RELATIVE ČECH-DOLBEAULT HOMOLOGY AND APPLICATIONS

NICOLETTA TARDINI

Abstract. We define the relative Dolbeault homology of a complex manifold with currents via a Čech approach and we prove its equivalence with the relative Čech-Dolbeault cohomology as defined in [7]. This definition is then used to compare the relative Dolbeault cohomology groups of two complex manifolds of the same dimension related by a suitable proper surjective holomorphic map. Finally, an application to blow-ups is considered.

Introduction

On a complex manifold an important global invariant is represented by the Dolbeault cohomology which can be described equivalently using complex differential forms or currents. In particular, this double interpretation was used fruitfully by R. O. Wells in [10] in order to compare the Dolbeault cohomology of two complex manifolds of the same dimensions related by a proper holomorphic surjective map. More precisely, he proved that if \( \pi : \tilde{X} \to X \) is a proper holomorphic surjective map between two complex manifolds of the same dimension then the induced map in cohomology

\[
\pi^* : H^{\bullet, \bullet}_\partial(X) \to H^{\bullet, \bullet}_\partial(\tilde{X})
\]

is injective. In particular, if \( X \) and \( \tilde{X} \) are compact we have the dimensional inequalities

\[
\text{h}^{\bullet, \bullet}_\partial(X) \leq \text{h}^{\bullet, \bullet}_\partial(\tilde{X})
\]

for their Hodge numbers. In fact, this result can be weaken to almost-complex manifolds as done in [9] for pseudo-holomorphic maps, where the Dolbeault cohomology is replaced by the cohomology groups introduced by Li and Zhang [6].

The Dolbeault cohomology can also be interpreted via a Čech approach, as done in [7], and it turns out that this approach is also useful in order to define its relative version. The relative Čech-Dolbeault cohomology, which is equivalent to the relative Dolbeault cohomology as defined for instance in [5] (cf. [8]), can be used to describe the localization theory of characteristic classes ([7], [1]) and recently has found applications to the Sato hyperfunction theory [4].

In the present paper we define the relative Čech-Dolbeault homology of a complex manifold in terms of currents, and we prove in Theorem 2.5 that this description is equivalent to Suwa’s one using complex differential forms. This different interpretation is used in Theorem 3.1 in order to prove a Wells-type result for relative cohomology, in particular we prove the following

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Date: December 4, 2018.

2010 Mathematics Subject Classification. 32Q99, 32H99.

Key words and phrases. Dolbeault cohomology, relative cohomology, currents.

Partially supported by SIR2014 project RBH14DYEB “Analytic aspects in complex and hypercomplex geometry” and by GNSAGA of INdAM.
Theorem 0.1. Let $\pi : \tilde{X} \to X$ be a proper, surjective, holomorphic map between two complex manifolds of the same dimension. Suppose that $X$ is connected and let $S$ and $\tilde{S}$ be closed complex submanifolds of $X$ and $\tilde{X}$ respectively, such that $\pi(\tilde{S}) \subset S$ and $\pi(\tilde{X} \setminus \tilde{S}) \subset X \setminus S$.

Then,

$$\pi^* : H^p,q_c(X, X \setminus S) \to H^p,q_c(\tilde{X}, \tilde{X} \setminus \tilde{S})$$

is injective for any $p,q$.

Differently from the classical Dolbeault theory, a dimension inequality in this setting cannot be expected even if $X$ and $\tilde{X}$ are compact, indeed in general the relative Dolbeault cohomology groups of a compact complex manifold are infinite-dimensional.

Notice that the hypothesis in the previous Theorem are satisfied by modifications. In particular, if $\tau : \tilde{X} \to X$ is the blow-up of a complex manifold $X$ along a closed submanifold $Z$ then the previous assumptions are satisfied with $S = Z$ and $\tilde{S} = E := \pi^{-1}(Z)$ the exceptional divisor. Then, as a consequence (cf. Remark 3.2) the Dolbeault cohomology of $\tilde{X}$ can be expressed in terms of the Dolbeault cohomology of $X$ and the relative cohomology, more precisely there are isomorphisms

$$H^p,q_c(\tilde{X}) \cong \tau^*H^p,q_c(X) \oplus \frac{H^p,q_c(\tilde{X} \setminus \tilde{S})}{\tau^*H^p,q_c(X \setminus Z)}$$

for any $p,q$.

