ON POSITIVE SOLUTION TO A SECOND ORDER ELLIPTIC EQUATION WITH A SINGULAR NONLINEARITY

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Abstract. We consider a second order elliptic equation with measurable bounded coefficients
\[(a_{ij}(x)u_{x_i})_{x_j} + p(x)|x|^s u^{-\sigma} = 0, \quad x \in \Omega \setminus \{O\},\]
where \(\sigma > 0\), \(s\) is any real number, and \(\Omega \subset \mathbb{R}^n\), \(n \geq 3\) is a bounded domain, which contains the origin \(O\).

The aim of this paper is to establish existence, nonexistence and behavior of positive weak solutions near the isolated singularity \(O\).

1. Introduction. Semi-linear elliptic equations has been a subject of many researches lately, and it is well known that properties of equation depend essentially on the character of nonlinear terms. This paper is devoted to a second order elliptic equation
\[(a_{ij}(x)u_{x_i})_{x_j} + p(x)|x|^s u^{-\sigma} = 0, \quad x \in \Omega \setminus \{O\}, \quad (1.1)\]
where \(\Omega \subset \mathbb{R}^n\), \(n \geq 3\) is a bounded domain, which contains the origin \(O\). We assume \(\sigma > 0\), \(s \in \mathbb{R}\), all coefficients to be measurable bounded functions, \(0 < C_1 < p(x) < C_2\), \(a_{ij}(x) = a_{ji}(x)\), and the uniformly elliptic condition is satisfied:
\[\lambda^{-1}|\zeta|^2 \leq a_{ij}(x)\zeta_i \zeta_j \leq \lambda|\zeta|^2,\]
where \(\lambda > 0\), \(\forall \zeta \in \mathbb{R}^n\) a.e. in \(\Omega\).

It is easy to see that a nonlinear term becomes singular at \(O\) when \(s\) is negative, and also as \(u \to 0\). We will consider only positive weak solutions of the equation, which we define below.

Definition 1.1. We say that a function \(u(x) \in W^1_2(\Omega \setminus \{O\})\), such that \(u(x)^{-1} \in L^\infty_{loc}(\Omega \setminus \{O\})\), and \(u(x) > 0\) a.e. is a weak solution to (1.1) if \(\forall \delta > 0\) and \(\forall \xi(x) \in W^1_2(\Omega \setminus B_\delta(0))\), \(\xi(x) \geq 0\), where \(B_\delta(0) = \{x \in \mathbb{R}^n : |x| < \delta\}\),
\[
\int_\Omega a_{ij}(x)u_{x_i}\xi_{x_j}dx = \int_\Omega p(x)|x|^s u^{-\sigma}\xi dx. \quad (1.2)
\]

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The main purpose of this work is to find existence and nonexistence conditions for a positive weak solution near an isolated singular point, and also to study the behavior of the solution in the neighborhood of this point.

The model equation can be written in the form

\[ \Delta u + p(x)u^{-\sigma} = 0, \quad x \in \Omega \setminus \{O\}. \]

It was considered earlier by some authors from the point of view of existence of entire positive solution (see, for instance, [1],[2], the list is not complete). In particular, for \( n \geq 3 \), the existence of a solution \( u(x) \in C^2_{\text{loc}}(\mathbb{R}^n) \) such that

\[ 0 < C_1|x|^{2-n} \leq u(x) \leq C_2|x|^{2-n}, \quad |x| > R \]

was proved under the conditions

\[ 0 < cf(|x|) \leq p(x) \leq f(|x|) \quad \text{as} \quad x \neq 0, \]

where \( 0 < c < 1, p(x) \in C^\alpha_{\text{loc}}(\mathbb{R}^n), 0 < \alpha < 1, f(t) \in C(\mathbb{R}) \) and

\[ \int_1^\infty \frac{t^{n-1+\sigma(n-2)}f(t)}{t} dt < \infty. \]

Their methods of proofs used essentially the presence of the Laplacian in the equation.

Recently an existence and nonexistence problem was studied in [4] for a general equation (1.1). Solutions of the equation were considered in an outer domain, and the results of [4] correspond with those we present here near an isolated singular point. It can be verified by using a generalization of Kelvin’s transformation for (1.1) [7]. However the methods of proofs are different. Also we study a behavior of a positive solution near the singularity.

