RECENT DEVELOPMENTS
IN THE PINCH TECHNIQUE *

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ABSTRACT
Some of the most important theoretical and phenomenological aspects of the
pinch technique are presented, and several recent developments are briefly re-
viewed.

1. General considerations

The pinch technique (PT) is an algorithm that allows the construction of modified
gauge independent (g.i.) off-shell n-point functions, through the order by order
rearrangement of Feynman graphs contributing to a certain physical and therefore
ostensibly g.i. process, such as an S-matrix element or a Wilson loop. The PT
was originally introduced in an attempt to device a consistent truncation scheme for
the Schwinger-Dyson equations (SDE) that govern the dynamics of gauge theories.
These equations are inherently non-perturbative and could in principle provide im-
portant information about a plethora of phenomena in non-Abelian gauge theories
not captured by perturbation theory. In practice however, one is severely limited in
exploiting them, mainly because they constitute an infinite set of coupled non-linear
integral equations. Even though the need for a truncation scheme is evident, partic-
ular care is needed for respecting the crucial property of gauge invariance. Indeed,
the SDE are conventionally built out of gauge dependent Green’s functions. Since
the mechanism of gauge cancellation is very subtle and involves a delicate conspiracy
of terms coming from all orders, a casual truncation of the series often gives rise to
gauge dependent approximations for ostensibly g.i. quantities. The PT attempts to
address this problem in its root, namely the building blocks of the SDE. According to
this approach, the Feynman graphs contributing to a given gauge invariant process
are rearranged into new propagators and vertices where the gauge dependence has
been reduced to an absolute minimum, namely that of the free gluon propagator.
The proper self-energy of the new propagator and the new vertices are themselves g.i.
and as it turns out so are the SDE governing these new Green’s functions. These new
SDE are in general more complicated than the usual ones because of the presence of

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extra terms which enforce gauge invariance. Nonetheless, it is possible to truncate them, usually by keeping only a few terms of a dressed loop expansion, and maintain exact gauge invariance, while at the same time accommodating non-perturbative effects. One very important aspect of gauge invariance in the context of SDE is that the Green’s functions defined via the PT satisfy tree-level Ward identities. This feature is very important since it enables the cancellation of the final gauge dependences stemming from the free parts of the gluon propagators entering in the expressions for the SDE.

The systematic derivation of such a series for QCD has been the focal point of extensive research. In a ghost free gauge, the usual SDE for quarkless QCD are build out of three basic quantities: the gluon propagator $\Delta$, the three gluon vertex $\Gamma_3$, and the four gluon vertex $\Gamma_4$. One then considers the effective potential $\Omega$, a functional of the three fundamental Green’s functions $\hat{\Delta}$ and then extremizes independently the variations of $\Omega(\Delta, \Gamma_3, \Gamma_4)$ with respect to $\Delta$, $\Gamma_3$, and $\Gamma_4$, e.g. $\frac{\delta \Omega}{\delta \Delta} = 0$, $\frac{\delta \Omega}{\delta \Gamma_3} = 0$, and $\frac{\delta \Omega}{\delta \Gamma_4} = 0$. The resulting expressions will be the corresponding SDE for $\Delta$, $\Gamma_3$ and $\Gamma_4$. In such a picture the solutions to the SDE will in general be gauge dependent in a non-trivial way. If one could solve the entire renormalized set of SDE and then substitute the resulting gauge dependent solutions $\hat{\Delta}$, $\hat{\Gamma}_3$ and $\hat{\Gamma}_4$ back into $\Omega(\Delta, \Gamma_3, \Gamma_4)$ and calculate its value, $\Omega(\hat{\Delta}, \hat{\Gamma}_3, \hat{\Gamma}_4)$, the final answer would be g.i., since $\Omega$ is a physical quantity (vacuum energy). The way this gauge cancellations would manifest themselves is complicated and involves non-trivial mixing of all orders. However, since solving the entire series is practically impossible, some form of truncation is necessary. The minimum requirement for any such truncation scheme must be that the solutions of the truncated SDE, when substituted into $\Omega$, should still preserve its gauge invariance. Unfortunately this is not the case if one truncates the series without a particular guiding principle. The alternative approach that has been proposed is to demand from the beginning that the effective potential $\Omega(\hat{\Delta}, \hat{\Gamma}_3, \hat{\Gamma}_4)$, as well as the individual expressions for the self-energy $\hat{\varphi}$, for $\hat{\Gamma}_3$ and for $\hat{\Gamma}_4$, should be g.i. order by order in the dressed loop expansion (we use hats to indicate that these expressions are in general different from their conventionally derived unhatted counterparts). Assuming that $\hat{\varphi}$, $\hat{\Gamma}_3$ and $\hat{\Gamma}_4$ are individually g.i. is not sufficient however to guarantee the order by order gauge independence of $\Omega$, because there is a residual dependence on the gauge fixing parameter coming from the free part of the propagators $\hat{\Delta}$ entering in the expression for $\Omega$. The necessary and sufficient condition for the order by order cancellation of the residual gauge dependence is that the renormalized self energy $\hat{\Pi}_{\mu\nu}$ is transverse, e.g.

