DYNAMICS OF TRAVELING WAVES OF VARIABLE ORDER
HYPERBOLIC LIOUVILLE EQUATION:
REGULATION AND CONTROL

EMILE FRANC DOUNGMO GOUFO*
Department of Mathematical Sciences, University of South Africa
Florida, 0003, South Africa

ABDON ATANGANA
Institute for Groundwater Studies, University of the Free State
Bloemfontein, 9300, South Africa

ABSTRACT. Traveling waves remain significant in Applied Sciences mostly because they involve the movement of energy carrier particles. In this paper, traveling waves described by a generalized system, the fractional variable order hyperbolic Liouville model is solved numerically by means of Crank-Nicholson scheme. Detailed analysis are performed and prove that the numerical method is stable and converges. Simulations reveal that the model’s variable order derivate (a function of time and position variables) has a considerable impact on the dynamics of the whole system. It influences the movement and the shape of the resulting waves including their amplitude, their wavelength as well as their compression and rarefaction processes. Such a variable order derivative becomes, due to these results, a substantial parameter and non-constant tool for the regulation and control of models describing wave motion.

1. Introduction, motivation and model setting. There exists in the literature of traveling wave theory, a series of partial differential equations (PDEs) that are all of second order in the initial value variable and are very important in applied theory with applications to real life phenomena. The non-linearity of such models forces the scientists to adopt various approaches to address their solvability. Hence, their numerical approximations are often the best practical way to go. Usually traveling waves, when subjected to a certain number of conditions (physical or mathematical) exhibit similar behaviors. These are related to propagation, transmittance, reflection, diffraction, interference, transmission medium, refraction, absorption, dispersion, polarization, etc. They involve the movement of sequence of particles that pursue the same simple harmonic trajectory, so that the particle in movement begins to follow the one just before it. Those particles are most of the time carriers of energy and we can understand why traveling wave motions are the common ways to transfer energy from one point to another using various properties

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* Corresponding author: franckemile2006@yahoo.ca.
of the medium they apply to. An example of water wave is depicted by Fig. 1 and shows particles (in red) being propagated from one location to another across the water. Successive positions held by the same particle are marked by blue dots and seem to be on a straight line (rectilinearity). The topic of energy transfer has been trending throughout this decade with scientists looking for new and innovative approaches to optimize the use of the energy in a lasting and sustainable way.

On the other side, real life phenomena described by models with fractional differentiation or variable order derivative have been analyzed in number of works [2, 3, 13, 17, 27, 14, 37, 38, 42] with such a derivative proven to have evident memory properties which vary along with time and spatial variable. The fractional differentiation or the variable order derivative has turned out to play the role of one of the model’s important parameter able to change de dynamics of the system and ultimately control it [7, 1, 15, 16, 17, 35, 30, 20]. Controlling the dynamic of a system in applied science is already very important and being able to use a simple and easy parameter like the variable order derivative to do it can only be a prowess. Hence, in the same momentum to push the ongoing investigations a bit farther, we explore the possibility of extending to the concept of variable order differentiation to the traveling wave process issued from the hyperbolic Liouville model. A comprehensive analysis related to the stability and the convergence of such a model is given and supported by numerical simulations. The influence of the derivative on the dynamics of hyperbolic Liouville model will be assessed. This model belongs to the family of differential equations with exponential non-linearities and reads as [36]

\[
D_{tt}^{\sigma(x,t)}u = a^2 \frac{\partial^2 u}{\partial x^2} + b e^{\beta u}, \quad 1 < \sigma(x,t) \leq 2 \quad 0 \leq x \leq L, \quad t \geq 0. \tag{1}
\]

where \(a, b\) and \(\beta\) are arbitrary real constants, \(L > 0\) and \(D_{tt}^{\sigma(x,t)}\) the fractional differentiation of variable order \(\sigma(x,t)\) depending on the time \(t\) and the position variable \(x\). More details about the definition and significance of the operator \(D_{tt}\) is given in the next section. To find initial and boundary conditions for model (1), more considerations are to be done. If we initialize \(\sigma(x,t) = 2\) then, model (1) becomes the classical hyperbolic Liouville system

\[
\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} + b e^{\beta u}. \tag{2}
\]

**Remark 1.1.** Assuming that the function \(u(x,t)\) is a solution of the model (2), thus the following functions are also solutions of the equation:

- \(u_1 = u(\pm C_1 x + C_2, \pm C_1 t + C_3) + (2/\beta) \ln |C_1|\)
- \(u_2 = u(x \coth \xi + at \sinh \xi, \quad t \coth \xi + (x/a) \sinh \xi),\)

where \(C_1, C_2, C_3\) and \(\xi\) are arbitrary real constants.

Hence, an analytical solution of (2) is given as [36, Section 3.2]:

\[
\tilde{u}(x,t) = (1/\beta) \ln \left( \frac{2(a^2 A^2 - B^2)}{b^2 \cosh^2(Ax + Bt + C)} \right) \tag{3}
\]

where \(A, B\) and \(C\) are arbitrary real constants. Recall that the traveling wave’s property of solution (3) is due to the combination \(Ax + Bt + C\). Whence, we can consider

\[
\varpi(x) = \tilde{u}(x,0) = (1/\beta) \ln \left( \frac{2(a^2 A^2 - B^2)}{b^2 \cosh^2(Ax + C)} \right) \tag{4}
\]
Figure 1. An example of water wave showing particles (in red) being propagated, after each oscillation, from one location to another across the water. Successive positions held by the same particle are marked by blue dots and appear to be rectilinear.

as one initial condition for model (1). Another initial condition is obtained by differentiating (3) with respect to time $t$ and initializing at $t = 0$. This leads to

$$\frac{\partial}{\partial t} \tilde{u}(x,t)|_{t=0} = \frac{(B/\beta)}{2(a^2A^2 - B^2)} \left( \frac{-4(a^2A^2 - B^2)}{b\beta \cosh^{2}(A \lambda + C)} \right) \sinh(Ax + C)$$

