Topological invariants of symmetry-protected and symmetry-enriched topological phases of interacting bosons or fermions

Xiao-Gang Wen\textsuperscript{1,2,3}

\textsuperscript{1}Perimeter Institute for Theoretical Physics, Waterloo, Ontario, N2L 2Y5 Canada
\textsuperscript{2}Department of Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139, USA
\textsuperscript{3}Institute for Advanced Study, Tsinghua University, Beijing, 100084, P. R. China

Recently, it was realized that quantum states of matter can be classified as long-range entangled (LRE) states (i.e. the topologically ordered states) and short-range entangled (SRE) states. The SRE states with a symmetry \(SG\) [named as symmetry-protected topological (SPT) states] are shown to be classified by group cohomology class \(\mathcal{H}^d(SG, \mathbb{F}/\mathbb{Z})\) in \(d\)-dimensional space-time. The LRE states with a symmetry \(SG\) are named as symmetry-enriched topological (SET) states. One class of SET states are described by weak-coupling gauge theories with gauge group \(GG\) and quantized topological terms. Those SET states (i.e. the quantized topological terms) are classified \(\mathcal{H}^d(PSG, \mathbb{F}/\mathbb{Z})\) in \(d\) space-time dimensions, where the projective symmetry group \(PSG\) is an extension of \(SG\) by \(GG\): \(SG = PSG/GG\). In this paper, we study the physical properties of those SPT/SET states, such as the fractionalization of the quantum numbers of the global symmetry. Those physical properties are topological invariants of the SPT/SET states that allow us to experimentally or numerically detect those SPT/SET states, i.e. to measure the elements in \(\mathcal{H}^d(PSG, \mathbb{F}/\mathbb{Z})\) that label different SPT/SET states.

CONTENTS

I. Introduction 1

II. A duality relation between the SPT and the SET phases 3

A. A simple formal description 3
B. Exactly soluble gauge theory with a finite gauge group \(GG\) and a global symmetry group \(SG\) 3
   1. Discretize space-time 3
   2. Lattice gauge theory with a global symmetry 4

III. Physical properties and topological invariants of SPT states 5

A. A general discussion 5
B. Bosonic \(Z_2\) SPT phases 6
   1. Topological invariants in \((0+1)\)D 6
   2. Monodromy defect 6
   3. Topological invariant in \((2+1)\)D 6
C. Bosonic \(U(1)\) SPT phases 8
   1. Topological invariants in \((0+1)\)D 8
   2. Topological invariants in \((2+1)\)D 8
   3. Topological invariants in \((4+1)\)D 9
D. Fermionic \(U_f(1)\) SPT phases 9
   1. Symmetry in fermionic systems 9
   2. Topological invariant for fermionic \(U_f(1)\) SPT phases 9
E. Fermionic \(Z_2^f\) SPT phases 10

IV. Topological invariants of SPT states with symmetry \(G = GG \times SG\) 10

A. Bosonic \(U(1) \times \tilde{U}(1)\) SPT phases 10
B. Fermionic \(U(1) \times U_f(1)\) SPT phases 11
C. A general discussion for the case \(G = GG \times SG\) 11
D. An example with \(SG = U(1)\) and \(GG = U(1)\) 12
   1. Topological invariants in \((2+1)\)D 12
   2. Topological invariants in \((4+1)\)D 13
E. Bosonic \(Z_2^G \times Z_2^GG\) SPT states 13
   1. Topological invariants in \((2+1)\)D 13
   2. Topological invariants in \((3+1)\)D 14

3. Topological invariants in \((1+1)\)D 14
F. Bosonic \(U(1) \times Z_2\) SPT phases 14
G. Fermionic \(U(1) \times Z_2^f\) SPT phases 14
   1. Topological invariants in \((2+1)\)D 14
   2. Topological invariants in \((3+1)\)D 15
H. Fermionic \(Z_2 \times Z_2^f\) SPT states 15

V. Summary 16

A. Group cohomology theory 16
   1. Homogeneous group cocycle 16
   2. Nonhomogeneous group cocycle 17
   3. “Normalized” cocycles 17
B. The K"unneth formula 17
C. Lyndon-Hochschild-Serre spectral sequence 18

References 18

I. INTRODUCTION

For a long time, we thought that Landau symmetry breaking theory\textsuperscript{1–3} describes all phases and phase transitions. In 1989, through a theoretical study of high \(T_c\) superconducting model, we realized that there exists a new kind of orders – topological order – which cannot be described by Landau symmetry breaking theory.\textsuperscript{4–6} Recently, it was found that topological orders are related to long range entanglements.\textsuperscript{7,8} In fact, we can regard topological order as pattern of long range entanglements defined through local unitary (LU) transformations.\textsuperscript{9–12} The notion of topological orders and quantum entanglements leads to a more general and also more detailed picture of phases and phase transitions (see Fig. 1).\textsuperscript{9} For gapped quantum systems without any symmetry, their
quantum phases can be divided into two classes: short-range entangled (SRE) states and long-range entangled (LRE) states.

SRE states are states that can be transformed into direct product states via LU transformations. All SRE states can be transformed into each other via LU transformations. So all SRE states belong to the same phase (see Fig. 1a), i.e. all SRE states can continuously deform into each other without closing energy gap and without phase transition.

LRE states are states that cannot be transformed into direct product states via LU transformations. It turns out that, in general, different LRE states cannot be connected to each other through LU transformations. The LRE states that are not connected via LU transformations represent different quantum phases. Those different quantum phases are nothing but the topologically ordered phases. Chiral spin liquids, fractional quantum Hall states, torsional Hall states, etc are examples of topologically ordered phases.

The possible topological orders are very rich. The mathematical foundation of topological orders is closely related to tensor category theory\textsuperscript{9,10,24,25} and simple current algebra.\textsuperscript{20,26} Using this point of view, we have developed a systematic and quantitative framework for non-chiral topological orders in 2D interacting boson and fermion systems.\textsuperscript{9,10,25} Also for chiral 2D topological orders with only Abelian statistics, we find that we can use integer K-matrices to describe them.\textsuperscript{27–32}

For gapped quantum systems with symmetry, the structure of phase diagram is even richer (see Fig. 1b). Even SRE states now can belong to different phases. One class of non-trivial SRE phases for Hamiltonians with symmetry is the Landau symmetry breaking phases.

![FIG. 1: (Color online) (a) The possible gapped phases for a class of Hamiltonians $H(g_1, g_2)$ without any symmetry. (b) The possible gapped phases for the class of Hamiltonians $H_{\text{symm}}(g_1, g_2)$ with a symmetry. The yellow regions in (a) and (b) represent the phases with long range entanglement. Each phase is labeled by its entanglement properties and symmetry breaking properties. SRE stands for short range entanglement, LRE for long range entanglement, SB for symmetry breaking, SY for no symmetry breaking. SB-SRE phases are the Landau symmetry breaking phases. The SY-SRE phases are the SPT phases. The SY-LRE phases are the SET phases.](image)

But even SRE states that do not break the symmetry of the Hamiltonians can belong to different phases. The 1D Haldane phase for spin-1 chain\textsuperscript{33–36} and topological insulators\textsuperscript{37–42} are non-trivial examples of phases with short range entanglements that do not break any symmetry. We will call this kind of phases symmetry-protected trivial (SPT) phases or symmetry-protected topological (SPT) phases.\textsuperscript{35,36} Note that the SPT phases have no long range entanglements and have trivial topological orders.

It turns out that there is no gapped bosonic LRE state in 1D.\textsuperscript{11} So all 1D gapped bosonic states are either symmetry breaking states or SPT states. This realization led to a complete classification of all 1D gapped bosonic quantum phases.\textsuperscript{43–45}

In Ref. 46 and 47, the classification of 1D SPT phases is generalized to any dimensions: For gapped bosonic systems in $d$ space-time dimensions with an on-site symmetry $SG$, the SPT phases that do not break the symmetry $SG$ are classified by the elements in $\mathcal{H}^d[SG, \mathbb{R}/\mathbb{Z}]$ – the group cohomology class of the symmetry group $SG$. We see that we have a systematic understanding of SRE states with symmetry.\textsuperscript{48–50}

For gapped LRE states with symmetry, the possible quantum phases should be much richer than SRE states. We may call those phases Symmetry Enriched Topological (SET) phases. Projective symmetry group (PSG) was introduced to study the SET phases.\textsuperscript{51–53} The PSG describes how the quantum numbers of the symmetry group $SG$ get fractionalized on the gauge excitations.\textsuperscript{52} When the gauge group $GG$ is Abelian, the PSG description of the SET phases can be expressed in terms of group cohomology: The different SET states with symmetry $SG$ and gauge group $GG$ can be (partially) described by $\mathcal{H}^d(SG, GG)$.\textsuperscript{54}

One class of SET states in $d$ space-time dimensions with global symmetry $SG$ are described by weak-coupling gauge theories with gauge group $GG$ and quantized topological terms (assuming the weak-coupling gauge theories are gapped, that can happen when the space-time dimension $d = 3$ or when $d > 3$ and the gauge group $GG$ is finite). Those SET states (i.e. the quantized topological terms) are classified by the elements in $\mathcal{H}^d(PSG, \mathbb{R}/\mathbb{Z})$,\textsuperscript{55,56} where the group $PSG$ is an extension of $SG$ by $GG$: $SG = PSG/\mathbb{Z}$. Or in other words, we have a short exact sequence

$$1 \to GG \to PSG \to SG \to 1.$$  \hspace{1cm} (1)

We will denote $PSG$ as $PSG = GG \times SG$. Many examples of the SET states can be found in Ref. 48, 51, 57–59.

Although we have a systematic understanding of SPT phases and some of the SET phases in term of $\mathcal{H}^d(SG, \mathbb{R}/\mathbb{Z})$ and $\mathcal{H}^d(PSG, \mathbb{R}/\mathbb{Z})$, however, those results do not tell us to how experimentally or numerically measure the elements in $\mathcal{H}^d(SG, \mathbb{R}/\mathbb{Z})$ or $\mathcal{H}^d(PSG, \mathbb{R}/\mathbb{Z})$ that label the different SPT or SET phases. We do not know, even given an exact ground state wave function, how to determine which SPT or SET phase the ground state
belongs to. In this paper, we will address this important question. We will find physical ways to detect different SPT/SET phases and to measure the elements in $\mathcal{H}^d(SG, \mathbb{R}/\mathbb{Z})$ or $\mathcal{H}^d(PSG, \mathbb{R}/\mathbb{Z})$. This is achieved by gauging the symmetry group $SG$ by coupling the $SG$ quantum numbers to a $SG$ gauge potential $A^{SG}$. Note that $A^{SG}$ is treated as a non-fluctuating probe field. By study the topological response of the system to various $SG$ gauge configurations, we can measure the elements in $\mathcal{H}^d(SG, \mathbb{R}/\mathbb{Z})$ or $\mathcal{H}^d(PSG, \mathbb{R}/\mathbb{Z})$. Those topological response are the measurable topological invariants that characterize the SPT/SET phases.

II. A DUALITY RELATION BETWEEN THE SPT AND THE SET PHASES

There is a duality relation between the SPT and the SET phases described by weak-coupling gauge field.\cite{55,56,60} We first give a simple formal description of such a duality relation. Then we will give an exact description for finite gauge groups.

