Index theoretic characterization of $d$-wave superconductors in the vortex state

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We employ index theoretic methods to study analytically the low energy spectrum of a lattice $d$-wave superconductor in the vortex lattice state. This allows us to compare singly quantized $\hbar c/2e$ and doubly quantized $\hbar c/e$ vortices, the first of which must always be accompanied by $Z_2$ branch cuts. For an inversion symmetric vortex lattice and in the presence of particle-hole symmetry we prove an index theorem that imposes a lower bound on the number of zero energy modes. Generic cases are constructed in which this bound exceeds the number of zero modes of an equivalent lattice of doubly quantized vortices, despite the identical point group symmetries. The quasiparticle spectrum around the zero modes is doubly degenerate and exhibits a Dirac-like dispersion, with velocities that become universal functions of $\Delta_0/t$ in the limit of low magnetic field. For weak particle-hole symmetry breaking, the gapped state can be characterized by a topological quantum number, related to spin Hall conductivity, which generally differs in the cases of the $\hbar c/2e$ and $\hbar c/e$ vortex lattices.

Understanding the interactions among fermionic quasiparticles and topological excitations of two dimensional superconductors constitutes one of the main challenges in modern condensed matter physics. It is almost certainly true that at least some of the peculiar phenomena observed in the underdoped cuprates are related to the existence of nodal fermionic excitations of a $d_{x^2-y^2}$-wave superconductor with low superfluid density, which is eventually susceptible to dephasing by proliferation of mobile vortices $[1]$.

When an external magnetic field induces a finite density of vortices which form an Abrikosov lattice, the motion of vortices can be neglected and the problem of the quasiparticle spectrum is simplified. The initial theoretical analysis was based on numerical computations $[2]$, semiclassical approximation $[3]$, and scaling arguments $[4]$. As pointed out by Franz and Tesanovic $[5, 6, 7]$, the quasiparticles interact with the $\hbar c/2e$ vortices by both the (semiclassical) coupling of the charge currents to the superflow as well as a purely quantum coupling of the spin currents to the $Z_2$ branch cuts emanating from the vortices. Marinelli, Halperin, and Simon $[8]$ then showed that within the linearized approximation the spectrum at each node remains gapless to all orders of perturbation theory in phase gradients. This result was rederived by an insightful symmetry analysis in Ref. $[9]$, where it was concluded that the effect of the $\hbar c/2e$ vortices is merely a possible renormalization of the nodal quasiparticle velocities, but that the presence and particularly the number of the zero modes is the same as in the lattice of doubly quantized $\hbar c/e$ vortices. Various topological aspects of the problem were discussed in $[8, 9, 11, 12]$.

In this Letter, we revisit the problem of the Fermionic (quasiparticle) spectrum of a lattice $d$-wave superconductor in the vortex state. Our main result is that there can arise a fundamental topological difference between the singly and doubly quantized vortex lattice which is beyond perturbative analysis $[8, 9]$. In the case of an inversion symmetric vortex lattice, this difference relates to the non-trivial transformation properties of the $Z_2$ branch cuts under inversion, absent for $\hbar c/e$ vortices.

Our point of departure is the inversion symmetric vortex lattice state of the $d$-wave superconductor on a tight-binding lattice with perfect particle-hole (p-h) symmetry. We use the p-h symmetry to conduct non-perturbative analysis of the low energy spectrum and find conditions sufficient for the existence of zero energy nodal states. Armed with this analysis, we study weak breaking of the p-h symmetry, which gaps the nodal points, and show that the resulting state is topologically non-trivial. It is characterized by a topological quantum number proportional to the spin-Hall conductivity $[3, 9, 11, 12]$.

Remarkably, in cases when the branch cuts transform non-trivially under inversion, the number of the zero modes is doubled for $\hbar c/2e$ vortices relative to the $\hbar c/e$ counterpart, despite identical discrete translational sym- metry of both vortex lattices in question. The difference also holds for a lattice of $\hbar c/2e$ vortices, when the branch cuts are ignored. We demonstrate this by proving a form of an index theorem $[13, 14]$, which puts a lower bound on the number of zero modes, and by explicit analytical and numerical solutions, which reveal that the lower bound is, in fact, saturated.

