GENERALISED MONOPOLE EQUATIONS ON KÄHLER SURFACES

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Abstract. In this article, we establish a Hitchin-Kobayashi type correspondence for generalised Seiberg-Witten monopole equations on Kähler surfaces. We show that the “stability” criterion we obtain, for the existence of solutions, coincides with that of the usual Seiberg-Witten monopole equations. This enables us to construct a map from the moduli space of solutions to the generalised equations to effective divisors.

1. Introduction

In this article, we study a generalisation of the Seiberg-Witten (SW) monopole equations on a Kähler surface. Let \((X, g_X)\) be a smooth, oriented, four-dimensional Riemannian manifold. Fix a Spin\(^c\)-structure \(Q \to X\). Spinor bundles are vector bundles associated to \(Q\), with respect to a certain standard action on the vector space of quaternions \(\mathbb{H}\). The idea behind the generalisation is to replace the spinor representation \(\mathbb{H}\) with a hyperKähler manifold \((M, g_M, I_1, I_2, I_3)\) admitting certain symmetries. Generalised spinors are the sections of the associated fiber bundle. It is then possible to construct a non-linear Dirac operator, acting on the sections of the fiber-bundle. The operator is a first order, non-linear elliptic operator. This is the essence of the generalisation of Seiberg-Witten (GSW) monopole equations. An appropriate replacement of the quadratic map, which maps spinors to self-dual 2-forms on \(X\), gives the GSW monopole equations. The generalisation was first introduced by C. Taubes \([1]\) in three dimensions. It was extended to four dimensions by V. Pidstrygach \([2]\). However, such generalisations of the Dirac operator were already known to physicists and have been used in the study of gauged \(\sigma\)-sigma models \([3], [4]\).

On a Kähler surface, the generalised monopole equations were studied by R. Waldmüller \([5]\) and K. Strokorb \([6]\) in their Diploma thesis and by A. Haydys \([7]\) in his Ph.D thesis. The equations reduce to a system of twisted, symplectic vortex equations (see \([7]\), Sec. 4.2). The latter are a system of vortex-like equations with values in a symplectic manifold \((F, \omega)\) and can be defined over any compact Kähler manifold. The equations were discovered independently by I. Mundet i Riera \([8]\) and K. Cieliębak, A.R. Gaio and D. Salamon \([9]\). I. Mundet i Riera obtained a Hitchin-Kobayashi-type correspondence when \(F\) is Kähler. The correspondence relates the spaces of solutions up to real and complex gauge transformation.

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and coincides with the notion of “stability” which arises in the construction of algebraic moduli space, by using Geometric Invariant Theory (GIT).

The aim of this article is to explicitly evaluate the stability condition for the (Abelian) GSW equations on a Kähler surface, for a large class of hyperKähler manifolds, admitting a hyperKähler potential. We show that the condition can be reduced to the existence and the uniqueness of solutions to Kazdan-Warner equation. This, however, coincides with condition for the existence of solutions to the usual SW monopole equations.

Given this, it is tempting to ask if there exists a map between moduli space of gauge-equivalent solutions to GSW and effective divisors on $X$? Section 4 provides an affirmative answer to this question.

2. HyperKähler manifolds

Let $(M, g_M, I_1, I_2, I_3)$ be a $4n$-dimensional hyperKähler manifold. Let $\text{Sp}(1)$ denote the group of unit quaternions and $\mathfrak{sp}(1)$ its Lie algebra. As a matter of convenience, we think of the complex structures as covariantly constant endomorphisms of $TM$ with values in $\mathfrak{sp}(1)^* = (\mathfrak{Im}(\mathbb{H}))^*$

$$I \in \Gamma(M, \text{End}(TM) \otimes \mathfrak{sp}(1)^*), \quad I_\xi := \xi_1 I_1 + \xi_2 I_2 + \xi_3 I_3, \quad \xi \in \mathfrak{sp}(1). \quad (2.1)$$

It is easy to see that $M$ has an entire family of Kähler structures parametrized by $S^2 \subset \mathfrak{Im}(\mathbb{H})$. Let $\omega_i$, $i = 1, 2, 3$, denote the Kähler 2-forms associated to $I_1, I_2, I_3$. Combining the three Kähler 2-forms, we define a single, $\mathfrak{sp}(1)$-valued 2-form

$$\omega := i \omega_1 + j \omega_2 + k \omega_3.$$ 

Suppose that a Lie group $G$ acts isometrically on $M$. Let $\mathfrak{g}$ denote its Lie algebra. We will denote by

$$K^M : \mathfrak{g} \longrightarrow \Gamma(M, TM), \quad \gamma \longmapsto K^M_\gamma$$

the Killing vector field on $M$ due to $\gamma$.

**Definition 1.** An isometric (left) action of $\text{Sp}(1)$ on $M$ is said to be **permuting** if

$$(L_q)^* \omega = \overline{q} \omega q, \quad q \in \text{Sp}(1).$$

In other words, the induced action of $\text{Sp}(1)$, on the two-sphere of complex structures, is the standard action of $\text{Sp}(1)$ on $S^2$.

