Investigation of the circular geodesics in a rotating charged black hole in presence of perfect fluid dark matter

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Abstract. We give a charged black hole solution in perfect fluid dark matter (PFDM). The metric corresponding to the rotating avatar of the black hole solution is obtained by incorporating the Newman-Janis algorithm. We then compute two type of circular geodesics, namely, the null geodesics and time-like geodesics for the mentioned spacetime geometry. For the case of time-like geodesics we consider massive particles characterized by charge with values $q = 0$ and $q \neq 0$. The effective potentials of the corresponding circular geodesics has also been studied briefly. We then continue the subsequent analysis by graphically representing the collective effects of the black hole parameters, namely, the charge of the black hole ($Q$), spin parameter ($a$) and the PFDM parameter ($\alpha$) on the energy ($E$), angular momentum ($L$) and effective potential ($V_{\text{eff}}$) of the concerned particle. We then discuss the Penrose process in order to study the negative energy particles which have possible existence within the ergosphere and which in turn leads to the energy gain of the emitting particle.
1 Introduction

The study of the black holes, one of the most fascinating objects in the universe has been playing the key role in the study about the observational aspects of the geometric theory of relativity. Although the general theory of relativity is still not able to grasp the whole story of the black holes, it turns out that upto a great extent the results of general relativity match with observations which urges us to work following the same. The supermassive black holes are believed to be the central core of all the galaxies and are responsible for holding the entire galaxy together. Recently the shadow images of the M87 \[1\] provided the first direct evidence for the existence of black holes and this motivates us to study some interesting aspects of black holes. It is difficult to study black holes directly and measure any of its properties like Hawking temperature, etc. So the study of spacetime structure and thereby the geodesic motion of various types of particles in it’s vicinity, helps to gain knowledge regarding the features of black holes. A vast amount of study has been done on the geodesic motion around black holes and it is very hard to mention them all, yet some of them has to be looked upon due to their novelty. In \[2, 3\], the study of geodesics around Schwarzchild and Kerr black holes can be found in detail. Besides that, geodesic motion of massive particles have been studied around conformal Schwarzchild black hole in \[4\]. The study of geodesics in regular Hayward BH \[5\], Kerr-Sen BH \[6, 7\], Kerr-Newman-Taub-NUT BH \[8\], generic black hole coupled to non-linear electrodynamics \[9\] and Schwarzchild black hole in quintessence \[10\] are worth mentioning. Since the observer in reality is far away from the black hole so theoretically one places the observer at infinity from which all planes are identical and hence for convenience and effectiveness the geodesics in the equatorial plane of the black holes are able to capture the whole story. The geodesics in equatorial plane for dyonic Kerr-Newman black hole has been studied in \[11\] and that for quintessential rotating black hole with and without cosmological constant (\(\Lambda\)) has been studied in \[12\] and \[13\] respectively. Also the
study of geodesics for distorted static BH can be found in [14] along with particle motion in Kerr spacetime in [15] and Kerr pierced by cosmic string in [16]. Besides massive test particles, the study of null geodesics demands more attention since they consume an extensive area of research now-a-days after the discovery of black hole shadow image recently by The Event Horizon Telescope collaboration [1]. The motivation to study the null geodesics is related to the complete understanding about the photon sphere which in turn leads to the formation of the black hole shadow. There are many studies on the null geodesics and thereby on the shadow. Some of them can be found in [7],[17]–[24]. Apart from the study of photons and massive particles, researchers have shown considerable interest in the study of charged particles due to the presence of the Maxwell fields which leads to additional interacting terms. The study of charged particles around Kerr Newman black hole has been carried out in [25]-[28]. Also the motion of the charged particles in case of black holes immersed in external electromagnetic fields has been studied in [29]–[33].

Another interesting thing in this context is dark matter and dark energy which seeks our attention by its own right. Dark matter is supposed to fill about 27% of the universe along with dark energy (68%) and ordinary matter(5%). Due to the indirect observations of dark matter, it is assumed to be present everywhere within and outside the observable universe and the amount increases as we move away from the galactic center (from the observation of galaxy rotation curves). There are different models of dark matter which are very useful to explain large and small scale structure of the universe [34]—[36]. Besides, dark matter in most theories is assumed to be present around black holes as halos and black hole shadows has been calculated in those cases as in [37, 38]. Also quintessential dark matter solutions were studied by Kiselev [40]. In recent times dark matter has been studied using perfect fluid model as in [42]. This model has gained attention recently and helps to identify the impact of the dark matter on various observables related to the black holes. There has been study of shadows in perfect fluid dark matter in case of rotating black holes with and without cosmological constant [43, 44] along with non-rotating charged black holes [45]. Dark matter is composed of non-baryonic matter and being present around black holes can contribute to the effective mass of the black hole system. The geodesics around black hole in PFDM (perfect fluid dark matter) should in principle have an appreciable influence due to the presence of extra matter and hence can led to interesting observations. Besides, black holes are sources of extreme energy which can be gained as theorised by Penrose [46]. The process has maximum efficiency at the event horizon of the black hole and requires particles of negative energy and angular momentum to be absorbed by the black hole which requires the presence of ergosphere, the place where the process takes place. There are different studies of Penrose process and thereby the extraction of energy from the black hole as in [47]–[50] where energy extraction for spinning particles have been studied and also it is found that the efficiency of the Penrose process increases in case of higher dimensions. In this paper we study the Penrose process for Kerr-Newman black holes in asymptotically flat spacetime in presence of the PFDM. We can briefly describe the plan for this paper in the following points below.

In this work, we give a static, charged black hole solution surrounded by perfect fluid dark matter (PFDM). We obtain the rotating version of the black hole solution by incorporating the Newman-Janis algorithm. The variation of the event horizon with respect to the PFDM parameter $\alpha$ has been observed. We study the geodesics corresponding to both massless and massive particles in the vicinity of the rotating charged black hole with PFDM. We then study the circular geodesics of photons and use them to determine the radius of unstable photon orbits. Furthermore, we look into how the dark matter parameter $\alpha$ as well as the spin ($a$)
and charge \((Q)\) of the black hole affect the photon radius. In case of massive particles, we compute the energy \((E)\) and the angular momentum \((L)\) of the particle in addition with the effective potential \((V_{eff})\). The effects of the PFDM parameter \(\alpha\), spin \((a)\) and charge \((Q)\) on these computed entities, has been observed briefly. We then study the Penrose process. We wish to study how PFDM parameter \(\alpha\) affects the size of the ergosphere, the negative energy of the particle and thereby the efficiency of the Penrose process.

The paper is organised as follows. In section 2 we discuss the charged black hole spacetime in perfect fluid dark matter. In section 3, we introduce rotation in the black hole by using the Newman-Janis algorithm. In section 4, we then discuss the circular geodesics of various particles (photons, uncharged and charged massive particles) and then we show the nature of potential of the black hole in section 5. Later we discuss the Penrose process in detail in section 6 and conclude by summarising the results in section 7. In this paper we have assumed \(\hbar = c = G = 1\).

2 Charged black hole in perfect fluid dark matter

We consider a \((3 + 1)\)-dimensional gravity theory minimally coupled with a U(1) gauge field, in presence of a perfect fluid dark matter (PFDM). The action can be written in the following form \([36, 39-41, 45]\)

\[
S = \int d^4x \sqrt{-g} \left( \frac{R}{16\pi G} + \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \mathcal{L}_{DM} \right) \tag{2.1}
\]

where, \(R\) is the Ricci scalar and \(G\) is the Newton’s gravitational constant. \(F_{\mu\nu}\) is the Maxwell field strength which is related with the electromagnetic potential \(A_\mu\) as \(F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\) and \(\mathcal{L}_{DM}\) gives the Lagrangian density for the perfect fluid dark matter. Extremizing the action we get the Einstein field equations as

\[
R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G (T^{M}_{\mu\nu} + T^{DM}_{\mu\nu}). \tag{2.2}
\]

