String Vacua with $N = 2$ Supersymmetry in Four Dimensions

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Abstract

In this article we review four-dimensional string vacua with $N = 2$ space-time supersymmetry. In particular, we will discuss several aspects of the string-string duality between the heterotic string, compactified on $K3 \times T^2$, and the type II superstring compactified on a Calabi-Yau three-fold. We investigate the massless supersymmetric spectra, showing agreement for a large class of dual heterotic/type II string pairs. Some emphasis is given to non-perturbative heterotic phenomena, such as non-perturbative transitions among different vacua and strong coupling singularities, and to their geometric Calabi-Yau description on the type II side. We compare the effective $N = 2$ supergravity actions of dual heterotic/type II string compactifications, and show that the $N = 2$ prepotentials and also higher order gravitational couplings nicely agree in the weak heterotic coupling limit. Finally we consider extremal black hole solutions of $N = 2$ supergravity which arise in the context of heterotic or type II $N = 2$ compactifications. For the type II backgrounds we show how the entropies of these black holes depend on the topological data of the underlying Calabi-Yau spaces; we also construct massless black holes which are relevant for the conifold transition among different Calabi-Yau vacua.

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1 Introduction

Several types of strong-weak coupling duality symmetries in string theories were explored and established during recent years (for recent review see [1]). For example, the heterotic string compactified to four dimensions on a six-dimensional torus $T^6$ exhibits $S$-duality [2], which is the stringy version of the Montonen-Olive electric/magnetic duality symmetry in $N = 4$ supersymmetric field theories. $S$-duality transforms perturbative, electrically charged states into non-perturbative solitonic states with non-vanishing magnetic charges. To give a slightly more general definition, $S$-duality is a non-perturbative symmetry which relates the weak coupling regime of one particular theory to the strong coupling regime of the same theory: $g \leftrightarrow \frac{1}{g}$; hence $S$-duality can be regarded as a self-duality within a particular string model. In this spirit, also the uncompactified IIB superstring possesses $S$-duality [3], where $S$-duality transforms the BPS p-branes in the Neveu-Schwarz (NS) sector into the Dirichlet (D) p-branes of the Ramond (R) sector (for recent introductions into D branes see for example [4]).

Second, there exist also string duality symmetries which relate one string construction, say at weak coupling, to a seemingly different, but eventually equivalent string construction at strong coupling. The first and very basic example of this string-string duality symmetry is the duality between the heterotic string, compactified to six dimensions on $T^4$, and the type IIA superstring, compactified on $K3$ [3, 6, 7]. To establish the string-string duality symmetry one has to take into account the full perturbative and non-perturbative BPS spectra of the dual models. In general, the perturbative states of one string construction are mapped to non-perturbative states of the dual string construction, and vice versa, by the string-string duality symmetry. In fact one can show that the type IIA string contains the heterotic string as a soliton [8], and one obtains the following duality relation among the heterotic and type IIA string couplings in six dimensions: $g_H = \frac{1}{g_{IIA}}$.

Finally, there are duality symmetries which go beyond the perturbatively known string constructions. Namely, Witten [7] has given very strong evidence that the type IIA string in ten dimensions is dual to 11-dimensional supergravity compactified on a circle of radius $R_{11}$, where the Kaluza-Klein modes of 11-dimensional supergravity correspond to the BPS Dirichlet 0-branes of the type IIA string. The full quantum version of 11-dimensional supergravity is now called $M$-theory; $M$-theory on circle of radius $R$ is supposed to describe the full strongly coupled type IIA superstring, with the following duality relation among the IIA string coupling and $R_{11}$: $R_{11} \sim g_{IIA}^{2/3}$. In addition, one can also show [9] that $M$-theory compactified on the semicircle $S^1/Z_2$ corresponds to the strongly coupled heterotic string with gauge group $E_8 \times E_8$. Including also all the other string theories, it seems nowadays clear that all five known consistent string constructions, the two heterotic strings, the type IIA,B superstrings and the type I superstring, plus 11-dimensional supergravity are related by some kind of duality transformation such that they are all equivalent and on equal footing. The underlying unifying quantum theory is $M$-theory, which contains all the six theories as different ways to perform a weak coupling limit.
However a fundamental definition of $M$-theory is up to now not clear. One promising attempt to define $M$-theory is given by a Hamiltonian theory of non-commuting matrices, known as M(atrix)-theory [10]. In particular these matrices describe the dynamics of the non-perturbative D 0-branes of the type IIA superstring. It is important to note that in M(atrix)-theory space-time is not fundamental, but arises only in the limit where the matrices are commuting.

In order to fully establish the unity of all string vacua one must also show that there exist continuous transitions among different string compactifications, i.e. string backgrounds. For example, in the study of the moduli spaces of type II strings compactified on Calabi-Yau three-folds it became clear that the topologies of the Calabi-Yau spaces can be continuously deformed into each other; therefore many, or even possibly all, Calabi-Yau vacua are just branches of a larger universal moduli space. Calabi-Yau phase transitions contain for example the conifold transition in the complex structure moduli space of type IIB superstrings. The conifold transitions occur at those points in the moduli space where certain 3-cycles shrink to zero size and then are blown up as two cycles, changing in this way the Hodge numbers. The physical understanding of the conifold transition was provided by Strominger [11]; at the conifold point a BPS hypermultiplet black hole becomes massless, being responsible for the singularity in the moduli space metric of the $N = 2$ vector multiplets at this point.

In this paper we will review the string-string duality between dual string pairs in four dimensions with $N = 2$ space-time supersymmetry. In particular we will concentrate on the duality between certain $N = 2$ string vacua [12, 13], namely the heterotic string compactified on $K3 \times T^2$, which is dual to the type IIA string on a particular Calabi-Yau background, as it was first discussed by Kachru and Vafa [12]. Using this $N = 2$ string-string duality symmetry, many interesting non-perturbative phenomena can be studied. In particular, effects which are non-perturbative on the heterotic side, follow from classical geometrical considerations on the type II side, where various branes are wrapped around the internal cycles of the underlying Calabi-Yau space. In this way, performing a suitable field theory limit, this approach provides a nice geometrical understanding [13, 16] of non-perturbative effects in supersymmetric field theories, for example ala Seiberg/Witten [17] (for reviews on non-perturbative effects in $N = 2$ supersymmetric gauge theories and their string origin see [18]).

Obviously, for a candidate dual string pair the massless spectrum has to match. In addition, more quantitative checks are possible by comparing the effective interactions of the massless modes. Specifically the holomorphic couplings of the $N = 2$ vector multiplets are determined by the $N = 2$ prepotential of $N = 2$ special geometry (for a review on the holomorphic couplings in $N = 2$ string vacua see [19]). We will in section 3 show that the heterotic and the type IIA prepotentials agree [20, 21] for several dual pairs in a certain corner of the vector multiplet moduli space, which corresponds to the heterotic

\footnote{In addition, four-dimensional string models with $N = 2$ space-time are given by type I compactifications on $K3 \times T^2$; this leads then to a $N = 2$ heterotic/type II/type I string triality [14].}
weak coupling limit. Moreover explicit comparisons \cite{22, 23, 24, 25, 20, 21, 26} of higher derivative vector-gravitational couplings provide an additional successful confirmation of the \( N = 2 \) string-string duality.

In section 4 we will study the extremal BPS saturated solutions \cite{27, 28, 29, 30, 31, 32, 33} of the four-dimensional \( \mathcal{N}=2 \) supergravity coupled to \( \mathcal{N}=2 \) vector multiplets as the effective langrangian of the \( N = 2 \) string models. In general one obtains extremal, supersymmetric, charged black holes that allow, besides the non-trivial four-dimensional black hole metric, for non-constant moduli fields. We will determine the space-time dependence of the black hole metrics, of the scalar fields and of the gauge fields and also the macroscopic Bekenstein-Hawking black hole entropies from the prepotential of \( \mathcal{N}=2 \) special geometry. In case of type IIA Calabi-Yau compactifications, the black hole entropies therefore depend on the topological data of the underlying Calabi-Yau spaces, such as intersection numbers and rational instanton numbers. In the dual heterotic vacua, these contributions correspond to perturbative as well non-perturbative corrections to the black hole entropies. The \( N = 2 \) black hole solutions are also relevant for transitions between different type II vacua on Calabi-Yau spaces in four dimensions. We will discuss the type IIB conifold transition where one of the complex structure moduli fields is small. The following generic picture will emerge. If one moves through a non-singular space time one varies at the same time the radii of the cycles of the Calabi-Yau. At any point in space time the Calabi-Yau looks differently. We will find that for vanishing 3-cycles in IIB compactifications, our solution corresponds to a massless BPS states.

\section{\( \mathcal{N}=2 \) Heterotic/Type II String Duality}

In this chapter we will compare the spectra of \( E_8 \times E_8 \) heterotic string compactifications on \( K3 \times T^2 \) with the dual type IIA compactifications on a suitably chosen Calabi-Yau 3-fold. On the heterotic side, the different vacua are essentially characterized by different choices of the \( E_8 \times E_8 \) gauge bundle. These choices lead to a large variety of different spectra in four dimensions. On the type II side, the different ways to compactify are defined by the choice of the Calabi-Yau 3-fold \( X^3 \). We will discuss how the heterotic data of specifying the gauge bundle will map on the Calabi-Yau data of the dual type II model. It turns out that the Calabi-Yau spaces that are dual to perturbative heterotic string vacua are \( K3 \)-fibrations over a two sphere \( P^1 \) \cite{34, 35}.

Since the toroidal \( T^2 \) compactification on the heterotic side leads to a universal sector, the heterotic data can be already specified by considering the heterotic compactification on \( K3 \) to six dimensions. Many of the interesting perturbative and non-perturbative effects like gauge symmetry enhancement and transitions can be already understood in six dimensions. On the type II side one can also construct the corresponding dual models in six dimensions by considering \( F \)-theory compactified on the same Calabi-Yau space \( X^3 \) which has to be an elliptic fibration over an two-dimensional base \( B^2 \).
The $N = 2$ heterotic/type II string duality can be explained, at least in an heuristic way, from the $N = 4$ string-string duality between the heterotic string on $T^6$ and the type IIA string on $K3 \times T^2$. This more simple string duality already contains the main clues of how the non-Abelian gauge symmetry enhancement is realized on the type II side. Hence, to give an understanding of this important phenomenon and to see the how the $N = 2$ string duality is related to the $N = 4$ string duality, let us briefly compare the heterotic string on $T^6$ with type IIA on $K3 \times T^2$.

### 2.1 The $N = 4$ heterotic/type II string duality

The heterotic string on $T^6$ contains as its massless spectrum first the $N = 4$ supergravity multiplet, which includes six graviphotons plus the complex dilaton axion field, $S_H = e^{-\phi_H} + i a$, and second 22 $N = 4$ vector multiplet with 22 $U(1)$ vector boson and 132 moduli fields. Among the 132 moduli there is the complex $T_H$ field whose real part corresponds to the Kähler class of a $T^2$, which describes, say, the compactification from six to four dimensions. The well known Narain moduli space, including the $S_H$-field, is locally given by

$$
\mathcal{M}_{N=4} = \frac{SO(6,22)}{SO(6) \times SO(22)} \otimes \left( \frac{SU(1,1)}{U(1)} \right)_{S_H}.
$$

Due to the presence of the duality transformations $\mathcal{M}_{N=4}$ has to be moded by the discrete duality group $\Gamma = SO(2,22;\mathbb{Z})_T \times SL(2,\mathbb{Z})_S$, where the first factor corresponds to the perturbative $T$-duality group, and the second factor is the non-perturbative $S$-duality group. The gauge group, at generic points in the moduli space, is given by $G = U(1)^{22} \times U(1)^6$, where the last factor belongs to the six graviphotons. At special points in the moduli space, which correspond to fix-points of the $T$-duality group, there is a perturbative gauge symmetry enhancement to a non-Abelian gauge group of maximal rank 22. The non-Abelian charged gauge bosons are given by internal momentum and winding states; these states hence belong to short perturbative BPS multiplets of the $N = 4$ supersymmetry algebra with central charges.

The type IIA string compactified on $K3 \times T^2$ precisely leads to same massless spectrum with the same moduli space as the heterotic string discussed before. However now, in contrast to the heterotic case, the $N = 4$ gravity multiplet contains the $T_{IIA}$-modulus of two-torus $T^2$. On the other hand, the complex type IIA dilaton field $S_{IIA}$ sits in one of the 22 vector multiplets, where the 132 scalars divide themselves into 84 Neveu-Schwarz scalars and 48 Ramond scalar fields. This means that perturbative effects in the heterotic string can be non-perturbative in the type II string, and vice versa. This exchange between the $S$ and $T$ fields implies that the string-string duality relation between the heterotic and type couplings reads:

$$
S_{H(IIA)} = T_{IIA(H)}.
$$

(2)
This relation can be easily understood from the strong-weak coupling string-string duality relation in six-dimensions, $\phi^6_H = -\phi^6_{IIA}$, and by noting that the four-dimensional dilaton and the six-dimensional dilaton are related as

$$\text{Re}S/\text{Re}T = e^{-\phi^6}.$$  \hspace{1cm} (3)

It is also instructive to recognize that the subspace of $\mathcal{M}_{N=4}$ which is spanned by the 84 NS scalar fields is given by the coset $\frac{SO(1,20)}{SO(4) \times SO(20)} \otimes \frac{SO(2,2)}{SO(2) \times SO(2)}$. This is just the moduli space of $K3 \times T^2$.

Let us now come to the type IIA gauge symmetries. The perturbative gauge group is always $U(1)^{22} \times U(1)^6$; four graviphotons are from the NS-NS sector and all remaining $U(1)$ vector fields come from the R-R sector. Specifically the additional 24 R-R vector fields arise from the ten-dimensional R-R gauge fields $A_M$ and $A_{MNP}$. One of them is the four-dimensional component of $A_M$, and the remaining 23 come from expressing $A_{MNP}$ as exterior product of a four-dimensional one-form gauge potential times each of the 22+1 harmonic two-forms of $K3 \times T^2$. To discuss the non-Abelian gauge symmetry enhancement in the type IIA string, we have to find the charged BPS states which couple to 22 Abelian R-R gauge bosons. These cannot be found within the perturbative type II spectrum, but they are given by the non-perturbative electric D 2-branes which can be wrapped around the 22 homology two-cycles of $K3$. They lead in four dimensions to electrically charged BPS black holes with 22 different electric charges. The masses of the BPS particles is proportional to the area of the $K3$ two-cycles. Therefore massless non-Abelian gauge bosons arise at those points in the $K3$ moduli space where the $K3$ degenerates, and the two-cycles shrink to zero sizes. This is the non-perturbative version of the perturbative Frenkel-Kac mechanism in conformal field theory. In fact, the degenerate $K3$’s allow for an ADE classification, and the intersection form in the cohomology of $K3$ is just given by $E_8 \oplus E_8 \oplus H \oplus H \oplus H$ ($H$ being the hyperbolic plane). This same observation also led to the discovery of the heterotic string as a soliton of the type IIA string in six dimensions.

The $N = 2$ string duality between the heterotic string on $K3 \times T^2$ and the type IIA string on $X^3$ can be made plausible from the $N = 4$ string duality by the adiabatic extension principle. Namely, starting from the $N = 4$ duality, replace the common $T^2$ factor by the two sphere $P^1$, and let the rest of the space vary as a fibre over the base $P^1$. This is the fibre wise application of the $N = 4$ string duality. On the type IIA side, the resulting space is a $K3$ fibred Calabi-Yau space $X^3$ with base $P^1$; the various ways to perform this $K3$-fibration results in the variety of different type II $N = 2$ vacua. On the heterotic side, we deal with a $T^2$ varying over the base $P^1$; this is nothing else than $K3$ represented as an elliptic fibration. The different heterotic models arise by specifying how the fibration, which breaks $N = 4$ supersymmetry to $N = 2$, acts on the heterotic gauge bundle. One has to emphasize that this adiabatic argument is somewhat heuristic; nevertheless one can show that for every perturbative $N = 2$ heterotic vacuum on $K3 \times T^2$ the dual Calabi-Yau space has to be a $K3$ fibration, as we will discuss in the following sections.
2.2 The heterotic string on $K3 \times T^2$

2.2.1 Spectrum in Six Dimensions

Now let us discuss in more detail the spectrum of the $E_8 \times E_8$ heterotic string compactified on $K3 \times T^2$. We start with the compactification on $K3$ which leads to (0,1) supergravity in six dimensions. In general, the 6-dimensional massless spectrum contains first the gravity supermultiplet with $g_{\mu\nu}$ and $B_{\mu\nu}$ as bosonic components. The massless matter fields are given by $N_6$ vector multiplets each with a six-dimensional vector field, by $N_H$ hypermultiplets containing each four real scalar fields, and by $N_T$ tensor multiplets consisting each of a real scalar plus a self-dual antisymmetric tensor $B_{\mu\nu}^+$. Since the effective field theory is a chiral supergravity there are strong constraints due to anomaly matching conditions. The anomaly 8-form is given by the expression

$$I_8 = \alpha \text{tr} R^4 + \beta (\text{tr} R^2)^2 + \gamma \text{tr} R^2 \text{tr} F^2 + \delta (\text{tr} F^2)^2. \quad (4)$$

The requirement of absence of gravitational anomalies, i.e. $\alpha = 0$, demands that

$$N_H - N_V^6 + 29 N_T = 273. \quad (5)$$

Then the anomaly 8-form factorizes, $I_8 = I_4 \wedge \tilde{I}_4$, $I_4 = \text{tr} R^2 - \sum_a v_a \text{tr} F^2_a$, $\tilde{I}_4 = \text{tr} R^2 - \sum_a \tilde{v}_a \text{tr} F^2_a$, where the coefficient $v_a$, $\tilde{v}_a$ are gauge group dependent constants $[10]$, and the remaining anomalies can be cancelled by the Green-Schwarz mechanism $[11]$. For that purpose the three form $H = dB + \omega^L - \sum_a v_a \omega^Y_a$ must be well defined, i.e. $\int_{K3} H = 0$, and the following condition must hold:

$$\sum_a n_a = \sum_a \int (\text{tr} F^2_a) = \int_{K3} \text{tr} R^2 = 24. \quad (6)$$

Here $n_a$ ($a = 1, 2$) are the numbers of $E_8 \times E_8$ instantons turned on. So we see that the gauge bundle has to be non-trivial, and the gauge group $E_8 \times E_8$ will be broken to some subgroup $G_{n_1}^1 \times G_{n_2}^2$ by the gauge instantons. This unbroken group depends on the distribution of the instanton numbers $n_1$, $n_2$, and also on the values of hypermultiplet moduli, as we will discuss later. The number $N_V^6$ is then simply given by the dimension of this unbroken group.

