Generic scaling relation in the scalar $\phi^4$ model.

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Abstract

The results of analysis of the one–loop spectrum of anomalous dimensions of composite operators in the scalar $\phi^4$ model are presented. We give the rigorous constructive proof of the hypothesis on the hierarchical structure of the spectrum of anomalous dimensions — the naive sum of any two anomalous dimensions generates a limit point in the spectrum. Arguments in favor of the nonperturbative character of this result and the possible ways of a generalization to other field theories are briefly discussed.
1 Introduction

The great interest to the renormalization of composite operators is mainly motivated by the successful application of the operator product expansion (OPE) method [1] for the description of the processes of deep inelastic scattering in QCD. It is well known now that the behavior of the structure functions at large \( Q^2 \) is closely related to the spectrum of anomalous dimensions of composite operators [2]. The mixing matrix describing the renormalization of the operators in QCD is well known for any set of the latters [3]. Unfortunately, the size of the mixing matrix for the operators with given spin \( j \) and twist greater than 2 increases very fast with the growth of \( j \), so the problem of the calculation of anomalous dimensions seems to admit only numerical solution for not too large values of \( j \).

The first attempt to analyze the spectrum of anomalous dimensions for the particular class of twist–3 operators in the limit \( j \to \infty \) has been undertaken in the paper [4]. In this work the analytic solution for the minimal anomalous dimension has been obtained. The exact solution as well as the results of the numerical study of the spectrum allow to suggest that the asymptotic behavior of anomalous dimensions of twist–3 operators in some sense is determined by the spectrum of those with twist–2.

It should be stressed that the troubles connected with the calculation of the spectrum of anomalous dimensions are not the peculiarity of QCD only. One encounters the same problems and in more simple theories. In the recent papers [5, 6, 7, 8] the analysis of the spectrum of anomalous dimensions in \( O(N) – \) vector model in \( 4 – \epsilon \) dimensions has been carried out. Due to relative simplicity of this model it is appeared possible to obtain the exact solution of the eigenvalue problem for some classes of composite operators (see refs. [5, 7]). The consideration of these solutions together with the results of the numerical analysis fulfilled in [6] for the wide class of operators leads to the same conclusion that the asymptotic of anomalous dimensions of the operators with given twist in the \( j \to \infty \) limit is determined by the dimensions of the operators with a smaller twist [4, 8]. It was supposed in [8] that spectrum of anomalous dimensions has a ”hierarchical” structure; this means that the sum of any two points of the spectrum is the limit point of the latter.

In the present paper we investigate the large spin asymptotic of one–loop anomalous dimensions of the spatially traceless and symmetric composite operators in the scalar \( \phi^4 \) in \( 4 – \epsilon \) dimensions. The approach used here is not specific only for this model and admits the straightforward generalization for other theories.

Before proceeding with calculations we would like to discuss the main troubles which arise in the course of the analysis of the spectrum of anomalous dimensions of large spin operators. It is easy to understand that the source of all difficulties is the mixing problem. Indeed, a long time ago C. Callan and D. Gross proved a very strong statement concerning anomalous dimensions of the twist-2 operators [8] for which the mixing problem is absent. They obtained that in all orders
of the perturbation theory the anomalous dimension \( \lambda_l \) of the operator \( \phi \partial_{\mu_1} \ldots \partial_{\mu_l} \phi \) tends to \( 2\lambda_\phi \) at \( l \to \infty \) (\( \lambda_\phi \) is the anomalous dimension of the field \( \phi \)).

Let us see what prevents the generalization of this result, even on the one loop level, for the case of the operators of higher twists. Although to calculate the mixing matrix is not very difficult, the extraction of the information about eigenvalues of the latter needs a lot of work. Really, if one has not any idea about structure of eigenvectors the only way to obtain eigenvalues is to solve a characteristic equation. But it is almost a hopeless task. However, let us imagine that one has a guess on the form of an eigenfunction; then there are no problem with the evaluating the corresponding eigenvalue. (Note, that the exact solutions in \([4, 5, 7]\) were obtained precisely in this manner.) Thus the more promising strategy is to guess the approximate structure of eigenfunctions in the "asymptotic" region. But the simple criterion to determine that given vector is close to some eigenvector exists only for hermitian matrices (see sec.2).

Thus for the successful analysis of the asymptotic behavior of anomalous dimensions two ingredients – the hermiticity of the mixing matrix and the true choice of test vector – are essential. It is not evident that first condition can be satisfied at all. But for the model under consideration one can choose the scalar product in a such way that a mixing matrix will be hermitian \([5, 6]\). Some arguments in favor that it can be done in a general case will be given in Sec. 4. As to the choice of the trial vector this will be discussed below.

Henceforth, taking in mind the considerations given above, we shall carry out the analysis of the asymptotic behavior of anomalous dimensions for the whole class of the symmetric and traceless operators in the scalar \( \phi^4 \) theory.

The paper is organized in the following way: in sec.2 we shall introduce notations, derive some formulae and give the exact formulation of the problem; the sec. 3 is devoted to the proof of the theorem about asymptotic behavior of anomalous dimension, which is the main result of this paper; in the last section we discuss the obtained results.

## 2 Preliminary remarks

It had been shown In the papers \([3, 6]\) that the problem of calculation of anomalous dimensions of the traceless and symmetric composite operators in the scalar \( \phi^4 \) theory in the one–loop approximation is equivalent to the eigenvalue problem for the hermitian operator \( H \) acting on a Fock space \( \mathcal{H} \):

\[
H = \frac{1}{2} \sum_{n=0}^{\infty} \frac{1}{n+1} h_n^\dagger h_n = \frac{1}{2} \sum_{n=0}^{\infty} \frac{1}{n+1} \sum_{i=0}^{n} a_i^\dagger a_{n-i}^\dagger \sum_{j=0}^{n} a_j a_{n-j}. 
\]  
(2.1)

Here \( a_i^\dagger, a_i \) are the creation and annihilation operators with the standard commutation relations \([a_i, a_k^\dagger] = \delta_{ik} \). The eigenvalues of \( H \) and the anomalous dimensions of composite operators are simply related: \( \gamma_{an} = \epsilon/3 \cdot \lambda + O(\epsilon^2) \). There is also a one to one correspondence between the
eigenvectors of H and the multiplicatively renormalized composite operators \( [6] \).

