Equivariant quantization on quotients of simple Lie groups by reductive subgroups of maximal rank

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We consider a class of homogeneous manifolds including all semisimple coadjoint orbits. We describe manifolds of that class admitting deformation quantizations equivariant under the action of $G$ and the corresponding quantum group. We also classify Poisson brackets relating to such quantizations.

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1 Introduction

A quantum manifold is obtained from a usual manifold by replacing the original commutative function algebra with a deformed non-commutative algebra. If a manifold $M$ is equipped with an action of a Lie group $G$, i.e. $M$ is a $G$-manifold, it is natural to consider deformation quantizations of the function algebra which are equivariant with respect to the action of the group $G$ or the corresponding quantum group $U_h(g)$, where $g$ is the Lie algebra of $G$.

We consider here homogeneous $G$-manifolds of type $M = G/K$, where $G$ is a simple connected Lie group over $\mathbb{C}$ and $K$ its reductive subgroup of maximal rank, i.e. $K$ contains a maximal torus of $G$. The Lie algebra of $K$, $\mathfrak{k}$, is generated by a Cartan subalgebra of $g$ and the set of contained in $\mathfrak{k}$ root vectors. Such a class of manifolds includes semisimple coadjoint orbits of $G$ in $g^\ast$. For a coadjoint orbit, $\mathfrak{k}$ is a Levi subalgebra generated by simple root vectors.

The notion of $G$-equivariant quantization is very natural and supposed to be known. So, we only give the definition of $U_h(g)$-equivariant quantization we admit in the paper.

Definition 1.1. Let $M$ be a $G$-manifold. A $U_h(g)$-equivariant quantization on $M$ is the $\mathbb{C}[h]$-module $\mathcal{F}(M)[[h]]$ endowed with a deformed multiplication $\mu_h$ such that:

(i) $\mu_h$ defines a star-product on $M$, i.e. is of the form $\mu_h = \mu + h\mu_1 + h^2\mu_2 + \cdots$, where $\mu$ is the usual commutative multiplication on $\mathcal{F}(M)$ and $\mu_i$, $i \geq 1$, are bidifferential operators on $M$ vanishing on constants;

(ii) the multiplication $\mu_h$ is $U_h(g)$-equivariant, i.e.

$$x\mu_h(a \otimes b) = \mu_h(\Delta_h(x)(a \otimes b)), \text{ for any } x \in U_h(g), \ a, b \in A_h,$$

where $\Delta_h$ denotes the comultiplication in $U_h(g)$. We also suppose that the action of $U_h(g)$ on $\mathcal{F}(M)[[h]]$ is the natural extension of the initial action of $U(g)$ on $\mathcal{F}(M)$ in
the following sense. Since \( \mathfrak{g} \) is simple, \( U_h(\mathfrak{g}) \) is isomorphic to \( U(\mathfrak{g})[[h]] \) as an algebra, and the natural action of \( U(\mathfrak{g})[[h]] \) on \( \mathcal{F}(M)[[h]] \) is induced from the action of \( U(\mathfrak{g}) \) on \( \mathcal{F}(M) \) by \( \mathbb{C}[[h]] \)-linearity.

The element \( \nu \) defined as \( \nu(a, b) = \mu_1(a, b) - \mu_1(b, a) \) for \( a, b \in \mathcal{F}(M) \) is a Poisson bracket on \( M \). In particular, \( \nu \) is a bivector field on \( M \). We call \( \mu_h \) a quantization corresponding to the Poisson bracket \( \nu \). One can prove, \([9]\), that the Poisson bracket of any \( U_h(\mathfrak{g}) \)-equivariant quantization has the form

\[
\nu = s - r_M, \tag{1}
\]

where \( r_M \) is the bivector field which is the image of the classical \( r \)-matrix \( r \in \wedge^2 \mathfrak{g} \) related to \( U_h(\mathfrak{g}) \) via the action map \( \mathfrak{g} \rightarrow \text{Vect}(M) \), and \( s \) is a \( G \)-invariant bivector field on \( M \) satisfying

\[
[s, s] = -\varphi_M. \tag{2}
\]

By \( \varphi_M \) we denote the 3-vector field induced on \( M \) by the \( \mathfrak{g} \)-invariant element \( \varphi = [r, r] \in \wedge^3 \mathfrak{g} \) and by \( [,] \) the Schouten bracket of two polyvector fields. We call a Poisson bracket of the form \([1]\) admissible.

