Conformal correlators of mixed-symmetry tensors

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Abstract: We generalize the embedding formalism for conformal field theories to the case of general operators with mixed symmetry. The index-free notation encoding symmetric tensors as polynomials in an auxiliary polarization vector is extended to mixed-symmetry tensors by introducing a new commuting or anticommuting polarization vector for each row or column in the Young diagram that describes the index symmetries of the tensor. We determine the tensor structures that are allowed in $n$-point conformal correlation functions and give an algorithm for counting them in terms of tensor product coefficients. We show, with an example, how the new formalism can be used to compute conformal blocks of arbitrary external fields for the exchange of any conformal primary and its descendants. The matching between the number of tensor structures in conformal field theory correlators of operators in $d$ dimensions and massive scattering amplitudes in $d+1$ dimensions is also seen to carry over to mixed-symmetry tensors.

Keywords: CFT, conformal field theory, correlation function, embedding space
1 Introduction

The study of Conformal Field Theories (CFTs) is among the most important subjects in theoretical physics, with implications to critical phenomena, particle physics and, in the light of the AdS/CFT duality, quantum gravity. In past years we have witnessed a revival in the study of CFTs in dimensions higher than two. This study is considerably more difficult than the two-dimensional case, where the conformal group possesses an infinitely dimensional extension given by the Virasoro algebra, which leads to many known exactly solvable models. On one hand, the conformal bootstrap program [1, 2] applied to higher dimensional CFTs, revived in [3], has already shown its merits by providing the most accurate computation to date of 3D Ising model critical exponents [4, 5]. On the other hand, the most studied case
of the AdS/CFT duality [6] considers $\mathcal{N} = 4$ Super Yang-Mills, which is a four-dimensional CFT. In particular, this theory is believed to be integrable in the planar limit [7, 8], thus providing the first example of an exactly solvable 4D gauge theory.

To advance in the conformal bootstrap program, as well as our understanding of AdS/CFT, it is necessary to further develop analytic and computational techniques to deal with arbitrary tensor primary fields. In $d$ dimensions, these fields are classified by the unitary irreducible representations of the conformal group $SO(d+1,1)$, which are labeled by the conformal dimension $\Delta$ and by an irreducible representation (irrep) of $SO(d)$. A first step in this direction was made in [9], where symmetric tensors of arbitrary spin were studied in detail. The goal of this paper is to extend this work by considering $SO(d)$ tensors with mixed symmetry.

We shall start, in section two, with the general classification of irreducible tensor representations of $SO(d)$, which can be represented by Young diagrams. This is a well known subject, which we shall review in order to introduce the reader to the necessary formalism. We will then see how to encode, in general, mixed-symmetry tensors in terms of polynomials of polarization vectors. To encode their mixed symmetry it is necessary to employ a combination of Grassmann valued and ordinary commuting polarizations. Actual computations simplify considerably if fields that live in $d$-dimensional Euclidean space are embedded in an auxiliary $(d+2)$-dimensional Minkowski space, where the conformal group $SO(d+1,1)$ acts linearly as the usual Lorentz transformations. We shall see that this formalism can be easily extended to include mixed-symmetry tensors by encoding them in polynomials of polarization vectors in the embedding space.

In section three we show how to construct CFT $n$-point correlation functions of arbitrary tensors. The formalism is presented in general terms, but we shall give a number of simple examples for two-, three- and four-point functions, so the reader can appreciate the simplicity and efficiency of the method. We will also describe the general case of $n$-point functions.

As an application of the new formalism we consider, in section four, the problem of computing conformal blocks for any desired external primary fields, describing the exchange of an arbitrary conformal primary and its descendants. With the help of shadow operators, these conformal blocks can be written as an integral of three point functions, leading to an expression of the conformal blocks in terms of a finite number of integrals, which can be expressed in terms of hypergeometric functions for even dimensions [10]. To see the method at work, we shall consider explicitly the example of the four point function of two scalars and two vectors, exchanging a mixed-symmetry tensor of rank three.

In section five we show that the number of tensor structures in CFT correlators of non-conserved mixed-symmetry tensors in $d$ dimensions matches that of massive scattering amplitudes in $d + 1$ dimensions, as expected.

Section six presents final comments.
2 Mixed-symmetry tensors

2.1 Parametrizing Young diagrams

In this paper the traceless irreducible tensor representations of $SO(d)$ are considered. These representations are enumerated mostly\footnote{There is a one-to-one correspondence between traceless irreducible tensor representations of $SO(d)$ and the standard Young tableaux satisfying (2.1) except for the case $d = 2n, h^\lambda = n$ [11]. In this case the representation with the symmetry corresponding to $\lambda$ can be decomposed further using the Levi-Civita tensor and is therefore not irreducible. A well-known example is the decomposition of the two-form in four dimensions into self-dual and anti-self-dual parts.} by standard Young tableaux (tableaux with increasing entries in each row and column), which encode the (anti-) symmetry of the tensors under permutation of their indices.

There are two different ways to parametrize the shape of a Young diagram $\lambda$. The first is by giving a partition $l^\lambda = (l^\lambda_1, l^\lambda_2, \ldots)$ containing the lengths of the rows, $l^\lambda_i$ being the length of the $i$-th row. The diagram that is obtained from $\lambda$ by exchanging rows and columns is called the transpose $\lambda^t$. The partition $h^\lambda$ describes the column heights of $\lambda$ and is the conjugate partition to $l^\lambda, l^\lambda_t \equiv h^\lambda = (h^\lambda_1, h^\lambda_2, \ldots)$. A second way to describe the shape of a Young diagram is by its Dynkin label $\lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_{h^\lambda_1}\}$, which lists the numbers $\lambda_i$ of columns with $i$ boxes. Apart from the exception mentioned in the footnote, the Young diagram $\lambda$ labels an irrep of $SO(d)$ if and only if its overall height $h^\lambda_1$ does not exceed the rank of the Lie algebra corresponding to $SO(d)$

$$h^\lambda_1 \leq \left\lfloor \frac{d}{2} \right\rfloor = \begin{cases} \frac{d}{2}, & d \text{ even}, \\ \frac{d-1}{2}, & d \text{ odd}. \end{cases} \tag{2.1}$$

The total number of boxes is denoted by $|\lambda|$

$$|\lambda| = \sum_i i \lambda_i = \sum_i l^\lambda_i = \sum_i h^\lambda_i. \tag{2.2}$$

It will be useful to label the number of rows with more than one box $n^\lambda_Z$ and the number of columns with more than one box $n^\lambda_{\Theta}$.

$$n^\lambda_Z = \sum_{i=2}^{l^\lambda_t} \lambda_i, \quad n^\lambda_{\Theta} = \sum_{i=2}^{h^\lambda} \lambda_i. \tag{2.3}$$

All of this is best illustrated by the following example:

\begin{align*}
\lambda &= [2, 1, 0, 2], & |\lambda| &= 12, \\
l^\lambda &= (5, 3, 2, 2), & n^\lambda_Z &= 4, \\
h^\lambda &= (4, 4, 2, 1, 1), & n^\lambda_{\Theta} &= 3. \tag{2.4}
\end{align*}

The $\lambda$ on $l^\lambda, h^\lambda, n^\lambda_Z$ and $n^\lambda_{\Theta}$ will frequently be omitted or replaced by $i$ if the Young diagram is of shape $\lambda_i.$


2.2 Birdtracks and Grassmann variables

Probably the best way to think about mixed-symmetry tensors is in terms of birdtrack notation\(^2\) where index contractions are simply drawn as lines

\[ = \delta^{a_1 b_1} \delta^{a_2 b_2} \]  

(2.5)

Symmetrization and antisymmetrization are indicated by the symbols

\[
\begin{align*}
\vdots & \vdots n = \frac{1}{n!} \left\{ \begin{array}{c}
\vdots \vdots n + \vdots \vdots n + \vdots \vdots n + \ldots \\
\vdots \vdots n - \vdots \vdots n - \vdots \vdots n - \ldots
\end{array} \right\}, \\
\vdots & \vdots n = \frac{1}{n!} \left\{ \begin{array}{c}
\vdots \vdots n + \vdots \vdots n + \vdots \vdots n + \ldots \\
\vdots \vdots n - \vdots \vdots n - \vdots \vdots n - \ldots
\end{array} \right\}.
\end{align*}
\]

(2.6)

This notation has the advantage that it makes it immediately visible when terms are vanishing because two or more symmetric indices are antisymmetrized or vice versa

\[
\vdots = 0.
\]

(2.7)

Furthermore birdtracks can be diagrammatically transformed, for example using that repeated (anti)symmetrizations of subsets of indices have no effect

\[
\vdots \begin{array}{c}
\vdots \vdots n \\
\vdots \vdots n
\end{array} = \vdots \begin{array}{c}
\vdots \vdots n \\
\vdots \vdots n
\end{array}, \quad \begin{array}{c}
\vdots \vdots n \\
\vdots \vdots n
\end{array} = \begin{array}{c}
\vdots \vdots n \\
\vdots \vdots n
\end{array}.
\]

(2.8)

A symmetrized contraction of \(n\) indices is generated by the \(n\)-th derivative of \(n\) components of an auxiliary vector \(z\),

\[
\vdots \begin{array}{c}
\vdots \vdots n \\
\vdots \vdots n
\end{array} = \frac{1}{n!} \partial_{z_{a_1}} \ldots \partial_{z_{a_n}} z_{b_1} \ldots z_{b_n}.
\]

(2.9)

Antisymmetrization works analogously with an auxiliary vector in Grassmann variables \(\theta\),

\[
\vdots \begin{array}{c}
\vdots \vdots n \\
\vdots \vdots n
\end{array} = \frac{1}{n!} \partial_{\theta_{a_1}} \ldots \partial_{\theta_{a_n}} \theta_{b_1} \ldots \theta_{b_n}.
\]

(2.10)

The Grassmann variables are anticommuting in the sense that

\[
\theta^{(p)}_a \theta^{(q)}_b = (-1)^{\delta^{pq}} \theta^{(q)}_b \theta^{(p)}_a.
\]

(2.11)

Here an additional label \((p)\) was introduced to allow for several independent antisymmetrizations at the same time. Derivatives with respect to Grassmann variables are implied to be right derivatives,

\[
\partial_{\theta^{(p)}} \theta^{(q)}_a \theta^{(r)}_b = \delta^{r q} \delta^{a b} \theta^{(p)} + (-1)^{\delta^{pq}} \delta^{r p} \delta^{ca} \theta^{(q)}.
\]

(2.12)

\(^2\) See [12] for a beautiful group theory book entirely in terms of birdtracks.
2.3 Young symmetrization and antisymmetric basis

The symmetry of a Young tableau is imposed on a tensor via Young symmetrizers. Each row of the diagram corresponds to a symmetrization and each column corresponds to an antisymmetrization. This can be nicely illustrated by an example, following [12]. A symmetrizer given by the Young tableau $\lambda$ creates the tensor $T^\lambda$ with appropriate symmetry from a generic tensor $T$.

$$
\lambda = \begin{array}{cccc}
1 & 2 & 3 & 4 \\
5 & 6 & & \\
7 & & & \\
\end{array} \quad \rightarrow \quad T^\lambda = \begin{array}{c}
a_1 \\
\vdots \\
a_7 \\
\end{array}.
$$

(2.13)

This tensor has the manifest symmetry properties

$$
T^\lambda_{a_1a_2a_3a_4a_5a_6a_7} = T^\lambda_{(a_1a_2a_3)(a_4a_5a_6)} ,
$$

(2.14)

but there are also less obvious symmetries caused by the antisymmetrizations. Due to the manifest symmetries, $T^\lambda$ is said to belong to the symmetric basis. The antisymmetric basis is obtained by changing the order of symmetrization and antisymmetrization

$$
\lambda' = \begin{array}{cccc}
1 & 4 & 6 & 7 \\
2 & 5 & & \\
3 & & & \\
\end{array} \quad \rightarrow \quad T^{\lambda'} = \begin{array}{c}
a_1 \\
\vdots \\
a_7 \\
\end{array}.
$$

(2.15)

Here we have manifest antisymmetry

$$
T^{\lambda'}_{a_1a_2a_3a_4a_5a_6a_7} = T^{\lambda'}_{[a_1a_2a_3][a_4a_5][a_6a_7]} .
$$

(2.16)

The only reason we used a different Young tableau for this second example is to spare us from having to cross lines on the right hand side of the birdtrack diagram. We will in this paper work only in the antisymmetric basis and therefore consider for each Young diagram only the single tableau where the boxes are enumerated column by column, as in (2.15) and omit writing the numbers. The tensors corresponding to different standard tableaux (and therefore different $SO(d)$ irreps) can be obtained simply by commutation of indices.

It may also be instructive to see how the non-explicit index symmetries manifest themselves explicitly, again in the antisymmetric representation. To this end assign different labels to each anticommuting group of indices

$$
f_{a_1...a_h b_1...b_k c_1...c_l} = f_{[a_1...a_h][b_1...b_k][c_1...c_l]...} = f_{[a_1...a_h][b_1...b_k][c_1...c_l]...[g_1...g_{h_l}]} .
$$

(2.17)

Apart from the antisymmetry, the Young symmetrization implies that the antisymmetrization of any of the indices $b$ with all the $a$ vanishes, as well as the antisymmetrization of any of the...
c with all indices \( a \) or all \( b \) and so forth [13]. Explicitly this means that

\[
f_{[a_1...a_k][b_1...b_k][c_1...c_h3]...[g_{h_1}]} \quad (2.18)
\]

\[
= f_{[b_1a_2...a_k][a_1b_2...b_k][c_1]...+f_{[a_1b_1a_3...a_k][a_2b_2...b_k][c_1]...+...+f_{[a_1...a_{h_1}1b_1][a_{h_1}b_2...b_k][c_1]...}}
\]

\[
= f_{[c_1a_2...a_k][b_1...b_k][c_1]...+f_{[a_1c_1a_3...a_k][b_1...b_k][a_2c_2]...+...+f_{[a_1...a_{h_1}1c_1][b_1...b_k][a_{h_1}c_2]...}}
\]

Furthermore, the tensors are symmetric under exchange of complete groups of antisymmetric indices if the corresponding columns in the Young diagram are of equal height, e.g. for \( h_2 = h_3 \),

\[
f_{[a_1...a_{h_1}][b_1...b_{h_2}][c_1...c_{h_3}]...[g_{h_1}]} = f_{[a_1...a_{h_1}][c_1...c_{h_3}][b_1...b_{h_2}]...[g_{h_1}]} . \quad (2.19)
\]

### 2.4 Encoding mixed-symmetry tensors by polynomials

To encode a mixed-symmetry tensor by a polynomial, the strategy is to contract it with a tensor with the same mixed symmetry, which is built out of auxiliary polarizations. To construct a Young symmetrized tensor in the antisymmetric basis out of auxiliary vectors, one can start with a set of polarizations that is already symmetrized so that only the antisymmetrization is left to do. For the example (2.15), the following tensor depending on the auxiliary vectors \( z^{(1)} \), \( z^{(2)} \) and \( z^{(3)} \) is appropriately symmetrized

\[
\begin{array}{c}
\begin{array}{c}
\vdots \\
\end{array}
\end{array}
\]

Using (2.10) to encode the antisymmetrization, (2.20) can be written as

\[
\frac{1}{3!2!} \left( z^{(1)} \cdot \partial_{\Theta^{(1)}} \right) \left( z^{(1)} \cdot \partial_{\Theta^{(2)}} \right) \left( z^{(1)} \cdot \partial_{\Theta^{(3)}} \right) \left( z^{(1)} \cdot \partial_{\Theta^{(4)}} \right) \left( z^{(1)} \cdot \partial_{\Theta^{(1)}} \right) \left( z^{(2)} \cdot \partial_{\Theta^{(2)}} \right) \left( z^{(3)} \cdot \partial_{\Theta^{(1)}} \right) \theta^{(1)}_{a_1} \theta^{(1)}_{a_2} \theta^{(1)}_{a_3} \theta^{(1)}_{a_4} \theta^{(2)}_{a_5} \theta^{(2)}_{a_6} \theta^{(3)}_{a_7} \theta^{(4)}_{a_8} . \quad (2.21)
\]

This can be shortened by avoiding the introduction of polarizations that appear only once and hence do not cause any (anti-)symmetrization, i.e. doing explicitly the derivatives in the polarizations \( \Theta^{(3)} \) and \( \Theta^{(4)} \),

\[
\left( z^{(1)} \cdot \partial_{\Theta^{(3)}} \right) \left( z^{(1)} \cdot \partial_{\Theta^{(4)}} \right) \theta^{(3)}_{a_6} \theta^{(4)}_{a_7} = z^{(1)}_{a_6} z^{(1)}_{a_7} . \quad (2.22)
\]

After this step the symmetry in the indices \( a_6 \) and \( a_7 \) is manifest. Likewise, \( z^{(3)} \) that appears only once in this example through the derivative \( \left( z^{(3)} \cdot \partial_{\Theta^{(1)}} \right) \), does not encode any symmetry. More generally, for diagrams with more than one row of length one the action of such derivatives hides antisymmetry. We shall therefore omit these derivative terms, with the result that the encoding polynomial will depend not only on symmetric polarizations, but also on \( \Theta^{(1)} \), therefore making antisymmetrization explicit on the indices corresponding to all rows of length one.

