SUMMABLE SOLUTIONS OF SOME PARTIAL DIFFERENTIAL EQUATIONS AND GENERALISED INTEGRAL MEANS

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Abstract. We describe partial differential operators for which we can construct generalised integral means satisfying Pizzetti-type formulas. Using these formulas we give a new characterisation of summability of formal power series solutions to some multidimensional partial differential equations in terms of holomorphic properties of generalised integral means of the Cauchy data.

1. Introduction

We consider the initial value problem for a multidimensional linear partial differential equation with constant coefficients

$$(\partial_t - P(\partial_z)) u = 0, \quad u(0, z) = \varphi(z),$$

where $t \in \mathbb{C}$, $z \in \mathbb{C}^n$, $P(\partial_z) \in \mathbb{C}[\partial_z]$ is a differential operator of order greater than 1 with complex coefficients and $\varphi$ is holomorphic in a complex neighbourhood of the origin. The unique formal power series solution $\hat{u}$ of (1) is in general divergent. So, it is natural to ask about sufficient and necessary conditions (expressed in terms of the Cauchy data $\varphi$) under which the formal solution $\hat{u}$ is convergent or, more generally, summable.

By the general theory (see [2] and [5]) it is easy to find the sufficient condition for summability of $\hat{u}$ in terms of the analytic continuation property of the Cauchy data $\varphi$. Moreover, if the spatial variable $z$ is one-dimensional then, due to a certain symmetry between the variables $t$ and $z$, one can show that the sufficient condition is also necessary (see [5] and [7]).

But if the spatial variable $z$ is multidimensional then such a sufficient condition is too strong to be necessary. For this reason it is of interest to find weaker conditions, which characterise analytic or summable solutions of (1).

Such conditions were found in the special case $P(\partial_z) = \Delta_z = \sum_{k=1}^n \partial_{z_k}^2$ when (1) is the Cauchy problem for the multidimensional heat equation. Namely, in this case Lysik [4] proved that $\hat{u}$ is convergent if and only if the integral mean of $\varphi$ over the closed ball $B(x, r)$ or the sphere $S(x, r)$, as a function of the radius $r$, extends to an entire function of exponential order at most 2. Moreover, the author [6] showed that $\hat{u}$ is 1-summable in a direction $d$ if and only if the integral mean of $\varphi$ over the closed ball $B(x, r)$ or
the sphere $S(x, r)$, can be analytically continued to infinity in some sectors bisected by $d/2$ and $\pi + d/2$ with respect to $r$, and this continuation is of exponential order at most 2 as $r$ tends to infinity. These characterisations are based on the Pizzetti formulas which give the expansions of the integral means in terms of the radius $r$.

In the paper we extend the results [4] and [6] to more general multidimensional partial differential operators $P(\partial_z)$. The main tools used in the paper are generalised integral means and Pizzetti-type formulas.

We show that if a generalised integral mean of $\varphi$ satisfies a Pizzetti-type formula for the operator $P(\partial_z)$ then we are able to characterise convergent (and under some additional conditions also summable) solutions of (1) in terms of the generalised integral means (Theorem 1). For this reason it is important to describe such operators $P(\partial_z)$, for which we can find generalised integral means satisfying a Pizzetti-type formula. In the paper we construct such generalised integral means for every homogeneous operator with real coefficients of order 1 (Proposition 6) and for every elliptic homogeneous operator of order 2 with real coefficients (Theorem 2), respectively) denotes the real bisected by $d/2$ and $\pi + d/2$ with respect to $r$, and this continuation is of exponential order at most 2 as $r$ tends to infinity. In the last section we extend the notion of generalised integral means to the complex case. Then we are able to construct complex generalised integral means satisfying Pizzetti-type formulas for $P(\partial_z) = Q^s(\partial_z)$, where $Q(\partial_z)$ is a homogeneous operator of order $p \leq 2$ and $s \in \mathbb{N}$. In particular, as corollaries we obtain characterisations of analytic and summable solutions of (1) in terms of generalised integral means in the case when $P(\partial_z) = \sum_{i,j=1}^n a_{ij}\partial_{z_i z_j}$ is a homogeneous elliptic operator of order 2 with real coefficients (Theorem 2). We also prove that it is impossible to find a generalised integral mean satisfying a Pizzetti-type formula for any homogeneous (Theorem 3) or quasi-homogeneous (Theorem 4) operator of order $p > 2$.

In the last section we extend the notion of generalised integral means to the complex case. Then we are able to construct complex generalised integral means satisfying Pizzetti-type formulas for $P(\partial_z) = Q^s(\partial_z)$, where $Q(\partial_z)$ is a homogeneous operator of order $p \leq 2$ and $s \in \mathbb{N}$. In particular, as corollaries we obtain characterisations of analytic and summable solutions of (1) in terms of generalised integral means in the case when $P(\partial_z) = \sum_{i,j=1}^n a_{ij}\partial_{z_i z_j}$ is a homogeneous elliptic operator of order 2 with real coefficients (Theorem 2). We also prove that it is impossible to find a generalised integral mean satisfying a Pizzetti-type formula for any homogeneous (Theorem 3) or quasi-homogeneous (Theorem 4) operator of order $p > 2$.

2. Notation

Throughout the paper $B(x, r)$ ($S(x, r)$, respectively) denotes the real closed ball (sphere, respectively) with centre at $x \in \mathbb{R}^n$ and radius $r > 0$. Moreover, the complex disc in $\mathbb{C}^n$ with centre at the origin and radius $r > 0$ is denoted by $D_r^n := \{z \in \mathbb{C}^n : |z| < r\}$. To simplify notation, we write $D_r$ instead of $D_r^1$. If the radius $r$ is not essential, then we denote it briefly by $D^n$ (resp. $D$).

The Pochhammer symbol is defined for non-negative integers $k$ and complex numbers $a$ as $(a)_0 := 1$ and $(a)_k := a(a+1) \cdots (a+k-1)$ for $k \in \mathbb{N}$.

A sector in a direction $d \in \mathbb{R}$ with an opening $\varepsilon > 0$ in the universal covering space $\mathbb{C} \setminus \{0\}$ of $\mathbb{C} \setminus \{0\}$ is defined by

$$S_d(\varepsilon) := \{z \in \mathbb{C} \setminus \{0\} : z = re^{i\theta}, d - \varepsilon/2 < \theta < d + \varepsilon/2, r > 0\}.$$
Moreover, if the value of opening angle $\varepsilon$ is not essential, then we denote it briefly by $S_d$.

Analogously, by a \textit{disc-sector in a direction $d \in \mathbb{R}$ with an opening $\varepsilon > 0$ and radius $r_1 > 0$ we mean a domain $\hat{S}_d(\varepsilon; r_1) := S_d(\varepsilon) \cup D_{r_1}$. If the values of $\varepsilon$ and $r_1$ are not essential, we write it $\hat{S}_d$ for brevity (i.e. $\hat{S}_d = S_d \cup D$).

By $\mathcal{O}(G)$ we understand the space of holomorphic functions on a domain $G \subseteq \mathbb{C}^n$.

The space of formal power series $\hat{u}(t, z) = \sum_{j=0}^{\infty} u_j(z)t^j$ with $u_j(z) \in \mathcal{O}(D^n)$ is denoted by $\mathcal{O}(D^n)[[t]]$ or shortly by $\mathcal{O}[[t]]$. We use the “hat” notation (i.e. $\hat{u}$) to denote the formal power series. If the formal power series $\hat{u}$ is convergent, we denote its sum by $u$.

3. \textbf{Moment functions and k-summability}

We recall the notion of moment methods introduced by Balser \cite{1} and next we use them to define the Gevrey order and the k-summability. For more details we refer the reader to \cite{1}.