The boundary conditions are considered as

$$\varphi_1(t) = u(0,t) = \tilde{u}(0,t) = \tilde{u}(x,t) = \left( \frac{1}{\beta} \right) \ln \left( \frac{2(a^2A^2 - B^2)}{b\beta \cosh^{2}(Bt + C)} \right)$$

$$\varphi_2(t) = u(k\pi,t) = \tilde{u}(L,t) = \left( \frac{1}{\beta} \right) \ln \left( \frac{2(a^2A^2 - B^2)}{b\beta \cosh^{2}(AL + Bt + C)} \right)$$
where \( L > 0 \), \( \varphi_1 \) and \( \varphi_2 \) are real functions. Without loss of generality we can fix \( A = 0 \) and find \( B_0 \) and \( C_0 \) such that
\[
\frac{2a^2A^2 - B^2}{b\beta \cosh^2(AL + Bt + C)} = \frac{-2B_0^2}{b\beta \cosh^2(B_0t + C_0)} = 1. \tag{7}
\]
Therefore, we are left with vanishing boundary conditions and constant initial conditions and (1), (4) to (4) lead to the final variable order hyperbolic Liouville model:
\[
D_t^{\sigma(x,t)} u = a^2 \frac{\partial^2 u}{\partial x^2} + b\sigma u, \quad 1 < \sigma(x,t) \leq 2, \quad 0 \leq x \leq L, \quad t \geq 0. \tag{8}
\]
with the initial conditions
\[
u(x,0) = \varpi(x) = \varpi_0 \quad \text{and} \quad \frac{\partial}{\partial t} u(x,0) = \varpi_1 \tag{9}
\]
and boundary conditions
\[
u(0,t) = \varphi_1(t) = \varphi_2(t) = 0 \tag{10}
\]
where \( \varpi_0 \), \( \varpi_1 \) are real constant functions. For reasons of simplicity we take \( \varpi_1 = 0 \) for the rest of this analysis.

1.1. A quick review on the concept of the variable order derivative. There are several types of definitions for derivatives with fractional order and they include, among others, the variable order derivative (VOD). Such types of models have been discussed in number of research articles [2, 3, 13, 15, 16, 18, 22, 35, 30, 20]. A special attention has been given to the reasonable memory properties that characterize the variable order derivative and that strongly depend on the time or space coordinates [27, 37, 38, 42, 4, 39, 10, 8, 9]. The variable order definition takes its foundation on the Riemann-Liouville integral of variable order as illustrated by the following:

**Definition 1.1** (VOD in the sense of Riemann-Liouville). Let \( c, e > 0 \) with \( c < e \), and let \( AC[c,e] \) denote the space of absolutely continuous functions on \( [c,e] \). Consider the function \( \sigma : [c,e] \times \mathbb{R}_+ \ni (x,t) \mapsto \mathbb{R} \) such that \( 0 < \sigma(x,t) < 1 \) and the function \( u \in L_1[c,e] \). The left Riemann-Liouville fractional integral and derivative of \( u \) of variable order \( \sigma(\cdot, \cdot) \) are defined respectively as
\[
R_cD_t^{\sigma(\cdot, \cdot)} u(t) = \frac{d}{dt} \int_c^t \frac{1}{\Gamma(\sigma(x,t))}(t-\zeta)^{\sigma(x,t)-1}u(\zeta)d\zeta, \quad \text{with} \quad t > c
\]
and
\[
R_cD_t^{\sigma(\cdot, \cdot)} u(t) = \frac{d}{dt} \int_c^t \frac{1}{\Gamma(1-\sigma(x,t))}(t-\zeta)^{-\sigma(x,t)}u(\zeta)d\zeta, \quad \text{with} \quad t > c,
\]
provided that \( R_cD_t^{\sigma(\cdot, \cdot)} u \in AC[c,e] \) Similarly, the right Riemann-Liouville fractional integral and derivative of \( u \) of variable order \( \sigma(\cdot, \cdot) \) are defined respectively as
\[
R_dD_t^{\sigma(\cdot, \cdot)} u(t) = \frac{d}{dt} \int_t^e \frac{1}{\Gamma(\sigma(x,t))}(t-\zeta)^{\sigma(x,t)-1}u(\zeta)d\zeta, \quad \text{with} \quad e > t
\]
and
\[
R_dD_t^{\sigma(\cdot, \cdot)} u(t) = \frac{d}{dt} \int_t^e \frac{1}{\Gamma(1-\sigma(x,t))}(t-\zeta)^{-\sigma(x,t)}u(\zeta)d\zeta, \quad \text{with} \quad e > t,
\]
provided that \( R_cD_t^{\sigma(\cdot, \cdot)} u \in AC[c,e] \).

Hence, we can also define the fractional VOD in the sense of Caputo as follows
Definition 1.2 (VOD in the sense of Caputo). Let $c, e > 0$ with $c < e$ and consider the function $\sigma : [c, e] \times \mathbb{R}_+ \ni (x, t) \mapsto \mathbb{R}$ such that $0 < \sigma(x, t) < 1$ and the function $u \in L_1[c, e]$. The left and right Caputo fractional derivatives of $u$ of variable order $\sigma(\cdot, \cdot)$ are defined respectively as

\[
\mathcal{D}^\sigma_{c+} D^\sigma_{t+} u(t) = \int_c^t \frac{1}{\Gamma(1-\sigma(x, t))} (t-\zeta)^{-\sigma(x, t)} u'(\zeta) d\zeta, \quad \text{with } t > c,
\]

and

\[
\mathcal{D}^\sigma_{c} D^\sigma_{t-} u(t) = \int_t^e \frac{1}{\Gamma(1-\sigma(x, t))} (t-\zeta)^{-\sigma(x, t)} u'(\zeta) d\zeta, \quad \text{with } e > t,
\]

provided that $\mathcal{R} D^\sigma_{t} u(t) \in AC[c, e]$.

Recall that the derivative in the Definition 1.2 can become the Caputo variable order differential operator in the special case of $\sigma = \sigma(t)$ leading to the statement:

Definition 1.3. Assume that $\sigma : \mathbb{R} \ni t \mapsto \mathbb{R}_+$ is a continuous function so that $0 < \sigma(t) \leq 1$, $u : \mathbb{R} \ni t \mapsto \mathbb{R}$ is a continuous and differentiable function. Let $H > 0$, the variable order differentiation of $u$ in $[0, H)$ reads as:

\[
CV D^{\sigma(t)} u(t) = \frac{1}{\Gamma(1-\sigma(t))} \int_0^t (t-\zeta)^{-\sigma(t)} u'(\zeta) d\zeta, \quad \text{with } t \in [0, H). \quad (11)
\]

Note that this operator vanishes when it is applied to a constant function and it is an important property that makes it very popular in Applied Sciences in general and this work in particular.

