THE KOBAYASHI BALLS OF (C-)CONVEX DOMAINS

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ABSTRACT. A pure geometric description of the Kobayashi balls of (C-)convex domains is given in terms of the so-called minimal basis.

1. INTRODUCTION AND RESULTS

Let $D$ be a domain in $\mathbb{C}^n$. Denote by $c_D$ and $l_D$ the Carathéodory distance and the Lempert function of $D$, respectively:

$$c_D(z, w) = \sup\{\tanh^{-1}|f(w)| : f \in \mathcal{O}(D, \mathbb{D}), \text{ with } f(z) = 0\},$$

$$l_D(z, w) = \inf\{\tanh^{-1} |\alpha| : \exists \varphi \in \mathcal{O}(\mathbb{D}, D) \text{ with } \varphi(0) = z, \varphi(\alpha) = w\},$$

where $\mathbb{D}$ is the unit disc. The Kobayashi distance $k_D$ is the largest pseudodistance not exceeding $l_D$.

We are interested in a description of the Kobayashi balls near boundary points of convex and, more generally, C-convex domains in terms of parameters that reflect the geometry of the boundary. Such a description is

The first results in this direction can be found in [2, Theorems 1 and 5.1], where the strongly pseudoconvex case in $\mathbb{C}^n$ and the weakly pseudoconvex finite type case in $\mathbb{C}^2$ are discussed with applications to invariant forms of the Fatou type theorems (for the boundary values). The weakly pseudoconvex finite type case in $\mathbb{C}^2$, as well as the convex finite type case in $\mathbb{C}^n$, are treated in [6, Propositions 8.8 and 8.9] as byproducts of long considerations. The strongly pseudoconvex case in $\mathbb{C}^n$ and the weakly pseudoconvex finite type in $\mathbb{C}^2$ are particular cases of the pseudoconvex Levi corank one case which is considered in [3, Theorem 1.3]. The behavior of the Kobayashi balls in all the mentioned

2010 Mathematics Subject Classification. 32F17, 32F45.

Key words and phrases. Carathéodory distance, Kobayashi distance, convex domain, C-convex domain.

The second named author is supported by international PhD programme “Geometry and Topology in Physical Models” of the Foundation for Polish Science. The paper was prepared during her visit to the Institute of Mathematics and Informatics, Bulgarian Academy of Science, October 2013 - April 2014.

1See also [14] for complex ellipsoids.
results is given in terms of the Levi geometry of the boundary which is assumed smooth and bounded.

Our aim is to describe the Kobayashi balls of \((\mathbb{C}-)\)convex domains (not necessarily smooth and bounded) in terms of the so-called minimal basis (cf. [4, 9, 12]). The constants that appear depend only on the radius of the balls and the dimension of the domains. The respective proof is short and pure geometric. The obtained result covers [6, Propositions 8.8 and 8.9].

Assume that \(D\) contains no complex lines. Let \(q \in D\) and \(d_D(q) = \text{dist}(q, \partial D)\). Choose \(q^1 \in \partial D\) so that \(\tau_1(q) := \|q^1 - q\| = d_D(q)\). Put \(H_1 = q + \text{span}(q^1 - q)\) and \(D_1 = D \cap H_1\). Let \(q^2 \in \partial D_1\) so that \(\tau_2(q) := \|q^2 - q\| = d_{D_1}(q)\). Put \(H_2 = q + \text{span}(q^1 - q, q^2 - q)\) and \(D_2 = D \cap H_2\) and so on. Thus we get an orthonormal basis of the vectors \(e_j = \frac{q_j - q}{\|q_j - q\|}\), \(1 \leq j \leq n\), which is called minimal for \(D\) at \(q\), and positive numbers \(\tau_1(q) \leq \tau_2(q) \leq \cdots \leq \tau_n(q)\) (the basis and the numbers are not uniquely determined). After rotation we may assume that \(e_1, e_2, \ldots, e_n\) is the standard basis of \(\mathbb{C}^n\).

Recall now that a open set \(D\) in \(\mathbb{C}^n\) is said to be (cf. [1]):

- **\(\mathbb{C}\)-convex** if any non-empty intersection with a complex line is a simply connected domain.
- **linearly (weakly linearly convex)** convex if for any \(p \in \mathbb{C}^n \setminus D\) \(p \in \partial D\) there exists a complex hyperplane through \(a\) which does not intersect \(D\).

