Dijkgraaf-Vafa theory as large-$N$ reduction

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Abstract

We construct a large-$N$ twisted reduced model of the four-dimensional super Yang-Mills theory coupled to one adjoint matter. We first consider a non-commutative version of the four-dimensional superspace, and then give the mapping rule between matrices and functions on this space explicitly. The supersymmetry is realized as a part of the internal $U(\infty)$ gauge symmetry in this reduced model. Our reduced model can be compared with the Dijkgraaf-Vafa theory that claims the low-energy glueball superpotential of the original gauge theory is governed by a simple one-matrix model. We show that their claim can be regarded as the large-$N$ reduction in the sense that the one-matrix model they proposed can be identified with our reduced model. The map between matrices and functions enables us to make direct identities between the free energies and correlators of the gauge theory and the matrix model. As a by-product, we can give a natural explanation for the unconventional treatment of the one-matrix model in the Dijkgraaf-Vafa theory where eigenvalues lie around the top of the potential.

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1 Introduction

Reduction of dynamical degrees of freedom has played a central role and has been paid much attention in physics. It sometimes reveals not only an essential structure of complicated systems, but their fundamental degrees of freedom. For example, the renormalization group [1], the basic idea of which is the reduction of degrees of freedom by the block spin transformation, gives us insights into the universality of quantum field theory. Another example is the large-$N$ reduction [2, 3]. This states that in the large-$N$ limit gauge theories in any dimensions are in a sense equivalent. Thus it can be regarded as a universality of the large-$N$ field theories. Furthermore, the reduced model brings some insights into the fundamental degrees of freedom of string theory. For example, there are a few kinds of large-$N$ reduced models which are conjectured to be nonperturbative formulations of string/M theory. One is the Matrix theory [4], which is the large-$N$ reduced model in one dimension. There, the fundamental degrees of freedom are the D-particles whose space-time coordinates are described by large-$N$ matrices. Another prototype is the IIB matrix model [5], which is the large-$N$ reduced model in zero dimension. Here the eigenvalues of matrices may be regarded as the space-time points themselves [6].

Recently Dijkgraaf and Vafa have made a claim that the exact low-energy superpotential for $\mathcal{N} = 1$ gauge theories can be obtained by the perturbative computations in simple matrix models [7]. There, only planar diagrams of the matrix models contribute to the results, even if the large-$N$ limit is not taken in the original gauge theories. Though this claim is motivated by topological strings [8], it can be proved purely by the gauge theory considerations in [9, 10]. Among others, in [10] a proof of the Dijkgraaf-Vafa theory is presented transparently by comparing the Schwinger-Dyson equations of the gauge theory and the matrix model.

At first sight, the Dijkgraaf-Vafa theory is another kind of the reduction of degrees of freedom, because it arises not from the large-$N$ limit, but from the supersymmetry as shown in [10]. However, in this paper, we show that the Dijkgraaf-Vafa theory can be regarded as the large-$N$ reduction. The idea is quite simple; we first consider the noncommutative supersymmetric gauge theory, and express it in terms of matrices. Here the noncommutativity does not contribute to the holomorphic quantities which appear in the Dijkgraaf-Vafa theory. Because the original gauge theory is defined on the superspace, we need to consider the noncommutative superspace where the fermionic
coordinates are also noncommutative. As a consequence, the original gauge theory is mapped to a supermatrix model. We show that this model is nothing but the matrix model that Dijkgraaf-Vafa considered.

This paper is organized as follows. In section 2, we review the Schwinger-Dyson approach of the Dijkgraaf-Vafa theory, where we slightly modify the argument in [10]. In particular, we clarify the origin of the Konishi anomaly [12], which plays an important role in our argument as well. In section 3, we review the basic facts on the relationship between noncommutative gauge theories and matrix models [11]. In section 4, we construct the large-$N$ twisted reduced model [3] of the noncommutative supersymmetric gauge theory. Then we consider the noncommutative superspace and the gauge theory defined on it. We show that it is mapped to a supermatrix model. In section 5, we find a direct relation between the correlation functions and free energies of the supersymmetric gauge theory and the supermatrix model. Then we show that our supermatrix model captures the low-energy superpotential and incorporates the Dijkgraaf-Vafa theory. The point here is that we can make a direct map between the supersymmetric gauge theory and the supermatrix model. Section 6 is devoted to discussions. In appendix A, we give a derivation of the Konishi anomaly on the bosonic noncommutative space.

2 Review of the Schwinger-Dyson approach

We consider $\mathcal{N} = 1$ $U(n)$ gauge theory coupled to an adjoint matter $\Phi$. According to the Dijkgraaf-Vafa theory, the prepotential of this theory is identified with the free energy of a large $\tilde{N}$ one-matrix model.

In this section, we slightly modify the proof of [10] using the Schwinger-Dyson equations. In this approach, the Konishi anomaly enters as a result of the regularization of $\delta^4(0)\delta^2(0)$, the value of the $\delta$-function at the origin of the superspace, that appears in the Schwinger-Dyson equations. In section 5, this quantity plays an important role to connect the field theory correlation functions with those of the matrix model.
The action of the $U(n)$ gauge theory is given by

$$ S = \int d^4x d^2\theta d^2\bar{\theta} \, \text{tr} \left( e^{-V} \bar{\Phi} e V \Phi \right) $$

$$ + \int d^4x d^2\theta \, \text{tr} \left( W(\Phi) \right) $$

$$ + \int d^4x d^2\theta \, 2\pi i \tau \, \text{tr} \left( W^\alpha W_\alpha \right) + \text{c.c..} \quad (2.1) $$

Here $\Phi$ is a chiral superfield in the adjoint representation of $U(n)$, $\tau$ is the gauge coupling constant, $V$ is the vector superfield, $W_\alpha$ is the field strength

$$ W_\alpha = -\frac{1}{4} D D e^{-V} D_\alpha e^V, \quad (2.2) $$

and $W(\Phi)$ is a $(m+1)$-th order polynomial superpotential

$$ W(\Phi) = \sum_{k=0}^{m} \frac{g_k}{k+1} \Phi^{k+1}. \quad (2.3) $$

This theory is invariant under the translation $W_\alpha \mapsto W_\alpha - 8\pi \psi_\alpha$, where $\psi_\alpha$ is an anti-commuting c-number, because all fields are in the adjoint representation so that the $U(1)$ gauge field is decoupled. Owing to this symmetry, the low energy effective action $W_{\text{eff}}$ can be expressed by a prepotential $\mathcal{F}$

$$ W_{\text{eff}} = \int d^2\psi \, \mathcal{F}. \quad (2.4) $$

The $g_k$ dependence of $\mathcal{F}$ is given by the resolvent as follows. First by differentiating the partition function with respect to $g_k$, we obtain

$$ \frac{\partial}{\partial g_k} W_{\text{eff}} = \frac{\partial}{\partial g_k} \int d^2\psi \, \mathcal{F} = \left\langle \text{tr} \frac{\Phi^{k+1}}{k+1} \right\rangle. \quad (2.5) $$

If we introduce the resolvent

$$ \mathcal{R}(z) = \frac{1}{64\pi^2} \text{tr} \left( (W^\alpha - 8\pi \psi^\alpha)(W_\alpha - 8\pi \psi_\alpha) \frac{1}{z - \Phi} \right), \quad (2.6) $$

the right hand side is expressed as

$$ \left\langle \text{tr} \frac{\Phi^{k+1}}{k+1} \right\rangle = \frac{1}{2\pi i(k+1)} \int d^2\psi \int dz \, \left\langle \mathcal{R}(z) \right\rangle z^{k+1}. \quad (2.7) $$

By comparing (2.5) and (2.7), we find that the $g_k$ derivative of $\mathcal{F}$ can be expressed as

$$ \frac{\partial}{\partial g_k} \mathcal{F}(S_i) = \frac{1}{2\pi i(k+1)} \int dz \, \left\langle \mathcal{R}(z) \right\rangle z^{k+1}. \quad (2.8) $$
We can determine the prepotential by solving the Schwinger-Dyson equations up to some ambiguities, and in order to fix them, we impose the following $m$ conditions

$$S_i = \frac{1}{2\pi i} \oint_{C_i} dz \langle R(z) \rangle,$$  

(2.9)

where $C_i$ is a contour around the $i$-th critical point. Thus we obtain $F$ as a function of $S_i$.

Corresponding to the gauge theory (2.1), we consider the $U(\hat{N})$ one-matrix model given by

$$S_m = \frac{\hat{N}}{g_m} \text{Tr} W(\hat{\Phi}),$$  

(2.10)

where $W$ is the same polynomial potential as (2.3) and $g_m$ is an appropriate constant of dimension three that makes the action dimensionless.

The free energy of the matrix model is defined by

$$\exp\left(-\frac{\hat{N}^2 F_m}{g_m^2}\right) = \int d\hat{N}^2 \hat{\Phi} e^{-S_m}.$$  

(2.11)

Again the $g_k$ derivative of the free energy can be expressed by the resolvent as follows

$$\frac{\partial}{\partial g_k} F_m = \frac{1}{2\pi i (k+1)} \oint dz \langle R_m(z) \rangle z^{k+1},$$  

(2.12)

$$R_m(z) = \frac{g_m}{\hat{N}} \text{Tr} \frac{1}{z - \hat{\Phi}}.$$  

(2.13)

As we will see below, $R_m(z)$ obeys the same Schwinger-Dyson equation as $R(z)$. Therefore if we impose $m$ conditions given by

$$S_i = \frac{1}{2\pi i} \oint_{C_i} dz \langle R_m(z) \rangle,$$  

(2.14)

$F(S_i)$ and $F_m(S_i)$ become identical functions up to $g_k$ independent part.

