Extremal extensions of entanglement witnesses and their connection with UPB

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In this paper we describe a new connection between UPB (unextendable product bases) and P (positive) maps which are not CP (completely positive). We show that inner automorphisms of the set of P maps which are not CP, produce extremal extensions of these maps that help in entanglement detection. By constructing such an extension of the well-known Choi map, we strengthen its power to unearth PPT (positive under partial transpose) entangled states. We further show that the class of maps generated from the Choi map via an inner automorphism naturally detects the entanglement of states in the orthogonal complement of certain UPB. This brings out a hitherto undiscovered connection between the Choi map and UPB. We also show that certain other recently considered extremal extensions are obtainable by such extensions of the Choi map.

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I. INTRODUCTION

Quantum entanglement is a fundamentally new feature that emerges in the quantum world and its study remains a central theme in quantum theory. On the one hand, entanglement is responsible for the non-classical correlations leading to the violation of Bell type inequalities and on the other, it plays a key role in quantum algorithms giving them a clear advantage over their classical counterparts.

In the quantum mechanical description, the physical states of the system are represented by trace class operators denoted by $\rho$ on a complex Hilbert space $\mathcal{H}$. The dimension of this space can be finite or infinite and for the finite dimensional case, which concern us in this paper, we have $\mathcal{H} = \mathbb{C}^n$. If the rank of $\rho$ is 1 the state is pure otherwise it is mixed. The set of states forms a convex set with pure states being the extremal points.

For composite systems the Hilbert space is the tensor product of the Hilbert spaces of the individual systems. Thus the state space of a bipartite system is given by $\mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$. A bipartite state $\rho \in \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$ is called a separable state if and only if it can be written as

$$\rho = \sum_{j=1}^{n} p_j \rho_j^A \otimes \rho_j^B, \quad p_j > 0, \quad \sum_{j=1}^{n} p_j = 1.$$  \hspace{1cm} (1)

where $\rho_j^A$ and $\rho_j^B$ are states in the systems $A$ and $B$ respectively. If a state $\rho \in \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$ cannot be written in the above form, then it is an entangled state.

The central question in this field is to determine whether a given arbitrary (pure or mixed) bipartite state $\rho$ is entangled or separable. The problem has a simple solution for the case of pure states. A pure bipartite state is separable if and only if the reduced density operator obtained by tracing over one of the systems is pure. In fact the entropy of the reduced density operator can be used to quantify the amount of entanglement. However, for the case of mixed states such a characterization is not possible and only partial solutions are available. While there are methods to uncover entangled states, all of them involve one-way conditions whose violation indicates entanglement. The sum total of such conditions leading to a complete characterization of states is not available and the solution to this problem has remained elusive. A vast body of literature exists in this field and for a review see.

In this process of distinguishing entangled states from separable ones, the most important mathematical tool is provided by positive (P) maps which are not completely positive (CP). A map $\varphi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is said to be positive if it maps the set of positive operators in $\mathcal{B}(\mathcal{H})$ (denoted by $\mathcal{B}(\mathcal{H}_+)\to\mathcal{B}(\mathcal{H}_+)$) to itself. A positive map is said to be completely positive if the extension $1_d \otimes \varphi : \mathcal{B}(\mathbb{C}^d \otimes \mathcal{H}) \to \mathcal{B}(\mathbb{C}^d \otimes \mathcal{H})$ is a positive map for all $d \geq 1$. While CP maps show a remarkably simple representation due to Sudarshan, Kraus and Choi, P maps which are not CP are not easily characterizable. The fact that all separable states defined by Equation (1) remain positive when we apply P maps which are not CP to one of the systems, helps us to convert such maps into entanglement witnesses. Therefore, any state which turns into a non-state under the application of a P map which is not CP on one of the systems, has to be entangled. Thus such maps help us detect bipartite entangled states. The transpose operation was the first such P map which is not CP used to detect entanglement. Using the results of Arveson and Stormer, Woronowicz showed that in the dimensions $2 \otimes 2$ and $2 \otimes 3$ the transpose map is powerful enough to detect all entangled states and we need not use any other witnesses. For composite systems with dimensions larger than $2 \otimes 3$, the transpose is a useful tool to detect entangled states, however it does not detect all the entangled states. In higher dimensions, the states which are negative under partial transpose (NPT) are entangled while those which are positive under partial transpose (PPT) can either be separable or entangled. For the PPT entangled states there has to exist a P map (not CP) which
will convert them into a non-state and provide a witness for their entanglement. The Choi map is the first non-trivial example of a P map which is not CP and which unearths entanglement of PPT entangled states. Only a few more examples of such maps are available in the literature and the theory of such maps is far from complete. In the absence of a complete solution, discovering new families of PPT entangled states, finding new entanglement witnesses, and understanding their connections and structure is important.

