Possible spin liquid states with parton Fermi surfaces in the SU(3) ring-exchange model on the triangular lattice

Hsin-Hua Lai
National High Magnetic Field Laboratory, Florida State University, Tallahassee, Florida 32310, USA
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We consider a SU(3) ring-exchange model on a triangular lattice. Unlike the SU(2) case, under perturbation expansion of the SU(3) Hubbard model, the three-site ring exchange is present as well as the usual four-site ring exchange. Interestingly, the three-site ring exchange differs from the usual two-site and four-site exchanges by a minus sign and is ferromagnetic. We first present numerical site-factorized state studies on this model which shows a three-sublatticed order phase and a ferromagnetic phase. We further study the model using slave-fermion mean field in which we rewrite the exchange operators in terms of three flavors of fermions. We find the main competing trial states are the trimer state (triangular plaquette state) and the gapless U(1) spin liquid states with parton Fermi surfaces which include both the uniform zero-flux spin liquid state, and the uniform π-flux spin liquid state. Furthermore, we find there are possible pairing instabilities of the zero-flux (Fermi surface) spin liquid state toward a f-wave gapless (nodal) spin liquid state and the π-flux (Fermi surface) spin liquid state toward an interesting exotic s-wave “gapless” spin liquid state with two flavors of fermions paired up while one flavor of fermions remains gapless.

I. INTRODUCTION

In recent years, the cold atomic systems have become a powerful tool to realize strongly-correlated systems, including N-flavor Hubbard model on different lattices. Such systems consisting of several flavors of interacting fermions can be realized as different hyperfine states of alkali atoms or nuclear spin states of ytterbium or alkaline-earth atoms. A model Hamiltonian to describe such systems is the N-flavor fermionic Hubbard model,\(^{16,22}\)

\[
H = -t \sum_{\langle jk \rangle} \sum_{\alpha} \left[ c_{j\alpha}^\dagger c_{k\alpha} + \text{H.c.} \right] + U \sum_{j} n_{j\alpha} n_{j\beta},
\]

where \(\alpha, \beta\) run over the different flavors, \(\langle jk \rangle\) runs over pairs of nearest neighbors on the lattice, and \(j\) runs over all lattice sites.

The most common case is when \(N = 2\), which corresponds to the usual spin-1/2 Hubbard model. If we focus on the half-filling condition, i.e., when each site is occupied by exactly one fermion, the system undergoes metal-to-Mott insulator phase transition for sufficiently large repulsion \(U\). In experiments on cold atoms, the Mott insulating state of fermions have recently been observed at the low temperature compared with the Fermi temperature.\(^{9,10}\) In the large \(U\) regime, it is generally accepted that the ground state is well-captured by the usual anti-ferromagnetic (AFM) Heisenberg model and the charge degrees of freedom are completely frozen out. However, in the regime where \(U\) is not sufficiently large compared with the hopping strength \(t\), the ground state is not well-understood. It is possible that the strong charge fluctuations play important role for stabilizing “gapless” spin liquid phases in this regime.\(^{11–13}\)

For the more general case with \(N > 2\),\(^{14–22}\) if we focus on certain fillings, the Mott insulating states also emerge. In this case, the spin order is not understood even in the large \(U\) limit in which we can only focus on the Heisenberg-like (two-site exchange) Hamiltonian

\[
H_2 = J \sum_{\langle jk \rangle} P_{jk},
\]

where \(P_{jk}\) is so-called two-site exchange operator, which permutes the fermions between two nearest-neighbor sites as \(P_{jk}(\alpha, \beta) = |\delta, \alpha\rangle\), where the \(\alpha, \beta\) represent the spin states at sites \(j\) and \(k\). For \(N = 3\), there has been numerical evidence on such SU(3) Heisenberg model on a triangular lattice suggesting three-sublattice ordered ground state in this regime.\(^{26}\)

The situation becomes more complex if \(t/U \sim O(1)\), since in this regime the two-site exchange Hamiltonian is not sufficient to capture the essential physics and higher ordered contributions such as ring exchanges should be taken into account. In this paper, we focus on this regime and consider the SU(3) ring-exchange model on the triangular lattice. Motivated by the perturbative studies of the SU(3) Hubbard model at 1/3 filling, we include the “FM” three-site ring exchanges and the “AFM” four-site ring exchanges in the model whose Hamiltonian is given in Eq. \([3]\). With the interplay between the ring exchanges, we conjecture that the Quantum spin liquid (QSL) state can arise due to the strong frustration.

In this work, we first study the ordered phases using the site-factorized ansatz.\(^{26}\) We find that the phase diagram contains the three-sublattice ordered phase, similar to the phase found in Ref. \([23]\) and the FM state. We further study this model using slave-fermion trial states. After performing numerical full optimization of the trial energy, we consider the three ansatz states such as the uniform zero-flux and π-flux gapless (Fermi-surface) spin liquid state and the trimer (plaquette) state. Furthermore, we consider several pairing instabilities and find that there is a pairing instability of the zero-flux spin liquid state toward a f-wave (nodal) spin liquid state; there is a pairing instability of the uniform π-flux spin liquid state toward an interesting exotic s-wave spin liquid state with two flavors of fermions paired up while one flavor of fermions remains gapless.

