Research Article

On the Minimum Variable Connectivity Index of Unicyclic Graphs with a Given Order

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1. Introduction

All graphs that we discuss in the present study are simple, connected, undirected, and finite. For a graph \( G = (V, E) \), the number \( |V(G)| \) is called its order and \( |E(G)| \) is called its size. For a vertex \( v_1 \in V(G) \), denoted by \( N(v_1) \), the set of all those vertices of \( G \) are adjacent with \( v_1 \). The number \( d_{v_1} = |N(v_1)| \) is called degree of \( v_1 \). If \( d_{v_1} = 1 \), then \( v_1 \) is called a pendent vertex or a leaf. A graph of order \( n \) is known as an \( n \)-vertex graph. As usual, the \( n \)-vertex path and star graphs are denoted by \( P_n \) and \( S_n \), respectively. An \( n \)-vertex graph containing exactly one cycle is called a unicyclic graph. The class of all \( n \)-vertex unicyclic graphs is denoted by \( U_n \). The graph obtained from \( S_n \) by adding an edge is denoted by \( S_n^0 \). For the (chemical) graph theoretical notation and terminology that are not defined in this paper, we refer the reader to some standard books, such as [1–3].

To model the heteroatoms molecules, it is better to use the vertex-weighted graphs, which are the graphs whose one or more vertices are distinguished in some way from the rest of the vertices [4]. Let \( G \) be a vertex-weighted graph with the vertex set \( \{v_1, v_2, \ldots, v_n\} \) and the vertex weight \( w_i \) of the vertex \( v_i \) is for \( i = 1, 2, \ldots, n \). The augmented vertex-adjacency matrix of \( G \) is an \( n \times n \) matrix denoted by \( \tilde{A}(G) \) and is defined as \( \tilde{A}(G) = [a_{ij}]_{n \times n} \) where

\[
a_{ij} = \begin{cases} 1, & \text{if } v_iv_j \in E(G), \\ w_i, & \text{if } i = j, \\ 0, & \text{otherwise}. \end{cases}
\]

The variable connectivity index [5, 6], proposed by Randić, for the graph \( G \) is defined as

\[
\chi'(G) = \sum_{v_iv_j \in E(G)} \frac{1}{(d_{v_i} + w_i)(d_{v_j} + w_j)}
\]

We associate this index’s name with its inventor Randić by calling it as the variable Randić index. This index was actually introduced within the QSPR/QSAR (quantitative structure-property/activity relationship) studies of heteroatoms molecules. If \( G \) is the molecular graph of a homooatomic molecule, then \( w_1 = w_2 = \cdots = w_n = \gamma \) (say) and hence the variable Randić index \( \chi'(G) \) becomes
\[ 1^{\chi'}(G) = \sum_{v \in V(G)} \frac{1}{(d_v + \gamma)(d_v + \gamma)} \]  

(3)

In the rest of this paper, we denote this index by \( r_\gamma \), instead of \( 1^{\chi'} \). Clearly, if we take \( \gamma = 0 \) then the invariant \( r_\gamma \) is the classical Randić index [7, 8]. Details about the chemical applications of the variable Randić index can be found in [4, 7, 9–17] and related references listed therein. In [18], a mathematical study of the variable Randić index was initiated and it was proved that the star graph has the minimum variable Randić index among all trees of a fixed order \( n \), where \( n \geq 4 \). It needs to be mentioned here that the variable Randić index seems to have more chemical applications than the several well-known variable indices, see, for example, the variable indices considered in the papers [19–25].

For convenience, we introduce some further notation and terminology. Let \( A_{n,i,j}^k \) be the unicyclic graph constructed from a path \( P = u_0 u_1 \ldots u_l \) (\( l \geq 1 \)) by connecting \( k \) pendant vertices to \( u_0 \) and a cycle \( C_i, l = n - j - k \) to \( u_l \), respectively. Let \( A_{n,j}^i \) be the unicyclic graph constructed from cycle \( C_i, l = n - j \) by connecting \( k \) pendant vertices to one vertex of \( C_i \) and \( A_{n,j,k}^{i,i} \) be the unicyclic graph created from cycle \( C_n, n = i + j + k + l \) by attaching \( K_{1,i,j} \) and \( K_{1,j,k} \) to the vertices of \( C_i \). Let \( D \) be the such class of \( U_n \), whose every member has unique 3-cycle, and the vertices apart from 3-cycle are pendant vertices. Let \( F \) be the such class of \( U_n \), whose every member has unique 4-cycle, and the vertices apart from 4-cycle are pendant vertices and are joined to nonadjacent vertices of the unique 4-cycle.

