A DIVERGENCE THEOREM FOR PSEUDO-FINSLER SPACES

E. Minguzzi

Abstract

We study the divergence theorem on pseudo-Finsler spaces and obtain a completely
Finslerian version for spaces having a vanishing mean Cartan torsion. This result
helps to clarify the problem of energy-momentum conservation in Finsler gravity
theories.

1. Introduction

Two Finslerian divergence theorems have been discussed by H. Rund [1] and Z. Shen
[2,3]. While they both equate a certain volume integral on a domain of M with a surface
integral at the boundary, they have quite a different nature.

Let \( \pi: TM \to M \) be the tangent bundle, let \( E = TM \setminus 0 \) be the slit tangent bundle,
and let \( VE \) be the kernel of \( \pi_* \), namely the vertical tangent bundle. Let \( \mathcal{L}: TM \setminus 0 \to \mathbb{R} \) be the Finsler Lagrangian (for detailed definitions we refer the reader to Sec. 2),
and let \( g \) be its vertical Hessian, namely the Finsler metric. Rund’s theorem involves a vector field
\( Z: E \to TM \), a section \( s: M \to E \), and the Finslerian divergence of \( Z \) on \( E \) calculated
with the horizontal covariant derivative and pulled back to \( M \) using \( s \) (cf. Eq. (4)). It
has the drawback that the boundary term is not genuinely Finslerian, indeed, both the
normal to the hypersurface and the boundary form are deduced from the pullback metric
\( s^*g \). So there appear elements of Riemannian geometry.

The version by Shen is somewhat complementary [2,3, Theor. 2.4.2]. It is less Finslerian
for what concerns the vector field since it deals with a field \( X: M \to TM \) on the base, for
the computation of whose divergence the Finsler connection is not required. However,
the boundary term is genuinely Finslerian as it involves the notion of Finsler normal to
a hypersurface.

In this work we are going to elaborate a further version of the divergence theorem
in Finsler geometry. We shall first give a short derivation of Rund’s result and then we
shall show that for pseudo-Finsler spaces with vanishing mean Cartan torsion, \( I_a = 0 \),
it is possible to give a genuinely Finslerian divergence theorem in which both the vector
field and the boundary terms are Finslerian.

It must be recalled that by Deicke’s theorem [4,6] a Finsler space with vanishing mean
Cartan torsion is actually Riemannian. As a consequence, the mentioned result will be

Dipartimento di Matematica e Informatica “U. Dini”, Università degli Studi di Firenze,
Via S. Marta 3, I-50139 Firenze, Italy. e-mail: ettore.minguzzi@unifi.it
of interest only for pseudo-Finsler spaces of non-definite signature. It has been suggested by the author that Lorentz-Finsler spaces with zero mean Cartan torsion might be the appropriate objects of study in Finsler gravity theory [7–9], particularly because they have affine sphere indicatrixes and because, as in general relativity, they are uniquely determined by a spacetime volume form and a light cone distribution [10]. Therefore, it is expected that the results of this work could shed light on the problem of energy-momentum conservation in Finslerian extensions of general relativity.

2. Connections in pseudo-Finsler geometry

In this section we assume some familiarity with the notion of Finsler connection and of pullback connection [7,11–16]. Let us give some key coordinate expressions in order to fix terminology and notations. Let \( \{x^\mu\} \) be coordinates on a chart of \( M \) and let \( \{x^\mu, y^\nu\} \) be the induced coordinates on \( TM \). The Finsler Lagrangian is, by definition, positive homogeneous of degree two
\[
L(x, sy) = s^2 L(x, y), \quad \forall s > 0.
\]
Although we assumed that \( L \) is defined on the slit tangent bundle, this assumption can be relaxed, e.g. it could be defined on just a convex cone subbundle provided the next equations are evaluated on its domain, see [17,18] for a complete discussion. The Finsler metric is given by the Hessian
\[
g_{\mu\nu} = \frac{\partial^2 L}{\partial y^\mu \partial y^\nu}
\]
and is assumed non-degenerate. The Cartan torsion is
\[
C_{\alpha\beta\gamma} = \frac{1}{2} \frac{\partial}{\partial y^\alpha} g_{\beta\gamma}
\]
while the mean Cartan torsion is
\[
I_\gamma = g^{\alpha\beta} C_{\alpha\beta\gamma}.
\]

