ZEROS OF DYNAMICAL ZETA FUNCTIONS FOR HYPERBOLIC QUADRATIC MAPS

YUQIU FU

ABSTRACT. We prove that the dynamical zeta function $Z(s)$ associated to $z^2 + c$ with $c < -3.75$ has essential zero-free strips of size $1/2+$, that is, for every $\epsilon > 0$, there exist only finitely many zeros in the strip $\text{Re}(s) > 1/2 + \epsilon$. We also present some numerical plots of zeros of $Z(s)$ based on the method proposed in [JP02].

1. Introduction

We consider the rational map on the complex plane $f(z) = z^2 + c$. We let $\mathcal{J}$ be the Julia set associated to $f$. For $c < -2$, it can be shown (for example by using Theorem 2.2 of [Urb03]) that $f$ is hyperbolic, that is,

\begin{equation}
\inf_{z \in \mathcal{J}} \left\{ |(f^{(N)}(z))'| \right\} > 1
\end{equation}

for some $N > 0$. From now on we will assume $c < -2$.

The dynamical zeta function associated to the dynamical system $f$ can be defined in the following way [SZ04, Nau05].

We let $\mu_c = \sqrt{1 - 4c + 1}/2$ be the largest fixed point of $f$. Under our assumption $c < -2$ we can easily check that $2 < \mu_c < -c$. We choose $R \in (\mu_c, -c)$. Then we may verify that $f^{-1}(D(0, R)) \subset D(0, R)$. Here $D(0, R)$ is the disc of radius $R$ on the complex plane. Let $g_+(z) = \sqrt{z - c}$, $g_-(z) = -\sqrt{z - c}$ be the two inverse branches of $f$ on $D(0, R)$. Here the square root function is defined on $\mathbb{C} \setminus (-\infty, 0]$ and is real on the real axis. Then we have $g_i(D(0, R)) \subset D(0, R)$ for $i \in \{+, -\}$. We denote $g_i(D(0, R))$ by $D_i$ for $i \in \{+, -\}$. See Figure 1 for an illustration.

If we let $I_+ = [\sqrt{-\mu_c} - c, \mu_c]$, $I_- = [-\mu_c, -\sqrt{-\mu_c} - c]$, then it can be easily checked that $I_i \subset g_i(D(0, R))$, $g_i(I_j) \subset I_i$. We have $g_j(D_i) \subset D_j$. Also, when $|z| > \mu_c$, $f^{kN}(z)$ goes to infinity as $k \to \infty$ and therefore the Julia set of $f$ is given by

$$
\mathcal{J} = \bigcap_{k \geq 1} f^{-kN}(D(0, R)) = \bigcap_{k \geq 1} f^{-kN}(I_1 \cup I_2).
$$

In particular we have $\mathcal{J} \subset I_1 \cup I_2$.

Define $\mathcal{H}$ to be the Bergman space

$$
\mathcal{H} := \{ u \in L^2(D_+ \cup D_-) : u \text{ is holomorphic on } D_+ \cup D_- \}.
$$

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We define the Ruelle transfer operator $L_s : \mathcal{H} \rightarrow \mathcal{H}$ by
\[
L_s u(z) = \sum_{i \in \{+, -\}} [g'_i(z)]^s u(g_i(z)), \quad z \in D_j.
\]
Here $[g'_i]$ is the analytic continuation to $D_1 \cup D_2$ of $|g'_i|$ defined on $\mathbb{R} \cap (D_1 \cup D_2)$. $L$ is a trace class operator, and therefore we could define the dynamical zeta function by
\[
Z(s) = \det(I - L_s).
\]
If $s \in \mathbb{C}$ is a zero of the dynamical zeta function $Z(s)$, then there exists a nonzero $u \in \mathcal{H}$ such that $L_s u = u$. Our main theorem will be the following.

**Theorem 1.1.** Suppose $c < -3.75$. For every $\epsilon > 0$, there exists $M > 0$ such that for $s \in \mathbb{C}$ with $\text{Re}(s) > 1/2 + \epsilon$, $|\text{Im}(s)| > M$, and for every $u \in \mathcal{H}$,
\[
L_s u = u \Rightarrow u = 0.
\]

**Corollary 1.2.** Suppose $c < -3.75$. Then for every $\epsilon > 0$, $Z(s)$ has only finitely many zeros in the half plane $\text{Re}(s) > 1/2 + \epsilon$.

Let $\delta$ denote the Hausdorff dimension of the Julia set $J$ associated to $f$. It has been proved that $Z(s)$ has no zeros in the strip $\text{Re}(s) > \delta - \epsilon$ other than a simple zero at $s = \delta$, for some $\epsilon > 0$ — see [Nau05]. Therefore our result is only meaningful when $\delta > 1/2$. See Figure 2 for a plot of $\delta$ with respect to $c$. The graph shows that our result provides an improvement on the essential zero-free strips of $Z(s)$ for $c$ in, for example, the range $(-4.6, -3.75)$.

We will prove some properties of iterates of $f(z) = z^2 + c$ in Section 2, introduce a refined transfer operator in Section 3, and obtain some a priori bounds in Section 4. Finally we will complete the proof of Theorem 1.1 in Section 5. We will also present some numerical results in Section 6.
Figure 2. The solid curve in the plot shows how the Hausdorff dimension $\delta$ of the Julia set varies as $c$ varies. The dashed line is the line $\delta = 0.5$. For $c > -4.6$, we have $\delta > 0.5$.

**Notation.** We let $\mathcal{W}$ be the set of words generated by alphabets (or letters) $+, -$. That is, if we let $\mathcal{W}_n$ be the set of words of length $n$

$$\mathcal{W}_n := \{a_1 \cdots a_n : a_i \in \{+, -\}\},$$

then $\mathcal{W} = \bigcup_{n \geq 1} \mathcal{W}_n$. Let $|w|$ denote the length of the word $w$. For $w = w_1 \cdots w_n \in \mathcal{W}_n$, we let $g_w$ be the function

$$(1.2) \quad g_w = g_{w_n} \circ \cdots \circ g_{w_1}.$$  

For $m \leq n$, we define $w_{\leq m}$ to be the word $w_1 \cdots w_{m-1}$. We can define $w_{m_1 \leq \cdots \leq m}$ and other similar expressions in the obvious way.

$\hat{f}$ will denote the Fourier transform of $f$:

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-ix \cdot \xi} dx$$

and $\check{f}$ will denote the inverse Fourier transform such that $(\check{\hat{f}}) = f$. We let $\mathcal{F}_h(f)$ be the semiclassical Fourier transform of $h$, which is defined by $\mathcal{F}_h(f)(\xi) = \hat{f}(\xi/h)$.

We let $A \lesssim B$ denote the statement that there exists a constant $C > 0$ such that $A \leq CB$. The constant may depend on our dynamical system we set up in this section and various parameters chosen in sections below, which only depend on the dynamical system and $\epsilon$ (as in Theorem 1.1). In fact we can regard those constants as only depending on $c$ and $\epsilon$. We call such constants admissible. We let $B \gtrsim A$ be the statement $A \lesssim B$, and let $A \sim B$ be the statement that $A \lesssim B$ and $B \lesssim A$. We will also use the notation $A \lesssim_j B$ or $A = O_j(B)$ if there exist an admissible constant $C > 0$ and a constant $C_j$ which depends on $j$ such that $A \leq CC_j B$. Similarly, we define $A \gtrsim_j B$ and $A \sim_j B$. 


2. Properties of the Dynamical System \( f(z) = z^2 + c \)

We recall that the Julia set \( J \) is preserved under backward and forward iteration [CG13]:
\[
f(J) = J, \quad f^{-1}(J) = J.
\]

Due to hyperbolicity (1.1) we know that
\[
\sup_{|w|=N} \sup_{z \in J} |g'_w(z)| < 1.
\]

Since \( \sup_{|w|=N} |g'_w(z)| \) is bounded on \( D(0,R) \), there exists \( \delta > 0 \) such that
\[
(2.1) \sup_{|w|=N} \sup_{z \in J} |g'_w(z)| = \gamma < 1.
\]

Here \( J(\delta) = J + \overline{D(0,\delta)} \) is the closed \( \delta \) neighborhood of \( J \) on the complex plane. We denote \( J(\delta) \cap \mathbb{R} \) by \( I \). By choosing \( \delta \) sufficiently small, we can assure that \( J(\delta) \subset D(0,R) \setminus \{0\} \), and \( I \) is a finite union of disjoint closed intervals. The latter can be found in [SZ04, Proposition 3]. We write \( I = \bigcup_{\alpha=1}^{K} I^{\alpha} \), where \( I^{\alpha} \) are closed intervals.

Because \( g_w \) with \( |w| = N \) is strictly contracting on \( J(\delta) \) and \( g_w(J) \subset J \), we conclude that when \( |w| = N \), \( g_w : J(\delta) \rightarrow J(\delta) \) and in particular
\[
(2.2) \quad g_w : I \rightarrow I, \quad \text{if } |w| = N.
\]

Note that since \( \sup_{z \in D(0,R)} |g'_w(z)| \lesssim 1 \), we actually have
\[
|g'_w(z)| \lesssim |w|/N \text{ for every } z \in J(\delta).
\]

Consequently if \( n \) is large enough, we always have
\[
(2.3) \quad g_w : J(\delta) \rightarrow J(\delta), \quad g_w(I) \subset I, \quad \text{if } |w| = n.
\]

We will always assume that \( |w| \) is large enough such that the above mapping invariance holds.

Denote \( g_w(I^{\alpha}) \) by \( I^{\alpha}_w \), and let \( I_n = \bigcup_{|w|=n,\alpha} I^{\alpha}_w = \bigcup_{|w|=n} g_w(I) \). Then we have the following proposition.

