Estimates in Beurling–Helson type theorems.
Multidimensional case

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Abstract. We consider the spaces $A_p(T^m)$ of functions $f$ on the $m$-dimensional torus $T^m$ such that the sequence of the Fourier coefficients $\hat{f} = \{\hat{f}(k), k \in \mathbb{Z}^m\}$ belongs to $l^p(\mathbb{Z}^m)$, $1 \leq p < 2$. The norm on $A_p(T^m)$ is defined by $\|f\|_{A_p(T^m)} = \|\hat{f}\|_{l^p(\mathbb{Z}^m)}$. We study the rate of growth of the norms $\|e^{i\lambda \varphi}\|_{A_p(T^m)}$ as $|\lambda| \to \infty$, $\lambda \in \mathbb{R}$, for $C^1$-smooth real functions $\varphi$ on $T^m$ (the one-dimensional case was investigated by the author earlier). The lower estimates that we obtain have direct analogues for the spaces $A_p(\mathbb{R}^m)$.

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Introduction

Given any integrable function $f$ on the $m$-dimensional torus $T^m = \mathbb{R}^m/2\pi \mathbb{Z}^m$, $m \geq 1$ (where $\mathbb{R}$ is the real line, $\mathbb{Z}$ is the set of integers) consider its Fourier coefficients:
\[
\hat{f}(k) = \frac{1}{(2\pi)^m} \int_{T^m} f(t) e^{-i(k,t)} dt, \quad k \in \mathbb{Z}^m.
\]

Let $A_1(T^m)$ be the space of continuous functions $f$ on $T^m$ such that the sequence of Fourier coefficients $\hat{f} = \{\hat{f}(k), k \in \mathbb{Z}^m\}$ belongs to $l^1(\mathbb{Z}^m)$. For $1 < p \leq 2$ let $A_p(T^m)$ be the space of integrable functions $f$ on $T^m$ such that $\hat{f} \in l^p(\mathbb{Z}^m)$. Provided with the natural norms
\[
\|f\|_{A_p(T^m)} = \|\hat{f}\|_{l^p(\mathbb{Z}^m)} = \left( \sum_{k \in \mathbb{Z}^m} |\hat{f}(k)|^p \right)^{1/p}
\]
the spaces $A_p$ are Banach spaces ($1 \leq p \leq 2$). The space $A = A_1$ is a Banach algebra (with the usual multiplication of functions).

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According to the Beurling–Helson theorem [1] (see also [2]), if \( \varphi \) is a map of the circle \( \mathbb{T} \) into itself such that \( \|e^{in\varphi}\|_{A(T)} = O(1), \ n \in \mathbb{Z} \), then \( \varphi \) is linear (with integer tangent coefficient), i.e. \( \varphi(t) = kt + \varphi(0), \ k \in \mathbb{Z} \). A similar statement holds for the maps \( \varphi : \mathbb{T}^m \to \mathbb{T} \). This case easily reduces to the one-dimensional case.

Let \( C^n(\mathbb{T}^m) \) be the class of (complex-valued) functions on the torus \( \mathbb{T}^m \) such that all partial derivatives of order \( n \) are continuous.

In the present paper we study the growth of the norms \( \|e^{i\lambda \varphi}\|_{A_p(\mathbb{T}^m)} \) as \( \lambda \to \infty, \ \lambda \in \mathbb{R} \), for \( C^1 \) -smooth real functions \( \varphi \) on \( \mathbb{T}^m \). In the one-dimensional case we studied this question in [3]. The same paper contains a survey on the subject.

It is easy to show that for every \( C^1 \) -smooth (real) function \( \varphi \) on the circle \( \mathbb{T} \) we have
\[
\|e^{i\lambda \varphi}\|_{A_p(\mathbb{T})} = O(|\lambda|^{\frac{1}{p} - \frac{1}{2}}), \quad |\lambda| \to \infty, \quad \lambda \in \mathbb{R},
\]
for all \( p, \ 1 \leq p < 2 \).

On the other hand we have the Leibenson–Kahane–Alpár estimate ([4], [5], [2], [6]): if \( \varphi \in C^2(\mathbb{T}) \) is a nonconstant (which is equivalent, due to periodicity, to it being nonlinear) real function, then
\[
\|e^{i\lambda \varphi}\|_{A_p(\mathbb{T})} \geq c_p |\lambda|^{\frac{1}{p} - \frac{1}{2}}, \quad \lambda \in \mathbb{R},
\]
for all \( 1 \leq p < 2 \).

Thus, for every \( C^2 \) -smooth nonlinear real function \( \varphi \) on \( \mathbb{T} \) we have
\[
\|e^{i\lambda \varphi}\|_{A_p(\mathbb{T})} \simeq |\lambda|^{\frac{1}{p} - \frac{1}{2}}.
\]
In particular \( \|e^{i\lambda \varphi}\|_{A(\mathbb{T})} \simeq |\lambda|^{1/2} \).

The case of \( C^1 \) -smooth phase \( \varphi \) is essentially different from the \( C^2 \) -smooth case. As we showed in [3] (and earlier in [7]), for \( \varphi \in C^1(\mathbb{T}) \) the norms \( \|e^{i\lambda \varphi}\|_{A(\mathbb{T})} \) can grow rather slowly, namely, one can closely approach the \( O(|\log |\lambda||) \) -condition. If \( p > 1 \) the corresponding norms can even be bounded.

\[\text{2} \text{Actually estimate (1) holds even in the case when } \varphi \text{ is absolutely continuous with derivative in } L^2(\mathbb{T}) \text{ and, in particular, when } \varphi \text{ satisfies the Lipschitz condition of order 1 (see [2, Ch. VI, § 3] for } p = 1; \text{ for } 1 < p < 2 \text{ the estimate follows immediately by interpolation between } l^1 \text{ and } l^2).\]

\[\text{3} \text{We write } a(\lambda) \simeq b(\lambda) \text{ in the case when } a_1 \leq a(\lambda)/b(\lambda) \leq a_2 \text{ for all sufficiently large } |\lambda| \text{ (with constants } a_1, a_2 > 0 \text{ independent of } \lambda).\]
We note that the proof of the Leibenson–Kahane–Alpár estimate (2) is based on the van der Corput lemma and essentially uses nondegeneration of the curvature of a certain arc of the graph of \( \varphi \). This approach does not allow to consider the functions of smoothness less than \( C^2 \).

In the multidimensional case for the phase functions \( \varphi \) of smoothness \( C^2 \) (and higher) the behavior of the norms \( \|e^{i\lambda \varphi}\|_A \) was considered by Hedstrom [8]. As in the one-dimensional case it is easy to get an upper estimate; for instance, if \( \varphi \in C^\nu(\mathbb{T}^m), \ \nu > m/2, \ m \geq 2 \), then \( \|e^{i\lambda \varphi}\|_{A(\mathbb{T}^m)} = O(|\lambda|^{m/2}) \) (in [8] this estimate is obtained under somewhat different assumptions on smoothness). The same work [8] contains the following lower estimate: if \( \varphi \in C^2(\mathbb{T}^m) \) is a real function such that the determinant of the matrix of its second derivatives is not identically equal to zero, then

\[
\|e^{i\lambda \varphi}\|_{A(\mathbb{T}^m)} \geq c|\lambda|^{m/2}.
\]

This is proved by reduction to the one-dimensional case.

