Online Learning with Continuous Variations: Dynamic Regret and Reductions

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Abstract

Online learning is a powerful tool for analyzing iterative algorithms. However, the classic adversarial setup fails to capture regularity that can exist in practice. Motivated by this observation, we establish a new setup, called Continuous Online Learning (COL), where the gradient of online loss function changes continuously across rounds with respect to the learner’s decisions. We show that COL appropriately describes many interesting applications, from general equilibrium problems (EPs) to optimization in episodic MDPs. Using this new setup, we revisit the difficulty of sublinear dynamic regret. We prove a fundamental equivalence between achieving sublinear dynamic regret in COL and solving certain EPs. With this insight, we offer conditions for efficient algorithms that achieve sublinear dynamic regret, even when the losses are chosen adaptively without any a priori variation budget. Furthermore, we show for COL a reduction from dynamic regret to both static regret and convergence in the associated EP, allowing us to analyze the dynamic regret of many existing algorithms.

1 INTRODUCTION

Online learning (Gordon, 1999; Zinkevich, 2003), which studies the interactions between a learner (i.e. an algorithm) and an opponent through regret minimization, has proved to be a powerful framework for analyzing and designing iterative algorithms. However, while classic setups focus on bounding the worst case, many applications are not naturally adversarial.

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In this work, we aim to bridge this reality gap by establishing a new online learning setup that better captures certain regularity that appears in practical problems.

Formally, an online learning problem repeats the following steps: in round $n$, the learner plays a decision $x_n$ from a decision set $\mathcal{X}$, the opponent chooses a loss function $l_n : \mathcal{X} \to \mathbb{R}$ based on the decisions of the learner, and then information about $l_n$ (e.g. $\nabla l_n(x_n)$) is revealed to the learner for making the next decision. This abstract setup (Shalev-Shwartz et al., 2012; Hazan et al., 2016) studies the adversarial setting where $l_n$ can be almost arbitrarily chosen except for minor restrictions like convexity. Often the performance is measured relatively through static regret,

$$\text{Regret}^s_N := \sum_{n=1}^{N} l_n(x_n) - \min_{x \in \mathcal{X}} \sum_{n=1}^{N} l_n(x).$$  (1)

Recently, interest has emerged in algorithms that make decisions that are nearly optimal at every round. The regret is therefore measured on-the-fly and suitably named dynamic regret,

$$\text{Regret}^d_N := \sum_{n=1}^{N} l_n(x_n) - \sum_{n=1}^{N} l_n(x_n^*),$$  (2)

where $x_n^* \in \arg\min_{x \in \mathcal{X}} l_n(x)$. As dynamic regret by definition upper bounds static regret, minimizing dynamic regret is a more difficult problem.

While algorithms with sublinear static regret are well understood, the research on dynamic regret is relatively recent. As dynamic regret grows linearly in the adversarial setup, most papers (Zinkevich, 2003; Mokhtari et al., 2016; Yang et al., 2016; Dixit et al., 2019; Besbes et al., 2015; Jadabaie et al., 2015; Zhang et al., 2017) focus on how dynamic regret depends on certain variations of the loss sequence across rounds (such as the path variation $V_N = \sum_{n=1}^{N-1} \|x_n^* - x_{n+1}^*\|$).

Even if the algorithm does not require knowing the variation, the bound is still written in terms of it. While tight bounds have been established (Yang et al., 2016), their results do not always translate into conditions for achieving sublinear dynamic regret in practice, because the size (i.e. budget) of the variation can be difficult to verify beforehand. This is especially the
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case when the opponent is adaptive, responding to the learner’s decisions at each round. In these situations, it is unknown if existing results become vacuous or yield sublinear dynamic regret.

Motivated by the use of online learning to analyze iterative algorithms in practice, we consider a new setup we call Continuous Online Learning (COL), which directly models regularity in losses as part of the problem definition, as opposed to the classic adversarial setup that adds ad-hoc budgets. As we will see, this minor modification changes how regret and feedback interact and makes the quest of seeking sublinear dynamic regret well-defined and interpretable, even for adaptive opponents, without imposing variation budgets.

1.1 Definition of COL

A COL problem is defined as follows. We suppose that the opponent possesses a bifunction \( f : (x, x') \mapsto f_x(x') \in \mathbb{R} \), for \( x, x' \in \mathcal{X} \), that is unknown to the learner. This bifunction is used by the opponent to determine the per-round losses: in round \( n \), if the learner chooses \( x_n \), then the opponent responds with

\[ l_n(\cdot) = f_{x_n}(\cdot). \tag{3} \]

Finally, the learner suffers \( l_n(x_n) \) and receives feedback about \( l_n \). For \( f_x(x') \), we treat \( x \) as the query argument that proposes a question (i.e. an optimization objective \( f_x(\cdot) \)), and treat \( x' \) as the decision argument whose performance is evaluated. This bifunction \( f \) generally can be defined online as queried, with only the limitation that the same loss function \( f_x(\cdot) \) must be selected by the opponent whenever the learner plays the same decision \( x \). Thus, the opponent can be adaptive, but in response to only the learner’s current decision.

In addition to the restriction in (3), we impose regularity into \( f \) to relate \( l_n \) across rounds so that seeking sublinear dynamic regret becomes well defined.\(^1\)

**Definition 1.** We say an online learning problem is continuous if \( l_n \) is set as in (3) by a bifunction \( f \) satisfying, \( \forall x' \in \mathcal{X}, \nabla f_x(x') \) is a continuous map in \( x \).\(^2\)

The continuity structure in Definition 1 and the constraint (3) in COL limit the degree that losses can vary, making it possible for the learner to partially infer future losses from the past experiences.

The continuity may appear to restrict COL to purely deterministic settings, but adversity such as stochasticity can be incorporated via an important nuance in the relationship between loss and feedback. In the classic online learning setting, the adversity is incorporated in the loss: the losses \( l_n \) and decisions \( x_n \) may themselves be generated adversarially or stochastically and then they directly determine the feedback, e.g., given as full information (receiving \( l_n \) or \( \nabla l_n(x_n) \)) or bandit (just \( l_n(x_n) \)). The (expected) regret is then measured with respect to these intrinsically adversarial losses \( l_n \). By contrast, in COL, we always measure regret with respect to the true underlying bifunction \( l_n = f_{x_n} \). However, we give the opponent the freedom to add an additional stochastic or adversarial component into the feedback; e.g., in first-order feedback, the learner could receive \( g_n = \nabla l_n(x_n) + \xi_n \), where \( \xi_n \) is a probabilistically bounded and potentially adversarial vector, which can be used to model noise or bias in feedback. In other words, the COL setting models a true underlying loss with regularity, but allows the adversary to be modeled within the feedback. This addition is especially important for dynamic regret, as it allows us to always consider regret against the true \( f_{x_n} \), while incorporating the possibility of stochasticity.

1.2 Examples

At this point, the setup of COL may sound abstract, but this setting is in fact motivated by a general class of problems and iterative algorithms used in practice, some of which have been previously analyzed in the online learning setting. Generally, COL describes the trial-and-error principle, which attempts to achieve a difficult objective \( f_{x_n}(x) \) through iteratively constructing a sequence of simplified and related subproblems \( f_{x_n}(x) \), similar to majorize-minimize (MM) algorithms. Our first application of this kind is the use of iterative algorithms in solving (stochastic) equilibrium problems (EPs) (Bianchi and Schaible, 1996). EPs are a well-studied subject in mathematical programming, which includes optimization, saddle-point problems, variational inequality (VI) (Facchinei and Pang, 2007), fixed-point problems (FP), etc. Except for toy cases, these problems usually rely on using iterative algorithms to generate \( \epsilon \)-approximate solutions; interestingly, these algorithms often resemble known algorithms in online learning, such as mirror descent or Follow-the-Leader (FTL). In Sections 4 and 5, we will show how the residual function of these problems renders a natural choice of bifunction \( f \) in COL and how the regret of COL relates to its solution quality. In this example, it is particularly important to classify the adversary (e.g. due to bias or stochasticity) as feedback rather than as a loss function, to properly incorporate the continuity in the source problem.

Another class of interesting COL problems comes from optimization in episodic Markov decision processes

\(^1\)Otherwise the opponent can define \( f_x(\cdot) \) pointwise for each \( x \) to make \( l_n(x_n) - l_n(x_n^*) \) constant.

\(^2\)We define \( \nabla f_x(x') \) as the derivative with respect to \( x' \).
(MDPs). In online imitation learning (IL) (Ross et al., 2011), the learner optimizes a policy to mimic an expert policy $\pi^*$. In round $n$, the loss is $l_n(\pi) = \mathbb{E}_{s \sim d_n}[c(s, \pi, \pi^*)]$, where $d_n$ is the state distribution visited by running the learner’s policy $\pi_n$ in the MDP, and $c(s, \pi, \pi^*)$ is a cost that measures the difference between a policy $\pi$ and the expert $\pi^*$. This is a bifunction form where continuity exists due to expectation and feedback is noisy about $l_n$ (allowed by our feedback model). In fact, online IL is the main inspiration behind this research. An early analysis of IL was framed using the adversarial, static regret setup (Ross et al., 2011). Recently, results were refined through the use of continuity in the bifunction and dynamic regret (Cheng and Boots, 2018; Lee et al., 2018; Cheng et al., 2019b). This problem again highlights the importance of treating stochasticity as the feedback. We wish to measure regret with respect to the expected cost $l_n(\pi)$ which admits a continuous structure, but feedback only arrives via stochastic samples from the MDP. Structural prediction and system identification can be framed similarly (Ross and Bagnell, 2012; Venkatraman et al., 2015). Details, including new insights into the IL, can be found in Appendix F.

Lastly, we note that the classic fitted Q-iteration (Gordon, 1995; Riedmiller, 2005) for reinforcement learning also uses a similar setup. In the $n$th round, the loss can be written as $l_n(Q) = \mathbb{E}_{s, a \sim \mu(Q_n), \mu'(Q_n) \sim \mathcal{P}(s, a)}[\mathbb{E}_n(Q(s, a) - r(s, a) - \gamma \max_{a'} Q_n(s', a'))^2]$, where $\mu(Q_n)$ is the state-action distribution induced by running a policy $\pi(Q_n)$ based on the Q-function $Q_n$ of the learner, and $\mathcal{P}$ is the transition dynamics, $r$ is the reward, and $\gamma$ is the discount factor. Again this is a COL problem.

### 1.3 Main Results

The goal of this paper is to establish COL and to study, particularly, conditions and efficient algorithms for achieving sublinear dynamic regret. We choose not to pursue algorithms with fast static regret rates in COL, as there have been studies on how algorithms can systematically leverage continuity in COL to accelerate learning (Cheng et al., 2019b,a) although they are framed as online IL research. Knowledge of dynamic regret is less well-known, with the exception of Cheng and Boots (2018); Lee et al. (2018) (both also framed as online IL), which study the convergence of FTL and mirror descent, respectively.

Our first result shows that achieving sublinear dynamic regret in COL is equivalent to solving certain EP, VI, and FP problems that are known to be PPAD-complete (Daskalakis et al., 2009). In other words, we show that achieving sublinear dynamic regret that is polynomial in the dimension of the decision set can be extremely difficult.

Nevertheless, based on the solution concept of EP, VI, and FP, we show a reduction from monotone EPs to COL, and we present necessary conditions and sufficient conditions for achieving sublinear dynamic regret with polynomial dependency. Particularly, we show a reduction from sublinear dynamic regret to static regret and convergence to the solution of the EP/VI/FP. This reduction allows us to quickly derive non-asymptotic dynamic regret bounds of popular online learning algorithms based on their known static regret rates. Finally, we extend COL to consider partially adversarial loss and discuss open questions.

### 2 RELATED WORK

Much work in dynamic regret has focused on improving rates with respect to various measures of the loss sequence’s variation. Zinkevich (2003); Mokhtari et al. (2016) showed the dynamic regret of gradient descent in terms of the path variation. Other measures of variation such as functional variation (Besbes et al., 2015) and squared path variation (Zhang et al., 2017) have also been studied. While these algorithms may not need to know the variation size beforehand, their guarantees are still stated in terms of these variations. Therefore, these results can be difficult to interpret when the losses can be chosen adaptively.

To illustrate, consider the online IL problem. It is impossible to know the variation budget a priori because the loss observed at each round of IL is a function of the policy selected by the algorithm. This budget could easily be linear, if an algorithm selects very disparate policies, or it could be zero if the algorithm always naively returns the same policy. Thus, existing budget-based results cannot describe the convergence of an IL algorithm.

Our work is also closely related to that of Rakhlin and Sridharan (2013); Hall and Willett (2013), which consider predictable loss sequences, i.e., sequences that are presumed to be non-adversarial and admit improved regret rates. The former considers static regret for both full and partial information cases, and the latter considers a similar problem setting but for the dynamic regret case. These analyses, however, still require a known variation quantity in order to be interpretable.

By contrast, we leverage extra structures of COL to provide interpretable dynamic regret rates, without a priori constraints on the variation. That is, our rates are internally governed by the algorithms, rather than known to exist, but it is open as to if they belong to $P$.  

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Or some fixed distribution with sufficient excitation.  

4In short, they are NP problems whose solutions are
externally dictated by a variation budget. This problem setup is in some sense more difficult, as achieving sublinear dynamic regret requires that both the per-round losses and the loss variation, as a function of the learner’s decisions, be simultaneously small. Nonetheless, we can show conditions for sublinear dynamic regret using the bifunction structure in COL.

3 PRELIMINARIES

We review background, in particular VIs and EPs, for completeness (Facchinei and Pang, 2007; Bianchi and Schaible, 1996; Konnov and Laitinen, 2002).

Notation Throughout the paper, we reserve the notation $f$ to denote the bifunction that defines COL problems, and we assume $\mathcal{X} \subset \mathbb{R}^d$ is compact and convex, where $d \in \mathbb{N}_+$ is finite. We equip $\mathcal{X}$ with norm $\| \cdot \|$, which is not necessarily Euclidean, and write $\| \cdot \|_*$ to denote its dual norm. We denote its diameter by $D \mathcal{X} := \max_{x,x' \in \mathcal{X}} \| x - x' \|$. As in the usual online learning, we are particularly interested in the case where $f_x(\cdot)$ is convex and continuous. For simplicity, we will assume all functions are continuously differentiable, except for $f_x(x')$ as a function over the querying argument $x$, where $x' \in \mathcal{X}$. We will use $\nabla$ to denote gradients. In particular, for the bifunction $f$, we use $\nabla f$ to denote $\nabla f : x \mapsto \nabla f_x(x)$ and we recall, in the context of $f$, $\nabla$ is always with respect to the decision argument. Likewise, given $x \in \mathcal{X}$, we use $\nabla f_x$ to denote $\nabla f_x(\cdot)$. Note that the continuous differentiability of $f_x(\cdot)$ together with the continuity of $\nabla f(x)$ implies $\nabla f$ is continuous; the analyses below can be extended to the case where $\nabla f_x(\cdot)$ is a subdifferential.\footnote{Our proof can be extended to upper hemicontinuity for set-valued maps, such as subdifferentials.}

Convexity For $\mu \geq 0$, a function $h : \mathcal{X} \rightarrow \mathbb{R}$ is called $\mu$-strongly convex if it satisfies, for all $x, x' \in \mathcal{X}$, $h(x') \geq h(x) + \langle \nabla h(x), x' - x \rangle + \frac{\mu}{2} \| x - x' \|^2$. If $h$ satisfies above with $\mu = 0$, it is called convex. A function $h$ is called pseudo-convex if $\langle \nabla h(x), x' - x \rangle \geq 0$ implies $h(x') \geq h(x)$. These definitions have a natural inclusion: strongly convex functions are convex; convex functions are pseudo-convex. We say $h$ is $L$-smooth if $\nabla h$ is $L$-Lipschitz continuous, i.e., there is $L \in [0, \infty)$ such that $\| \nabla h(x) - \nabla h(x') \|_* \leq L \| x - x' \|$ for all $x, x' \in \mathcal{X}$. Finally, we will use Bregman divergence $B_R(x|z) := R(x') - R(x) - \langle \nabla R(x), x' - x \rangle$ to measure the difference between $x, x' \in \mathcal{X}$, where $R : \mathcal{X} \rightarrow \mathbb{R}$ is a $\mu$-strongly convex function with $\mu > 0$; by definition $B_R(\cdot|x)$ is also $\mu$-strongly convex.