Acknowledgements. The author would like to thank Daniele Angella, Tatsuo Suwa and Adriano Tomassini for many useful discussions and comments.

1. Preliminaries and notations

We start by fixing some notations and recalling some results about relative Čech-Dolbeault cohomology as presented in [7].

Čech-Dolbeault cohomology. Let $X$ be a complex manifold of complex dimension $n$. We denote with $A^{p,q}$ the space of smooth $(p,q)$-forms on $X$. Let $\mathcal{U} = \{U_0, U_1\}$ be an open covering of $X$ and consider

$$A^{p,q}(\mathcal{U}) := A^{p,q}(U_0) \oplus A^{p,q}(U_1) \oplus A^{p,q-1}(U_{01})$$

where by definition $U_{01} := U_0 \cap U_1$, with the differential operator $D : A^{p,q}(\mathcal{U}) \to A^{p,q+1}(\mathcal{U})$ defined on every element $(\xi_0, \xi_1, \xi_{01}) \in A^{p,q}(\mathcal{U})$ by

$$D(\xi_0, \xi_1, \xi_{01}) = (\overline{\partial}\xi_0, \overline{\partial}\xi_1, \xi_1 - \xi_0 - \overline{\partial}\xi_{01})$$

The Čech-Dolbeault cohomology associated to the covering $\mathcal{U}$ is defined by $H^{\bullet,\bullet}_D(\mathcal{U}) = \text{Ker } D/\text{Im } D$ (cf. [7] where this definition is given for an arbitrary open covering of $X$). In [7] it is proven that the morphism $A^{p,q}(X) \to A^{p,q}(\mathcal{U})$ given by $\xi \mapsto (\xi|_{U_0}, \xi|_{U_1}, 0)$ induces an isomorphism

$$H^{\bullet,\bullet}_D(X) \cong H^{\bullet,\bullet}_D(\mathcal{U}),$$
where $H^{p,q}_{\overline{\partial}}(X)$ denotes the Dolbeault cohomology of $X$. In particular, the definition does not depend on the choice of the covering of $X$. The inverse map is given by assigning to the class of $\xi = (\xi_0, \xi_1, \xi_{01})$ the class of global $\overline{\partial}$-closed form $\rho_0 \xi_0 + \rho_1 \xi_1 - \overline{\partial} \rho_0 \wedge \xi_{01}$, where $(\rho_0, \rho_1)$ is a partition of unity subordinate to the covering $\mathcal{U}$.

One can define naturally the cup product, integration on top-degree cohomology and the Kodaira-Serre duality and they turn out to be compatible with the above isomorphism (see [7] for more details).

**Relative Dolbeault cohomology.** Let $S$ be a closed set in $X$. We let $U_0 = X \setminus S$ and $U_1$ be an open neighborhood of $S$ in $X$ and consider the covering $\mathcal{U} = \{U_0, U_1\}$ of $X$.

We set, for any $p, q$,

$$A^{p,q}(\mathcal{U}, U_0) := \{ \xi \in A^{p,q}(\mathcal{U}) \mid \xi_0 = 0 \} = A^{p,q}(U_1) \oplus A^{p,q-1}(U_{01}).$$

Then $(A^{p,\bullet}(\mathcal{U}, \overline{\partial}), D)$ is a subcomplex of $(A^{p,\bullet}(\mathcal{U}), \overline{\partial})$. Let $H^{p,q}_D(\mathcal{U}, U_0)$ be the cohomology of $(A^{p,\bullet}(\mathcal{U}, U_0), \overline{\partial})$. From the exact sequence

$$0 \to A^{p,\bullet}(\mathcal{U}, U_0) \to A^{p,\bullet}(\mathcal{U}) \to A^{p,\bullet}(U_0) \to 0$$

where the first map is the inclusion and the second map is the projection on the first element, we obtain the long exact sequence in cohomology

$$\cdots \to H^{p,q-1}_D(U_{01}) \to H^{p,q}_D(\mathcal{U}, U_0) \to H^{p,q}_D(\mathcal{U}) \to H^{p,q}_D(U_0) \to \cdots.$$ 

Therefore, $H^{p,\bullet}_D(\mathcal{U}, U_0)$ is determined uniquely modulo canonical isomorphism. Thus we denote it with $H^{p,\bullet}_D(X, X \setminus S)$ and we call it the relative Dolbeault cohomology of $X$.