### 2.1 Schwinger-Dyson equations of the matrix model

In order to obtain the Schwinger-Dyson equations for $R_m$, we start from

$$\int d\hat{N}^2 \hat{\Phi} \text{Tr} \left( T^a \frac{1}{z - \hat{\Phi}} \right) e^{-S_m}.$$  

(2.15)
By shifting $\hat{\Phi} \mapsto \hat{\Phi} + \epsilon T^a$, we obtain
\[ 0 = \int d\hat{N}^2 \hat{\Phi} \text{Tr} \left( T^a \frac{1}{z - \hat{\Phi}} T^a \frac{1}{z - \hat{\Phi}} \right) e^{-S_m} \]
\[ - \frac{\hat{N}}{g_m} \int d\hat{N}^2 \hat{\Phi} \text{Tr} \left( T^a \frac{1}{z - \hat{\Phi}} \right) \text{Tr} \left( T^a W'(\hat{\Phi}) \right) e^{-S_m}. \]  
(2.16)
By using the completeness of the $U(\hat{N})$ Gell-Mann matrices
\[ \sum_a \text{Tr}(T^a X T^a Y) = \text{Tr} X \text{Tr} Y, \]
\[ \sum_a \text{Tr}(T^a X) \text{Tr}(T^a Y) = \text{Tr}(XY), \]  
(2.17)
the equation becomes
\[ 0 = \left\langle \text{Tr} \frac{1}{z - \hat{\Phi}} \text{Tr} \frac{1}{z - \hat{\Phi}} \right\rangle - \frac{\hat{N}}{g_m} \left\langle \text{Tr} \frac{1}{z - \hat{\Phi}} W'(\hat{\Phi}) \right\rangle. \]
Using the large $\hat{N}$ factorization, we obtain
\[ \left( \frac{g_m}{\hat{N}} \left\langle \text{Tr} \frac{1}{z - \hat{\Phi}} \right\rangle \right)^2 = \frac{g_m}{\hat{N}} \left\langle \text{Tr} \left( \frac{1}{z - \hat{\Phi}} W'(\hat{\Phi}) \right) \right\rangle, \]  
(2.18)
and the right hand side can be rewritten as
\[ \frac{g_m}{\hat{N}} \text{Tr} \frac{1}{z - \hat{\Phi}} \left( W'(\hat{\Phi}) - W'(z) + W'(z) \right) = \frac{g_m}{\hat{N}} \text{Tr} \frac{1}{z - \hat{\Phi}} \left( W'(\hat{\Phi}) - W'(z) \right) + R_m(z)W'(z). \]
Because the first term of the right hand side is the $(m - 1)$-th polynomial, (2.18) can be expressed as
\[ \frac{d^m}{dz^m} \left( R_m(z)^2 - W'(z)R_m(z) \right) = 0. \]  
(2.19)
This is an $m$th-order differential equation, and as we mentioned above, we need $m$ conditions (2.14) to fix the ambiguities. In the next subsection we show that the Schwinger-Dyson equation for $R(z)$ in the gauge theory is identical to (2.18).

### 2.2 Schwinger-Dyson equations of the gauge theory

As in the matrix model, we start from
\[ \int D\Phi \text{tr} \left( e^{a(W^\alpha(y', \theta') - 8\pi \psi^\alpha)(W^\alpha(y', \theta') - 8\pi \psi^\alpha)} \right) e^{-S}. \]  
(2.20)
Again by shifting
\[ \Phi(y, \theta) \mapsto \Phi(y, \theta) + \epsilon t^a \delta^4(y - y_0) \delta^2(\theta - \theta_0), \]
we obtain
\[
0 = \int \mathcal{D} \Phi \text{tr} \left( t^a \frac{(W^a - 8\pi \psi^a)(W_a - 8\pi \psi_a)}{z - \Phi(y', \theta')} \delta^4(y' - y_0) \delta^2(\theta' - \theta_0) t^a \frac{1}{z - \Phi(y', \theta')} \right) e^{-S} \\
- \int \mathcal{D} \Phi \text{tr} \left( t^a \frac{(W^a - 8\pi \psi^a)(W_a - 8\pi \psi_a)}{z - \Phi(y', \theta')} \right) \text{tr} (t^a W'(\Phi(y_0, \theta_0))) e^{-S} \\
+ \frac{1}{4} \int \mathcal{D} \Phi \text{tr} \left( t^a \frac{(W^a - 8\pi \psi^a)(W_a - 8\pi \psi_a)}{z - \Phi(y', \theta')} \right) \text{tr} (t^a D^2 \Phi(y_0, \theta_0, \bar{\theta}_0)) e^{-S}. \tag{2.21}
\]

If we take the limit \((y', \theta') \mapsto (y_0, \theta_0)\), the third term becomes zero because of the property of the chiral ring, and there is no difficulty in the second term. However, the first term involves a singular factor \(\delta^4(0)\delta^2(0)\), and we regularize it by the heat kernel method as shown in appendix A:

\[
\left. \frac{\delta \Phi^a(y_0, \theta_0)}{\delta \Phi^b(y, \theta)} \right|_{(y, \theta) \mapsto (y_0, \theta_0)} = \delta^b \delta^4(y - y_0) \delta^2(\theta - \theta_0) \left|_{(y, \theta) \mapsto (y_0, \theta_0)} \right. \\
= \frac{1}{64\pi^2} (W^a W_a)^b. \tag{2.22}
\]

Thus (2.21) becomes
\[
\frac{1}{64\pi^2} \left\langle \text{tr} t^a \frac{(W^a - 8\pi \psi^a)(W_a - 8\pi \psi_a)}{z - \Phi} [W^\beta, [W_\beta, t^a]] \frac{1}{z - \Phi} \right\rangle \\
- \left\langle \text{tr} \frac{(W^a - 8\pi \psi^a)(W_a - 8\pi \psi_a)}{z - \Phi} W'(\Phi) \right\rangle = 0 \tag{2.23}
\]

Again by the property of the chiral ring, terms containing more than two factors of \((W_a - 8\pi \psi_a)\) vanish. In order to use this property, we can shift \(W_a \mapsto W_a - 4\pi \psi_a\) in the Konishi anomaly, because such shifts of \(U(1)\) part do not affect the commutator. And by using the property of \(t^a\) and the factorization of the chiral ring, we obtain

\[
\left( \frac{1}{64\pi^2} \left\langle \text{tr} \frac{(W^a - 8\pi \psi^a)(W_a - 8\pi \psi_a)}{z - \Phi} \right\rangle \right)^2 \\
= \frac{1}{64\pi^2} \left\langle \text{tr} \frac{(W^a - 8\pi \psi^a)(W_a - 8\pi \psi_a)}{z - \Phi} W'(\Phi) \right\rangle. \tag{2.24}
\]
This form is exactly the same as (2.18), and as in the matrix model, we can rewrite it in term of $R(z)$. We obtain the same differential equation as (2.19), and also need $m$ conditions (2.9) to fix the ambiguities. Here we emphasize that the Konishi anomaly can be understood as a result of $\delta^4(0)\delta^2(0)$, which will play a crucial role in section 5.

3 The large-$N$ twisted reduced model

In this section, we give a brief review of the large-$N$ twisted reduced model. We first introduce the noncommutative space on which we define noncommutative field theory. Then we construct a mapping between field theory and matrix model.

3.1 Noncommutative space

In order to define a $D$-dimensional noncommutative space, we first consider a quantum mechanics of degrees of freedom $D/2$, which has $D/2$ momenta and $D/2$ coordinates. By taking appropriate linear combinations of them, we have operators $\hat{p}_\mu$ ($\mu = 1, \ldots, D$) that satisfy

$$[\hat{p}_\mu, \hat{p}_\nu] = iB_{\mu\nu},$$

where $B_{\mu\nu}$ is an antisymmetric tensor with real components, and $\text{rank} B = D$. Later we will see that (3.1) can be obtained as a classical solution of a large-$N$ matrix model. Let $C$ be the inverse matrix of $B$

$$C^{\mu\lambda} B_{\lambda\nu} = \delta^\mu_\nu,$$

and we define $\hat{x}^\mu$ by

$$\hat{x}^\mu = C^{\mu\nu} \hat{p}_\nu.$$ (3.3)

Then $\hat{x}^\mu$ and $\hat{p}_\nu$ satisfy the following commutation relations:

$$[\hat{x}^\mu, \hat{p}_\nu] = i\delta^\mu_\nu, \quad [\hat{x}^\mu, \hat{x}^\nu] = -iC^{\mu\nu}, \quad [\hat{p}_\mu, \hat{p}_\nu] = iB_{\mu\nu}.$$ (3.4)

We regard $\hat{x}^\mu$ ($\mu = 1, \ldots, D$) as the noncommutative coordinates of a $D$-dimensional noncommutative space, and consider a field theory defined on it. In fact, various gauge theories defined on this space are known to arise as the low-energy effective theory of string theory or M-theory [13]. In such a noncommutative field theory, fields or functions of $\hat{x}^\mu$ have one-to-one correspondence to operators in the original quantum mechanics via the Weyl ordering,

$$O(x) = \int \frac{d^Dk}{(2\pi)^D} e^{ik\cdot x} \tilde{O}(k) \leftrightarrow \hat{O} = \int \frac{d^Dk}{(2\pi)^D} e^{i\hat{k}\cdot \hat{x}} \tilde{O}(k).$$ (3.5)
Roughly speaking, the operator $\hat{O}$ corresponding to $O(x)$ can be regarded as $O(\hat{x})$. In this correspondence, a Hermitian operator corresponds to a real function. From (3.5), we can read the following mapping rule between functions on the noncommutative space and operators (matrices):

1. If $\hat{O}_1$ and $\hat{O}_2$ correspond to $O_1(x)$ and $O_2(x)$ respectively, $\hat{O}_1 \hat{O}_2$ corresponds to $O_1 \star O_2(x)$, where the $\star$-product is defined by

   \[
   O_1 \star O_2(x) = \exp \left( -\frac{i}{2} C^{\mu\nu} \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial y^\nu} \right) O_1(x)O_2(y) \bigg|_{y=x}. \tag{3.6}
   \]

2. If $\hat{O}$ corresponds to $O(x)$,

   \[
   \text{Tr} \, \hat{O} = \frac{1}{(2\pi)^{D/2} \sqrt{\det C}} \int d^D x \, O(x). \tag{3.7}
   \]

3. If $\hat{O}$ corresponds to $O(x)$, $[\hat{p}_\mu, \hat{O}]$ corresponds to $-i\partial_\mu O(x)$.

### 3.2 Noncommutative field theory

Now we construct a field theory defined on the noncommutative space, namely, noncommutative field theory. As the simplest example, we start with an infinite dimensional Hermitian matrix model

\[
S = (2\pi)^{D/2} \sqrt{\det C} \text{Tr} \left( -\frac{1}{2} [\hat{p}_\mu, \hat{\phi}]^2 + V(\hat{\phi}) \right). \tag{3.8}
\]

Here $\hat{\phi}$ and $\hat{p}$ are Hermitian operators acting on a vector space, and we assume that $\hat{p}$ form an irreducible representation of the algebra (3.1). Using the mapping rule described above, we can reinterpret this theory as a real scalar field theory defined on the noncommutative space

\[
S = \int d^D x \left( \frac{1}{2} (\partial_\mu \phi)^2 + V(\phi) \right). \tag{3.9}
\]

Here $\star$ means that every product is understood as the $\star$-product defined by (3.6). If we take a reducible representation of (3.1) such as $\hat{p}_\mu = \hat{p}_\mu^{(0)} \otimes 1_n$, where $\hat{p}_\mu^{(0)}$ is the irreducible representation of (3.1), and $1_n$ is the $n \times n$ unit matrix, (3.8) can be mapped to an $n \times n$ Hermitian matrix-valued scalar field theory

\[
S = \int d^D x \text{tr} \left( \frac{1}{2} (\partial_\mu \phi)^2 + V(\phi) \right). \tag{3.10}
\]
Next we turn to quantum aspects of the noncommutative field theory. As is well known, if we deduce the Feynman rule of (3.9), we have the noncommutative phase factor for each vertex arising from the $\ast$-product. Due to this phase factor, if external momenta are much larger than $|B| \equiv (\sqrt{\det B})^{1/D}$, only the planar diagrams survive [3], which means that in high momentum region the noncommutative field theory is equivalent to the large-$N$ theory. On the other hand, if external momenta are much smaller than $|B|$, this theory is at least classically equivalent to the ordinary field theory on the commutative space because the phase factor does not contribute. However, in quantum theory, the noncommutative field theory has an effective UV cutoff of order $1/|C_{\mu\nu}p_{\nu}|$ due to the phase factor, where $p$ is an external momentum. Therefore, if the theory does not have UV divergence at all as a field theory, we can take the low energy limit $p \to 0$ smoothly and the noncommutative field theory is reduced to the ordinary commutative field theory. However, if the theory has an UV divergence, it possibly violates this classical equivalence [14].

