CYCLIC HENKIN LOGIC

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Abstract. In this paper, we study Cyclic Henkin Logic $\text{CHL}$, a logic that can be described as provability logic without the third Löb condition, to wit, that provable implies provably provable (aka principle 4). The logic $\text{CHL}$ does have full modalised fixed points. We implement these fixed points using cyclic syntax, so that we can work just with the usual repertoire of connectives.

The main part of the paper is devoted to developing the logic on cyclic syntax. Many theorems, like the multiple fixed point theorem, become matter of course in this context. We submit that the use of cyclic syntax is of interest even for the study of classical Löb’s Logic. We show that a version of the de Jongh-Sambin algorithm can be seen as one half of a synonymy between $\text{GL}^\circ$, i.e. $\text{CHL}$ plus the third Löb Condition, and ordinary Löb’s Logic $\text{GL}$. Our development illustrates that an appropriate computation scheme for the algorithm is guard recursion.

We show how arithmetical interpretations work for the cyclic syntax. In an appendix, we give some further information about the arithmetical side of the equation.

1. Introduction

In the present paper, we study Cyclic Henkin Logic $\text{CHL}$. Our original interest in this logic was triggered by the development, in our paper [Vis19], of a class of provability predicates for which we do have Löb’s Rule but for which the third Löb condition, to wit, provable implies provably provable (aka principle 4), fails. It turns out that, in this context, the de Jongh-Sambin-Bernardi Theorem about the uniqueness of modalised fixed points does hold. Thus, a substantial amount of familiar reasoning from Löb’s Logic $\text{GL}$ is still present. On the other hand, in the absence of the third Löb condition, the de Jongh-Sambin Theorem about the explicit definability of fixed points fails. In fact, the most salient fixed point of them all, the Gödel fixed point, does not have an explicit definition.

So, what is the appropriate form for an appropriately weakened version of Löb’s Logic? Our intuition is that the logic in question should have full modalised fixed points. Given that design choice, in the light of non-explicit definability, there seem to be essentially three roads to follow: (i) we try to extend the repertoire of ordinary modal operators so that we do recover explicit definability for the enriched
we add a variable-binding fixed point operator $\mu p.\varphi$ that is only allowed when $\varphi$ is modalised in $p$; (iii) we work with cyclic syntax. We have not looked seriously at option (i), but it very well possible that it is not feasible in a reasonable way. Option (ii) will be worked out in a forthcoming paper by Tadeusz Litak and the author. In the present paper we study option (iii).

We think that the cyclic syntax has a wider interest than just the study of a weaker version of L"ob’s Logic. It also gives us a new way to look at L"ob’s Logic itself. We will show, for example, that a version of the de Jongh-Sambin algorithm to compute explicit fixed points can be viewed as the specification of one half of a synonymy between $GL^0$, i.e. CHL plus the third L"ob condition, and $GL$ in its original formulation. The specification of the algorithm shows that guard recursion is an appropriate way of thinking about this algorithm.

The logic CHL is synonymous with a corresponding theory, the Henkin Calculus or HC, which is formulated in a language with a variable-binding fixed point operator. This will be proved in a forthcoming paper by Tadeusz Litak and the author. In that same paper, we will prove that HC, and, thus, also CHL, is synonymous with the well-founded part of the $\mu$-calculus, i.e., the $\mu$-calculus plus the minimal Henkin sentence $\mu p.\top$. This last result is based on ideas from Johan van Benthem’s paper [VB06], which were extended in a paper by the author [Vis05]. Thus, CHL can be viewed as a treatment of the well-founded part of the $\mu$-calculus on a quite different syntax.

An obvious further step in the project of provability logic on cyclic syntax is to connect circular syntax with circular proofs as studied in, e.g., [Sha14] and [Sha20]. We have not explored this attractive possibility yet.

1.1. Plan of the Paper. A substantial part of the paper is devoted to carefully developing the system. We develop the syntax and introduce the appropriate principles of definition and proof concerning the syntax in Section 2. Then, we introduce CHL and work out the basic facts about the logic in Section 3. In Section 4 we prove the synonymy between $GL^0$ and $GL$. We study some further inter-theory relations in Section 5. Section 6 contains our development of arithmetical interpretations. Finally, Appendix A provides a somewhat closer look at the arithmetical side of the equation.

1.2. Prerequisites. Some knowledge of basic provability logic is helpful. The classical textbooks [Boo93] or [Smo85] are quite sufficient. However, there are many other good expositions available nowadays, like [Lin96], [JdJ98], [Sve00], [AB04].

It would also be good if the reader has at least seen the notion of bisimulation. Any modern introduction to modal logic or to computer science will explain this notion. The books [BE87] and [Acz88] also introduce the notion and also contain some material closely connected to the present paper.

The appendix contains some more advanced material on arithmetic, but the rest of the paper is independent of this.

2. Syntax

In this Section we provide the basics of our cyclic syntax. A major inspiration of our approach is the treatment of the paradoxes in the book [BE87].

In Subsection 2.1 we specify what kind of graphs we employ in our development. Then, in Subsection 2.2 we introduce the relevant notion of formula and develop
some basic proof methods and definition methods. Also, we prove a number of basic facts.

2.1. **Graphs.** Let a non-empty set of labels \( L \) be fixed. We have an arity function \( \text{ar} : L \to \omega \).

We need the following notion of graph: a directed pointed labeled graph with ordered successors. A graph \( G \) is given as a quadrupel \( \langle V, r, S, \lambda \rangle \). Here:

- \( V \) is the set of vertices or nodes. In our paper this set will always be finite.
- \( r \in V \) is the point or root.
- \( \lambda : V \to L \) is the labeling function. A vertex \( a \) with label \( a \) will be called an occurrence of \( a \).
- \( S : V \to V^* \), where \( V^* \) is the set of (finite) sequences of elements in \( V \) (including the empty sequence). We demand that \( \text{length}(S_a) = \text{ar}(\lambda a) \) (in \( G \)).

We write \( S_i(a) \) for \( (S(a))_i \), where \( i < \text{length}(Sa) \).

- Let \( \hat{S} \) be the relation given by \( a \hat{S} b \) iff there is an \( i < \text{length}(Sa) \) such that \( b = S_i a \). A path in the a graph is a sequence of vertices \( a_0 \hat{S} a_1 \hat{S} ... \hat{S} a_k \).

We demand that every vertex can be reached via a finite path from \( r \).

For many purposes the notion of path as defined here is sufficient. However it is also good to have the notion of directive path. To motivate this consider a graph with nodes \( a \) and \( b \) and suppose \( Sa = \langle b, b \rangle \) and \( Sb = \langle a \rangle \). We can have a path \( aba \) that takes the left turn and one that takes the right turn. To distinguish such possibilities, we define a directive path as a sequence \( a_0 \hat{S} a_1 \hat{S} ... \hat{S} a_k \), where \( i_j < \text{ar}(\lambda a_j) \) and \( a_{j+1} = S_i a_j \).

**Remark 2.1.** We opted for the present format for graphs since this is in accordance with the representation as co-algebra. Of course other formats are possible. ✓

A bisimulation between graphs \( G \) and \( G' \) is a relation \( R \) between \( V \) and \( V' \), such that:

i. If \( aRa' \), then \( \lambda a = \lambda a' \).

ii. If \( aRa' \), then \( Sa \) and \( S'a' \) are sequences with the same length \( \ell \) and we have \( S_i a \hat{S} S_i a' \), for all \( i < \ell \).

Two graphs are bisimilar if there is a bisimulation between them that relates their roots. We note that is follows that the bisimulation is total and surjective. We write \( \simeq \) for bisimilarity. We remind the reader of the well-known fact that bisimulations are closed under unions. Thus, there is a maximal bisimulation between any two graphs. Of course, this does not need to relate the roots.

An isomorphism between graphs is a bijective bisimulation that relates the roots. We write \( \cong \) for isomorphism. We will think of our graphs modulo isomorphism. Philosophically, we want to think about the graphs modulo bisimulation, however, it is technically convenient to have the more 'concrete' representations modulo isomorphism available.

**Remark 2.2.** If we would allow infinite graphs, we could define the canonical unraveling of \( G \) as the graph \( G' \) with as domain the directive paths in \( G \). The new successor and labeling functions are as expected. One can then show that two graphs are bisimilar iff their canonical unravelings are isomorphic. ✓

Rooted graphs are our default. On occasion we will also consider unrooted graphs. Of course these are just graphs minus the root. We will allow such graphs to contain...
disconnected parts. The definition of bisimulation remains the same without the condition for the roots. Instead we demand the relation to be total and surjective.

A cycle in a graph is a set of vertices \( C \) such that we can arrange the elements of \( C \) in a path \( a_0 \tilde{S} a_1 \tilde{S} \ldots \tilde{S} a_k \tilde{S} a_0 \). We demand that the \( a_j \) are pairwise distinct. Note that in our definition a cycle has no designated starting point. A vertex is a cycle vertex if it is on a cycle.

We will write \( c(\mathcal{G}) \) for the number of cycles in \( \mathcal{G} \).

Remark 2.3. There is also the notion of directive cycle. A directive cycle is a function \( \gamma \) from a set of vertices \( C \) to numbers, such that \( \gamma(a) < \text{ar}(\lambda a) \). We demand that we can arrange the elements of \( C \) in a directive path \( a_0 i_0 a_1 \ldots a_k i_{k-1} a_0 \), where \( i_j = \gamma(a_j) \). Here the \( a_j \) are pairwise distinct.

We define a number of operations on graphs.

- \( g(a, \mathcal{G}_0, \ldots, \mathcal{G}_{k-1}) \) is the result of taking the disjoint sum of the \( \mathcal{G}_0, \ldots, \mathcal{G}_{k-1} \) and adding a fresh root \( r \) with label \( a \) to this sum. We allow that \( k = 0 \) here.
- \( \mathcal{G} \downarrow a =: \mathcal{G}' \) is subgraph of \( \mathcal{G} \) generated by \( a \). It is defined as follows.
  - \( V' \) is the set of vertices that can be reached via \( a \) (possibly empty) path from \( a \).
  - \( r' := a \).
  - \( \lambda' \) is \( \lambda \) restricted to \( V' \).
  - \( S' \) is \( S \) restricted to \( V' \).
- Suppose the length of \( Sr \) is \( n \) and \( i < n \). Then \( su_i(\mathcal{G}) := \mathcal{G} \downarrow S_i r \).
- \( \text{min}(\mathcal{G}) \) is the result of dividing out the maximal auto-bisimulation of \( \mathcal{G} \).

We have the following obvious lemmas.

Lemma 2.4. Suppose \( \mathcal{G}_0 \simeq \mathcal{G}_1 \) via \( R \) and \( a_0 Ra_1 \). Then \( \mathcal{G}_0 \downarrow a_0 \simeq \mathcal{G}_1 \downarrow a_1 \) via the restriction of \( R \) to the nodes of \( \mathcal{G}_0 \downarrow a_0 \) and \( \mathcal{G}_1 \downarrow a_1 \).

Lemma 2.5. \( \text{min}(\mathcal{G}) \) is bisimulation minimal: all bisimulations on it are subsets of the identity relation on \( V \). Moreover, \( \mathcal{G} \simeq \mathcal{G}' \) iff \( \text{min}(\mathcal{G}) \cong \text{min}(\mathcal{G}') \).

Consider a graph \( \mathcal{G} \). Let \( W \subseteq V \). We say that \( W \) is a guard (for \( \mathcal{G} \)) if every cycle contains an element of \( W \).

Thus, we have guard-induction and guard-recursion in a guarded graph:

Lemma 2.6. Suppose \( W \) is a guard of \( \mathcal{G} \). We have:

i. Suppose we have a property \( P \) of vertices such that all \( g \in W \) have \( P \). Suppose further that if \( a \notin W \) and all \( a \)'s successors have \( P \), then \( a \) has \( P \). Then all vertices have \( P \).

ii. Suppose for every label \( a \), we have a function \( G_a : D^a(a) \rightarrow D \) and suppose \( F : W \rightarrow D \). Then there is a unique function \( H : V \rightarrow D \) such that \( H(a) = F(a) \) if \( a \in W \) and \( H(a) = G_a(H(S_0 a), \ldots, H(S_{k-1} a)) \) if \( a \notin W \) and \( \lambda a = a \) and \( k = \text{ar}(a) \).

Proof. Ad (i): Consider any \( a \) in \( \mathcal{G} \). Let \( B \) be the union of \( W \) with the set of leaves of \( \mathcal{G} \). Consider any vertex \( a \). Clearly in \( \mathcal{G} \downarrow a \), the intersection \( C \) of \( B \) with the vertices in \( \mathcal{G} \downarrow a \) forms a bar, i.e. every indefinitely prolonged path must eventually pass through an element of \( C \). So by bar-induction, the root \( a \) has property \( P \).

The proof of (ii) is similar using bar recursion. \( \square \)
Suppose $W$ is a guard for $G$ and $W'$ is a guard for $G'$. A bisimulation $R$ between $G$ and $G'$ is a $W,W'$-bisimulation if whenever $aRa'$, then $a \in W$ iff $a' \in W'$.

**Lemma 2.7.** Suppose for every label $a$, we have a function $G_a : D^{ar(a)} \to D$. Let $G$ be a guard for $G'$ and let $G'$ be a guard for $G'$. Suppose $R$ is a $G,G'$-bisimulation between $G$ and $G'$. (R does not need to be root-preserving.) Consider $F : G \to D$ and $F' : G' \to D$ such that whenever $a \in W$ and $aRa'$, we have $F(a) = F'(a')$. Let $H$ and $H'$ be the functions guaranteed to exist by Lemma 2.6. Then, for all $a \in V$ and $a' \in V'$, if $aRa'$, then $H(a) = H(a')$.

**Proof.** Let $P(a)$ be the property: for all $a'$ if $aRa'$, then $H(a) = H(a')$. Clearly we have $P$ on $W$. Moreover, it is easy to see that $P$ is preserved from the successors of $a$ to $a$. □

### 2.2. Formulas

We define our formulas. The totality of formulas will constitute our full language $L$. The set of labels $L$ for formulas is given by $\perp, \top, \neg, \Box, \land, \lor, \rightarrow, p_0, p_1, \ldots$, where $ar(\top) = ar(\perp) = ar(p_i) = 0$, $ar(\neg) = ar(\Box) = 1$, $ar(\land) = ar(\lor) = ar(\rightarrow) = 2$. As usual we use also $p,q,r,r',\ldots$ for propositional variables. A formula $\varphi$ is a graph for which $bo(\varphi)$, the set of $\Box$-occurrences, is a guard. We note that the set of $\Box$-occurrences on a cycle $bo^\circ(\varphi)$ also forms a guard.

