A Supersymmetric Solution in $N = 2$ Gauged Supergravity with the Universal Hypermultiplet

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Abstract

We present supersymmetric solutions for the theory of gauged supergravity in five dimensions obtained by gauging the shift symmetry of the axion of the universal hypermultiplet. This gauged theory can also be obtained by dimensionally reducing M-theory on a Calabi-Yau threefold with background flux. The solution found preserves half of the $N = 2$ supersymmetry, carries electric fields and has nontrivial scalar field representing the CY-volume. We comment on the possible solutions of more general hypermultiplet gauging.

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1 Introduction

The study of supersymmetric configurations in five-dimensional gauged supergravity theories has been receiving considerable attention in recent years. This study is relevant to AdS/CFT correspondence which relates string theory (or, for energies much smaller than the string scale, supergravity) on anti-de Sitter (AdS) spaces to supersymmetric conformal field theories residing on the boundary of AdS [1]. This provides a possibility to study the regime of large $N$ and large 't Hooft coupling of these field theories by means of classical supergravity solutions. It is hoped that this correspondence will eventually lead to a deeper insight into confinement in QCD, or at least into the strong coupling regime of gauge theories with fewer supersymmetries. Moreover, gauged supergravity models in five dimensions provide the framework which may incorporate smooth Randall-Sundrum brane worlds [2]. Most of the analysis so far has been focused on the study supersymmetric backgrounds in the gauged supergravity with vector multiplet gaugings [3, 4]. In these models, the so-called $U(1)$-gauged theories, the hypermultiplets decouple in the analysis of finding supersymmetric background solutions. Apart from the hyperbolic-string solutions [5], most of the known supersymmetric solutions contain naked singularities or have other peculiar properties. For example, the rotating charged solutions in the five dimensional gauged theories were found to violate causality and represent naked time machines [6]. Moreover, domain wall solutions have also been the subject of intense research activity motivated both by the interpretation of domain wall solutions as realizations of renormalization group flows in the dual field theory and by the attempt to find an embedding of Randall-Sundrum scenario in string or M-theory (see [7] and references therein).

Five dimensional supergravity theory needs a minimum of eight supercharges. The coupling of this theory to vector multiplets and its gaugings was formulated in [8, 9]. The gravity multiplet contains the graviton, the gravitino and the graviphoton. The abelian vector multiplets each contains a $U(1)$ vector field, a gaugino and a real scalar field. The scalar fields of the vector multiplets parameterize a space $M_V$ defining a very special geometry where all the couplings of the theory are determined in terms of a cubic form $F = C_{IJK} h^I h^J h^K$ [10].

The compactification of M-theory on Calabi-Yau three-fold [11] gives an effective field theory which is $N = 2$ five dimensional supergravity interacting with a number of hypermultiplets and vector multiplets with scalars parameterizing a manifold $M = M_V \times M_H$. In this framework, $h^I$ are related to the Kähler class moduli fields and $C_{IJK}$ are the topological intersection numbers. The hypermultiplets, each containing two hyperinos and four real scalar fields living on a quaternionic Kähler manifold $M_H$ [12, 13]. In string or M-theory compactification, there is at least one hypermultiplet which contains the Calabi-Yau volume. The geometry of this hypermultiplet is given by the coset space $SU(2,1)/U(2)$ [14]. Gauged supergravity theories can be obtained by gauging the isometries of the manifolds $M_V$ and $M_H$ [15] (see also [16] and references therein). If one compactifies M-theory on a Calabi-Yau threefold in the presence of $G$-flux, the effective theory will contain a superpotential which is related to the gauging of some global isometries of the scalar manifold [17, 18]. Due to the non-trivial flux and the Chern-Simons term in the
eleven dimensional supergravity, the axionic scalar of the hypermultiplet becomes charged and the theory obtained corresponds to gauging a combination of the R-symmetry and the axionic shift symmetry.

In general BPS solutions are characterized by their symmetries. Flat BPS domain walls have an $ISO(3,1)$ Poincare symmetry and the solution depends only on the transverse coordinate. In this paper, we will consider solutions which have $SO(4)$ rotational symmetry and where the metric depends only on the radial coordinate. Such supersymmetric solutions with hypermultiplet gauging have not been yet explored. In these cases, the hypermultiplets do not decouple from the theory and this seems to rule out the presence of BPS solutions, at least when the gauge fields are set to zero. Therefore, it seems that in order to obtain supersymmetric solutions, one needs to have non–trivial electric or magnetic fields. It is our purpose in this paper to touch upon this subject by considering the case where the axionic shift of the hypermultiplet is also gauged. This model comes from the dimensional reduction of M-theory on a Calabi-Yau manifold with background flux. We organize our work as follows. In section two, we review the gauged supergravity model we wish to study, the reader is referred to for a more detailed discussion. In section three we present a supersymmetric solution by examining the supersymmetry transformations of the fermionic fields as well as the equations of motion. Section four contains a summary and a discussion.

