On transversal and 2-packing numbers in uniform linear systems

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Abstract

A linear system is a pair \((P, \mathcal{L})\) where \(\mathcal{L}\) is a family of subsets on a ground finite set \(P\), such that \(|l \cap l'| \leq 1\), for every \(l, l' \in \mathcal{L}\). The elements of \(P\) and \(\mathcal{L}\) are called points and lines, respectively, and the linear system is called intersecting if any pair of lines intersect in exactly one point. A subset \(T\) of points of \(P\) is a transversal of \((P, \mathcal{L})\) if \(T\) intersects any line, and the transversal number, \(\tau(P, \mathcal{L})\), is the minimum order of a transversal. On the other hand, a 2-packing set of a linear system \((P, \mathcal{L})\) is a set \(R\) of lines, such that any three of them have a common point, then the 2-packing number of \((P, \mathcal{L})\), \(\nu_2(P, \mathcal{L})\), is the size of a maximum 2-packing set. It is known that the transversal number \(\tau(P, \mathcal{L})\) is bounded above by a quadratic function of \(\nu_2(P, \mathcal{L})\). An open problem is to characterize the families of linear systems which satisfies \(\tau(P, \mathcal{L}) \leq \lambda \nu_2(P, \mathcal{L})\), for some \(\lambda \geq 1\). In this paper, we give an infinite family of linear systems \((P, \mathcal{L})\) which satisfies \(\tau(P, \mathcal{L}) = \nu_2(P, \mathcal{L})\) with smallest possible cardinality of \(\mathcal{L}\), as well as some properties of \(r\)-uniform intersecting linear systems \((P, \mathcal{L})\), such that \(\tau(P, \mathcal{L}) = \nu_2(P, \mathcal{L}) = r\). Moreover, we state a characterization of 4-uniform intersecting linear systems \((P, \mathcal{L})\) with \(\tau(P, \mathcal{L}) = \nu_2(P, \mathcal{L}) = 4\).

Keywords. Linear systems, transversal number, 2-packing number, finite projective plane.

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1 Introduction

A linear system is a pair \((P, \mathcal{L})\) where \(\mathcal{L}\) is a family of subsets on a ground finite set \(P\), such that \(|l \cap l'| \leq 1\), for every pair of distinct subsets \(l, l' \in \mathcal{L}\). The linear system \((P, \mathcal{L})\) is intersecting if \(|l \cap l'| = 1\), for every pair of distinct subsets \(l, l' \in \mathcal{L}\). The elements of \(P\) and \(\mathcal{L}\) are called points and lines, respectively; a line with exactly \(r\) points is called a \(r\)-line, and the rank of \((P, \mathcal{L})\) is the maximum cardinality of a line in \((P, \mathcal{L})\), when all the lines of \((P, \mathcal{L})\) are \(r\) lines we have a \(r\)-uniform linear system. In this context, a simple graph is an 2-uniform linear system.

A subset \(T \subseteq P\) is a transversal (also called vertex cover or hitting set in many papers, as example \([7, 10, 11, 12, 13, 16–21]\)) of \((P, \mathcal{L})\) if for any line \(l \in \mathcal{L}\) satisfies \(T \cap l \neq \emptyset\). The transversal number of \((P, \mathcal{L})\), denoted by \(\tau(P, \mathcal{L})\), is the smallest possible cardinality of a transversal of \((P, \mathcal{L})\).

A subset \(R \subseteq \mathcal{L}\) is called 2-packing of \((P, \mathcal{L})\) if three elements are chosen in \(R\) then they are not incident in a common point. The 2-packing number of \((P, \mathcal{L})\), denoted by \(\nu_2(P, \mathcal{L})\), is the maximum number of a 2-packing of \((P, \mathcal{L})\).

There are many interesting works studying the relationship between these two parameters, for instance, in \([20]\), the authors propose the problem of bounding \(\tau(P, \mathcal{L})\) in terms of a function of \(\nu_2(P, \mathcal{L})\) for any linear system. In \([2]\), some authors of this paper and others proved that any linear system satisfies:

\[
\left\lfloor \frac{\nu_2}{2} \right\rfloor \leq \tau \leq \frac{\nu_2(\nu_2 - 1)}{2}.
\]

That is, the transversal number, \(\tau\), of any linear system is upper bounded by a quadratic function of their 2-packing number, \(\nu_2\).

