On Global Dynamics of Schrödinger Map Flows on Hyperbolic Planes Near Harmonic Maps

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Abstract: The results of this paper are twofold. In the first part, we prove that for Schrödinger map flows from hyperbolic planes to Riemannian surfaces with non-positive sectional curvatures, the harmonic maps which are holomorphic or anti-holomorphic of arbitrary size are asymptotically stable. In the second part, we prove that for Schrödinger map flows from hyperbolic planes into Kähler manifolds, the admissible harmonic maps of small size are asymptotically stable. The asymptotic stability results stated here contain two types: one is the convergence in $L^\infty$ as the previous works, the other is convergence to harmonic maps plus radiation terms in the energy space, which is new in literature of Schrödinger map flows without symmetric assumptions.

1. Introduction

Let $(\mathcal{M}, h)$ be a Riemannian manifold and $(\mathcal{N}, J, g)$ be a Kähler manifold, the Schrödinger map flow is a map $u : (x, t) \in \mathcal{M} \times \mathbb{R} \mapsto \mathcal{N}$ which satisfies

\[
\begin{aligned}
    u_t &= J(u) h^{jk} \tilde{\nabla}_j \partial_k u \\
    u_{|t=0} &= u_0,
\end{aligned}
\]

where $\tilde{\nabla}$ denotes the pullback covariant derivative on $u^* T\mathcal{N}$. When $\mathcal{N}$ is the 2 dimensional sphere, (1.1) plays a fundamental role in solid-state physics and is usually referred as the Landau-Lifshitz equation (LL) or the continuous isotropic Heisenberg spin model in physics literature. LL describes the dynamics of the magnetization field inside ferromagnetic material (Landau–Lifshitz [24]), and its various forms are also related to many other problems such as vortex motions [12], gauge theories, motions of membranes. The general Schrödinger map flow with Kähler targets is a natural geometric generalization of the Landau–Lifshitz equation, and was first studied by geometers at late 1990s (see [61]). In this paper, we consider the case where $\mathcal{M}$ is the hyperbolic plane and $\mathcal{N}$ is a Riemannian surface or more generally Kähler manifold.
The Schrödinger map flow (SMF) on Euclidean spaces has been intensively studied. The local well-posedness and small data global well-posedness theory of SMF on $\mathbb{R}^d$ were developed by [3–5, 9, 13, 20, 34, 35, 39, 41, 48, 55, 56]. The dynamical behaviors of SMF on $\mathbb{R}^d$ near harmonic maps were studied in the equivariant case. For the equivariant SMF from $\mathbb{R}^2$ into $\mathbb{S}^2$ with energy below the ground state and equivariant flows from $\mathbb{R}^2$ into $\mathbb{H}^2$ with initial data of finite energy, the global well-posedness and scattering in the gauge sense were proved by Bejenaru-Ionescu-Kenig-Tataru [6, 7]. For the equivariant SMF from $\mathbb{R}^2$ into $\mathbb{S}^2$ with initial data near the harmonic map, Gustafson-Kang-Tsai [15] proved asymptotic stability for $m \geq 4$. And later Gustafson-Nakanishi-Tsai [16] proved that 3-equivariant SMF is also in the stable regime and the 2-equivariant (dissipative) LL has winding oscillatory solutions. For the 1-equivariant SMF, Bejenaru–Tataru [8] proved the harmonic map is unstable in the energy space and stable in a stronger topology. Merle-Raphael-Rodnianski [40] and Perelman [46] built type II blow up solutions near the harmonic maps for 1-equivariant SMF.

For the curved background manifolds, the sequel pioneering works of Lawrie, Oh, Shahshahani [28–32] studied global dynamics of wave maps on hyperbolic spaces in the equivariant case. And the first non-equivariant data result on hyperbolic spaces was obtained by Lawrie-Oh-Shahshahani [31] where optimal small data global theory for high dimensional wave maps on hyperbolic spaces was established. More recently, Lawrie–Luhrmann–Oh–Shahshahani [26, 27] studied global dynamics of SMF under the non-equivariant perturbations of strongly linear stable equivariant harmonic maps from $\mathbb{H}^2$ to rotationally symmetric Riemannian surfaces. Their works exploit the delicate smoothing effects of Schrödinger equations. On the other hand, for wave maps on small perturbations of Euclidean spaces, Lawrie [25] studied small data global theory for high dimensions, and Gavrus–Jao–Tataru [14] established optimal local well-posedness in the energy critical case. We also mention the works of [33, 36, 37] where we proved asymptotic stability of harmonic maps under the wave map between hyperbolic planes. And for wave maps on product spaces of spheres and Euclidean spaces, Shatah, Tahvildar-Zadeh [51] and Shahshahani [49] studied orbital stability of stationary solutions.

In this paper, we study stability of non-equivariant harmonic maps under the Schrödinger map flow from $\mathbb{H}^2$ to Riemannian surfaces or general Kähler manifolds. The main result is the stability of holomorphic or anti-holomorphic maps from $\mathbb{H}^2$ to Riemannian surfaces.

In the following, all the maps studied are assumed or proved to have bounded images in $\mathcal{N}$. So for convenience, given the unperturbed harmonic map $Q : \mathcal{M} \to \mathcal{N}$ with bounded image, we fix $\mathcal{N}$ to be an open sub-manifold of $\mathcal{N}$ containing $Q(\mathcal{M})$. Moreover, let $\mathcal{P} : \mathcal{N} \to \mathbb{R}^N$ be one isometric embedding of $\mathcal{N}$ into Euclidean space $\mathbb{R}^N$. Moreover, for maps $u : \mathbb{H}^2 \to \mathcal{N}$ with image $Q(M)$ contained in $\mathcal{N}$ a.e., we define the corresponding extrinsic Sobolev space $\mathcal{H}^s_{Q}$ by the norm

$$\|u\|_{\mathcal{H}^s_{Q}} := \|\mathcal{P}(u) - \mathcal{P}(Q)\|_{\mathcal{H}^s(\mathbb{H}^2; \mathbb{R}^N)}.$$

**Definition 1.1.** A map $f$ from Kähler manifold $(\mathcal{M}_1, J_1, g_1)$ to Kähler manifold $(\mathcal{M}_2, J_2, g_2)$ is said to be holomorphic if

$$f_*(J_1X) = J_2f_*(X), \quad \forall X \in T\mathcal{M}_1,$$

or anti-holomorphic if

$$f_*(J_1X) = -J_2f_*(X), \quad \forall X \in T\mathcal{M}_1.$$
Note that both holomorphic and anti-holomorphic maps are harmonic maps.

Our main theorem is as follows.

**Theorem 1.1.** Let $\mathcal{M} = \mathbb{H}^2$, and $\mathcal{N}$ be a Riemannian surface with non-positive sectional curvatures. Assume that $Q : \mathcal{M} \to \mathcal{N}$ is either a holomorphic or an anti-holomorphic map such that the image $Q(\mathcal{M})$ is contained in a compact subset of $\mathcal{N}$. Given any $\delta > 0$, there exists a sufficiently small constant $\epsilon_* > 0$ such that if $u_0 \in H^3_\mathcal{Q}$ satisfies

$$\|u_0 - Q\|_{H^{2+\delta}} \leq \epsilon_*,$$  \hspace{1cm} (1.2)

then the Schrödinger map flow with initial data $u_0$ evolves into a global solution and converges to $Q$ as $t \to \infty$ in the following two senses:

- **On one side,** we have
  $$\lim_{t \to \infty} \|u(t) - Q\|_{L^\infty} = 0;$$  \hspace{1cm} (1.3)

- **On the other side,** there exist functions $f_1^1, f_2^2 : \mathcal{M} \to \mathbb{C}^N$ belonging to $H^1$ such that
  $$\lim_{t \to \infty} \|u - Q - \text{Re}(e^{it\Delta} f_1^1) - \text{Im}(e^{it\Delta} f_2^2)\|_{H^1_\mathcal{Q}} = 0,$$  \hspace{1cm} (1.4)

where we view $u$ and $Q$ as maps into $\mathbb{R}^N$.

**Remark 1.1.** Results as (1.3) were first established by Tao [58–60] for the wave map equation on $\mathbb{R}^2$, and later obtained by [27,31–33,37] in the setting of wave maps/ Schrödinger map flows on hyperbolic spaces. The type result (1.4) is new in the setting of non-equivariant Schrödinger map flows on both Euclidean spaces and curved base manifolds.

To understand (1.4), it is convenient to think of the trivial target $\mathcal{N} = \mathbb{R}^{2n}$. Then the complex structure is given by the matrix

$$J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$$

where $I_n$ denotes the $n \times n$ identity matrix. It is easy to check that the solution to Schrödinger map flow is now given by

$$u^j(t) = \text{Re}(e^{it\Delta} f_0^j), \quad u^{j+n}(t) = \text{Im}(e^{it\Delta} f_0^j), \quad j = 1, ..., n,$$

where we denote $u = (u^1, ..., u^{2n})$, and $f_0^j = u_0^j + iu_0^{j+n}$ with $j = 1, ..., n$ is defined via the initial data $u_0$.

**Remark 1.2.** The same arguments of Theorem 1.1 also refine our previous results [37] on stability of harmonic maps between $\mathbb{H}^2$ under the wave map evolutions. In fact, we can as well prove that given an admissible harmonic map $Q$ from $\mathbb{H}^2$ to non-positively curved Riemannian surfaces, the solution to the wave map equation with initial data $(u_0, u_1)$ satisfying $\|u_0 - Q\|_{H^2} + \|u_1\|_{H^1} \ll 1$, evolves to a global solution and scatters to $Q$ in the energy space, i.e., there exist some functions $(v_0, v_1) : \mathbb{H}^2 \to \mathbb{R}^N \times \mathbb{R}^N$ belonging to $H^1_\mathcal{Q} \times L^2_\mathcal{Q}$ such that
\[
\lim_{t \to \infty} \| (u, \partial_t u) - (Q, 0) - (v_L, \partial_t v_L) \|_{H^1_\times L^2_\times} = 0, \tag{1.5}
\]
where \( v_L \) denotes the solution of linear wave equation on \( \mathbb{H}^2 \) with initial data \((v_0, v_1)\):

\[
\begin{cases}
\partial_t^2 v_L - \Delta v_L = 0, \\
(v_L, \partial_t v_L)|_{t=0} = (v_0, v_1).
\end{cases}
\]

**Remark 1.3.** Holomorphic maps and anti-holomorphic maps are typical harmonic maps between Kähler manifolds. A remarkable observation of \[27\] reveals that linearized operators around holomorphic maps between Riemannian surfaces are self-adjoint. We focus on holomorphic maps and anti-holomorphic maps to take this convenience for large data. In several cases, one can show harmonic maps are either holomorphic or anti-holomorphic, see Siu-Yau \[53\], Siu \[52\] and references therein. One may see “Appendix B” for more background materials for holomorphic maps and anti-holomorphic maps.

**Remark 1.4.** The perturbation regularity we assume is \( H^{2+2\delta} \). In the work \[27\], for equivariant harmonic maps \( Q \) (which are especially holomorphic) and rotationally symmetric Riemannian surfaces \( \mathcal{N} \), the stability was proved for non-equivariant initial data \( u_0 \) satisfying \( \| u_0 - Q \|_{H^{1+2\delta}} + \| D_\Omega u_0 \|_{H^{1+2\delta}} \ll 1 \), where \( D_\Omega \) can be viewed as a derivative measuring how co-rotational \( u_0 \) is. We see the topology of perturbations assumed here is almost the same as that used in \[27\] in the angle direction and stronger in the radial direction. Moreover, \[Remark 1.8, [27]\] pointed out that non-equivariant harmonic maps \( Q \) seem to require new ideas in addition to \[27\]'s arguments. Theorem 1.1 here covers all holomorphic maps and anti-holomorphic maps of bounded images, which contain wide class of non-equivariant harmonic maps.

**Remark 1.5.** Without non-positive sectional curvature assumptions, Theorem 1.1 also holds by assuming that the corresponding linearized operator satisfies some spectral conditions, e.g. the strongly linear stable condition introduced by \[27\].

**Definition 1.2.** Let \((r, \theta)\) be the polar coordinates of \( \mathbb{H}^2 \) with metric \( dr^2 + \sinh^2 r \, d\theta^2 \). We call \( Q : \mathbb{H}^2 \to \mathcal{N} \) an admissible harmonic map, if \( \| \nabla^j Q \|_{L^\infty_\times L^2_\times} + \| e^r \nabla^j Q \|_{L^\infty_\times} \lesssim 1 \) for \( j = 1, 2, 3, l = 1, 2 \). \( Q \) is called a small admissible harmonic map if \( Q \) is admissible and \( \| \nabla Q \|_{L^\infty} \) is sufficiently small.

We remark that any holomorphic map or anti-holomorphic map from \( \mathbb{H}^2 \) to Kähler targets with bounded image is an admissible harmonic map.

**Theorem 1.2.** Let \( \mathcal{M} = \mathbb{H}^2 \), and \( \mathcal{N} \) be a compact 2n-dimensional Kähler manifold. Let \( Q : \mathbb{H}^2 \to \mathcal{N} \) be a small admissible harmonic map with bounded image. Given any \( \delta > 0 \), if the initial data \( u_0 \in \mathcal{H}^2_\Omega \) to (1.1) satisfies

\[
\| u_0 - Q \|_{H^{2+2\delta}} \ll \mu, \tag{1.6}
\]

with \( \mu > 0 \) being sufficiently small, then (1.1) has a global solution \( u(t) \) and as \( t \to \infty \) we have

\[
\lim_{t \to \infty} \| u(t, x) - Q(x) \|_{L^\infty} = 0
\]

\[
\lim_{t \to \infty} \| u(t) - Q - \sum_{j=1}^n \text{Re}(e^{it\Delta} f^j_+) - \sum_{j=1}^n \text{Im}(e^{it\Delta} g^j_+) \|_{H^1_\times} = 0
\]

for some functions \( f^1_+, ..., f^n_+, g^1_+, ..., g^n_+ : \mathcal{M} \to \mathbb{C}^N \) belonging to \( H^1_\times \).
Remark 1.6. It is interesting that the dynamics for flows defined on Euclidean spaces and curved spaces are typically different and of independent interest. This in the setting of dispersive geometric flows was first noticed by [32]. One example is the difference of solutions to the stationary problem. For instance, Lawrie-Oh-Shahshahani [32] showed there exists harmonic map $Q_s$ solutions to the stationary problem. For instance, Lawrie-Oh-Shahshahani [32] showed there exists harmonic map $Q_s$ from $\mathbb{H}^2$ to $S^2$ and $\mathbb{H}^2$ with energy $\lambda$ for any given $\lambda > 0$. However, there exists no non-trivial finite energy harmonic map from $\mathbb{R}^2$ to $\mathbb{H}^2$ and the minimal energy to support a non-trivial harmonic map from $\mathbb{R}^2$ to $S^2$ is $4\pi$. The difference on stationary solutions directly leads to distinct long time dynamics. For instance, the energy critical wave maps from $\mathbb{R}^2$ to $\mathbb{H}^2$ are proved to scattering to free waves in energy space, while wave maps from $\mathbb{H}^2$ to $\mathbb{H}^2$ are believed to scattering to one stationary solution in energy space.

1.1. Outline of proof. The whole proof is set up by the previous geometric linearization strategy based on Tao’s caloric gauge. This linearization scheme gives rise to a nonlinear Schrödinger equation governing the evolution of heat tension field $\phi_s$ and a nonlinear heat equation of the Schrödinger map tension field $Z$. In the holomorphic or anti-holomorphic setting, the resulting linearized operator is self-adjoint. The difficulty is to compensate the derivative loss caused by the magnetic term $A \cdot \nabla \phi_s$.

In this work, to compensate the derivative loss, we essentially use three key estimates: inhomogeneous Morawetz estimates for time dependent magnetic Schrödinger operators; Strichartz estimates for linearized operator $\mathbf{H}$; energy estimates. They are derived for different linear evolution equations and play different roles, see Sect. 1.2 for more detailed expositions.

1.2. Main ideas of Theorem 1.1 and main linear estimates. The Morawetz estimate we adopt reads as follows: Suppose that $A = A_j dx^j$ is a real valued one form on $\mathbb{H}^2$, and denote $\Delta_A, D_A$ to be

$$\Delta_A f = (\nabla_j + i A_j) h^{kj} (\nabla_k + i A_k) f, \quad D_A f = \partial_j f dx^j + i A_j f dx^j.$$ 

Let $u$ solve $i \partial_t u + \Delta_A u = F$, then there holds

$$\|e^{-\frac{t}{2}} \nabla u\|^2_{L^2_{t,x}} \lesssim \|(-\Delta)^{\frac{1}{4}} u\|^2_{L^\infty_t L^2_x} + \|\partial_t A\| \|u\|^2_{L^1_{t,x}} + \| |A| + |\nabla A| \| u\|^2_{L^1_{t,x}} + \| |D_A u| \| F\|^2_{L^1_{t,x}} + \| |A| + |\nabla A| \| u\| \|\nabla u\|_{L^1_{t,x}} + \| |u| \| F\|^2_{L^1_{t,x}}.$$ 

This behaves well in the sense that no weight nor derivative is imposed to the inhomogeneous term $F$. In application, $u$ is just $\phi_s$ and $A$ is the connection one form under the caloric gauge. Moreover, we remark that the derivative loss now indeed hides in $\partial_t A$, which is easier to handle than $\phi_s$, since the heat flow provides much better regularity for $A$ than $\phi_s$ itself.

The energy estimate is simple but useful: Let $u$ solve $i \partial_t u + \Delta_A u = F$, then there holds

$$\|\nabla u\|^2_{L^\infty_t L^2_x} \lesssim \|\nabla u\|_{L^1_{t,x}}^2 + \|A u\|^2_{L^\infty_t L^2_x} + \|\partial_t A\| \|D_A u\|^2_{L^1_{t,x}} + \|D_A F\| \|D_A u\|_{L^1_{t,x}} + \|Au\| \|\nabla u\|_{L^1_{t,x}}.$$ 

As the Morawetz estimates, the derivative loss is also hidden in $\partial_t A$. 


The third is Strichartz estimates of linearized operator \( H \), which enable us to control quadratic nonlinear terms. In fact, we have
\[
\| \phi_s \|_{L_t^q L_x^r} \lesssim \| \tilde{A} \| \| \nabla \phi_s \|_{L_t^1 L_x^2} + \text{lower derivative terms},
\]
where \( q \in [4, \frac{8}{3}] \), and \( \tilde{A} \) denotes the connection one form under the caloric gauge removing away the limit part, see Sect. 2.

We aim to use Morawetz estimates and energy estimates to control the above magnetic term \( \tilde{A} \cdot \nabla \phi_s \) in Strichartz estimates. Meanwhile, we expect to use Strichartz estimates to control the zero derivative terms in Morawetz estimates and energy estimates.

In order to make the three key linear estimates work, we need additional ideas. In fact, in the right hand side of both Morawetz estimates and energy estimates, there are quadratic terms which are of the same order as the left hand side. This is something troublesome for closing bootstrap arguments. However, one observes that all these terms contain at least one zero derivative term or two half derivative terms, which inspires us to set up the bootstrap assumption as (3.11–3.13). The two key points for this setting are:

(i) The smallness size of \( \| \nabla \phi_s \|_{L_t^q L_x^r} \), \( e^{-\frac{1}{4}t} \| \phi_s \|_{L_t^2 L_x^2} \), \( \| \phi_s \|_{L_t^q L_x^r} \) shall be decreasing in order;
(ii) The regularity loss measured by the power of \( s \) (the parameter \( \gamma \) in \( \omega_{\gamma}(s) \)) shall be the same for these three quantities, see (3.11–3.13).

The point (i) helps to control quadratic terms in Morawetz estimates and energy estimates, and the point (ii) enables us to close bootstrap via using Morawetz estimates and energy estimates to dominate magnetic terms in Strichartz estimates. We give a heuristic explanation of how this argument works in Sect. 3.2.

The type result of (1.4) for dispersive geometric flows with non-equivariant data only appeared in the wave map setting with \( Q \) being a fixed point. The result itself is of independent interest, especially it coincides with the soliton resolution conjecture for critical dispersive PDEs, which claims that solutions with bounded trajectories would decouple into either separated solitons and radiations or separated solitons and some weak limit solution. The proof of (1.4) consists of two parts, one is to derive Strichartz estimates, and the point (ii) enables us to close bootstrap via using Morawetz estimates for higher order derivatives of \( \phi_s \), the other is to establish the linear scattering theory of the linearized magnetic Schrödinger operators.

**Notations** We will use the notation \( a \lesssim b \) whenever there exists some positive constant \( C \) so that \( a \leqCb \). Similarly, we will use \( a \sim b \) if \( a \lesssim b \lesssim a \). For a linear operator \( T \) from Banach space \( X \) to Banach space \( Y \), we denote its operator norm by \( \|T\|_{L(X \to Y)} \). All the constants are denoted by \( C \) and they may change from line to line.

Let \( \mathcal{Z} \) be a manifold with connection \( D \). Given a frame \( \{e_a\} \) for \( T^* \mathcal{Z} \) and corresponding dual frame \( \{\xi_a\} \) for \( T^* \mathcal{Z} \), for an arbitrary \( (r, s) \)-type tensor \( \mathbb{T} \), we write
\[
D_{c_1}...D_{c_k} e_{a_1,...,a_r} = (DD...D\mathbb{T})(e_{c_1}, ..., e_{c_k}; \xi_{a_1}, ..., \xi_{a_r}, e_{b_1}, ..., e_{b_s}).
\]

Let \( \mathbb{R}^{1+2} \) be the \((1+2)\)-dimensional Minkowski space equipped with metric \( \mathbf{m} := -dy^0dy^0 + dy^1dy^1 + dy^2dy^2 \). The hyperbolic plane denoted by \( \mathbb{H}^2 \) is defined by
\[
\mathbb{H}^2 := \{ y \in \mathbb{R}^{1+2} : (y^0)^2 - (y^1)^2 - (y^2)^2 = 1, y^0 > 0 \},
\]
equipped with the pullback Riemannian metric \( h := \iota^* \mathbf{m} \), where \( \iota : \mathbb{H}^2 \to \mathbb{R}^{1+2} \) is the inclusion map.
Let \((\mathcal{M}, h)\) be a \(d\)-dimensional Riemannian manifold. The Riemannian curvature tensor on \((\mathcal{M}, h)\) denoted by \(R\) is defined by

\[
R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]}Z, \quad X, Y, Z \in T\mathcal{M}.
\]

And we also denote

\[
R(X, Y, Z, W) = h(R(Z, W)X, Y), \quad X, Y, Z, W \in T\mathcal{M}.
\]

For local coordinates \((x^1, \ldots, x^d)\) for \(\mathcal{M}\), the curvature tensor components are defined by

\[
R_{ijkl} := h(R(\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l}) \frac{\partial}{\partial x^j}, R(\frac{\partial}{\partial x^l}, \frac{\partial}{\partial x^j}) \frac{\partial}{\partial x^k} = R^l_{kij} \frac{\partial}{\partial x^j}.
\]

The Ricci tensor is defined by \(R_{ij} dx^i dx^j\) with

\[
R_{ij} = h^{kl} R_{iklj} = R^k_{ikj}.
\]

And the sectional curvature is defined via

\[
K(X, Y) := -\frac{R(X, Y, X, Y)}{\|X \wedge Y\|^2}, \quad \|X \wedge Y\|^2 := h(X, X)h(Y, Y) - h(X, y)^2, \quad X, Y \in T\mathcal{M}.
\]

We recall the skew-symmetry, symmetry, and first Bianchi identity of Riemannian curvature tensor as follows:

\[
R_{ijkl} = -R_{jikl} = -R_{ij lk},
\]

\[
R_{ijkl} = R_{klij},
\]

\[
R_{ijkl} + R_{lijk} + R_{kjli} = 0.
\]

The Riemannian curvature tensor on the target manifold \(\mathcal{N}\) will be denoted by \(\mathbf{R}\). In some cases, we use \(A \cdot B\) to denote \(A_j B^j\).

2. Preliminaries on Sobolev Inequalities and Caloric Gauges

2.1. Sobolev inequalities. In this part, we collect some preliminaries on function spaces and Sobolev inequalities on hyperbolic spaces.

Given a tensor \(\mathbf{T}\) defined on \(\mathbb{H}^2\), for \(k \in \mathbb{N}, p \in [1, \infty]\), the \(W^{k,p}\) norm is defined by

\[
\|\mathbf{T}\|_{W^{k,p}} := \left(\sum_{j=0}^{k} \int_{\mathbb{H}^2} |\nabla^j \mathbf{T}|^p d\vol_h\right)^{\frac{1}{p}},
\]

with standard modifications when \(p = \infty\).

For \(1 < p < \infty\), one has the equivalence relation:

\[
\|f\|_{W^{k,p}} \sim \|(-\Delta)^{k/2} f\|_{L^p}. \quad (2.1)
\]

If \(\gamma \geq 0\) is not an integer, we define the norm \(W^{\gamma,p}\) with \(p \in (1, \infty)\) to be

\[
\|f\|_{W^{\gamma,p}} := \|(-\Delta)^{\gamma/2} f\|_{L^p}.
\]
Lemma 2.1 ([1,27,31]). Let \( f \in C^\infty_c(\mathbb{H}^2; \mathbb{R}) \). Then for \( 1 \leq p \leq q \leq \infty \), \( 0 < \theta < 1 \), \( \frac{1}{q} - \frac{\theta}{2} = \frac{1}{p} \), the Gagliardo-Nirenberg inequality is
\[
\| f \|_{L^q} \lesssim \| \nabla f \|_{L^2}^{\theta} \| f \|_{L^p}^{1-\theta}. \tag{2.2}
\]
The Poincare inequality is
\[
\| f \|_{L^p} \lesssim \| \nabla f \|_{L^p}, \quad 1 < p < \infty. \tag{2.3}
\]
The \( L^p \) interpolation inequality holds as
\[
\| f \|_{L^q} \lesssim \| f \|_{L^p}^{1-\theta} \| (-\Delta)^{\alpha} f \|_{L^p}^{\theta}, \quad p \in (1, \infty), \quad 0 < \theta < \frac{2}{\alpha} \left( \frac{1}{q} - \frac{1}{p} \right) < 1.
\]
The Riesz transform is bounded in \( L^p \) for \( 1 < p < \infty \), i.e.,
\[
\| \nabla f \|_{L^p} \sim \| (-\Delta)^{\frac{1}{2}} f \|_{L^p}. \tag{2.4}
\]
The Sobolev product rule is
\[
\| fg \|_{H^\gamma} \lesssim \| f \|_{L^\infty} \| g \|_{H^\gamma} + \| g \|_{L^\infty} \| f \|_{H^\gamma}, \quad \gamma \geq 0.
\]
And we recall the general Sobolev inequality: Let \( 1 < p, q < \infty \) and \( \sigma_1, \sigma_2 \in \mathbb{R} \) such that \( \sigma_1 - \sigma_2 \geq 2/p - 2/q \geq 0 \). Then for all \( f \in C^\infty_c(\mathbb{H}^2; \mathbb{R}) \)
\[
\| (-\Delta)^{\sigma_2} f \|_{L^q} \lesssim \| (-\Delta)^{\sigma_1} f \|_{L^p}.
\]
The diamagnetic inequality known also as Kato’s inequality is as follows (see [31]): If \( T \) is a tension filed defined on \( \mathbb{H}^2 \), then in the distribution sense it holds that
\[
|\nabla |T|| \leq |\nabla T|.
\]

Recall that we always assume that \( \tilde{N} \) is an open sub-manifold of \( N \) which is isometrically embedded into \( \mathbb{R}^N \) by the map \( P \). For \( \gamma \geq 0 \), define \( \mathcal{H}_Q^\gamma \) to be
\[
\mathcal{H}_Q^\gamma := \{ u : \mathcal{M} \to \mathbb{R}^N : \| P(u) - P(Q) \|_{H^\gamma} < \infty, u(x) \in \tilde{N} \text{a.e.} \}
\]
For initial data \( u_0 \in \mathcal{H}_Q^3 \), we have the local well-posedness.

Lemma 2.2. Let \( k \geq 3 \) be an integer. If \( u_0 \in \mathcal{H}_Q^k \), then there exists a positive constant \( T_0 > 0 \) depending only on \( \| u_0 \|_{\mathcal{H}_Q^k} \) such that SMF with initial data \( u_0 \) has a unique local solution \( u \in C([0, T_0]; \mathcal{H}_Q^k) \).

Remark 2.1. McGahagan [39] introduced an approximate scheme for SMF via considering a generalized wave map equation. We remark that in the flat case \( \mathcal{M} = \mathbb{R}^d \), \( 1 \leq d \leq 3 \), Theorem 7.1 of Shatah, Struwe [50] gave a local theory for Cauchy problem of wave maps in \( H^2 \times H^1 \) by an energy method. The same energy arguments also yield a local well-posedness theory for the generalized wave map equation on \( \mathbb{H}^2 \) introduced above. Then our lemma follows by proving uniform Sobolev norms for approximate solutions.
2.2. Moving frames. Let $\mathbb{I}$ be $[0, T]$ or $[0, T] \times (0, \infty)$ for some $T > 0$. Let $\mathcal{N}$ be a $2n$-dimensional Kähler manifold. In the following, we make the convention that Greek indexes run in $\{1, \ldots, n\}$, and denote $\bar{\gamma} = \gamma + n$.

Since $\mathbb{I} \times \mathcal{M}$ with $\mathcal{M} = \mathbb{H}^2$ is contractible, there must exist global orthonormal frames for $u^*(T\mathcal{N})$. Using the complex structure one can assume the orthonormal frames are of the form

$$ E := \{ e_1(t, x), J e_1(t, x), \ldots, e_n(t, x), J e_n(t, x) \}. $$

Let $\psi_j = (\psi_j^1, \psi_j^\bar{1}, \ldots, \psi_j^n, \psi_j^{\bar{n}})$ for $j = 0, 1, 2$ be the components of $\partial_t u$ in the frame $E$:

$$ \psi_j^\gamma = \langle \partial_j u, e_\gamma \rangle, \psi_j^{\bar{\gamma}} = \langle \partial_j u, J e_\gamma \rangle, \gamma = 1, \ldots, n. \quad (2.6) $$

We always use 0 to represent $t$ in index. The isomorphism of $\mathbb{R}^{2n}$ to $\mathbb{C}^n$ induces a $\mathbb{C}^n$-valued 1-form $\phi_j dx^j$ defined by

$$ \phi_j^\gamma = \psi_j^\gamma + i \psi_j^{\bar{\gamma}}, \gamma = 1, \ldots, n. \quad (2.7) $$

Conversely, given a function $\phi : \mathbb{I} \times \mathcal{M} \to \mathbb{C}^n$, we associate it with a section $\phi e$ of the bundle $u^*(T\mathcal{N})$ via

$$ \phi \mapsto \phi e := \text{Re}(\phi^\gamma) e_\gamma + \text{Im}(\phi^\gamma) J e_\gamma, \quad (2.8) $$

where $(\phi^1, \ldots, \phi^n)$ denotes the components of $\phi$.