It can be easily shown that H commutes with the operator of particles number N and with the generators of \( SL(2, C) \) group \( S, S_+, S_- \):

\[
[S_-, S] = S_-, \quad [S_+, S] = -S_+, \quad [S_+, S_-] = 2S.
\] (2.2)

They can be written as:

\[
N = \sum_{j=0}^{\infty} a_j^\dagger a_j, \quad S = \sum_{j=0}^{\infty} (j + 1/2) \cdot a_j^\dagger a_j,
\] (2.3)

\[
S_- = \sum_{j=0}^{\infty} (j + 1) \cdot a_{j+1}^\dagger a_j, \quad S_+ = -\sum_{j=0}^{\infty} (j + 1) \cdot a_{j+1}^\dagger a_j.
\] (2.4)

Further, due to commutativity of H with \( SL(2, C) \) generators, each of the subspaces \( \mathcal{H}_n^l \) and \( \overline{\mathcal{H}}_n^l \in \mathcal{H}_n^l \) \((n, l = 0, \ldots, \infty )\):

\[
\mathcal{H}_n^l = \{ \psi \in \mathcal{H} | N\psi = n\psi, S\psi = (l + n/2)\psi \}, \quad \overline{\mathcal{H}}_n^l = \{ \psi \in \mathcal{H}_n^l | S_-\psi = 0 \}
\] (2.5)

are invariant subspaces of the operator H. Since every eigenvector from \( \mathcal{H}_n^l \) which is orthogonal to \( \overline{\mathcal{H}}_n^l \) has the form \([7]\):

\[
|\psi\rangle = \sum_k c_k S_+^k |\psi_\lambda\rangle, \quad |\psi_\lambda\rangle \in \overline{\mathcal{H}}_n^l,
\]

to obtain all spectrum of the operator H it is sufficient to solve the eigenvalue problem for H on each \( \overline{\mathcal{H}}_n^l \) separately.

Moreover, there exists a large subspace of the eigenvectors with zero eigenvalues in each \( \overline{\mathcal{H}}_n^l \). They have been completely described in ref. \([6]\) and will not be considered here.

As to nonzero eigenvalues, although at finite \( l \) the spectrum of H has a very complicated structure (the numerical results for particular values of \( n \) and \( l \) are given in refs. \([6, 8]\) ), at large \( l \) as it will be shown below considerable simplifications take place.

The main result of the present work can be formulated in the form of the following theorem:

**Theorem 1** Let eigenvectors \( \psi_1 \in \overline{\mathcal{H}}_n^l \) and \( \psi_2 \in \overline{\mathcal{H}}_m^s \) \((\psi_1 \neq \psi_2)\) of operator H have the eigenvalues \( \lambda_1 \) and \( \lambda_2 \) correspondingly. Then there exists a number \( L \) such, that for every \( l \geq L \), there exists eigenvector \( \psi_l^l \in \overline{\mathcal{H}}_{(m+n)}^l \) with the eigenvalue \( \lambda_l \) such that

\[
|\lambda_l - \lambda_1 - \lambda_2| \leq C\sqrt{\ln l}/l,
\] (2.6)

where \( C \) is some constant independent of \( l \). In the case when \( \psi_1 = \psi_2 \) the same inequality holds only for even \( l \geq L \).
The proof is based on the simple observation. Since any of subspaces $\mathcal{H}_n$ has a finite dimension, operator $H$ restricted on $\mathcal{H}_n$ has only pointlike spectrum. In this case it can be easily shown that if there is a vector $\psi$, for which the condition
\[ ||(H - \tilde{\lambda})\psi|| \leq \epsilon ||\psi|| \] (2.7)
is fulfilled, then there exists the eigenvector $\psi_\lambda (H\psi_\lambda = \lambda \psi_\lambda)$, such that $|\lambda - \tilde{\lambda}| \leq \epsilon$. Indeed, expanding a vector $\psi$ in the basis of the eigenvectors of $H$ $\psi = \sum_k c_k \psi_k$ we obtain:
\[ \epsilon ||\psi|| \geq ||(H - \tilde{\lambda})\psi|| = (\sum_k (\lambda_k - \tilde{\lambda})^2 c_k^2)^{1/2} \geq \min_k |\lambda_k - \tilde{\lambda}| \cdot ||\psi||. \]
So, to prove the theorem it is sufficient to find out in the each subspace $\mathcal{H}_{n+m}$ a vector which satisfies the corresponding inequality. Note, that for a nonhermitian matrix these arguments are not applicable.

Before to proceed to the proof we give another formulation of the eigenvalue problem for the operator $H$. Let us note that there exists the one to one correspondence between the vectors from $H_n$ and the symmetric homogeneous polynomials degree of $l$ of $n$ variables:
\[ |\Psi > = \sum_{\{j\}} c_{j_1,\ldots,j_n} a_{j_1}^\dagger \ldots a_{j_n}^\dagger |0\rangle \rightarrow \psi(z_1,\ldots,z_n) = \sum_{\{j\}} c_{j_1,\ldots,j_n} z_1^{j_1} \ldots z_n^{j_n}, \] (2.8)
the coefficient $c_{j_1,\ldots,j_n}$ being assumed totally symmetric. It is evident that this mapping can be continued to all space.

The operators $S, S_+, S_-$ in the $n$ - particles sector take the form
\[ S = \sum_{i=1}^n (z_i \partial_{z_i} + 1/2), \quad S_+ = \sum_{i=1}^n \partial_{z_i}, \quad S_- = - \sum_{i=1}^n (z_i^2 \partial_{z_i} + z_i). \] (2.9)
The operator $H$ in its turn can be represented as the sum of the two–particle hamiltonians:
\[ H = \sum_{i<k} H(z_i, z_k), \] (2.10)
where
\[ H(z_i, z_k)\psi(z_1,\ldots,z_n) = \int_0^1 d\alpha \psi(z_1,\ldots,\alpha z_i + (1-\alpha)z_k,\ldots,\alpha z_i + (1-\alpha)z_k,\ldots,z_n). \] (2.11)
It should be stressed that not only $H$, but every $H(z_i, z_k)$ commutes with $S, S_+, S_-$. For further calculations it is very convenient to put into correspondence to every function of $n$ variables the another one by the following formula:
\[ \psi(z_1,\ldots,z_n) = \sum_{\{j\}} c_{j_1,\ldots,j_n} z_1^{j_1} \ldots z_n^{j_n} \rightarrow \phi(z_1,\ldots,z_n) = \sum_{\{j\}} (j_1!\ldots j_n!)^{-1} c_{j_1,\ldots,j_n} z_1^{j_1} \ldots z_n^{j_n}. \] (2.12)
The function $\psi$ can be expressed in terms of $\phi$ in the compact form:
\[ \psi(z_1,\ldots,z_n) = \phi(\partial_{x_1},\ldots,\partial_{x_n}) \prod_{i=1}^n \frac{1}{(1-x_i z_i)} \bigg|_{x_1=\ldots=x_n=0} \] (2.13)
Then one obtains the following expression for the scalar product for two vectors from $\mathcal{H}_n^l$:

$$< \psi_1 | \psi_2 > = n! \cdot \phi(\partial z_1, \ldots, \partial z_n) \psi(z_1, \ldots, z_n)|_{z_1=\ldots=z_n=0}. \quad (2.14)$$

It is easy now to check that the operators $S$, $S_+$, $S_-$, $H$ on the space of the "conjugated" functions $\phi(z_1, \ldots, z_n)$ look as:

$$S = \sum_{i=1}^{n} (z_i \partial z_i + 1/2), \quad S_+ = \sum_{i=1}^{n} z_i, \quad S_- = - \sum_{i=1}^{n} (z_i \partial^2 z_i + \partial z_i), \quad (2.15)$$

$$H \phi(z_1, \ldots, z_n) = \sum_{i<k} \int_{0}^{1} d\alpha \phi(z_1, \ldots, (1-\alpha)(z_i + z_k), \ldots, \alpha(z_i + z_k), \ldots, z_n). \quad (2.16)$$

Up to now we assume the functions $\psi(z_1, \ldots, z_n)$ to be totally symmetric. But in the following we shall deal with nonsymmetric functions as well. To treat them on equal footing it is useful to enlarge the region of the definition of the operators $S$, $S_+$, $S_-$, $H$ up to the space of all polynomial functions $\mathcal{B} = \bigoplus_{n,l=0}^{\infty} \mathcal{B}_l^n$, where $\mathcal{B}_l^n$ is the linear space of the homogeneous polynomials of degree $l$ of $n$ variables with the scalar product given by eq. (2.14). Then the Fock space $\mathcal{H}$ will be isomorphic to the subspace of the symmetric functions of $\mathcal{B}$; and the subspace $\bar{\mathcal{H}}_l^n$ — to the subspace of the symmetric homogeneous translational invariant polynomials of degree $l$ of $n$ variables $\bar{\mathcal{B}}_l^n \subseteq \mathcal{B}_l^n$.

### 3 Proof of Theorem

#### 3.1 Part I

Let us consider two eigenvectors of $H$: $\psi_1 \in \bar{\mathcal{H}}_r^n$ and $\psi_2 \in \bar{\mathcal{H}}_s^m$ ($H\psi_1(2) = \lambda_1(2)\psi_1(2)$); and let $\psi_1(x_1, \ldots, x_n)$ and $\psi_2(y_1, \ldots, y_m)$ are the symmetric translation–invariant homogeneous polynomials corresponding to them, of degree $r$ and $s$ respectively. To prove the theorem it is enough to pick out in the subspace $\bar{\mathcal{B}}_{n+m}$ (or, the same, in the $\bar{\mathcal{H}}_{n+m}$) the function, for which the inequality (2.7) holds.

Let us consider the following function (nonsymmetric yet):

$$\psi^l(x, y) = \sum_{k=0}^{l} c_k (\text{Ad}^k S_+) \psi_1(x) (\text{Ad}^{l-k} S_+) \psi_2(y), \quad (3.1.1)$$

where $c_k = (-1)^k C_{k}^{r} \cdot C_{k+A}^{l} \cdot C_{k-A}^{l} \cdot C_{k+B}^{l} \cdot C_{k-B}^{l}$ (the binomial coefficient); $A = n + 2r - 1$; $B = m + 2s - 1$; $\text{Ad} S_+ = [S_+, \cdot]$; and for the brevity we used notations $\psi^l(x, y)$ for $\psi^l(x_1, \ldots, x_n, y_1, \ldots, y_m)$, and $\psi_{1}(x)$ for $\psi_{1}(x_1, \ldots, x_{n(m)})$.

Using eq. (2.2) it is straightforward to check that function $\psi$ given by eq. (3.1.1) is translational invariant, i.e. $S_\pm \psi^l(x, y) = 0$.

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To construct the test function we symmetrize $\psi^l(x,y)$ over all $x$ and $y$:

$$
\psi^l_S(x,y) = \text{Sym}(x,y)\psi^l(x,y) = \frac{n!m!}{(n+m)!} \sum_{k=0}^{m} \sum_{\{i_1<...<i_k\} \atop \{j_1<...<j_k\}} \psi_{(j_1...j_k)}^{(i_1...i_k)}(x,y),
$$

(3.1.2)

where $\psi_{(j_1...j_k)}^{(i_1...i_k)}(x,y)$ is obtained from $\psi^l(x,y)$ by interchanging $x_{i_1} \leftrightarrow y_{j_1}$, and so on. Also, without loss of generality, hereafter we take $n \geq m$. (For the cases $m=1$ and $n=1(2)$ the expression for the $\psi^l_S(x,y)$ yields the exact eigenfunctions, so one might hope that it will be a good approximation in other cases also.)