In order to find how a function of \(\nu_2(P, \mathcal{L})\) can bound \(\tau(P, \mathcal{L})\), the authors of \([10]\) using probabilistic methods to prove that \(\tau \leq \lambda \nu_2\) does not hold for any positive \(\lambda\). In particular, they exhibit the existence of \(k\)-uniform linear systems \((P, \mathcal{L})\) for which their transversal number is \(\tau(P, \mathcal{L}) = n - o(n)\) and their 2-packing number is upper bounded by \(\frac{2n}{k}\).

Nevertheless, there are some relevant works about families of linear systems in which their transversal numbers are upper bounded by a linear function of their 2-packing numbers. In \([1]\) the authors proved that if \((P, \mathcal{L})\) is a 2-uniform linear system, a simple graph, with \(|\mathcal{L}| > \nu_2(P, \mathcal{L})\) then \(\tau(P, \mathcal{L}) \leq \nu_2(P, \mathcal{L}) - 1;\)
moreover, they characterize the simple connected graphs that attain this upper bound and the lower bound given in Equation (1). In [2] was proved that the linear systems \((P, L)\) with \(|L| > \nu_2(P, L)\) and \(\nu_2(P, L) \in \{2, 3, 4\}\) satisfy \(\tau(P, L) \leq \nu_2(P, L)\); and when attain the equality, they are a special family of linear subsystems of the projective plane of order 3, \(\Pi_3\), with transversal and 2-packing numbers equal to 4. Moreover, they proved that \(\tau(\Pi_q) \leq \nu_2(\Pi_q)\) when \(\Pi_q = (P_q, L_q)\) is a projective plane of order \(q\), consequently the equality holds when \(q\) is odd.

The rest of this paper is structured as follows: In Section 2, we present a result about linear systems satisfying \(\tau \leq \nu_2 - 1\). In Section 3, we give an infinite family of linear systems such that \(\tau = \nu_2\) with smallest possible cardinality of lines. And, finally, in the last section, we presented some properties of the \(r\)-uniform linear systems, such that \(\tau = \nu_2 = r\), and we characterize the 4-uniform linear systems with \(\tau = \nu_2 = 4\).

2 On linear systems with \(\tau \leq \nu_2 - 1\)

Let \((P, L)\) be a linear system and \(p \in P\) be a point. It is denoted by \(L_p\) to the set of lines incident to \(p\). The degree of \(p\) is defined as \(\deg(p) = |L_p|\) and the maximum degree overall points of the linear systems is denoted by \(\Delta(P, L)\). A point of degrees 2 and 3 is called double and triple point, respectively, and two points \(p\) and \(q\) in \((P, L)\) are adjacent if there is a line \(l \in L\) with \(\{p, q\} \subseteq l\).

In this section, we generalize Proposition 2.1, Proposition 2.2, Lemma 2.1, Lemma 3.1 and Lemma 4.1 of [2] proving that a linear system \((P, L)\) with \(|L| > \nu_2(P, L)\) and “few” lines satisfies \(\tau(P, L) \leq \nu_2(P, L) - 1\). Notice that, through this paper, all linear systems \((P, L)\) are considered with \(|L| = \nu_2(P, L)\) if and only if \(\Delta(P, L) \leq 2\).

**Theorem 2.1.** Let \((P, L)\) be a linear system with \(p, q \in P\) be two points such that \(\deg(p) = \Delta(P, L)\) and \(\deg(q) = \max\{\deg(x) : x \in P \setminus \{p\}\}\). If \(|L| \leq \deg(p) + \deg(q) + \nu_2(P, L) - 3\), then \(\tau(P, L) \leq \nu_2(P, L) - 1\).

