Quantum Hall Fluid of Vortices in a Two Dimensional Array of Josephson Junctions

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Abstract

A two dimensional array of Josephson junctions in a magnetic field is considered. It is shown that the dynamics of the vortices in the array resembles that of electrons on a two–dimensional lattice put in a magnetic field perpendicular to the lattice. Under appropriate conditions, this resemblance results in the formation of a quantum Hall fluid of vortices. The bosonic nature of vortices and their long range logarithmic interaction make some of the properties of the vortices’ quantum Hall fluid different from those of the electronic one. Some of these differences are studied in detail. Finally, it is shown that a quantum Hall fluid of vortices manifests itself in a quantized Hall electronic transport in the array.

1. Introduction

This paper discusses a quantized Hall effect (QHE) state of vortices in a two dimensional (2D) array of Josephson junctions. Motivated by the analogy between Magnus force acting on a vortex moving in a two–dimensional ideal fluid and Lorenz force acting on a charge in a magnetic field, we study the transport of vortices in a Josephson junction array. In particular, we focus on the case in which the charging energy of the array is minimized when the number of Cooper pairs on each element of the array is not an integer. We find that for a certain range of parameters the vortices are expected to form a quantum Hall fluid, and the resistivity of the array is expected to show a QHE behavior. While some of the
The properties of the quantum Hall fluid formed by the vortices are similar to those of the well-known Laughlin fluid, formed by electrons in QHE systems, we find that the logarithmic interaction between the vortices leads to interesting modifications of other properties.

The paper is organized in the following way: in Section (2) we review the classical and quantum mechanical analogies between the 2D dynamics of charged particles under the effect of a magnetic field and that of vortices in a 2D fluid. These analogies, arising from the analogy between Magnus and Lorenz forces, motivate the introduction of the system we analyze – a Josephson junction array in a magnetic field, and the study of a quantized Hall effect in that system. Section (2) is concluded with a precise formulation of the problem to be studied. In Section (3) we analyze the transport of vortices in this array. We show that the dynamics of the vortices can be mapped on that of charged particles under the effect of a magnetic field, a lattice-induced periodic potential and a mutual interaction. The mutual interaction is composed of a logarithmic ”static” part as well as a velocity-dependent short ranged part. In Section (4) we analyze the quantized Hall effect associated with the transport of the vortices, and its observable consequences. In particular, we study the unique features of the QHE for logarithmically interacting particles. Conclusions are presented in Section (5).

The possibility of Quantum Hall phenomena in Josephson junction arrays was recently discussed in two other works, one of Odintsov and Nazarov [1], and the other of Choi [2]. The regime of parameters we consider is different from the ones considered by these authors. We comment briefly on this difference and its implications in section (2).

2. Transport of vortices in a Josephson junction array – introduction and motivation

The classical dynamics of vortices in two-dimensional ideal fluids is well known to resemble that of charged particles under the effect of a strong magnetic field [3]. An electron in a magnetic field is subject to Lorenz force, while a vortex in an ideal fluid is subject to
Magnus force. Both forces are proportional and perpendicular to the velocity. Two electrons in a strong magnetic field encircle each other, and so do two vortices in an ideal fluid. The dynamics of both are well approximated by an Hamiltonian that includes only a potential energy $V(x, y)$, where $x, y$ are the planar coordinates, and for which $x$ and $y$ are canonically conjugate. This approximation is known as the guiding center approximation for the electronic problem, and as Eulerian dynamics for the vortex problem. This close resemblance naturally raises the possibility of analogies between transport phenomena of electrons in a magnetic field and those of vortices in ideal fluids.

A vortex in a fluid can be viewed as an excitation in which each fluid particle is given an angular momentum $l$ relative to the vortex center. Consequently, the velocity field $\vec{v}(\vec{r})$ of the fluid satisfies $\int_{\Gamma} \vec{v} \cdot d\vec{l} = \frac{2\pi l}{m}$, where $m$ is the mass of a fluid particle, and $\Gamma$ is a curve that encloses the vortex center. When the vortex center moves with a velocity $\vec{u}$, and the fluid is at rest far away from the vortex center, the vortex center is subject to a Magnus force, given by $F_{\text{Magnus}} = 2\pi l n \vec{u} \times \hat{z}$, where $n$ is the number density of the fluid far away from the vortex core. Being both proportional and perpendicular to the velocity of the vortex center, Magnus force obviously resembles the Lorenz force acting on an electron moving in the $x - y$ plane under the effect of a magnetic field $B \hat{z}$. This Lorenz force is given by $F_{\text{Lorenz}} = \frac{eB}{c} \vec{u} \times \hat{z}$, where $\vec{u}$ is the velocity of the electron. Thus, the role played by the product $\frac{eB}{c}$ in the latter case is played by the product $2\pi l n$ in the former. While the fluid density plays a role analogous to that of a magnetic field, a fluid current plays a role analogous to that of an electric field. To see that, note that in a frame of reference in which the electron is at rest, the force it is subject to looks as if it arises from an electric field, given by $\frac{eB}{c} \vec{u} \times \hat{z}$. Similarly, in a frame of reference in which the vortex center is at rest, the force it is subject to seems to arise from the motion of the fluid. Since the fluid current density is $\vec{J} = n \vec{u}$, in the vortex rest frame the Magnus force is $F_{\text{Magnus}} = 2\pi l \vec{J} \times \hat{z}$, and $\vec{J} \times \hat{z}$ plays a role analogous to that of an electric field. Thus, while the fluid density affects the vortex dynamics in the same way a magnetic field affects electronic dynamics,
the fluid current plays the role of an electric field. Maxwell’s equation $\nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$ is then analogous to the continuity equation in the fluid $\nabla \cdot \vec{J} + \frac{\partial n}{\partial t} = 0$. \[5\] \[6\]

Quantum mechanics introduces two new ingredients to the analogies discussed above. The first is the quantum of angular momentum $l$, given by $\hbar$ (or alternatively, the quantum of vorticity, $\frac{\hbar}{m}$) \[7\]. The second is the quantization of the magnetic flux, the integral of the magnetic field over area. This quantization is most clearly seen through the Aharonov–Bohm effect \[8\]: the Aharonov–Bohm phase shift accumulated by an electron traversing a closed path in a magnetic field is $2\pi$ times the number of flux quanta it encircles. Combining these two ingredients together, one should expect a quantization associated with the integral of the number density over area, i.e., with the number of particles. This quantization should manifest itself in the phase accumulated by a vortex carrying a single quantum of angular momentum, $\hbar$, when it traverses a closed path in a fluid. Indeed, as shown first by Arovas, Schrieffer and Wilczek \[9\], such a vortex does accumulate a geometric (Berry) phase \[10\], and this phase is $2\pi$ times the number of fluid particles it encircles. The analog of a flux quantum is then a single fluid particle \[11\]. Note that the analogy between vortex dynamics in a fluid and electron dynamics in a magnetic field does not depend on the fluid being charged, and is valid for neutral fluids as well.

Quantum transport of 2D electrons in a magnetic field crucially depends on the electronic filling factor, the ratio between the density of conduction electrons and the density of flux quanta. For a very low filling factor, ($\ll 1$), electrons are expected to form a Wigner lattice. At higher filling factors, the quantized Hall effect takes place \[12\]. Similarly, we expect transport of vortices in a fluid to depend on a vortex ”filling factor”, the ratio between the density of vortices and the density of fluid particles. However, in continuous two dimensional fluids this ratio is usually much smaller than one, and the vortices indeed form an Abrikosov lattice.

How can the vortices ”filling factor” be made larger? In this work we make the vortex filling factor larger by considering a lattice structure. As is well known, all properties of
electrons on a lattice are invariant to the addition of a magnetic flux quantum to a lattice plaquette. Similarly, when we consider vortex transport on a lattice-structured fluid, we find all properties to be invariant to the addition of a single fluid particle to a lattice site. It is this periodicity that allows us to make the effective filling factor much larger than the ratio between the density of vortices and the density of fluid particles.

Based on the foregoing general considerations, we study in this paper a Josephson junction array in a magnetic field. Josephson junction arrays were extensively studied in recent years [13] [14]. The array we consider is composed of identical small super–conducting dots coupled by a nearest–neighbors Josephson coupling $E_J$, and by a capacitance matrix $\hat{C}$. For definiteness, we consider a square lattice of the superconducting dots. Generalizations to other lattices are straightforward. A perpendicular magnetic field induces vortices in the configuration of the superconducting phase. We denote the average number of vortices per lattice plaquette by $n_v$. Each of the dots carries a dynamical number of Cooper pairs, denoted by $n_i$ (for the $i$’th dot), as well as positive background charges. The charging energy of the array is minimized for a certain set of values of $n_i$, which we denote by $n_{x,i}$. (In our notation, charge is always expressed in units of $2e$, i.e., $n_i, n_{x,i}$ are dimensionless.) While the $n_i$’s are operators with integer eigenvalues, $n_{x,i}$ are real parameters, that are closely related to the chemical potential of the dots. In this work we consider the case in which for all sites $n_{x,i} = n_x$. The Hamiltonian describing the array is [14],

$$H = \frac{(2e)^2}{2} \sum_{ij} (n_i - n_x) \hat{C}_{ij}^{-1} (n_j - n_x) + E_J \sum_{\langle ij \rangle} (1 - \cos(\phi_i - \phi_j - \int_i^j \vec{A} \cdot d\vec{l}))$$ (1)

where $\sum_{\langle ij \rangle}$ denotes a sum over nearest neighbors, $n_i$ is the number of Cooper pairs on the $i$’th dot, $\phi_i$ is the phase of the superconducting order parameter on the $i$’th dot, $\vec{A}$ is the externally put vector potential and the integral is taken between the sites $i$ and $j$. A factor of $\frac{2e}{c}$ is understood to be absorbed in $\vec{A}$. The matrix $\hat{C}^{-1}$ is the inverse of the capacitance matrix $\hat{C}$. Generally, the matrix $\hat{C}$ includes elements coupling a dot to its nearest neighbors, to the substrate and to neighbors further away. The matrix elements of both $\hat{C}$ and $\hat{C}^{-1}$ are a function of the distance between the sites $i$ and $j$. For short distances the electrostatic
energy is determined by nearest neighbors capacitance only, and all other capacitances can be ignored. The \( ij \) matrix element of \( C^{-1} \) is then \( \frac{2\pi}{C_{nn}} \log |r_i - r_j| \), where \( C_{nn} \) is the nearest neighbors capacitance \([14]\). For large distances, the electrostatic interaction depends also on capacitance to the substrate and capacitance to neighbors further away. The inverse capacitance matrix then decays with the distance. Throughout most of our discussion we assume that the size of the array is small enough such that the charging energy is determined by nearest neighbors capacitance only. Then, the charging energy involves one energy scale, \( E_C \equiv \frac{e^2}{2C_{nn}} \). The effect of other capacitances is briefly discussed in section (3).

