Nonsymmetric Gravity Does Have Acceptable Global Asymptotics.

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Abstract

We consider the claim by Damour, Deser and McCarthy that nonsymmetric gravity theory has unacceptable global asymptotics. We explain why this claim is incorrect.

I. INTRODUCTION

In a series of papers, Damour, Deser and McCarthy (DDM) have claimed that the nonsymmetric gravitational theory (NGT) is theoretically inconsistent. They noted that a spurious “gauge invariance” in the linearised version of NGT did not generalise to curved spacetime, from which they concluded that ghost excitations would occur. However, the “gauge invariance” in question is simply an artifact of the linearisation, and plays no role in ensuring the conservation of the true Noether charges of the theory. The loss of such an invariance is as unimportant as its existence.

They went on to argue that generic solutions suffered unacceptable asymptotic behaviour at future null infinity, \( I^+ \). This argument was based on the supposed behaviour of a Lagrange multiplier field which, in fact, can and should be eliminated from the field equations. We explained the pitfalls of such a treatment in ref. [4], and we shall expand on our original explanation in this note. It has since been shown that general radiative solutions can be found with good asymptotic behaviour at \( I^+ \) [5,6].

Damour, Deser and McCarthy now accept that good asymptotic behaviour can be found at \( I^+ \), but claim this can only be achieved at the expense of bad asymptotic behaviour at \( I^- \). They base this claim on a lemma which states that any solution of an inhomogeneous wave equation which falls off faster than \( 1/r \) at \( I^+ \) must be an advanced solution. We point out in this note that the lemma they use is not applicable to the system of equations being studied, since they apply Green's theorem to a hyperbolic system while neglecting lower order differential constraints on the boundary of integration. Consequently, their assertion that NGT has bad global asymptotics is unfounded.
II. ANALYSIS OF THE FIELD EQUATIONS

For ease of comparison with the work of DDM, we shall adopt their unconventional notation for the field variables. Performing an expansion in powers of the anti-symmetric field $B_{\mu\nu}$ about a fixed symmetric background $G_{\mu\nu}$, the NGT field equations \[7\] read to first order:

$$R_{(\mu\nu)}(G) = 0,$$  \hspace{1cm} (2.1)

$$\nabla^\alpha \nabla_\alpha B_{\mu\nu} - 2R^\alpha_{\mu\nu\beta}B_{\alpha\beta} = \frac{2}{3}(\partial_\mu \Gamma_\nu - \partial_\nu \Gamma_\mu),$$  \hspace{1cm} (2.2)

$$\nabla^\nu B_{\mu\nu} = 0,$$  \hspace{1cm} (2.3)

which may be supplemented by the gauge choice:

$$\nabla^\mu \Gamma_\mu = 0.$$  \hspace{1cm} (2.4)

The covariant derivative, $\nabla$, is that of the background metric $G_{\mu\nu}$, and the vector $\Gamma_\mu$ is a non-dynamical Lagrange multiplier. This Lagrange multiplier appears in the linearised NGT Lagrangian density via the term:

$$L_\Gamma = \sqrt{-G} B^{\mu\nu}(\partial_\nu \Gamma_\mu - \partial_\mu \Gamma_\nu),$$  \hspace{1cm} (2.5)

which gives rise to the field equation (2.3). In the language of field theory, we see that $\Gamma_\nu$ does not have a propagator, since the Lagrangian cannot be formulated to have a kinetic energy term for $\Gamma_\nu$. As such, it makes no sense to talk about $\Gamma_\nu$ having retarded or advanced propagator solutions. We shall return to this crucial point later.

