The asymptotic strong Feller property does not imply the e-property for Markov-Feller semigroups

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Abstract

T. Szarek [Stud. Math. 189 (2008), § 4] have discussed the relationship between two important notions concerning Markov semigroups: the asymptotic strong Feller property and the e-property, asserting that the former property implies the latter one. In this very short note we rectify this issue exhibiting a simple example of a Markov-Feller semigroup enjoying the asymptotic strong Feller property, for which the e-property is not satisfied. (See also the comment on a connection between the asymptotic strong Feller property and the e-property by T. Szarek, D. Worm [ETDS 32 (2012), § 1]). Additionally we give a very simple example – in comparing with the one given by T. Szarek [Stud. Math. 189 (2008), § 4] – showing that also the converse implication does not hold.

1 Introduction

In recent years, a great deal of attention has been focused on properties of probability measures invariant with respect to Markov semigroups and in particular on the problem of the uniqueness of such measures. First important criterion for their uniqueness was given by R. Khas’minskii (see [8] or [1, Proposition 4.1.1, p. 42]), who proved it for a strongly Feller semigroup under the assumption of its irreducibility. Throughout the last half of a century, this criterion turned out to be very useful for applications in analysis of SDEs of various types. Recently, motivated by the study of properties of the 2D Navier-Stokes equations with degenerate stochastic forcing, M. Hairer and J. C. Mattingly have developed the Khas’minskii’s ideas introducing a brand new Feller-type condition. They have called it an asymptotic strong Feller condition and used it to show the uniqueness of invariant probability measures (see [3]). The uniqueness of invariant probability measures has also been examined by T. Szarek in [13], whose results, in comparing with the Hairer’s and Mattingly’s work, have taken advantage of the e-property and the overlapping supports condition instead of the asymptotic strong Feller property. His motivation for [13] was connected with his earlier achievement of proving the existence of an invariant probability measure for a discrete-time Markov semigroup acting on a Polish space and possessing the e-property, i.e. being an e-chain (see [12]). Let us mention that in

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that way a major breakthrough in the field was made, as earlier there was no handy and general enough criterion for the existence of an invariant measure. On the occasion of his research, T. Szarek has discussed the relation between the asymptotic strong Feller property and the e-property, claiming in [13, § 4] that asymptotic strong Feller operators possess an equicontinuous dual operator (i.e. the e-property). In a more recent paper, [14, § 1], its authors have formulated a weaker assertion: “it seems that all known examples of Markov processes with the asymptotic strong Feller property satisfy the e-property as well”. The aim of this note is to provide the missing example of a Markov process with the asymptotic strong Feller property which does not satisfy the e-property. Our example of such process may be built on every nontrivial normed vector space, no matter finite or infinite dimensional. The example is based on the construction of a discontinuous nonlinear solution to the Cauchy equation given by G. Hamel in [1] and on pieces of information about this solution which may be found in [11].

The rest of the paper is divided into two sections: in first of them we introduce our notation and terminology and in the second one, in Theorem 2 we present the example of a Markov-Feller semigroup with the asymptotic strong Feller property, for which the e-property does not hold. Actually we show that this example does not satisfy the asymptotic e-property, which is a variant of the e-property, slightly general than the original condition. Let us mention that the asymptotic e-property has been introduced in [7] and it seems that it may be a convenient tool to examine Feller as well as some non-Feller processes (for a recent study of non-Feller processes see [6]). In the last section we also put a converse example of a Markov-Feller semigroup with the e-property and without the asymptotic strong Feller property.