Since our main interest in this study is focused on transport phenomena of vortices, we constrain ourselves to arrays in which \( E_J > E_C \). In that regime of parameters vortices are mobile enough not to be trapped within plaquettes, but their rest energy is large enough such that quantum fluctuations of vortex–antivortex pair production can be neglected. Arrays in which \( E_J > E_C \) were studied experimentally by van der Zant et.al. \([15]\), and were found to show a magnetic field tuned transition from an almost superconducting state to an almost insulating state. At weak magnetic fields the density of vortices is low, and their ground state is the Abrikosov lattice. The array is then superconducting. The transition to the insulating state, at a critical value of the magnetic field, is interpreted as caused by a transition of the vortices from a lattice phase to a correlated super–fluid–like phase \([11]\) \([15]\).

As mentioned in section (1), QHE phenomena in Josephson junction arrays were discussed in two recent preprints. The first, by Odintsov and Nazarov \([1]\), focuses on the regime \( E_C \gg E_J \), and discusses a quantum Hall fluid of Cooper–pairs. The second, by Choi \([2]\), focuses on the regime \( E_J \gg E_C \), and discusses a quantum Hall fluid of vortices. The quantum fluid we discuss in this paper has some similarity to the one discussed by Choi. However, the difference in the regime discussed, as well as our detailed study of the effect of the logarithmic vortex–vortex interaction, make some of our conclusions different from those of Choi.

Due to the lattice structure of the array, the spectrum and eigenstates of the Hamiltonian
are manifestly periodic with respect to $n_x$, with the period of one Cooper–pair ($n_x = 1$). This periodicity is similar to the periodicity of the spectrum of electrons on a lattice with respect to the addition of one flux quantum per plaquette. Thus, although the ratio between the density of vortices and the density of charges in the system is very small, the physically meaningful ratio is the ratio of $\bar{n}_v$ to $(n_x - \lfloor n_x \rfloor)$ (where $\lfloor n_x \rfloor$ is the largest integer smaller than $n_x$), and this ratio is not necessarily small. Following this observation, we limit ourselves from now on to the case $0 \leq n_x < 1$.

Having described in detail the Josephson junction array to be considered, we conclude this section by formulating precisely the question to be studied, namely, how do the physical properties of the array depend on the ratio between the vortex density $\bar{n}_v$ and the charge density $n_x$? We start our examination of that question by deriving an effective action for the vortices in the array.

3. The effective action for the vortices

The Hamiltonian (1) describes the Josephson junctions array in terms of the sets of variables $\{n_i\}, \{\phi_i\}$. In this section we derive an equivalent description of the array in terms of the vortex density $\rho_{\text{vor}}$, the vortex current $\vec{J}_{\text{vor}}$ and gauge fields the vortices interact with. Our goals in attempting to derive this description are three-fold. The first goal is to verify the validity of our assertion that vortices are subject to Magnus force, and that $n_x$ plays a role analogous to that of a magnetic field in electronic dynamics. The second goal is to study the mutual interactions between vortices. The third goal is to estimate the mass of the vortices. The first two goals are relatively easy to achieve. Estimating the mass of the vortex, however, turns out to be a harder task, which we are able to handle only approximately.

The effective action for vortices in a Josephson junction array was first discussed by Eckern and Schmid, who considered the Hamiltonian (1), with $n_x = 0$. More generally, the effective action for singularities in the phase configuration of a complex field was discussed
in various other contexts in physics. A particularly convenient method for the derivation of such an action is the "duality transformation", developed and used by Jose et.al. [16], Berezhinskii [17], Peskin [18], Fisher and Lee [19] and others. This method was applied to analyze the motion of vortices in Josephson junction arrays (again, for the case \( n_x = 0 \)) by Fazio, Geigenmuller and Schon [20].

In our derivation of the effective action, we follow Fazio, Geigenmuller and Schon [20] by applying the duality transformation to obtain an effective action for vortices on a lattice. The action resulting from the duality transformation (Eq. (3) below) describes the vortices as bosons on a lattice interacting with an externally put vector potential, as well as with a dynamical vector potential. The externally put vector potential, which we denote by \( \vec{K}_{ext} \), satisfies \( \vec{\nabla} \times \vec{K}_{ext} = 2\pi\hbar n_x \). The interaction with the dynamical vector potential mediates a vortex–vortex interaction, composed of two parts. The first part is the familiar logarithmic interaction. Its strength is proportional to the Josephson energy \( E_J \). The second part, induced mostly by the charging energy of the array, is a short ranged velocity–velocity interaction. The latter makes the vortices massive, since it includes a self interaction term, quadratic in the vortex velocity. However, the mass defined by this interaction is a "bare mass", that does not take into account the periodic potential exerted on the vortices by the lattice. Generally speaking, the periodic potential changes the bare mass into an effective band mass. In an attempt to estimate the band mass we write the continuum limit of the vortices action. In the continuum language, vortices are massive particles interacting with an external vector potential, with a periodic lattice potential and with one another. Our analysis of this rather complicated dynamics follows the way the dynamics of electrons on a lattice is analyzed. We start by neglecting vortex–vortex interactions. We are then faced with a single particle problem, in which a massive vortex interacts with a static vector potential \( \vec{K}_{ext} \), and with a periodic lattice potential. This problem is identical to the problem of an electron under the effect of a uniform magnetic field and a lattice periodic potential, whose solution is well known. When \( n_x \ll 1 \) the effect of the periodic potential can be accounted for by changing the "bare mass" to an effective mass. We limit ourselves to this
case, and estimate the resulting effective mass. Then, we incorporate the vortex–vortex
interactions back into the action.

Before turning into the details of the derivation sketched in the last paragraph, we pause
to define a notation. We denote 3–vectors by bold–faced letters, and their two spatial
components by vector arrows. The electromagnetic potential is then \( \mathbf{A} = (A_0, A_x, A_y) = (A_0, \vec{A}) \). We number array sites by a subscript \( i \). The bond connecting a site \( i \) to its neighbor
on the right side is denoted by the subscript \( i, x \). Similarly, the bond connecting the \( i \)’th site to the site above it is denoted by the subscript \( i, y \). The difference operator \( \vec{\Delta} \), a discretized version of \( \vec{\nabla} \), is defined accordingly. When operating on a scalar \( \phi \), for example,
the \( x \)–component of \( \vec{\Delta} \) is \( \phi_j - \phi_i \) where \( j \) is the neighbor to the right side of \( i \).

Our derivation of the effective action starts by considering the part
ition function
\[
Z = \text{tr} e^{-\beta H} = \int D\{n\} \int D\{\phi\} e^{-\frac{1}{\hbar} S(\{n(t)\}, \{\phi(t)\})}
\]

where the action \( S(\{n(t)\}, \{\phi(t)\}) \) is defined by (1) and the
Hamiltonian is given by (1). The path integral is to include all paths satisfying \( n_i(\beta) = n_i(0) \)
and \( \phi_i(0) = \phi_i(\beta) \). The variables \( n_i \) are integers and therefore the path integral has to be
performed stepwise [21]. We limit ourselves to zero temperature, i.e., \( \beta = \infty \).

The first step of the derivation follows closely previous works [20], and is therefore given
in Appendix A. Using the Villain approximation and the duality transformation method,
the path integral over the charge and phase degrees of freedom, \( n_i \) and \( \phi_i \), is transformed
to a path integral over a 3–component integer field \( J^\text{vor} \) describing the vortex charge and
density, and a 3–component real gauge field \( K \), to which \( J^\text{vor} \) is coupled. This gauge field
describes the charge degrees of freedom, to which it is related through its derivatives. The
field strengths associated with this gauge field, \( \frac{1}{2\pi\hbar} \epsilon^{\mu\nu\sigma} \partial_\mu K_{i,\nu} \), are the Cooper–pair currents
and density on the \( i \)’th site. In terms of \( J^\text{vor} \) and \( K \), and in a gauge in which \( \vec{\Delta} \cdot \vec{K} = 0 \) the
action is given by,
\[ S^{\text{vor}} = \sum_i \left\{ i(\rho_i^{\text{vor}} - \vec{n}_i)K_{i,0} + i\vec{J}_i^{\text{vor}} \cdot (\vec{K}_i + \vec{K}_i^{\text{ext}}) + \frac{1}{8\pi^2E_J}[:(\Delta_i\vec{K}_i)^2 + (\Delta K_{0i})^2] \right\} \]

(3)

where \( \vec{\Delta} \times \vec{K}^{\text{ext}} = 2\pi \hbar n_x \). This action describes the vortices as bosons on a lattice, interacting with an externally put gauge field \( \vec{K}^{\text{ext}} \) whose spatial curl is a constant, given by \( 2\pi \hbar n_x \), and with a dynamical gauge field \( K \). As expected from the similarity between Magnus and Lorenz forces, a moving vortex is affected by the Cooper–pairs on the dots in the same manner a charged particle is affected by a magnetic field. Moreover, the Josephson currents between the dots affect the vortices in the same way an electric field affects charged particles. The last three terms of the action include the self energy of the field \( K \). They are simply understood once the relation between \( K \) and the Cooper–pairs currents and densities is taken into account. The first two are the kinetic energy of the Josephson currents (the transverse part of that current is \( \frac{e}{\pi \hbar} \vec{\Delta} K_0 \), and \( \frac{e}{\pi \hbar} \vec{\Delta} \) is the longitudinal part). The last term is the charging energy (the net charge on the \( i \)'th dot is \( \frac{1}{2\pi \hbar} \vec{\Delta} \times \vec{K}_i \)). The transverse part of the current satisfies a two dimensional Gauss law \( \vec{\Delta}^2 K_0 = 4\pi^2 E_J \rho^{\text{vor}} \) and mediates a logarithmic interaction between the vortices. The excitation spectrum of \( \vec{K} \) is the spectrum of longitudinal oscillations of the Cooper–pairs, i.e., the plasma spectrum of the array.