**Definition 2.** An isometric action of a Lie group $G$ on $M$ is **tri-holomorphic** (or *hyperKähler*), if it fixes the 2-sphere of complex structures

$$\mathcal{L}_{K^M_\eta} \omega = 0, \quad \eta \in \mathfrak{g}.$$ 

If, in addition, the $G$ action is Hamiltonian with respect to each $\omega_i$, then the action is said to be **tri-Hamiltonian** (or hyperHamiltonian). We can define a single $G$-equivariant
hyperKähler moment map

$$\mu : M \longrightarrow \mathfrak{sp}(1)^* \otimes \text{Lie}(G)^* = \mathfrak{sp}(1)^* \otimes \mathfrak{g}^*,$$

(2.2)

by combining the three moment maps into one

$$\mu = i\mu_1 + j\mu_2 + k\mu_3.$$

Amongst the class of hyperKähler manifolds, which admit a permuting $\text{Sp}(1)$-action, there are those that also admit a hyperKähler potential; i.e, a smooth map $\rho_0 : M \longrightarrow \mathbb{R}^+$, which is simultaneously a Kähler potential for each $\omega_i$. Swann [10] shows that for such hyperKähler manifolds, the permuting $\text{Sp}(1)$ action can be extended to a homothetic action of $\mathbb{R}^+ \subset \mathbb{H}^*$:

$$\langle L_r \rangle^* g_M(\cdot, \cdot) = r^2 g_M(\cdot, \cdot).$$

### Definition 3.
A quaternionic-Kähler manifold is a $4n$-dimensional Riemannian manifold whose holonomy is contained in $\text{Sp}(n)\text{Sp}(1) := (\text{Sp}(n) \times \text{Sp}(1)) / \pm 1$.

### Theorem 2.1 ([10]).
Let $M$ be a hyperKähler manifold admitting a hyperKähler potential $\rho_0$. Then $\rho_0^{-1}(c)/\text{Sp}(1) := N$ is a quaternionic-Kähler manifold of positive scalar curvature.

On the other hand, starting with a quaternionic-Kähler manifold $N$ of positive scalar curvature, Swann’s construction produces a hyperKähler manifold $U(N)$ with a permuting $\text{Sp}(1)$-action and a hyperKähler potential. This is the total space of the fiber bundle $U(N) \longrightarrow N$ with a typical fiber $\mathbb{H}^*/(\mathbb{Z}/2\mathbb{Z})$. Moreover, Swann shows that any action of a Lie group $G$ on $N$ which preserves the quaternionic-Kähler structure, can be lifted to a tri-Hamiltonian action of $G$ on $U(N)$. In this case, the moment map has a simple expression (see Sec. 3.3 of [11]):

$$\langle \mu, \xi \otimes \eta \rangle = -\frac{1}{2} g_M(K^M_\xi, K^M_\eta), \, \xi \in \mathfrak{sp}(1) \text{ and } \eta \in \mathfrak{g}. \quad (2.3)$$

Examples for compact, quaternionic-Kähler manifolds, with positive scalar curvature are given by Wolf spaces. These are compact, homogeneous, quaternionic-Kähler manifolds classified by Wolf [12] and Alekseevskii [13]. The list includes quaternionic projective spaces $\mathbb{H}P^n = \frac{\text{Sp}(n+1)}{\text{Sp}(n) \times \text{Sp}(1)}$, complex Grassmannians $X^n = \frac{\text{SU}(n)}{\text{SU}(n-2) \times \text{U}(2)}$, real Grassmannians $Y^n = \frac{\text{SO}(n)}{\text{SO}(n-4) \times \text{SO}(4)}$, etc. The associated manifolds $U(N)$ are certain co-adjoint orbits of complex simple Lie groups (see [10]).

### 2.1. Target hyperKähler manifold.
Let $(M, g_M, I_1, I_2, I_3)$ be a $4n$-dimensional hyperKähler manifold. Suppose that there is an isometric action of $\text{U}(1)$ on $M$, that preserves $\omega_1$ and rotates $\omega_2$ and $\omega_3$; in other words, if $X$ is the Killing vector field on $M$ that generates the action, then

$$\mathcal{L}_X \omega_1 = 0, \, \mathcal{L}_X \omega_2 = -\omega_3, \, \mathcal{L}_X \omega_3 = \omega_2.$$
Such an action is called a *rotating action* of $U(1)$. This notion was introduced by N. Hitchin, A. Karlhede, U. Lindström and M. Roček [14] . Henceforth, we will refer to such a hyperKähler manifold as a *target hyperKähler manifold*.

**Example 1.** Consider the flat quaternionic space $\mathbb{H}^n$. If we write $\mathbb{H}^n = \mathbb{C}^n \oplus j\mathbb{C}^n$, we have

$$\omega_1 = \frac{1}{2} \sum_{l=1}^{n} dz_l \wedge d\bar{z}_l + dw_l \wedge d\bar{w}_l, \quad \omega_c := \omega_2 + i\omega_3 = \sum_{l=1}^{n} dz_l \wedge dw_l.$$ 

Then the circle action $(e^{i\theta}, (z, w)) \mapsto (z, e^{i\theta} \cdot w)$ is a rotating action, fixing $\omega_1$.

**Example 2.** Given $U(N)$ for some $N$ of positive scalar curvature, the stabilizer $U(1)_r \subset \text{Sp}(1)$ of $I_1$ gives the requisite rotating action.

