Generalized Brouncker’s continued fractions and their logarithmic derivatives

O.Y. Kushel
kushel@mail.ru

Institut für Mathematik, MA 4-5, Technische Universität Berlin,
D-10623 Berlin, Germany

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Abstract
In this paper, we study the continued fraction $y(s, r)$ which satisfies the equation
$$y(s, r) y(s + 2r, r) = (s + 1)(s + 2r - 1)$$
for $r > \frac{1}{2}$. This continued fraction is a generalization of the Brouncker’s continued fraction $b(s)$. We extend the formulas for the first and the second logarithmic derivatives of $b(s)$ to the case of $y(s, r)$. The asymptotic series for $y(s, r)$ at $\infty$ are also studied. The generalizations of some Ramanujan’s formulas are presented.

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1 Introduction

The Brouncker’s continued fraction $b(s) = s + \infty \prod_{n=1}^{\infty} \left( \frac{(2n - 1)^2}{2s} \right)$ still attracts the attention of researchers due to its role in the theory of orthogonal polynomials and its relations to the Gamma and Beta functions (see [3], [5]–[7]). Recall the following theorem of Brouncker describing the properties of $b(s)$ (see [5], p. 145, Theorem 3.16).

**Theorem 1.1 (Brouncker)** Let $b(s)$ be a function on $(0, +\infty)$ satisfying the functional equation $b(s)b(s + 2) = (s + 1)^2$ and the inequality $s < b(s)$ for $s > C$, where $C$ is some constant. Then

$$b(s) = (s + 1) \prod_{n=1}^{\infty} \left( \frac{s + 4n - 3}{s + 4n - 1} \right) = s + \infty \prod_{n=1}^{\infty} \left( \frac{(2n - 1)^2}{2s} \right)$$

for every positive $s$.

Ramanujan discovered the formula, expressing the Brouncker’s continued fraction in terms of the Gamma function (see [5], p. 153, Theorem 3.25).

**Theorem 1.2 (Ramanujan)** For every $s > 0$

$$b(s) = s + \infty \prod_{n=1}^{\infty} \left( \frac{(2n - 1)^2}{2s} \right) = 4 \left( \frac{\Gamma \left( \frac{3+s}{4} \right)}{\Gamma \left( \frac{1+s}{4} \right)} \right)^2.$$

The following extension of Brouncker’s theorem (Theorem 1) was obtained by Euler (see [5], p. 180, Theorem 4.17).
Theorem 1.3 (Euler) Let \( y(s,r) \) be a positive continuous function satisfying the inequality \( s < y(s,r) \) and the equation

\[
y(s,r)y(s+2r,r) = (s+1)(s+2r-1)
\]

for any \( s > 0, r > \frac{1}{2} \). Then

\[
y(s,r) = (s+1) \prod_{n=0}^{\infty} \frac{(s+2r-1+4nr)(s+4r+1+4nr)}{(s+2r+1+4nr)(s+4r-1+4nr)} = s + \sum_{n=1}^{\infty} \left( \frac{(2n-1)^2r^2-(r-1)^2}{2s} \right).
\]

In [5] Ramanujan’s theorem (Theorem 2) was extended to the case of the continued fraction \( y(s,r) \) (see [5], p. 220, ex. 4.22).

Theorem 1.4 For every \( s > 0, r > \frac{1}{2} \)

\[
y(s,r) = s + \sum_{n=1}^{\infty} \left( \frac{(2n-1)^2r^2-(r-1)^2}{2s} \right) = 4r^2 \Gamma\left(\frac{s+2r+1}{4r}\right) \Gamma\left(\frac{s+4r-1}{4r}\right) \Gamma\left(\frac{4n+1}{4r}\right) \Gamma\left(\frac{4n+3}{4r}\right).
\]

The following exact continued fraction representation for the first logarithmic derivative of \( b(s) \)

\[
\frac{b'}{b}(s) = \frac{1}{s + \sum_{n=1}^{\infty} \left( \frac{1}{n} \right)}
\]

allows one to obtain the exponential representation for \( b(s) \) (see [5], p. 192, Theorem 4.25).

Theorem 1.5 For \( s > 0 \)

\[
s + \sum_{n=1}^{\infty} \left( \frac{(2n-1)^2}{2s} \right) = 8\pi^2 \Gamma^4\left(\frac{1}{4}\right) \exp \left\{ \int_{0}^{s} \frac{dt}{t + \sum_{n=1}^{\infty} \left( \frac{1}{n} \right)} \right\}.
\]

In this paper, we represent the first logarithmic derivative of \( y(s,r) \) in the form of the sum of two continued fractions (see Section 4, Corollary 3). For \( s > |r-1|, r > \frac{1}{2} \)

\[
\frac{\partial}{\partial s} \ln y(s,r) = f_1(s,r) + f_2(s,r),
\]

where

\[
f_1(s,r) = \frac{1}{2 - 2r + 2s + 2 \sum_{n=1}^{\infty} \left( \frac{1}{n^2 + r^2} \right)} \quad (1)
\]

and

\[
f_2(s,r) = \frac{1}{2r - 2 + 2s + 2 \sum_{n=1}^{\infty} \left( \frac{1}{n^2 + r^2} \right)} \quad (2)
\]

Then we extend Theorem 5 to the case of \( y(s,r) \) (see Section 5, Theorem 9).
Theorem 1.9 For \( s > |r - 1|, r > \frac{1}{2} \)

\[
y(s, r) = s + \sum_{n=1}^{\infty} \left( \frac{(2n-1)^2r^2 - (r-1)^2}{2s} \right) = 8\pi r^{1-\frac{1}{4}} \Gamma^2 \left( \frac{1}{4} \frac{r}{2r} \right) \cot \left( \frac{\pi}{4r} \right) \exp \left\{ \int_{0}^{s} (f_1(t, r) + f_2(t, r)) dt \right\},
\]

where \( f_1(t, r) \) and \( f_2(t, r) \) are given by formulas (1) and (2), respectively.