In order to solve the field equations, we need to take the divergence and cyclic curl of (2.2), while remembering that the solutions to these higher derivative equations are constrained to be solutions to the primary lower derivative equation, (2.2). The equations for $B_{\mu\nu}$ and $\Gamma_\mu$ then read

$$\nabla^\nu B_{\mu\nu} = 0,$$  \hspace{1cm} (2.6)

$$\nabla^\alpha \nabla_\alpha B_{(\mu\nu,\kappa)} - 2\nabla_{(\kappa}(R^\alpha_{\mu\nu}\beta)B_{\alpha\beta)} + \nabla_{\alpha}(R^\alpha_{(\kappa\mu\nu}\beta)B_{\nu)\beta}) = 0,$$  \hspace{1cm} (2.7)

$$\nabla^\alpha \nabla_\alpha \Gamma_\nu = -3\nabla^\mu(R^\alpha_{\mu\nu\beta}B_{\alpha\beta}).$$  \hspace{1cm} (2.8)

The first two sets of equations, (2.6, 2.7), represent six equations for the six $B_{\mu\nu}$. These six equations fully determine $B_{\mu\nu}$ with no reference to the Lagrange multiplier $\Gamma_\nu$, which is what one expects from a system of equations with a Lagrange multiplier. The last equation, (2.8), can be used to solve for the Lagrange multiplier $\Gamma_\nu$ once the six $B_{\mu\nu}$ are known. We note that on its own, (2.8) can only determine the LHS of (2.2) to be $2/3(\nabla_\mu \Gamma_\nu - \nabla_\nu \Gamma_\mu) + F_{\mu\nu}$, where $F_{\mu\nu}$ is any skew tensor that satisfies $\nabla^\nu F_{\mu\nu} = 0$.

In their analysis, DDM concentrate their attention on first solving for the Lagrange multiplier $\Gamma_\nu$ via equation (2.8). This approach is fraught with problems. Firstly, the $\nabla R B$ “source term” is an unknown function unless you have already solved for $B_{\mu\nu}$. DDM fail to check whether their eventual solution for $B_{\mu\nu}$ is a self-consistent solution of this equation. It is not. Secondly, the hyperbolic differential operator $\nabla^\alpha \nabla_\alpha$ in (2.8) demands propagating,
retarded and advanced $1/r$ Green’s function solutions for $\Gamma_\nu$. This leads to a distorted physical picture, as the primary field equation, (2.2), for $\Gamma_\nu$ is not a wave equation. Wave solutions for $\Gamma_\mu$ play no part in the physics of NGT.

Additionally, DDM base their arguments on the Green’s function for the flat-space d’Alembertian $\Box$ instead of the operator $\nabla^\alpha \nabla_\alpha$, while at the same time keeping the source term $\nabla RB$. This treatment is inconsistent since $(\nabla^\alpha \nabla_\alpha - \Box)\Gamma \sim \mathcal{O}(R\Gamma)$ is of the same order as the source term.

We shall see that the most important of these errors is their failure to correctly treat the spurious solutions for $\Gamma_\nu$, which are manufactured in taking the divergence of the field equations.

III. THE EXPLICIT FAULTS IN DDM’S ARGUMENT

While the reasons given above more than suffice to invalidate the proof given by DDM, it is instructive to see the failings of their arguments shown explicitly. Indeed, one needs to look no further than the static case to illustrate the errors in their analysis.

For static systems, the uniqueness theorem for inhomogeneous wave equations cited by DDM \[8\] reduces to the statement that $\Gamma_\mu$ will fall-off as $1/r$ at spatial or null infinity.

Explicitly, we find for the component $\Gamma_t$ that (2.8) becomes

\[
\left[\nabla^2 - \frac{2M}{r^2} \left(r \frac{\partial^2}{\partial r^2} + 2 \frac{\partial}{\partial r}\right)\right] \Gamma_t = 3\nabla^\mu (R^\alpha_\mu \beta B_{\alpha\beta}),
\]

where $\nabla^2$ is the usual flat-space Laplacian for a scalar field and $M$ is the mass associated with the Schwarzschild background. If we follow the argument of DDM - by neglecting the fact that the source term is unknown and the operator acting on $\Gamma_t$ is not just the flat-space Laplacian - then we conclude from the uniqueness theorem that $\Gamma_t \sim 1/r$.