\textbf{Definition 1.} A function $u(t, z) \in \mathcal{O}(\hat{S}_d(\varepsilon; r_1) \times D^n)$ is of exponential growth of order at most $k > 0$ as $t \to \infty$ in $\hat{S}_d(\varepsilon; r_1)$ if for every $\tilde{r} \in (0, r)$, $\tilde{r}_1 \in (0, r_1)$ and every $\tilde{\varepsilon} \in (0, \varepsilon)$ there exist $A, B < \infty$ such that

$$\max_{|z| \leq \tilde{r}} |u(t, z)| \leq Ae^{B|t|^k} \quad \text{for} \quad t \in \hat{S}_d(\tilde{\varepsilon}; \tilde{r}_1).$$

The space of such functions is denoted by $\mathcal{O}^k(\hat{S}_d(\varepsilon; r_1) \times D^n)$. If the values of $\varepsilon$, $r_1$ and $r$ are not essential, we write it $\mathcal{O}^k(\hat{S}_d \times D^n)$ for brevity.

\textbf{Definition 2} (see [1, Section 5.5]). A pair of functions $e_m$ and $E_m$ is said to be \textit{kernel functions of order $k$ ($k > 1/2$)} if they have the following properties:

1. $e_m \in \mathcal{O}(S_0(\pi/k))$, $e_m(z)/z$ is integrable at the origin, $e_m(x) \in \mathbb{R}_+$ for $x \in \mathbb{R}_+$ and $e_m$ is exponentially flat of order $k$ in $S_0(\pi/k)$ (i.e. $\forall \varepsilon > 0 \exists A, B > 0$ such that $|e_m(z)| \leq Ae^{-|z|/B}$ for $z \in S_0(\pi/k - \varepsilon)$).
2. $E_m \in \mathcal{O}^k(\mathbb{C})$ and $E_m(1/z)/z$ is integrable at the origin in $S_\pi(2\pi - \pi/k)$.
3. The connection between $e_m$ and $E_m$ is given by the \textit{corresponding moment function} $m$ of order $1/k$ as follows. The function $m$ is defined in terms of $e_m$ by

$$m(u) := \int_0^\infty x^{u-1}e_m(x)dx \quad \text{for} \quad \text{Re } u \geq 0$$

and the kernel function $E_m$ has the power series expansion

$$E_m(z) = \sum_{j=0}^{\infty} \frac{z^j}{m(j)} \quad \text{for} \quad z \in \mathbb{C}.$$
Definition 3 (see [1, Section 5.6]). A function \( e_m \) is called a kernel function of order \( k > 0 \) if we can find a pair of kernel functions \( \tilde{e}_m \) and \( E_m \) of order \( pk > 1/2 \) (for some \( p \in \mathbb{N} \)) so that
\[
e_m(z) = e_{\tilde{m}}(z^{1/p})/p \quad \text{for} \quad z \in S(0, \pi/k).
\]
For the kernel function \( e_m \) of order \( k > 0 \) we define the corresponding moment function \( m \) of order \( 1/k > 0 \) by (2) and the kernel function \( E_m \) of order \( k > 0 \) by (3).

Remark 1. Observe that by Definitions 2 and 3 we have
\[
m(u) = \tilde{m}(pu) \quad \text{and} \quad E_m(z) = \sum_{j=0}^{\infty} \frac{z^j}{m(j)} = \sum_{j=0}^{\infty} \frac{z^j}{\tilde{m}(jp)}.
\]
Hence if \( \tilde{m} \) is a moment function of order \( s > 0 \) then \( m(u) := \tilde{m}(pu) \) is a moment function of order \( ps \).

Example 1. For any \( a \geq 0, b \geq 1 \) and \( k > 0 \) we can construct the following examples of kernel functions \( e_m \) and \( E_m \) of orders \( k > 0 \) with the corresponding moment function \( m \) of order \( 1/k \) satisfying Definition 2 or 3:
- \( e_m(z) = akz^k e^{-z^k} \),
- \( m(u) = a\Gamma(b+u/k) \), where \( \Gamma \) is the gamma function,
- \( E_m(z) = \frac{1}{a} \sum_{j=0}^{\infty} z^{j^k} =: E_{1/k}(z) \), where \( E_{1/k} \) is the Mittag-Leffler function of index \( 1/k \).
In particular for \( a = b = 1 \) we get the kernel functions and the corresponding moment function, which are used in the classical theory of \( k \)-summability.
- \( e_m(z) = kze^{-z^k} \),
- \( m(u) = \Gamma(1+u/k) \),
- \( E_m(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(1+j/k)} =: E_{1/k}(z) \), where \( E_{1/k} \) is the Mittag-Leffler function of index \( 1/k \).

For any \( s > 0 \) we will denote by \( \Gamma_s \) the function \( \Gamma_s(u) := \Gamma(1+su) \), which is the crucial example of a moment function of order \( s \).

By [1, Theorems 31 and 32] we obtain the following proposition, which allows us to construct new moment functions.

Proposition 1. Let \( m_1, m_2 \) be moment functions of orders \( s_1, s_2 > 0 \) respectively. Then
- \( m_1m_2 \) is a moment function of order \( s_1 + s_2 \),
- \( m_1/m_2 \) is a moment function of order \( s_1 - s_2 \) for \( s_1 > s_2 \).

In particular we have

Corollary 1. We assume that \( s > 1 \). Then \( m_1 \) is a moment function of order \( s \) if and only if \( m_2 = m_1/\Gamma_1 \) is a moment function of order \( s - 1 \).

Every moment function \( m \) of order \( s > 0 \) has the same growth as \( \Gamma_s \). Precisely speaking, we have

Proposition 2 (see [1, Section 5.5]). If \( m \) is a moment function of order \( s > 0 \) then there exist constants \( c, C > 0 \) such that
\[
c^j \Gamma_s(j) \leq m(j) \leq C^j \Gamma_s(j) \quad \text{for every} \quad j \in \mathbb{N}.
\]
More generally we say

**Definition 4.** If \( s > 0 \) and a function \( m \) satisfies (4) then \( m \) is called a function of order \( s \).

We also define some class of functions of order \( s \), which contains moment functions of order \( s \).

**Definition 5.** If there exists a moment function \( m_2 \) of order \( s \) > 0 and polynomials \( p_1(j), \ p_2(j) \) and \( c > 0 \) such that a function \( m_1 \) of order \( s \) satisfies

\[
(5) \quad p_1(j) m_1(j) = p_2(j) m_2(j) c^j \quad \text{for every} \quad j \in \mathbb{N}
\]

then \( m_1 \) is called a generalised moment function of order \( s \).

The importance of the assumption (5) comes from

**Proposition 3.** We assume that \( m_1(j) \) and \( m_2(j) \) are sequences satisfying (5), \( k > 0 \), \( d \in \mathbb{R} \) and \( \hat{F}_i(t) = \sum_{j=0}^{\infty} \frac{a_i j^j}{m_i(j)} t^j \) for \( i = 1, 2 \). Then \( F_i(t) \in \mathcal{O}^k(\hat{S}_d) \) if and only if \( F_2(t) \in \mathcal{O}^k(\hat{S}_d) \).

**Proof.** Since \( m_1(j) \) and \( m_2(j) \) satisfy (5), there exist polynomials \( p_1(j), \ p_2(j) \) and \( c > 0 \) such that

\[
\frac{p_1(j)}{m_2(j)} = \frac{p_2(j)}{m_1(j)} c^j \quad \text{for every} \quad j \in \mathbb{N}.
\]

Hence \( m_1(j) \) and \( m_2(j) \) have the same order and consequently also \( \hat{F}_1(t) \) and \( \hat{F}_2(t) \) have the same Gevrey order, so in particular \( \hat{F}_i(t) \in \mathbb{C}[[t]]_0 \) if and only if \( \hat{F}_2(t) \in \mathbb{C}[[t]]_0 \). Moreover, if \( \hat{F}_i(t) \in \mathbb{C}[[t]]_0 \) \( (i = 1, 2) \) then their sums \( F_i(t) \) are well defined and satisfy

\[
p_1(t \partial_t) F_2(t) = \sum_{j=0}^{\infty} \frac{p_1(j)a_j}{m_2(j)} t^j = \sum_{j=0}^{\infty} \frac{p_2(j)a_j c^j}{m_1(j)} t^j = p_2(t \partial_t) F_1(ct).
\]

It means that \( F_1(t) \in \mathcal{O}^k(\hat{S}_d) \) if and only if \( F_2(t) \in \mathcal{O}^k(\hat{S}_d) \). \( \Box \)

We use moment functions to define moment Borel transforms, the Gevrey order and the Borel summability.