Fixed-Point Problems Let $T : \mathcal{X} \rightarrow 2^\mathcal{X}$ be a point-to-set map, where $2^\mathcal{X}$ denotes the power set of $\mathcal{X}$. A fixed-point problem $FP(\mathcal{X}, T)$ aims to find a point $x^* \in \mathcal{X}$ such that $x^* \in T(x^*)$. Suppose $T$ is $\lambda$-Lipschitz. It is called non-expansive if $\lambda = 1$ and $\lambda$-contractive if $\lambda < 1$.

Variational Inequalities VIs study equilibria defined by vector-valued maps. Let $F : \mathcal{X} \rightarrow \mathbb{R}^d$ be a point-to-point map. The problems $VI(\mathcal{X}, F)$ and $DVI(\mathcal{X}, F)$ aim to find $x^* \in \mathcal{X}$ and $x_* \in \mathcal{X}$, respectively, such that the following conditions are satisfied:

$$
VI : \langle F(x^*), x - x^* \rangle \geq 0, \quad \forall x \in \mathcal{X} \\
DVI : \langle F(x), x - x_* \rangle \geq 0, \quad \forall x \in \mathcal{X}
$$

VIs and DVIs are also known as Stampacchia and Minty VIs, respectively (Facchinei and Pang, 2007). The difficulty of solving VIs depends on the property of $F$. For $\mu \geq 0$, $F$ is called $\mu$-strongly monotone if $\forall x,x' \in \mathcal{X}$. $(F(x) - F(x'), x - x') \geq \mu \| x - x' \|^2$. If $F$ satisfies the above with $\mu = 0$, $F$ is called monotone. $F$ is called pseudo-monotone if $\langle F(x'), x - x' \rangle \geq 0$ implies $\langle F(x), x - x' \rangle \geq 0$ for $x, x' \in \mathcal{X}$. It is known that the gradient of a (strongly/pseudo) convex function is (strongly/pseudo) monotone.

Equilibrium Problems EPs further generalize VIs. Let $\Phi : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}$ be a bifunction such that $\Phi(x, x) \geq 0$. The problems $EP(\mathcal{X}, \Phi)$ and $DEP(\mathcal{X}, \Phi)$ aim to find $x^*, x_* \in \mathcal{X}$, respectively, such that:

$$
EP : \Phi(x^*, x) \geq 0, \quad \forall x \in \mathcal{X} \\
DEP : \Phi(x, x_*) \leq 0, \quad \forall x \in \mathcal{X}.
$$

By definition, we have $VI(\mathcal{X}, F) = EP(\mathcal{X}, \Phi)$ if we define $\Phi(x, x') = \langle F(x), x' - x \rangle$.

We can also define monotonicity properties for EPs. For $\mu \geq 0$, $\Phi$ is called $\mu$-strongly monotone if for $\forall x, x' \in \mathcal{X}$. $\Phi(x, x') + \Phi(x', x) \leq -\mu \| x - x' \|^2$. It is called monotone if it satisfies the above with $\mu = 0$. Similarly, $\Phi$ is called pseudo-monotone if $\Phi(x, x') \geq 0$ implies $\Phi(x', x) \leq 0$ for $x, x' \in \mathcal{X}$. One can verify that these definitions are consistent with the ones for VIs.

Primal and Dual Solutions We establish some basics of the solution concepts of EPs. As VIs are a special case of EPs, these results can be applied to VIs too. First, we have a basic relationship between the solution sets, $X^*$ of EP and $x_*$ of DEP.
**Proposition 1.** (Bianchi and Schaible, 1996) If Φ is pseudo-monotone, $X^* \subseteq X_*$. If Φ(·, x) is continuous $\forall x \in \mathcal{X}$, $X_* \subseteq X^*$.

The proposition states that a dual solution is always a primal solution when the problem is continuous, and a primal solution is a dual solution when the problem is pseudo-monotone. Intuitively, we can think of the primal solutions $X^*$ as local solutions and the dual solutions $X_*$ as global solutions. In particular for VIs, if $F$ is a gradient of some, even nonconvex, function, any solution in $X_*$ is a global minimum; any local minimum of a pseudo-convex function is a global minimum (Konnov and Laitinen, 2002).

We note, however, that Proposition 1 does not directly ensure that the solution sets are non-empty. The existence of primal solutions $X^*$ has been extensively studied. Here we include a basic result that is sufficient for the scope of our online learning problems with compact and convex $\mathcal{X}$.

**Proposition 2.** (Bianchi and Schaible, 1996) If $\Phi(x, \cdot)$ is convex and $\Phi(\cdot, x)$ is continuous $\forall x \in \mathcal{X}$, then $X^*$ is non-empty.

Analogous results have been established for VIs and FP as well. If $F$ and $T$ are continuous then solutions exist for both VI($\mathcal{X}, F$) and FP($\mathcal{X}, T$), respectively (Facchinei and Pang, 2007). On the contrary, the existence of dual solutions $X_*$ is mostly based on assumptions. For example, by Proposition 1, $X_*$ is non-empty when the problem is pseudo-monotone. Uniqueness can be established with stronger conditions.

**Proposition 3.** (Konnov and Laitinen, 2002) If the conditions of Proposition 2 are met and Φ is strongly monotone, then the solution to EP($\mathcal{X}, \Phi$) is unique.

**4. EQUIVALENCE AND HARDNESS**

We first ask what extra information the COL formulation entails. We present this result as an equivalence between achieving sublinear dynamic in COL and solving various mathematical programming problems.

**Theorem 1.** Let $f$ be given in Definition 1. Suppose $f_x(\cdot)$ is convex and continuous. The following problems are equivalent:

1. Achieving sublinear dynamic regret w.r.t. $f$.
2. VI($\mathcal{X}, F$) where $F(x) = \nabla f_x(x)$.
3. EP($\mathcal{X}, \Phi$) where $\Phi(x, x') = f_x(x') - f_x(x)$.
4. FP($\mathcal{X}, T$) where $T(x) = \arg \min_{x' \in \mathcal{X}} f_x(x')$.

Therefore, if there is an algorithm that achieves sublinear dynamic regret that in poly($d$), then it solves all PPAD problems in polynomial time.

Theorem 1 says that, because of the existence of a hidden bifunction, achieving sublinear dynamic regret is essentially equivalent to finding an equilibrium $x^* \in X^*$, in which $X^*$ denotes the set of solutions of the EP/VI/FP problems in Theorem 1. Therefore, a necessary condition for sublinear dynamic regret is that $X^*$ is non-empty. Fortunately, this is true for our problem definition by Proposition 2.

Moreover, it suggests that extra structure on COL is necessary for algorithms to achieve sublinear dynamic regret that depends polynomially on $d$ (the dimension of $\mathcal{X}$). The requirement of polynomial dependency is important to properly define the problem. Without it, sublinear dynamic regret can be achieved already at least asymptotically, e.g. by simply discretizing $\mathcal{X}$ (as $\mathcal{X}$ is compact and $\nabla f$ is continuous) and grid-searching, albeit with an exponentially large constant.

Due to space limitation, we defer the proof of Theorem 1 to Appendix A, along with other proofs for this section. But we highlight the key idea is to prove that the gap function $\rho(x) := f_x(x) - \min_{x' \in \mathcal{X}} f_x(x')$ can be used as a residual function for the above EP/VI/FP in Theorem 1. In particular, we note that, for the $\Phi$ in Theorem 1, $\rho(x)$ is equivalent to a residual function $r_{ep}(x) := \max_{x' \in \mathcal{X}} -\Phi(x, x')$ used in the EP literature.

Below we discuss sufficient conditions on $f$ based on the equivalence between problems in Theorem 1, so that the EP/VI/FP in Theorem 1 becomes better structured and hence allows efficient algorithms.

**4.1 EP and VI Perspectives**

We first discuss some structures on $f$ such that the VI/EP in Theorem 1 can be efficiently solved. From the literature, we learn that the existence of dual solutions is a common prerequisite to design efficient algorithms (Konnov, 2007; Dang and Lan, 2015; Burachik and Millán, 2016; Lin et al., 2018). For example, convergence guarantees on combined relaxation methods (Konnov, 2007) for VIs rely on the assumption that the dual solution set is non-empty. Here we discuss some sufficient conditions for having a non-empty dual solution set, which by Proposition 1 and Definition 1 is a subset of the primal solution set.

By Proposition 1 and 2, a sufficient condition for non-empty $X_*$ is pseudo-monotonicity of $F$ or $\Phi$ (which we recall is a consequence of monotonicity). For our problem, the dual solutions of the EP and VI are different, while their primal solutions $X^*$ are the same.

**Proposition 4.** Let $X_*$ and $X_*$ be the solutions to DVI($\mathcal{X}, F$) and DEP($\mathcal{X}, \Phi$), respectively, where $F$ and $\Phi$ are defined in Theorem 1. Then $X_* \subseteq X_*$. The converse is true if $f_x(\cdot)$ is linear $\forall x \in \mathcal{X}$.
Proposition 4 shows that, for our problem, pseudo-monotonicity of $\Phi$ is stronger than that of $F$. This is intuitive: as the pseudo-monotonicity of $\Phi$ implies that there is $x_*$ such that $f_\alpha(x_*) \leq f_\alpha(x)$, i.e. a decision argument that is consistently better than the querying argument under the latter’s own question, whereas the pseudo-monotonicity of $F$ merely requires the intersection of the half spaces of $\mathcal{X}$ cut by $\nabla f_\alpha(x)$ to be non-empty. Another sufficient assumption for non-empty $X_*$ of VIs is that $\mathcal{X}$ is sufficiently strongly convex. This condition has recently been used to show fast convergence of mirror descent and conditional gradient descent (Garber and Hazan, 2015; Veliov and Vuong, 2017). We leave this discussion to Appendix B.

The above assumptions, however, are sometimes hard to verify for COL. Here we define a subclass of COL and provide constructive (but restrictive) conditions.

**Definition 2.** We say a COL problem with $f$ is $(\alpha, \beta)$-regular if for some $\alpha, \beta \in [0, \infty)$, $\forall x \in \mathcal{X}$,

1. $f_\alpha(\cdot)$ is a $\alpha$-strongly convex function.

2. $\nabla f_\alpha(x)$ is a $\beta$-Lipschitz continuous map.

We call $\beta$ the regularity constant; for short, we will also say $\nabla f$ is $\beta$-regular and $f$ is $(\alpha, \beta)$-regular. We note that $\beta$ is different from the Lipschitz constant of $\nabla f_\alpha(\cdot)$. The constant $\beta$ defines the degree of online components; in particular, when $\beta = 0$ the learning problem becomes offline. Based on $(\alpha, \beta)$-regularity, we have a sufficient condition to monotonicity.

**Proposition 5.** $\nabla f$ is $(\alpha - \beta)$-strongly monotone.

Proposition 5 shows if $\nabla f_\alpha(\cdot)$ does not change too fast with $x$, then $\nabla f$ is strongly monotone in the sense of VI, implying $X^* = X_*$ is equal to a singleton (but not necessarily the existence of $X_*$). Strong monotonicity also implies fast linear convergence is possible for deterministic feedback (Facchinei and Pang, 2007). When $\alpha = \beta$, it implies at least monotonicity, by which we know $X_*$ is non-empty.

We emphasize that the condition $\alpha \geq \beta$ is not necessary for monotonicity. The monotonicity condition of $\nabla f$ more precisely results from the monotonicity of $\nabla f_\alpha(x)$ and $\nabla f_\alpha(x')$, as $(\nabla f_\alpha(x) - \nabla f_\alpha(x'), x - x') = (\nabla f_\alpha(x) - \nabla f_\alpha(x'), x - x') + (\nabla f_\alpha(x') - \nabla f_\alpha(x'), x - y)$. From this decomposition, we can observe that as long as the sum of $\nabla f_\alpha(x')$ and $\nabla f_\alpha(x)$ is monotone for any $x, x' \in \mathcal{X}$, then $\nabla f$ is monotone. In the definition of $(\alpha, \beta)$-regular problems, no condition is imposed on $\nabla f_\alpha(x)$, so we need $\alpha \geq \beta$ in Proposition 5.

### 4.2 Fixed-point Perspective

We can also study the feasibility of sublinear dynamic regret from the perspective of the FP in Theorem 1. Here again we consider $(\alpha, \beta)$-regular problems.

**Proposition 6.** Let $\alpha > 0$. If $\alpha > \beta$, then $T$ is $\frac{\beta}{\alpha}$-contractive; if $\alpha = \beta$, $T$ is non-expansive.

We see again that the ratio $\frac{\beta}{\alpha}$ plays an important role in rating the difficulty of the problem. When $\alpha > \beta$, an efficient algorithm for obtaining the the fixed point solution is readily available (i.e. by contraction).

An alternative interpretation is that $x_n^n$ changes at a slower rate than $x_n$ when $\alpha > \beta$ with respect to $\| \cdot \|$.

### 5 MONOTONE EP AS COL

After understanding the structures that determine the difficulty of COL, we describe a converse result of Theorem 1, which converts monotone EPs into COL. Here we assume that $\Phi(x, \cdot)$ is convex.

**Theorem 2.** Let $EP(\mathcal{X}, \Phi)$ be monotone with $\Phi(x, x) = 0$. Consider COL with $f_\alpha(x') = \Phi(x, x')$. Let $\{x_n\}_{n=1}^N$ be any sequence of decisions and define $\hat{x}_N := \frac{1}{N} \sum_{n=1}^N x_n$. It holds that $r_{dep}(\hat{x}_N) \leq \frac{1}{N} \text{Regret}_N^r$, where $r_{dep}(x') := \max_{x \in \mathcal{X}} \Phi(x, x')$ is the dual residual.

Theorem 2 shows monotone EPs can be solved by achieving sublinear static regret in COL, at least in terms of the dual residual. Below we relate bounds on the dual residual back to the primal residual, which we recall is given as $r_{ep}(x) := \max_{x \in \mathcal{X}} -\Phi(x, x')$.

**Theorem 3.** Suppose $\Phi(x, \cdot)$ is $L$-Lipschitz, $\forall x \in \mathcal{X}$. If $\Phi$ satisfies $\Phi(x, x') = -\Phi(x', x)$, i.e. $\Phi$ is skew-symmetric, then $r_{ep}(x) = r_{dep}(x)$. Otherwise,

1. For $x \in \mathcal{X}$ such that $r_{dep}(x) \leq 2LD_x$, it holds $r_{ep}(x) \leq 2\sqrt{2LD_x} \sqrt{r_{dep}(x)}$.

2. If $\Phi(x, \cdot)$ is in addition $\mu$-strongly convex with $\mu > 0$, for $x \in \mathcal{X}$ such that $r_{dep}(x) \leq L^2/\mu$, it holds $r_{ep}(x) \leq 2.8(L^2/\mu)^{1/3} r_{dep}(x)^{2/3}$.

We can view the above results as a generalization of the classic reduction from convex optimization and Blackwell approachability to no-regret learning (Abernethy et al., 2011). Generally, the rate of primal residual converges slower than the dual residual. However, when the problem is skew-symmetric (which is true for EPs coming from optimization and saddle-point problems; see Appendix C), we recover the classic results. In this case, we can show $r_{ep}(\hat{x}_N) = r_{dep}(\hat{x}_N) \leq \frac{1}{N} \text{Regret}_N^r \leq \frac{1}{N} \sum_{n=1}^N r_{ep}(x_n)$.
These results complement the discussion in Section 4.1, as monotonicity implies the dual solution set $X_{\star}$ is non-empty. Namely, these monotone EPs constitute a class of source problems of COL for which efficient algorithms are available. Proofs and further discussions of this reduction are given in Appendix C.

6 REDUCTION BY REGULARITY

Inspired by Theorem 1, we present a reduction from minimizing dynamic regret to minimizing static regret and convergence to $X^\star$. Intuitively, this is possible, because Theorem 1 suggests achieving sublinear dynamic regret should not be harder than finding $x^\star \in X^\star$. Define $\text{Regret}_d^N(x^\star) := \sum_{n=1}^N l_n(x_n) - l_n(x^\star) \leq \text{Regret}_s^N(x^\star)$.

**Theorem 4.** Let $x^\star \in X^\star$ and $\Delta_n := ||x_n - x^\star||$. If $f$ is $(\alpha, \beta)$-regular, then for all $N$,

\[
\text{Regret}_d^N \leq \min\{G \sum_{n=1}^N \Delta_n, \text{Regret}_s^N(x^\star)\}
+ \sum_{n=1}^N \min\{\beta D_X \Delta_n, \frac{\beta^2}{2\alpha} \Delta_n^2\}
\]

If further $X_{\star\star}$ of the dual EP is non-empty, $\text{Regret}_d^N \geq \frac{\beta}{\alpha} \sum_{n=1}^N ||x_n^\star - x_n||^2$, where $x_n \in X_{\star\star} \subseteq X^\star$.

