Slowly decaying classical fields, unitarity, and gauge invariance

Dennis D. Dietrich

The Niels Bohr Institute, Copenhagen, Denmark
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In classical external gauge fields that fall off less fast than the inverse of the evolution parameter (time) of the system the implementability of a unitary perturbative scattering operator (S-matrix) is not guaranteed, although the field goes to zero. The importance of this point is exposed for the counter-example of low-dimensionally expanding systems. The issues of gauge invariance and of the interpretation of the evolution at intermediate times are also intricately linked to that point.

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I. INTRODUCTION

This note deals with questions appearing frequently when considering processes in classical fields. The principal question is whether there exists a well-defined description for a generic scattering process in the presence of a given background. After clarifying this point at least for asymptotically early or late times, one may ask if information about the system can be extracted also at intermediate times. Here, the required criteria are exhibited and analysed for slowly decaying fields. In the course of the investigation this will also necessitate a discussion of gauge invariance.

A classical field emerges as the expectation value of a quantised bosonic field of the underlying quantum field theory. In situations where the occupation number of the bosonic field modes is very large, the commutator of the bosonic field creation and annihilation operators is subleading. The dynamics of the classical field capture a major part of the dynamics of the physical system. Quantum effects are parametrically smaller. In abelian field theories at weak coupling, the leading quantum effects are due to fermions and antifermions. In non-abelian field theories, quantum fluctuations of the bosonic field are parametrically equally favoured. The currents induced by the movement of these quanta in the classical field modify the current term in the equations of motion for the classical field. Frequently this quantum effect, the so-called back-reaction, is neglected in what is known as the external field approximation. While quantisation effects are regarded as parametrically suppressed in the systems considered, they might still fundamentally alter the systems’ description beyond a mere correction to observables.

The present article addresses this point for the characterisation of a system with a slowly decaying external field in terms of free particles. In the present sense, a slowly decreasing field means a gauge field which does decay to zero but not faster than the inverse of the evolution parameter (time). In many cases, a change of gauge can lead to a faster decay. Here, however, we will consider situations where this is not possible. For example, one can imagine a volume expanding in one spatial direction with a given constant total energy content in the classical field. Then, the corresponding energy density will decrease inversely with time, and the components of the field tensor like the inverse of the square root of time. This is also the fastest possible decay for the largest component of the gauge field in any gauge. Note that here, in the spirit of perturbation theory, the situation for asymptotically free particles is studied. For other asymptotic bases, the picture can be different. Actually, a change of basis can often cure some of the problems one encounters as in the case of the slow spatial decay (long-range interaction) of the field in Coulomb scattering.

Beyond the external field approximation, the fall-off of the fields is accelerated by processes subsumed under back-reactions. For instance, accelerating or even producing particles by vacuum polarisation requires energy which must come from the classical field. Naturally, one can also imagine situations where energy is transferred from the quantum sector to the classical field but in the present setting, where the classical field is to carry the bulk of the energy initially, this contribution is relatively suppressed. As the incorporation of the back-reaction is usually a formidable task, one tends to work within the framework of the external field approximation. Then, however, one must also pay attention to potential fundamental problems such as the non-unitarity of the theory.

In order to shed more light on these points, section addresses the issue of unitarity in slowly decaying external fields. It is investigated whether a unitary implementation of the scattering operator for fermions and antifermions exists in the presence of a slowly decreasing field. (Although the implementation of the scattering operator in Fock space is an operator in a Hilbert space, it will be called by its traditional name S-matrix, in what follows.) As shall be discussed below, this depends on whether the number of particles produced from the vacuum in the presence of the field is finite or not. The requirement that the field should decay is a much weaker condition than the known set of sufficient conditions guaranteeing a finite number of produced particles. In fact, fields that decay slowly do not satisfy such conditions. However, the minimal sufficient conditions are not known. Thereby, on the one hand, the situation is un-
clear in many physically interesting systems, and, on the other, has to be investigated separately for single or small classes of problems. The set of backgrounds which allow determination of the free particle content at any intermediate time is even more restricted than the set of backgrounds which permit a unitary perturbative S-matrix and does not contain slowly decaying fields. In section II A the existence of the Møller wave operators in slowly decaying fields is investigated in one spatial and one temporal dimension. In section II B the existence of a unitary S-matrix is studied by explicitly calculating the expectation value of the number of massive fermions produced in the presence of a slowly decreasing field for three spatial and one temporal dimension. Calculational details are given in appendix A.

