Edge States in Gravity and Black Hole Physics

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Abstract

We show in the context of Einstein gravity that the removal of a spatial region leads to the appearance of an infinite set of observables and their associated edge states localized at its boundary. Such a boundary occurs in certain approaches to the physics of black holes like the one based on the membrane paradigm. The edge states can contribute to black hole entropy in these models. A “complementarity principle” is also shown to emerge whereby certain “edge” observables are accessible only to certain observers. The physical significance of edge observables and their states is discussed using their similarities to the corresponding quantities in the quantum Hall effect. The coupling of the edge states to the bulk gravitational field is demonstrated in the context of (2+1) dimensional gravity.
I. INTRODUCTION

Since the discovery of black hole radiation by Hawking [1,2], the eventual fate of information falling into a black hole has remained an unsolved puzzle. The different scenarios that have been proposed can be summarized in the following two classes:

(1) Information going into a black hole is irretrievably lost so that unitary quantum theory breaks down [3].

(2) Information that was thought to be “lost” reappears in some form thus saving conventional quantum physics [4].

Quantum black hole physics is usually studied in a semiclassical approximation. Typically it is shown that processes like Hawking radiation and the scattering of matter fields by black holes are governed by physics occurring close to the event horizon [5].

There have been several interesting proposals [5–7] to treat the black hole horizon as a membrane with dynamical degrees of freedom attached to it. It is found that for any external stationary observer, the black hole can be well approximated by hypothesizing a stretched membrane having certain classical properties, surrounding the hole at a small distance outside its event horizon. A physical justification for this procedure is that particles classically can never leave the interior of the black hole or reach it from outside in a finite time according to any such observer. Thus it seems that any such person can study the quantization of the system after removing the interior of the black hole and replacing it by a membrane. The manifold in this approach thus has a boundary.

In this paper, we study the quantum physics of black holes for such observers, or more generally features of quantum gravity on manifolds with spatial boundaries. We show that the presence of the boundary necessarily leads to an infinite set of observables which are completely localized at this boundary in the absence of anomalies. [The important issue of anomalies and their significance is also addressed in this paper.] These are obtained here in analogy to “edge” observables in gauge theories defined on manifolds with boundaries [8,9]. Such observables, defined in the context of gauge theories, have enjoyed good physical
interpretation in many examples of condensed matter physics. For instance, it has been known for some time that many of the essential features of the fractional quantum Hall effect (FQHE) are captured purely by the existence of these edge observables [10]. We will discuss more about this analogy with the quantum Hall effect (QHE) later on in this paper. From previous work dealing with the construction of these observables for pure gauge theories [9] and also from the construction exhibited in this paper, one realizes that the edge observables are independent of observables defined in the bulk and that they commute with the Hamiltonian. This is quite interesting because it means that, no matter what the quantum theory of the bulk observables is, there is always a Hilbert space living entirely on the edge. One can then talk of excitations which only involve the edge and leave the bulk unaffected in the absence of any coupling between them.

Hence, in spite of the deep problems in quantizing gravity, one can still examine the relatively simpler task of quantization of the gravitational edge observables and the associated edge states at the boundary (which, for a black hole, is $S^2 \times \mathbb{R}$). In this way we can also hope to make meaningful physical predictions because of the above mentioned claim that the quantum aspects of a black hole are linked to the existence of an imaginary boundary near the event horizon.

It merits emphasis that the stretched membrane [8] is to be thought of as surrounding the black hole at a small but finite distance outside its event horizon. For this reason, our boundary will also assumed to be situated slightly away from the actual event horizon. There is also a good technical reason for this assumption since the induced metric becomes degenerate on the event horizon.

It is worth pointing out one other important observation emerging from the analysis of Section 3. We notice the fact that even though there exists an infinite set of observables at the edge of the black hole, not all of these are relevant for all the observers. In fact, a sort of complementarity principle can be shown according to which the existence of certain observables as “good” observables will preclude some others from being so. The choice of the “good” observables is dictated by physical considerations. Thus, for example, we
shall see that for an asymptotic observer, physics will dictate that there exists an infinite sub-family of “good” edge observables whereas for an observer stationed close to the event horizon, only a finite number out of this infinite set survive as “good” observables. This difference between the notion of “good” observables for these two observers certainly needs to be understood more carefully since this may be at the heart of many of the conceptual difficulties concerning the quantum physics of black holes. [ In this connection, see [5], where a different kind of complementarity principle has been discussed. ]