The notion of fractional VOD is seen as the generalization of constant order derivative (COD). Moreover, it has an exceptional memory properties that vary according to the space position and time. An illustration can clearly be seen in the following example: Let us consider the differential model given by

\[
\partial_t u(x, t) = CV D^{1-\sigma(t)} X(u(x, t)), \quad 0 < \sigma(t) \leq 1, \quad 0 \leq x \leq L, \quad t \geq 0,
\]

with $CV D^{\sigma(t)}$ representing the time fractional VOD given here above. Then, this equation can easily be interpreted by the fact that the partial derivative $\partial_t u(x, t)$ is strongly dependent on part of the function $u(x, t)$’s history. As a proof, we have the integral form of $CV D^{1-\sigma(t)} X(u(x, t))$ and when applying the anti-derivative $CV D^{\sigma(t)-1}$ to both of its sides leads to an expression more-or-less close to

\[
u = CV D^{\sigma(t)} X(u).
\]

Any reader interested in more properties and interpretations of this type fractional VOD may consult the papers \cite{3, 13, 15, 16, 27, 37, 38, 42, 18, 22, 35, 30, 20} and the references therein.

2. Numerical settings and solvability. The numerical scheme of dynamical systems with fractional VOD in space and time describing processes like dispersion, advection and diffusion have been recently discussed in several articles \cite{38, 42, 18, 2, 26, 47}. In some of those analysis, clear expression of the schemes were explicitly given. In the same time several other authors have investigated the case where the derivative order is constant in time and space variable \cite{29, 41, 43, 46}. For instance, a difference scheme for a time fractional diffusion system was implicitly given in \cite{6} while a method of weighted average finite difference was defined in \cite{45} and also applicable to the time fractional diffusion process. Other authors \cite{11, 34, 19}
The fractional VOD is discretized in terms of the time variable by setting $t_j = j\tau$ with $0 \leq j \leq N$, $N\tau = T$ and $x_k = ks$, with $0 \leq k \leq M$, $Ms = L$ with $N$ and $M$ representing grid points, $t$ the time and $s$ the step size. For the Crank–Nicholson scheme, the first and second order derivative can be discretized in terms of the space variable $x$ as follows:

$$\frac{\partial u}{\partial x} = \frac{1}{4}\left(\frac{[u(x_{k+1}, t_{j+1}) - u(x_{k-1}, t_{j+1})] + [u(x_{k+1}, t_{j}) - u(x_{k-1}, t_{j})]}{s}\right) + O(s)$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{2} \times \frac{[u(x_{k+1}, t_{j+1}) - 2u(x_k, t_{j+1}) + u(x_{k-1}, t_{j+1})] + [u(x_{k+1}, t_{j}) - 2u(x_k, t_{j}) + u(x_{k-1}, t_{j})]}{s^2} + O(s^2)$$

$$u = \frac{u(x_k, t_{j+1}) + u(x_k, t_{j})}{2}.$$  

Similarly, the first and second order derivative can be discretized in terms of the time variable $t$ as follows:

$$\frac{\partial u}{\partial t} = \frac{1}{4}\left(\frac{[u(x_{k+1}, t_{j+1}) - u(x_{k-1}, t_{j+1})] + [u(x_{k+1}, t_{j}) - u(x_{k-1}, t_{j})]}{2\tau}\right) + O(\tau)$$

$$\frac{\partial^2 u}{\partial t^2} = \frac{1}{2} \times \frac{[u(x_{k+1}, t_{j+1}) - 2u(x_k, t_{j+1}) + u(x_{k-1}, t_{j+1})] + [u(x_{k+1}, t_{j}) - 2u(x_k, t_{j}) + u(x_{k-1}, t_{j}-1)]}{\tau^2} + O(\tau^2)$$

The fractional VOD is discretized in terms of the time variable $t$ as follows:

$$\frac{\partial^{\sigma_k^{j+1}} u(x_k, t_{j+1})}{\partial t^{\sigma_k^{j+1}}} = \frac{\tau^{-\sigma_k^{j+1}}}{\Gamma(2 - \sigma_k^{j+1})} \times \left( u(x_k, t_{j+1}) - u(x_k, t_{j}) + \sum_{i=1}^{j} [u(x_k, t_{j+1-i}) - u(x_k, t_{j-i})] [((i+1)^{1-\sigma_k^{j+1}} - i^{1-\sigma_k^{j+1}})] \right).$$

Now we substitute all the previous equations into the system (8) to get

$$\frac{\tau^{-\sigma_k^{j+1}}}{\Gamma(2 - \sigma_k^{j+1})} (u(x_k, t_{j+1}) - u(x_k, t_{j})) + \frac{\tau^{-\sigma_k^{j+1}}}{\Gamma(2 - \sigma_k^{j+1})} \left( \sum_{i=1}^{j} [u(x_k, t_{j+1-i}) - u(x_k, t_{j-i})] [((i+1)^{1-\sigma_k^{j+1}} - i^{1-\sigma_k^{j+1}})] \right)$$
Proposition 2.1. The Crank–Nicholson technique described in the previous section to address the solvability of the hyperbolic Liouville model with fractional VOD, is stable.

For more clarity in the analysis, we adopt the following notations:

\[ u^{k,j} = u(x_k, t_j), \quad Q_i^{k,j+1} = (i + 1)^{1-\sigma_{k}^{j+1}} - (i)^{1-\sigma_{k}^{j+1}}, \]

\[ E^{k,j+1} = \frac{a^2 \sigma_{k}^{j+1} \Gamma(2 - \sigma_{k}^{j+1})}{2s^2}, \quad F^{k,j} = b \sigma_{k}^{j+1} \Gamma(2 - \sigma_{k}^{j+1}), \]

\[ Y(q) = \frac{\beta^q}{2^q q!} \]

\[ \vartheta_{i}^{k,j+1} = Q_{i}^{k,j+1} - Q_{i}^{k,j+1} \quad \text{with} \quad \vartheta_{0}^{k,j+1} = 1. \]

Hence, the model (18) is reduced to become

\[ u^{k,j+1} - u^{k,j} + \sum_{i=1}^{j} [u^{k,j+1-i} - u^{k,j-i}]Q_{i}^{k,j+1} \]

\[ = E^{k,j+1} (u^{k+1,j+1} + 2u^{k,j+1} + u^{k-1,j+1} + u^{k+1,j} - 2u^{k,j} + u^{k-1,j}) \]

\[ + F^{k,j} \sum_{q=0}^{\infty} Y(q)(1 - \vartheta_{j}^{k,j+1}) (u^{k,j+1} + u^{k,j})^q, \]

equivalent to

\[ u^{k,j+1} - u^{k,j} + \sum_{i=1}^{j} [u^{k,j+1-i} - u^{k,j-i}]Q_{i}^{k,j+1} \]

\[ = E^{k,j+1} (u^{k+1,j+1} + 2u^{k,j+1} + u^{k-1,j+1} + u^{k+1,j} - 2u^{k,j} + u^{k-1,j}) \]

\[ + F^{k,j} + F^{k,j} Y(1) (u^{k,j+1} + u^{k,j}) + \sum_{q=2}^{\infty} Y(q)(1 - \vartheta_{j}^{k,j+1}) (u^{k,j+1} + u^{k,j})^q. \]

