Connections under Symplectic Reduction

by

Izu Vaisman

Dedicated to the memory of Prof. Gheorghe Vrăîceanu

ABSTRACT. In this note, we give conditions which ensure the reduction of a symplectic connection in the process of a Marsden-Weinstein reduction and of the reduction of a presymplectic manifold.

Symplectic reduction is a technique which produces new symplectic manifolds from either symplectic manifolds with symmetries or presymplectic manifolds [2]. The aim of this note is to formulate conditions which ensure that a compatible connection on the initial manifold induces a compatible connection on the reduced manifold. Everything in this note is of the class $C^{\infty}$.

For me, this subject is related with the name of Vrăîceanu because my Ph.D. thesis [4] was on the subject of symplectic connections, and Vrăîceanu was one of the referees.

1 Marsden-Weinstein Reduction

We refer the reader to [2] for the original Marsden-Weinstein reduction theorem. In this theorem, the basic configuration consists of: (i) a symplectic manifold $M^{2n}$ with the symplectic form $\omega$; (ii) a Hamiltonian action $\Phi : G \times M \to M$ of a Lie group $G$ on $M$ with an equivariant moment map $J : M \to \mathfrak{g}^*$ (the dual space of the Lie algebra $\mathfrak{g}$ of $G$); (iii) the level set...
$C := J^{-1}(\xi)$ of a non critical value $\xi \in \mathcal{G}^*$. Then, if $G_\xi$ is the isotropy subgroup of $\xi$ for the coadjoint action of $G$ on $\mathcal{G}^*$, the group $G_\xi$ acts on $C$, and, if $N := C/G_\xi$ is a manifold, $\omega$ induces a symplectic structure $\omega'$ on $N$.

Let us assume that $(M, \omega)$ is also equipped with a symplectic connection $\nabla$, i.e., a torsionless linear connection on $M$ such that $\nabla \omega = 0$ (e.g., [5]). In order to formulate a corresponding reduction theorem, we need the following notions: a) if the action $\Phi$ preserves the connection $\nabla$ we say that $\Phi$ is an affine action [1], b) a submanifold $C \subseteq M$ such that $T_C$ is preserved by $\nabla$-parallel translations along paths in $C$ is called a self-parallel submanifold.

Then, we get

1.1 Theorem. Let $(M, \omega)$ be a symplectic manifold with a symplectic connection $\nabla$ and an affine Hamiltonian action $\Phi$ of the Lie group $G$ which has the equivariant moment map $J$. Let $\xi \in \mathcal{G}^*$ be a non critical value of $J$ such that $C := J^{-1}(\xi)$ is $\nabla$-self-parallel, and such that the reduced symplectic manifold $(N = C/G_\xi, \omega')$ exists. Then $\nabla$ induces a well defined symplectic connection $\nabla'$ on $(N, \omega')$.

Proof. Since $C$ is self-parallel, the values of $\nabla_X Y$ are tangent to $C$ whenever $X, Y \in \Gamma T C$ ($\Gamma$ denotes spaces of global cross sections of bundles), and $\nabla$ may also be seen as a linear connection on $C$.

On the other hand, the connected components of the orbits of $G_\xi$ in $C$ define an isotropic foliation $\mathcal{I}$ such that $T \mathcal{I} = (T C)^\perp \cap T C$, and, since $\nabla$ is a symplectic connection, the bundle $T \mathcal{I}$ is parallel with respect to $\nabla$.

Now, we will use the connection $\nabla$ to define an induced connection $\nabla'$ on $N$ as follows. If $X, Y \in \Gamma TN$, we have $X = \pi_* X$, $Y = \pi_* Y$, where $\pi : C \to C/G_\xi$ is the natural projection and $X, Y$ are $G_\xi$-invariant vector fields on $C$ defined up to a term in $\Gamma T \mathcal{I}$. Accordingly, we shall take

\[(1.1) \quad \nabla'_X Y = \pi_*(\nabla_X Y),\]

and show that the result is well defined. The reasons for that are:

1) the result of (1.1) does not change if $Y \mapsto Y + Z$, $Z \in \Gamma T \mathcal{I}$, because $T \mathcal{I}$ is $\nabla$-parallel;

2) the same is true if $X \mapsto X + Z$, $Z \in \Gamma T \mathcal{I}$, since

\[(1.2) \quad \nabla_Z Y = \nabla_Y Z + [Y, Z] \in \Gamma T \mathcal{I};\]