$$q^\mu \hat{\Pi}_{\mu\nu} = 0 , \quad (1)$$

order by order in the dressed expansion. It turns out that Eq(1) can be satisfied as long as $\hat{\varphi}$, $\hat{\Gamma}_3$ and $\hat{\Gamma}_4$ satisfy the following Ward identities:

$$q_1^\mu \hat{\Gamma}_{\mu\nu\alpha}(q_1, q_2, q_3) = t_{\nu\alpha}(q_2)\hat{\varphi}^{-1}(q_2) - t_{\nu\alpha}(q_3)\hat{\varphi}^{-1}(q_3) , \quad (2)$$
and

$$q_\mu \hat{\Gamma}^{abcd}_{\mu\nu\alpha\beta} = f_{abp} \hat{\Gamma}^{cdp}_{\nu\alpha\beta} (q_1 + q_2, q_3, q_4) + \text{c.p.}, \quad (3)$$

with $t_{\mu\nu} = \hat{d}^{-1}(q) = q^2 - \hat{\Pi}(q)$, $f^{abc}$ the structure constants of the gauge group, and the abbreviation c.p. in the r.h.s. stands for cyclic permutations. If Eq(2) and Eq(3) are satisfied, then $\Omega$ is manifestly g.i. order by order in the dressed loop expansion and so are the SDE generated after its variation. In particular, one should extremize independently the variations of $\Omega(\hat{d}, \hat{\Gamma}_3, \hat{\Gamma}_4)$ with respect to $\hat{d}$, $\hat{\Gamma}_3$, and $\hat{\Gamma}_4$, e.g.

$$\frac{\delta \Omega}{\delta \hat{d}} = 0, \quad \frac{\delta \Omega}{\delta \hat{\Gamma}_3} = 0, \quad \frac{\delta \Omega}{\delta \hat{\Gamma}_4} = 0,$$

imposing Eq(3) as an additional constraint. Once solved these equations will give rise to g.i. $\hat{d}$, $\hat{\Gamma}_3$, and $\hat{\Gamma}_4$. Although this program has been laid out conceptually, its practical implementation is as yet incomplete. One thing is certain however: if Green’s functions with the properties described above can arise out of a self-consistent treatment of QCD, one should be able to construct Green’s functions with the same properties at the level of ordinary perturbation theory after appropriate rearrangement of Feynman graphs. The PT accomplishes this task by providing the systematic algorithm needed to recover the desired Green’s functions order by order in perturbation theory. So, g.i. two, three, and four- gluon vertices have already been constructed via the PT at one-loop, and they satisfy the Ward identities of Eq(1)-Eq(3).

2. The pinch technique

The simplest example that demonstrates how the PT works is the gluon two point function. Consider the $S$-matrix element $T$ for the elastic scattering such as $q_1 \bar{q}_2 \rightarrow q_1 \bar{q}_2$, where $q_1, q_2$ are two on-shell test quarks with masses $m_1$ and $m_2$. To any order in perturbation theory $T$ is independent of the gauge fixing parameter $\xi$. On the other hand, as an explicit calculation shows, the conventionally defined proper self-energy depends on $\xi$. At the one loop level this dependence is canceled by contributions from other graphs, which, at first glance, do not seem to be propagator-like. That this cancellation must occur and can be employed to define a g.i. self-energy, is evident from the decomposition:

$$T(s, t, m_1, m_2) = T_0(t, \xi) + T_1(t, m_1, \xi) + T_2(t, m_2, \xi) + T_3(s, t, m_1, m_2, \xi), \quad (4)$$

where the function $T_0(t, \xi)$ depends kinematically only on the Mandelstam variable $t = -(p_1 - p_1)^2 = -q^2$, and not on $s = (p_1 + p_2)^2$ or on the external masses. Typically, self-energy, vertex, and box diagrams contribute to $T_0$, $T_1$, $T_2$, and $T_3$, respectively. Such contributions are $\xi$ dependent, in general. However, as the sum $T(s, t, m_1, m_2)$ is g.i., it is easy to show that Eq(4) can be recast in the form

$$T(s, t, m_1, m_2) = \hat{T}_0(t) + \hat{T}_1(t, m_1) + \hat{T}_2(t, m_2) + \hat{T}_3(s, t, m_1, m_2), \quad (5)$$

where the $\hat{T}_i$ ($i = 0, 1, 2, 3$) are individually $\xi$-independent. The propagator-like parts of vertex and box graphs which enforce the gauge independence of $T_0(t)$, are
called pinch parts. They emerge every time a gluon propagator or an elementary three-gluon vertex contributes a longitudinal $k_\mu$ to the original graph’s numerator. The action of such a term is to trigger an elementary Ward identity of the form $k' = (p+ k - m) - (p - m)$ when it gets contracted with a $\gamma$ matrix. The first term removes (pinches out) the internal fermion propagator, whereas the second vanishes on shell. From the g.i. functions $\hat{T}_i$ ($i = 1, 2, 3$) one may now extract a g.i. effective gluon ($G$) self-energy $\hat{\Pi}_{\mu\nu}(q)$, g.i. $Gq_i\bar{q}_i$ vertices $\hat{\Gamma}^{(i)}_\mu$, and a g.i. box $\hat{B}$, in the following way:

$$
\hat{T}_0 = g^2 \bar{u}_1 \gamma^\mu u_1 ((\frac{1}{q^2}) \hat{\Pi}_{\mu\nu}(q)(\frac{1}{q^2})) \bar{u}_2 \gamma^\nu u_2 ,
\hat{T}_1 = g^2 \bar{u}_1 \hat{\Gamma}_{\nu}^{(1)} u_1 (\frac{1}{q^2}) \bar{u}_2 \gamma^\nu u_2 ,
\hat{T}_2 = g^2 \bar{u}_1 \gamma^\mu u_1 (\frac{1}{q^2}) \bar{u}_2 \hat{\Gamma}_{\nu}^{(2)} u_2 ,
\hat{T}_3 = \hat{B} ,
$$

where $u_i$ are the external spinors, and $g$ is the gauge coupling. Since all hatted quantities in the above formula are g.i., their explicit form may be calculated using any value of the gauge-fixing parameter $\xi$, as long as one properly identifies and allots all relevant pinch contributions. The choice $\xi = 1$ simplifies the calculations significantly, since it eliminates the longitudinal part of the gluon propagator. Therefore, for $\xi = 1$ the pinch contributions originate only from momenta carried by the elementary three-gluon vertex. The one-loop expression for $\hat{\Pi}_{\mu\nu}(q)$ is given by

$$
\hat{\Pi}_{\mu\nu}(q) = \Pi_{\mu\nu}^{(\xi=1)}(q) + t_{\mu\nu} \Pi^P(q) ,
$$

and

$$
\Pi^P(q) = -2i c_a g^2 \int \frac{f_n}{k^2(q + k)^2} ,
$$

where $f_n \equiv \int \frac{d^4 k}{(2\pi)^4}$ is the dimensionally regularized loop integral, and $c_a$ is the quadratic Casimir operator for the adjoint representation [for $SU(N)$, $c_a = N$] After integration and renormalization we find

$$
\Pi^P(q) = -2c_a \left( \frac{g^2}{16\pi^2} \right) \ln\left( \frac{-q^2}{\mu^2} \right) .
$$

Adding this to the Feynman-gauge proper self-energy

$$
\Pi_{\mu\nu}^{(\xi=1)}(q) = -[\frac{5}{3} c_a \left( \frac{g^2}{16\pi^2} \right) \ln\left( \frac{-q^2}{\mu^2} \right)] t_{\mu\nu} ,
$$

we find for $\hat{\Pi}_{\mu\nu}(q)$

$$
\hat{\Pi}_{\mu\nu}(q) = -bg^2 \ln\left( \frac{-q^2}{\mu^2} \right) t_{\mu\nu} ,
$$
where \( b = \frac{11c}{4\pi^2} \) is the coefficient of \(-g^3\) in the usual \( \beta \) function.

This procedure can be extended to an arbitrary \( n \)-point function; of particular physical interest are the g.i. three and four point functions \( \hat{\Gamma}_{\mu\nu\alpha} \) and \( \hat{\Gamma}_{\mu\nu\alpha\beta} \). Finally, the generalization of the PT to the case of non-conserved external currents is technically more involved, but conceptually straightforward.

3. The current algebra formulation of the pinch technique

We now present an alternative formulation of the PT introduced in the context of the standard model. In this approach the interaction of gauge bosons with external fermions is expressed in terms of current correlation functions, i.e. matrix elements of Fourier transforms of time-ordered products of current operators. This is particularly economical because these amplitudes automatically include several closely related Feynman diagrams. When one of the current operators is contracted with the appropriate four-momentum, a Ward identity is triggered. The pinch part is then identified with the contributions involving the equal-time commutators in the Ward identities, and therefore involve amplitudes in which the number of current operators has been decreased by one or more. A basic ingredient in this formulation are the following equal-time commutators:

\[
\delta(x_0 - y_0)[J_0^\mu W(x), J_\nu^\mu W(y)] = c^2 J_0^\mu W(x) \delta^4(x - y), \\
\delta(x_0 - y_0)[J_0^\mu W(x), J_\nu^\dagger W(y)] = -J_3^\mu(x) \delta^4(x - y), \\
\delta(x_0 - y_0)[J_0^\mu W(x), J_\nu^\gamma W(y)] = J_0^\mu W(x) \delta^4(x - y), \\
\delta(x_0 - y_0)[J_0^\mu W(x), J_\nu^V W(y)] = 0,
\]

where \( J_3^\mu \equiv 2(J_2^\mu + s^2 J_5^\nu) \) and \( V, V' \in \{ \gamma, Z \} \). To demonstrate the method with an example, consider the vertex \( \Gamma_\mu \), where now the gauge particles in the loop are \( W \) instead of gluons and the incoming and outgoing fermions are massless. It can be written as follows (with \( \xi = 1 \)):

\[
\Gamma_\mu = \int \frac{d^4k}{2\pi^2} \Gamma_{\mu\alpha\beta}(q, k, -k - q) \int d^4x e^{ikx} < f|T^*[J_0^\alpha W(x)J_0^\beta W(0)]|i > .
\]

When an appropriate momentum, say \( k_\alpha \), from the vertex is pushed into the integral over \( dx \), it gets transformed into a covariant derivative \( \frac{d}{dx_\alpha} \) acting on the time ordered product \( < f|T^*[J_0^\alpha W(x)J_0^\beta W(0)]|i > \). After using current conservation and differentiating the \( \theta \)-function terms, implicit in the definition of the \( T^* \) product, we end up with the left-hand side of the second of Eq(13). So, the contribution of each such term is proportional to the matrix element of a single current operator, namely \( < f|J_3^\mu |i > \); this is precisely the pinch part. Calling \( \Gamma_\mu^P \) the total pinch contribution from the \( \Gamma_\mu \) of Eq(13), we find that

\[
\Gamma_\mu^P = -g^3 cIWW(Q^2) < f|J_3^\mu |i > ,
\]

\( (14) \)
where
\[
I_{ij}(q) = i \int \frac{d^4k}{2\pi^4} \frac{1}{(k^2 - M_i^2)((k + q)^2 - M_j^2)} .
\] (15)

Obviously, the integral in Eq(15) is the generalization of the QCD expression Eq(8) to the case of massive gauge bosons.