**Definition 6.** Let \( m \) be a moment function of order \( s > 0 \) (or, more generally, a function of order \( s > 0 \)). Then the linear operator \( \mathcal{B}_m : \mathcal{O}[[t]] \to \mathcal{O}[[t]] \) defined by

\[
\mathcal{B}_m \left( \sum_{j=0}^{\infty} u_j t^j \right) := \sum_{j=0}^{\infty} \frac{u_j}{m(j)} t^j
\]

is called the \( m \)-moment Borel transform of order \( s \).

We define the Gevrey order of formal power series as follows

**Definition 7.** Let \( s \) > 0. Then \( \hat{u} \in \mathcal{O}(D^n)[[t]] \) is called a formal power series of Gevrey order \( s \) if there exists a disc \( D \subset \mathbb{C} \) with centre at the origin such that \( \mathcal{B}_1 \hat{u} \in \mathcal{O}(D \times D^n) \). The space of formal power series of Gevrey order \( s \) is denoted by \( \mathcal{O}(D^n)[[t]]_s \).
Remark 2. By Proposition 1 [1], we may replace $\Gamma_s$ in Definition 7 by any function $m$ of order $s$.

Now we are ready to define the $k$-summability of formal power series (see Balser [1]).

**Definition 8.** Let $k > 0$ and $d \in \mathbb{R}$. Then $\hat{u} \in \mathcal{O}(D^n)[[t]]$ is called $k$-
summable in a direction $d$ if there exists a disc-sector $\hat{S}_d$ in a direction $d$ such that $\mathcal{B}_{\Gamma_{1/k}} \hat{u}(t, z) \in \mathcal{O}^k(\hat{S}_d \times D^n)$.

By [1 Proposition 13] and Definition 8 we may also characterise convergent series $\hat{u}$ in terms of $\mathcal{B}_{\Gamma_{1/k}} \hat{u}$ as follows

**Proposition 4.** Let $k > 0$ and $\hat{u} \in \mathcal{O}(D^n)[[t]]$. Then $\hat{u}(t, z)$ converges for sufficiently small $|t|$ if and only if $\mathcal{B}_{\Gamma_{1/k}} \hat{u}(t, z) \in \mathcal{O}^k(\mathbb{C} \times D^n)$.

Remark 3. By the general theory of moment summability (see [1 Section 6.5 and Theorem 38]) and by Proposition 3, we may replace $\Gamma_{1/k}$ in Definition 8 by any moment function $m$ of order $1/k$ or even by any generalised moment function of order $1/k$.

By Proposition 4 and by Remarks 2–3 we conclude that

**Proposition 5.** Let $k > 0$, $d \in \mathbb{R}$, $m$ be a function of order $1/k$ and $\hat{u} = \sum_{j=0}^{\infty} u_j(z) t^j \in \mathcal{O}(D^n)[[t]]$. Then $\hat{u}$ is convergent for sufficiently small $|t|$ if and only if $\sum_{j=0}^{\infty} \frac{u_j(z)}{m(j)} t^j \in \mathcal{O}^k(\mathbb{C} \times D^n)$.

Moreover, if additionally $m$ is a generalised moment function of order $1/k$, then $\hat{u}$ is $k$-summable in a direction $d$ if and only if there exists a disc-sector $\hat{S}_d$ in a direction $d$ such that $\sum_{j=0}^{\infty} \frac{u_j(z)}{m(j)} t^j \in \mathcal{O}^k(\hat{S}_d \times D^n)$.

### 4. Generalised integral means

In this section we introduce the notion of generalised integral means. To this end we take

**Definition 9.** Let $\mu$ be a finite complex Borel measure supported in the closed ball $B(0, R)$ (for some $R > 0$) in $\mathbb{R}^n$ of total mass 1 (i.e. $\int_{\mathbb{R}^n} d\mu(y) = 1$). Moreover we assume that $\varphi$ is a continuous function on a domain $\Omega \subset \mathbb{R}^n$, $x \in \Omega$ and $0 < r < \text{dist}(x, \partial \Omega)/R$. Then the generalised integral mean $M_\mu(\varphi; x, r)$ of $\varphi$ is defined by

$$M_\mu(\varphi; x, r) := \int_{\mathbb{R}^n} \varphi(x + ry) \, d\mu(y).$$

**Remark 4.** Let us assume that

$$\mu_B(y) = \frac{dy}{\alpha(n)} \quad \text{and} \quad \mu_S(y) = \frac{dS(y)}{n\alpha(n)},$$

where $\alpha(n) = \pi^{n/2}/\Gamma(1 + n/2)$ is the volume of the $n$-dimensional unit ball $B(0, 1)$, $dy$ is the Lebesgue measure on $B(0, 1)$ and $dS(y)$ is the surface measure on $S(0, 1)$. Then

$$M_{\mu_B}(\varphi; x, r) = \int_{B(0, 1)} \varphi(x + yr) \, d\mu_B(y) = \int_{B(0, 1)} \varphi(x + yr) \, dy.$$
and
\[ M_{\mu_S}(\varphi; x, r) = \int_{s(0,1)} \varphi(x + yr) \, d\mu_S(y) = \int_{s(0,1)} \varphi(z + yr) \, dS(y) \]
are respectively the solid and spherical means of \( \varphi \).

**Remark 5.** We can extend Definition \( \text{[1]} \) to the complex case replacing in \( \text{[6]} \) the real variables \( x, r \) by the complex ones \( z, t \). In this way we define by \( \text{[6]} \) the generalised integral mean \( M_{\mu}(\varphi; z, t) \) for \( z \in \mathbb{C}^n \) and \( t \in \mathbb{C} \).

The crucial role in our considerations play the extensions of the Pizzetti formulas, which hold for some generalised integral means \( M_{\mu}(\varphi; z, t) \). Precisely

**Definition 10.** Let \( \varphi(z) \in \mathcal{O}(D^n) \) and \( P(\partial_z) \in \mathbb{C}[\partial_z] \) be a homogeneous partial differential operator of order \( p \) with constant coefficients. We say that a generalised integral mean \( M_{\mu}(\varphi; z, t) \) satisfies a Pizzetti-type formula for the operator \( P(\partial_z) \) if

\[ M_{\mu}(\varphi; z, t) = \sum_{j=0}^{\infty} \frac{P^j(\partial_z)\varphi(z)}{m(j)} t^j \]

for some function \( m \) of order \( p > 0 \) with \( m(0) = 1 \).

Our aim is to describe such generalised integral means \( M_{\mu}(\varphi; z, t) \) which satisfy Pizzetti-type formulas for some operators \( P(\partial_z) \).

**Remark 6.** Observe that a generalised integral mean \( M_{\mu}(\varphi; z, t) \) satisfying the Pizzetti-type formula \( \text{[8]} \) for the operator \( P(\partial_z) \) has the following natural properties:

a) \( \lim_{t \to 0} M_{\mu}(\varphi; z, t) = M_{\mu}(\varphi; z, 0) = \varphi(z) \).

b) The mean-value property for \( P(\partial_z) \): If \( P(\partial_z)\varphi(z) = 0 \) then \( \varphi(z) = M_{\mu}(\varphi; z, t) \) for any \( z \in D^n \) and \( t \in \mathbb{C} \).

c) Converse to the mean-value property for \( P(\partial_z) \): If \( \varphi(z) = M_{\mu}(\varphi; z, t) \) for any \( z \in D^n \) and \( t > 0 \) then \( P(\partial_z)\varphi(z) = 0 \).

