A FLOW APPROACH TO THE PRESCRIBED GAUSSIAN CURVATURE PROBLEM IN $\mathbb{H}^{n+1}$

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Abstract. In this paper, we study the following prescribed Gaussian curvature problem

$$K = \frac{\tilde{f}(\theta)}{\phi(\rho)^{n-2} \sqrt{\phi(\rho)^2 + |\nabla \rho|^2}},$$

a generalization of the Alexandrov problem ($\alpha = n + 1$) in hyperbolic space, where $\tilde{f}$ is a smooth positive function on $S^n$, $\rho$ is the radial function of the hypersurface, $\phi(\rho) = \sinh \rho$ and $K$ is the Gauss curvature. By a flow approach, we obtain the existence and uniqueness of solutions to the above equations when $\alpha \geq n + 1$. Our argument provides a parabolic proof in smooth category for the Alexandrov problem in $\mathbb{H}^{n+1}$. We also consider the cases $2 < \alpha \leq n + 1$ under the evenness assumption of $\tilde{f}$ and prove the existence of solutions to the above equations.

1. INTRODUCTION

The Alexandrov problem proposed by A. D. Alexandrov [1] is of great significance to the study of convex bodies. It quests for the existence of closed convex hypersurfaces with prescribed volume element of the Gaussian image in the Euclidean space. Let $\hat{M}^n$ be the boundary of some convex domains containing a neighborhood of the origin in $\mathbb{R}^{n+1}$, which can be written as a radial graph over $S^n$, i.e., $\hat{M}^n = \{R(\theta) = r(\theta)\theta | \theta \in S^n\}$ with induced metric $\hat{g}_{ij}$. Denote by $\hat{\nu}(Y) : \hat{M}^n \to S^n$ the generalized Gauss map. For smooth $\hat{M}^n$, $\hat{\nu}(Y)$ is the unit outward normal vector at $Y \in \hat{M}^n$. The Alexandrov problem is to reconstruct $\hat{M}^n$ by given integral Gaussian curvature

$$\mu(\omega) = |\nu(R(\omega))|$$

for a nonnegative completely additive function $\mu$ on the set of Borel subsets $\omega$ of $S^n$. Furthermore, if $\hat{M}^n$ is at least $C^2$, then

$$|\nu(R(\omega))| = \int_{R(\omega)} \hat{K} \, dv_{\hat{M}^n} = \int_{\omega} \hat{K} \sqrt{\det(\hat{g}_{ij})} \, d\theta_{S^n}$$

where $\hat{K}$ is the Gauss curvature of $\hat{M}^n$ and $d\theta_{S^n}$ is the standard measure on $S^n$. If $\mu$ is given by integrating a function, we write it as

$$\mu(\omega) = \int_{\omega} \tilde{f} \, d\theta_{S^n}.$$
Then the Alexandrov problem can be reduced to the following fully nonlinear partial differential equation

\[ \hat{K} = \frac{\tilde{f}}{r^{n-1}\sqrt{r^2 + |\nabla r|^2}}. \]

The existence of regular solutions to this equation was solved by Pogorelev [22] for surfaces and Oliker [20] for higher dimensional cases. For more related interesting studies of the Alexandrov problem in \( \mathbb{R}^{n+1} \), one can refer to [11, 27].

Naturally, similar prescribed Gaussian curvature problems of hypersurface \( M^n \) in \( \mathbb{H}^{n+1} \) were studied in [20, 21], where the given function defined on \( M^n \) need constrained conditions to ensure \( C^0 \) estimate. Recently, Yang [31] studied the Alexandrov problem in the hyperbolic space. He considered the hypersurface \( M^n \) in \( \mathbb{H}^{n+1} \) whose Gauss curvature measures were prescribed via a radial map. Let \( M^n \) be the boundary of some convex body in \( \mathbb{H}^{n+1} \) enclosing the origin, we can parametrize it as a graph of the radial function \( \rho(\theta) \), such that \( M^n = \{ R(\theta) = (\rho(\theta), \theta) : \rho : S^n \to \mathbb{R}^+, \theta \in S^n \} \). If \( M^n \) is at least \( C^2 \), similar to the Euclidean space, then we can define the prescribed Gaussian curvature measure problem by

\[ \int_{R(\omega)} K d\nu_{M^n} = \int_\omega \tilde{f} d\theta_{S^n}. \]

where \( \tilde{f} \) is a given positive function on \( S^n \). By the coordinate transformation, we write the both sides of the above integrals on any Borel set \( \omega \) of \( S^n \) as

\[ \int_\omega K \sqrt{\det(g_{ij})} d\theta_{S^n} = \int_\omega \tilde{f} d\theta_{S^n}. \]

Then this curvature measure problem is reduced to the following fully nonlinear PDE

\[ K = \frac{\tilde{f}(\theta)}{\phi(\rho)^{n-1}\sqrt{\phi(\rho)^2 + |\nabla \rho|^2}} \quad \text{on} \quad S^n. \quad (1.1) \]

Yang proved the existence and uniqueness of (1.1) with the condition \( \inf \tilde{f} > 1 \). In his study [31], he mentioned that whether the solution to (1.1) exists with \( \tilde{f} \) endowed with other geometric conditions is still an open question. Note that the condition \( \inf \tilde{f} > 1 \) in [31] is only used in the \( C^0 \) estimate. In the studies of prescribed curvature measure problem, \( C^0 \) estimates are difficult in most cases, but are also more geometric. In Theorem 1.2 and Corollary 1.2, we weaken the condition of \( \tilde{f} \) and prove the existence of solutions to (1.1) under the evenness assumption by deriving a delicate \( C^0 \) estimate. Besides, in this paper we consider the following more general prescribed Gaussian curvature problems which correspond to the following fully nonlinear PDE

\[ K = \frac{\tilde{f}(\theta)}{\phi(\rho)^{n-2}\sqrt{\phi(\rho)^2 + |\nabla \rho|^2}} \quad \text{on} \quad S^n. \quad (1.2) \]

Motivated by the flow studied by Li-Sheng-Wang [18] in the Euclidean space, we provide a curvature flow approach to (1.2) in hyperbolic space. Write \( f(\theta) = \tilde{f}(\theta)^{-1} \). Let \( M_0 \) be a smooth closed uniformly convex hypersurface in \( \mathbb{H}^{n+1} \) enclosing the origin. In this paper, we
study the following sort of flow

\[
\begin{aligned}
\frac{\partial}{\partial t} X(x,t) &= - \phi(\rho)^{\alpha} f(\theta) K(x,t) \nu(x,t) + V(x,t), \\
X(\cdot, 0) &= X_0(\cdot),
\end{aligned}
\]  

(1.3)

where \( \alpha \geq n + 1 \) is a constant. Here we regard \( \mathbb{H}^{n+1} \) as a warped product space. Any point \( X \in \mathbb{H}^{n+1} \) can be parametrized by \( X = (\rho, \theta) \in \mathbb{R}^+ \times S^n \). Then \( f \) is a smooth positive function defined on \( S^n \), \( \phi(\rho) = \sinh \rho \), \( K \) is the Gauss curvature of the flow hypersurface \( M_t \), \( \nu \) is the unit outward normal at \( X(x,t) \) and \( V = \sinh \rho \partial_\rho \) is a conformal Killing vector field on \( \mathbb{H}^{n+1} \). Equivalently, up to a tangential diffeomorphism the flow (1.3) can be written as follows:

\[
\partial_t X = (-\phi(\rho)^{\alpha} f(\theta) K(x,t) + u(x,t)) \nu(x,t),
\]  

(1.4)

where \( u = \langle V, \nu \rangle \) is the support function of \( M_t \). Note that the following elliptic equation

\[
\phi(\rho)^{\alpha} K = \tilde{f}(\theta) u \quad \text{on } S^n
\]  

(1.5)

(that is exactly (1.2)) remains invariant under (1.3).

In this paper, we prove the following results.

**Theorem 1.1.** Let \( M_0 \) be a smooth, closed, uniformly convex hypersurface in hyperbolic space \( \mathbb{H}^{n+1} \) enclosing the origin. Suppose that \( f \) is a smooth positive function on \( S^n \). If

(i) \( \alpha > n + 1 \) or
(ii) \( \alpha = n + 1 \) and \( f < 1 \),

then the flow (1.3) has a unique smooth uniformly convex solution \( M_t \) for all time \( t > 0 \). When \( t \to \infty \), \( M_t \) converges smoothly to the unique smooth solution of (1.2).

**Corollary 1.1.** Suppose that \( \tilde{f} \) is a smooth positive function on \( S^n \). Then there is a unique smooth uniformly convex and origin-symmetric hypersurface \( M^n \) in \( \mathbb{H}^{n+1} \), such that it satisfies (1.2) under one of the following two assumptions,

(i) \( \alpha > n + 1 \) or
(ii) \( \alpha = n + 1 \) and \( \inf \tilde{f} > 1 \).

**Remark 1.1.** Case (ii) in Corollary 1.1 was proved by Fengrui Yang in [31, Theorem 6].

**Theorem 1.2.** Let \( M_0 \) be a smooth, closed, uniformly convex and origin-symmetric hypersurface in hyperbolic space \( \mathbb{H}^{n+1} \). Assume \( \alpha = n + 1 \), and \( f \) is a smooth positive even function on \( S^n \) satisfying

\[
\int_{S^n} f(\theta)^{-1} d\theta_{S^n} > |S^n|.
\]

Then the flow (1.3) has a smooth uniformly convex solution for all time \( t > 0 \), and converges smoothly to the unique smooth even solution of (1.1).

**Corollary 1.2.** Suppose \( \tilde{f} \) is a smooth positive even function on \( S^n \) satisfying

\[
\int_{S^n} \tilde{f}(\theta) d\theta_{S^n} > |S^n|.
\]

Then there is a unique smooth uniformly convex and origin-symmetric hypersurface \( M^n \) in \( \mathbb{H}^{n+1} \), such that it satisfies (1.1).

**Remark 1.2.** In Corollary 1.2, we weaken the condition of \( \tilde{f} \) in Case (ii) of Theorem 1.2, from \( \inf \tilde{f} > 1 \) to \( \tilde{f}(\theta) d\theta_{S^n} > |S^n| \) and obtain the existence of the even Alexandrov problem in hyperbolic space.
When $2 < \alpha \leq n + 1$, we consider the following flow

$$
\begin{cases}
\frac{\partial}{\partial t}X(x, t) = -\phi(\rho)^{\alpha}f(\theta)K(x, t)\nu(x, t) + \eta(t)V(x, t), \\
X(\cdot, 0) = X_0(\cdot),
\end{cases}
\tag{1.6}
$$

where $\eta(t) = \int_{\mathbb{S}^n} \frac{K}{u} \phi^{n+1}\theta_{\mathbb{S}^n}$ and obtain the existence of the following equation

$$
\phi(\rho)^{\alpha}K = c\tilde{f}(\theta)u \quad \text{on } \mathbb{S}^n.
\tag{1.7}
$$

**Theorem 1.3.** Let $M_0$ be as in Theorem 1.2. Assume $2 < \alpha \leq n + 1$, and $f$ is a smooth positive even function on $\mathbb{S}^n$. Then the flow (1.6) has a smooth uniformly convex solution for all time $t > 0$. When $t \to \infty$, $M_t$ converges smoothly to the smooth solution of (1.7) for some positive constant $c$ in a subsequence.

**Corollary 1.3.** Suppose that $\tilde{f}$ is a smooth positive even function on $\mathbb{S}^n$. If $2 < \alpha < n + 1$, then there is a smooth uniformly convex and origin-symmetric hypersurface $M_n$ in $\mathbb{H}^{n+1}$, such that it satisfies (1.7) for some positive constant $c$.

Curvature flows in hyperbolic space have been studied extensively in recent years. In these studies, constrained flows were introduced to prove geometric inequalities, see, e.g., [3, 14, 16, 25, 28]. Convergence results for inverse curvature flows were obtained in [9, 17, 23, 24, 29]. Furthermore, volume preserving curvature flows in hyperbolic space have been studied, see for example [5, 19]. All these flows in hyperbolic space have the same limiting shape in common, i.e., they all become round. When $f \equiv 1$, case (i) in Theorem 1.1 was proved by Fang Hong in [12], in which he proved that the flow (1.3) converges smoothly to a geodesic sphere. For the first time, we introduce the function $f$ in flows (1.3), (1.6) and derive the convergence results of solutions to (1.2), (1.7), which build a bridge between curvature flows in hyperbolic space and solutions to the elliptic equation. By using the Klein model (see also in [5, 7, 30]), we project the hyperbolic flow (1.3) to the Euclidean space and obtain the projection flow (5.14). We discover the monotone function (5.16) along (5.14) and derive the asymptotic convergence result of (1.3). For $\alpha \leq n + 1$, we design the flow (1.6) and deduce the convergence result by deriving a delicate $C^0$ estimate.

This paper is organized as follows. In Section 2, we collect some properties of star-shaped hypersurfaces in hyperbolic space, derive some evolution equations of various geometric quantities along (1.3), (1.6) and show that the flows can be reduced to a scalar parabolic PDE for the radial function. In Section 3, we prove $C^0$, $C^1$ estimates when $\alpha \geq n + 1$ and show that the hypersurface preserves star-shaped along the flow (1.3). In Section 4, we obtain the uniform bound of the Gauss curvature $K$ which implies the short time existence of the flow (1.3). By using a new auxiliary function, we obtain the uniform bound of the principal curvatures of $M_t$ and establish the a priori estimates for the long time existence of (1.3). In Section 5, we study the asymptotic behaviour of the flow (1.3) by projecting $M_t$ to the Euclidean space, prove the uniqueness of the solution to (1.2) and complete the proof of Theorem 1.1. In Section 6, we complete the proof of Theorem 1.2 by deriving a delicate $C^0$ estimate. In Section 7, we study the normalized flow (1.6) under the evenness assumption when $2 < \alpha \leq n + 1$ and complete the proof of Theorem 1.3.
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2. Preliminaries

In this paper, we fix a point \( o \in \mathbb{H}^{n+1} \) and consider the polar geodesic coordinates centered at \( o \) and regard \( \mathbb{H}^{n+1} \) as a warped product space \( [0, +\infty) \times \mathbb{S}^n \) equipped with Riemannian metric

\[
g_{\mathbb{H}^{n+1}} = d\rho^2 + \phi(\rho)^2 g_{\mathbb{S}^n}
\]

where \( \phi(\rho) = \sinh \rho \) and \( g_{\mathbb{S}^n} \) is the standard metric on the unit sphere \( \mathbb{S}^n \). Denote

\[
\Phi(\rho) = \int_0^\rho \sinh s \, ds = \phi'(\rho) - 1.
\]

The conformal Killing vector field can be written as \( DV = \phi' g_{\mathbb{H}^{n+1}} \).