3. **Generalised Seiberg-Witten equations on Kähler surface**

Fix a target hyperKähler manifold $M$ and assume that there is a tri-Hamiltonian action of $U(1)$ on $M$ that commutes with the rotating $U(1)_r$-action. To distinguish this group action from the rotating one, we denote this group by $U(1)_0$. Therefore, $M$ has a *rotating action* of $\mathbb{T}^2 = U(1)_r \times_{\mathbb{Z}/2\mathbb{Z}} U(1)_0$.

Let $X$ be a Kähler surface, and let $\omega_X$ be the Kähler 2-form. The Kähler structure on $X$ determines the reduction of its SO(4)-frame bundle to a principal $U(2)$-bundle $P_{U(2)}$. More precisely, $U(2) = (U(1)_r \times \text{Sp}(1)_-)/(\mathbb{Z}/2\mathbb{Z})$, where $U(1)_r \subset \text{Sp}(1)_+$ is the stabilizer of the complex structure $R_0$ in the SO(4) = $(\text{Sp}(1)_+ \times \text{Sp}(1)_-)/(\mathbb{Z}/2\mathbb{Z})$-representation $\mathbb{R}^4 \cong \mathbb{H}$.

The $U(1)$-bundle $P_r := P_{U(2)}/\text{Sp}(1)_-$ is precisely the one associated to the anti-canonical line bundle of $X$. Fix an auxiliary principal $U(1)$-bundle $P_0$ over $X$ and define the $\mathbb{T}^2$-bundle $P_{T^2} := (P_r \times_X P_0)/(\mathbb{Z}/2\mathbb{Z})$.

**Generalised spinors** are $\mathbb{T}^2$-equivariant maps

$$\text{Map}(P_{T^2}, M)^{\mathbb{T}^2} \cong \Gamma(X, M), \quad \text{where } M = P_{T^2} \times_{\mathbb{T}^2} M.$$ 

The Levi-Civita connection on $X$ defines a connection on $P_{U(2)}$. Therefore, a connection $A$ on $P_0$ and the Levi-Civita connection together determine a unique connection $A$ on $P_{T^2}$. A spinor $u : P_{T^2} \rightarrow M$ and the connection $A$ determine a $\mathbb{T}^2$-equivariant map $K^M_A|_u : TP_{T^2} \rightarrow TM$ as follows. For any $v \in T_p P_{T^2}$, take

$$K^M_A(v)|_u := K^M_A(v)|_{u(p)} \in T_{u(p)}M$$

(note that $A(v) \in \mathfrak{t}^2 := \text{Lie}(\mathbb{T}^2)$).

The differential of $u$ is also $\mathbb{T}^2$-equivariant. We define the covariant derivative of $u \in \text{Map}(P_{T^2}, M)^{\mathbb{T}^2}$ with respect to $A$ to be the one-form $D_A u \in \Omega^1(P_{T^2}, u^*TM)^{\mathbb{T}^2}$

$$D_A u = du + K^M_A|_u.$$ 

This is an equivariant, horizontal one-form on $P_{T^2}$. Indeed, for any $\xi \in \mathfrak{t}^2$, we have

$$D_A u \left(K^{P_{T^2}}_\xi\right) = du \left(K^{P_{T^2}}_\xi\right) + K^M_A\left(K^{P_{T^2}}_\xi\right)|_u = -K^M_\xi|_u + K^M_\xi|_u = 0.$$
Therefore, $D_Au$ descends to a one-form on $X$ with values in $(u^*TM)/\mathbb{T}^2$. Denote by $\bar{\partial}_A u$, the $(0,1)$-part of this 1-form, meaning

$$
\bar{\partial}_A u = \frac{1}{2} \left( D_A u - I_1 \circ D_A u \circ \tilde{I}_X \right),
$$

where $\tilde{I}_X$ is the lift of the complex structure $I_X$ to the horizontal subspace $\mathcal{H}_A \subset TP_{\mathbb{T}}$.

Note that in defining the $\bar{\partial}_A$ operator, we treat $M$ as a Kähler manifold with respect to complex structure $-I_1$.

Denote by $A(P_0)$ the space of connections on $P_0$. Define the configuration space

$$
\mathcal{C} := \text{Map} (P_{\mathbb{T}}, M) \times A(P_0).
$$

(3.1)

Let $G := \text{Map} (X, U(1))$ be the infinite-dimensional gauge group. Then, the configuration space carries a (right) action of $G$.

**Theorem 3.1** ([7]). Let $(X, \omega_X)$ be a Kähler surface. Then for a pair $(u, A) \in \mathcal{C}$, the perturbed, GSW equations on $X$ reduce to the following system:

$$
\begin{aligned}
\bar{\partial}_A u &= 0 \\
\Lambda_{\omega_X} F_A + i\mu_1 \circ u + it &= 0, \ t \in \mathbb{R} \\
\mu_c \circ u &= 0, \ F_A^{0,2} = 0
\end{aligned}
$$

(3.2)

where $F_A$ is the curvature of $A$, and $\mu_c$ is the complex moment map $\mu_2 + i\mu_3$. Moreover, these equations are invariant under the action of $G$.

### 4. A Hitchin-Kobayashi-type correspondence

In this section, we establish a Hitchin-Kobayashi-type correspondence for the solutions of (3.2). To understand what we mean by this, observe that the configuration space is naturally a Kähler manifold, with respect to the complex structure induced by the complex structures $I_X$ on $X$ and $I_1$ on $M$ (see [8], Sec. 2.3). The (right) action of $G$ on $\mathcal{C}$ extends in a natural way to the action of its complexification $G^C = \text{Map} (X, \mathbb{C}^*)$, with respect to the induced complex structure.