A. A simple formal description

To understand such the duality between the SPT and the SET phases, we note that a SPT state with symmetry $G$ in $d$-dimensional space-time $M$ can be described by a non-linear $\sigma$-model with $G$ as the target space

$$S = \int_M d^d x \left[ \frac{1}{\lambda_s} [\partial g(x^\mu)]^2 + iW_{\text{top}}(g) \right].$$

in large $\lambda_s$ limit. Here we triangulate the $d$-dimensional space-time manifold $M$ to make it a random lattice or a $d$-dimensional complex, and $g(x^\mu)$ live on the vertices of the complex: $g(x^\mu) = \{ g_i \}$ where $i$ labels the vertices (the lattice sites). So $\int d^d x$ is in fact a sum over lattice sites and $\partial$ is the lattice difference operator. The above action $S$ actually defines a lattice theory. $W_{\text{top}}[g(x^\mu)]$ is a lattice topological term which satisfy

$$\int_M d^d x \, W_{\text{top}}(\{ g_i \}) = 0 \text{ mod } 2\pi,$$

where $\{ g_i \}$ are the group cocycles. Thus the lattice topological term $W_{\text{top}}(\{ g_i \})$ is defined and classified by the elements (the cocycles) in $\mathcal{H}^d(G, \mathbb{R}/\mathbb{Z})$.\cite{47,61} This is why the bosonic SPT states are classified by $\mathcal{H}^d(G, \mathbb{R}/\mathbb{Z})$.

If $G$ contains a normal subgroup $GG \subset G$, we can "gauge" $GG$ to obtain a gauge theory in the bulk

$$S = \int d^d x \left[ \frac{[\partial g(x^\mu)]^2}{\lambda_s} + \frac{\text{Tr}(F_{\mu\nu})^2}{\lambda} + iW_{\text{top}}^{\text{gauge}}(g, A) \right],$$

where $A$ is the $GG$ gauge potential. When $\lambda$ is small the above theory is a weak-coupling gauge theory with a gauge group $GG$ and a global symmetry group $SG = G/\langle GG \rangle$.

The topological term $W_{\text{top}}^{\text{gauge}}(g, A)$ in the gauge theory is a generalization of the Chern-Simons term,\cite{32,33,64} which is obtained by "gauging" the topological term $W_{\text{top}}(g)$ in the non-linear $\sigma$-model. The two topological terms $W_{\text{top}}^{\text{gauge}}(g, A)$ and $W_{\text{top}}(g)$ are directly related when $A$ is a pure gauge:

$$W_{\text{top}}^{\text{gauge}}(g, A) = W_{\text{top}}[h(x)g(x)],$$

where $A = h^{-1}\partial h, \, h \in GG$.\cite{5} (A more detailed description of the two topological terms $W_{\text{top}}(g)$ and $W_{\text{top}}^{\text{gauge}}(g, A)$ on lattice can be found in Ref. 63 and 64. See also the next section.) So the topological term $W_{\text{top}}^{\text{gauge}}(g, A)$ in the gauge theory is also classified by same $\mathcal{H}^d(G, \mathbb{R}/\mathbb{Z})$ that classifies $W_{\text{top}}(g)$. (We like to remark that although both topological terms $W_{\text{top}}(g)$ and $W_{\text{top}}^{\text{gauge}}(A)$ are classified by the same $\mathcal{H}^d(G, \mathbb{R}/\mathbb{Z})$, when $\mathcal{H}^d(G, \mathbb{R}/\mathbb{Z}) = \mathbb{Z}$, the correspondence can be tricky: for a topological term $W_{\text{top}}(g)$ that corresponds to an integer $k$ in $\mathcal{H}^d(G, \mathbb{R}/\mathbb{Z})$, its corresponding topological term $W_{\text{top}}^{\text{gauge}}(g, A)$ may correspond to an integer $nk$ in $\mathcal{H}^d(G, \mathbb{R}/\mathbb{Z})$. However, for finite group $G$, the correspondence is one-to-one.)

When the space-time dimensions $d = 3$ or when $d > 3$ and $GG$ is a finite group, the theory (4) is gapped in $\lambda_s \to \infty$ and $\lambda \to 0$ limit, which describe a SET phase with symmetry group $SG$ and gauge group $GG$. Such SET phase are classified by $\mathcal{H}^d(G, \mathbb{R}/\mathbb{Z})$.

B. Exactly soluble gauge theory with a finite gauge group $GG$ and a global symmetry group $SG$

To understand the above formal results more rigorously, we would like to review the exactly soluble models of weak-coupling gauge theories with a finite gauge group $GG$ and a global symmetry group $SG$. The exactly soluble models were introduced in Ref. 20, 56, 60, and 65. The exactly soluble models is defined on a space-time lattice, or more precisely, a triangulation of the space-time. So we will start by describing such a triangulation.

1. Discretize space-time

Let $M_{\text{tri}}$ be a triangulation of the $d$-dimensional space-time. We will call the triangulation $M_{\text{tri}}$ as a space-time complex, and a cell in the complex as a simplex.
In order to define a generic lattice theory on the space-time complex $M_{\text{tri}}$, it is important to give the vertices of each simplex a local order. A nice local scheme to order the vertices is given by a branching structure.\textsuperscript{37,61,66} A branching structure is a choice of orientation of each edge in the $d$-dimensional complex so that there is no oriented loop on any triangle (see Fig. 2).

The branching structure induces a local order of the vertices on each simplex. The first vertex of a simplex is the vertex with no incoming edges, and the second vertex is the vertex with only one incoming edge, etc. So the simplex in Fig. 2a has the following vertex ordering: 0, 1, 2, 3.

The branching structure also gives the simplex (and its sub simplexes) an orientation denoted by $s_{ij\ldots k} = \pm 1$. Fig. 2 illustrates two 3-simplexes with opposite orientations $s_{0123} = 1$ and $s_{0123} = \ast$. The red arrows indicate the orientations of the 2-simplexes which are the sub simplexes of the 3-simplexes. The black arrows on the edges indicate the orientations of the 1-simplexes.

2. Lattice gauge theory with a global symmetry

To define a lattice gauge theory with a gauge group $GG$ and a global symmetry group $SG$, let $G$ be an extension of $SG$ by $GG$: $G = GG \times SG$. Here we will assume $GG$ to be a finite group.

In our lattice gauge theory, the degrees of freedom on the vertices of the space-time complex, is described by $g_i \in G$ where $i$ labels the vertices. The gauge degrees of freedom are on the edges $ij$ which are described by $h_{ij} \in GG$.

The action amplitude $e^{-S}$ for a d-cell $(ij\ldots k)$ is complex function of $g_i$ and $h_{ij}$: $V_{ij\ldots k}(\{h_{ij}\}, \{g_i\})$. The total action amplitude $e^{-S}$ for configuration (or a path) is given by

$$e^{-S} = \prod_{(ij\ldots k)} [V_{ij\ldots k}(\{h_{ij}\}, \{g_i\})]^{s_{ij\ldots k}}$$

(6)

where $\prod_{(ij\ldots k)}$ is the product over all the d-cells $(ij\ldots k)$. Note that the contribution from a d-cell $(ij\ldots k)$ is $V_{ij\ldots k}(\{h_{ij}\}, \{g_i\})$ or $V_{ij\ldots k}^*(\{h_{ij}\}, \{g_i\})$ depending on the orientation $s_{ij\ldots k}$ of the cell. Our lattice theory is defined by following imaginary-time path integral (or partition function)

$$Z = \sum_{\{h_{ij}\}, \{g_i\}} \prod_{(ij\ldots k)} [V_{ij\ldots k}(\{h_{ij}\}, \{g_i\})]^{s_{ij\ldots k}}$$

(7)

If the above action amplitude $\prod_{(ij\ldots k)} [V_{ij\ldots k}(\{h_{ij}\}, \{g_i\})]^{s_{ij\ldots k}}$ on closed space-time complex $(\partial M_{\text{tri}} = \emptyset)$ is invariant under the gauge transformation

$$h_{ij} \rightarrow g^{-1}_{ij}h_{ij}g_{ij}, g_i \rightarrow g'_i = gg_i, g \in G,$$

then the action amplitude $V_{ij\ldots k}(\{h_{ij}\}, \{g_i\})$ defines a gauge theory of gauge group $GG$. If the action amplitude is invariant under the global transformation

$$h_{ij} \rightarrow h'_{ij} = gh_{ij}g^{-1}, g_i \rightarrow g'_i = gg_i, g \in G,$$

then the action amplitude $V_{ij\ldots k}(\{h_{ij}\}, \{g_i\})$ defines a $GG$ lattice gauge theory with a global symmetry $SG = G/GG$. (We need to mod out $GG$ since when $h \in GG$, it is a part of gauge transformation which does not change the physical states, instead of a global symmetry transformation which change a physical state to another one.)

Using a cocycle $\nu_d(g_0, g_1, \ldots, g_d) \in \mathcal{H}^d(G, \mathbb{R}/{\mathbb{Z}})$, $g_i \in G$ ($\nu_d(g_0, g_1, \ldots, g_d)$ is a real function over $G^{d+1}$), we can construct an action amplitude $V_{ij\ldots k}(\{h_{ij}\}, \{g_i\})$ that define a gauge theory with gauge group $SG$ and global symmetry $SG$. The gauge theory action amplitude is obtained from $\nu_d(g_0, g_1, \ldots, g_d)$ as

$$V_{01\ldots d}(\{h_{ij}\}, \{g_i\}) = 0, \text{if } h_{ij}h_{jk} \neq h_{ik}$$

$$V_{01\ldots d}(\{h_{ij}\}, \{g_i\}) = e^{2\pi i \omega_d(g_0^{-1}h_{01}g_1^{-1}, \ldots, g_d^{-1}h_{d-1}g_d)}, \text{if } h_{ij}h_{jk} = h_{ik},$$

where $h_i$ are given by

$$h_0 = 1, \ h_1 = h_0h_{01}, \ h_2 = h_1h_{12}, \ h_3 = h_2h_{23}, \cdots$$

and $\omega_d$ is the nonhomogenous cocycle that corresponds to $\nu_d$

$$\omega_d(h_{01}, h_{12}, \cdots, h_{d-1}d) = \nu_d(h_0, h_1, \cdots, h_d).$$

To see the above action amplitude defines a $GG$ lattice gauge theory with a global symmetry $SG$, we note that the cocycle satisfies the cocycle condition

$$\nu_d(g_0, g_1, \ldots, g_d) = \nu_d(g_0, g_g, \ldots, g_d) \mod 1, \ g \in G$$

$$\sum_i \nu_d(g_0, \ldots, \hat{g}_i, \ldots, g_{d+1}) = 0 \mod 1$$

(13)}
TOPOLOGICAL INVARIANTS OF SPT STATES

III. PHYSICAL PROPERTIES AND TOPOLOGICAL INVARIANTS OF SPT STATES

Because of the duality relation between the SPT states and the SET states described by weak-coupling gauge theories, in this paper, we will mainly discuss the physical properties and the topological invariants of the SPT state. The physical properties and the topological invariants of the SET states can be obtained from the physical properties and the topological invariants of corresponding SPT states via the duality relation.