In addition one can consider also the non-perturbative background of $n_5$ heterotic 5-branes turned on $[12]$. In this case the $E_8 \times E_8$ heterotic string spectrum on $K3$ is determined by the three integers $n_1$, $n_2$ and $n_5$, and the anomaly condition eq.(5) has to modified in the following way $[13]$:

$$n_1 + n_2 + n_5 = 24. \quad (7)$$

Without 5-branes, i.e. for perturbative vacua with $n_5 = 0$, the number of tensor multiplets is just one, where the corresponding scalar field is the heterotic dilaton $\phi_H^6$. However considering the non-perturbative contribution of heterotic 5-branes, there are additional
tensor multiplets in the massless spectrum \[12\], since on the world sheet theory of the 5-brane lives a massless tensor field. Hence

\[ N_T = 1 + n_5. \]

Thus for non-trivial 5-brane backgrounds, the Coulomb branch of the six-dimensional theory is characterized by a real \((1 + n_5)\)-dimensional moduli space, parametrized by the scalar field vev’s of the tensor multiplets. The Higgs branch of the theory is parametrized by the scalar fields of the hypermultiplets. The moduli space of the hypermultiplets is a quaternionic space, and its quaternionic dimension, i.e the number of hypermultiplets is given by

\[ N_H = 20 + n_5 + \dim Q \mathcal{M}_{n_1}^{\text{inst}}[H_1] + \dim Q \mathcal{M}_{n_2}^{\text{inst}}[H_2]; \]

in this formula, the first term comes from the 20 moduli of \(K3\). The second term arises, since the position of each five brane on \(K3\) is parametrized by a hypermultiplet. The last two terms denote the dimensions of the quaternionic instanton moduli space of the embedded \(E_8\) instantons. Specifically, if \(E_8\) is broken to the group \(G\) by \(n\) instantons, the instanton moduli space has the dimension

\[ \dim Q \mathcal{M}_{n}^{\text{inst}}[H] = nc_2(H) - \dim H, \]

where \(H\) is the commutant of \(G\) in \(E_8\), and \(c_2(H)\) is the quadratic Casimir of \(H\).

Let us discuss the unbroken gauge group in little bit more detail. For \(n \geq 10\), \(E_8\) is completely broken at an arbitrary point in the hypermultiplet moduli space. However at special loci in the hypermultiplet moduli space the instantons fit into a subgroup of \(E_8\), such that there is an unbroken gauge group left over. This effect is nothing else than the (reverse) Higgs effect in field theory, i.e. breaking the gauge group by the vev’s of the charged hypermultiplets. For example, if the instantons fit into a \(E_7\) subgroup, a \(SU(2)\) gauge symmetry gets restored; the number of hypermultiplets necessary to be tuned to open a \(SU(2)\) can be easily read off from eq.\[\text{(10)}\]:

\[ \Delta N_H(SU(2)) = \dim Q \mathcal{M}_{n}^{\text{inst}}[E_8] - \dim Q \mathcal{M}_{n}^{\text{inst}}[E_7] = 12n - 115. \]

On the other hand, for \(n < 10\) there are not enough instantons to break \(E_8\) completely; so in this case there is a terminal gauge group which is given by \(G = E_8, E_7, E_6, SO(8)\) for \(n = 0, 4, 6, 8\). (No smooth \(E_8\) instantons exist for \(n = 1, 2, 3\).) Again at special loci in the hypermultiplet moduli these gauge groups can be further enhanced.

As already mentioned the gauge group breaking by the \(E_8\) instantons can be equally described by the standard Higgs effect. Here the starting point is to consider \(SU(2)\) gauge bundles on \(K3\) with instanton numbers \((n_1, n_2)\). These break \(E_8 \times E_8\) to \(E_7 \times E_7\) with the following hypermultiplet spectrum (in addition there are 20 singlets, being the \(K3\) moduli, plus \(n_5\) singlets from the 5-branes):

\[ \frac{1}{2} (n_1 - 4)(56, 1) + \frac{1}{2} (n_2 - 4)(1, 56) + (2(n_1 + n_2) - 6)(1, 1). \]
The $E_7 \times E_7$ gauge group can now be broken by giving vev’s to the charged hypermultiplets. For $n \geq 10$, $E_7$ can be completely Higgsed through the chain $E_7 \rightarrow E_6 \rightarrow SO(10) \rightarrow SU(5) \rightarrow SU(4) \rightarrow SU(3) \rightarrow SU(2) \rightarrow 1$. The dimensions of the corresponding Higgs moduli spaces precisely agree with the dimensions of the $E_8$ instanton moduli spaces, given in eq.(10). On the other hand, for $n < 10$, there are not enough charged hypermultiplets for a complete gauge symmetry breakdown, and one ends at the already quoted terminal gauge groups. Let us summarize the six-dimensional heterotic spectrum in the following small table for the case of no five branes, i.e. $n_5 = 0$, $n_1 = 12 + k$, $n_2 = 12 - k$, assuming that the gauge groups are Higgsed as far as possible:

| $k$  | $G_2$ | $N_H$ |
|------|-------|-------|
| 0,1,2| 1     | 244   |
| 4    | $SO(8)$ | 272   |
| 6    | $E_6$ | 322   |
| 8    | $E_7$ | 377   |
| 12   | $E_8$ | 492   |

Table 1: Perturbative heterotic spectra in 6 dimensions.

As expected, all spectra are in agreement with gravitational anomaly matching condition eq.(5).

2.2.2 Spectrum in Four Dimensions

Upon compactification from six to four dimensions on $T^2$, the massless heterotic spectrum can be obtained in a straightforward way. Beside the massless $N = 2$ supergravity multiplet, including one $U(1)$ graviphoton field, there are $N_V$ massless vector and $N_H$ massless hypermultiplets. The structure and the number of the hypermultiplets is unchanged in comparison to six-dimensions; the hypermultiplets parametrize the quaternionic moduli space $\mathcal{M}_H$. On the other hand, a four-dimensional $N = 2$ vector multiplet contains in contrast to six dimensions, a complex scalar field. Therefore the massless vector multiplets give rise to a new complex (special) Kähler moduli space $\mathcal{M}_V$ of complex dimension $N_V$, parametrizing the Coulomb branch of the theory. Giving vevs to the complex scalar fields the non-Abelian gauge group is broken to the maximal Abelian subgroup $U(1)^{\text{rank}G}$. So let us determine the number, $N_V$, of four-dimensional Abelian vector fields from the six-dimensional spectrum. First the six-dimensional unbroken gauge group $G_1 \times G_2$ provides
rank\((G_1 \times G_2)\) vector multiplets; the corresponding \(U(1)\) (Wilson line) moduli are denoted by \(V_i\). Second from the \(T^2\) compactification one obtains two more \(U(1)\) vector multiplets, called \(T\) and \(U\), whose complex scalar components describe the shape of the internal \(T^2\). Finally each tensor multiplet in six-dimensions leads to a \(U(1)\) vector field. The one which contains the dilaton \(\phi^4\) is commonly denoted by \(S\). Additional \(U(1)\) vectors from six-dimensional tensor fields are of non-perturbative nature. So in total we obtain that

\[
N_V = \text{rank}(G_1 \times G_2) + 2 + n_5. \tag{13}
\]

The total moduli space in four dimension is given by the direct product of \(M_V\) and \(M_H\), \(M = M_V \otimes M_H\), because due to \(N = 2\) supersymmetry the kinetic terms do not mix hyper with vector multiplets. For perturbative vacua with no non-perturbative vector fields, the classical vector moduli space can be simply obtained by truncating the \(N = 4\) Narain moduli space to its \(N = 2\) subsector:

\[
M_V = \frac{SO(2, N_V - 1)}{SO(2) \times SO(N_V - 1)} \otimes \left( \frac{SU(1,1)}{SU(1)} \right)_S, \tag{14}
\]

where the relevant target space duality group has still to be modded out. Along the Coulomb branch, at special points in \(M_V\) the Abelian gauge group \(U(1)^{N_V}\) is classically enhanced to a non-Abelian gauge group, where the rank is always preserved. Consider the simplest case, namely the perturbative vacua with the three vector fields \(S, T\) and \(U\), i.e. \(N_V = 3\), which originate from the six-dimensional models with \(n_5 = 0\) and completely broken gauge groups \((k = 0, 1, 2)\). The perturbative target space duality group which acts on the moduli \(T\) and \(U\) is given by \(\Gamma = SO(2,2;\mathbb{Z}) = SL(2,\mathbb{Z})_T \times SL(2,\mathbb{Z})_U \times \mathbb{Z}_2^{T+U}\). At the fixed point/lines of \(\Gamma\) the \(U(1)^2\) gauge symmetry which belongs to \(T\) and \(U\) gets enhanced. The charged states which can become massless at the critical points/lines are given by elementary BPS vector multiplets, whose BPS masses are equal to the central charge of the \(N = 2\) supersymmetric algebra. One finds that at the line \(T = U\) the enhanced gauge group is \(SU(2) \times U(1)\), for \(T = U = 1\) there is a further enhancement to \(SU(2)^2\), and at the point \(T = U = e^{i\pi/6}\) the enhanced group is \(SU(3)\). Also note that particular elements of the duality group \(\Gamma\), namely the fixed point transformations, correspond to the elements of the Weyl group of the enhanced gauge groups. For example the exchange \(T \leftrightarrow U\) is just the Weyl reflection of the \(SU(2)\) which appears at the line \(T = U\).

It is important to emphasize that the perturbative gauge boson enhancement on special loci of \(M_V\) in general disappears when discussing all non-perturbative corrections to the moduli space \(M_V\). Non-perturbatively, according to Seiberg/Witten \[17\], the points of enhanced gauge symmetries will split, and there will be special loci where BPS hyper multiplets, namely BPS monopoles and dyons, become massless. By giving vevs to these massless hyper multiplets (monopole condensation) non-perturbative transitions to a different vacuum can take place under certain circumstances (when the gauge theory also possesses charged hypermultiplets); during this transition \(N_V\) is reduced, but \(N_H\) is increased. In the dual type II models the loci of massless monopoles and dyon are given by
the conifold loci of the dual Calabi-Yau spaces, and the possible transitions correspond to conifold transitions where the topology of the underlying Calabi-Yau space gets changed.

Now we turn to the hypermultiplet moduli space $M_H$. Just as in six dimensions, tuning a certain number of hyper multiplets to special values leads to a perturbative gauge symmetry enhancement, but now also with enlarged rank. This effect opens the possibility to perform perturbative Higgs transitions when going to the Coulomb phase of the newly obtained gauge group. During this Higgs transition $N_V$ increases, whereas $N_H$ is reduced. Of course also the reverse Higgs transition is possible. For example, if we start from the three models with a completely broken gauge group ($k = 0, 1, 2$) a perturbative $SU(2)$ can be opened from the original $E_8^1$ by tuning $12k+29$ hyper multiplets to special values. Then, via the Higgs transition we can reach heterotic vacua with $N_V = 4$ and $N_H = 215 - 12k$.

So in conclusion, a large number of four dimensional heterotic starting vacua can be obtained via Higgs transitions in the heterotic gauge group $G_1 \times G_2$. For example, consider again the embedding of $(12 + k, 12 - k)$ instantons in $E_8 \times E_8$, and perform Higgs transitions in the first gauge group factor $G_1$ (i.e. assume that $G_2$ is still broken as far as possible). Then the four-dimensional spectra $(N_V, N_H)$ of perturbative heterotic string vacua can be computed by the techniques described above and are summarized in the following two tables (in the first horizontal lines, the unbroken groups $G_1$ are denoted, and the first vertical line shows the different values $k$):

| $k$ | $SU(2)$ | $SU(3)$ | $SU(4)$ | $SU(5)$ | $SO(10)$ | $E_6$ | $E_7$ |
|-----|---------|---------|---------|---------|----------|-------|-------|
| 0   | (3,244) | (4,215) | (5,198) | (6,183) | (7,168)  | (8,165) | (9,160) | (10,153) |
| 1   | (3,244) | (4,203) | (5,180) | (6,161) | (7,143)  | (8,139) | (9,133) | (10,125) |
| 2   | (3,244) | (4,191) | (5,162) | (6,139) | (7,118)  | (8,113) | (9,106) | (10,97)  |
| 4   | (7,272) | (8,195) | (9,154) | (10,123)| (11,96)  | (12,89) | (13,80) | (14,69)  |
| 6   | (9,322) | (10,221)| (11,168)| (12,129)| (13,96)  | (14,87) | (15,76) | (16,63)  |
| 8   | (10,377)| (11,252)| (12,187)| (13,140)| (14,101)| (15,90) | (16,77) | (17,62)  |
| 12  | (11,492)| (12,319)| (13,230)| (14,167)| (15,116)| (16,101)| (17,84) | (18,65)  |

Table 2: Perturbative heterotic spectra in 4 dimensions.

Of course this list could be easily completed (making a three-dimensional plot) by considering also the possible Higgs transitions in the second gauge group factor.
2.2.3 Non-perturbative Effects

In this paragraph we will discuss a few non-perturbative effects which play an important role at singular points in the heterotic moduli spaces, where non-perturbative BPS states become massless. In particular we first want to adress the question how non-perturbative transitions between heterotic vacua with different \( E_8 \times E_8 \) instanton numbers \((n_1, n_2)\) can occur, i.e. how non-perturbative effects allow for vertical transitions in table (2) (Remember that the horizontal transitions are realized by the perturbative Higgs transitions.) Then we want to discuss singularities at strong heterotic coupling and the related issue of strong coupling transitions to non-perturbative heterotic vacua with frozen dilaton field. Finally we will discuss non-perturbative \( S \)-duality symmetries which are expected to be present in some particular heterotic models.

We will start the discussion in six dimensions. Singularities in the heterotic moduli spaces can either occur at arbitrary, i.e. also at weak, or at strong heterotic coupling. Considering the first possibility, it can happen \([16]\) that at some special locus in the hypermultiplet moduli space \( M_H \) one or several of the gauge instantons shrink to zero sizes such that the gauge bundle becomes singular. This singularity occurs at arbitrarily weak coupling due to the absence of communication between the hyper and the tensor moduli fields, but nevertheless it is a non-perturbative effect which has no conformal field theory description. The corresponding singularity in the effective field theory is caused by BPS states which become ‘massless’ at the special locus. For the case of the \( SO(32) \) heterotic string the extra BPS states are given by non-perturbative gauge fields which become massless when an instanton shrinks to zero size \([16]\). In this way the rank of the gauge group can be enhanced non-perturbatively beyond the bound set by conformal field theory. On the other hand, for the case of the \( E_8 \times E_8 \) heterotic string it was argued that in general no vector fields become massless at the loci of zero size instantons, but a non-critical string becomes tensionless \([12]\), where the tension of this string is controlled by the vevs of the hypermultiplets. The world sheet description of this non-critical string is characterized by \((0,4)\) world sheet supersymmetry which supports \((0,1)\) supersymmetry in the six-dimensional target space. On the world sheet of the tensionless string there lives a massless antisymmetric tensor field \( B^+_{\mu\nu} \); hence at the special locus of \( M_H \) there will be new massless tensor multiplets in six dimensions. The gravitational anomaly constraint eq.(5) implies that in order to get one extra massless tensor, i.e. to shrink one \( E_8 \) instanton to zero size, 29 hypermultiplets have to be tuned to special values. The emergence of the additional tensor multiplets can be also seen from the Green-Schwarz anomaly condition eq.(7). Namely, shrinking one \( E_8 \) instanton to zero size, i.e. \( n_i \to n_i - 1 \), equation (7) demands that \( n_5 \to n_5 + 1 \), i.e. we get one extra massless tensor. So this transition leads to a new non-perturbative phase with extra tensor multiplets, and the new non-perturbative Coulomb branch is parametrized by vevs of the additional scalars in the tensor multiplets.

Now it also becomes rather clear how the non-perturbative transitions between different perturbative vacua with different instanton numbers \( (12+k, 12-k) \) can be realized. First
one goes to the special locus, where, say, one $E_8^1$ instanton becomes small; so $n_1 \to 11 + k$. At the second step one blows up one instanton in $E_8^2$, i.e. $n_2 \to 13 - k$. The net-effect of these two transitions with an intermediate non-perturbative phase is clearly given by the change $k \to k - 1$. The (intermediate) Coulomb branch with non-perturbative tensor fields has no geometrical description in terms of the uncompactified ten-dimensional heterotic string theory. Instead the transition between different perturbative vacua has a very nice explanation in $M$-theory, compactified on $K3 \times S^1/Z_2$. The two $E_8$ gauge factors are not sitting in the bulk of 11-dimensional space-time, but are on two different “end of the world” 9-branes. The location of the $n_5$ 5-branes, which fill 6-dimensional space-time, on $K3 \times S^1/Z_2$ are labeled by five real parameters, namely one hypermultiplet (the location on $K3$) and a tensor multiplet (the coordinate on $S^1/Z_2$). The singularity from the zero size $E_8$ instanton can be interpreted as resulting from a 5-brane stuck to the 9-brane. Then in the new Coulomb phase, the 5-brane leaves the boundary, so the instanton is converted into a 5-brane which is emitted into the bulk. The 5-brane can now move to the other end 9-brane and can be converted into an instanton again. So in this way all values of $(n_1, n_2)$ are connected in $M$-theory.

In four dimensions this effect describes a transition to a non-perturbative Coulomb branch with additional massless non-perturbative vector multiplets. The effective interactions of the non-perturbative vector fields are given by special Chern-Simons couplings, which can be also described by Poincare dualizing the vector multiplets to $N = 2$ four-dimensional vector-tensor multiplets. Models with several non-perturbative vector multiplets and their couplings have been discussed in [47, 48]. In the next section we will discuss how these transition, which are non-perturbative on the heterotic side, are realized on the dual type II side by a change in the topology of the underlying Calabi-Yau space.

