Scalar Field Cosmologies Hidden Within the Nonlinear Schrödinger Equation

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The nonlinear, cubic Schrödinger (NLS) equation has numerous physical applications, but in general is very difficult to solve. Nonetheless, under certain circumstances parameters quantifying the width, momentum and energy of the wavefunction evolve under a closed set of ordinary differential equations. It is shown that for the case of the radial, two dimensional NLS equation, such evolution equations may be mapped directly onto the cosmological Friedmann equations for a spatially flat and isotropic universe sourced by a self-interacting scalar field and a barotropic perfect fluid. Consequently, algorithms for finding exact solutions are presented and the scaling solutions determined. A form-invariance of the wavefunction evolution equations is identified. The analysis has direct applications to anisotropic Bose-Einstein condensation. The Ermakov-Pinney equation plays a central role in establishing the correspondence between the quantum-mechanical and gravitational systems.

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I. INTRODUCTION

In the absence of a consistent, non-perturbative theory of quantum gravity, it is important to establish connections between quantum mechanical and gravitational systems. One route towards such a goal is to uncover direct links between the underlying equations of motion for different models.

From the quantum mechanical perspective, an important family of equations are nonlinear Schrödinger (NLS) equations, which arise for example in the fields of Bose-Einstein condensation [1–5] and quantum optics [6]. In \( d \) spatial dimensions, the NLS equation with a \( \text{U}(1) \)-invariant interaction term has the generic form:

\[
\begin{align*}
\frac{i}{\partial \tau} u &= -\frac{1}{2} \nabla^2 u + V(r, \tau)u + \nu(\tau)|u|^{2(m-1)}u, \\
\end{align*}
\]

where \( \nabla^2 \) is the \( d \)-dimensional spatial Laplacian, \( V(r, \tau) \) represents a time-dependent potential, \( \nu(\tau) \) is a time-dependent coupling parameter and \( m \) is a constant. Due to the nonlinearities involved, it is difficult (if not impossible) to find exact solutions to such an equation. Nonetheless, insight into the properties of the wavefunction can be gained by employing the ‘moment method’ [7–11]. In this approach, physical quantities such as the width, momentum and energy of the quantum system can be defined in terms of integral relations involving the wavefunction. These ‘moments’ satisfy a set of coupled, linear, first-order ordinary differential equations (ODEs) that derive from the NLS equation.

On the other hand, cosmology provides a natural environment for the study of gravitational physics. The Friedmann equations (the time-time and space-space components of the Einstein field equations) are a cornerstone of modern cosmology and relate the expansion of the universe to the total energy density contained within it. Of particular interest are spatially isotropic cosmological models that are sourced by a mixture of self-interacting scalar fields and perfect (barotropic) fluids. Such models play an important role in the inflationary scenario of the very early universe and are a leading candidate for the origin of the dark energy that dominates the universe at the current epoch (for reviews, see, e.g., [12–15]). Scalar fields also arise ubiquitously in cosmologies inspired by superstring/M-theory (see Refs. [16, 17] and references therein).

The purpose of the present paper is to uncover a connection between the dynamics associated with such cosmological models and the radially symmetric, cubic NLS equation. The link is made possible due to the central role played by a nonlinear, second-order ODE of the form

\[
\frac{d^2 X(\tau)}{d\tau^2} + \lambda(\tau)X(\tau) = \frac{Q}{X^3(\tau)},
\]

where \( X(\tau) \) and \( \lambda(\tau) \) are functions of the independent variable \( \tau \) and \( Q \) is a constant. Eq. (2) is known as the Ermakov-Pinney (EP) equation [18–20]. It plays a fundamental role in diverse branches of mathematical and theoretical physics,
ranging from nonlinear optics, nonlinear elasticity, molecular structures, Bose-Einstein condensation, cosmology and quantum cosmology. (A brief survey of the algebraic properties and physical applications of the EP equation can be found in Refs. \cite{21, 22}) Its role in cosmology has been discussed by a number of authors \cite{23, 28}.

For the classes of quantum and cosmological models we consider, the physical interpretations of the dependent and independent variables in Eq. (2) differ. This allows a dictionary relating different variables to be established. It is then found that the equations of motion for the quantum and gravitational models are equivalent, in the sense that there exists a one-to-one correspondence that maps the cosmological Friedmann equations onto the evolution equations of the wavefunction moments (and vice-versa). In other words, the field equations for the two systems are formally interchangeable.

Consequently, analytical techniques that have been developed previously for studying the early- and late-time dynamics of the universe can be employed to gain insight into the nature of the wavefunction of the NLS equation. Conversely, nonlinear quantum dynamics can be employed to develop an alternative formulation of the Einstein equations.