### 3.3 Noncommutative gauge theory

If we consider the gauge theory on the noncommutative space in the same way, we find that the corresponding matrix model is nothing but the large-$N$ twisted reduced model [3]. In order to see this, we consider the noncommutative $U(n)$ gauge theory coupled to a fermion in the adjoint representation,

$$S = \int d^Dx \left( \frac{1}{g^2} \text{tr} \left( -\frac{1}{4} F_{\mu\nu}^2 - \frac{i}{2} \bar{\psi} \Gamma^\mu [D_\mu, \psi] \right) \right)_{\ast}. \quad (3.11)$$

The corresponding matrix model is obtained via the mapping rule as

$$S = (2\pi)^{D/2} \sqrt{\det C} \frac{1}{g^2} \text{Tr} \left( \frac{1}{4} [\hat{p}_\mu + \hat{a}_\mu, \hat{p}_\nu + \hat{a}_\nu]^2 + \frac{1}{2} \bar{\psi} \Gamma^\mu [\hat{p}_\mu + \hat{a}_\mu, \psi] \right), \quad (3.12)$$

up to some ambiguities coming from the ordering. Here $\hat{p}_\mu = \hat{p}_\mu^{(0)} \otimes 1_n$ and the trace is taken over both the representation space of $\hat{p}^{(0)}$ and $n \times n$ matrix. If we define

$$\hat{A}_\mu = \hat{p}_\mu + \hat{a}_\mu, \quad (3.13)$$

this action can be rewritten as

$$S = (2\pi)^{D/2} \sqrt{\det C} \frac{1}{g^2} \text{Tr} \left( \frac{1}{4} [\hat{A}_\mu, \hat{A}_\nu]^2 + \frac{1}{2} \bar{\psi} \Gamma^\mu [\hat{A}_\mu, \psi] \right). \quad (3.14)$$

As a result, $\hat{p}_\mu$ dependence disappears in (3.14). Instead, it has a classical solution $\hat{A}_\mu = \hat{p}_\mu$ where $\hat{p}$ satisfies (3.1) and if we expand (3.14) around it as (3.13), we recover (3.12)
or, equivalently, the noncommutative gauge theory (3.11) [11]. (3.14) is the dimensional reduction of the $U(\infty)$ gauge theory with an adjoint matter to the zero dimension. This is nothing but the large-$N$ reduced model, and the expansion around the noncommutative background $\hat{A}_\mu = \hat{p}_\mu$ is known as the twisted reduced model.

4 Supersymmetric large-$N$ twisted reduced model

Now we construct the large-$N$ twisted reduced model of the supersymmetric gauge theory with an adjoint matter. We do this in the following two steps:

step1 We first describe the supersymmetric gauge theory on the noncommutative space in terms of superfield. At this stage, the four-dimensional bosonic coordinates $x^\mu$ become noncommutative, while the fermionic coordinates $\theta^\alpha, \bar{\theta}^{\dot{\alpha}}$ remain intact. As a result, each component of the superfield corresponds to a large-$N$ matrix.

step2 Next we make the fermionic coordinates $\theta^\alpha, \bar{\theta}^{\dot{\alpha}}$ noncommutative. As a result, a superfield corresponds to a supermatrix. Namely, all components are encoded into a single supermatrix.

4.1 Large-$N$ reduction via superfield

We are interested in the $U(n)$ gauge theory with one adjoint matter (2.1). Before considering a noncommutative version of this theory, we rewrite this action in terms of fields appropriate for the large-$N$ reduction. When we concentrate on the chiral superfields as in Dijkgraaf-Vafa theory, convenient coordinates are given by $y^\mu = x^\mu + i\theta^\sigma \bar{\theta}^\mu$. In fact, a solution of the chiral condition $\bar{D}\Phi(x, \theta, \bar{\theta}) = 0$ is in general given by

$$
\Phi(x, \theta, \bar{\theta}) = \Phi(y)(x + i\theta^\sigma \bar{\theta}, \theta) = \exp(i\theta^\sigma \bar{\theta} \partial_\mu) \Phi(y)(x, \theta) \exp(-i\theta^\sigma \bar{\theta} \partial_\mu),
$$

(4.1)

where the superscript $(y)$ indicates the representation in terms of $y, \theta$ and $\bar{\theta}$. The advantage of the $y$-representation is that a chiral superfield $\Phi(y)$ does not have $\bar{\theta}$ component as above and that if we expand $\Phi(y)(y, \theta)$ with respect to $\theta$ as

$$
\Phi(y)(y, \theta) = \phi(y) + \sqrt{2}\theta \psi(y) + \theta \theta F(y),
$$

(4.2)

4Rigorously, fermionic coordinates become non-anticommutative. However, we call them ‘noncommutative’ fermionic coordinates for simplicity.
all components $\phi(y)$, $\psi(y)$, $F(y)$ are independent, arbitrary functions of $y$. However, the natural coordinate for which we can introduce the noncommutativity is not $y^\mu$ but $x^\mu$. Therefore, we rewrite the original action in terms of $\Phi(y)(x, \theta)$ appearing in (4.1). Similarly, we define an antichiral superfield $\bar{\Phi}(y)(x, \bar{\theta})$ by

$$\bar{\Phi}(x, \theta, \bar{\theta}) = \bar{\Phi}^{(y)}(x, \bar{\theta}) \exp(-i\theta\sigma^\mu \bar{\theta}\partial_\mu) \Phi(y)(x, \theta) \exp(i\theta\sigma^\mu \bar{\theta}\partial_\mu).$$

(4.3)

Then the kinetic term of the matter field can be rewritten as

$$\text{tr}(\bar{\Phi} e^V \Phi e^{-V}) = \text{tr}(\bar{\Phi}^{(y)} e^{i\theta\sigma^\mu \bar{\theta}\partial_\mu} e^V e^{i\theta\sigma^\mu \bar{\theta}\partial_\mu} \Phi(y) e^{-i\theta\sigma^\mu \bar{\theta}\partial_\mu} e^{-V} e^{-i\theta\sigma^\mu \bar{\theta}\partial_\mu}).$$

(4.4)

This motivates us to define a new vector superfield

$$e^V(x, \theta, \bar{\theta}) \equiv \exp(i\theta\sigma^\mu \bar{\theta}\partial_\mu) e^V(x, \theta, \bar{\theta}) \exp(i\theta\sigma^\mu \bar{\theta}\partial_\mu).$$

(4.5)

Note that it is not a similarity transformation like (4.1), and $V(x, \theta, \bar{\theta})$ is no longer a function but a first-order differential operator. Obviously, $V^\dagger = V$. Thus the kinetic term becomes

$$\text{tr}(\bar{\Phi} e^V \Phi e^{-V}) = \text{tr}(\bar{\Phi}^{(y)} e^V \Phi(y) e^{-V}).$$

(4.6)

Next we consider the kinetic term of the gauge field in (2.1), which is written in terms of the field strength

$$W_\alpha(x, \theta, \bar{\theta}) = -\frac{1}{4} \bar{D} D e^{-V(x, \theta, \bar{\theta})} D_\alpha e^V(x, \theta, \bar{\theta}).$$

(4.7)

It is worth noticing that this equation can be regarded as an equation for differential operators acting on the space of chiral superfields, as is the case with the field strength in the ordinary gauge theories. Namely, the action of the differential operator in the right-hand side of (4.7) on any chiral superfield is equal to the multiplication of $W_\alpha$. Because $W_\alpha$ is a chiral superfield, (4.1) tempts us to define $W^{(y)}(x, \theta)$ as

$$W_\alpha(x, \theta, \bar{\theta}) = W^{(y)}_\alpha(x + i\theta\sigma^\mu \bar{\theta}, \theta) \exp(-i\theta\sigma^\mu \bar{\theta}\partial_\mu).$$

(4.8)

In fact, $W^{(y)}_\alpha$ is exactly the field strength constructed from $V$ defined in (4.5) in the same way as in (4.7):

$$W^{(y)}_\alpha = e^{-A} W_\alpha e^A$$

$$= -\frac{1}{4} (e^{-A} D e^A)(e^{-A} D e^A)(e^{-A} e^{-V} e^{-A})(e^A D_\alpha e^{-A})(e^A e^V e^A)$$

$$= -\frac{1}{4} \bar{D} D e^{-V} D_\alpha e^V.$$

(4.9)
where $A = i \theta \sigma^\mu \bar{\theta} \partial_\mu$, and
\[
D_\alpha = \exp(i \theta \sigma^\mu \bar{\theta} \partial_\mu) D_\alpha \exp(-i \theta \sigma^\mu \bar{\theta} \partial_\mu) = \frac{\partial}{\partial \theta^\alpha},
\]
\[
\bar{D}_{\dot{\alpha}} = \exp(-i \theta \sigma^\mu \bar{\theta} \partial_\mu) \bar{D}_{\dot{\alpha}} \exp(i \theta \sigma^\mu \bar{\theta} \partial_\mu) = -\frac{\partial}{\partial \bar{\theta}^{\dot{\alpha}}},
\]
are natural differential operators on the new chiral or antichiral superfields, $\Phi(y)$ and $\bar{\Phi}(y^\dagger)$.