Together with integration theory the relative Dolbeault cohomology has been used to study the localization of characteristic classes (cf. [7], [1]) and has found more recent applications to hyperfunction theory (cf. [4]).

If $X$ and $\tilde{X}$ are complex manifolds, $S$ and $\tilde{S}$ are closed subsets in $X$ and $\tilde{X}$ respectively and $f : \tilde{X} \to X$ is a holomorphic map such that $f(\tilde{S}) \subset S$ and $f(\tilde{X} \setminus \tilde{S}) \subset f(X \setminus S)$, then $f$ induces a natural map in relative cohomology. Indeed, setting $U_0 := X \setminus S$, $\tilde{U}_0 := \tilde{X} \setminus \tilde{S}$ and let $U_1$, $\tilde{U}_1$ be open neighborhoods of $S$ and $\tilde{S}$ in $X$ and $\tilde{X}$ respectively, chosen in such a way that $f(\tilde{U}_1) \subset U_1$. Take the open coverings $\mathcal{U} := \{U_0, U_1\}$ and $\tilde{\mathcal{U}} := \{\tilde{U}_0, \tilde{U}_1\}$ of $X$ and $\tilde{X}$ respectively, then we have a homomorphism

$$f^* : A^{\bullet,\bullet}(\mathcal{U}, U_0) \to A^{\bullet,\bullet}(\tilde{\mathcal{U}}, \tilde{U}_0)$$

defined on every element $(\xi_1, \xi_{01}) \in A^{\bullet,\bullet}(\mathcal{U}, U_0)$ as

$$f^*(\xi_1, \xi_{01}) := (f^*\xi_1, f^*\xi_{01})$$

which induces a homomorphism in relative cohomology

$$f^* : H^{\bullet,\bullet}_D(X, X \setminus S) \to H^{\bullet,\bullet}_D(\tilde{X}, \tilde{X} \setminus \tilde{S}).$$

## 2. Relative Čech-Dolbeault homology

In this Section we describe a Čech interpretation of the Dolbeault homology and we give a definition for its relative counterpart. Finally in Theorem 2.5 we prove that
the relative Dolbeault cohomology can be computed equivalently using forms and currents.

**Čech-Dolbeault homology.** Before defining the relative Dolbeault homology we discuss a Čech interpretation of the Dolbeault homology. Let \( X \) be a complex manifold of complex dimension \( n \) and denote with \( \mathcal{K}^{p,q}(X) = \mathcal{K}_{n-p,n-q}(X) \) the space of currents of bidegree \((p, q)\), or equivalently of bidegree \((n - p, n - q)\), on \( X \), namely the topological dual of the space of \((n - p, n - q)\)-forms with compact support in \( X \).

The differential operator \( \overline{\partial} : \mathcal{K}^{p,q}(X) \to \mathcal{K}^{p,q+1}(X) \) is defined as usual, for any \( T \in \mathcal{K}^{p,q}(X), \varphi \in A^{n-p,n-q-1}(X) \) with compact support, as

\[
\langle \overline{\partial} T, \varphi \rangle := (-1)^{p+q+1} \langle T, \overline{\partial} \varphi \rangle ,
\]

where \(- , -\) stands for the duality pairing. Then for any \( p \), we denote with \( H^p_{\overline{\partial}K}(X) \) the cohomology of the complex \((\mathcal{K}^{p,\bullet}(X), \overline{\partial})\) which is called the Dolbeault homology of \( X \).

Now let \( \mathcal{U} = \{ U_\alpha \}_{\alpha \in I} \) be an open covering of \( X \) where \( I \) is an ordered set and let \( I^{(r)} := \{ (\alpha_0, \ldots, \alpha_r) \mid \alpha_0 < \cdots < \alpha_r, \alpha_\nu \in I \} \). We set, for any \( r, p, q \),

\[
B^r(\mathcal{U}, \mathcal{K}^{p,q}) := \Pi_{(\alpha_0, \ldots, \alpha_r) \in I^{(r)}} \mathcal{K}^{p,q}(U_{\alpha_0 \cdots \alpha_r}),
\]

where, by definition, \( U_{\alpha_0 \cdots \alpha_r} := U_{\alpha_0} \cap \cdots \cap U_{\alpha_r} \), and we define the boundary operator as

\[
\delta : B^r(\mathcal{U}, \mathcal{K}^{p,q}) \to B^{r+1}(\mathcal{U}, \mathcal{K}^{p,q})
\]

