New superintegrable models with position-dependent mass from Bertrand’s Theorem on curved spaces

A Ballesteros\textsuperscript{1}, A Enciso\textsuperscript{2}, F J Herranz\textsuperscript{1}, O Ragnisco\textsuperscript{3} and D Riglioni\textsuperscript{3}

\textsuperscript{1} Departamento de Física, Universidad de Burgos, E-09001 Burgos, Spain
\textsuperscript{2} Departamento de Física Teórica II, Universidad Complutense, E-28040 Madrid
\textsuperscript{3} Dipartimento di Fisica, Università di Roma Tre and Instituto Nazionale di Fisica Nucleare sezione di Roma Tre, Via Vasca Navale 84, I-00146 Roma, Italy

E-mail: angelb@ubu.es, aenciso@fis.ucm.es, fjherranz@ubu.es, ragnisco@fis.uniroma3.it, riglioni@fis.uniroma3.it

Abstract. A generalized version of Bertrand’s theorem on spherically symmetric curved spaces is presented. This result is based on the classification of (3+1)-dimensional (Lorentzian) Bertrand spacetimes, that gives rise to two families of Hamiltonian systems defined on certain 3-dimensional (Riemannian) spaces. These two systems are shown to be either the Kepler or the oscillator potentials on the corresponding Bertrand spaces, and both of them are maximally superintegrable. Afterwards, the relationship between such Bertrand Hamiltonians and position-dependent mass systems is explicitly established. These results are illustrated through the example of a superintegrable (nonlinear) oscillator on a Bertrand-Darboux space, whose quantization and physical features are also briefly addressed.

1. Introduction
Bertrand’s theorem, which dates back to the XIX century [1], is a landmark result in classical mechanics characterizing the Kepler and harmonic oscillator potentials in terms of their qualitative dynamics. More precisely [2], Bertrand’s theorem asserts that any spherically symmetric natural Hamiltonian system $H = \frac{1}{2}|p|^2 + V(|q|)$ in (a subset of) $\mathbb{R}^3$ that has a stable circular trajectory passing through each point in its configuration space and all whose bounded trajectories are closed is either a harmonic oscillator ($V(r) = A/r^2 + B$) or a Kepler system ($V(r) = A/r + B$).

Surprisingly, the classical theorem of Bertrand found a natural extension in and application to general relativity some fifteen years ago thanks to a remarkable paper of Perlick [3]. Indeed, the author undertook the classification of all Bertrand spacetimes, which, roughly speaking, are spherically symmetric and static spacetimes whose timelike geodesics satisfy properties analogous to those of the trajectories of the harmonic oscillator or Kepler systems. The connection between timelike geodesics in spacetime and the trajectories of a classical Hamiltonian system is that, if one writes the Lorentzian metric as

$$\eta = g_{ij}(q) \, dq^i \, dq^j - V^{-1}(q) \, dt^2,$$

where $g$ is a Riemannian metric on a 3-manifold, the timelike geodesics in spacetime are naturally related to the trajectories of the Hamiltonian system

$$H = g^{ij}(q)p_ip_j + V(q).$$

Published under licence by IOP Publishing Ltd
Perlick’s classification of Bertrand spacetimes consisted of two multi-parametric families. When one additionally imposes that the spacial part of the metric be Euclidean, Perlick’s result becomes tantamount to the classical Bertrand’s theorem. Moreover, the general case includes a number of other systems that have received considerable attention, particularly in connection with integrable monopole motion and the existence of generalized Runge–Lenz vectors. For instance, Perlick’s classification includes spacetimes constructed over the 3-sphere, the hyperbolic 3-space and the Iwai–Katayama spaces [4, 5], which generalize the Taub–NUT spacetime.

In this paper we review several recent results related to this problem and we also establish a natural connection between curved Bertrand systems and position-dependent mass (PDM) Hamiltonians [6]–[20]. We begin by showing, in the next section, that the rather complicated families of Bertrand spacetimes admit a strikingly simple physical interpretation [21]: they correspond to either an intrinsic oscillator or an intrinsic Kepler system. Next we address in section 3 the superintegrability of the associated Hamiltonian system and recall the construction of additional integrals of motion of Runge–Lenz type, which settled in a satisfactory way a problem with a large body of related literature (cf. [22, 23] and references therein). This allows us to state an optimal version of Bertrand’s theorem on Riemannian manifolds. Furthermore, we present in section 4 the application of the above results to PDM Hamiltonians by rewriting the previous Bertrand Hamiltonians in terms of a variable mass function. All of these results are explicitly illustrated by discussing in section 5 the $N$-dimensional (ND) version of one of the most interesting Bertrand Hamiltonians: the Darboux III Hamiltonian [24]. The associated quantum mechanical problem together with some possible physical applications of these new PDM integrable models are also considered in the last section.