The first and third equations of (3.2) are invariant under the action of $G^C$ whereas the second equation is invariant only under the action of $G$. One would like to know if and when there exists a transformation $g \in G^C$, such that

$$
\Lambda_{\omega_X} F_{g \cdot A} + i\mu_1 \circ (g \cdot u) + it = 0.
$$

The necessary and sufficient conditions for the existence of such a gauge transformation lies at the heart of Hitchin-Kobayashi correspondence. The condition coincides with the notion of stability that arises in the algebraic construction of the moduli space of solutions to various gauge-theoretic equations using Geometric Invariant Theory (GIT).

One can also view this from the point of view of Kempf-Ness theory in infinite dimensions. The action of the gauge group $G$ on $\mathcal{C}$ is holomorphic with the associated infinite dimensional
moment map given by

\[ \Upsilon_t(u, A) := i\Lambda_\omega X F_A + i\mu_1 \circ u + it. \]

Let \( A^{1,1}(P_0) \subset A(P_0) \) be the space connections on \( P_0 \), whose curvature is of the form \((1, 1)\). Then \( C^{1,1} := \text{Map}(P T^2, U(N))^T \times A^{1,1}(P_0) \) is a complex subvariety of \( C \), with a holomorphic action of \( \mathcal{G} \), that extends to an action of \( \mathcal{G}^C \). The statement of correspondence now narrows down to asking when does a \( \mathcal{G}^C \)-orbit in \( C^{1,1} \) intersect the zero of the infinite-dimensional moment map. This is a common paradigm in gauge theory and was pioneered by M. Atiyah and R. Bott [15]. It has been used in several other contexts, most notably by Donaldson [16], [17] and by Uhlenbeck and Yau [18] to relate stable vector bundles over complex manifolds with Hermitian-Einstein vector bundles. The idea has been subsequently used in the study of various other gauge theoretic equations [19], [20], [21]. However, since we are interested in existence of solutions to (3.2), we will additionally demand that the first equation of (3.2) also be satisfied.

A more general criterion of stability has been obtained by Mundet i Riera [8] for Kähler vortex equations over compact Kähler manifolds. These are a system of vortex-like equations with values in a Kähler manifold \((F, \omega)\) and can be defined over any compact Kähler manifold. In general, this criterion is not easy to evaluate. However, for target hyper-Kähler manifolds with a hyper-Kähler potential, we show that the condition reduces to Kazdan-Warner equations. The existence of a hyper-Kähler potential lies at the heart of this computation.

To effect our computations below, we need the completion of the configuration space and the gauge group in an appropriate \((k, p)\)-Sobolev norm. We will assume that the Sobolev exponent \( k - \frac{4}{p} > 0 \). It is in this setting that we evaluate the stability criterion. The assumption is implicit in our computations that follow. For details on Sobolev completion of maps between manifolds, we refer the reader to Subsection 4.1, Appendix B of [22].

In order to give a clear picture of our construction, we begin by considering the simplest possible generalisation.

### 4.1. A simpler case: Seiberg-Witten with multiple spinors.

Let \( M \) be the flat quaternionic space \( \text{Hom}_C(C^n, \mathbb{H}) \), where \( \mathbb{H} \) is regarded as a complex vector space with respect to the complex structure \( R_i \). The standard complex volume form on \( C^n \) defines a complex linear isomorphism \( (C^n)^* \cong C^n \). We can therefore identify \( M \cong \mathbb{H} \otimes_C C^n \cong \mathbb{H}^n \). The natural action of the group \( \text{SU}(n) \times \text{U}(1)_0 \) on \( \mathbb{H} \otimes_C C^n \) corresponds to an action

\[ \text{SU}(n) \times \text{U}(1)_0 \hookrightarrow \text{U}(n) \hookrightarrow \text{Sp}(n) \twoheadrightarrow \mathbb{H}^n. \]

For \( \alpha, \beta \in C^n \), the moment map associated to the \( \text{U}(1)_0 \)-action is given by

\[ \mu_1(\alpha + \beta j) = -\sum_{i=1}^n |\alpha_i|^2 - |\beta_i|^2, \quad \mu_c(\alpha + \beta j) = -\sum_{i=1}^n \langle \alpha_i, \beta_i \rangle, \]

where \( \alpha + \beta j = \sum_{i=1}^n \alpha_i s_i + j \sum_{i=1}^n \beta_i s_{n+i} \), and \( \{ s_i \} \) is the complex spinor basis.
The rotating action of $U(1)_r \subset Sp(1)_+$ on $\mathbb{H} \otimes_{\mathbb{C}} \mathbb{C}^n$ is

$$z \cdot (\alpha, \beta) \mapsto (\alpha, \beta \overline{z}).$$

Fix a principal $SU(n)$ bundle $P_{SU(n)}$ over $X$, and denote by $Q$ the principal bundle $P_{T^2 \times X} P_{SU(n)}$. As a representation of $T^2 \times SU(n)$, note that $M$ decomposes as

$$M = \mathbb{C}^n \oplus W \otimes_{\mathbb{C}} \mathbb{C}^n,$$

where $W$ is the representation of $U(1)_r$ on $\Lambda^{0,2}(\mathbb{R}^4)^* \cong \mathbb{C}$. In particular, the associated fiber bundle $P_{U(1)_r \times U(1)} W$ is the anti-canonical line bundle $K^{-1}_X$ over $X$. Therefore, any spinor $u \in \text{Map}(Q, M)^{T^2 \times SU(n)}$ decomposes into two components $f$ and $g$. In terms of the complex basis $\{s_i\}$, we can write a spinor $u = \sum_{i=1}^n f_i s_i + \sum_{i=1}^n g_i s_i$.

