MATRIX FACTORIZATIONS
FOR COMPLETE INTERSECTIONS
AND MINIMAL FREE RESOLUTIONS

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Abstract: We describe the asymptotic structure of minimal free resolutions over complete intersections. Suppose that $S$ is a regular local ring. Matrix factorizations of a non-zerodivisor $f$, as introduced in [Ei1], allow one to describe the minimal free resolutions, both over $S$ and over the hypersurface ring $R = S/(f)$, of modules that are high syzygies over $R$. We introduce the concept of a matrix factorization of a regular sequence $f_1, \ldots, f_c$, which allows us to describe the minimal free resolutions, both over $S$ and over the complete intersection ring $R = S/(f_1, \ldots, f_c)$, of modules that are high syzygies over $R$.

1. Introduction

A matrix factorization of a non-zerodivisor $f$ in a ring $S$ is a pair of square matrices $(d, h)$ such that $dh = hd = f \cdot I$, where $I$ is an identity matrix. Invented to study minimal free resolutions over hypersurfaces [Ei1], they have also found extensive applications in the representation theory of Cohen-Macaulay modules, singularity theory, knot theory, Hodge theory, cluster algebras, and mathematical physics. A complete intersection of codimension $c > 1$ can be treated as a family of hypersurfaces, but this idea fails to capture the important application to minimal free resolutions. We introduce the notion of a matrix factorization for a complete intersection that does allow us to analyze minimal free resolutions.

There has been a great deal of interest in the minimal free resolution of a finitely generated module $N$ over a complete intersection ring $R = S/(f_1, \ldots, f_c)$, where $S$ is a regular local ring with residue field $k$ and $f_1, \ldots, f_c$ is a regular sequence. Part of this interest comes from a connection with group cohomology: the group algebra of an elementary abelian $p$-group in characteristic $p > 0$ is a complete intersection. Perhaps with this motivation Tate [Ta] in 1957, described the minimal $R$-free resolution of $k$. In 1969 Shamash [Sh] constructed a non-minimal free resolution of $N$. In 1974 Gulliksen [Gu] showed that Ext$_R(N, k)$ can be regarded as a finitely generated graded module over a polynomial ring $\mathcal{R} = k[\chi_1, \ldots, \chi_c]$. The main numerical invariants of the minimal free resolution of $N$ are the Betti numbers $b_i^R(N) := \dim_k \text{Ext}_R(N, k)$, which are the ranks of the free modules in the resolution. Gulliksen’s result implies that the Poincar’e series $\sum_i b_i^R(N)x^i$ is rational, and the denominator divides $(1 - x^2)^c$, [Gu]. It follows that the even Betti numbers $b_{2i}^R(N)$ are eventually given by a polynomial in $i$, and similarly for the odd Betti numbers. In 1989 Avramov [Av] proved that the two polynomials have the same leading coefficient, and he also extended constructions from group cohomology to
the general case. In 1997 Avramov, Gasharov and Peeva [AGP] gave further restrictions on the Betti numbers, establishing in particular that the Betti sequence \( \{b_i^R(N)\}_{i \geq q} \) is either strictly increasing or constant for \( q \gg 0 \). Examples in [Ei1] show that, as with the Betti numbers, minimal free resolutions over a complete intersection can have intricate structure, but exhibit stable patterns when sufficiently truncated.

The theory of matrix factorizations entered the picture in 1980, when Eisenbud [Ei1] introduced them to describe the minimal free resolutions of modules that are high syzygies over hypersurface rings—the case \( c = 1 \). In Commutative Algebra matrix factorizations were used by Backelin-Herzog-Ulrich [BHU], Buchweitz-Greuel-Schreyer [BGS], Dao-Huneke [DH], Hochster [Ho], Huneke-Wiegand [HW], Knörrer [Kn].

Minimal free resolutions of high syzygies over a codimension two complete intersection were constructed by Avramov and Buchweitz in [AB], using special properties present only in that case. The structure of minimal free resolutions of high syzygies over complete intersections of higher codimension remained mysterious until now, though nonminimal resolutions have been known, from the work of Shamash [Sh], for over forty years.

We will now describe our new definition of matrix factorization and then outline the main results of this paper.

**What is a Matrix Factorization?**

If \( 0 \neq f \in S \) is an element in a commutative ring then a matrix factorization of \( f \) is a pair of maps of finitely generated free modules

\[
A_0 \xrightarrow{h} A_1 \xrightarrow{d} A_0
\]

such that the diagram

\[
\begin{array}{ccc}
A_1 & \xrightarrow{f} & A_0 \\
\downarrow{d} & & \downarrow{h} \\
A_0 & \xrightarrow{f} & A_1
\end{array}
\]

commutes or, equivalently:

\[
\begin{align*}
\text{d}h &= f \cdot \text{Id}_{A_0} \\
\text{h}d &= f \cdot \text{Id}_{A_1}.
\end{align*}
\]

We call such a pair \((d,h)\) a codimension 1 matrix factorization. If \( f \) is a non-zerodivisor and \( S \) is local, then the matrix factorization describes the minimal free resolutions of \( M := \text{Coker}(d) \) over the rings \( S \) and \( R := S/(f) \); if \( M \) has no direct summand then the resolutions are:

\[
\begin{align*}
0 &\longrightarrow A_1 \xrightarrow{d} A_0 \longrightarrow M \longrightarrow 0 \quad \text{over } S; \quad \text{and} \\
\cdots &\longrightarrow R \otimes A_1 \xrightarrow{d} R \otimes A_0 \xrightarrow{h} R \otimes A_0 \longrightarrow R \otimes A_1 \xrightarrow{d} R \otimes A_0 \longrightarrow M \longrightarrow 0 \quad \text{over } R.
\end{align*}
\]

Minimal free resolutions of all sufficiently high syzygies over a hypersurface ring are always of this form by [Ei1]. To extend the theory to higher codimensions, we make a new definition:
Definition 1.2. Codimension \(c\) Matrix Factorization: Let \(f_1, \ldots, f_c \in S\) be elements of a commutative ring, and set \(R = S/(f_1, \ldots, f_c)\). A matrix factorization \((d, h)\) with respect to \(f_1, \ldots, f_c\) is:

1. A pair of free finitely generated \(S\)-modules \(A_0, A_1\) with filtrations
   \[0 \subseteq A_s(1) \subseteq \cdots \subseteq A_s(c) = A_s,\quad \text{for } s = 0, 1,\]
   such that each \(A_s(p - 1)\) is a free summand of \(A_s(p)\);
2. A pair of maps \(d, h\) preserving filtrations,
   \[
   \bigoplus_{q=1}^c A_0(q) \xrightarrow{h} A_1 \xrightarrow{d} A_0,
   \]
   where we regard \(\bigoplus_q A_0(q)\) as filtered by the submodules \(\bigoplus_{q \leq p} A_0(q)\);
   such that, writing
   \[A_0(p) \xrightarrow{h_p} A_1(p) \xrightarrow{d_p} A_0(p)\]
   for the induced maps, the diagrams

   \[
   \begin{array}{c}
   A_1(p) \\
   \downarrow d_p
   \end{array}
   \begin{array}{c}
   A_0(p) \\
   \downarrow h_p \\
   A_1(p)
   \end{array}
   \begin{array}{c}
   A_0(p)
   \end{array}
   \begin{array}{c}
   \downarrow d_p
   \end{array}
   \begin{array}{c}
   A_1(p)/A_1(p - 1)
   \end{array}
   \]

   \[
   \begin{array}{c}
   A_1(p) \\
   \downarrow d_p
   \end{array}
   \begin{array}{c}
   A_0(p) \\
   \downarrow h_p \\
   A_1(p)
   \end{array}
   \begin{array}{c}
   A_0(p)
   \end{array}
   \begin{array}{c}
   \downarrow d_p
   \end{array}
   \begin{array}{c}
   A_1(p)/A_1(p - 1)
   \end{array}
   \]

   commute modulo \((f_1, \ldots, f_{p-1})\) for all \(p\); or, equivalently,
   
   (a) \(d_p h_p \equiv f_p \text{Id}_{A_0(p)} \mod (f_1, \ldots, f_{p-1}) A_0(p)\);
   
   (b) \(\pi_p h_p d_p \equiv f_p \pi_p \mod (f_1, \ldots, f_{p-1})(A_1(p)/A_1(p - 1))\),
   where \(\pi_p\) denotes the projection \(A_1(p) \longrightarrow A_1(p)/A_1(p - 1)\).

We define the module of the matrix factorization \((d, h)\) to be \(M := \text{Coker}(R \otimes d)\). We refer to modules of this form as matrix factorization modules.

In Section 13, we show that a homomorphism of matrix factorization modules induces a morphism of the whole matrix factorization structure; see Definition 13.1 and Theorem 13.2 for details. In Section 12 we show that our constructions yield functors to stable categories of Cohen-Macaulay modules.

For each \(1 \leq p \leq c\), we have a matrix factorization \((d_p, (h_1| \cdots | h_p))\) with respect to \(f_1, \ldots, f_p\), where \((h_1| \cdots | h_p)\) denotes the concatenation of the matrices \(h_1, \ldots, h_p\) and thus a matrix factorization module \(M(p) = \text{Coker}(S/(f_1, \ldots, f_p) \otimes d_p)\). This allows us to do induction on \(p\).

If \(S\) is local, then we call the matrix factorization minimal if \(d\) and \(h\) are minimal (that is, the image of each map is contained in the maximal ideal times the target).

Example 1.3. Let \(S = k[a, b, x, y]\) over a field \(k\), and consider the complete intersection \(R = S/(xa, yb)\). Let \(N = R/(x, y)\). The module \(N\) is a maximal Cohen-Macaulay \(R\)-module. The earliest syzygy of \(N\) that is a matrix factorization module is the third
syzygy $M$. We can describe the matrix factorization for $M$ as follows. After choosing a splitting $A_s(2) = A_s(1) \oplus B_s(2)$, we can represent the map $d$ as

$$
\begin{array}{ccc}
A_1(1) = B_1(1) = S^2 & \xrightarrow{\begin{pmatrix} a & 0 \\ y & x \end{pmatrix}} & A_0(1) = B_0(1) = S^2 \\
\downarrow & & \downarrow \\
B_1(2) = S^2 & \xrightarrow{\begin{pmatrix} 0 & -b \\ y & x \end{pmatrix}} & B_0(2) = S.
\end{array}
$$

The pair of maps

$$d_1 : A_1(1) \xrightarrow{\begin{pmatrix} a & 0 \\ y & x \end{pmatrix}} A_0(1) \quad \text{and} \quad h_1 : A_0(1) \xrightarrow{\begin{pmatrix} x & 0 \\ -y & a \end{pmatrix}} A_1(1)$$

is a matrix factorization for the element $xa$ since $d_1 h_1 = h_1 d_1 = xa \text{Id}$. The map $h_2 : A_0 = A_0(2) \rightarrow A_1 = A_1(2)$ is given by the matrix

$$h_2 = \begin{pmatrix}
0 & b & 0 \\
0 & 0 & 0 \\
x & 0 & b \\
-y & a & 0
\end{pmatrix}, \quad \text{and} \quad d_2 = \begin{pmatrix}
a & 0 & 0 & -b \\
y & x & 0 & 0 \\
0 & y & x & 0
\end{pmatrix}.$$ 

Hence

$$d_2 h_2 = \begin{pmatrix}
yb & 0 & 0 \\
0 & yb & 0 \\
0 & xa & yb
\end{pmatrix} \quad \text{and} \quad h_2 d_2 = \begin{pmatrix}
yb & xb & 0 & 0 \\
0 & 0 & 0 & 0 \\
- & - & - & - \\
xa & 0 & yb & 0 \\
0 & xa & 0 & yb
\end{pmatrix}.$$ 

Thus $d_2 h_2$ is congruent, modulo $(xa)$, to $yb \text{Id}$. Furthermore, condition (b) of Definition 1.2 is the statement that the two bottom rows in the latter matrix are congruent modulo $(xa)$ to $yb \pi_2$.

Our theory complements the classical theory of matrix factorizations, which has many applications: Starting with Kapustin and Li [KL], who followed an idea of Kontsevich, physicists discovered amazing connections with string theory — see [As] for a survey. A major advance was made by Orlov [Or1, Or3, Or4, Or5], who showed that matrix factorizations could be used to study Kontsevich’s homological mirror symmetry by giving a new description of singularity categories. Matrix factorizations have also proven useful for the study of cluster tilting [DH], Cohen-Macaulay modules and singularity theory [BGS, BHU, CH, Kn], Hodge theory [BFK], Khovanov-Rozansky homology [KR1, KR2], moduli of curves [PV2], quiver and group representations [AM, Av, KST, Re], and other topics, for example, [BDFIK, CM, DM, Dy, Ho, HW, Is, PV1, Se, Sei, Sh]. Orlov [Or2] and subsequent authors [Bu, BW, PV2, St] have studied modules over a complete intersection $S/(f_1, \ldots, f_c)$ by reducing to families of codimension 1 matrix factorizations.
over the hypersurface $\sum z_i f_i = 0$ in the projective space $\mathbb{P}^{c-1}_S$, where the $z_i$ are the homogeneous coordinates of $\mathbb{P}^{c-1}_S$. By contrast, our theory is focused on understanding the minimal free resolutions of such modules.

In the rest of the introduction we focus on the case when $S$ is a regular local ring and $R = S/(f_1, \ldots, f_c)$ is a complete intersection, although most of our results are proved in greater generality. We will keep the notation of Definition 1.2 throughout the introduction.

High Syzygies are Matrix Factorization Modules

The next result was the key motivation for our definition of a matrix factorization. A more precise version of this result is proved in Corollary 9.3.

**Theorem 1.4.** Let $S$ be a regular local ring with infinite residue field, and let $I \subset S$ be an ideal generated by a regular sequence of length $c$. If $N$ is a finitely generated $R := S/I$-module and $M$ is a sufficiently high syzygy of $N$ over $R = S/I$, then $M$ is the matrix factorization module of a minimal matrix factorization $(d, h)$ with respect to a generic choice of generators $f_1, \ldots, f_c$ of $I$. Moreover $d \otimes_R$ and $h \otimes_R$ are the first two differentials in the minimal free resolution of $M$ over $R$.

The meaning of “a sufficiently high syzygy” is explained in Section 7, where we introduce a class of $R$-modules that we call pre-stable syzygies and show that they have the property given in Theorem 1.4. Given an $R$-module $N$ we give in Corollary 9.3 a sufficient condition, in terms of $\text{Ext}_R(N, k)$, for the $r$-th syzygy module of $N$ to be pre-stable. We also explain more about the genericity condition. Over a local Gorenstein ring, we introduce the concept of a stable syzygy in Section 7 and discuss it in Section 10.

Minimal $R$-free and $S$-free Resolutions

Theorem 1.4 shows that in order to understand the asymptotic behavior of minimal free resolutions over the complete intersection $R$ it suffices to construct the resolutions of matrix factorization modules. This is accomplished by Construction 5.1 and Theorem 5.2.

The finite minimal free resolution over $S$ of a matrix factorization module is given by Construction 3.3 and Theorem 3.4. Here is an outline of the codimension 2 case: Let $(d, h)$ be a codimension 2 matrix factorization. We first choose splittings $A_s(2) = B_s(1) \oplus B_s(2)$. Since $d(B_1(1)) \subset B_0(1)$, we can represent the differential $d$ as

$$
\begin{align*}
B(1) : & \quad B_1(1) \xrightarrow{b_1} B_0(1) \\
& \quad \downarrow \psi_2 \\
B(2) : & \quad B_1(2) \xrightarrow{b_2} B_0(2),
\end{align*}
$$

which may be thought of as a map of two-term complexes $\psi_2 : B(2)[-1] \rightarrow B(1)$. This extends to a map of complexes $K(f_1) \otimes B(2)[-1] \rightarrow B(1)$, as in the following diagram:
Theorem 3.4 asserts that this is the minimal $S$-free resolution of the matrix factorization module $M = \text{Coker}(S/(f_1, f_2) \otimes d)$.

Strong restrictions on the finite minimal $S$-free resolution of a high syzygy $M$ over the complete intersection $S/(f_1, \ldots, f_c)$ follow from our results: for example, by Corollary 3.12 the minimal presentation matrix of $M$ must include $c - 1$ columns of the form
\[
\begin{pmatrix}
    f_1 & \cdots & f_{c-1} \\
    0 & \cdots & 0 \\
    \vdots & \vdots & \vdots \\
    0 & \cdots & 0
\end{pmatrix}
\]
for a generic choice of $f_1, \ldots, f_c$. For instance, in Example 1.3, the presentation matrix of $M$ is
\[
\begin{pmatrix}
    a & 0 & 0 & -b & 0 \\
    y & x & 0 & 0 & 0 \\
    0 & 0 & y & x & xa
\end{pmatrix},
\]
and the last column is of the desired type. There are numerical restrictions as well; see Corollary 9.15 and the remark following it.

Every maximal Cohen-Macaulay $S/(f_1)$-module is a pre-stable syzygy, but this is not true in higher codimension — one must go further back in the syzygy chain. This is not surprising, since every $S$-module of finite length is a maximal Cohen-Macaulay module over an artinian complete intersection, and it seems hopeless to characterize the minimal free resolutions of all such modules.

In Corollary 3.9 and Corollary 5.7 we get formulas for the Betti numbers of a matrix factorization module over $S$ and over $R$ respectively. Furthermore, the vector spaces $\text{Ext}_S^i(M, k)$ and $\text{Ext}_R^i(M, k)$ can be expressed as follows.

**Corollary 1.5.** Suppose that $f_1, \ldots, f_c$ is a regular sequence in a regular local ring $S$ with infinite residue field $k$, so that $R = S/(f_1, \ldots, f_c)$ is a local complete intersection. Let $M$ be the module of a minimal matrix factorization $(d, h)$ with respect to $f_1, \ldots, f_c$. Using notation as in Definition 1.2, for $s = 0, 1$, choose splittings $A_s(p) = A_s(p-1) \oplus B_s(p)$ for $s = 0, 1$, so $A_s(p) = \oplus_{1 \leq q \leq p} B_s(q)$. Set $B(p) = B_1(p) \oplus B_0(p)$, where we think of
as placed in homological degree $s$. There are decompositions

\[
\begin{align*}
\Ext_S(M,k) & \cong \bigoplus_{p=1}^{c} k\langle e_1, \ldots, e_{p-1} \rangle \otimes \Hom_S(B(p), k) \\
\Ext_R(M,k) & \cong \bigoplus_{p=1}^{c} k[\chi_p, \ldots, \chi_c] \otimes \Hom_S(B(p), k),
\end{align*}
\]

as vector spaces, where $k\langle e_1, \ldots, e_{p-1} \rangle$ denotes the exterior algebra and $k[\chi_p, \ldots, \chi_c]$ denotes the polynomial ring.

This is proved in Proposition 3.18 and Corollary 5.6, where we also explain how the given decompositions reflect certain natural actions of the exterior and symmetric algebras on the graded modules $\Ext_S(M,k)$ and $\Ext_R(M,k)$.

**Syzygies over intermediate quotient rings**

For each $0 \leq p \leq c$ set $R(p) := S/(f_1, \ldots, f_p)$. In the case of a codimension 1 matrix factorization $(d,h)$, one can use the data of the matrix factorization to describe two minimal free resolutions, as explained in (1.1). In the case of a codimension $c$ matrix factorization we construct the minimal free resolutions of a matrix factorization module over all $c+1$ rings $S = R(0), S/(f_1) = R(1), \ldots, S/(f_1, \ldots, f_c) = R(c)$. See Theorem 6.4 for the intermediate cases.

By Definition 1.2 a matrix factorization module $M$ with respect to the regular sequence $f_1, \ldots, f_c$ determines, for each $p \leq c$, a matrix factorization $R(p)$-module $M(p)$ with respect to $f_1, \ldots, f_p$. In the notation and hypotheses as in Theorem 1.4, Corollary 10.5 shows that

\[
M(p-1) = \Syz_{2}^{R(p-1)}(\Cosyz_{2}^{R(p)}(M(p))),
\]

where $\Syz(-)$ and $\Cosyz(-)$ denote syzygy and cosyzygy, respectively. Furthermore, Corollary 10.6 says that if we replace $M$ by its first syzygy, then all the modules $M(p)$ are replaced by their first syzygies:

\[
(\Syz_{1}^{R(p)}(M(p)))(p-1) = \Syz_{1}^{R(p-1)}(M(p-1)).
\]

Theorem 11.1 expresses the modules $M(p)$ as syzygies of $Y := \Cosyz_{c+1}^{R}(M)$ over the intermediate rings $R(p)$ as follows:

\[
\Syz_{c+1}^{R(p)}(Y) \cong M(p) \quad \text{for } p \geq 0.
\]

The package CompleteIntersectionResolutions, distributed with Macaulay2 [M2], starting with version 1.6, can compute examples of many of the constructions in this paper.

2. **Notation and Conventions**

Unless otherwise stated, in the rest of the paper **all rings are assumed commutative and Noetherian, and all modules are assumed finitely generated.**
If $S$ is a local ring with maximal ideal $\mathfrak{m}$ then a map of $S$-modules is called minimal if its image is contained in $\mathfrak{m}$ times the target.

To distinguish a matrix factorization for one element from the general concept, we will refer to the former as a codimension 1 matrix factorization or a hypersurface matrix factorization.

We will frequently use the following notation.

**Notation 2.1.** A matrix factorization

$$\begin{align*}
(d : A_1 \longrightarrow A_0, \ h : \oplus_{p=1}^c A_0(p) \longrightarrow A_1)
\end{align*}$$

with respect to $f_1, \ldots, f_c$ as in Definition 1.2 involves the following data:

- a ring $S$ over which $A_0$ and $A_1$ are free modules;
- for $1 \leq p \leq c$, the rings $R(p) := S/(f_1, \ldots, f_p)$, and in particular $R = R(c)$;
- for $s = 0, 1$, the filtrations $0 = A_s(0) \subseteq \cdots \subseteq A_s(c) = A_s$, preserved by $d$;
- the induced maps

$$A_0(p) \overset{h_p}{\longrightarrow} A_1(p) \overset{d_p}{\longrightarrow} A_0(p);$$

- the quotients $B_s(p) = A_s(p)/A_s(p - 1)$ and the projections $\pi_p : A_1(p) \longrightarrow B_1(p)$;
- the two-term complexes induced by $d$:

$$\begin{align*}
A(p) : & \begin{array}{c} A_1(p) \end{array} \overset{d_p}{\longrightarrow} A_0(p) \\
B(p) : & \begin{array}{c} B_1(p) \end{array} \overset{b_p}{\longrightarrow} B_0(p)
\end{align*}$$

- the modules

$$M(p) = \text{Coker}(R(p) \otimes d_p : R(p) \otimes A_1(p) \longrightarrow R(p) \otimes A_0(p));$$

and in particular, the matrix factorization module $M = M(c)$ of $(d, h)$.

We sometimes write $h = (h_1| \cdots |h_c)$. We say that the matrix factorization is trivial if $A_1 = A_0 = 0$.

If $1 \leq p \leq c$ then $d_p$ together with the maps $h_q$ for $q \leq p$, is a matrix factorization with respect to $f_1, \ldots, f_p$; we write it as $(d_p, h(p))$, where $h(p) = (h_1| \cdots |h_p)$. We call $(d_1, h_1)$ the codimension 1 part of the matrix factorization; $(d_1, h_1)$ is a hypersurface matrix factorization for $f_1$ over $S$ (it could be trivial). If $q \geq 1$ is the smallest number such that $A(q) \neq 0$ and $R' = S/(f_1, \ldots, f_{q-1})$, then writing $\cdot'$ for $R' \otimes \cdot$, the maps

$$b'_q : B_1(q)' \longrightarrow B_0(q)' \quad \text{and} \quad h'_q : B_0(q)' \longrightarrow B_1(q)'$$

form a hypersurface matrix factorization for the element $f_q \in R'$. We call it the top nonzero part of the matrix factorization $(d, h)$.

For each $0 \leq p \leq c$ set $R(p) := S/(f_1, \ldots, f_p)$. The matrix factorization module $M(p) = \text{Coker}(R(p) \otimes d_p)$ is an $R(p)$-module.

Next, we make some conventions about complexes.

We write $U[-a]$ for the shifted complex, with $U[-a]_i = U_{i+a}$ and differential $(-1)^a d$.

Let $(W, \partial^W)$ and $(Y, \partial^Y)$ be complexes. The complex $W \otimes Y$ has differential

$$\partial^W \otimes Y_q = \sum_{i+j=q} ((-1)^j \partial^W_i \otimes \text{Id} + \text{Id} \otimes \partial^Y_j).$$
A map of complexes $\gamma : W[a] \to Y$ is homotopic to 0 if there exists a map $\alpha : W[a + 1] \to Y$ such that

$$\gamma = \partial Y \alpha - \alpha \partial W[a + 1] = \partial Y \alpha - (-1)^a \alpha \partial W.$$ 

If $\varphi : W[-1] \to Y$ is a map of complexes, so that $-\varphi \partial W = \partial Y \varphi$, then the mapping cone $\operatorname{Cone}(\varphi)$ is the complex $\operatorname{Cone}(\varphi) = Y \oplus W$ with modules $\operatorname{Cone}(\varphi)_i = Y_i \oplus W_i$ and differential

$$\begin{pmatrix} Y_i & W_i \\ W_{i-1} & \varphi_{i-1} \end{pmatrix}.$$

If $f$ is an element in a ring $S$ then we write $K(f)$ for the two-term Koszul complex $f : \epsilon S \to S$, where we think of $\epsilon$ as an exterior variable. If $(W, \partial)$ is any complex of $S$-modules we write $K(f) \otimes W = \epsilon W \oplus W$; it is the mapping cone of the map $W \to W$ that is $(-1)^i f : W_i \to W_i$.