2. Bertrand spacetimes

To begin with, let us recall Perlick’s definition of a Bertrand spacetime [3]. We will consider a spherically symmetric, static spacetime ($M \times \mathbb{R}, \eta$), where $M$ is a 3-manifold. This ensures that the Lorentzian metric $\eta$ can be written as

$$\eta = h(r)^2 \, dr^2 + r^2 (d\theta^2 + \sin^2 \theta \, d\varphi^2) - \frac{dt^2}{V(r)},$$

where $V$ is a smooth scalar function and

$$g = h(r)^2 \, dr^2 + r^2 (d\theta^2 + \sin^2 \theta \, d\varphi^2)$$

defines a Riemannian metric on $M$. Following Perlick, by a trajectory in spacetime we mean the projection of an inextendible timelike geodesic to a (fixed but otherwise arbitrary) constant time leaf $M\{t_0\}$. This terminology is motivated by the fact that a trajectory in spacetime actually corresponds to a trajectory (in configuration space) of the Hamiltonian

$$H = \frac{1}{2} g^{ij} p_i p_j + V = \frac{1}{2} \left( \frac{p_r^2}{h(r)^2} + \frac{p_{\theta}^2}{r^2} + \frac{p_{\varphi}^2}{r^2 \sin^2 \theta} \right) + V(r)$$

in $M$. As customary, $g^{ij}$ is the inverse matrix to $g_{ij}$, $p \in T^* M$ is the momentum and $(p_r, p_{\theta}, p_{\varphi})$ are the conjugate momenta of the coordinates $(r, \theta, \varphi)$.

**Definition 1.** The Lorentzian 4-manifold $(M \times \mathbb{R}, \eta)$ is a Bertrand spacetime if:

(i) There is a circular ($r = \text{const.}$) trajectory passing through each point of $M$.

(ii) The above circular trajectories are stable, that is, any initial condition sufficiently close to that of a circular trajectory gives a periodic trajectory.
Perlick’s classification of all Bertrand spacetimes [3] can then be stated as follows:

**Theorem 2** (Perlick). The metric of a Bertrand spacetime can be expressed in exactly one of the following forms:

(i) **Type I**:

\[ g = \frac{m^2 dr^2}{n^2 (1 + K r^2)} + r^2 (d\theta^2 + \sin^2 \theta \, d\varphi^2), \quad V = \sqrt{r^{-2} + K + G}. \]

(ii) **Type II**:

\[ g = \frac{2m^2 \left(1 - Dr^2 \pm \sqrt{(1 - Dr^2)^2 - K r^4}\right)}{n^2 ((1 - Dr^2)^2 - K r^4)} \, dr^2 + r^2 (d\theta^2 + \sin^2 \theta \, d\varphi^2), \quad V = G + r^2 \left(1 - Dr^2 \pm \sqrt{(1 - Dr^2)^2 - K r^4}\right)^{-1}. \]

Here \( m, n \) are coprime nonnegative integers and \( D, G, K \) are real constants.

Notice that in [3, 21] this result was expressed through the rational number \( \beta = n/m \).

To provide an interpretation of the above (rather involved) formulas for the Bertrand spacetimes, it is convenient to define the concept of harmonic oscillator and Kepler potential in any spherically symmetric 3-manifold. To this end, let us use the notation (1) for a spherically symmetric metric on \( \mathcal{M} \) and consider its associated Laplacian, which we denote by \( \Delta_g \). It is standard that if \( u(r) \) is function on \( \mathcal{M} \) that only depends on the radial coordinate, then its Laplacian is also radial and given by

\[ \Delta_g u(r) = \frac{1}{r^2 h(r)} \frac{d}{dr} \left( \frac{r^2}{h(r)} \frac{d u}{dr} \right). \]

Then the symmetric Green function \( u(r) \) is obtained as the solution of the equation \( \Delta_g u(r) = 0 \) on \( \mathcal{M}\setminus\{0\} \), namely

\[ u(r) = \int_a^r \frac{h(r')}{r'^2} \, dr'. \quad (3) \]

As the Kepler potential in 3D Euclidean space is simply the radial Green function \( u(r) \) of the Laplacian and the harmonic oscillator is its inverse square, it is natural to make the following definition.

**Definition 3.** The Kepler and the harmonic oscillator potentials in \( (\mathcal{M}, g) \) are respectively given by the radial functions

\[ V_K(r) = A_1 \left( \int_a^r r'^{-2} h(r') \, dr' + B_1 \right), \quad V_O(r) = A_2 \left( \int_a^r r'^{-2} h(r') \, dr' + B_2 \right)^{-2}, \]

where \( A_j \) and \( B_j \) (\( j = 1, 2 \)) are constants.

This definition is obviously valid in higher dimensions as well.

**Example 4.** Let \( (\mathcal{M}, g) \) be the simply connected 3D space of constant sectional curvature \( \kappa \). In this case the metric has the form (1) with \( h(r)^2 = 1/(1 - \kappa r^2) \), so that the corresponding Kepler and harmonic oscillator potentials are

\[ V_K = \sqrt{r^{-2} - \kappa}, \quad V_O = (r^{-2} - \kappa)^{-1}, \]

up to additive and multiplicative constants. In terms of the radial geodesic distance \( \rho_\kappa \) to the origin point \( r = 0 \), that is, \( r = \sin(\sqrt{\kappa} \rho_\kappa)/\sqrt{\kappa} \) [21], these can be rewritten as

\[ V_K = \sqrt{\kappa} \cot (\sqrt{\kappa} \rho_\kappa), \quad V_O = \tan^2(\sqrt{\kappa} \rho_\kappa)/\kappa, \]

thus reproducing the known prescriptions for the sphere and the hyperbolic space [25, 26].
This readily gives the following interpretation of the Bertrand spacetimes [21]:

**Theorem 5.** In a type I (resp. type II) Bertrand spacetime, $V$ is the intrinsic Kepler (resp. harmonic oscillator) potential associated with $g$.