The paper is organized as follows. In Sec. \([II]\) we define explicitly the model Hamiltonian we will study in the paper. In
Sec. II A we use the site-factorized ansatz to study the ordered states in this ring-exchange model. In Sec. II B we use the slave-fermion representation to rephrase the SU(3) Hamiltonian in terms of the three-flavor fermionic Hamiltonian and perform the fermionic mean-field treatment of the model. We further study the possible pairing instability in the spin liquids regime. In Sec. III we conclude with some discussions.

II. SU(3) MODEL WITH RING EXCHANGE TERMS

The model Hamiltonian we consider is

\[ H_{SU(3)} = J \sum_{\langle \tau \tau \rangle} P_{12} - K_3 \sum_{\langle \tau \tau \rangle} [P_{123} + \text{H.c.}] + K_4 \sum_{\langle \tau \tau \rangle} [P_{1234} + \text{H.c.}], \]

with \( \tau \) running over all the bonds on the lattice; \( P_{123} \) running over all the triangles, up- and down-triangles, on the lattice, while \( \langle \tau \tau \rangle \) are the sites on the triangles labeled counterclockwise; \( P_{1234} \) running over all the rhombi (for one site, there are three rhombi associated with it—blue, green and yellow shaded rhombi, see Fig. 1) and \( \langle \tau \tau \rangle \) are the sites on a rhombus labeled counterclockwise. \( P_{12} \) is the nearest-neighbor two-site exchange operator, \( P_{123} \) is the three-site spin ring exchange operator, which permutes the fermions on the triangles as \( P_{jkl} |\alpha,\beta,\gamma\rangle = |\gamma,\alpha,\beta\rangle \), and \( P_{1234} \) is the four-site spin ring exchange operator, with \( P_{jklm} |\alpha,\beta,\gamma,\eta\rangle = |\eta,\alpha,\beta,\gamma\rangle \), Fig. 1. The couplings \( J \), \( K_3 \), and \( K_4 \) can be obtained from perturbative analysis of SU(3) Hubbard model at \( 1/3 \) filling and the leading-order terms are

\[ J = \frac{2t^2}{U}, \]
\[ K_3 = \frac{6t^3}{U^2}, \]
\[ K_4 = \frac{20t^4}{U^3}. \]

Without the ring exchange terms, previous studies of the site-factorized ansatz on the triangular lattice predicted a three-sublattice ordered state\(^{26-28}\) which was recently confirmed by Density Matrix Renormalization Group (DMRG) and infinite Projected Entangled-Pair States (iPEPS) analysis\(^{22}\).

Recently, Ref. 29 did variational studies on the SU(3) model with three-site ring exchange, and they found the FM state and the three-sublattice ordered state in the regime of FM three-site ring exchanges. On the AFM side of the three-site ring exchanges, they found an interesting \( d_x + id_y \) spin liquid state with a gapless parton Fermi surface.

However, the situation is not clear if both the three-site and four-site ring exchange terms are included. In this section, we will focus on the regime where the three-site ring exchange is FM and the four-site ring exchange is AFM, motivated by the perturbative studies of SU(3) Hubbard model. In order to study the ordered phases, below in Sec. II A we first present our studies using the site-factorized states. Later in Sec. II B we will present our studies using the slave-fermion trial states.
A. Site-factorized state studies

In this subsection, we consider the site-factorized state defined as

$$|s⟩ = \prod_j |x_j⟩, \quad (7)$$

with

$$|x_j⟩ \equiv a_j |x⟩_j + b_j |y⟩_j + c_j |z⟩_j, \quad (8)$$

where we fix the overall phase by setting the phase of $a_j$ to be zero such that $a_j \in \mathbb{R}$ and $b_j, c_j \in \mathbb{C}$ and $|a_j|^2 + |b_j|^2 + |c_j|^2 = 1$. Above, we used the usual time-reversal invariant basis of the SU(3) fundamental representation, defined as

$$|x⟩ = \frac{|1⟩ - |1⟩}{\sqrt{2}}; \quad |y⟩ = \frac{|1⟩ + |1⟩}{\sqrt{2}}; \quad |z⟩ = -i|0⟩, \quad (9)$$

with $|S^z = ±1⟩ \equiv |±1⟩$ and $|S^z = 0⟩ \equiv |0⟩$. According to the parametrization of the $|x_j⟩$ vector along with the constraint, at each site there are 4 independent parameters. 

In this subsection, we follow the approach similar to the one outlined in Ref. [32] for the spin $S = 1$. We write the spin operators in terms of three flavors of fermionic spinons, $f^α$.

$$S^α_j = \sum_{\beta, \gamma} e^{αβγ} f^β_j f^γ_j, \quad \sum_α f^α_j f^α_j = 1, \quad (10)$$

with $α, β, γ \in \{x, y, z\}$ and $j$ is the site label. We note that the fermionic spinon $f$ is different to the electron $e$ in the Hubbard model. Conceptually, we are focusing on the insulating side of the metal-to-Mott insulator phase transition, in which the charge degrees of freedom are localized. In the mean field level, one assumes that the spinons do not interact with one another and are hopping freely on the 2D lattice. The mean field Hamiltonian would have the spinons hopping in zero magnetic field, and the ground state would correspond to filling up a spinon Fermi sea. In doing this one has artificially enlarged the Hilbert space, since the spinon hopping Hamiltonian allows for unoccupied and doubly-occupied sites, which have no meaning in terms of the spin model of interest. It is thus necessary to project back down into the physical Hilbert space for the spin model, restricting the spinons to single occupancy.

