The Dynamics of Knotted Strings Attached to D-Branes

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Abstract

We extend the general solution to the Cauchy problem for the relativistic closed string (Phys. Lett. \textbf{B404} (1997) 57-65) to the case of open strings attached to Dp-branes, including the cases where the initial data has a knotlike topology. We use this extended solution to derive intrinsic dynamical properties of open and closed relativistic strings attached to Dp-branes. We also study the singularity structure and the oscillating periods of this extended solution.

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String theory is an important field of research due to the rich phenomena and physical concepts associated to nonlinear field theories. Pioneering work in general nonlinear field theory goes back to Born and Infeld where the two-dimensional scalar equation of nonlinear electrodynamics is the equation of motion for a string in Minkowsky space [1-3]. More recently, it has been recognized that spinning massless strings of finite length in Minkowsky space behave like particles with nonzero rest mass and intrinsic spin [4-7].

In the general context of nonlinear field theories, it has been analytically shown that knotlike string configurations appear as stable solutions of the evolution equations [8]. Recently, Faddeev and Niemi [9-10] have shown, with methods of high performance computing, that knotlike configurations appear as stable solitons in certain relativistic quantum field theories.

Several problems related with the dynamical evolution of strings and in particular knotted strings remain unsolved. In the context of the theory of galaxy formation, strings formed at a phase transition very early in the history of the universe might provide the density perturbations to start the condensation process. Different estimates for the temperature transitions exist depending on the dynamical configuration of strings: strings with a random configuration [11] or strings with stable closed loops [12].

All these questions have an answer if explicit solutions for nonlinear equations of motion supporting arbitrary string configurations and topologies at some initial time exist. In fact, the general Cauchy problem for the free motion of a closed string, knotted or not, in the Minkowski space $M^{3+1}$ has been solved [8]. It has been shown that (1) Initially static closed strings always oscillate with the period $T = \ell/2c$, where $\ell$ is the string length; (2) Oscillating knotted strings show singular behavior leading to the simple link for times $t = (2n+1)\ell/4c$, with $n = 0, 1, \ldots$; (3) Knotted strings can stretch infinitely; (4) Infinite strings have solitonic like behaviour in the sense that localized string perturbations propagate without losing coherence [13]. In the case of collision of two opposite velocity packets the coherence is lost, but after collision both packets acquire their initial shape and propagate freely with the same velocity.

When dealing with open strings, boundary conditions on the coordinates must be considered, with the possibility of choosing these as either of Dirichlet or Neumann type. Appropriately choosing these boundary conditions, we can construct D-branes, which are extended objects with the property that strings can end on them (see [14-15], and references therein). One important feature of D-brane theory relies on the hypothesis that if black holes are described as a collection of D-branes with attached strings, the Bekenstein-Hawking entropy computed from the classical black hole solution agrees with the black hole entropy given by the number of states of strings and D-branes (see [16] and references therein). A general proof of this result in four dimensional spacetime is, however, still lacking.
The aim of this paper is to describe the dynamics of strings attached to Dp-branes in Minkowsky space $M^{3+1}$, with $p = 0, 1, 2, 3$, including knotted string configurations.

We consider strings in Minkowsky space $M^{3+1}$ with metric $d\ell^2 = dt^2 - dx^2 - dy^2 - dz^2 = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2$, and units such that $c = 1$. The motion of strings in $M^{3+1}$ is described by the Nambu action

$$A = -m \int \int dp \ dq \sqrt{(\partial_p \mathbf{r} \cdot \partial_q \mathbf{r})^2 - (\partial_p \mathbf{r})^2(\partial_q \mathbf{r})^2} := \int \int dp \ dq \ L$$

where $p$ and $q$ parameterize the surface $S$ spanned by the string in $M^{3+1}$, $m$ is a constant of proportionality, $\mathbf{r} = (x^0(p, q), x^1(p, q), x^2(p, q), x^3(p, q))$ is the four dimensional position vector and $L$ is the Lagrangean density of the string.