The corresponding expression for the "conjugate" function $\phi^l(x,y)$ looks more simple. Really, taking into account that $S_+\phi(....) = (x_1 + \cdots + x_n)\phi(....)$ one obtains:

$$
\phi^l(x,y) = \sum_{k=0}^{l} c_k (x_1 + \cdots + x_n)^k (y_1 + \cdots + y_m)^{l-k} \phi_1(x)\phi_2(y) = \phi_1(x)\phi_2(y)K(a,b) \exp a(\sum_{i=1}^{n} x_i) \exp b(\sum_{j=1}^{m} x_j)|_{a=b=0},
$$

(3.1.3)

where

$$
K(a,b) \equiv \sum_{k=0}^{l} c_k \partial^k_a \partial^k_b.
$$

(3.1.4)

Then, with the help of eqs. (2.13), (3.1.3) the following representation for the function $\psi^l(x,y)$ can be derived:

$$
\psi^l(x,y) = \phi_1(\partial_x)\phi_2(\partial_y)K(a,b) \prod_{i,j} \frac{1}{(1 - x_i(a + \xi_i))(1 - y_j(b + \eta_j))}|_{(a,b,\xi,\eta)=0}.
$$

(3.1.5)

Now we are going to show that the inequality (2.7) with $\lambda = \lambda_1 + \lambda_2$ and $\epsilon = C\sqrt{\ln l/l}$ holds for the function $\psi^l_S(x,y)$. As it was mentioned before this is sufficient to prove the theorem. Our first task is the calculation of the norm of the function $\psi^l_S(x,y)$. To be more precise, in the rest of this subsection we obtain the estimate from below of the norm of $\psi^l_S(x,y)$ for large values of $l$.

Using eqs. (2.14), (3.1.3) and taking into account that $\psi^l_S(x,y)$ is totally symmetric, one gets:

$$
||\psi^l_S||^2 = n!m! \sum_{k=0}^{m} \sum_{\{i_1<...<i_k\} \atop \{j_1<...<j_k\}} \phi(\partial_x, \partial_y)\psi_{(j_1...j_k)}^{(i_1...i_k)}(x,y) = n!m! \sum_{k=0}^{m} C_k^m A_{l}^{(k)}.
$$

(3.1.6)

The coefficients $A_{l}^{(k)}$ are given by the formula:

$$
A_{l}^{(k)} = N \phi(\partial_x, \partial_y)\psi_{(1...k)}^{(1...k)}(x,y) = N \phi_1(\partial_x)\phi_2(\partial_y)K(a,b)\psi_{(1...k)}^{(1...k)}(x,y + b - a),
$$

(3.1.7)

where the symbol $N$ denotes that in the end of calculation all arguments must be set to zero.
The substitution of (3.1.3), (3.1.5) into (3.1.7) yields:

\[ A^{(k)}_l = N \phi_1(\partial_x)\phi_2(\partial_y) \phi_1(\partial_x)\phi_2(\partial_y) K(a, b) K(\bar{a}, \bar{b}) \left[ \prod_{i=k+1}^m \frac{1}{1 - (\bar{y}_i + \bar{b} - \bar{a})(y_i + b)} \right] \left[ \prod_{i=1}^k \frac{1}{1 - (\bar{y}_i + \bar{b} - \bar{a})(x_i + a)} \right] \left[ \prod_{i=k+1}^m \frac{1}{1 - x_i(y_i + b)} \right]. \]  

(3.1.8)

The expression in the square brackets of (3.1.8) depends only on the difference \( \bar{b} - \bar{a} \), hence

\[ K(\bar{a}, \bar{b}) \left[ \prod_{i=1}^k (-1)^k c_k \right] = \left[ \prod_{i=k+1}^m \frac{1}{1 - y_i - b} \right] \left[ \prod_{i=1}^k \frac{1}{1 - x_i - a} \right] \].

(3.1.9)

In the resulting expression the dependence on \( \bar{b} \) can be factorized after the appropriate rescaling of the variables \( x, y, x, y, a, b \). At last, taking advantage of eq. (2.13) and remembering that the function \( \psi_1, \psi_2 \) (but not \( \phi(\ldots) \)) are translation invariant, one gets:

\[ A^{(l)}_p = ZN \phi_1(\partial_x)\phi_2(\partial_y) K(a, b) F_k(a, b, x, y), \]  

(3.1.10)

where \( Z = l! \sum_{k=0}^l c_k (-1)^k = l! C_{l+A+B}^{2l+A+B} \) and

\[ F_k(a, b, x, y) = \left[ \psi_1(y_1, \ldots, y_k, x_{k+1} + a - b, \ldots, x_n + a - b) \prod_{i=1}^k \frac{1}{1 - x_i - a} \prod_{i=k+1}^m \frac{1}{1 - y_i - b} \right] \left[ (1 - a)^{-s} \psi_2(\frac{x_1}{1 - x_1 - a}, \ldots, \frac{x_k}{1 - x_k - a}, \frac{y_{k+1} + b - a}{1 - y_{k+1} - b}, \ldots, \frac{y_m + b - a}{1 - y_m - b}) \right]. \]

(3.1.11)

It is easy to understand that after the differentiation with respect to \( x_i, y_j \) the resulting expression will have form:

\[ A^{(k)}_l = ZN K(a, b) \sum_{n_1, n_2, n_3} C_{n_1, n_2, n_3}^k \frac{(a - b)^{n_1}}{(1 - a)^{n_2}(1 - b)^{n_3}} = Z \sum_{n_1, n_2, n_3} C_{n_1, n_2, n_3}^k A_{n_1, n_2, n_3}^{k}(l), \]  

(3.1.11)

the summation over \( n_1, n_2, n_3 \) being carried out in the limits, which as well as the coefficients \( C_{n_1, n_2, n_3}^k \) are independent from the parameter \( l \). Thus, all dependence on \( l \) of \( A^{(k)}_l \), except for the trivial factor \( Z \), is contained in the coefficients \( A_{n_1, n_2, n_3}^{k} \).

Our further strategy is the following: first of all we shall obtain the result for the quantity \( A^{(0)}_l \) and for \( A^{(m)}_l \). (These terms give the main contributions to the norm of the vector \( \psi_S \)). Then, we shall show (it will be done in Appendix A) that for all other \( A^{(k)}_l \) \((1 \leq k \leq m - 1)\) for which we are not able to get the exact result, the ratio \( A^{(k)}_l / A^{(0)}_l \) tends to zero as \( 1/l^2 \) at least.