4. Phenomenological applications

In this section we present some of the most important phenomenological applications of the PT.

4.1. Neutrino electromagnetic form factor

It has been known since the early days of gauge theories with spontaneous symmetry breaking that both the electric and magnetic form factors of fermions, \(F_1(q^2)\) and \(F_2(q^2)\), respectively, defined by
\[
\Gamma_\mu = \gamma_\mu F_1(q^2) + \frac{i}{2m} \sigma_{\mu\nu} q^\nu F_2(q^2),
\] (16)

with \(\sigma_{\mu\nu} \equiv \frac{i}{2}[\gamma_\mu, \gamma_\nu]\) are gauge dependent for general values of the momentum transfer \(q^2\). It is only at \(q^2 = 0\) when the gauge dependence drops so that \(F_1(0)\) can be identified with the fermion charge, and \(F_2(0)\) with the anomalous magnetic moment. In the context of the standard model the effective electromagnetic form factor \(F(q^2)\) of the neutrino has been a long-standing puzzle. It has been argued [11] that the neutrino mean-square radius \(<r^2>\) and \(F(q^2)\) are related by
\[
<r^2> = 6 \left. \frac{dF(q^2)}{dq^2} \right|_{q^2=0} ,
\] (17)

but it was soon realized that the conventional definition of \(F(q^2)\) would give rise to gauge-dependent and divergent expressions for \(<r^2>\). This, of course, comes as no surprise. There is indeed no a priori reason why even if \(F(q^2)\) is g.i. at \(q^2 = 0\), its derivative will be also.

The root of the problem lies in the fact that, although everyone agrees that the Feynman diagrams are just convenient visualizations of a complex underlying formalism, the prevailing attitude is to treat them as individually inseparable entities. According to this logic, a Feynman diagram either contributes to \(F(q^2)\) in its entirety or it does not contribute at all. This sort of logic is not part of the PT; certain diagrams, not relevant to the definition of \(F(q^2)\) at first glance, contain pieces which cannot be distinguished from the contributions of the regular graphs and must therefore be included. It is precisely the inclusion of these contributions which renders the answer g.i. and finite.[12]
4.2. Top magnetic dipole moment

One of the most efficient ways to study top quarks will be to pair-produce them in future very energetic $e^+e^-$ colliders, through the reaction $e^+e^- \to t\bar{t}$. In general, the leptonic nature of the target allows for clean signals. In addition, due to their large masses, the produced top quarks are expected to decay weakly ($t\bar{t} \to bW^+\bar{b}W^-$, with subsequent leptonic decays of the $W$), before hadronization takes place; therefore electroweak properties of the top can be studied in detail and QCD corrections can be reliably evaluated in the context of perturbation theory, when the energy of the collider is well above the threshold for $t\bar{t}$ production.

The standard method for extracting theoretical information out of such an experiment is to evaluate in the context of a specific gauge theory, such as the SM or its extensions, all Feynman graphs contributing to the process $e^+e^- \to t\bar{t}$, up to a given order in perturbation theory, compute the value of an appropriately chosen observable, such as the cross-section or the production rate, and then compare it with the value obtained experimentally. An alternative approach is to parametrize amplitudes in terms of form factors of particles. The main motivation of such a method is to isolate possible new physics in a particular sub-amplitude, assuming that the rest of the dynamics has already been successfully tested in previous experiments. Adopting the latter approach, Atwood and Soni presented a phenomenological analysis for determining the magnetic and electric dipole moment form factors of the top quark in upcoming $e^+e^- \to t\bar{t}$ experiments. Such form factors are defined through the following Lorentz decomposition of the $Vt\bar{t}$ vertex, where $V$ represents a boson (a $\gamma$ or $Z$ in our case) coupled to the conserved leptonic current:

$$\Gamma^V_\mu(q^2) = \gamma_\mu F^V_1(q^2) + \frac{i\sigma_{\mu\nu}q^\nu}{2m_t} F^V_2(q^2) + \gamma_\mu\gamma_5 F^V_3(q^2) + \frac{i\sigma_{\mu\nu}q^\nu}{2m_t} \gamma_5 F^V_4(q^2),$$