### 3. Bertrand’s theorem in 3D curved spaces

Let us now consider the maximal superintegrability (MS) of the Bertrand Hamiltonians (2). It is well known that, the Bertrand Hamiltonians being spherically symmetric, to establish their MS it suffices to obtain a functionally independent additional integral. As it turns out, the most convenient way to obtain this additional integral is as a ‘generalized Runge–Lenz tensor’.

Indeed, Bertrand Hamiltonians are somehow similar to the usual harmonic oscillator and Kepler systems in Euclidean space. It is classical that the MS of the Kepler system can be readily proved using that the Runge–Lenz vector is conserved [27]. In the case of the harmonic oscillator, there is no natural way of defining a conserved Runge–Lenz vector, but the elements of the symmetric Fradkin matrix $C_{ij} = 2\omega^2 q_i q_j + p_i p_j$ are constants of the motion and encode the main algebraic properties of the model [28]. A straightforward computation shows that the standard Kepler system is obtained from the type I Bertrand Hamiltonian by setting $K = 0$ and $n = m = 1$, while the harmonic oscillator is the type II Hamiltonian with parameters $K = D = 0$, $n = 2$ and $m = 1$. The statement of the following key theorem, which proves the MS of Bertrand Hamiltonians [22], is therefore not surprising:

**Theorem 6.** There exists a (nontrivial) rank-$n$ symmetric tensor field invariant under the flow of the Bertrand Hamiltonian, where $n$ is the parameter introduced in theorem 2.

The proof of this theorem has been fully discussed in [22]. Here we want to stress that the parameter $n$ plays a crucial role in the construction of the first integrals. The basic observation, which goes back to Fradkin [29], is that any spherically symmetric Hamiltonian $H_0 = \frac{1}{2}|p|^2 + U(|q|)$ preserves the unit vector field

$$a = \frac{\cos \varphi}{r} q + \frac{\sin \varphi}{r J} q \times (q \times p), \quad J = p_\varphi = r^2 \dot{\varphi}.$$ 

Of course, this only shows the existence of a local additional integral, which is trivial in view of the flow-box theorem, but the point is that this provides a bona fide global first integral provided one can express $\cos \varphi$ and $\sin \varphi / J$ in terms of $p$ and $q$. In the case of the Kepler problem, this readily yields a conserved vector field which is essentially the Runge–Lenz vector divided by its norm. In the case of the harmonic oscillator, the above quantities are not well defined functions of $p$ and $q$, but they do define an analogous bi-valued conserved vector field. By taking the two-fold symmetrized tensor product of this vector field one can remove this indeterminacy, and this procedure yields an invariant 2-tensor essentially analogous to the conserved matrix $C_{ij}$ mentioned above. The proof of theorem 6 follows this line of thought a bit further. Ultimately, the role $n$ plays in the proof merely reflects the properties of the trajectories of the Hamiltonian (studied in [3, 21]), which depend crucially on the numbers $n$ and $m$. It should be noticed that the dependence of the constants of motion on the momenta could be extremely complicated, which is the reason why only a few among the Bertrand Hamiltonians had previously been identified as MS systems.

To summarize, let us state the complete version of the optimal extension of Bertrand’s theorem to Riemannian manifolds [22]:

**Theorem 7.** Let $H = \frac{1}{2}g^{ij}p_ip_j + V$ be a Bertrand Hamiltonian, i.e., a spherically symmetric, natural Hamiltonian system on a Riemannian 3-manifold $(M, g)$ that has a stable circular trajectory passing through each point in its configuration space and whose bounded trajectories are all closed. Then the following statements hold:
(i) The metric $g$ and the potential $V$ are of the form given in theorem 2 for some coprime positive integers $n, m$.

(ii) The potential is the intrinsic Kepler or the harmonic oscillator potential of $(M,g)$.

(iii) $H$ is superintegrable. More precisely, there exists a nontrivial rank $n$ tensor field which is invariant and plays the role of the Runge–Lenz vector.

4. Curved Bertrand systems as classical PDM Hamiltonians

So far, by starting from Perlick’s classification of (3+1)D Bertrand spacetimes, we have obtained two families of MS Hamiltonians on 3D Riemannian manifolds, which are either of Kepler or oscillator type. These results can also be translated into the language of PDM systems [6]–[20].

