A \( q \)-analogue of the type A Dunkl operator and integral kernel

T.H. Baker\( ^* \) and P.J. Forrester\( ^\dagger \)

Department of Mathematics, University of Melbourne, Parkville, Victoria 3052, Australia

We introduce the \( q \)-analogue of the type A Dunkl operators, which are a set of degree–lowering operators on the space of polynomials in \( n \) variables. This allows the construction of raising/lowering operators with a simple action on non-symmetric Macdonald polynomials. A bilinear series of non-symmetric Macdonald polynomials is introduced as a \( q \)-analogue of the type A Dunkl integral kernel \( K_A(x; y) \). The aforementioned operators are used to show that the function satisfies \( q \)-analogues of the fundamental properties of \( K_A(x; y) \).

1 Introduction

The purpose of this paper is to obtain \( q \)-analogues of some fundamental results concerning type A integral kernels \( K_A(x; y) \) appearing in the works of Dunkl \[3, 4, 5, 6\]. The kernel \( K_A \) allows a multidimensional analogue of the Fourier transform to be constructed, and plays a pivotal role in recent studies of generalized Hermite polynomials \[1, 2\]. A necessary prerequisite will be to give a suitable definition of the \( q \)-analogue of the type A Dunkl operator

\[ d_i := \frac{\partial}{\partial x_i} + \frac{1}{\alpha} \sum_{j \neq i} \frac{1}{x_i - x_j} (1 - s_{ij}) \] (1.1)

where \( s_{ij} \) acts on functions of \( x := (x_1, \ldots, x_n) \) by interchanging the variables \( x_i \) and \( x_j \).

For the purpose of comparison with later results, we recall some results concerning the kernel \( K_A(x; y) \) and the Dunkl operator \( d_i \). The former is a bilinear series in non-symmetric Jack polynomials, denoted \( E_\eta(x) \), which themselves are eigenfunctions of the Cherednik operators \[3\]

\[ \xi_i = \alpha x_i \frac{\partial}{\partial x_i} + \sum_{p<i} \frac{x_i}{x_i - x_p} (1 - s_{ip}) + \sum_{p>i} \frac{x_p}{x_i - x_p} (1 - s_{ip}) + 1 - i \] (1.2)

The Dunkl operator \( d_i \) is related to the Cherednik operator \( \xi_i \) via

\[ d_i = \frac{1}{\alpha x_i} \left( \xi_i + n - 1 - \sum_{p>i} s_{ip} \right) \] (1.3)

and this will be our starting point in defining an suitable \( q \)-analogue of \( d_i \).

Following Sahi \[17\], for a node \( s = (i, j) \) in a composition \( \eta := (\eta_1, \eta_2, \ldots, \eta_n) \in \mathbb{N}^n \), define the arm length \( a(s) \), arm colength \( a'(s) \), leg length \( l(s) \) and leg colength \( l'(s) \) by

\[ a(s) = \eta_i - j \quad l(s) = \# \{ k > i | j \leq \eta_k \leq \eta_i \} + \# \{ k < i | j \leq \eta_k + 1 \leq \eta_i \} \]
\[ a'(s) = j - 1 \quad l'(s) = \# \{ k > i | \eta_k > \eta_i \} + \# \{ k < i | \eta_k \geq \eta_i \} \] (1.4)

\*email: tbaker@maths.mu.oz.au; supported by the ARC
\dagger email: matpjf@maths.mu.oz.au; supported by the ARC
Using these, define constants

\[ d_\eta := \prod_{s \in \eta} (\alpha(a(s) + 1) + l(s) + 1) \quad d'_\eta := \prod_{s \in \eta} (\alpha(a(s) + 1) + l(s)) \]

\[ e_\eta := \prod_{s \in \eta} (\alpha(a'(s) + 1) + n - l'(s)) \]

With these constants, the type A kernel is defined as \footnote{2}

\[ \mathcal{K}_A(x; y) = \sum_\eta \alpha^{\|\eta\|} d_\eta d'_\eta E_\eta(x) E_\eta(y) \]

(1.6)

Set \( s_i := s_{i,i+1} \) for \( 1 \leq i \leq n - 1 \). The following raising/lowering operators

\[ \Phi := x_n s_{n-1} \cdots s_2 s_1 = s_{n-1} \cdots s_i s_{i-1} \cdots s_1 \]  
\[ \hat{\Phi} := d_1 s_1 s_2 \cdots s_{n-1} = s_1 s_2 \cdots s_{i-1} d_i s_is_{i+1} \cdots s_{n-1} \]

(1.7)  
(1.8)

have a very simple action on the non-symmetric Jack polynomials \( E_\eta(x) \) \footnote{10, 2},

\[ \Phi E_\eta = E_{\Phi\eta} \]  
\[ \hat{\Phi} E_\eta = \frac{1}{\alpha} d''_{\Phi\eta} E_{\hat{\Phi}\eta} \]

(1.9)  
(1.10)

where \( \Phi\eta := (\eta_2, \eta_3, \ldots, \eta_n, \eta_1 + 1) \) and \( \hat{\Phi}\eta := (\eta_n - 1, \eta_1, \eta_2, \ldots, \eta_{n-1}) \).

The fundamental properties of the kernel \( \mathcal{K}_A(x; y) \) are given by the following result \footnote{3}, Theorem 3.8

**Theorem 1.1** The function \( \mathcal{K}_A(x; y) \) possesses the following properties

\[ (a) \quad s_i^{(y)} \mathcal{K}_A(x; y) = s_i^{(x)} \mathcal{K}_A(x; y) \]
\[ (b) \quad \hat{\Phi}^{(y)} \mathcal{K}_A(x; y) = \Phi^{(x)} \mathcal{K}_A(x; y) \]
\[ (c) \quad d_i^{(y)} \mathcal{K}_A(x; y) = x_i \mathcal{K}_A(x; y) \]

The above kernel has a symmetric counterpart \( \mathcal{F}_0(x; y) \) which itself is expressed in terms of the symmetric Jack polynomials \( P^{(\alpha)}_\lambda(x) \) \footnote{13}. The symmetric Jack polynomials can be expressed in terms of their non-symmetric siblings \( E_\eta(x) \) by \footnote{17}

\[ P^{(\alpha)}_\kappa(x) = d'_\kappa \sum_\eta \frac{1}{d''_{\eta\eta}} E_\eta(x) \]

(1.11)

where the sum is over distinct permutations \( \eta \) of the partition \( \kappa \). They can also be obtained by symmetrization \footnote{2}.