The fermion representation $\{f^x, f^y, f^z\}$ in Eq. (10) can be related to the usual fermion representation $\{f^+_{J}, f^-_{J}, f_0\}$, where $f^±_J$ carry $S^z$ quantum number $±1$ and $f_0$ carries $S^z$ quantum number 0, based on Eq. (9):

$$f^x = -i \sqrt{2} f^+_{1} - f^-_{-1}, \quad f^y = 1 \sqrt{2} f^+_{1} + f^-_{-1}, \quad f^z = i f_0. \quad (11)$$

In $\{f^±_J, f_0\}$ basis, the spin operator can be represented as

$$S^+ \equiv S^x + i S^y = \sqrt{2} \left(f^+_1 f^-_{0} + f^1_0 f^-_{-1}\right), \quad (12)$$

$$S^z = f^+_1 f^-_{-1} - f^-_{-1} f^-_{1}. \quad (13)$$

The exchange operators in terms of fermions are

$$P_{jk} = \sum_{αβ} f^α_j f^β_k f^β_1 f^α_k, \quad (14)$$

$$P_{jkl} = \sum_{αβγ} f^α_j f^β_k f^β_l f^γ_j f^γ_l, \quad (15)$$

$$P_{jklm} = \sum_{αβγη} f^α_j f^β_k f^β_l f^γ_j f^γ_l f^η_m f^η_m, \quad (16)$$

where $α, β, γ, \eta = x, y, z$. 

FM phase are aligned with each other, say $|z⟩$ for all sites. The energy in the FM phase is $⟨H_{SU(3)}⟩_{FM} = 3J - 4K_3 + 6K_4$. The boundary between these two phases can be determined analytically from the condition $3J - 4K_3 + 6K_4 = 0$. For $K_4 = 0$, the transition point is at 3/4, which is consistent with our numerical studies.
The Hamiltonian, Eq. (3), can be re-expressed as

\[ H_{SU(3)} = J \sum_{\alpha} \sum_{j} f_1^\dagger f_2 f_3 f_4^\dagger - K_3 \sum_{\alpha} \sum_{\beta \gamma} \left[ f_1^\dagger f_2 f_3 f_4^\dagger + H.c. \right] + K_4 \sum_{\alpha} \sum_{\beta \gamma} \left[ f_1^\dagger f_2 f_3 f_4^\dagger + H.c. \right] \]  

Below, we will calculate the trial energies using the slave fermion trial states. We start from the case without considering pairing instability. We find that the main competing states are what we call the “trimer” state, the uniform π-flux spin liquid state, and the zero-flux spin liquid state. Later, focusing on the regime in which spin liquid states are the optimal slave-fermion states, we consider the possible pairing instabilities. We find that there is a possible pairing instability of the zero-flux spin liquid state toward a f-wave (nodal) spin liquid state and a pairing instability of the π-flux spin liquid state toward an exotic s-wave spin liquid state with two flavors of fermions paired up while one flavor of fermions remains gapless.

\[ E_{MF} = -J \sum_{\alpha} \sum_{j} \chi_{12}^\alpha_j^2 - K_3 \sum_{\alpha} \sum_{\beta \gamma} \left[ \chi_{12}^\alpha \chi_{23}^\beta \chi_{31}^\gamma - \left( n_1^\alpha \chi_{23}^\beta \chi_{32}^\gamma + n_2^\alpha \chi_{31}^\beta \chi_{13}^\gamma + n_3^\alpha \chi_{12}^\beta \chi_{21}^\gamma \right) + \chi_{13}^\alpha \chi_{32}^\beta \chi_{21}^\gamma \right] + H.c. \right] + K_4 \sum_{\alpha} \sum_{\beta \gamma} \left[ \chi_{13}^\alpha \chi_{34}^\beta \chi_{41}^\gamma + \chi_{12}^\alpha \chi_{23}^\beta \chi_{31}^\gamma \right] - \sum_{\alpha \beta} \left( n_1^\alpha n_2^\alpha \chi_{24}^\beta \chi_{43}^\gamma + n_1^\alpha n_4^\alpha \chi_{23}^\beta \chi_{32}^\gamma + n_3^\alpha n_4^\alpha \chi_{14}^\beta \chi_{43}^\gamma + n_2^\alpha n_3^\alpha \chi_{14}^\beta \chi_{43}^\gamma + \chi_{12}^\alpha \chi_{23}^\beta \chi_{31}^\gamma \right) + \sum_{\alpha \beta \gamma} \left( \chi_{12}^\alpha \chi_{23}^\beta \chi_{34}^\gamma + \chi_{13}^\alpha \chi_{32}^\beta \chi_{43}^\gamma + n_1^\alpha \chi_{14}^\beta \chi_{32}^\gamma \chi_{21}^\gamma \right) - \sum_{\alpha \beta \gamma} \left( \chi_{13}^\alpha \chi_{34}^\beta \chi_{42}^\gamma \chi_{21}^\gamma \right) + H.c. \right] . \]

Above we defined \( \langle f_{j k}^{(1)} f_{k l}^{(1)} \rangle_{\text{trial}} \).

