Phase–locking of a Nonlinear Optical Cavity via Rocking: 
Transmuting Vortices into Phase Patterns

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Abstract

We report experimental observation of the conversion of a phase–invariant nonlinear system into a phase–locked one via the mechanism of rocking [de Valcárcel and Staliunas, Phys. Rev. E 67, 026604 (2003)]. This conversion results in that vortices of the phase–invariant system are being replaced by phase patterns such as domain walls. The experiment is carried out on a photorefractive oscillator in a two-wave mixing configuration. A model for the experimental device is given that reproduces the observed behavior.

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Introduction.—The symmetry properties of the order parameter phase in extended nonlinear systems determine the type of localized structures supported by these systems. On the one hand phase-invariant systems display vortices, which are phase defects of the order parameter, around which the phase changes in $2\pi$, forcing the order parameter to be null at the vortex center. Self-oscillatory systems represent a paradigm of such behavior as the oscillating state reached after a homogeneous Hopf bifurcation can have any phase (the system is autonomous). Examples of such systems are certain chemical reaction (like the Belousov-Zhabotinsky reaction) and several optical systems (like lasers, nondegenerate optical parametric oscillators, OPOs, and nondegenerate wave mixing photorefractive oscillators). On the other hand there are nonlinear systems with broken phase invariance that, obviously, cannot support vortices. Among them, especially interesting systems are those displaying phase bistability. Such systems support a different type of localized structure, the domain wall, which connects spatial regions where the order parameter passes from one homogeneous state, $F_0$, to the equivalent, symmetric state, $-F_0$, so that the phase changes by $\pi$ from one side of the wall to the other one. A paradigm of such type of phase-bistable system is the degenerate OPO.

The question we address here is: Is it possible to convert a phase-invariant system into a phase-bistable one by some simple external action? In other words, can a system displaying vortices be forced to display domain walls? An old and well known answer to these questions consists in the periodic forcing of a self-oscillatory system at a frequency around two times its natural oscillation frequency $\omega_0$ (resonance 2 : 1). Under that type of (so called) parametric driving the originally phase-invariant system is predicted and experimentally observed to transform into a phase-bistable system, exhibiting domain walls. But while parametric driving is useful in many contexts it is not so in general in nonlinear optics. For instance, a laser emitting at frequency $\omega_0$ is insensitive to a forcing at $2\omega_0$ because the gain line is extremely narrow as compared with the magnitude of the optical frequency $\omega_0$. Yet, a new mechanism coined rocking has been recently proposed to overcome such limitation of the usual parametric driving. Rocking consists in forcing a self-oscillatory system around its natural oscillation frequency (resonance 1 : 1) so that the forcing amplitude is periodically modulated in time at a frequency $\omega \ll \omega_0$. Thus rocking is a multifrequency forcing (in its simplest case, a sinusoidal modulation of the forcing amplitude, it is a bichromatic forcing) around the resonance 1 : 1 of a self-oscillatory system. We want to remark that unlike the
parametric driving, any self-oscillatory system (including lasers and any nonlinear optical system) should be sensitive to rocking as this forcing acts on the main resonance of the system.

In order to gain insight into the rocking idea, consider a particle in a 2D space with coordinates \((x, y)\) under the action of a potential with the shape of a sombrero having a maximum at the origin \((0, 0)\) and a degenerated minimum at \(x^2 + y^2 = r_{\text{min}}^2\). This is a phase-invariant system, what means that the equilibrium position is phase(a ngle)-degenerated. Suppose now that this potential is rocked along the \(x\)-direction (e.g., by adding a term \(x \sin \omega t\) to the original potential). In this case the phase degeneracy is obviously broken and now the system tends to be around \((x = 0, y = \pm r_{\text{min}})\): The system is now phase-bistable. This pictorial image illustrates the physical rationale behind the rocking idea: The coordinates of the fictitious particle correspond to the real and imaginary parts of the complex amplitude of oscillations of the self-oscillatory system, whose evolution derives from a sombrero-like potential in the simplest case, as originally introduced in [5] through an analysis of a complex Ginzburg-Landau equation, which is the simplest model for spatially extended self-oscillatory systems [1]. As shown in [3], under the action of rocking the order parameter of the system \(F(r, t)\) (e.g., the laser electric field complex amplitude) develops two components, \(F(r, t) = F_\omega(t) + i\psi(r, t)\), where \(F_\omega\) is a \(\frac{2\pi}{\omega}\)-periodic function of time that follows the modulation, at frequency \(\omega\), of the forcing amplitude and \(\psi\) is a phase-bistable, reduced order parameter that can display domain walls, phase domains, and phase domain solitons [3]. Even if the original prediction is based on a complex Ginzburg-Landau model, rocking is expected to be of wide applicability as the phase-bistability mechanism introduced above seems to be quite model independent.