If the label of the root is not a variable, we call it the main connective. We say that $\varphi$ is modalised in $p$ if every path from the root to an occurrence of $p$ contains a $\Box$-occurrence.

**Remark 2.8.** We can very well model our formulas in the hereditarily finite sets of non-well-founded set theory $\text{AFA}$ with the labels as $u$-elements. Only the guardedness condition is perhaps somewhat unnatural in this context. □

We define some operations on formulas:

- We identify $\top$ with $g(\top)$ and, similarly, for $\perp$ and the $p_i$.\footnote{The convenient confusion between labels and operations has its limits should be treated with some care. We will warn the reader when to tread carefully.}
- $\neg \varphi := g(\neg, \varphi)$ and, similarly, for $\Box$.
- $(\varphi \land \psi) := g(\land, \varphi, \psi)$ and, similarly, for $\lor$ and $\rightarrow$.
- $F p.\varphi$ is the result of identifying the root with all vertices labeled $p$, where one keeps the label of the root. This is only allowed when $\varphi$ is modalised in $p$, since otherwise the resulting graph will not be a formula. It is easily seen that in the resulting graph all cycles contain a $\Box$-occurrence, as desired.
- $\varphi[p_0 : \psi_0, \ldots, p_{k-1} : \psi_{k-1}]$ is the result of the following operation. First, we form the disjoint union of $\varphi$ and the $\psi_i$. Then, simultaneously, we identify the vertices labeled $p_i$ in (the disjoint copy of) $\varphi$ with the root of (the disjoint copy of) $\psi_i$, where we keep the label of the root of $\psi_i$ and discard the label $p_i$.

**Remark 2.9.** Suppose $\varphi$ is modalised in $p$ and $q$. Then $F p.F q.\varphi \equiv F p.\varphi[q : p]$. Thus, we see that one of the costs of the graph approach is that this principle is built in. Of course, one may also consider it as a bonus. □

The operations are safe for bisimulation:
Lemma 2.10. Bisimilarity between formulas is a congruence relation for the operations in the above list.

Proof. We just do the case of $Fp$. Suppose $\varphi$ is bisimilar to $\varphi'$. Let $R$ be the witnessing bisimulation. We define $R'$ between $Fp\varphi$ and $Fp\varphi'$ simply as $R$ restricted to the vertices not labeled by $p$ in $\varphi$. We claim that $R'$ is a root-preserving bisimulation. Consider any $a, a'$ with $aR'a'$. Consider $Sa$ in $Fp\varphi$. The only difference with $Sa$ in $\varphi$ is that all vertices labeled $p$ are now replaced by the root with the label of the root. Similarly, for $S'a'$ in $Fp\varphi'$. However, since $R$ was a root-preserving bisimulation we see that now $R'$ is.

We will sometimes write $\sigma, \tau, \ldots$, for substitutions. If the substitution $\sigma$ is $[q_0 : \psi_0, \ldots, q_{k-1} : \psi_{k-1}]$ and $\tau$ is $[r_0 : \chi_0, \ldots, r_{m-1} : \chi_{m-1}]$, then

$$\sigma \star \tau := [q_0 : \psi_0, \ldots, q_{k-1} : \psi_{k-1}, r_0 : \chi_0, \ldots, r_{m-1} : \chi_{m-1}].$$

This only makes sense if the $q_i$ and the $r_j$ are pairwise disjoint. We note an important insight.

Lemma 2.11. i. $\star$ is associative, assuming that the three domains of the substitutions are pairwise disjoint.

ii. Suppose the $q_i$ are disjoint from the domain of $\tau$. Then,

$$\varphi[q_0 : \psi_0, \ldots, q_{k-1} : \psi_{k-1}]^\tau \equiv \varphi([q_0 : \psi_0 \tau, \ldots, q_{k-1} : \psi_{k-1} \tau] \star \tau).$$

Suppose $p$ does not occur in $\varphi$. We define the following operation:

- $(\varphi \xi p) := \varphi'$ is obtained as follows. If the root of $\varphi$ is not on a cycle $\varphi' := \varphi$. Otherwise, we take an $r^*$ that is not in $V$, the set of vertices of $\varphi$. Let $V' := V \cup \{r^*\}$ and $\lambda' := \lambda \cup \{r^*, p\}$. We define $f : V \rightarrow V'$ by $f(r) = r^*$ and $f(a) = a$, if $a \neq r$. As usual, we write $f(a_0, \ldots, a_{k-1})$ for $(a_0, \ldots, a_{k-1})$. We take $S'a := fSa$ if $a \neq r^*$ and $Sr^* := \varepsilon$.

So, $(\varphi \xi p)$ is the result of redirecting all incoming arrows of the root, if there are any, to a new vertex labeled $p$. The new vertex, of course, does not have outgoing arrows. We note that, whether $p$ occurs in $\varphi$ or not, $(\varphi \xi p)$ is modalised in $p$.

A good heuristic, in case the root is on a cycle, is to view $(\varphi \xi p)$ as a non-deterministic sub-formula of $\varphi$. The number of nodes increases, so in the sense this ‘subformula’ is larger than the original formula. On the other hand, trivially, the number of cycles decreases, so in that sense the ‘subformula’ is smaller. This last feature will be quite useful in the paper.

It will be convenient to write $(\varphi \xi \psi)$ for $(\varphi \xi p)[p : \psi]$.

We enumerate some useful facts about the operation $(\cdot \xi \cdot)$.

Lemma 2.12. Suppose the root of $\varphi$ is a cycle vertex and $p$ does not occur in $\varphi$. Suppose $p$ is modalised in $\psi$. Then,

i. $c(\varphi \xi p) < c(\varphi)$.

ii. $Fp(\varphi \xi p) \equiv \varphi$.

iii. $(\varphi \xi \varphi) \simeq \varphi$.

iv. $Fp \psi \simeq \psi[p : Fp \psi]$.

Remark 2.13. The operation $(\cdot \xi \cdot)$ has to be treated with great care since we may have $\varphi \simeq \varphi'$ but $(\varphi \xi p) \not\simeq (\varphi' \xi p)$. This can be easily seen from the fact that we can always unravel a formula a bit to a bisimilar one of which the root is not on a cycle.

$\square$
Interestingly, there is something like uniqueness of fixed points modulo bisimulation, which gives an intriguing analogy with the de Jongh-Sambin-Bernardi Theorem.

**Theorem 2.14.** Suppose $p$ is modalised in $\varphi$ and $\psi \simeq \varphi[p : \psi]$. Then $\psi \simeq Fp.\varphi$.

*Proof.* Let $E$ be the embedding of the vertices of $\varphi$ that are not occurrences of $p$ into $\varphi[p : \psi]$. Let $F$ be the embedding of the vertices of $\psi$ into the vertices of the substituted copy of $\psi$ in $\varphi[p : \psi]$. We note that $F$ is a (non-root-preserving) bisimulation between $\psi$ and $\varphi[p : \psi]$.

Let $R$ be the maximal bisimulation between $\varphi[p : \psi]$ and $\psi$. We claim that $R^* := E; R$ is a root-preserving bisimulation between $Fp.\varphi$ and $\psi$. (Here ‘;' is composition in the order of reading.)

Clearly $R^*$ connects the roots of $Fp.\varphi$ and $\psi$ (since the root of $\varphi$ cannot be a $p$-occurrence). Consider any node $a$ of $Fp.\varphi$. Then $a$ is, by definition, a non-$p$-occurrence in $\varphi$. Suppose $aRb$, say $aEcRb$. Let $a' := S_{Fp.\varphi}a$ and $b' := S_{\varphi[p:\psi]}b$, $c' := S_{\psi,b,c}, d' := S_{\varphi,b,a}$.

Case 1: Suppose $d'$ is not a $p$-occurrence in $\varphi$. In this case $d' = d'E c'$. Finally, because $cRb$ it follows that $c'Rb'$ and, hence, $a'Rb'$.

Case 2: Suppose $d'$ is a $p$-occurrence in $\varphi$. In this case $d' = r_{F\varphi[p:\psi]}E c'Rb'$. Moreover, $c'$ is the root of the substituted copy of $\psi$ in $\varphi[p : \psi]$ and $c'Rb'$. We have $r_{\varphi[p:\psi]}R_{\psi}Fc'Rb'$. Since, $R; F; R$ is a bisimulation, it is contained in $R$, so $r_{\varphi[p:\psi]}Rb'$. It follows that $a'R_{\varphi[p:\psi]}Rb'$, i.o.w., $a'R^*b'$.

We may conclude that $Fp.\varphi \simeq \psi$. \qed

We note that the above proof does not use the full guard condition. It just uses that the root is not a $p$-occurrence.

The fixed point theorem as given in Lemma 2.12 has a easy generalisation to systems of equations. Suppose $E$ is a system of equations, i.e., a function from a finite set $Q$ of variables to formulas $\varphi_q$. We form a directed (unlabeled) graph (without ordered successors) $G_E$ with domain $Q$ where we have an arrow from $q$ to $q'$ precisely if $\varphi_q$ is not modalised in $q'$. We say that $E$ is *modalised* if $G_E$ is acyclic. This condition generalises the usual one: $\varphi$ is modalised in $p$ iff the equation $p \mapsto \varphi$ is modalised.

We want to solve $E$. This means that we want to find a function $F : q \mapsto \psi_q$ on $Q$, such that $\psi_q \simeq \varphi_q F$, for all $q \in Q$, where in the right-hand-side we view $F$ as a substitution. We demand that the $q' \in Q$ do not occur in the $\psi_q$.

We define a new unrooted graph $\Psi$ as follows. We define a sub-graph $G^\varphi_E$ of $G_E$ as follows. We have an arrow from $q$ to $q'$ if $\lambda_{\varphi_q}(r_{\varphi_{q'}}) = q'$. This new graph is clearly non-cyclic and every $q$ in $Q$ has at most one outgoing arrow. Let $\end(q)$ be the variable at the end of the unique outgoing path from $q$.

We have the following definitions:

- $V_\Psi$ is the set of all pairs $\langle \varphi_q, a \rangle$, where $q \in Q$ and $a \in V_{\varphi_q}$ and $\lambda_{\varphi_q}(a) \notin Q$.
- $\lambda_\Psi(\langle \varphi_q, a \rangle) := \lambda_{\varphi_q}(a)$.
- Suppose $\lambda_{\varphi_q}(a) = q' \in Q$. Then, $\idf(\langle \varphi_q, a \rangle) := \langle \varphi_{\end(q')}, r_{\varphi_{\end(q')}}\rangle$. In all other cases, $\idf(\langle \varphi_q, a \rangle) := \langle \varphi_q, a \rangle$. We note that the label of a value of $\idf$ cannot be in $Q$.
- Suppose $S_{\varphi_q,a} = \langle b_0, \ldots, b_{n-1} \rangle$. Then,
  
  $S_{\Psi}(\varphi_q, a) := \langle \idf(\langle \varphi_q, b_0 \rangle), \ldots, \idf(\langle \varphi_q, b_{n-1} \rangle) \rangle$. 
In a sense the \( r_q \) are multiple roots of \( \Psi \).

\[ E := F, \] where \( q_F := \Psi \downarrow r_q \). We write \( E^q \) for \( E^q q \).

**Theorem 2.15.** \( E^q \) is a solution of \( E \).

**Proof.** Let \( \varphi_q, \psi_q \) and \( F \) be as above. Without loss of generality we may assume that the vertices of \( \Psi \) are disjoint from the vertices of the \( \varphi_q \). We define a bisimulation between \( \psi_q \) and \( \varphi_q F \) as follows:

\[ \alpha R \beta \iff \alpha = \beta \text{ or } (\alpha = \langle \varphi_q, a \rangle \text{ and } \beta = a). \]

We note that, in case \( \varphi_q = q' \in Q \), the second disjunct cannot become active. In this case, \( \psi_q \) and \( \varphi_q F \) will be identical.

Suppose that \( \varphi_q \) is not in \( Q \). We consider the case where \( \alpha = \langle \varphi_q, a \rangle \) and \( \beta = a \).

We need an auxiliary definition. Suppose \( \lambda \varphi_q (a) = q' \in Q \). Then, \( \text{idfy}_0 (a) := \langle \varphi_q, r_q \rangle \). In all other cases, \( \text{idfy}_0 (a) := a \). We lift the relation \( R \) to sequences in the obvious way.

Let \( S_{\varphi_q} a = \langle b_0, \ldots, b_{n-1} \rangle \). Then,

\[ S_{\psi_q} (\alpha) = \langle \text{idfy}(\langle \varphi_q, b_0 \rangle), \ldots, \text{idfy}(\langle \varphi_q, b_{n-1} \rangle) \rangle \\
R \langle \text{idfy}_0 (b_0), \ldots, \text{idfy}_0 (b_{n-1}) \rangle = S_{\varphi_q} F a \]

3. **Cyclic Henkin Logic**

In this section we develop the logic CHL. We choose to develop it for itself and not as part of a wider class of logics, even if, from a systematic standpoint, that would be better. The reason is simply the desire not to overburden the presentation. We will touch on the broader perspective in Section 4 and, specifically, in Subsection 4.1.

Subsection 3.1 provides the basic development of CHL. Curiously, the central result of the subsection is inter-substitution of equivalents, Theorem 3.6. All further development rests on this central result. In Subsection 3.2, we consider some alternative axiomatisations. Finally, in Subsection 3.3, we provide the Kripke semantics for CHL.

3.1. **Basic Development.** Cyclic Henkin Logic or CHL is axiomatised as follows.

- **chl1.** If \( \vdash \varphi \) and \( \vdash \varphi \rightarrow \psi \), then \( \vdash \psi \).
- **chl2.** If \( \vdash \varphi \), then \( \vdash \square \varphi \).
- **chl3.** We have all substitution instances of propositional tautologies. Here we think of these tautologies as given by the usual parse trees.
- **chl4.** \( \vdash \square (\varphi \rightarrow \psi) \rightarrow (\square \varphi \rightarrow \square \psi) \).
- **chl5.** Suppose \( \varphi \simeq \psi \). Then \( \vdash \varphi \leftrightarrow \psi \).
- **chl6.** If \( \vdash \square \varphi \rightarrow \varphi \), then \( \vdash \varphi \).

We note that if we think of our formulas as the result of dividing out bisimularity, then Axiom Scheme chl6 becomes superfluous. That is good since it is the only principle that does not have the standard form of an axiom scheme.

Our first five axioms amount to something like cyclic \( K \). Löb’s Rule provides this system with bite. We remind the reader that ordinary \( K \) is closed under Löb’s Rule, so it is the presence of circularity that makes the rule powerful.