We now provide justification for the above claims. Consider the Hamiltonian $H = H_0 - \mu N$, where

$$H_0 = \sum_{\mathbf{r}\mathbf{r}'} \left[ -t_{\mathbf{r}\mathbf{r}'} c_{\mathbf{r}'\sigma}^\dagger c_{\mathbf{r}\sigma} + \frac{\Delta_{\mathbf{r}'\mathbf{r}}}{2} \left( c_{\mathbf{r}'\uparrow}^\dagger c_{\mathbf{r}\uparrow} + c_{\mathbf{r}'\downarrow}^\dagger c_{\mathbf{r}\downarrow} + h.c. \right) \right],$$

(1)

where the sum is over the nearest neighbors $\langle \mathbf{r}\mathbf{r}' \rangle$ of the (underlying) tight-binding lattice. The hopping integral $t_{\mathbf{r}\mathbf{r}'} = t \exp(-iA_{\mathbf{r}\mathbf{r}'})$; in the symmetric gauge the magnetic flux $\Phi$ enters the Peierls factor via $A_{\mathbf{r}\mathbf{r}+\mathbf{y}} = y\Phi/2$ and $A_{\mathbf{r}\mathbf{r}+\mathbf{y}'} = -x\Phi/2$, and the $d$-wave pairing field in
the vortex lattice state reads $\Delta_{rr'} = \eta_{rr'} \Delta_0 \exp(i\theta_{rr'})$; 
$\eta_\delta = (+ - )$ if $|\delta|$ $x$ $y$. The bond phase factors $\exp(i\theta_{rr'})$ result from a self-consistent calculation, but the main topological feature of $\theta_{rr'}$ is its $2\pi$ winding around the magnetic field induced vortices. As the initial Ansatz for the self-consistent solution, we choose the lattice bond variables $\theta_{rr'}$ as follows: first we solve the continuum equations \( \nabla \times \nabla \phi(r) = 2\pi \sum \delta(r-r_i) \) and \( \nabla \times \nabla \phi(r) = 0 \) where $r_i$ denotes the vortex positions which in this Letter form Abrikosov lattice. These conditions determine $\phi(r)$ up to $\phi_0 + \gamma_0 \cdot r$, which is fixed by requiring zero overall current. Next, we define the site variables $e^{i\theta_r}$ as the value of $\exp(i\phi(r))$ at each site, and the initial bond variable $e^{i\theta_{rr'}}$ is taken to be the geometric mean of two neighboring site variables: $e^{i\theta_{rr'}} = (e^{i\phi_r} + e^{i\phi_{r'}})/|e^{i\phi_r} + e^{i\phi_{r'}}|$. This choice guarantees that if the vortices in $e^{i\theta_{rr'}}$ reside inside plaquettes, so will the vortices in $e^{i\theta_r}$.

The diagonalization of the Hamiltonian (1) is equivalent to the solution of the Bogoliubov-de Gennes (BdG) equation \( \hat{H}_0\psi_r = E\psi_r \) where the lattice operator \[
\hat{H}_0 = \left( \hat{\mathcal{E}}_r - \mu \right) \hat{\Delta}_r \left( \hat{\Delta}_r^* - \hat{\mathcal{E}}_r^* + \mu \right). \tag{2}
\]

Both $\hat{\mathcal{E}}_r$ and $\hat{\Delta}_r$ are defined through their action on a wavefunction at the lattice site $r$ as $\hat{\mathcal{E}}_r \psi_r = -t \sum_{\delta \neq \pm x, \pm y} e^{-i\mathbf{A}_{rr'} \cdot \mathbf{r}_{r'}}$ and $\hat{\Delta}_r \psi_r = \Delta_0 \sum_{\delta} e^{i\theta_{rr'} + \eta_{rr'} \cdot \mathbf{r}_{r'}}$. The BdG Hamiltonian $\hat{H}_0$, as well as Pauli matrices, act on the two component Nambu spinor $\psi_r = [\psi_r, \bar{\psi}_r]^T$.