The organization of the paper is as follows. In Section 2, we briefly review the canonical formulation of pure gravity in (3+1) dimensions for the convenience of the reader, following the approach of [11] closely. In Section 3, we use this formalism in (3+1) dimensions for a manifold obtained by removing a spatial ball. In black hole physics, this ball may be regarded as enclosing the hole including its event horizon in its interior, its boundary being the membrane. We show that, in such a spacetime, we are naturally led to certain observables that are confined purely to the edge. In Section 4, we discuss the analogy of the aforementioned edge observables with similar observables arising in condensed matter physics, notably in the QHE. In particular, we argue that even though formally the edge observables do not mix with the other observables living in the bulk, this ceases to be the case especially when we have an anomalous coupling between the edge and the bulk. Such a coupling is not a matter of choice, it being required from very general arguments of gauge invariance in the case of the QHE [10]. The surprise is that this analogy extends even to the case of gravity, where it turns out that an anomalous coupling between the edge and the exterior is forced on us if we require general coordinate invariance (diffeomorphism invariance). In Section 5, we explicitly demonstrate that such a coupling indeed appears when we are dealing with (2+1) gravity. The final Section 6 outlines an argument suggesting that the edge and bulk states couple in (3+1) dimensions as well. This argument is unfortunately incomplete as we lack a satisfactory description of edge dynamics.
II. CANONICAL FORMULATION OF THE EINSTEIN-HILBERT ACTION

Consider a four-manifold $M$ which is topologically $\Sigma \times \mathbb{R}$ and let its time-slices $\Sigma_t$ be parametrized by $t$. Hereafter we assume that $\Sigma_t$ is diffeomorphic to the exterior of a ball $\mathcal{B}_3$ in $\mathbb{R}^3$. Let $t^a$ be the vector field whose affine parameter is $t$ and let $n_a$ denote the unit normal to the surfaces $\Sigma_t$ in the direction of increasing $t$. A metric $g_{ab}$ on $M$ induces a metric $q_{ab}$ on $\Sigma_t$. Since $\Sigma_t$ is the spatial slice at time $t$, we need $q_{ab}$ to have +++ signature. Then $n_a$, being normal to $\Sigma_t$, will be timelike and (say) future-directed. Thus we have the relation

$$q_{ab} = g_{ab} + n_a n_b. \quad (2.1)$$

It gives

$$g_{ab} t^b = N_a + \Gamma n_a, \quad (2.2)$$

$N^a$ being tangent to $\Sigma_t$. Here $\Gamma$ and $N^a$ are commonly referred as the ‘lapse’ function and the ‘shift’ vector field respectively [11].

Since we want to interpret $t^a$ as the vector field along which the variables defined on $\Sigma_t$ evolve, it has to be timelike and future-directed. Thus

$$\Gamma^2 - N^a N_a > 0, \quad \Gamma > 0. \quad (2.3)$$

Furthermore, since we are interested in asymptotically flat spacetimes, the appropriate asymptotic conditions to impose on the lapse and shift, in order that $t^a$ reduces to a timelike Killing vector field normal to the spatial slice at spatial infinity, are

$$\Gamma \to 1, \quad N_a \to 0 \quad (2.4)$$

The Einstein-Hilbert action is

$$S = \int (-g)^{\frac{1}{2}} (4) R \, d^4x, \quad (2.5)$$
where \( g \) is the determinant of \( g_{ab} \) and \( ^{(4)}R \) is the Ricci scalar of the four-manifold. It gives the Lagrangian

\[
L' = \int d^3x \ (-g)^{\frac{1}{2}} \ ( ^{(4)}R).
\] (2.6)

at time \( t \). Here \((x_1, x_2, x_3)\) are coordinates on \( \Sigma_t \) which we identify hereafter with \( \Sigma \).

Now apart from surface terms, this \( L' \) is the same as

\[
L = \int_{\Sigma} d^3x \ q^{\frac{3}{2}}( ^{(3)}R + K^{ab}K_{ab} - K^2),
\] (2.7)

where \( q \) is the determinant of \( q_{ab} \), \( ^{(3)}R \) is its Ricci scalar and

\[
K_{ab} = \frac{1}{2\Gamma}(\mathcal{L}_tq_{ab} - \mathcal{L}_Nq_{ab}),
\]

\[
K = q^{ab}K_{ab} = K^a_a
\] (2.8)

The advantage of using (2.7) over (2.6) is that (2.7) is a functional only of \( q_{ab} \) and \( \dot{q}_{ab} (= \mathcal{L}_tq_{ab}) \) whereas (2.6) depends also on \( \ddot{q}_{ab} \).