In § 1 we obtain Theorem 1, which is the main result of the present paper. In this theorem we give lower estimates for the norms \( \|e^{i\lambda \varphi}\|_{A_p(\mathbb{T}^m)} \) for \( C^1 \)-smooth real functions \( \varphi \) on the torus \( \mathbb{T}^m \). We assume that the range \( \nabla \varphi(\mathbb{T}^m) \) of the gradient \( \nabla \varphi \) of \( \varphi \) has positive (Lebesgue) measure; in this case we say that the gradient of \( \varphi \) is nondegenerate. The proof of the theorem is based on the natural modification of the method that we used earlier for the one-dimensional case in [3]. It can be called the concentration of Fourier transform large values method. We also obtain local versions of the lower estimates in the spaces \( A_p(\mathbb{T}^m) \) (Theorem 1') and \( A_p(\mathbb{R}^m) \) (Theorem 1''), where \( A_p(\mathbb{R}^m) \) is the space of functions \( f \) on \( \mathbb{R}^m \) such that the Fourier transform \( \hat{f} \) belongs to \( L^p(\mathbb{R}^m) \).

We note that in the one-dimensional case we have the obvious inclusion \( C^1(\mathbb{T}) \subseteq A(\mathbb{T}) \subseteq A_p(\mathbb{T}) \). At the same time for the torus of dimension \( m \geq 3 \) the class of \( C^1 \)-smoothness does not guarantee that a function belongs to all classes \( A_p, 1 \leq p < 2 \). (Smoothness conditions that depend on dimension and imply that a function belongs to the classes \( A_p(\mathbb{T}^m) \) are well known, see § 3). This is why we naturally consider the lower estimates of the norms \( \|e^{i\lambda \varphi}\|_{A_p(\mathbb{T}^m)} \) for \( \lambda \in \Lambda(\varphi, p) \), where for a function \( \varphi \) on \( \mathbb{T}^m \) the set \( \Lambda(\varphi, p) \) is the set of all those \( \lambda \in \mathbb{R} \) for which \( e^{i\lambda \varphi} \in A_p(\mathbb{T}^m) \).
Here, in Introduction, we note only the corollaries of Theorem 1. Let \( \varphi \in C^1(\mathbb{T}^m) \) be a real function such that its gradient is nondegenerate and satisfies the Lipschitz condition of order \( \alpha, \ 0 < \alpha \leq 1 \); then (Corollary 1) for \( 1 \leq p < 1 + \alpha \) we have
\[
\| e^{i\lambda \varphi} \|_{A^p(\mathbb{T}^m)} \geq c_p |\lambda|^m \left( \frac{\alpha}{p} - \frac{1}{1 + \alpha} \right), \quad \lambda \in \Lambda(\varphi, p).
\]
In particular (Corollary 2), we see that if the gradient of a phase function \( \varphi \) is nondegenerate and satisfies the Lipschitz condition of order 1, then
\[
\| e^{i\lambda \varphi} \|_{A^p(\mathbb{T}^m)} \geq c_p |\lambda|^m \left( \frac{1}{p} - \frac{1}{2} \right), \quad \lambda \in \Lambda(\varphi, p).
\]
(5)

Putting here \( p = 1 \) we see that every such function \( \varphi \) satisfies (4) for all \( \lambda \in \Lambda(\varphi, 1) \).

In § 2 for each class \( C^{1,\omega}(\mathbb{T}^m) \) (see the definition of the classes \( C^{1,\omega} \) at the end of the introduction) we construct a real function \( \varphi \in C^{1,\omega}(\mathbb{T}^m) \) such that it has nondegenerate gradient and the norms \( \| e^{i\lambda \varphi} \|_{A^p(\mathbb{T}^m)} \) grow very slowly. (For the one-dimensional case we did this in [3]. The general case can be easily deduced from the one-dimensional case.) Thus we show (see Theorem 2 and Corollaries 3 and 4) that the lower estimates obtained in § 1 are close to being sharp and in certain cases are sharp.

In § 3 using quite standard methods we obtain the multidimensional version of estimate (1), namely (Theorem 3): if \( \varphi \) is a sufficiently smooth (depending on dimension) real function on \( \mathbb{T}^m \), then \( \| e^{i\lambda \varphi} \|_{A^p(\mathbb{T}^m)} = O(\lambda |\lambda|^{m(1/p - 1/2)}) \). Hence taking our estimate (5) into account, we obtain the multidimensional version of relation (3), namely (Theorem 4): if \( \varphi \) is sufficiently smooth and its gradient is nondegenerate, then
\[
\| e^{i\lambda \varphi} \|_{A^p(\mathbb{T}^m)} \simeq |\lambda|^{m(1/p - 1/2)},
\]
in particular \( \| e^{i\lambda \varphi} \|_{A^\infty(\mathbb{T}^m)} \simeq |\lambda|^{m/2} \).

We use the following notation. Let \( V \) be a domain in \( \mathbb{R}^m \) and let \( g \) be a function on \( V \). We define the modulus of continuity of \( g \) by
\[
\omega(V, g, \delta) = \sup_{|t_1 - t_2| \leq \delta} |g(t_1) - g(t_2)|, \quad \delta \geq 0,
\]
where \( |x| \) is the length of a vector \( x \in \mathbb{R}^m \). If \( V = \mathbb{R}^m \), then we just write \( \omega(g, \delta) \). Let \( \omega \) be a given continuous nondecreasing function on \([0, +\infty)\), \( \omega(0) = 0 \).
0. The class $\text{Lip}_\omega(V)$ consists of functions $g$ on $V$ satisfying $\omega(V, g, \delta) = O(\omega(\delta))$, $\delta \to +0$. The class $C^{1,\omega}(V)$ consists of functions $f$ on $V$ such that all derivatives of the first order $\partial f/\partial t_j$, $j = 1, 2, \ldots, m$, belong to $\text{Lip}_\omega(V)$. Certainly for a (real) function $\varphi$ on $V$ the condition $\varphi \in C^{1,\omega}(V)$ means that $\omega(V, \nabla \varphi, \delta) = O(\omega(\delta))$, $\delta \to +0$, where

$$\omega(V, \nabla \varphi, \delta) = \sup_{|t_1 - t_2| \leq \delta} |\nabla \varphi(t_1) - \nabla \varphi(t_2)|, \quad \delta \geq 0,$$