5. Formal power series solutions

We will use generalised integral means satisfying Pizzetti-type formulas to characterise summable formal power series solutions of the Cauchy problem

\[ (\partial_t - P(\partial_z))u = 0, \quad u(0, z) = \varphi(z) \in \mathcal{O}(D^n), \]

where \( t \in \mathbb{C}, z \in \mathbb{C}^n \) and \( P(\partial_z) \in \mathbb{C}[\partial_z] \) is a homogeneous partial differential operator of order \( p > 1 \) with constant coefficients. The unique formal power solution of \( \text{[9]} \) is given by

\[ \hat{u}(t, z) = \sum_{j=0}^{\infty} \frac{P^j(\partial_z)\varphi(z)}{j!} t^j \in \mathcal{O}(D^n)[[t]]. \]
Since $P(\partial_z)$ is an operator of order $p$, for every $r_0 \in (0, r)$ there exist $A, B < \infty$ such that
\[
\sup_{|z| < r_0} \left| \frac{P_j(\partial_z)\varphi(z)}{j!} \right| \leq AB^j(j!)^{p-1} \quad \text{for every} \quad j \in \mathbb{N}_0.
\]
It means that $\hat{u}(t, z)$ is a formal series of Gevrey order $p - 1$. So, it is natural to ask about $\frac{1}{p-1}$-summability of $\hat{u}(t, z)$. By Proposition 5 and Corollary 1 we conclude

Theorem 1. Let $P(\partial_z) \in \mathbb{C}[\partial_z]$ be a homogeneous partial differential operator of order $p > 1$ with constant coefficients and $M_\mu(\varphi; z, t)$ be a generalised integral mean satisfying a Pizzetti-type formula (5) for the operator $P(\partial_z)$. Then the formal solution of (2) is convergent for sufficiently small $|t|$ if and only if the generalised integral mean $M_\mu(\varphi; z, t)$ satisfies
\[
M_\mu(\varphi; z, t) \in \mathcal{O}_{\frac{p}{p-1}}(D^n \times \mathbb{C}) \quad \text{for} \quad k = 0, \ldots, p - 1.
\]
Moreover, if we additionally assume that $m$ is a generalised moment function of order $p$ then the formal solution of (9) is $\frac{1}{p-1}$-summable in a direction $d \in \mathbb{R}$ if and only if there exist disc-sectors $\hat{S}_{\frac{2k\pi}{p}}$ in the directions $\frac{2k\pi}{p}$ (for $k = 0, \ldots, p - 1$) such that the generalised integral mean $M_\mu(\varphi; z, t)$ satisfies
\[
M_\mu(\varphi; z, t) \in \mathcal{O}_{\frac{p}{p-1}}(D^n \times \hat{S}_{\frac{2k\pi}{p}}) \quad \text{for} \quad k = 0, \ldots, p - 1.
\]

6. First and second order homogeneous operators

In this section we describe generalised integral means satisfying Pizzetti-type formulas (5) for the homogeneous operators of first and second order.

First we assume that $P(\partial_z)$ is a general homogeneous first order operator with real coefficients. For such operators we have

Proposition 6. Let $P(\partial_z) = \sum_{k=1}^n a_k \partial_{z_k}$ with $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$ and $\mu_{\delta, a}(y) = \delta(y - a)$, where $\delta$ is the Dirac delta. Then the generalised integral mean $M_{\mu_{\delta, a}}(\varphi; z, t)$ satisfies a Pizzetti-type formula
\[
M_{\mu_{\delta, a}}(\varphi; z, t) = \varphi(z + at) = \sum_{j=0}^{\infty} \frac{P_j(\partial_z)\varphi(z)}{j!} t^j.
\]

Proof. By the definition
\[
M_{\mu_{\delta, a}}(\varphi; z, t) = \int_{\mathbb{R}^n} \varphi(z + ty) d\mu_{\delta, a}(y) = \varphi(z + at).
\]
Moreover, by the Taylor formula for the function $t \mapsto \varphi(z + at)$ we conclude that
\[
\varphi(z + at) = \sum_{j=0}^{\infty} \frac{d^j}{dt^j}\varphi(z + at)|_{t=0} t^j = \sum_{j=0}^{\infty} \frac{P_j(\partial_z)\varphi(z)}{j!} t^j
\]
and (10) holds. \qed
For the Laplace operator $P(\partial_z) = \sum_{k=1}^{n} \partial_{z_k}^2 = \Delta_z$ we get the classical formulas introduced for the first time in dimension $n = 3$ by Pizzetti [9]. Namely, we have

**Proposition 7** ([1] Theorem 3.1). We assume that the measures $\mu_B(y)$ and $\mu_S(y)$ are defined by [7] and the sequences of numbers $m_B$ and $m_S$ are given by

$$m_B(j) = 4^j(n/2 + 1)j! \quad \text{and} \quad m_S(j) = 4^j(n/2)j!.$$  

Then the solid $M_B(\varphi; z, t)$ and spherical $M_S(\varphi; z, t)$ means satisfy the Pizzetti formulas

$$M_B(\varphi; z, t) = \int_{B(0,1)} \varphi(z + ty) \, dy = \sum_{j=0}^{\infty} \frac{\Delta_j^j \varphi(z)}{m_B(j)} t^{2j}$$

and

$$M_S(\varphi; z, t) = \int_{S(0,1)} \varphi(z + ty) \, dS(y) = \sum_{j=0}^{\infty} \frac{\Delta_j^j \varphi(z)}{m_S(j)} t^{2j}.$$  

Moreover, using the Pizzetti formulas we conclude

**Proposition 8** ([1] Theorems 4.5 and 4.6 and [6] Theorem 4.1). Let $d \in \mathbb{R}$ and $\hat{u} \in \mathcal{O}[[t]]$ be the formal solution of the $n$-dimensional complex heat equation

$$(\partial_t - \Delta_z) u(t, z) = 0, \quad u(0, z) = \varphi(z) \in \mathcal{O}(D^n).$$

Then:

- $u \in \mathcal{O}(D^{n+1})$ if and only if $M_B(\varphi; z, t) \in \mathcal{O}^2(D^n \times \mathbb{C})$, and if and only if $M_S(\varphi; z, t) \in \mathcal{O}^2(D^n \times \mathbb{C})$,
- $\hat{u}$ is 1-summable in the direction $d$ if and only if $M_B(\varphi; z, t) \in \mathcal{O}^2(D^n \times (\hat{S}_{d/2} \cup \hat{S}_{d/2+\pi}))$, and if and only if $M_S(\varphi; z, t) \in \mathcal{O}^2(D^n \times (\hat{S}_{d/2} \cup \hat{S}_{d/2+\pi})).$

