ASYMPTOTIC SPREADING OF TIME PERIODIC COMPETITION DIFFUSION SYSTEMS

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ABSTRACT. This paper deals with the asymptotic spreading of time periodic Lotka-Volterra competition diffusion systems, which formulates the coinvasion-coexistence process. By combining auxiliary systems with comparison principle, some results on asymptotic spreading are established. Our conclusions indicate that the coinvasions of two competitors may be successful, and the interspecific competitions slow the invasion speed of one species.

1. Introduction. Since the pioneer work of Aronson and Weinberger [1], much attention has been paid to the asymptotic spreading of parabolic type equations. When the asymptotic spreading is concerned, it is a Cauchy problem, and one basic index is the spreading speed. In literature, many important results have been obtained by monotone semiflows, we refer to Liang and Zhao [14], Lui [19], Weinberger [29, 30] and Weinberger et al. [32, 33] for some abstract results. Moreover, these results were applied to many classical models, which implies some important threshold dynamical behaviors in population dynamics. When a system is local cooperative, some conclusions were established by constructing two auxiliary cooperative systems which admits the same spreading speed, see [5, 11, 27, 31, 37, 38] as well as their applications. Note that in some noncooperative systems, their dynamical behavior can be investigated by utilizing the theory of monotone semiflows after a linear transformation, see Lewis et al. [12] for a Lotka-Volterra competitive model.

However, since the limitation of sources and energy transfer, many classical diffusion models are not (local) cooperative, e.g., competitive and predator-prey systems [8, 20, 21, 22, 25]. When it comes to the coinvasion-coexistence process of competitive systems [25], the asymptotic spreading was studied by Lin and Li [17] and Lin et al. [18]. It should be noted that if we make the linear transformation as that in [12], the desired equilibria are not ordered such that the theory of monotone semiflows cannot be applied to investigate the coinvasion-coexistence process. For the predator-prey systems, we refer to Lin [15] and Pan [23, 24]. Besides these, Du et al. [4], Guo and Wu [9], Wang and Zhang [28] considered the asymptotic spreading of a competitive system with free boundaries.

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In fact, the properties of media and environment may also effect the evolution of populations. From Xin [35], we can find some challenges and new phenomena of propagation theory in heterogeneous media. Indeed, some above abstract results also can be applied to evolutionary systems with periodic coefficients, for instance, Liang et al. [13], Yu and Zhao [39].

In this paper, we investigate the asymptotic spreading of the following time periodic Lotka-Volterra competition system

$$\begin{align*}
\frac{\partial u(t,x)}{\partial t} &= d_1(t)\Delta u(t,x) + u(t,x)[r_1(t) - a_1(t)u(t,x) - b_1(t)v(t,x)], \\
\frac{\partial v(t,x)}{\partial t} &= d_2(t)\Delta v(t,x) + v(t,x)[r_2(t) - a_2(t)u(t,x) - b_2(t)v(t,x)],
\end{align*}$$

where $t > 0, x \in \mathbb{R}$, $u(t,x)$ and $v(t,x)$ denote the population densities at time $t$ and location $x$ of the two competitors, respectively. For an integrable function $f(t)$ with periodic $T > 0$, we define $\overline{f} = \frac{1}{T} \int_0^T f(t) dt$. In addition, the coefficients in (1) satisfy

$$(A1): d_i(t), r_i(t), a_i(t), b_i(t) \in C^2(\mathbb{R}, \mathbb{R}), i = 1, 2,$$ are $T$-periodic functions for some $\theta \in (0, 1), T > 0$. Moreover, $d_i(t) > 0, a_i(t) > 0, a_2(t) \geq 0, b_1(t) \geq 0, b_2(t) > 0, t \in [0, T], r_i > 0, i = 1, 2.$

When (A1) holds, the corresponding kinetic system of (1) has a trivial solution $(0, 0)$ and two nonnegative semi-trivial periodic solutions. Moreover, it may admit a positive periodic solution with more conditions. If we consider the interspecific exclusive process of (1), then it can be rewritten into a cooperative evolutionary system, and this process was investigated by the viewpoint of traveling wave solutions [2, 40, 41] and entire solutions [3].

Different from the interspecific exclusive process in [2, 40, 41], in this paper, we shall investigate the asymptotic spreading of (1) with coinvasion-coexistence process, where both $u, v$ are invaders [25]. Just like what we have mentioned, the monotonicity is one of the main difficulty. Furthermore, since (1) may not have a constant steady state, the estimation of long time behavior of solutions is another difficulty.

In what follows, we consider the spreading speeds of (1) which involves trivial equilibrium and the positive periodic solution. More precisely, by using the method similar to [17, 18], some estimates of invasion speeds of the two species are given. Of course, because of time periodicity, the construction of auxiliary equations/functions will be more complex than that in [17, 18]. Moreover, after the invasion is successful, it is difficult to estimate long time behavior of the solution. Very recently, Lin et al. [16] studied the traveling wave solutions of periodic systems. To obtain the limit behaviors of traveling wave solutions, [16] utilized the stability of positive periodic solutions as well as the locality of basic solution of heat equation. With the idea of [16], we confirm that $(u, v)$ converges to the positive periodic solution once the invasion is successful.

The remained of this paper is organized as follows. Some preliminaries are listed in Section 2. In Section 3, we give the main results as well as the corresponding proof for (1), and show some significant explanations in the population dynamics.

