The Lagrange Inversion Theorem in the Smooth Case

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Abstract: The classical Lagrange inversion theorem is a concrete, explicit form of the implicit function theorem for real analytic functions. The authors derive a suitable version of this result for \( C^{\infty} \) functions. Along the way, they find a new asymptotic for smooth functions.

1 The Problem

The implicit function theorem has a long and colorful history. Finding its provenance in considerations of problems of celestial mechanics (as studied by Lagrange and Cauchy, among others), the result was at first a rather primitive observation about monotone functions on \( \mathbb{R}^1 \). Over time, the result was extended to \( N \) variables, and the monotonicity hypothesis was replaced by the now more familiar assumption of nondegeneracy of the Jacobian at a point (see [KPb] for a more detailed history).

It is easy to imagine that there were a number of vestigial forms of the implicit function theorem that historically preceded the crisp result that can be found in textbooks today. One of these is the so-called Lagrange inversion theorem. That is the topic of the present paper. The classical Lagrange theorem is about analytic functions. Our purpose here is to extend the result to \( C^k \) or \( C^{\infty} \) functions.

The form of the Lagrange inversion theorem that we will consider in this paper concerns solving

\[ y = x + f(y) \] (1)

for \( y \) as a function of \( x \). Lagrange’s result is that, if \( f(0) = 0 \), \( |f'(0)| < 1 \),

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1 The authors are happy to thank the American Institute of Mathematics for its hospitality during the writing of this paper.

2 Lagrange’s inversion formula can be found in [LAG]. The date of the memoir is 1768, but it was published in 1770 in *Histoire de L’Académie Royal des Sciences et Belles-Lettres*. 

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and \( f \) is real analytic, then
\[
y = x + \sum_{n=1}^{\infty} \frac{1}{n!} \left( \frac{d}{dx} \right)^{n-1} \{ f^n(x) \} = x + f(x) + \sum_{n=1}^{\infty} \frac{1}{(n+1)!} \left( \frac{d}{dx} \right)^n \{ f^{n+1}(x) \} .
\]

One can obtain (2) from [KPb; Theorem 2.3.1] by complexifying \( f \), substituting \( \phi(z) = f(z) \), and taking \( \psi(z) \equiv z, t = 1 \). A version of the Lagrange theorem in the context of real analysis, for smooth functions, is to be found in [GRO]. The arguments presented there seem to be incomplete, and the statement of the result incorrect. The purpose of the present paper is to explore these matters further and to set the record straight.

We will find it useful to combine the Lagrange inversion formula with the formula of Faá di Bruno.\(^{3}\) The formula of Faá di Bruno tells us that if \( I \) and \( J \) are open intervals in \( \mathbb{R} \) and \( f : I \to J \) and \( h : J \to \mathbb{R} \) are \( C^\infty \) functions, then the \( n \)th derivative of \( h \circ f \) is given by
\[
(h \circ f)^{(n)}(t) = \sum \frac{n!}{k_1! k_2! \cdots k_n!} h^{(k)}(f(t)) \left( \frac{f^{(1)}(t)}{1!} \right)^{k_1} \left( \frac{f^{(2)}(t)}{2!} \right)^{k_2} \cdots \left( \frac{f^{(n)}(t)}{n!} \right)^{k_n},
\]
where \( k = k_1 + k_2 + \cdots + k_n \) and the sum is taken over all \( k_1, k_2, \ldots, k_n \) for which \( k_1 + 2k_2 + \cdots + nk_n = n \).

In particular, when \( h(t) = t^{n+1} \), we have
\[
h^{(k)}(t) = \begin{cases} 
    \left[ \frac{(n+1)!}{(n+1-k)!} \right] t^{n+1-k} & \text{if } k \leq n+1, \\
    0 & \text{if } n+1 < k.
\end{cases}
\]

Faá di Bruno’s formula then becomes
\[
\left( \frac{d}{dx} \right)^n \{ f^{n+1}(x) \}(t) = \sum \frac{n! (n+1)!}{k_0! k_1! \cdots k_n!} \left( \frac{f^{(0)}(t)}{0!} \right)^{k_0} \left( \frac{f^{(1)}(t)}{1!} \right)^{k_1} \cdots \left( \frac{f^{(n)}(t)}{n!} \right)^{k_n},
\]