There is also an exact integral representation of \( b'(s) \) (see [5], p. 191, Formula 4.71). For \( s > 0 \)

\[
b'(s) = \frac{1}{s + \sum_{n=1}^{\infty} \left( \frac{n^2}{2s} \right)^2} = 2 \int_{0}^{+\infty} e^{-sx} dx.
\]

Theorem 5 together with (3) imply the following asymptotic relation, which holds for \( b(s) \) as \( s \to +\infty \) (see [5], p. 192, Corollary 4.26).

\[
b(s) = s + \sum_{n=1}^{\infty} \left( \frac{(2n-1)^2}{2s} \right) \sim s \exp \left\{ -\sum_{k=1}^{\infty} \frac{E_{2k}}{2ks^{2k}} \right\},
\]

where \( E_{2k} \) are the Euler’s numbers. Here the asymptotic power series \( \exp \left\{ -\sum_{k=1}^{\infty} \frac{E_{2k}}{2ks^{2k}} \right\} \) arises from replacing \( x \) in the formal power series \( \exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \) by \( \sum_{k=1}^{\infty} \frac{E_{2k}}{2ks^{2k}} \) and combining coefficients afterwards (on the possibility of such a substitution see [9], p. 15, Theorem 124, see also [2], p. 15).

We obtain exact integral representations for both the continued fractions (1) and (2) (see Section 3, Lemma 4). For \( s > |r - 1|, r > \frac{1}{2} \)

\[
\frac{1}{2 - 2r + 2s + \sum_{n=1}^{\infty} \left( \frac{n^2 r^2}{2s} \right)} = \frac{1}{2r} \int_{0}^{+\infty} e^{-s \frac{x+s}{x+r+s}} dx.
\]

\[
\frac{1}{2r - 2 + 2s + \sum_{n=1}^{\infty} \left( \frac{n^2 r^2}{2s} \right)} = \frac{1}{2r} \int_{0}^{+\infty} e^{-s \frac{x+s}{x+r+s}} dx.
\]

These two formulas together with Theorem 9 allows us to obtain the asymptotic expansion for \( y(s, r) \) at infinity (see Section 6, Theorem 10), using Euler’s methods.

\[
y(s, r) = s + \sum_{n=1}^{\infty} \left( \frac{(2n-1)^2r^2 - (r-1)^2}{2s} \right) \sim s \exp \left\{ -\sum_{k=0}^{\infty} \frac{\sum_{n=1}^{\infty} \left( \frac{(2n-r)^2(r-1)^{2k}E_{2(n-k)}}{2n s^{2n}} \right)}{2n s^{2n}} \right\}.
\]
Let us introduce the notation:

\[ s^2 - 1 + \frac{4 \times 1^2}{s^2 - 1 + 1} + \frac{4 \times 2^2}{s^2 - 1 + 1} + \cdots = s^2 - 1 + \sum_{n=1}^{\infty} \left( \frac{4n^2}{1 + s^2 - 1} \right). \]

Ramanujan stated the following formula for the second logarithmic derivative of \( b(s) \), which was proved later by Perron (see [5], p. 231, Formula (5.6), see also [8]).

**Theorem 1.6 (Ramanujan’s formula)** For \( s > 1 \)

\[ \frac{d^2}{ds^2} \ln b(s) = - \int_0^{\infty} x e^{-sx} dx = - \frac{1}{s^2 - 1 + \sum_{n=1}^{\infty} \left( \frac{4n^2}{1 + \frac{4n^2}{s^2 - 1}} \right)}. \]  

We obtain the corresponding formula for the second logarithmic derivative of \( y(s,r) \).

**Theorem 1.12** For \( s > \max(1, 2r - 1), r > \frac{1}{2} \)

\[ \frac{\partial^2}{\partial s^2} \ln y(s,r) = - \frac{1}{2r^2} \int_0^{\infty} x \left( e^{-\frac{1-r-1+s}{r}x} + e^{-\frac{r-1+s}{r}x} \right) \cosh x \, dx = -h_1(s,r) - h_2(s,r), \]

where

\[ h_1(s,r) = \frac{1}{2(1-2r+s)(1+s) + 2 \sum_{n=1}^{\infty} \left( \frac{4n^2 r^2}{1 + \frac{4n^2 r^2}{(1-2r+s)(1+s)}} \right)}. \]

\[ h_2(s,r) = \frac{1}{2(2r - 1 + s)(s - 1) + 2 \sum_{n=1}^{\infty} \left( \frac{4n^2 r^2}{1 + \frac{4n^2 r^2}{(2r-1+s)(s-1)}} \right)}. \]

2 **Functional equations for logarithmic derivatives of** \( y(s, r) \)

Let us recall the following statement, which will be used later (see [5], p. 152, Lemma 3.23).

**Lemma 2.1** Let \( g(s) \) be a monotonic function on \((0, \infty)\), vanishing at infinity, and \( a > 0 \) be a positive number. Then the functional equation \( f(s) + f(s+a) = g(s) \) has a unique solution, vanishing at infinity, given by the formula

\[ f(s) = \sum_{n=0}^{\infty} (-1)^n g(s + na). \]

Let us prove two following statements for the first and the second logarithmic derivatives of \( y(s, r) \).
Lemma 2.2  The functional equation

\[
f(s, r) + f(s + 2r, r) = \frac{1}{s + 1} + \frac{1}{s + 2r - 1} = \frac{2(s + r)}{(s + 1)(s + 2r - 1)}
\]  

(5)

has a unique solution, satisfying \(\lim_{s \to \infty} f(s, r) = 0\), which is

\[
f(s, r) = \frac{\partial}{\partial s} (\ln y)(s, r).
\]

Proof. The equality \(y(s, r)y(s + 2r, r) = (s + 1)(s + 2r - 1)\) implies

\[
\ln(y(s, r)y(s + 2r, r)) = \ln((s + 1)(s + 2r - 1));
\]

\[
\ln y(s, r) + \ln y(s + 2r, r) = \ln(s + 1) + \ln(s + 2r - 1)).
\]

