On the rainbow connection number of triangle-net graph

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Abstract. Let $G$ be an arbitrary non-trivial connected graph. For every two vertices $u$ and $v$ in $G$, a $(u,v)$-path in $G$ is called a rainbow $(u,v)$-path if all edges are colored differently. Next, a rainbow $(u,v)$-geodesic in $G$ is a rainbow $(u,v)$-path of length $d(u,v)$. Graph $G$ is rainbow connected if for every two vertices $u, v \in V(G)$, there exists a rainbow $(u,v)$-path. If there exists a rainbow $(u,v)$-geodesic in $G$ for every two vertices $u, v \in V(G)$ then $G$ is strongly rainbow connected. The rainbow connection number $rc(G)$ is the minimum number of colors needed to make $G$ rainbow connected, while the strong rainbow connection number $src(G)$ is the minimum number of colors needed to make $G$ strongly rainbow connected. Let $T_{rn}$ be the generalized triangle-ladder graph for $n \geq 2$. The triangle-net graph, denoted by $H = (T_{rn})^{m}$, is constructed by taking $m$ homogeneous generalized triangle-ladder graphs and identifying their terminal vertices, for $m \geq 2$. This paper determined the rainbow connection number of the triangle-net graph and the upper bound of the strong rainbow connection number of the graph.

1. Introduction

The concept of rainbow connection was introduced by Chartrand et al. [1]. For an arbitrary nontrivial connected graph $G = (V,E)$ and some positive integer $k$, define $c : E(G) \to \{1,2,\cdots,k\}$ as an edge-coloring of $G$, where the adjacent edges can be colored the same. Let $u$ and $v$ be two vertices in $G$. A $(u,v)$-path in $G$ is called a rainbow $(u,v)$-path if all edges are colored differently. Graph $G$ is rainbow connected if for every two vertices $u, v \in V(G)$, there exists a rainbow $(u,v)$-path. In this case, the coloring $c$ is a rainbow coloring. If $c$ is a rainbow coloring with $k$ colors, then $c$ is a rainbow $k$-coloring. The rainbow connection number $rc(G)$ is the smallest $k$ such that $G$ has a rainbow $k$-coloring. If the diameter of $G$, $diam(G) = \max\{d(u,v)|u,v \in V(G)\}$, where $d(u,v)$ denotes the distance between $u$ and $v$, then it is clear that $diam(G) \leq rc(G)$.

Let $c$ be an edge-coloring of a nontrivial connected graph $G$. For two vertices $u, v \in V(G)$, a rainbow $(u,v)$-geodesic in $G$ is a rainbow $(u,v)$-path of length $d(u,v)$. Graph $G$ is strongly rainbow connected if for every two vertices $u, v \in V(G)$, there exists a rainbow $(u,v)$-geodesic. In this case, the coloring $c$ is a strong rainbow coloring of $G$. If $c$ is a strong rainbow coloring with $k$ colors, then $c$ is a strong rainbow $k$-coloring. The strong rainbow connection number $src(G)$ is the smallest $k$ such that $G$ has a strong rainbow $k$-coloring. Therefore, it is clear that $rc(G) \leq src(G)$.

1 corresponding author
Let \( q \) be the size of a nontrivial connected graph \( G \). If every edge of \( G \) is colored differently, then the coloring is a rainbow coloring and also strong rainbow coloring. Therefore, the following inequality holds.

\[
diam(G) \leq rc(G) \leq src(G) \leq q.
\]

In [1], Chartrand et al. stated some characterizations of connected graphs \( G \) with some specified values of \( rc(G) = 1, 2, \) and \( q \). In the same paper, they gave the rainbow connection number and strong rainbow connection for path, cycle, wheel, complete graph, complete bipartite graph, and complete multipartite graph. Let \( L_n \) be the triangle ladder graph for \( n \geq 2 \). The diamond graph \( Br_n \), for \( n \geq 2 \), is constructed by adding one vertex, denoted by \( v \), and adding \( n \) edges from \( v \) to the vertices in the first layer of \( L_n \). Shulhany and Salman [7] gave the exact number of the rainbow and strong rainbow connection number of the diamond graph \( Br_n \).