**Remark 7.** Observe that

$$m_B(j) = \begin{cases} p_{B1}(j)(2j)! & \text{for odd } n \\ p_{B2}(j)4^j(j)!^2 & \text{for even } n \end{cases}$$

with polynomials $p_{B1}(j) = \frac{1}{n!}(2j + n)(2j + n - 2) \cdots (2j + 1)$ and $p_{B2}(j) = \frac{1}{n!!}(2j + n)(2j + n - 2) \cdots (2j + 2)$. Analogously

$$m_S(j) = \begin{cases} p_{S1}(j)(2j)! & \text{for odd } n \\ p_{S2}(j)4^j(j)!^2 & \text{for even } n \end{cases}$$

with polynomials $p_{S1}(j) = \frac{1}{(n-2)!!}(2j + n - 2)(2j + n - 4) \cdots (2j + 1)$ and $p_{S2}(j) = \frac{1}{(n-2)!!}(2j + n - 2)(2j + n - 4) \cdots (2j + 2)$. Moreover by Example [1] and Proposition [1] we see that $\Gamma(1 + 2u)$ and $\Gamma^2(1 + u)$ are moment functions of order 2. Hence $m_B(u)$ and $m_S(u)$ are generalised moment functions of order 2 and Proposition [8] is an easy consequence of Theorem [1] and Proposition [7].
We may generalise the above results to the case when $P(\partial_z) = \sum_{i,j=1}^n a_{ij} \partial_{z_i z_j}^2$ is a homogeneous elliptic operator of order 2 with real coefficients. So, we may assume that $A = (a_{ij})_{i,j=1}^n$ is a symmetric nonsingular positive-definite real matrix. Then there exists an orthogonal real matrix $C$ such that $\Lambda = C^TAC$ is a diagonal matrix with positive entries on the main diagonal. Then

$$P(\partial_z)\varphi(z) = \Delta_w \hat{\varphi}(w) \quad \text{for} \quad w = \Lambda^{-1/2} C^T z \quad \text{and} \quad \hat{\varphi}(w) = \varphi(C \Lambda^{1/2} w).$$

Applying the above change of variables to Proposition \ref{theo:generalised_integral_means} we get

**Proposition 9.** Let us assume that

$$\mu_B(y) = \frac{dy}{|C\Lambda^{1/2}| \alpha(n)} \quad \text{and} \quad \mu_S(y) = \frac{dS(y)}{|C\Lambda^{1/2}| \alpha(n)},$$

where $|C\Lambda^{1/2}|$ is the determinant of the matrix $C\Lambda^{1/2}$, $\alpha(n) = \pi^{n/2} / \Gamma(1 + n/2)$ is the volume of the $n$-dimensional unit ball $B(0,1)$, $dy$ is the Lebesgue measure on $C\Lambda^{1/2}(B(0,1))$ and $dS(y)$ is the surface measure on $C\Lambda^{1/2}(S(0,1))$. Then the generalised solid $M_{\mu_B}(\varphi; z, t)$ and spherical $M_{\mu_S}(\varphi; z, t)$ means satisfy Pizzetti-type formulas

$$M_{\mu_B}(\varphi; z, t) = \int_{C\Lambda^{1/2}(B(0,1))} \varphi(z + ty) \, dy = \sum_{j=0}^\infty \frac{P(\partial_z)^j \varphi(z)}{m_B(j)} t^{2j}$$

and

$$M_{\mu_S}(\varphi; z, t) = \int_{C\Lambda^{1/2}(S(0,1))} \varphi(z + ty) \, dS(y) = \sum_{j=0}^\infty \frac{P(\partial_z)^j \varphi(z)}{m_S(j)} t^{2j},$$

where $m_B(j)$ and $m_S(j)$ are defined by \eqref{eq:coefficients}.

Hence, as in Proposition \ref{theo:generalised_integral_means} we conclude

**Theorem 2.** Let $d \in \mathbb{R}$, $P(\partial_z) = \sum_{i,j=1}^n a_{ij} \partial_{z_i z_j}^2$ be a homogeneous elliptic operator of order 2 with real coefficients and $\hat{u}$ be the formal solution of

$$(\partial_t - P(\partial_z)) u(t, z) = 0, \quad u(0, z) = \varphi(z) \in \mathcal{O}(D^n),$$

Moreover, let $M_{\mu_B}(\varphi; z, t)$ and $M_{\mu_S}(\varphi; z, t)$ be generalised integral means with measures defined by \eqref{eq:coefficients}. Then:

- $u \in \mathcal{O}(D^{n+1})$ if and only if $M_{\mu_B}(\varphi; z, t) \in \mathcal{O}^2(D^n \times \mathbb{C})$, and if and only if $M_{\mu_S}(\varphi; z, t) \in \mathcal{O}^2(D^n \times \mathbb{C})$,

- $\hat{u}$ is 1-summable in the direction $d$ if and only if $M_{\mu_B}(\varphi; z, t) \in \mathcal{O}^2(D^n \times (\tilde{S}_{d/2} \cup \tilde{S}_{d/2+\pi}))$, and if and only if $M_{\mu_S}(\varphi; z, t) \in \mathcal{O}^2(D^n \times (\tilde{S}_{d/2} \cup \tilde{S}_{d/2+\pi}))$.

7. **Higher order homogeneous operators**

In the previous section we have found the generalised integral means satisfying Pizzetti-type formulas \eqref{eq:coefficients} for first order homogeneous operators and second order elliptic homogeneous operators. In this section we will consider the homogeneous operators $P(\partial_z)$ of order $p > 2$. We will show
that for such operators it is impossible to find any generalised integral means satisfying \((8)\). To this end we use the following results:

**Lemma 1** (Zalcman theorem [11, Theorem 1]). Let \(\mu\) be a finite complex Borel measure of compact support on \(\mathbb{R}^n\),

\[
F(\zeta) = \int_{\mathbb{R}^n} e^{-i\zeta \cdot y} d\mu(y)
\]

be the Fourier-Laplace transform of \(\mu\) and \(\varphi \in \mathcal{O}(D^n)\). Then for each \(z \in D^n\) we have

\[
M_\mu(\varphi; z, t) = \int_{\mathbb{R}^n} \varphi(z + yt) d\mu(y) = F(it\partial_z)\varphi(z)
\]

for all \(t \in \mathbb{C}\) for which the integral exists and the right-hand side converges.

**Lemma 2** (Paley-Wiener-Schwartz theorem [3, Theorem 1.7.7]). \(F(z) \in \mathcal{O}(C^n)\) is a Fourier-Laplace transform of a distribution \(v\) with compact support \((\text{supp } v \subset B(0, r))\) if and only if there exist \(C > 0\) and \(N \in \mathbb{N}\) such that

\[
|F(\zeta)| \leq C(1 + |\zeta|)^N e^{r|\text{Im } (\zeta)|}.
\]

**Lemma 3** (Phragmén-Lindelöf theorem, [8]). Let \(d \in \mathbb{R}, \alpha < \pi\) and \(F \in \mathcal{O}^1(S_d(\alpha))\). We assume that \(|F(\zeta)| < M\) for \(\zeta \in \partial S_d(\alpha)\). Then \(|F(\zeta)| < M\) for \(\zeta \in S_d(\alpha)\).

No we are ready to state the main result of this section:

**Theorem 3.** If \(p > 2\), \(P(\partial_z)\) is a homogeneous operator of order \(p\) and \(m\) is a function of order \(p\) then

\[
\sum_{j=0}^{\infty} \frac{P_j(\partial_z)\varphi(z)}{m(j)} t^{pj}
\]

is not a generalised integral mean for any Borel measure \(\mu\) of compact support in \(\mathbb{R}^n\).

**Proof.** First observe that for every homogeneous operator \(P(\partial_z)\) of order \(p\)

\[
F(it\partial_z)\varphi(z) = \sum_{j=0}^{\infty} \frac{P_j(\partial_z)\varphi(z)}{m(j)} t^{pj}
\]

if and only if

\[
F(\zeta) = \sum_{j=0}^{\infty} \frac{P_j(-i\zeta)}{m(j)}.
\]

Hence by Lemmas 1 and 2 it is sufficient to show that the function \(F(\zeta)\) defined by \((14)\) does not satisfy the estimation \((13)\).