Example 1. A non-trivial example of affine sphere spacetime is
\[
\mathcal{L} = -\frac{2}{3\sqrt{a}} \left( \left( \frac{1}{2} dt + \sqrt{\frac{3}{2}} a(t) dz \right)^2 \right)^{1/4} \left( \left( \sqrt{\frac{3}{2}} dt - \frac{1}{2} a(t) dz \right)^2 - a^2(t)(dx^2 + dy^2) \right)^{3/4}
\]
which in the low velocity limit (with respect to an observer whose velocity field is \( u \propto \partial_t \)) gives the general relativistic Friedmann metric in the flat space section case (i.e. \( k = 0 \))
\[
g = -dt^2 + a(t)^2(dx^2 + dy^2 + dz^2).
\]
Here \( a(t) \) is the scale factor of the Universe. Many other examples of affine sphere spacetimes and a discussion can be found in [9]. There are really plenty of affine sphere spacetimes since there are plenty of cone structures and volume forms. If the light cones have ellipsoidal sections one recovers the usual Lorentzian spacetimes [10]. We remark that in what follows we do not assume a Lorentzian signature for the Finsler metric, though this is certainly the most interesting case.

A non-linear connection is a splitting of the tangent space \( TE = V E \oplus HE \) into vertical and horizontal bundles. A basis for the horizontal space is given by
\[
\left\{ \frac{\delta}{\delta x^\mu} \right\}, \quad \frac{\delta}{\delta x^\mu} = \frac{\partial}{\partial x^\mu} - N^\nu_{\mu}(x, y) \frac{\partial}{\partial y^\nu},
\]
where the coefficients \( N^\nu_{\mu}(x, y) \) define the non-linear connection and have suitable transformation properties under change of coordinates. The covariant derivative for the non-linear connection is defined as follows. Given a section \( s: U \to E, U \subset M \),
\[
D_\xi s^\alpha = \left( \frac{\partial s^\alpha}{\partial x^\mu} + N^\alpha_{\mu}(x, s(x)) \right) \xi^\mu.
\]
Usually one considers the non-linear connection determined by a spray as follows

\[ N^\mu_\alpha := G^\mu_\alpha := \frac{\partial G^\mu}{\partial y^\alpha}, \]

where

\[ 2G^\alpha(x, y) = g^\alpha\beta \left( \frac{\partial^2 \mathcal{L}}{\partial x^\gamma \partial y^\beta} y^\gamma - \frac{\partial \mathcal{L}}{\partial x^\beta} \right). \]

This will be our choice also.

The Finsler connections are splittings of the vertical bundle \( \pi: \dot{E} \rightarrow E \). In coordinates they are triplets of coefficients \( (N^\beta_\alpha, H^\beta_\alpha, V^\beta_\alpha) \) with suitable transformations properties. We shall assume that they are regular, namely \( H^\beta_\alpha = N^\alpha_\gamma \), and symmetric in the lower indices (torsionless). The Berwald, Cartan, Chern-Rund and Hashiguchi connections are of this type [7, 11, 12, 14–16].

Each Finsler connection determines two covariant derivatives \( \nabla^H \) and \( \nabla^V \) respectively being obtained from \( \nabla_X \) whenever \( X \) is the horizontal or the vertical lift of a vector \( X \in TM \). The horizontal covariant derivative \( \nabla^H \) is determined by local connection coefficients \( H^\mu_\alpha_\beta(x, y) \), in particular over a vector field \( Z: E \rightarrow \dot{E} \simeq TM \) it acts as follows

\[ (\nabla^H \alpha Z)^\beta = \frac{\delta Z^\beta}{\delta x^\alpha} + H^\beta_\alpha Z^\mu. \]

The Berwald connection is determined by \( V^\alpha_\beta_\gamma = 0 \) and

\[ H^\alpha_\mu_\nu := G^\alpha_\mu_\nu := \frac{\partial}{\partial y^\nu} G^\alpha_\mu, \]

