Entanglement, Identical Particles and the Uncertainty Principle

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Abstract A new uncertainty relation (UR) is obtained for a system of N identical pure entangled particles if we use symmetrized observables when deriving the inequality. This new expression can be written in a form where we identify a term which explicitly shows the quantum correlations among the particles that constitute the system. For the particular cases of two and three particles, making use of the Schwarz inequality, we obtain new lower bounds for the UR that are different from the standard one.

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1 Introduction

One of the most important consequences of Quantum Mechanics (QM) is a theoretical limit imposed on the simultaneous measurement of canonically conjugated variables. This limitation, first derived by Heisenberg,[1] departs from the classical belief that we can, in principle, reduce the uncertainties in the measurements just by building more and more accurate measurement devices.

Many experiments have checked, using a variety of canonically conjugated variables, the validity of the UR. No one has ever verified a violation of the inequality that imposes lower bounds in the product of the dispersions of the canonical variables.

Recently, however, an experiment[2] has suggested that we could have a violation of the UR. A possible explanation for that “violation” is that the Heisenberg uncertainty relation (HUR), as we know it, derived for a single particle, cannot be consistently applied when we deal with more than one particle at the same time.[8] Indeed, the data from the experiment of Ref. [2], and then used to compute the dispersions in position and momentum, come from simultaneous measurements on two identical and entangled particles. In this sense, there is no violation of the HUR since in its original derivation the quantum correlation between identical particles were not taken into account.

In this article we want to formalize this idea by deriving a general UR for a pure quantum state describing N identical and entangled particles, which explicitly shows the correlations among the particles that constitute the system. Here two particles are considered correlated if for some pair of observables the quantum covariance function defined in Eq. (14) is non zero. Since we deal with pure states, any possible correlation is called quantum correlation or equivalently entanglement. Another assumption in the derivation of this general UR is that we are dealing with identical particles. Putting together all these pieces, we arrive at a generalized UR that explains the experimental result of Ref. [2], the main motivation for this paper. We also apply the general UR to the case of two and three identical particles, where, using the Schwarz inequality, we obtain new lower bounds for the UR valid when we deal with identical and entangled particles.

2 Identical and Entangled Particles

2.1 Identical Particles

It is well known that when we deal with a system of N identical particles we should use (anti-) symmetrized wave functions to describe an ensemble of (fermions) bosons. Less known, however, is the fact that we should also use physical observables,[4–5] defined as those that commute with the permutation operators of the system. Mathematically, a physical observable must satisfy the following commutation relation,

$$[\mathcal{O}, \mathcal{P}] = 0,$$  \hspace{1cm} (1)

where $\mathcal{O}$ is an observable and $\mathcal{P}$ is any permutation operator of the system.

It is worth mentioning that physical observables are symmetrical operators. These operators are the only quantities that should be symmetrical to satisfy the symmetrization postulate.[4–5] Manipulating them properly we can arrive at any physical quantity of interest, being it symmetrical or not. Physical quantities, for instance, are real numbers which are not subjected to the symmetrization postulate. Therefore, mean values such as the dispersion in position and momentum for one particle, i.e. $\Delta Q_1$
and $\Delta P_1$, are legitimate real numbers that can be calculated from mathematical manipulations of symmetrical operators.

### 2.2 Pure Entangled Particles

The Hilbert space $\mathcal{H}$ of a system of $N$ particles is the direct product of the Hilbert spaces $\mathcal{H}_i$ associated with each particle individually,

$$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_N = \bigotimes_{i=1}^{N} \mathcal{H}_i .$$

The general state $|\Psi\rangle$ which describes a system of $N$ pure identical and entangled particles is the tensor product of $N$ kets, each one belonging to an individual particle. It means that we have a non-factorizable state$^1$

$$|\Psi\rangle \neq |\psi_1\rangle \otimes \cdots \otimes |\psi_N\rangle ,$$

which implies that Eq. (3) cannot be written as

$$|\Psi\rangle \neq \sum_{i_1} c_{i_1} |u_{i_1}\rangle_1 \otimes \cdots \otimes \sum_{i_N} c_{i_N} |u_{i_N}\rangle_N ,$$

where $c_{i_1}, \ldots, c_{i_N}$ are the expansion coefficients of the non entangled state $|\psi_1\rangle \otimes \cdots \otimes |\psi_N\rangle$ in the base $\{|u_{i_1}\rangle_1, \ldots, |u_{i_N}\rangle_N\}$.