In this paper, we characterize the collection of unicyclic graphs on \( n \) vertices that minimize variable Randić index. We further show that, for \( \gamma \geq 0 \), \( S_1^n \) has minimum variable Randić index among the collection \( U_n \) and

\[
\psi(G) \geq \frac{n - 3}{(1 + \gamma)(n - 1 + \gamma)} + \frac{\gamma}{(2 + \gamma)(n - 1 + \gamma)}
\]

(4)

where equality holds if and only if \( G \equiv S_1^n \).

2. Transformations Which Decrease the Variable Randić Index

We introduce some transformations and prove some lemmas to establish main results.

**Lemma 1.** If \( \gamma \geq 0 \) and \( s, t \geq 1 \), then the function \( \Psi \) defined as

\[
\Psi(y, s, t) = \frac{s}{\sqrt{1 + y}} \left( \frac{1}{\sqrt{s + 2 + y}} - \frac{1}{\sqrt{s + t + 3 + y}} \right) + \frac{t}{\sqrt{1 + y}} \left( \frac{1}{\sqrt{t + 2 + y}} - \frac{1}{\sqrt{s + t + 3 + y}} \right)
\]

(5)

\[
= \frac{1}{\sqrt{(s + 2 + y)(t + 2 + y)}}
\]

is positive valued.

**Proof.** We note that the function \( \partial \Psi / \partial s \) is strictly increasing in \( t \) on the interval \((1, \infty)\) because

\[
\frac{\partial \Psi}{\partial s} = \frac{1}{\sqrt{1 + y}} \left( \frac{1}{\sqrt{s + 2 + y}} - \frac{1}{\sqrt{s + t + 3 + y}} \right)
\]

\[
= \frac{1}{2\sqrt{1 + y}(s + t + 3 + y)^{3/2}} - \frac{1}{(s + 2 + y)^{3/2}}
\]

\[
- \frac{1}{(s + t + 3 + y)^{3/2}}
\]

(6)

Also, it can be seen that the value of the function \( \partial \Psi / \partial s \) at \( t = 1 \) is \( > 0 \), which implies that the function \( \partial \Psi / \partial s \) is positive valued for \( t > 1 \), and hence the function \( \Psi \) is strictly increasing w.r.t. \( s \) on the interval \((1, \infty)\). In our case \( \Psi(y, s, t) = \Psi(y, s, t) \), the function \( \Psi \) is also strictly increasing w.r.t. \( t \) on the interval \((1, \infty)\). Hence, \( \Psi(y, s, t) > 0 \) for all \( s, t > 1 \) and \( \gamma \geq 0 \).

**Lemma 2.** If \( \gamma \geq 0 \) and \( s, t \geq 1 \), then the function \( \Phi \) defined as

\[
\Phi(y, s, t) = \frac{s}{\sqrt{1 + y}} \left( \frac{1}{\sqrt{s + 2 + y}} - \frac{1}{\sqrt{s + t + 4 + y}} \right) + \frac{t}{\sqrt{1 + y}} \left( \frac{1}{\sqrt{t + 2 + y}} - \frac{1}{\sqrt{s + t + 4 + y}} \right)
\]

(7)

\[
= \frac{1}{\sqrt{(s + 2 + y)(t + 2 + y)}}
\]

is positive valued.

The proof of Lemma 2 is analogous to the proof of Lemma 1.