Translational symmetry: While the vortex positions are periodic, the Hamiltonian (2) is invariant only if the discrete translations are followed by a gauge transformation (magnetic translations). However, as shown in Refs. [8, 9], $\hat{H}_0$ can be transformed into a periodic Hamiltonian by a singular gauge transformation, $\mathcal{U} = \exp(\frac{i}{2} \mathbf{b} \times \mathbf{\sigma} \phi_0)$. On a lattice, the phase factors $e^{\pm i\phi_{rr'}}$ must be defined with care. First, connect pairs of the singly quantized vortices by branch cuts (see Fig.1). Then, choose a reference point $r_0$ at which $b_0$ is either one of the two solutions to $b_0^2 = e^{i\phi_{r_0}}$, and set $e^{\pm i\phi_{r_0}} = b_0$. Now, let $r$ be a site neighboring $r_0$ not connected by a bond that crosses a branch cut, and determine $e^{i\phi_{rr'}}$ by the solution to $b_0^2 = e^{i\phi_{r}}$ closer to $e^{\pm i\phi_{r_0}}$, i.e. minimizing $|b_0 - b|$. Next, set $r$ as a reference point and determine the phase of its neighborhoods in the same way. This procedure defines $e^{\pm i\phi_{rr'}}$ on all sites of the lattice uniquely. It also ensures that the difference of the phases of $e^{i\phi_{rr'}}$ and $e^{i\phi_{rr'}}$ lies inside $(-\frac{\pi}{2}, \frac{\pi}{2})$ mod $2\pi$ on all bonds, except across the bonds crossing the branch cut where it is more than $\frac{\pi}{2}$. It follows that 
\[
e^{i\theta_{rr'}} e^{-i\phi_{rr'}} e^{-i\phi_{rr'}} = \frac{e^{i\phi_r} + e^{i\phi_{r'}}}{|e^{i\phi_r} + e^{i\phi_{r'}}|} e^{-i\phi_{rr'}} e^{-i\phi_{rr'}} = \delta_{2,rr'}
\]

where $\delta_{2,rr'} = 1$ on each bond except the ones crossing the branch cut where $\delta_{2,rr'} = -1$. The transformed Hamiltonian, $\tilde{H}_0 = \mathcal{U}^{-1} \hat{H}_0 \mathcal{U}$, is now 
\[
\tilde{H}_0 = \sigma_3 (\hat{\mathcal{E}}_r - \mu) + \sigma_1 \hat{\Delta}_r, \tag{3}
\]

where the transformed lattice operators satisfy 
\[
\tilde{\mathcal{E}}_r \psi_r = -t \sum_{\delta \neq \pm x, \pm y} \delta_{2,rr'} \psi_{r+\delta}, \tag{4}
\]
\[
\tilde{\Delta}_r \psi_r = \Delta_0 \sum_{\delta} \delta_{2,rr'} \psi_{r+\delta}. \tag{5}
\]

The physical superfluid velocity enters via the factor $e^{i\Delta_{rr'}} / |1 + e^{i\phi_{rr'}}| e^{-i\mathbf{A}_{rr'}}$ and describes the lattice version of the semicircular (Doppler) effect. On the other hand, the effect of the $z_2$ field is purely quantum mechanical and tied to the $hc/2e$ flux quantization. It would be absent for doubly quantized $hc/2e$ vortices.

The resulting Hamiltonian $\tilde{H}_0$ is invariant under discrete translations by $\ell_x, \ell_y$ defining the magnetic unit cell, due to the periodicity of $V_{rr'}[13]$ and our periodic choice of the branch cuts. Thus, it can be diagonalized in the Bloch basis. Extracting the crystal wavevector $k$ from the Bloch wavefunctions gives $\mathcal{H}(k) = e^{-ikr} \tilde{H}_0 e^{ikr}$ acting on the Hilbert space of periodic Nambu spinors. Inversion symmetry: Our subsequent analysis will focus on $\mathcal{H}(k)$. For a unit cell with $\ell_x, \ell_y$ sites, there are $2\ell_x \ell_y$ eigenvalues for each $k$ in the first Brillouin zone. Consider now an inversion symmetric vortex lattice for which $e^{i\phi_{rr'}} = e^{i\phi_{r'-r}}$. The inversion operator $I$, defined by its action on the wavefunctions $I \psi_r = \psi_{-r}$, transforms the Hamiltonian according to $I \tilde{H}(k) I = \mathcal{H}(-k)$. The important point is that $\mathcal{H}(-k)$ does not always equal to $\mathcal{H}(-k)$, because the branch-cut can transform in a non-trivial way under inversion. A square vortex lattice depicted in Fig. 1 represents such a case. However, the branch cut can be restored by an additional $Z_2$ gauge transformation $\gamma_r$, where $\gamma_r = 1 \forall r \in C$, and

