Topological String, Matrix Integral, 
and Singularity Theory

T. Nakatsu,\textsuperscript{1}† A. Kato,\textsuperscript{2} M. Noumi\textsuperscript{2} and T. Takebe\textsuperscript{3}

\textsuperscript{1}Research Institute for Mathematical Sciences, Kyoto University 
Sakyo-ku, Kyoto 606, Japan
\textsuperscript{2}Department of Mathematical Sciences, University of Tokyo 
Komaba, Meguro-ku, Tokyo 153, Japan
\textsuperscript{3}Department of Mathematical Sciences, University of Tokyo 
Hongo, Bunkyo-ku, Tokyo 113, Japan

ABSTRACT

We study the relation between topological string theory and singularity theory 
using the partition function of $A_{N-1}$ topological string defined by matrix integral of 
Kontsevich type. Genus expansion of the free energy is considered, and the genus 
g = 0 contribution is shown to be described by a special solution of \textit{N}-reduced 
dispersionless KP system. We show a universal correspondences between the time 
variables of dispersionless KP hierarchy and the flat coordinates associated with 
versal deformations of simple singularities of type $A$. We also study the behavior 
of topological matter theory on the sphere in a topological gravity background, to 
clarify the role of the topological string in the singularity theory. Finally we make 
some comment on gravitational phase transition.

\textsuperscript{*} Partly supported by the Grant-in-Aid for Scientific Research from the Ministry 
of Education, Science and Culture (No.04-2597, No.05854014, No.05740083).
\textsuperscript{†} A Fellow of the Japan Society of the Promotion of Science for Japanese Junior 
Scientists.
Recently much attention has been paid to the understanding of topological strings in terms of Gauss-Manin systems associated with versal deformations of simple singularities \cite{1},\cite{2}. At present, however, it is not yet clear what is the interrelation between the Gauss-Manin structure and the integrable hierarchy \cite{3},\cite{4} which also plays an important role in topological strings.

In this letter, by using a more fundamental framework, we describe the relation between topological string and singularity theory. We study $A_{N-1}$ topological string which is defined by matrix integral of Kontsevich type. By splitting its free energy into the contribution of each genus, the genus $g=0$ contribution is shown to be described by a special solution of $N$-reduced dispersionless KP system. We also establish a universal correspondence between the time variables of dispersionless KP hierarchy and the flat coordinates associated with versal deformations of simple singularities of type $A$. The combination of these two results clarifies the relation between the $A_{N-1}$ topological string and a simple singularity of type $A_{N-1}$. In order to search the role of the topological string in the singularity theory, we study the behavior of the gravitational primary fields on the sphere ($g=0$) in a fixed topological gravity background. The possibility of gravitational phase transition of the matter theory is strongly suggested.

In this letter we will describe our results briefly. Their derivations will be given in a separate publication \cite{5}.

Let us begin by some conceptual observations on $A_{N-1}$ topological string. The $A_{N-1}$ topological string is the coupled system of $A_{N-1}$ topological minimal matter and topological gravity \cite{6}. The physical observables are denoted by $\sigma_{nN+m}$ ($n \in \mathbb{Z}_{\geq 0}, 1 \leq m \leq N-1$). $\sigma_1 = P, \sigma_2, \ldots, \sigma_{N-1}$ are gravitational primary fields and $\sigma_{nN+m}$ are the $n$-th gravitational descendants of $\sigma_m$.

The free energy $F$ of the $A_{N-1}$ topological string should be given by:

$$F(\{t_k\}) = \sum_{\{k_i\},\{d_i\},s} \langle \sigma_{k_1}^{d_1} \cdots \sigma_{k_s}^{d_s} \rangle \frac{t_{k_1}^{d_1} \cdots t_{k_s}^{d_s}}{d_1! \cdots d_s!},$$

(1)

where the correlation function $\langle \sigma_{k_1}^{d_1} \cdots \sigma_{k_s}^{d_s} \rangle$ should be described by the intersection theory on the compactified moduli space $\overline{M}_{g,s}$ of Riemann surface $\Sigma$. 