**Lemma 3.** If \( \gamma \geq 0 \), \( t > 2 \), and \( r > 1 \), it holds that
\[ \Omega (r, t, y) = \frac{(r - 1)(t - 1)}{\sqrt{(1 + y)(r + y)(t + y)(r + t + y)}} \]

\[ \times \left( \frac{1}{\sqrt{r + y + \sqrt{r + t - 1 + y}} - \frac{1}{\sqrt{(r + y)(t + y) + \sqrt{(1 + y)(r + t - 1 + y)}}}} \right) \]

\[ = \frac{(r - 1)(t - 1)}{\sqrt{(1 + y)(r + y)(t + y)(r + t - 1 + y)}} \]

\[ \times \left( \frac{\sqrt{r + y} (\sqrt{(r + y)(t + y)} + \sqrt{(1 + y)(r + t - 1 + y)})}{\sqrt{r + y + \sqrt{r + t - 1 + y}} (\sqrt{(r + y)(t + y)} + \sqrt{(1 + y)(r + t - 1 + y)})} \right) \]

\[ - \left( \frac{\sqrt{r + y} + \sqrt{r + t - 1 + y}}{(r + y + \sqrt{r + t - 1 + y}) (\sqrt{(r + y)(t + y)} + \sqrt{(1 + y)(r + t - 1 + y)})} \right) \]

Since \( y \geq 0, t > 2 \) so \( t + y > 0 \), also \( (\sqrt{1 + y} \sqrt{t + y} - 1) > 0 \). Hence, \( \Omega (r, t, y) > 0 \).

Proof. Let

\[ \text{Lemma 3.} \]

\[ \text{Lemma 4. For } y \geq 0, \text{ it holds that} \]

\[ f(y) = \frac{2}{\sqrt{2 + y} \left( \frac{1}{\sqrt{y + 2}} - \frac{1}{\sqrt{y + 3}} \right)} + \frac{1}{\sqrt{(2 + y)(2 + y)}} - \frac{1}{\sqrt{(1 + y)(y + 3)}} > 0. \]

Lemma 5. For \( y \geq 0 \), it holds that

\[ f(y) = \frac{1}{\sqrt{1 + y} \left( \frac{1}{\sqrt{y + 3}} - \frac{1}{\sqrt{y + 4}} \right)} + \frac{2}{\sqrt{2 + y} \left( \frac{1}{\sqrt{y + 3}} - \frac{1}{\sqrt{y + 4}} \right)} + \frac{1}{\sqrt{(2 + y)(2 + y)}} - \frac{1}{\sqrt{(1 + y)(y + 4)}} > 0. \]

The proof of Lemma 5 is analogous to the proof of Lemma 3.

Now, we use three transformations, which will reduce the variable Randić index.

2.1. Transformation 1. Let \( G \in U_n \) and \( uv \) be an edge of \( G \) such that \( d_v = r \geq 2 \). \( N(v) = \{u, w_1, w_2, \ldots, w_r-1\} \), where \( w_1, w_2, \ldots, w_{r-1} \) are leaves.

Define \( G = G - \{vw_1, vw_2, \ldots, vw_{r-1}\} + \{uw_1, uw_2, \ldots, uw_{r-1}\} \). \( G \) and \( G \) are depicted in Figure 1.
Lemma 6. Let $G, \hat{G} \in U_n$. By applying Transformation 1 and for $\gamma \geq 0$, we have
\[
\gamma R_{\gamma}(G) > \gamma R_{\gamma}(\hat{G}).
\]  
Proof. Let $d_u = t \geq 3$:

\[
\gamma R_{\gamma}(G) - \gamma R_{\gamma}(\hat{G}) > \frac{d_v - 1}{\sqrt{(1 + \gamma)(d_v + \gamma)}} + \frac{1}{\sqrt{(d_v + \gamma)(d_u + \gamma)}} - \frac{d_v}{\sqrt{(1 + \gamma)(d_v + d_u - 1 + \gamma)}}
\]

\[
\gamma R_{\gamma}(G) - \gamma R_{\gamma}(\hat{G}) > \frac{d_v - 1}{\sqrt{1 + \gamma}}\left(\frac{1}{\sqrt{d_v + \gamma}} - \frac{1}{\sqrt{d_u + d_v - 1 + \gamma}}\right) + \frac{1}{\sqrt{(d_v + \gamma)(d_u + \gamma)}} - \frac{1}{\sqrt{(1 + \gamma)(d_v + d_u - 1 + \gamma)}}
\]

\[
\gamma R_{\gamma}(G) - \gamma R_{\gamma}(\hat{G}) > \frac{r - 1}{\sqrt{1 + \gamma}}\left(\frac{1}{\sqrt{r + \gamma}} - \frac{1}{\sqrt{r + r - 1 + \gamma}}\right) + \frac{1}{\sqrt{(r + \gamma)(r + \gamma)}} - \frac{1}{\sqrt{(1 + \gamma)(r + r - 1 + \gamma)}} > 0.
\]

Using Lemma 3, one can see that inequality (13) holds.