Section III is concerned with the issue of gauge invariance. The description of the production of free fermion-antifermion-pairs by vacuum polarisation in the presence of classical fields with identified particle and antiparticle momenta per se depends on the chosen gauge. As explained below, a gauge independent interpretation can be given to the results. However, for slowly decaying field configurations this is more complicated to achieve. Section IV summarises the results.

II. UNITARITY

A physical system can be described by the notion of in- and out-going asymptotic particles if the scattering operator s acting on the Hilbert space h, \( s : h \to h \), has a unitary implementation in the form of the operator S (S-matrix) in the Fock space \( F \) constructed from the corresponding asymptotic states: \( S : F \to F \). When talking about processes with in- and out-going free particles one must check whether the unitary (perturbative) S-matrix for free states exists in the case under consideration.

A more stringent requirement on the relative system than the existence of the unitary S-matrix is the implementability of unitary evolution at any intermediate time. For fermions, this means that the time-development operator, \( U(x_0, y_0) \), describes the system's evolution based on the initial vacuum states and that at any, in general not asymptotically large or small, time the particle content of the system is known. The operator \( U(x_0, y_0) \) evolves the wave-function solutions \( \psi \) of the Dirac equation

\[
i\partial_t \psi(t) = H(t) \psi(t),
\]

according to:

\[
\psi(x_0) = U(x_0, y_0)\psi(y_0).
\]

\( H(t) \) is the Dirac Hamiltonian. The set of backgrounds for which this is actually possible is much too restricted to include slowly decaying fields. (The set of sufficient conditions given in \( \text{II} \) admits only an \( A_0 \) component on a compact support. Thereby no magnetic fields are included.) Therefore, projections onto asymptotic states at an intermediate time do not yield the number of particles in the system at that time for the situations addressed here. Frequently, it is impossible even to define a particle in the presence of an external field for non-asymptotic times.

The conditions for the existence of the Møller wave operators are less stringent. They are defined as limits of products of full evolution operators \( U \) with free evolution operators \( U_0 \) for infinitely early or late times:

\[
W_\pm(x_0) := \lim_{y_0 \to \pm \infty} U(x_0, y_0)U_0(y_0, x_0).
\]

\( U_0 \) is the evolution operator for the free Dirac equation:

\[
i\partial_t \psi_0(t) = H_0 \psi_0(t).
\]

If the Møller wave operators exist, the single-particle scattering operator exists as well and can be expressed with the help of the former:

\[
s = W_+^*W_-.
\]

If it exists, \( s \) describes the infinite initial and final time limit of the evolution. It contains the information on how the initial state evolves from infinitely early times to infinitely late times. For time-dependent Hamiltonians the Møller operators and \( s \) depend on the time \( x_0 \). The time-dependence reduces to a pure phase factor for the matrix elements \( \langle m|S|n \rangle \) (see below). This phase drops out of the absolute squares of the matrix elements, which are the observable quantities. Therefore, the time-dependence of the relative objects is not denoted.