A. A general discussion

Let us consider a system with symmetry group $G$ in $d$ space-time dimensions. The ground state of the system is a SPT state described by an element $\nu_d$ in $\mathcal{H}^d(G, \mathbb{R}/\mathbb{Z})$. But how to physically measure $\nu_d$? Here we will propose to measure $\nu_d$ by “gauging” the symmetry $G$, i.e. by introducing a $G$ gauge potential $A_\mu(x^\nu)$ to couple to the quantum numbers of $G$. The $G$ gauge potential is a fixed probe field. So $A_\mu$ is not a dynamical field. It is like local coupling constants in the theory. We like to consider how the system responds to various $G$ gauge configurations described by $A_\mu$. We will show that the topological responses allow us to fully measure the cocycle $\nu_d$ that characterizes the SPT phase. Those topological responses are the topological invariants that we are looking for.

There are several topological responses that we can use to construct topological invariants:

1. If the $G$ gauge configuration $A_\mu(x^\nu)$ is time independent and is invariant under a subgroup $GG$ of $G$: $A_\mu(x^\nu) = h^{-1}A_\mu(x^\nu)h$, $h \in GG$, then we can study the conserved $GG$ quantum number of the ground state under such gauge configuration. Some times, the ground states may be generate and may a higher dimensional representation of $GG$.

In particular, we can remove $n$ identical regions $D(i)$, $i = 1, \cdots, n$, from the space $M_{d-1}$ to get a $(d-1)$-dimensional manifold $M'_{d-1}$ with $n$ “holes”. Then we consider a flat $G$ gauge configuration $A_\mu(x^\nu)$ on $M'_{d-1}$ such that the gauge fields near the boundary of those holes, $\partial D(i)$, are identical. We then measure the conserved $GG$ quantum number one the ground state for such $G$ gauge configuration. We will see that the $GG$ quantum number may not be multiples of $n$, indicating a non-trivial SPT phases.

2. We start with a $G$ gauge configuration $A_\mu(x^\nu)$ in space, and then use an element $h \in GG \subset G$ to transform $A_\mu(x^\nu)$ to $A_\nu(x^\nu) = h^{-1}A_\mu(x^\nu)h$. Let $|h\rangle$ be the ground state of the system with the gauge configuration described by $A_\nu(x^\nu)$. Now, we allow $h$ to be time dependent and derive the effective theory for $h$. The effective theory is obtained from the coherent state $|h\rangle$ using the coherent state path integral approach, where the phase-space Lagrangian is given by

$$L(h, \dot{h}) = i\langle h | \frac{d}{dt} |h\rangle - \langle h | H(A^b) |h\rangle$$

where $H(A^b)$ is the Hamiltonian with $A_\nu(x^\nu)$ gauge configuration. Note that $\langle h | H(A^b) |h\rangle$ is independent of $h$. This will allow us to determine the $GG$ quantum number of the ground state. Again, we consider space with $n$ identical holes and consider only flat $G$ gauge configurations.

3. We may choose the space to have a form $M_k \times M_{d-k-1}$ where $M_k$ is a closed $k$-dimensional manifold or a closed $k$-dimensional manifold with $n$ identical holes. $M_{d-k-1}$ is a closed $(d-k-1)$-dimensional manifold. We then put a $G$ gauge configuration $A_\mu(x^\nu)$ on $M_k$, or a flat $G$ gauge configuration on $M_k$ if $M_k$ has $n$ holes. In the large $M_{d-k-1}$ limit, our system can be viewed as a system in $(d-k-1)$-dimensional space with a symmetry $GG$, where $GG \subset G$ is formed by the symmetry transformations that leave the $G$ gauge configuration invariant. The ground state of the system is a SPT state characterized by cocycles in $\mathcal{H}^{d-k}(GG, \mathbb{R}/\mathbb{Z})$.

4. The above topological responses can be easily measured in a Hamiltonian formulation of the system. In the imaginary-time path-integral formulation of the system where the space-time manifold $M_d$ can...
have an arbitrary topology, we can use a most general construction of topological invariants. We simply put a nearly-flat $G$ gauge configuration on a closed space-time manifold $M_d$ and evaluate the path integral. We will obtain a partition function $Z(M_d, A_\mu)$ which is a function of the space-time topology $M_d$ and the nearly-flat gauge configuration $A_\mu$. In the limit of the large volume $V$ of the space-time, $Z(M_d, A_\mu)$ has a form

$$Z(M_d, A_\mu) = e^{-\int_0^V Z_{\text{top}}(M_d, A_\mu)}, \quad (15)$$

where $Z_{\text{top}}(M_d, A_\mu)$ is independent of the volume $V$. $Z_{\text{top}}(M_d, A_\mu)$ is a topological invariant that allows us to fully measure the elements in $\mathcal{H}^d(G, \mathbb{R}/\mathbb{Z})$ that classify the SPT phases.\textsuperscript{62-64} In fact, $Z_{\text{top}}(M_d, A_\mu)$ is the partition function for the pure topological term $W_{\text{top}}^\text{gauge}(g, A)$ in eqn. (4).

We like to point out an element in the free part of $\mathcal{H}^d(G, \mathbb{R}/\mathbb{Z})$ gives rise to a Chern-Simons term in $Z_{\text{top}}(M_d, A_\mu)$. An element in the torsion part of $\mathcal{H}^d(G, \mathbb{R}/\mathbb{Z})$ gives rise to a topological term in $Z_{\text{top}}(M_d, A_\mu)$ whose value is independent of small perturbations of $A_\mu$.\textsuperscript{63}

In the following, we will illustrate the above construction of topological invariants using some simple examples. We will show that the constructed topological invariants can fully characterize those SPT phases.

**B. Bosonic $Z_2$ SPT phases**

1. **Topological invariants in (0+1)D**

In 1-dimensional space-time, the bosonic SPT states with symmetry $Z_2 = \{1, -1\}$ are classified by the cocycles in $\mathcal{H}^1(Z_2, \mathbb{R}/\mathbb{Z}) = Z_2$. How to measure the cocycles in $\mathcal{H}^1(Z_2, \mathbb{R}/\mathbb{Z})$? What is the measurable topological invariant that allows us to characterize the $Z_2$ SPT states?

The non-trivial cocycle in $\mathcal{H}^1(Z_2, \mathbb{R}/\mathbb{Z})$ is given by

$$\omega_1(1) = 0, \quad \omega_1(-1) = 1/2. \quad (16)$$

Let us assume the space-time is a circle $S_1$ formed by a ring of vertices labeled by $i$. A flat $Z_2$ gauge configuration on $S_1$ is given by $Z_2$ group elements $g_i, g_{i+1}$ on each link $(i, i+1)$. The topological part of the partition function for such a flat $Z_2$ gauge configuration is given by the cocycle $\omega_1$

$$Z_{\text{top}}(S_1, A_\mu) = e^{i 2\pi \sum_i \omega_1(g_i, g_{i+1})}. \quad (17)$$

We note that the above $\omega_1(g_i, g_{i+1})$ is a torsion element in $\mathcal{H}^1(Z_2, \mathbb{R}/\mathbb{Z})$. So it gives rise to a quantized topological term $Z_{\text{top}}(S_1, A_\mu)$:

$$Z_{\text{top}}(S_1, A_\mu) = 1, \quad \text{if} \quad \prod_i g_i, g_{i+1} = 1,$$

$$Z_{\text{top}}(S_1, A_\mu) = -1, \quad \text{if} \quad \prod_i g_i, g_{i+1} = -1. \quad (18)$$

Such a partition function is a topological invariant. Its non-trivial dependence on the total $Z_2$ flux through the circle, $\prod_i g_i, g_{i+1}$, implies that the SPT state is non-trivial.

The above partition function also implies that the ground state of the system carries a non-trivial $Z_2$ quantum number. Thus the non-trivial $Z_2$ quantum number of the ground state also measure the non-trivial cocycle in $\mathcal{H}^1(Z_2, \mathbb{R}/\mathbb{Z})$.

In 3-dimensional space-time, the bosonic $Z_2$ SPT states are classified by the cocycles in $\mathcal{H}^3(Z_2, \mathbb{R}/\mathbb{Z}) = Z_2$. To find the topological invariants for such a case, let us introduce the notion of monodromy defect.\textsuperscript{60}

2. **Monodromy defect**

Let us assume that the 2D lattice Hamiltonian for the $Z_2$ SPT state has a form (see Fig. 3)

$$H = \sum_{(ijk)} H_{ijk}, \quad (19)$$

where $\sum_{(ijk)}$ sums over all the triangles in Fig. 3 and $H_{ijk}$ acts on the states on site-$i$, site-$j$, and site-$k$: $(g_{i}, g_{j}, g_{k})$. (Note that the states on site-$i$ are labeled by $g_i \in Z_2$.) $H$ and $H_{ijk}$ are invariant under the global $Z_2$ transformations.

Let us perform a $Z_2$ transformation only in the shaded region in Fig. 3. Such a transformation will change $H$ to $H'$. However, only the Hamiltonian terms on the triangles $(ijk)$ across the boundary are changed from $H_{ijk}$ to $H'_{ijk}$. Since the $Z_2$ transformation is an unitary transformation, $H$ and $H'$ have the same energy spectrum. In other words the boundary in Fig. 3 (described by $H'_{ijk}$’s) do not cost any energy.

Now let us consider a Hamiltonian on a lattice with a “cut” (see Fig. 4)

$$\tilde{H} = \sum_{(ijk)}' H_{ijk} + \sum_{(ijk)}^\text{cut} H_{ijk} \quad (20)$$

where $\sum_{(ijk)}'$ sums over the triangles not on the cut and $\sum_{(ijk)}^\text{cut}$ sums over the triangles that are divided into disconnected pieces by the cut. The triangles at the ends of the cut have no Hamiltonian terms. We note that the cut carries no energy. Only the ends of cut cost energies. Thus we say that the cut corresponds to two monodromy defects. The Hamiltonian $\tilde{H}$ defines the two monodromy defects.

We also like to point that the above procedure to obtain $\tilde{H}$ is actually the “gauging” of the $Z_2$ symmetry. $\tilde{H}$ is a gauged Hamiltonian that contain two $Z_2$ vortices at the ends of the cut.

3. **Topological invariant in (2+1)D**

The topological invariant to detect the cocycle in $\mathcal{H}^3(Z_2, \mathbb{R}/\mathbb{Z})$ is the $Z_2$ quantum number of two identical
monodromy defects (see Fig. 4). Note that both monodromy defects or $Z_2$ vortices correspond to the same kind of $\triangle$ triangles.

To calculate the $Z_2$ quantum number of two monodromy defects (or two $Z_2$ vortices), we need to compare the phases of the ground state wave function for configurations $\{g_i\}$ and $\{gg_i\}$. Such a phase difference is given by the evolution from $\{g_i\}$ to $\{gg_i\}$. In the gauged theory, such a evolution is given by a $Z_2$ gauge configuration on space-time where the $Z_2$ gauge fields $g_{ij} \in Z_2 = \{1, -1\}$ on the spatial links are the same on the two time slices: $g_{01} = g_{01'}$, $g_{12} = g_{12'}$, $g_{02} = g_{02'}$, and the $Z_2$ gauge fields in the time links are given by $g_{00'} = g_{11'} = g_{22'} = g$ (see Fig. 5). The $Z_2$ gauge field on the other links are determined by the zero-flux condition $g_{ij}g_{jk} = g_{ik}$.