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Thus, the slightly less elegant, but more pragmatic Young symmetric polarization we use for the example at hand will be the polynomial in $z \equiv (z^{(1)}, z^{(2)}, \theta^{(1)})$ given by
\[
\left((z^{(1)} \cdot \partial_{\theta^{(1)}}) \left((z^{(1)} \cdot \partial_{\theta^{(2)}}) \left((z^{(2)} \cdot \partial_{\theta^{(1)}}) \left((z^{(2)} \cdot \partial_{\theta^{(2)}}) \theta_{a_1}^{(1)} \theta_{a_2}^{(1)} \theta_{a_3}^{(1)} \theta_{a_4}^{(2)} \theta_{a_5}^{(2)} z_{a_6}^{(1)} z_{a_7}^{(1)},
\right)\right)\right)\right)\right).
\]
(2.23)
which is quartic in $z^{(1)}$, quadratic in $z^{(2)}$ and linear in $\theta^{(1)}$, as appropriate for a Young tableau with lengths of rows given by $l^\lambda = (4, 2, 1)$. This Young symmetric polarization is obtained by acting with derivatives of the type $(z^{(p)} \cdot \partial_{\theta^{(q)}})$ on a polynomial in $\theta \equiv (\theta^{(1)}, \theta^{(2)}, z^{(1)})$, cubic in $\theta^{(1)}$, quadratic in $\theta^{(2)}$ and quadratic in $z^{(1)}$, as appropriate for a Young tableau with lengths of columns given by $h^\lambda = (3, 2, 1, 1)$. A tensor with components $f^{a_1 \ldots a_7}$ in the irreps of this example will then be encoded by the polynomial
\[
f(z) \equiv \left((z^{(1)} \cdot \partial_{\theta^{(1)}}) \left((z^{(1)} \cdot \partial_{\theta^{(2)}}) \left((z^{(2)} \cdot \partial_{\theta^{(1)}}) \left((z^{(2)} \cdot \partial_{\theta^{(2)}}) \tilde{f}(\theta),\n\right)\right)\right)\right),
\]
(2.24)
where
\[
\tilde{f}(\theta) \equiv \theta_{a_1}^{(1)} \theta_{a_2}^{(1)} \theta_{a_3}^{(1)} \theta_{a_4}^{(2)} \theta_{a_5}^{(2)} z_{a_6}^{(1)} z_{a_7}^{(1)} f^{a_1 \ldots a_7}.
\]
(2.25)
In general we shall consider $n_\theta$ anticommuting and $n_Z$ commuting polarization vectors for a given tensor operator. A convenient notation for the mostly anticommuting polarizations which are first contracted to the tensor is
\[
\theta \equiv \left(\theta^{(1)}, \theta^{(2)}, \ldots, \theta^{(n_\theta)}, z^{(1)}\right).
\]
(2.26)
In cases where there are no columns with one box the last entry is absent and $\theta$ contains only anti-commuting polarizations. Similarly, we will write for the mostly commuting polarizations on which the final encoding polynomial depends
\[
z \equiv \left(z^{(1)}, z^{(2)}, \ldots, z^{(n_Z)}, \theta^{(1)}\right).
\]
(2.27)
Again, in cases where there are no rows with one box the last entry is absent and $z$ contains only commuting polarizations. Generalizing the previous example, we have that a tensor $f^{a_1 \ldots a_{|\lambda|}}$ in the irrep $\lambda$ is encoded by the polynomial
\[
f(z) \equiv \prod_{p=1}^{n_Z} \prod_{q=1}^{\min(l_p, n_\theta)} (z^{(p)} \cdot \partial_{\theta^{(q)}}) \tilde{f}(\theta),
\]
(2.28)
where
\[
\tilde{f}(\theta) \equiv \theta_{a_1}^{(1)} \ldots \theta_{a_{l_1}}^{(1)} \theta_{a_{l_1+1}}^{(2)} \ldots \theta_{a_{l_1+h_2}}^{(2)}
\]
\[
\ldots \theta_{a_{l_{n_\theta}}}^{(n_\theta)} \theta_{a_{l_{n_\theta}+h_{n_\theta}}-1}^{(1)} \ldots \theta_{a_{l_{n_\theta}+h_{n_\theta}}-1}^{(n_\theta)} z_{a_{l_1+1}}^{(1)} \ldots z_{a_{l_1+1}}^{(1)} f^{a_1 \ldots a_{|\lambda|}}.
\]
(2.29)
When there are more than one row with one box, the dependence of $\tilde{f}(z)$ on $\theta^{(1)}$ makes manifest the antisymmetry of the indices corresponding to such boxes. Likewise, when there are more than one column with one box, the dependence of $\tilde{f}(\theta)$ on $z^{(1)}$ makes manifest the symmetry of the indices corresponding to such boxes.
The condition that \( f^{a_1 \ldots a_\lambda} \) is traceless can be used to choose the polarizations to have vanishing products
\[
f^{a_1 \ldots a_\lambda} \text{ traceless} \iff \tilde{f}(\theta) |_{\theta^{(\mu)} \theta^{(\nu)} = \theta^{(\nu)} \theta^{(\mu)} , z^{(1)} = z^{(1)} = 0} ,
\]
\[
\iff f(z) |_{z^{(\mu)} z^{(\nu)} = z^{(\nu)} z^{(\mu)} = 0} .
\]
This means that all terms in the tensor proportional to Kronecker deltas \( \delta^{a_i a_j} \) are discarded. They have to be restored by projection to traceless tensors if one wishes to extract the tensor from the polynomial.

To extract the tensor \( f^{a_1 \ldots a_\lambda} \) back from the polynomials one can simply restore the indices by acting with \( |\lambda\rangle \) derivatives on the polarizations and then project to the irreducible representation \( \lambda \) with the projector \( \pi^{a_1 \ldots a_\lambda | b_1 \ldots b_\lambda} \),
\[
f^{a_1 \ldots a_\lambda} = \pi^{a_1 \ldots a_\lambda | b_1 \ldots b_\lambda} \frac{1}{h_1!} \partial_{\theta_{b_1}}(1) \ldots \frac{1}{h_2!} \partial_{\theta_{b_{h_1}}} \partial_{\theta_{b_{h_1}+1}}(2) \ldots \partial_{\theta_{b_{h_1}+k_2}} \frac{1}{\lambda_1!} \partial_{z_{b_1}^{(1)}}(1) \ldots \partial_{z_{b_1+1}^{(1)}}(1) \ldots \partial_{z_{b_{\lambda}^{(1)}}}(1) f(\theta) ,
\]
\[
= \pi^{a_1 \ldots a_\lambda | b_1 \ldots b_\lambda} \frac{1}{\lambda_1!} \partial_{z_{b_1}^{(1)}}(1) \ldots \partial_{z_{b_1+1}^{(1)}}(1) \ldots \partial_{z_{b_\lambda}^{(1)}}(1) f(z) .
\]
The normalizations can be explained as follows. When extracting the components \( f^{a_1 \ldots a_\lambda} \) from the polynomial \( f(\theta) \) all that happens is the antisymmetrization of a tensor which is already in the antisymmetric basis. For each set of antisymmetric indices every generated term is the same and the normalization factor only has to cancel the number of terms. Going from \( f(z) \) to \( f^{a_1 \ldots a_\lambda} \) involves a Young projection of a tensor that is already Young symmetrized. Therefore the normalization \( \lambda_1(\lambda) \) is that of the Young projectors, which are given in [12]. It is computed from the shape of \( \lambda \) by a hook rule. Write into each box of a Young diagram the number of boxes to its right and below, including the box itself. The product of all numbers is \( \lambda_1(\lambda) \). For example,
\[
\lambda_1(\lambda) = 6 \cdot 4 \cdot 2 \cdot 1 .
\]

As far as we are aware an explicit general formula for the projector \( \pi^{a_1 \ldots a_\lambda | b_1 \ldots b_\lambda} \) is only known for symmetric tensors [14]. For the simplest mixed-symmetry tensor \( \boxdot \) the projector is [12]
\[
\pi^{a_1 a_2 a_3 | b_1 b_2 b_3} = \frac{4}{3} a_1 a_2 a_3 b_1 b_2 b_3 - \frac{2}{d-1} a_1 a_2 a_3 b_1 b_2 b_3 .
\]
Let \( f^{a_1 a_2 a_3} \) and \( g^{b_1 b_2 b_3} \) be two tensors in the irrep \( \boxdot \) and
\[
f(z) = f(z, \theta) = (\theta_{a_1} z_{a_2} z_{a_3} - \theta_{a_2} z_{a_1} z_{a_3}) f^{a_1 a_2 a_3} |_{z^2 = z \theta = 0} ,
\]
\[
g(z) = g(z, \theta) = (\theta_{a_1} z_{a_2} z_{a_3} - \theta_{a_2} z_{a_1} z_{a_3}) g^{a_1 a_2 a_3} |_{z^2 = z \theta = 0} ,
\]
\[
(2.35)
\]
their encoding polynomials. We would like to know how to contract these tensors using
directly the polynomials. The antisymmetrization in the projector (2.34) is already done in
the construction of the polynomials, only the symmetrization and subtraction of the trace is
left to do. This can be done by introducing a differential operator $D_z^a$ that satisfies

$$D_z^{a_1} D_z^{a_2} z^{b_1} z^{b_2} = \frac{1}{4} \left( \frac{2}{3} \left( \delta^{a_1 b_1} \delta^{a_2 b_2} + \delta^{a_1 b_2} \delta^{a_2 b_1} \right) - \frac{2}{d-1} \delta^{a_1 a_2} \delta^{b_1 b_2} \right),$$

(2.36)

where the factor $\frac{1}{4}$ normalizes the antisymmetrizations. $D_z^a$ can be found to be

$$D_z^a = \frac{1}{\sqrt{6}} \left( \frac{\partial}{\partial z^a} - \frac{3}{2(d-1)} z^a \frac{\partial^2}{\partial z \cdot \partial z} \right).$$

(2.37)

The contraction of the two traceless tensors can then be expressed in terms of the encoding
polynomials as

$$f^{a_1 a_2 a_3} g_{a_1 a_2 a_3} = f(D_z, \partial_\theta) g(z, \theta).$$

(2.38)

This is entirely analogous to the situation of symmetric traceless tensors, but now the explicit
form of the projector and corresponding differential operator acting on the polarization vectors
is not known in general. We will assume that there exists for every irrep $\lambda$ a set of differential
operators

$$D_z = \left( D_z^{(1)}, \ldots, D_z^{(n_Z)}, D_{\theta(1)}^{(n_Z)}, \ldots, D_{\theta(n_Z+1)} \right),$$

(2.39)

that reproduces the projector in this way. We have no proof that every projector can be
expressed like this. If nothing else it is a notation that allows us to write any contraction as

$$f^{a_1 \ldots a_\lambda} g_{a_1 \ldots a_\lambda} = f(D_z) g(z).$$

(2.40)

We postpone a more general treatment of the projectors to traceless mixed-symmetry tensors
to a subsequent paper.

### 2.5 Tensors in embedding space

To work out the constrains conformal symmetry imposes on correlation functions of tensor
operators, it is convenient to use the embedding formalism. The idea, which dates back
to Dirac [15], is to lift the problem to the embedding space $\mathbb{M}^{d+2}$ where the conformal
group $SO(d + 1, 1)$ acts linearly as standard Lorentz transformations in $(d + 2)$-dimensional
Minkowski space. Let $P \in \mathbb{M}^{d+2}$ be a point in this embedding space. Points in physical space
are identified with light-rays, i.e. with null vectors in $\mathbb{M}^{d+2}$ up to rescalings,

$$P^2 = 0, \quad P \sim \alpha P \quad (\alpha > 0).$$

(2.41)

Then, a specific choice of conformal frame corresponds to a specific section of the light cone.
In particular, for a CFT on $d$-dimensional Euclidean space $\mathbb{R}^d$, we consider the Poincaré
section of the light-cone

$$P^A = (P^+, P^-, P^a) = (1, x^2, x^a),$$

(2.42)
where we are using light-cone coordinates with metric
\[
P_1 \cdot P_2 = \eta_{AB} P_1^A P_2^B = -\frac{1}{2} (P_1^+ P_2^- + P_1^- P_2^+) + \delta_{ab} P_1^a P_2^b.
\] (2.43)

For example, it is simple to see that the Euclidean distance between two points in \( \mathbb{R}^d \) is written in the embedding space as \(-2P_1 \cdot P_2 = (x_1 - x_2)^2\). It will later be abbreviated by \(P_{ij} \equiv -2P_i \cdot P_j\). In general, \(SO(d + 1, 1)\) Lorentz transformations map the light-cone into itself and, by the identification (2.42), define the action of the conformal group in physical space. A more thorough discussion of the embedding formalism can be seen in [9, 16], whose notation we follow here.