### 3 New Uncertainty Relation

#### 3.1 $N$ Particles

We know that any two operators, $A$ and $B$, satisfy the following inequality,

$$(\Delta A)^2 (\Delta B)^2 \geq \frac{|\langle[A,B]\rangle|^2}{4} ,$$

where

$$\Delta A = \sqrt{\langle \psi | A^2 |\psi\rangle - \langle \psi | A |\psi\rangle^2} ,$$

$$\Delta B = \sqrt{\langle \psi | B^2 |\psi\rangle - \langle \psi | B |\psi\rangle^2} ,$$

are the root mean square deviation (dispersion) of the operators $A$ and $B$, for a given state $|\psi\rangle$, respectively. We now require that when deriving the uncertainty relation of an ensemble of $N$ pure identical and entangled particles we should satisfy the conditions stated above about entangled and identical particles, namely:

(i) Use only physical observables.

(ii) Use non-factorizable states.

With only these two assumptions, which represent mathematically, in the simplest form, the fact that we are dealing with a system of $N$ identical and entangled particles, we shall obtain a new uncertainty relation that generalizes the traditional HUR.

Let us begin the derivation of this generalized uncertainty relation (GUR) by defining the physical observables that we should use in Eq. (6). In the state space $\mathcal{H}_i$ of the particle $i$ we have the usual position and momentum observables, $Q_i$ and $P_i$, respectively. When treating a system of $N$ particles we use the extended observables defined below (from now on, we restrict ourselves, for simplicity, to the one-dimensional case):

$$Q_i = I_i \otimes \cdots \otimes Q_i \otimes \cdots \otimes I_N ,$$

$$P_i = I_i \otimes \cdots \otimes P_i \otimes \cdots \otimes I_N ,$$

where $I_i$ is the identity operator in the state space of particle $i$. It is easily shown that these observables do not commute with the permutation operators defined in the state space of the $N$ particle system. We need, then, to create a pair of physical observables that should be used in the derivation of the uncertainty relation. The simplest pair of physical observables is written as

$$Q = Q_1 + \cdots + Q_N ,$$

$$P = P_1 + \cdots + P_N ,$$

where $Q_i$ and $P_i$ are the extended observables defined in Eqs. (7) and (8). Using these two physical observables in Eq. (6) we get

$$(\Delta Q)^2 (\Delta P)^2 \geq \frac{|\langle Q, P \rangle|^2}{4} .$$

But the commutator of $Q$ and $P$ is

$$[Q, P] = [Q_1, P_1] + \cdots + [Q_N, Q_N] = i N\hbar .$$

Then, Eq. (11) becomes

$$(\Delta Q)^2 (\Delta P)^2 \geq \frac{N^2 \hbar^2}{4} .$$

If we define the quantum covariance function (QCF) for the position and for the momentum $a_8$[8]

$$C_Q(i, j) = \langle Q_i Q_j \rangle - \langle Q_i \rangle \langle Q_j \rangle ,$$

$$C_P(i, j) = \langle P_i P_j \rangle - \langle P_i \rangle \langle P_j \rangle ,$$

Eq. (13) becomes

$$\sum_{i,j=1}^{N} C_Q(i, j) \sum_{i,j=1}^{N} C_P(i, j) \geq \frac{N^2 \hbar^2}{4} .$$

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$^1$Entanglement is a partition dependent concept. For example, in the state $|u_1\rangle_1 \otimes (|u_2\rangle_2 |u_2\rangle_3 + |u_2\rangle_2 |u_1\rangle_3)$ particles 1 and 2 are not entangled while 2 and 3 are. The definition of entanglement used here considers states like this to be entangled.
But since \( C_Q(i,i) = (\Delta Q_i)^2 \) and \( C_P(i,i) = (\Delta P_i)^2 \), we can write Eq. (16) as

\[
\left( \sum_{i=1}^{N} (\Delta Q_i)^2 + \sum_{i \neq j=1}^{N} C_Q(i,j) \right) \times \left( \sum_{i=1}^{N} (\Delta P_i)^2 + \sum_{i \neq j=1}^{N} C_P(i,j) \right) \geq \frac{N^2 \hbar^2}{4}.
\] 

(17)