2 The Gauged Supergravity Model

In the following, we will discuss the five dimensional gauged supergravity theory with one hypermultiplet only. A hypermultiplet is always present in any Calabi-Yau (CY) compactification of M-theory and type II string theory. For example, compactifying M-theory or type IIA string theory on a rigid CY (i.e. $h_{2,1} = 0$) leads to an $N = 2$ theory with a single hypermultiplet, the so-called universal hypermultiplet (the term “universal” is slightly misleading for compactifications with $h_{2,1} > 0$, see section 4.4.3). The bosonic part of the action for the universal hypermultiplet which can be derived by dimensionally reducing eleven dimensional supergravity is given by

$$S_{\text{hyper}} = -\int d^5x\sqrt{h} \left( (\partial_{\mu}\phi)^2 + e^{2\phi}((\partial_{\mu}\chi_1)^2 + (\partial_{\mu}\chi_2)^2) + \frac{1}{4}e^{4\phi}(\partial_{\mu}a + (\chi_1\partial_{\mu}\chi_2 - \chi_2\partial_{\mu}\chi_1))^2 \right).$$ (2.1)

Classically, the scalar fields of this action parameterize the group coset manifold

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1 Curved domain walls in four dimensional $N = 2$ gauged supergravity were recently discussed in [15].

2 The moduli space receives gravitational correction which depends on the Euler character of the Calabi-Yau threefold, this can however be absorbed by a field redefinition [22].
SU(2, 1)/U(2). This can be easily seen if one defines the new complex coordinates
\[ S = e^{-2\phi} + ia + \chi_1^2 + \chi_2^2, \quad C = \chi_1 + i\chi_2. \] (2.2)
The moduli space is Kähler with Kähler potential
\[ K = -\frac{1}{2} \ln \left( \frac{S + \bar{S}}{2} - |C|^2 \right). \] (2.3)
In the appendix, we gather all the necessary formulae of the quaternionic geometry defined by the universal hypermultiplet as well as the various quantities resulting from gauging the axionic shift symmetry. In addition to the hypermultiplet action, we also have the \( N = 2 \) supergravity multiplet whose bosonic action is given by
\[ S_{\text{grav}} = -\int d^5x \sqrt{-\hat{h}} \left( \frac{1}{2} R + \frac{1}{4} F_{\mu
u} F^{\mu\nu} \right) + \frac{1}{6\sqrt{6}} \int d^5x \epsilon^{\mu\nu\rho\sigma\tau} F_{\mu\nu} F_{\rho\sigma} A_{\tau}. \] (2.4)
The supersymmetry transformations of the ungauged theory for the fermions are given by \([12, 13]\)
\[ \delta \psi_{\mu i} = \partial_{\mu} \epsilon_i + \frac{1}{4} \omega^{ab}_{\mu} \gamma_{ab} \epsilon_i \partial_{\mu} q^u (P^j_i)_u \epsilon_j + \frac{i}{4\sqrt{6}} (\gamma_{\mu\rho\tau} - 4 \gamma_{\mu\rho} \gamma_{\tau}) \epsilon_i F^{\mu\rho \tau}, \]
\[ \delta \zeta^A = -\frac{i}{2} f^{A}_{ib} \gamma^\mu \epsilon_i \partial_{\mu} q^u. \] (2.5)
The gauged \( N = 2 \) supergravity theory can be obtained by gauging the \( R \)-symmetry as well as isometries in the hypermultiplet manifold. The isometry in the hypermultiplet moduli space is associated with a Killing vector \( k^u \)
\[ q^u \rightarrow q^u + \varepsilon k^u(q). \] (2.6)
The \( N = 2 \) supersymmetry demands that \( k^u \) is determined in terms of a triplet of Killing prepotentials \( P^i_j \)
\[ k^u (K_{ij})^i_j = \partial_{v} P^i_j + [p, P^i_j]; \] (2.7)
where \((K_{ij})^i_j\) is the \( SU(2) \) triplet of Kähler forms and \( p^i_j \) is the \( SU(2) \) connection. In order to restore supersymmetry in the gauged theory, the bosonic part of the action \([3]\) of the ungauged theory gets modified in two ways: Firstly the derivatives and connections get replaced by gauge covariant ones and secondly a potential term gets added to the action \( S_{\text{pot}} = -\int d^5x \sqrt{-\hat{h}} V \) where
\[ V(q) = -g^2 P_{ij} P^{ij} + 2g^2 N_{iA} N^{iA}. \] (2.8)