For this purpose, we remark that Bertrand Hamiltonians are initially expressed in the form (2), that is, $H = \frac{1}{2} g^{ij} p_i p_j + V(r)$, with the potential $V$ being determined by theorem 2. Hence it is necessary to rewrite the Hamiltonian in terms of a variable mass $M(|q|)$ in the form

$$ H = \frac{p^2}{2M(|q|)} + V(|q|). \tag{4} $$

At this point we stress that the ‘radial’ Bertrand coordinate $r$ is by no means $|q|$. Therefore, the translation has to be achieved by defining the appropriate change of coordinates $r \leftrightarrow |q|$ in the underlying metric (1) of the Bertrand Hamiltonians (as in example 4), thus giving rise to a conformally flat metric:

$$ g = h(r)^2 dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) = f(|q|)^2 dq^2. \tag{5} $$

This yields the relations

$$ r = |q| f(|q|), \quad f(|q|) d|q| = h(r) dr, \quad \frac{1}{|q|} d|q| = \frac{h(r)}{r} dr. \tag{6} $$

Consequently, the variable mass can thus be related to the conformal factor by setting

$$ M(|q|) = m_0 f(|q|)^2 = \frac{m_0 r^2}{q^2}, \tag{7} $$

where $m_0$ is a positive real constant that hereafter we shall fix to 1. In the following we present this relationship for each of the two types of Bertrand Hamiltonians.

4.1. Type I: Bertrand-Kepler Hamiltonians

In this case, by taking into account that

$$ h(r) = \frac{m}{n \sqrt{1 + K r^2}} $$

and by applying relations (6) we obtain that

$$ |q| = \left( \frac{r}{1 + \sqrt{1 + K r^2}} \right)^{m/n}, \quad r = \frac{2}{|q|^{-\langle n/m \rangle} - K |q|^{\langle n/m \rangle}}. $$

Therefore the PDM function is given by

$$ M(|q|) = \frac{4}{(|q|^{-\langle n/m \rangle} - K |q|^{\langle n/m \rangle})^2 q^2}, \tag{8} $$

and the resulting PDM Hamiltonian provided by theorem 2 turns out to be

$$ H = \frac{1}{8} \left( |q|^{-\langle n/m \rangle} - K |q|^{\langle n/m \rangle} \right)^2 q^2 p^2 + A \left( K + \frac{1}{4} \left( |q|^{-\langle n/m \rangle} - K |q|^{\langle n/m \rangle} \right)^2 \right)^{1/2}. \tag{9} $$
Example 8. Riemannian spaces of constant sectional curvature $\kappa$ arise in the Bertrand Hamiltonians of type I when $n = m = 1$ and $K = -\kappa$ [21]. Hence the variable mass (8) and the Hamiltonian (9) reduce to

$$M(|q|) = \frac{4}{(1 + \kappa q^2)^2}, \quad H_\kappa = \frac{1}{8} (1 + \kappa q^2)^2 p^2 + A_1 \frac{1 - \kappa q^2}{2|q|},$$

which is the known Kepler system written in Poincaré coordinates [25, 30] on the spherical ($\kappa > 0$), hyperbolic ($\kappa < 0$) and Euclidean ($\kappa = 0$) spaces. Clearly, we can scale the Hamiltonian as $4H_\kappa$. The corresponding Runge–Lenz vector in these coordinates can be found in [25, 30].

4.2. Type II: Bertrand-oscillator Hamiltonians

Now the function $h(r)$ is given by

$$h(r) = \frac{m\sqrt{2}}{n} \left( \frac{1 - Dr^2 \pm \sqrt{(1 - Dr^2)^2 - Kr^4}}{(1 - Dr^2)^2 - Kr^4} \right)^{1/2},$$

and the relation between the variables $|q|$ and $r$ defined through (6) can be obtained as

$$|q| = \exp \left\{ r u(r) - \int_r^\infty u(r')dr' \right\},$$

(11)

(up to an additive constant coming from the integral) where $u(r)$ is the Green function (3). For these type II systems $u(r)$ reads as

$$u(r) = \pm \frac{m\sqrt{2}}{nr} \left( 1 - Dr^2 \pm \sqrt{(1 - Dr^2)^2 - Kr^4} \right)^{1/2}.$$ 

Therefore, the explicit general result for $M(|q|) = r^2/q^2$ is quite cumbersome and we omit it here. However, for some particular cases it adopts a simple form, as it is shown in the sequel.

Example 9. The three Riemannian spaces of constant curvature $\kappa$ now appear by setting $n = 2$, $m = 1$, $K = 0$, $D = \kappa$ and by taking the positive sign within the two possibilities ‘±’ in (10) [21]. In this way we find that

$$h(r) = \frac{1}{\sqrt{1 - \kappa r^2}}, \quad u(r) = -\frac{\sqrt{1 - \kappa r^2}}{r},$$

so that

$$r = \frac{2|q|}{1 + \kappa q^2}, \quad |q| = \frac{r}{1 + \sqrt{1 - \kappa r^2}},$$

provided that we have dropped an additive constant $\ln 2$ in the integral (11). Then, as expected, we obtain the same variable mass as in the previous example and the corresponding Bertrand oscillator in Poincaré variables is given by

$$H_\kappa = \frac{1}{8} (1 + \kappa q^2)^2 p^2 + A_2 \frac{2q^2}{(1 - \kappa q^2)^2}.$$