(18)

with $q^2 = s$ the Mandelstam variable associated to the squared energy of the center of mass. In the above decomposition, $F^V_2$ is the magnetic dipole moment (MDM) and $F^V_4$ is the electric dipole moment (EDM) form factor. In particular, $F^V_2$ defined at $q^2 = 0$ is the usual definition of the anomalous magnetic moment. In the case of the top quark production, clearly $q^2 \geq 4m_t^2$. Within the SM the tree-level value for both $F^V_2$ and $F^V_4$ is zero. The upshot of the analysis was that the dependence of the differential cross section for the reaction $e^+e^- \to t\bar{t}$ on the real and imaginary (absorptive) parts of the MDM and EDM form factors, for an incoming photon or $Z$, can be singled out individually, through a set of optimally chosen physical observables. The theoretical prediction for these observables is obtained by calculating the tree-level amplitude for $e^+e^- \to t\bar{t}$, using the $Vt\bar{t}$ vertices of Eq.(18), instead of the usual tree-level vertices. Clearly, the effective vertex of Eq.(18) can only be used for tree-level computations, since its inclusion in loops will give rise to non-renormalizable
divergences. The result of such a tree-level computation is g.i., if one assumes that the quantities $F_i^V$ do not depend explicitly on the gauge-fixing parameter $\xi$. Indeed, in that case the only dependence on $\xi$ is proportional to the longitudinal part of the $\gamma$ or $Z$ propagator, and therefore vanishes, as long as the leptonic current is conserved ($m_e = 0$). The final answer is expressed in terms of $F_2^V$ and $F_4^V$, which at this level are treated as free parameters. Comparison of these expressions with the experimentally obtained quantities, can yield, after appropriate fitting, the experimental values of $F_2^V$ and $F_4^V$. Clearly, before any possible non-zero experimental values for $F_2^V$ and $F_4^V$ can be attributed to Physics beyond the SM, one ought to first take into account the contributions induced by quantum corrections from the SM. So, $F_2^V$ becomes non-zero through one-loop quantum corrections, whereas $F_4^V$, which violates CP, receives its first non-vanishing contributions at three loops. Such contributions are traditionally extracted from the one-loop corrections to the $\gamma t\bar{t}$ and $Zt\bar{t}$ vertices; clearly the resulting amplitude is of the desired form, namely a bare $\gamma$ or $Z$ propagator multiplied by a vertex of the form of Eq.(18). However, as it was pointed out already in the classic paper by Fujikawa, Lee, and Sanda, off-shell form factors of fermions are in general gauge dependent quantities. In the context of the $R_\xi$ gauges, for example, a residual dependence on the gauge-fixing parameter $\xi$ survives in the final expressions of form factors, when $q^2 \neq 0$. Obviously, in the case of $e^+e^-$ annihilation into heavy fermions, the value of $q^2$ must be above the heavy fermion threshold ($q^2 \geq 4m_t^2$, in our case). Consequently, the intermediate photon and $Z$ are far off-shell, and therefore, MDM and EDM form factors may in general be gauge-dependent and not suitable for comparison with experiment. This gauge dependence was computed and turned out to be numerically very strong; its presence distorts not only the quantitative but also the qualitative behavior of the answer. More specifically, unphysical thresholds are introduced, and the numerical dominance of perturbative QCD, which is present in the gauge g.i. treatment, is totally washed out. Of particular interest is the fact that the popular unitary gauge (the limit of the $R_\xi$ gauges as $\xi \to \infty$) gives a completely wrong answer. This analysis indicates that the gauge dependence established is a serious pathology and may lead to erroneous conclusions. Applying the PT to the case of the MDM form factor computed in a general $R_\xi$ gauge means that one has to identify vertex-like contributions contained in box diagrams, which, when added to to the usual vertex graphs, render the result $\xi$-independent. Interestingly enough, the g.i. answer so obtained coincides with the one derived when only the usual vertex graphs are considered (without contributions from boxes), but are evaluated in a special gauge, namely the Feynman gauge ($\xi = 1$).
4.3. The $S$, $T$, and $U$ parameters

One of the most frequently used parametrizations of the leading contributions of electroweak radiative corrections is in terms of the $S$, $T$, and $U$ parameters. As was shown by Degrassi, Kniehl, and Sirlin in the context of the standard model, these parameters become infested with gauge-dependences, as soon as the bosonic contributions to the one-loop self-energies are taken into account. In addition, these quantities are in general ultraviolet divergent, unless one happens to work within a very special class of gauges, namely those satisfying the constraint

$$\xi_W = \xi_WS^2 + \xi_ZC^2\theta.$$  \hspace{1cm} (19)

The above shortcomings may be circumvented if one defines the $S$, $T$, and $U$ parameters through the g.i. PT self-energies $W_W$, $Z_Z$, $\gamma_Z$, and $\gamma\gamma$.

5. Anomalous gauge boson couplings

A new and largely unexplored frontier on which the ongoing search for new physics will soon focus is the study of the structure of the three-boson couplings. In particular one expects to probe directly the non-Abelian nature of the standard model at LEP2 (and NLC), through the process $e^+e^- \rightarrow W^+W^-$. A general parametrization of the trilinear gauge boson vertex for on-shell $W$s and off-shell $V = \gamma, Z$ is

$$\Gamma_{\mu\alpha\beta}^V = -ig_V \left[ f \left( 2g_{\alpha\beta}\Delta_\mu + 4(g_{\alpha\mu}Q_\beta - g_{\beta\mu}Q_\alpha) \right) + 2\Delta_{\kappa V}(g_{\alpha\mu}Q_\beta - g_{\beta\mu}Q_\alpha) \right] + \frac{4\Delta_{Q V}}{M_W}(\Delta_{\mu\alpha\beta}Q_\gamma - \frac{1}{2}Q^2g_{\alpha\mu}\Delta_\mu) + ... \right],$$  \hspace{1cm} (20)