We focus on applying the ‘Hamilton-Jacobi’ (HJ) formalism of the cosmological field equations \cite{30, 36} to the moment equations of motion. In this present-work, the scalar field is viewed as the independent dynamical variable rather than cosmic time. This allows the second-order Friedmann equations to be expressed explicitly as a first-order system. Within the context of the present-work, we present an alternative way of expressing the moment evolution equations in terms of a coupled, first-order system of ODEs. A form-invariance of the ODE’s is uncovered, in the sense that the set of ODEs is invariant under a non-local transformation group of the variables \cite{37}.

The paper is organized as follows. Sections \textbf{II} and \textbf{III} briefly review scalar field cosmology and the moment method, respectively. The dictionary between different cosmological and quantum-mechanical models is established in Section \textbf{IV} The HJ approach to cosmology is applied in Section \textbf{V} and the underlying form-invariance of the differential equations made manifest. In Section \textbf{VI} we present algorithms for finding exact solutions and determine the analogues of cosmological scaling solutions. We conclude in Section \textbf{VII} with a discussion.

Unless otherwise stated, units are chosen such that $\hbar = c = 1$ and $m_P = \sqrt{8\pi}$, where $m_P$ is the Planck mass.

\section*{II. COSMOLOGICAL FRIEDMANN EQUATIONS}

The field equations for the spatially flat, isotropic, Friedmann-Robertson-Walker (FRW) universe sourced by a barotropic perfect fluid and a minimally coupled scalar field $\phi$ self-interacting through a potential $W(\phi)$ are given by the Friedmann equation

$$3H^2 = \frac{1}{2} \left( \frac{d\phi}{dt} \right)^2 + W(\phi) + \frac{D}{a^n},$$

and the scalar field equation of motion:

$$\frac{d^2\phi}{dt^2} + 3H \frac{d\phi}{dt} + \frac{dW}{d\phi} = 0,$$

where $H \equiv d\ln a/ dt$ is the Hubble parameter of the scale factor $a(t)$, $t$ denotes cosmic time, $\rho_{\text{mat}} = Da^{-n}$ is the energy density of the fluid with an equation of state $p_{\text{mat}} = [(n - 3)/3] \rho_{\text{mat}}$ and $\{D, n\}$ are constants. Within the context of Einstein gravity, causality implies the bounds $D > 0$ and $0 \leq n \leq 6$. For $n < 0$ (corresponding to ‘phantom’ matter), the null energy condition is violated, $\rho_{\text{mat}} + p_{\text{mat}} < 0$. For $n > 6$ (corresponding to ultra-stiff matter), the speed of sound in the fluid exceeds the speed of light, $c_s = \sqrt{dp/d\rho} > 1$. For $D < 0$, the weak energy condition is violated, $\rho_{\text{mat}} < 0$. However, since we are interested in establishing correspondences between different systems, we leave the fluid parameters unspecified in what follows. Indeed, it is worth noting that quantum cosmological considerations typically lead to corrections to the standard Friedmann equation, where $D$ is effectively negative. This is the case for example in braneworld \cite{38} and loop quantum cosmology \cite{39} scenarios, where a term proportional to $-\rho^2_{\text{mat}}$ generically arises. The above equations also apply to spatially curved FRW universes ($n = 2$) and the anisotropic, Bianchi I cosmology ($n = 6$).

Differentiating the Friedmann equation (4) and substituting for Eq. (4) yields the Raychaudhuri equation:

$$\frac{1}{a} \frac{d^2 a}{dt^2} - \frac{1}{a^2} \left( \frac{da}{dt} \right)^2 = -\frac{1}{2} \left( \frac{d\phi}{dt} \right)^2 + \frac{nD}{3a^n}.$$

We may rewrite this equation by defining an effective scale factor:

$$b \equiv a^{n/2}$$ (6)
and a new time parameter through the relation:

\[
\frac{d}{dt} = a^{n/2} \frac{d}{d\tau}.
\] (7)

It follows that Eq. (5) transforms into the EP equation (23):

\[
\frac{d^2 b}{d\tau^2} + \frac{n}{4} \left( \frac{d\phi}{d\tau} \right)^2 b = -Dn^2 \frac{1}{12} b^3.
\] (8)

The inclusion of a perfect fluid in the energy-momentum tensor \( (D \neq 0) \) results in the nonlinear cubic term on the right-hand side of Eq. (8).

Finally, we note for future reference that the scalar field equation (4) can be written as the first-order differential equation:

\[
\frac{d\rho_\phi}{dt} = -3H \left( \frac{d\phi}{dt} \right)^2,
\] (9)

where \( \rho_\phi \equiv \frac{1}{2} \left( \frac{d\phi}{dt} \right)^2 + W(\phi) \) represents the energy density of the field.