The exactly soluble $Z_2$ SPT model is described by a path integral defined by a cocycle in $\mathcal{H}^3(G, \mathbb{R}/\mathbb{Z})$. The path integral amplitude on the space-time complex in Fig. 5 is given by the product of three nonhomogeneous cocycles on the three tetrahedrons that form the complex:

$$U(g; g_{01'}, g_{12'}, g_{02'}) = \frac{e^{i2\pi \omega_3(g_{01'}g_{12'}g_{22'})} e^{i2\pi \omega_3(g_{00'}g_{01'}g_{12'})} e^{i2\pi \omega_3(g_{11'}g_{12'}g_{22'})}}{e^{i2\pi \omega_3(g_{01}g_{12}g_{22})}}$$

(21)

The non-trivial element in $\mathcal{H}^4(G, \mathbb{R}/\mathbb{Z})$ can be described by a nonhomogeneous cocycle

$$\omega_3(-1, -1, -1) = 1/2, \quad \omega_3(g_{01}, g_{12}, g_{23}) = 0 \text{ otherwise.}$$

(22)

We find that (see Fig. 6)

$$U(1, g_{01}, g_{12}, g_{02}) = 1,$$
$$U(-1, -1, -1, g_{02}) = -1,$$
$$U(-1, g_{01}, g_{12}, g_{02}) = 1 \text{ otherwise.}$$

(23)

The total $Z_2$ representation is given by

$$U(g) = \prod_{(ijk)} U^{s_{ijk}}(g; g_{01}, g_{12}, g_{02}),$$

(24)

where $s_{ijk}$ describes the orientation of the triangle $(ijk)$, and $\prod_{(ijk)}$ is a product over all the triangles that are not monodromy defects ($i.e.$ contain no $Z_2$-flux).

This allows us to show that two identical $Z_2$ vortices $\triangle$ and $\triangle$ have a total $Z_2$-charge 1 (see Fig. 4).

While two non-identical $Z_2$ vortices $\triangle$ and $\triangle$ have a total $Z_2$-charge 0 (see Fig. 7). Thus, we can say that the $Z_2$ vortex $\triangle$ has a $Z_2$-charge 1/2, while the $Z_2$ vortex $\triangle$ has a $Z_2$-charge $-1/2$. The fractional $Z_2$-charge on the $Z_2$ vortices (i.e. the monodromy defects) is our
topological invariant. Such a topological invariant can be measured by detecting an odd total $Z_2$-charge on two identical $Z_2$ vortices (i.e. on two identical monodromy defects).

We can easily generalize the above construction to obtain the topological invariant for $Z_n$ SPT states in 3-dimensional space-time. We simply need to consider $n$ identical $Z_n$ monodromy defects on a close 2D space and measure the $Z_n$-charge of the ground state.

We can also generalize the above construction to 5-dimensional space-time where $Z_n$ SPT states are classified by $H^5(Z_n, \mathbb{R}/\mathbb{Z}) = Z_n$. We choose the 4D space to have a topology $M_2 \times M'_2$ where $M_2$ and $M'_2$ are two close 2D manifolds. We then create $n$ identical $Z_n$ monodromy defects on $M'_2$. In the small $M'_2$ limit, we may view our 4D $Z_n$ SPT state on $M_2 \times M'_2$ as a 2D $Z_n$ SPT state on $M_2$ which is classified by $H^3(Z_n, \mathbb{R}/\mathbb{Z})$. In the above we have just discussed how to detect the cocycles in $H^3(Z_n, \mathbb{R}/\mathbb{Z})$, by just creating $n$ identical $Z_n$ monodromy defects on $M_2$, and then measure the $Z_n$-charge of the ground state. So the cocycles in $H^3(Z_n, \mathbb{R}/\mathbb{Z})$ can be measured by creating $n$ identical $Z_n$ monodromy defects on $M_2$ and $n$ identical $Z_n$ monodromy defects on $M'_2$. Then we measure the $Z_n$-charge of the corresponding ground state.

The above construction of $Z_n$ topological invariant is motivated by the following mathematical result. First $H^{2k+1}(Z_n, \mathbb{R}/\mathbb{Z}) \simeq H^{2k+2}(Z_n, \mathbb{Z})$. The generating cocycle $c_{2k+2}$ in $H^{2k+2}(Z_n, \mathbb{Z})$ can be expressed as a wedge product $c_{2k+2} = c_2 \wedge c_2 \wedge \cdots \wedge c_2$ where $c_2$ is the generating cocycle in $H^2(Z_n, \mathbb{Z})$. Since $H^2(Z_n, \mathbb{Z}) \simeq H^1(Z_n, \mathbb{R}/\mathbb{Z})$, we can replace one of $c_2$ in $c_{2k+2} = c_2 \wedge c_2 \wedge \cdots \wedge c_2$ by $\theta_1$ in $H^1(Z_n, \mathbb{R}/\mathbb{Z})$, and write $c_{2k+2} = \theta_1 \wedge c_2 \wedge \cdots \wedge c_2$. Note that $c_2 \wedge \cdots \wedge c_2$ describes the topological gauge configuration on $2k$ dimensional space, while $\theta_1$ describes the 1D representation of $Z_n$. This motivates us to use a $Z_n$ gauge configuration on $2k$ dimensional space to generate a non-trivial $Z_n$-charge in the ground state. In the next section, we use the similar idea to construct the topological invariant for bosonic $U(1)$ SPT states.

C. Bosonic $U(1)$ SPT phases

1. Topological invariants in (0+1)D

In 1-dimensional space-time, the bosonic SPT states with symmetry $U(1) = \{e^{i\theta}\}$ are classified by the cocycles in $H^1(U(1), \mathbb{R}/\mathbb{Z}) = Z$. Let us first study the topological invariant from the topological partition function.

A non-trivial cocycle in $H^1(Z_2, \mathbb{R}/\mathbb{Z}) = Z$ labeled integer $k$ is given by

$$\omega_1(e^{i\theta}) = e^{ik\theta}. \quad (25)$$

Let us assume the space-time is a circle $S_1$ formed by a ring of vertices labeled by $i$. A flat $U(1)$ gauge configuration on $S_1$ is given the $U(1)$ group elements $e^{i\theta_{i,i+1}}$ on each link $(i, i+1)$. The topological part of the partition function for such a flat $U(1)$ gauge configuration is determined by the above cocycle $\omega_1$

$$Z_{\text{top}}(S_1, A_\mu) = e^{i2\pi \sum_i \omega_1(\theta_{i,i+1})}. \quad (26)$$

We note that the above $\omega_1(\theta_{i,i+1})$ is a free element in $H^1(Z_2, \mathbb{R}/\mathbb{Z})$. So it gives rise to a Chern-Simons-type topological term $Z_{\text{top}}(S_1, A_\mu)$:

$$Z_{\text{top}}(S_1, A_\mu) = e^{i k \sum_i \theta_{i,i+1}} = e^{i k \oint dt A_0} \quad (27)$$

(Not that $\oint dt A_0$ is the $U(1)$ Chern-Simons term in 1D.) Such a partition function is a topological invariant. Its non-trivial dependence on the total $U(1)$ flux through the circle, $\sum_i \theta_{i,i+1} = \oint dt A_0$, implies that the SPT state is non-trivial.

The above partition function also implies that the ground state of the system carries a $U(1)$ quantum number $k$. Thus the non-trivial $U(1)$ quantum number $k$ of the ground state also measure the non-trivial cocycle in $H^1(U(1), \mathbb{R}/\mathbb{Z})$.

2. Topological invariants in (2+1)D

In 3-dimensional space-time, the bosonic $U(1)$ SPT states are classified by the cocycles in $H^3(U(1), \mathbb{R}/\mathbb{Z}) = Z$. How to measure the cocycles in $H^3(U(1), \mathbb{R}/\mathbb{Z})$? One way is to “gauge” the $U(1)$ symmetry and put the “gauged” system on a 2D closed space $M_2$. We choose a $U(1)$ gauge configuration on $M_2$ such that there is a unit of $U(1)$-flux. We then measure the $U(1)$-charge $k$ of the ground state on $M_2$. We will show that $k$ is an even integer and $k/2 \in Z$ is the topological invariant that characterizes the $U(1)$ SPT states. In fact, such a topological invariant is actually the quantized Hall conductance, which is quantized as an even integer $\sigma_{xy} = \frac{k}{2\pi}$.\cite{8, 49, 67-69}

To show the above result, let us use the following $U(1) \times U(1)$ Chern-Simons theory to describe the $U(1)$ SPT state\cite{49, 69}

$$L = \frac{1}{4\pi} K_{IJ} a_{I\mu} \partial_\mu a_{J\lambda} e^{\mu \nu \lambda} + \frac{1}{2\pi} q_1 A_\mu \partial_\mu a_{I\lambda} e^{\mu \nu \lambda} + \cdots$$

(28)
with the $K$-matrix and the charge vector $q$:\cite{27,28,30}

$$K = \begin{pmatrix} 0 & 1 \\ 1 & 2 - k \end{pmatrix}, \quad q = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad k = \text{even.}$$  \tag{29}

The Hall conductance is given by

$$\sigma_{xy} = (2\pi)^{-1} q^T K^{-1} q = \frac{k}{2\pi}. \tag{30}$$

If we write the topological partition function as $Z_{\text{top}}(M_4, A_\mu) = \exp \int d^4x L_{\text{top}}$, the above Hall conductance implies that topological partition function is given by a 3D Chern-Simons term

$$L_{\text{top}} = \frac{k}{4\pi} A_\mu \partial_\mu A_\lambda \epsilon^{\mu\nu\lambda} \tag{31}$$

### 3. Topological invariants in (4+1)D

In 5-dimensional space-time, the bosonic $U(1)$ SPT states are also classified by $\mathcal{H}^3(U(1), \mathbb{R}/\mathbb{Z}) = \mathbb{Z}$. Again, one can “gauged” the $U(1)$ symmetry and put the “gauged” system on a 4D closed space $M_4$. We choose a $U(1)$ gauge configuration on $M_4$ such that

$$\int_{M_4} F^2 \frac{8\pi^2}{k} = 1, \tag{32}$$

where $F$ is the two-form $U(1)$ gauge field strength. We then measure the $U(1)$-charge $k$ of the ground state induced by the $U(1)$ gauge configuration. Again, we can show that $k$ is even and $k/2$ is the topological invariant of the $U(1)$ SPT state in 5-dimensional space-time. $k/2$ measures the cocycles in $\mathcal{H}^2(U(1), \mathbb{R}/\mathbb{Z})$.

### D. Fermionic $U^{f}(1)$ SPT phases

Although the topological invariant described above is motivated by the group cohomology theory that classifies the bosonic SPT states, however, the obtained topological invariant can be used to characterize/define/classify fermionic SPT phases. The general theory of interacting fermionic SPT phases is not as well developed as the bosonic SPT states. (A general theory of free fermion SPT phases were developed in Ref. 70–72, which include the noninteracting topological insulators\cite{37–42,73} and the noninteracting topological superconductors,\cite{74–78}). The first attempt was made in Ref. 79 where a group super-cohomology theory was developed. However, the group super-cohomology theory can only describe a subset of fermionic SPT phases. A more general theory is needed to describe all fermionic SPT phases. We hope the study of the topological invariants may help to develop this more general theory.