2
of genus \( g \) with \( s \) punctures \( \mathbb{1} \), where \( g \) is fixed by the relation:

\[
2(g - 1)(N + 1) = \sum_{i=1}^{s} d_i \{ k_i - (N + 1) \}.
\]  

(2)

This is nothing but the ghost number conservation law for the system.

Now we define the free energy \( \mathcal{F}(\{t_k\}) \) in eq (I) by the following matrix integral of Kontsevich type:

\[
\begin{align*}
&\quad \mathcal{F}(\{t_k\}) := \lim_{M \to \infty} Z_N^{(M)}(\Lambda), \\
&= \int dX \exp \left[ - \text{Tr} \left\{ \frac{(X + \Lambda)^{N+1}}{N+1} - \frac{\Lambda^{N+1}}{N+1} + \Lambda^N X \right\} \right],
\end{align*}
\]

(3)

\[
Z_N^{(M)}(\Lambda) := \int dX \exp \left[ - \sum_{k=0}^{N-1} \text{Tr} \left\{ \Lambda^k X \Lambda^{N-1-k} X \right\} \right],
\]

(4)

where \( X \) is a \( M \times M \) hermitian matrix and \( \Lambda = \text{diag} (\lambda_1, \ldots, \lambda_M) \). The parameters \( t_k \) are related with \( \Lambda \) by \( t_k = \frac{1}{k} \text{Tr} \Lambda^{-k} \), which can be treated as independent variables when \( M \) is sufficiently large. The matrix integral in (I) is studied in \( \mathbb{3} \) \( (N = 2 \) case) and in \( \mathbb{4} \) \( (N \geq 2 \) case). Especially they showed that \( e^\mathcal{F} \) is a \( \tau \)-function of \( N \)-reduced KP system and that, after an appropriate shift of the marginal parameter \( t_{N+1} \), it satisfies the following Virasoro condition \( \mathbb{7} \) :

\[
0 = \mathcal{L}_n e^{\mathcal{F}(\{t_k\})} = \sum_{k \geq 1} k t_k \frac{\partial}{\partial t_{k+Nn}} e^{\mathcal{F}(\{t_k\})} + \frac{1}{2} \sum_{k+l=n} \frac{\partial}{\partial t_k} \frac{\partial}{\partial t_l} e^{\mathcal{F}(\{t_k\})}
\]

\[
+ \frac{1}{2} \sum_{k+l=N} k l t_k t_l e^{\mathcal{F}(\{t_k\})} \delta_{n+1,0} + \frac{N^2 - 1}{24} e^{\mathcal{F}(\{t_k\})} \delta_{n,0}
\]

(5)

where \( n \geq -1 \).

Let us combine the above result with our first observation. By applying the relation (2) to (I) we can split \( \mathcal{F}(\{t_k\}) \) in eq (3) into the contribution from each genus;

\[
\mathcal{F}(\{t_k\}) = \sum_{g=0}^{\infty} \mathcal{F}_g(\{t_k\}),
\]

(6)
where $\mathcal{F}_g$ is the free energy of the $A_{N-1}$ topological string on Riemann surface of genus $g$. The Virasoro constraints (3) on $\mathcal{F}$ give us the nonlinear coupled conditions on each $\mathcal{F}_g$;

$$0 = c^{(n)}_g(\mathcal{F}_{g}, \mathcal{F}_{g-1}, \ldots, \mathcal{F}_0; \{t_k\}), \quad (7)$$

where $n \geq -1$ and $g \geq 0$.