**Remark 1.2.** It is easy to see that the admissible Poisson brackets are precisely those which makes \( M \) a Poisson homogeneous manifold with respect to the Drinfeld–Sklyanin Poisson bracket \( r^r - r^l \) on \( G \).

The admissible brackets are also related to the dynamical CYBE \([3], [4]\).

It is convenient to introduce the following

**Definition 1.3.** A \( G \)-invariant bivector field \( s \) on \( M \) is called a \( \varphi \)-bracket if it satisfies \([2]\).

The first result of the paper is a classification of invariant and admissible Poisson brackets on homogeneous manifolds under consideration. Equation \([1]\) reduces the classification of admissible brackets on \( M \) to a classification of \( \varphi \)-brackets on \( M \).

The second result of the paper is that for any admissible Poisson bracket on \( M \) there exists the corresponding \( U_h(\mathfrak{g}) \)-equivariant quantization. Note that there exists a \( G \)-invariant connection on \( M \), since the stabilizer of a point of \( M \) semisimply acts on \( \mathfrak{g} \). Moreover, since \( M \) is \( G \)-homogeneous, an invariant Poisson bracket on \( M \) has the same rank at any point of \( M \). Applying, for example, the well known Fedosov method, we obtain that for any \( G \)-invariant Poisson bracket on \( M \) the corresponding \( G \)-equivariant quantization always exists.

Among other works relevant to our study, we would like to mention the following. J. Donin and D. Gurevich \([5]\) proved that on a semisimple coadjoint orbit there exists a quantization of the Sklyanin–Drinfeld Poisson bracket. J. Donin and S. Shnider \([6]\) constructed one and two parameter equivariant quantization on any symmetric manifold. J. Donin, D. Gurevich, and S. Shnider \([2]\) solved the problem of one and two parameter equivariant quantization on any semisimple coadjoint orbit, including a complete classification of corresponding Poisson brackets on them.
The paper is organized as follows. In Section 2, we introduce a group $\Gamma_k$ related to a reductive subalgebra $\mathfrak{k} \subset \mathfrak{g}$ of maximal rank. It is the quotient of the lattice spanned by all roots of $\mathfrak{g}$ by the sublattice spanned by roots of $\mathfrak{k}$. We prove a statement about how the images of roots are lying in $\Gamma_k$.

In Section 3, using this statement we give a classification of invariant Poisson brackets and $\varphi$-brackets on $M$.

In Section 4, we show that those brackets can be quantized.

2 Reductive subalgebras of maximal rank and group $\Gamma_k$

Let $\mathfrak{g}$ be a reductive Lie algebra over $\mathbb{C}$. Let us fix a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and denote by $\Omega \subset \mathfrak{h}^*$ the root system of $\mathfrak{g}$. We are interested in reductive subalgebras, $\mathfrak{k}$, in $\mathfrak{g}$ containing $\mathfrak{h}$, which are called subalgebras of maximal rank. To any subalgebra of this type, it corresponds a subsystem of roots, $P$. Namely, $\alpha \in P$ if and only if the root vector $E_\alpha \in \mathfrak{k}$. Note that $\mathfrak{k}$ is generated by $\mathfrak{h}$ and the root vectors $E_\alpha$, $\alpha \in P$.