The later equality can be rearranged to become

\[ u^{k,j+1} [1 + 2E^{k,j+1} - F^{k,j} Y(1)] = u^{k,j} [1 - 2E^{k,j+1} + F^{k,j} Y(1)] \]

\[ - \sum_{i=1}^{j} [u^{k,j+1-i} - u^{k,j-i}]Q_{i}^{k,j+1} \]

\[ + E^{k,j+1} (u^{k+1,j+1} + u^{k-1,j+1} + u^{k+1,j} + u^{k-1,j}) + F^{k,j} \]

\[ + \sum_{q=2}^{\infty} Y(q)(1 - \vartheta_{j}^{k,j+1}) (u^{k,j+1} + u^{k,j})^q. \]

2.1. Stability of the Crank–Nicholson scheme applied to the hyperbolic Liouville model with fractional VOD.

Proposition 2.1. The Crank–Nicholson technique described in the previous section to address the solvability of the hyperbolic Liouville model with fractional VOD, is stable.
Before beginning to prove this proposition, we have make following observations:

**Remark 2.1.**
- $1 > 1 - \sigma_i^{k,j+1} \geq 0$, for all $k$, $j > 0$
- $1 > Q_{i,j}^{k,j+1} \geq 0$, for all $k = 1, 2, \cdots, M - 1$
- $\sum_{i=0}^{j-1} Q_{i,j+1}^{k,j+1} = 1 - \varphi_{i,j}^{k,j+1} + 1$, for $k = 1, 2, \cdots, M - 1$
- $1 > \varphi_{i,j}^{k,j+1} \geq 0$
- The coefficients $E^{k,j+1}$, $F^{k,j}$ are non-negative for all $k = 1, 2, \cdots, M - 1$.

Hence, the proof can follow.

**Proof.** Let $U^{k,j} = U(x_k, t_j)$ denote the approximated solution evaluated at the point $(x_k, t_j)$, $k = 1, 2, \cdots, M - 1$, $j = 1, 2, \cdots, N$. Define $\nu^{k,j} = u^{k,j} - U^{k,j}$ and $\nu^j = [\nu^{1,j}, \nu^{2,j}, \cdots, \nu^{M-1,j}]^T$ assumed to verify the equality

$$
\nu^j(x) = \begin{cases} 
\nu^{k,j} & \text{for } x_k - \frac{s}{2} < x < x_k + \frac{s}{2}, \\
0 & \text{for } L - \frac{s}{2} < x \leq L.
\end{cases}
$$

(23)

Therefore, the quantity $\nu^j(x)$ is an expansible function into a Fourier series that leads to

$$
\nu^j(x) = \sum_{r=-\infty}^{+\infty} \delta_j(r) e^{2\pi r j/L}
$$

(24)

with $\delta$ representing the Dirac Delta function and the underlined sign ($\underline{i}$) is the complex number ($\underline{i}$)$^2 = -1$ not to be mixed up with the subscript ($i$).

$$
\delta_j(x) = (1/L) \int_0^L \lambda^j(x) e^{2\pi r x/L} dr
$$

(25)

It is possible to show thanks to the references [6, 25] that

$$
\| \lambda^2 \|^2 = \sum_{r=-\infty}^{+\infty} \| \delta_j(r) \|^2.
$$

(26)

Hence, the error being made during the application of the Crank–Nicholson scheme to the hyperbolic Liouville model with fractional VOD defined here above satisfies the following equation

$$
\begin{aligned}
\nu^{k,j+1} \left[ 1 + 2E^{k,j+1} - F^{k,j} Y(1) \right] &= \nu^{k,j} \left[ 1 - 2E^{k,j+1} + F^{k,j} Y(1) \right] \\
- \sum_{i=1}^{j} [\nu^{k,j+1-i} - \nu^{k,j-i}] \nu^{k,j} \\
+ E^{k,j+1} \left( \nu^{k+1,j+1} + \nu^{k-1,j+1} + \nu^{k+1,j} + \nu^{k-1,j} \right) + F^{k,j} + \\
\sum_{q=2}^{\infty} Y(q) (1 - \varphi^{k,j+1}_j) \left( \nu^{k,j+1} + \nu^{k,j} \right)^q
\end{aligned}
$$

(27)

Assuming that $\nu^{k,j}$ takes the form of the Dirac Delta-exponential in the equation (27) as

$$
\nu^{k,j} = \delta_j e^{\eta kl}
$$

(28)

where $\eta$ is a real number representing spatial wave number. Substituting (28) into (27) leads to the following iterative equations:
For \( j = 0 \),
\[
\delta_1 \left[ 1 + 16 \sin^2 \left( \frac{\eta s}{2} \right) E^{k,1} \right.
+ 4 \left( \partial_i^{k,0} \nu^{k,0} + \nu_i^{k,1} \omega_{0k} + \sum_{q=2}^{\infty} Y(q)(1 - \vartheta_j^{k,j+1}) \left( \nu^{k,1} + \nu_i^{k,0} \nu^{k,0} \right)^q \right) \sin^2 \left( \frac{\eta s}{2} \right) \right] 
= \delta_0 \left[ 1 - 16 \sin^2 \left( \frac{\eta s}{2} \right) E^{k,1} \right.
- 4 \left( \partial_i^{k,0} \nu^{k,0} + \nu_i^{k,1} \omega_{0k} + \sum_{q=2}^{\infty} Y(q)(1 - \vartheta_j^{k,j+1}) \left( \nu^{k,1} + \nu_i^{k,0} \nu^{k,0} \right)^q \right) \sin^2 \left( \frac{\eta s}{2} \right) \right]
\]

where \( \omega_{0k} = \omega_0(x_k) \) and for \( j > 0 \),
\[
\delta_{j+1} \left[ 1 + 16 \sin^2 \left( \frac{\eta s}{2} \right) E^{k,j+1} + 4 \left( \sum_{i=1}^{j} \left( \nu^{k,j+1-i} - \nu^{k,j-i} \right) \partial_i^{k,j+1} \right) \sin^2 \left( \frac{\eta s}{2} \right) \right]
+ 4 \left( \sum_{q=2}^{\infty} Y(q)(1 - \vartheta_j^{k,j+1}) \left( \nu^{k,j+1} + \nu^{k,j} \right)^q \right) \sin^2 \left( \frac{\eta s}{2} \right) \right]
= \delta_j \left[ 1 + 16 \sin^2 \left( \frac{\eta s}{2} \right) E^{k,j+1} + 4 \left( \sum_{i=1}^{j} \left( \nu^{k,j+1-i} - \nu^{k,j-i} \right) \vartheta_j^{k,j} \right) \sin^2 \left( \frac{\eta s}{2} \right) \right]
- 4 \left( \sum_{q=2}^{\infty} Y(q)(1 - \vartheta_j^{k,j+1}) \left( \nu^{k,j+1} + \nu^{k,j} \right)^q \right) \sin^2 \left( \frac{\eta s}{2} \right) \right] 
- \sum_{i=0}^{j-1} \vartheta_j^{k,j+1} \delta_{j-i} + \vartheta_j^{k,j+1} \delta_0.
\]