(1.2) is justified by the fact that $\nabla$ has no torsion and preserves $T \mathcal{I}$, and by the fact that, locally, $Z = \sum_i \varphi_i Z_i$ where $\varphi_i \in C^\infty(C)$ and $Z_i \in \Gamma T \mathcal{I}$.
are infinitesimal actions of elements of the Lie algebra $\mathfrak{g}_\xi$ on $C$, therefore, $[Y, Z] = \sum_i (Y \varphi_i) Z_i \in \Gamma T\mathcal{I}$ because of the invariance of $Y$;

3) in (1.1), the vector field $\nabla_X Y$ is projectable since the action $\Phi$ of $G$ is affine, which means that, $\forall g \in G$, the transformation $\Phi_g(x) := \Phi(g, x) (x \in M)$ satisfies the condition

$$(1.3) \quad \Phi_g(\nabla_X Y) = \nabla_{\Phi_g X}(\Phi_g Y).$$

Q.e.d.

It would be interesting to have conditions which imply the fact that the level submanifold $C$ of Theorem 1.1 is self-parallel. We can give one such condition which, unfortunately, is not simple to use. Namely, let us agree to say that the moment map $J$ is affine if, whenever $J$ is constant along a path $\gamma$ in $M$, the differential $J_\gamma$ is constant along any $\nabla$-horizontal lift $\tilde{\gamma}$ of $\gamma$ to $TM$. It is easy to see that if $J$ is an affine moment map, $C = J^{-1}(\xi)$ is self-parallel for all the non critical values $\xi \in \mathcal{G}^*$. Indeed, for $Y \in \Gamma TM$ with $Y/C \in \Gamma TC$, and if $X_0 \in T_{x_0}C (x_0 \in C)$, one has

$$(\nabla_{X_0} Y)_{x_0} = \lim_{t \to 0} \frac{1}{t} [\tau_0^t(Y(x(t))) - Y(x_0)],$$

where $x(t)$ is a path in $M$ such that $x(0) = x_0$, $\dot{x}(0) = X_0$, and $\tau^t_0 : T_{x(t)}M \to T_{x_0}M$ is the $\nabla$-parallel translation along $x(t)$. Accordingly, if $x(t)$ is in $C$ then, because of the affine character of $J$, $(\nabla_{X_0} Y)_{x_0} \in T_{x_0}C$, and the submanifold $C$ is self-parallel.

An example where Theorem 1.1 applies will be given at the end of the next section.

2 Cotangent Bundles

An important example of a Hamiltonian action as required in reduction theory is provided by the natural lift $\Phi^*$ of an action $\Phi$ of a Lie group $G$ on a manifold $P$ to the cotangent bundle $T^*P$. The latter is endowed with the canonical symplectic structure $\omega = -d\lambda$, where $\lambda$ is the Lieouville 1-form

$$(2.1) \quad \lambda_\alpha(X) = \langle \alpha, \pi_* X \rangle, \quad \alpha \in T^*P, X \in T_\alpha T^*P, \pi : T^*P \to P,$$

and $\Phi^*$ has the moment map $J : T^*P \to \mathcal{G}^*$ given by

$$(2.2) \quad \langle J(\alpha), A \rangle = \langle \alpha, A_P(\pi(\alpha)) \rangle,$$
where $\alpha$ is as in (2.1), $A \in G$, the Lie algebra of $G$, and $A_P$ is the infinitesimal transformation defined by $A$ on $P$.

In this section, we assume that the action $\Phi$ is affine with respect to a torsionless connection $\nabla^P$ on $P$, and show how to get a symplectic connection $\nabla$ on $T^*P$ with respect to which the action $\Phi^*$ is affine. This sets the scene for possible applications of Theorem 1.1.

Let $V$ be the vertical distribution tangent to the fibers of $M := T^*P$, and $H$ be the horizontal distribution of the connection $\nabla^P$ on $M$. The following formula defines a connection $\nabla^H$ on the vector bundle $H$:

\[
\nabla^H_X Y = \begin{cases} 
pr_H [X, Y] & \text{if } X \in \Gamma V, Y \in \Gamma H, \\
\overset{\sim}{\nabla}^P_{\pi_*X \pi_*Y} & \text{if } X \in \Gamma_{pr}H, Y \in \Gamma_{pr}H,
\end{cases}
\]