**Proof** Let \(p, q \in P\) be two points as in the theorem, and let \(L'' = L \setminus \{L_p \cup L_q\}\), which implies that \(|L''| \leq \nu_2(P, L) - 2\). Assume that \(|L''| = \nu_2(P, L) - 2\) \((L_p \cap L_q \neq \emptyset)\), otherwise, the following set \(\{p, q\} \cup \{a_l : a_l\text{ is any point of } l \in L''\}\) is a transversal of \((P, L)\) of cardinality at most \(\nu_2(P, L) - 1\), and the statement
holds. Suppose that \( \mathcal{L}'' = \{L_1, \ldots, L_{\nu_2 - 2}\} \) is a set of pairwise disjoint lines because, in otherwise, they induce at least a double point, \( x \in P \), hence the following set of points \( \{p, q, x\} \cup \{a_l : l \in \mathcal{L}'' \setminus \mathcal{L}'\} \), where \( a_l \) is any point of \( l \), is a transversal of \((P, \mathcal{L})\) of cardinality at most \( \nu_2(P, \mathcal{L}) - 1 \), and the statement holds.

Let \( l_q \in \mathcal{L}_q \setminus \{l_{p,q}\} \) be a fixed line and let \( l_p \) be any line of \( \mathcal{L}_p \setminus \{l_{p,q}\} \), where \( l_{p,q} \) is the line containing to \( p \) and \( q \) (since \( \mathcal{L}_p \cap \mathcal{L}_q \neq \emptyset \)). Then \( l_p \cap l_q \neq \emptyset \), since the \( l_q \) induce a triple point on the following 2-packing \( \mathcal{L}'' \cup \{l_p, l_{p,q}\} \), which implies that there exists a line \( l_{p,q} \in \mathcal{L}'' \) with \( l_q \cap l_p \cap l_{p,q} \neq \emptyset \), and hence \( l_p \cap l_q \neq \emptyset \). Consequently, \( \deg (q) = \Delta (P, \mathcal{L}) \) and \( \Delta (P, \mathcal{L}) \leq \nu_2 (P, \mathcal{L}) - 1 \) (since \( \deg (p) - 1 \leq \nu_2 (P, \mathcal{L}) - 2 \)). Therefore, the following set:

\[
\{l_p \cap L_i : i = 1, \ldots, \Delta - 1\} \cup \{a_{\Delta}, \ldots, a_{\nu_2 - 2}\} \cup \{p\},
\]

where \( a_i \) is any point of \( L_i \), for \( i = \Delta, \ldots, \nu_2 - 2 \), is a transversal of \((P, \mathcal{L})\) of the cardinality at most \( \nu_2 (P, \mathcal{L}) - 1 \), and the statement holds.

\[ \square \]

### 3 A family of uniform linear systems with \( \tau = \nu_2 \)

In this section, we exhibit an infinite family of linear systems \((P, \mathcal{L})\) with two points of maximum degree and \( |\mathcal{L}| = 2\Delta (P, \mathcal{L}) + \nu_2 (P, \mathcal{L}) - 2 \) with \( \tau (P, \mathcal{L}) = \nu_2 (P, \mathcal{L}) \). It is immediately, by Theorem 2.1, that \( \tau (P, \mathcal{L}) \leq \nu_2 (P, \mathcal{L}) - 1 \) for linear systems with less lines.

In the remainder of this paper, \((\Gamma, +)\) is an additive Abelian group with neutral element \( e \). Moreover, if \( \sum_{g \in \Gamma} g = e \), then the group is called neutral sum group. In the following, every group \((\Gamma, +)\) is a neutral sum group, such that \( 2g \neq e \), for all \( g \in \Gamma \setminus \{e\} \). As an example of this type of groups we have \((\mathbb{Z}_n, +)\), for \( n \geq 3 \) odd.