2. Preliminaries. We first recall some classical results in this section. Define a complete metric space $\mathcal{C}$ by

$$\mathcal{C} = \{(u(x), v(x)) | (u, v) : \mathbb{R} \to \mathbb{R}^2 \text{ is uniformly continuous and bounded}\}$$
equipped with the compact open topology and maximum norm $\| \cdot \|$. Denote $C^+$ as

$$ C^+ = \{(u, v) : (u, v) \in C \text{ and } (u(x), v(x)) \geq 0 \text{ for all } x \in \mathbb{R} \}. $$

Moreover, if $a \leq b \in \mathbb{R}^2$, then

$$ C_{[a, b]} = \{(u, v) : (u, v) \in C \text{ and } a \leq (u(x), v(x)) \leq b \text{ for all } x \in \mathbb{R} \}. $$

The corresponding kinetic system of (1) is

$$ \begin{cases} u'(t) = u(t) \left[r_1(t) - a_1(t)u(t) - b_1(t)v(t)\right], t > 0, \\ v'(t) = v(t) \left[r_2(t) - a_2(t)u(t) - b_2(t)v(t)\right], t > 0. \end{cases} \tag{2} $$

Suppose that (A1) holds, then (2) has a trivial solution $(0, 0)$ and two nonnegative semi-trivial periodic solutions $(p(t), 0)$ and $(0, q(t))$, where

$$ \begin{cases} p(t) = \frac{\int_0^T e^{\int_s^T r_1(\tau) d\tau} a_1(s) ds}{1 + \int_0^T e^{\int_s^T r_1(\tau) d\tau} a_1(s) ds} \cdot p_0 = \frac{\int_0^T e^{\int_s^T r_1(\tau) d\tau} a_1(s) ds - 1}{e^{\int_s^T r_1(\tau) d\tau} a_1(s) ds} > 0, \\ q(t) = \frac{\int_0^T e^{\int_s^T r_2(\tau) d\tau} b_2(s) ds}{1 + \int_0^T e^{\int_s^T r_2(\tau) d\tau} b_2(s) ds} \cdot q_0 = \frac{\int_0^T e^{\int_s^T r_2(\tau) d\tau} b_2(s) ds - 1}{e^{\int_s^T r_2(\tau) d\tau} b_2(s) ds} > 0. \end{cases} $$

Further assume

(A2): $\int_0^T (r_1(t) - b_1(t)q(t)) dt > 0$, $\int_0^T (r_2(t) - a_2(t)p(t)) dt > 0$,

then (2) also admits a positive periodic solution $(u^*(t), v^*(t))$ satisfying

$$ 0 < u^*(t) \leq p(t), \quad 0 < v^*(t) \leq q(t). $$

In what follows, we say that $(u^*(t), v^*(t))$ is asymptotic stable if $(u(t), v(t))$ in (2) satisfies

$$ \lim_{t \to \infty} \left[|u(t) - u^*(t)| + |v(t) - v^*(t)|\right] = 0 $$

with the initial value $u(0) > 0, v(0) > 0$. To our knowledge, there are many sufficient conditions (e.g., $\tau_1 > \max_{t \in [0, T]} (b_1(t)/a_1(t)) \tau_2$, $\tau_2 > \max_{t \in [0, T]} (a_2(t)/a_1(t)) \tau_1$) on the asymptotic stability of $(u^*(t), v^*(t))$ in (2) (see [10, 34, 42]), and we shall not list precise conditions on the stability of $(u^*(t), v^*(t))$.

In order to obtain the spreading speed of competition systems described by (1), we should investigate the following initial value problem

$$ \begin{cases} \frac{\partial u(t, x)}{\partial t} = d_1(t) \Delta u(t, x) + u(t, x) \left[r_1(t) - a_1(t)u(t, x) - b_1(t)v(t, x)\right], \\ \frac{\partial v(t, x)}{\partial t} = d_2(t) \Delta v(t, x) + v(t, x) \left[r_2(t) - a_2(t)u(t, x) - b_2(t)v(t, x)\right], \\ (u(0, x), v(0, x)) = (u(x), v(x)) \in C_{[0, K]}, \end{cases} \tag{3} $$

in which $x \in \mathbb{R}, t \in (0, +\infty), K = (p_0, q_0)$.

Obviously, (3) can be analysed by the classical theory of reaction-diffusion systems [7, 36]. Furthermore, the solution of (3) satisfies $(0, 0) \leq (u(t, x), v(t, x)) \leq (p(t), q(t))$. However, it is difficult to construct a pair of super and sub-solutions which is continuously differentiable in $t$ and twice continuously differentiable in $x$. By Fife and Tang [7, Definition 4 and Remark 2], we introduce an admissible pair of weaker super and sub-solutions of (3) as follows.

**Definition 2.1.** Define vector functions

$$ \hat{u}(t, x) = \min \{ \pi_1(t, x), \ldots, \pi_n(t, x) \}, \hat{v}(t, x) = \min \{ \pi_1(t, x), \ldots, \pi_n(t, x) \}, $$

$$ \hat{u}(t, x) = \max \{ \mu_1(t, x), \ldots, \mu_n(t, x) \}, \hat{v}(t, x) = \max \{ \mu_1(t, x), \ldots, \mu_n(t, x) \}, $$

where $\pi_i, \pi_i, \mu_i, \mu_i \in C^{1,2}([0, T']) \times \mathbb{R}, i = 1, \ldots, n$ with $T' > 0$ and satisfy
corresponding kinetic equation of (4) has a unique positive periodic solution

Lemmas 3.1. In terms of Liang et al \[13, \text{Theorem 2.1}\], Hess \[10\] and Zhao \[42\], the

(3) for any given \((t_0, x_0)\), then there exists a neighborhood \(B_0(t_0, x_0)\) of \((t_0, x_0)\) such that

\[
\frac{\partial \bar{u}_i(t, x)}{\partial t} \geq d_1(t) \Delta \bar{u}_i(t, x) + \bar{u}_i(t, x) [r_1(t) - a_1(t) \bar{u}_i(t, x) - b_1(t) \bar{v}(t, x)],
\]

where \((t, x) \in B_0(t_0, x_0)\);

(2) for any given \((t_0, x_0)\), if \(\hat{v}(t_0, x_0) = \bar{v}_i(t_0, x_0)\) for some \(i \in \{1, \cdots, n\}\), then there exists a neighborhood \(B_0(t_0, x_0)\) of \((t_0, x_0)\) such that

\[
\frac{\partial \bar{v}_i(t, x)}{\partial t} \geq d_2(t) \Delta \bar{v}_i(t, x) + \bar{v}_i(t, x) [r_2(t) - a_2(t) \hat{u}(t, x) - b_2(t) \bar{v}(t, x)],
\]

where \((t, x) \in B_0(t_0, x_0)\);