\[ \text{Sym} E_\eta(x) = \frac{n! e_\eta}{d''_\eta P^{(\alpha)}_\eta(1^n)} P^{(\alpha)}_{\eta^+}(x) \]

(1.12)

where Sym denotes the operation of symmetrization of the variables \( x_1, \ldots, x_n \) and \( \eta^+ \) denotes the (unique) partition associated with \( \eta \), obtained by permuting its entries. It was shown in \footnote{3} that the symmetric kernel

\[ \mathcal{F}_0(x; y) := \sum_\kappa \frac{\alpha^{\|\kappa\|} P^{(\alpha)}_\kappa(x) P^{(\alpha)}_\kappa(y)}{d'_\kappa} \frac{1}{P^{(\alpha)}_\kappa(1^n)} \]
can be obtained from the non-symmetric kernel $K_A(x;y)$ via symmetrization:

$$\text{Sym}^{(x)} K_A(x;y) = n! \mathcal{F}_0(x;y).$$  \hfill (1.13)

In this work, we shall be concerned with providing $q$-analogues of the above results. After introducing preliminary results and notations dealing with non-symmetric Macdonald polynomials $E_q(x;q,t)$ and (type $A$) affine Hecke algebras, we proceed to define an analogue of the Dunkl operator (1.1) and show that they form a mutually commuting set of degree-lowering operators. We then construct the analogue of Knop and Sahi’s raising operator $\Phi$ given by (1.7), as well as its lowering counterpart $\hat{\Phi}$, demonstrating their simple action on $E_q(x;q,t)$. Finally, we construct a $q$-analogue of the kernel $K_A(x;y)$ and derive the corresponding version of Theorem 1.1. Symmetrization of this kernel is then shown to recover the well-known symmetric version $\mathcal{F}_0(x;y;q,t)$ \cite{13,14}.

2 Preliminaries

We begin by presenting the standard realization of the affine Hecke algebra on the space of polynomials in $n$ variables (see e.g. \cite{1,11}). Let $\tau_i$ be the $q$-shift operator in the variable $x_i$, so that

$$(\tau_i f)(x_1, \ldots, x_i, \ldots, x_n) = f(x_1, \ldots, qx_i, \ldots, x_n).$$

The Demazure-Lustig operators are defined by

$$T_i = t + \frac{tx_i - x_{i+1}}{x_i - x_{i+1}} (s_i - 1) \quad i = 1, \ldots, n - 1$$

and

$$T_0 = t + \frac{qx_n - x_1}{qx_n - x_1} (s_0 - 1)$$

where $s_0 := s_1 \tau_1 \tau_n^{-1}$. For future reference, we note the following action of $T_i$, $1 \leq i \leq n - 1$ on the monomial $x_i^a x_{i+1}^b$

$$T_i x_i^a x_{i+1}^b = \begin{cases} (1-t)x_i^{a-1}x_{i+1}^{b+1} + \cdots + (1-t)x_i^{b+1}x_{i+1}^{a-1} + x_i^a x_{i+1}^b & a > b \\ tx_i^a x_{i+1}^b & a = b \\ (t-1)x_i^a x_{i+1}^{b-1} + \cdots + (t-1)x_i^{b-1}x_{i+1}^a + tx_i^a x_{i+1}^b & a < b \end{cases}$$ \hfill (2.3)

In addition to the operators $T_i$, define

$$\omega := s_{n-1} \cdots s_2 s_1 \tau_1 = s_{n-1} \cdots s_i \tau_i s_{i-1} \cdots s_1.$$  

The affine Hecke algebra is then generated by elements $T_i$, $0 \leq i \leq n - 1$ and $\omega$, satisfying the relations

$$\begin{align*}
(T_i - t)(T_i + 1) &= 0 \\
T_i T_{i+1} T_i &= T_{i+1} T_i T_{i+1} \\
T_i T_j &= T_j T_i \quad |i - j| \geq 2 \\
\omega T_i &= T_{i-1} \omega
\end{align*}$$  \hfill (2.4-2.7)

From the quadratic relation (2.4), we have the identity

$$T_i^{-1} = t^{-1} - 1 + t^{-1} T_i.$$  \hfill (2.8)
Useful relations between the operators $\omega$, $T_i$ and $x_i$, $1 \leq i \leq n - 1$, include

\begin{align}
T_i^{-1} x_{i+1} &= t^{-1} x_i T_i \\
T_i x_i &= t x_{i+1} T_i^{-1} \\
\omega x_i &= q x_{n+1}
\end{align}

(2.9) (2.10) (2.11)

We can define operators \[ Y_i = t^{-n+1} T_1 \cdots T_{n-1} \omega T_i^{-1} \cdots T_{i-1}^{-1}, \]

where

\[ \omega x_i = q x_{n+1} \quad \omega x_{i+1} = x_i \omega \quad 1 \leq i \leq n - 1 \]

(2.12)

which commute amongst themselves: $[Y_i, Y_j] = 0$, $1 \leq i, j \leq n$. They also possess the following relations with the operators $T_i$

\[ T_i Y_{i+1} T_i = t Y_i \]

(2.13)

while the following relations with $x_n$ will be needed in Section 4

\[ Y_i x_n = x_n Y_i + t^{-n+1} (1 - t) x_n T_1 \cdots T_{n-1} \omega T_i^{-1} \cdots T_{i-1}^{-1} \]

(2.14)

\[ Y_n x_n = q t^{-n+1} x_n \omega T_1 \cdots T_{n-1} \]

(2.15)

These identities follow from a direct calculation involving (2.11), (2.10) and (2.9).

