A Newly Constant-Thrust Control Method Based on Parameter Estimation

Yujin Zhang\textsuperscript{1} and Chao Han\textsuperscript{1*}

\textsuperscript{1}School of Astronautics, Beihang University, Beijing 100191, China

E-mail: hanchao@buaa.edu.cn

Abstract. Finite-thrust control technology is widely applied to space missions such as orbital transfer and rendezvous. However, traditional low-thrust method requires time-varying acceleration for achieving transfer and rendezvous. In this study, a newly piecewise constant thrust method is proposed for orbital transfer problems considering $J_2$ perturbation. To rapidly obtain the solution, UKF parameter estimation method is adopted, which transform the original problem into a parameter estimation problem. Numerical simulations are conducted to demonstrate the validation of the proposed method. It proves to be a practical choice for engineering applications.

1. Introduction

With the increasing demand of transfer missions [1,2], lots of related works have been conducted, which can be divided into impulse-thrust methods and continuous-thrust methods. The Lambert method is the most studied impulse method for deal with two-point boundary value problem (TPBVP) [3-10]. However, these impulse approaches require instantaneous change of velocity vector, which limited engineering application.

In aspect of the continuous-thrust methods, the position and velocity changes continuously based on perturbation equations. The solution of the problem depends on the boundary conditions such as transfer time and the magnitude of thrust. Unfortunately, it has not a closed-form solution in general. The semi-analytical methods, as approximate or analytical solutions with simplified assumptions, have gained wide attention. Petropoulos and Longuski [11,12] made a great contribution for the ‘shape-based method’ and proposed the exponential sinusoid for solving TPBVP. Furthermore, other scholars presented different shape-based methods such as fifth- and sixth-order inverse polynomials [13]. However, because of the assumption of the shape, the solutions may not exist.

The main contribution of this paper is to propose a newly piecewise constant thrust method for solving the transfer problems, which could reduce the demand for the engine in the space missions. This paper is organized as follows: In Sec. II, the mathematical model for constant thrust control are introduced. In Sec. III, a newly piecewise constant thrust method is proposed. In Sec. IV, the effectiveness of the methods above is examined through several numerical simulations. Finally, Sec. V summarizes the paper.

2. Mathematical Model of Two-section Thrust Control
The classical perturbation equations related to \( X = (a, e, i, \Omega, \omega, M) \) for two-body case can be expressed as:

\[
\begin{align*}
\frac{da}{dt} & = \frac{2a'v}{\mu} f_1, \\
\frac{de}{dt} & = \frac{1}{v} \left[ 2(e + \cos \theta) f_1 - \frac{r}{a} \sin \theta f_2 \right], \\
\frac{di}{dt} & = \frac{r}{h} \cos(\omega + \theta) f_h, \\
\frac{d\Omega}{dt} & = \frac{r}{h \sin i} \sin(\omega + \theta) f_h, \\
\frac{d\omega}{dt} & = \frac{1}{v} \left[ 2 \sin \theta f_1 + \frac{2e}{v} \cos \theta f_2 \right] - \frac{r \sin(\omega + \theta) \cos i}{h \sin i} f_h, \\
\frac{dM}{dt} & = n - \frac{dM_f}{dE} = n - \frac{h}{a v} \left[ 2(1 - e^2) \sin \theta f_1 + \frac{r}{a} \cos \theta f_2 \right],
\end{align*}
\]

(1)

Where \( p = a(1 - e^2), h = \sqrt{\mu p}, b = a \sqrt{1 - e^2}, r = a(1 - e \cos E), v = \sqrt{\mu(2/r - 1/a)}, \) and \( f = (f_1, f_2, f_3)^T \) is perturbation acceleration in the TNH frame [8]. The so-called two-section constant thrust means that the magnitude and direction of the applied thrust remain unchanged in each section \( f_{12} = l_{t_1, t_2} f + l_{t_1, t_2} f_2 \):

\[
\begin{align*}
f_{12} f_1 f_2 & = (f_1, f_{a1}, f_{\alpha})^T, t \in [t_1, t_2], t_{in} = t \frac{t_1 + t_2}{2}, \\
f_1 & = f_1 (f_1, f_{a1}, f_{\alpha})^T, t \in [t_1, t_2].
\end{align*}
\]

(2)

The constant thrust control problem related to the initial and target orbital elements \( X_0 \) and \( X_f \) at \( t_0 \) and \( t_f \) can be described by equations (2) below:

\[
X_0 - X_f = X(f_{12}, t_f) - X_0 \Leftrightarrow F(f_{12}) = 0
\]

(3)

Where \( X(f_{12}, t_f) \) denote the final orbital elements during the thrust control \( f_{12} \). In general, non-linear equations (3) \( F(f_{12}) = 0 \) has no analytic solution, which only can be solved numerically. Moreover, equations (1-3) can be extended with \( J_2 \) terms to consider \( J_2 \) perturbation. To rapidly obtain the solution, UKF parameter estimation method [14] is adopted, which transform the original problem into a parameter estimation problem. The observation error is defined as \( q_k = d_k - F(w_k) \), and the parameter estimation problem can be expressed as:

\[
\begin{align*}
w_{k+1} &= w_k + r_k, \\
0 &= -q_k + e_k
\end{align*}
\]

