Quantum stirring as a sensitive probe of 1D superfluidity

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We propose quantum stirring with a laser beam as a probe of superfluid behavior for a strongly interacting one-dimensional Bose gas confined to a ring. Within the Luttinger liquid theory framework, we calculate the fraction of stirred particles per period as a function of the stirring velocity, the interaction strength and the coupling between the stirring beam and the bosons. The fraction of stirred particles allows to probe superfluidity of the system. We find that it crosses over at a critical velocity, lower than the sound one, from a characteristic power law at high velocities to a constant at low velocities. Some experimental issues on quantum stirring in ring-trapped condensates are discussed.

Progress in the ability to manipulate low-dimensional ultracold atomic gases has stimulated the interest in fundamental properties of one-dimensional (1D) Bose liquids 1,2,3. A Bose-Einstein condensate (BEC) of an atomic gas is known to exhibit superfluidity. Experiments have confirmed the superfluid behavior by demonstrating a critical velocity below which a laser beam could be moved through the gas without causing excitations 4,5,6, and an irrotational flow through the creation of vortices 7 and vortex lattices 8 in both rotating and nonrotating traps. For a Bose-Einstein condensate in a toroidal trap the observation of a persistent flow has also been reported 8.

Quantum pumping offers another way of inducing particle transfer without creating excitations. In pumping, periodic (ac) perturbations of the system yield a dc current. Indeed, this current may be entirely adiabatic as long as the external perturbations are slow enough such that the system always remains in the instantaneous ground state. The number of particles transferred in each cycle is then independent of the pumping period T and the integral of the current over a period is quantized for a clean infinite periodic system with filled bands 9,10.

Up to now, spectacular precision of quantization of the pumped current has been achieved in experiments with nano-electronics devices 11.

Quantum pumping is intimately connected to quantum stirring. Quantum stirring is accomplished by the transverse irrotational flow through the creation of vortices 6 and vortex lattices 7 in both rotating and nonrotating traps. We focus here on stirring a 1D Bose gas confined to a ring. Within the Luttinger liquid theory framework, we calculate the fraction of stirred particles per period as a function of the stirring velocity 12,13,14.

We consider N bosons of mass m confined onto a 1D ring of circumference L, with contact interactions $v(x-x') = g\delta(x-x')$ at zero temperature. The long-wavelength behavior of this system at distances larger than the cutoff length $\alpha = 1/\rho_0 = L/N$ is described by the Luttinger liquid Hamiltonian in terms of the density and phase fluctuation modes of the bosonic field 12,13,14:

$$H_0 = \frac{\hbar}{2\pi} \int dx \left[ \frac{v_s}{K} (\nabla \phi(x))^2 + v_s K (\nabla \theta(x))^2 \right],$$

(1)

where the field $\phi(x)$ is related to the particle density according to

$$\rho(x) = [\rho_0 - \frac{1}{\pi} \nabla \phi(x)] \sum_{p=-\infty}^{\infty} e^{i2p(\pi\rho_0 x - \phi(x))},$$

(2)

the field $\theta(x)$ corresponds to the phase of the superfluid, and we have $[\phi(x), \nabla \theta(x')] = i\delta(x-x')$. In the case of repulsive contact interaction between bosons, the Luttinger parameters $v_s$ and $K$ used in (1) are obtained 12,14 by: $v_s K = \frac{\pi \hbar \omega}{m}$, as follows from galilean invariance, and $\frac{1}{K} = \frac{\pi}{\rho_0}$ in the weak coupling limit, while $\frac{1}{K} = \frac{\pi}{\rho_0} \left(1 - \frac{8\pi^{3/2} m^{1/2}}{\hbar^{3/2}}\right)$ in the strong coupling limit.