We will use the notation \( \Gamma \vdash_{\text{CHL}} \varphi \), where \( \Gamma \) is a finite set of formulas for: \( \text{CHL} \vdash \bigwedge \Gamma \rightarrow \varphi \). Trivially, we have the deduction rule for this notion.

The following theorem tells us that CHL is indeed a logic.
Theorem 3.1. If CHL ⊨ ϕ, then CHL ⊨ ϕ[p₀ : ψ₀, . . . , pₖ₋₁ : ψₖ₋₁].

Proof. By a simple induction on proofs. In the case of chl5, this uses the safety of substitution. □

Theorem 3.2. i. CHL ⊨ ϕ ↔ (ϕ ⊦ ϕ).

ii. CHL ⊨ F p.ψ ↔ ψ[p : F p.ψ], assuming that ψ is modalised in p.

Proof. This is immediate by the fact that ϕ ≃ (ϕ ⊦ ϕ) and F p.ψ ≃ ϕ[p : F p.ψ]. □

We define □⁺ ϕ := F p.□(ϕ ∧ p), where p does not occur in ϕ.

Theorem 3.3. We have:

i. If CHL ⊨ □ϕ, then CHL ⊨ □⁺ ϕ.

ii. If CHL ⊨ ϕ, then CHL ⊨ □⁺ ϕ. (Necessitation for □⁺)

iii. If CHL ⊨ □⁺ ϕ → ϕ, then CHL ⊨ ϕ. (LR for □⁺)

Proof. We verify (i). Suppose CHL ⊨ □ϕ. We have:

□⁺ ϕ ⊨ CHL □(ϕ ∧ □⁺ ϕ)

□⁺ ϕ ⊨ CHL □⁺ ϕ

Hence, by Löb’s rule, ⊨ CHL □⁺ ϕ.

(ii) is immediate from (i).

We verify (iii). Suppose □⁺ ϕ ⊨ CHL ϕ. Then, □(ϕ ∧ □⁺ ϕ) ⊨ CHL ϕ ∧ □⁺ ϕ. By LR, we find CHL ⊨ ϕ.

Theorem 3.4. □⁺ satisfies Löb’s Logic GL over CHL.

Proof. We have L1, i.e. necessitation, by Theorem 3.3(ii). We verify L2.

□⁺(ϕ → ψ) → (□⁺ ϕ → □⁺ ψ)) ⊨ CHL □⁺(ϕ → ψ) → □⁺((ϕ → ψ) ∧ □⁺(ϕ → ψ))

□⁺((ϕ → ψ) → (□⁺ ϕ → □⁺ ψ))

□⁺(ϕ → ψ) → □⁺(ϕ ∧ □⁺ ϕ)

□⁺(ϕ ∧ □⁺ ϕ → (ψ ∧ □⁺ ψ))

□⁺(ψ ∧ □⁺ ϕ) → □⁺(ϕ ∧ □⁺ ψ))

□⁺(ϕ → □⁺ ψ)

So, by Löb’s Rule, we are done. We verify L3.

□⁺(ϕ → □⁺ □⁺ ϕ) ⊨ CHL □⁺ ϕ → □⁺ □⁺ ϕ

□⁺(ϕ ∧ □⁺ □⁺ ϕ) → □⁺ □⁺ □⁺ ϕ

By Löb’s Rule we are done. We verify Löb’s Principle L4. We have:

□⁺(□⁺(ϕ → ϕ) → □⁺ ϕ) ⊨ CHL □⁺(□⁺ ϕ → ϕ) → □⁺ □⁺(□⁺ ϕ → ϕ)

□⁺ □⁺(□⁺ ϕ → ϕ) → □⁺ □⁺ ϕ

By LR for □⁺ (Theorem 3.3(iii)), we are done. □
We write \(\Box \varphi\) for \(\varphi \land \Box \varphi\) and, similarly for \(\Box^*\).

**Corollary 3.5.** We have strengthened Löb’s Rule for \(\Box^*\) over CHL, i.e.,

\[
\text{if } \bigwedge_{i<n} \Box^* \psi_i, \bigwedge_{j<m} \Box^* \chi_j \vdash \Box \varphi \rightarrow \varphi, \text{ then } \bigwedge_{i<n} \Box^* \psi_i, \bigwedge_{j<m} \Box^* \chi_j \vdash \varphi.
\]

We may simply copy the usual derivation of strengthened Löb’s Rule from Löb’s Principle. For completeness we repeat the argument.

**Proof.** Suppose \(\bigwedge_{i<n} \Box^* \psi_i, \bigwedge_{j<m} \Box^* \chi_j \vdash \Box \varphi \rightarrow \varphi\). Then,

\[
(\dagger) \quad \bigwedge_{i<n} \Box^* \psi_i, \bigwedge_{j<m} \Box^* \chi_j \vdash \Box^* \varphi \rightarrow \varphi.
\]

It follows that \(\bigwedge_{i<n} \Box^* \psi_i, \bigwedge_{j<m} \Box^* \chi_j \vdash \Box^* (\Box^* \varphi \rightarrow \varphi)\) and, hence, that

\[
(\ddagger) \quad \bigwedge_{i<n} \Box^* \psi_i, \bigwedge_{j<m} \Box^* \chi_j \vdash \Box^* \varphi.
\]

Combining (\(\dagger\)) and (\(\ddagger\)), we find: \(\bigwedge_{i<n} \Box^* \psi_i, \bigwedge_{j<m} \Box^* \chi_j \vdash \varphi\) \(\Box\).

Now we are ready to prove a principle that will be a central tool: a strong form of substitution of equivalents.

**Theorem 3.6.** Suppose that the variables \(s_i\), for \(i < n\) and \(r_j\) for \(j < m\) are pairwise distinct; and that \(\varphi\) is modalised in the \(r_i\). Then, we have:

\[
\bigwedge_{i<n} \Box^*(\psi_i \leftrightarrow \chi_i), \bigwedge_{j<m} \Box^*(\theta_j \leftrightarrow \rho_j) \vdash_{\text{CHL}} \varphi[s := \vec{\psi}, r := \vec{\theta}] \leftrightarrow \varphi[s := \vec{\chi}, r := \vec{\rho}].
\]

**Proof.** Suppose a formula \(\psi\) and substitutions \(\sigma\) and \(\tau\) on \(q_0, \ldots, q_{k-1}\) are given. We consider a conjunction \(\alpha = \bigwedge_{i<k} \Delta_i(q_i \sigma \leftrightarrow q_i \tau)\), where \(\Delta_i\) is either \(\Box^*\) or \(\Box^*\).

We say that \(\alpha\) is acceptable for \(\psi, \sigma, \tau\) if, whenever \(\Delta_i\) is \(\Box^*\), then \(\psi\) is modalised in \(q_{i}\).

We prove: for every \(\varphi\), we have \(P(\varphi)\), where:

\[
P(\varphi) : \quad \leftrightarrow \quad \text{for every } q_i, \text{for every } \sigma, \tau \text{ on } q_i,
\]

\[
\quad \text{for every } \alpha \text{ acceptable for } \varphi, \sigma, \tau, \text{ we have } \alpha \vdash \varphi \leftrightarrow \varphi \tau.
\]

The proof is by course-of-values induction for \(c(\varphi)\). As a sub-induction we use guard-induction using bo\(^*\)(\(\varphi\)), the set of cycle \(\Box\)-occurrences, as a guard. So, in the sub-induction we prove the induction property for \(\varphi \downarrow a\).

We treat the examples of conjunction and box, splitting the second case into the sub-cases that the occurrence is not on a cycle or is on a cycle.

Suppose \(a\) is labeled \(\land\). Suppose \(\alpha\) is acceptable for \((\varphi \downarrow a), \sigma, \tau\). Clearly, \(\alpha\) is also acceptable for the \((\varphi \downarrow S_{i}a), \sigma, \tau\). We have, by the safety of substitution, \(\varphi \downarrow a \simeq (\varphi \downarrow S_{0}a \land \varphi \downarrow S_{1}a)\) and, hence, \((\varphi \downarrow a) \sigma \simeq ((\varphi \downarrow S_{0}a) \sigma \land (\varphi \downarrow S_{1}a) \sigma)\) and, similarly, for \(\tau\). So, assuming the desired property for the \(\varphi \downarrow S_{i}a\), we are done.

Suppose \(a\) is labeled with a box, but not on a cycle. We have \(\varphi \downarrow a \simeq \Box(\varphi \downarrow S_{i}a)\).

Let \(\alpha\) be acceptable for \((\varphi \downarrow a), \sigma, \tau\). Let \(\alpha^* := \bigwedge_{i<k} \Box^*(q_i \sigma \leftrightarrow q_i \tau)\). Clearly, \(\alpha^*\) is acceptable for \((\varphi \downarrow S_{i}a), \sigma, \tau\). By guard-induction, we may assume: \(\alpha^* \vdash_{\text{CHL}} (\varphi \downarrow b) \sigma \leftrightarrow (\varphi \downarrow b) \tau\). It follows, by K-reasoning, that

\[
\Box \alpha^* \vdash_{\text{CHL}} \Box(\varphi \downarrow b) \sigma \leftrightarrow \Box(\varphi \downarrow b) \tau.
\]

Moreover, \(\alpha \vdash_{\text{CHL}} \Box \alpha^*\). So, \(\alpha \vdash_{\text{CHL}} (\Box(\varphi \downarrow b)) \sigma \leftrightarrow (\Box(\varphi \downarrow b)) \tau\), as desired.
Suppose $a$ is a box-occurrence on a cycle, i.e., $a$ is in the chosen guard. Let $\varphi_0$ be $\varphi \Downarrow a$. Suppose $\alpha$ is acceptable for $\varphi_0, \sigma, \tau$. We choose $p$ not in $\varphi_0$, nor in the $\psi_i, \chi_i, \theta_j$ and $\rho_j$ and distinct from the $q_i$ and $r_j$. We have:

$$
\varphi_0 \sigma \simeq (\varphi_0 \Downarrow p) [p : \varphi_0] \sigma \\
\varphi_0 \tau \simeq (\varphi_0 \Downarrow p) (\sigma \star [p : \varphi_0] \tau).
$$

Similarly, $\varphi_0 \sigma \simeq (\varphi_0 \Downarrow p) (\sigma \star [p : \varphi_0] \tau)$.

Since $c(\varphi_0 \Downarrow p) < c(\varphi_0) \leq c(\varphi)$, we may apply the main induction hypothesis.

Since $(\varphi_0 \Downarrow p)$ is modalised in $p$, we find that $\alpha \land \Box (\varphi_0 \Downarrow p)$ is acceptable for $(\varphi_0 \Downarrow p), \sigma \star [p : \varphi_0] \tau$. Thus, by the main induction hypothesis:

$$
\alpha, \Box (\varphi_0 \Downarrow p) \Downarrow CHL \varphi_0 \sigma \leftrightarrow (\varphi_0 \Downarrow p) \Downarrow \varphi_0 \tau
$$

By the strengthened Löb’s Rule, we may omit the assumption $\Box (\varphi_0 \Downarrow p)$ and we are done.

**Theorem 3.7.** Suppose $\varphi$ and $\psi$ are modalised in $p$ and $CHL \vdash \varphi \leftrightarrow \psi$, then $CHL \vdash F.p.\varphi \leftrightarrow F.p.\psi$.

**Proof.** Suppose $CHL \vdash \varphi \leftrightarrow \psi$. Then,

$$
\Box (F.p.\varphi \leftrightarrow F.p.\psi) \Downarrow CHL F.p.\varphi \leftrightarrow \varphi[p : F.p.\varphi] \\
\leftrightarrow \psi[p : F.p.\varphi] \\
\leftrightarrow \psi[p : F.p.\psi] \\
\leftrightarrow F.p.\psi
$$

So, by Löb’s rule, we are done. 

We prove the de Jong-Sambin-Bernardi Theorem about the uniqueness of fixed points.

**Theorem 3.8.** Suppose $\varphi$ is modalised in $p$. Then

$$
CHL \vdash \Box (p \leftrightarrow \varphi) \to (p \leftrightarrow F.p.\varphi).
$$

**Proof.** We have:

$$
\Box (p \leftrightarrow \varphi) \Downarrow CHL \Box (p \leftrightarrow F.p.\varphi) \to (p \leftrightarrow \varphi) \\
\leftrightarrow \varphi[p : F.p.\varphi] \\
\leftrightarrow F.p.\varphi
$$

By the Strengthened Löb’s Rule, we may omit the assumption $\Box (p \leftrightarrow F.p.\varphi)$. 

We generalise the de Jong-Sambin-Bernardi Theorem to systems of equations as follows.

**Theorem 3.9.** Suppose $E$ is modalised. Then,

$$
\bigwedge_{q \in Q} \Box (q \leftrightarrow qE) \Downarrow CHL \bigwedge_{q \in Q} (q \leftrightarrow F_q.\varphi).
$$
Proof. We write \( \varphi_q := qE \) and \( \psi_q := I_q E \) and \( F := I E \). Let \( \chi := \bigwedge_{q \in Q} [q \leftrightarrow \varphi_q] \) and \( \rho := \Box \bigwedge_{q \in Q} (q \leftrightarrow \psi_q) \). We prove \( \chi, \rho \vdash q \leftrightarrow \varphi_q \), by induction on \( G_E \).

Let \( Q_q \) be the set of all \( q' \) that can be reached from \( q \) in \( G_E \) via a non-empty path. We suppose we have the desired result for all \( q' \) in \( Q_q \). Let \( F_q \) be restriction of \( F \) to the \( Q_q \). It follows that we have:

\[
\chi, \rho \vdash CHL q \leftrightarrow \varphi_q \leftrightarrow \varphi_q F_q \leftrightarrow \varphi_q F \leftrightarrow \psi_q
\]

The first equivalence is by \( \chi \). The second equivalence follows by the combination of \( \rho \) and the induction hypothesis. The third equivalence follows by \( \rho \) in combination with the fact that all variables from \( Q \) in \( \varphi_q F_q \) are guarded: the only variables that \( \varphi_q \) can 'see' in \( G_E \) have been removed by the substitution. Moreover, no variables from \( Q \) occur in the \( \psi_q' \) that are substituted. The fourth and fifth equivalence are immediate.