The fact that the operators $Y_i$ mutually commute implies that they possess a set of simultaneous eigenfunctions, the non-symmetric Macdonald polynomials. Specifically, let $<$ denote the partial order on compositions $\eta$ defined for $\eta \neq \nu$ by

\[ \nu < \eta \iff \nu^+ < \eta^+ \quad \text{or in the case } \nu^+ = \eta^+ \nu < \eta \]

where $<$ is the usual dominance order for $n$-tuples, i.e. $\nu < \eta$ iff $\sum_{i=1}^p (\eta_i - \nu_i) \geq 0$, for all $1 \leq p \leq n$. Then the non-symmetric Macdonald polynomials $E_\eta(x; q, t)$ can be defined by the conditions

\[ E_\eta(x; q, t) = x^\eta + \sum_{\nu < \eta} b_\nu x^\nu \]

(2.16)

where

\[ \bar{\eta}_i = \alpha \eta_i - \# \{ k < i \mid \eta_k \geq \eta_i \} - \# \{ k > i \mid \eta_k > \eta_i \} \]

(2.17)

Define the analogue of the constants (2.15) by (2.16)

\[ d_\eta(q, t) := \prod_{s \in \eta} \left( 1 - q^{a(s)} t^{l(s)} \right) \]

\[ d'_\eta(q, t) := \prod_{s \in \eta} \left( 1 - q^{a(s)+1} t^{l(s)} \right) \]

\[ e_\eta(q, t) := \prod_{s \in \eta} \left( 1 - q^{a(s)+1} t^{l(s)} \right) \]

(2.18)

Certain properties of these coefficients follow immediately from [7, Lemmas 4.1, 4.2]

**Lemma 2.1** We have

\[ \frac{d_\phi(q, t)}{d_\eta(q, t)} = e_\phi(q, t) \]

(2.19)

\[ \frac{d'_\phi(q, t)}{d'_\eta(q, t)} = 1 - q t^{n+1} \]

(2.20)

\[ \frac{d_{s, \eta}(q, t)}{d_\eta(q, t)} = \frac{1 - t^{\bar{\eta}_i + 1}}{1 - t^{\bar{\eta}_i, \eta}} \]

(2.21)

\[ \frac{d'_{s, \eta}(q, t)}{d'_\eta(q, t)} = \frac{1 - t^{\bar{\eta}_i + 1}}{1 - t^{\bar{\eta}_i, \eta}} \quad \text{for } \eta_i > \eta_{i+1} \]

(2.22)

\[ \text{The normalization is chosen such that as } q \to 1, (1 - Y_i)/(1 - q) \to \xi_i/\alpha \text{ with } \xi_i \text{ given by (2.2)} \]
3 \ q-Dunkl operators

We introduce the \( q \)-Dunkl operators \( D_i \) for \( 1 \leq i \leq n \) according to

\[
D_i := x_i^{-1} \left( 1 - t^{n-1} \left[ 1 + (t^{-1} - 1) \sum_{j=i+1}^{n} t^{j-i} T_{ij}^{-1} \right] Y_i \right)
\]  (3.1)

where for \( i < j \),

\[
T_{ij}^{-1} := T_{i+1}^{-1} T_{i+1}^{-1} \cdots T_{j-2}^{-1} T_{j-1}^{-1} T_{j-1}^{-1} \cdots T_{i-1}^{-1}
\]

Note that as \( q \to 1 \) in (3.1), since \( T_{ij}^{-1} \to s_{ij} \) and \( (1 - Y_i)/(1 - q) \to \xi_i/\alpha \), we recover the type A Dunkl operators due to the relation (3.3):

\[
\lim_{q \to 1} \frac{D_i}{1 - q} = d_i
\]

The relations between the operators \( D_i \) and the elements \( T_i, \omega \) of the affine Hecke algebra are given by the following two lemmas,

Lemma 3.1

\[
T_i D_{i+1} = t D_i T_{i+1}^{-1}, \quad T_i D_i = D_{i+1} T_i + (t - 1) D_i \quad 1 \leq i \leq n - 1 \]  (3.2)

\[
[T_i, D_j] = 0 \quad j \neq i, i + 1 \]  (3.3)

Proof. First note that the second relation in (3.2) follows from the first relation by multiplying the latter on the left by \( T_i \), on the right by \( T_i \), and then using (2.8).

From (2.8) it follows that \( T_i^{-1} \) obeys the quadratic relation

\[
T_i^{-2} + (1 - t^{-1}) T_i^{-1} - t^{-1} = 0.
\]  (3.4)

Multiply the relation \( t^{-1} Y_{i+1} = T_i^{-1} Y_i T_i^{-1} \) (which follows directly from (2.13)) on the left by \( T_i^{-1} \) and apply (3.4) to give

\[
T_i^{-1} Y_{i+1} = (t^{-1} - 1) Y_{i+1} + Y_i T_i^{-1}.
\]  (3.5)

From the definition (3.1) and the relations \( T_i x_i^{-1} = tx_i^{-1} T_i^{-1} \) (which follows from the first equation in (2.10)), (2.13) and (3.5), the first relation in (3.2) can be deduced.

Turning to (3.3), note that a convenient representation of \( D_j \) for \( j < n \) in terms of

\[
D_n := x_n^{-1} (1 - t^{n-1} Y_n)
\]  (3.6)

is

\[
D_j = t^{-n+j} T_j \cdots T_{n-1} D_n T_{n-1} \cdots T_j.
\]  (3.7)

Thus for \( i < j - 1 \) it follows that \( [T_i, D_j] = 0 \). For \( j - 1 \leq i \leq n - 1 \) we have

\[
T_i D_j = t^{-n+j} T_j \cdots T_{n-1} T_{i-1} D_n T_{n-1} \cdots T_j \]
\[
= t^{-n+j} T_j \cdots T_{n-1} D_n T_{i-1}, T_{n-1} \cdots T_j \]
\[
= D_j T_i,
\]

where the first equality follows from (2.3), the second from the already established commutativity of \( T_i, D_j \) for \( i < j - 1 \), and the final equality follows by further use of (2.4). \( \square \)
Lemma 3.2 We have
\[ \omega D_{i+1} = D_i \omega \quad 1 \leq i \leq n - 1 \]  
\[ q \omega D_1 = D_n \omega \]  

Proof. We first prove (3.8) in the special case \( i = n - 1 \). To this end, note that for \( i \geq 2 \),
\[ \omega Y_i = Y_{i-1} \omega + (1-t)T_{i-1} \cdots T_{n-2}T_{n-1}^{-1}T_{n-2}^{-1} \cdots T_{i-1}^{-1}Y_{i-1} \omega \]
which follows from using (2.7) to shift the operator \( \omega \) to the right. In particular
\[ \omega Y_n = Y_{n-1} \omega + (1-t)T_{n-1}^{-1}Y_{n-1} \omega. \]

The use of (2.11) and (3.10) and the explicit expression for \( D_n \) given by (3.6) allows one to show that \( \omega D_n = D_{n-1} \omega \). For the cases \( i < n - 1 \) the result follows from the case \( i = n - 1 \) and the representation of \( D_i \) in terms of \( D_n \) given by (3.7) since
\[ \omega D_{i+1} = t^{-n+i+1} \omega T_{i+1} \cdots T_{n-1}D_nT_{n-1} \cdots T_{i+1} \]
\[ = t^{-n+i+1}T_i \cdots T_{n-2}D_{n-1}T_{n-2} \cdots T_i \omega \]
\[ = t^{-n+i+1}T_i \cdots T_{n-2} \left( t^{-1}T_{n-1}D_nT_{n-1} \right) T_{n-2} \cdots T_i \omega = D_i \omega \]

