ADM and Bondi four-momenta for the ultrarelativistic Schwarzschild black hole

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Abstract

We argue that it is possible to assign Bondi as well as ADM four-momentum to the ultrarelativistic limit of the Schwarzschild black hole in agreement to what is expected on physical grounds: The Bondi-momentum is lightlike and equal to the ADM-momentum up to the retarded time when particle and radiation escape to infinity and drops to zero thereafter, leaving flat space behind.
Introduction

Black holes, specifically the Schwarzschild geometry, are prototypes of isolated systems. This intuitive concept is mathematically captured in the notion of asymptotic flatness [1], which states that “at large distances” the geometry approaches flat space. Precisely this notion of asymptotic Minkowski space allows to define asymptotic symmetries, specifically asymptotic translations. Its corresponding charges represent the total energy-momentum at spatial infinity and the energy-momentum left in the system at a given moment of retarded time at null infinity. For the Schwarzschild geometry, ADM and Bondi momenta agree, since there is no radiation due to the static nature of the spacetime.

In the present article we consider the ultrarelativistic limit of the Schwarzschild black hole [2], which is an impulsive, plane-fronted wave with parallel rays [3]. It represents the gravitational field as it appears to an asymptotic observer moving with ultrarelativistic speed (in fact at the speed of light) relative to black hole. In this limit the gravitational field becomes completely concentrated on a null hyperplane, spacetime being flat above and below the pulse. A closer look at the wave profile reveals that it decays logarithmically along spacelike directions in the null plane thus leaving little hope for asymptotic flatness. On the other hand from its construction as limiting boost of Schwarzschild it seems physically reasonable to assign the corresponding ultrarelativistic (null) limit of the ADM momentum to the AS-geometry. At this point it is important to note that both geometries belong to the Kerr-Schild class, which is characterized by a geometric decomposition of the metric into a flat part and the tensor square of a geodetic null vector-field. Moreover, the limiting process can be considered as a one-parameter family of geometries within this class [4]. We will make use of the flat part of the decomposition to define asymptotic translations. Application of this procedure to the Schwarzschild geometry produces the correct, well-known result of a constant timelike Bondi and ADM momentum. Encouraged by this we extend the procedure to the limit geometry: A careful distributional treatment produces a lightlike ADM momentum. Similar calculations, taking the limit in lightlike directions, give a Bondi momentum equal to ADM up to the instant of retarded time where the pulse reaches infinity. For later retarded times the Bondi momentum is zero, showing that the energy was completely radiated away. The physical picture just described is a direct result of the aforementioned calculation. It is also possible to use (generalized)
conformal techniques \[3, 4\], which endow asymptotically flat spacetimes with certain boundaries and allow to interpret the energy four-momenta as integrals over two-dimensional cross-sections of these boundaries. However, we will refrain from the presentation of this construction in the present essay, reserving the full calculations for a subsequent publication \[5\].

**Energy integrals for the ultrarelativistic Schwarzschild black hole**

To give an illustration of our approach let us start from the Schwarzschild geometry

\[
g_{ab} = \eta_{ab} + f \, k_a k_b \quad k^a k^b \eta_{ab} = 0 \quad (k \nabla)k^a = (k \partial)k^a = 0
\]

\[
f = \frac{2m}{r} \quad k^a = \partial^a_t + \partial^a_r
\]

which has been written in a form that explicitly exhibits it as a member of the Kerr-Schild class. The coordinates \(t, r, \theta, \phi\) denote standard spherical polar coordinates in Minkowski space. Arbitrary translations with respect to the flat part of the decomposition \(\alpha^a\), \(\partial_b \alpha^a = 0\), are asymptotically Killing, as can be seen from

\[
\nabla^{(a} \alpha^{b)} = C^{(b}{}^c_{a} \alpha^c = \frac{1}{2} ((\alpha \partial)(f k^a k^b) - f (k \partial) f (\alpha k) k^a k^b),
\]

which behaves like \(1/r^2\). The Bondi four-momentum is defined by the Komar expression \[6\], which evaluates the curl of \(\alpha^a\) on a two-sphere moving along a surface of constant retarded time to null infinity

\[
P^\text{Bondi}_a \alpha^a = \lim_{v \to \infty} \frac{1}{2} \int_{S^2_{vu}} \nabla^{[a} \alpha^{b]} \epsilon_{abcd} = -8\pi m \alpha^t,
\]

whereas the ADM four-momentum is given by an Ashtekar-Hansen type expression \[8\]

\[
P^\text{ADM}_a \alpha^a = \lim_{r \to \infty} \frac{1}{2} \int_{S^2_{tr}} R_{abmn} \alpha^a x^b \epsilon^{mn} \epsilon_{cd} = -8\pi m \alpha^t,
\]

where by a slight abuse of notation the Lie-Algebra of translation has been identified with its representation as vector-fields. Here and in the following
\( u, v \) will denote the retarded and advanced coordinates associated with \( t, r \).