We have shown that:

\[
\chi, \Box \bigwedge_{q \in Q} (q \leftrightarrow \varphi_q) \vdash CHL \bigwedge_{q \in Q} (q \leftrightarrow \psi_q).
\]

So, by the Strengthened Löb’s Rule, we are done. \( \Box \)

Consider a formula \( \varphi \). We assign to each \( \Box \)-occurrence \( a \) a propositional variable \( q_a \), where the \( q_a \) are pairwise distinct and also distinct from the propositional variables of \( \varphi \). We map the nodes of \( \varphi \) to formulas of the language of ordinary modal logic, i.e., the cycle-free formulas, as follows:

- \( E_\varphi(a) := q_a \), if \( a \) is a \( \Box \)-occurrence.
- \( E_\varphi(a) := (E_\varphi(S_0a) \land E_\varphi(S_1a)) \), if \( a \) is a \( \land \)-occurrence. Similarly, for the other connectives and for the propositional variable-occurrences in \( \varphi \).

Our definition is correct by guard-recursion. We write \( \psi_a \) for \( E_\varphi(a) \).

**Theorem 3.10.** Let \( G \) be the set of \( \Box \)-occurrences of \( \varphi \). We have:

\[
\bigwedge_{a \in G} [q \leftrightarrow \Box \psi_{S_0a}] \vdash CHL \psi_r \leftrightarrow \varphi.
\]

**Proof.** We show by guard induction with \( \text{bo}(\varphi) \) as guard that, for all nodes \( b \), we have:

\[
\Box \bigwedge_{e \in V} (\psi_e \leftrightarrow \varphi \downarrow e), \bigwedge_{a \in G} [q \leftrightarrow \Box \psi_{S_0a}] \vdash \psi_b \leftrightarrow \varphi \downarrow b.
\]

We first treat the case of conjunction. Suppose \( b \) is a \( \land \)-occurrence. We assume our desired conclusion for \( S_0b \) and \( S_1b \). We have:

\[
\Box \bigwedge_{e \in V} (\psi_e \leftrightarrow \varphi \downarrow e), \bigwedge_{a \in G} [q \leftrightarrow \Box \psi_{S_0a}] \vdash CHL \psi_b \leftrightarrow (\psi_{S_0b} \land \psi_{S_1b}) \leftrightarrow (\varphi \downarrow S_0b) \land (\varphi \downarrow S_1b) \leftrightarrow \varphi \downarrow b
\]
The reasoning for the propositional variable-occurrences of $\varphi$ and for the other non-box connectives is similar. Suppose $b$ is a $\square$-occurrence. We have:

$$\square \bigwedge_{e \in V} (\psi_e \leftrightarrow \varphi \downarrow e), \bigwedge_{a \in G} \square^*(q_a \leftrightarrow \square \psi_{S_0 a}) \vdash_{\text{CHL}} q_b \leftrightarrow \square \psi_{S_0 b}$$

So we find:

$$\square \bigwedge_{e \in V} (\psi_e \leftrightarrow \varphi \downarrow e), \bigwedge_{a \in G} \square^*(q_a \leftrightarrow \square \psi_{S_0 a}) \vdash_{\text{CHL}} \bigwedge_{e \in V} (\psi_e \leftrightarrow \varphi \downarrow e)$$

We apply the strengthened version of Löb’s Rule to obtain:

$$\bigwedge_{a \in G} \square^*(q_a \leftrightarrow \square \psi_{S_0 a}) \vdash_{\text{CHL}} \bigwedge_{e \in V} (\psi_e \leftrightarrow \varphi \downarrow e)$$

For this last insight, the desired conclusion is immediate. $\square$

**Remark 3.11.** One would hope that the same kind of treatment we give here for CHL would be also possible for the $\mu$-calculus. Formulas are defined in the same way, only the guarding constraint is replaced by the following constraint. Consider any directive cycle $\gamma : C \rightarrow \omega$ in $\varphi$. Consider the set $X$ consisting of all occurrences $a$ of $\neg$ in $C$ plus all occurrences $b$ of $\rightarrow$ in $C$ such that $\gamma(b) = 0$. We demand that $X$ has an even number of elements. In stead of Löb’s Rule one would have the Minimality Rule: if $\vdash (\varphi \downarrow \alpha) \rightarrow \alpha$, then $\vdash \varphi \rightarrow \alpha$. I have no idea how the details of this will work out. What replaces guard-induction and recursion? $\square$

### 3.2. Alternative Axiomatisations

We provide some alternative axiomatisations for CHL. We consider the following axioms an rules.

- **IPE:** Suppose $\varphi$ and $\psi$ are modalised in $p$ and $\vdash \varphi \leftrightarrow \psi$, then $\vdash F p. \varphi \leftrightarrow F p. \psi$ (Intersubstitutivity of Provable Equivalents).
- **N*:** If $\vdash \varphi$, then $\vdash \square^* \varphi$.
- **H:** $\vdash H$, where $H = F p. \square p$.

We define the following theories:

- $K$ is given by chl1-5.
- $K^\circ$ is $K$ plus IPE.
- $CHL_0$ is $K^\circ$ plus H.
- $CHL_1$ is $K^\circ$ plus N*.

Here $K^\circ$ is the reasonable circular version of $K$. We will show that over this theory Löb’s rule is equivalent with the Henkin sentence (as axiom).

**Theorem 3.12.** CHL$_0$ and CHL$_1$ prove the same theorems as CHL.

**Proof.** To prove inclusion of CHL$_1$ in CHL$_0$, we show that CHL$_0$ is closed under N*. Suppose $\vdash_{\text{CHL}_0} \varphi$. It follows that $\vdash_{\text{CHL}_0} \square (p \land \varphi) \leftrightarrow \square p$. Hence, by IPE, we have $\vdash_{\text{CHL}_0} F p. \square (\varphi \land p) \leftrightarrow F p. \square p$, i.e., $\vdash_{\text{CHL}_0} \square^* \varphi \leftrightarrow H$. So, by axiom H, we find $\vdash_{\text{CHL}_0} \square^* \varphi$.

To prove inclusion of CHL in CHL$_1$ it is sufficient to show that CHL$_1$ is closed under Löb’s Rule. This uses a well-known proof of Löb’s Rule. Suppose $\vdash_{\text{CHL}_1} \ldots$
\( \square \varphi \rightarrow \varphi \). Let \( \nu := F q \square (q \rightarrow \varphi) \), where \( q \) does not occur in \( \varphi \). We have:

\[
\text{CHL}_1 \vdash \nu \rightarrow \square (\nu \rightarrow \varphi) \\
\rightarrow \square (\nu \rightarrow \varphi) \land \square (\nu \rightarrow \varphi)) \\
\rightarrow \square (\nu \rightarrow \varphi) \land \nu) \\
\rightarrow \square \varphi \\
\rightarrow \varphi
\]

So, we have \((a)\) \( \text{CHL}_1 \vdash \nu \rightarrow \varphi \). Hence, by \( \mathbb{N}^* \), we find \( \text{CHL}_1 \vdash \square (\nu \rightarrow \varphi) \). It follows that \((b)\) \( \text{CHL}_1 \vdash \nu \). Combining \((a)\) and \((b)\), we find: \( \vdash \varphi \).

The inclusion of \( \text{CHL}_0 \) in \( \text{CHL} \) follows from our previous results. \( \Box \)

3.3. Kripke Semantics. A Kripke model for \( \text{CHL} \) is given by a triple \( \langle W, \sqsubseteq, f \rangle \), where \( W \) is a non-empty set of worlds, \( \sqsubseteq \) is an acyclic binary relation on \( W \) and \( f : W \times \text{Prop} \rightarrow \{0,1\} \) where \( \text{Prop} \) is the set of propositional variables.

Consider any formula \( \varphi \). We define \( \text{Ev}_\varphi : W \times V \rightarrow \{0,1\} \) as follows. We define \( \text{Ev}_\varphi (w, \cdot) \) assuming that we already have defined \( \text{Ev}_\varphi (w', \cdot) \) for all \( w' \sqsubseteq w \). We use guard-recursion w.r.t. \( \text{bo} (\varphi) \) as guard.

- If \( a \) is an occurrence of \( p \), then \( \text{Ev}_\varphi (w, a) = f(w, p) \).
- If \( a \) is an occurrence of \( \wedge \), then

\[
\text{Ev}_\varphi (w, a) = \min (\text{Ev}_\varphi (w, S_0 a), \text{Ev}_\varphi (w, S_1 a))
\]

and, similarly for the other non-box connectives.

- If \( a \) is a \( \square \)-occurrence, we set \( \text{Ev}_\varphi (w, a) = 1 \) iff, for all \( w' \sqsubseteq w \), we have \( \text{Ev}_\varphi (w', S_0 a) = 1 \). We take \( \text{Ev}_\varphi (w, a) = 0 \), otherwise.

We note that this definition works since in the clause for box do not call on values for \( w \) but on the previously defined values for \( w' \sqsubseteq w \).

We define \( \llbracket \varphi \rrbracket (w) := \text{Ev}_\varphi (w, r_\varphi) \) and we write \( w \vdash \varphi \) for \( \llbracket \varphi \rrbracket (w) = 1 \). Since our guard is preserved by bisimulation, we find that if \( \varphi \simeq \varphi' \), then \( \llbracket \varphi \rrbracket (w) = \llbracket \varphi' \rrbracket (w) \).

We easily verify the usual clauses like \( w \vdash \varphi \land \psi \) iff \( w \vdash \varphi \) and \( w \vdash \psi \). Using this the validity of \( \text{CHL} \) is immediate.

Remark 3.13. We can derive the Kripke completeness theorem for \( \text{CHL} \) if finite acyclic models in two ways. The first is using the synonymy of \( \text{CHL} \) and the Henkin Calculus \( \text{HC} \), which is essentially \( \text{CHL} \) on standard syntax using a fixed point operator. We can prove the Kripke completeness for \( \text{HC} \), for example, by showing that \( \text{HC} \) is synonymous to the well-founded part of the \( \mu \)-calculus and invoking the Kripke completeness theorem for the \( \mu \)-calculus. We will give the details of this argument in a sequel to this paper by Tadeusz Litak and myself. Alternatively, we can use the Kripke Completeness of \( \text{WfL} \) and use the reduction of \( \text{CHL} \) to that theory: see Subsection \([5,2]\) for more detail. Of course, it would be much nicer to give a direct proof of the completeness theorem in terms of the circular syntax. To do this remains open. \( \Box \)

4. Löb’s Logic meets Cyclic Syntax

The counterpart of \( \text{CHL} \) in cycle-free language is the Henkin Calculus \( \text{HC} \). This calculus employs a variable-binding fixed point operator in the object language. The logics \( \text{CHL} \) and \( \text{HC} \) are synonymous/definitionally equivalent. We will establish this fact in a later paper in which \( \text{HC} \) is developed. Here we will treat the simpler case
of the synonymy of GL², the extension of CHL with the transitivity axiom scheme L₃, aka 4, and Löb’s Logic GL.

In Subsection 4.1, we set up a modest framework for a province in which both theories live and some basics of comparing theories living in that province. In Subsection 4.2, the logic GL° is introduced and in Subsection 4.3 we do the same for ordinary GL (as it appears in our framework). In Subsection 4.4, we develop the de Jongh-Sambin algorithm as one half of the witness of a synonymy between GL² and GL.

4.1. Languages, Translations and Interpretations. In this section we look at a restricted class of logics and employ a very restricted framework of interpretations. The interpretations are something like □-preserving logic-interpretations. However, in this section we will call them ℓ-interpretations.

A language in this section will simply be a sub-set of our full language that is closed under (i) bisimilarity, (ii) the propositional variables and the syntactic operations associated with the connectives, (iii) subformulas and (iv) substitution. The minimal language is L, the set of all acyclic formulas. We may view L as the ordinary language of modal logic since each bisimulation equivalence class of an acyclic formula has a unique finite tree (modulo isomorphism) as normal form.

We define HL(L) as the logic in the language L that is axiomatised by chl₁–6. One easily sees that this definition makes sense. We note that CHL = HL(L°) and that K proves the same theorems as HL(L). A logic Λ will be a rule-preserving extension of one of the HL(L) in the same language by schematic rules.

A salient language is L°. This is the language generated by the variables, the logical connectives including □ and ◻. We define WfL := HL(L°).

Here is an important observation.

Observation 4.1. Suppose L is a language that extends L°. Then Theorems 3.1, 3.2, 3.3, 3.4, 3.6 and Corollary 3.5 remain valid when we replace CHL by HL(L).

We can simply check that these results do not use closure under the operation ◻p.(·).

A local translation T of a formula ϕ into a logic Λ is a mapping from Vϕ to the language of Λ, such that if a is $-occurrence for a connective $ of arity n or a variable treated as having arity 0, where

Λ ⊢ T(a) ↔ $T(S₀a),...,$T(Sₙ₋ₑa)).

Strictly speaking the local translation is given as the triple of the formula, the function and the logic.

Theorem 4.2. Consider any logic Λ. Suppose T is a local translation of of ϕ in Λ and T’ is a local translation of of ϕ’ in Λ. Suppose further that R is a bisimulation between ϕ and ϕ’, where R does not need to be root-preserving. Suppose aRa’.

Then, Λ ⊢ T(a) ↔ T’(a’).

Proof. Let χ := \bigwedge_{bRa'} (T(b) ↔ T'(b')). We prove by guard-induction on □-occurrences d in ϕ that, if dRd’, then □χ ⊢ Λ T(d) ↔ T'(d’). The cases of the non-box occurrences are trivial. In case d is a □-occurrence, we have:

□χ ⊢ T(d) ↔ □T(S₀d) ↔ □T'(S₀d') ↔ T'(d')
It follows that $\square \chi \vdash_{\Lambda} \chi$. So, by Löb’s rule, we are done.

A global translation $\mathcal{G}$ of a language $\mathcal{L}$ into a logic $\Lambda$ is a function from $\mathcal{L}$ to the language of $\Lambda$ that commutes modulo $\Lambda$-provable equivalence with the propositional variables and the connectives. Strictly speaking a global translation is given as the triple of language, function and logic. We usually omit the ‘global’ of ‘global translation’.

We collect some trivial insights.

**Theorem 4.3.**

i. Suppose $\mathcal{G}$ is a global translation of $\mathcal{L}$ into $\Lambda$. Let $\varphi$ be an $\mathcal{L}$-formula. For $a \in V_\varphi$, we define $T(a) := \mathcal{G}(\varphi \downarrow a)$. Then, $T$ is a local translation of $\varphi$ in $\Lambda$.

ii. Suppose every $\varphi$ in $\mathcal{L}$ has a local translation $T(\varphi)$ in $\Lambda$. Then $\mathcal{G}$ with $\mathcal{G}(\varphi) := T(\varphi)(r_\varphi)$ is a global translation of $\mathcal{L}$ in $\Lambda$.

iii. If $\mathcal{G}$ and $\mathcal{G}'$ are global translations of $\mathcal{L}$ in $\Lambda$. Then, for each $\varphi$, we have $\Lambda \vdash \mathcal{G}(\varphi) \leftrightarrow \mathcal{G}'(\varphi)$.