Let \( n = 2k + 1 \), with \( k \) a positive integer, and \((\Gamma, +)\) be a neutral sum group of order \( n \). Let:

\[
\mathcal{L} = \{L_g : g \in \Gamma \setminus \{e\}\}, \text{ where } L_g = \{(h, g) : h \in \Gamma\},
\]

for \( g \in \Gamma \setminus \{e\} \), and:

\[
\mathcal{L}_p = \{l_{p_g} : g \in \Gamma\}, \text{ where } l_{p_g} = \{(g, h) : h \in \Gamma \setminus \{e\}\} \cup \{p\},
\]

for \( g \in \Gamma \), and \( \mathcal{L}_q = \{l_{q_g} : g \in \Gamma\} \), where:

\[
l_{q_g} = \{(h, f_g(h)) : h \in \Gamma, f_g(h) = h + g \text{ with } f_g(h) \neq e\} \cup \{q\},
\]
Hence, the set of lines $\mathcal{L}$ is a set of pairwise disjoint lines with $|\mathcal{L}| = n - 1$ and each line of $\mathcal{L}$ has $n$ points. On the other hand, $\mathcal{L}_p$ and $\mathcal{L}_q$ are set of lines incidents to $p$ and $q$, respectively, with $|\mathcal{L}_p| = |\mathcal{L}_q| = n$, and each line of $\mathcal{L}_p \cup \mathcal{L}_q$ has $n$ points. Moreover, this set of lines satisfies that, giving $l_{pa} \in \mathcal{L}_p$ there exists an unique $l_{qb} \in \mathcal{L}_q$ with $l_{pa} \cap l_{qb} = \emptyset$, otherwise, there exits $l_{pa} \in \mathcal{L}_p$ such that $l_{pa} \cap l_{qb} \neq \emptyset$, for all $l_{qb} \in \mathcal{L}_q$, which implies that $a + b \in \Gamma \setminus \{e\}$, for all $b \in \Gamma$, which is a contradiction.

The linear system $(P_n, \mathcal{L}_n)$ with $P_n = (\Gamma \times \Gamma \setminus \{e\}) \cup \{p, q\}$ and $\mathcal{L}_n = \mathcal{L} \cup \mathcal{L}_p \cup \mathcal{L}_q$, denoted by $\mathcal{C}_{n,n+1}$, is an $n$-uniform linear system with $n(n-1) + 2$ points and $3n - 1$ lines. Notice that, this linear system has 2 points of degree $n$ (points $p$ and $q$) and $n(n-1)$ points of degree 3.

A linear subsystem $(P', \mathcal{L}')$ of a linear system $(P, \mathcal{L})$ satisfies that for any line $l' \in \mathcal{L}'$ there exists a line $l \in \mathcal{L}$ such that $l' = l \cap P'$, where $P' \subset P$. Given a linear system $(P, \mathcal{L})$ and a point $p \in P$, the linear system obtained from $(P, \mathcal{L})$ by deleting the point $p$ is the linear system $(P', \mathcal{L}')$ induced by $\mathcal{L}' = \{l \setminus \{p\} : l \in \mathcal{L}\}$. On the other hand, given a linear system $(P, \mathcal{L})$ and a line $l \in \mathcal{L}$, the linear system obtained from $(P, \mathcal{L})$ by deleting the line $l$ is the linear system $(P', \mathcal{L}')$ induced by $\mathcal{L}' = \mathcal{L} \setminus \{l\}$. The linear systems $(P, \mathcal{L})$ and $(Q, \mathcal{M})$ are isomorphic, denoted by $(P, \mathcal{L}) \simeq (Q, \mathcal{M})$, if after deleting the points of degree 1 or 0 from both, the systems $(P, \mathcal{L})$ and $(Q, \mathcal{M})$ are isomorphic as hypergraphs (see [4]).

It is important to state that in the rest of this paper it is considered linear systems $(P, \mathcal{L})$ without points of degree one because, if $(P, \mathcal{L})$ is a linear system which has all lines with at least two points of degree 2 or more, and $(P', \mathcal{L}')$ is the linear system obtained from $(P, \mathcal{L})$ by deleting all points of degree one, then they are essentially the same linear system because it is not difficult to prove that transversal and 2-packing numbers of both coincide (see [2]).