To calculate \( A^{(0)}_l \) it is sufficient to note that the expression for \( F_0(a, b, x, y) \) (eq. (3.1.10)) after the appropriate shift of the arguments in the functions \( \psi_1 \) and \( \psi_2 \) reads:

\[ \left[ \psi_1(x_1, \ldots, x_n) \prod_{i=1}^m \frac{1}{1 - y_i - b} (1 - b)^{-s} \psi_2(\frac{y_1}{1 - y_1 - b}, \ldots, \frac{y_m}{1 - y_m - b}) \right]. \]  

(3.1.12)
Then carrying out the differentiation with respect to \(x_i, y_j\) in eq. (3.1.9) one obtains:

\[
A_l^{(0)} = Z(m!n!)^{-1}||\psi_1||^2||\psi_2||^2N K(a, b)(1 - b)^{-(2s + m)} = (m!n!)^{-1}A(l),
\]

(3.1.13)

where

\[
A(l) = (2l + A + B)!l!(l + A + B)!/(l + A)!l!(l + B)! A! B!
\]

(3.1.14)

The evaluation of \(A_l^n\) in the case when \(n = m = k\) differs from the considered above only in the interchange of variables \(x, a \leftrightarrow y, b\) in (3.1.12). Then the straightforward calculations yield:

\[
A_l^{(m)} = A_l^{(0)} \delta_{\psi_1, \psi_2}.
\]

(3.1.15)

Here, we take into account that \(\psi_1, \psi_2\) are the eigenfunctions of the self-adjoint operator. Thus, when \(l\) is odd and \(\psi_1 = \psi_2\) these two contributions \((A_l^{(0)}\) and \(A_l^{(m)})\) cancel each other. But, as one can easily see from eq. (3.1.4), the function \(\psi^l(x, y)\) is identically equal to zero in this case.

In the case when \(k = m\) and \(m < n\) eq. (3.1.11) for \(A_l^{(m)}\) reads:

\[
A_l^{(m)} = ZN K(a, b) \sum_{z=s}^{r-s} c_z z^s (a - b)^{2s + m} = ZN K(a, b) \sum_{k=0}^{r-s} c_z z^s (1 - b)^k / (1 - a)^{2s + m + k}.
\]

(3.1.16)

After some algebra one obtains:

\[
A_l^{(m)} = Z(l + A + B)! \sum_{k=0}^{r-s} c_k z^s l!(l + B + k - i)! / l!(l + A - i)! \leq c_A l^{n - m} \leq c_A l.
\]

(3.1.17)

The similar calculations (see Appendix A for details) in the case of \(0 < k < m\) give:

\[
|A_l^{(k)}| \leq c_A l^{n - m} l^2,
\]

(3.1.18)

where \(\alpha_k\) are some constants. Then, taking into account (3.1.14), (3.1.15), (3.1.17), (3.1.18), one obtains:

\[
||\psi_S^l||^2 = (1 + (-1)^l \delta_{\psi_1, \psi_2}) A(l)(1 + O(1/l)).
\]

(3.1.19)

### 3.2 Part II

To complete the proof of the theorem it is remained to obtain the following inequality:

\[
\epsilon(l) = ||\delta H \psi_S^l||^2 = ||(H - \lambda_1 - \lambda_2) \psi_S^l||^2 \leq C \ln l/l^2 A(l)
\]

(3.2.1)

for \(l \geq l_0\).

To do this let us, first of all, substitute the expression (3.2.1) for \(\psi_S^l\) in (3.2.1). Then, taking into account the invariance of the operator \(H\) under any transposition of its arguments (see eq. (2.10)) one gets:

\[
(\epsilon(l))^{1/2} = \frac{n!m!}{(n + m)!} \sum_{k=0}^{m} \sum_{\{i_1 < \ldots < i_k\}} \delta H \psi_S^{(i_1, \ldots, i_k)}(x, y) \leq ||\delta H \psi^l(x, y)||.
\]

(3.2.2)
Further, from the eqs. (2.10), (3.1.1) the equality
\[ \delta H\psi^j(x, y) = \sum_{i,k} H(x_i, y_k)\psi^j(x, y) \] (3.2.3)
immediately follows. At last, taking into consideration that
\[ \|H(x_i, y_k)\psi^j(x, y)\| = \|H(x_1, y_1)\psi^j(x, y)\| \]
and
\[ \|H(x_1, y_1)\psi^j(x, y)\| \geq \psi^j(x, y)H(x_1, y_1)\psi^j(x, y) > \]
we obtain the following estimate of \( \epsilon(l) \):
\[ \epsilon(l) \leq (mn)^2 < \psi^j(x, y)H(x_1, y_1)\psi^j(x, y) > . \] (3.2.4)

Our next purpose is to obtain the expression for the matrix element in (3.2.4) similar to that for \( A^k_l \) (eq. (3.1.11)). This matrix element, with the help of formulae (3.1.3), (3.1.5), can be represented in the following form:
\[ < \psi^l(x, y)H(x_1, y_1)\psi^l(x, y) > = Z(n + m)!K(a, b)\phi_1(\partial_x)\phi_2(\partial_y)\prod_{i=2}^{n} \frac{1}{1 - ax_i} \prod_{i=2}^{m} \frac{1}{1 - by_i} \int_0^1 ds \frac{1}{1 - a\theta(s)} \frac{1}{1 - b\theta(s)} \psi_1(\frac{\theta(s)}{1 - a\theta(s)}) \psi_2(\frac{\theta(s)}{1 - b\theta(s)}) \bigg|_{x=0, y=1, a=b=0} \] (3.2.5)
where \( \theta(s) = sx_1 + (1 - s)y_1 \). After the differentiation with respect to \( x_i, y_j, i, j > 1 \) one gets:
\[ < \ldots > = \sum_{n_1=0}^{r} \sum_{n_2=0}^{r} \sum_{s_1=0}^{s} \sum_{s_2=0}^{s} \sum_{m_1=0}^{s_1} \sum_{m_2=0}^{s_2} Z \alpha_{m_1, m_2, m_3}(l) \] (3.2.6)
where
\[ \alpha_{m_1, m_2, m_3}(l) = ZK(a, b) \left[ \frac{a^{n_2-n_1}}{(1 - b)^{\beta-m_2}} \partial_x^{n_1} \partial_y^{m_1} \int_0^1 ds \frac{\theta^{n_2-m_2}}{(1 - a\theta)^{n_2+1}(1 - b\theta)^{m_2+1}} \right]_{x=0, y=1, a=b=0} \] (3.2.7)
and \( \beta = s + m_3 + m - 1 \). Note that the coefficients \( \alpha_{m_1, m_2, m_3} \) in eq. (3.2.6) do not depend on \( l \).

