Nonclassical mathematical model for the mechanical analysis of additive manufacturing viscoelastic materials on rotating cylindrical substrates

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Abstract. Technological processes of additive manufacturing viscoelastic aging axisymmetric material layers on the inner surface of a rotating rigid cylindrical substrate by means of deposition of supplementary material plies are modelled. The added plies may be preliminary tensed or compressed in the circumferential direction. A mathematical model based on general approaches of mathematical theory of accreted solids is proposed for describing the deformation process of the being formed material layer under the action of centrifugal forces in assumption of small plain strain and in quasistatic approximation. The corresponding closed-form expressions containing quadratures are constructed for describing the technological stresses evolution in the manufactured layer during and after the process of its accretion.

1. Introduction

A deal of technological processes are accompanied by an increase in the size and, possibly, by a change in the shape of solids due to attaching additional material to them, that is by accretion, or growth of these solids. Processes of such a kind are used to be called additive.

It is obvious that when studying additive processes one should take into account the kinematic and force features of the new material gradual inflow to the surface of the accreted solid simultaneously with loadings acting on this solid in the considered technological process. However, such accounting is impossible in principle to implement correctly in the framework of the classical mechanics of solids, even if we consider the traditional equations and boundary conditions in a domain variable in time. This is due to the fact that the classical formulation of the mechanical problems always implies the existence for the entire solid of such a configuration in which there are no any stresses, i.e. a natural configuration. Meanwhile, an accreted solid does not have in general case any natural configuration (unlike, for example, a solid exposed to removal of material), because while some of material elements have already deformed with the solid, others have not yet been incorporated into it. As a consequence of this fundamental fact, the problems on mechanical behaviour of accreted solids should possess a number of specific features and constitute a special class of problems in solid mechanics.

Development of common approaches to mathematical formulation of such nonclassical problems and constructing general methods of these problems solution, as well as studying...
various growth processes on the basis of the mechanical problems for different accreted solids solved due to the elaborated techniques present works [1–20]. Cited studies demonstrate a great number of specific examples and discuss a variety of new mechanical effects discovered thanks to the obtained solutions of the nonclassical mechanical problems.

2. The mechanical problem under study
This study is devoted to modelling the processes of gradual deposing material layers of uniform thickness on the inner surface of an axisymmetric cylindrical substrate rotating around its axis with the angular velocity $\omega(t)$ arbitrarily changing with time $t$. The current inner radius of the being formed material layer will be denoted by $a(t)$. Another radius of the being formed material layer, i.e. the radius of the used substrate working face will be denoted by $a_0$.

It is assumed that when deposing the material the rate of its inflow in the circumferential direction is incomparably higher than the rate of its inflow in the radial direction. This assumption makes it possible to simulate the considered deposing process as an axisymmetric process in which the material layer is being accreted simultaneously throughout its whole inner cylindrical surface. We will term this surface the accretion surface. We will consider the material accretion process as continuous, that means, an infinitely thin additional material ply joins to the accreted solid each infinitely small period of time. In this case the dependence of the accretion surface radius $a$ on the time $t$ is a continuous function.

In this paper we will consider the case of small strain arisen in the accreted material layer. So it makes no sense to take into account the strain constituent in the time-dependance $a(t)$, and we get the variable radius $a$ of the being formed layer decreasing in the course of the time $t$ purely due to inflow of the additional material to the accretion surface. The dependence $a(t)$ can be considered as a prescribed program of attaching the new material to the layer, which is implemented in the simulated process. Thus, in the present study the dependence $a(t)$ is a given continuous function, strictly monotonic in those time intervals in which the material is deposed, and constant in those time intervals in which the inflow of additional material to the formed layer is temporarily or finally stopped.

In the here proposed model the possible compliance of the substrate on which the material layer is being formed is not taken into consideration, and the substrate is considered to be absolutely rigid. So the working radius of the substrate $a_0$ is supposed to be a constant in the modelled technological process.

A sufficiently slow time variation of the substrate rotational speed is assumed: $|\dot{\omega}(t)| \ll \omega^2(t)$ (we will denote the derivative of any function of the only time-variable $t$ by a dot on the top). Thereby the tangential inertia forces of rotation are negligible if comparing to centrifugal forces.

It is also supposed that the potentially possible dynamic effects from the attaching the additional material to the accretion surface of the being formed layer are insignificant, and therefore the forces of inertia of its deformation can also be neglected in comparison with the centrifugal forces of inertia and the problem can be considered in a quasi-static statement.

We assume that the processes and effects which accompany the continuous inclusion of the additional substance in the composition of the being formed cylindrical solid lead to the appearance of some circumferential initial stresses in this solid near the accretion surface, and these stresses can arbitrary change with the radius $\rho$ of the currently added infinitesimal circular material ply, but not with the angular position $\varphi$ on this ply. That is the newly added plies can undergo preliminary uniform tension or compression in the hoop direction.