By means of Delta sequence limit \( \lim_{j \to \infty} \delta_j(x) = \delta(x), \ x \in \mathbb{R} \), the equation \( (30) \) becomes
\[
\delta_{j+1} \left[ 1 + 16 \sin^2 \left( \frac{\eta s}{2} \right) E^{k,j+1} + 4 \left( \sum_{q=2}^{\infty} Y(q)(1 - \vartheta_j^{k,j+1}) \left( \nu^{k,j+1} + \nu^{k,j} \right)^q \right) \sin^2 \left( \frac{\eta s}{2} \right) \right]
= \delta_j \left[ 1 + 16 \sin^2 \left( \frac{\eta s}{2} \right) E^{k,j+1} - 4 \left( \sum_{q=2}^{\infty} Y(q)(1 - \vartheta_j^{k,j+1}) \left( \nu^{k,j+1} + \nu^{k,j} \right)^q \right) \sin^2 \left( \frac{\eta s}{2} \right) \right]
- \sum_{i=0}^{j-1} \vartheta_j^{k,j+1} \delta_{j-i} + \vartheta_j^{k,j+1} \delta_0.
\]

The two equations \( (29) \) and \( (31) \) are respectively equivalent to
\[
\delta_1 = \delta_0
\]
\[
\left[ 1 - 16 \sin^2 \left( \frac{\eta s}{2} \right) E^{k,1} + 4 \left( \sum_{i=1}^{2} \left( \nu^{k,j+1-i} - \nu^{k,j-i} \right) \vartheta_j^{k,j+1} \right) \sin^2 \left( \frac{\eta s}{2} \right) \right]
+ \left[ 1 + 16 \sin^2 \left( \frac{\eta s}{2} \right) E^{k,1} + 4 \left( \sum_{i=1}^{2} \left( \nu^{k,j+1-i} - \nu^{k,j-i} \right) \vartheta_j^{k,j+1} \right) \sin^2 \left( \frac{\eta s}{2} \right) \right]
\]

and
\[
\delta_{j+1}
= \delta_j \left[ 1 - 16 \sin^2 \left( \frac{\eta s}{2} \right) E^{k,j+1} - 4 \left( \sum_{i=1}^{2} \left( \nu^{k,j+1-i} - \nu^{k,j-i} \right) \vartheta_j^{k,j+1} \right) \sin^2 \left( \frac{\eta s}{2} \right) \right]
+ \left[ 1 + 16 \sin^2 \left( \frac{\eta s}{2} \right) E^{k,j+1} + 4 \left( \sum_{i=1}^{2} \left( \nu^{k,j+1-i} - \nu^{k,j-i} \right) \vartheta_j^{k,j+1} \right) \sin^2 \left( \frac{\eta s}{2} \right) \right]
\]
where the triangle inequality was used. The iterative assumption leads to

\[
\sum_{j=0}^{j-1} \theta_j^{k,j+1} \delta_{j-1} + \theta_j^{k,j+1} \delta_0
\]

\[
1 + 16 \sin^2 \left( \frac{\eta_q}{2} \right) \left( \frac{\nu}{4} \right)^2 E^{k,j+1} + 4 \left( \sum_{q=2}^{\infty} Y(q) \left( 1 - \theta_j^{k,j+1} \right) \left( \nu^{k,j+1} + \nu^{k,j} \right)^q \right) \sin^2 \left( \frac{\eta_q}{2} \right)
\]

to complete this stability proof, we show in the following lines, that the stability condition

\[
|\delta_j| \leq |\delta_0|
\]

holds for both equations (32) and (33). Thus, we proceed by mathematical induction on \( j \).

For \( j = 1 \), then by the Remark 2.1 it is clear that

\[
|\delta_1| = |\delta_0| \times
\]

\[
1 - 16 \sin^2 \left( \frac{\eta_q}{2} \right) \left( \frac{\nu}{4} \right)^2 E^{k,1} + 4 \left( \sum_{q=2}^{\infty} Y(q) \left( 1 - \theta_q^{k,1} \right) \left( \nu^{k,1} + \nu^{k,0} \right)^q \right) \sin^2 \left( \frac{\eta_q}{2} \right)
\]

\[
\leq |\delta_0| \times
\]

\[
1 - 16 \sin^2 \left( \frac{\eta_q}{2} \right) \left( \frac{\nu}{4} \right)^2 E^{k,1} + 4 \left( \sum_{q=2}^{\infty} Y(q) \left( 1 - \theta_q^{k,1} \right) \left( \nu^{k,1} + \nu^{k,0} \right)^q \right) \sin^2 \left( \frac{\eta_q}{2} \right)
\]

\[
= |\delta_0|,
\]

which proves the desired stability condition. We can now assume that the condition holds for any \( p = 2, 3, \ldots, j \). Whence,

\[
|\delta_{j+1}|
\]

\[
= |\delta_j| - 16 \sin^2 \left( \frac{\eta_q}{2} \right) \left( \frac{\nu}{4} \right)^2 E^{k,j+1} - 4 \left( \sum_{q=2}^{\infty} Y(q) \left( 1 - \theta_q^{k,j+1} \right) \left( \nu^{k,j+1} + \nu^{k,j} \right)^q \right) \sin^2 \left( \frac{\eta_q}{2} \right)
\]

\[
1 + 16 \sin^2 \left( \frac{\eta_q}{2} \right) \left( \frac{\nu}{4} \right)^2 E^{k,j+1} + 4 \left( \sum_{q=2}^{\infty} Y(q) \left( 1 - \theta_q^{k,j+1} \right) \left( \nu^{k,j+1} + \nu^{k,j} \right)^q \right) \sin^2 \left( \frac{\eta_q}{2} \right)
\]