The criterion for when a unitary S-matrix can be constructed is formulated most conveniently after carrying out a decomposition. The projectors \( P_\pm \) serve to project onto the positive and negative energy sectors of the Hilbert space \( h : P_\pm h = h_\pm \). With their help, the scattering operator \( s \) and thereby its implementation S on the Fock space \( F \) can be decomposed according to

\[
S_{\pm \pm} := P_\pm SP_\pm,
\]

where the first and second sign on each side of the equality are linked. Hereafter, the upper or the lower signs have to be chosen in all corresponding expressions. The Shale-Stinespring criterion \( \text{II} \) now says that a unitary S-matrix exists iff \( S_{\pm \pm} \) are Hilbert-Schmidt operators, i.e. have a finite Hilbert-Schmidt norm:

\[
\sum_n \langle n|S^\dagger_{\pm \pm}S_{\pm \pm}|n \rangle \in \mathbb{R}^+,
\]

where the \( |n \rangle \) form a complete set of states. Interestingly, after inserting a complete set of states between the two operators in the previous expression one sees that the criterion has a direct physical interpretation because:

\[
\sum_{m, n} |\langle m|S_{\pm \pm}|n \rangle|^2 = \sum_m N_m,
\]
where \( N_m \) is the number of particles/antiparticles (upper/lower signs) in mode \( m \) produced from the vacuum \( \mathbb{R} \); that is only if the initial state is a vacuum. In other words, if a unitary \( S \)-matrix is to exist, the number of (anti)particles produced from the vacuum must be finite in every single mode \( m \) and in total.

In fact, in order to describe the system’s particle content at any intermediate point in time, the evolution operator \( U \) must also be a Hilbert-Schmidt operator. This, however, constrains the possible forms of the field much more severely (see above) than in the case of the Möller wave operators and the scattering operator.

A final important remark is due at this point: In systems in which the perturbative \( S \)-matrix (i.e. the one formulated for free states) does not exist, an \( S \)-matrix based on different asymptotic states may well exist. A prominent example is the scattering in unscreened Coulomb fields. When formulated with free states it is plagued by infinite corrections of the order \( e^{-\lambda r} \) for massless fermions in a slowly decaying abelian field. Therefore, the Möller wave operator \( W_+ \) does not exist.

### B. Unitarity of the \( S \)-matrix

In addition to the results for the Möller operators, we now calculate explicitly the number of particles produced in the presence of a slowly decaying field. As mentioned above, the theory possesses a unitary formulation only if this number is finite. To this end, the expectation value of the number of produced pairs is to be calculated in the special field:

\[
A_\mu = A_\mu(t) = a \ g_{3\mu} \theta(t)/\sqrt{\mathcal{I}},
\]

This expectation value can be written as the double phase-space integral,

\[
\langle n \rangle = \int \frac{d^3 p}{2(2\pi)^3 \omega_p} \frac{d^3 q}{2(2\pi)^3 \omega_q} |M_{q,p}|^2,
\]

of the absolute square of the amplitude:

\[
M_{q,p} = \lim_{y_0 \to \pm \infty} \int d^3 x d^3 y \rho_q^0(x) G_R(x,y) \gamma^0 \psi_p(y),
\]

which is equal to the overlap of an incoming free antiparticle \( \psi_p(y) \) with an outgoing free particle \( \phi_q(x) \) after propagation through the field \( \rho \). \( G_R(x,y) \) represents the retarded propagator to all orders in the field and \( \omega_p := \sqrt{|p|^2 + m^2} \). In Eq. (16) the summation over all discrete degrees of freedom is implicit.

In order to make the link, we note that the double phase-space integral in Eq. (16) corresponds to the summation over the complete sets of states in Eq. (5) and that the amplitude \( 17 \) is equivalent to the matrix element in Eq. (3). The homogeneous solution \( G_H(x,y) \) of the Dirac equation in Eq. (7) of \( 11 \) is equivalent to the evolution operator \( U(x_0,y_0) \) if the former is taken at \( \vec{x} = \vec{y} \) and the derivative operators acting on \( \vec{y} \) instead of \( \vec{x} \). Based on the homogeneous solution \( G_H(x,y) \), the retarded propagator \( G_R(x,y) \) appearing in the amplitude \( 17 \) can be expressed as \( 16 \):

\[
iG_R(x,y) \gamma^0 = + G_H(x,y) \delta^{(3)}(\vec{x} - \vec{y}) \theta(x_0 - y_0).
\]