A translation $\mathcal{G}$ is an $\ell$-translation if it commutes with substitution. This means that, for all formulas $\varphi$ and all $\mathcal{L}$-substitutions $\sigma$, we have $\Lambda \vdash \mathcal{G}(\varphi \sigma) \leftrightarrow \mathcal{G}(\varphi)(G \circ \sigma)$.

A interpretation $\mathcal{K} : \Lambda \rightarrow \Lambda'$ is given as a triple of $\Lambda$, a function $\mathcal{G}$, and $\Lambda'$, where $\mathcal{G}$ is a translation from the language of $\Lambda$ into $\Lambda'$. We demand that if $\Lambda \vdash \varphi$, then $\Lambda' \vdash \mathcal{G}(\varphi)$.

We show that in $\text{GL}^\circ$ the modalities $\Box$ and $\Box^\circ$ coincide.

**Theorem 4.4.** $\text{GL}^\circ \vdash \Box^\circ \varphi \leftrightarrow \Box \varphi$.

**Proof.** We have:

$$
\Box(\Box^\circ \varphi \leftrightarrow \Box \varphi) \vdash_{\text{GL}^\circ} \Box^\circ \varphi \leftrightarrow \Box(\varphi \land \Box^\circ \varphi) \\
\leftrightarrow \Box(\varphi \land \Box \varphi) \\
\leftrightarrow \Box \varphi.
$$
Thus, all insights that we accumulated for \( \mathbf{C} \) in CHL transfer to \( \square \) in GL\(^{\circ} \).

Here is a GL\(^{\circ} \) reformulation of a well-known insight due to Dick de Jongh.

**Lemma 4.5.** Suppose the root of \( \psi \) is an \( \square \)-occurrence. Then, we have \( \psi \equiv^\circ (\psi \nabla T) \).

**Proof.** If the root of \( \psi \) is not on a cycle, this is trivial. So, suppose the root is on a cycle. Let \( \chi := s \Diamond (\psi) \). We have \( \psi \equiv^\circ \Box \chi \), and hence \( GL^{\circ} \vdash \psi \rightarrow \Box \psi \). Similarly, for \( (\psi \nabla T) \). Suppose \( p \) does not occur in \( \psi \). We have:

\[
\psi \vdash_{GL^{\circ}} \Box (\psi \nabla p)[p : \psi]
\]

\[
\vdash_{GL^{\circ}} (\psi \nabla p)[p : T]
\]

We also have:

\[
\Box (\psi \nabla T) \vdash_{GL^{\circ}} (\psi \nabla p)[p : T]
\]

By the Strenghsted L"ob's rule, we find \( (\psi \nabla T) \vdash_{GL^{\circ}} \psi \). \( \square \)

4.3. L"ob's Logic without Cycles. In our context we may define L"ob's Logic \( GL \) simply as \( HL(\mathbb{L}) \) plus \( L3 \). We note that this is \( K \) plus L"ob's Rule plus \( L3 \). We will employ the usual facts about \( GL \) and especially the de Jongh result:

**Lemma 4.6.** \( GL \vdash (\Box \varphi)[p : T] \leftrightarrow (\Box \varphi)[p : (\Box \varphi)[p : T]] \).

4.4. The de Jongh-Sambin Interpretation. We define functions \( j_2 \) and \( j_2^* \). Our aim is to show that \( j_2^*(\varphi, \cdot) \) is a local \( t \)-translation of \( \varphi \) to \( GL \).

* Suppose \( a \) is not in \( bo^{\circ}(\varphi) \). We treat the case of a \( \land \)-occurrence, the other cases being similar. We take \( j_2^*(\varphi, a) := (j_2^*(\varphi, a_0) \land j_2^*(\varphi, a_1) \land \cdots) \).

* Suppose \( a \) is in \( bo^{\circ}(\varphi) \). Then, \( j_2^*(\varphi, a) := j_2^*((\varphi \downarrow a \nabla T), a) \).

* \( j_2(\varphi) := j_2^*(\varphi, r_\varphi) \).

We have:

**Lemma 4.7.** \( j_2^*(\varphi, a) \) is in \( \mathbb{L} \).

**Lemma 4.8.** \( j_2^*(\varphi, a) = j_2(\varphi \downarrow a) \).

The desired results follow trivially by course of values induction on \( c(\varphi) \) and by guard induction on \( bo^{\circ}(\varphi) \).

We prove a result on commutation with substitution:

**Lemma 4.9.** Let \( \sigma \) be a substitution on \( Q \). Then, for \( a \in V_\varphi \), we have \( j_2^*(\varphi \sigma, a) \simeq j_2^*(\varphi, a)(j_2 \circ \sigma) \).

**Proof.** The proof is by course of values induction on \( c(\varphi) \) and by guard induction on \( bo^{\circ}(\varphi) \).

Suppose \( a \notin bo^{\circ}(\varphi) \). Suppose, e.g., \( a \) is a \( \land \)-occurrence. We have:

\[
j_2^*(\varphi \sigma, a) = (j_2^*(\varphi \sigma, a_0) \land j_2^*(\varphi \sigma, a_1)) \simeq (j_2^*(\varphi, a_0)(j_2 \circ \sigma) \land j_2^*(\varphi, a_1)(j_2 \circ \sigma)) \simeq j_2^*(\varphi, a)(j_2 \circ \sigma) \]
The cases where $a$ is an occurrence of a variable not in $Q$ or where $a$ is an occurrence of another connective are similar. Suppose $a$ is an occurrence of $q \in Q$. We have:

\[
 js^*(\varphi \sigma, a) \equiv js(\varphi \downarrow a) \\
= js(q \sigma) \\
= q(js \circ \sigma) \\
= js(q)(js \circ \sigma) \\
= js(\varphi \downarrow a)(js \circ \sigma) \\
\equiv js^*(\varphi, a)(js \circ \sigma)
\]

Suppose $a \in bo^0(\varphi)$. We have:

\[
 js^*(\varphi \sigma, a) = js^*((\varphi \downarrow a) \uplus \top, a) \\
= js^*((\varphi \downarrow a) \uplus \top \sigma, a) \\
\equiv js^*((\varphi \downarrow a) \uplus \top)(js \circ \sigma) \\
= js^*(\varphi, a)(js \circ \sigma)
\]

We write $\equiv^0$ for provable equivalence in $GL^0$ and $\equiv$ for provable equivalence in $GL$.

For the proof of our main insight, Theorem 4.11, we need a lemma, that is a strengthening of Theorem 4.2. Consider formulas $\varphi$ and $\varphi'$. Let $F : V_{\varphi} \rightarrow L$ and let $F' : V_{\varphi'} \rightarrow L$. We define $C_{\varphi}(F)$ as the conjunction of formulas $F(a) \leftrightarrow \$$(F(S_0 a), \ldots, F(S_{n-1} a))$, where $\$ is the label of $a$ in $\varphi$ and where $\$ is $n$-ary. We treat the variable as a 0-ary operation here.

**Lemma 4.10.** Suppose $R$ is a bisimulation between $\varphi$ and $\varphi'$ and $aRa'$. Then,

i. $\Box C_{\varphi}(F) \land \Box C_{\varphi'}(F') \vdash_{GL} F(a) \leftrightarrow F(a')$.

ii. $\Box C_{\varphi}(F) \land \Box C_{\varphi'}(F') \vdash_{GL} \Box F(a) \leftrightarrow F(a')$.

**Proof.** The proof of (i) is an immediate adaptation of the proof of Theorem 4.2, replacing Löb’s Rule by the Strengthened Löb’s Rule. Item (ii) is immediate from (i).

**Theorem 4.11.** $js^*(\varphi, \cdot)$ is a local translation of $\varphi$ into $GL$.

**Proof.** We employ course of values induction on the number of cycles in $\varphi$ and, then, guard induction on $bo^0(\varphi)$. The only non-trivial case is where $a \in bo^0(\varphi)$. So, suppose $a \in bo^0(\varphi)$. We have $js^*(\varphi, a) = js^*((\varphi \downarrow a) \uplus \top, a)$. Let us write $\psi_b := js^*(\varphi, b)$, for $b \in V_\varphi$ and $\psi_c := js^*((\varphi \downarrow a) \uplus p, c)$, for $c \in V_{(\varphi \downarrow a) \uplus p}$, where $p$ is a fresh variable. Let $S'$ be the successor function of $((\varphi \downarrow a) \uplus p)$.

We have (\downarrow):

\[
\begin{align*}
\psi_a &= js^*((\varphi \downarrow a) \uplus p)[p : \top, a] \\
&= \psi'_a[p : \top] \\
&= (\Box \psi'_S a)[p : \top] \\
&= (\Box \psi'_S a)[p : (\Box \psi'_S a)[p : \top]] \\
&= (\Box \psi'_S a)[p : \psi_a] \\
&= (\Box \psi'_S a)[p : \psi_a]
\end{align*}
\]
Here the second step is by Lemma 4.9. The third step uses the fact that $a$ is not on a cycle in $(\langle \varphi \downarrow a \rangle \downarrow p)[p : T]$. The fourth step uses Lemma 4.6.

We now define $F$ from $V_{\varphi,a}$ to $L$ as follows. $F(a) := \psi_a$ and $F(b) := \psi'_b[p : \psi_a]$ if $a \neq b$. We prove that $F$ is a local translation from $\varphi \downarrow a$ to GL. The case of $a$ is by (†). Suppose $b$ is a $\dagger$-occurrence unequal to $a$, where $\dagger$ is $n$-ary. By the induction hypothesis, we have $GL \vdash \psi'_b \leftrightarrow S(\psi'_{S_0b})$ and hence $GL \vdash \psi'_b[p : \psi_a] \leftrightarrow S(\psi'_{S_0b}[p : \psi_a])$. We note that:

1. $\psi'_b[p : \psi_a] = F(b)$;
2. if $S_1b$ is not a $\ddagger$-occurrence, then $\psi'_{S_1b}[p : \psi_a] = \psi'_{S_0b}[p : \psi_a] = F(S_1b)$;
3. if $S_1b$ is a $\ddagger$-occurrence, then $\psi'_{S_1b}[p : \psi_a] = \psi_a = F(a) = F(S_1b)$.

Thus, we have $GL \vdash C_{\varphi,a}(F)$ and, hence, (†) $GL \vdash \Box C_{\varphi,a}(F)$.

We define $G$ from $V_{\varphi}$ to the acyclic formulas by $G(b) := js^*(\varphi, b)$. It follows that:

$\Box C_{\varphi}(G) \vdash GL \Box C_{\varphi,a}(F) \land \Box C_{\varphi}(G)$
$\vdash GL F(S_0a) \leftrightarrow G(S_0a)$
$\vdash GL F(S_0a) \leftrightarrow G(S_0a)$
$\vdash GL G(a) \leftrightarrow G(S_0a)$

Here the first step is by (†), the second step is by Lemma 4.10(ii) and the fourth step is by (†).

Since $a$ was an arbitrary $\Box$-occurrence on a cycle, we find $\Box C_{\varphi}(G) \vdash GL C_{\varphi}(G)$. Ergo, $GL \vdash C(G)$. Thus, $G = js^*(\varphi, \cdot)$ is a local translation of $\varphi$ to GL.

\begin{theorem}
js carries an $\ell$-interpretation $JS$ of $GL^o$ in $GL$.
\end{theorem}

\begin{proof}
By Theorem 4.11, $js^*(\varphi, \cdot)$ is a local translation of $\varphi$ in $GL$. It follows that $js$ is a translation of $L^\circ$ into $GL$. Our translation is an $\ell$-translation by Lemma 4.9.

So we need just to verify the translations of the axioms and rules of $GL^\circ$ in $GL$. However, the translations of the axioms and rules of $GL^\circ$ except ch5, are all instances of the same axioms and rules of $GL$, modulo $GL$-provable equivalence. The axiom ch5 follows by Theorem 4.12.
\end{proof}

The identical translation $emb$ of $L$ into $GL^o$ clearly carries an $\ell$-interpretation of $GL$ in $GL^o$. So, the pair $JS$, Emb forms an $\ell$-syonymy. Since the arrows of a synonymy are faithful, it follows that $GL^o$ is conservative over $GL$. In other words, $GL$ is the acyclic fragment of $GL^o$.

Consider any acyclic $\varphi$. Clearly, $GL \vdash js(\varphi) \leftrightarrow \varphi$. (Inspection shows that we even have $dj(\varphi) \simeq \varphi$.) Suppose $p$ is modalsised in $\varphi$. We have: $GL^o \vdash f p.\varphi \leftrightarrow \varphi[p : f p.\varphi]$. It follows that $GL \vdash js(f p.\varphi) \leftrightarrow \varphi[p : js(f p.\varphi)]$. Thus $js(f p.\varphi)$ is a de Jongh-Sambin explicit fixed point of $\varphi$.

\begin{remark}
Of course, we could also develop the synonymy by using the known de Jongh-Sambin Theorem for $GL$. However, the advantage of the present set-up is that we can see the algorithm stated using guard recursion.
\end{remark}
5. Further Inter-theory Relations

We develop the relations of CHL to two other theories, to wit, Multiple Fixed Point Theory MFT and Well-foundedness Logic \( WfL \) (or: \( HL(L^\circ) \)). In Subsection 5.1, we discuss MFT. Our result on MFT will play a role in the definition of arithmetical interpretations. In Subsection 5.2, we address \( WfL \). Our result in that subsection is an ingredient of one possible Kripke completeness proof for CHL.

5.1. Multiple Fixed Point Theory. We define the following theory MFT in \( L \) extended with fresh constants. We have \( K \) plus Löb’s Rule plus, for every system of equations \( \mathcal{E} \) on a set of fixed-point variables \( Q \) for the modal language without constants, constants \( c_{\mathcal{E},q} \) and axioms stating that these constants solve \( \mathcal{E} \). Let us say that an extension \( \Theta \) of MFT \( ( \text{in the same language} ) \) is a strong extension if it is closed under necessitation and Löb’s Rule. Here we do not demand that \( \Theta \) is closed under substitution: it is a theory not a logic.

We adapt the notion of local translation to the new setting in the following way. In the CHL-MFT direction we allow formulas containing the constants. In the MFT-CHL connection we allow the constants to be interpreted by formulas. Similarly, for global translations.

We have an immediate adaptation of Theorem 4.2 to the slightly modified setting that we formulate here for completeness.

