RULED SURFACES OF FINITE TYPE IN 3-DIMENSIONAL HEISENBERG GROUP

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Abstract. In this paper, on the first, we prove $\Delta r = 2H$ where $\Delta$ is the Laplacian operator, $r = (r_1, r_2, r_3)$ the position vector field and $H$ is the mean curvature vector field of a surface $S$ in the 3-dimensional Heisenberg group $H_3$. In the second, we classify the ruled surfaces by straight geodesic lines, which are of finite type in $H_3$. The straight geodesic lines belong to $\ker \omega$, where $\omega$ is the Darboux form.

Keywords
Heisenberg group, ruled minimal surface, finite type surface, straight geodesic lines, Laplacian operator.

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1 Introduction

1.1 Submanifolds of finite type in 3-dimensional space

Finite type submanifolds were introduced by B.-Y. Chen. A submanifold $M^n$ of an Euclidean space $E^{n+p}$ is said to be of finite type if each component of its position vector field $r$ can be written as a finite sum of eigenfunctions of the Laplacian $\Delta$ of $M^n$, i.e. if $r = r_o + r_1 + r_2 + ... + r_k$ where $r_o$ is a constant and $r_1, r_2, ..., r_k$ non constant maps such that $\Delta r_i = \lambda_i r_i, \lambda_i \in \mathbb{R}, i = 1, \ldots, k$. If, all eigenvalues $\lambda_i, i = 1, \ldots, k$ are different, $M^n$ is said to be of $k$-type.

The interesting examples of finite type submanifolds are some surfaces of 3-dimensional spaces of the Euclidean space $E^3$.

During 1980 – 1990, [6, 7, 8, 9] B.-Y. Chen introduced the notion of finite type immersion in the dimensional Euclidean space $E^m$, in the pseudo Euclidean space and in the riemannian manifolds. In the Euclidean space $E^3$, this notion is, in one way, a natural extension of the notion of minimal surfaces, which will be itself, an extension of totally geodesic surfaces.

If we represent a such hypersurface in $E^m$ for $m = 3$ we have in $E^3$

$$S : r(x, y) = (r_1(x, y), r_2(x, y), r_3(x, y)), (x, y) \in D \subseteq \mathbb{R}^2,$$

It is well known that in $E^3$, for all regular surface $S$

$$\Delta r = -2H$$

where $\Delta$ is the Laplace operator and $H$ is the mean curvature vector field of $S$ The components $(r_1, r_2, r_3)$ of $r$ are differentiable functions so that

$$\Delta r(x, y) = (\Delta r_1(x, y), \Delta r_2(x, y), \Delta r_3(x, y))$$

From (1.2), minimal surfaces and spheres verify $\Delta H = \lambda H, \lambda \in \mathbb{R}$. (1.2) shows that $S$ is minimal surface of $E^3$, if and only if, $r_i, i = 1, 2, 3$ are harmonic.
If the matrix \((g_{ij})\) consists on the components of the induced any metric on \(S\), and \((g^{ij})\) its inverse and \(D = \det (g_{ij})\); the Laplacian (Beltrami’s operator) \(\Delta\) on \(S\) is given by

\[
\Delta = \frac{1}{\sqrt{|D|}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x^i} \left( \sqrt{|D|} g^{ij} \frac{\partial}{\partial x^j} \right).
\]

The classification of 1-type submanifolds of Euclidean space \(\mathbb{E}^3\) was done in 1966 by T. Takahashi [21]. He proved that the submanifolds in \(\mathbb{R}^m\) satisfy the differential equation

\[
\Delta r = \lambda r,
\]

for some real number \(\lambda\), if and only if, either the submanifold is a minimal submanifold of \(\mathbb{R}^m\), (\(\lambda = 0\)), or a hypersphere of \(\mathbb{R}^m\) centered at the origin (\(\lambda \neq 0\)). O. J. Garay [15] generalized (1.5) where he studied hypersurfaces in \(\mathbb{R}^m\), not necessarily associated to the same eigenvalue. He linked and considered hypersurfaces in \(\mathbb{R}^m\) satisfying the differential equation

\[
\Delta r = Ar, A \in M(m, \mathbb{R}).
\]

Next, F. Dillen, J. Pas and L. Verstraelen [10] observed and proposed the study of submanifolds of \(\mathbb{R}^m\) satisfying the following equation

\[
\Delta r = Ar + B, B \in \mathbb{R}^m.
\]

The same and others authors studied several problems on link to the subject of finite type particular surfaces like translation surfaces, the quadrics, surfaces of revolution, helicoidal surfaces,...

The study of this notion of finite type was extended for surfaces in Euclidean 3-space \(\mathbb{E}^3\) of specific form such that their Gauss map \(N\) satisfies an analogous or similar equation \(\Delta N = AN, A \in M(m, \mathbb{R})\).

In the same way, many authors studied particular surfaces of finite type in the Euclidean, pseudo Euclidean and Lorentz-Minkowski 3-dimensional space satisfying the differential equation \(\Delta^{II} r = Ar, \Delta^{III} r = Ar\), where \(\Delta^{II}, \Delta^{III}\) are respectively the Laplace operator with respect to the second and third fundamental form which are not degenerated. For these works of different authors, see the references and therein. Among from others, we have B.-Y. Chen, L. J. Alias, A. Ferrandez, C. Baikoussis, D. E. Blair, Piccinni, S. M. Choi, Kim Y. H., Yoon, D. W, P. Lucas... The list of authors working in the subject is very long and certainly not closed.

In our work, first, we proof that (1.2) stays true in Heisenberg space \(\mathbb{H}_3\) and classify ruled surfaces by the straight lines as geodesics of the Heisenberg group \(\mathbb{H}_3\) which are of finite type. Explicitly, we search ruled surfaces by the straight lines geodesics of \(\mathbb{H}_3\) which satisfy

\[
\Delta r_i = \lambda_i r_i, \lambda_i \in \mathbb{R}, i = 1, 2, 3.
\]

## 1.2 Minimal surfaces in 3-dimensional space.

Let \(z = f(x, y)\) be a graph of a regular surface \(S\) in the Euclidean space \(\mathbb{E}^3\). \(S\) is minimal surface if \(f\) satisfies

\[
f_{xx} (1 + f_y^2) - 2f_x f_y f_{xy} + f_{yy} (1 + f_x^2) = 0, f_x = \frac{\partial f}{\partial x}, \ldots
\]
This equation obtained by J. L. Lagrange in 1776 where he used the variational calculus. In 1775, J. B. Meusnier gave a geometric interpretation to this, said that $S$ is minimal if its mean curvature function $H \equiv 0$. Some particular solutions have been given by the same or others authors until 1866 when Weierstrass solved (1.8).