1. Symmetry in fermionic systems

A fermionic system always has a $Z_2^f$ symmetry generated by $P_f \equiv (-)^{N_F}$ where $N_F$ is the total fermion number. Let us use $G_f$ to denote the full symmetry group of the fermion system. $G_f$ always contain $Z_2^f$ as a normal subgroup. Let $G_b \equiv G_f/Z_2^f$ which represents the “bosonic” symmetry. We see that $G_f$ is an extension of $G_b$ by $Z_2^f$, described by the short exact sequence:

$$1 \to Z_2^f \to G_f \to G_b \to 1. \tag{33}$$

People some times use $G_b$ to describe the symmetry in fermionic systems and some times use $G_f$ to describe the symmetry. Both $G_b$ and $G_f$ do not contain the full information about the symmetry properties of a fermion system. In this paper we will use the short exact sequence (33) to describe the symmetry of a fermion system. However, for simplicity, we will use $G_f$ to refer the symmetry in fermionic systems. Note that when we say that a fermion system has a $G_f$ symmetry, we imply that we also know how $Z_2^f$ is embedded in $G_f$ as a normal subgroup. We know that $P_f$ always commute with any elements in $G_f$:

$$[P_f, g] = 0, \quad g \in G_f. \tag{34}$$

2. Topological invariant for fermionic $U^{f}(1)$ SPT phases

In this section, we are going to discuss the topological invariant for fermionic SPT states with a full symmetry group $G_f = U^{f}(1)$, which contains $Z_2^f$ as a subgroup such that odd $U^{f}(1)$-charges are always fermions. We will use the topological invariant developed in the last section to study fermionic SPT states with a $U^{f}(1)$ symmetry in 3-dimensional space-time. To construct the topological invariance, we first “gauged” the $U^{f}(1)$ symmetry, and then put the fermion system on a 2D close space $M_2$ with a $U^{f}(1)$ gauge configuration that carries a unit of the gauge flux $\int_{M_2} F^2 = 1$. We then measure the $U^{f}(1)$-charge $k$ of the ground state on $M_2$ induced by the $U^{f}(1)$ gauge configuration. Such a $U^{f}(1)$-charge is a topological invariant that can be used to characterize the fermionic $U^{f}(1)$ SPT phases.

Do we have other topological invariant? We may choose $M_2 = S_1 \times S_1$ (where $S_d$ is a $d$-dimensional sphere). However, on $S_1 \times S_1$ we do not have additional discrete topological $U^{f}(1)$ gauge configurations except those described by the $U^{f}(1)$-flux $\int_{M_2} F^2$ discussed above. (We need discrete topological gauge configurations to induce discrete $U^{f}(1)$ charges.) This suggests that we do not have other topological invariant and the fermionic $U^{f}(1)$ SPT states are classified by integers $Z$. In fact, the integer $k$ is nothing but the integral quantized Hall conductance $\sigma_{xy} = \frac{k}{2\pi}$.

The above just show that every fermionic $U^{f}(1)$ SPT state can be characterized by an integer $k$. But we do
not know if every integer \( k \) can be realized by a fermionic \( U^f(1) \) SPT state or not. To answer this question, we note that a fermionic \( U^f(1) \) SPT state is an Abelian state. So it can described by a \( U(1) \times \cdots \times U(1) \) Chern-Simons theory with a \( K \)-matrix and a charge vector \( q \).\(^{30}\) Let us first assume that the \( K \)-matrix is two dimensional. In this case, the fermionic \( U^f(1) \) SPT state must be described by a \( U(1) \times U(1) \) Chern-Simons theory in eqn. \((28)\) with the \( K \)-matrix and the charge vector \( q \) of the form\(^{30}\)

\[
K = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad q = \begin{pmatrix} 2m_1 + 1 \\ 2m_2 + 1 \end{pmatrix}, \quad m_{1,2} = \text{integers}.
\]

(35)

We require the elements of \( q \) to be odd integers since odd \( U^f(1) \)-charges are always fermions. The Hall conductance is given by

\[
\sigma_{xy} = (2\pi)^{-1} q^T K^{-1} q = \frac{4|m_1(m_1 + 1) - m_2(m_2 + 1)|}{2\pi}.
\]

(36)

We find that the Hall conductance for fermionic \( U^f(1) \) SPT states are always quantized as 8 times an integer. This result is valid even if we consider higher dimensional \( K \)-matrices.

It is interesting to see that the potential topological invariants for bosonic \( U(1) \) SPT states are integers (the integrally quantized Hall conductances). But the actual topological invariants are even integers. Similarly, the potential topological invariants for fermionic \( U^f(1) \) SPT states are also integers (the integrally quantized Hall conductances). However, the actual topological invariants are 8 times integers.

E. Fermionic \( Z^f_2 \) SPT phases

To understand the fermionic \( Z^f_2 \) SPT phases in 3-dimensional space-time, let us construct their topological invariants. We again create two identical \( Z^f_2 \) monodromy defects on a closed 2D space. We then measure the \( P_f \) quantum number for ground state with the two identical \( Z^f_2 \) monodromy defects. So the potential topological invariants \( k_2 \) are elements in \( Z_2 \). But what are the actual topological invariants?

We may view a fermion \( U^f(1) \) SPT phase discussed above as a \( Z^f_2 \) SPT phase by viewing the \( \pi U^f(1) \) rotation as \( P_f \). In this case the topological invariants \( k \) for the \( U^f(1) \) SPT phases become the topological invariants \( k_2 \) for \( Z^f_2 \) SPT phases: \( k_2 = k \mod 2 \). To see this result, we note that \( k \) in the induced \( U^f(1) \)-charge by a unit of \( U^f(1) \)-flux. A unit of \( U^f(1) \) flux can be viewed as two identical \( Z^f_2 \) vortex. So the induced \( Z^f_2 \) charge is \( k_2 = k \mod 2 \).

Since \( k = 0 \mod 8 \). Therefore fermion \( U^f(1) \) SPT phases always correspond to a trivial \( Z^f_2 \) SPT phase. We fail to get any non-trivial fermionic \( Z^f_2 \) SPT phases.

We like to point out that the induced \( P_f \) quantum numbers by two identical \( Z^f_2 \) monodromy defects are not the only type of topological invariants. There exist a new type of topological invariants: two identical \( Z^f_2 \) monodromy defects may induce topological degeneracy,\(^7\) with different degenerate states carrying different \( P_f \) quantum numbers. This new type of topological invariants is realized by a \( p + ip \) state where \( 2N \) identical \( Z^f_2 \) monodromy defects induce \( 2^N \) topologically degenerate ground states. Those topologically degenerate ground states are described by \( 2N \) Majorana zero modes which correspond to \( N \) zero-energy orbitals for complex fermions.\(^{75,80}\) But the \( p + ip \) state have an intrinsic topological order which is not a fermionic SPT state.

To summarize, although the fermionic \( Z^f_2 \) SPT phases in 3-dimensional space-time have two types of potential topological invariants, so far we cannot find any fermionic SPT phases that give rise to non-trivial topological invariants. This suggests that there is no non-trivial fermionic \( Z^f_2 \) SPT phases in 3-dimensional space-time. Let us use \( fSPT^3_{Z^f_2} \) to denote the Abelian group that classifies the fermionic SPT phases with full symmetry group \( G_f \) in \( d \)-dimensional space-time. The above result can written as \( fSPT^3_{Z^f_2} = 0 \).

We also have \( fSPT^1_{Z^f_2} = \mathbb{Z}_2 \). The two fermionic SPT phases correspond to 0-dimensional ground state with non fermion and one fermion. One can also show that \( fSPT^2_{Z^f_2} = 0 \).\(^{79}\)

IV. TOPOLOGICAL INVARIANTS OF SPT STATES WITH SYMMETRY \( G = GG \times SG \)

A. Bosonic \( U(1) \times \tilde{U}(1) \) SPT phases

In this section, we are going to discuss the topological invariant for bosonic \( U(1) \times \tilde{U}(1) \) SPT states in 3-dimensional space-time. To construct the topological invariance, we first “gauge” the \( U(1) \times \tilde{U}(1) \) symmetry, and then put the boson system on a 2D close space \( M_2 \) with a \( U(1) \times \tilde{U}(1) \) gauge configuration \((A_\mu, \tilde{A}_\mu)\) that carries a unit of the \( U(1) \) gauge flux \( \int_{M_2} \tilde{F} = 1 \). We then measure the \( U(1) \)-charge \( c_{11} \) and the \( \tilde{U}(1) \)-charge \( c_{12} \) of the ground state. Next, we put another \( U(1) \times \tilde{U}(1) \) gauge configuration on \( M_2 \) with a unit of the \( \tilde{U}(1) \) gauge flux \( \int_{M_2} F \equiv 1 \), then measure the \( U(1) \)-charge \( c_{21} \) and the \( \tilde{U}(1) \)-charge \( c_{22} \). We can use \( c_{ij} \) to form a two by two integer matrix \( C \). So an integer matrix \( C \) is a potential topological invariant for fermionic \( U(1) \times \tilde{U}(1) \) SPT phases in 3-dimensional space-time.

But what are the actual topological invariants? To answer this question, let us consider the following \( U(1) \times U(1) \) Chern-Simons theory that describe the bosonic
$U(1) \times \tilde{U}(1)$ SPT state

$$
\mathcal{L} = \frac{1}{4\pi} K_{IJ} a_I a_J \partial_\mu a_{IJ} e^{\mu\lambda} + \frac{1}{2\pi} q_{1,I} A_\mu \partial_\mu a_{IJ} e^{\mu\lambda} + \frac{1}{2\pi} q_{2,I} \tilde{A}_\mu \partial_\mu a_{IJ} e^{\mu\lambda} + \cdots
$$

(37)

with the $K$-matrix and two charge vectors $q_1, q_2$:

$$
K = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad q_1 = \begin{pmatrix} m_1 \\ m_2 \end{pmatrix}, \quad q_2 = \begin{pmatrix} m_3 \\ m_4 \end{pmatrix},
$$

$m_i$ = integers.

(38)

The topological invariant $C$ is given by

$$
C = \left( q_i^T K^{-1} q_j \right).
$$

(39)

Since stacking two SPT states with topological invariants $C_1$ and $C_2$ gives us a SPT state with a topological invariant $C_1 + C_2$, so the actual topological invariants form a vector space. We find that the actual topological invariants form a three-dimensional vector space with basis vectors

$$
C_1 = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}, \quad C_2 = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}, \quad C_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
$$

(40)

So the bosonic $U(1) \times \tilde{U}(1)$ SPT phases in 3-dimensional space-time are classified by three integers $Z^3$.

**B. Fermionic $U(1) \times U^f(1)$ SPT phases**

Now let us discuss the topological invariant for fermionic SPT states in 3-dimensional space-time, which has a full symmetry group $G_f = U(1) \times U^f(1)$ (with $Z_2^f$ as a subgroup where odd $U^f(1)$-charges are always fermions). To construct the topological invariance, we again “gauge” the $U(1) \times U^f(1)$ symmetry, and then put the fermion system on a 2D close space $M_2$ with a $U(1) \times U^f(1)$ gauge configuration that carries a unit of the $U(1)$ gauge flux $\int_{M_2} F = 1$. We then measure the $U(1)$-charge $c_{11}$ and the $U^f(1)$-charge $c_{12}$ of the ground state on $M_2$ induced by the $U(1)$ gauge flux. Next, we put another $U(1) \times U^f(1)$ gauge configuration on $M_2$ with a unit of the $U^f(1)$ gauge flux $\int_{M_2} \tilde{F} = 1$, then measure the $U(1)$ charge $c_{21}$ and the $U^f(1)$-charge $c_{22}$. So an integer matrix $C$ formed by $c_{ij}$ is a potential topological invariant for fermionic $U(1) \times U^f(1)$ SPT phases in 3-dimensional space-time.