Our next step is a formulation of the continuum limit of the action (3). When doing that, two points should be handled carefully. The first is the periodic potential exerted by the lattice on the vortex. This potential was studied in detail by Lobb, Abraham and Tinkham [22]. Since Currents do not flow uniformly within the array, the energy cost associated with a creation of a vortex depends on the position of its center within a plaquette (i.e., on the precise distribution of the currents circulating its core). This energy cost is periodic with respect to a lattice spacing of the array, and is independent of the sign of the vorticity. The origin of this potential can be visualized using the analogy with 2D electrostatics. In that analogy, a vortex is analogous to a charge in a two–dimensional world. A vortex on a lattice is then analogous to a charge in a two dimensional world in which the dielectric
constant varies periodically with position. The electrostatic energy of such a charge varies periodically with position, too, and is independent of the sign of the charge. This energy cost can then be interpreted as a periodic potential exerted by the lattice on the vortices. The characteristic energy scale for that potential is $E_J$. Its amplitude and functional form were studied in Ref. 22. The amplitude was found to be $0.2E_J$ and $0.05E_J$ for square and triangular lattices, respectively.

A convenient way to incorporate the periodic dependence of the vortex potential energy on the position of the vortex core within a plaquette is by replacing the Josephson energy $E_J$ in the action (3) by a periodically space dependent function $\epsilon_J(\vec{r})$, that is non–zero only along lattice bonds. The period of $\epsilon_J(\vec{r})$ is obviously the lattice spacing. The energy cost involved with the Josephson currents then becomes \( \int d\vec{r} \frac{1}{8\pi^2\epsilon_J(\vec{r})} \left[ (\vec{\nabla}K_0(\vec{r},t))^2 + (\dot{\vec{K}}(\vec{r},t))^2 \right] \), and that energy cost confines the currents to the lattice bonds. The effect of the spatial dependence of $\epsilon_J(\vec{r})$ on the interactions mediated by $K_0$, $\vec{K}$ is discussed below.

The second point to be handled carefully when transforming to a a continuum action is the short distance cut–off on the capacitance matrix. The model we employ does not attempt to describe statics and dynamics of Cooper–pairs within superconducting dots. Thus, its continuum version should not allow for excitations of $\vec{K}$ at wavelengths shorter than the lattice spacing. This constraint is taken into account by introducing a high wave–vector cut–off to the capacitance matrix, as it was done in Ref. 14.

Taking into account the two points discussed above, the continuum limit of the action (3) is,

\[
S = \int dt \int d\vec{r} \left\{ i[\rho^{vor}(\vec{r},t) - \vec{\pi}_v]K_0(\vec{r},t) + i\vec{J}(\vec{r},t) \cdot [\vec{K}(\vec{r},t) + \vec{K}^{ext}(\vec{r})] \\
+ \frac{1}{8\pi^2\epsilon_J(\vec{r})} \left[ (\vec{\nabla}K_0(\vec{r},t))^2 + (\dot{\vec{K}}(\vec{r},t))^2 \right] \\
+ \frac{e^2}{2\pi^2\hbar} \int dt \int d\vec{r} \int d\vec{r}' \left[ \vec{\nabla} \times \vec{K}(\vec{r},t) \right] \hat{C}^{-1}(\vec{r} - \vec{r}') \left[ \vec{\nabla} \times \vec{K}(\vec{r}',t) \right]
\]

When the fields $K_0$, $\vec{K}$ are integrated out they mediate mutual interactions between...
vortices and self interactions of a vortex with itself. The spatial dependence of $\epsilon_J(\vec{r})$ does not significantly affect mutual interactions between vortices whose distance is much larger than one lattice spacing. It does, however, potentially affect the self interaction.

Consider the action associated with a single vortex. As discussed above, due to the spatial dependence of $\epsilon_J(\vec{r})$, the interaction of the vortex with $K_0$ yields a periodically space dependent potential energy [22]. The coupling to $\vec{K}$ results in a kinetic energy. To see that, note that the vortex current corresponding to a moving vortex whose center is at $\vec{r}_0(t)$ is $\vec{J}^{\text{vor}} = \dot{\vec{r}}_0(t)\delta(\vec{r} - \vec{r}_0(t))$. Substituting this expression in the action (4), we find the part of the action that depends on the vortex velocity, $\dot{\vec{r}}_0$, and the gauge field it interacts with, $\vec{K}$, to be

$$i \int dt \dot{\vec{r}}_0(t) \cdot \vec{K}(\vec{r}_0, t)$$

$$+ \int dt \left\{ \int d\vec{r} \frac{1}{8\pi^2\epsilon_J(\vec{r})} (\vec{K}(\vec{r}, t))^2 + \frac{e^2}{2\pi^2\hbar} \int d\vec{r} \int d\vec{r}' \left[ \vec{\nabla} \times \vec{K}(\vec{r}, t) \right] \tilde{C}^{-1}(\vec{r} - \vec{r}') \left[ \vec{\nabla} \times \vec{K}(\vec{r}', t) \right] \right\}$$

(5)

The gauge field $\vec{K}$ can be integrated out, with the resulting effective action for the vortex velocity $\dot{\vec{r}}_0$ being non-local in time. However, as pointed out by Eckern and Schmid [14] and by Fazio et al. [20], the time non-locality can be neglected as long as the characteristic frequencies involved in $\dot{\vec{r}}_0(t)$ are smaller than the Josephson plasma frequency $\frac{1}{\hbar}\sqrt{8E_JE_C}$. This neglect is possible due to the gap in the excitation spectrum of $\vec{K}$, a gap that makes the time non-locality short ranged [23]. Having neglected the time non-locality, we find the effective action for the vortex velocity $\dot{\vec{r}}_0$ to be

$$\int dt \frac{1}{2} m_{\text{bare}} \dot{\vec{r}}_0(t)^2$$

(6)

where $m_{\text{bare}}$, the vortex bare mass, is defined by $m_{\text{bare}} = \frac{\pi^2\hbar^2}{4E_C}$. Thus, the interaction of the vortex with $\vec{K}$ results in a kinetic term.

Having integrated out both $K_0$ and $\vec{K}$ we have turned the single vortex action into an action of a charged particle interacting with a periodic potential and a magnetic field.
$2\pi \hbar n_x$. For $n_x \ll 1$ the effect of the periodic potential is to change the bare mass into an effective band mass. Since the lowest band is the relevant one for bosons, the band mass is always larger than the bare one [24]. The effective band mass for $n_x = 0$ was studied both theoretically and numerically by Geigenmuller [25] and by Fazio et al. [20] (see also references therein). While a qualitative estimate of the mass is easy to arrive at, a quantitative determination depends on the precise details of the periodic potential, and is therefore hard to obtain. Qualitatively, the tight binding limit, in which $E_J \gg E_C$, is distinguished from the weak periodic potential limit, in which the opposite condition applies. In the former, the effective band mass is

$$m_{\text{band}} \sim \hbar^2 \sqrt{\frac{\alpha_1}{E_J E_C}} e^{\sqrt{\frac{E_J}{E_C}}}$$

(7)

where $\alpha_{1,2}$ are numbers of order unity [20] [25]. In the latter,

$$m_{\text{band}} \sim m_{\text{bare}} \left(1 + \left(\frac{E_J}{E_C}\right)^2\right)$$

(8)

The regime of parameters we are interested in lies between the two limits. It is therefore reasonable to assume that the band mass is larger than, but of the order of, the bare one.

We now turn to discuss the many vortices configuration. As we have argued above, the discreteness of the array does not significantly affect the mutual interaction between vortices. Therefore, for the study of this interaction we may replace $\epsilon_J(\vec{r})$ by $E_J$. Then, the action (4) can be written in momentum space as

$$S = \int dt \int \frac{d\vec{q}}{(2\pi)^2} \left\{ i \rho_{-\vec{q}}^{\text{vor}} - \pi_{\text{vor}}(\vec{q}) \right\} \mathcal{K}_{0\vec{q}} + i \tilde{J}_{\text{vor}} \cdot (\tilde{\mathcal{K}}_{-\vec{q}} + \tilde{\mathcal{K}}_{\text{ext}}) + \frac{1}{8\pi E_J} [||\tilde{q}\mathcal{K}_{0\vec{q}}||^2 + |\tilde{\mathcal{K}}_{\vec{q}}|^2]$$

$$+ \frac{\epsilon^2}{2\hbar^2} |\tilde{q} \times \tilde{\mathcal{K}}_{\vec{q}}|^2 \hat{C}^{-1}(\tilde{q})$$

(9)

where $\rho_{\vec{q}}^{\text{vor}}, J_{\vec{q}}^{\text{vor}}, \tilde{\mathcal{K}}_{\vec{q}}, \tilde{\mathcal{K}}_{\text{ext}}, \hat{C}^{-1}(\tilde{q})$ are the Fourier transforms of the corresponding quantities. The momentum space representation used in Eq. (9) is convenient for the integration of the field $\mathcal{K}$. The integration out of the time component, $\mathcal{K}_0$, yields a density–density interaction between the vortices, of the form $\frac{1}{2} \int d\vec{q} E_J |\rho_{\vec{q}}^{\text{vor}} - \pi_{\text{vor}}(\vec{q})|^2$, where $q \equiv |\vec{q}|$. When transformed back to real space, this interaction is
\[ \pi E_J \int dr \int dr' (\rho_{\text{vor}}(\vec{r}) - \bar{\rho}_{\text{v}}) \log |\vec{r} - \vec{r}'| (\rho_{\text{vor}}(\vec{r}') - \bar{\rho}_{\text{v}}) \]  

(10)

Similarly, the integration of \( \vec{K} \) yields a current–current interaction between the vortices. In the gauge we use, \( \vec{K}_{\vec{q}} \) has only transverse components. It therefore mediates an interaction only between the transverse component of the vortex currents. Integrating out \( \vec{K} \), and neglecting again the slight non–locality in time, we find that the current–current interaction is given, in momentum space, by

\[ \frac{\hbar^2}{8e^2} \int \frac{d\vec{q}}{q^2 \tilde{C}^{-1}(q)} |J_{\perp \vec{q}}^{\text{vor}}|^2 = \frac{\hbar^2}{16E_C} \int d\vec{q} |J_{\perp \vec{q}}^{\text{vor}}|^2 \]  

(11)

where \( J_{\perp \vec{q}}^{\text{vor}} \equiv \vec{q} \times \vec{J}_{\vec{q}}^{\text{vor}} \) is the transverse component of \( \vec{J}_{\vec{q}}^{\text{vor}} \), and the integral over \( \vec{q} \) is cut–off at \( q = 2\pi \). The current–current interaction is described in real space as,

\[ \frac{\hbar^2}{16E_C} \int d\vec{r} \int d\vec{r}' \int d\vec{q} \left[ J_{\perp \vec{q}}^{\text{vor}}(\vec{r}) \times \vec{q} \right] \left[ J_{\perp \vec{q}}^{\text{vor}}(\vec{r}') \times \vec{q} \right] \frac{1}{q^2} e^{i\vec{q} \cdot (\vec{r} - \vec{r}')} \]  

(12)

As pointed out in Ref. [14], this current–current interaction includes both a self interaction term, that assigns a mass \( m_{\text{bare}} \) to each vortex, and a velocity–velocity interaction between different vortices. The former was discussed in the context of a single vortex. The latter is short ranged. For large separations, it is inversely proportional to the square of the distance between the interacting vortices.