is the modulus of continuity of the gradient $\nabla \varphi$ of $\varphi$. The class $C^{1,\omega}(\mathbb{T}^m)$ consists of functions that are $2\pi$-periodic with respect to each variable and belong to $C^{1,\omega}(\mathbb{R}^m)$. For $0 < \alpha \leq 1$ we write $C^{1,\alpha}$ instead of $C^{1,\delta_\alpha}$. Generally, for an arbitrary $\nu = 0, 1, 2, \ldots$ and $0 < \alpha \leq 1$ let $C^{\nu,\alpha}(\mathbb{T}^m)$ be the class of functions $f$ on $\mathbb{T}^m$ such that $f$ is $\nu$ times differentiable and all partial derivatives of order $\nu$ of $f$ (for $\nu = 0$ the function $f$ itself) satisfy the Lipschitz condition of order $\alpha$ (i.e. belong to $\text{Lip}_{\delta_\alpha}$). It is convenient to put $C^{\nu,0} = C^\nu$. For an arbitrary measurable set $E$ in $\mathbb{T}^m$ or in $\mathbb{R}^m$ by $|E|$ we denote its Lebesgue measure. By $(x, y)$ we denote the usual inner product of vectors $x$ and $y$ in $\mathbb{R}^m$ (or of vectors $x \in \mathbb{Z}^m$, $y \in \mathbb{T}^m$). If $W$ is a set in $\mathbb{R}^m$ and $\lambda \in \mathbb{R}$, then we put $\lambda W = \{\lambda x : x \in W\}$. In the usual way we identify integrable functions on the torus $\mathbb{T}^m$ with integrable functions on the cube $[0, 2\pi]^m$. By $c, c_p, c(p, \varphi), c_m$, etc. we denote various positive constants which may depend only on $p, \varphi$ and the dimension $m$.

§ 1. Lower estimates

As we indicated in Introduction, given a function $\varphi$ on the torus $\mathbb{T}^m$ we denote by $\Lambda(\varphi, p)$ the set of all those $\lambda \in \mathbb{R}$ for which $e^{i\lambda \varphi} \in A_p(\mathbb{T}^m)$.

**Theorem 1.** Let $1 \leq p < 2$. Let $\varphi \in C^{1,\omega}(\mathbb{T}^m)$ be a real function. Suppose that the gradient $\nabla \varphi$ is nondegenerate that is the set $\nabla \varphi(\mathbb{T}^m)$ is of positive measure. Then

$$\|e^{i\lambda \varphi}\|_{A_p(\mathbb{T}^m)} \geq c \left(|\lambda|^{1/p} \chi^{-1} \left(\frac{1}{|\lambda|}\right)\right)^m, \quad \lambda \in \Lambda(\varphi, p), \quad |\lambda| \geq 1,$$

where $\chi^{-1}$ is the function inverse to $\chi(\delta) = \delta \omega(\delta)$ and $c = c(p, \varphi) > 0$ is independent of $\lambda$. 

5
In the one-dimensional case we obtained this theorem in [3]. (For each $C^1$-smooth function $\varphi$ on $\mathbb{T}$ we have $\Lambda(\varphi, p) = \mathbb{R}$ for all $p \geq 1$. Nondegeneration of the gradient in the one-dimensional case means nonlinearity of $\varphi$ which due to periodicity is equivalent to the condition that $\varphi \not\equiv \text{const}$.)

Theorem 1 immediately implies the following corollary.

**Corollary 1.** Let $0 < \alpha \leq 1$. Let $\varphi \in C^{1, \alpha}(\mathbb{T}^m)$ be a real function with nondegenerate gradient. Then for all $p$, $1 \leq p < 1 + \alpha$, we have

$$\|e^{i\lambda \varphi}\|_{A_p(\mathbb{T}^m)} \geq c_p |\lambda|^m \left(\frac{\frac{1}{p} - \frac{1}{1+\alpha}}{1+\alpha}\right), \quad \lambda \in \Lambda(\varphi, p).$$

In particular $\|e^{i\lambda \varphi}\|_{A(\mathbb{T}^m)} \geq c |\lambda|^{\frac{m}{1+\alpha}}$, $\lambda \in \Lambda(\varphi, 1)$.

We especially note the case of a $C^2$-smooth phase and even the more general case of a $C^{1,1}$-smooth phase.

**Corollary 2.** Let $\varphi \in C^{1,1}(\mathbb{T}^m)$ be a real function with nondegenerate gradient. Then for all $p$, $1 \leq p < 2$, we have

$$\|e^{i\lambda \varphi}\|_{A_p(\mathbb{T}^m)} \geq c_p |\lambda|^m \left(\frac{\frac{1}{p} - \frac{1}{2}}{2}\right), \quad \lambda \in \Lambda(\varphi, p).$$

In particular $\|e^{i\lambda \varphi}\|_{A(\mathbb{T}^m)} \geq c |\lambda|^{m/2}$, $\lambda \in \Lambda(\varphi, 1)$.

We shall see that a local version of Theorem 1 also holds. Let $E$ be an arbitrary set contained in $[0, 2\pi]^m$ or, more generally, contained in some cube with edges of length $2\pi$ parallel to coordinate axes. We say that a function $f$ defined on $E$ belongs to $A_p(\mathbb{T}^m, E)$ if there exists a function $F \in A_p(\mathbb{T}^m)$ such that its restriction $F|_E$ to the set $E$ coincides with $f$. We put

$$\|f\|_{A_p(\mathbb{T}^m, E)} = \inf_{F|_E = f} \|F\|_{A_p(\mathbb{T}^m)}.$$

As in Theorem 1, everywhere below $\chi^{-1}$ is the function inverse to $\chi(\delta) = \delta \omega(\delta)$.

**Theorem 1’.** Let $1 \leq p < 2$. Let $V$ be a domain in $[0, 2\pi]^m$. Let $\varphi \in C^{1, \omega}(V)$ be a real function. Suppose that the gradient $\nabla \varphi$ is nondegenerate
on $V$, that is, the set $\nabla \varphi(V)$ is of positive measure. Then

$$\|e^{i\lambda \varphi}\|_{A_p(T^m, V)} \geq c \left( |\lambda|^{1/p} \chi^{-1}(1/|\lambda|) \right)^m$$

for all those $\lambda \in \mathbb{R}$, $|\lambda| \geq 1$, for which $e^{i\lambda \varphi} \in A_p(T^m, V)$.

The local version of Corollary 1 (as well as of Corollary 2) is obvious.

Technically it will be convenient to deal with the spaces $A_p$ in nonperiodic case. (The reason for this is that in distinction with the one-dimensional case, for a $C^1$-smooth function of several variables the range of its gradient can be very complicated.)