It has been shown that for a composite system with dimension more than $2 \otimes 3$, one can construct a set of orthogonal product states spanning a subspace, such that there is no product state in the orthogonal complement of this subspace. This implies that any state in the orthogonal complement of this set is an entangled state. Since these states are by construction PPT, this allows one to construct families of PPT entangled states. Such families of PPT entangled states should become non-states when some P map which is not CP acts on one of the subsystems, thereby revealing their entanglement. Examples of such maps, although existing in literature, are somewhat contrived. We show that these families of PPT entangled states in the orthogonal complement of UPB have a connection with the Choi map. This connection is new and provides insights into the Choi map as well as UPB.

In this work we consider the extremal extension of the positive maps. We show that this changes their ability to detect entanglement. We begin with the Choi map which is an extremal map and construct its extremal extensions using appropriate automorphisms. To the best of our knowledge, the Choi map and its extension are the only examples of maps which are unital, extremal and exposed. Our extensions based on automorphisms preserve the extremality and exposedness and we can always restrict the extensions to a sub-class of automorphisms to preserve the unital nature of the map. The family of extremal extensions thus generated are expected to be able to reveal entanglement of new classes of states. We then define a one-parameter family of such extremal extensions and show that for a certain value of the parameter, the map is able to implicate the entanglement of states in the orthogonal complement of UPB arising out of the TILES and PYRAMID constructions. This demonstrates that, where the original Choi maps fail to reveal entanglement of states based on UPB, their extremal extensions succeed.

The material in this paper is arranged as follows: In Section II we describe the construction of extremal extensions of the P maps which are not CP. We show how the extensions preserve extremality and how only inner automorphisms are useful in the context of entanglement detection. In Section III we take the example of the Choi map and construct its extremal extensions using the method described in Section II while restricting ourselves to $3 \otimes 3$ systems. We define a very interesting family of these extensions where the quantum operation is restricted to a one-parameter subgroup of $SU(3)$. We show that this particular family, for certain values of the parameters, is able to detect entanglement of PPT entangled states in orthogonal complement of UPB for TILES and PYRAMID constructions. In Section IV we describe three more examples of automorphisms demonstrating the usefulness of the formulation. Section V contains some concluding remarks.

II. EXTREMAL EXTENSIONS OF POSITIVE MAPS

In this section, starting with a P map (which is not CP) and a CP map, we construct a composite map. This composite map turns out to be extremal if the original map is extremal and under certain conditions has more power to detect entanglement as compared to the original map. Consider $\varphi : B(\mathcal{H}) \rightarrow B(\mathcal{H})$ to be a positive indecomposable map. For any $A \in GL_3(C)$ we can define a map

$$A : B(\mathcal{H}) \rightarrow B(\mathcal{H})$$

$$X \mapsto AXA^\dagger$$

For $X \in B(\mathcal{H})$ (2)

Clearly, $A$ is a CP map. Note that we are using the same symbol $A$ for the $GL_3(C)$ element and the corresponding map. To make it a valid quantum operation, we impose the condition $AA^\dagger \leq I$ where $I$ denotes the identity element of $B(\mathcal{H})$.