![FIG 1: Magnetic unit cell (for $\ell = 6$) containing two $hc/2e$ vortices represented by two solid (red) circles. The tight-binding lattice is shown by the thin (black) lines, the $(Z_2)$ branch-cut connecting the vortices is shown by a dashed (blue) line. The invasion of the branch cut about the midpoint between the vortices defines the two regions $C$ (black dots) and $C^*$ (white dots). The gauge factor $\gamma(r)$ defined in the text, is 1 for the points in $C$ and -1 for the points in $C^*$.](image-url)
\(\gamma_r = -1\ \forall\ r \in \mathbb{C}^*\). The set \(\mathbb{C}^*\) is defined by the points inside the area enclosed by the branch cut and its inverted image, and \(\mathbb{C}^*\) is its complement (see Fig. 4). Thus, 
\(\gamma_r \mathcal{H}(\mathbf{k}) 1 \tau = \mathcal{H}(-\mathbf{k})\). Moreover, the spectrum of \(\mathcal{H}(\mathbf{k})\) is symmetric under \(E \to -E\) for any \(\mu\) [7].

**Index theorem:** For \(\mu = 0\) we can define
\[
\mathcal{P} = (-1)^{x+y} \gamma_r \mathbb{I},
\]
where \(\mathbb{I}\) is the unity in the Nambu space and the function \((-1)^{x+y}\) changes sign on every other site. Clearly, \(\mathcal{P}^2 = 1\). In addition this operator satisfies
\[
\mathcal{P} \mathcal{H}(\mathbf{k}) \mathcal{P} = -\mathcal{H}(-\mathbf{k}),
\]
and, in particular, at \(\mathbf{k} = 0\), \(\mathcal{P} \mathcal{H}(0) \mathcal{P} = -\mathcal{H}(0)\). Therefore, if there are zero modes at \(\mathbf{k} = 0\), they can be chosen to be eigenstates of \(\mathcal{P}\) with eigenvalues \(\pm 1\). At \(\mathbf{k} = 0\), for any eigenstate \(\psi_{E,r}\) of \(\mathcal{H}(0)\) with energy \(E \neq 0\), we can generate an eigenstate \(\psi_{-E,r} = \mathcal{P} \psi_{E,r}\) with energy \(-E\). Therefore, the only diagonal matrix elements of \(\mathcal{P}\) in the basis of the eigenstates of \(\mathcal{H}(0)\) come from the zero-energy states. Let us denote by \(n_\pm\) the number of zero-energy states with \(\mathcal{P}\)-eigenvalue \(\pm 1\). Then, 
\(\text{Tr} \mathcal{P} = n_+ - n_-\). Therefore, \(\text{Tr} \mathcal{P}\) constitutes the lower bound on the number of zero energy states of \(\mathcal{H}(0)\).

Since \(\text{Tr} \mathcal{P}\) is independent of the basis, we can compute the trace in the coordinate basis. Consider again the square vortex lattice depicted in Fig. 1 with two \(hc/2e\) vortices per unit cell with the primitive vortex cell 45° relative to the tight-binding lattice and the magnetic length \(\ell = 2(2n + 1)\). Then, there are only four spatial points which remain invariant under inversion and contribute to the trace. We listed the values of \(\mathcal{P}\) in Table II from which it follows that \(\text{Tr} \mathcal{P} = 4\). Therefore, the index of \(\mathcal{H}(0)\) is 4 and it has at least 4 zero-energy eigenstates. Same argument holds for \(\mathcal{H}(\mathbf{k}^*)\) where \(\mathbf{k}^* \in \{(0, 0), (\pi/\ell, 0), (\pi/\ell, \pi/\ell), (0, \pi/\ell)\}\), with the only modification being \(\mathcal{P}_{k^*} = (-1)^{x+y} e^{G^* r \gamma_r} \mathbb{I}_{12}\), where \(G^* = -2k^*\). So for \(l = 2(2n + 1)\) there are at least 16 zero energy eigenstates in the first Brillouin zone.

Consider now the same vortex arrangement, but for \(hc/e\) vortex lattice (or for \(hc/2e\) but ignoring the \(Z_2\) branch cuts). Then, \(\gamma_r = 1\) everywhere and \(\text{Tr} \mathcal{P}_{k^*} = 0\) except at M point \((\mathbf{k}^* = (\pi/\ell, \pi/\ell))\) where \(\text{Tr} \mathcal{P}_{k^*} = 8\), i.e. the index theorem guarantees only 8 zero energy states. This is the same number of zeros as in the absence of magnetic field. Thus, while the effects of the superflow are perturbative, the effect of the branch cuts is non-perturbative. The symmetries discussed exist only if the bonds change sign exactly. Clearly, 16 zero energy states cannot be reached by perturbation theory around a state with 8 zero energy states, such as the \(d\)-wave superconductor in the absence of the magnetic field.

