Low-complexity BER computation for coherent detection of orthogonal signals

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Introduction: Digital modulations can be broadly categorized into two classes: signaling schemes for high-rate bandlimited communications, such as quadrature-amplitude modulation (QAM), and signaling schemes for low-rate low-power communications, such as frequency-shift keying (FSK) [1]. This second set is of paramount importance for Internet of Things (IoT) applications [2, 3], where the low-power requirement is often necessary, for example, for green communications. Among low-rate low-power modulations, it is well known that orthogonal signaling schemes are optimal, because they asymptotically approach the Shannon limit [1]. Orthogonal signaling includes not only FSK [4–7], whose orthogonality is in the frequency domain, but also other modulation schemes that impose the orthogonality condition in other domains, such as pulse-position modulation (PPM) [8, 9], which is orthogonal in the time domain, and chirp modulation [4, 10–14], which is orthogonal in the Fresnel domain [15].

A communication device can self-assess the quality of the received signal, for instance, by estimating the signal-to-noise ratio (SNR) or the bit error rate (BER). While the SNR is a measure to be estimated, the BER is a more precise indicator of performance and is often a function of the SNR. For coherent detection of orthogonal signals with additive white Gaussian noise (AWGN), the BER is expressed in an integral form as a function of the SNR [1]. Unfortunately, this integral form cannot be expressed in terms of simpler functions like the exponential or the Gaussian Q function [1]. Although numerical integration is nowadays easy to be performed by modern computers and laptops, the same operation is quite cumbersome in low-complexity IoT devices. If IoT devices can avoid numerical integration, useful resources would be saved for sensing and transmission. Hence, it is of interest to derive approximated BER expressions that can be easily computed by low-complexity IoT devices.

In the literature, since the exact result is available, only few research studies have tried to derive a simple BER approximation for coherent detection of orthogonal signals. For instance, the studies concerning the error rate performance of FSK have focused on upper bounds (UBs) only [16], on quaternary constellations only [17], and on non-orthogonal signaling only [5]. On the other hand, the investigation on the performance of chirp modulation (also known as chirp spread spectrum or LoRa) has considered non-coherent detection only [18–20], large constellation sizes only (larger than or equal to 64) [13, 21], and non-orthogonal waveforms only [22]. A simple BER approximation valid for coherent detection of orthogonal signaling with moderate constellation sizes is still missing.

This letter presents a simple approach that enables a low-complexity BER computation for coherently detected orthogonal signals in AWGN channels. We start from a few BER values known in advance. First, we choose a convenient parametric BER expression. Second, we compute the unknown parameters by fitting the chosen BER expression to the known BER values. The choice of the parametric BER expression is the main novelty of our approach, which simplifies the subsequent fitting procedure. The numerical comparison between the approximated BER and the exact BER shows the good accuracy of the proposed low-complexity method, for several constellation sizes. The proposed approach allows the BER self-computation in low-complexity IoT devices that use orthogonal signals.

Exact BER for orthogonal signals: For coherently detected orthogonal signals in AWGN, the probability of symbol error \( P_e \) is expressed by [1, p. 262]

\[
P_e(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[ 1 - Q(\sqrt{y}) \right] e^{-\frac{(y - \gamma)^2}{2}} dy,
\]

where \( Q(y) \) is the Gaussian Q function, \( M \) is the number of symbols, and \( y = E_s/N_0 \) is the SNR per symbol, defined as the ratio between the signal energy \( E_s \) and the one-sided AWGN power spectral density \( N_0 \). Using (1), the probability of bit error (also known as BER) is given by

\[
P_b(y) = k P_e(y),
\]

where \( k = M/(2M - 2) \) is the average number of erroneous bits for each erroneous symbol. The computation of (1) and (2) requires numerical integration and hence is not suitable for low-complexity IoT devices.