Differentiating by \(s\), we obtain

\[
\frac{\partial}{\partial s} y(s, r) + \frac{\partial}{\partial s} y(s + 2r, r) = \frac{1}{s + 1} + \frac{1}{s + 2r - 1}.
\]

(6)

The function \(f(s) = \frac{\partial}{\partial s} y(s, r)\) satisfy the conditions of Lemma 1 with \(a = 2r\) and \(g(s) = \frac{2(s + r)}{(s + 1)(s + 2r - 1)}\). Applying Lemma 1, we complete the proof. □

Let us examine two equations:

\[
f_1(s, r) + f_1(s + 2r, r) = \frac{1}{s + 1}
\]

(7)

and

\[
f_2(s, r) + f_2(s + 2r, r) = \frac{1}{s + 2r - 1}.
\]

(8)

Both of them satisfy the conditions of Lemma 1 with \(a = 2r\), \(g(s) = \frac{1}{s + 1}\) and \(g(s) = \frac{1}{s + 2r - 1}\), respectively. So, applying Lemma 1, we obtain, that the solution \(f_1(s, r)\) of equation (7) which satisfies \(\lim_{s \to \infty} f_1(s, r) = 0\) is unique. The solution \(f_2(s, r)\) of equation (8) which satisfies \(\lim_{s \to \infty} f_2(s, r) = 0\) is also unique. Since their sum \(f_1(s, r) + f_2(s, r)\) satisfies equation (5), we have from Lemma 2, that

\[
\frac{\partial}{\partial s} (\ln y)(s, r) = f_1(s, r) + f_2(s, r),
\]

where \(f_1(s, r)\) and \(f_2(s, r)\) are the solutions of (7) and (8), respectively, vanishing as \(s \to +\infty\).

Lemma 2.3  The functional equation

\[
f(s, r) + f(s + 2r, r) = -\frac{1}{(s + 1)^2} - \frac{1}{(s + 2r - 1)^2}
\]

(9)

has a unique solution, satisfying \(\lim_{s \to \infty} f(s, r) = 0\), which is

\[
f(s, r) = \frac{\partial^2}{\partial^2 s} (\ln y)(s, r).
\]

5
Proof. Differentiate equation (6) once again by $s$:

$$
\frac{\partial}{\partial s} \left( \frac{\partial}{\partial s} y(s, r) \right) + \frac{\partial}{\partial s} \left( \frac{\partial}{\partial s} y(s + 2r, r) \right) = -\frac{1}{(s + 1)^2} - \frac{1}{(s + 2r - 1)^2}.
$$

Applying Lemma 1 with $a = 2r$ and $g(s) = -\frac{1}{(s + 1)^2} - \frac{1}{(s + 2r - 1)^2}$, we complete the proof. □

Repeating the above reasoning, we obtain that

$$
-\frac{\partial^2}{\partial s^2} \ln y(s, r) = h_1(s, r) + h_2(s, r),
$$

where $h_1(s)$ is the unique solution of the equation

$$
h_1(s, r) + h_1(s + 2r, r) = \frac{1}{(s + 1)^2},
$$

satisfying $\lim_{s \to \infty} h_1(s, r) = 0$ and $h_2(s)$ is the unique solution of the equation

$$
h_2(s, r) + h_2(s + 2r, r) = \frac{1}{(s + 2r - 1)^2},
$$

satisfying $\lim_{s \to \infty} h_2(s, r) = 0.$

3 Exact integral representation for certain type continued fractions

To begin, we formulate the following result by Euler (see [4] and [5], p. 191, Theorem 4.24).

Theorem 3.1 For $s > 0$

$$
\frac{1}{s + \sum_{n=1}^{\infty} \left( \frac{n^2}{s} \right)} = 2 \int_{0}^{1} \frac{x^r dx}{1 + x^2}.
$$

Corollary 3.1 For $s > 0$

$$
\frac{1}{s + \sum_{n=1}^{\infty} \left( \frac{n^2}{s} \right)} = \int_{0}^{\infty} \frac{e^{-sx} dx}{\cosh x}.
$$

Let us formulate and prove the following lemma.

Lemma 3.1 Let $\varphi(s, r)$ be an arbitrary real-valued function of $s$ and $r$. Then for $r > 0$, $\varphi(s, r) > 1$.

$$
\frac{1}{2\varphi(s, r) + 2 \sum_{n=1}^{\infty} \left( \frac{n^2}{\varphi(s, r)} \right)} = \frac{1}{r} \int_{0}^{1} \frac{x^{\varphi(s, r) - 1}}{1 + x^2} dx.
$$

1. Actually we have the condition $\frac{\varphi(s, r)}{r} > 0$ but since in the conditions of Theorem 3 $r > \frac{1}{2}$ we restrict ourselves to the case $r > 0$.  

6
Proof. Examine equality (13). Using the substitution $s := \frac{\varphi(s, r)}{r}$, where $\varphi(s, r)$ is an arbitrary real-valued function of $s$ and $r$, we obtain the equality, which is correct for all $s$, $r$ satisfying $\varphi(s, r) > 0$, $r > 0$.

$$\frac{1}{\varphi(s, r) + \sum_{n=1}^{\infty} \left( \frac{n^2}{\varphi(s, r)} \right)} = 2 \int_{0}^{1} \frac{\hat{x}(s, r)}{1 + x^2} \, dx.$$  

$$\frac{1}{2\varphi(s, r) + 2r \sum_{n=1}^{\infty} \left( \frac{n^2}{\varphi(s, r)} \right)} = \frac{r}{\pi} \int_{0}^{\frac{\pi}{2}} \frac{\hat{x}(s, r)}{\cosh x} \, dx.$$  