\[
\leq |\delta_j| - 16 \sin^2 \left( \frac{\eta_q}{2} \right) \left( \frac{\nu}{4} \right)^2 E^{k,j+1} - 4 \left( \sum_{q=2}^{\infty} Y(q) \left( 1 - \theta_q^{k,j+1} \right) \left( \nu^{k,j+1} + \nu^{k,j} \right)^q \right) \sin^2 \left( \frac{\eta_q}{2} \right)
\]

\[
1 + 16 \sin^2 \left( \frac{\eta_q}{2} \right) \left( \frac{\nu}{4} \right)^2 E^{k,j+1} + 4 \left( \sum_{q=2}^{\infty} Y(q) \left( 1 - \theta_q^{k,j+1} \right) \left( \nu^{k,j+1} + \nu^{k,j} \right)^q \right) \sin^2 \left( \frac{\eta_q}{2} \right)
\]

\[
+ \sum_{j=0}^{j-1} |\delta_{j+1}| + |\theta_j^{k,j+1}| |\delta_0|
\]

\[
(35)
\]

where the triangle inequality was used. The iterative assumption leads to

\[
|\delta_{j+1}|
\]

\[
\leq |\delta_0| - 16 \sin^2 \left( \frac{\eta_q}{2} \right) \left( \frac{\nu}{4} \right)^2 E^{k,j+1} - 4 \left( \sum_{q=2}^{\infty} Y(q) \left( 1 - \theta_q^{k,j+1} \right) \left( \nu^{k,j+1} + \nu^{k,j} \right)^q \right) \sin^2 \left( \frac{\eta_q}{2} \right)
\]

\[
1 + 16 \sin^2 \left( \frac{\eta_q}{2} \right) \left( \frac{\nu}{4} \right)^2 E^{k,j+1} + 4 \left( \sum_{q=2}^{\infty} Y(q) \left( 1 - \theta_q^{k,j+1} \right) \left( \nu^{k,j+1} + \nu^{k,j} \right)^q \right) \sin^2 \left( \frac{\eta_q}{2} \right)
\]

\[
+ \sum_{j=0}^{j-1} |\delta_{j+1}| + |\theta_j^{k,j+1}| |\delta_0|
\]

\[
(36)
\]

where the triangle inequality was used. The iterative assumption leads to

\[
|\delta_{j+1}|
\]

\[
\leq |\delta_0| - 16 \sin^2 \left( \frac{\eta_q}{2} \right) \left( \frac{\nu}{4} \right)^2 E^{k,j+1} - 4 \left( \sum_{q=2}^{\infty} Y(q) \left( 1 - \theta_q^{k,j+1} \right) \left( \nu^{k,j+1} + \nu^{k,j} \right)^q \right) \sin^2 \left( \frac{\eta_q}{2} \right)
\]

\[
1 + 16 \sin^2 \left( \frac{\eta_q}{2} \right) \left( \frac{\nu}{4} \right)^2 E^{k,j+1} + 4 \left( \sum_{q=2}^{\infty} Y(q) \left( 1 - \theta_q^{k,j+1} \right) \left( \nu^{k,j+1} + \nu^{k,j} \right)^q \right) \sin^2 \left( \frac{\eta_q}{2} \right)
\]

\[
+ \sum_{j=0}^{j-1} |\delta_{j+1}| + |\theta_j^{k,j+1}| |\delta_0|
\]

\[
(37)
\]

where the triangle inequality was used. The iterative assumption leads to

\[
|\delta_{j+1}|
\]

\[
\leq |\delta_0| - 16 \sin^2 \left( \frac{\eta_q}{2} \right) \left( \frac{\nu}{4} \right)^2 E^{k,j+1} - 4 \left( \sum_{q=2}^{\infty} Y(q) \left( 1 - \theta_q^{k,j+1} \right) \left( \nu^{k,j+1} + \nu^{k,j} \right)^q \right) \sin^2 \left( \frac{\eta_q}{2} \right)
\]

\[
1 + 16 \sin^2 \left( \frac{\eta_q}{2} \right) \left( \frac{\nu}{4} \right)^2 E^{k,j+1} + 4 \left( \sum_{q=2}^{\infty} Y(q) \left( 1 - \theta_q^{k,j+1} \right) \left( \nu^{k,j+1} + \nu^{k,j} \right)^q \right) \sin^2 \left( \frac{\eta_q}{2} \right)
\]

\[
+ \sum_{j=0}^{j-1} |\delta_{j+1}| + |\theta_j^{k,j+1}| |\delta_0|
\]

\[
(37)
\]

where the triangle inequality was used. The iterative assumption leads to

\[
|\delta_{j+1}|
\]

\[
\leq |\delta_0|
\]

\[
(33)
\]

where the triangle inequality was used. The iterative assumption leads to

\[
|\delta_{j+1}|
\]

\[
\leq |\delta_0|
\]

\[
(33)
\]

where the triangle inequality was used. The iterative assumption leads to

\[
|\delta_{j+1}|
\]

\[
(33)
\]
and this proves the stability of the Crank–Nicholson scheme applied to our hyperbolic Liouville model with fractional VOD given here above.

The convergence of the scheme follows.

2.2. Convergence analysis of the Crank–Nicholson scheme applied to our hyperbolic Liouville model with fractional VOD. In order to successfully prove the convergence of the scheme, it is necessary to make observation that there exist constants $K_1, K_2, K_3, K_4, K_5 > 0$ that can be used to transform the models (12) to (17) into the following expressions

$$\frac{\partial u}{\partial x} + sK_1 = \frac{1}{4}\left( [u(x_{k+1}, t_{j+1}) - u(x_{k-1}, t_{j+1})] + [u(x_{k+1}, t_j) - u(x_{k-1}, t_j)] \right)$$

$$\frac{\partial^2 u}{\partial x^2} + s^2K_2 = \left( \frac{u(x_{k+1}, t_{j+1}) - 2u(x_k, t_{j+1}) + u(x_{k-1}, t_{j+1})}{s^2} \right) + \frac{1}{2}\left( \frac{u(x_{k+1}, t_j) - 2u(x_k, t_j) + u(x_{k-1}, t_j)}{s^2} \right)$$

$$\frac{\partial u}{\partial t} + \tau K_3 = \frac{1}{4}\left( [u(x_{k+1}, t_{j+1}) - u(x_{k-1}, t_{j+1})] + [u(x_{k+1}, t_j) - u(x_{k-1}, t_j)] \right) \frac{1}{\tau}$$

$$\frac{\partial^2 u}{\partial t^2} + \tau^2K_4 = \left( \frac{u(x_{k+1}, t_{j+1}) - 2u(x_k, t_{j+1}) + u(x_{k-1}, t_{j+1})}{\tau^2} \right) + \frac{1}{2}\left( \frac{u(x_k, t_{j+1}) - 2u(x_k, t_j) + u(x_k, t_{j-1})}{\tau^2} \right)$$

$$\frac{\partial^{\sigma_k+1} u(x_k, t_{j+1})}{\partial t^{\sigma_k+1}} + \tau K_5 = \frac{\tau^{-\sigma_k+1}}{\Gamma(2-\sigma_k+1)}$$

$$\times \left( u(x_k, t_{j+1}) - u(x_k, t_j) + \sum_{i=1}^{j} [u(x_k, t_{j+1-i}) - u(x_k, t_{j-i})][(i+1)^{1-\sigma_k+1} - (i)^{1-\sigma_k+1}] \right)$$