Let $A_p(\mathbb{R}^m)$, where $1 \leq p \leq \infty$, be the space of tempered distributions $f$ on $\mathbb{R}^m$ such that the Fourier transform $\hat{f}$ belongs to $L^p(\mathbb{R}^m)$. We put

$$\|f\|_{A_p(\mathbb{R}^m)} = \|\hat{f}\|_{L^p(\mathbb{R}^m)} = \left( \int_{\mathbb{R}^m} |\hat{f}(u)|^p du \right)^{1/p}.$$  

For $1 \leq p \leq 2$ each distribution that belongs to $A_p$ is actually a function in $L^q$, $1/p + 1/q = 1$. For $p = 1$ we naturally assume that $f$ is continuous.

We choose the normalization factor of the Fourier transform so that

$$\hat{f}(u) = \frac{1}{(2\pi)^m} \int_{\mathbb{R}^m} f(t) e^{-i(u,t)} dt, \quad u \in \mathbb{R}^m,$$

for $f \in L^1(\mathbb{R}^m)$.

We often write $A$ instead of $A_1$.

Let us define the local spaces $A_p$ in nonperiodic case.

Let $E \subseteq \mathbb{R}^m$ be an arbitrary set. We say that a function $f$ defined on $E$ belongs to $A_p(\mathbb{R}^m, E)$ if there exists a function $F \in A_p(\mathbb{R}^m)$ such that its restriction $F|_E$ to the set $E$ coincides with $f$. The norm on $A_p(\mathbb{R}^m, E)$ is defined in the natural way:

$$\|f\|_{A_p(\mathbb{R}^m, E)} = \inf_{F|_E = f} \|F\|_{A_p(\mathbb{R}^m)}.$$

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5In this connection we note the following question. Let $V$ be a domain in $\mathbb{R}^m$ and let $\varphi \in C^1(V)$ be a real function. Suppose that the set $\nabla \varphi(V)$ is of positive measure. Is it true then that this set has nonempty interior? For $m = 2$ the answer to this question is positive [9]. For $m \geq 3$ the answer is unknown.

6We note that in [1] the notation $A$ stands for the space of (inverse) Fourier transforms of measures on $\mathbb{R}$. We follow the notation which is common nowadays (see e.g. [2]).
Theorem 1". Let $1 \leq p < 2$. Let $V$ be a domain in $\mathbb{R}^m$. Let $\varphi \in C^{1,\omega}(V)$ be a real function on $V$. Suppose that the gradient $\nabla \varphi$ is nondegenerate on $V$ that is the set $\nabla \varphi(V)$ is of positive measure. Then

$$
\|e^{i\lambda \varphi}\|_{A_p(\mathbb{R}^m, V)} \leq c \left( |\lambda|^{1/p} \frac{1}{|\lambda|} \right)^m
$$

for all those $\lambda \in \mathbb{R}$, $|\lambda| \geq 1$, for which $e^{i\lambda \varphi} \in A_p(\mathbb{R}^m, V)$.

Theorem 1, which is the main result of this section, follows immediately from Theorem 1'. Let us show that in turn Theorem 1' follows from Theorem 1"; then we shall prove Theorem 1" itself.

Fix $p$, $1 \leq p < 2$. For each $\lambda$ such that $e^{i\lambda \varphi} \in A_p(\mathbb{T}^m, V)$ consider an arbitrary $2\pi$-periodic (with respect to each variable) extension $F_\lambda \in A_p(\mathbb{T}^m)$ of the function $e^{i\lambda \varphi}$ from $V$ to $\mathbb{R}^m$.

Under the assumptions of Theorem 1' we can find a cube $I \subset V$ with edges parallel to the coordinate axes such that together with its closure it is contained in the interior of the cube $[0, 2\pi]^m$ and the gradient $\nabla \varphi$ is nondegenerate on $I$. Let $\chi$ be an infinitely differentiable function on $\mathbb{R}^m$ equal to 1 on $I$ and equal to 0 on the compliment $\mathbb{R}^m \setminus [0, 2\pi]^m$. We have $\chi \in A(\mathbb{R}^m)$. Put $c_0 = \|\chi\|_{A(\mathbb{R}^m)}$. For each $u \in \mathbb{R}^m$ define the function $e_u$ on $\mathbb{R}^m$ by $e_u(t) = e^{i(u, t)}$, $t \in \mathbb{R}^m$. We have $\|e_u \chi\|_{A(\mathbb{R}^m)} = c_0$.

For an arbitrary function $h$ on $\mathbb{R}^m$ that vanishes outside of the cube $[0, 2\pi]^m$ let $\tilde{h}$ be its $2\pi$-periodic with respect to each variable extension from $[0, 2\pi]^m$ to $\mathbb{R}^m$. It is well known (see e.g. [10, § 50]) that if $h \in A(\mathbb{R}^m)$ then $\tilde{h} \in A(\mathbb{T}^m)$ and $\|\tilde{h}\|_{A(\mathbb{T}^m)} \leq c \|h\|_{A(\mathbb{R}^m)}$.

We put $g_u = e_u \chi$. For each $u \in \mathbb{R}^m$ we have $\|g_u\|_{A(\mathbb{T}^m)} \leq c_1$ whence $\|g_u F_\lambda\|_{A_p(\mathbb{T}^m)} \leq c_1 \|F_\lambda\|_{A_p(\mathbb{T}^m)}$. It is also clear that $\|g_u F_\lambda\|_{A_p(\mathbb{T}^m)}$ is a measurable function of $u$.

The function $\chi F_\lambda$ coincides with $e^{i\lambda \varphi}$ on $I$. Thus,

$$
\|e^{i\lambda \varphi}\|_{A_p(\mathbb{R}^m, I)} \leq \|\chi F_\lambda\|_{A_p(\mathbb{R})} = \sum_{k \in \mathbb{Z}^m} \int_{[0, 1]^m} |\chi F_\lambda(u + k)|^p \, du
$$

$$
= \int_{[0, 1]^m} \sum_{k \in \mathbb{Z}^m} |e^{-u} \tilde{\chi} F_\lambda(k)|^p \, du = \int_{[0, 1]^m} \|g_u F_\lambda\|_{A_p(\mathbb{T}^m)}^p \, du \leq c_1 \|F_\lambda\|_{A_p(\mathbb{T}^m)}^p.
$$

It remains to use Theorem 1".
Proof of Theorem 1''. We can find a closed cube $I \subseteq V$ with edges parallel to the coordinate axes such that its image $W = \nabla \varphi(I)$ is of positive measure. The set $W$ is bounded.