**Proposition 2.1.** If \( n \) is large enough (larger than some admissible constant) then \( \{I^{\alpha}_w : \alpha, |w| = n\} \) is a collection of disjoint closed intervals contained in \( I \).

**Proof.** Since \( I^{\alpha} \) are closed intervals and \( g_w \) are continuous, \( I^{\alpha}_w \) are closed intervals. (2.3) implies that \( I^{\alpha}_w \subset I \) if \( |w| \) is large enough. To show they are disjoint, we suppose \( g_w(z_1) = g_w(z_2) \) for some \( z_1 \in I^{\alpha_1}, z_2 \in I^{\alpha_2} \) and \( |w| = |w'| = n \). If \( w = w' \), then since \( g_+, g_- \) are injective on \( I \), we conclude that \( z_1 = z_2 \). If \( w \neq w' \), we let \( j \) be the largest integer such that the \( j \)th letter of \( w \) differs from that of \( w' \). As before by injectivity of \( g_\pm \) on \( I \), we conclude \( g_{w,z_1}(z_1) = g_{w',z_2}(z_2) \). However, this cannot happen because \( g_+(I) \) and \( g_-(I) \) are disjoint. \( \square \)

We remark the following distortion theorem, which can be found in for instance [CG13].

**Theorem 2.2.** Suppose \( F \) is holomorphic and injective on the disc \( D(0,\delta_0) \) for some \( \delta_0 > 0 \), and \( |F'(0)| = M \). Then
\[
M \frac{1 - |z|/\delta_0}{(1 + |z|/\delta_0)^2} \leq |F'(z)| \leq M \frac{1 + |z|/\delta_0}{(1 - |z|/\delta_0)^2}.
\]

Note that there exists \( R' < R \) such that \( J(\delta) \subset D(0,R') \subset D(0,R) \). Since \( g_+, g_- \) are injective on \( D \), \( g_w \) is injective on \( D \) for every word \( w \). Therefore we obtain the following corollary.
Corollary 2.3. There exists an admissible constant $C > 0$ such that for every word $w$,

$$C^{-1}|g'_w(z_1)| \leq |g'_w(z_2)| \leq C|g'_w(z_1)|$$

for every $z_1, z_2 \in D(0, R')$.

Proof. By Theorem 2.2 there exists an admissible constant $C_1 > 0$ such that for every word $w$,

$$C_1^{-1}|g'_w(z)| \leq |g'_w(0)| \leq C_1|g'_w(z)|$$

for every $z \in D(0, R')$. Taking $C = C_1^2$ completes the proof. □

We let $\gamma_w = \sup_{x \in J(\delta)} |g'_w(z)|$. Then Corollary 2.3 implies

$$g'_w(z) \sim \gamma_w$$

for every $z \in J(\delta)$.

Also, (2.3) implies

$$\gamma_w \lesssim w/|w|.$$

Choose $R'' < R'$ such that $J(\delta) \subset D(0, R'')$. The above corollary combined with the Cauchy integral formula implies the following.

**Proposition 2.4.** For every integer $k \geq 2$, every $w$, and every $z \in D(0, R'')$, we have

$$|g^{(k)}_w(z)| \lesssim_k |g'_w(z)|.$$

Proof. Choose $R''' \in [R'', R']$. By the Cauchy integral formula we have, for every $z \in D(0, R''')$

$$g^{(k)}_w(z) = \frac{(k-1)!}{2\pi i} \int_{\partial D(0, R''')} \frac{g_w(x)}{(x-z)^{k}} \, dx.$$

Since $d(z, \partial D(0, R'''')) \geq R'''' - R''' \geq 1$, we have

$$|g^{(k)}_w(z)| \lesssim_k \sup_{z \in \partial D(0, R''')} |g'_w(x)| \sim |g'_w(z)|,$$

where the last step is due to Corollary 2.3. □

Besides the above observations, in this section we are mainly interested in the phase function, defined for every $w \in W$,

$$\phi_w(z) = \log |g'_w(z)| \quad \text{for } z \in J(\delta).$$

Let $w, v$ be two words and let $m$ be the largest integer such that $v_{<m} = w_{<m}$.\footnote{We use the convention that if $w_{\leq 1} \neq v_{\leq 1}$, then $m = 1$. If $|w| > |v|$ and $w_{\leq |v|} = v$, then we let $m = |v| + 1$.} If $m \leq \min\{|w|, |v|\} - 1$, then we write $w \sim v$. Obviously $w \sim v$ if and only if $v \sim w$. The following proposition establishes separation of the phase functions $\partial_x \phi_w(x)$.

**Proposition 2.5.** Suppose $w \sim v$. If $c < -2 - \sqrt{5}$ then by choosing $\delta$ small enough depending only on $c$, we have for every $x \in I$

$$|\partial_x \phi_w(x) - \partial_x \phi_v(x)| \gtrsim |g^{(m+1)}_w(x)|,$$

Here $m$ is the largest integer such that $v_{<m} = w_{<m}$.\footnote{We use the convention that if $w_{\leq 1} \neq v_{\leq 1}$, then $m = 1$. If $|w| > |v|$ and $w_{\leq |v|} = v$, then we let $m = |v| + 1$.}
Proof. We let \( n = |w| \) and \( k = |v| \). By the inverse function theorem we can compute

\[
g'_w(x) = \frac{1}{2g_{w}(x)}, \quad g''_w(x) = \frac{-1}{4(g_w(x))^3}.
\]

If we let \( x_j = g_{w,j}(x) \), then by the chain rule we have

\[
g'_w(x) = g'_{w_n}(x_{n-1}) \cdots g'_{w_1}(x_0)
\]

where \( x_0 := x \). So by the definition of the phase function (2.4) we have

\[
\partial_x \phi_w(x) = \partial_x (\log |g'_{w_n}(x_{n-1})|) + \cdots + \partial_x (\log |g'_{w_1}(x_0)|) = \frac{g''_{w_n}(x_{n-1})x'_{n-1}(x) + \cdots + g''_{w_1}(x_0)}{g_{w_n}(x_{n-1})}
\]

Since \( x_j' = g'_{w_j}(x_{j-1}) \cdots g'_{w_1}(x_0) \), we conclude the formula

\[
(2.5) \quad \partial_x \phi_w(x) = \frac{g''_{w_n}(x_{n-1})g'_{w_{n-1}}(x_{n-2}) \cdots g'_{w_1}(x_0)}{g_{w_n}(x_{n-1})} + \cdots + \frac{g''_{w_1}(x_0)}{g_{w_1}(x_0)}
\]

where by definition \( s_i = -1/(2^ix_1 \cdots x_i^2) \). Similarly we can write

\[
\partial_x \phi_v(x) = -\frac{1}{2^ky_1 \cdots y_{k-1}y_k} - \cdots - \frac{1}{2y_1^2}
\]

where \( y_j = g'_{v_j}(x) \) and \( t_i = -1/(2^iy_1 \cdots y_i^2) \).

Since \( m \) is the largest integer such that \( w_{<m} = v_{<m} \), we have \( x_i = y_i \) for \( 1 \leq i < m \), and \( x_m = -y_m \). Therefore \( s_i = t_i \) for \( 1 \leq i < m \), and \( s_{m+1}, t_{m+1} \) differ in sign. We write

\[
(2.6) \quad \partial_x \phi_w(x) - \partial_x \phi_v(x) = (s_{m+1} + \cdots + s_n) - (t_{m+1} + \cdots + t_k)
\]

\[
= (s_{m+1} - t_{m+1}) + (s_{m+2} + \cdots + s_n) - (t_{m+2} + \cdots + t_k).
\]

We claim that if \( \delta \) is chosen small enough depending on \( c \), then

\[
|s_{m+2} + \cdots + s_n| < |s_{m+1}|, \quad |t_{m+2} + \cdots + t_k| < \beta|t_{m+1}|,
\]

where \( \beta \in (0,1) \) is some admissible constant. We group consecutive terms with alternating signs in the sequence \( s_{m+2}, \ldots, s_n \), that is,

\[
s_{m+2} + \cdots + s_n = s'_1 + \cdots + s'_n,
\]

where \( s'_j = s_{i_j} + \cdots + s_{i_j+1-1} \) is a sum of consecutive terms, \( s_{i_j+l} \) and \( s_{i_j+l+1} \) differ in sign for \( 0 \leq l \leq i_{j+1} - i_j - 2 \), and \( s_{i_j-1} \) and \( s_{i_j} \) have the same sign.

Note that

\[
\frac{s_{i+1}}{s_i} = \frac{2^i x_1 \cdots x_i^2}{2^{i+1} x_1 \cdots x_i x_{i+1}^2} = \frac{x_i}{2(x_i - c)}.
\]

Our assumption \( c < -2 - \sqrt{5} \) in particular implies that \( c < -3.75 \). We observe that when \( c < -3.75 \)

\[
(2.7) \quad \sup_{x \in I_s \cup I_-} \left| \frac{x}{2(x - c)} \right| < \theta < 1.
\]
Here $\theta$ can be any number larger than $\frac{\zeta}{2(\zeta_c - c)}$ and less than 1. Recalling that $\mathcal{J} \subset I_+ \cup I_-$ and $I = \mathcal{J}(\delta) \cap I$, we therefore conclude that if we choose $\delta$ sufficiently small depending on $c$, then we have

$$\sup_{x \in I} \left| \frac{x}{2(x - c)} \right| \leq \theta < 1.$$ 

Therefore $|s_{i+1}/s_i| \leq \theta < 1$. Since $s_j'$ has the same sign as $s_j$, and $|s_j'| \leq |s_{ij}|$.