Contrary, let us suppose that \(F(\zeta)\) satisfies \((13)\). We introduce an auxiliary function

\[
f(t) := F(t, \ldots, t) \quad \text{for} \quad t \in \mathbb{C}.
\]
By (13), if \( \arg t = 0 \) then the function \( f(t) \) has a polynomial growth at infinity, i.e. there exist \( C > 0 \) and \( N \in \mathbb{N} \) such that

\[
|f(t)| \leq C(1 + |t|)^N
\]

for every \( t \in \mathbb{C} \) with \( \arg t = 0 \). Since \( P(\cdot) \) is a homogeneous polynomial of order \( p \), \( f(t) \) is invariant under the rotation of the angle \( \frac{2\pi}{p} \) (i.e. \( f(t) = f(te^{\frac{2\pi}{p}}) \) for every \( t \in \mathbb{C} \)). It means that \( f(t) \) satisfies (16) also for every \( t \in \mathbb{C} \) such that \( \arg t = \frac{2k\pi}{p} \) for some \( k \in \{0, \ldots, p-1\} \).

Next we fix the sector \( \hat{S}_k := S_{\frac{2k+1}{p}}(\frac{2\pi}{p}) \) and \( a_k \in \mathbb{C} \setminus \hat{S}_k \) for \( k = 0, \ldots, p-1 \). Then the function

\[
f_k(t) := \frac{f(t)}{(t-a_k)^N} \quad \text{for} \quad t \in \mathbb{C} \setminus \hat{S}_k
\]

satisfies the following conditions:

a) there exists \( M_k > 0 \) such that \( |f_k(t)| \leq M_k \) for every \( t \in \partial \hat{S}_k \),

b) \( f_k(t) \in O^1(\hat{S}_k) \).

Moreover, the opening of \( \hat{S}_k \) is equal to \( \frac{2\pi}{p} \) and \( \frac{2\pi}{p} < \pi \) for \( p > 2 \). It means that the assumptions of Lemma 3 are satisfied. Consequently, by Lemma 3

\[
|f_k(t)| \leq M_k \quad \text{for every} \quad t \in \hat{S}_k \quad \text{and} \quad k = 0, \ldots, p-1.
\]

Hence there exist \( \tilde{M}_k < \infty \) (\( k = 0, \ldots, p-1 \)) satisfying

\[
|f(t)| \leq \tilde{M}_k(1 + |t|)^N \quad \text{for every} \quad t \in \mathbb{C} \setminus \hat{S}_k \quad \text{and} \quad k = 0, \ldots, p-1.
\]

Taking \( M = \sup\{\tilde{M}_0, \ldots, \tilde{M}_{p-1}\} \) we conclude that

\[
|f(t)| \leq M(1 + |t|)^N \quad \text{for every} \quad t \in \mathbb{C}.
\]

On the other hand, if \( P(-i, \ldots, -i) = r e^{i\phi}, m(j) \leq AB^j(pj)! \) and

\[
t = e^{\frac{i\phi}{p}}R \quad \text{for} \quad R > 0,
\]

then

\[
f(t) = \sum_{j=0}^{\infty} e^{-i\phi j} \frac{P_j(-i, \ldots, -i) R^{pj}}{m(j)} = \sum_{j=0}^{\infty} \frac{(R \sqrt{r/B})^{pj}}{m(j)} \geq A \sum_{j=0}^{\infty} \frac{(R \sqrt{r/B})^{pj}}{(pj)!} \geq a e^{bR}
\]

for some \( a, b > 0 \). Hence \( f(t) \) is of exponential growth as \( R \to \infty \), contrary to (17). It means that \( F(\zeta) \) does not satisfy (13), which completes the proof. \( \square \)
8. Quasi-homogeneous operators of higher order

We extend the results of the previous section to the case of quasi-homogeneous operators using the ideas of Pokrovskii [10].

**Definition 11.** Let \( N = (N_1, \ldots, N_n) \in \mathbb{N}_0^n \) and \( p \in \mathbb{N}_0 \). A polynomial \( P(\zeta), \zeta \in \mathbb{C}^n \), (respectively an operator \( P(\partial_z) \)) is said to be quasi-homogeneous of type \( N \) and of order \( p \) if \( P(\zeta) = \sum a_k \zeta^k \neq 0 \), where the sum is taken over the set of all multi-indices \( k = (k_1, \ldots, k_n) \in \mathbb{N}_0^n \) such that \( |kN| = k_1N_1 + \cdots + k_nN_n = p \) and \( \zeta^k := \zeta_1^{k_1} \cdots \zeta_n^{k_n} \).

Observe that every homogeneous operator is quasi-homogeneous of type \((1, \ldots, 1)\). More interesting example is given by the heat operator \( P(\partial_z) = \partial_{z_1}^2 - \partial_{z_2}^2 - \cdots - \partial_{z_n}^2 \), which is a quasi-homogeneous operator of type \((2, 1, \ldots, 1)\) and of order 2.

For quasi-homogeneous operators it is convenient to consider the following \( N \)-version of generalised integral means

**Definition 12.** Let \( \mu \) be a finite complex Borel measure supported in the closed ball \( B(0, R) \) (for some \( R > 0 \)) in \( \mathbb{R}^n \) of total mass 1 (i.e. \( \int_{\mathbb{R}^n} d\mu(y) = 1 \)). Moreover we assume that \( \varphi \) is a continuous function on a domain \( \Omega \subset \mathbb{R}^n \), \( x \in \Omega \) and \( r > 0 \) such small that \( x + r^N y \in \Omega \) for very \( y \in B(0, R) \), where \( r^N y = r^{N_1} y_1 + \cdots + r^{N_n} y_n \). Then the \( N \)-generalised integral mean \( M_{\mu}^N(\varphi; x, r) \) of \( \varphi \) is defined by

\[
M_{\mu}^N(\varphi; x, r) := \int_{\mathbb{R}^n} \varphi(x + r^N y) d\mu(y).
\]

For the \( N \)-generalised integral means we have the following version of Lemma [10].

**Lemma 4 (see [10]).** Let \( \mu \) be a finite complex Borel measure of compact support on \( \mathbb{R}^n \),

\[
F(\zeta) = \int_{\mathbb{R}^n} e^{-i\zeta \cdot y} d\mu(y)
\]

be the Fourier-Laplace transform of \( \mu \) and \( \varphi \in \mathcal{O}(D^n) \). Then for each \( z \in D^n \) we have

\[
M_{\mu}^N(\varphi; z, t) = \int_{\mathbb{R}^n} \varphi(z + t^N y) d\mu(y) = F(it^N \partial_z) \varphi(z)
\]

for all \( t \in \mathbb{C} \) for which the integral exists and the right-hand side converges.

So, we can extend Theorem [8] to the quasi-homogeneous operators as follows

**Theorem 4.** If \( p > 2 \), \( P(\partial_z) \) is a quasi-homogeneous operator of type \( N \) and of order \( p \) and \( m \) is a function of order \( p \) then

\[
\sum_{j=0}^{\infty} \frac{P^j(\partial_z) \varphi(z)}{m(j)} t^{pj}
\]

is not an \( N \)-generalised integral mean for any Borel measure \( \mu \) of compact support in \( \mathbb{R}^n \).
Proof. First, similarly to the proof of Theorem 3, we observe that for quasi-homogeneous operators $P(\partial_z)$ of type $N$ and of order $p$ 

$$F(it^N\partial_z)\varphi(z) = \sum_{j=0}^{\infty} \frac{P^j(\partial_z)\varphi(z)}{m(j)} t^{pj}$$

if and only if 

$$(18) \quad F(\zeta) = \sum_{j=0}^{\infty} \frac{P^j(-i\zeta)}{m(j)}$$

so by Lemmas 2 and 4 it is sufficient to show that $F(\zeta)$ given by (18) does not satisfy (13). To this end we repeat the proof of Theorem 3 replacing (15) by 

$$f(t) := F(t^N) = F(t^{N_1}, \ldots, t^{N_n}) \quad \text{for} \quad t \in \mathbb{C}.$$ 

9. Complex generalised integral means

In this section we extend the definition of generalised integral means to the complex case. Namely we have

Definition 13. Let $\mu$ be a finite complex Borel measure supported in the closed disc $\overline{D}_R$ (for some $R > 0$) in $\mathbb{C}^n$ of total mass 1 (i.e. $\int_{\mathbb{C}^n} d\mu(y) = 1$). Moreover we assume that $\varphi$ is a continuous function on a domain $G \subset \mathbb{C}^n$, $z \in \Omega$ and $0 < r < \text{dist}(z, \partial\Omega)/R$. Then the complex generalised integral mean $M_\mu(\varphi; z, r)$ of $\varphi$ is defined by 

$$M_\mu(\varphi; z, r) := \int_{\mathbb{C}^n} \varphi(z + ry) \, d\mu(y).$$

Using complex generalised integral means we are able to extend the results about homogeneous operators of first and second order.