We can then define the two automorphisms as the compositions

$$\varphi \circ A = \varphi_A$$

$$A \circ \varphi = \varphi^A$$

The former is called inner automorphism while the latter is called outer automorphism. The outer automorphism is not useful for us as it does not strengthen the entanglement detection capability of $\varphi$. However as we will see below and in the next sections, the inner automorphism is useful.

It is worth noting that the set of positive maps is a convex set and can be described by its ‘extremal points’, in our case ‘extremal maps’. A positive map $h$ is said to be extremal, when for any decomposition $h = h_1 + h_2$, where $h_1$ and $h_2$ are positive maps, $h_i = \lambda_i h_i$, where $\lambda_1 \geq 0$ and $\lambda_1 + \lambda_2 = 1$.

Theorem 1 For any positive map $\varphi : B(\mathcal{H}) \rightarrow B(\mathcal{H})$, and for any full rank operator $A$, (such that $AA^\dagger \leq I$) $\varphi_A$ is a positive map. Moreover, if $\varphi$ is not completely positive and extremal, so is the map $\varphi_A$.

Proof: The map $A : X \mapsto AXA^\dagger$, when $A$ is a non-singular operator defines an automorphism on $B(\mathcal{H})$. If $X$ is Hermitian, so is $AXA^\dagger$ and if $X$ is positive, so is the image as the map $A$ is completely positive. Thus the map $A \circ \varphi = \varphi^A$.
$A$ is a bijection map from the set of positive semi-definite operators onto itself.

Let $\varphi$ be a $P$ but not CP map. Assume that $\varphi_A$ is a CP map. Then by Kraus decomposition, there exists a finite set of operators $\{V_i\}$ which represents the map and we can write for any $X \in \mathcal{B}(\mathcal{H})$

$$\varphi_A(X) = \sum_i V_i X V_i^\dagger.$$  \hfill (4)

Now $\varphi(X) = \varphi_A(A^{-1}X A^{-1})$ since $A$ is a non-singular operator. We thus have

$$\varphi(X) = \varphi_A(A^{-1}X A^{-1}) = \sum_i V_i A^{-1}X A^{-1}V_i^\dagger.$$  \hfill (5)

implying that $\varphi$ is a CP map. This is a contradiction. Hence $\varphi_A$ is a $P$ but not CP map.

For the second part, let $\varphi$ be extremal and let us assume that $\varphi_A$ is not extremal. Then there exist positive maps $\varphi_1$ and $\varphi_2$ so that $\varphi_A = \varphi_1 + \varphi_2$. Using a similar argument as above we can write

$$\varphi(X) = \varphi_A(A^{-1}X A^{-1})$$

$$= \varphi_1(A^{-1}X A^{-1}) + \varphi_2(A^{-1}X A^{-1})$$

$$= \varphi_{A^{-1}}(X) + \varphi_{A^{-1}}(X).$$  \hfill (6)

But the map $\varphi$ is an extremal map. By definition of extremality, if $\varphi = \varphi_1 + \varphi_2$, where $\varphi_i$ are positive maps, then $\varphi_i = \lambda_i \varphi$, where $\lambda_1 + \lambda_2 = 1$. Hence

$$\varphi_{A^{-1}} = \lambda_i \varphi \Rightarrow \varphi_{A^{-1}} A = \lambda_i \varphi \circ A\Rightarrow \varphi_i = \lambda_i \varphi_A.$$  \hfill (7)

Hence $\varphi_A$ is an extremal map.

A special case of interest is when $A$ is unitary which we denote by $U$. A number of special results are available for this case. By the Russo–Dye theorem (see [31]) we can show that for any unitary operator $U$,

$$\|\varphi^U\| = \|\varphi_U\| = \|\varphi\|.$$  \hfill (8)

It is obvious that if $\varphi$ is unital, so are $\varphi_U$ and $\varphi^U$.

