SPACEABILITY ON SOME CLASSES OF BANACH SPACES

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Abstract. In this paper, we study spaceability of subsets of generalized Orlicz and Lebesgue spaces associated to a Banach function space. Also, we give some sufficient conditions for spaceability of subsets of a general Banach space which improves an important result on this topic. As an application, it is shown that the set of all bounded linear operators which are not positive semidefinite on a separable Hilbert space is spaceable.

1. Introduction

A subset of a topological vector space is called spaceable if its union with the singleton \{0\} contains a closed infinite-dimensional linear subspace. This concept was introduced in [11, 1] and so far has been considered by many researchers. As a useful tool, L. Bernal-González and M.O. Cabrera in [5, Theorem 2.2] give some sufficient conditions for spaceability of the complement of a cone in a Banach function space. This result covers some important ones proved in [8, 9]. By this tool, in [21, 22] it is shown that the set \( \mathcal{M}_q^p(\mathbb{R}^n) \setminus \bigcup_{q < r \leq p} \mathcal{M}_r^p(\mathbb{R}^n) \) is spaceable in the Morrey space \( \mathcal{M}_q^p(\mathbb{R}^n) \), if \( 0 < q < p < \infty \). Also, technically it is also proved that \( w\mathcal{M}_q^p(\mathbb{R}^n) \setminus \mathcal{M}_q^p(\mathbb{R}^n) \) is spaceable in the weak Morrey space \( w\mathcal{M}_q^p(\mathbb{R}^n) \). In [14, Theorem 3.3] D. Kitson and R. M. Timoney present another nice sufficient condition for a set to be spaceable in a Fréchet space. This topic has been studied in the context of some special sequence and function spaces in several papers (see [3, 4, 7, 8, 9, 12, 24] for example).

In this paper, we focus on generalized Orlicz and Lebesgue spaces \( X^\Phi \) and \( X^p \) associated to a Banach function space \( X \), where \( \Phi \) is a Young function and \( p \geq 1 \). These structures were studied in [10, 13, 19, 23] and contains usual Orlicz and Lebesgue spaces. Inspiring [8, 9] and as an extension of [5, Theorem 3.3] we prove that if \( X \) is a solid Banach function space, \( \inf\{\|\chi_E\|_X : E \in \mathcal{A}_0\} = 0 \), and \( \sup\{\|\chi_E\|_X : E \in \mathcal{A}_\infty\} < \infty \), where \( \mathcal{A}_\infty := \{E \in \mathcal{A} : \chi_E \in X\} \), then for each \( p \geq 1 \), \( X^p - \bigcup_{p < q} X^q \) is spaceable in \( X^p \). This result is concluded from the technical Lemma 3.3 which is a generalization of [16, Theorem 14.22]. In sequel, we give some necessary condition for inclusion of two generalized
Orlicz space (as a generalization of [20, Theorem 3 page 155]), and then prove that if the Young function $\Phi_2$ is not stronger than the other one $\Phi_1$, then $X^{\Phi_2} - X^{\Phi_1}$ is spaceable in $X^{\Phi_2}$. Finally, we give an abstract improvement of [5, Theorem 2.2]. To emphasize the capacity of the obtained result, we apply it to show that if $X$ is a solid Banach function space on $\Omega$ and $\inf\{\|x_E\|_X : E \in \mathcal{A}_0\} = 0$, then for each $1 \leq p, q < r$, the set $\{(f, g) \in X^p \times X^q : fg \notin X^r\}$ is spaceable in $X^p \times X^q$. As another application, we prove that the set of all bounded linear operators which are not positive semidefinite on a separable Hilbert space is spaceable. Moreover, it is shown that if $K$ is a two sided ideal cone in $B(\mathcal{H})$ and there exists a sequence of mutually disjoint subsets $\{J_n\}_{n \in \mathbb{N}}$ of $\mathbb{N}$ satisfying the condition $P_{J_n}K P_{J_n} \neq P_{J_n}B(\mathcal{H})P_{J_n}$ for all $n \in \mathbb{N}$, then $B(\mathcal{H}) - K$ is spaceable in $B(\mathcal{H})$, where $\mathcal{H}$ is a separable Hilbert space with an orthonormal basis $\{e_j\}_{j \in \mathbb{N}}$, and $P_{J_n}$ is the orthogonal projection on the closed linear span of $\{e_j\}_{j \in J_n}$.

2. Preliminaries

In sequel, $(\Omega, \mathcal{A}, \mu)$ is always a $\sigma$-finite measure space, and $\mathcal{M}_0(\Omega)$ is set of all $\mathcal{A}$-measurable complex-valued functions on $\Omega$.

A linear subspace $X$ of $\mathcal{M}_0(\Omega)$ equipped with a given norm $\| \cdot \|_X$ is called a Banach function space on $\Omega$ if $(X, \| \cdot \|_X)$ is a Banach space. It is called solid if for each $f \in X$ and $g \in \mathcal{M}_0(\Omega)$ we have $g \in X$ and $\|g\|_X \leq \|f\|_X$ whenever $|g| \leq |f|$ a.e.

A convex function $\Phi : [0, \infty) \to [0, \infty)$ is called a Young function if $\Phi(0) = \lim_{x \to 0} \Phi(x) = 0$ and $\lim_{x \to \infty} \Phi(x) = \infty$. Let $X$ be a Banach function space on $\Omega$. For each $f \in \mathcal{M}_0(\Omega)$ we put

$$\|f\|_\Phi := \inf \left\{ \lambda > 0 : \Phi\left(\frac{|f|}{\lambda}\right) \in X, \left\|\Phi\left(\frac{|f|}{\lambda}\right)\right\|_X \leq 1 \right\}.$$ (2.1)

Then, the set of all $f \in \mathcal{M}_0(\Omega)$ with $\|f\|_\Phi < \infty$ is denoted by $X^{\Phi}$.