In the following text we denote

\[ Lu \equiv (a_{ij}(x)u_{x_i})_{x_j}. \]

2. A priory estimates.

Lemma 2.1. If \( u(x) \) is a positive weak solution of equation (1.1) in \( \Omega \setminus \{O\} \), then

\[ u(x) \geq C_0|x|^{\frac{1+\gamma}{1-\gamma}} \]

where the constant \( C_0 = C_0(\lambda, n, \sigma, a_{ij}, p) > 0 \) does not depend on \( u \).

Proof. First we make a replacement

\[ u = v^\gamma, \quad \xi = v^{1-\gamma}\phi, \quad \phi \in C^\alpha(\Omega \setminus B_\delta(0)), \]

where \( \gamma < 0, 0 \leq \phi \leq 1 \). Substituting these functions into (1.2) we obtain

\[ - \int_\Omega a_{ij}v_{x_i}\phi_{x_j}dx - (1-\gamma) \int_\Omega v^{-1}a_{ij}v_{x_i}v_{x_j}\phi dx + \frac{1}{\gamma} \int_\Omega p(x)|x|^{\sigma}v^{1-\gamma(1+\sigma)}\phi dx = 0. \]
Hence
\[- \int_\Omega a_{ij} v_x \phi_{x_j} \, dx + \frac{1}{\gamma} \int_\Omega p(x)|x|^s v^{1-\gamma(1+\sigma)} \phi \, dx \geq 0.\]
That means
\[Lv \geq - \frac{1}{\gamma} p(x)|x|^s v^{1-\gamma(1+\sigma)},\]
where \( q = 1 - \gamma(1 + \sigma) > 1 \), and there exists a constant \( \alpha_0 > 0 \), not depending on \( v \) such that
\[v(x) \leq \alpha_0 |x|^{\frac{s+2}{1+\sigma}}.\]
This follows from Keller-Osserman’s type of estimate proved in [3] for \( s = 0 \), which can be extended without a problem for any real \( s \).
Finally,
\[u(x) \geq \alpha_0^\gamma |x|^{\frac{s+2}{1+\sigma}},\]
and \( C_0 = \alpha_0^\gamma \).
Therefore the lemma is proved. \( \square \)

**Remark 2.1.** The estimate in Lemma 2.1 is sharp. Indeed, the equation
\[\Delta u + |x|^s u^{-\sigma} = 0\]
possesses a solution
\[u = K |x|^{\frac{2+s}{1+\sigma}}\]
with
\[K = \left( -\frac{2+s}{1+\sigma} \left( \frac{2+s}{1+\sigma} \frac{s}{1+\sigma} + n - 2 \right) \right)^{-\frac{1}{1+\sigma}}\]
if
\[2 - n < \frac{2+s}{1+\sigma} < 0.\]
There is a singularity of the solution at \( O \).

**Lemma 2.2.** If \( u(x) > 0 \) is a solution of equation (1.1) in \( \Omega \setminus \{O\} \), then
\[u(x) \leq C |x|^{2-n}.\]
where the constant \( C > 0 \) does not depend on \( u \).

**Proof.** Without loss of generality we suppose that \( B_1(0) \subset \Omega \). Let us denote \( m_\rho = \inf_{|x| = \rho} u(x) \), and rewrite the equation as the following
\[Lu + q(x)u = 0,\]
where
\[q(x) = p(x)|x|^s u^{-(1+\sigma)} \leq C_0 |x|^{-2}.\]
The last estimate follows from Lemma 2.1. Then from Harnack inequality we have
\[m_\rho \leq u|_{|x| = \rho} \leq K m_\rho,\]
where the positive constant $K$ doesn’t depend on $\rho$.

Let $\Gamma(x)$ be a fundamental solution, such that

$$L\Gamma(x) = 0, \quad x \in B_1(0) \setminus \{O\},$$

and

$$\Gamma(x)|_{|x|=1} = 0.$$

Such $\Gamma(x)$ exists and satisfies the inequalities

$$C_1|x|^{2-n} \leq \Gamma(x) \leq C_2|x|^{2-n}$$

with some positive constants $C_1$ and $C_2$ [7]. Then

$$\Gamma(x)|_{|x|=\rho} \leq \frac{C_2\rho^{2-n}}{m_\rho} u(x)|_{|x|=\rho}.$$

Since

$$L \left( \frac{C_2\rho^{2-n}}{m_\rho} u(x) - \Gamma(x) \right) < 0, \quad \rho \leq |x| \leq 1$$

the maximum principle gives

$$\Gamma(x) \leq \frac{C_2\rho^{2-n}}{m_\rho} u(x) \quad \text{in} \quad \rho \leq |x| \leq 1.$$