This is the GUR that we should use when working with a system of \( N \) entangled and identical particles, our main result in this paper. Note that since we are dealing with pure states, entanglement manifests itself by the non-vanishing QCF.\(^7\) We clearly see that in this expression we get two terms for each physical observable. For the physical observable \( Q \), \( \sum_{i=1}^{N} (\Delta Q_i)^2 \) is the sum of the square of the dispersions in position for each particle that constitutes the system. The other one, \( \sum_{i \neq j=1}^{N} C_Q(i,j) \), is the term that shows the quantum correlations among all the particles that belong to the system. If we could factorize the state describing the system this term would be zero and we would recover the HUR. It is worth noting that dealing with mixed states it is possible to have at the same time a non-vanishing QCF and a separable state. Since here we are interested only in pure states, a vanishing QCF always implies separable (non-entangled) states.\(^7\)

Let us show explicitly that we do indeed get the usual HUR if the QCF are all zero. In this case, \( \sum_{i \neq j=1}^{N} C_Q(i,j) = 0 \) and we have \( N \) independent equal particles, namely, \( (\Delta Q_i)^2 = (\Delta Q_j)^2 \) and \( (\Delta P_i)^2 = (\Delta P_j)^2 \) for any \( i \) and \( j \). Inserting these facts into Eq. (17) we readily see that it implies \( \Delta Q_i \Delta P_i \geq \hbar/2 \), for all particles (any \( i \)). This last equation is the usual HUR.

### 3.2 Two Particles

We now restrict the GUR for the case of two particles. In this scenario Eq. (17) becomes

\[
\left( \frac{(\Delta Q_1)^2}{2} + \frac{(\Delta Q_2)^2}{2} + (\langle Q_1 Q_2 \rangle - \langle Q_1 \rangle \langle Q_2 \rangle) \right) \times \left( \frac{(\Delta P_1)^2}{2} + \frac{(\Delta P_2)^2}{2} + (\langle P_1 P_2 \rangle - \langle P_1 \rangle \langle P_2 \rangle) \right) \geq \frac{\hbar^2}{4}.
\] 

(18)

We simplify further Eq. (18) invoking the Schwarz inequality,

\[
\langle \varphi_1 | \varphi_2 \rangle \langle \varphi_2 | \varphi_1 \rangle \geq \langle \varphi_1 | \varphi_2 \rangle \langle \varphi_2 | \varphi_1 \rangle,
\]

(19)

where \( |\varphi_1\rangle \) and \( |\varphi_2\rangle \) are two generic states. Since Eq. (19) is valid for any state, it is valid for the two states below,

\[
|\varphi_1\rangle = (Q_1 \pm Q_2)|\psi\rangle,
\]

(20)

\[
|\varphi_2\rangle = |\psi\rangle,
\]

(21)

where \( |\psi\rangle \) represents the normalized wave function that describes our system of two particles. Hence, applying Eq. (19) for these two states we get

\[
|\langle Q_1 Q_2 \rangle - \langle Q_1 \rangle \langle Q_2 \rangle| \leq \frac{(\Delta Q_1)^2}{2} + \frac{(\Delta Q_2)^2}{2}.
\] 

(22)

The same reasoning can be used for the momentum operator and we get a similar expression,

\[
|\langle P_1 P_2 \rangle - \langle P_1 \rangle \langle P_2 \rangle| \leq \frac{(\Delta P_1)^2}{2} + \frac{(\Delta P_2)^2}{2}.
\] 

(23)

Analyzing Eqs. (22) and (23) we see that the absolute value of the QCF for the position and for the momentum is always less than the mean of the dispersions in position and momentum of the two particles. Hence, we have an upper bound for the QCF. Note that if we are dealing with non-entangled states the left hand side of Eqs. (22) and (23) are always zero. In this situation Eq. (18) is equivalent to the usual HUR and furnishes no new result. In other words, entanglement is essential to obtain non trivial upper bounds for the QCF which imply the new results that follow.

Substituting these upper bounds in Eq. (18) we get

\[
|\langle Q_1 Q_2 \rangle - \langle Q_1 \rangle \langle Q_2 \rangle| \leq \frac{\hbar^2}{4}.
\]

(24)

This last expression should be the correct uncertainty relation when treating an entangled pair of identical particles.

An interesting case arises when we produce an entangled pair of identical particles with the same dispersion in position and in momentum. Using these assumptions, i.e. \( \Delta Q_1 = \Delta Q_2 \) and \( \Delta P_1 = \Delta P_2 \), Eq. (24) becomes

\[
\Delta Q_1 \Delta P_1 \geq \frac{\hbar}{4},
\]

(25)

where \( i = 1, 2 \). Here we see that the GUR can be twice smaller than the traditional HUR.