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\(^{3}\)In the above action, \( \phi \) is associated with the volume of the CY, \( a \) comes from the dual of the four-form of eleven dimensional supergravity, \( F = dA_3 \), and \( C \) corresponds to the expectation values of \( A_3 \).

\(^{4}\)Since we are interested in bosonic BPS solutions, we will neglect fermionic mass terms induced by the gauging.
The supersymmetry transformations also get modified \[16\]

\[\begin{align*}
\delta \psi_{\mu i} &= D_{\mu} \epsilon_i + \frac{i}{4\sqrt{6}} (\gamma_{\mu\nu\rho} - 4\gamma_{\mu\nu}\gamma_{\rho}) \epsilon_i F^\nu\rho + \frac{i}{\sqrt{6}} g \gamma_{\mu} \epsilon^j P_{ij}, \\
\delta \zeta^A &= -\frac{i}{2} f_{\mu}^A \gamma_{\mu} \epsilon^i D_{\mu} q^u + g \epsilon^i N_i^A.
\end{align*}\] (2.9)

Where

\[D_{\mu} \epsilon_i = \partial_{\mu} \epsilon_i + \frac{1}{4} \omega_{\mu}^{ab} \gamma_{ab} \epsilon_i + \partial_{\mu} q^u (p_i^j) u \epsilon_j + g k^a A_{\mu} (p_i^j) u \epsilon_j + g A_{\mu} P^j_i \epsilon_j.\] (2.10)

There are many different isometries which can be gauged for the universal hypermultiplet. Here we will consider the gauging of the shift symmetry of the axion \(a \to a + \text{const.}\) This gauging has special significance since, as mentioned in the introduction, corresponds to turning on fluxes on the internal Calabi-Yau threefold \[17, 18\].

3 The Supersymmetry Preserving Solution

In this section we present a field configuration which preserves half of the \(N = 2\) supersymmetries. Domain wall solutions of this model have been discussed in \[17, 18\]. As an ansatz, we will only keep the component \(A_t\) of the graviphoton potential and the volume scalar \(\phi\) as the dynamic fields. We should point out that our analysis is equally valid for the general gauged supergravity models with vector multiplets with constant scalar fields. The most general ansatz for the metric which has \(SO(4)\) symmetry in spherical coordinates is given by

\[ds^2 = -e^{2V} (dt)^2 + e^{2W} (dr^2 + f^2 r^2 (d\theta^2 + \sin^2 \theta d\phi^2 + \cos^2 \theta d\psi^2)).\] (3.1)

The F"unfbein and its inverse can be chosen as

\[\begin{align*}
e_0^t &= e^V, & e_1^r &= e^W, & e_2^\theta &= e^W f r, & e_3^\phi &= e^W f r \sin \theta, & e_4^\psi &= e^W f r \cos \theta, \\
e_0^t &= e^{-V}, & e_1^r &= e^{-W}, & e_2^\theta &= \frac{e^{-W}}{f r}, & e_3^\phi &= \frac{e^{-W}}{f r \sin \theta}, & e_4^\psi &= \frac{e^{-W}}{f r \cos \theta}.
\end{align*}\] (3.2)

The non-vanishing components of the spin connection are given by

\[\begin{align*}
\omega_{01}^{12} &= \partial_r V e^{V-W}, \\
\omega_{02}^{12} &= -(f r \partial_r W + r \partial_r f + f), \\
\omega_{03}^{12} &= -(f r \partial_r W + r \partial_r f + f) \sin \theta, \\
\omega_{04}^{12} &= -(f r \partial_r W + r \partial_r f + f) \cos \theta, \\
\omega_{23}^{12} &= -\cos \theta, \\
\omega_{24}^{12} &= \sin \theta.
\end{align*}\] (3.3)

\[\text{See} [24] \text{for an exhaustive treatment of the possible gaugings. Note that for the translational gauging considered in our paper the last two terms in (2.10) cancel.}\]
As a first step, we consider the supersymmetry transformations of the hyperinos. Allowing only a non-trivial \( \phi \), these transformations for our ansatz give,