The slave-fermion trial states which conserve the translational symmetry we consider in this work are the uniform zero-flux state and uniform π-flux state, which represents the spin liquid states with uniform \( \phi = \theta_{jk} = 0 \) and \( \phi = \theta_{jk} = \pi \) respectively for all nearest \( < j k > \). The fermions in both of these trial states hop isotropically on the lattice, \( t_{jk} \cong t \), and therefore the expectation values of \( \chi_{jk} \) are expected to be isotropic, \( \chi_{jk} \simeq \chi^\alpha = \text{const} \). Below we list the numerical values of the energy per site in the two states

\[ E_{MF}^{(\phi=0)} = -0.7672J + 0.4482K_3 - 0.4521K_4, \]
\[ E_{MF}^{(\phi=\pi)} = -0.4395J + 0.8352K_3 - 0.7949K_4. \]

We can observe that the optimal translationally invariant slave-fermion state favors \( \phi = 0 \) state when \( K_3 > 0 \) and \( \phi = \pi \) state when \( K_3 < 0 \). The distinguishability between these two uniform-flux spin liquid states can be related to the origin of the \( K_\alpha \) which arises from the third-order perturbation of the SU(3)Hubbard model at 1/3 filling. However, \( K_4 \)
does not distinguish these two uniform-flux (U(1)) spin liquid
states from this perspective.

Besides the translationally invariant state, we also consider
what we call the “trimer” state. Fig. 3 shows one example
of the configuration of such a state in which the non-zero \( t_{jk} \)
form non-overlapping trimer covering of the lattice. These
states break translational invariance, and any trimer covering
produces such a state. Such states can have lower Heisenberg
exchange energy. The occupied bonds attain the maximal ex-
ppectation value which is found analytically |\( \chi_{jk}^3 \)|\( _{max} = n_j^3 = 1/3 \). Their contribution can be sufficient to produce the low-
est total energy and such states are expected to be the lowest-
energy states with \( K_3 = 0 \) and \( K_4 = 0 \).

\[
E_{\text{trimer}}^{MF} = -J + 0.5926K_3 - 0.3704K_4. \tag{23}
\]

The energies of different slave-fermion trial states are func-
tions of \( K_3/J \) and \( K_4/J \) and it is expected that different optimal
trial state is realized in different parameter regime. In or-
der to clearly show the cross between the energies of different
mean-field ansatz states as functions of \( K_3/J \) and \( K_4/J \), we
plot Eqs. (21) with \( J = 1 \) in the limit of either \( K_3 = 0 \)
or \( K_4 = 0 \).

Figure 4 shows the energies of different mean-field states as a function of \( K_3/J \) with \( K_4 = 0 \) and as a function of
\( K_4/J \) with \( K_3 = 0 \). For the former, Fig. 4(a) clearly shows
that at \( K_3, K_4 = 0 \) the trimer state is the lowest energy
state followed by the uniform zero-flux state and \( \pi \)-flux state.

When \( K_3 \) is gradually increased, the energy line of the zero-
flux state crosses that of the trimer state and the zero-flux
state becomes the lowest energy state at \( K_3/J \approx 1.61 \). On
the other hand, for the latter, Fig. 4(b) shows that the energy line of \( \pi \)-flux state first crosses that of the zero-flux state at
\( K_4/J \approx 0.96 \) and then crosses that of trimer state. The zero-
flux state becomes the lowest energy state at \( K_4/J \approx 1.32 \).

For a complete phase diagram, we numerically determine the
ground states in the \( K_3 - K_4 \) parameter regimes and the result
is summarized in the mean-field phase diagram, Fig. 5.

In order to check if the mean-field ansatz states we consid-
ered are sufficient to describe the physics in this model, we
perform numerically “full optimization” of the mean-field en-
ergy, Eq. (20), on a triangular lattice with \( 100 \times 100 \) 3-site unit
cells, by treating \( \chi_{jk} \)-s and \( \theta_{jk} \)-s as varying variables. In the
numerical optimization, there are in total 18 variables, 9 \( \chi_{jk} \)
and 9 \( \theta_{jk} \), and we take \( t_{jk} = 1, \mu_j^3 = \mu \). Numerics suggest
that the above trial states are the three optimal states.

Before leaving this section, we want to remark that the
trimer state is a singlet state around a triangular plaquette, and
we can write down the exact singlet wave function in a closed
form as
\[
|\psi_{\text{trimer}}\rangle = \sum_{\alpha,\beta,\gamma} \frac{\epsilon^{\alpha\beta\gamma}}{\sqrt{6}} |\alpha,\beta,\gamma\rangle.
\]
with $\alpha = x, y, z$. With the trimer wave function, we can calculate the energy per site
\[
E_{\psi_{\text{trimer}}} = -\frac{1}{3} J - \frac{4}{27} K_3 + \frac{2}{9} K_4.
\]
For $K_3 = 0$ and $K_4 = 0$, the exact trimer state energy is very close to the variational energy of U(1) spin liquid which is $-0.34J$ [23], but much higher than the three-sublattice order state energy obtained by DMRG which is roughly $-0.678J$ [24]. However, such a plaquette state can be stabilized with finite $K_3$ before reaching the FM state. To see this, we can compare the energy of the FM state as shown analytically in the end of Sec. [1A] with that of the exact trimer state energy. Knowing the energy of the FM state, $E_{FM} = 3J - 4K_3 + 6K_4$, we can see that the trimer state indeed has the lower energy than that of the FM state when $K_3 < 45/52 + 3K_4/2 \simeq 0.86 + 1.5K_4$ with $J = 1$.