We also observe that as $I_+ = [\sqrt{-\zeta_c - c}, \zeta_c]$, the following inequality holds

$$\sup_{x \in I_+} \left| \frac{x}{2(x - c)} \right| \leq \left| \frac{\zeta_c}{2(\zeta_c - c)} \right| < \eta < 1/4.$$ 

Here $\eta$ can be any number larger than $\frac{\zeta_c}{2(\zeta_c - c)}$ and less than 1/4. By definition $s_{i-1}$ and $s_{ij}$ have the same sign, which implies that $x_{i-1} > 0$ and therefore $x_{i-1} \in I_+$. Again if $\delta$ is sufficiently small depending on $c$ then we have

$$\frac{s_{ij}}{s_{i-1}} \leq \eta.$$ 

Hence combining (2.8) and (2.9) and using the triangle inequality we conclude that

$$|s_{m+2} + \cdots + s_n| \leq \sum_j |s_j'| \leq \sum_{j=0}^{\infty} \eta^j |s'_1| \leq \frac{1}{1 - \eta} \theta|s_{m+1}|.$$ 

Under our assumption that $c < -2 - \sqrt{5}$, we can choose $\eta, \theta$ depending on $c$ and then $\delta$ sufficiently small depending on $\eta, \theta, c$ such that

$$\frac{1}{1 - \eta} \theta < 1.$$ 

In fact we only need to guarantee that

$$\frac{\zeta_c}{2(\zeta_c - c)} \frac{1}{1 - \zeta_c/(2(\zeta_c - c))} < 1,$$

which under our general assumption $c < -2$ is equivalent to $c < -2 - \sqrt{5}$. Therefore there exists an admissible constant $\beta \in (0, 1)$ such that

$$|s_{m+2} + \cdots + s_n| < \beta|s_{m+1}|.$$ 

Similarly we have

$$|t_{m+2} + \cdots + t_k| < \beta|t_{m+1}|.$$ 

Therefore the above two estimates, (2.6) together with the triangle inequality imply that

$$|\partial_x \phi_w(x) - \partial_x \phi_v(x)| \geq (1 - \beta)(|s_{m+1}| + |t_{m+1}|).$$ 

Recalling that

$$s_{m+1} = -1/((2^{m+1} x_1 \cdots x_m x_{m+1}^2) = -\frac{g'_{w \leq m+1}(x)}{2x_{m+1}},$$

we have $|s_{m+1}| \geq |g'_{w \leq m+1}(x)|$. Similarly we have $|t_{m+1}| \geq |g'_{v \leq m+1}(x)|$. We therefore conclude that

$$|\partial_x \phi_w(x) - \partial_x \phi_v(x)| \geq |g'_{w \leq m+1}(x)| + |g'_{v \leq m+1}(x)| \geq |g'_{w \leq m+1}(x)|.$$

\qed
We remark that by definition of \( m \), \(|g'_{w \leq m+1}(x)| \sim |g'_{v \leq m+1}(x)|\) for \( x \in I \).

In fact we can do better with a little more effort.

**Proposition 2.6.** Suppose \( w \sim v \). If \( c < -3.75 \) then by choosing \( \delta \) small enough depending only on \( c \), we have for every \( x \in I \)

\[
|\partial_x \phi_w(x) - \partial_x \phi_v(x)| \geq |g'_{w \leq m+1}(x)|.
\]

Here \( m \) is the largest integer such that \( v_{<m} = w_{<m} \).

**Proof.** Still we let \( n = |w|, \ k = |v| \) and let \( m \) be the largest integer such that \( w_{<m} = v_{<m} \). Under the assumption \( w \sim v \) we have \( m \leq \min\{n, k\} - 1 \).

If we examine the proof of Proposition 2.5, we see that under the assumption \( c < -3.75 \), the range of \( c \) should be determined by the case

\[
x_{m+1} \in I_-, \quad x_{m+i} \in I_+ \quad \text{for every } i \geq 2, \quad y_{m+1} \in I_-, \quad y_{m+i} \in I_+ \quad \text{for every } i \geq 2.
\]

To be more precise, for a fixed \( x \in I \), if the word \( w \) makes this case happen, the ratio between the alternating sum \( \sum_j s'_j \) and \( s_{m+1} \) is the smallest in absolute value among all possible words \( w \) with the same length, and this smallest ratio decreases as the length of the words increases when \( m \) is fixed.

To check the claim in the previous paragraph, we fix \( x \in I \) and \( m \). Let \( i_0 \) denote the first positive integer such that \( x_{m+i_0} \in I_- \). If such an \( i_0 \) does not exist, then \( x_{m+i} \in I_+ \) for every \( i \geq 1 \), which implies \( \sum_j s'_j \) has the same sign as \( s_{m+1} \), and hence the ratio \( \langle \sum_j s'_j \rangle / s_{m+1} \geq 0 \). Now suppose such an \( i_0 \) does exist. We first observe that

\[
\inf_{x \in I_+} \left| \frac{x}{2(x-c)} \right| > \frac{1}{2} > \sup_{x \in I_-} \left| \frac{x}{2(x-c)} \right|.
\]

Let \( \tilde{w} \) denote the word with the same length as \( w \), with \( \tilde{w}_{m+1} = - \), \( \tilde{w}_{m+i} = + \) for \( i \geq 2 \). We write \( \tilde{x}_i \), \( \tilde{s}_{m+i} \) and \( \tilde{s}'_j \) for the corresponding expression defined as \( x_i \), \( s_{m+i} \) and \( s'_j \) for \( w \). If \( i_0 = 1 \), then (2.11) implies that \( \tilde{s}_m = s_{m+1} \), \( |\tilde{s}_{m+i}| \geq |s_{m+i}| \) for every \( i \geq 2 \), and therefore \( \langle \sum_j \tilde{s}'_j \rangle / \tilde{s}_{m+i_0} \leq \langle \sum_j s'_j \rangle / s_{m+i_0} \). If \( i_0 > 1 \), then \( s_{m+i_0} \) has the same sign as \( s_{m+i_1} \), and \( s'_j / s_{m+i_0} \geq 0 \), and therefore \( \langle \sum_{j>1} s'_j \rangle / s_{m+i_0} \geq \langle \sum_{j>1} s'_j \rangle / s_{m+i_0} \). The previous analysis implies for the ratio \( \langle \sum_{j>1} s'_j \rangle / s_{m+i_0} \) to be the smallest among all possible words with the same length and the first \( m + i_0 - 1 \) letters, we must have \( w_{m+i_0+1} = + \) for every \( i \geq 1 \). The argument below will show that then under the condition \( c < -3.75 \), \( s_{m+i_0} + \sum_{j>1} s'_j \) has the same sign as \( s_{m+i_0} \). Since \( s_{m+i_0} \) has the same sign as \( s_{m+1} \), we have \( \langle \sum_j s'_j \rangle / s_{m+1} \geq \langle \sum_{j>1} s'_j \rangle / s_{m+1} \geq 0 \). On the other hand, \( \langle \sum_j s'_j \rangle / s_{m+i_0} \leq 0 \).

This completes the proof of the claim in the previous paragraph.

Now we return to the proof of the proposition. By our assumption \( x_m \) and \( y_m \) differ in sign. Without loss of generality we may assume that \( x_m \in I_- \) and \( y_m \in I_+ \). As we have more information about where \( x_m, x_{m+1}, x_{m+2} \) lie, when estimating \( |s_{m+2} / s_{m+1}|, |s_{m+3} / s_{m+2}|, |t_{m+2} / t_{m+1}|, |t_{m+3} / t_{m+2}| \), we can do better than simply bounding them by \( \eta \) or \( \theta \). In fact, as

\[
g_+(I_-) = \left[ \sqrt{-\zeta_c - c}, g_+(-\sqrt{-\zeta_c - c}) \right], \quad g_+(I_-) = \left[ g_+(-\sqrt{-\zeta_c - c}), -\sqrt{-\zeta_c - c} \right],
\]

we have

\[
\frac{s_{m+2}}{s_{m+1}} \leq \frac{-g_+(-\sqrt{-\zeta_c - c})}{2g_+(-\sqrt{-\zeta_c - c} - c)} + o(1),
\]

\[
\frac{s_{m+3}}{s_{m+2}} \leq \frac{g_+(-\sqrt{-\zeta_c - c})}{2g_+(\sqrt{-\zeta_c - c} - c)} + o(1).
\]
Therefore $c (2.12)$ a term which goes to 0 as $\delta \to 0$. Observe that
\[ g_+ (g_- (I_+)) = \sqrt{-\zeta_c - c}, g_+ (g_- (-\sqrt{-\zeta_c - c})). \]
So we obtain
\[ \frac{|t_{m+3}|}{t_{m+2}} \leq \frac{g_+ (g_- \sqrt{-\zeta_c - c})}{2 (g_+ (g_- \sqrt{-\zeta_c - c}) - c)} + o(1). \]
We let $\theta_0 = \frac{\zeta_c}{2 (-\zeta_c - c)}$ and $\eta_0 = \frac{\zeta_c}{2 (-\zeta_c - c)}$. Since $s_m = t_m$, we also have
\[ \frac{\sqrt{-\zeta_c - c}}{\eta_0} + o(1). \]
Note that in fact
\[ |g_{w_{\leq m+1}} (x)| \sim |g_{v_{\leq m+1}} (x)| \sim \text{max} \{|g_{w_{\leq m+1}} (x)|, |g_{v_{\leq m+1}} (x)|\} \]
as $g_{w_{\leq m}} (x) = g_{v_{\leq m}} (x)$. So if we have
\[ |s_{m+1}| - |s_{m+2} + \cdots | + |t_{m+1}| - |t_{m+2} + \cdots | \geq \beta \min \{|s_{m+1}|, |t_{m+1}|\} \]
for some admissible constant $\beta > 0$, then $|\partial_x \phi_w(x) - \partial_x \phi_v(x)| \gtrsim |g_{w_{\leq m+1}} (x)|$. In our case $|t_{m+1}| \leq |s_{m+1}|$ (when $\delta$ is sufficiently small depending on $c$) since when $c < -3.75$, $\frac{\sqrt{-\zeta_c - c}}{\eta_0} > 1$. Therefore $c$ can be any real number less than $-3.75$ such that
\[ (2.12) \quad \left( \theta_0 \left( 1 + \frac{g_+ (g_- (-\lambda))}{2 (g_+ (g_- (-\lambda)) - c)} \frac{1}{1 - \eta_0} \right) - 1 \right) \]
\[ + \frac{2 (\lambda - c)}{\eta_0} \left( \frac{-g_- (\lambda)}{2 (g_- (\lambda) - c)} \left( 1 + \frac{g_+ (\lambda)}{2 (g_+ (\lambda) - c)} \frac{1}{1 - \eta_0} \right) - 1 \right) < 0, \]
where $\lambda = -\sqrt{-\zeta_c - c}$. We can check that (2.12) always holds when $c < -3.75$. \qed

We also have the following observation on the separation of $\phi''_w$.