with $g_{\gamma} = gs$, $g_Z = gc$, where $g$ is the $SU(2)$ gauge coupling, $s \equiv \sin\theta_W$ and $c \equiv \cos\theta_W$, and the ellipses denote omission of C, P, or T violating terms. The four-momenta Q and $\Delta$ are related to the incoming momenta $q$, $p_1$ and $p_2$ of the gauge bosons $V$, $W^-$ and $W^+$ respectively, by $q = 2Q$, $p_1 = \Delta - Q$ and $p_2 = -\Delta - Q$. The form factors $\Delta_{\kappa V}$ and $\Delta_{Q V}$, also defined as $\Delta_{\kappa V} = \kappa_V + \lambda_V - 1$, and $\Delta_{Q V} = -2\lambda_V$, are compatible with C, P, and T invariance, and are related to the magnetic dipole moment $\mu_W$ and the electric quadrupole moment $Q_W$, by the following expressions:

$$\mu_W = \frac{e}{2M_W}(2 + \Delta_{\kappa V}) \right), \hspace{0.5cm} Q_W = -\frac{e}{M_W^2}(1 + \Delta_{\kappa V} + \Delta_{Q V}) \right].$$  \hspace{1cm} (21)

In the context of the standard model, their canonical, tree level values are $f = 1$ and $\Delta_{\kappa V} = \Delta_{Q V} = 0$. To determine the radiative corrections to these quantities one must cast the resulting one-loop expressions in the following form:

$$\Gamma_{\mu\alpha\beta}^V = -ig_V[a_1^Vg_{\alpha\beta}\Delta_\mu + a_2^V(g_{\alpha\mu}Q_\beta - g_{\beta\mu}Q_\alpha) + a_3^V\Delta_{\mu\alpha\beta}Q_\gamma],$$  \hspace{1cm} (22)
where $a_1^V$, $a_2^V$, and $a_3^V$ are complicated functions of the momentum transfer $Q^2$, and the masses of the particles appearing in the loops. It then follows that $\Delta \kappa_V$ and $\Delta Q_V$ are given by the following expressions:

$$\Delta \kappa_V = \frac{1}{2} (a_2^V - 2a_1^V - Q^2 a_3^V), \quad \Delta Q_V = \frac{M_W^2}{4} a_3^V.$$  \hfill (23)

Calculating the one-loop expressions for $\Delta \kappa_V$ and $\Delta Q_V$ is a non-trivial task, both from the technical and the conceptual point of view. If one calculates just the Feynman diagrams contributing to the $\gamma W^+ W^-$ vertex and then extracts from them the contributions to $\Delta \kappa_\gamma$ and $\Delta Q_\gamma$, one arrives at expressions that are plagued with several pathologies, gauge-dependence being one of them. Indeed, even if the two $W$ are considered to be on shell, since the incoming photon is not, there is no a priori reason why a g.i. answer should emerge. In the context of the renormalizable $R_\xi$ gauges the final answer depends on the choice of the gauge fixing parameter $\xi$, which enters into the one-loop calculations through the gauge-boson propagators ($W, Z, \gamma$, and unphysical Higgs particles). In addition, as shown by an explicit calculation performed in the Feynman gauge ($\xi = 1$), the answer for $\Delta \kappa_\gamma$ is infrared divergent and violates perturbative unitarity, e.g. it grows monotonically for $Q^2 \to \infty$. All the above pathologies may be circumvented if one adopts the PT. The application of the PT gives rise to new expressions, $\hat{\Delta} \kappa_\gamma$ and $\hat{\Delta} Q_\gamma$, which are gauge fixing parameter $\xi$ independent, ultraviolet and infrared finite, and well behaved for large momentum transfers $Q^2$.

Using carets to denote the g.i. one-loop contributions, we have

$$\hat{\Delta} \kappa_\gamma = \Delta \kappa_\gamma^{(\xi=1)} + \Delta \kappa_\gamma^P,$$  \hfill (24)

and

$$\hat{\Delta} Q_\gamma = \Delta Q_\gamma^{(\xi=1)} + \Delta Q_\gamma^P,$$  \hfill (25)

where $\Delta Q_\gamma^{(\xi=1)}$ and $\Delta Q_\gamma^{(\xi=1)}$ are the contributions of the usual vertex diagrams in the Feynman gauge, whereas $\Delta Q_\gamma^P$ and $\Delta Q_\gamma^P$ the analogous contributions from the pinch parts. A straightforward calculation yields:

$$\Delta \kappa_\gamma^P = -\frac{1}{2} \frac{Q^2}{M_W^2} \sum_V \frac{\alpha_V}{\pi} \int_0^1 da \int_0^1 (at-1) \frac{(at-1)}{\bar{L}_V^2},$$  \hfill (26)

and

$$\Delta Q_\gamma^P = 0.$$

where

$$L_V^2 = t^2 - t^2 a (1-a) \frac{4Q^2}{M_W^2} + (1-t) \frac{M_W^2}{M_W^2}.$$  \hfill (28)

We observe that $\Delta \kappa_\gamma^P$ contains an infrared divergent term, stemming from the double integral shown above, when $V = \gamma$. This term cancels exactly against a similar
infrared divergent piece contained in \( \Delta \kappa_\gamma^{(\xi=1)} \), thus rendering \( \hat{\Delta} \kappa_\gamma \) infrared finite. After the infrared pieces have been cancelled, one notices that the remaining contribution of \( \Delta \kappa_\gamma^P \) decreases monotonically as \( Q^2 \to \pm \infty \); due to the difference in relative signs this contribution cancels asymptotically against the monotonically increasing contribution from \( \Delta \kappa_\gamma^{(\xi=1)} \). Thus by including the pinch part the unitarity of \( \hat{\Delta} \kappa_\gamma \) is restored and \( \hat{\Delta} \kappa_\gamma \to 0 \) for large values of \( Q^2 \). It would be interesting to determine how these quantities could be directly extracted from future \( e^+e^- \) experiments.