Now we like to adress the question whether it is possible to obtain heterotic string compactifications with no perturbative tensor multiplets in six-dimensions at all, i.e. with no $S$-field in four dimensions? This question is related to the second type of singularity, which occurs only at strong coupling, i.e. for some special value of the dilaton field. So let us investigate the nature of the strong coupling singularity on more detail. Namely it is very closely related to the gauge kinetic term in six dimensions, given as

$$L_{\text{kin}} = - \sum_a (v_a e^{-\phi_6/2} + \tilde{v}_a e^{\phi_6/2}) \text{tr} F_a^2, \quad (15)$$

where the second term proportional to $\tilde{v}_a$ is a one-loop contribution. Since $\tilde{v}_a$ can be negative (the tree level term $v_a$ is always positive), it follows that at the critical string coupling

$$e^{\phi_6} = \frac{v_a}{\tilde{v}_a} \quad (16)$$

the gauge coupling of one gauge group factor diverges. In this case of infinite six-dimensional gauge coupling one deals with non-trivial gauge dynamics in the infra-red; as it was argued in [42] the singularity is again related to a non-critical string becoming tensionless. In fact in six dimensions, the $(0,1)$ supersymmetry algebra allows for a vector
central charge carried by BPS strings which couple electrically or magnetically to the $B$ field. For a single $B$-field, the tension obeys the BPS bound

$$T \geq |n_e e^{\phi/2} + n_m e^{-\phi/2}|,$$

which approaches zero for the critical coupling $e^{\phi_c} = \frac{n_m}{n_e}$.

Let us see what is happening at the critical couplings $\phi_c$, starting from the perturbative vacua with $E_8 \times E_8$ instanton numbers $(12 + k, 12 - k)$. In these models the ratios $\tilde{v}_a/v_a$ are determined as

$$\frac{\tilde{v}_1}{v_1} = -\frac{\tilde{v}_2}{v_2} = \frac{k}{2}.$$  \hfill (18)

Assuming $k \geq 0$, the gauge coupling of $G_1$ is always finite, whereas the other gauge coupling diverges at

$$e^{-\phi_c} = \frac{k}{2}.$$  \hfill (19)

Compactifying further to four dimensions on $T^2$ with Kähler modulus $T$, the singularity occurs at the line

$$S = \frac{k}{2} T.$$  \hfill (20)

Let us briefly analyze [42] for which values of $k$ there could by a strong coupling transition to a Higgs branch with no tensor multiplet at all. In such a branch there is only the anti-self-dual $B_{\mu\nu}$ field in the supergravity multiplet, and anomaly cancelation requires that the anomaly 8-form $I_8$ in eq.(4) not only factorizes but must be a perfect square. Using the expressions eq.(18) for $\tilde{v}_a/v_a$ this is only possible for $k = 1, 4$. For other values of $k$ there is no Higgs branch without dilaton. The $k = 4$ model, i.e. the model with instanton distribution $(16, 8)$, has unbroken (minimal) gauge group $SO(8)$ in six dimensions; this model is known [49] to be $T$-dual to the $SO(32)$ heterotic string with instanton number $n = 24$.

Finally we now discuss the presence of non-perturbative $S$-duality symmetries in $N = 2$ heterotic models. First consider the $k = 0$ model with symmetric $(12, 12)$ instanton embedding. If we un-Higgs some part of the gauge group, we get $\tilde{v}_1 = \tilde{v}_2 = 0$; it is obvious that there is no strong coupling singularity for this model. So one can continuously extrapolate from weak coupling to strong coupling without hitting any phase transition point. This suggests that there might be a strong-weak coupling $S$-duality for this model. Indeed going to the $M$-theory description of the heterotic string one can strongly argue that this model has a heterotic strong-weak coupling $S$-duality symmetry \( \phi^6 \leftrightarrow -\phi^6 \).  \hfill (21)

The dual heterotic string comes from wrapping the heterotic 5-brane around $K3$ which is consistently only possible for symmetric instanton embedding. However, since $v_a \neq 0$, $\tilde{v}_a = 0$ there is no manifest self-duality, but the heterotic strong-weak coupling duality requires the existence of non-perturbative gauge bosons which have $v_a = 0$ and $\tilde{v}_a \neq 0$.  \hfill (21)
So the heterotic strong-weak coupling duality of this model exchanges perturbative with non-perturbative gauge fields. The enhancement loci in $\mathcal{M}_H$ of the perturbative and non-perturbative gauge bosons are symmetric but not identical. Therefore the strong-weak coupling duality acts on the hyper multiplet moduli in a non-trivial way, i.e. the $K3$ gets changed under the duality transformation. Remember that for the $(12,12)$ model one has to tune 29 hypermultiplets to open a perturbative $SU(2)$ gauge group. So it takes the same number of parameters to open an perturbative $SU(2)$ gauge group as is needed to shrink an $E_8$ or $SO(32)$ instanton. This observation again supports the strong-weak coupling $S$-duality of this model; the appearance of non-perturbative gauge bosons in case of small $E_8$ instantons is naturally expected since the $E_8 \times E_8$ heterotic string with symmetric instanton embedding is $T$-dual to the $SO(32)$ type I string where zero size instantons lead to a non-perturbative $SU(2)$ gauge symmetry.

Upon further compactification to four dimensions on $T^2$ the strong-weak coupling duality eq. (21) becomes a non-perturbative symmetry under the exchange of the $S$ and the $T$ field:

$$S \leftrightarrow T$$

(22)

The presence of this non-perturbative exchange symmetry is in contrast to the $N = 4$ supersymmetric vacua where, as discussed in section (2.1), the exchange of $S$ and $T$ is not a symmetry of the heterotic string but maps the $N = 4$ heterotic string onto the dual type II string. In the dual $N = 2$ type II models, the non-perturbative $S$-$T$ exchange symmetry will find a nice geometrical explanation by considering the corresponding Calabi-Yau spaces. The $S$-$T$ exchange symmetry implies that for $T \to \infty$ there is a modular symmetry $SL(2, Z)_S$. Moreover, since in the perturbative region $S \to \infty$ there was a perturbative gauge symmetry enhancement at $T = U$, there is a non-perturbative $SU(2)$ enhancement at the line $S = U$ for $T \to \infty$ [27].

Finally let us discuss the model $k = 2$, i.e. with instanton embedding $(14,10)$ which also allows for complete Higgsing of the gauge gauge group. We will see that this model is in fact very closely related to the previously studied case with symmetric $(12,12)$ instanton embedding. For $k = 2$, we get $\tilde{v}_2/v_2 = -1$; so the possible strong coupling singularity is at $\phi^6_c = 0$. However this singularity is non-generic since it requires in addition to tune also one hypermultiplet modulus field. Therefore there is no strong coupling transition possible at this point. The reason for the absence of a transition is that the tensionless string which appears at the point $\phi^6_c = 0$ carries $(4,4)$ world sheet supersymmetry and hence supports $(2,0)$ supersymmetry in the target space in contrast to the cases considered before (though the full space-time configuration has still $(0,1)$ supersymmetry). These non-critical $(4,4)$ strings become tensionless if one tunes five real parameters, i.e. one tensor multiplet plus one hypermultiplet. The singularity can be avoided keeping the hypermultiplet at generic values. The uniqueness of six-dimensional $(0,2)$ supergravity implies the absence of a phase transition; so no new Higgs branch can emerge at the singular locus. These observations suggest that like the $(12,12)$ model also the $(14,10)$ model possesses the strong-weak coupling $S$-duality symmetry eq. (21), as it was noted in...
This symmetry is possible since the singularity occurs at the self-dual point of the duality transformation. However for the (14,10) model the $S$-duality does not require the existence of non-perturbative gauge bosons since $v_1 = \tilde{v}_1$. Hence in this sense, the model is really self-dual. Compactifying further to four dimensions it immediately follows that the (14,10) model also possesses a $S$-$T$ exchange symmetry. In four dimensions at the singular locus $S = T$ there is no tensionless string but a massless non-perturbative $SU(2)$ gauge boson together with one massless adjoint $N = 2$ hypermultiplet. (This becomes more clear in the dual type II picture.) Therefore this gauge theory is finite, because the $\beta$-function coefficient is proportional to $N_V - N_H$.

We have seen that the symmetric (12,12) model and the (14,10) models have many common features such as the possibility to Higgs the gauge group completely, the absence of a strong coupling singularity and the presence of the non-perturbative $S$-duality symmetry. This suggest that these two models might be equivalent and perhaps related by some sort of $T$-duality symmetry. In fact, using the dual type II or the $F$-theory picture of these two models, it will become evident that they are indeed completely equivalent after taking into account all non-perturbative effects. There are on the same moduli space, or more precisely, the moduli space of the (14,10) model is a subspace of the moduli space of the (12,12) model. In other words, upon restricting certain moduli of the (12,12) model to a particular domain, the (14,10) model is fully covered. It follows that tuning one hypermultiplet, also the (12,12) model acquires a singularity at the line $S = T$.

2.3 The type IIA string on $CY^3$

2.3.1 $K3$-fibrations

The compactification of the ten-dimensional type IIA/B superstring on a Calabi-Yau threefold leads to string vacua with $N = 2$ space-time supersymmetry in four dimensions. The massless spectrum is determined by the cohomology of the three-fold, namely by the two Hodge number $h^{(1,1)}$ and $h^{(2,1)}$. As it is known, the type IIA/B compactifications are related by mirror symmetry, which means that the type IIA superstring on the Calabi-Yau space $X^3$ is equivalently described by compactifying the type IIB on the mirror manifold $\tilde{X}^3$ which has exchanged Hodge numbers: $\tilde{h}^{(1,1)} = h^{(2,1)}$, $\tilde{h}^{(2,1)} = h^{(1,1)}$. We will mainly use the type IIA description in the following; however the type IIB picture is very useful, for example, for computing the low-energy effective action or for describing the conifold transition. The massless $N = 2$ spectrum of the type IIA superstring compactified on $X^3$ contains first the $N = 2$ supergravity multiplet including the $U(1)$ graviphoton field, which originates from the ten dimensional R-R gauge field $A_M$. Second, there are $h^{(1,1)}$ $U(1)$ vector multiplets:

$$N_V = h^{(1,1)}. \quad (23)$$

Their $h^{(1,1)}$ complex scalar moduli in the NS-NS sector correspond to the deformations of the Kähler form $J$ of $X^3$ plus the internal $B_{MN}$ fields; the $h^{(1,1)}$ $U(1)$ R-R vectors originate
from the ten-dimensional 3-form gauge potential $A_{MNP}$ with two indices in the internal space. So the Abelian gauge symmetry including the graviphoton is given by $U(1)^{h(1,1)+1}$. Finally there are $h^{(2,1)} + 1$ massless $N = 2$ hypermultiplets:

$$N_H = h^{(2,1)} + 1.$$  \hspace{1cm} (24)

$h^{(2,1)}$ of them correspond to the complex structure deformations of $X^3$, where the two additional R-R scalar degrees of freedom, needed to fill an $N = 2$ hypermultiplet, come from $A_{MNP}$ with all indices in the internal direction. The additional hypermultiplet contains together with the NS-NS axion field $a$ the four-dimensional dilaton $\phi_{IIA}$ plus two more R-R scalar fields.

It is clear that for a dual heterotic/type IIA $N = 2$ string pair the number of massless states have to match. The fact that the IIA dilaton belongs to an hypermultiplet is in apparent contrast to the heterotic dual where the dilaton belongs to a vector multiplet. Therefore the heterotic/type II duality relations will identify a particular internal type II Calabi-Yau modulus, $t_s$, with the heterotic dilaton field $S_H$, and in turn one of the heterotic hypermultiplet moduli with the type II dilaton. Moreover the effective actions, i.e. the moduli spaces have to agree, i.e. $\mathcal{M}_{V}^{IIA} = \mathcal{M}_{V}^{IIA}$, $\mathcal{M}_{H}^{IIA} = \mathcal{M}_{H}^{IIA}$. This will be the main subject of section 3. However let us already mention that the difference in the structure of the heterotic/type IIA multiplets with respect to the dilaton field, whose vev sets the string coupling constant, has very important consequences for the effective couplings. Namely due to the $N = 2$ non-renormalization theorems which forbid mixings in the kinetic energies of the vector and hypermultiplets, it follows that $\mathcal{M}_{V}^{IIA}$ does not receive any quantum corrections at all, whereas on the heterotic side $\mathcal{M}_{V}^{H}$ receives perturbative as well as non-perturbative corrections. This very powerful observation will enable us to determine non-perturbative corrections to the classical heterotic vector moduli space by computing the purely classical space $\mathcal{M}_{V}^{IIA}$ of the dual type IIA model. Obviously, the converse statement is true for $\mathcal{M}_{H}$. However computing the purely classical heterotic space $\mathcal{M}_{H}$ is very complicated, and not many results exist in this direction.

Now let us discuss what is the general structure of a Calabi-Yau space $X^3$ such that there exist a dual perturbative heterotic vacuum. We have to require that in a particular corner of the Calabi-Yau Kähler moduli space, which corresponds to the limit $S_H \to \infty$, we must see all known classical heterotic effects, like the classical moduli space eq.(14), the target space duality symmetries, the classical enhancement of the Abelian gauge group at special points in the moduli space and, finally, the possibility to perform Higgs transitions through points of enhanced gauge symmetries. Let us start with the information about $X^3$ we can deduce from the form of the classical special Kähler moduli space eq.(14). Consider the cohomology of $X^3$ and introduce a basis of $H_4(X^3, \mathbb{Z})$, the 4-cycles $D_A$, $A = 1, \ldots, h^{(1,1)}$. Dual to these 4-cycles we have a basis of 2-forms, $E_A$, generating $H^2(X^3, \mathbb{Z})$. Then we expand the Kähler form $J$ and the internal $B$-field as

$$iB + J = \sum_{A=1}^{h^{(1,1)}} t_A E_A,$$  \hspace{1cm} (25)
where the complex numbers $t_A = iB_A + J_A$ denote the complex Kähler moduli. The geometrical moduli space $\mathcal{M}_V$ is bounded by the wall of the Kähler cone, $\sigma(K) = \{ \sum_A J_A E_A | J_A > 0 \}$. We will take $t_1$ to correspond to the heterotic dilaton field $S$, i.e. $t_1 = t_S$. Next consider the classical cubic ‘Yukawa’ couplings $C_{ABC}$ between the Kähler moduli $t_A$. They are given in terms of the classical intersection numbers among the 4-cycles:

$$C_{ABC} = \sharp(D_A \cap D_B \cap D_C). \quad (26)$$

As we will explain more in section 3 the intersection numbers determine the metric of the special Kähler moduli space $\mathcal{M}_V$, in the limit where the Kähler form of $X^3$ is large, i.e. in the limit $\alpha'/R^2 \to 0$. This is the limit where the world-sheet instantons do not give any contribution to the moduli space metric. The essence of this discussion is that we can read off from the classical moduli space eq.(14) those topological intersection numbers which give the dominant contribution in the limit $t_S \to \infty$. The result is

$$C_{111} = 0,$$

$$C_{11a} = 0, \quad a = 2, \ldots, h^{(1,1)}$$

$$C_{1ab} = \eta_{ab}, \quad a, b = 2, \ldots, h^{(1,1)} \quad (27)$$

where $\eta_{ab}$ is a matrix of non-zero determinant and signature $(+, - , \ldots, -)$. Without going into the details of the proof [35] this specific form of the intersection number implies that the Calabi-Yau space $X^3$ must be $K3$ fibration, $X^3 \to P^1$, where the $K3$ is varying over a two-sphere $P^1$. The Kähler class, i.e. the size of the base $P^1$, is just given by the modulus $t_S$. This means that the weak coupling limit on the heterotic side corresponds on the type IIA side to the limit where the $K3$ is fibred over a large base space $P^1$. This limit looks like a decompactification to six dimensions, i.e. type IIA on $K3$, but with the difference that only part of the spectrum survives after the fibration. Now it also becomes clear that the modular target space duality symmetries arise in the type IIA compactifications in the limit of large $P^1$, since the $K3$ possesses modular properties as discussed in section (2.1).

Knowing that $X^3$ is a $K3$ fibration, we now like to analyze more closely the second cohomology $H^2(X^3)$ or equivalently $H_4(X^3)$; $h^{(1,1)}$ gets contributions from the cohomology of the base $P^1$ as well as from the cohomology of the fibre $K3$:

$$h^{(1,1)} = 1 + \rho + \sharp(\text{degenerate fibres}). \quad (28)$$

Let us discuss the three terms which appear in this formula.

(i) First the Kähler class $t_S$ will contribute to $h^{(1,1)}$.

(ii) Next consider a 2-cycle of the fibre $K3$ and “sweep out” a 4-cycle in $X^3$ by transporting it around the base $P^1$. This can be done in a consistent way if the $K3$ 2-cycle is monodromy invariant. The number of monodromy invariant 2-cycles in $K3$ is given

\[ X^3 \text{ can also be a } T^4 \text{ fibration over } P^1. \]
by the so-called Picard number $\rho$, where $\rho$ depends on how the $K3$ is fibred over the $P^1$. For Calabi-Yau spaces whose heterotic dual comes from a $K3 \times T^2$ compactification, it follows that $\rho \geq 2$, where the three universal Kähler moduli always correspond to the heterotic fields $S$, $T$ and $U$. In this case $X^3$ is also an elliptic fibration over a complex two-dimensional space $B^2$ (see the F-theory section).

(iii) There can be a degenerate fibre which is a reducible divisor in $X^3$. It turns out that these classes cannot be understood in perturbative heterotic string vacua.

Now consider very briefly the complex structure deformations of $X^3$. First, $h^{(2,1)}$ receives contributions from varying the complex structure of the $K3$ fibre; this number is given by $20 - \rho$. So we see that with increasing number of monodromy invariant two cycles the $K3$ becomes more and more rigid, i.e. it provides less complex structure deformations. Second the deformations of the fibration contribute to $h^{(2,1)}$. So in total one gets

$$h^{(2,1)} = 20 - \rho + \sharp(\text{fibration deformations}). \quad (29)$$

In the limit $t_S \to \infty$ one recovers the perturbative gauge symmetry enhancement of the heterotic string at special points in the moduli space. Since we have to take the large $P^1$ limit, we are dealing almost with the $N = 4$ situation of type IIA compactification on $K3$. The charged gauge bosons are again given by non-perturbative D2-branes which can wrap around the 2-cycles of $K3$. They become massless if the $K3$-fibre degenerates by shrinking the sizes the 2-cycles. However the difference compared to the $N = 4$ situation comes from possible global obstructions due to the $K3$ fibration. Namely the wrapping of D2-branes only around monodromy invariant 2-cycles can lead to physical BPS states. This means that not the whole intersection lattice $\Lambda$ of $K3$ can lead to a non-Abelian gauge group enhancement, but only the monodromy invariant part of $\Lambda$. This is the so-called Picard lattice $\Lambda_\rho$, whose rank is given by the Picard number $\rho$. Only $\Lambda_\rho$ corresponds to the visible gauge group in case the $K3$-fibre degenerates. The Picard lattice has signature $(+, -, \ldots, -)$, and $\eta_{ab}$ is simply the natural inner product of $\Lambda_\rho$.