Example 2.1. Choose a system of simple roots $\Pi \subset \Omega$. Fix a subset $\Sigma \subset \Pi$. Let us put $P = \mathbb{Z}(\Sigma) \cap \Omega$, where $\mathbb{Z}(\Sigma)$ denote the free abelian group (lattice) spanned by $\Sigma$. Let $\mathfrak{k}$ be the Lie subalgebra of $\mathfrak{g}$ generated by the root vectors $E_\alpha$ for all $\alpha \in P$ and by the Cartan subalgebra $\mathfrak{h}$. Such $\mathfrak{k}$ is called a Levi subalgebra of $\mathfrak{g}$.

Example 2.2. Choose a system of simple roots $\Pi \subset \Omega$. Fix a root $\alpha \in \Pi$ and a positive integer $n$. Consider the set $P$ of all roots in $\Omega$ whose coefficient before $\alpha$ in their expansion in basis $\Pi$ is divisible by $n$. Let $\mathfrak{k}$ be the Lie subalgebra of $\mathfrak{g}$ generated by the root vectors $E_\alpha$ for all $\alpha \in P$ and by the Cartan subalgebra $\mathfrak{h}$.

The following is a modified version of the well known Dynkin theorem, [7].

Theorem 2.3. For any reductive subalgebra $\mathfrak{k} \subset \mathfrak{g}$ of maximal rank, there exists a chain $\mathfrak{g} \supset \mathfrak{k}_1 \supset \mathfrak{k}_2 \supset \cdots \supset \mathfrak{k}_k = \mathfrak{k}$ of Lie subalgebras such that the subalgebra $\mathfrak{k}_j$ is obtained from $\mathfrak{k}_{j-1}$ by applying one of the procedures described in Examples 2.1 and 2.2.

Let us set $\Gamma_k = \mathbb{Z}(\Omega)/\mathbb{Z}(P)$ and denote by $\text{Tor}(\Gamma_k)$ the subgroup of elements of finite order in $\Gamma_k$. The following theorem describes the group $\text{Tor}(\Gamma_k)$ and images of roots in it.

Proposition 2.4. (I) The group $\Gamma_k$ is free, i.e. $\text{Tor}(\Gamma_k) = 0$, if and only if $\mathfrak{k}$ is a Levi subalgebra.

(II) $\text{Tor}(\Gamma_k)$ is cyclic if and only if $\mathfrak{k}$ is obtained as in Theorem 2.3 applying the procedure of Example 2.2 no more than once.

(III) If $\text{Tor}(\Gamma_k)$ is cyclic then any its non-zero element is the image of a root.

(IV) If $\text{Tor}(\Gamma_k)$ is non-cyclic, then the kernel of any character of it contains the image of a root.
Proof. (I) is obvious, since in this case \( P \) is generated by a set of roots, \( P' \), lying in a linear subspace of the vector space spanned by \( \Omega \). But it is known that such a \( P' \) can be included in a set of simple roots.

To prove other points, we use the structure theory of simple Lie algebras. First of all, we come to the following conclusions. Any cyclic summand in \( \text{Tor}(\Gamma_k) \) is of order \( \leq 6 \). If \( g \) is a classical simple Lie algebra, the group \( \text{Tor}(\Gamma_k) \) is isomorphic to a direct sum of copies of \( \mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z} \). For exceptional simple Lie algebras \( g \), also may be \( \text{Tor}(\Gamma_k) = \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_5, \mathbb{Z}_2 \oplus \mathbb{Z}_3, \mathbb{Z}_2 \oplus \mathbb{Z}_4, \) or \( \mathbb{Z}_3 \oplus \mathbb{Z}_3 \). Using this structure of \( \text{Tor}(\Gamma_k) \), the proof of the theorem can be completed by a direct computation.