As in [10, Theorem 4.11], $(X^\Phi, \| \cdot \|_\Phi)$ is a Banach function space on $\Omega$ (two functions in $X^\Phi$ which are equal almost everywhere is considered same). For each $p \geq 1$, the function $\Phi_0$ defined by $\Phi_0(x) := x^p$ for all $x \geq 0$, is a Young function. Then, we denote $X^p := X^{\Phi_0}$ and $\| \cdot \|_p := \| \cdot \|_{\Phi_0}$. In particular, if $X := L^1(\mu)$, then $X^\Phi = L^\Phi(\mu)$ and $X^p = L^p(\mu)$, the classical Orlicz and Lebesgue spaces.

3. Main Results

A subset $S$ of a Banach space $Y$ is called spaceable in $Y$ if $S \cup \{0\}$ contains a closed infinite-dimensional subspace of $Y$. In this section, first we study the spaceability of special subsets of $X^p$. As in [23], for each function $f$ in $\mathcal{M}_0(\Omega)$ we denote $E_f := \{x \in \Omega : f(x) \neq 0\}$.

Remark 3.1. Recall from [5] that a Banach function space $(\mathcal{E}, \| \cdot \|)$ on $\Omega$ is a PCS-space if for each sequence $(f_n)$ with $f_n \to f$ in $\mathcal{E}$, there is a subsequence $(f_{n_k})$ of $(f_n)$ such that $f_{n_k} \to f$ a.e. This property plays a key role in the
subject spaceability. For instance, see Theorem 3.2 below as a main result on this topic. A Banach function space $X$ on a σ-finite measure space $(\Omega, \mathcal{A}, \mu)$ is PCS-space if and only if the embedding of $X$ into $\mathcal{M}_0(\Omega)$ is continuous, where $\mathcal{M}_0(\Omega)$ is equipped with the topology of convergence in measure on finite measure subsets. If $X$ is a solid quasi-Banach function space on a σ-finite measure space then the embedding $X$ in $\mathcal{M}_0(\Omega)$ is always continuous, see [17, Proposition 2.2 (i)] for the finite measure case.

Next, we recall a result which was proved in [5, Theorem 2.2].

**Theorem 3.2.** Let $(E, \| \cdot \|)$ be a Banach function space on $\Omega$ and $B$ be a nonempty subset of $E$ such that:

1. $E$ is a PCS-space;
2. there is a constant $k > 0$ such that $\|f + g\| \geq k \|f\|$ for all $f, g \in E$ with $E_f \cap E_g = \emptyset$;
3. $B$ is a cone;
4. if $f, g \in E$ such that $f + g \in B$ and $E_f \cap E_g = \emptyset$ then $f, g \in B$;
5. there is a sequence $\{f_n\}_{n=1}^{\infty} \subseteq E - B$ such that for each distinct $m, n \in \mathbb{N}$, $E_{f_n} \cap E_{f_m} = \emptyset$.

Then, $E - B$ is spaceable in $E$.

For sequel, we need the next result which is a generalization of [16, Theorem 14.22]. The main idea for the proof comes from [16, Theorem 14.22] but details are different. The item (a) in this theorem is a more general version of the relation (α) in [5]. Denote

$$A_0 := \{E \in A : 0 < \mu(E) \text{ and } \chi_E \in X\}.$$ 

**Lemma 3.3.** Let $X$ be a solid Banach function space. Then, the followings are equivalent:

(a) $\inf \{\|\chi_E\|_X : E \in A_0\} = 0$.

(b) There exists a sequence $\{A_n\}_{n=1}^{\infty}$ in $A_0$ such that $A_n \cap A_m = \emptyset$ for all distinct $m, n \in \mathbb{N}$ and

$$0 < \|\chi_{A_n}\|_X \leq \frac{1}{2n}, \quad (n \in \mathbb{N}).$$

**Proof.** (b) $\Rightarrow$ (a): Let $\{A_n\}_{n=1}^{\infty}$ be a sequence in $A$ which satisfies in (b). Let $1 \leq p < q < \infty$. We claim that $X^p \not\subset X^q$. By the assumptions, we can write

$$\sum_{n=1}^{\infty} \left\| n \|\chi_{A_n}\|_X^{\frac{p}{q}} \cdot \chi_{A_n} \right\|_X \leq \sum_{n=1}^{\infty} n \|\chi_{A_n}\|_X^{1 - \frac{p}{q}} \leq \sum_{n=1}^{\infty} n(2^{\frac{1}{q}} - 1)n < \infty. \quad (3.1)$$

Set

$$f := \sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{p}}} \|\chi_{A_n}\|_X^{\frac{1}{q}} \cdot \chi_{A_n} \quad \text{and} \quad S_N := \sum_{n=1}^{N} \frac{1}{n^{\frac{1}{p}}} \|\chi_{A_n}\|_X^{\frac{1}{q}} \cdot \chi_{A_n}$$