3. The minimal $S$-free resolution of a matrix factorization module

We will use the notation in 2.1 throughout this section. Suppose that $M$ is the module of a matrix factorization $(d, h)$ with respect to a regular sequence $f_1, \ldots, f_p$ in a local ring $S$. Theorem 3.4 expresses the minimal $S$-free resolution of $M$ as an iterated mapping cone of Koszul extensions, which we will now define in 3.1. We say that a complex $(U, d)$ is a left complex if $U_j = 0$ for $j < 0$; thus for example the free resolution of a module is a left complex.

**Definition 3.1.** Let $S$ be a ring. Let $B$ and $L$ be $S$-free left complexes, and let $\psi : B[-1] \to L$ be a map of complexes. Note that $\psi$ is zero on $B_0$. Denote $K(f_1, \ldots, f_p) = K(f_1, \ldots, f_p)$ the Koszul complex on $f_1, \ldots, f_p \in S$. An $(f_1, \ldots, f_p)$-Koszul extension of $\psi$ is a map of complexes $\Psi : K \otimes B[-1] \to L$ extending

$$K_0 \otimes B[-1] = B[-1] \xrightarrow{\psi} L$$

whose restriction to $K \otimes B_0$ is zero.

The next proposition shows that Koszul extensions exist in the case we will use.

**Proposition 3.2.** Let $f_1, \ldots, f_p$ be elements of a ring $S$. Let $L$ be a free resolution of an $S$-module $N$ annihilated by $f_1, \ldots, f_p$. Let $\psi : B[-1] \to L$ be a map from an $S$-free left complex $B$.

1. There exists an $(f_1, \ldots, f_p)$-Koszul extension of $\psi$.
2. If $S$ is local, the elements $f_i$ are in the maximal ideal, $L$ is minimal, and the map $\psi$ is minimal, then every Koszul extension of $\psi$ is minimal.

**Proof:** Set $K = K(f_1, \ldots, f_p)$, and let $\varphi : K \otimes L \to L$ be any map extending the identity map $S/(f_1, \ldots, f_p) \otimes N \to N$. The map $\varphi$ composed with the tensor product map $\text{Id}_K \otimes \psi$ is a Koszul extension, proving existence. For the second statement, note that if $\psi$ is minimal, then so is the Koszul extension we have constructed. Since any two
extensions of a map from a free complex to a resolution are homotopic, it follows that every Koszul extension is minimal. \[\square\]

We can now describe our construction of an $S$-free resolution of a matrix factorization module.

**Construction 3.3.** Let $(d, h)$ be a matrix factorization with respect to a regular sequence $f_1, \ldots, f_c$ in a ring $S$. Using notation as in 2.1, we choose splittings $A_s(p) = A_s(p-1) \oplus B_s(p)$ for $s = 0, 1$, so $A_s(p) = \oplus_{1 \leq q \leq p} B_s(q)$, and denote by $\psi_p$ the component of $d_p$ mapping $B_1(p)$ to $A_0(p-1)$.

- Set $L(1) := B(1)$, a free resolution of $M(1)$ with zero-th term $B_0(1) = A_0(1)$.
- For $p \geq 2$, suppose that $L(p-1)$ is an $S$-free resolution of $M(p-1)$ with zero-th term $L_0(p-1) = A_0(p-1)$. Let

$$\psi'_p : B(p)[-1] \longrightarrow L(p-1)$$

be the map of complexes induced by $\psi_p : B_1(p) \longrightarrow A_0(p-1)$, and let

$$\Psi_p : K(f_1, \ldots, f_{p-1}) \otimes B(p)[-1] \longrightarrow L(p-1)$$

be an $(f_1, \ldots, f_{p-1})$-Koszul extension. Set $L(p) = \text{Cone}(\Psi_p)$.

The following theorem implies that $H_0(L(p)) = M(p)$, so that the construction can be carried through to $L(c)$. Note that $L(c)$ has a filtration with successive quotients of the form $K(f_1, \ldots, f_{c-1}) \otimes B(p)$.

**Theorem 3.4.** With notation and hypotheses as in 3.3 the complex $L(p)$ is an $S$-free resolution of $M(p)$ for $p = 1, \ldots, c$. Moreover, if $S$ is local and $(d, h)$ is minimal, then the resolution $L(p)$ is minimal.

**Example 3.5.** Here is the case of codimension 2. After choosing splittings $A_s(2) = B_s(1) \oplus B_s(2)$, a matrix factorization $(d, h)$ for a regular sequence $f_1, f_2 \in S$ is a diagram of free $S$-modules.
where \( d \) has components \( b_1, b_2, \psi_2 \), and for some \( C, D \) we have

\[
\begin{align*}
 dh_1 &= f_1 \text{Id on } B_0(1) \\
 h_1 d &= f_1 \text{Id on } B_1(1) \\
 dh_2 &= f_2 \text{Id} + f_1 C \text{ on } B_0(1) \oplus B_0(2) \\
 \pi_2 h_2 d_2 &= f_2 \pi_2 + f_1 D \pi_2 \text{ on } B_1(1) \oplus B_1(2). 
\end{align*}
\] (3.6)

By Theorem 3.4, we may write an \( S \)-free resolution of the matrix factorization module \( M = \text{Coker}(S/(f_1, f_2) \otimes d) \) as

\[
\begin{align*}
 &\xrightarrow{h_1} B_1(1) \xrightarrow{b_1} B_0(1) \\
 &\xrightarrow{h_2} B_1(2) \xrightarrow{b_2} B_0(2) \\
 &\xrightarrow{-f_1} B_1(2) \xrightarrow{f_1} B_0(2) \\
 &\xrightarrow{b_2} B_0(2)
\end{align*}
\]

The homotopy for \( f_1 \) is shown with red arrows, and the homotopy for \( f_2 \) is not shown.

Before giving the proof of Theorem 3.4 we exhibit some consequences for the structure of modules that can be expressed as matrix factorization modules. We keep notation as in 2.1.

**Corollary 3.8.** With notation and hypotheses as in 3.3, if in addition \( S \) is local and the matrix factorization is minimal, then the minimal \( S \)-free resolution of \( M \) has a filtration by minimal \( S \)-free resolutions of the modules \( M(p) := \text{Coker}(S/(f_1, \ldots, f_p) \otimes d_p) \), whose successive quotients are the complexes

\[
\text{K}(f_1, \ldots, f_{p-1}) \otimes_S B(p).
\]

**Corollary 3.9.** With notation and hypotheses as in 3.3, if in addition \( S \) is local and the matrix factorization \((d, h)\) is minimal, then the Poincaré series of the module \( M \) of the matrix factorization \((d, h)\) is

\[
\mathcal{P}_M^S(x) = \sum_{1 \leq p \leq e} (1 + x)^{p-1} (x \text{rank } (B_1(p)) + \text{rank } (B_0(p))).
\]

**Corollary 3.10.** With notation and hypotheses as in 3.3, if \( M(p) \neq 0 \) then its projective dimension over \( S \) is \( p \), and \( f_{p+1} \) is a non-zerodivisor on \( M(p) \). If \( S \) is a local Cohen-Macaulay ring then the module \( M(p) \) is a maximal Cohen-Macaulay \( R(p) \)-module.
Proof: The resolution $L(p)$ has length $p$, and no module annihilated by a regular sequence of length $p$ can have projective dimension $< p$. The Cohen-Macaulay statement follows from this and the Auslander-Buchsbaum formula.

Suppose that $f_{p+1}$ is a zerodivisor on $M(p)$. Hence, $f_{p+1}$ is contained in a minimal prime $n$ over $\text{ann}_S(M(p))$. Since $f_1, \ldots, f_p$ annihilate $M(p)$, they are contained in $n$ as well. Therefore, the height of $n$ is $\geq p+1$. The projective dimension of $M(p)n$ over $S_n$ is less or equal to $p$, so it is strictly less than $\dim(S_n)$. Thus the minimal $S_n$-free resolution of $M(p)n$ is a complex of length $< \dim(S_n)$ and its homology $M(p)n$ has finite length. This is a contradiction by the New Intersection Theorem, cf. [PW].

**Corollary 3.11.** With notation and hypotheses as in 3.3, if in addition $S$ is local and the matrix factorization is minimal, then $M(p)$ has no $R(p)$-free summands.

Proof: If $M(p)$ had an $R(p)$-free summand, then with respect to suitable bases the minimal presentation matrix $R(p) \otimes d_p$ of $M(p)$ would have a row of zeros. Thus a matrix representing $R(p-1) \otimes d_p$ would have a row of elements divisible by $f_p$. Composing with $h_p$, we see that a matrix representing $R(p-1) \otimes d_ph_p$ would have a row of elements in $mf_p$. However $R(p-1) \otimes (d_ph_p) = f_pId$, a contradiction.

The following result shows that matrix factorization modules are quite special. Looking ahead to Corollary 9.3, we see that it can be applied to any $S$ module that is a sufficiently high syzygy over $R$.

**Corollary 3.12.** With notation and hypotheses as in 3.3, suppose in addition that $S$ is local and that the matrix factorization $(d,h)$ is minimal, and let $n = \sum_p \text{rank} B_0(p)$, the rank of the target of $d$. In a suitable basis, the minimal presentation matrix of the matrix factorization module $M$ consists of the matrix $d$ concatenated with an $(n \times \sum_p (p-1) \text{rank} B_0(p))$-matrix that is the direct sum of matrices of the form

$$
(f_1 \ldots f_{p-1}) \otimes \text{Id}_{B_0(p)} = 
\begin{pmatrix}
  f_1 & \ldots & f_{p-1} & 0 & \cdots & 0 & \cdots & 0 \\
  0 & \cdots & 0 & f_1 & \cdots & f_{p-1} & \cdots & 0 \\
  0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 \\
  \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
  0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & f_1 & \cdots & f_{p-1}
\end{pmatrix}
$$

We remark that a similar property holds for all matrices of the differential in the minimal free resolution of $M$.

Proof: In the notation of Construction 3.3, the given direct sum is the part of the map $L_1(c) \rightarrow L_0(c)$ that corresponds to

$$
\bigoplus_p (K(f_1, \ldots, f_{p-1}))_1 \otimes B_0(p) \rightarrow \bigoplus_p B_0(p).
$$

Theorem 3.4 and Corollary 3.11 allow us to express the Betti numbers of a matrix factorization module in terms of the ranks of the modules $B_s(p)$. Recall that if $S$ is a local ring with residue field $k$ then the Betti numbers of a module $N$ over $S$ are $b^S_i(N) = \dim_k(\text{Tor}^S_i(N,k))$. They are often studied via the Poincaré series:

$$
\mathcal{P}_N^S(x) = \sum_{i \geq 0} b^S_i(N) x^i.
$$
Corollary 3.9 makes it worthwhile to ask whether there are interesting restrictions on the ranks of the $B_s(p)$. Here is a first result in this direction:

**Corollary 3.13.** With notation and hypotheses as in 3.3, suppose in addition that $S$ is local and Cohen-Macaulay and that the matrix factorization $(d,h)$ is minimal. If $B_1(p) = 0$ for some $p$, then $B_1(q) = B_0(q) = 0$ for all $q \leq p$.

**Proof:** Suppose that $B_1(p) = 0$. If $B_0(p) \neq 0$ then $M(p)$ would have a free summand, contradicting Corollary 3.11, so $B_0(p) = 0$ as well. It follows that $h_p$ restricts to a map $A_0(p-1) \rightarrow A_1(p-1)$, and thus $M(p-1)$ is annihilated by $f_p$. However, if $M(p-1) \neq 0$ then by Corollary 3.10 it would be a maximal Cohen-Macaulay module over the ring $R(p-1)$, and this is a contradiction. Thus $M(p-1) = 0$, so $B_s(q) = 0$ for $q \leq p$. □

**Example 3.14.** Let $S = k[x,y,z]$ and let $f_1, f_2$ be the regular sequence $xz, y^2$. We give an example of a matrix factorization with respect to $f_1, f_2$ such that $B_1(2) \neq 0$, but $B_0(2) = 0$. If

$$B_1(1) = S^2 \xrightarrow{(z, -y)} B_0(1) = S^2$$

$$B_1(2) = S \xrightarrow{0} B_0(2) = 0,$$

and

$$h_1 = \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \quad \text{and} \quad h_2 = \begin{pmatrix} 0 & 0 \\ -y & 0 \\ x & y \end{pmatrix},$$

then $(d,h)$ is a matrix factorization.

In the case of matrix factorizations that come from high syzygies (stable matrix factorizations) Corollary 3.13 can be strengthened further: $B_0(p) = 0$ implies $B_1(p) = 0$ as well; see Corollary 9.14. This is not the case in general, as the above example shows.

**Proof of Theorem 3.4:** The minimality statement follows at once from the construction and Proposition 3.2(2). Thus it suffices to prove the first statement.

Note that $d_1 = b_1$. The equations in the definition of a matrix factorization imply in particular that $h_1b_1 = b_1h_1 = f_1 \text{Id}$, so $b_1$ is a monomorphism. Note that $\text{Coker}(d_1)$ is annihilated by $f_1$. Thus $L(1) = B(1)$ is an $S$-free resolution of $M(1) = \text{Coker}(R(1) \otimes d_1) = \text{Coker}(d_1)$.

To complete the proof we do induction on $p$. By induction hypothesis

$$L(p-1) : \cdots \rightarrow L_1(p-1) \rightarrow L_0(p-1)$$

is a free resolution of $M(p-1)$. Since $L_0(p-1) = A_0(p-1)$, the map $\psi_p$ defines a morphism of complexes $\psi'_p : B(p)[-1] \rightarrow L(p-1)$ and thus a mapping cone $13$
\[ \cdots \longrightarrow L_2(p-1) \longrightarrow L_1(p-1) \longrightarrow L_0(p-1) \longrightarrow \]

\[ B_1(p) \xrightarrow{\psi_p} B_0(p). \]

To simplify the notation, denote by \( K \) the Koszul complex \( K(f_1, \ldots, f_{p-1}) \) of \( f_1, \ldots, f_{p-1} \), and write \( \kappa_i : \wedge^i S^{p-1} \longrightarrow \wedge^{i-1} S^{p-1} \) for its differential. Also, set \( B_s := B_s(p) \) and \( B : B_1 \xrightarrow{b_1} B_0 \).

Since \( M(p-1) \) is annihilated by \( (f_1, \ldots, f_{p-1}) \), Proposition 3.2 shows that there exists a Koszul extension \( \Psi_p : K \otimes B[-1] \longrightarrow L(p-1) \) of \( \psi'_p \). Let \( (L(p), e) \) be the mapping cone of \( \Psi_p \), and note that the zero-th terms of \( L(p) \) is \( L_0 = L_0(p-1) \oplus B_0 = A_0(p) \). We will show that \( L(p) \) is a resolution of \( M(p) \).

We first show that \( H_0(L(p)) = \text{Coker}(e_1) = M(p) \). If we drop the columns corresponding to \( B_1 \) from a matrix for \( e_1 \) we get a presentation of \( M(p-1) \oplus (R(p-1) \otimes B_0(p-1)) \), so \( \text{Coker}(e_1) \) is annihilated by \( (f_1, \ldots, f_{p-1}) \). Moreover, the map \( h_p : A_0(p) \longrightarrow A_1(p) \subset L_1(p) \) defines a homotopy for multiplication by \( f_p \) modulo \( (f_1, \ldots, f_{p-1}) \), so \( \text{Coker}(e_1) \) is annihilated by \( f_p \) as well. Thus \( \text{Coker}(e_1) = \text{Coker}(R(p) \otimes e_1) = M(p) \) as required.

We next analyze the homology of the complex \( K \otimes B \). It is isomorphic to \( B \otimes K \), which is the mapping cone of the map \( (-1)^i b_1 \otimes \text{Id} : B_1[-1] \otimes K_i \longrightarrow B_0 \otimes K_i \), so there is a long exact sequence

\[ \cdots \longrightarrow H_i(K \otimes B_1) \longrightarrow H_i(K \otimes B_0) \longrightarrow H_i(K \otimes B) \longrightarrow H_{i-1}(K \otimes B_1) \longrightarrow \cdots . \]

Since \( K \otimes B_0 \) is a resolution of \( R(p-1) \otimes B_0 \), we see that \( H_i(K \otimes B_0) = 0 \) for \( i > 1 \) and there is a four-term exact sequence

\[ 0 \longrightarrow H_1(K \otimes B) \longrightarrow R(p-1) \otimes B_1 \xrightarrow{R(p-1) \otimes b_1} R(p-1) \otimes B_0 \longrightarrow H_0(K \otimes B) \longrightarrow 0. \]

Since \( L(p) \) is the mapping cone of \( \Psi_p \), we have a long exact sequence in homology of the form

\[ \cdots \longrightarrow H_i(L(p-1)) \longrightarrow H_i(L(p)) \longrightarrow H_i(K \otimes B) \xrightarrow{\Psi_p} H_{i-1}(L(p-1)) \longrightarrow \cdots , \]

so from the vanishing of the \( H_i(K \otimes B) \) for \( i > 1 \) we see that \( H_i(L(p)) = 0 \) for \( i > 1 \).

It remains to prove only that \( H_1(L(p)) = 0 \), or equivalently that the map

\[ \Psi_p : H_1(K \otimes B) \longrightarrow H_0(L(p-1)) = M(p-1) \]

is a monomorphism. From the four-term exact sequence above we see that

\[ H_1(K \otimes B) = \text{Ker}(R(p-1) \otimes b_1). \]

Also note that by construction the map

\[ \Psi_p : \text{Ker}(R(p-1) \otimes b_1) \longrightarrow H_0(L(p-1)) = \text{Coker}(R(p-1) \otimes d_{p-1}) \]

is induced by

\[ \psi_p : R(p-1) \otimes B_1(p) \longrightarrow R(p-1) \otimes A_0(p-1). \]

Since \( \mathcal{A}_0(p-1) = \mathcal{A}_0(p-1), \) the proof is finished by the next Lemma 3.15, which we will use again in Section 5. \( \square \)
Lemma 3.15. With notation and hypotheses as in Construction 3.3, \( \psi_p \) induces a monomorphism from \( \text{Ker}(R(p-1) \otimes b_p) \) to \( \text{Coker}(R(p-1) \otimes d_{p-1}) \).

Proof: To simplify notation we write \( \overline{\cdot} \) for \( R(p-1) \otimes \cdot \). Consider the diagram:

\[
\begin{array}{ccc}
u \in \overline{A}_1(p) & \xrightarrow{\overline{\psi}_p} & \overline{A}_0(p) \\
& \overline{d}_{p-1} & \\
v \in \overline{B}_1(p) & \xrightarrow{\overline{b}_p} & \overline{B}_0(p).
\end{array}
\]

We must show that if \( v \in \text{Ker} \overline{b}_p \) and \( \overline{\psi}_p(v) = \overline{d}_{p-1}(u) \) for some \( u \in \overline{A}_1(p-1) \), then \( v = 0 \).

Write \( \overline{\pi}_p \) for the projection of \( \overline{A}_1(p) = \overline{A}_1(p-1) \oplus \overline{B}_1(p) \) to \( \overline{B}_1(p) \), and note that \( \overline{d}_p \) is the sum of the three maps in the diagram above. Our equations say that \( d_p(-u,v) = 0 \).

By condition (b) in Definition 1.2,

\[
f_p v = f_p \overline{\pi}_p(-u,v) = \overline{\pi}_p \overline{d}_p(-u,v) = 0.
\]

Since \( f_p \) is a non-zerodivisor in \( R(p-1) \), it follows that \( v = 0 \).

Recall the following well-known result.

Lemma 3.16. If \( W \) is a complex of \( S \)-modules with a homotopy \( \sigma \) for multiplication by an element \( f \in S \), and \( U \) is any complex of \( S \)-modules, then \( W \otimes U \) has a homotopy for multiplication by \( f \) defined by

\[
\tau_{ij} : W_i \otimes U_j \rightarrow \bigoplus_{q+r=i+j+1} W_q \otimes U_r
\]

whose components are 0 except for

\[(-1)^j \sigma_i \otimes \text{Id} : W_i \otimes U_j \rightarrow W_{i+1} \otimes U_j.\]

Construction 3.17. With notation and hypotheses as in 3.3, the homotopies for multiplication by the \( \sigma_i \) on the minimal free resolution \( L := L(c) \) of \( M \) anticommute up to homotopy; that is, if \( \sigma_i : L \rightarrow L[-1] \) denotes a homotopy for \( f_i \) on \( L \), then \( \sigma_i^2 : L \rightarrow L[-2] \) and \( \sigma_i \sigma_j - \sigma_j \sigma_i : L \rightarrow L[-2] \) are homotopic to 0. Thus these maps induce an action of the exterior algebra \( k \langle e_1, \ldots, e_r \rangle \) on the graded vector space \( \text{Tor}_S^i(M,k) \) in which multiplication by \( e_i \) takes \( \text{Tor}_j^S(M,k) \) to \( \text{Tor}_{j+i}^S(M,k) \).

On the other hand, by Lemma 3.16, the standard homotopies for \( f_1, \ldots, f_{p-1} \) on the Koszul complex \( K(f_1, \ldots, f_{p-1}) \) extend to homotopies on \( K(f_1, \ldots, f_{p-1}) \otimes S B(p) \). The underlying free module of \( K \) is the exterior algebra on generators \( e_i \) corresponding to the \( f_i \). Set \( B(p) = B_0(p) \oplus B_1(p) \), and thus we get the module \( k \langle e_1, \ldots, e_{p-1} \rangle \otimes S B(p) \).

The next results explain the relation between these exterior algebra actions.

Proposition 3.18. With notation and hypotheses as in 3.3, we have

\[
\text{Tor}_S^i(M,k) \cong \bigoplus_{p=1}^c k \langle e_1, \ldots, e_{p-1} \rangle \otimes S B(p)
\]
as vector spaces in such a way that, for \( i \leq p-1 \), the action of \( e_i \) on \( \text{Tor}^S(M,k) \) preserves the summand \( k\langle e_1, \ldots, e_{p-1} \rangle \otimes_S B(p) \) and acts on it via the action on the first factor.

Taking duals, we see that

\[
\text{Ext}^i_S(M,k) \cong \bigoplus_{p=1}^\infty \text{Hom}_k(k\langle e_1, \ldots, e_{p-1} \rangle \otimes_S B(p), k)
\]

Further, the \( p \)-th term on the right hand side is isomorphic to

\[
k\langle e_1, \ldots, e_{p-1} \rangle \otimes_k \text{Hom}_S(B(p), k)
\]

up to a shift in degree. This establishes the first formula in Corollary 1.5.

For the proof of Proposition 3.18 we will use the following well-known result.

**Lemma 3.19.** If \( \alpha, \varphi : W[-1] \to Y \) are maps of complexes over \( S \) that are homotopic by a homotopy \( \theta : W \to Y \), then the mapping cones \( \text{Cone}(\alpha) \) and \( \text{Cone}(\varphi) \) are isomorphic by the isomorphism

\[
\begin{pmatrix}
\text{Id} & \theta \\
0 & \text{Id}
\end{pmatrix},
\]

and it induces the identity on \( Y \subset \text{Cone}(\alpha) \) and on \( W = \text{Cone}(\alpha)/Y \). \( \square \)

**Lemma 3.20.** Suppose that \( \psi : W[-1] \to Y \) is a map of complexes over \( S \), and \( \nu \) is a homotopy for multiplication by \( f \in S \) on \( Y \).

(1) The map

\[
\Psi_i = ((-1)^i-1 \nu_{i-2} \psi_{i-2}, \psi_{i-1}) : eW_{i-1} \oplus W_i \to Y_{i-1}
\]

defines an \( f \)-Koszul extension of \( \psi \). Thus, we have the following diagram of the complex \( \text{Cone}(\Psi) \), where we have denoted \( \partial \) and \( \delta \) the differentials in \( Y \) and \( W \), respectively:

\[
\begin{array}{cccccccc}
\partial & Y_{i+1} & \to & \partial & Y_i & \to & \partial & Y_{i-1} & \to \\
(-1)^i \psi & \downarrow & \nu_{i} \psi & \downarrow & (-1)^{i-1} \psi & \downarrow & (-1)^i f & \downarrow & (-1)^{i-1} f \\
\delta & W_{i+1} & \to & \delta & W_i & \to & \delta & W_{i-1} & \to \\
\delta & eW_i & \to & \delta & eW_{i-1} & \to & \delta & eW_{i-2} & \to \\
\end{array}
\]

(2) If \( -\tau \) is a homotopy for \( \nu^2 \sim 0 \) on \( Y \), then

\[
\xi := (\tau \psi : eW_{i-1} \to Y_{i+1}, (-1)^i : W_i \to eW_i, \nu : Y_i \to Y_{i+1})
\]

is a homotopy for multiplication by \( f \) on the mapping cone \( \text{Cone}(\Psi) \). We have the following diagram of that homotopy:
The restriction of the homotopy $\xi$ on $Y$ is the given homotopy $\nu$.