In this Letter we give the first experimental evidence of rocking induced phase-bistability and the associated formation of phase domains and domain walls. Our system is a (nondegenerate) two-wave mixing photorefractive oscillator (PRO), which highly resembles a laser from the nonlinear dynamics viewpoint. Rocking is done by injecting an amplitude modulated laser beam into the resonator. We show that in the absence of rocking (free running configuration) the PRO exhibits vortices, which are converted into phase domains under the action of rocking. Experiments performed under a quasi 1D transverse geometry (in order to avoid curvature effects) show that the system displays domain walls in a window of rocking modulation frequencies, in agreement with [5]. A model for the experimental de-
vice is presented that reproduces the experimental findings. In a limit, such model reduces to a parametrically driven complex Swift-Hohenberg equation, which represents a generalization of \[5\] to other types of order parameter equations and hence enlarges the range of applicability of rocking.

**Experimental setup.** The PRO, Fig. 1, consists of a photorefractive BaTiO$_3$ crystal placed inside a (near) self-imaging ring resonator \[6\] of cavity length 1.2m (the resonator free-spectral range is 250MHz), similar to that used in \[7\]. The effective cavity length is approximately 2cm, which is actively stabilized by means of piezomirror PZT1 in order to have a precise control on cavity detuning (the difference between the frequencies of the pump field and that of the closest cavity mode). The crystal is pumped by a singlemode 514nm Ar$^+$ laser with a power around 100mW cm$^{-2}$. An amplitude modulated beam (the rocking beam) coherent to the pump field is injected into the cavity. The c+ crystal axis is oriented so that gain is maximized when all three beams (pump, oscillation and injection) are extraordinarily polarized. The intracavity slit D is placed in a Fourier plane in order to make the system quasi 1D in the transverse dimension \[8\]. Finally, there is the possibility of injecting a tilted coherent beam in order to ”write” domain walls as in \[9, 10\].

The rocking beam must be an amplitude modulated field of zero mean: We chose to use a field of constant amplitude, whose phase changes exactly by $\pi$ in every half period. We do this by injecting into the cavity a beam coming from the pumping laser, after being reflected on piezomirror PZT2, Fig. 1, which is moved periodically back and forth by half a wavelength (the modulation frequency is on the order of 1Hz). This way the rocking has a pure amplitude modulation (only the sign of the field amplitude changes). This operation must be very precise as phase jumps different to $\pi$ do not produce the desired result.

**Experimental results.** The cavity length is chosen for the system be in (almost) exact resonance. This is not mandatory but is the simplest way for the rocking field be resonant with the intracavity field (see below the model section). When the rocking beam is off and the intracavity slit is open, the system spontaneously forms vortices in the output field, as expected \[11\]. Under these conditions, the application of a rocking beam is able to transform vortices into phase domains, as shown in Fig. 2. This result is a direct demonstration of the rocking induced phase-bistability.