### 3.3 Three Particles

For three particles Eq. (17) reads

\[
\left( \sum_{i=1}^{3} (\Delta Q_i)^2 + \sum_{i \neq j=1}^{3} C_Q(i,j) \right) \times \left( \sum_{i=1}^{3} (\Delta P_i)^2 + \sum_{i \neq j=1}^{3} C_P(i,j) \right) \geq \frac{9 \hbar^2}{4}.
\] 

(26)

Again we can apply the Schwarz inequality to simplify the above expression. We now substitute the following two states in Eq. (19),

\[
|\varphi_1\rangle = (a_1 Q_1 + a_2 Q_2 + a_3 Q_3)|\psi\rangle,
\]

(27)

\[
|\varphi_2\rangle = |\psi\rangle,
\]

(28)

where \( a_i = \pm 1, i = 1, 2, 3 \).

Thus, the Schwarz inequality becomes

\[
((a_1 Q_1 + a_2 Q_2 + a_3 Q_3)^2) \geq (a_1 Q_1 + a_2 Q_2 + a_3 Q_3)^2.
\]

(29)

Manipulating Eq. (29) and an equivalent expression for the momentum we get the following inequality (see Ap-
pendix A):

\[
\left( \sum_{i=1}^{3} (\Delta Q_i)^2 \right) \left( \sum_{i=1}^{3} (\Delta P_i)^2 \right) \geq \frac{9\hbar^2}{64}.
\]

This is the GUR that we should use when working with three identical entangled particles. An interesting case arises when all the three identical and entangled particles are prepared with the same dispersions in position and in momentum. Therefore, if

\[
\begin{align*}
\Delta Q_1 &= \Delta Q_2 = \Delta Q_3, \\
\Delta P_1 &= \Delta P_2 = \Delta P_3,
\end{align*}
\]

Eq. (30) becomes

\[
\Delta Q_1 \Delta P_1 \geq \frac{\hbar}{8},
\]

where \(i = 1, 2, 3\).

In this case the departure from HUR is more drastic. We have a lower bound in the product of the dispersions four times smaller than that furnished by the HUR. Here, the conditions given in Eqs. (31) and (32) are also achieved via a triple coincidence measurement on a maximally entangled tripartite state.

### 4 The Kim and Shih’s Experiment

The experiment of Ref. [2] dealt with a quantum system satisfying the two assumptions we used to arrive at the GUR. Indeed, in Ref. [2] identical and entangled pairs of photons were produced by spontaneous parametric down conversion (SPDC) with the subsequent measurement of their position and momentum in such a way that the quantum correlation between them was not destroyed, which was achieved by the use of the “ghost image” experimental technique.\(^9\) Also, coincidence circuits were employed to guarantee that the detection of two photons corresponded to the two entangled photons produced in the SPDC process and only events (two photon detection) within a very narrow window of time (coincidence) were selected as valid events.

Note that without the coincidence measurements, we would have spurious events, those in which we are detecting pairs of photons that are not correlated. In this case, i.e., without post-selecting the valid events, no correlation exists between the photons and we would get the usual HUR. Also, if we let both particles/photons interact with something else in an uncontrollable way we destroy the entanglement/correlation between them (or average the correlation out) and we should also recover the HUR.

Our goal now is to analyze the experiment of Ref. [2] in the light of the new GUR here presented. Looking at Fig. 5 of Ref. [2], we note two sets of data/courses. The wider curve is the standard single-slit diffraction pattern for a single particle, which can be employed to infer the particle’s dispersion/uncertainty in momentum when it is localized to a precision given by the width of the slit (the dispersion/uncertainty in position). This curve was obtained sending pairs of photons, created as described above, to physical slits of width 0.16 mm. A detector \(D_1\), after the slit receiving photon 1, was fixed at the position zero (center of the slit) while detector \(D_2\), after the slit receiving photon 2, was scanned along a direction perpendicular to the momentum of the incoming photon. The data was obtained only when coincidence events occurred (simultaneous detection of photons at \(D_1\) and \(D_2\)) and they can be used to infer the dispersion/uncertainty in the momentum perpendicular to the direction of incidence of photon 2. Note that in this case both photons interact in an uncontrollable way with their slits, destroying any possible quantum correlation they might have had prior to entering the slits. That is why the usual HUR applies and, according to the authors of Ref. [2], their data is compatible with the minimum value of the HUR, i.e., \(\Delta Q_2 \Delta P_2 = \hbar/2\).

