Prediction of short time qubit readout via measurement of the next quantum jump of a coupled damped driven harmonic oscillator

Massimo Porrati\textsuperscript{a} and Seth Putterman\textsuperscript{b}

\textsuperscript{a} CCPP, Department of Physics, NYU, 726 Broadway, New York NY 10003, USA
\textsuperscript{b} Department of Physics and Astronomy University of California Los Angeles, CA 90095-1547 USA

Abstract

The dynamics of the next quantum jump for a qubit [two level system] coupled to a readout resonator [damped driven harmonic oscillator] is calculated. A quantum mechanical treatment of readout resonator reveals non exponential short time behavior which could facilitate detection of the state of the qubit faster than the resonator lifetime.
A quantum system that is both driven and observed will execute deterministic changes in the amplitudes of occupation of its various levels $|i\rangle$ that are interrupted by quantum jumps. In order to measure the time $t_j$ between successive quantum jump one has to be able to measure that a jump has not happened for times $t < t_j$. For an atom driven to fluorescence by a laser a measurement of the next quantum jump implies the ability to determine that no fluorescent photon have been emitted during the interval between jumps. Such a measurement does not imply that the quantum system is closed and unitary during this interval. Instead, the ability to make such a null measurement makes a small irreversible, though phase coherent change, in the wave function. This weak, non projective measurement dramatically changes the temporal dynamics of the quantum system. In particular one can observe long periods of intermittency $[1–4]$ where the phase coherence enables the reversal of a quantum jump before it occurs $[5]$.

Recently the next quantum jump has been measured for a few level transmon qubit that is dispersively coupled to a read-out resonator. For the purpose of our analysis the transmon is to be thought of as an atom with discrete levels and the resonator to which it is coupled is approximated as a damped driven quantized harmonic oscillator. In our analysis the resonator is not treated as a classical measuring apparatus. Instead the resonator and atom are calculated as a coupled quantum system $[4]$. A key result of this approach is that for times short compared to the decay time $1/\kappa$ of the cavity the probability $W(t)$ that a jump has not occurred for a time $t$ is not distributed as an exponential or Poisson distribution but as

$$W(t) \sim \exp(-\bar{n}\kappa^3 t^3/12),$$  \hspace{1cm} (1) 

where $\bar{n}$ is a measure of the strength of the drive. ($\bar{n}$ is the occupation number that characterizes the steady state of the damped driven oscillator when treated as a classical system.) The probability that the next jump will take place in the interval $[t, t + dt]$ is:

$$D(t) = -\frac{dW}{dt} dt.$$  \hspace{1cm} (2) 

In the limit of large $\bar{n}$ $[1,2]$ imply that the next quantum jump can occur on a time scale shorter than the lifetime, $1/\kappa$, of the resonator. To the extent that the resonator is a proxy for the qubit, a measurement of the next jump yields information on the state of the qubit on this short timescale. We also calculate how the coupling to the resonator determines the effective

$1$ Including the resonator in the quantum system has been used often in the theory of quantum measurement by photodetectors. While we are aware of applications to quadrature, homodyne and heterodyne measurements, we are not aware of it being applied to either next-photon detection or to the computation of lifetimes of atomic states from first principles. A few references particularly relevant to our work are the reviews $[6,7]$ and refs. $[8,9]$. 

1
lifetime of the excited states of the atom. We work in the limit of perfect detection, in which the damping coefficient, $\kappa$, is due only to measurement. We also assume the validity of the dispersive approximation [10].

