Entanglement is the most distinctive feature of quantum mechanics [1], and represents the essential resource for quantum information processing [2]. One of the most striking predictions of quantum theory is that entanglement may exist up to the macroscopic realm. In reality, the entanglement of systems with several degrees of freedom is countered by fast interaction with the environment, and the deepest essence of such decoherence process has not yet been unraveled. Experiments along this line might clarify the mechanisms underlying decoherence, or even hint at a fundamental restriction of the quantum predictions, that would define a quantum-classical boundary. Within a more applied perspective, entanglement at the macroscopic scale – especially in integrated solid-state microstructures – will be required for quantum information storage and communication [3]. Recently, several groundbreaking experiments have succeeded in producing entanglement of macroscopic degrees of freedom, involving ultracold atoms [4–6], photons [7, 8], and collective electronic excitations [9].

A promising route to macroscopic entanglement is optomechanics [10–13]. In an optomechanical system, electromagnetic modes of an optical cavity are coupled to one or more mechanical oscillators via radiation pressure and mechanical backaction. Several theoretical studies have suggested the generation nonclassical mechanical states [14, 15], and more specifically entangled states of two mechanical modes [16–24]. Recently, an entangled state of optical phonons of two distant bulk diamond crystals has been produced and detected [25].

Here, we demonstrate a full protocol for heralded entanglement generation and readout of two mechanical modes in a realistic optomechanical system. The protocol is specifically studied having in mind two localized flexural modes of a L3 photonic crystal cavity, for which significant optomechanical coupling strengths and close-valued mechanical frequencies have been demonstrated [24]. Silicon photonic crystal cavities are increasingly considered as an ideal building block of an integrated technological platform for quantum photonics, and the present protocol might indicate a way to the generation and storage of quantum information. It can however equally be applied to other optomechanical systems, for which the state-of-the-art parameters are steadily improving [27–31].

We consider the system sketched in Fig.1(a) composed of an optical cavity with a mode (\(\hat{a}\) operator) resonant at frequency \(\omega_c\), optomechanically coupled to two harmonic mechanical modes (\(\hat{b}_{1,2}\) operators) of resonant frequencies \(\Omega_{1,2}\). The system Hamiltonian reads

\[
\hat{H}_s = \hbar \omega_c \hat{a}^{\dagger} \hat{a} + \sum_{j=1,2} \left[ \hbar \Omega_j \hat{b}_j^{\dagger} \hat{b}_j - g_j \hat{a}^{\dagger} \hat{a} (\hat{b}_j^{\dagger} + \hat{b}_j) \right] + F(t) \hat{a}^{\dagger} + F^*(t) \hat{a}.
\]

Here, \(g_{1,2}\) are the single-photon optomechanical coupling constants, and the cavity is driven by a classical field \(F(t)\). We assume parameters in the range of those measured in Ref. [26], i.e. \(\Omega_1/2\pi = 700\) MHz, \(\Omega_2/2\pi = 980\) MHz, \(g_1/2\pi = 72\) kHz, \(g_2/2\pi = 84\) kHz, and mechanical loss rates \(\gamma_1/2\pi = 4.4\) MHz and \(\gamma_2/2\pi = 5.4\) MHz. Selective Stokes and anti-Stokes driving however requires the system to be in the well resolved sideband limit. Hence, we propose the use of a last-generation sili-
con L3 photonic crystal cavity, for which we recently
demonstrated quality factors up to two mil-
lion \[32, 33\]. We therefore assume \(\omega_c/2\pi = 194\) THz,
corresponding to \(\lambda = 1550\) nm, and an optical loss rate
\(\kappa/2\pi = 200\) MHz. Fig. 1(b) shows the computed optical
spectrum consisting of the main cavity resonance,
and two pairs of sidebands associated to Stokes and anti-
Stokes processes.