For example, the constraints $c^{(n)}_{g=0,1}$ read as follows;

$$0 = c^{(n)}_{g=0} = \sum_{k \geq 1} k t_k \frac{\partial F_0}{\partial t_{k+Nn}} + \frac{1}{2} \sum_{k+l=nN} \frac{\partial F_0}{\partial t_k} \frac{\partial F_0}{\partial t_l} + \frac{1}{2} \sum_{k+l=N} k t_l t_{n+1,0}$$

$$0 = c^{(n)}_{g=1} = \sum_{k \geq 1} k t_k \frac{\partial F_0}{\partial t_{k+Nn}} + \sum_{k+l=nN} \frac{\partial F_0}{\partial t_k} \frac{\partial F_1}{\partial t_l} + \frac{1}{2} \sum_{k+l=N} \frac{\partial^2 F_0}{\partial t_k \partial t_l} + \frac{N^2 - 1}{24} \delta_{n,0} \quad (8)$$

Let us study the constraints (8) on $\mathcal{F}_{g=0}$. It is important to note that these constraints characterize a $\tau$-function of $N$-reduced dispersionless KP system [8].

Dispersionless KP hierarchy is a “quasi-classical” ($\hbar \to 0$) limit of KP hierarchy in the following substitution;

$$[\hbar \partial_x, x] = \hbar \to \{p, x\} = 1, \quad (10)$$

where the Poisson bracket is defined by

$$\{f, g\} = \frac{\partial f}{\partial p} \frac{\partial g}{\partial x} - \frac{\partial f}{\partial x} \frac{\partial g}{\partial p} \quad (11)$$

In the paper [8] dispersionless KP hierarchy is formulated as an analogue of Orlov’s improved Lax formalism of KP hierarchy [10]. The former is given by the following system of equations;

$$\frac{\partial L}{\partial t_n} = \{B_n, L\}, \quad (n \geq 1)$$

$$\frac{\partial M}{\partial t_n} = \{B_n, M\}, \quad (n \geq 1)$$

$$\{L, M\} = 1, \quad (12)$$

4
where $L, M$ are defined by

\[
L := p + \sum_{i=1}^{\infty} u_{i+1}(t)p^{-i},
\]

\[
M := L + \sum_{i=1}^{\infty} it_i L^{i-1} + \sum_{i=1}^{\infty} v_{i+1}(t) L^{-i-1}
\]  
(13)

and $B_n$ are given as truncation to positive powers of $p$;

\[
B_n := (L^n)_+.
\]  
(14)

The solution of the equations (12) is completely characterized by a Riemann-Hilbert problem related to the area preserving diffeomorphism on two plane \[8\]. Especially the following constraints specify an unique solution $(L, M)$ of the equations (12);

\[
(P^n Q^m)_- = 0 \quad (n \geq -1, m \geq 0)
\]  
(15)

where

\[
P = \frac{1}{N} L^N, \quad Q = L^{1-N} M.
\]  
(16)

In the case of $m = 0$ the condition (15) are equivalent to $(L^N)_- = 0$, that is, $N$-reduction of the system. What about the case of $m \geq 1$ ? Let us introduce $\tau^{d.1.KP} (\{t_k\})$, a $\tau$-function of dispersionless KP hierarchy \[8\], through the relation

\[
v_{n+1} = \frac{\partial \log \tau^{d.1.KP}}{\partial t_n} \quad (n \geq 1).
\]  
(17)

Then we can show that in the case of $m = 1$ the constraints (15) on $\log \tau^{d.1.KP}$ precisely coincide with $c_{g=0}^{(n)} = 0 \quad (n \geq -1)$ constraints on $\mathcal{F}_{g=0}$ given in eq (8). Hence the following relation holds;

\[
\mathcal{F}_{g=0} = \log \tau^{d.1.KP},
\]  
(18)

where the $\tau$-function $\tau^{d.1.KP}$ is the solution of (17).

At this stage let us give some comments.