The Lagrangian \( L \) gives the following expression for the conjugate momentum \( p^{ab} \):

\[
p^{ab} = \frac{\delta L}{\delta \dot{q}_{ab}} = \sqrt{q}(K^{ab} - Kq^{ab}).
\] (2.9)

We can now do a Legendre transform to go to the Hamiltonian. Since \( L \) does not contain terms with \( \dot{\Gamma} \) and \( \dot{N}^a \), the corresponding momenta are constrained to be zero. They also generate secondary constraints. The Hamiltonian (again up to surface terms) and the secondary constraints are

\[
H' = \int_{\Sigma} d^3x \ p^{ab}\dot{q}_{ab} - L
\]

\[
= \int_{\Sigma} d^3x \ \Gamma [-q^{\frac{1}{2}}( ^{(3)}R + q^{-\frac{1}{2}}(p^{ab}p_{ab} - \frac{p^2}{2}))] + \int_{\Sigma} d^3x \ p^{ab}\mathcal{L}_Nq_{ab},
\] (2.10)

\[
D_a p^{ab} \approx 0
\] (2.11)

\[
-q^{\frac{1}{2}}( ^{(3)}R + q^{-\frac{1}{2}}(p^{ab}p_{ab} - \frac{p^2}{2}) \approx 0
\] (2.12)
The primary constraints have vanishing Poisson brackets (PB’s) with \((2.11),(2.12)\) which also do not contain \(\Gamma\) and \(N^a\). The former can therefore be ignored. The secondary constraints do not generate further constraints, their PB’s with the Hamiltonian being weakly zero. They are also first class.

In evaluating the Poisson brackets of the constraints amongst themselves and for finding their action on the phase space it is of course necessary to smear them with test functions so that they become differentiable \([12]\). The vector constraint is to be smeared with a form \(V_a\) that vanishes at the boundaries of the manifold and the scalar constraint is to be smeared with a test function \(S\) that vanishes (along with derivatives) at the boundaries. The boundaries here are the boundary \(\partial \mathcal{B}_3\) of \(\mathcal{B}_3\) and spatial infinity. The smeared constraints are

\[
V_V(q,p) = -2 \int_\Sigma \, d^3x \, V_a D_b p^{ab} \approx 0, \quad (2.13)
\]

\[
S_S(q,p) = \int_\Sigma \, d^3x \, S[-q^{\frac{1}{2}} (3) R + q^{-\frac{1}{2}} (p^{ab} p_{ab} - \frac{p^2}{2})] \approx 0, \quad (2.14)
\]

where

\[
V_a|_{\partial \Sigma} = 0 \quad (2.15)
\]

\[
S|_{\partial \Sigma} = 0, \quad D_a S|_{\partial \Sigma} = 0 \quad (2.16)
\]

The above conditions on the form \(V_a\) and the function \(S\) follow purely from requiring differentiability in the phase space variables \(q_{ab}\) and \(p^{ab}\) of \((2.13)\) and \((2.14)\).

The PB’s among the constraints are

\[
\{V_{V_1}, V_{V_2}\} = V_{[V_1, V_2]},
\]

\[
\{V_V, S_S\} = S_{L_V S_S},
\]

\[
\{S_{S_1}, S_{S_2}\} = \nu_{S_1 D S_2 - S_2 D S_1}. \quad (2.17)
\]

Clearly, the advantage that \((2.7)\) had over \((2.6)\) would be maintained if we add a surface term to \((2.7)\) which is a functional only of \(q_{ab}\) and not of \(\dot{q}_{ab}\) \([12]\). In fact, \((2.10)\) is meaningless as it stands because the boundary condition \((2.4)\) is different from those required of the test
functions $S$ in (2.16). Thus we need to add suitable surface terms to make the Hamiltonian, for example, differentiable.

If the Hamiltonian $H$ obtained after including these surface terms is to describe evolution for an asymptotic observer, $\Gamma$ and $N_a$ have to satisfy the asymptotic conditions specified in (2.4) while both of them have to vanish at the boundary surrounding the black hole interior. This is because in the reference frame of the asymptotic observer, the observer’s own time runs at unit rate while the time on the horizon of the black hole has “stopped”. We can thus write, using (2.13),

$$H = \int_{\Sigma} d^3 x \Gamma \left[ -q^{\frac{1}{2}} (3) R + q^{-\frac{1}{2}} \left( p^{ob} p_{ab} - \frac{p^2}{2} \right) \right] + \text{(surface terms)},$$

(2.18)

$$\Gamma|_{\partial \Sigma} = 0; \quad \Gamma|_{\text{spatial infinity}} = 1$$

(2.19)

where the surface terms are to be chosen so that $H$ becomes differentiable in $q_{ab}$ and $p^{ob}$. We will further discuss these terms below.