After having discussed the general structure of the Calabi-Yau spaces which are dual to perturbative $N = 2$ heterotic string vacua, let us now consider explicit examples of $K3$ fibrations and try to identify them with the heterotic models which we have discussed in the previous section. In the following, we will mainly concentrate on those Calabi-Yau spaces which can be represented as hypersurfaces in weighted projected spaces. Consider first the four-dimensional heterotic vacua with $k = 2, 4, 6, 8, 12$ and completely Higgsed gauge group (first column in table 2.) The numbers $(N_V, N_H)$ of these models are precisely matched by the Hodge numbers of the $K3$ fibrations given as hypersurfaces of degree $6k + 12$ in $WP_4^{(1,1,k,2k+4,3k+6)}(6k + 12)$ \[14\]. The corresponding $K3$ fibres can also be represented as hypersurfaces of degree $3k + 6$ in $WP_3^{(1,\frac{k}{2},k+2,\frac{3k+3}{2})}(3k + 6)$. The Picard numbers of these $K3$'s are given by $\rho = 2, 6, 8, 9, 10$ for $k = 2, 4, 6, 8, 12$ respectively. Un-Higgsing $SU(r)$ ($r = 2, 3, 4$) gauge group factors (see columns 2 - 4 in table 2), the dual Calabi-Yau space can be again given by hypersurfaces in weighted projected spaces.
In summary, the Calabi-Yau models $X^3$ which can be represented as hypersurfaces in weighted projective spaces and which are the type IIA duals to the heterotic compactifications with $k = 2, 4, 6, 8, 12$ and $r = 1, 2, 3, 4$ are listed in the following table. The corresponding Hodge numbers can be read off from the heterotic spectra $(N_V, N_H)$ using eqs. (23) and (24).

| $r$ | $X^3$ |
|-----|-------|
| 1   | $WP_4^{(1,1,k,2k+4,3k+6)}(6k + 12)$ |
| 2   | $WP_4^{(1,1,k,k+4,2k+6)}(4k + 12)$ |
| 3   | $WP_4^{(1,1,k,k+4,k+6)}(3k + 12)$ |
| 4   | $WP_5^{(1,1,k,k+4,k+6,k+8)}(2k + 8, 2k + 12)$ |

Table 3: Dual Calabi-Yau spaces as hypersurfaces in weighted projected spaces.

So far we are missing the type IIA duals of many of the heterotic spectra in table 2, namely all models with $k = 0, 1$ and the models with larger un-Higgsed gauge group than $SU(4)$. These models cannot be simply represented by an hypersurface in a weighted projective space; however they can be constructed in terms of reflexive polyhedra using the techniques of toric geometry [44]. In addition, all Calabi-Yau’s dual to the heterotic models with completely Higgsed gauge group can be written as an elliptic fibration in the Weierstraß form (see next subsection).

To be specific let us discuss the Calabi-Yau space $WP_4^{(1,1,2,8,12)}(24)$ in more detail [54, 12, 34]. The corresponding Hodge numbers are $h^{(1,1)} = 3$ (i.e. $\rho = 2$) and $h^{(2,1)} = 243$. The three Kähler moduli are $t_1$, $t_2$ and $t_3$. This IIA model is dual to the heterotic string with instanton embedding $(n_1, n_2) = (14, 10)$ and completely Higgsed gauge group. The three heterotic vector fields were denoted by $S$, $T$ and $U$. Therefore this model is called the $S - T - U$ model. The identification between the Calabi-Yau Kähler parameters and the heterotic moduli is given as (see the next chapter about the prepotential):

$$t_1 = S - T, \quad t_2 = U, \quad t_3 = T - U.$$  \hfill (30)

The positivity of the Kähler cone implies that the heterotic variables are ordered as $S > T > U$.

The defining polynomial of the Calabi-Yau space is

$$p = \frac{1}{24}(z_1^{24} + z_2^{24} + 2z_3^{12} + 8z_4^3 + 12z_5^2) - \psi_0 z_1 z_2 z_3 z_4 z_5 - \frac{1}{6} \psi_1 (z_1 z_2 z_3)^6 - \frac{1}{12} \psi_2 (z_1 z_2)^{12}. \hfill (31)$$

The exist many more $K3$-fibred Calabi-Yau spaces (see [34, 53]). Some of them could be shown to possess a dual heterotic vacuum ($B, C$ chains) which is however not a $K3 \times T^2$ compactification with $(12 + k, 12 - k)$ instanton embedding.
In this equation we have included three deformation parameters $\psi_0, \psi_1$ and $\psi_2$ which denote three of the complex structure deformations of this Calabi-Yau; furthermore there are 239 additional allowed deformations which could be added to $\psi$; however one complex structure deformation of $X^3$ cannot be represented by a deformation of this defining polynomial. In the type IIB compactification on the mirror space $\tilde{X}^3$ with $h^{(1,1)} = 243$ and $h^{(2,1)} = 3$ the defining polynomial $p$ contains all possible allowed deformations. The relation between the the Kähler parameters $t_A$ of the type IIA model on $X^3$ and the complex structure parameters $\psi_A$ of the type IIB model on $\tilde{X}^3$ can by found by using the mirror symmetry between these two three-folds [54].

At special points in the moduli space the Calabi-Yau degenerates and acquires a conifold singularity. The degeneration takes place on the locus where the Calabi-Yau discriminant $\Delta$ is vanishing:

$$\Delta = (y - 1) \times \frac{(1 - z)^2 - yz^2}{z^2} \times \frac{((1 - x)^2 - z)^2 - yz^2}{z^2} = \Delta_y \times \Delta_z \times \Delta_x; \quad (32)$$

here $x = -\psi_0^2/\psi_1, \ y = 1/\psi^2_2$ and $z = \psi_2/\psi_1^2$ are functions of the three vector fields $S, T, U$. For $y \to 0$, $\Delta$ degenerates into quadratic factors that will provide the loci where precisely the perturbative heterotic gauge symmetry enhancement takes place. Therefore it is natural to make the identication $y = e^{-2\pi S_{uv}}$, where $S_{uv}$ is a redefined dilaton field which is invariant under the target space duality transformations [53] (see also next chapter). Moreover it turns out that in the weak coupling limit $y = 0$ the mirror map can be explicitly solved, and $x$ and $z$ can be written in terms of the elliptic $j$-function as

$$x = \frac{1}{864} \left( \frac{j(T)j(U)}{j(T) + j(U) - 1728} + \frac{\sqrt{j(T)j(U)(j(T) - 1728)(j(U) - 1728)}}{j(T) + j(U) - 1728} \right),$$

$$z = 864^2 \frac{x^2}{j(T)j(U)}. \quad (33)$$

So in this limit one recovers the perturbative duality symmetry $SL(2, \mathbb{Z})_T \times SL(2, \mathbb{Z})_U \times \mathbb{Z}_2^{T+U}$ from the modular properties of the elliptic $j$-function, as we have expected since $X^3$ is a $K3$-fibration. Using the expressions eq. (33), we can easily recover the vanishing of the discriminant locus $\Delta$ at the lines of classical gauge symmetry enhancement. First at the line $T = U$, i.e. $j(T) = j(U), \ z = 1$ and $x = \frac{1}{864}j(T)$ and hence $\Delta_z = 0$ and $\sqrt{\Delta_z} = \frac{4j(T)(j(T) - 1728)}{1728^2}$. The points $x = 0$ ($T = e^{i\pi/6}$), $x = 2$ ($T = 1$), where $\Delta_x = 0$, correspond to the points of further enhanced gauge symmetries $SU(3)$ or $SU(2)^2$ respectively.

Let us now briefly discuss what is happening if we go to finite coupling, i.e. taking into account quantum corrections on the heterotic side. For finite $y$ the quadratic degenerations of $\Delta_x$ and $\Delta_z$ are lifted and the classical loci of enhanced gauge symmetries split into two separated singular lines. Moreover, the classical target space duality symmetries disappear for finite couplings. In particular the Weyl reflections, like the $T \leftrightarrow U$ exchange, are not any more quantum symmetries. These effects are precisely what is happening in
$N = 2$ supersymmetric field theories, as discovered by Seiberg and Witten \cite{7}. The singular lines at $\Delta_x = 0$ or $\Delta_z = 0$, where the Calabi-Yau has a conifold singularity for finite $y$, correspond to the loci of massless monopoles and dyons. So there is no gauge symmetry enhancement at $T = U$ in the quantum case, but at $T - U = \pm \Lambda$ one gets massless monopoles or dyons. In fact one can verify that the perturbative monodromies from the duality transformations split into monopole times dyon like monodromies \cite{56}, and that (in a somewhat simpler model with $h^{(1,1)} = 2$ and only classical $SU(2)$ gauge symmetry enhancement) that the monodromies around the conifold points precisely agree with the monodromies around the points of massless monopoles and massless dyons after performing suitable field theory limit $\alpha' \to 0$ \cite{57}. Moreover, considering only the local geometry around the singularities, the full Seiberg-Witten curves can be reconstructed. In a type IIB picture the conifold points are described by those loci in $\mathcal{M}_V$ where certain 3-cycles shrink to zero sizes. The massless monopoles and dyons of Seiberg and Witten correspond to BPS self-dual 3-branes which are wrapped around the singular 3-cycles. In the mirror type IIA models, the Seiberg-Witten effect is then translated into the wrapping of 2-branes around isolated 2-cycles which shrink to zero with respect to the quantum ($\alpha'$) corrected metric. In this way, string theory provides a very beautiful geometrical description of non-perturbative effects in field theory, an approach which is called ‘Geometrical Engineering of Field Theories’ \cite{58}. The $S$-$T$-$U$ model possesses a further singularity at $y = 1$ where $\Delta_y = 0$. This locus does not intersect with the Seiberg-Witten locus, and the corresponding singularity cannot be found in field theory. This Calabi-Yau singularity corresponds to the strong coupling singularity of the heterotic (14,10) model at the line $S = T$, which we have discussed before. The geometric singularity is a genus one curve of $A_1$ singularity \cite{59}, predicting that $SU(2)$ gauge bosons plus an adjoint hypermultiplet become massless at the singularity. To reproduce this singularity on the Calabi-Yau space, one has to go to a locus of codimension one in $\mathcal{M}_V$ ($S = T$), but in addition also to a locus of codimension one in $\mathcal{M}_H$, namely to the codimension one space of those complex structures which correspond to the polynomial deformations of $p$. This strong coupling effect has no field theory explanation and is therefore an entirely new stringy effect. Analyzing further the symmetries of this Calabi-Yau compactification it turns out \cite{54} that, instead of the classical $T$-dualities, the model is symmetric under the quantum symmetry $t_1 \to -t_1$, $t_3 \to t_1 + t_3$. This symmetry is just the non-perturbative exchange symmetry $S \leftrightarrow T$. It means that this Calabi-Yau has two equivalent $K3$-fibrations \cite{61}, where the base $P^1_S$ gets exchanged with the $P^1_T$ that is inside the $K3$, regarded as an elliptic fibration over $P^1_T$. After having discussed this specific Calabi-Yau model in some detail we now want to investigate how the perturbative heterotic Higgs transitions are realized in the dual type II Calabi-Yau models. As explained, such transitions go through enhanced non-Abelian gauge symmetries with charged matter fields. When the charged spectrum is such that the theory contains a Higgs branch with broken gauge symmetry, the transition describes a topology change to the moduli space of another Calabi-Yau manifold. These transitions are in general called extremal transitions which generalize the notion of the conifold
transitions. In the type IIA compactifications $h^{(1,1)}$ gets reduced whereas $h^{(2,1)}$ increases. (We called this the reverse Higgs transition in the heterotic section.) These transitions can occur at weak coupling using perturbative non-Abelian gauge groups or also at strong coupling when a non-perturbative non-Abelian gauge groups appears as we discussed in the last paragraph. In the type IIA picture such transitions occur at the boundary of the (quantum) Kähler cone where a 2-cycle shrinks to zero size. In the mirror type IIB picture, if a 3-cycle shrinks to a curve (rather than two a point which leads to a massless hypermultiplet) one obtains enhanced gauge symmetries. More precisely, if the local geometry is an ALE space with $A_N$ singularity over a curve of genus $g$ one obtains an enhanced gauge $SU(N+1)$ symmetry with $g$ hypermultiplets in the adjoint representation \[61, 59, 62\]. As an example \[63\], consider the type IIA compactification on the Calabi-Yau $WP_{4}^{(1^{1},1^{2},6^{10})}(20)$ with $h^{(1,1)} = 4$ and $h^{(2,1)} = 190$. Its heterotic dual is the $K3 \times T^2$ compactification with $(14, 10)$ instanton distribution and an additional $SU(2)$ from the first $E_8$ which is generically broken to $U(1)$. By the mechanism described before this Calabi-Yau has a Higgs transition at weak coupling to the $S - T - U$ Calabi-Yau $WP_{4}^{(1^{1},2^{8},12)}(24)$ with Hodge numbers $h^{(1,1)} = 3$ and $h^{(2,1)} = 243$.

### 2.3.2 Six-dimensions – $F$-theory on elliptic $CY^3$

Now we are interested in how the non-perturbative transitions which connect heterotic vacua with different instanton number $k$ are realized on the type II side. In addition we also want to discuss the dual description of the strong coupling transition to heterotic models with no dilaton multiplet. As discussed at the end of section (2.2) both effects are related to the appearance of tensionless strings and occur already in the heterotic string on $K3$ in six dimensions with $(0,1)$ supersymmetry. Therefore it would be very useful to get a six-dimensional formulation of the dual type II strings. One way to achieve this would be perform a limit in the Calabi-Yau moduli space where the Kähler modulus $t_T$ which corresponds to the heterotic $T$-field gets large. However this limit is not very easy to analyze. Instead it turns out to be much more instructive to consider $F$-theory compactifications of the type IIB string to six dimensions.

Let us very briefly recall the idea behind $F$-theory. In \[64\] Vafa conjectured that the II B superstring theory should be regarded as the toroidal compactification of twelve–dimensional $F$–theory. If one is slightly less ambitious and does not want to enter the enormous difficulties in formulating a consistent 12-dimensional theory, $F$-theory can be regarded as to provide new non-perturbative type IIB vacua on D–manifolds in which the complexified coupling varies over the internal space. These compactifications then have a beautiful geometric interpretation as compactifications of $F$-theory on elliptically fibred manifolds, where the fibre encodes the behaviour of the coupling, the base is the D–manifold, and the points where the fibre degenerates specifies the positions of the D 7-branes in it. Compactifications of $F$–theory on elliptic Calabi–Yau two-folds (the K3), three-folds and four-folds can be argued to be dual to certain heterotic string theories in...
8,6 and 4 dimensions and have provided new insights into the relation between geometric singularities and perturbative as well as non–perturbative gauge symmetry enhancement and into the structure of moduli spaces.

Let us consider in more detail $F$-theory on a Calabi-Yau three-fold $X^3$ which leads to $(0,1)$ supergravity in six dimensions. This is supposed to be dual to the heterotic string on $K3$. The Calabi-Yau space $X^3$ is assumed to be an elliptic fibration, $X^3 \to B^2$, over a two-dimensional base space $B^2$. The precise fibration data on the $F$-theory side will map to the gauge bundle data on the heterotic side, in particular to the $E_8 \times E_8$ instanton numbers. Upon further compactification to four dimensions on $T^2$, i.e. considering $F$-theory on $X^3 \times T^2$, we obtain $N = 2$ supersymmetric models which are dual to the heterotic string on $K3 \times T^2$. Using the adiabatic argument one is then lead to the conclusion that $F$-theory on $X^3 \times T^2$ is equivalent to the type IIA string on the very same Calabi-Yau space $X^3$.

Let us discuss [64, 65] how the Hodge numbers of the elliptic $X^3$ determine spectrum of the $F$-theory compactifications. In six dimensions the number of tensor multiplets $N_T$ is given by the number of Kähler deformations of the (complex) two-dimensional type IIB base $B^2$ except for the overall volume of $B^2$:

$$N_T = h^{(1,1)}(B^2) - 1.$$  \text{(34)}

These tensor fields become Abelian $N = 2$ vector fields upon further $T^2$ compactification to four dimensions. Since the four-dimensional $F$-theory is equivalent to the type IIA string on $X^3$ it follows that the number of four-dimensional Abelian vector fields in the Coulomb phase is given by $N_T + r(V) + 2 = h^{(1,1)}(X^3)$, where $r(V)$ is the rank of the six-dimensional gauge group, and the additional two vector fields arise from the toroidal compactification. This then leads to the following equation for $r(V)$:

$$r(V) = h^{(1,1)}(X^3) - h^{(1,1)}(B^2) - 1.$$  \text{(35)}

Finally, the number of hypermultiplets $H$, which are neutral under the Abelian gauge group, is given in four as well as in six dimensions by the number of complex deformations of $X^3$ plus the freedom in varying the the size of the base $B^2$:

$$N_H = h^{(2,1)}(X^3) + 1.$$  \text{(36)}

Now we are interested in those $F$-theory compactifications which are dual to the perturbative heterotic string vacua on $K3$ with $n_5 = 0$ and $(n_1, n_2) = (12 + k, 12 - k)$ $E_8 \times E_8$ instanton embedding. So we expect a one parameters family of elliptic Calabi-Yau spaces $X^3_k$, to provide the dual $F$-theory models. At the same time the $X^3_k$ must be also $K3$ fibrations over $P^1_b$, $X^3_k \to P^1_b$, where the $K3$ fibre itself should an elliptic fibration over $P^1_f$, $K3 \to P^1_f$. Putting these observations together it is suggested that at least locally the (complex) two-dimensional base space $B^2$ is a product of $P^1_b \times P^1_f$. This can be formulated [65] in a more precise way with the result that $X^3_k$ must be an elliptic fibration over the
base \( B^2 = F_k \), which is the \( k \)-th Hirzebruch surface. These surfaces are all \( P^1_f \) fibrations over \( P^1_b \), and they are distinguished by how the \( P^1 \)'s are twisted. For example, \( F_0 \) is just the direct product \( P^1_b \times P^1_f \). For all \( F_k \), \( h^{(1,1)}(F_k) \) is universally given by \( h^{(1,1)}(F_k) = 2 \). Therefore one immediately gets that \( N_T = 1 \), which corresponds to the universal heterotic dilaton tensor multiplet in six dimensions. The Hodge numbers of \( X^3_k \) can be read off from the first column of table 2.