1. The condition (15) in the case of $m \geq 2$ are equal to those on $\mathcal{F}_{g=0}$ obtained by the genus expansion (8) from the $W$-constraints which $\mathcal{F}$ defined in eq (4) satisfies [4].
2. By comparing two equations (2) and (10), the genus expansion is nothing but the $\bar{h}$ expansion;

$$F = \sum_{g=0}^{\infty} \bar{h}^{\chi_g} F_g$$

(19)

where $\chi_g = 2(g - 1)$ is the Euler characteristic of the genus $g$ Riemann surface.

3. By using the technique of dispersionless KP hierarchy, we can show that $c_{g=0}^{(n=1)}$ implies the recursion relation on punctured sphere [11]:

$$\frac{\partial^3 F_0}{\partial t_m \partial t_i \partial t_j} = \frac{1}{1(N - l)} \frac{\partial^2 F_0}{\partial t_{m-N} \partial t_l \partial t_{N-i} \partial t_j},$$

(20)

where $m \geq N + 1$.

4. By the identification (18), we can check that the following form of $F_{g=1}$ is the solution of $c_{g=1}^{(n)} = 0$ (see eq (9));

$$F_{g=1} = \frac{1}{24} \log \left[ \det \left( \frac{\partial^3 F_0}{\partial t_i \partial t_j} \right) \right],$$

(21)

which is exactly the conjectured formula (up to an additive constant) given in [11].

In order to proceed further, let us change our perspective. We shall study the perturbed $A_{N-1}$ topological minimal matter on the sphere in the presence of constant background of gravity descendants. Namely we treat the gravitational descendants as external fields with some fixed strength. Let $\alpha = (\alpha_{N+1}, \alpha_{N+2}, \cdots)$ be a sequence of constants where only a finite number of $\alpha_i$ are non zero. We introduce the following $(N - 1)$ dimensional subspace $V_\alpha$ of the infinite dimensional parameter space (full phase space);

$$V_\alpha = \{(t_1, t_2, \cdots, t_{N-1}, t_{N+1} = \alpha_{N+1}, t_{N+2} = \alpha_{N+2}, \cdots)\},$$

(22)

where $t_k (k \geq N + 1)$ are constrained to the value $\alpha_k$. The conventional small phase space of the perturbed $A_{N-1}$ topological minimal matter on the sphere [12] is given by $V_{\alpha_0}$ with $\alpha_0 := (\frac{1}{N+1}, 0, 0, \cdots)$ [8, 9].

1 The derivation of the above recursion relation in the framework of dispersionless KP hierarchy was first given by Prof. K. Takasaki.
We shall take the following Legendre transformed variables:

\[
q^{(\alpha)}_{i+1} := -\frac{1}{i} \left. \frac{\partial^2 F_{g=0}}{\partial t_i \partial t_i} \right|_{V_{\alpha}} \quad (1 \leq i \leq N - 1). \tag{23}
\]

Notice that, via \(0 = C^{(n-1)}_{g=0} \) \[^{[9]}\], we can write down \(q^{(\alpha)}_{i+1}\) perturbatively in terms of \(t_1, \ldots, t_{N-1}\). Especially, on \(V_{\alpha_0}\), \(q_{i+1} := q^{(\alpha_0)}_{i+1}\) get the following forms:

\[
q_{i+1} = (N - i)t_{N-i} \quad (1 \leq i \leq N - 1), \tag{24}
\]

that is, \(q^{(\alpha)}_{i+1}\) reduce to the time variables. So we may take \(q^{(\alpha)}\) as the coordinates of the subspace \(\alpha \sim \alpha_0\).