2 Preliminaries

As we want to make this note self-contained, we recall all the notions concerning properties of Markov semigroups which we will need in the sequel. The basic structure for our study is a nonempty metric space \((X, g)\) and the set \(\mathcal{M}_g(X)\) of all finite positive Borel measures on \(X\). These measures will be used to integrate functions from the set \(B_b(X)\) of all real bounded Borel functions and from its subset \(C_b(X)\) of all real bounded continuous functions. A family \((\mathcal{P}_t)_{t \geq 0}\) is called a Markov semigroup provided it is a semigroup of positively homogeneous and additive maps \(\mathcal{P}_t : \mathcal{M}_g(X) \to \mathcal{M}_g(X)\) which preserve measures, i.e. the equality \(\mathcal{P}_t \mu(X) = \mu(X)\) is fulfilled for every \(\mu \in \mathcal{M}_g(X)\) and \(t \geq 0\). A Markov semigroup \((\mathcal{P}_t)_{t \geq 0}\) is called a Markov-Feller semigroup provided there exists a semigroup \((\mathcal{U}_t)_{t \geq 0}\) of maps \(\mathcal{U}_t : B_b(X) \to B_b(X)\) such that \(\mathcal{U}_t C_b(X) \subset C_b(X)\) and \((\mathcal{U}_t)\) dual to \((\mathcal{P}_t)\). This last condition means that \(\int_X \mathcal{U}_t \varphi \, d\mu = \int_X \varphi \, d\mathcal{P}_t \mu\) for every \(t \geq 0, \mu \in \mathcal{M}_g(X)\) and \(\varphi \in B_b(X)\). A useful example of a Markov semigroup is a Perron-Frobenius semigroup \((\mathcal{P}_t)_{t \geq 0}\) corresponding to a given semigroup of Borel maps \((S_t : X \to X)_{t \geq 0}\), i.e. such a family that

\[
(1) \quad \mathcal{P}_t \mu(A) = \mu \left( S_t^{-1}(A) \right) \quad \text{for } t \geq 0, \mu \in \mathcal{M}_g(X), A \in B_X,
\]

where \(\delta_y\) denotes, as usual, the probabilistic measure concentrated at \(y \in X\) and \(B_X\) – the family of all Borel subsets of \(X\). We will use \(\delta_x\) later on. Notice that if a semigroup \((S_t : X \to X)_{t \geq 0}\) consists of continuous maps, then the corresponding Perron-Frobenius semigroup is a Markov-Feller one (see Chapter 12 of [9]).

Now we recall the stability-related concepts which we would like to compare. We start with two conditions concerning equicontinuity. In what follows \(L_b(X)\) denotes the set of all real Lipschitz-continuous bounded
The asymptotic strong Feller property does not imply the e-property

functions. A Markov semigroup \( (\mathcal{P}_t : \mathcal{M}_\varphi(X) \to \mathcal{M}_\varphi(X))_{t \geq 0} \) is said to have the \textit{e-property} at \( y \in X \) if for every \( f \in L_b(X) \), the family \( (\mathcal{U}_t f)_{t \geq 0} \) is equicontinuous at \( y \in X \), i.e.

\[
\lim_{x \to y} \sup_{t \geq 0} |\mathcal{U}_t f(y) - \mathcal{U}_t f(x)| = 0.
\]

It is said to have the \textit{asymptotic e-property} at \( y \in X \) if for every \( f \in L_b(X) \)

\[
(2) \quad \lim_{x \to y} \limsup_{t \to \infty} |\mathcal{U}_t f(y) - \mathcal{U}_t f(x)| = 0.
\]

We say that the semigroup \( (\mathcal{P}_t)_{t \geq 0} \) has the \textit{(asymptotic) e-property} provided the appropriate property holds at every \( y \in X \). If a Markov semigroup possess the e-property, the Markov chain it generates is called an \textit{e-chain}. This last name was popularized thanks to \cite{10}. Clearly, if a Markov semigroup \( (\mathcal{P}_t)_{t \geq 0} \) has the e-property at \( y \in X \), it has the asymptotic e-property at \( y \) as well. The converse implication does not hold, see \cite[Example 7]{7}.