Note that convexity \(\Rightarrow\) \(\mathbb{C}\)-convexity \(\Rightarrow\) linear convexity \(\Rightarrow\) weak linear convexity (cf. [1, Theorem 2.3.9 ii]) for the second implication). Moreover, in the case of \(C^1\)-smooth bounded domains the last three notions coincide (cf. [1, Corollary 2.5.6]).

In view of this remark and the inequalities \(c_D \leq k_D \leq l_D\), we have the following quantitative information about the Carathéodory/Kobayashi/Lempert balls of \((\mathbb{C}\text{-})\)convex domains.\(^2\)

**Theorem 1.** Let \(D\) be a domain in \(\mathbb{C}^n\), containing no complex lines, and \(q \in D\). Assume that the standard basis of \(\mathbb{C}^n\) is minimal for \(D\) at \(q\). Let \(r > 0\).

(i) If \(D\) is weakly linearly convex, then

\[
\max_{1 \leq j \leq n} \frac{|z_j - q_j|}{\tau_j(q)} < \frac{e^{2r} - 1}{n(e^{2r} + 1)} \Rightarrow \sum_{j=1}^{n} \frac{|z_j - q_j|}{\tau_j(q)} < \frac{e^{2r} - 1}{e^{2r} + 1} \Rightarrow z \in D \text{ and } l_D(q, z) < r.
\]

\(^2\)By the Lempert theorem, \(c_D = k_D = l_D\) in the convex case, as well as in the bounded \(C^2\)-smooth \(C^2\)-convex case (cf. [11]).
(ii) If $D$ is convex, then $c_D(q, z) < r$ implies $\max_{1 \leq j \leq n} \frac{|z_j - q_j|}{\tau_j(q)} < e^{2r} - 1$.

(iii) If $D$ is $\mathbb{C}$-convex, then $c_D(q, z) < r$ implies $\max_{1 \leq j \leq n} \frac{|z_j - q_j|}{\tau_j(q)} < e^{4r} - 1$.

So there exist constants $c' = c'(r, n)$ and $c'' = c''(r)$ such that

\[ D(q_1, c' \tau_1(q)) \times \cdots \times D(q_n, c' \tau_n(q)) \subset \text{kob}_D(q, r) \]
\[ \subset D(q_1, c'' \tau_1(q)) \times \cdots \times D(q_n, c'' \tau_n(q)), \]

where $\text{kob}_D(q, r)$ is the Kobayashi ball $\{ z \in D : k(q, z) < r \}$ and $D(p, r) = \{ z \in \mathbb{C} : |z - p| < r \}$. By [4, Lemma 3.10], the sizes of these polydiscs are comparable (in terms of small/big constant depending on $D$) with the sizes of polydiscs in [3, 6] arising from the Levi geometry of the boundary. Thus Proposition 1 extends [6, Propositions 8.9].

Note also that if $D$ is a proper $\mathbb{C}$-convex domain in $\mathbb{C}^n$ containing complex line, then it is biholomorphic to $D' \times \mathbb{C}^{n-k}$, where $D'$ is a bounded domain in $\mathbb{C}^k$, $0 < k < n$. (cf. Proposition 3 and the preceding remark in [10]). So $\tau_k(q) < \infty = \tau_{k+1}(q)$ and it is easy to see that Theorem 1 remains true.

To prove Theorem 1, we need the planar cases of following

**Proposition 2.** (i) Let $D$ be proper convex domain in $\mathbb{C}^n$. Then (cf. [13, (2)])

\[ c_D(z, w) \geq \frac{1}{2} \log \frac{d_D(z)}{d_D(w)}. \]

Moreover, if $n = 1$, then

\[ c_D(z, w) \geq \frac{1}{2} \log \left( 1 + \frac{|z - w|}{d_D(w)} \right). \]

(ii) Let $D$ be proper $\mathbb{C}$-convex domain in $\mathbb{C}^n$. Then

\[ c_D(z, w) \geq \frac{1}{4} \log \frac{d_D(z)}{d_D(w)}. \]

Moreover, if $n = 1$, then

\[ c_D(z, w) \geq \frac{1}{4} \log \left( 1 + \frac{|z - w|}{d_D(w)} \right). \]

The constants $1/2$ and $1/4$ are sharp as the examples $D = \mathbb{D}$ and $D = \mathbb{C} \setminus \mathbb{R}^+$ show. Note that in the $\mathbb{C}$-convex case the weaker estimate

\[ c_D(z, w) \geq \frac{1}{4} \log \frac{d_D(z)}{4d_D(w)} \]
is contained in [13, Proposition 2] Theorem 1 has a local version.