(3) for any given \((t_0, x_0)\), if \(\hat{u}(t_0, x_0) = \bar{u}_i(t_0, x_0)\) for some \(i \in \{1, \cdots, n\}\), then there exists a neighborhood \(B_0(t_0, x_0)\) of \((t_0, x_0)\) such that

\[
\frac{\partial \bar{u}_i(t, x)}{\partial t} \leq d_1(t) \Delta \bar{u}_i(t, x) + \bar{u}_i(t, x) [r_1(t) - a_1(t) \bar{u}_i(t, x) - b_1(t) \bar{v}(t, x)],
\]

where \((t, x) \in B_0(t_0, x_0)\);

(4) for any given \((t_0, x_0)\), if \(\hat{v}(t_0, x_0) = \bar{v}_i(t_0, x_0)\) for some \(i \in \{1, \cdots, n\}\), then there exists a neighborhood \(B_0(t_0, x_0)\) of \((t_0, x_0)\) such that

\[
\frac{\partial \bar{v}_i(t, x)}{\partial t} \leq d_2(t) \Delta \bar{v}_i(t, x) + \bar{v}_i(t, x) [r_2(t) - a_2(t) \hat{u}(t, x) - b_2(t) \bar{v}(t, x)],
\]

where \((t, x) \in B_0(t_0, x_0)\);

(5) \((0, 0) \leq (\hat{u}(t, x), \hat{v}(t, x)) \leq (\bar{u}(t, x), \bar{v}(t, x)) \leq (p(t), q(t))\) for \(x \in \mathbb{R}, t \in [0, T^*]\), and \((\hat{u}(0, x), \hat{v}(0, x)) \leq (u(x), v(x)) \leq (\bar{u}(0, x), \bar{v}(0, x))\) with \(x \in \mathbb{R}\).

Then \((\hat{u}, \hat{v}), (\bar{u}, \bar{v})\) are said to be a pair of weaker super and sub-solutions of (3).

Thanks to comparison principle, the following lemma is clear.

**Lemma 2.2.** Assume that \((\hat{u}(t, x), \hat{v}(t, x)), (\bar{u}(t, x), \bar{v}(t, x))\) are a pair of weaker super and sub-solutions of (3) with \(t \in [0, T^*], x \in \mathbb{R}\). Then there is a unique solution of (3) satisfying \((\hat{u}(t, x), \hat{u}(t, x)) \leq (u(t, x), v(t, x)) \leq (\bar{u}(t, x), \bar{v}(t, x))\) for all \(t \in [0, T^*], x \in \mathbb{R}\).

### 3. Main results

In this section, the initial value problem (3) will be analyzed. Firstly, we consider the following initial value problem involving Fisher nonlinearity

\[
\begin{aligned}
\frac{\partial z(t, x)}{\partial t} &= d(t) \Delta z(t, x) + z(t, x) [r(t) - a(t) z(t, x)], \quad t > 0, x \in \mathbb{R}, \\
\quad z(0, x) &= z(x), \quad x \in \mathbb{R},
\end{aligned}
\]

in which \(r(t), d(t), a(t) \in C^2(\mathbb{R}, \mathbb{R})\) are \(T\)-periodic functions with some \(T > 0\) and \(\theta \in (0, 1)\). Moreover, \(d(t) > 0, a(t) > 0, t \in [0, T]\) and \(\tau > 0\). Then the corresponding kinetic equation of (4) has a unique positive periodic solution \(\beta(t)\).

Let \(z(x) \in [0, \beta(0)]\) be a bounded and continuous function with non-empty compact support. In terms of Liang et al [13, Theorem 2.1], Hess [10] and Zhao [42], the following results hold.

**Lemma 3.1.** (a) The unique \(T\)-periodic solution \(\beta(t)\) is globally stable and attracts all solutions with \(z(x) > 0\).
(b) Assume that \( z(t, x) \) is a solution of (4). Then

\[
\lim_{t \to \infty} \inf_{|x| < (2\sqrt{d} \cdot \tau - \epsilon)t} (z(t, x) - \beta(t)) = \lim_{t \to \infty} \sup_{|x| < (2\sqrt{d} \cdot \tau - \epsilon)t} (z(t, x) - \beta(t)) = 0,
\]

\[
\lim_{t \to \infty} \sup_{|x| > (2\sqrt{d} \cdot \tau + \epsilon)t} z(t, x) = 0
\]

for any given \( \epsilon > 0 \).

**Remark 1.** Assume that the constants \( r > 0, \sigma > 0 \) are fixed and \( z(t, x) \) is a solution of (4), if \( z(x) > \sigma \) for \( |x - x_0| \leq r \) with some \( x_0 \in \mathbb{R} \), then for any \( \epsilon > 0 \), there is a \( T_0 = T_0(\epsilon) \) such that \( z(t, x_0) > \beta(t) - \epsilon, t > T_0 \).

Under the assumption (A2), it is clear that there are positive periodic functions \( \alpha(t), \gamma(t) \) satisfying

\[
\alpha'(t) = \alpha(t)[r_1(t) - b_1(t)q(t) - a_1(t)\alpha(t)],
\]

and

\[
\gamma'(t) = \gamma(t)[r_2(t) - a_2(t)p(t) - b_2(t)\gamma(t)].
\]

Define positive constants as follows

\[
c_1 = 2 \sqrt{d_1 \cdot \tau_1}, \quad c_2 = 2 \sqrt{d_2 \cdot \tau_2}, \quad c_3 = 2 \sqrt{d_2 \cdot (\tau_2 - \alpha_2 \gamma)},
\]

\[
c_4 = 2 \sqrt{d_2 (\tau_2 - \alpha_2 \gamma)}, \quad c_5 = 2 \sqrt{d_1 \cdot (\tau_1 - b_1 q)},
\]

we state the main results for (1) here.

**Theorem 3.2.** Assume that (A1) holds. If \( u(x), v(x) \) have non-empty compact supports, then the solution \( (u(t, x), v(t, x)) \) of (3) is well defined for all \((t, x) \in (0, +\infty) \times \mathbb{R}\) and satisfies

\[
\lim_{t \to \infty} \sup_{|x| > (c_1 + \epsilon)t} u(t, x) = \lim_{t \to \infty} \sup_{|x| > (c_2 + \epsilon)t} v(t, x) = 0
\]

for any given \( \epsilon > 0 \). Further suppose that (A2) holds, then we can obtain the following properties.