Eqs. (10) and (12) both neglected the effect of the lattice structure of the array on the vortex–vortex interactions. Similar to the common practice in the analysis of electrons on a lattice, we assume that the sole effect of the lattice is to modify the single vortex mass from the bare mass \( m_{\text{bare}} \) to the effective band mass \( m_{\text{band}} \). The current–current interaction in Eq. (12) includes a self interaction that assigns a mass \( m_{\text{bare}} \) to each vortex. To account for the modification of the mass by the lattice, we add another kinetic term to the action, of the form \( M^* \int dr \frac{\vec{J}_{\text{vor}}(\vec{r})^2}{2\rho_{\text{vor}}(\vec{r})} \), where \( M^* \equiv m_{\text{band}} - m_{\text{bare}} \). Altogether, then, the effective vortices action becomes,
\[ S_{\text{vor}}(\vec{J}_{\text{vor}}) = \int dt \left\{ \int d\vec{r} \left[ M^{*} \frac{\vec{J}_{\text{vor}}(\vec{r})^2}{2\rho_{\text{vor}}(\vec{r})} + i\vec{J}_{\text{vor}}(\vec{r}) \cdot \vec{K}_{\text{ext}}(\vec{r}) \right] + \frac{\hbar^2}{16E_C} \int d\vec{r} \int d\vec{r}' \left[ \int d\vec{q} \left[ \vec{J}_{\text{vor}}(\vec{r}) \times \vec{q} \right] \left[ \vec{J}_{\text{vor}}(\vec{r}') \times \vec{q} \right] \frac{1}{|\vec{q}|^2} e^{i\vec{q} \cdot (\vec{r} - \vec{r}')} \right] \right\} + \pi E_J (\rho_{\text{vor}}(\vec{r}) - \bar{n}_v) \log |\vec{r} - \vec{r}') (\rho_{\text{vor}}(\vec{r}') - \bar{n}_v) \right\} \] (13)

Eq. (13) is a concise description of the dynamics of the vortices, since the only dynamical fields it includes are those of the vortices. It describes the vortices as interacting particles of mass \( m_{\text{band}} \) and an average density \( \bar{n}_v \), under the effect of a "magnetic field" \( 2\pi\hbar n_x \). The vortices "filling factor" is then indeed \( \frac{\bar{n}_v}{n_x} \).

The current–current interaction term in the action (13) is somewhat inconvenient for calculations. Thus, in our analysis of the quantum Hall fluid of vortices in the next section we choose to reintroduce \( \vec{K} \), and consider the vortices as particles of mass \( M^{*} \) interacting with a dynamical vector potential \( \vec{K} \), as well as with \( \vec{K}_{\text{ext}} \) and with one another.

We conclude this section with a few remarks regarding the dependence of its results on the form of the capacitance matrix. The capacitance matrix determines the bare mass of the vortex (see Eqs. (5) and (6)) and the form of the vortex current–current interaction (see Eq. (12)). So far we considered a capacitance matrix that includes only nearest neighbors coupling. The inverse capacitance matrix describes then a two dimensional Coulomb interaction between Cooper–pairs on the superconducting dots. In Fourier space, it is proportional to \( \frac{1}{q^2} \). Inclusion of capacitances to the ground and/or capacitance between dots that are not nearest neighbors result in a screening of that interaction. Then, at small \( q \), \( \hat{C}^{-1}(q) \propto q^{-\alpha} \), with \( 0 \leq \alpha < 2 \). This screening has two consequences. First, the kinetic energy cost involved in a vortex motion, i.e., its bare mass, is affected. Second, the excitation spectrum of \( \vec{K} \) becomes gapless. We now examine these consequences. Consider a vortex moving in a constant velocity \( \vec{v} \). As seen from Eq. (4), a moving vortex acts like a source for the vector potential \( \vec{K} \). The (imaginary time) wave equation for \( \vec{K} \) can be derived from Eq. (5). In Fourier space its solution is,
\[ K_{\perp} \bar{q}, \omega = i v_{\perp} \frac{\delta(\omega - \bar{q} \cdot \bar{v})}{4\pi^2 E_J} + \frac{\epsilon^2}{\pi^2 \hbar^2 q^2} C(q) \]  

(14)

where \( v_{\perp} \equiv \frac{\bar{q} \times \bar{v}}{q} \) is the transverse part of the velocity vector. The longitudinal component of \( \bar{K} \) vanishes in the gauge we use. The kinetic energy cost associated with the motion of a vortex is the energy cost of the fields \( \hat{\bar{K}} \) and \( \hat{\nabla} \times \hat{\bar{K}} \) it creates. It is composed of two parts. The first, \( \int d\bar{r} \frac{1}{8\pi^2 E_J} \hat{\bar{K}}^2 \), is the kinetic energy cost of the longitudinal currents created by the motion of the vortex. The second, \( \frac{\epsilon^2}{2\pi^2 \hbar^2} \int d\bar{r} \int d\bar{r}' \left[ \hat{\nabla} \times \hat{\bar{K}}(\bar{r}) \right] \hat{C}^{-1}(\bar{r} - \bar{r}') \left[ \hat{\nabla} \times \hat{\bar{K}}(\bar{r}') \right] \) is the cost in charging energy. A moving vortex induces voltage drops between the superconducting dots, and those result in a charging energy cost, determined by the capacitance matrix. The first energy cost is proportional to \( v^4 \), while the second is proportional to \( v^2 \). Thus, the bare mass is determined by the charging energy. Transforming Eq. (14) to real space, and substituting into the expression for the charging energy, we observe that the charging energy is finite as long as \( \alpha > 0 \), and diverges logarithmically with the system size for \( \alpha = 0 \). The gapless excitations of \( \bar{K} \), characteristic of \( \alpha < 2 \), play a role when vortices accelerate or decelerate. Then, the coupling of the vortices to these excitations (the ”spin waves” \[20\]) becomes a weak mechanism for dissipation of a vortex kinetic energy \[14\]. For the present context we note that the effect of a weak dissipative mechanism on the quantized Hall effect was studied by Hanna and Lee \[26\]. While some properties of the effect are affected by such a mechanism, its main features are not.

4. The quantum Hall fluid of vortices

4.1 General discussion

In the previous section we established the mapping of the vortex dynamics in a Josephson junction array on the problem of interacting charged particles in a magnetic field. We have also identified the ratio \( \frac{\pi v}{n_e} \) as the vortices filling factor. In this section we examine the formation of QHE fluid state of vortices at appropriate filling factors. While so far we have emphasized the similarities between the dynamics of the vortices and that of electrons in
a magnetic field, in this section we must study the differences between the two. We begin
by a general discussion of two of the differences. Then, we turn in the next subsection
to a detailed calculation, using the Chern Simon Landau Ginzburg approach to the QHE,
developed by Zhang, Hansson, Kivelson and Lee [27].

The first difference is in the statistics: while vortices are bosons, electrons are fermions.
This difference changes the values of the "magic" filling factors, and eliminates the possibility
of a QHE in the absence of interactions. The filling factors at which bosons form quantum
Hall fluids are $\frac{p}{q}$, where $p, q$ are integers, and one of them is even [28]. Fermi liquids of the
type discussed by Halperin, Lee and Read [29] form at filling factors $\frac{1}{2n+1}$, where $n$ is an
integer.

The second difference is in the interaction: the logarithmic interaction between vortices
is of longer range than the Coulomb interaction between electrons. This difference leads to
a modification of the quantized Hall conductance, a modification of the charge of Laughlin’s
quasiparticles, and, perhaps most interestingly, to a modification of one of the diagonal
elements of the linear response function. While for a short range interaction these diagonal
elements vanish in the long wavelength low frequency limit ($\vec{q}, \omega \to 0$), reflecting the lack of
longitudinal dc conductance in the QHE state, we find that the logarithmic interaction makes
one of the diagonal elements non–zero. In fact, rather than describing insulator–type zero
longitudinal response, as expected from a QHE system, this element describes a longitudinal
response of the type usually associated with a superconductor. These consequences of the
logarithmic interaction are all derived in detail in the next subsection, where we also study
the difference between the linear response function and the conductivity. In this subsection
we preceed the derivation by a discussion of a thought experiment that makes the role of
the logarithmic interaction physically transparent. The thought experiment we consider
was extensively used in the study of the Quantized Hall Effect, e.g., by Laughlin [30] and
Halperin [31], and was proved very useful in understanding various aspects of the effect.

Consider a "conventional" electronic quantum Hall system, in which a disk shaped two
dimensional electron gas (2DEG) is put in a strong magnetic field, and a thin solenoid
threads the disk at its center. The flux through the solenoid is time dependent, and is denoted by $\Phi(t)$. If the electrons on the disk are in a QHE state, the current density at any point is perpendicular to the total electric field at that point. The time dependence of the flux induces an electric field in the azimuthal direction, given by $\frac{c}{2\pi r c} \dot{\Phi}(t)$, where $c$ is the speed of light, and $r$ is the distance from the center. Due to the finite Hall conductance, this electric field induces a radial current, and, consequently, a charge accumulation at the center of the disk. The charge accumulated at the center during the interval $0 < t < t_0$ is given by $\frac{\sigma_{xy}}{\Phi_0} (\Phi(t_0) - \Phi(0))$ (where $\sigma_{xy}$ is the dimensionless Hall conductivity and $\Phi_0 \equiv \frac{hc}{e}$ is the flux quantum). This charge accumulation, in turn, creates a radial electric field. Now, if the electrons interact via a Coulomb interaction, the radial electric field is proportional to $\frac{1}{r^2}$, i.e., it decays faster than the azimuthal one. Then, far away from the center the electric field is predominantly azimuthal, and the currents are predominantly radial. However, if the electrons interact via a logarithmic interaction, both the radial and azimuthal components of the electric field are inversely proportional to $r$, and thus their ratio is independent of $r$. The current then has both radial and azimuthal components, and their ratio is independent of $r$, too. Moreover, the azimuthal component of the current is proportional to the flux at the center, and not to its time derivative.