**Proof:** (1) and (2) are proven by “chasing” the given diagrams, using the formulas

$$
\partial \psi = \psi \delta \quad \text{(since $\psi$ is a map of complexes)}
$$

$$
\nu^2 + \partial \tau + \tau \partial = 0 \quad \text{(by the definition of $\tau$)}
$$

$$
\nu^2 \psi + \partial \tau \psi + \tau \psi \delta = 0 \quad \text{(by the above two formulas)}.
$$

**Proof of Proposition 3.18**

The result follows from the iterative Construction 3.3 of $L(c)$ together with Lemmas 3.19 and 3.20. It suffices to show the desired property at step $c$.

Let $K = K(f_1, \ldots, f_{c-1})$ be the Koszul complex, and note that the underlying free module of $K$ is the exterior algebra on generators $e_i$ corresponding to the $f_i$. For $i \leq c-1$, set

$$
K(i) := K(f_1) \otimes \cdots \otimes \tilde{K}(f_i) \otimes \cdots \otimes K(f_{c-1}).
$$

Set $B := B(c)$ and $C := L(c-1)$. Note that for each $i \leq c-1$ we have a homotopy $\xi_i$ for $f_i$ on $C$.

Consider the map $\Psi_c : K \otimes B[-1] \rightarrow C$ from Construction 3.3, and denote $\nu(i) : K(i) \otimes B[-1] \rightarrow C$ the restriction of $\Psi_c$. There is an isomorphism $K \cong K(f_i) \otimes K(i)$ and thus for each $2 \leq i \leq c-1$ we can make an $f_i$-Koszul extension

$$
\alpha_i : K \otimes B[-1] \rightarrow C
$$

of the map $\nu(i)$ using $\xi_i$ and the formula in Lemma 3.20(1). Using Lemma 3.20(2) we can produce a homotopy $\sigma_i$ for $f_i$ on $\text{Cone}(\alpha_i) \cong L(c)$ that induces multiplication by $e_i$ on $(\text{Cone}(\alpha_i)/C) \otimes k \cong K \otimes B[-1] \otimes k$.

We can choose the minimal free resolution $L(c)$ to be $\text{Cone}(\alpha_1)$. The maps $\alpha_i$ are homotopic because they are extensions of the same map $\psi'_c : B(c)[-1] \rightarrow C$ in the notation of Construction 3.3. Therefore, by Lemma 3.19 there is an isomorphism $\text{Cone}(\alpha_1) \rightarrow \text{Cone}(\alpha_i)$ such that the composition

$$
K \otimes B[-1] \cong \text{Cone}(\alpha_1)/C \rightarrow \text{Cone}(\alpha_i)/C \cong K \otimes B[-1]
$$

is the identity. Using this isomorphism to transport all the $\sigma_i$ to $\text{Cone}(\alpha_1)$, we see that there is a family of homotopies $\sigma_i$ for $f_i$ on $L(c) = \text{Cone}(\alpha_1)$ such that $\sigma_i$ induces multiplication by $e_i$ on $k\langle e_1, \ldots, e_{p-1} \rangle \otimes_S B(c) \subset \text{Tor}^S(M, k)$. 

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4. Resolutions with a surjective CI operator

We begin by recalling the definition of CI operators. Suppose that $f_1, \ldots, f_c \in S$ is a regular sequence and $(V, \partial)$ is a complex of free modules over $R = S/(f_1, \ldots, f_c)$. Suppose that $\tilde{V}$ is a lifting of $V$ to $S$, that is, a sequence of free modules $\tilde{V}_i$ and maps $\tilde{\partial}_i : \tilde{V}_{i+1} \rightarrow \tilde{V}_i$ such that $\partial = R \otimes \tilde{\partial}$. Since $\partial^2 = 0$ we can choose maps $\tilde{t}_j : \tilde{V}_{i+1} \rightarrow \tilde{V}_i$, where $1 \leq j \leq c$, such that

$$\tilde{\partial}^2 = \sum_{j=1}^c f_j \tilde{t}_j.$$ 

We set

$$t_j := R \otimes \tilde{t}_j.$$ 

Since

$$\sum_{j=1}^c f_j \tilde{t}_j \tilde{\partial} = \tilde{\partial}^3 = \sum_{j=1}^c f_j \tilde{\partial} \tilde{t}_j,$$

and the $f_i$ form a regular sequence, we see that each $t_j$ commutes with $\tilde{\partial}$, and thus the $t_j$ define a map of complexes $\tilde{V}[-2] \rightarrow V$, [Ei1, 1.1]. In the case $c = 1$, we have $\tilde{\partial}^2 = f_1 \tilde{t}_1$ and we sometimes write $\tilde{t}_1 = \frac{1}{f_1} \tilde{\partial}^2$ and call it the lifted CI operator.

[Ei1, 1.2 and 1.5] shows that the operators $t_j$ are, up to homotopy, independent of the choice of liftings. They are called the CI operators (sometimes called Eisenbud operators) associated to the sequence $f_1, \ldots, f_c$.

We next recall the definition of higher homotopies and the Shamash construction. The version for a single element is due to Shamash [Sh]; [Ei2] treats the more general case of a collection of elements.

**Definition 4.1.** Let $f_1, \ldots, f_c \in S$, and $G$ be a free complex of $S$-modules. We denote $a = (a_1, \ldots, a_c)$, where each $a_i \geq 0$ is an integer, and set $|a| = \sum a_i$. A system of higher homotopies $\sigma$ for $f_1, \ldots, f_c$ on $G$ is a collection of maps $\sigma_a : G \rightarrow G[-2|a| + 1]$ of the underlying modules such that the following three conditions are satisfied:

1. $\sigma_0$ is the differential on $G$.
2. For each $1 \leq i \leq c$, the map $\sigma_0 \sigma_{e_i} + \sigma_{e_i} \sigma_0$ is multiplication by $f_i$ on $G$, where $e_i$ is the $i$-th standard vector.
3. If $a$ is a multi-index with $|a| \geq 2$, then $\sum_{b+s=a} \sigma_b \sigma_s = 0$.

A system of higher homotopies $\sigma$ for one element $f \in S$ on $G$ consists of maps $\sigma_j : G \rightarrow G[-2j + 1]$ for $j = 0, 1, \ldots$, and will be denoted $\{\sigma_j\}$.

**Proposition 4.2.** [Ei2, Sh] If $G$ is a free resolution of an $S$-module annihilated by elements $f_1, \ldots, f_c \in S$, then there exists a system of higher homotopies on $G$ for $f_1, \ldots, f_c$.

For the reader’s convenience we present a short proof following [Sh]:

**Proof:** It is well-known that homotopies $\sigma_{e_i}$ satisfying (2) in Definition 4.1 exist. Equation (3) in 4.1 can be written as

$$d\sigma_a = - \sum_{\substack{b+s=a \\text{and} \ b \neq 0}} \sigma_b \sigma_s.$$
As \( G \) is a free resolution, in order to show by induction on \( a \) and on the homological degree that the desired \( \sigma_a \) exists, it suffices to show that the right-hand side is annihilated by \( d \). Indeed,

\[
- \sum_{b \neq 0} (d \sigma_b)\sigma_a = \sum_{b \neq 0} \sum_{m \neq b} \sigma_r \sigma_m \sigma_s - \sum_{\{i : e_i < a\}} f_i \sigma_a - e_i \\
= \sum_{m \neq s \neq a} \sigma_r \sigma_m \sigma_s - \sum_{\{i : e_i < a\}} f_i \sigma_a - e_i = - \sum_{\{i : e_i < a\}} f_i \sigma_a - e_i + \sum_{r \neq 0} \sigma_r \left( \sum_{m \neq s \neq a - r} \sigma_m \sigma_s \right) \\
= \sum_{r \neq 0} \sigma_r \left( \sum_{m \neq s \neq a - r} \sigma_m \sigma_s \right) + \sum_{\{i : e_i < a\}} \sigma_a - e_i (\sigma e_i \sigma a + \sigma e_i \sigma - f_i) = 0,
\]

where the first and the last equalities hold by induction hypothesis. \( \square \)

Construction 4.3. (cf. [Ei1, Section 7]) Suppose that \( f_1, \ldots, f_c \) are elements in a ring \( S \), and that \( G \) is a free complex over \( S \) with a system \( \sigma \) of higher homotopies. This gives rise to a new complex \( \text{Sh}(G, \sigma) \). To define it, we will write \( S\{y_1, \ldots, y_c\} \) for the divided power algebra over \( S \) on variables \( y_1, \ldots, y_c \); thus,

\[
S\{y_1, \ldots, y_c\} \cong \text{Hom}_S (S[t_1, \ldots, t_c], S) = \oplus S y_1^{(i_1)} \cdots y_c^{(i_c)}
\]

where the \( y_1^{(i_1)} \cdots y_c^{(i_c)} \) form the dual basis to the monomial basis of the polynomial ring \( S[t_1, \ldots, t_c] \). We will use the fact that \( S\{y_1, \ldots, y_c\} \) is an \( S[t_1, \ldots, t_c] \)-module with action \( t_j y_j^{(i)} = y_j^{(i-1)} \) (see [Ei3, Appendix 2]).

Set \( R = S/(f_1, \ldots, f_c) \). The graded module

\[
S\{y_1, \ldots, y_c\} \otimes G \otimes R,
\]

where each \( y_i \) has degree 2, becomes a free complex over \( R \) when equipped with the differential

\[
\delta := \sum t^a \otimes \sigma_a \otimes R.
\]

This complex is called the Shamash complex and denoted \( \text{Sh}(G, \sigma) \).

In the case when we consider only one element \( f \in S \), we denote the divided power algebra by \( S\{y\} \), where the \( y^{(i)} \) form the dual basis to the basis \( t^i \) of the polynomial ring \( S[t] \).

Proposition 4.4. [Ei1, Sh] Let \( f_1, \ldots, f_c \) be a regular sequence in a ring \( S \), and let \( N \) be a module over \( R := S/(f_1, \ldots, f_c) \). If \( G \) is an \( S \)-free resolution of \( N \) and \( \sigma \) is a system of higher homotopies for \( f_1, \ldots, f_c \) on \( G \), then \( \text{Sh}(G, \sigma) \) is an \( R \)-free resolution of \( N \).

Construction 4.5. The \( k[\chi_1, \ldots, \chi_c] \)-structure on \( \text{Ext}_S(M, k) \). [Ei1, 1.2 and 1.5] shows that the CI operators are, up to homotopy, independent of the choice of liftings, and also that they commute up to homotopy. If \( S \) is local with maximal ideal \( m \) and residue field \( k \), and \( V \) is an \( R \)-free resolution of an \( R \)-module \( N \), then the CI operators \( t_j \) induce well-defined, commutative maps \( \chi_j \) on \( \text{Ext}_R(N, k) \), and thus make \( \text{Ext}_R(N, k) \) into a module over the polynomial ring \( R := k[\chi_1, \ldots, \chi_c] \), where the variables \( \chi_j \) have degree 2. The \( \chi_j \) are also called CI operators. By [Ei1, Proposition 1.2], the action of \( \chi_j \) can be defined using any CI operators on any \( R \)-free resolution of \( N \). Because the
χ_j have degree 2, we may split any ℛ-module into even degree and odd degree parts; in particular, we write

$$\text{Ext}_R(N, k) = \text{Ext}^{\text{even}}_R(N, k) \oplus \text{Ext}^{\text{odd}}_R(N, k).$$

A version of the following result was first proved in [Gu] by Gulliksen, who used a different construction of operators on Ext. Other constructions of operators were introduced and used by Avramov [Av], Avramov-Sun [AS], Eisenbud [Ei1], and Mehta [Me]. The relations between these constructions were explained by Avramov and Sun [AS]. We will use only the construction from [Ei1] outlined at the beginning of this section. Using that construction, we provide a new and short proof of the following result.

**Theorem 4.6.** [AS, Ei1, Gu] Let $f_1, \ldots, f_c$ be a regular sequence in a local ring $S$ with residue field $k$, and set $R = S/(f_1, \ldots, f_c)$. If $N$ is an $R$-module with finite projective dimension over $S$, then the action of the CI operators makes $\text{Ext}_R(N, k)$ into a finitely generated $R := k[\chi_1, \ldots, \chi_c]$-module.

**Proof:** Let $G$ be a finite $S$-free resolution of $N$. By Proposition 4.2, there exists a system of higher homotopies on $G$. Proposition 4.4 shows that $\text{Sh}(G, \sigma)$ is an $R$-free resolution of $N$. Consider its dual. By [Ei1, Theorem 7.2] (also see Construction 4.7), the CI operators can be chosen to act on $\text{Sh}(G, \sigma)$ as multiplication by the variables, and thus they commute. By the construction of the Shamash resolution, it is clear that $\text{Hom}_R(\text{Sh}(G, \sigma), k)$ is a finitely generated module over $R$. As the CI operators commute with the differential, it follows that both the kernel and the image of the differential are submodules, so they are finitely generated as well. Thus, so is the quotient module $\text{Ext}_R(N, k)$.

In this paper we will use higher homotopies and the Shamash construction for one element $f \in S$. We focus on that case in the rest of the section.

**Construction 4.7.** Suppose that $f \in S$, and that $(G, \partial)$ is a free complex over $S$ with a system $\sigma$ of higher homotopies. We use the notation in Construction 4.3. The standard lifting $\tilde{\text{Sh}}(G, \sigma)$ of the Shamash complex to $S$ is $S \{y\} \otimes G$ with the maps $\tilde{\delta} = \sum t^j \otimes \sigma_j$. In particular, $\tilde{\delta}|_G = \partial$, so of course $\tilde{\delta}^2|_G = \partial^2 = 0$. Moreover, the equations of Definition 4.1 say precisely that, $\tilde{\delta}^2$ acts on the complementary summand $G' = \oplus_{i>0} y^{(i)} G$ by $ft$; that is, it sends each $y^{(i)} G$ isomorphically to $f y^{(i-1)} G$. Thus

$$\tilde{\delta}^2 = ft \otimes 1.$$

The standard CI operator for $f$ on $\text{Sh}(G, \sigma)$ is $t \otimes 1$. Note that $t : \text{Sh}(G, \sigma) \rightarrow \text{Sh}(G, \sigma)[2]$ is surjective, and is split by the map sending $y^{(i)} u \in S \{y\} \otimes G \otimes S/(f)$ to $y^{(i+1)} u$. Also, the standard lifted CI operator $\tilde{t} := t \otimes 1 : \tilde{\text{Sh}}(G, \sigma) \rightarrow \tilde{\text{Sh}}(G, \sigma)$ commutes with the lifting $\tilde{\delta} = \sum t^j \otimes \sigma_j$ of the differential $\delta$.

We will use the following modified version of Proposition 4.4:

**Proposition 4.8.** Let $G$ be a complex of $S$-free modules with a system of higher homotopies $\sigma$ for a non-zerodivisor $f$ in a ring $S$. If $F = \text{Sh}(G, \sigma)$, then $H_j(F) = 0$ for all
\[ 0 < j \leq i \text{ if and only if } H_j(\tilde{G}) = 0 \text{ for all } j \leq i. \]

In particular, \( \text{Sh}(\tilde{G}, \sigma) \) is an \( S/(f) \)-free resolution of a module \( N \) if and only if \( \tilde{G} \) is an \( S \)-free resolution of \( N \).

**Proof:**

We first show that (without any exactness hypothesis) \( H_0(\tilde{G}) = H_0(F) \). Since the standard lifted CI operator \( \tilde{t} : \tilde{F}_i \to \tilde{F}_{i-2} \) is surjective, \( f \) annihilates \( N := \text{Coker}(\tilde{\delta} : \tilde{F}_1 \to \tilde{F}_0) \), and thus \( N = \text{Coker}(\delta : F_1 \to F_0) = H_0(F) \). But for \( i \leq 1 \) we have \( \tilde{G}_i = \tilde{F}_i \), so \( H_0(\tilde{G}) = H_0(F) \) as required.

Set \( \overline{G} = R \otimes \tilde{G} \). We now use the short exact sequences of complexes

\[
0 \to \overline{G} \to F \xrightarrow{t} F[2] \to 0
\]

which yield long exact sequences

\[
\cdots \to H_{j-1}(F) \to H_j(\overline{G}) \to H_j(F) \to H_{j-2}(F) \to H_{j-1}(\overline{G}) \to \cdots
\] (4.9)

\[
\cdots \to H_{j+1}(\overline{G}) \to H_j(\tilde{G}) \xrightarrow{f} H_j(\tilde{G}) \to H_j(\overline{G}) \to H_{j-1}(\tilde{G}) \to \cdots
\] (4.10)

respectively. Since \( \sigma_1 \) is a homotopy for \( f \) on \( \tilde{G} \), the latter sequence breaks up into short exact sequences

\[
0 \to H_j(\tilde{G}) \to H_j(\overline{G}) \to H_{j-1}(\tilde{G}) \to 0.
\] (4.11)

First, assume that \( H_j(F) = 0 \) for \( 1 \leq j \leq i \). From the long exact sequence (4.9) we conclude that \( H_j(\tilde{G}) = 0 \) for \( 2 \leq j \leq i \), and then (4.11) implies that \( H_j(\overline{G}) = 0 \) for \( 1 \leq j \leq i \).

Conversely, suppose that \( H_j(\overline{G}) = 0 \) for \( 1 \leq j \leq i \). It is well known that if we apply the Shamash construction to a resolution then we get a resolution, but since the bound \( i \) is not usually present we give an argument:

Assume that \( H_j(\tilde{G}) = 0 \) for \( 1 \leq j \leq i \). By (4.11) it follows that \( H_j(\overline{G}) = 0 \) for \( 2 \leq j \leq i \). Applying (4.9), we conclude that \( H_j(F) \cong H_{j-2}(F) \) for \( 3 \leq j \leq s \). Hence, it suffices to prove that \( H_1(F) = H_2(F) = 0 \).

We will prove that \( H_1(F) = 0 \). Let \( g_1 \in \tilde{G}_1 \) be an element that reduces modulo \( f \) to \( \overline{g}_1 \). We have

\[
\tilde{\delta}(g_1) = fg_0 = \tilde{\delta}\sigma_1(g_0)
\]

for some \( g_0 \in G_0 \). Thus \( g_1 - \sigma_1(g_0) \in \text{Ker}(\tilde{\delta}) \) is a cycle in \( \tilde{G} \). Since \( H_1(\tilde{G}) = 0 \), we must have \( g_1 - \sigma_1(g_0) = \tilde{\delta}(g_2) \) for some \( g_2 \in \tilde{G}_2 \). Using the isomorphism \( \tilde{F}_2 = \tilde{G}_2 \oplus \tilde{G}_0 \) we see that

\[
g_1 = \sigma_1(g_0) + \tilde{\delta}(g_2) = \tilde{\delta}(g_0 + g_2).
\]

It follows that \( \overline{g}_1 = \delta(\overline{g}_0 + \overline{g}_2) \) is a boundary in \( F \), as required.

Finally, we show that \( H_2(F) = 0 \). Part of (4.9) is the exact sequence

\[
H_2(\tilde{G}) \to H_2(F) \to H_0(F) \xrightarrow{\beta} H_1(\tilde{G}) \to H_1(F).
\]

Since \( H_2(\overline{G}) = 0 \), it suffices to show that the map marked \( \beta \) is a monomorphism. But we already showed that \( H_1(F) = 0 \), so \( \beta \) is an epimorphism. Since its source and target
are isomorphic finitely generated modules over the ring \( S \), this implies that it is an isomorphism, whence \( H_2(F) = 0 \).

It follows from Theorem 4.6 that CI operators on the resolutions of high syzygies over complete intersections are often surjective, in a sense we will make precise. To prepare for the study of this situation, we consider what can be said when a CI operator is surjective.

**Proposition 4.12.** Let \( f \in S \) be a non-zerodivisor in a ring \( S \), and let

\[
(F, \delta) : \cdots \longrightarrow F_i \xrightarrow{\delta_i} F_{i-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{\delta_1} F_0
\]

be a complex of free \( R := S/(f) \)-modules. Let \((\tilde{F}, \tilde{\delta})\) be a lifting of \((F, \delta)\) to \( S \). Set

\[
\tilde{t} := (1/f)^2 : \tilde{F} \longrightarrow \tilde{F}[2],
\]

\[
\tilde{G} = \text{Ker}(\tilde{t}).
\]

Suppose that \( \tilde{t} \) is surjective. Then:

1. [Ei1, Theorem 8.1] The maps \( \tilde{\delta} : \tilde{F}_i \longrightarrow \tilde{F}_{i-1} \) induce maps \( \tilde{\delta} : \tilde{G}_i \longrightarrow \tilde{G}_{i-1} \), and

\[
\tilde{\delta} : \cdots \longrightarrow \tilde{G}_{i+1} \xrightarrow{\tilde{\delta}_{i+1}} \tilde{G}_i \longrightarrow \cdots \longrightarrow \tilde{G}_1 \xrightarrow{\tilde{\delta}_1} \tilde{G}_0
\]

is an \( S \)-free complex. If \( S \) is local and \( F \) is minimal, then so is \( \tilde{G} \).

2. We may write \( \tilde{F}_i = \bigoplus_{j \geq 0} \tilde{G}_{i-2j} \) in such a way that the lifted CI operator \( \tilde{t} \) consists of the projections

\[
\tilde{F}_i = \bigoplus_{0 \leq j \leq i/2} \tilde{G}_{i-2j} \xrightarrow{\tilde{t}_j} \bigoplus_{0 \leq j \leq (i-2)/2} \tilde{G}_{i-2j} = \tilde{F}_{i-2}.
\]

If \( \sigma_j : \tilde{G}_{i-2j} \longrightarrow \tilde{G}_{i-1} \) denotes the appropriate component of the map \( \tilde{\delta} : \tilde{F}_i \longrightarrow \tilde{F}_{i-1} \), then \( \sigma = \{\sigma_j\} \) is a system of higher homotopies on \( \tilde{G} \), and \( F \equiv \text{Sh}(\tilde{G}, \sigma) \).

**Proof:** (2): Since the maps \( \tilde{t} \) are surjective, it follows inductively that we may write \( \tilde{F}_i \) and \( \tilde{t} \) in the given form. The component corresponding to \( \tilde{G}_{i-2j} \longrightarrow \tilde{G}_{i-1} \) in \( \tilde{\delta} : \tilde{F}_m \longrightarrow \tilde{F}_{m-1} \) is the same for any \( m \) with \( m \geq i - 2j \) and \( m \equiv i \) mod(2) because \( \tilde{\delta} \) commutes with \( \tilde{t} \). The condition that \( \sigma \) is a sequence of higher homotopies is equivalent to the condition that \( \tilde{\delta}^2 = f\tilde{t} \), as one sees by direct computation. It is now immediate that \( F \equiv \text{Sh}(\tilde{G}, \sigma) \).

**Corollary 4.13.** With hypotheses and notation as in Proposition 4.12, suppose in addition that \( S \) is a local ring and that \((F, \delta)\) is a minimal \( R \)-free resolution of \( N \). The minimal \( S \)-free resolution of \( N \) is \((\tilde{G}, \tilde{\delta})\) = Ker(\(\tilde{t}\)). If we split the epimorphisms \( t : F_i \longrightarrow F_{i-2} \) and correspondingly write \( F_i = \tilde{G}_i \oplus F_{i-2} \) then the differential \( \delta : F_i \longrightarrow F_{i-1} \) has the form

\[
\delta_i = \begin{pmatrix} \tilde{G}_i & F_{i-2} \\ \tilde{\delta}_i & \varphi_i \end{pmatrix}.
\]

\(\square\)
As an immediate consequence of Propositions 4.8 and 4.12 we obtain a result of Avramov-Gasharow-Peeva; their proof relies on the spectral sequence proof of [AGP, Theorem 4.3].

Corollary 4.14. [AGP, Proposition 6.2] Let \( f \in S \) be a non-zerodivisor in a local ring. If \( N \) is a module over \( S/(f) \) then the CI operator \( \chi \) corresponding to \( f \) is a non-zerodivisor on \( \text{Ext}_S(N,k) \) if and only if the minimal \( S/(f) \)-free resolution of \( N \) is obtained by a Shamash construction applied to the minimal free resolution of \( N \) over \( S \).

Proof: Nakayama’s Lemma shows that the CI operator \( t : \mathbf{F}[-2] \to \mathbf{F} \) is surjective if and only if the operator \( \chi : \text{Ext}_R(N,k) \to \text{Ext}_R(N,k) \) is injective.