In case of \( F \)-theory on \( K3 \), the 8-dimensional heterotic dilaton is just related to the Kähler class (size) \( K_f \) of \( P^1_f \) \[ e^{\phi_8} H = K_f \]. Therefore in six dimensions one gets

\[
e^{\phi_6} = \frac{e^{\phi_8}}{K_b} = \frac{K_f}{K_b},
\]

(37)

where \( K_b \) denotes the Kähler class of \( P^1_b \). The possible values for the heterotic coupling constant are restricted by the boundaries of the Kähler cone of the Hirzebruch surface \( F_k \). They lead to the inequalities \( K_f \geq 0 \) and

\[
e^{-\phi_6} = \frac{K_b}{K_f} \geq \frac{k}{2}.
\]

(38)

This precisely coincides with those values for the string couplings (see eq.(19)) where one gets the strong coupling singularities, and tensionless strings appear. In the \( F \)-theory, i.e IIB context, the tensionless strings arise by wrapping the D 3-brane around a 2-cycle which collapses at the boundary of the Kähler cone. We will come back to this issue when discussion the strong transitions in the \( F \)-theory picture. For \( k = 0 \) the compactification is completely invariant under the exchange of \( K_b \) and \( K_f \), since we are dealing with a direct product \( F_0 = P^1_b \times P^1_f \), i.e there is a double \( K3 \) fibration structure. This symmetry obviously implies the strong-weak coupling S-duality \( \phi_6 \leftrightarrow -\phi_6 \).

So far have only determined the rank of the six-dimensional gauge group by eq.(35). In general the gauge group will contain also a non-Abelian part, either generically without tuning moduli (for \( k > 2 \)) or at specific loci (for \( k = 0, 1, 2 \)). The corresponding gauge symmetries can be determined by analyzing the singularities of the \( K3 \) fibre of \( X^3_k \). For that purpose it is very useful to describe the elliptically fibred Calabi-Yau spaces \( X^3_k \) in the Weierstrass form \[\text{[65]}\):

\[
X^3_k: \quad y^2 = x^3 + \sum_{n=-4}^{4} f_{8-nk}(z_1)z_2^{4-n}x + \sum_{n=-6}^{6} g_{12-nk}(z_1)z_2^{6-n}.
\]

(39)

Here \( f_{8-nk}(z_1), g_{12-nk}(z_1) \) are polynomials of degree \( 8 - nk, 12 - nk \) respectively, where the polynomials with negative degrees are identically set to zero. From this equation we see that the Calabi-Yau threefolds \( X^3_k \) are indeed \( K3 \) fibrations over \( P^1_b \) with coordinate \( z_1 \); the \( K3 \) fibres themselves are elliptic fibrations over the \( P^1_f \) with coordinate \( z_2 \). The Hodge numbers \( h^{(2,1)}(X^3_k) \), which count the number of complex structure deformations of \( X^3_k \), are given by the the number of parameters of the curve (39) minus the number of
possible reparametrizations, which are given by 7 for \( k = 0 \) and by \( k + 6 \) for \( k > 2 \). The non-Abelian gauge symmetries are determined by the singularities of the curve (39) and were analyzed in detail in [66]. For example, for \( k = 0, 1, 2 \) it is easy to see that the elliptic curve (39) is generically non-singular. Only tuning some parameters of the polynomials \( f \) and \( g \) to special values, the curve will become singular. These \( F \)-theory singularities correspond to the perturbative gauge symmetry enhancement in the dual heterotic models. On the other hand, for the cases \( k > 2 \) the curve (39) always contains generic singularities, since on the heterotic side the gauge group cannot be completely Higgsed. Note that the appearance of massless matter, i.e. massless hypermultiplets also can be analyzed from extra singularities.

Let us now discuss phase transitions in the Calabi-Yau topology which correspond to non-perturbative transitions on the dual heterotic side. Recall that these transitions occur at singular loci where the number of massless tensor multiplets gets changed. Specifically, if we shrink an \( E_8 \) instanton to zero size we trade 29 hypermultiplets to get one additional tensor field. On the other hand, during the strong transition (\( k = 1, 4 \)) we were loosing the perturbative tensor field, ending up in a vacuum with no tensor field at all. Since the number of tensor fields in \( F \)-theory is entirely related to the number of Kähler parameters of the base \( B^2 \) (see eq. (34)) we learn that these transitions have to due with a change in the topology of the base \( B^2 \). First, to increase the number of Kähler parameters of \( B^2 \) one blows up one or several points in \( B^2 \). For example, blowing up one point in \( F_k \) one obtains one additional tensor field whose scalar vev controls the Kähler class of the blown up.

On the other hand, to loose tensor multiplets one considers extremal transitions, where the Kähler parameters we wish to tune in order to approach the transition point are the Kähler parameters of the base \( B^2 \). In general, extremal transition between Calabi-Yau spaces take place going to the boundary of the Kähler cone. Then the Calabi-Yau space degenerates in one of the following three ways [67]: (i) A two cycle collapses to a point. (ii) A complex divisor (4-cycle) collapses either to a curve or to a point. Case (i) corresponds to a topology change via a flop transition between two different but birationally equivalent Calabi-Yau spaces with same \( h^{(1,1)} \), where by moving through the wall a new geometrical Kähler cone is reached; the union of all geometrical Kähler cones is called the extended Kähler cone. During the flop transition one Kähler modulus, say \( t_2 \) changes its sign and the size of the new 2-cycle is given by \( -t_2 \). This has the effect that the intersection numbers change by the new term \( C_{222} = -\frac{1}{6} \) [68]. (Several concrete examples of flop transitions in particular Calabi-Yau spaces where investigated in [67].) Case (ii) on the other hands corresponds to the strong transition with reduced \( h^{(1,1)} \). In case of elliptic Calabi-Yau manifolds over \( F_k \) the strong transition point occurs where a rational curve \((P^1)\) shrinks to zero size (blow down), and that curve has self-intersection number \(-k\).

\footnote{However in four-dimensional type IIA compactifications on a Calabi-Yau space, the flop transition is in fact not a sharp transition since one can turn on the axionic components of the complex moduli fields. Moreover in four-dimensions there are in general non-geometrical phases outside the extended Kähler cone due to the effect of world sheet instantons.}
We now can understand how the transition among two elliptic Calabi-Yau spaces $X^3_k$ and $X^3_{k+1} \pm 1$ takes place. Namely we can go from the base $F_k$ to the new base $F_{k+1}$ by blowing up one point in $F_k$ and then blowing down one curve of self-intersection number -1; this corresponds to the shrinking of one $E_8$ instanton and creating a new instanton in the other $E_8$.

The strong coupling transition to a phase with no dilaton corresponds to the blowing down of a curve in $F_k$. Specifically for $k = 1$ we blow down a curve of self-intersection number -1, where it turns out that before the blowing down one has to do a flop transition. This describes the transition $F_1 \to P^2$. Since $h^{(1,1)}(P^2) = 1$, eq.(34) implies $N_T = 0$, in agreement with the strong coupling transition. Similarly, for $k = 4$ the topology change describes the transition $F_4 \to P^2$.

3 The Effective $N = 2$ Action

So far we have mainly compared the massless spectra and some of the massive BPS states in dual heterotic/type II string pairs. Now we will discuss the four-dimensional effective $N = 2$ supergravity action of the heterotic string on $K3 \times T^2$ and of the type IIA superstring on a Calabi-Yau space $X^3$. In general a lot of important informations about the field dependence of the effective couplings is gained by studying the massive BPS spectra of the theory. In particular, it turns out that many holomorphic Wilsonian couplings are BPS dominated, i.e. that they can be computed by integrating over the massive BPS states [69, 70, 71]. At the singular lines in the moduli space where some of the BPS states become massless (for example massless gauge bosons), the effective Wilsonian description breaks down. This breakdown is signalled by singularities in the field dependend effective couplings at the loci of massless BPS states. For example, the perturbative Abelian gauge couplings exhibit logarithmic singularities at the loci of enhanced non-Abelian gauge symmetries, where the coefficients of the logarithmic terms are just given by the $\beta$-function coefficients of the enhanced gauge group. This is nothing else that the known threshold effect in the field theory running of the gauge coupling constants.

Let us briefly recall the structure of the effective couplings with respect to their dilaton dependence. The form of the field dependent couplings is strongly constrained by the holomorphy of the Wilsonian part of the effective action. First, since the dilaton multiplet $S_H$ is one of the heterotic vector fields, the holomorphic vector couplings on the heterotic side do depend on $S_H$; they receive classical (of order $S_H^0$), one loop (of order $S_H^1$) plus also non-perturbative (or order $e^{-S_H}$) contributions.

On the other hand, the dilaton belongs to a hypermultiplet in the four-dimensional type II vacua. This implies that the type II vector couplings (gauge couplings plus moduli space metric) do not depend on the type II dilaton and are purely classical in string perturbation theory. Higher derivative type II vector couplings will get perturbative genus $g$ corrections,
but no non-perturbative contributions at all. In type IIA vacua, the vector couplings will receive non-perturbative world-sheet corrections (of order $e^{-R^2/\alpha'}$), since the Kähler class of the Calabi-Yau space $X^3$ belongs to the vector multiplets. However in the mirror type IIB models on $\tilde{X}^3$, the Kähler class is an hypermultiplet, and hence all vector couplings follow entirely from classical type IIB geometry.

Switching to the effective couplings among the $N = 2$ hypermultiplets, the heterotic couplings are classical whereas the type II effective action will receive perturbative as well as non-perturbative corrections. However we will not discuss the hypermultiplet effective action in the following.

The purpose of the comparison of the $N = 2$ heterotic/type II effective actions is two-fold. First one can obtain very powerful and non-trivial checks of the $N = 2$ string-string duality for candidate dual string pairs. Considering the couplings of the $N = 2$ vector multiplets, these checks are possible in the region of small heterotic coupling, $S_H \to \infty$, where one can compute the effective heterotic string action from string perturbation theory. Indeed, the string-string duality has been sucessfully tested by the comparison of lower-order gauge and gravitational couplings $[24, 20, 21]$ of the perturbative heterotic string with the corresponding couplings of the dual type-II string. That is, it was shown that the perturbative heterotic prepotential $\mathcal{F}$ and the function $\mathcal{F}_1$ (which specifies the non-minimal gravitational interactions involving the square of the Riemann tensor) agree with the corresponding type-II functions in the limit where one specific Kähler-class modulus of the underlying Calabi-Yau space becomes large. A set of interesting relations between certain topological Calabi–Yau data (intersection numbers, rational and elliptic instanton numbers) and various modular forms, which appear at the perturbative heterotic string, has emerged when performing these tests $[20, 21]$.

Second the string-string duality can be used to derive the form of the non-perturbative interactions on the heterotic side from geometrical considerations on the dual type II side. Via the $N = 2$ string-string duality, the topological Calabi-Yau data, in particular the world-sheet instanton numbers which determine the perturbative type II couplings, translate into the non-perturbative space-time instanton numbers on the heterotic side. Since the world sheet instanton effects in the type IIA compactifications can be computed entirely from classical geometry in the corresponding type IIB vacua by utilizing the the mirror map, one is calling this map between world-sheet and space-time instantons the second quantized mirror symmetry $[13]$.

3.1 $N = 2$ Special Geometry

The vector couplings of $N = 2$ supersymmetric Yang-Mills theory are encoded in a holomorphic function $F(X)$, where the $X$ denote the complex scalar fields of the vector supermultiplets. With local supersymmetry this function depends on one extra field, in order to incorporate the graviphoton. The theory can then be encoded in terms of a holomorphic
function $F(X)$ which is homogeneous of second degree and depends on complex fields $X^I$ with $I = 0, 1, \ldots, N_V$. An intrinsic definition of special Kähler manifold can be given\cite{72} in terms of a flat $(2n + 2)$-dimensional symplectic bundle over the $(2n)$-dimensional Kähler-Hodge manifold, with the covariantly holomorphic sections

$$V = \left( \frac{X^I}{F_I} \right), \quad I = 0, \ldots, N_V. \quad (40)$$

Usually the $F_I$ can be expressed in terms of a holomorphic prepotential $F(X)$, homogeneous of degree two, via $F_I = \partial F(X)/\partial X^I$. The $N_V$ physical scalar fields of this system parametrize an $N_V$-dimensional complex hypersurface, defined by the condition that the section satisfies a constraint $\langle \nabla, V \rangle \equiv \nabla^T \Omega V = -i$, with $\Omega$ the antisymmetric matrix $\Omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. The embedding of this hypersurface can be described in terms of $N_V$ complex coordinates $z^A$ ($A = 1, \ldots, N_V$) by letting the $X^I$ be proportional to some holomorphic sections $X^I(z)$ of the complex projective space. In terms of these sections the $X^I$ read

$$X^I = e^{\frac{i}{2}K(z, \bar{z})} X^I(z), \quad (41)$$

where $K(z, \bar{z})$ is the Kähler potential, to be introduced below. The resulting geometry for the space of physical scalar fields belonging to vector multiplets of an $N = 2$ supergravity theory is a special Kähler geometry, with a Kähler metric $g_{AB} = \partial_A \partial_B K(z, \bar{z})$, being the metric of the moduli space $\mathcal{M}_V$, following from a Kähler potential of the special form

$$K(z, \bar{z}) = -\log \left( i \bar{X}^I(z) F_I(X^I(z)) - i X^I(z) \bar{F}_I(X^I(\bar{z})) \right). \quad (42)$$

The holomorphic sections transform under projective transformations $X^I(z) \to e^{f(z)} X^I(z)$, which induce a Kähler transformation on the Kähler potential $K$ and a $U(1)$ transformation on the section $V$,

$$K(z, \bar{z}) \rightarrow K(z, \bar{z}) - f(z) - \bar{f}(\bar{z}), \quad V(z, \bar{z}) \rightarrow e^{\frac{i}{2}(f(z) - \bar{f}(\bar{z}))} V(z, \bar{z}). \quad (43)$$

A convenient choice of inhomogeneous coordinates $z^A$ are the special coordinates, defined by $X^0(z) = 1, X^A(z) = z^A, A = 1, \ldots, N_V$. In this parameterization the Kähler potential can be written as

$$K(z, \bar{z}) = -\log \left( 2(\mathcal{F} + \bar{\mathcal{F}}) - (z^A - \bar{z}^A)(\mathcal{F}_A - \bar{\mathcal{F}}_A) \right), \quad (44)$$

where $\mathcal{F}(z) = i(X^0)^{-2} F(X)$.

Besides $V$, the magnetic/electric field strengths $(F_{\mu I}^I, G_{\mu I}^I)$ also constitute a symplectic vector. Here $G_{\mu I}^I$ is generally defined by $G_{\mu I}^I(x) = -4i g^{-1/2} \delta S/\delta F^{+\mu I}$. Consequently, also the corresponding magnetic/electric charges $Q = (p^I, q_I)$ transform as a symplectic vector. The Lagrangian terms containing the kinetic energies of the gauge fields are

$$4\pi L^{\text{gauge}} = -\frac{i}{8} (\nabla_I J^I F_{\mu I}^{+\mu J} - \nabla_I J^I F^{-\mu J}_\mu), \quad (45)$$
where $F_{\mu\nu}^{\pm I}$ denote the selfdual and anti-selfdual field-strength components and the field dependend $N_{IJ}(z, \bar{z})$, which are essentially given by the second derivative of the prepotential, comprises the inverse gauge couplings $g_{IJ}^{-2} = \frac{i}{16\pi}(N_{IJ} - \bar{N}_{IJ})$ and the generalized $\theta$ angles $\theta_{IJ} = \frac{\pi}{2}(N_{IJ} + \bar{N}_{IJ})$.

### 3.2 The Heterotic Prepotential

Now, we will discuss the heterotic $N = 2$ models, obtained by compactifying the $E_8 \times E_8$ string on $K3 \times T_2$. The moduli $z^A (A = 1, \ldots, N_V)$ comprise the dilaton $S$, the two toroidal moduli $T$ and $U$ as well as Wilson lines $V^i (i = 1, \ldots, N_V - 3)$:

$$S = -iz^1, T = -iz^2, U = -iz^3, V^i = -iz^{i+3}. \quad (46)$$

We will, in the following, collectively denote the moduli $T, U$ and $V^i$ by $T^a$, so that $a = 2, \ldots, N_V$. For this class of models, the heterotic prepotential has the form

$$F_{\text{het}} = -S T^a \eta_{ab} T^b + h(T^a) + f^{\text{NP}}(e^{-2\pi S}, T^a), \quad (47)$$

where $T^a \eta_{ab} T^b = T^2 T^3 - \sum_{I=4}^{N_V}(T^I)^2$. The first term in (47) is the classical part of the heterotic prepotential, $h(T^a)$ denotes the one-loop contribution and $f^{\text{NP}}$ is the non-perturbative part, which is exponentially suppressed for small coupling.

The classical prepotential leads to the metric of the special Kähler manifold eq.(14) with corresponding tree-level Kähler potential

$$K = -\log[(S + \bar{S})] - \log[(T^a + \bar{T}^a)\eta_{ab}(T^b + \bar{T}^b)]. \quad (48)$$

Note that the classical ‘Yukawa’ couplings $F_{ABC} = \frac{\partial F}{\partial z^A \partial z^B \partial z^C}$ have precisely the same form as the Calabi-Yau intersection numbers $C_{ABC}$ in the limit $t_S \to \infty$ (see eq.(26)). Remember that this matching of ‘Yukawa’ couplings in the weak coupling limit was one of the reasons to consider $K3$ fibrations on the dual type II side.