**Theorem 5.1.** Suppose \( R \) is a bisimulation between \( \varphi \) and \( \varphi' \) and suppose \( \mathcal{T} \) is a local translation of \( \varphi \) in \( \Theta \) and \( \mathcal{T}' \) is a local interpretation of \( \varphi' \) in \( \Theta \). Suppose further that, for any variable \( p \), if \( a \) and \( a' \) are occurrences of \( p \), then \( \Theta \vdash \mathcal{T}(a) \leftrightarrow \mathcal{T}'(a') \).

Our next order of business is to prove the existence of a local translation of \( \varphi \). In Section 4 we introduced the mapping \( E \). For convenience, we repeat the definition here. Consider a formula \( \varphi \). We assign to each \( \Box \)-occurrence \( a \) a propositional variable \( q_a \), where the \( q_a \) are pairwise distinct and also distinct from the propositional variables of \( \varphi \). We map the nodes of \( \varphi \) to formulas of the language of ordinary modal logic as follows:

- \( E_{\varphi}(a) := q_a \), if \( a \) is a \( \Box \)-occurrence.
- \( E_{\varphi}(a) := (E_{\varphi}(S_0a) \land E_{\varphi}(S_1a)) \), if \( a \) is a \( \land \)-occurrence. Similarly, for the other connectives and for the propositional variable-occurrences in \( \varphi \).

We write \( \psi_a \) for \( E_{\varphi}(a) \). We note that \( \psi_a \) is cycle-free.

Let \( \mathcal{E} \) be defined by \( q_a \mathcal{E} := \Box E_{\varphi}(S_0a) \), where \( a \) is an \( \Box \)-occurrence in \( \varphi \). In MFT this system of equations has a solution, say \( q_a \mathcal{F} := c_a \). Finally we define, for every \( b \in V_{\varphi} \), the mapping \( \text{cyco}^*_{\varphi} \) by \( \text{cyco}^*_{\varphi}(b) := \psi_b \mathcal{F} \).

**Lemma 5.2.** \( \text{cyco}^*_{\varphi} \) is a local translation of \( \varphi \) in MFT.

**Proof.** The cases where \( b \) is not a \( \Box \)-occurrence are simple. Suppose \( b \) is an \( \Box \)-occurrence. We need that MFT \( \vdash \psi_b \mathcal{F} \leftrightarrow \Box \psi_{S_0b} \mathcal{F} \). However, this is precisely MFT \( \vdash c_b \leftrightarrow \Box \psi_{S_0b} \mathcal{F} \), the promised solution of \( \mathcal{E} \).

It follows that \( \text{cyco} \) defined by \( \text{cyco}(\varphi) := \text{cyco}^*_{\varphi}(r_{\varphi}) \) is a global translation of \( L^\circ \) in MFT. Using Theorem 5.1, we now find:

**Theorem 5.3.** There is an interpretation \( \text{CyCo} \) based on \( \text{cyco} \) of CHL in MFT.
In the other direction we define a translation \( \text{cocy} \) that commutes with propositional variables and connectives and that sends a constant \( c_q \) in introduced for a system of equations \( E \) to \( F \). The solution of \( E \) for \( q \) as guaranteed by Theorem \( 2.14 \). We find:

**Theorem 5.4.** There is an interpretation \( \text{CoCy} \) based on \( \text{cocy} \) of MFT in CHL. Moreover, this interpretation is unique.

**Proof.** The verification that \( \text{cocy} \) carries an interpretation is entirely as expected. For the uniqueness we use Theorem \( 5.9 \). \( \Box \)

Using the analogue of Theorem \( \ref{5.9} \) in MFT, we find:

**Theorem 5.5.** \( \text{CyCo} \) and \( \text{CoCy} \) form a synonymy.

We note that the synonymy we are looking at here is a synonymy of theories not logics. It is a form of sameness weaker than the sameness of \( \text{GL}^0 \) and \( \text{GL} \).

5.2. **Well-Foundedness Logic.** It is interesting to note that the global translations \( JS \) and \( \text{Emb} \) still carry a synonymy between WfL plus \( L^3 \) and \( \text{GL} \) when we restrict \( JS \) to \( L^* \). In this case we simply have \( JS(\Box \varphi) = \Box JS(\varphi) \wedge \top \).

We borrow a result form a forthcoming paper with Tadeusz Litak: we have the announced result, we have:

**Announced Theorem 5.6.** Consider any \( L \)-formula \( \varphi \). We use \( \psi_a := E_\varphi(a) \) as in Subsection \( \ref{5.7} \). We have:

\[
\text{CHL} \vdash \varphi \iff \bigwedge_{a \in \text{bo}(\varphi)} \Box^*(q_a \leftrightarrow \Box \psi_{S_0 a}) \vdash_{\text{WfL}} \psi_r.
\]

**Proof.** Suppose \( \bigwedge_{a \in \text{bo}(\varphi)} \Box^*(q_a \leftrightarrow \Box \psi_{S_0 a}) \vdash_{\text{WfL}} \psi_r \). Then, a fortiori,

\[
\bigwedge_{a \in \text{bo}(\varphi)} \Box^*(q_a \leftrightarrow \Box \psi_{S_0 a}) \vdash_{\text{CHL}} \psi_r.
\]

Then, by Theorem \( \ref{5.10} \) we have: (†) \( \bigwedge_{a \in \text{bo}(\varphi)} \Box^*(q_a \leftrightarrow \Box \psi_{S_0 a}) \vdash_{\text{CHL}} \varphi \). We note that \( \varphi \) does not contain occurrences of the \( q_a \). By the Simultaneous Fixed Point Theorem \( \ref{2.15} \) we have CHL-verifiable solutions \( \chi_a \) of the equations in the antecedent conjunction. Substituting these for the \( q_a \), we find CHL \( \vdash \varphi \).

Conversely, suppose \( \bigwedge_{a \in \text{bo}(\varphi)} \Box^*(q_a \leftrightarrow \Box \psi_{S_0 a}) \not\vdash_{\text{WfL}} \psi_r \). By the announced Completeness Theorem for WfL, we have an acyclic finite Kripke model \( K \) with root \( r \) such that \( r \not\models \bigwedge_{a \in \text{bo}(\varphi)} \Box^*(q_a \leftrightarrow \Box \psi_{S_0 a}) \) and \( r \not\models \psi_r \). Clearly the forcing relation for \( L^* \) is the restriction of the forcing relation on \( L^0 \) to \( L^* \). Since \( K \) forces CHL it follows, by Theorem \( \ref{3.10} \) that \( r \not\models \varphi \). So, \( \text{CHL} \not\models \varphi \). \( \Box \)

**Announced Theorem 5.7.** We have completeness for CHL in finite acyclic Kripke models.

**Proof.** Suppose \( \text{CHL} \not\models \varphi \). Then, \( \bigwedge_{a \in \text{bo}(\varphi)} \Box^*(q_a \leftrightarrow \Box \psi_{S_0 a}) \not\vdash_{\text{WfL}} \psi_r \). So, we have, by the announced completeness theorem for WfL, a Kripke model \( K \) with root \( r \) such that \( r \models \bigwedge_{a \in \text{bo}(\varphi)} \Box^*(q_a \leftrightarrow \Box \psi_{S_0 a}) \) and \( r \not\models \psi_r \). It follows by Theorem \( \ref{3.10} \) that \( r \not\models \varphi \). \( \Box \)
6. ARITHMETICAL INTERPRETATIONS

In this section, we introduce the notion of *arithmetical interpretation* and verify its basic properties.

We fix a theory $U$, an interpretation $N$ of $S^1_2$ in $U$. We suppose that we have a provability predicate $\text{bew}$ for numerals in $N$ that satisfies L1.2 plus Löb’s Rule. We write $\models A$ for $\text{bew}(\ulcorner A \urcorner)$.

**Remark 6.1.** In our paper [Vis19], we show that if we have a predicate that uniformly semi-represents a given axiom set of $U$ (w.r.t. $N$) in $U$. The predicate $\pro{\phi}{\alpha}$, where $\pro$ is a standard provability predicate, has the desired properties we ask of $\text{bew}$. We discuss these ideas in some detail in Appendix A.5.

A translation of $\phi$ in $U$ (for $\text{bew}$) is a mapping $\mathcal{I}$ from $V_\phi$ to $U$-sentences, such that:

- If $a$ is an occurrence of $\wedge$ then $U \vdash \mathcal{I}(a) \leftrightarrow (\mathcal{I}(S0a) \wedge \mathcal{I}(S1a))$. Similarly for the other non-box connectives.
- If $a$ is an occurrence of $\Box$ and $S^a = (b)$, then $U \vdash \mathcal{I}(a) \leftrightarrow \Box \mathcal{I}(S0a)$.
- If $a$ and $a'$ are occurrences of $p$, then $U \vdash \mathcal{I}(a) \leftrightarrow \mathcal{I}(a')$.

An interpretation of $L^0$ in $U$ is a mapping $\mathcal{U}$ from $L^0$ to $U$-sentences that commutes modulo $U$-provability with the propositional connectives and the commutes with $\models$ in the sense that $U \vdash \mathcal{U}(\phi) \leftrightarrow \mathcal{U}(\phi)$.

Modulo $U$-provability, arithmetical translations are preserved under bisimulation.

**Theorem 6.2.** Suppose $R$ is a bisimulation between $\phi$ and $\phi'$ and suppose $\mathcal{I}$ is a translation of $\phi$ in $U$ and $\mathcal{I}'$ is a translation of $\phi'$ in $U$. Suppose further that, for any variable $p$, if $a$ and $a'$ are occurrences of $p$, then $U \vdash \mathcal{I}(a) \leftrightarrow \mathcal{I}'(a')$. Then, whenever $aRa'$, we have $U \vdash \mathcal{I}(a) \leftrightarrow \mathcal{I}'(a')$.

The proof is just a minor variation of the proof of Theorem 4.2. We note that it follows that, if we have translations $\mathcal{I}_\phi$ for each $\phi \in L^0$, we can base an interpretation on them by defining $\mathcal{U}(\phi) := \mathcal{I}_\phi(r_\phi)$. The Uniqueness Theorem guarantees that the local pieces add up to a coherent whole.

As a preparation of the theorem concerning the existence of arithmetical translations, we first remind the reader of the simultaneous fixed point lemma and provide two proofs of it.

**Lemma 6.3.** Suppose $N$ interprets $S^1_2$ in $U$. Consider $U$-formulas $A_i(x_0, \ldots, x_{n-1}, \vec{y})$ for $i < n$. Here the variables $x_i$ range over the domain of $N$. Then, there are formulas $B_0(\vec{y}), \ldots, B_{n-1}(\vec{y})$ such that

$$U \vdash B_i(\vec{y}) \leftrightarrow A_i(\ulcorner B_0(\vec{y}) \urcorner, \ldots, \ulcorner B_{n-1}(\vec{y}) \urcorner, \vec{y}),$$

for $i < n$.

**First Proof.** The proof is by induction on $n$. By the usual fixed point lemma with parameters we find $C_0(x_1, \ldots, x_{n-1}, \vec{y})$ such that

$$U \vdash C_0(x_1, \ldots, x_{n-1}, \vec{y}) \leftrightarrow A_0(\ulcorner C_0(\vec{x}_1, \ldots, \vec{x}_{n-1}, \vec{y}) \urcorner, x_1, \ldots, x_{n-1}, \vec{y}).$$

We assume here that the variables of a formula or term are among the variables that are displayed.

We employ the Smoryński dot notation.
Now we define:

\[ C_{i+1}(x_1, \ldots, x_{n-1}, \vec{y}) := A_{i+1}(\check{C}_0(\check{x}_1, \ldots, \check{x}_{n-1}, \check{\vec{y}}), \check{x}_1, \ldots, \check{x}_{n-1}, \check{\vec{y}}). \]

We solve the system \( C_1, \ldots, C_{n-1} \) as is guaranteed by the induction hypothesis, resulting in \( B_1, \ldots, B_{n-1} \). Finally we set:

\[ B_0(\vec{y}) := C_0(\check{B}_1(\check{\vec{y}}), \ldots, \check{B}_{n-1}(\check{\vec{y}}), \check{\vec{y}}). \]

It is now easy to see that the \( B_i \) form the desired simultaneous fixed point. \( \square \)

Second Proof. Let us write \( x[w] \) for the term representing the result of substituting the numeral of \( w \) for the variable \( z \) in the formula represented by \( x \). We define \( D(z, x, \vec{y}) := \bigvee_{i<n}(z = \check{i} \land A_i(x[\vec{0}], \ldots, x[n-1], \vec{y})). \) Let \( E(z, \vec{y}) \) be the fixed point of \( D \) w.r.t. \( x \). Then, we can take \( B_i(\check{\vec{y}}) := E(\check{i}, \check{\vec{y}}). \)

The notion of interpretation for MFT in \( U \) is the obvious one, since MFT has acyclic syntax. Using the simultaneous fixed point lemma, we find:

**Theorem 6.4.** For every \( f \) from the propositional variables \( U \)-sentences, there is an interpretation \( FA_f \) based on \( fa_f \) of MFT in \( U \) such that \( fa_f(p) = f(p) \).

**Theorem 6.5.** For every \( f \) from the propositional variables \( U \)-sentences, there is an interpretation \( HA_f \) based on a translation \( ha_f \) of CHL in \( U \) such that \( fa_f(p) = f(p) \). This interpretation is unique modulo provable equivalence.

**Proof.** We take \( ha_f := fa_f \circ \text{cyco.} \) For the verification of chl5 and for uniqueness, we use Theorem 6.2. \( \square \)

We note that we can view \( ha_f \) as a mapping of formulas modulo bisimulation to elements of the diagonalised algebra of \( U \).

If \( U \) satisfies some further schematic logical principle, it is easy to see that \( ha_f \) carries an interpretation \( H \) of CHL plus that logical principle. So, e.g., if \( \square \) satisfies L3 in \( U \), we have in interpretation \( H \) based on \( ha_f \) of GL° in \( U \), etcetera.

**Remark 6.6.** Clearly, we can use Theorem 4.12 to prove arithmetical completeness for GL° in a \( \Sigma^0_1 \)-sound extension \( U \) of Elementary Arithmetic for Fefermanian provability with respect to an elementary representation \( \alpha \) of the axiom set. We can do it directly. It is somewhat remarkable that (a lifted version of) Theorem 6.5 is not needed in the proof. The Solovay construction delivers translations with the desired properties directly.