**Proposition 3.** Let $D$ be a domain in $\mathbb{C}^n$ whose boundary contains no affine discs through $a \in \partial D$. Assume that the standard basis of $\mathbb{C}^n$ is minimal for $D$ at $q \in D$. Let $r > r' > 0$.

(i) If $D$ is weakly linearly convex near $a$, then

$$\max_{1 \leq j \leq n} \frac{|z_j - q_j|}{\tau_j(q)} < \frac{e^{2r} - 1}{ne^{2r} + 1} \Rightarrow \sum_{j=1}^{n} \frac{|z_j - q_j|}{\tau_j(q)} < \frac{e^{2r} - 1}{e^{2r} + 1}$$

$$\Rightarrow z \in D \text{ and } l_D(q, z) < r.$$

for $q$ sufficiently close to $a$.

(ii) If $D$ is convex near $a$, then $k_D(q, z) < r'$ implies $\max_{1 \leq j \leq n} \frac{|z_j - q_j|}{\tau_j(q)} < e^{2r} - 1$ for $q$ sufficiently close to $a$.

(iii) If $D$ is $C^1$-convex near $a$ and bounded, then $l_D(q, z) < r'$ implies $\max_{1 \leq j \leq n} \frac{|z_j - q_j|}{\tau_j(q)} < e^{4r} - 1$ for $q$ sufficiently close to $a$.

By any of the above three notions of convexity near $a$ we mean that there exists a neighborhood $U$ of $a$ such that $D \cap U$ is an open set with the respective global convexity.

Note that in the convex case, as well as in the $C^1$-smooth $\mathbb{C}$-convex case, if $\partial D$ contains no affine discs through $a$, then $\partial D$ contains no analytic discs through $a$ (cf. [12, Proposition 7]).

2. Proofs

**Proof of Theorem 1.** (i) Since $D$ contains the discs $D(q_1, \tau_1(q)), \ldots, D(q_n, \tau_n(q))$ (lying in the respective coordinate complex planes), it contains their convex hull

$$C = \{ \xi \in \mathbb{C}^n : h(\xi) = \sum_{j=1}^{n} \frac{|\xi_j - q_j|}{\tau_j(q)} < 1 \}$$

(cf. [12, Lemma 15]). Then

$$l_D(q, z) \leq l_C(q, z) = \tanh^{-1} h(z)$$

(cf. [5, Proposition 3.1.10]) which implies (i).

Before proving (ii) and (iii) note that by ($\mathbb{C}$)-convexity and the construction of the minimal basis there exists a complex hyperplane $q^{i+1} + W_j$ through $q^{i+1}$ that is disjoint from $D$, $j = 0, \ldots, n - 1$. It is not difficult to see that $W_j$ is given by the equation

$$\alpha_{j,1}\xi_1 + \cdots + \alpha_{j,j}\xi_j + \xi_{j+1} = 0.$$
Let $\Lambda : \mathbb{C}^n \to \mathbb{C}^n$ be the linear mapping with matrix whose rows are given by the vectors $(\alpha_{j,1}, \ldots, \alpha_{j,j}, 1, 0, \ldots, 0)$. Set $\Lambda_q(\zeta) = q + \Lambda(\zeta - q)$. Note that $G = \Lambda_q(D)$ is a \((\mathbb{C}-)\)convex domain. Denote by $G_j$ the projection of $G$ onto $j$-th coordinate plane. Then $G \subset G' = G_1 \times \cdots \times G_n$ and the product formula for the Carathéodory distance (cf. [5, Theorem 9.5]) implies that
\[
(1) \quad c_D(q, z) \geq c_{G'}(q, \Lambda_q(z)) = \max_{1 \leq j \leq n} c_{G_j}(q_j, z_j).
\]
Observe also that $d_{G_j}(q_j) = \tau_j(q)$.

(ii) If $D$ is a convex domain, then $G_j$ is a convex domain. Hence, by Proposition 2 (i),
\[
c_{G_j}(q_j, z_j) \geq \frac{1}{2} \log \left(1 + \frac{|z_j - q_j|}{\tau_j(q)}\right)
\]
and (ii) follows from here and (1).