\(^{3}\)Faá di Bruno’s formula first appeared in [FDB]—for a proof, see Section 1.3 of [KPa].
where \( k_0 = (n + 1) - (k_1 + k_2 + \cdots + k_n) \) and the sum is taken over all \( k_1, k_2, \ldots, k_n \) for which \( k_1 + 2k_2 + \cdots + nk_n = n \). Accordingly, Lagrange’s inversion formula can be written

\[
y = x + f(x) + \sum_{n=1}^{\infty} n! \sum_{i=0}^{n} \frac{1}{k_i!} \left( \frac{f^{(i)}(x)}{i!} \right)^{k_i},
\]

(5)

where the inner sum is taken over all \( k_1, k_2, \ldots, k_n \) for which \( k_1 + 2k_2 + \cdots + nk_n = n \) and where \( k_0 \) is defined by \( k_0 = (n + 1) - (k_1 + k_2 + \cdots + k_n) \).

We can rewrite (5) as

\[
y = x + f(x) + \sum_{(k_0, \ldots, k_m) \in \Xi} C_{k_0, \ldots, k_m} \cdot \left[ f(x) \right]^{k_0} \cdot \prod_{i=1}^{m} \left[ f^{(i)}(x) \right]^{k_i},
\]

(6)

where

\[
\Xi = \bigcup_{n=1}^{\infty} \bigcup_{m=1}^{n} \left\{ (k_0, k_1, \ldots, k_m) : 0 \leq k_i, \text{ for } i = 1, \ldots, m - 1, \\
1 \leq k_m, \\
k_0 = (n + 1) - (k_1 + k_2 + \cdots + k_m), \\
k_1 + 2k_2 + \cdots + mk_m = n \right\},
\]

and where

\[
C_{k_0, \ldots, k_m} = \frac{(k_1 + 2k_2 + \cdots + mk_m)!}{k_0!} \left( \prod_{i=1}^{m} k_i! \right)^{k_i},
\]

for \((k_0, \ldots, k_m) \in \Xi\). Note that all the \( C_{k_0, \ldots, k_m} \) are positive.

2 A Counterexample in the \( C^\infty \) Case

We will now show that the Lagrange inversion formula is not true in the \( C^\infty \) category.

To begin with, we let \( f \) be a real analytic function defined on an open interval containing \([0, 1]\) and such that

(a) \( f(0) = 0 \),

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(b) $0 < f'(0) < 1$.

(c) for each non-negative integer $m$, $0 < f^{(m)}(t)$ holds for all $0 < t$.

Note that, by (b), we have

$$0 < f(t) < t \quad \text{for} \quad t \in (0, 1). \quad (7)$$

Many such functions exist. For example, we could take $f(x) = \lambda e^x - \lambda$, where $0 < \lambda < 1$.

Since $f$ is real analytic, the Lagrange inversion formula is valid in an open interval $I$ containing 0.

Next, we will define sequences $\{a_\ell\}$ and $\{b_\ell\}$ in $I$ that satisfy

$$b_0 > a_0 > b_1 > a_1 > \cdots b_\ell > a_\ell > \cdots > 0$$

$$b_\ell = a_\ell + f(b_\ell).$$

In fact, we may choose any $b_0$ in $I$ with $0 < b_0 < 1$. We set $a_0 = b_0 - f(b_0)$, noting that (7) implies that $0 < a_0 < b_0$. Proceeding inductively, we observe that if $b_0 > a_0 > b_1 > a_1 > \cdots b_\ell > a_\ell$ have already been chosen, then we may choose any $b_{\ell+1}$ with $0 < b_{\ell+1} < a_\ell$ and set

$$a_{\ell+1} = b_{\ell+1} - f(b_{\ell+1}).$$

Again we use (7) to conclude that $0 < a_{\ell+1} < b_{\ell+1}$.

The Lagrange inversion formula (6) tells us that

$$b_\ell = a_\ell + f(a_\ell) + \sum_{(k_0, \ldots, k_m) \in \Xi} C_{k_0, \ldots, k_m} \cdot \left[ f(a_\ell) \right]^{k_0} \prod_{i=1}^{m} \left[ f^{(i)}(a_\ell) \right]^{k_i} \quad (8)$$

holds, for $\ell = 0, 1, \ldots$.

Now we consider a $C^\infty$ function $g : \mathbb{R} \to \mathbb{R}$ such that

(d) $g(b_\ell) = 0$ for $\ell = 0, 1, \ldots$,

(e) $g(a_\ell) > 0$ for $\ell = 0, 1, \ldots$,

(f) $g^{(m)}(a_\ell) \geq 0$ for $\ell = 0, 1, \ldots$ and $m = 1, 2, \ldots$. 