III. OBSERVABLES LIVING AT THE EDGE

The construction of edge observables uses a trick that we have already employed in the previous Section. It is illustrated by the distinction between the test functions $\Gamma$ and $S$ for the Hamiltonian (2.18) and the smeared constraint (2.14). Since $H$ becomes a first class constraint if $\Gamma$ goes to zero at the boundaries, we can regard $H$ as localized at the boundaries. We can, in a similar fashion, construct more edge observables by allowing the quantities analogous to $S$ and $V_a$, denoted now by $T$ and $W_a$ respectively, to vary as arbitrarily as possible at $\partial \Sigma$ such that (2.13) and (2.14) turn out to be differentiable after adding suitable surface terms. They will then generate good canonical transformations and will be well-defined in the canonical framework. Such observables are edge observables because they depend only on the boundary values taken by the test forms/functions (and their derivatives in the latter case) [9,12]. This can be seen from the fact that the difference of two of these observables with different smearing forms/functions which coincide (along
with derivatives in the case of the latter) only at the boundaries is a constraint. This is as in the case of $H$. Also these observables are gauge invariant having weakly zero PB’s with the constraints. This will be shown later.

We will first look at the edge observables that come from the vector constraint. To do this, we first rewrite the vector constraint in (2.13) after a partial integration as

$$\mathcal{V}_V = \int_{\Sigma} d^3x \ q_{ab} \mathcal{L}_V p^{ab}. \quad (3.1)$$

In the above, let us replace $V$ by $W$ where $W$ is any vector field. We require of $W$ that, at the boundaries of the manifold, it is tangential to the boundary. Then it can be verified that the quantity so obtained, namely

$$\mathcal{D}_W = \int_{\Sigma} d^3x \ q_{ab} \mathcal{L}_W p^{ab}. \quad (3.2)$$

continues to be differentiable in both $q_{ab}$ and $p^{ab}$. It furthermore has weakly zero PB’s with the constraints:

$$\{\mathcal{D}_W, \mathcal{V}_V\} = \mathcal{V}_{[W,V]},$$

$$\{\mathcal{D}_W, \mathcal{S}_S\} = \mathcal{S}_{\mathcal{L}_W S}. \quad (3.3)$$

The right hand sides in these equations are constraints and hence weakly zero because their respective test fields are easily verified to satisfy the conditions (2.15) and (2.16).

The algebra of observables generated by $\mathcal{D}_W$ is seen to be

$$\{\mathcal{D}_{W_1}, \mathcal{D}_{W_2}\} = \mathcal{D}_{[W_1, W_2]}. \quad (3.4)$$

We are interested in observables which are supported at the edge corresponding to the event horizon rather than those which are supported at spatial infinity. We will therefore hereafter assume that $W$ is non-zero only at the inner boundary and vanishes like $V$ at the boundary at infinity.

We next define the observables which can be got from a partial integration of the scalar constraint. Let us first look at the scalar constraint $\mathcal{S}_S$:
\[
\mathcal{S}_S = \int_\Sigma d^3x \left[ -q^{\frac{3}{2}} (3)R + q^{-\frac{3}{2}}(p^{ab}p_{ab} - \frac{p^2}{2}) \right].
\] (3.5)

The above is clearly differentiable in \( p^{ab} \). As for differentiability in \( q_{ab} \), it can be verified that a variation of \( q_{ab} \) induces surface terms in its variation. They vanish only if the test functions \( S \) satisfy (2.16). The condition on their derivatives emerges because variation of \( (3)R \) contains second derivatives of the variation of the metric \( q_{ab} \). The boundary condition in (2.16) on \( S \) are in fact got from this requirement of differentiability of \( \mathcal{S}_S \).