Theorem 4 roughly shows that when $x^\star$ exists (e.g. given by the sufficient conditions in the previous section), it provides a stabilizing effect to the problem, so the dynamic regret behaves almost like the static regret when the decisions are around $x^\star$.

This relationship can be used as a powerful tool for understanding the dynamic regret of existing algorithms designed for EPs, VIs, and FPs. These include, e.g., mirror descent (Beck and Teboulle, 2003), mirror-prox ( Nemirovski, 2004; Juditsky et al., 2011), conditional gradient descent (Jaggi, 2013), Mann iteration (Mann, 1953), etc. Interestingly, many of these are also standard tools in online learning, with static regret bounds that are well known (Hazan et al., 2016). We can apply Theorem 4 in different ways, depending on the known convergence of an algorithm. For algorithms whose convergence rate of $\Delta_n$ to zero is known, Theorem 4 essentially shows that their dynamic regret is at most $O(\sum_{n=1}^N \Delta_n)$. For the algorithms with only known static regret bounds, we can use a corollary.

**Corollary 1.** If $f$ is $(\alpha, \beta)$-regular and $\alpha > \beta$, it holds that $\text{Regret}_s^N \leq \text{Regret}_s^N(x^\star) + \frac{\beta^2 \text{Regret}_s^N(x^\star)}{2\alpha(\alpha - \beta)}$, where $\text{Regret}_s^N(x^\star)$ denotes the static regret of the linear online learning problem with $l_n(x) = \langle \nabla f_n(x_n), x \rangle$.

The purpose of Corollary 1 is not to give a tight bound, but to show that for nicer problems with $\alpha > \beta$, achieving sublinear dynamic regret is not harder than achieving sublinear static regret. For tighter bounds, we still refer to Theorem 4 to leverage the equilibrium convergence. We note that the results in Section 5 and here concern different classes of COL in general, because $\alpha > \beta$ does not necessarily imply the EP($X, \Phi$) is monotone, but only VI($X, F$) unless $f_x(\cdot)$ is linear.

Finally, we remark Theorem 4 is directly applicable to expected dynamic regret (the right-hand side of the inequality will be replaced by its expectation) when the learner only has access to stochastic feedback, because the COL setup in non-anticipating. Similarly, high-probability bounds can be obtained based on martingale convergence theorems, as in (Cesa-Bianchi et al., 2004). In these cases, we note that the regret is defined with respect to $l_n$ in COL, not the sampled losses.

6.1 Example Algorithms

We showcase applications of Theorem 4. These bounds are non-asymptotic and depend polynomially on $d$. Also, these algorithms do not need to know $\alpha$ and $\beta$, except to set the stepsize upper bound for first-order methods. Please refer to Appendix D for the proofs.

6.1.1 Functional Feedback

We first consider the simple greedy update, which sets $x_{n+1} = \arg\min_{x \in X} l_n(x)$. By Proposition 6 and Theorem 4, we see that if $\alpha > \beta$, it has $\text{Regret}_d^N = O(1)$. For $\alpha = \beta$, we can use algorithms for non-expansive fixed-point problems (Mann, 1953).

**Proposition 7.** For $\alpha = \beta$, there is an algorithm that achieves sublinear dynamic regret in $\text{poly}(d)$.

6.1.2 Exact First-order Feedback

Next we use the reduction in Theorem 4 to derive dynamic regret bounds for mirror descent, under deterministic first-order feedback. We recall that mirror descent with step size $\eta_n > 0$ follows

\[
x_{n+1} = \arg\min_{x \in X} \langle \eta_n g_n, x \rangle + B_R(x\|x_n) \tag{4}
\]

where $g_n$ is feedback direction, $B_R$ is a Bregman divergence with respect to some $\text{1-weakly convex function } R$. Here we assume additionally that $f_x(\cdot)$ is $\gamma$-smooth with $\gamma > 0$ for all $x \in X$.

**Proposition 8.** Let $f$ be $(\alpha, \beta)$-regular and $f_x(\cdot)$ be $\gamma$-smooth, $\forall x \in X$. Let $R$ be $1$-strongly convex and $L$-smooth. If $\alpha > \beta$, $g_n = \nabla l_n(x_n)$, and $\eta_n < \frac{2(\alpha - \beta)}{(\gamma + \beta)^2}$, then, for some $0 < \nu < 1$, $\text{Regret}_d^N \leq (G + \beta D_X)\sqrt{2B_R(x^\star\|x_1)} \sum_{n=1}^N \nu^{n-1} = O(1)$ for (4).

6.1.3 Stochastic & Adversarial Feedback

We now consider stochastic and adversarial cases in COL. As discussed, these are directly handled in the
feedback, while the (expected) regret is still measured against the true underlying bifunction. Importantly, we make the subtle assumption that bifunction $f$ is fixed before learning. We consider mirror descent in (4) with additive stochastic and adversarial feedback given as $g_n = \nabla l_n(x_n) + \epsilon_n + \xi_n$, where $\epsilon_n \in \mathbb{R}^d$ is zero-mean noise with $\mathbb{E} [\|\epsilon_n\|^2] < \infty$ and $\xi_n \in \mathbb{R}^d$ is a bounded adversarial bias. The component $\epsilon_n$ can come from observing a stochastic loss $l_n(x; \zeta_n)$ with random variable $\zeta_n$, when the true loss is $l_n(x) = \mathbb{E}_{\zeta_n}[l_n(x; \zeta_n)]$ (i.e. $\nabla l_n(x_n; \zeta_n) = \nabla l_n(x_n) + \epsilon_n$). On the other hand the adversarial component $\xi_n$ can describe extra bias in computation. We consider the expected dynamic regret $\mathbb{E}[\text{Regret}_N] = \mathbb{E}[\sum_{n=1}^N l_n(x_n) - \min_{x \in \mathcal{X}} l_n(x)]$, where the expectation is over $\epsilon_n$. Define $\Xi := \sum_{n=1}^N \|\xi_n\|_\infty$. By reduction to static regret in Corollary 1, we have the following proposition.

**Proposition 9.** If $f$ is fixed before learning, $\alpha > \beta$ and $\eta_n = \frac{1}{\sqrt{n}}$, then mirror descent with $g_n = \nabla l_n(x_n) + \epsilon_n + \xi_n$ has $\mathbb{E}[\text{Regret}_N] = O(\sqrt{N} + \Xi)$.

**6.2 Remark**

Essentially, our finding indicates that the feasibility of sublinear dynamic regret is related to a problem’s properties. For example, the difficulty of the problem depends largely on the ratio $\frac{\alpha}{\beta}$ when there is no other directional information about $\nabla f(x)$, such as monotonicity. When $\beta \leq \alpha$, we have shown efficient algorithms are possible. But, for $\beta > \alpha$, we are not aware of any efficient algorithm. If one exists, it would solve all ($\alpha, \beta$)-regular problems, which, in turn, would efficiently solve all EP/VI/FP problems as we can formulate them into the problem of solving COL problems with sublinear dynamic regret by Theorem 1.

**7 EXTENSIONS**

The COL framework reveals some core properties of dynamic regret. However, while we allow feedback to be adversarial, we still assume that the same loss function $f_x(\cdot)$ must be returned by the bifunction for the same query argument $x \in \mathcal{X}$. Therefore, COL does not capture time-varying situations where the opponent’s strategy can change across rounds. Also, this constraint allows the learner to potentially enumerate the opponent. Here we relax (3) and define a generalization of COL. The proofs of this section are included in Appendix E.

**Definition 3.** We say an online learning problem is ($\alpha, \beta$)-predictable with $\alpha, \beta \in [0, \infty)$ if $\forall x \in \mathcal{X}$,

1. $l_n(\cdot)$ is an $\alpha$-strongly convex function.

2. $\|\nabla l_n(x) - \nabla l_{n-1}(x)\|_\infty \leq \beta \|x_n - x_{n-1}\| + o_n$, where $o_n \in [0, \infty)$ and $\sum_{n=1}^N o_n = A_N = o(N)$.

This problem generalizes COL along two directions: 1) it makes the problem non-stationary; 2) it allows adversarial components within a sublinear budget inside the loss function. We note that the second condition above is different from having adversarial feedback, e.g., in Section 6.1.3, because the regret now is measured with respect to the adversarial loss as opposed to those generated by a fixed bifunction. This new condition can make achieving sublinear dynamic regret considerably harder.

Let us further discuss the relationship between ($\alpha, \beta$)-predictable and ($\alpha, \beta$)-regular problems. First, a contraction property like Proposition 6 still holds.

**Proposition 10.** For ($\alpha, \beta$)-predictable problems with $\alpha > 0$, $\|x_n - x_{n-1}\| \leq \frac{\beta}{\alpha} \|x_n - x_{n-1}\| + \frac{2\alpha}{\alpha}$.

Proposition 10 shows that when functional feedback is available and $\frac{\beta}{\alpha} < 1$, sublinear dynamic regret can be achieved, e.g., by a greedy update. However, one fundamental difference between predictable problems and COL problems is the lack of equilibrium $x^*$, which is the foundation of the reduction in Theorem 4. This makes achieving sublinear dynamic regret much harder when functional feedback is unavailable or when $\alpha = \beta$. Using Proposition 10, we establish some preliminary results below.

**Theorem 5.** Let $\frac{\beta}{\alpha} < \frac{\alpha}{2\sqrt{\gamma}}$. For ($\alpha, \beta$)-predictable problems, if $l_n(\cdot)$ is $\gamma$-smooth and $R$ is $1$-strongly convex and $L$-smooth, then mirror descent with deterministic feedback and step size $\eta = \frac{\alpha}{2L^2}$ achieves $\text{Regret}_N = O(1 + A_N + \sqrt{NA_N})$.

We find that, in Theorem 5, mirror descent must maintain a sufficiently large step size in predictable problems, unlike COL problems which allow for decaying step size. When $\alpha = \beta$, we can show that sublinear dynamic regret is possible under functional feedback.

**Theorem 6.** For $\alpha = \beta$, if $A_{\infty} < \infty$ and $\|\cdot\|$ is the Euclidean norm, then there is an algorithm with functional feedback achieving sublinear dynamic regret. For $d = 1$ and $a_n = 0$ for all $n$, sublinear dynamic regret is possible regardless of $\alpha, \beta$.

We do not know, however, whether sublinear dynamic regret is feasible when $\alpha = \beta$ and $A_{\infty} = \infty$. We conjecture this is infeasible when the feedback is only first-order, as mirror descent is insufficient to solve monotone problems using the last iterate (Facchinei and Pang, 2007) which contain COL with $\alpha = \beta$ (a simpler case than predictable online learning with $\alpha = \beta$).
8 CONCLUSION

We present COL, a new class of online problems where the gradient varies continuously across rounds with respect to the learner’s decisions. We show that this setting can be equated with certain equilibrium problems (EPs). Leveraging this insight, we present a reduction from monotone EPs to COL, and show necessary conditions and sufficient conditions for achieving sublinear dynamic regret. Furthermore, we show a reduction from dynamic regret to static regret and the convergence to equilibrium points.

There are several directions for future research on this topic. Our current analyses focus on classical algorithms in online learning. We suspect that the use of adaptive or optimistic methods can accelerate convergence to equilibria, if some coarse model can be estimated. In addition, although we present some preliminary results showing the possibility for interpretable dynamic regret rates in predictable online learning, further refinement and understanding the corresponding lower bounds remain important future work. Finally, while the current formulations restrict the loss to be determined solely by the learner’s current decision, extending the discussion to history-dependent bifunctions is an interesting topic.

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A Complete Proofs of Section 4

A.1 Proof of Theorem 1

A.1.1 Highlight

The key idea to proving Theorem 1 is that the gap function \( \rho(x) := f_x(x) - \min_{x' \in X} f_x(x') \) can be used as a residual function for the above EP/VI/FP in Theorem 1. That is, \( \rho(x) \) is non-negative, computable in polynomial time (it is a convex program), and \( \rho(x) = 0 \) if and only if \( x \in X^* \) (because \( f_x(\cdot) \) is convex \( \forall x \in X \)). Therefore, to show Theorem 1, we only need to prove that solving one of these problems is equivalent to achieving sublinear dynamic regret.

First, suppose an algorithm generates a sequence \( \{x_n \in X\} \) such that \( \lim_{n \to \infty} x_n = x^* \), for some \( x^* \in X^* \). To show this implies \( \{x_n \in X\} \) has sublinear dynamic regret, we first show \( \lim_{n \to \infty} x_n = x^* \). Then define \( \rho_n = \rho(x_n) \). Because \( \lim_{n \to \infty} \rho_n = 0 \), we have \( \text{Regret}_N = \sum_{n=1}^N \rho_n = o(N) \).

Next, we prove the opposite direction. Suppose an algorithm generates a sequence \( \{x_n \in X\} \) with sublinear dynamic regret. This implies that \( \hat{\rho}_N := \min_n \rho_n \leq \frac{1}{N} \sum_{n=1}^N \rho_n \) is in \( o(1) \) and non-increasing. Thus, \( \lim_{N \to \infty} \hat{\rho}_N = 0 \). As \( \rho \) is a proper residual, the algorithm solves the EP/VI/FP problem by returning the decision associated with \( \hat{\rho}_N \).

The proof of PPAD-completeness is based on converting the residual of a Brouwer’s fixed-point problem to a bifunction, and use the solution along with \( \hat{\rho}_N \) above as the approximate solution.

Note that the gap function \( \rho \), despite motivated by dynamic regret here, corresponds to a natural gap function \( r_{EP}(x) := \max_{x' \in X} -\Phi(x, x') \) used in the EP literature, showing again a close connection between the dynamic regret and the EP in Theorem 1. Nonetheless, \( \rho(x) \) is not conventional for VIs and FPs. Below we relate \( \rho(x) \) to some standard residuals of VIs and FPs under a stronger assumption on \( f \).

**Proposition 11.** For \( \epsilon > 0 \), consider some \( x_\epsilon \in X \) such that \( \rho(x_\epsilon) \leq \epsilon \). If \( f_x(\cdot) \) is \( \alpha \)-strongly convex, then \( \lim_{\epsilon \to 0} \langle \nabla f_{x_\epsilon}(x), x - x_\epsilon \rangle \geq 0, \forall x \in X \), and \( \lim_{\epsilon \to 0} \|x_\epsilon - T(x_\epsilon)\| = 0 \).

A.1.2 Full proof

Now we give the details of the steps above.

We first show the solutions sets of the EP, the VI, and the FP are identical.

\[ 2. \implies 3. \]

Let \( x_{VI}^* \in X \) be a solution to \( \text{VI}(X, F) \) where \( F(x) = \nabla f_x(x) \). That is, for all \( x \in X \), \( \langle \nabla f_{x_{VI}}(x_{VI}), x - x_{VI} \rangle \geq 0 \). The sufficient first-order condition for optimality implies that \( x_{VI}^* \) is optimal for \( f_{x_{VI}} \). Therefore, \( f_{x_{VI}}(x_{VI}) \leq f_{x_{VI}}(x) \) for all \( x \in X \), meaning that \( x_{VI}^* \) is also a solution to \( \text{EP}(X, \Phi) \) where \( \Phi(x, x') = f_x(x') - f_x(x) \).

\[ 3. \implies 4. \]

Let \( x_{EP}^* \in X \) be a solution to \( \text{EP}(X, \Phi) \). By definition, it satisfies \( f_{x_{EP}}(x_{EP}^*) \leq f_{x_{EP}}(x) \) for all \( x \in X \), which implies \( x_{EP}^* = \arg\min_{x \in X} f_{x_{EP}}(x) = T(x_{EP}^*) \). Therefore, \( x_{EP}^* \) is a also solution to \( \text{FP}(X, T) \), where \( T(x') = \arg\min_{x \in X} f_x(x) \).

\[ 4. \implies 2. \]

If \( x_{FP}^* \) is a solution to \( \text{FP}(X, T) \), then \( x_{FP}^* = \arg\min_{x \in X} f_{x_{FP}}(x) \). By the necessary first-order condition for optimality, we have \( \langle \nabla f_{x_{FP}}(x_{FP}^*), x - x_{FP}^* \rangle \geq 0 \) for all \( x \in X \). Therefore \( x_{FP}^* \) is also a solution to \( \text{VI}(X, F) \) where \( F(x) = \nabla f_x(x) \).