\[
(\delta T)_{\alpha_0 \cdots \alpha_r+1} := \sum_{\nu=0}^{r+1} (-1)^\nu T_{\alpha_0 \cdots \hat{\alpha}_\nu \cdots \alpha_{r+1}}
\]

where all the currents \( T_{\alpha_0 \cdots \hat{\alpha}_\nu \cdots \alpha_{r+1}} \) have to be restricted to \( U_{\alpha_0 \cdots \alpha_{r+1}} \). Moreover, setting \( \overline{\partial} : B^r(\mathcal{U}, \mathcal{K}^{p,q}) \to B^r(\mathcal{U}, \mathcal{K}^{p,q+1}) \) the extension of the operator \( \overline{\partial} \) to every components one gets that \( B^\bullet(\mathcal{U}, \mathcal{K}^{p,\bullet}) \) endowed with the operators \( \delta \) and \( \overline{\partial} \) is a double complex. The associated total complex will be denoted with \((\mathcal{K}^{p,\bullet}(\mathcal{U}), \overline{\partial}K)\), namely

\[
\mathcal{K}^{p,q}(\mathcal{U}) := \oplus_{s+r=q} B^r(\mathcal{U}, \mathcal{K}^{p,s})
\]

and

\[
\overline{\partial}K(T_{\alpha_0 \cdots \alpha_r}) = \sum_{\nu=0}^{r} (-1)^\nu T_{\alpha_0 \cdots \hat{\alpha}_\nu \cdots \alpha_r} + (-1)^r \overline{\partial} T_{\alpha_0 \cdots \alpha_r} .
\]

In particular, for \( r = 1 \) we have

\[
\overline{\partial}K(T_{\alpha_0 \alpha_1}) := T_{\alpha_1} - T_{\alpha_0} - \overline{\partial} T_{\alpha_0 \alpha_1} .
\]

**Definition 2.1.** The cohomology of the complex \((\mathcal{K}^{p,\bullet}(\mathcal{U}), \overline{\partial}K)\) will be denoted with \( H^*_{\overline{\partial}K}(\mathcal{U}) \) and will be called Čech Dolbeault homology associated to the covering \( \mathcal{U} \).

This definition does not depend on the open covering, indeed one has the following

**Theorem 2.2.** The restriction map \( \mathcal{K}^{p,q}(X) \to B^0(\mathcal{U}, \mathcal{K}^{p,q}) \) induces a natural isomorphism

\[
H^p_{\overline{\partial}K}(\mathcal{U}) \to H^p_{\overline{\partial}K}(\mathcal{U}).
\]
Proof. The argument works exactly as in [7], since the complex of \((p, q)\)-currents on \(U\) is acyclic, being a \(C^\infty(X)\)-module. For completeness we recall here the proof. We consider the first spectral sequence associated to the double complex \(B^\bullet(\mathcal{U}, \mathcal{K}^{p, \bullet})\). In particular, at the second page one has
\[ E_2^{pq} := H^q_\partial H^p_\delta (B^\bullet(\mathcal{U}, \mathcal{K}^{p, \bullet})) \]
which converges to \(H^{p+q}_\partial(\mathcal{U})\). By the acyclic property, for \(r > 0\), \(H^{p, q}_\partial(\mathcal{U}, \mathcal{K}^{p, \bullet}) = 0\) and for \(r = 0\), \(H^0_\partial(\mathcal{U}, \mathcal{K}^{p, \bullet}(X)) = \mathcal{K}^{p, \bullet}(X)\). Therefore, \(H^p_\partial(\mathcal{U}) \simeq E_2^{q, 0} \simeq H^{p, q}_\partial(\mathcal{U})\).