Let $E$ denote the trivial complex vector bundle over base manifold $\mathbb{I} \times \mathbb{H}^2$ with fiber $\mathbb{C}^n$. Then $u$ induces a covariant derivative on $E$ defined by

$$ \nabla_j (\phi e) = (D_j \phi) e, $$

where $\nabla$ denotes the induced Levi-Civita connection on $u^*(T\mathcal{N})$. We can also decompose $D$ as

$$ D_j = \nabla_j + A_j, $$

where $\nabla$ denotes the covariant derivative on $\mathbb{I} \times \mathcal{M}$ with trivial product metric, and $A_j$, which takes values in $n \times n$ complex matrices, is given by

$$ (A_j)_{\beta, \alpha} := [A_j]_\beta^\alpha + i [A_j]_\alpha^{\bar{\beta}}, \quad [A_j]_m^l := \langle \nabla_j e_l, e_m \rangle. $$

Recall that $\phi$ defined in (2.7) and (2.6) is a 1-form on $\mathbb{I} \times \mathcal{M}$ with values in $\mathbb{C}^n$,

$$ \phi = \phi_t dt + \phi_j dx^j, $$

and $A := A_t dt + A_j dx^j$ is a matrix valued 1-form on $\mathbb{I} \times \mathcal{M}$. The torsion free identity and the commutator identity hold as follows:

$$ D_k \phi_j = D_j \phi_k $$

$$ ([D_k, D_j] \phi) e = ((\nabla_k A_j - \nabla_j A_k + [A_k, A_j]) \phi) e $$

$$ = R(\partial_k u, \partial_j u)(\phi e). $$
Schematically, we write

\[ [D_k, D_j] = \mathcal{R}(\phi_k, \phi_j). \]

With the notations given above, (1.1) can be written as

\[ \phi_t = i \sum_{j=1}^{2} D^j \phi_j. \]  

(2.9)

2.3. Caloric gauge. The caloric gauge was first introduced by Tao [57,58] in the wave map setting. Later, it was applied to various geometric PDEs, see [5,31,42–45,54] for instance. Roughly speaking, Tao’s caloric gauge in our setting is to parallel transport the frame fixed on the limit harmonic map along the heat flow to the original map \( u(t) : M \to N \). Note that the heat flow from \( H^2 \) to \( N \) with initial data near reasonable harmonic maps \( Q \) evolves to a global solution and converges to the unperturbed harmonic map \( Q \) as heat time goes to infinity, see Lemma 10.2. Here, Tao’s caloric gauge can be defined as follows.

**Definition 2.1.** Assume \( Q \) is an admissible harmonic map. Let \( u(t, x) : [0, T] \times M \to N \) be a solution of (1.1) in \( C([0, T]; \mathcal{H}^2_Q) \). For a given orthonormal frame \( E^\infty := \{ e^\infty_\alpha, Je^\infty_\alpha \}_{\alpha=1}^{n} \) for \( Q^*TN \), a caloric gauge is a tuple consisting of a map \( v : \mathbb{R}^+ \times [0, T] \times M \to N \) and an orthonormal frame \( E(v(s, t, x)) := \{ e_1, Je_2, ..., e_n, Je_n \} \) such that

\[
\begin{align*}
\partial_s v &= h^{jk} \tilde{\nabla}_j \partial_k v \\
v(0, t, x) &= u(t, x)
\end{align*}
\]  

(2.10)

and

\[
\begin{align*}
\tilde{\nabla}_s e_k &= 0, \quad k = 1, ..., n \\
\lim_{s \to \infty} e_k &= e^\infty_k.
\end{align*}
\]  

(2.11)

Here, in (2.10) we denote \( \tilde{\nabla} \) the induced covariant derivative on \( v^*TN \) with abuse of notations.

Let \( u : [0, T] \times H^2 \to N \) be a map such that \( u \in C([0, T]; \mathcal{H}^2_Q) \). Then given \( t \in [0, T] \), our previous works [33,37] show the heat flow equation (2.10) with initial data \( u(t) \) has a global solution \( v(s, t, x) \) and converges to \( Q \) as \( s \to \infty \). Let \( \psi_s = (\psi^1_s, \psi^n_s, \psi^n_s) \) be the components of \( \partial_s v \) in the frame \( E \):

\[ \psi'_{s} = \partial_s v, \quad \psi_{s} = \partial_s v, \quad \psi_{y} = \partial_s v, \quad \gamma = 1, ..., n. \]

Define the corresponding \( \mathbb{C}^n \) valued function \( \phi_s \) by \( \phi_{s}^\alpha = \psi^\alpha_s + i \psi_{s}^\alpha, \alpha = 1, ..., n. \) \( \phi_s \) is usually called the heat tension field.

The caloric gauge in Definition 2.1 does exist and we can decompose the connection coefficients according to the gauge.
Lemma 2.3. Let $u$ be a solution of (1.1) in $C([0, T]; \mathcal{H}_Q^2)$. For any fixed frame $E^\infty := \{e^\infty_\alpha, J e^\infty_\alpha\}_{\alpha=1}^n$ for $\mathcal{Q}^*TN$, there exists a unique corresponding caloric gauge defined in Definition 2.1. Moreover, denote $E = \{e_\alpha, J e_\alpha\}_{\alpha=1}^n$ the caloric gauge, then we have for $j = 1, 2$

$$\lim_{s \to \infty} [A_j]^k_l(s, t, x) = \langle \tilde{\nabla} j e^\infty_k(x), e^\infty_l(x) \rangle$$
$$\lim_{s \to \infty} A_l(s, t, x) = 0,$$

where $\tilde{\nabla}$ denotes the induced connection on $\mathcal{Q}^*TN$. Particularly, denoting $A^\infty_j$ the limit coefficient matrix, i.e.,

$$(A^\infty_j)^\beta_\alpha := [A^\infty_j]^\beta_\alpha + i [A^\infty_j]^\beta_\alpha, \quad [A^\infty_j]^k_l := \langle \tilde{\nabla} j e^\infty_k, e^\infty_l \rangle |_{\mathcal{Q}(x)},$$

we have for $j = 1, 2, s > 0$,

$$A_j(s, t, x) = - \int_s^\infty \mathcal{R}(v(s'))(\phi_s, \phi_j)ds' + A^\infty_j,$$
$$A_l(s, t, x) = - \int_s^\infty \mathcal{R}(v(s'))(\phi_s, \phi_l)ds'.$$

Remark 2.2. We adopt some notations for convenience: Lemma 2.3 shows $A_j$ can be decomposed into the limit part and the effective part:

$$A_j(s, t, x) = A^\infty_j + \tilde{A}_j(s, t, x); \quad \tilde{A}_j := - \int_s^\infty \mathcal{R}(v(s'))(\phi_s, \phi_j)ds'.$$

Similarly, we split $\phi_i$ into

$$\phi_i = \phi^\infty_i + \tilde{\phi}_i,$$
$$\tilde{\phi}_i = - \int_s^\infty \partial_s \phi_i ds'.$$

where the $\beta$-component of $\phi^\infty_j$ is

$$\left\langle \partial_j Q, e^\infty_\beta \right\rangle + i \left\langle \partial_j Q, J e^\infty_\beta \right\rangle.$$

We shall use the following evolution equations of heat tension field and Schrödinger map tension field.

Lemma 2.4. We have for any $s \geq 0$,

$$\phi_s = D^j \phi_j. \quad (2.12)$$

Define Schrödinger map tension field to be $Z := \phi_t - i\phi_s$. Then, the Schrödinger map tension field $Z$ in the heat direction satisfies

$$\partial_s Z = \Delta_A Z + \mathcal{R}(Z, \phi^j) \phi_j - i \mathcal{R}(\phi_s, \phi^j) \phi_j + \mathcal{R}(i \phi_s, \phi^j) \phi_j. \quad (2.13)$$

Along the Schrödinger direction, the heat tension field $\phi_s$ satisfies the nonlinear Schrödinger equation:

$$i D^j \phi_s + \Delta_A \phi_s = i \partial_s Z - \mathcal{R}(\phi_s, \phi^j) \phi_j. \quad (2.14)$$


3. Proof of Theorem 1.1, Part I: Estimates of Heat Tension Field \( \phi_s \)

Given \( \mathcal{N} \) and \( Q \) satisfying Theorem 1.1, let \( u \) be a solution to SMF in \( C([0, T]; \mathcal{H}^2_Q) \). Denote \( v \) the solution to heat flow with initial data \( u \). Let \( \{e_\alpha, Je_\alpha\}_{\alpha=1}^n \) be the corresponding caloric gauge with prescribed limit gauge \( \{e_\infty^\alpha, Je_\infty^\alpha\}_{\alpha=1}^n \) for \( Q^*T\mathcal{N} \). Denote the induced connections on \( Q^*T\mathcal{N}, v^*T\mathcal{N} \) by \( \tilde{\nabla} \) and \( \nabla \) respectively. For Riemannian surface targets \( \mathcal{N} \) with Gauss curvature \( \kappa : \mathcal{N} \to \mathbb{R} \), the connection 1-form \( A \) can be corresponded to two real valued functions \( A_1, A_2 \):

\[
A_j = \langle \tilde{\nabla}_j e_1, Je_1 \rangle,
\]

and the induced connection on the trivial complex vector bundle \( \mathcal{M} \times \mathbb{I} \times \mathbb{C} \) now reads as

\[
D_j = \nabla_j + iA_j.
\]

The commutator identity is

\[
[D_j, D_k] \psi = i \kappa(v) \text{Im}(\phi_j \phi_k) \psi.
\]

And the identities in Lemmas 2.3 and 2.4 now can be formulated as

\[
A_j(s, t, x) = -\int_s^\infty \kappa \text{Im}(\phi_s \phi_j) ds' + A_j^\infty
\]

\[
\partial_x Z = \Delta_A Z + i \kappa \phi^j \phi_j \bar{\phi}_s - i \kappa \text{Im}(\phi^j \bar{Z}) \phi_j, \quad Z(0, t, x) = 0
\]

\[
i \partial_s \phi_s + \Delta_A \phi_s = A_1 \phi_s + i \partial_s Z + i \kappa \text{Im}(\phi^j \bar{\phi}_s) \phi_j
\]

\[
\partial_s \phi_s = \Delta_A \phi_s - i \kappa \text{Im}(\phi^j \bar{\phi}_s) \phi_j
\]

\[
\kappa \text{Im}(\phi^j \phi_j) = \kappa^\infty \text{Im}(\phi^j \phi_j) + \tilde{\kappa} \text{Im}(\phi^j \phi_j),
\]

where we denote

\[
\kappa^\infty(x) = \kappa(Q(x)), \quad \tilde{\kappa} = \kappa(v) - \kappa^\infty.
\]

The connection \( A \) now can be decomposed as

\[
A_j = \tilde{A}_j + A_j^\infty
\]

\[
\tilde{A}_j = A_j^{lin} + A_j^{qua}
\]

\[
A_j^{lin} = \kappa^\infty \int_s^\infty \text{Im}(\phi_j \phi_s^\infty) ds'
\]

\[
A_j^{qua} = \int_s^\infty \tilde{\kappa} \text{Im}(\phi_j \phi_s^\infty) ds' + \int_s^\infty \kappa \text{Im}(\phi_j \phi_s) ds'.
\]

As mentioned in Remark 1.2, the linearized operator is self-adjoint.

Lemma 3.1. Assume that \( Q \) is a holomorphic map or an anti-holomorphic map from \( \mathbb{H}^2 \) to Riemannian surface \( \mathcal{N} \), and \( Q \) has a bounded image. Then the operator \( H \) defined by

\[
Hf := \Delta_A f - \kappa^\infty h^{jk} \text{Im}(\phi_k^\infty \bar{f}) \phi_j^\infty
\]

is self-adjoint in \( L^2 \) with domain \( H^2 \). If \( \mathcal{N} \) is non-positively curved, then \( H \) is non-negative.
For more details on holomorphic maps and anti-holomorphic maps, one may read Appendix B. And we remark that $A_\infty, \phi_\infty$ and their covariant derivatives decay exponentially, see Lemma 10.1 in Appendix B. In addition, Proposition 5.1 in Sect. 5 presents more properties of $H$.

**Remark 3.1.** The remarkable observation of [27] shows the linear term $h_{jk}^\infty \phi_k^\infty \phi_j^\infty \bar{\phi}_s$ in the RHS of (3.4) vanishes for holomorphic maps $Q$. This also holds for anti-holomorphic maps, see Appendix B for some discussions.

3.1. Bootstrap. In this part, we explain the main bootstrap procedure by assuming the three key estimates in Sect. 1.2.

Fix $L \geq 100$. Given $\gamma \in \mathbb{R}$, define the functions $\omega_\gamma(s) : [0, \infty) \to \mathbb{R}^+$, $\tilde{\omega}_\gamma(s) : [0, \infty) \to \mathbb{R}^+$ to be

$$\omega_\gamma(s) = \begin{cases} s^\gamma, & s \in [0, 1) \\ s^L, & s \in [1, \infty) \end{cases}$$

$$\tilde{\omega}_\gamma(s) = \begin{cases} s^\gamma, & s \in [0, 1) \\ s^{L-1}, & s \in [1, \infty) \end{cases}$$

Let $0 < \delta < \frac{1}{2}$ be a fixed constant. Also, define the function $\Theta_\gamma(s, s') : \{ (s, s') \in \mathbb{R}^+ \times \mathbb{R}^+ : s \geq s' \} \to \mathbb{R}^+$:

$$\Theta_\gamma(s, s') = \begin{cases} (s - s')^{-\gamma}, & s - s' \in (0, 1) \\ e^{-\rho_\delta(s-s')}, & s - s' \in [1, \infty) \end{cases}$$

where $\rho_\delta > 0$ depends only on the given constant $\delta \in (0, \frac{1}{2})$. Denote

$$M := \sum_{j=0}^3 \| \nabla^j A_\infty \|_{L^2_t L^\infty_x \cap L^\infty_x} + \sum_{j=0}^2 \| \nabla^j \phi_\infty \|_{L^2_t L^2_x \cap L^\infty_x} + \sum_{j=0}^2 \sup_{y \in \tilde{N}} | D^j \kappa(y) |,$$

where $D$ denotes the covariant derivative with respect to the Riemannian metric of $\tilde{N}$.

Let $\epsilon_1, \epsilon_\ast > 0$ be sufficiently small satisfying

$$\epsilon_\ast \leq \epsilon_1^3; \quad \epsilon_1 \frac{1}{30} (M^4 + 1) \leq 1.$$

Our main result is

**Proposition 3.1.** Assume that $u_0$ satisfies Theorem 1.1. Suppose that $u \in C([0, T_\ast); \mathcal{H}_Q^3)$. Then we have

$$\sup_{s>0} \omega_\frac{1}{2-\delta}(s) \| \phi_s \|_{L^2_t L^\frac{8}{3} \cap L^\infty_x (0, T_\ast) \times \mathbb{H}^2} \leq \epsilon_1^{\frac{1}{3}} \quad (3.8)$$

$$\sup_{s>0} \omega_\frac{1}{2-\delta}(s) \| e^{-\frac{1}{2} \tau} \nabla \phi_s \|_{L^2_t L^2_x (0, T_\ast) \times \mathbb{H}^2} \leq \epsilon_1^{\frac{1}{3}} \quad (3.9)$$

$$\sup_{s>0} \omega_\frac{1}{2-\delta}(s) \| \nabla \phi_s \|_{L^\infty_t L^2_x (0, T_\ast) \times \mathbb{H}^2} \leq \epsilon_1^{\frac{1}{3}}. \quad (3.10)$$
In order to prove Prop. 3.1, we rely on a bootstrap argument. Fix
\[ \frac{1}{2} + \frac{1}{50} < \alpha < \beta + \frac{1}{100} < \frac{1}{2} (1 + \alpha) < 1. \]

Assume that \( T \) is the maximal positive time such that for all \( 0 \leq T < T \), \( q \in [4, \frac{8}{3}] \), there hold
\[
\begin{align*}
\sup_{s > 0} \omega_{\frac{1}{2} - \delta}(s) \| \phi_s \|_{L_{x}^{2} L_{t}^{q} \cap L_{x}^{1} L_{t}^{q}(\mathbb{R}^2 \times \mathbb{R}^2)} & \leq \epsilon_1, \\
\sup_{s > 0} \omega_{\frac{1}{2} - \delta}(s) \| e^{-\frac{1}{2} f} \nabla \phi_s \|_{L_{x}^{2} L_{t}^{q}(\mathbb{R}^2 \times \mathbb{R}^2)} & \leq \epsilon_1^\beta, \\
\sup_{s > 0} \omega_{\frac{1}{2} - \delta}(s) \| \nabla \phi_s \|_{L_{x}^{2} L_{t}^{q}(\mathbb{R}^2 \times \mathbb{R}^2)} & \leq \epsilon_1^{\alpha}.
\end{align*}
\] (3.11) (3.12) (3.13)

The following lemma and the local theory Lemma 2.2 show \( T > 0 \).

**Lemma 3.2.** Assume that \( u \in C([0, T_0]; \mathcal{H}^1_2) \) be a solution to SMF. Then for any \( 0 < T < T_0 \), one has
\[
\sup_{s > 0} \omega_{0}(s) \| \phi_s \|_{L_{x}^{\infty} H_{x}^{1}(\mathbb{R}^2)} \lesssim \epsilon_*. \] (3.14)

Lemma 3.2 is a corollary of Lemma 10.2 in Appendix B.

By Sobolev embedding and local well-posedness of SMF, we see Lemma 3.2 indeed yields \( T > 0 \).

In order to prove \( T = T_\epsilon \), we apply Strichartz estimates, Morawetz estimates and energy estimates to the equation (3.5) to conclude that

**Proposition 3.2.** With the above assumptions (3.11–3.13), we have for any \( T \in (0, T) \), \( q \in [4, \frac{8}{3}] \) that
\[
\begin{align*}
\sup_{s > 0} \omega_{\frac{1}{2} - \delta}(s) \| \phi_s \|_{L_{x}^{2} L_{t}^{q} \cap L_{x}^{1} L_{t}^{q}(\mathbb{R}^2 \times \mathbb{R}^2)} & \lesssim M \epsilon_1^{1+\alpha}, \\
\sup_{s > 0} \omega_{\frac{1}{2} - \delta}(s) \| e^{-\frac{1}{2} f} \nabla \phi_s \|_{L_{x}^{2} L_{t}^{q}(\mathbb{R}^2 \times \mathbb{R}^2)} & \lesssim M \epsilon_1^{\frac{1}{4}(1+\alpha)}, \\
\sup_{s > 0} \omega_{\frac{1}{2} - \delta}(s) \| \nabla \phi_s \|_{L_{x}^{2} L_{t}^{q}(\mathbb{R}^2 \times \mathbb{R}^2)} & \lesssim M \epsilon_1.
\end{align*}
\] (3.15) (3.16) (3.17)

If Proposition 3.2 is done, with Lemma 3.2 and bootstrap we finish the proof of Proposition 3.1. Then it is not hard to prove that the solution \( u \) of SMF converges to harmonic maps in \( L_{x}^{\infty} \) norm by Proposition 3.1 (see Sects. 4.1 and 4.2). The resolution in energy space will be proved in Sect. 4.3.

### 3.2. Main idea and outline of proof to Proposition 3.2

In this subsection, we give a heuristic explanation for how one uses the three key linear estimates to prove Proposition 3.2. Assuming \( \mathcal{N} \) is of constant curvature and omitting lower order terms, we can roughly write

\[
\begin{align*}
A_{lin} & \approx \int_{S}^{\infty} \phi \phi_s ds', \\
\tilde{\phi} & \approx \int_{S}^{\infty} \nabla \phi_s ds', \\
A_{qua} & \approx \int_{S}^{\infty} \tilde{\phi} \phi_s ds', \\
A_t & \approx \int_{S}^{\infty} \phi_t \phi_s ds', \\
Z(s) & \approx \int_{0}^{s} e^{iHs} \tilde{\phi} \phi_s ds' + \int_{0}^{s} e^{iHs} \tilde{\phi} \phi_s ds', \\
i \phi_t & \approx \phi_s - Z,
\end{align*}
\] (H1) (3.18)
where \( e^r |\nabla^j \phi^\infty | \lesssim 1 \) for \( j = 0, 1, 2, 3 \). Since \( \phi_s \) in the heat direction fulfills a nonlinear heat equation, we can expect that \( \phi_s \) behaves like \( e^{i \Delta} \phi_s \) (\( s=0 \)). So heuristically one has the parabolic estimates of \( \phi_s \) by the assumptions (3.11–3.13):

\[
\begin{align*}
\sup_{s > 0} \omega_{\frac{1}{2} - \delta}(s) \| \nabla^j \phi_s \|_{L_t^2 L_x^2 \cap L_t^\infty L_x^2 ([0,T] \times \mathbb{R}^2)} & \lesssim \epsilon_1 \\
\sup_{s > 0} \omega_{\frac{1}{2} - \delta}(s) \| e^{-\frac{i}{2} r} \nabla^j \phi_s \|_{L_t^2 L_x^2 ([0,T] \times \mathbb{R}^2)} & \lesssim \epsilon_1^\beta \\
\sup_{s > 0} \omega_{\frac{1}{2} - \delta}(s) \| \nabla^j \phi_s \|_{L_t^\infty L_x^2 ([0,T] \times \mathbb{R}^2)} & \lesssim \epsilon_1^\alpha.
\end{align*}
\]

Then by (H1), heuristically we expect the parabolic estimates of \( A^{lin}, A^{qua}, Z, \tilde{\phi}, \phi_s \) (H2)

\[
\left\{ \begin{array}{l}
\sup_{s > 0} \omega_{0}(s) \| A^{qua} \|_{L_t^1 L_x^{\infty}} + \sup_{s > 0} \omega_{\frac{1}{2} - \delta}(s) \| \partial_s Z \|_{L_t^1 L_x^2} \lesssim \epsilon_1^2 \\
\sup_{s > 0} \omega_{0}(s) \| \nabla A^{qua} \|_{L_t^1 L_x^4} \lesssim \epsilon_1^{1+\alpha} \\
\sup_{s > 0} \omega_{0}(s) \| e^{\frac{i}{2} r} A^{lin} \|_{L_t^2 L_x^2 \cap L_t^\infty L_x^2} \lesssim \epsilon_1 \\
\sup_{s > 0} \omega_{\frac{1}{2} - \delta}(s) \| \nabla Z \|_{L_t^1 L_x^2} \lesssim \epsilon_1^{\alpha+1} \\
\sup_{s > 0} \omega_{0}(s) \| \tilde{\phi} \|_{L_t^\infty L_x^2} \lesssim \epsilon_1^5 \\
\sup_{s > 0} \omega_{0}(s) \| \tilde{\phi} \|_{L_t^2 L_x^2} \lesssim \epsilon_1.
\end{array} \right.
\]

Now, we present how to bound the gradient terms of \( \nabla \phi_s \) in the three key linear estimates mentioned in Sect. 1. First, the Strichartz estimates in Lemma 7.4 show

\[
\| \phi_s \|_{L_t^\infty L_x^2 \cap L_t^2 L_x^q} \lesssim \| A^{lin} \nabla^j \phi_s \|_{L_t^1 L_x^2} + \| \partial_s Z \|_{L_t^1 L_x^2} + \text{zero derivative order terms of } \phi_s.
\]

for \( q \in [4, \frac{8}{3}] \). Then by (H2) and assumptions (3.11–3.13), we have

\[
\begin{align*}
\| \phi_s \|_{L_t^\infty L_x^2 \cap L_t^2 L_x^q} & \lesssim e^{\frac{i}{2} r} A^{lin} \| \nabla^j \phi_s \|_{L_t^1 L_x^2} + e^{-\frac{i}{2} r} \nabla \phi_s \|_{L_t^1 L_x^2} + \| A^{qua} \|_{L_t^1 L_x^2} \| \nabla \phi_s \|_{L_t^\infty L_x^2} + \| \partial_s Z \|_{L_t^1 L_x^2} + \ldots \\
& \lesssim \omega_{\frac{1}{2} - \delta}(s) (\epsilon_1^{\alpha+1} + \epsilon_1^{2+\alpha} + \epsilon_1^2),
\end{align*}
\]

which provides admissible bounds for \( \| \phi_s \|_{L_t^\infty L_x^2 \cap L_t^2 L_x^q} \).

Second, the energy estimate in Proposition 6.1, (H2) and assumptions (3.11–3.13) give

\[
\begin{align*}
\| \nabla \phi_s \|_{L_t^\infty L_x^2}^2 & \lesssim \| A \phi_s \|_{L_t^1 L_x^4}^2 + \| \partial_s Z \|_{L_t^1 L_x^2} \| \nabla \phi_s \|_{L_t^1 L_x^4} \\
& + \| \tilde{\phi} \|_{L_t^2 L_x^2} \| \nabla \phi_s \|_{L_t^1 L_x^4}^2 + \| \tilde{\phi} \|_{L_t^1 L_x^4} \| \nabla \phi_s \|_{L_t^1 L_x^4} + \ldots \\
& \lesssim \| A \|_{L_t^1 L_x^4} \| \phi_s \|_{L_t^2 L_x^2} \| \nabla \phi_s \|_{L_t^2 L_x^2} + \| \partial_s Z \|_{L_t^1 L_x^2} \| \phi_s \|_{L_t^\infty L_x^2} \\
& + \| \tilde{\phi} \|_{L_t^2 L_x^2} \| \nabla \phi_s \|_{L_t^2 L_x^2} e^{-\frac{i}{2} r} \nabla \phi_s \|_{L_t^2 L_x^2} + \| \tilde{\phi} \|_{L_t^1 L_x^2} \| \nabla \phi_s \|_{L_t^2 L_x^2} + \ldots \\
& \leq \omega_{\frac{1}{2} - \delta}(s) (\epsilon_1^2 + \epsilon_1^{2+\alpha} + \epsilon_1^{1+\alpha+\beta} + \epsilon_1^{2\alpha+2}).
\end{align*}
\]
Thus
\[ \omega_{\frac{1}{2} - \delta} (s) \| \nabla \phi_s \|_{L_t^\infty L_x^2} \lesssim \epsilon_1. \]

Finally, we dominate the gradient terms \( \nabla \phi_s \) in the Morawetz estimate (Corollary 6.1) as follows:
\[
\| e^{-\frac{T}{2} r} \nabla \phi_s \|_{L_t^2 L_x^2}^2 \\
\lesssim \| (-\Delta)^{\frac{1}{4}} \phi_s (|r| = 0) \|_{L_x^2}^2 + \| \nabla A \|_{L_t^1 L_x^1}^2 + \| \nabla \phi_s \|_{L_t^4 L_x^4}^2 + \| \nabla \phi_s \|_{L_t^1 L_x^1}^2 + \| \partial_x Z \|_{L_t^1 L_x^1} + \| \phi \|_{L_t^1 L_x^1} \| \nabla \phi_s \|_{L_t^1 L_x^1} + \ldots
\]

These are easier to dominate compared with the above energy estimate since only \( \| \nabla \phi_s \| \) appears in each term. By interpolation,
\[
\| (-\Delta)^{\frac{1}{2}} \phi_s (|r| = 0) \|_{L_x^2}^2 \lesssim \omega_{\frac{1}{2} - \delta}^{-2} \epsilon_1^{1+\alpha}.
\]

By (H2) and assumptions (3.11–3.13), one finds
\[
\| \nabla A \|_{L_t^1 L_x^1} \| \nabla \phi_s \|_{L_t^1 L_x^1} \lesssim \| e^{\frac{T}{2} r} \nabla A^{\infty} \|_{L_t^\infty L_x^2} \| e^{-\frac{T}{2} r} \nabla \phi_s \|_{L_t^2 L_x^2} \| \phi_s \|_{L_t^2 L_x^2}
\]
\[
+ \| e^{\frac{T}{2} r} \nabla A^{fin} \|_{L_t^2 L_x^4} \| \phi_s \|_{L_t^2 L_x^4} \| e^{-\frac{T}{2} r} \nabla \phi_s \|_{L_t^2 L_x^2}
\]
\[
+ \| \nabla A_{qua} \|_{L_t^2 L_x^4} \| \nabla \phi_s \|_{L_t^\infty L_x^4} \| \phi_s \|_{L_t^2 L_x^4}
\]
\[
\lesssim \omega_{\frac{1}{2} - \delta}^{-2} \left( \epsilon_1^{1+\beta} + \epsilon_1^{2+\beta} + \epsilon_1^{3+2\alpha} \right),
\]

and
\[
\| \partial_x Z \|_{L_t^1 L_x^1} \| \nabla \phi_s \|_{L_t^1 L_x^1} + \| \phi \|_{L_t^1 L_x^1} \| \nabla \phi_s \|_{L_t^1 L_x^1}
\]
\[
\lesssim \| \partial_x Z \|_{L_t^2 L_x^2} \| \nabla \phi_s \|_{L_t^\infty L_x^1} + \| \phi \|_{L_t^\infty L_x^1} \| e^{\frac{T}{2} r} \phi^{\infty} \|_{L_t^\infty L_x^4} \| e^{-\frac{T}{2} r} \nabla \phi_s \|_{L_t^2 L_x^2} \| \phi_s \|_{L_t^2 L_x^2}
\]
\[
+ \| \tilde{\phi} \|_{L_t^2 L_x^2}^2 \| \nabla \phi_s \|_{L_t^\infty L_x^2} \| \phi_s \|_{L_t^\infty L_x^2}
\]
\[
\lesssim \omega_{\frac{1}{2} - \delta}^{-2} \left( \epsilon_1^{2+3\alpha} + \epsilon_1^{1+\beta} + \epsilon_1^{3+2\alpha} \right).
\]

Thus
\[
\omega_{\frac{1}{2} - \delta} (s) \| e^{-\frac{T}{2} r} \nabla \phi_s \|_{L_t^2 L_x^2} \lesssim \epsilon_1^\frac{1}{2} (1+\alpha).
\]

Therefore, we have heuristically obtained Proposition 3.2. The concrete details are in the next subsection.

**Remark 3.2.** We remark that this argument does not work in the Euclidean space \( \mathcal{M} = \mathbb{R}^2 \) due to the less available Strichartz pairs. In fact, in the energy estimates, we need to control

\[ \| A_{qua} \|_{L_t^1 L_x^1}. \]

The only effective and available norm for \( \nabla \phi_s \) term to bound \( \| A_{qua} \|_{L_t^1 L_x^1} \) is the \( L_t^\infty L_x^2 \) norm. So one expects to use \( \| A_{qua} \|_{L_t^1 L_x^\infty} \), which further forces us to use the \( \| \phi_s \|_{L_t^2 L_x^P} \) norm. But no endpoint Strichartz estimate is available for \( \mathbb{R}^2 \).
3.3. Proof of Proposition 3.2. Slightly different from Sect. 2, we make the convention that the notation $\phi$ now denotes the 1-form $\phi = \phi_j dx^j$, and $A$ now denotes the 1-form $A = A_j dx^j$. Note that now neither $\phi$ nor $A$ includes the time component $\phi_t$, $A_t$. And the same holds for $\tilde{A}^{lin}$, $\tilde{A}^{qua}$, $\tilde{A}$, $\tilde{\phi}$.