### III. NONLINEAR SCHRÖDINGER EQUATION AND THE MOMENT METHOD

In this paper, we focus on the two-dimensional, radially symmetric form of Eq. (1) with a cubic interaction term \( (m = 2) \) and an harmonic potential given by \( V(r, \tau) = \lambda(\tau) r^2/2 \), where \( \lambda(\tau) \) is a time-dependent frequency. We further assume that the parameter \( \nu \) is constant. In this case, Eq. (1) reduces to

\[
i \frac{\partial u}{\partial \tau} = -\frac{1}{2r} \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) + \frac{1}{2} \lambda(\tau) r^2 u + \nu |u|^2 u.
\] (10)

We refer to the independent variable, \( \tau \), as ‘laboratory time’ throughout the discussion. Within the context of Bose-Einstein condensation (3–5), Eq. (10) is known as the Gross-Pitaevskii equation (1, 2). It determines the dynamics when the condensate is tightly bound along one direction by the trapping potential \( V(r, \tau) \), in which case a quasi-two-dimensional system is established. The parameter \( \nu \) is proportional to the \( s \)-wave scattering length that quantifies the strength of the atomic interactions.

The moments associated with the wavefunction are defined by the integral quantities (8):

\[
I_1(\tau) = \int |u|^2 d^2 x,
\] (11)

\[
I_2(\tau) = \int |u|^2 r^2 d^2 x,
\] (12)

\[
I_3(\tau) = i \int \left( u \frac{\partial u^*}{\partial r} - u^* \frac{\partial u}{\partial r} \right) r d^2 x,
\] (13)

\[
I_4(\tau) = \frac{1}{2} \int \left( |\nabla u|^2 + \nu |u|^4 \right) d^2 x,
\] (14)

respectively, where \( d^2 x = r dr d\theta \) and integration over the angular coordinate yields a factor of \( 2\pi \). A star denotes complex conjugate. The moments \( I_j \) \((j = 2, 3, 4) \) quantify the width, radial momentum and energy of the quantum configuration, respectively.

Differentiating Eqs. (11)-(14) with respect to laboratory time, substituting Eq. (10) and its complex conjugate, and integrating by parts a number of times yields (after some algebra) a set of coupled, first-order, linear ODEs (8):

\[
\frac{dI_1}{d\tau} = 0
\] (15)

\[
\frac{dI_2}{d\tau} = I_3,
\] (16)

\[
\frac{dI_3}{d\tau} = -2\lambda(\tau)I_2 + 4I_4,
\] (17)

\[
\frac{dI_4}{d\tau} = -\frac{1}{2} \lambda(\tau)I_3.
\] (18)
Eqs. (15)-(18) represent a set of closed evolution laws and admit an invariant under time evolution:

\[
Q = 2I_2I_4 - \frac{1}{4}I_2^2, \tag{19}
\]

as may be verified by direct differentiation and substitution. This constant may be negative for sufficiently negative \(\nu\).

Given the constraint equation (19), the system (15)-(18) can be reduced to a single, second-order differential equation

\[
\frac{d^2I_2}{d\tau^2} - \frac{1}{2I_2} \left( \frac{dI_2}{d\tau} \right)^2 + 2\lambda(\tau)I_2 = \frac{2Q}{I_2} \tag{20}
\]

and Eq. (20) transforms into the EP equation

\[
\frac{d^2X}{d\tau^2} + \lambda(\tau)X = \frac{Q}{X^3}, \tag{21}
\]

where \(X(\tau) \equiv I_2^{1/2}\) determines the width of the wavepacket. Hence, solutions to the moment equations (16)-(18) follow directly from solutions to the EP equation (21) (and vice-versa). Since the \(L^2\)-norm of the wavefunction is conserved, the amplitude of the wavefunction is approximately \(A \sim 1/I_2\), i.e., the inverse square of the width of the wavepacket.

IV. A DICTIONARY BETWEEN QUANTUM-MECHANICAL AND COSMOLOGICAL SYSTEMS

A direct comparison between the ‘cosmological’ EP equation (8) and the ‘Schrödinger’ EP equation (21) immediately suggests that a formal correspondence may be established between appropriate cosmic and wavefunction parameters. Specifically, we identify

\[
I_2(\tau(t)) = a^n(t), \tag{22}
\]

\[
\lambda(\tau(t)) = \frac{n}{4} \left( \frac{d\phi}{dt} \right)^2, \tag{23}
\]

\[
Q = -\frac{Dn^2}{12}, \tag{24}
\]

where the relation between cosmic and laboratory times is given by integrating Eq. (7):

\[
t(t) = \int^t \frac{d\tau}{\sqrt{I_2(\tau)}}, \quad \tau(t) = \int^t dt\ a^{n/2}(t). \tag{25}
\]

It then follows from Eqs. (7) and (16) that

\[
I_3(\tau(t)) = na^{n/2}(t)H(t) \tag{26}
\]

and further insight may be gained if we identify the moment \(I_4\) with the energy density of the scalar field such that

\[
I_4(\tau(t)) = \frac{n^2}{24}\rho_\phi(t). \tag{27}
\]

It may now be verified after substitution of Eqs. (22), (26) and (27) into the constraint equation (19) that the latter corresponds precisely to the Friedmann equation (3). Furthermore, some straightforward algebra implies that when the above correspondence is applied, the equation of motion for the moment \(I_4\), Eq. (14), transforms directly into the scalar field equation (9) and that the corresponding equation for \(I_3\), Eq. (13), transforms directly into the Raychaudhuri equation (5). The evolution equation for \(I_2\), Eq. (12), corresponds to the definition of the Hubble parameter, \(H \equiv d\ln a/dt\).