**Proposition 2.7.** Suppose $w \sim v, c < -2$. By choosing $\delta$ small enough depending on $c$, we have for every $x \in \mathcal{F} (\delta)$
\[ |\phi''_w (x) - \phi''_v (x)| \lesssim |g_{w_{\leq m+1}} (x)|. \]
Here $m$ is the largest integer such that $v_{\leq m} = w_{\leq m}$.

**Proof.** We recall that if $|w| = n_1, |v| = n_2$, then
\[ \phi'_w (x) = - \left( \frac{g'_{w_1} (x)}{y_{n_2}} + \cdots + \frac{g'_{w_{\leq 1}} (x)}{y_1} \right). \]
Differentiating both sides of the above equation we obtain
\[ (2.13) \quad \phi''_w (x) = - \left( \frac{g''_{w_1} (x)}{y_{n_2}} + \frac{-g'_w (x) g'_{w_1} (x)}{y_{n_2}} + \cdots + \frac{g''_{w_{\leq 1}} (x) g'_{w_{\leq 1}} (x)}{y_1} + \frac{-g'_w (x) g'_{w_{\leq 1}} (x) g''_{w_{\leq 1}} (x)}{y_1^2} \right). \]
Since $w_{\leq m} = v_{\leq m}$, we have
\begin{equation}
(2.14) \quad \phi''_w(x) - \phi''_v(x) = \left( \frac{g''_w(x)}{y_{n_2}} - \frac{g''_v(x)g'_{y_{n_2}}(x)}{y_{n_2}^2} + \cdots + \frac{g''_{v_{\leq m+1}}(x)}{y_{m+1}} + \frac{-g''_{v_{\leq m+1}}(x)g'_{v_{\leq m+1}}(x)}{y_{m+1}^2} \right) - \\
\left( \frac{g''_w(x)}{x_{n_1}} - \frac{g''_v(x)g'_{x_{n_1}}(x)}{x_{n_1}^2} + \cdots + \frac{g''_{w_{\leq m+1}}(x)}{x_{m+1}} + \frac{-g''_{w_{\leq m+1}}(x)g'_{w_{\leq m+1}}(x)}{x_{m+1}^2} \right).
\end{equation}

We have the identity, for a general word $w$ and $z \in \mathcal{J}(\delta)$,
\begin{equation}
(2.15) \quad g''_w(z) = g'_w(z) \left( -\frac{g'_w(z)}{z_{n_2}} + \cdots + \frac{-g'_{w_{\leq 1}}(z)}{z_1} \right)
\end{equation}
where $z_j := g_{w_{\leq j}}(z)$. This is because

\[
g''_w(z) = \left( \frac{1}{(2z_n) \cdots (2z_1)} \right)' = -\frac{2g''_w(z)}{(2z_n)^2(2z_{n-1}) \cdots (2z_1)} + \cdots + \frac{-2g'_{w_{\leq 1}}(z)}{(2z_n) \cdots (2z_2)(2z_1)^2} = g'_w(z) \left( \frac{-g'_w(z)}{z_n} - \cdots - \frac{g'_{w_{\leq 1}}(z)}{z_1} \right).
\]

Noting that $\gamma_w \leq \gamma_{|w|/N}$, $\sum_{j=1}^{\infty} \gamma_j^j/N \lesssim 1$, and $\mathcal{J}(\delta)$ is away from 0, we conclude by substituting (2.15) into (2.14) that
\[
|\phi''_w(x) - \phi''_v(x)| \lesssim |g'_{w_{\leq m+1}}(x)| + |g'_{v_{\leq m+1}}(x)| \lesssim |g'_{w_{\leq m+1}}(x)|.
\]
The last inequality is due to $|g'_{w_{\leq m+1}}(x)| \sim |g'_{v_{\leq m+1}}(x)|$ for every $x \in I$. \hfill $\square$

As a corollary of the above proposition and the Cauchy integral formula, we have the following corollary.

**Corollary 2.8.** Suppose $w \sim v$, $c < -2$, and $k \geq 2$. Then by choosing $\delta$ small enough depending on $c$, we have for every compact set $\bar{I} \subset \mathbb{R}$ contained in the interior of $I$,
\[
|\phi^{(k)}_w(x) - \phi^{(k)}_v(x)| \lesssim_{k, \bar{I}} |g'_{w_{\leq m+1}}(x)|
\]
for every $x \in \bar{I}$. Here $m$ is the largest integer such that $v_{\leq m} = w_{\leq m}$.

**Proof.** Fix a compact $\bar{I}$ contained in the interior of $I$. There exists $\delta_0 > 0$ depending on $\bar{I}$ such that the closure of the $\delta_0$-neighborhood of $\bar{I}$ is contained in $\mathcal{J}(\delta)$. By the Cauchy integral formula we have for $x \in \bar{I}$,
\[
\phi^{(k)}_w(x) - \phi^{(k)}_v(x) = \frac{(k-2)!}{2\pi i} \int_{\partial D(x, \delta_0)} \frac{\phi''_w(z) - \phi''_v(z)}{(z-x)^{k-1}} \, dz.
\]
Therefore, by Proposition 2.7 and Corollary 2.3 we conclude
\[
|\phi^{(k)}_w(x) - \phi^{(k)}_v(x)| \lesssim_{k, \bar{I}} |g'_{w_{\leq m+1}}(x)|.
\]
\hfill $\square$

From now on we will always assume that $\delta$ is sufficiently small (depending only on $c$) such that Proposition 2.6, 2.7 and Corollary 2.8 hold.
3. Refined Transfer Operator

Our strategy of the proof will be deriving a contradiction from the identity \( L_s u = u \). A naive attempt will be applying iterates of \( L_s \) and therefore obtaining \( L^n_s u = u \), which we hope would necessarily fail if \( n \) is large enough. This method indeed works if our dynamical system is simple enough. However, to tackle our problem here we need to introduce a refined transfer operator \( L_{Z,s} \), which has the property that \( L_s u = u \Rightarrow L_{Z,s} u = u \) so that in some sense it is a generalization of simple iterates \( L^n_s \).

Definition 3.1. We say \( Z \subset W \) is a partition if \( Z \) is finite and there exists \( M > 0 \) such that for every word \( w \) with length greater than \( M \), there exists a unique \( v \in Z \) such that \( w_{\leq m} = v \) for some \( m \geq 1 \).

Remark. We observe that the set

\[
Z(\tau) := \{ w \in W : |I_w| < \tau, \quad |I_{w \circ j}| \geq \tau \text{ for every } 1 \leq j \leq |w| - 1 \}
\]

is a partition for every \( \tau > 0 \). The reason why \( Z(\tau) \) is a partition is the fact that \( g_w \) is eventually contracting in the sense that \( \sup_{z \in J(\delta)} |g_w(z)| \leq \gamma^{|w|/N} \).

We let \( L_{Z,s} \) be the operator

\[
L_{Z,s} u(z) = \sum_{w \in Z} [g'_w(z)]^s u(g_w(z)).
\]

Proposition 3.1. Suppose \( Z \) is a partition. If \( L_s u = u \), then

\[
L_{Z,s} u = u.
\]

Proof. We prove by induction on \( K := \sum_{w \in Z} |w| \).

If \( K = 2 \), then \( Z \) is a partition if and only if \( Z = \{ +, - \} \) and therefore \( L_{Z,s} u = L_s u = u \).

Suppose our proposition holds for \( K \leq K_1 \) where \( K_1 \geq 2 \). Let \( Z \) be a partition with \( \sum_{w \in Z} |w| = K_1 + 1 \) and let \( v = v_1 \cdots v_{n+1} \) be a longest word in \( Z \). Since \( Z \) is a partition and \( v \) is one of the longest words in \( Z \), if we let \( v' \) and \( v'' \) be words \( v_1 \cdots v_k \) and \( v_{k+1} \cdots v_{n+1} \) respectively, then \( v', v'' \in Z \). Define \( Z' \) to be the set \( Z \setminus \{ v', v'' \} \cup v_{\leq n} \). Then \( Z' \) is a partition with \( \sum_{w \in Z'} |w| \leq K \). So by induction hypothesis,

\[
L_s u \Rightarrow L_{Z',s} u = u.
\]

Observe that

\[
L_{Z,s} u(z) - L_{Z',s} u(z) = (g'_v(z))^s u(g_v(z)) + (g'_v(z))^s u(g_v(z)) - (g'_{v_{\leq n}}(z))^s u(g_{v_{\leq n}}(z)),
\]

but the right hand side of the above equality is zero by the chain rule if \( L_s u = u \) which in particular implies that \( L_s u(g_{v_{\leq n}}(z)) = u(g_{v_{\leq n}}(z)) \). \( \square \)

Due to our distortion estimate Corollary 2.3 we know that \( |g'_w(z)| \) are comparable on the region \( J(\delta) \). We therefore obtain the following estimate on \( |g'_w| \) for \( w \in Z(\tau) \).