Let \( X^* \) denote their common solution sets. To finish the proof of equivalence in Theorem 1, we only need to show that converging to \( X^* \) is equivalent to achieving sublinear dynamic regret.

\[ \bullet \] Suppose there is an algorithm that generates a sequence \( \{x_n \in X\} \) such that \( \lim_{n \to \infty} x_n = x^* \), for some \( x^* \in X^* \). To show this implies \( \{x_n \in X\} \) has sublinear dynamic regret, we need a continuity lemma.
Lemma 1. \( \lim_{x\to x^* \in \mathcal{X}} \rho(x) = 0. \)

**Proof.** Let \( \bar{x} \in T(x) \). Using convexity, we can derive that
\[
\rho(x) = f_x(x) - f_x(\bar{x}) \leq \langle \nabla f_x(x), x - \bar{x} \rangle \\
\leq \langle \nabla f_x(x^*), x - \bar{x} \rangle + \|\nabla f_x(x^*) - \nabla f_x(x)\| \|x - \bar{x}\| \\
\leq \langle \nabla f_x(x^*), x^* - \bar{x} \rangle + \|\nabla f_x(x^*)\| \|x - x^*\| + \|\nabla f_x(x^*) - \nabla f_x(x)\| \|x - \bar{x}\| \\
\leq \|\nabla f_x(x^*)\| \|x - x^*\| + \|\nabla f_x(x^*) - \nabla f_x(x)\| \|x - \bar{x}\|
\]
where the second and the third inequalities are due to Cauchy-Schwarz inequality, and the last inequality is due to that \( x^* \) solves VI(\( \mathcal{X}, \nabla f \)). By continuity of \( \nabla f \), the above upper bound vanishes as \( x \to x^* \). \( \square \)

For short hand, let us define \( \rho_n = \rho(x_n) \); we can then write \( \text{Regret}_N^d = \sum_{n=1}^N \rho_n \). By Lemma 1, \( \lim_{n \to \infty} x = x^* \) implies that \( \lim_{n \to \infty} \rho_n = 0 \). Finally, we show by contradiction that \( \lim_{n \to \infty} \rho_n = 0 \) implies \( \text{Regret}_N^d = o(N) \). Suppose the dynamic regret is linear. Then \( c > 0 \) exists such that there is a subsequence \( \{\rho_{n_i}\} \) satisfying \( \rho_{n_i} \geq c \) for all \( n_i \). However, this contradicts with \( \lim_{n \to \infty} \rho_n = 0 \).

*We can also prove the opposite direction. Suppose an algorithm generates a sequence \( \{x_n \in \mathcal{X}\} \) with sublinear dynamic regret. This implies that \( \hat{\rho} := \min_{\rho_n} \rho_n \leq \frac{1}{N} \sum_{n=1}^N \rho_n \) is in \( o(N) \) and non-increasing. Thus, \( \lim_{N \to \infty} \hat{\rho} = 0 \) and the algorithm solves the VI/EP/FP problem because \( \rho \) is a residual. Alternatively, we may view \( \rho \) as a Lyapunov-like function. The sequence of minimizers \( \bar{x}_N = \min_{x_n} \rho(x_n) \) are confined to the level sets of \( \rho \), which converge to the zero-level set. Since \( \mathcal{X} \) is compact, \( \bar{x}_N \) converges to this set.*

Finally, we show the PPAD-completeness by proving that achieving sublinear dynamic regret with polynomial dependency on \( d \) implies solving a Brouwer’s problem (finding a fixed point of a continuous point-to-point map on a convex compact set). Because Brouwer’s problem is known to be PPAD-complete Daskalakis et al. (2009), we can use this algorithm to solve all PPAD problems.

Given a Brouwer’s problem on \( \mathcal{X} \) with some continuous map \( T \). We can define the bifunction \( f \) as \( f_{x'}(x) = \frac{1}{2} \|x - T(x')\|^2 \), where \( \| \cdot \|^2 \) is Euclidean. Obviously, this \( f \) satisfies Definition 1, and its gap function is zero at \( x^* \) if and only \( x^* \) is a solution to the Brouwer’s problem. Suppose we have an algorithm that achieves sublinear dynamic regret for continuous online learning. We can use the definition \( \hat{\rho}_N \) in the proof above to return a solution whose gap function is less than \( \frac{1}{N} \epsilon^2 \), which implies an \( \epsilon \)-approximate solution to Brouwer’s problem (i.e. \( \|x - T(x)\| \leq \epsilon \)). If the dynamic regret depends polynomially on \( d \), we have such an \( N \) in \( poly(d) \), which implies solving any Brouwer’s problem in polynomial time.

**A.1.3 Proof of Proposition 11**

For the VI problem, let \( x^*_n = T(x_n) \) and notice that
\[
\frac{\alpha}{2} \|x_n - x^*_n\|^2 \leq f_{x_n}(x_n) - f_{x_n}(x^*_n) \leq \epsilon \tag{5}
\]
for some \( \alpha > 0 \). Therefore, for any \( x \in \mathcal{X} \),
\[
\langle \nabla f_{x_n}(x), x - x_n \rangle \geq \langle \nabla f_{x_n}(x^*_n), x - x_n \rangle - \|\nabla f_{x_n}(x^*_n) - \nabla f_{x_n}(x)\| \|x - x_n\| \\
\geq \langle \nabla f_{x_n}(x^*_n), x - x_n \rangle - \|\nabla f_{x_n}(x^*_n)\| \|x - x^*_n\| - \|\nabla f_{x_n}(x^*_n) - \nabla f_{x_n}(x)\| \|x - x_n\| \\
\geq -\|\nabla f_{x_n}(x^*_n)\| \|x - x^*_n\| - \|\nabla f_{x_n}(x^*_n) - \nabla f_{x_n}(x)\| \|x - x_n\|
\]
Since \( \|x_n - x^*_n\|^2 \leq \frac{\alpha}{2} \epsilon \), by continuity of \( \nabla f_{x_n} \), it satisfies that \( \lim_{x \to 0} \langle \nabla f_{x_n}(x), x - x_n \rangle \geq 0, \forall x \in \mathcal{X} \).

For the fixed-point problem, similarly by (5), we see that \( \lim_{x \to 0} \|x_n - T(x_n)\| = 0 \)

**A.2 Proofs of Proposition 4**

**Proof of Proposition 4.** Let \( x_* \in X^* \). It holds that \( \forall x \in \mathcal{X}, \ 0 \geq \Phi(x, x_*) = f_{x_*(x)} - f_{x}(x) \geq \langle \nabla f_{x}(x), x_*(x) - x \rangle \), which implies \( x_* \in \mathcal{X}^* \). The condition for the converse case is obvious. \( \square \)
A.3 Proof of Proposition 5

Because $\nabla f_x$ is $\alpha$-strongly monotone, we can derive
\[
\langle \nabla f_x(x) - \nabla f_y(y), x - y \rangle = \langle \nabla f_x(x) - \nabla f_y(y), x - y \rangle + \langle \nabla f_x(y) - \nabla f_y(y), x - y \rangle \\
\geq \alpha \| x - y \|^2 - \| \nabla f_x(y) - \nabla f_y(y) \| \cdot \| x - y \| \\
\geq (\alpha - \beta) \| x - y \|^2
\]
\[
\forall x, y \in \mathcal{X}, \text{ where the last step is due to } \beta\text{-regularity.}
\]

A.4 Proof of Proposition 6

The result follows immediately from the following lemma.

Lemma 2. Suppose $f$ is $(\alpha, \beta)$-regular with $\alpha > 0$. Then $F$ in Theorem 1 is point-valued and $\frac{\alpha}{\beta}$-Lipschitz.

Proof. Let $x^* = F(x)$ and $y^* = F(y)$ for some $x, y \in \mathcal{X}$. By strong convexity, $x^*$ and $y^*$ are unique, and $\nabla f_x(\cdot)$ is $\alpha$-strongly monotone; therefore it holds that
\[
\langle \nabla f_x(y^*), y^* - x^* \rangle \geq \langle \nabla f_x(x^*), y^* - x^* \rangle + \alpha \| x^* - y^* \|^2 \\
\geq \alpha \| x^* - y^* \|^2
\]

Since $y^*$ satisfies $\langle \nabla f_y(y^*), x^* - y^* \rangle \geq 0$, the above inequality implies that
\[
\alpha \| x^* - y^* \|^2 \leq \langle \nabla f_x(y^*), y^* - x^* \rangle \\
\leq \langle \nabla f_x(y^*) - \nabla f_y(y^*), y^* - x^* \rangle \\
\leq \| \nabla f_x(y^*) - \nabla f_y(y^*) \| \cdot \| y^* - x^* \| \\
\leq \beta \| x - y \| \| y^* - x^* \| \\
\]

Rearranging the inequality gives the statement. \qed

B Dual Solution and Strongly Convex Sets

We show when the strong convexity property of $\mathcal{X}$ implies the existence of dual solution for VIs. We first recall the definition of strongly convex sets.

Definition 4. Let $\alpha_\mathcal{X} \geq 0$. A set $\mathcal{X}$ is called $\alpha_\mathcal{X}$-strongly convex if, for any $x, x' \in \mathcal{X}$ and $\lambda \in [0, 1]$, it holds that, for all unit vector $v$, $\lambda x + (1 - \lambda) x' + \frac{\alpha_\mathcal{X}(1 - \lambda)}{2} \| x - x' \|^2 v \in \mathcal{X}$.

When $\alpha_\mathcal{X} = 0$, the definition reduces to usual convexity. Also, we see that this definition implies $\alpha_\mathcal{X} \leq \frac{1}{\beta_\mathcal{X}}$. In other words, larger sets are less strongly convex. This can also be seen from the lemma below.

Lemma 3. (Journee et al., 2010, Theorem 12) Let $f$ be non-negative, $\alpha$-strongly convex, and $\beta$-smooth on a Euclidean space. Then the set $\{ x | f(x) \leq r \}$ is $\frac{\alpha}{\sqrt{2}\beta}$-strongly convex.

Here we present the existence result.

Proposition 12. Let $x^* \in X^*$. If $\mathcal{X}$ is $\alpha_\mathcal{X}$-strongly convex $\forall x \in \mathcal{X}$, it holds that $\langle F(x^*), x - x^* \rangle \geq \frac{\alpha_\mathcal{X}}{2} \| x - x^* \|^2 \| F(x^*) \|$. If further $F$ is $L$-Lipschitz, this implies $\langle F(x), x - x^* \rangle \geq \left( \frac{\alpha_\mathcal{X}}{2} \| F(x^*) \| \right) \| x - x^* \|^2$, i.e. when $\alpha_\mathcal{X} \geq \frac{2L}{\| F(x^*) \|}$, $x^* \in X^*$.

Proof of Proposition 12. Let $g = F(x^*)$. Let $y = \lambda x + (1 - \lambda)x^*$ and $d = -\lambda(1 - \lambda) \| x - y \|^2 v$, for some $\lambda \in [0, 1]$ and some unit vector $v$ to be decided later. By $\alpha_\mathcal{X}$-strongly convexity of $\mathcal{X}$, we have $y + d \in \mathcal{X}$. We can derive
\[
\langle g, x - x^* \rangle = \langle g, x - y - d \rangle + \langle g, y + d - x^* \rangle \\
\geq \langle g, x - y \rangle - \langle g, d \rangle \\
= (1 - \lambda) \langle g, x - x^* \rangle - \langle g, d \rangle
\]
which implies $\langle g, x - x^* \rangle \geq \frac{-\langle g, d \rangle}{\lambda} = (1 - \lambda) \frac{\alpha_\mathcal{X}}{2} \| x - x^* \|^2 \langle g, v \rangle$. Since we are free to choose $\lambda$ and $v$, we can set $\lambda = 0$ and $v = \arg \max_{v, \| v \| \leq 1} \langle g, v \rangle$, which yields the inequality in the statement. \qed
C Complete Proofs of Section 5

In this section, we describe a general strategy to reduce monotone equilibrium problems (EPs) to continuous online learning (COL) problems. This reduction can be viewed as refinement and generalization of the classic reduction from convex optimization to adversarial online learning and that from saddle-point problem to two-player adversarial online learning. In comparison, our reduction

1. results in a single-player online learning problem, which allows for unified algorithm design
2. considers potential continuous relationship of the losses between different rounds through the setup of COL, which leads to a predictable online problem amenable to acceleration techniques, such as (Rakhlin and Sridharan, 2013; Juditsky et al., 2011; Cheng et al., 2019a).
3. and extends the concept to general convex problems, namely, monotone EPs, which includes of course convex optimization and convex-concave saddle-point problems but also fixed-point problems (FPs), variational inequalities (VIs), etc.

The results here are summarized as Theorem 2 and Theorem 3.

Here we further suppose $\Phi(x, x) = 0$ in the definition of EP. This is not a strong condition. First all the common source problems in introduced below in Appendix C.1.1 satisfy this condition. Generally, suppose we have some EP problem with $\Phi'(x, x) > 0$ for some $x$. We can define $\Phi(x, x) = \Phi'(x, x') - \Phi'(x, x)$. Then the solution of $EP(\mathcal{X}, \Phi)$ are subset of the solution $EP(\mathcal{X}, \Phi')$. In other words, allowing $\Phi(x, x) > 0$ only makes problem easier. We note that the below reduction and discussion can easily be extended to work directly with EPs with $\Phi(x, x) > 0$ by defining instead $f_x(x') = \Phi(x, x') - \Phi(x, x)$, but this will make the presentation less clean.

C.1 Background: Equilibrium Problems (EPs)

Let $\mathcal{X}$ be a compact and convex set in a finite dimensional space. Let $F : x \times x' \mapsto \Phi(x, x')$ be a bifunction\(^7\) that is continuous in the first argument, convex in the second argument, and satisfies $\Phi(x, x) = 0$.\(^8\) The problem $EP(\mathcal{X}, F)$ aims to find $x^* \in \mathcal{X}$ such that

$$
\Phi(x^*, x) \geq 0, \quad \forall x \in \mathcal{X}
$$

Its dual problem $DEP(\mathcal{X}, F)$ finds $x_{**} \in \mathcal{X}$ such that

$$
\Phi(x, x_{**}) \leq 0, \quad \forall x \in \mathcal{X}
$$

Based on the problem’s definition, a natural residual (or gap function) of $EP(\mathcal{X}, F)$ is

$$
r_{ep}(x) := - \min_{x' \in \mathcal{X}} \Phi(x, x')
$$

which says the degree that the inequality in the EP definition is violated. A residual for $DEP(\mathcal{X}, F)$ can be defined similarly as

$$
r_{dep}(x') := \max_{x \in \mathcal{X}} \Phi(x, x')
$$

Sometimes EPs are called maxInf (or minSup) problems (Jofrée and Wets, 2014), because

$$
x^* \in \arg \min_{x \in \mathcal{X}} r_{ep}(x) = \arg \max_{x \in \mathcal{X}} \min_{x' \in \mathcal{X}} \Phi(x, x')
$$

In a special case, when $\Phi(\cdot, x)$ is concave. It reduces to a saddle-point problem.

We say a bifunction $F$ is monotone if it satisfies

$$
\Phi(x, x') + \Phi(x', x) \leq 0,
$$

\(^7\)We impose convexity and continuity to simplify the setup; similar results hold for subdifferentials and Lipschitz continuity defined based on hemi-continuity.

\(^8\)As discussed, we concern only EP with $\Phi(x, x) = 0$ here
and we say $F$ is skew-symmetric if
\[ \Phi(x, x') = -\Phi(x, x'), \]
which implies $F$ is monotone. Finally, we say the problem $\text{EP}(\mathcal{X}, F)$ is monotone, if its bifunction $F$ is monotone.

### C.1.1 Examples

We review some source problems of EPs. Please refer to e.g. (Jofré and Wets, 2014; Konnov and Schaible, 2000) for a more complete survey.

**Convex Optimization**

Consider $\min_{x \in \mathcal{X}} h(x)$ where $h$ is convex. We can simply define
\[ \Phi(x, x') = h(x') - h(x) \]
which is a skew-symmetric (and therefore monotone) bifunction.