Let us apply the equivalence transform with the parameters $r_0 = 1$, $r_n = r$, $n = 1, 2, \ldots$ to the continued fraction on the left-hand side. This results the formula:

$$\frac{1}{2\varphi(s, r) + 2r \sum_{n=1}^{\infty} \left( \frac{n^2}{\varphi(s, r)} \right)} = \frac{1}{r} \int_{0}^{1} \frac{x \hat{x}(s, r)}{1 + x^2} \, dx.$$  

□

Corollary 3.2 For $r > 0$, $\varphi(s, r) > 0$

$$\frac{1}{\varphi(s, r) + \sum_{n=1}^{\infty} \left( \frac{n^2}{\varphi(s, r)} \right)} = \frac{1}{r} \int_{0}^{\frac{\pi}{2}} \frac{\hat{x}(s, r)}{\cosh x} \, dx.$$  

Proof. It is enough for the proof to use the substitution $x := e^{-x}$. □

Example. Let $\varphi(s, r) = s + \sin r$, $s = 1$, $r = \frac{\pi}{2}$. Then

$$\frac{1}{2 + \sum_{n=1}^{\infty} \left( \frac{n^2}{\varphi(s, r)} \right)} = \frac{2}{\pi} \int_{0}^{\frac{\pi}{2}} \frac{\hat{x}(s, r)}{\cosh x} \, dx.$$  

Using the equivalence transformation with the parameters $r_0 = 1$, $r_n = 2$, $n = 1, 2, \ldots$, we get

$$\frac{1}{4 + \sum_{n=1}^{\infty} \left( \frac{n^2}{4} \right)} = \frac{1}{\pi} \int_{0}^{\frac{\pi}{2}} \frac{\hat{x}(s, r)}{\cosh x} \, dx.$$  

4 Functional equations for certain type continued fractions

Now let us formulate and prove the following theorem.

Theorem 4.1 Let $\varphi(s, r) = s + \psi(r)$, where $\psi(r)$ is an arbitrary real-valued function of $r$. Then for $r > 0$, $s > -\psi(r)$ the continued fraction of the form

$$f(s, r) = \frac{1}{2\varphi(s, r) + 2r \sum_{n=1}^{\infty} \left( \frac{n^2}{\varphi(s, r)} \right)}$$

7
is the unique solution of the functional equation
\[ f(s, r) + f(s + 2r, r) = \frac{1}{\varphi(s, r) + r}, \quad (16) \]
satisfying \( \lim_{s \to \infty} f(s, r) = 0 \).

**Proof.** Examine the series expansion for the right-hand side of equation (15):
\[
\frac{1}{r} \int_0^1 \frac{x \varphi(s+x)}{1 + x^2} \, dx = \frac{1}{r} \sum_{n=0}^{\infty} (-1)^n \int_0^1 x^{2n+\varphi(s+x)} \, dx = \frac{1}{r} \sum_{n=0}^{\infty} \frac{(-1)^n}{\varphi(s, r) + 2n + 1} = \\
= \sum_{n=0}^{\infty} \frac{(-1)^n}{\varphi(s, r) + 2rn + r}.
\]

It follows from Lemma 1, that the unique solution of equation (16) satisfying \( \lim_{s \to \infty} f(s, r) = 0 \) is given by the formula:
\[
f(s, r) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\varphi(s, r) + 2rn + r}.
\]
Since \( \varphi(s, r) = s + \psi(r) \), we have
\[
f(s, r) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\varphi(s + 2rn, r) + r} = \sum_{n=0}^{\infty} \frac{(-1)^n}{\varphi(s, r) + 2rn + r} = \\
= \frac{1}{r} \int_0^1 \frac{x \varphi(s+x)}{1 + x^2} \, dx = \frac{1}{2\varphi(s, r) + 2 K \left( \frac{n^2 + 1}{s+r} \right)}.
\]

by Lemma 4. □

**Corollary 4.1** For \( s > |r - 1|, r > 0 \) functional equation (7) has a unique solution satisfying \( \lim_{s \to 0} f_1(s, r) = 0 \) which is
\[
f_1(s, r) = \frac{1}{2 - 2r + 2s + 2 K \left( \frac{n^2 + 1}{s+r} \right)}.
\]

Functional equation (8) also has a unique solution satisfying \( \lim_{s \to 0} f_2(s, r) = 0 \) which is
\[
f_2(s, r) = \frac{1}{2r - 2 + 2s + 2 K \left( \frac{n^2 + 1}{s+r} \right)}.
\]

**Proof.** From the equality \( \frac{1}{s+1} = \frac{1}{\varphi_1(s, r) + r} \) we obtain \( \varphi_1(s, r) = s + 1 - r \). Since \( \varphi_1(s, r) \) satisfies the conditions of Theorem 8, we obtain, that the continued fraction \( f_1(s, r) \) is the solution of (7). By analogy, from the equality \( \frac{1}{s+2r-1} = \frac{1}{\varphi_2(s, r) + r} \) we obtain \( \varphi_2(s, r) = s + r - 1 \), which also satisfies the conditions of Theorem 8. Applying Theorem 8 again, we obtain that the continued fraction \( f_2(s, r) \) is the solution of (8). □
5 The exponential formula for generalized Brouncker’s continued fraction

Theorem 5.1 For $s > |r - 1|$, $r > \frac{1}{2}$

$$y(s, r) = s + \sum_{n=1}^{\infty} \left( \frac{(2n - 1)^2 r^2 - (r - 1)^2}{2s} \right) =$$

$$= 8\pi r^{1 - \frac{1}{2}} \frac{\Gamma^2(\frac{1}{4}r)}{\Gamma(\frac{1}{4}r)} \cot\left( \frac{\pi}{4r} \right) \exp\left\{ \int_0^s f_1(t, r)dt \right\} \exp\left\{ \int_0^s f_2(t, r)dt \right\},$$

where the continued fractions $f_1(s, r)$ and $f_2(s, r)$ are defined by equations (17) and (18), respectively.