The two components of the current and the charge accumulated in the center can be calculated using classical equations of motion, since in the absence of impurities, the classical and quantum mechanical calculations coincide. Consider, therefore, the hydrodynamical equation of motion of a fluid of electrons in a magnetic field, whose electronic density and velocity fields are denoted by $\rho(\vec{r})$ and $\vec{v}(\vec{r})$, respectively. Assuming a uniform positive background charge density $\overline{\rho}$ on the disk, this equation of motion is

$$m\rho(\vec{r})\ddot{\vec{v}}(\vec{r}) = -\rho(\vec{r})\vec{v}(\vec{r}) \times \vec{B} - \int d\vec{r}'\vec{\nabla}V_{e-e}(\vec{r} - \vec{r}')(\rho(\vec{r}') - \overline{\rho})\rho(\vec{r}) + \frac{\Phi}{2\pi r^2}\rho(\vec{r})$$

(15)

where $\ddot{\vec{v}}(\vec{r})$ is the complete time derivative of the velocity field, $V_{e-e}$ is the electron–electron interaction potential, and we use a system of units where $e = c = 1$. The initial conditions corresponding to the scenario discussed in the previous paragraph are $\rho(\vec{r}) = \overline{\rho}$, $\vec{v}(\vec{r}) = 0$.
and $\Phi(t = 0) = 0$. Due to the circular symmetry of both Eq. (15) and its initial and boundary conditions, the current and density remain circularly symmetric when the flux is turned on, and the electron–electron interaction term can be written as $V'_{e-e}(r)Q(r)\rho(r)$ where $Q(r) \equiv \int_0^r dr'2\pi r'(\rho(r') - \bar{\rho})$ is the net charge within a distance $r$ from the origin, and $V'_{e-e}(r) \equiv \frac{\partial V_{e-e}(r)}{\partial r}$. The conservation of charge constraint implies $\dot{Q}(r) = -2\pi r\rho(r)v_r(r)$, where $v_r$ is the radial component of $\vec{v}$. The azimuthal component of Eq. (15) can therefore be written as,

$$2\pi r\rho(r)\dot{v}_\phi = \frac{\rho(r)}{m} \dot{\Phi} - \omega_c \dot{Q}$$

with $\omega_c \equiv \frac{B}{m}$. For values of $r$ far away from the center but not close to the edge the density $\rho(r)$ remains approximately equal to $\bar{\rho}$ all along the process, and thus $Q(r)$ is $r$–independent. Within that approximation, and for such values of $r$, the azimuthal equation can be integrated and substituted in the radial one. The latter then becomes,

$$\bar{\rho} \ddot{v}_r = \frac{\omega_c^2 Q}{2\pi r} - \frac{\bar{\rho} \omega_c \Phi}{m 2\pi r} - \frac{1}{m} V'_{e-e}(r)Q(r)\bar{\rho} + \frac{1}{r} \bar{\rho} \bar{v}_\phi(r)^2$$

where the last term is the centrifugal force. Suppose now that the flux $\Phi$ is turned adiabatically on from zero to $\Phi(t_0)$ in the interval $0 < t < t_0$. For times $t \gg t_0$ the velocity field is purely azimuthal and $\dot{v}_r = 0$. Then, if the potential gradient $\vec{\nabla}V_{e-e}$ decays faster than $\frac{1}{r}$, so does also the azimuthal velocity $v_\phi$, and

$$Q = \nu \frac{\Phi(t_0)}{\Phi_0}$$

where $\nu = \frac{\bar{\rho}}{\bar{\Phi}}$ is the filling factor. If $\Phi(t_0) = \Phi_0$ then the charge accumulated near the origin is the charge of Laughlin’s quasiparticle, namely $-e\nu$.

However, in the case of a logarithmic interaction, $V_{e-e}(r) = -V_0 \log r$,

$$v_\phi = \frac{\Phi}{2\pi m \hbar \omega_c + V_0 \nu} \frac{V_0 \nu}{\nu}$$

$$Q = \frac{\Phi}{\Phi_0} \nu \frac{1}{\hbar \omega_c}$$
The azimuthal current is then indeed inversely proportional to the distance from the origin, and proportional to the flux $\Phi$. In fact, the current is related to the vector potential created by the solenoid, $\vec{A}^{sol}$, via a London–type equation

$$\vec{\nabla} \times \vec{J} = \frac{\bar{\rho}}{m} \frac{V_0}{\hbar \omega_c} + V_0 \nu \vec{\nabla} \times \vec{A}^{sol}$$

(20)

Thus, the longitudinal response of the electrons on the disk to the vector potential created by the solenoid resembles the longitudinal response of a two dimensional superconducting disk in a similar situation.

Two conclusions can be drawn from the above thought experiment. First, for logarithmically interacting particles, the charge of the Laughlin quasiparticle does not equal the filling factor, but depends on the interaction. And second, the transverse part of the linear response function, relating a transverse current to an externally applied transverse vector potential, resembles that of a superconductor. Since this response function is proportional to the transverse current–current correlation function, the latter should be expected to resemble a superconductor, too. Both conclusions are substantiated in the next subsection, and are applied to the study of the quantum Hall fluid of vortices.

4.2 A study of the vortices QHE state by the Chern Simon Landau Ginzburg approach

In this subsection we use the Chern Simons Landau Ginzburg approach to further analyze the properties of the quantized Hall fluid of vortices formed at appropriate values of $\frac{n_v}{n_x}$. We start by writing a Landau–Ginzburg action that describes the dynamics of the vortices. We then perform a Chern–Simons singular gauge transformation that attaches an even number of fictitious Cooper pairs to each vortex. The resulting action, in which the order parameter describes transformed ”composite” bosons, is convenient for a saddle point analysis. We find the uniform density saddle point that describes a superfluid of composite bosons. By expanding the action to quadratic order around that saddle point we calculate the response function of the vortices to an external probing field. This response function, denoted by $\hat{\Sigma}$,
is the ratio of the vortex density and current $J^{\text{vor}}$ to an infinitesimal probing field $K^p$ applied externally to the system. The matrix $\hat{\Sigma}$ is calculated by adding an external probing field $K^p$ to the Lagrangian, and integrating out all the other fields to obtain an effective Lagrangian $L^{\text{eff}}$ in terms of $K^p$ only. The three components of $J^{\text{vor}}$ are then given by $J^{\text{vor}}_{\alpha} = -\frac{\partial L^{\text{eff}}}{\partial K^p_{\alpha}}$, and the elements of $\hat{\Sigma}$ are

$$\Sigma_{\alpha\beta} = \frac{\partial J^{\text{vor}}_{\alpha}}{\partial K^p_{\beta}} \bigg|_{K^p=0} = -\frac{\partial}{\partial K^p_{\alpha}} \frac{\partial}{\partial K^p_{\beta}} L^{\text{eff}}(K^p) \bigg|_{K^p=0}$$

(21)

The physical meaning of $\hat{\Sigma}$, as well as the important distinction between the response to $K^p$ and the response to the total field $K^p + K$, are discussed after the calculation is presented.

The Landau–Ginzburg action that describes the properties of the vortices as they were found in section (3) is,

$$S_{\text{LG}}(\tilde{\psi}, K) = \int dt \left\{ \int d\vec{r} \left[ \hbar \tilde{\psi}^* \partial_t \tilde{\psi} + \frac{1}{2M} \left( (i\hbar \vec{\nabla} - K^{\text{ext}} - K) \tilde{\psi} \right)^2 + \frac{1}{8\pi^2 E_J} \vec{K}^2 \right] + \int d\vec{r} \int d\vec{r}' \left[ \pi E_J \left( |\tilde{\psi}(\vec{r})|^2 - \pi_e \right) \log |\vec{r} - \vec{r}'| \left( |\tilde{\psi}(\vec{r}')|^2 - \pi_e \right) \right] + \frac{e^2}{2\pi^2 \hbar} \left[ \vec{\nabla} \times \vec{K}(\vec{r}) \right] \cdot \left[ \vec{\nabla} \times \vec{K}(\vec{r}') \right] \right\}$$

(22)

The fields $\tilde{\psi}, K^{\text{ext}}, K$ all depend on $\vec{r}$ and on $t$. For the brevity of the expressions we omit this dependence whenever this omission does not lead to confusion. The field $\tilde{\psi}$, the order parameter for the vortices, satisfies bosonic commutation relations.