To prove (3.9), first note that repeated use of (3.4) yields that, for \( i < j \)
\[ I_{ij}^{-1} := \left( T_i^{-1} T_{i+1}^{-1} \cdots T_{j-1}^{-1} T_j^{-1} \right) \]
\[ = t^{i-j} \cdot (t^{-1} - 1) \sum_{p=i+1}^{j+1} t^{p-j-1} T_{ip}^{-1} \]  

(3.11)

It then follows that
\[ Y_n \omega = \omega T_1^{-1} \cdots T_{n-1}^{-1} \omega = t^n \omega I_{1,n-1}^{-1} Y_1 \]
and so from this and (3.11), we have
\[ D_n \omega = x_n^{-1} \left( 1 - t^{n-1}Y_n \right) \omega = x_n^{-1} \omega \left( 1 - t^{2n-2} I_{1,n-1}^{-1} Y_1 \right) \]
\[ = q \omega x_n^{-1} \left( 1 - t^{n-1} \left[ 1 + (t^{-1} - 1) \sum_{p=2}^{n} t^{p-1} T_{ip}^{-1} \right] Y_1 \right) = q \omega D_1 \]

\[ \square \]

Remarks.

1. The final relations between the operators \( D_i \), \( 1 \leq i \leq n \) and the generators of the affine Hecke algebra are the ones involving the generator \( T_0 \). These takes the form
\[ T_0 D_i = q^{-1} i D_i T_0^{-1}, \quad T_0 D_n = q D_1 T_0 + (t-1)D_n \]
\[ [T_0, D_i] = 0, \quad 2 \leq i \leq n - 1 \]
which follow immediately using the fact that \( T_0 = \omega T_1 \omega^{-1} \) along with Lemmas 3.1 and 3.2.

2. It follows from (3.11) that
\[ D_i = x_i^{-1} \left( 1 - t^{2n-i-1} I_{i,n-1}^{-1} Y_i \right) \]
\[ = x_i^{-1} \left( 1 - t^{n-1} T_i^{-1} \cdots T_{n-1}^{-1} \omega T_1^{-1} \cdots T_{i-1}^{-1} \right) \]

(3.12)

providing an alternative definition of the \( q \)-Dunkl operators.

Another set of relations which shall be needed later on is an analogue of \([1\) Lemma 3.1]
Lemma 3.3 We have

\[ [D_i, Y_j] = \begin{cases} \ t^{i-j}(1-t)Y_j T_{ij} D_j & i < j \\ \ t^{i-j}(1-t)Y_j T_{ji} D_i & i > j \end{cases} \] (3.13)

\[ D_i Y_i - q Y_i D_i = (t-1)\sum_{p=1}^{n} t^{-p+i}Y_p T_{ip} D_p + q(t-1)Y_i \sum_{p=1}^{i-1} t^{p-i} T_{ip} D_i \] (3.14)

Proof. We start with (3.13) when \( i < j \). In this case, we can use Lemmas 3.1 and 3.2 to shuffle the \( D_i \) to the right to get

\[ D_i Y_j = t^{-n+j} T_j \cdots T_{n-1} \omega T_1^{-1} \cdots T_{i-1}^{-1} D_{i+1} T_i^{-1} \cdots T_{j-1}^{-1} \]

\[ = Y_j D_i + t^{-n+j}(t-1) T_j \cdots T_{n-1} \omega T_1^{-1} \cdots T_{i-1}^{-1} D_{i+1} T_i^{-1} \cdots T_{j-1}^{-1} \]

where we have used the fact that \( D_{i+1} T_i^{-1} = T_i^{-1} D_i + (t-1) D_{i+1} \) (which follows directly from (3.2) ). Using Lemma 3.1 on the second of these terms to move \( D_{i+1} \) to the right results in the term \( t^{i-j}(1-t)Y_j T_{ij} D_j \) as required. The proof in the case \( i > j \) is somewhat similar.

To prove (3.14) we must also consider 2 cases: \( 1 \leq i < n \) and \( i = n \). For the case \( 1 \leq i < n \), we use the identity

\[ D_i T_i = T_i D_{i+1} + (t-1)D_i \]

(which can be obtained from Lemma 3.3) to move the operator \( D_i \) to the right in the expression

\[ D_i Y_i = t^{-n+i}D_i T_1 \cdots T_{n-1} \omega T_1^{-1} \cdots T_{i-1}^{-1} \]

and thus obtain

\[ D_i Y_i = (t-1)\sum_{p=i+1}^{n} t^{-p+i}Y_p T_{ip} D_p + qt^{-i+1}Y_i T_{i-1} \cdots T_1 T_1 \cdots T_{i-1} D_i \]

The second term in the above expression can be simplified using a result similar to (3.11), namely that for \( i < j \)

\[ T_j T_{j-1} \cdots T_i T_{i-1} = t^{j-i+1} + (t-1)\sum_{p=i}^{j} t^{p-i} T_{p,j+1} \]

and the stated result follows. The case \( i = n \) is derived similarly. \( \square \)

In the rest of this section we shall show that the \( q \)-Dunkl operators \( D_i \) commute amongst themselves. We do this by showing that all the \( D_i \) commute with \( D_n \) (recall that \( D_n \) has the simplest form amongst all the \( q \)-Dunkl operators) from which the general result follows swiftly.

First note that for \( i < n \)

\[ Y_i x_n^{-1} = x_n^{-1}Y_i + t^{n-i+1}(t-1)x_n^{-1} T_{in}^{-1} Y_i \] (3.15)

which follows from pulling \( x_n^{-1} \) to the left using

\[ T_i x_n^{-1} = tx_n^{-1} T_i^{-1} \quad \quad \quad \quad \quad T_i x_n^{-1} = x_n^{-1} T_i + (t-1)x_i \] (3.16)

(which themselves follow from (2.9) and (2.10) ). Using (3.15) and (3.16), a series of manipulations yields the relation

\[ Y_n T_{in} x_n^{-1} = t^{2(n-i)-1} x_n^{-1} T_{in}^{-1} Y_i \]
Using this and Lemma 3.3 we can rewrite the commutator \([D_i, Y_n]\) in the form
\[
[D_i, Y_n] = t^{n-i}(t^{-1} - 1)x_i^{-1}T_{in}^{-1}Y_i(1 - t^{n-1})Y_n
\tag{3.17}
\]

Another result we need is
\[
\left[T_{in}^{-1}, x_n^{-1}\right] = (x_i^{-1} - x_n^{-1})T_{in}^{-1} + (t^{-1} - 1)x_n^{-1}I_{i,n-1}
\]
\[
+ (t^{-1} - 1) \sum_{p=i+1}^{n-1} x_p^{-1}T_p^{-1}\cdots T_{p-1}^{-1}T_{n-1}^{-1}\cdots T_i^{-1}
\tag{3.18}
\]
which can be derived by repeated use of
\[
T_{i}^{-1}x_{i+1}^{-1} = x_{i}^{-1}T_{i}^{-1} + (t^{-1} - 1)x_{i+1}^{-1}
\]
(which itself is derived from \(3.16\)).

