Approximation of mutual information in a bipartite quantum state under single-party decoherence

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Abstract. We consider the problem of approximate calculation of quantum mutual information in a bipartite quantum state where one of subsystems goes through a decoherence process (quantum channel). The approximation employs a reduced density matrix of the subsystem affected by decoherence and the initial value of mutual information before decoherence. We test the approximation on sets of randomly generated qubit-qudit mixed states for three types of qubit decoherence channels: dephasing, depolarizing, and damping. We consider how approximation accuracy is related to decoherence type, decoherence strength, and the dimension of the qudit subsystem.

Decoherence, that is, an uncontrolled interaction of a quantum system with its environment [1, 2], becomes one of the main obstacles to practical realization of quantum information technologies. Study of decoherence includes different problems such as quantitative characterisation [3] and finding efficient ways for protecting quantum states [4, 5].

In this paper we consider decoherence of a subsystem within a bipartite system. In this case, decoherence usually leads to destruction of correlation between subsystems, which is a vital resource for realization of quantum computation and communication. In particular, we consider an approach to a simplified calculation of quantum mutual information, that is, a measure of total correlation (classical and quantum) between subsystems.

Consider an arbitrary finite dimensional quantum state of a bipartite system $AB$, given by its density matrix $\rho_{AB}$. Let us introduce a decoherence process of the subsystem $A$ described by a completely positive trace-preserving map $\Phi$ acting in the state space of the subsystem $B$.

Then the state of the whole system after decoherence takes the form

$$\rho_{\text{dec}}^{AB} = (\Phi \otimes \text{Id})[\rho_{AB}],$$

where $\text{Id}$ is an identical channel acting in the state space of the subsystem $B$.

Next, we introduce the definition of quantum mutual information for an arbitrary bipartite state $\sigma_{AB}$:

$$I[\sigma_{AB}] = S[\sigma_A] + S[\sigma_B] - S[\sigma_{AB}],$$

where $\sigma_A = \text{Tr}_B\sigma_{AB}$ and $\sigma_B = \text{Tr}_A\sigma_{AB}$ are corresponding reduced density matrices of the subsystems $A$ and $B$ of the state $\sigma_{AB}$, and

$$S[\sigma] = -\text{Tr}[\sigma \log \sigma]$$
is von Neumann entropy.

To obtain an approximate expression for calculating the quantity $I_{AB}^{\text{dec}} \equiv I[ho_{AB}^{\text{dec}}]$, that is, mutual information after the decoherence process, we employ a spectral decomposition of the density matrix $\rho_A = \text{Tr}_B \rho_{AB}$ of the initial state $\rho_A$:

$$\rho_A = \sum_i \lambda_i \vert \psi_i \rangle_A \langle \psi_i \vert,$$

where $\vert \psi_i \rangle_A$ and $\lambda_i$ denote eigenvectors and eigenvalues of the matrix $\rho_A$. Next, we employ a purification of the state $\rho_A$ given by

$$\vert \Psi \rangle_{AR} = \sum_i \sqrt{\lambda_i} \vert \psi_i \rangle_A \otimes \vert \psi_i \rangle_R,$$

where $R$ is a reference subsystem with a state space identical to the one of $A$. It is easy to see that $\text{Tr}_R \vert \Psi \rangle_{AR} \langle \Psi \vert = \rho_A$.

Then, we can consider the state realized after the decoherence of the subsystem of the pure state $\vert \Psi \rangle_{AR}$:

$$\rho_{AR}^{\text{dec}} = (\Phi \otimes \text{Id}) [\vert \Psi \rangle_{AR} \langle \Psi \vert].$$

We denote its mutual information as $I_{AR}^{\text{dec}} = I[\rho_{AR}^{\text{dec}}]$. Finally, we consider the following hypothesis

$$I_{AB}^{\text{dec}} / I_{AR} \approx I_{AB} / I_{AR},$$

where $I_{AB} \equiv I[\rho_{AB}]$ and $I_{AR} \equiv I[\vert \Psi \rangle_{AR} \langle \Psi \vert]$ are initial values of the mutual information in the states of $AB$ and $AR$ before decoherence. The intuition behind (7) is that a decoherence process acting on $A$ destroys approximately the same portion of the initial mutual information both in $\rho_{AB}$ and the purified version $\vert \Psi \rangle_{AR}$.