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[We consider in Lemma 2.1 and Remark 2.2 why such a function $g$ exists.] Then we have

$$b_\ell = a_\ell + (f + g)(b_\ell).$$

(9)

If the Lagrange inversion formula held for the function $f + g$ in any neighborhood of 0, then it would hold at infinitely many of the pairs $(a_\ell, b_\ell)$ in (9). For such a pair $(a_\ell, b_\ell)$, the Lagrange inversion formula for the function $f + g$ would tell us that

$$b_\ell = a_\ell + (f + g)(a_\ell) + \sum_{(k_0, \ldots, k_m) \in \Xi} C_{k_0, \ldots, k_m} \cdot \left[ (f + g)(a_\ell) \right]^{k_0} \cdot \prod_{i=1}^{m} \left[ (f + g)^{(i)}(a_\ell) \right]^{k_i}$$

holds. But, in fact, (10) fails to hold for any choice of $\ell$. This is so because, while the lefthand sides of (8) and (10) are equal, the righthand side of (10) consists of the righthand side of (8) plus nonnegative terms, among which is the positive term $g(a_\ell)$.

Finally, it remains to show that a function $g$ satisfying the above conditions exists, but this fact follows from the next lemma.

**Lemma 2.1** Let $\{x_n\}_{n=1}^\infty$ be a strictly decreasing sequence of positive numbers with $\lim_{n \to \infty} x_n = 0$. If, for each $n = 1, 2, \ldots$, there is given a sequence $\{\sigma_{n,k}\}_{k=0}^\infty \in \{-1, 0, 1\}$, then there exists a $C^\infty$ function $g : \mathbb{R} \to \mathbb{R}$ such that

$$\text{sgn} \left[ g^{(k)}(x_n) \right] = \sigma_{n,k}, \text{ for } n = 1, 2, \ldots \text{ and } k = 0, 1, \ldots,$$

(11)

$$g^{(k)}(0) = 0, \text{ for } n = 1, 2, \ldots.$$  

(12)

**Proof.** Let $I_1, I_2, \ldots$ be pairwise disjoint open intervals with $x_n \in I_n$. Assume also that $I_1$ is bounded. The other intervals are automatically bounded, because $I_1 \subset (x_{n+1}, x_{n-1})$ holds for $n = 2, 3, \ldots$.

An old theorem of Émile Borel (see page 44 of [BOR] or else [KPa]) tells us that, for each $n$, there exists a $C^\infty$ function $\phi_n$ with

$$\phi_n^{(k)}(x_n) = \sigma_{n,k}, \text{ for } k = 0, 1, \ldots$$

(13)

We may assume that $\phi_n$ has compact support contained in $I_n$.

We will define

$$g(x) = \sum_{k=1}^\infty \xi_n^{-1} \phi_n(x),$$

(14)
where the positive numbers $\xi_n$ will be chosen large enough to insure that $g$ is $C^\infty$. Specifically, it is easy to see that the choices
\[ \xi_n = e^{1/x_{n+1}} \cdot \sup \{ |\phi_m^{(k)}(x)| : x \in I_m, \ 1 \leq m \leq n, \ 0 \leq k \leq n \} \]
will suffice to guarantee that $g$ is $C^\infty$ and that (12) holds.

It is clear that (13) and (14) insure that (11) holds.

**Remark 2.2** Lemma 2.1 is applied to construct a function $g$ satisfying (d), (e), and (f) by setting
\[ x_{2\ell+1} = b_{\ell}, \ \sigma_{2\ell+1,0} = 0, \ x_{2\ell+2} = a_{\ell}, \ \sigma_{2\ell+2,0} = 1, \text{ for } \ell = 0, 1, 2, \ldots \]
and setting
\[ \sigma_{n,k} = 0 \text{ for } n = 1, 2, \ldots \text{ and } k = 1, 2, \ldots. \]

### 3 A Positive Result in the $C^N$ Case

It is reasonable to conjecture—and we will show it to be true—that, if $f$ is $C^N$, then the function $y$ satisfying $y = x + f(y)$ is well approximated by the truncated Lagrange formula of order $N$. By the later we mean that part of
\[ x + \sum_{n=1}^{\infty} \frac{1}{n!} \left( \frac{d}{dx} \right)^{n-1} \{ f^n(x) \} \]
that only contains derivatives of $f$ of order $N$ or smaller (the truncated Lagrange formula is given below in (27)).