where tilde denotes the operation of taking the horizontal lift, $pr$ stands for projectable, and projectability is with respect to $\pi$. If $Y$ of the second line of (2.3) is of the general form $\sum_i \varphi_i Y_i$, where $\varphi_i \in C^\infty(M)$, $Y_i \in \Gamma_{pr}H$, $\nabla^H_X Y$ is to be derived from (2.3) by means of the general properties of a covariant derivative. Notice also that the first line of (2.3) is equivalent to $\nabla^H_X Y = 0$ for all $X \in \Gamma V$ and for all $Y \in \Gamma_{pr}H$. It is easy to understand that (2.3) indeed provides a connection on $H$; in fact, it is a Bott connection with respect to the foliation by the fibers of $T^*P$.

The connection $\nabla^H$ transposes to a connection $\nabla^{H^*}$ on the dual bundle $H^*$ of $H$. On the other hand, the musical morphism $\sharp_\omega := \flat_\omega^{-1}$ is an isomorphism between $H^*$ and $V$ (e.g., [6]). Hence, $\nabla^V := \sharp_\omega \circ \nabla^{H^*} \circ \flat_\omega$ is a well defined connection on the vector bundle $V$.

Accordingly, we obtain a natural lift $\nabla^M := \nabla^H \oplus \nabla^V$ of the connection $\nabla^P$ to a connection on $T^*P$. This connection satisfies the condition $\nabla^M \omega = 0$. Indeed, let $(x^i)$ be local coordinates on $P$ such that

\[
\frac{\partial}{\partial x^j} = \Gamma_{ij}^k \frac{\partial}{\partial x^k} \quad (\Gamma_{ji}^k = \Gamma_{ij}^k),
\]

and let $(y_i)$ be the corresponding covector coordinates. Then, $(x^i, y_i)$ are local coordinates on $M$, and [6]

\[
\mathcal{V} = \text{span} \left\{ \frac{\partial}{\partial y_i} \right\}, \quad \mathcal{H} = \text{span} \left\{ X_i := \frac{\partial}{\partial x^i} + \Gamma_{ik}^s y_s \frac{\partial}{\partial y_k} \right\},
\]

\[
\mathcal{V}^* = \text{span} \{ \theta_i := dy_i - \Gamma_{ik}^s y_s dx^k \}, \quad \mathcal{H}^* = \text{span} \{ dx^i \},
\]

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In all these formulas the Einstein summation convention is used.

Accordingly, we get the following local equations of $\nabla^M$

\begin{align}
\nabla^M_{\frac{\partial}{\partial y_i}} X_j &= 0, \\
\nabla^M_{\frac{\partial}{\partial y_j}} \frac{\partial}{\partial y_i} &= 0,
\end{align}

and $\nabla^M \omega = 0$ follows by straightforward computations.

Furthermore, if we come back to the $\nabla^P$-affine action $\Phi$, its affine character implies the invariance of the horizontal distribution $\mathcal{H}$ by $\Phi^*$. Then, if we use $Y \in \Gamma_{pr} \mathcal{H}$ on the first line of (2.3) it is easy to check that the connection $\nabla^M$ satisfies condition (1.3). Therefore, $M$ is endowed with the $\omega$-compatible connection $\nabla^M$ and the affine Hamiltonian action $\Phi^*$.

This is not yet the required situation since $\nabla^M$ may have torsion [3]. To fix this, let

\[ ^t \nabla^M_X Y := \nabla^M_Y X + [X, Y] \]

be the transposed connection, and

\[ \overset{\circ}{\nabla} := \frac{1}{2}(\nabla_X Y + ^t \nabla_X Y) \]

be the connection on $M$ known as the symmetric part of the connection $\nabla^M$. Then $\overset{\circ}{\nabla}$ is a torsionless connection which satisfies the condition (1.3), hence, it is preserved by the action $\Phi^*$ but, it is no more symplectic.

However, there exists a well known transformation of such a connection which yields a symplectic one. Namely (e.g., [3]):

\begin{align}
\nabla_X Y &= \overset{\circ}{\nabla}_X Y + A(X, Y),
\end{align}

where $A(X, Y)$ is defined by the relation

\begin{align}
\omega(A(X, Y), Z) &= \frac{1}{2}(\overset{\circ}{\nabla}_X \omega)(Y, Z) + \frac{1}{6}\{((\overset{\circ}{\nabla}_Y \omega)(X, Z) + (\overset{\circ}{\nabla}_Z \omega)(X, Y)\}.
\end{align}
Since both $\tilde{\nabla}$ and $\omega$ are invariant by the action $\Phi^*$ of $G$, the connection $\nabla$ of (2.9) satisfies condition (1.3). Therefore, the action $\Phi^*$ is affine with respect to the connection $\nabla$, and we are done.