What is the relation between the physics on \(V_{\alpha_0}\) and on \(V_{\alpha \neq \alpha_0}\)? Because of (23) and (24) it may be measured by the Jacobian associated with the change of coordinates \(q^{(\alpha_0)} \rightarrow q^{(\alpha)}\), det \(\left. \frac{\partial q^{(\alpha)}_{i+1}}{\partial t_j}\right|_{1 \leq i,j \leq N-1}\), that is,

\[
\det \left. \left( \frac{\partial^3 F_{g=0}}{\partial t_i \partial t_i \partial t_j} \right) \right|_{1 \leq i,j \leq N-1} = e^{24F_{g=0}} \Big|_{V_{\alpha}}, \tag{25}
\]

where we use the relation (21). The metric \(\eta\) of the perturbed \(A_{N-1}\) topological minimal matter is given by \(\eta_{ij} := \left. \frac{\partial^3 F_{g=0}}{\partial t_i \partial t_i \partial t_j} \right|_{V_{\alpha}}\) and it is deformed as \(\alpha\) changes from \(\alpha_0\) into \(\alpha_1\). Let us assume that the R.H.S of (25) vanishes at \(V_{\alpha_1}\). Then, finally at \(\alpha = \alpha_1\), the deformed metric \(\eta_{ij}\) becomes degenerate. This phenomena can be described as “gravitational phase transition” of the perturbed \(A_{N-1}\) topological minimal matter, which may be interpreted as a simple example of “BRST symmetry breaking” introduced in \[^{[13]}\]. We also note, because of the relation (25), the \(\tau\)-function \[^{[4]}\] becomes zero at \(\alpha_1\);

\[
e^F \big|_{V_{\alpha_1}} = 0. \tag{26}
\]

This property reflects the analyticity of the wave function \[^{[14]}\] associated with \(e^F \) \[^{[4]}\], which predicts the above scenario.

It may be helpful to give the following remarks.

In topological field theories, the physical Hilbert spaces are finite dimensional, and it appears strange to speak of their “phase transition” which is,
as is often explained, the cooperative phenomena of infinitely many degrees of freedom. However, we all know that mean field theories, although correlation effect are totally neglected, can explain the phase transition in a very simple manner. Let us take Ising model, for example. The free energy of the system is a function of the external magnetic field and the temperature. The convexity of the free energy enables us to express the free energy in terms of the magnetization. The phase transition occurs at the point where this Legendre transformation becomes singular. This is exactly what is happening in the topological field theory coupled to topological gravity. In the case of minimal topological matter theory without gravity, free energy is a simple polynomial and no phase transition occurs. In this sense, it may be called “gravitational phase transition”.

The perturbed $A_{N-1}$ topological minimal matter on $V_{\alpha}$ (on the sphere) is known to be deeply related with a versal deformation of a simple singularity of type $A_{N-1}$ [12]. Let us investigate this relation in our framework. Up to now, the perturbed $A_{N-1}$ topological minimal matter on $V_{\alpha}$ is described using the terminology of $N$-reduced dispersionless KP system. In order to step into deformations of $A_{N-1}$ simple singularity, we should make clear the correspondence between these two objects. For this purpose we introduce a concept of hierarchy into simple singularities of type $A$ [13], which will build a bridge between them.

Consider a formal Laurent series;

$$L(p) := p + \sum_{i \geq 1} u_{i+1} p^{-i}, \quad (27)$$

where $u = (u_2, u_3, \cdots)$ is a sequence of countably many variables. Then one can find a unique Laurent series;

$$p(L) := L + \sum_{i \geq 1} q_{i+1} L^{-i}, \quad (28)$$

such that $p \circ L(y) = y$ and $L \circ p(y) = y$. Hence we obtain a new sequence $q = (q_2, q_3, \cdots)$ of countably many variables. It should be noted that any time evolution such as (13) is not yet assumed in the Laurent series $L$ in eq (27). Define the following set of polynomials;

$$\phi_i(p) := \frac{1}{i + 1} \partial_p (L^{N+1})_+ \quad (i \geq 0), \quad (29)$$
which satisfies;

\[
\operatorname{res} \left\{ \phi_i \phi_j \frac{dp}{\phi_k} \right\} = \delta_{i+j,k-1} \quad (0 \leq i, j \leq k - 1).
\] (30)