From a dynamical point of view, therefore, the gravitational Friedmann equations are equivalent to the coupled evolution laws of the wavefunction moments. In this sense, classical cosmic dynamics is contained within the quantum-mechanical dynamics of the nonlinear, cubic Schrödinger equation (and vice-versa). In other words, the cosmological Friedmann equations can be directly transformed into the equations of motion for the wavefunction moments after appropriate redefinitions of the dependent and independent variables. The correspondence is summarized in Tables I and II.
Table I: A dictionary between the cosmological and wavefunction variables. The cosmological scale factor, Hubble parameter and scalar field energy density are related to the width, momentum and energy of the quantum configuration, respectively. Cosmic time, \( t \), and laboratory time, \( \tau \), are related by Eq. (25). The fraction of the scalar field energy density relative to the total cosmic density can be expressed as a combination of all three wavefunction moments.

| Cosmological Parameter | Wavefunction Moments |
|------------------------|----------------------|
| \( a \)                | \( I_2^{1/n} \)       |
| \( H \)                | \( I_3/(nI_2^{1/2}) \) |
| \( \rho_\phi \)        | \( 24I_4/n^2 \)      |
| \( t \)                | \( \int d\tau I_2^{-1/2} \) |
| \( \Omega_\phi \)      | \( 8I_2I_4/I_3^2 \)  |

Table II: A direct link between the Einstein field equations for a spatially isotropic and flat universe sourced by a self-interacting scalar field and the evolution laws for the wavefunction moments of the nonlinear Schrödinger equation. The equation of motion for \( I_2 \) is equivalent to the definition of the Hubble parameter.

From the cosmological perspective, an important physical parameter is the fraction of the scalar field energy density relative to the total energy density. This is quantified in terms of the ratio \( \Omega_\phi \equiv \rho_\phi/(3H^2) \). In the present context, \( \Omega_\phi \) can be related to a combination of the three moments:

\[
\Omega_\phi = 8\frac{I_2I_4}{I_3^2}, \quad I_3^2 = -\frac{4Q}{1 - \Omega_\phi}.
\]

The above discussion is interesting because it implies that the techniques that have been developed for analyzing the dynamics of scalar field and perfect fluid cosmologies in a variety of different settings can be carried over to study the dynamics of the wavefunction of the cubic Schrödinger equation (and vice-versa). This is the topic of the next Section.

V. HAMILTON-JACOBI FORMULATION AND FORM-INVARiance OF THE MOMENT EVOLUTION EQUATIONS

The cosmological field equations \(^{[2]}\text{(3)}\)-\(^{[2]}\text{(5)}\) can be written in an alternative first-order form by interpreting the scalar field as the effective dynamical variable of the system \(^{[2]}\text{(30)}\). Assuming the field to be evolving monotonically with cosmic time (as is the case, for example, for inflationary models where the field rolls slowly down its potential), Eq. \(^{[2]}\text{(4)}\) can be expressed as

\[
\frac{d\rho_\phi}{d\phi} = -3H \frac{d\phi}{dt}.
\]

From the definition of the Hubble parameter, it then follows that \( 3nH^2 = -\rho_\phi'\chi'/\chi \), where \( \chi \equiv a^n \) and a prime denotes \( d/d\phi \) in this and following Sections. Substituting this expression into Eq. \(^{[2]}\text{(3)}\) implies that the Friedmann equation can be rewritten in the ‘Hamilton-Jacobi’ form:

\[
\frac{d\rho_\phi(\phi)}{d\phi} \frac{d\chi(\phi)}{d\phi} + n\rho_\phi(\phi)\chi(\phi) = -nD.
\]

On the other hand, invoking the dictionary summarized in Table I implies that the scalar field equation of motion \(^{[2]}\text{(29)}\) can also be expressed as

\[
\frac{dI_4}{d\phi} = -\frac{n}{8} I_3 \frac{d\phi}{d\tau}
\]
and multiplying both sides of this expression by $dI_2/d\phi$ yields the relation

$$I_3^2 = \frac{8}{n} \frac{dI_2}{d\phi} \frac{dI_4}{d\phi},$$

where we have employed Eq. (16). As a result, Eq. (19) transforms to

$$\frac{dI_2}{d\phi} \frac{dI_4}{d\phi} + nI_2I_4 = \frac{nQ}{2}.$$  \hspace{1cm} (33)

We have therefore rewritten the moment evolution equations (17)-(19) in the form (31)-(33), where the new independent variable is related to laboratory time by integrating Eq. (23):

$$\phi(\tau) = \frac{2}{\sqrt{n}} \int^\tau \sqrt{\lambda(\tau)}.$$  \hspace{1cm} (34)

We choose the positive root without loss of generality, since the negative root corresponds to a time-reversal. Assuming the scalar field to be a real-valued variable implies $\lambda < 0$ if $n < 0$.