Fix $c > 0$ so that

$$\omega(I, \nabla \varphi, \delta) \leq c \omega(\delta), \quad \delta \geq 0.$$  

For each $\lambda > 0$ choose $\delta_\lambda > 0$ so that

$$\chi(m^{1/2}2\delta_\lambda) = \frac{1}{2c\lambda}.$$  

(6)

For $\varepsilon > 0$ let $\Delta_\varepsilon$ mean the “triangle” function supported on the interval $(-\varepsilon, \varepsilon)$, that is the function on $\mathbb{R}$ defined by

$$\Delta_\varepsilon(t) = \max (1 - \frac{|t|}{\varepsilon}, 0), \quad t \in \mathbb{R},$$

and for an arbitrary interval $J \subseteq \mathbb{R}$ let $\Delta_J$ be the triangle function supported on $J$ that is $\Delta_J(t) = \Delta_{|J|/2}(t - c_J)$, where $c_J$ is the center of the interval $J$ (and $|J|$ is its length). Let then $J$ be a cube in $\mathbb{R}^m$ with edges parallel to coordinate axes, $J = J_1 \times J_2 \times \ldots \times J_m$. We define the triangle function $\Delta_J$ supported on $J$ as follows

$$\Delta_J(t) = \Delta_{J_1}(t_1)\Delta_{J_2}(t_2)\ldots\Delta_{J_m}(t_m), \quad t = (t_1, t_2, \ldots, t_m) \in \mathbb{R}^m.$$  

We shall use the following lemma.

**Lemma 1.** Let $\lambda > 0$ be sufficiently large. Let $F_\lambda$ be an arbitrary function on $\mathbb{R}^m$ that coincides with $e^{i\lambda \varphi}$ on $V$. Then for each $u \in \lambda W$ there exists a cube $I_{\lambda, u} \subseteq I$ with edges of length $2\delta_\lambda$ parallel to coordinate axes, such that

$$|\langle \Delta_{I_{\lambda, u}} F_\lambda \rangle^\wedge(u)| \geq c_m \delta_\lambda^m,$$

where $c_m > 0$ depends only on the dimension $m$.

**Proof.** Denote by $a$ the length of the edge of the cube $I$. We shall assume that $\lambda > 0$ is so large that

$$2\delta_\lambda < a.$$  

(7)
Take an arbitrary \( u \in \lambda W \). We can find a point \( t_{\lambda,u} \in I \) such that \( \nabla \varphi(t_{\lambda,u}) = \lambda^{-1}u \). Let \( I_{\lambda,u} \subseteq I \) be a cube (a closed one if necessary) with edges of length \( 2\delta_\lambda \) parallel to coordinate axes such that it contains the point \( t_{\lambda,u} \) (see (7)). Consider the following linear function:

\[
\varphi_{\lambda,u}(t) = \varphi(t_{\lambda,u}) + (\lambda^{-1}u, t - t_{\lambda,u}), \quad t \in \mathbb{R}^m.
\]

If \( t \in I_{\lambda,u} \) then for some point \( \theta \in I_{\lambda,u} \) we shall have

\[
\varphi(t) - \varphi(t_{\lambda,u}) = (\nabla \varphi(\theta), t - t_{\lambda,u})
\]

(\( \theta \) lies on the straight segment that joins \( t \) with \( t_{\lambda,u} \)) and therefore

\[
|\varphi(t) - \varphi_{\lambda,u}(t)| = |\varphi(t) - \varphi(t_{\lambda,u}) - (\nabla \varphi(t_{\lambda,u}), t - t_{\lambda,u})|
\]

\[
= |(\nabla \varphi(\theta) - \nabla \varphi(t_{\lambda,u}), t - t_{\lambda,u})| \leq |\nabla \varphi(\theta) - \nabla \varphi(t_{\lambda,u})||t - t_{\lambda,u}|
\]

\[
\leq \omega(I, \nabla \varphi, m^{1/2}2\delta_\lambda)m^{1/2}2\delta_\lambda \leq c\chi(m^{1/2}2\delta_\lambda).
\]

Hence, taking into account (6), we see that

\[
|e^{i\lambda \varphi(t)} - e^{i\lambda \varphi_{\lambda,u}(t)}| \leq |\lambda \varphi(t) - \lambda \varphi_{\lambda,u}(t)| \leq \lambda c\chi(m^{1/2}2\delta_\lambda) = \frac{1}{2}, \quad t \in I_{\lambda,u}.
\]

Using this estimate we obtain

\[
|\left(\Delta I_{\lambda,u}F_\lambda\right)^\wedge(u) - \left(\Delta I_{\lambda,u}e^{i\lambda \varphi_{\lambda,u}}\right)^\wedge(u)| \leq \frac{1}{(2\pi)^m} \int_{I_{\lambda,u}} |\Delta I_{\lambda,u}(t)|e^{i\lambda \varphi(t)} - e^{i\lambda \varphi_{\lambda,u}(t)}|dt
\]

\[
\leq \frac{1}{2} \cdot \frac{1}{(2\pi)^m} \int_{\mathbb{R}^m} \Delta I_{\lambda,u}(t)dt = \frac{1}{2}\Delta I_{\lambda,u}(0).
\]

At the same time

\[
|\left(\Delta I_{\lambda,u}e^{i\lambda \varphi_{\lambda,u}}\right)^\wedge(u)| = \frac{1}{(2\pi)^m} \int_{\mathbb{R}^m} \Delta I_{\lambda,u}(t)e^{i\lambda \varphi_{\lambda,u}(t) - (u,t)}dt = \Delta I_{\lambda,u}(0).
\]

Thus

\[
\left|\left(\Delta I_{\lambda,u}F_\lambda\right)^\wedge(u)\right| \geq \frac{1}{2}\Delta I_{\lambda,u}(0) = c_m\delta_\lambda^m.
\]

The lemma is proved.
Fix $p$, $1 \leq p < 2$. Everywhere below we assume that the frequencies $\lambda$ are such that $e^{i\lambda \varphi} \in A_p(\mathbb{R}^m, V)$. We can assume that the set of these $\lambda$ is unbounded, otherwise there is nothing to prove. We can also assume that these frequencies $\lambda$ are positive (the complex conjugation does not affect the norm of a function in $A_p$). For every such $\lambda$ let $F_\lambda$ be an extension of the function $e^{i\lambda \varphi}$ from $V$ to $\mathbb{R}^m$ satisfying

$$\|F_\lambda\|_{A_p(\mathbb{R}^m)} \leq 2\|e^{i\lambda \varphi}\|_{A_p(\mathbb{R}^m, V).} \quad (8)$$

Let $I_{\lambda, u}$ be the corresponding cubes whose existence (for all sufficiently large $\lambda$) is established in Lemma 1.

Define functions $g_\lambda$ by

$$g_\lambda = (|(F_\lambda)|)\check{\ },$$

where $\check{\ }$ means the inverse Fourier transform. We have $g_\lambda \in A_p(\mathbb{R}^m)$.