The following is the heat estimates for $e^{sH}$. (see Proposition 5.1 in Sect. 5 for the proof)

**Lemma 3.3.** For any $1 < q < p \leq \infty$, there exists $\delta_q > 0$ such that

$$
\| (-\Delta)^{\gamma} e^{sH} f \|_{L^p_x} \lesssim e^{-\delta_q s} s^{-\gamma - \frac{1}{q} + \frac{1}{p}} \| f \|_{L^q_x}. \tag{3.19}
$$

**Lemma 3.4 (Parabolic Estimates of $\phi_s$).** Let (3.11–3.13) hold. Assume that for any $q \in [\frac{4}{\delta}, \frac{8}{\delta}]$

$$
\sup_{s>0} \tilde{\omega}_0(s) \| \tilde{A} \|_{L^\infty_{t,x}} \leq \epsilon_1 \tag{3.20}
$$

$$
\sup_{s>0} \tilde{\omega}_0(s) \| \nabla \tilde{A} \|_{L^\infty_t L^\infty_x} \leq \epsilon_1 \tag{3.21}
$$

$$
\sup_{s>0} \tilde{\omega}_0(s) \| \tilde{\phi} \|_{L^\infty_t L^\infty_x} \leq \epsilon_1 \tag{3.22}
$$

$$
\sup_{s>0} \tilde{\omega}_0(s) \| \nabla \tilde{\phi} \|_{L^\infty_t L^2_x} \leq \epsilon_1 \tag{3.23}
$$

$$
\sup_{s>0} \tilde{\omega}_1 \sqrt{2-\delta}(s) \| \nabla \tilde{\phi} \|_{L^2_t L^q_x} \leq \epsilon_1 \tag{3.24}
$$

Let $0 < T < T$, then for any $p \in [2, \infty]$, $q \in [\frac{4}{\delta}, \frac{8}{\delta}]$, we have

$$
\sup_{s>0} \omega_{1-\delta}(s) \| \nabla \phi_s \|_{L^2_t L^q_x} \lesssim_M \epsilon_1 \tag{3.25}
$$

$$
\sup_{s>0} \omega_{3-\delta}(s) \| \Delta \phi_s \|_{L^2_t L^q_x} \lesssim_M \epsilon_1 \tag{3.26}
$$

$$
\sup_{s>0} \omega_{\frac{1}{2}-\frac{7}{8}\delta}(s) \| \phi_s \|_{L^\infty_t L^\infty_x} \lesssim_M \epsilon_1 \tag{3.27}
$$

$$
\sup_{s>0} \omega_{1-\frac{7}{8}\delta}(s) \| \nabla \phi_s \|_{L^2_t L^\infty_x} \lesssim_M \epsilon_1 \tag{3.28}
$$

$$
\sup_{s>0} \omega_{\frac{1}{2}-\frac{7}{8}\delta}(s) \| \Delta \phi_s \|_{L^2_t L^\infty_x} \lesssim_M \epsilon_1 \tag{3.29}
$$

and

$$
\sup_{s>0} \omega_{1-\delta}(s) \| \Delta \phi_s \|_{L^\infty_t L^2_x} \lesssim_M \epsilon_1^q \tag{3.30}
$$

$$
\sup_{s>0} \omega_{\frac{3}{2}-\frac{1}{2}\delta - \frac{1}{p}}(s) \| (-\Delta)^{1+\frac{1}{2}\delta} \phi_s \|_{L^\infty_t L^p_x} \lesssim_M \epsilon_1^q \tag{3.31}
$$

$$
\sup_{s>0} \omega_{\frac{3}{2}-\delta - \frac{1}{p}}(s) \| (-\Delta) \phi_s \|_{L^\infty_t L^p_x} \lesssim_M \epsilon_1^q. \tag{3.32}
$$
Proof. Step 1.1 We obtain from (3.22), (3.23), (3.20) that

\[ \|du\|_{L^\infty_t L_x^2} + \|\nabla du\|_{L^\infty_t L_x^2} \lesssim M. \]

Then Lemma 10.3 shows

\[ \|dv\|_{L^\infty_t L_x^2} + \|\nabla dv\|_{L^\infty_t L_x^2} + \sup_{s > 0} \min(1, s^{1/2}) \|\nabla^2 dv\|_{L^\infty_t L_x^2} \lesssim M. \]  
\[ (3.33) \]

\[ \sup_{s > 0} \xi(s) \|\partial_s v\|_{L^\infty_t L_x^2} + \sup_{s > 0} \xi s^{1/2}(s) \|\nabla \partial_s v\|_{L^\infty_t L_x^2} \lesssim M, \]  
\[ (3.34) \]

where \( \xi(s) := 1_{s \in (0,1)}s^{1/2} + 1_{s \geq 1}e^{\rho_0 s}, \rho_0 > 0 \). Meanwhile, recall that in the intrinsic form one has

\[ |\nabla \tilde{A}| \lesssim \int_s^\infty |\nabla dv||\partial_s v|ds' + \int_s^\infty |dv||\nabla \partial_s v|ds' \]
\[ \|\nabla^2 \tilde{A}\| \lesssim \int_s^\infty |\nabla^2 dv||\partial_s v|ds' + \int_s^\infty |\nabla dv||\nabla \partial_s v|ds' + \int_s^\infty |dv||\nabla^2 \partial_s v|ds'. \]  
\[ (3.35) \]

Hence, we conclude

\[ \sup_{s > 0} \tilde{\omega}(s) \|\nabla^2 \tilde{A}\|_{L^\infty_t L_x^2} \lesssim M. \]  
\[ (3.36) \]

Step 1.2 Observe that

\[ |\tilde{k}| \lesssim \int_s^\infty |\partial_s \kappa(v)|ds' \lesssim \int_s^\infty |D\kappa||\partial_s v|ds' \]
\[ \lesssim M \int_s^\infty |\phi|ds', \]  
\[ (3.37) \]
\[ (3.38) \]

where \( D \) denotes the covariant derivative with respect to the Riemannian metric of \( \mathcal{N} \). Thus (3.37) and (3.34) show

\[ \|\tilde{k}\|_{L^\infty_t L_x^2} \lesssim M \int_s^\infty \xi^{-1}(s')ds' \lesssim M e^{-\rho_0 s} \]
\[ (3.39) \]
\[ \|\tilde{k}\|_{L^\infty_t L_x^2} \lesssim M \int_s^\infty \xi^{-1}_0(s')ds' \lesssim M e^{-\rho_0 s}. \]  
\[ (3.40) \]

Also, we have

\[ |\nabla \tilde{k}| \lesssim \int_s^\infty (|D^2\kappa||\partial_s v||\partial_x v| + |D\kappa||\nabla \partial_s v|ds' \]
\[ \lesssim M \int_s^\infty |\phi||\phi_x| + |\nabla \phi_x| + |A||\phi_x|ds'. \]  
\[ (3.41) \]
\[ (3.42) \]
Hence (3.41), (3.33) and (3.34) give

\[ \| \nabla \tilde{\kappa} \|_{L_t^\infty L_x^2} \lesssim M \int_s^\infty \left[ \frac{\zeta_1^{-1}(s') + \zeta_1^{-1}(s')}{} \right] ds' \lesssim M 1_{s \in (0, 1]} \ln s + e^{-\rho_0 s} 1_{s \geq 1} \]  
\[ (3.43) \]

\[ \| \nabla \tilde{\kappa} \|_{L_t^\infty L_x^2} \lesssim M \int_s^\infty \left[ \frac{\zeta_0^{-1}(s') + \zeta_1^{-1}(s')}{} \right] ds' \lesssim M e^{-\rho_0 s}. \]  
\[ (3.44) \]

**Step 2.** Recall the evolution equation of \( \phi_s \) in the heat direction:

\[ \partial_s \phi_s = \Delta_A \phi_s - i \kappa \text{Im}(\phi^j \overline{\phi_s}) \phi_j. \]  
\[ (3.45) \]

As mentioned before, the RHS of (3.45) is in fact

\[ \mathbf{H} \phi_s + 2i \tilde{A}_j \nabla^j \phi_s + i \nabla^j \tilde{A}_j \phi_s - 2 A_j^\infty \tilde{A}^j \phi_s - \tilde{A}_j \tilde{A}^j \phi_s - i \kappa \text{Im}(\tilde{\phi}^j \overline{\phi_s}) \phi_j^\infty \]

\[ - i \kappa \text{Im}(\phi_j^\infty \overline{\phi_s}) \tilde{\phi}^j - i \kappa \text{Im}(\phi^j \overline{\phi_s}) \tilde{\phi}_j - \kappa^k \text{Im}(\phi_k^\infty \overline{\phi_s}) \phi_j^\infty, \]

and we emphasize that \( \mathbf{H} \phi_s \) is the linear part and the other above terms are at least quadratic. Then by Lemma 3.3, (3.20–3.22), we get

\[ \| \nabla \phi_s \|_{L_t^2 L_x^q} \lesssim M \int_s^\infty \Theta_{\frac{1}{2}}(s, s') \| \tilde{A} \|_{L_t^\infty L_x^q} \| \phi_s \|_{L_t^2 L_x^q} ds' \]

\[ + \int_s^\infty \Theta_{\frac{1}{2}}(s, s') \| \nabla \tilde{A} \|_{L_t^\infty L_x^q} \| \phi_s \|_{L_t^2 L_x^q} ds' \]

\[ + \int_s^\infty \Theta_{\frac{1}{2}}(s, s') \| \tilde{\phi} \|_{L_t^\infty L_x^q} \| \phi_s \|_{L_t^2 L_x^q} ds' \]

\[ + \int_s^\infty \Theta_{\frac{1}{2}}(s, s') \| \tilde{k} \|_{L_t^\infty L_x^q} \| \phi_s \|_{L_t^2 L_x^q} ds' \]

\[ + \int_s^\infty \Theta_{\frac{1}{2}}(s, s') \| \tilde{\kappa} \|_{L_t^\infty L_x^q} \| \phi_s \|_{L_t^2 L_x^q} ds' \]

\[ + \int_s^\infty \Theta_{\frac{1}{2}}(s, s') \| \tilde{\kappa} \|_{L_t^\infty L_x^q} \| \nabla \phi_s \|_{L_t^2 L_x^q} ds' + \Theta_{\frac{1}{2}}(s, s') \| \phi_s \|_{L_t^2 L_x^q} \]

\[ \lesssim \epsilon_1 \Theta_{\frac{1}{2}}(s, s') \| \nabla \phi_s \|_{L_t^2 L_x^q} ds' + \epsilon_1 \Theta_{\frac{1}{2}}(s, s') \| \phi_s \|_{L_t^2 L_x^q} ds' \]

\[ + \epsilon_1 \Theta_{\frac{1}{2}}(s, s') \| \nabla \phi_s \|_{L_t^2 L_x^q} ds'. \]  
\[ (3.46) \]

Set

\[ B(s) := \sup_{0<s'<s} \omega_{\frac{1}{2} - \frac{s'}{s}} \| \nabla \phi_s \|_{L_t^2 L_x^q}. \]

Then by (3.11), (3.40), assumptions (3.20–3.24) and (3.46), we get

\[ B(s) \lesssim_M \epsilon_1 + \epsilon_1^\frac{1}{4} B(s). \]

Thus \( B(s) \lesssim \epsilon_1 \), which gives (3.25).
Now, let’s prove (3.30). Again by Lemma 3.3 and (3.20–3.22), we get
\[
\| \Delta \phi_s \|_{L_t^\infty L_x^2} \lesssim \int_0^s \Theta_1 (s, s') \left( (\| \nabla^2 \tilde{A} \|_{L_t^\infty L_x^2} + \| \nabla \tilde{k} \|_{L_t^\infty L_x^2} + \| \nabla \tilde{A} \|_{L_t^\infty L_x^\infty}) \| \phi_s \|_{L_t^\infty L_x^2} ds'
\right)
\]
\[
+ \int_0^s \Theta_1 (s, s') \| \nabla \phi_s \|_{L_t^\infty L_x^2} ds'
\]
\[
+ \int_0^s \Theta_1 (s, s') \left( (\| \nabla \tilde{A} \|_{L_t^\infty L_x^2} + \| \tilde{k} \|_{L_t^\infty L_x^2} + \| \tilde{A} \|_{L_t^\infty L_x^\infty}) \| \phi_s \|_{L_t^\infty L_x^2} ds'
\right)
\]
\[
+ \int_0^s \Theta_1 (s, s') \| \tilde{A} \|_{L_t^\infty L_x^\infty} \| \nabla^2 \phi_s \|_{L_t^\infty L_x^2} ds' + \Theta_1 (s, s') \| \tilde{\phi} \|_{L_t^\infty L_x^2} ds' + \Theta_1 (s, \frac{s}{2}) \| \nabla \phi_s (\frac{s}{2}, t) \|_{L_t^\infty L_x^2}.
\]
(3.47)

Then by assumptions (3.20–3.24), (3.36), estimates of \( \tilde{k} \) in (3.40), (3.44), and Sobolev embeddings, we obtain that the function \( B'(s) \) defined by
\[
B'(s) := \sup_{0<s'<s} \omega_{1-\delta} (s') \| \nabla^2 \phi_s (s') \|_{L_t^\infty L_x^2}
\]
satisfies
\[
B'(s) \lesssim \epsilon_1^q + \epsilon_1^q B'(s).
\]
Thus \( B'(s) \lesssim \epsilon_1^q \), which gives (3.30).

Similarly, one can prove (3.31), (3.32) by additionally applying \( \| e^{\lambda H} f \|_{L_t^p} \lesssim s^{-\frac{1}{2}} + \frac{1}{p} \| f \|_{L_t^2} \).

By Sobolev interpolation inequalities, we infer from (3.31), (3.30), (3.13) that
\[
\sup_{s>0} \omega_{1-\frac{1}{q}} \| \phi_s \|_{L_t^\infty L_x^2} \lesssim \epsilon_1^q \quad (3.49)
\]
\[
\sup_{s>0} \omega_{1-\frac{1}{q}} \| \nabla \phi_s \|_{L_t^\infty L_x^2} \lesssim \epsilon_1^q. \quad (3.50)
\]

Now, we prove (3.26). Lemma 3.3 and (3.20–3.22) yield
\[
\| \Delta \phi_s \|_{L_t^2 L_x^q} \lesssim_M \int_0^s \Theta_1 (s, s') \left( (\| \nabla^2 A \|_{L_t^\infty L_x^2} + \| \nabla \tilde{k} \|_{L_t^\infty L_x^2} + \| \nabla \tilde{A} \|_{L_t^\infty L_x^\infty}) \| \phi_s \|_{L_t^2 L_x^q} ds'
\right)
\]
\[
+ \int_0^s \Theta_1 (s, s') \left( (\| \nabla^2 A \|_{L_t^\infty L_x^2} + \| \nabla \tilde{A} \|_{L_t^\infty L_x^2} + \| \nabla \tilde{k} \|_{L_t^\infty L_x^\infty}) \| \phi_s \|_{L_t^2 L_x^q} ds'
\right)
\]
\[
+ \int_0^s \Theta_1 (s, s') \| \nabla \phi_s \|_{L_t^2 L_x^q} ds' + \Theta_1 (s, s') \| \nabla \phi_s (\frac{s}{2}, t) \|_{L_t^2 L_x^q}.
\]
Set
\[
\hat{B}(s) := \sup_{0 < s' < s} \omega^{\frac{3}{2} - \delta}(s') \| \Delta \phi_s(s') \|_{L_t^2 L_x^q}.
\]

Then by assumptions (3.20–3.24), estimates of \( \tilde{\kappa} \) in (3.40), (3.44), (3.36), (3.11) and (3.25), we get
\[
\hat{B}(s) \lesssim \epsilon_1 + \epsilon_1^{\frac{1}{4}} \hat{B}(s).
\]

Thus \( \hat{B}(s) \lesssim \epsilon_1 \), which gives (3.26).

(3.27), (3.28) and (3.29) follow by the same arguments as (3.25), (3.26) via additively applying \( \| e^{itH} f \|_{L_t^\infty} \lesssim s^{-\frac{\delta}{2}} \| f \|_{L_t^8 L_x^{\tilde{q}}} \).

Lemma 3.5 (Close parabolic estimates of \( \tilde{A}, \tilde{\phi} \)). Let (3.11–3.13) hold. Assume that (3.20), (3.21), (3.22), (3.23), (3.24) hold. Let \( 0 < T < T' \), then we have for any \( \tilde{q} \in [4, \infty], q \in [4, \frac{8}{7}], p \in [2, \infty] \),
\[
\| \tilde{\phi} \|_{L_t^\infty L_x^q} \lesssim M (3.51)
\]
\[
\sup_{s > 0} \tilde{\omega}_0(s) \| \tilde{\phi} \|_{L_t^\infty L_x^q L_t^p} \lesssim M \epsilon_1^q (3.52)
\]
\[
\sup_{s > 0} \tilde{\omega}_{\text{max}(0, \frac{1}{2} - \delta - \frac{1}{p})}(s) \| \nabla \tilde{\phi} \|_{L_t^\infty L_x^p} \lesssim M \epsilon_1^q (3.53)
\]
\[
\sup_{s > 0} \tilde{\omega}_0(s) \| \tilde{\phi} \|_{L_t^2 L_x^q L_t^p} \lesssim M \epsilon_1 (3.54)
\]
\[
\sup_{s > 0} \tilde{\omega}_{\frac{1}{2} - \delta}(s) \| \nabla \tilde{\phi} \|_{L_t^2 L_x^q} \lesssim M \epsilon_1 (3.55)
\]
\[
\sup_{s > 0} \tilde{\omega}_{\frac{1}{2} - \frac{7}{8} \delta}(s) \| \nabla \tilde{\phi} \|_{L_t^2 L_x^q} \lesssim M \epsilon_1 (3.56)
\]

And the connection satisfies
\[
\| A \|_{L_t^\infty L_x^q} \lesssim M 1 (3.57)
\]
\[
\sup_{s > 0} \tilde{\omega}_0(s) \| \tilde{A} \|_{L_t^\infty L_x^q} + \sup_{s > 0} \tilde{\omega}_0(s) \| \nabla \tilde{A} \|_{L_t^\infty L_x^q} \lesssim M \epsilon_1^q, (3.58)
\]

where the implicit constant is at most of \((1 + M)^2\) order.

Proof. (3.51) and (3.57) follow directly by (3.20), (3.22).

By the identity
\[
\tilde{\phi}_j = - \int_{s'}^\infty \partial_j \phi_s + i A_j \phi_s ds' = - \int_{s'}^\infty \partial_j \phi_s + i A_j^\infty \phi_s + i \tilde{A}_j \phi_s ds' (3.59)
\]

and (3.13), (3.20), Poincare inequality, the \( L_t^\infty L_x^q \) bounds stated in (3.52) follows.

(3.49) and (3.50) give
\[
\int_s^\infty \| \phi_s \|_{L_t^\infty L_x^q} ds' \lesssim \epsilon_1^q (3.60)
\]
\[
\int_s^\infty \| \nabla \phi_s \|_{L_t^\infty L_x^q} ds' \lesssim \epsilon_1^q. (3.61)
\]
Thus due to (3.59) and (3.20), the $L^\infty_{t,x}$ bounds stated in (3.52) follows.

By (3.25), (3.27), (3.28), (3.20) and (3.59), one easily deduces (3.54).

Using (3.59), we see

$$|\nabla \tilde{\phi}| \lesssim \int_s^\infty |\nabla^2 \phi_s| + |A||\nabla \phi_s| + |\nabla A||\phi_s|ds'. \quad (3.62)$$

By (3.32), we get (3.53). And (3.56) follows by (3.28), (3.29), (3.27).

Recall that $\tilde{\phi}$ is given by

$$\tilde{\phi}_j = \int_s^\infty \kappa \Im(\tilde{\phi}_j \phi_s)ds'. \quad (3.63)$$

Then the $\|\tilde{\phi}\|_{L^\infty_{t,x}}$ in (3.58) follows by (3.51) and (3.60). Again due to (3.63), we get

$$|\nabla \tilde{\phi}| \lesssim \int_s^\infty (|\phi|^2|\phi_s| + |\phi_s||\nabla \phi| + |\nabla \phi_s||\phi|)ds'. \quad (3.64)$$

Then the $\|\nabla \tilde{\phi}\|_{L^\infty_{t,x}}$ bound stated in (3.58) follows by (3.49), (3.50), (3.51) and (3.53).

\[\square\]

**Proposition 3.3** (Parabolic estimates of $\phi_s$). Assume that (3.11–3.13) hold. Then for $0 < T < T$, we have (3.25–3.32), and there also hold (3.51–3.56), (3.57–3.58). Furthermore, one has for $q \in [4, \frac{8}{3}]$

$$\sup_{s>0} \omega_0(s) \|\tilde{\kappa}\|_{L^q_{t,x} \cap L^\infty_{t} L^2_x} \lesssim_M \epsilon_1^\alpha \quad (3.65)$$

$$\sup_{s>0} \omega_0(s) \|\nabla \tilde{\kappa}\|_{L^q_{t,x} \cap L^\infty_{t} L^2_x} \lesssim_M \epsilon_1^\alpha \quad (3.66)$$

$$\sup_{s>0} \omega_0(s) \|\tilde{\kappa}\|_{L^2_t L^q_x \cap L^2_t L^\infty_x} \lesssim_M \epsilon_1 \quad (3.67)$$

$$\sup_{s>0} \omega_0(s) \|\nabla \tilde{\kappa}\|_{L^2_t L^q_x \cap L^2_t L^\infty_x} \lesssim_M \epsilon_1 \quad (3.68)$$

$$\sup_{s>0} \omega_0(s) \|A^{qaa}\|_{L^1_t L^2_x \cap L^1_t L^\infty_x} \lesssim_M \epsilon_1^2 \quad (3.69)$$

$$\sup_{s>0} \omega_0(s) \|\nabla A^{qaa}\|_{L^1_t L^2_x \cap L^1_t L^\infty_x} \lesssim_M \epsilon_1^{1+\alpha} \quad (3.70)$$

**Proof.** By bootstrap argument and Lemma 3.5, we see (3.20–3.24) hold for all $s > 0$, $T \in (0, T)$. Then Lemma 3.4 shows (3.51–3.56) hold for all $s > 0$, $T \in (0, T)$. Meanwhile, Lemma 3.5 also implies that in fact (3.51–3.54), (3.57–3.58) hold for any $s > 0$, $T \in (0, T)$.

The rest is to prove (3.65–3.70). Recalling (3.38), we see (3.65) follows from Lemma 3.4.

By (3.42), the $L^\infty_{t,x}$ bound in (3.66) follows by (3.51), (3.49), (3.50). Moreover, the $L^\infty_{t,x} L^2_t$ bounds in (3.66) and the $L^2_t L^q_x \cap L^2_t L^\infty_x$ bounds in (3.67), (3.68) follow by corresponding ones in Lemma 3.4 and Lemma 3.5 as well.

Recall that

$$A^{qaa} = \int_s^\infty \kappa \Im(\tilde{\phi}_s \phi_s)ds' + \int_s^\infty \tilde{\kappa} \Im(\phi^\infty \phi_s)ds'. \quad (3.71)$$
Then (3.69) follows by (3.54), (3.11), (3.67) and \(\|\phi_t\|_{L_t^2 L_x^\infty}\) bounds in Lemma 3.4.

By (3.71), one has
\[
|\nabla A^{q\mu a}| \lesssim M \int_0^\infty (|\nabla \tilde{\phi}||\phi_x| + |\nabla \phi_x||\tilde{\phi} + |\phi_x||\tilde{\phi}||\phi|)ds' + \int_0^\infty (|\nabla \tilde{k}||\phi_x| + |\nabla \phi_x||\tilde{k} + |\phi_x||\tilde{k}||\phi|)ds'.
\]
Thus (3.70) follows by (3.67), (3.68), Lemma 3.5 and Lemma 3.4.

**Lemma 3.6 (Parabolic estimates of \(Z\)).** Assuming that (3.11–3.13) hold, for \(0 < T < T'\), and any \(q \in [4, \frac{8}{3}]\), \(p \in [2, \infty]\), we have
\[
\sup_{s>0} \omega^{- \frac{1}{2} - \delta} (s) \|Z\|_{L_t^\infty L_x^2} \lesssim M \epsilon_1^{2\alpha} \tag{3.72}
\]
\[
\sup_{s>0} \omega^{- \frac{1}{2} - \frac{1}{p} - \delta} (s) \|Z\|_{L_t^\infty L_x^p} \lesssim M \epsilon_1^{2\alpha} \tag{3.73}
\]
\[
\sup_{s>0} \omega^{- \frac{1}{2} - \delta} (s) \|Z\|_{L_t^1 L_x^2} \lesssim M \epsilon_1^{2} \tag{3.74}
\]
\[
\sup_{s>0} \omega^{- \delta} (s) \|\nabla Z\|_{L_t^1 L_x^2} \lesssim M \epsilon_1^{2} \tag{3.75}
\]
\[
\sup_{s>0} \omega^{- \frac{1}{2} - \delta} (s) \|\nabla^2 Z\|_{L_t^1 L_x^2} \lesssim M \epsilon_1^{2} \tag{3.76}
\]
\[
\sup_{s>0} \omega^{- \frac{1}{2} - \delta} (s) \|Z\|_{L_t^2 L_x^q} \lesssim M \epsilon_1^{1+\alpha} \tag{3.77}
\]
\[
\sup_{s>0} \omega^{- \delta} (s) \|\nabla Z\|_{L_t^2 L_x^q} \lesssim M \epsilon_1^{1+\alpha} \tag{3.78}
\]
\[
\sup_{s>0} \omega^{- \frac{1}{2} - \delta} (s) \|\nabla^2 Z\|_{L_t^2 L_x^q} \lesssim M \epsilon_1^{1+\alpha} \tag{3.79}
\]

**Proof.** Recall that \(Z\) satisfies (3.4):
\[
\partial_s Z = \Delta_A Z - i\kappa \text{Im}(\phi^j \bar{Z}) \phi_j + i\kappa \phi^j \phi_j \phi_x, \quad Z(0, t, x) = 0. \tag{3.80}
\]
As mentioned before, the RHS of (3.80) is in fact
\[
\textbf{HZ} + 2i \bar{A}^j \nabla^j Z + i \nabla^j \bar{A}^j Z - 2\bar{A}^j \bar{A}^j Z - \bar{A}^j \bar{A}^j Z - i\kappa \text{Im}(\phi^j \bar{Z}) \phi_j^\infty
\]
\[
- i\kappa \text{Im}(\phi^j \bar{Z}) \phi_j^j - i\kappa \text{Im}(\phi^j \bar{Z}) \phi_j - i\kappa h^{kj} \text{Im}(\phi^j \bar{Z}) \phi_k \phi_j^\infty
\]
\[
+ 2i\kappa h^{kj} \phi^\infty \phi_j \phi_k \phi_x + i\kappa \phi^j \phi_j \phi_x,
\]
for which we remark that \(\textbf{HZ}\) is the only linear part and the other above terms are at least quadratic, and the only troublesome term is \(\nabla^j \bar{A}^j Z\), which will be dominated by smoothing effects and the Leibnitz formula.

Thus Lemma 3.3, Duhamel principle, and the \(L_t^\infty\) bounds of \(\tilde{\phi}, \phi, \nabla \tilde{\phi}\) in Proposition 3.3 (see (3.52), (3.57–3.58), (3.65)) give
\[
\|Z\|_{L_t^1 L_x^2} \lesssim M \int_0^s \Theta_0 (s, s') (\|\tilde{A}\|_{L_t^\infty} + \|\nabla \tilde{A}\|_{L_t^\infty} + \|\phi\|_{L_t^\infty}) \|Z\|_{L_t^1 L_x^2} ds' + \int_0^s \Theta_1 (s, s') \|\nabla \tilde{A}\|_{L_t^\infty} \|Z\|_{L_t^1 L_x^2} ds'
\]
\[
+ \int_0^s \Theta_2 (s, s') (\|\tilde{k}\|_{L_t^1 L_x^4} + \|\phi\|_{L_t^2 L_x^4}) \|\phi_x\|_{L_t^2 L_x^4} ds'.
\]
where the in the last line we also applied $L^2_t L^4_x$ norms of $\tilde{\kappa}, \tilde{\phi}$ in Proposition 3.3. Thus we get

$$\|Z\|_{L^1_t L^2_x} \lesssim_M \epsilon_1^2 \min(s^{1+\delta}, s^{-L}),$$

which gives (3.74). Again using Lemma 3.3, Duhamel principle, one obtains

$$\| \nabla Z \|_{L^1_t L^2_x} \lesssim_M \int_0^s \Theta \left( \frac{1}{2}, (s, s') \right) (\| \tilde{A} \|_{L^\infty} + \| \tilde{\phi} \|_{L^\infty_t L^2_x}) \|Z\|_{L^1_t L^2_x} ds'$$

$$+ \int_0^s \Theta \left( \frac{1}{2}, (s, s') \right) \| \tilde{A} \|_{L^\infty_t L^2_x} \| \nabla Z \|_{L^1_t L^2_x} ds'$$

$$+ \int_0^s \Theta \left( \frac{1}{2}, (s, s') \right) (\| \nabla \tilde{\kappa} \|_{L^2_t L^4_x} + \| \tilde{\phi} \|_{L^2_t L^4_x}) \| \phi_s \|_{L^2_t L^4_x} ds'.$$

Define $B_1(s)$ to be

$$B_1(s) := \sup_{0 < \tau < s} \max(\tau^{-\delta}, \tau^L) \| \nabla Z(\tau) \|_{L^1_t L^2_x}.$$

Then we arrive at

$$B_1(s) \lesssim \epsilon_1^2 + \epsilon_1^\alpha B_1(s),$$

from which (3.75) follows. Similarly one has (3.72), (3.73).

For $\Delta Z$, we use

$$\| \Delta Z \|_{L^1_t L^2_x} \lesssim \int_0^s \Theta \left( \frac{1}{2}, (s, s') \right) (\| \tilde{A} \|_{L^\infty} + \| \nabla \tilde{A} \|_{L^\infty} + \| \nabla \tilde{A} \|_{L^\infty_t L^2_x}) \|Z\|_{L^1_t L^2_x} ds'$$

$$+ \| \tilde{\phi} \|_{L^\infty_t L^2_x} \| \nabla \tilde{\phi} \|_{L^\infty_t L^2_x} \| \nabla Z \|_{L^1_t L^2_x} ds'$$

$$+ \int_0^s \Theta \left( \frac{1}{2}, (s, s') \right) (\| \tilde{A} \|_{L^\infty_t L^2_x} + \| \tilde{\phi} \|_{L^\infty_t L^2_x}) \| \phi_s \|_{L^2_t L^4_x} ds'$$

$$+ \int_0^s \Theta \left( \frac{1}{2}, (s, s') \right) (\| \nabla \tilde{\kappa} \|_{L^2_t L^4_x} + \| \tilde{\phi} \|_{L^2_t L^4_x}) \| \phi_s \|_{L^2_t L^4_x} \| \nabla \phi_s \|_{L^2_t L^4_x} ds'.$$

Noting (3.55), (3.36), (3.76) then follows by (2.1) and Proposition 3.3.