Example 3.1. Let $\Gamma = \mathbb{Z}_3$. The linear system $\mathcal{C}_{3,4} = (P_3, \mathcal{L}_3)$ has as set of points to $P_3 = \{(0,1), (1,1), (2,1), (0,2), (1,2), (2,2)\} \cup \{p\} \cup \{q\}$ and as set of lines to $\mathcal{L}_3 = \mathcal{L} \cup \mathcal{L}_p \cup \mathcal{L}_q$, where

\[
\mathcal{L} = \{(0,1), (1,1), (2,1)\}, \{(0,2), (1,2), (2,2)\}, \\
\mathcal{L}_p = \{(0,1), (0,2), p\}, \{(1,1), (1,2), p\}, \{(2,1), (2,2), p\}, \\
\mathcal{L}_q = \{(1,1), (2,2), q\}, \{(0,1), (1,2), q\}, \{(0,2), (2,1), q\}
\]
Figure 1: Linear system $C_{3,4} = (P_3, \mathcal{L}_3)$.

and depicted in Figure 1 This linear system is isomorphic to the linear system giving in Figure 3, which is the linear system with the less number of lines and maximum degree 3 such that $\tau = \nu_2 = 4$.

Proposition 3.1. The linear system $C_{n,n+1}$ satisfies that:

$$\tau(C_{n,n+1}) = n + 1$$

Proof Notice that $\tau(C_{n,n+1}) \leq n + 1$ since $\{x_g : x_g \text{ is any point of } L_g \in \mathcal{L}\} \cup \{p,q\}$ is a transversal of $C_{n,n+1}$. To prove that $\tau(P_n, \mathcal{L}_n) \geq n + 1$, suppose on the contrary that $\tau(P_n, \mathcal{L}_n) = n$. If $T$ is a transversal of cardinality $n$ then $T \subseteq \Gamma \times \Gamma \setminus \{e\}$, i.e., $p, q \notin T$ because, in other case, if $p \in T$ then, by the Pigeonhole principle, there is a line $l_{q_n} \in \mathcal{L}_q$ such that $T \cap l_{q_n} = \emptyset$, since $\deg(q) = n$, which is a contradiction, unless that $q \in T$, which implies that there exists $L \in \mathcal{L}$ such that $L \cap T = \emptyset$ (because $|\mathcal{L}| = n - 1$), which is also a contradiction. Therefore $T \subseteq \Gamma \times \Gamma \setminus \{e\}$.

Suppose that:

$$T = \{(h_0, f_{g_0}(h_0)), \ldots, (h_{n-1}, f_{g_{n-1}}(h_{n-1}))\},$$

where $\{h_0, \ldots, h_{n-1}\} = \{g_0, \ldots, g_{n-1}\} = \Gamma$ and $f_{g_i} = h_i + g_i \neq e$, for $i = 0, \ldots, n - 1$. Then:

$$\sum_{i=0}^{n-1} f_{h_i}(g_i) = \sum_{i=0}^{n-1} (g_i + h_i) = \sum_{i=0}^{n-1} g_i + \sum_{i=0}^{n-1} h_i = e,$$

since $\sum_{g \in \Gamma} g = \sum_{g \in \Gamma \setminus \{e\}} g = e$, which implies that there exists $f_{h_j}(g_j) \in T$ that
satisfies \( f_{h_j}(g_j) = e \), which is a contradiction, and consequently \( \tau(C_{n,n+1}) = n + 1 \).

\[ \square \]

**Proposition 3.2.** The linear system \( C_{n,n+1} \) satisfies that:

\[ \nu_2(C_{n,n+1}) = n + 1 \]

**Proof** Notice that \( \nu_2(C_{n,n+1}) \geq n + 1 \) because, for any two lines \( l_{p_1}, l_{p_2} \in \mathcal{L}_p \), \( \mathcal{L} \cup \{l_{p_1}, l_{p_2}\} \) is a 2-packing. To prove that \( \nu_2(C_{n,n+1}) \leq n + 1 \), suppose on the contrary that \( \nu_2(C_{n,n+1}) = n + 2 \), and that \( R \) is a maximum 2-packing of size \( n + 2 \), we analyze to cases:

**Case (i):** Suppose that \( R = \mathcal{L} \cup \{l_{p_1}, l_{p_2}, l_{q_1}\} \), where \( l_{p_1}, l_{p_2} \in \mathcal{L}_p \) and \( l_{q_1} \in \mathcal{L}_q \); since there is an unique line \( l_p \in \mathcal{L}_p \) which intersect to \( l_{q_1} \), then we assume that \( l_{p_1} \cap l_{q_1} \neq \emptyset \). By construction of \( C_{n,n+1} \) there exists \( L \in \mathcal{L} \) that satisfies \( l_{p_1} \cap l_{q_1} \cap L = \emptyset \), inducing a triple point, which is a contradiction.

**Case (ii):** Let \( k \) be an element of \( \Gamma \setminus \{e\} \) and \( R = \{l_{p_1}, l_{p_2}, l_{q_1}, l_{q_2}\} \cup \mathcal{L} \setminus \{L_k\} \) with \( l_{p_1}, l_{p_2} \in \mathcal{L}_p \) and \( l_{q_1}, l_{q_2} \in \mathcal{L}_q \), without loss of generality, suppose that \( l_{p_1} \cap l_{q_1} \neq \emptyset \), \( l_{p_1} \cap l_{q_2} \neq \emptyset \), \( l_{p_1} \cap l_{q_d} = \emptyset \) and \( l_{p_2} \cap l_{q_c} = \emptyset \), otherwise, \( R \) is not a 2-packing. It is claimed that there exists \( L \in \mathcal{L} \setminus \{L_k\} \) such that either \( l_{p_1} \cap l_{q_1} \cap L \neq \emptyset \) or \( l_{p_1} \cap l_{q_d} \cap L \neq \emptyset \), which implies that \( R \) induce a triple point, which is contradiction and hence \( \nu_2(C_{n,n+1}) = n + 1 \). To verify the claim suppose on the contrary that every \( L \in \mathcal{L} \setminus \{L_k\} \) satisfies \( l_{p_1} \cap l_{q_1} \cap L = \emptyset \) and \( l_{p_1} \cap l_{q_d} \cap L = \emptyset \). It means that \( l_{p_1} \cap l_{q_1} \cap L_k \neq \emptyset \) and \( l_{p_2} \cap l_{q_d} \cap L_k \neq \emptyset \). By construction of \( C_{n,n+1} \) it follows that:

\[
\begin{align*}
  l_{p_i} & = \{(i, x) : x \in \Gamma \setminus \{e\}\}, \text{ for all } i \in \Gamma, \\
  l_{q_j} & = \{(x, x+j) : x \in \Gamma \setminus \{e\} \text{ and } x+j \neq e\}, \text{ for all } j \in \Gamma, \text{ and} \\
  L_k & = \{(x,k) : x \in \Gamma\}.
\end{align*}
\]

If \( l_{p_1} \cap l_{q_1} \cap L_k \neq \emptyset \) and \( l_{p_2} \cap l_{q_d} \cap L_k \neq \emptyset \), then \( a+c = b+d = k \). On the other hand, as \( l_{p_1} \cap l_{q_d} = \emptyset \) and \( l_{p_2} \cap l_{q_c} = \emptyset \), then \( a+d = b+c = e \). As a consequence of \( a+c = b+d = k \) and \( a+d = b+c = e \) we obtain \( 2k = e \), which is a contradiction. Therefore, \( \nu_2(C_{n,n+1}) = n + 1 \).

\[ \square \]

Hence, by **Proposition 3.1** and **Proposition 3.2** it was proved that:
Theorem 3.2. Let $n = 2k + 1$, with $k \in \mathbb{N}$, then

$$\tau(C_{n,n+1}) = \nu_2(C_{n,n+1}) = n + 1,$$

with smallest possible cardinality of lines.