It is stated the task to trace the evolution of the stress-strain state of the material layer being additively formed in the above described manner, under the influence of the centrifugal inertia forces caused by the layer rotational motion together with the rigid substrate, in the approximation of the plane strain state.
3. The material governing equations

In the model proposed in this paper, we consider a linear viscoelastic uniform and uniformly aging isotropic material with the same constant (independent of either the material age or the time elapsed since the application of loads to it) Poisson’s ratio $\nu = \text{const}$ for instantaneous elastic strain and developing over time creep strain. For the given material the relationship between the stress tensor $T$ and the small strain tensor $E$ at each point $r$ of the solid at any time moment $t$, calculated from the instant of the material origination, has the form [21, 22]:

$$T^\rho(r, t) = 2E(r, t) + (\kappa - 1) 1 \text{tr} E(r, t), \quad g^\rho(r, t) = \frac{g(r, t)}{G(t)} - \int_{\tau_0(r)}^t \frac{g(r, \tau)}{G(\tau)} K(t, \tau) d\tau. \quad (1)$$

Here $1$ is the unit tensor of the second rank, $\kappa = 1/(1 - 2\nu)$ is the material constant, $G(t)$ is the elastic shear modulus of the material at its age $t$, and $K(t, \tau)$ is the creep kernel. The latter can be expressed through various characteristics of the material, describing its behaviour in any elementary stress state. For example, using the characteristics for the pure shear state it will be $K(t, \tau) = G(\tau) \partial \Delta(t, \tau)/\partial \tau$, where $\Delta(t, \tau) = 1/G(\tau) + \Omega(t, \tau)$, and $\Omega(t, \tau)$ is the creep measure for the pure shear, $\Omega(t, \tau) \equiv 0$. The function $\Delta(t, \tau)$ describes the evolution over time $t$ of specific (per unit of the acting shear stress) shear strain caused by the constant stress state of pure shear created at the time moment $\tau > 0$. We will call the tensor $T^\rho$ standing on the left hand side of constitutive relation (1) the operator stress tensor.

The lower limit of integration in (1) has the meaning of the time moment of occurrence of the stress state in the neighborhood of the considered point of the solid. Since the accreted solid is increased with new material elements already during the process of its deformation, the moment of occurrence of stresses at the points $r$ of such a solid will change from point to point and be set by a certain function $\tau_0(r)$, which is taken into account in constitutive law (1). In the modelled axisymmetric additive processes of the material deposition on a rigid cylindrical substrate we have $\tau_0(r) \equiv \tau_0(\rho)$, where the value $\tau_0(\rho)$ should be considered as the moment of joining the material ply of a radius $\rho$ to the accreted solid.

The constitutive relations similar to (1) are commonly applied to describe the mechanical behaviour of such wide used engineering materials as, for example, concrete and polymers. Note that state equation (1) contains as a special case the state equation of the isotropic linearly elastic material. This case is obtained by taking $\Omega(t, \tau) \equiv 0$ and $G(t) \equiv \text{const}$.

4. Nonclassical boundary value problem statement

Let the deposition of the material on the inner surface of a cylindrical substrate rotating around its axis begins at some time moment $t = t_1 > 0$. The further deposition process may be interrupted at arbitrary time moments with arbitrarily long pauses when the inflow of additional material to the material layer having already been formed on the substrate is temporarily stopped. Such a piecewise continuous process of accreting the material layer finally comes to the end after a certain number $N$ of stages of continuous depositing the additional material. The ready-formed layer may still continue to rotate rigidly coupled with the substrate.

Let us associate with the rotating around its axis axisymmetric cylindrical rigid substrate a polar cylindrical coordinate system $(\rho, \varphi, z)$ with the orthonormal frame $\{e_\rho, e_\varphi, e_z\}$, where: $e_z = \text{const}$ is the direction along the axis of rotation, $z$ is the longitudinal coordinate measured in this direction; $e_\rho = e_\rho(\varphi)$ and $e_\varphi = e_\varphi(\varphi)$ are the radial and circumferential directions in the cross-section of a rotating cylindrical solid formed on the substrate, respectively, $\rho$ and $\varphi$ are the polar radius and the polar angle in the cross-section. In this movable noninertial base the standard equilibrium equation is valid for the considered accreted solid:

$$\nabla \cdot T(r, t) + f(r, t) = 0 \quad (2)$$
where the centrifugal forces of inertia with the volume density 
\[ f(\mathbf{r}, t) = e_{\rho}(\varphi) \mu \rho \omega^2(t) \] (3)
play a part of body forces, \( \mu = \text{const} \) is the mass density of the material used.