In the above equation, \(T^{M}_{\mu\nu}\) represents the energy-momentum tensor corresponding to the ordinary matter (in this case the Maxwell field). This can be denoted as \([55]\]

\[
(T^{M}_{\mu\nu}) = \begin{cases} \lambda_\theta^\theta T^\theta_{\theta} = \text{diag} \left( -\frac{Q^2}{8\pi G r^2}, -\frac{Q^2}{8\pi G r^2}, \frac{Q^2}{8\pi G r^2}, \frac{Q^2}{8\pi G r^2} \right) \end{cases} \tag{2.3}
\]

where \(Q\) is the electric charge. On the other hand \(T^{DM}_{\mu\nu}\) corresponds to the energy-momentum tensor of the perfect fluid dark matter \([51]\). It is specified as

\[
(T^{DM}_{\mu\nu}) = \begin{cases} \lambda_\theta^\theta T^{DM}_{\theta\theta} = \text{diag} (-\rho, P, P, P) \end{cases} \tag{2.4}
\]

where \(\rho\) and \(P\) correspond to density and pressure of the perfect fluid dark matter. Following the approach given in \([41, 51]\), we further consider

\[
(T^{DM}_{\theta\theta}) = (T^{DM}_{\theta\theta}) (1 - \epsilon) \tag{2.5}
\]

with \(\epsilon\) being a constant. By substituting the components (from eq.(2.4)) in eq.(2.5), we obtain the equation of state for PFDM to be \([51]\)

\[
\frac{P}{\rho} = (\epsilon - 1). \tag{2.6}
\]
In order to obtain a static, spherically symmetric solution, we assume an ansatz metric of the form
\[ ds^2 = -e^\nu dt^2 + e^\lambda dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \] (2.7)
with \( \nu = -\lambda \) and \( \nu, \lambda \) being functions of \( r \) only. Now, the Einstein field equations read
\[ e^{-\lambda} \left( \frac{1}{r^2} - \frac{\lambda'}{r} \right) - \frac{1}{r^2} = 8\pi G \left( -\rho - \frac{Q^2}{8\pi Gr^4} \right) \]
\[ e^{-\lambda} \left( \frac{1}{r^2} + \frac{\nu'}{r} \right) - \frac{1}{r^2} = 8\pi G \left( P - \frac{Q^2}{8\pi Gr^4} \right) \]
\[ \frac{e^{-\lambda}}{2} \left( \nu' + \frac{\nu'}{2} + \frac{\nu' - \lambda'}{r} - \frac{\nu' \lambda'}{2} \right) = 8\pi G \left( P + \frac{Q^2}{8\pi Gr^4} \right) \] (2.8)
where the prime (') denotes derivative with respect to radial coordinates \( (r) \). The first and third equations can be rearranged to the form
\[ e^{-\lambda} \left( \frac{1}{r^2} - \frac{\lambda'}{r} \right) - \frac{1}{r^2} + \frac{Q^2}{r^4} = -8\pi G \rho \]
\[ \frac{e^{-\lambda}}{2} \left( \nu' + \frac{\nu'}{2} + \frac{\nu' - \lambda'}{r} - \frac{\nu' \lambda'}{2} \right) - \frac{Q^2}{r^4} = 8\pi G P . \] (2.9)
By using the equation of state for the PFDM (given in eq.(2.6)) in the above eq.(s) and taking their ratio we get
\[ \frac{e^{-\lambda}}{2} \left( \nu' + \frac{\nu'}{2} + \frac{\nu' - \lambda'}{r} - \frac{\nu' \lambda'}{2} \right) - \frac{Q^2}{r^4} = (1 - \epsilon) \left[ e^{-\lambda} \left( \frac{1}{r^2} - \frac{\lambda'}{r} \right) - \frac{1}{r^2} + \frac{Q^2}{r^4} \right] . \] (2.10)
In order to solve the above equation, we set \( \nu = -\lambda = \ln(1 - U) \) where \( U \equiv U(r) \). This in turn simplifies the above equation to the following form
\[ \frac{U''}{2} + \frac{U'}{r} + (\epsilon - 1) \frac{U}{r^2} + (2 - \epsilon) \frac{Q^2}{r^4} = 0 . \] (2.11)
Eq.(2.11) can be solved for different values of \( \epsilon \) [41]. However we are particularly interested in the solution for \( \epsilon = \frac{3}{2} \) [39, 41]. For \( \epsilon = \frac{3}{2} \), eq.(2.11) reduces to the following form
\[ r^2 U'' + 3r U' + U + \frac{Q^2}{r^2} = 0 . \] (2.12)
The solution of the above equation is obtained to be
\[ U(r) = \frac{r_s}{r} - \frac{Q^2}{r^2} - \frac{\alpha}{r} \ln \left( \frac{r}{|\alpha|} \right) \] (2.13)
where \( r_s \) and \( \alpha \) are integration constants. In order to evaluate \( r_s \), we set \( Q = 0 \) and \( \alpha = 0 \). In this limit, by utilizing the weak field approximation, \( r_s \) is obtained to be \( 2GM \). Thus the lapse function becomes
\[ f(r) = e^\nu = e^{-\lambda} = e^{\ln(1 - U)} = 1 - U = 1 - \frac{2GM}{r} + \frac{Q^2}{r^2} + \frac{\alpha}{r} \ln \left( \frac{r}{|\alpha|} \right) \] (2.14)
corresponding to the following metric of a static, spherically symmetric, charged black hole in PFDM
\[ ds^2 = -f(r)dt^2 + \frac{1}{f(r)} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 . \] (2.15)
3 Rotating charged black hole in perfect fluid dark matter: Newman-Janis algorithm

From the point of view of a more realistic set up, inclusion of spin parameter in the black hole metric is necessary. In order to incorporate spin \(a\), we shall use the Newman-Janis algorithm \cite{52} which is by far the easiest and most effective technique to include the spin parameter to any metric without cosmological constant. Before that we would like to consider a general metric and follow the approach accordingly as in \cite{53}. Let us assume a metric of the form

\[
ds^2 = -f(r)dt^2 + \frac{1}{g(r)}dr^2 + h(r)\left(d\theta^2 + \sin^2\theta d\phi^2\right). \tag{3.1}\]

Then the metric can be rewritten in the Eddington-Finkelstein coordinates \((u)\) by the transformation

\[
dt = du + \frac{dr}{\sqrt{f(r)g(r)}} \tag{3.2}\]

which modifies the metric as

\[
ds^2 = -f(r)du^2 - 2\sqrt{\frac{f(r)}{g(r)}}dudr + h(r)\left(d\theta^2 + \sin^2\theta d\phi^2\right). \tag{3.3}\]

We now introduce the null tetrads \(Z^\mu = (l^\mu, n^\mu, m^\mu, \overline{m}^\mu)\) in terms of which the metric tensor can be written as

\[
g^{\mu\nu} = -l^\mu n^\nu - l^\nu n^\mu + m^\mu \overline{m}^\nu + m^\nu \overline{m}^\mu \tag{3.4}\]

with the relation between the tetrad components as \(l^\mu n_\mu = -m^\mu \overline{m}_\mu = 1\) and all others are zero. Now in order to get the inverse metric, we need to represent the tetrad components in terms of \((u, r, \theta, \phi)\). This reads

\[
l^\mu = \delta_r^\mu, \quad n^\mu = \sqrt{\frac{g(r)}{f(r)}} \delta_u^\mu - \frac{g(r)}{2} \delta_r^\mu, \quad m^\mu = \frac{1}{\sqrt{2h(r)}} \left(\delta_\theta^\mu + \frac{i}{\sin\theta} \delta_\phi^\mu\right). \tag{3.5}\]

To incorporate spin to the metric, we make the transformation of the form \(u \to u' = u - ia\cos\theta\) and \(r \to r' = r + ia\cos\theta\) which transforms the null tetrads, and the transformation relations look like

\[
Z'_{\beta}^\mu = \frac{\partial x'^\mu}{\partial x^\nu} Z_{\beta}^\nu. \tag{3.6}\]

Here \(\mu\) denotes the components of the tetrads along the directions \(u, r, \theta, \phi\) and \(\beta\) denotes the tetrads \((l, n, m, \overline{m})\). So the transformation results in a new set of tetrads which read

\[
l'^\mu = \delta_r^\mu; \quad n'^\mu = \sqrt{\frac{g(r)}{f(r)}} \delta_u^\mu - \frac{f(r)}{2} \delta_r^\mu; \quad m'^\mu = \frac{1}{\sqrt{2h(r)}} \left(ia\sin\theta \left(\delta_u^\mu - \delta_r^\mu\right) + \delta_\theta^\mu + \frac{i}{\sin\theta} \delta_\phi^\mu\right). \tag{3.7}\]
Since \( r \) and \( u \) got transformed, hence the components of the metric tensor will also change. So the modified functions are \( f(r) \to F(r, \theta) \), \( g(r) \to G(r, \theta) \) and \( h(r) \to H(r, \theta) \). The non-zero components of the inverse metric tensor are as follows:

\[
\begin{align*}
&g^{uu} = \frac{a^2 \sin^2 \theta}{H}; \\
&g^{rr} = G + \frac{a^2 \sin^2 \theta}{H}; \\
&g^{\phi \phi} = g^{rr} = -\sqrt{\frac{G}{F}} - \frac{a^2 \sin^2 \theta}{H}; \\
&g^{\theta \theta} = \frac{1}{H}; \\
&g^{\phi u} = g^{\phi r} = \frac{a}{H}; \\
&g^{r \phi} = -\frac{a}{H}.
\end{align*}
\]