Both the Chern-Rund and the Cartan connections are such that \( \nabla^H g = 0 \), hence

\[ H^\alpha_\beta_\gamma := \Gamma^\alpha_\beta_\gamma := \frac{1}{2} g^{\alpha\sigma} \left( \frac{\delta}{\delta x^\beta} g_{\sigma\gamma} + \frac{\delta}{\delta x^\gamma} g_{\sigma\beta} - \frac{\delta}{\delta x^\sigma} g_{\beta\gamma} \right), \tag{1} \]

however, for the former \( V^\alpha_\beta_\gamma = 0 \) while for the latter \( V^\alpha_\beta_\gamma = C^\alpha_\beta_\gamma \). We shall denote with \( \nabla^{HC} \) the horizontal covariant derivative for the Cartan (Chern-Rund) connection, and with \( \nabla^{HB} \) the horizontal covariant derivative for the Berwald connection. The difference \( G^\alpha_\beta_\gamma - \Gamma^\alpha_\beta_\gamma = L^\alpha_\beta_\gamma \) is the Landsberg tensor [3, 7].

3. The divergence theorem

Let \( s: M \rightarrow TM \{0 \) be a section, and let us consider the pullback metric \( s^* g \). Its components are \( (s^* g)_{\alpha\beta}(x) = g_{\alpha\beta}(x, s(x)) \). Let \( \nabla^s g \) be the Levi-Civita connection of \( s^* g \), and let \( \hat{\nabla} \) be the pullback of a Finsler connection. Its coefficients are [12 Sec. 4.1.1] [13 Eq. (3.7)]

\[ \hat{\nabla}^s \gamma_\alpha_\beta = H^\gamma_\alpha_\beta(x, s(x)) + V^\gamma_\alpha_\beta(x, s(x)) D_\alpha s^\mu, \]

where \( H^\gamma_\alpha_\beta \) and \( V^\gamma_\alpha_\beta \) are the coefficients of the horizontal and vertical covariant derivatives of the Finsler connection. Particularly simple are the coefficients of the pullback \( \hat{\nabla}^{CLR} \)
of the Chern-Rund connection, namely $\Gamma^\alpha_{\beta\gamma}(x, s(x))$. Since the connection coefficients on $M$ transform with the usual law, their difference is a tensor which we wish to calculate.

**Proposition 1.** For every $X, Y: M \to TM$

$$\nabla^* s^\alpha Y - \nabla_X^{\text{ChR}} Y = C(D_Y s, X) + C(D_X s, Y) - g(C(X, Y), Ds)^\alpha. \quad (2)$$

It is understood that in this formula $C$ stands for $s^* C$ and $\sharp: T^* M \to TM$ is the musical isomorphism given by the metric $s^* g$.

**Proof.** With obvious meaning of the notation, and lowering the upper index of the connection to the left, we have at $(x, s(x))$

$$2g_{\alpha\delta} \left[ (\nabla^* s^g)^\alpha_{\beta\gamma} - \Gamma^\alpha_{\beta\gamma}(x, s(x)) \right] = \frac{\partial}{\partial x^\beta} (s^g)_{\beta\gamma} + \frac{\partial}{\partial x^\gamma} (s^g)_{\delta\beta} - \frac{\partial}{\partial x^\delta} (s^g)_{\beta\gamma} - 2\Gamma_{\delta\beta\gamma}$$

$$= (\frac{\partial}{\partial x^\beta} g_{\beta\gamma} + \frac{\partial}{\partial x^\gamma} g_{\delta\beta} - \frac{\partial}{\partial x^\delta} g_{\beta\gamma})(x, s(x)) - 2\Gamma_{\delta\beta\gamma} + 2C_{\mu\delta\gamma} \frac{\partial s^\mu}{\partial x^\gamma}$$

$$+ 2C_{\mu\beta\gamma} \frac{\partial s^\mu}{\partial x^\delta} - 2C_{\mu\beta\delta} \frac{\partial s^\mu}{\partial x^\gamma}$$

$$= (\frac{\delta}{\delta x^\beta} g_{\beta\gamma} + \frac{\delta}{\delta x^\gamma} g_{\delta\beta} - \frac{\delta}{\delta x^\delta} g_{\beta\gamma})(x, s(x)) - 2\Gamma_{\delta\beta\gamma}$$

$$+ 2C_{\mu\delta\gamma} D_\beta s^\mu + 2C_{\mu\beta\delta} D_\gamma s^\mu - 2C_{\mu\beta\gamma} D_\delta s^\mu$$

Thus

$$\nabla^* s^g \Gamma^\alpha_{\beta\gamma}(x, s(x)) = C^\alpha_{\mu\beta} D_\gamma s^\mu + C^\alpha_{\mu\delta} D_\gamma s^\mu - C^\alpha_{\mu\beta\gamma} g^\delta D_\delta s^\mu$$

Equation (2) clarifies that the connection of the pullback metric is neither the pullback of the Chern-Rund connection, nor the pullback of the Cartan connection (known as Barthel connection [13, Theor. 1]).