Suppose $\rho < 1/2$, and fix now $|x| = 1/2$, then

$$m_\rho \leq C_3\rho^{2-n},$$

where the constant $C_3$ does not depend on $\rho$. Using again Harnack inequality we complete the proof of the lemma:

$$u(x) \leq C|x|^{2-n},$$

$C = C_3K$. \quad \square

**Remark 2.2.** From Harnack inequality it follows that $m_\rho = \inf_{|x|=\rho} u(x)$ and $M_\rho = \sup_{|x|=\rho} u(x)$ must have the same rate of growth as $\rho \to 0$.

### 3. Existence and nonexistence results.

**Theorem 3.1.** Let $\frac{4+2\sigma}{1+\sigma} \leq 2 - n$. Then there is no positive solution of (1.1) in any $\Omega \setminus \{O\}$.

**Proof.** If there exists a solution of equation (1.1) $u(x) > 0$ in $\Omega \setminus \{O\}$, then Lemma 2.1 and Lemma 2.2 give that

$$C_0|x|^{\frac{2+s}{1+\sigma}} \leq u(x) \leq C|x|^{2-n},$$

which leads to a contradiction when

$$\frac{2+s}{1+\sigma} < 2 - n.$$
Assume now \( \frac{2 + s}{1 + \sigma} = 2 - n \).

If there exists a solution \( u(x) \) then \( u \to \infty \) as \( |x| \to 0 \). Integrating the equation (1.1) in \( D = \{ x : 1 < u < M \} \) by part, we get

\[
0 < \lambda \int_{\{u=M\}} |\nabla u| dS \leq \int_{\{u=M\}} a_{ij} u_{x_i} \frac{u_{x_j}}{|\nabla u|} dS = \int_{\{u=1\}} a_{ij} u_{x_i} \frac{u_{x_j}}{|\nabla u|} dS - \int_D p(x)|x|^s u^{-\sigma} dx.
\]

The first term in the right hand side is fixed, the second is equivalent to a divergent integral

\[
c \int |x|^{-1} d|x|,
\]

as \( M \to \infty \), and then the right hand side is negative while the left hand side is positive, which is impossible.

Therefore there is no solution, and the theorem is proved. \( \square \)

**Theorem 3.2.** Let \( 2 - n < \frac{s+2}{1+\sigma} \). Then there exists a positive solution of (1.1), such that

\[
K_1|x|^{2-n} \leq u(x) \leq K_2|x|^{2-n},
\]

where \( K_1, K_2 > 0 \).

**Proof.** Let \( \Gamma(x) \) be a fundamental solution [6]

\[
L\Gamma(x) = 0, \quad x \in \Omega \setminus \{O\},
\]

such that

\[
0 < C_1|x|^{2-n} \leq \Gamma(x) \leq C_2|x|^{2-n}.
\]

Then \( \Gamma(x) \) is a subsolution for (1.1)

\[
L\Gamma + p(x)|x|^s \Gamma^{-\sigma} \geq 0.
\]

Let us consider a linear equation

\[
L\Gamma_\alpha + p(x)|x|^{-2+\alpha} \Gamma_\alpha = 0,
\]

where \( \alpha > 0 \) will be chosen later. This equation possesses a solution \( \Gamma_\alpha(x) \) such that

\[
K_\alpha^{-1}|x|^{2-n} \leq \Gamma_\alpha \leq K_\alpha|x|^{2-n},
\]

where \( K_\alpha > 0 \) [8]. Then using the estimate \( 2 - n < \frac{s+2}{1+\sigma} \) we obtain

\[
L\Gamma_\alpha + p(x)|x|^s \Gamma_\alpha^{-\sigma} = L\Gamma_\alpha + p(x)|x|^s \Gamma_\alpha^{-\sigma-1} \Gamma_\alpha \leq L\Gamma_\alpha + p(x)|x|^{s-(\sigma+1)(2-n)} \Gamma_\alpha = 0,
\]