Lemma 3.2. Suppose \( w \in Z(\tau) \). Then

\[
|g'_w(z)| \sim \tau
\]

for every \( z \in J(\delta) \). In particular we have \( \gamma_w \sim \tau \).

\( ^2 \)Our definition of \( L_{Z,s} \) differs from that in [DZ20] since in [DZ20], \( g_w \) would be \( g_{w_1} \circ \cdots \circ g_{w_n} \) instead of \( g_{w_n} \circ \cdots \circ g_{w_1} \) as in (1.2).
Proof. Recall that $I_w^a = g_w(I^a)$ and for fixed $w$, $I_w^a$ are disjoint closed intervals. Since $|g'_w(x)| \sim 1$ when $w \in \{+, -\}$, $x \in I$, the definition of $Z(\tau)$ and Lagrange’s mean value theorem imply that for every $w \in Z(\tau)$, we have $|I_w| \sim \tau$.

Similarly due to Lagrange’s mean value theorem, the above observation implies that there exists $x \in I$ such that $|g'_w(x)| \sim \tau$. Hence Corollary 2.3 shows that $|g'_w(z)| \sim \tau$ for every $z \in \mathcal{J}(\delta)$. □

We also have the following “almost orthogonality” property.

Lemma 3.3. For every $w \in Z(\tau)$, we have
\begin{equation}
|\{v \in Z(\tau) : |v| \geq |w| \text{ and } w \sim v\}| \lesssim 1.
\end{equation}
Here $|A|$ denotes the cardinality of the set $A$, and $w \sim v$ means the negation of $w \sim v$.

Proof. Suppose $w, v \in Z$, $|v| \geq |w|$, and $w \sim v$. We let $|w| = n$. Then we must have $v_{\leq n-1} = w_{\leq n-1}$.

Due to Lemma 3.2, $|g'_w(z)| \sim \tau$ for every $z \in \mathcal{J}(\delta)$. Therefore $|g_{v_{\leq n-1}}(z)| = |g_{w_{\leq n-1}}(z)| \sim \tau$ for every $z \in \mathcal{J}(\delta)$. Again due to Lemma 3.2 we know that $|g'_w(z)| \sim \tau$ for every $z \in \mathcal{J}(\delta)$. Since $g_w$ is eventually contracting on $\mathcal{J}(\delta)$, we conclude that $|v| - n \lesssim 1$. Hence
\begin{equation}
|\{v \in Z(\tau) : |v| \geq |w| \text{ and } w \sim v\}| \lesssim 1.
\end{equation}
□

We remark that Lemma 3.3 also shows that when $\tau$ is sufficiently small, for every $z \in I$,
\begin{equation}
|\{w \in Z(\tau) : z \in g_w(I)\}| \lesssim 1,
\end{equation}

since by the proof of Proposition 2.1, $z \in g_w(I)$ and $z \in g_v(I)$ imply $w \sim v$.

For a word $w$ with length $n > N$, we let $\tilde{w}$ be the word $w_{\leq n-N}$, which is obtained by removing the last $N$ letters of $w$. Since for every $w \in Z(\tau)$, $|\{v \in Z(\tau) : z = \tilde{w}\}| \lesssim 1$, we have for every $z \in I$,
\begin{equation}
|\{w \in Z(\tau) : z \in g_w(I)\}| \lesssim 1.
\end{equation}

4. A Priori Bounds

In this section we fix $\epsilon > 0$ and suppose $\text{Re}(s) > 1/2 + \epsilon$ and $\mathcal{L}_s u = u$ for some $u \in \mathcal{H}$. We will establish some a priori bounds on the function $u$. Let $h = 1/|\text{Im}(s)|$. Recall that $N$ is defined by (1.1) and depends only on $c$.

Proposition 4.1. We have
\begin{enumerate}
\item[(a)] $\sup_{x \in I_N} |u(x)| \lesssim h^{-1/2} \|u\|_{L^2(I)}$;
\item[(b)] $\|u^{(k)}\|_{L^2(I_N)} \lesssim_k h^{-k} \|u\|_{L^2(I)}$, for every $k \in \mathbb{N}$.
\end{enumerate}

To prove the above two bounds, we will follow the argument in [DZ20]. We let $D_N$ be the set $\bigcup_{|w|=N} g_w(\mathcal{J}(\delta))$. Recall that $I_N = \bigcup_{|w|=N} g_w(I) = D_N \cap \mathbb{R}$. Recalling that $g_w$ with $|w| = N$ are strictly contracting on $\mathcal{J}(\delta)$, and map $\mathcal{J}$ to $\mathcal{J}$, we conclude that
\begin{equation}
d(D_N, \partial \mathcal{J}(\delta)) \geq (1 - \gamma)\delta.
\end{equation}
Here $d(A, B)$ is the Euclidean distance between two subsets $A, B$ of $\mathbb{R}^n$.

Without loss of generality we may assume that $h > 0$ from now on. We introduce the following weight function
\begin{equation}
w_K : \mathbb{C} \to (0, \infty), \quad z \mapsto e^{-K|\text{Im}(z)|}/h.
\end{equation}
By Lemma 2.3 in [DZ20], there exists $c \in (0, 1]$ such that

\[(4.2) \quad \sup_{z \in D_N} |w_K(z)u(z)| \leq \left( \sup_{z \in I} |w_K(z)u(z)| \right)^c \left( \sup_{z \in \mathcal{J}(\delta)} |w_K(z)u(z)| \right)^{1-c}.
\]

Note that $w_K(z)$ equals to 1 when $z$ is real.

Since $\mathcal{L}_x^N u = u$, for every $z \in \mathcal{J}(\delta)$ we have

\[
|w_K(z)u(z)| = \left| w_K(z) \sum_{|w|=N} g'_w(z)^s u(g_w(z)) \right|
\leq \gamma \Re(s) \sum_{|w|=N} w_K(z)e^{-\operatorname{Arg}(g'_w(z))/h}|u(g_w(z))|
\leq \left( \sum_{|w|=N} \frac{w_K(z)}{w_K(g_w(z))} e^{-\operatorname{Arg}(g'_w(z))/h} \right) \sup_{z \in D_N} |w_K(z)u(z)|
\leq e^{-K|\operatorname{Im}(z)|(1-\gamma) - \operatorname{arg}(g'_w(z))} \sup_{z \in D_N} |w_K(z)u(z)|.
\]

Observe that since $\operatorname{Arg}(g'_w(z)) = 0$ when $\operatorname{Im}(z) = 0$, we can choose an admissible $K > 10$ sufficiently large such that

\[
\sup_{z \in \mathcal{J}(\delta)} (-K|\operatorname{Im}(z)|(1-\gamma) - \operatorname{arg}(g'_w(z))) \leq 0,
\]

which therefore implies that

\[
\sup_{z \in \mathcal{J}(\delta)} |w_K(z)u(z)| \leq \sup_{z \in D_N} |w_K(z)u(z)|.
\]

Combining the above estimate with (4.2) we therefore conclude

\[(4.3) \quad \sup_{z \in \mathcal{J}(\delta)} |w_K(z)u(z)| \leq \sup_{z \in I} |w_K(z)u(z)| = \sup_{z \in I} |u(z)|.
\]

We let $\chi : \mathbb{R} \to [0, 1]$ be a smooth cutoff function satisfying $\chi = 1$ on a neighborhood of $I_N$ and $\chi$ is supported on $I$. This is possible because of (4.1), and we can further assume that

\[
\sup_{x \in \mathbb{R}} |\partial_x^j \chi(x)| \leq \delta^j (1-\gamma)^j.
\]

We let $\tilde{\chi} \in C^\infty_c(\mathcal{J}(\delta))$ be an almost analytic continuation of $\chi$ to the complex plane satisfying

\[
\tilde{\chi} = \chi \text{ on } \mathbb{R}, \quad \sup_{z \in \mathbb{C}} |\partial_{\overline{\tau}} \tilde{\chi}(z)| \leq j |\operatorname{Im}(z)|^j \text{ for every } j \geq 0.
\]

A construction of such an almost analytic continuation can be found in [DS99, Chapter 8].

Let $u_N(x) = \chi(x)u(x)$. We claim that for every $j > 0$,

\[(4.4) \quad |\mathcal{F}_h(u_N)(\xi)| \leq j h^{-j} \sup_{I} |u| \text{ when } |\xi| \geq 2K.
\]

By definition

\[
\mathcal{F}_h(u_N)(\xi) = \int_{\mathbb{R}} e^{-ix\xi/h} u(x)\tilde{\chi}(x)dx.
\]

We assume that $\xi \geq 2K$. The case $\xi \leq -2K$ can be similarly treated. By Green’s formula we have

\[
\mathcal{F}_h(u_N)(\xi) = -\int_{\mathcal{J}(\delta) \cap \{\operatorname{Im}(z) \leq 0\}} u(z)e^{-iz\xi/h} \partial_z \tilde{\chi} d\tau dz.
\]
Therefore by (4.3), for every $j$ we have

$$|\mathcal{F}_h(u_N)(\xi)| \lesssim_j \sup_{z \in I} |u(z)| \int_{(\delta(h) \cap \{\text{Im}(z) \leq 0\})} e^{\xi \text{Im}(z)/h - K \text{Im}(z)/h} (-\text{Im}(z))^{j} d\xi dz$$

$$\lesssim_j \sup_{z \in I} |u(z)| \int_{0}^{\infty} e^{y(K-\xi)/hy} dy$$

$$\lesssim_j \sup_{z \in I} |u(z)| \int_{0}^{\infty} e^{-\xi y/(2h)} y^{j} dy$$

$$\lesssim_j \sup_{z \in I} |u(z)| h^j \xi^{-j}$$

when $\xi \geq 2K$. Hence (4.4) is proved.