The other curve, the narrower one, was obtained in such a way that all the conditions listed here to the validity of the GUR were satisfied. Indeed, we have identical and entangled particles and, most importantly, a situation where the quantum correlation between those photons are not destroyed. This is achieved by sending only one of the photons, photon 1, to a physical slit. Photon number 2, on the other hand, is sent to a virtual slit, what the authors of Ref. [2] called the “ghost image” of the physical slit of photon 1. This virtual slit is achieved by properly inserting a lens in the optical path of the photons in such a way that it creates an image of the physical slit of photon 1 exactly in the position where the physical slit of photon 2 was located when the data of the wider curve was obtained. According to the authors of Ref. [2], in this arrangement both photons uncertainty in position are still given by the width of the slit, even though photon 2 does not pass through a physical slit. We can see that this is true because the photons emerging the SPDC process are entangled in momentum.\(^2\) Moreover, since photon 2 does not interact with a physical slit, it is still correlated to photon 1 and in this case we should apply the GUR and not the HUR to analyze the product of the dispersions/uncertainties in position and momentum of photon 2. In other words, the GUR predicts that \(\Delta Q_2 \Delta P_2 \geq \hbar/4\).

We have used the tildes over the quantum uncertainties to differentiate the dispersions associated to the narrower curve from the wider one.

Returning to the data given by Fig. 5 of Ref. [2], we can estimate \(\Delta \tilde{Q}_2 \Delta \tilde{P}_2\) for the actual experiment if we note that the uncertainty in momentum is proportional to the spread of the diffraction pattern.\(^10\) Following Ref. [10], we set the uncertainty in momentum as proportional to the position of the first minimum of the sinc-square functions fitting the data. For the wider curve,
the first minimum is located about 2.15 mm while for
the narrower one it is roughly at 1.25 mm. Therefore,
\( \Delta \tilde{P}_2/\Delta P_2 \approx 1.25/2.15 \). Now, noting that for the wider
curve we have \( \Delta Q_2 \Delta P_2 = \hbar/2 \) and that \( \Delta Q_2 = \Delta \tilde{Q}_2 \),
we immediately get \( \Delta Q_2 \Delta \tilde{P}_2 \approx 0.29 \hbar \). This number is
slightly higher than \( \hbar/4 \approx 0.25 \hbar \), the minimum value pred-
icted by the GUR, and clearly lower than the minimum
value allowed by the HUR, i.e., \( 0.5 \hbar \).

Finally, it is worth mentioning that the GUR pre-
sented here also predicts that for the first experiment all
QCF’s should be zero, i.e., \( \langle Q_1 Q_2 \rangle - \langle Q_1 \rangle \langle Q_2 \rangle = 0 \) and
\( \langle P_1 P_2 \rangle - \langle P_1 \rangle \langle P_2 \rangle = 0 \). For the second experi-
ment, however, at least the QCF for the momenta must be non-zero,
since we have entanglement between the momenta of
the photons. Also, the values of all QCF’s must be compat-
ible with Eq. (18). Unfortunately, with the data provided
in Ref. [2] we cannot compute those functions.

5 Discussion

The above results deserve careful attention. They can-
not be considered a violation of the Heisenberg uncer-
tainty principle. We still cannot measure simultaneously
the position and momentum (or any pair of canonically
conjugated observables) of a particle. What we have ac-
tually shown was that if we prepare a system of identical
and entangled particles and use coincidence circuits to
detect all the constituents of the system in a way that do not
destroy the correlation between the particles, we can ob-
tain \( \Delta Q_1 \Delta P_1 < \hbar/2 \) for the product of the dispersions in
position and in momentum for at least one of the particles.
We should keep in mind that the GUR, which implies for
some special cases \( \Delta Q_1 \Delta P_1 < \hbar/2 \), was deduced in the
framework of QM, with no additional postulates.

In other words, QM does not forbid \( \Delta Q_1 \Delta P_1 < \hbar/2 \)
for the cases of identical and entangled particles in coin-
cidence measurements. All the well known previous de-
ductions of UR’s were made considering only an isolated
particle. What we have reinforced here is that things are
not so simple and straightforward when we have two or
more particles that are quantum correlated. Our results
show that what is believed to be valid for one particle can-
not be always trivially extended to systems of more than
one particle in some special experimental conditions.