Restricting our derivation to the "fundamental" fractions $\frac{1}{\eta}$, where $\eta$ is an even number, our first step in analysing the action (22) is the Chern Simon singular gauge transformation, in which the field $\tilde{\psi}(\vec{r}, t)$ is transformed to

$$\psi(\vec{r}, t) = e^{i\eta} \int d\vec{r}' \arg(\vec{r} - \vec{r}') |\tilde{\psi}(\vec{r}', t)|^2 \tilde{\psi}(\vec{r}, t)$$

(23)

where $\arg(\vec{r} - \vec{r}')$ is the angle the vector $\vec{r} - \vec{r}'$ forms with the $x$ axis. Since $\eta$ is an even integer the field $\psi$ has the same statistics as $\tilde{\psi}$, i.e., bosonic. Note that $|\tilde{\psi}(\vec{r}, t)| = |\psi(\vec{r}, t)|$. The singular gauge transformation shifts the phase of the field. Denoting the phase of $\psi(\tilde{\psi})$
by \( \theta(\tilde{\theta}) \), the Chern–Simons transformation amounts to \( \bar{\nabla}\theta(\vec{r},t) = \bar{\nabla}\tilde{\theta}(\vec{r},t) - \frac{1}{\hbar}\vec{a}(\vec{r},t) \) where \( \vec{a}(\vec{r},t) \equiv \hbar\eta \int d^2\vec{r}' \frac{\bar{\nabla} \times (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^2} |\tilde{\psi}(\vec{r}',t)|^2 \). The Chern–Simons field \( \vec{a} \) has a gauge freedom, which we fix below. Writing \( \psi(\vec{r},t) \equiv \sqrt{n_v(\vec{r},t)} e^{i\theta(\vec{r},t)} \), the above Landau–Ginzburg functional becomes

\[
S_{\text{LG}}(n_v, \theta, K, a) = \int dt \left\{ \int d^2\vec{r} \left[ in_v(\hbar \partial_\theta - a_0) + \frac{n_v}{2M^*}(\hbar \bar{\nabla}\theta - K_{\text{ext}} - K + \vec{a})^2 + \frac{\hbar^2}{2M^*}(\bar{\nabla} \sqrt{n_v})^2 + \frac{1}{8\pi^2 E_J} \dot{K}^2 + \frac{i}{4\pi\hbar} \epsilon^{\mu\nu\sigma} a_\mu \partial_\nu a_\sigma \right] 
+ \int d^2\vec{r} \int d^2\vec{r}' \left[ \pi E_J(n_v(\vec{r}) - \bar{n}_v) \log |\vec{r} - \vec{r}'|(n_v(\vec{r}') - \bar{n}_v) 
+ \frac{e^2}{2\pi^2\hbar^2} [\bar{\nabla} \times \vec{K}(\vec{r})] \dot{\vec{K}}^{-1}(\vec{r} - \vec{r}') [\bar{\nabla} \times \vec{K}(\vec{r}')] \right] \right\}
\]

(24)

While the Euler–Lagrange equations of motion for the original field \( \tilde{\psi} \) required its phase to be multiply valued (for a minimization of the kinetic energy), the equations of motion for the transformed field \( \psi \) allow for a solution in which the magnitude and the phase of the field are constant. It is straightforward to see that the action (24) is minimized when,

\[
n_v(\vec{r},t) = \bar{n}_v \tag{25}
\]

\[
\bar{\nabla}\theta(\vec{r},t) = 0 \tag{26}
\]

\[
\vec{a}(\vec{r},t) = \vec{K}_{\text{ext}}(\vec{r}) \tag{27}
\]

\[
K(\vec{r},t) = 0 \tag{28}
\]

This minimum of the action describes a state in which the vortex density is constant on the average and the Chern–Simons field \( \vec{a} \) cancels \( K_{\text{ext}} \) on the average. We now expand the action around these minimum values. Around the saddle point the phase \( \theta \) is singly valued, and thus we can choose a gauge in which \( \theta = 0 \) identically, and the field \( \psi \) is real.
Writing, then, \( n_v = \bar{n}_v + \delta n_v, \psi = \sqrt{\bar{n}_v + \frac{\delta n_v}{2\sqrt{\bar{n}_v}}} \) and \( \bar{a} = \bar{K}^{ext} + \delta \bar{a}, \) we find that the quadratic deviations from the extermum point (25)-(28) are described by the following action,

\[
S_{LG}(\delta n_v, \bar{K}, \bar{a}) \approx \int dt \int d\vec{r} \int d\vec{r} [i a_0 \delta n_v + \frac{\delta n_v}{2M^*} (\bar{K} + \delta \bar{a})^2 + \frac{\hbar^2}{8\pi M^*} (\bar{\nabla} \delta n_v)^2 + \frac{1}{8\pi^2 E_j} \bar{K}^2 \delta n_v + \frac{i}{4\pi \hbar} e^{\mu \sigma} a_\mu \partial_v a_\sigma]
\]

+ \int dt \int d\vec{r} \int d\vec{r} \left[ \pi E_j \delta n_v (\vec{r}) \log |\vec{r} - \vec{r}'| \delta n_v (\vec{r}') + \frac{\epsilon_d^2}{2\pi^2 \hbar^2} (\vec{\nabla} \times \bar{K} (\vec{r})) \frac{1}{\bar{C}} (\vec{r} - \vec{r}') (\vec{\nabla} \times \bar{K} (\vec{r}')) \right]

(29)

For the calculation of the conductivity matrix and the correlations functions, we imagine coupling the vortices to an infinitesimal 3–vector probing field \( \bar{K}_p. \) Naturally, linear response functions are more conveniently described in Fourier space. Thus, we write the action (29) in the presence of the probing field \( \bar{K}_p, \) in Fourier space, as

\[
S_{LG}(\psi, \bar{K}, \bar{a}, \bar{K}_p) \approx \int \frac{d\hat{q}}{2\pi} \int \frac{d\hat{q}}{2\pi} \left[ i (a_0 (\hat{q}) + \bar{K}_p^0 (\hat{q})) \delta n_v (-\hat{q}) + \frac{\hbar v}{2M^*} |\bar{K}_p (\hat{q}) + \bar{K} (\hat{q}) + \delta \bar{a} (\hat{q})|^2
\]

+ \frac{\hbar^2}{8\pi M^*} |\bar{\nabla} n_v (\hat{q})|^2 + \frac{1}{8\pi^2 E_j} |\omega \bar{K}_p (\hat{q})|^2 + \frac{i}{4\pi \hbar} e^{\mu \sigma} a_\mu (\hat{q}) q_\nu a_\sigma (\hat{q})

+ \frac{2\pi^2 E_j}{\epsilon_d^2} |\delta n_v (\hat{q})|^2 + \frac{\epsilon_d^2}{2\pi^2 \hbar^2} |\bar{q} \times \bar{K} (\hat{q})|^2 \bar{C}^{-1} (\hat{q}) \right]

(30)

where \( \hat{q}_0 \equiv \omega. \) We choose the Coulomb gauge for \( \bar{K}_p, \) i.e., \( \bar{q} \cdot \bar{K}_p (\hat{q}) = 0. \) Thus, the probing field becomes a 2–component vector. Since the three components of \( \mathbf{J}^{vor} \) are constrained by the conservation of vorticity, \( \mathbf{J}^{vor} \) is effectively a two–component vetor, too, and \( \Sigma_{\alpha \beta} \) is a \( 2 \times 2 \) matrix. The indices \( \alpha \) and \( \beta \) take the values 0 (for the time component) and \( \perp \) (for the component perpendicular to \( \hat{q} \)). The integration of the fields \( \delta n_v, \bar{K}, \bar{a} \) is easily carried out, since (30) is quadratic in all fields. The resulting effective Lagrangian is,

\[
L^{eff} (\bar{K}_p) = \frac{(\bar{K}_p)^2}{2} \left[ \frac{M^* \omega^2}{2\pi \hbar^2} + \frac{\hbar q^2}{8\pi M^* \hbar^2} \right] + \frac{\hbar q^2 (\bar{K}_p^0)^2}{2D} - iq \bar{K}_p^0 \bar{K}_p^0
\]

where \( D \equiv 2\pi \hbar \left[ \frac{M^*}{\hbar^2} + \frac{1}{4\hbar^2 \epsilon_d^2} \right]^{-1}. \) In the limit of \( q, \omega \rightarrow 0 \) and for \( \bar{C} (\hat{q}) = C_{nq^2} \)

\[
D = 2\pi \hbar \left[ \frac{M^*}{\hbar^2} + \frac{1}{2m_{bare}} \right]^{-1}
\]

23
Eq. (21) expresses the matrix elements of $\hat{\Sigma}$ in terms of second derivatives of this effective Lagrangian with respect to $K_0^p$ and $K_{\perp}^p$. Each of the four components of the matrix $\Sigma_{\alpha\beta}$ warrants a short discussion. First, the transverse component of the vortex current and the transverse component of the gauge field $K^p$ are related, in the limit $\vec{q}, \omega \rightarrow 0$, by

$$J_{\perp}^{\text{vor}} = -\frac{2\pi E_J D}{4\pi^2 \hbar \eta E_J + 2\pi \hbar^2 \eta^2 D^p} K_{\perp}^p$$

(32)

This London–type of relation was anticipated by Eq. (20). It is characteristic of superconductors, and is very different from the insulating behaviour characteristic of the diagonal components of the response functions in QHE systems. This difference results from the static vortex–vortex interaction being of a long range. Defering the discussion of the effect of Eq. (32) on the longitudinal conductivity to a later stage, we now point out its effect on the vortex current–current correlation function. By the fluctuation–dissipation theorem,

$$\langle J_{\perp}^{\text{vor}} J_{\perp}^{\text{vor}} \rangle_{\vec{q}, \omega} = \text{Im} \Sigma_{\perp \perp} (\vec{q}, \omega)$$

$$\int d\omega' \mathcal{P}(\frac{1}{\omega' - \omega}) \langle J_{\perp}^{\text{vor}} J_{\perp}^{\text{vor}} \rangle_{\vec{q}, \omega'} = \text{Re} \Sigma_{\perp \perp} (\vec{q}, \omega)$$

(33)

where $\mathcal{P}$ denotes the principal part of the integral, and the second line is an application of Kramers–Kronig relations [32]. For an insulator, $\text{Re} \Sigma(q, \omega) \propto \omega^2$ when $\omega \rightarrow 0$. This is also the case for QHE systems with short range interactions [33]. For a superconductor $\text{Re} \hat{\Sigma}(\vec{q}, \omega)$ approaches a constant in the $\omega \rightarrow 0$ limit. As we now see, so is also the case for a QHE system in which the interactions are logarithmic. In the particular problem we study, this constant is $-\frac{2\pi E_J D}{4\pi^2 \hbar \eta E_J + 2\pi \hbar^2 \eta^2 D^p}$.

Second, we note that the compressibility of the vortex fluid vanishes in the limit $\vec{q}, \omega \rightarrow 0$, as is manifested by the absence of low frequency poles in the density–density correlation function. Like its electronic analog, the quantum Hall fluid of vortices is incompressible.

Third, the Hall component of the linear response function is given, in the limit $\vec{q}, \omega \rightarrow 0$, by,

$$\Sigma_{0,\perp} = \frac{-iq}{\frac{4\pi^2 E_J}{D} + 2\pi \hbar \eta}$$

(34)

24
If the vortex–vortex interaction was of shorter range, the long wavelength limit of $\Sigma_{0,\perp}$ would satisfy $\Sigma_{0,\perp} = \frac{iq}{2\pi\hbar}$, seemingly demonstrating the quantization of the Hall conductivity \[27\].