5. The minimal \( R \)-free resolution of a matrix factorization module

Let \( (d,h) \) be a matrix factorization with respect to a regular sequence \( f_1,\ldots,f_c \) in a ring \( S \), and \( R = S/(f_1,\ldots,f_c) \). We will describe an \( R \)-free resolution of the matrix factorization module \( M \) that is minimal when \( S \) is local and \( (d,h) \) is minimal.

Construction 5.1. Let \( (d,h) \) be a matrix factorization with respect to a regular sequence \( f_1,\ldots,f_c \) in a ring \( S \). Using notation as in 2.1, choose splittings \( A_s(p) = A_s(p-1) \oplus B_s(p) \) for \( s = 0,1 \), so \( A_s(p) = \oplus_{1 \leq q \leq p} B_s(q) \), and write \( \psi_p \) for the component of \( d_p \) mapping \( B_1(p) \) to \( A_0(p-1) \). Set

\[
\mathbf{A}(p): A_1(p) \xrightarrow{d_p} A_0(p) \quad \text{and} \quad \mathbf{B}(p): B_1(p) \xrightarrow{b_p} B_0(p).
\]

- Set \( U(1) = B(1) \), and note that \( h_1 \) is a homotopy for \( f_1 \). Set \( T(1) := \text{Sh}(U(1),h_1) \). Its beginning is the complex \( R(1) \otimes A(1) \).
- Given an \( R(p-1) \)-free resolution \( T(p-1) \) of \( M(p-1) \) with beginning \( R(p-1) \otimes A(p-1) \), let

\[
\Psi_p : R(p-1) \otimes B(p)[-1] \to T(p-1)
\]

be the map of complexes induced by \( \psi_p : B_1(p) \to A_0(p-1) \). Set

\[
U(p) := \text{Cone} (\Psi_p).
\]

We will show that \( U(p) \) is an \( R(p-1) \)-free resolution of \( M(p) \). Thus we can choose a system of higher homotopies \( \sigma(p) \) for \( f_p \) on \( U(p) \) that begins with \( d_p \) (that is, \( \sigma(p)_0 = d_p \) and

\[
R(p-1) \otimes h_p : R(p-1) \otimes A_0(p) \to R(p-1) \otimes A_1(p).
\]

Set

\[
T(p) := \text{Sh}(U(p),\sigma(p)).
\]

The underlying graded module of \( T(p) \) is \( U(p) = \text{Cone}(\Psi_p) \) tensored with a divided power algebra on a variable \( y_p \) of degree 2. Its first differential is

\[
R(p) \otimes \mathbf{A}(p) : R(p) \otimes A_1(p) \xrightarrow{R(p) \otimes d_p} R(p) \otimes A_0(p),
\]
which is the presentation of \( M(p) \). We see by induction on \( p \) that the term \( T_j(p) \) of homological degree \( j \) in \( T(p) \) is a direct sum of the form

\[
T_j(p) = \bigoplus y_{q_1}^{(a_1)} \cdots y_{q_i}^{(a_i)} B_s(q) \otimes R(p)
\]

where the sum is over all terms with

\[
p \geq q_1 > q_2 > \cdots > q_i \geq q \geq 1,
\]

\[
a_m > 0 \text{ for } 1 \leq m \leq i,
\]

\[
j = s + \sum_{1 \leq m \leq i} 2a_m.
\]

We say that an element \( y_{q_1}^{(a_1)} \cdots y_{q_i}^{(a_i)} v \) with \( v \in B_s(q) \) and \( a_1 > 0 \) is admissible of weight \( q_1 \), and we make the convention that the admissible elements in \( B_s(q) \) have weight 0.

The complex \( T(c) \) is thus filtered by:

\[
T(0) := 0 \subseteq R \otimes T(1) \subseteq \cdots \subseteq R \otimes T(p - 1) \subseteq T(c),
\]

where \( R \otimes T(p) \) is the subcomplex spanned by elements of weight \( \leq p \) with with \( v \in B_s(q) \) for \( q \leq p \).

**Theorem 5.2.** With notation and hypotheses as in 5.1:

1. The complex \( T(p) \) is an \( R(p) \)-free resolution of \( M(p) \) whose first differential is \( R(p) \otimes d_p \) and whose second differential is

\[
R(p) \otimes \left( \left( \bigoplus_{q \leq p} A_0(q) \right) \xrightarrow{h} A_1(p) \right),
\]

where the \( q \)-th component of \( h \) is \( h_q : A_0(q) \to A_1(q) \to A_1(p) \).

2. If \( S \) is local then \( T(p) \) is the minimal free resolution of \( M(p) \) if and only if the matrix factorization \( (d_p, h(p) = (h_1| \cdots |h_p)) \) (see 2.1 for notation) is minimal.

**Proof of Theorem 5.2(1):** We do induction on \( p \). To start the induction, note that \( U(1) \) is the two-term complex \( A(1) = B(1) \). By hypothesis, its differential \( d_1 \) and homotopy \( h_1 \) form a hypersurface matrix factorization for \( f_1 \), and \( T(1) \) has the form

\[
T(1) : \ R(1) \otimes \left( \cdots \xrightarrow{h_1} A_1(1) \xrightarrow{d_1} A_0(1) \xrightarrow{h_1} A_1(1) \xrightarrow{d_1} A_0(1) \right).
\]

Inductively, suppose that \( p \geq 2 \), and that

\[
T(p - 1) : \cdots \to T_2 \to T_1 \to T_0
\]

is an \( R(p - 1) \)-free resolution of \( M(p - 1) \) whose first two maps are as claimed. We write \( \overline{\cdot} \) for \( R(p - 1) \otimes \cdot \). It follows that the first map of \( U(p) \) is

\[
\overline{d}(p) : \overline{A}_1(p) = T_1 \oplus \overline{B}_1(p) \to \overline{A}_0(p) = T_0 \oplus \overline{B}_0(p).
\]

Since \( R(p - 1) \otimes (d_p h_p) = f_p \text{Id}_{A_0(p)} \) we may take \( R(p - 1) \otimes h_p \) to be the start of a system of higher homotopies \( \sigma(p) \) for \( f_p \) on \( R(p - 1) \otimes U(p) \). It follows from the definition that the first two maps in \( T(p) = \text{Sh}(U(p), \sigma(p)) \) are as asserted.

By Proposition 4.8, the Shamash construction takes an \( R(p - 1) \)-free resolution to an \( R(p) \)-free resolution of the same module. Thus for the induction it suffices to show
that \( U(p) \) is an \( R(p-1) \)-free resolution of \( M(p) \). Since the first map of \( U(p) \) is \( \overline{d}(p) \), and since \( \overline{h}(p) \) is a homotopy for \( f_p \), we see at once that

\[
H_0(U(p)) = \text{Coker}(\overline{d}(p)) = \text{Coker}(R(p) \otimes d_p) = M(p).
\]

To prove that \( U(p) \) is a resolution, note first that \( U(p)_{\geq 2} = T(p-1)_{\geq 2} \), and the image of \( U(p)_2 = T(p-1)_2 \) is contained in the summand \( T(p-1)_1 \subseteq U(p)_1 \), so \( H_i(U(p)) = H_i(T(p-1)) = 0 \) for \( i \geq 2 \). Thus it suffices to prove that \( H_1(U(p)) = 0 \).

Let \( (y, v) \in U(p)_1 = T(p-1)_1 \oplus \overline{B}(p)_1 \) be a cycle in \( U(p) \). Thus, \( \overline{b}_p(v) = 0 \) and \( \overline{\psi}_p(v) = -\overline{d}_{p-1}(y) \). By Lemma 3.15, we conclude that \( v = 0 \).

For the proof of part (2) of Theorem 5.2 we will use the form of the resolutions \( T(p) \) to make a special lifting of the differentials to \( S \), and thus to produce especially “nice” CI operators. We pause in the proof of Theorem 5.2 to describe this construction and deduce some consequences.

**Proposition 5.3.** With notation and hypotheses as in 5.1, there exists a lifting of the filtration \( T(1) \subseteq \cdots \subseteq T(c) \) to a filtration \( \tilde{T}(1) \subseteq \cdots \subseteq \tilde{T}(c) \) over \( S \), and a lifting \( \tilde{\delta} \) of the differential \( \delta \) in \( T(c) \) to \( S \) with lifted CI operators \( \tilde{t}_1, \ldots, \tilde{t}_c \) on \( \tilde{T}(c) \) such that for every \( 1 \leq p \leq c \):

1. Both \( \tilde{\delta} \) and \( \tilde{t}_p \) preserve \( \tilde{T}(p) \), and \( \tilde{t}_p|_{\tilde{T}(p)} \) commutes with \( \tilde{\delta}|_{\tilde{T}(p)} \) on \( \tilde{T}(p) \).
2. The CI operator \( \tilde{t}_p \) vanishes on the subcomplex \( R \otimes U(p) \) and induces an isomorphism from \( R \otimes T(p)_j/U(p)_j \) to \( R \otimes T(p)_{j-2} \) that sends an admissible element \( y_{q_1}^{(a_1)} \cdots y_{q_i}^{(a_i)} v \) with \( q_1 = p \) to \( y_{q_1}^{(a_1-1)} \cdots y_{q_i}^{(a_i)} v \).

**Proof:** If \( p = 1 \) the result is obvious. Thus we may assume by induction that liftings

\[
0 \subseteq \tilde{T}(1) \subseteq \cdots \subseteq \tilde{T}(p-1),
\]

\( \tilde{\delta}(p-1) \) and \( \tilde{t}_1, \ldots, \tilde{t}_{p-1} \) on \( \tilde{T}(p-1) \) satisfying the Proposition have been constructed. We use the maps \( \tilde{\psi}_p \) and \( \tilde{b}_p \) from the definition of the matrix factorization to construct a lifting of \( U(p) \) from the given lifting of \( T(p-1) \). In addition, we choose liftings \( \tilde{\delta} \) of the maps (other than the differential) in the system of higher homotopies \( \sigma(p) \) for \( f_p \) on \( U(p) \).

By construction, \( T(p) = \text{Sh}(U(p), \sigma(p)) \), so we take the standard lifting to \( S \) from 4.7, that is, take \( \tilde{T}(p) = \oplus_{\ell \geq 0} y_{\ell}^{(p)} \tilde{U}(p) \) with lifting of the differential \( \tilde{\delta} = \sum t^j \otimes \tilde{\sigma}_j \), where \( t \) is the dual variable to \( y_p \).

By Construction 4.7 it follows that, modulo \( (f_1, \ldots, f_{p-1}) \), the map \( \tilde{\delta} \) vanishes on \( \tilde{U}(p) \) and induces \( f_p \) times the projection \( \tilde{T}_j(p)/\tilde{U}_j(p) \rightarrow \tilde{T}_{j-2}(p) \).

We choose \( \tilde{t}_p \) to be the standard lifted CI operator, which vanishes on \( \tilde{U}(p) \) and is the projection \( \tilde{T}_j(p)/\tilde{U}_j(p) \rightarrow \tilde{T}_{j-2}(p) \). Then \( \tilde{\delta}_{i-2}\tilde{t}_p = \tilde{t}_p \tilde{\delta}_i \) by construction; see 4.7.

Recall that \( \tilde{\delta}|_{\tilde{T}(p-1)} \) is the lifting \( \tilde{\delta}(p-1) \) given by induction. Therefore, from \( \tilde{\delta} \) we can choose maps \( \tilde{t}_1, \ldots, \tilde{t}_{p-1} \) on \( \tilde{T}(p) \) that extend the maps \( \tilde{t}_1, \ldots, \tilde{t}_{p-1} \) given by induction on \( \tilde{T}(p-1) \subseteq \tilde{U}(p) \).

The CI operators commute up to homotopy, and it is an open conjecture from [Ei1] (see also [AGP, Section 9]) that they can be chosen to commute when restricted to the
minimal free resolution of a high syzygy in the local case. Proposition 5.3 allows us to give a partial answer, based on the following general criterion.

**Proposition 5.4.** Let \( f_1, \ldots, f_c \) be a regular sequence in a local ring \( S \), and let \( R = S/(f_1, \ldots, f_c) \). Suppose that \((\mathcal{F}, \delta)\) is a complex over \( R \) with lifting \((\mathcal{F}, \bar{\delta})\) to \( S \), and let \( t_1, \ldots, t_c \) on \( \mathcal{F} \) define CI operators corresponding to \( f_1, \ldots, f_c \). If, for some \( j \), \( \bar{t}_j \) commutes with \( \bar{\delta}^2 \), then \( t_j \) commutes with each \( t_i \).

**Proof:** Since \( \bar{\delta}^2 = \sum f_i \bar{t}_i \) by definition, we have \( \sum f_i \bar{t}_j \bar{t}_i = \sum f_i \bar{t}_i \bar{t}_j \), or equivalently \( \sum f_i (t_j t_i - t_i t_j) = 0 \). Since \( f_1, \ldots, f_c \) is a regular sequence it follows that \( \bar{t}_j \bar{t}_i - \bar{t}_i \bar{t}_j \) is zero modulo \((f_1, \ldots, f_c)\) for each \( i \).

As an immediate consequence, we have:

**Corollary 5.5.** Suppose that \( S \) is local. With CI operators on \( T(p) \) chosen as in Proposition 5.3 the operator \( t_p \) commutes on \( T(p) \) with each \( t_i \) for \( i < p \).

**Corollary 5.6.** Let \( k[\chi_1, \ldots, \chi_c] \) act on \( \text{Ext}_R(M, k) \) as in Construction 4.5. There is an isomorphism

\[
\text{Ext}_R(M, k) \cong \bigoplus_{p=1}^c k[\chi_p, \ldots, \chi_c] \otimes_k \text{Hom}_S(B(p), k)
\]

of vector spaces such that, for \( i \geq p \), \( \chi_i \) preserves the summand

\[ k[\chi_p, \ldots, \chi_c] \otimes_k \text{Hom}_S(B(p), k) \]

and acts on it via the action on the first factor.

**Proof:** Since \( T(c) \) is a minimal free resolution of \( M \), the \( k[\chi_1, \ldots, \chi_c] \)-module \( \text{Ext}_R(M, k) \) is isomorphic to \( \text{Hom}_R(T(c), k) \). Using the decomposition in (5.1) we see that the underlying graded free module of \( \text{Hom}_R(T(c), k) \) is

\[
\bigoplus_p k[\chi_p, \ldots, \chi_c] \otimes_k \text{Hom}_S(B(p), k).
\]

From part (2) of Proposition 5.3 we see that, for \( i \geq p \), the action of \( \chi_i \) on the summand \( k[\chi_p, \ldots, \chi_c] \otimes_k \text{Hom}_S(B(p), k) \) is via the natural action on the first factor.

Corollary 5.6 provides a standard decomposition of \( \text{Ext}_R(M, k) \) in the sense of [EP].

We will complete the proof of Theorem 5.2:

**Proof of Theorem 5.2(2):** We suppose that \( S \) is local with maximal ideal \( \mathfrak{m} \). If the resolution \( T(p) \) is minimal then it follows at once from the description of the first two maps that \((d, h)\) is minimal. We will prove the converse by induction on \( p \).

If \( p = 1 \) then \( T(1) \) is the periodic resolution

\[
T(1) : \quad \cdots \xrightarrow{h_1} A_1 \xrightarrow{d_1} A_0 \xrightarrow{h_1} A_1 \xrightarrow{d_1} A_0
\]

and only involves the maps \((d_1, h_1)\); this is obviously minimal if and only if \( d_1 \) and \( h_1 \) are minimal.
Now suppose that \( p > 1 \) and that \( T(q) \) is minimal for \( q < p \). Let \( \delta_i : T_i(p) \to T_{i-1}(p) \) be the differential of \( T(p) \). We will prove minimality of \( \delta_i \) by a second induction, on \( i \), starting with \( i = 1, 2 \).

Recall that the underlying graded module of \( T(p) = \text{Sh}(U(p), \sigma) \) is the divided power algebra \( S\{y_p\} = \sum_i S y_p^{(i)} \) tensored with the underlying module of \( R(p) \otimes U(p) \). Thus the beginning of the resolution \( T(p) \) has the form

\[
\cdots \to R(p) \otimes y_p A_0(p) \oplus R(p) \otimes T_2(p-1) \xrightarrow{\delta_2} R(p) \otimes A_1(p) \xrightarrow{\delta_1} R(p) \otimes A_0(p).
\]

The map \( \delta_1 \) is induced by \( d_p \), which is minimal by hypothesis. Further, \( \delta_2 = (h_p, \partial_2) \) where the map \( \partial_2 \) is the differential of \( T(p-1) \) tensored with \( R(p) \). The map \( h_p \) is minimal by hypothesis, and \( \partial \) is minimal by induction on \( p \), so \( \delta_2 \) is minimal as well.

Now suppose that \( j \geq 2 \) and that \( \delta_i \) is minimal for \( i \leq j \). We must show that \( \delta_{j+1} \) is minimal, that is, \( \delta_{j+1}(w) \in mT_j(p) \) for any \( w \in T_{j+1}(p) \). By Construction 5.1, \( \delta_{j+1}(w) \) can be written uniquely as a sum of admissible elements of the form

\[
y_{q_1}^{(a_1)} \cdots y_{q_i}^{(a_i)} v
\]

with \( 0 \neq v \in B_s(q) \) and

\[
p \geq q_1 > q_2 > \cdots > q_i \geq q \geq 1, \\
a_m > 0 \quad \text{for} \quad 1 \leq m \leq i, \\
j = s + \sum_{1 \leq m \leq i} 2a_m.
\]

If \( \delta_{j+1}(w) \notin mT_j(p) \) then there exists a summand \( y_{q_1}^{(a_1)} \cdots y_{q_i}^{(a_i)} v \) in this expression that is not in \( mT_j(p) \). Since \( \delta_{j+1}(w) \) has homological degree \( j \geq 2 \), the weight of this summand must be \( > 0 \), that is, a factor \( y_{q_1}^{(a_1)} \) must be present.

Choose such a summand with weight \( q'_1 \) as large as possible. We choose \( t_{q'_1} \) as in Proposition 5.3. Then \( t_{q'_1} \) sends every admissible element of weight \( < q'_1 \) to zero. The admissible summands of \( \delta_{j+1}(w) \) with weight \( > q'_1 \) can be ignored since they are in \( mT_{j-2}(p) \). By Proposition 5.3 it follows that \( t_{q'_1} \delta_{j+1}(w) \notin mT_{j-2}(p) \). Since

\[
t_{q'_1} \delta_{j+1}(w) = \delta_{j-1} t_{q'_1}(w),
\]

this contradicts the induction hypothesis.

\[\ \]

Gulliksen [Gu] proved that the Poincaré series of \( M \) over \( R \) is \( \mathcal{P}_M^R(x) = g(x)(1-x^2)^{-c} \) for some \( g(x) \in \mathbb{Z}[x] \). It follows that the Betti numbers are eventually given by two polynomials, and Avramov [Av, Theorem 4.1] showed that they have the same degree and the same leading coefficient. We can make this very explicit.

**Corollary 5.7.** With notation and hypotheses as in 5.1, if in addition \( S \) is local and the matrix factorization \( (d, h) \) is minimal, then:

1. The Poincaré series of \( M \) over \( R \) is

\[
\mathcal{P}_M^R(x) = \sum_{1 \leq p \leq c} \frac{1}{(1-x^2)^{p-c+1}} (x \text{ rank } (B_1(p)) + \text{ rank } (B_0(p))).
\]

\[\]
The Betti numbers of $M$ over $R$ are given by the following two polynomials in $z$:

$$b_{2z}^R(M) = \sum_{1 \leq p \leq c} \binom{c - p + z}{c - p} \operatorname{rank}(B_0(p))$$

$$b_{2z+1}^R(M) = \sum_{1 \leq p \leq c} \binom{c - p + z}{c - p} \operatorname{rank}(B_1(p)).$$

**Proof:** For (2), recall that the Hilbert function of $k[Z_p, \ldots, Z_c]$ is $g_\rho(z) = (c-p+z).$ □

Recall that the complexity of an $R$-module $N$ is defined to be

$$\text{cx}_R(N) = \inf\{q \geq 0 \mid \text{there exists a } w \in R \text{ such that } b_i^R(N) \leq wi^{q-1} \text{ for } i \gg 0\}.$$

If the complexity of $N$ is $\mu$ then, as noted above, $\dim_k \operatorname{Ext}^i_R(N, k) = (\beta/(\mu - 1)!i^{\mu-1} + O(i^{\mu-2})$ for $i \gg 0$. Following [AB, 7.3] $\beta$ is called the Betti degree of $N$ and denoted $\text{Bdeg}(N)$; this is the multiplicity of the module $\operatorname{Ext}^i_{R\text{even}}(N, L)$, which is equal to the multiplicity of the module $\operatorname{Ext}^i_{Rd}(N, L)$.

**Corollary 5.8.** With notation and hypotheses as in 5.1, suppose in addition that $S$ is local. Suppose that $(d, h)$ is a minimal matrix factorization, and set

$$\gamma = \min\{p \mid B_1(p) \neq 0\}.$$

The complexity of $M := M(c)$ is

$$\text{cx}_R M = c - \gamma + 1.$$

Moreover, $B_0(p) = 0$ for $p < \gamma$, and the Betti degree of $M$ is

$$\text{Bdeg}(M) = \operatorname{rank}(B_1(\gamma)) = \operatorname{rank}(B_0(\gamma)).$$

If in addition $S$ is Cohen-Macaulay, then $\operatorname{rank}(B_1(p)) > 0$ for every $\gamma \leq p \leq c$.

**Proof:** By Corollary 3.11, $B_1(p) = 0$ implies that $B_0(p) = 0$. Hence the Betti degree of $N$ is equal to $\min\{p \mid B_1(p) \neq 0\}$ and $B_0(p) = 0$ for $p < \gamma$.

The equality $\operatorname{rank}(B_1(\gamma)) = \operatorname{rank}(B_0(\gamma))$ follows from the fact that $M(\gamma)$ is annihilated by $f_\gamma$, and has minimal free resolution $B_1(\gamma) \xrightarrow{b_\gamma} B_0(\gamma)$ over $S/(f_1, \ldots, f_{\gamma-1})$.

Corollary 3.13 implies that $\operatorname{rank}(B_1(p)) > 0$ for every $\gamma \leq p \leq c$, when $S$ is Cohen-Macaulay. □

### 6. Resolutions over intermediate rings

Using a slight extension of the definition of a matrix factorization we can describe the resolutions of the modules $M(p)$ over any of the rings $R(q)$ with $q < p$.

**Definition 6.1.** A generalized matrix factorization over a ring $S$ with respect to a regular sequence $f_1, \ldots, f_c \in S$ is a pair of maps $(d, h)$ satisfying the definition of a matrix factorization except that we drop the assumption that $A(0) = 0$, so that we have a map of free modules $A_1(0) \xrightarrow{b_0} A_0(1)$. We do not require the existence of a map $h_0$. 

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Construction 6.2. Let \((d,h)\) be a generalized matrix factorization with respect to a regular sequence \(f_1, \ldots, f_c\) in a ring \(S\). Using notation as in 2.1, we choose splittings \(A_s(p) = A_s(p-1) \oplus B_s(p)\) for \(s = 0, 1\), and write \(\psi_p\) for the component of \(d_p\) mapping \(B_1(p)\) to \(A_0(p-1)\).

- Let \(V\) be a free resolution of the module \(\text{Coker}(b_0)\) over \(S\), and set \(Q(0) := V\).

- Let
  \[
  \Psi_1 : B(1)[-1] \rightarrow Q(0)
  \]
  be the map of complexes induced by \(\psi_1 : B_1(1) \rightarrow A_0(0)\), and set \(Q(1) = \text{Cone}(\Psi_1)\).

- For \(p \geq 2\), suppose that an \(S\)-free resolution \(Q(p-1)\) of \(M(p-1)\) with first term \(Q_0(p-1) = A_0(p-1)\) has been constructed. Let
  \[
  \psi_p : B(p)[-1] \rightarrow L(p-1)
  \]
  be the map of complexes induced by \(\psi_p : B_1(p) \rightarrow A_0(p-1)\), and let
  \[
  \Psi_p : K(f_1, \ldots, f_{p-1}) \otimes B(p)[-1] \rightarrow Q(p-1)
  \]
  be an \((f_1, \ldots, f_{p-1})\)-Koszul extension. Set \(Q(p) = \text{Cone}(\Psi_p)\).

The proof of Theorem 3.4 can be applied in this situation and yields the following result.

**Proposition 6.3.** Let \((d,h)\) be a generalized matrix factorization over a ring \(S\), and let \(V\) be a free resolution of the module \(\text{Coker}(b_0)\) over \(S\). For each \(p\), the complex \(Q(p)\), constructed in 6.2, is an \(S\)-free resolution of the module \(M(p)\). If the ring \(S\) is local then the resulting free resolution is minimal if and only if \((d,h)\) and \(V\) are minimal. \(\Box\)

**Theorem 6.4.** Let \((d,h)\) be a matrix factorization. Fix a number \(1 \leq j \leq c-1\). Let \(T(j)\) be the free resolution of \(M(j)\) over the ring \(R(j) = S/(f_1, \ldots, f_j)\) given by Theorem 5.1. Let \((d', h')\) be the generalized matrix factorization over the ring \(R(j)\) with

\[
A_s(0) = R(j) \otimes (\oplus_{1 \leq q \leq j} A_s(q)) \quad \text{and} \quad d'_0 = R(j) \otimes d_j,
\]

for \(p > j\), \(A_s(p) = R(j) \otimes A_s(p + j)\) and \(d'_p = R(j) \otimes d_{p+j}\),

for \(s = 0, 1\) and maps induces by \((d,h)\). Then \(M'(0) = M(j)\).