Our task is to find differentiable observables using formal partial integration in (3.5). We proceed as follows. Consider (3.5) with \( S \) replaced by \( T \). \( T \) now does not have to satisfy the boundary conditions satisfied by \( S \). We have already seen above that differentiability in \( q_{ab} \) of (3.5) requires \( S \) along with its derivatives to vanish at the boundary. Here we instead keep track of the variations to see if we can cancel them with other surface terms. The only term in this expression that requires careful scrutiny is

\[
\int_\Sigma d^3x \left[ -q^{\frac{3}{2}} (3)R \right].
\] (3.6)

The change in above term due to a variation \( \delta q_{ab} \) is

\[
-\int_\Sigma d^3x \left[ q^{\frac{3}{2}} \left( \frac{1}{2} (3)R q^{ab} - (3)R^{ab} \right) \delta q_{ab} - \int d^3x T q^{\frac{3}{2}} \left[ D^a D^b (\delta q_{ab}) - D^a (q^{cd} D_a \delta q_{cd}) \right] \right].
\] (3.7)

Since the second term above contains derivatives of \( \delta q_{ab} \), (3.6) is not differentiable with respect to \( q_{ab} \).

Suppose now that

\[
\delta q_{ab}\big|_{\partial \Sigma} = 0,
\] (3.8)

\[
D_a T\big|_{\partial \Sigma} = 0.
\] (3.9)

[ Note that (3.9) implies that \( T \) at the boundary goes to a constant which can be non-zero.]

The terms involving derivatives of \( \delta q_{ab} \) in (3.7) give rise to surface terms in the variation. These surface terms are now exactly cancelled by the variation of

\[
-2 \int_{\partial \Sigma} TK \sqrt{h}
\]
where $K_{ab}$ and $h_{ab}$ are respectively the extrinsic curvature and the induced metric of the boundary $\partial \Sigma$. Thus so long as the conditions (3.8) and (3.9) above are met, we can define an edge observable of the form

$$H = \mathcal{H}_T = \int_{\Sigma} d^3x \left[ -q^{3/2} R + q^{1/2}(p^{ab}p_{ab} - \frac{p^2}{2}) \right] - 2 \int_{\partial \Sigma} d^2x \ h^{1/2}TK. \quad (3.10)$$

Note that the Hamiltonian $H$ defined in (2.18) is just $\mathcal{H}_\Gamma$:

$$H = \mathcal{H}_\Gamma$$

Thus $H$ is also an edge observable as indicated in the last Section except that it is confined to the asymptotic boundary rather than the inner boundary ($\Gamma$ being non-vanishing only at spatial infinity).

In contrast, $T$ goes to a constant at the boundary surrounding the hole. When this boundary value is non-zero, the observable $\mathcal{H}_T$ acts as a Hamiltonian which non-trivially evolves observables at the inner boundary. Such an observable is physically required by any stationary observer who is sufficiently close to the event horizon (and outside it) because it will function as the time evolution operator of this observer.

The interesting fact is that all the other edge observables $D_W$ defined earlier cease to be well-defined for this observer. The reason is that we required also the condition (3.9) to make $\mathcal{H}_T$ well-defined. Such a condition is clearly incompatible with the existence of the observables $D_W$ because the latter generate non-trivial diffeomorphisms at the boundary. (In some special cases, there may exist certain symmetries of the background metric so that the boundary possesses Killing vector fields. In those cases, we are still permitted those edge observables that generate diffeomorphisms along the Killing vector fields of the boundary. However, it is nevertheless generically true that only a finite set out of the potentially infinite set of edge observables survives as observables for this person.)

Conversely, for the asymptotic observer, there is no necessity for introducing $\mathcal{H}_T$ as a well-defined observable. In fact, this observable is quite unphysical for this observer because it performs “time evolution” at a boundary which from this person’s viewpoint
has “stopped” evolving because of gravitational time dilation. Thus, for this observer, the infinite observables generating diffeomorphisms at the inner boundary do exist as well-defined observables.

Such a difference between two types of stationary observers seems to be similar to the black hole complementarity principle recently envisaged by [5]. In [5], the complementarity is between a stationary external observer and an in-falling observer, whereas here the complementarity is between two stationary external observers, one of them located at spatial infinity and the other located close to the event horizon. The result found here may be significant because it has arisen purely out of technical reasons having to do with general requirements of compatibility of test function spaces suitable for these observers.