(3.77), (3.78), (3.79) follow by the same way as (3.74) and $L^\infty_{t,x}$ bounds of $\tilde{A}, \tilde{\phi}, \nabla \tilde{A}$.

\[ \Box \]

**Lemma 3.7** (Improved Parabolic estimates of $Z$ near $s = 0$). Assume that (3.11–3.13) hold. There exists a sufficiently small constant $\nu$ depending only on $M$ such that for all $0 < T < T$

$$\sup_{0 < s < \nu} s^{1-\delta} \| e^{-\frac{1}{2}r} \nabla^2 \phi_s \|_{L^2_t L^2_x} \lesssim_M \epsilon_1^\beta$$

(3.81)

$$\sup_{s > 0} \omega_{\frac{1}{2}-\delta}(s) \| \nabla^3 Z \|_{L^1_t L^2_x} \lesssim_M \epsilon_1^{\alpha+1}$$

(3.82)
Proof. **Step 0.** In this step, we prove (3.81). Recall the evolution equation of $φ_s$ in the heat direction:

$$\partial_s φ_s - Δ φ_s = 2i A_j \nabla^j φ_s - A_j A^j φ_s + i \nabla^j A_j φ_s - i κ \text{Im}(φ^j \overline{φ_s}) φ_j. \quad (3.83)$$

Denote $G$ the RHS of (3.83). Lemma 9.3 and Duhamel principle give

$$\| e^{-\frac{1}{2}r} \nabla^2 φ_s \|_{L^2_t L^2_x} \lesssim \int_{\frac{1}{2}}^s (s - s')^{-\frac{1}{2}} (\| e^{-\frac{1}{2}r} \nabla G \|_{L^2_t L^2_x}) ds'$$

$$+ s^{-\frac{1}{2}} (\| e^{-\frac{1}{2}r} \nabla φ_s (\frac{s}{2}) \|_{L^2_t L^2_x})$$

$$\lesssim \int_{\frac{1}{2}}^s (s - s')^{-\frac{1}{2}} (\| e^{-\frac{1}{2}r} \nabla G \|_{L^2_t L^2_x})$$

$$+ \| G \|_{L^2_t L^4_x}) ds' + s^{-\frac{1}{2}} (\| e^{-\frac{1}{2}r} \nabla φ_s (\frac{s}{2}) \|_{L^2_t L^2_x} + \| φ_s (\frac{s}{2}) \|_{L^2_t L^4_x}).$$

The last three terms are admissible by assumptions (3.11–3.13) and Proposition 3.3. It suffices to bound the $\nabla G$ term. In fact, by $L^∞_{t,x}$ bounds of $\tilde{A}, \tilde{\phi}, \nabla \tilde{A}$, we have

$$\int_{\frac{1}{2}}^s (s - s')^{-\frac{1}{2}} (\| e^{-\frac{1}{2}r} \nabla φ_s \|_{L^2_t L^2_x}) ds'$$

$$\lesssim \int_{\frac{1}{2}}^s (s - s')^{-\frac{1}{2}} (\| A \|_{L^∞_t W^1_x})$$

$$+ \| φ \|_{L^∞_t W^1_x}) \| e^{-\frac{1}{2}r} φ_s \|_{L^2_t L^2_x} ds'$$

$$+ \int_{\frac{1}{2}}^s (s - s')^{-\frac{1}{2}} \left( (\| φ \|_{L^∞_t} + \| A \|_{L^∞_t W^1_x}) \| e^{-\frac{1}{2}r} \nabla φ_s \|_{L^2_t L^2_x} + \| A \|_{L^∞_t} \| e^{-\frac{1}{2}r} \nabla φ \|_{L^2_t L^2_x} \right) ds'.$$

Set

$$\tilde{B}(s) := \sup_{0 < s' < s} \omega_1(s') \| e^{-\frac{1}{2}r} \nabla^2 φ(s') \|_{L^2_t L^2_x}.$$ 

Then for $0 < v \ll 1$, we get for all $s \in (0, v)$ that

$$\tilde{B}(s) \lesssim_M e_1^β + v^\frac{1}{2} \tilde{B}(s).$$

Thus $\sup_{s \in (0, v)} \tilde{B}(s) \lesssim e^β_1$, which gives (3.81).

**Step 1.** In addition to Proposition 3.3, we will also need

$$\omega_{\frac{1}{2} - \delta}(s) (\| \nabla^3 φ_s \|_{L^∞_t L^2_x} + \| \nabla^2 φ_s \|_{L^∞_t L^∞_x}) + \omega_{2 - \delta}(s) \| \nabla^3 φ \|_{L^∞_t L^∞_x} \lesssim_M e_1^α \quad (3.84)$$

$$\tilde{ω}_{1 - \delta}(s) \| \nabla^3 \tilde{A} \|_{L^∞_t L^∞_x} + \tilde{ω}_{\frac{1}{2} - \delta}(s) \| \nabla^2 \tilde{A} \|_{L^∞_t L^∞_x} \lesssim_M e_1^α \quad (3.85)$$

$$\tilde{ω}_{1 - \delta}(s) \| \nabla^2 \tilde{φ} \|_{L^∞_t L^∞_x} + \tilde{ω}_{\frac{1}{2} - \delta}(s) \| \nabla \tilde{φ} \|_{L^∞_t L^∞_x} \lesssim_M e_1^α \quad (3.86)$$

$$\tilde{ω}_{1 - \delta}(s) \| \nabla^2 \tilde{k} \|_{L^∞_t L^∞_x} \lesssim_M e_1^α \quad (3.87)$$

$$\omega_{2 - \delta}(s) \| \nabla^3 φ \|_{L^2_t L^4_x} \lesssim_M e_1. \quad (3.88)$$

The estimates (3.84–3.88) are just parabolic estimates and follow by the same way as Lemma 3.4, Lemma 3.5 and Proposition 3.3. Note that all the smoothing indexes except
that in (3.88), i.e. the parameter $\gamma$ in $\omega_\gamma(s)$, are calculated based on a-prior bounds on $\|\nabla \phi_s\|_{L_t^1 L_x^4}$. This also coincides with the $\alpha$ power in $\epsilon_1^a$ in the RHS of (3.84–3.87). (3.88) was obtained from a-prior bounds on $\|\phi_s\|_{L_t^2 L_x^4}$.

**Step 2.** We will also use

$$\tilde{\omega}_{1-\delta}(s) \|e^{-\frac{1}{2}r^2} \nabla^2 \bar{\phi}\|_{L_{t,x}^2} + \tilde{\omega}_0(s) \|e^{-\frac{1}{2}r^2} \nabla \bar{\phi}\|_{L_{t,x}^2} \lesssim_M \epsilon_1^\beta$$  \hspace{1cm} (3.89)

$$\sup_{s \in (0,1)} s^{\frac{3}{2}-\delta} \|e^{-\frac{1}{2}r^2} \nabla \phi_s\|_{L_{t,x}^2} \lesssim_M \epsilon_1^\alpha. \hspace{1cm} (3.90)$$

The bound (3.90) follows by similar arguments of Step 0. To verify (3.89), we use (3.90) for $s \in (0,1)$ and $L_t^2 L_x^4$ bounds of $\|\nabla^j \phi_s\|_{L_t^2 L_x^4}$, $j = 1, 2, 3$, for $s \geq 1$ via

$$\| f e^{-\frac{1}{2}r} \|_{L_{t,x}^2} \lesssim \| f \|_{L_t^2 L_x^4}.$$  

**Step 3.** Recall that $Z$ satisfies (3.4):

$$\partial_s Z = \Delta_A Z - i\kappa \Im(\phi^j \bar{Z})\phi_j + i\kappa \phi^j \bar{\phi}_j, \quad Z(0, t, x) = 0. \hspace{1cm} (3.91)$$

And the RHS of (3.91) is indeed

$$H Z + 2i \tilde{A}_j \nabla^j Z + i \nabla^j \tilde{A}_j Z - 2A_j^\infty \tilde{A}_j^\infty Z - \tilde{A}_j \tilde{A}_j^\infty Z - i\kappa \Im(\phi^j \bar{Z})\phi_j^\infty - i\kappa \Im(\phi^j \bar{\tilde{Z}})\phi_j - i\kappa h_{kj} \Im(\phi_j^\infty \bar{Z})\phi_k^\infty + 2i\kappa h_{kj} \phi_j^\infty \bar{\phi}_k \bar{\phi}_s + i\kappa \phi^j \bar{\phi}_j \bar{\phi}_s.$$  

The key to obtain (3.82) rather than $\sup_{s \in (0,1)} \omega_{1-\delta}(s) \|\nabla^3 Z\|_{L_t^1 L_x^3} \lesssim_M \epsilon_1^2$ is to adopt more delicate estimates for the terms $i\kappa h_{kj} \phi_j^\infty \bar{\phi}_k \bar{\phi}_s + i\kappa \phi^j \bar{\phi}_j \bar{\phi}_s$. In fact, using Lemma 3.3, Duhamel principle, one obtains

$$\|\nabla^3 Z\|_{L_t^1 L_x^3} \lesssim \int_0^s (s - s')^{-\frac{1}{2}} \| A \|_{L_t^\infty L_x^2} \|\nabla^3 Z\|_{L_t^1 L_x^3} ds' + \int_0^s (s - s')^{-\frac{1}{2}} \| A \|_{L_t^\infty W_x^1,\infty} \|\nabla^2 Z\|_{L_t^1 L_x^3} ds' + \int_0^s (s - s')^{-\frac{1}{2}} \| A \|_{L_t^\infty W_x^2,\infty} \|\nabla Z\|_{L_t^1 L_x^3} ds' + \int_0^s (s - s')^{-\frac{1}{2}} \| A \|_{L_t^\infty W_x^3,\infty} \| Z\|_{L_t^1 L_x^3} ds'$$

$$+ \int_0^s (s - s')^{-\frac{1}{2}} \left( \sum_{a+j+k+l=2} \|\nabla^a k \| \|\nabla^j \phi^\infty \| \|\nabla^k \bar{\phi}\| \|\nabla l \phi_s\|_{L_t^1 L_x^3} \right) ds' + \int_0^s (s - s')^{-\frac{1}{2}} \left( \sum_{a+j+k+l=2} \|\nabla^a k \| \|\nabla^j \bar{\phi}\| \|\nabla^k \bar{\phi}\| \|\nabla l \phi_s\|_{L_t^1 L_x^3} \right) ds'$$

$$:= G_1 + G_2 + G_3 + G_4 + G_5 + G_6.$$
By (3.84–3.87) and Lemma 3.6, we get for any $0 < s < 1$

$$G_2 + G_3 + G_4 \lesssim \epsilon_1^2 \left( \int_0^s (s - \tau)^{-\frac{1}{2}} \tau^{-\frac{1}{2} + \delta} \tau^{-\frac{1}{2} + \delta} d\tau + \int_0^s (s - \tau)^{-\frac{1}{2}} \tau^\delta d\tau + \int_0^s (s - \tau)^{-\frac{1}{2}} \tau^{-1 + \delta} \tau^\frac{1}{2} + \delta d\tau \right) \lesssim s^{-\frac{1}{2} + \delta} \epsilon_1^2.$$

Now, let’s deal with $G_5$. By (3.84–3.87), (3.89) and (3.81), we have

$$\sum_{a + j + k = 1} \|\nabla^a \kappa\|\nabla^j \phi \|\nabla^k \tilde{\phi}\|\nabla \phi_s\| L_{1, x}^1 \lesssim \epsilon_1^1 \|\nabla^2 \phi_s\| L_{1, x}^1 \|\kappa\| L_{2, x}^2 \lesssim s^{-1+\delta} \epsilon_1^{1+\beta}$$

and

$$\sum_{j=0,1, a=0,1} \|\nabla^a \kappa\|\nabla^k \phi \|\nabla^j \tilde{\phi}\|\nabla \phi_s\| L_{1, x}^1 \lesssim \sum_{j=0,1} \|\nabla^2 \phi_s\| L_{1, x}^1 \|\kappa\| L_{2, x}^2 \lesssim s^{-1+\delta} \epsilon_1^{1+\beta}$$

Therefore, for $s \in (0, 1)$, $G_5$ is bounded by

$$G_5 \lesssim \epsilon_1^{1+\alpha} \int_0^s (s - \tau)^{-\frac{1}{2}} \tau^{-1+\delta} d\tau \lesssim s^{-\frac{1}{2} + \delta} \epsilon_1^{1+\alpha}.$$

We turn to estimate $G_6$. By (3.84–3.87), (3.89) and (3.81), we have

$$\sum_{a + j + k = 1} \|\nabla^a \kappa\|\nabla^j \phi \|\nabla^k \tilde{\phi}\|\nabla \phi_s\| L_{1, x}^1 \lesssim \|\nabla \phi_s\| L_{1, x}^1 \|\nabla^2 \phi_s\| L_{2, x}^2 \|\kappa\| L_{2, x}^2 \|\tilde{\phi}\| L_{2, x}^2 \lesssim s^{-1+\delta} \epsilon_1^{2+\alpha}.$$
and
\[ \sum_{a,k,j=0,1} \| \nabla^a \kappa \| \| \nabla^j \tilde{\phi} \| \| \nabla^k \tilde{\phi} \| \| \phi_s \| \|_{L^1_t L^2_x} \]
\[ \lesssim \| (\nabla) \tilde{\phi} \|_{L^\infty_t L^{\frac{4}{3}}_x} + \| (\nabla) \tilde{\phi} \|_{L^2_t L^4_x} + \| (\nabla) \tilde{\phi} \|_{L^\infty_t L^{\frac{4}{3}}_x} \| \phi_s \|_{L^2_t L^4_x} \]
\[ + \| (\nabla) \tilde{\phi} \|_{L^\infty_t L^{\frac{4}{3}}_x} \| \tilde{\phi} \|_{L^2_t L^4_x} \| \phi_s \|_{L^2_t L^4_x} \]
\[ \lesssim s^{-\frac{1}{2}+\delta} \epsilon_1^{2+\alpha}, \]
and
\[ \| \nabla^2 \kappa \| \| \tilde{\phi} \| \| \phi_s \| \|_{L^1_t L^2_x} \lesssim \| \nabla^2 \kappa \|_{L^\infty_t L^{\frac{4}{3}}_x} \| \tilde{\phi} \|_{L^\infty_t L^{\frac{4}{3}}_x} \| \phi_s \|_{L^2_t L^4_x} \lesssim s^{-\frac{1}{2}+\delta} \epsilon_1^{2+\alpha}. \]

Thus, we have proved
\[ \| \nabla^3 Z \|_{L^1_t L^2_x} \lesssim \int_0^s (s - s')^{-\frac{1}{2}} \| A \|_{L^\infty_t} \| \nabla^3 Z \|_{L^1_t L^2_x} + \epsilon^{\alpha+1} \int_0^s (s - \tau)^{-\frac{1}{2}} \tau^{-1+\delta} d\tau \]
\[ \lesssim M \int_0^s (s - s')^{-\frac{1}{2}} \| \nabla^3 Z \|_{L^1_t L^2_x} + \epsilon^{\alpha+1} s^{-\frac{1}{2}+\delta}. \]

Set
\[ B_3(s) = \sup_{0 < s' < s} s^{\frac{1}{2}-\delta} \| \nabla^3 Z \|_{L^1_t L^2_x}. \]

Then for \( \nu > 0 \) sufficiently small, we have for any \( 0 < s < \nu \),
\[ B_3(s) \leq \nu^{\frac{1}{2}} B_3(s) + \epsilon_1^{\alpha+1}. \]

Hence, \( B_3(\nu) \lesssim \epsilon_1^{\alpha+1} \), i.e.
\[ \sup_{0 < s < \nu} s^{\frac{1}{2}-\delta} \| \nabla^3 Z \|_{L^1_t L^2_x} \lesssim \epsilon_1^{\alpha+1}. \]

**Step 4.** It remains to verify the \( s \geq \nu \) part for \( \| \nabla^3 Z \|_{L^1_t L^2_x} \). Similar to Lemma 3.6, one has
\[ \sup_{s > 0} \omega_{1-\delta}(s) \| \nabla^3 Z \|_{L^1_t L^2_x} \lesssim \epsilon_1^2, \]
which is sufficient to give (3.82) for \( s > \nu \).

\[ \square \]

Lemmas 3.6, 3.7, (3.80) and Proposition 3.3 give the following final version of parabolic estimates of \( Z \).

**Corollary 3.1.** Assume that (3.11–3.13) hold. Then for all \( 0 < T < T' \)
\[ \sup_{s > 0} \omega_{\frac{1}{2}-\delta}(s) \| \partial_s Z \|_{L^1_t L^2_x} \lesssim M \epsilon_1^2 \]
\[ \sup_{s > 0} \omega_{\frac{1}{2}-\delta}(s) \| \nabla \partial_s Z \|_{L^1_t L^4_x} \lesssim M \epsilon_1^{\alpha+1} \]
\[ \sup_{s > 0} \omega_{\frac{1}{2}-\delta}(s) \| \partial_s Z \|_{L^2_t L^4_x} \lesssim M \epsilon_1^{1+\alpha}. \]
Lemma 3.8 (Estimates of $\phi_t$ and $A_t$). Assume that (3.11–3.13) hold. Then for $0 < T < T$, any $q \in [4, \frac{8}{5}]$, $p \in [2, \infty[$, we have

\[
\sup_{s > 0} \omega_1 \frac{1}{2} - \delta(s) \|\phi_t\|_{L^2_t L^p_x} \lesssim M \epsilon_1^{2\alpha} \tag{3.92}
\]

\[
\sup_{s > 0} \omega_3 \frac{1}{2} - \delta(s) \|\nabla \phi_t\|_{L^2_t L^p_x} \lesssim M \epsilon_1^{2\alpha} \tag{3.93}
\]

\[
\sup_{s > 0} \omega_1 \frac{1}{2} - \frac{1}{p} \delta(s) \|\nabla \phi_t\|_{L^\infty_t L^p_x} \lesssim M \epsilon_1^{2\alpha}. \tag{3.94}
\]

And $A_t$, $\nabla A_t$ satisfy

\[
\sup_{s > 0} \omega_0(s) \|A_t\|_{L^1_t L^\infty_x} \lesssim M \epsilon_1^2 \tag{3.95}
\]

\[
\sup_{s > 0} \omega_0(s) \|A_t\|_{L^2_t L^4_x} \lesssim M \epsilon_1^2 \tag{3.96}
\]

\[
\sup_{s > 0} \omega_0(s) \|A_t\|_{L^\infty_t L^4_x} \lesssim M \epsilon_1^{2\alpha} \tag{3.97}
\]

\[
\sup_{s > 0} \omega_0(s) \|\nabla A_t\|_{L^2_t \frac{2}{1-\delta} L^\infty_x} \lesssim M \epsilon_1^2. \tag{3.98}
\]

Proof. Recall that $Z = \phi_s - i \phi_t$. Then (3.93), (3.94) and (3.92) follow by Lemma 3.6 and Proposition 3.3. One can verify that

\[
|A_t| \lesssim \int_s^\infty |\phi_t| |\phi_s| ds' \tag{3.99}
\]

\[
|\nabla A_t| \lesssim \int_s^\infty (|\phi_s| |\phi| |\phi_t| + |\nabla \phi_t| |\phi_s| + |\phi_t| |\nabla \phi_s|) ds'. \tag{3.100}
\]

Then, (3.97) follows directly from (3.92–3.94) and Proposition 3.3. Moreover, (3.95), (3.96) are dominated by

\[
\|A_t\|_{L^1_t L^\infty_x} \lesssim \int_s^\infty \|\phi_t\|_{L^2_t L^\infty_x} \|\phi_s\|_{L^2_t L^\infty_x} ds' \tag{3.101}
\]

\[
\|A_t\|_{L^2_t L^4_x} \lesssim \int_s^\infty \|\phi_t\|_{L^2_t L^4_x} \|\phi_s\|_{L^\infty_t L^4_x} ds'. \tag{3.102}
\]

The rest (3.98) is bounded as

\[
\|\nabla A_t\|_{L^2_t \frac{2}{1-\delta} L^\infty_x} \lesssim \int_s^\infty \|\nabla \phi_t\|_{L^\infty_t L^\frac{8}{5} x} \|\phi_s\|_{L^\frac{8}{5} L^\frac{8}{5} x} ds' + \int_s^\infty \|\nabla \phi_s\|_{L^\infty_t L^\frac{8}{5} x} \|\phi_t\|_{L^\frac{8}{5} L^\frac{8}{5} x} ds' \tag{3.103}
\]

\[
+ \int_s^\infty \|\phi_t\|_{L^\frac{8}{5} L^\frac{8}{5} x} \|\phi_s\|_{L^\infty_t L^\frac{8}{5} x} ds'. \tag{3.104}
\]

Thus by (3.92–3.94), Lemma 3.6 and Proposition 3.3, we obtain (3.95–3.98).
3.4. Evolution along the Schrödinger direction. The proof of Proposition 3.2 will be divided into four lemmas.

First, we deal with \( \phi_s(s, 0, x) \). Lemma 10.2 in Appendix B gives

**Lemma 3.9.** For initial data \( u_0 \) in Theorem 1.1, there holds

\[
\sup_{s > 0} \omega_0(s) \| \phi_s(s, 0, x) \|_{L^2_x} \lesssim \epsilon_* \quad (3.99)
\]

\[
\sup_{s > 0} \omega_1(s) \| \phi_s(s, 0, x) \|_{H^1_x} \lesssim \epsilon_* \quad (3.100)
\]

\[
\sup_{s > 0} \omega_2(s) \| \phi_s(s, 0, x) \|_{H^1_x} \lesssim \epsilon_* \quad (3.101)
\]

**Lemma 3.10.** With the above assumption (3.11–3.13), for any \( q \in [4, \frac{8}{3}] \), one has

\[
\sup_{s > 0} \omega_3(s) \| \phi_s(s, 0, x) \|_{L^q_T L^q_x \cap L^q_T([0, T] \times \mathbb{R}^2)} \lesssim \epsilon_*^{q+1}. \quad (3.102)
\]

**Proof.** Let’s first rewrite the equation of \( \phi_s \) in (3.5):

\[
i \partial_t \phi_s + \mathbf{H} \phi_s = A_1 \phi_s + \Delta \phi_s - 2A_j^\infty \tilde{\phi}_j \phi_s + i \kappa \text{Im}(\tilde{\phi}_j \phi_s) \phi_j
\]

\[
+ i \kappa \text{Im}(\tilde{\phi}_j \phi_s) \phi^j + i \kappa \text{Im}(\phi_j \phi_s) \phi^j
\]

\[
+ i \tilde{\kappa} h \hat{k} \text{Im}(\tilde{\phi}_j \phi_s) \phi^j + i \partial_s Z,
\]

where for simplicity we write \( \Delta \tilde{\phi} f = \Delta \tilde{\phi} f - \Delta f \).

Let \( \mathbf{N} \) denote the nonlinearity, i.e.

\[
i \partial_t \phi_s + \mathbf{H} \phi_s = \mathbf{N}. \quad (3.103)
\]

Applying the endpoint Strichartz estimates of Lemma 7.4 for \( \mathbf{H} \) to (3.103) gives

\[
\| \phi_s \|_{L^q_T L^q_x \cap L^q_T \mathbb{R}^2} \lesssim \| \mathbf{N} \|_{L^1_T \mathbb{R}^2} + \| \phi_s(s, 0, x) \|_{L^2_x},
\]

where we omit the integral domain \( (0, T) \times \mathbb{R}^2 \) for all the above involved norms for simplicity.

The initial data term \( \| \phi_s(s, 0, x) \|_{L^2_x} \) is admissible by Lemma 3.9. Now, let’s consider the three classes of terms in \( \mathbf{N} \). The first class is quadratic terms in \( \phi_s \):

\[
\mathbf{N}_1 := 2i A_j^\text{lin} \nabla \phi_j + i \nabla \phi_j A_j^\text{lin} \phi_s - A^\text{lin} \cdot A^\text{lin} \phi_s - 2A^\infty \cdot A^\text{lin} \phi_s
\]

\[
+ \kappa^\infty \text{Im}(\tilde{\phi}_j \phi_s) \phi^j + \kappa^\infty \text{Im}(\phi_j \phi_s) \phi^j + \tilde{\kappa} h \hat{k} \text{Im}(\phi_j \phi_s) \phi^j.
\]

The second is cubic and higher order terms in \( \phi_s \):

\[
\mathbf{N}_2 := A_1 \phi_s + 2i A_j^\text{qua} \nabla \phi_j + i \nabla \phi_j A_j^\text{qua} \phi_s - A^\text{qua} \cdot A^\text{qua} \phi_s
\]

\[
- 2A^\infty \cdot A^\text{lin} \phi_s - 2A^\infty \cdot A^\text{qua} \phi_s - 2A^\text{qua} \cdot A^\text{lin} \phi_s - A^\text{lin} \cdot A^\text{lin} \phi_s
\]

\[
+ i \kappa \text{Im}(\tilde{\phi}_j \phi_s) \phi^j + i \tilde{\kappa} \text{Im}(\phi_j \phi_s) \phi^j + i \kappa \text{Im}(\phi_j \phi_s) \phi^j + i \tilde{\kappa} \text{Im}(\phi_j \phi_s) \phi^j.
\]

The third is the \( Z \) term:

\[
\mathbf{N}_3 := i \partial_s Z.
\]
For the quadratic terms, we observe that each term in $N_1$ has a decay weight $A^\infty$ or $\phi^\infty$. Thus the gradient term of $\phi_3$ can be controlled by Morawetz estimates bootstrap assumption (3.12). We pick the most troublesome gradient term as the candidate:

$$\|A^{lin} \cdot \nabla \phi \|_{L_t^1 L_x^\infty} \leq \|e^{\frac{1}{2}r} A^{lin} \|_{L_t^1 L_x^\infty} e^{-\frac{1}{2}r} \nabla \phi_3 \|_{L_t^1 L_x^\infty} \lesssim \omega_1^{-1} (s) \epsilon_1^{1+\beta},$$

where we applied (3.28) to bound $A^{lin}$. Similarly, we have

$$\|A_t \phi_3 \|_{L_t^1 L_x^\infty} \lesssim \|A_t \|_{L_t^2 L_x^4} \|\phi \|_{L_t^2 L_x^4} \lesssim \omega_1^{-1} (s) \epsilon_1^3$$

$$\|A^{\infty} \cdot A^{lin} \phi \|_{L_t^1 L_x^\infty} \lesssim \|\phi_3 \|_{L_t^2 L_x^4} \|A^{lin} \|_{L_t^1 L_x^\infty} \lesssim \omega_1^{-1} (s) \epsilon_1^1$$

$$\|\nabla j A^{lin} \phi \|_{L_t^1 L_x^\infty} \lesssim \|\phi_3 \|_{L_t^2 L_x^4} \|\nabla A^{lin} \|_{L_t^1 L_x^\infty} \lesssim \omega_1^{-1} (s) \epsilon_1^{\alpha+1}$$

$$\|\kappa^{\infty} \text{Im}(\tilde{\phi}^j \tilde{\phi}) \|_{L_t^1 L_x^\infty} \lesssim \|\tilde{\phi} \|_{L_t^2 L_x^4} \|\phi_3 \|_{L_t^2 L_x^4} \lesssim \omega_1^{-1} (s) \epsilon_1^2$$

$$\|\tilde{\kappa}^{\infty} \text{Im}(\tilde{\phi}^j \tilde{\phi}) \|_{L_t^1 L_x^\infty} \lesssim \|\tilde{\kappa} \|_{L_t^2 L_x^4} \|\phi_3 \|_{L_t^2 L_x^4} \lesssim \omega_1^{-1} (s) \epsilon_1^2,$$

where we applied Proposition 3.3 and Lemma 3.8.

For the cubic or higher order terms involved in $N_2$, if there exists gradient term of $\phi_3$, one uses $\|\nabla \phi_3 \|_{L_t^\infty L_x^2}$ to control $\nabla \phi_3$ and $L_t^1 L_x^\infty$ norms to control the other quadratic or higher order terms. If there exists no $\nabla \phi_3$, one directly uses $L_t^2 L_x^4$ to bound two of them and use $L_t^\infty L_x^\infty$ to dominate the others. In fact, for the gradient term we have

$$\|A^{qua} \cdot \nabla \phi_3 \|_{L_t^1 L_x^2} \lesssim \|\nabla \phi_3 \|_{L_t^\infty L_x^2} \|A^{qua} \|_{L_t^1 L_x^\infty} \lesssim \omega_1^{-1} (s) \epsilon_1^{\alpha+2}.$$

And for the other terms by Proposition 3.3 we have

$$\|A^{qua} \cdot A^{qua} \phi_3 \|_{L_t^1 L_x^2} \lesssim \|A^{qua} \|_{L_t^2 L_x^4} \|\phi_3 \|_{L_t^2 L_x^4} \|A^{qua} \|_{L_t^1 L_x^\infty} \lesssim \omega_1^{-1} (s) \epsilon_1^{2\alpha+2}$$

$$\|A^{qua} \cdot A^{lin} \phi_3 \|_{L_t^1 L_x^2} \lesssim \|A^{qua} \|_{L_t^2 L_x^4} \|\phi_3 \|_{L_t^2 L_x^4} \|A^{lin} \|_{L_t^1 L_x^\infty} \lesssim \omega_1^{-1} (s) \epsilon_1^{\alpha+1}$$

$$\|A^{qua} \cdot A^{\infty} \phi_3 \|_{L_t^1 L_x^2} \lesssim \|A^{qua} \|_{L_t^2 L_x^4} \|\phi_3 \|_{L_t^\infty L_x^2} \|A^{\infty} \|_{L_t^1 L_x^\infty} \lesssim \omega_1^{-1} (s) \epsilon_1^{3\alpha+2}$$

$$\|A^{lin} \cdot A^{lin} \phi_3 \|_{L_t^1 L_x^2} \lesssim \|A^{lin} \|_{L_t^2 L_x^4} \|\phi_3 \|_{L_t^2 L_x^4} \|A^{lin} \|_{L_t^1 L_x^\infty} \lesssim \omega_1^{-1} (s) \epsilon_1^{\alpha+2}$$

$$\|\nabla j A^{qua} \phi_3 \|_{L_t^1 L_x^2} \lesssim \|A^{qua} \|_{L_t^2 L_x^4} \|\phi_3 \|_{L_t^2 L_x^4} \|A^{qua} \|_{L_t^1 L_x^\infty} \lesssim \omega_1^{-1} (s) \epsilon_1^{\alpha+2},$$

and

$$\|\kappa \text{Im}(\tilde{\phi}^j \tilde{\phi}) \|_{L_t^1 L_x^\infty} \lesssim \|\tilde{\phi} \|_{L_t^2 L_x^4} \|\phi_3 \|_{L_t^2 L_x^4} \|\phi \|_{L_t^\infty L_x^\infty} \lesssim \omega_1^{-1} (s) \epsilon_1^3$$

$$\|\tilde{\kappa} \text{Im}(\tilde{\phi}^j \tilde{\phi}) \|_{L_t^1 L_x^\infty} \lesssim \|\tilde{\kappa} \|_{L_t^2 L_x^4} \|\phi_3 \|_{L_t^2 L_x^4} \|\phi \|_{L_t^\infty L_x^\infty} \lesssim \omega_1^{-1} (s) \epsilon_1^3$$

$$\|\kappa \text{Im}(\tilde{\phi}^j \tilde{\phi}) \|_{L_t^1 L_x^\infty} \lesssim \|\tilde{\phi} \|_{L_t^2 L_x^4} \|\phi_3 \|_{L_t^2 L_x^4} \|\phi \|_{L_t^\infty L_x^\infty} \lesssim \omega_1^{-1} (s) \epsilon_1^3$$

$N_3$ is direct by applying Corollary 3.1.
Lemma 3.11. With the above assumption (3.11–3.13), we have
\[
\sup_{s > 0} \omega_{\frac{1}{2} - \delta}^2(s) \| \nabla \phi_s \|_{L^\infty_t L^2_x([0,T] \times \mathbb{H}^2)} \lesssim \epsilon_1.
\] (3.104)

Proof. To apply the energy estimates, we use the following suitable equation for \( \phi_s \)
\[
(i \partial_t + \Delta_A) \phi_s = A_t \phi_s + i \kappa \text{Im}(\overline{\phi} \phi_s) + i \kappa \text{Im}(\phi \phi_s^\infty) + i \kappa \text{Im}(\overline{\phi} \phi_s^\infty) + i \partial_s Z.
\] (3.105)

And the energy estimates of Proposition 6.1 give
\[
\|
\nabla \phi_s \|^2_{L^\infty_t L^2_x} \lesssim \| A \phi_s \|^2_{L^\infty_t L^2_x} + \| \nabla \phi_s(s, 0, x) \|^2_{L^2} + \| \partial_t A \| \phi_s \| D_A \phi_s \|_{L^1_{t,x}} + \| D_A F \| \| D_A \phi_s \|_{L^1_{t,x}},
\] (3.106)

where \( F \) denotes the RHS of (3.105). The initial data term \( \| \nabla \phi_s(s, 0, x) \|_{L^2_x} \) is admissible by Lemma 3.9.