1. Construction 6.2, starting from the \(R(j)\) free resolution \(Q(0) := T(j)\) of \(M'(0) = M(j)\), produces a free resolution \(Q(c-j)\) of \(M\) over \(R(j)\).

2. If \(S\) is local and \((d,h)\) is minimal, then the resolution \(Q(c-j)\) is minimal. In that case, the Poincaré series of \(M\) over \(R(j)\) is

\[
\mathcal{P}_M^{R(j)}(x) = \left( \sum_{1 \leq p \leq j} \frac{1}{(1-x^2)^{p-j-1}} \left( x \text{rank}(B_1(p)) + \text{rank}(B_0(p)) \right) \right) \left( \sum_{j+1 \leq p \leq c} (1+x)^{p-j-1} \left( x \text{rank}(B_1(p)) + \text{rank}(B_0(p)) \right) \right).
\]

**Proof:** First, we apply Theorem 5.1, which gives the resolution \(T(j)\) of \(M(j)\) over the ring \(R(j)\). Then we apply Proposition 6.3. \(\Box\)
7. Pre-stable Syzygies and Generic CI Operators

Our goal in this section and Section 9 is to show that every sufficiently high syzygy over a complete intersection is a matrix factorization module. In this section we introduce the concepts of pre-stable syzygy and stable syzygy. We will see that any sufficiently high syzygy in a minimal free resolution over a local complete intersection ring is a stable syzygy. In Section 9 we will show that a pre-stable syzygy is a matrix factorization module.

Definition 7.1. Suppose that \( f_1, \ldots, f_c \) is a regular sequence in a local ring \( S \), and set \( R = S/(f_1, \ldots, f_c) \). We define the concept of a pre-stable syzygy recursively: We say that an \( R \)-module \( M \) is a pre-stable syzygy with respect to \( f_1, \ldots, f_c \) if either \( c = 0 \) and \( M = 0 \), or \( c \geq 1 \) and the following conditions are satisfied:

1. There exists a minimal \( R \)-free resolution \( (F, \delta) \) of an \( R \)-module of finite projective dimension over \( S \) with a surjective CI operator \( t_c \) on \( F \) and such that \( M = \text{Ker}(\delta_1) \);
2. If \( \tilde{\delta}_1 \) is a lifting of \( \delta_1 \) to \( \tilde{R} := S/(f_1, \ldots, f_{c-1}) \), then \( \tilde{M} := \text{Ker}(\tilde{\delta}_1) \) is a pre-stable syzygy with respect to \( f_1, \ldots, f_{c-1} \).

We say that a pre-stable syzygy is stable if the module resolved by \( F \) in Condition (1) in 7.1 is maximal Cohen-Macaulay and the module \( \tilde{M} \) in Condition (2) is a stable syzygy.

Remark 7.2. The property of being pre-stable is independent of choices: Condition (1) of the definition is independent of the choice of \( t_c \) because \( t_c \) is uniquely defined up to homotopy, and \( F \) is assumed minimal. Condition (2) is independent of the choice of the lifting of \( \delta_1 \) because, if we write \( L \) for the module resolved by \( F \), then \( \text{Ker}(\tilde{\delta}_1) \) is the second syzygy of \( L \) over \( \tilde{R} \) by Propositions 4.8 and 4.12.

Note that if \( M \) is a pre-stable syzygy, then by (1) it follows that \( M \) has finite projective dimension over \( S \).

The property described in Definition 7.1 is preserved under taking syzygies:

Proposition 7.3. Suppose that \( f_1, \ldots, f_c \) is a regular sequence in a local ring \( S \), and set \( R = S/(f_1, \ldots, f_c) \). If \( M \) is a pre-stable syzygy over \( R \), then \( \text{Syz}^R_1(M) \) is pre-stable as well. If \( M \) is a stable syzygy over \( R \), then so is \( \text{Syz}^R_1(M) \).

Proof: Let \( (F, \delta) \) be a minimal \( R \)-free resolution of a module \( L \) such that \( M = \text{Ker}(\delta_1) \) and the conditions in Definition 7.1 are satisfied. Lifting \( F \) to \( \tilde{F} \) over \( \tilde{R} := S/(f_1, \ldots, f_{c-1}) \) and using the hypothesis that \( S \) is local, we see that the lifted CI operator \( \tilde{t_c} \) is surjective on \( \tilde{F} \). By Propositions 4.8 and 4.12, \( \tilde{G} := \text{Ker}(\tilde{t_c}) \) is the minimal free resolution of the module \( L \) over \( \tilde{R} \).

Let \( M' = \text{Syz}^R_1(M) \) and let \( L' = \text{Syz}^R_1(L) \), so that \( F' = F_{\geq 1}[-1] \) is the minimal free resolution of \( L' \). Clearly \( t_c|_{F'} \) is surjective. The shifted truncation \( \tilde{F}' := F_{\geq 1}[-1] \) is a lifting of \( F' \), and \( \tilde{G}' := \text{Ker}(\tilde{t_c}|_{\tilde{F}'}) \) is a minimal free resolution of \( L' \) over \( \tilde{R} \). The complex
\[ \overline{G}'_{\geq 2} \text{ agrees (up to the sign of the differential) with } \overline{G}[-1]_{\geq 2}: \]

\begin{align*}
(7.4) \\
\overline{G} : & \ldots \rightarrow \overline{G}_4 \rightarrow \overline{G}_3 \rightarrow \overline{G}_2 \rightarrow \overline{F}_1 \overset{\delta_1}{\rightarrow} \overline{F}_0 \\
\overline{G}' : & \ldots \rightarrow \overline{G}_4 \rightarrow \overline{G}_3 \rightarrow \overline{F}_2 \overset{\delta_2}{\rightarrow} \overline{F}_1,
\end{align*}

Thus \( \text{Ker}(\overline{\delta}_2) = \text{Syz}_R^1(\text{Ker}(\overline{\delta}_1)) \). Since \( \text{Ker}(\overline{\delta}_1) \) is a pre-stable syzygy, we can apply the induction hypothesis to conclude that \( \text{Ker}(\overline{\delta}_2) \) is pre-stable.

The last statement in the proposition follows from the observation that if \( L \) is a maximal Cohen-Macaulay \( R \)-module, then so is \( L' \).

The next result shows that in the codimension 1 case, pre-stable syzygies are the same as codimension 1 matrix factorizations.

**Proposition 7.5.** Let \( f \in S \) be a non-zerodivisor in a local ring and set \( R = S/(f) \). The following conditions on an \( R \)-module \( M \) are equivalent:

1. \( M \) is a pre-stable syzygy with respect to \( f \).
2. \( M \) has projective dimension 1 as an \( S \)-module.
3. The minimal \( R \)-free resolution of \( M \) comes from a codimension 1 matrix factorization of \( f \) over \( S \).

**Proof:** (1) \( \Rightarrow \) (2): Let \( F \) be a minimal free resolution satisfying condition (1) in Definition 7.1. By Proposition 7.3 and its proof and notation, \( \text{Syz}_R^2(M) \) is a pre-stable syzygy, and thus the free resolution \( \overline{G}' : \ldots \rightarrow \overline{G}_4 \rightarrow \overline{F}_3 \rightarrow \overline{F}_2 \) (which is the kernel of the lifting of the CI operator \( t_c \) on the minimal free resolution \( F_{\geq 2} \) of \( M \)) is zero in degrees \( \geq 4 \). Since \( \overline{G}' \) is the minimal free resolution (up to a shift) of \( M \) over \( S \), the projective dimension of \( M \) over \( S \) is 1.

(2) \( \Rightarrow \) (3): If \( M \) has projective dimension 1 then \( M \) is the cokernel of a square matrix over \( S \), and the homotopy for multiplication by \( f \) defines the matrix factorization.

(3) \( \Rightarrow \) (1): Continuing the periodic free resolution of \( M \) as an \( R \)-module two steps to the right we get a minimal free resolution \( F \) of a module \( L \cong M \) on which the CI operators are surjective, and also injective on \( F_{\geq 2} \). It follows that \( \text{Ker}(\overline{\delta}_1) = 0 \) in the notation of Definition 7.1, so it is pre-stable. 

We now return to the situation of Theorem 4.6: Let \( N \) be an \( R \)-module with finite projective dimension over \( S \). We regard \( E := \text{Ext}_R(N,k) \) as a module over \( \mathcal{R} = k[\chi_1, \ldots, \chi_c] \), where \( \chi_j \) have degree 2. Since we think of degrees in \( E \) as cohomological degrees, we write \( E[a] \) for the shifted module whose degree \( i \) component is \( E^{i+a} = \text{Ext}^{i+a}_R(N,k) \). If \( M \) is the \( r \)-th syzygy module of \( N \) then \( \text{Ext}_R(M,k) = \text{Ext}^{2r}_R(N,k)[-r] \).

Recall that the Castelnuovo-Mumford regularity \( \text{reg} \ E \) is defined as

\[ \text{reg} \ E = \max_{0 \leq i \leq c} \left\{ i + \max \left\{ j \mid H^j_i(\chi_1, \ldots, \chi_c)(E) \neq 0 \right\} \right\}. \]

Since the generators of \( \mathcal{R} \) have degree 2, some care is necessary. Note that if \( \text{Ext}^{odd}_R(N,k) \neq 0 \) then \( E = \text{Ext}_R(N,k) \) can never have regularity \( \leq 0 \), since it is generated in degrees \( \geq 0 \) and the odd part cannot be generated by the even part. Thus we will often have recourse to the condition \( \text{reg} \ Ext_R(N,k) = 1 \). On the other hand, many things
work as usual. If we split $E$ into even and odd parts, $E = E^{\text{even}} \oplus E^{\text{odd}}$ we have
$$\text{reg } E = \max(\text{reg } E^{\text{even}}, \text{reg } E^{\text{odd}})$$ as usual. Also, if $\chi_c$ is a non-zerodivisor on $E$ then
$$\text{reg}(E/\chi_cE) = \text{reg } E.$$

**Theorem 7.6.** Suppose that $f_1, \ldots, f_c$ is a regular sequence in a local ring $S$ with infinite residue field $k$, and set $R = S/(f_1, \ldots, f_c)$. Let $N$ be an $R$-module with finite projective dimension over $S$, and let $L$ be the minimal $R$-free resolution of $N$. There exists a non-empty Zariski open dense set $Z$ of upper-triangular matrices $(\alpha_{i,j})$ with entries in $k$, such that for every
$$r \geq 2c - 1 + \text{reg}(\text{Ext}_R(N, k))$$
the syzygy module $\text{Syz}^R(N)$ is pre-stable with respect to the regular sequence $f'_1, \ldots, f'_c$ with
$$f'_i = f_i + \sum_{j>i} \alpha_{i,j} f_j.$$

To prepare for the proof of Theorem 7.6 we will explain the property of the regular sequence $f'_1, \ldots, f'_c$ that we will use. Recall that a sequence of elements $\chi'_1, \chi'_{c-1}, \ldots, \chi'_1 \in \mathcal{R}$ is said to be an almost regular sequence on a graded module $E$ if, for $q = c, \ldots, 1$, the submodule of elements of $E/(\chi'_q, \ldots, \chi'_1)E$ annihilated by $\chi'_q$ is of finite length.

We will use the following lemma with $E = \text{Ext}_R(N, k)$.

**Lemma 7.7.** Suppose that $E$ is a non-zero graded module of regularity $\leq 1$ over $\mathcal{R} = k[\chi_1, \ldots, \chi_c]$. The element $\chi_c$ is almost regular on $E$ if and only if $\chi_c$ is a non-zerodivisor on $E^{\geq 2}[2]$ (equivalently, $\chi_c$ is a non-zerodivisor on $E^{\geq 2}$).

More generally, if we set $E(c) = E$ and $E(j - 1) = E(j)^{\geq 2}[2]/\chi_j E(j)^{\geq 2}[2]$ for $j \leq c$, then the sequence $\chi_c, \ldots, \chi_1$ is almost regular on $E$ if and only if $\chi_j$ is a non-zerodivisor on $E(j)^{\geq 2}[2]$ for every $j$. In that case $\text{reg } E(i) \leq 1$.

**Proof:** By definition the element $\chi_c$ is almost regular on $E$ if the submodule $P$ of $E$ of elements annihilated by $\chi_c$ has finite length. Since $\text{reg}(E) \leq 1$, all such elements must be contained in $E^{\leq 1}$. Hence, $\chi_c$ is a non-zerodivisor on $E^{\geq 2}$.

Conversely, if $\chi_c$ is a non-zerodivisor on $E^{\geq 2}$ then $P \subseteq E^{\leq 1}$ so $P$ has finite length. Therefore, $\chi_c$ is almost regular on $E$.

Thus $\chi_c$ is almost regular if and only if it is a non-zerodivisor on $E^{\geq 2}$ as claimed.

If $\chi_c$ is a non-zerodivisor on $E^{\geq 2}$, then
$$\text{reg}(E^{\geq 2}/\chi_c E^{\geq 2}) = \text{reg}(E^{\geq 2}) \leq 3,$$
whence $\text{reg}(E(c - 1)) \leq 1$. By induction, $\chi_{c-1}, \ldots, \chi_1$ is an almost regular sequence on $E(c - 1)$ if and only if $\chi_j$ is a non-zerodivisor on $E(j)^{\geq 2}[2]$ for every $j < c$, as claimed.

The following result is a well-known consequence of the “Prime Avoidance Lemma” (see for example [Ei3, Lemma 3.3] for Prime Avoidance):

**Lemma 7.8.** If $k$ is an infinite field and $E$ is a graded module over the polynomial ring $\mathcal{R} = k[\chi_1, \ldots, \chi_c]$, then there exists a non-empty Zariski open dense set $\mathcal{Y}$ of lower-triangular matrices $(\nu_{i,j})$ with entries in $k$, such that the sequence of elements $\chi'_c, \ldots, \chi'_1$ with
$$\chi'_i = \chi_i + \sum_{j<i} \nu_{i,j} \chi_j$$
is almost regular on $E$.

Again let $f_1, \ldots, f_c$ be a regular sequence in a local ring $S$ with infinite residue field $k$ and maximal ideal $\mathfrak{m}$, and set $R = S/(f_1, \ldots, f_c)$. Let $N$ be an $R$-module with
finite projective dimension over $S$, and let $L$ be the minimal $R$-free resolution of $N$. Suppose we have CI operators defined by a lifting $\bar{L}$. If we make a change of generators of $(f_1, \ldots, f_c)$ using an invertible matrix $\alpha$ and $f'_i = \sum j \alpha_{i,j} f_j$ with $\alpha_{i,j} \in S$, then the lifted CI operators on the lifting $\bar{L}$ change as follows:

$$\bar{\partial}^2 = \sum_i f'_i \bar{t}_i = \sum_i \left( \sum_j \alpha_{i,j} f_j \right) \bar{t}_i = \sum_j f_j \left( \sum_i \alpha_{i,j} \bar{t}_i \right).$$

So the CI operators corresponding to the sequence $f_1, \ldots, f_c$ are expressed as $t_j = \sum_i \alpha_{i,j} \bar{t}_i$. Thus, if we make a change of generators of the ideal $(f_1, \ldots, f_c)$ using a matrix $\alpha$ then the CI operators transform by the inverse of the transpose of $\alpha$. Another way to see this is from the fact that $R = k[\chi_1, \ldots, \chi_c]$ can be identified with the symmetric algebra of the dual of the vector space $(f_1, \ldots, f_c)/m(f_1, \ldots, f_c)$.

In view of this observation, Lemmas 7.7 and 7.8 can be translated as follows:

**Proposition 7.9.** Let $f_1, \ldots, f_c \in S$ be a regular sequence in a local ring with infinite residue field $k$, and set $R := S/(f_1, \ldots, f_c)$. Let $N$ be an $R$-module of finite projective dimension over $S$, and set $E := \text{Ext}_R(N, k)$.

1. [Av, Ei1] There exists a non-empty Zariski open dense set $\mathcal{Z}$ of upper-triangular matrices $\overline{\pi} = (\pi_{i,j})$ with entries in $k$, such that if $\alpha = (\alpha_{i,j})$ is any matrix over $S$ that reduces to $\overline{\pi}$ modulo the maximal ideal of $S$, then the sequence $f'_1, \ldots, f'_c$ with $f'_i = f_i + \sum j \alpha_{i,j} f_j$ corresponds to a sequence of CI operators $\chi'_c, \ldots, \chi'_1$ that is almost regular on $E$.

2. Furthermore, for such $\chi'_i$ we have the following property. Set $E(c) = E$ and $E(i - 1) = E(i) \otimes [2]/(\chi'_i E(i) \otimes [2])$ for $i \leq c$. Suppose $\text{reg}(E) \leq 1$. Set $\nu = (\alpha^v)^{-1}$. Then $\chi'_c$ is a non-zerodivisor on $\text{Ext}_R^{\geq 2}(N, k)$, and more generally $\chi'_i = \sum j \nu_{i,j} \chi_j$ is a non-zerodivisor on $E(i) \otimes [2]$ for every $i$.

We say that $f'_1, \ldots, f'_c$ with $f'_i = f_i + \sum j \alpha_{i,j} f_j$ are generic for $N$ if $(\alpha_{i,j}) \in \mathcal{Z}$ in the sense above.

**Proof of Theorem 7.6:** To simplify the notation, we may begin by replacing $N$ by its $(\text{reg}(\text{Ext}_R(N, k)) - 1)$-st syzygy, and assume that $\text{reg}(\text{Ext}_R(N, k)) = 1$. After a general change of $f_1, \ldots, f_c$ we may also assume, by Lemma 7.7, that $\chi_c, \ldots, \chi_1$ is an almost regular sequence on $\text{Ext}_R(N, k)$. By Proposition 7.3 it suffices to treat the case $r = 2c$. Set $M = \text{Syz}_R^2(N)$.

Let $(F, \delta)$ be the minimal free resolution of $N' := \text{Syz}_R^2(N)$, so that $M = \ker(\delta_{2c-3})$. Since $N$ has finite projective dimension over $S$, the module $N'$ also has finite projective dimension over $S$.

Let $(F, \delta)$ be a lifting of $F$ to $\tilde{R} := S/(f_1, \ldots, f_{c-1})$, and let $\tilde{t}_c$ be the lifted CI operator. Set $(\tilde{G}, \tilde{\delta}) = \ker(\tilde{t}_c)$. By Proposition 7.9, $\chi_c$ is a monomorphism on $\text{Ext}_R(N', k) = \text{Ext}_R^{\geq 2}(N, k)[2]$. Since $\chi_c$ is induced by $t_c$, Nakayama’s Lemma implies that $t_c$ is surjective, so in particular $F_{\geq 2c-2} \longrightarrow F_{\geq 2c-4}$ is surjective, as required for Condition (1) in 7.1 for $c > 1$.

Using Nakayama’s Lemma again, we see that the lifted CI operator $\tilde{t}_c$ is also an epimorphism. Propositions 4.8 and 4.12 show that $\tilde{G}$ is a minimal free resolution of $N'$.
over \( \tilde{R} \), and \( \mathbf{F} \) is obtained from \( \mathbf{G} \) by the Shamash construction 4.3. Hence \( \text{Ext}_{\tilde{R}}(N', k) = \text{Ext}_R(N', k) / \chi_c \text{Ext}_R(N', k) \), and therefore

\[
\text{Ext}_{\tilde{R}}(N', k) = \left( \text{Ext}^{\geq 2}_R(N, k) / \chi_c \text{Ext}^{\geq 2}_R(N, k) \right)[2].
\]

By Proposition 7.9 we conclude that \( \text{Ext}^{\geq 2}_{\tilde{R}}(N', k) \) has regularity \( \leq 1 \) over \( k[\chi_1, \ldots, \chi_{c-1}] \).

Suppose now that \( c = 1 \), so that \( M = N' \) is the second syzygy of \( N \). In this case \( \tilde{R} = S \), and by hypothesis \( M = N' \) has finite projective dimension over \( S \). Therefore, \( \text{Ext}_S(M, k) \) is a module of finite length. Since it has regularity \( \leq 1 \) (as a module over \( k \)), it follows that it is zero except in degrees \( \leq 1 \), that is, the projective dimension of \( M \) over \( \tilde{R} \) is \( \leq 1 \). By Proposition 7.5, \( M \) is a pre-stable syzygy.

Next suppose that \( c > 1 \). We have \( \text{Ker}(\tilde{\delta}_{2c-3}) = \text{Syz}^{\tilde{R}}_{2c-1}(N') \), and by induction on \( c \) this is a pre-stable syzygy, verifying Condition (2) in 7.1. Thus \( M \) is a pre-stable syzygy.

**Remark 7.10.** Some caution is necessary if we wish to work in the graded case (see for example [Pe] for graded resolutions). Suppose that \( S = k[x_1, \ldots, x_n] \) is a graded polynomial ring with generators \( x_i \) in degree 1, let \( f_1, \ldots, f_c \) be a regular sequence of homogeneous elements, and let \( N \) be a graded module. When all the \( f_i \) have the same degree, so that a general linear scalar combination of them is still homogeneous, then Proposition 7.9 and Theorem 7.6 hold for \( E = \text{Ext}_R(N, k) \) verbatim, without first localizing at the maximal ideal. However when the \( f_1, \ldots, f_c \) have distinct degrees, there may be no homogeneous linear combination of the \( f_j \) that corresponds to an eventually surjective CI operator, as can be seen from the following example. Let \( R = k[x, y] / (x^2, y^3) \) and consider the module \( N = R/x \oplus R/y \). Over the local ring \( S_{(x, y)}/(x^2, y^3) \) the CI operator corresponding to \( x^2 + y^3 \) is eventually surjective. However, the minimal \( R \)-free resolution of \( N \) is the direct sum of the free resolutions of \( R/x \) and \( R/y \). The CI operator corresponding to \( x^2 \) vanishes on the minimal free resolution of \( R/y \). The CI operator corresponding to \( y^3 \) vanishes on the minimal free resolution of \( R/x \), and thus the CI operator corresponding to \( y^3 + ax^3 + bx^2y \), for any \( a, b \), does too.

8. The Box complex

Suppose that \( f \in S \) is a non-zerodivisor. Given an \( S \)-free resolution of an \( S/(f) \)-module \( L \) and a homotopy for \( f \), we will construct an \( S \)-free resolution of the second syzygy \( \text{Syz}^{S/(f)}_2(L) \) of \( L \) as an \( S/(f) \)-module, and also a homotopy for \( f \) on it.

**Box Construction 8.1.** Suppose that \( f \in S \) is a non-zerodivisor, and that

\[
Y : \cdots \rightarrow Y_4 \xrightarrow{\partial_4} Y_3 \xrightarrow{\partial_3} Y_2 \xrightarrow{\partial_2} Y_1 \xrightarrow{\partial_1} Y_0
\]

\[
\theta_3 \\quad \theta_2 \\quad \theta_1 \\quad \theta_0
\]

\[
\tau_1 \\quad \tau_0
\]
is an $S$-free resolution of a module $L$ annihilated by $f$, with homotopies $\{\theta_i : Y_i \rightarrow Y_{i+1}\}_{i \geq 0}$ and higher homotopies $\tau_0 : Y_0 \rightarrow Y_3$ and $\tau_1 : Y_1 \rightarrow Y_4$ for $f$, so that $\partial_3 \tau_0 + \theta_1 \theta_0 = 0$ and $\tau_0 \partial_1 + \theta_2 \theta_1 + \partial_4 \tau_1 = 0$. We call the mapping cone

$$\text{Box}(Y) : \rightarrow Y_4 \xrightarrow{\partial_4} Y_3 \xrightarrow{\partial_3} Y_2$$

(8.3)

$$\oplus \psi \oplus$$

$$Y_1 \xrightarrow{\partial_1} Y_0$$

of the map $\psi := \theta_1 : Y_{\leq 1}[1] \rightarrow Y_{\geq 2}$ the box complex and denote it $\text{Box}(Y)$.

**Box Proposition 8.4.** With notation as above, the box complex $\text{Box}(Y)$ is an $S$-free resolution of the module $\text{Ker}(S/(f) \otimes Y_1 \xrightarrow{S/(f)\otimes \theta_1} S/(f) \otimes Y_0)$, the second $S/(f)$-syzygy of $L$. Moreover, the maps

$$(\theta_2 \tau_0, \theta_1 \theta_0), (\theta_3, \tau_1), \theta_4, \ldots$$

(8.5)

give a homotopy for multiplication by $f$ on $\text{Box}(Y)$ as shown in diagram (8.6):

A similar formula yields a full system of higher homotopies on $\text{Box}(Y)$ from higher homotopies on $Y$, but we will not need this.