We can also define (following the VI given by its optimality condition)
\[ \Phi(x, x') = \langle \nabla h(x), x' - x \rangle. \]

We can easily verify that this construction is also monotone
\[ \Phi(x, x') + \Phi(x', x) = \langle \nabla h(x), x' - x \rangle + \langle \nabla h(x'), x - x' \rangle = \langle \nabla h(x) - \nabla h(x'), x' - x \rangle \leq 0. \]

Suppose $h$ is $\mu$-strongly convex. We can also consider
\[ \Phi(x, x') = \langle \nabla h(x), x' - x \rangle + \frac{\mu'}{2} \| x' - x \|^2 \]
where $\mu' \leq \mu$. Such $F$ is still monotone:
\[ \Phi(x, x') + \Phi(x', x) = \langle \nabla h(x) - \nabla h(x'), x' - x \rangle + \mu' \| x' - x \|^2 \leq 0. \]

**Saddle-Point Problem**

Let $\mathcal{U}$ and $\mathcal{V}$ to convex and compact sets in a finite dimensional space. Consider a convex-concave saddle point problem
\[ \min_{u \in \mathcal{U}} \max_{v \in \mathcal{V}} \phi(u, v) \]
in which $\phi$ is continuous, $\phi(\cdot, y)$ is convex, and $\phi(x, \cdot)$ is concave. It is well known that in this case
\[ \min_{u \in \mathcal{U}} \max_{v \in \mathcal{V}} \phi(u, v) = \max_{v \in \mathcal{V}} \min_{u \in \mathcal{U}} \phi(u, v) =: \phi^*. \]

We can define a EP by the bifunction
\[ \Phi(x, x') := -\phi(u, v') + \phi(u', v). \]
By definition we have the skew symmetry property, which implies monotonicity.

**Variational Inequality**

A VI with a vector-valued map $F$ finds $x^* \in \mathcal{X}$ such that
\[ \langle F(x^*), x - x^* \rangle \geq 0, \quad \forall x \in \mathcal{X}. \]
To turn that into a EP, we can simply define
\[ \Phi(x, x') = \langle F(x), x' - x \rangle. \]
Mixed Variational Inequality (MVI) MVI considers problems that finds $x^* \in \mathcal{X}$ such that
\[ h(x) - h(x^*) + \langle F(x^*), x - x^* \rangle \geq 0, \quad \forall x \in \mathcal{X}. \]

Following the previous idea, we can define its equivalent EP through the bifunction
\[ \Phi(x, x') = h(x') - h(x) + \langle F(x), x' - x \rangle \]

C.2 More insights into residuals of primal and dual EPs

We derive further relationships between primal and dual EPs. These properties will be useful for understanding the reduction introduced in the next section.

C.2.1 Monotonicity

By the definition of monotonicity, $\Phi(x, x') + \Phi(x', x) \leq 0$, we can relate the primal and the dual residuals: for $\hat{x} \in \mathcal{X}$,
\[ r_{dep}(\hat{x}) = \max_{x \in \mathcal{X}} \Phi(x, \hat{x}) \leq \max_{x \in \mathcal{X}} -\Phi(\hat{x}, x) = r_{ep}(\hat{x}) \]

Let $X^*$ and $X_{**}$ be the solution sets of the EP and DEP, respectively. In other words, for monotone EPs, $X^* \subseteq X_{**}$.

C.2.2 Continuity

When $\Phi(\cdot, x)$ is continuous, it can be shown that $X^* \subseteq X_{**}$ (Konnov and Schaible, 2000) (this can be relaxed to hemi-continuity). Below we relate the primal and the dual residuals in this case. It implies that the convergence rate of the primal residual is slower than the dual residual.

**Proposition 13.** Suppose $\Phi(\cdot, x)$ is $L$-Lipschitz continuous for any $x \in \mathcal{X}$ and $\max_{x, x' \in \mathcal{X}} \|x - x'\| \leq D$. If $r_{dep}(x) \leq 2LD$, the $r_{ep}(x) \leq 2\sqrt{2LD/r_{dep}(x)}$.

Suppose in addition $\Phi(x, \cdot)$ is $\mu$-strongly convex with $\mu > 0$. If $r_{dep}(x) \leq \frac{L^2}{\mu}$, we can remove the dependency on $D$ and show $r_{ep}(x) \leq 2.8(\frac{L^2}{\mu})^{1/3}r_{dep}(x)^{2/3}$.

**Proof.** Let $y \in \mathcal{X}$ be arbitrary. Define $z = \tau x + (1 - \tau)y$, where $\tau \in [0, 1]$. Suppose $x$ is an $\epsilon$-approximate dual solution, i.e.,
\[ r_{dep}(x) = \max_{x' \in \mathcal{X}} \Phi(x', x) = \epsilon \]

By convexity and $\Phi(z, z) = 0$, we can write
\[ \epsilon \geq \Phi(z, x) = \Phi(z, x) - \Phi(z, z) \geq \Phi(z, x) - \tau \Phi(z, x) - (1 - \tau)\Phi(z, y) = (1 - \tau)(\Phi(z, x) - \Phi(z, y)) \]

Using this, we can then show
\[ -\Phi(x, y) = -\Phi(x, y) + \Phi(z, y) + (\Phi(z, x) - \Phi(z, y)) - \Phi(z, x) + \Phi(x, x) \leq |\Phi(z, y) - \Phi(x, y)| + |\Phi(z, x) - \Phi(z, y)| + \Phi(z, x) - \Phi(z, y) \leq 2(1 - \tau)L\|x - y\| + \Phi(z, x) - \Phi(z, y) \]
\[ \leq 2(1 - \tau)L\|x - y\| + \frac{\epsilon}{1 - \tau} \quad (\because \text{Lipschitz condition}) \]
\[ \leq 2(1 - \tau)LD + \frac{\epsilon}{1 - \tau} \quad (\because \text{The inequality above}) \]

Assume $\epsilon \leq 2LD$ and let $(1 - \tau) = \sqrt{\frac{\epsilon}{2LD}}$, which satisfies $\tau \in [0, 1]$. Then
\[ -\Phi(x, y) \leq 2\sqrt{2LD\epsilon} \]
When we have $\mu$-strong convexity, we have a tighter bound

$$
\epsilon \geq \Phi(z, x) - \Phi(z, z) \geq \Phi(z, x) - \tau \Phi(z, x) - (1 - \tau) \Phi(z, y) + \frac{\mu \tau (1 - \tau)}{2} \|x - y\|^2
$$

$$
= (1 - \tau)(\Phi(z, x) - \Phi(z, y)) + \frac{\mu \tau (1 - \tau)}{2} \|x - y\|^2
$$

Using this, we can instead show (following similar steps as above)

$$
-\Phi(x, y) \leq 2(1 - \tau) L \|x - y\| + \Phi(z, x) - \Phi(z, y)
$$

$$
\leq 2(1 - \tau) L \|x - y\| + \frac{\epsilon}{1 - \tau} - \frac{\mu \tau}{2} \|x - y\|^2
$$

$$
\leq \frac{\epsilon}{1 - \tau} + \frac{2 L^2 (1 - \tau)^2}{\mu \tau}
$$

where the last inequality is simply $bx - \frac{a}{2} x^2 \leq \frac{b^2}{2a}$ for $a > 0$. Assume $\epsilon \leq \frac{L^2}{\mu} =: \frac{K}{2}$ and let $(1 - \tau) = (\frac{\epsilon}{K})^{1/3} \in [0, 1]$.

We have the following inequality, where the last step uses $\epsilon \leq \frac{K}{2}$.

$$
-\Phi(x, y) \leq \frac{\epsilon}{1 - \tau} + \frac{2 L^2 (1 - \tau)^2}{\mu \tau} = \epsilon^{2/3} K^{1/3} \left(1 + \frac{1}{1 - (\frac{\epsilon}{K})^{1/3}}\right) \leq 2.2 \epsilon^{2/3} K^{1/3}
$$

C.2.3 Equivalence between primal and dual EPs.

An interesting special case of EP is those with skew-symmetric bifunctions, i.e.

$$
\Phi(x, x') = -\Phi(x', x)
$$

In this case, the EP and the DEP become identical

$$
(DEP) \quad \Phi(x, x^{**}) \leq 0 \iff -\Phi(x^{**}, x) \leq 0 \iff \Phi(x^{**}, x) \geq 0 \quad (EP)
$$

and we have $X^* = X^{**}$ and naturally matching residuals

$$
r_{dep}(\hat{x}) = r_{ep}(\hat{x}).
$$

Recall from the results of the previous two subsections, generally, when $\Phi(\cdot, x)$ is Lipschitz and $F$ is monotone (but not skew-symmetric), we have $X^* = X^{**}$ (as known before) but only $(\Phi(\cdot, \cdot)$ is convex)

$$
r_{dep}(x) \leq r_{ep}(x) \leq \sqrt{2LD} \sqrt{r_{dep}(x)}
$$

(8)

or $(\Phi(\cdot, \cdot)$ is $\mu$-strongly convex)

$$
r_{dep}(x) \leq r_{ep}(x) \leq 2.8 \left(\frac{L^2}{\mu}\right)^{1/3} r_{dep}(x)^{2/3}
$$

C.2.4 Relationship with VIs

We can reduce a EP into a VI problem. We observe that if a point $x^* \in \mathcal{X}$ satisfies

$$
\Phi(x^*, x) \geq 0, \quad \forall x \in \mathcal{X}
$$

if only if

$$
\nabla_2 \Phi(x^*, x^*)^\top (x - x^*) \geq 0, \quad \forall x \in \mathcal{X}
$$
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(i.e. \(x^*\) is a global minimum of the function \(\Phi(x^*, \cdot)\)), where \(\nabla_2\) denotes the partial derivative with respect to the second argument. Therefore, EP(\(X, \Phi\)) is equivalent to VI(\(X, F\))

\[
\text{find } x^* \in X \quad \text{s.t.} \quad (F(x), x' - x) \geq 0, \quad \forall x' \in X
\]

if we define \(F\) as

\[
F : x \in X \mapsto F(x) = \nabla_2 \Phi(x, x)
\]

(9)

In a sense, this VI problem is a linearization of the EP problem. In other words, VIs are EPs whose bifunction satisfies that \(\Phi(x, \cdot)\) is linear.

By the definition in (9), we can show that

\[
r_{dv}^d(\hat{x}) \leq r_{dep}(\hat{x}) \quad \text{and} \quad r_{ep}(\hat{x}) \leq r_{vi}(\hat{x})
\]

And if \(\Phi\) is monotone, then \(F = \nabla_2 \Phi(x, x)\) is monotone (though the opposite is not true), because

\[
(F(x), x' - x) = (\nabla_2 \Phi(x, x), x' - x) \leq \Phi(x, x') \quad (\therefore \text{Convexity})
\]

\[
\leq -\Phi(x', x) \quad (\therefore \text{Monotonicity})
\]

\[
\leq (\nabla_2 \Phi(x', x'), x' - x) = (F(x'), x' - x) \quad (\therefore \text{Convexity})
\]

Note the converse is not true, unless \(\Phi(x, \cdot)\) is linear.

C.3 Reduction from Equilibrium Problems to Continuous Online Learning

Now we present the general reduction strategy. Given a EP \((X, \Phi)\), we propose to define a COL problem by identifying

\[
f_x(x') = \Phi(x, x')
\]

We can see that this definition is consistent with Theorem 1: due to \(\Phi(x, x) = 0\), it satisfies

\[
f_x(x') - f_x(x) = \Phi(x, x') - \Phi(x, x) = \Phi(x, x')
\]

Therefore, we can say a COL is normalized if \(f_x(x) = 0\). In this case, \(f\) and \(\Phi\) are interchangeable.

Below we relate the dynamic regret \(\text{Regret}^d_N := \sum_{n=1}^N f_{x_n}(x_n) - \min_{x \in X} f_{x_n}(x)\) and the static regret \(\text{Regret}^s_N := \sum_{n=1}^N f_{x_n}(x_n) - \min_{x \in X} \sum_{n=1}^N f_{x_n}(x)\) of this problem to the convergence to the EP’s solution; note that the above definitions use the fact that in COL \(I_n(x) = f_{x_n}(x)\).

C.3.1 Dynamic Regret and Primal Residual

We first observe that each instant term in the dynamic regret of this COL problem is exactly the residual function:

\[
f_{x_n}(x_n) - \min_{x \in X} f_{x_n}(x) = -\Phi(x_n, x) = r_{ep}(x_n)
\]

Therefore, the average dynamic regret describes the rate the gap function converges to zero:

\[
\sum_{n=1}^N r_{ep}(x_n) = \sum_{n=1}^N f_{x_n}(x_n) - \min_{x \in X} f_{x_n}(x) = \text{Regret}^d_N
\]

Note that the above relationship holds also for weighted dynamic regret. In general, it means that if the average dynamic regret converges, then the last iterate must converge to the solution set of the EP (since the residual is non-negative.)
C.3.2 Static Regret and Dual Residual of Monotone EPs

Next we relate the weighted static regret to the dual residual of the EP. Let \( \{w_n\} \) be such that \( w_n > 0 \). Let \( \hat{x}_N = \frac{1}{x_{1,N}} \sum_{n=1}^{N} w_n x_n \) for some \( \{x_n \in X\}_{n=1}^{N} \), where we define \( w_{1:N} := \sum_{n=1}^{N} w_n \). We can derive

\[
\begin{align*}
    r_{dep}(\hat{x}_N) &= \max_{x \in X} \Phi(x, \hat{x}_N) \\
    \leq& \max_{x \in X} \frac{1}{w_{1:N}} \sum_{n=1}^{N} w_n \Phi(x, x_n) \\
    \leq& \max_{x \in X} \frac{1}{w_{1:N}} \sum_{n=1}^{N} - w_n \Phi(x_n, x) \\
    =& - \min_{x \in X} \frac{1}{w_{1:N}} \sum_{n=1}^{N} w_n \Phi(x_n, x) \\
    =& \frac{1}{w_{1:N}} \left( \sum_{n=1}^{N} w_n f_n(x_n) - \min_{x \in X} \sum_{n=1}^{N} w_n f_n(x) \right) \\
    =& \frac{\text{Regret}_D(w)}{w_{1:N}}
\end{align*}
\]

Note that the inequality \( r_{dep}(\hat{x}_N) \leq \frac{\text{Regret}_D(w)}{w_{1:N}} \) holds for any sequence \( \{x_n\} \) and \( \{w_n\} \). Interestingly, by (8), we see that by the definition of regrets and the property of monotonicity and local Lipschitz continuity, it holds that

\[
\frac{r_{ep}(\hat{x}_N)^2}{2LD} \leq r_{dep}(\hat{x}_N) \leq \frac{\text{Regret}_N^s(w)}{w_{1:N}} \leq \frac{\text{Regret}_N^d(w)}{w_{1:N}} \leq \sum_{n=1}^{N} \frac{w_n r_{ep}(x_n)}{w_{1:N}}
\]

where \( L \) is the Lipschitz constant of \( \Phi(\cdot, x) \) and \( D \) is the size of \( X \).

C.4 Summary

Let us summarize the insights gained from the above discussions.

1. We can reduce EP(\( X, \Phi \)) with monotone \( \Phi \) to the COL problem with \( l_n(x) = \Phi(x_n, x) \)

2. In this COL, the convergence in (weighted) average dynamic regret implies the convergence of the last iterate to the primal solution set. The convergence in (weighted) average static regret implies the convergence of the (weighted) average decision to the dual solution set.

3. Because any dual solution is a primal solution when \( \Phi(\cdot, x) \) is continuous, this implies the (weighted) average solution above also converges to the primal solution set. Particularly, if the problem is Lipschitz, we can show \( r_{ep} \leq O(\sqrt{r_{dep}}) \) and therefore we can also quantify the exact quality of \( \hat{x}_N \) in terms of the primal EP (though it results in a slower rate).

4. When the problem is skew-symmetric (as in the case of common reductions from optimization and saddle-point problems), we have exactly \( r_{ep} = r_{dep} \). This means the average static regret rate directly implies the quality of \( \hat{x}_N \) in terms of the primal residual, without rate degradation.

D Complete Proofs of Section 6

D.1 Proof of Theorem 4

The main idea is based on the decomposition that

\[
\text{Regret}_N^s = \sum_{n=1}^{N} f_{x_n}(x_n) - f_{x_n}(x^*) + \sum_{n=1}^{N} f_{x_n}(x^*) - f_{x_n}(x_n^*)
\]
For the first term, \( \sum_{n=1}^{N} f_{x_n}(x_n) - f_{x_n}(x^*) = \text{Regret}_N(x^*) \leq \text{Regret}_N^* \) and \( f_{x_n}(x_n) - f_{x_n}(x^*) \leq (\nabla f_{x_n}(x_n), x_n - x^*) \leq G\Delta_n \). For the second term, we derive

\[
\begin{align*}
\frac{1}{\alpha} & \leq f_{x_n}(x^*) - f_{x_n}(x_n) \\
& \leq \langle \nabla f_{x_n}(x^*), x^* - x_n^* \rangle - \frac{\alpha}{2} \|x^* - x_n^*\|^2 \\
& \leq \langle \nabla f_{x_n}(x^*) - \nabla f_{x_n}(x^*), x^* - x_n^* \rangle - \frac{\alpha}{2} \|x^* - x_n^*\|^2 \\
& \leq \|\nabla f_{x_n}(x^*) - \nabla f_{x_n}(x^*)\| \|x^* - x_n^*\| - \frac{\alpha}{2} \|x^* - x_n^*\|^2 \\
& \leq \beta \|x_n^* - x^*\| \|x^* - x_n^*\| - \frac{\alpha}{2} \|x^* - x_n^*\|^2 \\
& \leq \min \{\beta D\|x_n - x^*\|, \frac{\beta^2}{2\alpha} \|x_n - x^*\|^2\}
\end{align*}
\]

in which the second inequality is due to that \( x^* \in X^* \) and the fourth inequality is due to \( \beta \)-regularity. Combining the two terms gives the upper bound. For the lower bound, we notice that when \( x_\bullet \in X_\bullet \), we have \( f_{x_n}(x_n) - f_{x_n}(x^*) \geq 0 \). Since by Proposition 1 \( x_\bullet \in X_\bullet \) is also true, we can use (10) and the fact that \( f_{x_n}(x_\bullet) - f_{x_n}(x^*_n) \geq \frac{\eta}{2} \|x_\bullet - x^*_n\|^2 \) to derive the lower bound.