3 Classification of invariant Poisson brackets and \( \varphi \)-brackets on \( M \)

In this section, we suppose that \( G \) is a simple connected Lie group and \( K \) a reductive subgroup of maximal rank. We are going to describe invariant bivector fields, \( s \), on \( M \) satisfying the equation

\[
[s, s] = \kappa^2 \varphi_M,
\]

where \( \kappa \in \mathbb{C} \). Note that if \( s \) satisfies \( \ref{3} \) with \( \kappa^2 = 0 \), then it defines an invariant Poisson bracket, while if \( s \) satisfies \( \ref{3} \) with \( \kappa^2 = -1 \) then it defines a \( \varphi \)-bracket on \( M \).

Consider the natural projection \( \pi : G \to M = G/K \). It induces the map \( \pi_* : \mathfrak{g} \to T_o \) where \( T_o \) is the tangent space to \( M \) at the point \( o \) being the image of unity. Since the ad-action of \( \mathfrak{k} \) on \( \mathfrak{g} \) is semisimple, there exists an \( \text{ad}(\mathfrak{k}) \)-invariant subspace \( \mathfrak{m} \) complementary to \( \mathfrak{k} \), and one can identify \( T_o \) and \( \mathfrak{m} \) by means of \( \pi_* \). It is easy to see that subspace \( \mathfrak{m} \) is uniquely defined and has a basis formed by elements \( E_\gamma, \gamma \in \Omega \setminus P \). Here \( E_\gamma \) are root vectors satisfying \( (E_\gamma, E_{-\gamma}) = 1 \) for the Killing form \( (\cdot, \cdot) \).

It follows from the above that invariant bivector fields on \( M \) correspond to \( \mathfrak{k} \)-invariant bivectors of \( \wedge^2 \mathfrak{m} \). It is proven in \( \ref{2} \) and \( \ref{8} \) that any bivector of \( \wedge^2 \mathfrak{m} \) is \( \mathfrak{k} \)-invariant if and only if it has the form

\[
\sum_{\alpha \in \Omega \setminus P} c(\overline{\alpha}) E_\alpha \wedge E_{-\alpha},
\]

where \( \overline{\alpha} \) is the image of \( \alpha \) in \( \Gamma_k \) and coefficients \( c(\overline{\alpha}) \) satisfy the condition \( c(\overline{\alpha}) = -c(-\overline{\alpha}) \). This means that for invariant bivectors the coefficients before terms \( E_\alpha \wedge E_{-\alpha} \) and \( E_\beta \wedge E_{-\beta} \) are the same if \( \overline{\alpha} = \overline{\beta} \).

The following result is proved in \( \ref{2} \).

**Proposition 3.1.** A bivector field \( s \) presented in the form \( \ref{3} \) satisfies \( \ref{3} \) if and only if the coefficients \( c(\overline{\beta}) \) obey the following condition: if \( \overline{\alpha} + \overline{\beta} \in \Gamma_k \) then

\[
c(\overline{\alpha} + \overline{\beta})(c(\overline{\alpha}) + c(\overline{\beta})) = c(\overline{\alpha})c(\overline{\beta}) + \kappa^2.
\]

Using the previous proposition, one can obtain the following description of \( \varphi \)-brackets and invariant Poisson brackets on \( M \), \( \ref{12} \). See also \( \ref{9} \), where this is done for admissible brackets on coadjoint orbits.
Proposition 3.2. A \( \varphi \)-bracket on \( M \) is determined by the following data: a subgroup \( \Psi \subset \Gamma_k \) such that \( \Gamma_k/\Psi \) is free, a group homomorphism \( \chi : \Psi \to \mathbb{C} \setminus \{0\} \) such that \( \chi(\overline{\alpha}) \neq 1 \) for \( \overline{\alpha} \) being the image of a root, and a linear ordering in the group \( \Gamma_k/\Psi \). The coefficients \( c(\overline{\alpha}) \), \( \overline{\alpha} \in \Gamma_k \), from (4) are given by the formula:

\[
    c(\overline{\alpha}) = \frac{\chi(\overline{\alpha}) + 1}{\chi(\overline{\alpha}) - 1} \quad \text{for } \overline{\alpha} \in \Psi,
\]

where \( \chi(\overline{\alpha}) \neq 1 \) for \( \overline{\alpha} \) being the image of a root.