The system is described by the amplitudes $C_{G,n}; C_{B,n}$ for the resonator to be in level $n$ and the atom to be in its ground state $G$ or excited state $B$, subject to the condition that there has been no quantum jump between a reset at $t = 0$ and time $t$. The probability that the next jump has not occurred during an interval of time $t'$ is then

$$W(t) = \sum_{n=0}^{\infty} \left[ |C_{G,n}(t)|^2 + |C_{B,n}(t)|^2 \right].$$  \hspace{1cm} (3)$$

The temporal response [1] already appears in the quantum dynamics for the next quantum jump of the driven damped resonator by itself. In this case we consider the atom in state $G$ and the resonator being driven at its resonant frequency $\omega_C$ when the atom is in its ground state. In the rotating wave approximation the equation of motion for the resonator amplitudes is:

$$\frac{dC_{G,n}}{dt} = \Gamma \sqrt{n} C_{G,n-1} - \sqrt{n+1} C_{G,n} - \frac{\kappa n}{2} C_{G,n},$$  \hspace{1cm} (4)$$

where the externally imposed drive is $\Gamma = \kappa \sqrt{\bar{n}}/2$. In the absence of damping, $\kappa$, the initial state $C_{G,n}(0) = 1$ is lifted by $\Gamma$ to higher levels and the norm is preserved. But with damping the norm of the wave function –projected onto the space that has not jumped– is decreasing. This is contained in the exact solution to (4):

$$C_{G,n}(t) = e^{\beta(t)} \frac{\alpha(t)^n}{\sqrt{n!}},$$  \hspace{1cm} (5)$$

$$\beta(t) = -\frac{\kappa}{2} \bar{n} \left[ t + \frac{2}{\kappa} \left( e^{-\kappa t/2} - 1 \right) \right];$$  \hspace{1cm} (6)$$

$$\frac{d\beta}{dt} = -\Gamma \alpha(t),$$  \hspace{1cm} (7)$$

$$W(t) = \exp[\beta + \beta^* + |\alpha|^2].$$  \hspace{1cm} (8)$$

This exact solution is the exponential of an exponential. In the limit of small times $W(t)$ is given by (1), and the average time between the initial condition and the first quantum jump between some level of the resonator is:

$$t_j = \int_0^{\infty} t \frac{dW}{dt} \, dt.$$

At short times $\bar{t}_j \sim a_0(3/\kappa \Gamma^2)^{1/3}$ where $a_0 = \Gamma(1/3)/3$. If $\bar{n} \gg 1$, $\kappa \bar{t}_j < 1$ and the dynamics of the next quantum jump is dominated by the non-exponential behavior in [1]. For $\kappa t > 1$, $W \to W_0 \exp(-\kappa \bar{n} t)$ which is exponentially distributed.
If the drive frequency \( \omega_D \) is detuned from the natural resonant frequency \( \omega_C \) so that, \( \omega_D = \omega_C - |\chi| \), then the RWA yields the equations of motion of the resonator when the atom is in \( G \):

\[
\frac{dC_{G,n}}{dt} = i n|\chi| C_{G,n} + \frac{\kappa \sqrt{n}}{2} [\sqrt{n}C_{G,n-1} - \sqrt{n+1}C_{G,n}] - \frac{n\kappa}{2} C_{G,n}.
\] (10)

The solution is given by (7) and:

\[
\alpha = \frac{i\Gamma}{i(\kappa/2) + \chi} \{ 1 - \exp[-(\kappa/2)t + i\chi t] \},
\] (11)

\[
\frac{dW}{dt} = -\kappa \alpha^* W(t).
\]

If \( \kappa/\chi \sim 1 \) the detuning is irrelevant, so the new interesting limit is \( \kappa/\chi \ll 1 \). If in addition \( \kappa t \gg 1 \):

\[
W(t) \sim \exp\left(-\Gamma^2/\chi^2 - \Gamma^2\kappa t/\chi^2 \right),
\] (12)

which is the exponential in time decay typical of an excited quantum system with discrete levels.