We need a lemma.

**Lemma 3.1** Suppose $f(x)$ is $N$ times continuously differentiable, $N \geq 1$, in a neighborhood of $x = 0$ with $f(0) = 0$ and $|f'(0)| < 1$. Let $P(x)$ be the Taylor polynomial of degree $N$ for $f$ at 0. If $y = y(x)$ satisfies
\[ y = x + f(y) \]
in a neighborhood of 0, then
\[ y - \left[ x + \sum_{n=1}^{\infty} \frac{1}{n!} \left( \frac{d}{dx} \right)^{n-1} \{ P^n(x) \} \right] = o(x^N) \]
as $x \to 0$. 

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Remark 3.2 Because $P(x)$ is the Taylor polynomial of degree $N$ for $f$ at 0, only derivatives of $f$ of order $N$ or less are involved in the construction of $P$. Thus we see that the expression

$$x + \sum_{n=1}^{\infty} \frac{1}{n!} \left( \frac{d}{dx} \right)^{n-1} \{P^n(x)\}$$

appearing in the lemma involves only derivatives of $f$ of order $N$ or less, but note that those derivatives are evaluated at 0 rather than at $x$ as are the derivatives in Lagrange’s formula.

Proof of the Lemma. By Taylor’s theorem, we may write

$$f(x) = P(x) + R(x), \quad (15)$$

where

$$R(x) = o(x^N) \text{ as } x \to 0. \quad (16)$$

Since $f$ is at least $C^1$ near 0, we can apply the implicit function theorem to see that there exists a function $y = y(x)$ satisfying

$$y = x + f(y). \quad (17)$$

Implicit differentiation shows us that $y'(0) = 1/[1 - f'(0)]$. Thus we have

$$y = O(x) \text{ as } x \to 0. \quad (18)$$

Combining (15) and (17), we have

$$y = x + P(y) + R(y). \quad (19)$$

Since $P(x)$ is an analytic function, we may apply the Lagrange inversion theorem to see that the function $z = z(x)$ given by

$$z = x + \sum_{n=1}^{\infty} \frac{1}{n!} \left( \frac{d}{dx} \right)^{n-1} \{P^n(x)\} \quad (20)$$

satisfies

$$z = x + P(z). \quad (21)$$

As before, we have

$$z = O(x) \text{ as } x \to 0. \quad (22)$$

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Note that (20) gives us
\[
y - \left( x + \sum_{n=1}^{\infty} \frac{1}{n!} \left( \frac{d}{dx} \right)^{n-1} \{ P^n(x) \} \right) = y - z. \tag{23}
\]

Now consider an \( x \) near 0 for which \( y(x) \neq z(x) \). Applying the mean value theorem to
\[ Q(t) = t - P(t) \]
on the interval \( [\min\{y, z\}, \max\{y, z\}] \), we obtain \( \xi \) in that interval with
\[
(y - z) - [P(y) - P(z)] = Q(y) - Q(z) = Q'(\xi)(y - z) = [1 - P'(\xi)](y - z). \tag{24}
\]
By (19), (21), and (24), we have
\[
R(y) = [y - x - P(y)] - [z - x - P(z)] = (y - z) - [P(y) - P(z)] = [1 - P'(\xi)](y - z). \tag{25}
\]
By (16) and (18), we have \( R(y) = o(x^N) \), so we see from (25) that
\[
[1 - P'(\xi)](y - z) = o(x^N). \tag{26}
\]
Since \( P'(0) = f'(0) < 1 \) and since \( \xi \to 0 \) as \( x \to 0 \) [by (18) and (22)], we conclude from (26) that \( y - z = o(x^N) \). The result follows from (23). \( \blacksquare \)