for all $N \in \mathbb{N}$. By the relation (3.1) the sequence $\{S_N\}_{N=1}^{\infty}$ is Cauchy in $X$ and so it converges to some $g \in X$ with norm topology, because $X$ is complete.
Now, by Remark 3.1, there exists a subsequence of \( \{S_N^p\}_{N=1}^\infty \) that converges to \( g \) a.e. Therefore,
\[
g = \sum_{n=1}^\infty n\|X_{A_n}\|_X^{\frac{1}{p}} \cdot X_{A_n} = \left( \sum_{n=1}^\infty n^{\frac{1}{p}} \|X_{A_n}\|_X^{\frac{1}{p}} \cdot X_{A_n} \right)^p = f^p \text{ a.e.}
\]
This implies that \( |f|^p \in X \), and so \( f \in X^p \). On the other hand, in contrast, let \( f \in X^q \). Then, since \( X \) is solid we have
\[
\|f\|_q = \|f^q\|_X = \left\| \left( \sum_{n=1}^\infty n^{\frac{1}{p}} \|X_{A_n}\|_X^{\frac{1}{p}} \cdot X_{A_n} \right)^q \right\|_X \geq \left\| k^{\frac{q}{p}} \|X_{A_n}\|_X^{-1} \cdot X_{A_n} \right\|_X = k^{\frac{q}{p}}
\]
for all \( k \in \mathbb{N} \), and this implies that \( \|f\|_q = \infty \), a contradiction. Hence, \( f \in X^p - X^q \). Now, thanks to [23, Theorem 2.1] we have \( \inf\{\|X_E\|_X : E \in \mathcal{A}_0\} = 0 \).

(a) \( \Rightarrow \) (b): Let \( \inf\{\|X_E\|_X : E \in \mathcal{A}_0\} = 0 \). For each \( A \in \mathcal{A} \) put
\[
\mathcal{K}(A) := \inf\{\|X_B\|_X : B \in \mathcal{A}_0, B \subseteq A\}.
\]
Clearly,
1. if \( A_1, A_2 \in \mathcal{A} \) and \( A_1 \subseteq A_2 \), then \( \mathcal{K}(A_2) \leq \mathcal{K}(A_1) \), and
2. for each \( C, B \in \mathcal{A} \) with \( B \subseteq C \), if \( \mathcal{K}(B), \mathcal{K}(C-B) > 0 \), then \( \mathcal{K}(C) > 0 \).

Note that (2) holds since for each \( E \in \mathcal{A}_0 \), if \( E \subseteq C \), then \( \|X_E\|_X \geq \min\{\mathcal{K}(B), \mathcal{K}(C-B)\} \).

For each \( A \in \mathcal{A} \) we put
\[
\mathcal{K}'(A) := \sup\{\|X_B\|_X : B \in \mathcal{A}_0, B \subseteq A\}.
\]
Similar to the proof of [16, Theorem 14.22] with different details, one can prove that

- if \( C \in \mathcal{A} \) and \( \mathcal{K}(C) = 0 \), then for each \( \epsilon > 0 \) there exists \( A \in \mathcal{A}_0 \) such that \( A \subseteq C \), \( 0 < \|X_A\|_X < \min\{\epsilon, \mathcal{K}'(C)\} \) and \( \mathcal{K}(C-A) = 0 \).

Indeed, let \( \mathcal{K}(C) = 0 \). Then, there exists a set \( B \subseteq C \) such that \( 0 < \|X_B\|_X < \min\{\epsilon, \mathcal{K}'(C)\} \). If \( \mathcal{K}(C-B) = 0 \) we set \( A := B \). If \( \mathcal{K}(C-B) > 0 \), by (2) we have \( \mathcal{K}(B) = 0 \), and so there is a set \( D \in \mathcal{A}_0 \) such that \( D \subseteq B \) and \( 0 < \|X_D\|_X = \|X_B\|_X \). In this situation, because of (2) we have \( \mathcal{K}(D) = 0 \) or \( \mathcal{K}(B-D) = 0 \), and then by (1) it would be enough to set \( A := B - D \) or \( A := D \), respectively.

Now, since \( \inf\{\|X_E\|_X : E \in \mathcal{A}_0\} = 0 \), we have \( \mathcal{K}(\Omega) = 0 \). So, there exists \( A_1 \in \mathcal{A}_0 \) such that \( 0 < \|X_{A_1}\|_X < \min\{\frac{1}{2}, \mathcal{K}'(\Omega)\} \) and \( \mathcal{K}(\Omega - A_1) = 0 \). Setting \( C := \Omega - A_1 \) in the above fact, there exists \( A_2 \in \mathcal{A}_0 \) such that \( A_2 \subseteq \Omega - A_1 \),
\[
0 < \|X_{A_2}\|_X < \min\{\frac{1}{2^2}, \mathcal{K}'(\Omega - A_1)\},
\]
and \( \mathcal{K}(\Omega - (A_1 \cup A_2)) = \mathcal{K}((\Omega - A_1) - A_2) = 0 \). By continuing this method, the desired sequence in (b) is obtained. \( \square \)
Now, we can give one of the main results of this section.

**Theorem 3.4.** Let $X$ be a solid Banach function space and $\inf\{\|\chi_E\|_X : E \in \mathcal{A}_0\} = 0$. Also, assume that $\sup\{\|\chi_E\|_X : E \in \mathcal{A}_\infty\} < \infty$, where $\mathcal{A}_\infty := \{E \in \mathcal{A} : \chi_E \in X\}$. Then, for each $p \geq 1$, $X_{p\text{-}\text{strict}} := X^p - \bigcup_{p < q} X^q$ is spaceable in $X^p$.