Note that $D$ and $\bar{D}$ do not contain $\partial_\mu$ because their similarity transformations in (4.10) are inverse to each other. Note also that (4.9) can again be regarded as an equation for differential operators acting on the space of the chiral superfields $\Phi(y)(x, \theta)$. Using $W(\alpha)$, the kinetic term of the gauge field becomes
\[
\text{tr}(W^\alpha W_\alpha) = \text{tr}(W(y)^\alpha W(y)_\alpha).
\] (4.11)

Now we make the bosonic coordinates $x^\mu$ noncommutative, which amounts to replacing all products appearing in (2.1) with the $*$-product defined in (3.6):
\[
S_{NC} = \int d^4x d^2\theta d^2\bar{\theta} \left( \text{tr} (\hat{\Phi} e^V \hat{\Phi} e^{-V}) \right)_* + \int d^4x d^2\theta \left( \text{tr} (W^{\alpha} W_\alpha) \right)_* + \int d^4x d^2\theta \left( \text{tr} W(\Phi) \right)_* + c.c.
\]
\[
= \int d^4x d^2\theta d^2\bar{\theta} \left( \text{tr} (\hat{\Phi}^{(y)} e^V \hat{\Phi}^{(y)} e^{-V}) \right)_* + \int d^4x d^2\theta \left( \text{tr} (W^{(y)} W^{(y)}_\alpha) \right)_* + \int d^4x d^2\theta \left( \text{tr} W(\Phi(y)) \right)_* + c.c.,
\] (4.12)

Following the prescription given in subsection 3.1, we can express it in terms of matrices. We first introduce the noncommutative space-time coordinate $\hat{x}^\mu$ and $\hat{p}_\nu$ that satisfy (3.4). Then by the mapping rule given in subsection 3.1, we have matrix variables corresponding to the chiral superfield, antichiral superfield, vector superfield, and field strength, respectively,
\[
\hat{\Phi}(\theta) \leftrightarrow \Phi(y)(x, \theta),
\]
\[
\hat{\Phi}(\bar{\theta}) \leftrightarrow \Phi(y^\dagger)(x, \bar{\theta}),
\]
\[
\hat{V}(\theta, \bar{\theta}) \leftrightarrow V(x, \theta, \bar{\theta}),
\]
\[
\hat{W}_\alpha(\theta) \leftrightarrow W^{(y)}(x, \theta).
\] (4.13)

The action (4.12) is rewritten as $S_{red}$ given by
\[
S_{red} = (2\pi)^2 \sqrt{\det C} \left\{ \int d^2\theta \int d^2\bar{\theta} \text{Tr} (\hat{\Phi}(\theta) e^{V(\theta, \bar{\theta})} \hat{\Phi}(\theta) e^{-V(\theta, \bar{\theta})}) + \int d^2\theta \int d^2\bar{\theta} \text{Tr} (\hat{W}_\alpha(\theta) \hat{W}_\alpha(\theta)) + \int d^2\theta \text{Tr} W(\hat{\Phi}(\theta)) + c.c. \right\},
\] (4.14)
where
\[ \hat{W}_\alpha = -\frac{1}{4} \bar{D} \hat{D} e^{-\hat{\nu}} D_\alpha e^{\hat{\nu}}, \]
(4.15)
and \( \text{Tr} \) is taken over both \( U(n) \) group and the representation space of (3.4). As seen in (3.14), this is nothing but the large-\( N \) twisted reduced model of the original theory (2.1). It should be noted that \( S_{NC} = S_{\text{red}} \) holds as an identity.

### 4.2 Properties of the supersymmetric reduced model

In this subsection we discuss some interesting properties of the supersymmetric reduced model (4.14).

First, as is the case with the ordinary large-\( N \) reduced model (3.14), it does not have background dependence at all. In general, as we have seen in the previous section, \( \hat{p}_\mu \) appears in the action through the mapping rule \( -i \partial_\mu \leftrightarrow \text{ad} \hat{p}_\mu \), where \( \text{ad} \hat{O} \) denotes the adjoint action of \( \hat{O} \). However, our action does not have explicit \( \hat{p}_\mu \) dependence. In fact, the \( x^\mu \) derivatives do not appear in the definition of \( \hat{D}, \bar{D} \) and \( \hat{\Phi}(y) \), as shown in (4.10) and (4.2). Moreover, the equation of motion of (4.14) for the vector superfield \( \hat{\nu} \) is given by
\[ D_\alpha e^{\hat{\nu}} \hat{W}^\alpha e^{-\hat{\nu}} = 0, \]
(4.16)
which has a special solution
\[ e^{\hat{\nu}} = e^{2 \hat{A}}, \]
(4.17)
where \( \hat{A} = -\theta \sigma^\mu \bar{\theta} \hat{p}_\mu \). As is evident from the construction in the previous subsection, if we expand \( e^{\hat{\nu}} \) around this background as
\[ e^{\hat{\nu}} = e^{\hat{A}} e^{\hat{\nu}'} e^{\hat{\nu}'}, \]
(4.18)
the action (4.14) becomes
\[
\frac{S_{\text{red}}}{(2\pi)^2 \sqrt{\text{det} C}} = \int d^2 \theta \int d^2 \bar{\theta} \text{Tr}(\hat{\Phi}' e^{\hat{\nu}'} \hat{\Phi}' e^{-\hat{\nu}'}) \\
+ \int d^2 \theta \int d^2 \bar{\theta} \text{Tr}(\hat{W}^\alpha \hat{W}_\alpha) + \int d^2 \theta \text{Tr} W(\hat{\Phi}') + \text{c.c.}, \]
(4.19)
where
\begin{align*}
\hat{\Phi}' &= e^{\hat{A}} \Phi e^{-\hat{A}}, \\
\hat{\bar{\Phi}}' &= e^{-\hat{A}} \bar{\Phi} e^{\hat{A}}, \\
W_{\alpha}' &= e^{\hat{A}} \hat{W}_{\alpha} e^{-\hat{A}} = -\frac{1}{4} D D e^{-\hat{V}'} D_{\alpha} e^{\hat{V}'}, \\
D_{\alpha} &= e^{-\hat{A}} D_{\alpha} e^{\hat{A}} = \frac{\partial}{\partial \theta^\alpha} - (\sigma^\mu \bar{\theta})_{\alpha} \hat{P}_\mu, \\
\bar{D}_{\dot{\alpha}} &= e^{\hat{A}} \bar{D}_{\dot{\alpha}} e^{-\hat{A}} = -\frac{\partial}{\partial \bar{\theta}^{\dot{\alpha}}} + (\theta \sigma^\mu)_{\dot{\alpha}} \hat{P}_\mu.
\end{align*}

By using the mapping rule given in subsection 3.1, we recover the noncommutative supersymmetric gauge theory (4.12), where \( \hat{\Phi}' \), \( \hat{\bar{\Phi}}' \) and \( \hat{V}' \) are mapped to \( \Phi \), \( \bar{\Phi} \) and \( V \), respectively. This is a supersymmetric analog of what happens in the bosonic twisted reduced model discussed in subsection 3.3. In particular, \( \hat{V} \) in (4.18) is a supersymmetric analog of \( \hat{A}_\mu \) given in (3.13). Indeed, it is easy to compute the components of \( \hat{V} \) in (4.18) and to find that after the usual rescaling \( \hat{V}' \to 2 \hat{V}' \), the \( \theta \sigma^\mu \bar{\theta} \) component of \( \hat{V} \) is given by \(-2 \hat{A}_\mu = -2(\hat{p}_\mu + \hat{a}_\mu)\), where \(-\hat{a}_\mu \) is the \( \theta \sigma^\mu \bar{\theta} \)-component of \( \hat{V}' \). Similarly, \( \hat{W}_{\alpha} \) corresponds to \( \hat{F}_{\mu\nu} = [\hat{A}_\mu, \hat{A}_\nu] \) in the bosonic twisted reduced model.

Next we discuss the symmetry of the supersymmetric reduced model. The action (4.14) is manifestly invariant under the following transformation:
\begin{align*}
\hat{\Phi} &\to e^{-i\hat{A}} \Phi e^{i\hat{A}}, \\
\hat{\bar{\Phi}} &\to e^{-i\hat{A}^\dagger} \bar{\Phi} e^{i\hat{A}^\dagger}, \\
e^{\hat{V}} &\to e^{-i\hat{A}} e^{\hat{V}} e^{i\hat{A}},
\end{align*}

where \( \hat{A} \) is an arbitrary chiral superfield, \( \bar{D}\hat{A} = 0 \). This symmetry is the counterpart of the ordinary gauge symmetry of the supersymmetric gauge theory (2.1). Remarkably, this symmetry includes the supersymmetry of the corresponding noncommutative gauge theory (4.12). In this sense, in the twisted reduced model (4.14), the gauge symmetry and the supersymmetry are unified. This fact can be shown as follows: take the background (4.17) and make the expansion around it as (4.18), then we get the action (4.19). In terms of the fields appearing in (4.19), the gauge transformation becomes
\begin{align*}
\hat{\Phi}' &\to e^{-i\hat{A}'} \hat{\Phi}' e^{i\hat{A}'}, \\
\hat{\bar{\Phi}}' &\to e^{-i\hat{A}^\dagger'} \hat{\bar{\Phi}}' e^{i\hat{A}^\dagger'},
\end{align*}

\begin{align*}
e^{\hat{V}'} &\to e^{-i\hat{A}^\dagger'} \hat{e} \hat{V}' e^{i\hat{A}^\dagger'},
\end{align*}
where $e^{\hat{A}'} = e^{\hat{A}} e^{\hat{A}} e^{-\hat{A}}$. Note that if $\hat{A}$ is chiral, namely, $\mathcal{D}_\alpha \hat{A} = 0$, then $\hat{A}'$ is chiral, namely, $\hat{D}_\alpha \hat{A}' = 0$. Now we consider a particular gauge transformation (4.21) with $\hat{A}$ given by

$$
\hat{A} = \xi^\alpha \frac{\partial}{\partial \theta^\alpha} + \bar{\xi}^{\dot{\alpha}} \left( -\frac{\partial}{\partial \bar{\theta}^{\dot{\alpha}}} - 2(\theta \sigma^\mu)_{\dot{\alpha}} \hat{p}_\mu \right).
$$

If we expand the theory around the background (4.17), this symmetry becomes the gauge symmetry (4.22) with $\hat{A}'$ given by

$$
\hat{A}' = \xi^\alpha \left( \frac{\partial}{\partial \theta^\alpha} + (\sigma^\mu \bar{\theta})_{\dot{\alpha}} \hat{p}_\mu \right) + \bar{\xi}^{\dot{\alpha}} \left( -\frac{\partial}{\partial \bar{\theta}^{\dot{\alpha}}} - (\theta \sigma^\mu)_{\dot{\alpha}} \hat{p}_\mu \right) + \lambda,
$$

where $\lambda$ is a complex number. Because $\hat{A}'^\dagger = \hat{A}'$, the infinitesimal form of the gauge transformation (4.22) is given by

$$
\delta \hat{\Phi}' = \text{ad}(-i\hat{A}') \hat{\Phi}' = (-i\xi^\alpha Q_\alpha - i\bar{\xi}^{\dot{\alpha}} \bar{Q}_{\dot{\alpha}}) \hat{\Phi}',
$$

$$
\delta \hat{\bar{\Phi}}' = \text{ad}(-i\hat{A}'^\dagger) \hat{\bar{\Phi}}' = (-i\xi^\alpha Q_\alpha - i\bar{\xi}^{\dot{\alpha}} \bar{Q}_{\dot{\alpha}}) \hat{\bar{\Phi}}',
$$

$$
\delta V' = \text{ad}(-i\hat{A}') V' = (-i\xi^\alpha Q_\alpha - i\bar{\xi}^{\dot{\alpha}} \bar{Q}_{\dot{\alpha}}) V',
$$

where

$$
Q_\alpha = \frac{\partial}{\partial \theta^\alpha} + (\sigma^\mu \bar{\theta})_{\dot{\alpha}} \text{ad} \hat{p}_\mu,
$$

$$
\bar{Q}_{\dot{\alpha}} = -\frac{\partial}{\partial \bar{\theta}^{\dot{\alpha}}} - (\theta \sigma^\mu)_{\dot{\alpha}} \text{ad} \hat{p}_\mu.
$$

Note that the transformation law for $V'$ becomes a similarity transformation due to the Hermiticity of $\hat{A}'$. (4.26) are equivalent to the ordinary supercharges in the noncommutative gauge theory (4.12) via the mapping rule $\text{ad} \hat{p}_\mu \leftrightarrow -i\partial/\partial x^\mu$. Therefore, we have shown that once we expand the original model (4.14) around the background (4.17), we get the noncommutative gauge theory (4.12) and its supersymmetry originates from the gauge symmetry (4.21) of the original model. In the ordinary field theory, what makes difference between the gauge symmetry and the supersymmetry is that the former is generated by functions of $x^\mu$, while the latter by the derivative $\partial/\partial x^\mu$, $\partial/\partial \theta^\alpha$ and $\partial/\partial \bar{\theta}^{\dot{\alpha}}$. However, in the large-$N$ twisted reduced model, or in the noncommutative space, there is no definite difference between the ‘coordinate’ and the ‘momentum’ as we can see from eq. (3.3). This is the reason why the gauge symmetry and the supersymmetry are unified in (4.14).