(i) For any given \( \epsilon \in (0, c_1) \), if \( c_2 < c_5 \), then

\[
\lim_{t \to \infty} \inf_{|x| < (c_1 - \epsilon)t} (u(t, x) - \alpha(t)) \geq 0.
\]

(ii) For any given \( \epsilon > 0 \), if \( c_2 < c_5 \) and \( c_1 > c_2 + c_4 \), then

\[
\lim_{t \to \infty} \sup_{|x| > (c_4 + \epsilon)t} v(t, x) = 0.
\]

(iii) For any given \( \epsilon \in (0, (c_1 - c_2)/2) \), if \( c_2 < c_5 \), then

\[
\lim_{t \to \infty} \sup_{(c_2 + \epsilon)t < |x| < (c_1 - \epsilon)t} |u(t, x) - p(t)| = 0.
\]

(iv) For any given \( \epsilon \in (0, c_3) \), if \( (u^*(t), v^*(t)) \) is asymptotic stable and \( c_2 < c_5 \), then

\[
\lim_{t \to \infty} \sup_{|x| < (c_3 - \epsilon)t} [u(t, x) - u^*(t)] + |v(t, x) - v^*(t)| = 0.
\]
Similarly, one has

\[
\liminf_{t \to \infty} \inf_{|x| < (c-\epsilon)t} u(t, x) > 0, \liminf_{t \to \infty} \inf_{|x| < (c-\epsilon)t} v(t, x) > 0
\]

for any given \( \epsilon \in (0, c) \). If \((u^*(t), v^*(t))\) is asymptotic stable, then

\[
\limsup_{t \to \infty} \sup_{|x| < (c-\epsilon)t} \|u(t, x) - u^*(t)\| + |v(t, x) - v^*(t)| = 0.
\]

Now, we begin to prove Theorem 3.2. For the sake of simplicity, the proof of Theorem 3.2 will be split into the following lemmas.

**Lemma 3.3.** Assume that (A1) holds and \( \epsilon > 0 \) is given. Then

\[
\limsup_{t \to \infty} \sup_{|x| > (c_1+\epsilon)t} u(t, x) = \limsup_{t \to \infty} \sup_{|x| > (c_2+\epsilon)t} v(t, x) = 0.
\]

**Proof.** Note that \( u(t, x) \) is a sub-solution of

\[
\begin{align*}
\frac{\partial u(t, x)}{\partial t} &= d_1(t) \Delta u(t, x) + \alpha(t), \\
0 &= u(x),
\end{align*}
\]

then comparison principle and Lemma 3.1 imply that

\[
\lim_{t \to \infty} \sup_{|x| > (c_1+\epsilon)t} u(t, x) = 0.
\]

Similarly, one has

\[
\lim_{t \to \infty} \sup_{|x| > (c_2+\epsilon)t} v(t, x) = 0.
\]

The proof is complete. \(\square\)

**Lemma 3.4.** Assume that (A1), (A2) hold. If \( \epsilon \in (0, c_5) \) is given, then

\[
\liminf_{t \to \infty} \inf_{|x| < (c_3-\epsilon)t} (u(t, x) - \alpha(t)) \geq 0.
\]

Moreover, if \( \epsilon \in (0, c_3) \) is given, then

\[
\liminf_{t \to \infty} \inf_{|x| < (c_4-\epsilon)t} (v(t, x) - \gamma(t)) \geq 0.
\]

**Proof.** Note that \( v(t, x) \leq q(t) \) for any \( (t, x) \in (0, \infty) \times \mathbb{R} \), then

\[
\frac{\partial v(t, x)}{\partial t} \geq d_1(t) \Delta v(t, x) + \alpha(t).
\]

Since (A2) holds, then Lemma 3.1 indicates that

\[
\liminf_{t \to \infty} \inf_{|x| < (c_5-\epsilon)t} (u(t, x) - \alpha(t)) \geq 0.
\]

Together with the fact that \( u(t, x) \leq p(t) \) for any \( (t, x) \in (0, \infty) \times \mathbb{R} \), we can conclude the proof. \(\square\)

**Lemma 3.5.** Assume that (A1), (A2) hold and \( \epsilon \in (0, c_1) \) is given. If \( c_2 < c_5 \), then

\[
\liminf_{t \to \infty} \inf_{|x| < (c_1-\epsilon)t} u(t, x) > 0.
\]

**Proof.** From Lemmas 3.3 and 3.4, if \( c_2 < c_5 \), then for any \( \epsilon' > 0 \), there exists a large number \( T_1 \) such that

(i): \( \sup_{2|\epsilon| < (c_2+c_5)T} \{ v(t, x)/u(t, x) \} < 2q(t)/\alpha(t), t > T_1 \),

(ii): \( \sup_{2|\epsilon| < (2c_2+c_5)T} v(t, x) < \epsilon', t > T_1 \).
Let $\epsilon' > 0$ be sufficient small such that

$$2\sqrt{d_1 \cdot \left( r_1 - \bar{r}_1 \epsilon' \right)} > c_1 - \epsilon.$$ 

Therefore, we have

$$\frac{\partial u(t, x)}{\partial t} \geq d_1(t)\Delta u(t, x) + u(t, x) \left[ r_1(t) - b_1(t)\epsilon' - \frac{a_1(t)\alpha(t) + 2b_1(t)q(t)}{\alpha(t)} u(t, x) \right]$$ 

for all $t > T_1, x \in \mathbb{R}$. Lemma 3.1 implies $\liminf_{t \to \infty} \inf_{|x| < (c_1 - \epsilon) t} u(t, x) \geq 0$, the proof is complete. \hfill $\Box$

**Lemma 3.6.** Suppose that Lemma 3.5 holds. Then

$$\liminf_{t \to \infty} \inf_{|x| < (c_1 - \epsilon) t} (u(t, x) - \alpha(t)) \geq 0.$$ 