We now obtain the non-zero components of the metric:

\[
\begin{align*}
&g_{uu} = -F; \\
&g_{ur} = g_{ru} = -\sqrt{\frac{F}{G}}; \\
&g_{u \phi} = g_{\phi u} = \left(-a\sqrt{\frac{F}{G}} + aF\right) \sin^2 \theta.
\end{align*}
\]

The resulting metric reads

\[
ds^2 = -Fd\alpha^2 - 2\sqrt{\frac{F}{G}} dudr + 2a \sin^2 \theta \left(F - \sqrt{\frac{F}{G}}\right) dud\phi + 2a \sin^2 \theta \sqrt{\frac{F}{G}} dr d\phi \\
+ \frac{H}{G} d\theta^2 + \sin^2 \theta \left[H + a^2 \sin^2 \theta \left(2\sqrt{\frac{F}{G}} - F\right)\right] d\phi^2.
\]  
(3.7)

Now we need to remove \( u \) and express the metric in the Boyer-Lindquist coordinates for which the necessary transformation is \( du = dt + \xi_1(r) dr \) and \( d\phi = d\phi + \xi_2(r) dr \). Only the diagonal elements and \( dt d\phi \) component of the metric survives, rest are zero. This leads to the values of \( \xi_1 \) and \( \xi_2 \) as

\[
\begin{align*}
\xi_1(r) &= -\left(\sqrt{\frac{F}{G}}H + a^2 \sin^2 \theta\right) \left(GH + a^2 \sin^2 \theta\right)^{-1}; \\
\xi_2(r) &= \frac{-a}{G\left(GH + a^2 \sin^2 \theta\right)^{-1}}.
\end{align*}
\]

and the metric becomes

\[
ds^2 = -F dt^2 + \left(\frac{H}{G\left(GH + a^2 \sin^2 \theta\right)^{-1}}\right) dr^2 + \frac{H}{G\left(GH + a^2 \sin^2 \theta\right)^{-1}} d\theta^2 \\
+ \sin^2 \theta \left[H + a^2 \sin^2 \theta \left(2\sqrt{\frac{F}{G}} - F\right)\right] d\phi^2.
\]  
(3.8)

Now in order to keep the terms in the metric real, we use the transformation \( r^p \to \frac{r^{p+2}}{r^2 + a^2 \cos^2 \theta} = \frac{r^p + 2}{r^2}; \ p \geq 0 \) following [54]. This approach of incorporating spin \( a \) into the terms, leads to the modified functions of the form

\[
\begin{align*}
f(r) &\to F(r) = 1 - \frac{2Mr}{\rho^2} + \frac{Q^2}{\rho^2} + \frac{\alpha r}{\rho^2} \ln \left(\frac{r}{|a|}\right) \\
g(r) &\to G(r) = 1 - \frac{2Mr}{\rho^2} + \frac{Q^2}{\rho^2} + \frac{\alpha r}{\rho^2} \ln \left(\frac{r}{|a|}\right)
\end{align*}
\]
Using the above relations, we obtain the final expression for the metric of rotating charged black hole in PFDM to be

\[
ds^2 = -\frac{1}{\rho^2} (\Delta - \alpha^2 \sin^2 \theta) dt^2 + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 - \frac{2a \sin^2 \theta}{\rho^2} \left[ 2Mr - Q^2 - \alpha r \ln \left( \frac{r}{|\alpha|} \right) \right] dtd\phi \\
+ \sin^2 \theta \left[ r^2 + a^2 + \frac{a^2 \sin^2 \theta}{\rho^2} \left( 2Mr - Q^2 - \alpha r \ln \left( \frac{r}{|\alpha|} \right) \right) \right] d\phi^2
\]

with \( \Delta = r^2 + a^2 - 2Mr + Q^2 + \alpha r \ln \left( \frac{r}{|\alpha|} \right) \) and \( \rho^2 = r^2 + a^2 \cos^2 \theta \).

4 Circular geodesics

As we have mentioned previously, we are interested for an observer far away (theoretically at infinity) from the black hole. This motivates to confine ourselves only for the case of geodesics in the equatorial plane (\( \theta = \frac{\pi}{2} \)). The consideration of equatorial plane simplifies the following functions as

\[
\Delta = r^2 + a^2 - 2Mr + Q^2 + \alpha r \ln \left( \frac{r}{|\alpha|} \right) ; \quad \rho^2 = r^2 + a^2 \cos^2 \theta.
\]

The logarithmic term with parameter \( \alpha \) corresponds to the contribution of the PFDM to the charged black hole metric. This term can be positive or negative depending on the sign of \( \alpha \). In [51], the authors have discussed about relevance of both positive and negative values of \( \alpha \) but here we are interested only in positive values (\( \alpha > 0 \)). The allowed maximum value for \( \alpha \) is \( \alpha_{\text{max}} = 2M [43, 51] \). For the sake of simplicity, we will consider \( M = 1 \) in our subsequent analysis. The analysis with negative values of \( \alpha \) can also be done by following the subsequent analysis. The PFDM term influences the structure of the spacetime and hence the trajectories of the geodesics.

| \( \alpha \) | 0.1 | 0.2 | 0.3 | 0.4 | 0.47 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1.0 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| \( r_{h+} \) | 1.5 | 1.37 | 1.30 | 1.27 | 1.265 | 1.267 | 1.28 | 1.30 | 1.34 | 1.37 | 1.415 |
| \( r_{h-} \) | 0.2 | 0.185 | 0.17 | 0.1525 | 0.14 | 0.135 | 0.12 | 0.105 | 0.095 | 0.085 | 0.075 |

Table 1. Values of inner (\( r_{h-} \)) and outer (\( r_{h+} \)) horizon of the black hole for various values of \( \alpha \) at \( a = 0.5 \), \( Q = 0.3 \).

In Table 1, we show the variations in the values of the inner horizon (\( r_{h-} \)) and outer horizon (\( r_{h+} \)) of the black hole for various values of \( \alpha \) at constant spin parameter \( a = 0.5 \) and charge \( Q = 0.3 \) of the black hole. We find that the outer horizon (\( r_{h+} \)) decreases with the increase in the value of \( \alpha \). This behavior can be observed up to certain critical value \( \alpha_c \) (which in this case \( \alpha_c \approx 0.47 \)), however, after this critical value (\( \alpha_c \)), the outer horizon (\( r_{h+} \)) starts to increase slowly. On the other hand with the increase in the value of \( \alpha \), the value of the inner horizon (\( r_{h-} \)) decreases. This critical value \( \alpha_c \) can be interpreted as the point of reflection. The point of reflection can be found by plotting \( \Delta(r) \) with \( r \) at fixed values of spin (\( a \)) and
Table 2. Critical values of PFDM parameter ($\alpha_c$) for valid combinations of spin ($a$) and charge ($Q$) of the black hole.

| $a$  | $Q$  | $\alpha_c$ |
|------|------|------------|
| 0.1  | 0.0  | 0.602      |
| 0.1  | 0.8  | 0.454      |
| 0.2  | 0.0  | 0.596      |
| 0.2  | 0.8  | 0.432      |
| 0.3  | 0.0  | 0.574      |
| 0.3  | 0.8  | 0.402      |
| 0.4  | 0.0  | 0.552      |
| 0.4  | 0.7  | 0.454      |

charge ($Q$) of the black hole. With increase in $a$, the outer event horizon ($r_{bh+}$) reduces and at some value of $a$ starts to increase. That value of $a$ gives the point of reflection ($\alpha_c$).

This effect has also been observed for the shadow of the rotating black hole with PFDM [44]. Due to such effect observed on event horizon of black hole, we are interested in analysing different properties of the black hole spacetime in terms of the nature of particles in two ranges of $a$, namely, the lower range $a < \alpha_c$ and the higher range $a > \alpha_c$. This apparent increase in size of the black hole may be assigned to the fact that after a critical value ($\alpha_c$) of the dark matter, it contributes to the effective mass of the black hole system. It can be explained by the fact that dark matter acts as a point mass distribution. So as mentioned in [44], we consider that the total system consists of two parts - one is the original BH with mass $M$ and the other effective black hole due to the dark matter with mass $M'$. When the PFDM parameter $a$ is less than the critical value $\alpha_c$, then the dark matter hinders the original BH system, hence the effective horizon is less than $2M$. But as $a$ gradually increases and becomes $a > \alpha_c$, the total system is dominated by the dark matter component. Thus, the event horizon effectively increases. Hence we observe such effects in the system concerned.