This is used in the derivation of the final necessary ingredient

**Lemma 3.4** We have for \(i < n\)
\[
\left[D_i, x_n^{-1}\right] = t^{2n-i-1}(t^{-1} - 1)x_i^{-1}x_n^{-1}T_{in}^{-1}Y_i
\]

**Proof.** Write \(D_i = x_i^{-1}(1 - t^{n-1}A_iY_i)\) where
\[
A_i := 1 + (t^{-1} - 1) \sum_{j=i+1}^{n} t^{j-i}T_{ij}^{-1}
\]
Then we have
\[
\left[D_i, x_n^{-1}\right] = \left[x_i^{-1}\left(1 - t^{n-1}A_iY_i\right), x_n^{-1}\right]
\]
\[
= -t^{n-i}x_i^{-1}\left([A_i, x_n^{-1}]Y_i + A_i\left[Y_i, x_n^{-1}\right]\right)
\tag{3.19}
\]
Note that
\[
[A_i, x_n^{-1}] = (t^{-1} - 1)t^{n-i}\left[T_{in}^{-1}, x_n^{-1}\right]
\tag{3.20}
\]
while from \((3.15)\) we have
\[
A_i\left[Y_i, x_n^{-1}\right] = (1 - t^{-1})t^{n-i}\left(x_i^{-1}T_{in}^{-1} + (t^{-1} - 1) \sum_{j=i+1}^{n} t^{j-i}T_{ij}^{-1}x_i^{-1}T_{in}^{-1}\right)Y_i
\tag{3.21}
\]
However, for \(i < j\) application of the relation \(T_{i}^{-1}x_{i}^{-1} = t^{-1}x_{i+1}^{-1}T_{i}\) tells us that
\[
T_{ij}^{-1}x_{i}^{-1} = t^{-j+i}x_{j}^{-1}T_{i}^{-1}\cdots T_{j-2}^{-1}T_{j-1}T_{j-2}\cdots T_{i}
\]
so that
\[
T_{ij}^{-1}x_{i}^{-1}T_{in}^{-1} = \begin{cases} t^{-j+i}x_{j}^{-1}T_{i}^{-1}\cdots T_{j-2}^{-1}T_{j-1}T_{j-2}\cdots T_{i} & j < n \\ t^{-n+i}x_{n}^{-1}I_{i,n-2}^{-1} & j = n \end{cases}
\]
Substituting this into \((3.21)\) and hence into \((3.19)\) along with \((3.20)\) (after using \((3.18)\) ) yields the result.

**Proposition 3.5** We have
\[
[D_i, D_j] = 0 \quad 1 \leq i, j \leq n
\]

**Proof.** Consider first the case \(j = n\). In this case
\[
[D_i, D_n] = \left[D_i, x_n^{-1}\left(1 - t^{n-1}Y_n\right)\right]
\]
\[
= \left[D_i, x_n^{-1}\right]\left(1 - t^{n-1}Y_n\right) - t^{n-1}x_n^{-1}[D_i, Y_n] = 0
\]
thanks to \((3.17)\) and Lemma 3.4. The general result now follows using the representation \((3.4)\) for \(D_j\) in terms of \(D_n\).
4 Raising/lowering operators

The \(q\)-analogue of the raising operator \(\Phi\) (recall (1.7)) introduced by Knop and Sahi \cite{10} is defined as
\[
\Phi_q := x_n T_{n-1}^{-1} \cdots T_2^{-1} T_1^{-1} = t^{-n+i} T_{n-1}^{-1} \cdots T_i x_i T_{i-1}^{-1} \cdots T_1^{-1} \tag{4.1}
\]
This operator enjoys the following properties

**Proposition 4.1**

(a) \(Y_j \Phi_q = \Phi_q Y_{j+1}\) \(1 \leq j \leq n - 1\)

(b) \(Y_n \Phi_q = q \Phi_q Y_1\)

**Proof.** To prove (a), first note from (2.14) that
\[
Y_j \Phi_q = Y_j x_n T_{n-1}^{-1} \cdots T_1^{-1} = B_1 + B_2
\]
where
\[
B_1 = x_n Y_j T_{n-1}^{-1} \cdots T_1^{-1} = \Phi_q Y_{j+1} + (1 - t^{-1}) x_n T_{n-1}^{-1} T_{j+1}^{-1} \cdots T_1^{-1} Y_j + 1
\]
and
\[
B_2 = t^{-n+j} (1 - t) x_n T_j \cdots T_{n-2} \omega T_1^{-1} \cdots T_{j-1}^{-1} T_{n-1}^{-1} \cdots T_1^{-1}
\]
A careful inspection of the second term occurring in \(B_1\) shows that it cancels with \(B_2\), whence the result.

Similar considerations follow for (b), with the aid of (2.15). \(\square\)

The analogue of (1.9) is given by the following

**Corollary 4.2** The operator \(\Phi_q\) acts on non-symmetric Macdonald polynomials in the following manner
\[
\Phi_q E_{\eta}(x; q, t) = t^{-\#(\eta \leq n)} E_{\Phi q}(x; q, t)
\]
where \(\Phi \eta := (\eta_2, \eta_3, \ldots, \eta_n, \eta_1 + 1)\).

**Proof.** From the previous Proposition, it is clear that \(\Phi_q E_{\eta}(x; q, t)\) is a constant multiple of \(E_{\Phi \eta}(x; q, t)\) as they are both eigenfunctions of the operators \(Y_i\) with the same set of eigenvalues. The multiple is deduced by means of examining the coefficient of the leading term \(x^{\Phi \eta}\) in the expansion of \(\Phi_q E_{\eta}(x; q, t)\) with the aid of (2.8) and (2.3). \(\square\)

The definition of the lowering operator analogous to (1.8) makes use of the \(q\)-Dunkl operator introduced in Section 3:
\[
\Phi_q = T_1 T_2 \cdots T_{n-1} D_n = t^{n-i} T_1 \cdots T_{i-1} D_i T_{i-1}^{-1} \cdots T_{n-1}^{-1} \tag{4.2}
\]
This operator acts as a shift operator for the operators \(Y_i\).