As the $p$ and $q$ parameters are arbitrary, we can choose $p$ and $q$ such that, [2] and [8],

$$(\partial_p \mathbf{r})^2 = 0, \quad (\partial_q \mathbf{r})^2 = 0 \quad (2)$$

where $p = p(t, \sigma)$ and $q = q(t, \sigma)$. The parameter $\sigma$ describes the string shape in the proper reference frame of the string. Under these conditions, the equation of motion for string perturbations is

$$\partial_p \partial_q \mathbf{r} = 0 \quad (3)$$

provided the constraint equations (2) hold.

For initial string positions and velocities described by functions $a^k(\sigma)$ and $b^k(\sigma)$ such that

$$x^k(0, \sigma) = a^k(\sigma)$$

$$\partial_t x^k(t, \sigma)|_{t=0} = b^k(\sigma), \quad k = 1, 2, 3 \quad (4)$$

the general solution of the Cauchy problem, including the cases of knotted topologies, is, [8],

$$t(p, q) = \frac{1}{2} \int_p^q \Pi(s) ds$$

$$x^k(p, q) = \frac{1}{2}(a^k(p) + a^k(q)) + \frac{1}{2} \int_p^q b^k(s) \Pi(s) ds, \quad k = 1, 2, 3 \quad (5)$$

where

$$\Pi(s) = \left( \frac{\sum_{i=1}^{3}(\partial_s a^i(s))^2}{1 - \sum_{i=1}^{3}(b^i(s))^2} \right)^{1/2} \quad (6)$$

provided

$$0 \leq \sum_{i=1}^{3}(b^i(\sigma))^2 < 1 \quad (7a)$$

$$\sum_{i=1}^{3} b^i(\sigma) \partial_\sigma a^i(\sigma) = (\partial_t \mathbf{r} \cdot \partial_\sigma \mathbf{r})|_{t=0} = 0 \quad (7b)$$
and \( p = q = \sigma \), for \( t = 0 \). In the string proper reference frame \((t, x^1, x^2, x^3)\), the Minkowsky distance between string points is negative. Therefore, the string lies on a space-like three-dimensional surface in \( M^{3+1} \).

One of the consequences of the solution of the Cauchy problem for the relativistic string is related with the impossibility of having infinite or closed rotating strings, \textit{defined for all times}. We exemplify this effect with two examples. Suppose that we have initial Cauchy data, \( a^1(\sigma) = \cos(\sigma), a^2(\sigma) = \sin(\sigma), a^3(\sigma) = 0, b^1(\sigma) = -\omega \sin(\sigma), b^2(\sigma) = \omega \cos(\sigma) \) and \( b^3(\sigma) = 0 \), representing a rotating closed loop with angular velocity \( \omega \), at \( t = 0 \). For the Cauchy problem to be well defined we must verify (7). But by (7b), it follows that \( \omega = 0 \), and we cannot have simple closed rotating relativistic strings.

In the case of an infinite string we take as initial data \( a^1(\sigma) = \sigma, a^2(\sigma) = 0, a^3(\sigma) = 0, b^1(\sigma) = 0, b^2(\sigma) = f(\sigma) \) and \( b^3(\sigma) = 0 \), where \( f(\sigma) \) is a strictly increasing even function. By (7a), we obtain the condition \(|f(\sigma)| < 1\), which is not compatible with the initial requirement for the function \( f(\sigma) \). For finite open strings, \( f(\sigma) \) must be extended to \( \sigma \in \mathbb{R} \) (see below), and so we still cannot have rotation.

We note that, according to Scherk [5], we can relate the classical spin of the string, \( J \), to the mass squared of the string, \( M^2 \), through the inequality \( J \leq \alpha' M^2 \), where \( \alpha' \) is a constant related to \( m \) in (1). Moreover, the equality is reached only for a rigid rotating string. Thus, our previous discussion shows that the equality is not compatible with the solution of the Cauchy problem.

Choosing the new string world sheet parameterization,

\[
\nu = \frac{p + q}{2}, \quad \tau = \frac{q - p}{2}
\]

string solutions are now

\[
t(\tau, \nu) = \frac{1}{2} \int_{\nu - \tau}^{\nu + \tau} \Pi(s) ds
\]

\[
x^k(\tau, \nu) = \frac{1}{2} (a^k(\nu - \tau) + a^k(\nu + \tau)) + \frac{1}{2} \int_{\nu - \tau}^{\nu + \tau} b^k(s) \Pi(s) ds, \quad k = 1, 2, 3
\]

In the new parameters \((\tau, \nu)\), the equation of motion of the string becomes

\[
(\partial_{\tau\tau} - \partial_{\nu\nu}) \mathbf{r} = 0
\]

with constraints

\[
(\partial_\nu \mathbf{r})^2 + (\partial_\tau \mathbf{r})^2 = 0, \quad \partial_\nu \mathbf{r} \cdot \partial_\tau \mathbf{r} = 0
\]

For \( t = 0 \) and \( p = q = \sigma, \tau = 0 \) and \( p = q = \sigma = \nu \).