In Table 2, we show the values of $\alpha_c$ for various valid combinations of spin and charge which has been used in the subsequent analysis. In Table 2, we fix the spin $a$ of the black hole at a particular value, then we obtain the values of $\alpha_c$ for various values of charge $Q$. We observe that the obtained values of $\alpha_c$ lies within the range $\alpha_c \in [0.4 - 0.602]$ depending upon the values of spin ($a$) and charge ($Q$). On the basis of these values, we define the lower range of values of $\alpha$ which are less than these values and higher range of values of $\alpha$ which are greater than these values.

In order to continue the analysis for circular geodesics, we consider a particle with Lagrangian $L = \frac{1}{2}g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu$ where $\dot{x}^\mu = u^\mu = \frac{dx^\mu}{d\lambda}$ is the four-velocity obtained by undertaking the derivative of spacetime position ($x^\mu$) with respect to the affine parameter $\lambda$. The affine parameter corresponds to the proper time ($\tau$) of the massive particles in case of timelike geodesics. The Lagrangian is expressed in terms of metric and we observe that the metric coefficients are independent of $t$ and $\phi$, hence the metric is invariant along those directions (directions of symmetry) which results into conserved quantities $E$ and $L$. These two quantities physically represent the specific energy (energy per unit mass) and angular momentum (angular momentum per unit mass) of the particle respectively with respect to a stationary observer at relatively infinite distance. In terms of these quantities, the geodesic equations of $t$ and $\phi$
takes the following form

\[ i = \frac{1}{r^2} \left[ \frac{r^2 + a^2}{\Delta} \left( E(r^2 + a^2) - aL \right) + a(L - aE) \right] \quad (4.1) \]

\[ \dot{\phi} = \frac{1}{r^2} \left[ \frac{a}{\Delta} \left( E(r^2 + a^2) - aL \right) + (L - aE) \right]. \quad (4.2) \]

The Hamiltonian \( H = p_0 \dot{x}^0 - \mathcal{L} \) therefore reads

\[ 2H = p_0 \dot{x}^0 + p_1 \dot{x}^1 + p_3 \dot{x}^3 = -Et + L \dot{\phi} + \frac{r^2}{\Delta} \dot{r}^2 = \text{constant} = \epsilon. \quad (4.3) \]

with \( \epsilon = -1, 0, 1 \) for timelike, null and spacelike geodesics respectively. We are mainly interested in the first two types of geodesics which are physically relevant. Substituting the values of \( \dot{r} \) and \( \dot{\phi} \), we obtain the geodesic equation for \( r \). This reads

\[ \dot{r}^2 = \frac{1}{r^4} \left[ \left( E(r^2 + a^2) - aL \right)^2 - \Delta(L - aE)^2 \right] + \frac{\Delta}{r^2} \epsilon. \quad (4.4) \]

The radial equation is very useful for the analysis of the circular geodesics and also for the computation of the effective potential.

### 4.1 Null geodesics

In case of null geodesics \( \epsilon = 0 \), hence the radial equation becomes

\[ \dot{r}^2 = \frac{1}{r^4} \left[ \left( E(r^2 + a^2) - aL \right)^2 - \Delta(L - aE)^2 \right] = F(r). \quad (4.5) \]

For the sake of convenience we define \( \frac{L}{E} = D \) as the impact parameter which reduces two constants into one. In terms of the impact parameter, the above equation becomes

\[ \dot{r}^2 = \frac{E^2}{r^2} \left[ r^2 + \frac{2M}{r} (a - D)^2 - \frac{Q^2}{r^2} (a - D)^2 - \frac{\alpha}{r} \ln \left( \frac{r}{\alpha} \right) + (a^2 - D^2) \right]. \quad (4.6) \]

In general \( D \neq a \), but for the trivial case considering \( D = a \), we get the geodesic equations to be

\[ \frac{dt}{d\lambda} = \frac{r^2 + a^2}{\Delta} E ; \quad \frac{d\phi}{d\lambda} = \frac{a}{\Delta} E ; \quad \frac{dr}{d\lambda} = \pm E. \quad (4.7) \]

For the general case \((D \neq a)\), we aim to find the circular photon orbits subject to the conditions \( F(r) = F'(r) = 0 \). These two conditions yield

\[ r^2_p + \frac{2M}{r_p} (a - D)^2 - \frac{Q^2}{r_p^2} (a - D)^2 - \frac{\alpha}{r_p} \ln \left( \frac{r_p}{\alpha} \right) (a - D)^2 + (a^2 - D^2) = 0 \quad (4.8) \]

\[ 2r_p - \frac{2M}{r_p^2} (a - D)^2 + \frac{Q^2}{r_p} (a - D)^2 + \frac{\alpha}{r_p} \ln \left( \frac{r_p}{\alpha} \right) (a - D)^2 - \frac{\alpha}{r_p} (a - D)^2 = 0. \quad (4.9) \]

Solving for \( D \) from eq.(4.9), we get

\[ D = a \mp \sqrt{\frac{2r^5_p}{2Mr^2_p - 2Q^2r_p - \alpha r^3_p \ln \left( \frac{r_p}{\alpha} \right) + \alpha r^2_p}}. \quad (4.10) \]
\[ a=0.5, Q=0.3 \] 
\[
\begin{array}{c|c}
\alpha & r_{p1} \\
0.1 & 2.95 \\
0.2 & 2.755 \\
0.3 & 2.605 \\
0.4 & 2.515 \\
0.9 & 2.47 \\
1.0 & 2.505 \\
1.1 & 2.545 \\
1.2 & 2.59 \\
\end{array}
\]

\[
\begin{array}{c|c|c}
\alpha & a & r_{p1} \\
0.2 & 0.1 & 2.39 \\
0.2 & 0.4 & 2.67 \\
0.2 & 0.7 & 2.92 \\
0.2 & 1.0 & 3.15 \\
1.0 & 0.1 & 2.28 \\
1.0 & 0.4 & 2.45 \\
1.0 & 0.7 & 2.61 \\
1.0 & 1.0 & 2.745 \\
\end{array}
\]

\[
\begin{array}{c|c|c}
\alpha & Q & r_{p1} \\
0.2 & 0.0 & 2.82 \\
0.2 & 0.3 & 2.75 \\
0.2 & 0.6 & 2.54 \\
0.2 & 0.9 & 2.40 \\
1.0 & 0.0 & 2.55 \\
1.0 & 0.3 & 2.5 \\
1.0 & 0.6 & 2.34 \\
1.0 & 0.9 & 2.1 \\
\end{array}
\]

Table 3. Radius \( (r_{p1}) \) of the co-rotating (prograde) photon orbits.

\[
\begin{array}{c|c}
\alpha & r_{p2} \\
0.1 & 1.85 \\
0.2 & 1.69 \\
0.3 & 1.62 \\
0.4 & 1.60 \\
0.9 & 1.81 \\
1.0 & 1.87 \\
1.1 & 1.94 \\
1.2 & 2.05 \\
\end{array}
\]

\[
\begin{array}{c|c|c}
\alpha & a & r_{p2} \\
0.2 & 0.1 & 2.18 \\
0.2 & 0.4 & 1.83 \\
0.2 & 0.7 & 1.345 \\
1.0 & 0.1 & 2.15 \\
1.0 & 0.4 & 1.95 \\
1.0 & 0.7 & 1.7 \\
\end{array}
\]

\[
\begin{array}{c|c|c}
\alpha & Q & r_{p2} \\
0.2 & 0.0 & 1.78 \\
0.2 & 0.3 & 1.7 \\
0.2 & 0.6 & 1.4 \\
1.0 & 0.0 & 1.9 \\
1.0 & 0.3 & 1.87 \\
1.0 & 0.6 & 1.7 \\
\end{array}
\]

Table 4. Radius \( (r_{p2}) \) of the counter-rotating (retrograde) photon orbits.

Here the signs \( \mp \) corresponds to the counter rotation (-) and the co-rotation (+) of the orbits along with the black hole. By substituting the expression for \( D \) in eq.\((4.8)\), we obtain the following equation which leads to the radius of the photon orbits

\[
6Mr_p^3 - 4Q^2r_p^2 - 3\alpha r_p^3 \ln \left( \frac{r_p}{|\alpha|} \right) + \alpha r_p^3 r_p^2 \pm 2a \sqrt{2r_p^4 \left( 2Mr_p - 2Q^2 - \alpha r_p \ln \left( \frac{r_p}{|\alpha|} \right) \right)} = 0.
\]

(4.11)

The \( \pm \) signs denote the co-rotating and the counter rotating photon orbits. Tables (3) and (4) show the photon sphere radius \( (r_p) \) with the variation in spin \( (a) \), charge \( (Q) \) and the parameter \( (\alpha) \) which gives the weightage of the dark matter. Here \( r_{p1} \) and \( r_{p2} \) correspond to the co-rotating and counter rotating photon sphere radius.