**Proposition 4.3**

(a) \(Y_j \Phi_q = \Phi_q Y_{j-1}\) \(2 \leq j \leq n\)

(b) \(Y_n \Phi_q = q^{-1} \Phi_q Y_n\)
Proof. We begin with (a). Note that for \( j > 2 \) we have
\[
T_{j}^{-1} T_1 \cdots T_{n-1} = T_1 \cdots T_{n-1} T_{j-1}^{-1}.
\]
Thus
\[
Y_j \hat{\Phi}_q = t^{-n+j} T_j \cdots T_{n-1} \omega T_1^{-1} \cdots T_{j-1}^{-1} (T_1 \cdots T_n) D_n
\]
\[
= t^{j-2} D_1 \left( T_j \cdots T_{n-1} T_1^{-1} \cdots T_{n-2}^{-1} \right) \omega T_1^{-1} \cdots T_{j-2}^{-1}
\]
where we have used (4.3) along with Lemmas 3.1, 3.2 and (2.7) to make the necessary manipulations to get it into the above form. However we can rewrite the term in the parenthesis occurring in (4.4) as
\[
T_j \cdots T_{n-1} T_1^{-1} \cdots T_{n-2}^{-1} = T_1^{-1} \cdots T_{n-1} T_{j-1} \cdots T_n
\]
which, when substituted back into (4.4), yields the requisite result. The proof of (b) is almost immediate from (3.9).

As a consequence, we have the analogue of (1.10)

**Corollary 4.4** The action of \( \hat{\Phi}_q \) on non-symmetric Macdonald polynomials is given by
\[
\hat{\Phi}_q E_\eta(x; q, t) = \# \{ \eta_i < \eta_n \} \frac{d\eta'(q, t)}{d\hat{\Phi}_\eta(q, t)} E_{\hat{\Phi}_\eta}(x; q, t)
\]
where \( \hat{\Phi}_\eta := (\eta_n - 1, \eta_1, \eta_2, \ldots, \eta_{n-1}) \).

**Proof.** From Proposition 4.3 we have that \( \hat{\Phi}_q E_\eta(x; q, t) \) is a multiple of \( E_{\hat{\Phi}_\eta}(x; q, t) \). An examination of the leading term \( x^{\hat{\Phi}_\eta} \) in the expansion of \( \hat{\Phi}_q E_\eta(x; q, t) \) using (2.3) and the explicit form (3.6) for \( D_n \) tells us that
\[
\hat{\Phi}_q E_\eta(x; q, t) = \# \{ \eta_i < \eta_n \} \left( 1 - t^{n-1+\eta_n} \right) E_{\hat{\Phi}_\eta}(x; q, t).
\]
However from Lemma 2.1 we know that
\[
\frac{d\eta'(q, t)}{d\hat{\Phi}_\eta(q, t)} = 1 - t^{n-1+\eta_n}
\]
whence the result.

**5 Kernels**

We introduce the \( q \)-analogue of the Dunkl kernel \( K_A(x; y) \) by
\[
K_A(x; y; q, t) = \sum_\eta \frac{d\eta(q, t)}{d\eta'(q, t) e_\eta(q, t)} E_\eta(x; q, t) E_\eta(y; q^{-1}, t^{-1})
\]
This function reduces to \( K_A(x; y) \) as \( q \to 1 \) (although \( K_A(x; y; q, t) \neq K_A(y; x; q, t) \) in general) and satisfies generalizations of Theorem 1.1 and (1.13). To establish these generalizations requires properties of the operators \( D_i, \Phi_q \) and \( \hat{\Phi}_q \) obtained above, as well as some additional
properties of the operators $T_i$ ($1 \leq i \leq n - 1$). The first such property required relates to the action of the operators $T_i^{\pm 1}$ on the non-symmetric Macdonald polynomials.

\[
T_i E_\eta = \begin{cases} 
\left( \frac{t - 1}{1 - t^{\delta i}} \right) E_\eta + t E_{s_i \eta} & \eta_i < \eta_{i+1} \\
t E_\eta & \eta_i = \eta_{i+1} \\
\left( \frac{t - 1}{1 - t^{\delta i}} \right) E_\eta + \frac{(1 - t^{\delta i+1})(1 - t^{\delta i-1})}{(1 - t^{\delta i})^2} E_{s_i \eta} & \eta_i > \eta_{i+1}
\end{cases} \tag{5.2}
\]

\[
T_i^{-1} E_\eta = \begin{cases} 
\left( \frac{t - 1}{1 - t^{\delta i}} \right) E_\eta + E_{s_i \eta} & \eta_i < \eta_{i+1} \\
t^{-1} E_\eta & \eta_i = \eta_{i+1} \\
\left( \frac{t - 1}{1 - t^{\delta i}} \right) E_\eta + t^{-1} \frac{(1 - t^{\delta i+1})(1 - t^{\delta i-1})}{(1 - t^{\delta i})^2} E_{s_i \eta} & \eta_i > \eta_{i+1}
\end{cases} \tag{5.3}
\]

where $\delta i := \bar{\eta}_i - \bar{\eta}_{i+1}$.

Now define the involution $\tilde{\cdot}$ as acting on operators or functions by sending $q \to q^{-1}$, $t \to t^{-1}$. The following lemma is the analogue of [2, Lemma 3.7]

**Lemma 5.1** Let $F(x, y) = \sum A_\eta E_\eta(x; q, t)E_\eta(y; q^{-1}, t^{-1})$. Then

\[
(T_i^{\pm 1}(x) F(x, y) = T_i^{-1}(y)) F(x, y)
\]

if and only if the coefficients $A_\eta$ satisfy

\[
A_{s_i \eta} = \begin{cases} 
\frac{(1 - t^{\delta i})^2}{(1 - t^{\delta i+1})(1 - t^{\delta i-1})} A_\eta & \eta_i > \eta_{i+1} \\
\frac{(1 - t^{\delta i+1})(1 - t^{\delta i-1})}{(1 - t^{\delta i})^2} A_\eta & \eta_i < \eta_{i+1}
\end{cases} \tag{5.5}
\]

Moreover, these two conditions on $A_\eta$ are equivalent.