**Proof.** From Lemma 3.5, there exist $\sigma_1 > 0$ and $T_2 > 0$ such that

$$\inf_{|x| \leq (c_1 - \frac{\epsilon}{2}) t_1} (u(t_1, x) - \sigma_1) > 0, t_1 \geq T_2,$$

which follows that $u(t_1, x) > \sigma_1 > 0$ for all $t_1 \geq T_2$ and $|x| \leq (c_1 - \epsilon/2)t_1$. Remark 1 indicates that for any constant $c_1 \in (0, \min_t \alpha(t))$, if $|x| \leq (c_1 - \epsilon/2)t_1 - r_1$ with some positive number $r_1 > 0$, then there exists $T_3 > 0$ such that

$$u(t, x) \geq \alpha(t) - \epsilon_1, t \in [t_1 + T_3, t_1 + T_3 + T].$$

Furthermore, for any given constant $\epsilon > 0$, we can find an integer $T_4 > 0$ such that

$$(c_1 - \epsilon) t < (c_1 - \frac{\epsilon}{2}) t_1 - r_1,$$

where $t_1 \geq T_4$ and $t \in [t_1, t_1 + T]$. Take $t_1 > \max\{T_2 + T_3, T_4\}$ and $t \in [t_1, t_1 + T]$, then

$$u(t, x) \geq \alpha(t) - \epsilon_1, |x| < (c_1 - \epsilon)t.$$ 

Therefore, one has $u(t, x) \geq \alpha(t) - \epsilon_1$ for any $t > \max\{T_2 + T_3, T_4\}$ and $|x| < (c_1 - \epsilon)t$, which implies that

$$\inf_{|x| < (c_1 - \epsilon)t} (u(t, x) - \alpha(t) + \epsilon_1) \geq 0, t > \max\{T_2 + T_3, T_4\}.$$ 

Because of the arbitrariness of $\epsilon_1$, we complete the proof. \hfill $\Box$

**Lemma 3.7.** Assume that (A1), (A2) hold and $\epsilon > 0$ is given. If $c_2 < c_5$ and $c_1 > c_2 + c_4$, then

$$\lim_{t \to \infty} \sup_{|x| > (c_4 + \epsilon) t} v(t, x) = 0.$$ 

**Proof.** It suffices to consider the case $c_1 - \epsilon > c_2 + c_4$ with any given $\epsilon > 0$ small enough. Let $\delta > 0$ be such that

$$(c_4 + \epsilon/2)^2 = 4d_2 \left( \frac{\bar{r}_2}{a_2} \alpha + \delta a_2 \right)$$

and $\lambda_1, \lambda_2$ be defined by

$$\lambda_1 = \frac{c_2}{2d_2} = \sqrt{\frac{\bar{r}_2}{d_2}}, \lambda_2 = \frac{c_4 + \frac{\epsilon}{2}}{2d_2}.$$
Given a positive number

\[ M = (\lambda_1 + \lambda_2) \int_0^T |d_2(\tau) - \bar{d}_2| d\tau + \frac{1}{(\lambda_1 - \lambda_2)} \int_0^T |a_2(\tau)\alpha(\tau) - \bar{a}_2\alpha| d\tau \]

\[ + \frac{1}{2(\lambda_1 - \lambda_2)} \int_0^T |a_2(\tau) - \bar{a}_2| d\tau + 1, \]

then Lemma 3.6 implies that there is a constant \( T^* > 0 \) such that

(a): \( \inf_{|x| < (c_1 - \frac{\delta}{2})} u(t, x) \geq \alpha(t) - \delta/2, t > T^*; \)

(b): \( (c_1 - c_2 - c_4 - \epsilon)T^* > M. \)

Define continuous functions

\[ \nabla(t, x) = \min\{q(t), e^{\lambda_1(\pm x + c_2 t) + t_2 + \int_0^t [d_2(\tau)\lambda_1^2 + r_2(\tau) - \bar{d}_2\lambda_1^2 - \bar{r}_2] d\tau}\}, \]

and

\[ \nabla(t, x) = \min\{q(t), e^{\lambda_1(\pm x + c_2 t) + t_2 + \int_0^t [d_2(\tau)\lambda_1^2 + r_2(\tau) - \bar{d}_2\lambda_1^2 - \bar{r}_2] d\tau}, \]

\[ e^{\gamma_2(x + c_4 + \frac{\delta}{2}) t + t_2 + \int_0^t [d_2(\tau)\lambda_1^2 + r_2(\tau) - \sigma(\tau) - \bar{d}_2\lambda_1^2 - \sigma r_2 + \sigma] d\tau}\}, \]

where \( \sigma(t) = a_2(t)(\alpha(t) - \delta/2) \) and \( t_2 > 0 \) is a constant such that

\[ \nabla(T^*, x) \geq v(T^*, x) \] for all \( x \in \mathbb{R} \).

It should be noted that for \( t_2 > 0 \) large enough such that \( v(x) \leq \nabla(0, x) \), then \( \nabla(t, x) \) is a super-solution of

\[ \begin{cases} \frac{\partial \omega(t, x)}{\partial t} = \Delta \omega(t, x) + \omega(t, x)[r_2(t) - b_2(t)\omega(t, x)], & (t, x) \in (0, +\infty) \times \mathbb{R}, \\ \omega(0, x) = v(x), & x \in \mathbb{R}, \end{cases} \]

and so \( v(t, x) \leq \nabla(t, x) \) for all \( (t, x) \in (0, +\infty) \times \mathbb{R} \). Moreover, for each \( t > 0 \) and \( c_2 > c_4 \), we have

\[ \nabla(t, x) = e^{\lambda_1(\pm x + c_2 t) + t_2 + \int_0^t [d_2(\tau)\lambda_1^2 + r_2(\tau) - \bar{d}_2\lambda_1^2 - \bar{r}_2] d\tau} \] as \( x \to \pm\infty \),

which indicates the existence of \( t_2 \). Further construct continuous functions

\[ \overline{u}(t, x) = p(t), u(t, x) = w(t, x), v(t, x) = 0, \]

in which \( w(t, x) \) is given by

\[ \begin{cases} \frac{\partial w(t, x)}{\partial t} = d_1(t)\Delta w(t, x) + w(t, x) \left[r_2(t) - a_1(t)w(t, x) - b_1(t)\nabla(t, x)\right], \\ w(0, x) = u(x). \end{cases} \]

Lemmas 3.4-3.6 imply that \( w(t, x) \) satisfies the inequality (a) if \( t > T^* \).