Step 0. By Corollary 3.1, we obtain that
\[
\| D_A \partial_s Z \| \| D_A \phi_s \|_{L^1_{t,L^1_t}} \lesssim (\| \nabla \partial_t Z \|_{L^1_x L^2_t} + \| A \|_{L^\infty_{t,x}} \| \partial_s Z \|_{L^1_t L^2_x}) (\| \nabla \phi_s \|_{L^\infty_t L^2_x} + \| A \|_{L^\infty_{t,x}} \| \phi_s \|_{L^\infty_t L^2_x}) \lesssim \epsilon_1^{2a+1} \omega_{\frac{1}{2} - \delta}^2(s).
\]

And it is easy to see
\[
\| A \phi_s \|^2_{L^\infty_t L^2_x} \lesssim \| \phi_s \|^2_{L^\infty_t L^2_x} \lesssim M \epsilon_1^2 \omega_{\frac{1}{2} - \delta}^2(s).
\]

Step 1. Let’s calculate \( \partial_t A \). In fact, one has
\[
\partial_t A_i = \int_s^\infty (\partial_t \kappa) \text{Im}(\phi_i \overline{\phi}_s) ds' + \int_s^\infty \kappa \text{Im}(\partial_t \phi_i \overline{\phi}_s) ds' + \int_s^\infty \kappa \text{Im}(\phi_i \partial_t \overline{\phi}_s) ds'.
\]
Recall that \( D_i \phi_j = D_j \phi_i \). Then by (3.6),
\[
| \partial_t \phi | \leq | A | | \phi_i | + | \phi |, \quad | \partial_t \phi_s | \leq | A_t | | \phi_s | + | \phi |^2 | \phi_s | + | \Delta_A \phi_s | + | \partial_s Z |,
\]
where we emphasize again that \( \phi \) does not contain the time component in this section. And \( \partial_t \kappa \) is point-wisely bounded as
\[
| \partial_t \kappa | \lesssim | D \kappa | | \partial_t v | \lesssim M | \phi_t |,
\]
where \( D \) denotes the covariant derivative corresponding to the Riemannian metric of \( \mathcal{N} \). In a summary, we have
\[
| \partial_t A | \lesssim \int_s^\infty | \phi_t | | \phi | | \phi_s | ds' + \int_s^\infty | \phi_s | (| A | | \phi_t | + | \nabla \phi_t |) ds' + \int_s^\infty | \phi_t | | A_t | | \phi_s | + | \phi_t |^2 | \phi_s | + | \Delta_A \phi_s | ds' + \int_s^\infty | \phi_t | | \partial_s Z | ds'.
\]
Let’s rearrange these upper-bounds for \( \partial_t A \) to be
\[
| \partial_t A | \leq I_1 + I_2 + I_3 + I_4,
\]
where $I_1, I_2, I_3, I_4$ are defined by

\[
I_1 := \int_s^\infty |\phi^\infty| |\Delta_A \phi_s| ds' \\
I_2 := \int_s^\infty |\phi'| |\phi||\phi_s| ds' + \int_s^\infty |\phi_s| |A||\phi| ds' + \int_s^\infty |\phi|(|A| |\phi_s| + |\phi_x|^2 |\phi_s|) ds'
\]

\[
I_3 := \int_s^\infty |\phi_s| |\nabla \phi| ds'
\]

\[
I_4 := \int_s^\infty \|\phi||\partial_s Z||ds'.
\]

We see $I_2$ is at least quadratic. By interpolation and decay of $\phi^\infty$, we get from Proposition 3.3 that

\[
\|e^{\frac{1}{2}t} I_1\|_{L_x^\infty L_t^4} \lesssim \int_s^\infty (\delta \delta^{-\alpha} + 1) \delta^{-\frac{1}{2}} \|\phi^\infty\|_{L_x^\infty} e_1^\alpha ds \lesssim e_1^\alpha.
\]  

(3.107)

Meanwhile, by Proposition 3.3 and Lemma 3.8 we have

\[
\|I_2\|_{L_t^\infty L_x^4} \lesssim \int_s^\infty \|\phi_t\|_{L_t^\infty L_x^4} \|\phi_s\|_{L_t^\infty L_x^4} (\|\phi\|_{L_t^\infty L_x^4} + \|A\|_{L_t^\infty L_x^4}) ds' \\
+ \int_s^\infty (\|\phi\|_{L_t^\infty L_x} \|A_t\|_{L_t^\infty L_x} + \|\phi\|_{L_t^\infty L_x}^2) \|\phi_s\|_{L_t^\infty L_x} \|L_t^\infty L_x^4\| ds' \\
\lesssim e_1^{1+\alpha}.
\]  

(3.108)

The $I_3$ term is bounded by Lemma 3.8 as

\[
\|I_3\|_{L_t^\infty L_x^4} \lesssim \int_s^\infty \|\phi_t\|_{L_t^\infty L_x^4} \|\nabla \phi_t\|_{L_t^\infty L_x^4} \|\phi_s\|_{L_t^\infty L_x^4} ds' \lesssim e_1^{1+\alpha}.
\]  

(3.109)

The $I_4$ term is dominated by Corollary 3.1 and the $\|\phi\|_{L_t^\infty L_x^4}$ bound:

\[
\|I_4\|_{L_t^\infty L_x^4} \lesssim e_1^{1+\alpha}.
\]  

(3.110)

**Step 2.** In this step we bound $\|\partial_t A\|_{L_t^{1,x}}$. By Step 1, we see

\[
\|\partial_t A\|_{L_t^{1,x}} \|\phi_x\|_{L_t^{1,x}} \|\nabla \phi_x\|_{L_t^{1,x}} \lesssim e^{\frac{1}{2}t} \|I_1\|_{L_t^{\infty} L_x^{\frac{2}{3}}} + \|I_2\|_{L_t^{\infty} L_x^{\frac{2}{3}}} \|\phi_s\|_{L_t^{\infty} L_x^{\frac{2}{3}}} \|\nabla \phi_s\|_{L_t^{\infty} L_x^{\frac{2}{3}}} \\
+ \|I_3\|_{L_t^{\infty} L_x^{\frac{2}{3}}} \|\phi_s\|_{L_t^{\infty} L_x^{\frac{2}{3}}} \|\nabla \phi_s\|_{L_t^{\infty} L_x^{\frac{2}{3}}} \\
+ \|I_4\|_{L_t^{\infty} L_x^{\frac{2}{3}}} \|\phi_s\|_{L_t^{\infty} L_x^{\frac{2}{3}}} \|\nabla \phi_s\|_{L_t^{\infty} L_x^{\frac{2}{3}}}.
\]

Then by (3.107)–(3.110), we obtain

\[
\|\partial_t A\|_{L_t^{1,x}} \|\phi_x\|_{L_t^{1,x}} \|\nabla \phi_x\|_{L_t^{1,x}} \lesssim e_1^{1+2\alpha}.
\]
The left \(|\|\partial_t A\|\|\phi_s\|\|A\|\phi_s\|\|L_{1,x}^1\|\) is much easier, and we conclude for this step that
\[
|\|\partial_t A\|\|\phi_s\|\|D_A\phi_s\|\|L_{1,x}^1\| \lesssim \omega^{-2\delta}(s)\epsilon_1^{1+2\alpha}.
\]

**Step 3.** Now, let’s bound \(|\|D_A L\|\|D_A \phi_s\|\|L_{1,x}^1\|\), where \(L\) denotes the RHS of (3.105) with \(i\partial_s Z\) ruling out. Let’s consider the top derivative term, i.e. \(|\|\nabla L\|\|\nabla \phi_s\|\|L_{1,x}^1\|\). As before, we divide \(L\) into three parts:

\[
L_1 := i\bar{\kappa}\Im h^{kj}(\phi_j^\infty \phi_s^\infty) + i\kappa^{\infty}\Im (\bar{\phi}^j \phi_s^\infty) + i\kappa^{\infty} h^{kj}\Im (\phi_k^\infty \bar{\phi}_s^\infty) \phi_j
\]

\[
L_2 := i\bar{\kappa}\Im (\bar{\phi}^j \phi_s^\infty) + i\kappa^{\infty}\Im (\phi_j^\infty \bar{\phi}_s^\infty) \phi_j + i\bar{\kappa}\Im (\phi_j^\infty \phi_s^\infty) \phi_j + i\kappa^{\infty}\Im (\bar{\phi}^j \bar{\phi}_s^\infty) \phi_j
\]

\[
L_3 := A_t \phi_s.
\]

The \(|\|L_1\|\|\nabla \phi_s\|\|L_{1,x}^1\|\) norm is bounded by
\[
\begin{align*}
|\|\nabla L_1\|\|\nabla \phi_s\|\|L_{1,x}^1\| & \lesssim \|e^{\frac{1}{2}t} |\|\nabla L_1\|\|L_{1,x}^1\| \|e^{-\frac{1}{2}r}|\|\nabla \phi_s\|\|L_{1,x}^1\| \\
& \lesssim \|\nabla \tilde{\kappa}\|_{L_t^\infty L_{x}^{\frac{3}{4}}} \|\phi_s\|_{L_t^2 L_{x}^4} \|e^{-\frac{1}{2}r}|\|\nabla \phi_s\|\|L_{1,x}^1\| \\
& \quad + \|\nabla \bar{\phi}\|_{L_t^{\infty} L_{x}^{\frac{1}{4}}} \|\phi_s\|_{L_t^2 L_{x}^4} \|e^{-\frac{1}{2}r}|\|\nabla \phi_s\|\|L_{1,x}^1\| \\
& \quad + \left(\|\tilde{\kappa}\|_{L_t^2 L_{x}^{\infty}} + \|\tilde{\phi}\|_{L_t^2 L_{x}^{\infty}}\right) \|\nabla \phi_s\|_{L_t^\infty L_{x}^2} \|e^{-\frac{1}{2}r}|\|\nabla \phi_s\|\|L_{1,x}^1\|.
\end{align*}
\]

The \(|\|L_2\|\|\nabla \phi_s\|\|L_{1,x}^1\|\) norm is bounded by
\[
\begin{align*}
|\|\nabla L_2\|\|\nabla \phi_s\|\|L_{1,x}^1\| & \lesssim |\|\nabla L_2\|\|L_t^2 L_{x}^2 \|\nabla \phi_s\|\|L_t^\infty L_{x}^2 \\
& \lesssim \|\nabla \tilde{\kappa}\|_{L_t^{\infty} L_{x}^{\frac{3}{4}}} \|\tilde{\phi}\|_{L_t^2 L_{x}^2} \|\phi_s\|_{L_t^2 L_{x}^4} \|\nabla \phi_s\|_{L_t^\infty L_{x}^2} \\
& \quad + \|\tilde{\kappa}\|_{L_t^2 L_{x}^{\infty}} \|\nabla \bar{\phi}\|_{L_t^{\infty} L_{x}^{\frac{1}{4}}} \|\phi_s\|_{L_t^2 L_{x}^4} \|\nabla \phi_s\|_{L_t^\infty L_{x}^2} \\
& \quad + \|\tilde{\phi}\|_{L_t^2 L_{x}^2} \|\nabla \bar{\phi}\|_{L_t^{\infty} L_{x}^{\frac{1}{4}}} \|\phi_s\|_{L_t^2 L_{x}^4} \|\nabla \phi_s\|_{L_t^\infty L_{x}^2} \\
& \quad + \|\tilde{\phi}\|_{L_t^2 L_{x}^2} \|\nabla \phi_s\|_{L_t^\infty L_{x}^2} \\
& \quad + \|\tilde{\phi}\|_{L_t^2 L_{x}^2} \|\nabla \phi_s\|_{L_t^\infty L_{x}^2}.
\end{align*}
\]

The \(|\|L_3\|\|\nabla \phi_s\|\|L_{1,x}^1\|\) norm is bounded by
\[
\begin{align*}
|\|\nabla L_3\|\|\nabla \phi_s\|\|L_{1,x}^1\| & \lesssim |\|A_t\|\|L_t^1 L_{x}^\infty \|\nabla \phi_s\|_{L_t^2 L_{x}^2} + |\|A_t\|\|L_t^{\infty} L_{x}^{\frac{3}{4}} \|\phi_s\|_{L_t^2 L_{x}^4} \|\nabla \phi_s\|_{L_t^\infty L_{x}^2}.
\end{align*}
\]

Hence, by Lemma 3.8 and Proposition 3.3, we get
\[
|\|\nabla L\|\|\nabla \phi_s\|\|L_{1,x}^1\| \lesssim \omega^{-2\delta}(s)\epsilon_1^{1+2\alpha}.
\]

The other lower derivative order terms in \(|\|D_A L\|\|D_A \phi_s\|\|L_{1,x}^1\|\) are easier to dominate and we conclude that
\[
|\|D_A L\|\|D_A \phi_s\|\|L_{1,x}^1\| \lesssim \omega^{-2\delta}(s)\epsilon_1^{1+2\alpha}.
\]

Combining these estimates together, the desired result follows by (3.106). □
Lemma 3.12. With the above assumption (3.11–3.13), we have
\[ \sup_{s > 0} \omega_{1/2 - \delta}(s) \| e^{-2^{1/2} r} |\nabla \phi_s \|_{L_t^2 L_x^2 ([0, T] \times \mathbb{R}^2)} \lesssim M \epsilon_1^{\frac{1}{2}(1+\alpha)} . \]  
(3.111)

Proof. To apply Morawetz estimates, we use the following equation of \( \phi_s \):
\[ (i \partial_t + \Delta) \phi_s = A_1 \phi_s + i \kappa \text{Im}(\phi^j \phi^\ell) \phi_j + i \partial_s Z. \]

Since \( \| A \|_{L_{t,x}^\infty} \lesssim 1 \), the Morawetz estimates in Corollary 6.1 give
\[ \| e^{-2^{1/2} r} |\nabla \phi_s \|_{L_t^2 L_x^2} \lesssim \| (-\Delta) \frac{1}{2} \phi_s \|_{L_t^\infty L_x^2} + \| (|A| + |\nabla A|) \phi_s \|_{L_t^1 L_x^4} + \| \partial_s Z \|_{L_t^\infty L_x^4} + \| A_1 \phi_s \|_{L_t^1 L_x^4} + \| \kappa \text{Im}(\phi^j \phi^\ell) \phi_j \|_{L_t^1 L_x^4} \]
\[ + \| \partial_s A \|_{L_t^1 L_x^4} \phi_s \|_{L_t^2 L_x^4} + \| \phi_s \|_{L_t^2 L_x^4} \phi_s \|_{L_t^2 L_x^4} \lesssim \epsilon_1^{1+\alpha} \omega_{1/2 - \delta} . \]

For the other terms, we have
\[ \| (|A| + |\nabla A|) \phi_s \|_{L_t^1 L_x^4} \lesssim \| (|A^\infty| + |\nabla A^\infty|) e^{\frac{1}{2} r} \| L_t^2 L_x^4 \| \phi_s \|_{L_t^1 L_x^4} + \| e^{-\frac{1}{2} r} |\nabla \phi_s \|_{L_t^2 L_x^4} \]
\[ + \| A_1 \phi_s \|_{L_t^1 L_x^4} \phi_s \|_{L_t^2 L_x^4} \| \text{Im}(\phi^j \phi^\ell) \phi_j \|_{L_t^1 L_x^4} \]
\[ + \| A \|_{L_t^\infty L_x^4} \phi_s \|_{L_t^2 L_x^4} + \| A \|_{L_t^\infty L_x^4} \phi_s \|_{L_t^2 L_x^4} \| \phi_s \|_{L_t^2 L_x^4} \]
\[ + \| \partial_s A \|_{L_t^1 L_x^4} \phi_s \|_{L_t^2 L_x^4} + \| A \|_{L_t^\infty L_x^4} \phi_s \|_{L_t^2 L_x^4} \phi_s \|_{L_t^2 L_x^4} \| \phi_s \|_{L_t^2 L_x^4} \lesssim \epsilon_1^{1+\alpha} \omega_{1/2 - \delta} . \]

And similar arguments as Lemma 3.11 yield
\[ \| \kappa \text{Im}(\phi^j \phi^\ell) \phi_j \|_{L_t^1 L_x^4} \lesssim \| \phi_s \|_{L_t^2 L_x^4} \| \phi_s \|_{L_t^2 L_x^4} \| e^{-\frac{1}{2} r} |\nabla \phi_s \|_{L_t^2 L_x^4} \]
\[ + \| \phi_s \|_{L_t^2 L_x^4} |\phi_s \|_{L_t^2 L_x^4} \| \phi_s \|_{L_t^2 L_x^4} \| \phi_s \|_{L_t^2 L_x^4} \| \phi_s \|_{L_t^2 L_x^4} \| \phi_s \|_{L_t^2 L_x^4} \| \phi_s \|_{L_t^2 L_x^4} \lesssim \epsilon_1^{1+\alpha} \omega_{1/2 - \delta} . \]

Also, we have
\[ \| \phi_s \|_{L_t^2 L_x^4} \| \phi_s \|_{L_t^2 L_x^4} \| e^{-\frac{1}{2} r} |\nabla \phi_s \|_{L_t^2 L_x^4} \lesssim \| A \|_{L_t^\infty W_x^{1,2}} \| \phi_s \|_{L_t^2 L_x^4} \]
\[ + \| \phi_s \|_{L_t^2 L_x^4} |\phi_s \|_{L_t^2 L_x^4} \| \phi_s \|_{L_t^2 L_x^4} \| \phi_s \|_{L_t^2 L_x^4} |\phi_s \|_{L_t^2 L_x^4} \| \partial_s Z \|_{L_t^2 L_x^4} \]
\[ + \| A \|_{L_t^\infty \| \phi_s \|_{L_t^2 L_x^4} \| \phi_s \|_{L_t^2 L_x^4} \| \phi_s \|_{L_t^2 L_x^4} \| \partial_s Z \|_{L_t^2 L_x^4} \]
\[ + \| A \|_{L_t^\infty \| \phi_s \|_{L_t^2 L_x^4} \| \phi_s \|_{L_t^2 L_x^4} \| \phi_s \|_{L_t^2 L_x^4} \| \partial_s Z \|_{L_t^2 L_x^4} \lesssim \epsilon_1^{1+\alpha} \omega_{1/2 - \delta} . \]

End of proof of Proposition 3.2.

Hence, Proposition 3.2 follows by bootstrap and Lemma 3.10, Lemma 3.11, Lemma 3.12.
4. Proof of Theorem 1.1, Part II: Asymptotic Behaviors

4.1. Global existence. The following Lemma 4.1 states a blow-up criterion for SMF. The corresponding result is well-known in 2D heat flows. In the case $\mathcal{M} = \mathbb{R}^2$, without claiming any originality we gave a detailed proof for the analogous result as Lemma 4.1 in [ Appendix B,[34]]. Since the proof in [34] is purely an energy argument, it is easy to give a parallel proof for $\mathcal{M} = H^2$.

**Lemma 4.1.** Let $u \in C([0, T); \mathcal{H}^3_Q)$ solve SMF, and assume that there exists some constant $C > 0$ such that

$$\|\nabla u\|_{L^\infty_t, t, x} \leq C.$$

Then there exists some $\rho > 0$ such that $u \in C([0, T + \rho); \mathcal{H}^3_Q)$.

Now, let’s use Lemma 4.1 to show $u$ is global. Assume that $u \in C([0, T_*); \mathcal{H}^3_Q)$, and $T_* > 0$ is the maximum lifespan. Proposition 3.2 has shown all the bounds obtained in Sect. 3 hold uniformly on $t \in [0, T_*)$. Particularly, (3.61) holds, and thus

$$\int_0^\infty \|\phi_s\|_{L_t^\infty, W^1_t, \infty} ds' \leq C,$$  \hspace{1cm} (4.1)

for some $C > 0$.

Recall that

$$\phi_j(0, t, x) = \phi_j^\infty - \int_0^\infty (\partial_j \phi_s + iA_j \phi_s) ds'.$$  \hspace{1cm} (4.2)

Then (4.1) and the bound of $\|A\|_{L^\infty}$ show

$$\|\nabla u\|_{L^\infty_t, t, x} \leq C,$$

which by Lemma 4.1 implies that $u$ can be continuously extended beyond $T_*$, thus contradicting with the maximum of $T_*$. Therefore, $u$ is global and belongs to $C([0, T]; \mathcal{H}^k_Q)$ for any $T > 0$, $k \in \mathbb{Z}_+$, as long as $u_0$ verifies Theorem 1.1 and $u_0 \in \mathcal{H}^k_Q$.

4.2. Convergence to harmonic map in $L^\infty_x$. The convergence of $u$ to harmonic maps in $L^\infty_x$ is now standard. By applying

$$\|u - Q\|_{L^\infty_x} \leq \int_0^\infty \|\phi_s\|_{L^\infty_t} ds',$$  \hspace{1cm} (4.3)

to prove (1.3) it suffices to show

$$\lim_{t \to \infty} \int_0^\infty \|\phi_s\|_{L^\infty_t} ds' = 0.$$  \hspace{1cm} (4.4)

One may see [27,31,36] for details of proving (4.4).
4.3. Resolution in energy spaces. In this part, we prove the resolution of SMF solutions in the energy space claimed in Theorem 1.1. The proof is divided into three parts.

**Lemma 4.2.** We have

\[ \sup_{s > 0} \omega_{1-\delta}(s) \| \nabla^3 Z \|_{L^1_t L^\infty_x} \lesssim 1 \]  
\[ \sup_{s > 0} \omega_{1-\delta}(s) \| e^{-\frac{1}{2}r} \nabla^2 \phi_s \|_{L^1_t L^2_x} \lesssim 1. \]

**Proof.** (4.5) follows by the same arguments as Lemma 3.6, if one has proved

\[ \| \nabla^3 A \|_{L^\infty_{t,x}} \lesssim \max(s^{-1}, 1). \]  

To prove (4.7), it suffices to apply Lemma 10.3 in Appendix B and formulas like (3.35).

Next, let’s prove (4.6). We introduce two cutoff functions. Set \( \mu_1, \mu_2 > 0 \). Let \( \vartheta_{\mu_1} : \mathbb{H}^2 \to [0, 1] \) be a smooth function which equals 1 for \( r > 3\mu_1 \) and vanishes for \( 0 \leq r \leq \mu_1 \). Let \( \nu_{\mu_2} : \mathbb{H}^2 \to [0, 1] \) be a smooth function which vanishes for \( r > 3\mu_2 \) and equals 1 for \( 0 \leq r \leq \mu_2 \).

Then by (3.45), \( e^{-\frac{1}{2}r} \vartheta_{\mu_1} \phi_s \) satisfies

\[ (\partial_s - \Delta)[e^{-\frac{1}{2}r} \vartheta_{\mu_1} \phi_s] = -2ie^{-\frac{1}{2}r} \vartheta_{\mu_1} A^\infty \cdot \nabla \phi_s - 2\nabla(e^{-\frac{1}{2}r} \vartheta_{\mu_1}) \cdot \nabla \phi_s + L, \]

where \( L \) denotes terms involved with zero order derivative of \( \phi_s \). We remark that since \( \vartheta_{\mu_1} \) is supported in \( r > 2\mu_1, e^{-\frac{1}{2}r} \) is now smooth. Then, by smoothing estimates for heat equations and the equivalence relation (2.1), we get

\[ \| \nabla^2(e^{-\frac{1}{2}r} \vartheta_{\mu_1} \phi_s) \|_{L^2_{t,x}} \lesssim \Theta_1(s, \frac{s}{2}) \| \nabla(e^{-\frac{1}{2}r} \vartheta_{\mu_1} \phi_s(s, t, x)) \|_{L^2_{t,x}} \]
\[ + C_{\mu_1} \int_{\frac{s}{2}}^s \Theta_1(s, s') \| \nabla^2 \phi_{s'} \|_{L^2_t L^4_x} ds' \]
\[ + C_{\mu_1} \int_{\frac{s}{2}}^s \Theta_1(s, s')(\| \nabla \phi_{s'} \|_{L^2_t L^4_x} + \| \phi_{s'} \|_{L^2_t L^4_x}) ds', \]

which by Proposition 3.3, further gives

\[ \sup_{s > 0} \omega_{1-\delta}(s) \| \nabla^2(e^{-\frac{1}{2}r} \vartheta_{\mu_1} \phi_s) \|_{L^2_{t,x}} \lesssim C_{\mu_1}. \]

This, together with chain rule and the previous bounds

\[ \sup_{s > 0} \omega_{1-\delta}(s)(\| e^{-\frac{1}{2}r} \nabla \phi_s \|_{L^2_{t,x}} + \| \phi_s \|_{L^2_t L^4_x}) \lesssim 1, \]

indeed provides out ball estimates for (4.6). It remains to consider bounds near the origin. Since \( e^{-r} \sim 1 \) for \( r \in [0, 1] \), it suffices to bound \( \| \nu_{\mu_2} \nabla^2 \phi_s \|_{L^2_{t,x}} \). Then by the same arguments as above, one has

\[ \sup_{s > 0} \omega_{1-\delta}(s) \| \nu_{\mu_2} \nabla^2 \phi_s \|_{L^2_{t,x}} \lesssim C_{\mu_2}. \]

Thus, (4.6) has been verified. \( \square \)
Lemma 4.3. There exist functions $v_1, v_2 : \mathbb{H}^2 \rightarrow \mathbb{R}^N$ and $f : \mathbb{H}^2 \rightarrow \mathbb{C}$, such that
\[
\lim_{t \to \infty} \| \nabla u - \nabla Q - v_1 \text{Re}(e^{it\Delta} \nabla f) - v_2 \text{Im}(e^{it\Delta} \nabla f) \|_{L_x^2} = 0 \tag{4.8}
\]

$v_1 \perp v_2$, $|v_1| = |v_2| = 1$, $f \in H^1$, where we view $u$ and $Q$ as maps into $\mathbb{R}^N$.

Proof. Recall that $\phi_s$ satisfies
\[
i \partial_t \phi_s + H \phi_s = N.
\]
Applying $(-H)^{\frac{1}{2}}$ to above equation and using Duhamel principle, we see
\[
e^{-itH}(-H)^{\frac{1}{2}}\phi_s(t) = (-H)^{\frac{1}{2}}\phi_s(0) - i \int_0^t e^{-i\tau H}(-H)^{\frac{1}{2}}N \, d\tau.
\]
Using Lemma 4.2, Proposition 3.3 and (5.6), one can verify
\[
\sup_{s > 0} \omega_1 - \delta(s) \| (-H)^{\frac{1}{2}}N \|_{L_x^1 L_x^2} \lesssim \sup_{s > 0} \omega_1 - \delta(s) \| \nabla N \|_{L_x^1 L_x^2} \lesssim 1.
\]
Let
\[
h_s := (-H)^{\frac{1}{2}}\phi_s(s, 0, x) - i \int_0^\infty e^{-\tau H}(-H)^{\frac{1}{2}}N \, d\tau.
\]
Then there holds
\[
\sup_{s > 0} \omega_1 - \delta(s) \| h_s \|_{L_x^2} \lesssim 1. \tag{4.9}
\]
Thus we arrive at
\[
\lim_{t \to \infty} \sup_{s > 0} \omega_1 - \delta(s) \| (-H)^{\frac{1}{2}}\phi_s(t) - e^{itH}h_s \|_{L_x^2} = 0. \tag{4.10}
\]
Set
\[
g_s := (-H)^{-\frac{1}{2}}h_s.
\]
Then, since $(-H)^{-\frac{1}{2}}$ is bounded from $L_x^2$ to $H_x^1$ (see (5.7)), we get from (4.9) that
\[
\sup_{s > 0} \omega_1 - \delta(s) \| g_s \|_{H_x^1} \lesssim 1. \tag{4.11}
\]
Thus applying the equivalence relation $\| \nabla f \|_{L_x^2} \sim \| (-H)^{\frac{1}{2}} \|_{L_x^2}$ gives
\[
\| \nabla(\phi_s - e^{itH}g_s) \|_{L_x^2} \lesssim \| (-H)^{\frac{1}{2}} (\phi_s - e^{itH}g_s) \|_{L_x^2}.
\]
And one infers from (4.10) that
\[
\lim_{t \to \infty} \sup_{s > 0} \omega_1 - \delta(s) \| \nabla(\phi_s - e^{itH}g_s) \|_{L_x^2} = 0. \tag{4.12}
\]
Recall that
\[ \phi_i = \phi^\infty_i - \int_s^\infty (\partial_i \phi_s + iA_i \phi_s)ds. \]

By \(|A| \in L_{i,3}^\infty L_x^2\), we deduce from (4.4) that
\[ \lim_{t \to \infty} \int_0^\infty \|A\|_{L_x^2}ds' = 0. \]

Thus (4.12) implies
\[ \lim_{t \to \infty} \|\phi - \phi^\infty - \nabla \int_0^\infty e^{itH}g_sds\|_{L_x^2} = 0. \]

Moving \(e^{itH}\) outside of the integral (it is reasonable by (4.11)), we see
\[ \lim_{t \to \infty} \|\phi - \phi^\infty - \nabla e^{itH} \int_0^\infty g_sds\|_{L_x^2} = 0. \]

Set
\[ f_1 = \int_0^\infty g_sds'. \]

By Lemma 4.7, there exists a function \(f \in H^1\) such that
\[ \lim_{t \to \infty} \|\nabla[e^{itH}f_1 - e^{itA}f]\|_{L_x^2} = 0. \]

Hence, we have arrived at
\[ \lim_{t \to \infty} \|\phi - \phi^\infty - e^{itA} \nabla f\|_{L_x^2} = 0. \tag{4.13} \]

Recall that
\[ \partial_j u = \text{Re}(\phi_j)e_1 + \text{Im}(\phi_j)e_2, \tag{4.14} \]

and \(\mathcal{P}\) denotes the isometric embedding of \(\tilde{\mathcal{N}}\) into \(\mathbb{R}^N\). By caloric gauge condition,
\[ \|d\mathcal{P}e_j - d\mathcal{P}e^\infty_j\| \lesssim \int_0^\infty |\phi_s|ds', \tag{4.15} \]

which combined with (4.4) implies
\[ \lim_{t \to \infty} \|d\mathcal{P}e_j - d\mathcal{P}e^\infty_j\|_{L_x^2} = 0. \]

Then we see
\[ \|d\mathcal{P}\nabla u - d\mathcal{P}[\text{Re}(\phi^\infty)e_1 + \text{Im}(\phi^\infty)e_2]\]
\[ - d\mathcal{P}[\text{Re}(e^{itA} \nabla f)e_1 + \text{Im}(e^{itA} \nabla f)e_2]\|_{L_x^2} \]
\[ \lesssim \|\phi - \phi^\infty - e^{itA} \nabla f\|_{L_x^2} + \|\phi^\infty + e^{itA} \nabla f\|d\mathcal{P}e - d\mathcal{P}e^\infty\|_{L_x^2} \]
\[ + \|\phi - \phi^\infty - e^{itA} \nabla f\|d\mathcal{P}e - d\mathcal{P}e^\infty\|_{L_x^2}. \tag{4.16} \]

(4.17)
By (4.13), it is direct to see the first term in the RHS of (4.16) and the term in the RHS of (4.17) tend to zero as $t \to \infty$.