D.2 Proof of Corollary 1

By Proposition 5, \( \nabla f \) is \( (\alpha - \beta) \)-strongly monotone, implying \( \langle \nabla f_{x_n}(x_n), x_n - x^* \rangle \geq (\alpha - \beta)\Delta_n^2 \), where we recall that \( \Delta_n = \|x_n - x^*\| \) and \( x^* \in X^* \). Because \( \sum_{n=1}^{N} \langle \nabla f_{x_n}(x_n), x_n - x^* \rangle = \text{Regret}_N(x^*) \leq \text{Regret}_N^* \), we have by Theorem 4 the inequality in the statement.

D.3 Proof of Proposition 7

In this case, by Proposition 6, \( T \) is non-expansive. We know that, e.g., Mann iteration (Mann, 1953), i.e., for \( \eta_n \in (0, 1) \) we set

\[
x_{n+1} = \eta_n x_n + (1 - \eta_n)x_n^*, \tag{11}
\]

converges to some \( x^* \in X^* \); in view of (11), the greedy is update is equivalent to Mann iteration with \( \eta_n = 1 \). As Mann iteration converges in general Hilbert space, by Theorem 1, it has sublinear dynamic regret with some constant that is polynomial in \( d \).

D.4 Proof of Proposition 8

We first establish a simple lemma related to the smoothness of \( \nabla f_{x}(x) \) and then a result on the convergence of the Bregman divergence \( B_R(x_n||x^*) \). The purpose of the second lemma is to establish essentially a contraction showing that the distance between the equilibrium point \( x^* \) and \( x_n \) strictly decreases.

Lemma 4. If, \( \forall x \in X, \nabla f(x) \) is \( \beta \)-Lipschitz continuous and \( f_x(\cdot) \) is \( \gamma \)-smooth, then, for any \( x, y \in X \),

\[
\|\nabla f_x(x) - \nabla f_y(y)\| \leq (\gamma + \beta)\|x - y\|.
\]

Proof. For any \( x, y \in X \), it holds that

\[
\begin{align*}
\|\nabla f_x(x) - \nabla f_y(y)\| & \leq \|\nabla f_x(x) - \nabla f_y(x) + \nabla f_y(x) - \nabla f_y(y)\| \\
& \leq \|\nabla f_x(x) - \nabla f_y(x)\| + \|\nabla f_y(x) - \nabla f_y(y)\| \\
& \leq \beta \|x - y\| + \gamma \|x - y\|.
\end{align*}
\]

The last inequality uses \( \beta \)-regularity and \( \gamma \)-smoothness of \( \nabla f(x) \) and \( f_y(\cdot) \), respectively.

Lemma 5. If \( f \) is \( (\alpha, \beta) \)-regular, \( f_x(\cdot) \) is \( \gamma \)-smooth for all \( x \in X \), and \( R \) is 1-strongly convex and \( L \)-smooth, then for the online mirror descent algorithm it holds that

\[
B_R(x^*||x_n) \leq (1 - 2\eta(\alpha - \beta)L^{-1} + \eta^2(\gamma + \beta)^2)^{n-1} B_R(x^*||x_1).
\]
Proof. By the mirror descent update rule in (4), \( \langle \eta \nabla f_{x_n}(x_n) + \nabla R(x_{n+1}) - \nabla R(x_n), x^* - x_{n+1} \rangle \geq 0 \). Since \( x^* \in X \), \( \langle \eta \nabla f_{x_n}(x^*), x_{n+1} - x^* \rangle \geq 0 \). Combining these inequalities yields \( \eta \langle \nabla f_{x_n}(x_n) - \nabla f_{x^*}(x^*), x_{n+1} - x^* \rangle \leq (\nabla R(x_{n+1}) - \nabla R(x_n), x^* - x_{n+1}) \). Then by the three-point equality of the Bregman divergence, we have

\[
B_R(x^* \| x_{n+1}) \leq B_R(x^* \| x_n) - B_R(x_{n+1} \| x_n) - \eta \langle \nabla f_{x_n}(x_n) - \nabla f_{x^*}(x^*), x_{n+1} - x^* \rangle.
\]

Because of the \((\alpha - \beta)\)-strong monotonicity of \( \nabla f_x(x) \), the above inequality implies

\[
B_R(x^* \| x_{n+1}) \leq B_R(x^* \| x_n) - B_R(x_{n+1} \| x_n) - \eta \langle \nabla f_{x_n}(x_n) - \nabla f_{x^*}(x^*), x_{n+1} - x_n \rangle
- \eta \langle \nabla f_{x_n}(x_n) - \nabla f_{x^*}(x^*), x_{n+1} - x^* \rangle
\leq B_R(x^* \| x_n) - B_R(x_{n+1} \| x_n) - \eta \langle \nabla f_{x_n}(x_n) - \nabla f_{x^*}(x^*), x_{n+1} - x_n \rangle - \eta (\alpha - \beta) \| x^* - x_n \|^2
\leq B_R(x^* \| x_n) + \frac{\eta^2 (\gamma + \beta)^2}{2} \| x^* - x_n \|^2 - \eta (\alpha - \beta) \| x^* - x_n \|^2
\leq (1 + \eta^2 (\gamma + \beta)^2 - 2\eta (\alpha - \beta) L^{-1}) B_R(x^* \| x_n).
\]

The third inequality results from the Cauchy-Schwarz inequality followed by maximizing over \( \| x_{n+1} - x_n \| \) and then applying Lemma 4. The last inequality uses the fact that \( R \) is 1-strongly convex and \( L \)-smooth.

If \( \alpha > \beta \) and \( \eta \) is chosen such that \( \eta < \frac{2(\alpha - \beta)}{L(\gamma + \beta)^2} \), we can see that the online mirror descent algorithm guarantees linear convergence of \( B_R(x^* \| x_n) \) to zero with rate \( (1 - 2\eta (\alpha - \beta) L^{-1} + \frac{\eta^2 (\gamma + \beta)^2}{2}) \in (0, 1) \). By strong convexity, we have,

\[
\Delta_n = \| x^* - x_n \| \leq \sqrt{2 B_R(x^* \| x_n)}
\leq \sqrt{2} \left( 1 + \eta^2 (\gamma + \beta)^2 - 2\eta (\alpha - \beta) L^{-1} \right)^{\frac{n+1}{2}} B_R(x^* \| x_0)^{1/2}.
\]

The proposition follows immediately from combining this result and Theorem 4.

D.5 Proof of Proposition 9

Recall that \( g_n = \nabla l_u(x_n) + \epsilon_n + \xi_n \). As discussed previously, we assume there exist constants \( 0 \leq \sigma, \kappa < \infty \) such that \( \mathbb{E} \left[ \| \epsilon_n \|^2 \right] \leq \sigma^2 \) and \( \| \xi_n \|^2 \leq \kappa^2 \) for all \( n \). The mirror descent update rule is given by

\[
x_{n+1} = \arg \min_{x \in X} \langle \eta_n g_n, x \rangle + B_R(x \| x_n).
\]

We use Corollary 1 along with known results for the static regret to bound the dynamic regret in the stochastic case. The main idea of the proof is to show the result for the linearized losses. By convexity, this can be used to bound both terms in Corollary 1.

Let \( u \) be any fixed vector in \( X \), chosen independent of the learner’s decisions \( x_1, \ldots, x_n \). The first-order condition for optimality of (12) yields \( \langle \eta_n g_n, x_{n+1} - u \rangle \leq \langle u - x_{n+1}, \nabla R(x_{n+1}) - \nabla R(x_n) \rangle \). We use this condition to bound the linearized losses as in the proof of Proposition 8. We can bound the linearized losses by the magnitude of the stochastic gradients and Bregman divergences between \( u \) and the learner’s decisions:

\[
\langle g_n, x_n - u \rangle \leq \frac{1}{\eta_n} \langle u - x_{n+1}, \nabla R(x_{n+1}) - \nabla R(x_n) \rangle + \langle g_n, x_n - x_{n+1} \rangle
= \frac{1}{\eta_n} B_R(u \| x_n) - \frac{1}{\eta_n} B_R(u \| x_{n+1}) - \frac{1}{\eta_n} B_R(x_{n+1} \| x_n) + \langle g_n, x_n - x_{n+1} \rangle
\leq \frac{1}{\eta_n} B_R(u \| x_n) - \frac{1}{\eta_n} B_R(u \| x_{n+1}) - \frac{1}{2 \eta_n} \| x_n - x_{n+1} \|^2 + \| g_n \| \| x_n - x_{n+1} \|
\leq \frac{1}{\eta_n} B_R(u \| x_n) - \frac{1}{\eta_n} B_R(u \| x_{n+1}) + \frac{\eta_n}{2} \| g_n \|^2.
\]

The first inequality follows from adding \( \langle g_n, x_n - x_{n+1} \rangle \) to both sides of the inequality from the first-order condition for optimality. The equality uses the three-point equality of the Bregman divergence. The second
inequality follows from the Cauchy-Schwarz inequality and the fact that $\frac{1}{2}\|x_n - x_{n+1}\|^2 \leq B_R(x_{n+1} \| x_n)$ due to the 1-strong convexity of $R$. The last inequality maximizes over $\|x_n - x_{n+1}\|$. Define $R = \sup_{w_1,w_2 \in \mathcal{X}} B_R(w_1 \| w_2)$, which is bounded. Note that $\mathbb{E}[\|g_n\|^2] \leq 3(G^2 + \sigma^2 + \kappa^2)$. Therefore, summing from $n = 1$ to $N$, it holds for any $u \in \mathcal{X}$ selected before learning,

$$
\mathbb{E}\left[\sum_{n=1}^{N} \langle g_n, x_n - u \rangle\right] \leq \mathbb{E}\left[\sum_{n=1}^{N} \left( \frac{1}{\eta_n} - \frac{1}{\eta_{n-1}} \right) R + \frac{3}{2} (G^2 + \sigma^2 + \kappa^2) \eta_n \right]
$$

After rearrangement, we have

$$
\mathbb{E}\left[\sum_{n=1}^{N} \langle \nabla l_n(x_n) + \epsilon_n, x_n - u \rangle\right] \leq \mathbb{E}\left[\sum_{n=1}^{N} \left( \frac{1}{\eta_n} - \frac{1}{\eta_{n-1}} \right) R + \frac{3}{2} (G^2 + \sigma^2 + \kappa^2) \eta_n + D_X \| \xi_n \| \right].
$$

Choosing $\eta_n = \frac{1}{\sqrt{n}}$, $\eta_n = \eta_1$, and $u = x^*$ (because $x^*$ is fixed for a fixed $f$ selected before learning) yields

$$
\mathbb{E}\left[\sum_{n=1}^{N} \langle \nabla l_n(x_n) + \epsilon_n, x_n - x^* \rangle\right] = O(\sqrt{N} + \Xi).$

Because of the law of total expectation and that $x_n$ does not depend on $\epsilon_n$, we have $\mathbb{E}[\text{Regret}^*_n(x^*)] = \mathbb{E}\left[\sum_{n=1}^{N} \langle \nabla l_n(x_n) + \epsilon_n, x_n - x^* \rangle\right]$. Further, by convexity, it follows

$$
\mathbb{E}[\text{Regret}^*_n(x^*)] \leq \mathbb{E}[\text{Regret}^*_1(x^*)].
$$

Then, we may apply Corollary 1 to obtain the result. Note that there is no requirement that $R$ is smooth.

### E. Complete Proofs of Section 7

#### E.1 Proof of Proposition 10

Because $\nabla l_n(\cdot)$ is $\alpha$-strongly monotone, it holds

$$
\langle \nabla l_n(x_{n-1}^*), x_{n-1}^* - x_n^* \rangle \geq \alpha \|x_{n-1}^* - x_n^*\|^2
$$

Since $y^*$ satisfies $\langle \nabla l_{n-1}(x_{n-1}^*), x_n^* - x_{n-1}^* \rangle \geq 0$, the above inequality implies that

$$
\alpha \|x_n^* - x_{n-1}^*\|^2 \leq \langle \nabla l_n(x_{n-1}^*) - \nabla l_{n-1}(x_{n-1}^*), x_{n-1}^* - x_n^* \rangle \\
\leq (\beta \|x_n - x_{n-1}\| + a_n) \|x_{n-1}^* - x_n^*\|
$$

Rearranging the inequality gives the statement.

#### E.2 Proof of Theorem 5

For convenience, define $\lambda := \frac{\beta}{\alpha}$. Recall that, by the mirror descent update rule, the first-order conditions for optimality of both $x_{n+1}$ and $x_n^*$ yield, for all $x \in \mathcal{X}$,

$$
\langle \eta \nabla l_n(x_n), x - x_{n+1} \rangle \geq \langle \nabla R(x_n) - \nabla R(x_{n+1}), x - x_{n+1} \rangle \\
\langle \nabla l_n(x_n^*), x - x_n^* \rangle \geq 0.
$$

The proof requires many intermediate steps, which we arrange in a series of lemmas that typically follow from each other in order. Ultimately, we aim to achieve a result that resembles a contraction as done in Proposition 8 but with additional terms due to the adversarial component of the predictable problem. We begin with general bounds on the Bregman divergence between the learner’s decisions and the optimal decisions.

**Lemma 6.** At round $n$, for an $(\alpha, \beta)$-predictable problem under the mirror descent algorithm, if $l_n$ is $\gamma$-smooth and $R$ is $1$-strongly convex and $L$-smooth, then it holds that

$$
B_R(x_{n+1}^* \| x_n^*) \leq B_R(x_{n+1}^* \| x_n^*) + B_R(x_n^* \| x_{n+1}) + \lambda \|x_{n+1} - x_n\| \| \nabla R(x_n^*) - \nabla R(x_{n+1}) \|_* + \frac{a_n}{\alpha} \| \nabla R(x_n^*) - \nabla R(x_{n+1}) \|_*
$$

and, in the next round,

$$
B_R(x_n^* \| x_{n+1}) \leq B_R(x_n^* \| x_n) - B_R(x_{n+1}^* \| x_n) - \alpha \eta \|x_n - x_n^*\|^2 + \eta \gamma \|x_n - x_n^*\| \|x_{n+1} - x_n\|.
$$
Proof. The first result uses the basic three-point equality of the Bregman divergence followed by the Cauchy-Schwarz inequality and Proposition 10. Note that this first part of the lemma does not require that $x_n$ is generated from a mirror descent algorithm:

$$BR(x_{n+1}^*||x_{n+1}) = BR(x_{n+1}^*||x_n^*) + BR(x_n^*||x_{n+1}) + (x_{n+1}^* - x_n^*, \nabla R(x_n^*) - \nabla R(x_{n+1}))$$

$$\leq BR(x_{n+1}^*||x_n^*) + BR(x_n^*||x_{n+1}) + \|x_{n+1}^* - x_n^*\| \|\nabla R(x_n^*) - \nabla R(x_{n+1})\|_\ast$$

$$\leq BR(x_{n+1}^*||x_n^*) + BR(x_n^*||x_{n+1})$$

$$\quad + \lambda \|x_{n+1} - x_n\| \|\nabla R(x_n^*) - \nabla R(x_{n+1})\|_\ast + \frac{\alpha_n}{\alpha} \|\nabla R(x_n^*) - \nabla R(x_{n+1})\|_\ast.$$

For the second part of the lemma, we require using the first-order conditions of optimality of both $x_{n+1}$ for the mirror descent update and $x_n^*$ for $l_n$:

$$BR(x_n^*||x_{n+1}) = BR(x_n^*||x_n) - BR(x_n^*||x_{n+1}) + (x_n^* - x_{n+1}, \nabla R(x_n) - \nabla R(x_{n+1}))$$

$$\leq BR(x_n^*||x_n) - BR(x_{n+1}^*||x_{n+1}) + \eta(\nabla l_n(x_n^*) - \nabla l_n(x_{n+1}))$$

$$\quad + \eta \|\nabla l_n(x_n^*) - \nabla l_n(x_{n+1}), x_{n+1} - x_n^*\|$$

$$\leq BR(x_n^*||x_n) - BR(x_{n+1}^*||x_{n+1}) - \alpha \eta \|x_n - x_{n+1}\|^2 + \eta \gamma \|x_n - x_{n+1}\|^2 \|x_{n+1} - x_n\|.$$

The first line again applies the three-point equality of the Bregman divergence. The second line combines both first-order optimality conditions to bound the inner product. The last inequality uses the strong convexity of $l_n$ to bound $\eta \|\nabla l_n(x_n^*) - \nabla l_n(x_{n+1}), x_n - x_{n+1}^*\| \leq -\alpha \eta \|x_n - x_{n+1}\|^2$ and the Cauchy-Schwarz inequality along with the smoothness of $l_n$ to bound the other inner product.