The integrals $C_{ii} \equiv I_i$ in the diagonal of the conserved matrix $C_{ij}$ can be found in [25].
5. The Darboux III oscillator

In what follows we focus our attention on a particular system of the family of Bertrand Hamiltonians of type II, the so called Darboux III oscillator. The underlying Bertrand space is the 3D version of the Darboux surface of type III [31, 32], for which an ND spherically symmetric generalization was constructed in [33, 34]. Such a 3D Darboux-Bertrand space corresponds to choose the + sign in (10) and to set \( n = 2, \ m = 1, \ K = D^2 \) and \( D = -2\lambda \), where \( \lambda \) is a real parameter. This yields

\[
h(r) = \frac{1 + \sqrt{1 + 4\lambda r^2}}{2\sqrt{1 + 4\lambda r^2}}, \quad u(r) = -\left(\frac{1 + \sqrt{1 + 4\lambda r^2}}{2r}\right).
\]

Therefore the transformations between the radial variables \( r \) and \( |q| \) turn out to be

\[
r = |q|\sqrt{1 + \lambda q^2}, \quad |q| = \left(\frac{\sqrt{1 + 4\lambda r^2} - 1}{2\lambda}\right)^{1/2}.
\]

Then the variable mass function reads \( M(|q|) = 1 + \lambda q^2 \) and the resulting Darboux Hamiltonian is given by

\[
H_\lambda = \frac{p^2}{2(1 + \lambda q^2)} + \frac{\omega^2 q^2}{2(1 + \lambda q^2)}, \quad (12)
\]

where we have written \( A_2 = \omega^2/2 \). According to section 3, the 3D Hamiltonian \( H_\lambda \) is a MS system, since it is endowed with a conserved matrix \( C_{ij} \) (a curved Fradkin tensor). In fact, this result can directly be extended to arbitrary dimension \( N \), as it has been proven in [24, 35]:

**Theorem 10.** (i) The Hamiltonian \( H_\lambda \) (12), for any dimension \( N \) and for any real value of \( \lambda \), is endowed with the following constants of motion.

- \((2N - 3)\) angular momentum integrals:

  \[
  C^{(m)} = \sum_{1 \leq i < j \leq m} (q_ip_j - q_jp_i)^2, \quad C_{(m)} = \sum_{N-m<i<j\leq N} (q_ip_j - q_jp_i)^2, \quad (13)
  \]

  where \( m = 2, \ldots, \ N \) and \( C^{(N)} = C_{(N)} \).

- \( N^2 \) integrals given by the components of the ND curved Fradkin tensor:

  \[
  C_{ij} = p_ip_j - (2\lambda H_\lambda(q, p) - \omega^2)q_iq_j, \quad (14)
  \]

where \( i, j = 1, \ldots, \ N \) and such that \( H_\lambda = \frac{1}{2} \sum_{i=1}^{N} C_{ii} \).

(ii) Each of the three sets \( \{H_\lambda, C^{(m)}\}, \{H_\lambda, C_{(m)}\} \) \( (m = 2, \ldots, \ N) \) and \( \{C_{ii}\} \) \( (i = 1, \ldots, \ N) \) is formed by \( N \) functionally independent functions in involution.

(iii) The set \( \{H_\lambda, C^{(m)}, C_{(m)}, C_{ii}\} \) for \( m = 2, \ldots, \ N \) with a fixed index \( i \) is constituted by \( 2N - 1 \) functionally independent functions.

We remark that the constants of motion (13) and (14) can also be obtained [35] from the free Euclidean motion by means of a Stäckel transform or coupling constant metamorphosis (see [36, 37] and references therein).

It is also worth stressing that although the above statement holds for any real value of \( \lambda \), the specific resulting system does depend on such a value, in such a manner that \( H_\lambda \) comprises, in fact, three different nonlinear physical systems [35]:
The quantization problem for Bertrand Hamiltonians arises as a challenging research program, let us consider the quantum Cartesian coordinates and momenta, ˆq, ˆp, with Lie brackets and differential representation given by

\[ [\hat{q}_i, \hat{p}_j] = i\hbar \delta_{ij}, \quad \hat{q}_i = q_i, \quad \hat{p}_i = -i\hbar \frac{\partial}{\partial q_i}, \quad \Delta = \frac{\partial^2}{\partial^2 q_1} + \cdots + \frac{\partial^2}{\partial^2 q_N}. \]

Our aim now is to construct the quantum mechanical counterpart of the ND classical Hamiltonian (12): \( H_\lambda(q, p) \rightarrow \hat{H}_\lambda(\hat{q}, \hat{p}) \). As it is well known, the crucial point is to obtain the quantum analogue of the kinetic term, since there is an order ambiguity in its quantization. This task can be faced by applying three different quantization procedures [38]: (i) the ‘Schrödinger quantization’; (ii) the Laplace–Beltrami quantization (which makes use of the Laplace operator on curved spaces); and (iii) a PDM quantization.