Due to the required embedding of the $T$-duality group into the $N = 2$ symplectic transformations, it follows [53, 73] that the heterotic one-loop prepotential $h(T^a)$ must obey well-defined transformation rules under this group. The function $h(T^a)$ leads to the following modified Kähler potential [55], which represents the full perturbative contribution,

$$K = -\log[(S + \bar{S}) + V_{GS}(T^a, \bar{T}^a)] - \log[(T^a + \bar{T}^a)\eta_{ab}(T^b + \bar{T}^b)], \quad (49)$$

where

$$V_{GS} = \left(2(h + \bar{h}) - (T^a + \bar{T}^a)(\partial_{T^a} h + \partial_{T^a} \bar{h})\right)/(T^a + \bar{T}^a)\eta_{ab}(T^b + \bar{T}^b) \quad (50)$$

is the Green-Schwarz term describing the mixing of the dilaton with the moduli $T^a$. Due to this mixing, the $S$-field is not any more invariant under the target-space duality
Evaluating the instanton embedding coefficients of particular Jacobi modular forms \[21\] it can be explicitly computed in string perturbation theory.

K of the Abelian gauge couplings will exhibit logarithmic singularities due to the additional of the one-loop prepotential. At the loci of enhanced non-Abelian gauge symmetries some specific heterotic compactifications on \(K3 \times T^2\). Namely we will consider the models with instanton embedding \(k = 0, 1, 2\), and will first start with those models with \(N_V = 4\) and \(N_H = 215 - 12k\). The four vector multiplets are denoted by \(S\), \(T\), \(U\) and the Wilson line \(V\). Later we will discuss the Higgs transition via an enhanced \(SU(2)\) gauge group to the \(S - T - U\) models with \(N_V = 3\) and \(N_H = 244\). For the class of \(S-T-U-V\) models considered here, the one-loop prepotential is given by

\[
h = p_n(T, U, V) - \frac{1}{4\pi^2} \sum_{n,l,b \geq 2} c_k(4nl - b^2) Li_3(e^{inT + ilU + ibV}),
\]

where \(c = \frac{c_n(0) \zeta(3)}{8\pi^4}\) and \(e[x] = \exp 2\pi ix\). The coefficients \(c_k(4nl - b^2)\) are the expansion coefficients of particular Jacobi modular forms \[21\]

\[
\frac{1}{\Delta(\tau)} \left( \frac{12 + k}{24} E_6(\tau) E_{4,1}(\tau, z) + E_4(\tau) \frac{12 - k}{24} E_{6,1}(\tau, z) \right) = \sum_{n,b} c_k(4n - b^2) q^n r^b,
\]

\(\Delta = \eta^{24}(\tau), q = e^{2\pi i \tau}, r = e^{2\pi iz}\). This expression is very closely related to the elliptic genus of \(K3, \frac{1}{\eta_0} Tr R F(-1)^F q^{L_0 - c/24} \tilde{q}^{L_0 - c/24}\), with gauge bundle characterized by the integer \(k\); it can be explicitly computed in string perturbation theory.

\(p_k\) is a cubic polynomial of the form \[71, 21\]

\[
p_k(T, U, V) = -\frac{1}{3} U^3 - \left( \frac{4}{3} + k \right) V^3 + (1 + \frac{1}{2}k)UV^2 + \frac{1}{2}kTV^2.
\]

The (Wilsonian) Abelian gauge threshold functions are related to the second derivatives of the one-loop prepotential. At the loci of enhanced non-Abelian gauge symmetries some of the Abelian gauge couplings will exhibit logarithmic singularities due to the additional

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A Siegel modular form \(F(T, U, V)\), which is an automorphic function of the discrete group \(Sp(4, \mathbb{Z})\) of weight \(n\), has a Fourier expansion with respect to its variable \(iU\), \(F(T, U, V) = \sum_{m=0}^{\infty} \phi_{n,m}(T, V) s^m\), where \(s = e[iU], e[x] = \exp 2\pi ix\). Each of the \(\phi_{n,m}(T, V)\) is a Jacobi form of weight \(n\) and index \(m\). A Jacobi form \(\phi_{n,m}(T, V)\) of index \(m\) has in turn an expansion \(\phi(T, V) = \sum_{m=0}^{\infty} \sum_{l \in \mathbb{Z}} q^{c(n,l)} r^l\), where \(q = e[iT], r = e[iV]\). Consider, for instance, the Eisenstein series, which have the expansion \(E_n(T, U, V) = E_n(T) - \frac{2\pi}{B_n} E_{n,1}(T, V) s + \mathcal{O}(s^2)\). The explicit expansion coefficients of \(E_{4,1}\) and \(E_{6,1}\) are given in \[21\]. In addition, a review of several properties of Siegel and Jacobi modular forms is given in the appendix of \[23\].
massless states. First consider $\partial_T \partial_U h$. At the line $T = U$ one $U(1)$ is extended to $SU(2)$ without additional massless hypermultiplets. It can be easily checked that, as $T \to U$, 
\[
\partial_T \partial_U h = -\frac{1}{\pi} \log(T - U) ,
\] (55)
as it should. The Siegel modular form which vanishes on the $T = U$ locus and has modular weight 0 is given by $\overline{c_{30}}^{2}/c_{12}^{5}$. It can be shown that, as $V \to 0$, 
\[
\frac{C_{30}^{2}}{C_{12}^{5}} \to (j(T) - j(U))^{2} ,
\] (56)
up to a normalization constant. Hence one deduces that 
\[
\partial_T \partial_U h = -\frac{1}{2\pi} \log \frac{C_{30}^{2}}{C_{12}^{5}} + \text{regular}.
\] (57)

On the other hand, at the locus $V = 0$, a different $U(1)$ gets enhanced to $SU(2)$, and at the same time $N'_H = 12 + 32$ hyper multiplets, being doublets of $SU(2)$, become massless. The Siegel modular form which vanishes on the $V = 0$ locus and has modular weight 0 is given by $\overline{c_{5}}^{2}/c_{12}^{5/2}$, and can derive that 
\[
-\frac{1}{4} \partial_{V}^{2} h = \frac{3}{4\pi}(2 + k) \log \left( \frac{C_{5}}{C_{12}^{5/2}} \right)^{2} + \text{regular}
\] (58)

The coefficient of the log in this equation is just the $N = 2 \beta$-function coefficient of $SU(2)$ with $12k + 32$ massless doublets.

Let us briefly investigate the Higgs transition to the $S-T-U$ models with $N_V = 3$ and $N_H = 244$. This transition takes place at the locus $V = 0$ giving vev’s to the $N'_H$ new massless hypermultiplets (3 of them will eaten by the $SU(2)$ gauge bosons). It follows that the effective action of the $S-T-U$ models can be obtained from the $S-T-U-V$ prepotential by setting $V = 0$ in eq.(52). Then, the sum over $b$ in (52) yields independently from $k$ the coefficients of the 3 parameter model, 
\[
c_{STU}(nl) = \sum_{b} c_{k}(4nl - b^{2}) .
\] (59)

Via the properties of the Jacobi functions, we get that 
\[
\frac{1}{\Delta} E_{4}(\tau)E_{6}(\tau) = \sum_{n \geq -1} c_{STU}(n)q^n .
\] (60)

This result can be independently checked by computing $c_{STU}(n)$ directly in the $S-T-U$ model [20].
The one-loop gauge couplings of the $S$-$T$-$U$ models exhibit logarithmic singularities at the locus $T = U$ [74, 75, 73]:

$$\partial_T \partial_U h = -\frac{1}{\pi} \log(j(T) - j(U)) + \text{regular.} \quad (61)$$

Moreover the modular transformation properties of $h$ allow us to uniquely determine the Yukawa couplings [55, 73]; e.g.

$$\partial^3_T h = \frac{2}{\pi} \eta^2(U)(j(T) - j(U)). \quad (62)$$

By performing the $q$-expansion of this expression, this equation can be checked by taking the third derivative of the prepotential eq.(52), and also expanding it up to some order in $q$.

### 3.3 The Type-IIA Prepotential and the Heterotic/type IIA Map

As already mentioned, the prepotential in type-IIA Calabi–Yau compactifications, which depends on the Kähler-class moduli $t_A = -iz^A \ (A = 1, \ldots, N_V = h_{1,1})$, is of purely classical origin. The type-IIA prepotential has the following general structure [74, 54]:

$$F^{II} = -\frac{1}{6} C_{ABC} t_A t_B t_C - \frac{\chi \zeta(3)}{2(2\pi)^3} + \frac{1}{(2\pi)^3} \sum_{d_1, \ldots, d_h} n^r_{d_1, \ldots, d_h} Li_3(e[i\sum_A d_A t_A]), \quad (63)$$

where we work inside the Kähler cone $Re t_A \geq 0$. The cubic part of the type-IIA prepotential is given in terms of the classical intersection numbers $C_{ABC}$ and the Euler number $\chi$, whereas the coefficients $n^r_{d_1, \ldots, d_h}$ of the exponential terms denote the rational instanton numbers of genus 0 and multi degree $d_A$. They can be computed by the mirror map studying the corresponding type IIB compactification on the mirror Calabi-Yau space. The rational instantons are related to non-perturbative effects on the world-sheet, namely they count the embeddings of the genus 0 world-sheet into the Calabi-Yau space; their contributions disappear in large radius limit $\alpha'/R \rightarrow 0$. Hence, in the limit of large Kähler class moduli, $t_A \rightarrow \infty$, only the classical part, related to the intersection numbers, survives.

To be specific let us consider those three Calabi-Yau space with $h^{(1,1)} = 4$ and $h^{(2,1)} = 214 - 12k$, $k = 0, 1, 2$. Recall that the $k = 2$ Calabi-Yau can be constructed as hypersurface in $W^{1,1,2,6,10}_4$ (20). The classical intersection numbers are given in [14, 63], and the cubic part of the type II prepotentials for these three models can be written as follows

$$-F^{II}_{cubic} = t_1(t_2^2 + 2t_2t_3 + 4t_2t_4 + 2t_3t_4 + 3t_4^2) + \frac{4}{3} t_2^3 + 8t_2^2t_4 + \frac{k}{2} t_3^2t_4 + (1 + \frac{k}{2})t_2^2t_3 + 2(k + 2)t_2t_3t_4 + k t_3^2t_4 + (14 - k)t_2t_4^2 + (4 + k)t_3t_4^2 + (8 - k)t_4^3. \quad (64)$$
In order to match (64) with the cubic part of the heterotic prepotential given in (52), we have to perform the following identification of type II and heterotic moduli

\[
\begin{align*}
t_1 &= S - \frac{k}{2} T - (1 - \frac{k}{2}) U, \\
t_2 &= U - 2 V, \\
t_3 &= T - U, \\
t_4 &= V ,
\end{align*}
\] (65)

Since we are working inside the Kähler cone \( \text{Re} t_A \geq 0 \), the heterotic variables \( S, T, U \) and \( V \) have to obey the following inequalities:

\[
T \geq U \geq 2 V \geq 0, \quad S \geq \frac{k}{2} T - (1 - \frac{k}{2}) U.
\] (66)

Next, let us consider the contributions of the world sheet instantons to the type II prepotential of the 4 parameter models. The \( n_{d_1,d_2,d_3,d_4} \) denote the rational instanton numbers. The heterotic weak coupling limit \( S \to \infty \) corresponds to the large Kähler class limit \( t_1 \to \infty \). In this limit, only the instanton numbers with \( d_1 = 0 \) contribute in the above sum. Using the identification \( nT + lU + bV = d_1 t_1 + d_3 t_3 + d_4 t_4 \), it follows that \( n = d_3 \), \( l = d_2 - d_3 \), \( b = d_4 - 2d_2 \). Then, the instanton part of the prepotential turns into

\[
\mathcal{F}_{\text{inst}}^{II} = \frac{1}{(2\pi)^3} \sum_{n,l,b} n_{0,l+n,n,b+2l+2n}^r L i_3(\frac{2\pi n T + l U + b V}{n T + l U + b V}).
\] (67)

Matching this expression with eq.(52) shows that the rational instanton numbers have to satisfy the following constraint

\[
n_{0,l+n,n,b+2l+2n}^r = -2c_k(4nl - b^2) .
\] (68)

This relation is very non-trivial and connects the rational instanton numbers to the coefficients of the modular functions which determine the heterotic prepotential. For concreteness, above relation (68) can be successfully checked for the \( k = 2 \) model, which is based on the Calabi-Yau space \( WP_4^{1,1,2,6,10} \) (20), using the instanton numbers given in [63].

Let us discuss some properties of the heterotic/type IIA map in a slightly simpler situation, namely in the \( S-T-U \) models; the Higgs transition from \( h^{(1,1)} = 4 \) to the \( h^{(1,1)} = 3 \) models takes place at the boundary of the Kähler cone \( t_4 = 0 \); the cubic part of the prepotential of the \( S-T-U \) models is then simply obtained by setting \( t_4 = 0 \) in eqs.(64) and (65). In addition, the instanton numbers \( n_{d_1,d_2,d_3}^r \) of the \( S-T-U \) model are given by [63]

\[
n_{d_1,d_2,d_3}^r = \sum_b n_{d_1,d_2,d_3,b}^r ,
\] (69)

where the summation range over \( b \) is finite. In order that the heterotic and the type IIA prepotentials match in the weak coupling limit \( S,t_1 \to \infty \), the instanton numbers \( n_{0,t+n,n}^r \)
must be related to the coefficients \(c_{STU}(n)\) of the modular form \(E_4E_6/\Delta\) in the following way \([71, 20]\)
\[
n_{0,l+n,n}^r = -2c_{STU}(ln).
\] (70)

Again one can check by direct computation of some instanton numbers that this relation is satisfied for the \(k = 2\) model based on the Calabi-Yau space \(WP^{(1,1,2,8,12)}(24)\). In addition one can show \([12, 22]\) for this model that the Calabi-Yau ‘Yukawa couplings’ in the weak coupling limit agree with the heterotic Yukawa couplings eq. (62), as predicted from the K3 fibration of this Calabi-Yau.

By inspection of the instanton numbers the quantum symmetries of the \(S-T-U\) vacua become manifest. Specifically, for the \(k = 2\) model, two symmetries among the instanton numbers are discovered \([34]\), namely \(n_{0,d_2,d_3}^r = n_{0,d_2,d_3}^r\) and \(n_{d_1,d_2,d_3}^r = n_{d_3-d_1,d_2,d_3}^r\). So the theory is invariant under the following two shifts
\[
t_2 \to t_2 + t_3, t_3 \to -t_3, \quad \text{for} \quad t_1 = \infty
\]
\[
t_1 \to -t_1, t_3 \to t_1 + t_3.
\] (71)

Using the identification \(t_1 = S - T, t_2 = U\) and \(t_3 = T - U\), the first symmetry corresponds to the perturbative exchange symmetry \(T \leftrightarrow U\) for \(S \to \infty\), whereas the second is the non-perturbative exchange symmetry \(S \leftrightarrow T\). The line \(S = T\) is the border of the Kähler cone, \(\text{Ret}_1 \geq 0\), of the \(k = 2\) model and is the locus of the strong coupling singularity.

Furthermore comparing the intersection numbers (see eq. (24)) and the instanton numbers of the \(k = 0\) and \(k = 2\) \(S-T-U\) models it can be seen that the prepotentials of these two models become equivalent upon the substitution \(t_1 \to t_1 + t_3\). In fact, the \(k = 0\) model is completely invariant under the exchange of \(t_1\) and \(t_3\), which corresponds to an exchange of the two \(P^1\)’s which serve as the base of the two alternative K3 fibrations. In the \(k = 0\) model the identification among the heterotic and type IIA variables is given as \(t_1 = S - U, t_2 = U\) and \(t_3 = T - U\). This shows that the Kähler cone of the \(k = 0\) model contains both regions \(S > T\) and \(S < T\) in agreement with the absence of the strong coupling singularity along the line \(S = T\). It means that the Kähler cone of the \(k = 2\) model is a subcone of the Kähler cone of the \(k = 0\) model. Considering also the \(k = 1\) model, there would be also a linear transformation, \(t_1 \to t_1 + \frac{1}{2}t_3\), which maps the intersection numbers of the \(k = 1\) model and the \(k = 2\) model into each other. However, since this transformation contains half-integer coefficients, the instanton parts of the prepotentials do not agree. Therefore the \(k = 1\) vacumm is physically different from the \(k = 0, 2\) vacua after including the world-sheet instanton effects.

3.4 The gravitational coupling \(F(1)\)

The quantitative tests of the heterotic/type II string-string duality can be continued by considering higher order couplings in the effective \(N = 2\) supergravity action. The higher derivative couplings of vector multiplets \(X\) to the Weyl multiplet \(W\) can be expressed as
a power series
\[ F(X, W^2) = \sum_{g=0}^{\infty} F^{(g)}(X) [W^2]^g. \] (72)

\( F^{(0)} \) is just the prepotential discussed before. We will restrict the discussion to the function \( F^{(1)} \) which appears as the field dependent coupling of the gravitational term \( R^2 \), \( R \) being the Riemmann tensor, in the effective action. Higher order terms and a recursion between terms of different order in \( g \) were discussed in [23, 74, 26]. For simplicity, we will restrict the discussion to the \( S-T-U \)-models with \( k = 0, 1, 2 \). Explicit results on the \( S-T-U-V \) models can be found in [21].

The heterotic perturbative (tree level plus one loop) Wilsonian gravitational coupling for the \( S-T-U \) model is given by [74, 22]
\[ F^{(1)} = 24S_{\text{inv}} - \frac{b_{\text{grav}}}{\pi} \log \eta(T) \eta(U) + \frac{2}{\pi} \log(j(T) - j(U)), \] (73)
where \( S_{\text{inv}} = S - \sigma(T, U), \sigma(T, U) = -\frac{1}{2} \partial_{T} \partial_{U} R + \frac{1}{8} L(T, U), L(T, U) = -\frac{4}{\pi} \log(j(T) - j(U)). \) \( F^{(1)} \) is singular at the line \( T = U \) due to the additional massless states at \( T = U \). The term proportional \( b_{\text{grav}} \) is needed in order to cancel the gravitational anomaly due to the massless states. Using some product formula for \( j(T) - j(U) \), the function \( F^{(1)} \) can be written [20] as a power series expansion with coefficients \( \tilde{c}_1 \), where the coefficients \( \tilde{c}_1 \) are given by [74] \( \frac{E_2(\tau)E_4(\tau)E_6(\tau)}{\Delta(\tau)} = \sum \tilde{c}_1(n) q^n \).