It is well known that the function $\Delta_\varepsilon$ belongs to $A(\mathbb{R})$ and has nonnegative Fourier transform. Hence $\Delta_{(-\varepsilon, \varepsilon)^m} \geq 0$ and since for an arbitrary cube $J \subseteq \mathbb{R}^m$ with edges of length $2\varepsilon$ parallel to coordinate axes the function $\Delta_J$ is obtained from $\Delta_{(-\varepsilon, \varepsilon)^m}$ by shift, we have $|\hat{\Delta}_J| = \Delta_{(-\varepsilon, \varepsilon)^m}$. Therefore (* means convolution),

$$|\langle \Delta_{I_{\lambda, u}} F_\lambda \rangle (u) | = |\langle \Delta_{I_{\lambda, u}} \rangle \check{\ } \ast (F_\lambda) \check{\ } (u) | \leq |\langle \Delta_{I_{\lambda, u}} \rangle \check{\ } | \ast |(F_\lambda)| (u) =$$

$$= (\Delta_{(-\delta_\lambda, \delta_\lambda)^m}) \check{\ } \ast \hat{g}_\lambda (u) = (\Delta_{(-\delta_\lambda, \delta_\lambda)^m} g_\lambda) \check{\ } (u)$$

for almost all $u \in \mathbb{R}^m$. (We used the standard facts on the convolution of functions in $L^1$ with functions in $L^p$, see e.g. [11, Ch. I, § 2].)

Thus, according to Lemma 1, we see that if $\lambda > 0$ is sufficiently large, then for almost all $u \in \lambda W$ we have

$$c_m \delta_\lambda^m \leq (\Delta_{(-\delta_\lambda, \delta_\lambda)^m} g_\lambda) \check{\ } (u). \quad (9)$$

Since $\|\Delta_{(-\varepsilon, \varepsilon)^m}\|_{A(\mathbb{R}^m)} = \Delta_{(-\varepsilon, \varepsilon)^m}(0) = 1$, it follows that for every function $f \in A_p(\mathbb{R}^m)$ and an arbitrary $\varepsilon > 0$ we have

$$\|\Delta_{(-\varepsilon, \varepsilon)^m} f\|_{A_p(\mathbb{R}^m)} \leq \|f\|_{A_p(\mathbb{R}^m)}.$$

So, raising inequality (9) to the power $p$ and integrating over $u \in \lambda W$, we see that

$$(c_m^p \delta_\lambda^m |\lambda W|)^{1/p} \leq \left( \int_{\lambda W} |(\Delta_{(-\delta_\lambda, \delta_\lambda)^m} g_\lambda)^{\check{\ }} (u)|^p du \right)^{1/p}$$

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for all sufficiently large \( \lambda \).

Hence, taking (8) into account we obtain

\[
c_m \delta^m \lambda^{m/p} |W|^{1/p} \leq 2 \|e^{i\lambda \varphi}\|_{A_p(T^m, V)}.
\]

It remains only to note that condition (6) implies \( \delta \lambda \geq c \chi^{-1}(1/\lambda) \). The theorem is proved.

**Remark.** Let \( D \) be a bounded domain in \( \mathbb{R}^n \), \( n \geq 2 \). Consider its characteristic function \( 1_D \), i.e., the function that takes value \( 1_D(t) = 1 \) for \( t \in D \) and value \( 1_D(t) = 0 \) for \( t \notin D \). One can show that if the boundary \( \partial D \) of a domain \( D \) is \( C^2 \)-smooth, then \( 1_D \in A_p(\mathbb{R}^n) \) for \( p > \frac{2n}{n+1} \) and \( 1_D \notin A_p(\mathbb{R}^n) \) for \( p \leq \frac{2n}{n+1} \). At the same time in the more general case of domains with \( C^1 \)-smooth boundary the Fourier transform of the characteristic function may behave in an essentially different way. The author constructed a domain \( D \subset \mathbb{R}^2 \) with \( C^1 \)-smooth boundary such that \( 1_D \in A_p(\mathbb{R}^2) \) for all \( p > 1 \). Theorem 1" plays the key role in the study of the question for which domains \( D \subset \mathbb{R}^n \) with \( C^1 \)-smooth boundary we have inclusion \( 1_D \in A_p(\mathbb{R}^n) \). Our results will be presented in another publication.

§ 2. Slow growth of \( \|e^{i\lambda \varphi}\|_{A_p(T^m)} \)

In the work [3] for each given class \( C^{1, \omega}(T) \) (under certain simple assumption imposed on \( \omega \)) we constructed a nontrivial real function \( \varphi \in C^{1, \omega}(T) \) with slow growth of the norms \( \|e^{i\lambda \varphi}\|_{A_p(T)} \). Namely, put (as above \( \chi^{-1} \) is the function inverse to \( \chi(\delta) = \delta \omega(\delta) \))

\[
\Theta_1(y) = \frac{y}{\log y} \chi^{-1} \left( \frac{\log y}{y} \right)^2
\]

and for \( 1 < p < 2 \) put

\[
\Theta_p(y) = \left( \int_1^y \left( \chi^{-1} \left( \frac{1}{\tau} \right) \right)^p d\tau \right)^{1/p}.
\]

Let \( \omega \) satisfies condition \( \omega(2\delta) < 2\omega(\delta) \) for all sufficiently small \( \delta > 0 \). Then there exists a real nowhere linear, i.e. not linear on any interval,
function $\varphi \in C^{1,\omega}(\mathbb{T})$ such that for all $p$, $1 \leq p < 2$, we have $\|e^{i\lambda \varphi}\|_{A_p(\mathbb{T})} = O((\Theta_p(\lambda))^{m(1/2 + 1/\alpha)}) = O((\log |\lambda|)^{m(1/2 + 1/\alpha)})$.

This result on the slow growth easily transfers to the case of a torus of an arbitrary dimension. Let us say that the gradient of a function $\varphi$ is nowhere degenerate if it is nondegenerate on any open set, that is if for every open set $V \subseteq \mathbb{R}^m$ we have $|\nabla \varphi(V)| > 0$.

**Theorem 2.** Let $\omega(2\delta) < 2\omega(\delta)$ for all sufficiently small $\delta > 0$. There exists a real function $\varphi \in C^{1,\omega}(\mathbb{T}^m)$ with nowhere degenerate gradient such that for all $p$, $1 \leq p < 2$, we have

$$
\|e^{i\lambda \varphi}\|_{A_p(\mathbb{T}^m)} = O((\Theta_p(\lambda))^{m(1 + 1/\alpha)}); \quad |\lambda| \to \infty, \quad \lambda \in \mathbb{R}.
$$

The multidimensional version follows from the one-dimensional. Indeed, if $\varphi_0 \in C^{1,\omega}(\mathbb{T})$ is a real nowhere linear function, then, taking $\varphi(t) = \varphi_0(t_1) + \varphi_0(t_2) + \ldots + \varphi_0(t_m)$, $t = (t_1, t_2, \ldots, t_m) \in \mathbb{T}^m$, we have a function $\varphi \in C^{1,\omega}(\mathbb{T}^m)$ with nowhere degenerate gradient and it remains only to note that $\|e^{i\lambda \varphi}\|_{A_p(\mathbb{T}^m)} = (\|e^{i\lambda \varphi_0}\|_{A_p(\mathbb{T})})^m$.