Proof. According to Corollary 3, the continued fractions $f_1(s, r)$ and $f_2(s, r)$ satisfy equations (7) and (8), respectively. Hence applying Lemma 2 we obtain,

$$\frac{\partial}{\partial s} \ln y(s, r) = \frac{1}{2 - 2r + 2s + 2 \sum_{n=1}^{\infty} \left( \frac{n^2 r^2}{2s - r + s} \right)} + \frac{1}{2r - 2 + 2s + 2 \sum_{n=1}^{\infty} \left( \frac{n^2 r^2}{2s - r + s} \right)}.$$

Integrating the obtained differential equation, we get

$$\ln y(s, r) = \int_0^s (f_1(t, r) + f_2(t, r))dt + C(r)$$

$$y(s, r) = C(r) \exp\left\{ \int_0^s (f_1(t, r) + f_2(t, r))dt \right\},$$

where $C(r)$ is a function of $r$.

It is easy to see, that $C(r) = y(0, r)$. Let us calculate $y(0, r)$, using Theorem 4. At first let us recall some well-known formulas for the Gamma function. These are the duplication formula

$$\Gamma(z)\Gamma(z + \frac{1}{2}) = 2^{1 - 2z} \sqrt{\pi} \Gamma(2z),$$

and the Euler’s reflection formula

$$\Gamma(1 - z)\Gamma(z) = \frac{\pi}{\sin(\pi z)}.$$

Since by definition $\Gamma(z + 1) = z\Gamma(z)$, we have $\Gamma(1 - z) = -z\Gamma(-z)$ and rewrite the Euler’s reflection formula in the following form

$$\Gamma(z)\Gamma(-z) = -\frac{\pi}{z \sin(\pi z)}. \quad (19)$$

Using simple calculations we obtain, that

$$y(0, r) = 4r \frac{\Gamma(\frac{2r + 1}{4})\Gamma(\frac{4r - 1}{4})}{\Gamma(\frac{r}{2})\Gamma(\frac{2r - 1}{4})} + 4r \frac{\Gamma(\frac{1}{2} + \frac{1}{4}r)\Gamma(1 - \frac{1}{4}r)}{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2} - \frac{1}{4}r)} = \ldots$$
\[
\Gamma(\frac{1}{2} + \frac{1}{4r})\Gamma(\frac{1}{4r}) = 2^{1 - \frac{1}{2r}} \sqrt{\pi} \Gamma(\frac{1}{2r})
\]

and
\[
\Gamma(\frac{1}{2} - \frac{1}{4r})\Gamma(-\frac{1}{4r}) = 2^{1 + \frac{1}{2r}} \sqrt{\pi} \Gamma(-\frac{1}{2r}),
\]

we have
\[
\ldots = 4r^{\frac{1}{2} - \frac{1}{2r}} \sqrt{\pi} \Gamma(\frac{1}{4r}) (1 - \frac{1}{4r}) \Gamma(-\frac{1}{4r}) = 4r \frac{\Gamma(\frac{1}{4r}) \Gamma(1 - \frac{1}{4r}) \Gamma(-\frac{1}{4r})}{\Gamma(\frac{1}{4r}) 2^{1 + \frac{1}{2r}} \sqrt{\pi} \Gamma(-\frac{1}{2r})} =
\]
\[
= 4r \frac{\Gamma(\frac{1}{4r}) \pi \Gamma(-\frac{1}{4r})}{\Gamma(\frac{1}{4r}) 2^{1 + \frac{1}{2r}} \sin(\frac{\pi}{4r})} = \ldots
\]

Using (19) we obtain
\[
\ldots = 4r \frac{\Gamma(\frac{1}{4r}) \pi \Gamma(-\frac{1}{4r}) \Gamma(\frac{1}{4r})}{\Gamma(\frac{1}{4r}) 2^{1 + \frac{1}{2r}} \sin(\frac{\pi}{4r})} = -4r \frac{\Gamma(\frac{1}{4r}) \pi^2}{\Gamma(\frac{1}{4r}) 2^{1 + \frac{1}{2r}} \sin(\frac{\pi}{4r}) \sin(\frac{\pi}{4r})} =
\]
\[
= -16r^2 \frac{\Gamma^2(\frac{1}{4r}) \pi^2}{\Gamma(\frac{1}{4r}) 2^{1 + \frac{1}{2r}} \sin(\frac{\pi}{4r}) \sin(\frac{\pi}{4r})} = 16r^2 \frac{\Gamma^2(\frac{1}{4r}) \pi^2}{\Gamma(\frac{1}{4r}) 2^{1 + \frac{1}{2r}} \pi \sin(\frac{\pi}{4r}) \sin(\frac{\pi}{4r})} =
\]
\[
= 8\pi r \frac{\Gamma^2(\frac{1}{4r}) 2 \sin(\frac{\pi}{4r}) \cos(\frac{\pi}{4r})}{\Gamma(\frac{1}{4r}) 2^{1 + \frac{1}{2r}} \cot(\frac{\pi}{4r})}.
\]

\[
\Box
\]

**Corollary 5.1** For \( s > 0 \)
\[
s + \frac{\infty}{n=1} \left(\frac{2n - 1)^2}{2s}\right) = \frac{8\pi^2}{\Gamma^4(\frac{1}{4})} \exp \left\{ \frac{\int_0^s dt}{t + \frac{\infty}{n=1} \left(\frac{a^2}{n^2}\right)} \right\}.
\]

**Proof.** Just put \( r = 1 \) and observe that
\[
\frac{1}{2s} + \frac{\infty}{n=1} \left(\frac{\pi^2}{n^2}\right) = \frac{2}{2s} + \frac{\infty}{n=1} \left(\frac{\pi^2}{n^2}\right) = \frac{1}{s} + \frac{\infty}{n=1} \left(\frac{\pi^2}{n^2}\right).
\]

\[
\Box
\]