6. Recent developments

In the previous sections we presented the theoretical motivation for introducing the PT, and discussed its spectacular success on issues related to electroweak phenomenology. Several important issues remain however open. Most noticeably, it is crucial to establish on much firmer ground the physical significance of the PT amplitudes, address issues of uniqueness, extend the PT beyond one-loop, and explore the possibility of directly extracting the PT form factors from future experiments. Some of the above questions have been addressed in a series of relatively recent papers. When reviewing these most recent developments we will give additional emphasis on conceptual rather than technical issues. In particular, we will discuss the application of the PT in the context of the unitary gauge, the connection between the PT and the background field method, the large mass limit of the S-matrix, and the process independence of the PT results.

6.1. The unitary gauge

Since the early days of spontaneously broken non-Abelian gauge theories, the unitary gauge has been known to give rise to renormalizable S-matrix elements, but to Green’s functions that are non-renormalizable in the sense that their divergent parts cannot be removed by the usual mass and field-renormalization counterterms.\(^{[2]}\) Even though this situation may be considered acceptable from the physical point of view, the inability to define renormalizable Green’s functions has always been a theoretical shortcoming of the unitary gauge. For example, the self-energies, vertex and box diagrams are divergent, and gauge-boson propagators cannot be consistently defined for arbitrary values of \( q^2 \) beyond the tree-level. The application of the PT to the unitary gauge calculations\(^{[2]}\) systematically reorganizes the one-loop S-matrix contributions into kinematically distinct pieces (propagators, vertices, boxes) that can be renormalized with the usual counter-terms characteristic of a renormalizable theory. The aforementioned shortcomings associated with the unitary gauge are thus circumvented. Furthermore, the renormalizable amplitudes obtained in this fashion are identical to those calculated in the \( R_\xi \) gauges.\(^{[3]}\)

It should be emphasized that the above results, are by no means obvious. The
point is that the unitary gauge can be obtained from the $R_\xi$ gauges if the limit $\xi \to \infty$ is taken before Feynman integrals are performed. Thus, there is no obvious guarantee that when the PT is applied directly to the highly divergent amplitudes characteristic of the unitary gauge calculations, it will lead to the same $\xi$-independent self-energies, vertices, and boxes derived in the $R_\xi$ framework.

6.2. The connection with the background field method

Recently, a connection between the background field method (BFM)\(^{23}\) and the S-matrix PT,\(^{24}\) and subsequently the intrinsic PT\(^{25}\) has been advertised. In particular, it was shown that when QCD is quantized in the context of BFM, the conventional $n$-point functions, calculated with the BFM Feynman rules, coincide with those obtained via the PT, for the special value $\xi_Q = 1$ of the gauge fixing parameter $\xi_Q$, used to gauge fix the quantum field. For any other value of $\xi_Q$ the resulting expressions differ from those obtained via the PT. However, the BFM $n$-point functions, for any choice of $\xi_Q$, satisfy exactly the same Ward identities as the PT $n$-point functions (Eq(3) for example). Based on these observations, it was argued\(^{24}\) that the PT is but a special case of the BFM, and represent one out of an infinite number of equivalent choices, parameterized by the values chosen for $\xi_Q$. This misleading point of view originates from the erroneous impression that in the context of the BFM Green’s functions should be rendered g.i. automatically. So, the naive expectation was that Green’s functions calculated within the BFM should be completely g.i., and identical to the corresponding PT Green’s functions. Therefore, when at the end of the calculation it was realized that, contrary to the initial expectation, a residual dependence on $\xi_Q$ persists, there was an attempt to assign a physical significance to this dependence. In particular, it was argued\(^{24}\) that the residual $\xi_Q$ dependence is a trade-off for the (presumably) intrinsic arbitrariness of the PT in defining $n$-point functions. There is no a priori reason however, why the Green’s functions of the BFM should not be gauge-dependent; indeed, the requirement of gauge-invariance with respect to the background field does not imply gauge-invariance with respect to the quantum field. In particular, the choice $\xi_Q = 1$ acquires a special meaning in the context of the BFM, only because it coincides with the result of a g.i. calculation, namely that of the PT.