\[
\begin{align*}
\delta \zeta^1 &= -\frac{i}{4} e^{-W} e^{2\phi} \partial_r (e^{-2\phi}) \gamma_1 \epsilon^2 - \frac{1}{4} g e^{-V} e^{2\phi} A_t \gamma_0 \epsilon^2 + i \frac{\sqrt{6}}{8} g e^{2\phi} \epsilon^2,
\delta \zeta^2 &= -\frac{i}{4} e^{-W} e^{2\phi} \partial_r (e^{-2\phi}) \gamma_1 \epsilon^1 + \frac{1}{4} g e^{-V} e^{2\phi} A_t \gamma_0 \epsilon^1 - i \frac{\sqrt{6}}{8} g e^{2\phi} \epsilon^1.
\end{align*}
\]

Here and in the following all functions will only depend on the radial parameter \( r \) and derivatives with respect to \( r \) will be denoted by primes. The hyperino transformations vanish for half the supersymmetries if \( \epsilon^1, \epsilon^2 \) satisfy

\[
(i a \gamma_0 + b \gamma_1) \epsilon^1 = \epsilon^1, \quad (i a \gamma_0 - b \gamma_1) \epsilon^2 = \epsilon^2.
\]

The time component of the gravitino supersymmetry transformation is given by

\[
\delta \psi^1_t = \left[ \partial_t + \frac{1}{2} V' e^{-W} \gamma_0 \gamma_1 + \frac{i}{\sqrt{2}} e^{-W} A_t' \gamma_1 + \frac{1}{4 \sqrt{6}} g e^{2\phi} e^V \gamma_0 \right] \epsilon^1.
\]

Here and in the following we will only display the \( \delta \psi^1 \) variations and drop the spinor index, the \( \delta \psi^2 \) work in the same way after adjusting some signs. Assuming that \( \partial_t \epsilon = 0 \), the projection gives the following condition

\[
A_t = c_1 e^V.
\]

This implies that \( a = -\sqrt{\frac{2}{3}} c_1 \) is a constant, and therefore \( b \) is a constant too and one finds that

\[
(e^{-2\phi})' = c_2 g e^W,
\]

\text{(3.10)}
where \( b = -\sqrt{\frac{2}{3}c_2} \). The second condition coming from \( \delta \psi_t \) can then be brought into the following form

\[
V' = -\frac{g}{2\sqrt{6}b} e^{2\phi} e^W. \tag{3.11}
\]

Using these relations the \( r \) component of the \( \delta \psi_\mu \) can be simplified and gives

\[
\delta \psi_r = \left( \partial_r - \frac{1}{2} V' \right) \epsilon, \tag{3.12}
\]

which can easily be integrated. The other components of the gravitino supersymmetry variations are

\[
\begin{align*}
\left[ \partial_\theta - \frac{1}{2} e^{-W} (e^W f r)' \gamma_{12} - \frac{i}{2\sqrt{6}} f r e^{-V} A'_t \gamma_{103} + \frac{1}{4\sqrt{6}} g e^{2\phi} e^W f r \gamma_2 \right] & \epsilon = 0, \\
\left[ \partial_\phi - \frac{1}{2} \sin \theta \left( e^{-W} (e^W f r)' \gamma_{13} + \frac{if r}{\sqrt{6}} e^{-V} A'_t \gamma_{103} - \frac{g}{2\sqrt{6}} e^{2\phi} e^W f r \gamma_3 \right) - \frac{1}{2} \cos \theta \gamma_{23} \right] & \epsilon = 0, \\
\left[ \partial_\psi - \frac{1}{2} \cos \theta \left( e^{-W} (e^W f r)' \gamma_{14} + \frac{if r}{\sqrt{6}} e^{-V} A'_t \gamma_{104} - \frac{g}{2\sqrt{6}} e^{2\phi} e^W f r \gamma_4 \right) + \frac{1}{2} \sin \theta \gamma_{24} \right] & \epsilon = 0.
\end{align*}
\tag{3.13}
\]

Integrability conditions \([\nabla_\mu, \nabla_\nu] \epsilon = 0\), fix all other functional relations and constants. One gets

\[
fr e^W e^{2\phi} = k, \tag{3.14}
\]

where \( k \) is a constant. One also obtains the relation

\[
b = -\frac{1}{2\sqrt{6}c_2}, \tag{3.15}
\]

which implies that the dilaton \( \phi \) and metric function \( V \) are related by

\[
V = -2\phi. \tag{3.16}
\]