The second result also leads to a natural corollary that will be useful later in the full proof.

**Corollary 2.** Under the same conditions as Lemma 6, it holds that

$$BR(x_n^*||x_{n+1}) = (1 - 2\alpha \eta L^{-1} + \eta^2 \gamma^2) BR(x_n^*||x_n).$$

Proof. We start with the first inequality of Lemma 6 and then maximize over $\|x_{n+1} - x_n\|^2$. Finally, we apply the strong convexity and smoothness of $R$ to achieve the result:

$$BR(x_n^*||x_{n+1}) \leq BR(x_n^*||x_n) - BR(x_{n+1}^*||x_n) - \alpha \eta \|x_n - x_{n+1}\|^2 + \eta \gamma \|x_n - x_{n+1}\| \|x_{n+1} - x_n\|$$

$$\leq (1 - 2\alpha \eta L^{-1}) BR(x_n^*||x_n) - \frac{1}{2} \|x_{n+1} - x_n\|^2 + \eta \gamma \|x_n - x_{n+1}\| \|x_{n+1} - x_n\|$$

$$\leq (1 - 2\alpha \eta L^{-1}) BR(x_n^*||x_n) + \eta^2 \gamma^2 BR(x_n^*||x_n) = (1 - 2\alpha \eta L^{-1} + \eta^2 \gamma^2) BR(x_n^*||x_n).$$

We can combine both results of Lemma 6 in order to show

$$BR(x_n^*||x_{n+1}) \leq BR(x_n^*||x_{n+1}) + \lambda \|x_{n+1} - x_n\| \|\nabla R(x_n^*) - \nabla R(x_{n+1})\| + \frac{\alpha_n}{\alpha} \|\nabla R(x_n^*) - \nabla R(x_{n+1})\|$$

$$\quad + BR(x_n^*||x_n) - BR(x_{n+1}^*||x_{n+1}) - \alpha \eta \|x_n - x_{n+1}\|^2 + \eta \gamma \|x_n - x_{n+1}\| \|x_{n+1} - x_n\|.$$

Some of the terms in the above inequality can be grouped and bounded above. By $L$-smoothness of $R$, we have $BR(x_{n+1}^*||x_n) \leq \frac{1}{2} \|x_{n+1}^* - x_n\|^2 \leq \frac{1}{2} \left(\lambda^2 \|x_{n+1} - x_n\|^2 + \frac{\alpha_n}{\alpha^2} \|\nabla R(x_{n+1}) - \nabla R(x_n)\|^2\right)$. Because, $R$ is 1-strongly convex, $L \geq 1$; therefore, the previous inequality can be bounded from above using $L^2$ instead of $L$. While this artificially worsens the bound, it will be useful for simplifying the conditions sufficient for sublinear dynamic regret. 1-strong convexity of $R$ also gives us $-BR(x_{n+1}, x_n) \leq -\frac{1}{2} \|x_{n+1} - x_n\|^2$. Applying these upper bounds and then aggregating terms yields

$$BR(x_n^*||x_{n+1}) \leq \frac{-(1 - L^2 \lambda^2)}{2} \|x_n - x_{n+1}\|^2 + (\lambda \|\nabla R(x_n) - \nabla R(x_{n+1})\| + \eta \gamma \|x_n - x_{n+1}\| \|x_n - x_{n+1}\|$$

$$\quad + BR(x_n^*||x_n) - \alpha \eta \|x_n - x_{n+1}\|^2 + \frac{\alpha_n}{\alpha} \|\nabla R(x_n^*) - \nabla R(x_{n+1})\| + \frac{\alpha_n L}{2 \alpha^2} \|x_n - x_{n+1}\| + \frac{\alpha_n L \lambda}{\alpha} \|x_n - x_{n+1}\|$$

$$\quad \leq \frac{-(1 - L^2 \lambda^2)}{2} \|x_n - x_{n+1}\|^2 + (\lambda \|\nabla R(x_n) - \nabla R(x_{n+1})\| + \eta \gamma \|x_n - x_{n+1}\| \|x_n - x_{n+1}\|$$

$$\quad + BR(x_n^*||x_n) - \alpha \eta \|x_n - x_{n+1}\|^2 + \frac{\alpha_n}{\alpha} \|\nabla R(x_n^*) - \nabla R(x_{n+1})\| + \frac{\alpha_n L}{2 \alpha^2} \|x_n - x_{n+1}\| + \frac{\alpha_n L \lambda}{\alpha} \|x_n - x_{n+1}\|$$

$$\quad \leq \frac{-(1 - L^2 \lambda^2)}{2} \|x_n - x_{n+1}\|^2 + (\lambda \|\nabla R(x_n) - \nabla R(x_{n+1})\| + \eta \gamma \|x_n - x_{n+1}\| \|x_n - x_{n+1}\|$$
The quantity on the right hand size of the above inequality is in fact smaller than 2 by optimizing over choices of $\alpha$. It is sufficient to have the condition for a contraction be:

$$\text{condition required to guarantee the contraction is stricter than requiring that } L \alpha < \lambda.$$ 

Alternatively, $\eta = \frac{\alpha}{2L\gamma}$. The third inequality follows from maximizing over $\|x_n - x_{n+1}\|$ and then applying $(a + b)^2 \leq 2a^2 + 2b^2$ for any $a, b \in \mathbb{R}$. For this operation, we require that $L^2 \lambda^2 < 1$. The fourth inequality uses $L$-smoothness of $R$. The last inequality uses the fact that $R$ is 1-strongly convex to bound the squared normed differences by the Bregman divergence.

We then use Corollary 2 to bound this result on $B_R(x_{n+1}^*||x_{n+1})$ in terms of only $B_R(x_n^*||x_n)$ and the appropriate constants:

$$B_R(x_{n+1}^*||x_{n+1}) \leq \frac{2L^2\lambda^2 B_R(x_n^*||x_{n+1}) + 2\eta^2\gamma^2 B_R(x_n^*||x_n)}{1 - L^2\lambda^2} + B_R(x_n^*||x_n) - \alpha \eta \|x_n - x_n^*\|^2 + \zeta_n$$

$$\leq \frac{2L^2\lambda^2}{1 - L^2\lambda^2} (1 - 2\alpha \eta L^{-1} + \eta^2 \gamma^2) B_R(x_n^*||x_n) + \frac{2\eta^2\gamma^2}{1 - L^2\lambda^2} B_R(x_n^*||x_n)$$

$$+ B_R(x_n^*||x_n) - 2\alpha \eta L^{-1} B_R(x_n^*||x_n) + \zeta_n$$

$$= \left(1 - 2\alpha \eta L^{-1} + \frac{2\eta^2\gamma^2}{1 - L^2\lambda^2} + \frac{4L^2\lambda^2\alpha \eta}{1 - L^2\lambda^2} + \frac{2L^2\lambda^2\eta^2\gamma^2}{1 - L^2\lambda^2}\right) B_R(x_n^*||x_n) + \zeta_n$$

$$= \left(1 + \frac{L^2\lambda^2}{1 - L^2\lambda^2}\right) \left(1 - 2\alpha \eta L^{-1} + \eta^2 \gamma^2\right) B_R(x_n^*||x_n) + \zeta_n.$$ 

Thus, we have arrived at an inequality that resembles a contraction. However, the stepsize $\eta > 0$ may be chosen such that it minimizes the factor in front of the Bregman divergence. This can be achieved, but it requires that additional constraints are put on the value of $\lambda$. 

**Lemma 7.** If $\lambda < \frac{\alpha}{2L\gamma}$ and $\eta = \frac{\alpha}{2L\gamma}$, then

$$\left(1 - 2\alpha \eta L^{-1} + \eta^2 \gamma^2\right) < 1$$

**Proof.** By optimizing over choices of $\eta$, it can be seen that

$$1 - 2\alpha \eta L^{-1} + \eta^2 \gamma^2 \geq 1 - \frac{\alpha^2}{2L^2\gamma^2},$$

where $\eta$ is chosen to be $\frac{\alpha}{2L\gamma}$. Therefore, in order to realize a contraction, we must have

$$1 > \left(1 + \frac{L^2\lambda^2}{1 - L^2\lambda^2}\right) \left(1 - \frac{\alpha^2}{2L^2\gamma^2}\right).$$

Alternatively,

$$0 > 2L^2\lambda^2 - \frac{\alpha^2}{2L^2\gamma^2} = \frac{\lambda^2 - \alpha^2}{2\gamma^2}.$$ 

The quantity on the right hand size of the above inequality is in fact smaller than $2L^2\lambda^2 - \frac{\alpha^2}{2L\gamma}$, meaning that it is sufficient to have the condition for a contraction be: $\frac{\alpha}{2L\gamma} > \lambda$. \(\square\)

Note that $\frac{\alpha}{2L\gamma} < 1$ since $L \geq 1$ and $\gamma \geq \alpha$ by the definitions of smoothness of $R$ and $l_n$, respectively. Thus, this condition required to guarantee the contraction is stricter than requiring that $\lambda < 1$. If this condition is satisfied.
and if we set $\eta = \frac{\rho}{2L\gamma}$, then we can further examine the contraction in terms of constants that depend only on the properties of $l_n$ and $R$:

$$BR(x_{n+1}^*\|x_{n+1}) \leq \left(1 + \frac{L^2\lambda^2}{1 - L^2\lambda^2}\right)(1 - 2\alpha\eta L^{-1} + 2\eta^2\gamma^2)BR(x_n^*\|x_n) + \zeta_n$$

$$= \left(1 - \frac{\alpha^2}{2L^2\gamma^2}\right)BR(x_n^*\|x_n) + \zeta_n.$$

It is easily verified that the factor in front of the Bregman divergence on the right side is less than 1 and greater than $\frac{1}{2}$. By applying the above inequality recursively, we can derive the inequality below

$$\frac{1}{2}\|x_n - x_n^*\|^2 \leq BR(x_n^*\|x_n) \leq \rho^{n-1}BR(x_1^*\|x_1) + \sum_{k=1}^{n-1}\rho^{n-k-1}\zeta_k,$$

where $\rho = \left(1 + \frac{L^2\lambda^2}{1 - L^2\lambda^2}\right)(1 - 2\alpha\eta L^{-1} + 2\eta^2\gamma^2) < 1$. Therefore the dynamic regret can be bounded as

$$\text{Regret}_N^d = \sum_{n=1}^{N}f_n(x_n) - f_n(x_n^*) \leq G\sum_{n=1}^{N}\|x_n - x_n^*\|$$

$$\leq \sqrt{2GBR(x_1^*\|x_1)^{1/2}\sum_{n=1}^{N}\rho^{n-1} + \sqrt{2G}\sum_{n=2}^{N}\sum_{k=1}^{n-1}\rho^{n-k-1}\zeta_k}^{1/2}$$

$$\leq \sqrt{2GBR(x_1^*\|x_1)^{1/2}\sum_{n=1}^{N}\rho^{n-1} + \sqrt{2G}\sum_{n=2}^{N}\sum_{k=1}^{n-1}\rho^{n-k-1}\zeta_k^{1/2}},$$

where both inequalities use the fact that for $a, b > 0$, $a + b \leq a + b + 2\sqrt{ab} = (\sqrt{a} + \sqrt{b})^2$. The left-hand term is clearly bounded above by a constant since $\sqrt{\rho} < 1$. Analysis of the right-hand term is not as obvious, so we establish the following lemma independently.

**Lemma 8.** If $\rho < 1$ and $\zeta_n = \frac{a_n LD\alpha}{\alpha} (1 + \lambda) + \frac{a_n^2 L}{2\alpha^2}$, then it holds that

$$\sqrt{2}\sum_{n=1}^{N}\sum_{k=1}^{n-1}\rho^{n-k-1}\zeta_k^{1/2} = O(AN + \sqrt{AN}).$$

**Proof.**

$$\sum_{n=2}^{N}\sum_{k=1}^{n-1}\rho^{n-k-1}\zeta_k^{1/2} = \sum_{n=1}^{N-1}\zeta_n^{1/2}\left(1 + \rho^2 + \ldots + \rho^{N-1-n}\right) \leq \frac{1}{1 - \sqrt{\rho}}\sum_{n=1}^{N-1}\sqrt{\zeta_n}.$$

The last inequality upper bounds the finite geometric series with the value of the infinite geometric series since again $\sqrt{\rho} < 1$ for each $k$. Recall that $\zeta_n$ was defined as

$$\zeta_n = \frac{a_n LD\alpha}{\alpha} (1 + \lambda) + \frac{a_n^2 L}{2\alpha^2}.$$

Therefore, the over the square roots can be bounded:

$$\sum_{n=1}^{N-1}\sqrt{\zeta_n} \leq \sqrt{\frac{LD\alpha}{\alpha}(1 + \lambda)\sum_{n=1}^{N-1}\sqrt{a_n} + \alpha^{-1}\frac{L}{2}\sum_{n=1}^{N-1}a_n.}$$

While the right-hand summation is simply the definition of $A_{N-1}$, the left-hand summation yields $\sum_{n=1}^{N-1}\sqrt{a_n} \leq \sqrt{(N - 1)A_{N-1}}$. Then the total dynamic regret has order $\text{Regret}_N^d = O(1 + AN + \sqrt{AN})$. 

Then the total dynamic regret has order $\text{Regret}_N^d = O(1 + AN + \sqrt{AN})$.
Online Learning with Continuous Variations

E.3 Proof of Theorem 6

E.3.1 Euclidean Space with \( \frac{\beta}{\alpha} = 1 \)

The proof first requires a result from analysis on the convergence of sequences that are nearly monotonic.

**Lemma 9.** Let \((a_n)_{n \in \mathbb{N}} \subset \mathbb{R}\) and \((b_n)_{n \in \mathbb{N}} \subset \mathbb{R}\) be two sequences satisfying \(b_n \geq 0\) and \(\sum_{k=1}^{n} a_k < \infty\ \forall n \in \mathbb{N}\). If \(b_{n+1} \leq b_n + a_n\), then the sequence \(b_n\) converges.

**Proof.** Define \(u_1 := b_1\) and \(u_n := b_n - \sum_{k=1}^{n-1} a_k\). Note that \(u_1 = b_1 \geq b_2 - a_1 = u_2\). Recursively, \(b_n - a_{n-1} \leq b_{n-1} \implies b_n - \sum_{k=1}^{n-1} a_k \leq b_{n-1} - \sum_{k=1}^{n-2} a_k\). Therefore, \(u_n \leq u_{n+1}\). Note that \((u_n)_{n \in \mathbb{N}}\) is bounded below because \(b_n \geq 0\) and \(\sum_{k=1}^{n} a_k < \infty\). This implies that \((u_n)_{n \in \mathbb{N}}\) converges. Because \((\sum_{k=1}^{n} a_k)_{n \in \mathbb{N}}\) also converges, \((b_n)_{n \in \mathbb{N}}\) must converge.

The majority of the proof follows a similar line of reasoning as a standard result in the field of discrete-time pursuit-evasion games Alexander et al. (2006). Let \(\| \cdot \|\) denote the Euclidean norm. We aim to show that if the distance between the learner’s decision \(x_n\) and the optimal decision \(x_n^*\) does not converge to zero, then they travel unbounded in a straight line, which is a contradiction.