Before we go on to the next Section, certain points about these edge observables are worth noting. One triviality to be noted is that they commute with $H (= H_T \text{ in (3.11)})$ as mentioned in the introduction. Therefore the eigenstates of $H$ will have an infinite degeneracy corresponding to the edge states carrying the representation of the diffeomorphism algebra (3.4). One possibility to explore is whether this infinite degeneracy of states for any given energy level causes a “particle” to be in any one of these states with equal likelihood and whether this randomness is what leads to the entropy of the black hole. If such is the case, it would be very interesting to examine if the logarithm of the degeneracy actually increases directly in proportion to the area [14]. Clearly some form of regularization is required in this calculation, since the degeneracy is infinite to begin with and it does not make sense to talk about the dependence of an infinite quantity on area. As this way of arriving at an entropy does not involve matter anywhere, it would be purely intrinsic to the gravitational degrees of freedom.

**IV. THE QUANTUM HALL EFFECT**

As promised, we devote this section to a well-studied condensed matter system. Later on we will borrow the ideas used here to analyze the gravity systems. All of what we are going
to describe below for the Hall system is well-understood by workers in the field of quantum Hall effect (QHE) [15].

Classical Hall effect is the phenomenon of a longitudinal electric field causing a transverse current in the presence of a perpendicular magnetic field. A simple effective action that describes the physics is the Chern-Simons action added on to the usual electromagnetic action:

$$S_{\text{bulk}} = \int_{\mathcal{M}} d^3x \left[ -\frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu} - \frac{\sigma_H}{2} \epsilon_{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda \right]$$ (4.1)

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

Here $e^2$ is a constant related to the inverse of the “effective thickness” of the Hall sample, while our metric is $(-1, +1, +1)$ diagonal.

The equation of motion of the above action is

$$\frac{1}{e^2} \partial_\mu F^{\mu\nu} = \sigma_H \epsilon^{\nu\rho\lambda} F_{\rho\lambda}$$ (4.2)

Thus we see that the current $j^\nu$ for this theory is to be identified with $\sigma_H \epsilon^{\nu\rho\lambda} \partial_\rho A_\lambda$. In particular,

$$j^i = -\sigma_H \epsilon^{ij} E_j$$ (4.3)

Thus the $\sigma_H$ that appears as the coefficient of the Chern-Simons term is the same as the Hall conductivity.

The connection of the above system with edge observables is also well-known. The latter arise when we confine the above theory to a finite geometry (as is appropriate for any physical Hall sample). From very general arguments first forwarded by Halperin [16], the existence of chiral edge currents at the boundary can then be established.

Now naively what happens is that the theory in the bulk (described by (4.1)) does not communicate with the theory describing these chiral currents at the edge. It is then not clear how these edge currents can have any role in the description of bulk phenomena.
What saves the situation, however, is the fundamental requirement of gauge invariance \[1\]. Thus the action (4.1) under gauge transformation \( A \to A + d\alpha \) changes by the surface term

\[
\frac{\sigma_H}{2} \int_{\partial \mathcal{M}} d\alpha \wedge A \tag{4.4}
\]

But, physics is gauge invariant. Therefore it must be that there is a theory at the boundary describing the chiral edge currents which is also gauge non-invariant such that the total action \( S_{\text{tot}} = S_{\text{bulk}} + S_{\text{edge}} \) is itself gauge invariant. This line of argument \[1\] then leads us to the action

\[
S_{\text{tot}} = S_{\text{CS}} + \frac{\sigma_H}{2} \int_{\partial \mathcal{M}} d\phi \wedge A - \frac{\sigma_H}{4} \int_{\partial \mathcal{M}} D_\mu \phi \, D^\mu \phi. \tag{4.5}
\]

\[
D_\mu \phi = \partial_\mu \phi + A_\mu. \tag{4.6}
\]

The field \( \phi \) under a gauge transformation transforms as

\[
\phi \to \phi - \alpha. \tag{4.7}
\]

so that

\[
D\phi \equiv d\phi + A \to D\phi \tag{4.8}
\]

and

\[
S_{\text{tot}} \to S_{\text{tot}}. \tag{4.9}
\]

The second term in (4.5) is the term which restores gauge invariance. Notice that it is the anomalous coupling alluded to in the Introduction. The last term is a kinetic energy term, it is required if the theory at the edge is to give rise to a chiral theory. [See \[1\] for details on how this action can be used very fruitfully to obtain results in the FQHE.]

All of the above material is standard in the literature on the Hall effect.

Before dealing with Einstein gravity in (3+1) dimensions, let us deal with a simple toy model in (2+1) dimensions in Section V in order to get ideas fixed.
V. HOW THE EDGE COUPLES TO THE BULK IN (2+1) GRAVITY

As a toy model, we now show the explicit coupling between the “edge” and the exterior in the case of 2+1 dimensional gravity. The model considered below is related to the one considered in [17] and more recently in Carlip [18]. See also [19].