A large detuning is one of the conditions under which a dispersive Hamiltonian approximates the photon-cavity dynamics [10] so this is anyway the regime of greatest physical interest to us. For large dispersion and short time [ \( \kappa t < 1 \)]:

\[
W(t) \sim \exp\left[-\frac{\kappa^3 n}{2\chi^2} \left( t - \frac{\sin \chi t}{\chi} \right) \right].
\] (13)

When additionally \( \chi t < 1 \) this again gives the distribution of dark times in Eq. (1). The percentage of jumps to happen during the time when Eq. (1) applies is

\[
\frac{\kappa^3 n}{12\chi^3}.
\]

When the resonator is coupled to an atom there is the possibility of Rabi flopping at a frequency \( \Omega \) between the ground state \( G \) and the excited state \( B \). For the dispersively coupled system the resonators natural frequency depends upon the state of the atom. When the atom is in \( G \) the resonant frequency of the oscillator is \( \omega_C \) and when the atom is in \( B \) the resonant frequency is \( \omega_C - |\chi| \). Following the experimental arrangement [5] the external drive is tuned to the upper level [or bright level] resonant frequency so that: \( \omega_D = \omega_C - |\chi| \). In this case the equations that give the amplitude for the next quantum jump are:

\[
\frac{dC_{B,n}}{dt} = i\Omega C_{G,n} + \Gamma [\sqrt{n}C_{B,n-1} - \sqrt{n+1}C_{B,n}] - \frac{n\kappa}{2} C_{B,n}
\] (14)

\[
\frac{dC_{G,n}}{dt} = i\Omega^* C_{B,n} + \left[ \frac{dC_{G,n}}{dt} \right]_R
\] (15)

where \( [dC_{G,n}/dt]_R \) is given by the rhs of (10). A closed form approximation of the solution to (15) can be obtained via multi-time scale analysis in the limit of large dispersion. In the
limit where $\Gamma^2/\chi^2 \ll 1$, $|C_{G,2}/C_{G,1}| \ll 1$ and we neglect $C_{G,n}$ for $n > 1$. This leads to the next jump equations:

\[
\begin{align*}
\frac{dC_{B,0}}{dt} &= i\Omega C_{G,0} - \Gamma C_{B,1}, \\
\frac{dC_{G,0}}{dt} &= i\Omega^* C_{B,0} - \Gamma C_{B,1}, \\
\frac{dC_{G,1}}{dt} &= (i\chi - \kappa/2)C_{G,1} + \Gamma C_{G,0}.
\end{align*}
\]

These equations can be closed by noting that:

\[C_{B,1}(t) = \int ds \frac{dC_{B,0}}{ds} \alpha(t-s) \exp[\beta(t-s)].\]  

More generally in the multiscale approximation the occupation of the excited resonator levels $|B,n\rangle$ is driven by the occupation of $|B,0\rangle$:

\[|B(t)\rangle = \sum C_{B,n}(t)|B,n\rangle = \int ds \frac{dC_{B,0}}{ds} \exp[\alpha(t-s)a^\dagger + \beta(t-s)]|0\rangle.\]

Using (6) for $\beta$ in combination with the saddle point method which can be applied when $\bar{n} \gg 1$, yields: $C_{B,1} = \sqrt{2/\pi}C_{B,0}$. The eigenvalues of (16-18) are determined by the solution to the cubic equation:

\[\lambda(\lambda + \beta_B/2)(\lambda - i\chi + \kappa/2) + \Omega^2(\lambda - i\chi + \kappa/2) + \Gamma^2(\lambda + \beta_B/2) = 0,\]

where $\beta_B/2 = \sqrt{2/\pi}\Gamma$. For large dispersion the next jump can come from either of the transitions $|G,1\rangle \to |G,0\rangle$ or $|B,1\rangle \to |B,0\rangle$. In order that the relative probability of a jump originating from the excited $B$ state be large and furthermore take place on a long time scale we must require that

\[\beta_B^2/4 > \Omega^2 > \kappa\beta_B\Gamma^2/4\chi^2.\]