Introduce their generating function by

\[
\phi(p; \lambda) := \sum_{i \geq 0} \phi_i(p) \lambda^i,
\] (31)

and then we can see that it satisfies the fundamental relation [15];

\[
\lambda \mu \operatorname{res} \left\{ \frac{\phi(\lambda) \phi(\mu) L}{\partial p L} dp \right\} = (\lambda \partial_\lambda + \mu \partial_\mu) \log \left( 1 - \lambda \mu \sum_{n \geq 2} q_n \frac{\lambda^{n-1} - \mu^{n-1}}{\lambda - \mu} \right).
\] (32)

On the other hand, in dispersionless KP hierarchy, any \( \tau \)-function (17) enjoys the same property;

\[
\sum_{n,m \geq 1} \lambda^n \mu^m \frac{\partial_{t_n} \partial_{t_m}}{n \ m} \log \tau^{d.l.KP}
\]

\[
= \log \left( 1 + \lambda \mu \sum_{n \geq 1} \frac{\lambda^n - \mu^n}{\lambda - \mu} \frac{\partial_{t_n} \partial_{t_1}}{n} \log \tau^{d.l.KP} \right),
\] (33)

which is the “quasi-classical” limit of the corresponding relation in KP hierarchy. By comparing (32) with (33) we are naturally lead to establish the universal correspondences;

\[
q_{n+1} = -\frac{1}{n} \partial_{t_n} \partial_{t_1} \log \tau^{d.l.KP}
\] (34)

\[
\operatorname{res} \left\{ \phi_{m-1} \phi_{n-1} L \frac{dp}{\partial_p L} \right\} = \frac{mn}{m+n} \partial_{t_m} \partial_{t_m} \log \tau^{d.l.KP}
\] (35)

Notice that these correspondences hold for arbitrary \( \tau \)-function of dispersionless KP hierarchy (independent of \( N \)). Any \( u = (u_1, u_2, \cdots) \) in (27) can be parameterized by \( t = (t_1, t_2, \cdots) \) (KP-times) with choosing a \( \tau \)-function which satisfies (34), (35). Of course we can say it in the reverse order. That
is, we can get a \( \tau \) function of dispersionless KP hierarchy by imposing the conditions (34),(35) on \( u \) and \( t \).

With the above correspondences (34),(35) we can easily clarify the relation between the perturbed \( A_{N-1} \) topological minimal matter on \( V_{\alpha_0} \) and a versal deformation of a simple singularity of type \( A_{N-1} \). Impose the following condition on \( L(p) \) (27);

\[
L^N = (L^N)_+ = p^N + a_2p^{N-2} + \cdots + a_{N-1}p + a_N, \tag{36}
\]

which can be regarded as a versal deformation of the singularity \( p^N = 0 \). With this constraint \( q = (q_2, q_3, \cdots) \) in (28) are weighted homogeneous polynomials with respect to \( a_2, \cdots, a_N \). Especially \( y = (y_2, \cdots, y_N) \), the flat coordinates [16] of this versal deformation, can be taken as [15];

\[
y_i = -Nq_i \quad (2 \leq i \leq N). \tag{37}
\]

On \( V_{\alpha_0} \), applying the correspondence (34) to (18), this flat coordinate system can be rephrased as;

\[
y_i = -N(N+1-i)t_{N+1-i}. \tag{38}
\]

Notice that, by combining the relations (34),(37), we can describe the perturbed \( A_{N-1} \) minimal matter on \( V_{\alpha \neq \alpha_0} \) in terms of singularity theory [5].