Assume that there exists a connection $B$ on $P_{SU(n)}$ compatible with the holomorphic structure, which means that $F^{0,2}_B = 0$. Fix such a connection $B$. The configuration space is given by

$$C = \text{Map}(Q, M)^{T^2 \times SU(n)} \times A(P_0).$$

For a pair $(u, A) \in C$, the (perturbed) SW equations with multiple spinors, on a compact, Kähler surface $X$ are

$$\begin{cases}
\sum_{i=1}^n \overline{\partial}_{A \otimes B} f_i + \partial_{A \otimes B} g_i = 0 \\
\Lambda_{\omega_X} F_A - i \left( \sum_{i=1}^n |f_i|^2 - |g_i|^2 \right) + it = 0, \ t \in \mathbb{R} \\
\sum_{i=1}^n \langle f_i, g_i \rangle = 0, \ F^{0,2}_A = 0
\end{cases} \quad (4.1)$$

where $\langle \cdot, \cdot \rangle$ is the standard Hermitian inner product on $\mathbb{C}^n$.

The Kähler structure on the configuration space is induced by the complex structures $I_X$ on $X$ and $R_T$ on $\mathbb{H}^n$. The moment map for the holomorphic action of $G$ on $C$ is

$$\Upsilon_t(u, A) = \Lambda_{\omega_X} F_A - i \left( \sum_{i=1}^n |f_i|^2 - |g_i|^2 \right) + it.$$

Define $\mathcal{H}^{1,1} \subset C$ to be the complex subvariety

$$\mathcal{H}^{1,1} = \left\{ (u, A) \in C \mid \overline{\partial}_{A \otimes B} u = 0 \text{ and } F^{0,2}_A = 0 \right\}.$$

The moduli space of solutions to (4.1) is a Kähler submanifold of $\Upsilon_t^{-1}(0)/G$, given by

$$\mathcal{M}(B, g_X) := \left( \mathcal{H}^{1,1} \cap \Upsilon_t^{-1}(0) \cap \left\{ (u, A) \in C \mid \sum_{i=1}^n \langle f_i, g_i \rangle = 0 \right\} \right) / G.$$
Let $\mathcal{H}^{ss} = \left\{ (u, A) \in \mathcal{H}^{1,1} \mid (f_1, f_2, \cdots, f_n) \neq 0 \text{ and } \sum_{i=1}^{n} \langle f_i, g_i \rangle = 0 \right\} \subset \mathcal{H}^{1,1}$.

**Theorem 4.1** ([23]). The moduli space of solutions to (4.1) has a holomorphic description

$$\mathcal{M}(B, g_X) \cong \mathcal{H}^{ss}/G^C.$$ 

The above correspondence reduces to a Kazdan-Warner type equation, which gives the necessary condition for existence of solutions to be

$$t > \frac{4\pi}{\text{vol}(X)} \deg_{\omega_X} P_0.$$  

Recall that the degree of the bundle $P_0$ is given by

$$\deg_{\omega_X} P_0 = \frac{i}{2\pi} \int_X \Lambda_{\omega_X} F_A.$$ 

A choice of a large enough $t$ ensures that $(f_1, f_2, \cdots, f_n) \neq 0$. Bryan and Wentworth obtained the above correspondence when $P_{SU(n)}$ is trivial and $B$ is a trivial connection. However, a verbatim argument carries over to the case when $P_{SU(n)}$ is non-trivial.

The statement of Theorem 4.1 is an infinite dimensional analogue of a finite-dimensional principal. Namely, suppose that we are given a smooth, projective variety $W$, with a holomorphic action of a reductive Lie group $G^C$. Let $\mu$ denote the moment map for the action of $G$ on $W$. Then the algebraic and the symplectic quotients agree; i.e,

$$\mu^{-1}(0)/G \cong W^{ss}/G^C$$

where $W^{ss}$ is a dense open set.

Under the assumption (4.2) we will now establish a map from $\mathcal{M}(B, g_X)$ to the moduli space of solutions to the usual SW monopole equations

$$\begin{cases} 
\overline{J_A} \alpha = 0, \beta = 0 \\
\Lambda_{\omega_X} F_A - i \left( \frac{|\alpha|^2}{2} - t \right) = 0, \ t \in R \\
\langle \alpha, \beta \rangle = 0 \\
F_A^{0,2} = 0
\end{cases}$$  

on $X$. We denote the latter moduli space by $\mathcal{M}^{SW}(g_X)$.

**4.1.1. SW with multiple spinors $\Rightarrow$ SW.** We will denote by $u$ the equivariant map $\text{Map}(Q, \mathbb{H}^n)^{T^2 \times SU(n)}$ and by $\phi$ a positive spinor; i.e, a $T^2$-equivariant map $\text{Map}(P_{T^2}, \mathbb{H})^{T^2}$.