Here is a sketch of how this works. Consider any tail-model \( K \) for GL/GL°. See [Vis84] for this notion. Let \( X \) be a finite or cofinite set of nodes. We write \([X]\) for (an appropriate paraphrase of) \( \exists x \in X \ell = x \), where \( \ell \) is the limit statement constructed by Solovay (for \( K \) and for \( \square_n \)). We can now show: \( U \vdash [X \land Y] \leftrightarrow [X] \land [Y] \) and, similarly, for the other non-modal connectives. We write

\[ \square X := \{y \mid \forall x (y \sqsubseteq x \Rightarrow x \in X)\}. \]

We have: \( U \vdash [\square X] \leftrightarrow \square_n[X]. \)

Consider any \( \varphi \in L^\circ \). We define \( S^\circ_{L,\varphi}(a) := \{x \mid x \models \varphi \downarrow a\} \). (Here we need to check that \( \{x \mid x \models \psi\} \) is always finite or co-finite not just for \( L \) as is proven in [Vis84], but also for \( L^\circ \).) It is easy to see that we can base an interpretations
$S_K$ on the $S^a_{K^c}$ and that these interpretations witness the desired arithmetical completeness for $GL^c$.

Thus, in this proof we use the Magari Algebra of $K$ to replace the use of $MFT$ in the proof of Theorem 6.5. We interpret $GL^c$ in the Magari Algebra of $K$ via our result on evaluation in Kripke models and we interpret $K$ in $U$ via the standard Solovay argument as applied to tail models.

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Appendix A. Arithmetical Matters

In this appendix, we summarise the relevant ideas from our paper [Vis19] and prove some additional results that connect that paper to the present one.

In Subsection A.1 we provide some basic insights. In Subsection A.2 we reprove the version of the Second Incompleteness Theorem from [Vis19] using the notations of the present paper. In Subsection A.3 we revisit the transitive closure modality $\Box$ and show that some of its properties also hold globally. Moreover, we show that, under certain conditions, the Fefermanianness of the input modality is preserved to its transitive closure. In Subsection A.4 we discuss the operations of Craigification and Smoothening. Finally, we consider an example with some remarkable properties in Subsection A.5.

A.1. Preliminaries. We consider theories $T$ in predicate logic of finite signature. We allow $T$ to have any complexity. We always assume that $T$ is equipped with an interpretation $N : S^a_2 \to T$. Unless stated otherwise, displayed variables range over the domain of $N$. In other words, we pretend that $T$ is an arithmetical theory.
Moreover, we will use single variables to range over the domain of $N$, even if $N$ might be multi-dimensional.

Our treatment is, at places, somewhat dependent on details of the coding. Let us simply assume that we base our coding on an alphabet that contains (at least) the signs of the language plus some extra brackets "[" and "]". Our Gödel numbering is given by the length-first ordering of strings from this alphabet. We code finite sets of expressions as (the Gödel numbers of) strings of expressions, so $\lceil A_0 A_1 \ldots A_{k-1} \rceil$ and finite sets of numbers as the finite sets of their efficient numerals. We represent proofs with assumptions as strings of formulas where some formulas, the assumptions, are enclosed between square brackets. We code finite functions $B_i \mapsto n_i$ from sentences to numbers as strings of the form $B_0 \overline{n_0} \ldots B_m \overline{n_m}$. Etcetera. Of course, most of these details are immaterial. The main things we need are properties like the following: the function that sends $n$ to the Gödel number of its efficient numeral, $\lceil \overline{n} \rceil$, is $p$-time; the assumption set of a proof can be efficiently extracted from the proof; et cetera.

Consider a $T$-predicate bew. We write $\square A$ for $\text{bew}(\lceil A \rceil)$. Here the numerals are defined w.r.t. $N$. We say that $\text{bew}$ is a provability predicate if it satisfies the first two Löb conditions w.r.t. $T$.

L1. If $T \vdash A$, then $T \vdash \square A$.
L2. $T \vdash (\square A \land \square (A \rightarrow B)) \rightarrow \square B$.

A special class of provability predicates are the Feferman-style predicates. This works as follows. We fix arithmetisations of $\text{proof}^T(p, y)$ and $\text{ass}(p)$, where $\text{proof}^T(p, y)$ is a good arithmetisation of: $p$ codes a proof in predicate logic of $y$ (of the ambient signature) from assumptions in $\text{ass}(p)$. We write:

- $\text{proof}_x(p, y)$ for $(\text{proof}^T(p, y) \land \forall z \in \text{ass}(p) \alpha(z))$,
- $\text{prov}_\alpha(y)$ for $\exists p \text{proof}_x(p, y)$,
- $\square_\alpha A$ for $\text{prov}_\alpha(\lceil \alpha \rceil)$.

Now consider the theory $T$ with axiom set $X$. We suppose we have an interpretation $N$ of $S^T_L$ in $T$. Suppose that $\alpha$ semi-numerates $X$ in $T$, i.e., if $A \in X$, then $T \vdash \alpha(\lceil A \rceil)$. In these circumstances, it is easy to see that $\text{prov}_\alpha^N$ is a provability predicate for $T$. We say that $\text{prov}_\alpha^N$ is a Fefermanian provability predicate. We note that $\alpha$ need not be of the form $\beta^N$, where $\beta$ is an arithmetical predicate.

If $X_0$ is a finite set of $T$-sentences, we write $[X_0]$ for $\bigvee_{B \in X_0} \lceil B \rceil$. We write $[A]$ for $\{[A]\}$. We note that $\square_{\bot}$, $\square_\top$ and $\square_\perp$ all represent provability in predicate logic of the ambient signature.

Here are some further definitions.

- $\text{bew}$ is an LR-provability predicate for $T$ if, it satisfies (L2) and $T$ is closed under Löb's rule LR for $\text{bew}$: if $T \vdash \square A \rightarrow A$, then $T \vdash A$.
- $\text{bew}$ is a uniform provability predicate for $T$, if it satisfies the following three principles
  - $\text{L}^u_1$. Whenever $T \vdash A$, there is finite set of $T$-sentences $X_0$ such that, $X_0 \vdash A$ and, for each $B \in X_0$, we have $T \vdash B$ and $X_0 \vdash \square B$.
  - $\text{L}^u_2$. $T \vdash \forall b (\text{prov}_0^N(b) \rightarrow \text{bew}(b))$.
  - $\text{L}^u_3$. $T \vdash \forall a \forall b (\text{bew}(a) \land \text{bew}(\text{imp}(a, b))) \rightarrow \text{bew}(b)$.
- $\text{bew}$ is a global provability predicate, if it satisfies L1 and $\text{L}^u_2$ and $\text{L}^u_3$.

We provide some basic results on these notions.
Theorem A.1. Suppose $\text{bew}$ is an LR-provability predicate for $T$. Then, $\text{bew}$ is a provability predicate for $T$.

Proof. The proof is just a minor adaptation of an argument due to Dick de Jongh. Suppose $\Box$ is an LR-provability predicate for $T$. Suppose $T \vdash A$. Then, by $K$-reasoning, $T \vdash \Box(A \land \Box A) \rightarrow (A \land \Box A)$. So, by Löb’s rule, we have $T \vdash \Box A$.

Theorem A.2. If $\text{bew}$ is a uniform provability predicate, it is a global provability predicate. If $\text{bew}$ is a global provability predicate, it is a provability predicate.

Proof. We verify e.g. that a uniform provability predicate satisfies $L_1$. Suppose $T \vdash A$. We find $X_0 = \{B_0, \ldots, B_{n-1}\}$ as promised. We have $X_0 \vdash A$ and, hence, $T \vdash \Box_X^\mathcal{N} \bigwedge X_0 \rightarrow A$. It follows that $T \vdash \Box C$. We also have $T \vdash \Box B_i$ for $i < n$, so by repeated application of $L_1\text{m} 3$, we find $T \vdash \Box A$.

The next theorem is obvious.

Theorem A.3. Any Fefermanian provability predicate is global.

A.2. A Version of the Second Incompleteness Theorem. We present the relevant result of [Vis19] in the terminology of the present paper.

Theorem A.4. Suppose $\Box$ is a uniform provability predicate for $T$. Then, $\Box$ is an LR-provability predicate for $T$.

Proof. Let $\Box$ be a uniform provability predicate for $T$. Suppose $T \vdash \Box A \rightarrow A$. Let $C$ be the conjunction of the following statements:

- $(\bigwedge S_3^1)^\mathcal{N}$ (we assume that the axioms of $S_3^1$ include the axioms of identity),
- $\forall b \left[ \text{prov}_{X_0}^\mathcal{N}(b) \rightarrow \text{bew}(b) \right]$,  
- $\forall a \forall b \left[ (\text{bew}(a) \land \text{bew}(\text{imp}(a, b))) \rightarrow \text{bew}(b) \right]$, 
- $\Box A \rightarrow A$.

Let $X_0$ be as promised for $C$ in the definition of uniformity. We have $X_0 \vdash C$. Hence,

$$ X_0 \vdash \Box_{X_0}^\mathcal{N} A \rightarrow \Box_{X_0}^\mathcal{N} \left( \bigwedge X_0 \rightarrow A \right) \rightarrow \Box \left( \bigwedge X_0 \rightarrow A \right) \rightarrow \Box A \rightarrow A $$

So, we find that $X_0 \vdash \Box_{X_0}^\mathcal{N} A \rightarrow A$. Since we have Löb’s Rule for finitely axiomatised theories with standard axiomatisation (w.r.t. an interpretation $N$ of $S_3^1$), we find $X_0 \vdash A$ and, hence, $T \vdash A$.

Open Question A.5. It would be interesting to have an example of a provability predicate for which we have the Second Incompleteness Theorem, but not Löb’s Rule.

It would be interesting to have an example of an LR-provability predicate that is not uniform.
A.3. Global Properties of the Transitive Closure Modality. We show that a number of desirable properties can be lifted from \textsc{bew} to \textsc{bew}*. Suppose \textsc{bew} is an LR-provability predicate for \textsc{T} w.r.t. \textsc{N}, where \textsc{N} is an interpretation of \textsc{S}^1_2. Let \(a, b, c, \ldots\) range over codes of sentences. We use \textsc{conj}, \textsc{imp}, etcetera, for the arithmetisations of the obvious syntactical operations.

By the fixed point lemma with parameters we find \textsc{bew}*, such that:

\[
\bullet \ T \vdash \textsc{bew}^*(a) \leftrightarrow \textsc{bew}^\mathcal{L}(\textsc{conj}(a, \textsc{bew}^*(\bar{a}\bar{\gamma}))).
\]

This is just our previous definition but now not locally for each sentence but for all sentences at once. The definition is unproblematic using the fixed point construction with parameters. We note that by the uniqueness result our global definition will locally coincide with whatever way we implemented the local definitions.

**Theorem A.6.** Suppose \textsc{bew} is a global LR-provability predicate for \textsc{T} w.r.t. \textsc{N}, where \textsc{N} is an interpretation of \textsc{S}^1_2. Then, \textsc{bew}* is a global LR-provability predicate w.r.t. \textsc{N}.

**Proof.** We assume the conditions of the theorem. We can copy the reasoning of the proof of Theorem 3.3 to show that \textsc{bew} is closed under necessitation and Löb’s Rule.

We verify \textsc{L}^m2 for \textsc{bew}*. Let \(A := (\bigwedge \textsc{S}^1_2)^\mathcal{N}\), where we assume \textsc{S}^1_2 to include the theory of identity. Reason in \textsc{T}. Suppose \textsc{prov}^\mathcal{N}_0(b). By the formalisation in \textsc{S}^1_2 of global \(\exists^1\Sigma_1\)-completeness, we find \textsc{prov}_0(\textsc{imp}(\neg A\neg, \neg \textsc{prov}^\mathcal{N}_0(b))). It follows that \textsc{bew}(\textsc{imp}(\neg A\neg, \neg \textsc{prov}^\mathcal{N}_0(b))). Since, by necessitation, we also have \textsc{bew}(\neg A\neg). So, we find, by \textsc{L}^m3 for \textsc{bew}, that \textsc{bew}(\neg \textsc{prov}^\mathcal{N}_0(b)). We leave \textsc{T}. We have shown:

\[
T \vdash \forall b (\textsc{prov}^\mathcal{N}_0(b) \rightarrow \textsc{bew}(\neg \textsc{prov}^\mathcal{N}_0(b))).
\]

It follows that:

\[
\square (\forall a (\textsc{prov}^\mathcal{N}_0(a) \rightarrow \textsc{bew}^*(a))) \vdash_T \textsc{prov}^\mathcal{N}_0(b) \rightarrow (\textsc{bew}(b) \land \textsc{bew}(\neg \textsc{prov}^\mathcal{N}_0(b)))
\]

\[
\rightarrow (\textsc{bew}(b) \land \textsc{bew}^*(\neg \textsc{prov}^\mathcal{N}_0(b)))
\]

\[
\rightarrow \textsc{bew}(\textsc{conj}(b, \textsc{bew}^*(\neg \textsc{prov}^\mathcal{N}_0(b))))
\]

\[
\rightarrow \textsc{bew}^*(b).
\]

We may conclude that \(\square (\forall a (\textsc{prov}^\mathcal{N}_0(a) \rightarrow \textsc{bew}^*(a))) \vdash_T \forall a (\textsc{prov}^\mathcal{N}_0(a) \rightarrow \textsc{bew}^*(a)),\) hence, by Löb’s Rule, \(T \vdash \forall a (\textsc{prov}^\mathcal{N}_0(a) \rightarrow \textsc{bew}^*(a)).\)

We verify \textsc{L}^m3. Let \(B := \forall a \forall b ((\textsc{bew}^*(a) \land \textsc{bew}^*(\textsc{imp}(a, b))) \rightarrow \textsc{bew}^*(b)).\) We have:

\[
\square B \vdash (\textsc{bew}^*(a) \land \textsc{bew}^*(\textsc{imp}(a, b))) \rightarrow \textsc{bew}(\textsc{conj}(a, \textsc{bew}^*(\neg \textsc{prov}^\mathcal{N}_0(b)))) \land
\]

\[
\textsc{bew}(\textsc{conj}(\textsc{imp}(a, b), \textsc{bew}^*(\textsc{imp}(a, b))))
\]

\[
\rightarrow \textsc{bew}(a) \land \textsc{bew}(\textsc{imp}(a, b)) \land
\]

\[
\textsc{bew}(\textsc{conj}(\textsc{bew}^*(a), \textsc{bew}^*(\textsc{imp}(a, b))))
\]

\[
\rightarrow \textsc{bew}(b) \land \textsc{bew}(\textsc{bew}^*(b))
\]

\[
\rightarrow \textsc{bew}(\textsc{conj}(b, \textsc{bew}^*(b)))
\]

\[
\rightarrow \textsc{bew}^*(b)
\]

It follows that \(\square B \vdash T.\) So, by Löb’s Rule, we have \(T \vdash B.\)

We can now show that the (\cdot)*-operation preserves Fefermianniess.
Theorem A.7. Suppose $\text{prov}_N^N$ is an LR provability predicate. Then $(\text{prov}_N^N)^*$ is also a Fefermanian provability predicate.