In particular Proposition 6 holds in the case when $P(\partial_z)$ is a general homogeneous operator of order one with complex variables (i.e. $P(\partial_z) = \sum_{j=1}^{n} a_j \partial_{z_j}$, for some $a = (a_1, \ldots, a_n) \in \mathbb{C}^n$).

Analogously, Proposition 9 and Theorem 2 are valid for general homogeneous operators $P(\partial_z)$ of order two with complex coefficients (i.e. $P(\partial_z) = \sum_{i,j=1}^{n} a_{ij} \partial^2_{z_i z_j}$, where $A = (a_{ij})_{i,j=1}^{n}$ is a symmetric nonsingular complex matrix). In this case a diagonal matrix $\Lambda = C^T A C$ is not necessary positive and consequently the measures $\mu_B$ and $\mu_S$ defined by (12) are supported respectively in the complex sets $C A^{1/2}(B(0, 1))$ and $C A^{1/2}(S(0, 1))$.

As an example we take the $n$-dimensional complex wave operator 

$$P(\partial_z) = \partial^2_{z_1} - \sum_{k=2}^{n} \partial^2_{z_k} = \partial^2_{z_1} - \Delta_{z'} = \Box_z,$$

where $z = (z_1, z_2, \ldots, z_n) = (z_1, z') \in \mathbb{C}^n$. Next we assume that $\mu_{B_1}(y) = \frac{dy}{\alpha(n)}$ and $\mu_{S_1}(y) = \frac{dS(y)}{\alpha(n)}$, where $dy$ is the Lebesgue measure on $B_1(0, 1) = \{(x_1+iy_1, \ldots, x_n+iy_n) \in \mathbb{C}^n : x_1 = x_2 = \cdots = x_n = 0, \ y_1^2+y_2^2+\cdots+y_n^2 \leq 1\}$.
and $dS(y)$ is the surface measure on 

$$S_1(0,1) = \{(x_1+iy_1, \ldots, x_n+iy_n) \in \mathbb{C}^n : y_1 = x_2 = \cdots = x_n = 0, x_1^2+y_2^2+\cdots+y_n^2 = 1\}.$$ 

Then the generalised solid $M_{\mu_{B_1}}(\varphi; z, t)$ and spherical $M_{\mu_{S_1}}(\varphi; z, t)$ means defined by

$$M_{\mu_{B_1}}(\varphi; z, t) = \int_{B_1(0,1)} \varphi(z + ty) \, dy \quad \text{and} \quad M_{\mu_{S_1}}(\varphi; z, t) = \int_{S_1(0,1)} \varphi(z + ty) \, dS(y)$$

satisfy Pizzetti-type formulas for the complex wave operator $\Box_z$

$$M_{\mu_{B_1}}(\varphi; z, t) = \sum_{j=0}^{\infty} \frac{\Box_j^2 \varphi(z)}{m_B(j)} t^{2j} \quad \text{and} \quad M_{\mu_{S_1}}(\varphi; z, t) = \sum_{j=0}^{\infty} \frac{\Box_j \varphi(z)}{m_S(j)} t^{2j},$$

where $m_B(j)$ and $m_S(j)$ are defined by (11). 

So by Theorem 1 we get

**Corollary 2.** Let $d \in \mathbb{R}$ and $\hat{u} = \hat{u}(t, z) \in \mathcal{O}[[t]]$ be the formal solution of the initial value problem

$$(\partial_t - \Box_z)u = 0, \quad u(0, z) = \varphi(z) \in \mathcal{O}(D^n).$$

Then we conclude that:

- $u \in \mathcal{O}(D^{n+1})$ if and only if $M_{\mu_{B_1}}(\varphi; z, t) \in \mathcal{O}^2(D^n \times \mathbb{C})$, and if and only if $M_{\mu_{S_1}}(\varphi; z, t) \in \mathcal{O}^2(D^n \times \mathbb{C})$.
- $\hat{u}$ is 1-summable in the direction $d$ if and only if $M_{\mu_{B_1}}(\varphi; z, t) \in \mathcal{O}^2(D^n \times (\hat{S}_{d/2} \cup \hat{S}_{d/2+\pi}))$, and if and only if $M_{\mu_{S_1}}(\varphi; z, t) \in \mathcal{O}^2(D^n \times (\hat{S}_{d/2} \cup \hat{S}_{d/2+\pi}))$.

Using complex generalised integral means we also obtain

**Theorem 5.** Let $p, s \in \mathbb{N}$. We assume that a complex generalised integral mean satisfies a Pizzetti-type formula

$$M_{\mu_B}(\varphi; z, t) = \int_B \varphi(z + ty) \, d\mu_B(y) = \sum_{j=0}^{\infty} \frac{Q^j(\partial_z)\varphi(z)}{m(j)} t^{pj}$$

for some measure $\mu_B$ supported in $B \subset \mathbb{C}^n$, for some homogeneous operator $Q(\partial_z)$ of order $p$ and for some function $m$ of order $p$. Moreover, we assume that the measure $\mu_B$ is defined by

$$\mu_B^s(y) := \frac{1}{s} \sum_{k=0}^{s-1} \mu_B(e^{\frac{2\pi i k}{p}} y).$$

Then the complex generalised integral mean $M_{\mu_B^s}(\varphi; z, t)$ satisfies a Pizzetti-type formula for the operator $Q^s(\partial_z)$

$$M_{\mu_B^s}(\varphi; z, t) = \frac{1}{s} \sum_{k=0}^{s-1} \int_{e^{\frac{2\pi i k}{p}} B} \varphi(z + ty) \, d\mu_B(y) = \sum_{j=0}^{\infty} \frac{Q^s j(\partial_z)\varphi(z)}{m(s)} t^{pj}.$$
Moreover, if \( \hat{u} = \hat{u}(t, z) \in \mathcal{O}[[t]] \) is the formal solution of the initial value problem
\[
(\partial_t - Q^*(\partial_z))u = 0, \quad u(0, z) = \varphi(z) \in \mathcal{O}(D^n),
\]
then we conclude that \( u \in \mathcal{O}(D^{n+1}) \) if and only if \( M_{\mu_B^*}(\varphi; z, t) \in \mathcal{O}_{\frac{2m}{ps}}(D^n \times \mathbb{C}) \).

If we additionally assume that \( d \in \mathbb{R} \) and \( m \) is a generalised moment function of order \( p \) then \( \hat{u} \) is \((ps - 1)^{-1}\)-summable in the direction \( d \) if and only if \( M_{\mu_B^*}(\varphi; z, t) \in \mathcal{O}_{\frac{2m}{ps}}(D^n \times \hat{S}_{\frac{4+2k}{ps}}) \) for \( k = 0, \ldots, ps - 1 \).

**Proof.** We have
\[
M_{\mu_B^*}(\varphi; z, t) = \frac{1}{s} \sum_{k=0}^{s-1} \sum_{j=0}^{\infty} \frac{Q^j(\partial_z)\varphi(z)}{m(j)} (e^{\frac{2\pi i}{s} p^j})^j = \sum_{j=0}^{\infty} \frac{Q^j(\partial_z)\varphi(z)}{m(s_j)} p^{sj},
\]
which proves the first part of the theorem. The proof is completed by applying Theorem [1] and Remark [1].