The author construct examples of minimal surfaces with the choice of two complex functions so that the parameters in representation of $S$ are isothermal.

Since, more mathematicians work and study different directions on minimal surfaces and several questions therein, like to found minimal surfaces in other spaces, like in Lorentz space raised but unsuccessfully..., until in 1982 when Do Carmo M. found an original and beautiful minimal surface.

S. N. Bernstein, proved (1915 − 1917) that "there are no other complete graphs, except the planes, which are minimal surfaces in $\mathbb{E}^3$.

In 1990, a natural question raised for the $\mathbb{H}_3$, are there, as $\mathbb{E}^3$, minimal surfaces in $\mathbb{H}_3$?

This space is the 3-dimensional Heisenberg group which can be seen as $\mathbb{R}^3$ equipped with a riemannian metric

$$ds^2_{\mathbb{H}_3} = dx^2 + dy^2 + \left(dx + \frac{1}{2}(ydx - xdy)\right)^2.$$  

To know deeply the geometry of $\mathbb{H}_3$, in 1991, the author [2] searched and wrote the equation of minimal surfaces as a graph of functions $z = f(x, y)$

$$(1.9) \quad f_{xx} \left(1 + \left(f_y - \frac{1}{2}\right)^2\right) - 2f_{xy} \left(f_x + \frac{1}{2}\right) \left(f_y - \frac{1}{2}\right) + f_{yy} \left(1 + \left(f_x + \frac{1}{2}\right)^2\right) = 0,$$

and gave some particular solutions [2]. Among these, ones are already minimal surfaces in $\mathbb{E}^3$ like planes, helicoids. The other particular solution $f(x, y) = \frac{x^2}{2}$ of (1.9), which is not solution of (1.8), is the hyperbolic paraboloid which have a very important part as a minimal surface in $\mathbb{H}_3$.

In an other work, T. Sari together with the author [3] gave a complete description of minimal surfaces ruled by straight lines as geodesics of $\mathbb{H}_3$ and lines. By the analogous method used by Weirstrass process to solve (1.8), F. Mercuri, S. Montaldo and P. Piu solved (1.9), [20]. Other works of different authors followed. Bernstein’s theorem is not valid in $\mathbb{H}_3$ since $z = \frac{x^2}{2}$ is a complete minimal surface, other the plane.

2 Preleminaries

2.1 Heisenberg space

- Heisenberg group $\mathbb{H}_3$ is a two-step nilpotent Lie group which is a subgroup of linear group $Gl(3, \mathbb{R})$. It is known as a quantic physic model strongly studied by the theoritical physicists and mathematicians. It is a 3-dimensional riemannian manifold equiped with the riemannian metric

$$(2.1) \quad ds^2_{\mathbb{H}_3} = dx^2 + dy^2 + \omega^2$$

3
where \( \omega = dz + \frac{1}{2}(ydx - xdy) \), is an Pfaffian form, known as Darboux form, it is also an contact form.

- As a Lie group, \( \mathbb{H}_3 \) acts by left translations keeping invariant \( ds^2_{\mathbb{H}_3} \)
  \[
  \mathbb{H}_3 = \left\{ \begin{pmatrix} x & z & 0 \\ 0 & y & 0 \\ 0 & 0 & 1 \end{pmatrix}, (x, y, z) \in \mathbb{R}^3 \right\}.
  \]

The one to one map
\[
\left( \begin{array}{c} x \\ y \\ z \end{array} \right) \mapsto \left( \begin{array}{c} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{array} \right)
\]
induces a group structure.

- \( \mathbb{H}_3 \) is equipped with the no commutatif group structure \((\ast)\) given by
  \[
  (x, y, z) \ast (x', y', z') = (x + x', y + y', z + z' + \frac{1}{2}(x'y - xy')),
  \]
  \( \forall (x, y, z), (x', y', z') \in \mathbb{R}^3. \)

- These 1-forms \( dx, dy, \omega \) in \( ds^2_{\mathbb{H}_3} \) are invariant by left translations in \( \mathbb{H}_3 \) and by rotations about \((Oz)\) axis. The left invariant orthonormal coframe is associate with the orthonormal left invariant frame

\[
\{ \partial_x, \partial_y, \partial_z \} \text{ denote the natural vector fields in } \mathbb{R}^3. \text{ The Lie brackets are }
\]
\[
[e_2, e_3] = [e_3, e_1] = 0, [e_1, e_2] = e_3.
\]

- The Levi-Civita connection denoted \( \tilde{\nabla} \) of \( \mathbb{H}_3 \) is given by
  \[
  \begin{pmatrix}
  \tilde{\nabla}_{e_1} e_1 \\
  \tilde{\nabla}_{e_1} e_2 \\
  \tilde{\nabla}_{e_1} e_3
  \end{pmatrix} = \begin{pmatrix}
  0 \\
  \frac{1}{2} e_3 \\
  -\frac{1}{2} e_2
  \end{pmatrix},
  \begin{pmatrix}
  \tilde{\nabla}_{e_2} e_1 \\
  \tilde{\nabla}_{e_2} e_2 \\
  \tilde{\nabla}_{e_2} e_3
  \end{pmatrix} = \begin{pmatrix}
  -\frac{1}{2} e_3 \\
  0 \\
  \frac{1}{2} e_1
  \end{pmatrix},
  \begin{pmatrix}
  \tilde{\nabla}_{e_3} e_1 \\
  \tilde{\nabla}_{e_3} e_2 \\
  \tilde{\nabla}_{e_3} e_3
  \end{pmatrix} = \begin{pmatrix}
  -\frac{1}{2} e_2 \\
  \frac{1}{2} e_1 \\
  0
  \end{pmatrix}.
  \]

- Let \( \mathcal{S} \) in \( \mathbb{H}_3 \) be an orientable surface and \( r : \mathcal{S} \rightarrow \mathbb{H}_3 \) an immersion. Denote by \( \nabla \) the induced Levi-Civita connection on \( \mathcal{S} \). If \( \mathbf{N} \) is the unit normal vector on \( \mathcal{S} \), we have the well known Gauss and Weingarten formulae for riemannian manifold and hypersurfaces

\[
\begin{cases}
  \tilde{\nabla}_X Y = \nabla_X Y + h(X, Y)\mathbf{N}, h(X, Y) = g \left( \tilde{\nabla}_X Y, \mathbf{N} \right) \\
  \tilde{\nabla}_X \mathbf{N} = -A_X
  \end{cases}
\]
Euclidean metrics.