Further, positivity under partial transpose is invariant under inner unitary automorphism. In other words, for the transpose map $T$ and any unitary operator $U$, $(I \otimes T)\rho \geq 0$ implies $(I \otimes T)\rho \geq 0$ for any state $\rho$.

This can be proved as follows: Let $\rho \in \mathcal{B}(\mathcal{H} \otimes \mathcal{H})$ be a PPT state. Let us write $\rho = ((\rho_{ij}))$ in the block form, where for each $i$ and $j$, $\rho_{ij} \in \mathcal{B}(\mathcal{H})$. Then $(I \otimes T)\rho = ((T(\rho_{ij})) = ((\rho^T_{ji}))$. Hence

$$(1 \otimes T)\rho = ((T(U\rho_{ij}U^\dagger))$$

$$= ((U^T T(\rho_{ij} U^\dagger)))$$

$$= (I \otimes \overline{U})(1 \otimes T)\rho(I \otimes \overline{U})^\dagger.$$  \hfill (9)

Where $U = ((u_{ij}))$ and its complex conjugate $\overline{U} = ((\overline{u_{ij}}))$ are unitary operators. Since eigenvalues remain invariant under unitary transformations (local unitary in our case), the result follows.

**Theorem 2**

1. For any positive map $\varphi : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$, and any unitary operator $U$, the outer automorphism $\varphi^U$ is a positive map.

2. Any entangled state $\rho$ detected by $\varphi^U$ is detected by $\varphi$ and vice versa.

**Proof:** Let $x \in \mathcal{B}(\mathcal{H})$ be any positive semi-definite Hermitian operator. Since $\varphi$ is positive, $\varphi(x) \geq 0$. Since the unitary operators do not change eigenvalues, we have $U\varphi(x)U^\dagger \geq 0$, i.e. $\varphi^U = U(x) \circ \varphi \geq 0$. Hence $\varphi^U$ is a positive map.

For the second part, notice that the eigenvalues are invariant under unitary operators. Hence,

$$(1 \otimes \varphi)\rho \geq 0 \iff (I \otimes U)(1 \otimes \varphi)\rho(I \otimes U)^\dagger \geq 0$$

$$\iff (1 \otimes U\varphi(U)^\dagger)\rho \geq 0$$

$$\iff (1 \otimes \varphi^U)\rho \geq 0.$$  \hfill (10)

This means that for the entanglement detection application, unitary outer automorphisms are not useful and therefore we should focus only on the inner automorphism.

In the next section we discuss the power of such extensions. We will consider PPT entangled states discovered through UPB construction due to Bennett et. al. [32] and apply one-parameter sub-families of unitary inner automorphisms to them.

### III. EXTENSIONS OF CHOI MAP AND UPB CONSTRUCTION

#### A. The Choi Map

The first non-trivial example of a $P$ map which is not CP and can provide a witness for the entanglement of some PPT entangled states was discovered by Choi [13]. This map comes in two variants and they are defined on a 3-dim Hilbert space as follows:

$$\varphi_{C_1} : ((x_{ij})) \rightarrow \frac{1}{2} \begin{pmatrix} x_{11} + x_{22} & -x_{12} & -x_{13} \\ -x_{21} & x_{22} + x_{33} & -x_{23} \\ -x_{31} & -x_{32} & x_{33} + x_{11} \end{pmatrix}$$

(11)

and

$$\varphi_{C_2} : ((x_{ij})) \rightarrow \frac{1}{2} \begin{pmatrix} x_{11} + x_{33} & -x_{12} & -x_{13} \\ -x_{21} & x_{22} + x_{11} & -x_{23} \\ -x_{31} & -x_{32} & x_{33} + x_{22} \end{pmatrix}$$

(12)

Both these maps as defined in (11) and (12) are useful in unearthing entanglement of PPT entangled states and are extremal points in the space of maps [33]. There are only a few examples of extremal maps and apart from Choi maps, there have been extensions of Choi maps by Kye [15] which were shown to be extremal by Osaka [16]. We are interested in unitary inner automorphisms of the
Choi maps which are defined as the composition $\varphi_{C_{1,2}} \circ U$ where $U \in SU(3)$ is a unitary operator. For every $U \in SU(3)$ we have an extremal map generated from the Choi map. For example, for every one-parameter subgroup of $SU(3)$ we will have a family of maps which can help us unearth entanglement of PPT entangled states.