On the other hand, for \(\ell = 2(2n)\), \(\text{Tr} \mathcal{P} = 0\) in both cases, because there are no lattice points invariant under inversion. The index theorem thus does not guarantee any zero energy states, i.e. the spectrum is gapped.

**Degenerate Dirac cones:** Around each of the zeros in the magnetic Brillouin zone, the energy vanishes linearly with the deviation \(\delta \mathbf{k}\) from the degeneracy points. This follows from the standard \(k \cdot p\) perturbation theory arguments as well as the fact that at \(\mu = 0\) all the bands are doubly degenerate [7]. Expanding the Hamiltonian \(\mathcal{H}(\delta \mathbf{k})\) in the basis of the zero energy states near \(\mathbf{k} = 0\) we find
\[
\mathcal{H}_{eff} = \mathcal{V}_+ \delta k_+ \Gamma_1 + \mathcal{V}_- \delta k_- \Gamma_2,
\]
where \(\Gamma_{1,2}\) are 4×4 Dirac gamma matrices and \(\delta k_\pm = (k_x \pm k_y)/\sqrt{2}\). The spectrum near \(\mathbf{k}^* = 0\) thus consists of two doubly degenerate Dirac cones with energies
\[
\mathcal{E}_0 = \pm t \sqrt{\delta k_+^2 + \delta k_-^2}.
\]
The new nodal velocities \(\mathcal{V}_\pm\) must be determined from the wavefunctions of the degenerate multiplet. For the square vortex lattice in Fig. 1 we found analytically that for \(\ell = 2\), \(\mathcal{V}_\pm^2 = 2(1 + \alpha^2) \pm \frac{2\alpha^2}{1 + \alpha^2} (\alpha^2 - 2 + \sqrt{2})\), where the bare Dirac cone anisotropy \(\alpha = \Delta_0/\ell\). Eq. (4) holds also near \(\mathbf{k}^* = (\pi/\ell, \pi/\ell)\), where near \((\pi/\ell, 0)\) and \((0, \pi/\ell)\), \(\delta k_\pm \to \pm \delta k_\perp\) as required by the fourfold symmetry. While \(\ell = 2\) corresponds to an unrealistically large magnetic field for physical parameters, \(\ell = 2\) is well representative of the sequence \(\ell = 4n + 2\) and has the virtue of being analytically soluble. We verified numerically that the qualitative results discussed below are indeed the same along the sequence \(\ell = 4n + 2\), for integer \(n\). It is not difficult to see that the effective Hamiltonian [8] has the same form for any \(\ell = 4n + 2\). In addition, in the limit of large \(n\), the Dirac velocities \(\mathcal{V}_\pm\) become universal functions of \(\alpha\) in accordance with Simon-Lee scaling [4]. We verified this numerically (see Fig. 2).
Hamiltonian of a 2+1 dimensional massive Dirac particle. By the $E \rightarrow -E$ symmetry of the spectrum, the negative energy states are all occupied. It is well known, that in 2+1 dimensions, in the representation where $v_+ > 0$, the sign of the mass determines the Hall conductivity $\sigma_{xy} = -\frac{1}{2} \text{sgn}(m_D) \frac{\sigma_x}{\sigma_y}$. In the case at hand, the charge is not conserved due to broken U(1) symmetry of the superconductor, but the component of the spin along the magnetic field is conserved, and so spin Hall conductivity, $\sigma_{xy}$, is quantized. We verified that all 8 of the Dirac particles have the same chirality, i.e. their additive contributions are the same. In order to determine the total value of $\sigma_{xy}$, generically, it is not enough to know the low energy part of the spectrum. Only the change of $\sigma_{xy}$ can thus be determined. Nevertheless, in this case, at $\mu = 0$, $\sigma_{xy} = 0$ due to the particle-hole symmetry and since the change is $\pm 8$, $\sigma_{xy} = \pm 4$ (in appropriate units). We verified this numerically as well. The resulting state at $\mu \neq 0$ remains distinguishable from the case of $hc/e$ vortex lattice, where $|\sigma_{xy}| \leq 2$.