Before presenting the details of the protocol, let us
illustrate its general principle through a simplified anal-
ysis based on pure states: We denote by \(|n_c, n_1, n_2\rangle\) the
number state with \(n_c\) photons and, respectively, \(n_1\) and
\(n_2\) quanta in the first and second mechanical mode. We
assume that the system is initially in its vacuum state
\(|\psi_0\rangle = |000\rangle\). The writing phase consists in driving the
system with a weak optical pulse resonant with the two
Stark sidebands, as sketched in Fig. 1(b) (black curve).
Two Stokes processes will take place, each leaving aphonon in a mechanical mode and a photon in the cavity
mode. For a weak enough pulse, the dominant terms in
the final state will be the vacuum, followed by the state
with one photon:

\[
|\psi_1\rangle = \alpha_1|000\rangle + \beta_1|1_c\rangle \left( |10\rangle + e^{i\phi_s}|01\rangle \right) + \ldots ,
\] (2)

where \(\phi_s\) is a phase, that characterizes the coherent super-
position of the two Raman processes. After the writing pulse has fully decayed, the heralding is performed by
detecting the emission spectrally filtered at \(\omega_c\). If a pho-
ton is detected, then the vacuum component is projected
out giving:

\[
|\psi_2\rangle = \beta_2|0_c\rangle \left( |10\rangle + e^{i\phi(t)}|01\rangle \right) + \ldots ,
\] (3)

where the phase \(\phi(t) = \phi_s - (\Omega_2 - \Omega_1)t\) evolves due to
the nondegeneracy of the two mechanical modes. In the
limit where higher-occupation states have negligible am-
plitudes, \(3\) is a maximally entangled state of the two
mechanical modes. Conditioned to the detection of a
heralding photon, a readout is performed by driving the
system with a second optical pulse resonant with the two
anti-Stokes sidebands, as sketched in Fig. 1(b) (red curve).
As for the writing phase, a coherent superposition of two anti-Stokes processes will bring the system to:

\[
|\psi_3\rangle = \alpha_3 \left( 1 + e^{i\phi'(t)} \right) |1,00\rangle + \beta_3|0_c\rangle \left( |10\rangle + e^{i\psi(t)}|01\rangle \right) + \ldots ,
\] (4)

where \(\phi'(t) = \phi(t) + \phi_{as}\) accounts, as above, for the relative
phase of the two anti-Stokes processes. If emission at \(\omega_c\) is again detected, after the readout pulse de-
cayed, the emitted intensity will be \(I \propto |1 + e^{i\phi'(t)}|^2\),
i.e. it will display a full-contrast interference pattern as a
function of the photon detection time. If instead of \(3\) one has a fully separable state \(|\psi_2\rangle =
\beta_2|0_c\rangle \left( |0\rangle + e^{i\phi_1}|1\rangle \right) \left( |0\rangle + e^{i\phi_2}|1\rangle \right)\), then the visibility of
the interference pattern will be of only 50%.

This protocol follows in the footsteps of the procedure
proposed in Refs. \[14\] \[15\] to produce a mechanical Fock state.
Similarly to other schemes studied in the past, \[17\] \[25\] it is based on the original DLCZ proposal \[3\] for
entangling atomic ensembles. An important distinctive
feature of the present proposal however, is that the light
emitted by the two Stokes processes must not be mixed
in a beam-splitter in order to erase the which-path infor-
mation, as the two processes emit directly into the same
mode.

The interference pattern in the emitted light is not
in itself an entanglement witness. In particular, a sim-
ilar interference pattern might arise from classical field
amplitudes, provided they have the right mutual phase
relation. However – similarly to seminal quantum optics
experiments \[34\] \[35\] – an interference pattern, combined
with the knowledge that the average mode occupation
fulfils \(|\bar{n}_m| \ll 1\), is a solid indication of the occurrence of
a nonclassical state.