Now we give, after \cite{3}, the definition of the asymptotic strong Feller property. Firstly we need to introduce some technical notions concerning pseudo-metrics. For a pseudo-metric \( \sigma \) on \( X \), by \( B_\sigma(x, \gamma) \) we denote the open \( \sigma \)-ball with the radius \( \gamma > 0 \) and the center \( x \in X \), i.e. \( B_\sigma(x, \gamma) = \{ y \in X : \sigma(x, y) < \gamma \} \). By \( \sigma^y \) we denote the function \( \sigma^y : X \to \mathbb{R} \) defined by the formula

\[
\sigma^y(x) = \sigma(x, y) \quad \text{for} \quad x \in X.
\]

Notice that for every \( y \in X \), \( \sigma^y \in Lip_\sigma(1) \), where

\[
\text{Lip}_\sigma(1) = \{ \varphi : X \to \mathbb{R} \mid \forall x, y \in X : |\varphi(x) - \varphi(y)| \leq \sigma(x, y) \}.
\]

An increasing sequence \( (\varrho_n : X \times X \to [0, \infty))_{n \in \mathbb{N}} \) of pseudo-metrics continuous with respect to \( (X, \varrho) \) is called a \textit{totally separating system of pseudo-metrics} if \( \varrho_n(x, y) \to 1 \) for every \( x \neq y \). Every \( \varrho \)-continuous pseudo-metric \( \sigma \) on \( X \) may be extended to \( \mathcal{M}_\varphi(X) \) by the formula

\[
\|\mu_1 - \mu_2\|_\sigma = \sup_{\varphi \in \text{Lip}_\sigma(1)} \int_X \varphi d(\mu_1 - \mu_2).
\]

This pseudo-metric on \( \mathcal{M}_\varphi(X) \) is well known under the name the \textit{Wasserstein pseudo-metric}.

A Markov semigroup \( (\mathcal{P}_t : \mathcal{M}_\varphi(X) \to \mathcal{M}_\varphi(X))_{t \geq 0} \) of operators defined for a metric space \( (X, \varrho) \) is said to have the \textit{asymptotic strong Feller property} at \( y \in X \) provided there are: a sequence \( (t_n)_{n \in \mathbb{N}} \) of positive reals and a totally separating system of pseudo-metrics \( (\varrho_n)_{n \in \mathbb{N}} \) on \( X \) such that

\[
\lim_{\gamma \to 0^+} \lim_{n \to \infty} \sup_{x \in B_\varrho(y, \gamma)} \|\mathcal{P}_{t_n} \delta_y - \mathcal{P}_{t_n} \delta_x\|_{\varrho_n} = 0.
\]

It is said to have the \textit{asymptotic strong Feller property} if the above condition holds at every \( y \in X \).

In practice (see e.g. \cite{3, 5, 16}), to verify that certain Markov semigroup satisfies the asymptotic strong Feller property one usually takes \( t_n = n \) and \( \varrho_n = 1 \wedge a_n \varrho \) for some fixed sequence \( (a_n) \) of positive reals with \( a_n \uparrow \infty \). Our choice of \( (t_n) \) and \( (\varrho_n) \) in the proof of Theorem 2 is similar to this one.

We end this section with providing some information on Hamel bases which will be needed in our proof of Theorem 2. A set \( \mathcal{B} \subset \mathbb{R} \) is called a \textit{Hamel basis} for \( \mathbb{R} \) if every element of \( \mathbb{R} \) is a unique finite rational linear combination of elements of \( \mathcal{B} \). The existence of a Hamel basis is guaranteed by the axiom of choice (see e.g. \cite[135, p. 132]{4}). The Hamel bases are useful for constructions of functions with nontypical properties, as the following observation, taken from \cite[Theorem 1.6, p.10]{11}, shows.
**Theorem 1.** If $\mathbb{B}$ is a Hamel basis for $\mathbb{R}$ and $g : \mathbb{B} \rightarrow \mathbb{R}$ is an arbitrary function then there exists a function $f_g : \mathbb{R} \rightarrow \mathbb{R}$ which satisfies the Cauchy equation (i.e. $f_g(x) + f_g(y) = f_g(x + y)$ for all $x, y \in \mathbb{R}$, i.e. $f_g$ is additive) and such that $f_g|_{\mathbb{B}} = g|_{\mathbb{B}}$.