It is obvious that for the entire cylindrical solid \( a(t) < \rho < a_0 \) having been formed by the instant \( t \) a smooth velocity vector field \( \mathbf{v}(\mathbf{r}, t) \) of this solid deformation motion is determined. Due to axial and mirror symmetry of the considered process and the plane strain state of the solid, the velocity field has the following representation in the above introduced rotating coordinate system: 
\[ \mathbf{v}(\mathbf{r}, t) = e_{\rho}(\varphi) v(\rho, t) \]
where \( v \) is the rate of the growing layer points displacement in the radial direction owing to the layer deformation. On the being formed layer boundary fastened to the rigid substrate the condition of immobility of this boundary in the rotating coordinate system must be set:
\[ \mathbf{v}(\mathbf{r}, t) = 0 \quad \text{if} \quad \rho = a_0. \] (4)

For formulating a mechanical problem in terms of velocities it is necessary to enter into consideration the strain velocity tensor
\[ \mathbf{D} = (\nabla \mathbf{v}^T + \nabla \mathbf{v})/2. \] (5)
We can formulate the analogue of defining relation (1) for the velocity characteristics of the deformation process, i.e. for tensor \( \mathbf{D} \) and the operator stress velocity tensor \( \mathbf{S} = \partial \mathbf{T}^0/\partial t \) [1]:
\[ \mathbf{S}(\mathbf{r}, t) = 2\mathbf{D}(\mathbf{r}, t) + (\kappa - 1) \mathbf{1} \text{tr} \mathbf{D}(\mathbf{r}, t). \] (6)

The differential equation for tensor \( \mathbf{S} \) in the occupied with the solid variable domain \( a(t) < \rho < a_0 \) can be obtained by acting on equilibrium equation (2) with the linear integral operator of viscoelasticity \( \circ \) from (1) and by differentiating the result with respect to time \( t \). Here it is important to note that in general case the specified integral operator and the divergence operator \( \partial \) are not permutable because of the dependence of the lower integration limit in (1) on the considered solid point \( \mathbf{r} \). However it can be shown following Manzhirov [1] that for the processes in question these operators commute in relation to the stress tensor \( \mathbf{T} \), that is the permutation \( \nabla \cdot \mathbf{T}^0 = \nabla \cdot \mathbf{T}^0 \) will be fair in the entire region occupied at the given time moment by the whole accreted solid. That means the opportunity to write out the analogue of equilibrium equation (2) for the operator stress tensor: 
\[ \nabla \cdot \mathbf{T}^0(\mathbf{r}, t) = \mathbf{0}, \]
and therefore, for the operator stress velocity tensor:
\[ \nabla \cdot \mathbf{S}(\mathbf{r}, t) + \partial \mathbf{T}^0(\mathbf{r}, t)/\partial t = \mathbf{0}. \] (7)

In view of the transition in the mathematical problem formulation from the operator stresses \( \mathbf{T}^0 \) to their time derivatives \( \mathbf{S} \) it is necessary for the closure of the problem statement to set some initial conditions for the operator stresses. This can be done taking into account the initial stresses in each newly attached material ply described by the stress tensor
\[ \mathbf{T}(\mathbf{r}, \tau_0(\mathbf{r})) = \mathbf{T}_0(\mathbf{r}) \] (8)
inside the growing solid under consideration close to its accretion surface, where \( \mathbf{T}_0(\mathbf{r}) \) is a known tensor of initial stresses in the being formed solid at its point \( \mathbf{r} \) at the time instant \( t = \tau_0(\mathbf{r}) \) of joining this point to the solid. The tensor-function \( \mathbf{T}_0(\mathbf{r}) \) sets the stress state of the newly added material immediately after its attachment to the growing solid. This state emerges by uniform circumferential tension–compression of the plies attached to the layer being formed and is determined by the following:
\[ \mathbf{T}_0(\mathbf{r}) = e_{\varphi}(\varphi) e_{\varphi}(\varphi) s(\rho), \] (9)
where the given hoop stress \( s \) is taken in the present study as a known function of the joined ply radius \( \rho \). Note that tensor (9) matches with the stress-free current accretion surface during the accretion process: \( \mathbf{e}_p(\varphi) \cdot \mathbf{T}_0(\mathbf{r}) = 0 \).

Initial condition (8) with representation (9) implies the following condition for the operator stress velocity tensor \( \mathbf{S} \) on the moving due to the inflow of additional material boundary \( \rho = a(t) \) of the growing layer in any time interval of its continuous growth:

\[
\mathbf{e}_p(\varphi) \cdot \mathbf{S}(\mathbf{r}, t) = -\mathbf{e}_p(\varphi) q(t), \quad q(t) = \left[ \frac{s(a(t))}{a(t)} - \mu \omega^2(t) a(t) \right] \frac{\dot{a}(t)}{G(t)}, \quad \text{if} \quad \rho = a(t). \tag{10}
\]

We omit here the proof of this fact because of the limited content of the paper.