The Model.— The model considered here is based on
Hamiltonian \([1]\) where the driving field is characterized
by two pulsed excitations \(F(t) = F_W(t) + F_R(t)\) where

\[
F_j(t) = A_j e^{-(t-t_j)^2/\sigma_j^2} e^{-i\omega_j t}, \quad j = W, R
\] (5)

together with the readout and readout procedures. Here, \(\omega_j, t_j, \sigma_j,\) and \(A_j\) are the central frequency, central
time, bandwidth, and amplitude, respectively for the two
pulses.

The protocol relies on photon detection spectrally fil-
tered at the cavity frequency \(\omega_c\). A computationally very
effective way to model spectral filtering consists in in-
roducing an additional harmonic oscillator, representing
the degrees of freedom of the narrow-band detector \[36\].
Its Hamiltonian is expressed as

\[
\hat{\mathcal{H}}_d = \hat{\omega}_d \hat{d} \hat{d}^\dagger ,
\]

and the frequency \(\omega_d\) and damping rate \(\kappa_d\) define the frequency
and pass-band of the filter. The global Hamiltonian reads

\[
\hat{\mathcal{H}} = \hat{\mathcal{H}}_h + \hat{\mathcal{H}}_d.
\]

In presence of coupling to the environment, the sys-
tem dynamics is governed by the master equation for the
density matrix, which in the Lindblad form \[37\] reads

\[
\frac{d\hat{\rho}}{dt} = -i [\hat{\mathcal{H}}, \hat{\rho}] - \frac{\kappa}{2} D [\hat{a}] \hat{\rho} - \frac{\kappa_d}{2} D [\hat{d}] \hat{\rho} + \frac{\zeta}{2} D [\hat{a}, \hat{d}] \hat{\rho} - \bar{n}_{th} \sum_j \frac{\gamma_j}{2} D [\hat{b}_j] \hat{\rho} - (\bar{n}_{th} + 1) D [\hat{b}_j^\dagger] \hat{\rho} .
\] (6)

Here, \(D[\hat{a}] \hat{\rho} = \hat{a} \hat{\rho} \hat{a}^\dagger + \hat{a}^\dagger \hat{\rho} \hat{a} - 2\hat{\rho} \hat{a} \hat{a}^\dagger\) describe the coupling
to the environment at rates \(\kappa, \kappa_d\) and \(\gamma_j\) for
the cavity, detector and mechanical modes respectively.
\(\bar{n}_{th} = [\exp(\Omega/k_BT) - 1]^{-1}\) is the mean thermal phonon
number associated to a mechanical frequency \(\Omega\). Con-
trarily to Ref. \[36\], we assume here a dissipative, thus
irreversible, cavity to detector coupling, described by
\(D[\hat{a}, \hat{d}] \hat{\rho} = [\hat{a} \hat{\rho}, \hat{d}^\dagger] + [\hat{d} \hat{\rho}, \hat{a}^\dagger]\). This approach prevents any
unwanted coherent oscillations between the two modes and therefore allows to consider an arbitrary coupling strength $\zeta$. Here, $\kappa_d$ acts as the bandwidth of the filter. From now on, we shall consider $\omega_d = \omega_c$ and $\kappa_d = \zeta = 0.1\kappa$.

Equation (2) was solved numerically: For the write phase, the equation was directly solved on a finite dimensional Hilbert space, restricted to a maximum number of quanta for each mode. The readout phase is characterized by a significantly stronger driving field. Hence, for this phase each field was characterized as the sum of a classical and a quantum fluctuation component. For the cavity field we define $\hat{a}_c = \langle \hat{a} \rangle + \delta \hat{a}$, and analogously for the other fields [38].

**Write**— The write step starts with the arrival of the first pulse, $F_W(t)$. We set the parameters to $A_W = 2.5\kappa$ low enough to ensure a negligible two photon occupation $t_W = 10/\kappa$, $\sigma_W = 2.5/\kappa$ and $\omega_W = \omega_c + (\Omega_1 + \Omega_2)/2$ which allows the simultaneous excitation of both Stokes sidebands. The cavity, mechanical, and detector average occupations $n_c(t) = \langle \hat{a}_c \rangle$, $n_{b_j}(t) = \langle \hat{b}_j \rangle$ and $n_d(t) = \langle \hat{d} \rangle$ are plotted in Fig. 2(a), assuming that the heralding photon has not been detected. The plot shows that, after the writing pulse, a small average occupation of the mechanical modes is produced through a Stokes Raman process, and then decays as a result of mechanical damping.