2.1. Hypersurfaces in hyperbolic space. Let \( M^n \) be a closed hypersurface in \( \mathbb{H}^{n+1} \) and \( \{x^1, \cdots, x^n\} \) be a local coordinate system of \( M^n \). We regard \( \nu \) as the unit outward normal vector field of \( M^n \). We denote by the induced metric \( g_{ij} = g(X_i, X_j) \) and the second fundamental form \( h_{ij} = h(X_i, X_j) \) of \( M^n \), where the second fundamental form is defined by \( h(X, Y) = \langle \nabla_X \nu, Y \rangle \) with any two tangent vector fields \( X, Y \in TM^n \). The Weingarten matrix is regarded as \( W = \{h_{ij}\} = \{h_{ik}g^{kj}\} \), where \( \{g^{ij}\} \) is the inverse matrix of \( \{g_{ij}\} \). The principal curvature \( \kappa = (\kappa_1, \cdots, \kappa_n) \) of \( M^n \) are eigenvalues of \( W \). Let \( f(\kappa) \) be a symmetric function of the principal curvatures \( \kappa = (\kappa_1, \kappa_2, \cdots, \kappa_n) \). There exists a function \( F(W) \) defined on the Weingarten matrix, such that \( F(W) = f(\kappa) \). Since \( h_{ij} = \sum_k h_{ik} g^{kj} \), \( F \) can be viewed as a function \( \hat{F}(h_{ij}, g_{ij}) \) defined on the second fundamental form \( \{h_{ij}\} \) and the metric \( \{g_{ij}\} \).

In the subsequent article, we denote

\[
\hat{F}^{pq}(W) := \frac{\partial \hat{F}}{\partial h_{pq}}(h_{ij}, g_{ij}), \quad \hat{F}^{pq,rs}(W) := \frac{\partial^2 \hat{F}}{\partial h_{pq} \partial h_{rs}}(h_{ij}, g_{ij}).
\]

Here we collect some formulas of hypersurface in hyperbolic space (see [10, 13]).

Lemma 2.1. Let \( (M^n, g) \) be a smooth hypersurface in \( \mathbb{H}^{n+1} \). Then we have

\[
\nabla_i \Phi = \langle V, X_i \rangle, \quad \nabla_j \nabla_i \Phi = \phi' g_{ij} - uh_{ij}.
\]

The support function \( u = \langle V, \nu \rangle \) satisfies

\[
\nabla_i u = \langle V, X_k \rangle h_{ik}^1, \quad \nabla_j \nabla_i u = \langle V, \nabla h_{ij} \rangle + \phi' h_{ij} - uh_{ij} h_{kj} \]

where \( \nabla \) is the Levi-Civita connection on \( M^n \) with respect to the induced metric and \( \{X_1, \cdots, X_n\} \) is a basis of the tangent space of \( M^n \).

Then we have the first and second derivatives of the distance function \( \rho \).

Corollary 2.1.

\[
\nabla_i \rho = \frac{\langle V, X_i \rangle}{\phi}, \quad \nabla_j \nabla_i \rho = \frac{\phi'}{\phi} (g_{ij} - \nabla_j \rho \nabla_i \rho) - \frac{uh_{ij}}{\phi}. \tag{2.4}
\]

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Proof. Observe that
\[ \nabla_i \Phi = \phi \nabla_i \rho, \quad \nabla_j \nabla_i \Phi = \phi \nabla_j \nabla_i \rho + \phi' \nabla_j \rho \nabla_i \rho. \]
Combining with (2.2) we get (2.4) by a direct calculation. \qed

2.2. Evolution equations. For convenience, we consider the following flow
\[ \frac{\partial}{\partial t} X(x, t) = -\Theta \nu(x, t) + \tilde{\eta}(t)V \] (2.5)
where \( \Theta = \phi(\rho)^{\alpha} f(\theta) K \) and the global term \( \tilde{\eta}(t) \) is a function of time \( t \). For \( \tilde{\eta}(t) \equiv 1 \), (2.5) is the flow (1.3); for \( \tilde{\eta}(t) = \eta(t) \), (2.5) is the flow (1.6).

Lemma 2.2. Along the flow (2.5), we have the following evolution equations (also see [12, 29]). The induced metric evolves by
\[ \frac{\partial}{\partial t} g_{ij} = -2\Theta h_{ij} + 2\phi' \tilde{\eta}(t) g_{ij}. \] (2.6)
The support function evolves by
\[ \frac{\partial}{\partial t} u = -\phi' \Theta + \phi' \tilde{\eta}(t) u + \langle V, \nabla \Theta \rangle. \] (2.7)
The second fundamental form evolves by
\[ \frac{\partial}{\partial t} h_{ij} = \nabla_i \nabla_j \Theta + \Theta h_{k} h_{j} - \tilde{\eta}(t) \phi' h_{ij} + (\tilde{\eta}(t) u - \Theta) \delta_{ij} \] (2.8)
where \( \nabla \) is the Levi-Civita connection of the induced metric on \( M_t \).

Proof. By a direct calculation, we have
\[
\frac{\partial}{\partial t} g_{ij} = \partial_t \langle \partial_i X, \partial_j X \rangle \\
= \langle D_i (-\Theta \nu + \tilde{\eta}(t)V), \partial_j X \rangle + \langle \partial_i X, D_j (-\Theta \nu + \tilde{\eta}(t)V) \rangle \\
= -\Theta \langle \langle D_i \nu, \partial_j X \rangle + \langle \partial_i X, D_j \nu \rangle \rangle + 2\tilde{\eta}(t) \phi' g_{ij} \\
= -2\Theta h_{ij} + 2\tilde{\eta}(t) \phi' g_{ij}.
\]
Since \( \partial_t \nu \) is tangential,
\[
\frac{\partial}{\partial t} \nu = \langle \partial_t \nu, \partial_j X \rangle g^{\ell} \partial_\ell X \\
= -\langle \nu, \partial_j (-\Theta \nu + \tilde{\eta}(t)V) \rangle g^{\ell} \partial_\ell X \\
= \partial_j \Theta g^{\ell} \partial_\ell X = \nabla \Theta.
\] (2.9)
Using (2.9), we obtain the evolution of the support function \( u \) as follows:
\[
\frac{\partial}{\partial t} u = \partial_t \langle V, \nu \rangle = \langle -\phi' \Theta \nu + \phi' \tilde{\eta}(t)V, \nu \rangle + \langle V, \nabla \Theta \rangle \\
= -\phi' \Theta + \phi' \tilde{\eta}(t) u + \langle V, \nabla \Theta \rangle.
\]
Now we calculate the evolution of \( h_{ij} \)
\[
\frac{\partial}{\partial t} h_{ij} = -\partial_t \langle D_{\partial_i X} \partial_j X, \nu \rangle \\
= -\langle D_{\partial_i X} D_{\partial_j X} (-\Theta \nu + \tilde{\eta}(t)V), \nu \rangle - R^{p=n+1} (\partial_i X, \partial_j X, \partial_j X, \nu) - \langle D_{\partial_i X} \partial_j X, \nabla \Theta \rangle
\]
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\[
= \partial_i \partial_j \Theta - \Theta (h^2)_{ij} + \bar{\eta}(t) \phi' h_{ij} + (\bar{\eta}(t) u - \Theta) g_{ij} - \langle \nabla_{\partial_i X} \partial_j X, \nabla \Theta \rangle
\]

\[
= \nabla_i \nabla_j \Theta - \Theta h^k_i h_{kj} + \bar{\eta}(t) \phi' h_{ij} + (\bar{\eta}(t) u - \Theta) g_{ij}.
\]

From (2.6), we have

\[
\frac{\partial}{\partial t} g^{ij} = g^{il} (\partial_t g_{lm}) g^{mj} = 2 \Theta h_{ij} - 2 \bar{\eta}(t) \phi' g^{ij}.
\]

Thus

\[
\frac{\partial}{\partial t} h_{ij} = \partial_i h_{il} g^{lj} + h_{il} \partial_t g^{lj} = \nabla_i \nabla_j \Theta + \Theta h^k_i h_{kj} - \bar{\eta}(t) \phi' h_{ij} + (\bar{\eta}(t) u - \Theta) \delta^{ij}.
\]

\[
\tag{2.10}
\]

2.3. **Parametrization by radial graph.** For a closed star-shaped hypersurface $M^n \subset \mathbb{H}^{n+1}$, we can parametrize it as a graph of the radial function $\rho(\theta) : S^n \to \mathbb{R}$, i.e.,

\[
M^n = \{ (\rho(\theta), \theta) : \rho : S^n \to \mathbb{R}^+, \theta \in S^n \}
\]

where $\theta = (\theta^1, \ldots, \theta^n)$ is a local normal coordinate system of $S^n$ and $\rho$ is a smooth function on $S^n$. Let $f_i = \nabla_i f$, $f_{ij} = \nabla^2_{ij} f$, where $\nabla$ is the Levi-Civita connection on $S^n$ with respect to the standard metric $g_{S^n}$.

The tangent space of $M^n$ is spanned by (also see [6])

\[
X_i = \rho_i \partial_\rho + \partial_{\theta_i}
\]

and the unit outward normal vector is

\[
\nu = \frac{\partial_\rho - \rho^i \partial_{\theta_i}}{w},
\]

where we set

\[
w = \sqrt{1 + \frac{\|
abla \rho\|^2}{\phi^2}}.
\]

Then the support function and the induced metric can be expressed as

\[
u = \phi \sqrt{\frac{\phi^2}{\phi^2 + \|
abla \rho\|^2}},
\]

\[
g_{ij} = \phi^2 \delta_{ij} + \rho_i \rho_j, \quad g^{ij} = \frac{1}{\phi^2} (\delta^{ij} - \frac{\rho_i \rho_j}{\phi^2 + \|
abla \rho\|^2}).
\]

The second fundamental form is given by

\[
h_{ij} = \frac{-\phi \rho_{ij} + 2 \phi' \rho_i \rho_j + \phi^2 \phi' \delta_{ij}}{\sqrt{\phi^2 + \|
abla \rho\|^2}}
\]

and we have the Weingarten matrix

\[
h_i^j = \frac{1}{\phi^2 \sqrt{\phi^2 + \|
abla \rho\|^2}} \left( \delta^{ik} - \frac{\rho_i \rho_k}{\phi^2 + \|
abla \rho\|^2} \right) \left( -\phi \rho_{ki} + 2 \phi' \rho_k \rho_i + \phi^2 \phi' \delta_{ki} \right).
\]

\[
\tag{2.16}
\]
Similar to [18, p. 901], the flow \((2.5)\) can be written as a scalar parabolic PDE for the radial function

\[
\begin{aligned}
\frac{\partial}{\partial t} \rho(\theta, t) &= -\phi(\rho)\alpha f(\theta)wK + \tilde{\eta}(t)\phi(\theta, t), \quad \text{for } (\theta, t) \in \mathbb{S}^n \times [0, +\infty), \\
\rho(\cdot, 0) &= \rho_0(\cdot),
\end{aligned}
\tag{2.17}
\]

where \(w\) is the function defined in \((2.12)\).

3. \(C^0\) AND \(C^1\) ESTIMATES

In this section, we establish the \(C^0\) and \(C^1\) estimates of the flow \((1.3)\) for the proof of Theorem 1.1. Especially, we show that the flow hypersurface \(M_t\) preserves star-shaped along \((1.3)\).

3.1. \(C^0\) ESTIMATE. In this subsection, we will show that the radial function \(\rho\) of \((1.3)\) has uniform lower bound depending on \(\min f\) and \(\max f\) and \(\rho(\cdot, 0)\) is the radial function defined in \((1.3)\).

**Lemma 3.1.** Let \(\rho(\cdot, t)\) be a smooth, positive, uniformly convex solution to \((2.17)\) on \(\mathbb{S}^n \times [0, T)\) provided \(\tilde{\eta}(t) \equiv 1\). If \(\alpha > n + 1\), or \(\alpha = n + 1\) with \(f < 1\), then there is a positive constant \(C\) depending only on \(n\), \(\max f\), \(\inf f\) and the initial hypersurface \(M_0\), such that

\[
\frac{1}{C} \leq \rho(\cdot, t) \leq C, \quad \forall t \in [0, T).
\]

**Proof.** Fix time \(t\) and suppose that \(\rho\) attains its maximum at point \((p_0, t)\). At \((p_0, t)\), we have \(|\nabla \rho| = 0\) and \(\rho_{ij} \leq 0\). From \((2.16)\),

\[
h_i^j \geq \frac{\phi'}{\phi} \delta_i^j.
\tag{3.1}
\]

Inserting \((3.1)\) into \((2.17)\), we obtain

\[
\partial_t \rho \leq -\phi^{\alpha-n} \phi^n f + \phi = \phi(-\phi^{\alpha-n-1} \phi^n f + 1) \leq \phi(-\phi^{\alpha-1} f + 1),
\tag{3.2}
\]

where we use the fact \(\frac{\phi'}{\phi} \geq 1\). Since \(\alpha \geq n + 1\), if \(\phi \leq (\min f)^{-\frac{1}{\alpha-1}}\) for all \(t \geq 0\), then we obtain the uniform upper bound of \(\rho\). If there is some \(t_0\), such that \(\phi > (\min f)^{-\frac{1}{\alpha-1}}\), then we obtain \(\rho \leq \max_{\mathbb{S}^n} \rho(\cdot, 0)\) by \((3.2)\). Thus, \(\rho\) obtains the uniform upper bound with respect to the positive lower bound of \(f\) on \(\mathbb{S}^n\) and the initial hypersurface.

Suppose \(\rho\) attains its spatial minimum at point \((q_0, t)\). Similarly, at \((q_0, t)\), we have

\[
h_i^j \leq \frac{\phi'}{\phi} \delta_i^j
\tag{3.3}
\]

and

\[
\partial_t \rho \geq -\phi^{\alpha-n} \phi^n f + \phi = \phi(-\phi^{\alpha-n-1} \phi^n f + 1).
\tag{3.4}
\]

When \(\alpha > n + 1\), \(\phi^{\alpha-n-1} \phi^n = (\sinh \rho)^{\alpha-n-1}(\cosh \rho)^n \to 0\) as \(\rho \to 0\). When \(\alpha = n + 1\), \(\phi^{\alpha-n-1} \phi^n = (\cosh \rho)^n \to 1\) as \(\rho \to 0\). Hence if \(\phi^{\alpha-n-1} \phi^n \geq \frac{1}{\max f}\) for all \(t \geq 0\), then we obtain the uniform lower bound of \(\rho\) provided \(\alpha > n + 1\) or \(\alpha = n + 1\) with the assumption \(f < 1\). If there is some \(t_0\), such that \(\phi^{\alpha-n-1} \phi^n < \frac{1}{\max f}\), we obtain \(\rho \geq \min_{\mathbb{S}^n} \rho(\cdot, 0)\) by \((3.4)\). Thus \(\rho\) has the uniform lower bound depending on \(\min f\) and the initial hypersurface. \(\square\)
3.2. $C^1$ estimate. In this subsection, we derive a uniform upper bound of the gradient of $\rho$ by using the approach in [8, 29].