**Corollary 1.** For every vector field $Y: M \to TM$,

$$\nabla^* s \cdot Y = \nabla_X^{\text{ChR}} \cdot Y + s^* I(D_Y s).$$

We need a lemma to remove the derivative of the pullback connection in favor of the derivative of the horizontal connection.

**Lemma 1.** Let $Z: TM \setminus 0 \to TM$ and let $s: M \to TM \setminus 0$, then

$$\nabla_X^{\text{ChR}} s^* Z = s^*(\nabla_X^H Z) + s^*(\frac{\partial Z^\mu}{\partial y^\beta}) D_X s^\beta \frac{\partial}{\partial x^\mu} \quad (3)$$
Proof. It follows from
\[
\frac{\partial}{\partial x^\gamma} Z^\mu(x, s(x)) = \frac{\partial Z^\mu}{\partial x^\gamma} + \frac{\partial Z^\mu}{\partial y^\beta} \frac{\partial s^\beta}{\partial x^\gamma} = \left( \frac{\partial Z^\mu}{\partial x^\gamma} - N^\beta_{\gamma \gamma} \frac{\partial Z^\mu}{\partial y^\beta} \right) + \frac{\partial Z^\mu}{\partial y^\beta} \left( \frac{\partial s^\beta}{\partial x^\gamma} + N^\beta_{\gamma \gamma} \right) = s^* \left( \frac{\partial Z^\mu}{\partial x^\gamma} \right) + s^* \left( \frac{\partial Z^\mu}{\partial y^\beta} \right) D_{\gamma s}^\beta,
\]
adding \( \Gamma^\nu_{\gamma \gamma}(x, s(x)) Z^\mu(x, s(x)) \) to both sides. \( \square \)

As a consequence we obtain the following expression for the divergence, see also [1, Eq. 3.10].

**Theorem 1.** Let \( Z: TM\setminus 0 \to TM \) and let \( s: M \to TM\setminus 0 \), then

\[
\nabla^s g \cdot s^* Z = s^* (\nabla^{HC} \cdot Z) + s^* I(D_{s^* Z} s) + s^* \left( \frac{\partial Z^\mu}{\partial y^\beta} \right) D_{\mu s^\beta} \tag{4}
\]

Integrating we recover Rund’s divergence theorem [1, Eq. (3.17)]

\[
\int_D \left( s^* (\nabla^{HC} \cdot Z) + s^* I(D_{s^* Z} s) + s^* \left( \frac{\partial Z^\mu}{\partial y^\beta} \right) D_{\mu s^\beta} \right) \mu(s) = \int_{\partial D} s^* g(s^* Z, n^R(s)) \nu^R(s) \tag{5}
\]

where \( \mu(s) = \sqrt{\det s^* g} \, dz \) is the canonical volume of the Riemannian metric \( s^* g \), \( \nu^R(s) \) is the canonical volume on the hypersurface \( \partial D \) associated to the induced metric \( s^* g \), and \( n^R(s) \) is the outward normal to \( \partial D \) according to \( s^* g \) (hence \( T_p \partial D = \ker g_{s(p)}(n^R(s), \cdot) \)). The letter \( R \) stands for “Rund’s”.

The problem with this formulation is, of course, that in this way one would get a boundary term in which the normal is determined according to \( s^* g \) and the area form according to the metric induced by \( s^* g \) on the boundary. It is really a (pseudo-)
Riemannian divergence theorem in disguise rather than a Finslerian divergence theorem.

We are now going to use Eq. (4) in order to get a genuinely Finslerian divergence theorem in integral form. By this we mean that the pullback metric and its Levi-Civita connection will not appear, and also the area form and normal at the boundary will be Finslerian.