It is well-known [20] that the standard Einstein-Hilbert action on a three-manifold admits a reformulation as an $ISO(2,1)$-gauge theory with the Chern-Simons action

$$ S = \kappa \int_{\mathcal{M}} \text{tr} \left( A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right) \quad (5.1) $$

where

$$ A = e^a P_a + \omega^m J_m \equiv e \cdot P + \omega \cdot J. \quad (5.2) $$

and $\kappa$ is a constant which we hereafter set equal to one. We will use the form and explicit index notations interchangeably. Here $A, e^a, \omega^m$ are one-forms. Also $P_a$ and $J_m$ are the translation and Lorentz group generators respectively. They satisfy the $ISO(2,1)$ Lie algebra relations

$$ [P_a, P_b] = 0, $$

$$ [P_a, J_m] = \epsilon_{amb} P_b, $$

$$ [J_m, J_n] = \epsilon_{mnk} J_k. \quad (5.3) $$

Also, the trace appearing in (5.1) is given by,

$$ \text{tr} \left( P_a P_b \right) = 0, $$

$$ \text{tr} \left( P_a J_m \right) = \delta_{am}, $$

$$ \text{tr} \left( J_m J_n \right) = 0. \quad (5.4) $$

The action (5.1) is not invariant under arbitrary gauge transformations of the form

$$ A \rightarrow A' = u A u^{-1} + u \, du^{-1}. \quad (5.5) $$
To remedy this situation, we introduce fields $g$ valued in the $ISO(2,1)$ group transforming under $u$ as

$$g \rightarrow u g$$  \hspace{1cm} (5.6)

We can now define the “covariant” derivative

$$Dg \equiv dg + Ag,$$  \hspace{1cm} (5.7)

which has the transformation property

$$Dg \rightarrow uDg,$$  \hspace{1cm} (5.8)

which allows us to construct the gauge invariant one-form

$$A \equiv g^{-1}Dg = g^{-1}dg + g^{-1}Ag.$$  \hspace{1cm} (5.9)

We can now construct the gauge invariant “Chern-Simons”-like action $S_{inv}$ simply by replacing $A$ by $A$ in (5.1) [21]:

$$S_{inv}(A) = \int_{\mathcal{M}} \text{tr} \left[ A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right].$$  \hspace{1cm} (5.10)

It can be expanded, using (5.9), as follows:

$$S_{inv} = S_{CS}(A) - \frac{1}{3} \int_{\mathcal{M}} \text{tr} (g^{-1}dg)^3 - \int_{\partial \mathcal{M}} \text{tr} (dgg^{-1} \wedge A).$$  \hspace{1cm} (5.11)

Here the first term is the usual Chern-Simons action, the second term is the Wess-Zumino-Novikov-Witten (WZNW) term and the last term gives the anomalous coupling between the variable $g$ and the gauge field $A$. We have also assumed that $\mathcal{M}$ has boundary $\partial \mathcal{M}$, arising from the boundaries of its spatial cuts [21].

The form of such a gauge invariant action (5.11) is the same for arbitrary $G$. We now specialize it to the group $ISO(2,1)$. Let $H = \{h\}$ be its subgroup of translations and let $\{w\}$ denote its $SO(2,1)$ subgroup. Any $g \in ISO(2,1)$ can then be written as

$$g = hw = e^\chi w,$$

$$h \equiv e^\chi.$$  \hspace{1cm} (5.12)
Using the fact that $h^{-1} dh \equiv d\chi$, we can now reduce the WZNW term to

\[
\int_M \text{tr} \left( g^{-1} dg \right)^3 = \int_M \left[ 3 \text{tr} \left( d\chi \wedge d(w^{-1}) \right) + \text{tr} \left( dw^{-1} \right)^3 \right] \\
= \int_M \left[ -3 \text{tr} \left( d\chi \wedge dw^{-1} \right) \right] \\
= -3 \int_{\partial M} \text{tr} \left( d\chi \wedge dw^{-1} \right) .
\] (5.13)

Here we have used (5.4) to get rid of the term involving $w$ only.