A constant equal to one. It is an important characteristic of endomorphism with respect to the metric on \( S \) in invariant Heisenberg metrics \( ds \), its Riemannian metric \( H \). The mean curvature of the immersion \( r \) is \( H = \frac{1}{2}tr(A) \).

- \( H_3 \) is a homogenous Riemannian manifold after Euclidean space \( \mathbb{E}^3 \), the sphere \( S^3 \) and hyperbolic space \( \mathbb{H}^3 \). Its isometry group is 4-dimensional and it is isomorphic to semi direct sum \( SO(2) \triangleright \mathbb{R}^3 \). Explicitly, the component of identity is the affine group of transformations of \( \mathbb{R}^3 \)

\[
\begin{pmatrix}
  x \\
  y \\
  z
\end{pmatrix} \rightarrow \begin{pmatrix}
\cos \theta & -\sin \theta & 0 \\
\sin \theta & \cos \theta & 0 \\
A & B & 1
\end{pmatrix} \begin{pmatrix}
x \\
y \\
z
\end{pmatrix} + \begin{pmatrix}
a \\
b \\
c
\end{pmatrix}
\]

where \( A = \frac{1}{2}(a \sin \theta - b \cos \theta) \), \( B = \frac{1}{2}(a \cos \theta + b \sin \theta) \).

The isometry group of \( \mathbb{H}_3 \) is described by \( \{(\theta, a, b, c) ; \theta, a, b, c \in \mathbb{R}\} \). It contains the rotations \( (\theta, 0, 0, 0) \) about \( (Oz) \) and the left translations \( (0, a, b, c) \).

**Remarks**

1. The left translations of the isometry group of \( \mathbb{H}_3 \) don’t keep left invariant Euclidean metrics. As the same, the Euclidean translations in \( \mathbb{H}_3 \) don’t keep invariant Heisenberg metrics \( ds^2_{\mathbb{H}_3} \).

2. The determinant of matrix of \( \mathbb{H}_3 \), and the one associated to the metric is constant equal to one. It is an important characteristic of \( \mathbb{H}_3 \) and especially for its Riemannian metric \( ds^2_{\mathbb{H}_3} \).

- As in \( \mathbb{E}^3 \), let \( z = f(x,y) \) be a graph of a regular surface \( S \) in \( \mathbb{H}_3 \) with its position vector field

\[
r(x, y) = (x, y, f(x, y)); (x, y) \in \mathcal{D} \subseteq \mathbb{R}^2.
\]

The first fundamental form of \( S \) is obtained like trace of \( ds^2_{\mathbb{H}_3} \) on \( S \),

\[
ds^2_{\mathbb{H}_3, S} = g_{i,j} = dx^2 + dy^2 + (f_x dx + f_y dy + \frac{1}{2}(ydx - xdy))^2
\]

\[
= (1 + P^2) dx^2 + 2PQ dx dy + (1 + Q^2) dy^2.
\]

where \( P = f_x + \frac{y}{2}, Q = f_y - \frac{x}{2} \).

- A basis of tangent space \( T_p S \) on \( p \in S \) associated to

\[
S : r(x, y) = (x, y, f(x, y))
\]

is given by

\[(2.3)\]

\[
\begin{align*}
x_x = (1, 0, f_x) = \partial_x + f_x \partial z = c_1 + Pe_3 \\
y_y = (0, 1, f_y) = \partial_y + f_y \partial z = c_2 + Qe_4
\end{align*}
\]

The coefficients of the first fundamental form of \( S \) are

\[(2.4)\]

\[
\begin{align*}
E = g(r_x, r_x) = 1 + P^2; G = g(r_y, r_y) = 1 + Q^2 \\
F = g(r_x, r_y) = PQ; EG - F^2 = 1 + P^2 + Q^2 = W^2
\end{align*}
\]

- The unit normal vector field \( \mathcal{N} \) on \( S \) is
In order to compute the coefficients of the second fundamental form of $S$, we have to calculate the following

\[
\begin{align*}
\{ r_{xx} &= \nabla_{r_x} r_x = -P e_2 + f_{xx} e_3 \\
\{ r_{xy} &= \nabla_{r_x} r_y = \nabla_{r_y} r_x = \frac{\partial}{\partial x} e_1 - \frac{\partial}{\partial y} e_2 + f_{xy} e_3 \\
\{ r_{yy} &= \nabla_{r_y} r_y = Q e_1 + f_{yy} e_3 \\
\end{align*}
\]

which imply that the coefficients of the second fundamental form of $S$ are given by

\[
\begin{align*}
L &= -g(\nabla_{r_x} r_x, N) = \frac{f_{xx} + PQ}{W} \\
M &= -g(\nabla_{r_x} r_y, N) = \frac{f_{xy} + \frac{1}{2} Q^2 - \frac{1}{2} P^2}{W} \\
N &= -g(\nabla_{r_y} r_y, N) = \frac{f_{yy} - PQ}{W} \\
\end{align*}
\]

In [3] we have some expressions which simplify the computation

\[
\begin{align*}
P_x &= f_{xx}, Q_y = f_{yy}, P_y = f_{xy} + \frac{1}{2}, Q_x = f_{xy} - \frac{1}{2}, P_y - Q_x = 1 \\
\begin{align*}
P_y + Q_x &= 2f_{xy}; Q &= -uP, u^2 P_x + u P_y + u Q_x + Q_y = 0 \\
\end{align*}
\]

If we put $H_1 = EN - 2FM + GL$, the mean curvature $H$ of $S$, with the help of (2.7), is

\[
\begin{align*}
H &= \frac{H_1}{W}, H_1 = \frac{1}{W} (P_x + Q_y) = \frac{1}{W} (f_{xx} + f_{yy}) \\
\end{align*}
\]

2.2 Characterization of minimal surfaces in $\mathbb{H}_3$

We proved in [3], the theorem 2. The minimal surfaces of $\mathbb{H}_3$ ruled by straight geodesic lines are locally the graph of harmonic functions. In the next theorem, we prove, $S$ is a minimal surfaces in $\mathbb{H}_3$, if and only if its coordinate functions are harmonic.