**Lemma 3.2.** Let $\rho(\cdot, t)$ be a smooth, positive, uniformly convex solution to (2.17) on $\mathbb{S}^n \times [0, T)$. Based on the results of Lemma 3.1, we have

$$|\nabla \rho| \leq C,$$

where $C$ only depends on the uniform upper and lower bounds of $\rho$.

**Proof.** Fix some time $t$ and consider the auxiliary function $Q = \log w + \beta \rho$, where we assume $\beta = -2 \tanh(\rho_{\min})$ from Lemma 3.1. At the maximal point of $Q$, we have

$$\gamma_{\rho(i)} + \beta \rho_i = 0,$$  \hspace{1cm} (3.5)

where we regard $\gamma(\rho) = \log(1 - \frac{2}{2e+1})$ and $\frac{d\rho}{d\rho} = \frac{1}{\phi}$. Then (2.12), (2.14), (2.15), (2.16) becomes

$$g_{ij} = \phi^2(\delta_{ij} + \gamma_i \gamma_j), \quad g^{ij} = \frac{1}{\phi^2}(\delta^{ij} - \gamma^i \gamma^j),$$  \hspace{1cm} (3.6)

$$w = \sqrt{1 + |\nabla \gamma|^2}, \quad h_{ij} = \frac{\phi'}{\phi} g_{ij} - \frac{\phi}{w} \gamma_{ij}.$$  \hspace{1cm} (3.7)

The Weingarten matrix turns into

$$h_{ij} = h_{ik} g^{kj} = \frac{\phi'}{\phi w} \delta_{ij} - \frac{\phi}{w} \gamma_{ij}.$$  \hspace{1cm} (3.8)

Inserting (3.8) into (3.5) and multiplying $w^2 \rho^j$ on the both sides of (3.5), we obtain

$$w \left( \frac{\phi'}{\phi w} \delta_{ij} - h_{ij} \right) g_{ij} \gamma_i \rho^j + \beta w^2 |\nabla \rho|^2 = 0.$$  \hspace{1cm} (3.9)

By a direct calculation, from (3.6) and (3.7), we have

$$g_{lj} \gamma_l = \phi \rho_j + |\nabla \rho|^2 \rho_j = \phi w^2 \rho_j.$$  \hspace{1cm} (3.10)

This together with (3.9) implies that

$$(\beta + \frac{\phi'}{\phi}) |\nabla \rho|^2 = w h_{ij} \rho_j \rho^i.$$  \hspace{1cm} (3.11)

Since $\{h_{ij}\}$ is positive-definite and $\beta + \frac{\phi'}{\phi} < 0$, we obtain $\nabla \rho = 0$ at the maximal point of $Q$. Thus $Q_{\max} \leq \beta \rho_{\min} < 0$. And

$$|\nabla \rho| \leq \sinh(\rho_{\max}) \sqrt{e^4 \rho_{\max} \tanh(\rho_{\min})} - 1.$$  \hspace{1cm} □

**Remark 3.1.** When the hypersurface is uniformly convex in hyperbolic space, the gradient estimate of $\rho$ follows from the uniform upper and lower bounds of the radial function $\rho$.

**Corollary 3.1.** Based on the results of Lemma 3.1, along the flow (1.3), the hypersurface $M_t$ preserves star-shaped and the support function $u$ satisfies

$$\frac{1}{C} \leq u \leq C, \quad \forall t \in [0, T),$$

for some constant $C > 0$, where $C$ only depends on $C_0$ estimate.
Proof. Recall (2.13),
\[ u = \frac{\phi}{\sqrt{1 + \frac{|\nabla \rho|^2}{\phi^2}}} \]

The upper and lower bounds of \( u \) follows from Lemma 3.1 and 3.2. Besides, we have
\[ \langle \partial_\rho, \nu \rangle = \frac{u}{\phi} = \frac{1}{w} = \frac{1}{\sqrt{1 + \frac{|\nabla \rho|^2}{\phi^2}}} \geq \frac{1}{C'} \]
for some \( C' > 0 \) depends on \( \max |\nabla \rho| \) and \( \min \rho \). Thus the hypersurface preserves star-shaped along the flow (2.5).

Remark 3.2. When the hypersurface is uniformly convex in hyperbolic space, the upper and lower bounds of the support function of \( u \) follow from the uniform bound of the radial function \( \rho \).

4. \( C^2 \) estimates

In this section, let us assume that we have already obtained the uniform upper and lower bounds of the radial function \( \rho \) and the global term \( \tilde{\eta}(t) \) along the flow (2.5). From Lemma 3.2 and Corollary 3.1, the uniform bound of \( \rho \) implies the upper and lower bounds of \( w \) and the support function \( u \). We shall establish the \( C^2 \) estimate under this assumption.

4.1. The bounds of \( K \). In this subsection, we show that \( K \) is bounded from above and below along (2.5).

Lemma 4.1. Along (2.5), there is a constant \( c > 0 \) depending on \( M_0, \alpha, n \) and the uniform bounds of \( \rho, \tilde{\eta}(t) \) and \( f \), such that
\[ K \geq c. \]

Proof. First, we calculate the evolution equation of \( \Theta = \phi^\alpha f K \)
\[ \frac{\partial}{\partial t} \Theta = \phi^\alpha f \partial_t K + K f \partial_t (\phi^\alpha) = \alpha \phi^{\alpha-1} \phi' \partial_t \rho f K + \phi^\alpha f \frac{\partial K}{\partial h_{ij}} \partial_t h_{ij}. \] (4.1)

Differentiating \( \langle V, V \rangle = \phi^2 \), we obtain
\[ \phi \phi' \partial_t \rho = \langle \partial_t V, V \rangle = \phi' \langle \partial_t X, V \rangle. \] (4.2)

By (2.5), we have
\[ \partial_t \rho = \frac{\langle \partial_t X, V \rangle}{\phi} = \frac{-\Theta u + \tilde{\eta}(t)\phi^2}{\phi} = \frac{-\Theta}{w} + \tilde{\eta}(t)\phi. \] (4.3)
Inserting (2.8) and (4.3) into (4.1), we get

\[
\frac{\partial}{\partial t} \Theta = -\alpha \phi^{\alpha-1} \frac{\phi' \Theta}{\phi} f K + \alpha \bar{\eta}(t) \phi' \Theta + \phi^\alpha f \frac{\partial K}{\partial \delta^j} \left( \nabla_i \nabla_j \Theta + \Theta h_i^k h_j^k - \phi' \bar{\eta}(t) h_i^j + (\bar{\eta}(t) u - \Theta) \delta^j \right)
\]

\[
= \phi^\alpha f K^{ij} \Theta_{ij} - \alpha \Theta^2 \frac{\phi'}{\phi} + (\alpha - n) \bar{\eta}(t) \phi' \Theta + (\bar{\eta}(t) u - \Theta) \phi^\alpha f \sigma_{n-1} + \Theta^2 H,
\]

where we use \( \frac{\partial K}{\partial h_i^j} \delta^j = \sigma_{n-1} \) and \( \frac{\partial K}{\partial h_i^j} h_i^k h_k^j = KH \) in the last equality. At the spatial minimum point of \( \Theta \) on \( M_t \), we have \( \phi^\alpha f K^{ij} \Theta_{ij} \geq 0 \). We can assume \( \Theta < \inf \bar{\eta}(t) \min_{[0, T]} \frac{n}{2} \) from Corollary 3.1 without loss of generality. Hence,

\[
\frac{d}{dt} \Theta_{\min}(t) \geq -\alpha \Theta\min \frac{\phi'}{\phi} + (\alpha - n) \bar{\eta}(t) \phi \Theta_{\min} + \inf \bar{\eta}(t) \left( u_{\min} \right) \frac{n-1}{2} \phi^\alpha f \Theta_{\min}^{n-1}
\]

\[
\geq -c_1 \Theta_{\min}^2(t) - c_2 \Theta_{\min} + c_3 \Theta_{\min}^{n-1}
\]

for some constant \( c_1, c_2, c_3 > 0 \), where all of them depend on \( \alpha, n \) and the uniform upper and lower bounds of \( \rho, \bar{\eta}(t) \) and \( f \). Hence there is a positive constant \( c_4 \) depending only on \( c_1, c_2, c_3 \), such that if \( \Theta_{\min}(t) \in (0, c_4) \), we have \( \frac{d}{dt} \Theta_{\min}(t) > 0 \). Therefore \( \Theta_{\min}(t) \geq \min \{ \Theta_{\min}(0), c_4, \inf \bar{\eta}(t) \min_{[0, T]} \frac{n}{2} \} \). Since \( \rho \) and \( f \) are bounded from above, \( K = \phi^{-\alpha} f^{-1} \Theta \) is bounded from below by some positive constant \( c \). □

**Lemma 4.2.** Along (2.5), there is a constant \( C > 0 \) depending only on \( M_0, \alpha, n \) and the uniform bounds of \( \rho, \bar{\eta}(t), \inf \| f \|_{C^1} \), such that

\[ K \leq C. \]

**Proof.** Let \( Q = \log \Theta - \log (u - a) \), where \( a = \frac{1}{2} \inf_{M \times [0, T]} u \). Recall (2.7),

\[
\frac{\partial}{\partial t} u = -\phi' \Theta + \phi' \bar{\eta}(t) u + \langle V, \nabla \Theta \rangle
\]

\[
= -\phi' (\Theta - \bar{\eta}(t) u) + \alpha \phi' \frac{\Theta}{\phi} \langle V, \nabla \rho \rangle + \frac{\langle V, \nabla f \rangle \Theta}{f} + \frac{\langle V, \nabla K, V \rangle \Theta}{K}
\]

\[
= -\phi' (\Theta - \bar{\eta}(t) u) + \alpha \phi' \| \nabla \rho \|^2 \Theta + \langle V, \nabla \log f \rangle \Theta + \langle \nabla \log K, V \rangle \Theta.
\]

Combining with (2.3), we obtain the evolution of \( u \)

\[
\partial_t u = \phi^\alpha f K^{ij} u_{ij} - \phi' ((n + 1) \Theta - \bar{\eta}(t) u) + \alpha \phi' \| \nabla \rho \|^2 \Theta + \frac{\langle V, \nabla f \rangle}{f} \Theta + u \phi^\alpha f K^{ij} h_i^k h_k^j,
\]

where we use the Codazzi equation in the equality. \( |\nabla f(X)| \) can be estimated as

\[
|\nabla f(X)| \leq C(\min \rho, \| \rho \|_{C^1(S^n)}, \| f \|_{C^1(S^n)}).
\]

At the spatial maximum point \((p_0, t)\) of \( Q \) on \( M_t \), we have

\[
\frac{\nabla \Theta}{\Theta} = \frac{\nabla u}{u - a}.
\]

(4.8)
Inserting (4.6) and (4.8) into (4.4), we obtain
\[ \partial_t Q = \frac{\partial_t \Theta}{\Theta} - \frac{\partial_t u}{u - a} \]
\[ = \frac{\dot{K}^{ij} \Theta_{ij}}{K} - \frac{\alpha \Theta \phi'}{w \phi} + \frac{(\alpha - n)\tilde{\eta}(t)\phi'}{(\alpha - n)\tilde{\eta}(t)u - \Theta} + \frac{\sigma_{n-1}}{K} + \Theta H \]
\[ - \frac{\phi^a f \dot{K}^{ij} u_{ij}}{u - a} + \phi' \left( \frac{n + 1}{u - a} \right) u_{ij} \Theta - \tilde{\eta}(t)u - \Theta \phi' \frac{\sigma_{n-1}}{K} + \Theta H \]
\[ \leq \phi^a f \dot{K}^{ij} Q_{ij} + (\alpha - n)\tilde{\eta}(t)\phi' + \phi' \left( \frac{n + 1}{u - a} \right) u_{ij} \Theta - \tilde{\eta}(t)u - \Theta \phi' \frac{\sigma_{n-1}}{K} + \Theta H \]
\[ \leq \phi^a f \dot{K}^{ij} Q_{ij} + (\alpha - n)\tilde{\eta}(t)\phi' + \phi' \left( \frac{n + 1}{u - a} \right) u_{ij} \Theta - \tilde{\eta}(t)u - \Theta \phi' \frac{\sigma_{n-1}}{K} + \Theta H \]
\[ \leq \phi^a f \dot{K}^{ij} Q_{ij} + (\alpha - n)\tilde{\eta}(t)\phi' + \phi' \left( \frac{n + 1}{u - a} \right) u_{ij} \Theta - \tilde{\eta}(t)u - \Theta \phi' \frac{\sigma_{n-1}}{K} + \Theta H \]
\[ \leq \phi^a f \dot{K}^{ij} Q_{ij} + (\alpha - n)\tilde{\eta}(t)\phi' + \phi' \left( \frac{n + 1}{u - a} \right) u_{ij} \Theta - \tilde{\eta}(t)u - \Theta \phi' \frac{\sigma_{n-1}}{K} + \Theta H \]
\[ \leq \phi^a f \dot{K}^{ij} Q_{ij} + (\alpha - n)\tilde{\eta}(t)\phi' + \phi' \left( \frac{n + 1}{u - a} \right) u_{ij} \Theta - \tilde{\eta}(t)u - \Theta \phi' \frac{\sigma_{n-1}}{K} + \Theta H \]
where we assume \( \Theta > \max \eta \max M_{x \in [0, T]} u \) from Corollary 3.1 without loss of generality. Here we also use the Newton-Maclaurin inequality \( H \geq nK^2 \) and the Cauchy-Schwartz inequality \( \langle V, \nabla f \rangle \leq \phi |\nabla f| \) to obtain the last inequality. Then we get
\[ \partial_t Q \leq C_1 + C_2 \Theta - C_3 \Theta^{n+1} \]
(4.9)
for some \( C_1, C_2, C_3 > 0 \) depending on \( \alpha, n, \min f \) and \( \|f\|_{C^1} \), the uniform bounds of \( \rho \) and \( \tilde{\eta}(t) \). Besides, there is a constant \( C_4 \) depending on the uniform upper and lower bounds of \( u \), such that
\[ \frac{1}{C_4} \leq \Theta = (u - a) e^Q \leq C_4 e^Q \]
(4.10)
We have
\[ \partial_t Q \leq C_1 + C_2 C_4 e^Q - C_3 (C_4)^{-\frac{n+1}{n}} e^{\frac{n+1}{n} Q} \]
Therefore, \( Q \leq \max \{ C_5, Q_{\max}(0), \max \eta \max M_{x \in [0, T]} u \} \) where \( C_5 \) is a positive constant depending on \( C_1, C_2, C_3 \) and \( C_4 \). Hence \( K = \phi^{-\alpha} f^{-1}(u - a) e^Q \) is bounded from above by some positive constant \( C \) depending only on \( M_0, \alpha, n \min f, \|f\|_{C^1} \) and the uniform bounds of \( \rho \) and \( \tilde{\eta}(t) \).