Eq. (33) is a key result of the paper and is interesting for a number of reasons. It is immediately apparent that is is invariant under the simultaneous transformation, $I_2(\phi) \leftrightarrow I_4(\phi)$, that interchanges the width and energy moments (when both are expressed explicitly as functions of the variable, $\phi$). In other words, given a solution \{I_2(\phi), I_4(\phi)\}, we may generate a new ‘dual’ solution \{I_2(\phi), \tilde{I}_4(\phi)\}, where

$$\tilde{I}_2(\phi) = I_4(\phi), \quad \tilde{I}_4(\phi) = I_2(\phi).$$

Moreover, it follows from Eq. (19) that the radial momentum is a singlet under this transformation:

$$\tilde{I}_3(\phi) = I_3(\phi).$$

Eqs. (35) and (36) represent a ‘form-invariance’ of the moment evolution equations, in the sense that Eqs. (32) and (33) preserve their analytical form under the transformation (35)-(36). The invariance becomes manifest when these equations are expressed in terms of the appropriate independent variable, in this case, $\phi$. The two solutions are not equivalent, however, since the potential, $\lambda(\phi)$, is not a singlet. The functional forms of the potential frequencies that generate the solution pair are related through Eq. (31) and it follows that

$$\tilde{\lambda}(\phi) = \frac{4}{\lambda(\phi)}.$$  \hspace{1cm} (37)

Eq. (31) further implies that the relationship between the scalar field and laboratory time changes in the dual solution. Labelling the new functional dependence by $\tilde{\phi}(\tilde{\phi})$, we deduce that

$$\tilde{\phi}(\tilde{\phi}) = \frac{1}{2} \int^{\phi} \frac{I_4'}{(2I_2I_4 - Q)^{1/2}} = -\frac{1}{2} \int d\tau \lambda(\tau).$$

Since the derivatives $I_2'$ and $I_4'$ have opposite signs, the interchange $I_2(\phi) \leftrightarrow I_4(\phi)$ maps an expanding configuration onto a collapsing one, and vice-versa. The analogous cosmological transformation would relate an expanding universe to a contracting one [10]. In this sense, Eqs. (35)-(36) represent an implosion/explosion duality of the quantum system.

Given the relation (38) between the dual ‘laboratory’ times $\tau$ and $\tilde{\tau}$, it is now straightforward to verify that the set of coupled, moment evolution equations (12)-(19) are form-invariant under a redefinition of the dependent and independent variables such that

$$\tilde{I}_2(\tilde{\tau}) = I_4(\tau), \quad \tilde{I}_3(\tilde{\tau}) = I_3(\tau), \quad \tilde{I}_4(\tilde{\tau}) = I_2(\tau),$$

$$\tilde{\lambda}(\tilde{\tau}) = \frac{4}{\lambda(\tau)}, \quad \frac{d}{d\tilde{\tau}} = -2 \frac{d}{\lambda(\tau) d\tau}.$$  \hspace{1cm} (39)

It should be emphasized that whilst the redefinitions (39) leave the equation of motion for $I_3(\tau)$ invariant, they interchange the evolution laws for $I_2(\tau)$ and $I_4(\tau)$. Hence, given a solution set \{I_j(\tau), \lambda(\tau)\}, we may generate a new solution set \{I_j(\tilde{\tau}), \tilde{\lambda}(\tilde{\tau})\}. This can be done analytically if the integral (38) can be evaluated and inverted.

An immediate consequence of Eqs. (39) is that the EP equation (2) exhibits the same form-invariance when the dependent variable is identified as $\sqrt{I_2(\tau)}$ and the moment equations of motion are satisfied. Moreover, the above discussion also applies when $Q = 0$ and therefore to the corresponding LS equation.