Remark 2.3. Since the Dolbeault cohomology can be computed using currents, it follows by the previous Theorem that there is also an isomorphism with the Dolbeault cohomology of \(X\), namely
\[ H^p_{\partial}(X) \simeq H^p_{\partial, k}(X) \simeq H^p_{\partial, k}(X). \]
Since to every \((p, q)\)-form \(\varphi\) on \(X\) we can associate a \((p, q)\)-current \(i(\varphi) := \int_X \varphi \wedge \cdot\), then we have a natural injective map
\[ i : A^{p,q}(\mathcal{U}) \to \mathcal{K}^{p,q}(\mathcal{U}) \]
where by definition (see [7] for more details)
\[ A^{p,q}(\mathcal{U}) := \oplus_{s+r=q} \left( \Pi_{(\alpha_0, \ldots, \alpha_r) \in I(r)} A^{p,s}(U_{\alpha_0, \ldots, \alpha_r}) \right). \]

Relative Čech-Dolbeault homology. Now we define the relative Čech Dolbeault homology. Let \(S\) be a closed set in \(X\). We let \(U_0 = X\setminus S\) and \(U_1\) be an open neighborhood of \(S\) in \(X\) and consider the open covering \(U = \{U_0, U_1\}\) of \(X\). In particular, in this situation we have
\[ \mathcal{K}^{p,q}(U_0) = B^0(\mathcal{U}, \mathcal{K}^{p,q}) \oplus B^1(\mathcal{U}, \mathcal{K}^{p,q-1}) \]
and an element of \(\mathcal{K}^{p,q}(U_0)\) can be written as a triple \((T_0, T_1, T_{01})\) where \(T_0\) is a \((p, q)\)-current on \(U_0\), \(T_1\) is a \((p, q)\)-current on \(U_1\) and \(T_{01}\) is a \((p, q-1)\)-current on \(U_{01}\).

We set
\[ \mathcal{K}^{p,q}(U, U_0) := \{ T \in \mathcal{K}^{p,q}(U) \mid T_0 = 0\} = \mathcal{K}^{p,q}(U_1) \oplus \mathcal{K}^{p,q-1}(U_{01}). \]
Then \(\mathcal{K}^{p,\bullet}(U, U_0)\) is a subcomplex of \(\mathcal{K}^{p,\bullet}(U)\). Therefore we have the following short exact sequence
\[ 0 \to \mathcal{K}^{p,\bullet}(U, U_0) \to \mathcal{K}^{p,\bullet}(U) \to \mathcal{K}^{p,\bullet}(U_0) \to 0 \]
where the first map is the inclusion and the second map is the projection on the first factor and clearly by definition
\[ \tilde{D}_\partial : \mathcal{K}^{p,q}(U, U_0) \to \mathcal{K}^{p,q+1}(U, U_0). \]
We denote with \(H^{p,q}_\partial(\mathcal{U}, U_0)\) the associated cohomology. We get the following long exact sequence in homology
\[ \cdots \to H^{p,q-1}_\partial(U_0) \xrightarrow{\delta} H^{p,q}_\partial(\mathcal{U}, U_0) \xrightarrow{i^*} H^{p,q}_\partial(\mathcal{U}) \xrightarrow{i^*} H^{p,q}_\partial(U_0) \to \cdots. \]

By Theorem 2.2 we see that \(H^{p,q}_\partial(\mathcal{U}, U_0)\) is determined uniquely modulo canonical isomorphisms.
**Definition 2.4.** In the above situation we set \( H^{p,q}_{D^c}(X, X \setminus S) := H^{p,q}_{D^c}(U, X \setminus S) \) and we call it the relative Čech-Dolbeault homology of \( X \) with respect to \( X \setminus S \).

Notice that we have an injective map \( i : A^{p,q}(U, U_0) \to K^{p,q}(U, U_0) \) which induces naturally a map \( H^{p,q}_D(X, X \setminus S) \to H^{p,q}_{D^c}(X, X \setminus S) \) from the relative Dolbeault cohomology to the relative Dolbeault homology.

**Theorem 2.5.** The map

\[
H^{p,q}_D(X, X \setminus S) \to H^{p,q}_{D^c}(X, X \setminus S)
\]

is an isomorphism.

**Proof.** We have the following commutative diagram between forms and currents

\[
\begin{array}{ccc}
0 & \longrightarrow & A^{p,q}(U, U_0) \longrightarrow A^{p,q}(U) \longrightarrow A^{p,q}(U_0) \longrightarrow 0 \\
\downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & K^{p,q}(U, U_0) \longrightarrow K^{p,q}(U) \longrightarrow K^{p,q}(U_0) \longrightarrow 0
\end{array}
\]

which induces the commutative diagram with exact rows

\[
\begin{array}{ccc}
\cdots & \longrightarrow & H^{p,q-1}_D(X) \longrightarrow H^{p,q-1}_D(X \setminus S) \longrightarrow H^{p,q}_D(X, X \setminus S) \longrightarrow H^{p,q}_D(X) \longrightarrow H^{p,q}_D(X \setminus S) \longrightarrow \cdots \\
\uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\
\cdots & \longrightarrow & H^{p,q-1}_{D^c}(X) \longrightarrow H^{p,q-1}_{D^c}(X \setminus S) \longrightarrow H^{p,q}_{D^c}(X, X \setminus S) \longrightarrow H^{p,q}_{D^c}(X) \longrightarrow H^{p,q}_{D^c}(X \setminus S) \longrightarrow \cdots
\end{array}
\]