**Example.** Putting \( r = 2 \) into the statement of Theorem 9 and calculating
\[
\cot(\frac{\pi}{8}) = \frac{\sin(\frac{\pi}{4})}{1 - \cos(\frac{\pi}{8})} = \sqrt{2} + 1, \text{ we obtain for } s > 1
\]
\[
s + \frac{\infty}{n=1} \left(\frac{4(2n - 1)^2 - 1}{2s}\right) = 16\pi(2 + \sqrt{2}) \frac{\Gamma^2(\frac{1}{4})}{\Gamma^4(\frac{1}{4})} \times
\]
\[
\times \exp \left\{ \frac{\int_0^s dt}{2t - 2 + \frac{\infty}{n=1} \left(\frac{a^2}{n^2}\right)} \right\} \exp \left\{ \frac{\int_0^s dt}{2t + 2 + \frac{\infty}{n=1} \left(\frac{a^2}{n^2}\right)} \right\}.
\]

10
6 Generalized Brouncker’s continued fraction and its asymptotic series

Let us recall the following lemma (see [1], p. 614, also [2], p. 150, Lemma 3.21).

**Lemma 6.1 (Watson)** Let $f$ be a function on $(0, +\infty)$, such that $|f(t)| < M$ for $t > \epsilon$ and $f(t) = \sum_{k=0}^{\infty} c_k t^k$, $0 < t < 2\epsilon$. Then

$$
\int_0^{+\infty} f(t) e^{-st} dt \sim \sum_{k=0}^{\infty} \frac{k!c_k}{q^{k+1}}, \quad s \to +\infty
$$

is the asymptotic expansion for the Laplace transform of $f$.

Let us write the asymptotic expansions for both continued fractions (17) and (18). Applying Corollary 2, we get the following formulas for $s > |r - 1|$, $r > 0$:

$$
\frac{1}{2 - 2r + 2s + 2K \frac{n^2x^2}{(r-1+s)}} = \frac{1}{2r} \int_0^{+\infty} e^{-x^{r-1+x}} \frac{dx}{\cosh x};
$$

$$
\frac{1}{2r - 2 + 2s + 2K \frac{n^2x^2}{(r-1+s)}} = \frac{1}{2r} \int_0^{+\infty} e^{-x^{r-1+x}} \frac{dx}{\cosh x}.
$$

(20)

(21)

Examine equation (20). Write the right-hand side of equation (20) in the following form:

$$
\frac{1}{2r} \int_0^{+\infty} e^{-x^{r-1+x}} \frac{dx}{\cosh x} = \frac{1}{2r} \int_0^{+\infty} e^{-x^{r-1+x}} \frac{1}{\cosh x} e^{-x^{r-1+x}} dx.
$$

(22)

Repeating the reasoning from [2], p. 92, we obtain:

$$
\frac{1}{\cosh x} = \sum_{n=0}^{\infty} \frac{E_n}{n!} x^n,
$$

where $E_n$ are the Euler’s numbers;

$$
e^{r-1+x} = \sum_{n=0}^{\infty} \frac{(r-1)^n}{r^n n!} x^n.
$$

Using the rules of series multiplication, we get:

$$
\frac{e^{r-1+x}}{\cosh x} = \left( \sum_{n=0}^{\infty} \frac{(r-1)^n}{r^n n!} x^n \right) \left( \sum_{n=0}^{\infty} \frac{E_n}{n!} x^n \right) = \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} \frac{(r-1)^k}{r^k k! (n-k)!} E_{n-k} \right) x^n.
$$

Applying Watson’s lemma 5 to (22) with $f(x) = e^{r-1+x}$, we obtain:

$$
\frac{1}{2r} \int_0^{+\infty} e^{-x^{r-1+x}} \frac{dx}{\cosh x} \sim \frac{1}{2r} \sum_{n=0}^{\infty} \frac{n!}{\sum_{k=0}^{n} \frac{(r-1)^k}{r^k k! (n-k)!} \left( \frac{E_{n-k}}{s^{n+1}} \right)} \frac{r^{n+1}}{s^{n+1}}
$$
as $s \to \infty$. Since \( \frac{n!}{k!(n-k)!} = \binom{n}{k} \), we have

\[
\frac{1}{2 - 2r + 2s + 2 \sum_{n=1}^{\infty} \binom{n}{k} (n-k) E_{n-k}} \sim \frac{1}{2} \sum_{k=0}^{\infty} \binom{n}{k} (r-1)^k E_{n-k} \quad \text{as } s \to \infty.
\] (23)

Analogically, we obtain the following asymptotic expansion for (21):

\[
\frac{1}{2 - 2r + 2s + 2 \sum_{n=1}^{\infty} \binom{n}{k} (1-r)^k E_{n-k}} \sim \frac{1}{2} \sum_{k=0}^{\infty} \binom{n}{k} (1-r)^k E_{n-k} \quad \text{as } s \to \infty.
\] (24)

**Theorem 6.1** The following asymptotic relation holds as $s \to +\infty$:

\[
s + \frac{\sum_{n=1}^{\infty} \binom{n}{k} (2n-1)^2 r^2 - (r-1)^2}{2s} \sim s \exp \left\{ - \frac{\sum_{n=1}^{\infty} \binom{n}{k} (2n-1)^2 r^2 (n-k) E_{2(n-k)}}{2n s^2 n} \right\}.
\] (25)

**Proof.** By Theorem 3, the left-hand side of (25) is divisible by $(s+1)$. Theorem 9 implies, that the continued fraction $y(s, r)$ can be written as

\[
y(s, r) = (s+1) y(0, r) \exp \left\{ \int_0^{+\infty} \gamma_1(t, r) dt \right\} \exp \left\{ - \int_s^{+\infty} \gamma_1(t, r) dt \right\} \times \exp \left\{ \int_0^{+\infty} \gamma_2(t, r) dt \right\} \exp \left\{ - \int_s^{+\infty} \gamma_2(t, r) dt \right\},
\]

where

\[
\gamma_1(t, r) = \frac{1}{2 - 2r + 2t + 2 \sum_{n=1}^{\infty} \binom{n}{k} (n-k) E_{n-k}} - \frac{1}{2(1+t)}.
\]

\[
\gamma_2(t, r) = \frac{1}{2r - 2t + 2 + 2 \sum_{n=1}^{\infty} \binom{n}{k} (n-k) E_{n-k}} - \frac{1}{2(1+t)}.
\]