Let \( \mu \) be a volume form on \( M \) and let \( S \subset M \) be a hypersurface with normal \( n \), meaning by this that at every \( p \in S \), \( T_p S = \ker g_n(n, \cdot) \). Here we are using the fact that whenever the manifold dimension is at least 3 the Legendre map \( \ell: v \mapsto g_n(v, \cdot) \) is a diffeomorphism of \( T_p M\setminus 0 \). The author provided a proof in [17, Theor. 6] and an independent and different one can be found in Ruzhansky and Sugimoto [19]. In 2 dimensions the signature of \( g \) can only be Riemannian or Lorentzian. In the former case it is well known that \( \ell \) is a diffeomorphism, in the latter case it is not difficult to show that \( \ell \) is surjective but not necessarily injective. So in more than two dimensions the normal exists and is unique up to scaling, while in 2 dimensions the normal exists but is non-unique, as there can be many with different directions. Observe that the section
s does not enter the definition of $n$, so this normal is different from Rund’s $n^R(s)$. Let $X$ be a field transverse to $S$, then the volume form induced on $S$ is

$$\nu = \frac{1}{-g_n(n, X)} i_X \mu, \quad (6)$$

and is evaluated only on vectors tangent to $S$. It is easy to show that it is independent of the choice of transverse field $X$ and depends only on the scale of $n$. If the signature of $g$ is Lorentzian, namely $(-, +, \cdots, +)$, and $S$ is spacelike the scale can be fixed with $g_n(n, n) = -1$, in which case $\nu_S = i^n \mu$.

Let $D$ be a domain such that $\partial D = S$. By the divergence theorem

$$\int_D \text{div}_\mu X \mu = \int_D d i_X \mu = \int_S i_X \mu = - \int_S g_n(n, X) \nu.$$ 

Now, if the mean Cartan torsion $I_\alpha := g^{\mu \nu} C_{\mu \nu \alpha} = \frac{1}{2} \frac{\partial}{\partial y^\alpha} \log |\det g_{\mu \nu}|$

vanishes, as the determinant does not depend on the fiber variable, we can define on $M$ the usual natural volume form $\mu = \sqrt{|\det g|} dx = \sqrt{|\det s^* g|} dx$, thus using Theor. 1 we arrive at the next result.

**Theorem 2.** Let $(M, \mathcal{L})$ be a pseudo-Finsler space for which the mean Cartan torsion vanishes. Let $Z : T M \setminus 0 \to T M$ and let $s : M \to T M \setminus 0$, then

$$\int_D \left\{ s^*(\nabla^{HC} \cdot Z) + s^*(\frac{\partial Z^\mu}{\partial y^\beta}) D_\mu s^\beta \right\} \mu = - \int_{\partial D} g_n(n, s^* Z) \nu. \quad (7)$$

The normal $n$ and the form $\nu$ do not depend on the section $s$.

**Remark 4.** This theorem does not follow from Rund’s divergence theorem [1, Eq. (3.17)] (see Eq. (5)) by setting $I_\alpha = 0$ since the normal and induced volume form have been obtained following a rather different geometrical argument. Observe that in Rund’s theorem the boundary term would still depend on the section, so it is interesting that it can be replaced by the more transparent form given by Theorem 2.

**Remark 5.** Apparently the theorem privileges the Chern-Rund or the Cartan connections but it is not really so. In fact, in the divergence $\nabla^{HC} \cdot Z$ the connection coefficients enter only in the combination [7, Eq. (59)] $\Gamma^n = \frac{1}{2} \frac{\partial}{\partial x} \ln \sqrt{|\det g|}$. Recall that $G^\mu_{\alpha \beta} = \Gamma^\alpha_{\alpha \beta} + L^\alpha_{\alpha \beta}$. We have shown [7, Eq. (52)] that whenever $I_\alpha = 0$ we have $J_\alpha := L^\alpha_{\alpha \mu} = 0$, thus $G^\mu_{\alpha \mu} = \Gamma^\mu_{\alpha \mu}$ and we can replace $\nabla^{HC} \cdot Z$ with $\nabla^{HB} \cdot Z$.