Since the black hole horizon shows different behaviour in different range of PFDM parameter \( \alpha \), so we have analysed the different geodesics and the corresponding characteristics of the particles in two separate ranges. One is the low range of \( \alpha \) where \( \alpha < \alpha_c \) and the other is the range \( \alpha > \alpha_c \).

- From the above Table(s) (3),(4), it is quite clear that for the lower range of values of \( \alpha < \alpha_c \), the photon radius corresponding to both the co-rotating and the counter-rotating orbits decreases with increase in PFDM parameter.

- In the higher range of values of \( \alpha > \alpha_c \), increase in \( \alpha \) increases the photon radius as is evident from the nature of the outer event horizon radius \( (r_{h+}) \).
• With the increase in the value of spin parameter \( (a) \) of the black hole, the radius of the co-rotating orbits increases while that of the counter rotating orbits decreases for both \( \alpha > \alpha_c \) and \( \alpha < \alpha_c \). Since the spin parameter \( (a) \) of the black hole assists the co-rotation and opposes the counter rotation, thus we observe such feature of photon radius.

• The presence of the charge \( (Q) \) also affects the radius of the photon sphere as can be observed from the Tables (3), (4). With the increase in the value of charge \( (Q) \), the photon sphere radius decreases both for prograde and retrograde orbits with \( \alpha = 0.2 \) \( (\alpha < \alpha_c) \) and \( \alpha = 1.0 \) \( (\alpha > \alpha_c) \).

• Besides we also find that the radius of prograde orbits are larger than the retrograde orbits of the photons moving around the black hole.

4.2 Time-like geodesics

In this case, we consider massive particles and this consideration makes the geodesics time-like hence \( \epsilon = -1 \). This results in the modification of the radial equation (4.4) into the form

\[
\dot{r}^2 = \left[ E^2 + \frac{2M}{r^3}(aE-L)^2 - \frac{Q^2}{r^7}(aE-L)^2 - \frac{\alpha}{r^3} \ln \left( \frac{r}{|\alpha|} \right) + \frac{1}{r^2} (a^2E^2 - L^2) \right] - \frac{\Delta}{r^2} = F(r). \tag{4.12}
\]

Again for the trivial case, \( L = aE \) which simplifies the radial equation to the form

\[
\frac{dr}{d\tau} = \left( E^2 - \frac{\Delta}{r^2} \right)^{-\frac{1}{2}}. \tag{4.13}
\]

So the proper time can be evaluated as

\[
\tau = \int \left( E^2 - \frac{\Delta}{r^2} \right)^{-\frac{1}{2}} \, dr. \tag{4.14}
\]

We now confine ourselves for the general case \( (L \neq aE) \). Our aim is to calculate and show the variation of the energy \( (E) \) and the angular momentum \( (L) \) of the particle with variation in parameter \( \alpha \). So we proceed by assuming \( x = L - aE \) and rewrite the above equation in terms of \( x \) and \( E \). Upon imposing the conditions for circular orbits \( (F(r) = F'(r) = 0) \), we obtain

\[
F(r) = x^2 \left( a^2 - \Delta \right) + r^4 E^2 - 2aEr^2 x - \Delta r^2 = 0 \tag{4.15}
\]

\[
F'(r) = 4r^3 E^2 - 4aErx - 2\Delta r - \Delta'(r^2 + x^2) = 0. \tag{4.16}
\]

Solving the above equations, we obtain the expression for \( x \). Using the obtained expression of \( x \), we determine the energy \( (E) \) and angular momentum \( (L) \) of the particle.

The expression for \( E \) is obtained from eq.(s)\( (4.15) \), \( (4.16) \) as

\[
E = \frac{1}{r^4 a^2 x} \left[ \left( a^2 - \Delta + \frac{r \Delta'}{4} \right) x^2 + \left( \frac{\Delta'}{4} r^3 - \frac{\Delta}{2} r^2 \right) \right]. \tag{4.17}
\]

On replacing \( E \) in eq\( (4.15) \), we get an equation in \( x \) as

\[
\left[ 4 \left( \Delta - a^2 - \frac{r \Delta'}{4} \right)^2 - 4a^2 \left( a^2 - \Delta + \frac{r \Delta'}{2} \right) \right] x^4 + \left[ \left( 4a^2 - 4\Delta + r \Delta' \right) \right. \\
\times \left. \left( \frac{r^3 \Delta'}{2} - r^2 \Delta \right) - 2r^3 a^2 \Delta' \right] x^2 + \left[ r^2 \Delta - \frac{r^3 \Delta'}{2} \right]^2 = 0. \tag{4.18}
\]
The equation is quadratic in $x^2$ with the discriminant

$$\Delta_D = 16a^2\Delta^2 r^4 \left[a^2 - \Delta + \frac{r\Delta'}{2}\right].$$

(4.19)

Now we can factorize the coefficient of $x^4$ as

$$4\left(\Delta - a^2 - \frac{r\Delta'}{4}\right)^2 - 4a^2\left(a^2 - \Delta + \frac{r\Delta'}{2}\right) = \mathcal{F}_+ \mathcal{F}_-$$

(4.20)

where

$$\mathcal{F}_\pm = 2\left(\Delta - a^2 - \frac{r\Delta'}{4}\right) \pm 2a\sqrt{\left(a^2 - \Delta + \frac{r\Delta'}{2}\right)}.$$  

(4.21)

Incorporating the above, the solution for $x^2$ reads

$$x^2 = r^2 \frac{\left(\Delta \mathcal{F}_\pm - \mathcal{F}_+ \mathcal{F}_-\right)}{\mathcal{F}_+ \mathcal{F}_-} = r^2 \frac{\left(\Delta - \mathcal{F}_\mp\right)}{\mathcal{F}_\mp}$$

(4.22)

where we consider $\mathcal{F}_- \equiv \mathcal{F}_\pm$ and $\mathcal{F}_+ \equiv \mathcal{F}_\pm$. The solution for $x$ becomes

$$x = \pm \frac{r}{\sqrt{\mathcal{F}_+}} \left[a \pm \sqrt{a^2 - \Delta + \frac{r\Delta'}{2}}\right].$$

(4.23)

Figure 1. Plots for energy of massive particles co-rotating with the black hole with variation in $a$, $Q$ and $\alpha$. 

\[\begin{array}{c}
(a) \ a = 0.5, \ Q = 0.3 \\
(b) \ a = 0.5, \ Q = 0.3 \\
(c) \ a = 0.2, \ Q = 0.3 \\
(d) \ a = 0.5, \ \alpha = 0.2
\end{array}\]
Replacing the values of $x$, the expression for energy becomes

$$E = \frac{1}{\sqrt{F + r'}} \left[ \Delta - a \left( a \pm \sqrt{a^2 - \Delta + \frac{r'\Delta'}{2}} \right) \right]$$

and that of angular momentum becomes

$$L = \frac{1}{\sqrt{F + r'}} \left[ a \left( \Delta - a^2 - r^2 \right) \mp \left( r^2 + a^2 \right) \sqrt{a^2 - \Delta + \frac{r'\Delta'}{2}} \right].$$

The plots of the above expressions for energy ($E$) and angular momentum ($L$) with respect to the radial distance ($r$) gives a firm idea about the orbits and thereby motions around the black hole which in effect gives an impression about the spacetime structure around the black hole. Besides in our case it helps us to get an idea about the impact of the surrounding dark matter on the geodesics.

- In Fig. (1), we find that the energy ($E$) of the co-rotating particle falls with distance ($r$) from the black hole.

- While the particle is near the black hole, it is assisted by the spin ($a$) of the black hole and its energy is thereby increased but as it starts to move away, the energy following eq. (5.1) behaves partially like the potential ($V_{eff}$) where we showed only a portion of the plot where the energy falls in order to distinguish both (co and counter rotating) type of particle(s) energy and hence the motion.

![Figure 2](image)

**Figure 2.** Plots of energy of the massive particles counter-rotating with respect to the black hole with variation in $\alpha$.

- The effect of dark matter ($\alpha$) is similar for both lower values ($\alpha < \alpha_c$) and higher values ($\alpha > \alpha_c$). In both cases energy increases with increase in the effective amount of dark matter in the system.