Therefore, we take $u(x_k, t_j), \ k = 1, 2, \cdots, M, \ j = 1, 2, \cdots, N - 1,$ to be the exact solution of our hyperbolic Liouville problem with fractional VOD at the point.
\((x_k, t_j)\). Take \(S^{k, j} = u(x_k, t_j) - u^k, j\) and \(S^j = [S^{0, j}, S^{1, j}, S^{2, j}, \ldots, S^{M-1, j}]^T\). Therefore, (27) becomes, for \(j = 0,\)
\[
S^{k, 1} \left[ 1 + 2E^{k, 1} - F^{k, 0} Y(1) \varpi_{0k} \right] - \left( S^{k+1, 1} - S^{k-1, 1} \right) E^{k, 1}
+ \sum_{q=2}^{\infty} Y(q)(1 - \vartheta^{k+1}) \left( S^{k, 1} + S^{k, 0} \right)^q = \Lambda^{k, 1}
\]
(44)
where \(\varpi_{0k} = \varpi_0(x_k)\) and for \(j > 0,\)
\[
S^{k, j+1} \left[ 1 + 2E^{k, j+1} - F^{k, j} Y(1) \right] - \left( S^{k+1, j+1} - S^{k-1, j+1} \right) E^{k, j+1}
+ \sum_{q=2}^{\infty} Y(q)(1 - \vartheta^{k+1}) \left( S^{k, j+1} + S^{k, j} \right)^q - S^{k, j} F^{k, j} Y(1) = \Lambda^{k, j+1}
\]
(45)
Here the expression of the term \(\Lambda^{k, j+1}\) that is used is given as
\[
\Lambda^{k, j+1} = u(x_k, t_{j+1}) - u(x_k, t_j)
+ \frac{\tau^{-\sigma^{j+1}}}{\Gamma(2 - \sigma^{j+1})} \left( \sum_{i=1}^{j} [u(x_k, t_{j+1-i}) - u(x_k, t_{j-i})][(i+1)^{1-\sigma^{j+1}} - (i)^{1-\sigma^{j+1}}] \right)
- \frac{a^2}{2s^2} (u(x_{k+1}, t_{j+1}) - 2u(x_k, t_{j+1}) + u(x_{k-1}, t_{j+1}) + u(x_{k+1}, t_j) - 2u(x_k, t_j) + u(x_{k-1}, t_j))
- b \sum_{q=0}^{\infty} \frac{q^q}{q!} (u(x_k, t_{j+1}) + u(x_k, t_j))^q .
\]

Whence, the four equations (39), (41), (43) and (45) lead to
\[
\Lambda_i^{k, j+1} \leq C \left( s^2 \tau^{\sigma^j} + 16r^{1+\sigma^{j+1}} \right) ,
\]
(46)
where \(C > 0\) is constant. We can interpret this inequality by saying that \(\Lambda_i^{k, j+1}\) grows no faster than \(s^2 \tau^{\sigma^j} + 16r^{1+\sigma^{j+1}}\) does. Any reader interested in knowing more details on the Caputo related error analysis for the present scheme might consult the papers [12, 25] and the references therein. This allows us to make the following statement:

**Proposition 2.2.** Considering the Crank-Nicholson scheme applied here above to solve the hyperbolic Liouville model with fractional VOD (8) to (10), the following convergence condition holds:
\[
\| S^{j+1} \|_{\infty} \leq C \left( s^2 \tau^{\sigma^j} + 16r^{1+\sigma^{j+1}} \right)^{-1} \left( \vartheta^{k, j+1} \right)^{-1}
\]

with \(j = 0, 1, \ldots, N-1\) and \(\| S^j \|_{\infty} = \max_{1 \leq k \leq M-1} | S^{k, j} | \) and \(C > 0\) is a constant. Furthermore, we have
\[
\sigma^{j+1} = \begin{cases} 
\max_{1 \leq k \leq M-1} \sigma^j_k, & \text{if } \tau > 1 \\
\min_{1 \leq k \leq M-1} \sigma^j_k, & \text{if } \tau < 1 
\end{cases}
\]
(47)

**Proof.** We proceed again, as done in the previous section, by mathematical induction on \(j\).
Then, when \( j = 0 \), and referring to the equation (46) and the Remark 2.1, we have

\[
|S^{k,1}| 
\leq |S^{k,1}| \left[ 1 + 2E^{k,1} - F_{k,j}Y(1) \right] - \left( S^{k+1,1,j+1} - S^{k-1,j+1} \right) F_{k,j+1} \\
+ \sum_{q=2}^{\infty} Y(q)(1 - \vartheta^{k,j+1}_q) \left( S^{k,j+1} + S^{k,0} \right)^q \\
= |\Lambda^{k,1}| \\
\leq C \left( s^2 \tau^{\sigma^k} + 16 \tau^{1+\sigma^k} \right) \left( \vartheta^{k,1}_i \right)^{-1}
\]

and the desired convergence condition holds. We can now assume that the same convergence condition holds for any \( p = 2,3,\cdots,j \), meaning that the inequality

\[
\|S^{p+1}\|_\infty \leq C \left( s^2 \tau^{\sigma^k} + 16 \tau^{1+\sigma^k} \right) \left( \vartheta^{k,p+1}_i \right)^{-1}
\]

holds.