In particular, we obtain

**Corollary 3.** We assume that \( s \in \mathbb{N}, P(\partial_z) = \Delta^s_z = (\sum_{k=1}^{n} \partial_{z_k}^2)^s \) is the \( s \)-Laplace operator,
\[
\mu_B^s(y) = \frac{1}{s} \sum_{k=0}^{s-1} \mu_B(e^{\frac{2\pi i}{s} p^j} y) \quad \text{and} \quad \mu_S^s(y) = \frac{1}{s} \sum_{k=0}^{s-1} \mu_S(e^{\frac{2\pi i}{s} p^j} y),
\]
where the measures \( \mu_B(y) \) and \( \mu_S(y) \) are defined by \([7]\).

Then the generalised solid \( M_{\mu_B^s}(\varphi; z, t) \) and spherical \( M_{\mu_S^s}(\varphi; z, t) \) means satisfy Pizzetti-type formulas for the operator \( \Delta^s_z \)
\[
M_{\mu_B^s}(\varphi; z, t) = \int_{\bigcup_{k=0}^{s-1} e^{\frac{2\pi i}{s} p^j} B(0,1)} \varphi(z + ty) \, dy = \sum_{j=0}^{\infty} \frac{\Delta^s_z \varphi(z)}{m_B(s_j)} t^{2sj},
\]
and
\[
M_{\mu_S^s}(\varphi; z, t) = \int_{\bigcup_{k=0}^{s-1} e^{\frac{2\pi i}{s} p^j} S(0,1)} \varphi(z + ty) \, dS(y) = \sum_{j=0}^{\infty} \frac{\Delta^s_z \varphi(z)}{m_S(s_j)} t^{2sj},
\]
where \( m_B \) and \( m_S \) are given by \([11]\).

Moreover, if we additionally assume that \( d \in \mathbb{R} \) and \( \hat{u} = \hat{u}(t, z) \in \mathcal{O}[[t]] \) is the formal solution of the initial value problem
\[
(\partial_t - \Delta^s_z)u = 0, \quad u(0, z) = \varphi(z) \in \mathcal{O}(D^n),
\]
then we conclude that
- \( u \in \mathcal{O}(D^{n+1}) \) if and only if \( M_{\mu_B^s}(\varphi; z, t) \in \mathcal{O}_{\frac{2m}{ps}}(D^n \times \mathbb{C}) \), and if and only if \( M_{\mu_S^s}(\varphi; z, t) \in \mathcal{O}_{\frac{2m}{ps}}(D^n \times \mathbb{C}) \),
- \( \hat{u} \) is \((2s - 1)^{-1}\)-summable in the direction \( d \) if and only if \( M_{\mu_B^s}(\varphi; z, t) \in \mathcal{O}_{\frac{2m}{ps}}(D^n \times \hat{S}_{\frac{4+2k}{ps}}) \) for \( k = 0, \ldots, 2s-1 \), and if and only if \( M_{\mu_S^s}(\varphi; z, t) \in \mathcal{O}_{\frac{2m}{ps}}(D^n \times \hat{S}_{\frac{4+2k}{ps}}) \) for \( k = 0, \ldots, 2s - 1 \).
Corollary 5. We assume that satisfy Pizzetti-type formulas for diagonal matrix \( \Lambda \) where the measures \( B \) and \( S \) are defined by (12).

Then the complex generalised integral mean \( M_{\mu_B}(\varphi; z, t) \) satisfies a Pizzetti-type formula for \( P(\partial_z) \)

\[
M_{\mu_B}(\varphi; z, t) = \frac{1}{s} \sum_{k=0}^{s-1} \varphi(z + e^{\frac{2\pi i}{s}k} at) = \sum_{j=0}^{\infty} \frac{P_j(\partial_z)\varphi(z)}{(sj)!} t^j.
\]

If we additionally assume that \( s > 1 \), \( d \in \mathbb{R} \) and \( \hat{u} = \hat{u}(t, z) \in \mathcal{O}[[t]] \) is the formal solution of the initial value problem

\[
(\partial_t - P(\partial_z))u = 0, \quad u(0, z) = \varphi(z) \in \mathcal{O}(D^n),
\]

then we conclude that

- \( u \in \mathcal{O}(D^{n+1}) \) if and only if \( M_{\mu_B}(\varphi; z, t) \in \mathcal{O}^{\frac{2\pi i}{s}}(D^n \times \mathbb{C}) \),
- \( \hat{u} \) is \((s-1)^{-1}\)-summable in the direction \( d \) if and only if \( M_{\mu_B}(\varphi; z, t) \in \mathcal{O}^{\frac{2\pi i}{s}}(D^n \times S_{\frac{2\pi i}{s}}) \) for \( k = 0, \ldots, s-1 \).

Corollary 5. We assume that \( s \in \mathbb{N} \), \( P(\partial_z) = (\sum_{i,j=1}^{n} a_{ij} \partial_{z_i} \partial_{z_j})^s \), where \( A = (a_{ij})_{i,j=1}^{n} \) is a symmetric nonsingular complex matrix with the corresponding diagonal matrix \( \Lambda \) (i.e., \( \Lambda = C^T A C \) for some orthogonal complex matrix \( C \)). We also assume that

\[
\mu_B^s(y) = \frac{1}{s} \sum_{k=0}^{s-1} \mu_B(e^{\frac{2\pi i}{s}k} y) \quad \text{and} \quad \mu_S^s(y) = \frac{1}{s} \sum_{k=0}^{s-1} \mu_S(e^{\frac{2\pi i}{s}k} y),
\]

where the measures \( \mu_B \) and \( \mu_S \) are defined by (12).

Then the complex generalised integral means \( M_{\mu_B^s}(\varphi; z, t) \) and \( M_{\mu_S^s}(\varphi; z, t) \) satisfy Pizzetti-type formulas for \( P(\partial_z) \)

\[
M_{\mu_B^s}(\varphi; z, t) = \int_{\mathbb{C}^n} \varphi(z + ty) \, d\mu_B^s(y) = \sum_{j=0}^{\infty} \frac{P_j(\partial_z)\varphi(z)}{m_B(sj)} t^{2sj}
\]

and

\[
M_{\mu_S^s}(\varphi; z, t) = \int_{\mathbb{C}^n} \varphi(z + ty) \, d\mu_S^s(y) = \sum_{j=0}^{\infty} \frac{P_j(\partial_z)\varphi(z)}{m_S(sj)} t^{2sj},
\]

where the sequences of numbers \( m_B \) and \( m_S \) are given by (17).

If we additionally assume that \( d \in \mathbb{R} \) and \( \hat{u} = \hat{u}(t, z) \in \mathcal{O}[[t]] \) is the formal solution of the initial value problem

\[
(\partial_t - P(\partial_z))u = 0, \quad u(0, z) = \varphi(z) \in \mathcal{O}(D^n),
\]

then we conclude that

- \( u \in \mathcal{O}(D^{n+1}) \) if and only if \( M_{\mu_B^s}(\varphi; z, t) \in \mathcal{O}^{\frac{2\pi i}{s}}(D^n \times \mathbb{C}) \), and if
- \( u \in \mathcal{O}(D^{n+1}) \) if and only if \( M_{\mu_S^s}(\varphi; z, t) \in \mathcal{O}^{\frac{2\pi i}{s}}(D^n \times \mathbb{C}) \),
\[ \hat{u} \text{ is } (2s-1)^{-1}\text{-summable in the direction } d \text{ if and only if } M_{\mu_d}(\varphi; z, t) \in O_{2s-1}(D^n \times \hat{S}_{2s+2k\pi}) \text{ for } k = 0, \ldots, 2s-1, \text{ and if and only if } M_{\mu_d}(\varphi; z, t) \in O_{2s-1}(D^n \times \hat{S}_{2s+2k\pi}) \text{ for } k = 0, \ldots, 2s-1. \]

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