2.2.1 Theorem

The Beltrami formula in $\mathbb{H}_3$ is given by $\Delta r = 2H$, $\Delta$ is the Laplacian of the surface with respect the first fundamental form of the surface $S: r(x, y)$ in $\mathbb{H}_3$, $H$ is the mean curvature vector field of $S$.

**Proof**

We have explicitly the Beltrami operator (1.4)

\[
\Delta = \frac{1}{W} \left( \frac{\partial}{\partial x} \left( W g^{11} \frac{\partial}{\partial x} + W g^{12} \frac{\partial}{\partial y} \right) + \frac{\partial}{\partial y} \left( W g^{21} \frac{\partial}{\partial x} + W g^{22} \frac{\partial}{\partial y} \right) \right). \\
\]

The matrix associated to the first fundamental form of $S$ is

\[
(g_{ij}) = \begin{pmatrix} E & F \\ F & G \end{pmatrix}, (g^{ij}) = \begin{pmatrix} G & \frac{F}{EG - F^2} \\ \frac{F}{EG - F^2} & \frac{1}{EG - F^2} \end{pmatrix}, \det (g_{ij}) = W^2. \\
\]

The operator $\Delta$ turns out to

\[
\Delta = \frac{1}{W} \left[ \left( \frac{\partial}{\partial x} \left( G \frac{\partial}{\partial x} - F \frac{\partial}{\partial y} \right) \right) + \left( \frac{\partial}{\partial y} \left( -F \frac{\partial}{\partial x} + E \frac{\partial}{\partial y} \right) \right) \right]. \\
\]

The first part of $\Delta$
The second part

\[
\Delta_2 = -\frac{1}{W^2} \left( \frac{\partial}{\partial y} \left( \frac{G \frac{\partial}{\partial x} - F \frac{\partial}{\partial y}}{W^2} \right) \right)
\]

Thus, \( \Delta \) turns out to

\[
\Delta = -\frac{1}{W^2} \left( G \frac{\partial^2}{\partial x^2} - 2F \frac{\partial^2}{\partial x \partial y} + E \frac{\partial^2}{\partial y^2} \right) - (Q + PQ_y - QQ_x) \frac{\partial}{\partial y} + ((P + QQ_y) F) \frac{\partial}{\partial y}
\]

With the help of (2.3) – (2.7)

\[
\Delta = -\frac{1}{W^2} \left( (FP \gamma - GPP \gamma + FQQ \gamma - GQQ \gamma) e_1
\right.
\]

\[
+ (FPX + FQQX - EPP \gamma - EQQ \gamma) e_2
\]

\[
+ (FP2P \gamma - GP2P \gamma + FQ2Q \gamma - Q2EQ \gamma)
\]

\[
\left. + (FPX + FQQ \gamma - GPX - PQ \gamma) e_3 \right).
\]

The computation gives

\[
\Delta r = 2H \sum \left( P_x + Q_y + Q^2P_x + P^2Q_y - PQ - 2PQQ_x \right) = 2H.
\]

Remark

We can, also write the Beltrami operator \( \Delta \) in the form

\[
\Delta = -\frac{1}{W^2} \left( W^2 \left( G \frac{\partial^2}{\partial x^2} - 2F \frac{\partial^2}{\partial x \partial y} + E \frac{\partial^2}{\partial y^2} \right) \right) - (PH_1 + QW^2) \frac{\partial}{\partial x} + (-QH_1 + PW^2) \frac{\partial}{\partial y}
\]

Thus, by a straightforward computation, and using (2.3) – (2.8), we obtain

\[
\Delta r = -\frac{1}{W^2} \left( Gr_{xx} - 2Fr_{xy} + Er_{yy} - Qr_x + Pr_y\right) + \frac{1}{W^2} H_1 (Pr_x + Qr_y)
\]

which give (2.10).

Hence (1.2) is verified in \( \mathbb{H}_3 \). \( S \) is a minimal surface in \( \mathbb{H}_3 \), if only if the functions \( r_i, i = 1, 2, 3 \) are harmonic. We conclude, as in the Euclidean space \( \mathbb{E}^3 \), all the minimal surfaces in \( \mathbb{H}_3 \) are of finite type.
2.3 Ruled surfaces by geodesic lines in $\mathbb{H}_3$.

In [17], Th. Hangan solved the Euler-Lagrange system of geodesics in $\mathbb{H}_3$ and gave explicitly the solutions. Among these, the author obtained a straight lines which are in $\ker \omega$, $\omega$ is the Darboux form.

After writing (1.9) and giving some particular solutions in [2], in 1992, T. Sari together with the author gave a complete description of minimal surfaces ruled by straight lines as geodesics and straight lines in $\mathbb{H}_3$ [3].

In fact and to summarise, if we get the geodesic $\Gamma$ coming from a point $p = (x(0), y(0), z(0))$ of $\mathbb{H}_3$ and tangent to the vector $v = (x'(0), y'(0), z'(0))$ of $T_p \mathbb{H}_3$, $\Gamma$ is an straight line in the two cases, when $v$ is perpendicular which says that $\Gamma$ is a perpendicular line or $v$ is in $\ker \omega$, saying that $\Gamma$ is a line in $\ker \omega$.

Explicitly, let a surface $S$ in $\mathbb{H}_3$ parametrized by

$$(t, s) \rightarrow \alpha(t) + s \beta(t), \alpha(t) \in \mathbb{R}^3, \beta(t) \in \mathbb{R}^3 \setminus \{0\}.$$  

$S$ is ruled by the straight line $L$ stem from the point $\alpha(t)$ and has $\beta(t)$ as director vector.