B. The TILES construction

The unextendable product basis, the ‘TILES’ construction was proposed by Bennett et. al. [29]. Given a composite system with Hilbert space $\mathbb{C}^3 \otimes \mathbb{C}^3$, we consider the normalized orthogonal states

$$|\psi_0\rangle = \frac{1}{\sqrt{2}} |0\rangle (|0\rangle - |1\rangle), \quad |\psi_1\rangle = \frac{1}{\sqrt{2}} |1\rangle (|1\rangle - |2\rangle),$$

$$|\psi_2\rangle = \frac{1}{\sqrt{2}} (|0\rangle - |1\rangle) |2\rangle, \quad |\psi_3\rangle = \frac{1}{\sqrt{2}} (|1\rangle - |2\rangle) |0\rangle,$$

$$|\psi_4\rangle = \frac{1}{3} (|0\rangle + |1\rangle + |2\rangle) (|0\rangle + |1\rangle + |2\rangle)$$  \hspace{1cm} (13)

Bennet et. al. showed that there is no product state in the orthogonal complement of these states. Therefore, the state

$$\rho = \frac{1}{4} \left( I_9 - \sum_{i=0}^{4} |\psi_i\rangle \langle \psi_i| \right).$$

is entangled. Further, by construction this state is PPT and therefore we have a PPT entangled state. We can apply the maps $I \otimes \varphi_{C_{1,2}}$ to the state and it turns out that the state remains positive and does not reveal its entanglement. Consider a one-parameter family of extremal extensions of the Choi maps $\varphi_{C_{1,2}}(\theta) = \varphi_{C_{1,2}} \circ U(\theta)$ with

$$U(\theta) = \begin{pmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{pmatrix}.$$ \hspace{1cm} (15)

These two families of maps defined via the unitary inner automorphism can now be tried on the PPT entangled states defined in Equation (14) to see if they can reveal its entanglement. We apply the maps $I \otimes \varphi_{C_{1,2}}(\theta)$ to the state defined in Equation (14).

$$I_3 \otimes \varphi_{C_{1,2}}(\theta) : \rho \rightarrow \rho'_1(\theta)$$  \hspace{1cm} (16)

We compute the eigen values of $\rho'_1(\theta)$ and $\rho'_2(\theta)$. It turns out that the smallest eigen value becomes negative for a range of $\theta$ values indicating that the resultant operator is not a state, thereby revealing the entanglement of the original state $\rho$. The plot of minimum eigen values of $\rho'_1(\theta)$ and $\rho'_2(\theta)$ are shown in Figure 1. The upper graph corresponds to the case $\rho'_1(\theta)$ and the lower one corresponds to the case $\rho'_2(\theta)$.

Both families of maps are able to reveal the entanglement of the state $\rho$ defined in Equation (14). However the $\theta$ ranges for which the map reveals the entanglement are different in each case. The lower graph can be superimposed on the upper graph by a shift of $\pi/2$ in $\theta$. In each graph the straight lines show the positive minimum eigen value obtained after application of the corresponding non-modified Choi map.

C. The PYRAMID construction

Another interesting UPB construction for the $3 \otimes 3$ Hilbert space is the PYRAMID construction [29]. We first define five vectors in a three dimensional Hilbert space as:

$$v_j = N \left( \cos \frac{2\pi j}{5}, \sin \frac{2\pi j}{5}, h \right) \quad j = 0, \cdots, 4;$$ \hspace{1cm} (17)

where $h = \frac{1}{2} \sqrt{1 + \sqrt{5}}$ and $N = \frac{2}{\sqrt{\sqrt{5} + 2}}$. Using these vectors we define the UPB set as

$$|\psi_j\rangle = |v_{j}\rangle \otimes |v_{2j \mod 5}\rangle, \quad j = 0, \cdots, 4.$$ \hspace{1cm} (18)