(iii) If $D$ is a $\mathbb{C}$-convex domain, then $G_j$ is a simple connected domain (cf. [1, Theorem 2.3.6]). Hence, by Proposition 2 (ii),
\[
c_{G_j}(q_j, z_j) \geq \frac{1}{4} \log \left(1 + \frac{|z_j - q_j|}{\tau_j(q)}\right)
\]
and (iii) follows from here and (1).

**Proof of Proposition 2.** After translation and rotation, we may assume that $0 \in \partial D$ and $w = (d_D(w), 0, \ldots, 0)$.

(i) We have that $D \subset \Pi^+ = \{\zeta \in \mathbb{C}^n : \text{Re } \zeta_1 > 0\}$ and hence
\[
c_D(z, w) \geq c_{\Pi^+}(z, w) = \tanh^{-1} \frac{|z_1 - w_1|}{|z_1 + w_1|} \geq \tanh^{-1} \frac{|z_1 - w_1|}{|z_1 - w_1| + 2d_D(w)} = \frac{1}{2} \log \left(1 + \frac{|z_1 - w_1|}{d_D(w)}\right).
\]

(ii) It follows by weak linear convexity that $D \cap \{\zeta_1 \in \mathbb{C}^n : \zeta_1 = 0\} = \emptyset$. Denote by $D_1$ the projection of $D$ onto the $\zeta_1$-plane. Let $\gamma_D$ the Carathéodory metric of a domain $G$ in $\mathbb{C}^k$ :
\[
\gamma_G(\zeta; X) = \sup\{|f'(\zeta)X| : f \in \mathcal{O}(G, \mathbb{D})\}, \quad \zeta \in G, \ X \in \mathbb{C}^k.
\]
The Kőbe $1/4$ theorem implies that
\[
\gamma_{D_1}(\zeta_1; e_1) \geq \frac{1}{4d_{D_1}(\zeta_1)} \geq \frac{1}{4|\zeta_1|}.
\]
Since $D_1$ is a simply connected domain (cf. [1, Theorem 2.3.6]), then
\[
c_D(z, w) \geq c_{D_1}(z_1, w_1) = \inf_s \int_0^1 \gamma_{D_1}(s(t); s'(t)) dt \geq \frac{1}{4} \inf_s \int_0^1 \left|\frac{s'(t)}{s(t)}\right| dt,
\]
where the infimum is taken over all smooth curves \( s : [0, 1] \rightarrow D_1 \) with \( s(0) = z_1 \) and \( s(1) = w_1 \) (cf. [5]).

Set now
\[
d(\zeta_1, \eta_1) = \log \max(1 + |1 - \zeta_1/\eta_1|, 1 + |1 - \eta_1/\zeta_1|).
\]

It is easy to check that \( d \) is a distance on \( \mathbb{C}_+^3 \) with “derivative”
\[
\lim_{\lambda \to 0} \frac{d(\zeta_1, \zeta_1 + \lambda)}{|\lambda|} = \frac{1}{|\zeta_1|}.
\]

Then (cf. [5, Lemma 4.3.3 (d)])
\[
\inf_s \int_0^1 \frac{|s'(t)|}{|s(t)|} \, dt \geq d(z_1, w_1)
\]
and hence
\[
c_D(z, w) \geq \frac{1}{4} d(z_1, w_1) \geq \frac{1}{4} \log \left( 1 + \frac{|z_1 - w_1|}{d_D(w)} \right).
\]

**Proof of Proposition 3.** (i) Using Theorem 1(i), it is enough to show that \( \lim_{q \to a} \tau_n(q) = 0. \) Assume the contrary. Then there exists a sequence of points \( (q^j) \to a \) such that \( (\tau_n(q^j)) \to \epsilon > 0 \) and \( (e^j) \to e, \) where \( e^j \) is the last vector of the minimal basis for \( D \) at \( q^j. \) We may find a bounded neighborhood \( U \) of \( a \) such that \( D \cap U \) is a weakly linearly convex open set. Shrinking \( \varepsilon \) (if necessary), it follows that the \( e \)-directional disc \( \Delta \) with center \( q \) and radius \( \varepsilon \) is a limit of affine discs in \( D \cap U. \) Since \( D \cap U \) is a taut open set (cf. [11, Proposition 1.5]), then \( \Delta \subset \partial D, \) a contradiction.

(ii) Having in mind Theorem 1 (ii), it is enough to show the following.