Before applying the operator \( K(a, b) = \sum_{k=0}^l c_k \theta^k a^{p-k} b^{q-k} \) to the expression in the square brackets it is convenient to rewrite the latter in more suitable for this purpose form:
\[ [\ldots] = (n_2!m_2!)^{-1} \sum_{k=0}^l c_k \theta^k \left[ \frac{a^{n_2-n_1}}{(1 - b)^{\beta-m_2}} \int_0^1 ds \frac{\theta^{n_2-m_2}}{(1 - a\theta)^{n_2+1}(1 - b\theta)^{m_2+1}} \right]_{x=0, y=1, a=b=0} \] (3.2.8)
Now all differentiations with respect to \( a \) and \( b \) can be easily fulfilled:
\[ \frac{\partial^k}{a^k} \left[ \frac{a^{n_2-n_1}}{(1 - a\theta)^{n_2+1}(1 - b\theta)^{m_2+1}} \right]_{a=0} = (k + n_1)!s^{k+n_1} \partial_x^{n_2-n_1} x^k \]
\[
\partial_b^{l-k} \frac{1}{(1-b)^\beta} \partial_b^{m_2} \frac{1}{(1-sb)} = \frac{s^{m_2}(l-k+\beta)!}{\Gamma(\beta-m_2)} \int_0^1 d\alpha \alpha^{\beta-m_2-1}(1-\alpha)^{m_2}[\alpha + (1-\alpha)s]^{l-k}.
\]

At last, after a representation of the ratio of the factorials like \((k+n_1)!/(k+A)!\) in the form \(1/\Gamma(A-n_1-1)\int_0^1 du u^{k+n_1}(1-u)^{A-n_1-1}\) the summation over \(k\) becomes trivial and we obtain the following expression for \(\tilde{a}_{m_1,m_2,m_3}^{n_1n_2}(l)\):

\[
\tilde{a}_{m_1,m_2,m_3}^{n_1n_2}(l) = A(l)\tilde{a}_{m_1,m_2,m_3}^{n_1n_2}(l),
\]

where

\[
a_{m_1,m_2,m_3}^{n_1n_2}(l) = \frac{1}{\Gamma(A-n_1)\Gamma(\beta-m_2)\Gamma(\beta-\beta)} \int_0^1 ds \int_0^1 ds du v^{n_1}(1-u)^{A-n_1-1} s^{m_1} (1-s)^{n_1+m_1} s^{n_1+m_2} v^{1}(1-v)^{B-\beta-1}\alpha^{\beta-m_2-1}(1-\alpha)^{m_2} [v(\alpha + (1-\alpha)s) - su]^l_{x=1}.
\]

Note, that when the arguments of \(\Gamma\) – functions become equal to zero, the following evident changes must be done: \(1/\Gamma(A-n_1)\int_0^1 du (1-u)^{A-n_1-1} \rightarrow \int_0^1 du \delta(1-u)\) if \(A = n_1\), and so on. With the account of eq. (3.2.9) eq. (3.2.7) reads:

\[
< \psi^l(x,y)H(x_1,y_1)\psi^l(x,y) >= A(l) \sum_{n_1=0}^r \sum_{n_2=0}^r \sum_{m_1=0}^{s-m_3} \sum_{m_2=0}^{s-m_3} \sum_{m_3=0}^{s-m_3} c_{m_1,m_2,m_3}^{n_1n_2} a_{m_1,m_2,m_3}^{n_1n_2}(l).
\]

Thus to prove the inequality (3.2.11) for \(\epsilon(l)\) we should only show that the coefficients \(a_{m_1,m_2,m_3}^{n_1n_2}(l)\) tend to zero not slowly than \(ln/l^2\) at \(l \rightarrow \infty\).

First of all, let us consider the cases when at least one \(\Gamma\) function in (3.2.10) has zero argument. (It is worth to remind that \(A = 2r + n - 1\), \(B = 2s + m - 1\); \(n, m (m \leq n)\) are the numbers of variables and \(r, s\) – degrees of the translational invariant polynomials \(\psi_1\) and \(\psi_2\) correspondingly).

1. \(A = n_1\). It is easy to understand that equality \(A-n_1 = 0\) is possible only when \(n = 1, r = 0, m = 1, s = 0\). Then one immediately obtains that arguments of two other \(\Gamma\) functions are also zero, and the corresponding integral (see eq. (3.2.10) and the note to it) is zero.

2. \(\beta - m_2 = 0, n > 1 (0 < A - n_1)\). In this case one obtains \(m = 1, s = 0, B - \beta = 0\). Since two of \(\Gamma\) functions have arguments equal to zero, the integration over \(v\) and \(\alpha\) are removed. After this it is trivial to check that \(a(l)\) tends to zero as \(1/l^2\) at \(l \rightarrow \infty\).

3. \(B - \beta = 0\) and \(1 < m \leq n\). Evidently, this is possible only when \(m_1 = 0\) and \(m_3 = s\). Again one integration (over \(v\)) is removed. To calculate \(a(l)\) first of all let us write \(\partial_x^{n_1} \) in (3.2.11) as \(u^{n_2-n_1} \partial_u^{n_2-n_1} \) and carry out the integration by parts as over \(u\) as well as over \(s\). Note, that the boundary terms do not appear when \(m_1 = 0\). Then it is clear that integrand represent by itself the product of two functions, one of which, \([\alpha + (1-\alpha)s] - su\]^l, is positive definite in the region of integration and other is the sum of the monoms like \(s^u(1-s)^{s} \alpha^2...\) with finite coefficients. But since \(0 \leq u, \alpha \leq 1\) this sum can be limited by some constant which is independent of \(l\).

Then, taking into account this remark, one obtains the following estimate of \(a(l)\):

\[
|a_{m_1,m_2,m_3}^{n_1n_2}(l)| \leq C \int_0^1 \int_0^1 ds \int_0^1 du [(\alpha + (1-\alpha)s) - su]^l =
\]

\[\text{(3.2.12)}\]
\[
\frac{C}{(l+1)} \int_0^1 dsdu \frac{(1-su)^{l+1} - s^{l+1}(1-u)^{l+1}}{1-s} = \frac{2C}{(l+1)(l+2)} [\psi(l+2) - \psi(1)],
\]

where \( \psi(x) = \partial_x \ln \Gamma(x) \).