Proof of Proposition 4.1. To prove (a) we estimate, using (4.4), Hölder’s inequality and Plancherel’s theorem that

(4.5) \( \|u_N\|_{L^\infty(\mathbb{R})} \lesssim \frac{1}{h} \|\mathcal{F}_h(u_N)\|_{L^1(\mathbb{R})} \)

\( \lesssim \frac{1}{h} (\|\mathcal{F}_h(u_N)\|_{L^1(-2K,2K)} + \|\mathcal{F}_h(u_N)\|_{L^1(\mathbb{R}\setminus(-2K,2K))}) \)

\( \lesssim \frac{1}{h} (\|\mathcal{F}_h(u_N)\|_{L^2(-2K,2K)} + \mathcal{O}(h^\infty) \sup_{x \in I} |u(x)|) \)

\( \lesssim \frac{1}{h} (h^{1/2} \|u_N\|_{L^2(\mathbb{R})} + \mathcal{O}(h^\infty) \sup_{x \in I} |u(x)|) \)

\( \lesssim h^{-1/2} \|u\|_{L^2(I)} + \mathcal{O}(h^\infty) \sup_{x \in I} |u(x)|. \)

Here $\mathcal{O}(h^\infty)$ denotes a nonnegative term $O$ such that for every $j > 0$ there exists an admissible constant $C_j$ such that $O \leq C_j h^j$. Noting that by $\mathcal{L}_h^N u = u$ we have

(4.6) \( \sup_{x \in I} |u(x)| \lesssim \sup_{x \in I_N} |u(x)|. \)

Therefore combining this and (4.5) we get

\( \|u_N\|_{L^\infty(\mathbb{R})} \lesssim h^{-1/2} \|u\|_{L^2(I)}, \)

when $h$ is smaller than some admissible constant, which we will assume from now on. Since $u_N$ agrees with $u$ on $I_N$, (a) is proved.

To prove (b) we estimate similarly:

(4.7) \( \|u^{(k)}\|_{L^2(I_N)} \lesssim \|u^{(k)}\|_{L^2(\mathbb{R})} \)

\( \lesssim \|\xi^k \hat{u}_N(z)\|_{L^2(\mathbb{R})} \)

\( \lesssim \frac{1}{h^{k+\frac{1}{2}}} \|\xi^k \mathcal{F}_h(u_N)(\xi)\|_{L^2(\mathbb{R})} \)

\( \lesssim \frac{1}{h^{k+\frac{1}{2}}} \|\mathcal{F}_h(u_N)\|_{L^2(\mathbb{R})} + \mathcal{O}(h^\infty) \sup_{x \in I} |u(x)| \)

\( \lesssim \frac{1}{h^{k+\frac{1}{2}}} \|u\|_{L^2(I)} + \mathcal{O}(h^\infty) \sup_{x \in I} |u(x)|. \)
Due to (a) and (4.6) we conclude
\[ \|u^{(k)}\|_{L^2(I_N)} \lesssim_k \frac{1}{h^k} \|u\|_{L^2(I)} \]
if \( h \) is smaller than some admissible constant.

For a word \( w \) with length \( n > N \), we let \( w \) be the word \( w_{\leq n-N} \), which is obtained by removing the last \( N \) letters of \( w \).

Let \( \rho \in (0,1) \) be fixed. We observe that if \( h \) is small enough (smaller than some admissible constant), then \( |w| > N \) for every \( w \in Z(h^\rho) \), and \( I_w \subset I_N \). From now on we will always assume so.

**Proposition 4.2.** For every word \( w \in Z(h^\rho) \) we have
\[
\begin{align*}
\text{(a) } & \sup_{x \in I_w} |u(x)| \lesssim h^{-1/2} \|u\|_{L^2(I_w)} + O(h^\infty) \|u\|_{L^2(I)}; \\
\text{(b) } & \|u^{(k)}\|_{L^2(I_w)} \lesssim_k h^{-k} \|u\|_{L^2(I_w)} + O(h^\infty) \|u\|_{L^2(I)}.
\end{align*}
\]

To prove the above proposition, one can either repeat the argument in proving Proposition 4.1, by choosing a smooth cut-off function \( \chi_w \in C_c^\infty(I_w) \) which equals to 1 on \( I_w \) and then estimate \( \|\chi_w u\|_{L^\infty(\mathbb{R})}, \|\chi_w u^{(k)}\|_{L^2(\mathbb{R})} \).

Here we give a proof by utilizing the already established inequality (4.4).

**Proof of Proposition 4.2.** We let \( \chi_w \in C_c^\infty(I_w) \) be a smooth cut-off function which equals to 1 on \( I_w \). Due to the strict contracting property (2.1), we can choose \( \chi_w \) in a way such that
\[
\sup_{|\xi| > 1} |h^{-\rho} \widehat{\chi_w}(\xi)| \lesssim_j h^{-j\rho} |\xi|^{-j}
\]
and \( \|\chi_w\|_{L^1(\mathbb{R})} \lesssim 1 \).

As we have seen in the proof of Proposition 4.1, to prove (a) and (b) it suffices to show that
\[
|{\mathcal F}_h(\chi_w u_N)(\xi)| \lesssim_j h^j |\xi|^{-j} \|u\|_{L^2(I)} \text{ when } |\xi| \geq 4K.
\]

To show the above inequality we observe that
\[
{\mathcal F}_h(\chi_w u_N)(\xi) = \frac{1}{2\pi} \widehat{\chi_w} * \widehat{u_N}(\xi/h) = \frac{1}{2h\pi} \int_{\mathbb{R}} {\mathcal F}_h(u_N)(\xi - \eta) \widehat{\chi_w}(\eta/h) d\eta = A + B,
\]
where
\[
A = \int_{\mathbb{R}\setminus(-\frac{\xi}{2},\frac{\xi}{2})} {\mathcal F}_h(u_N)(\xi - \eta) \widehat{\chi_w}(\eta/h) d\eta,
\]
\[
B = \int_{(-\frac{\xi}{2},\frac{\xi}{2})} {\mathcal F}_h(u_N)(\xi - \eta) \widehat{\chi_w}(\eta/h) d\eta.
\]

Suppose \( |\xi| \geq 3K \). Due to (4.8), our assumption \( \rho < 1 \) and the fact that \( \|{\mathcal F}_h(u_N)\|_{L^\infty} \lesssim \|u_N\|_{L^1} \lesssim \|u\|_{L^2(I)} \), we obtain
\[
|A| \lesssim_j \|u\|_{L^2(I)} |\xi|^{-j} h^j.
\]

Due to (4.4), our assumption \( \rho < 1 \) and the fact that \( \|\widehat{\chi_w}\|_{L^\infty} \lesssim \|\chi_w\|_{L^1} \lesssim 1 \), we have
\[
|B| \lesssim_j \|u\|_{L^2(I)} |\xi|^{-j} h^j.
\]

Hence (4.9) is proved and so is the proposition. \( \square \)
5. Proof of Theorem 1.1

Suppose \( c < -3.75 \) and fix \( \epsilon > 0 \). Let \( u \) be an element of \( \mathcal{H} \setminus \{0\} \) satisfying \( \mathcal{L}_s u = u \) with \( \text{Re}(s) > 1/2 + \epsilon \).

Let \( I_0, I \) be compact subsets of \( I \) such that \( \tilde{I} \) is contained in the interior of \( I \), \( I_0 \) is contained in the interior of \( \tilde{I} \), and \( I_0 \) contains an open neighborhood of \( J \). For every \( w \in Z(h^\rho) \), let \( \chi_w \in C_0^\infty(\mathbb{R}) \) satisfy \( \chi_w(x) = 1 \) for \( x \in g_w(I_0) \), \( \text{supp} \chi_w(x) \subset g_w(I) \), and

\[
|\partial^j \chi_w(x)| \lesssim h^{-j\rho} \quad \text{for every } j \in \mathbb{N}.
\]

As before we let \( h = 1/\text{Im}(s) \) and assume that \( h > 0 \). For every \( \rho \in (0, 1) \) we have

\[
\mathcal{L}_{Z(h^\rho)} u = u.
\]

Because of our choice of \( \chi_w \), we have

\[
\left\| \frac{\partial^j \chi_w(x)}{\partial^j \mathcal{H}} \right\|_{L^2(I_0)} \leq \left\| \sum_{w \in Z(h^\rho)} |g_w'(x)|^2 u(g_w(x)) \chi_w(g_w(x)) \right\|_{L^2(I)}^2.
\]

For \( w, w' \in Z(h^\rho) \) with \( w \sim w' \), we let \( d(w, w') \) be the largest integer such that \( w < d(w, w') = w' < d(w, w') \).

Let \( \gamma(w, w') \) denote \( \gamma_{<d(w, w')} \). We write

\[
\left\| \sum_{w \in Z(h^\rho)} |g_w'(x)|^2 u(g_w(x)) \chi_w(g_w(x)) \right\|_{L^2(I)}^2 = T + \sum_{j = -M_1}^{M_2} T_j,
\]

where

\[
T = \sum_{w, w' \in Z(h^\rho), w \sim w'} \int_I |g_w'(x)|^2 |g_w'(x)|^2 u(g_w(x)) u(g_w'(x)) \chi_w(g_w(x)) \chi_w'(g_w'(x)) dx,
\]

\[
T_j = \sum_{w, w' \in Z(h^\rho), w \sim w', \gamma(w, w') \in \{2^j, 2^{j+1}\}} \int_I |g_w'(x)|^2 |g_w'(x)|^2 u(g_w(x)) u(g_w'(x)) \chi_w(g_w(x)) \chi_w'(g_w'(x)) dx,
\]

and \( M_2 \in \mathbb{N} \) is an admissible constant, \( M_1 \in \mathbb{N} \) satisfies \( 2^{-M_1} \sim h^\rho \).