Let us now consider a mixed-symmetry tensor primary field of dimension \(\Delta\). This field will have components \(f^{a_1...a_{|\lambda|}}(x)\) with symmetries given by the Young diagram \(\lambda\). We wish to express it in terms of a field on the embedding space. This new tensor field will have components \(F^{A_1...A_{|\lambda|}}(P)\) with the same symmetries as the physical tensor, it should be defined on the light cone \(P^2 = 0\) and it should be homogeneous of degree \(-\Delta\),
\[
F_{A_1...A_{|\lambda|}}(\alpha P) = \alpha^{-\Delta} F_{A_1...A_{|\lambda|}}(P), \quad \alpha > 0.
\] (2.44)

It should also obey the transversality condition
\[
P^A_i F_{A_1...A_{|\lambda|}} = 0.
\] (2.45)

Components of the physical tensor are then obtained by projecting into physical space by
\[
f_{a_1...a_{|\lambda|}} = \frac{\partial P_{A_1}}{\partial x_{a_1}} \cdots \frac{\partial P_{A_{|\lambda|}}}{\partial x_{a_{|\lambda|}}} F_{A_1...A_{|\lambda|}}.
\] (2.46)

Next we wish to encode the tensor in the embedding space \(F^{A_1...A_{|\lambda|}}(P)\) by a polynomial. The discussion is entirely analogous to that of the previous section, only that now the tensor will be a polynomial \(F(P, Z)\) in the embedding space polarization vectors
\[
Z \equiv \left(Z^{(1)}, Z^{(2)}, \ldots, Z^{(n_Z)}, \Theta^{(1)}\right).
\] (2.47)

Explicitly, the polynomial \(F(P, Z)\) is given by
\[
F(P, Z) \equiv \prod_{p=1}^{n_Z} \prod_{q=1}^{\min(l_p, n_{\Theta})} \left(Z^{(p)} : \partial_{\Theta^{(q)}}\right) \tilde{F}(P, \Theta),
\] (2.48)

where
\[
\tilde{F}(P, \Theta) \equiv \Theta^{(1)}_{A_1} \cdots \Theta^{(1)}_{A_{h_1}} \Theta^{(2)}_{A_{h_1}+1} \cdots \Theta^{(2)}_{A_{h_1}+h_2} \\
\cdots \Theta^{(n_{\Theta})}_{A_{h_1}+\cdots+h_{n_{\Theta}}-1+1} \cdots \Theta^{(n_{\Theta})}_{A_{h_1}+\cdots+h_{n_{\Theta}}} Z^{(1)}_{A_{|\lambda|-\lambda_1+1}} \cdots Z^{(1)}_{A_{|\lambda|}} F^{A_1...A_{|\lambda|}}(P),
\] (2.49)

with
\[
\Theta \equiv \left(\Theta^{(1)}, \Theta^{(2)}, \ldots, \Theta^{(n_{\Theta})}, Z^{(1)}\right).
\] (2.50)
For traceless transverse tensors one can, without loss of information, drop products of any two polarizations or of one polarization and the corresponding embedding space coordinate, i.e.

\[ F^{A_1 \ldots A_{|\lambda|}}(P) \text{ traceless & transverse} \leftrightarrow F^{(p)}(P, \Theta)|_{\Theta^{(p)} \cdot \Theta^{(q)} = \Theta^{(p)} \cdot Z^{(1)} = Z^{(1)} = 0, P = 0} , \]

\[ \leftrightarrow F(P, Z)|_{Z^{(p)} \cdot \Theta^{(1)} = Z^{(p)} \cdot \Theta^{(1)} = 0} , \]  

(2.51)

This means that transverse polynomials satisfy the transversality condition

\[ F(P, Z + cP) = F(P, Z) , \]  

(2.52)

for any set \( c = (c_1, \ldots, c_{n_Z}, \gamma) \) of \( n_Z \) commuting numbers \( c_i \) and one anti-commuting number \( \gamma \).

It is also possible to relate the polynomial \( f(x, z) \) to the embedding polynomial \( F(P, Z) \), as well as \( \tilde{f}(x, \theta) \) to \( \tilde{F}(P, \Theta) \). The procedure is entirely analogous to that described in [9]: in the case of the Poincaré patch where \( P_x = (1, x^2, x) \), each embedding polarization can be written as

\[ Z^{(p)}_{z,x} = \left( 0, 2x \cdot z^{(p)}, \pi^{(p)} \right) \quad \text{and} \quad \Theta^{(p)}_{\theta,x} = \left( 0, 2x \cdot \theta^{(p)}, \theta^{(p)} \right) , \]  

(2.53)

so that the relation between the polynomials is simply

\[ f(x, z) = F(P_x, Z_{z,x}) \quad \text{and} \quad \tilde{f}(x, \theta) = \tilde{F}(P_x, \Theta_{\theta,x}) . \]  

(2.54)

The projector \( \pi_{A_1 \ldots A_{|\lambda|} | B_1 \ldots B_{|\lambda|} }^{a_1 \ldots a_{|\lambda|} b_1 \ldots b_{|\lambda|} } \) to the irrep \( \lambda \) lifts to the projector \( \Pi_{A_1 \ldots A_{|\lambda|} | B_1 \ldots B_{|\lambda|} }^{a_1 \ldots a_{|\lambda|} b_1 \ldots b_{|\lambda|} } \) in embedding space. The only case we need here is when it is inserted between two transverse tensors, since we will always work with polynomials that are transverse.3 In this case \( \Pi_{A_1 \ldots A_{|\lambda|} | B_1 \ldots B_{|\lambda|} }^{a_1 \ldots a_{|\lambda|} b_1 \ldots b_{|\lambda|} } \) is obtained from \( \pi_{A_1 \ldots A_{|\lambda|} | B_1 \ldots B_{|\lambda|} }^{a_1 \ldots a_{|\lambda|} b_1 \ldots b_{|\lambda|} } \) by replacing all Kronecker deltas \( \delta^{a_i b_j} \), \( \delta^{a_i a_j} \) and \( \delta^{b_i b_j} \) by embedding space metrics \( \eta_{A_i B_j}, \eta_{A_i A_j} \) and \( \eta_{B_i B_j} \). This implies that the operators \( D_z \) can also be carried over to embedding space by replacing \( z \) by \( Z \), when they are used between two transverse polynomials. The contraction of two traceless transverse tensors \( F^{A_1 A_2 A_3} \) and \( G^{B_1 B_2 B_3} \) in the irrep \( \mathbb{P} \) is, as in the example (2.38), given by

\[ F^{A_1 A_2 A_3} G_{A_1 A_2 A_3} = F(D_Z, \partial \theta) G(Z, \Theta) , \]  

(2.55)

with

\[ D^A_Z = \frac{1}{\sqrt{6}} \left( \frac{\partial}{\partial Z_A} - \frac{3}{2(d-1)} Z^A \frac{\partial^2}{\partial Z \cdot \partial Z} \right) . \]  

(2.56)

In general, contractions will be written as

\[ F^{A_1 \ldots A_{|\lambda|}} G_{A_1 \ldots A_{|\lambda|}} = F(D_Z) G(Z) . \]  

(2.57)

---

3 The general form of \( \Pi_{\lambda} \) can be obtained analogously as it was done for symmetric tensors in [9].
3 Correlation functions

In this section we address the main kinematic problem that is to be solved when thinking about correlation functions of arbitrary tensor irreps: to count all independent tensor structures.

3.1 Tensor-product coefficients

One part of the problem is finding all the possible ways a given set of mixed-symmetry tensors can be contracted. A more mathematical way to pose this question is to ask for the multiplicity of the scalar representation in the tensor product of the tensors in question. Fortunately, this problem is already solved. Here we shall review the relevant results for our purposes; for a comprehensive introduction to the general properties of tensor-product coefficients see [17].

Let \( G = SU(n), SO(n) \) or \( Sp(n) \) and \( \lambda, \mu, \nu \) irreducible \( G \)-modules which are enumerated by the standard Young tableaux. These are the vector spaces of tensors with the index symmetries described in Section 2.4. They will often be called representations instead of modules in the following. \( \lambda^* \) denotes the vector space dual to \( \lambda \), i.e. if \( \lambda \) contains tensors with lower indices, \( \lambda^* \) contains tensors with upper indices. Upper and lower indices can be contracted and the result will then transform under \( G \) as indicated by the remaining indices.

Let \( N_{\lambda \mu}^\nu \) be the tensor-product coefficients of \( G \). They count the multiplicity with which the irrep \( \nu \) appears in the tensor product of \( \lambda \) and \( \mu \)

\[
\lambda \otimes \mu = \bigoplus_{\nu} N_{\lambda \mu}^\nu \nu,
\]

and satisfy

\[
N_{\lambda}^\mu \nu = \delta_{\lambda}^\nu, \quad N_{\lambda \lambda^*}^\bullet = 1, \quad N_{\lambda \mu}^\nu = N_{\lambda \mu^*}^\nu, \quad \tag{3.2}
\]

where \( \bullet \) denotes the scalar representation. Let \( N_{\lambda \mu \nu} \) denote the multiplicity of the scalar representation in the triple product

\[
\lambda \otimes \mu \otimes \nu = N_{\lambda \mu \nu} \bullet \oplus \text{other irreps}. \quad \tag{3.3}
\]

This notation has the advantage of being symmetric in its three labels and contains the same information due to

\[
N_{\lambda \mu}^\nu = N_{\lambda \mu^*}^\nu. \quad \tag{3.4}
\]

The multiplicity of the scalar representation in products of more than three tensors can be found by recursively using (3.1).

3.1.1 Unitary groups

When specializing to \( G = SU(n) \) the tensor-product coefficients are the famous Littlewood-Richardson coefficients \( c_{\lambda \mu}^\nu \):

\[
N_{\lambda \mu}^\nu = c_{\lambda \mu}^\nu \quad \text{for } G = SU(n). \quad \tag{3.5}
\]
The only allowed contraction in this group is between upper and lower indices, so the number of indices adds up when the tensor product between two tensors with lower indices is formed

\[ c_{\lambda \mu}^{\nu} = 0 \quad \text{for} \quad |\lambda| + |\mu| \neq |\nu|. \]  

(3.6)

This implies that the product of three tensors can only contain the scalar representation if one of them is in a dual representation relative to the other two. This can be illustrated by the following schematic contraction of tensor indices

\[ T_{\lambda} T_{\mu} T_{\nu} \propto c_{\lambda \mu}^{\nu} \bullet. \]  

(3.7)

The coefficients \( c_{\lambda \mu}^{\nu} \) can be calculated using the Littlewood-Richardson rule.\(^4\)

For simple examples one can often find the possible contractions for a given tensor product quickly using birdtracks. For example, one can easily convince oneself that the only two inequivalent ways to contract \( \lambda = \mu = \mathbb{F} \) and \( \nu^* = \mathbb{F}^* \) are

\[ T_{\lambda} T_{\mu} T_{\nu}, \quad (3.8) \]

and

\[ T_{\lambda} T_{\mu} T_{\nu}. \quad (3.9) \]

The Littlewood-Richardson coefficient is thus \( c_{\lambda \mu}^{\nu} = 2 \).

### 3.1.2 Orthogonal and symplectic groups

Following the reasoning of [12], the orthogonal and symplectic groups can be obtained from the unitary groups by accommodating for the fact that these groups have by definition additional group invariants. For \( SO(n) \) this is a symmetric quadratic form \( g_{ab} \) and its inverse \( g^{ab} \), while for \( Sp(n) \) the invariant is skew symmetric \( f_{ab} = -f_{ba} \). In both cases these invariants can be used to raise and lower indices, which implies that the distinction between the two becomes unnecessary. Any two indices can be contracted and this leads to different tensor-product coefficients

\[ N_{\lambda \mu \nu} = b_{\lambda \mu \nu} \quad \text{for} \quad G \in \{ SO(n), Sp(n) \}. \]  

(3.10)

\(^4\)The algorithm has been implemented for instance in Anders Buch’s lrcalc program, which is available at [http://www.math.rutgers.edu/~asbuch/lrcalc/](http://www.math.rutgers.edu/~asbuch/lrcalc/).
It is not hard to convince oneself that the counting of tensor structures here can be broken down to the counting that was relevant in the SU\(_{3n}\) case where the restriction |\(\lambda\) + |\(\mu\)| = |\(\nu\)| applied. The following figure shows how three sets of indices can be contracted with each other, by first dividing each set of indices into two,

\[
\sum_{\rho,\sigma,\gamma} P_\rho T_\lambda P_\sigma T_\mu P_\gamma T_\nu \propto b_{\lambda\mu\nu} \bullet, \quad (3.11)
\]

where P\(_\rho\) is a projector to the irrep \(\rho\), and so on. The number of tensor structures obtained in such a way is

\[
b_{\lambda\mu\nu} = \sum_{\rho,\sigma,\gamma} c_\rho^\lambda c_\sigma^\mu c_\gamma^\nu.
\]

(3.12)

This formula is known as the Newell-Littlewood formula [18, 19] and holds if the sum of the heights of two of the three irreps \(\lambda, \mu\) and \(\nu\) does not exceed \(\left\lfloor \frac{d}{2} \right\rfloor\), i.e. for

\[
h_1^\lambda + h_1^\mu + h_1^\nu - \max(h_1^\lambda, h_1^\mu, h_1^\nu) \leq \left\lfloor \frac{d}{2} \right\rfloor.
\]

(3.13)

Otherwise even the tensor product of the two irreps with the smallest \(h_1\) contains Young diagrams that violate (2.1) and hence do not correspond to irreps of SO(d). In this case (3.12) can be used anyway by transforming these Young diagrams into diagrams that correspond to irreps using modification rules [11] and taking the additional contributions that arise in this way into account. Then also the statement (3.10) that the tensor-product coefficients are the same for SO(2n), SO(2n + 1) and Sp(n) does not hold true anymore. For simplicity, we will assume (3.13) to be satisfied.

A notation that will be used below is the restriction of a tensor product to irreps that have the same number of indices as before. This operation will be denoted with square brackets and amounts to using the SU\(_n\) Littlewood-Richardson coefficients as tensor-product coefficients,

\[
[\lambda \otimes \mu] \equiv \bigoplus_\nu b_{\lambda\mu\nu} \big|_{|\nu|=|\lambda|+|\mu|} = \bigoplus_\nu c_{\lambda\mu}^\nu \nu.
\]

(3.14)

The second equality can be found for instance in [11]. To wrap up this section consider the following example

\[
[\lambda \otimes \mu \otimes \nu] \otimes \rho \otimes \sigma = \sum_{\gamma,\kappa} c_{\lambda\mu}^\gamma c_{\gamma\nu}^\kappa b_{\kappa\rho\sigma} \bullet \oplus \text{other irreps}.
\]

(3.15)
3.2 Two-point functions

Unitary irreducible representations of the conformal group $SO(d+1,1)$ will be labeled by $\chi \equiv [\lambda, \Delta]$, where $\Delta$ is the conformal dimension and $\lambda$ an irreducible representation of $SO(d)$. The two-point function of the primary corresponding to $\chi$ is, up to a normalization constant, a tensor depending on two points in the embedding space with components

$$G^{A_1 \ldots A_{|\lambda|}, B_1 \ldots B_{|\lambda|}}(P_1, P_2).$$

(3.16)

It is encoded, as described above, by a polynomial

$$G_\chi(P_1, P_2; Z_1, Z_2) = \prod_{p=1}^{n_Z} \min(l_p, n_\Theta) \prod_{q=1}^{l_p} \left(Z_1^{(p)} \cdot \partial_{\Theta_1^{(q)}} \right) \left(Z_2^{(p)} \cdot \partial_{\Theta_2^{(q)}} \right) G_\chi(P_1, P_2; \Theta_1, \Theta_2),$$

(3.17)

where

$$G_\chi(P_1, P_2; \Theta_1, \Theta_2) = \Theta_1^{(1)} \ldots \Theta_1^{(l_1)} \ldots Z_1^{(1)} \ldots \Theta_2^{(1)} \ldots \Theta_2^{(l_2)} \ldots Z_2^{(1)} \ldots G^{A_1 \ldots A_{|\lambda|}, B_1 \ldots B_{|\lambda|}}(P_1, P_2).$$

(3.18)

To construct the two-point function one has to find $\bar{G}_\chi$, which is subject to the following conditions. Firstly, it is homogeneous of degree $-\Delta$ in the embedding space coordinates

$$\bar{G}_\chi(\{\alpha_i P_i; \Theta_i\}) = (\alpha_1 \alpha_2)^{-\Delta} \bar{G}_\chi(\{P_i; \Theta_i\}).$$

(3.19)

for $\alpha_i$ arbitrary positive constants. Secondly, it is a polynomial in the polarizations with degrees given by the shape of the Young diagram $\lambda$,

$$\bar{G}_\chi(\{P_i; \beta_i \Theta_i\}) = \left(\beta_1^{(1)} \beta_2^{(1)} \right)^{h_1} \ldots \left(\beta_1^{(n_\Theta)} \beta_2^{(n_\Theta)} \right)^{h_\Theta} \left(\beta_1^{(Z)} \beta_2^{(Z)} \right)^{\lambda_1} \bar{G}_\chi(\{P_i; \Theta_i\}),$$

(3.20)

where we defined

$$\beta_i \Theta_i = \left(\beta_i^{(1)} \Theta_i^{(1)}, \ldots, \beta_i^{(n_\Theta)} \Theta_i^{(n_\Theta)}, \beta_i^{(Z)} Z_i^{(1)} \right),$$

(3.21)

for arbitrary (commuting) constants $\beta_i^{(p)}$.