The second step of the writing procedure consists in heralding the formation of an entangled state, through the detection at time $t_P > t_W$ of a photon at the cavity frequency. This projects the full system onto the subspace where one photon is present in the detector mode, i.e.

$$\hat{\rho}_P = \frac{\hat{P}_1 \hat{\rho}(t_P)}{\text{tr}[\hat{P}_1 \hat{\rho}(t_P)]},$$

where $\hat{P}_1 = \hat{I}_{\hat{a}} \otimes \hat{I}_{\hat{b}_1} \otimes \hat{I}_{\hat{b}_2} \otimes |1_d\rangle \langle 1_d|$

$$\hat{P}_1 = \hat{I}_{\hat{a}} \otimes \hat{I}_{\hat{b}_1} \otimes \hat{I}_{\hat{b}_2} \otimes |1_d\rangle \langle 1_d|$$

Readout — Readout is accomplished by the second pulse that excites the two anti-Stokes sidebands. The pulse parameters are $A_R = 150\kappa$, $\sigma_R = 3.5/\kappa$, and $\omega_R = \omega_c - (\Omega_1 + \Omega_2)/2$, while the readout time $t_R$ is always set after the heralding time $t_P$. On a first analysis of the protocol, we set the readout pulse amplitude so that $\langle \hat{b}_j \rangle \ll 1$. This is to avoid the onset of a large, classical mechanical field amplitude that might produce the final interference pattern in the detector even in the absence of entanglement. Figs. 3(a),(b) show respectively the classical field and the corresponding quantum fluctuation intensities as a function of time for a typical readout process. The plots show that the classical components to the mechanical fields are negligible as compared to the fluctuation part originally created by the heralding. Same holds for the detector field. In Fig. 3(c), the intensity at the detector is plotted as a function of the real and readout times. After a first strong signal orig-
inating from the classical field created in the cavity by the readout pulse, the actual anti-Stokes signal is left, and a clear interference pattern is observed as a function of $t_R$, which is the signature of entanglement between the two mechanical modes. A similar interference pattern is present in the zero delay two-photon correlation $g^{(2)}(t) = \langle \hat{d} \hat{d} \hat{d} \hat{d} \rangle / \langle \hat{d} \hat{d} \rangle^2$ [35], highlighting the non-classicality associated with the interference. Fig. 3(d) shows the normalized interference pattern, taken along the oblique dashed line of panel (c). The intensity at the detector (red) has a visibility $V = 0.97$, while the intracavity field shows a lower visibility, $V = 0.82$, due to the fact that the projection (7) is carried out at the detector, thus leaving a finite component in the state with $n_c > 1$. The black curve shows the interference pattern at the detector when assuming a fully separable state as the initial condition of the readout phase. Its visibility $V \approx 0.5$ sets the lower bound for the existence of entanglement.

Fig. 3(b) shows that the rate of emitted photons after the readout pulse is very low. This, combined with the heralding rate, results in an extremely low success rate for the whole write-readout protocol. This rate would obviously increase with the magnitude of the optomechanical coupling. Another possibility consists in increasing the intensity of both write and readout pulses. For the write pulse, care must be taken to keep the amplitude of the components with two and higher number of mechanical quanta negligible, as compared to those of the mechanical states $|01\rangle$ and $|10\rangle$. When instead increasing the readout pulse intensity, a transient classical field amplitude of the mechanical modes will arise, exceeding the quantum fluctuation component. Our simulations however show [38] that this amplitude decays rapidly with the readout pulse, and the mechanical Bell state is eventually restored, similarly to recent experimental protocols on micro-macro entanglement [7].