### 4.3 Noncommutative superspace and supermatrix model

As mentioned in the beginning of this section, the next task is to introduce the noncommutative fermionic coordinates as well as the bosonic coordinates. Then it is expected that a field
depending on the noncommutative fermionic coordinates $\theta$ or $\bar{\theta}$ is also mapped to a matrix, as a field on the noncommutative bosonic coordinates $\hat{x}^\mu$ becomes the large-$N$ matrix. It is shown that a field on the noncommutative superspace is described by a supermatrix.

We begin with introducing a noncommutativity into the fermionic coordinates as

$$\{\hat{\theta}^\alpha, \hat{\bar{\theta}}^\beta\} = \gamma^\alpha_{\beta}, \quad \{\hat{\bar{\theta}}^\dot{\alpha}, \hat{\bar{\theta}}^\dot{\beta}\} = \gamma^{* \dot{\alpha}}_{\dot{\beta}},$$

where $\gamma^\alpha_{\beta}$ is a symmetric matrix. In what follows, we consider only $\hat{\theta}^\alpha$ part because $\hat{\bar{\theta}}^\beta$ can be treated in the same way by replacing $\gamma^\alpha_{\beta}$ with $\gamma^{* \dot{\alpha}}_{\dot{\beta}}$. By using the $SL(2, \mathbb{C})$ transformation, $\gamma^\alpha_{\beta}$ can be taken in the following form without loss of generality:

$$\gamma^\alpha_{\beta} = \begin{pmatrix} \gamma & 0 \\ 0 & \gamma \end{pmatrix}, \quad \gamma \in \mathbb{C}. \tag{4.28}$$

In this case, $\hat{\theta}^\alpha$ can be represented in terms of Pauli matrices as

$$\hat{\theta}^1 = \sqrt{\gamma} \sigma^1, \quad \hat{\theta}^2 = \sqrt{\gamma} \sigma^2. \tag{4.29}$$

Let $\beta$ be the inverse matrix of $\gamma$

$$\gamma^\alpha_{\gamma \beta} = \delta^\alpha_{\beta}, \tag{4.30}$$

and define $\hat{\pi}_\alpha$ by

$$\hat{\pi}_\alpha = \beta_{\alpha \beta} \hat{\theta}^\beta. \tag{4.31}$$

Then $\hat{\theta}^\alpha$ and $\hat{\pi}_\beta$ satisfy the following anticommutation relations:

$$\{\hat{\theta}^\alpha, \hat{\pi}_\beta\} = \delta^\alpha_{\beta}, \quad \{\hat{\theta}^\alpha, \hat{\theta}^\beta\} = \gamma^\alpha_{\beta}, \quad \{\hat{\pi}_\alpha, \hat{\pi}_\beta\} = \beta_{\alpha \beta}. \tag{4.32}$$

As in the case of the bosonic noncommutative space, we regard $\hat{\theta}^\alpha$ as the noncommutative fermionic coordinates and make a correspondence between a function on this space and an operator (a matrix) via the Weyl ordering:

$$O(\theta) = \int d^2 \kappa \ e^{i \theta^\alpha \kappa_{\alpha}} \hat{O}(\kappa) \leftrightarrow \hat{O} = \int d^2 \kappa \ e^{i \hat{\theta}^\alpha \kappa_{\alpha}} \hat{O}(\kappa). \tag{4.33}$$

As before, the operator $\hat{O}$ is nothing but the Weyl ordered form of $O(\theta)$.

It is interesting to consider what corresponds to the fermionic integration $\int d^2 \theta$ in the space of operators under the correspondence (4.33). In general, a function of $\theta$ can be expanded as

$$\Phi(\theta) = \phi + \sqrt{2} \theta^\alpha \psi_{\alpha} + \theta \theta F$$

$$= \phi + \sqrt{2} \theta^\alpha \psi_{\alpha} - 2 \theta^1 \theta^2 F. \tag{4.34}$$
then $\int d^2 \theta \Phi(\theta) = F$. On the other hand, the operator corresponding to $\Phi(\theta)$ is given by its Weyl ordered form

$$
\Phi(\hat{\theta}) = \phi + \sqrt{2} \hat{\theta}^\alpha \psi_\alpha - (\hat{\theta}^1 \hat{\theta}^2 - \hat{\theta}^2 \hat{\theta}^1) F
= \phi + \sqrt{2} \hat{\theta}^\alpha \psi_\alpha + \hat{\theta} F.
$$

(4.35)

Because we have fixed the representation of $\hat{\theta}$ as (4.29), $\hat{\theta}^1 \hat{\theta}^2 - \hat{\theta}^2 \hat{\theta}^1 = 2i \gamma \sigma^3$ and therefore, if we define a $\text{Str}_\theta$ as

$$
\text{Str}_\theta(\hat{\Phi}) \equiv 2 \text{Tr}(\sigma^3 \hat{\Phi}),
$$

(4.36)

then

$$
\text{Str}_\theta(\hat{\Phi}) = -8i \gamma F = -8i \gamma \int d^2 \theta \Phi(\theta).
$$

(4.37)

Thus as in the case of $\hat{x}^\mu$, it is easy to derive the following mapping rule from (4.33):

1. If $\hat{O}_1$ and $\hat{O}_2$ correspond to $O_1(\theta)$ and $O_2(\theta)$ respectively, $\hat{O}_1 \hat{O}_2$ corresponds to $O_1 \star O_2(\theta)$, where the fermionic $\star$-product is defined by,

$$
O_1 \star O_2(\theta) = \exp \left( -\frac{1}{2} \gamma^{\alpha \beta} \frac{\partial}{\partial \theta^\alpha} \frac{\partial}{\partial \theta^\beta} \right) O_1(\theta) O_2(\theta') \bigg|_{\theta' = \theta}.
$$

(4.38)

2. If $\hat{O}$ corresponds to $O(\theta)$,

$$
\text{Str}_\theta(\hat{O}) = -8i \sqrt{\det \gamma} \int d^2 \theta \ O(\theta).
$$

(4.39)

3. If $\hat{O}$ corresponds to $O(\theta)$, $[\hat{\pi}_\alpha, \hat{O}]$ corresponds to $\partial/\partial \theta^\alpha O(\theta)$, where the commutator or anticommutator is taken according to the statistics of $\hat{O}$.

Now we define the large-$N$ twisted reduced model on the noncommutative superspace. First we replace the product in (4.14) with the $\star$-product in the space of $\hat{\theta}$ and $\hat{\bar{\theta}}$ defined above. We then rewrite the action using the mapping rule given above, and obtain

$$
S_{\text{smm}} = \frac{i(2\pi)^2 \sqrt{\det C}}{8 \sqrt{\det \gamma}} \text{Str}_{x \otimes \theta \otimes \theta}(\hat{\Phi} e^\hat{\nu} \hat{\Phi} e^{-\hat{\nu}})
+ \frac{i(2\pi)^2 \sqrt{\det C}}{8 \sqrt{\det \gamma}} \{2\pi i \tau \text{Str}_{x \otimes \theta}(\hat{W}^\alpha \hat{W}_\alpha) + \text{Str}_{x \otimes \theta}(W(\hat{\Phi}))\} + \text{c.c.},
$$

(4.40)

where

$$
\hat{W}_\alpha = -\frac{1}{4} \text{ad} \hat{\pi}_\alpha \text{ad} \hat{\pi}^\alpha e^{-\hat{\nu}} \text{ad} \hat{\pi}_\alpha e^{\hat{\nu}},
$$

(4.41)
and $\text{ad} \hat{\pi}_\alpha$ is defined by

$$\text{ad} \hat{\pi}_\alpha \hat{O} = \begin{cases} [\hat{\pi}_\alpha, O] & \text{for even } \hat{O}, \\ \{\hat{\pi}_\alpha, O\} & \text{for odd } \hat{O}, \end{cases}$$

(4.42)

and $\text{ad} \hat{\pi}_\alpha$ is similarly defined. $\text{Str}_{x \otimes \theta \otimes \bar{\theta}}$ means taking trace in the bosonic space of $\hat{x}^\mu$ and supertraces in the fermionic spaces of $\hat{\theta}$ and $\hat{\bar{\theta}}$. Here, as usual in the large-$N$ reduced model, the $U(n)$ gauge group and the bosonic noncommutative space are unified. Similarly, $\text{Str}_{x \otimes \theta}$ and $\text{Str}_{x \otimes \bar{\theta}}$ can be defined unambiguously.$^5$ In the supermatrix model (4.40), the chiral or antichiral condition becomes

$$\text{ad} \hat{\pi}_\alpha \hat{O} = 0, \quad \text{ad} \hat{\pi}_\alpha \hat{O} = 0,$$

(4.43)

which indicate that $\hat{O}$ does not have $\hat{\theta}$ dependence or $\hat{\bar{\theta}}$ dependence, respectively. It is evident by the mapping rule that $\hat{\Phi}$ and $\hat{W}_\alpha$ in (4.40) are chiral supermatrices, while $\hat{\Phi}$ is an antichiral supermatrix. It is also obvious by construction that in the fermionic commutative limit $\gamma, \gamma^* \to 0$, supermatrices in (4.40) tend to corresponding fields in (4.12) as follows:

$$\hat{\Phi} \to \Phi^{(y)}(x, \theta),$$

$$\hat{\Phi} \to \Phi^{(y)}(x, \bar{\theta}),$$

$$\hat{V} \to V(x, \theta, \bar{\theta}),$$

$$\hat{W}_\alpha \to W^{(y)}_\alpha(x, \theta).$$

(4.44)

5 Dijkgraaf-Vafa theory as the large-$N$ reduction

In this section we show that the Dijkgraaf-Vafa theory can be understood in terms of the large-$N$ reduced model.

To begin with, we note that the holomorphic quantities we have discussed in section 2 in the original gauge theory are not affected by the bosonic noncommutativity $C^{\mu\nu}$. These quantities carry zero external momenta, and do not have UV divergences. Therefore we expect that they do not depend on the bosonic noncommutativity $C^{\mu\nu}$ for the reason explained in subsection 3.2. In fact, as shown in [9, 10], in the perturbative expansion, only the planar

$^5$It is likely that by expanding (4.40) around a classical solution such as (4.17), we can obtain a field theory on the noncommutative superspace where every product is defined by the combination of the bosonic $\ast$ and fermionic $\star$ product. However, it does not seem straightforward to generalize the classical solution (4.17) to $\gamma \neq 0$ case. Moreover, it is easy to find that if we expand (4.40) around (4.17), (4.40) is not simply reduced to the ordinary noncommutative gauge theory because there appear terms which cannot be interpreted as a local field in the noncommutative field theory. Of course in the limit $\gamma, \gamma^* \to 0$ this theory is reduced to the noncommutative gauge theory (4.12).
diagrams contribute to them\textsuperscript{6}. It indicates that they have no dependence on $C^{\mu\nu}$, because the noncommutative phase factors cancel in planar diagrams [3]. Therefore, as far as the holomorphic quantities which appears in the Dijkgraaf-Vafa theory are concerned, the same results can be obtained, even if we use the noncommutative version of the original theory (4.12) or equivalently, its large-$N$ reduced model (4.14). This further implies that we can compute them via the supermatrix model (4.40), if we take the commutative limit $\gamma \to 0$, $\gamma^* \to 0$ of the fermionic coordinates. In this section we discuss how to do this.