Up to isometry of $\mathbb{H}_3$ and an appropriate choice of the curve $\alpha(t)$, we obtain two families of ruled minimal surfaces by geodesic lines [3] which are

$1^o) \alpha(t) = (t, a(t), 0), \beta(t) = (0, 0, 1)$

$2^o) \alpha(t) = (t, 0, a(t)), \beta(t) = (u(t), 1, \frac{1}{2})$

where $a(t)$ and $u(t)$ are differentiable functions of $t$.

For convenience, in the following, we used the notations of [3].

3 Ruled surfaces by geodesic lines satisfying $\Delta r_i = \lambda_i r_i$ in $\mathbb{H}_3$.

3.1 The first family of ruled surfaces by geodesic lines satisfying $\Delta r_i = \lambda_i r_i$

Take a regular surface $S : r(t, s)$ in $\mathbb{H}_3$. The Laplacian operator with respect to the first fundamental form for $S$ is, for each coordinate $r_i, i = 1, 2, 3$

(3.1) $\Delta r_i = -\frac{1}{r_i}\left(\left(\frac{G_{r_i r_i} - F_{r_i r_i}}{W} \right)_{t} + \left(\frac{F_{r_i r_j} + E_{r_i r_j}}{W} \right)_{s} \right) = \lambda_i r_i$

If we denote by $S_1$ the first ruled surface

$S_1 : r(t, s) = (r_1(t, s), r_2(t, s), r_3(t, s)) = (t, a(t), 0) + s(0, 0, 1) = (t, a(t), s)$

we have [3], by (2.3), (2.6), the coefficients of the fundamental forms

$$\begin{align*}
E &= 1 + (a')^2 + \frac{1}{4} (a - ta')^2, G = 1, F = \frac{1}{2} (a - ta') \cdot W^2 = 1 + (a')^2 \\
L &= a'' - \frac{1}{2} (a - ta') (1 + (a')^2), M = -\frac{1}{2} (1 + (a')^2), N = 0.
\end{align*}$$

The mean curvature of $S_1 : r(t, s)$ is

$$H = \frac{EN + GL - 2FM}{2(EG - F^2)} = \frac{a''}{2(1 + (a')^2)} = \frac{a''}{2W^2}.$$
The system (3.1) turns out for each \( i = 1, 2, 3 \) to

- For \( r_1(t, s) = t, r_{1,t} = 1, r_{1,t} = 0, r_{1,s} = 0, r_{1,s} = 0, \)
  \[
  Gr_{1,t} - Fr_{1,s} = 1, -Fr_{1,t} + Er_{1,s} = -\frac{1}{2}(a - ta'),
  \Delta r_1(t, s) = \frac{2a'''}{2(1+(a')^2)^2} = \frac{2H}{W^2}.
  \]

- For \( r_2(t, s) = a(t), r_{2,t} = a'; r_{2,s} = 0, \)
  \[
  Gr_{2,t} - Fr_{2,s} = a', -Fr_{2,t} + Er_{2,s} = -\frac{1}{2}(a - ta') a',
  \Delta r_2(t, s) = \frac{a''}{(1+(a')^2)^2} = \frac{-2H}{W^2}.
  \]

- For \( r_3(t, s) = s, r_{3,t} = 0, r_{3,s} = 1, \)
  \[
  Gr_{3,t} - Fr_{3,s} = -\frac{1}{2}(a - ta'), -Fr_{3,t} + Er_{3,s} = 1 + (a')^2 + \frac{1}{2}(a - ta')^2,
  \Delta r_3(t, s) = \frac{-1}{2(1+(a')^2)^2} a'' (t + aa') = \frac{-((t+aa')H)}{W^2}.
  \]

Then the system (1.7) with help of (3.1) turns out to

\[
\begin{aligned}
\frac{2H}{W^2} &= \lambda_1 t \\ -\frac{H}{W^2} &= \lambda_2 a(t) \\ -\frac{(t+aa')H}{W^2} &= \lambda_3 s
\end{aligned}
\]

Therefore, the problem of classifying the ruled surfaces \( r(t, s) = (t, a(t), s) \) by a straight geodesics in \( \mathbb{H}_3 \) satisfying (1.7) is reduced to the integration of the ordinary differential equations of (3.2).

Next we study it according to the constants \( \lambda_i, i = 1, 2, 3 \).

The unique case is \( \lambda_3 = 0 \) since the two parts of (3) are independants. The system (3.2) turn to

\[
\begin{aligned}
\frac{2H}{W^2} &= \lambda_1 t \\ \frac{-H}{W^2} &= \lambda_2 a(t) \\ \frac{(t+aa')H}{W^2} &= 0
\end{aligned}
\]

In the last form, (3) implies \( H \equiv 0 \) or \( aa' = -t \)

i) If \( H \equiv 0 \), then we get \( \lambda_1 = \lambda_2 = 0 \) and the surface \( S_1 : r(t, s) = (t, a(t), s) \) is minimal.

ii) If \( H \neq 0 \) and \( aa' + t = 0 \), from which we have

\[
aa' = -t \implies a^2 = -t^2 + c, a(t) = \pm \sqrt{c - t^2}, c \in \mathbb{R}^+, t \in [-c, c].
\]

By multiplying (2) by \( a' \) and using \( aa' = -t \), (3.2.a) became

\[
\begin{aligned}
\frac{2H}{W^2} &= -\lambda_1 aa' \\ -\frac{H}{W^2} &= \lambda_2 aa' \\ \frac{(t+aa')H}{W^2} &= 0
\end{aligned}
\]

and the sum \((1) + (2) = 0 \) which implies \( \lambda_1 = \lambda_2 \).