The corresponding PPT entangled state is obtained by substituting the UPB states given in Equation (18) above.
into Equation (14). We carry out an identical analysis to the TILES case and find that the entanglement of this state is again detected by the modified Choi maps. The plots are shown in Figure 2 where the minimum eigenvalue is displayed as a function of \( \theta \). The operators \( \rho_1, \rho_2(\theta) \) are obtained from the PPT entangled states in the orthogonal complement of the PYRAMID UPB construction by the action of families of extremal extensions of two Choi maps on the second system. The negativity of the minimum eigenvalues shows that the map is able to detect entanglement of the states. The straight line in each graph shows the minimum eigenvalue in the case of the original Choi map which remains positive and therefore does not reveal the entanglement.

![Figure 2](image)

**FIG. 2.** Plot of minimum eigen value of operators \( \rho_1, \rho_2(\theta) \) as a function of \( \theta \). The operators \( \rho_1, \rho_2(\theta) \) are obtained from the PPT entangled states in the orthogonal complement of the PYRAMID UPB construction by the action of families of extremal extensions of two Choi maps on the second system. The straight line in each graph shows the minimum eigenvalue in the case of the original Choi map which remains positive and therefore does not reveal the entanglement.

In the third example we turn to a generalization of the Choi map defined by Cho et. al. [14] as

\[
\varphi_m((x_{ij})) \mapsto \frac{1}{2} \times \begin{bmatrix}
    ax_{11} + bx_{22} + cx_{33} & -x_{12} & -x_{13} \\
    -x_{21} & ax_{22} + bx_{33} + cx_{11} & -x_{23} \\
    -x_{31} & -x_{32} & ax_{33} + bx_{11} + cx_{33}
\end{bmatrix}
\]

(22)

where \( a, b, c \) satisfy certain conditions given in detail in their paper.

IV. FURTHER EXAMPLES OF EXTERNAL EXTENSIONS

To demonstrate the usefulness of the extensions based on automorphisms we describe below three insightful results. The first result is that the two maps due to Choi described in Equations (11) and (12) naturally get connected via a combination of inner and outer unitary automorphisms. The map \( \varphi_{C_1} \) thus gets related to \( \varphi_{C_2} \).

\[
\varphi_{C_1} = U \left( \frac{3\pi}{2} \right) \circ \varphi_{C_2} \circ U \left( \frac{\pi}{2} \right).
\]

(19)

Secondly, the construction that we had described in a recent paper where we had generated extremal maps as candidate entanglement witnesses from the existing ones turns out to be a non-unitary inner automorphism [34]. To describe this connection we consider an extremal positive in-decomposable map \( \varphi : B(\mathbb{C}^n) \rightarrow B(\mathbb{C}^n) \) and the corresponding bi-quadratic

\[
F \left( \begin{array}{c}
X \\
Y
\end{array} \right) = \begin{bmatrix}
X_1 & \cdots & X_n \\
Y_1 & \cdots & Y_n
\end{bmatrix} = \langle Y|\varphi(|X\rangle\langle X|)|Y\rangle, \quad |X\rangle = (x_1, \cdots, x_n)^t, \quad |Y\rangle = (y_1, \cdots, y_n)^t
\]

\( \theta \) denotes the transpose, and \( x_i, y_j \) are real parameters. A map \( \varphi \) is positive and extremal if and only if the corresponding real bi-quadratic form is positive and extremal. In-decomposability of the map implies that the form \( F \) cannot be written as a sum of square of quadratic forms. It was shown in [34] that for any set of \( n \) non zero real parameters \( a_1, \cdots, a_n \); the form \( G \left( \begin{array}{c}
x_1 & \cdots & x_n \\
y_1 & \cdots & y_n
\end{array} \right) = \begin{bmatrix}
a_1x_1 & \cdots & a_nx_n \\
ay_1 & \cdots & y_n
\end{bmatrix} \) is also an extremal positive form.