### 5.1 Equivalence of the correlation function

In order to express the correlation functions in (2.1) in terms of the supermatrix model (4.40), we use the following simple but important equations:

\begin{align}
\delta^4(\hat{x} - x)^2 &= \frac{1}{\pi^4 \det C}, \\
\delta^2(\hat{\theta} - \theta)^2 &= -4 \det \gamma. 
\end{align}

The proof is straightforward, if we use the definitions
\begin{align}
\delta^4(\hat{x} - x) &= \int \frac{d^4k}{(2\pi)^4} e^{ik(\hat{x} - x)}, \\
\delta^2(\hat{\theta} - \theta) &= 4 \int d^2\kappa e^{i(\hat{\theta} - \theta)^2}. 
\end{align}

and take $\lim_{x\to x} \delta^4(\hat{x} - x)\delta^4(\hat{x} - y)$ and $\lim_{x\to y} \delta^2(\hat{\theta} - \theta)\delta^2(\hat{\theta} - \theta')$. In the commutative limit $C \to 0$ of the bosonic coordinates, the usual result in the bosonic commutative space $\delta^4(0) = \infty$ is reproduced:

\begin{equation}
\delta^4(0)\delta^4(\hat{x} - x) = \delta^4(\hat{x} - x)^2 \to \infty. 
\end{equation}

Similarly, in the commutative limit $\gamma \to 0$ of the fermionic coordinates, we have

\begin{equation}
\delta^2(0)\delta^2(\hat{\theta} - \theta) = \delta^2(\hat{\theta} - \theta)^2 \to 0, 
\end{equation}

which is the usual result in the commutative fermionic space. Eqs.(5.1) are quite peculiar to the noncommutative space which is essentially regularized by the noncommutativity and gives the finite result in nature. From (5.1), we can derive an identity

\begin{equation}
\left( \frac{i(2\pi)^2 \sqrt{\det C}}{8 \sqrt{\det \gamma}} \delta^4(\hat{x} - x)\delta^2(\hat{\theta} - \theta) \right)^2 = 1. 
\end{equation}

\textsuperscript{6}This fact is a consequence of the chiral ring [10]. It is easy to check that this structure persists in the bosonic noncommutative gauge theory (4.12).
On the other hand, if a chiral superfield \( O^{(y)}(x, \theta) \) in the bosonic noncommutative gauge theory (4.12) corresponds to a chiral supermatrix \( \hat{O} \) in the supermatrix model (4.40) in the \( \gamma \to 0 \) limit, we obtain by the mapping rule

\[
\frac{i(2\pi)^2 \sqrt{\det C}}{8\sqrt{\det \gamma}} \text{Str}_{x \otimes \theta} (\hat{O} \delta^4(\hat{x} - x) \delta^2(\hat{\theta} - \theta)) \\
\to \int d^4 x' d^2 \theta' \, \text{tr} \left( O^{(y)}(x', \theta') \delta^4(x' - x) \delta^2(\theta' - \theta) \right) \\
= \text{tr} \left( O^{(y)}(x, \theta) \right), \quad \text{as } \gamma \to 0,
\]

(5.6)

where the trace is taken over the \( U(n) \) group. Therefore, in \( \gamma \to 0 \) limit, the operator in the left-hand side corresponds to the local field in (4.12). Namely, the action \( \text{Str}_{x \otimes \theta} (\delta^4(\hat{x} - x) \delta^2(\hat{\theta} - \theta) \cdot) \) on a supermatrix essentially evaluates the corresponding field at \( x, \theta \) in the noncommutative field theory side.

In the supermatrix model (4.40), a fundamental correlator is the resolvent,

\[
\left\langle \text{Str}_{x \otimes \theta} \frac{1}{z - \Phi} \right\rangle.
\]

(5.7)

Using (5.5) and (5.6), we can find what kind of field in (4.12) corresponds to (5.7) in the \( \gamma \to 0 \) limit as follows:

\[
\frac{8\sqrt{\det \gamma}}{i(2\pi)^2 \sqrt{\det C}} \text{Str}_{x \otimes \theta} \left( \frac{1}{z - \Phi} \right) \\
= \frac{8\sqrt{\det \gamma}}{i(2\pi)^2 \sqrt{\det C}} \left[ \frac{i(2\pi)^2 \sqrt{\det C}}{8\sqrt{\det \gamma}} \right]^2 \text{Str}_{x \otimes \theta} \left( \frac{1}{z - \Phi} \delta^4(\hat{x} - x) \delta^2(\hat{\theta} - \theta)^2 \right) \\
= \frac{i(2\pi)^2 \sqrt{\det C}}{8\sqrt{\det \gamma}} \text{Str}_{x \otimes \theta} \left( \frac{1}{z - \Phi} \delta^4(0) \delta^2(0) \delta^4(\hat{x} - x) \delta^2(\hat{\theta} - \theta) \right) \\
\to \frac{1}{64\pi^2} \text{tr} \left( W^{(y) \alpha}(x, \theta) W^{(y) \alpha}(x, \theta) \frac{1}{z - \Phi^{(y)}(x, \theta)} \right)_s, \quad \text{as } \gamma \to 0
\]

(5.8)

where we have used the Konishi anomaly [12] in the bosonic noncommutative space

\[
\lim_{\gamma \to 0} \delta^4(0) \delta^2(0) = \frac{1}{64\pi^2} \hat{W}^{\alpha} \hat{W}_{\alpha}.
\]

(5.9)

In appendix A, we give a derivation of this equation. Because \( \lim_{\gamma \to 0} S_{\text{smm}} = S_{\text{red}} = S_{\text{NC}} \), we thus conclude

\[
\lim_{\gamma \to 0} \frac{8\sqrt{\det \gamma}}{i(2\pi)^2 \sqrt{\det C}} \left\langle \text{Str}_{x \otimes \theta} \left( \frac{1}{z - \Phi} \right) \right\rangle = \frac{1}{64\pi^2} \left\langle \text{tr} \left( W^{(y) \alpha}(x, \theta) W^{(y) \alpha}(x, \theta) \frac{1}{z - \Phi^{(y)}(x, \theta)} \right)_s \right\rangle_{\text{NC}}.
\]

(5.10)
where the subscript $NC$ indicates the correlation function in the theory with the bosonic noncommutativity (4.12).

From the point of view of the supermatrix model (4.40), holomorphic quantities such as (5.7) are determined by the holomorphic part of the action. In particular, they do not depend on the kinetic term of the chiral superfield $\hat{\Phi}$ (the first term) in (4.40) and we can neglect it in the computation of (5.7). Once we do it, it is evident that the kinetic term of the vector superfield $\hat{V}$ (the second term) can be also neglected because $\hat{\Phi}$ and $\hat{V}$ are now decoupled. Thus the holomorphic potential term

$$S_{smm}^{hol} = \frac{i(2\pi)^2\sqrt{\det C}}{8\sqrt{\det \gamma}} \text{Str}_{x\otimes \theta}(W(\hat{\Phi})), \quad (5.11)$$

is only the relevant term to (5.7). This fact can be explicitly checked if we consider the Schwinger-Dyson equation for (5.7) in (4.40) where the kinetic terms of $\hat{\Phi}$ and $\hat{V}$ do not play any roles. Thus as far as (5.7) is concerned, we can further reduce the action from (4.40) to (5.11).

Here we make a remark about a relation between $\sqrt{\det C}$ and the rank of the supermatrix. Suppose we represent the Heisenberg algebra (3.1) by the $N \times N$ matrix, where we take the large-$N$ limit at the end. Then the matrices in the twisted reduced model has rank $\hat{N} = nN$. Of course, as we have seen in subsection 3.3, there is no notion of $n$ and $N$ in the twisted reduced model itself. It is the background $\hat{p}_\mu = \hat{p}_\mu^{(0)} \otimes 1_n$ that brings the notion of the rank of the gauge group $n$ and that of the noncommutative space $N$ in the model. As is well known, from the point of view of the twisted reduced model, $\det C$ is proportional to $\hat{N}$ as

$$\sqrt{\det C} = \frac{\hat{N}}{(2\pi)^{D/2} \Lambda^D}. \quad (5.12)$$

This can be seen by considering the minimal twist configuration for $\hat{p}_\mu$, which is the basic classical solution in the twisted reduced model and satisfies

$$e^{iap^{(i)}_\mu} e^{iap^{(i)}_\mu} = e^{iap^{(i)}_\mu} e^{iap^{(i)}_\mu} e^{-i2\pi \frac{2\pi}{\hat{N}_i}.} \quad (5.13)$$

Here $a = 1/\Lambda$ is the lattice spacing, $i$ ($i = 1, \ldots, D/2$) is the label of the pair of the direction subject to the twist, and $\hat{N}_i$ is the rank of the matrix $\hat{p}^{(i)}_\mu$. Therefore we have

$$a^2B_{\mu\nu} = \frac{2\pi}{\hat{N}_i}, \quad (5.14)$$

which leads to (5.12) by using $\hat{N} = \Pi_{i=1}^{D/2} \hat{N}_i$. Eq.(5.12) can also be understood as follows. In the reduced model, we first fix a UV cutoff $\Lambda$. A matrix with rank $\hat{N}$ describes $\hat{N}$ degrees of
freedom because $\text{tr} \ 1_{\hat{N}} = \hat{N}$. Each degree of freedom has the mass dimension 1 as seen from (3.13) and has a volume $\sim \sqrt{\det B}$ in the momentum space due to (3.1), which effectively gives the IR cutoff. Thus we get

$$\Lambda^D \sim \hat{N} \sqrt{\det B},$$

(5.15)
which is consistent with (5.12). In the large-$\hat{N}$ limit, the volume of each degree of freedom in the momentum space becomes small, and therefore the IR cutoff in the momentum space tends to zero. This agrees with the remark we made in subsection 3.2 that it is the high energy region much larger than $|B|$ that the description by the large-$N$ field theory becomes good.