The result reflects the staticity of the gravitational field, i.e. the absence of gravitational waves carrying away mass-energy. Let us now turn to the ultrarelativistic version of the Schwarzschild field. This geometry belongs to a subfamily of the Kerr-Schild class, the so-called plane-fronted gravitational waves with parallel rays (pp-waves), characterized by the existence of a covariantly constant vector field \( p^a \)

\[
g_{ab} = \eta_{ab} + f p_a p_b \quad p^a p^b \eta_{ab} = 0 \quad \nabla_a p^b = \partial_a p^b = 0 \quad (p \partial) f = 0
\]

\( f = \mu \delta(t-z) \log \rho \quad p^a = \partial_t^a + \partial_z^a \)

\( \rho, z \) are related to \( r, \theta \) like cylindrical to spherical polar coordinates. Due to the impulsive nature of the wave profile, i.e. \( f \sim \delta(t-z) \) the spacetime is flat almost everywhere. We will therefore once again use the translations relative to the flat part of the decomposition to calculate the Bondi four-momentum as its corresponding curl,

\[
\nabla[^a \alpha^b] = (p\alpha) \partial[^a \alpha^b] p^b, \quad \frac{1}{2} \nabla[^a \alpha^b] \epsilon_{abcd} = (p\alpha) \mu \delta(t-z) p^c d\phi_d;
\]

\[
P_{a}^{\text{Bondi}} \alpha^a = \lim_{v \to \infty} \frac{1}{2} \int_{S^{2}_{vu}} \nabla[^a \alpha^b] \epsilon_{abcd}
\]

\[
= (p\alpha) \lim_{v \to \infty} \int \mu \delta \left( \frac{u}{2} (1 + \cos \theta) + \frac{v}{2} (1 - \cos \theta) \right) \frac{v - u}{2} \sin \theta d\theta d\phi
\]

\[
= 2\pi \mu \Theta(-u) p_a \alpha^a,
\]

where \( \Theta \) denotes the Heaviside function.

A closer look at the integrand of the ADM-expression

\[
R_{abcd} \alpha^a \alpha^b = \frac{1}{2} ((x \partial) \partial_d f p_c - (x \partial) \partial_c f p_d)(p\alpha) - ((\alpha \partial) \partial_d f p_c (\alpha \partial) \partial_c f p_d)(p x))
\]

\[
= -\frac{1}{2} (p\alpha) (p_c \partial_d f - p_d \partial_c f),
\]
which made use of the negative homogeneity of $\partial_a f$ and the identity $(px)\delta'(px) = -\delta(px)$, reveals that it coincides with the Komar expression. Therefore

$$P_a^{ADM} \alpha^a = \lim_{t \to \infty} \frac{1}{2} \int_{S^2_r} \nabla \epsilon_{abcd} = (p\alpha) \lim_{t \to \infty} \int_0^{\mu} \delta(t - r \cos \theta) r \sin \theta d\theta d\phi = 2\pi \mu p \alpha^a.$$  

It is precisely the negative homogeneity of the $\delta$-function, i.e. $\delta(\lambda x) = \lambda^{-1} \delta(x)$ $\lambda > 0$, which guarantees the convergence of the above integrals. This is also the reason why the slow logarithmic decay on the null hyperplane does not spoil the asymptotic behavior.

**Concluding remarks**

Since the gravitational field of the ultrarelativistic Schwarzschild geometry is concentrated on a null hyperplane it necessarily distributional in nature. We were able to show that this property is responsible for the convergence of the integrals of the total energy and the energy left in the system after the emission of gravitational radiation. Viewed in this context the above geometry is a nice example for a system that completely radiates off all energy leaving flat space behind. Due to its close connection with the Schwarzschild black hole the limit geometry plays a distinguished role within the class of impulsive pp-waves. Waves with a profile proportional to higher multipoles are asymptotically flat but have zero energy-momentum [9]. A closer look reveals, however, that they necessarily violate the (weak) energy condition. On the other hand for pp-waves that do not fall off, e.g. plane waves, the energy-momentum is undefined. Finally, we would like to comment on the theorem that the ADM and Bondi momenta cannot be null [9, 10]. In this context it is important to remark that the spacetime, due to its distributional nature, does not obey the conditions prerequisite to this conclusion, i.e. that it represents singular initial data. Nevertheless, we do believe that the simple physical interpretation of the result shows that this geometry represents a sensible (idealized) physical system, which is asymptotically flat in a distributional setting. Actually, the situation is reminiscent of the symmetry classification of pp-waves by Jordan Ehlers and Kundt, which assigns the
ultrarelativistic Schwarzschild geometry a two-parametric symmetry group. However, precisely the distributional nature of the profile shows that the symmetry group is four-parametric as it is for Schwarzschild [1].

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