When the interaction goes to zero $K$ goes to infinity, while $K = 1$ for infinitely strong hard-core interactions (Tonks-Girardeau limit), where the problem can be solved by mapping onto a gas of noninteracting fermions 15. In this regime $2\pi \rho_0 \to 2k_F$, with $k_F$ being the Fermi wavevector of the corresponding mapped spinless
fermions. The long-wavelength properties of 1D dipolar gases are also described by Eq. \(1\) with \(K < 1\) [19].

We describe next the effect of a barrier moving with velocity \(V\) through the fluid by introducing the time-dependent potential \(U(x,t) = U_0 \delta(x-Vt)\). In an experiment this could be realized e.g. by stirring the gas by a blue-detuned laser beam. The Hamiltonian acquires an explicitly time-dependent term which couples to the density:

\[
\delta H(t) = \int dx U(x,t) \rho(x). \tag{3}
\]

Using Eq. \(2\) for the density and keeping only the lowest, most relevant harmonics we may rewrite \(3\) as

\[
\delta H(t) = U_0[\rho_0 - \frac{1}{\pi} \nabla \phi(Vt) + 2\rho_0 \cos(2\pi \rho_0 Vt - 2\phi(Vt))]. \tag{4}
\]

The term proportional to \(\nabla \phi\) is analogous to a slowly varying chemical potential and can be absorbed in \(H\) by a redefinition of the field \(\phi\), \(\phi \rightarrow \phi - (K/v_s) \int dx U(x')\), while the last leading term in Eq. \(4\) represents scattering of the bosons off the barrier with momentum close to \(\pm 2\pi \rho_0\). In the Tonks Girardeau[18] limit it describes the backscattering of right-movers into left-movers, i.e. processes with momentum close to \(\pm 2k_F\). During its motion the barrier drags along a part of the bosons. We are interested in the stirred fraction \(N_{\text{stir}}/N\) i.e. the fraction of particles transported per period \(T = L/V\) by the moving barrier, and related to the particle current as \(N_{\text{stir}} = \frac{1}{2\pi} \int_0^T dt I(t)\). If the barrier height is infinitely large, the fraction of stirred particles per period is quantized, i.e. \(N_{\text{stir}}/N = 1\), independently of the interaction strength. If the barrier height is finite, the stirred fraction is in general smaller than one and we show that it is related to the degree of correlations in the system. We analyze perturbatively the regimes of weak and large barrier for arbitrary interaction strength and treat exactly the Tonks-Girardeau regime.

**Weak barrier** - In the weak barrier limit we perform a perturbative analysis of the current generated by the stirring Hamiltonian \(\delta H\). As customary in Luttinger liquid formalism we introduce the particle density of right(left) movers related to the fields \(\theta(x)\) and \(\phi(x)\) as \(\rho_{RL}(x) \approx \frac{\pi}{2} + \frac{1}{\pi} [\nabla\theta(x) + \nabla \phi(x)]\). The particle current at low energy is \(J(x) \sim \nabla \theta(x)\); since it involves the difference in the number of right and left movers, the term proportional to \(\nabla \phi\) in Eq. \(4\), which does not distinguish between left and right movers, plays no role in generating the particle current. On the contrary, the third, backscattering term in Eq. \(4\), can lead to the generation of a current which we define of backscattering, \(I_b\). In fact, addition of the moving-barrier potential breaks the continuous chiral symmetry [21] violating the conservation of the axial charge \(N_R - N_L\), where \(N_{RL} = \int dx \rho_{RL}(x)\). In the lowest order perturbation theory the backscattering current is given by \(I_b^0 = \frac{\pi}{2}[N_L, \delta H] = -iU_0 [N_R, \delta H]\).