In the proof of Lemma 4.1 and Lemma 4.2 we also obtain the uniform bounds of \( \Theta \).

**Corollary 4.1.** Along (2.5), \( \Theta = \phi^a f K \) is uniformly bounded, i.e.,
\[ \frac{1}{C} < \Theta(x, t) < C \]
for some constant \( C > 0 \) that only depends on \( M_0, \alpha, n \min f, \|f\|_{C^1} \) and the uniform bounds of \( \rho \) and \( \tilde{\eta}(t) \).

We established the \( C^0, C^1 \) estimates in Section 3 along (1.3) for the proof of Theorem 1.1. Due to Lemma 4.1 and Lemma 4.2, we have the following Corollary.

**Corollary 4.2.** Let \( M_0 \) be as in Theorem 1.1. When \( \alpha > n+1 \) or \( \alpha = n+1 \) with \( \max f < 1 \), there is a positive constant \( C \) depending only on \( M_0, \alpha, n, \min f \) and \( \|f\|_{C^1} \) along the flow (1.3), such that
\[ \frac{1}{C} \leq K \leq C \]
(4.11)
4.2. The bound of principal curvatures. In this subsection, we shall show the principal curvatures of $M$ is uniformly bounded along the flow (2.5) under the assumption in section 4.

Lemma 4.3. Along (2.5), there is a positive constant $C$ depending on $M_0$, $\alpha$, $n$, $\min f$, $\|f\|_{C^2(S^n)}$ and the uniform bounds of $\rho$, $\bar{\eta}$, such that the principal curvatures satisfy

$$\frac{1}{C} \leq \kappa_i(\cdot, t) \leq C, \quad \forall \ t \in [0, T) \text{ and } i = 1, 2, \cdots, n.$$

**Proof.** Denote by $\lambda(x, t)$ the maximal principal radii at $X(x, t)$. Let $A$ be a positive constant to be determined later. Denote by $Q' = \log \lambda(x, t) + A \rho$. Fix an arbitrary time $T_0 \in (0, T)$. Assume that $Q'$ attains its maximum at $(x_0(t), t)$ provided $t \in [0, T_0)$. We then introduce a normal coordinate system $\{\partial_i\}$ around $(x_0(t), t)$, such that $\nabla_{\partial_i}X(x_0(t), t) = 0$ for all $i, j = 1, 2, \ldots, n$ and $h_{ij}(x_0(t), t) = \kappa_i(x_0(t), t) \delta_{ij}$. Further, we can choose $\partial_1|_{(x_0, t_0)}$ as the eigenvector with respect to $\lambda(x_0, t_0)$, i.e., $\lambda(x_0, t_0) = \tilde{h}^{11}(x_0, t_0)$. Assume that $\{\tilde{h}^{ij}\}$ is the inverse matrix of $\{h_{ij}\}$. Clearly

$$\lambda(x, t) = \max \{\tilde{h}^{ij}(x, t) \xi_i \xi_j | g^{ij}(x, t) \xi_i \xi_j = 1\} \tag{4.12}$$

For the continuity, using this coordinate system we consider the auxiliary equation $Q = \log v + A \rho$, where $v = \frac{\tilde{h}^{11}}{\tilde{g}^{11}}$. Note that $v(x, t) \leq \lambda(x, t)$ from (4.12) and $v(x_0, t_0) = \lambda(x_0, t_0)$. Thus, $Q(x, t) \leq Q(x_0, t_0)$ for all $t \in [0, T_0)$. Now we can calculate the derivatives of $v$ at $(x_0, t_0)$ as follows:

$$\partial_i v = -(\tilde{h}^{11})^2 \partial_i h_{11} + \tilde{h}^{11} \partial_i g_{11} = -\left(\tilde{h}^{11}\right)^2 \partial_i h_{11}, \tag{4.13}$$

$$\nabla_i v = \frac{\tilde{h}_q}{\tilde{g}_{pq}} \nabla_q h_p = \tilde{h}_1 \tilde{h}^{1q} \nabla_q h_p,$$ \hspace{1cm} \(\nabla_i v(x_0, t_0) = -(\tilde{h}^{11})^2 \nabla_i h_{11}. \tag{4.14}$$

Then

$$\nabla_j \nabla_i v = \nabla_j (-\tilde{h}_1 \tilde{h}^{1q} \nabla_q h_p) = -\nabla_j \tilde{h}_1 \tilde{h}^{1q} \nabla_q h_p - \tilde{h}_1 \nabla_j \tilde{h}^{1q} \nabla_q h_p - \tilde{h}_1 \tilde{h}^{1q} \nabla_j \nabla_q h_p = \tilde{h}_1 \tilde{h}^{1q} \nabla_j \nabla_q h_p + \tilde{h}_1 \tilde{h}^{1q} \nabla_j \nabla_q h_p - \tilde{h}_1 \tilde{h}^{1q} \nabla_j \nabla_q h_p$$

and

$$\nabla_j \nabla_i v(x_0, t_0) = -\left(\tilde{h}^{11}\right)^2 \nabla_j \nabla_i h_{11} + 2(\tilde{h}^{11})^2 \tilde{h}^{pp} \nabla_p h_p \nabla_i h_{1p}. \tag{4.15}$$

Here we denote by $\{\tilde{h}^{ij}\}$ the inverse matrix of $\{h^{ij}\}$, i.e., $\{\tilde{h}^{ij}\} = \{\tilde{h}^{ij} g_{kj}\}$. Then We calculate the first term in (2.8)

$$\nabla_j \nabla_i \Theta = \phi^\alpha f \nabla_j \nabla_i K + K \nabla_j \nabla_i (\phi^\alpha f) + \nabla_i (\phi^\alpha f) \nabla_j K + \nabla_j (\phi^\alpha f) \nabla_i K$$

$$= \phi^\alpha f \tilde{K} h_{pq} h_{prij} + \phi^\alpha f \tilde{K} h_{pq} h_{rrij} + fK \nabla_j \nabla_i (\phi^\alpha) + K \phi^\alpha \nabla_j \nabla_i f + K \nabla_j \phi^\alpha \nabla_i f + K \phi^\alpha \nabla_j \nabla_i f$$

$$+ K \nabla_j \phi^\alpha \nabla_i f + f \nabla_i (\phi^\alpha) \nabla_j K + f \nabla_j (\phi^\alpha) \nabla_i K + \phi^\alpha \nabla_j f \nabla_i K + \phi^\alpha \nabla_i f \nabla_j K. \tag{4.16}$$
Due to the Codazzi equation and Ricci identity, we have

\[
\dot{K}^p q h_{pq ij} = \dot{K}^{pq} h_{p q ij} = \dot{K}^{pq} (h_{p q ij} + h_{ip} R_{l q j} + h_{li} R_{pq j}) \\
= \dot{K}^{pq} (h_{ij pq} + h_{tp} h_{iq} h_{ij} - h_{tp} h_{ij} h_{iq} + h_{li} h_{iq} h_{pq} - h_{ti} h_{ij} h_{pq} \\
- h_{pq} h_{ij} + h_{jp} h_{pi} - h_{qi} h_{pj} + h_{ij} h_{pq}) \\
\tag{4.17}
\]

It’s direct to calculate

\[
\dot{K}^{pq r s h_{pq i} h_{r s j}} = \frac{\partial (\dot{K}^{pq} h_{pq i})}{\partial h_{r s}} h_{pq i} h_{r s j} = \dot{K}^{pq} h_{pq i} h_{r s j} - \dot{K}^{pq} h_{pq i} h_{rs j}. \tag{4.18}
\]

Hence by (4.16) at \((x_0, t_0)\),

\[
\nabla_j \nabla_i \Theta = \phi^o f \dot{K}^{pq} h_{ij pq} + \phi^o f K H h_{ij} - n \phi^o f K K h_{ij} - n \phi^o f K \delta h_{ij} + \phi^o f \dot{K}^{pp h} h_{ij} + \phi^o f \frac{\nabla_i K \nabla_j K}{K} \\
- \phi^o f K \hat{h}_{pp} h_{pq} h_{pq i} h_{pq j} + \alpha (\alpha - 1) \phi^{o-2} h^2 f K \nabla_i \rho \nabla_j \rho + \alpha \phi^{o-1} \phi' f K \nabla_j \nabla_i \rho \\
+ \phi^o K \nabla_j \nabla_i f + \alpha \phi^{o-1} \phi' K \nabla_i \rho \nabla_j f + \alpha \phi^{o-1} \phi' K \nabla_j \rho \nabla_i f \\
+ \alpha \phi^{o-1} \phi' f \nabla_i \rho \nabla_j K + \alpha \phi^{o-1} \phi' f \nabla_j \rho \nabla_i K + \phi^o \nabla_i f \nabla_j K + \phi^o \nabla_j f \nabla_i K. \tag{4.19}
\]

Direct computation gives

\[
\nabla_j \nabla_i Q = \frac{\nabla_j \nabla_i v}{v} - \frac{\nabla_i v \nabla_j v}{v^2} + A \nabla_j \nabla_i \rho \\
= - \dot{h}^{11} \nabla_j \nabla_i h_{11} + 2 \dot{h}^{11} \hat{h}^{pp} \nabla_i h_{11 p} \nabla_j h_{11 p} - (\dot{h}^{11})^2 \nabla_i h_{11} \nabla_j h_{11} + A \nabla_j \nabla_i \rho. \tag{4.20}
\]

Recall (2.8) and (4.3). By (4.13) and (4.19), we obtain

\[
\partial_t Q = \frac{\partial h^{11} \overline{\Theta}}{h^{11}} + A \partial_t \rho \\
= - \dot{h}^{11} \left( \nabla_i \nabla_1 \Theta + \Theta h^{11} - \phi' \tilde{\eta}(t) h_{11} + (\tilde{\eta}(t) u - \Theta) \right) - A \frac{\Theta}{w} + A \tilde{\eta}(t) \phi \\
= - \dot{h}^{11} \left( \phi^o f \dot{K}^{pq} h_{11 pq} + \phi^o f K H h_{11} - n \phi^o f K (h_{11})^2 - n \phi^o f K + \phi^o f \dot{K}^{pp h} h_{11} \\
+ \frac{\phi^o f (\nabla_1 K \nabla_1 K)}{K} - \phi^o f K \hat{h}_{pp} h_{pq} (h_{pq i})^2 + \alpha (\alpha - 1) \phi^{o-2} h^2 f K + \alpha \phi^{o-1} \phi' f K \nabla_1 \nabla_1 \rho \\
+ \frac{\phi^o K \nabla_1 \nabla_1 f + 2 \alpha \phi^{o-1} \phi' K \nabla_1 \rho \nabla_1 f + 2 \alpha \phi^{o-1} \phi' f \nabla_1 \rho \nabla_1 K + 2 \phi^o \nabla_1 f \nabla_1 K}{w} \\
- \Theta h_{11} + \phi' \tilde{\eta}(t) - (\tilde{\eta}(t) u - \Theta) \dot{h}^{11} - A \frac{\Theta}{w} + A \tilde{\eta}(t) \phi. \tag{4.21}
\]
Substituting (4.20) into (4.21), we get
\[
\partial_t Q = \phi f \tilde{K}^{ij} \nabla_j \nabla_i Q - 2 \phi f K \tilde{h}^{ij} \tilde{h}^{pp} \nabla_i h_{1p} \nabla_j h_{1p} + \phi f K \tilde{h}^{ij} (\tilde{h}^{11})^2 \nabla_i h_{11} \nabla_j h_{11}
- A \phi f \tilde{K}^{ij} \nabla_j \nabla_i \rho - \phi f K \rho + n \phi f K h_{11} + n \phi f K \tilde{h}^{11} - \phi f K \tilde{h}^{pp} - \phi f (\frac{\nabla_1 K}{K})^2 \tilde{h}^{11}
+ \phi f K \tilde{h}^{11} \tilde{h}^{pp} \tilde{h}^{qq} (h_{pq})^2 - \alpha (\alpha - 1) \phi f (\nabla_1 \rho)^2 f K \tilde{h}^{11} - \alpha \phi f [f K \tilde{h}^{11} \nabla_1 \rho f - 2 \phi f \tilde{h}^{11} \nabla_1 \rho \nabla_1 K]
- 2 \phi f \tilde{h}^{11} \nabla_1 f \nabla_1 K - \Theta h_{11} + \phi f \tilde{h}(t) - (\tilde{h}(t) u - \Theta) \tilde{h}^{11} - A \frac{\Theta}{\omega} + A \tilde{h}(t) \phi.
\] (4.22)

Dividing (4.22) by \( \Theta \) on both sides, we obtain at \((x_0, t_0)\)
\[
\frac{\partial_t Q}{\Theta} \leq \tilde{h}^{ij} \nabla_j \nabla_i Q - A \tilde{h}^{ii} \left( \frac{\rho f}{\phi} (1 - (\nabla_1 \rho)^2) - \frac{u h_{ii}}{\phi} \right) + n h_{11} + (n + 1) \tilde{h}^{11}
- \alpha (\alpha - 1) \left( \frac{\rho f}{\phi} \right)^2 (\nabla_1 \rho)^2 \tilde{h}^{11} - \alpha \frac{\rho f}{\phi} \tilde{h}^{11} \left( \frac{\rho f}{\phi} (1 - (\nabla_1 \rho)^2) - \frac{u h_{11}}{\phi} \right)
- \frac{\nabla_1 \nabla_1 f}{f} \tilde{h}^{11}
- 2 \alpha \frac{\rho f}{\phi} \tilde{h}^{11} \nabla_1 \rho \nabla_1 f + 2 \alpha^2 \frac{\rho f}{\phi} (\nabla_1 \rho)^2 \tilde{h}^{11} + 2 (\nabla_1 \log f)^2 \tilde{h}^{11} + \frac{\rho f}{\Theta} \tilde{h}(t) + A \frac{\tilde{h}(t) \phi}{\Theta}.
\] (4.23)

where we wipe off some nonpositive terms. Recall (2.4), we have at \((x_0, t_0)\)
\[
|\nabla \rho|^2 = \frac{\sum_{i=1}^{n} (X_i, V_i)^2}{\phi^2} = \frac{|V|^2}{\phi^2} - \frac{(\nu, V)^2}{\phi^2} \leq 1 - \left( \frac{u}{\phi} \right)^2 < 1 - c_1
\] (4.24)

where \( c_1 > 0 \) depends on the lower bound of \( u \) and the upper bound of \( \rho \). Now substituting (2.4) into (4.23) and using the Cauchy-Schwartz inequality, we obtain at \((x_0, t_0)\)
\[
\frac{\partial_t Q}{\Theta} \leq \tilde{h}^{ij} \nabla_j \nabla_i Q - A \tilde{h}^{ii} \left( \frac{\rho f}{\phi} (1 - (\nabla_1 \rho)^2) - \frac{u h_{ii}}{\phi} \right) + n h_{11} + (n + 1) \tilde{h}^{11}
- \alpha (\alpha - 1) \left( \frac{\rho f}{\phi} \right)^2 (\nabla_1 \rho)^2 \tilde{h}^{11} - \alpha \frac{\rho f}{\phi} \tilde{h}^{11} \left( \frac{\rho f}{\phi} (1 - (\nabla_1 \rho)^2) - \frac{u h_{11}}{\phi} \right)
- \frac{\nabla_1 \nabla_1 f}{f} \tilde{h}^{11}
- 2 \alpha \frac{\rho f}{\phi} \tilde{h}^{11} \nabla_1 \rho \nabla_1 f + 2 \alpha^2 \frac{\rho f}{\phi} (\nabla_1 \rho)^2 \tilde{h}^{11} + 2 (\nabla_1 \log f)^2 \tilde{h}^{11} + \frac{\rho f}{\Theta} \tilde{h}(t) + A \frac{\tilde{h}(t) \phi}{\Theta}.
\] (4.25)