We employ the above ‘Hamilton-Jacobi’ formalism for the moment evolution equations in the following Section to find analytical solutions of the system in terms of quadratures.
VI. ALGORITHMS FOR FINDING EXACT ANALYTICAL SOLUTIONS

A. General Solution in terms of Quadratures

Eq. (33) can be solved in full generality in terms of quadratures for either \( I_2(\phi) \) or \( I_4(\phi) \) and their corresponding first derivatives. For example, the width of the configuration associated with the wavefunction is given in terms of its energy by

\[
I_2(\phi) = \exp \left[ -n \int d\phi \frac{I_2}{I_4^2} \right] \times \left[ \Pi_2 + \frac{nQ}{2} \int d\phi \frac{1}{I_4} \exp \left( n \int d\phi \frac{I_4}{I_2} \right) \right],
\]  

(40)

where \( \Pi_2 \) is an arbitrary integration constant. A similar expression arises for the energy of the wavepacket in terms of its width:

\[
I_4(\phi) = \exp \left[ -n \int d\phi \frac{I_2}{I_4} \right] \times \left[ \Pi_4 + \frac{nQ}{2} \int d\phi \frac{1}{I_2} \exp \left( n \int d\phi \frac{I_2}{I_4} \right) \right]
\]  

(41)

for arbitrary constant \( \Pi_4 \).

Thus, the width of the wavepacket can in principle be determined parametrically if the functional form of \( I_4(\phi) \) is specified. In this case, the corresponding expression for the momentum is given by

\[
I_3(\phi) = \pm 2\sqrt{2I_2(\phi)I_4(\phi)} - Q
\]  

(42)

and the potential frequency follows from Eqs. (51) - (52):

\[
\lambda(\phi) = -2 \frac{I_4}{I_2} = \frac{4}{n} \frac{I_2^2}{(2I_2 I_4 - Q)}.
\]  

(43)

Finally, laboratory time is determined parametrically by

\[
\tau(\phi) - \tau_0 = \sqrt{\frac{n}{8}} \int d\phi \left( \frac{I_4}{I_2^2} \right)^{1/2} = \frac{n}{4} \int d\phi \sqrt{\frac{2I_2 I_4 - Q}{I_4}},
\]  

(44)

where we have denoted the integration constant by \( \tau_0 \).

We have therefore expressed the general solution to the moment evolution equations (10) - (18) in parametric form in terms of quadratures with respect to a single (to be specified) function, \( I_4(\phi) \), and its first derivative with respect to the new independent variable, \( \phi \). (The solution is general since there are three independent constants \( \{Q, \Pi, \tau_0\} \).

This suggests an algorithm to find exact solutions to the moment equations. Specify the functional form of the configuration energy as a function of the field \( \phi \) and potential \( \lambda(\phi) \) follow directly from Eqs. (12) and (13), respectively. The dependence of this set of physical parameters on laboratory time then follows by integrating and inverting Eq. (44). The primary challenge in this approach is to successfully complete the last step in the iteration. Nonetheless, we present a solution to the equations of motion by employing this method in the Appendix.

An alternative approach is to first specify the energy density explicitly as a function of the width, i.e., choose an appropriate functional dependence \( I_4 = F[I_2(\phi)] \). Eq. (33) then reduces to an ODE of the form

\[
\left( \frac{dI_2}{d\phi} \right)^2 + n \frac{F}{F^*} I_2 = \frac{nQ}{2} \frac{1}{F^*},
\]  

(45)

where \( F^* = dF/dI_2 \). For cases where Eq. (45) is solvable, the algorithm then proceeds as above.

It should be emphasized that even in cases where a solution in terms of laboratory or cosmic time can not be deduced analytically, important information regarding the dynamics of a given system can still be inferred in terms of the independent variable, \( \phi \). To illustrate this point, let us consider the ansatz

\[
I_4(\phi) = C I_2^{-\beta}(\phi),
\]  

(46)

where \( \{C, \beta\} \) are arbitrary constants and let us further assume that \( n = 1 \) (for algebraic simplicity) and that \( Q > 0 \) (a similar analysis can be performed when \( Q < 0 \)). Eq. (18) reduces to

\[
\frac{\left( \frac{dI_2}{d\phi} \right)^2}{\frac{1}{\beta} \left( I_2^2 - \frac{Q}{2C} I_2^{1+\beta} \right)}
\]  

(47)
and defining a new dependent variable, $z$, such that
\[ I_2 \equiv \left( \frac{2C}{Q} \right)^{1/(\beta-1)} [\operatorname{sech} z]^{2/(\beta-1)} \]  
(48)

then transforms Eq. [47] into the trivial condition
\[ z' = \xi, \quad \xi \equiv \frac{\beta - 1}{2\sqrt{\beta}}, \quad \beta \neq \{0, 1\}. \]  
(49)

Hence, the general solution to Eq. [47] is given by
\[ I_2(\phi) = \left( \frac{2C}{Q} \right)^{1/(\beta-1)} [\operatorname{sech} \xi(\phi - \phi_0)]^{2/(\beta-1)}. \]  
(50)

Since $\phi$ is a monotonically varying function of time, the solution asymptotes from $I_2(-\infty)$ to $I_2(+\infty)$. This represents a configuration that is initially expanding, reaches a point of maximal expansion, and then proceeds to recollapse.