In general, a flat coordinate system was introduced as a special coordinate system in order to study the Gauss-Manin system associated with a versal deformation of an isolated singularity [17]. A detailed study of these Gauss-Manin systems is given in [18]. What is the relation between the \( A_{N-1} \) type Gauss-Manin system and the \( A_{N-1} \) topological string? It may shed some insights on topological string theory. Via the correspondences (34), (35) we may extend the \( A_{N-1} \) Gauss-Manin system in terms of topological string. The discriminant \( \Delta \) of (37);

\[
\Delta := \det \left( \text{res} \left\{ \frac{\phi_i \phi_{N-2-j}L}{\partial_pL}\frac{dp}{dp} \right\} \right)_{0 \leq i,j \leq N-2} \tag{39}
\]

satisfies the equations [15];

\[
\theta_k(\Delta) = \frac{\partial^k}{\partial y_k} \Delta \quad (2 \leq k \leq N), \tag{40}
\]
where $\theta_k$ ($2 \leq k \leq N$) are the logarithmic vector fields given by:

$$
\theta_k := \sum_{i=2}^{N} \text{res} \left\{ \phi_{N-i} \phi_{i-2}L \frac{\partial}{\partial y_k} \right\} \frac{\partial}{\partial y_k}, \quad (41)
$$

and $\hat{\tau}$ is

$$
\hat{\tau} := \sum_{i=0}^{N-2} \text{res} \left\{ \phi_i \phi_{N-2-i}L \frac{\partial}{\partial y_k} \right\}. \quad (42)
$$

We expect that these quantities have meaning in the $A_{N-1}$ topological string theory through the correspondences (34), (35) and (37).

We would like to thank Prof. K. Takasaki, Prof. T. Shiota, and Dr. A. Nagai for useful discussions.

References

[1] A. Lossev, “Descendants constructed from matter field in topological Landau-Ginzburg theories coupled to topological gravity”, IPI-MINN-92-40-T (1992).

[2] T. Eguchi, H. Kanno, Y. Yamada and S. -K. Yang, Phys. Lett. B305 (1993) 235.

[3] M. L. Kontsevich, Comm. Math. Phys. 147 (1992) 1.

[4] S. Kharchev, A. Marshakov, A. Mironov, A. Morozov and A. Zabrodin, Nucl. Phys. B380 (1991) 181; C. Itzkson and J. B. Zuber, Int. Journ. Mod. Phys. A7 (1992) 5661; S. Kharchev, A. Marshakov, A. Mironov and A. Morozov, Mod. Phys. Lett. A8 (1993) 1047.

[5] A. Kato, T. Nakatsu, M. Noumi and T. Takabe, in preparation.

[6] E. Witten, Nucl. Phys. B371 (1992) 191.

[7] R. Dijkgraaf, E. Verlinde and H. Verlinde, Nucl. Phys. B348 (1991) 435; M. Fukuma, H. Kawai and R. Nakayama, Int. Journ. Mod. Phys. A6 (1992) 1385.
[8] K. Takasaki and T. Takebe, Int. Journ. Mod. Phys. A7. suppl. 1B (1992) 889; “Quasiclassical Limit of KP Hierarchy, W-Symmetries and Free Fermions”, preprint KUCP-0050/92, [hep-th/9207081].

[9] I. M. Krichever, Comm. Math. Phys. 143 (1992) 415.

[10] P. G. Grinevich and A. Yu. Orlov, in “Problems of Modern Quantum Field Theory”, Springer-Verlag (1989).

[11] R. Dijkgraaf and E. Witten, Nucl. Phys. B342 (1992) 486.

[12] R. Dijkgraaf, E. Verlinde and H. Verlinde, Nucl. Phys. B352 (1991) 59.

[13] E. Witten, Comm. Math. Phys. 118 (1988) 411.

[14] T. Shiota, private communication.

[15] S. Ishiura and M. Noumi, Proc. Japan. Acad. 58 (1982) 13, 62; “Gauss-Manin systems of type A” RIMS koukyuroku 459 (1982) 29 (in Japanese).

[16] K. Saito, T. Yano and J. Sekiguchi, Comm. Algebra. 8 (1980) 373.

[17] K. Saito, J. Fac. Sci. Uni. Tokyo. Sec. IA. 28 (1982) 775.

[18] M. Noumi, Tokyo. J. Math. 7 (1984) 1.