But what are the actual topological invariants? Let us consider the following $U(1) \times U(1)$ Chern-Simons theory that describe the fermionic $U(1) \times U^f(1)$ SPT state

$$
\mathcal{L} = \frac{1}{4\pi} K_{IJ} a_I a_J \partial_\mu a_{IJ} e^{\mu\lambda} + \frac{1}{2\pi} q_{1,I} A_\mu \partial_\mu a_{IJ} e^{\mu\lambda} + \frac{1}{2\pi} q_{2,I} \tilde{A}_\mu \partial_\mu a_{IJ} e^{\mu\lambda} + \cdots
$$

(41)

with the $K$-matrix and two charge vectors $q_1, q_2$:

$$
K = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad q_1 = \begin{pmatrix} m_1 \\ m_2 \end{pmatrix}, \quad q_2 = \begin{pmatrix} m_3 \\ m_4 \end{pmatrix},
$$

$m_{3,4} = \text{odd integers}.$

(42)

The requirement “$m_{3,4} = \text{odd integers}”$ comes from the fact that odd $U^f(1)$-charges are always fermions. The topological invariant $C$ is given by

$$
C = \left( q_i^T K^{-1} q_j \right).
$$

(43)

We find that the actual topological invariants form a three-dimensional vector space with basis vectors

$$
C_1 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad C_2 = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}, \quad C_3 = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}.
$$

(44)

So the fermionic $U(1) \times U^f(1)$ SPT phases in 3-dimensional space-time are classified by three integers $Z^3$.

**C. A general discussion for the case $G = GG \times SG$**

In the appendix, we show that that (see eqn. (C6))

$$
\mathcal{H}^d(G, R/Z) = \oplus_{k=0}^d \mathcal{H}^k[SG, \mathcal{H}^d-k(GG, R/Z)].
$$

(45)

This means that we can use $(y_{0}, \cdots, y_{d})$ to label each element of $\mathcal{H}^d(G, R/Z)$ where $y_{k} \in \mathcal{H}^k[SG, \mathcal{H}^d-k(GG, R/Z)]$. Here we like to discuss how to physically measure each $y_{k}$.

First, we notice that $\mathcal{H}^d-k(GG, R/Z)$ classify the bosonic SPT phases in $(d-k)$-dimensional space-time. To stress this point, we rewrite $\mathcal{H}^d-k(GG, R/Z)$ as $bSPT_{GG}^{d-k}$, and rewrite above decomposition as

$$
\mathcal{H}^d(G, R/Z) = \oplus_{k=0}^d \mathcal{H}^k[SG, bSPT_{GG}^{d-k}].
$$

(46)

Since $bSPT_{GG}^{d-k}$ is a direct sum of $Z$’s and $Z_n$’s, $\mathcal{H}^k[SG, bSPT_{GG}^{d-k}]$ is direct sum of $\mathcal{H}^k[SG, Z]$’s and $\mathcal{H}^k[SG, Z_n]$’s. Such a structure motivates the following construction of topological invariants that allow us to measure $y_{k}$.

Following the idea in Ref. 60, we first gauge the group $SG$ to obtain a gauge theory with gauge group $SG$. However, the gauge potential for $SG$ are treated as fixed classical background without any fluctuations. In other words, the gauge field for $SG$ is a non-fluctuating probe field that couples to the $SG$ quantum numbers. We then, examine the properties of our model with such a non-fluctuating $SG$ gauge field as a background.

We then choose the space-time manifold to have a form $M_k \times M_{d-k}$ where $M_k$ has $k$ dimensions and $M_{d-k}$ has $d-k$ dimensions. We assume the $SG$ gauge configuration to be constant on $M_{d-k}$. Such a $SG$ gauge configuration can be viewed as a gauge configuration on $M_k$. Now we
assume that $M_k$ is very small, and our system can be viewed as a system on $M_{d-k}$ which has a $GG$ symmetry. The ground state of such a $GG$ symmetric system is $GG$ SPT state on $M_{d-k}$ which is labeled by an element in $h_{d-k}(GG, R/Z)$. This way, we obtain a function $\tilde{y}_k$ that maps a $SG$ gauge configuration on $M_k$ to an element in $H^{d-k}(GG, R/Z)$. In the above, we have discussed how to measure such an element physically when $GG = U(1)$. We choose $y_k$ (or $\tilde{y}_k$) is given by

$$\omega_k(s_{01}, s_{12}, \cdots, s_{k-1,k}) \in H^{d-k}(GG, R/Z),$$

where $s_{ij} \in SG$ live on the edges of the $k$-cell which describe a $SG$ gauge configuration on the $k$-cell. If we sum over the contributions from all the $k$-cells in $M_k$, we will obtain the above $\tilde{y}_k$ function that maps an $SG$ gauge configuration on $M_k$ to an element in $H^{d-k}(GG, R/Z)$.

The key issue that we need to show is whether the function $\tilde{y}_k$ allows us to fully detect $y_k \in \mathcal{H}^k[SG, H^{d-k}(GG, R/Z)]$, i.e. whether different $y_k$ always lead to different $\tilde{y}_k$. We can show that this is indeed the case using the classifying space. Let $BSG$ be the classifying space of $SG$. We know that the group cocycles in $\mathcal{H}^k[SG, H^{d-k}(GG, R/Z)]$ can be one-to-one represented by the topological cocycles in $\mathcal{H}^k[BSG, H^{d-k}(GG, R/Z)]$. We know that a topological cocycle $y_k^B$ in $\mathcal{H}^k[BSG, H^{d-k}(GG, R/Z)]$ gives rise to a function that maps all the $k$ cycles in $BSG$ to $\mathcal{H}^{d-k}(GG, R/Z)$. And such a function can fully detect the cocycle $y_k^B$ (i.e. different cocycles always lead to different mappings). We also know that each $k$ cycles in $BSG$ can be viewed as an embedding map from a $k$-dimensional space-time $M_k$ to $BSG$, and each embedding map define a $SG$ gauge configuration on $M_k$. Thus the topological cocycle $y_k^B$ is actually a function that maps a $SG$ gauge configuration in space-time to $\mathcal{H}^{d-k}(GG, R/Z)$, and such a mapping can fully detect $y_k^B$. All the $k$ cycles in $BSG$ can be continuously deformed into a particular type of $k$ cycles where all the vertices on the $k$-cycle occupy one point in $BSG$. The $y_k^B$ that maps the $k$ cycles to $\mathcal{H}^{d-k}(GG, R/Z)$ is a constant under such a deformation. $y_k^B$, when restricted on the $k$-cycles whose vertices all occupy one point, become the map $\tilde{y}_k$. This way, we show that the function $\tilde{y}_k$ can fully detect the group cocycles $y_k$ in $\mathcal{H}^k[SG, H^{d-k}(GG, R/Z)]$. This is how we can measure $y_k$.

In the above we see that each embedding map from $k$-dimensional space-time $M_k$ to $BSG$ define a $SG$ gauge configuration on $M_k$. This relation tells us how to choose the $SG$ gauge configurations on $M_k$ so that we can fully measure $y_k$. We choose the $SG$ gauge configurations on $M_k$ that come from the embedding maps from $M_k$ to $BSG$ such that the images are the non-trivial $k$-cycles in $BSG$. 

D. An example with $SG = U(1)$ and $GG = U(1)$

1. Topological invariants in $(2+1)D$

Let us reconsider the bosonic SPT states with symmetry $G = U^S(1) \times U^G(1)$ (i.e. $SG = U(1)$ and $GG = U(1)$) in 3 space-time dimensions. Such SPT states are classified by $H^3(G, R/Z)$.

$$H^3(G, R/Z) = \{0\}$$

Let us consider the bosonic SPT states with symmetry $G = U^S(1) \times U^G(1)$.

$$H^3(G, R/Z) = \{0\}$$

So the topological partition function $Z_{top}(M_d, \Lambda_d) = e^{i \int d^4x L_{top}}$ is given by

$$L_{top} = -\frac{\delta A_{GG}}{2\pi} \bar{F}_{GG} + O(\delta A_{GG}^2) + \cdots$$

If we turn on one unit of $U^G(1)$ flux on $S_2$ described by a background field $\Lambda_{GG}$, the above topological terms become (with $A_{GG} = \delta A_{GG} + \Lambda_{GG}$):

$$L_{top} = -\frac{2\delta A_{GG}}{2\pi} \bar{F}_{GG} + O(\delta A_{GG}^2) + \cdots$$

where $A_{SG}$ is the gauge field on $S_2$.

Thus we can fully measure $y_{top}$ through the topological terms in $L_{top}$.
which implies that one unit of $U^{GG}(1)$-flux on $S_2$ will induce 2y_0 unit of $U^{GG}(1)$-charge. The factor 2 agrees with the result of even-integer-quantized Hall conductance obtained before.

2. Topological invariants in (4+1)D

Next, we consider bosonic $U^{SG}(1) \times U^{GG}(1)$ SPT states in (4+1)D. The SPT states are classified by

$$\mathcal{H}^3(G, \mathbb{R}/\mathbb{Z}) = \oplus_{k=0}^{\infty} \mathcal{H}^3[U(1)^{SG}, H^{d-k}(U(1)^{GG}, \mathbb{R}/\mathbb{Z})]$$

with

$$\mathcal{H}^3(U^{GG}(1), \mathbb{R}/\mathbb{Z}) = Z = \{y_0\},$$
$$\mathcal{H}^2[U^{SG}(1), H^{3}(U^{GG}(1), \mathbb{R}/\mathbb{Z})] = Z = \{y_2\},$$
$$\mathcal{H}^4[U^{SG}(1), H^{1}(U^{GG}(1), \mathbb{R}/\mathbb{Z})] = Z = \{y_4\},$$
$$\mathcal{H}^5(U^{SG}(1), \mathbb{R}/\mathbb{Z}) = Z = \{y_5\}.$$  (55)

The topological terms labeled by $y_k$ are the Chern-Simons terms:

$$\mathcal{L}_{\text{top}} = \frac{y_0}{(2\pi)^2} A_{GG} F_{GG}^2 + \frac{y_2}{(2\pi)^2} A_{SG} F_{SG}^2 + \frac{y_4}{(2\pi)^2} A_{SG} F_{SG}^2 + \frac{y_5}{(2\pi)^2} A_{SG} F_{SG}^2.$$  (56)

which gives rise to the topological partition function

$$Z_{\text{top}}(M_d, A_\mu) = e^{i \int d^4x \mathcal{L}_{\text{top}}}.$$

To measure $y_2$, we choose a space-time manifold of the form $M_d \times M'_2 \times S_1$ (where $S_1$ is the time direction). We put a $SG$ gauge field on space $M_2$ such that $\int_{M_2} \frac{1}{2\pi} F_{SG} = 1$. In the small $M_2$ limit, our theory reduces to a $GG$-gauge theory on $M'_2 \times S_1$ described by $y_2$ in $\mathcal{H}^3[U^{GG}(1), \mathbb{R}/\mathbb{Z}]$. We can then put a $GG$ gauge field on space $M_2$ such that $\int_{M_2} \frac{1}{2\pi} F_{GG} = 1$. Such a configuration will induce 2y_2 unit of $U^{GG}(1)$-charges.