- The effect of spin ($a$) both for low ($a = 0.2$) and high ($a = 1.0$) values of the parameter $\alpha$ results in the increment of the energy of the particle which means an assist by the rotation of the black hole.
• On the other hand, increase in charge \((Q)\) results in decrease in the energy of the particle.

• The energy of the particle in all the cases can be found to be approaching towards unity as it is the energy of the particle at infinity observed by a stationary observer.

In Fig.(2), we find that energy \((E)\) of the counter-rotating particle increases with distance \((r)\) from the black hole and as it moves away it slowly approaches towards unity. As the particle is closer it is opposed by the black hole rotation and hence it has less energy.

• As the particle moves away, the intensity of opposing falls and the particle has positive energy due to rotation in circular orbit.

• In this case, the effect of dark matter \((\alpha)\) is similar both for low values \((\alpha < \alpha_c)\) and high values \((\alpha > \alpha_c)\). In both cases (varying \(\alpha\)) energy increases with increase in the effective amount of dark matter in the system.

• The increase in spin \((a)\) of the black hole both for low \((\alpha=0.2)\) and high \((\alpha=1.0)\) values (not shown) of the parameter \(\alpha\) results in decrement of energy of the particle as is evident from Fig.(3), whereas the increase in charge \((Q)\) results in increase in energy of the particle.

The plots of the angular momentum present a completely different picture with respect to the different range (higher and lower values) of the parameter \(\alpha\) of PFDM.

• Fig.(4) shows the angular momentum of the particles moving in prograde (co-rotating) orbits. The plots show that in the lower range values of \(\alpha\) \((\alpha < \alpha_c)\), increasing the value of \(\alpha\) decreases the angular momentum whereas the reverse is observed in case of the higher range values of \(\alpha\) \((\alpha > \alpha_c)\).

• The different behaviour of the angular momentum of the particle is due to the nature of dark matter which gets reflected on the particle trajectories. Also the variation of the angular momentum \((L)\) with respect to the spin \((a)\) shows that \(L\) increases with the increment in the value of the spin of the black hole for \(\alpha > \alpha_c\) whereas there is very slight variation in case of \(\alpha < \alpha_c\).

• The effect of the charge \((Q)\) shows that increase in the charge of the black hole decreases the angular momentum \((L)\) of the particles with prograde orbits and the rate of reduction increases with further increment in the value of the charge \((Q)\).

• The angular momentum of the particles in the retrograde orbits are negative since they move opposite to the direction of rotation of the black hole as shown in Fig.(5).
\[ \alpha = 0.1 \]
\[ \alpha = 0.2 \]
\[ \alpha = 0.3 \]
\[ \alpha = 0.4 \]

\[ r \]
\[ L \]
\[ a = 0.5, Q = 0.3 \]
\[ r \]
\[ L \]
\[ a = 0.5, Q = 0.3 \]

**Figure 4.** Plots of angular momentum of the massive particles co-rotating with the black hole with variation in \( \alpha \) with fixed spin(\( a \)) and charge(\( Q \)).

**Figure 5.** Plots of angular momentum of the massive particles counter-rotating with the black hole with variation in \( \alpha \) with fixed values of spin(\( a \)) and charge(\( Q \)).

- As the particle moves away (\( r \) increases) from the black hole the angular momentum increases (in the opposite direction which in turn increases the negative value) for \( \alpha < \alpha_c \) and decreases for \( \alpha > \alpha_c \).
- As can be seen from the plots that for \( \alpha < \alpha_c \), the increment in the value of \( \alpha \) reduces the angular momentum of the massive test particle. However, for \( \alpha > \alpha_c \), it increases the angular momentum.
- Also with increase in the value of the spin (\( a \)) of the black hole, angular momentum increases for \( \alpha < \alpha_c \) and decreases for \( \alpha > \alpha_c \).
- The increment in the charge (\( Q \)) of the black hole reduces the angular momentum of the particle for both \( \alpha < \alpha_c \) and \( \alpha > \alpha_c \).

### 4.3 Geodesics of charged particle

After studying the null-geodesics and geodesics of the chargeless massive particles, we move on to study the geodesic motion of the massive charged particles. Incorporating the interactions
of the gauge fields, the Hamiltonian of the particle in this case gets modified to
\[ 2\mathcal{H} = g^{\mu\nu} \left( p_\mu + qA_\mu \right) \left( p_\nu + qA_\nu \right) = \epsilon = -1 \] (4.26)

with the electromagnetic potential for charged spinning black hole coupled to PFDM given by [33]
\[ A = A_\mu dx^\mu = \frac{Qr}{\rho^2} \left( dt - a \sin^2 \theta \, d\phi \right). \] (4.27)

Using the Legendre transformation \( \mathcal{H} = p_\mu \dot{x}^\mu - L \), we obtain the Lagrangian of the particle as
\[ L = \frac{1}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu - q A_\mu \dot{x}^\mu \] (4.28)

where \( q \) denotes the charge of the particle. Also we consider the particle to have unit mass \( (m = 1) \) for the sake of simplicity. Using the symmetry of the metric we compute the conserved quantities \( E \) and \( L \) which have same physical meaning as mentioned previously. Also since we are interested in the equatorial geodesics, we use \( \theta = \frac{\pi}{2} \) and \( \dot{\theta} = 0 \) which leads to following the geodesic equations
\[ \dot{\theta} = \frac{1}{r^2} \left[ \frac{\Delta}{a} \left( E - \frac{qQ}{r} \right) \left( r^2 + a^2 \right) - a \left( L - \frac{qaQ}{r} \right) \right] + a \left( \frac{L - qaQ}{r} \right) - a \left( E - \frac{qQ}{r} \right) \] (4.29)

\[ \dot{r}^2 = \frac{\Delta}{r^2} + \left( \frac{E - \frac{qQ}{r}}{r^4} \right)^2 \left[ \left( r^2 + a^2 \right)^2 - a^2 \Delta \right] - \frac{2a}{r^4} \left( r^2 + a^2 - \Delta \right) \left( E - \frac{qQ}{r} \right) \left( L - \frac{qaQ}{r} \right) \] (4.31)

\[ - \frac{1}{r^4} \left( \Delta - a^2 \right) \left( L - \frac{qaQ}{r} \right)^2. \] (4.31)

\[ \dot{\phi} = \frac{1}{r^2} \left[ \frac{\Delta}{a} \left( E - \frac{qQ}{r} \right) \left( r^2 + a^2 \right) - a \left( L - \frac{qaQ}{r} \right) \right] + a \left( \frac{L - qaQ}{r} \right) - a \left( E - \frac{qQ}{r} \right) \] (4.30)

\[ \dot{\theta} = \frac{1}{r^2} \left[ \frac{\Delta}{a} \left( E - \frac{qQ}{r} \right) \left( r^2 + a^2 \right) - a \left( L - \frac{qaQ}{r} \right) \right] + a \left( \frac{L - qaQ}{r} \right) - a \left( E - \frac{qQ}{r} \right) \] (4.29)

\[ \dot{r}^2 = \frac{\Delta}{r^2} + \left( \frac{E - \frac{qQ}{r}}{r^4} \right)^2 \left[ \left( r^2 + a^2 \right)^2 - a^2 \Delta \right] - \frac{2a}{r^4} \left( r^2 + a^2 - \Delta \right) \left( E - \frac{qQ}{r} \right) \left( L - \frac{qaQ}{r} \right) \] (4.31)

Figure 6. Plots for energy and angular momentum of charged particles co-rotating with the black hole with varying \( q \) for \( \alpha = 0.2 \), \( Q = 0.3 \) and \( a = 0.5 \).
In order to determine the circular orbits, we use the conditions $F(r) = F'(r) = 0$ where

$$F(r) = -\frac{\Delta}{r^2} + \left(\frac{E - qQ}{r^4}\right)^2 \left[(r^2 + a^2)^2 - a^2\Delta\right] - \frac{2a}{r^4} \left[(r^2 + a^2) - \Delta\right] \left(E - \frac{qQ}{r}\right) \left(L - \frac{qaQ}{r}\right)$$

$$- \frac{1}{r^4} \left(\Delta - a^2\right) \left(L - \frac{qaQ}{r}\right)^2.$$  

(4.32)

Figure 7. Plots for energy and angular momentum of charged particles counter-rotating with the black hole with varying $q$ for $\alpha=0.2$, $Q=0.3$ and $a=0.5$.