(4)

The estimation algorithm is simply discussed below, which is detailly described in previous works [14]. First, obtain initialization parameter:

\[
\hat{w}_0 = E[w], \\
P_0 = E[(w - \hat{w}_0)(w - \hat{w}_0)^T]
\]

(5)

Then, update the state and calculate sigma points:
\[\dot{\omega}_k = \omega_{k-1}\]
\[\dot{R}_k = \left(\hat{\omega}_{BSL}^{-1}\right) P_{nk}\]
\[P_{nk} = P_{nk} + \dot{R}_k\]
\[W_{jnk} = \left[\ddot{w}_k + \sqrt{P_{nk} - \dot{w}_k - \gamma\sqrt{P_{nk}}}\right]
\[D_{jnk} = F(W_{jnk})\]
\[\ddot{a}_k = \sum_{i=0}^{2} W_{ni}^{\text{lo}} D_{jnk}\]

Finally, update observation information:
\[P_{nk} = \sum_{i=0}^{2} W_{ni}^{\text{lo}} [D_{jnk} - \ddot{a}_k] [D_{jnk} - \ddot{a}_k]^T + R_k^{\prime}\]
\[P_{nk} = \sum_{i=0}^{2} W_{ni}^{\text{lo}} [W_{jnk} - \ddot{w}_k] [D_{jnk} - \ddot{a}_k]^T\]

3. A Newly Piecewise Constant Thrust Method Based on Lambert Method

Satellite orbit transfer is of great value in aerospace engineering. In this paper, for two-point boundary value problem with \(J_2\) perturbation considered, a newly two-section constant thrust method combined with Lambert method is proposed. Firstly, based on Lambert method, a double-impulse solution can be obtained, which provides a reference trajectory. Then, each impulse solution thrust can be converted into a two-section constant thrust solution respectively. Hence, by a total of four segments of constant thrust control, the controlled spacecraft successfully achieves a fixed-time orbit transfer. As illustrated in Fig.1, \(A_1A_2\) and \(B_1B_2\) denote initial orbit and target orbit, where \(A_1\) and \(B_1\) is initial state and final state respectively. Moreover, \(A_1C_1C_2B_1\) is the Lambert transfer orbit. If the impulse at \(A_1\) is replaced by a two-section constant thrust, the controlled spacecraft will follow \(A_1C_1\) (red curve) and arrives at \(C_1\) the same time. Similarly, the spacecraft can move along the red curve \(C_2B_1\), imposed by another two-section constant thrust instead of an impulse at \(B_1\). By applying a total of four segments of constant thrust control, the spacecraft achieves a fixed time transfer.

![Figure 1. Double two-section constant thrust controls combined with Lambert transfer](image-url)

The flowchart of the algorithm is as follows:
1. Solve Lambert solution with \(J_2\) perturbation considered, then the Lambert transfer \(A_1C_1C_2B_1\) is obtained.
2. Set the thrust time respectively, and then the target state of the first time \(C_1\) and the initial state of the second time \(C_2\) are determined, as illustrated in Fig.1.
3. Based on the known state of \(A_1\) and \(C_1\), two-point boundary value equations (3) for \(A_1C_1\) are obtained. Similarly, two-point boundary value equations \(C_2B_1\) can be derived.
The two-section constant solutions of the equations for $A_1C_1$ and $C_2B_1$ are obtained through UKF parameter in equations (4-7).

### 4. Simulation Results

To validate the efficiency and accuracy of the piecewise constant thrust method, a sequence of separate numerical simulations is carried out in this paper, including LEO, two MEO and GEO. In addition, the influence of $J_2$ perturbation on orbits is considered. Tables 1-4 show the input parameters and results of each scenario. The position error is less than $10^{-5}$ and the velocity error is also less than $10^{-5}$. Thus, the proposed constant thrust method successfully realizes different transfer missions. Fig.2 shows the trajectories of LEO, MEO1, MEO2 and GEO transfer with piecewise constant thrust control.

Table 1. Input parameters and results of LEO transfer

| Initial state | Target state |
|---------------|--------------|
| Semi-major axis, m/s | 7093730 | 7093050 |
| Eccentricity | 0.00064 | 0.00023 |
| Inclination, deg | 86.504 | 86.501 |
| Ascending node, deg | 177.214 | 177.210 |
| Argument of Perigee, deg | 108.671 | 85.727 |
| Mean anomaly, deg | 301.465 | 18.862 |
| Transfer time, s | 900 |
| Position error, m | $10^{-5}$ |
| Velocity error, m/s | $10^{-5}$ |