We recall by Lemma 3.2 we have

\[
|g_w'(x)| \sim \gamma_w \sim h^\rho
\]

for every \( w \in Z(h^\rho) \) and \( x \in I \). In the following discussion, all words will be assumed to be in \( Z(h^\rho) \).

5.1. Estimating \( |T| \). First, we estimate \( |T| \). By Hölder’s inequality, and the change of variable formula,

\[
\left\| |g_w'(x)|^2 u(g_w(x)) \chi_w(g_w(x)) \right\|_{L^2(I)}^2 \leq \gamma_w^2 \text{Re}(s)^{-1} \int_I |g_w'(x)|^2 |u(g_w(x))|^2 \chi_w^2(g_w(x)) dx
\]

\[
\leq h^{\rho(2\text{Re}(s) - 1)} \int_\mathbb{R} |u(x)|^2 \chi_w^2(g_w(x)) dx
\]

\[
\leq h^{\rho(2\text{Re}(s) - 1)} \|u\chi_w\|_{L^2(\mathbb{R})}^2.
\]
Similarly, we have
\[ \|g'_w(x)\|^\ast u(g_w(x))\chi_w(x)\|_{L^2(I)} \|g'_w(x)\|^\ast u(g_w'(x))\chi_w'(x)\|_{L^2(I)} \lesssim h^{(2\Re(s) - 1)}\|u\chi_w\|_{L^2(\mathbb{R})}\|u\chi_w'\|_{L^2(\mathbb{R})}. \]

Previous estimates combined with Lemma 3.3 and the Cauchy-Schwartz inequality show that
\[
|T| \lesssim h^{(2\Re(s) - 1)}\rho \sum_{w \in Z(h^\rho)} \|u\chi_w\|_{L^2(\mathbb{R})}^2 \lesssim h^{(2\Re(s) - 1)}\rho \|u\|_{L^2(I)}^2,
\]
where the last inequality is due to (3.2).

5.2. Estimating $|T_j|$. Now we estimate $|T_j|$. Because of Proposition 2.6, for every pair of words $(w, w')$ with $w \sim w'$, we can write
\[
\int_I |g'_w(x)|^\ast |g'_w(x)|^\ast u(g_w(x))u(g_w'(x))\chi_w(g_w(x))\chi_w'(g_w'(x))dx
\]
\[
= \int_I |g'_w(x)|^{\Re(s)}|g'_w(x)|^{\Re(s)} u(g_w(x))u(g_w'(x))\chi_w(g_w(x))\chi_w'(g_w'(x))e^{i(\phi_w(x) - \phi_w'(x))/h} dx
\]
\[
= \int_I |g'_w(x)|^{\Re(s)}|g'_w(x)|^{\Re(s)} u(g_w(x))u(g_w'(x))\chi_w(g_w(x))\chi_w'(g_w'(x)) \left( \frac{h}{i(\phi_w'(x) - \phi_w'(x))} \partial_x \right)^k
\]
\[
eq A_{w, w'}
\]

Fix $w, w' \in Z(h^\rho)$ with $w \sim w'$. To shorten the notation, we denote
\[
b(x) := \frac{1}{\phi_w'(x) - \phi_w'(x)},
\]
and
\[
G(x) := |g'_w(x)|^{\Re(s)}|g'_w(x)|^{\Re(s)} u(g_w(x))u(g_w'(x))\chi_w(g_w(x))\chi_w'(g_w'(x)).
\]
Implicitly $b, G$ depend on $w, w'$. We can then rewrite $A_{w, w'}$ as
\[
A_{w, w'} = \left( -\frac{h}{i} \right)^k \int_I e^{i(\phi_w(x) - \phi_w'(x))/h}(\partial_x(b(x)))^k G(x) dx,
\]
where $(\partial_x(b(x)))$ is the operator $f \mapsto \partial_x(b(x))f(x)$.

We write
\[
B_k(x) := (\partial_x(b(x)))^k G(x).
\]

So
\[
A_{w, w'} = \left( -\frac{h}{i} \right)^k \int_I e^{i(\phi_w(x) - \phi_w'(x))/h} B_k(x) dx.
\]
5.2.1. Estimate of $|B_k(x)|$. Applying the chain rule repeatedly we see that $B_k(x)$ is a sum of $O_k(1)$ many terms of the form

\begin{equation}
(\partial_x^j b)^{k_0} \cdots (\partial_x b)^{k_1} b^{k_0} \partial_x^j G(x),
\end{equation}

where $j_k, \ldots, j_1, j_0, J$ are nonnegative integers satisfying

\begin{equation}
k j_k + (k - 1) j_{k-1} + \cdots + j_1 + J = k,
\end{equation}

\begin{equation}
\sum_{j_k} j_k + \sum_{j_1} j_1 + j_0 = k.
\end{equation}

We claim that for $x \in \tilde{I}$,

\begin{equation}
|\partial_x^j b(x)| \lesssim \frac{1}{\gamma(w, w')^k} \frac{1}{\gamma(w, w')^k} |\partial_x^j G(x)|.
\end{equation}

In fact, for a nonnegative integer $m$, we have

\[
\partial_x^m b(x) = \partial_x^m \left( \frac{1}{\phi'_w(x) - \phi'_w(x)} \right)
\]

is a sum of $O_m(1)$ many terms of the form $X(\phi'_w(x) - \phi'_w(x))^y(-1)^{y-1}(y-1)!$, where $y$ is an integer satisfying $1 \leq y \leq m + 1$, and $X$ is a product of $(y-1)$ many terms in $\{\phi'_w(x) - \phi'_w(x) : 2 \leq l \leq m + 1\}$. Because of Proposition 2.6 and Corollary 2.8 we have for $x \in I$,

\[
\|\partial_x^m b\|_{L^\infty(\tilde{I})} \lesssim \frac{1}{\gamma(w, w')^m} \frac{1}{\gamma(w, w')^m} \sim \frac{1}{\gamma(w, w')}.
\]

Recalling (5.6), we conclude (5.7).

Now we estimate $|\partial_x^j G(x)|$. $\partial_x^j G(x)$ is a sum of $O_k(1)$ many terms of the form

\begin{equation}
\partial_x^{l_1} (|g'_w(x)|)^{r_1} |g'_w(x)|^{r_2} |g'_w(x)|^{r_3} \partial_x^{l_2} (u(g'_w(x))) \partial_x^{l_3} (\chi_w(g'_w(x)) \chi_w(g'_w(x)))
\end{equation}

where $l_1, l_2, l_3$ are nonnegative integers satisfying $l_1 + l_2 + l_3 = J$. Because of Proposition 2.4, we have for $x \in I$,

\[
|\partial_x^{l_1} (|g'_w(x)|)^{r_1} |g'_w(x)|^{r_2} |g'_w(x)|^{r_3} |g'_w(x)|^{r_4} |_k |g'_w(x)|^{r_5} |g'_w(x)|^{r_6} |g'_w(x)|^{r_7} |g'_w(x)|^{r_8} |g'_w(x)|^{r_9} |g'_w(x)|^{r_10} |g'_w(x)|^{r_11} |g'_w(x)|^{r_12} |g'_w(x)|^{r_13} |g'_w(x)|^{r_14} |g'_w(x)|^{r_15} |g'_w(x)|^{r_16} |g'_w(x)|^{r_17} |g'_w(x)|^{r_18} |g'_w(x)|^{r_19} |g'_w(x)|^{r_20} \lesssim \frac{1}{\gamma(w, w')}.
\]

where $l_4, l_5$ are nonnegative integers. Because of Proposition 2.4 and (5.1), we have for $x \in I$,

\[
|\partial_x^J (\chi_w(g'_w(x)) \chi_w(g'_w(x)))| \lesssim 1.
\]

Since $|g'_w(x)| \sim |g'_w(x)| \sim h^p$ for $x \in \tilde{I}$, combining the above three estimates we obtain for $x \in \tilde{I}$,

\begin{equation}
|\partial_x^j G(x)| \lesssim_k \sup_{0 \leq l_2 \leq J} \sum_{l_4 + l_5 \leq l_2} h^p(l_4 + l_5 + 2\Re(s) - 1) |g'_w(x)|^{1/2} |g'_w(x)|^{1/2} |u^{(l_4)}(g'_w(x))|^{1/2} |u^{(l_5)}(g'_w(x))|^{1/2} |g'_w(x)|^{1/2} |g'_w(x)|^{1/2} |u^{(l_4)}(g'_w(x))|^{1/2} |u^{(l_5)}(g'_w(x))|^{1/2}.
\end{equation}

Combining (5.7) and (5.9), we have for $x \in \tilde{I}$,

\begin{equation}
|B_{w, w'}(x)| \lesssim_k \frac{1}{\gamma(w, w')^k} \sum_{l_2 = 0}^{l_2} \sum_{l_4 + l_5 \leq l_2} h^p(l_4 + l_5 + 2\Re(s) - 1) |g'_w(x)|^{1/2} |g'_w(x)|^{1/2} |u^{(l_4)}(g'_w(x))|^{1/2} |u^{(l_5)}(g'_w(x))|^{1/2}.
\end{equation}
5.2.2. Estimate of $|A_{w,w'}|$. As a consequence of (5.10), we have
\[ |A_{w,w'}| \lesssim_k \frac{h^k}{\gamma(w,w')^k} k \sum_{l_2=0}^{k} \sum_{l_4+l_5+2\Re(s)-1} h^\rho(l_4+l_5+2\Re(s)-1) \left| \int I g'_{w}(x)^{1/2} \left| g'_{w'}(x)^{1/2} |u^{(l_4)}(g_w(x))| |u^{(l_5)}(g_{w'}(x))| dx. \right| \]