Firstly, we verify that \((\overline{u}, \nabla)\) and \((u, v)\) are a pair of weaker super and sub-solutions of (3) with \( t > T^* \), \( x \in \mathbb{R} \). By the construction of \( \nabla(t, x) \), it is clear that \( \overline{u}, u, v \) satisfy Definition 2.1. Therefore, we only need to show that \( \nabla \) satisfies the inequality in Definition 2.1.

If \( t > T^* \), then \( \overline{u}(t, x) \geq 0 \), and if \( v(t, x) = q(t) \), we have

\[ \begin{align*} \overline{v}_t - d_2(t)\overline{v}_{xx} - \overline{v} \left[r_2(t) - a_2(t)\overline{u} - b_2(t)\overline{v}\right] & \geq \overline{v}_t - d_2(t)\overline{v}_{xx} - \overline{v} \left[r_2(t) - b_2(t)\overline{v}\right] \\ & = 0. \end{align*} \]
If \( \varphi(t, x) = e^{\lambda_1(x + c_2 t) + t_2 + \int_0^t [d_2(r) \lambda^2_2 + r_2(t) - \varphi_2^2] \, dr} \), then the positivity of \( \varphi \) indicates that
\[
\begin{align*}
\varphi_t - d_2(t)\varphi_{xx} - \varphi [r_2(t) - a_2(t)\varphi - b_2(t)\varphi] & \geq \varphi_t - d_2(t)\varphi_{xx} - \varphi [r_2(t) - a_2(t)\varphi] \\
& = (\lambda_1 c_2 + d_2(t)\lambda^2_2 - \bar{d}_2 \lambda^2_2 + r_2(t) - \bar{r}_2) \bar{\varphi} \\
& - d_2(t)\lambda^2_2 \bar{\varphi} - \varphi [r_2(t) - b_2(t)\bar{\varphi}] \\
& = b_2(t)\bar{\varphi}^2 \\
& \geq 0.
\end{align*}
\]

We now prove the result if
\[
\varphi(t, x) = e^{\lambda_2(x + (c_4 + \frac{\epsilon}{2}) t) + t_2 + \int_0^t [d_2(r) \lambda^2_2 + r_2(t) - \sigma(t) - \bar{d}_2 \lambda^2_2 - \bar{r}_2 + \bar{\sigma}] \, dr},
\]
(5)

A direct calculations yields
\[
e^{\lambda_2(x + (c_4 + \frac{\epsilon}{2}) t) + t_2 + \int_0^t [d_2(r) \lambda^2_2 + r_2(t) - \sigma(t) - \bar{d}_2 \lambda^2_2 - \bar{r}_2 + \bar{\sigma}] \, dr} < e^{\lambda_1(x + c_2 t) + t_2 + \int_0^t [d_2(r) \lambda^2_2 + r_2(t) - \bar{d}_2 \lambda^2_2 - \bar{r}_2] \, dr}.
\]

According to the definitions of \( \lambda_1 \) and \( \lambda_2 \), we can deduce that
\[
(\lambda_2 - \lambda_1)x < (\lambda_1 c_2 - \lambda_2 (c_4 + \epsilon/2)) t \\
+ (\lambda_2^2 - \lambda_2^2) \int_0^t (d_2(r) - \bar{d}_2) \, dr + \int_0^t (\sigma(r) - \bar{\sigma}) \, dr,
\]
which further implies that
\[
-x < (c_2 + c_4 + \epsilon/2) t \\
+ (\lambda_1 + \lambda_2) \int_0^t (d_2(r) - \bar{d}_2) \, dr + \frac{1}{(\lambda_1 - \lambda_2)} \int_0^t (\sigma(r) - \bar{\sigma}) \, dr.
\]

Since \( d_2(t), a_2(t) \) and \( \alpha(t) \) are periodic functions, then \( \int_0^t (d_2(r) - \bar{d}_2) \, dr \) and \( \int_0^t (\sigma(r) - \bar{\sigma}) \, dr \) are uniformly bounded in \( t \). Furthermore, the following inequality holds
\[
(\lambda_1 + \lambda_2) \int_0^t (d_2(r) - \bar{d}_2) \, dr + \frac{1}{(\lambda_1 - \lambda_2)} \int_0^t (\sigma(r) - \bar{\sigma}) \, dr \leq M
\]
for all \( t > 0 \) with \( \delta \) sufficient small. It then follows that \( -x < (c_1 - \epsilon/2) t \) for any fixed \( \epsilon > 0 \) small enough and \( t > T^* \), which suggests that (a) holds.

Because of the positivity of \( \varphi \), then
\[
\begin{align*}
\varphi_t - d_2(t)\varphi_{xx} - \varphi [r_2(t) - a_2(t)\varphi - b_2(t)\varphi] & \geq \varphi_t - d_2(t)\varphi_{xx} - \varphi [r_2(t) - a_2(t)\varphi] \\
& = (\lambda_2 (c_4 + \frac{\epsilon}{2}) + d_2(t)\lambda^2_2 - \bar{d}_2 \lambda^2_2 + r_2(t) - \bar{r}_2 - \sigma(t) + \bar{\sigma}) \bar{\varphi} \\
& - d_2(t)\lambda^2_2 \bar{\varphi} - \varphi [r_2(t) - \sigma(t)] \\
& \geq 0.
\end{align*}
\]

Moreover, Lemma 2.2 implies that
\[
(\varphi(t, x), \varphi(t, x)) \leq (u(t, x), v(t, x)) \leq (\varphi(t, x), \varphi(t, x))
\]
for all \( t > T^*, x \in \mathbb{R} \). The proof is complete. \( \square \)
Lemma 3.8. Assume that (A1), (A2) hold and \( \epsilon \in (0, (c_1 - c_2)/2) \) is given. If \( c_2 < c_5 \), then
\[
\limsup_{t \to \infty} \sup_{|x| < (c_1 - \epsilon)t} |u(t, x) - p(t)| = 0. \tag{6}
\]