Finally, $\bar{G}_\chi$ has to be transverse

$$\bar{G}_\chi(\{P_i; \Theta_i + \gamma_i P_i\}) = \bar{G}_\chi(\{P_i; \Theta_i\}).$$

(3.22)

where

$$\gamma_i = \left(\gamma_i^{(1)}, \ldots, \gamma_i^{(n_\Theta)}, c_i \right),$$

(3.23)

is a set of $n_Z$ anticommuting numbers and one commuting number. This last condition has to be satisfied modulo $O(P^2)$ terms. An identically transverse function $\bar{G}_\chi$ can be obtained by dropping terms proportional to $\Theta^{(p)} \cdot \Theta^{(q)}$ and $\Theta^{(p)} \cdot P$, where $p = 1, \ldots, n_\Theta, Z$. Notice
that we are using the notation $\Theta^{(Z)} = Z^{(1)}$ to make equations more compact. We are left to constructing $G^\chi$ from the tensors

$$C_{iAB}^{(p)} = \Theta_{iA}^{(p)} P_{iB} - \Theta_{iB}^{(p)} P_{iA} = \begin{cases} 
\Theta_{iA}^{(p)} P_{iB} - \Theta_{iB}^{(p)} P_{iA}, & p = 1, \ldots, n_\Theta, \\
Z_{iA}^{(1)} P_{iB} - Z_{iB}^{(1)} P_{iA}, & p = Z,
\end{cases} \quad (3.24)$$

with $i = 1, 2$. Contracting two such tensors with the same index $i$ leads to terms of the type that do not appear in transverse functions, so the only possible terms are traces of a string of $C$’s with alternating $i$’s, i.e. of the form

$$\text{Tr} \left( C_1^{(p)} \cdot C_2^{(q)} \cdots C_1^{(r)} \cdot C_2^{(s)} \right), \quad (3.25)$$

the shortest one being

$$H_{ij}^{(p,q)} = \text{Tr} \left( C_i^{(p)} \cdot C_j^{(q)} \right) = 2 \left( (P_j \cdot \Theta_i^{(p)}) (P_i \cdot \Theta_j^{(q)}) - (\Theta_i^{(p)} \cdot \Theta_j^{(q)}) (P_i \cdot P_j) \right). \quad (3.26)$$

Note that both $p$ and/or $q$ can also take the value $Z$, for which case they describe the commuting polarization $Z^{(1)}$. Here and in the following calculations we shall use this notation.

Traces of more than two alternating $C_1$’s and $C_2$’s can always be expressed in terms of $H_{12}^{(p,q)}$. This can be seen by considering

$$\left( C_1^{(p)} \cdot C_2^{(q)} \cdot C_1^{(r)} \cdot C_2^{(s)} \right)_{AB} = \frac{1}{2} \left( C_{1AC}^{(p)} H_{21}^{\{q,r\}} C_{2CB}^{(s)} + P_{1A} P_{2B} R \right), \quad (3.27)$$

where

$$R = \Theta_{1A}^{(p)} H_{21}^{(q,r)} \Theta_{2A}^{(s)} + C_{1AB}^{(p)} \left( \Theta_2^{(q)} \cdot \Theta_1^{(r)} \right) C_{2BA}^{(s)} - \left( \Theta_1^{(p)} \cdot \Theta_2^{(q)} \right) H_{12}^{(r,s)} - H_{12}^{(p,q)} \left( \Theta_1^{(r)} \cdot \Theta_2^{(s)} \right)$$

$$+ 2 (P_1 \cdot P_2) \left[ \Theta_{1A}^{(p)} \left( \Theta_2^{(q)} \cdot \Theta_1^{(r)} \right) \Theta_{2A}^{(s)} - \left( \Theta_1^{(p)} \cdot \Theta_2^{(q)} \right) \left( \Theta_1^{(r)} \cdot \Theta_2^{(s)} \right) \right], \quad (3.28)$$

which satisfies

$$(P_1 \cdot P_2) R = \frac{1}{2} \left( H_{12}^{(p,q)} H_{12}^{(r,s)} - C_{1AB}^{(p)} H_{21}^{\{q,r\}} C_{2BA}^{(s)} \right). \quad (3.29)$$

Using also that

$$\left( P_2 \cdot C_1^{(p)} \cdot C_2^{(q)} \right)_A = \frac{1}{2} H_{12}^{(p,q)} P_{2A}, \quad (3.30)$$

one sees that multiplying (3.27) by any number of factors $C_1 \cdot C_2$ produces only more terms of the same structure that turn into products of $H_{12}$’s when the trace is closed.

Naively one could imagine that the different ways to distribute polarizations among $H_{12}$’s lead to different tensor structures, e.g. for the diagram $\boxplus$ one could consider

$$\left( H_{12}^{(1,1)} H_{12}^{(2,2)} \right)^2, \quad H_{12}^{(1,1)} H_{12}^{(2,2)} H_{12}^{(1,2)} H_{12}^{(2,1)} \quad \text{and} \quad \left( H_{12}^{(1,2)} H_{12}^{(2,1)} \right)^2. \quad (3.31)$$

However, the tensor product of two copies of an irrep contains the scalar representation with multiplicity one, as written in (3.2), so there can be only one tensor structure for each two-point function. Indeed, all possible ways to distribute the polarizations among the $H_{12}$’s lead
to the same result after Young symmetrization (this can be checked explicitly by considering (3.17)). With the weights of coordinates and polarizations being fixed by (3.19) and (3.20), we choose a convenient set of $H_{12}$’s and find that the unique tensor structure for the two-point function is given by (3.17) with

$$G_\chi(P_1, P_2; \Theta_1, \Theta_2) = \frac{1}{(P_{12})^{\Delta+|\lambda|}} \prod_{r=1}^{n_\Theta} \left( H_{12}^{(r,r)} \right)^{h_r} \left( H_{12}^{(Z,Z)} \right)^{\lambda_1}. \quad (3.32)$$

### 3.2.1 Example: $p$-form field

As an example, let us write explicitly the two-point function of a $p$-form field. The Young diagram of a $p$-form field consists of one column of $p$ boxes, therefore $|\lambda| = p$, $n_Z = 0$ and $n_\Theta = 1$. Since there are no rows with more than one box and hence there are no indices to symmetrize, there is no need to introduce commuting polarizations. There is a single anti-commuting polarization vector, which we denote by $\Theta$. The correlation function can be read off from (3.32) to be

$$G_\chi(P_1, P_2; \Theta_1, \Theta_2) = \frac{\left( H_{12}^{(\Theta,\Theta)} \right)^p}{(P_{12})^{\Delta+2p}} \left( \left( \Theta_1 \cdot \Theta_2 \right) - \frac{(P_2 \cdot \Theta_1)(P_1 \cdot \Theta_2)}{P_1 \cdot P_2} \right)^p. \quad (3.33)$$

Then, using the maps (2.53) and (2.54), it is simple to find the polynomial $\bar{g}_\chi(x_1, x_2; \theta_1, \theta_2)$ that describes this tensor structure in physical space.

Note also that, acting with the $\partial_\Theta$ derivatives $\partial_{\Theta_1}^{a_1} \ldots \partial_{\Theta_1}^{a_p} \partial_{\Theta_2}^{b_1} \ldots \partial_{\Theta_2}^{b_p}$, one can write explicitly the components of the tensor in the embedding space as

$$G^{A_1 \ldots A_p, B_1 \ldots B_p}_\chi = \frac{1}{(P_{12})^{\Delta}} \delta_{C_1 \ldots C_p}^{A_1 \ldots A_p} \delta_{D_1 \ldots D_p}^{B_1 \ldots B_p} \prod_{k=1}^p \left( \delta_{C_k D_k} - \frac{P_2 C_k P_1 D_k}{P_1 \cdot P_2} \right), \quad (3.34)$$

whose projection to physical space gives the components

$$g^{a_1 \ldots a_p, b_1 \ldots b_p}_\chi = \frac{1}{(x_{12})^{\Delta}} \delta_{c_1 \ldots c_p}^{a_1 \ldots a_p} \delta_{d_1 \ldots d_p}^{b_1 \ldots b_p} \prod_{k=1}^p \left( \delta_{c_k d_k} - 2 \frac{(x_{12})^{c_k} (x_{12})^{d_k}}{x_{12}} \right), \quad (3.35)$$

where $x_{12} = x_1 - x_2$.

### 3.2.2 Example: Smallest hook diagram

As another example let us consider the irrep corresponding to the diagram $\mathbb{F}$. This is the simplest example where the Young symmetrization operator appears. Here we have $n_Z = 1$ and $n_\Theta = 1$, with polarization vectors $Z = (Z, \Theta)$ and $\Theta = (\Theta, Z)$. Thus, the polynomials encoding the tensor structure for the two-point function of these operators are

$$G_\chi(P_1, P_2; \Theta_1, \Theta_2) = \frac{1}{(P_{12})^{\Delta+3}} \left( H_{12}^{(\Theta,\Theta)} \right)^2 H_{12}^{(Z,Z)}, \quad (3.36)$$

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and
\[ G_\chi(P_1, P_2; Z_1, Z_2) = \left( Z_1 \cdot \partial_{\Theta_1} \right) \left( Z_2 \cdot \partial_{\Theta_2} \right) \tilde{G}_\chi(P_1, P_2; \Theta_1, \Theta_2) = \frac{2}{(P_{12})^{d+3}} \left( H^{(\Theta, \Theta)}_{12}(Z, Z) - H^{(\Theta, Z)}_{12}(Z, \Theta) \right) H^{(Z, Z)}_{12}. \] (3.37)

It is now a simple exercise to derive the components of the physical tensor associated to this polynomial. We shall not pursue this here, and work instead with embedding polynomials.

3.3 Three-point functions

Next we consider the tensor structures allowed in a three-point functions with each operator in the \( SO(d + 1, 1) \) irrep labelled by \( \chi_j \equiv [\lambda_j, \Delta_j] \), for \( j = 1, 2, 3 \). Such three-point function is conveniently written as
\[ G_{\chi_1, \chi_2, \chi_3}(\{P_i, Z_i\}) = \prod_{j=1}^3 \prod_{p=1}^{n_j^l} \prod_{q=1}^{n_j^r} \left( Z_j^{(p)} \cdot \partial_{\Theta_j^{(q)}} \right) \frac{\bar{Q}_{\lambda_1, \lambda_2, \lambda_3}(\{P_i, \Theta_i\})}{(P_{12})^{\tau_1 + \tau_2 - \tau_3} \left( P_{23} \right)^{\tau_2 + \tau_3 - \tau_1} \left( P_{31} \right)^{\tau_3 + \tau_1 - \tau_2}}, \] (3.38)

where \( \tau_i = \Delta_i + |\lambda_i| \). The factor in the denominator was included to give \( \bar{Q}_{\lambda_1, \lambda_2, \lambda_3} \) the same overall weight in embedding space coordinates as in polarizations which simplifies its construction out of building blocks that have the same property. The conditions on \( \bar{Q}_{\lambda_1, \lambda_2, \lambda_3} \) are otherwise analogous to (3.19-3.22), i.e.
\[ \bar{Q}_{\lambda_1, \lambda_2, \lambda_3}(\{\alpha_i P_i; \beta_i(\Theta_i + \gamma_i P_i)\}) = \bar{Q}_{\lambda_1, \lambda_2, \lambda_3}(\{P_i; \Theta_i\}) \prod_i (\alpha_i)^{\lambda_i} \left( \beta_i^{(1)} \right)^{h_i^1} \ldots \left( \beta_i^{(n_i)} \right)^{h_i^{n_i}} \left( \beta_i(Z) \right)^{\lambda_i}. \] (3.39)

In addition to \( H_{ij}^{(p,q)} \), given in (3.26), there is now another building block that can appear in the polynomial \( \bar{Q}_{\lambda_1, \lambda_2, \lambda_3}(\{P_i; \Theta_i\}) \), which is
\[ V^{(p)}_{i,j,k} \equiv \frac{P_j \cdot C_i^{(p)} \cdot P_k}{P_j \cdot P_k} = \frac{P_j \cdot \Theta_i^{(p)} \left( P_i \cdot P_k \right) - \left( P_j \cdot P_i \right) \left( \Theta_i^{(p)} \cdot P_k \right)}{P_j \cdot P_k}. \] (3.40)

Because of the property \( V_{i,j,k} = -V_{i,k,j} \) there is only one independent \( V \) for each operator \( i \). They will be denoted
\[ V_1^{(p)} = V_{1,23}^{(p)}, \quad V_2^{(p)} = V_{2,31}^{(p)}, \quad V_3^{(p)} = V_{3,12}^{(p)}. \] (3.41)

Other terms of the form \( P \cdot C \cdot \ldots \cdot C \cdot P \) can always be expressed in terms of \( V_i^{(p)} \) and \( H_{ij}^{(p,q)} \) due to (3.30). One could imagine that traces of more than two \( C \)'s result in independent terms, but it was proven in [9] that this is not the case. This means that the three-point function can be completely constructed out of \( V_i^{(p)} \) and \( H_{ij}^{(p,q)} \).

Let us first consider the terms in the polynomial \( \bar{Q}_{\lambda_1, \lambda_2, \lambda_3}(\{P_i; \Theta_i\}) \) that are constructed only out of \( H_{ij}^{(p,q)} \)'s. The number of independent tensor structures that can arise from such
terms is given by the tensor-structure coefficients \(b^{\lambda_1,\lambda_2,\lambda_3}\) of \(SO(d)\), which were introduced in Section 3.1.2. We shall denote by \(\bar{W}_{\lambda_1,\lambda_2,\lambda_3}\) the linear combination (with arbitrary coefficients) of these \(b^{\lambda_1,\lambda_2,\lambda_3}\) combinations of the \(H_{ij}^{(p,q)}\)'s that lead to independent tensor structures and scale as in (3.39). Such a function can easily be constructed for any example by constructing terms and checking if they give rise to independent tensor structures after the full Young symmetrization.