**Proof.** We shall show that the conditions of Theorem 3.2 hold with $\mathcal{E} := X^p$ and $B = \bigcup_{p < q} X^q$. Note that since $\Omega \in \mathcal{A}_0$, for each $q > p$ we have $X^q \subset X^p$ thanks to [23, Theorem 2.4]. Clearly, $B$ is a cone because each $X^q$ is a linear space, and $X^p$ is a PCS-space by Remark 3.1. Also, the condition (2) in Theorem 3.2 holds since $X$ is solid. For the condition (4), let $f, g \in X^p$ with $E_f \cap E_g = \emptyset$ and $f + g \in B$. Then, there exists $q > p$ such that $f + g \in X^q$. We have

$$|f|^q, |g|^q \leq |f|^q + |g|^q = |f + g|^q \in X,$$

and this implies that $f, g \in X^q \subset B$. At the end, we show that the condition (5) in Theorem 3.2 hold. The main idea for the proof of this part comes from [5, Theorem 3.3]. Since $\inf\{\|\chi_E\|_X : E \in \mathcal{A}_0\} = 0$, by Lemma 3.3 there exists a sequence $\{A_n\}_{n=1}^\infty$ in $\mathcal{A}$ with pairwise disjoint terms such that $0 < \|\chi_{A_n}\|_X \leq \frac{1}{2^n}$ for all $n \in \mathbb{N}$. As in [5, Theorem 3.3], for each $n \in \mathbb{N}$, we choose a strictly increasing sequence $\{p_{n,k}\}_{k=1}^\infty$ of natural numbers such that $k \leq p_{n,k}$ for all $n, k \in \mathbb{N}$ and the elements of family $\{p_{n,k}\}_{k=1}^\infty : n \in \mathbb{N}$ are mutually disjoint. For each $n, k, m \in \mathbb{N}$ we put

$$\alpha_{n,k} := \frac{1}{(k \log(1 + k))^2\|\chi_{A_{p_{n,k}}}\|_X}$$

and $S_{n,m} := \sum_{k=1}^m \alpha_{n,k} \chi_{A_{p_{n,k}}}$. Since $\sum_{k=1}^\infty \|\alpha_{n,k} \chi_{A_{p_{n,k}}}\|_X = \sum_{k=1}^\infty \frac{1}{k \log(1 + k)} < \infty$, the sequence $\{|S_{n,m}|^p\}_{m=1}^\infty$ is Cauchy and so convergence in $X$ for all $n \in \mathbb{N}$. Now, we have

$$\lim_{m \to \infty} |S_{n,m}|^p = \sum_{k=1}^\infty \alpha_{n,k} \chi_{A_{p_{n,k}}}$$

in $X$ because $X$ is a PCS-space (see Remark 3.1). In particular, we have $f_n^p = \sum_{k=1}^\infty \alpha_{n,k} \chi_{A_{p_{n,k}}} \in X$, where

$$f_n := \sum_{k=1}^\infty \alpha_{n,k} \chi_{A_{p_{n,k}}}, \quad (n \in \mathbb{N}).$$

In fact, we have $\{f_n\}_{n=1}^\infty \subseteq X^p$ with $E_{f_n} \cap E_{f_m} = \emptyset$ for all distinct $m, n \in \mathbb{N}$. On the other hand, for each $q > p$, if $f_n \in X^q$, then

$$\|f_n\|_{X^q} = \left(\|f_n\|^q_{X^q}\right)^{\frac{1}{q}} = \left\| \sum_{k=1}^\infty \alpha_{n,k} \chi_{A_{p_{n,k}}} \right\|_{X^q}^{\frac{1}{q}}.$$
≥ \left\| \alpha_{n,k}^q \chi_{A_{p_n,k}} \right\|_{X}^{\frac{1}{q}}

≥ \left( \frac{2(\frac{2}{p} - 1)p_{n,k}}{k \tau \log(1 + k)^{\frac{2}{p}}} \right)^{\frac{1}{q}}

≥ \left( 2(\frac{2}{p} - 1)k \right)^{\frac{1}{q}}.

So, since \( k \in \mathbb{N} \) is arbitrary, we have \( \| f_n \|_{X^q} = \infty \), a contradiction. Therefore, \( \{ f_n \}_{n=1}^{\infty} \subseteq X^p_{\text{strict}} \) and the proof is complete. \( \square \)

Next, an extension of the main part of [20, Theorem 3 page 155] is proved.

Motivated by definition of a diffuse set for a measure (see [20, page 46]), we initiate the following concept. For each \( E \subseteq \Omega \), denote \( A_E := \{ A \subseteq E : A \in \mathcal{A} \} \).

**Definition 3.5.** A set \( E \in \mathcal{A} \) is called diffuse for a Banach function space \( X \) if \( \chi_E \in X \) and for each \( Y \in A_E \) and \( 0 \leq \alpha \leq \| \chi_Y \|_X \) there exists some \( F \in A_Y \) such that \( \| \chi_F \|_X = \alpha \).

The main idea for proof of the next result comes from [20, Theorem 3 page 155], but the details are different because the situation is more general.

**Theorem 3.6.** Let \( \Phi_1, \Phi_2 \) be two strictly increasing continuous Young functions. If there exists a diffuse set \( E \in \mathcal{A}_\infty \) for \( X \) with \( \mu(E) > 0 \), then the inclusion \( X^{\Phi_2} \subseteq X^{\Phi_1} \) implies that \( \Phi_1 \prec \Phi_2 \).

**Proof.** Let the assumptions hold and \( X^{\Phi_2} \subseteq X^{\Phi_1} \). In contrast, assume that \( \Phi_1 \nprec \Phi_2 \). Then, there exists an increasing sequence \( \{ a_n \}_{n=1}^{\infty} \) in \( (0, \infty) \) such that \( \lim_{n \to \infty} a_n = \infty \) and

\[
\Phi_1(a_n) > n^2 \Phi_2(n^2 a_n), \quad (n \in \mathbb{N}).
\]  

(3.2)

Since \( \sum_{n=1}^{\infty} \frac{\Phi_2(a_1)\| \chi_E \|_X}{2^n \Phi_2(n^2 a_n)} < \| \chi_E \|_X \), there exists \( E_0 \in \mathcal{A}_E \) such that

\[
\| \chi_{E_0} \|_X = \sum_{n=1}^{\infty} \frac{\Phi_2(a_1)\| \chi_E \|_X}{2^n \Phi_2(n^2 a_n)},
\]

because \( E \) is a diffuse set for \( X \). Inductively, one can find a pairwise disjoint sequence \( \{ E_n \}_{n=1}^{\infty} \) in \( \mathcal{A}_{E_0} \) such that

\[
\| \chi_{E_n} \|_X = \frac{\Phi_2(a_1)\| \chi_E \|_X}{2^n \Phi_2(n^2 a_n)}, \quad (n \in \mathbb{N}).
\]