Lemma 7. Let $G, \hat{G} \in U_n$ such that $|G| = |\hat{G}|$. By applying Transformation 2 and for $\gamma \geq 0$, we have
\[
\gamma R_{\gamma}(G) > \gamma R_{\gamma}(\hat{G}).
\]

Proof. Since

\[
\gamma R_{\gamma}(G) = \frac{r - 2}{\sqrt{(y + 2)(y + 2)}} + \frac{n - r}{\sqrt{(y + 1)(n - r + y + 2)}} + \frac{2}{\sqrt{(y + 2)(n - r + y + 2)}}
\]

\[
\gamma R_{\gamma}(\hat{G}) = \frac{r - 3}{\sqrt{(y + 2)(y + 2)}} + \frac{n - r + 1}{\sqrt{(y + 1)(n - r + y + 3)}} + \frac{2}{\sqrt{(y + 2)(n - r + y + 3)}}
\]

\[
\gamma R_{\gamma}(G) - \gamma R_{\gamma}(\hat{G}) = \frac{n - r}{\sqrt{1 + \gamma}}\left(\frac{1}{\sqrt{n - r + y + 2}} - \frac{1}{\sqrt{n - r + y + 3}}\right)
\]

\[
+ \frac{2}{\sqrt{2 + \gamma}}\left(\frac{1}{\sqrt{n - r + y + 2}} - \frac{1}{\sqrt{n - r + y + 3}}\right)
\]

\[
+ \frac{1}{\sqrt{(2 + \gamma)(2 + y)}} - \frac{1}{\sqrt{(1 + \gamma)(n - r + y + 3)}}.
\]
Now, there will be three cases.

Case I: if \(n = r\), then

\[
\gamma R_{\gamma}(G) - \gamma R_{\gamma}(\hat{G}) = \frac{2}{\sqrt{2 + \gamma}} \left( \frac{1}{\sqrt{\gamma + 2}} - \frac{1}{\sqrt{\gamma + 3}} \right) + \frac{1}{\sqrt{(2 + \gamma)(2 + \gamma)}} - \frac{1}{\sqrt{(1 + \gamma)(\gamma + 3)}}
\]

(16)

Using Lemma 4, one can see that relation (16) is positive valued.

Case II: if \(n - r = 1\), then

\[
\gamma R_{\gamma}(G) - \gamma R_{\gamma}(\hat{G}) = \frac{1}{\sqrt{1 + \gamma}} \left( \frac{1}{\sqrt{n - r + \gamma + 2}} - \frac{1}{\sqrt{n - r + \gamma + 3}} \right) + \frac{2}{\sqrt{2 + \gamma}} \left( \frac{1}{\sqrt{n - r + \gamma + 2}} - \frac{1}{\sqrt{n - r + \gamma + 3}} \right) + \frac{1}{\sqrt{(2 + \gamma)(2 + \gamma)}} - \frac{1}{\sqrt{(1 + \gamma)(n - r + \gamma + 3)}}
\]

Using Lemma 5, one can see that relation (17) is positive valued.