Using asymptotic expansions (23), (24) and the expansion

\[
\frac{1}{(1+t)} = \frac{1}{t(1+\frac{1}{t})} \sim \frac{1}{t} \sum_{n=0}^{\infty} (-1)^n \frac{1}{t^n} = \sum_{n=0}^{\infty} (-1)^n \frac{1}{t^{n+1}} \quad t \to +\infty
\]

we obtain

\[
\gamma_1(t, r) \sim \frac{1}{2} \sum_{n=0}^{\infty} \binom{n}{k} (r-1)^k E_{n-k} \frac{1}{t^{n+1}} - \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n \frac{1}{t^{n+1}} = \ldots
\]
Since the numerator of the null’s term in the first sum is equal to $E_0 = 1$,

\[
\ldots = \frac{1}{2} \sum_{n=1}^{\infty} \frac{\sum_{k=0}^{n} \binom{n}{k}(r - 1)^k r^{n-k} E_{n-k} - (-1)^n}{t^{n+1}}. \quad t \to +\infty
\]

Analogically,

\[
\gamma_2(t, r) \sim \frac{1}{2} \sum_{n=1}^{\infty} \frac{\sum_{k=0}^{n} \binom{n}{k}(1 - r)^k r^{n-k} E_{n-k} - (-1)^n}{t^{n+1}}. \quad t \to +\infty
\]

Integrating this over $(s, +\infty)$, we obtain

\[
\int_{s}^{+\infty} \gamma_1(t, r) dt \sim \frac{1}{2} \sum_{n=1}^{\infty} \frac{\sum_{k=0}^{n} \binom{n}{k}(r - 1)^k r^{n-k} E_{n-k} - (-1)^n}{ns^n}. \quad t \to +\infty
\]

\[
\int_{s}^{+\infty} \gamma_2(t, r) dt \sim \frac{1}{2} \sum_{n=1}^{\infty} \frac{\sum_{k=0}^{n} \binom{n}{k}(1 - r)^k r^{n-k} E_{n-k} - (-1)^n}{ns^n}. \quad t \to +\infty
\]

Since $y(s, r) \sim s$ as $s \to +\infty$, we conclude that

\[
y(0, r) \exp \left\{ \int_{0}^{+\infty} \gamma_1(t, r) dt \right\} \exp \left\{ \int_{0}^{+\infty} \gamma_2(t, r) dt \right\} = 1
\]

and

\[
y(s, r) \sim (s + 1) \exp \left\{ -\frac{1}{2} \sum_{n=1}^{\infty} \frac{\sum_{k=0}^{n} \binom{n}{k}(r - 1)^k r^{n-k} E_{n-k} - (-1)^n}{ns^n} \right\} \times
\]

\[
\times \exp \left\{ \frac{1}{2} \sum_{n=1}^{\infty} \frac{\sum_{k=0}^{n} \binom{n}{k}(1 - r)^k r^{n-k} E_{n-k} - (-1)^n}{ns^n} \right\}
\]

Using the equality $\sum_{n=1}^{\infty} \frac{(-1)^n}{ns^n} = - \ln \left( \frac{s + 1}{s} \right)$, as $s > 1$, we obtain that

\[
y(s, r) \sim s \exp \left\{ -\frac{1}{2} \sum_{n=1}^{\infty} \frac{\sum_{k=0}^{n} \binom{n}{k}((r - 1)^k + (1 - r)^k) r^{n-k} E_{n-k}}{ns^n} \right\} =
\]

\[
= [ \text{since}(1 - r)^k = (-1)^k(r - 1)^k ] =
\]

\[
s \exp \left\{ -\sum_{n=1}^{\infty} \frac{\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k}(r - 1)^{2k} r^{n-2k} E_{n-2k}}{ns^n} \right\}
\]

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The proof is completed by observing that all the Euler’s numbers with odd parameters $E_1, E_3, E_5, \ldots$ are equal to zero.

**Example.** Putting $r = 2$ we obtain

$$s + \sum_{n=1}^{\infty} \left( \frac{4(2n-1)^2 - 1}{2s} \right)^\frac{2n}{n} \sim s \exp \left\{ -\sum_{n=1}^{\infty} \frac{1}{2n}s^{2n} \right\},$$

as $s \to +\infty$. Computations with the first few Euler’s numbers $E_0 = 1, E_1 = 0, E_2 = -1, E_3 = 0, E_4 = 5, E_5 = 0, E_6 = -61$ shows that

$$s + \sum_{n=1}^{\infty} \left( \frac{4(2n-1)^2 - 1}{2s} \right)^\frac{2n}{n} \sim s \exp \left\{ \sum_{n=1}^{\infty} \frac{1}{2s}s^{2n} \right\}.$$

Writing the first terms of the expansion of $e^x$

$$e^x \sim 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + O(x^4)$$

and substituting $x = \frac{3}{2s^2} \frac{57}{4s^4} + \frac{2763}{6s^6} + O \left( \frac{1}{s^8} \right)$ we obtain the first terms of the expansion:

$$s + \sum_{n=1}^{\infty} \left( \frac{4(2n-1)^2 - 1}{2s} \right)^\frac{2n}{n} \sim s + \frac{3}{2s} \frac{105}{8s^8} + \frac{7035}{16s^6} + O \left( \frac{1}{s^7} \right).$$

### 7 Ramanujan’s formula and its generalization

Our generalization of Ramanujan’s formula (4) requires some preliminary results. The first of them is the following theorem.