(3.3)

So, setting \( f := \sum_{n=1}^{\infty} na_n \chi_{E_n} \), we have

\[
\sum_{n=1}^{\infty} \frac{\Phi_2(n^2 a_n)\| \chi_{E_n} \|_X}{2^n} = \sum_{n=1}^{\infty} \frac{\Phi_2(a_1)\| \chi_E \|_X}{2^n} = \Phi_2(a_1)\| \chi_E \|_X < \infty.
\]
This implies that the sequence
\[
\left( \sum_{n=1}^{k} \Phi_2(n^2a_n)\chi_{E_n} \right)_k
\]
is Cauchy and so convergent in \(X\). But, by Remark 3.1, the convergence point is \(\Phi_2(f) = \sum_{n=1}^{\infty} \Phi_2(n^2a_n)\chi_{E_n}\). So, \(f \in X^{\Phi_2}\).

On the other hand, let \(\alpha > 0\) be arbitrary. In contrast, let \(\Phi_1(\alpha f) \in X\).

Fix a number \(m \in \mathbb{N}\) such that \(\frac{1}{n} < \alpha\) for all \(n \geq m\). Then, thanks to the relations (3.2) and (3.3) we have
\[
\|\Phi_1(\alpha f)\|_X = \left\| \sum_{n=1}^{\infty} \Phi_1(\alpha na_n)\chi_{E_n} \right\|_X \\
\geq \Phi_1(\alpha ka_k)\|\chi_{E_k}\|_X \\
\geq \Phi_1(ak)\|\chi_{E_k}\|_X \\
\geq k\Phi_2(a_1)\|\chi_{E}\|_X.
\]
for all \(k \geq m\), and so \(\|\Phi_1(\alpha f)\|_X = \infty\), a contradiction. This shows that \(f \notin X^{\Phi_1}\), and the proof is complete. \(\square\)

**Corollary 3.7.** Under the assumptions of Theorem 3.6, if \(\Phi_1 \neq \Phi_2\), then \(X^{\Phi_2} - X^{\Phi_1}\) is spaceable in \(X^{\Phi_2}\).

**Proof.** Let for each \(n \in \mathbb{N}\), \(N_n\) be a strictly increasing sequence of natural numbers and \(\{N_n\}_{n=1}^{\infty}\) be a partition of \(\mathbb{N}\). Then, similar to the proof of Theorem 3.6 it would be routine to construct a sequence \((f_n)\) in \(X^{\Phi_2} - X^{\Phi_1}\) such that for each distinct \(m, n \in \mathbb{N}\), \(E_{f_n} \cap E_{f_m} = \emptyset\). Now, easily by Theorem 3.2 the statement is proved. \(\square\)

4. Some New Sufficient Conditions With Applications

In this section, first we give an abstract version of Theorem 3.2 and then present some applications regarding Cartesian product of \(X^p\) spaces and also the space of bounded linear operators on a Hilbert space. For this we need to initiate the next concept.

**Definition 4.1.** Let \(\mathcal{E}\) be a topological vector space. We say that a relation \(\sim\) on \(\mathcal{E}\) has property \((D)\) if the following conditions hold:

1. If \((x_n)\) is a sequence in \(\mathcal{E}\) such that \(x_n \sim x_m\) for all distinct index \(m, n\), then for each disjoint finite subsets \(A, B\) of \(\mathbb{N}\) we have
   \[
   \sum_{n \in A} \alpha_n x_n \sim \sum_{m \in B} \beta_m x_m,
   \]
   where \(\alpha_n\) and \(\beta_m\)'s are arbitrary scalars.
2. If a sequence \((x_n)\) converges to \(x\) in \(\mathcal{E}\) and for some \(y \in \mathcal{E}\), \(x_n \sim y\) for all \(n \in \mathbb{N}\), then \(x \sim y\).
We recall that a sequence \((x_n)\) in a Banach space \(E\) is called a basic sequence
if for each \(x\) in \(\text{span}\{x_1,x_2,\ldots,\}\), the closed linear span of \(\{x_1,x_2,\ldots,\}\), there are unique scalars \(\alpha_1,\alpha_2,\ldots\) such that \(x = \lim_{n} \sum_{k=1}^{n} \alpha_k x_k\) in \(E\). Note that, by [2, Proposition 1, Chapter II], \((x_n)\) is a basic sequence if and only if there is a constant \(k > 0\) such that for each \(m, n\) with \(m \geq n\) and each scalars \(\alpha_1,\ldots,\alpha_m, \left|\sum_{j=1}^{n} \alpha_j x_j\right| \leq k \left|\sum_{j=1}^{m} \alpha_j x_j\right|\). In this paper (as in [5]) a subset \(B\) of a vector space is called a cone if for each scalar \(c\), \(cB \subseteq B\).

**Theorem 4.2.** Let \((E, \| \cdot \|)\) be a Banach space, \(\sim\) be a relation on \(E\) with property \((D)\), and \(K\) be a nonempty subset of \(E\). Assume that:

1. there is a constant \(k > 0\) such that \(\|x + y\| \geq k \|x\|\) for all \(x, y \in E\) with \(x \sim y\);
2. \(K\) is a cone;
3. if \(x, y \in E\) such that \(x + y \in K\) and \(x \sim y\) then \(x, y \in K\);
4. there is an infinite sequence \(\{x_n\}_{n=1}^{\infty} \subseteq E - K\) such that for each distinct \(m, n \in \mathbb{N}\), \(x_m \sim x_n\).