Next, in [9], Yulianti et al. constructed the generalized triangle ladder graph, denoted by \( Tr_n \) for \( n \geq 2 \), by adding some triangle ladder graph to the upper layer of the diamond graph. They determined the rainbow and the strong rainbow connection number of the generalized triangle ladder graph \( Tr_n \) for \( n \geq 2 \).

There are some nice results regarding the rainbow and strong rainbow connection number of new graphs coming from some operation of graphs. The rainbow and strong rainbow connection numbers for Cartesian product and corona operations of several connected graphs were summarized in a survey by Li and Sun in [5]. Next, in [8], Sy et al. determined the rainbow connection number and strong rainbow connection number of fan and sun graphs. Following this result, Rao and Murali [6] considered the rainbow connection numbers of the line, middle and total graph of the sunlet graph, while the strong rainbow connection number of the graphs were given by Zhao et al. [10]. Fitriani and Salman [4] gave the upperbound and the lowerbound for the rainbow and strong rainbow connection numbers of amalgamation of certain graphs. In this paper, we determined the exact value of the rainbow connection number of the triangle-net graph, i.e the amalgamation of homogeneous generalized triangle-ladder graphs, denoted by \( H = (Tr_n)^m \), for \( m, n \geq 2 \), and the upperbound of the strong rainbow connection number of the graph.

2. The Triangle-net Graph

The generalized triangle-ladder graph \( Tr_n \), for \( n \geq 2 \), was first introduced by Yulianti et al. in [9]. The graph is constructed from the triangle ladder graph \( L_n \) and diamond graph \( Br_n \) as given in Figure 1. The vertex set and the edge set of \( Tr_n \) are as follows.

\[
V(Tr_n) = \{v\} \cup \{v_{a,j} \mid 1 \leq a \leq n, 1 \leq j \leq n - a\},
\]

\[
E(Tr_n) = \{v_{1,a}\} \cup \{v_{k,l}v_{k,l+1} \mid 1 \leq k \leq n - 1, 1 \leq l \leq n - k\} \tag{1}
\]

\[
\cup \{v_{l,k}v_{l+1,k} \mid 1 \leq k \leq n - 1, 1 \leq l \leq n - k\}
\]

\[
\cup \{v_{t,s}v_{t+1,s-1} \mid 2 \leq s \leq n, 1 \leq t \leq s - 1\}.
\]

In [9], Yulianti et al. determined the rainbow and the strong rainbow connection number of \( Tr_n \) in the following theorem.

**Theorem 1** [9] Let \( Tr_n \) be a generalized triangle-ladder graph and \( n \geq 2 \). Then

\[
diam(Tr_n) = rc(Tr_n) = src(Tr_n) = n.
\]

The definition of the amalgamation of some arbitrary connected graphs is given as follows. Other definition and terminologies of graphs are taken from Diestel [3].
Figure 1. The triangle ladder graph $L_n$, the diamond graph $Br_n$ and the generalized triangle ladder graph $Tr_{n,i}$, for $n \geq 2$

**Definition 2** [2] For some positive integer $m \geq 2$, let $\{G_i \mid 1 \leq i \leq m\}$ be the collection of some nontrivial connected graphs. Let $x_i$ be the fixed vertex in $G_i$, called the terminal vertex of $G_i$. The amalgamation of $\{G_i \mid 1 \leq i \leq m\}$, denoted by $Amal(G_i, x_i \mid 1 \leq i \leq m)$ is a graph formed by taking all $G_i$ and identifying their terminals. If $G_1 = G_2 = \cdots = G_m = G$, for some connected graph $G$, then the operation of amalgamation is called the amalgamation of homogeneous graphs.

Next, Fitriani and Salman [4] determined the lower bound and upper bound for rainbow connection number of the amalgamation of arbitrary connected graphs in the following theorem.