Two dimensional phase domains are transient structures due to curvature \[5\]. Hence we performed a series of experiments under quasi 1D conditions (by narrowing the intracavity
slit), which allow stable domain walls in phase bistable systems \cite{9}. These domain walls are however unstable in 1D phase invariant systems: If a domain wall is injected into the free running PRO, Fig. 3(a), we observe that the domain wall vanishes with time, Fig. 3(b), and is replaced by a spatially uniform state, Fig. 3(c). On the contrary, if the PRO is rocked completely different results are obtained. Now an injected domain wall, Fig. 3(e), remains stable and fixed, Figs. 3(f) and (g). (The amplitude modulation frequency of the rocking beam was 1.5Hz and its intensity was of the same order as that of the output field in the absence of rocking.) Figure 3(h) evidences that we are in the presence of an Ising wall \cite{2,9}. The structure is robust and this proofs that the system is now phase bistable. The ability of rocking to sustain domain walls is thus proven.

There remains to clarify which is the effect on the performance of the system of changing the modulation frequency and amplitude of the rocking beam. We have observed that the frequency of modulation cannot be too small or too large: For modulation frequencies of 0.1Hz or below rocking cannot sustain the injected domain wall. The same happens for modulation frequencies greater than 10Hz. It seems that modulations from 1 to 3Hz are optimal, which are on the order of the inverse of the photorefractive grating decay time. While we do not have definite measurements, we can say that rocking is effective in a window of rocking intensities as well.

All previous features are in agreement with the predictions of Ref. \cite{5}. The theory is however based on a Ginzburg-Landau model, which should not be valid near resonance \cite{16,17,18}, the case considered here. In order to justify theoretically the results on a more firm basis we thus proceed to model the PRO.

**Theory.**– We adopt the two-wave mixing PRO model for a purely diffusive photorefractive crystal \cite{12}, such as BaTiO$_3$, generalized to account for the injection of the rocking beam (details will be given elsewhere). The model equations, suitably normalized, can be written as:

\[ \sigma^{-1} \partial_t F = -(1 + i\Delta) F + i\alpha \nabla^2 F + N + F_R(t), \]  

\[ \partial_t N = -N + g \frac{F}{1 + |F|^2}, \]

where \( F(\mathbf{r},t) \) is the slowly varying envelope of the intracavity field, \( N(\mathbf{r},t) \) is the complex amplitude of the photorefractive nonlinear grating, \( \mathbf{r} = (x,y) \) are the transverse coordinates,
\[ \nabla^2 = \partial_x^2 + \partial_y^2, \quad \sigma = \kappa \tau \] is the product of the cavity linewidth \( \kappa \) with the photorefractive response time \( \tau \) \((\sigma \gtrsim 10^8 \text{ under typical conditions, and } \tau \sim 1\text{s})\), \( t \) is time measured in units of \( \tau \), the detuning \( \Delta = (\omega_C - \omega_P) / \kappa \) (\( \omega_P \) and \( \omega_C \) are the frequencies of the pump and of its nearest cavity longitudinal mode, respectively), \( a \) is the diffraction coefficient (that depends upon geometry \([8]\) and can take either sign), \( F_R \) is the complex envelope of the rocking (injected) field, and \( g \) is the (real) gain parameter that depends on crystal parameters and on the geometry of the interaction. The actual intracavity field, \( \mathcal{E} \), and rocking field, \( \mathcal{E}_R \), read \( \mathcal{E} = \text{Re} F E_P e^{-i\omega_P \tau t} \) and \( \mathcal{E}_R = \text{Re} F_R E_P e^{-i\omega_P \tau t} \), respectively, where \( E_P \) and \( \omega_P \) are the complex amplitude and the angular frequency of the pumping laser field. Note that \( E_P \) acts just as a scaling factor (it is unrelated with the gain parameter \( g \)) \([12, 13]\).