**Theorem 7.1** Let $\varphi(s, r)$ be an arbitrary real-valued function of $s$ and $r$. Then for $r > 0$, $\varphi(s, r) > r$

$$\frac{\varphi(s, r) - r^2}{r^2} + \sum_{n=1}^{\infty} \frac{1}{r^2} \int_0^\infty \frac{xe^{-x} \varphi(s, r)}{\cosh x} dx = \frac{1}{r^2} \int_0^\infty \frac{xe^{-x} \varphi(s, r)}{\cosh x} dx.$$  \hspace{1cm} (26)

**Proof.** Examine equality (4) with the substitution $s := \frac{\varphi(s, r)}{r}$, where $\varphi(s, r)$ is an arbitrary real-valued function of $s$ and $r$. Then we obtain the following formula for $\varphi(s, r) > r, r > 0$:
Apply the equivalence transform with the parameters $r_0 = 1$, $r_n = r^2$, $n = 1, 2, \ldots$ to the continued fraction on the left-hand side. This results the formula:

$$
\frac{1}{\varphi^2(s, r) - r^2 + \sum_{n=1}^{\infty} \left( \frac{4n^2}{r^2} + \frac{4n^2}{\varphi^2(s, r)} - r^2 \right)} = \frac{1}{r^2} \int_0^{\infty} xe^{-x \varphi(s, r)} \cosh x \, dx.
$$

Using simple calculations:

$$
\frac{1}{\varphi^2(s, r) - r^2 + \sum_{n=1}^{\infty} \left( \frac{4n^2}{r^2} + \frac{4n^2}{\varphi^2(s, r)} - r^2 \right)} = \frac{1}{r^2} \int_0^{\infty} xe^{-x \varphi(s, r)} \cosh x \, dx.
$$

Let us prove the following lemma, which describes the derivative of the continued fraction

$$
f(s, r) = \frac{1}{\varphi(s, r) + \sum_{n=1}^{\infty} \left( \frac{a_n^2 r^2}{\varphi(s, r)} \right)}.
$$

**Lemma 7.1** Let $\varphi(s, r) = s + \psi(r)$, where $\psi(r)$ is an arbitrary real-valued function of $r$. Then for $r > 0$, $s > r - \psi(r)$

$$
\frac{\partial}{\partial s} f(s, r) = -\frac{1}{\varphi^2(s, r) - r^2 + \sum_{n=1}^{\infty} \left( \frac{4n^2 r^2}{1} + \frac{4n^2 r^2}{\varphi^2(s, r) - r^2} \right)},
$$

where

$$
f(s, r) = \frac{1}{\varphi(s, r) + \sum_{n=1}^{\infty} \left( \frac{a_n^2 r^2}{\varphi(s, r)} \right)}.
$$

**Proof.** Using Corollary 2, we obtain the equality

$$
f(s, r) = \frac{1}{\varphi(s, r) + \sum_{n=1}^{\infty} \left( \frac{a_n^2 r^2}{\varphi(s, r)} \right)} = \frac{1}{r} \int_0^{\infty} e^{-x \varphi(s, r)} \cosh x \, dx.
$$

Differentiating this equality by $s$ and changing the sign, we obtain:

$$
-\frac{\partial}{\partial s} f(s, r) = \frac{1}{r^2} \int_0^{\infty} xe^{-x \varphi(s, r)} \cosh x \, dx,
$$

which exactly coincide with the right-hand side of (26). □
Corollary 7.1 For \( r > 0, s > \max(1, 2r - 1) \)
\[
\frac{\partial}{\partial s} f_1(s, r) = -\frac{1}{2(1 - 2r + s)(1 + s) + 2 \sum_{n=1}^{\infty} \frac{4n^2 r^2}{1 - (1 - 2r + s)(1 + s)}},
\]
where
\[
f_1(s, r) = \frac{1}{2 - 2r + 2s + 2 \sum_{n=1}^{\infty} \frac{n^2 r^2}{1 - r + s}}.
\]
\[
\frac{\partial}{\partial s} f_2(s, r) = -\frac{1}{2(2r - 1 + s)(s - 1) + 2 \sum_{n=1}^{\infty} \frac{4n^2 r^2}{(2r - 1 + s)(s - 1)}}.
\]
where
\[
f_2(s, r) = \frac{1}{2r - 2 + 2s + 2 \sum_{n=1}^{\infty} \frac{n^2 r^2}{r - 1 + s}}.
\]

Example. Put \( \varphi(s, r) = s + \sin r, \ r = \frac{\pi}{2} \). Then for \( s > \frac{\pi}{2} - 1 \) we have
\[
f'(s) = -\frac{1}{(s + 1)^2 - \frac{\pi^2}{4} + \sum_{n=1}^{\infty} \frac{n^2 \pi^2}{1 + (s + 1)^2 - \frac{\pi^2}{4}}},
\]
where
\[
f(s) = \frac{2}{2s + 2 + \sum_{n=1}^{\infty} \frac{n^2 \pi^2}{2(s + 1)}}.
\]

Theorem 7.2 For \( s > \max(1, 2r - 1), r > \frac{1}{2} \)
\[
\frac{\partial^2}{\partial s^2} \frac{1}{y}(s, r) = -\frac{1}{2r^2} \int_0^\infty x(e^{-\frac{1}{r} + x} + e^{-\frac{r - 1}{r} + x}) \cosh x \ dx = -h_1(s, r) - h_2(s, r),
\]
where
\[
h_1(s, r) = \frac{1}{2(1 - 2r + s)(1 + s) + 2 \sum_{n=1}^{\infty} \frac{4n^2 r^2}{1 - (1 - 2r + s)(1 + s)}},
\]
\[
h_2(s, r) = \frac{1}{2(2r - 1 + s)(s - 1) + 2 \sum_{n=1}^{\infty} \frac{4n^2 r^2}{(2r - 1 + s)(s - 1)}}.
\]

Proof. The proof comes out from Equality 10 and Corollary 5.

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