In our specific case, by using the bosonized expression of the density operators and of the stirring Hamiltonian, the resulting backscattering-current operator is \(I_b^0 = i\gamma(t)\tilde{n}(t) - \text{h.c.}\), where \(\gamma(t) = U_0 e^{2\pi \rho_0 V t} \) and \(\tilde{n} \sim \rho_0 e^{2\phi(Vt)}\), and it is characterized by the backscattering frequency \(\omega_b = 2\pi \rho_0 V\). Linear response theory yields the backscattering current to second order in the barrier strength \(U_0\) as

\[
I_b \approx i \int_{-\infty}^t dt' \langle [I_b^0(t), \delta H(t')] \rangle_{\phi} \tag{5}
\]

and turns out to be related to the Fourier transform of the Green’s function of the backscattering operator \(e^{2\phi(Vt)}\) at the characteristic frequency \(\omega_b\). In the thermodynamic limit \(N, L \rightarrow \infty\) with \(\rho_0 = N/L\) constant and for small stirring velocity, the resulting backscattering current is given by

\[
I_b \approx \frac{(2\pi)^{2K-1}}{\Gamma(2K)} \frac{U_0^2}{(hv_s)^2} \left( \frac{V}{v_s} \right)^{2K-2} 2\pi \rho_0 V, \tag{6}
\]

with \(\Gamma\) being the Euler Gamma function. The fraction of stirred particles is readily obtained from the backscattering current according to \(I_b = N_{\text{stir}}/N = I_b/\omega_b\). In the Tonks-Girardeau limit \(K \rightarrow 1\), Eq. \(6\) yields \(N_{\text{stir}}/N \propto (U_0/v_s)^2\), i.e. the result is independent of the frequency \(\omega_b\) and hence adiabatic. \(\omega_b\) in the small \(\omega_b\) limit this result is in agreement with the exact calculation of the fraction of stirred particles, as shown below. Note that as the Luttinger liquid theory is an effective low-energy model, it describes correctly the system at frequencies \(\omega_b < 2\pi v_s/\alpha\), hence the expression \(6\) is valid only if \(V < v_s\), and cannot treat the supersonic regime. By recalling that the power-law dependence in Eq. \(6\) originates from the excitation of sound waves in the quasi-one-dimensional geometry, we can also determine the smallest velocity for which Eq. \(6\) holds in the case of a ring of finite length. In this case no excitations are possible below \(V_{\text{low}} = v_s/N \sim \pi \hbar/mL\) corresponding to the momentum of the lowest bosonic mode on the ring. The value of \(V_{\text{low}}\) found agrees with the one obtained by a Gross-Pitaevskii approach for \(K \gg 1\) [23]. Thus as a main result we find that at the critical velocity \(V_{\text{low}}\) the fraction of stirred particles crosses from a power-law to a constant (adiabatic regime). Note also that the adiabatically stirred fraction decreases with decreasing interaction strength as \(1/\Gamma(2K)\): when \(K\) grows, the system becomes more superfluid, hence the interaction with the external barrier decreases and \(N_{\text{stir}}/N \rightarrow 0\).

The results obtained above are consistent with a treatment based on the perturbative renormalization group (RG) approach [24]. In this approach the scaling of the potential \(U_0\) with frequency \(\omega\) is obtained from the flow equation \(dU_0/dz = (K-1)U_0\) where \(dz = d\omega/\omega\). As a function of \(K\), two regimes are distinguished. When \(K > 1\) the barrier is irrelevant: \(U_0\) decreases as \(\omega\) is
increased from $v_s/\alpha$ down to $\omega_b \sim V/\alpha$. For an infinite system, $U_0$ and hence $N_{\text{stir}}/N$ scale to zero as $\omega_b \to 0$; for a finite system the RG procedure should stopped when $\omega_b \sim v_s/L$, i.e. for $V \sim v_s/N$; this is the regime where we find a residual adiabatically stirred fraction, independent of $V$. For $K < 1$, e.g. in the dipolar gas, $U_0$ and hence $N_{\text{stir}}/N$ grow under RG, i.e. the barrier is a relevant perturbation. This is shown in the right panel of Fig. 1. Perturbation theory breaks down when $N_{\text{stir}}/N \sim 1$, i.e. at the velocity $V_{t_0}^* = v_s(U_0/h\nu_s)^{1/K}$ and the RG flow must be stopped. The behavior beyond this breakdown point is described by an effective weak-link tunneling model [24].