Let $(u, A)$ be a solution to (4.1) and suppose that there exists a $\phi = \alpha + \beta$ such that

$$\frac{i}{2} (|\alpha|^2 - |\beta|^2) = i \left( \sum_{i=1}^{n} \frac{|f_i|^2 - |g_i|^2}{2} \right), \ \langle \alpha, \beta \rangle = 0.$$ 

Here $\phi = \alpha + \beta$ is the usual decomposition of the spinor on a Kähler surface. Owing to (4.2), a non-trivial $\phi$, satisfying the above equations, always exists. Moreover, the condition
(4.2) also implies that any solution $\phi$ to the monopole equations will have $\beta = 0$. Therefore, without loss of generality, we may assume that $\beta = 0$. Therefore, pair $(\phi, A)$ satisfies
\[
\begin{cases}
\Lambda_{\omega, x} F_A - \frac{1}{2} \left( |\alpha|^2 - t \right) = 0 \\
F^0_A = 0.
\end{cases}
\]

**Lemma 4.2.** The spinor $\phi$ satisfying (4.4) is holomorphic; i.e.,
\[\overline{\partial} A\alpha = 0.\]

**Proof.** We have
\[
d \left( \frac{|\alpha|^2}{2} \right) = d \left( \sum_{i=1}^{n} \frac{|f_i|^2 - |g_i|^2}{2} \right).
\]

Computing the left hand side:
\[
\langle \alpha, D_A\alpha \rangle_R = \frac{1}{2} \left( \langle \alpha, \overline{\partial} A\alpha \rangle_R + \langle \alpha, \partial A\alpha \rangle_R \right)
\]
where $\langle \cdot, \cdot \rangle_R$ denotes the real part of the respective Hermitian inner products. Similarly, on the right hand side we have
\[
d \left( \sum_{i=1}^{n} \frac{|f_i|^2 - |g_i|^2}{2} \right) = \sum_{i=1}^{n} \frac{1}{2} \left( \langle f_i, \overline{\partial} A\otimes B f_i \rangle_R + \langle f_i, \partial A\otimes B f_i \rangle_R - \langle g_i, \partial A\otimes B g_i \rangle_R \right)
\]

Equating the $(0,1)$-parts on both the sides, we get $\langle \alpha, \overline{\partial} A\alpha \rangle_R = \sum_{i=1}^{n} \langle f_i, \overline{\partial} A\otimes B f_i \rangle_R$. The equation $\sum_{i=1}^{n} \langle f_i, g_i \rangle = 0$ implies
\[
d^* \sum_{i=1}^{n} \langle f_i, g_i \rangle = \sum_{i=1}^{n} \langle f_i, \overline{\partial} A\otimes B g_i \rangle = 0 \implies \sum_{i=1}^{n} \langle f_i, \overline{\partial} A\otimes B g_i \rangle_R = 0.
\]

Together, this gives
\[
\langle \alpha, \overline{\partial} A\alpha \rangle_R = \sum_{i=1}^{n} \langle f_i, \overline{\partial} A\otimes B f_i + \overline{\partial} A\otimes B g_i \rangle_R = \left( \sum_{i=1}^{n} \langle f_i \rangle_R + \sum_{i=1}^{n} \overline{\partial} A\otimes B f_i + \overline{\partial} A\otimes B g_i \rangle_R \right).
\]

The last equality follows from the fact that $\langle f_i, \overline{\partial} A\otimes B f_j \rangle = \langle f_i, \overline{\partial} A\otimes B g_j \rangle$ for $i \neq j$. Therefore, from (4.1), we conclude that $\langle \alpha, \overline{\partial} A\alpha \rangle_R = 0$ and so, the $(0,2)$-form
\[
\overline{\partial} \langle \alpha, \overline{\partial} A\alpha \rangle = \langle \overline{\partial} A\alpha \wedge \overline{\partial} A\alpha \rangle = 0.
\]

The statement of the Lemma follows. \qed

In particular, we have shown that $(A, \phi)$ is a solution to the monopole equations (4.3). The uniqueness of the solution $(\phi, A)$ is easily seen.
4.2. General case: Swann bundles. We will now implement the above program in a more general setting where \( \mathbb{H}^n \) is replaced by a more general target hyperKähler manifold, with a hyperKähler potential; i.e., the total space of a Swann bundle. The strategy for the program is the same as that for multi-monopole equations discussed in the previous subsection.