2. With pairing instability

So far, we have ignored the possible pairing instabilities in the Fermi surface spin liquid states discussed above. Here we want to address this issue. We now know that besides the trimer state, there are actually two spin liquid states—zero-flux spin liquid state and $\pi$-flux spin liquid state. Both of these spin liquid states are gapless and contain a single or multiple parton Fermi surfaces. Focusing on these regimes, we take the pairing mechanism into consideration and the trial Hamiltonian becomes
\[
H_{\text{trial}} = -\sum_{\langle jk \rangle} \sum_{\alpha,\beta} \left[ t_{jk} \delta^{\alpha\beta} f^{\alpha}_{j} f^{\beta}_{k} + \Delta_{jk} \delta^{\alpha\beta} f^{\alpha\dagger}_{j} f^{\beta}_{k} \right] + \text{H.c.}
\]
with the constraint, $n_{j}^\alpha = \langle f^{\alpha}_{j} f^{\alpha\dagger}_{j} \rangle = 1/3$ and $t_{jk}^\phi = \pm t_{jk}$ for $\phi = 0/\pi$. For clarity, from now on we will replace the bond labeling $\langle jk \rangle$ by $(r, r + \delta r)$, with $r$ running over all lattice sites and $\delta r = 1, 2, 3$ are shown in Fig. [1]. We abbreviated the sum over all $\delta r = 1, 2, 3$ as $\delta r$.

In these regimes, the optimal state are uniform flux states which suggests uniform trial hopping amplitude $t_{jk} \simeq t \equiv 1$ and the uniform expectation values of hopping functions, $\langle f^{\alpha\dagger}(r) f^{\beta}(r + \delta r) \rangle = \delta^{\alpha\beta} \chi^{\alpha\beta}(\delta r)$. Furthermore, in these regimes, the hopping functions, $\chi^{\alpha\beta}(\delta r)$, should be real and are numerically confirmed. Below, we consider two pairing cases: Case (1) corresponds to pairing within the same flavor of fermions. This requires the orbital angular momentum quantum number to be $l = 1, 3, \ldots$ corresponding to $p_x + ip_y$, $f$-wave, ...pairing states; Case (2) corresponds to BCS-type pairing with different flavor of fermions. This pairing requires $l = 0, 2, \ldots$ corresponding to $s$-wave, $d_x + id_y, \ldots$ pairing states [29, 30]. Below, we discuss the mathematical set up for each case separately.

Case (1): pairing ansatz with $\Delta^{\alpha\beta} = \delta^{\alpha\beta} \Delta^\alpha$.

The pairing functions we consider are $\langle f^{\alpha}(r) f^{\beta}(r + \delta r) \rangle = \delta^{\alpha\beta} \Delta^\alpha(\delta r)$. By symmetry arguments, the model conserves spatial rotational symmetry and as already explained above, the angular momentum of pairing functions should be odd, $l = 1, 3, \ldots$, corresponding to $p_x + ip_y$, $f$-wave pairing...... The ansatz can be further simplified to be $\chi^{\alpha}(\delta r) = \chi^{\alpha}(\delta r') = \chi^{\alpha}(\delta r') \equiv \chi^{\alpha}$, and $\Delta^{\alpha}(\delta r) = \Delta^{\alpha}(\delta r') \equiv \Delta^{\alpha}$ in addition, because the vectors $\vec{f} = \{f_x, f_y, f_z\}$ and $\vec{f}^\dagger = \{f_x^\dagger, f_y^\dagger, f_z^\dagger\}$ transform as a three-dimensional vectors under spin rotation, we expect $\chi_x = \chi_y = \chi_z = \chi$ and $\Delta^x = \Delta^y = \Delta^z$. In this pairing scheme, the $SU(3)$ is broken down to $SO(3)$. It is straightforward to diagonalize the mean-field Hamiltonian
\[
H_{\text{trial}} = \sum_{\alpha} \sum_{k \in BZ} E_{\alpha}(k) a^{\alpha\dagger}(k) a^{\alpha}(k),
\]
where $a^{\alpha}(k)$ are Bogoliubov quasiparticles satisfying the transformation
\[
\begin{pmatrix}
    f^{\alpha}(k) \\
    f^{\alpha\dagger}(-k)
\end{pmatrix} =
\begin{pmatrix}
    u_k^{\alpha} & -u_k^{\alpha*} \\
    v_k^{\alpha} & u_k^{\alpha*}
\end{pmatrix}
\begin{pmatrix}
    a^{\alpha}(k) \\
    a^{\alpha\dagger}(k)
\end{pmatrix},
\]
with
\[
|u_k^{\alpha}|^2 = \frac{1}{2} \left[ 1 + \frac{\xi_{\alpha}(k)}{E_{\alpha}(k)} \right],
\]
\[
|v_k^{\alpha}|^2 = \frac{1}{2} \left[ 1 - \frac{\xi_{\alpha}(k)}{E_{\alpha}(k)} \right].
\]
The ground state can be written as