We now consider the case where strings have finite lengths and the end points are connected to a \( p \)-dimensional hypersurface, a Dp-brane, with \( p = 0, 1, 2, 3 \). To fix ideas,
we first consider the case where strings have fixed extreme points — the D0-brane. In this case, we consider that string extreme points occur for $\nu = 0$ and $\nu = \pi$. The motion of the attached string can then be described by a set of initial position and velocities, $\bar{a}^k(\nu)$ and $\bar{b}^k(\nu)$ as in (4), with $\nu \in [0, \pi]$, together with the spatial Dirichlet boundary conditions:

$$x^k(\tau, \nu = 0) = X^k_1(\cdot), \quad x^k(\tau, \nu = \pi) = X^k_2(\cdot), \quad k = 1, 2, 3$$ (12)

where $X^k_1(\cdot)$ and $X^k_2(\cdot)$ are arbitrary functions associated to the D0-branes. For example, for a static D0-brane, the functions $X^k(\cdot)$ are constants. For a dynamic D0-brane, we can have $X^k_i = X^k_i(\xi)$, where $\xi$ is a new parameter describing the dynamics of the attachment points. In general, we can assume that $\xi = \xi(\tau)$.

For a D3-brane, the string has moving end points and we must choose spatial Neumann boundary conditions

$$\partial_\nu x^k(\tau, \nu)|_{\nu=0} = 0, \quad \partial_\nu x^k(\tau, \nu)|_{\nu=\pi} = 0, \quad k = 1, 2, 3$$ (13)

In Dirichlet, Neumann and mixed cases, we have an additional Neumann boundary condition on the temporal coordinate, [14-15], $\partial_\nu t(\tau, \nu)|_{\nu=0, \pi} = \frac{1}{2}(\Pi(\nu + \tau) - \Pi(\nu - \tau))|_{\nu=0, \pi} = 0$. As we will see below, this is always true as $\Pi(s)$ is an even function around the extreme points of the string. Therefore, a generic Dp-brane has $p+1$ Neumann boundary conditions and $(3 - p)$ Dirichlet boundary conditions, in the $(\tau, \nu)$ parameterization.

Let us now extend the general solution (9) of the Cauchy problem for the D0-brane case. Suppose that (9) holds in the finite length string case. Therefore, by (12), for $\nu = 0$ and $\nu = \pi$,

$$x^k(\tau, 0) = \frac{1}{2}(a^k(-\tau) + a^k(\tau)) + \frac{1}{2} \int_{-\pi}^\tau b^k(s)\Pi(s)ds = X^k_1(\xi)$$

$$x^k(\tau, \pi) = \frac{1}{2}(a^k(\pi - \tau) + a^k(\pi + \tau)) + \frac{1}{2} \int_{\pi-\tau}^{\pi+\tau} b^k(s)\Pi(s)ds = X^k_2(\xi), \quad k = 1, 2, 3$$ (14)

Suppose now that, $b^k(s)\Pi(s)$ is an odd function around the two extreme points of the string, $s = 0$ and $s = \pi$. Then, relations (14) simplify to

$$a^k(-\tau) = -a^k(\tau) + 2X^k_1(\xi)$$

$$a^k(\pi + \tau) = -a^k(\pi - \tau) + 2X^k_2(\xi), \quad k = 1, 2, 3$$ (15)

for every $\tau$. As (15) must hold for $\tau \in \mathbb{R}$, the functions $a^k(\nu)$ can be constructed if we know the values of $a^k(\nu)$ in the interval $[0, \pi]$. Therefore, given the initial string position $\bar{a}^k(\nu)$, with $\nu \in [0, \pi]$, $\bar{a}^k(0) = X^k_1$ and $\bar{a}^k(\pi) = X^k_2$, it is always possible to extend $\bar{a}^k(\nu)$ to $a^k(\nu)$ as odd functions around the points $X^k_1$ and $X^k_2$. This extension is easily obtained with $a^k(\nu) = \bar{a}^k(\nu)$ for $\nu \in [0, \pi]$, relations (15) with $\tau \to \nu$, and

$$a^k(\nu + n\pi) = a^k(\nu + (n - 2)\pi) - 2X^k_1 + 2X^k_2, \quad k = 1, 2, 3$$ (16)
derived by induction from (15).