All the constants are thus fixed and given by

\[
c_2^2 = \frac{1}{4}, \quad c_1^2 = \frac{5}{4}, \quad a^2 = \frac{5}{6}, \quad b^2 = \frac{1}{6}, \quad k^2 g^2 = \frac{32}{15}. \tag{3.17}
\]

So far we have determined a configuration which preserves half of supersymmetry. The gauge field \( A_t \) and the scalar \( \phi \) depend on the metric which can be expressed in terms of the function \( V \), these relations are summarized by
\begin{align}
A_t & = c_1 e^V, \\
(e^{-2\phi})' & = (e^V)' = c_2 g e^W, \\
fre^W & = k e^V.
\end{align}

(3.18)

In order to fix the space-time dependence of the configuration, one needs to solve the equations of motion and determine the function \( V \).

Einstein’s equation reads,

\[ G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} h_{\mu\nu} R = T_{\mu\nu}, \]

(3.19)

where, the stress-energy tensor is given by

\[ T_{\mu\nu} = \frac{2}{\sqrt{-h}} \frac{\delta S}{\delta h_{\mu\nu}} = g_{\mu\nu} D_\mu q^u D_\nu q^v - \frac{1}{2} h_{\mu\nu} g_{uv} D_\rho q^u D^\rho q^v + F_{\mu\nu}^2 - \frac{1}{4} h_{\mu\nu} F^2 - h_{\mu\nu} V. \]

(3.20)

Since we look at a purely electric configuration the Chern-Simons term in the \( N = 2 \) supergravity Lagrangian is not important and the gauge field and the scalar field equations of motion, are respectively given by (we set \( a = C = 0 \))

\begin{align}
\frac{1}{\sqrt{-h}} \partial_\mu \left( \sqrt{-h} F^{\mu\nu} \right) - g_2 e^{4\phi} A^\nu & = 0, \\
\frac{1}{\sqrt{-h}} \partial_\nu \left( \sqrt{-h} h^{\mu\nu} \partial_\nu \phi \right) - \frac{1}{2} g_2 e^{4\phi} A_\mu A^\mu - \frac{1}{4} g_2^2 e^{4\phi} & = 0.
\end{align}

(3.21)

(3.22)

It is customary that the supersymmetric property of the solution determines the relations among the various fields of the configuration, i.e., an off-shell solution which can only be fixed by solving the equations of motion. For example, in the ungauged and the \( U(1) \)-gauged theory, solving the gauge field equations fixes the solution in terms of harmonic functions \([25, 4]\). The harmonic functions are clearly solutions of a second order differential equations and can not be determined from the first order differential equations resulting from the vanishing of the supersymmetry transformations \([25]\).

Here due to the second term in the gauge field equation of motion, it is clear that our solution can not be determined in terms of a harmonic function. As a matter of fact, it can be checked that all the above equations of motion are satisfied for any choice of the function \( e^V \). This might come as a surprise. However, the BPS configuration we found is of the following form

\[ ds^2 = -e^{2V} dt^2 + \frac{4}{g_2^2} (\partial_r e^V)^2 dr^2 + e^{2V} k^2 dS_3^2, \]

(3.23)
\[ A_t = c_1 e^V, \quad e^{-2\phi} = e^V. \quad (3.24) \]

By inspecting the metric (3.23) it is clear why the equations of motion did not determine \( V \). This is because different \( V \) correspond to different choice of radial coordinates. Indeed defining a new coordinate \( u(r) = e^V \), we find that the solution in the new coordinates is given by

\[ ds^2 = -u^2 dt^2 + \frac{4}{g^2} du^2 + k^2 u^2 d\Omega^2_{S_3} \quad (3.25) \]
and

\[ A_t = c_1 u, \quad e^{-2\phi} = u. \quad (3.26) \]

The solution has several peculiar properties. For example, the field strength \( F_{ut} = c_1 \) is constant and the charge is given by

\[ q = \oint *F = c_1 c_2 g k^3 u^2, \quad (3.27) \]
diverges as \( u \to \infty \). Since \( e^{2\phi} \) parameterizes the volume of the Calabi-Yau manifold, we see that the volume becomes infinite at \( u = 0 \) and goes to zero at \( u \to \infty \). The Ricci scalar is given by \( R = -\frac{3}{8} \frac{1}{u^2} \) and the metric is singular at \( u = 0 \). Note that for the values of the constants given in (3.17), the spatial part of the metric has the form of a cone. It is an open question whether the singular BPS solution we found has a proper interpretation in M-theory, we note however that the solution shares some of its properties with other BPS solutions in gauged supergravities.