On the other hand, as we have explained in subsection 3.2, in the low energy region much smaller than $|B|$, the description via the noncommutative field theory becomes good in the sense that it is well approximated by its commutative limit. In this case, the noncommutativity in the coordinate space brings an effective UV cutoff, and it is convenient to consider in the coordinate space. In order to go to the description by the noncommutative field theory, we have taken the background $\hat{p}_\mu = \hat{p}^{(0)}_\mu \otimes 1_n$, and expanded the theory around it. Then our space-time consists of $N$ (not $\hat{N}$) unit cells of volume $\sim \sqrt{\det C}$. Therefore the total volume $V$ is given by

$$V \sim \frac{\hat{N} \sqrt{\det C}}{n}.$$

(5.16)

Turning back to our model (5.11), this observation leads us to define

$$\frac{i(2\pi)^2 \sqrt{\det C}}{8\sqrt{\det \gamma}} = \frac{\hat{N}}{g_m},$$

(5.17)
where we have introduced a formal parameter $g_m$ with the mass dimension 3 on the dimensional grounds, and have used (5.12) because we are now at the standpoint of the matrix model. Various factors such as $\sqrt{\det \gamma}$ have been absorbed in the definition of $g_m$, and (5.11) becomes

$$S_{\text{smm}}^{\text{hol}} = \frac{\hat{N}}{g_m} \text{Str}_{x \otimes \theta} (W(\hat{\Phi})).$$

(5.18)
We can start from this action, and compute the $\gamma \to 0$ limit of the resolvent

$$\frac{g_m}{N} \left\langle \text{Str}_{x \otimes \theta} \left( \frac{1}{z - \Phi} \right) \right\rangle.$$

(5.19)
As a matter of fact, the $g_m$ dependence disappears, if we express the resolvent in terms of $S_i$’s constructed from (5.19), which indicates that the result has no explicit dependence on $C$, $\gamma$ and $\gamma^*$. It can be checked directly by considering the Schwinger-Dyson equation for
(5.19) in the one-supermatrix model (5.18). Therefore, we can take the commutative limit $C \to 0$ of (5.10) to obtain
\[
\frac{g_m}{N} \left\langle \text{Str}_{x^2} \left( \frac{1}{z - \hat{\Phi}} \right) \right\rangle = \frac{1}{64\pi^2} \left\langle \text{tr} \left( \frac{W^\alpha(x,\theta)W_\alpha(x,\theta)}{z - \Phi(x,\theta)} \right) \right\rangle,
\]
where the correlation function in the right-hand side is the one in the original gauge theory (2.1). This argument supports the observation given at the beginning of this section. There, we have noted that the holomorphic quantities without UV divergence in the Dijkgraaf-Vafa theory are not influenced by the bosonic noncommutativity $C^{\mu\nu}$ from the point of view given in subsection 3.2 or, more explicitly, from that of the perturbation theory. Thus we establish the equivalence between the resolvent (5.19) of the one-supermatrix model (5.18) and the correlation function in the right-hand side of eq.(5.20) in the supersymmetric gauge theory (2.1). This is nothing but the Dijkgraaf-Vafa theory, except that we should consider the one-supermatrix model rather than the ordinary Hermitian one-matrix model. Later we will discuss this point in more detail. In fact, it is pointed out in [15] that the effective superpotential of the gauge theory can be computed by a supermatrix model. Note that we have seen the Dijkgraaf-Vafa theory by constructing a direct mapping (5.20) between the correlators of the gauge theory and the supermatrix model, instead of comparing the formal structures of the Schwinger-Dyson equation.

### 5.2 Equivalence of the free energy

In this section we show that in the limit $\gamma, \gamma^* \to 0$, the free energy of the supermatrix model (5.18) becomes the prepotential of the original gauge theory (2.1).

We define the free energy of the supermatrix model (5.18) by
\[
\exp \left( -\frac{\hat{N}^2}{g_m^2} F_m \right) = \int d^{N^2} \hat{\Phi} \exp \left( -\frac{\hat{N}}{g_m} \text{Str}_{x\otimes\theta} (W(\hat{\Phi})) \right).
\]
It is easy to check that $F_m$ is equal to the the holomorphic part of the free energy $F_{smm}$ of the large-$N$ reduced model (4.40)
\[
\exp \left( -\frac{\hat{N}^2}{g_m^2} F_{smm} \right) = \int d^{\hat{N}^2} \hat{\Phi} \int d^{\hat{N}^2} \hat{V} \exp \left( -\frac{\hat{N}}{g_m} \{2\pi i\tau \text{Str}_{x\otimes\theta} (\hat{W}^\alpha \hat{W}_\alpha) + \text{Str}_{x\otimes\theta} (W(\hat{\Phi}))\} \right). \quad (5.22)
\]
Here we have omitted the kinetic term and the anti-holomorphic term from (4.40), because they do not contribute to the holomorphic part of the free energy due to the holomorphy. Then $\hat{\Phi}$ and $\hat{\mathcal{V}}$ are decoupled from each other, and the integration over $\hat{\mathcal{V}}$ can be performed to yield just a constant. Thus we obtain $F_{\text{smm}} = F_m$. Here we make a remark on the decoupling of $\hat{\mathcal{V}}$. In the supermatrix model (4.40) with $\gamma, \gamma^* \neq 0$, the holomorphic part of the free energy $F_{\text{smm}}$ has no UV divergence, even if we turn off the kinetic term. And once we do so, it is evident that $\hat{\mathcal{V}}$ is decoupled from $\hat{\Phi}$. On the other hand, in the $\gamma, \gamma^* \to 0$ limit, we have (4.12) or (4.14), in which the holomorphic part of the free energy becomes UV divergent if we drop the kinetic term, and we should introduce a regularization if we want to do so. In other words, the kinetic term plays the role of the regularization. And in general it is possible that a regularization induces a coupling between $\hat{\Phi}$ and $\hat{\mathcal{V}}$, which is universal in the sense that it does not depend on the detail of the regularization scheme. We can see that this is indeed the case in (5.8), where the operators that consist of $\hat{\Phi}$ are affected by the Konishi anomaly (5.9) in the $\gamma \to 0$ limit. In fact, as shown in (5.1), the left-hand side of (5.9) is finite when $\gamma \neq 0$. However in the $\gamma \to 0$ limit it needs some regularization which is the origin of the noncommutative Konishi anomaly (5.9) as we show in appendix A. This is also the case when we consider correlation functions. When we compute correlation functions of holomorphic quantities such as the resolvent (5.7) in the supermatrix model (4.40), it is sufficient to consider the simplified supermatrix model (5.18). However, when we take $\gamma \to 0$ limit, we should take account of the Konishi anomaly in (4.12) and (4.14). In fact, eq.(5.10) prescribes how it appears in the correlation function in the $\gamma \to 0$ limit in these theories.

It immediately follows from (5.21) and (5.22) that

$$
\frac{\partial F_m}{\partial g_k} = \frac{\partial F_{\text{smm}}}{\partial g_k} = \frac{1}{k + 1} \frac{g_m}{N} \left\langle \text{Str}_{x \otimes \bar{g}} \hat{\Phi}^{k+1} \right\rangle. \quad (5.23)
$$

As shown in section 2, the prepotential $F$ in the original gauge theory (2.1) satisfies

$$
\frac{\partial F}{\partial g_k} \bigg|_{\psi=0} = \frac{1}{k + 1} \frac{1}{64\pi^2} \left\langle \text{tr} \left( W^\alpha W_\alpha \hat{\Phi}^{k+1} \right) \right\rangle. \quad (5.24)
$$

Because in (5.20) we have shown directly the equivalence between the generating functions of (5.23) and (5.24), we find

$$
\frac{\partial F_m}{\partial g_k} = \left. \frac{\partial F}{\partial g_k} \right|_{\psi=0}. \quad (5.25)
$$

More precisely, $F$ is a function of $g_k$ and $S_i$ where $S_i$ is defined by

$$
S_i = \frac{1}{2\pi i} \oint_{C_i} dz \frac{1}{64\pi^2} \left\langle \text{tr} \left( W^\alpha W_\alpha \right) \right\rangle. \quad (5.26)
$$
From (5.20) we find that this quantity is expressed by the matrix model as

\[ S_i = \frac{1}{2\pi i} \oint_{C_i} dz \frac{g_m}{N} \left\langle \text{Str}_x \theta \left( \frac{1}{z - \Phi} \right) \right\rangle = \frac{g_m \hat{N}_i}{N}, \tag{5.27} \]

where \( \hat{N}_i \) is the number of eigenvalues of \( \Phi \) near the \( i^{th} \) critical point. Note that in our supermatrix model, \( \hat{N}_i \) can take negative values, on which we will make some comments in the next subsection. We emphasize that we have shown (5.25) as an identity. \( \mathcal{F} \) and \( F_m \) are the same quantity. The only difference is the way they are represented.

In addition to this correspondence, we have a rather unconventional relation. Because we have derived \( \lim_{\gamma \to 0} S_{\text{smm}} = S_{\text{NC}} \), we can obtain the effective potential, or free energy of the noncommutative gauge theory (4.12) directly from the free energy \( F_{\text{smm}} = F_m \) of the matrix model (4.40) by taking the \( \gamma \to 0 \) limit. Moreover, the effective potential of (4.12) is independent of the bosonic noncommutativity \( C^{\mu\nu} \) as shown at the beginning of this section. It is hence the same as that of the commutative theory (2.1). Therefore we obtain the following relation between the effective potential \( W_{\text{eff}} \) in the original gauge theory (2.1) and the free energy \( F_m \) of the simplified supermatrix model (5.18):

\[ \exp \left( - \int d^4 x d^2 \theta \ W_{\text{eff}} \right) = \exp \left( - \frac{\hat{N}_i^2}{g_m^2} F_m \right). \tag{5.28} \]

Returning to the original noncommutativities (5.17), we find

\[ \int d^4 x d^2 \theta \ W_{\text{eff}} = \left( \frac{i(2\pi)^2 \sqrt{\det C}}{8\sqrt{\det \gamma}} \right)^2 F_m, \tag{5.29} \]

which seems different from the claim of the Dijkgraaf-Vafa theory

\[ \mathcal{F} = F_m. \tag{5.30} \]

Somehow the naive use of the mapping rule gives not (5.30) but (5.29). This suggests that \( W_{\text{eff}} \) is related to \( \mathcal{F} \) in an unconventional way through \( F_m \). It would be interesting to clarify the meaning of this relation.