For \( F_R = 0 \) the model is equivalent to that in \([12]\), which holds the continuous symmetry \((F, N) \to (F e^{i\phi}, N e^{i\phi})\). Hence the system is phase invariant in the absence of forcing. This free running PRO model has two main solutions \([12]\): The trivial solution \( F = N = 0 \), and the family of traveling-wave solutions (parametrized by the wavevector \( k \)) \( F = \sqrt{g/g_0 - 1} e^{i(k \cdot r - \Omega t)} \) and \( N = (1 + i\Omega) F \), with \( g_0 = 1 + \Omega^2 \), and \( \Omega = \frac{\omega}{\sigma \tau} (\Delta + ak^2) \to \Delta + ak^2 \) as \( \sigma \gg 1 \). One easily sees that: (i) for \( \Delta/a > 0 \) the oscillation threshold is minimum for \( k = 0 \) (on–axis emission), occurs at \( g = 1 + \Delta^2 \), and the frequency of the generated field is shifted by \( \Omega = \Delta \) from that of the pump beam; and (ii) for \( \Delta/a < 0 \) the threshold is minimum for \( k = \sqrt{-\Delta/a} \) (off–axis emission), occurs at \( g = 1 \), and there is no frequency shift \((\Omega = 0)\). We checked these features in our experiment. In particular cavity resonance was determined by interfering the emitted field with a reference coming from the pumping laser: The cavity length at which the beating frequency \((\Omega)\) passes from zero to a non null value (or vice versa) corresponds to exact resonance \([12]\). These facts, together with the phase invariance of the model, make the PRO in the two-wave mixing configuration a system largely equivalent to a large aspect ratio laser \([12,14]\).

In the following we use \( F_R = R \cos \omega t \) \((R \text{ real without loss of generality})\), which is the simplest form of amplitude modulation and corresponds to a bichromatic injected signal. In order to give analytical evidence of the rocking induced phase-bistability in the PRO model we perform an asymptotic expansion of Eqs. \([12]\) based on the of multiple scales technique \([15]\). In order to approach the experimental conditions we assume small detuning, and large cavity linewidth: We take \( \Delta = O(\varepsilon) \), \( \sigma = O(\varepsilon^{-4}) \) with \( \varepsilon \) a smallness parameter \((0 < \varepsilon \ll 1)\) (the final result does not depend on the precise scaling for \( \sigma \), whenever \( \sigma \gg 1 \) is assumed).
We further assume that gain is close to threshold, \( g = 1 + O(\varepsilon^2) \), and that the rocking parameters verify \( R = O(\varepsilon) \), \( \omega = O(1) \). The analysis, similar to that performed in laser [16] and nondegenerate OPO [17, 18] models, yields

\[
N = \left[ 1 + i(\Delta - a\nabla^2) \right] F - R \cos \omega t,
\]

\[
F(\mathbf{r}, t) = F_\omega(t) + i\psi(\mathbf{r}, t) + O(\varepsilon^3),
\]

\[
F_\omega(t) = \frac{R}{\omega} \left[ (1 - 2i\Delta) \sin \omega t + \left(1 - i\Delta \frac{\omega^2 - 1}{\omega^2} \right) \omega \cos \omega t \right],
\]

the order parameter \( \psi \) verifying a complex Swift-Hohenberg equation with parametric gain:

\[
\partial_t \psi = \gamma \psi^* + (g - 1 - 2\gamma) \psi - |\psi|^2 \psi
+ i \left( a\nabla^2 - \Delta \right) \psi - \left( a\nabla^2 - \Delta \right)^2 \psi,
\]

where \( \gamma = \frac{R^2 + \omega^2}{\omega^2} \) is the rocking parameter (note that \( \gamma \geq \frac{R^2}{2} \)). In the absence of rocking \( (\gamma = 0) \), Eq. (4) is phase invariant and is isomorphic to those describing lasers [16] and nondegenerate OPOs [17, 18], as well as highly resembles that for drift-type PROs [14]. The role of rocking is clearly appreciated in Eq. (4): It introduces a phase-sensitive gain (first term in the r.h.s.) that breaks the original phase invariance of the undriven system down to the discrete one \( \psi \rightarrow -\psi \). Thus the PRO becomes phase bistable. The spatially uniform steady states of Eq. (4) are given by \( \psi = \pm |\psi| e^{i\phi} \), \( |\psi|^2 = \mu - 2\gamma + \sqrt{\gamma^2 - \Delta^2} \), \( \mu = g - 1 - \Delta^2 \), \( e^{2i\phi} = \frac{\sqrt{\gamma^2 - \Delta^2 - i\Delta}}{\gamma} \) (other two, intrinsically unstable states exist as well, which we do not consider). These phase-locked states exist if \( |\Delta| \leq \gamma \leq \gamma_{\text{max}} \) \( (\gamma_{\text{max}} = \frac{2\mu}{3} + \frac{1}{3}\sqrt{\mu^2 - 3\Delta^2}) \) as far as \( \mu \geq \sqrt{3} |\Delta| \). These inequalities imply analogous ones for the rocking intensity, \( R^2 \), or the rocking modulation frequency, \( \omega \), depending on which parameter \( (\omega \text{ or } R) \) is kept fixed. This prediction supports the experimental findings described above. Finally note that as \( \gamma \geq \frac{R^2}{2} \) the rocked states will exist only if \( R^2 \leq 2\gamma_{\text{max}} \), and \( \mu \geq \sqrt{3} |\Delta| \), for any \( \omega \).