The late-time behaviour of the gauge field is most important for particles with long wavelengths. Therefore, the behaviour of the expectation value for small energies has to be investigated. As can be seen by putting Eqs. (A3), (A5), (A7), and (A9) into Eq. (A3) and the latter into Eq. (A2), the number of pairs produced in the presence of the field \( A \) at low energies is infinite [up to finite corrections of the order \( O(\omega^2/\mathcal{I}^2) \)] because of the divergent expressions \( A9 \), \( A5 \), and \( A7 \) as well as the
pole in Eq. (A1). For this reason, there exists no unitary perturbative $S$-matrix in this system. This is consistent with the observation that the Møller wave operators are ill-defined. Actually, in this context the existence of the Møller operators can be seen as a necessary condition.

In order to understand how this result emerges and how to overcome the inconsistency, let us investigate the same quantity in the regularised field:

$$A_\mu = A_\mu(t) = a \, g_{3\mu} \, \theta(t) \, e^{-\beta \sqrt{t} / \sqrt{t}}. \tag{19}$$

In this case, all Möller wave operators exist because the limits (12) exist. For this field one finds by introducing Eqs. (A11) and (A12) into (A3) and subsequently into (A2) that

$$\frac{(2\pi)^3 \, d(n)}{V \, d^3p} = 8 \, m_p^2 \, \omega^2 \left[ 4 \frac{a}{\beta} - \sin \left( \frac{2 \, a}{\beta} \right) \right]^2 + O \left( \frac{\omega^2}{\beta^4} \right),$$

which is finite for any non-zero value of the parameter $\beta$. The problem, that a unitary $S$-matrix cannot be constructed due to a violation of the criterion (1) at small energies, is absent. Further, no meaningful limit $\beta \to 0$ exists because the first term in the square brackets diverges and the second is non-analytic. Subleading terms, which become important for decreasing $\beta$ at fixed $\omega$, finally lead to the small energy limit of Eq. (A8), which is infinite. Thus, we again see that no meaning can be given to the unregularised situation defined by the field $\tilde{A}_\mu$.

This calculation can also be carried out for boost-invariant field configurations, and the principal outcome will be the same. While the longitudinal length $L$ here factors out as part of the three-volume $V$, the result there becomes independent of the rapidity $y$.

The regularisation of the result by exponentially decaying functions leads to the introduction of the decay parameter $\beta$ on which the final result now depends. One cannot eliminate it right away without encountering the same problems as before. One must either give a physical meaning to the parameter, or one must calculate the effect of the back-reaction in this regularised version. After including the back-reaction in an appropriate way, the limit $\beta \to 0$ of the expectation value should be finite because the mathematical inconsistency is ultimately not a problem of physics but of computational ability, analytical as well as numerical.

Yet another word of caution is due in connection with these so-called asymptotic switching constructions. For example, while an *eternally* constant field tensor does not produce any particles (12), particle production is described when temporal damping is included and the limit of no damping is taken at the end (12). In general, the introduction of the switching function can fundamentally alter the physical content of the calculation. Another time-independent example that requires regularisation is the Coulomb field, e.g. leading to its Yukawa screened form.

One more source for the artificial alteration of the result by the above manipulations occurs if the decay parameter is assigned a physically motivated value which turns out to lead to rapid decrease. Then, the characteristics of the unregularised field can be hidden and the unduly fast decay of the gauge field can lead to an unduly large field tensor through its derivative. With a cut-off at $t_{\max}$, this is even more obvious because it induces a $\delta$-peak in the field tensor: $\tilde{E}(t, \vec{x}) = \tilde{A}(t_{\max}, \vec{x}) \delta(t - t_{\max})$. For a slowly decaying function, this contribution cannot safely be left uncompensated and should be estimated. In addition, such spikes contain all frequencies and lead to a distortion of the spectrum. Note, however, that situations without magnetic field can be described entirely by the $A_0$ component of the gauge field, if an adequate gauge is chosen. Then the spike with its shortcomings is absent. This circumstance is reminiscent of the sufficient conditions for the unitary implementability of the evolution operator $U$ (see section II B 2). Nevertheless, even for $\tilde{A} = 0$, an abrupt cutoff leads to differences in observables like, for example, the number of produced particles. Ultimately, these originate from the fact that with, respectively without the switching one finds oneself in different Fock spaces.