**Claim 1.** Let \( U \) be a neighborhood of \( a \) such that \( D \cap U \) is convex. There exist neighborhoods \( W \subset V \subset U \) of \( a \) such that if \( q \in D \cap W \) and \( k_D(q, z) < r', \) then \( z \in V \) and \( pk_{D \cap U}(q, z) \leq k_D(q, z), \) where \( p = r'/r. \)

To prove this claim, recall that \( k_D \) is the integrated form of the Kobayashi metric
\[
\kappa_D(\zeta; X) = \inf \{ |\alpha| : \exists \varphi \in \mathcal{O}(\mathbb{D}, D) \text{ with } \varphi(0) = z, \alpha \varphi'(0) = X \}
\]

Let \( a, b, c \in \mathbb{C}_+ \) and \( d_1 = 1 - a/b, d_2 = 1 - b/c, d_3 = 1 - a/c. \) We may assume that \( d(a, c) = \log(1 + |d_3|). \) Then
\[
d(a, b) + d(b, c) \geq \log(1 + |d_1|) + \log(1 + |d_2|)
\]
Fix an $\varepsilon > 0$. Then we may find a smooth curve $s : [0, 1] \to D$ such that $s(0) = q$, $s(1) = z$ and
\[ k_D(q, z) + \varepsilon > \int_0^1 \kappa_D(s(t); s'(t)) dt. \]

Since $D \cap U$ is convex and its boundary contains no affine discs through $a$, then $a$ is a peak point for $D \cap U$ (cf. [8, Theorem 6]). Hence the strong localization property for the Kobayashi metric holds (cf. [7, Theorem 1 and Corollary 2]). So there exists a neighborhood $V \subset U$ of $a$ such that
\[ \kappa_D(\zeta; X) \geq p\kappa_{D \cap U}(\zeta; X), \quad \zeta \in D \cap V, \quad X \in \mathbb{C}^n. \]

Set $t' = \sup\{ t : s([0, t]) \subset V \}$ and $z' = s(t')$. Then
\[ r' + \varepsilon > k_D(q, z) + \varepsilon > \int_0^{t'} \kappa_D(s(t); s'(t)) dt \]
\[ \geq p \int_0^{t'} \kappa_{D \cap U}(s(t); s'(t)) dt \geq pk_{D \cap U}(q, z') \geq pc_{D \cap U}(q, z'). \]

Taking a peak function for $D \cap U$ at $a$ as a competitor in the definition of $c_{D \cap U}$, it follows that
\[ \liminf_{q \to a} c_{D \cap U}(q, \zeta) = +\infty. \]

Therefore, we may find a neighborhood $W \subset V$ such that if $q \in W$, then $z' \in V$. Therefore $z' = z$ and the claim follows by letting $\varepsilon \to 0$.

(iii) Note the proof of [11, Proposition 1.5] implies the tautness of $C$-convex domains. Then, in view of Theorem 1 (iii), it suffices to show the following.

**Claim 2.** Let $U$ be a neighborhood of $a$ such that $D \cap U$ is taut. There exist neighborhoods $W \subset V \subset U$ of $a$ such that if $q \in D \cap W$ and $l_D(q, z) < r'$, then $z \in V$ and $pl_{D \cap U}(q, z) \leq l_D(q, z)$, where $p = r'/r$.

We point out that, in contrast to (ii), we do not know if $a$ is a local peak point.

It is easy to see that Claim 2 will be a consequence of

**Claim 2’.** If $(\varphi_j) \subset \mathcal{O}(\mathbb{D}, D)$ and $\varphi_j(0) \to a$, then $\varphi_j \Rightarrow a$.

To prove Claim 2’, assume the contrary. Since $D$ is bounded, then, passing to a subsequence (if necessary), we may suppose that $\varphi_j \Rightarrow \varphi \in \mathcal{O}(\mathbb{D}, D)$ and $\varphi \neq a$. Using again that $D$ is bounded, we may find an $s \in (0, 1)$ such that $\varphi_j(s\mathbb{D}) \subset U$ for any $j$. It follows by the tautness of $D \cap U$ that $\varphi(s\mathbb{D}) \in \partial D$. Since $\partial D$ contains no affine discs through $a \in \partial D$, we get similarly to the proof of [12, Proposition 7]
that \( \varphi(sD) = \{a\} \). Then the identity principle implies that \( \varphi = a \) which is a contradiction.

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