It is also worth mentioning that possibly there exists
one experimental confirmation of the GUR by Kim and
Shih,[2] which motivated this article. We wanted to find
a theoretical explanation to the results of Ref. [2] without
invoking any particularity of the experimental setup but
the fact that we have a pure bipartite entangled system of
two identical photons and coincidence measurements that
do not destroy or average out the quantum correlation
between the entangled photons.

Finally, we insist that new experiments with more than
two particles or with other kinds of canonically con-
jugated operators are most welcome.

6 Conclusion

In this article we tried to show that when dealing with
\( N \) identical and entangled particles we should use a new
UR. This new UR is a natural consequence of two rea-
sonable assumptions we make when treating such system,
that is: use only non-factorizable states and physical ob-
servables to deduce the UR.

The first assumption is a definition of entanglement
for pure states\[4,6\] and the second one appears because we are
dealing with identical particles.[4–5]

In some special conditions we have shown that for two
and three particles it is possible to obtain UR’s that have
different lower bounds than that furnished by the tradi-
tional HUR, reinforcing that in QM the physics of systems
of more than one particle is not a trivial extension of the
physics of single particle systems.

Appendix A: Proof of Eq. (30)

We now present a detailed deduction of Eq. (30). First,
we expand Eq. (29) using the fact that \( \Delta (Q_i^2) = 1 \). This leads to
\[
\sum_{i=1}^{3} \langle Q_i^2 \rangle + \sum_{i \neq j=1}^{3} a_i a_j \langle Q_i Q_j \rangle
\geq \sum_{i=1}^{3} \langle Q_i \rangle^2 + \sum_{i \neq j=1}^{3} a_i a_j \langle Q_i \rangle \langle Q_j \rangle. \tag{A1}
\]
Using the fact that \( (\Delta Q_i)^2 = (Q_i^2) - (Q_i) \), Eq. (A1) re-
duces to
\[
\sum_{i=1}^{3} (\Delta Q_i)^2 \geq - \sum_{i \neq j=1}^{3} a_i a_j C_Q(i,j). \tag{A2}
\]
For \( a_1 = 1 \) and \( a_2 = a_3 = -1 \) Eq. (A2) is
\[
\sum_{i=1}^{3} (\Delta Q_i)^2 \geq 2(C_Q(1,2) + C_Q(1,3) - C_Q(2,3)). \tag{A3}
\]
For \( a_2 = 1 \) and \( a_1 = a_3 = -1 \) Eq. (A2) is
\[
\sum_{i=1}^{3} (\Delta Q_i)^2 \geq 2(-C_Q(1,2) - C_Q(1,3) + C_Q(2,3)). \tag{A4}
\]
For \( a_3 = 1 \) and \( a_1 = a_2 = -1 \) Eq. (A2) is
\[
\sum_{i=1}^{3} (\Delta Q_i)^2 \geq 2(-C_Q(1,2) + C_Q(1,3) + C_Q(2,3)). \tag{A5}
\]
Adding Eqs. (A3), (A4), and (A5) we get
\[
2(C_Q(1,2) + C_Q(1,3) + C_Q(2,3)) \leq 3 \sum_{i=1}^{3} (\Delta Q_i)^2. \tag{A6}
\]
Now, for $a_1 = a_2 = a_3 = 1$, Eq. (A2) is

$$2(C_Q(1, 2) + C_Q(1, 3) + C_Q(2, 3)) \geq -\sum_{i=1}^{3} (\Delta Q_i)^2. \quad (A7)$$

Combining Eqs. (A6) and (A7) we get the following expression

$$-\sum_{i=1}^{3} (\Delta Q_i)^2 \leq \sum_{i \neq j=1}^{3} C_Q(i, j) \leq 3 \sum_{i=1}^{3} (\Delta Q_i)^2. \quad (A8)$$

A similar procedure leads to an equivalent expression for the momentum:

$$-\sum_{i=1}^{3} (\Delta P_i)^2 \leq \sum_{i \neq j=1}^{3} C_P(i, j) \leq 3 \sum_{i=1}^{3} (\Delta P_i)^2. \quad (A9)$$

These two inequalities furnish upper bounds for the QCF’s that appear in the GUR for three identical particles. Substituting the upper bounds of Eqs. (A8) and (A9) in Eq. (26) we arrive at Eq. (30),

$$\left( \sum_{i=1}^{3} (\Delta Q_i)^2 \right) \left( \sum_{i=1}^{3} (\Delta P_i)^2 \right) \geq \frac{9\hbar^2}{64}. \quad (A10)$$

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