3.1 Straight line systems

A straight line representation on $\mathbb{R}^2$ of a linear system $(P, L)$ maps each point $x \in P$ to a point $p(x)$ of $\mathbb{R}^2$, and each line $L \in L$ to a straight line segment $l(L)$ of $\mathbb{R}^2$ in such a way that for each point $x \in P$ and line $L \in L$ satisfies $p(x) \in l(L)$ if and only if $x \in L$, and for each pair of distinct lines $L, L' \in L$ satisfies $l(L) \cap l(L') = \{p(x) : x \in L \cap L'\}$. A straight line system $(P, L)$ is a linear system, such that it has a straight line representation on $\mathbb{R}^2$. In [2] was proved that the linear system $C_{3,4}$ is not a straight one. The Levi graph of a linear system $(P, L)$, denoted by $B(P, L)$, is a bipartite graph with vertex set $V = P \cup L$, where two vertices $p \in P$, and $L \in L$ are adjacent if and only if $p \in L$.

In the same way as in [2] and according to [15], any straight line system is Zykov-planar, see also [23]. Zykov proposed to represent the lines of a set system by a subset of the faces of a planar map on $\mathbb{R}^2$, i.e., a set system $(X, F)$ is Zykov-planar if there exists a planar graph $G$ (not necessarily a simple graph) such that $V(G) = X$ and $G$ can be drawn in the plane with faces of $G$ two-colored (say red and blue) so that there exists a bijection between the red faces of $G$ and the subsets of $F$ such that a point $x$ is incident with a red face if and only if $x$ is incident with the corresponding subset. In [22] was shown that the Zykov’s definition is equivalent to the following: A set system $(X, F)$ is Zykov-planar if and only if the Levi graph $B(X, F)$ is planar. It is well-known that for any planar graph $G$ the size of $G$, $|E(G)|$, is upper bounded by $\frac{k(|V(G)|-2)}{2}$ (see [5] page 135, exercise 9.3.1 (a)), where $k$ is the girth of $G$ (the length of a shortest cycle contained in the graph $G$). It is not difficult to prove that the Levi graph $B(C_{n,n+1})$ of $C_{n,n+1}$ is not a planar graph, since the size of the girth of $B(C_{n,n+1})$ is 6, it follows:

$$3n^2 - n = |E(C_{n,n+1})| > \frac{3(n^2 + 2n - 1)}{2},$$

for all $n \geq 3$. Therefore, the linear system $C_{n,n+1}$ is not a straight line system.

Finally, as a Corollary of Theorem 2.1 we have the following:
Corollary 3.1. Let \((P, L)\) be a straight line system with \(p, q \in P\) be two points such that \(\text{deg}(p) = \Delta(P, L)\) and \(\text{deg}(q) = \max\{\text{deg}(x) : x \in P \setminus \{p\}\}\). If \(|L| \leq \text{deg}(p) + \text{deg}(q) + \nu_2(P, L) - 3\), then \(\tau(P, L) \leq \nu_2(P, L) - 1\).

4 Intersecting \(r\)-uniform linear systems with \(\tau = \nu_2 = r\)

In this subsection, we give some properties of \(r\)-uniform linear systems that satisfies \(\tau = \nu_2 = r\) as well as a characterization of \(4\)-uniform linear systems with \(\tau = \nu_2 = 4\).

Let \(L_r\) be the family of intersecting linear systems \((P, L)\) of rank \(r\) that satisfies \(\tau(P, L) = \nu_2(P, L) = r\), then we have the following lemma:

Lemma 4.1. Each element of \(L_r\) is an \(r\)-uniform linear system.

Proof. Let consider \((P, L) \in L_r\) and \(l \in L\) any line of \((P, L)\). It is clear that \(T = \{p \in l : \text{deg}(p) \geq 2\}\) is a transversal of \((P, L)\). Hence \(r = \tau(P, L) \leq |T| \leq r\), which implies that \(|l| = r\), for all \(l \in L\). Moreover, \(\text{deg}(p) \geq 2\), for all \(p \in l\) and \(l \in L\).

In [8] was proved the following:

Lemma 4.2. [8] Let \((P, L)\) be an \(r\)-uniform intersecting linear system then every edge of \((P, L)\) has at most one vertex of degree 2. Moreover \(\Delta(P, L) \leq r\).