Case III: if \(n - r \geq 2\), then

\[
\gamma R_{\gamma}(G) - \gamma R_{\gamma}(\hat{G}) = \frac{n - r}{\sqrt{1 + \gamma}} \left( \frac{1}{\sqrt{n - r + \gamma + 2}} - \frac{1}{\sqrt{n - r + \gamma + 3}} \right) + \frac{2}{\sqrt{2 + \gamma}} \left( \frac{1}{\sqrt{n - r + \gamma + 2}} - \frac{1}{\sqrt{n - r + \gamma + 3}} \right) + \frac{1}{\sqrt{(2 + \gamma)(2 + \gamma)}} - \frac{1}{\sqrt{(1 + \gamma)(n - r + \gamma + 3)}} \geq 0.
\]

Since \(\frac{n - r}{\sqrt{1 + \gamma}} \left( \frac{1}{\sqrt{n - r + \gamma + 2}} - \frac{1}{\sqrt{n - r + \gamma + 3}} \right) > 0\) and

\[
\frac{2}{\sqrt{2 + \gamma}} \left( \frac{1}{\sqrt{n - r + \gamma + 2}} - \frac{1}{\sqrt{n - r + \gamma + 3}} \right) > 0.
\]

(18) \(\square\)

2.3. Transformation 3. Let \(G \in U_n\) with \(|G| = n\) and \(u, v \in V(G)\) are on the unique cycle of \(G\). Pendant neighbors of \(u\) and \(v\) are \(u_i\), where \(1 \leq i \leq s\), and \(v_j\), where \(1 \leq j \leq t\), respectively. Assume that the path between \(u\) and \(v\) on cycle is \(P_{uv}\) such that \(|E(P_{uv})| = r\) and \(|E(P_{uv})|\geq 2\). Construct \(\hat{G}\) from \(G\) by removing the edges \(uv_i\) and \(vv_j\), reducing the path \(P_{uv}\) into one vertex \(u(v)\) and attaching a star \(K_{1, r+s+t-1}\) to \(u(v)\), by making sure that \(|\hat{G}| = |G|\). \(G\) and \(\hat{G}\) are depicted in Figure 2.

**Lemma 8.** Let \(G, \hat{G} \in U_n\) such that \(|\hat{G}| = |G|\). By applying Transformation 3 and for \(r \geq 0\), we have

\[
\gamma R_{\gamma}(G) > \gamma R_{\gamma}(\hat{G}).
\]

(19)

**Proof**

Case I: let \(r = 1\) and \(uv \in E(G)\), then

\[
\gamma R_{\gamma}(G) - \gamma R_{\gamma}(\hat{G}) > \frac{s}{\sqrt{1 + \gamma}} \left( \frac{1}{\sqrt{s + 2 + \gamma}} - \frac{1}{\sqrt{s + 3 + \gamma}} \right) + \frac{t}{\sqrt{1 + \gamma}} \left( \frac{1}{\sqrt{t + 2 + \gamma}} - \frac{1}{\sqrt{t + 3 + \gamma}} \right) + \frac{r}{\sqrt{(2 + \gamma)(2 + \gamma)}} - \frac{1}{\sqrt{(1 + \gamma)(s + t + 3 + \gamma)}}
\]

Using Lemma 1, one can see that relation (18) is positive valued.

Case II: now let \(r \geq 2\), then

\[
\gamma R_{\gamma}(G) - \gamma R_{\gamma}(\hat{G}) > \frac{s}{\sqrt{1 + \gamma}} \left( \frac{1}{\sqrt{s + 2 + \gamma}} - \frac{1}{\sqrt{s + 3 + \gamma}} \right) + \frac{t}{\sqrt{1 + \gamma}} \left( \frac{1}{\sqrt{t + 2 + \gamma}} - \frac{1}{\sqrt{t + 3 + \gamma}} \right) + \frac{r}{\sqrt{(2 + \gamma)(2 + \gamma)}} - \frac{1}{\sqrt{(1 + \gamma)(s + t + r + 2 + \gamma)}}
\]

(20)

Since \(r \geq 2\) and keeping in mind the fact that...
\[ \Psi(r, s, t) = \frac{s + t + r}{\sqrt{(1 + \gamma)(s + t + r + 2 + \gamma)}} \] (22)

is an increasing function because
\[ \frac{\partial \Psi}{\partial r} = \frac{s + t + r + 4 + 2\gamma}{2\sqrt{\gamma + 1}(s + t + r + 2 + \gamma)^{3/2}} > 0. \] (23)