In the dual type IIA string, the gravitational coupling \( F^{(1)} \) arises at the one-loop level (genus one), and is determined by the rational instanton numbers \( n^r \) as well as by the elliptic instanton numbers \( n^e \):
\[ F^{(1)} = -i \sum_{A=1}^{h^{(1,1)}_A} t_A c_{2A} - \frac{1}{\pi} \sum_{n} \left( 12n^e_{d_i, \ldots, d_h} \log(\tilde{\eta}(\prod_{A=1}^{h^{(1,1)}_A} q^d_A)) + n^r_{d_i, \ldots, d_h} \log(1 - \prod_{A=1}^{h^{(1,1)}_A} q^d_A) \right). \] (74)
Here \( \tilde{\eta}(q) = \prod_{m=1}^{\infty} (1 - q^m) \), the \( c_{2A} \) are the 2nd Chern numbers of the Calabi-Yau space and the \( n^e_{d_i, \ldots, d_A} \) are the elliptic instanton numbers which count elliptic genus one curves embedded into the Calabi-Yau space. In the weak coupling limit \( S \rightarrow \infty \), the heterotic and the type IIA \( F^{(1)} \) functions must agree. By this requirement one can derive [20] the following interesting relation among rational, elliptic instanton numbers and \( \tilde{c}_1 \) coefficients:
\[ 12 \sum_{i=1}^{s} n^e_{0, d_i, d_i^*} + n^r_{0, l + k, k} = -2\tilde{c}_1(kl), \] (75)
where \( s \) is the number of common divisors \( m_i \) \( (i = 1, \ldots, s) \) of \( d_2 = k + l \) and \( d_3 = k \) with \( d_{2,3} = d_{2,3}/m_i \) (where \( m_2 = 1 \)). Considering again the 3 parameter Calabi-Yau \( WP_4^{(1,1,2,8,12)} \) (24) one can explicitly compute some of the elliptic instanton numbers and check in this way that the relation (75) indeed holds.
4 $N = 2$ Black Hole Solutions and their Entropies

One of the celebrated successes within the recent non-perturbative understanding of string theory and M-theory is the matching of the thermodynamic Bekenstein-Hawking black-hole entropy with the microscopic entropy based on the counting of the relevant D brane configurations which carry the same charges as the black hole \[78\]. This comparison works nicely for type-II string, respectively, M-theory backgrounds which break half of the supersymmetries, i.e. exhibit $N = 4$ supersymmetry in four dimensions. For example, consider the type IIA superstring compactified on $K3 \times T^2$. Then, the intersection of three D 4-branes, whose spatial parts of their world volumes are wrapped $p^A$ times ($A = 1, 2, 3$) around the internal 4-cycles, together with $q_0$ D 0-branes leads to a four-dimensional black hole with electric charge $q_0$ and magnetic charges $p^A$. This black hole has non-vanishing event horizon $A$, and the corresponding Bekenstein-Hawking entropy is given by the following expression:

$$S_{BH} = \frac{A}{4} = 2\pi \sqrt{q_0 p^1 p^2 p^3}. \quad (76)$$

Here we will discuss black hole solutions and corrections to this formula which arise in string backgrounds with $N = 2$ supersymmetry in four dimensions. In particular, studying IIA compactifications on a (complex) 3-dimensional Calabi-Yau space, corrections arise due to the internal geometry and topology of the Calabi-Yau manifold \[29\]. In the large volume limit of the Calabi-Yau space, the Calabi-Yau intersection numbers enter the entropy formula, since the D 4-branes are now wrapped around the non-trivial Calabi-Yau 4-cycles. Furthermore, in case of generic radii also world-sheet instanton corrections, encoded by the rational instanton numbers determine the black hole solutions and their entropies. In addition, we will consider corrections to the $N = 2$ black hole entropies which arise from higher-derivative terms involving higher-order products of the Riemann tensor and the vector field strengths \[32, 33\].

From the point of view of the low energy effective lagrangian the wrapped D branes correspond to extremal charged black hole solutions of the effective $N = 2$ supergravity. Extremal, charged, $N = 2$ black holes, their entropies and also the corresponding brane configurations were discussed in several recent papers \[27, 28, 29, 30, 31, 32, 33\]. One of the key features of extremal $N = 2$ black-hole solutions is that the moduli depend in general on $r$, but show a fixed-point behaviour at the horizon. This fixed-point behaviour is implied by the fact that, at the horizon, full $N = 2$ supersymmetry is restored; at the horizon the metric is equal to the Bertotti-Robinson metric, corresponding to the $AdS_2 \times S^2$ geometry. The extremal black hole can be regarded as a soliton solution which interpolates between two fully $N = 2$ supersymmetric vacua, namely corresponding to $AdS_2 \times S^2$ at the horizon and flat Minkowski spacetime at spatial infinity.
4.1 Extremal black holes in $N = 2$ supergravity

Now we want to discuss the static $N = 2$ black hole solutions of the lowest order $N = 2$ supergravity action which we introduced in section (3.1). In particular we have seen that in the context of special geometry the two fundamental objects of the $N = 2$ supergravity action are given by two symplectic vectors, namely the period vector $V = (X^I, F^I)$ and the magnetic/electric field strength vector $(F^I_{\mu\nu}, G_{\mu\nu I})$ respectively by $V = (X^I, F_I)$ and the magnetic/electric charge vector $Q = (p^I, q^I)$.

In terms of these symplectic vectors the stationary solutions have been discussed in [30, 33]. Here one solved the generalized Maxwell equations in terms of $2N_V + 2$ harmonic functions, which therefore also transform as a symplectic vector $(m, n = 1, 2, 3)$,

$$F^I_{mn} = \frac{1}{2} \epsilon_{mnp} \partial_p \tilde{H}^I(r), \quad G_{mnI} = \frac{1}{2} \epsilon_{mnp} \partial_p H_I(r).$$

The harmonic functions can be parametrized as

$$\tilde{H}^I(r) = \tilde{h}^I + \frac{p^I}{r}, \quad H_I(r) = h_I + \frac{q^I}{r},$$

and we write the corresponding symplectic vector as $H(r) = (\tilde{H}^I(r), H_I(r)) = h + Q/r$.

Next we want to find the solutions for the four-dimensional black hole metric and for the scalar moduli fields $z^A = X^A/X^0$, which break half of the $N = 2$ supersymmetries. For the metric we make the ansatz [79]

$$ds^2 = -e^{2U} dt^2 + e^{-2U} dx^m dx^m,$$

where $U$ is a function of the radial coordinate $r = \sqrt{x^m x^m}$. As shown in [30] the Killing spinor equations follow from the symplectic covariance of the vectors $V$ and $H$, namely these vectors should satisfy a certain proportionality relation. The simplest possibility is to assume that $V$ and $H$ are directly proportional to each other. Because $H$ is real and invariant under U(1) transformations, there is a complex proportionality factor, which we denote by $Z$. Hence we define a U(1)-invariant symplectic vector (here we use the homogeneity property of the function $F$),

$$\Pi = \bar{Z} V = (Y^I, F_I(Y)),$$

so that $Y^I = \bar{Z} X^I$, and set

$$\Pi(r) - \bar{\Pi}(r) = i H(r).$$

The solutions of this set of algebraic equations, which we call stabilization equations and which depend on the particular choice of the prepotential, fully determine the form of the extremal BPS black holes. Specifically, the solution for the black hole metric is given by the symplectically invariant ansatz

$$e^{-2U(r)} = Z(r) \bar{Z}(r),$$

39
where $Z(r)$ is determined as
\[ Z(r) = -H_I(r) X^I + \tilde{H}^I(r) F_I(X), \quad |Z(r)|^2 = i\langle \Pi(r), \Pi(r) \rangle. \quad (83) \]
The stabilization equations equations (81) also determine the $r$-dependence of the scalar moduli fields:
\[ z^A(r) = Y^A(r)/Y^0(r). \quad (84) \]
So the constants $(\tilde{h}^I, h_I)$ just determine the asymptotic values of the scalars at $r = \infty$. In order to obtain an asymptotically flat metric with standard normalization, these constants must fullfill some constraints. Near the horizon ($r \approx 0$), (81) takes the form used in [29] and $Z$ becomes proportional to the holomorphic BPS mass $\mathcal{M}(z) = q_I X^I(z) - p^I F_I(X(z))$.

It can be shown that the solution preserves half the supersymmetries, except at the horizon and at spatial infinity, where supersymmetry is unbroken.

From the form of the static solution at the horizon ($r \to 0$) we can easily derive its macroscopic entropy. Specifically the Bekenstein-Hawking entropy is given by
\[ S_{\text{BH}} = \pi (r^2 e^{-2U})_{r=0} = \pi (r^2 Z\bar{Z})_{r=0} = i\pi \left( Y^I_{\text{hor}} F_I(Y_{\text{hor}}) - \tilde{F}_I(\bar{Y}_{\text{hor}}) Y^{I}_{\text{hor}} \right), \quad (85) \]
where the symplectic vector $\Pi$ at the horizon,
\[ \Pi(r) \approx 0 \frac{\Pi_{\text{hor}}}{r}, \quad Y^I(r) \approx 0 \frac{Y^I_{\text{hor}}}{r}, \quad (86) \]
is determined by the following set of stabilization equations:
\[ \Pi_{\text{hor}} - \bar{\Pi}_{\text{hor}} = iQ. \quad (87) \]
So we see that the entropy as well as the scalar fields $z^A_{\text{hor}} = Y^A_{\text{hor}}/Y^0_{\text{hor}}$ depend only on the magnetic/electric charges $(p^I, q_I)$. It is useful to note that the set of stabilization equations (87) is equivalent to the minimization of $Z$ with respect to the moduli fields [27].

4.2 N = 2, Type II Calabi-Yau superstring vacua

4.2.1 The large radius limit of type IIA Calabi-Yau compactifications

As an example, consider a type-IIA compactification on a Calabi-Yau 3-fold. The number of vector superfields is given as $N_V = h^{(1,1)}$. As discussed in section (3.3) the prepotential, which is purely classical, contains the Calabi-Yau intersection numbers of the 4-cycles, $C_{ABC}$, and, as $\alpha'$-corrections, the Euler number $\chi_4$ and the rational instanton numbers $n^r$. The corresponding term in the effective supergravity action comes from an higher derivative $R^4$ term in ten dimensions.
Hence the black-hole solutions will depend in general on all these topological quantities \[29\]. However, for a large Calabi-Yau volume, i.e. large values of the Kähler class moduli fields \(z^A = X^A / X^0\), only the part from the intersection numbers survives and the \(N = 2\) prepotential is given by

\[
F(Y) = D_{ABC} \frac{Y^AY^BY^C}{Y^0}, \quad D_{ABC} = -\frac{1}{6} C_{ABC}.
\]  

(88)

Based on this prepotential we consider in the following a class of non-axionic black-hole solutions (that is, solutions with purely imaginary moduli fields \(z^A\), i.e. real moduli fields \(t_A\)) with only non-vanishing charges \(q_0\) and \(p^A (A = 1, \ldots, h^{(1,1)})\). So only the harmonic functions \(H_0(r)\) and \(\tilde{H}^A(r)\) are nonvanishing. This charged configuration corresponds, in the type-IIA compactification, to the intersection of three D4-branes, wrapped over the internal Calabi-Yau 4-cycles and hence carrying magnetic charges \(p^A\), plus one D0-brane with electric charge \(q_0\). For the configuration indicated above, the solutions of the stabilization equations eq.(81) have the following form:

\[
Y^0(r) = \frac{\lambda(r)}{2}, \quad Y^A(r) = i \frac{\tilde{H}^A(r)}{2}
\]  

(89)

where the function \(\lambda(r)\) is given by

\[
\lambda(r) = \sqrt{D_{ABC} \tilde{H}^A(r) \tilde{H}^B(r) \tilde{H}^C(r) / H_0(r)}.
\]  

(90)

Then the four-dimensional metric of the extremal black-hole is given by

\[
e^{-2U(r)} = 2 \sqrt{H_0(r) D_{ABC} \tilde{H}^A(r) \tilde{H}^B(r) \tilde{H}^C(r)},
\]  

(91)

and the scalar fields have the following radius dependence:

\[
z^A(r) = i \tilde{H}^A(r) \sqrt{H_0(r) D_{ABC} \tilde{H}^A(r) \tilde{H}^B(r) \tilde{H}^C(r)}.
\]  

(92)

The scalars at the horizon are determined as

\[
z^A_{\text{hor}} = \frac{Y^A_{\text{hor}}}{Y^0_{\text{hor}}}, \quad Y^A_{\text{hor}} = \frac{1}{2} i p^A, \quad Y^0_{\text{hor}} = \frac{1}{2} \sqrt{\frac{D}{q_0}}, \quad D = D_{ABC} p^A p^B p^C.
\]  

(93)

Finally, the corresponding macroscopic entropy takes the form \[29\]

\[
S_{\text{BH}} = 2\pi \sqrt{q_0 D}.
\]  

(94)
4.2.2 Topology change at the conifold point in type IIB Calabi-Yau compactifications

Now consider type IIB compactifications on a Calabi-Yau space; hence $N_V = h^{(2,1)}$. The symplectic vector $V$ corresponds to the period integrals over the Calabi-Yau 3-cycles $\gamma_I$ and $\delta_I$:

$$F_I = \int_{\gamma_I} \Omega, \quad X^I = \int_{\delta_I} \Omega$$

(95)

The conifold transitions occur at those points in the moduli space where certain 3-cycles shrink to zero size and then blowing them up as two cycles, changing in this way the Hodge numbers. In the following we will discuss the most simple situation with periods $X^0$ and $X^1$ (together with $F_0$ and $F_1$), where $X^1$ vanishes at the conifold point and $X^0$ remains finite. Near this point the prepotential can be expanded as

$$F = -i (Y^0)^2 \left( c + \frac{1}{4\pi} \left( \frac{Y^1}{Y^0} \right)^2 \log \frac{Y^1}{Y^0} + \text{(analytic terms)} \right).$$

(96)

$c = \chi(3)/2(2\pi)^3$.

Let us look on the structure of the space-time solution near the conifold point. For simplification, we will consider again axion-free black holes with non-vanishing charges $q_0$ and $p^1$. As solution of the stabilization equations (81) we get

$$Y^0(r) = \frac{\lambda(r)}{2}, \quad Y^1(r) = i \frac{\tilde{H}^1(r)}{2}$$

(97)

where now the function $\lambda(r)$ is given by

$$\lambda(r) = \frac{H_0(r)}{2c} - \frac{(\tilde{H}^1(r))^2}{4\pi H_0(r)} + \mathcal{O}((\tilde{H}^1(r))^4).$$

(98)

Keeping only the first correction, we obtain for the function $e^{-2U}$ in the metric

$$e^{-2U(r)} = \frac{H_0^2(r)}{4c} + \frac{(\tilde{H}^1(r))^2}{4\pi} \log \frac{2c \tilde{H}^1(r)}{H_0(r)}.$$ 

(99)

Hence, we obtained a non-singular metric, also at the points where 3-cycles vanish ($\tilde{H}^1 = 0$).

As a special case consider the black hole solution with only non-vanishing charge $p^1$, i.e. $q_0 = 0$. Furthermore we also set $\tilde{h}^1 = 0$. Then we obtain a massless black hole with metric and scalar field $z^1$ as follows:

$$e^{-2U(r)} = 1 + \frac{(p^1)^2}{8\pi r^2} \log \frac{c(p^1)^2}{r^2} \pm .., \quad z^1 = i \frac{\sqrt{c} p^1}{r(1 \mp ...)},$$

(100)
where we used $\rho_0 = 2\sqrt{c}$, in order to have an asymptotic Minkowski space and $\pm \cdots$ indicate higher powers in $1/r$. This black hole solutions corresponds to a single D 3-brane which is wrapped around the shrinking 3-cycle. It is the dual to the electric massless BPS state discussed by Strominger. The main property of this massless solution is that it carries only one charge and has a shrinking internal 3-cycle at spatial infinity ($z^1 \to 0$ for $r \to \infty$). Hence the Calabi-Yau space degenerates at $r = \infty$ where the topology change can take place.

4.3 Higher curvature corrections and microscopic $N=2$ entropy

The $N=2$ black holes together with their entropies which were considered so far, appeared as solutions of the equations of motion of $N=2$ Maxwell-Einstein supergravity action, where the bosonic part of the action contains terms with at most two space-time derivatives (i.e., the Einstein action, gauge kinetic terms and the scalar non-linear $\sigma$-model). However, the $N=2$ effective action of strings and M-theory contains in addition an infinite number of higher-derivative terms involving higher-order products of the Riemann tensor and the vector field strengths, such as nonminimal gravitational couplings $R^2$. A particularly interesting subset of these couplings in $N=2$ supergravity can be again described by a holomorphic function $F(X, W^2) = \sum_{g=0}^{\infty} F^{(g)}(X^I) W^{2g}$, as discussed in section (3.4). Using the superconformal calculus, $N=2$ black-hole solutions for higher order $N=2$ supergravity based on the holomorphic function $F(X, W^2)$ were studied in ref. [32, 33]. As a result of these investigations, the Bekenstein-Hawking entropy of $N=2$ black holes will receive corrections due to the higher derivative terms in the effective $N=2$ supergravity action. In case of type IIA Calabi-Yau compactifications the higher order entropies will depend on additional topological Calabi-Yau data, such as the second Chern numbers or the elliptic instanton numbers, since the functions $F^{(g)}$ depend on these data (for $F^{(1)}$ see eq. (74)). These corrections are essential to match the macroscopic Bekenstein-Hawking entropy with the microscopic black hole entropy, which comes from counting the corresponding D-brane degrees of freedom. In fact, for the large radius limit of type IIA, Calabi-Yau compactifications, the microscopic entropy was recently computed [32]:

$$S_{\text{micro}} = 2\pi \sqrt{q_0 D \left(1 + \frac{c_{2A} p^A}{6D}\right)}.$$  \hfill (101)

The $c_{2A}$ are the second Chern class numbers of the Calabi-Yau 3-fold. As shown in [32, 33], expanding this expression to lowest order in $c_{2A}$,

$$S_{\text{micro}} = 2\pi \sqrt{q_0 D} + 2\pi \frac{1}{12} c_{2A} p^A \sqrt{q_0} D + \cdots$$  \hfill (102)

the correction agrees with a correction to the effective action involving $R^2$ type terms with a coefficient function $F^{(1)}(z^A) = -\frac{1}{24} c_{2A} z^A$ (see eq. (74)). In order to match the full microscopic $N=2$ entropy from the effective action, more work is still required.
5 Summary

In this article we have reviewed the string-string duality between string vacua with $N = 2$ supersymmetry or, respectively, with $N = 1$ supersymmetry in six dimensions. All the results presented here support the hypothesis of the $N = 2$ string-string duality symmetry. Beyond checking the string-string duality for a large class of heterotic/type II vacua, new insights into non-perturbative phenomena in compactified string theory are gained. Many of these results have a very beautiful description in terms of Calabi-Yau geometry. It appears that there is a huge web of $N = 2$ string theories, continuously connecting many, perhaps all, $N = 2$ vacua in four dimensions. Especially, in the study of the moduli spaces of type II strings compactified on Calabi-Yau three-folds it became clear that the topologies of the Calabi-Yau spaces can be continuously deformed into each other. The transitions among different vacua are made possible by BPS states which become massless at some special loci in the moduli spaces where the transitions take place. In the geometric type II Calabi-Yau description these BPS states originate from wrapping the ten-dimensional D p-branes around internal cycles of the Calabi-Yau space. Seen from the four-dimensional point of view, the BPS states correspond to supersymmetric charged black hole solutions of the effective $N = 2$ supergravity action.