We further comment on the difference between the two expressions below.

The qualitative effect the logarithmic interaction has on the transverse and Hall components of the linear response function raises the following question: does the conductivity of the system we study have the properties of the conductivity matrix of a QHE system, namely, zero longitudinal conductivity and quantized Hall conductivity? To answer this question, we clarify the relation between the response function $\Sigma_{\alpha\beta}$ and the vortex conductivity matrix. A similar relation was discussed, in the context of the QHE, by Halperin \[31\], Laughlin \[34\], Halperin Lee and Read \[29\] and Simon and Halperin \[35\]. The transport of vortices in the array is probed by externally applied (number) density and current of Cooper–pairs, given, respectively, by $\frac{1}{2\pi\hbar} \vec{V} \times \mathcal{K}^p$, and $-\frac{1}{2\pi\hbar} \vec{V} \mathcal{K}^p_0 - \frac{1}{2\pi\hbar} \dot{\mathcal{K}}^p$. The matrix $\hat{\Sigma}$ is defined such that $J^{\text{vor}} = \hat{\Sigma} \mathcal{K}^p$. However, the vortices themselves contribute to the Cooper–pair density and current, with the most trivial contribution being the circulation of current around each vortex center. The total Cooper–pair density and current are therefore given by the derivatives of a total gauge field, composed of the probing field $\mathcal{K}^p$ and the field induced by the vortex density and current, denoted by $\mathcal{K}^{\text{ind}}$. The latter is proportional to the vortex density and current $J^{\text{vor}}$. Thus, we can define a matrix $\hat{V}$ such that

$$\mathcal{K}^{\text{ind}} \equiv \hat{V} J^{\text{vor}} = \hat{V} \hat{\Sigma} \mathcal{K}^p \tag{35}$$

Consequently, the total field is $\mathcal{K}^{\text{tot}} = (1 + \hat{V} \hat{\Sigma}) \mathcal{K}^p$, and

$$J^{\text{vor}} = \hat{\Sigma} (1 + \hat{V} \hat{\Sigma})^{-1} \mathcal{K}^{\text{tot}} \tag{36}$$

Thus, the matrix $\hat{\Sigma} (1 + \hat{V} \hat{\Sigma})^{-1}$ relates $\mathcal{K}^{\text{tot}}$ to the vector $(\rho^{\text{vor}} \ J^{\text{vor}}_{\perp})$. The vortex conductivity matrix $\sigma^{\text{vor}}$, relating the vortex current $(J^{\text{vor}}_{\parallel} \ J^{\text{vor}}_{\perp})$ to the total driving force vector

$$\begin{pmatrix}
-i \frac{q}{2\pi\hbar} \mathcal{K}_0^{\text{tot}} \\
-i \frac{\omega}{2\pi\hbar} \dot{\mathcal{K}}^{\text{tot}}
\end{pmatrix}$$

is then,
\[ \sigma^{\text{vor}} = \begin{pmatrix} -\frac{\omega}{q} & 0 \\ 0 & 0 \end{pmatrix} \Sigma(1 + \hat{V} \hat{\Sigma})^{-1} \begin{pmatrix} \frac{i}{2\pi} & 0 \\ 0 & \frac{i}{2\pi} \end{pmatrix} \]  

(37)

where the leftmost matrix converts \( \rho^{\text{vor}} \) to \( J^{\text{vor}} \).

Eq. (37) defines the vortex conductivity matrix in terms of the matrices \( \hat{\Sigma} \) and \( \hat{V} \). The matrix \( \hat{\Sigma} \) is defined by Eqs. (21) and (31). The matrix \( \hat{V} \), relating \( J^{\text{vor}} \) to \( \mathcal{K}^{\text{ind}} \), is specified by the action (30) to be

\[ \hat{V} = \begin{pmatrix} \frac{4\pi^2 E_j}{q^2} & 0 \\ 0 & \frac{1}{\pi^2} \frac{q^2}{\hbar^2} \mathcal{C}^{-1}(q) \end{pmatrix} \]  

(38)

The upper left element describes the field \( \mathcal{K}_0 \) created by a vortex density \( \rho^{\text{vor}} \). The gradient of that field is the transverse current circulating around the vortex center. The bottom right element describes the field \( \mathcal{K}_\bot \) created by a transverse vortex current \( J^{\text{vor}}_\bot \), and is obtained from Eq. (30) by taking its derivative with respect to \( \mathcal{K}_\bot(q) \).

Substituting the matrices \( \hat{\Sigma} \) and \( \hat{V} \) to Eq. (37), we find, to leading order in \( q, \omega \),

\[ \hat{\sigma}^{\text{vor}} = \begin{pmatrix} \frac{i \omega M^*}{q^2 n_w} & \frac{1}{\eta} \\ -\frac{1}{\eta} & -\frac{i \omega M^*}{q^2 n_w} \end{pmatrix} \]  

(39)

In the limit \( \vec{q}, \omega \to 0 \) the diagonal terms vanish, and the vortex current satisfies

\[ \vec{J}^{\text{vor}} = -\frac{i}{\eta 2\pi \hbar} \hat{z} \times (q\mathcal{K}_0 + \omega \hat{K}) \]  

(40)

Eqs. (38) and (40) describe a quantized Hall effect: the current is purely perpendicular to the total ”driving force”, and the Hall conductivity is quantized. Contrasting Eqs. (32) and (34) with Eq. (40) we can finally summarize the effect of the logarithmic interaction on the linear response of the system: the dc conductivity, which is the \( q, \omega \to 0 \) response to the total driving force, has the usual form of the quantum Hall conductivity, and is unaffected by the interaction. The correlation functions, on the other hand, determined by the response to the
externally applied driving force, are affected by the interactions even in the $q, \omega \to 0$ limit, with the most notable effect being on the transverse current–current correlation function.

We conclude this section by relating the vortices conductivity, calculated above, to the electric conductivity and resistivity, which are the quantities typically measured in experiments. The electric conductivity is the matrix relating voltage drops (or, in the continuum limit, electric fields) between superconducting dots to the electric Josephson current flowing in the array. The electrostatic potential at a point $\vec{r}$ is given by $\frac{e}{\pi \hbar} \int d\vec{r}' \hat{C}^{-1}(-\vec{q}) \hat{\nabla} \times \vec{K}(-\vec{q})$. Thus, in Fourier space the $\vec{q}$ component of the electrostatic potential is $i \frac{e}{\pi \hbar} \hat{C}^{-1}(-\vec{q}) \vec{K}(-\vec{q})$ and the longitudinal electric field is $-i \frac{e}{\pi \hbar} \hat{q} \hat{C}^{-1}(-\vec{q}) \vec{K}(-\vec{q})$. Now, by deriving an equation of motion for $\vec{K}$ from the action (9), we see that a dc transverse vortex current $\vec{J}_{q\perp}^{\text{vor}}$ creates a field $\vec{K}_{q\perp}$ given by

$$\vec{J}_{q\perp}^{\text{vor}} = \frac{e^2}{\pi^2 \hbar e^2} \hat{q}^2 \hat{C}^{-1}(q) \vec{K}_{q\perp} \tag{41}$$

i.e., a transverse vortex current $\vec{J}_{q\perp}^{\text{vor}}$ induces a longitudinal electric field $\frac{\pi \hbar}{e} \vec{J}_{q\perp}^{\text{vor}}$. A similar argument regarding the relation of the longitudinal vortex current to the transverse electric field leads to the conclusion that a vortex current $\vec{J}_{q\parallel}^{\text{vor}}$ creates an electric field $\frac{\pi \hbar}{e} \hat{\vec{z}} \times \vec{J}_{q\perp}^{\text{vor}}$. The Josephson charge current, on the other hand, is $-i \frac{e}{\pi \hbar} \hat{\vec{z}} \times (\vec{q} \vec{K}_0 + \omega \vec{K}_\perp)$, i.e., it is proportional to the ”driving force” acting on the vortices. Thus, the matrix relating the Josephson current to the electric field is proportional to the matrix relating the driving force acting on the vortices to the vortex current, or, explicitly,

$$\rho_{el} = \frac{2\pi \hbar}{(2e)^2} \sigma_{vor} \tag{42}$$

where $\rho_{el}$ is the electric resistivity matrix of the array [2]. This result can be simply concluded from Eq. (40). The right hand side of that equation is the Josephson current, divided by $2e$. The left hand side is proportional and perpendicular to the electric field. The electric field is then proportional and perpendicular to the Josephson current, with the proportionality constant being $\frac{2\pi \hbar}{(2e)^2} \eta$. The quantum Hall fluid of vortices manifests itself in electronic properties of the array – the longitudinal electric resistivity vanishes, and the Hall electric resistivity is quantized.
5. Conclusions

In the previous sections we presented a study of the transport of vortices in an array of Josephson junctions described by the Hamiltonian (1). In particular, we focused on a quantum Hall fluid formed by the vortices at appropriate values of $\frac{\bar{n}_v}{n_x}$. In this section we summarize the results of this study, and comment on a few open questions.

Our study was motivated by the analogy between Magnus force acting on vortices and Lorenz force acting on charges in a magnetic field. In this analogy, fluid density plays a role analogous to a magnetic field, and fluid current density plays a role analogous to an electric field. Quantum mechanics extends the analogy further: a fluid particle is found to play a role analogous to that of a flux quantum. This analogy motivates the search for a quantized Hall effect for the vortices. The vortices’ filling factor is identified with the ratio of the vortex density to the fluid density. This ratio is very small for superconducting films, and it is this smallness that motivates the study of the Josephson junction array. Due to the periodicity of the spectrum of the Hamiltonian (1) with respect to the parameter $n_x$, the effective filling factor becomes $\frac{\bar{n}_v}{n_x \text{ (mod 1)}}$, which can be made of order unity.

The dynamics of the vortices in a Josephson junction array was studied in section (3). It is found to be that of massive interacting charged particles under the effect of a magnetic field and a periodic potential. The magnetic field is $2\pi \hbar n_x$. The effect of the periodic potential is taken into account in an effective mass approximation, changing the mass from a bare mass to an effective band mass. The effective mass is exponentially large for $E_J \gg E_C$, and of the order of $\frac{\pi^2 \hbar^2}{4E_C}$ for $E_J \approx E_C$. Being interested in a phenomenon resulting from a motion of vortices, we obviously consider the latter regime. The mutual interaction between vortices consists of a velocity independent logarithmic interaction, whose strength is proportional to $E_J$, and a short ranged velocity–velocity interaction.