We will use this theorem below. Notice that the extension $f_g : \mathbb{R} \rightarrow \mathbb{R}$ of a given function $g : \mathbb{B} \rightarrow \mathbb{R}$ is unique, which is guaranteed by [1] Theorem 1.5, p.9.

### 3 Examples of Markov-Feller semigroups

**Theorem 2.** If $(X, \| \cdot \|)$ is a normed vector space over $\mathbb{R}$ or $\mathbb{C}$, $X \neq \{0\}$ and $\varrho$ is a metric induced by $\| \cdot \|$ then there exists a Markov-Feller semigroup $(\mathcal{P}_t : \mathcal{M}_\varrho(X) \rightarrow \mathcal{M}_\varrho(X))_{t \geq 0}$ which does have the asymptotic strong Feller property and does not have the asymptotic e-property (and a fortiori the e-property) at every $y \in X$.

**Proof.** Let $\mathbb{B} \subset \mathbb{R}$ be a Hamel basis for $\mathbb{R}$. Fix any $b_1, b_2 \in \mathbb{B}$ and $g : \mathbb{B} \rightarrow \mathbb{R}$ such that $b_1 \cdot g(b_1) < 0 < b_2 \cdot g(b_2)$. Let $f_g$ be the additive extension of $g$, existing by Theorem [1]. Notice that since $\frac{f_g(b_1)}{|b_1|} < 0 < \frac{f_g(b_2)}{|b_2|}$, $f_g$ is nonlinear and hence (highly) discontinuous.

Let $(X, \| \cdot \|)$ be any fixed nontrivial (i.e. $\neq \{0\}$) normed vector space over $\mathbb{R}$ or $\mathbb{C}$ and let $\varrho$ be a metric induced by $\| \cdot \|$. Next, let

$$S_t(x) = e^{f_g(|b_1|^t)}x \quad \text{for } t \geq 0, x \in X.$$ 

As $f_g$ is additive, $(S_t : X \rightarrow X)_{t \geq 0}$ is a semigroup and hence it generates, via [1], the Perron-Frobenius semigroup $(\mathcal{P}_t)_{t \geq 0}$. Since $S_t$ is continuous for each $t \geq 0$, $(\mathcal{P}_t)_{t \geq 0}$ is a Markov-Feller semigroup, with the dual semigroup $(\mathcal{U}_t)_{t \geq 0}$ given by the formula

$$\mathcal{U}_t \varphi(x) = \varphi\left( e^{f_g(|b_1|^t)}x \right) \quad \text{for } t \geq 0, x \in X, \varphi \in B_b(X).$$

Firstly we shall show that $(\mathcal{P}_t)_{t \geq 0}$ satisfies the asymptotic strong Feller property with $t_n = n$ and $\varrho_n = 1 \wedge n \varrho$. As $\text{Lip}_{\varrho_n}(1) \subset \{ \varphi | \frac{\varphi}{n} \in \text{Lip}_{\varrho}(1) \}$, we obtain

$$\| \mathcal{P}_n \delta_y - \mathcal{P}_n \delta_x \|_{\varrho_n} \leq \sup_{\varphi \in \text{Lip}_{\varrho}(1)} n \| \mathcal{U}_n \varphi(x) - \mathcal{U}_n \varphi(y) \| \leq n e^{n f_g(|b_1|)} \varrho(x, y)$$

for every $x, y \in X$. Hence, since $f_g(|b_1|) < 0$, we have

$$\sup_{x \in B_{\varrho}(y, \gamma)} \| \mathcal{P}_n \delta_y - \mathcal{P}_n \delta_x \|_{\varrho_n} \leq \gamma n e^{n f_g(|b_1|)} \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

for every $y \in X$ and the asymptotic strong Feller property at arbitrarily chosen $y \in X$ follows.