Now, as a corollary of the above construction and of Theorem 1.1, we have

2.1 Proposition. Let $P$ be an arbitrary manifold with a linear connection $\nabla^P$, and an affine action $\Phi$ of a Lie group $G$. Then, the cotangent bundle $T^*P$ with the canonical symplectic structure $\omega$ has a well defined symplectic connection $\nabla$ (the lift of $\nabla^P$) which is preserved by the lift $\Phi^*$ of $\Phi$. Furthermore, if there exists $\xi \in G^*$ which is non critical for the natural moment map $J$ of $\Phi^*$ (given by (2.2)), and such that $J^{-1}(\xi)$ is $\nabla$-self-parallel, the connection $\nabla$ reduces to a symplectic connection of the reduced manifold $J^{-1}(\xi)/G_\xi$ (if the latter exists).

We end this section by a simple example which illustrates both Proposition 2.1 and Theorem 1.1. Take $P = \mathbb{R}^n$ with the natural flat torsionless connection $\nabla$. Then, any affine subgroup on $\mathbb{R}^n$ is affine with respect to $\nabla$, and we will use the group $G$ given by

$$
(2.11) \quad \tilde{x}^a = sx^a + t^a, \quad \tilde{x}^u = x^u,
$$

where $a = 1, \ldots, h \leq n$, $u = h + 1, \ldots n$, $(x^a, x^u)$ are natural coordinates in $\mathbb{R}^n$, and $s, t^a \in \mathbb{R}$, $s \neq 0$.

Then, $T^*P = \mathbb{R}^{2n}$ with coordinates $(x^i, y_i)$ ($i = 1, \ldots, n$), and with the canonical symplectic form (2.7). Furthermore, in (2.4) $\Gamma^k_{ji} = 0$, and it follows from (2.8) that the lifted connection $\nabla$ is exactly the natural flat torsionless connection of $\mathbb{R}^{2n}$. The action defined by (2.11) on covector coordinates $(y_i)$ is given by

$$
(2.12) \quad \tilde{y}_a = \frac{1}{s}y_a, \quad \tilde{y}_u = y_u.
$$

Thus, the resulting action on $\mathbb{R}^{2n}$ is affine with respect to the flat connection $\nabla$.

An infinitesimal action of the Lie algebra $\mathcal{G}$ of $G$ is obtained from (2.11) by taking $s = 1 + \epsilon\sigma$, $t^a = \epsilon\tau^a$, where $(\sigma, \tau^a)$ is a vector in $\mathcal{G}$, then, taking the derivative with respect to $\epsilon$ at $\epsilon = 0$. 
Now, (2.2) shows that the moment map $J$ has the expression

\[
J(x, y) = \left( \sum_{a=1}^{h} x^a y_a, y_1, ..., y_h \right) \in G^*,
\]

and, for instance, it follows that $\xi = (0, 1, ..., 1) \in G^*$ is a non critical value of $J$. The corresponding level set $J^{-1}(\xi)$ has the equations

\[
y_1 = 1, ..., y_h = 1, \sum_{a=1}^{h} x^a = 0,
\]

and it is a plane of dimension $2n - h - 1$ in $\mathbb{R}^{2n}$. Therefore, $J^{-1}(\xi)$ is self-parallel with respect to the flat connection $\nabla$. From (2.12), (2.13), we see that the isotropy subgroup $G_\xi$ consists of the transformations (2.11) where $s = 1$ i.e., translations of the coordinates $(x^a)$ such that $\sum_{a=1}^{h} x^a = 0$ (notation of (2.11)). Accordingly, the orbits of $G_\xi$ are the $(h - 1)$-dimensional planes through the points of $J^{-1}(\xi)$, where the coordinates in the planes are $(x^a)$ with $\sum_{a=1}^{h} x^a = 0$. Therefore, we have a quotient manifold

\[
J^{-1}(\xi)/G_\xi = \mathbb{R}^{2(n-h)} = \{(x^u, y_u)_{u=1}^{n-h+1}\}.
\]

The restriction of the form $\omega$ to $J^{-1}(\xi)$ is $\omega' = \sum_{u=1}^{n} dx^u \wedge dy_u$, and this also is the expression of the reduced symplectic form induced by $\omega$. We see that we are in a situation where Proposition 2.1 and Theorem 1.1 apply, and the reduced symplectic connection is again the natural flat torsionless connection of $\mathbb{R}^{2(n-h)}$.