In those time intervals when the inflow of additional material to the additively formed rotating layer is ad interim or definitively halted, the condition of loading absence on the fixed accretion surface \( \rho = a(t) \equiv \text{const} \) is set by the equality \( \mathbf{e}_p(\varphi) \cdot \mathbf{T}(\mathbf{r}, t) = 0 \) which is valid at each point \( \mathbf{r} \) of this surface, starting from the moment \( t = \tau_0(\mathbf{r}) \) of this point inclusion into the solid composition. From this equality it obviously follows that condition (10) formally remains in force in the absence of an inflow of new material to the formed solid, that is, in any time interval of constancy of the function \( a(t) \), i.e. in intervals of naught derivative \( \dot{a}(t) \).

Thus, the entire process of piecewise continuous accreting an aging viscoelastic solid under consideration, including an arbitrary prolonged period of time after the end of accreting, will be described by the boundary value problem consisting of equation (7), (3), valid in the time-varying region \( a(t) < \rho < a_0 \), supplemented with relations (6), (5) and boundary conditions (4), (10). The time variable \( t \in (t_1, +\infty) \) is a parameter in this boundary problem.

5. Stress evolution in the manufactured material

We can construct the exact analytical solution of the above formulated boundary value problem. The corresponding evolution with time \( t \) of the operator stress velocities, at each point \( \mathbf{r} \) of the considered piecewise continuously accreted aging viscoelastic material layer on the time beam \( t > \tau_0(\mathbf{r}) \) covering the entire deformation history of the neighborhood of a given point \( \mathbf{r} \) since it is in the composition of this layer, may be represented in the following closed explicit form:

\[
\mathbf{S}(\mathbf{r}, t) = \mathbf{e}_p(\varphi) \mathbf{e}_p(\varphi) S_\rho(\rho, t) + \mathbf{e}_\varphi(\varphi) \mathbf{e}_\varphi(\varphi) S_\varphi(\rho, t) + \mathbf{e}_z \mathbf{e}_z \nu \left[ S_\rho(\rho, t) + S_\varphi(\rho, t) \right],
\]

\[
S_\rho(\rho, t) = -\Psi^\pm(\rho, t) + \frac{1}{\zeta + a^2(t)/\rho^2} \Psi^\pm(a_0, t) - \frac{\zeta + a^2(t)/a_0^2}{\zeta + a^2(t)/a_0^2} q(t),
\]

\[
\Psi^\pm(\rho, t) = \frac{2\mu}{\zeta + 1} \int_{a(t)}^{\rho} \left[ \frac{\omega^2(\tau_0(\xi))}{2} \frac{\partial \Omega(t, \tau_0(\xi))}{\partial t} + \frac{\omega(t) \dot{\omega}(t)}{G(t)} + \int_{\tau_0(\xi)}^{t} \omega(\tau) \dot{\omega}(\tau) \frac{\partial \Omega(t, \tau)}{\partial \tau} d\tau \right] \left( \alpha \pm \frac{\xi^2}{\rho^2} \right) d\xi.
\]

From initial condition (8) for the stress tensor we obtain according to the form of the integral operator ( ) the following initial condition for the operator stress tensor at the time moments \( t = \tau_0(\mathbf{r}) \): \( \mathbf{T}(\mathbf{r}, \tau_0(\mathbf{r})) = \mathbf{T}_0(\mathbf{r})/G(\tau_0(\mathbf{r})) \). After that we can reconstruct the entire time evolution of the operator stress tensor by the known operator stress velocity tensor evolution using the integration procedure: \( \mathbf{T}(\mathbf{r}, t) = \mathbf{T}_0(\mathbf{r})/G(\tau_0(\mathbf{r})) + \int_{\tau_0(\mathbf{r})}^{t} \mathbf{S}(\mathbf{r}, \tau) d\tau \).

As soon as at each point \( \mathbf{r} \) of the ready-made material layer we know the evolution of the tensor \( \mathbf{T}(\mathbf{r}, t) \) since the moment \( t = \tau_0(\mathbf{r}) \) of stresses appearance at the given point, we can find the entire time-evolution of the stress tensor \( \mathbf{T}(\mathbf{r}, t) \) at the point \( \mathbf{r} \) by the integral transformation inverse to the transformation ( ): \( \mathbf{T}(\mathbf{r}, t) = \mathbf{T}(\mathbf{r}, t) + \int_{\tau_0(\mathbf{r})}^{t} \mathbf{T}(\mathbf{r}, \tau) R(t, \tau) d\tau \) where the function \( R(t, \tau) \), called the relaxation kernel, is the resolvent of the kernel \( K(t, \tau) \).
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