Whence, triangle inequality implies

\[
|S^{k,j+1}| \leq \left| S^{k,j+1} \left[ 1 + 2E^{k,j+1} - F_{k,j}Y(1) \right] - \left( S^{k+1,j+1} - S^{k-1,j+1} \right) F_{k,j+1} \\
+ \sum_{q=2}^{\infty} Y(q)(1 - \vartheta^{k,j+1}_q) \left( S^{k,j+1} + S^{k,0} \right)^q \right| \\
\leq |S^{k,j+1}| \left[ 1 + 2E^{k,j+1} - F_{k,j}Y(1) \right] + \left( |S^{k+1,j+1} - S^{k-1,j+1}| \right) |E^{k,j+1}| \\
+ \sum_{q=2}^{\infty} Y(q)(1 - \vartheta^{k,j+1}_q) \left( S^{k,j+1} + S^{k,0} \right)^q - S^{k,j} F_{k,j} Y(1)
\]

Due to the fact that \( \|S^j\|_\infty = \max_{1 \leq k \leq M-1} |S^{k,j}| \) hence, (49) becomes

\[
|S^{k,j+1}| 
\leq |S^{k,j+1}| \left[ 1 + 2E^{k,j+1} - F_{k,j}Y(1) \right] + |S^{k+1,j+1}| |E^{k,j+1}| + |S^{k-1,j+1}| |E^{k,j+1}| \\
+ \|S^j\|_\infty \sum_{q=2}^{\infty} Y(q)(1 - \vartheta^{k,j+1}_q) \left( S^{k,j+1} + S^{k,0} \right)^q - S^{k,j} F_{k,j} Y(1)
\]

By the recurrence assumption and knowing that \( \sum_{i=0}^{j-1} Q^{k,j+1}_{i+1} = 1 - \vartheta^{k,j+1}_i \), for \( k = 1,2,\cdots,M-1 \) (Remark 2.1) together with (19); \( \vartheta^{k,j+1}_0 = 1 \) yield

\[
|S^{k,j+1}| 
\leq \Lambda^{k,j+1}_i + \|S^j\|_\infty \sum_{q=2}^{\infty} Y(q)(1 - \vartheta^{k,j+1}_q) \left( S^{k,j+1} + S^{k,0} \right)^q - S^{k,j} F_{k,j} Y(1) \\
\leq \tilde{C} \left( s^2 \tau^{\sigma^k} + 16 \tau^{1+\sigma^k} \right) \\
+ \|S^j\|_\infty \sum_{q=2}^{\infty} Y(q)(1 - \vartheta^{k,j+1}_q) \left( S^{k,j+1} + S^{k,0} \right)^q - S^{k,j} F_{k,j} Y(1) \\
\leq \tilde{C} M \left( s^2 \tau^{\sigma^k} + 16 \tau^{1+\sigma^k} \right) \left( Y(q) \vartheta^{k,j+1}_i + M \vartheta^{k,j+1}_i - Y(q) \vartheta^{k,j+1}_i \right) \left( \vartheta^{k,j+1}_i \right)^{-1} \\
\leq C \left( s^2 \tau^{\sigma^k} + 16 \tau^{1+\sigma^k} \right) \left( \vartheta^{k,j+1}_i \right)^{-1}
\]
2.3. Graphical approximations for the hyperbolic Liouville model with fractional VOD. Now, we can use the previous scheme to perform in this section numerical simulations for the variable order hyperbolic Liouville dynamics. The full model reads as

\[
D^{\sigma(x,t)}_t u = a^2 \frac{\partial^2 u}{\partial x^2} + be^{\beta u}, \quad 1 < \sigma(x,t) \leq 2 \quad 0 \leq x \leq L, \; t \geq 0. \tag{50}
\]

with the initial conditions

\[
u(x,0) = \varpi(x) = \varpi_0 \quad \text{and} \quad \frac{\partial}{\partial t} u(x,0) = \varpi_1 \tag{51}
\]

and boundary conditions

\[
u(0,t) = \varphi_1(t) = \varphi_2(t) = 0 \tag{52}
\]

where the unknowns keep the same definitions as in (8)–(10).

2.4. Two-dimensional (2D) plot comparison with analytical solution. We start by providing a plot in 2D showing a comparison between the analytical solution and the numerical scheme developed above. For that we fixe the variable order \( \sigma(x,t) = 2 \) (the standard Newton integer derivative case) and take \( \varpi_0 = 1/7 \). This yields the analytical solution for (8) given by

\[
\tilde{u}(x,t) = \frac{1}{\beta} \ln \left( \frac{2(a^2A^2 - B^2)}{b\beta \cosh^2(Ax + Bt + C)} \right)
\]

Choosing the parameter values \( A = 2 \), \( B = -1 \), \( \beta = 1 \), \( a = b = 1 \), \( C = 0 \), and also taking \( \tau = \frac{2}{100} \), \( s = \frac{1}{100} \) and \( N = 40 \). We obtain the representation in Fig. 2 depicting the desired comparison that seems identical point per point. The graphs are done in terms of the variable position \( x \) for some fixed time \( t \), namely \( t_1 = 0 \), \( t_2 = 0.55 \), \( t_3 = 0.9 \). The dynamics in Fig. 3 (with the same parameter values as those mentioned above) show different shapes, movements and displacements for the resulting traveling waves, and this, depending on the chosen value for the VOD. For instance, in Fig. 3 (a) and (b) delineated respectively for the variable orders \( \sigma(x,t) = 2 \cos(t^2 + x^2) \) and \( \sigma(x,t) = 2 - \cos(tx) \), the traveling waves show amplitudes larger and wavelengths shorter than those in Fig. 3 (c) and (d) delineated respectively for \( \sigma(x,t) = 1.80 \) (the constant fractional case) and \( \sigma(x,t) = 2 \). Further more each traveling wave happen to exhibit its unique compression or rarefaction process that changes from one to another depending again on the value of the variable order derivative \( \sigma(x,t) \).

2.5. Concluding remarks. Any research work that can improve the way energy is carried and propagated remain important for the scientific community around the world. The process of traveling waves is identified as one of energy carriers. Hence, the process of traveling waves described by a generalized system, the hyperbolic Liouville model with the variable order derivative (VOD) has been solved numerically by means of Crank-Nicholson method. Stability and convergence results for the scheme have been proven in details before providing some numerical approximations. It has been revealed, thanks to the simulations, that the dynamics of the whole system is strongly influenced and even modified by its variable order derivative which is a function of time and space. The figures showed various movements for the waves and different shapes of the resulting waves including the
amplitude, the wavelength. Moreover, the compression and rarefaction processes also happen to be more or less different depending on the values of the VOD. These results are great observations in the sense that the VOD of mathematical models like the hyperbolic Liouville model can become a substantial parameter and a non-constant valuable tool essential for the modulation and control of models' dynamics describing wave motion. These observations will certainly contribute efficiently to the ongoing research on the optimization of energy transfer.

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Figure 3. An example of plot illustrating the dependence of traveling waves on the VOD. It shows different shapes, movements and displacements for resulting traveling waves for the model (8)–(10). Traveling waves show amplitudes larger and wavelengths shorter in (a) and (b) compared to (c) and (d). The compression and rarefaction processes also change from one traveling wave to another as the VOD varies.
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