In this limit the eigenvalues are approximately:

\[\lambda_1 = -\beta_B/2; \quad \lambda_2 = -\gamma; \quad \lambda_3 = i\chi - \kappa/2\]

where $\gamma = 2|\Omega|^2/\beta_B$. The coefficient that controls the effective lifetime of the upper level of an ideal qubit is determined by the coupling to the resonator drive. The solution to (16,21) displays motion on two well separated time scales: the short time scale $2/\beta_B$ and the long time scale $1/\gamma$. If the next jump has not occurred on the short time scale then there is a lull and a warning that the jump will take place on the long time scale $1/\gamma$. The time to the next jump for the 2 level system coupled to the cavity depends on the norm of the system (3):

\[W(t) = \langle B(t)|B(t)\rangle + |C_G(t)|^2.\]
For large $\bar{n}$ there is again a regime of time-scales with non-exponential behavior, this time for the 2 level system. When $\bar{n}^{1/6} > \Gamma t > 1$, the rate of the next quantum jump becomes:

$$-\frac{dW}{dt} = 2\gamma W - 2\gamma \exp \left[ -\bar{n}(\kappa t)^3/12 \right].$$

(25)

Figure 1: The graph of the logarithmic norm of the system vs $\tau = \kappa t$: $\frac{\log(W(\tau))}{\bar{n}} = -\tau + 2\alpha(\tau)/\sqrt{\bar{n}} + \alpha^2(\tau)/\bar{n}$. It shows that the time evolution changes from $\tau^3$ to $3 - \tau$ at times $t = O(1/\kappa)$. A precise determination of the short-time behavior of the norm and the next quantum jump rate is essential for qubit readouts at times $t \lesssim 1/\kappa$.

Exponential time evolution of continuously observed systems appeared in previous work on quantum measurement theory and experiment of cavity-atom systems, see especially [11,12]. We are not aware of previous works exhibiting time evolution $O[\exp(t^3)]$ as in our eq. (25). We
propose that this short time behavior appears because we evaluate the amplitude for the next jump as compared to taking an average over the occupation levels of the resonator/cavity.

For the 2 level atom dispersively coupled to a resonator a measurement of the next jump of the resonant cavity yields information on the state of the qubit. If the qubit is in $|B\rangle$ and the drive frequency is on resonance for this state then the expected time to the next jump is $\bar{t}_j \sim (12/\bar{n})^{1/3}/\kappa$. For sufficiently large $\bar{n}$ this time is shorter than $1/\kappa$. The norm $W$ for the state which has not yet jumped is plotted in Figure [1]. If instead of being in $|B\rangle$ the qubit was in state $|G\rangle$ when the drive is turned on there is also a chance of recording a jump. In the limit of large dispersion this “error” rate, $\epsilon$ is approximately the ratio of the change in the norms for the different initial conditions:

$$\epsilon \sim \frac{1 - W(\chi, t_j)}{1 - W(0, t_j)}.$$  \hfill (26)

For $\kappa t_j < 1$, $\epsilon \sim 2\kappa t_j \Gamma^2/\chi^2 \sim [\kappa \bar{n}^{1/3}/\chi]^2$ where dispersion is sufficiently large that $\epsilon \ll 1$. For large dispersion the time dependence of the probability for next photon emission goes through large oscillations. Figure [2] displays the scaled logarithmic decrement $Y$ of the norm as a function of time for $\chi/\kappa = 5$:

$$Y = -\frac{1 + (2\chi/\kappa)^2}{\bar{n}W} \frac{dW}{d\tau} = \frac{1 + (2\chi/\kappa)^2}{\bar{n}} \alpha \alpha^* = [1 - e^{-\tau/2}]^2 + 4e^{-\tau/2} \sin^2(\chi \tau/2\kappa).$$  \hfill (27)