The $y_4$ term can be measured by putting a $SG$ gauge field on space $M_2$ such that $\int_{M_2} \frac{1}{2\pi} F_{SG} = 1$ and a $GG$ gauge field on space $M'_2$ such that $\int_{M'_2} \frac{1}{2\pi} F_{GG} = 1$ will induce 2y_2 unit of $U^{GG}(1)$-charges.

The $y_5$ term can be measured by putting a $SG$ gauge field on space $M_4$ such that $\int_{M_4} \frac{1}{2\pi} F_{SG} = 1$. Such a $SG$ gauge configuration will induce a 2y_4 unit of the $U^{GG}(1)$-charges. The $SG$ gauge configuration will also induce a 6y_0 unit of the $U^{SG}(1)$-charges.

E. Bosonic $Z^{SG}_2 \times Z^{GG}_2$ SPT states

1. Topological invariants in (2+1)D

Next, let us consider SPT states with symmetry $G = Z^{SG}_2 \times Z^{GG}_2$ in 2+1 dimensions. Such a theory was studied in Ref. 56 using $U(1) \times U(1)$ Chern-Simons theory. The $Z^{SG}_2 \times Z^{GG}_2$ SPT states are classified by $\mathcal{H}^3(G, \mathbb{R}/\mathbb{Z})$, which has the following decomposition

$$\mathcal{H}^3(G, \mathbb{R}/\mathbb{Z}) = \oplus_{k=0}^{\infty} \mathcal{H}^3[Z^{SG}_2, H^{d-k}(Z^{GG}_2, \mathbb{R}/\mathbb{Z})]$$
$$= \mathcal{H}^3(Z^{GG}_2, \mathbb{R}/\mathbb{Z}) \oplus \mathcal{H}^3[Z^{SG}_2, H^1(Z^{GG}_2, \mathbb{R}/\mathbb{Z})] \oplus \mathcal{H}^3(Z^{SG}_2, \mathbb{R}/\mathbb{Z}),$$  (57)

with

$$\mathcal{H}^3(Z^{SG}_2, \mathbb{R}/\mathbb{Z}) = Z_2 = \{y_0\},$$
$$\mathcal{H}^3(Z^{GG}_2, \mathbb{R}/\mathbb{Z}) = Z_2 = \{y_2\},$$
$$\mathcal{H}^3(Z^{SG}_2, \mathbb{R}/\mathbb{Z}) = Z_2 = \{y_3\}.$$  (58)

$y_0$ labels different 2+1D $Z^{SG}_2$ SPT states and $y_3$ labels different 2+1D $Z^{GG}_2$ SPT states. To measure $y_k$, we may create two identical $Z^{SG}_2$ monodromy defects on a closed 2D space. We then measure the induced $Z^{GG}_2$-charge, which measures $y_3$. We can also measure the induced $Z^{SG}_2$-charge, which measures $y_2$.

To understand why measuring the induced $Z^{SG}_2$-charges and $Z^{GG}_2$ charges allow us to measure $y_3$ and $y_2$, let us start with the dual gauge theory description of the $Z^{SG}_2 \times Z^{GG}_2$ SPT state: The total Lagrangian has a form

$$\mathcal{L} + W_{\text{top}} = \frac{1}{4\pi} K_{IJ} a^I_\mu \theta I a^J_\lambda + \ldots$$  (59)

with

$$K = \begin{pmatrix} 2y_3 & 2y_2 & 0 \\ 2 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}.$$  (60)

Two K-matrices $K_1$ and $K_2$ are equivalent $K_1 \sim K_2$ (i.e. give rise to the same theory) if $K_1 = U^T K_2 U$ for an integer matrix with det($U$) = ±1. We find $K(y_3, y_2, y_0) \sim K(y_3 + 2, y_2, y_0) \sim K(y_3, y_2 + 2, y_0) \sim K(y_3, y_2, y_0 + 2)$. Thus only $y_3, y_2, y_0 = 0, 1$ give rise to inequivalent K-matrices.

A particle carrying $l_I a^I_\mu$-charge will have a statistics

$$\theta_I = \pi l_I (K^{-1})^{IJ} l_J.$$  (61)

A particle carrying $l_I a^I_\mu$-charge will have a mutual statistics with a particle carrying $l_I a^I_\mu$-charge:

$$\theta_{IJ} = 2\pi l_I (K^{-1})^{IJ} l_J.$$  (62)

A particle with a unit of $Z^{SG}_2$-charge is described by a particle with a unit $a^\mu_\mu$-charge. A particle with a unit of $Z^{GG}_2$-charge is described by a particle with a unit $a^\mu_\mu$-charge. Using

$$K^{-1} = \frac{1}{4} \begin{pmatrix} 0 & 2 & 0 & 0 \\ 2 & -2y_3 & 0 & -y_2 \\ 0 & 0 & 0 & 2 \\ 0 & -y_2 & 2 & -2y_0 \end{pmatrix},$$  (63)
we find that the $Z_2^{SG}$-charge (the unit $a_1^1$-charge) and the $Z_2^{GG}$ gauge charge (the unit $a_3^1$-charge) are always bosonic.

Since a $Z_2^{SG}$-charge has a mutual statistics $\pi$ with a unit $a_3^1$-charge, thus a unit $a_2^1$-charge correspond to a $Z_2^{SG}$ monodromy defect. Similarly, a unit $a_3^1$-charge correspond to a $Z_2^{GG}$ monodromy defect. We notice that a $Z_2^{SG}$ monodromy defect always correspond to 1/2 units of $a_3^1$-flux and a $Z_2^{GG}$ monodromy defect always correspond to 1/2 units of $a_3^1$-flux.

Let us move a $Z_2^{SG}$ monodromy defect (described by $(l_1) = (0, 0, 0, 0)$) around a $Z_2^{SG}$ monodromy defect (described by $(l_1) = (0, 1, 0, 0)$). From eqn. (62), we see that such a motion will induce a phase $\frac{\pi}{2}$. Thus a $Z_2^{SG}$ monodromy defect carries $-y_2/2$ $Z_2^{GG}$-charges, and two identical $Z_2^{SG}$-monodromy defect carries $y_2$ $Z_2^{GG}$-charges.

Similarly, moving a $Z_2^{SG}$ monodromy defect around another $Z_2^{SG}$ monodromy defect induce a phase $-y_3\pi$. However, the phase $-y_3\pi$ has two contributions: one from the $Z_2^{SG}$-charge of the first monodromy defect going around the $Z_2^{SG}$ flux of the second monodromy defect, and the other from the $Z_2^{SG}$ flux of the first monodromy defect going around the $Z_2^{SG}$-charge of the second monodromy defect. Since each contribution is $-y_3\pi/2$, so each $Z_2^{SG}$ monodromy defect carries $-y_2/2$ $Z_2^{GG}$-charges, and two identical $Z_2^{SG}$-monodromy defect carries $y_3$ $Z_2^{SG}$-charges.

2. Topological invariants in (3+1)D

In the above examples, we see that measuring topological responses give rise to a complete set of topological invariants which fully characterize the SPT states. We believe this is true in general. Next we will use this idea to study the $Z_2^{SG} \times Z_2^{GG}$ SPT states in (3+1)D, which are classified by $\mathcal{H}^i(G, \mathbb{R}/\mathbb{Z})$, which has the following decomposition

$$\mathcal{H}^i(G, \mathbb{R}/\mathbb{Z}) = \oplus_{k=0}^3 \mathcal{H}^k[Z_2^{SG}, \mathcal{H}^{d-k}(Z_2^{GG}, \mathbb{R}/\mathbb{Z})]$$

$$\oplus \mathcal{H}^1[Z_2^{SG}, \mathcal{H}^1(Z_2^{GG}, \mathbb{R}/\mathbb{Z})]$$

with

$$\mathcal{H}^1[Z_2^{SG}, \mathcal{H}^1(Z_2^{GG}, \mathbb{R}/\mathbb{Z})] = \mathbb{Z}_2 = \{y_1\},$$

$$\mathcal{H}^3[Z_2^{SG}, \mathcal{H}^1(Z_2^{GG}, \mathbb{R}/\mathbb{Z})] = \mathbb{Z}_2 = \{y_3\}.$$  
(64)

To measure $y_1$, we choose the space to be $S_1 \times M_2$. We then create a $Z_2^{SG}$ twist boundary condition on $S_1$ (which measure $\mathcal{H}^1(Z_2^{SG}, Z_2)$). In the small $S_1$, the SPT state on $S_1 \times M_2$ reduces to SPT state on $M_2$ which is described by $\mathcal{H}^3(Z_2^{GG}, \mathbb{R}/\mathbb{Z})$. The elements in $\mathcal{H}^3(Z_2^{GG}, \mathbb{R}/\mathbb{Z})$ can be measured by measuring the $Z_2^{GG}$-charge induced by two identical $Z_2^{GG}$ monodromy defects on $M_2$. Thus $y_1$ is the $Z_2^{GG}$ charge on space $S_1 \times M_2$ induced by two identical $Z_2^{GG}$ monodromy defects on $M_2$ and a $Z_2^{SG}$ twist boundary condition on $S_1$.

3. Topological invariants in (1+1)D

The topological invariants for bosonic $G = Z_2^{SG} \times Z_2^{GG}$ SPT states in (1+1)D have a similar structure, but much simpler. The SPT states are classified by $\mathcal{H}^i(G, \mathbb{R}/\mathbb{Z})$, which has the following decomposition

$$\mathcal{H}^i(G, \mathbb{R}/\mathbb{Z}) = \oplus_{k=0}^2 \mathcal{H}^k[Z_2^{SG}, \mathcal{H}^{d-k}(Z_2^{GG}, \mathbb{R}/\mathbb{Z})]$$

$$= \mathcal{H}^1[Z_2^{SG}, \mathcal{H}^1(Z_2^{GG}, \mathbb{R}/\mathbb{Z})] = \mathbb{Z}_2 = \{y_1\}.$$  
(66)

To measure $y_1$, we choose the space to be $S_1$ and create a $Z_2^{SG}$ twist boundary condition on $S_1$ (which measure $\mathcal{H}^1(Z_2^{SC}, M_2)$). Then we measure the induced $Z_2^{GG}$-charge on $S_1$, which gives rise to $y_1$.