4. At last, we consider the case when all arguments of \( \Gamma \) functions in eq. (3.2.10) are greater than zero. As in the previous case it is convenient to replace \( \partial^{n_2-n_1}_x \) with \( u^{n_2-n_1} \partial^{n_2-n_1}_u \) and fulfil the integration over \( u \) and \( s \) by parts. But the boundary terms arise now with the integration over \( s \) at upper bound \( (s = 1) \). However, it is not hard to show that each of them decreases as \( 1/l^2 \) at \( l \to \infty \). (All calculations practically repeat those given in Appendix A)

The last term to be calculated has the form:

\[
I(l) = \int \int \int dsd\alpha du dv A(s, \alpha, u, v)[v(\alpha + (1-\alpha)s) - su]^l,
\]

where \( A(s, \alpha, u, v) \) is some polynomial of variables \( s, \alpha, u, v \), such that \( A(s, \alpha, u, v) < C \) in the domain \( 0 \leq s, \alpha, u, v \leq 1 \). Then for even \( l \) one obtains:

\[
I(l) \leq C \int_0^1 \ldots \int_0^1 dsd\alpha du dv [v(\alpha + (1-\alpha)s) - su]^l.
\]

If \( p \) is odd, let us divide the domain of the integration into two regions \( \Omega_+ \) and \( \Omega_- \) — in which the function \( [v(\alpha + (1-\alpha)s) - su] \) is either positive or negative. It is straightforward to get that \( \Omega_+, \Omega_- \) are defined by the following conditions:

\[
\Omega_+: \quad (u \leq v) \text{ or } (v \leq u, \; s \leq v/u, \; s(u-v)/v(1-s) \leq \alpha \leq 1); \\
\Omega_-: \quad (v \leq u); \text{ and } (s \geq v/u \text{ or } (s \leq v/u, \; 0 \leq \alpha \leq s(u-v)/v(1-s))).
\]

Then integral \( I(l) \) is estimated as:

\[
I(l) \leq C_+ \int_{\Omega_+} dsd\alpha du dv [v(\alpha + (1-\alpha)s) - su]^l - C_- \int_{\Omega_-} dsd\alpha du dv [v(\alpha + (1-\alpha)s) - su]^l. \]  (3.2.15)

The evaluation of the corresponding integrals does not cause any troubles and leads to the following result:

\[
I(l) \leq C \ln l/l^2. \]  (3.2.16)

Then taking into account eqs. (3.2.1), (3.2.4), (3.2.11) and (3.1.19) one concludes that there exist such constants \( L \) and \( C \) that inequality

\[
\|(H - \lambda_1 - \lambda_2) \psi^l_S\|^2 \leq C \ln l/l^2 \|\psi^l_S\|^2
\]

holds for all \( l \geq L \). This inequality, as it has been shown in sec. 2 guarantees the existence of the eigenvector of the operator \( H \) with the eigenvalue satisfying the eq. (2.6). \( \blacksquare \)
4 Conclusion

The theorem proven in the previous section provides a number of consequences for the spectrum of the operator \( H \):

- Every point of the spectrum is either a limit point of the latter or an exact eigenvalue of infinite multiplicity;
- Any finite sum of eigenvalues and limit points of the spectrum is a limit point again.

These statements directly follow from the theorem.

Further, let us denote by \( S_n \) the spectrum of operator \( H \) restricted on \( n \)– particle sector of Fock space, \((N\psi = n\psi)\) and by \( \bar{S}_n \) the set of the limit points of \( S_n \). Then it is easy to see that the definite relations (“hierarchical structures”) between \( S_n \), \( \bar{S}_n \) \((n = 2, \ldots \infty)\) exist. Indeed, let \( \Sigma_n \) is the set of all possible sums of \( s_{i_1} + s_{i_2} + \cdots + s_{i_m} \) type, where \( s_{i_k} \in S_k \) and \( i_1 + \cdots + i_m \leq n \), \( i_1 \leq i_2 \leq n \cdots \leq i_m \). Then one can easily conclude that the following relations

\[ \Sigma_n \subset \bar{S}_n \]

are valid. For \( n = 2, 3 \) the more strong equations \( \Sigma_n = \bar{S}_n \) hold, but the examination of this conjecture for a general \( n \) requires additional analysis.

So one can see that a number of interesting properties are specific to one–loop spectrum of anomalous dimensions of composite operators in \( \phi^4 \) theory. Of course, two question arise: which of these (one–loop) properties survive in a higher order of the perturbation theory? And, in what extent they are conditioned by the peculiarity of \( \phi^4 \) theory?

As to the first question we can only adduce some arguments in favor of a nonperturbative character of the obtained results. In the paper \[8\] the spectrum of critical exponents of the \( N \)–vector model in \( 4 - \epsilon \) dimensions was investigated to the second order in \( \epsilon \). In this work it was obtained that some one–loop properties of the spectrum, a generic class of degeneracies \[6, 7\] in particular, are lifted in two–loop order. However, the results of the numerical analysis of critical exponents carried out for the operators with number of fields \( \leq 4 \) distinctly show that a limit points structure of the spectrum is preserved.

The other evidence in favor of this hypothesis can be found in the works of K. Lang and W. Rühl \[10\]. They investigated the spectrum of critical exponents in the nonlinear \( \sigma \) model in \( 2 < d < 4 \) dimensions in the first order of \( 1/N \) expansion. The results for various classes of composite operators \[10\] display the existence of a similar limit points structures in this model for whole range \( 2 < d < 4 \). Since the critical exponents should be consistent in \( 1/N \) expansion for \( \sigma \) model and the \( 4 - \epsilon \) expansion for \( (\phi^2)^2 \) model one may expect this property of the spectrum to hold to all orders in the \( \epsilon \) in the latter.