To proceed, it is natural to consider the nature of the solution when $\beta = 1$ in ansatz [46]. It is clear from Eq. [45] that $I_2$ depends exponentially on the scalar field in this case. To understand the physical significance of this solution, we first consider a specific class of cosmological solutions in the following Subsection.

### B. Scaling Solutions

One class of cosmological solutions that are of particular interest are ‘scaling solutions’, whereby the energy densities of the scalar field and perfect fluid vary at the same rate as the universe evolves. (See, e.g., Ref. [11] for an exhaustive list of references). Indeed, these solutions describe a universe that expands (or contracts) as if it were sourced only by the perfect fluid and the scalar field is said to ‘track’ the fluid. These solutions are important because they typically represent early- or late-time attractors and repellors (critical points in the phase space) for more general solutions and they thereby allow one to determine the asymptotic behaviour and stability of more complicated models.

Cosmological scaling solutions are characterized by the condition $\Omega_\phi = \text{constant}$ with $0 < n \leq 6$. This is equivalent to the conditions $I_3 = \text{constant}$ and $I_2 I_4 = \text{constant}$, i.e., to ansatz [46] with $\beta = 1$. It follows from Eq. [48] that $I_2'$ is also $\phi$-invariant. This implies that $I_2(\phi)$ and $I_4(\phi)$ are both purely exponential functions of the scalar field. Consequently, the integration constant in solution [40] must vanish, $\Pi_2 = 0$. The wavefunction analogue of the cosmological scaling solution is therefore readily deduced from Eqs. [42] and [43] and we find that
\[ I_2(\phi) = 2 \left( \frac{Q}{\Omega_\phi - 1} \right)^{1/2} e^{\sqrt{n/\Omega_\phi} \phi}, \quad I_4(\phi) = \frac{\Omega_\phi}{4} \left( \frac{Q}{\Omega_\phi - 1} \right)^{1/2} e^{-\sqrt{n/\Omega_\phi} \phi}. \]  
(51)

Laboratory time is related to the value of the scalar field via $\phi = \sqrt{\Omega_\phi/n} \ln(\tau - \tau_0)$, where $\tau_0$ is an arbitrary constant. Hence,
\[ I_2(\tau) = 2 \left( \frac{Q}{\Omega_\phi - 1} \right)^{1/2} (\tau - \tau_0), \quad I_4(\tau) = \frac{\Omega_\phi}{4} \left( \frac{Q}{\Omega_\phi - 1} \right)^{1/2} \frac{1}{\tau - \tau_0}, \quad \lambda(\tau) = \frac{\Omega_\phi}{4} \frac{1}{(\tau - \tau_0)^2}. \]  
(52)

Finally, we note that the dual solution generated from the form-invariance transformation of Section [V] is given by
\[ \tilde{I}_2(\tilde{\tau}) = 2 \left( \frac{Q}{\Omega_\phi - 1} \right)^{1/2} (\tilde{\tau} - \tilde{\tau}_0), \quad \tilde{I}_4(\tilde{\tau}) = \frac{\Omega_\phi}{4} \left( \frac{Q}{\Omega_\phi - 1} \right)^{1/2} \frac{1}{\tilde{\tau} - \tilde{\tau}_0}, \quad \tilde{\lambda}(\tilde{\tau}) = \frac{\Omega_\phi}{4} \frac{1}{(\tilde{\tau} - \tilde{\tau}_0)^2}, \]  
(53)

where the time-parameters are related by $\tilde{\tau} - \tilde{\tau}_0 = \Omega_\phi/[8(\tau - \tau_0)]$. In this sense, the scaling solution is the self-dual solution of the system [31]-[33].

### VII. CONCLUSION AND DISCUSSION

In this paper, we have established a direct link between the dynamics of two apparently disparate systems, namely between the dynamics arising from the radially-symmetric, cubic Schrödinger equation and the cosmological Friedmann
equations sourced by a self-interacting scalar field and barotropic perfect fluid. The correspondence is established between the moments of the wavefunction and the cosmological parameters. Whilst similar analogies between the physical variables of the two systems have been considered previously in the literature for spatially curved and anisotropic Bianchi I models [25, 28, 29], we have made the correspondence precise at the level of the equations of motion (field equations). This correspondence is summarized in Tables I and II. In effect, the dictionary between the two systems is equivalent to a change of dependent variables and a reparametrization of cosmic and laboratory times.

By analogy with the Hamilton-Jacobi formalism of scalar field cosmology, we have expressed the moment evolution equations in an alternative, first-order form. This uncovered a form-invariance in the ODEs, generally relating expanding configurations to contracting ones. It also allows parametric solutions to the EP equation to be found and we introduced algorithms for finding solutions with some worked examples. We emphasize that for consistency, the scalar field must be a monotonically varying function of laboratory (cosmic) time in this formalism. Consequently, the trapping potential frequency \( \lambda(\tau) \) can not pass through zero.