As an example consider one of the first combinations of irreps where the Littlewood-Richardson coefficient is larger than one, \(\lambda_1 = \lambda_2 = \Box, \lambda_3 = \Box\). The corresponding Littlewood-Richardson coefficient is \(b^{\Box\Box\Box} = 2\). Indeed, there are two combinations of the \(H_{ij}^{(p,q)}\) that lead to different tensor structures

\[
(Z_1 \cdot \partial_{\Theta_1}) (Z_2 \cdot \partial_{\Theta_2}) \left( H_{12}^{(\Theta,\Theta)} \right)^2 H_{13}^{(Z,Z)} H_{23}^{(Z,Z)} = 2 \left( H_{12}^{(Z,Z)} H_{12}^{(\Theta,\Theta)} - H_{12}^{(\Theta,Z)} H_{12}^{(Z,\Theta)} \right) H_{13}^{(Z,Z)} H_{23}^{(Z,Z)},
\]

\[
(Z_1 \cdot \partial_{\Theta_1}) (Z_2 \cdot \partial_{\Theta_2}) H_{12}^{(Z,Z)} H_{12}^{(\Theta,\Theta)} H_{13}^{(\Theta,Z)} H_{13}^{(\Theta,Z)} = H_{12}^{(Z,Z)} \left[ \left( H_{12}^{(\Theta,\Theta)} H_{13}^{(Z,Z)} - H_{12}^{(\Theta,Z)} H_{13}^{(\Theta,Z)} \right) H_{23}^{(Z,Z)} \right.
\]

\[
\left. + \left( H_{12}^{(Z,Z)} H_{13}^{(\Theta,\Theta)} - H_{12}^{(\Theta,Z)} H_{13}^{(\Theta,Z)} \right) H_{23}^{(\Theta,Z)} \right].
\]

Thus, we conclude that for this example

\[
\bar{W}_{\Box\Box\Box} = C_1 \left( H_{12}^{(\Theta,\Theta)} \right)^2 H_{13}^{(Z,Z)} H_{23}^{(Z,Z)} + C_2 H_{12}^{(Z,Z)} H_{12}^{(\Theta,\Theta)} H_{13}^{(\Theta,Z)} H_{23}^{(\Theta,Z)},
\]

with \(C_1\) and \(C_2\) constants.

Next we describe how to construct the general terms containing both \(H_{ij}^{(p,q)}\)'s and \(V_i^{(p)}\)'s. A given term may have an arbitrary number of \(V_i^{(p)}\)'s. However, since for \(p \in \{1, \ldots, n_\Theta\}\) the \(V_i^{(p)}\) are linear in the Grassmann variables and inherit their property,

\[
V_i^{(p)} V_j^{(q)} = (-1)^{\delta pq \delta ij} V_j^{(q)} V_i^{(p)}, \quad p, q \in \{1, \ldots, n_\Theta\},
\]

each \(V_i^{(p)}\) can appear only once in a given term.\(^5\) Next, for each \(V_i^{(p)}\) in a given set of \(V_i^{(p)}\)'s, we remove a box from the \(p\)-th column of the Young diagram \(\lambda_i\). Since each \(V_i^{(p)}\) can only appear once, at most one box can be removed from each column of the Young diagram. To illustrate this, the boxes that may be removed from the following Young diagram are shaded

\[
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\]

\[\text{.}\]

It is enough to consider the cases where the result after removing boxes is again a Young diagram, i.e. to remove only boxes that do not have a remaining box to their right. This\(^5\)A simple corollary is the well-known fact that two scalar operators couple only to fully symmetric representations.
is due to the symmetry of mixed-symmetry tensors under exchange of columns of the same height in the corresponding Young tableau (2.19).

Thus, a given set of $V_i^{(p)}$'s is associated with three new representations denoted by $\tilde{\lambda}_i$. Finally, this set of $V_i^{(p)}$'s should be multiplied by $\tilde{W}_{\tilde{\lambda}_1, \tilde{\lambda}_2, \tilde{\lambda}_3}$, the polynomial of $H_{ij}^{(p,q)}$'s constructed from the corresponding reduced representations $\tilde{\lambda}_i$. However, the representations $\tilde{\lambda}_i$ are not necessarily irreducible. If the $\tilde{\lambda}_i$ are irreducible representations they each correspond to a single Young diagram and $\tilde{W}_{\tilde{\lambda}_1, \tilde{\lambda}_2, \tilde{\lambda}_3}$ consists of $b^{\tilde{\lambda}_1} \cdot \tilde{\lambda}_2 \cdot \tilde{\lambda}_3$ tensor structures. Next we explain how to find the representations $\tilde{\lambda}_i$ and count the tensor structures in the general case.

After boxes have been removed from a Young diagram, there are two types of distinguishable columns, the ones where a box has been removed and the untouched ones. An example would be

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The symmetry that is implied by having columns in the same Young diagram is no longer existent. To perform the counting of tensor structures, this has to be taken into account by splitting the Young diagram into two, each containing one type of columns. When taking the tensor product of these two Young diagrams, there is the special requirement that no indices are contracted between the two, since that would amount to taking contractions between polarizations that belong to the same irrep. This means that the restricted tensor product introduced in (3.14) has to be used

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Removal of boxes in this way can be parametrized by defining a subtraction from the Dynkin label

$$
\tilde{\lambda} \equiv \lambda - m = \left[ \lambda_1 - m_1, \lambda_2 - m_2, \ldots, \lambda_{h_{\lambda}} - m_{h_{\lambda}} \right] \otimes [m_2, m_3, \ldots, m_{h_{\lambda}}]
$$

$$
\equiv \left[ (\lambda - m)' \otimes [m]' \right],
$$

(3.48)

where

$$
m = (m_1, \ldots, m_{h_{\lambda}}), \quad m_u \in \{0, 1, \ldots, \lambda_u\},
$$

(3.49)

and $m_u$ counts how many boxes are removed from columns of height $u$. In this notation the example in (3.47) becomes

$$
[2, 2, 1] - (0, 1, 1) = \left[ [2, 1] \otimes [1, 1] \right]
$$

$$
= [4, 0, 1] \oplus [3, 2] \oplus [3, 0, 0, 1] \oplus 2 \cdot [2, 1, 1] \oplus [1, 3] \oplus [1, 1, 0, 1] \oplus [1, 0, 2] \oplus [0, 2, 1].
$$

(3.50)
We start with a simple example of a two-form, a vector and a scalar,

\[ \sum_{m} \equiv \sum_{m_{1}=0}^{\lambda} \sum_{m_{2}=0}^{\lambda_{1}} \sum_{m_{3}=0}^{\lambda_{2}} \ldots \sum_{m_{n}^{i}=0}^{\lambda_{n}^{i}}. \]  

(3.51)

All this means that \( \tilde{Q}_{\lambda_{1},\lambda_{2},\lambda_{3}} \) is of the form

\[ \tilde{Q}_{\lambda_{1},\lambda_{2},\lambda_{3}} = \sum_{m_{1},m_{2},m_{3}} 3! \prod_{i=1}^{3} \left( V_{i}^{(l_{i}^{u}+1-m_{u}^{i})} \ldots V_{i}^{(l_{i}^{u}-1)} V_{i}^{(l_{i}^{u})} \right) \left( V_{i}^{(Z)} \right)^{m_{1}^{i}} \tilde{W}_{\lambda_{1},\lambda_{2},\lambda_{3}}, \]  

(3.52)

where we recall that \( \tilde{\lambda}_{i} = \lambda_{i} - m^{i} \). The problem of finding the number of independent tensor structures in \( \tilde{W}_{\lambda_{1},\lambda_{2},\lambda_{3}} \) reduces to finding the multiplicity of the scalar representation in the tensor product

\[ \left( \lambda_{1} - m^{1} \otimes \lambda_{2} - m^{2} \otimes \lambda_{3} - m^{3} \right) = \left[ (\lambda_{1} - m^{1})' \otimes [m^{1}]' \right] \otimes \left[ (\lambda_{2} - m^{2})' \otimes [m^{2}]' \right] \otimes \left[ (\lambda_{3} - m^{3})' \otimes [m^{3}]' \right]. \]  

(3.53)

The number of tensor structures in (3.52) is thus

\[ \sum_{m_{1},m_{2},m_{3}} \sum_{\mu,\nu,\rho} c_{(\lambda_{1} - m^{1})',[m^{1}]',[m^{1}]'}^{\mu} c_{(\lambda_{2} - m^{2})',[m^{2}]',[m^{2}]'}^{\nu} c_{(\lambda_{3} - m^{3})',[m^{3}]',[m^{3}]'}^{\rho} \delta^{\mu\nu\rho}. \]  

(3.54)

Next we analyze some examples.

### 3.3.1 Example: Two-form-Vector-Scalar

We start with a simple example of a two-form, a vector and a scalar, \( \lambda_{1} = \Box, \lambda_{2} = \mathbf{v}, \lambda_{3} = \bullet \). As already explained for the two-point function of a \( p \)-form, there is no need to introduce commuting polarizations for the two-form. Also, for the vector, there is obviously no need to introduce any symmetrization or antisymmetrization. It has \( n_{Z_{2}} = n_{\Theta_{2}} = 0 \), therefore one can freely choose whether to use \( Z_{2} \) or \( \Theta_{2} \) as polarization. In this case the only possible tensor structure has \( m^{1} = (0,1), m^{2} = m^{3} = (0) \), so the three-point function is of the form

\[ G_{\lambda_{1},\lambda_{2},\lambda_{3}}(P_{1},P_{2},P_{3};\Theta_{1},Z_{2}) = \frac{V_{1}(\Theta) H_{12}^{(\Theta,Z)}}{(P_{12})^{\Delta_{1}+\Delta_{2}+\Delta_{3}+3} / 2 (P_{23})^{\Delta_{2}+\Delta_{1}+\Delta_{3}+1} / 2 (P_{31})^{\Delta_{3}+\Delta_{1}+\Delta_{2}+1} / 2} \]  

(3.55)

\[ = -4 \left( (P_{2} \cdot \Theta_{1})(P_{1} \cdot P_{3}) - (P_{2} \cdot P_{1})(\Theta_{1} \cdot P_{3}) \right) \left( (P_{2} \cdot \Theta_{1})(P_{1} \cdot Z_{2}) - (\Theta_{1} \cdot Z_{2})(P_{1} \cdot P_{2}) \right) \]  

(3.56)

\[ = \frac{(-4)(P_{2} \cdot \Theta_{1})(P_{1} \cdot P_{3}) - (P_{2} \cdot P_{1})(\Theta_{1} \cdot P_{3})}{(P_{12})^{\Delta_{1}+\Delta_{2}+\Delta_{3}+3} / 2 (P_{23})^{\Delta_{2}+\Delta_{1}+\Delta_{3}+1} / 2 (P_{31})^{\Delta_{3}+\Delta_{1}+\Delta_{2}+1} / 2}. \]

It is now a mechanical computation to act on this polynomial with the derivatives \( \partial_{\Theta_{1}} \partial_{\Theta_{1}} \partial_{Z_{2}} \) to obtain the components of the corresponding tensor in the embedding space.
3.3.2 Example: Two-form-Vector-Vector

Next we consider the three-point function of a two-form and two vectors, \( \lambda_1 = \mathcal{F} \), \( \lambda_2 = \lambda_3 = \mathcal{E} \). In this case there are three possible tensor structures that can be created,

\[
\begin{align*}
  m^1 &= (0, 0), \quad m^2 = m^3 = (0) \quad \rightarrow \quad H_{12}^{(\Theta, Z)} H_{13}^{(\Theta, Z)}, \\
  m^1 &= (0, 1), \quad m^2 = (1), \quad m^3 = (0) \quad \rightarrow \quad V_1^{(\Theta)} V_2^{(Z)} H_{13}^{(\Theta, Z)}, \\
  m^1 &= (0, 1), \quad m^3 = (1), \quad m^2 = (0) \quad \rightarrow \quad V_1^{(\Theta)} V_3^{(Z)} H_{12}^{(\Theta, Z)}.
\end{align*}
\]

This leads to the three-point function

\[
G_{\chi_1, \chi_2, \chi_3}(\{P_i\}; \Theta_1, Z_2, Z_3) = \frac{C_1 H_{12}^{(\Theta, Z)} H_{13}^{(\Theta, Z)} + C_2 V_1^{(\Theta)} V_2^{(Z)} H_{13}^{(\Theta, Z)} + C_3 V_1^{(\Theta)} V_3^{(Z)} H_{12}^{(\Theta, Z)}}{(P_{12})^{\frac{\Delta_1 + \Delta_2 - \Delta_3 + 2}{2}} (P_{23})^{\frac{\Delta_2 + \Delta_3 - \Delta_1}{2}} (P_{31})^{\frac{\Delta_3 + \Delta_1 - \Delta_2 + 2}{2}}},
\]
with \( C_1 \), \( C_2 \) and \( C_3 \) constants.

3.3.3 Example: Hook-Scalar-Vector

The polynomial that encodes the correlator of a small hook diagram \( \lambda_1 = \mathcal{F} \), a scalar \( \lambda_2 = \bullet \) and a vector \( \lambda_3 = \mathcal{E} \) consists of a single tensor structure. Recall that for the small hook diagram we have \( n_{Z_1} = 1 \) and \( n_{\Theta_1} = 1 \), with polarization vectors \( Z_1 = (Z_1, \Theta_1) \) and \( \Theta_1 = (\Theta_1, Z_1) \), so the tensor structure is obtained by acting with a derivative \( Z_1 \cdot \partial_{\Theta_1} \) on a polynomial of the \( V_i^{(p)} \)'s and \( H_i^{(p,q)} \)'s. In this case the single tensor structure has the form

\[
G_{\chi_1, \chi_2, \chi_3}(\{P_i\}; Z_1, Z_3) = \frac{(Z_1 \cdot \partial_{\Theta_1}) V_1^{(\Theta)} V_1^{(Z)} H_{13}^{(\Theta, Z)}}{(P_{12})^{\frac{\Delta_1 + \Delta_2 - \Delta_3 + 2}{2}} (P_{23})^{\frac{\Delta_2 + \Delta_3 - \Delta_1}{2}} (P_{31})^{\frac{\Delta_3 + \Delta_1 - \Delta_2 + 2}{2}}} - \left( V_1^{(Z)} \right)^2 H_{13}^{(\Theta, Z)}.
\]

3.3.4 Example: Hook-Spin 2-Vector

Let us now consider an example that involves the split of a Young diagram as in (3.47). We consider the case \( \lambda_1 = \mathcal{F} \), \( \lambda_2 = \mathcal{R} \), \( \lambda_3 = \mathcal{E} \). With the shorthand \( \check{\lambda} = \lambda - m^i \), Table 1 contains all possible tensor structures in this case. In this example all tensor-product coefficients \( b^{\mu \nu \rho} \) and \( c_{\mu \nu \rho} \) are either 0 or 1. The third line in the table contains two independent factors in \( \tilde{W}_{\check{\lambda}_1, \check{\lambda}_2, \check{\lambda}_3} \) because

\[
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\end{array} - (0, 1) = \begin{array}{c}
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\square \otimes \square \\
\end{array} = \begin{array}{c}
\begin{array}{c}
\square \oplus \square \\
\end{array}
\end{array},
\end{array}
\]

and the resulting irreps can both be tensored with two copies of \( \mathcal{E} \) to form a scalar.