Then, \(E - K\) is spaceable in \(E\).

**Proof.** The main idea of the proof comes from [5, Theorem 2.2]. Indeed, applying condition \((D)\) in Definition 4.1 and thanks to [2, Proposition 1, Chapter II] one can see that the sequence \((x_n)\) in assumption (4) is a basic sequence, and this shows that \((x_n)\) is linearly independent. Let \(0 \neq x \in \text{span}\{x_1,x_2,\ldots,\}\). Then, since by definition of basic sequences, there exist unique scalars \(\alpha_1,\alpha_2,\ldots\) such that \(x = \sum_{n=1}^{\infty} \alpha_n x_n\). Put \(N := \min\{n \in \mathbb{N} : \alpha_n \neq 0\}\). So, \(x = \alpha_N x_N + y\), where \(y := \lim_{n \to \infty} \sum_{n=N+1}^{m} \alpha_n x_n\). Again, applying both conditions in Definition 4.1 we have \(x_N \sim y\). In contrast, if \(x \in K\), then by the assumptions (3) and (2) we have \(x_N \in K\), a contradiction. Therefore, \((E - K) \cup \{0\}\) contains the closed infinite-dimensional space \(\text{span}\{x_1,x_2,\ldots,\}\), and this completes the proof. \(\Box\)

**Remark 4.3.** We mention that this theorem is a generalization of [5, Theorem 2.2] (Theorem 3.2). Just note that for each Banach function space \(X\), the relation \(\sim\) defined by

\[ f \sim g \text{ if and only if } E_f \cap E_g = \emptyset \]

for all \(f, g \in X\), has the property \((D)\).

Applying Theorem 4.2, we give the next result which could not be concluded from [5, Theorem 2.2].

**Theorem 4.4.** Let \(X\) be a solid Banach function space on \(\Omega\) and assume that \(\inf\{\|\chi_E\|_X : E \in \mathcal{A}_0\} = 0\). Then, for each \(1 \leq p, q < r\), the set \(\{(f,g) \in X^p \times X^q : fg \notin X^r\}\) is spaceable in \(X^p \times X^q\).
Proof. Let \( 1 \leq p, q < r \). By Lemma 3.3, there is a sequence \( \{A_n\}_{n=1}^{\infty} \) in \( A_0 \) with disjoint terms such that \( 0 < \|\chi_{A_n}\|_X \leq \frac{1}{n} \) for all \( n \in \mathbb{N} \). We define
\[
j := \sum_{n=1}^{\infty} \|\chi_{A_n}\|_X^{-1} \cdot \chi_{A_n}.
\]
Then, similar to the proof of Lemma 3.3, one can see that \( j \in X^p \cap X^q \). In contrast, if \( j^2 \in X^r \), then we have
\[
\|j^2\|_{X^r} = \|j^2r\|_X
= \left\| \left( \sum_{n=1}^{\infty} \|\chi_{A_n}\|_X^{-2} \cdot \chi_{A_n} \right)^r \right\|_X
\geq \left\| \|\chi_{A_k}\|_X^{-2} \cdot \chi_{A_k} \right\|_X \geq 2^k
\]
for all \( k \in \mathbb{N} \), a contradiction. This implies that setting
\[K := \{(f, g) \in X^p \times X^q : fg \in X^r\},\]
we have \((j, j) \in (X^p \times X^q) - K\). Put \( h := j^2 \). By the above relations, it would be standard to find a sequence \( (F_n) \) such that for each distinct \( m, n \in \mathbb{N} \), \( F_n \cap F_m = \emptyset \), and \( h \chi_{F_n} \notin X^r \). This implies that \((j \chi_{F_n}, j \chi_{F_n}) \in (X^p \times X^q) - K\) for all \( n \in \mathbb{N} \). Finally, note that the relation \( \sim \) defined by
\[
(f_1, g_1) \sim (f_2, g_2) \text{ if and only if } E_{f_1} \cap E_{f_2} = E_{g_1} \cap E_{g_2} = \emptyset
\]
for all \( f_1 \in X^p \) and \( g_i \in X^q \) \((i = 1, 2)\), satisfies the condition \((D)\). Applying Theorem 4.2, the proof is complete. \( \square \)

Let \( \mathcal{H} \) be a separable infinite dimensional Hilbert space and \( \{e_j\}_{j \in \mathbb{N}} \) be an orthonormal basis for \( \mathcal{H} \). For each non-empty subset \( J \subseteq \mathbb{N} \) we let \( P_J \) denote the orthogonal projection onto \( \overline{\text{span}} \{e_j\}_{j \in J} \). For each \( T, S \in B(\mathcal{H}) \), the space of all bounded linear operators on \( \mathcal{H} \), we say that \( T \sim S \) if there exist two disjoint subsets \( J_1, J_2 \subseteq \mathbb{N} \) such that \( P_{J_1}TP_{J_2} = T \) and \( P_{J_2}SP_{J_2} = S \). With these notations, we give the next lemma.

Lemma 4.5. The relation \( \sim \) on \( B(\mathcal{H}) \) defined above has the property \((D)\).