$$|GS⟩ = \prod_{\alpha=x,y,z} \prod_{k} \left[ u_{k}^{\alpha} + v_{k}^{\alpha} f^{\alpha†}(k) f^{\alpha}(−k) \right]|vac⟩, (30)$$

where we define

$$\xi_{\alpha}(k) = − \sum_{\vec{e}} 2 \cos(\vec{k} \cdot \vec{e}) − \mu, \quad (31)$$

$$\hat{\Delta}_{\alpha}(k) \equiv \sum_{\vec{e}} i \Delta \sin(\vec{k} \cdot \vec{e}), \quad (32)$$

$$E_{\alpha}(k) = \sqrt{\left(\xi_{\alpha}(k)/2\right)^2 + |\hat{\Delta}_{\alpha}(k)|^2}. \quad (33)$$

We can see there are three degenerate bands in this case.

**Case (2):** pairing ansatz with $\Delta^{\alpha\alpha} = 0$, $\Delta^{\alpha\beta}_{\alpha\neq \beta} \neq 0$.

We consider the pairing function of the form, $\langle f^{\alpha}(r) f^{\beta}(r+\vec{e}_v) \rangle_{\alpha\neq \beta} = \Delta^{\alpha\beta}(\vec{e}_v)$. It seems there are three pairing functions we need to consider, $\Delta^{xy}$, $\Delta^{yz}$, and $\Delta^{zx}$, but in this SU(3) symmetric model, we can perform a global gauge transformation to make $\Delta^{yz}$, $\Delta^{zx} = 0$ as long as the length of the vector formed by $\Delta$ is conserved, $|\Delta^{xy}|^2 + |\Delta^{xz}|^2 = \Delta^2$ = constant. If this pairing state is energetically favored, there is always one flavor of gapless fermions in this SU(3) system which we choose to be $f^z$.

From now on, we will set $\Delta^{yz} = \Delta^{zx} = 0$ and $\Delta^{xy} = \sqrt{3} \Delta$. In this gauge choice, the symmetry breaking process is more apparent. The symmetry breaking is SU(3) → SU(2) × U(1). The SU(2) symmetry is generated by the pseudo-spin doublet $f^x$ and $f^y$ and the U(1) is generated by the gapless $f^z$ fermion. After Bogoliubov transformation, the ground state is

$$|GS⟩ = \prod_k \left[ u_{k}^{xy} + v_{k}^{xy} f^{x†}(k) f^{y†}(−k) \right]|vac⟩ \otimes (34)$$

$$\otimes \sum_{\xi < 0} f^{z†}(k)|vac⟩, \quad (35)$$

where the second part, Eq. (35), is the wave function of the free $f^z$ fermion, and $u_{k}^{xy}$ and $v_{k}^{xy}$ have the same expression as in Eq. (29) with $\xi^{x}(k) = \xi^{y}(k) = \xi^{z}(k) \equiv \xi(k)$ the same to Eq. (31) and

$$\hat{\Delta}^{xy}(k) \equiv \sum_{\vec{e}} \sqrt{3} \Delta \cos(\vec{k} \cdot \vec{e}), \quad (36)$$

$$E^{xy}(k) = \sqrt{\left(\xi(k)/2\right)^2 + |\hat{\Delta}^{xy}(k)|^2}. \quad (37)$$

We can see in the SU(3)-symmetric point, the energy bands always show one gapless branch corresponding to one flavor of gapless fermions.

With the two pairing ansatzes above, we focus on the regimes of the zero-flux spin liquid state and the $\pi$-flux spin liquid state in Fig. 5. We again perform full Wick contractions and ignore the pure constant density terms. The trial energy expression after Wick contractions is too complex to write out explicitly. We test all different pairing ansatze above and numerically calculate the trial energies with the optimal pairing functions $\Delta$ in the uniform-flux spin liquids regime on the triangular lattice with $300 \times 300$ sites.

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**III. DISCUSSION**

We study the SU(3) ring-exchange model with “FM” three site ring exchanges and “AFM” four-site ring exchanges. We first use the site-factorized ansatz to study the model and find the three-sublattice ordered states, FM states in a large regime of $K_3/K_4 > 0$.

In the slave-fermion trial states studies, we find the main competing states are the trimer state, uniform $\pi$-flux spin liquid state and the zero-flux spin liquid. The trimer state is strongly suppressed by increasing the strength of the ring exchanges. We also find that the zero-flux state has a possible...
pairing instability toward a $f$-wave gapless (nodal) spin liquid and the $\pi$-flux spin liquid state has a pairing instability toward an exotic $s$-wave spin liquid state with $f^\parallel$ pairing with $f^\perp$ fermions while $f^\perp$ fermions remain gapless.