### III. GAUGE INVARIANCE

We start the discussion of gauge invariance for situations in which gauges can be found such that the Møller wave operators exist. The gauges for which this is the case for a given system are connected by transformations, $\omega(x)$, which approach unity sufficiently fast for early and late times. In all such gauges, the amplitude (17) remains the same. This is true because the propagator $G_R(x, y)$ in Eq. (10) transforms under an arbitrary gauge transformation $\Omega(x)$ like

$$G_R(x, y) \to \Omega^\dagger(x) G_R(x, y) \Omega(y). \tag{20}$$

The gauge transformations at all intermediate times $z_0$ with $x_0 > z_0 > y_0$ drop out. Due to the limits taken in Eq. (17), the amplitude $M_{q, \rho}$ does not change because

$$\lim_{x_0 \to +\infty} \omega^\dagger(x) G_R(x, y) \omega(y) = \lim_{y_0 \to -\infty} G_R(x, y). \tag{21}$$

Now, one possibility is to say that the gauge invariance of the theory is tantamount to invariance under transformations $\omega(x)$. Alternatively, one can allow all gauge transformations but define the asymptotic states as

$$\tilde{\psi}_p(x) := \tilde{\Omega}(x) \psi_p(x), \tag{22}$$

where $\tilde{\Omega}(x)$ is a gauge transformation that for very early and very late times leads to the same pure gauge field as the transformation $\Omega(x)$ (10, 11). In principle
The issues of unitarity and gauge invariance have been discussed for asymptotically free particles in slowly decaying backgrounds. Here the emphasis is on fermions. Note, however, that in non-abelian theories bosonic quantum fluctuations are parametrically equally favoured as the fermions and must be taken into account in a complete treatment of the system.

The projection of the wave function onto free states at non-asymptotic times does not yield the free particle content at that time but can only be interpreted as a cut-off of the field with all the usual caveats caused by the induced delta-spike in the field tensor.

The regularisation of a slowly decaying field may be an absolute necessity because it does not allow for a unitary formulation of the theory in the first place. In other words, it has to be given a faster decay either by means of the incorporation of neglected phenomena or artificially. Therefore, a gauge invariant interpretation of the result is more difficult than in situations with a fast decaying field because the regularisation has to be carried out in a gauge invariant manner.

Hence, the conclusion must be that the description of the scattering of free particles in slowly decaying backgrounds is, in general, not a well-defined problem. Usually this is caused by the neglect of effects which are important for the modelling of the system. In the present case, their inclusion would ensure a sufficiently fast decrease of the classical field. Here, the required effect that comes to mind first is the inclusion of the back-reaction. In the introduction we mentioned low-dimensional expanding systems as examples of environments in which slowly decreasing fields appear. In that case the small number of dimensions into which the field expands may also be an approximation. Dropping this constraint can be a practical step which alleviates the difficulties by accelerating the gauge field’s decay.