**Proof:** The following straightforward computation shows that the maps in (8.5) are homotopies for $f$ on $\text{Box}(Y)$:

$$(\partial_3 \theta_1 + \theta_1 \theta_0, \theta_2 \tau_0, \theta_3) = \left(\begin{array}{ccc} \partial_3 \theta_2 & \theta_1 \partial_2 & \partial_3 \tau_0 + \theta_1 \theta_0 \\ \partial_2 \theta_0 & \theta_1 \partial_2 & \partial_3 \theta_0 \\ \partial_1 \theta_0 & \theta_1 \partial_2 & \partial_3 \theta_0 \end{array} \right) = \left(\begin{array}{c} f \\ 0 \\ f \end{array} \right)$$

(8.7)

$$(\partial_1 \theta_1 + \theta_1 \theta_0, \theta_2 \tau_0, \theta_3) = \left(\begin{array}{ccc} \partial_1 \theta_3 & \theta_1 \partial_3 & \theta_2 \theta_1 + \theta_2 \theta_0 + \partial_1 \tau_1 \\ \partial_2 \theta_0 & \theta_1 \partial_3 & \theta_2 \theta_1 + \theta_0 \partial_1 \\ \partial_1 \theta_0 & \theta_1 \partial_3 & \theta_2 \theta_1 + \theta_0 \partial_1 \end{array} \right) = \left(\begin{array}{c} f \\ 0 \\ f \end{array} \right).$$

Next we will prove that $\text{Box}(Y)$ is a resolution. There is a short exact sequence of complexes

$$0 \rightarrow Y_{\geq 2} \rightarrow \text{Box}(Y) \rightarrow Y_{\leq 1} \rightarrow 0,$$
so \( H_i(\text{Box}(Y)) = H_i(Y_{\geq 2}) = 0 \) for \( i \geq 2 \) since \( Y_{\leq 1} \) is a two-term complex. If \((v, w) \in Y_3 \oplus Y_1\) is a cycle, then applying the homotopy maps in (8.5) we get \((fv, fw) = (\partial_4 \theta_3(v) + \partial_4 \tau_1(w), 0)\). Since \( f \) is a non-zerodivisor, it follows that \( w = 0 \). Thus \( v \) is a cycle in \( Y_{\geq 2} \), which is acyclic, so \( v \) is a boundary in \( Y_{\geq 2} \). Hence, the complex \( \text{Box}(Y) \) is acyclic.

To simplify notation, we write \( \bar{w} \) for the functor \( S/(f) \otimes - \) and set \( \bar{\psi} = \theta_1 \). To complete the proof we will show that \( H_0(\text{Box}(Y)) = \text{Ker}(\bar{\partial}_1 : \bar{Y}_1 \rightarrow \bar{Y}_0) \). Since we have a homotopy for \( f \) on \( Y \), we see that \( f \) annihilates the module resolved by \( Y \). Therefore, \( H_0(Y) = H_1(Y) \). The complex \( \text{Box}(Y) \) is the mapping cone \( \text{Cone}(\bar{\psi} \otimes S/(f)) \), where \( \bar{\psi} = \psi \otimes S/(f) \), so there is an exact sequence of complexes

\[
0 \rightarrow \bar{Y}_{\geq 2} \rightarrow \text{Box}(Y) \rightarrow \bar{Y}_{\leq 1} \rightarrow 0.
\]

Since \( Y \) is a resolution, \( H_0(Y_{\geq 2}) \) is contained in the free \( S \)-module \( Y_1 \). Thus \( f \) is a non-zerodivisor on \( H_0(Y_{\geq 2}) \) and \( \bar{Y}_{\geq 2} \) is acyclic. Therefore, the long exact sequence for the mapping cone yields

\[
0 \rightarrow H_1(\text{Cone}(\bar{\psi})) \rightarrow H_1(\bar{Y}_{\leq 1}) \xrightarrow{\bar{\psi}} H_0(\bar{Y}_{\geq 2}) \rightarrow 0.
\]

It suffices to prove that the map induced on homology by \( \bar{\psi} \) is 0. Let \( u \in Y_1 \) be such that \( \bar{w} \in \text{Ker}(\bar{\partial}_1) \), so \( \partial_1(u) = fy \) for some \( y \in Y_0 \). We also have \( fy = \partial_1 \theta_0(y) \), so \( u - \theta_0(y) \in \text{Ker}(\partial_1) \). Since \( Y \) is acyclic \( u = \theta_0(y) + \partial_2(z) \) for some \( z \in Y_2 \). Applying \( \psi \) we get

\[
\psi(u) = \theta_1 \theta_0(y) + \theta_1 \partial_2(z) = -\partial_3 \tau_0(y) + (fz - \partial_3 \theta_2(z)) = -\partial_3(\tau_0(y) + \theta_2(z)) + fz,
\]

so the map induced on homology by \( \bar{\psi} \) is 0 as desired. \( \square \)

Proposition 8.4 has a partial converse that we will use in the proof of Theorem 10.3.

**Proposition 8.8.** Let \( f \in S \) be a non-zerodivisor and set \( R = S/(f) \). Let

\[
\begin{array}{ccccccccc}
\cdots & \rightarrow & Y_1 & \xrightarrow{\partial_4} & Y_3 & \xrightarrow{\partial_3} & Y_2 & \xrightarrow{\psi} & \oplus \rightarrow & Y_1 & \xrightarrow{\partial_1} & Y_0 \\
\text{Box}(Y) : & & & & & & & & & & & \\
\end{array}
\]

be an \( S \)-free resolution of a module annihilated by \( f \). Set \( \theta_1 := \psi \), and with notation as in diagram (8.6), suppose that

\[
\begin{pmatrix}
\theta_2 \\
\tau_0
\end{pmatrix}
\]

is the first map of a homotopy for multiplication by \( f \) on \( \text{Box}(Y) \). If the cokernels of \( \partial_2 \) and of \( \partial_3 \) are \( f \)-torsion free, then the following complex is exact:

\[
\begin{array}{cccccccccc}
\cdots & \rightarrow & Y_4 & \xrightarrow{\partial_4} & Y_3 & \xrightarrow{\partial_3} & Y_2 & \xrightarrow{\partial_2} & Y_1 & \xrightarrow{\partial_1} & Y_0, \\
\end{array}
\]

and there are homotopies for \( f \) as in (8.2).
 Proof: We first show that the sequence is a complex. The equation \( \partial_3 \partial_4 = 0 \) follows from our hypothesis. Let \((\theta_3, \tau_1) : Y_3 \oplus Y_1 \to Y_2 \) be the next map in the homotopy for \( f \). To show that \( \partial_2 \partial_3 = 0 \) and \( \partial_1 \partial_2 = 0 \), use the homotopy equations

\[
\begin{align*}
0\theta_3 + \partial_2 \partial_3 &= 0 : Y_3 \to Y_1 \\
\partial_1 \partial_2 &= 0 : Y_2 \to Y_0 .
\end{align*}
\]

The equalities in (8.7) imply that \( \theta_0 : Y_0 \to Y_1, \psi = \theta_1 : Y_1 \to Y_2, \theta_2 : Y_2 \to Y_3, \) and \( \theta_3 : Y_3 \to Y_4 \) form the beginning of a homotopy for \( f \) on (8.9). Thus (8.9) becomes exact after inverting \( f \). The exactness of (8.9) is equivalent to the statement that the induced maps \( \text{Coker}(\partial_3) \to Y_1 \) and \( \text{Coker}(\partial_2) \to Y_2 \) are monomorphisms. Since this is true after inverting \( f \), and since the cokernels are \( f \)-torsion free by hypothesis, exactness holds before inverting \( f \) as well.

9. From Syzygies to Matrix Factorizations

Matrix factorizations arising from pre-stable syzygies have an additional property. We introduce the concept of a pre-stable matrix factorizations, which captures that property.

Definition 9.1. A a matrix factorization \((d, h)\) is pre-stable if, in the notation of 2.1, for each \( p = 1, \ldots, c \) the element \( f_p \) is a non-zerodivisor on the cokernel of the composite map

\[
R(p - 1) \otimes A_0(p - 1) \otimes A_0(p) \xrightarrow{h_p} R(p - 1) \otimes A_1(p) \xrightarrow{\pi_p} R(p - 1) \otimes B_1(p).
\]

If \( S \) is Cohen-Macaulay then we say that the matrix factorization \((d, h)\) is stable if the cokernel of the composite map above is a maximal Cohen-Macaulay \( R(p - 1) \)-module.

The advantage of stable matrix factorizations over pre-stable matrix factorizations is that if \( g \in S \) is an element such that \( g, f_1, \ldots, f_c \) is a regular sequence and \((d, h)\) is a stable matrix factorization, then \((S/(g) \otimes d, S/(g) \otimes h)\) is again a stable matrix factorization. We do not know of pre-stable matrix factorizations that are not stable.

Theorem 9.2. Suppose that \( f_1, \ldots, f_c \) is a regular sequence in a local ring \( S \), and set \( R = S/(f_1, \ldots, f_c) \). If \( M \) is a pre-stable syzygy over \( R \) with respect to \( f_1, \ldots, f_c \), then \( M \) is the module of a pre-stable minimal matrix factorization \((d, h)\) such that \( d \) and \( h \) are liftings to \( S \) of the first two differentials in the minimal \( R \)-free resolution of \( M \). If \( M \) is a stable syzygy, then \((d, h)\) is stable as well.

Combining Theorem 9.2 and Theorem 7.6 we obtain the following more precise version of Theorem 1.4 in the introduction.

Corollary 9.3. Suppose that \( f_1, \ldots, f_c \) is a regular sequence in a local ring \( S \) with infinite residue field \( k \), and set \( R = S/(f_1, \ldots, f_c) \). Let \( N \) be an \( R \)-module with finite projective dimension over \( S \). There exists a non-empty Zariski open dense set \( Z \) of matrices \((\alpha_{i,j})\) with entries in \( k \) such that for every

\[
r \geq 2c - 1 + \text{reg}(\text{Ext}_R(N, k))
\]

the syzygy \( \text{Syz}_c^R(N) \) is the module of a minimal pre-stable matrix factorization with respect to the regular sequence \( \{ f'_i = \sum_j \alpha_{i,j} f_j \} \).  

\[\square\]
Proof of Theorem 9.2: The proof is by induction on \( c \). If \( c = 0 \), then \( M = 0 \) so we are done.

Suppose \( c \geq 1 \). We use the notation of Definition 7.1. By assumption, the CI operator \( t_c \) is surjective on a minimal \( R \)-free resolution \((F, \delta)\) of a module \( L \) of which \( M \) is the second syzygy. Let \((\tilde{F}, \tilde{\delta})\) be a lifting of \((F, \delta)\) to \( R' = S/(f_1, \ldots, f_{c-1}) \). Since \( S \) is local, the lifted CI operator \( \tilde{t}_c := (1/f_c)\tilde{\delta}^2 \) is also surjective, and we set \((\tilde{G}, \tilde{\delta}) := \text{Ker}(\tilde{t}_c)\).

By Propositions 4.12, \( F \) is the result of applying the Shamash construction to \( \tilde{G} \). Let \( \tilde{B}_1(c) \) and \( \tilde{B}_0(c) \) be the liftings to \( R' \) of \( F_1 \) and \( F_0 \) respectively. By Propositions 4.8 and 4.12 the minimal \( R' \)-free resolution of \( L \) has the form

\[
\cdots \to \tilde{G}_4 \xrightarrow{\tilde{\partial}_4} \tilde{G}_3 \xrightarrow{\tilde{\partial}_3} \tilde{G}_2 \xrightarrow{\tilde{\partial}_2} \tilde{G}_1(c) \xrightarrow{\tilde{\partial}_1} \tilde{B}_1(c) \to \tilde{B}_0(c),
\]

where \( \tilde{b} := \tilde{\partial}_1, \tilde{\partial}_2, \tilde{\partial}_3, \tilde{\partial}_4 \) are the liftings of the differential in \( F \).

Since \( L \) is annihilated by \( f_c \) there exist homotopy maps \( \tilde{\theta}_0, \tilde{\psi} := \tilde{\theta}_1, \tilde{\theta}_2 \) and a higher homotopy \( \tilde{\tau}_0 \) so that on

\[
\cdots \to \tilde{G}_4 \xrightarrow{\tilde{\partial}_4} \tilde{G}_3 \xrightarrow{\tilde{\partial}_3} \tilde{G}_2 \xrightarrow{\tilde{\partial}_2} \tilde{G}_1(c) \xrightarrow{\tilde{\partial}_1} \tilde{B}_1(c) \to \tilde{B}_0(c)
\]

we have

\[
\begin{align*}
\tilde{\partial}_1\tilde{\theta}_0 &= f_c Id \\
\tilde{\partial}_2\tilde{\theta}_1 + \tilde{\theta}_0\tilde{\partial}_1 &= f_c Id \\
\tilde{\partial}_3\tilde{\theta}_2 + \tilde{\theta}_1\tilde{\partial}_2 &= f_c Id \\
\tilde{\partial}_4\tilde{\theta}_0 + \tilde{\theta}_1\tilde{\theta}_0 &= 0.
\end{align*}
\]

Proposition 8.4 implies that the minimal free resolution of \( M \) over \( R' \) has the form

\[
\cdots \to \tilde{G}_4 \xrightarrow{\tilde{\partial}_4} \tilde{G}_3 \xrightarrow{\tilde{\partial}_3} \tilde{G}_2 \xrightarrow{\tilde{\partial}_2} \tilde{G}_1(c) \xrightarrow{\tilde{\partial}_1} \tilde{B}_1(c) \to \tilde{B}_0(c)
\]

Using this structure we change the lifting of the differential \( \delta_3 \) so that \( \tilde{\delta}_3 = \begin{pmatrix} \tilde{\partial}_3 & \tilde{\psi} \\ 0 & \tilde{b} \end{pmatrix} \).

Note that the differential \( \tilde{\partial} \) on \( \tilde{G}_{\geq 2} \) has not changed.

Set \( M' = \text{Coker}(\tilde{G}_{\geq 2}) = \text{Syz}_{R'}^2(L) \). Since \( M' \) is a pre-stable syzygy, the induction hypothesis implies that \( M' \) is the module of a matrix factorization \((d', h')\) with respect to \( f_1, \ldots, f_{c-1} \) so that the differential \( \tilde{G}_3 \to \tilde{G}_2 \) is \( \tilde{\delta}_3 = d' \otimes R' \) and the differential \( \tilde{G}_4 \to \tilde{G}_3 \) is \( \tilde{\partial}_4 = h' \otimes R' \). Thus, there exist free \( S \)-modules \( A'_1(c-1) \) and \( A'_0(c-1) \) with filtrations so that \( \tilde{G}_3 = A'_1(c-1) \otimes R' \) and \( \tilde{G}_2 = A'_0(c-1) \otimes R' \).
We can now define a matrix factorization for $M$. Let $B_1(c)$ and $B_0(c)$ be free $S$-modules such that $\tilde{B}_0(c) = B_0(c) \otimes R'$ and $\tilde{B}_1(c) = B_1(c) \otimes R'$. For $s = 0, 1$, we consider free $S$-modules $A_1$ and $A_0$ with filtrations such that $A_s(p) = A'_s(p)$ for $1 \leq p \leq c - 1$ and $A_s(c) = A'_s(c) - 1 \oplus B_s(c)$. We define the map $d : A_1 \to A_0$ to be

$$
A_1(c) = A_1(c - 1) \oplus B_1(c) \xrightarrow{(d', \psi)} A_0(c - 1) \oplus B_0(c) = A_0(c)
$$

(9.8)

where $b_c$ and $\psi_c$ are arbitrary lifts to $S$ of $\tilde{b}$ and $\tilde{\psi}$. For every $1 \leq p \leq c - 1$, we set $h_p = h'_p$. Furthermore, we define $h_c : A_0(c) = A_0 \to A_1(c) = A_1$ to be

$$
A_0(c) = A_0(c - 1) \oplus B_0(c) \xrightarrow{(\theta_2, \theta_0)} A_1(c - 1) \oplus B_1(c) = A_1(c)
$$

(9.9)

where $\theta_2, \theta_0, \tau_0$ are arbitrary lifts to $S$ of $\tilde{\theta}_2, \tilde{\theta}_0, \tilde{\tau}_0$ respectively.

We must verify conditions (a) and (b) of Definition 1.2. Since $(d', h')$ is a matrix factorization, we need only check

$$
dh_c \equiv f_c \text{Id}_{A_0(c)} \mod(f_1, \ldots, f_{c - 1})A_0(c)
$$

$$
\pi_c h_c d \equiv f_c \pi_c \mod(f_1, \ldots, f_{c - 1})B_1(c).
$$

Condition (a) holds because

$$
\begin{pmatrix} d' & \psi \\ 0 & b_c \end{pmatrix} \begin{pmatrix} \theta_2 & \tau_0 \\ \partial_2 & \theta_0 \end{pmatrix} = \begin{pmatrix} d'\theta_2 + \theta_1 \partial_2 & d' \tau_0 + \theta_1 \theta_0 \\ \partial_1 \theta_2 & \partial_1 \theta_0 \end{pmatrix} \equiv \begin{pmatrix} f_c & 0 \\ 0 & f_c \end{pmatrix}
$$

by (9.6). Similarly, Condition (b) is verified by the computation

$$
\begin{pmatrix} \theta_2 & \tau_0 \\ \partial_2 & \theta_0 \end{pmatrix} \begin{pmatrix} d' & \psi \\ 0 & b_c \end{pmatrix} = \begin{pmatrix} \theta_2 d' & \theta_2 \theta_1 + \tau_0 \partial_1 \\ \partial_2 d' & \partial_2 \theta_1 + \theta_0 \partial_1 \end{pmatrix} \equiv \begin{pmatrix} * & * \\ 0 & f_c \end{pmatrix}.
$$

Next we show that the matrix factorization that we have constructed is pre-stable. Consider the complex (9.4), which is a free resolution of $L$ over $R'$. It follows that

$$
\text{Coker}(\tilde{A}_0(c - 1) \xrightarrow{\tilde{\partial}} \tilde{B}_1(c)) \cong \text{Im}(\tilde{\partial}_1) \subset \tilde{B}_0(c)
$$

has no $f_c$-torsion, verifying the pre-stability condition.

It remains to show that $d$ and $h$ are liftings to $S$ of the first two differentials in the minimal $R'$-free resolution of $M$.

By (9.5) and Proposition 8.4 we have the following homotopies on the minimal $R'$-free resolution of $M$:

$$
(9.10)
$$

$$
\begin{array}{c}
G_1 \xrightarrow{\tilde{\partial}_1} G_2 \oplus B_0(c) \\
\tilde{B}_1(c) \xrightarrow{\tilde{\partial}_1} \tilde{B}_0(c).
\end{array}
$$
The minimal $R$-free resolution of $M$ is obtained from the resolution above by applying
the Shamash construction. Hence, the first two differentials are

$$ R \otimes \begin{pmatrix} \partial_3 & \psi \\ 0 & b \end{pmatrix} \quad \text{and} \quad R \otimes \begin{pmatrix} \partial_4 & \partial_2 \\ 0 & \partial_0 \end{pmatrix}. $$

By induction hypothesis $\partial_3 = R' \otimes d_{c-1}$ and $\partial_4 = R' \otimes h(c-1)$. By the construction of $d$ and $h$ in (9.8), (9.9) we see that $R \otimes d$ and $R \otimes h$ are the first two differentials in the minimal $R$-free resolution of $M$.

Finally, we will prove that if $M$ is a stable syzygy, then $(d, h)$ is stable as well. By construction (9.9) $\partial_2$ is the composite map $A_0(p-1) \hookrightarrow A_0(p) \xrightarrow{h_p} A_1(p) \xrightarrow{\pi_p} B_1(p).$ By (9.4) it follows that if $L$ is a maximal Cohen-Macaulay $R$-module, then $\text{Coker}(\partial_2)$ is a maximal Cohen-Macaulay $R'$-module, verifying the stability condition for a matrix factorization over $R(p-1)$. By induction, it follows that $(d, h)$ is stable.

**Remark 9.11.** In order to capture structure when minimality is not present, Definition 7.1 can be modified as follows. We extend the definition of syzygies to non-minimal free resolutions: if $(F, \delta)$ is an $R$-free resolution of an $R$-module $P$, then we define $\text{Syz}_r F(P) = \text{Im}(\delta_r)$.

We say that $M$ is a pre-stable syzygy in $F$ with respect to $f_1, \ldots, f_c$ if either $c = 0$ and $M = 0$, or $c \geq 1$ and there exists a lifting $(\bar{F}, \bar{\delta})$ of $(F, \delta)$ to $R' = S/(f_1, \ldots, f_{c-1})$ such that the CI operator $\bar{\tau} := (1/f_c)\bar{\delta}^2$ is surjective and, setting $G, \delta := \text{Ker}(\bar{\tau})$, the module $\text{Im}(\bar{\delta}_r)$ is pre-stable in $G_{>2}$ with respect to $f_1, \ldots, f_{c-1}$.

With minor modifications, the proof of Theorem 9.2 yields the following result: Let $F$ be an $R$-free resolution. If $M$ is a pre-stable $r$-th syzygy in $F$ with respect to $f_1, \ldots, f_c$ then $M$ is the module of a pre-stable matrix factorization $(d, h)$ such that $d$ and $h$ are liftings to $S$ of the consecutive differentials $\delta_{r+1}$ and $\delta_{r+2}$ in $F$. If $F$ is minimal then the matrix factorization is minimal.

We can use the concept of pre-stable syzygy and Proposition 8.8 in order to build the minimal free resolutions of the modules $\text{Coker}(R(p-1) \otimes b_p)$:

**Proposition 9.12.** Let $(d, h)$ be a minimal pre-stable matrix factorization for a regular sequence $f_1, \ldots, f_c$ in a local ring $S$, and use the notation of 2.1. For every $p \leq c$, set $R(p) = S/(f_1, \ldots, f_p)$ and $D(p) = \text{Coker}(R(p-1) \otimes b_p)$. Then $D(p) = \text{Coker}(R(p) \otimes b_p)$. Let $T(p)$ be the minimal $R(p)$-free resolution of $M(p)$ from Construction 5.1 and Theorem 5.2. The minimal $R(p-1)$-free resolution of $D(p)$ is

$$ V(p-1): \quad T(p-1) \longrightarrow R(p-1) \otimes B_1(p) \xrightarrow{R(p-1) \otimes b_p} R(p-1) \otimes B_0(p), $$

where the second differential is induced by the composite map

$$ \delta: A_0(p-1) \hookrightarrow A_0(p) \xrightarrow{h_p} A_1(p) \xrightarrow{\pi_p} B_1(p). $$
The minimal $R(p)$-free resolution of $D(p)$ is
\[ W(p) : \quad T(p) \rightarrow R(p) \otimes B_1(p) \xrightarrow{R(p-1) \otimes b_p} R(p) \otimes B_0(p), \]
where the second differential is given by the Shamash construction applied to $V(p-1)_{\leq 3}$.

Proof: By Theorem 5.2 (using the notation in that theorem) the complex $T(p)$ is an $R(p)$-free resolution of $M(p)$. By Theorem 6.4, the minimal $R(p-1)$-free resolution of $M(p)$ is
\[ T(p-1)_2 \rightarrow T(p-1)_1 \rightarrow T(p-1)_0 \]
\[ \oplus \quad \oplus \quad \oplus \]
\[ R(p-1) \otimes B_1(p) \rightarrow R(p) \otimes B_0(p). \]

Since $f_p$ is a non-zerodivisor on $M(p-1)$ by Corollary 3.10 and since the matrix factorization is pre-stable, we can apply Proposition 8.8, where the homotopies $\theta_i$ and $\tau_i$ for $f_p$ are chosen to be the appropriate components of the map $R(p-1) \otimes h_p$. We get the minimal $R(p-1)$-free resolution
\[ V(p-1) : \quad T(p-1) \rightarrow R(p-1) \otimes B_1(p) \xrightarrow{R(p-1) \otimes b_p} R(p-1) \otimes B_0(p), \]
where the second differential is induced by the composite map
\[ \delta : A_0(p-1) \hookrightarrow A_0(p) \xrightarrow{h_p} A_1(p) \xrightarrow{\pi_p} B_1(p). \]

Since we have a homotopy for $f_p$ on $R(p-1) \otimes B_1(p) \rightarrow R(p-1) \otimes B_0(p)$ it follows that $D(p) = \text{Coker}(R(p) \otimes b_p)$.

We next apply the Shamash construction to the following diagram with homotopies:
\[ V(p-1)_{\leq 3} : \quad A_1(p-1) \xrightarrow{d_{p-1}} A_0(p-1) \xrightarrow{\partial_2} B_1(p) \xrightarrow{\partial_1 = \partial_1'} B_0(p), \]
where $\partial$ stands for $R(p-1) \otimes \cdot$. By Proposition 4.8 we obtain an exact sequence
\[ R(p) \otimes A_1(p) \rightarrow R(p) \otimes A_0(p) \rightarrow R(p) \otimes B_1(p) \rightarrow R(p) \otimes B_0(p). \]
It is minimal since $\theta_0$ is induced by $h_p$. The leftmost differential
\[ R(p) \otimes A_1(p) \xrightarrow{R(p) \otimes b_p} R(p) \otimes A_0(p) \]
coincides with the first differential in $T(p)$. \qed
The following result (stated somewhat differently) and the idea of the proof are from [AGP, Theorem 7.3]. We will use it in Corollary 9.14 in order to obtain numerical information about pre-stable matrix factorizations.