**Theorem 3** [4] For some positive integer $m \geq 2$, let $\{G_1, G_2, \cdots, G_m\}$ be the collection of nontrivial arbitrary connected graphs, and each $G_i$ has a terminal vertex $x_i$. If $H = Amal(G_i, x_i \mid 1 \leq i \leq m)$ then

$$diam(H) \leq rc(H) \leq \sum_{i=1}^{m} rc(G_i).$$

Using Definition 2, the triangle-net graph, i.e the amalgamation of homogeneous generalized triangle-ladder graph is constructed. Similar with (1), define the vertex and edge sets of the $i^{th}$ $Tr_{n,i}$, denoted by $Tr_{n,i}$ for $1 \leq i \leq m$, as follows.

$$V(Tr_{n,i}) = \{v_i \cup \{v_{i,a,j} \mid 1 \leq a \leq n, 1 \leq j \leq n-a\},$$

$$E(Tr_{n,i}) = \{v_i, v_{i,a} \} \cup \{v_{i,k,l}v_{i,k,l+1} \mid 1 \leq k \leq n-1, 1 \leq l \leq n-k\}$$

$$\cup \{v_{i,l,k}v_{i,l+1,k} \mid 1 \leq k \leq n-1, 1 \leq l \leq n-k\}$$

$$\cup \{v_{i,s,l}v_{i,t+1,s-1} \mid 2 \leq s \leq n, 1 \leq t \leq s-1\}. \quad (2)$$

Let $v_i$ be the terminal vertex of $Tr_{n,i}$ for $1 \leq i \leq m$, and denote the identified vertex as $v$. Then the triangle-net graph, denoted by $H = (Tr_{n})^m$ for $m, n \geq 2$, has vertex set and edge set as follows.

$$V(H) = \{v \cup \{v_{i,a,j} \mid 1 \leq i \leq m, 1 \leq a \leq n, 1 \leq j \leq n-a\},$$

$$E(H) = \{vv_{i,1,a} \} \cup \{v_{i,k,l}v_{i,k,l+1} \mid 1 \leq i \leq m, 1 \leq k \leq n-1, 1 \leq l \leq n-k\}$$

$$\cup \{v_{i,l,k}v_{i,l+1,k} \mid 1 \leq i \leq m, 1 \leq k \leq n-1, 1 \leq l \leq n-k\}$$

$$\cup \{v_{i,s,l}v_{i,t+1,s-1} \mid 1 \leq i \leq m, 2 \leq s \leq n, 1 \leq t \leq s-1\}. \quad (3)$$

Graph $H$ is given in Figure 2. It is easy to see that $diam(H) = 2n$ for $n \geq 2$. 

\[\text{Figure 2. Graph representation of} \ Tr_{n,i} \text{and} \ Tr_{n}\]
3. The Rainbow Connection Number of Triangle-net Graph

Theorem 4 gives the rainbow connection number of $H$, while Theorem 5 states the upper bound of strong rainbow connection number of $H$.

**Theorem 4** For some positive integers $m, n \geq 2$, let $H = (Tr_n)^m$ for $m, n \geq 2$ be the triangle-net graph. The rainbow connection number of $H$ is $rc(H) = 2n$.

**Proof.** Let $m$ and $n$ be some positive integers, $m, n \geq 2$. The vertex set and edge set of $H$ were defined in (3). Because $\text{diam}(H) = 2n$, then from Theorem 3, it is clear that $rc(H) \geq \text{diam}(H) = 2n$.

Next, it will be shown that $rc(H) \leq 2n$. Define an edge-coloring $c : E(H) \rightarrow \{1, 2, \cdots, 2n\}$ as follows.

$$
c(e) = \begin{cases} 
1, & e = vv_i,1, \text{ for } 1 \leq i \leq m, \\
2s, & e = vv_i,1,s, \text{ for } 1 \leq i \leq m, 2 \leq s \leq n, \\
k + 1, & e = v_i k, v_{i,k,l+1}, \text{ for } 1 \leq i \leq m, 1 \leq k \leq n - 1, 1 \leq l \leq n - k, \\
k + 2l, & e = v_i k, l+1 v_{i,k,l+1,l}, \text{ for } 1 \leq i \leq m, 1 \leq k \leq n - 1, 1 \leq l \leq n - k, \\
l + k, & e = v_i l, v_{i,l,k+1,k}, \text{ for } 1 \leq i \leq m, 1 \leq k \leq n - 1, 1 \leq l \leq n - k.
\end{cases}
$$