3.4 Four-point functions

Starting from four-point functions, correlation functions can depend on functions of the conformally invariant cross-ratios. For four points, there are two cross-ratios that can be defined to be

\[
u = \frac{P_{12} P_{34}}{P_{13} P_{24}}, \quad \nu = \frac{P_{14} P_{23}}{P_{13} P_{24}}.
\]
Then a generic four-point function can be written as
\[
G_{\lambda_1,\lambda_2,\lambda_3,\lambda_4}(\{P_i, Z_i\}) = \left(\frac{P_{32}}{P_{13}}\right)^{\tau_1 \cdot \tau_2} \left(\frac{P_{34}}{P_{14}}\right)^{\tau_3 \cdot \tau_4} \times \\
\prod_{j=1}^{4} \prod_{p=1}^{n_j} \prod_{q=1}^{\min(l_p, n_{q})} \left(\frac{Z_j^{(p)} \cdot \partial_{z_j^{(p)}}}{(P_{12})^{\frac{1}{2}} (P_{34})^{\frac{1}{2}}}ight) \sum_{k} f_k(u, v) Q^{(k)}_{\lambda_1,\lambda_2,\lambda_3,\lambda_4}(\{P_i, \Theta_j\}),
\]
where \(\tau_i = \Delta_i + |\lambda_i|\) and the sum over \(k\) runs over all independent tensor structures. Each tensor structure is multiplied by a function of the cross-ratios \(f_k(u, v)\) and the pre-factor is chosen in such a way that each \(Q^{(k)}_{\lambda_1,\lambda_2,\lambda_3,\lambda_4}\) scales analogously to (3.39),
\[
Q^{(k)}_{\lambda_1,\lambda_2,\lambda_3,\lambda_4}(\{\alpha_i P_i; \beta_i(\Theta_j + \gamma_i P_i)\})
\]
\[
= \tilde{Q}^{(k)}_{\lambda_1,\lambda_2,\lambda_3,\lambda_4}(\{P_i; \Theta_j\}) \prod_i (\alpha_i)^{|\lambda_i|} (\beta_i^{(1)})^{n_i_1} \cdots (\beta_i^{(n_{i_1})})^{n_i_{1}} (\beta_i^{(Z)})^{(\lambda_i)_1}.
\]

For an \(n\)-point function, \(n - 2\) of the building blocks \(V^{(p)}_{i,j,k}\) are linearly independent [9]. In the case of \(n = 4\), this is due to the relations \(V^{(p)}_{i,j,k} = -V^{(p)}_{i,k,j}\) and
\[
(P_2 \cdot P_3)(P_1 \cdot P_4)V^{(p)}_{1,2,3} + (P_3 \cdot P_4)(P_1 \cdot P_2)V^{(p)}_{1,3,4} + (P_4 \cdot P_2)(P_1 \cdot P_3)V^{(p)}_{1,4,2} = 0.
\]
In general one can choose for instance the basis
\[
\mathcal{V}^{(p)}_{ij} \equiv V^{(p)}_{i,(i+1)j}, \quad j \in \{i + 2, i + 3, \ldots, i + n - 1\},
\]
where the external point labels \(i\) and \(j\) are meant to be interpreted modulo \(n\).

The possibility to have two \(V^{(p)}_{i,j,k}\) building blocks for each combination of \(i\) and \(p\) implies that the procedure of removing boxes from Young diagrams described in the previous section
has to be iterated. To this end, let $m^{ij}$ count the number of $v_{ij}^{(p)}$ building blocks, as $m^i$ did count $v_i^{(p)}$ s before. The number of independent tensor structures is given by the multiplicity of the scalar representation in the tensor product

$$
\left( \sum_{m^{14}} \left( \sum_{m^{13}} \lambda_1 - m^{13} \right) - m^{14} \right) \otimes \left( \sum_{m^{21}} \left( \sum_{m^{24}} \lambda_2 - m^{24} \right) - m^{21} \right) \otimes \left( \sum_{m^{32}} \left( \sum_{m^{31}} \lambda_3 - m^{31} \right) - m^{32} \right) \otimes \left( \sum_{m^{43}} \left( \sum_{m^{42}} \lambda_4 - m^{42} \right) - m^{43} \right),
$$

where the definition of the $m$ sums over a representation is given in (3.51). Here the upper bounds of the outer sums are not specified because the first subtraction of an $m$ may lead to a direct product of irreps. The outer sums should then be done accordingly. For example, if the first subtraction by $m^{13}$ results in a direct sum of irreps $\mu$ that appear with multiplicity $n_{\mu}$,

$$
\sum_{m^{13}} \lambda_1 - m^{13} = \bigoplus_{\mu} n_{\mu} \mu,
$$

the sum over $m^{14}$ in (3.65) has to be

$$
\sum_{m^{14}} \left( \sum_{m^{13}} \lambda_1 - m^{13} \right) - m^{14} = \bigoplus_{\mu} n_{\mu} \sum_{m^{14}} \mu - m^{14}.
$$

### 3.4.1 Example: Scalar-Vector-Scalar-Vector

As an example, table 2 lists the tensor structures in a four-point function of irreps $\lambda_1 = \lambda_3 = \bullet$ and $\lambda_2 = \lambda_4 = \square$, which were already given in [9, 16].

| $m^{24}$ | $m^{21}$ | $m^{42}$ | $m^{43}$ | tensor structure |
|----------|----------|----------|----------|-----------------|
| (0)      | (0)      | (0)      | (0)      | $H_{24}^{(Z,Z)}$ |
| (0)      | (1)      | (0)      | (1)      | $\gamma_{24}^{(Z)}$ |
| (0)      | (1)      | (1)      | (0)      | $\gamma_{42}^{(Z)}$ |
| (1)      | (0)      | (0)      | (1)      | $\gamma_{24}^{(Z)}$ |
| (1)      | (0)      | (1)      | (0)      | $\gamma_{42}^{(Z)}$ |

Table 2. All tensor structures in a four-point function of irreps $\bullet, \square$. 

### 3.4.2 Example: Hook-Vector-Scalar-Scalar

This example involves the split of a Young diagram during the first subtraction of an $m$. To see how this works, let us consider for the irreps $\lambda_1 = \overline{1}, \lambda_2 = \square, \lambda_3 = \lambda_4 = \bullet$, only the case $m^{13} = (0,1)$ and $m^{24} = 0$, for which the remainder of the hook diagram splits, as in (3.59), into $\overline{\square}$ and $\overline{\bullet}$. Table 3 shows how iterating the counting algorithm leads to three tensor structures.
Table 3. Tensor structures with $m_{13} = (0, 1), m_{24} = 0$ in a four-point function of irreps $\mathcal{F}, \Box, \bullet, \bullet$.

3.4.3 Example: Vector-Vector-Vector-Vector

Finally, the correlation function of four vectors is an example that illustrates how the tensor product generates the number of possible contractions between $H$’s. To this end, consider only the tensor structures for $m_{ij} = 0$. The number of such tensor structures is calculated using the $SO(d)$ tensor product

$$
\Box \otimes \Box \otimes \Box \otimes \Box = (\Box \oplus \Box \oplus \bullet) \otimes (\Box \oplus \Box \oplus \bullet) = 3 \bullet \oplus \text{other irreps}.
$$

Correspondingly, there are three tensor structures that can be built out of $H$’s, namely

$$
H_{12}^{(Z,Z)} H_{34}^{(Z,Z)}, \quad H_{13}^{(Z,Z)} H_{24}^{(Z,Z)} \quad \text{and} \quad H_{14}^{(Z,Z)} H_{23}^{(Z,Z)}.
$$

Of course there are $3!2^2$ other structures with two $V$’s and one $H$ and $2^4$ with four $V$’s. Thus in this case there are 43 independent tensor structures.

3.5 $n$-point functions

Let us comment briefly on the general construction of $n$-point functions. It is analogous to the construction of four-point functions. Generically one can write,

$$
G_{\chi_1, \ldots, \chi_n} (\{P_i; Z_i\}) = \prod_{g<h} P_{gh}^{-\alpha_{gh}} \prod_{j=1}^{n} \prod_{p=1}^{n} \prod_{q=1}^{\min(p,n)} (Z_{ij}^{(p)} \cdot \partial^{(q)}_{ij}) \sum_{k} f_k(u_a) \tilde{Q}^{(k)}_{\chi_1, \ldots, \chi_n} (\{P_i; \Theta_i\}),
$$

where $u_a$ are the $n(n-3)/2$ independent conformally invariant cross-ratios,

$$
\alpha_{gh} = \frac{\tau_g + \tau_h}{n-2} - \frac{1}{(n-1)(n-2)} \sum_{i=1}^{n} \tau_i,
$$

and the pre-factor is again chosen to let the functions $\tilde{Q}^{(k)}_{\chi_1, \ldots, \chi_n}$ scale in the already familiar way

$$
\tilde{Q}^{(k)}_{\chi_1, \ldots, \chi_n} (\{\alpha_i P_i; \beta_i (\Theta_i + \gamma_i P_i)\})
= \tilde{Q}^{(k)}_{\chi_1, \ldots, \chi_n} (\{P_i; \Theta_i\}) \prod_{i} (\alpha_i)^{h_i} \beta_i^{(n_i)} (\beta_i (Z))^{(h_i)}.
$$

These functions are again constructed from the building blocks $H_{ij}^{(p)}$ and $V_{ij}^{(p)}$ defined in (3.26) and (3.64). The counting of tensor structures is done as described for four-point functions.
in the previous section, but now the removal of boxes from the Young diagrams has to be iterated \( n - 2 \) times, since there are that many independent \( V_{ij}^{(p)} \) building blocks for each \( i \).

4 Conformal blocks

In this section we shall show how the above methods can be used to compute the conformal blocks for arbitrary irreducible tensor representations of the conformal group. The basic idea is that a conformal block in the channel \( \mathcal{O}_1 \mathcal{O}_2 \to \mathcal{O}_3 \mathcal{O}_4 \), can be written as a conformal integral of the product of the 3-point function of the operators \( \mathcal{O}_1, \mathcal{O}_2 \) and the exchanged operator \( \mathcal{O} \) of dimension \( \Delta \), times the 3-point function of the operators \( \mathcal{O}_3, \mathcal{O}_4 \) and the shadow of the exchanged operator \( \tilde{\mathcal{O}} \) of dimension \( d - \Delta \) [14, 20, 21]. This method makes use of the shadow formalism of [22–25]. In practice however one needs to remove from the final expression the contribution of the shadow operator exchange to the conformal block, which has the wrong OPE limit. This can be done rather efficiently by doing a monodromy projection of the above conformal integral, as proposed in [10].

Conformal blocks are known for many cases involving external scalar operators and the exchange of spin \( l \) symmetric tensors. These results can be reused for correlators of external spin \( l \) operators by acting with differential operators on the conformal blocks for external scalars [27], but new exchanged tensor representations can not be taken care of in this way. Here we will follow closely the approach detailed in [10] to compute the conformal blocks, and show with a non-trivial example that the embedding methods here presented can be used to compute conformal blocks with external and exchanged operators in an arbitrary tensor representation of the conformal group.

The idea is to define a projector to the conformal multiplet of a given operator which, when inserted into a four-point function, produces the conformal partial wave for the exchange of that operator (and its descendants). For an operator \( \mathcal{O} \) with conformal dimension \( \Delta \) this projector has the form

\[
|\mathcal{O}| = \frac{1}{N_\mathcal{O}} \int D^d P_0 D^d P_5 |\mathcal{O}(P_0, DZ_0)\rangle \langle \mathcal{O}(P_0, Z_0)\mathcal{O}(P_5, DZ_5)\rangle|_{\Delta \to \tilde{\Delta}} \langle \mathcal{O}(P_5, Z_5)\rangle.
\]  

(4.1)

Note that we are schematically representing the index contraction of \( \mathcal{O} \) with a differential operator acting on the polarization vectors, as explained in (2.57). The integrals appearing here are called conformal integrals and defined as

\[
\int D^d P = \frac{1}{\text{Vol} \ GL(1, \mathbb{R})^+} \int_{P^+ + P^- \geq 0} d^{d+2} P \, \delta(P^2).
\]

(4.2)

Explicit solutions of these integrals are known for all functions that appear in the computation of conformal blocks.

\[\text{Such split of the operator and its shadow exchanges can also be done using the Mellin space representation of the conformal partial wave [26].}\]
The projector (4.1) can be more compactly expressed in terms of the shadow operator $\tilde{\mathcal{O}}$ which is in the same $SO(d)$ irrep as $\mathcal{O}$ and has conformal dimension $\tilde{\Delta} = d - \Delta$,

$$|\mathcal{O}| = \frac{1}{N_\mathcal{O}} \int D^d P_0 \langle \mathcal{O}(P_0, D Z_0) \rangle \langle \tilde{\mathcal{O}}(P_0, Z_0) \rangle ,$$

(4.3)

where

$$\langle \tilde{\mathcal{O}}(P_0, Z_0) \rangle = \int D^d P_5 \langle \mathcal{O}(P_0, Z_0) \mathcal{O}(P_5, D Z_5) \rangle |_{\Delta \to \tilde{\Delta}} \langle \mathcal{O}(P_5, Z_5) \rangle .$$

(4.4)

Consider for simplicity the case where the three-point functions have only one tensor structure. Inserting $|\mathcal{O}|$ into a four-point function one obtains the conformal partial wave $W_\mathcal{O}$

$$W_\mathcal{O} = \langle \mathcal{O}_1(P_1, Z_1) \mathcal{O}_2(P_2, Z_2) | \mathcal{O} \mathcal{O}_3(P_3, Z_3) \mathcal{O}_4(P_4, Z_4) \rangle$$

$$= \frac{1}{N_\mathcal{O}} \int D^d P_0 \langle \mathcal{O}_1(P_1, Z_1) \mathcal{O}_2(P_2, Z_2) \mathcal{O}(P_0, D Z_0) \rangle \langle \tilde{\mathcal{O}}(P_0, Z_0) \mathcal{O}_3(P_3, Z_3) \mathcal{O}_4(P_4, Z_4) \rangle .$$

(4.5)

Since $\tilde{\mathcal{O}}$ is in the same $SO(d)$ irrep as $\mathcal{O}$, three-point functions containing either of them must be equal, up to an overall constant and to the conformal dimensions of the operators, i.e.

$$\langle \tilde{\mathcal{O}}(P_0, Z_0) \mathcal{O}_3(P_3, Z_3) \mathcal{O}_4(P_4, Z_4) \rangle = S_\Delta \langle \mathcal{O}(P_0, Z_0) \mathcal{O}_3(P_3, Z_3) \mathcal{O}_4(P_4, Z_4) \rangle |_{\Delta \to \tilde{\Delta}} .$$

(4.6)

This constant $S_\Delta$ is calculated by using the definition of the shadow operator (4.4) and by computing the corresponding conformal integral. The constant $N_\mathcal{O}$ in (4.5) can then be calculated by demanding that $|\mathcal{O}|$ acts trivially when inserted into a three-point function, i.e. requiring

$$\langle \mathcal{O}(P_0, Z_0) | \mathcal{O} \mathcal{O}_3(P_3, Z_3) \mathcal{O}_4(P_4, Z_4) \rangle = \langle \mathcal{O}(P_0, Z_0) \mathcal{O}_3(P_3, Z_3) \mathcal{O}_4(P_4, Z_4) \rangle .$$

(4.7)

Using (4.3) and (4.4) one sees that this insertion amounts to doing the shadow transformation twice, hence with (4.6) we have

$$\langle \mathcal{O}(P_0, Z_0) | \mathcal{O} \mathcal{O}_3(P_3, Z_3) \mathcal{O}_4(P_4, Z_4) \rangle = \frac{1}{N_\mathcal{O}} \langle \tilde{\mathcal{O}}(P_0, Z_0) \mathcal{O}_3(P_3, Z_3) \mathcal{O}_4(P_4, Z_4) \rangle$$

$$= \frac{S_\Delta S_{\tilde{\Delta}}}{N_\mathcal{O}} \langle \mathcal{O}(P_0, Z_0) \mathcal{O}_3(P_3, Z_3) \mathcal{O}_4(P_4, Z_4) \rangle ,$$

(4.8)

and thus $N_\mathcal{O} = S_\Delta S_{\tilde{\Delta}}$.