If we then do as DDM suggest, and feed this information into (2.2) we find

\[
\left[\left(\nabla^2 + \frac{2}{r} \frac{\partial}{\partial r} + \frac{2}{r^2}\right) - \frac{2M}{r} \left(\frac{\partial^2}{\partial r^2} + \frac{3}{r} \frac{\partial}{\partial r} + \frac{4}{r^2}\right)\right] B_{tr} = \frac{2}{3} \frac{\partial \Gamma_t}{\partial r} \sim 1/r^2,
\]

from which we conclude $B_{tr} \sim r^0$, exactly as predicted by DDM.

The above analysis would seem to provide a perfect example of DDM’s claim that NGT has unacceptable asymptotic behaviour. Let us now look at what has been forgotten. Firstly, we have neglected the field equation (2.5) which demands

\[
\nabla^\alpha B_{\alpha\tau} = 0,
\]

so that $B_{tr} \sim r^{-2}$. DDM’s putative solution, $B_{tr} \sim r^0$, is not a solution to (3.3). Putting $B_{tr} = l^2/r^2$ (where $l^2$ is a constant of integration) into (2.2) gives

\[
-\frac{8Ml^2}{r^5} \sim \frac{1}{r^2},
\]

which clearly excludes the $1/r$ behaviour for $\Gamma_t$ suggested by (3.1). This is simply an example of a higher order derivative equation, (3.1), having solutions which are incompatible with
the lower order equation, \( (3.2) \), from which it came. Restoring the term \( \partial_t \Gamma \) in \( (3.2) \), we see that the true solution for \( \Gamma \) has \( \Gamma_t = \frac{3Ml}{r^4} \).

The above calculation has now faithfully reproduced the exact NGT static solution to first order in \( B_{\mu\nu} \) - a solution which represents a non-trivial extension of the Schwarzschild solution of General Relativity (GR) with good asymptotic behaviour for \( B_{\mu\nu} \). The putative \( 1/r \) Green’s function solution for \( \Gamma \) was simply an artifact produced by taking additional derivatives of the field equations.

Another way of seeing what went wrong with DDM’s argument is to recall that we are free to add a tensor of integration, \( F_{\mu\nu} \), to the LHS of \( (3.2) \), where \( F_{\mu\nu} \) is a solution of \( \nabla^\alpha F_{\alpha\beta\mu} = 0 \). In this case, we may add \( F_{tr} \sim \frac{1}{r^2} \) to the LHS of \( (3.2) \) with the coefficient chosen to eliminate the \( 1/r^2 \) term given by the curl of \( \Gamma_t \). In this way, \( B_{tr} \) is no longer driven to behave as \( r^0 \). From this argument we see that the putative \( 1/r \) solution for \( \Gamma_t \) was a purely homogeneous solution which must be chosen to vanish.

Generalising our analysis to the time-dependent case we find a very similar picture. Again the higher-derivative equation for \( \Gamma \) admits homogeneous solutions with \( 1/r \) fall-off, while the lower order constraints on \( \Gamma \) demand that these solutions be discarded.

We shall demonstrate this in the context of wave solutions on a radiative, axi-symmetric GR background [5]. To leading order, the GR background is described by \( M(u, \theta) \) and \( c(u, \theta) \), where \( u = t - r \) is retarded time, the mass associated with the background is given by the angular average of \( M \) and the time rate of change of this mass is given by the angular average of \( -(\partial_u c)^2 \). We shall only sketch the main steps in solving the equations, as the full solution is derived in detail in ref. [5].