3 Presymplectic Manifolds

From the geometrical point of view, the basic configuration where symplectic reduction is encountered is that of a presymplectic manifold i.e., a $(2n + p)$-dimensional differentiable manifold $M$ endowed with a closed 2-form $\omega$ of constant rank $2n$. Then, the $p$-dimensional distribution

\[
I := \text{span}\{X \in \Gamma TM / i(X)\omega = 0\}
\]

is integrable, and yields a foliation $\mathcal{I}$, $T\mathcal{I} = I$, called the characteristic foliation. Furthermore, if the space of leaves $Q = M/\mathcal{I}$ is a Hausdorff manifold,
Q is endowed with the reduced symplectic form $\omega'$ which is the projection of $\omega$, and $(Q, \omega')$ is the reduced symplectic manifold of $(M, \omega)$ \[^2\].

Let us define a presymplectic connection on a presymplectic manifold $(M, \omega)$ as being a linear connection $\nabla$ on $M$ which is $\omega$-compatible i.e., $\nabla \omega = 0$, and its torsion $T_\nabla$ takes values in the vector bundle $I$. Then, let us assume that $(M, \omega)$ is endowed with a presymplectic connection $\nabla$. We want to find conditions which ensure the existence of a corresponding induced symplectic connection on the reduced manifold $(Q, \omega')$. We begin with

3.1 Proposition. Let $V$ be a differentiable manifold endowed with a foliation $\mathcal{F}$ and with a linear connection $\nabla$ such that: i) $\mathcal{F}$ is parallel with respect to $\nabla$ (i.e., $\nabla(T\mathcal{F}) \subseteq T\mathcal{F}$), ii) the torsion $T_\nabla$ of $\nabla$ takes values in $T\mathcal{F}$, iii) the curvature $R_\nabla$ of the connection $\nabla$ satisfies the condition

\[(3.1) \quad R_\nabla(Z, X)Y \in \Gamma T\mathcal{F}, \quad \forall Z \in \Gamma T\mathcal{F}, \forall X, Y \in \Gamma TV.\]

Then $\nabla$ induces an $\mathcal{F}$-projectable connection on the normal bundle $\nu\mathcal{F}$ of the foliation $\mathcal{F}$.

Proof. By definition, $\nu\mathcal{F} = TV/T\mathcal{F}$, and a cross section $\sigma \in \Gamma \nu\mathcal{F}$ is an equivalence class $\sigma = [Y]_\mathcal{F}$ where $Y \in \Gamma TV$, and $Y_1, Y_2$ yield the same class $\sigma$ iff $Y_1 - Y_2 \in \Gamma T\mathcal{F}$. Accordingly, the result of the formula

\[(3.2) \quad \nabla'_X \sigma := [\nabla_X Y]_\mathcal{F} \quad (X \in \Gamma TV)\]

is independent of the choice of $Y$ (hypothesis i)), and $\nabla'$ is a well defined connection on $\nu\mathcal{F}$.

We want to show that $\nabla'$ is $\mathcal{F}$-projectable i.e., it projects to connections of the local slice spaces of $\mathcal{F}$. The conditions for this are: a) if $\sigma$ is a projectable cross section and $X \in \Gamma T\mathcal{F}$ then $\nabla'_X \sigma = 0$, b) if $\sigma$ is a projectable cross section of $\nu\mathcal{F}$ and $X$ is a projectable vector field then $\nabla'_X \sigma$ is a projectable cross section of $\nu\mathcal{F}$ \[^3\]. We recall that the projectability of $\sigma$ means that the vector fields which represent $\sigma$ are projectable, which, in turn, means that their flow preserves the foliation $\mathcal{F}$.

Condition a) easily follows from i) and ii).