Let

$$v_1 = dP(e_1^\infty), \quad v_2 = dP(e_2^\infty).$$  \hfill (4.18)

To finish our proof, it suffices to prove

$$\lim_{t \to \infty} \| |\phi_\infty + e^{it\Delta} \nabla f| |dP e - dP e^\infty| \|_{L^2_x} = 0.$$  \hfill (4.19)

In fact, (4.15) and Hölder inequality give

$$\| |\phi_\infty + e^{it\Delta} \nabla f| |dP e - dP e^\infty| \|_{L^2_x} \lesssim (\| \phi_\infty \|_{L^2_x} + \| \nabla f \|_{L^2_x}) \|dP e - dP e^\infty\|_{L^\infty_x} \lesssim \int_0^\infty \| \phi_s \|_{L^\infty_x} ds'.$$

Then (4.19) follows by (4.4).

Therefore, (4.8) follows and we have accomplished the proof. \qed

**Lemma 4.4.** There exist functions $f_1^+, f_2^+: \mathbb{H}^2 \to \mathbb{C}^N$ belonging to $H^1$, such that

$$\lim_{t \to \infty} \| u - Q - \text{Re}(e^{it\Delta} f_1^+) - \text{Im}(e^{it\Delta} f_2^+)\|_{H^1_x} = 0.$$  \hfill (4.20)

where we view $u$ and $Q$ as maps into $\mathbb{R}^N$.

**Proof.** By Lemma 4.3,

$$\lim_{t \to \infty} \| \nabla u - \nabla Q - v_1 \text{Re}(e^{it\Delta} \nabla f) - v_2 \text{Im}(e^{it\Delta} \nabla f)\|_{L^2_x} = 0.$$  \hfill (4.21)

where $v_1$, $v_2$ are given by (4.18). By density argument and dispersive estimates, we infer from

$$\| \nabla v_1 \|_{L^4_x} + \| \nabla v_2 \|_{L^4_x} \lesssim \| \phi_\infty \|_{L^4_x} + \| A_\infty \|_{L^4_x} \lesssim 1,$$

that there holds

$$\lim_{t \to \infty} \sum_{j=1}^2 \| \nabla v_j \| e^{it\Delta} f \|_{L^2_x} = 0.$$  \hfill (4.22)

Hence (4.21) and (4.22) give

$$\lim_{t \to \infty} \| u - Q - v_1 \text{Re}(e^{it\Delta} f) - v_2 \text{Im}(e^{it\Delta} f)\|_{H^1_x} = 0.$$  \hfill (4.23)

We claim that there exist functions $f_1^+, f_2^+: \mathbb{H}^2 \to \mathbb{C}^N$ belonging to $H^1$ such that

$$\lim_{t \to \infty} \| v_1 e^{it\Delta} f - e^{it\Delta} f_1^+\|_{H^1_x} = 0$$  \hfill (4.24)

$$\lim_{t \to \infty} \| v_2 e^{it\Delta} f - e^{it\Delta} f_2^+\|_{H^1_x} = 0.$$  \hfill (4.25)

If (4.24–4.25) are done, the desired result (4.20) follows by (4.23). We will prove (4.24) in detail and (4.25) follows by the same way.
Now, let’s prove (4.24). \( v_1 e^{it\Delta} f \) solves the equation
\[
[i \partial_t + \Delta](v_1 e^{it\Delta} f) = \Delta v_1 e^{it\Delta} f + 2 \nabla v_1 \cdot \nabla e^{it\Delta} f.
\]

Thus Duhamel principle yields
\[
e^{-it\Delta}(v_1 e^{it\Delta} f) = v_1 f - i \int_0^t e^{-i\tau\Delta}[(\Delta v_1) e^{i\tau\Delta} f + 2 \nabla v_1 \cdot \nabla e^{i\tau\Delta} f] d\tau.
\]

By endpoint Strichartz estimates of \( e^{it\Delta} \) and the equivalence relation \( \|(-\Delta)^{\frac{1}{2}} g\|_{L^2} \sim \|\nabla g\|_{L^2} \), we deduce that
\[
\|(-\Delta)^{\frac{1}{2}} \left[ \int_{t_1}^{t_2} e^{-i\tau\Delta}[(\Delta v_1) e^{i\tau\Delta} f + 2 \nabla v_1 \cdot \nabla e^{i\tau\Delta} f] d\tau \right]\|_{L^2_x^2} \\
\lesssim \|\nabla[(\Delta v_1) e^{i\tau\Delta} f + 2 \nabla v_1 \cdot \nabla e^{i\tau\Delta} f]\|_{L^2_t \times L^2_x}^{\frac{3}{4}} \\
\lesssim \sum_{j=1}^3 \|\nabla^j v_1\|_{L^2_x} \sum_{k=0}^1 \|\nabla^k e^{i\tau\Delta} f\|_{L^2_t \times L^4_x}
\]

where the integral domains are \( (\tau, x) \in [t_1, t_2] \times \mathbb{H}^2 \). Hence, there exists a function \( g_+^1 : \mathbb{H}^2 \to \mathbb{C}^N \) belonging to \( L^2 \) such that
\[
\lim_{t \to \infty} \|(-\Delta)^{\frac{1}{2}} (v_1 e^{it\Delta} f) - e^{-it\Delta} g_+^1\|_{L^2_x} = 0.
\]

Letting \( f_1^+ = (-\Delta)^{-\frac{1}{2}} g_+^1 \), using Sobolev embedding we obtain (4.24). \( \square \)

**End of Proof to Theorem 1.1**

Until now, we have proved SMF with initial data \( u_0 \) evolves to a global solution and the two convergence results (1.3) and (1.4) hold.

### 4.4. Linear scattering theory

In the proof of Lemma 4.3, we applied the linear scattering results for \( H \) stated in Lemma 4.7 below. In this subsection, let’s prove Lemma 4.7. Its proof will be divided into three lemmas.

**Lemma 4.5** (Scattering in \( L^2 \)). Let \( f \in L^2(\mathbb{H}^2) \), then there exists a function \( g \in L^2(\mathbb{H}^2) \) such that
\[
\lim_{t \to \infty} \|e^{itH} f - e^{it\Delta} g\|_{L^2_x} = 0. \tag{4.26}
\]

**Proof.** By a density argument, it suffices to consider \( f \in H^1 \). The function \( e^{-it\Delta} e^{itH} f \) can be written as
\[
e^{-it\Delta} e^{itH} f = f - i \int_0^t e^{-i\tau\Delta} (2i A_j^\infty \nabla j - A^\infty \cdot A^\infty \\
+ i \nabla j A_j^\infty + \kappa^\infty |\phi^\infty|^2)(e^{i\tau H} f) d\tau.
\]
By endpoint Strichartz estimates for $e^{it\Delta}$, one has

$$\| \int_{t_1}^{t_2} e^{-it\Delta} (2i A_j^{\infty} \nabla^j - A^{\infty} \cdot A^{\infty} + i \nabla^j A_j^{\infty} + \kappa^{\infty} |\phi^{\infty}|^2) (e^{it \mathbf{H}} f) d\tau \|_{L^4_x}^2$$

$$\lesssim \| 2i A_j^{\infty} \nabla^j - A^{\infty} \cdot A^{\infty} + i \nabla^j A_j^{\infty} + \kappa^{\infty} |\phi^{\infty}|^2) (e^{it \mathbf{H}} f) \|_{L^2_x}^4$$

Lemma 4.6

Thus applying resolvent estimates in Lemma 5.1 and H"older inequality yields

$$\| e^{-\alpha |\nabla|} (e^{it \mathbf{H}} f) \|_{L^2_x L^4_x} \leq e^{-\alpha |\nabla|} A^{\infty} \|_{L^\infty_x L^4_x} + \| e^{\alpha |\nabla|} |\phi^{\infty}|^2 \|_{L^\infty_x L^4_x}$$

where we take $0 < \alpha' \ll 1$, and all the integral domains in the involved norms are $[t_1, t_2] \times \mathbb{R}^2$. Then since $f \in H^1$, we get by Lemma 7.2 that (4.27) converges to 0 as $t_2 \to t_1 \to \infty$. Hence, $e^{-it\Delta} e^{it \mathbf{H}} f$ converges in $L^2$ as $t \to \infty$, and thus setting

$$g := \lim_{t \to \infty} e^{-it\Delta} e^{it \mathbf{H}} f$$

suffices for the desired result.

\[ \square \]

Lemma 4.6 (Asymptotic vanishing). Let $f \in H^1(\mathbb{R}^2)$. We have

$$\lim_{t \to \infty} \| (-\mathbf{H})^{\frac{1}{2}} - (-\Delta)^{\frac{1}{2}} e^{it \Delta} f \|_{L^2_x} = 0$$

(4.28)

$$\lim_{t \to \infty} \| (-\mathbf{H})^{-\frac{1}{2}} - (-\Delta)^{-\frac{1}{2}} e^{it \Delta} f \|_{H^1_x} = 0.$$  

(4.29)

Proof. Write

$$\mathbf{H} = \Delta + V + X^j \nabla^j$$

(4.30)

where $V$ denotes the electric potential and $X$ denotes the magnetic field. Recall the resolvent identity

$$(-\mathbf{H} + \lambda)^{-1} - (-\Delta + \lambda)^{-1} = (-\mathbf{H} + \lambda)^{-1} (V + X) (-\Delta + \lambda)^{-1},$$

(4.31)

and the Balakrishnan formula for self-adjoint and non-negative operators, i.e.

$$T^{-\frac{1}{2}} h := c_1 \int_0^\infty \lambda^{-\frac{1}{2}} (T + \lambda)^{-1} T h d\lambda.$$  

Then direct calculations give

$$(-\mathbf{H})^{\frac{1}{2}} h - (-\Delta)^{\frac{1}{2}} h = -c_1 \int_0^\infty \lambda^{\frac{1}{2}} (\mathbf{H} + \lambda)^{-1} (V + X) (-\Delta + \lambda)^{-1} h d\lambda.$$  

(4.32)

Thus applying resolvent estimates in Lemma 5.1 and H"older inequality yields

$$\| (-\mathbf{H})^{\frac{1}{2}} - (-\Delta)^{\frac{1}{2}} h \|_{L^2_x} \lesssim \int_0^\infty \lambda^{\frac{1}{2}} (\mathbf{H} + \lambda)^{-1} \| V \|_{L^2_x} \| (-\Delta + \lambda)^{-1} \|_{L^4_x \to L^4_x} \| h \|_{L^4_x} d\lambda$$

$$+ \int_0^\infty \lambda^{\frac{1}{2}} (\mathbf{H} + \lambda)^{-1} \| X \|_{L^2_x} \| \Delta + \lambda)^{-1} \|_{L^4_x \to L^4_x} \| h \|_{L^4_x} d\lambda$$

$$\lesssim \| h \|_{L^4_x}.$$
Hence, we get
\[
\|[(\mathbf{-H})^{\frac{1}{2}} - (-\Delta)^{\frac{1}{2}}] e^{it\Delta} f\|_{L^2_x} \lesssim \|e^{it\Delta} f\|_{L^2_x}.
\] (4.33)

Also, using \(\|X\|_{L^\infty} + \|V\|_{L^\infty} \lesssim 1\), we have
\[
\|[(\mathbf{-H})^{\frac{1}{2}} - (-\Delta)^{\frac{1}{2}}] e^{it\Delta} f\|_{L^2_x} \lesssim \|e^{it\Delta} f\|_{L^2_x}.
\] (4.34)

Now, let’s prove (4.28) by density argument. Assume that \(f_n \in C_c^\infty(\mathbb{R}^2)\) are a sequence of functions such that \(f_n \to f\) in \(H^1\). Then (4.33) and (4.34) show
\[
\|[(\mathbf{-H})^{\frac{1}{2}} - (-\Delta)^{\frac{1}{2}}] e^{it\Delta} f\|_{L^2_x} \lesssim \|e^{it\Delta} (f_n - f)\|_{L^2_x}.
\]
(4.35)

Then for any small constant \(\eta > 0\), fixing a sufficiently large \(n\) such that the first term on the RHS of (4.35) is smaller than \(\eta\) and then using dispersive estimates for the second term on the RHS implies that (4.35) tends to zero as \(t \to \infty\).

(4.29) follows by the same way with (4.32) replaced by
\[
(\mathbf{-H})^{\frac{1}{2}} h - (-\Delta)^{\frac{1}{2}} h = c_2 \int_0^\infty \lambda^{-\frac{1}{2}} (\mathbf{H} + \lambda)^{-1} (V + X)(-\Delta + \lambda)^{-1} h d\lambda.
\]

Now, we are ready to prove the scattering in \(H^1\).

**Lemma 4.7.** Let \(f \in H^1(\mathbb{R}^2)\), then there exists a function \(g \in H^1(\mathbb{R}^2)\) such that
\[
\lim_{t \to \infty} \|e^{it\mathbf{H}} f - e^{it\Delta} g\|_{H^1_x} = 0.
\] (4.36)

**Proof.** Given \(f \in H^1\), applying Lemma 4.5 to \((\mathbf{-H})^{\frac{1}{2}} f\) shows there exists a function \(g_1 \in L^2_x\) such that
\[
\lim_{t \to \infty} \|e^{it\Delta} g_1 - e^{it\mathbf{H}} (\mathbf{-H})^{\frac{1}{2}} f\|_{L^2_x} = 0.
\]

Since \((\mathbf{-H})^{-\frac{1}{2}}\) is bounded from \(L^2\) to \(H^1\) (see (5.7)), we have
\[
\lim_{t \to \infty} \|(-\Delta)^{-\frac{1}{2}} e^{it\Delta} g_1 - e^{it\mathbf{H}} f\|_{H^1_x} = 0.
\] (4.37)

By (4.29) of Lemma 4.6, one also has
\[
\lim_{t \to \infty} \|[(\mathbf{-H})^{\frac{1}{2}} - (-\Delta)^{\frac{1}{2}}] e^{it\Delta} g_1\|_{H^1_x} = 0.
\] (4.38)

Thus (4.37) and (4.38) yield
\[
\lim_{t \to \infty} \|e^{it\Delta} (-\Delta)^{-\frac{1}{2}} g_1 - e^{it\mathbf{H}} f\|_{H^1_x} = 0.
\]

Set
\[
g := (-\Delta)^{-\frac{1}{2}} g_1,
\]
then (4.36) holds. 
\(\square\)
5. Resolvent Estimates and Equivalent Norms

Recall that the resolvent of $H$ is denoted by

$$R(\lambda, H) := (-H + \lambda)^{-1}.$$

By applying the results in our previous paper [33], we obtain the following estimates for $H$.

**Proposition 5.1** (Uniform Resolvent Estimates for $H$).

- **Fixing** $0 < \alpha' \ll 1$, for any $\lambda \in \mathbb{R}$, there holds

  $$\sup_{0 < \varepsilon < 1} \| e^{-\alpha' r} R(\lambda + i\varepsilon, H)e^{-\alpha' r} \|_{L^2 \to L^2} \lesssim 1. \quad (5.1)$$

- The operator $H$ is self-adjoint in $L^2$ and has only absolutely continuous spectrum, i.e. $\sigma(H) = \sigma_{ac}(H)$.

- **(Resolvent Estimates for $\lambda \geq 0$)** The operator $H$ satisfies the resolvent estimates stated in Lemma 5.1 below.

- **(Smoothing Estimates for $e^{sH}$)** For any $1 < q < p \leq \infty$, $\gamma \in [0, 1)$, and any $s > 0$, we have for some $\delta_q > 0$

  $$\| (-\Delta)^{\gamma/2} e^{sH} f \|_{L^p_q} \leq C s^{-\gamma} s^{\frac{1}{p} - \frac{1}{q}} e^{-\delta_q s} \| f \|_{L^q_q}. \quad (5.2)$$

- **(Exact equivalence in $L^p$ and weighted $L^2$)** Let $1 < p < \infty$, $0 < \gamma < 2$, $0 < \alpha' \ll 1$, then we have

  $$\| (-H)^{-\gamma/2} f \|_{L^p_\gamma} \lesssim \| f \|_{L^p_\gamma}. \quad (5.3)$$

  $$\| (-H)^{-\gamma/2} f \|_{e^{\alpha' r} L^2_\gamma} \lesssim \| f \|_{e^{\alpha' r} L^2_\gamma}. \quad (5.4)$$

  $$\| (-\Delta)^{\gamma/2} f \|_{L^p_\gamma} \lesssim \| (-H)^{\gamma/2} f \|_{L^p_\gamma}. \quad (5.5)$$

  $$\| (-H)^{\gamma/2} f \|_{L^p_\gamma} \lesssim \| (-\Delta)^{\gamma/2} f \|_{L^p_\gamma}. \quad (5.6)$$

where we denote

$$\| f \|_{e^{\alpha' r} L^2_\gamma} = \| e^{-\alpha' r} f \|_{L^2_\gamma}.$$

- **(Sobolev embedding of $H$)** For $0 < \gamma < 2$, we have

  $$\| (-H)^{-\gamma/2} f \|_{H^\gamma_\gamma} \lesssim \| f \|_{L^2_\gamma}. \quad (5.7)$$

**Proof.** (5.1) follows by our previous paper [33]. We remark that in [33] we use Coulomb gauge for $Q^*TN$ to rule out possible bottom resonance of $H$, but [26] proved that the bottom resonance does not emerge no matter what gauge one uses. Hence, (5.1) also holds without taking Coulomb gauge for $Q^*TN$.

By [Theorem XIII.19, [48]] and the continuity of spectral projection operators, to prove the spectrum is absolutely continuous, it suffices to prove for any bounded interval $(a, b)$ and any $g \in C^\infty_c$

$$\sup_{0 < \varepsilon < 1} \int_a^b |\Im \langle g, R(\tau + i\varepsilon, H)g \rangle|^2 \, d\tau < \infty. \quad (5.8)$$
Using (5.1) we have for any $\varepsilon > 0$

$$\left| \langle g, R(\tau + i\varepsilon, H)g \rangle \right| = \left| \left\langle e^{\alpha r}, e^{-\alpha r} R(\tau + i\varepsilon, H)e^{-\alpha r} e^{\alpha r} g \right\rangle \right| \lesssim \| g \|_{L^2}^2 < \infty,$$

which leads to (5.8). Meanwhile, by Weyl’s criterion, $\sigma_{ess}(H) = \left[ \frac{1}{4}, \infty \right)$. Thus, we see 

$$\sigma(H) = \sigma_{ac}(H) = \left[ \frac{1}{4}, \infty \right).$$

(5.2) indeed follows by [33] and [27]. In [33], we proved the same bounds as (5.2) for $s \geq 1$. For $s \in [0, 1]$, we proved

$$\| e^{sH}f \|_{L^p_t} \lesssim s^{-\left( \frac{1}{q} - \frac{1}{p} \right)} \| f \|_{L^q_t}.$$

And for $s \in [0, 1]$, [27] proved

$$\| (-\Delta)^{\nu} e^{sH}f \|_{L^p_t} \lesssim s^{-\nu} \| f \|_{L^p_t}.$$

Then (5.2) follows by the inequality

$$\| (-\Delta)^{\nu} e^{sH}f \|_{L^p_t} \lesssim \| (-\Delta)^{\nu} e^{\frac{1}{4}sH} \|_{L^p_t \rightarrow L^p_t} \| e^{\frac{1}{4}sH} \|_{L^p_t \rightarrow L^p_t} \| f \|_{L^p_t}.$$

Now let’s prove (5.3), (5.4). By the identity

$$\int_0^\infty \lambda^{-\nu} (\lambda + T)^{-1} d\lambda = \frac{\sin(\pi \nu)}{\pi} \int_0^\infty \lambda^{-\nu} (\lambda + T)^{-1} d\lambda,$$

which holds provided that $\nu \in (0, 1)$ and $T$ is a non-negative self-adjoint operator, we find it suffices to show

$$\int_0^\infty \lambda^{-\nu} \| (\lambda - H)^{-1} \|_{L^p \rightarrow L^p} d\lambda \lesssim 1 \quad (5.9)$$

$$\int_0^\infty \lambda^{-\nu} \| (\lambda - H)^{-1} \|_{e^{\alpha r}L^2 \rightarrow e^{\alpha r}L^2} d\lambda \lesssim 1 \quad (5.10)$$

Applying the change of variables $\lambda = \sigma^2 - \frac{1}{4}$, (5.9–5.10) follow directly from Lemma 5.1 below.

In our previous work [33], we proved for $s \in (0, 2)$, $p \in (1, \infty)$,

$$\| (-\Delta)^{\frac{s}{2}} f \|_{L^p_t} \lesssim \| (-H)^{\frac{s}{2}} f \|_{L^p_t} + \| f \|_{L^p_t} \quad (5.11)$$

$$\| (-H)^{\frac{s}{2}} f \|_{L^p_t} \lesssim \| (-\Delta)^{\frac{s}{2}} f \|_{L^p_t} + \| f \|_{L^p_t}. \quad (5.12)$$

Then (5.5) follows by (5.3) and (5.11), and (5.6) follows by Sobolev embedding and (5.12).

(5.7) follows by (5.5) and the fact

$$\| f \|_{H^s_t} \sim \| (-\Delta)^{\frac{s}{2}} f \|_{L^2_t}.$$
Lemma 5.1. For all $\sigma \geq \frac{1}{2}$, $p \in [2, \infty)$, we have
\[
\|(-\Delta + \sigma^2 - \frac{1}{4})^{-1}\|_{L^p \to L^p} \lesssim \min(1, \sigma^{-2}) \tag{5.13}
\]
\[
\|\nabla(-\Delta + \sigma^2 - \frac{1}{4})^{-1}\|_{L^p \to L^p} \lesssim \min(1, \sigma^{-1}) \tag{5.14}
\]
\[
\|(-H + \sigma^2 - \frac{1}{4})^{-1}\|_{L^p \to L^p} \lesssim \min(1, \sigma^{-2}) \tag{5.15}
\]
\[
\|\nabla(-H + \sigma^2 - \frac{1}{4})^{-1}\|_{L^p \to L^p} \lesssim \min(1, \sigma^{-1}). \tag{5.16}
\]

And for $0 < \alpha' \ll 1$, we also have
\[
\|e^{-\alpha' r}(-H + \sigma^2 - \frac{1}{4})^{-1}e^{-\alpha' r}\|_{L^2 \to L^2} \lesssim \min(1, \sigma^{-2}). \tag{5.17}
\]
\[
\|e^{-\alpha' r}\nabla(-H + \sigma^2 - \frac{1}{4})^{-1}e^{-\alpha' r}\|_{L^2 \to L^2} \lesssim \min(1, \sigma^{-1}). \tag{5.18}
\]

Proof. (5.13), (5.14) were proved in [Lemma 3.7, [33]]. (5.15) was obtained in [(6.18), [33]]. Now we prove (5.16).

Denote $H = \Delta + W$. Before going on, we first point out that $W(-\Delta + \lambda)^{-1}$ is a compact operator in $L^p_x$ when $\lambda$ lies in the right half complex plane. In fact, for any $\Re \lambda \geq 0$, $f \in L^p_x$, by (5.14) and (5.13), we see $W(-\Delta + \lambda)^{-1}f \in L^p_x$ with the bound:
\[
\|W(-\Delta + \lambda)^{-1}f\|_{L^p_x} \lesssim \|f\|_{L^p_x}. \tag{5.19}
\]

And furthermore, by applying (5.13–5.14) and the identity
\[-\Delta(-\Delta + \lambda)^{-1} = I - \lambda(-\Delta + \lambda)^{-1},
\]
we obtain
\[
\|\nabla \left(W(-\Delta + \lambda)^{-1}f\right)\|_{L^p_x} \lesssim (1 + |\lambda|)\|f\|_{L^p_x}. \tag{5.20}
\]

Therefore, $W(-\Delta + \lambda)^{-1} \in \mathcal{L}(L^p_x, W^{1,p})$. Meanwhile, using the decay of $A^\infty, \phi^\infty$ at infinity, we see for any $\epsilon > 0$ there exists $R_\epsilon$ such that
\[
\|W(-\Delta + \lambda)^{-1}f\|_{L^p_x(r \geq R_\epsilon)} \leq \epsilon. \tag{5.21}
\]

Then since Sobolev embedding $W^{1,p} \hookrightarrow L^p$ is compact in bounded domains, we conclude $W(-\Delta + \lambda)^{-1}$ is compact.

Second, we claim that

Claim 5.1. For any $\Re \lambda \geq 0$, the operator $I + W(-\Delta + \lambda)^{-1}$ is invertible in $\mathcal{L}(L^p_x, L^p_x)$ and is analytic w.r.t $\lambda \in \{z \in \mathbb{C} : \Re z \geq 0\}$.

We prove Claim 5.1 by contradiction. Assume that there exists some $\lambda_0 \in \{z \in \mathbb{C} : \Re z \geq 0\}$ such that $I + W(-\Delta + \lambda)^{-1}$ is not invertible in $L^p_x$. Then by Fredholm’s alternative, there exists $f \in L^p_x$ such that
\[
f + W(-\Delta + \lambda_0)^{-1}f = 0 \tag{5.22}
\]
Since \( W(\Delta + \lambda)^{-1} \in \mathcal{L}(L^p, W^{1,p}) \), we see \( g := (\Delta + \lambda_0)^{-1}f \in W^{1,p} \). Then by Hölder and \( p \geq 2 \), we infer that \( Wg \in L^2 \). Then by (5.22), \( f \in L^2 \) and thus \( g \in W^{2,2} \) because \( -\lambda_0 \in \rho(\Delta) \). Since (5.22) shows \( -Hg + \lambda_0g = 0 \), and now \( g \in W^{2,2} \), we see \( g \) is an eigenfunction of \( -\Delta \) with eigenvalue \( -\lambda_0 \). This implies \( g = 0 \) since \( -\lambda_0 \in \rho(\Delta) \). Hence Claim 5.1 follows.

Now, we are ready to prove (5.16). By the formal identity
\[
(-H + \lambda)^{-1} = (\Delta + \lambda)^{-1}(I + W(\Delta + \lambda)^{-1})^{-1},
\]
and (5.14), it suffices to prove \((I + W(\Delta + \lambda)^{-1})^{-1}\) is bounded in \( L^p \) with a uniform bound:
\[
\sup_{\lambda \geq 0} \| (I + W(\Delta + \lambda)^{-1})^{-1} \|_{\mathcal{L}(L^p_x, L^p_x)} \lesssim 1.
\]
Claim 5.1 shows for any bounded interval \([0, R_0]\) there uniformly holds
\[
\sup_{\lambda \in [0, R_0]} \| (I + W(\Delta + \lambda)^{-1})^{-1} \|_{\mathcal{L}(L^p_x, L^p_x)} \lesssim C(R_0).
\]
By (5.13), (5.14) for \( R_0 \gg 1 \) and \( \lambda \geq R_0 \),
\[
\| W(\Delta + \lambda)^{-1} \|_{\mathcal{L}(L^p_x, L^p_x)} \lesssim \frac{1}{2}.
\]
Then by Neumann series argument we see \( \| (I + W(\Delta + \lambda)^{-1})^{-1} \|_{\mathcal{L}(L^p_x, L^p_x)} \lesssim 2 \) for \( \lambda \geq R_0 \), which combined with (5.25) yields (5.24). Therefore (5.16) is obtained.