For practical purposes the inclusion of the back-reaction is a difficult task. It is seldom possible to incorporate it into the calculations exactly because, ultimately, this amounts to solving the full theory. A standard loop expansion around the classical limit will, in general, also be plagued by divergences in every order of the series. Let this not be misunderstood; the classical expectation value for the bosonic field in itself is consistent even though it might not provide an accurate picture of the system. Additionally, in a situation where a sizeable expectation value for the bosonic field is present the expansion around zero field fails. The problems emerge when one has to go beyond the classical limit. Already at the prequantum level, concretely for classical Dirac particles in the background field, problems can arise in form of the non-existence of the Møller operators. Through the successive inclusion of quantum fluctuations the framework becomes yet more prone to inconsistencies. One observes that the description within the usual framework cannot be executed straightforwardly in situations which

Now let us continue by turning to slowly decaying fields. As has been discussed in section 4 a slowly decaying field can prevent a well-defined description of the system. In that case a regularisation has to be carried out to have a meaningful theory. Regularisations on the basis of the gauge field can be carried out in any gauge whence they are not unique and, in general, hinder a gauge invariant interpretation of the result. For this reason the regularisation must be carried out for gauge covariant objects like the field tensor because they transform homogeneously. In a thus regularised theory it is again possible to follow the steps explained in section 3 for obtaining a gauge invariant interpretation of the result. This means for example that, for a given regularisation carried out for the field tensor, any gauge that leads to a fast decaying field should lead to the same result for observables for free particles.

The most drastic regularisation is the cut-off. If applied to a slowly decaying gauge field it leads to significant artefacts and is therefore highly dangerous as discussed in section 4. If used for the field tensor, the gauge field is switched abruptly to a pure gauge. Although the pure gauge field corresponds to a vanishing field tensor, it must decay in time if a projection onto free states is to be used finally. For a given situation, it is not clear that there exist gauges in which the continuity of the gauge field at the cut-off and a sufficiently fast decay for large times can be achieved simultaneously. The continuity of the gauge field at the cut-off is again required to avoid cut-off artefacts. An induced spike in the field tensor leads to unnatural effects. But even without the spike, discontinuities in the gauge field have an influence on observables.
seem to be most natural. In other words, in situations in which the classical gauge field varies (decays) slowly, it provides an appropriate approximate description of the system. However, in this case, the standard procedure for the incorporation of quantum effects beyond the classical approximation is ill-defined. To the contrary, if the field varies (decays) rapidly the classical approximation is less good. Nevertheless, the quantum corrections can be calculated within the standard approach.

In physics, other similar conceptual problems are known, for example within the Thomas-Fermi method. The description of an almost homogeneous electron gas by means of a gradient expansion around the homogeneous limit is flawed already for arbitrarily small perturbations. This is the case although, in this setting, the homogeneous approximation is very good. The conditions for the applicability of the gradient expansion require low-amplitude, short-wavelength perturbations and forbid them at the same time [16].

Finally, a loop expansion has to be amended by a self-consistency condition in order to become viable.

However, commonly used iterative procedures which are adapted for numerical studies are frequently not transparent when it comes to understanding their physical content. Consequently, it is worth considering a fundamental reorganisation of the expansion scheme.

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**APPENDIX A: EXPECTATION VALUE FOR THE NUMBER OF PRODUCED PAIRS**

Making use of the implicit definition of the momentum transfer function $T$:

$$G(x, y) = G^0(x - y) + \int d^4 \xi d^4 \eta G^0(x - \xi)T(\xi, \eta)G^0(\eta - y),$$

the amplitude [17] can be reexpressed so that the expectation value [16] in a purely time dependent field becomes:

$$\frac{4(2\pi)^3 d(n)}{V} = tr \left\{ T_R(+k_0, +\vec{k}; -k_0, +\vec{k}) \frac{\gamma^0 \omega - \vec{\gamma} \cdot \vec{k} - m}{\omega} - T_A(-k_0, +\vec{k}; +k_0, +\vec{k}) \frac{\gamma^0 \omega + \vec{\gamma} \cdot \vec{k} + m}{\omega} \right\}.$$  \hspace{1cm} (A2)

Due to the relation $T_A(p, q) = -T_R^\ast(q, p)$ it does suffice to give, for example, the retarded $T$ in a field $A_\mu(x) = g_{\mu3} A_3(t)$:

$$T_R = a \gamma^3 T^B - i a^2 \sum_n \gamma^3 [T_+^{(n)} \rho^+ + T_-^{(n)} \rho^-] - i \gamma^0 (\vec{\gamma} \cdot \vec{k}_\perp + m)n \gamma^0 \gamma^3$$  \hspace{1cm} (A3)