Beginning with the wave equations for \( \Gamma_\mu \) we find

\[
\Box \Gamma_u + \mathcal{O}(R\Gamma) = -3\nabla^\mu (R^\alpha_\mu B_{\alpha\beta}) ,
\]

where \( \Box \) is the usual flat-space d’Alembertian for a scalar field, and the extra background terms of order \( R\Gamma \) are given by

\[
\mathcal{O}(R\Gamma) = \frac{2}{r} \left[ M \frac{\partial^2 \Gamma_u}{\partial r^2} + \frac{\partial M}{\partial u} \frac{\partial \Gamma_r}{\partial r} \right] + \frac{2}{r^2} \left[ \left( 4M + 2c + 2\frac{\partial c}{\partial u} - \frac{\partial^2 c}{\partial u^2} - 3\cot \theta \frac{\partial c}{\partial \theta} \right) \frac{\partial \Gamma_u}{\partial r} + \frac{c^2}{2} \frac{\partial^2 \Gamma_u}{\partial r \partial u} \right.
\]

\[
+ \left. \frac{c^2}{2} \frac{\partial^2 \Gamma_u}{\partial r \partial u} - \frac{2}{r} \left( \frac{\partial^2 c}{\partial u^2} + 2\cot \theta \frac{\partial c}{\partial u} \right) \frac{\partial \Gamma_u}{\partial r} \right]
\]

\[
- 2M \frac{\partial \Gamma_r}{\partial u} - 2 \left( \frac{\partial c}{\partial u} \right)^2 \Gamma_r - \left. \frac{c}{\partial u} \frac{\partial \Gamma_r}{\partial r} \right] + \ldots .
\]

Although it is inconsistent to drop these terms while keeping the \( RB \) source term, we shall follow the method proposed by DDM and drop them anyway. The wave equation for \( \Gamma_r \) then reads

\[
\left[ \Box - \frac{2}{r} \frac{\partial}{\partial r} - \frac{2}{r^2} + \frac{2}{r} \frac{\partial}{\partial t} \right] \Gamma_r + \left[ \frac{2}{r} \frac{\partial}{\partial r} + \frac{2}{r^2} \right] \Gamma_u = -3\nabla^\mu (R^\alpha_\mu B_{\alpha\beta}) .
\]

When combined with the gauge condition \( \nabla^\alpha \Gamma_\alpha = 0 \) and the wave equation for \( \Gamma_\theta \), we find that the above equations have the usual retarded wave solutions \( \Gamma_u = f(u, \theta)/r \), \( \Gamma_r = \)
\[ g(u, \theta)/r^2 \] and \( \Gamma_\theta = h(u, \theta) \) (in orthonormal coordinates this means \( \Gamma_\theta \sim 1/r \)). Since the source terms are at present unknown, we cannot give explicit forms for \( f, g \) and \( h \), although the gauge condition does demand \( f + \partial_\theta g = \partial_\theta h + h \cot \theta \).

Turning to the \((ur)\) component of (2.2) we find

\[
\left[ \Box - \frac{2}{r} \frac{\partial}{\partial r} - \frac{2}{r^2} - \frac{2}{r} \frac{\partial}{\partial t} \right] B_{ur} + \mathcal{O}(RB) = \frac{2}{3} \left( \frac{\partial \Gamma_r}{\partial u} - \frac{\partial \Gamma_u}{\partial r} \right) \]

\[
= \frac{2 \partial_\theta (h \sin \theta)}{3r^2 \sin \theta}. \tag{3.8}
\]