Proof. Suppose that \( \{T_n\}_{n \in \mathbb{N}} \) is a sequence in \( B(\mathcal{H}) \) such that for each distinct \( m, n \) we have \( T_n \sim T_m \). Let \( A := \{n_1, \ldots, n_k\} \) and \( B := \{m_1, \ldots, m_l\} \) be two disjoint finite subsets of \( \mathbb{N} \). Then, for each \( n \in A \) and \( m \in B \), there exist some disjoint subsets \( J_{(n, m)}, J'_{(n, m)} \subseteq \mathbb{N} \) such that
\[
P_{J_{(n, m)}} T_n P_{J_{(n, m)}} = T_n \quad \text{and} \quad P_{J'_{(n, m)}} T_m P_{J'_{(n, m)}} = T_m. \tag{4.1}
\]
By [6, Chapter 2, Section 8, Theorem 4], we have
\[
P_{\bigcap_{r=1}^{l} J_{(n, m_r)}} = P_{J_{(n, m_1)}} P_{J_{(n, m_2)}} \cdots P_{J_{(n, m_l)}} \tag{4.2}
\]
for all \( n \in A \). Then, (4.1) implies that
\[
P_{\bigcap_{r=1}^{l} J_{(n, m_r)}} T_n P_{\bigcap_{r=1}^{l} J_{(n, m_r)}} = T_n \tag{4.3}
\]
for all \( n \in A \). Indeed, by (4.2) we have
\[
P_{\cap_{r=1}^k J_{(n,m_r)}} P_n P_{\cap_{r=1}^k J_{(n,m_r)}} = P_{J_{(n,m_1)}} P_{J_{(n,m_{r-1})}} \cdots P_{J_{(n,m_1)}} P_{J_{(n,m_2)}} \cdots P_{J_{(n,m_l)}} = T_n
\]
for all \( n \in A \). Put
\[
E := \bigcup_{i=1}^k \bigcap_{r=1}^l J_{(n_i,m_r)}.
\]
Then, by [6, Chapter 2, Section 8, Corollary 5], we have
\[
P_E P_{\cap_{r=1}^k J_{(n_i,m_r)}} = P_{\cap_{r=1}^k J_{(n_i,m_r)}} P_E = P_{\cap_{r=1}^k J_{(n_i,m_r)}}
\]
for every \( i \in \{1, \ldots, k\} \). Because of (4.3),
\[
P_E T_{n_i} P_E = T_{n_i}
\]
for all \( i \in \{1, \ldots, k\} \). This implies that \( P_E (\sum_{i=1}^k \alpha_i T_{n_i}) P_E = \sum_{i=1}^k \alpha_i T_{n_i} \) for all \( \alpha_1, \ldots, \alpha_k \in \mathbb{C} \). Similarly, setting \( F := \bigcup_{r=1}^k \bigcap_{l=1}^{l_r} J_{(n_l,m_r)} \) we have
\[
P_F (\sum_{i=1}^{l_r} \beta_i T_{m_r}) P_F = \sum_{i=1}^{l_r} \beta_i T_{m_r}
\]
for all \( \beta_1, \ldots, \beta_{l_r} \in \mathbb{C} \). Now, since \( J_{(n_l,m_r)} \cap J_{(n_l,m_r)} = \emptyset \) for each \( i \in \{1, \ldots, k\} \) and \( r \in \{1, \ldots, l\} \), easily we have \( E \cap F = \emptyset \). Therefore, \( \sim \) satisfies the condition (1) in Definition 4.1. Next, suppose that \( S \in B(\mathcal{H}) \) and \( \{T_n\}_{n \in \mathbb{N}} \) is a sequence in \( B(\mathcal{H}) \) such that \( T_n \to T \), in operator norm, for some \( T \in B(\mathcal{H}) \), and \( T_n \sim S \) for all \( n \). Then, for each \( n \in \mathbb{N} \), there are disjoint subsets \( J_n, J'_n \subseteq \mathbb{N} \) such that
\[
P_{J_n} T_n P_{J_n} = T_n \quad \text{and} \quad P_{J'_n} S P_{J'_n} = S.
\] (4.4)
Again, by [6, Chapter 2, Section 8, Theorem 4] for each \( n \in \mathbb{N} \) we have
\[
P_{\bigcap_{m=1}^n J_m} P_{\bigcap_{m=1}^n J_m} = P_{J_m} \cdots P_{J_1} S P_{J_1} \cdots P_{J_n} = S.
\] (4.5)
Now, the sequence \( \{P_{\bigcap_{m=1}^n J_m}\}_{n \in \mathbb{N}} \) is a non-increasing sequence of orthogonal projections, hence by [6, Chapter 2, Section 8, Theorem 6],
\[
s - \lim_{n \to \infty} P_{\bigcap_{m=1}^n J_m} = P_{\bigcap_{m=1}^n J_m},
\]
where \( s - \lim \) means the limit in the strong operator topology. From (4.5) and thanks to [6, Chapter 2, Section 5, Theorem 2] we have
\[
S = s - \lim_{n \to \infty} (P_{\bigcap_{m=1}^n J_m} S P_{\bigcap_{m=1}^n J_m}) = (s - \lim_{n \to \infty} P_{\bigcap_{m=1}^n J_m}) S (s - \lim_{n \to \infty} P_{\bigcap_{m=1}^n J_m}) = P_{\bigcap_{m=1}^n J_m} S P_{\bigcap_{m=1}^n J_m}.
\]
By (4.4) and [6, Chapter 2, Section 8, Corollary 5] we have
\[
P_{\bigcup_{m=1}^\infty J_m} T_n P_{\bigcup_{m=1}^\infty J_m} = T_n
\]
for all \( n \in \mathbb{N} \). Letting \( n \to \infty \) we get
\[
P_{\bigcup_{m=1}^\infty J_m} T P_{\bigcup_{m=1}^\infty J_m} = T,
\]
and this completes the proof because \((\bigcup_{m=1}^{\infty} J_m) \cap (\bigcap_{m=1}^{\infty} J'_m) = \emptyset\).

\[ \Box \]

**Definition 4.6.** Let \( K \) be a cone in \( B(\mathcal{H}) \). We denote
\[
\tilde{K} := \bigcup_{J \subseteq \mathbb{N}} P_J K P_J,
\]
where \( P_J K P_J := \{ P_J T P_J : T \in K \} \).