UB on the BER for orthogonal signals: To avoid the numerical integration in (1), bounds and approximations have been proposed in [16,17]. The Hughes UB on the symbol error probability is given by [16]

\[
P_e(Hughes) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[ 1 - Q(\sqrt{y}) \right]^{M-1} dy,
\]

leading to the BER UB

\[
P_b(Hughes) = k P_e(Hughes) = \frac{M}{2M-2} \left[ 1 - Q(\sqrt{y}) \right]^{M-1}. \tag{4}
\]

which is valid for any \( M \). This bound is more accurate than the union UB [16], expressed by

\[
P_e(\text{union}) = (M - 1) Q(\sqrt{y}). \tag{5}
\]

\[
P_b(\text{union}) = k P_e(\text{union}) = \frac{M}{2} Q(\sqrt{y}). \tag{6}
\]

Tighter bounds on (1) and accurate approximations have been proposed in [17], but, differently from (3) and (5), the expressions in [17] are valid for \( M = 4 \) only.

Proposed BER computation approach: We aim at low-complexity approximations of (1) and (2), but with improved accuracy with respect to (3) and (4). Our key idea is to express the approximated BER \( P_{b,\text{approx}}(y) \) as the product of two functions: a known low-complexity decreasing function \( D(y) \), which represents a very rough approximation of the BER curve \( P_b(y) \), and a parametric function \( R(y) \) that tries to correct the misalignment between the exact BER curve \( P_b(y) \) and the decreasing function \( D(y) \). This model can be expressed by

\[
P_{b,\text{approx}}(y) = R(y) D(y). \tag{7}
\]

Concerning the choice of the decreasing function \( D(y) \), several alternatives are possible, such as

\[
D(y) = D_1(y) = P_e(Hughes(y)). \tag{8}
\]

\[
D(y) = D_2(y) = P_b(\text{union})(y) = \frac{M}{2} Q(\sqrt{y}). \tag{9}
\]

\[
D(y) = D_3(y) = \frac{M}{2} e^{-0.4774y - 0.4483}\sqrt{y - 0.0949}. \tag{10}
\]

Equations (8) and (9) are taken from the bounds (4) and (6), respectively, while (10) is obtained from (9) by applying an exponential approximation of the Gaussian Q function proposed by Lopez-Benitez and Casadevall in [23]. Instead of [23], other simple approximations of the
Gaussian Q function could be employed, such as those proposed in [24–29]. Note that the functions in (8)–(10) do not require numerical integra-
tion and have the following two properties:

\[
\lim_{y \to \infty} \frac{D(y)}{P_b(y)} = 1, \\
\lim_{y \to \infty} \frac{D(y)}{P_b(y)} = c_i = \begin{cases} 
M(2^{M-1} - 1) & \text{if } i = 1, \\
\frac{M}{2} & \text{if } i = 2, \\
M e^{-0.9049} & \text{if } i = 3.
\end{cases}
\]

For the correction function \( R(y) = R_i(y) \), we need a parametric func-
tion with two properties related to (11) and (12): \( \lim_{y \to \infty} R(y) = 1 \) and \( \lim_{y \to 0} R(y) = 1/c_i \). We propose a rational function expressed by

\[
R(y) = R_i(y; \mathbf{p}) = \frac{y^j + n_1 y^2 + n_0}{y^j + d_3 y^2 + d_1 y + c_i b_0},
\]

where \( \mathbf{p} = [n_2, n_1, n_0, d_2, d_1]^T \) is the vector of real parameters to be opti-
mized. The rational form of \( R(y) \) in (13) permits a simple computation of the parameter vector \( \mathbf{p} \), as explained in the next section. Then, we will show that five parameters in \( \mathbf{p} \) lead to a sufficient accuracy.

**Computation of the parameters:** The parameters in \( \mathbf{p} \) can be pre-
computed using least-squares (LS) fitting, starting from a few values of the exact BER \( P_b(y_j) \), with \( j = 1, \ldots, F \), already known in advance or computed using numerical integration.

First, we calculate \( D(y) \) to determine \( r_j = R(y_j) = P_b(y_j)/D(y_j) \). Second, (13) can be rewritten as

\[
\mathbf{a}_j^T \mathbf{p} = b_j, \quad j = 1, \ldots, F,
\]

where \( \mathbf{a}_j = [y_j^2, y_j, 1, -c_r y_j, -r_j y_j^2, -r_j y_j]^T \) and \( b_j = (r_j - 1)y_j^2 \).