It is very difficult to find an exact solution for $E$ and $L$ from these conditions. So we proceed by assuming $\frac{q}{m} \ll 1$ (particle with small specific charge) and approximately write down the following solutions by incorporating Taylor expansion about $\frac{q}{m} = 0$ [26]

$$E(q) = E(0) + qE'(0) + O(q^2) + ...$$  

(4.33)

$$L(q) = L(0) + qL'(0) + O(q^2) + ... .$$  

(4.34)

The approximate solutions of $E$ and $L$ satisfy the condition of circular orbits given as $F(r) = F'(r) = 0$. We display the plots.

- The plots of energy ($E$) show that for both $\alpha < \alpha_c$ and $\alpha > \alpha_c$, the increase in the value of the charge of the particle $q$ from $-0.3$ to $0.3$, decreases the energy of co-rotating...
particles whereas there is an increase in case of counter rotating particles. In both cases the energy tends towards unity.

- In case of angular momentum, we observe that for $\alpha = 0.2$ the angular momentum of both co-rotating (+ve increase) and counter rotating (−ve increase) particles increases with the increase in the value of the charge $q$ of the particle.

- However, for $\alpha = 1.0$ we find that with increase in the value of charge $q$, angular momentum ($L$) for co-rotating and counter rotating particles decreases.

- The observations depict that the particle’s charge $q$ responses differently depending on the dark matter. Also it implies that the particle with more charge ($q$) is hindered more if the intensity of dark matter increases.

5 Nature of the effective potential

In this section we study the effective potential which results in both stable and unstable orbits depending upon the condition $\frac{\partial^2 V_{\text{eff}}}{\partial r^2} > 0$ or $\frac{\partial^2 V_{\text{eff}}}{\partial r^2} < 0$ respectively. The stable and unstable orbits correspond to the local minima and maxima of the potential which we obtain from the radial geodesic equations. The potential depends upon the following parameters, the charge of the black hole ($Q$), spin parameter ($a$), the dark matter parameter $\alpha$ and on the charge of the particle ($q$). The effective potential in case of the massive particles obtained from the corresponding radial geodesic equation eq.(4.12) reads

$$\dot{r}^2 + V_{\text{eff}} = E^2$$

(5.1)

where the effective potential $V_{\text{eff}}$ is given by

$$V_{\text{eff}} = -\frac{2M}{r^3} (aE - L)^2 + \frac{Q^2}{r^4} (aE - L)^2 + \frac{\alpha}{r^3} \ln \left( \frac{r}{|\alpha|} \right) - \frac{1}{r^2} \left( a^2 E^2 - L^2 \right) + \Delta .$$

(5.2)

For circular geodesics, the particle moves in a circular trajectory of constant radius $r$ which implies $\dot{r} = 0$. In case of photons, the effective potential as obtained from eq.(4.5) takes the form

$$V_{\text{eff}} = -\frac{2M}{r^3} (aE - L)^2 + \frac{Q^2}{r^4} (aE - L)^2 + \frac{\alpha}{r^3} \ln \left( \frac{r}{|\alpha|} \right) - \frac{1}{r^2} \left( a^2 E^2 - L^2 \right) .$$

(5.3)
Also for massive particles with charge \( q \), the effective potential takes the following form obtained using eq. (4.31)

\[
V_{\text{eff}} = E^2 + \frac{(\Delta - a^2) \left( L - \frac{aqQ}{r^4} \right)^2}{r^4} + 2a \frac{(a^2 - \Delta + r^2) \left( E - \frac{qQ}{r^4} \right) \left( L - \frac{aqQ}{r^4} \right)}{r^4} - \frac{\left( (a^2 + r^2)^2 - a^2 \Delta \right) \left( E - \frac{qQ}{r^4} \right)}{r^4} + \frac{\Delta}{r^2}. \tag{5.4}
\]
In the trivial case when $L = aE$, the radial equations and effective potentials become

$$
\dot{r}_{\text{massive}} = \pm \sqrt{E^2 - \frac{\Delta}{r^2}} \ ; \ \dot{r}_{\text{null}} = \pm E 
$$

(5.5)

$$
V_{\text{massive}} = \frac{\Delta}{r^2} \ ; \ V_{\text{null}} = 0.
$$

(5.6)

In Figures (10),(11), we show the plots of the effective potential for both null and timelike geodesics and observe the effects of $a, Q, L$ and $\alpha$ on it. The first set of plot Fig.(10) represents the effective potential for null geodesic particles.

**Figure 11.** Plots of effective potential for timelike geodesics with variation in $a$, $Q$, $L$ and $\alpha$. 
We find that for $\alpha < \alpha_c$, the potential increases with increase in the amount of dark matter around the black hole, whereas for $\alpha > \alpha_c$ the effective potential falls with increase in the PFDM parameter $\alpha$.

Also we find that the potential increases with increase in the spin ($a$) and charge ($Q$) of the black hole both for $\alpha < \alpha_c$ and $\alpha > \alpha_c$.

The maxima of the potential in both of these cases shifts towards smaller values of $r$. The plots shown for null particles has been computed by setting $E = 1$ (particles with unit energy).

Besides we observe that with increase in angular momentum ($L$) of the particle, the effective potential rises in both cases ($\alpha < \alpha_c$ and $\alpha > \alpha_c$) and the maxima shifts towards larger radial distance $r$.

Fig. (11) shows the effective potential for massive particles. In this case we find that with the increase in parameter $\alpha$, the effective potential rises sharply for $\alpha < \alpha_c$ but it decreases in case of $\alpha > \alpha_c$.

The spin parameter ($a$) of the black hole is suitable for the increment of the potential and thereby $V_{eff}$ increases with increase in the value of $a$ for both $\alpha < \alpha_c$ and $\alpha > \alpha_c$.

Similar to the effect of the spin parameter, increase in the value of the charge of the black hole ($Q$) increases $V_{eff}$ for both $\alpha < \alpha_c$ and $\alpha > \alpha_c$.

Also the potential increases steadily with increase in the angular momentum ($L$) for both cases of $\alpha < \alpha_c$ and $\alpha > \alpha_c$.

Here also the plots are shown for the massive particle with unit energy ($E = 1$). The nature of the shift of maxima with $a$, $Q$ and $L$ is similar to the case of massless particles.

The effective potential in case of the null particles finally approaches zero whereas for massive particles approaches unity which can be observed from the plots.

6 Penrose process

Black hole is a vessel of extreme energy and there are many processes theorised which are responsible to gain energy from the black hole. One of them is the Penrose process named after Roger Penrose who proposed the mechanism in [46]. In case of rotating black hole a region gets created between the outer event horizon ($g^{rr} = 0$) and the stationary limit surface ($g_{tt} = 0$). These two surfaces meet at the poles and have largest separation in the equatorial plane. This varying annular region is known as the ergosphere. In case of static black hole this region vanishes. The speciality of this region is that the Killing vector $\frac{\partial}{\partial t}$ which has a unit norm as observed by a stationary observer at infinity becomes spacelike within the region. The symmetry of the metric with change in the said Killing vector results in energy conservation and hence the energy in this region can be negative. This fact can be utilised to gain energy from the black hole.

Let a particle (uncharged) with positive energy fall into this ergoregion and split into two particle, one with positive energy and the other with negative energy. The negative energy particle is absorbed by the black hole and that with positive energy comes out of the black
hole having more energy than the particle that entered the black hole and hence resulting in energy gain.

The condition of negative energy of the particle can be found using the condition of circular orbits. The equation with $\dot{r} = 0$ results in

$$E^2 \left[ (r^2 + a^2)^2 - a^2 \Delta \right] - E \left[ 2aL (r^2 + a^2 - \Delta) + L^2 (a^2 - \Delta) + \Delta r^2 \epsilon \right] = 0$$  \hspace{1cm} (6.1)

which can be solved for both $E$ and $L$ as given by

$$E = \frac{aL (r^2 + a^2 - \Delta) \pm r \sqrt{\Delta \left[ r^2 L^2 - \epsilon \left( (r^2 + a^2)^2 - a^2 \Delta \right) \right]}}{(r^2 + a^2)^2 - a^2 \Delta}$$  \hspace{1cm} (6.2)

$$L = \frac{aE (r^2 + a^2 - \Delta) \pm r \sqrt{\Delta \left[ r^2 E^2 - \epsilon \left( (r^2 + a^2)^2 - a^2 \Delta \right) \right]}}{(a^2 - \Delta)}.$$  \hspace{1cm} (6.3)

If one assumes positive sign in eq.(6.2) along with the condition

$$a^2 L^2 (r^2 + a^2 - \Delta)^2 > \Delta r^2 \left[ r^2 L^2 - \epsilon \left( (r^2 + a^2)^2 - a^2 \Delta \right) \right]$$  \hspace{1cm} (6.4)

and $L < 0$, then $E < 0$, i.e., particle with negative energy is possible. This gives the idea that $E < 0$ is possible for $L < 0$ which is the case for counter rotating orbits. The negative energy particle following counter rotating orbits must lie within the ergoregion. The plots of negative energy with the variation in different parameters are shown below.