In view of the mapping of the dynamics of the vortices on that of massive interacting charged particles in a magnetic field, the existence of a quantum Hall fluid phase is to be expected. In Section (4) we examine some of the properties of that phase, but we leave unexplored some other important properties. Most notable among the latter are the regime
of vortices filling factors at which the quantum Hall fluid is the lowest energy state, and the energy gap for excitations above that fluid.

Our study of the quantum Hall fluid is performed by means of the Chern Simon Landau Ginzburg approach to the quantum Hall effect. When \( \frac{\tilde{n}_v}{n_x} = \frac{1}{\eta} \) with \( \eta \) being an even integer, the vortices Landau–Ginzburg action is found to have a saddle point corresponding to a quantum Hall fluid. By hierarchical construction such saddle points can be found for \( \frac{\tilde{n}_v}{n_x} = \frac{p}{q} \), with \( p, q \) being one even and one odd integer. The properties of the vortices quantum Hall fluid are studied within a quadratic expansion of the action around the corresponding saddle point. We find that the vortices conductivity matrix shows a typical QHE behavior, i.e., zero diagonal elements and quantized non–diagonal elements. However, we find the \( \vec{q}, \omega \rightarrow 0 \) limit of the current–current correlation functions in the ground state to be different from those of a typical quantum Hall state, due to the long range logarithmic interaction. In particular, the transverse current–current correlation function is predicted to behave like that of a superconductor, rather than an insulator. For large arrays (larger than an effective London penetration length) the vortex–vortex interaction is screened. Then, both the conductivity matrix and the correlation functions are expected to behave, in the \( dc \) limit, like those of a typical quantum Hall state.

A necessary condition for the quantum Hall fluid to be the lowest energy state is presumably that the ground state at \( n_x = 0 \) and \( \tilde{n}_v \neq 0 \) (infinite filling factor) is a superfluid of vortices, i.e., an insulator. The observation, by van der Zant et. al. [15], of a magnetic field tuned transition points at the regime of parameters in which this condition is satisfied, namely, \( E_J \approx E_C \) and \( 0.3 > \tilde{n}_v > 0.15 \). In this regime of parameters we expect the quantum Hall fluid to be the ground state at large filling factors, and the Abrikosov lattice to be the ground state at small filling factors. This expectation is based on the phase diagram of a two dimensional electron gas. For the latter, if the ground state at zero magnetic field is a Fermi liquid, then the ground state at large filling factors (\( \tilde{n}_v > 0.2 \)) is the quantum Hall fluid and the ground state at low filling factors is the Wigner lattice.
Appendix A – The Villain approximation and the duality transformation

The starting point of this appendix is the expression of the partition function as a path integral over the phase and number sets of variables \( \{\phi_i\}, \{n_i\} \), Eq. (2). Using the Villain approximation and the duality transformation we transform that path integral to a path integral over an integer 3-component vector field \( \mathbf{J}^{\text{vor}} \), describing the vortex density and current, and a real 3-component vector gauge field, \( \mathcal{K} \). The action in terms of \( \mathbf{J}^{\text{vor}} \) and \( \mathcal{K} \), to be derived below, is given by Eq. (3). The following derivation follows the method of Fazio et.al. [20]

In the Villain approximation the imaginary time integral is done in discrete steps, where the size of each step, denoted by \( \tau_0 \), is of the order of the inverse Josephson plasma frequency \( \omega_J \equiv \hbar^{-1} \sqrt{8E_JE_C} \). Each term in the Josephson energy part of the path integral is approximated by a Villain form (we put \( \hbar = 1 \) throughout the appendix, and restore its value in the final formula),

\[
e^{-\tau_0 E_J(1-\cos(\phi_{ij}-A_{ij}))} \approx \sum_{v_{ij}} \sum_{-\infty}^{\infty} e^{-\frac{1}{2} \tau_0 E_J(\phi_{ij}-A_{ij}+2\pi v_{ij})^2} \]

This approximation is valid for \( E_J \tau_0 > 1 \), and gets better as \( E_J \tau_0 \) gets larger. However, it retains the most important feature of the Josephson energy, the periodicity with respect to \( \phi \), for all values of \( E_J \tau_0 \). Altogether, then, the approximation we discuss holds for \( E_J > E_C \).

The significance of the field \( \mathbf{v}_i \) can be understood by noting that \( \mathbf{\Delta} \times \mathbf{v}_i \) describes the density of vortices [36]. As for the real variable \( p_{ij} \), as shown below, it describes the Josephson current along the bond \( ij \). Since the Josephson energy includes a sum over all lattice bonds, the Villain approximation introduces a variable \( p_{ij} \) to each lattice bond. Thus, we can regard \( p \) as a vector defined for each lattice site, such that \( p_{ix} \) corresponds to the bond \( i, x \) and \( p_{iy} \) corresponds to the bond \( i, y \). Similarly, the difference \( \phi_i - \phi_j \), the integral \( \int_{i}^{j} \mathbf{A} \cdot dl \) and the variables \( v_{ij} \) can be regarded as vectors \( \mathbf{\Delta} \phi_i, \mathbf{A}_i \) and \( \mathbf{v}_i \).

Next we apply the Poisson resummation formula to the \( n_i \) dependent part of the action.
By doing that we make the \( n_i \) variables real numbers rather than integers and add a time component to the integer–valued vector field \( \vec{v}_i \). The partition function then becomes,

\[
Z = \sum_{\{v_i(t)\}} \int D\{n_i(t)\} \int D\{\vec{p}_i(t)\} \int D\{\vec{\phi}_i(t)\}
\]

\[
\exp \left\{ \int_0^\beta dt \left[ \sum_i i n_i (\dot{\phi}_i + 2\pi v_0) - \frac{(2\pi)^2}{2} \sum_{ij} (n_i - n_x) \hat{C}^{-1}_{ij} (n_j - n_x) \right.ight.
\]

\[
- \sum_i \frac{\vec{p}_i^2}{2E} + i\vec{p}_i \cdot (\vec{\Delta}\phi_i - \vec{A}_i + 2\pi \vec{v}_i) \right\} \quad (44)
\]

where the path integral should be performed stepwise \([21]\). For the brevity of this expression we omitted the explicit time dependence of \( n_i, \phi_i, \vec{p}_i, \vec{v}_i \) in the stepwise integrated action. This form allows us to understand the physical significance of \( \vec{p}_i \). The only \( \vec{A} \)–dependent term in the action is \(-i\vec{p}_i \cdot \vec{A}_i \). The derivative of the Lagrangian with respect to \( \vec{A} \) is the current. Thus, \( \vec{p}_i \) is the Josephson current flowing through the site \( i \).

The path integral over the phase variables \( \phi_i(t) \) can now be performed. The phase \( \phi_i \) at the site \( i \) is coupled to the charge \( n_i \) (via the term \( in_i \dot{\phi}_i \)) and to the vector \( \vec{p}_i \) at the site \( i \) and its nearest neighbors. The integration over \( \phi_i \) yields conservation of charge constraint on the integration over \( n_i, \vec{p}_i \), in the form

\[
\Delta_i n_i + \vec{\Delta} \cdot \vec{p}_i = 0 \quad (45)
\]

where the definition \( \Delta_i n_i \equiv \frac{1}{\tau_0} [n_i(t + \tau_0) - n_i(t)] \) makes the difference operator \( \Delta \) a 3–vector.

The constraint \((45)\) is nothing but a discretized form of a two dimensional conservation of charge equation. Like the latter, it can be solved by defining a 3–vector field \( \mathcal{K} \) that satisfies,

\[
\epsilon^{\alpha\beta\mu} \Delta_{\alpha} \mathcal{K}_{i,\beta} = 2\pi p_{i,\mu} \quad (46)
\]

where \( p_i \equiv (n_i, p_{i,x}, p_{i,y}) \). The three components of \( \mathcal{K}_i \) are real, like those of \( p_i \). The definition \((46)\) of \( \mathcal{K}_i \) is not unique and it becomes unique only when a gauge is fixed. The partition function is, of course, independent of that gauge. The constrained path integral

31
over \( p_i \) is replaced now by path integrals over \( K_i \), constrained by the gauge condition. Here we choose to work in the Coulomb gauge, in which \( \vec{\Delta} \cdot \vec{K} = 0 \). In that gauge the partition function becomes,

\[
Z = \sum_{v_i(t)} \int D\{K_i\} \exp \int_0^\beta dt \left[ -\frac{e^2}{2\pi^2} \sum_{ij} (\vec{\Delta} \times \vec{K}_i) \hat{C}_{ij}^{-1} (\vec{\Delta} \times \vec{K}_j) \right. \\
\left. - \sum_i \frac{1}{8\pi^2 E_j} (\Delta_i \vec{K}_i^2 + \vec{\Delta} K_{0i}^2) + i \sum_i [\vec{\Delta} \times (K + K^{ext})]_i \cdot (v_i - \frac{1}{2\pi} A_i) \right]
\]

(47)

where \( \vec{K}^{ext} \) is defined by \( \vec{\Delta} \times \vec{K}^{ext} = 2\pi n_x \). We are now one step away from having an effective action for the vortices. The remaining step is an integration by parts of the last term in the action in (47). After performing that integration, the following action is obtained:

\[
S^{vor} = \int_0^\beta dt \sum_i \left\{ i(\rho^{vor}_i - \bar{n}_v) K_{0i} + i J^{vor}_i \cdot (\vec{K}_i + \vec{K}^{ext}) \\
\right. \\
\left. + \frac{e^2}{2\pi^2 \hbar^2} \sum_j (\vec{\Delta} \times \vec{K}_i) \hat{C}_{ij}^{-1} (\vec{\Delta} \times \vec{K}_j) + \frac{1}{8\pi^2 E_j} ((\Delta_i \vec{K}_i)^2 + (\vec{\Delta} K_{0i})^2) \right\}
\]

(48)

where the vortex 3–vector current \( J^{vor} \) is defined as \( J^{vor} = \Delta \times \nu \), the average density of vortices is given by \( \bar{n}_v = B \Phi_0 \) and the value of \( \hbar \) has been restored. Equation (48) is the starting point of the discussion in section (3).

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