We stress that if we impose that the quantum Hamiltonian \( \hat{H}_\lambda \) keeps the MS property (that is, the existence of \( 2N - 2 \) algebraically independent operators that commute with \( \hat{H}_\lambda \) ), then only the Schrödinger quantization yields, in a direct way, to fulfill this condition. Nevertheless, the Laplace–Beltrami and PDM quantizations also lead to MS quantum Hamiltonians once an additional ‘pure’ quantum potential term is added to the initial quantum Hamiltonian, and such potential terms are related through gauge transformations to the Schrödinger quantization. The resulting MS Schrödinger quantization of \( H_\lambda (12) \) is characterized as follows [39] (this result is worth to be compared with theorem 10).

**Theorem 11.** Let \( \hat{H}_\lambda \) be the ND quantum Bertrand-Darboux Hamiltonian given by

\[ \hat{H}_\lambda = \frac{1}{2(1 + \lambda q^2)} \hat{p}^2 + \frac{\omega^2 q^2}{2(1 + \lambda q^2)} = \frac{1}{2(1 + \lambda q^2)} (-\hbar^2 \Delta + \omega^2 q^2). \]  

For any real value of \( \lambda \) the following statements hold:

(i) \( H_\lambda \) commutes with the following observables:

- \( (2N - 3) \) quantum angular momentum operators,

\[ \hat{C}^{(m)} = \sum_{1 \leq i < j \leq m} (\hat{q}_i \hat{p}_j - \hat{q}_j \hat{p}_i)^2, \quad \hat{C}^{(m)} = \sum_{N-m < i < j \leq N} (\hat{q}_i \hat{p}_j - \hat{q}_j \hat{p}_i)^2, \]

where \( m = 2, \ldots, N \) and \( \hat{C}^{(N)} = \hat{C}^{(N)} \).

**Nonlinear hyperbolic oscillator.** For \( \lambda > 0 \) the Darboux space is the complete Riemannian manifold \( \mathcal{M}^N = (\mathbb{R}^N, g) \) with metric \( g_{ij} = (1 + \lambda q^2) \delta_{ij} \). The scalar curvature \( R(|q|) \) has a minimum at the origin \( R(0) = -2\lambda N(N - 1) \), which coincides with the scalar curvature of the ND hyperbolic space with negative constant sectional curvature \( \kappa = -2\lambda \).

**Nonlinear spherical oscillator.** For \( \lambda < 0 \) we firstly consider the interior Darboux space defined by \( \mathcal{M}^N = (B_q, g) \), where \( g_{ij} = (1 - |\lambda q|^2) \delta_{ij} \) and \( B_q \) denotes the ball centered at 0 of radius \( |q|_c = 1/\sqrt{|\lambda|} \) (the critical value for the metric and for \( H_\lambda \)). Now \( R(0) = 2|\lambda|N(N - 1) \) is exactly the scalar curvature of the ND spherical space with positive constant sectional curvature \( \kappa = +2|\lambda| \).

**Nonlinear exterior potential.** For \( \lambda < 0 \) we can also consider the exterior Darboux space defined by \( \mathcal{M}^N = (\mathbb{R}^N \setminus \overline{B}_q, g) \); this implies to reverse the sign of the metric and, therefore, of the Hamiltonian itself, namely, \( g_{ij} = (|\lambda q|^2 - 1) \delta_{ij} \). This system can naturally be interpreted as an infinite barrier potential rather than an oscillator one.

6. Quantization of the Darboux III oscillator

The quantization problem for Bertrand Hamiltonians arises as a challenging research program, whose first steps in the case of the Darboux III oscillator are summarized as follows.

Let us consider the quantum Cartesian coordinates and momenta, ˆq, ˆp, with Lie brackets and differential representation given by

\[ [\hat{q}_i, \hat{p}_j] = i\hbar \delta_{ij}, \quad \hat{q}_i = q_i, \quad \hat{p}_i = -i\hbar \frac{\partial}{\partial q_i}, \quad \Delta = \frac{\partial^2}{\partial^2 q_1} + \cdots + \frac{\partial^2}{\partial^2 q_N}. \]

Our aim now is to construct the quantum mechanical counterpart of the ND classical Hamiltonian (12): \( H_\lambda(q, p) \rightarrow \hat{H}_\lambda(\hat{q}, \hat{p}) \). As it is well known, the crucial point is to obtain the quantum analogue of the kinetic term, since there is an order ambiguity in its quantization. This task can be faced by applying three different quantization procedures [38]: (i) the ‘Schrödinger quantization’; (ii) the Laplace–Beltrami quantization (which makes use of the Laplace operator on curved spaces); and (iii) a PDM quantization.