As it was mentioned before, our solution is also valid in the presence of vector multiplets with trivial scalars. In this case the gauge fields are given by

\[ A_I^t = c_1 h^I e^V, \quad (3.28) \]
and we set \( \alpha_I h^I = 1 \), where \( \alpha_I \) is the flux vector \( [18] \). (For this ansatz, the gaugino supersymmetry variation vanishes). Generalization of our results to four dimensions is straightforward since the hypermultiplet sector in four dimensions is the same as in five dimensions. In four dimensions, we have \( \phi = -\phi_4 \) where \( \phi_4 \) is the four dimensional dilaton.

\[ \text{4 Discussion} \]

In this paper we have studied supersymmetric configurations for a gauged supergravity theory coming from the compactification of M-theory on a Calabi-Yau threefold with a background flux. This effective gauged theory correspond to a supergravity theory where the axionic shift of the universal hypermultiplet is also gauged. The peculiar
configuration found is perhaps related to the “run-away” nature of the scalar potential. The domain wall solution for the gauging considered in our paper was discussed in [18] and has a similar singular behavior. However, one would hope to get smooth and better behaved black hole solutions for more general gauging and potentials with some fixed points. It is thus of interest to study more general gaugings like those discussed in [26, 24]. Also one should be less restrictive and study configurations with more non-trivial scalar fields of the universal hypermultiplet and vector multiplets. In general, the connection piece in $D_\mu$, as well as $P_{ij}$, $f^A_{iu}$ and $N^A_i$ in the hyperino variations, which were our starting point to determine the projection condition on the Killing spinors all define $2 \times 2$ matrices. Therefore, for the general gauging one would expect to find more complicated projectors which are not diagonal in the indices $i = 1, 2$ of the supersymmetry parameters $\epsilon^i$. However, the conditions for the existence of the projectors as well as the compatibility of the various supersymmetry transformations are very restrictive. BPS solutions for other types of gauging is currently under investigation. Other cases with non-trivial hypermultiplet dynamics, are the instanton solutions of the (Euclideanized) $N = 2$ supergravity [27]. It would be interesting to see whether BPS instanton solutions exist in the gauged supergravity.

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Appendix A: The universal hypermultiplet

In this appendix we gather some useful formulae for the universal hypermultiplet and its quaternionic geometry, for more details see [14, 22, 23].

The universal hypermultiplet moduli space is determined by the Kähler potential

$$\phi = -\frac{1}{2} \ln \left( \frac{1}{2}(S + \bar{S} - 2|C|^2) \right). \quad (A.1)$$

Using the coordinates $q^u = (S, \bar{S}, C, \bar{C})$, the metric components are

$$g_{SS} = \frac{1}{4} e^{4\phi}, \quad g_{SC} = -\frac{1}{2} C e^{4\phi}, \quad g_{\bar{S}C} = -\frac{1}{2} \bar{C} e^{4\phi}, \quad g_{C\bar{C}} = e^{2\phi} + C \bar{C} e^{4\phi}. \quad (A.2)$$

The metric can be given by is given by

$$ds^2 = u \otimes \bar{u} + v \otimes \bar{v}, \quad (A.3)$$

where we have introduced the vielbein forms by [14, 22].
\[ u = e^\phi dC, \]
\[ v = e^{2\phi} \left( \frac{dS}{2} - \bar{C} dC \right), \] (A.4)
satisfying
\[ du = \frac{1}{2} u \wedge (v + \bar{v}), \quad dv = v \wedge \bar{v} + u \wedge \bar{u}, \quad d\phi = -\frac{1}{2} (v + \bar{v}). \] (A.5)

One can define the vielbein \( f_i^A \) where \( i, A \) are \( SU(2) \) indices.
\[ f_1^1 = u, \quad f_1^2 = \bar{v}, \quad f_2^1 = v, \quad f_2^2 = -\bar{u}. \] (A.6)

In components, we have \( f_i^A = (f_i^A)_u dq^u \) with
\[ (f_1^1)_C = -(f_2^2)_C = e^\phi, \]
\[ (f_1^2)_S = (f_2^1)_S = \frac{1}{2} e^{2\phi}, \]
\[ (f_1^2)_\bar{C} = -e^{2\phi} C, \]
\[ (f_2^1)_C = -e^{2\phi} \bar{C}. \] (A.7)