6. Morawetz Estimates

Let \((\mathcal{M}, h)\) be a \( d\)-dimensional Riemannian manifold without boundary. We use \( \langle f, g \rangle \) to denote the inner product in \( L^2 \), i.e.
\[
\langle f, g \rangle := \int_{\mathcal{M}} f \bar{g} \text{dvol}_h.
\]

**Theorem 6.1** (Morawetz estimates for magnetic Schrödinger). Let \((\mathcal{M}, h)\) be a \( d\)-dimensional Riemannian manifold without boundary. Let \( A \) be a real valued time-dependent 1-form on \( \mathcal{M} \). Assume that \( x \mapsto a(x) : \mathcal{M} \to \mathbb{R} \) is a \( C^2 \) real valued function. Let \( u \) solve
\[
i \partial_t u + \Delta_A u = F.
\]
Set
\[
M(t) = i \int_{\mathcal{M}} \langle u, 2 \nabla a : D_A u + u \Delta a \rangle \text{dvol}_h.
\]
Then there holds
\[
\frac{d}{dt} M(t) = 4(\nabla^i \nabla_k a \nabla_j u, \nabla^k u) - \langle u \Delta^2 a, u \rangle + 2\langle u, \nabla^j a \partial_t A_j u \rangle \\
+ O\int_{\mathcal{M}} \bigg[ |u|^2 |\nabla A||A| + \left( |u|^2 |\Delta a| + |u||\nabla u||\nabla a| \right) (|\nabla A| + |A|^2) \bigg] \text{dvol}_h \\\n+ O\int_{\mathcal{M}} |u|^2 |\nabla a||A||\nabla A| \text{dvol}_h \rangle + O\int_{\mathcal{M}} |u|^2 |\nabla^j a \nabla_j u A_k| \text{dvol}_h \rangle \\
+ 2\Re \langle F, 2\nabla a \cdot D_A u + u \Delta a \rangle. \tag{6.3}
\]

\textbf{Proof.} It is easy to see for any smooth function \( f \)
\[
Tf := [\Delta A, a] f = 2\nabla a \cdot D_A f + (\Delta a) f. \tag{6.4}
\]

Then we have
\[
\frac{d}{dt} M(t) = 2\langle u, \nabla^j a \partial_t A_j u \rangle + \langle -\Delta A u + F, Tu \rangle + \langle u, T(\Delta A u - F) \rangle \\
= \langle u, [T, \Delta A] u \rangle + \langle F, Tu \rangle - \langle u, TF \rangle + 2\langle u, \nabla^j a \partial_t A_j u \rangle. \tag{6.5}
\]

Now we calculate \([T, \Delta A] \):
\[
[T, \Delta A] f = [\Delta a, \Delta A] f + 2[\nabla a \cdot D_A, \Delta A] f, \tag{6.6}
\]

By (6.4), the first right side term of (6.6) is
\[
[\Delta a, \Delta A] f = -2(\nabla \Delta a) \cdot D_A f - (\Delta^2 a) f. \tag{6.7}
\]

For the second right side term of (6.6), expanding \( \Delta_A = \Delta + 2i A^l \nabla_j + i \nabla^l A_l - A_k A^k \) we obtain
\[
2[\nabla a \cdot D_A, \Delta A] f \\
= -2 \Delta_A (\nabla a \cdot D_A f) + 2 \nabla a \cdot D_A (\Delta_A f) \\
= -2(\Delta + 2i A^j \nabla_j) (\nabla a \cdot D_A f) + 2 \nabla a \cdot D_A (\Delta f + 2i A \cdot \nabla f) \\
+ 2f \nabla a \cdot \nabla (i \nabla^l A_l - A^k A_k) \\
= -2[\Delta, \nabla a \cdot D_A] f - 4i[A \cdot \nabla, \nabla a \cdot D_A] f + 2f \nabla a \cdot \nabla (i \nabla^l A_l - A^k A_k). \tag{6.8}
\]

We calculate \([A \cdot \nabla, \nabla a \cdot D_A] \) first.
\[
A^l \nabla_j [\nabla^l a(\nabla_l f + iA_l f)] = A^j \nabla_j \nabla^l a \nabla_l f + A^j \nabla^l a \nabla_j \nabla_l f \\
+ i \nabla^l a A^j \nabla_j A_l f + i \nabla^l a A_l A^j \nabla_j f \\
\nabla^l a(\nabla_l + iA_l)(A^j \nabla_j f) = \nabla^l a \nabla_l A^j \nabla_j f + \nabla^l a A^j \nabla_l \nabla_j f + i \nabla^l a A_l A^j \nabla_j f
\]

Since the Hessian of a scaler valued function is symmetric, we thus summarize that
\[
[A \cdot \nabla, \nabla a \cdot D_A] f = A^j \nabla_j \nabla^l a \nabla_l f + i \nabla^l a A^j \nabla_j A_l f - \nabla^l a \nabla_l A^j \nabla_j f.
\]

\textbf{Proof.} It is easy to see for any smooth function \( f \)
\[
Tf := [\Delta A, a] f = 2\nabla a \cdot D_A f + (\Delta a) f. \tag{6.4}
\]

Then we have
\[
\frac{d}{dt} M(t) = 2\langle u, \nabla^j a \partial_t A_j u \rangle + \langle -\Delta A u + F, Tu \rangle + \langle u, T(\Delta A u - F) \rangle \\
= \langle u, [T, \Delta A] u \rangle + \langle F, Tu \rangle - \langle u, TF \rangle + 2\langle u, \nabla^j a \partial_t A_j u \rangle. \tag{6.5}
\]

Now we calculate \([T, \Delta A] \):
\[
[T, \Delta A] f = [\Delta a, \Delta A] f + 2[\nabla a \cdot D_A, \Delta A] f, \tag{6.6}
\]

By (6.4), the first right side term of (6.6) is
\[
[\Delta a, \Delta A] f = -2(\nabla \Delta a) \cdot D_A f - (\Delta^2 a) f. \tag{6.7}
\]

For the second right side term of (6.6), expanding \( \Delta_A = \Delta + 2i A^j \nabla_j + i \nabla^l A_l - A_k A^k \) we obtain
\[
2[\nabla a \cdot D_A, \Delta A] f \\
= -2 \Delta_A (\nabla a \cdot D_A f) + 2 \nabla a \cdot D_A (\Delta_A f) \\
= -2(\Delta + 2i A^j \nabla_j) (\nabla a \cdot D_A f) + 2 \nabla a \cdot D_A (\Delta f + 2i A \cdot \nabla f) \\
+ 2f \nabla a \cdot \nabla (i \nabla^l A_l - A^k A_k) \\
= -2[\Delta, \nabla a \cdot D_A] f - 4i[A \cdot \nabla, \nabla a \cdot D_A] f + 2f \nabla a \cdot \nabla (i \nabla^l A_l - A^k A_k). \tag{6.8}
\]

We calculate \([A \cdot \nabla, \nabla a \cdot D_A] \) first.
\[
A^l \nabla_j [\nabla^l a(\nabla_l f + iA_l f)] = A^j \nabla_j \nabla^l a \nabla_l f + A^j \nabla^l a \nabla_j \nabla_l f \\
+ i \nabla^l a A^j \nabla_j A_l f + i \nabla^l a A_l A^j \nabla_j f \\
\nabla^l a(\nabla_l + iA_l)(A^j \nabla_j f) = \nabla^l a \nabla_l A^j \nabla_j f + \nabla^l a A^j \nabla_l \nabla_j f + i \nabla^l a A_l A^j \nabla_j f
\]

Since the Hessian of a scaler valued function is symmetric, we thus summarize that
\[
[A \cdot \nabla, \nabla a \cdot D_A] f = A^j \nabla_j \nabla^l a \nabla_l f + i \nabla^l a A^j \nabla_j A_l f - \nabla^l a \nabla_l A^j \nabla_j f.
\]
Now we calculate the first term in (6.8). In fact, one has

$$\Delta(\nabla a \cdot D_A f) = \nabla^k \nabla^l a \nabla_i \nabla_l f + i \nabla^k \nabla_k (\nabla^l a A_l f)$$

$$= \nabla^l a \nabla^k \nabla_k \nabla_l f + \nabla^k \nabla^l a \nabla_i \nabla_l f + 2\nabla^k \nabla^l a \nabla_k \nabla_l f +$$

$$+ i \nabla^k \nabla_k (\nabla^l a A_l f) + 2i \nabla^k (\nabla^l a A_l) \nabla_k f + i \nabla^l a A_l \nabla k f$$

$$= \nabla^l a \nabla_i \Delta f + \nabla^l a R^k_{lkm} \nabla^m f + \nabla^k \nabla^l a \nabla_i \nabla_l f + 2\nabla^k \nabla^l a \nabla_k \nabla_l f +$$

$$+ i \Delta(\nabla^l a A_l) f + 2i \nabla^k (\nabla^l a A_l) \nabla_k f + i \nabla^l a A_l \Delta f,$$

and

$$\nabla a \cdot D_A(\Delta f) = \nabla^l a \nabla_i \Delta f + i \nabla^l a A_l \Delta f.$$

Hence we get

$$[\Delta, \nabla a \cdot D_A] f = \nabla^l a R^k_{lkm} \nabla^m f + \nabla^k \nabla^l a \nabla_i \nabla_l f + 2\nabla^k \nabla^l a \nabla_k \nabla_l f$$

$$+ i \Delta(\nabla^l a A_l) f + 2i \nabla^k (\nabla^l a A_l) \nabla_k f.$$

Therefore, by (6.6), (6.7), (6.8),

$$[T, \Delta_A] f$$

$$= -2(\nabla \Delta a) \cdot D_A f - (\Delta^2 a) f - 2\nabla^l a R^k_{lkm} \nabla^m f$$

$$- 2\nabla^k \nabla^l a \nabla_i \nabla_l f - 4\nabla^k \nabla^l a \nabla_k \nabla_l f$$

$$- 2i \Delta(\nabla^l a A_l) f - 4i \nabla^k (\nabla^l a A_l) \nabla_k f - 4i A^j \nabla j \nabla^l a \nabla_l f +$$

$$+ 4i \nabla^l a \nabla_l A^j \nabla_j f$$

$$+ 2f \nabla^m a \nabla_m (i \nabla^l A_l - A^k A_k).$$  (6.9)

So the term $\langle u, [T, \Delta_A] u \rangle$ in (6.5) now expands as

$$\langle u, [T, \Delta_A] u \rangle = -4 \langle u, \nabla^k \nabla^l a \nabla_k \nabla_i \nabla_l u \rangle - 2 \langle u, \nabla^k \nabla^l a \nabla_i \nabla_l u \rangle - 2 \langle u, (\nabla \Delta a) \cdot D_A u \rangle$$

$$- 2 \langle u, \nabla^l a R^k_{lkm} \nabla^m u \rangle + 4i \langle u, A^j \nabla j \nabla^l a \nabla_i \nabla_l u \rangle - 4i \langle u, \nabla^l a \nabla_i A^j \nabla_j u \rangle - \langle u, (\Delta^2 a) u \rangle$$

$$+ 2i \langle u, \Delta(\nabla^l a A_l) u \rangle + 4i \langle u, \nabla^k (\nabla^l a A_l) \nabla_k u \rangle + 4 \langle u, \nabla^l a A^j \nabla_j A_l u \rangle$$

$$+ 2 \langle u, \nabla^m a \nabla_m (i \nabla^l A_l - A^k A_k) \rangle.$$  (6.10)

We now aim to use integration by parts to obtain simpler formula for (6.10). By integration by parts and commutator between covariant derivatives, we first note

$$- 4 \langle u, \nabla^k \nabla^l a \nabla_k \nabla_i \nabla_l u \rangle = -4 \langle u, \nabla_k (\nabla^k \nabla^l a \nabla_i \nabla_l u) \rangle + 4 \langle u, \nabla_k u \nabla_k \nabla^l a \rangle$$

$$= 4 \langle \nabla_k u, \nabla^k \nabla^l a \nabla_i \nabla_l u \rangle + 4 \langle u, \nabla^l u \nabla_i (\nabla a) \rangle + 4 \langle u, R^k_{lkm} \nabla^l u \nabla^m a \rangle.$$  (6.10)
By integration by parts, the first term in the third line of the RHS of (6.10) becomes
\[ 2i \langle u, \Delta (\nabla^l a A_l) u \rangle = -2i \langle \nabla_k u, \nabla^k (\nabla^l a A_l) \nabla_k u \rangle - 2i \langle u, \nabla^k (\nabla^l a A_l) \nabla_k u \rangle \]
\[ = -2i \langle \nabla_k u, \nabla^k \nabla^l a A_l u \rangle - 2i \langle u, \nabla^k \nabla^l a A_l \nabla_k u \rangle - 2i \langle \nabla_k u, \nabla^l a \nabla^k A_l \nabla_k u \rangle - 2i \langle u, \nabla^l a \nabla^k A_l \nabla_k u \rangle. \]

Similarly, the last term in the RHS of (6.10) equals
\[ 2 \langle u, u \nabla^m a \nabla_m (i \nabla^l A_l - A^k A_k) \rangle = -2 \langle v_m u, u \nabla^m a (i \nabla^l A_l - A^k A_k) \rangle - 2 \langle u, \nabla_m u \nabla^m a (i \nabla^l A_l - A^k A_k) \rangle - 2 \langle u, u \Delta a (i \nabla^l A_l - A^k A_k) \rangle. \]

Combining all these together implies the RHS of (6.10) becomes
\[
\begin{align*}
4 \langle \nabla_k u, \nabla^k \nabla^l a \nabla_l u \rangle + 2 \langle u, R^k_{lkm} \nabla^l a \nabla^m a \rangle + 2i \langle u, \nabla^l \Delta a A_l J u \rangle + 2i \langle u, \nabla^l a R^k_{lkm} \nabla^m u \rangle + 4i \langle u, A_l \nabla^l \nabla^m a \nabla_l u \rangle &+ 4i \langle u, \nabla^l a \nabla^l A_j \nabla_l \nabla u \rangle - 4i \langle u, (\Delta^2 a) u \rangle \\
-2 \langle u, \nabla^l a \nabla^l a \nabla_l u \rangle - 2i \langle u, \nabla^l \nabla^2 a A_l \nabla_l u \rangle &+ 2i \langle u, \nabla^l A_l \nabla^l a A_l \nabla_k u \rangle - 2 \langle u, \nabla_m u \nabla^m a (i \nabla^l A_l - A^k A_k) \rangle - 2 \langle u, u \Delta a (i \nabla^l A_l - A^k A_k) \rangle. 
\end{align*}
\]

(6.11)

Since the Ricci tensor is symmetric and \( R^k_{lkm} = R_{lm} \), we observe that
\[(a1) + (a2) = 0. \]

(6.12)

Therefore, (6.3) follows by checking each term in the RHS of (6.11) and (6.12).

\[ \square \]

**Corollary 6.1.** Assume that \( \|A\|_{L^\infty_{t,x}} \lesssim 1 \). Let \( \mathcal{M} = \mathbb{H}^2 \) in Theorem 6.1, then the Morawetz estimates hold:
\[
\| e^{-\frac{t}{2}} \nabla u \|^2_{L^2_{t,x}} \lesssim \| (-\Delta)^{\frac{1}{2}} u \|^2_{L^\infty_{t}L^2_x} + \| |\partial_t A| |u|^2 \|_{L^1_{t,x}} + \| (\|A\| + |\nabla A|) |u| |\nabla u| \|_{L^1_{t,x}} + \| |u|^2 (|A| + |\nabla A|) \|_{L^1_{t}L^1_x} + \| |D_A u| |F| \|_{L^1_{t,x}} + \| |u| |F| \|_{L^1_{t,x}}
\]

(6.13)

**Proof.** As [21], we take \( a \) to be a radial function such that \( \Delta a = 1 \), then \( a \) is given by

\[
a(r) = \int_0^r \left( \frac{1}{\sinh \tau} \int_0^\tau \sinh s ds \right) d\tau, \]

(6.14)

and

\[
\partial_r a(r) = \frac{1}{\sinh r} \int_0^r \sinh s ds = \tanh \left( \frac{r}{2} \right)
\]

(6.15)

\[
\partial_r^2 a(r) = \frac{1}{2 \cosh^2 \frac{r}{2}}.
\]

(6.16)
By simple bilinear argument (see Lemma 9.2) one obtains \( M(t) \) defined in (6.2) satisfies
\[
|M(t)| \lesssim \|(-\Delta)^{\frac{1}{4}} u\|_{L^2_t}^2.
\] (6.17)

Then integrating (6.3) in \( t \in [t_1, t_2] \) gives
\[
\sup_{t \in [t_1, t_2]} \|(-\Delta)^{\frac{1}{4}} u(t)\|_{L^2_t}^2 \\
\geq 4 \int_{t_1}^{t_2} \int_{\mathbb{H}^2} \nabla^k \nabla^j a \nabla_j u \nabla_k u \text{dvol}_h dt \\
- C \int_{t_1}^{t_2} \int_{\mathbb{H}^2} |\nabla^k \nabla^j a \nabla_j u A_k u| \text{dvol}_h dt \\
- C \|\partial_t A\|_{L^1_t L^1_t((t_1, t_2) \times \mathbb{H}^2)} - C \|u \nabla u\|_{L^1_t L^1_t((t_1, t_2) \times \mathbb{H}^2)}
\]
(6.18) is the leading term and it can be expanded as
\[
\int_{t_1}^{t_2} \int_{\mathbb{H}^2} \nabla^k \nabla^j a \nabla_j u \nabla_k u \text{dvol}_h dt \\
= \int_{t_1}^{t_2} \int_{\mathbb{H}^2} \frac{1}{2} \frac{\cosh^2 r}{\sinh^3 r} |\partial_r u|^2 + \frac{\cosh r}{\sinh^3 r} \tanh \frac{r}{2} |\partial_\theta u|^2 \text{dvol}_h dt \\
\geq \int_{t_1}^{t_2} \int_{\mathbb{H}^2} e^{-r} |\nabla u|^2 \text{dvol}_h dt + \int_{t_1}^{t_2} \int_{\mathbb{H}^2} h^{\theta \theta} |\partial_\theta u|^2 \text{dvol}_h dt.
\]

Similarly (6.19) is dominated by
\[
\int_{t_1}^{t_2} \int_{\mathbb{H}^2} |\nabla^k \nabla^j a \nabla_j u A_k u| \text{dvol}_h dt \\
\lesssim \int_{t_1}^{t_2} \int_{\mathbb{H}^2} \frac{1}{2} \frac{\cosh^2 r}{\sinh^3 r} |u| |\partial_r u| |A_r| + \frac{\cosh r}{\sinh^3 r} \tanh \frac{r}{2} |\partial_\theta u| |A_\theta| |u| \text{dvol}_h dt \\
\leq \eta \int_{t_1}^{t_2} \int_{\mathbb{H}^2} [e^{-r} h^{\theta \theta} |\partial_r u|^2 + h^{\theta \theta} |\partial_\theta u|^2] \text{dvol}_h dt \\
+ C \eta^{-1} \int_{t_1}^{t_2} \int_{\mathbb{H}^2} |u|^2 [e^{-r} h^{\theta \theta} |A_r|^2 + h^{\theta \theta} |A_\theta|^2] \text{dvol}_h dt.
\]

Thus by choosing \( \eta \) sufficiently small, we can achieve
\[
4 \int_{t_1}^{t_2} \int_{\mathbb{H}^2} \nabla^k \nabla^j a \nabla_j u \nabla_k u \text{dvol}_h dt - C \int_{t_1}^{t_2} \int_{\mathbb{H}^2} |\nabla^k \nabla^j a \nabla_j u A_k u| \text{dvol}_h dt \\
\geq -\tilde{C} \int_{t_1}^{t_2} \int_{\mathbb{H}^2} |u|^2 |A|^2 \text{dvol}_h dt.
\]

Then the desired result follows by combing the above together. \( \square \)
6.1. Energy estimates.

**Proposition 6.1.** Denote $B = B_j(s, t, x) dx^j$ be a 1-form on $\mathbb{H}^2$. Assume that $u$ solves linear Schrödinger equation corresponding to $B$:

$$
\begin{align*}
\left\{ \begin{array}{l}
i \partial_t u + \Delta u = F \\
u(0, x) = u_0(x),
\end{array} \right.
\end{align*}
$$

(6.20)

Then there holds

$$
\| \nabla u \|_{L^\infty_t L^2_x([0,T] \times \mathbb{H}^2)} \lesssim \| |B|_u \|_{L^\infty_t L^2_x} + \| \nabla u_0 \|_{L^2_x} + \| D_B F D_B u \|_{L^1_t L^1_x([0,T] \times \mathbb{H}^2)},
$$

(6.21)

with implicit constant independent of $T > 0$.

**Proof.** This follows directly by calculating

$$
d \frac{1}{dt} \int_{\mathbb{H}^2} D_B \bar{u} \overline{D_B u} d\text{vol}_h
$$

and integration by parts. \qed

7. Strichartz Estimates

Let us recall the Strichartz Estimates for linear free Schrödinger equations on $\mathbb{H}^2$.

For free Schrödinger equations on $\mathbb{H}^n$, Anker-Pierfelice [1] proved the following dispersive estimates:

$$
\| e^{i t \Delta} f \|_{L^q(\mathbb{H}^n)} \lesssim t^{-\max(0, \frac{n}{q'}, \frac{n}{q} - 1)} \| f \|_{L^{\tilde{q}'}}, \ 0 < |t| < 1
$$

$$
\| e^{i t \Delta} f \|_{L^q(\mathbb{H}^n)} \lesssim t^{-\frac{n}{2}} \| f \|_{L^{\tilde{q}'}, \ |t| \geq 1},
$$

provided that $q, \tilde{q} \in (2, \infty)$, and $\tilde{q}'$ denotes the conjugate of $\tilde{q}$.

Define the admissible pair $(p, q)$ to be

$$
\left\{ \left( \frac{1}{p}, \frac{1}{q} \right) \in (0, \frac{1}{2}) \times (0, \frac{1}{2}) : \frac{2}{p} + \frac{2}{q} \geq 1 \right\} \cup \left\{ \left( \frac{1}{p}, \frac{1}{q} \right) = (0, \frac{1}{2}) \right\}.
$$

(7.1)

Anker-Pierfelice [1] proved the following Strichartz estimates on $\mathbb{H}^2$: Let $u$ solve the equation

$$
\begin{align*}
\left\{ \begin{array}{l}
i \partial_t u + \Delta u = F \\
u(0, x) = u_0 \in L^2,
\end{array} \right.
\end{align*}
$$

then for all admissible pairs $(p, q)$, $(\tilde{p}, \tilde{q})$ in (7.1), one has

$$
\| u \|_{L^p_t L^q_x} \lesssim \| u_0 \|_{L^2_x} + \| F \|_{L^p_t L^q_x}.
$$

The start point to endpoint Strichartz estimates for magnetic Schrödinger operators is the following estimates. The wave version of (7.3) was obtained by our previous work [36] for magnetic wave equations.
Lemma 7.1. Let $u$ solve the equation
\begin{equation}
\begin{aligned}
i \partial_t u + \Delta u &= F \\
u(0, x) &= 0.
\end{aligned}
\end{equation}

Let $\alpha' > 0$, then for any admissible pair $(p, q)$ one has
\begin{equation}
\|(-\Delta)^{\frac{1}{4}} u\|_{L_t^p L_x^q} \lesssim \|e^{\alpha' r} F\|_{L_t^2 x}.
\end{equation}

Remark 7.1. The $\mathbb{R}^n$ version of (7.3) was obtained by Ionescu-Kenig [19]. The proof of Lemma 7.1 relies on Keel-Tao’s bilinear arguments and kernel estimates for frequency localized Schrödinger propagators analogous to [2]. The proof is presented in “Appendix C”. One may see our previous work [36] for the wave analogy.

We also recall the following Kato smoothing estimates of $H$ as a corollary of [26].

Lemma 7.2. Let $u$ solve the equation
\begin{equation}
\begin{aligned}
i \partial_t u + Hu &= 0 \\
u(0, x) &= u_0.
\end{aligned}
\end{equation}

Then for admissible pair $(p, q)$ one has
\begin{equation}
\|e^{-\alpha' r} \nabla u\|_{L_t^2 L_x^2} + \|e^{-\alpha' r} u\|_{L_t^2 L_x^2} \lesssim \|(-\Delta)^{\frac{1}{4}} u_0\|_{L_x^2},
\end{equation}
provided that $\alpha' > 0$.

Proof. As a corollary of the local smoothing estimates by [26], one has for $\alpha' > 0$
\begin{equation}
\|e^{-\alpha' r} (-\Delta)^{\frac{1}{4}} u\|_{L_t^2 L_x^2} \lesssim \|u_0\|_{L_x^2}.
\end{equation}

Then using the equivalence of $(-\Delta)^{\frac{1}{4}}$ and $(-H)^{\frac{1}{4}}$ in weighted $L^2$ space (see Proposition 5.1) yields
\begin{equation}
\|e^{-\alpha' r} (-H)^{\frac{1}{4}} u\|_{L_t^2 L_x^2} \lesssim \|u_0\|_{L_x^2}.
\end{equation}

Therefore, applying $(-H)^{\frac{1}{4}}$ to (7.4) shows
\begin{equation}
\|e^{-\alpha' r} (-H)^{\frac{1}{4}} u\|_{L_t^2 L_x^2} \lesssim \|(-H)^{\frac{1}{4}} u_0\|_{L_x^2}.
\end{equation}

Then the first RHS term of (7.5) follows by applying the equivalence of $(-\Delta)^{\frac{1}{4}}$ and $(-H)^{\frac{1}{4}}$ in weighted space and $L^2$. The second RHS term of (7.5) follows by applying Poincare inequality to $e^{-\alpha' r} u$ with $0 < \alpha' \ll 1$.

Lemma 7.3. Let $u$ solve the equation
\begin{equation}
\begin{aligned}
i \partial_t u + Hu &= 0 \\
u(0, x) &= u_0.
\end{aligned}
\end{equation}

Then for admissible pairs $(p, q)$ we have
\begin{equation}
\|u\|_{L_t^p L_x^q(\mathbb{R} \times \mathbb{H}^2)} \lesssim \|u_0\|_{L_x^2}.
\end{equation}
Proof. Step 1. We first prove the following claim: Claim A. Let \( u \) solve the equation
\[
\begin{cases}
i \partial_t u + \mathbf{H} u = 0 \\
 u(0, x) = u_0,
\end{cases}
\]
then for admissible pairs \((p, q)\) there holds that
\[
\|(-\Delta)^{1/4} u\|_{L^p_t L^q_x} \lesssim \|(-\Delta)^{1/4} u_0\|_{L^2_x}.
\] (7.8)

Let’s prove (7.8). Applying Lemma 7.1 and Duhamel principle, we obtain
\[
\|(-\Delta)^{1/4} u\|_{L^p_t L^q_x([0, T] \times \mathbb{H}^2)} \lesssim \|(-\Delta)^{1/4} u_0\|_{L^2_x} + \left\| e^{2\alpha' r |A^\infty|} \right\|_{L^\infty_x} \| e^{-\alpha' \nabla u}\|_{L^2_{t,x}} \\
+ \left\| (|\phi^\infty|^2 + |A^\infty|^2 + |\nabla A^\infty|^2) e^{2\alpha' r}\right\|_{L^\infty_x} \| e^{-\alpha' u}\|_{L^2_{t,x}},
\]
where we take \(0 < \alpha' \ll 1\). Then by Lemma 7.2,
\[
\|(-\Delta)^{1/4} u\|_{L^p_t L^q_x([0, T] \times \mathbb{H}^2)} \lesssim \|(-\Delta)^{1/4} u_0\|_{L^2_x}.
\]

Step 2. If \( u \) solves (7.6), then \( v := (-\mathbf{H})^{-1/4} u(t) \) solves
\[
i \partial_t v + \mathbf{H} v = 0.
\] (7.9)
Thus Claim A of Step 1 gives
\[
\|(-\Delta)^{1/4} v\|_{L^p_t L^q_x([0, T] \times \mathbb{H}^2)} \lesssim \|(-\Delta)^{1/4} v_0\|_{L^2_x}.
\]
Applying the equivalence of \((-\Delta)^{1/4}\) and \((-\mathbf{H})^{1/4}\) in \(L^q_x\) (see Prop. 5.1) gives (7.7). \(\square\)

By Christ–Kiselev Lemma, Lemma 7.3 immediately yields

Lemma 7.4. Let \( u \) solve the equation
\[
\begin{cases}
i \partial_t u + \mathbf{H} u = F \\
 u(0, x) = u_0(x).
\end{cases}
\] (7.10)
Then for any admissible pair \((p, q)\) one has
\[
\|u\|_{L^p_t L^q_x([0, T] \times \mathbb{H}^2)} \lesssim \|u_0\|_{L^2_x} + \|F\|_{L^1_t L^q_x([0, T] \times \mathbb{H}^2)},
\] (7.11)
where the implicit constant is independent of \(T\).
8. Proof of Kähler Targets

The proof for Kähler targets is the almost same as Riemannian surface targets. The main estimates we rely on are energy estimates in Proposition 6.1, Morawetz estimates in Corollary 6.1 (where \( A_j \) now shall be matrix-valued functions rather than scalar-valued functions), and endpoint Strichartz estimates for \( e^{it\Delta} \) of [1].

We point out the differences: (i) Since we treat small harmonic maps in Kähler targets case, Strichartz estimates for \( e^{it\Delta} \) is enough (particularly, the \( \phi^l \phi^j \bar{\phi}_s \) term in the RHS of (2.13) causes no trouble); (ii) The curvature part in Kähler targets case shall be treated slightly different from Riemannian surface.

The curvature term in (2.14) can be schematically written as

\[
\begin{align*}
\text{Re}[\mathcal{R}(v)(\phi_i, \phi)\phi_j]^\alpha &= \psi_i^{l_1} \psi_j^{l_2} \psi_j^{l_3} \mathcal{R}(e_{i_1}, e_{i_2}, e_{i_3}, e_{\alpha}) \\
\text{Im}[\mathcal{R}(v)(\phi_i, \phi)\phi_j]^\alpha &= \psi_i^{l_1} \psi_j^{l_2} \psi_j^{l_3} \mathcal{R}(e_{i_1}, e_{i_2}, e_{i_3}, e_{\alpha+n})
\end{align*}
\]

By caloric condition \( \tilde{\nabla}_s e_i = 0 \), we further have

\[
\mathcal{R}(e_{i_1}, e_{i_2}, e_{i_3}, e_{i_4}) = \text{limit} - \int_s^\infty \phi^k_s \mathcal{D}(e_k; e_{i_1}, ..., e_{i_4})ds'
\]

where “limit” refers to the limit of the RHS as \( s \to \infty \). For simplicity we denote all terms like

\[
\int_s^\infty \phi^k_s \mathcal{D}(e_k; e_{i_1}, ..., e_{i_4})ds'
\]

for various indexes \( i_1, ..., i_4, k \) by the same notation \( \tilde{\mathcal{G}} \). The “limit” for different \( e_{i_1}, e_{i_2}, e_{i_3}, e_{i_4} \) will denoted by \( \Gamma \) for simplicity. Moreover, the cubic product \( \psi_i^{l_1} \psi_j^{l_2} \psi_j^{l_3} \) for various indexes \( i_1, ..., i_3 \) will be denoted as the same notation

\[
\phi_i \diamond \phi \diamond \phi_j.
\]

Using these notations, (2.14) now reads as

\[
i\mathcal{D}, \phi_s + \Delta_{\Delta} \phi_s = i\partial_s Z + \Gamma \phi^i \diamond \phi_s \diamond \phi_j + \tilde{\mathcal{G}} \phi^i \diamond \phi_s \diamond \phi_j. \tag{8.1}
\]

In the following lemma, we collect the point-wise estimates related with curvatures.

**Lemma 8.1.** For compact Kähler targets, we have under the caloric gauge that

\[
|\tilde{\mathcal{G}}| \lesssim \int_s^\infty |\phi_s|ds'
\]

\[
|\nabla \tilde{\mathcal{G}}| \lesssim \int_s^\infty (|\nabla \phi_s| + |A||\phi_s|)ds'
\]

\[
|\partial_t \tilde{\mathcal{G}}| \lesssim \int_s^\infty (|\partial_t \phi_s| + |A_t||\phi_s|)ds'.
\]

Using these estimates and repeating the proof of Theorem 1.1 yields Theorem 1.2.
9. Appendix A

The estimate of the heat semigroup in $\mathbb{H}^2$ is as follows.

**Lemma 9.1** ([10,11,27,38]). For $1 \leq r \leq p \leq \infty$, $\alpha \in [0, 1]$, $1 < q < \infty$, the heat semigroup on $\mathbb{H}^2$ denoted by $e^{-s\Delta}$ satisfies

$$\|e^{-s\Delta}(-\Delta)^{\alpha}f\|_{L^q_x} \lesssim e^{-\rho_q s}\|f\|_{L^q_x}, \quad (9.1)$$

for any $0 < \rho_q \leq \frac{1}{2} \min(\frac{1}{q}, 1 - \frac{1}{q})$. And for $f \in L^2$ it holds that

$$\int_0^\infty \|e^{s\Delta}f\|_{L^\infty_x}^2 ds \lesssim \|f\|_{L^2}^2.$$  

**Lemma 9.2.** Let $M = \mathbb{H}^2$. Assume that $\|A\|_{L^\infty_{t,x}} \lesssim 1$, then the bilinear form of $M$ defined by (6.2) satisfies

$$|M(f, g)| \lesssim \|f\|_{H^{1/2}_x} \|g\|_{H^{1/2}_x}.$$ 

**Proof.** Recall that

$$M(f, g) = i \int_{\mathbb{H}^2} (f, (2\nabla a \cdot DA + \Delta a)g)_C dvol_h.$$ 

Then by Poincare’s inequality and $\Delta a = 1$, $|\nabla a| \leq 1$ we have

$$|M(f, g)| \leq \|f\|_{L^2_x} \|g\|_{L^2_x} + 2\|f\|_{L^2_x} \|\nabla g\|_{L^2_x} + 2\|A\|_{L^\infty} \|f\|_{L^2_x} \|g\|_{L^2_x} \leq \left( \frac{5}{2} + \|A\|_{L^\infty} \right) \|f\|_{L^2_x} \|\nabla g\|_{L^2_x}.$$ 

And by integration by parts, one also has

$$|M(g, f)| \leq 2 \|\nabla f\|_{L^2_x} \|g\|_{L^2_x} + 3\|f\|_{L^2_x} \|g\|_{L^2_x} + 2\|A\|_{L^\infty} \|f\|_{L^2_x} \|g\|_{L^2_x} \lesssim (1 + \|A\|_{L^\infty}) \|g\|_{L^2_x} \|\nabla f\|_{L^2_x}.$$ 

Thus

$$|M(f, g)| \lesssim (1 + \|A\|_{L^\infty}) \|f\|_{L^2} \|\nabla g\|_{L^2},$$

$$|M(g, f)| \lesssim (1 + \|A\|_{L^\infty}) \|g\|_{L^2} \|\nabla f\|_{L^2},$$

which gives the desired result by bilinear interpolation. \qed

**Lemma 9.3.** Let $dr^2 + \sinh^2 r d\theta^2$ be the polar coordinates of $\mathbb{H}^2$. There holds

$$\sup_{s \in [0,1]} \left( \|e^{-\frac{1}{2}r} e^{s\Delta} f\|_{L^1_x} + \|e^{-\frac{1}{2}r} e^{s\Delta} f\|_{L^2_x} + s^\frac{1}{2} \|e^{-\frac{1}{2}r} \nabla e^{s\Delta} f\|_{L^2_x} \right) \lesssim \|e^{-\frac{1}{2}r} \nabla f\|_{L^2_x} + \|e^{-\frac{1}{2}r} f\|_{L^2_x}. \quad (9.2)$$
Proof. Without loss of generality, we assume \( f \) is real valued, otherwise we treat the real and imaginary parts separately. First, let’s prove

\[
\| e^{-\frac{1}{2}r} \nabla^2 f \|_{L_x^2} \lesssim \| e^{-\frac{1}{2}r} \Delta f \|_{L_x^2} + \| e^{-\frac{1}{2}r} \nabla f \|_{L_x^2} + \| e^{-\frac{1}{2}r} f \|_{L_x^2}. \tag{9.3}
\]

Note that the function \( r \) is not smooth at \( r = 0 \). It is convenient to introduce a smooth weight function \( e^{\chi(r)} \) which is equivalent to \( e^{-\frac{1}{2}r} \). To find this kind of function, we choose \( \chi : y \in \mathbb{R} \mapsto \chi(y) \in \mathbb{R} \) to be a smooth function which equals 1 when \( |y| \leq 1 \) and equals \(-\frac{1}{2}y\) for \( |y| \geq 4 \). Then the function \( \chi(r) \) satisfies

\[
\| \nabla \chi(r) \|_{L_x^\infty(\mathbb{H}^2)} + \| \nabla^2 \chi(r) \|_{L_x^\infty(\mathbb{H}^2)} \lesssim 1.
\]

Hence, point-wisely one has

\[
|\nabla e^{\chi(r)}| + |\nabla^2 e^{\chi(r)}| \leq C e^{\chi(r)}, \quad e^{\chi(r)} \sim e^{-\frac{1}{2}r}, \tag{9.4}
\]

where \( C > 0 \) is a universal constant.