We note that it is not legitimate to compare the phase diagram obtained from the site-factorized state studies, Fig. 2 with those obtained from the mean-field slave-fermion trial state studies, Figs. 3,6. It is more appropriate to compare the energetics of the site-factorized states with those of the Gutzwiller-projected states, which can be obtained by performing Variational Monte Carlo (VMC) studies and Gutzwiller projection on the mean-field states we obtain in this paper, which is beyond the scope of the paper. Besides, in the slave-fermion trial state studies, we only focus on the non-magnetic trial states and fix the number density of each flavor to be equal, $n^f_j = \langle f_j^{\alpha \dagger} f_j^\alpha \rangle = 1/3$. In principle, we can also consider the magnetic ordered states by making the number density of each flavor per site different as, for example, $\langle f_j^{\alpha \dagger} f_j^\alpha \rangle = 1$, and $\langle f_j^{\alpha \dagger} f_j^\alpha \rangle = 0 = \langle f_j^{\alpha \dagger} f_j^\alpha \rangle$. However, similar to the previous discussion, we still need to perform Gutzwiller projection on all these fermionic mean-field states (magnetic ordered states and non-magnetic states) in order to compare the energies of different states even within the slave-fermion trial studies.

For further clarifying in which regime the spin liquid states are more robust and qualitatively relate the phase diagrams of the site-factorized studies and mean-field slave-fermion trial state studies, Figs. 2, and Figs. 3,6, we follow Ref. 23 to perform linear flavor wave theory (LFWT), an extension of spin wave theory to SU(N) model, on the three-sublattice ordered state. We find that the energy per site after taking the quantum fluctuations into account is

$$E_{\text{3-subl}_LFWT} \sim -0.6295J + 0.5462K_3 + 0.2568K_4,$$ (38)

which is consistent with the result in Ref. 23 at $K_3, K_4 \to 0$. We can see that even though the quantum fluctuations lower the (two-site exchange) Heisenberg energy from 0 to −0.63J, the ring exchange energies increase. At strong $K_3$ and $K_4$, it is expected that quantum fluctuations eventually destroy the three-sublattice ordered state. On the other hand, since FM state in Fig. 2 is an exact eigenstate of the SU(3)-ring exchange Hamiltonian, the FM is a much stable phase and the interesting quantum spin liquid states are unlikely to arise in the FM regime. Therefore, in this SU(3)-ring exchange model, we expect that the interesting quantum spin liquid states can arise in the large $K_3, K_4$ parameter regime of the three-sublattice ordered state.

The gapless spin liquid states have different properties and can be distinguished, at least in the mean-field picture. For clarity, below we will consider specifically the spin correlations, $\langle \tilde{S}_j \cdot \tilde{S}_k \rangle_{\text{conn}}$, and the (nematic) correlations of diagonal elements of traceless quadrupolar tensor, $\langle Q_{k}^{\alpha \alpha} Q_{k'}^{\alpha \alpha} \rangle_{\text{conn}}$ with $Q_{ij}^{\alpha \beta} = \langle S_i^\alpha S_j^\beta + S_i^\beta S_j^\alpha \rangle/2 - 2\delta^{\alpha \beta}/3$. Above we defined the connected correlations $\langle \mathcal{O}_j \mathcal{O}_k \rangle_{\text{conn}} \equiv \langle \mathcal{O}_j \mathcal{O}_k \rangle - \langle \mathcal{O}_j \rangle \langle \mathcal{O}_k \rangle$, with $\mathcal{O}_j = \tilde{S}_j$ or $Q_j$. Before jumping into the discussions of the properties of different spin liquid states, we first note that the correlation functions of the diagonal elements of the quadrupolar tensor can be expressed as

$$\langle Q_{j}^{\alpha \alpha} Q_{k}^{\alpha \alpha} \rangle_{\text{conn}} = \frac{4}{9} - \left| \langle f_j^{\alpha \dagger} f_k^\alpha \rangle \right|^2,$$ (39)

where above we used the identity, $Q_{ij}^{\alpha \beta} = \delta^{\alpha \beta} - f_j^{\alpha \dagger} f_j^\alpha$ with the constraint $\sum_{j} f_j^{\alpha \dagger} f_j^\alpha = 1$. We can see the first constant is universal for different spin liquid states, but the second contribution is qualitatively different in different spin liquid states can be used for characterizing different spin liquid state. For simplicity of discussion, we define $\langle Q_{j}^{\alpha \alpha} Q_{k}^{\alpha \alpha} \rangle_{\text{conn}}$. We will also discuss the thermal properties of each spin liquid state.

**Properties of uniform-flux (U(1)) spin liquid states:** The uniform zero-flux and $\pi$-flux spin liquid states are both U(1) (Fermi-surface) spin liquid states. The zero-flux spin liquid possesses a single partron Fermi pocket in the center and the $\pi$-flux spin liquid possesses two partron Fermi pockets near the hexagonal Brillouin zone. In these two different U(1) spin liquid states,

$$\langle \tilde{S}_j \cdot \tilde{S}_k \rangle_{\text{conn}} \sim \langle Q_{k}^{\alpha \alpha} Q_{k}^{\alpha \alpha} \rangle_{\text{conn}} \sim \frac{1 + \cos (\langle \mathbf{k}_{\text{FR}} - \mathbf{k}_{\text{FL}} \rangle \cdot \langle \mathbf{r}_k - \mathbf{r}_j \rangle)}{|\mathbf{r}_k - \mathbf{r}_j|^3}$$ (40)

where the $\mathbf{k}_{\text{FR/L}}$ represent the momenta of the right patch and the left patch of the Fermi surface for an observable direction. Since the two Fermi surface spin liquid states have different geometric information of Fermi surfaces, the corresponding wave vectors are quantitatively different and can be detected by studying the corresponding spin structure factors. Since the uniform-flux spin liquid states have gapless partron Fermi surface(s), we expect to see linear-temperature dependent specific heat ($C_v \propto T$) and thermal conductivity ($\kappa \propto T$).