Define \( \mathcal{H}^{1,1} = \{ (u, A) \in \mathcal{C} \mid \overline{\partial}_A u = 0 \) and \( F_A^{0,2} = 0 \}. \) Then, the moduli space of solutions to (3.2) is once again a Kähler submanifold of \( \Upsilon_t^{-1}(0)/\mathcal{G} \), given by

\[
\mathcal{M} = \left( \mathcal{H}^{1,1} \cap \Upsilon_t^{-1}(0) \cap \left\{ u \in \text{Map}(P_{\mathbb{T}^2}, \mathcal{U}(N))^{\mathbb{T}^2} \mid \mu_c \circ u = 0 \right\} \right) / \mathcal{G}.
\]

Let \( F_0 \subset \mathcal{U}(N) \) be the fixed-point set of the \( U(1)_0 \) action on \( \mathcal{U}(N) \). Consider the dense open subset of \( \mathcal{H}^{1,1} \)

\[
\mathcal{H}^{ss} := \{ (u, A) \in \mathcal{H}^{1,1} \mid u(P_{\mathbb{T}^2}) \not\subset F_0 \}.
\]

**Theorem 4.3 (Hitchin-Kobayashi correspondence).** Let \( (u, A) \in \mathcal{H}^{1,1} \) and assume that \( t > \frac{4\pi}{\text{vol}(X)} \deg_{\omega_X} P_0 \). Then, the moduli space \( \mathcal{M} \) is non-empty. Moreover, \( \mathcal{M} \) has a holomorphic description

\[
\mathcal{M} \cong \mathcal{H}^{ss}/\mathcal{G}^C.
\]

**Proof.** Our aim is to find conditions under which there exists a \( g \in \mathcal{G}^C \) such that

\[
\Upsilon_t(g \cdot u, g \cdot A) = 0.
\]

Consider an element \( e^f \in \mathcal{G}^C \). If \( f \) is purely imaginary, then \( e^f \in \mathcal{G} \). Since the equations are invariant under \( \mathcal{G} \), we consider the case when \( f \) is real. Now the complexified gauge group \( \mathcal{G}^C \) acts on \( \mathcal{H}^{1,1} \) as

\[
e^f \cdot A \mapsto A + \partial \overline{f} - \overline{\partial} f \text{ and } e^f \cdot u, \text{ for } e^f \in \mathcal{G}^C.
\]

So \( F_{e^f \cdot A} = F_A + \overline{\partial} f - \partial \overline{f} \). From (2.3), the moment map component \( \mu_1 \circ u \) can be written down explicitly as

\[
\mu_1 \circ u = -\frac{1}{2} g_M(K^M_{\xi_1}|_u, K^M_{\xi_1}|_u)
\]

where \( \xi_1 \in S^2 \subset \mathfrak{sp}(1) \) is the basis element fixed by the rotating \( U(1)_u \)-action. Owing to the homothetic \( \mathbb{R}^* \)-action on \( \mathcal{U}(N) \), we have

\[
\mu_1 \circ (e^f \cdot u) = -e^{2f} \frac{1}{2} g_M(K^M_{\xi_1}|_u, K^M_{\xi_1}|_u).
\]

Observe that \( \frac{1}{2} g_M(K^M_{\xi_1}|_u, K^M_{\xi_1}|_u) : P_{\mathbb{T}^2} \to \mathbb{R} \) is \( \mathbb{T}^2 \)-invariant and so we can think of it as a smooth, real-valued function on \( X \). For simplicity, let \( a(u) = \frac{1}{2} g_M(K^M_{\xi_1}|_u, K^M_{\xi_1}|_u) \). We can therefore write

\[
\Lambda_{\omega_X} F_{e^f \cdot A} + i \mu_1 \circ (e^f \cdot u) + it = \Lambda_{\omega_X} (\overline{\partial} f - \partial \overline{f}) + \frac{i}{2} e^{2f} a(u) + \Lambda_{\omega_X} F_A + it.
\]

Hence, in order to find a \( g \in \mathcal{G}^C \) such that \( \Upsilon_t(g \cdot u, g \cdot A) = 0 \), we need to solve

\[
\Delta_X f + e^{2f} a(u) = (t - 2i \Lambda_{\omega_X} F_A)
\]

(4.5) where \( \Delta_X \) is the positive definite Laplacian on \( X \). Let \( w = t - 2i \Lambda_{\omega_X} F_A \).

We now recall a result of [24] which will be used.
Lemma 4.4 (Kazdan-Warner). Let $X$ be a compact Riemannian manifold, and let $B$ and $w$ be smooth functions on $X$ with $B$ being positive outside of a measure zero set and $\int_X w > 0$. Let $\Delta_X = -2i\partial\overline{\partial}$ be the negative definite Laplacian on $X$. Then the equation

$$\Delta_X f + B(x) e^{2f} - w = 0$$

has a unique solution.

The condition $\int_X w > 0$ translates to fixing a $t > \frac{4\pi}{\text{vol}(X)} \deg_{\omega_X} P_0$. It follows that there exists a unique solution to (4.5). The statement of the theorem follows.\[\square\]

A technical requirement in Lemma 4.4 is that the function $B$ be a positive function, outside of a measure zero set. A priori, it is unclear why this should hold for an abstract map $a(u)$. However, in the following section, we will show that solutions to (3.2) determine a unique solution to (4.3). This in turn will imply that $a(u) = |\alpha|^2$. Therefore, the technicality is automatically satisfied for $t > \frac{4\pi}{\text{vol}(X)} \deg_{\omega_X} P_0$.

4.2.1. Solutions to GSW $\Rightarrow$ SW. Assume that $t > \frac{4\pi}{\text{vol}(X)} \deg_{\omega_X} P_0$ and let $(u, A)$ be a solution to (3.2). Moreover, let $\phi$ be a usual spinor, satisfying

$$\begin{cases} \Lambda_{\omega_X} F_A - i \left( \frac{\omega^2}{2} - t \right) = 0, \ t \in \mathbb{R} \\ F^0_A = 0 \end{cases}$$

In particular, $\mu_1 \circ u = \frac{|\alpha|^2}{2}$. Once again, owing to the fact that $t > \frac{4\pi}{\text{vol}(X)} \deg_{\omega_X} P_0$, such a spinor always exists and has $\beta = 0$, where $\phi = \alpha + \beta$. The second condition is already satisfied since $(u, A)$ is a solution to (3.2).