In a series of numerical experiments we have checked the above predictions and have found that they remain valid even far from the asymptotic limit described by Eq. (4), as it happens in the original proposal [5]. In order to have additional comparison between theory and experiment, we present in Fig. 4 some numerical simulations of Eqs. (1,2) for \( \sigma = 10^2 \) (large, but not extremely large in order to avoid stiffness problems), \( \Delta = 0 \), \( g = 2 \) (we estimate that the gain in the experiment is about 100% above threshold), \( \omega = 2\pi \) and \( R = 0.5 \). Comparison with Fig. 3 shows that results are very similar to the experimental ones. A message from this theoretical treatment arises: Phase invariant systems described by order parameter equations of different nature (like the Ginzburg-Landau and the Swift-Hohenberg complex models) behave similarly under the influence of rocking.
In conclusion, we have experimentally demonstrated, and theoretically justified, the ability of the rocking mechanism introduced in [5] to generate phase-bistable states in otherwise phase invariant systems. The studied system, a PRO in two-wave mixing configuration, highly resembles laser systems from the nonlinear dynamics viewpoint. Hence the results put forward in this Letter should motivate similar experiments in laser systems, which could have potential applications in the field of information technologies.

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Figure captions

**Fig. 1.** Scheme of the experimental setup. A photorefractive BaTiO$_3$ crystal is placed inside the quasi self-imaging resonator formed by mirrors M1, M2 and M3, piezomirror PZT1 and the four lenses of focal length $f$. D is a diaphragm, and PZT2 is a piezomirror driven by a square-wave signal, plot $V$ vs. time, that produces the rocking beam.

**Fig. 2.** Interferometric snapshots of vortices (a) existing in the free running PRO and phase domain structure (b) generated in the rocked PRO at resonance. Magnifications of the interferometric images in the position of (c) a vortex showing the annihilation of the interference fringes and (d) of a part of the domain wall showing the $\pi$ phase jump.

**Fig. 3.** Experimental snapshots of injected domain walls (DW) close to cavity resonance. (a)-(c): Disappearance of a DW in the free running PRO (the time interval between snapshots is 5s). (e)-(g): Stabilization of the DW in the rocked PRO (the rocking frequency is 1.5Hz and the time interval between snapshots is 15s). (d) and (h): Horizontal cuts showing the field amplitude (a.u.) and phase corresponding to snapshots (b) and (f), respectively. The snapshots transverse dimension is 1.2mm.

**Fig. 4.** As Fig. 3 but from numerical simulations of Eqs. (1), (2). Parameters: $\sigma = 10^2$, $\Delta = 0$, $g = 2$, $\omega = 2\pi$, and $R = 0.5$. The length of the horizontal dimension is $21\sqrt{a}$ in (a)-(c) and (e)-(g), $14\sqrt{a}$ in (d) and $9.3\sqrt{a}$ in (h).
Figure 1

A Esteban-Martin et al, Rocking of a nonlinear optical...
Figure 2

A. Esteban-Martín et al, *Rocking of a nonlinear optical...*
Figure 3

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Figure 4

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