It should be emphasized that to the extend that the BFM $n$-point functions display a residual (even though mild) $\xi_Q$-dependence, one still has to apply the PT algorithm, in order to obtain a unique g.i. answer. So, the PT results can be recovered for every value of $\xi_Q$ as long as one properly identifies the relevant pinch contributions concealed in the rest of the graphs contributing to the $S$-matrix element\(^{26}\). These contributions vanish for $\xi_Q = 1$, but are non-vanishing for any other value of $\xi_Q$. In the case of the gluon self-energy, for example, one has to identify the propagator-like parts of boxes and vertex graphs and, according to the PT rules, append them to the conventional self-energy expressions. After this procedure is completed, a unique
PT result for the self-energy emerges, regardless of the gauge fixing procedure (BFM, \( R_\xi \), light-cone, etc), or the value of the gauge fixing parameter (\( \xi_Q \), \( \xi \), \( n_\mu \), etc) used. From the point of view of the PT, there is no real conceptual difference between a theory quantized in the \( R_\xi \) gauge or in the BFM. Indeed, in the PT framework the crucial quantity is the S-matrix, whose uniqueness and gauge independence is systematically exploited, in order to extract g.i. sub-amplitudes. Even though these sub-amplitudes have not yet been associated with specific physical observables, there are several indications supporting such a possibility. As it was recently realized, for example, the PT expression for the gluon self-energy coincides with the renormalized static quark-antiquark potential, in the limit of very heavy quark masses. The BFM, regardless of any calculational simplifications it may cause, is bound to give rise to the same S-matrix elements, order by order in perturbation theory. It is therefore not surprising that the application of the PT gives exactly the same answers in the BFM, as in any other gauge fixing procedure. The difference between various gauge fixing procedures is only operational. From that point of view one could alternatively say that the Feynman gauge (\( \xi_Q = 1 \)) in the BFM has the special property (at least at one-loop) of giving zero total pinch contribution. To see that we recall that the main characteristics of the Feynman rules in the BFM are that the gauge fixing parameters for the background (classical) and the quantum fields are different (\( \xi_C \) and \( \xi_Q \) respectively), the three and four-gluon vertices are \( \xi_Q \)-dependent at tree-level, and the couplings to the ghosts are modified (they are however \( \xi_Q \)-independent). In particular, the three-gluon vertex assumes the form

\begin{equation}
\Gamma^{(0)}_{\mu \nu \alpha} = (1 - \frac{\xi_Q}{\xi_C}) \Gamma^{P}_{\mu \nu \alpha} + \Gamma^{F}_{\mu \nu \alpha},
\end{equation}

(29)

where \( \Gamma^{P}_{\mu \nu \alpha} = (q + k)_{\nu} g_{\mu \alpha} + k_{\mu} g_{\nu \alpha} \) gives rise to pinch parts, when contracted with \( \gamma \) matrices, whereas \( \Gamma^{F}_{\mu \nu \alpha} = 2q_{\mu} g_{\nu \alpha} - 2q_{\nu} g_{\mu \alpha} - (2k + q)_{\alpha} g_{\mu \nu} \) cannot pinch. Clearly, it vanishes for \( \xi_Q = 1 \), and so do the longitudinal parts of the gluon propagators; therefore pinching in this gauge is zero.

6.3. The large mass limit of the S-matrix

An important open question is if the g.i. quantities extracted via the PT correspond to physical quantities. Using Eq(11), it is straightforward to verify that, the one-loop expression for \( \hat{T}_1 \) is:

\begin{equation}
\hat{T}_1 = \bar{u}_1 \gamma_\mu u_1 \{ \frac{g^2}{q^2[1 + bg^2 \ln(\frac{q^2}{\mu^2})]} \} \bar{u}_2 \gamma^\mu u_2,
\end{equation}

(30)

where \( u_i \) are the external quark spinors. Thus, up to the kinematic factor \( \frac{1}{q^2} \), the r.h.s. of Eq(30) is the one-loop running coupling. Equivalently, the expression of Eq(30) is the Fourier transform of the static quark-antiquark potential, in the limit
of very heavy quark masses.\textsuperscript{26} Clearly, the quark-antiquark potential is a physical quantity, which, at least in principle, can be extracted from experiment, or measured on the lattice. In fact, as was recently realized,\textsuperscript{28} when one computes the one-loop contribution to the scattering amplitude $q\bar{q} \to q\bar{q}$ of quarks with mass $M$, retaining leading terms in $\frac{q^2}{M^2}$, one arrives again at the expression of Eq\textsuperscript{(30)}. So in principle, one can extract the quantity of Eq\textsuperscript{(30)} from a scattering process, in which the momentum transfer $q^2$ is considerably larger than the QCD mass $\Lambda^2$, so that perturbation theory will be reliable, and, at the same time, significantly smaller than the mass of the external quarks, so that the sub-leading corrections of order $O\left(\frac{q^2}{M^2}\right)$ can be safely neglected. Top-quark scattering, for example, could provide a physical process, where the above requirements are simultaneously met. The above observations led to the conjecture that the PT expressions for the gluonic $n$-point functions correspond to the static potential of a system of $n$ heavy quarks.\textsuperscript{27}

6.4. Process independence of the pinch technique

The most recent development addresses the issue of process-independence of the PT results. In particular, the g.i. $n$-point functions obtained by the application of the S-matrix PT do not depend on the particular process employed (fermion + fermion $\to$ fermion + fermion, fermion + fermion $\to$ gluon + gluon, gluon + gluon $\to$ gluon + gluon, etc.), and are in that sense universal. This fact can be seen with an explicit calculation, where one can be convinced that the only quantities entering in the definition of the g.i. self-energies are just the gauge group structure constants; therefore, the only difference from process to process is the external group matrices associated with external-leg wave functions, which are, of course, immaterial to the definition of the things inside. The fact that the PT gives rise to process-independent results has been recently proved by N. J. Watson via detailed calculations for a wide variety of cases.

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