Theorem 4.5. A spinor $\phi$ satisfying the above equation is holomorphic; i.e.,

$$\overline{\partial}_A \alpha = 0.$$ 

Proof. Observe that

$$d \left( \frac{|\alpha|^2}{2} \right) = d(a(u)) = D_A(\mu_1 \circ u) = d\mu_1(u)(D_A u).$$

We can split the 1-forms on left-hand side and right-hand side into its $(0,1)$ and $(1,0)$-components and equate them to get

$$\langle \alpha, \overline{\partial}_A \alpha \rangle = d\mu_1(u)(\overline{\partial}_A u) = 0.$$ 

Then, arguing as before, we have $|\overline{\partial}_A \alpha|^2 = 0$. The statement of the theorem follows. Therefore, $(\phi, A)$ is a solution to SW monopole equations (4.3).\[\square\]
4.3. **From SW to GSW.** It is possible to prove the converse of Theorem 4.5 and Lemma 4.2. In other words, starting with a solution to vortex equations, it is possible to construct a solution to generalised equations.

Assume that $t > \frac{4\pi}{\text{vol}(X)} \deg_{\omega_X} P_0$, so that there exists a pair $(\phi, A)$ satisfying (4.3). Fix $M$ to be either $\mathbb{H}^n$ or $\mathcal{U}(N)$ for some $N$. The given condition then implies that there must exist a generalised spinor $u$, such that

$$\Lambda_{\omega_X} F_A + i \mu_1 \circ u + it = 0 . \quad (4.6)$$

To show that $(u, A)$ is a solution to (3.2), we must show that $\partial_A u = 0$. From (4.3), we know that $\partial_A \alpha = 0$. Since $(u, A)$ satisfies (4.6) we also know that $\mu_1 \circ u = \frac{|\alpha|^2}{2}$. In particular, we have $d(\mu_1 \circ u) = d \left( \frac{|\alpha|^2}{2} \right)$. Equating the $(1,0)$ parts on both sides we get that $d\mu_1 (\bar{\partial}_A u) = 0$. If $\bar{\partial}_A u$ is not identically zero, then $\bar{\partial}_A u(p) \in \ker d\mu_1 (u(p)) \subset T_{u(p)} M$ for every $p \in T_p P_0$, which in turn implies that $\mu_1 \circ u(p) = 0$ for every $p \in P_0$. In particular we have $\mu_1 \circ u = 0$. But this is a contradiction since $\alpha \neq 0$. It must therefore be the case that $\bar{\partial}_A u = 0$. In conclusion, $(u, A)$ is a solution to (3.2).

5. **Maps between moduli spaces**

In both the cases discussed above, over a Kähler surface, we get an explicit description of the map from the moduli space of solutions to the generalised equations to that of the usual SW monopole equations. More precisely,

$$\Pi : \mathcal{M}(g_X, M) \longrightarrow \mathcal{M}^{SW}(g_X), \quad [(u, A)] \longmapsto [(\phi, A)], \quad \text{where } \mu_1 \circ u = \frac{|\alpha|^2}{2}. \quad (5.1)$$

The fiber of the map is the set of all solutions $u$, which are holomorphic with respect to $A$ and $\mu_1 \circ u = \frac{|\alpha|^2}{2}$. Since the solutions to SW are in one-to-one correspondence with effective divisors, $\Pi$ maps a solution to the generalised SW equations to an effective divisor $D$, given by the zeroes of the function $a(u)$.

A more general version of the correspondence (5.1) was studied by the second author in [25]. Namely, the following theorem was proved:

**Theorem 5.1.** On a compact Riemannian manifold $X$, suppose that there exists a solution of the GSW equations. The composition $\mu \circ u$ defines a self-dual 2-form on $X$, which we denote by $\Omega$. Then, away from the set of degenerate points of $\Omega$, the equations (3.2) can be expressed as a second order PDE in terms of $\Omega$:

$$\nabla^* \nabla \Omega = -\left( \frac{s}{2} + |\Omega|^2 \right) \Omega - 2\langle d\Omega + *d|\Omega| , N_\Omega \rangle + \frac{1}{2} \left( \frac{|d\Omega|^2}{|\Omega|^2} - |N_\Omega|^2 \right) \Omega$$

$$+ \frac{1}{2} \left( |d|\Omega| |^2 + 2\langle d|\Omega| , *d\Omega \rangle \right) \frac{\Omega}{|\Omega|^2}. \quad (5.2)$$

This is a generalisation of Donaldson’s result [26], who showed that the solutions to the usual SW equations are in one-to-one correspondence with self-dual 2-forms satisfying (5.2),
away from the singular set. It follows that there is a map between the moduli spaces of solutions to the GSW and SW equations. On a Kähler surface, the equation (5.2) reduces to a PDE for a single function, which is reminiscent of the formulation of two-dimensional vortex equations by Jaffe and Taubes [27]. In the Kähler situation, (5.2) gives an alternate description of the map Π.

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