By (2.1) and (9.4), we see

\[
\| \nabla^2 (e^{\chi(r)} f) \|_{L_x^2} \lesssim \| (-\Delta)(e^{\chi(r)} f) \|_{L_x^2} \lesssim \| e^{\chi(r)} \Delta f \|_{L_x^2} + \sum_{j=0}^{1} \| e^{\chi(r)} \nabla^j f \|_{L_x^2}.
\]

Then (9.3) follows by further expanding \( \nabla^2 (e^{\chi(r)} f) \) and (9.4).

Second, let’s prove (9.2). Denote \( v = e^{s\Delta} f \). By integration by parts, we compute

\[
\frac{d}{ds} \| e^{\chi(r)} \Delta v \|_{L_x^2}^2 = \int_{\mathbb{H}^2} |\Delta v|^2 e^{2\chi(r)} dvol_h + 2 \int_{\mathbb{H}^2} se^{2\chi(r)} \Delta v \Delta \partial_h v dvol_h
\]

\[
= \int_{\mathbb{H}^2} |\Delta v|^2 e^{2\chi(r)} dvol_h - 2 \int_{\mathbb{H}^2} se^{2\chi(r)} |\nabla \Delta v|^2 dvol_h
\]

\[+ O(\int_{\mathbb{H}^2} se^{2\chi(r)} |\nabla v| |\Delta v| dvol_h).\]

Similarly, one has

\[
\frac{d}{ds} \| e^{\chi(r)} |\nabla v| \|_{L_x^2}^2 = 2 \int_{\mathbb{H}^2} e^{2\chi(r)} \nabla_j v \nabla^j \partial_h v dvol_h
\]

\[= -2 \int_{\mathbb{H}^2} e^{2\chi(r)} |\nabla^2 v| dvol_h
\]

\[+ O(\int_{\mathbb{H}^2} e^{2\chi(r)} [|\nabla v|^2 + |\nabla v| |\nabla^2 v|] dvol_h).\]

And

\[
\frac{d}{ds} \| e^{\chi(r)} v \|_{L_x^2}^2 = 2 \int_{\mathbb{H}^2} e^{2\chi(r)} v \partial_h v dvol_h
\]

\[= -2 \int_{\mathbb{H}^2} e^{2\chi(r)} |\nabla v|^2 dvol_h + O(\int_{\mathbb{H}^2} e^{2\chi(r)} |\nabla v| |v| dvol_h).\]
Let
\[ g(s) = \| e^{X(r)} v \|_{L^2_x}^2 + \lambda \| e^{X(r)} \nabla v \|_{L^2_x}^2 + s \| e^{X(r)} \Delta v \|_{L^2_x}^2. \]
Then by (9.3), we see for \( \lambda \) sufficiently large and any \( s \in (0, 1) \) there holds
\[ \frac{d}{ds} g(s) \lesssim g(s). \]
Thus it follows by Gronwall inequality that
\[ \sup_{s \in [0,1]} \left( \| e^{-\frac{1}{2} r \nabla e^s \Delta f \|_{L^2_x} + \| e^{-\frac{1}{2} r} e^s \Delta f \|_{L^2_x} + s \frac{1}{2} \| e^{-\frac{1}{2} r} \Delta e^s f \|_{L^2_x} \right) \]
\[ \lesssim \| e^{-\frac{1}{2} r \nabla f \|_{L^2_x} + \| e^{-\frac{1}{2} r} f \|_{L^2_x}, \]
which yields (9.2) by (9.3).

10. Appendix B

We say a few words on holomorphic and anti-holomorphic maps. A map \( f \) from Kähler manifold \((M_1, J_1, g_1)\) to Kähler manifold \((M_2, J_2, g_2)\) is said to be holomorphic if
\[ f_*(J_1 X) = J_2 f_*(X), \quad \forall X \in T M_1, \quad (10.1) \]
or anti-holomorphic if
\[ f_*(J_1 X) = -J_2 f_*(X), \quad \forall X \in T M_1. \quad (10.2) \]
Both holomorphic and anti-holomorphic maps are harmonic maps. Assume that \( M_1, M_2 \) are Riemannian surfaces. Fixing local complex coordinates for \( M_1, M_2 \), one may treat \( f \) as a complex valued function \( f(z, \bar{z}) \), then holomorphic maps refer to
\[ \frac{\partial}{\partial z} f = 0, \]
and anti-holomorphic maps refer to
\[ \frac{\partial}{\partial \bar{z}} f = 0. \]

From the above discussions, when \( M = \mathcal{N} = \mathbb{H}^2 \), it is easy to see any analytic function from Poincare disk into bounded subset of Poincare disk induces a holomorphic map satisfying assumptions of Theorem 1.1. Moreover, the conjugate of any analytic function from Poincare disk into bounded subset of Poincare disk also gives an anti-holomorphic map satisfying assumptions of Theorem 1.1.

For orthonormal frames \((e'_1, e'_2)\) for \( M_1 \), and orthonormal frames \((e_1, e_2)\) for \( M_2 \), satisfying \( e'_2 = J_1 e'_1, e_2 = J_2 e_1 \), letting \( \phi_j = \langle \nabla_{e'_j} f, e_1 \rangle + i \langle \nabla_{e'_j} f, e_2 \rangle \), \((10.1)\) then reads as
\[ \phi_1 = i \phi_2, \]
while \((10.2)\) reads as
\[ \phi_1 = -i \phi_2. \]
Remark 10.1. Using the above two identities, it is easy to check the \( i \kappa^\infty \text{Im}(\phi^j \overline{\phi}_j) \phi_j \) part of linearized operator \( H \) in Lemma 3.1 is self-adjoint. Moreover, when the sectional curvature \( \kappa^\infty \) is non-positive, \( \phi_s \rightarrow -i \kappa^\infty \text{Im}(\phi^j \overline{\phi}_j) \phi_j \) is a non-negative operator.

Lemma 10.1. Let \( Q \) be a holomorphic map or an anti-holomorphic map from \( \mathbb{H}^2 \) to \( \mathcal{N} \) with bounded image. Fix any orthonormal frame \( \{e^\infty_i, Je^\infty_i\}_{i=1}^n \) for \( Q^*\mathcal{T}\mathcal{N} \). Then we have

\[
|A^\infty| \lesssim e^{-r}, \quad |\phi^\infty| \lesssim e^{-r} \tag{10.3}
\]

\[
|\nabla^j A^\infty| \lesssim e^{-(j+1)r}, \quad |\nabla^j \phi^\infty| \lesssim e^{-(j+1)r}, \quad j \in \mathbb{Z}_+, \tag{10.4}
\]

where \( r \) denotes the radial variable in polar coordinates for \( \mathbb{H}^2 \).

Proof. It is convenient to work with Poincare disk model for \( \mathbb{H}^2 := \{z \in \mathbb{C} : |z| < 1\} \) equipped with metric \( 4(1 - |z|^2)^{-2}(dx^2 + dy^2) \). Then there holds \( r = \ln(\frac{|z^2|}{1-|z|^2}) \).

Now, (10.3), (10.4) follow by direct calculations, see [38] for instance.

Lemma 10.2. Let \( Q : \mathbb{H}^2 \to \mathcal{N} \) be a harmonic map in Theorem 1.1 or Theorem 1.2. Let \( \delta > 0, \sigma \geq 1 \). Assume that \( v_0 : \mathbb{H}^2 \to \mathcal{N} \) satisfies

\[
\|v_0 - Q\|_{H^{\sigma+2\delta}} \leq \epsilon \ll 1.
\]

Then \( v_0 \) evolves to a global solution \( v : \mathbb{R}^+ \times \mathbb{H}^2 \to \mathcal{N} \) of heat flow, and \( v(s, x) \) converges to \( Q(x) \) uniformly on \( \mathbb{H}^2 \). Moreover, for \( \gamma \geq 0 \) there holds

\[
\sup_{s>0} \omega_1^2(\gamma-\sigma-\delta)(s) \|\phi_s\|_{H^\gamma_x} \lesssim \epsilon.
\]

Proof. The proof is based on the extrinsic formulation of heat flows and the caloric gauge. We recommend the reader to read the nice presentation of [Sec. 4, [27]] where the results for \( \sigma = 1 \) were established. Small modifications of arguments of [27] yield the desired result.

Lemma 10.3. Let \( v_0 : \mathbb{H}^2 \to \mathcal{N} \) be a map such that

\[
\|dv_0\|_{L^2_1 \cap L^\infty_1} + \|\nabla dv_0\|_{L^2_1} \leq K.
\]

Then if \( \mathcal{N} \) is negatively curved or \( K \) is small, \( v_0 \) evolves to a global solution \( v : \mathbb{R}^+ \times \mathbb{H}^2 \to \mathcal{N} \) to heat flow, and \( v(s, x) \) satisfies

\[
\|\nabla dv\|_{L^2_1} + \min(s^\frac{1}{2}, 1)\|\nabla^2 dv\|_{L^2_1} + \min(s, 1)\|\nabla^2 dv\|_{L^\infty_1} + \min(s^\frac{3}{2}, 1)\|\nabla^3 dv\|_{L^\infty_1} \lesssim K 1
\]

\[
\sup_{s>0} \xi^\frac{1}{2}(s)\|\partial_s v\|_{L^\infty_1} + \sup_{s>0} \xi^\frac{1}{2}(s)\|\nabla_\perp \partial_s v\|_{L^\infty_1} + \sup_{s>0} \xi^\frac{1}{2}(s)\|\nabla^2 \partial_s v\|_{L^\infty_1} + \sup_{s>0} \xi^\frac{1}{2}(s)\|\nabla^3 \partial_s v\|_{L^\infty_1} \lesssim K 1.
\]

where \( \xi_\gamma(s) := 1_{s \in (0,1]} s^{\gamma} + 1_{s \geq 1} e^{\rho s}, \rho_0 > 0 \).

Proof. If \( K \) is small, the proof follows by Poincare inequality, Harnack inequality and Bochner-Weitzenbock formulas. If \( K \) is not small, we assume \( \mathcal{N} \) is non-positively curved. Then the Bochner-Weitzenbock can be improved, and the desired results follow by the same arguments as our previous works [33,37].
In this part, we prove Lemma 7.1. Let’s first recall background materials on harmonic analysis on $\mathbb{H}^2$. We refer to Helgason [18] for geometry analysis and harmonic analysis on hyperbolic spaces $\mathbb{H}^n$. The notations adopted here exactly coincide with Koornwinder [23]. $\mathbb{H}^2$ can be realized as the symmetric space $G/K$, where $G = SO_0(1, 2)$, and $K = SO(2)$. In geodesic polar coordinates on $\mathbb{H}^2$, the Laplace-Beltrami operator is given by

$$\Delta := \partial_r^2 + \coth r \partial_r + \sinh^{-2} r \partial^2_{\theta}.$$ 

The spherical functions $\psi_\lambda$ on $\mathbb{H}^2$ are normalized radial eigenfunctions of $\Delta$:

$$\Delta \psi_\lambda = -(\lambda^2 + \frac{1}{4}) \psi_\lambda, \quad \psi_\lambda(0) = 1.$$ 

And they can be represented as

$$\psi_\lambda = \int_K e^{-(\frac{1}{2} + i\lambda) H(a-r,k)} dk$$

$$= \text{const.} \int_{-r}^{r} (\cosh r - \cosh s)^{-\frac{1}{2}} e^{-i\lambda s} ds.$$ 

We also have the Harish-Chandra expansion of $\psi_\lambda$, i.e.

$$\psi_\lambda = \mathbf{c}(\lambda) \Phi_\lambda(r) + \mathbf{c}(-\lambda) \Phi_{-\lambda}(r), \quad r > 0, \lambda \in \mathbb{C} \setminus \mathbb{Z},$$

where $\mathbf{c}(\lambda)$ denotes the Harish-Chandra c-function given by

$$\mathbf{c}(\lambda) = \text{const.} \frac{\Gamma(i\lambda)}{\Gamma(i\lambda + \frac{1}{2})},$$

and $\Phi_\lambda(r)$ is

$$\Phi_\lambda(r) = \text{const.} (\sinh r)^{-\frac{1}{2}} e^{i\lambda r} \sum_{j=0}^{\infty} \Gamma_j(\lambda) e^{-2jr},$$

with $\{\Gamma_j(\lambda)\}$ defined by the recurrence formula

$$\Gamma_0(\lambda) = 1, \quad \Gamma_j(\lambda) = -\frac{1}{4k(k-i\lambda)} \sum_{l=0}^{j-1} (k-l)\Gamma_l(\lambda).$$

For $\{\Gamma_j(\lambda)\}, [2]$ proved that there exists some $\nu > 0$

$$|\partial_\lambda \Gamma_j(\lambda)| \leq C_l k^\nu (1 + |\lambda|)^{-l-1}, \quad \lambda \in \mathbb{R}. \quad (11.4)$$

For reader’s convenience, we recall the following lemma of [2] whose proof is based on the Kunze-Stein phenomenon.
Lemma 11.1 (Lemma 5.1,[2]). For any radial function \( h \) on \( \mathbb{H}^2 \), any \( 2 \leq s, k < \infty \), and \( g \in L^k(\mathbb{H}^2) \), there holds
\[
\|g \ast h\|_{L^s} \lesssim \|g\|_{L^k} \left\{ \int_0^\infty (\psi_0(r))^{\frac{5}{2}} |h(r)|^P \sinh r dr \right\}^{1/P},
\]
where \( S = \frac{2 \min\{s, k\}}{s + k} \), \( P = \frac{sk}{k + s} \), and \( \psi_0 \) is the spherical function defined above.

Let \( \chi(0) \) be a smooth function defined on \( \mathbb{R} \) such that \( \chi(0) \) vanishes for \( |\lambda| \geq 4 \), and \( \chi(0) \) equals 1 for \( |\lambda| \leq 2 \). Let \( \chi(0) = 1 - \chi(\lambda) \). Define the kernel of low frequency and high frequency Schrödinger propagators as follows:
\[
w_{l}^{low, \sigma}(r) = \int_0^\infty \chi(0)(\lambda) \left( \lambda^2 + \frac{1}{4} \right)^{\sigma} |e(\lambda)|^{-2} \psi_\lambda(r) e^{it \lambda^2 + \frac{1}{4}} d\lambda \tag{11.5}
\]
\[
w_{l}^{high, \sigma}(r) = \int_0^\infty \chi(\lambda)(\lambda^2 + \frac{1}{4})^{\sigma} |e(\lambda)|^{-2} \psi_\lambda(r) e^{it \lambda^2 + \frac{1}{4}} d\lambda, \tag{11.6}
\]
where the integral (11.6) is understood in the sense of oscillatory integrals.

Lemma 11.2. Let \( \sigma \in \mathbb{C} \). If \( r \in (0, \frac{1}{2}|t|) \), then for any \( |t| \geq 1, N \geq 1 \),
\[
|w_{l}^{high, \sigma}(r)| \lesssim N^{-1} |t|^{-N} \psi_0(r).
\]
If \( r \in [\frac{1}{2}|t|, \infty) \), then for any \( |t| \geq 1, N \geq 1 \),
\[
|w_{l}^{high, \sigma}(r)| \lesssim N^{-1} |t|^{-N} e^{-\frac{1}{4}r} (1 + r)^N.
\]

Proof. Case 1. We consider \( r \in [\frac{1}{2}|t|, \infty), |t| \geq 1 \). In this case, we directly apply (11.6). By the fact
\[
\left| \partial_\lambda [t(\lambda^2 + \frac{1}{4})^{-1}] \right| \geq |\lambda t| \geq |t|,
\]
and applying integration by parts for \( L_\sigma := [\max(0, \Re(\sigma))] + 4 \) times, we obtain
\[
|w_{l}^{high, \sigma}(r)| \lesssim t^{-L_\sigma} \int_0^\infty \sum_{0 \leq j \leq L_\sigma} \lambda^{-L_\sigma - j} \left| \partial_\lambda^{-L_\sigma - j} \left[ \chi(0)(\lambda^2 + \frac{1}{4})^{\sigma} |e(\lambda)|^{-2} \psi_\lambda(r) \right] \right| d\lambda.
\]
Hence, applying the point-wise estimates in (11.4), (11.3), (11.2) yields
\[
|w_{l}^{high, \sigma}(r)| \lesssim t^{-L_\sigma} e^{-\frac{1}{4}r} (1 + r)^{L_\sigma}.
\]
The same arguments show that applying integration by parts for \( N \geq L_\sigma \) times gives
\[
|w_{l}^{high, \sigma}(r)| \leq C_N |t|^{-N} e^{-\frac{1}{4}r} (1 + r)^N.
\]

Case 2. We consider \( r \in (0, \frac{1}{2}|t|), |t| \geq 1 \). By (11.1), \( w_{l}^{high, \sigma}(r) \) defined by (11.6) now reads as
\[
w_{l}^{high, \sigma}(r) = \int_K e^{-\frac{1}{2}H(a_\sigma, k)} dK \int_0^\infty b(\lambda) e^{it(\lambda^2 + \frac{1}{4}) - iH(a_\sigma, k)\lambda} d\lambda, \tag{11.7}
\]
where $b(\lambda)$ is

$$b(\lambda) = \chi_\infty(\lambda)(\lambda^2 + \frac{1}{4})^\sigma |e(\lambda)|^{-2}.$$ 

If $r \in (0, \frac{1}{2}|t|)$, $|t| \geq 1$, then in the support of $b(\lambda)$ one has

$$|\partial_\lambda [t(\lambda^2 + \frac{1}{4}) - H(a_{-r} k) \lambda]| \geq 2|t| - |H(a_{-r} k)| \geq |t|.$$ 

Then by stationary phase method, we get for any $N \geq 1$, there exists $C_N > 0$ such that

$$|\int_0^\infty b(\lambda)e^{it(\lambda^2 + \frac{1}{4})-iH(a_{-r} k)\lambda}d\lambda| \leq C_N|t|^{-N}.$$ 

Thus (11.7) gives

$$|w_{t}^{high,\sigma}(r)| \leq C_N|t|^{-N}\int_K e^{-\frac{1}{2}H(a_{-r} k)}dk \lesssim C_N|t|^{-N}\psi_0(r).$$

Lemma 11.3. Let $\sigma \in \mathbb{C}$. If $r \in (0, \frac{1}{2}|t|)$, then for any $|t| \geq 1$,

$$|w_{t}^{low,\sigma}(r)| \lesssim |t|^{-\frac{1}{2}}\psi_0(r).$$

If $r \in [\frac{1}{2}|t|, \infty)$, then for any $t \geq 1$,

$$|w_{t}^{low,\sigma}(r)| \lesssim |t|^{-\frac{1}{2}}e^{-\frac{1}{2}r(1+r)^2}.$$ 

Proof. Case 1. $r \in (0, \frac{1}{2}|t|)$, $|t| \geq 1$. By (11.1), $w_{t}^{low,\sigma}(r)$ defined by (11.5) now reads as

$$w_{t}^{low,\sigma}(r) = \int_K e^{-\frac{1}{2}H(a_{-r} k)}dk \int_0^\infty \tilde{b}(\lambda)e^{it(\lambda^2 + \frac{1}{4})-iH(a_{-r} k)\lambda}d\lambda,$$

where $\tilde{b}(\lambda)$ is

$$\tilde{b}(\lambda) = \chi_0(\lambda)(\lambda^2 + \frac{1}{4})^\sigma |e(\lambda)|^{-2}.$$ 

Since one has

$$|\partial_\lambda^2 [t(\lambda^2 + \frac{1}{4}) - H(a_{-r} k) \lambda]| \geq 2|t|,$$

by stationary phrase method, for $r \in (0, \frac{1}{2}|t|)$ we get

$$|\int_0^\infty \tilde{b}(\lambda)e^{it(\lambda^2 + \frac{1}{4})-iH(a_{-r} k)\lambda}d\lambda| \lesssim t^{-\frac{1}{2}}.$$ 

Thus (11.9) shows

$$|w_{t}^{low,\sigma}(r)| \lesssim t^{-\frac{1}{2}} \int_K e^{-\frac{1}{2}H(a_{-r} k)}dk \lesssim t^{-\frac{1}{2}}\psi_0(r).$$
Case 2. $r \in \left[ \frac{1}{2}|t|, \infty \right)$, $|t| \geq 1$. In this case, we apply (11.5) to get

$$w^{\text{low}, \sigma}_t(r) = \int_0^\infty \chi_0(\lambda)(\lambda^2 + \frac{1}{4})^{\sigma} e^{it(\lambda^2 + \frac{1}{4})}(I_+^0 + I_-^0) d\lambda,$$

(11.10)

where

$$I_\pm^0 := c(\lambda)e^{\pm i \lambda r}(\sinh r)^{-\frac{1}{2}}(\sum_{j=0}^{\infty} \Gamma_j(\pm \lambda))e^{-kr}.$$  

Using the point-wise estimates in (11.4), (11.3), (11.2) and the fact

$$|\partial_\lambda^2 t(\lambda^2 + \frac{1}{4})| \geq 2|t|,$$

we obtain by stationary phrase method that for $r$ in Case 2, the RHS of (11.10) is bounded as

$$|w^{\text{low}, \sigma}_t(r)| \lesssim t^{-\frac{1}{2}} e^{-\frac{1}{2}r} (1 + r)^{2/3}.$$  

$\square$

The bound (11.8) needs to be refined.

Lemma 11.4. Let $\sigma \in \mathbb{R}$, $\gamma \in (1, \frac{3}{2})$, $t \geq 1$. For $p, q \in (2, \infty)$, there holds

$$\|w^{\text{low}, \sigma}_t(r) \ast f\|_{L^p_x} \lesssim |t|^{-\gamma} \|f\|_{L^q_x}.$$  

Proof. Step 1. For $|t| \geq 1$, $p, q \in (2, \infty)$, [Theorem 3.4, [1]] implies

$$\|s_t(r) \ast f\|_{L^p_x} \lesssim |t|^{-\frac{3}{2}} \|f\|_{L^q_x},$$

(11.11)

where $s_t(r)$ equals $w^{\text{high}, 0} + w^{\text{low}, 0}$ in our notations. And one infers from Lemma 11.1 and Lemma 11.2 that for any $\sigma_1 \in \mathbb{C}$, $p, q \in (2, \infty)$, $|t| \geq 1$, $N \geq 1$,

$$\|w^{\text{high}, \sigma_1}_t(r) \ast f\|_{L^p_x} \lesssim N |t|^{-N} \|f\|_{L^q_x}.$$  

(11.12)

Therefore, (11.11) and (11.12) give

$$\|w^{\text{low}, \sigma}_t(r) \ast f\|_{L^p_x} \lesssim |t|^{-\frac{3}{2}} \|f\|_{L^q_x},$$

(11.13)

for $p, q \in (2, \infty)$, $|t| \geq 1$. Meanwhile, Lemma 11.3 gives for any $\sigma' \in \mathbb{C}$, $p, q \in (2, \infty)$, $|t| \geq 1$,

$$\|w^{\text{low}, \sigma'}_t(r) \ast f\|_{L^p_x} \lesssim |t|^{-\frac{1}{2}} \|f\|_{L^q_x}.$$  

(11.14)

Given $\sigma \in \mathbb{R}$, take $\sigma'$ to be

$$\sigma' = \frac{2\sigma}{3 - 2\gamma}.$$  

Then by Sobolev interpolation, we obtain from (11.13), (11.14) that

$$\|w^{\text{low}, \sigma}_t(r) \ast f\|_{L^p_x} \lesssim \|w^{\text{low}, 0}_t(r) \ast f\|_{L^p_x}^{\frac{\gamma}{2}} \|w^{\text{low}, \sigma'}_t(r) \ast f\|_{L^p_x}^{\frac{3 - \gamma}{2}} \lesssim |t|^{-\gamma} \|f\|_{L^q_x}.$$  

$\square$
Now, Lemma 7.1 follows by Lemma 11.4, (11.12), Strichartz estimates of [1], Kato smoothing estimates of Kaizuka [22] and Keel-Tao’s bilinear arguments. The details are given below.

**Proof of Lemma 7.1.** **Step 1. Non-endpoint Results.** First, we have a non-endpoint result, i.e.,

$$\|(-\Delta)^{\frac{1}{2}}u\|_{L^p_t L^q_x} \lesssim \|e^{\alpha r} F\|_{L^2_t L^2_x},$$

(11.15)

where \((p, q)\) is an admissible pair and \(p > 2\). The proof of (11.15) follows from the Christ-Kiselev lemma, Strichartz estimates and the dual form of Kato’s smoothing effects for \(e^{it\Delta}\).

**Step 2. Bilinear Argument for Endpoint.** Along with our previous work [36], it suffices to prove the following claim:

**Claim B.** There exists a constant \(\beta(q) > 0\) such that for all \(k, j \in \mathbb{Z}\)

$$\int_{\mathbb{R}} \int_{\mathbb{R}^2} \int_{t-2^j \leq s \leq t-2^{j-1}} G(t) e^{i(t-s)\Delta} (-\Delta)^{\frac{1}{2}} F(s) ds dx dt$$

$$\lesssim 2^{-\beta j} \|e^{\alpha r} F\|_{L^2_t L^2_x} \|G\|_{L^2_t L^q_x'},$$

(11.16)

if \(F(s)\) and \(G(t)\) are supported on a time interval of size \(2^j\) on \([t, s) : t - 2^j \leq s \leq t - 2^{j-1}\).

**Step 2.1. Sum of Negative \(j\).** For \(j \leq 0, q \in (2, \infty)\), choose \(m \in (2, \infty)\) to be slightly larger than 2. Then, Hölder and similar arguments as (11.15) give that, for \(\frac{1}{m} \geq \frac{1}{2} - \frac{1}{q}\) (then \((m, q)\) is an admissible pair), the LHS of (11.16) is bounded by

$$\|\int_{t-2^j \leq s \leq t-2^{j-1}} e^{i(t-s)\Delta} (-\Delta)^{\frac{1}{2}} F(s) ds\|_{L^m_t L^q_x} \|G(t)\|_{L^{m'}_t L^{q'}_x'}$$

$$\lesssim \|e^{\alpha r} F(t)\|_{L^2_t L^2_x} \|G(t)\|_{L^{m'}_t L^{q'}_x'} \lesssim \|e^{\alpha r} F(t)\|_{L^2_t L^2_x} \|G(t)\|_{L^2_t L^{q'}_x} 2^j \left(\frac{1}{m} - \frac{1}{2}\right),$$

where in the last line we used the fact that the time support of \(G(t)\) is of size \(2^j\). Therefore, (11.16) is done when \(j \leq 0\).

**Step 2.2. High Frequency and Low Frequency for Positive \(j\).** Decompose the LHS of (11.16) into

$$\int_{\mathbb{R}} \int_{\mathbb{R}^2} \int_{t-2^j \leq s \leq t-2^{j-1}} G(t) w_{t-s}^{\text{high}, \frac{1}{2}} F(s) ds dx dt$$

(11.17)

$$+ \int_{\mathbb{R}} \int_{\mathbb{R}^2} \int_{t-2^j \leq s \leq t-2^{j-1}} G(t) w_{t-s}^{\text{low}, \frac{1}{2}} F(s) ds dx dt.$$  

(11.18)

Then Hölder inequalities give for any \(q \in (2, \infty)\), the (11.17) term is bounded by

$$\|\int_{t-2^j \leq s \leq t-2^{j-1}} w_{t-s}^{\text{high}, \frac{1}{2}} F(s) ds\|_{L^\infty_t L^q_x} \|G\|_{L^1_t L^{q'}_x} \lesssim_N 2^{-Nj} \|F\|_{L^1_t L^m_x} \|G(s)\|_{L^2_t L^{q'}_x'}$$

$$\lesssim_N 2^{-Nj} 2^j \|F\|_{L^2_t L^m'} \|G(s)\|_{L^2_t L^{q'}_x'} \lesssim_N 2^{-Nj} 2^j \|e^{\alpha r} F\|_{L^2_t L^2_x} \|G(s)\|_{L^2_t L^{q'}_x'},$$
where we chose some $m > 2$ such that $e^{-\alpha r} \in L_2^{\frac{2m}{m-2}}$, and applied (11.12). This is admissible for (11.16) by choosing large $N$. Similarly, by Hölder inequalities and Lemma 11.4, we also see for any $q \in (2, \infty)$, the (11.18) term is dominated by

$$\int_{t-2^j \leq s \leq t-2^{j-1}} u_{\text{low}}^\frac{1}{p} \cdot F(s) ds \|_{L_2^\infty L_2^q} \|_{L_1^m L_1^{q'}} \lesssim 2^{-\gamma j} \| F \|_{L_1^m L_1^{q'}} \| G(s) \|_{L_1^m L_1^{q'}} \lesssim 2^{-\gamma j} \| e^{\alpha r} F \|_{L_1^m L_1^{q'}} \| G(s) \|_{L_1^m L_1^{q'}} ,$$

where we again chose some $m > 2$ such that $e^{-\alpha r} \in L_2^{\frac{2m}{m-2}}$. This is admissible for (11.16) by choosing $\gamma > 1$.

\[\Box\]

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