Lemma 4.3. [8] Let \((P, L)\) be an \(r\)-uniform intersecting linear system then
\[3(r - 1) \leq |L| \leq r^2 - r + 1.\]

Hence, by Theorem 2.1 and Lemma 4.3 it follows:

Corollary 4.1. If \((P, L) \in L_r\) then \(3(r - 1) + 1 \leq |L| \leq r^2 - r + 1\).

In [2] was proved that the linear systems \((P, L)\) with \(|L| > \nu_2(P, L)\) and \(\nu_3(P, L) \in \{2, 3, 4\}\) satisfy \(\tau(P, L) \leq \nu_2(P, L)\); and when attain the equality, they are a special family of linear subsystems of the projective plane of order \(3\), \(\Pi_3\) (some of them \(4\)-uniform intersecting linear systems) with transversal and \(2\)-packing numbers equal to \(4\). Recall that a finite projective plane (or merely
projective plane) is a linear system satisfying that any pair of points have a common line, any pair of lines have a common point and there exist four points in general position (there are not three collinear points). It is well known that, if \((P, L)\) is a projective plane, there exists a number \(q \in \mathbb{N}\), called order of projective plane, such that every point (line, respectively) of \((P, L)\) is incident to exactly \(q+1\) lines (points, respectively), and \((P, L)\) contains exactly \(q^2 + q + 1\) points (lines, respectively). In addition to this, it is well known that projective planes of order \(q\), denoted by \(\Pi_q\), exist when \(q\) is a power prime. For more information about the existence and the unicity of projective planes see, for instance, [3, 6].

Given a linear system \((P, L)\), a triangle \(\mathcal{T}\) of \((P, L)\), is the linear subsystem of \((P, L)\) induced by three points in general position (non collinear) and the three lines induced by them. In [2] was defined \(\mathcal{C} = (P_C, L_C)\) to be the linear system obtained from \(\Pi_3\) by deleting \(\mathcal{T}\); also there was defined \(\mathcal{C}_{4,4}\) to be the family of linear systems \((P, L)\) with \(\nu_2(P, L) = 4\), such that:

i) \(\mathcal{C}\) is a linear subsystem of \((P, L)\); and

ii) \((P, L)\) is a linear subsystem of \(\Pi_3\),

this is \(\mathcal{C}_{4,4} = \{(P, L) : \mathcal{C} \subseteq (P, L) \subseteq \Pi_3 \text{ and } \nu_2(P, L) = 4\}\).

Hence, the authors proved the following:

**Theorem 4.1.** [2] Let \((P, L)\) be a linear system with \(\nu_2(P, L) = 4\). Then, \(\tau(P, L) = \nu_2(P, L) = 4\) if and only if \((P, L) \in \mathcal{C}_{4,4}\).

Now, consider the projective plane \(\Pi_3\) and a triangle \(\mathcal{T}\) of \(\Pi_3\) (see (a) of Figure 2). Define \(\hat{\mathcal{C}} = (P_C, L_C)\) to be the linear subsystem induced by \(L_C = L\backslash \mathcal{T}\) (see (b) of Figure 2). The linear system \(\hat{\mathcal{C}} = (P_C, L_C)\) just defined has ten points and ten lines. Define \(\hat{\mathcal{C}}_{4,4}\) to be the family of 4-uniform intersecting linear systems \((P, L)\) with \(\nu_2(P, L) = 4\), such that:

i) \(\hat{\mathcal{C}}\) is a linear subsystem of \((P, L)\); and

ii) \((P, L)\) is a linear subsystem of \(\Pi_3\),

It is clear that \(\hat{\mathcal{C}}_{4,4} \subseteq \mathcal{C}_{4,4}\) and each linear system \((P, L) \in \hat{\mathcal{C}}_{4,4}\) is an 4-uniform intersecting linear system. Hence

**Corollary 4.2.** \((P, L) \in \mathbb{L}_4\) if and only if \((P, L) \in \hat{\mathcal{C}}_{4,4}\).
Figure 2: (a) Projective plane of order 3, $\Pi_3$ and (b) Linear system obtained from $\Pi_3$ by deleting the lines of the triangle $\mathcal{T}$.

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