It will be shown that for every two vertices $x$ and $y$ in $H$, there is some rainbow $(x, y)$-path between them. First, for any two vertices $x, y \in V(H)$, if $xy \in E(H)$, then there exists a rainbow $(x, y)$-path in $H$. Next, let $d(x, y) \geq 2$. Let $p, q, r, s \in \{1, 2, \cdots, n\}$ and $i, l \in \{1, 2, \cdots, m\}$. Consider the following cases.

1. If $x = v$ and $y = v_{i,r,s}$, then the rainbow $(x, y)$-path is

$$
x = v, v_{i,1,r+s-1}, v_{i,2,r+s-2}, v_{i,3,r+s-3}, \cdots, v_{i,r-1,s+1}, v_{i,r,s} = y.
$$

2. If $x = v_{i,p,q}$ and $y = v_{i,r,s}$, for $i = l$ and $p \leq r$, then consider the following subcases.
Then, consider the following cases.

(a) If \( p = r \) and \( q < s \), then the rainbow \((x, y)\)-path is

\[
x = v_{i,p,q}, v_{i,p,q+1}, v_{i,p,q+2}, \ldots, v_{i,p,s-1}, v_{i,p,s} = y,
\]

(b) If \( p < r \) and \( p + q < r + s \), then the rainbow \((x, y)\)-path is

\[
x = v_{i,p,q}, v_{i,p,q+1}, \ldots, v_{i,p,t}, v_{i,p+1,t-1}, v_{i,p+2,t-2}, \ldots, v_{i,r,s} = y,
\]

where \( t = r + s - p \).

(c) If \( p < r \) and \( p + q > r + s \), then the rainbow \((x, y)\)-path is

\[
x = v_{i,p,q}, v_{i,p+1,q-1}, \ldots, v_{i,r,h}, v_{i,r,h-1}, v_{i,r,h-2}, \ldots, v_{i,r,s} = y,
\]

where \( h = p + q - r \).

(d) If \( p < r \) and \( p + q = r + s \), then the rainbow \((x, y)\)-path is

\[
x = v_{i,p,q}, v_{i,p+1,q-1}, \ldots, v_{i,r,s} = y.
\]

(3) If \( x = v_{i,p,q} \) and \( y = v_{i,r,s} \) for \( i \leq l \), then consider the following subcases.

(a) If \( p + q \leq r + s \), then the rainbow \((x, y)\)-path is

\[
x = v_{i,p,q}, v_{i,p,q-1}, \ldots, v_{i,p,1}, v_{i,p-1,1}, v_{i,p-2,1}, \ldots, v_{i,1,1}, v_{v,1,1,n}, v_{v,2,n-1},
\]

\[
\ldots, v_{1,r,n+1-r}, v_{l,r,n-r}, v_{l,r,n-r-1}, v_{l,r,n-2-r}, v_{l,r,s} = y.
\]

(b) If \( p + q > r + s \), then the rainbow \((x, y)\)-path is

\[
x = v_{i,p,q}, v_{i,p,q+1}, \ldots, v_{i,p,n-(p-1)}, v_{i,p-1,n-(p-2)}, \ldots, v_{i,1,n}, v_{v,1,1,n}, v_{v,2,1},
\]

\[
\ldots, v_{l,r,1}, v_{l,r,2}, \ldots, v_{l,r,s} = y.
\]

Because there exists some rainbow \((x, y)\)-path for every \( x, y \in V(H) \) then \( c \) is the rainbow 2\( n \)-coloring of \( H \). Therefore, \( rc(H) \leq 2n \).

The upper bound for strong rainbow connection number of \( H \) is given in the following theorem.