**Proof.** Equation (5.4) consists of two separate equations; only $T_i^{(x)} F = T_i^{-1}(y) F$ will be established as the other case follows from (2.8). The proof is similar to that given for [2, Lemma 3.7]. Split the sum in $T_i^{(x)} F(x, y)$ according to whether $\eta_i < \eta_{i+1}$, $\eta_i = \eta_{i+1}$ or $\eta_i > \eta_{i+1}$. Apply (5.2) and collect coefficients of $E_\eta(x; q, t)$. Also, to work out the action of $T_i^{-1}(y)$ on $E_\eta(y; q^{-1}, t^{-1})$ (and hence on $F(x, y)$), set $t \to t^{-1}$ in (5.3). The two sides of (5.4) are equal if and only if (5.5) holds.

With this result at our disposal, the $q$-analogue of Theorem 1.1 can now be given.

**Theorem 5.2** The function $K_q(x; y)$ possesses the following properties:

\[(a) \quad (T_i^{\pm 1}(x) K_A(x; y; q, t) = T_i^{-1}(y) K_A(x; y; q, t)) \]

\[(b) \quad \Phi_q^{(x)} K_A(x; y; q, t) = \Phi_q^{-1}(y) K_A(x; y; q, t)) \]

\[(c) \quad D_i^{(x)} K_A(x; y; q, t) = y_i K_A(x; y; q, t)) \]

**Proof.**

(a) From Lemma 5.1, the constants $A_\eta = \frac{d_{q}(q, t)}{d_{q}(q, t)e_{q}(q, t)}$ satisfy the conditions of Lemma 5.1.
When acting on symmetric functions

(b) From Lemma 2.1 and Corollaries 4.2, 4.4 we have hence the result.

(c) From (4.1) we have

\[ x = t^{n-i} T_{i}^{-1} \cdots T_{i-1} \]

where the second form follows from applying the involution \( \sim \) to the first form. Also, from (4.2)

\[ D_i = t^{-n+i} T_{i-1}^{-1} \cdots T_{i-1} \Phi_q T_{n-1} \cdots T_i \]

The result now follows from these two expressions for \( x, D_i \) by means of (a), (b) and the fact that operators acting on different sets of variables commute.

It remains to present the analogue of (1.13). For the \( q \)-analogue of Sym, Macdonald [14] has introduced the operator

\[ U^+ := \sum_{w \in S_n} T_w \]

(5.6)

where \( w = s_{i_1} \cdots s_{i_p} \) (\( 1 \leq i_1, \ldots, i_p \leq n-1 \)) is the reduced decomposition in terms of elementary transpositions of each element of \( S_n \) and

\[ T_w = T_{i_1} \cdots T_{i_p} \]

(5.7)

The operators \( T_i \) used by Macdonald satisfy \( (T_i - t)(T_i + t^{-1}) = 0 \) as distinct from (2.4); consequently in [14] \( U^+ \) is defined with \( T_w \) multiplied by \( t^\ell(w) \), where \( \ell(w) \) is the length of the permutation \( w \), i.e. the number of elementary transpositions in its reduced decomposition. As noted in [14], use of (2.4) and (2.5) shows that \( T_i U^+ = t U^+ \) which from the definition (2.1) implies that \( U^+ f \) is symmetric in \( x_1, \ldots, x_n \). In particular, for some proportionality constant \( a_q(q, t) \), we must have

\[ U^+ E_q(x; q, t) = a_q(q, t) P_{q^+}(x; q, t) \]

(5.8)

where \( P_{q^+} \) denotes the symmetric Macdonald polynomial normalized so that the coefficient of the leading term is unity.

Our interest here is the action of \( U^+ \) on the kernel \( K_A(x; y; q, t) \). For this we require Theorem 5.2 (a), (b) and (5.8) as well as the result of the following lemma.

**Lemma 5.3** Define

\[ (1-q)E_{0,m} := \sum_{i=m}^{n} A_{i,m} \frac{(1-\tau_i)}{x_i} \quad \text{where} \quad A_{i,m} := \prod_{j=m}^{n} \frac{t x_i - x_j}{x_i - x_j}, \]

When acting on symmetric functions

\[ \sum_{i=m}^{n} D_i = (1-q)E_{0,m} \]

(5.9)
Proof. When acting on symmetric functions, we see from (3.7) and (2.1) that
\[ D_j = T_j \ldots T_{n-1} x_n^{-1} (1 - \tau_n). \]
In particular \( D_n = x_n^{-1} (1 - \tau_n) \) so (5.9) is true for \( m = n \). Thus by induction (7.9) is equivalent to the statement that
\[ D_{m-1} = (1 - q)(E_{0,m-1} - E_{0,m}) \]
Noting that
\[ A_{i,m} = \left(1 + (t - 1) \frac{x_i}{x_i - x_{m-1}}\right) A_{i,m} \]
we see that (5.11) can be rewritten to read
\[ D_{m-1} = A_{m-1,m-1} \frac{1 - \tau_{m-1}}{x_{m-1}} + (t - 1) \sum_{i=m}^{n} \frac{x_i}{x_i - x_{m-1}} A_{i,m} \frac{1 - \tau_i}{x_i} := R_{m-1} \]
Since \( R_n = D_n \), and from (5.10) \( T_j^{-1} D_j = D_{j+1} \), to establish (5.12) it suffices to show
\[ T_{m-1}^{-1} R_{m-1} = R_m \]
This can be verified by direct calculation using (5.12) and (2.8).

We are now ready to calculate the action of \( U^+ \) on \( K_A(x; y; q, t) \).