We also have
\[ df_1^1 = \frac{1}{2} u \wedge (v + \bar{v}), \]
\[ df_1^2 = -(v \wedge \bar{v} + u \wedge \bar{u}), \]
\[ df_2^2 = -\frac{1}{2} \bar{u} \wedge (v + \bar{v}), \]
\[ df_2^1 = (v \wedge \bar{v} + u \wedge \bar{u}). \] (A.8)

Using the Cartan structure equation, one can then find that the \( SU(2) = Sp(1) \) connection appearing in the gravitino supersymmetry transformations is given by
\[ p = \begin{pmatrix} \frac{1}{4} (v - \bar{v}) & -u \\ \bar{u} & \frac{1}{4} (\bar{v} - v) \end{pmatrix}. \] (A.9)

In components, the connections are
\[ p_S = -p_{\bar{S}} = \frac{1}{8} \begin{pmatrix} e^{2\phi} & 0 \\ 0 & -e^{2\phi} \end{pmatrix}, \]
\[ p_C = \frac{1}{4} \begin{pmatrix} -e^{2\phi} \bar{C} & -4e^\phi \\ 0 & e^{2\phi} \bar{C} \end{pmatrix}, \]
\[ p_{\bar{C}} = \frac{1}{4} \begin{pmatrix} e^{2\phi} C & 0 \\ 4e^\phi & -e^{2\phi} C \end{pmatrix}. \] (A.10)
The covariantly constant $SU(2)$ triplet of Kähler is given by

$$K^i_j = \begin{pmatrix} -\frac{1}{2}(u \wedge \bar{u} - v \wedge \bar{v}) & \bar{u} \wedge v & \frac{1}{2}(u \wedge \bar{u} - v \wedge \bar{v}) \\ \bar{u} \wedge v & \frac{1}{2}(u \wedge \bar{u} - v \wedge \bar{v}) & -u \wedge \bar{v} \\ \frac{1}{2}(u \wedge \bar{u} - v \wedge \bar{v}) & -u \wedge \bar{v} & \bar{u} \wedge v \end{pmatrix}$$ (A.11)

or in components

$$K_{SS} = \frac{1}{8} \begin{pmatrix} e^{4\phi} & 0 \\ 0 & -e^{4\phi} \end{pmatrix},$$

$$K_{CC} = \frac{1}{2} \begin{pmatrix} -e^{2\phi}(1 - e^{2\phi} \bar{C}C) & 2e^{3\phi}C \\ 2e^{3\phi} \bar{C} & e^{2\phi}(1 - e^{2\phi} C\bar{C}) \end{pmatrix},$$

$$K_{SC} = \frac{1}{4} \begin{pmatrix} -e^{4\phi}C & 0 \\ 0 & -2e^{3\phi} \bar{C} \end{pmatrix},$$

$$K_{\bar{S}C} = \frac{1}{4} \begin{pmatrix} e^{4\phi} \bar{C} & 2e^{3\phi} \\ e^{4\phi} \bar{C} & -2e^{3\phi} \end{pmatrix}$$ (A.12)

and it satisfies

$$dp + p \wedge p = K.$$ (A.13)

Which is equivalent to the statement that the geometry of the universal hypermultiplet scalars is quaternionic.

**Appendix B: Gauging the shift symmetry**

Given a Killing vector of the hypermultiplet manifold $k^a$, the $SU(2)$ prepotential is determined by the equation

$$k^u K_{uv} = \partial_v P + [p_v, P].$$ (B.1)

The Killing vector associated with shifts of the axion $a$ is given by

$$k^a = (i, -i, 0, 0)$$ (B.2)

The prepotential related by (B.1) to this Killing vector is given by

$$P^i_j = \begin{pmatrix} i^{-1}e^{2\phi} & 0 \\ 0 & i\frac{1}{4}e^{2\phi} \end{pmatrix}$$ (B.3)

The potential produced by the gauging is given by

$$V = -2g^2 P_{ij} P^{ij} + 2g^2 N_{iA} N^{iA},$$ (B.4)
where $N_{iA}$ is defined by

$$N_{iA} = \frac{\sqrt{6}}{4} k^u f^A_i.$$  \hspace{1cm} (B.5)

Using the definition of $P_{ij}$ and the fact that $f^A_i f^A_v = g_{uv}$ we find

$$V = \frac{g^2}{8} e^{4\phi}. \hspace{1cm} (B.6)$$
References

[1] J. M. Maldacena, Adv. Theor. Math. Phys. 2 (1998) 231; Int. J. Theor. Phys. 38 (1999) 1113; E. Witten, Adv. Theor. Math. Phys. 2 (1998) 253; S. S. Gubser, I. R. Klebanov and A. M. Polyakov, Phys. Lett. B428 (1998) 105; O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri and Y. Oz, Phys. Rept. 323 (2000) 183.