In the same manner (or directly from Theorem 2, taking Corollary 1 into account) we obtain the corollaries below, which are the multidimensional versions of Corollaries 2, 3 from our paper [3].

**Corollary 3.** Let $0 < \alpha < 1$. There exists a real function $\varphi \in C^{1,\alpha}(\mathbb{T}^m)$ with nowhere degenerate gradient such that

(i) $\|e^{i\lambda \varphi}\|_{A_p(\mathbb{T}^m)} = O((\lambda)^{m(1 + 1/\alpha)(1 - \alpha)/(1 + \alpha)})$;

(ii) $\|e^{i\lambda \varphi}\|_{A_p(\mathbb{T}^m)} \simeq |\lambda|^m \left(\frac{1}{p - \frac{1}{1 + \alpha}}\right)$ for $1 < p < 1 + \alpha$,

$\|e^{i\lambda \varphi}\|_{A_p(\mathbb{T}^m)} \simeq 1$ for $1 + \alpha < p < 2$,

$\|e^{i\lambda \varphi}\|_{A_p(\mathbb{T}^m)} = O((\lambda)^{m/p})$ for $p = 1 + \alpha$.

In particular, we see that for $1 < p < 1 + \alpha$ the estimate in Corollary 1 of the present paper is sharp.
Corollary 4. Let $\gamma(\lambda) \geq 0$ and $\gamma(\lambda) \to +\infty$ as $\lambda \to +\infty$. There exists a real function $\varphi \in C^1(\mathbb{T}^m)$ with nowhere degenerate gradient such that
\[
\|e^{i\lambda \varphi}\|_{A(\mathbb{T}^m)} = O(|\lambda|^{\nu}(\log |\lambda|)^m).
\]

§ 3. Upper estimates

Let $1 \leq p < 2$. As we noted in Introduction, if $\varphi$ is a real function on the circle $\mathbb{T}$ satisfying Lipschitz condition of order 1, then $\|e^{i\lambda \varphi}\|_{A_p(\mathbb{T})} = O(|\lambda|^{1/p-1/2})$ and due to Leibenson–Kahane–Alpár estimate (2) we have $\|e^{i\lambda \varphi}\|_{A_p(\mathbb{T})} \simeq |\lambda|^{1/p-1/2}$ for every nonlinear real function $\varphi \in C^2(\mathbb{T})$ (actually this holds even under assumption that $\varphi \in C^{1,1}(\mathbb{T})$, see Corollary 2 for $m = 1$). Here we shall obtain similar results in the case of torus of an arbitrary dimension.

Recall the known smoothness condition that guarantees that a function belongs to the classes $A_p(\mathbb{T}^m)$, $1 \leq p < 2$, namely: if $f \in C^{\nu,\alpha}(\mathbb{T}^m)$ and $\nu + \alpha > m(1/p - 1/2)$, then $f \in A_p(\mathbb{T}^m)$. It is known that this condition is sharp in the sense that for $\nu + \alpha = m(1/p - 1/2)$ the inclusion $C^{\nu,\alpha}(\mathbb{T}^m) \subseteq A_p(\mathbb{T}^m)$ fails.\footnote{In the one-dimensional case for $p = 1$ the corresponding results are due to Bernstein (see [12, Ch. VI, Theorem 3.1]), the generalization to the case $1 < p < 2$ was obtained by Szász (see [12, Ch. VI, Theorem 3.10]), in the multidimensional case the sufficiency of the indicated condition was obtained by Szász and Minakshisundaram [13] (see also remark of Bochner [14]). The sharpness of the indicated smoothness condition in the multidimensional case was established by Weinger [15].}

We shall show that the following theorem holds.

**Theorem 3.** Let $1 \leq p < 2$ and let $0 \leq \alpha \leq 1$. Let $\varphi \in C^{\nu,\alpha}(\mathbb{T}^m)$ be a real function. Suppose that $\nu + \alpha \geq 1$, $\nu + \alpha > m(1/p - 1/2)$. Then
\[
\|e^{i\lambda \varphi}\|_{A_p(\mathbb{T}^m)} = O(|\lambda|^{m(1/p-1/2)}), \quad |\lambda| \to \infty, \quad \lambda \in \mathbb{R}.
\]

Under assumption that $\nu + \alpha \geq 2$, we have $C^{\nu,\alpha} \subseteq C^{1,1}$, so, using Corollary 2 and Theorem 3, we see that the following theorem holds.
Theorem 4. Let $1 \leq p < 2$ and let $0 \leq \alpha \leq 1$. Let $\varphi \in C^{\nu,\alpha}(\mathbb{T}^m)$ be a real function with nondegenerate gradient. Suppose that $\nu + \alpha \geq 2$, $\nu + \alpha > m(1/p - 1/2)$. Then

$$
\|e^{i\lambda \varphi}\|_{A^p(\mathbb{T}^m)} \simeq |\lambda|^{m(1/p - 1/2)}, \quad |\lambda| \to \infty, \quad \lambda \in \mathbb{R}.
$$

In particular, if $\nu + \alpha \geq 2$, $\nu + \alpha > m/2$, then

$$
\|e^{i\lambda \varphi}\|_{A(\mathbb{T}^m)} \simeq |\lambda|^{m/2}.
$$

Proof of Theorem 3. In the usual way we define the norm on the space $L^2(\mathbb{T}^m)$ and the norm on the space $C(\mathbb{T}^m)$ of continuous functions on $\mathbb{T}^m$:

$$
\|f\|_{L^2(\mathbb{T}^m)} = \left(\frac{1}{(2\pi)^m} \int_{\mathbb{T}^m} |f(t)|^2 dt\right)^{1/2}, \quad \|g\|_{C(\mathbb{T}^m)} = \sup_{t \in \mathbb{T}^m} |g(t)|.
$$

We define the norm on $C^{\nu,\alpha}(\mathbb{T}^m)$ by

$$
\|f\|_{C^{\nu,\alpha}(\mathbb{T}^m)} = \max_{0 \leq \gamma_1 + \gamma_2 + \ldots + \gamma_m \leq \nu} \|D_{\gamma_1,\gamma_2,\ldots,\gamma_m} f\|_{C(\mathbb{T}^m)} +
$$

$$
+ \max_{\gamma_1 + \gamma_2 + \ldots + \gamma_m = \nu} \sup_{\delta > 0} \frac{1}{\delta^\alpha} \omega(D_{\gamma_1,\gamma_2,\ldots,\gamma_m} f, \delta),
$$

where

$$
D_{\gamma_1,\gamma_2,\ldots,\gamma_m} f(t) = \frac{\partial^{\gamma_1 + \gamma_2 + \ldots + \gamma_m} f}{\partial t_1^{\gamma_1} \partial t_2^{\gamma_2} \ldots \partial t_m^{\gamma_m}}, \quad t = (t_1, t_2, \ldots, t_m).
$$

It is easy to see that for $\nu = 0, 1, 2, \ldots$ and $0 \leq \alpha \leq 1$ we have

$$
\|fg\|_{C^{\nu,\alpha}(\mathbb{T}^m)} \leq c\|f\|_{C^{\nu,\alpha}(\mathbb{T}^m)}\|g\|_{C^{\nu,\alpha}(\mathbb{T}^m)}
$$

where $c = c(m, \nu, \alpha) > 0$ is independent of $f$ and $g$.