Proof. For any given \( \epsilon > 0 \), if \( c_2 < c_5 \), Lemmas 3.3 and 3.6 imply that
\[
\liminf_{t \to \infty} \inf_{|x| < (c_1 - \epsilon)t} (u(t, x) - \alpha(t)) \geq 0, \quad \limsup_{t \to \infty} \sup_{|x| > (c_2 + \epsilon)t} v(t, x) = 0,
\]
which further indicates that
\[
\liminf_{t \to \infty} \inf_{(c_2 + \epsilon)t < |x| < (c_1 - \epsilon)t} (u(t, x) - \alpha(t)) \geq 0,
\]
\[
\limsup_{t \to \infty} \sup_{(c_2 + \epsilon)t < |x| < (c_1 - \epsilon)t} v(t, x) = 0.
\]
Hence, for any \( \epsilon' > 0 \), there exists \( T_5 > 0 \) such that
\[
\sup_{(c_2 + \epsilon)t < |x| < (c_1 - \epsilon)t} v(t, x) < \epsilon', \quad \inf_{(c_2 + \epsilon)t < |x| < (c_1 - \epsilon)t} u(t, x) \geq \alpha(t)/2
\]
for \( t \geq T_5 \). It then follows that
\[
\frac{\partial u(t, x)}{\partial t} \geq d_1(t) \Delta u(t, x) + u(t, x) [r_1(t) - b_1(t) \epsilon' - a_1(t)u(t, x)]
\]
for all \( t \geq T_5 \) and \( (c_2 + \epsilon)t < |x| < (c_1 - \epsilon)t \). Denote by \( p^*(t) \) the unique positive periodic solution of
\[
(p^*)'(t) = p^*(t) [r_1(t) - b_1(t) \epsilon' - a_1(t)p^*(t)],
\]
then \( \lim_{\epsilon' \to 0} \sup_{t \in [0, T]} |p^*(t) - p(t)| = 0 \).

For any given \( \epsilon_1 > 0 \), let \( \epsilon' > 0 \) be small enough such that
\[
|p(t) - p^*(t)| < \frac{\epsilon_1}{4},
\]
and \( \theta \in (0, 1) \) be sufficient small such that
\[
|p(t) - (1 - 2\theta)p^*(t)| < \frac{\epsilon_1}{2}.
\]
Further take \( \nu \in (0, 1) \) such that
\[
(1 - \nu) \max_{t \in \mathbb{R}} p(t) \leq \min_{t \in \mathbb{R}} \frac{\alpha(t)}{2}.
\]
Define a continuous function in \([-R, R]\) by
\[
u(t, x) := (1 - \theta)(1 - \nu e^{-\lambda t}) \cos \left( \frac{\pi x}{2R} \right) p^*(t), t \geq 0.
\]
Then there exist \( \lambda > 0, R > 0 \) such that
\[
\begin{aligned}
\left\{ \begin{array}{l}
\frac{\partial \nu(t, x)}{\partial t} \leq d_1(t) \Delta \nu(t, x) + \nu(t, x) [r_1(t) - b_1(t) \epsilon' - a_1(t)\nu(t, x)], \\
\nu(t, \pm R) = 0, \\
\nu(0, x) = (1 - \theta)(1 - \nu) \cos \left( \frac{\pi x}{2R} \right) p^*(0) \leq \frac{\alpha(t)}{2}
\end{array} \right.
\tag{7}
\end{aligned}
\]
for \((t, x) \in (0, +\infty) \times [-R, R]\). More precisely, we have
\[
\frac{\partial u(t, x)}{\partial t} - d_1(t)\Delta u(t, x) - \alpha(t)\left[u(t, x) - b_1(t)e^\alpha - a_1(t)u(t, x)\right]
\leq \nu\lambda e^{-\lambda t} \left[p^*(t) + (1 - \theta)(1 - \nu e^{-\lambda t}) \cos \left(\frac{\pi x}{2R}\right) \left(p^*\right)'(t)\right]
\leq \nu\lambda e^{-\lambda t} \left[p^*(t) + \left(\frac{\pi}{2R}\right)^2 d_1(t)\right]
\leq \nu\lambda e^{-\lambda t} \left[p^*(t) + \left(\frac{\pi}{2R}\right)^2 d_1(t) - a_1(t)p^*(t)\right]
\leq 0
\]
if
\[
\nu\lambda e^{-\lambda t} + \left(1 - \nu e^{-\lambda t}\right) \left[a_1(t)\alpha(t) + \left(\frac{\pi}{2R}\right)^2 d_1(t) - a_1(t)p^*(t)\right] \leq 0
\quad (8)
\]
or
\[
\nu\lambda e^{-\lambda t} + a_1(t)(1 - \theta)(1 - \nu e^{-\lambda t}) \cos \left(\frac{\pi x}{2R}\right) p^*(t)
\leq (1 - \nu e^{-\lambda t}) \left[a_1(t)p^*(t) - \left(\frac{\pi}{2R}\right)^2 d_1(t)\right].
\]
By the definition of \(p^*(t)\), (8) holds when
\[
\nu\lambda e^{-\lambda t} \leq (1 - \nu e^{-\lambda t}) \left[\theta a_1(t)p^*(t) - \left(\frac{\pi}{2R}\right)^2 d_1(t)\right].
\]
Take
\[
R = \pi \sqrt{\frac{\max_{t \in \mathbb{R}} d_1(t)}{2\theta \min_{t \in \mathbb{R}} [a_1(t)p^*(t)]}}
\]
and if
\[
\lambda \leq \frac{\theta(1 - \nu)}{2\nu} \min_{t \in \mathbb{R}} [a_1(t)p^*(t)],
\]
then (7) is straightforward and there exists \(T' > 0\) such that
\[
u(0) > (1 - \theta)p^*(t), t \geq T'.
\]
Take \(T_5 > 0\) such that \(R < eT_5/4\). For any given \(t_1 > T_5\) and \((c_2 + \frac{3\pi}{4})t_1 < |x| < (c_1 - \frac{\pi}{4})t_1\), we have
\[
\left\{\begin{array}{l}
\frac{\partial u(s, y)}{\partial s} \geq d_1(s)\Delta u(s, y) + u(s, y) \left[r_1(s) - b_1(s)e^\alpha - a_1(s)u(s, y)\right], \\
u(s, y) > \frac{a(s)}{2}
\end{array}\right.
\]
for all \(s \in [t_1, t_1 + T']\) and \(|y - x| \leq R\). By the comparison principle, we have
\[
u(t_1 + T', y) \geq u(t_1 + T', y)
\]
with \(y\) defined above. Note that \(\lambda > 0\) is independent of \(t\), then
\[
u(t_1 + T', x) \geq (1 - \theta)p^*(t_1 + T')
\]
In fact, Lemma 3.8 also reflects the locality of parabolic type operator.