where

\[
-2 + \alpha = s - (\sigma + 1)(2 - n), \quad \alpha > 0
\]
Hence, $\Gamma \alpha$ is a super-solution for (1.1), and therefore exists a positive solution $u$ in $\Omega \setminus \{O\}$ such that

$$\Gamma(x) \leq u(x) \leq \Gamma \alpha(x).$$

Finally,

$$C_1|x|^{2-n} \leq u(x) \leq K_\alpha|x|^{2-n},$$

and taking $K_1 = C_1$, and $K_2 = K_\alpha$ we complete the proof. \hfill \Box

4. Properties of solutions in the case $2 + s \geq 0$.

**Theorem 4.1.** Let $s + 2 > 0$, $u(x)$ is a positive solution to (1.1) in $\Omega \setminus \{O\}$. Then either $u$ is a solution of (1.1) in $\Omega$ and $u(0) > 0$ or

$$K_1|x|^{2-n} \leq u(x) \leq K_2|x|^{2-n},$$

where $K_1, K_2 > 0$.

**Proof.** It follows from Theorem 3.1 that there exists a solution $u(x)$ such that

$$K_1|x|^{2-n} \leq u(x) \leq K_2|x|^{2-n},$$

where $K_1, K_2 > 0$. Due to Remark 2.2 and the estimate from Lemma 2.2 it suffices to show that the singularity at $O$ is removable if $u(x) = o(|x|^{-n})$ as $x \to 0$.

Let $u(x)$ be a solution in $B_1 \setminus \{O\}$, and $v_1$ be a solution of the problem

\[
\begin{cases}
Lv_1 = 0, & x \in B_1 \\
v_1|_{\partial B_1} = u|_{\partial B_1}.
\end{cases}
\]

The function $v_1$ is a subsolution for the equation (1.1)

$$Lv_1 + p(x)|x|^s v_1^{-\sigma} \geq 0. \quad (4.1)$$

Let’s consider

\[
\begin{cases}
Lv_2 + p(x)|x|^s v_2^{-\sigma} = 0, & x \in B_1(0) \\
v_2|_{\partial B_1(0)} = u|_{\partial B_1(0)}.
\end{cases}
\]  \hfill (4.2)

There is a solution $v_2 > 0$ and $v_2 \in C(B_1(0))$, since $2 + s > 0$ [5]. From (4.1) and (4.2) we obtain

$$L(v_2 - v_1) < 0,$$

$$v_1|_{\partial B_1(0)} = v_2|_{\partial B_1(0)},$$

and

$$v_2 - v_1 \geq 0, \quad x \in B_1(0).$$

So, $v_2$ is a supersolution for (1.1)

$$Lv_2 + p(x)|x|^s v_2^{-\sigma} \leq 0.$$

And therefore, there exists a solution $v > 0$ of the problem

\[
\begin{cases}
Lv + p(x)|x|^s v^{-\sigma} = 0, & x \in B_1(0) \\
v|_{\partial B_1(0)} = u|_{\partial B_1(0)}.
\end{cases}
\]  \hfill (4.3)
Consider now \( w = u - v \).

From (1.1) and (4.3) we have
\[
Lw = p(x)|x|^s \frac{u^\sigma - v^\sigma}{(uv)^\sigma}, \quad x \in B_1 \setminus \{O\},
\]
and
\[
\text{sign}(u^\sigma - v^\sigma) = \text{sign}(u - v) = \text{sign} \ w.
\]

Let’s show \( w \equiv 0 \) in \( B_1 \setminus \{O\} \). Suppose there is a point \( x_0 \in B_1 \setminus \{O\} \), such that \( w(x_0) > 0 \), and denote \( G = \{x \in B_1 \setminus \{O\} : w > 0\} \).