Using the maximum principle, by (4.24) at \((x_0, t_0)\), we have
\[
\frac{\partial_t Q}{\Theta} \leq - A c_1 \tilde{h}^{11} \frac{\rho f}{\phi} + A \left( \frac{n u}{\phi} + \frac{\tilde{h}(t) \phi}{\Theta} \right) + n h_{11} + (n + 1) \tilde{h}^{11} - \alpha (\alpha - 1) \left( \frac{\rho f}{\phi} \right)^2 (\nabla_1 \rho)^2 \tilde{h}^{11}
- \alpha \left( \frac{\rho f}{\phi} \right)^2 (1 - \rho_1^2) \tilde{h}^{11} + \alpha \frac{u \rho f}{\phi^2} - \frac{\nabla_1 \nabla_1 f}{f} \tilde{h}^{11} - 2 \alpha \frac{\rho f}{\phi} \tilde{h}^{11} \nabla_1 \rho \nabla_1 f
\] (4.26)

Write \( X \in \mathbb{H}^{n+1} \) as \( X = (\rho, \theta) \in \mathbb{R}^+ \times \mathbb{S}^n \). Then we extend \( f \) to \( (\mathbb{H}^{n+1}, g_{\mathbb{H}^{n+1}}) \) as
\[
f(X) = f(\theta).
\] (4.27)

We have by Reilly formula,
\[
D_j D_i f = \nabla_j \nabla_i f + h_{ij} D_v f.
\] (4.28)
where \( D \) is the standard Levi-Civita connection with respect to \( g_{\mathbb{H}^{n+1}} \). Then at \( (x_0, t_0) \), \( \nabla_1 \nabla_1 f \) can be further estimated as
\[
|\nabla_1 \nabla_1 f(x_0, t_0)| \leq C (\max \rho, \min \rho, \|f\|_{C^2(\mathbb{S}^n)}) (1 + h_{11}(x_0, t_0)). \tag{4.29}
\]
By (4.7) and (4.28), We obtain at \( (x_0, t_0) \)
\[
0 \leq \frac{\partial_\theta Q}{\Theta} \leq -\left( \frac{Ac_1}{2} - c_2 \right) \tilde{h}_1 - A \left( \frac{c_1 \tilde{h}_1}{2} - c_3 \right) + c_4 + c_5 \frac{1}{\tilde{h}_1} \tag{4.30}
\]
for some positive constants \( c_1, c_2, c_3, c_4, c_5 \) depends on the uniform upper and lower bounds of the \( \rho, \tilde{\eta}, u \) from Corollary 3.1, \( K \) from Lemma 4.1-4.2 and \( \alpha, n, \inf f, \|f\|_{C^2(\mathbb{S}^n)} \). We deduce from (4.30) that if we choose \( A = \frac{2\kappa}{c_1} \), \( \lambda(x_0, t_0) \) can’t be too large, i.e., \( \lambda(x_0, t_0) \) has a uniform upper bound \( C(c_i, i = 1, \cdots, 5) \) independent of time \( T_0 \). Hence we have the uniform upper bound of the principal radii of \( M_t \), which means that the principal curvatures are bounded from below by some positive constant \( c_6 \). Meanwhile, by Lemma 4.2, one can get
\[
C \geq K \geq c_6^{-1} \kappa_{\text{max}}
\]
for some constant \( C \) from Lemma 4.2. Hence, the principal curvatures are bounded from above. This completes the proof of Lemma 4.3. \( \square \)

Now we have obtained the a priori estimates of the flow (1.3). By Lemma 3.1, Lemma 3.2 and Lemma 4.1, these flows have short time existence. Using the \( C^2 \) estimate given in Lemma 4.3, due to [2, Theorem 6], we obtain the \( C^{2, \lambda} \) estimate of the scalar equation (2.17). Then the standard regularity theory implies the estimates for higher order derivatives. Hence, we obtain the long time existence and regularity for these flows.

**Theorem 4.1.** Let \( M_0 \) be as in Theorem 1.1. Assume that \( f \) is a smooth positive function on \( \mathbb{S}^n \). If \( \alpha > n + 1 \) or \( \alpha = n + 1 \) and \( f \) satisfies \( f < 1 \), then the smooth uniformly convex solution to the flow (1.3) exists for all time \( t \in [0, +\infty) \) and there is a constant \( C_{m, \lambda} > 0 \) depending on \( M_0, \alpha, n, m, \lambda, \) and \( f \) such that
\[
\|\rho\|_{C^{m, \lambda}(\mathbb{S}^n \times [0, +\infty))} \leq C_{m, \lambda}.
\]

**Remark 4.1.** If the Gauss curvature \( K \) in flow (1.3) is replaced by the \( k \)-th mean curvature \( \sigma_k \) of the hypersurface, using the similar argument we can obtain the same long time existence results as in Theorem 4.1.

In other words, there is a subsequence of time \( \{t_i\} \), such that \( \{M_{t_i}\} \) converge to a smooth positive uniformly convex hypersurface \( M_\infty \) in \( C^\infty \) topology. Next we shall see \( M_\infty \) is a solution to (1.2).

5. **Proof of Theorem 1.1**

In this section, we consider the hyperbolic space as the hyperboloid in Lorentz space \( \mathbb{R}^{n+1,1} \), where
\[
\mathbb{H}^{n+1} = \{(x_1, \cdots, x_{n+1}, x_{n+2}) \in \mathbb{R}^{n+1,1} | \sum_{i=1}^{n+1} x_i^2 - x_{n+2}^2 = -1, x_{n+2} > 0 \}.
\]
Let \( p = (0, \cdots, 0, 1) \) and \( L_p \) be the tangent plane of \( \mathbb{H}^{n+1} \) at \( p \). Denote the projection from \( \mathbb{H}^{n+1} \) to \( L_p \) as
\[
\pi_p : \mathbb{H}^{n+1} \to L_p
\]
\[
z = (x_1, \cdots, x_{n+1}, x_{n+2}) \mapsto \frac{z}{-\langle z, p \rangle} = (\frac{x_1}{x_{n+2}}, \cdots, \frac{x_{n+1}}{x_{n+2}}, 1).
\]
Since \( \sum_{i=1}^{n+1} \left( \frac{x_i}{x_{n+2}} \right)^2 < 1 \) on \( \pi_p(\mathbb{H}^{n+1}) \), \( \pi_p(\mathbb{H}^{n+1}) \) is contained in the unit ball \( B_p^{n+1}(1) \) centered at \( p \). Recall that \( \rho(q) \) is the geodesic distance between \( q \) and \( p \) in \( \mathbb{H}^{n+1} \). If we regard \( r \) as the Euclidean distance from \( \pi(q) \) to \( p \) in \( B_p^{n+1}(1) \), we have the following relation
\[
r(\theta) = \tanh \rho(\theta).
\] (5.1)

Direct computation gives
\[
\frac{1}{(\cosh \rho)^2} = 1 - r^2,
\] (5.2)
\[
\sinh \rho = \frac{r}{\sqrt{1 - r^2}},
\] (5.3)
\[
|\nabla \rho|^2 = \frac{|\nabla r|^2}{(1 - r^2)^2},
\] (5.4)
\[
w = \sqrt{1 + \frac{|\nabla \rho|^2}{\phi^2}} = \frac{r}{\hat{u}} \sqrt{\frac{1 - \hat{u}^2}{1 - r^2}},
\] (5.5)
\[
u = \frac{\hat{u}}{\sqrt{1 - \hat{u}^2}}.
\] (5.6)

Here we can see \( u > 0 \) as long as \( 1 > \hat{u} > 0 \), which means that \( \pi_p(M_t) \) remains star-shaped as long as \( M_t \) does. From (5.1), we have the scalar equation of the radial graph of \( \pi_p(M_t) \) under the flow (2.5) (\( \tilde{\eta} \equiv 1 \)),
\[
\partial_t r(\theta, t) = \partial_p(\tanh \rho) \partial_t \rho = \frac{-w\phi(\rho)^2 f(\theta) K}{(\cosh \rho)^2} + \frac{\phi}{(\cosh \rho)^2}
\] (5.7)
where we denote by \( (r(\theta), \theta) \) the radial graph of \( \pi_p(M_t) \). Similar to (2.13)-(2.15), we have in \( \hat{M}_t = \pi(M_t) \),
\[
\dot{\hat{u}} = \frac{r}{\sqrt{r^2 + |\nabla r|^2}} \left( \partial_r - \frac{\nabla r}{r^2} \right),
\] (5.8)
\[
\hat{\nu} = \frac{r^2}{\sqrt{r^2 + |\nabla r|^2}},
\] (5.9)
\[
\hat{g}_{ij} = r_i r_j + r^2 \delta_{ij}
\] (5.10)
and
\[
\hat{h}_{ij} = \frac{-rr_{ij} + 2r_ir_j + r^2\delta_{ij}}{\sqrt{r^2 + |\nabla r|^2}}. \tag{5.11}
\]
Therefore,
\[
\hat{K} = \frac{\det \hat{h}_{ij}}{\det \hat{g}_{ij}} = \frac{\det(-rr_{ij} + 2r_ir_j + r^2\delta_{ij})}{(r^2 + |\nabla r|^2)^{n+2}}. \tag{5.12}
\]
Recall (2.14) and (2.15). Substituting (5.1) in (5.12), we have (see [7])
\[
K = \hat{K} \left(\frac{(r^2 + |\nabla r|^2)}{r^2 + |\nabla r|^2}\right)^{n+2} = \hat{K} \left(\frac{1 - r^2}{1 - \hat{u}^2}\right)^{n+2}. \tag{5.13}
\]
Now we calculate the scalar parabolic PDE of the support function \( \hat{u} \) of \( \pi_p(M_t) \). Substituting (5.2), (5.3), (5.5) and (5.13) into (5.7), we have
\[
\partial_t \hat{u}(x, t) = \frac{\hat{u}}{r} \partial_r r(\theta, t) = (-\phi(\rho)^{\alpha} f(\theta) K + \phi)(1 - r^2) \frac{\hat{u}}{r} = -r^{\alpha} f(\theta) (1 - r^2)^{-\frac{n+3-\alpha}{2}} (1 - \hat{u}^2)^{-\frac{n+1}{2}} + \hat{u} \sqrt{1 - r^2}. \tag{5.14}
\]
Denote by \( \psi(r) = r^{\alpha}(1 - r^2)^{\frac{n+2-\alpha}{2}} \) and\( \varphi(\hat{u}) = \hat{u}^{-1}(1 - \hat{u}^2)^{-\frac{n+1}{2}} \) (5.14) becomes
\[
\partial_t \hat{u} = -\psi(r) \sqrt{1 - r^2} f(\theta) \varphi(\hat{u}) \hat{u} \hat{K} + \hat{u} \sqrt{1 - r^2}. \tag{5.15}
\]
Let \( \Psi = \int_a^r \psi^{-1}(s)s^n ds, \quad \Omega = \int_a^\hat{u} \varphi(s) ds \) and
\[
Q(t) = \int_{S^n} \Psi f^{-1} d\theta_{S^n} - \int_{S^n} \Omega d\sigma_{S^n}. \tag{5.16}
\]
where \( 0 < a < 1 \) some constant to be chosen later for convenience. \( a = \frac{1}{2} \min \frac{r}{M_0} = \frac{1}{2} \min \frac{u}{M_0} \).

We have the following properties of \( Q \) along (5.14) (referring to [15, 18]).

**Lemma 5.1.** Along (5.14), \( Q(t) \) is non-decreasing and the equality holds if and only if \( \pi_p(M_t) \) satisfies the following equation
\[
r^{\alpha}(1 - r^2)^{\frac{n+2-\alpha}{2}} f \hat{K} = \hat{u}(1 - \hat{u}^2)^{\frac{n+1}{2}}. \tag{5.17}
\]

**Proof.**
\[
Q'(t) = \partial_t \int_{S^n} \Psi f^{-1} d\theta_{S^n} - \partial_t \int_{S^n} \Omega d\sigma_{S^n}
= \int_{S^n} \psi^{-1}(r) r^n \partial_r r f^{-1} d\theta_{S^n} - \int_{S^n} \varphi(\hat{u}) \partial_t \hat{u} d\sigma_{S^n}. \tag{5.18}
\]
Since \( \frac{\partial_r r(\theta, t)}{r} = \frac{\partial_r \hat{u}(x, t)}{\hat{u}} \) and \( r^{n+1} d\theta_{S^n} = \frac{d\sigma_{S^n}}{K} \), (5.18) becomes
\[
Q'(t) = \int_{S^n} \left( \psi^{-1}(r) f^{-1}(\theta) \hat{K}^{-1} - \varphi(\hat{u}) \right) \partial_t \hat{u} d\sigma_{S^n}
= \int_{S^n} \left( \psi^{-1}(r) f^{-1}(\theta) \hat{K}^{-1} - \varphi(\hat{u}) \right)^2 \psi(r) \sqrt{1 - r^2} f(\theta) \hat{u} \hat{K} d\sigma_{S^n}
\geq 0. \tag{5.19}
\]
Here the equality holds if and only if \( \varphi(r)\psi(\hat{u})f(\theta)\hat{K} \equiv 1. \)

By Lemma 3.1, we obtain the uniform upper and lower bounds of \( \rho \) in the hyperbolic space. Thus from (5.1), there exists \( 0 < c_1 < c_2 < 1 \) independent of time, such that \( c_1 < r < c_2 \) and \( c_1 < \hat{u} < c_2. \) From (5.16), \( Q(t) \) is uniformly bounded independent of time, i.e., there exists a constant \( C > 0, \) such that \( |Q(t)| \leq C. \) Since

\[
Q(t) - Q(0) = \int_0^t \int_{S^\infty} \left( \psi^{-1}(r)f^{-1}(\theta)\hat{K}^{-1} - \varphi(\hat{u}) \right)^2 \psi(r)\sqrt{1-r^2f(\theta)}\hat{K}d\sigma_{S^n}dt,
\]

we obtain

\[
\int_0^{\infty} \int_{S^\infty} \left( \psi^{-1}(r)f^{-1}(\theta)\hat{K}^{-1} - \varphi(\hat{u}) \right)^2 \psi(r)\sqrt{1-r^2f(\theta)}\hat{K}d\sigma_{S^n}dt < \infty.
\]

By Theorem 4.1, there exists a subsequence \( \{t_i\} \), such that \( M_{t_i} \) converge smoothly to \( M_\infty \) and

\[
\int_{S^n} \left( \psi^{-1}(r)f^{-1}(\theta)\hat{K}^{-1} - \varphi(\hat{u}) \right)^2 d\sigma_{S^n} \rightarrow 0
\]
as \( t_i \rightarrow \infty \), since \( \hat{K} \) is uniformly bounded from below by (5.13) and Lemma 4.1. Thus \( M_\infty \) satisfies

\[
\psi(r)f(\theta)\hat{K} = \varphi(\hat{u})^{-1}.
\]

Combining with (5.1), (5.2), (5.3) etc., we obtain \( M_\infty \) satisfies (1.2).