**Temperature and pure dephasing.**— We have additionally studied the influence on the entanglement of a finite thermal occupation of the two mechanical modes. Both the visibility $V$ and the concurrence $C$ preserve values exceeding $0.7$ up to $\bar{n}_{th} = 0.1$, corresponding to a temperature of roughly $T \approx 4 \text{ mK}$. This temperature might be obtained by a cycle of laser cooling prior to the actual writing step, thanks to the well resolved sideband limit. [15]

Finally, entanglement can obviously be suppressed by decoherence processes. Here, we study the effect of pure dephasing acting on the optical degree of freedom. This can be included in the model by introducing a pure-dephasing term $-\eta/2D [\hat{a}^{\dagger} \hat{a}] \hat{\rho}$ to Eq. 2. The main source of pure dephasing is expected to be given by optomechanical coupling to other mechanical modes of the structure. The pure dephasing rate resulting from a single mechanical mode with frequency $\Omega$, mechanical damping $\gamma$, and zero-photon optomechanical coupling $g$ is estimated by $\frac{1}{11} \eta \sim (g/\Omega)^2 (2\bar{n}_{th} + 1)^2 \gamma$ [38]. Assuming $T = 4 \text{ K}$, and the optomechanical parameters measured in Ref. [20] for a photonic crystal cavity, an estimate of the total pure dephasing rate gives, as a conservative upper bound, $\eta < 10^{-7}\kappa$. In this range, the computed values of $C$ and $V$ stay well above 0.5, suggesting the robustness of this entanglement to pure dephasing mechanisms. An analysis of the entanglement as a function of $\bar{n}_{th}$ and $\eta$ is presented in Ref. [38].

**Conclusion.**— We have proposed a realistic protocol for the preparation and readout of mechanical Bell states in a silicon photonic crystal cavity. The cavity quality factor, mechanical mode frequencies, mechanical damping rates and optomechanical coupling strengths all correspond to experimentally demonstrated values for a state-of-the-art system of this kind. The protocol assumes optical cooling of the mechanical modes well within the accessible range, and entanglement is shown to survive the pure dephasing rate expected for this system. The present analysis thus represents a realistic and detailed proposal for an experiment where macroscopic entanglement of two acoustic vibrational modes of
a semiconductor microstructure would be produced and detected. An extension of the present analysis to more than two mechanical modes or to several write pulses, might indicate the way to the generation of more complex nonclassical states such as GHZ or NOON states.

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Supplemental Material

We detail here the semiclassical model deployed for readout procedure. We provide some additional information on the second order correlation function during readout, demonstrating the single photon emission, and on the impact of temperature and pure dephasing. Finally we discuss the potential improvement of the single photon emission rates.

READOUT SEMICLASSICAL MODEL

The readout procedure was modeled by distinguishing the quantum fluctuations $\delta \alpha$, $\delta \bar{d}$ and $\delta \bar{b}_j$ from the classical field amplitudes $\alpha = \langle \hat{a} \rangle$, $\delta = \langle \hat{d} \rangle$ and $\beta_j = \langle \hat{b}_j \rangle$. More specifically, the total fields are expressed as $\hat{a} = \alpha + \delta \hat{a}$, $\hat{d} = \delta + \delta \bar{d}$ and $\hat{b}_j = \beta_j + \delta \hat{b}_j$. This procedure allows modeling the effect of a large amplitude of the driving field, as assumed in the readout phase. An intense driving field typically induces large average occupations of the modes under study. Of these, only a small part is contributed from quantum fluctuations, while the largest contribution is accounted for by the classical-field-component, similarly to the textbook case of a displaced quantum oscillator. Hence, the separation into the two contributions makes the numerical analysis possible while still restricting the Hilbert space to a reasonably small number of fluctuation quanta. In such a displaced representation, where the classical fields act as the new vacuum, the system Hamiltonian reads

$$\hat{H} = \sum_{j=1,2} \left[ 2 g_j \text{Re} (\beta_j) \hat{a}^\dagger \hat{a} + \hbar \Omega_j \hat{b}_j^\dagger \hat{b}_j + g_j (\alpha^* \hat{a} + \alpha \hat{a}^\dagger) \left( \hat{b}_j^\dagger + \hat{b}_j \right) \right] + g_j \hat{a}^\dagger \hat{a} \left( \hat{b}_j^\dagger + \hat{b}_j \right)$$