Hence the corresponding map denoted by \( \varphi(a_1, \cdots, a_n) \) is an extremal in-decomposable positive map. We had used this extended class of maps to unearth the entanglement of a new class of PPT entangled states [34]. It turns out that this extremal extension can be recast as an inner automorphism of the original map given below

\[
\varphi(a_1, \cdots, a_n) = \varphi \circ A
\]

(20)

where \( A \) is an operator given by the diagonal matrix

\[
A = \text{Diag}(a_1, a_2 \cdots, a_n)
\]

(21)

This is clearly a non-unitary inner automorphism and connects our earlier result with the present formulation.

In the third example we turn to a generalization of the Choi map defined by Cho et al. [14] as

\[
\varphi_m((x_{ij})) \mapsto \frac{1}{2} \times \begin{bmatrix}
    ax_{11} + bx_{22} + cx_{33} & -x_{12} & -x_{13} \\
    -x_{21} & ax_{22} + bx_{33} + cx_{11} & -x_{23} \\
    -x_{31} & -x_{32} & ax_{33} + bx_{11} + cx_{33}
\end{bmatrix}
\]

(22)

where \( a, b, c \) satisfy certain conditions given in detail in their paper.

It has been shown by Ha and Kye [37] that a sub-class of the above family of maps, given by

\[
0 < a < 1, \quad a + b + c = 2, \quad bc = (1 - a)^2
\]

are extremal maps. It has been further shown [36, 37] that these extremal maps can be written as a one-parameter family of maps \( \varphi_t \) with \( 0 \leq t < \infty \).
parameters \(a(t), b(t)\) and \(c(t)\) are given by
\[
a(t) = \frac{(1 - t)^2}{1 - t + t^2}, \quad b(t) = \frac{t^2}{1 - t + t^2}, \quad c(t) = \frac{1}{1 - t + t^2}.
\]

We have \(\varphi_{t=0} = \varphi_{C_1}\), \(\varphi_{t=\infty} = \varphi_{C_2}\) while \(\varphi_{t=1}\) is a decomposable map. Using the unitary automorphism defined through the one-parameter family of unitary transformations given in Equation (13), we are able to relate the maps in the interval \([0, 1]\) to maps in the interval \([1, \infty)\) as follows:
\[
\varphi_t = U \left( \frac{3\pi}{2} \right) \circ \varphi_{\frac{1}{2}} \circ U \left( \frac{\pi}{2} \right).
\]

This means that we need to consider only the maps in the interval \([0, 1]\) if we are interested in using them as entanglement witnesses and the others can be generated via the automorphism given above. The above examples show that the automorphisms provide us with a way to connect various seemingly unrelated maps.

V. CONCLUDING REMARKS

In this paper we have described extremal extensions of P maps which are not CP via their composition with quantum operations. Two kinds of automorphisms are described and it is shown that only one of them, namely, the inner automorphism has the ability to enhance the entanglement detection power of the original map. This construction opens up new possibilities of extremal extensions of P maps which are CP. Focusing on the famous Choi map and its extensions via a one-parameter family of unitary transformations, we have discovered a useful and interesting connection with UPB. We discover that for a certain parameter range the map begins to unearth the entanglement of states in the orthogonal complement of UPB.

The exposedness of maps has been discussed and used in the entanglement context in a recent interesting development [35]. It turns out that the automorphisms described in our work preserve the exposed property and thus if we start with an exposed map we can construct families of exposed maps. In this context extensions of external exposed maps have also been considered by Sarbicki and Chruściński [36].

In the context of UPB there is a way to interpolate between TILES and PYRAMID [30]. This possibility provides us with a rich variety of PPT entangled states. The possibility of detecting these states with extensions of already known P but not CP maps or implicating the non-CP character of certain maps using these states will be taken up elsewhere. There could be interesting consequences of these results in higher dimensions and they will also be taken up elsewhere.

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