In order to discuss the Penrose process in detail, we must start by considering an uncharged particle of energy $E_0$ entering the ergosphere and let it breaks down into two photons with energies $E_1$ and $E_2$. Let the angular momentum of the particles be $L_0$ (entering), $L_1$ (leaving) and $L_2$ (captured). Also let the energy of the particle entering the ergosphere be $E_0 = 1$.

Hence the angular momentum of the particles are

$$L_0 = \frac{a \left( r^2 + a^2 - \Delta \right) + r \sqrt{\Delta \left[ r^2 + \left( \Delta - a^2 \right) \right]}}{(a^2 - \Delta)}$$  \hspace{1cm} (6.5)

$$L_1 = \frac{aE_1 \left( r^2 + a^2 - \Delta \right) + r \sqrt{\Delta \left( r^2 E_1^2 \right)}}{(a^2 - \Delta)} = b_1 E_1$$  \hspace{1cm} (6.6)

$$L_2 = \frac{aE_2 \left( r^2 + a^2 - \Delta \right) - r \sqrt{\Delta \left( r^2 E_2^2 \right)}}{(a^2 - \Delta)} = b_2 E_2$$  \hspace{1cm} (6.7)

where

$$b_1 = \frac{a \left( r^2 + a^2 - \Delta \right) + r^2 \sqrt{\Delta}}{(a^2 - \Delta)} \quad ; \quad b_2 = \frac{a \left( r^2 + a^2 - \Delta \right) - r^2 \sqrt{\Delta}}{(a^2 - \Delta)}.$$  \hspace{1cm} (6.8)
By conservation of energy and angular momentum we get

\[ E_0 = E_1 + E_2 = 1 \quad ; \quad L_0 = b_1 E_1 + b_2 E_2 . \]  

(S.9)

Solving for \( E_1 \) and \( E_2 \) we obtain

\[ E_1 = \frac{1}{2} \left[ 1 + \sqrt{\frac{r^2 + \Delta - a^2}{r^2}} \right] \]  

(S.10)

\[ E_2 = \frac{1}{2} \left[ 1 - \sqrt{\frac{r^2 + \Delta - a^2}{r^2}} \right] \]  

(S.11)

where \( E_1 \) and \( E_2 \) corresponds to the positive and negative energies of the two particles. Thus the particle with energy \( E_2 \) is captured by the black hole while that with energy \( E_1 \) comes out of the black hole resulting in an energy gain of

\[ \Delta E = E_1 - 1 = \frac{1}{2} \left[ \sqrt{\frac{r^2 + \Delta - a^2}{r^2}} - 1 \right] = -E_2 . \]  

(S.12)

In the limit \( a \to 0 \), the ergosphere vanishes and the region of ergosphere corresponds to event horizon with radius \( r_{h+} \) and hence \( \Delta = 0 \) and we get energy gain \( \Delta E = 0 \), \( E_1 = 1 \) and \( E_2 = 0 \) and hence no particle with negative energy exists.
The plots of negative energy and energy gain from the black hole are shown above. The plots depict how the negative energy states depend on the parameters characterising the black hole spacetime.

- From Fig. (12) we find that for $\alpha < \alpha_c$, the negative energy increases with increase in $\alpha$ and similar is true for $\alpha > \alpha_c$ also. However the change is less prominent for higher values.

- Also it is noticeable that with the increase in spin ($a$), the negative energy increases though the effect is very feeble.

- The influence of charge ($Q$) and angular momentum ($L$) are firmly observed, where for both large and small constant values of $\alpha$, the increase in the charge and the negative angular momentum increases the negative energy of the particle quite impressively which results in the fact that the particle absorbed by the black hole will have higher negative energy and will lead to increased energy gain from the black hole.

The energy gain from the black hole via., the Penrose process is astrophysically very important and significant. Also indirectly more the negative energy absorbed by the black hole more is the gain in positive energy via Penrose process. The plots of energy gain in Fig.(13) show us the impact of different black hole parameters on the proportion of increment or decrement of the gain in the energy.
We find that energy gain increases with increase in the PFDM parameter $\alpha$ for both $\alpha < \alpha_c$ and $\alpha > \alpha_c$. The increment is more significant for lower range values of PFDM parameter.

With increase in the value of the charge ($Q$), energy gain increases for both $\alpha < \alpha_c$ and $\alpha > \alpha_c$.

7 Summary and Conclusion

We now summarise our findings. We have made some interesting observations in this paper. First of all we give a static, charged black hole solution in PFDM. Furthermore, we have incorporated the Newman-Janis algorithm in order to compute the metric corresponding to a rotating, charged black hole surrounded by perfect fluid dark matter. Our initial study on the event horizon radius of the mentioned black hole reveals that the PFDM parameter $\alpha$ creates a noticeable influence on both outer and inner event horizons. We observe that there exists a certain value $\alpha_c$ (at a constant value of spin $a$ and charge $Q$) upto which the outer event horizon radius ($r_{h+}$) decreases with the increase in the value of $\alpha$. However, after $\alpha_c$, surprisingly $r_{h+}$ starts to increase with increasing value of $\alpha$. On the basis of this critical value, we define two range of values for $\alpha$. We speculate that it might be due to the fact that the dark matter contributes to the effective mass of the black hole system. Then we look for the radius of photon spheres for both prograde and retrograde orbits and found that with the increase in $\alpha$ the radius decreases for $\alpha < \alpha_c$ and increases for $\alpha < \alpha_c$. Besides we also found that increase in the value of the spin parameter ($a$) increases the radius of the photon orbits.

Then we observed the energy ($E$) and angular momentum ($L$) for massive particles moving in prograde and retrograde orbits. The energy ($E$) of the particle in prograde orbits decreases whereas that of retrograde orbits increases with increase in the radial distance ($r$) from the black hole and gets close to unity as the particle approaches infinity. The increment in the values of $\alpha$ and spin ($a$) increases the energy of the particle considerably for the prograde orbits. On the other hand in case of retrograde orbits, the energy ($E$) increases with the increase in $\alpha$ but falls with the increasing value of the spin ($a$). This is because when the particle spins along the black hole, the black hole helps its motion whereas in the reverse case it opposes. The most important observation is in the case of angular momentum of the black hole which decreases with the increase in value of $\alpha$ for $\alpha < \alpha_c$ and increases for $\alpha > \alpha_c$ for both types of orbits. Also we observe that with increase in the spin ($a$) of the black hole, the angular momentum ($L$) of the particle increases for co-rotating particles (+ ve increase) for both $\alpha < \alpha_c$ and $\alpha > \alpha_c$. But for counter-rotating particles, the angular momentum ($L$) rises (-ve increase) for $\alpha < \alpha_c$ while it decreases for $\alpha > \alpha_c$. We found that with the increase in the charge ($Q$) of the black hole, the angular momentum ($L$) of the particle decreases.

We have then studied charged particles. In case of charged particles we analyse both the energy ($E$) and angular momentum ($L$) with the variation in the charge ($q$) of the particle and found that with the increase in the value of charge ($q$), the energy falls in prograde orbits and increases in case of retrograde orbits. Besides we observe that angular momentum ($L$) increases with increasing $q$ for $\alpha < \alpha_c$ and falls for $\alpha > \alpha_c$. It is observed that the effective potential of the black hole for photons and massive particles, increases with the increasing spin ($a$) and charge ($Q$) of the black hole as also with angular momentum ($L$) of the particle. The change is quite sharp with the change in the angular momentum ($L$). Also with increase in the value of $\alpha$ ($\alpha < \alpha_c$), the potential increases whereas for $\alpha > \alpha_c$, it decreases slightly.
The potential of the black hole for the charged particle is analysed with variation in $q$ and we found that potential increases with increase in the charge from $-0.3$ to $0.3$.

Finally, we studied the Penrose process. The negative energy particles are very important with respect to the idea of energy gain from black hole and we observed that negative energy considerably increases with increase in negative angular momentum (counter-rotating particle) and also with increase in the charge ($Q$) of the black hole. The effect of dark matter on the negative energy is less pronounced even though negative energy slightly increases. More the negative energy of the particle absorbed by the black hole, more is the gain, and we found that the energy gain via Penrose process increases due to the presence of dark matter in the system. We also observed that more the black hole charge ($Q$), more is the energy gain and hence more efficient is the Penrose process.

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