While the above analysis pertains to the initial Ansatz for $e^{i\theta_{dr}}$, the main conclusions depend only on the translational and inversion invariance of $\Delta_{xy} \times \gamma_2 \tau_2 e^{-i\delta \phi_x} e^{-i\delta \phi_y}$ which we verified holds at each step of the self-consistent iteration for a model with the nearest neighbor attraction that stabilizes d-wave pairing.

Finally, we conclude with comments about experiments. Recently it was observed that the low temperature thermal conductivity, $\kappa_{xx}$, of the ultraclean YBa$_2$Cu$_3$O$_7$ showed pronounced deviations from the semiclassical predictions. Rather, it was qualitatively consistent with the onset of a new universal $\kappa_{xx}$, distinct from its zero field value. As observed by Franz and one of us (O.V), the universal renormalization of the nodal velocities, shown in Fig. 2 could provide a qualitative explanation. Presently, measurements of $\kappa_{xy}$ extend to about 10K which is too high to observe the quantization discussed.

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FIG. 2: Upper panel: The energy dispersion in the gapless (nodal) ($\mu = 0$) and gapped ($\mu \neq 0$) phases. Only the positive energies are shown as the spectrum is symmetric under $E \rightarrow -E$. For $\mu = 0$ and $\ell = 4n + 2$ each (red) band is doubly degenerate and there are 16 zero modes in the Brillouin zone. Finite $\mu$ splits the degeneracy and the low energy spectrum corresponds to 8 massive Dirac fermions, each contributing equally to the spin Hall conductivity $\sigma_{xy} = \pm \frac{\hbar}{2}$. Lower panel: The renormalized nodal velocities normalized to their value at $\ell = 2$ as functions of $\ell = 4n + 2$, approach the universal value dependent on $\alpha = \Delta_0/\ell$ only. Here $\alpha = 0.1$. P-h symmetry breaking and topology of the gaps: For small but finite $\mu$, all the degeneracies are lifted Fig. 2, but the spectrum is still symmetric under $E \rightarrow -E$ for each $k$ point due to the inversion symmetry of the vortex lattice. At low energies we have

$$E = \pm \sqrt{\tilde{E}_0^2 + \mu^2} \pm |\mu| \sqrt{\tilde{E}^2 + \tilde{m}_0^2},$$

(10)

where the $\pm$ signs are uncorrelated, $\tilde{m}_0$ and $\tilde{m}_0$ are functions of $\ell$ and $\alpha$, and $\tilde{E}^2 = \ell^2 \left( \tilde{\gamma}_0^2 \delta k_+^2 + \tilde{\gamma}_0^2 \delta k_-^2 \right)$. As shown in Fig. 2, at each $k^*$ there is a local maximum in the lowest positive band of the spectrum; the local minimum is split and moved to $\{\delta k_+ = \delta k_- = (\pm k_{min}, 0)\}$ near $(0,0)$ and $(\pi/\ell, \pi/\ell)$, and $\{\delta k_+ = \delta k_- = (0, \pm k_{min})\}$ near $(\pi/\ell, 0)$ and $(0, \pi/\ell)$, where $k_{min} = \frac{\mu e^2}{2 \hbar^2 v_+ \sqrt{\tilde{\gamma}_0^2 - \frac{4 \tilde{m}_0^2}{\ell^2}}}.$

In the vicinity of each of the 8 band-gap minima, the spectrum is equivalent to the 2+1 dimensional massive Dirac particle, with mass $m_D$ given by the gap minimum $E_{min}$. To see that, around each of the eight lowest energy minima, project onto the lowest energy basis, $|\pm\rangle$, such that $H(k_{min})|\pm\rangle = \pm E_{min} |\pm\rangle$, and expand the resulting effective Hamiltonian in $\delta k$:

$$\tilde{H}_{eff} = E_{min} \tau_3 + v_+ \delta k_+ \tau_1 + v_- \delta k_- \tau_2,$$

(11)

$\tau_i$ are the Pauli matrices acting on the 2-dimensional subspace spanned by $|\pm\rangle$. Clearly, $\tilde{H}$ is equivalent to the

$\Delta_{xy} \times \gamma_2 \tau_2 e^{-i\delta \phi_x} e^{-i\delta \phi_y}$
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