As \( a^k(\sigma) \) is an odd extension of \( \bar{a}^k(\sigma) \), and if \( b^k(\nu) \) is an odd extension of \( \bar{b}^k(\nu) \), with \( b^k(\nu = 0) = b^k(\nu = \pi) = 0 \), by (6), \( \Pi(s) \) is even. Therefore, \( b^k(s)\Pi(s) \) is also an odd function around the extreme points of the string, as we have initially assumed.

Hence, we have proved that for given initial functions \( \bar{a}^k(\nu) \) and \( \bar{b}^k(\nu) \), the string attached to a static or dynamic D0-brane evolves according to solutions (9), where \( a^k(\nu) \) and \( b^k(\nu) \) are odd extensions of \( \bar{a}^k(\nu) \) and \( \bar{b}^k(\nu) \) around the string end points and \( b^k(0) = b^k(\pi) = 0 \). These odd extensions are calculated through (15) and (16).

For the case of a string attached to a D3-brane we impose the Neumann boundary conditions (13) to the solution (9) for the motion of the relativistic string. Introducing (9) into (13), we obtain,

\[
\begin{align*}
\dot{a}^k(\tau) + \dot{a}^k(\tau + \pi) + b^k(\tau)\Pi(\tau) - b^k(\tau + \pi)\Pi(\tau) = 0, \\
\dot{a}^k(\tau) + \dot{a}^k(\tau + \pi) + b^k(\tau + \pi)\Pi(\tau + \pi) - b^k(\tau + \pi)\Pi(\tau + \pi) = 0, \\
&\quad k = 1, 2, 3
\end{align*}
\]

(17)

where, \( \dot{a}^k(\tau) = \partial_\tau a^k(s)|_{s=\tau} \). Now we consider that the initial Cauchy data is specified by functions \( \bar{a}^k(\nu) \) and \( \bar{b}^k(\nu) \), with \( \nu \in [0, \pi] \), as in (4). Let us suppose further that the product functions \( b^k(\nu)\Pi(\nu) \) are even around the points \( \nu = 0 \) and \( \nu = \pi \). Hence, by (17),

\[
\begin{align*}
\dot{a}^k(\tau) &= -\dot{a}^k(\tau) \\
\dot{a}^k(\tau + \pi) &= -\dot{a}^k(\tau - \pi) \\
\dot{a}^k(\tau + n\pi) &= \dot{a}^k(\tau + (n - 2)\pi), \\
&\quad k = 1, 2, 3
\end{align*}
\]

(18)

But if \( \dot{a}^k \) are odd functions around \( \tau = 0 \) and \( \tau = \pi \), as (18) implies, we can construct \( a^k(\nu) \) by the even extension of \( \bar{a}^k(\nu) \) around the points \( \nu = 0 \) and \( \nu = \pi \). Therefore by (18) we have

\[
\begin{align*}
a^k(-\nu) &= a^k(\nu) \\
a^k(\nu + \pi) &= a^k(\pi - \nu) \\
a^k(\nu + n\pi) &= a^k(\nu + (n - 2)\pi), \\
&\quad k = 1, 2, 3
\end{align*}
\]

(19)

where \( a^k(\nu) = \bar{a}^k(\nu) \) for \( \nu \in [0, \pi] \). Under these conditions, with \( b^k(\nu) \) even, around \( \nu = 0 \) and \( \nu = \pi \), the function \( b^k(\nu)\Pi(\nu) \) is also even around the extreme points of the string, as we have initially assumed.

Therefore, (9) is also a solution of the D3-brane problem, where \( a^k(\nu) \) and \( b^k(\nu) \) are even extensions of \( \bar{a}^k(\nu) \) and \( \bar{b}^k(\nu) \) around the string end points. The even extension of the initial functions are calculated according to the recurrence relations (19), for both \( a^k(\nu) \) and \( b^k(\nu) \).