Since the Dolbeault cohomology of \( X \) and \( X \setminus S \) can be computed equivalently using differential forms or currents we have that the first two and the last two vertical maps are isomorphisms. By the five lemma we obtain the same conclusion for the central vertical morphism. \( \square \)

Now let \( \pi : \tilde{X} \longrightarrow X \) be a proper holomorphic map between two complex manifolds and let \( S \) be a closed subset of \( X \) and \( \tilde{S} \) a closed subset of \( \tilde{X} \) and take \( U_0 = X \setminus S \) and \( U_1 \) a neighborhood of \( S \) in \( X \), similarly \( \tilde{U}_0 = \tilde{X} \setminus \tilde{S} \) and \( \tilde{U}_1 \) a neighborhood of \( \tilde{S} \) in \( \tilde{X} \). Suppose that \( \pi(S) \subset S \), \( \pi(U_0) \subset U_0 \) and \( \pi(U_1) \subset U_1 \). Then \( \mathcal{U} := \{U_0, U_1\} \) and \( \tilde{\mathcal{U}} := \{\tilde{U}_0, \tilde{U}_1\} \) are compatible open coverings of \( X \) and \( \tilde{X} \) respectively. We define the **push-forward** \( \pi_* : K^{p,q}_{\mathcal{U}}(\tilde{U}, \tilde{U}_0) \to K^{p,q}(U, U_0) \) namely

\[
\pi_* : K^{p,q}(\tilde{U}_1) \oplus K^{p,q-1}(\tilde{U}_0) \to K^{p,q}(U_1) \oplus K^{p,q-1}(U_0)
\]

as

\[
(T_1, T_0) \mapsto (\pi_* T_1, \pi_* T_0).
\]

In particular, the push-forward commutes with the differential \( D_{\mathcal{U}} \), indeed

\[
\pi_* D_{\mathcal{U}}(T_1, T_0) = (\pi_* \overline{\partial} T_1, \pi_* T_1 - \pi_* \overline{\partial} T_0) = (\overline{\partial} \pi_* T_1, \pi_* T_1 - \overline{\partial} \pi_* T_0) = D_{\mathcal{U}} \pi_* (T_1, T_0),
\]
where in the second equality we use that \( \pi_* \) commutes with the operator \( \overline{\partial} \). Hence, the push-forward induces a map in relative Dolbeault homology

\[
\pi_* : H^{\bullet,\bullet}_{D\bar{K}}(\tilde{X}, \tilde{X} \setminus \tilde{S}) \to H^{\bullet,\bullet}_{D\bar{K}}(X, X \setminus S).
\]

We will use this map in the next Section in order to prove that under suitable assumptions the pull-back map in relative cohomology is injective.

3. Comparisons via proper holomorphic surjective maps

In this Section we study a Wells-type result for relative Dolbeault cohomology, in particular we prove the following

**Theorem 3.1.** Let \( \pi : \tilde{X} \to X \) be a proper, surjective, holomorphic map between two complex manifolds of the same dimension. Suppose that \( X \) is connected and let \( S \) and \( \tilde{S} \) be closed complex submanifolds of \( X \) and \( \tilde{X} \) respectively, such that \( \pi(\tilde{S}) \subset S \) and \( \pi(\tilde{X} \setminus \tilde{S}) \subset X \setminus S \).

Then,

\[
\pi^* : H^{p,q}_D(X, X \setminus S) \to H^{p,q}_D(\tilde{X}, \tilde{X} \setminus \tilde{S})
\]

is injective for any \( p, q \).