**Proposition 9.13.** Let \( f \in S \) be a non-zerodivisor in a local ring \( S \), and let \( F \) be a minimal free resolution of a nonzero module over \( S/(f) \). If the CI operator \( t : F_2 \rightarrow F_0 \) corresponding to \( f \) is surjective, then \( \text{rank} \ (F_1) \geq \text{rank} \ (F_0) \), and if equality holds then \( F \) is periodic of period 2 (that is, \( \text{Syz}_2^{S/(f)}(L) \cong L \) where \( L = H_0(F) \)). In the latter case, the ranks of the free modules \( F_i \) are constant.

**Proof:** We lift the first two steps of \( F \) to \( S \) as \( \tilde{F}_2 \xrightarrow{\tilde{\delta}_2} \tilde{F}_1 \xrightarrow{\tilde{\delta}_1} \tilde{F}_0 \), so that \( \tilde{\delta}_1 \tilde{\delta}_2 = ft \). Since \( t \) is surjective and \( f \) is in the maximal ideal, \( \tilde{t} \) is surjective. Thus the image of \( \tilde{\delta}_1 \) contains \( f \tilde{F}_0 \), and it follows that \( \text{rank} \ (\tilde{\delta}_1) = \text{rank} \ (\tilde{F}_0) \). In particular, \( \text{rank} \ (F_1) \geq \text{rank} \ (F_0) \). In case of equality \( \tilde{\delta}_1 \) is a monomorphism, and we can factor the multiplication by \( f \) on \( \tilde{F}_0 \) as \( \tilde{\delta}_1 \tilde{u}_1 \) for some \( \tilde{u}_1 \) — a matrix factorization of \( f \). Thus the cokernel of \( \delta_1 \) is resolved by the periodic resolution coming from this matrix factorization, so \( F \) is periodic. Then the ranks of the free modules \( F_i \) are constant by [Ei1, Proposition 5.3]. \( \square \)

Using Proposition 9.13, we get a stronger version of Corollary 3.13 for pre-stable matrix factorizations.

**Corollary 9.14.** Let \((d, h)\) be a minimal pre-stable matrix factorization, and use the notation of 2.1. Let \( \gamma \) be the minimal number such that \( A(\gamma) \neq 0 \). Then \( \text{cx}_R(M) = c - \gamma + 1 \) and

\[
\begin{align*}
\text{rank} \ (B_1(p)) &= \text{rank} \ (B_0(p)) = 0 \quad \text{for every } 1 \leq p \leq \gamma - 1 \\
\text{rank} \ (B_1(\gamma)) &= \text{rank} \ (B_0(\gamma)) > 0 \\
\text{rank} \ (B_1(p)) &= \text{rank} \ (B_0(p)) > 0 \quad \text{for every } \gamma + 1 \leq p \leq c.
\end{align*}
\]

The multiplicity of \( \text{Ext}^{2\gamma} \) (equal to the multiplicity of \( \text{Ext}^{2\gamma} \) and called the Betti degree) is the size of the hypersurface matrix factorization that is the top non-zero part of the matrix factorization \((d, h)\).

For every \( p \leq \gamma - 1 \), the projective dimension of \( M \) over \( R(p) \) is finite and we have the equality of Poincaré series

\[ P^R_M(p)(x) = (1 + x)^p \ P^S_M(x). \]

**Proof:** By Corollary 5.7(2) it follows that \( \text{cx}_R(M) = c - \gamma + 1 \). The definition of matrix factorization shows that \((d, h, \gamma)\) is a codimension 1 matrix factorization, and hence \( \text{rank} \ (B_1(\gamma)) = \text{rank} \ (B_0(\gamma)) > 0 \).

For \( \gamma + 1 \leq p \leq c \), we apply Proposition 9.12. Since \( V(p-1) \) is a free resolution, it follows that \( B_0(p) = 0 \) if \( B_1(p) = 0 \). But \( B_0(p) = B_1(p) = 0 \) implies that \( M(p) = M(p-1) \) which contradicts to the fact that \( f_p \) annihilates \( M(p) \) and \( f_p \) is a non-zerodivisor on \( M(p-1) \) by Corollary 3.10. Hence, \( B_1(p) \neq 0 \). Since the map \( h_p \) is minimal and the free resolution \( V(p-1) \) is minimal, it follows that \( B_0(p) \neq 0 \). The inequality
rank \( (B_1(p)) > \text{rank} \ (B_0(p)) \) follows from Proposition 9.13 since the free resolution \( W(p) \) has a surjective CI operator.

It follows at once that the matrix factorization in Example 3.14 is not pre-stable. But in fact Corollary 9.14 implies stronger restrictions on the Betti numbers in the finite resolution of modules that are pre-stable syzygies:

**Corollary 9.15.** If \( M \) is a pre-stable syzygy of complexity \( \zeta \) with respect to the regular sequence \( f_1, \ldots, f_c \) in a local ring \( S \) and \( b^S_i(M) \) denotes the \( i \)-th Betti number of \( M \) as an \( S \)-module, then

\[
\begin{align*}
\bar{b}_0^S(M) & \geq \zeta \\
\bar{b}_1^S(M) & \geq (c - \zeta + 1)\bar{b}_0^S(M) + \frac{\zeta(\zeta + 1)}{2} - 1
\end{align*}
\]

**Proof:** Set \( \gamma = c - \zeta + 1 \). By Theorem 3.4, Theorem 5.2, and Corollary 9.14 we get

\[
b_0^S(M) = \bar{b}_0^S(M) = \sum_{p=\gamma}^c \text{rank} \ B_0(p) \geq \zeta
\]

and

\[
b_1^S(M) = \bar{b}_1^S(M) + (c - \zeta)\bar{b}_0^S(M) = \bar{b}_1^S(M) + (c - \zeta)\bar{b}_0^S(M)
\]

\[
b_1^{R(\gamma)}(M) = \sum_{p=\gamma}^c \text{rank} \ B_1(p) + \sum_{p=\gamma}^{c-1} (c - p)\text{rank} \ B_0(p)
\]

\[
\geq (c - \gamma + 1 - 1 + \sum_{p=\gamma}^c \text{rank} \ B_0(p)) + \sum_{p=\gamma}^{c-1} (c - p)\text{rank} \ B_0(p)
\]

\[
= \zeta - 1 + \bar{b}_0^S(M) + \sum_{p=\gamma}^{c-1} (c - p)\text{rank} \ B_0(p).
\]

Therefore,

\[
b_1^S(M) = (c - \zeta + 1)\bar{b}_0^S(M) + \zeta - 1 + \sum_{p=\gamma}^{c-1} (c - p)\text{rank} \ B_0(p)
\]

\[
\geq (c - \zeta + 1)\bar{b}_0^S(M) + \zeta - 1 + \left(\frac{\zeta}{2}\right)
\]

\[
= (c - \zeta + 1)\bar{b}_0^S(M) + \frac{\zeta(\zeta + 1)}{2} - 1.
\]

For example, a pre-stable syzygy module of complexity \( \geq 2 \) cannot be cyclic and cannot have \( b_1^S(M) = \bar{b}_0^S(M) + 1 \).

10. **Stable Syzygies in the Local Gorenstein case**

In this section \( S \) will denote a local Gorenstein ring. We write \( f_1, \ldots, f_c \) for a regular sequence in \( S \) and \( R = S/(f_1, \ldots, f_c) \). Thus \( R \) is also a Gorenstein ring. In this setting
Proposition 10.2. takes a particularly canonical form: and only if either \( c_k \) over \( R \) on all of \( T \) Tate resolution of \( R \) integer. collection of functors \( \hat{\cdot} \) and residue field \( m \) matters are simplified by the fact that a maximal Cohen-Macaulay module is, in a canonical way, an \( m \)-th syzygy for any \( m \).

When \( M \) is a maximal Cohen-Macaulay \( S \)-module we let \( \text{Cosyz}_j^S(M) \) be the dual of the \( j \)-th syzygy of \( M^* := \text{Hom}_S(M, S) \). When we speak of syzygies or cosyzygies, we will implicitly suppose that they are taken with respect to a minimal resolution. The following result is well-known.

Cosyzygy Lemma 10.1. Let \( S \) be a local Gorenstein ring.

1. If \( M \) is a maximal Cohen-Macaulay \( S \)-module, then \( M^* \) is a maximal Cohen-Macaulay \( S \)-module, \( M \) is reflexive, and \( \text{Ext}^i_S(M, S) = 0 \) for all \( i > 0 \).
2. If \( M \) is the first syzygy module in a minimal free resolution of a maximal Cohen-Macaulay \( S \)-module, then \( M \) has no free summands.
3. If \( M \) is a maximal Cohen-Macaulay module without free summands, then
   \[
   M \cong \text{Syz}_j^S(\text{Cosyz}_j^S(M)) \cong \text{Cosyz}_j^S(\text{Syz}_j^S(M))
   \]
   for every \( j \geq 0 \), and \( N := \text{Cosyz}_j^S(M) \) is the unique maximal Cohen-Macaulay \( S \)-module \( N \) without free summands such that \( M \) is isomorphic to \( \text{Syz}_j^S(N) \).

Proof Sketch: After replacing \( S \) by its completion we may choose a regular local ring \( S' \subseteq S \) over which \( S \) is finite, and we have \( \text{Ext}_S(M, S) = \text{Ext}_{S'}(M, S') \), and \( M \) is free over \( S' \). Part (2) is obvious over an artinian ring, and the general case follows by factoring out a maximal regular sequence. The first statement of (3) follows from the vanishing of \( \text{Ext} \), and the second part follows from the first.

When \( M \) is a maximal Cohen-Macaulay module over the Gorenstein ring \( S \), we define the Tate resolution of \( M \) to be the doubly infinite free complex \( T \) without homology that results from splicing the minimal free resolution of \( M \) with the dual of the minimal free resolution of \( M^* \). If \( N \) is also an \( S \)-module then the stable \( \text{Ext} \) is by definition the collection of functors \( \overset{j}{\text{Ext}}(M, N) \), the \( j \)-th homology of \( \text{Hom}(T, N) \); here \( j \) can be any integer.

Let \( f_1, \ldots, f_c \) be a regular sequence in a Gorenstein local ring \( S \) with maximal ideal \( \mathfrak{m} \) and residue field \( k \). Set \( R = S/(f_1, \ldots, f_c) \). Let \( M \) be a maximal Cohen-Macaulay \( R \)-module with no free summands and finite projective dimension over \( S \). If \( T \) is the Tate resolution of \( M \) over \( R \), the CI operators corresponding to \( f_1, \ldots, f_c \) are defined on all of \( T \), so that \( \overset{\geq j}{\text{Ext}}_R(M, k) := \oplus_i \overset{-i}{\text{Ext}}_R^j(M, k) \) becomes a graded module over \( \mathcal{R} = k[\chi_1, \ldots, \chi_c] \). Then \( \overset{\geq j}{\text{Ext}}_R^j(M, k) = \text{Ext}_R(\text{Cosyz}_j^R(M), k)[j] \) is a finitely generated module over \( \mathcal{R} \) for any integer \( j \). In this case the definition of a stable syzygy (Definition 7.1) takes a particularly canonical form:

Proposition 10.2. With hypotheses as above, \( M \) is stable with respect to \( f_1, \ldots, f_c \) if and only if either \( c = 0 \) and \( M = 0 \), or the following two conditions are satisfied:

1. \( \chi_c \) is a non-zerodivisor on \( \overset{\geq -2}{\text{Ext}}_R^j(M, k) \).
2. \( \text{Syz}_j^R(\text{Cosyz}_j^R(M)) \) is a stable syzygy with respect to \( f_1, \ldots, f_{c-1} \in S' \), where \( R' = S/(f_1, \ldots, f_{c-1}) \).
Proof: \( \text{Ext}_R^{\geq -2}(M,k) \) is, up to a shift in grading, the same as \( \text{Ext}_R(\text{Cosyz}_2^R(M),k) \), and \( \text{Cosyz}_2^R(M) \) is the only maximal Cohen-Macaulay module of which \( M \) could be the second syzygy.

We will show that stable syzygies all come from stable matrix factorizations.

**Theorem 10.3.** Let \( f_1, \ldots, f_c \) be a regular sequence in a Gorenstein local ring \( S \), and set \( R = S/(f_1, \ldots, f_c) \). An \( R \)-module \( M \) is a stable syzygy if and only if it is the module of a minimal stable matrix factorization with respect to \( f_1, \ldots, f_c \).

We postpone the proof to give a necessary homological construction:

**Proposition 10.4.** Let \( f_1, \ldots, f_c \) be a regular sequence in a Gorenstein local ring \( S \), and set \( R = S/(f_1, \ldots, f_c) \). Let \( M \) be the module of a minimal stable matrix factorization \((d,h)\). Then

\[
\text{Cosyz}_2^R(M(p)) = \text{Coker}(R(p) \otimes b_p) = \text{Coker}(R(p-1) \otimes b_p).
\]

In the notation of Proposition 9.12, the minimal \( R(p-1) \)-free resolution of \( \text{Cosyz}_2^R(M(p)) \) is \( V(p-1) \), and the minimal \( R(p) \)-free resolution of \( \text{Cosyz}_2^R(M(p)) \) is \( W(p) \).

Proof: We apply Proposition 9.12. As the matrix factorization is stable, we conclude that the depth of the \( R(p-1) \)-module \( \text{Coker}(R(p-1) \otimes b_p) \) is one less than that of a maximal Cohen-Macaulay \( R(p-1) \)-module. Therefore, it is a maximal Cohen-Macaulay \( R(p) \)-module. The free resolution \( W \) implies that \( \text{Cosyz}_2^R(M(p)) = \text{Coker}(R(p) \otimes b_p) \).

**Corollary 10.5.** Let \( f_1, \ldots, f_c \) be a regular sequence in a Gorenstein local ring \( S \), and set \( R = S/(f_1, \ldots, f_c) \). If \( M \) is the module of a minimal stable matrix factorization with respect to \( f_1, \ldots, f_c \), then

\[
M(p-1) \cong \text{Syz}_2^R(M(p)).
\]

Proof: For each \( p = 1, \ldots, c \), by Proposition 10.4 we have

\[
M(p-1) = \text{Coker}(R(p-1) \otimes d_{p-1}) = \text{Syz}_2^R(M(p)) = \text{Syz}_2^R(M(p)).
\]

where as usual \( d_p : A_1(p) \rightarrow A_0(p) \) denotes the restriction of \( d : A_1 \rightarrow A_0 \).

Proof of Theorem 10.3: Theorem 9.2 shows that a stable syzygy yields a stable matrix factorization.

Conversely, let \( M \) be the module of a minimal stable matrix factorization \((d,h)\). Use notation as in 2.1. By Proposition 10.4 and in its notation, \( W(p) \) is the minimal \( R \)-free resolution of \( \text{Cosyz}_2^R(M(p)) = \text{Coker}(R(p) \otimes b_p) \). We have a surjective CI operator \( t_c \) on \( W(p) \) because on the one hand, we have it on \( T(p) \) and on the other hand \( W(p)_{\leq 3} \) is given by the Shamash construction so we have a surjective standard CI operator on \( W(p)_{\leq 3} \). Furthermore, the standard lifting of \( W(p) \) to \( R(p-1) \) starts with \( V(p-1)_{\leq 3} \), so in the notation of Definition 7.1 we get \( \text{Ker}(\delta_1) = M(p-1) \), which is stable by induction hypothesis.
Corollary 10.6. Let \( f_1, \ldots, f_c \) be a regular sequence in a Gorenstein local ring \( S \), and set \( R = S/(f_1, \ldots, f_c) \). Let \( M \) be a stable syzygy with a stable minimal matrix factorization \( (d, h) \). For every \( p = 1, \ldots, c \) we have
\[
(Syz^R_1(M(p)))(p - 1) = Syz^R_1(M(p - 1)).
\]

Proof: By induction, it will suffice to prove this assertion for \( M = M(c) \).

The syzygy module \( Syz^R_1(M) \) is stable by Proposition 7.3. Recall the proof of Proposition 7.3 with \( L = \text{Cosyz}^R_2(M) \). The first and last equalities below are from Corollary 10.5, and then we apply (7.4) to get
\[
(Syz^R_1(M))(c - 1) = Syz^R_2(c - 1) \left( \text{Cosyz}^R_2(Syz^R_1(M)) \right)
\]
\[
= \text{Im}(\delta_3) = Syz^R_3(c - 1) \left( \text{Cosyz}^R_2(M) \right)
\]
\[
= Syz^R_1(c - 1) \left( \text{Cosyz}^R_2(M) \right)
\]
\[
= Syz^R_1(c - 1) \left( M(c - 1) \right),
\]
as desired. \( \square \)

Recall that if \( E \) is a graded \( \mathcal{R} \)-module then we define the \( \text{S2-ification} \) of \( E \), written \( \text{S2}(E) \), by the formula
\[
\text{S2}(E) = \bigoplus_{j \in \mathbb{Z}} H^0(\tilde{E}(j))
\]
where \( \tilde{E} \) denotes the coherent sheaf on projective space associated to \( E \).

Proposition 10.7. Suppose that \( R = S/I \), where \( S \) is a regular local ring and \( I \) is generated by a regular sequence, and let \( M \) be maximal Cohen-Macaulay \( R \)-module.

(1) If \( M \) is a stable syzygy then \( M \) has no free summand.

(2) Set \( E := \text{Ext}^{\geq -2}_R(M, k) \). If \( M \) is a stable syzygy, then \( \text{reg} E = -1 \), and \( E \) coincides with \( \text{S2}(E) \) in degrees \( \geq -2 \).

We could restate the last condition of (2) in terms of local cohomology by saying that \( H_{R_+}^1(E) \) is 0 in degree \( \geq -2 \).

Proof: (1): This follows at once from part (2) of Lemma 10.1.

(2): We do induction on \( c \). If \( c = 1 \) then \( E \) is free and generated in degrees \( -2 \) and \( -1 \), so the result is obvious, and we may suppose \( c > 1 \).

From Proposition 10.2 we see that \( \chi_c \) is a non-zerodivisor on \( E \), so
\[
\text{reg}(E) = \text{reg}(E/\chi_c E),
\]
and Corollary 4.14 shows \( (E/\chi_c E)^{\geq 0} = \text{Ext}^{\geq 0}_R(M', k) \), where \( M' = \text{Syz}^R_2(\text{Cosyz}^R_2(M)) \).

Since \( M' \) is stable, \( \chi_{c-1} \) is a non-zerodivisor on \( E' := \text{Ext}^{\geq -2}_R(M', k) \), and thus also on \( E'^{\geq 0} = (E/\chi_c E)^{\geq 0} \), so
\[
H^0_{(\chi_1, \ldots, \chi_c)}((E/\chi_c E)^{\geq 0}) = 0.
\]
Since the modules \( E' \), \( E'^{\geq 0} \) and \( E/\chi_c E \) differ by modules of finite length, they have the same \( i \)-th local cohomology for \( i \geq 1 \). By induction, \( \text{reg}(E') = -1 \), so \( \text{reg}(E/\chi_c E) = -1 \) as well, proving that \( \text{reg} E = -1 \).
Finally we show that $E$ agrees with $S2(E)$ in degrees $\geq -2$. Since $\chi_c$ is a non-zerodivisor on $E$, we see that $E$ is a submodule of $F := S2(E) \geq -2$. Because $\text{reg} \, E = -1$ the natural map $E \rightarrow S2(E)$ is surjective in degrees $\geq -1$.

Thus we need only prove that $E \rightarrow S2(E)$ is surjective in degree $-2$. By induction, $E^{\geq 0}/\chi_c E = \text{Ext}^{\geq 0}(M',k)$ has depth at least 1. But from the exact sequence

$$0 \rightarrow \chi_c F/\chi_c E \rightarrow E^{\geq 0}/\chi_c E \rightarrow E^{\geq 0}/\chi_c F \rightarrow 0$$

we see that the module of finite length $\chi_c F/\chi_c E$ is contained in $E^{\geq 0}/\chi_c E$, so $\chi_c F/\chi_c E = 0$. Since $\chi_c$ is a non-zerodivisor on $E$, and thus also on $F$, this implies that $F/E = 0$ as well.

\[\square\]

11. Syzygies over Intermediate Rings

In this section we suppose that $S$ is a Gorenstein ring. Let $I \subset S$ be an ideal generated by a regular sequence, and set $R = S/I$. Let $N'$ be a finitely generated $R$-module of finite projective dimension over $S$. If $M = \text{Syz}^R_2(N')$ is a sufficiently high syzygy, then by Theorem 7.6 and Theorem 9.2 $M$ comes from a matrix factorization with respect to a generic choice of generators $f_1, \ldots, f_c$ for the ideal $I$. Set $R(p) = S/(f_1, \ldots, f_p)$. The following result identifies the matrix factorization module $M(p)$ with the module $\text{Syz}^R_{c+1}(M)$.

**Theorem 11.1.** Let $f_1, \ldots, f_c$ be a regular sequence in a local Gorenstein ring $S$. Set $R(p) = S/(f_1, \ldots, f_p)$ and $R = R(c)$. Suppose that $M$ is a stable syzygy with stable matrix factorization $(d,h)$ with respect to $f_1, \ldots, f_c$. Let $N = \text{Cosyz}^R_{c+1}(M)$, and set $M(0) = 0$.

1. With notation as in 2.1,
   $$\text{Syz}^R_{c+1}(N) \cong M(p) \quad \text{for } p \geq 0.$$

2. We write $\nu_p$ for the map
   $$R(p) \otimes \text{Syz}^R_{(p-1)}(N) \xrightarrow{\nu_p} \text{Syz}^R_p(N),$$
   induced by the comparison map from the minimal $R(p-1)$-free resolution of $N$ to the minimal $R(p)$-free resolution of $N$ inducing the identity map on $N$ (this comparison map is unique up to homotopy). For each $p$, there is a short exact sequence
   $$0 \rightarrow R(p) \otimes M(p-1) \xrightarrow{\nu_p} M(p) \rightarrow \text{Cosyz}_2 M(p) \rightarrow 0.$$

For the proof of Theorem 11.1 we will make use of the following well-known result. For the reader’s convenience we sketch the proof. Write $\text{mod}(R)$ for the category of finitely generated $R$-modules and $\text{MCM}(R(p))$ for the stable category of maximal Cohen-Macaulay $R(p)$-modules, where the morphisms are morphisms in $\text{mod}(R(p))$ modulo those that factor through projectives. We say that $S$-modules $M, M'$ have a common syzygy if there exists a $j$ such that $\text{Syz}^S_j(M) \cong \text{Syz}^S_j(M')$ in $\text{MCM}(S)$.

**Lemma 11.2.** Suppose that $S$ is a Gorenstein ring and that $M, M'$ are $S$-modules. 47
(1) If $N, N'$ are $S$-modules and there are exact sequences
\[
0 \to M \to P_r \to \cdots \to P_0 \to N \to 0,
\]
\[
0 \to M' \to P'_r \to \cdots \to P'_0 \to N' \to 0
\]
such that each $P_i$ and each $P'_i$ is a module of finite projective dimension over $S$, then $M$ and $M'$ have a common syzygy if and only if $N$ and $N'$ have a common syzygy.

(2) If $M$ and $M'$ have a common syzygy and are both maximal Cohen-Macaulay $S$-modules then $M \cong M'$ in $\text{MCM}(S)$.

(3) If $M \cong M'$ in $\text{MCM}(S)$, the ring $S$ is local, and both $M$ and $M'$ are maximal Cohen-Macaulay $S$-modules without free summands, then $M \cong M'$ as $S$-modules.

**Proof:** (1): It suffices to do the case $r = 0$. Let $N_1 = \text{Ker}(P_0 \to N)$, and let $V$ be a free resolution of $N_1$. The mapping cone of a map from $V$ to a finite resolution of $P_0$ is a free resolution of $N$, so that for $i \gg 0$ we have $\text{Syz}_i^S(N) \cong \text{Syz}_{i-1}(N_1)$ in $\text{MCM}(S)$. By induction, for $i \gg 0$ the $(i-1-r)$-th syzygy of $M$ agrees with the $i$-th syzygy of $N$, and the same is true for $M'$ and $N'$.

(2): If $\text{Syz}_j^S(M) \cong \text{Syz}_j^S(M') \cong N$, then $M \cong \text{Cosyz}_j^S(N) \cong M'$ in $\text{MCM}(S)$.

(3): Let $M \xrightarrow{\alpha} M' \xrightarrow{\beta} M$ be inverse isomorphisms in $\text{MCM}(S)$. This means that $\beta\alpha = \text{Id}_M + \varphi\phi$, where $M \xrightarrow{\varphi} F \xrightarrow{\phi} M$ for some free module $F$. Since $S$ is local and $M$ has no free summand, $\varphi$ must have image inside the maximal ideal times $F$, and thus $\phi\varphi$ has image inside the maximal ideal times $M$. By Nakayama’s Lemma, $\beta\alpha$ is an epimorphism, and it follows that $\beta\alpha$ is an isomorphism. Since the same goes for $\alpha\beta$, we see that $M \cong M'$.