From this we would conclude \( B \) has the unacceptable asymptotic form \( B_{ur} \sim r^0 \), as promised by DDM. However, this is in conflict with the field equations (2.6) and (2.7) which demand to leading order that \( B_{ur} = l^2(u, \theta)/r^2 \). The lower order field equations for \( \Gamma_\nu \) again demand that the \( 1/r \) solution for \( \Gamma \) be dropped, as was the case for static solutions. In this case we require \( \partial_\theta (h \sin \theta) = 0 \) which implies \( h = 0 \) to ensure regularity on the polar axis. Notice that setting \( h(u, \theta) = 0 \) removes the only transverse, propagating component of \( \Gamma_\nu \), and that the remaining components of \( \Gamma_\nu \) are longitudinal and non-propagating, despite the \( 1/r \) fall-off for \( \Gamma_u \). Such important subtleties are lost in DDM’s analysis, for they treat \( \Gamma_\nu \) as a scalar. It is also worth noting from the full solution [5] that the one transverse component of \( B_{\mu \nu} \), \( B_{u\theta} \), does have a \( 1/r \) retarded wave solution. The physical fields, \( B_{\mu \nu} \), that we expect to propagate do, and the non-dynamical Lagrange multiplier fields, \( \Gamma_\nu \), do not. We would be surprised if we had found otherwise.

As in the static case, we see that the extra solutions for \( \Gamma_\nu \), which are generated by taking additional derivatives of the field equations, can be cancelled by the tensor of integration \( F_{\mu \nu} \). Another way of seeing this is to recognise that a consistent treatment of the “inhomogeneous wave equation” for \( \Gamma \) demands that we drop the source term \( \nabla RB \) when using the flat-space d’Alembertian for \( \Gamma \). Then we see that the solutions for \( \Gamma \) which are independent of the background parameters \( M \) and \( c \) are simply homogeneous solutions which we are free to discard.

The above collection of results allows us to address DDM’s lemma head-on. They claim that the fast fall-off of \( \Gamma \) at \( \mathcal{I}^+ \) comes about because our solutions for \( \Gamma \) are advanced solutions. This is clearly not the case, as our solutions for \( \Gamma \) are explicitly retarded solutions - despite the fact they do not describe transverse waves with \( 1/r \) fall-off. If we accept DDM’s contention that \( \Gamma \) is described by advanced Green’s functions, we find this leads to a contradiction. If \( \Gamma \), and hence \( B \), are functions of advanced time, \( v = t + r \), their lemma demands that \( \Gamma \) has propagating, \( 1/r \), solutions at \( \mathcal{I}^- \). By a simple time reversal of our solutions in terms of retarded time, with the added simplification that the GR background is static at \( \mathcal{I}^- \), we find from the wave equation for \( \Gamma \) that \( \Gamma_\theta = h(v, \theta)/r + \mathcal{O}(1/r^2) \), while the lower order equation (2.2), and the fact that \( B_{ur} = l^2(v, \theta)/r^2 \), demand \( h(v, \theta) = 0 \). This proves by contradiction that DDM’s lemma is not valid for \( \Gamma \).

In their latest paper [3], DDM go on to repeat the same flawed analysis they applied to the Lagrange multiplier \( \Gamma \), by taking two derivatives of (2.2) in order to discuss a fourth order wave equation for \( B_{\mu \nu} \). Again, the original, lower order field equations do not allow the badly behaved solutions of the higher order equations. All of the pitfalls that plague DDM’s analysis can be avoided, if one works from the outset with (2.6, 2.7) and uniquely.
solves for $B_{\mu\nu}$ without reference to the Lagrange multiplier $\Gamma$. This was the method first employed by Einstein and Straus when solving an analogous system of equations in Unified Field Theory \[9\]. Using the same method, we have found exact, radiative solutions, well behaved at $I^+$ and $I^-$ \[5,6\]. These solutions represent a non-trivial modification to the GR limit of this system and lead to the prediction that the quadrupole moment of a source will decrease more rapidly in NGT than GR.

To summarise, we have shown that DDM's claim that NGT has bad global asymptotics is invalid, since it is based on wave solutions for a Lagrange multiplier field which fail to solve the original, lower order field equations. This invalidates the use of their lemma, since it can only be used for fields with lower order constraints, when those constraints do not demand that the propagating modes vanish (as is the case in electromagnetism, for example). If these constraints are properly accounted for, we find that NGT has non-trivial radiative solutions with good global asymptotics.

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