Whether such a correspondence between a nonlinear quantum system and the classical Einstein equations is more than an intriguing mathematical analogy (or indeed no more than a coincidence) remains to be seen. It would be interesting to explore this issue further.

On the other hand, for arbitrary \( d \), the moments of the wavefunction are defined by

\[
I_2 = \int |u|^2 r^2 d^d x ,
\]

\[
I_3 = i \int \left( u \frac{\partial u^*}{\partial r} - u^* \frac{\partial u}{\partial r} \right) r d^d x ,
\]

\[
I_4 = \int \left( \frac{1}{2} |\nabla u|^2 + \frac{\nu}{m} (uu^*)^m \right) d^d x .
\]

By repeating the analysis of section III it can be shown that the time-evolution of these moments is determined by the set of equations

\[
\frac{dI_2}{d\tau} = I_3 ,
\]

\[
\frac{dI_3}{d\tau} = -2\lambda(\tau)I_2 + 4 \int d^d x \left( \frac{1}{2} |\nabla u|^2 + \nu \frac{d(m - 1)}{2m} |u|^{2m} \right) ,
\]

\[
\frac{dI_4}{d\tau} = -\frac{1}{2} \lambda(\tau)I_3 .
\]

It follows that the system reduces to that of the two-dimensional case [10]-[18] if the conformal condition (55) is satisfied. In this case, the moment evolution equations can be identified with the cosmological Friedmann equations as for the two-dimensional NLS equation. At this level, therefore, the correspondence seems to be associated with non-relativistic conformal field theory and it would be interesting to explore this possibility further.

In conclusion, it is well known that universes sourced by scalar fields and perfect barotropic fluids have played a central role in modern cosmology over recent decades. We have found that such models also have direct applications in studies of nonlinear quantum mechanics.
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APPENDIX: A WORKED EXAMPLE

As a worked example of the solution-generating techniques developed in Section VI let us consider the ansatz

\[ I_4(\phi) = A \sinh \omega \phi, \]  

(62)
where $A$ is an arbitrary constant and $n \equiv -\omega^2$. It follows after integrating Eq. (40) that the (square) width is given by

$$I_2(\phi) = \Pi_2 \cosh \omega \phi - \frac{Q}{2A} \sinh \omega \phi$$

(63)

and laboratory time is determined from Eq. (44) in terms of the quadrature

$$\tau(\phi) = \frac{n}{4A\omega} \int d\phi \sqrt{2A\Pi_2 \tanh \omega \phi - Q}.$$  

(64)

Although the integral (64) can be evaluated analytically, the result is not invertible in general. To proceed, therefore, we specify $|Q| = 2A\Pi_2$, where the modulus sign applies if $Q < 0$. It then follows that

$$I_2(\phi) = \Pi_2 e^{\omega \phi}$$

(65)

and the corresponding expression for the potential frequency derives from Eq. (43):

$$\lambda(\phi) = -\frac{4A^2}{|Q|} \frac{1}{1 + \tanh \omega \phi}.$$  

(66)

For illustrative purposes, let us consider the case $Q < 0$. We may then employ the standard integral given by

$$\int d\phi \sqrt{1 + \tanh \omega \phi} = \frac{\sqrt{2}}{\omega} \tanh^{-1} \left( \sqrt{1 + \tanh \omega \phi} \right)$$

(67)

to deduce from Eq. (64) that

$$\omega \phi = \tanh^{-1} \left[ 2 \tanh^2 \left( \sqrt{\frac{8A^2}{|Q|} (\tau - \tau_0)} \right) - 1 \right]$$

(68)

$$= \ln \sinh \left( \sqrt{\frac{8A^2}{|Q|} (\tau - \tau_0)} \right),$$

(69)

where $\tau_0$ is the arbitrary integration constant. Substituting Eq. (68) into the relevant expressions for the moments then leads (after appropriate trigonometric identities have been employed) to the full solution

$$\lambda(\tau) = -\frac{2A^2}{|Q|} \cotanh^2 \left( \sqrt{\frac{8A^2}{|Q|} (\tau - \tau_0)} \right),$$

(70)

$$I_2(\tau) = X^2(\tau) = \Pi_2 \sinh \left( \sqrt{\frac{8A^2}{|Q|} (\tau - \tau_0)} \right),$$

(71)

$$I_3(\tau) = \sqrt{2|Q|} \cosh \left( \sqrt{\frac{8A^2}{|Q|} (\tau - \tau_0)} \right),$$

(72)

$$I_4(\tau) = \frac{A}{2} \cosech \left( \sqrt{\frac{8A^2}{|Q|} \tau} \right) \left[ \cosh^2 \left( \sqrt{\frac{8A^2}{|Q|} (\tau - \tau_0)} \right) - 2 \right].$$

(73)