**Properties of f-wave gapless (nodal) spin liquid state:** This spin liquid state possesses gapless nodal points. Interestingly, even though the pairing break the original SU(3) symmetry, the SO(3) rotational symmetry related to the spin rotation is still preserved (because of the fact that $\tilde{f}$ and $\tilde{f}^\perp$ with $\tilde{f} = (f^x, f^y, f^z)$ transform as three-dimensional vectors). As far as the low-energy physics is concerned, this spin liquid state possesses

$$\langle \tilde{S}_j \cdot \tilde{S}_k \rangle_{\text{conn}} \sim \langle Q_{j}^{\alpha \alpha} Q_{k}^{\alpha \alpha} \rangle_{\text{conn}} \sim \frac{1 + \cos (|\mathbf{k}_c| \cdot \langle \mathbf{r}_k - \mathbf{r}_j \rangle)}{|\mathbf{r}_k - \mathbf{r}_j|^4}$$ (41)

with $\mathbf{k}_c$ being the vectors connecting different Dirac points and we ignore all the pre-factors of each term in the above equation. Since this spin liquid state contains gapless nodal points, it should possess square-temperature specific heat ($C_v \propto T^2$).

**Properties of the exotic s-wave spin liquid state:** This spin liquid state possesses short-ranged spin correlations due to the superconducting gap in the $x$-$y$ channel. In order to detect the gaplessness of such a spin liquid state, we can measure the correlation functions of different diagonal elements...
of the quadrupolar tensor. For example, the $Q^{zz}$ correlation will show the power-law behavior

$$\langle Q^z_j Q^{zz}_k \rangle^m \sim \frac{1 + \cos[(k_{FR} - k_{FL}) \cdot (r_k - r_j)]}{|r_k - r_j|^3},$$

(42)

with $k_{FR,FL}^m$ being the momenta of the right and left patch of the $f^z$ Fermi surface. But the correlations related to $Q^{zz}$ and $Q^{yy}$ exponentially decay. As for the thermal properties, we expect to see $C_v \propto T$ and $\kappa \propto T$ due to the gapless $f^z$ parton Fermi surface.

From the perspective of numerics, since there have been DMRG and iPEPS studies on the SU(3) Heisenberg model of three-flavor fermions on the triangular lattice which found the three-sublattice ordered state. We suggest possibly the interesting zero-flux spin liquid state be also detected in the DMRG and iPEPS studies on the SU(3) ring-exchange model.

From the view of cold atom experiments. Recently, the cold atom experiment demonstrated a method to be able to add an artificial tunable gauge potential to the system. With the tunable gauge potential, it may be possible to tune the sign of the three-site ring exchanges from FM to AFM. In that case, for strong four-site ring exchange, the main competing states are still the zero-flux and $\pi$-flux spin liquid states, with the trial energies similar to Eqs. (21)-(22) with $K_3 < 0$. However, for small $K_4$, the main competing slave-fermion trial states are the uniform $\pi$-flux spin liquid state with the trial energy, $E_{MF, \pi/3} \approx -0.8992J + 0.8543K_3 + 0.0241K_4$, and the $\pi$-flux spin liquid state. It may be interesting to explore the regime with AFM three-site ring exchange with $K_4 \sim 0$ and it may be possible to see a phase transition between these two spin liquid states by manipulating the artificial gauge potential in experiment.

Another interesting theoretical outlook is how the phase diagram, Fig. 6 evolves if we perturb the model away from the SU(3)-symmetric point. In this way, the model can be connected to other models, which can explain the gapless spin-1 spin liquids possibly realized in Ba$_3$NiSb$_2$O$_9$ or other theoretical spin-1 models.

In the present model studies, we treat the parameters $J$, $K_3$, and $K_4$ as three independent parameters. However, from the perspective of the perturbation studies on the Hubbard-to-Mott insulator transition, as pointed out in the beginning, the parameters $J$, $K_3$, and $K_4$ are actually not independent to each other. According to the mean-field phase diagram, Fig. 5, the interesting gapless spin liquid states are more likely to arise in the parameter regime with $K_3/J$, $K_4/J$ of the order one or greater, which means the system on the insulating side is closer to the transition point. In this regime, the perturbation theory breaks and higher order terms need to be included, which not only makes the theoretical analysis out of control but also points out the possibility that the experimental realization of the interesting gapless spin liquid states is out of reach. In order to have a well-controlled theoretical analysis to get access to such interesting gapless spin liquid states and to possibly shed light on the experimental realization in the cold atom systems, we would like to study a Fermi-Hubbard-type model on a two-leg ladder system with on-site or more extended repulsion in the future, similar to the analysis outlined in Ref. [46].

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The four-sublattice ordered state can be possibly analyzed numerically by considering the helical states. The helical state can be constructed by first choosing the site-factorized vectors $\vec{X}_l$ and $\vec{X}_{l+1}$ of two consecutive stripes (in the basis of $\{|x\rangle, |y\rangle, |z\rangle\}$, and we can write the next one as 
$$\vec{X}_{l+2} = \cos \theta \vec{X}_l + \sin \theta \vec{X}_l \times \vec{X}_{l+1}.$$