**Proof of Theorem 11.1:**

(1): By Corollary 3.10 $M(p)$ is a maximal Cohen-Macaulay $R(p)$-module, and by Corollary 3.11 it has no free summand. In particular, $N = \text{Syz}_c^{R(p)}(M)$ is well-defined and has no free summands. It follows that $\text{Syz}_c^{R(p)}(N)$ is a maximal Cohen-Macaulay $R(p)$-module and by the Cosyzygy Lemma 10.1 it has no free summands. By Lemma 11.2(3), it suffices to show that the maximal Cohen-Macaulay $R(p)$-modules $M(p)$ and $\text{Syz}_c^{R(p)}(N)$ have a syzygy in common over $R(p)$. We will do this by showing that each of these modules has an $R(p)$-syzygy in common with $M$.

Observe that $R$ has finite projective dimension over $R(p)$. Lemma 11.2(1) implies that, indeed, $M = \text{Syz}_c^{R(p)}(N)$ and $\text{Syz}_c^{R(p)}(N)$ have a common syzygy over $R(p)$.

We next compare $M = M(c)$ with $M(p)$. When $p > q$ the module $R(p)$ has finite projective dimension over $R(q)$. By Corollary 10.5, $M(p-1) = \text{Syz}_c^{R(p-1)}(\text{Cosyz}_2^{R(p)}(M(p)))$.

Applying Lemma 11.2(1) to an $R(p-1)$-free resolution of $\text{Cosyz}_2^{R(p)}(M(p))$ and to an $R(p)$-free resolution of $\text{Cosyz}_2^{R(p)}(M(p))$, we conclude that $M(p-1)$ and $M(p)$ have a common syzygy over each ring $R(q)$ with $q \leq p - 1$.

(2): For each $p$, let $T(p)$ be the minimal $R(p)$-free resolution of $M(p)$ and let $W(p)$ be the minimal $R(p)$-free resolution of $\text{Cosyz}_2^{R(p)}(M(p))$. See also Proposition 10.4. Since
\( M(p-1) \) is a maximal Cohen-Macaulay \( R(p-1) \)-module by Corollary 3.10, the minimal free resolution of \( R(p) \otimes M(p-1) \) as an \( R(p) \)-module is \( R(p) \otimes T(p-1) \).

Since \( M(p) \) is a stable syzygy, the CI operator \( t_p \) is surjective on \( W(p) \). Take a lifting \( \tilde{t}_p \) acting on a lifting of \( W(p) \) to \( R(p-1) \). The kernel of \( \tilde{t}_p \) is a minimal \( R(p-1) \)-free resolution \( \tilde{G} \) of \( \text{Cysy}_2^{R(p)}(M(p)) \). By Corollary 10.5, \( T(p-1) \) is isomorphic to \( \tilde{G}_{\geq 2}[-2] \). Thus we have a short exact sequence of minimal free resolutions

\[
0 \rightarrow R(p) \otimes T(p-1) \rightarrow T(p) \rightarrow W(p)[-2] \rightarrow 0,
\]

and this induces the desired short exact sequence of modules.

The last claim in the theorem follows from Corollary 10.6.

**Corollary 11.3.** With hypotheses as in Theorem 11.1, let \( M \) be a stable syzygy with respect to \( f_1, \ldots, f_c \), with stable matrix factorization \( (d, h) \). If we denote the codimension 1 part of \( (d, h) \) by \( (d_1, h_1) \), then the codimension 1 part of the matrix factorization of \( \text{Syz}_{1}^{R}(M) \) is \( (h_1, d_1) \).

**Proof:** If \( (d_1, h_1) \) is non-trivial, then the minimal \( R(1) \)-free resolution of \( M(1) = R(1) \otimes d_1 \) is periodic of the form

\[
\cdots \rightarrow F_4 \xrightarrow{d_1} F_3 \xrightarrow{h_1} F_2 \xrightarrow{d_1} F_1 \xrightarrow{h_1} F_0 \rightarrow 0.
\]

**Theorem 11.4.** Suppose that \( f_1, \ldots, f_c \) is a regular sequence in a Gorenstein local ring \( S \), and set \( R = S/(f_1, \ldots, f_c) \). Suppose that \( N \) is an \( R \)-module of finite projective dimension over \( S \). Assume that \( f_1, \ldots, f_c \) are generic with respect to \( N \). Denote \( \gamma := c - e_xR(N) + 1 \), where \( e_xR(N) \) is the complexity of \( N \) (see Corollary 5.8). Then:

1. The projective dimension of \( N \) over \( R(p) = S/(f_1, \ldots, f_p) \) is finite for \( p < \gamma \).
2. Choose a \( j \geq 1 \) large enough so that \( M := \text{Syz}_{j}^{R}(N) \) is a stable syzygy and \( \text{Syz}_{j}^{R(p)}(N) \) is a maximal Cohen-Macaulay \( R(p) \)-module for every \( p \leq \gamma \). The hypersurface matrix factorization for the periodic part of the minimal free resolution of \( N \) over \( S/(f_1, \ldots, f_1) \) is isomorphic to the top non-zero part of the matrix factorization of \( M \).

A version of (1) is proved in [Av, Theorem 3.9], [AGP, 5.8 and 5.9].

**Proof:** Choose \( M \) as in (2). By Corollary 5.8, \( M(p) = 0 \) for \( p < \gamma \). Apply Proposition 11.1 for \( p \leq \gamma \). The case \( p = \gamma \) establishes (2).

**Remark 11.5.** In particular, the above theorem shows that the codimension 1 matrix factorization that is obtained from a high \( S/(f_1) \)-syzygy of \( N \) agrees with the codimension 1 part of the matrix factorization for \( M \) over \( R \), and both codimension 1 matrix factorizations are trivial if the complexity of \( N \) is \( < c \), where \( M \) is a sufficiently high syzygy of \( N \) over \( R \).

12. Functoriality

From Theorem 11.1 it follows immediately that the matrix factorization construction induces functors on the stable module category. In this section we make the result explicit.
Let \( R(p) = S/(f_1, \ldots, f_p) \). If \( i > \dim R \) then the modules \( \text{Syz}^R_i(N) \) are maximal Cohen-Macaulay \( R(p) \)-modules. We define functors

\[
F_i : \text{mod}(R) \to \prod_p \text{Mor}(\text{MCM}(R(p)))
\]

taking \( N \) to the collection of morphisms

\[
R(p) \otimes \text{Syz}^R_i(N) \overset{\nu_p}{\to} \text{Syz}^R_i(N),
\]

where \( \nu_p \) is the comparison map defined in Theorem 11.1. The map \( \nu_p \) is unique up to homotopy, and thus yields a well-defined morphism in \( \text{MCM}(R(p)) \).

**Corollary 12.1.** With assumptions and notation as in Theorem 11.1, for each \( p = 1 \ldots c - 1 \) there exists a triangle in \( \text{MCM}(R(p + 1)) \) of the form

\[
\begin{array}{ccc}
R(p + 1) \otimes M(p) & \overset{\nu_p}{\to} & M(p + 1) \\
\downarrow & & \downarrow \\
M(p + 1)[-2] := \text{Cosyz}^R_{i-2}(M(p + 1))
\end{array}
\]

If \( M' \) is a first syzygy of \( M \), then the corresponding triangles for \( M' \) are obtained from the triangles for \( M \) by applying the shift (equivalently, taking first syzygy) operator to each \( M(p) \).

We remark that Theorem 11.1 implies that for \( i \geq c + 3 \) we get a triangle

\[
\begin{array}{ccc}
R(p) \otimes \text{Syz}^R_i(N) & \overset{\nu_p}{\to} & \text{Syz}^R_i(N) \\
\downarrow & & \downarrow \\
\text{Syz}^R_{i-2}(N)
\end{array}
\]

Let \( \text{MF}(f_1, \ldots, f_c) \) be the full subcategory of \( \text{MCM}(R) \) whose objects are stable equivalence classes of maximal Cohen-Macaulay modules that are stable syzygies with respect to \( f_1, \ldots, f_c \). We get a functor \( \mathcal{F} : \text{MF}(f_1, \ldots, f_c) \to \mathcal{C} \), where an object \( \mathcal{M} \) of \( \mathcal{C} \) is a collection of objects \( M(p) \in \text{MCM}(R(p)) \) for \( p = 1, \ldots, c \) that fit into triangles as in Corollary 12.1 in \( \text{MCM}(R(p + 1)) \) and whose morphisms \( \mathcal{M} = \{M(p)\} \to \mathcal{M}' = \{M'(p)\} \) are collections of morphisms \( \{(M(p) \to M'(p)) \in \text{MCM}(R(p))\} \) that commute with the morphisms in the triangles. Furthermore, if \( M' \) is the first syzygy of \( M \), then \( \mathcal{F}(M') \) is obtained from \( \mathcal{F}(M) \) by applying the shift (equivalently, taking first syzygy) operator in \( \text{MCM}(R(p)) \) to each \( M(p) \) and to each triangle.
13. Morphisms

In this section we introduce the concept of a morphism of matrix factorizations so that it preserves the structures described in Definition 1.2, and then we show that any homomorphism of matrix factorization modules induces a morphism of matrix factorizations.

Definition 13.1. A morphism of matrix factorizations $\alpha : (d, h) \rightarrow (d', h')$ is a triple of homomorphisms of free modules

$$\begin{align*}
\alpha_0 : A_0 &\longrightarrow A_0' \\
\alpha_1 : A_1 &\longrightarrow A_1' \\
\alpha_2 : \oplus_{p \leq c} A_0(p) &\longrightarrow \oplus_{p \leq c} A_0'(p)
\end{align*}$$

such that, for each $p$:

(a) $\alpha_s(A_s(p)) \subseteq A'_s(p)$ for $s = 0, 1$. We write $\alpha_s(p)$ for the restriction of $\alpha_s$ to $A_s(p)$.

(b) $\alpha_2(\oplus_{q \leq p} A_0(q)) \subseteq \oplus_{q \leq p} A_0'(q)$, and the component $A_0(p) \longrightarrow A_0'(p)$ of $\alpha_2$ is $\alpha_0(p)$.

We write $\alpha_2(p)$ for the restriction of $\alpha_2$ to $\oplus_{q \leq p} A_0(q)$.

(c) The diagram

$$
\begin{array}{c}
\oplus_{q \leq p} A_0(q) \xymatrix{\ar[r]^-{h} & A_1(p) \ar[r]^-{d_p} & A_0(p)} \\
\oplus_{q \leq p} A_0'(q) \xymatrix{\ar[r]^-{h'} & A_1'(p) \ar[r]^-{d'_p} & A_0'(p)}
\end{array}
$$

commutes modulo $(f_1, \ldots, f_{p-1})$.

Theorem 13.2. Suppose that $f_1, \ldots, f_c$ is a regular sequence in a Gorenstein local ring $S$, and set $R = S/(f_1, \ldots, f_c)$. Let $M$ and $M'$ be stable syzygies over $R$, and suppose $\zeta : M \longrightarrow M'$ is a morphisms of $R$-modules. With notation as in 2.1, let $M$ and $M'$ be matrix factorization modules of stable matrix factorizations $(d, h)$ and $(d', h')$, respectively. There exists a morphism of matrix factorizations $\alpha : (d, h) \rightarrow (d', h')$ such that the map induced on

$$M = \text{Coker}(R \otimes d) \longrightarrow \text{Coker}(R \otimes d') = M'$$

is $\zeta$.

We first establish a strong functoriality statement for the Shamash construction. Suppose that $G$ and $G'$ are $S$-free resolutions of $S$-modules $M$ and $M'$ annihilated by a non-zero divisor $f$, and $\zeta : M \longrightarrow M'$ is any homomorphism. If we choose systems of higher homotopies $\sigma$ and $\sigma'$ for $f$ on $G$ and $G'$ respectively, then the Shamash construction yields resolutions $\text{Sh}(G, \sigma)$ and $\text{Sh}(G', \sigma')$ of $M$ and $M$ over $R = S/(f)$, and thus there is a morphism of complexes $\tilde{\phi} : \text{Sh}(G, \sigma) \longrightarrow \text{Sh}(G', \sigma')$ covering $\zeta$. To prove the Theorem we need more: a morphism defined over $S$ that commutes with the maps in the “standard liftings” $\text{Sh}(G, \sigma)$ and $\text{Sh}(G', \sigma')$ (see Construction 4.7) and respects the natural filtrations of these modules. The following statement provides the required morphism.
Lemma 13.3. Let $S$ be a commutative ring, and let $\varphi_0 : (G,d) \to (G',d')$ be a map of $S$-free resolutions of modules annihilated by an element $f$. Given systems of higher homotopies $\sigma_j$ and $\sigma'_j$ on $G$ and $G'$, respectively, there exists a system of maps $\varphi_j$ of degree $2j$ from the underlying free module of $G$ to that of $G'$ such that, for every index $m$,

$$\sum_{i+j=m} (\sigma'_i \varphi_j - \varphi_j \sigma_i) = 0.$$ 

We say that $\{\varphi_j\}$ is a system of homotopy comparison maps if they satisfy the conditions in the lemma above.

Recall that a map of free complexes $\lambda : U \to W[-a]$ is a homotopy for a map $\rho : U \to W[-a+1]$ if $\delta \lambda - (-1)^a \lambda \delta = \rho$, where $\delta$ and $\delta$ are the differentials in $U$ and $W$ respectively. Since in Lemma 13.3 $\sigma_0$ and $\sigma'_0$ are the differentials $d$ and $d'$, the equation above in Lemma 13.3 says that, for each $m$, the map $\varphi_m$ is a homotopy for the sum

$$- \sum_{i+j=m, i>0, j>0} (\sigma'_i \varphi_j - \varphi_j \sigma_i).$$

PROOF: The desired condition on $\varphi_0$ is equivalent to the given hypothesis that $\varphi_0$ is a map of complexes. We proceed by induction on $m > 0$ and on homological degree to prove the existence of $\varphi_m$. The desired condition can be written as

$$d' \varphi_m = - \sum_{i+j=m, i>0} (\sigma'_i \varphi_j + \sum_{i+j=m} \varphi_j \sigma_i).$$

Since $G$ is a free resolution, it suffices to show that the right-hand side is annihilated by $d'$. Indeed,

$$- \sum_{i+j=m} (d' \sigma'_i) \varphi_j + \sum_{i+j=m} (d' \varphi_j) \sigma_i
= \sum_{i+j=m} \sum_{i' \neq 0} \sigma'_i \sigma'_w \varphi_j - f \varphi_{m-1} - \sum_{i+j=m} \sum_{i' \neq 0} \sigma'_q \varphi_q \sigma_i
+ \sum_{i+j=m} \sum_{i' \neq 0} \varphi_u \sigma_q \sigma_i
= \sum_{i+j=m} \sum_{i' \neq 0} \sigma'_i \sigma'_w \varphi_j - f \varphi_{m-1} - \sum_{i+j=m} \sum_{i' \neq 0} \sigma'_q \varphi_q \sigma_i
+ \sum_{i+j=m} \sum_{i' \neq 0} \varphi_u \sigma_q \sigma_i,$$

where the first equality holds by (3) in 4.1 and by the induction hypothesis. Reindexing the first summand by $v = q$, $w = i$ and $j = u$ we get

$$= \sum_{i+j=m, \sigma_q \neq 0} \sigma'_q \sigma'_u \varphi_q - f \varphi_{m-1} - \sum_{i+j=m, \sigma_q \neq 0} \sigma'_q \varphi_q \sigma_i
+ \sum_{i+j=m} \varphi_u \sigma_q \sigma_i
= - f \varphi_{m-1} + \sum_{q \neq 0} \sigma'_u \left( \sum_{i+j=m} \sigma'_q \varphi_q - \varphi_q \sigma_q \right)
+ \sum_{u} \varphi_u \left( \sum_{q \neq 0} \sigma_q \sigma_q \right)
= - f \varphi_{m-1} + \sum_{q \neq 0} \sigma'_u \varphi_q - \sum_{q \neq 0} \varphi_q \sigma_q
= - f \varphi_{m-1} + 0 + 0 + \varphi_{m-1} f = 0.$$
where the last equality holds by (3) in 4.1 and by induction hypothesis.

The next result reinterprets the conditions of Lemma 13.3 as defining a map between liftings of Shamash resolutions.

**Proposition 13.4.** Let $S$ be a commutative ring, and let $\mathbf{G}$ and $\mathbf{G}'$ be $S$-free resolutions with systems of higher homotopies $\sigma = \{\sigma_j\}$ and $\sigma' = \{\sigma'_j\}$ for $f \in S$, respectively. Suppose that $\{\varphi_j\}$ is a system of homotopy comparison maps for $\sigma$ and $\sigma'$. We use the standard lifting of the Shamash resolution defined in 4.7, and the notation established there. Denote by $\tilde{\varphi}$ the map with components

$$
\varphi_i : y^{(v)}G_j \rightarrow y^{(v-i)}G'_{j+2i}
$$

from the underlying graded free $S$-module of the standard lifting $\tilde{\text{Sh}}(\mathbf{G}, \sigma)$ of the Shamash resolution $\text{Sh}(\mathbf{G}, \sigma)$, to the underlying graded free $S$-module of the standard lifting $\tilde{\text{Sh}}(\mathbf{G}', \sigma')$ of the Shamash resolution $\text{Sh}(\mathbf{G}', \sigma')$. The maps $\tilde{\varphi}$ satisfy $\tilde{\delta}' \tilde{\varphi} = \tilde{\varphi} \tilde{\delta}$, where $\tilde{\delta}$ and $\tilde{\delta}'$ are the standard liftings of the differentials defined in 4.7.

**Proof:** Fix $a$ and $v$. We must show that the diagram

$$
\begin{array}{ccc}
\oplus_{0 \leq i \leq a} y^{(a-i)}G_{v+2i-1} & \xrightarrow{\delta} & \oplus_{0 \leq j \leq a} y^{(a-j)}G'_{v+2j-1} \\
\downarrow \tilde{\varphi} & & \downarrow \tilde{\varphi}' \\
\oplus_{0 \leq i \leq a} y^{(a-i)}G'_{v+2i-1} & \xrightarrow{\delta} & \oplus_{0 \leq j \leq a} y^{(a-j)}G_{v+2j-1}.
\end{array}
$$

commutes. Fix $0 \leq q \leq a$. The map $\tilde{\delta}' \tilde{\varphi} - \tilde{\varphi} \tilde{\delta}$ from $y^{(a)}G_v$ to $y^{(q)}G'_{v+2a-2q-1}$ is equal to $\sum_{i+j=a-q} (\sigma'_i \varphi_j - \varphi_j \sigma_i)$, which vanishes by Lemma 13.3.

**Remark 13.5.** A simple modification of the proof of Lemma 13.3 shows that systems of homotopy comparison maps also exist in the context of systems of higher homotopies for a regular sequence $f_1, \ldots, f_c$, not just in the case $c = 1$ as above, and one can interpret this in terms of Shamash resolutions as in Proposition 13.4 as well, but we do not need these refinements.

**Proof of Theorem 13.2:** The result is immediate for $c = 1$, so we proceed by induction on $c \geq 1$. Let $\overline{R} = S/(f_1, \ldots, f_{c-1})$. To simplify the notation, we will write $\overline{-}$ for $\overline{R} \otimes -$, and $\overline{=} -$ for $R \otimes -$. We will make use of our standard notation 2.1.

Since $(d, h)$ is stable we can extend the map $\overline{d}$ to a complex

$$
\overline{A}_1(c) \longrightarrow \overline{A}_0(c) \longrightarrow \overline{B}_1(c) \longrightarrow \overline{B}_0(c)
$$

that is the beginning of an $R$-free resolution $\mathbf{F}$ of $\text{Cosyz}_2^R(M)$, and there is a similar complex that is the beginning of the $R$-free resolution $\mathbf{F}'$ of $\text{Cosyz}_2^R(M')$. By stability these cosyzygy modules are maximal Cohen-Macaulay modules, so dualizing these complexes
we may use $\zeta(c) := \zeta: M \to M'$ to induce maps
\[ \eta: \text{Cosyz}_2^R(M) \to \text{Cosyz}_2^R(M') \]
\[ \lambda: F \to F'. \]

Moving to $\tilde{R}$, we have
\[ M(c - 1) = \text{Coker} \tilde{d}(c - 1) = \text{Syz}_2^\tilde{R}(\text{Cosyz}_2^R(M)) \]
\[ M'(c - 1) = \text{Coker} \tilde{d}'(c - 1) = \text{Syz}_2^\tilde{R}(\text{Cosyz}_2^R(M')). \]

We will use the notation and the construction in the proof of Theorem 9.2, where we produced an $\tilde{R}$-free resolution $V$ of $\text{Cosyz}_2^R(M)$, and various homotopies on it. Of course we have a similar resolution $V'$ of $\text{Cosyz}_2^R(M')$. See diagram (13.6). The map $\eta$ induces
\[ \tilde{\xi}: V \to V' \]
which in turn induces a map
\[ \zeta(c - 1): M(c - 1) \to M'(c - 1). \]
See diagram (13.6).

By induction, the map $\zeta(c - 1)$ is induced by a morphism of matrix factorizations with components
\[ \alpha_s(c - 1): A_s(c - 1) \to A'_s(c - 1) \]
for $s = 0, 1$ and
\[ \alpha_2(c - 1): \bigoplus_{q \leq c - 1} \tilde{A}_0(q) \to \bigoplus_{q \leq c - 1} \tilde{A}_0(q'). \]
By the conditions in 13.1, it follows that the first two squares on the left are commutative; clearly, the last square on the right is commutative as well. Since \( \tilde{\alpha}_0(c-1) \) induces the same map on \( M(c-1) \) as \( \xi \), the remaining square commutes. Therefore we can apply Lemma 13.3 and conclude that there exists a system of homotopy comparison maps, the first few of which are shown as \( \varphi_1 \) and \( \varphi_2 \) in diagram (13.6).

With notation as in (9.8) and (9.9) we may write the first two steps of the minimal \( \tilde{R} \)-free resolution \( U \) of \( M \) in the form given by the top three rows of diagram (13.7), where we have used a splitting \( A_0(c) = A_0(c-1) \oplus B_0(c) \) to split the left-hand term \( \oplus_{q \leq c} A_0(q) \) into three parts, \( \oplus_{q \leq c-1} A_0(q), A_0(c-1), \) and \( B_0(c) \); and similarly for \( M' \) and the bottom three rows. Straightforward computations using the definition of the system of homotopy comparison maps shows that diagram (13.7) commutes.

Next, we will construct the maps \( \alpha_0 \). We construct \( \alpha_0 \) by extending the map \( \alpha_0(c-1) \) already defined over \( S \) by taking \( \alpha_0|_{B_0(c)} \) to have as components arbitrary liftings to \( S \) of \( \tilde{\xi}_0 \) and \( \tilde{\varphi}_1 \). Similarly we take \( \alpha_1 \) to be the extension of \( \alpha_1(c-1) \) that has arbitrary liftings of \( \tilde{\xi}_1 \) and \( \tilde{\varphi}_1 \) as components. Finally, we take \( \alpha_2 \) to agree with \( \alpha_2(c-1) \) on \( \oplus_{q \leq c-1} A_0(q) \) and on the summand \( A_0(c) = A_0(c-1) \oplus B_0(c) \), to be the map given by

\[
\alpha_0(c-1) : A_0(c-1) \to A'_0(c-1)
\]

and arbitrary liftings

\[
\varphi_1 : A_0(c-1) \to \oplus_{q \leq c-1} A'_0(q) \quad \varphi_1 : B_0(c) \to A'_0(c-1) \quad \varphi_2 : B_0(c) \to \oplus_{q \leq c-1} A'_0(q)
\]

to \( S \) of \( \tilde{\varphi}_1, \tilde{\varphi}_2 \) and \( \tilde{\xi}_0 \).

(13.7)
It remains to show that $\pi_0 = R \otimes_S \alpha_0$ induces $\zeta : M \to M'$.

By Proposition 10.4 the minimal $R$-free resolutions $F$ and $F'$ of $\text{Cosyz}_2^R(M)$ and $\text{Cosyz}_2^R(M')$ have the form given in the following diagram.

$$
\begin{array}{cccc}
F : & \cdots & \overline{B}_0(c) \oplus \overline{A}_0(c-1) & \overline{B}_1(c) & \overline{B}_0(c) \\
& \vline & \lambda_2 & \lambda_1 = \overline{\xi}_1 & \lambda_0 = \overline{\xi}_0 \\
& \pi_0 & \downarrow & \downarrow & \\
F' : & \cdots & \overline{B}'_0(c) \oplus \overline{A}'_0(c-1) & \overline{B}'_1(c) & \overline{B}'_0(c) \\
\end{array}
$$

By definition the map of complexes $\lambda : F \to F'$ induces $\zeta : M \to M'$. Using Lemma 13.3, we see that the left-hand square of the diagram also commutes if we replace $\lambda_2$ with $\alpha_0$, and thus these two maps induce the same map $M \to M'$, concluding the proof.

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