Note that if \( K \) is a cone, then \( \tilde{K} \) is a cone as well. Moreover, \( P_J \tilde{K} P_J \subseteq \tilde{K} \) for all \( J \subseteq \mathbb{N} \), and in particular, \( K \subseteq \tilde{K} \).

**Theorem 4.7.** Let \( K \) be a cone in \( B(\mathcal{H}) \). If there exists a sequence of mutually disjoint subsets \( \{ J_n \}_{n \in \mathbb{N}} \) of \( \mathbb{N} \) satisfying that \( P_{J_n} \tilde{K} P_{J_n} \neq P_{J_n} B(\mathcal{H}) P_{J_n} \) for all \( n \in \mathbb{N} \), then \( B(\mathcal{H}) - \tilde{K} \) (and consequently \( B(\mathcal{H}) - K \)) is spaceable in \( B(\mathcal{H}) \).

The statement holds if we consider \( B_0(\mathcal{H}) \) instead of \( B(\mathcal{H}) \).

**Proof.** We show that the relation \( \sim \) defined before Lemma 4.5 satisfies the conditions in Theorem 4.2 regarding the cone \( \tilde{K} \). Suppose that \( T, S \in B(\mathcal{H}) \) with \( T \sim S \). Then, there exist disjoint subsets \( J, J' \subseteq \mathbb{N} \) such that \( P_J T P_J = T \) and \( P_{J'} S P_{J'} = S \). By disjointness of \( J \) and \( J' \), from [6, Chapter 2, Section 8, Theorem 2] we have \( P_J S P_J = P_J P_{J'} S P_{J'} P_J = 0 \). We get
\[
\|T + S\| \geq \|P_J\| \|T + S\| \|P_J\|
\geq \|P_J (T + S) P_J\|
= \|P_J T P_J\|
= \|T\|.
\]
This shows that the relation \( \sim \) satisfies the condition (1) of Theorem 4.2. Now, if in addition \( T + S \in \tilde{K} \), we have
\[
T = P_J T P_J
= P_J (T + S) P_J \in \tilde{K}.
\]
Similarly, \( S \in \tilde{K} \). So, the condition (2) in Theorem 4.2 holds with respect to the cone \( \tilde{K} \). Finally, consider the sequence \( \{ J_n \} \) of mutually disjoint subsets of \( \mathbb{N} \) which was described in the assumptions. So, for each \( n \) we can choose an operator \( T_n \in P_{J_n} B(\mathcal{H}) P_{J_n} - P_{J_n} K P_{J_n} \). Then, easily one can see that \( \{ T_n \}_{n \in \mathbb{N}} \subset B(\mathcal{H}) - \tilde{K} \) and for each distinct \( m, n \in \mathbb{N} \) we have
\[
T_n = P_{J_n} T_n P_{J_n} \sim P_{J_m} T_m P_{J_m} = T_m,
\]
and this completes the proof.

\[ \Box \]

**Corollary 4.8.** The set of all bounded linear operators on \( \mathcal{H} \) which are not positive-semidefinite, is spaceable in \( B(\mathcal{H}) \).

**Proof.** Let \( K \) be the set of all scalar multiplicative of positive semidefinite operators on \( \mathcal{H} \). Then, \( K \) is a cone and \( P_J K P_J \subseteq K \) for all \( J \subseteq \mathbb{N} \) and so \( \tilde{K} = K \). By some calculations one can see that the assumptions of Theorem 4.7 hold in this situation, and therefore \( B(\mathcal{H}) - K \) is spaceable. This implies that \( B(\mathcal{H}) - B_0(\mathcal{H}) \) is spaceable as well.

\[ \Box \]
The following result is directly concluded from Theorem 4.7.

**Corollary 4.9.** If $K$ is a two sided ideal cone in $B(H)$ and there exists a sequence of mutually disjoint subsets $\{J_n\}_{n \in \mathbb{N}}$ of $\mathbb{N}$ satisfying the condition $P_{J_n} K P_{J_n} \neq P_{J_n} B(H) P_{J_n}$ for all $n \in \mathbb{N}$, then $B(H) - K$ is spaceable in $B(H)$.

**Example 4.10.** $B_0(H)$, the space of all compact operators on $H$, is a two sided ideal cone in $B(H)$ which satisfies the requirements of Corollary 4.9. So, the set of all non-compact operators is a spaceable subset of $B(H)$.

**Remark 4.11.** By the same argument, $B_0(H) - (B_+ (H) \cap B_0(H))$ is spaceable in $B_0(H)$, where $B_+ (H)$ is the set of all positive semidefinite operators on $H$. One can also replace $B_0(H)$ by the real Banach space of all Hermitian operators on $H$.

**Remark 4.12.** Let $(B_1(H), \|\cdot\|_1)$ and $(B_2(H), \|\cdot\|_2)$ denote the Banach space of all trace-class operators equipped with the trace norm and the Banach space of all Hilbert-Schmidt operators equipped with the Hilbert-Schmidt norm, respectively. Since for every $T \in B(H)$, $S \in B_1(H)$ and $G \in B_2(H)$ we have $\|ST\|_1, \|TS\|_1 \leq \|T\| \|S\|_1$ and $\|TG\|_2, \|GT\|_2 \leq \|T\| \|G\|_2$, it is not hard to see that the similar argument as in the proof of Lemma 4.5 and Theorem 4.7 can be applied to deduce that $B_1(H) - (B_+ (H) \cap B_1(H))$ and $B_2(H) - (B_+ (H) \cap B_2(H))$ are spaceable in $B_1(H)$ and $B_2(H)$, respectively. Also, $B_0(H) - B_1(H)$ and $B_0(H) - B_2(H)$ are both spaceable in $B_0(H)$.

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