We have
\[ \nu^r R_l(G) - \nu^r R_l(\bar{G}) > \frac{s}{\sqrt{1 + \gamma}} \left( \frac{1}{\sqrt{s + 2 + \gamma}} - \frac{1}{\sqrt{s + t + 4 + \gamma}} \right) + \frac{t}{\sqrt{1 + \gamma}} \left( \frac{1}{\sqrt{t + 2 + \gamma}} - \frac{1}{\sqrt{s + t + 4 + \gamma}} \right) \]
\[ + \frac{1}{\sqrt{(s + 2 + \gamma)(2 + \gamma)}} - \frac{1}{\sqrt{(1 + \gamma)(s + t + 4 + \gamma)}} + \frac{1}{\sqrt{(t + 2 + \gamma)(2 + \gamma)}} - \frac{1}{\sqrt{(1 + \gamma)(s + t + 4 + \gamma)}} \] (24)

Using Lemma 2, one can see that relation (24) is positive valued.

Let \( D_{n,3}^{i,j,k} \) be the such class of \( D \) that is obtained from cycle \( C_{1,l} \), \( l = 3 \) by attaching \( K_{i,j}, K_{1,j} \), and \( K_{1,k} \) to the vertices of \( C_{1,l}, l = 3 \). Let \( F_{n,4} \) be the such class of \( F \) that is obtained from cycle \( C_{l}, l = 4 \) by attaching \( K_{1,i} \) and \( K_{1,s} \) to the non-adjacent vertices of \( C_{l}, l = 4 \).

**Lemma 9.** Let \( D_{n,3}^{i,j,k}, D_{n,3}^{i,j-1,k}, \) and \( D_{n,3}^{i+1,j-1,k} \) be three unicyclic graphs in \( D \) as in Figure 3. If \( i \geq j \geq 1 \), then
\[ \nu^r R_l(D_{n,3}^{i,j,k}) > \nu^r R_l(D_{n,3}^{i+1,j-1,k}), \] (25)
for \( \gamma \geq 0 \).

**Proof:** Let \( D_{n,3}^{i,j,k}, D_{n,3}^{i,j-1,k}, \) and \( D_{n,3}^{i+1,j-1,k} \) be three unicyclic graphs in \( D \) as in Figure 3.
where $\xi_1, \xi_2 \in (j + 1, j + 2)$. Similarly, we have

$$vR_c^{D_{i,j}^{i+1,j-1,k}} - vR_c^{D_{i,j}^{i,j-1,k}} = \frac{i + 1}{\sqrt{(1 + \gamma)(i + 3 + \gamma)}} - \frac{i}{\sqrt{(1 + \gamma)(i + 2 + \gamma)}}$$

$$+ \frac{1}{\sqrt{(i + 3 + \gamma)(j + 1 + \gamma)}} - \frac{1}{\sqrt{(i + 2 + \gamma)(j + 1 + \gamma)}}$$

$$+ \frac{1}{\sqrt{(k + 2 + \gamma)(i + 3 + \gamma)}} - \frac{1}{\sqrt{(k + 2 + \gamma)(i + 2 + \gamma)}}$$

$$vR_c^{D_{i,j}^{i+1,j-1,k}} - vR_c^{D_{i,j}^{i,j-1,k}} = \frac{1}{\sqrt{1 + \gamma}} \left( \sqrt{i + 3 + \gamma} - \sqrt{i + 2 + \gamma} \right)$$

$$+ \left( \frac{1}{\sqrt{j + 1 + \gamma}} + \frac{1}{\sqrt{k + 2 + \gamma}} - \frac{\gamma + 2}{\sqrt{(1 + \gamma)}} \left( \frac{1}{\sqrt{i + 3 + \gamma}} - \frac{1}{\sqrt{i + 2 + \gamma}} \right) \right)$$

$$vR_c^{D_{i,j}^{i+1,j-1,k}} - vR_c^{D_{i,j}^{i,j-1,k}} = \frac{1}{\sqrt{1 + \gamma}} \left( \frac{1}{2^{1/2}} \right) + \left( \frac{1}{\sqrt{j + 1 + \gamma}} + \frac{1}{\sqrt{k + 2 + \gamma}} - \frac{\gamma + 2}{\sqrt{(1 + \gamma)}} \left( \frac{-1}{2^{1/2}} \xi_3^{-1/2} \right) \right)$$

$$\left. vR_c^{D_{i,j}^{i+1,j-1,k}} - vR_c^{D_{i,j}^{i,j-1,k}} > vR_c^{D_{i,j}^{i+1,j-1,k}} - vR_c^{D_{i,j}^{i,j-1,k}} \right)$$

where $\xi_3 \in (i + 2, i + 3)$.