Now we shall examine the asymptotic e-property. Let arbitrary $y \in X$ be fixed together with a sequence $(x_n)$ of points converging to $y$ such that $\|y\| \neq \|x_n\|$ for every $n \in \mathbb{N}$. We check that [2] does not hold for $f = \varrho^0$. As every neighbourhood of $y$ contains some $x_n$ and for every $n \in \mathbb{N}$ we have

$$\limsup_{t \rightarrow \infty} \left| \mathcal{U}_t \varrho^0(y) - \mathcal{U}_t \varrho^0(x_n) \right| \geq \limsup_{t \uparrow \infty, t \in \{m \frac{1}{|b_1|} | m \in \mathbb{N} \}} \left| \mathcal{U}_t \varrho^0(y) - \mathcal{U}_t \varrho^0(x_n) \right| = \|y\| - \|x_n\| \limsup_{m \rightarrow \infty} e^{m f_g(|b_2|)} = \infty,$$

the asymptotic e-property does not hold at every $y \in X$. \qed
Example 3. We end this note with the very simple example of a Markov-Feller semigroup, which, conversely to the semigroup from the above proof of Theorem \[2\] does have the e-property and does not satisfy the asymptotic strong Feller property. It is a simplest possible example: the semigroup of identity operators – please compare it with the example from \[13\ § 4\]. Fix a metric space \((X, \varrho)\) with \(l(X) \neq \emptyset\), where \(l(X)\) denotes the set of all limit points of \(X\), i.e. the set consisted of all such \(x \in X\) that every neighbourhood of \(x\) contains at least two distinct points: \(x\) and another one. Consider \((\mathcal{P}_t)_{t \geq 0}\) defined by the formula

\[
\mathcal{P}_t \mu = \mu \quad \text{for } t \geq 0, \mu \in \mathcal{M}_\varrho(X).
\]

As identity maps \(\mathcal{U}_t = id_{\mathcal{B}_\varrho(X)} \) \(t \geq 0\) form a semigroup dual to \((\mathcal{P}_t)_{t \geq 0}\), this last semigroup is a Markov-Feller one and satisfies the e-property. To check that the asymptotic strong Feller property does not hold at every point belonging to \(l(X)\), we fix any sequence \((t_n)\) of positive reals and a totally separating system of pseudo-metric \((\varrho_n)\). Next choose arbitrarily \(y \in l(X)\) and \(\gamma > 0\). We shall see that

\[
\limsup_{n \to \infty} \sup_{x \in \mathcal{B}_{\varrho}(y, \gamma)} \| \mathcal{P}_{t_n} \delta_y - \mathcal{P}_{t_n} \delta_x \|_{\varrho_n} \geq \frac{1}{2}.
\]

Indeed, since \(\varrho_n^y \in \text{Lip}_{\varrho_n}(1)\) and there exist \(z \in \mathcal{B}_{\varrho}(y, \gamma)\) and \(n_0 \in \mathbb{N}\) such that \(\varrho_n(z, y) \geq \frac{1}{2}\) for \(n \geq n_0\), we have

\[
\sup_{x \in \mathcal{B}_{\varrho}(y, \gamma)} \| \mathcal{P}_{t_n} \delta_y - \mathcal{P}_{t_n} \delta_x \|_{\varrho_n} \geq \sup_{x \in \mathcal{B}_{\varrho}(y, \gamma)} | \varrho_n^y(x) - \varrho_n^y(y) | \geq \frac{1}{2}
\]

for \(n \geq n_0\). Hence (4) follows. The lack of the asymptotic strong Feller property is not surprising here provided we take into account the smoothing character of this condition, together with the motivation for this notion and its relation to the strong Feller property (see \[2\]). Clearly our semigroup \((3)\) is not strong Feller as well (this is also not strange – see \[3\] Remark 3.10)).

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