Proposition 3.3. An invariant Poisson bracket on \( M \) is determined by choosing a subgroup \( \Psi \subset \Gamma_k \) containing no torsion elements and a group homomorphism \( \lambda : \Psi \to \mathbb{C} \) such that \( \lambda(\overline{\alpha}) \neq 0 \) for \( \overline{\alpha} \) being the image of a root. The coefficients \( c(\overline{\alpha}) \) from (4) are given by the formula

\[
    c(\overline{\alpha}) = \begin{cases} 
    \frac{1}{\lambda(\overline{\alpha})} & \text{for } \overline{\alpha} \in \Psi, \\
    0 & \text{if the projection of } \overline{\alpha} \text{ in } \Gamma_k/\Psi \text{ is not zero}.
    \end{cases}
\]

Remark 3.4. Another description of Poisson brackets on quotients of \( G \) by reductive subgroups of maximal rank is given in [4].

Combining Proposition 2.4, 3.2, and 3.3, we obtain the following

Corollary 3.5. (I) A \( \varphi \)-bracket (and, therefore, an admissible Poisson bracket) on \( M \) exists if and only if \( \text{Tor}(\Gamma_k) \) is cyclic.

(II) A nonzero Poisson bracket on \( M \) exists if and only if \( \Gamma_k \) is infinite.

(III) If \( \Gamma_k \) is a finite non-cyclic group then there are neither invariant nor admissible Poisson brackets on \( M \). Therefore, such an \( M \) does not admit neither \( G \)- nor \( U_h(g) \)-equivariant quantization.

Proof. (I) If \( \text{Tor}(\Gamma_k) \) is non-cyclic then, in virtue of point (IV) of Proposition 2.4, \( \text{Ker}(\chi) \) must contain the image of a root, what is impossible by Proposition 3.2. On the other hand, if \( \text{Tor}(\Gamma_k) \) is cyclic, a required homomorphism \( \chi \) is obviously exists.

(II) If \( \text{Tor}(\Gamma_k) \) is finite, then \( \Psi \) from Proposition 3.3 must be finite too, hence \( \lambda(\Psi) = 0 \). It follows that \( \lambda \) applying to any image of a root gives zero.

(III) follows from (I) and (II).

Example 3.6. The manifold \( M = SO(13)/(SL(2) \times SL(2) \times SL(2) \times SL(2) \times SL(2) \times SL(2)) \), has \( \Gamma_k = \mathbb{Z}_2 \times \mathbb{Z}_2 \). Therefore, it satisfies the point (III) of the Corollary.
4 Quantization

**Theorem 4.1.** Let $G$ be a semisimple Lie group, $K$ a reductive subgroup of maximal rank. Then, for any admissible Poisson bracket on $M = G/K$ there exists a corresponding $U_h(g)$-equivariant quantization.

**Proof.** As follows from Corollary 3.5, one can suppose that $\text{Tor}(\Gamma_T)$ is cyclic. The case when $\text{Tor}(\Gamma_T) = 0$, i.e. $M$ is a semisimple coadjoint orbit, is considered in [2], while the case when $\Gamma_T$ is cyclic itself is considered in [8]. The complete proof of the theorem obtains combining methods of the both papers.

**Remark 4.2.** As was noticed in Introduction, for any invariant Poisson bracket on $M$ under consideration there exists a corresponding $G$-equivariant quantization.

**Question.** Describe compatible pairs consisting of invariant and admissible Poisson brackets on $M$. Does there exists for any compatible pair a two parameter $U_h(g)$-equivariant quantization? This problem was solved for semisimple coadjoint orbits in [8].

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