According to [12] for the field $A_\mu = g_{\mu3} a \theta(t) t^{-\frac{1}{2}}$ the coefficient functions $T^B$ (Born) and $T_\pm^{(n)}$ are given by:

$$T^B = \sqrt{\frac{\pi}{\omega}} \frac{1 + i}{2}$$  \hspace{1cm} (A4)

$$T_\pm^{(0)} = \int_0^\infty dx_0 \int_0^{x_0} dy_0 x_0^{-\frac{1}{2}} y_0^{-\frac{1}{2}} e^{\pm i \omega (x_0 + y_0)} e^{\pm i k_3 (x_0 - y_0)} e^{\pm 2 i a (x_0 - y_0) \frac{\omega}{a^2}} =$$

$$= 4 \int_0^1 dc [\pm 2i (1 - c)]^{-1} \frac{\partial}{\partial a} \frac{1}{2} \sqrt{-i \omega (1 + c^2) + i k_3 (1 - c^2)} \times$$

$$\times \exp \left\{ \frac{[\mp ia (1 - c)]^2}{-i \omega (1 + c^2) + i k_3 (1 - c^2)} \right\} \text{erfc} \left[ \frac{\mp ia (1 - c)}{\sqrt{-i \omega (1 + c^2) + i k_3 (1 - c^2)}} \right] =$$

$$= \left( \frac{\partial}{\partial a} a^{-1} \right) \left[ \int_0^1 dc (1 - c)^{-2} + \mathcal{O} \left( \frac{\omega^2}{a^2} \right) \right]$$  \hspace{1cm} (A5)

$$\int_0^1 dc (1 - c)^{-2} \to \infty$$  \hspace{1cm} (A6)
\[ T_{\pm}^{(1)} = \int_0^\infty dx_0 \int_0^{x_0} dy_0 x_0^{-\frac{1}{2}} y_0^{-\frac{1}{2}} e^{+i\omega(x_0+y_0)} e^{\pm ik_3(x_0-y_0)} e^{\pm 2ia(x_0-y_0)} \int_{y_0}^{x_0} dz_0 e^{\pm 2ik_3 z_0} e^{\mp 4ia z_0} = \]
\[ = 8 \int_0^1 dc \int_1^{c_2} d_2 c_2 \sqrt{\pi} \frac{\partial^5}{\partial a^5} \frac{1}{2} \sqrt{-i\omega(1+c_1^2) \mp ik_3(1+c_1^2-2c_2^2)} \times \]
\[ \times \exp \left\{ \frac{[\mp ia(1+c_1^2-2c_2^2)]^2}{-i\omega(1+c_1^2) \mp ik_3(1+c_1^2-2c_2^2)} \right\} \text{erfc} \left[ \frac{\mp ia(1+c_1^2)}{\sqrt{-i\omega(1+c_1^2) \mp ik_3(1+c_1^2-2c_2^2)}} \right] = \]
\[ = -\frac{1}{2} \left( \frac{\partial^5}{\partial a^5} a^{-1} \right) \left[ \int_0^1 dc \int_1^{c_2} d_2 c_2 (1+c_1-2c_2)^{-4} + O \left( \frac{\omega^2}{a^2} \right) \right] \] (A7)

\[ \int_0^1 dc \int_1^{c_2} d_2 c_2 (1+c_1-2c_2)^{-4} \to \infty \] (A8)