In fact, since \( H = \frac{a''}{2W^2}, aa' = -t, W^2 = 1 + (a')^2, \) (1) turns out to

\[
\left( \frac{1}{1+(a')^2} \right)' = \lambda_1 (a^2)' \iff \frac{1}{c} (c - t^2)' = \lambda_1 (c - t^2)'.
\]

We obtain \( \lambda_1 = \frac{1}{c} \). The same happen with (2). We conclude \( \lambda_1 = \lambda_2 = \frac{1}{c} \).
3.1.1 Theorem

Let $S_1$ be a surface ruled by straight lines as geodesics in the 3-dimensional $\mathbb{H}_3$ parametrized by

$$S_1 : r(t, s) = (r_1(t, s), r_2(t, s), r_3(t, s)) = (t, a(t), s).$$

Then, $r_i(t, s)$ satisfies the equation $\Delta r_i = \lambda_i r_i, \lambda_i \in \mathbb{R}, i = 1, 2, 3,$ up to isometry of $\mathbb{H}_3$, if and only if

1°) $S_1$ has zero mean curvature, $S_1 : r(t, s) = (t, a(t) = \alpha t + \beta, s); \alpha, \beta \in \mathbb{R},$

2°) The Euclidean circular cylinder

$$S_1 : r(t, s) = (t, a(t) = \pm \sqrt{c - t^2}, s); c \in \mathbb{R}^+, t \in [-c, c], s \in \mathbb{R}.$$

In [1], [9], [10], [14] the authors proved that, the only well known examples of surfaces of finite type in Euclidean space $\mathbb{E}^3$ are the sphere, the circular cylinder, and naturally, the minimal surfaces. They proved and gave the circular cylinder in Euclidean space as the only ruled surface.

We observe in the above theorem, that the circular cylinder, is the only ruled surface by a geodesics lines which stays of finite type in $\mathbb{H}_3$ besides minimal surfaces in $\mathbb{H}_3$ which satisfies $\Delta r_i = \lambda_i r_i, \lambda_i \in \mathbb{R}, i = 1, 2, 3$.

3.2 The second family of surfaces ruled by geodesics lines satisfying $\Delta \tilde{r}_i = \lambda_i \tilde{r}_i$

For the second family of ruled surfaces parametrized as ruled surface

$$S_2 : \tilde{r}(x, y) = (\tilde{r}_1(x, y) = x, \tilde{r}_2(x, y) = y, \tilde{r}_3(x, y) = z)$$

where $(x, y, z)$ are themselves expressed in the parameters $(t, s)$ as

$$S_2 : (x(t, s) = t + su(t), y(t, s) = s, z(t, s) = a(t) + \frac{ts}{su}) = (t, 0, a(t)) + s \left( u(t), 1, \frac{1}{u} \right).$$

The equation $x(t, s) = t + su(t)$ defines implicitly the function $t = t(x, y)$ which implies $S_2$, is locally as the graph of a differential function $f$. From these remarks, about $t(x, y)$ and $f(x, y)$ we have

$$dx = (1 + su') dt +uds, dy = ds, dz = f_x ((1 + su') dt +uds) + fy ds$$

where

$$t_x = \frac{1}{1+su'}, t_y = -ut_x; f_x = (a' + \frac{u}{2}) t_x, f_y = t_y (a' + \frac{u}{2}) + \frac{t}{2};$$

By help with (2.3) – (2.7), the coefficients of the fundamental forms,

$$\begin{cases}
E = 1 + \left( (a' + \frac{u}{2}) t_x + \frac{u}{2} \right)^2 \\
G = 1 + u^2 \left( (a' + \frac{u}{2}) t_x + \frac{u}{2} \right)^2 \\
F = -u \left( (a' + \frac{u}{2}) t_x + \frac{u}{2} \right)^2.
\end{cases}$$
The corresponding Laplacian operator for $S_2$ is, for each component $\tilde{r}_i(x, y)$

$$\Delta \tilde{r}_i = -\frac{1}{\sqrt{EG-F^2}} \left( \frac{G\tilde{r}_{ix} - F\tilde{r}_{iy}}{\sqrt{EG-F^2}} \right)_x + \left( \frac{F\tilde{r}_{ix} + E\tilde{r}_{iy}}{\sqrt{EG-F^2}} \right)_y = \lambda_i \tilde{r}_i \tag{3.3}$$

As in the previous paragraph, for each coordinate $\tilde{r}_i$, $i = 1, 2, 3$, we have

- For $\tilde{r}_1(x, y) = x(t, s) = t + yu(t)$,
  $$\tilde{r}_{1,x} = t_x + yu't_x = 1; \tilde{r}_{1,y} = t_y + uy't_y = 0$$
  $$G\tilde{r}_{1,x} - F\tilde{r}_{1,y} = 1 + Q^2, -F\tilde{r}_{1,x} + E\tilde{r}_{1,y} = -PQ,$$
  (3.3), turns out to, for $i = 1$

$$\Delta x = -\frac{1}{W} \left( \left( \frac{(1+Q^2)}{W} \right)_x + \left( \frac{-PQ}{W} \right)_y \right)$$

thus, by help with $Q = -uP$ we have

$$\Delta x = \frac{1}{W} \left( PP_x + PQ_y + P^3 Q_y + P^3 u^2 P_x - P^3 u^3 P_y + P^3 u^3 Q_x - Pu P_y + Pu Q_x + 2P^3 u Q_x \right).$$

- For $\tilde{r}_2(x, y) = y(t, s) = s, \tilde{r}_{2,x} = 0; \tilde{r}_{2,y} = 1,$
  $$G\tilde{r}_{2,x} - F\tilde{r}_{2,y} = -PQ, -F\tilde{r}_{2,x} + E\tilde{r}_{2,y} = 1 + P^2,$$
  (3.3) turns out for $i = 2$

$$\Delta y = -\frac{1}{W} \left( \left( \frac{-PQ}{W} \right)_x + \left( \frac{1+P^2}{W} \right)_y \right)$$

and we obtain, with the help of $Q = -uP$

$$\Delta y(t, s) = -\frac{1}{W} \left( PP_y - PQ_x + P^3 P_y - P^3 Q_x + 2P^3 u^2 P_y + P^3 u^3 P_x + Pu P_y + Pu Q_y + P^3 u Q_y \right).$$