**Weak link limit** - The large-barrier limit is equivalent to a ring cut at the position of the delta barrier, and we treat the residual tunneling $t_0$ between the two ends of the ring as a perturbation [24]. In this case the bosonized Hamiltonian corresponding to the hopping across the weak-link can be obtained by a duality transformations [24], $\phi \rightarrow \theta$ and is given by $\delta H \sim t_0 \cos(2\pi \theta(Vt))$. Its contribution to the particle current (tunneling current $I_t$) can be calculated in the linear response regime and its explicit expression for an infinitely long ring is:

$$I_t = \frac{(2\pi)^{2} - 1}{2b_{0}h\nu_{s}} \frac{(V/
u_{s})^{2}}{2\pi} \rho_{0}V.$$  

In the presence of tunneling the stirred fraction of particles is $N_{\text{stir}}/N = 1 - I_{t}/\omega_{b}$, where $I_{t}/\omega_{b}$ is the fraction of tunnelled particles, not stirred. In the hard-core limit ($K = 1$) the stirred current will be again linear in the frequency of the stirring. We thus recover the adiabatic limit. Under the RG flow, the tunneling becomes relevant for interacting bosons with contact repulsion ($K > 1$) therefore, upon decreasing the stirring velocity the effective tunneling strength increases, thereby decreasing the stirred particle fraction, again shown in the right panel of Fig. 1. Perturbation theory breaks down when the effective tunneling strength reaches unity and the RG flow must be stopped at $V_{t_0}^* = v_s(t_0/\rho_{0}h\nu_{s})^{2/K}$, then the stirred fraction of particles is governed by the previous weak barrier limit. The results for the dependence of the stirred fraction of particles on the velocity $V$ is shown in Fig. 1. The results explicitly show a difference in the regime with $K > 1$ (short-range interactions) and $K < 1$ (dipolar interactions). In the latter case the stirred fraction decreases at increasing velocities, where the system tends towards superfluid behavior. Since a dipolar gas is characterized by a quasi-crystal order phase at increasing density [19], the result can be interpreted as an inefficiency of the stirring in creating an excitation in the ordered state.

**Non-perturbative analysis** - In the Tonks-Giradeau limit ($K = 1$) a time-dependent Fermi-Bose (FB) mapping [18] is employed to generate exact solutions of the problem [22] and the current is calculated exactly. The time dependent version of the FB mapping permits to write the exact many-body wavefunction of $N$ impenetrable bosons on a ring as $\Psi_B(x_1,\ldots,x_N;t) = A(x_1,\ldots,x_N)\Psi_F(x_1,\ldots,x_N;t)$, where $A$ is a unit antisymmetric function $A(x_1,\ldots,x_N) = \prod_{1 \leq i < k \leq N} \text{sgn}(x_k - x_j)$, and $\Psi_F(x_1,\ldots,x_N;t) = C \det_{j=1}^{N} \psi_{j}(x_j,t)$ is the wave function for any ideal Fermi gas, $\psi_{j}(x,t)$ being the solutions of the one-body time-dependent Schrödinger equation in the external potential $U(x,t)$. Starting from the above many-body wavefunction, we evaluate the Tonks-Girardeau particle current density in terms of the one-body density matrix $\rho_1(x,y) = \int dx_2\ldots dx_N \Psi_B^{*}(x_1,\ldots,x_N;t)\Psi_B(y_1,\ldots,y_N;t)$ as $J(x) = -(\hbar/2m)\partial_{x}\rho_1(x+r/2,x-r/2)r=0$. Although $\rho_1(x,y)$ for a TG gas is very different from the one of a Fermi gas due to the presence of the mapping function $A$, we find that the latter has no effect on the current, which then coincides with the current of an ideal Fermi gas. In the adiabatic limit $V \leq \pi\hbar/mL$ the particle current and the stirred fraction produced by the slow variation of the stirring potential can then be evaluated by following the adiabatic expansion of Thouless for an