### 4.1 Example: Hook diagram exchange

As an example we will compute the conformal block $g^\Delta_T(u, v)$ for the exchange of the tensor $T$ of the irreducible representation $[\mathbf{F}, \Delta]$ in the correlation function of two scalars and two vectors $\langle \phi_1 J_2^\mu \phi_3 J_4^\nu \rangle$. The conformal partial wave is

$$W_T = \left( \begin{array}{c} P_{14} \\ P_{13} \end{array} \right) \frac{\Delta_{12}}{\Delta_{14}} \left( \begin{array}{c} P_{24} \\ P_{14} \end{array} \right) \frac{\Delta_{24}}{\Delta_{23}} \frac{g^\Delta_T(u, v)}{P_{12}^{\Delta_1 + \Delta_2} P_{34}^{\Delta_3 + \Delta_4}}$$

$$= \langle \phi_1 (P_1) J_2 (P_2, Z_2) | T | \phi_3 (P_3) J_4 (P_4, Z_4) \rangle$$

$$= \frac{1}{S_{\tilde{\Delta}}} \int D^d P_0 \langle \phi_1 (P_1) J_2 (P_2, Z_2) T (P_0, D Z_0, \partial \Theta_0) \rangle \langle T (P_0, Z_0, \Theta_0) \phi_3 (P_3) J_4 (P_4, Z_4) \rangle |_{\Delta \to \tilde{\Delta}} ,$$

(4.9)
where we recall that $u, v$ are the cross ratios defined in (3.60). The ingredients for this calculation are the two- and three-point functions from (3.37) and (3.58), for which we choose the normalizations

$$\langle T(P_1, Z_1, \Theta) T(P_2, Z_2, \Theta) \rangle = \frac{2 \left( H_{12}^{(\Theta, \Theta)} H_{12}^{(Z, Z)} H_{12}^{(Z, Z)} - H_{12}^{(\Theta, Z)} H_{12}^{(Z, \Theta)} H_{12}^{(Z, Z)} \right)}{(P_{12})^{\Delta + 3}},$$

(4.10)

$$\langle T(P_0, Z_0, \Theta_0) \phi_3(P_3) J_4(P_4, Z_4) \rangle = \frac{V_{0,34}^{(\Theta)} V_{0,34}^{(Z)} H_{04}^{(Z, Z)}}{(P_{34})^{\Delta_1 + \Delta_2} (P_{40})^{\Delta_1 + \Delta_2}},$$

the differential operator $D_Z$ from (2.56) which encodes the projection to the irrep $\mathbb{B}$, the constant $S_\Delta$ and the solution of the conformal integrals.

The constant $S_\Delta$ is computed using (4.6) and evaluating the conformal integral

$$\langle \tilde{T}(P_0, Z_0, \Theta_0) \phi_3(P_3) J_4(P_4, Z_4) \rangle = \int D^d P_5 \langle T(P_0, Z_0, \Theta_0) T(P_5, Z_5, \Theta_5) \rangle |_{\Delta \to \Delta} \langle T(P_5, Z_5, \Theta_5) \phi_3(P_3) J_4(P_4, Z_4) \rangle.$$

(4.11)

All the integrals here are of the type

$$\int D^d P_5 \frac{P_0^{A_1} \ldots P^n_{A_n}}{(P_{01})^a (P_{03})^b (P_{04})^c},$$

(4.12)

and their explicit solution can be found for instance in [28, 29]. Comparing (4.11) with the three-point function, the resulting constant is

$$S_\Delta = \frac{\pi^h(\Delta - 2) \Delta \Gamma(\Delta - h)}{\Gamma(\Delta + 2)} \frac{\Gamma \left( \frac{\Delta + \Delta_3 + 2}{2} \right) \Gamma \left( \frac{\Delta - \Delta_3 + 2}{2} \right)}{\Gamma \left( \frac{\Delta + \Delta_4 + 2}{2} \right) \Gamma \left( \frac{\Delta - \Delta_4 + 2}{2} \right)}.$$

(4.14)

Note that this is very similar to the corresponding constant for the exchange of the antisymmetric two-tensor $\mathbb{B}$ (2.21) which was calculated in [10].

To calculate the conformal partial wave (4.9) it is enough to know the conformal integrals

$$\int D^d P_0 \frac{(P_0 \cdot Z_2) (P_0 \cdot Z_4)}{(P_{01})^a (P_{02})^b (P_{03})^c (P_{04})^f}, \quad \int D^d P_0 \frac{(P_0 \cdot Z_2)}{(P_{01})^a (P_{02})^b (P_{03})^c (P_{04})^f}, \quad \int D^d P_0 \frac{1}{(P_{01})^a (P_{02})^b (P_{03})^c (P_{04})^f},$$

(4.15)

To give an impression of how these solutions look like here is the case $n = 1$

$$\int D^d P_0 \frac{P_3^A}{(P_{01})^a (P_{03})^b (P_{04})^c} = \frac{\Gamma \left( \frac{h + 1}{2} \right) \Gamma \left( \frac{c + 1}{2} \right)}{\Gamma(a) \Gamma(b) \Gamma(c)} \Gamma \left( \frac{2 h + c + 1}{2} \right) \frac{\pi^h}{(P_{34})^{\frac{h + 1}{2} - 1} (P_{01})^{\frac{h + 1}{2}} (P_{03})^{\frac{c + 1}{2} - 1} (P_{04})^{\frac{c + 1}{2} - 1}}{\frac{1}{2} (b + c - a + 1) + \frac{1}{2} (c + a - b + 1) + \frac{1}{2} (a + b - c + 1)}.$$

(4.13)
which much like the example (4.13) can be brought into a form where the polarizations are contracted with \(P_1, P_2, P_3, P_4\) or with each other. Just as in [10], after the monodromy projection to eliminate the shadow block, the final expression depends on functions of the cross ratios \(u, v\) given by

\[
J^{(i)}_{j,k,l} = \frac{\Gamma(h + i - f)\Gamma(f)\sin(\pi f)}{\sin(\pi(e + f - h - i))} \int_0^\infty \frac{dx}{x} \int_{x+1}^\infty \frac{dy}{y} \frac{x^by^c}{(y + vx - ux)^{h+1-f}(y - x - 1)^f}, \tag{4.16}
\]

with

\[
b = \alpha + i + j - 1, \\
e = \beta - \Delta + h + i - k - l, \\
f = 1 - \beta + h - k,
\]

and

\[
\alpha = \frac{\Delta - \Delta_{12} - 2}{2}, \quad \beta = \frac{\Delta + \Delta_{34} - 2}{2}, \tag{4.18}
\]

where \(\Delta_{ij} = \Delta_i - \Delta_j\) and \(h = d/2\). In even dimensions, \(h \in \mathbb{N}\), the functions \(J^{(i)}_{j,k,l}\) can be expressed in terms of \(2F_1\) hypergeometric functions, see [10].

Doing the computation we arrived at the following expression for the conformal block defined in (4.9),

\[
g^{\Delta}_{T}(u, v) = \frac{u^{\Delta/2-1}\Gamma(\Delta + 2)}{4P_{24}(\Delta - 2)\Delta(2h - 1)\Gamma(\alpha + 2)\Gamma(\beta + 2)\Gamma(\Delta - \alpha)\Gamma(\Delta + \beta)\Gamma(h - \Delta)} \times \left[ V_{2,14}^{(Z)} V_{4,12}^{(Z)} uF_1 + V_{2,14}^{(Z)} V_{4,23}^{(Z)} vF_2 + V_{2,34}^{(Z)} V_{4,12}^{(Z)} uF_3 + V_{2,34}^{(Z)} V_{4,23}^{(Z)} vF_4 + \frac{1}{2} H_{24}^{(Z,Z)} F_H \right]. \tag{4.19}
\]

As expected this conformal block is organized into tensor structures that are analogous to the ones discussed for this four-point correlator in Section 3.4.1. The functions \(F_i = F_i(u, v)\) depend of \(h, \Delta, \alpha\) and \(\beta\), and are expressed in terms of a finite number of the integrals \(J^{(i)}_{j,k,l}\) given in (4.16) above. For clarity of exposition we decided to present these functions in the Appendix A.\(^8\)

The example at hand shows that we have a well defined algorithm to compute any conformal block. However, before going on to compute even more complicated conformal blocks, it would be helpful to study the functions \(J^{(i)}_{j,k,l}\) in detail. Once the relations among them are better understood, it may well be that much shorter expressions for the conformal blocks are possible. We hope to return to this question.

### 4.2 Example: Two-form exchange

The conformal block for exchange of a two-form tensor \(F\), which corresponds to the irrep \(\mathcal{F}\), was computed analogously in [10], however the result contained a few typos which we now

\(^8\) A Mathematica notebook containing this result can be obtained from the authors upon request.
The normalizations for the two- and three-point functions of \([10]\) are in our notation
\[
\langle F(P_1, \Theta_1) F(P_2, \Theta_2) \rangle = \frac{1}{4} \left( \frac{H_{12}^{(\Theta, \Theta)}}{(P_{12})^{\Delta+2}} \right)^2,
\]
\[
\langle F(P_0, \Theta_0) \phi_3(P_3) J_4(P_4, Z_4) \rangle = \frac{V_{0,34}^{(\Theta)} H_{04}^{(\Theta, Z)}}{(P_{03})^{\Delta+\Delta_1-\Delta_4+1} (P_{34})^{\Delta_3+\Delta_4-\Delta_1+1} (P_{40})^{\Delta_4+\Delta_3-\Delta_1+3}},
\]
and the contraction of two-forms is now done using the normalized derivative \(\partial_\Theta/\sqrt{2}\). The constant \(S_\Delta\) is given by
\[
S_\Delta^F = \frac{\pi^h(\Delta-2) \Gamma(\Delta-h) \Gamma\left(\frac{\Delta+\Delta_1+1}{2}\right) \Gamma\left(\frac{\Delta-\Delta_1+1}{2}\right)}{4 \Gamma(\Delta+1) \Gamma\left(\frac{\Delta+\Delta_1+1}{2}\right) \Gamma\left(\frac{\Delta-\Delta_1+1}{2}\right)}.
\]
After doing carefully the conformal integrals we obtained a slightly shorter formula for this conformal block,
\[
g_\Delta^F(u, v) = \frac{2u^{\Delta/2-1/2} \Gamma(\Delta+1)}{F_{24}(2-\Delta) \Gamma(\alpha+1) \Gamma(\beta+1) \Gamma(\Delta-\alpha) \Gamma(\Delta-\beta) \Gamma(h-\Delta)}
\times \left[ V_{2,14}^{(Z)} V_{4,12}^{(Z)} u \left( (v-1) J_{0,1,1}^{(2)} + v(\beta-h+1) J_{1,2,2}^{(1)} + (\beta-\Delta+h) J_{1,1,2}^{(1)} \right) 
\right.
\left. + V_{2,14}^{(Z)} V_{4,12}^{(Z)} u (v-1) J_{1,1,1}^{(1)} + V_{2,34}^{(Z)} V_{4,12}^{(Z)} u \left( (v-1) J_{0,1,1}^{(2)} - \alpha J_{0,1,1}^{(1)} + (\beta-h+1) J_{1,1,1}^{(1)} - \alpha J_{1,1,1}^{(0)} \right) 
\right.
\left. + \frac{1}{2} H_{24}^{(Z, Z)} \left( - \frac{v-1}{h+1} J_{0,0,0}^{(2)} + J_{0,1,0}^{(2)} + J_{0,1,1}^{(2)} + J_{1,0,1}^{(2)} + v J_{1,1,1}^{(2)} + u J_{1,2,1}^{(2)} \right) 
\right.
\left. + (\beta-\Delta+h) (\alpha J_{1,1,1}^{(0)} + (\alpha-\Delta+1) J_{2,1,1}^{(0)} \right)
\left. + (\beta-h+1) (\alpha J_{1,2,1}^{(0)} + (\alpha-\Delta+1) J_{2,2,1}^{(0)} \right),
\]
where \(J_{j,k,l}^{(i)}\) is defined in \((4.16)\), but now with
\[
\alpha = \frac{\Delta-\Delta_{12}-1}{2}, \quad \beta = \frac{\Delta+\Delta_{34}-1}{2}.
\]
\footnote{We thank David Simmons-Duffin for correspondence on this point.}
To compare this to the corrected result of [10], we took into account the different definitions for $H_{ij}$ and $V_{i,jk}$ and used the three following identities which we checked numerically,

$$u((v - 1)J_{0,1,2}^{(2)} + v(\beta - h + 1)J_{1,2,2}^{(1)} + (\beta - \Delta + h)J_{1,1,2}^{(1)}) = (\beta - \Delta + h) \left( \alpha J_{1,1,1}^{(0)} - J_{1,0,1}^{(1)} \right) - v(\beta - h + 1) \left( J_{2,2,1}^{(0)}(\alpha - \Delta + 1) + J_{1,2,1}^{(1)} \right)$$

$$- \alpha J_{0,1,1}^{(1)} - 2v(\alpha + \beta - \Delta + 1)J_{1,1,1}^{(1)} - v(\alpha - \Delta + 1) \left( J_{1,1,0}^{(1)} + J_{2,1,1}^{(1)} \right)$$

$$- \alpha vJ_{0,1,0}^{(1)} + (v - 1) \left( vJ_{0,1,0}^{(2)} + vJ_{1,1,1}^{(2)} - J_{0,0,1}^{(2)} \right),$$

$$uJ_{0,1,1}^{(2)} = J_{0,0,0}^{(2)} + vJ_{0,1,0}^{(2)} - \alpha J_{0,1,0}^{(1)}.$$  \hspace{1cm} (4.25)

(\text{v - 1}) J_{0,1,1}^{(2)} - \alpha J_{0,1,1}^{(1)} + (\beta - h + 1) \left( vJ_{1,2,1}^{(1)} - \alpha J_{1,2,1}^{(1)} \right) + (\beta - \alpha + h - 1)J_{1,1,1}^{(1)}$$

$$= -(v - 1) \left( J_{0,1,0}^{(2)} + J_{1,1,1}^{(2)} \right) + \alpha vJ_{0,1,0}^{(1)} + (\beta - h + 1) \left( J_{1,2,1}^{(1)} + (\alpha - \Delta + 1)J_{2,2,1}^{(0)} \right)$$

$$+ (\alpha + \beta - \Delta + h)J_{1,1,1}^{(1)} + (\alpha - \Delta + 1) \left( J_{1,1,0}^{(1)} + J_{2,1,1}^{(1)} \right).$$

### 5. S-matrix rule for counting structures

The matching of tensor structures in CFT correlators and scattering amplitudes that was found for symmetric tensors in [9] straightforwardly generalizes to general irreps when considering non-conserved operators. The generalized statement is:  \textit{The number of independent structures in a correlation function of $n$ non-conserved operators of $SO(d)$ irreps $\lambda_1, \ldots, \lambda_n$ is equal to the number of independent structures in a $n$-point scattering amplitude of massive particles of the same irreps in $d + 1$ dimensional flat Minkowski space.}  