[2] L. Randall and R. Sundrum, Phys. Rev. Lett. 83 23 (1999) 4690.

[3] M. J. Duff, “TASI Lecture on branes, black holes and anti-de Sitter space”, hep-th/9912164.

[4] L. A. J. London, Nucl. Phys. B434 (1995) 709; K. Behrndt, A. H. Chamseddine and W. A. Sabra, Phys. Lett. B442 (1998) 97; K. Behrndt, M. Cvetic and W. A. Sabra, Nucl. Phys. B553 (1999) 317.

[5] A. H. Chamseddine and W. A. Sabra, Phys. Lett. B477 (2000) 329; D. Klemm and W. A. Sabra, Phys. Rev. D62 (2000) 024003; J. M. Maldacena and C. Nuñez, hep-th/0007018.

[6] D. Klemm and W. A. Sabra, Phys. Lett. B503 (2001) 147; JHEP 0102 (2001) 031; M. M. Caldarelli, D. Klemm and W. A. Sabra, hep-th/0010313.

[7] R. Kallosh and A. Linde, JHEP 0002 (2000) 005; A. Ceresole and G. Dall’Agata, hep-th/0101214.

[8] M. Günyaydin, G. Sierra and P.K. Townsend, Nucl. Phys. B242 (1984) 244; Phys. Lett. B133 (1983) 72.

[9] M. Günyaydin, G. Sierra and P.K. Townsend, Nucl. Phys. B253 (1985) 573; Phys. Lett. B144 (1984) 41.

[10] B. de Wit and A. Van Proeyen, Comm. Math. Phys. 149 (1992) 307.

[11] A. Cadavid, A. Ceresole, R. D’Auria and S. Ferrara, Phys. Lett. B357 (1995) 76.

[12] G. Sierra, Phys. Lett. B157 (1985) 379.

[13] J. Bagger and E. Witten, Nucl. Phys. B222 (1983) 1.

[14] S. Ferrara and S. Sabharwal, Nucl. Phys. B332 (1990) 317.

[15] L. Andrianopoli, M. Bertolini, A. Ceresole, R. D’Auria, S. Ferrara, P. Fre and T. Magri, J. Geom. Phys. 23 (1997) 111.

[16] A. Ceresole and G. Dall’Agata, Nucl. Phys. B585 (2000) 143.

[17] A. Lukas, B. A. Ovrut, K. S. Stelle and D. Waldram, Nucl. Phys. B552 (1999) 246; Phys. Rev. D59 (1999) 086001.
[18] K. Behrndt and S. Gukov, Nucl. Phys. B580 (2000) 225.

[19] K. Behrndt, G. Lopes Cardoso and D. Lüst, hep-th/0102128.

[20] C. Cecotti, S. Ferrara and L. Girardello, Int. J. Mod. Phys. A4 (1989) 2475.

[21] S. Aspinwall, hep-th/0001001.

[22] A. Strominger, Phys. Lett. B421 (1998) 139.

[23] K. Behrndt, I. Gaida, D. Lüst, S. Mahapatra and T. Mohaupt, Nucl. Phys. B 508 (1997) 659.

[24] K. Behrndt and M. Cvetic, hep-th/0101007.

[25] W. A. Sabra, Mod. Phys. Lett. A12 (1997) 2585; Nucl. Phys. B510 (1998) 247; K. Behrndt, D. Lüst and W. A. Sabra, Nucl. Phys. B510 (1998) 264; W. A. Sabra, Mod. Phys. Lett. A13 (1998) 239 ; A. Chamseddine and W. A. Sabra, Phys. Lett. B426 (1998) 36.

[26] K. Behrndt, C. Herrmann, J. Louis and S. Thomas, hep-th/0008112.

[27] K. Becker, M. Becker and A. Strominger, Nucl. Phys. B456 (1995) 130.
K. Becker and M. Becker, Nucl. Phys. B551 (1999) 102.
M. Gutperle and M. Spalinski, Nucl. Phys. B598 (2001) 509.
M. Gutperle and M. Spalinski, JHEP 0006 (2000) 037.