Table 2. Input parameters and results of MEO1 transfer

| Initial state | Target state |
|---------------|--------------|
| Semi-major axis, m/s | 18785426 | 18800000 |
| Eccentricity | 0.00064 | 0.00719 |
| Inclination, deg | 39.361 | 39.362 |
| Ascending node, deg | 225.001 | 225.006 |
| Argument of Perigee, deg | 146.654 | 150.613 |
| Mean anomaly, deg | 29.403 | 51.165 |
| Transfer time, s | 1600 |
| Position error, m | $10^{-5}$ |
| Velocity error, m/s | $10^{-5}$ |

Table 3. Input parameters and results of MEO2 transfer

| Initial state | Target state |
|---------------|--------------|
| Semi-major axis, m/s | 28080808 | 28092344 |
| Eccentricity | 0.00687 | 0.0065 |
| Inclination, deg | 54.982 | 54.981 |
| Ascending node, deg | 280.611 | 280.610 |
| Argument of Perigee, deg | 181.176 | 181.274 |
| Mean anomaly, deg | 155.301 | 169.045 |
| Transfer time, s | 1600 |
| Position error, m | $10^{-5}$ |
| Velocity error, m/s | $10^{-5}$ |
Table 4. Input parameters and results of GEO transfer

|                      | Initial state | Target state |
|----------------------|---------------|--------------|
| Semi-major axis, m/s | 42160070      | 42165014     |
| Eccentricity         | 0.00019       | 4.62e-05     |
| Inclination, deg     | 0.0138        | 0.0138       |
| Ascending node, deg  | 98.123        | 100.295      |
| Argument of Perigee, deg | 29.946    | 79.744       |
| Mean anomaly, deg    | 224.425       | 179.961      |
| Transfer time, s     | 1600          |              |
| Position error, m    | 10^{-5}       |              |
| Velocity error, m/s  | 10^{-5}       |              |

Figure 2. Trajectories of LEO, MEO and GEO transfer with piecewise constant thrust control
5. Conclusions
In this paper, based on UKF parameter estimation method, a newly piecewise constant thrust method is proposed to solve the problems of orbital transfers. Moreover, a sequence of separate numerical simulations is carried out to validate the efficiency and accuracy of the methods. Simulation results show that the constant thrust methods can achieve orbit transfers in a very small error. Compared with the conventional continuous-thrust control strategy, the proposed constant control strategy is much more convenient because in each section the thrust is a constant vector. Hence, the method proposed is a practical choice for engineering application.

6. Acknowledgements
This research is supported by National Natural Science Foundation of China under grants 41804165.

7. References
[1] J.A., Watson. “Statistically architected human spaceflight missions to Mars,” Acta Astronautica, Vol. 160, 2019, pp. 155-162.
[2] Battin, R. H., “Lambert's problem revisited,” AIAA Journal, Vol. 15, No. 5, 1977, pp. 44–46.
[3] Lee S., Ahn J. and Bang J., “Dynamic Selection of Zero-Revolution Lambert Algorithms Using Performance Comparison Map,” Aerospace Science and Technology, Vol. 51, April 2016, pp. 96–105.
[4] Wagner S., Wie B. and Kaplinger B., “Computational Solutions to Lambert’s Problem on Modern Graphics Processing Units,” Journal of Guidance, Control, and Dynamics, Vol. 38, No. 7, 2015, pp. 1305–1311.
[5] Zhang G., Zhou D., Mortari D. and Akella M. R., “Covariance Analysis of Lambert’s Problem via Lagrange’s Transfer-Time Formulation,” Aerospace Science and Technology, Vol. 77, June 2018, pp. 765–773.
[6] Zhang, G. “Terminal-Velocity-Based Lambert Algorithm,” Journal of Guidance, Control, and Dynamics, Vol. 43, No. 2, 2020, pp. 1-11.
[7] He, Q., Li, J., & Han, C., “Multiple-revolution solutions of the transverse-eccentricity-based lambert problem,” Journal of Guidance Control Dynamics, Vol. 33, No. 1, 2015, pp. 265–269.
[8] Li, J. “Research on continuous thrust orbit maneuver optimization method,” phd thesis, Beijing: Beihang University, 2013.
[9] Vatistas, G. H. et al., “Reverse Flow Radius in Vortex Chambers,” AIAA Journal, Vol. 24, 1986, pp. 1872-1873.
[10] Abdelkhalik, O. and Taheri, E., “Approximate on-off low-thrust space trajectories using fourier series,” Journal of Spacecraft & Rockets, Vol. 49, No. 5, 2013, pp. 962-965.
[11] Petropoulos, A. E., Longuski, J. M., & Vinh, N. X., “Shape-based analytic representations of low-thrust trajectories for gravity-assist applications,” Astrodynamics, 2000, pp. 563-581.
[12] Petropoulos, A. E., & Sims, J. A., “A review of some exact solutions to the planar equations of motion of a thrusting spacecraft,” 2002.
[13] Wall, B. J., & Conway, B. A., “Shape-based approach to low-thrust rendezvous trajectory design,” Journal of Guidance Control & Dynamics, Vol. 32, No. 1, 2009, pp. 95-101.
[14] Zhang, H. et al., “A UKF based design method for lunar free-return trajectories,” Aerospace Science and Technology, Vol. 48, 2016, pp. 272-280.