(1)

where we have simplified the notations as $\delta \bar{d} \rightarrow \delta$. Note that contrary to the usual linearisation procedure we keep here terms of all orders in $\hat{H}$. The quantum fluctuations dynamics is governed by the master equation

$$\frac{d\hat{\rho}}{dt} = -i \left[ \hat{H}, \hat{\rho} \right] - \frac{\kappa}{2} D \left[ \hat{a} \right] \hat{\rho} - \frac{\kappa d}{2} D \left[ \hat{d} \right] \hat{\rho} - \tilde{n}_{th} \sum_j \frac{\gamma_j}{2} D \left[ \hat{b}_j \right] \hat{\rho} - \left( \tilde{n}_{th} + 1 \right) D \left[ \hat{b}_j^\dagger \right] \hat{\rho}$$

(2)

that is coupled the time-evolution equations for the classical field components

$$i \dot{\alpha} = -i \frac{\kappa}{2} \alpha + 2\alpha \sum_{j=1,2} g_j \text{Re} (\beta_j) + F \left( t \right)$$

(3)

$$i \dot{\beta}_j = \left( \hbar \Omega_j - i \frac{\gamma_j}{2} \right) \beta_j + g_j |\alpha|^2$$

(4)

$$i \dot{\delta} = -i \frac{\kappa d}{2} \delta + i \frac{\kappa}{2} \alpha \delta$$

(5)

We assume the projected mechanical density matrix $\hat{\rho}_m$, occurring in the write phase at the time where the concurrence is maximal [see Fig.2(b)], as the initial condition for the dynamics in the readout phase. In this framework, the total average occupation of the cavity mode for example is evaluated as $n_c = \langle \hat{a}^\dagger \hat{a} \rangle + |\alpha|^2$, and similar expressions hold for the other modes.

SECOND ORDER CORRELATIONS

We show in Fig.S1(a),(b) the second order correlation functions at zero delay versus time of the cavity and detector fields, respectively defined as

$$g_c^{(2)} (t,t) = \frac{\langle \hat{a}^\dagger \hat{a}^\dagger \hat{a} \hat{a} \rangle}{\langle \hat{a}^\dagger \hat{a} \rangle^2} \quad \text{and} \quad g_d^{(2)} (t,t) = \frac{\langle \hat{d}^\dagger \hat{d} \hat{d} \hat{d} \rangle}{\langle \hat{d}^\dagger \hat{d} \rangle^2}$$

(6)

along the oblique cut highlighted in Fig.3(c). Here, the operators encompass the total classical and fluctuation contributions. We have superimposed the normalized intensity (blue line) and we clearly see the strong antibunching obtained at the intensity maximum. This antibunching is the signature of the emission of single photons as a result of the anti-Stokes raman process that destroys the mechanical Bell state as a result of the readout pulse.

IMPACT OF TEMPERATURE AND PURE DEPHASING

We have analyzed the impact of a finite temperature on the concurrence of the entangled mechanical states and on the fringes visibility. The results are presented in Fig.S2(a). We obtain $C \simeq 0.7$ at $\tilde{n}_{th} = 0.1$, which correspond to typical temperatures attainable with sideband cooling [15]. We clearly see the correlation between the visibility and the concurrence versus $\tilde{n}_{th}$. Indeed, the contribution of the mechanical Fock states with more than one quantum increases with temperature. It results
in a larger matrix element of the $|11\rangle\langle 11|$ component of $\rho_m$ which, as suggested in the main text, reduces entanglement and the fringe visibility.