Of even greater phenomenological interest than $N = 2$ string vacua is the investigation of string duality symmetries in four-dimensional string vacua with $N = 1$ space-time supersymmetry. One hopes in this way to get important nonperturbative information, like the computation of nonperturbative $N = 1$ superpotentials [80] and supersymmetry breaking, or the stringy reformulation of many effects in $N = 1$ field theory. In addition, it is a very important question, how transitions among $N = 1$ string vacua, possibly with a different number of chiral multiplets, take place [81]. String dual pairs with $N = 1$ supersymmetry in four dimensions are provided by comparing heterotic string vacua on Calabi-Yau three-folds, supplemented by the specification of a particular choice of the heterotic gauge bundle, with particular four-dimensional type IIB superstring vacua, which can be formulated as $F$ theory compactifications on elliptic Calabi-Yau four-folds [64, 82, 83]. In [83] we have constructed several $N = 1$ dual string pairs with identical massless spectra. Moreover, one can show that for a particular class of dual $N = 1$ string vacua the $N = 1$ superpotentials agree. These superpotentials are given by specific modular functions, very similar like the $N = 2$ prepotentials we discussed here. It would be interesting to gain further insights into some common features of $N = 2$ prepotentials and $N = 1$ superpotentials from string theory.

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References

[1] C. Vafa, hep-th/9702201; B. de Wit and J. Louis, hep-th/9801132; A. Sen, hep-th/9802051.

[2] A. Font, L. Ibáñez, D. Lüst and F. Quevedo, Phys. Lett. B 249 (1990) 35; S.–J. Rey, Phys. Rev. D 43 (1991) 526; A. Sen, Phys. Lett. B 303 (1993) 22, B 329 (1994) 217; J. Schwarz and A. Sen, Nucl. Phys. B 411 (1994) 35, hep-th/9304154.

[3] J.H. Schwarz, Phys. Lett. B 360 (1995) 13, hep-th/9508143.

[4] J. Polchinski, hep-th/9611050; C. Bachas, hep-th/9701019; L. Thorlacius, hep-th/9708078.

[5] M.J. Duff and R. Khuri, Nucl. Phys. B 411 (1994) 473, hep-th/9303142.

[6] C. M. Hull and P. Townsend, Nucl. Phys. B 438 (1995) 109, hep-th/9410167.

[7] E. Witten, Nucl. Phys. B 443 (1995) 85, hep-th/9503124.

[8] J. Harvey and A. Strominger, Nucl. Phys. B 449 (1995) 535, hep-th/9504047.

[9] P. Horava and E. Witten, Nucl. Phys. B 460 (1996) 506, hep-th/9510209.

[10] T. Banks, W. Fischler, S. Shenker and L. Susskind, Phys. Rev. D 55 (1997) 5112, hep-th/9610043.

[11] A. Strominger, Nucl. Phys. B 451 (1995) 96, hep-th/9504090.

[12] S. Kachru and C. Vafa, Nucl. Phys. B 450 (1995) 69, hep-th/9505103.

[13] S. Ferrara, J. Harvey, A. Strominger and C. Vafa, Phys. Lett. B 361 (1995) 59, hep-th/9505162.

[14] I. Antoniadis, C. Bachas, C. Fabre, H. Partouche and T.R. Taylor, Nucl. Phys. B 489 (1997) 160, hep-th/9608012.

[15] A. Klemm, W. Lerche, P. Mayr, C. Vafa and N. Warner, Nucl. Phys. B 477 (1996) 746, hep-th/9604034.

[16] E. Witten, Nucl. Phys. B 500 (1997) 3, hep-th/9703166.

[17] N. Seiberg and E. Witten, Nucl. Phys. B 426 (1994) 19, hep-th/9407087; Nucl. Phys. B 431 (1994) 484, hep-th/9408099.
[18] L. Alvarez-Gaume and S.F. Hassan, Fortschritte d. Physik 45 (1997) 159, hep-th/9701069; S. Ketov, Fortschritte d. Physik 45 (1997) 237, hep-th/9611209; W. Lerche, Fortschritte d. Physik 45 (1997) 293, hep-th/9611190.

[19] G. Curio, Fortschritte d. Physik 46 (1998) 75, hep-th/9708009.

[20] G. L. Cardoso, G. Curio, D. Lüst and T. Mohaupt, Phys. Lett. B 382 (1996) 241, hep-th/9603108.

[21] G. L. Cardoso, G. Curio and D. Lüst, Nucl. Phys. B 491 (1997) 147, hep-th/9608154.

[22] V. Kaplunovsky, J. Louis and S. Theisen, Phys. Lett. B 357 (1995) 71, hep-th/9506110.

[23] I. Antoniadis, E. Gava, K. S. Narain and T. R. Taylor, Nucl. Phys. B 455 (1995) 109, hep-th/9507115.

[24] G. Curio, Phys. Lett. B 366 (1996) 131, hep-th/9509042; Phys. Lett. B 368 (1996) 78, hep-th/9509146.

[25] G. L. Cardoso, G. Curio, D. Lüst, T. Mohaupt and S.-J. Rey, Nucl. Phys. B 464 (1996) 18, hep-th/9512129.

[26] B. de Wit, G. Lopes Cardoso, D. Lüst, T. Mohaupt and S.-J. Rey, Nucl. Phys. B 481 (1996) 353, hep-th/9607184.

[27] S. Ferrara, R. Kallosh and A. Strominger, Phys. Rev. D52 (1995) 5412, hep-th/9508072; S. Ferrara and R. Kallosh, Phys. Rev. D54 (1996) 1514, hep-th/9602136; Phys. Rev. D54 (1996) 1525, hep-th/9603090; S. Ferrara, G.W. Gibbons and R. Kallosh, Nucl. Phys. B500 (1997) 75, hep-th/9702103.

[28] R. Kallosh, M. Shmakova and W.K. Wong, Phys. Rev. D54 (1996) 6284, hep-th/9607077; K. Behrndt, R. Kallosh, J. Rahmfeld, M. Shmakova and W.K. Wong, Phys. Rev. D54 (1996) 6293, hep-th/9608059; G. Lopes Cardoso, D. Lüst and T. Mohaupt, Phys. Lett. B388 (1996) 266, hep-th/9608099; S.-J. Rey, Nucl. Phys. B508 (1997) 569, hep-th/9610157; K. Behrndt and T. Mohaupt, Phys. Rev. D56 (1997) 2211, hep-th/9611140; K. Behrndt and W. Sabra, Phys. Lett. B401 (1997) 258, hep-th/9702010; K. Behrndt and I. Gaida, Phys. Lett. B401 (1997) 263, hep-th/9702168; W. A. Sabra, Mod. Phys. Lett. A12 (1997) 2585, hep-th/9703101; K. Behrndt, G. Lopes Cardoso and I. Gaida, Nucl. Phys. B506 (1997) 267, hep-th/9704095.
[29] K. Behrndt, G. Lopes Cardoso, B. de Wit, R. Kallosh, D. Lüst and T. Mohaupt, Nucl. Phys. B488 (1997) 236, hep-th/9610105.

[30] W.A. Sabra, Nucl. Phys. B510 (1998) hep-th/9704147;
K. Behrndt, D. Lüst and W.A. Sabra, Nucl. Phys. B510 (1998) 264, hep-th/9705169.

[31] K. Behrndt, D. Lüst and W.A. Sabra, hep-th/9708065.

[32] J.M. Maldacena, A. Strominger and E. Witten, hep-th/9711053.

[33] K. Behrndt, G. Lopes Cardoso, B. de Wit, D. Lüst, T. Mohaupt and W. Sabra, hep-th/9801081.

[34] A. Klemm, W. Lerche and P. Mayr, Phys. Lett. B 357 (1995) 313, hep-th/9506112.

[35] P. S. Aspinwall and J. Louis, Phys. Lett. B 369 (1995) 233, hep-th/9510234.

[36] K.S. Narain, Phys. Lett. B 169 (1986) 41

[37] N. Seiberg, Nucl. Phys. B 303 (1988) 286.

[38] D. Lüst and S. Theisen, Int. J. Mod. Phys. A4 (1989) 4513;
S. Ferrara, D. Lüst and S. Theisen, Nucl. Phys. B 325 (1989) 501.

[39] C. Vafa and E. Witten, hep-th/9507051.

[40] J. Erler, J. Math. Phys. 35 (1993) 377, hep-th/9304104.

[41] M.B. Green and J.H. Schwarz, Nucl. Phys. B 255 (1985) 93.

[42] N. Seiberg and E. Witten, Nucl. Phys. B 471 (1996) 121, hep-th/9603003.

[43] A. Sagnotti, Phys. Lett. B 294 (1992) 196, hep-th/9210127.

[44] P. Candelas and A. Font, hep-th/9603170.

[45] G. Aldazabal, A. Font, L.E. Ibáñez and F. Quevedo, Nucl. Phys. B 461 (1996) 85, hep-th/9510093.

[46] E. Witten, Nucl. Phys. B 460 (1996) 541, hep-th/9511030.

[47] J. Louis, J. Sonnenschein, S. Theisen and S. Yankielowicz, Nucl. Phys. B 480 (1996) 185, hep-th/9606049.
[48] P. Claus, B. de Wit, M. Faux and P. Termonia, Nucl. Phys. B 491 (1997) 201, hep-th/9612203.

[49] M. Berkooz, R. G. Leigh, J. Polchinski, J.H. Schwarz, N. Seiberg and Edward Witten, Nucl. Phys. B 475 (1996) 115, hep-th/9605184.

[50] M.J. Duff, R. Minasian and E. Witten, Nucl. Phys. B 465 (1996) 413, hep-th/9601036.

[51] Aldazabal, A. Font, L.E. Ibanez and F. Quevedo, Phys. Lett. B 380 (1996) 33, hep-th/9602094.

[52] P. Aspinwall, Phys. Lett. B 371 (1996) 231, hep-th/9511171.

[53] S. Hosono, B.H. Lian and S.T. Yau, alg-geom/9603020.
G. Aldazabal, A. Font, L.E. Ibanez and A.M. Uranga, Nucl. Phys. B 492 (1997) 119, hep-th/9607121.
A.C. Avram, M. Kreuzer, M. Mandelberg and H. Skarke, Nucl. Phys. B 494 (1997) 567, hep-th/9610154.
P. Candelas, E. Perevalov and G. Rajesh, Nucl. Phys. B 502 (1997) 613, hep-th/9703148.

[54] S. Hosono, A. Klemm, S. Theisen and S.-T. Yau, Commun. Math. Phys. 167 (1995) 301.

[55] B. de Wit, V. Kaplunovsky, J. Louis and D. Lüst, Nucl. Phys. B 451 (1995) 53, hep-th/9504006.

[56] G. Lopes Cardoso, D. Lüst and T. Mohaupt, Nucl. Phys. B 455 (1995) 131, hep-th/9507113.

[57] S. Kachru, A. Klemm, W. Lerche, P. Mayr and C. Vafa, Nucl. Phys. B 459 (1996) 537, hep-th/9508152.
I. Antoniadis and H. Partouche, Nucl. Phys. B 460 (1996) 470, hep-th/9509009.

[58] S. Katz, A. Klemm and C.Vafa, Nucl. Phys. B 497 (1997) 173, hep-th/9609239.

[59] S. Katz, D. Morrison and M.R. Plesser, Nucl. Phys. B 477 (1996) 105, hep-th/9601108.

[60] P.S. Aspinwall and M. Gross, Phys. Lett. B 382 (1996) 81, hep-th/9602118.

[61] M. Bershadsky, C. Vafa and V. Sadov, Nucl. Phys. B 463 (1996) 398, hep-th/9510223.

[62] A. Klemm and P. Mayr, Nucl. Phys. B 469 (1996) 37, hep-th/9601014.
[63] P. Berglund, S. Katz, A. Klemm and P. Mayr, Nucl. Phys. B 483 (1997) 209, hep-th/9605154.
[64] C. Vafa, Nucl. Phys. B469 (1996) 493, hep-th/9602022.
[65] D. R. Morrison and C. Vafa, Nucl. Phys. B 473 (1996) 74, hep-th/9602114; Nucl. Phys. B 476 (1996) 437, hep-th/9603161.
[66] M. Bershadsky, K. Intriligator, S. Kachru, D. Morrison, V. Sadov and C. Vafa, Nucl. Phys. B 481 (1996) 215, hep-th/9605200.
[67] E. Witten, Nucl. Phys. B471 (1996) 195; hep-th/9603150.
[68] I. Antoniadis, S. Ferrara and T.R. Taylor, Nucl. Phys. B460 (1996) 489, hep-th/9511108.
[69] L. Dixon, V. Kaplunovsky and J. Louis, Nucl. Phys. B 355 (1991) 649.
[70] S. Ferrara, C. Kounnas, D. Lüst and F. Zwirner, Nucl. Phys. B 365 (1991) 431.
[71] J. A. Harvey and G. Moore, Nucl. Phys. B 463 (1996) 315, hep-th/9510182.
[72] B. de Wit and A. Van Proeyen, Nucl. Phys. B245 (1984) 89 and hep-th/9505097; S. Cecotti, S. Ferrara, and L. Girardello, Int. J. Mod. Phys. A4 (1989) 2475; A. Strominger, Commun. Math. Phys. 133 (1990) 163; S. Ferrara and A. Strominger, in Strings '89, ed. R. Arnowitt, R. Bryan, M. J. Duff, D. Nanopulos and C. N. Pope, World Scientific, Singapore, (1990) 245; P. Candelas and X. de la Ossa, Nucl. Phys. B355 (1991) 455; A. Ceresole, R. D'Auria, S. Ferrara and A. van Proeyen, Nucl. Phys. B444 (1995) 92; L. Andrianopoli, M. Bertolini, A. Ceresole, R. D'Auria, S. Ferrara and P. Frè, Nucl. Phys. B476 (1996) 397, hep-th/9603004; B. Craps, F. Roose, W. Troost and A. Van Proeyen, Nucl. Phys. B503 (1997) 565, hep-th/9703082.
[73] I. Antoniadis, S. Ferrara, E. Gava, K. S. Narain and T. R. Taylor, Nucl. Phys. B 447 (1995) 35, hep-th/9504034.
[74] G. L. Cardoso, D. Lüst and T. Mohaupt, Nucl. Phys. B 450 (1995) 115, hep-th/9412209.
[75] K. Förger and S. Stieberger, Nucl. Phys. B 514 (1998) 135, hep-th/9709004.
[76] P. Candelas, X. C. de la Ossa, P. S. Green and L. Parkes, Nucl. Phys. B 359 (1991) 21.
[77] B. de Wit, Nucl. Phys. Proc. Suppl. 49 (1996) 191, hep-th/9602060; Fortschritte d. Physik 44 (1996) 529.
[78] A. Strominger and C. Vafa, Phys. Lett. B379 (1996) 99, hep-th/9601029;
    C.G. Callan and J.M. Maldacena, Nucl. Phys. B472 (1996) 591, hep-th/9602043.

[79] K.P. Tod, Phys. Lett. 121B (1983) 241.

[80] E. Witten, Nucl. Phys. B 474 (1996) 343, hep-th/9604030.

[81] S. Kachru and E. Silverstein, Nucl. Phys. B 504 (1997) 272, hep-th/9704183;
    E. Sharpe, hep-th/9705210;
    M. Klein and J. Louis, hep-th/9707047, hep-th/9707212;
    I. Brunner, A. Hanany, A. Karch and D. Lüst, hep-th/9801017.

[82] S. Sethi, C. Vafa and E. Witten, Nucl. Phys. B 480 (1996) 213, hep-th/9606122;
    I. Brunner and R. Schimmrigk, Phys. Lett. B 387 (1996) 750, hep-th/9606148;
    I. Brunner, M. Lynker and R. Schimmrigk, Nucl. Phys. B 498 (1997) 156, hep-th/9610193;
    P. Mayr, Nucl. Phys. B 494 (1997) 489, hep-th/9610162;
    A. Klemm, B. Lian, S-S. Roan and S-T. Yau, hep-th/9701023;
    R. Friedman, J. Morgan and E. Witten, Commun. Math. Phys. 187 (1997) 679, hep-th/9701162;
    K. Mohri, hep-th/9701147;
    M. Bershadsky, A. Johansen, T. Pantev and V. Sadov, Nucl. Phys. B 505 (1997) 165, hep-th/9701163;
    G. Curio, Phys. Lett. B409 (1997) 185, hep-th/9705197;
    B. Andreas and G. Curio, hep-th/9706093;
    M. Bershadsky, T.M. Chiang, B.R. Greene, A. Johansen and C.I. Lazaroiu, hep-th/9712023;
    G. Curio and R. Donagi, hep-th/9801057;
    B. Andreas, hep-th/9802202.

[83] G. Curio and D. Lüst, Int. J. Mod. Phys. A12 (1997) 5847, hep-th/9703007;
    B. Andreas, G. Curio and D. Lüst, Nucl. Phys. B 507 (1997) 175, hep-th/9705174;
    D. Lüst, hep-th/9709222.