**Theorem 5** For some positive integers \( m, n \geq 2 \), let \( H = (Tr_n)^m \) be the triangle-net graph for \( m, n \geq 2 \). The upper bound of the strong rainbow connection number of \( H \) is

\[
src(H) \leq \left[ \frac{n}{3} \right] m + 2(n - 1).
\]

**Proof.** Define an edge-coloring \( c_1 : E(H) \to \{1, 2, \ldots, \left[ \frac{n}{3} \right] m + 2(n - 1)\} \) as follows.

\[
c_1(e) = \begin{cases} 
2n - 2 + \left[ \frac{n}{3} \right] + \left[ \frac{n}{3} \right] (i - 1), & e = v_{v,1,a}, \text{ for } 1 \leq i \leq m, 1 \leq a \leq n, \\
p + 1, & e = v_{v,k,p}v_{k,p+1}, \text{ for } 1 \leq i \leq m, 1 \leq p \leq n - i, \\
n + 1 - t, & e = v_{l,s,t}v_{i,s-1,t+1}, \text{ for } 1 \leq i \leq m, 2 \leq s \leq n, \\
k + l - 1, & e = v_{l,t,k}v_{l,t+1,k}, \text{ for } 1 \leq i \leq m, 1 \leq k \leq n - 1, \\
1 \leq l \leq n - k. 
\end{cases}
\]

Let \( x = v_{i,p,q} \) and \( y = v_{i,r,s} \) for \( p, q, r, s \in \{1, 2, \ldots, n\} \) and \( i, l \in \{1, 2, \ldots, m\} \). Let \( d(x, y) \geq 2 \).

Then, consider the following cases.
(1) If \( x = v \) and \( y = v_{i,p,q} \), then there exists a rainbow geodesic \((x, y)\)-path

\[
x = v, v_{i,1,q}, v_{i,2,q}, \ldots, v_{i,p,q} = y.
\]

(2) If \( x = v_{i,l,q} \) and \( y = v_{i,l,q+3} \) for \( 1 \leq q \leq n-3 \), then there exists a rainbow geodesic \((x, y)\)-path

\[
x = v_{i,l,q}, v, v_{i,l,q+3} = y.
\]

(3) If \( x = v_{i,p,q} \) and \( y = v_{l,r,s} \) for \( i \neq l \), then there exists a rainbow geodesic \((x, y)\)-path

\[
x = v_{i,p,q}, v_{i,p-1,q}, \ldots, v_{i,1,q}, v, v_{l,1,s+r-1}, v_{l,2,s+r-2}, \ldots, v_{l,r,s} = y.
\]

Because there exists some rainbow geodesic \((x, y)\)-path for every \( x, y \in V(H) \) then \( c_1 \) is the strong rainbow \( 2n \)-coloring of \( H \). Therefore,

\[
src(H) \leq \lceil \frac{n}{3} \rceil m + 2(n - 1).
\]

The edge-coloring for \( F = (Tr_4)^3 \) such that \( rc(F) = 8 \) and \( src(F) \leq 12 \) are given in Figure 3.

![Figure 3. The \( rc(F) \) and \( src(F) \) for \( F = (Tr_4)^3 \)](image)

4. Conclusion
Given \( m \) generalized triangle-ladder graphs \( Tr_n \), for \( m, n \geq 2 \), the Triangle-net graph, denoted by \((Tr_n)^m\) is constructed by identifying the terminal vertices \( v_i \) of the \( i^{th} \) generalized triangle-ladder graph \( Tr_n \), for \( 1 \leq i \leq m \), and denote the new vertices as \( v \). In this paper, it is determined that the rainbow connection number of the triangle-net graph \((Tr_n)^m\) for \( m, n \geq 2 \) is \( 2n \). Moreover, we stated that the upperbound of the strong rainbow connection number of the triangle-net graph is \( \lceil \frac{n}{3} \rceil m + 2(n - 1) \). Further research will be focused on finding the lowerbound of the strong rainbow connection number of the graph.

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