The gauge invariant action now reduces to

\[
S_{inv} = S_{CS}(A) + \int_{\partial M} \text{tr} \left[ d\chi \wedge (dw^{-1} - A) - hdw^{-1} h^{-1} \wedge A \right] .
\] (5.14)

This equation shows that the variable $g$ couples to the gauge field $A$ only at the edge, just as in QHE. However, as it stands, the field $g$ does not have a dynamics of its own. We therefore add a gauge-invariant kinetic energy term for the $g$ field to the action living only on the boundary $\partial M$:

\[
S_{\text{kin}} = \frac{1}{2} \int_{\partial M} d^2 x \sqrt{\det \eta} \text{tr} \left[ D_\mu g^{-1} D_\nu g \right].
\] (5.15)

Here $D_\mu g = \partial_\mu g + A_\mu g$ is the covariant derivative. Also, $\eta^{\mu\nu}$ is the metric on $\partial M$ induced by the metric on $M$.

Note that (5.15) is the natural analogue of the kinetic energy term in (4.5) for QHE.

Using (5.12) and (5.4), we can expand the action (5.13) as follows:

\[
S_{\text{kin}} = - \int_{\partial M} d^2 x \sqrt{\det \eta} \eta^{\mu\nu} \text{tr} \left[ \partial_\mu \chi \left( \partial_\nu w^{-1} + \frac{1}{2} (J \cdot \omega_\nu + \frac{1}{2} w^{-1} A_\nu w) \right) + e_\mu \omega_\nu + \frac{1}{2} \left( e_\mu \cdot P(w^{-1} \partial_\nu w + \partial_\nu w^{-1} + \partial_\mu w^{-1} \omega_\nu \cdot J, \chi) \right) \right].
\] (5.16)

The total action is the sum of (5.14) and (5.16). Notice that the dynamics of the Chern-Simons field $A$ is completely determined in $M$ while the fields $\chi$ and $w$ get their dynamics on the edge. Also they couple to $A$ via the anomalous term in (5.11) and the kinetic energy term (5.16). The edge variables thus couple to the bulk variables.
VI. FINAL REMARKS: THE EDGE/BULK CONNECTION IN (3+1) GRAVITY

We now attempt to repeat these arguments for the case of (3+1) gravity. We first observe that we have an analogue of (4.1) in the bulk (namely, the exterior of the black hole) which is the Einstein-Hilbert action (2.5). We also have an analogue of the edge currents which are just the edge observables defined in (3.2). Note that we now imagine ourselves to be in the asymptotic frame of reference (rather than in the frame of reference of an observer close to the horizon) for specificity so that we do have the infinite set of edge observables corresponding to spatial diffeomorphisms of the horizon.

There is also a notion of “gauge invariance” which here is the general coordinate or diffeomorphism invariance. Thus we have all the necessary ingredients that the Hall system had. We need to check just one last feature: namely, does the Einstein Hilbert action (2.5) suffer from diffeomorphism non-invariance? If it does, and if we can postulate an anomalous coupling to restore diffeomorphism invariance, then the analogy would be complete. This coupling will serve as the crucial missing link connecting the exterior and the surface of the black hole.

It is very interesting therefore the action (2.5) is not diffeomorphism invariant for a manifold with a boundary. The reason of course is very trivial. If we consider the Einstein-Hilbert action restricted to the exterior of the black hole and perform arbitrary diffeomorphisms, then

\[ \delta_v \int_{\Sigma \times \mathbb{R}} d^4x \sqrt{-g} \, g^{ab} R = - \int_{\partial \Sigma \times \mathbb{R}} d^3x \sqrt{-f} v^a u_a \, R. \]  

(6.1)

Here \( v^a \) is the vector field generating the above diffeomorphism, \( \delta_v \) induces the corresponding variations on fields, \( f_{ab} \) is the induced metric at the boundary and \( u_a \) is the unit normal at the boundary. [ See also [19]. ]

If \( v^a \) is tangential to the boundary, then the above variation (6.1) is zero and then the action is unchanged. But there does not seem to be any reason to constrain \( v^a \) to be tangential to the boundary (at the boundary) especially because we know that the full theory (which
includes also the interior) is diffeomorphism invariant under all possible diffeomorphisms. The above variation is non-zero for such diffeomorphisms. So we do need an anomalous coupling just as in the case of the Hall effect.

At this point we neither have an explicit action describing the dynamics at the edge nor the “anomalous” term which provides the coupling, but the essential point is that a fully diffeomorphism invariant effective theory must include these surface terms. These ideas suggest that any description about loss of information which looks purely at the states in the bulk is necessarily incomplete because there is also the surface action (and associated edge states) to contend with. Just as diffeomorphism non-invariance may be avoided by the presence of surface terms, it seems reasonable to suppose that information loss too may be avoided or mitigated by the presence of such terms.

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