The general case for Dp-branes, with \( p = 1, 2 \) is simply calculated through (15), (16) and (19), with \( p + 1 \) Neumann and \( (3 - p) \) Dirichlet boundary conditions.
In Figs. 1, 2, 3 and 4 we represent the time evolution of finite length strings attached to D0, D2 and D3-branes. The numerical implementation of the solutions of the Cauchy problem for initially static strings shows that we have always periodic motion with periods $T = \ell$, Figs. 1 and 4, or, $T = 2\ell$, Figs. 2 and 3, where $\ell$ is the string length at $t = 0$. This differs from the free closed string case where the period is always $T = \ell/2$, [8].

The singularity structure of string solutions is easily analysed with the techniques developed in [8]. In the case of finite open strings attached to two D0-branes no singularities occur, Fig. 1, as in the case of a finite free open string (attachment to a D3-brane), Fig. 4. However, in the cases of Figs. 2 and 3, singular solutions occur for times $t = (2n + 1)T/4$, with $n \geq 0$, as already observed in the free closed string case.

In Fig. 1, the initial string configuration has an expansion in Fourier series with only one non-zero eigenmode. However, for $t > 0$, as clearly seen from the figures, the eigenmodes become time $t(\tau, \nu)$ dependent and the energy flows from eigenmode to eigenmode, with periodicity $T$. This could have consequences for the quantization of open strings [7]. Generically, strings attached to D2-branes, Fig. 3, intersect the brane, introducing some difficulties in the analysis of systems made of strings attached to D$p$-branes, namely, we would need dynamical branes and/or interaction terms between the brane and the string.

General properties extracted from the figures indicate that for ”closed” strings (where the closure is performed by the brane) the period is $T = 2\ell$ and a singularity occurs at $t = \ell/2 = T/4$. In the case of Figs. 2a) and 3, the singularity is a single point; while in the case of Fig. 2b), the treefoil knot, the singularity is a collection of three points: the projection of the knot onto a planar surface. For ”open” strings the period is $T = \ell$ and no singularities seem to occur, Figs. 1 and 4.

In conclusion, we have solved explicitly the Cauchy problem for strings attached to D$p$-branes, with $p = 0, 1, 2, 3$. Initially static strings attached to D$p$-branes oscillate with periods $T = \ell/c$ or $T = 2\ell/c$, contrasting with the period $T = \ell/2c$ for the free closed string. Singularities occur throughout string motion, with more complex patterns than in the free closed string.

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Figure Captions

**Figure 1:** Time evolution of a finite length string attached to two D0-branes, and initial data $a^1(\nu) = \nu$, $a^2(\nu) = \sin(2\nu)$, $a^3(\nu) = 0$ and $b^k(\nu) = 0$. The motion is periodic with period $T = \ell$, where $\ell$ is the string length at $t = 0$. The two D0-branes are represented by large black dots with static space coordinates $(X_1^1 = 0, X_1^2 = 0, X_1^3 = 0)$ and $(X_2^1 = 1, X_2^2 = 0, X_2^3 = 0)$. We display a tubular neighborhood around the string in order to better depict the knot topology.

**Figure 2:** Time evolution of closed strings attached to a D0-brane. The motion is periodic with period $T = 2\ell$, where $\ell$ is the string length at $t = 0$. In case a) the initial string configuration is a closed loop and the string solutions shows a singularity for $t = \ell/2$. In case b) the initial string configuration is a treefoil knot and during time evolution the string solution has several singularities corresponding to the crossover of string strands, and in particular it also shows a singularity for $t = \ell/2$.

**Figure 3:** Time evolution of a finite length string attached to a D2-brane in the $(x, y)$ plane, and initial data $a^1(\nu) = \cos(\nu)$, $a^2(\nu) = \cos(\nu)$, $a^3(\nu) = \sin(\nu)$ and $b^k(\nu) = 0$. The motion is periodic with period $T = 2\ell$. For $t = \ell/2$, the string solution has a singularity, collapsing into the D2-brane.

**Figure 4:** Time evolution of a finite length string attached to a D3-brane. The motion is periodic with period $T = \ell$. 
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