For b), notice that $\forall Z \in \Gamma T\mathcal{F}, \forall X, Y \in \Gamma \text{pr} TM$ one has

$$\nabla_Z \nabla_X Y = \nabla_X \nabla_Z Y + \nabla_{[Z, X]} Y + R_\nabla(Z, X)Y \in \Gamma T\mathcal{F},$$
because of a) and of hypothesis iii). (In (3.1), we didn’t have to ask $X, Y \in \Gamma_{pr}TM$ since, $R_{\nabla}$ being a tensor, (3.1) is a pointwise condition.) Accordingly, and using i) and ii) again, we get

$$[Z, \nabla_X Y] = \nabla_Z \nabla_X Y - \nabla_{\nabla_X Y} Z - T_{\nabla}(Z, \nabla_X Y) \in \Gamma TF,$$

therefore, $\nabla_X Y \in \Gamma_{pr}TM$. Q.e.d.

Using Proposition 3.1 we get

3.2 Proposition. Let $(M, \omega)$ be a presymplectic manifold with the characteristic foliation $\mathcal{I}$, and let $\nabla$ be a presymplectic connection on $M$. Assume that the curvature of $\nabla$ satisfies the condition

$$(3.3) \quad i(R_{\nabla}(Z, X)Y)\omega = 0, \quad \forall Z \in \Gamma T\mathcal{I}, \forall X, Y \in \Gamma TM.$$

Then, if the reduced symplectic manifold $(Q = M/\mathcal{I}, \omega')$ exists, it has a symplectic connection $\nabla'$ induced by $\nabla$.

Proof. It is easy to see that $\nabla \omega = 0$ implies hypothesis i) of Proposition 3.1 for the foliation $\mathcal{I}$ on $M$. Hypothesis ii) holds because of the definition of a presymplectic connection, and iii) follows from (3.3). Therefore, there exists an induced projectable connection $\nabla'$ on $\nu\mathcal{I}$, which, obviously, projects to a connection on $Q$ that we also denote by $\nabla'$. From the definition of $\nabla'$ as given in the proof of Proposition 3.1, and since $T_{\nabla}$ takes values in $I$, it follows that, on $Q$, $\nabla'$ has no torsion and that $\nabla \omega = 0$ implies $\nabla' \omega' = 0$. Q.e.d.

We remember that hypothesis iii) of Proposition 3.1 is what ensures the projectability of the “$\mathcal{I}$-normal part” of $\nabla$. We want to notice that this property may also be expressed without the use of the normal bundle of $\mathcal{I}$. Namely, if $V$ is a manifold with a foliation $\mathcal{F}$, we will say that a linear connection $\nabla$ on $V$ is $\mathcal{F}$-adapted if $\forall X \in \Gamma T\mathcal{F}$ and $\forall Y \in \Gamma_{pr}TV$, $\nabla_X Y \in \Gamma T\mathcal{F}$. Then if, moreover, $\forall X, Y \in \Gamma_{pr}TV$ one has $\nabla_X Y \in \Gamma_{pr}TV$, we will say that the connection $\nabla$ is $\mathcal{F}$-projectable. These definitions are correct since they are compatible with multiplication of a vector field in $\Gamma \mathcal{F}$ by an arbitrary function, and with multiplication of $X, Y \in \Gamma_{pr}TV$ by a projectable function. It is easy to see that any foliated manifold has adapted connections, and, if the foliation has a transversal projectable connection, the manifold has a projectable connection.

On a presymplectic manifold $(M, \omega)$, any presymplectic connection $\nabla$ is adapted to the characteristic foliation $\mathcal{I}$. Indeed, $\mathcal{I}$ is $\nabla$-parallel, and the
condition which was imposed on $T\nabla$ yields

$$\nabla_X Y = \nabla_Y X + [X, Y] + T\nabla(X, Y) \in \Gamma TT,$$

$\forall X \in \Gamma T$, $\forall Y \in \Gamma pr TM$. Using this remark, it follows

3.3 Proposition. If the reduced symplectic manifold $(Q = M/I, \omega')$ of the presymplectic manifold $(M, \omega)$ exists, any $I$-projectable presymplectic connection $\nabla$ on $M$ induces a symplectic connection $\nabla'$ on $Q$.

Proof. Any vector fields $X, Y \in \Gamma TQ$ are projections of vector fields $X, Y \in \Gamma pr TM$, and if we put $\nabla'_X Y = \text{pr}(\nabla_X Y)$ we are done. Q.e.d.

Proposition 3.3 raises the question of existence of projectable presymplectic connections on a given presymplectic manifold $(M, \omega)$.

