F(\text{SC}^d_{a,\bullet})(1,m)$ such that $p_1(\gamma) = \alpha$. Pick a representative $(f,g,s,\tilde{\gamma})$ where $f : y \to (1,0,1,m)$, $y \leq 1$, $g : z \to y$, $s \in W(g)$ and $\tilde{\gamma} \in \text{SC}^d_{a,\bullet}(z)$. Consider $\nu \in \text{SC}^d_{a,\bullet}(1,m)$, which is represented by $(fg,W_{\infty}(f)s,\tilde{\gamma})$. Recall that the condition $y \leq 1$ means that each connected component of the young forest $y$ has at most one input whose color lives in $\{f,f_{\bullet}\}$. This implies that $f$ has at least one internal edge $i \in E(f)$. Thus $W_{\infty}(f)s(i) = \infty$ when $i$ is viewed as an internal edge in $fg$.

We know $\nu \gamma = i \alpha$, so $\nu \gamma$ is represented by some triple $(S,s',\tilde{\gamma})$ with $S \in \text{Trees}_{a,1}$, $s' \in W_{a}(S)$, and $\gamma \in \text{SC}_{a}(S)$. The relations in $\text{SC}^d_{a,\bullet}$ preserve edges of length $\infty$, so we must have $s'(i) = \infty$ for some $i \in E(S)$. We are assuming $t_{a}(i) < \infty$ for all $i \in E(T_{a})$, so the infinite edge in $S$ must be in $E_{a}(S)$. This implies $s' \in W_{a,1}(S)$. Moreover we cannot have such an infinite edge if $S = S_{j,0}$ for some vertex $j$ or $S = S_{1,0}$ for some internal edge $i$. Thus $S \in \text{Trees}_{a,1}$. This defines the map from $p^{-1}(\alpha)$ to the coend. We leave the remainder to the reader. \hfill \square

Lemma 55. For any functor $F : \text{Trees}_{a,1} \to \text{Top}$ the coend $W_{a,1} \otimes_{\text{Trees}_{a,1}} F$ is the homotopy colimit of $F$ over $\text{Trees}_{a,1}$. \hfill \square

Proof. The argument here is similar to the proof of lemma 51.

Corollary 56. If $t_{a} < 0$ and $n = 1$, then the fiber $p_{1}^{-1}(\alpha)$ is equivalent to $\partial_{h} |\text{SC}_{d}(T_{a})\alpha| \simeq S^{d-1}$.

Proof. By the same argument as in lemma 52, $\text{hocolim}_{\text{Trees}_{a,1}} \text{SC}_{a}$ is equivalent to $\text{colim}_{\text{Trees}_{a,1}} \text{SC}_{a}$. This is easily computed as the $h$-colored boundary of the composite of $\alpha$. \hfill \square

proof of proposition 79. Recall $\alpha$ is represented by $f : y \to (n,m)$, $y \leq 1$, $g : z_{a} \to y$, $t \in W(g)$ and $\bar{\alpha} \in \text{SC}_{d}(z_{a})$. By applying relations in $\text{SC}_{d}^{\infty}$ we may assume $0 < t < \infty$. We may think of $(g,t)$ as representing an element of $\text{SC}_{d}^{\infty}(y)$ which we can write as $(\alpha(j))_{j \in J_{a}}$. If $\alpha(j) \in \text{SC}_{d}^{\infty}(n_{j},m_{j})$ then $n_{j} \leq 1$. Clearly $p_{a}^{-1}(\alpha(j)) \simeq p^{-1}(\alpha(j)) \simeq *$ when $n_{j} = 0$. Since $t_{a}(j) < \infty$ we can use corollary 56 to conclude $p_{1}^{-1}(\alpha(j)) \simeq \partial_{h}[(\text{SC}_{d}(g)\alpha)(j)]$. The fiber $p_{1}^{-1}(\alpha)$ is equal to the colimit of the diagram

$$
\coprod_{i \in E(f)} |1_{h}| \Rightarrow \coprod_{j \in V(f)} p_{1}^{-1}(\alpha(j)),
$$

where one arrow is given by $|1_{h}| \simeq \partial_{t} |\alpha(y(i))| \rightarrow \partial_{h} [(\text{SC}_{d}(g)\alpha)(y(i))]$ and the other by $|1_{h}| \simeq \partial_{s} |\alpha(f(i))| \rightarrow \partial_{h} [(\text{SC}_{d}(g)\alpha)(y(i))]$. This colimit is clearly $(S^{d-1})_{/n} \simeq p^{-1}(i\alpha)$. \hfill \square
Figure 3: On the left is $\alpha \in \mathcal{SC}_\delta^{h\infty}(2,3)$. In the middle is $p^{-1}(\alpha)$, and on the right is $p^{-1}_1(\alpha)$. Both $p^{-1}(\alpha)$ and $p^{-1}_1(\alpha)$ have the homotopy type of a wedge of spheres, one for each disc in $\alpha$.

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