To answer the second question it is useful to understand what features of the model under consideration determine the properties of operator \( H \) — hermiticity, invariance under \( SL(2, C) \)
group, two-particle type of interaction — which were crucial for the proof of the theorem. First two properties are closely related to the conformal invariance of the \( \phi^4 \) model. It can be shown that a two-particle form of operator \( H \) and the conformal invariance of a theory lead to hermiticity of \( H \) in the scalar product given by eq. 2.14. (The relation between the functions \( \psi \) and \( \phi \) in the general case is given in ref. [12].) Further, it should be emphasized that the commutativity of \( H \) with \( S \) and \( S^+ \) reflects two simple facts: 1. Nontrivial mixing occurs (in \( \phi^4 \) theory) only between operators with equal number of fields. 2. The total derivative of an eigenoperator is an eigenoperator with the same anomalous dimension as well.

But if operator \( H \) is hermitian it must commute with the operator conjugated to \( S^+ \) as well. So the minimal group of invariance of \( H \) (\( SL(2, C) \) in our case) has three generators.

Thus, the method of analysis of anomalous dimensions of composite operators in the \( l \to \infty \) limit presented here is not peculiar for \( \phi^4 \) theory only and can be applied to other theories, which are conformal invariant at one–loop level.

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**Appendix A**

In this appendix we calculate the quantities \( A^{(k)}_l \) for \( 1 \leq k \leq m-1 \). Let us remind the representation for \( A^{(k)}_l \) (see eq. (3.1.11)): \[ A^{(k)}_l = Z \mathcal{N} K(a, b) \sum_{n_1, n_2, n_3} C^k_{n_1, n_2, n_3} \frac{(a-b)^{n_1}}{(1-a)^{n_2}(1-b)^{n_3}}. \] (A.1)

The summation over \( n_1, n_2, n_3 \) is carried out in the following range:

\[ n_1 = m_1 + m_2; \quad n_2 = k + s + m_1 + m_3; \quad n_3 = s + m - k + m_2 - m_3; \]

\[ 0 \leq m_1 \leq \min [s, r]; \quad 0 \leq m_2 \leq s; \quad 0 \leq m_3 \leq r - m_1. \] (A.2)

The more simple way to obtain these bounds from (3.1.10) is to treat \( (a-b), (1-a), (b) \) as independent variables.

In the course of calculation of the coefficient \( A^{(k)}_{n_1, n_2, n_3} = \mathcal{N} K(a, b)(a-b)^{n_1}(1-a)^{-n_2}(1-b)^{-n_3} \) we shall not look after the factors independent of \( l \). Then taking advantage of the Feynman’s formula for the product \( (1-a)^{-n_2}(1-b)^{-n_3} \) one obtains:

\[ A^{(k)}_{n_1, n_2, n_3} \sim \sum_{k=0}^{l} C_k \partial_a^k \partial_b^{(l-k)} \int_0^1 dx x^{n_2-1}(1-x)^{n_3-1} \frac{1}{[1-ax-(1-x)b]^{B+1}}. \] (A.3)
(let us remind that \( B = 2s + m - 1, \ A = 2n + r - 1, \) and \( m \leq n \)). After the integration by parts in (A.3) (note, that in the case \( r \leq s \) there are no boundary terms) and the differentiation with respect to \( a, b \) and \( x \) one has:

\[
A_{n_1,n_2,n_3}^{(k)} \sim \sum_{k=0}^{l} c_k(l + B)! \sum_{j=n_3-n_1-1}^{B-1} \alpha_j \int_0^1 dx x^k (1 - x)^{l-k+j} =
\]

\[
(l + B + A)! \sum_{j=n_3-n_1-1}^{B-1} \alpha_j \frac{(l + B)!}{(l + j + 1)!} \sum_{k=0}^{l} \frac{(-1)^k C^d_k}{(k+A)!} \frac{k!}{(l-k+B)!} \sim \quad (A.4)
\]

\[
\sim \sum_{j=n_3-n_1-1}^{B-1} \frac{(l + B)!}{(l + j + 1)!} \frac{(l + B + A)!}{(l + j + 1)!} \int_0^1 du dv u^{A-1} (1 - v)^j v^{B-j-1} (u - v)^l. \quad (A.5)
\]

Here, \( \alpha_j \) are some unessential constants. Since \(|(u - v)| \leq 1\) every integral in (A.5) tends to zero at \( l \to \infty \). For more precise estimate it is convenient to divide the domain of integration into two regions \((u \leq v \text{ and } v \leq u)\) and to rescale the variables \( v = ut \) \((u = vt)\) in each of two integrals. Then replacing all terms of \((1 - ut)^\alpha\) type by unit one obtains the answer in the form of the product of two beta functions. Taking into account that \( s - r + m - k - 1 \leq j \leq B - 1 \) and collecting all necessary terms one gets that the contribution from \( A_{n_1,n_2,n_3}^{(k)} \) to the \( A_l^{(k)} \) is of order \( A(l)/l^2 \).

Thus we have obtained the required result for the case \( r \leq s \). To do the same for the \( s \leq r \), first of all, note that representation of \( A_l^{(k)} \) in the form (3.1.11) is not unique. Indeed, the expansion of \((a - b)^{n_1}\) in series in \((1 - a) (1 - b)\) will lead to the redefinition of coefficients \( C^k_{n_1,n_2,n_3} \) and to the change of the limits of the summation. We use this freedom to represent \( A_l^{(k)} \) in the form in which all calculations for the case \( s \leq r \) can be done in the same manner as those for \( r \leq s \). Let us consider the eq. (3.1.4). It is obvious that due to the translation invariance of function \( \psi^l(x, y) \) it does not change by the shift of the variables \( \bar{y} + \bar{b} - \bar{a} \to \bar{y} \) and \( \bar{x} \to \bar{x} + \bar{a} - \bar{b} \). Then carrying out successively all operations as in the Sec.3, one arrive at the formulae (3.1.11), the summation being carried out in the following range:

\[
n_1 = m_1 + m_2; \quad n_2 = r + n - k + m_2 - m_3; \quad n_3 = k + r + m_1 + m_3; \quad 0 \leq m_1 \leq \min [s, r]; \quad 0 \leq m_2 \leq r; \quad 0 \leq m_3 \leq s - m_1. \quad (A.6)
\]

The calculation of the coefficients \( \tilde{A}_l^{k}_{n_1,n_2,n_3} \) in this case simply repeats the one given above.

Thus, for \( 1 \leq k \leq m - 1 \) one has the required inequality:

\[
|A_l^{(k)}| \leq \text{const} A_l^{(0)}/l^2.
\]
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