\[ T_{\pm}^{(2)} = \int_0^\infty dx_0 \int_0^{x_0} dy_0 x_0^{-\frac{1}{2}} y_0^{-\frac{1}{2}} e^{+i\omega(x_0+y_0)} e^{\pm ik_3(x_0-y_0)} e^{\pm 2ia(x_0-y_0)} \times \]
\[ \times \int_0^{x_0} d_2 z_0 e^{\pm 2ik_3 z_0} e^{\mp 4ia z_0} \int_{y_0}^{x_0} dt_0 e^{\pm 2ik_3 t_0} e^{\mp 4iat_0} = \]
\[ = 8 \int_0^1 dc \int_1^{c_2} d_2 c_2 \int_1^{c_3} d_3 c_3 [\pm 2i(1-c_1-2c_2+2c_3)]^{-5} \frac{\partial^5}{\partial a^5} \sqrt{-i\omega(1+c_1^2) \pm k_3(1-c_1^2-2c_2^2+2c_3^2)} \times \]
\[ \times \exp \left\{ \frac{[\pm 2ia(1-c_1-2c_2+2c_3)]^2}{-i\omega(1+c_1^2) \mp ik_3(1-c_1^2-2c_2^2+2c_3^2)} \right\} \text{erfc} \left[ \frac{\mp 2ia(1-c_1-2c_2+2c_3)}{\sqrt{-i\omega(1+c_1^2) \mp ik_3(1-c_1^2-2c_2^2+2c_3^2)}} \right] = \]
\[ = \frac{1}{4} \left( \frac{\partial^5}{\partial a^5} a^{-1} \right) \left[ \int_0^1 dc \int_1^{c_2} d_2 c_2 \int_1^{c_3} d_3 c_3 (1-c_1-2c_2+2c_3)^{-6} + O \left( \frac{\omega^2}{a^2} \right) \right] \] (A9)

\[ \int_0^1 dc \int_1^{c_2} d_2 c_2 \int_1^{c_3} d_3 c_3 (1-c_1-2c_2+2c_3)^{-6} \to \infty \] (A10)

According to [12], one finds for the coefficient functions in the regularised field \( A_\mu = g_\mu a \theta(t) t^{\frac{1}{2}} e^{-\beta t^{\frac{1}{2}}} \):

\[ T^B = \sqrt{\frac{\pi}{-2i\omega}} \exp \left( \frac{\beta^2}{-8i\omega} \right) \text{erfc} \left( \frac{\beta}{\sqrt{-8i\omega}} \right) = -\frac{2}{\beta} + O \left( \frac{\omega}{\beta^3} \right) \] (A11)

\[ T_{\pm}^{(0)} = \int_0^\infty dx_0 \int_0^{x_0} dy_0 x_0^{-\frac{1}{2}} y_0^{-\frac{1}{2}} e^{-\beta x_0^{\frac{1}{2}}} e^{-\beta y_0^{\frac{1}{2}}} e^{+i\omega(x_0-y_0)} e^{\pm 2i\mu(\omega(x_0-y_0)+\omega y_0)} e^{\pm 2i\mu(\mu+1)c}(e^{-\beta x_0^{\frac{1}{2}}} - e^{-\beta y_0^{\frac{1}{2}}}) = \]
\[ = 2 \sum_{\nu=0}^{\infty} \sum_{\mu=0}^{\infty} \frac{1}{\nu! \mu!} \frac{1}{\beta^\nu} \left( \mp 2i \frac{a}{\beta} \right)^\nu \int_0^1 dc \frac{1}{[(\nu+1)+(\mu+1)c]} \frac{\partial}{\partial \beta} \times \]
\[ \sqrt{\frac{i\pi}{\omega(1+c^2) \pm k_3(1-c^2)}} \exp \left( \frac{\beta[(\nu+1)+(\mu+1)c][2]^{\frac{3}{2}}}{-i\omega(1+c^2) \pm k_3(1-c^2)} \right) \text{erfc} \left( \frac{\beta[(\nu+1)+(\mu+1)c]}{-i\omega(1+c^2) \pm k_3(1-c^2)} \right) = \]
\[ = -a^{-2} \left( e^{\pm 2ia^{\frac{1}{2}}} - 1 \mp 2i \frac{a}{\beta} \right) + O \left( \frac{\omega}{\beta^3} \right) \] (A12)
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