Consider the following update rule which essentially amounts to a constrained greedy update:

\[
x_{n+1} = \frac{x_n + x_n^*}{2}
\]

\(x_{n+1}\) is well defined at each round because \(X\) is convex. Define \(c_n := \|x_n - x_n^*\|\). Then we have

\[
0 \leq c_{n+1} = \|x_{n+1} - x_{n+1}^*\| \\
\leq \|x_{n+1} - x_n^*\| + \|x_n^* - x_n\| \\
= \frac{1}{2}\|x_n - x_n^*\| + \|x_{n+1}^* - x_n\| \\
\leq \frac{1}{2}\|x_n - x_n^*\| + \|x_{n+1} - x_n\| + \frac{a_n}{\alpha} \quad (\because \text{Proposition 10}) \\
= \|x_n - x_n^*\| + \frac{a_n}{\alpha} = c_n + \frac{a_n}{\alpha}
\]

Because it is assumed that \(\sum_{n=1}^{\infty} a_n < \infty\), the sequences \((c_n)_{n \in \mathbb{N}}\) and \((a_n)_{n \in \mathbb{N}}\) satisfy the sufficient conditions of Lemma 9. Thus the sequence \((c_n)_{n \in \mathbb{N}}\) converges, so there exists a limit point \(C := \lim_{n \to \infty} c_n \geq 0\).Towards a contradiction, consider the case where \(C > 0\). We will prove that this leads the points to follow a straight line in the following lemma.

**Lemma 10.** Let \(\theta_n\) denote the angle between the vectors from \(x_n^*\) to \(x_{n+1}^*\) and from \(x_n^*\) to \(x_{n+1}\). If \(\lim_{n \to \infty} c_n > 0\), then \(\lim_{n \to \infty} \cos \theta_n = -1\).

**Proof.** At round \(n + 1\) we can write the distance between the learner’s decision and the optimal decision in terms of the previous round:

\[
C^2 = \lim_{n \to \infty} \|x_{n+1} - x_{n+1}^*\|^2 \\
= \lim_{n \to \infty} \left( \|x_{n+1} - x_n^*\|^2 + \|x_{n+1}^* - x_n\|^2 - 2\|x_{n+1} - x_n^*\|\|x_{n+1}^* - x_n\| \cos \theta_n \right) \\
\leq \lim_{n \to \infty} \left( \frac{1}{4}\|x_n - x_n^*\|^2 + \|x_n - x_{n+1}\|^2 + \frac{a_n^2}{\alpha^2} + 2\frac{a_n}{\alpha} \|x_n - x_{n+1}\| - 2\|x_{n+1} - x_n^*\|\|x_{n+1}^* - x_n\| \cos \theta_n \right) \\
= \lim_{n \to \infty} \left( \frac{1}{2}\|x_n - x_n^*\|^2 + \frac{a_n^2}{\alpha^2} + 2\frac{a_n}{\alpha} \|x_n - x_{n+1}\| - 2\|x_{n+1} - x_n^*\|\|x_{n+1}^* - x_n\| \cos \theta_n \right) \\
= \lim_{n \to \infty} \frac{1}{2}\|x_n - x_n^*\|^2 - 2\lim_{n \to \infty} \|x_{n+1} - x_n^*\|\|x_{n+1}^* - x_n^*\| \cos \theta_n \\
= \frac{1}{2} - 2\lim_{n \to \infty} \|x_{n+1} - x_n^*\|\|x_{n+1}^* - x_n^*\| \cos \theta_n
\]

Therefore, \(\lim_{n \to \infty} \cos \theta_n = -1\).
The first inequality follows because $\|x_{n+1} - x_n^*\| = \frac{1}{2}\|x_n - x_n^*\|$ and $\|x_{n+1}^* - x_n^*\| \leq \|x_{n+1} - x_n\| + \frac{a_n}{\alpha}$ due to Proposition 10. The next equality again uses $\|x_{n+1} - x_n^*\| = \frac{1}{2}\|x_n - x_n^*\|$. The second to last line follows from passing the limit through the sum, where we have $\lim_{n \to \infty} a_n = 0$ because $A_\infty < \infty$. That is, the inequality above implies

$$2 \lim_{n \to \infty} \|x_{n+1} - x_n^*\| \|x_{n+1}^* - x_n^*\| \cos \theta_n = -\frac{C^2}{2} < 0$$

which in turn implies $\lim_{n \to \infty} \cos \theta_n < 0$. This leads to an upper bound

$$-2 \lim_{n \to \infty} \|x_{n+1} - x_n^*\| \|x_{n+1}^* - x_n^*\| \cos \theta_n = \left(-2 \lim_{n \to \infty} \cos \theta_n \right) \lim_{n \to \infty} \|x_{n+1} - x_n^*\| \|x_{n+1}^* - x_n^*\|$$

$$\leq \left(-2 \lim_{n \to \infty} \cos \theta_n \right) \lim_{n \to \infty} \frac{1}{2} \|x_n - x_n^*\| \left(\|x_{n+1} - x_n\| + \frac{a_n}{\alpha}\right)$$

$$= -\frac{C^2}{2} \lim_{n \to \infty} \cos \theta_n$$

Combining these two inequalities, we can then conclude $C^2 \leq \frac{C^2}{2} - \frac{C^2}{2} \cos \theta \leq C^2$. A necessary condition in order for the bounds to be satisfied is $\cos \theta = -1$.  

When $C > 0$, Lemma 10 therefore implies the points $x_n, x_{n+1}, x_{n+1}^*, x_n^*$ are colinear in the limit. Thus, $\|x_n - x_{n+m}\|$ grows unbounded in $m$, which contradicts the compactness of $\mathcal{X}$. The alternative case must then be true: $C = \lim_{n \to \infty} \|x_n - x_n^*\| = 0$. The dynamic regret can then be bounded as:

$$\text{Regret}_{1}^d = \sum_{n=1}^{N} l_n(x_n) - l_n(x_n^*) \leq G \sum_{n=1}^{N} \|x_n - x_n^*\|$$

Since $\|x_N - x_N^*\| \to 0$, we know $\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \|x_n - x_n^*\| = 0$. Therefore, the dynamic regret is sublinear.

Note that this result does not reveal a rate of convergence, only that $\|x_n - x_n^*\|$ converges to zero, which is enough for sublinear dynamic regret.

### E.3.2 One-dimensional Space with arbitrary $\frac{\beta}{n}$

In the case where $d = 1$, we aim to prove sublinear dynamic regret regardless of $\alpha$ and $\beta$ by showing that $x_n$ essentially traps $x_n^*$ by taking conservative steps as before. Rather than the constraint being $|x_n - x_{n+1}| \leq \frac{1}{2}\|x_n - x_n^*\|$, we choose $x_{n+1}$ in the direction of $x_n^*$ subject to $|x_n - x_{n+1}| \leq \frac{1}{1+\lambda}\|x_n - x_n^*\|$. Specifically, we will use the following update rule:

$$x_{n+1} = \frac{\lambda x_n + x_n^*}{1 + \lambda}$$

(13)

Recall that sublinear dynamic regret is implied by $c_n := |x_n - x_n^*|$ converging to zero as $n \to \infty$. Therefore, below we will prove the above update rule results in $\lim_{n \to \infty} c_n = 0$. Like our discussions above, this implies achieving sublinear dynamic regret but not directly its rate.

Suppose at any time $|x_n - x_n^*| = 0$. Then we are done since the learner can repeated play the same decision without $x_n^*$ changing. Below we consider the case $|x_n - x_n^*| \neq 0$. We prove this by contradiction. First we observe that the update in (13) makes sure that, at any round, $x_{n+1}^*$ cannot switch to the opposite side of $x_n^*$ with respect to $x_{n+1}$ and $x_n$; namely it is guaranteed that $(x_{n+1}^* - x_{n+1})(x_n^* - x_n) \geq 0$ and $(x_{n+1}^* - x_n)(x_{n+1}^* - x_n) \geq 0$.

Towards a contradiction, suppose that there is some $C > 0$ such that $|x_n - x_n^*| \geq C$ for infinitely many $n$. Then $x_n$ at every round moves a distance of at least $\frac{1}{1+\lambda}$ in the same direction infinitely since $x_{n+1}^*$ always lies the same side of $x_{n+1}$ as $x_n^*$. This contradicts the compactness of $\mathcal{X}$. Therefore $|x_n - x_n^*|$ must converge to zero.

### F New Insights into Imitation Learning

In this section, we investigate an application of the COL framework in the sequential decision problem of online IL (Ross et al., 2011). We consider an episodic MDP with state space $\mathcal{S}$, action space $\mathcal{A}$, and finite horizon $H$. 

For any $s, s' \in \mathcal{S}$ and $a \in \mathcal{A}$, the transition dynamics $\mathcal{P}$ gives the conditional density, denoted by $\mathcal{P}(s'|s,a)$, of transitioning to $s'$ starting from state $s$ and applying action $a$. The reward of state $s$ and action $a$ is denoted as $r(s,a)$. A deterministic policy $\pi$ is a mapping from $\mathcal{S}$ to a density over $\mathcal{A}$. We suppose the MDP starts from some fixed initial state distribution. We denote the probability of being in state $s$ at time $t$ under policy $\pi$ as $d^\pi_t(s)$, and we define the average state distribution under $\pi$ as $d^\pi_T(s) = \frac{1}{T} \sum_{t=1}^T d^\pi_t(s)$.

In IL, we assume that $\mathcal{P}$ and $r$ are unknown to the learner, but, during training time, the learner is given access to an expert policy $\pi^*$ and full knowledge of a supervised learning loss function $c(s, \pi; \pi^*)$, defined for each state $s \in \mathcal{S}$. The objective of IL is to solve

$$\min_{\pi \in \Pi} \mathbb{E}_{s \sim d^\pi} [c(s, \pi; \pi^*)],$$

(14)

where $\Pi$ is the set of allowable parametric policies, which will be assumed to be convex. Note that it is often the case that $\pi^* \not\in \Pi$.

As $d^\pi$ is not known analytically, optimizing (14) directly leads to a reinforcement learning problem and therefore can be sample inefficient. Online IL, such as the popular DAgger algorithm Ross et al. (2011), bypasses this difficulty by reducing (14) into a sequence of supervised learning problems. Below we describe a general construction of online IL: at the $n$th iteration (1) execute the learner’s current policy $\pi_n$ in the MDP to collect state-action samples; (2) update $\pi_{n+1}$ with information of the stochastic approximation of $l_n(\pi) = \mathbb{E}_{d^{\pi_n}} [c(s, \pi; \pi^*)]$ based the samples collected in the first step. Importantly, we remark that in these empirical risks, the states are sampled according to $d^\pi_n$ of the learner’s policy.

The use of online learning to analyze online IL is well established (Ross et al., 2011). As studied in Cheng and Boots (2018); Lee et al. (2018), these online losses can be formulated as a bifunction, $l_n(\pi) = \mathbb{E}_{d^{\pi_n}} [c(s, \pi; \pi^*)]$, and the policy class $\Pi$ can be viewed as the decision set $\mathcal{X}$. Naturally, this online learning formulation results in many online IL algorithms resembling standard online learning algorithms, such as follow-the-leader (FTL), which uses full information feedback $l_n(\cdot) = \mathbb{E}_{s \sim d^{\pi_n}} [c(s, \cdot; \pi^*)]$ at each round (Ross et al., 2011), and mirror descent (Sun et al., 2017), which uses the first-order feedback $\nabla l_n(\pi_n) = \mathbb{E}_{d^{\pi_n}} [\nabla_{\pi_n} c(s, \pi_n; \pi^*)]$. This feedback can also be approximated by unbiased samples. The original work by Ross et al. (2011) analyzed FTL in the static regret case by immediate reductions to known static regret bounds of FTL. However, a crucial objective is understanding when these algorithms converge to useful solutions in terms of policy performance, which more recent work has attempted to address (Cheng and Boots, 2018; Lee et al., 2018; Cheng et al., 2019b).

According to these refined analyses, dynamic regret is a more appropriate solution concept to online IL when $\pi^* \not\in \Pi$, which is the common case in practice.

Below we frame online IL in the proposed COL framework and study its properties based on the properties of COL that we obtained in the previous sections. We have already shown that the per-round loss $l_n(\cdot)$ can be written as the evaluation of a bifunction $f_{\pi_n}(\cdot)$. This COL problem is actually an $(\alpha, \beta)$-regular COL problem when the expected supervised learning loss $\mathbb{E}_{s \sim d^{\pi_n}} [c(s, \pi; \pi^*)]$ is strongly convex in $\pi$ and the state distribution $d^\pi$ is Lipschitz continuous (see Ross et al. (2011); Cheng and Boots (2018); Lee et al. (2018)). We can then leverage our results in the COL framework to immediately answer an interesting question in the online IL problem.

**Proposition 14.** When $\alpha > \beta$, there exists a unique policy $\hat{\pi}$ that is optimal on its own distribution:

$$\mathbb{E}_{s \sim d^{\hat{\pi}}} [c(s, \hat{\pi}; \pi^*)] = \min_{\pi \in \Pi} \mathbb{E}_{s \sim d^{\pi_n}} [c(s, \pi; \pi^*)].$$

This result is immediate from the fact that $\alpha > \beta$ implies that $\nabla f_{\pi}(\pi)$ is a $\mu$-strongly monotone VI with $\mu = \beta - \alpha$ by Proposition 5. The VI is therefore guaranteed to have a unique solution (Facchinei and Pang, 2007).

Furthermore, we can improve upon the known conditions sufficient to find this policy through online gradient descent and give a non-asymptotic convergence guarantee through a reduction to strongly monotone VIs. We will additionally assume that $f$ is $\gamma$-smooth in $\pi$, satisfying $\|\nabla f_{\pi}(\pi_1) - \nabla f_{\pi}(\pi_2)\| \leq \gamma \|\pi_1 - \pi_2\|$ for any fixed query argument $\pi'$.

We then apply our results from Section 6.1. Specifically, we consider mirror descent with $B_R(\pi; \pi') = \frac{1}{2} \|\pi - \pi'\|^2$, which is equivalent to online gradient descent studied in Sun et al. (2017); Lee et al. (2018). Note that $R = \frac{1}{2} \|\pi\|^2$, which is 1-strongly convex and 1-smooth. Then, we apply Lemma 5.
Corollary 3. If $\alpha > \beta$ and the stepsize is chosen such that $\eta = \frac{\alpha - \beta}{(\gamma + \beta)^2}$, then, under the online gradient descent algorithm with deterministic feedback $g_n = \nabla l_n(\pi_n)$, it holds that

$$\|\pi_n - \hat{\pi}\|^2 \leq \left(1 - \frac{(\alpha - \beta)}{\gamma + \beta}\right)^{n-1} \|\pi_1 - \hat{\pi}\|^2$$

By Proposition 8, Regret$_N^d$ will therefore be sublinear (in fact, Regret$_N^d = O(1)$) and the policy converges linearly to the policy that is optimal on its own distribution, $\hat{\pi}$. The only condition required on the problem itself is $\alpha > \beta$ while the state-of-the-art sufficient condition of Lee et al. (2018) additionally requires $\frac{\alpha}{\gamma} > \frac{\beta}{\alpha}$. The result also gives a new non-asymptotic convergence rate to $\hat{\pi}$.

The above result only considers the case when the feedback is deterministic; i.e., there is no sampling error due to executing the policy on the MDP, and the risk $\mathbb{E}_{d^t} \mathbb{E}[c(s, \pi; \pi^*)]$ is known exactly at each round. While this is a standard starting point in analysis of online IL algorithms (Ross et al., 2011), we are also interested in the more realistic stochastic case, which has so far not been analyzed for the online gradient descent algorithm in online IL. It turns out that the COL framework can be easily leveraged here too to provide a sublinear dynamic regret bound.

At round $n$, we consider observing the empirical risk $\hat{l}_n(\pi) = \frac{1}{T} \sum_{t=1}^{T} c(s_t, \pi; \pi^*)$ where $s_t \sim d^t_\pi$. Note that $\mathbb{E}[\hat{l}_n(\pi)|\pi_n] = l_n(\pi)$ and it is easy to show that the first-order feedback $\nabla \hat{l}_n(\pi_n)$ can be modeled as the expected gradient with an additive zero-mean noise: $g_n = \nabla l_n(\pi_n) + \epsilon_n$. For simplicity, we assume $\mathbb{E}[\|\epsilon_n\|^2] < \infty$.

Corollary 4. If $\alpha > \beta$ and the stepsize is chosen as $\eta_n = \frac{1}{\sqrt{n}}$, then, under online gradient descent with stochastic feedback, it holds that $\mathbb{E}[\text{Regret}_N^d] = O(\sqrt{N})$.

This corollary follows from Proposition 9, which in turn leverages the reduction to static regret in Corollary 1.