**Proposition 5.4** We have
\[ U^+(x) K_A(x; y; q, t) = [n]! \cdot 0 F_0(x; y; q, t) \]
where \([n]! := \prod_{i=1}^{n} (1 - t^i)/(1 - t)\), and with \( \kappa \) denoting a partition and \( b(\kappa) := \sum_{i=1}^{n} (i - 1) \kappa_i \),
\[ 0 F_0(x; y; q, t) := \sum_{\kappa} d_\kappa(x; q, t) P(1, t, \ldots, t^{n-1}; q, t) P_{\kappa}(x; q, t) P_{\kappa}(y; q, t). \]

**Proof.** Applying \( U^+ \) to the \( x \)-variables in (5.1) gives
\[ U^+(x) K_A(x; y; q, t) = \sum_{\eta} d_{\eta}(x; q, t) a_{\eta}(x; q, t) P_{\eta^+}(x; q, t) E_{\eta}(y; q^{-1}, t^{-1}) \]
Repeating this operation and use of Theorem 5.2 (a) shows
\[ U^+(x) K_A(x; y; q, t) = \sum_{\eta^+} \alpha_{\eta^+}(q, t) P_{\eta^+}(x; q, t) P_{\eta^+}(y; q, t) \]
where we have used the fact [13] that \( P_{\eta^+}(y; q^{-1}, t^{-1}) = P_{\eta^+}(y; q, t) \). To specify \( \alpha_{\eta^+} \) sum Theorem 5.2 (b) over \( i \), apply \( U^+(x) \) to both sides of the resulting equation and commute its action to the right of \( \sum_i D_i \) on the l.h.s. (Lemma 3.1) shows that this is valid), and use Lemma 5.3 to substitute for \( \sum_i D_i \) to show
\[ (1 - q) E_{0,1}^{(x)} \sum_{\eta^+} \alpha_{\eta^+}(q, t) P_{\eta^+}(x; q, t) P_{\eta^+}(y; q, t) = p_1(y) \sum_{\eta^+} \alpha_{\eta^+}(q, t) P_{\eta^+}(x; q, t) P_{\eta^+}(y; q, t) \]
Recent results of Lassalle [12, Theorems 3 and 5] give the action of the operator \( E_{0,1}^{(x)} \) on \( P_{\eta^+}(x; q, t)/P_{\eta^+}(1, \ldots, t^{n-1}; q, t) \) and an expression for the product \( p_1(y) P_{\eta^+}(y; q, t)/d_{\eta}(q, t) \) in terms of generalized binomial coefficients. These formulas imply a recurrence for the quantity
\[ \alpha_{\eta^+} d'_{\eta^+} P_{\eta^+}(1, \ldots, t^{n-1}; q, t) \] 

with the unique solution \[ \alpha_{\eta^+} d'_{\eta^+} P_{\eta^+}(1, \ldots, t^{n-1}; q, t) = t^{b(\eta^+)} \alpha_0. \]

The fact that \[ U+1 = \sum_{w \in S_n} t^{l(w)} = [n]! \] gives \[ \alpha_0 = [n]! t. \]

As an application of Proposition 5.4 we can specify the proportionality constant \( a_\eta \) in (5.8).

For this we also require the formula [14, 16]

\[ P_{\kappa}(y; q, t) = d'_{\kappa}(q, t) \sum_{\eta: \eta + \kappa = 1} d'_{\eta}(q, t) E_{\eta}(y; q, t), \] (5.17)

which is the analogue of (1.11). We first substitute (5.16) on the l.h.s. of (5.14), then substitute for \( P_{\eta^+}(y; q, t) \) on the l.h.s. using \( P_{\eta^+}(y; q, t) = P_{\eta^+}(y; q^{-1}, t^{-1}) \) = RHS (5.17) \( q \mapsto q^{-1}, t \mapsto t^{-1} \). Next we note from the definition (2.17) that

\[ \frac{d'_{\eta^+}(q^{-1}, t^{-1})}{d'_{\eta}(q^{-1}, t^{-1})} = \prod_{s \in \eta^+, r \in \eta} t^{l(r) - l(s)} \frac{d'_{\eta^+}(q, t)}{d'_{\eta}(q, t)} \]

and equate coefficients of \( P_{\eta^+}(x; q, t) E_{\eta}(y; q^{-1}, t^{-1}) \) on both sides to conclude

\[ a_\eta(x; q, t) = [n]! t^{\sum_{r \in \eta} l(r)} \frac{e_{\eta}(q, t)}{P_{\eta^+}(1, \ldots, t^{n-1}; q, t) d_{\eta}(q, t)}. \] (5.18)

Substituting (5.18) in (5.8), we obtain the \( q \)-analogue of (1.12).

The function \( \alpha_0 F_0(x; y; q, t) \) appears in an unpublished manuscript of Macdonald [13], as well as the recent work of Lassalle [12]. Kaneko [8] introduced a similar function, with \( t^{b(\kappa)} \) in (5.15) replaced by \( (-1)^{|\kappa|} q^{b(\kappa')} \).

References

[1] T. H. Baker and P. J. Forrester. The Calogero–Sutherland model and polynomials with prescribed symmetry. solv-int/9609010, to appear in Nuc. Phys. B.

[2] T. H. Baker and P. J. Forrester. Non–symmetric Jack polynomials and integral kernels. q-alg/9612003.

[3] I. Cherednik. A unification of the Knizhnik–Zamolodchikov and Dunkl operators via affine Hecke algebras. Inv. Math., 106:411–432, 1991.

[4] C. F. Dunkl. Intertwining operators and polynomials associated with the symmetric group. Monat. Math. to appear.

[5] C. F. Dunkl. Differential-difference operators associated to reflection groups. Trans. Amer. Math. Soc., 311:167–183, 1989.

[6] C. F. Dunkl. Integral kernels with reflection group invariance. Canad. J. Math., 43:1213–1227, 1991.

[7] C. F. Dunkl. Hankel transforms associated to finite reflection groups. In D. St. Richards, editor, Contemp. Math., volume 138, pages 123–138, 1992.

[8] J. Kaneko. \( q \)-Selberg integrals and Macdonald polynomials. Ann. Sci. Éc. Norm. Sup. 4e série, 29:1086–1110, 1996.

[9] A. N. Kirillov and M. Noumi. Affine Hecke algebras and raising operators for Macdonald polynomials. q-alg/9605004.
[10] F. Knop and S. Sahi. A recursion and combinatorial formula for Jack polynomials. q-alg/9610016.

[11] L. Lapointe and L. Vinet. Rodrigues formulas for the Macdonald polynomials. q-alg/9607025.

[12] M. Lassalle. Coefﬁcients binomiaux généralisés et polynômes de Macdonald. preprint.

[13] I. G. Macdonald. Hypergeometric functions. Unpublished manuscript.

[14] I. G. Macdonald. Affine Hecke algebras and orthogonal polynomials. Séminaire Bourbaki, 47ème année, Publ. I. R. M. A. Strasbourg, 797, 1994-95.

[15] I. G. Macdonald. Symmetric functions and Hall polynomials. Oxford University Press, Oxford, 2nd edition, 1995.

[16] K. Mimachi and M. Noumi. A reproducing kernel for nonsymmetric Macdonald polynomials. q-alg/9610014.

[17] S. Sahi. A new scalar product for nonsymmetric Jack polynomials. Int. Math. Res. Not., 20:997–1004, 1996.