Finally, in Ref. [26], at least four extra mechanical modes where shown to exhibit a non-negligible optomechanical coupling to the cavity. These modes form side-bands that are spectrally unresolved from the main cavity line, and induce pure dephasing of the cavity mode. This can be taken into account adding a pure dephasing Lindblad term $-\eta/2D [\hat{a}^\dagger \hat{a}] \hat{\rho}$ to the master equation. An estimation of $\eta$ can be made [11], accounting for the collective effect of all other mechanical modes, according to

$$\eta = \sum_j (\bar{n}_{j,th} + 1) \gamma_j \frac{g_j^2}{\Omega_j^2}$$  \hspace{1cm} (7)

where $j$ runs over all extra mechanical modes and $\bar{n}_{j,th} = 1/\exp(h\Omega_j/k_B T) - 1$. Assuming an average mechanical quality factor of $Q = 400$ and the frequency and optomechanical coupling parameters measured in Ref. [26], we obtain an upper boundary as small as $\eta \simeq 10^{-7}$ for a bath temperature of $T = 1K$. The impact on the concurrence and visibility are reported in Fig. S2(b) for $\eta$ spanning several decades. We see that the concurrence drops faster than the fringes visibility. Indeed, pure dephasing mostly affects the photon/phonon correlations and therefore the projection efficiency during the write procedure. It translates into an increase in the $|00\rangle\langle 00|$ element of the reduced mechanical density matrix $\rho_m$ after the heralding.

**EMISSION RATES IMPROVEMENT**

Here, we discuss possible ways to increase the rate of successful heralding/readout events in the protocol. The discussion is developed separately for the write and readout phases:

**Write** — In Fig. 2(a),(b) we have seen that during the writing procedure, the maximum concurrence is obtained at a time where the average cavity occupation is of the order of $10^{-7}$, which in turn sets the heralding efficiency. The heralding rate could be improved with a larger optomechanical coupling (which would induce a faster cavity/mechanical correlation time) or by increasing the intensity of the write pulse. In the latter case, one has to keep in mind that a limit is set by the requirement that the density matrix elements relative to states with two or more mechanical quanta be negligible compared to the elements relative to states with zero or one quantum. The parameters used in our simulations correspond to a situation largely within this boundary, and therefore leave considerable room for improvement.

A third possibility is to compromise to a lower value of the concurrence. As seen in Figs.2(a) and (b) of the main manuscript, if for example a concurrence $C = 0.1$ is accepted, such a value occurs at earlier heralding times, where the average cavity occupation is about $2 \times 10^{-8}$. This is highlighted in the Fig. S3(a) showing the cavity occupation $n_c t$ versus $C(t)$.

**Readout** — In Fig. 3 (a) and (b) of the main manuscript, the emission rate of single readout photons in the region where the fluctuations are dominating, turns out to be of the order of $10^{-9}$, i.e. corresponding to an exceedingly low readout rate when combined to the heralding rate. The associated parameters where chosen in order to keep the classical contribution to the phonon fields negligible. We have however investigated larger
pump amplitude regimes where the classical contribution to the phonon fields dominates over the fluctuations. In Figs. S3 (b), (c), and (d) we have used a pump amplitude of $F_R = 5000\kappa$ while all the other parameters where left unchanged. The interference fringes shown in panel (b) are still marked and the fluctuation contribution to the average occupations (panel (d)) reaches $2 \times 10^{-4}$. We have verified that, while a large classical component to the mechanical fields develops (see panel (c)), the net effect of such component is simply to displace the mechanical Bell state temporarily. Once the readout pulse has decayed, this classical component also vanishes and the mechanical mode evolves back into the Bell state. Note finally that, for such strong readout field intensity, the system approaches the strong optomechanical coupling regime, corresponding to $g_j\alpha \sim \kappa$, and mechanical backaction on the cavity field becomes sizeable, as one can see in the panel (d).