By exploiting \( F \geq 5 \) values of \( y_j \), we obtain

\[
\mathbf{A} \mathbf{p} = \mathbf{b},
\]

where \( \mathbf{A} = [\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_F]^T \) and \( \mathbf{b} = [b_1, b_2, \ldots, b_F]^T \). Therefore, the five parameters in \( \mathbf{p} \) can be obtained by solving (15) in the LS sense, as expressed by

\[
\mathbf{p} = (\mathbf{A}^T \mathbf{A})^{-1} \mathbf{A}^T \mathbf{b}.
\]

Table 1 lists the parameters in \( \mathbf{p} \) obtained from (16) using the decreas-
ing function (9) for \( M \in \{4, 8, 16, 32, 64\} \), when \( F = 9 \). Specifically, the values in Table 1 have been obtained using the SNR values \( y_j = (j + 1) \) dB, for \( j = 1, \ldots, F \). In all the cases, the values of the five coefficients in \( \mathbf{p} \) are positive: Consequently, both the numerator and the denominator of (13) are cubic polynomials whose roots are either complex or nega-
tive. Therefore, (13) is meaningful in the whole range of SNR and not only for the selected \( F \) values \( y_j \) used in the LS fitting procedure.

Note that we used numerical integration to determine the parameters in \( \mathbf{p} \), but numerical integration is not necessary in low-complexity IoT devices. Indeed, an IoT device can store the values of Table 1 only, and can compute the approximated BER (7) using (9) and (13) only, without numerical integration. Similarly to Table 1, a numerical table could be simply obtained using (8) or (10) instead of (9). For instance, using (10), a low-complexity device would not require the calculation of the Gauss-
ian Q function. Specifically, for a known \( y \), the computational complex-
ity of the BER approximation (7) with (10) is given by 8 additions, 11 multipli-
cations, 1 division, 1 square root, and 1 exponentiation.

It is noteworthy that Table 1 omits the naive case \( M = 2 \). Indeed, for \( M = 2 \), the bound (6) coincides with the exact BER (2): since \( c_2 = 1 \), any choice with \( n_1 = d_1 \) and \( n_2 = d_2 \) would produce \( R(y) = 1 \) in (13), and consequently the BER approximation would coincide with the exact BER, which for \( M = 2 \) does not require numerical integration.

**BER approximation accuracy:** Figure 1 compares different BER ap-
proximations, for \( M \in \{4, 8, 16, 32, 64\} \), as a function of the SNR per bit, defined as \( y_b = E_b/N_0 = y/\log_2 M \). In Figure 1 and following, ‘Approx. by Hughes UB’ stands for the approximation (7) derived from the Hughes UB in (8), ‘Approx. by union UB’ stands for (7) based on the union UB in (9), while ‘Approx. by exponential’ stands for (7) with the exponential function in (10). All the three approximations use \( F = 9 \). From Figure 1, it is clear that all the three approximations adequately match the exact curve, for all cases of \( M \).

Figure 2 compares the relative error on the BER when \( M = 16 \). The relative error on the BER is defined as \( \rho = |P_b(\text{approx})/P_b(y) - P_b(\text{exact})/P_b(y)| \). It is evident that the relative error is \( \rho < 10^{-3} \) in the range of \( y \) between 2 dB and 10 dB, which is the SNR range selected in the LS fitting procedure. Among the three approximations, the one derived from the exponential function in (10) appears to be slightly more accurate than the other two. Anyway, in our opinion, also the approximations derived from the Hughes UB (8) and from the union UB (9) are good.

Figure 3 displays the accuracy of the approximation (7) based on the union UB (9), for different values of \( M \). For all values of \( M \), the relative error on the BER stays below \( 10^{-3} \) in the SNR fitting range (\( y \) between 2 dB and 10 dB). This confirms the good accuracy of this approxima-
tion for all values of \( M \). Figure 4 focuses on the parameter coefficients in Table 1 for \( M = 16 \). Specifically, Figure 4 shows that, when the coeffi-
cients of Table 1 are rounded to two decimal digits, the relative error on the BER remains below \( 10^{-3} \) in the SNR fitting range. Therefore, a low-complexity low-memory IoT device can achieve an accurate BER
approximation even using a reduced memory size for storing the parameter coefficients.  

Conclusion: We have presented a simple method that allows an accurate BER computation for orthogonal signals with coherent detection and AWGN. The obtained results enable a low-complexity performance self-assessment within IoT devices. Although the proposed method is applied to coherent detection of orthogonal signaling, the same idea can be applied also to other modulations, using few BER values known in advance.

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