We stress that if we impose that the quantum Hamiltonian \( \hat{H}_\lambda \) keeps the MS property (that is, the existence of \( 2N - 2 \) algebraically independent operators that commute with \( \hat{H}_\lambda \) ), then only the Schrödinger quantization yields, in a direct way, to fulfill this condition. Nevertheless, the Laplace–Beltrami and PDM quantizations also lead to MS quantum Hamiltonians once an additional ‘pure’ quantum potential term is added to the initial quantum Hamiltonian, and such potential terms are related through gauge transformations to the Schrödinger quantization. The resulting MS Schrödinger quantization of \( H_\lambda (12) \) is characterized as follows [39] (this result is worth to be compared with theorem 10).

\[ \hat{H}_\lambda = \frac{1}{2(1 + \lambda q^2)} \hat{p}^2 + \frac{\omega^2 q^2}{2(1 + \lambda q^2)} = \frac{1}{2(1 + \lambda q^2)} (-\hbar^2 \Delta + \omega^2 q^2). \]  

For any real value of \( \lambda \) the following statements hold:

(i) \( H_\lambda \) commutes with the following observables:

- \( (2N - 3) \) quantum angular momentum operators,

\[ \hat{C}^{(m)} = \sum_{1 \leq i < j \leq m} (\hat{q}_i \hat{p}_j - \hat{q}_j \hat{p}_i)^2, \quad \hat{C}^{(m)} = \sum_{N-m < i < j \leq N} (\hat{q}_i \hat{p}_j - \hat{q}_j \hat{p}_i)^2, \]

where \( m = 2, \ldots, N \) and \( \hat{C}^{(N)} = \hat{C}^{(N)} \).
N2 operators which form an ND quantum Fradkin tensor,
\[ \hat{C}_{ij} = \hat{p}_i \hat{p}_j - 2\lambda \hat{q}_i \hat{q}_j \hat{H}_\lambda (\mathbf{q}, \mathbf{p}) + \omega^2 \hat{q}_i \hat{q}_j, \]

where i, j = 1, …, N and such that \( \hat{H}_\lambda = \frac{1}{2} \sum_{i=1}^N \hat{C}_{ii} \).

(ii) Each of the three sets \( \{ \hat{H}_\lambda, \hat{C}^{(m)} \} \), \( \{ \hat{H}_\lambda, \hat{C}^{(m)} \} \) (m = 2, …, N) and \( \{ \hat{C}_{ii} \} \) (i = 1, …, N) is formed by N algebraically independent commuting observables.

(iii) The set \( \{ \hat{H}_\lambda, \hat{C}^{(m)}, \hat{C}^{(m)}(n), \hat{C}_{ii} \} \) for m = 2, …, N with a fixed index i is formed by 2N − 1 algebraically independent observables.

(iv) \( \hat{H}_\lambda \) is formally self-adjoint on the Hilbert space \( L^2(M^N) \), endowed with the scalar product
\[ \langle \Psi | \Phi \rangle_\lambda = \int_{M^N} \overline{\Psi(\mathbf{q})} \Phi(\mathbf{q})(1 + \lambda \mathbf{q}^2) d\mathbf{q}. \]

Clearly, the results of theorem 11 should be adapted to each of the three different systems described in the previous section. In particular, we consider here the quantum hyperbolic-type Hamiltonian, that has been fully solved in [39].

6.1. The nonlinear hyperbolic oscillator: quantum case
The quantum Hamiltonian \( \hat{H}_\lambda \) with \( \lambda > 0 \) has recently been shown to give rise to a new exactly solvable quantum model in N dimensions [39] which has both a discrete and a continuous spectrum. The discrete spectrum depends on a single principal quantum number \( n = 0, 1, 2 \ldots \)
and its eigenvalues are
\[ E_n = -\hbar^2 \lambda \left( n + \frac{N}{2} \right)^2 + \hbar \left( n + \frac{N}{2} \right) \sqrt{\hbar^2 \lambda^2 \left( n + \frac{N}{2} \right)^2 + \omega^2} \]
\[ = -\hbar^2 \lambda \left( n + \frac{N}{2} \right)^2 \left\{ 1 - \sqrt{1 + \frac{\omega^2}{\hbar^2 \lambda^2 \left( n + \frac{N}{2} \right)^2}} \right\}. \]
\[ (18) \]

Therefore, the degeneracy of this model is exactly the same as in the ND isotropic oscillator (whose spectrum is recovered under the limit \( \lambda \to 0 \)), as it should be expected from the beginning due to its MS property. Notice also that the bound states of this system are hydrogen-like since
\[ \lim_{n \to \infty} E_n = \frac{\omega^2}{2\lambda}, \quad \lim_{n \to \infty} (E_{n+1} - E_n) = 0. \]

Therefore \( \hat{H}_\lambda \) has an infinite number of eigenvalues contained in \( (0, \frac{\omega^2}{2\lambda}) \) and their only accumulation point is \( \frac{\omega^2}{2\lambda} \) which is, in turn, the bottom of the continuous spectrum given by \( \left[ \frac{\omega^2}{2\lambda}, \infty \right) \). The corresponding wave functions can be explicitly found in [39].