**Proof.** We denote with \( U_0 = X \setminus S \) and \( \tilde{U}_0 = \tilde{X} \setminus \tilde{S} \). Let \( U_1 \) be a neighborhood of \( S \) in \( X \) and \( \tilde{U}_1 \) be a neighborhood of \( \tilde{S} \) in \( \tilde{X} \) such that \( \pi(\tilde{U}_1) = U_1 \) and \( \pi(\tilde{U}_0) = U_0 \). Hence we consider the open coverings \( \mathcal{U} := \{U_0, U_1\} \) of \( X \) and \( \tilde{\mathcal{U}} := \{\tilde{U}_0, \tilde{U}_1\} \) of \( \tilde{X} \).

We take the following diagram, for any \( p, q \)

\[
\begin{array}{ccc}
A^{p,q}(U, U_0) & \xrightarrow{i} & K^{p,q}(U, U_0) \\
\pi^* \uparrow & & \downarrow \pi_* \\
A^{p,q}(\tilde{U}, \tilde{U}_0) & \xrightarrow{\tilde{i}} & K^{p,q}(\tilde{U}, \tilde{U}_0)
\end{array}
\]

or equivalently, by definition

\[
\begin{array}{ccc}
A^{p,q}(U_1) \oplus A^{p,q-1}(U_0) & \xrightarrow{i} & K^{p,q}(U_1) \oplus K^{p,q-1}(U_0) \\
\pi^* \uparrow & & \downarrow \pi_* \\
A^{p,q}(\tilde{U}_1) \oplus A^{p,q-1}(\tilde{U}_0) & \xrightarrow{\tilde{i}} & K^{p,q}(\tilde{U}_1) \oplus K^{p,q-1}(\tilde{U}_0)
\end{array}
\]

where \( \tilde{i} \) and \( i \) denote the natural injections of forms into currents. By [10, Lemma 2.1] we have that the diagram

\[
\begin{array}{ccc}
A^{p,q}(\tilde{U}_1) & \xrightarrow{i} & K^{p,q}(\tilde{U}_1) \\
\pi^* \uparrow & & \downarrow \pi_* \\
A^{p,q}(U_1) & \xrightarrow{i} & K^{p,q}(U_1)
\end{array}
\]
commutes up to a constant, more precisely we have $\mu \iota = \pi_s \tilde{i} \pi^*$, where $\mu$ is the degree of $\pi$. Recall that the degree of $\pi$ is defined as $\pi_s(1)$, where 1 is thought as a current on $\tilde{X}$, in particular it is a $d$-closed function on $X$. By the connectedness of $X$ we have that the degree of $\pi$ is constant on $X$. Similarly, we also have the commutativity up to $\mu$, of

$$
\begin{array}{ccc}
A^{p,q-1}(U_01) & \overset{i}{\rightarrow} & \mathcal{K}^{p,q-1}(U_01) \\
\pi^* & & \pi^*
\end{array}
\begin{array}{ccc}
A^{p,q-1}(U_01) & \overset{i}{\rightarrow} & \mathcal{K}^{p,q-1}(U_01). \\
\end{array}
$$

Therefore, one can pass to (co)homology considering the following diagram:

$$
\begin{array}{ccc}
H^{p,q}_D(\tilde{X},\tilde{X}\setminus\tilde{S}) & \overset{\tilde{i}^*}{\rightarrow} & H^{p,q}_{D\mathcal{K}}(\tilde{X},\tilde{X}\setminus\tilde{S}) \\
\pi^* & & \pi^*
\end{array}
\begin{array}{ccc}
H^{p,q}_D(X,X\setminus S) & \overset{i^*}{\rightarrow} & H^{p,q}_{D\mathcal{K}}(X,X\setminus S). \\
\end{array}
$$

We have shown in Theorem 2.5 that the maps $\tilde{i}^*$ and $i^*$ in this last diagram are isomorphisms.

Using this fact we prove that $\pi^*$ is injective, indeed let $a \in H^{p,q}(X,X\setminus S)$ and suppose that $\pi^*a = 0$, then $\mu i_s a = \pi_s \tilde{i} \pi^* a = 0$. Then $i_s a = 0$ and by injectivity we can conclude that $a = 0$, proving the assertion.

**Remark 3.2.** Notice that the hypothesis in the previous Theorem are satisfied by blow-ups. Indeed, let $X$ be a compact complex manifold and let $Z$ be a closed complex submanifold. Then, the blow-up $\tau : \tilde{X} \rightarrow X$ of $X$ along $Z$ satisfies the previous assumptions with $S = Z$ and $\tilde{S} = E := \pi^{-1}(Z)$ the exceptional divisor. Then, by Theorem 3.1 the induced map in relative cohomology

$$
\tau^* : H^{p,q}_D(X,X\setminus Z) \rightarrow H^{p,q}_D(\tilde{X},\tilde{X}\setminus E)
$$

is injective.