Now we show the uniqueness of the solution of (1.2) when \( \alpha \geq n+1 \). First we assume that there exists two solutions \( \rho_1 \) and \( \rho_2 \). Let \( \gamma(\rho) = \log(1-\frac{2}{e\rho+1}) \) and \( G = \gamma_1 - \gamma_2 \), then \( \nabla \gamma = \frac{\nabla \rho}{\rho} \).

Assume \( G \) attains its maximal point at \( \theta_0 \) and \( G(\theta_0) > 0 \), which implies \( \rho_1(\theta_0) > \rho_2(\theta_0) \). At \( \theta_0 \), we have

\[
\nabla \gamma_1 = \nabla \gamma_2
\]

and \( \nabla^2_{ij} G \leq 0 \), i.e.,

\[
\nabla^2_{ij} \gamma_1 \leq \nabla^2_{ij} \gamma_2.
\]

Substituting \( \gamma \) into (2.14) and (2.16), we have

\[
g_{ij} = \phi^2 \left( \delta_{ij} + \gamma_i \gamma_j \right), \quad g^{kj} = \frac{1}{\phi^2} \left( \delta^{kj} - \frac{\gamma_k \gamma_j}{1 + |\nabla \gamma|^2} \right)
\]

and

\[
h_{ik} = \frac{\phi}{\sqrt{1 + |\nabla \gamma|^2}} \left( -\gamma_{ik} + \phi' \gamma_i \gamma_k + \phi' \delta_{ik} \right).
\]

Plugging (5.25) and (5.26) in (1.2) we have

\[
\sigma_n(h_{ij}) = \frac{f}{\phi(\rho)^{n-1} \sqrt{1 + |\nabla \gamma|^2}},
\]

(5.27)
where
\[ h_{ij} = \frac{1}{\phi \sqrt{1 + |\nabla \gamma|^2}} (-\gamma_{ik} + \phi' \gamma_i \gamma_k + \phi' \delta_{ik}) \left( \delta^{kj} - \frac{\gamma_k \gamma_j}{1 + |\nabla \gamma|^2} \right). \] (5.28)

Then
\[ \sigma_n \left( \frac{1}{\sqrt{1 + |\nabla \gamma|^2}} (-\gamma_{ik} + \phi' \gamma_i \gamma_k + \phi' \delta_{ik}) \left( \delta^{kj} - \frac{\gamma_k \gamma_j}{1 + |\nabla \gamma|^2} \right) \right) = \frac{f}{\phi(\rho)^{\alpha-n-1}} \sqrt{1 + |\nabla \gamma|^2}. \] (5.29)

Using (5.23), we have at \( \theta \)
\[ \phi(\rho_1)^{\alpha-n-1} \sigma_n (-\gamma_{ik}) + \phi(\rho_1)'(\gamma_1)_i(\gamma_1)_k + \phi(\rho_1)' \delta_{ik} \]
\[ = \phi(\rho_2)^{\alpha-n-1} \sigma_n (-\gamma_{ik}) + \phi(\rho_2)'(\gamma_1)_i(\gamma_1)_k + \phi(\rho_2)' \delta_{ik}. \] (5.30)

Since \( \rho_1(\theta) > \rho_2(\theta) \) and both of the solutions are uniformly convex, by (5.23) and (5.24)
\[ -(\gamma_1)_ik + \phi'(\rho_1)(\gamma_1)_i(\gamma_1)_k + \phi'(\rho_1) \delta_{ik} > -(\gamma_2)_ik + \phi'(\rho_2)(\gamma_2)_i(\gamma_2)_k + \phi'(\rho_2) \delta_{ik} > 0. \] (5.31)

When \( \alpha \geq n + 1 \), we have
\[ \phi(\rho_1)^{\alpha-n-1} \sigma_n (-\gamma_{ik}) + \phi(\rho_1)'(\gamma_1)_i(\gamma_1)_k + \phi(\rho_1)' \delta_{ik} \]
\[ > \phi(\rho_2)^{\alpha-n-1} \sigma_n (-\gamma_{ik}) + \phi(\rho_2)'(\gamma_1)_i(\gamma_1)_k + \phi(\rho_2)' \delta_{ik} \]
which is contrary to (5.30). So \( G \leq 0 \), which implies \( \gamma_1 \equiv \gamma_2 \) and \( \rho_1 \equiv \rho_2 \). Now we complete the proof of Theorem 1.1.

6. Proof of Theorem 1.2

In this section, we consider the flow (1.3) for \( \alpha = n + 1 \) and the corresponding even Alexandrov problem in \( \mathbb{H}^{n+1} \). Throughout this section, if not specified, we regard \( c_i, C_i \) for \( i \in \mathbb{N} \) as some positive constants. By (2.17), when \( \alpha = n + 1 \) the scalar equation of the radial function \( \rho \) becomes
\[
\left\{
\begin{array}{l}
\frac{\partial}{\partial t} \rho(\theta, t) = - (\phi(\rho))^{n+1} f(\theta) w K + \phi(\theta, t), \quad \text{for} \ (\theta, t) \in S^n \times [0, +\infty), \\
\rho(\cdot, 0) = \rho_0(\cdot).
\end{array}
\right.
\] (6.1)

By the projection \( \pi_p \) in Section 5 and the relation between \( M_t \) and \( \pi_p(M_t) \), (5.1)-(5.6) and (5.13), we obtain the scalar parabolic PDE of the radial function \( r \) of \( \pi_p(M_t) \) along the flow (1.3)
\[ \partial_r r = -r^{n+2}(1 - r^2) \hat{\nu}^{-1}(1 - \hat{\nu}^2)^{-\frac{n+1}{2}} f K + r \sqrt{1 - r^2}. \] (6.2)

Then we have the following results.

**Lemma 6.1.** Let \( \rho \) be a smooth, positive, uniformly convex and origin-symmetric solution to (6.1) on \( S^n \times [0, T] \). If \( f \) is a smooth positive even function on \( S^n \) satisfying \( \int_{S^n} f^{-1} d\theta > |S^n| \), then there exists a positive constant \( C \) depending on \( n, \max f, \min f \) and the initial hypersurface, such that
\[ \frac{1}{C} \leq \rho \leq C, \quad \forall t \in [0, T]. \] (6.3)
Proof. The upper bound of \( \rho \) is obtained directly from the proof of Lemma 3.1, i.e., there exists a positive constant \( c_1 \), such that \( \rho \leq c_1 \). Combining with (5.3), we obtain that the radial function \( r \) of \( \pi_p(M_t) \) is away from 1, i.e., there exists a positive constant \( c_2 \), such that \( \hat{u} \leq r \leq c_2 < 1 \). Under the flow (6.2), we have the following monotone non-decreasing function by Lemma 5.1

\[
Q(t) = \int_{S^n} \int_{a}^{r} s^{-1}(1 - s^2)^{-1/2} f^{-1} ds d\theta_{S^n} - \int_{S^n} \int_{a}^{\hat{u}} s^{-1}(1 - s^2)^{-\frac{n+1}{2}} ds d\sigma_{S^n}. \tag{6.4}
\]

where we can choose \( a = c_2 \) without loss of generality. Then there exists a positive constant \( C_1 \) depending on the initial hypersurface of (1.3), such that

\[
-C_1 \leq Q(t) \leq -f_{\text{max}}^{-1} \int_{S^n} \int_{c_2}^{r} s^{-1} ds d\theta_{S^n} + (1 - c_2^2)^{-\frac{n+1}{2}} \int_{S^n} \int_{\hat{u}}^{c_2} s^{-1} ds d\sigma_{S^n} \]

\[
= -f_{\text{max}}^{-1} \int_{S^n} (\log c_2 - \log r) ds d\theta_{S^n} + (1 - c_2^2)^{-\frac{n+1}{2}} \int_{S^n} (\log c_2 - \log \hat{u}) ds d\sigma_{S^n} \tag{6.5}
\]

\[
\leq C_2 + f_{\text{max}}^{-1} \int_{S^n} \log r ds d\theta_{S^n} - (1 - c_2^2)^{-\frac{n+1}{2}} \int_{S^n} \log \hat{u} ds d\sigma_{S^n}.
\]

Assume \( r_{\text{min}}(t) \) is attained at \( \theta_0 = (1, \vec{0}) \) at any fixed time \( t \). Here we define by \( \vec{0} \) an \( n \)-dimensional zero vector. We parametrize any point \( \theta \in S^n \) as

\[
\theta = (\cos \theta_1, \sin \theta_1 \vec{x}), \tag{6.6}
\]

where \( 0 \leq \theta_1 \leq \pi \), and \( \vec{x} = (x_2, \cdots, x_{n+1}) \in S^{n-1} \) is an \( n \)-dimensional unit vector. We have \( \langle \theta, \theta_0 \rangle = \cos \theta_1 \). Then \( r(\theta, t) \leq \frac{r_{\text{min}}(t)}{\cos \theta_1} \) because the flow hypersurface \( M_t \) is strictly convex and origin-symmetric. Assume \( \hat{u}_{\text{max}}(t) \) is attained at \( \theta_0 \in S^n \). We have \( \hat{u}(v, t) \geq \hat{u}_{\text{max}}(t) |\langle v, v_0 \rangle| \) (referring to [26, p.44]). Take the direction of \( v_0 \) as \( x \)-axis and regard \( v_1 \) as the angle with the \( x \)-axis. We parametrize any point \( v \in S^n \) as

\[
v = (\cos v_1, \sin v_1 \vec{y}), \tag{6.7}
\]

where \( 0 \leq v_1 \leq \pi \), and \( \vec{y} = (y_2, \cdots, y_{n+1}) \in S^{n-1} \) is an \( n \)-dimensional unit vector. We have \( \langle v, v_0 \rangle = \cos v_1 \). By a direct calculation, the area measure \( d\sigma_{S^n} \) becomes

\[
d\sigma_{S^n} = (\sin v_1)^{n-1} dv_1 d\sigma_{S^{n-1}}. \tag{6.8}
\]

Similarly,

\[
d\theta_{S^n} = (\sin \theta_1)^{n-1} d\theta_1 d\theta_{S^{n-1}}. \tag{6.9}
\]

Then

\[
\int_{S^n} \log \hat{u} d\sigma_{S^n} \geq \int_{S^n} \log(\hat{u}_{\text{max}} |\cos v_1|) d\sigma_{S^n}
\]

\[
= 2 \int_{0}^{\pi} \int_{S^{n-1}} (\log \hat{u}_{\text{max}} + \log \cos v_1) (\sin v_1)^{n-1} dv_1 d\sigma_{S^{n-1}} \tag{6.10}
\]

\[
\geq |S^n| \log \hat{u}_{\text{max}} + 2 |S^{n-1}| \int_{0}^{\pi} \log \cos v_1 dv_1 \geq |S^n| \log \hat{u}_{\text{max}} - C_3.
\]
The second term in the last inequality is convergent, since $\log(\cos v_1) \geq \log \left( \frac{\pi}{4} - \frac{v_1}{2} \right)$ at $v_1 \in \left[ 0, \frac{\pi}{2} \right]$. Besides, we have

\[
\int_{\mathbb{S}^n} \log r \, d\theta_{\mathbb{S}^n} \leq \int_{\mathbb{S}^n} \log \frac{r_{\min}}{|\cos \theta_1|} \, d\theta_{\mathbb{S}^n}
\]

\[
= 2 \int_0^{\frac{\pi}{2}} \int_{\mathbb{S}^{n-1}} (\log r_{\min} - \log \cos \theta_1) (\sin \theta_1)^{n-1} \, d\theta_1 \, d\theta_{\mathbb{S}^{n-1}}
\]

\[
\leq |S^n| \log r_{\min} - 2|S^{n-1}| \int_0^{\frac{\pi}{2}} \log \cos \theta_1 \, d\theta_1 \leq |S^n| \log r_{\min} + C_3,
\]

where the second term in the last inequality is convergent for the same reason as (6.10). Plugging (6.10) and (6.11) in (6.5), we obtain

\[
-C_1 \leq C_4 + f_{\max}^{-1} |S^n| \log r_{\min} - (1 - c_2^2)^{-\frac{n+1}{2}} |S^n| \log \hat{u}_{\max}.
\]

Hence, there exists positive constants $c_3$ and $c_4$, such that

\[
r_{\min} \geq c_3 r_{\max}^{c_4},
\]

where $c_4 = f_{\max} (1 - c_2^2)^{-\frac{n+1}{2}}$.