**Theorem 3.3** Suppose \( f(x) \) is \( N \) times continuously differentiable, \( N \geq 1 \), in a neighborhood of \( x = 0 \) with \( f(0) = 0 \) and \( |f'(0)| < 1 \). If \( y = y(x) \) satisfies
\[
y = x + f(y)
\]
in a neighborhood of 0, then
\[
y - \left[ x + f(x) + \sum_{(k_0, \ldots, k_m) \in \Xi_N} C_{k_0, \ldots, k_m} \cdot [f(x)]^{k_0} \cdot \prod_{i=1}^{m} [f^{(i)}(x)]^{k_i} \right] = o(x^N)
\]
as \( x \to 0 \). Here

\[
\Xi_N = \bigcup_{n=1}^{\infty} \bigcup_{m=1}^{\min\{n,N\}} \left\{ (k_0, k_1, \ldots, k_m) : \begin{array}{l}
0 \leq k_i, \text{ for } i = 1, \ldots, m-1, \\
1 \leq k_m,
\end{array} \right. \\
\left. k_0 = (n+1) - (k_1 + k_2 + \cdots + k_m), \\
k_1 + 2k_2 + \cdots + mk_m = n \right\},
\]

and

\[
C_{k_0,\ldots,k_m} = \frac{(k_1 + 2k_2 + \cdots + mk_m)!}{k_0!} \left( \prod_{i=1}^{m} k_i! (i!)^{k_i} \right)^{-1}.
\]

**Remark 3.4** By (6), we see that the expression

\[
x + f(x) + \sum_{(k_0,\ldots,k_m)\in\Xi_N} C_{k_0,\ldots,k_m} \cdot [f(x)]^{k_0} \cdot \prod_{i=1}^{m} [f^{(i)}(x)]^{k_i},
\]

appearing in the theorem involves only derivatives of \( f \) of order \( N \) or smaller, and those derivatives are evaluated at \( x \) as are the derivatives in Lagrange’s formula. Thus (27) is the truncated Lagrange formula described at the beginning of this section.

**Proof of the Theorem.** As in Lemma 3.1, let \( P(x) \) be the Taylor polynomial of degree \( N \) for \( f \) at 0. The formula of Faá di Bruno tells us that

\[
x + \sum_{n=1}^{\infty} \frac{1}{n!} \left( \frac{d}{dx} \right)^{n-1} \{ P^n(x) \}
\]

\[
= x + P(x) + \sum_{(k_0,\ldots,k_m)\in\Xi} C_{k_0,\ldots,k_m} \cdot [P(x)]^{k_0} \cdot \prod_{i=1}^{m} [P^{(i)}(x)]^{k_i}.
\]

Since \( P \) is a polynomial of degree \( N \), we have \( P^{(i)}(x) \equiv 0 \) whenever \( i > N \). Thus we have

\[
x + P(x) + \sum_{(k_0,\ldots,k_m)\in\Xi} C_{k_0,\ldots,k_m} \cdot [P(x)]^{k_0} \cdot \prod_{i=1}^{m} [P^{(i)}(x)]^{k_i}
\]

\[
= x + P(x) + \sum_{(k_0,\ldots,k_m)\in\Xi_N} C_{k_0,\ldots,k_m} \cdot [P(x)]^{k_0} \cdot \prod_{i=1}^{m} [P^{(i)}(x)]^{k_i}.
\]
Applying Lemma 3.1, we see that

\[
o(x^N) = y - \left[ x + \sum_{n=1}^{\infty} \frac{1}{n!} \left( \frac{d}{dx} \right)^{n-1} \{P^n(x)\} \right]
\]

\[
= y - \left[ x + P(x) + \sum_{(k_0,\ldots,k_m)\in \Xi_N} C_{k_0,\ldots,k_m} \cdot \left[ P(x)^{k_0} \cdot \prod_{i=1}^{m} [P^{(i)}(x)]^{k_i} \right] \right]
\]

\[
= y - \left[ x + f(x) + \sum_{(k_0,\ldots,k_m)\in \Xi_N} C_{k_0,\ldots,k_m} \cdot \left[ f(x)^{k_0} \cdot \prod_{i=1}^{m} [f^{(i)}(x)]^{k_i} \right] \right]
\]

\[
+ [f(x) - P(x)]
\]

\[
+ \sum_{(k_0,\ldots,k_m)\in \Xi_N} C_{k_0,\ldots,k_m}
\]

\[
\cdot \left( \left[ f(x)^{k_0} \cdot \prod_{i=1}^{m} [f^{(i)}(x)]^{k_i} \right] - \left[ P(x)^{k_0} \cdot \prod_{i=1}^{m} [P^{(i)}(x)]^{k_i} \right] \right)
\]

Of course, we have \(f(x) - P(x) = o(x^N)\). So it will suffice to show that

\[
[f(x)^{k_0} \cdot \prod_{i=1}^{m} [f^{(i)}(x)]^{k_i}] - [P(x)^{k_0} \cdot \prod_{i=1}^{m} [P^{(i)}(x)]^{k_i}] = o(x^N)
\]