- For $\tilde{r}_3(x, y) = f(x, y) = a(t) + \frac{1}{2} P, \tilde{r}_{3,x} = P - \frac{1}{2} P, \tilde{r}_{3,y} = Q + \frac{1}{2} Q$,
  $$\left\{ \begin{array}{l}
  G\tilde{r}_{3,x} - F\tilde{r}_{3,y} = P - \frac{1}{2} P - \frac{1}{2} y PQ - \frac{1}{2} y Q^2, \\
  -F\tilde{r}_{3,x} + E\tilde{r}_{3,y} = Q + \frac{1}{2} x + \frac{1}{2} y PQ + \frac{1}{2} x P^2
  \end{array} \right.$$
  (3.3) turns out for $i = 3$

$$\Delta f(x, y) = -\frac{1}{W} \left( \left( \frac{P - \frac{1}{2}(y+\frac{y}{x}PQ+\frac{y}{x}Q^2)}{W} \right)_x + \left( \frac{Q+\frac{1}{2}(x+yPQ+\frac{x}{y} P^2)}{W} \right)_y \right)$$

A straightforward computation using (2.4) – (2.8) gives

$$= -\frac{1}{2W^4} \left( \begin{array}{c}
  2P_x + 2Q_y + 2P^2 u^2 P_x + 2P^2 u^2 P_y + P_x P_y - P_x Q_x + Py P_y + Py Q_y - P^3 u y + 2P^2 u P_y + 2P^2 u Q_x + P^3 x P_y - P^3 x Q_x + P^3 u^2 Q_x + P^3 u^2 P_x + P^3 u^3 x P_x + P^3 u^3 x P_y + P^3 u^3 x Q_y + P^3 u^3 Q_x + P^3 u^3 Q_y + P^3 u^3 Q_y + P^3 u^3 Q_y + P^3 u^3 Q_y + P^3 u^3 Q_y \end{array} \right)$$
which give the third equation
\[ \Delta f(x, y) = -\frac{1}{2W^2} \left( (P_x + P_y)(2 + Pux + yP) + PW^2(x - yu) \right) \]

and the system \( \Delta \tilde{\tau}_i = \tilde{\lambda}_i \tilde{\tau}_i, i = 1, 2, 3 \) turns out to
\[
\begin{align*}
(3.5) \quad & -\frac{P_x}{W} (W^2 - (P_x + P_y)) = \tilde{\lambda}_1 x \\
& -\frac{P_y}{W} (uW^2 - (P_x + P_y)) = \tilde{\lambda}_2 y \\
& -\frac{P}{W^2} ((P_x + P_y)(2 + Pux + yP) + P(x - yu)W^2) = \tilde{\lambda}_3 \tilde{\tau}_3(x, y)
\end{align*}
\]

or, by help of (2.7), (2.8) in the form
\[
\begin{align*}
(3.6.a) \quad & -\frac{1}{W} (Pu - 2HPW) = \tilde{\lambda}_1 x \\
& -\frac{1}{W} (P + 2uHPW) = \tilde{\lambda}_2 y \\
& -\frac{1}{W} \left( \frac{1}{2} P (x - uy) + HW(2 + Pux + Puy) \right) = \tilde{\lambda}_3 f(x, y)
\end{align*}
\]

Multiplying by \( y \) and by \( x \), (1), and (2) respectively in (3.6.a), we have
\[
\begin{align*}
(1)' & -\frac{1}{W} (yu - \lambda_1 xy) = \tilde{\lambda}_1 x y \\
(2)' & -\frac{1}{W} (x, y) = \tilde{\lambda}_2 y x
\end{align*}
\]

The left part of (3), by help of (1)' and (2)' turns out to
\[
-\frac{1}{W} \left( \frac{1}{2} P (x - uy) + HW(2 + Pux + Puy) \right) = -\frac{1}{2W^2} (P + 2PuxHW) + \frac{1}{2W^2} (Puy - 2PyHW) - \frac{2}{W^2} HW = \tilde{\lambda}_3 f(x, y).
\]

Thus (3), became
\[
\frac{2H}{W} = \left( \frac{\tilde{\lambda}_1}{2} - \frac{\tilde{\lambda}_2}{2} \right) xy - \tilde{\lambda}_3 f(x, y).
\]

Again, the system (3.6.a) becomes
\[
(3.6.b) \quad \begin{align*}
(1) & -\frac{P_x}{W} (u - 2HW) = \tilde{\lambda}_1 x \\
(2) & -\frac{P_y}{W} (1 + 2uHW) = \tilde{\lambda}_2 y \\
(3) & \tilde{\tau}_i = \tilde{\lambda}_2 - \tilde{\lambda}_1 \text{ } xy - 2\tilde{\lambda}_3 f(x, y)
\end{align*}
\]

Therefore, as the same in previous paragraph, the problem of classifying the ruled surfaces by a straight geodesics in \( \mathbb{H}_3 \) satisfying (1.7) is the integration of the ordinary differential equations of (3.6.b).

Next we study it according to the constants \( \tilde{\lambda}_i, i = 1, 2, 3 \).

With the help of (2.8), the system (3.6) turns to
\[
(3.6.c) \quad \begin{align*}
(1) & -\frac{P_x}{W} (u - H_1) = \tilde{\lambda}_1 x \\
(2) & -\frac{P_y}{W} (W + uH_1) = \tilde{\lambda}_2 y \\
(3) & \tilde{\lambda}_3 H_1 = \left( \tilde{\lambda}_2 - \tilde{\lambda}_1 \right) xy - 2\tilde{\lambda}_3 f
\end{align*}
\]

Case 1. Let \( \tilde{\lambda}_3 = 0 \).

i) If \( \tilde{\lambda}_1 = \tilde{\lambda}_2 \), (3) implies \( H_1 = 0 \) say \( H = 0 \). Thus \( S \) is minimal and consequently \( \tilde{\lambda}_1 = \tilde{\lambda}_2 = 0 \).

ii) If \( \tilde{\lambda}_1 = 0, \tilde{\lambda}_2 \neq 0 \). From (1) we have \( P = 0 \) or \( H_1 = uW \).

a.1) For \( P = 0 \), (2) give \( \tilde{\lambda}_2 = 0 \). The same begins if \( \tilde{\lambda}_2 = 0 \) and \( \tilde{\lambda}_1 \neq 0 \). We get contradiction.
Thus, we conclude the same as obtained in i), \( \tilde{\lambda}_1 = \tilde{\lambda}_2 = \tilde{\lambda}_3 = 0. \)

a.2) For \( H_1 = uW \) with \( P \neq 0 \), the equations (2) and (3) in (3.6.c) turns out to

\[
\left\{ \begin{array}{l}
-\frac{P}{uW} (1 + u^2) = \tilde{\lambda}_2 y \\
\frac{\partial}{\partial u} = \tilde{\lambda}_2 xy
\end{array} \right. \tag{2}
\]