![Figure 1](link-to-image)
ideal Fermi gas, i.e.

$$N_s = \frac{i}{2\pi m} \frac{h^2}{L} \int_0^\tau \sum_{\ell, j \neq 0} \frac{f_{\ell j}}{(\epsilon \ell - \epsilon j)} \left[ (\psi_j \psi_\ell) (\partial_x \psi_\ell \psi_j) + h.c. \right],$$

where $f_{\ell j}$ is the fermionic probability occupation function of the state $\ell, j$. In the case of a blue-detuned laser field piercing the ring at a position $x = 0$ and modelled by the potential $U_0 \delta(x)$, the appropriate orbitals $\psi_i(x, t)$ are the $L$-periodic free-particle energy eigenstates satisfying at $x = 0$ the cusp condition. The complete orthonormal set of even parity $\psi_n^+(x)$ and odd parity $\psi_n^-(x)$ eigenstates are $\psi_n^+(x) = (e^{i k_n x} + e^{-i k_n (x - L)})/N_n$ and $\psi_n^-(x) = \sqrt{2} \sin(2n\pi x/L)$, where $k_n$ are obtained from the transcendental equation $k_n \tan(k_n L/2) = mU_0/h^2$ (for $U_0 \to \infty$) we have $k_n = \pi(2n + 1)/L$, in agreement with [18] and $N_n = \sqrt{2L[1 + \sin(k_n L)]}$, with $n$ running from 1 to $\infty$. The $N$-fermion ground state is obtained by inserting the lowest $N$ orbitals into the determinant above (Fermi sea), and using the exact orbitals $\psi_n^{(\pm)}$ as instantaneous ground state we obtain from (8)

$$N_s = 64 \sum_{\ell, j} (f_{\ell j} - f_{j \ell}) \frac{\sin^2(k_n L/2)}{1 + \sin(k_n L)} \left( \frac{k_n L}{(k_n L)^2 - 4\pi^2 \ell^2} \right)^2$$

For a weak barrier by using the small-$U_0$ expression for $k_n$ we obtain $N_s/N \approx 0.32(U_0/hv_0)^2$, which scales as the $K = 1$ limit of the backscattered current in Eq. (6) because $v_s = \hbar k_F/m$ for $K = 1$.[22] For an infinitely strong barrier using the $U_0 \to \infty$ limit of $k_n$ it’s straightforward to verify that the particle transport is quantized[3], i.e. all the particles are dragged by the barrier and $N_{\text{stir}}/N = 1$. This is shown in Fig 2a where the stirred faction of particles is plotted as a function of the barrier strength.

Experimental issues on condensates in closed loop waveguide -A possible way of achieving experimentally an annular condensate with strong transverse confinement is to use a magnetic toroidal trap, as reported in [26, 27]. Experimentally the stirring of hydrodynamic flow in a BEC by a blue-detuned laser beam, has been analyzed by calorimetric method[4] and phase contrast imaging[5]. The onset of a drag force has been shown by the asymmetry in the density profile, defined as the difference between the peak column density in front and behind the laser beam, as a function of the stirring velocity above a critical velocity. The space integral of the density asymmetry is analogous to the fraction of stirred particles calculated above. Recently, the persistent flow of Bose-condensed atoms in a toroidal trap has also been observed[6]. A variant to such experiment by the addition of a cyclic moving plug beam could be a valuable realization of the present proposal.

In conclusion, superfluid flow in a ring geometry raises interesting new possibilities. With the use of a moving barrier acting as a quantum stirrer, the analog of quantization of particle transport for electron systems could be realized for a gas of atoms as an alternative probe of superfluidity.

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The prefactor of $N_{\text{att}}/N$ obtained from Eq. (6) is non-universal, as it depends on the short-distance cutoff $\alpha$, here set equal to $1/\rho_0$.

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