5.1. **Symmetry implies conservation**

Let us suppose that $Z$ is a vertical gradient, $Z_\gamma = \frac{\partial}{\partial y^\gamma} f$, $f : E \to \mathbb{R}$, and divergenceless in a horizontal Finslerian sense, namely $\nabla^{HC} \cdot Z = 0$. We want to show that any pregeodesic Killing vector field $s$ implies a conserved quantity.
If $s$ is a Finslerian Killing vector \cite{20, 21}
\begin{equation}
    g_{\delta\gamma} \nabla^\gamma_{\gamma} s^\delta + g_{\gamma\beta} \nabla^\beta_{\delta} s^\gamma + 2y^\beta (\nabla^\beta_{\gamma} s^\mu) C_{\gamma\delta\mu} = 0.
\end{equation}
which evaluated at the support vector $s$, and setting $D^\gamma = g^{\alpha\beta}(x, s(x))D_\alpha$, gives
\begin{equation}
    D^\gamma s^\delta + D^\delta s^\gamma + 2(D_s s^\mu) C_{\gamma\delta\mu} = 0.
\end{equation}

We can write
\begin{equation}
    s^*(\partial Z^\mu / \partial y^\delta) D_\mu s^\delta = s^*(g^{\mu\nu} \partial Z^\nu / \partial y^\delta - 2C_{\gamma\delta} Z^\nu) D_\mu s^\delta = s^*(\partial Z^\gamma / \partial y^\delta - 2C_{\gamma\delta} Z^\nu) C_{\gamma\delta\mu}(D_s s^\mu)
\end{equation}

If $s$ is pregeodesic, $D_s s \propto s$, this term vanishes and so, if $(M, \mathcal{Z})$ is a globally hyperbolic spacetime, by Theor. 2 the quantity
\begin{equation}
    E = \int_S g_n(n, s^* Z) \nu,
\end{equation}
is conserved, namely independent of the Cauchy hypersurface $S$, where $n$ is the normal to $S$ and $\nu$ is the induced volume. As a typical application, $Z$ will be the energy-momentum current, $s$ will represent an observer as a future-directed timelike field, and $E$ will be the total energy content of spacetime according to observer $s$.

If $s$ is normalized, $g_s(s, s) = -1$, the geodesic assumption is superfluous, indeed from Eq. (8) we obtain $2g_s(D_s s, \cdot) = -d(g_s(s, s)) = 0$, thus $s$ is geodesic.

The problem of energy-momentum conservation in Finsler gravity theories is notoriously difficult \cite{22–26}. Here we have given sufficient conditions on $Z$ that imply the existence of a conserved energy.

The divergence of $Z$ can be elaborated as follows, see \cite[Eq. (16)]{7}
\begin{equation}
    \nabla^\gamma_{\nu} Z^\mu = g^{\mu\nu} \nabla^\gamma_{\mu} Z^\nu = g^{\mu\nu}(\partial f / \partial x^\mu - \Gamma^\alpha_{\nu\mu} \partial f / \partial y^\alpha + I^\alpha_{\nu\mu} \partial f / \partial y^\alpha) = g^{\mu\nu}(\partial f / \partial y^\nu \partial f / \partial x^\mu + J^\alpha_{\nu\mu} \partial f / \partial y^\alpha)
\end{equation}

Since we assume $I^\alpha = 0$ which implies $J^\alpha = 0$ by \cite[Eq. (52)]{7}, the divergenceless condition can be equivalently written $\partial / \partial y^\nu (g^{\mu\nu} \partial f / \partial x^\mu) = 0$. In conclusion, the conditions on the field $Z$ can be equivalently expressed through one of the following conditions on $f$: (a) the vertical divergence of the horizontal gradient of $f$ vanishes; (b) the horizontal divergence of the vertical gradient of $f$ vanishes.

6. Conclusions

We have discussed the advantages and drawbacks of the known Finslerian divergence theorems. For pseudo-Finsler spaces having a vanishing mean Cartan torsion we have
provided a divergence theorem which might be regarded as genuinely Finslerian as it
deals with a vector field that might depend on the fiber coordinates, and a boundary
integral which involves the Finslerian normal and the Finslerian induced volume form.

In the last section we have shown how to use this theorem in order to construct
conserved quantities in pseudo-Finsler space. The conditions placed on the energy-
momentum current $Z$ might help to select the correct dynamical field equations for
Finsler gravity theory.

Acknowledgments

A preprint version of this work appeared in arXiv:1508.06053.