Proof. To simplify the presentation, we will omit the relativisation to $N$. We reason inside $N$ but for the fact that $\alpha$ is not an internal $N$-formula. Inspecting the fixed point construction, we see that $\text{prov}_N^N(\alpha)$ is of the form ($\S$) $\exists b (S_0(a, b) \land \text{prov}_N^N(\text{conj}(a, b)))$, where $S_0$ is an $\exists \Sigma^b_1$-formula that represents the relevant term, such that, $T$-verifiably, $\forall a \forall b (S_0(a, b) \leftrightarrow b = \langle \text{prov}_N^N(\hat{a}) \rangle)$. We will treat the existential quantifier of $\text{prov}_N^N$ in ($\S$) as giving the primary witness of $\text{prov}_N^N(\alpha)$.

To define $\alpha^*(\alpha)$, we, sloppily, use meta-notations in the object language. Thus we write, for example, $(A \land B)$ for $\text{conj}(a, b)$. I think the gain in readability outweighs the loss of precision. The reader just should remember that, locally, the roman capitals represent internal variables.

We take $\alpha^*(A)$ iff $A$ is of the form $(\overline{\neg} = \overline{\neg} \land B)$, where overlining gives us efficient numerals and where $p$ is a primary witness of $\Box^*_\gamma B$. In other words, $p$ is a witness of $\Box^*_\gamma (B \land \Box^*_\gamma B)$. The attentive reader will see that the definition of $\alpha^*$ is an incarnation of Craig’s trick.

We work in $T$. Suppose $\Box^*_\gamma B$. Let $p$ be a primary witness. Then, $(\overline{\neg} = \overline{\neg} \land B)$ is in $\alpha^*$. Hence trivially $\Box^*_\gamma B$.

Now suppose $\Box^*_\gamma B$. Let $q$ be a witnessing proof. Suppose $(\overline{\neg} = \overline{\neg} \land C_0), \ldots , (\overline{\neg} = \overline{\neg} \land C_k)$ are the (possibly non-standardly many) axioms used in $q$. We want to prove $\Box^*_\gamma B$. Let $D := (C_0 \rightarrow (C_1 \rightarrow \ldots (C_k \rightarrow B) \ldots ))$. We claim that, inside $\Box^*_\gamma$, we have:

I. $B$
II. $\Box^*_\gamma C_i$, for $i < k$,
III. $\Box^*_\gamma D$,
IV. $\forall a, b ((\text{prov}_N^N(\alpha) \land \text{prov}_N^N(\text{imp}(a, b))) \rightarrow \text{prov}_N^N(b))$.

(I) follows from the fact that we can effectively transform $q$ into an $\alpha$-proof of $B$. We have (II) by the definition of the $p_i$. We note that the finite set of witnesses of $\Box^*_\gamma C_i$ is not much larger than $q$. (III) follows from the fact that we can transform $q$ effectively into a witness of $\Box^*_\gamma D$. Since, by Theorem A.7, $\text{prov}_N^N$ is global, it follows that $\Box^*_\gamma D$ and, hence, $\Box^*_\gamma \Box^*_\gamma D$. Finally, we have (IV) since $\Box^*_\gamma$ is global combined with necessitation for $\Box^*_\gamma$.

Now we combine (II), (III) and (IV) to effectively find a witness of $\Box^*_\gamma, \Box^*_\gamma B$. So, combining this with (I), we find $\Box^*_\gamma (B \land \Box^*_\gamma B)$. Hence, we have $\Box^*_\gamma B$.

We have the following immediate corollary.

Corollary A.8. Suppose $\text{prov}_N^N$ is a Fefermanian provability predicate. Then, $\text{prov}_N^N$ is an LR-predicate for $T$ iff there is a Fefermanian predicate $\text{prov}_N^\gamma$, such that $T \vdash \Box^*_\gamma B \rightarrow \Box^*_\gamma B$, for all $T$-sentences $B$, and $\text{prov}_N^N$ satisfies Löb’s Logic over $T$.

We also have:

Corollary A.9. Suppose $\Box^*_\gamma$ is an LR-predicate for $T$. Then $T$ interprets $T + \Box^*_\gamma \bot$.

Proof. Since we have the Interpretation Existence Lemma for Fefermanian provability predicates, we have the result for $\text{prov}_N^\gamma$. Moreover $\Box^*_\gamma \bot$ is equivalent over $T$ to $\Box^*_\gamma \bot$, which is again equivalent to $\Box^*_\gamma \bot$. By the usual argument, we have $T$ interprets $T + \Box^*_\gamma \bot$. Hence, $T$ interprets $T + \Box^*_\gamma \bot$.  \[\square\]
A.4. **Craigification and Smoothening.** To prepare the reader for the example of Section A.5, we briefly discuss Craigification and smoothening. The main point is that smoothening does preserve the Feferman property of being a provability predicate of an axiom class.

In this subsection, we follow the sloppy ways of the proof of Theorem A.7 and use meta-notations in the object language. We also will suppress the superscript \( N \) that signals relativisation to the chosen numbers.

Consider \( \alpha(x) \) of the form \( \exists y \alpha_0(y, x) \). We can transform \( \alpha \) to its Craigification \( \alpha_{cr} \) as follows:

- \( \alpha_{cr}(A) \) iff, for some \( n < A \) and \( B < A \), we have \( A = (\pi = \pi \wedge B) \) and \( \alpha_0(n, B) \).

The smoothening \( \text{prov}^\text{sm}_\alpha \) is defined as follows:

- \( \text{prov}^\text{sm}_\alpha(A) \) iff \( \exists p \exists f (\text{proof}_0(p, A) \wedge \forall B \in \text{ass}(p) \alpha_0(f(B), B)) \).

Here \( 'f' \) ranges over finite functions coded as numbers.

We note that the smoothening takes the syntactic form of \( \alpha \) as input, so the notation is a bit misleading. The basic insight on the relationship between Craigification and smoothening is simply this:

**Theorem A.10.** \( T \vdash \forall y (\text{prov}^\text{sm}_\alpha(y) \leftrightarrow \text{prov}_{\alpha_{cr}}(y)) \).

In other words, smoothening preserves the Fefermanian character of a provability predicate.

We will not give the proof here. The main thing is seeing that given an \( \alpha \)-proof \( p \) and the finite function \( f \), we can construct in p-time the corresponding \( \alpha_{cr} \)-proof \( p^* \). Conversely, from \( p^* \) we can efficiently find both a \( p \) and \( f \)\(^4\).

A.5. **An Example.** The reader does have to glance through Section 6.2 of [Vis19] to understand what is going on in this subsection.

We use \( \beta \) as a standard representation of some single axiom that axiomatises EA. In the example we constructed an axiomatisation \( \sigma \) of EA with various properties. We will show that \( \text{prov}_\beta \) and \( \text{prov}^\text{sm}_\sigma \) and \( \text{prov}_\sigma^\ast \) and \( \text{prov}_\sigma \) are pairwise distinct over EA. Distinctness means that EA does not prove sameness.

We will also show that the Gödel sentence of the provability predicate defined is the example has an explicit representation.

A.5.1. **Inclusions.** We first note that since EA is finitely axiomatisable, \( \text{prov}_\sigma \) is a uniform provability predicate for EA and, hence, uniform. So, we do have Löb’s Rule and hence all the desirable properties of \( \Box \). We have:

- a. \( \text{EA} \vdash \forall x (\text{prov}_\beta(x) \rightarrow \text{prov}^\text{sm}_\sigma(x)) \)
- b. \( \text{EA} \vdash \forall x (\text{prov}^\text{sm}_\sigma(x) \rightarrow \text{prov}_\sigma^\ast(x)) \)
- c. \( \text{EA} \vdash \forall x (\text{prov}_\sigma^\ast(x) \rightarrow \text{prov}_\sigma(x)) \)

**Proof.** We prove (b). We note that \( \text{prov}^\text{sm}_\sigma(x) \) is \( \Sigma_1^0 \) and that hence we have

\[ \text{EA} \vdash \forall x (\text{prov}^\text{sm}_\sigma(x) \rightarrow \text{prov}_\sigma('\text{prov}^\text{sm}_\sigma(x))'). \]

\(^4\)We choose our coding in such a way that these transformations are feasible. Note that if we had e.g. the Ackermann coding for finite functions and a string style coding for sequences all this would get far less clear.
Let $A : \forall x (\text{prov}^\text{sm}_\sigma(x) \to \text{prov}^*_\sigma(x))$. We have:

1. $\text{EA} + \Box_\sigma A \vdash \text{prov}^\text{sm}_\sigma(a) \to (\text{prov}_\sigma(a) \land \text{prov}_\sigma(\lnot \text{prov}^\text{sm}_\sigma(\overline{x})))$
2. $\text{EA} + \Box_\sigma A \vdash (\text{prov}_\sigma(a) \land \text{prov}_\sigma(\lnot \text{prov}^*_\sigma(\overline{x})))$
3. $\text{EA} + \Box_\sigma A \vdash \text{prov}^*_\sigma(a)$

It follows that $\text{EA} \vdash \Box_\sigma A \to A$ and, hence, by Löb’s rule, $\text{EA} \vdash A$. 

A.5.2. Separations. We separate $\text{prov}_\beta$ from $\text{prov}^\text{sm}_\sigma$. Clearly, $\text{EA} + S^* \vdash \Box_\beta \ell_p \neq T$ and $\text{EA} + S^* \vdash \Box_\beta \ell_p \neq \Box_\beta \bot$. However, as is shown in [Vis19], $\text{EA} + S^* \not\vdash \Box_\beta \bot$. So, $\text{EA} + S^* \not\vdash \Box_\beta \ell_p \neq T$.

We separate $\text{prov}^\text{sm}_\sigma$ from $\text{prov}^*_\sigma$. In $\text{EA} + S^*$, smoothening does allow us to use more and more of the non-standardly finitely many axioms but never all in one proof. It now follows by a minor adaptation of the argument for Lemma 6.13(b) of [Vis19] that, over $\text{EA} + S^*$, we have that $\Box^\text{sm}_\sigma \bot$ is equivalent to $\Box_\beta \bot$. Moreover, by the results of [Vis19], we have the equivalence of $\Box^\text{sm}_\sigma \bot$ and $\Box_\beta \bot$. By a model theoretic argument analogous to the argument in [Vis19], we find that $\text{EA} + S^* \not\vdash \Box_\beta \bot \rightarrow \Box_\beta \bot$. So, $\text{EA} + S^*$ does not prove the equivalence of $\Box^\text{sm}_\sigma \bot$ and $\Box^*_\sigma \bot$.

We separate $\text{prov}^*_\sigma$ from $\text{prov}^*_\sigma$. Suppose we would have the equivalence of $\text{prov}^*_\sigma$ and $\text{prov}^*_\sigma$ over $\text{EA}$. It would follow that we have Löb’s Logic for $\Box_\sigma$. But this was refuted in [Vis19].

A.5.3. Explicit Gödel Sentence. We have seen that the Gödel sentence for $\Box_\sigma$ is unique over $\text{EA}$. But what is it? The next theorem answers this question.

Theorem A.11. The Gödel sentence of $\Box_\sigma$ is, modulo $\text{EA}$-provable equivalence, $\Diamond_\sigma \Box_\sigma \bot$.

Proof. Since we already have uniqueness, it is sufficient to show that $\Diamond_\sigma \Box_\sigma \bot$ is $\text{EA}$-provably equivalent to $\neg \Box_\sigma \Diamond_\sigma \Box_\sigma \bot$. In other words, we want to show that $\Box_\sigma \Diamond_\sigma \top$ is $\text{EA}$-provably equivalent to $\Box_\sigma \Diamond_\sigma \Box_\sigma \bot$.

We remind the reader of the following lemmas of [Vis19].

- **Lemma 6.9**: $\text{EA} + S^* \vdash \Box_\beta \neg S^*$.
- **Lemma 6.11**: $\text{EA} + S^* \vdash \Box_\sigma A \leftrightarrow \Box_\beta (\Diamond_\beta \top \rightarrow A)$.
- **Lemma 6.14(a)**: $\text{EA} + S^* \vdash \Box_\sigma \top$.
- **Lemma 6.14(b)**: $\text{EA} + \neg S^* \vdash \Box_\sigma \Diamond_\sigma \top \leftrightarrow \Box_\beta \bot$.

By Lemma 6.14(b), it follows that:

$\text{EA} + \neg S^* \vdash \Box_\sigma \Diamond_\sigma \Box_\sigma \bot \rightarrow \Box_\sigma \Diamond_\sigma \top \rightarrow \Box_\beta \bot \rightarrow \Box_\sigma \Diamond_\sigma \Box_\sigma \bot$

So, in $\text{EA} + \neg S^*$ we have the desired equivalence.

We show we also have the equivalence in $\text{EA} + S^*$. In the light of Lemma 6.14(a), it suffices to show $\text{EA} + S^* \vdash \Box_\sigma \Diamond_\sigma \Box_\sigma \bot$. We reason in $\text{EA} + S^*$. By 6.11, $\Box_\sigma \Diamond_\sigma \Box_\sigma \bot$ is equivalent to $\Box_\beta (\Diamond_\beta \top \rightarrow \Diamond_\sigma \Box_\beta \bot)$. By 6.9, we have (i) $\Box_\beta \neg S^*$. By applying necessitation to 6.14(b), we have

(ii) $\Box_\beta (\neg S^* \rightarrow (\Box_\sigma \Diamond_\sigma \top \leftrightarrow \Box_\beta \bot))$. 

By combining (i) and (ii), we find that $\Box_\beta \Diamond_\beta \bot$ is equivalent to $\Box_\beta (\Diamond_\beta \top \rightarrow \Diamond_\beta \top)$, which in its turn is equivalent to $\top$. So, we are done.

Open Question A.12. Can we give an example of a modalised fixed point that has no definable solution for the case of $\mathsf{EA}, \sigma$?

Can we given an example of a pair $\mathsf{EA}, \tau$, where $\tau$ is $\Sigma_1^0$ and $\Box_\tau$ is a provability predicate for $\mathsf{EA}$ such that the Gödel sentence is not explicitly definable? If not, what about the more general case?

Open Question A.13. Consider an $\Sigma_1^0$-predicate $\tau$ that axiomatises $\mathsf{EA}$ in $\mathsf{EA}$. The provability logic of $\Box_\tau$ contains $\mathsf{GL}$. However, prima facie, Solovay’s proof fails. Can we still prove that the logic is precisely $\mathsf{GL}$. What about the logic for $\Box_\sigma$ for the specific predicate $\sigma$ studied above?

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