First, we use the known technique of the almost symplectic case e.g., [5] to get all the connections which satisfy $\nabla_\omega = 0$. Let us fix a transversal distribution $S$ of the characteristic distribution $I$ ($TM = S \oplus I$), a connection $D^I$ on the vector bundle $I$, and a connection $D^S$ on the vector bundle $S$. Then, change the connection $D^S$ to the Bott connection [3]

$$D^{S, X' + X''}Y' = \text{pr}_S[X'', Y'] + D^S X', Y',$$

where $X', Y' \in \Gamma S$, $X'' \in \Gamma I$, and define the connection

$$D = D^S \oplus D^I$$

on $TM$. Furthermore, let us define $\Theta \in \Gamma End(TM \wedge TM, TM)$ by

$$\Theta(X'', Y'') = 0, \Theta(X', Y''') = 0, \Theta(X', Y') \in \Gamma S,$$

$$\omega(\Theta(X', Y''), Z') = \frac{1}{2}(D_X \omega)(Y', Z'),$$

$\forall X', Y' \in \Gamma S, \forall X'', Y'' \in \Gamma I$. (The result is well defined because $\omega$ is non degenerate on $S$.)

Using $\Theta$, we get a new connection on $M$ namely,

$$\nabla = D + \Theta,$$

and the evaluation of $(\nabla \omega)(X, Y)$ for $X, Y \in \Gamma pr TM$ yields $\nabla \omega = 0$. Hence, we obtained one $\omega$-compatible connection. Necessarily, all the others are given by

$$\nabla = \nabla + A,$$
where $A \in \Gamma \text{End}(TM \wedge TM, TM)$ satisfies

$$
\omega(A(X, Y), Z) + \omega(Y, A(X, Z)) = 0.
$$

The torsion of the connection $D$ given by (3.6) is

$$
T_D(X' + X'', Y' + Y'') = D^S_X Y' - D^S_Y X' - pr_S[X', Y'] + \text{term in } \Gamma I
$$

(notation of (3.5)). Hence, if we take $\hat{D} = pr_S \circ K$, where $K$ is a torsionless covariant derivative on $M$, the associated connection $D$ will have an $I$-valued torsion, and it will follow

$$
d\omega(X, Y, Z) = \sum_{\text{cycl.}(X,Y,Z)} (D_X \omega)(Y, Z) = 0.
$$

Furthermore, the torsion of the connection (3.9) is

$$
T_\nabla(X, Y) = T_D(X, Y) + \Theta(X, Y) - \Theta(Y, X) + A(X, Y) - A(Y, X),
$$

where $X, Y \in \Gamma TM$, $T_D(X, Y) \in \Gamma I$, and from (3.7), (3.12) we obtain

$$
\omega(T_\nabla(X', Y'), Z') = \omega(A(X', Y'), Z') - \omega(A(Y', X'), Z') - \frac{1}{2}(D_Z \omega)(X', Y'),
$$

where $X', Y', Z' \in \Gamma TS$. Then, if we ask $A(X'', Y) = 0$ for $X'' \in \Gamma I$, $Y \in \Gamma TM$, and $A(X', Y') \in \Gamma S$, and such that [3]

$$
\omega(A(X', Y'), Z') = \frac{1}{6}\{(D_Y \omega)(X', Z') + (D_Z \omega)(X', Y')\},
$$

we obtain a connection $\nabla$ which satisfies the condition

$$
\omega(T_\nabla(X', Y'), Z') = 0,
$$

therefore, $\nabla$ has an $I$-valued torsion.

With a closer look at this latter connection we see that, in fact, the following result holds
3.4 Proposition. Every presymplectic manifold \((M, \omega)\) has presymplectic connections. Moreover, the manifold has a projectable presymplectic connection with respect to the characteristic foliation \(\mathcal{I}\), iff the normal bundle of \(\mathcal{I}\) has a projectable connection.

Proof. The first part was already proven above. For the second part, if the normal bundle \(\nu\mathcal{I}\) has a projectable connection, we may use it as \(D^S\) of (3.5), and we see that the connection \(\nabla\) defined by the use of (3.15) is projectable. Conversely, if \(\nabla\) is a projectable presymplectic connection on \((M, \omega)\), and if we put

\[
D_X[Y]_x = [\nabla_X Y]_x \quad \forall X, Y \in \Gamma TM
\]

we get a projectable connection on \(\nu\mathcal{I}\). Q.e.d.

We recall that a foliation has a projectable connection on its normal bundle iff the Atiyah class of the foliation vanishes \([3]\).

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Department of Mathematics,
University of Haifa, Israel.
E-mail: vaisman@math.haifa.ac.il