Bearing in mind the fact that $i \geq j$, one can see that
Lemma 10. Let $F_{n,k}^{r,s}$, $F_{n,k}^{r,s-1}$, and $F_{n,k}^{r+1,s-1}$ be three unicyclic graphs in $F$ as in Figure 4. If $r \geq s \geq 1$, then

$$\nu_R^\gamma(F_{n,k}^{r,s}) > \nu_R^\gamma(F_{n,k}^{r,s-1}).$$

for $\gamma \geq 0$.

Proof. Let $\Psi(t) = (t/\sqrt{1+y} + 2/\sqrt{2+y})(1/\sqrt{1+y} + 2)$ and keeping in mind Figure 4, we have

$$\nu_R^\gamma(F_{n,k}^{r,s}) = \frac{r}{\sqrt{(1+y)(r+2+y)}} + \frac{s}{\sqrt{(1+y)(s+2+y)}} + \frac{2}{\sqrt{(1+y)(s+1+y)}} + \frac{2}{\sqrt{(2+y)(s+2+y)}} + \frac{2}{\sqrt{(2+y)(s+1+y)}}$$

$$\nu_R^\gamma(F_{n,k}^{r,s-1}) = \frac{r}{\sqrt{(1+y)(r+2+y)}} + \frac{s}{\sqrt{(1+y)(s+1+y)}} + \frac{2}{\sqrt{(2+y)(r+2+y)}} + \frac{2}{\sqrt{(2+y)(s+1+y)}} + \frac{2}{\sqrt{(2+y)(s+1+y)}}$$

$$\nu_R^\gamma(F_{n,k}^{r+1,s-1}) = \frac{r+1}{\sqrt{(1+y)(r+3+y)}} + \frac{s}{\sqrt{(1+y)(s+1+y)}} + \frac{2}{\sqrt{(2+y)(r+3+y)}} + \frac{2}{\sqrt{(2+y)(s+1+y)}} + \frac{2}{\sqrt{(2+y)(s+1+y)}}$$

$$\nu_R^\gamma(F_{n,k}^{r,s}) - \nu_R^\gamma(F_{n,k}^{r,s-1}) = \left(\frac{s}{\sqrt{1+y}} + \frac{2}{\sqrt{2+y}}\right)\frac{1}{\sqrt{s+2+y}} - \left(\frac{s-1}{\sqrt{1+y}} + \frac{2}{\sqrt{2+y}}\right)\frac{1}{\sqrt{s+1+y}},$$

where $\xi_1 \in (s+1, s+2)$. Similarly, we have

$$\nu_R^\gamma(F_{n,k}^{r,s}) - \nu_R^\gamma(F_{n,k}^{r,s-1}) = \Psi(s) - \Psi(s-1) = \Psi'(\xi_2),$$

where $\xi_2 \in (r+2, r+3)$.

Bearing in mind the fact that $r \geq s \geq 1, \Psi''(t) = -1/2(2(t+2+y)/\sqrt{1+y} - (3/2)[t/\sqrt{1+y} + 2/(\sqrt{2+y})] < 0$ for all $t > 0$. Also, $\Psi'(\xi_1) - \Psi'(\xi_2) > 0$, which implies that $\nu_R^\gamma(F_{n,k}^{r,s}) > \nu_R^\gamma(F_{n,k}^{r,s-1})$.

As a consequence of Lemmas 6–10, one has the following. \square

Theorem 1. Among all the unicyclic graphs of order $n \geq 4$, $S_n^L$ has the minimum variable Randić index for $\gamma \geq 0$ and its value is

$$\nu_R^\gamma(G) \geq \frac{n-3}{\sqrt{(1+y)(n-1+y)}} + \frac{2}{\sqrt{(2+y)(n-1+y)}} + \frac{1}{\sqrt{(2+y)(2+y)}}$$

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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