The purity $\vert \Psi \rangle_{AR} \langle \Psi \vert$ of the state implies that its mutual information is given by $I_{AR} = 2S_A$, where $S_A = S[\rho_A]$. Employing the hypothesis (7), we can introduce an approximation of mutual information after decoherence $I_{AB}^{\text{dec}}$ in the following way:

$$\tilde{I}_{AB}^{\text{dec}} = I_{AB} / 2S_A \cdots$$

We note that the value of $\tilde{I}_{AB}^{\text{dec}}$ is actually defined by the initial mutual information $I_{AB}$, the reduced density matrix of the subsystem affected by decoherence, and the particular decoherence channel. As a quantitative measure of the approximation (8) accuracy we introduce the relative deviation

$$\epsilon = \frac{I_{AB}^{\text{dec}} - \tilde{I}_{AB}^{\text{dec}}}{I_{AB}^{\text{dec}}}. $$

Next, we consider the approximation (8) in the cases where the particle is a qubit and is affected by one of the following channels [7, 8]:

(i) the dephasing channel

$$\Phi^{\text{dep}} (p) [\rho] = (1 - p) \rho + p (\ket{0} \langle 0 | \rho \ket{0} \langle 0 | + \ket{1} \langle 1 | \rho \ket{1} \langle 1 |).$$

(ii) the depolarizing channel

$$\Phi^{\text{pol}} (p) [\rho] = (1 - p) \rho + p / 4 \sum_{i=0}^{3} \sigma_i \rho \sigma_i^\dagger,$$

where $\sigma_i$ are the Pauli matrices.
Figure 1. Mean values of relative approximation accuracy $\varepsilon$ for randomly generated qubit-qudit mixed states and different types of qubit decoherence: dephasing (a), depolarization (b), and damping (c). “Error bars” stand for standard deviation, $d$ denotes qudit’s dimension, and $p$ stands for the strength of decoherence process. Each point is obtained from $N = 10^4$ random states.

(iii) the damping channel

$$
\Phi^{\text{damp}}(p)[\rho] = A_1 \rho A_1^\dagger + A_2 \rho A_2^\dagger, \quad A_1 = |0\rangle\langle 0| + \sqrt{1-p} |1\rangle\langle 1|, \quad A_2 = \sqrt{p} |1\rangle\langle 0| 
$$

(12)

Here $\{\sigma_i\}_{i=0}^3$ stands for standard Pauli matrices together with the identity matrix, and $p \in [0,1]$ defines the strength of the decoherence process.

To study the accuracy of the approximation (8) we consider randomly and uniformly (according to the Haar measure) generated [9] mixed states $\rho_{AB}$, where $A$ is a qubit, and $B$ is a qudit of the dimension $d$. Fig. 1 presents the results of our numerical simulation.

Let us consider the basic patterns that are true for all three decoherence types. Firstly, the standard deviation around the mean value grows with the decoherence strength $p$. Thereby the approximation proves to be valuable for relatively weak decoherence. Secondly, the accuracy of approximation increases when the dimension of the second subsystem grows. Thirdly, the approximation considered is characterised by a bias which depends on the particular decoherence type: for the dephasing and damping channels usually $\bar{I}_{AB}^{\text{decA}} > I_{AB}^{\text{decA}}$, which yields negative mean values of $\varepsilon$.

Comparing the results of approximation for different decoherence models, we see that the most accurate results are obtained for the depolarizing channel. The worst accuracy appears for the dephasing channel, where the relative deviation could exceed 100% for $p > 0.5$.

To conclude, we point out the main results of our work. We considered an approximation (8) for mutual information in a bipartite state, where one subsystem is affected by a decoherence channel. We obtain an approximate value using the initial mutual information value, the initial density matrix of the subsystem affected by decoherence, and the particular form of the decoherence process. We implemented our approximation for randomly generated sets of states and found out that (i) it has a bias that depends on the decoherence process, (ii) the approximation accuracy decreases with increasing decoherence strength, and (iii) the approximation accuracy increases when the second subsystem (not affected by decoherence) dimension grows. We expect that this method for approximate calculation of mutual information can be useful in assessing the possibilities of using unitary rotation operations over a system for protecting quantum correlations.

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