This is not surprising since particles have polarizations in irreps of the little group, which is $SO(d)$ for massive particles in $d + 1$ dimensions. The index-free notation introduced in Section 2.4 can be employed by simply using the same Young-symmetrized polarizations. Thus, an $n$-point scattering amplitude of irreps $\lambda_1, \ldots, \lambda_n$ and momenta $k_1, \ldots, k_n$ can be written as

$$A_{\lambda_1, \ldots, \lambda_n} \left( \{k_i, z_i\} \right) = \prod_{j=1}^{n} \prod_{p=1}^{n'_{\lambda_j}} \prod_{q=1}^{n'_{\lambda_j}} \left( z_{j}^{(p)} \cdot \partial_{\theta_{i}^{(q)}} \right) \sum_{k} f_{k}(v_{a}) \tilde{R}_{\lambda_1, \ldots, \lambda_n}^{(k)} \left( \{k_i, \theta_i\} \right),$$

where $f_{k}(v_{a})$ are functions of the $n(n - 3)/2$ independent Mandelstams $v_{a}$. The momenta and polarizations are vectors in $(d + 1)$-dimensional Minkowski space and the polarizations are transverse to the corresponding momenta

$$\theta_{i}^{(p)} \cdot k_i = z_{i}^{(p)} \cdot k_i = 0.$$  \hspace{1cm} (5.2)

The scaling in the polarization vectors is fixed by the condition that the complete polarization tensor appears linearly in the amplitude. This translates to the following scaling of \( \tilde{R}_{\lambda_1, \ldots, \lambda_n}^{(k)} \)
in the polarization vectors, which is equivalent to the one for CFT correlators in (3.72),
\[
\tilde{R}^{(k)}_{\lambda_1,\ldots,\lambda_n}(\{k_i, \theta_i\}) = \tilde{R}^{(k)}_{\lambda_1,\ldots,\lambda_n}(\{k_i, \theta_i\}) \prod_i \left( \beta_i^{(1)} \right)^{h_i^1} \ldots \left( \beta_i^{(n_i)} \right)^{h_i^{n_i}} \left( \beta_i^{(Z)} \right)^{(\lambda_i)}_1. 
\]
(5.3)

The functions \( \tilde{R}^{(k)}_{\lambda_1,\ldots,\lambda_n} \) can be constructed from the two kinds of building blocks
\[
\tilde{H}^{(p,q)}_{ij} \equiv \theta_i^{(p)} \cdot \theta_j^{(q)}, \quad \tilde{V}^{(p)}_{ij} \equiv \theta_i^{(p)} \cdot k_j,
\]
(5.4)

where \( \theta_i^{(p)} \) should be replaced by \( z_i^{(1)} \) for \( p = z \). There are \( n - 2 \) independent \( \tilde{V}^{(p)}_{ij} \)'s for each \( i \), because one of the possible terms vanishes due to the transversality condition (5.2) and another one can be eliminated using momentum conservation
\[
k_1 + k_2 + \ldots + k_n = 0.
\]
(5.5)

Furthermore, the building blocks depend in the same way on Grassmann polarizations as their counterparts \( H^{(p,q)}_{ij} \) and \( V^{(p)}_{ij} \) that appear in CFT correlators. Hence, there is a one-to-one correspondence between building blocks and the counting of tensor structures is the same as in CFT correlators.

A more thorough treatment of on-shell amplitudes of arbitrary \( SO(d) \) irreps (in the context of the open bosonic string) can be found in [30].

6 Concluding remarks

In this work we developed a formalism to elegantly describe irreducible tensor representations of \( SO(d) \) in terms of polynomials. With this formalism and the help of representation theory, tensor structures in CFT correlators and scattering amplitudes become tangible. We gave the algorithm for counting the number of independent tensor structures in any CFT correlator (or massive scattering amplitude) of bosonic operators (or particles), allowing for a systematic construction of the tensor structures for any given example.

The most obvious application for correlators of mixed-symmetry tensors is the construction of conformal blocks, which we reviewed using our new index-free notation. Once all conformal blocks appearing in a given correlator are known, it is possible to implement constraints that follow from conformal symmetry, using recent conformal bootstrap techniques, i.e. proving bounds on the CFT data (conformal dimensions \( \Delta_i \) and OPE coefficients) by use of linear programming [3]. Since there are no further assumptions, such bounds are universal, they hold for any CFT. Until now, in lack of conformal blocks for mixed-symmetry tensor exchange, this has only been done for correlators of scalar operators.

While we only computed one conformal block of mixed-symmetry tensor exchange in a correlator of two scalars and two vectors, it would be much more interesting to consider correlators of stress-tensors. This is because the stress-tensor appears in any CFT and thus could lead to truly universal bounds on CFT data. Another reason for interest in the stress-tensor is its connection to the graviton in AdS, via the AdS/CFT duality. As was pointed out
Table 4. Exchanged irreps in correlators of currents and stress-tensors, following the discussion of possible tensor structures for three-point functions in Section 3.3 and the construction of conformal blocks in Section 4.

| correlator                  | new exchanged $SO(d)$ irreps |
|-----------------------------|-------------------------------|
| $\langle \phi_1 \phi_2 \phi_3 \phi_4 \rangle$ |                               |
| $\langle \phi_1 J^\mu_2 \phi_3 J^\nu_4 \rangle$ |                               |
| $\langle J^\mu_1 J^\rho_2 J^\sigma_3 J^\tau_4 \rangle$ |                               |
| $\langle J^\mu_1 T^\nu_2 \cdot T^\rho_3 T^\lambda_4 \rangle$ |                               |
| $\langle T^{\mu\nu}_1 T^{\rho\sigma}_2 T^{\lambda\kappa}_3 T^{\tau\omega}_4 \rangle$ |                               |

already in [10], universal bounds on CFT data for external operators with spin may explain the weak gravity conjecture [31] or the bounds on $a$ and $c$ in [32].

With the insights about three-point correlators from this work it is easy to outline what needs to be done to compute all conformal blocks for the correlator of four stress-tensors. Table 4 contains all irreps that are exchanged in this correlator. Some conformal blocks can actually be written in terms of derivatives of conformal blocks for exchange of the same irrep in a simpler correlator, as it is the case for exchange of symmetric tensors [9]. For example, the conformal blocks for exchange of $\langle J^\mu_1 J^\rho_2 J^\sigma_3 J^\tau_4 \rangle$ in $\langle T^{\mu\nu}_1 T^{\rho\sigma}_2 T^{\lambda\kappa}_3 T^{\tau\omega}_4 \rangle$ are given by derivatives of the conformal blocks of $\langle \phi_1 \phi_2 \phi_3 \phi_4 \rangle$. For this reason, each line in the table displays for some correlator the irreps of exchanged operators that appear for the first time for that correlator. If one picks a correlator in one line and a single irrep from the same line, the computation of that conformal block is comparatively easy, since in those cases the three-point function between external operators and the exchanged operator has only one tensor structure. One can hope that the conformal blocks for all other cases are given by derivatives of those simpler cases.

An interesting generalisation of our work would be to extend the formalism to general spinor representations of $SO(d)$. This would complete the counting and construction of tensor structures for all CFT correlators and facilitate the conformal bootstrap for combinations of operators that imply exchange of operators with half-integer spin.

Finally, note that most discussions of higher spin fields in AdS focus on the case of spin $J$ symmetric tensors. However, it would be interesting to consider AdS fields dual to operators in arbitrary irreps of the conformal group. We expect that the techniques described in this paper can also be extended to the case of AdS fields, in the spirit of [33].
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A Functions in the conformal block for hook diagram exchange

The following are the functions appearing in the conformal block (4.19) for exchange of the primary in the irreducible representation $[\mathcal{P}, \Delta]$ in the correlator of two scalars and two vectors $\langle \phi_1 J^\mu_2 \phi_3 J^\nu_4 \rangle$.

\[ F_1 = (\alpha - \Delta + 1) \left[ (\beta - \Delta + h + 1) \left( (2h - 1) J_{1,1,1}^{(1)} (\beta - \Delta + h) - (\alpha + 1) J_{1,1,1}^{(1)} \right) 
- J_{1,1,1}^{(2)} ((2h - 1) (v - 1) + u) + 2(2h - 1) v J_{2,2,2}^{(1)} (-\beta + h - 1) \right]
+ (\beta - h + 1) v \left( (2h - 1) (v J_{2,3,2}^{(1)} (-\beta + h - 2) - (v - 1) J_{1,2,2}^{(2)} ) + u J_{1,2,2}^{(2)} \right) \]

\[ + (\alpha + 1) \left[ (\beta - \Delta + h + 1) \left( 2\alpha J_{1,2,1}^{(0)} (-\beta + h - 1) + (1 - 2h) J_{1,1,2}^{(1)} (\beta - \Delta + h) \right) 
- J_{1,2,2}^{(1)} ((-\beta + h - 1) (v - 1) + u) + J_{0,1,2}^{(2)} ((1 - 2h) (v - 1) + u) - \alpha J_{0,1,1}^{(1)} \right)
+ (\beta - h + 1) \left( J_{0,2,2}^{(2)} ((-2h - 1) (v - 1) - u) \right)
+ (2h - 1) v J_{1,3,2}^{(1)} (-\beta + h - 2) + \alpha J_{0,2,1}^{(1)} + v (\alpha - \Delta + 1) J_{1,2,1}^{(1)} \right] \]  \tag{A.1} \]

\[ F_2 = (\alpha - \Delta + 1) \left[ (\beta - \Delta + h + 1) \left( u J_{2,2,2}^{(1)} (-\beta + h - 1) + J_{1,1,1}^{(2)} ((2h - 1) (v - 1) + u) \right) 
+ (\beta - h + 1) v \left( (-\beta + h - 2) ((\alpha + 1) J_{2,3,1}^{(0)} + u J_{2,3,2}^{(1)} ) 
- J_{1,2,1}^{(2)} ((1 - 2h) (v - 1) + u) - (\alpha + 1) J_{1,2,0}^{(1)} \right) \right] \]
\[ F_3 = (\alpha - \Delta + 1) \left[ (\beta - \Delta + h + 1) \left( (\beta - h + 1) \left( uJ^{(1)}_{1,2,2} - J^{(0)}_{2,2,1}(\alpha - \Delta + 1) - \alpha J^{(0)}_{1,2,1} \right) - J^{(2)}_{0,1,1}((1 - 2h)(v - 1) + u) + J^{(1)}_{1,1,0}(\alpha - \Delta + 1) + \alpha J^{(1)}_{0,1,0} \right) + (\beta - h + 1) \left( - \alpha \left( J^{(0)}_{1,3,1}(\beta - h + 2) + J^{(1)}_{0,2,0} \right) + uJ^{(1)}_{1,3,2}(\beta - h + 2) + J^{(2)}_{0,2,1}((2h - 1)(v - 1) + u) \right) \right] \] (A.2)

\[ F_4 = (\alpha - \Delta + 1) \left[ (\beta - \Delta + h + 1) \left( (2h - 1) \left( J^{(1)}_{1,1,1}((1 - 2h)(v - 1) - u) - (2h - 1) \left( (\alpha + 1)J^{(0)}_{2,2,1}(-\beta + h - 1) + J^{(1)}_{2,1,1}(-\alpha + \beta + h - 2) \right) \right) + (\beta - h + 1)u \left( (2h - 1)(\beta - h + 2) \left( (\alpha + 1)J^{(0)}_{2,3,1} - vJ^{(1)}_{2,3,1} \right) - (2h - 1)J^{(1)}_{2,2,1}(\alpha + 2\beta - \Delta + 2h) + J^{(2)}_{1,2,1}((2h - 1)(v - 1) + u) \right) \right) \right] \] (A.3)

\[ \begin{align*}
F_4 &= (\alpha - \Delta + 1) \left[ (\beta - \Delta + h + 1) \left( (2h - 1) \left( J^{(1)}_{1,1,1}((1 - 2h)(v - 1) - u) - (2h - 1) \left( (\alpha + 1)J^{(0)}_{2,2,1}(-\beta + h - 1) + J^{(1)}_{2,1,1}(-\alpha + \beta + h - 2) \right) \right) + (\beta - h + 1) \left( (\alpha + 1)J^{(1)}_{1,1,1}(\alpha - \Delta - 2) - 2(\alpha + 1)J^{(1)}_{1,1,0} \right) + uJ^{(1)}_{2,2,1}(-\beta + h - 1) + J^{(2)}_{2,1,0}((2h - 1)(v - 1) + u) \right) \right) + (\beta - h + 1) \left( (\alpha + 1)J^{(1)}_{1,2,0}( - (2h - 1)(v + 1) + 2u) + uvJ^{(1)}_{2,3,1}(-\beta + h - 2) - vJ^{(2)}_{1,2,0}((2h - 1)(v - 1) + u) + (2h - 1)vJ^{(1)}_{2,2,0}(-\alpha + \Delta - 2) \right) \right) \right] \] (A.4)
\[ F_H = \frac{1}{h+1} \left\{ (\alpha - \Delta + 1) \left[ (\beta - \Delta + h + 1) \left( -((2h-1)(v-1)+u) \right) \times \left( J_{1,1,1}^{(2)} + J_{1,0,0}^{(2)} + J_{2,0,1}^{(2)} + uJ_{2,1,2}^{(2)} \right) \right] + (\beta - h + 1) \left( v(- (2h-1)(v-1)+u)(J_{1,1,1}^{(2)} + J_{2,2,2}^{(2)} + uJ_{2,2,1}^{(2)} + J_{1,1,0}^{(2)} \right) \right. \\
+ vJ_{2,1,1}^{(2)}(- (2h-1)(u - (v-1)(\Delta - 2(\beta + 1))) + (\Delta - 1)u \right] \\
+ (\alpha + 1) \left[ (\beta - \Delta + h + 1) \left( (u - (2h-1)(v-1))(J_{1,1,1}^{(2)}v + J_{0,0,0}^{(2)} + J_{0,1,1}^{(2)} + uJ_{1,1,2}^{(2)} \right) \right] + (\beta - h + 1) \left( - (uJ_{1,2,2}^{(2)} + J_{0,1,0}^{(2)} + J_{0,2,1}^{(2)} + J_{2,1,1}^{(2)})( (2h-1)(v-1) + u ) \right) \\
+ J_{0,1,1}^{(2)}((2h-1)(u + (v-1)(\Delta - 2(\beta + 1))) - (\Delta - 1)u \right] \\
+ J_{1,0,1}^{(2)}(\beta - \Delta + h + 1)((2h-1)(v-1)(\Delta - 2(\alpha + 1)) + \Delta u) \\
- vJ_{1,2,1}^{(2)}(- \beta - h + 1)((2h-1)(v-1)(\Delta - 2(\alpha + 1)) - \Delta u \right) \\
\left. + (2h-1) \left\{ (\alpha - \Delta + 1) \left[ (\beta - \Delta + h + 1) \left( + 2(\alpha + 1)J_{2,1,1}^{(0)}(\beta - \Delta + h) \right) \\
- (\alpha - \Delta + 2)(J_{3,1,1}^{(0)}(\beta - \Delta + h) + 2vJ_{3,2,1}^{(0)}(\beta - h + 1)) \right) \\
+ (\beta - h + 1) \left( v(- \beta + h - 2)(vJ_{3,3,1}^{(0)}(- \alpha + \Delta - 2) - 2(\alpha + 1)J_{2,3,1}^{(0)} \right) \right] \\
+ (\alpha + 1) \left[ (\beta - \Delta + h + 1)(\alpha(-2J_{1,2,1}^{(0)}(- \beta + h - 1) + J_{1,1,1}^{(0)}(\beta - \Delta + h)) \\
+ J_{1,3,1}^{(0)}(- \beta + h - 2)(- \beta + h - 1) \right] \right\} \right\} \\
+ 2(\alpha + 1)J_{2,2,1}^{(0)}(- \alpha + \Delta - 1)(- \beta + h - 1)(\beta - \Delta + h + 1)((2h-1)(v+1) - 2u) \tag{A.5} \]

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