REFERENCES

[1] H. Rund: A divergence theorem for Finsler metrics, *Monatsh. Math.* **79**, 233 (1975).
[2] Z. Shen: On Finsler geometry of submanifolds, *Math. Ann.* **311**, 549 (1998).
[3] Z. Shen: *Lectures on Finsler geometry*, World Scientific, Singapore, 2001.
[4] A. Deicke: Über die Finsler-Räume mit $A_i = 0$, *Arch. Math.* **4**, 45 (1953).
[5] F. Brickell: A new proof of Deicke's theorem on homogeneous functions, *Proc. Amer.
Math. Soc.* **16**, 190 (1965).
[6] D. Bao, S.-S. Chern, and Z. Shen: *An Introduction to Riemann-Finsler Geometry*,
Springer-Verlag, New York, 2000.
[7] E. Minguzzi: The connections of pseudo-Finsler spaces, *Int. J. Geom. Meth. Mod. Phys.*
**11**, 1460025 (2014), *Erratum ibid* 12 (2015) 1592001. arXiv:1405.0645.
[8] E. Minguzzi: How many futures on Finsler spacetime? *J. of Phys.: Conf. Ser.* **626**, 012029 (2015), Contribution to the proceedings of the conference 'DICE14, Space-Time-Matter-Quantum Mechanics...news on missing links', Castello Pasquini, Castiglioncello (Italy) September 15 - 19, 2014.
[9] E. Minguzzi: Affine sphere spacetimes which satisfy the relativity principle, *Phys. Rev. D*
**95**, 024019 (2017), arXiv:1702.06745.
[10] E. Minguzzi: Affine sphere relativity, *Commun. Math. Phys.* **350**, 749 (2017),
arXiv:1702.06739.
[11] M. Matsumoto: *Foundations of Finsler Geometry and special Finsler Spaces*, Kaseisha
Press, Tokio, 1986.
[12] P. L. Antonelli, R. S. Ingarden, and M. Matsumoto: *The Theory of Sprays and Finsler
Spaces with Applications in Physics and Biology*, Springer Science+Business Media, Dor-
drecht, 1993.
[13] R. S. Ingarden and M. Matsumoto: On the 1953 Barthel connection of a Finsler space and
its mathematical and physical interpretation, *Rep. Math. Phys.* **32**, 35 (1993).
[14] M. Anastasiei: Finsler connections in generalized Lagrange spaces, *Balkan J. Geom. Appl.*
**1**, 1 (1996).
[15] A. Bejancu and H. R. Farran: *Geometry of pseudo-Finsler submanifolds*, Kluwer Academic
Publishers, Dordrecht, 2000.
[16] J. Szilasi, R. L. Lovas, and D. Cs. Kertesz: *Connections, sprays and Finsler structures*, World Scientific, London, 2014.

[17] E. Minguzzi: Light cones in Finsler spacetime, *Commun. Math. Phys.* 334, 1529 (2015), arXiv:1403.7060.

[18] E. Minguzzi: An equivalence of Finslerian relativistic theories, *Rep. Math. Phys.* 77, 45 (2016), arXiv:1412.4228.

[19] M. Ruzhansky and M. Sugimoto: On global inversion of homogeneous maps, *Bull. Math. Sci.* 5, 13 (2015).

[20] M. S. Knebelman: Collineations and Motions in Generalized Spaces, *Amer. J. Math.* 51, 527 (1929).

[21] H. Rund: *The differential geometry of Finsler spaces* Springer-Verlag, Berlin, 1959.

[22] H. Rund: Über Finslersche Raume mit speziellen Krümmungseigenschaften, *Monatsh. Math.* 66, 241 (1962).

[23] H. Ishikawa: Einstein equation in lifted Finsler spaces, *Il Nuovo Cimento* 56, 252 (1980).

[24] S. Ikeda: On the conservation laws in the theory of fields in Finsler spaces, *J. Math. Phys.* 22, 1211 (1981).

[25] M. Anastasiei: Conservation laws in the \{V, H\}-bundles of general relativity, *Tensor, N. S.* 46, 323 (1987).

[26] S. F. Rutz: Symmetry in Finsler spaces, in *Finsler geometry (Seattle, WA, 1995)*, Vol. 196 of *Contemp. Math.* Amer. Math. Soc., Providence, RI, 1996, p. 289.