Finally, a short disgression on possible physical applications of this kind of exactly solvable quantum PDM Hamiltonians is in order. Firstly, we recall that the MIC-Kepler and oscillator potentials on the 3D sphere have been shown in [10] to be suitable as effective models for strong and weak confinement regimes in spherical quantum dots [40]. Secondly, it is worth stressing that a parabolic mass function has been proposed in [13, 14] in order to describe a 1D quantum well formed by a GaAs/AlxGa1-xAs heterostructure. In fact, if the concentration x grows in terms of a given spatial coordinate \( q_1 \) as \( x(q_1) = a q_1^2 \) and in this type of material \( m^*(x) = m_0(a + b x) \) [13], then an effective mass function of the type \( m^*(q_1) = m_0(a + b x q_1^2) \) arises. Thus, we have obtained a realistic quantum exactly solvable model coming from the Bertrand-oscillator potential on a hyperbolic space with non-constant curvature.
Acknowledgments

This work was partially supported by the Spanish MICINN under grants MTM2010-18556 and FIS2008-00209, by the Junta de Castilla y León (project GR224), by the Banco Santander–UCM (grant GR58/08-910556) and by the Italian–Spanish INFN–MICINN (project ACI2009-1083).

References

[1] Bertrand J 1873 *C. R. Math. Acad. Sci. Paris* **77** 849
[2] Féjoz J and Kaczmarek L 2004 *Ergod. Th. & Dynam. Sys.* **24** 1583
[3] Perlick V 1992 *Class. Quant. Grav.* **9** 1009
[4] Iwai T and Katayama N 1994 *J. Math. Phys.* **35** 2914
[5] Iwai T and Katayama N 1995 *J. Math. Phys.* **36** 1790
[6] von Roos O 1983 *Phys. Rev. B* **27** 7547
[7] Lévy-Leblond J M 1995 *Phys. Rev. A* **52** 1845
[8] Chetouani L, Dekar L and Hammann T F 1995 *Phys. Rev. A* **52** 82
[9] Plastino A R, Rigo A, Casas M, Gracias F and Plastino A 1999 *Phys. Rev. A* **60** 4318
[10] Gritsev V V and Kurochkin Yu A 2001 *Phys. Rev. B* **27** 7547
[11] Quesne C and Tkachuk V M 2004 *J. Phys. A: Math. Gen.* **37** 4267
[12] Bagchi B, Banerjee A, Quesne C and Tkachuk V M 2005 *J. Phys. A: Math. Gen.* **38** 2929
[13] Koc R, Koca M and Sahinoglu G 2005 *Eur. Phys. J. B* **48** 583
[14] Schmidt A G M 2006 *Phys. Lett. A* **353** 459
[15] Mustafa O and Mazharimousavi S H 2006 *Phys. Lett. A* **358** 259
[16] Quesne C 2006 Ann. Phys. **321** 1221
[17] Cruz y Cruz S, Negro J and Nieto L M 2007 *Phys. Lett. A* **369** 400
[18] Schmidt A G M, Azeredo A D and Gusso A 2008 *Phys. Lett. A* **372** 2774
[19] Midya B and Roy B 2009 *Phys. Lett. A* **373** 4117
[20] Lévia G and Özer O 2010 *J. Math. Phys.* **51** 092103
[21] Ballesteros A, Enciso A, Herranz F J and Ragnisco O 2008 *Class. Quant. Grav.* **25** 165005
[22] Ballesteros A, Enciso A, Herranz F J and Ragnisco O 2009 *Comm. Math. Phys.* **290** 1033
[23] Ngome J P 2009 *J. Math. Phys.* **51** 122901
[24] Ballesteros A, Enciso A, Herranz F J and Ragnisco O 2008 *Physica D* **237** (2008) 505
[25] Ballesteros A and Herranz F J 2007 *J. Phys. A: Math. Theor.* **40** F51
[26] Shchepetilov A V 2005 *J. Math. Phys.* **46** 114101
[27] Guillenin V and Sternberg S 1990 *Variations on a theme by Kepler* (Providence: AMS)
[28] Fradkin D M 1965 *Amer. J. Phys.* **33** 207
[29] Fradkin D M 1967 *Prog. Theor. Phys.* **37** 798
[30] Ballesteros A and Herranz F J 2009 *J. Phys. A: Math. Theor.* **42** 245203
[31] Koenigs G 1972 *Leçons sur la théorie générale des surfaces* vol 4, ed G Darboux (New York: Chelsea) p 368
[32] Kalnins E G, Kress J M, Miller W Jr and Winternitz P 2003 *J. Math. Phys.* **44** 5811
[33] Ballesteros A, Enciso A, Herranz F J and Ragnisco O 2007 *Phys. Lett. B* **652** 376
[34] Ballesteros A, Enciso A, Herranz F J and Ragnisco O 2009 *Ann. Phys.* **324** 1219
[35] Ballesteros A, Enciso A, Herranz F J, Ragnisco O and Riglioni D 2010 arXiv:1010.3358
[36] Hietarinta J, Grammaticos B, Dorizzi B and Ramani A 1984 *Phys. Rev. Lett.* **53** 1707
[37] Kalnins E G, Miller W Jr and Post S 2010 *J. Phys. A: Math. Theor.* **43** 035202
[38] Ballesteros A, Enciso A, Herranz F J, Ragnisco O and Riglioni D 2010 submitted to *Ann. Phys.*
[39] Ballesteros A, Enciso A, Herranz F J, Ragnisco O and Riglioni D 2010 arXiv:1007.1335
[40] Harrison P 2009 *Quantum wells, wires and dots* (New York: Wiley)