Dividing the both sides of (6.2) by $r \sqrt{1 - r^2 f}$, we have

\[
\frac{\partial r}{r \sqrt{1 - r^2 f}} = -r^{n+1} \sqrt{1 - r^2 \hat{u}^2} (1 - \hat{u}^2)^{-\frac{n+1}{2}} \hat{K} + f^{-1}(\theta).
\]

Integrating (6.14) over $\mathbb{S}^n$, we get

\[
\int_{\mathbb{S}^n} \frac{\partial r}{r \sqrt{1 - r^2 f}} \, d\theta_{\mathbb{S}^n} = - \int_{\mathbb{S}^n} r^{n+1} \sqrt{1 - r^2 \hat{u}^2} (1 - \hat{u}^2)^{-\frac{n+1}{2}} \hat{K} \, d\theta_{\mathbb{S}^n} + \int_{\mathbb{S}^n} f^{-1}(\theta) \, d\theta_{\mathbb{S}^n}
\]

\[
= - \int_{\mathbb{S}^n} r^{n+1} \sqrt{1 - r^2 (1 - \hat{u}^2)^{-\frac{n+1}{2}} d\sigma_{\mathbb{S}^n} + \int_{\mathbb{S}^n} f^{-1}(\theta) \, d\theta_{\mathbb{S}^n},
\]

where we use $r^{n+1} d\theta_{\mathbb{S}^n} = \frac{\hat{u}}{K} d\sigma_{\mathbb{S}^n}$ in the last equality. Note that the left hand side of (6.15) is

\[
\int_{\mathbb{S}^n} \frac{\partial r}{r \sqrt{1 - r^2 f}} \, d\theta_{\mathbb{S}^n} = \partial_t \int_{\mathbb{S}^n} \int_{c_2}^r s^{-1} (1 - s^2)^{-\frac{1}{2}} f^{-1} \, ds \, d\theta_{\mathbb{S}^n}.
\]

Besides, we have

\[
\int_{\mathbb{S}^n} \int_{c_2}^r s^{-1} (1 - s^2)^{-\frac{1}{2}} f^{-1} \, ds \, d\theta_{\mathbb{S}^n}
\]

\[
\leq - f_{\max}^{-1} \int_{\mathbb{S}^n} \int_{c_2}^{c_1} s^{-1} \, ds \, d\theta_{\mathbb{S}^n}
\]

\[
= - |S^n| f_{\max}^{-1} \log c_2 + f_{\max}^{-1} \int_{\mathbb{S}^n} \log r \, d\theta_{\mathbb{S}^n}
\]

\[
\leq C_5 + |S^n| f_{\max}^{-1} \log r_{\max}
\]
and
\[
\int_{S^n} \int_{c_2}^r s^{-1}(1 - s^2)^{-\frac{1}{2}} f^{-1} ds d\theta_{S^n} \\
\geq - (1 - c_2^2)^{-\frac{1}{2}} f_{\min}^{-1} \int_{S^n} \int_{c_2}^r s^{-1} ds d\theta_{S^n} \\
= - (1 - c_2^2)^{-\frac{1}{2}} |S^n| f_{\min}^{-1} \log c_2 + (1 - c_2^2)^{-\frac{1}{2}} f_{\min}^{-1} \int_{S^n} \log r d\theta_{S^n} \\
\geq (1 - c_2^2)^{-\frac{1}{2}} |S^n| f_{\min}^{-1} \log r_{\min}. \tag{6.18}
\]

Since \( r(\theta_1, t) = \sqrt{\hat{u}(v)^2 + |\nabla \hat{u}(v)|^2} \geq \hat{u}(v, t) \) on \( S^n \), we have
\[
\int_{S^n} \sqrt{1 - r^2(1 - \hat{u}^2)^{-\frac{n+1}{2}}} d\sigma_{S^n} \leq \int_{S^n} (1 - \hat{u}^2)^{-\frac{n+1}{2}} d\sigma_{S^n} \leq (1 - \hat{u}_{\max}^2)^{-\frac{n+1}{2}} |S^n|. \tag{6.19}
\]

When \( u_{\max} \to 0, (1 - \hat{u}_{\max}^2)^{-\frac{n+1}{2}} |S^n| \to |S^n| \). By (6.19) and the condition of \( f \), there exists a positive constant \( c_5 \), such that when \( u_{\max} \leq c_5 \), \( \int_{S^n} \sqrt{1 - r^2(1 - \hat{u}^2)^{-\frac{n+1}{2}}} d\sigma_{S^n} \leq \frac{1}{2}(1 + \int_{S^n} f^{-1}(\theta) d\theta_{S^n}) < \int_{S^n} f^{-1}(\theta) d\theta_{S^n} \). This together with (6.15) and (6.16) implies
\[
\partial_t \int_{S^n} \int_{c_2}^r s^{-1}(1 - s^2)^{-\frac{1}{2}} f^{-1} ds d\theta_{S^n} = - \int_{S^n} \sqrt{1 - r^2(1 - \hat{u}^2)^{-\frac{n+1}{2}}} d\sigma_{S^n} + \int_{S^n} f^{-1}(\theta) d\theta_{S^n} > 0. \tag{6.20}
\]

When \( \hat{u}_{\max} = c_5 \), inserting (6.13) into (6.18), we have
\[
\int_{S^n} \int_{c_2}^r s^{-1}(1 - s^2)^{-\frac{1}{2}} f^{-1} ds d\theta_{S^n} \geq (1 - c_2^2)^{-\frac{1}{2}} |S^n| f_{\min}^{-1} \log c_3 + c_4 (1 - c_2^2)^{-\frac{1}{2}} |S^n| f_{\min}^{-1} \log r_{\max} \\
\geq - C_6 + (1 - c_2^2)^{-\frac{n+1}{2}} |S^n| f_{\max} f_{\min}^{-1} \log c_5, \tag{6.21}
\]

where we use \( \max_{M_t} \hat{u} = \max_{M_t} r_{\max} \).

If the maximal radial function of the initial hypersurface satisfies \( \hat{u}_{\max}(0) > c_5 \), then once \( \hat{u}_{\max}(t) \leq c_5 \), by (6.17), (6.20) and (6.21) we obtain
\[
\log r_{\max} \geq - C_7 + (1 - c_2^2)^{-\frac{n+1}{2}} f_{\max} f_{\min}^{-1} \log c_5, \tag{6.22}
\]

which implies that there exists a positive constant \( c_6 = e^{C_7 c_5^{-1}(1-c_2^2)^{-\frac{n+1}{2}} f_{\max} f_{\min}^{-1}} < c_5 \), such that \( r_{\max} \geq c_6 \), until \( \hat{u}_{\max} > c_5 \) again.

If the initial hypersurface of (6.2) satisfies \( \hat{u}_{\max}(0) \leq c_5 \), then from (6.20), we note that \( \int_{S^n} \int_{c_2}^r s^{-1}(1 - s^2)^{-\frac{1}{2}} f^{-1} ds d\theta_{S^n} \) will monotone increasing till \( \hat{u}_{\max}(t) > c_5 \). This together with (6.17) implies that
\[
|S^n| f_{\max}^{-1} \log r_{\max} + C_5 \geq \int_{S^n} \int_{c_2}^r s^{-1}(1 - s^2)^{-\frac{1}{2}} f^{-1} ds \big|_{t=0. \tag{6.23}}
\]

Hence there exists a positive constant \( c_7 \), such that \( r_{\max} \geq c_7 \). Clearly, we can deduce from (6.17) that \( c_7 \leq \hat{u}_{\max}(0) \leq c_5 \). Since \( \hat{u}_{\max} = r_{\max} \) in \( M_t \), we obtain \( r_{\max}(t) \geq \min\{c_6, c_7\} \). All those constants only depend on \( n, f \) and the initial hypersurface \( M_0 \). By (6.13), there exists a positive constant \( c_8 \), such that \( r_{\min} \geq c_8 \). Thus the radial function \( r \) of \( M_t \) satisfying \( 0 < c_8 \leq r \leq c_2 < 1 \). By (5.3), we obtain the uniform bounds of \( \rho \) and complete the proof. \( \square \)
**Proof of Theorem 1.2** Similar to the proof of Theorem 1.1, by Lemma 6.1, Lemma 3.2 and Lemma 4.3, we establish the a priori estimates and obtain the long time existence of (1.3) for the case \( \alpha = n + 1 \). Using the same argument in Section 5, we show that the flow (1.3) converges smoothly to the unique smooth even solution of (1.2). Hence we complete the proof of Theorem 1.2.

**Remark 6.1.** Note that the conditions of \( \alpha \) and \( f \) in Theorem 1.1 and Theorem 1.2 are necessary to the convergence and asymptotic results of (1.3). When \( \alpha < n + 1 \), or when \( \alpha = n + 1 \) and \( f \geq 1 \) at the same time, consider a geodesic sphere with its radial function \( \rho \equiv c \). Note that \( \sinh \rho \to 0 \) and \( \cosh \rho \to 1 \) as \( \rho \to 0 \). When \( \alpha < n + 1 \), \( \sinh \rho \alpha - n - 1 \cosh \rho^n \to \infty \). When \( \alpha = n + 1 \), \( \sinh \rho \alpha - n - 1 \cosh \rho^n \to 1^+ \). Hence there exists a constant \( \rho_0 > 0 \), such that for any \( \rho \in (0, \rho_0) \), \( -\sinh \rho^{\alpha-1} \cosh \rho^n f + 1 < 0 \) when \( \alpha < n + 1 \), or when \( \alpha = n + 1 \) and \( f \geq 1 \) at the same time. Assume the initial hypersurface of the flow (1.3) is a geodesic sphere with its radial function \( \rho \in (0, \rho_0) \). Then combining (2.17) \((\tilde{\eta}(t) \equiv 1)\) and (3.1), we obtain

\[
\frac{\partial}{\partial t}\rho(\theta, t) = -\phi(\rho)^{\alpha} f(\theta) w \sigma_k + \phi(\theta, t) \\
= \phi(\rho) \left( -\phi(\rho)^{\alpha-1} \phi'(\rho)^n + 1 \right) \\
< 0.
\]

So the initial geodesic sphere keeps shrinking along (1.3).

However, for the cases \( \alpha > n + 1 \), the same initial geodesic sphere of (1.3) will expand until it converges smoothly to the solution of (1.2). For the case \( \alpha = n + 1 \), convergence results of the flow (1.3) need the assumption of \( f \) in addition.

7. **Proof of Theorem 1.3**

In this part, we consider the flow (1.6) for the cases \( 2 < \alpha \leq n + 1 \). First, we parametrize \( M_t \) as a graph of the radial function \( \rho(\theta, t) : S^n \times [0, T) \to \mathbb{R} \). By (2.17), the scalar parabolic PDE of the radial function turns to

\[
\frac{\partial}{\partial t}\rho(\theta, t) = -\phi(\rho)^{\alpha} f(\theta) K(x, t) w + \int_{S^n} \frac{K(u)}{\phi^{n+1} \phi} \frac{\phi^{n+1} d\theta_{S^n}}{\phi^{n+1-\alpha} f^{-1} d\theta_{S^n}} \phi. \tag{7.1}
\]

Throughout this section, if not specified, we will regard \( C_i \) and \( C_i' \) for \( i \in \mathbb{N} \) as some positive constants. Observe that along (1.6), the hypersurfaces have the following properties.

**Lemma 7.1.** Denote \( \Omega(\rho) = \int_b^{\rho(\theta, t)} (\sinh s)^{n-\alpha} ds \). Then along the flow (1.6),

\[
\int_{S^n} \frac{\Omega(\rho)}{\rho} d\theta_{S^n} = \text{const}, \quad \text{for } t \geq 0. \tag{7.2}
\]

Here \( \rho(t) \) is the radial function of \( M_t \).
Proof. By (7.1),
\[ \partial_t \int_{\mathbb{S}^n} \frac{\Omega(\rho)}{f} d\theta_{\mathbb{S}^n} = \int_{\mathbb{S}^n} \phi(\rho)^{n-\alpha} f^{-1} \partial_t \rho d\theta_{\mathbb{S}^n} = 0 \] (7.3)
where we use \( w = \frac{\phi}{u} \) in the last equality. \[ \square \]

Remark 7.1. Without loss of generality, we can choose \( b = \frac{1}{2} \min \rho \).

From Lemma 7.1, we have the following proposition.

**Proposition 7.1.** Along (7.1), the maximal radial functions \( \rho_{\text{max}}(t) \) of \( M_t \) have uniform lower bounds.

**Proof.** Observe that when \( \alpha < n + 1 \), since \( \cosh s > 0 \) and it is monotone increasing, we have
\[
\phi^{n+1-\alpha}(\rho_{\text{max}})|\mathbb{S}^n| - \phi^{n+1-\alpha}(b)|\mathbb{S}^n| \\
\geq \int_{\mathbb{S}^n} (\phi^{n+1-\alpha}(\rho) - \phi^{n+1-\alpha}(b)) d\theta_{\mathbb{S}^n} \\
= (n + 1 - \alpha) \int_{\mathbb{S}^n} \int_b^\rho (\sinh s)^{n-\alpha} \cosh s ds d\theta_{\mathbb{S}^n} \\
\geq (n + 1 - \alpha) \int_{\mathbb{S}^n} \int_b^\rho (\sinh s)^{n-\alpha} \cosh b ds d\theta_{\mathbb{S}^n} \geq c > 0.
\] (7.4)
Here we use Lemma 7.1 in the last inequality. The positive constant \( c \) depends on the initial hypersurface, \( \min f \) and \( a \). Similarly, when \( \alpha = n + 1 \), there exists a positive constant \( c' \), such that
\[
\log \phi(\rho_{\text{max}})|\mathbb{S}^n| - \log \phi(b)|\mathbb{S}^n| \\
\geq \int_{\mathbb{S}^n} (\log \phi(\rho) - \log \phi(b)) d\theta_{\mathbb{S}^n} \\
= \int_{\mathbb{S}^n} \int_b^\rho (\sinh s)^{-1} \cosh s ds d\theta_{\mathbb{S}^n} \\
\geq \int_{\mathbb{S}^n} \int_b^\rho (\sinh s)^{-1} \cosh b ds d\theta_{\mathbb{S}^n} \geq c' > 0.
\] (7.5)

If \( \rho_{\text{max}} \to 0 \), we have \( \phi(\rho_{\text{max}})^{n+1-\alpha} \to 0 \) and \( \log \phi(\rho_{\text{max}}) \to -\infty \), which is contrary to (7.4) and (7.5). Hence, we obtain the uniform lower bound of \( \phi_{\text{max}} \) and complete the proof. \[ \square \]

Using the projection \( \pi_p \) in Section 5 and the relation between \( M_t \) and \( \pi_p(M_t) \), (5.1)-(5.6) and (5.13), we obtain the scalar equation of the support function \( \hat{u} \) of \( \pi_p(M_t) \),
\[ \partial_t \hat{u} = -r^\alpha (1 - r^2)^{-\frac{n+1}{2}} (1 - \hat{u}^2)^{-\frac{n+1}{2}} f \hat{K} + \eta(t) \hat{u} \sqrt{1 - r^2}, \] (7.6)
where \( \eta(t) = \frac{\int \sqrt{1 - r^2}(1 - \hat{u}^2)^{-\frac{n+1}{2}} f \hat{K} d\sigma_{\mathbb{S}^n}}{\int \frac{r^{n+1-\alpha}}{(1 - r^2)^{\frac{n+1-\alpha}{2}} f^{-1} d\theta_{\mathbb{S}^n}}} \).

Under the flow (7.6), we have the following monotone function
\[ J(\hat{u}) = \int_{\mathbb{S}^n} \Psi(\hat{u}) d\sigma_{\mathbb{S}^n}, \] (7.7)
where \( \Psi(\hat{u}) = \int_a^{\hat{u}(x,t)} \frac{1}{s}(1 - s^2)^{-\frac{n+1}{2}} ds \). Without loss of generality, we can choose \( a = \frac{1}{2} \min \hat{u}. \)

**Lemma 7.2.** Along \((7.6)\), \( J(\hat{u}) \) is non-increasing and the equality holds if and only if \( \pi_p(M_t) \) satisfies the following equation

\[
 r^\alpha (1 - r^2)^{\frac{n+1}{2} - \frac{n+1}{2}} f \dot{K} = c \hat{u} \sqrt{1 - r^2} 
\]  

(7.8)

for some positive constant \( c \).