Therefore, one has the following commutative diagram with exact rows (cf. [2])

$$
\begin{array}{cccccccccccc}
\cdots & H^{p,q-1}_D(X,Z) & \rightarrow & H^{p,q}_D(X,X\setminus Z) & \rightarrow & H^{p,q}_D(X) & \rightarrow & H^{p,q}_D(X,X\setminus Z) & \rightarrow & H^{p,q+1}_D(X,X\setminus Z) & \cdots \\
& \downarrow{\tau^*} & & \downarrow{\tau^*} & & \downarrow{\tau^*} & & \downarrow{\tau^*} & & \downarrow{\tau^*} & \\
\cdots & H^{p,q-1}_D(\tilde{X},\tilde{E}) & \rightarrow & H^{p,q}_D(\tilde{X},\tilde{X}\setminus E) & \rightarrow & H^{p,q}_D(\tilde{X}) & \rightarrow & H^{p,q}_D(\tilde{X},\tilde{X}\setminus E) & \rightarrow & H^{p,q+1}_D(\tilde{X},\tilde{X}\setminus E) & \cdots
\end{array}
$$

where the maps

$$
\tau^* : H^{p,q}_D(X,Z) \rightarrow H^{p,q}_D(\tilde{X},\tilde{X}\setminus E)
$$

are isomorphisms, the maps

$$
\tau^* : H^{p,q}_D(X,X\setminus Z) \rightarrow H^{p,q}_D(\tilde{X},\tilde{X}\setminus E)
$$

are injective by [10] Theorem 3.1] and finally the maps

$$
\tau^* : H^{p,q}_D(X,X\setminus Z) \rightarrow H^{p,q}_D(\tilde{X},\tilde{X}\setminus E)
$$
are injective by Theorem 3.1. In particular, for instance by Lemme II.6] this implies that we have isomorphisms

$$H^{p,q}_{\overline{\partial}}(\tilde{X}) \simeq \tau^* H^{p,q}_{\overline{\partial}}(X) \oplus \frac{H^{p,q}_{\overline{\partial}}(\tilde{X}, \tilde{X} \setminus E)}{\tau^* H^{p,q}_{\overline{\partial}}(X, X \setminus Z)},$$

for any $p, q$.

References

[1] M. Abate, F. Bracci, T. Suwa, F. Tovena, Localization of Atiyah classes, Rev. Mat. Iberoam. 29 (2013), 547–578.
[2] D. Angella, T. Suwa, N. Tardini, A. Tomassini, Note on Dolbeault cohomology and Hodge structures up to bimeromorphisms, arXiv:1712.08889 [math.DG], 2017.
[3] A. Blanchard, Sur les variétés analytiques complexes. (French) Ann. Sci. Ecole Norm. Sup. (3) 73 (1956), 157–202.
[4] N. Honda, T. Izawa, T. Suwa, Sato hyperfunctions via relative Dolbeault cohomology, arXiv:1807.01831 [math.CV], 2018.
[5] C. Ida, A note on relative cohomology of complex manifolds, Bul. Stiint. Univ. Politeh. Timis. Ser. Mat. Fiz. 56(70) (2011), no. 2, 23–29.
[6] T.-J. Li, W. Zhang, Comparing tamed and compatible symplectic cones and cohomological properties of almost complex manifolds, Comm. Anal. Geom. 17 (2009), 651–683.
[7] T. Suwa, Čech-Dolbeault cohomology and the $\overline{\partial}$-Thom class, Singularities—Niigata—Toyama 2007, 321–340, Adv. Stud. Pure Math., 56, Math. Soc. Japan, Tokyo, 2009.
[8] T. Suwa, Representation of relative sheaf cohomology, arXiv:1810.06198 [math.AT], 2018.
[9] N. Tardini, A. Tomassini, On the cohomology of almost-complex and symplectic manifolds and proper surjective maps, Internat. J. Math. 27 no. 12, 1650103 (20 pages) (2016)
[10] R. O. Wells, Comparison of de Rham and Dolbeault cohomology for proper surjective mappings, Pacific J. Math. 53 (1974), 281–300.