Since \( \epsilon \) is arbitrary, we conclude the proof.

**Remark 2.** In fact, Lemma 3.8 also reflects the locality of parabolic type operator.

It should be noted that for (3), the asymptotic stability of \((u^*(t), v^*(t))\) can be deduced by that in (2) (see [6, 16, 26, 36]). In particular, if \((u^*(t), v^*(t))\) is asymptotic stable, then the following lemma is clear, of which based on the locality of basic solution of heat equation.

**Lemma 3.9.** Assume that \((u^*(t), v^*(t))\) is asymptotic stable. Let \( \sigma > 0 \) be a fixed constant. For any \( \epsilon > 0 \), there exists a sufficient large positive number \( r = r(\epsilon, \sigma) \) such that for any \( x_0 \in \mathbb{R}, \) if \( (u(x), v(x)) \geq (\sigma, \sigma) \) for \( |x - x_0| \leq r \), then there is a constant \( T_0 = T_0(\epsilon, \sigma) \) such that \(|u(t, x_0) - u^*(t)|, |v(t, x_0) - v^*(t)| \leq (\epsilon, \epsilon)\) with \( t \in [T_0, T_0 + T] \).

**Lemma 3.10.** Assume that (A1), (A2) hold and \( \epsilon \in (0, c_3) \) is given. If \((u^*(t), v^*(t))\) is asymptotic stable and \( c_2 < c_5 \), then

\[
\limsup_{t \to \infty} \sup_{|x| < (c_3 - \epsilon)t} |\alpha(t)| = 0.
\]

**Proof.** Since \( c_1 > c_3 \), for any given \( \epsilon \in (0, c_3) \), Lemmas 3.4 and 3.6 indicate that

\[
\liminf_{t \to \infty} \inf_{|x| < (c_3 - \epsilon)t} (u(t, x) - \alpha(t)) \geq 0,
\]

\[
\liminf_{t \to \infty} \inf_{|x| < (c_3 - \epsilon)t} (v(t, x) - \gamma(t)) \geq 0.
\]

Let \( \sigma_1 = \min\{\min_i \alpha(t)/2, \min_i \gamma(t)/2\} \). It is clear that there exists a positive constant \( T_8 > 0 \) such that \((u(t_1, x), v(t_1, x)) \geq (\sigma_1, \sigma_1) > (0, 0) \) for any \( t_1 \geq T_8, |x| < (c_3 - \epsilon/2)t_1 \). Hence, Lemma 3.9 implies that there exist \( T_0 > 0 \) and a sufficient large constant \( r_1 > 0 \) such that

\[
(|u(t, x) - u^*(t)|, |v(t, x) - v^*(t)|) \leq (\epsilon_1, \epsilon_1) \tag{9}
\]

for all \( t \in [t_1 + T_9, t_1 + T_9 + T] \) and \( |x| < (c_3 - \epsilon/2)t_1 - r_1 \). In addition, for any given \( \epsilon > 0 \) and the positive number \( r_1 \) fixed above, there is an integer \( T_{10} \) large enough such that

\[
(c_3 - \epsilon)t < (c_3 - \epsilon/2)t_1 - r_1,
\]

in which \( t_1 \geq T_{10} \) and \( t \in [t_1, t_1 + T] \). Therefore, for any \( t_1 \geq \max\{T_8 + T_9, T_{10}\}, u(t, x) \) and \( v(t, x) \) satisfy (9) with \( t \in [t_1, t_1 + T] \) and \( |x| < (c_3 - \epsilon)t \). Choosing \( t > \max\{T_8 + T_9, T_{10}\} \), one has

\[
(|u(t, x) - u^*(t)|, |v(t, x) - v^*(t)|) \leq (\epsilon_1, \epsilon_1), |x| \leq (c_3 - \epsilon)t.
\]

Since \( \epsilon_1 > 0 \) is arbitrary, we conclude the proof. \( \square \)
 According to the proof of Lemma 3.10, the following result is also evident.

**Lemma 3.11.** Assume that (A1), (A2) hold and there exists a constant \( c > 0 \) such that
\[
\liminf_{t \to \infty} \inf_{|x| < (c-t)u(t,x)} u(t,x) > 0, \quad \liminf_{t \to \infty} \inf_{|x| < (c-t)v(t,x)} v(t,x) > 0
\]
for any given \( \epsilon \in (0, c) \). If \((u^*(t), v^*(t))\) is asymptotic stable, then
\[
\limsup_{t \to \infty} \sup_{|x| < (c-t)} \left( |u(t,x) - u^*(t)| + |v(t,x) - v^*(t)| \right) = 0.
\]
This completes the proof of Theorem 3.2. Moreover, the following remark can be deduced from Lemma 3.1 and Theorem 3.2 easily.

**Remark 3.** Under proper conditions, the interspecific competition does not slow the spreading speed of \( u(t,x) \), but it does slow the spreading speed of \( v(t,x) \). Furthermore, the spreading speed of \( v \) is not less than \( 2\sqrt{d_2(r_2 - \bar{w})} \), which implies that the nontrivial effect of nonlinearities. Indeed, our results only indicate that the invasion speed for \( v(t,x) \) lies in a bounded interval, there may be some more precise results and we will study it in the future.

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