The corresponding surface for \( \tilde{\lambda}_2 \neq 0, \tilde{\lambda}_1 = \tilde{\lambda}_3 = 0 \), is \( f(x, y) \) solution of

\( f_x = -\frac{2ux}{1+u^2} - \frac{y}{2}. \)

We obtain the same as a.1), if \( \tilde{\lambda}_2 = 0, \tilde{\lambda}_1 \neq 0 \) when \( P = 0. \n
\)

a.3) For \( uH_1 = -W \) with \( P \neq 0 \), the equations (1) and (3) in (3.6.c) turns out to

\[
\left\{ \begin{array}{l}
-\frac{P}{uW} (1 + u^2) = \tilde{\lambda}_1 x \\
\frac{\partial}{\partial u} = -\tilde{\lambda}_1 xy
\end{array} \right. \tag{3}
\]

The corresponding surface for \( \tilde{\lambda}_1 \neq 0, \tilde{\lambda}_2 = \tilde{\lambda}_3 = 0 \), is \( f(x, y) \) solution of

\( f_x = \frac{2}{y(1+u^2)} - \frac{y}{2}. \)

iii) If \( \tilde{\lambda}_1 \neq \tilde{\lambda}_2, \tilde{\lambda}_1 \tilde{\lambda}_2 \neq 0 \). We put \( \tilde{\lambda}_2 - \tilde{\lambda}_1 = \tilde{\lambda} \). The equation (3) turn to

\[
f_{xx} + f_{yy} = \frac{1}{2} \tilde{\lambda}_2 xy \left( 1 + (f_x + \frac{y}{2})^2 (1 + u^2) \right)^2. \tag{3.9}
\]

Case 2. Let \( \tilde{\lambda}_3 \neq 0 \). We replay the system (3.6.c)

iii) a) If \( \tilde{\lambda}_1 = 0 \) we have \( P = 0 \), then \( \tilde{\lambda}_2 = 0 \). The same begins if \( \tilde{\lambda}_2 = 0 \), we obtain \( \tilde{\lambda}_1 = 0 \). Thus, we conclude \( \tilde{\lambda}_3 \neq 0, \tilde{\lambda}_1 = \tilde{\lambda}_2 = 0 \). Since \( P = 0 \) implies \( Q = -uP = 0 \) and (3) becomes

\[
\frac{\partial}{\partial u} (f_{xx} + f_{yy}) = 0 \Rightarrow -\tilde{\lambda}_3 f = 0.
\]

Thus \( \tilde{\lambda}_3 = 0 \). Contradiction, see Case 1, i).

b) Also, if \( \tilde{\lambda}_1 = 0 \), with \( P \neq 0 \), from (1) we have \( H_1 = uW \) which implies that (2) became

\[
-\frac{P}{uW} (u^2 + 1) = \tilde{\lambda}_2 y
\]

then \( \tilde{\lambda}_2 \neq 0 \). The equation (3) turn out to

\[
f_{xx} + f_{yy} = \frac{1}{2} \left( \tilde{\lambda}_2 xy - 2\tilde{\lambda}_3 f \right) \left( 1 + (f_x + \frac{y}{2})^2 (1 + u^2) \right)^2. \tag{3.10}
\]

c) If \( \tilde{\lambda}_2 = 0 \), with \( P \neq 0 \), from (2) we have \( uH_1 = -W \) which implies that (1) became

\[
-\frac{P}{uW} (u^2 + 1) = \tilde{\lambda}_1 x
\]

then \( \tilde{\lambda}_1 \neq 0 \). The equation (3) turn to

\[
f_{xx} + f_{yy} = -\frac{1}{2} \left( \tilde{\lambda}_1 xy + 2\tilde{\lambda}_3 f \right) \left( 1 + (f_x + \frac{y}{2})^2 (1 + u^2) \right)^2. \tag{3.11}
\]

iv) a) If \( \tilde{\lambda}_2 = \tilde{\lambda}_1 \neq 0 \). The equation (3) turn to

\[
f_{xx} + f_{yy} = -\tilde{\lambda}_3 f \left( 1 + (f_x + \frac{y}{2})^2 (1 + u^2) \right)^2. \tag{3.12}
\]
v) a) If $\tilde{\lambda}_2 = \tilde{\lambda}_1 = \tilde{\lambda}_3 = \tilde{\lambda} \neq 0$. The equation (3.3) turn to

$$f_{xx} + f_{yy} = -\tilde{\lambda} f \left( 1 + \left( f_x + f_y \right)^2 + \left( 1 + u^2 \right)^2 \right)$$

To solve and to give explicitly the solutions of (3.7) - (3.13) is, in my opinion, difficult. The solutions were in the forms of elliptic integrals which are expressed in Legendre forms, see [16] pp. 276-280.

### 3.2.1 Theorem

Let $S_2$ be a surface ruled by straight lines as geodesics in the 3-dimensional $\mathbb{H}_3$ parametrized by

$$S_2 : \tilde{r}(x, y) = (\tilde{r}_1(x, y) = x, \tilde{r}_2(x, y) = y, \tilde{r}_3(x, y) = f(x, y))$$

where $(x, y, z)$ are themselves expressed in the parameters $(t, s)$ as

$$S_2 : (x(t, s) = t + su(t), y(t, s) = s, f(x(t, s), y(t, s)) = a(t) + \frac{u}{2})$$

Then, $\tilde{r}_i$ satisfies the equation $\Delta \tilde{r}_i = \tilde{\lambda}_i \tilde{r}_i, \tilde{\lambda}_i \in \mathbb{R}, i = 1, 2, 3$, up to isometry of $\mathbb{H}_3$, if and only if

1°) $S_2$ has zero mean curvature, $S_2 : \tilde{r}(x(t, s), y(t, s)) = (t + su(t), s, a(t) + \frac{u}{2})$

2°) $S_2$ is the graphs, solutions of the obvious partial differential equations (3.7) - (3.13). Among those, there are of them, those of the 1-type, the 2-type and the 3-type surfaces.

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