There is a fundamental solution \( \Gamma(x) \) [6]:
\[
L \Gamma(x) = 0, \quad x \in B_1(0) \setminus \{O\},
\]
and
\[
0 < C_1|x|^{2-n} \leq \Gamma(x) \leq C_2|x|^{2-n}.
\]

Then, since \( u(x) = o(|x|^{2-n}) \) as \( x \to 0 \), \( \forall \epsilon > 0 \) there exists a ball \( B_\delta(0) \), such that \( x_0 \in B_1 \setminus B_\delta \) and
\[
w|_{\partial B_\delta} < \epsilon \Gamma|_{\partial B_\delta} > 0, \quad w|_{\partial G} = 0.
\]

Hence, taking in the account that
\[
L(w - \epsilon \Gamma) = p(x)|x|^s \frac{u^\sigma - v^\sigma}{(uv)^\sigma} > 0, \quad x \in G,
\]
we conclude from the maximum principle that
\[
w < \epsilon \Gamma, \quad x \in G \setminus B_\delta
\]
Taking now \( \epsilon \to 0 \) we obtain that \( w(x_0) = 0 \), which leads to a contradiction. Then we have \( w \equiv 0 \) in \( B_1 \setminus \{O\} \), and can put \( u(0) = v(0) \), where \( v(0) > 0 \).

The theorem is proved. \( \square \)

**Remark 4.1.** Using a method of super- and subsolutions we can show the existence of a bounded positive solution to (1.1) in the whole domain \( \Omega \).

**Theorem 4.2.** Any positive weak solution of (1.1) in \( \Omega \setminus \{O\} \) is unbounded when \( s + 2 = 0 \)

First we prove

**Lemma 4.1.** Let \( u(x) \) be a positive solution of the problem
\[
\begin{cases}
Lu + q(x)u = 0, & x \in B_1(0), \\
u|_{x=1} = C_0,
\end{cases}
\]
where \( q(x) > q > 0 \). Then there exists a constant \( \delta > 0 \) which doesn’t depend on \( u \) such that
\[
u(0) > C_0 + \delta.
\]
Proof of Lemma 4.1. Let us consider
\[ v = u - C_0. \]

It follows from the maximum principle that
\[ v \geq 0 \]
in \( B_1(0) \), and
\[ \begin{cases} Lv + q(x)u = 0, & x \in B_1(0), \\ v|_{|x|=1} = 0. \end{cases} \]

Hence \( u \geq C_0 \), and \( v(x) \) can be considered as a solution of the equation
\[ Lv = -f(x), \]
where
\[ f(x) \geq qC_0 > 0 \]
in \( B_1(0) \).

From the Green’s representation we have
\[ v(0) = -\int_{B_1} f(x)G(x, 0)dx, \]
where \( G(x, y) \) is the Green function for our problem. It is known that \( G(x, y) < 0 \) and there are constants \( C_1, C_2 > 0 \) such that
\[ \frac{C_1}{|x-y|^{n-2}} \leq -G(x, y) \leq \frac{C_2}{|x-y|^{n-2}} \]
in \( B_\varepsilon(0), \varepsilon \in (0, 1) \) [6]. Then
\[ v(0) = -\int_{B_1 \setminus B_\varepsilon} f(x)G(x, 0)dx - \int_{B_\varepsilon} f(x)G(x, 0)dx > \delta(q, \varepsilon, C_2) > 0. \]

Hence
\[ u(0) > C_0 + \delta, \]
and Lemma 4.1 is proved. \( \square \)

Proof of Theorem 4.2. As it follows from Lemma 4.1, if \( u > 0 \) is a solution of the equation
\[ Lu + |x|^{-2}p(x)u^{-\sigma} = 0, \]
then
\[ u > C_0 > 0. \]

Let us suppose
\[ C_0 < u < M, \]
and rewrite the equation in the form

\[ Lu + \frac{q(x)}{|x|^2} u = 0, \]

where

\[ q(x) = p(x) u^{-(\sigma+1)} > p_0 M^{-(\sigma+1)}. \]

Now we can apply Lemma 4.1 for any ball in \( \Omega \setminus \{O\} \). Let it be a ball \( B_{1/2}(x) \), where \( |x| = 1 \). Then\[ u(x) > C_0 + \delta \quad \forall |x| = 1. \]

After the scaling \[ y = 2^k x \]
we get the same equation with respect to \( y \), and then by applying Lemma 4.1 we get that \[ u(x) > C_0 + \delta \quad x \in B_1(0) \setminus \{O\}. \]

Continuing the procedure, we obtain

\[ u(x) > C_0 + 2^k \delta \quad \text{for } |x| = 2^{-k} \quad (4.4) \]

does not imply \( u(x) \to \infty \) as \( |x| \to 0 \). However, the desired contradiction is clear since \( u \) was assumed to be bounded while from (3.4) it follows the contrary.

Theorem 4.2 is proved. \( \square \)

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