Using this relation it is easy to verify that if $\nu + \alpha \geq 1$, then for each real function $\varphi \in C^{\nu,\alpha}(\mathbb{T}^m)$ we have the estimate

$$
\|e^{i\lambda \varphi}\|_{C^{\nu,\alpha}(\mathbb{T}^m)} \leq c|\lambda|^\nu, \quad \lambda \in \mathbb{R}, \quad |\lambda| \geq 1,
$$

where $c = c(\nu, \alpha, \varphi)$ is independent of $\lambda$. For $\nu = 1, 2, \ldots$ this can be verified by induction over $\nu$ with fixed $\alpha$. For $\nu = 0$ the condition $\nu + \alpha \geq 1$ yields $\alpha = 1$ and the required estimate also holds.

Thus it is clear that the statement of the theorem is an immediate consequence of the following simple lemma.
Lemma 2. Let $f \in C^{\nu,\alpha}(\mathbb{T}^m)$ and let $\nu + \alpha > m(1/p - 1/2)$, $1 \leq p < 2$. Then

$$\|f\|_{A_p(\mathbb{T}^m)} \leq c \|f\|_{C^{\nu,\alpha}(\mathbb{T}^m)}^{1-\tau} \|f\|_{L^2(\mathbb{T}^m)},$$

where

$$\tau = \frac{m(1/p - 1/2)}{\nu + \alpha}.$$

Proof. Let $S^{m-1}$ denote the unit sphere in $\mathbb{R}^m$ centered at 0. Let $\delta > 0$ and let $\xi \in S^{m-1}$. For each $j = 1, 2, \ldots, m$ consider a function

$$\frac{\partial^\nu f}{\partial t_j^\nu} (t + \delta \xi) - \frac{\partial^\nu f}{\partial t_j^\nu} (t - \delta \xi), \quad t = (t_1, t_2, \ldots, t_m) \in \mathbb{T}^m.$$

Writing the Parseval identity for this function we obtain

$$\frac{1}{(2\pi)^m} \int_{\mathbb{T}^m} \left| \frac{\partial^\nu f}{\partial t_j^\nu} (t + \delta \xi) - \frac{\partial^\nu f}{\partial t_j^\nu} (t - \delta \xi) \right|^2 dt = \sum_{k=(k_1, k_2, \ldots, k_m) \in \mathbb{Z}^m} k_j^{2\nu} |\hat{f}(k)|^2 4 \sin^2(\delta k, \xi).$$

Thus,

$$\sum_{k \in \mathbb{Z}^m} k_j^{2\nu} |\hat{f}(k)|^2 \sin^2(\delta k, \xi) \leq \|f\|_{C^{\nu,\alpha}}^2 \delta^{2\alpha}.$$ 

Summing over $j = 1, 2, \ldots, m$, we have

$$\sum_{k \in \mathbb{Z}^m} |k|^{2\nu} |\hat{f}(k)|^2 \sin^2(\delta k, \xi) \leq c \|f\|_{C^{\nu,\alpha}}^2 \delta^{2\alpha},$$

where $c = c(\nu, m) > 0$.

Let $|k| \leq 1/\delta$, then $|\langle \delta k, \xi \rangle| \leq 1$ whence $|\sin(\delta k, \xi)| \geq |\langle \delta k, \xi \rangle|/2$ and we see that

$$\sum_{|k| \leq 1/\delta} |k|^{2\nu} |\hat{f}(k)|^2 (\delta k, \xi)^2 \leq c \|f\|_{C^{\nu,\alpha}}^2 \delta^{2\alpha}.$$ 

Integrating this inequality over $\xi \in S^{m-1}$ (for $m = 1$ we just put $\xi = 1$) and taking into account the fact that for an arbitrary vector $v \in \mathbb{R}^m$

$$\int_{S^{m-1}} (v, \xi)^2 d\xi = c_m |v|^2,$$
we obtain

\[ \sum_{|k| \leq 1/\delta} |k|^{2\nu} |\hat{f}(k)|^2 \delta^2 |k|^2 \leq c \|f\|_{C^{\nu,\alpha}}^2 \delta^{2\alpha}.\]

Let \( \delta = 2^{-n}, \ n = 1, 2, \ldots \) We see that

\[ \sum_{2^{n-1} \leq |k| < 2^n} |\hat{f}(k)|^2 \leq c \|f\|_{C^{\nu,\alpha}}^2 2^{-2n(\nu + \alpha)}, \quad n = 1, 2, \ldots, \] (10)

with constant \( c > 0 \) independent of \( n \) and \( f \).

Using the Hölder inequality with \( p^* = 2/p, \ 1/p^* + 1/q^* = 1 \), we obtain from relation (10) that

\[ \sum_{2^{n-1} \leq |k| < 2^n} |\hat{f}(k)|^p \leq \left( \sum_{2^{n-1} \leq |k| < 2^n} |\hat{f}(k)|^{pp^*} \right)^{1/p^*} \left( \sum_{2^{n-1} \leq |k| < 2^n} 1 \right)^{1/q^*} \]

\[ \leq c \|f\|_{C^{\nu,\alpha}}^p 2^{-pn(\nu + \alpha)} (2^{nm})^{1-p/2} = c \|f\|_{C^{\nu,\alpha}}^p 2^{-np(\nu + \alpha)(1-\tau)}. \]

Hence it is clear that for an arbitrary \( B \geq 1 \) we have

\[ \sum_{|k| \geq B} |\hat{f}(k)|^p \leq c \|f\|_{C^{\nu,\alpha}}^p B^{-(\nu + \alpha)(1-\tau)}. \] (11)

At the same time it is obvious that for \( B \geq 1 \)

\[ \sum_{|k| < B} |\hat{f}(k)|^p \leq \left( \sum_{|k| < B} |\hat{f}(k)|^{pp^*} \right)^{1/p^*} \left( \sum_{|k| < B} 1 \right)^{1/q^*} \]

\[ \leq \|f\|_{L^p}^p c B^m(1-p/2) = c \|f\|_{L^p}^p B^{p(\nu + \alpha)}}. \] (12)

It remains to add relations (11), (12) and put

\[ B = \left( \frac{\|f\|_{C^{\nu,\alpha}}}{\|f\|_{L^2}} \right)^\frac{1}{\nu + \alpha}. \]

The lemma is proved. Theorem 3 is proved. Theorem 4 follows.
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