### 5.3 Supermatrix versus bosonic matrix

In this subsection we discuss how the supermatrix model we have obtained (5.18) is reconciled with the Dijkgraaf-Vafa theory, where the ordinary Hermitian matrix model is considered.
Let us start with a general discussion on supermatrix. A Hermitian supermatrix \( \hat{\Phi} \) is defined to have the following form:

\[
\hat{\Phi} = \begin{pmatrix} B_1 & F_1 \\ F_1^\dagger & B_2 \end{pmatrix},
\]

where \( B_1 \) and \( B_2 \) are \( n \times n \) and \( m \times m \) Hermitian matrices with Grassmann even entries, respectively, and \( F_1 \) is an \( n \times m \) complex matrix with Grassmann odd entries. And the supertrace is defined by

\[
\text{Str}(\hat{\Phi}) = \text{Tr} B_1 - \text{Tr} B_2.
\]

We then consider the Hermitian supermatrix model given by

\[
S = \frac{\hat{N}}{g_m} \text{Str}(W(\hat{\Phi})),
\]

where \( \hat{N} = n - m \). Using the \( U(n|m) \) symmetry, we can diagonalize \( \hat{\Phi} \) as

\[
\hat{\Phi} = U^\dagger \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \\ \mu_1 & & \\ & \ddots & \\ & & \mu_m \end{pmatrix} U.
\]

Then by rewriting (5.33) in terms of the eigenvalues \( \lambda_1, ..., \lambda_n, \mu_1, ..., \mu_m \), we obtain the effective action for eigenvalues

\[
S_{\text{eff}} = \frac{\hat{N}}{g_m} \sum_{i=1}^{n} W(\lambda_i) - \frac{\hat{N}}{g_m} \sum_{j=1}^{m} W(\mu_j) - \sum_{i<j} \log(\lambda_i - \lambda_j)^2 - \sum_{i<j} \log(\mu_i - \mu_j)^2 + \sum_{i,j} \log(\lambda_i - \mu_j)^2.
\]

Note that the sign of the second and last terms are opposite to the ordinary Hermitian one-matrix model. The former is due to the supertrace, while the latter due to fermionic measures. Using the eigenvalue densities for \( \lambda_i \) and \( \mu_j \)

\[
\rho(\lambda) = \frac{1}{n} \sum_{i=1}^{n} \delta(\lambda - \lambda_i), \quad \eta(\mu) = \frac{1}{m} \sum_{j=1}^{m} \delta(\mu - \mu_j),
\]
we can rewrite this action as

\[
S_{\text{eff}} = \frac{n \hat{N}}{g_m} \int d\lambda \, \rho(\lambda) W(\lambda) - \frac{m \hat{N}}{g_m} \int d\mu \, \eta(\mu) W(\mu)
- \frac{n^2}{2} \int d\lambda d\lambda' \rho(\lambda) \rho(\lambda') \log(\lambda - \lambda')^2
- \frac{m^2}{2} \int d\mu d\mu' \eta(\mu) \eta(\mu') \log(\mu - \mu')^2
+ nm \int d\lambda d\mu \rho(\lambda) \eta(\mu) \log(\lambda - \mu)^2.
\] (5.37)

If we introduce the ‘total’ eigenvalue density defined by

\[
\rho_0(\lambda) = \frac{n}{\hat{N}} \rho(\lambda) - \frac{m}{\hat{N}} \eta(\lambda),
\] (5.38)

we can further rewrite \(S_{\text{eff}}\) as

\[
S_{\text{eff}} = \frac{\hat{N}^2}{g_m} \int d\lambda \, \rho_0(\lambda) W(\lambda) - \frac{\hat{N}^2}{2} \int d\lambda d\lambda' \rho_0(\lambda) \rho_0(\lambda') \log(\lambda - \lambda')^2,
\] (5.39)

which is nothing but the effective action of eigenvalues of the ordinary Hermitian one-matrix model with the potential \(W\).

In this sense, we can consider the ordinary Hermitian matrix model (2.10) instead of (5.18), which agrees with the Dijkgraaf-Vafa theory. However, in the Dijkgraaf-Vafa theory one should formally consider the eigenvalues which lie around the top of the potential. From the point of view of the ordinary matrix model, this is nothing but introducing a ‘negative density’ of eigenvalues, which seems unnatural, although it is formally a solution of the Schwinger-Dyson equations. On the other hand, this problem does not exist in the supermatrix model (5.33). Namely, suppose that eigenvalues around a bottom of the potential are regarded as those of \(B_1\) (\(\lambda_i\) in (5.34)), while eigenvalues around a top of the potential as those of \(B_2\) (\(\mu_j\)-type in (5.34)). Then, due to the property of the supertrace, the eigenvalue density for the latter naturally appears in \(S_{\text{eff}}\) with negative sign as we have seen in (5.37). This corresponds to introducing a density with indefinite sign from the viewpoint of the ordinary matrix model as in (5.38). Note here that in the supermatrix model the eigenvalue density \(\eta(\lambda)\) itself introduced in (5.36) is a positive, well-defined function. This tempts us to conclude that the glueball superpotential in the original gauge theory is described by the one-supermatrix model (5.18) instead of (2.10) in a rigorous sense.

However, our supermatrix model (5.18) seems to have the following difficulty. If we represent the noncommutative fermionic space in terms of Pauli matrices as in (4.29), the
first and the second block of $\hat{\Phi}$ should have the same size, that is, $n = m$ in (5.31). Then from (5.27) we find that only the restricted domain where $\sum_i S_i = 0$ can be described in this case. This drawback might originate from the too simple choice of the fermionic noncommutativity (4.27) or the representation (4.29). Another possibility is that some of the eigenvalues might be considered to lie at infinity. Note that in the Dijkgraaf-Vafa theory, the extrema of the potential at infinity play no role if no eigenvalues lie around them. In this sense, there is indeed an ambiguity in the limiting procedure of the potential in the corresponding matrix model. It might be possible that if we take account of the kinetic term and the other terms, we can fix this ambiguity, and some eigenvalues are considered to be around the extrema at infinity. If this is the case, we can realize an arbitrary distribution of eigenvalues as a subset of the total distribution even if the total $S_i$ satisfies $\sum_i S_i = 0$. In any case, it would be important to examine how we should generalize our supermatrix model so that it can describe more generic distributions of eigenvalues. We believe that the supermatrix model has a definite meaning, because it naturally arises in the mapping from the gauge theory to the matrix model.

6 Discussions

Although we have understood essential part of the Dijkgraaf-Vafa theory in terms of the large-$N$ reduced model, some issues still remain to be clarified.

In the Dijkgraaf-Vafa theory, the prepotential plays an important role in constructing the effective potential. In the context of the field theory, it can be understood as a result of the decoupling of the overall $U(1)$ part. However, from the point of view of the large-$N$ reduced model, it seems difficult to separate the $U(1)$ part and find matrix variables that correspond to such fields as

$$w_\alpha = \frac{1}{8\pi} \text{tr} W_\alpha.$$  \hspace{1cm} (6.1)

In this sense, in the matrix model, the symmetry $W_\alpha \mapsto W_\alpha - 8\pi \psi_\alpha$ cannot be expressed manifestly. In fact, in Dijkgraaf-Vafa theory, it is conjectured that

$$\mathcal{F} = F_m,$$  \hspace{1cm} (6.2)

which we have not yet shown in the matrix model context. Although we can prove it for the $g_k$ dependent part (5.25), the reason for the full coincidence is still unclear, and our naive argument gives (5.29) instead of (6.2). It would be an interesting problem to see how these structures of the $\mathcal{N} = 2$ supersymmetry are hidden in the reduced model.
As we commented at the end of subsection 5.3, the apparent drawback of our supermatrix model is that it cannot describe arbitrary eigenvalue distributions. It is natural to expect that if we consider a more general noncommutative superspace we will have a supermatrix model in which the first and the second blocks have different sizes. It would be important to deepen our understanding of the gauge theory on a noncommutative superspace.

As for a generalization of our model, several directions can be anticipated; inclusion of a matter in the fundamental representation [16], other gauge groups, higher supersymmetries, and so on. It is expected that such generalizations help us to understand a generic structure of gauge theories on the noncommutative superspace, or supersymmetric twisted reduced models.

In light of the ordinary twisted reduced model discussed in 3.3, our supermatrix model (4.40) is still unsatisfactory, because it has a dependence on the fermionic background ad $\hat{\pi}_\alpha$. In order to make our model background-independent, it is necessary to introduce a gauge field associated with $\hat{\pi}_\alpha$. It would clarify the meaning of the fermionic noncommutativity $\gamma$ as a regularization, and of the Konishi anomaly on the noncommutative space [16]. It would be also a clue to resolve the problem on the supertrace mentioned above.

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A Noncommutative Konishi anomaly

In this appendix we derive the noncommutative Konishi anomaly (5.9) on the bosonic noncommutative space.

We consider the noncommutative gauge theory (4.12). As explained in section 2, the Konishi anomaly can be regarded as

$$\frac{\delta \Phi^i(x, \theta)}{\delta \Phi^j(x', \theta')} = \delta^i_j \delta^4(x - x') \delta^2(\theta - \theta') = \delta^i_j \langle x, \theta | x', \theta' \rangle,$$

(A.1)
in the limit $x' \to x$ and $\theta' \to \theta$. We should evaluate this in a gauge invariant way, and in order to do this, we use the covariant Laplacian given by

$$\Box_{\text{cov}} = \frac{1}{16} \bar{D}^2 e^{-V} D^2 e^V,$$  \hspace{1cm} (A.2)

where $V$ is the vector superfield. It is easy to check that (A.2) indeed transforms covariantly under the gauge transformation. In the $V \to 0$ limit, it becomes the ordinary Laplacian. (A.2) can be also derived by adding the mass term of the antichiral superfield $\bar{m}/2 \text{tr} \bar{\Phi} \Phi$ to (4.12) and performing the Gaussian integration with respect to $\Phi$ [9]. We evaluate $\langle x, \theta | x, \theta \rangle$ by the heat kernel method as follows:

$$\langle x, \theta | x, \theta \rangle = \lim_{\tau \to 0} \int d^4 k \int d^2 \kappa \langle x, \theta | e^{\tau \Box_{\text{cov}}} | k, \kappa \rangle \langle k, \kappa | x, \theta \rangle$$

$$= \lim_{\tau \to 0} \int \frac{d^4 k}{(2\pi)^4} 4 \int d^2 \kappa \left( \exp_{\tau} \left( \kappa \Box_{\text{cov}} \right) \right) e^{i k x + i \theta \kappa} e^{-i k x - i \theta \kappa},$$

$$= \lim_{\tau \to 0} \int \frac{d^4 k}{(2\pi)^4} 4 \int d^2 \kappa \exp_{\tau} \frac{\tau}{16} \left( -16 k^2 - \kappa^2 \bar{D}^2 - 8 i \kappa W \right.$$  $$- 4 i k_\mu \kappa \sigma^\mu \bar{\theta} \bar{D}^2 + \bar{D}^2 e^{-V} D^2 e^V + 16 k_\mu W \sigma^\mu \bar{\theta} \big).$$ \hspace{1cm} (A.3)

Next we expand the exponential. First we note that if we use $-\kappa^2 \bar{D}^2$ or $-4 i k_\mu \kappa \sigma^\mu \bar{\theta} \bar{D}^2$ in one of the factors in the expansion, it vanishes because at least one $\bar{D}$ acts on the other factors which are chiral. Thus we can drop these terms in the exponential. Due to the integration with respect to $\kappa$, it is sufficient to consider the terms which contain two $\kappa$’s in the expansion of the exponential. However, it is easy to see that if such terms contain $k$, they yield positive power of $\tau$ after the integration with respect to $k$ and hence vanish in the $\tau \to 0$ limit. Therefore, a nonvanishing contribution comes only from

$$\lim_{\tau \to 0} \int \frac{d^4 k}{(2\pi)^4} 4 \int d^2 \kappa \left( -\kappa^2 \frac{\tau^2}{2} \left( -\frac{i}{2} \kappa W \right) \right)^2 = \frac{1}{64 \pi^2} W^\alpha * W_\alpha.$$ \hspace{1cm} (A.4)

Obviously this result is valid also in the commutative limit $C^{\mu\nu} \to 0$. 

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