Applying Hölder’s inequality and the change of variable formula we obtain
\[ |A_{w,w'}| \lesssim_k \frac{h^k}{\gamma(w,w')^k} k \sum_{l_2=0}^{k} \sum_{l_4+l_5+2\Re(s)-1} h^\rho(l_4+l_5+2\Re(s)-1) \|u^{(l_4)}\|_{L^2(I_w)} \|u^{(l_5)}\|_{L^2(I_{w'})}. \]

By Proposition 4.2 we conclude
\[ |A_{w,w'}| \lesssim_k \frac{h^k}{\gamma(w,w')^k} k \sum_{l_2=0}^{k} \sum_{l_4+l_5+2\Re(s)-1} h^\rho(l_4+l_5+2\Re(s)-1) \|u^{(l_4)}\|_{L^2(I_w)} \|u^{(l_5)}\|_{L^2(I_{w'})} + O_k(h^\infty) \|u\|_{L^2(I)}. \]

5.2.3. Estimate of $|T_j|$. Finally, we sum up all $A_{w,w'}$ to get an estimate on $|T_j|$. For a fixed $w \in \mathbb{Z}(h^n)$, we have the trivial estimate
\[ \{w' \in \mathbb{Z}(h^n) : w \sim w', \gamma(w,w') \in [2^j, 2^{j+1})\} \lesssim \left( \frac{h^p}{2^j} \right)^{-C_1}, \]
where $C_1 > 0$ is an admissible constant (depending only on $N, \gamma$ and $c$). Consequently, by the Cauchy–Schwarz inequality we have
\[ \sum_{(w,w') : \gamma(w,w') \in \mathbb{Z}(h^n), \gamma(w,w') \in [2^j, 2^{j+1})} \|u\|_{L^2(I_w)} \|u\|_{L^2(I_{w'})} \lesssim \left( \frac{2^j}{h^\rho} \right)^{C_1} \sum_{w \in \mathbb{Z}(h^n)} \|u\|_{L^2(I_w)}^2 \lesssim \left( \frac{2^j}{h^\rho} \right)^{C_1} \|u\|_{L^2(I)}^2, \]
where the last inequality is due to (3.3). Therefore,
\[ \sum_{(w,w') : \gamma(w,w') \in \mathbb{Z}(h^n), \gamma(w,w') \in [2^j, 2^{j+1})} |A_{w,w'}| \lesssim_k 2^{-j} \left( \frac{2^j}{h^\rho} \right)^{C_1} h^\rho h^{(2\Re(s)-1)\rho} \|u\|_{L^2(I)}^2 + \left( \frac{1}{h^\rho} \right)^{C_1} \|u\|_{L^2(I)}^2 + O_k(h^\infty) \|u\|_{L^2(I)}^2, \]
which implies
\[ |T_j| \lesssim_k 2^{-j} \left( \frac{2^j}{h^\rho} \right)^{C_1} h^\rho h^{(2\Re(s)-1)\rho} \|u\|_{L^2(I)}^2 + h^{-2C_1\rho} 2^{C_1\rho} O_k(h^\infty) \|u\|_{L^2(I)}^2 \]
\[ = \left( \frac{h^p}{2^j} \right)^{-k-C_1} h^{(2\Re(s)-1)\rho} \|u\|_{L^2(I)}^2 + h^{-2C_1\rho} 2^{C_1\rho} O_k(h^\infty) \|u\|_{L^2(I)}^2. \]

5.3. Concluding the proof of Theorem 1.1. We now choose and fix $k > C_1$, so
\[ \sum_{j=-M_1}^{M_2} \left( h^p / 2^j \right)^{k-C_1} \sim \left( h^p / 2^{-M_1} \right)^{k-C_1} \sim 1, \]
where the last inequality is due to $2^{-M_1} \sim h^p$. Also, we have
\[ \sum_{j=-M_1}^{M_2} h^{-2C_1\rho} 2^{C_1\rho} O_k(h^\infty) \|u\|_{L^2(I)}^2 \sim h^{-2C_1\rho} O_k(h^\infty) \|u\|_{L^2(I)}^2 = O(h^\rho) \|u\|_{L^2(I)}^2, \]
where the last estimate is due to
\[ \sum_{j=-M_1}^{M_2} h^{-2C_1\rho} 2^{C_1\rho} O_k(h^\infty) \|u\|_{L^2(I)}^2 = O(h^\rho) \|u\|_{L^2(I)}^2, \]
So combining (5.4) and (5.14) we conclude
\[
|T + \sum_{j=-M_1}^{M_2} T_j| \leq |T| + \sum_{j=-M_1}^{M_2} |T_j| \lesssim h^{\operatorname{Re}(s)-1} \rho \|u\|^2_{L^2(I)} + O(h^\infty) \|u\|^2_{L^2(I)}.
\]
We recall (5.2), and therefore
\[
\|u\|_{L^2(I_0)}^2 \lesssim h^{\operatorname{Re}(s)-1} \rho \|u\|^2_{L^2(I)} + O(h^\infty) \|u\|^2_{L^2(I)}.
\]
Since \(g_w : I \to I\) is eventually contracting to \(J\), and \(I_0\) contains an open neighborhood of \(J\), there exists \(M_0 \in \mathbb{N}\) sufficiently large depending on \(I_0\) such that \(g_w(I) \subset I_0\) for every \(|w| = M_0\). Therefore, from the identity \(L^M u = u\) on \(I\), we have
\[
\|u\|_{L^2(I)} = \|L^M u\|_{L^2(I)} \lesssim \sum_{|w|=M_0} \|g_w'(x)|^s u(g_w(x))\|_{L^2(I)} \lesssim M_0 \|u\|_{L^2(I_0)},
\]
where the last inequality is by applying the change of variable formula.

Hence (5.15) implies
\[
\|u\|_{L^2(I_0)}^2 \lesssim h^{\operatorname{Re}(s)-1} \rho \|u\|^2_{L^2(I)} + O(h^\infty) \|u\|^2_{L^2(I)}.
\]
Since \(2\operatorname{Re}(s) - 1 > 2\epsilon\), when \(h\) is sufficiently small depending on \(\epsilon\), the above implies \(\|u\|^2_{L^2(I_0)} \leq \frac{1}{2} \|u\|^2_{L^2(I)}\) and hence \(\|u\|_{L^2(I_0)} = 0\). Since \(u \in \mathcal{H}\) is holomorphic, we conclude \(u = 0\). This proves Theorem 1.1.

Corollary 1.2 is an immediate consequence of Theorem 1.1, the fact that \(Z(s)\) is entire, and the identity principle.

6. SOME NUMERICAL RESULTS

We follow the method in [JP02], in which they proposed the following approximations to the dynamical zeta function \(Z(s)\):
\[
\Delta_N(s) = 1 + \sum_{n=1}^{N} \sum_{n_1 + \cdots + n_m = n} \frac{(-1)^m}{m!} \prod_{l=1}^{m} \frac{1}{n_l} \sum_{f_{n_1} z = z} \frac{\zeta(f_{n_1})^{m_1}}{|f_{n_1}|^{m_1}} \left(1 + \frac{1 - 2(f_{n_1})'(z)}{|(f_{n_1})'(z)|^2}\right)^{-1}.
\]
Here the summation over all ordered sequences of positive integers \((n_1, \ldots, n_m)\) such that \(n_1 + \cdots + n_m = n\). In [JP02], the authors used such approximations to compute the Hausdorff dimension \(\delta\) of the Julia set associated to \(z^2 + c\) (as well as the Hausdorff dimension of the Kleinian limit set), which corresponds to the largest real zero of \(Z(s)\). The convergence turns out to be very fast near the point \(s = \delta\) and for \(s\) with small imaginary part. Also, the smaller \(\delta_c\), the Hausdorff dimension of the Julia set associated with \(z^2 + c\), is, the faster \(\Delta_N\) converges. Note however that in our paper only the cases \(\delta_c > 1/2\) are of interest, so we need to compute \(\Delta_N\) with sufficiently large \(N\) in order to provide a good estimate for the zeta function. The correspondence between \(c\) and \(\delta_c\) is shown in figure 2.

The method proposed by Jenkinson and Pollicott applies in a much more general context. See [Bor14, Bor16] for computations of dynamical zeta functions for convex co-compact hyperbolic surfaces using this method. See also [SZ04] for computations in the case of Julia sets as we have here, but using a different method.

Here we present the plots of zeros of \(Z(s)\) for \(c = -2.8\) and \(c = -4\). Limited by the computing time, we only compute approximations up to \(N = 12\). It seems that \(N = 12\) could not provide an accurate pattern of the distribution of zeros (except for the region near \(s = \delta\) or near the real axis),
due to the slow convergence when $\delta$ is large. However, a trend is that as $N$ grows larger, zeros tend to move leftwards. Therefore the two approximated plots (Figure 3 and 4) we present here could be instructive in telling us whether the essential zero-free strips exist. In fact, these two figures show empirically that the essential zero-free strips $\text{Re}(s) > \frac{1}{2} + \epsilon$ do exist, or even zero-free strips exist. So we conjecture that essential zero-free strips $\text{Re}(s) > \frac{1}{2} + \epsilon$ exist for $c$ in some range in $(-\infty, -2)$ much larger than $(-\infty, -3.75)$ as we have proved here.
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