**Proof.** By \((7.6)\),

\[
 \partial_t J(\hat{u}) = \int_{S^n} \frac{1}{\hat{u}} (1 - \hat{u}^2)^{-\frac{n+1}{2}} \partial_t u d\sigma_{S^n} 
\]

\[
 = \left( \frac{\int \sqrt{1 - r^2(1 - \hat{u}^2)^{-\frac{n+1}{2}}} d\sigma_{S^n}}{\int r^{-\alpha} \hat{u} (1 - r^2)^{\frac{n+1}{2}-\frac{n+1}{2}} f \dot{K} d\sigma_{S^n}} \right)^2 - \int r^\alpha (1 - r^2)^{\frac{n+1}{2} - \frac{n+1}{2}} f \dot{K} d\sigma_{S^n} 
\]

\[
 \leq 0.
\]

(7.9)

Here we use Hölder inequality in the last inequality and the equality holds if and only if \((7.8)\) holds.

Next, we will show the radial function \( \rho(\cdot,t) \) is uniformly bounded along the flow \((1.6)\) under the following assumption.

**Proposition 7.2.** Let \( \rho \) be a smooth, positive, uniformly convex and origin-symmetric solution to \((7.1)\) on \( S^n \times [0,T) \). If \( 2 < \alpha \leq n + 1 \) and \( f \) is a positive even function on \( S^n \), then there exists a positive constant \( C \) depending on \( \alpha, n, f \) and the initial hypersurface, such that

\[
 \frac{1}{C} \leq \rho \leq C, \quad \forall t \in [0,T). 
\]

(7.10)

**Proof.** Firstly, we prove \( \rho \) has a uniform upper bound. Motivated by the proof in Lemma 3.1, we will show that

\[
 \int_{S^n} \frac{K}{u} \phi^{n+1} d\theta_{S^n} 
\]

is bounded from above by multiple \( \phi_{\max} \). By Lemma
7.2, there exists a positive constant \( C_1 > 0 \), such that
\[
C_1 \geq J(\tilde{u})
\]
\[
= \int_{S^a} \int_{\tilde{a}}^{\tilde{u}} \frac{1}{s} \left( (1 - s^2)^{-\frac{n+1}{2}} ds \right) d\sigma_{S^a}
\]
\[
\geq 2^{\frac{n+1}{2}} \int_{S^n \cap \{ |\tilde{u}| > a \}} \int_{\tilde{a}}^{\tilde{u}} \left( (1 - s)^{-\frac{n+1}{2}} ds \right) d\sigma_{S^a} - (1 - a^2)^{-\frac{n+1}{2}} \int_{S^n \cap \{ |\tilde{u}| \leq a \}} \int_{\tilde{a}}^{\tilde{u}} \frac{1}{s} ds \right) d\sigma_{S^a}
\]
\[
\geq C_2 \int_{S^n \cap \{ |\tilde{u}| > a \}} \left( (1 - \tilde{u})^{-\frac{n-1}{2}} - (1 - a)^{-\frac{n-1}{2}} \right) d\sigma_{S^n} - C_3 \int_{S^n \cap \{ |\tilde{u}| \leq a \}} (\log a - \log \tilde{u}) d\sigma_{S^n}
\]
\[
\geq -C_4 + C_2 \int_{S^n \cap \{ |\tilde{u}| > a \}} (1 - \tilde{u})^{-\frac{n-1}{2}} d\sigma_{S^n} - C_2 \int_{S^n \cap \{ |\tilde{u}| \leq a \}} (1 - \tilde{u})^{-\frac{n-1}{2}} d\sigma_{S^n} + C_3 \int_{S^n} \log \tilde{u} d\sigma_{S^n}
\]
\[
\geq -C_4 + C_2 \int_{S^n} (1 - \tilde{u})^{-\frac{n-1}{2}} d\sigma_{S^n} + C_3 \int_{S^n} \log \tilde{u} d\sigma_{S^n}.
\] (7.11)

Similar to the proof in (6.5), we have when \( n = 1 \)
\[
-C'_5 \int_{S^1} \log (1 - \tilde{u}) d\sigma_{S^1} + C'_6 \int_{S^1} \log \tilde{u} d\sigma_{S^1} \leq C'_7,
\] (7.12)

and when \( n \geq 2 \)
\[
C_5 \int_{S^n} (1 - \tilde{u})^{-\frac{n-1}{2}} d\sigma_{S^n} + C_6 \int_{S^n} \log \tilde{u} d\sigma_{S^n} \leq C_7.
\] (7.13)

Assume \( \tilde{u}_{\text{max}}(t) \) is attained at \( v_0 \in S^n \) at any fixed time \( t \). Because the flow hypersurfaces are uniformly convex and origin-symmetric, we have \( \tilde{u}(v) \geq \tilde{u}_{\text{max}}(v, v_0) \). Then
\[
\int_{S^n} \log \tilde{u} d\sigma_{S^n} \geq \int_{S^n} (\log \tilde{u}_{\text{max}} + \log |\langle v, v_0 \rangle|) d\sigma_{S^n}.
\] (7.14)

The second integral in (7.14) is convergent, for the same reason as (6.10). Thus by (7.12) and (7.13), there exists positive constants \( C_5 \) and \( C_5' \) such that, when \( n = 1 \)
\[
\int_{S^1} \log (1 - \tilde{u}^2) d\sigma_{S^1} \geq \int_{S^1} \log (1 - \tilde{u}) d\sigma_{S^1} \geq -C'_8,
\] (7.15)

and when \( n \geq 2 \)
\[
C_8 \geq \int_{S^n} (1 - \tilde{u})^{-\frac{n-1}{2}} d\sigma_{S^n} \geq \int_{S^n} (1 - \tilde{u}^2)^{-\frac{n-1}{2}} d\sigma_{S^n}.
\] (7.16)

(7.4) and (7.5) implies that there exists a constant \( c_1 > 0 \), such that \( \rho_{\text{max}} \geq c_1 \), i.e., by (5.3) there exists \( c_2 > 0 \), such that \( r \geq c_2 \). Recall the relation between \( M_t \) and \( \pi_p(M_t) \), (5.1)-(5.6) and (5.13). Note that \( r(\theta, t) = \sqrt{\tilde{u}^2 + |\nabla \tilde{u}|^2} \geq \tilde{u}(v, t) \) and \( \max_r = \max_{M_t} \tilde{u} \). At a fixed time
For the same reason as \((\mathbb{E} \leq \mathbb{H})\) is a rotation symmetric ellipsoid in \(\mathbb{R}^n\) we have \(u = \hat{u} = 1\). Hence when \(u \rightarrow 1\), \(\hat{u} \rightarrow 1\) satisfying \((\mathbb{E} \leq \mathbb{H})\). Let \(\{\theta, \hat{\theta}\} = \{\sin \theta, \sin \hat{\theta}\}\) be the support function of \(\mathbb{E}\). The uniform lower bound of \(\hat{u}^n\) is obtained directly by \((\mathbb{E} \leq \mathbb{H})\). At the maximal point of \(\rho\), inserting \((\mathbb{E} \leq \mathbb{H})\) into \((\mathbb{E})\), similar to the proof in Lemma 3.1, we have

\[
\partial_t \rho_{\max} \leq -\phi_s^n \rho \hat{u}^n\phi_{\max} f + \int_{\mathbb{S}^n} \frac{K}{u} \phi_{\max} \frac{\phi_{\max}}{\max} \leq \phi_{\max}^2 (-\phi_{\max}^2 f + C_7).
\]

Hence when \(\alpha > 2\), \(\phi = \sin \rho\) has an uniform upper bound. Thus by \((\mathbb{E})\) there exists a constant \(c_3 > 0\), such that \(r < c_3 \leq 1\).

In order to obtain the lower bound of \(\rho\), we parametrize any point \(\theta\) in \(\mathbb{S}^n\) as in \((\mathbb{E})\). Assume \(r_{\min}\) is attained at \(\theta_0 = (1, 0)\). We have \(\langle \theta, \hat{\theta}\rangle = \cos \theta_1\). Since \(r \leq c_3 < 1\), there exists a positive constant \(c_4\), such that \(\phi = \frac{r}{\sqrt{1 - r^2}} \leq c_4 r\). Denote by \(\delta = \sqrt{r_{\min}}\). For \(2 < \alpha < n + 1\), we have

\[
\int_{\mathbb{S}^n} \phi^n \sigma^n d\theta_{\mathbb{S}^n} \leq c_4^n \int_{S_1 = \mathbb{S}^n \setminus \{r \leq \delta\}} r \sigma^n d\theta_{\mathbb{S}^n} + c_4^n \int_{S_2 = \mathbb{S}^n \setminus \{r > \delta\}} r^{\alpha - 1} \sigma^n d\theta_{\mathbb{S}^n} \leq C_8 \delta^{\alpha - 1} \sigma^n + C_8 c_3^{\alpha - 1} |S_2|.
\]

We also have \(r(\theta) \leq \frac{r_{\min}(\theta_0)}{\cos \theta_1}\) because the flow hypersurface \(M_t\) are strictly convex and origin symmetric. Then \(S_2 = \{\theta | \cos \theta_1 < \sqrt{r_{\min}}\}\). Suppose \(r_{\min} \rightarrow 0\). Clearly, \(\delta = \sqrt{r_{\min}} \rightarrow 0\) and \(|S_2| \rightarrow 0\) at the same time. By \((\mathbb{E})\), we have \(\int_{\mathbb{S}^n} \phi_{\max}(\rho) d\theta_{\mathbb{S}^n} \rightarrow 0\), which is contrary to \((\mathbb{E})\). For \(\alpha = n + 1\), we have

\[
\int_{\mathbb{S}^n} \log \phi(\rho) d\theta_{\mathbb{S}^n} \leq \int_{\mathbb{S}^n} \log r(\theta) d\theta_{\mathbb{S}^n} + |S^n| \log c_4 \leq |S^n| \log r_{\min}(\theta_0) - \int_{\mathbb{S}^n} \log |\cos \theta_1| d\theta_{\mathbb{S}^n} + |S^n| \log c_4.
\]

For the same reason as \((\mathbb{E})\), the second term in \((\mathbb{E})\) is convergent. So \(\int_{\mathbb{S}^n} \log \phi(\rho) d\theta_{\mathbb{S}^n} \rightarrow -\infty\) as \(r_{\min} \rightarrow 0\), which is contrary to \((\mathbb{E})\). Hence we obtain the uniform lower bounds of \(r\) as well as \(\rho\) by \((\mathbb{E})\) and complete the proof. \(\square\)

Remark 7.2. Similar to the Euclidean space, we hope \((\mathbb{E})\) implies the uniform upper bound of the support function \(\hat{u}\) away from 1. Unfortunately, we find an example here with unbounded \(u\), i.e., \(\hat{u}\) is very close to 1 satisfying \((\mathbb{E})\). Suppose \(\mathbb{E}(e_1, e_2, \cdots, e_3)\) is a rotation symmetric ellipsoid in \(\mathbb{R}^{n+1}\) with its centroid in the origin represented by
\[
\frac{x_1^2}{e_1^2} + \frac{x_2^2}{e_2^2} + \cdots + \frac{x_n^2}{e_n^2} = 1.
\]
Here we assume \(0 < e_2 < e_1 < 1\). Then \(\hat{E}\) can be parametrized by \(x = (e_1 \cos \theta, e_2 \sin \theta \bar{x})\) where \(\bar{x} \in S^{n-1}\) is an \(n\)-dimensional unit vector. Correspondingly, the unit outward norm vector is
\[
\frac{(e_2 \cos \theta, e_1 \sin \theta \bar{x})}{\sqrt{e_2^2 \cos^2 \theta + e_1^2 \sin^2 \theta}} \tag{7.21}
\]
The support function becomes
\[
\hat{u}(\theta) = \frac{e_1 e_2}{\sqrt{e_2^2 \cos^2 \theta + e_1^2 \sin^2 \theta}} \tag{7.22}
\]
If we parametrize \(S^n\) as in (6.7), by comparing with (7.21) we have
\[
\tan v_1 = \frac{e_1}{e_2} \tan \theta. \tag{7.23}
\]
By (7.22) and (7.23), the support function can be parametrized by \(v_1\) as
\[
\hat{u}(v_1) = \frac{e_1 e_2 \sqrt{\tan v_1^2 + 1}}{\sqrt{e_2^2 + e_1^2 \tan v_1^2}} = \frac{e_1 \sqrt{\tan v_1^2 + 1}}{\sqrt{1 + \tan v_1^2}} \tag{7.24}
\]
\[
= \sqrt{\tan v_1^2 e_2^2 + e_1^2 \cos v_1} = \sqrt{e_1^2 - (e_1^2 - e_2^2) \sin v_1^2}.
\]
Substitute (6.7) into (7.15) and (7.16) respectively. When \(n = 1\), we obtain
\[
\int_{S^1} \log(1 - \hat{u}^2) d\sigma_{S^1} = \int_0^{2\pi} \log(1 - e_1^2 + (e_1^2 - e_2^2)(\sin v_1^2)) dv_1 \tag{7.25}
\]
\[
> 2 \int_0^{\pi} (\log(e_1^2 - e_2^2) + 2 \log \sin v_1^2) dv_1 \geq -C(e_1, e_2).
\]
When \(n \geq 2\)
\[
\int_{S^n} (1 - \hat{u}^2)^{-\frac{n-1}{2}} d\sigma_{S^n} \leq C(e_1, e_2, n). \tag{7.26}
\]
Now we construct a sequence of \(\hat{E}(e_1^{(n)}, e_2^{(n)})\) satisfying \(e_1^{(n)} \to 1\) and \(e_2^{(n)} \leq c < 1\). Note that by (5.6), \(\max u_n \to \infty\) as \(\max u_n = e_1 \to 1\), which implies that the preimages of \(\hat{E}(e_1^{(n)}, e_2^{(n)})\) in Hyperbolic space tend to \(\infty\). However, by inserting (7.25), (7.26) into (7.12)and (7.13), we show that \(\mathcal{F}(\hat{u})_{\hat{E}(e_1^{(n)}, e_2^{(n)})}\) still can be bounded from above by some positive constant \(C\) independent of how far \(e_1\) is away from 1.

**Proof of Theorem 1.3** Combining Lemma 3.2, we established the \(C^0, C^1\) estimates of the flow (1.6) under the evenness assumption. Employing (5.3) and (5.6), we obtain the uniform upper and lower bounds of \(\eta(t)\) in (7.6). Then the \(C^2\) estimate follows from Lemma 4.3. Thus we obtain the long time existence and regularity for the flow (1.6). Furthermore,
Lemma 7.2 and the similar argument in Section 5 imply the asymptotic behaviour of the flow (1.6) and we complete the proof of the Theorem 1.3.

□

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