(28)

holds, for each \((k_0,\ldots,k_m)\in \Xi_N\).
We rewrite the lefthand side of (28) as

\[
[f(x)]^{k_0} \cdot \prod_{i=1}^{m} [f(i)(x)]^{k_i} - [f(x)]^{k_0} \cdot \prod_{i=1}^{m-1} [f(i)(x)]^{k_i} \cdot [P^m(x)]^{k_m} \\
+ \cdots + \\
+ \left( [f(x)]^{k_0} \cdot \prod_{i=1}^{j} [f(i)(x)]^{k_i} \cdot \prod_{i=j+1}^{m} [P^i(x)]^{k_i} \\
- [f(x)]^{k_0} \cdot \prod_{i=1}^{j-1} [f(i)(x)]^{k_i} \cdot \prod_{i=j}^{m} [P^i(x)]^{k_i} \right) \\
+ \cdots + \\
+ [f(x)]^{k_0} \prod_{i=1}^{m} [P^i(x)]^{k_i} - [P(x)]^{k_0} \prod_{i=1}^{m} [P^i(x)]^{k_i}.
\]

Thus it will suffice to show that

\[
[f(x)]^{k_0} \left( [P^{j}(x)]^{k_j} - [f^{j}(x)]^{k_j} \right) = o(x^N)
\]

holds, for each \( j = 1, 2, \ldots, m \) for which \( k_j > 0 \), and that

\[
[f(x)]^{k_0} - [P(x)]^{k_0} = o(x^N).
\]

If \( j \in \{1, 2, \ldots, m\} \) and \( k_j > 0 \) holds, then we have

\[
[P^{j}(x)]^{k_j} - [f^{j}(x)]^{k_j} \\
= [P^{j}(x) - f^{j}(x)] \cdot \sum_{\ell=0}^{k_j-1} [P^{j}(x)]^{\ell} [f^{j}(x)]^{k_j-1-\ell} = o(x^{N-j}).
\]

Using \( f(x) = O(x^1) \) and \( P^{j}(x) - f^{j}(x) = o(x^{N-j}) \), we obtain

\[
[f(x)]^{k_0} \left( [P^{j}(x)]^{k_j} - [f^{j}(x)]^{k_j} \right) = o(x^{N-j+k_0}).
\]
By the definition of $k_0$ and using $n = k_1 + 2k_2 + \cdots + k_j + \cdots + mk_m$, we have

\[
N - j + k_0 \\
= N - j + (n + 1) - (k_1 + k_2 + \cdots + k_j + \cdots + k_m) \\
= N - j + 1 + k_2 + 2k_3 + \cdots + (j - 1)k_j + \cdots + (m - 1)k_m \\
= N + (j - 1)(k_j - 1) + \sum_{\ell=2,\ldots,m} (\ell - 1)k_\ell \geq N.
\]

Finally, we observe that

\[
\left[ f(x) \right]^{k_0} - \left[ P(x) \right]^{k_0} = \left( f(x) - P(x) \right) \sum_{i=1}^{k_0-1} \left[ f(x) \right]^{i} \left[ P(x) \right]^{k_0-i-1} \\
= o(x^N). \quad \blacksquare
\]

REFERENCES

[BOR] Émile Borel, Sur quelques points de la théorie des fonctions, Annales Scientifiques de l’École Normale Supérieure (3) 12 (1895), 9–55.

[FDB] Francesco Faá di Bruno, Note sur une nouvelle formule de calcul différentiel. Quarterly Journal of Pure and Applied Mathematics 1 (1857), 359–360.

[GRO] Nathaniel Grossman, A $C^\infty$ Lagrange inversion theorem, American Mathematical Monthly 112 (2005), 512–514.

[KPa] Steven G. Krantz & Harold R. Parks, A Primer of Real Analytic Functions, second edition, Birkhäuser, Boston, 2002.

[KPb] _______ The Implicit Function Theorem, Birkhäuser, Boston, 2002.

[LAG] Joseph Louis Lagrange, Nouvelle méthode pour résoudre les équations littérales par le moyen des séries, (Mémoires de l’Académie Royale des Sciences et Belles-Lettres de Berlin 24) Œuvres de Lagrange, volume 3, Gauthier-Villars, Paris, 1869, p. 25.