The time scale for determining the state of the qubit is shorter than the cavity lifetime when $\bar{n} > 12$. “Reducing the time required to distinguish qubit states with high fidelity is a critical goal in quantum-information science [13].” So we propose considering whether a method based upon the next jump of the quantum states of the resonator will yield a more effective determination of the state of the qubit, than methods which take longer than $1/\kappa$ [5]. We further remark that the resonator lifetime $1/\kappa$ differs from spontaneous decay in that it is not a first principles quantity. The resonator decay is an ensemble average of thermodynamic/transport properties of the cavity. The irreversible processes that determine $1/\kappa$ take place on much shorter timescales. This suggests the possibility of varying $\kappa$ on a time scale long compared to these micro-processes, but short compared to the long time scale of a quantum jump, $1/\gamma$. We suggest that such changes will affect the phase of the between-jumps wave function while leaving the time evolution coherent.

**Acknowledgments**

We wish to acknowledge valuable discussions with Hong Wen Jiang.
Figure 2: The scaled logarithmic decrement $Y$ of probability for next quantum jump as a function of time.

References

[1] M. Porrati and S. Putterman, “Wave-function collapse due to null measurements: The origin of intermittent atomic fluorescence,” Phys. Rev. A 36, 929 (1987).

[2] M. Porrati and S. Putterman, “Coherent intermittency in the resonant fluorescence of a multilevel atom,” Phys. Rev. A 39, 3010 (1989).

[3] T. Erber and S. Putterman, “Randomness in quantum mechanics: nature’s ultimate cryptogram,” Nature 318, 41 (1985).

[4] M. Porrati, N. Penthorn, J. Rooney, H.W. Jiang, S. Putterman, “Regimes of Coherent Intermittency in the Next Quantum Jump of a Multilevel System,” arXiv:1810.03225 [quant-ph].

[5] Z. K. Minev, S. O. Mundhada, S. Shankar, P. Reinhold, R. Gutirrez-Juregui, R. J. Schoelkopf, M. Mirrahimi, H. J. Carmichael and M. H. Devoret, “To catch and reverse a quantum jump mid-flight, Nature 570, 200 (2019) arXiv:1902.10355 [quant-ph]].

[6] H. J. Carmichael, An Open Systems Approach to Quantum Optics (Springer, Berlin, Heidelberg, 1993).
[7] H. M. Wiseman, “Quantum Trajectories and Quantum Measurement Theory,” Quantum Semiclass. Opt. 8, 252 (1996) [quant-ph/0302080].

[8] G. J. Milburn, “Quantum measurement theory of optical heterodyne detection,” Phys. Rev. A 36, 5271 (1987).

[9] H. M. Wiseman and G. J. Milburn, “Quantum measurement theory of optical heterodyne detections,” Phys. Rev. A 47, 642 (1993).

[10] A. Blais, R.-S. Huang, A. Wallraff, S. M. Girvin and R. J. Schoelkopf “Cavity quantum electrodynamics for superconducting electrical circuits: an architecture for quantum computation,” Phys. Rev. A 69, 062320 (2004) [cond-mat/0402216]

[11] A. Blais, R. S. Huang, A. Wallraff, S. M. Girvin, and R. J. Schoelkopf, “Qubit-photon interactions in a cavity: Measurement-induced dephasing and number splitting,” Phys. Rev. A 74, 042318 (2006) [cond-mat/0602322]

[12] J. Gambetta, A. Blais, M. Boissonneault, A. A. Houck, D. I. Schuster and S. M. Girvin “Quantum trajectory approach to circuit QED: Quantum jumps and the Zeno effect,” Phys. Rev. A 77, 012112 (2008) [arXiv:0709.4264 [cond-mat.mes-hall]].

[13] T. Walter, P. Kurpiers, S. Gasparinetti, P. Magnard, A. Potocnik, Y. Salath, M. Pechal, M. Mondal, M. Oppliger, C. Eichler, and A. Wallraff, “Rapid High-Fidelity Single-Shot Dispersive Readout of Superconducting Qubits,” Phys. Rev. Applied 7, 054020 (2017) [arXiv:1701.06933 [quant-ph]]