F. Bosonic $U(1) \times Z_2$ SPT phases

In this section, we like to consider SPT states with symmetry $G = U(1) \times Z_2$ in 2+1 dimensions. The $U(1) \times Z_2$ SPT states are classified by $\mathcal{H}^i(G, \mathbb{R}/\mathbb{Z})$, which has the following decomposition

$$\mathcal{H}^i(G, \mathbb{R}/\mathbb{Z}) = \oplus_{k=0}^3 \mathcal{H}^k[Z_2, \mathcal{H}^{d-k}(U(1), \mathbb{R}/\mathbb{Z})]$$

$$= \mathcal{H}^3(U(1), \mathbb{R}/\mathbb{Z}) \oplus \mathcal{H}^2[Z_2, \mathcal{H}^1(U(1), \mathbb{R}/\mathbb{Z})]$$

$$\oplus \mathcal{H}^1(Z_2, \mathbb{R}/\mathbb{Z}),$$  
(67)

with

$$\mathcal{H}^3(U(1), \mathbb{R}/\mathbb{Z}) = \mathbb{Z} = \{y_0\},$$

$$\mathcal{H}^2[Z_2, \mathcal{H}^1(U(1), \mathbb{R}/\mathbb{Z})] = \mathbb{Z}_2 = \{y_2\},$$

$$\mathcal{H}^1(Z_2, \mathbb{R}/\mathbb{Z}) = \mathbb{Z}_2 = \{y_3\}.$$  
(68)

$y_0$ labels different 2+1D $U(1)$ SPT states and $y_3$ labels different 2+1D $Z_2$ SPT states. To measure $y_k$, we may create two identical $Z_2^{SG}$ monodromy defects on a closed 2D space. We then measure the induced $Z_2$-charge, which measures $y_3$. We can also measure the induced $U(1)$-charge, which measures $y_2$ mod 2. Thus the bosonic $U(1) \times Z_2$ SPT phases is classified by $\mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ in (2+1)D.

G. Fermionic $U(1) \times Z_2^f$ SPT phases

1. Topological invariants in 2+1D

The fermionic $U(1) \times Z_2^f$ SPT phases can be realized by systems with two types of fermions, one carry the $U(1)$ charge and the other is neutral. To construct the topological invariants for the fermionic $U(1) \times Z_2^f$ SPT states, we again “gauge” the $U(1) \times Z_2^f$ symmetry, and then put the fermion system on a 2D close space $M_2$ with a $U(1) \times Z_2^f$ gauge configuration that carries a unit of the $U(1)$ gauge flux $\int_{M_2} F = 1$. We then measure the $U(1)$-charge $c_{11}$ and the $Z_2^f$-charge $c_{12}$ of the ground
state on $M_2$ induced by the $U(1)$ gauge flux. Next, we put another $U(1) \times Z_2^f$ gauge configuration on $M_2$ with no $U(1)$ flux but two identical $Z_2^f$ vortices, then measure the $U(1)$ charge $c_{21}$ (mod 2) and the $Z_2^f$-charge $c_{22}$. So an integer matrix $C$ formed by $c_{ij}$

$$C = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} \mod 2 & c_{22} \mod 2 \end{pmatrix}$$

(69)

is a potential topological invariant for fermionic $U(1) \times Z_2^f$ SPT phases in 3-dimensional space-time.

But which topological invariants can be realized? What are the actual topological invariants? One way to realize the fermionic $U(1) \times Z_2^f$ SPT phases is to view them the fermionic $U(1) \times U^f(1)$ SPT phases discussed in section IV B. Using the $U(1) \times U(1)$ Chern-Simons theory for the fermionic $U(1) \times U^f(1)$ SPT phases, we see that the following topological invariant

$$C_1 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

(70)

can be realized.

By binding the $U(1)$ charged fermion and neutral fermion to form a $U(1)$ charged boson, we can form other fermionic $U(1) \times Z_2^f$ SPT phases through the bosonic $U(1)$ SPT phases of the above bosonic bound states. This allows us to realize the following topological invariant

$$C'_1 = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$$

(71)

which is twice of $C_1$.

We may also assume that the fermionic $U(1) \times Z_2^f$ SPT phases are described by $y_k \in \mathcal{H}^k[U(1), f SPT_{Z_2^f}] k = 0, 1, 2,$ and $y_3 \in b SPT^3_U(U(1))$. ($y_3 \in b SPT^3_U(U(1))$ because $U(1)$ does not contain $Z_2^f$ and is a bosonic symmetry for the fermion bound states discussed above.) Using $f SPT^1_{Z_2^f} = Z_2$ and $f SPT^k_{Z_2^f} = 0$ for $k > 1$, we have

$$y_0 = 0, \quad y_1 = 0,$n $$y_2 \in \mathcal{H}^2[U(1), f SPT^1_{Z_2^f}] = \mathcal{H}^2[U(1), Z_2] = Z_2$$

$$y_3 \in b SPT^3_U(U(1)) = \mathcal{H}^3[U(1), \mathbb{R}/\mathbb{Z}] = \mathbb{Z}.$$ (72)

$y_2$ can be measured by putting a $U(1) \times Z_2^f$ gauge configuration that carries a unit of the $U(1)$ gauge flux $\int_{M_2} \epsilon = 1$ on a closed 2D space, and then measure the induced fermion numbers (i.e. the $Z_2^f$ charges). We see that $(y_2, y_3) = (1, 0)$ corresponds to the topological invariant $C_1$ discussed above, while $(y_2, y_3) = (0, 1)$ corresponds to the topological invariant $C'_1$.

We see that some of the fermionic $U(1) \times Z_2^f$ SPT phases are classified by $Z$ in 3-dimensional space-time, whose topological invariant is $C_1$ times an integer. It is likely that those are all the fermionic $U(1) \times Z_2^f$ SPT phases. The integer $Z$ that label the fermionic $U(1) \times Z_2^f$ SPT phases correspond to the integer Hall conductance. This result should be contrasted with the result for the fermionic $U^f(1)$ SPT phases discussed in section III D.

2. Topological invariants in 3+1D

Let us assume that the fermionic $U(1) \times Z_2^f$ SPT phases in 3+1D are described by $y_k \in \mathcal{H}^k[U(1), f SPT_{Z_2^f}^k]$ $k = 0, 1, 2, 3,$ and $y_4 \in b SPT^4_U(U(1))$ (since $U(1)$ does not contain $Z_2^f$ and is a bosonic symmetry for the fermion bound states discussed above). Using $f SPT^1_{Z_2^f} = Z_2$ and $f SPT^k_{Z_2^f} = 0$ for $k > 1$, we have

$$y_0 = 0, \quad y_1 = 0, \quad y_2 = 0,$n $$y_3 \in \mathcal{H}^3[U(1), f SPT^1_{Z_2^f}] = \mathcal{H}^3[U(1), Z_2] = 0$$

$$y_4 \in b SPT^3_{U(SU)} = \mathcal{H}^4[U(1), \mathbb{R}/\mathbb{Z}] = 0.$$ (73)

This suggests that the fermionic $U(1) \times Z_2^f$ SPT phases in 3+1D are always trivial.

H. Fermionic $Z_2 \times Z_2^f$ SPT states

Now, let us consider fermionic SPT states with full symmetry $Z_2 \times Z_2^f$ in 2+1 dimensions. This kind of fermionic SPT states were studied in Ref. 79 using group super-cohomology theory where four fermionic $Z_2 \times Z_2^f$ SPT states (including the trivial one) were constructed. They were also studied in Ref. 81 where 8 SPT states were obtained (see also Ref. 82 and 83). To construct topological invariants for the fermionic $Z_2 \times Z_2^f$ SPT states, we may create two identical $Z_2$ monodromy defects on a closed 2D space. We then measure the induced $Z_2$-charge $c_{11}$ and the $Z_2^f$-charge $c_{12}$. We then create two identical $Z_2^f$ monodromy defects, and measure the induced $Z_2$-charge $c_{21}$ and the $Z_2^f$-charge $c_{22}$. Note that $c_{ij} = c_{ji} = 0, 1$. Thus there are 8 potential different topological invariants.

But how many of them are actual topological invariants that can be realized by fermion systems? We may view the fermionic $U(1) \times U^f(1)$ SPT states discussed in section IV B as fermionic $Z_2 \times Z_2^f$ SPT states. We find that the $U(1) \times U^f(1)$ SPT states can realize a topological invariant

$$C_1 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \mod 2.$$ (74)

If we assume that the fermions form bound states, we will get a bosonic system with $Z_2$ symmetry. Such a bosonic system can realize a topological invariant

$$C_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mod 2.$$ (75)
as discussed in section III B. The two kinds of topological invariants $C_1$ and $C_2$ will give us four different kinds of fermionic $Z_2 \times Z_2^f$ SPT states, which are classified by $Z_2 \times Z_2$.

The topological invariant $C_1$ is realized by a fermion system where the $Z_2$-charged fermions form a $\nu = 1$ integer quantum Hall state and the $Z_2$-neutral fermions form a $\nu = -1$ integer quantum Hall state. We can have a new topological invariant which is realized by a fermion system where the $Z_2$-charged fermions form a $p + ip$ superconducting state and the $Z_2$-neutral fermions form a $p - ip$ superconducting state.\textsuperscript{75,80} We will denote the new topological invariant as $C_1/2$, since stacking two of the $(p + ip)/(p - ip)$ superconducting states will realize the topological invariant $C_1$. Stacking four of the $(p + ip)/(p - ip)$ superconducting states will realize the topological invariant $2C_1$ which is trivial. The above consideration suggests that fermionic $Z_2 \times Z_2^f$ SPT states are classified by $Z_4 \times Z_2$. However, Ref. 75 suggested one needs to stack eight of the $(p + ip)/(p - ip)$ superconducting states to obtain a trivial fermionic SPT states. This implies that fermionic $Z_2 \times Z_2^f$ SPT states are classified by $Z_8 \times Z_2$.

Let us examine the assumption that the fermionic $Z_2 \times Z_2^f$ SPT phases are described by $y_k \in \mathcal{H}^k[Z_2, fSPT^3_{Z_2^f}]$ $k = 0, 1, 2$, and $y_3 \in bSPT^3_{Z_2}$ (note that $Z_2$ does not contain $Z_2^f$ and is a bosonic symmetry for the fermion bound states discussed above). Using $fSPT^1_{Z_2} = Z_2$ and $fSPT^k_{Z_2^f} = 0$ for $k > 1$, we have

$$y_0 = 0, \quad y_1 = 0,$$
$$y_2 \in \mathcal{H}^2[Z_2, fSPT^1_{Z_2^f}] = \mathcal{H}^2[Z_2, Z_2] = Z_2,$$
$$y_3 \in bSPT^3_{Z_2} = \mathcal{H}^3[Z_2, \mathbb{R}/\mathbb{Z}] = Z_2. \quad (76)$$

$y_2$ can be measured by putting two identical $Z_2$ monodromy defects on an open 2D space, and then measure the induced fermion numbers (i.e. the $Z_2^f$ charges). The possible induced fermion numbers are 0 and 1, but there is another possibility where there are two degenerate ground states: one with no fermion and the other with one fermion. Let us denote the later possibility as $y_2 = 1/2$. We see that $(y_2, y_3) = (1, 0)$ corresponds to the topological invariant $C_1$ discussed above, $(y_2, y_3) = (1/2, 0)$ corresponds to the topological invariant $C_1/2$, and $(y_2, y_3) = (0, 1)$ corresponds to the topological invariant $C_2$. So the assumption that $y_2 \in \mathcal{H}^2[Z_2, fSPT^1_{Z_2^f}]$ is not correct. It should be generalized to $y_2 \in \mathcal{H}^2[Z_2, fSPT^1_{Z_2^f}] + \text{extra}$.

### V. SUMMARY

In this paper, we construct many topological invariants which allow us to physically measure the cocycles in $\mathcal{H}^d(G, \mathbb{R}/\mathbb{Z})$ fully that classify the SPT states and some of the SET states for interacting bosons and fermions. Those topological invariants also allow us to understand some of the SPT states for interacting fermions. We list those results in table I. In particular, whether the fermionic $Z_2 \times Z_2^f$ SPT states in 2+1D are classified by $Z_4 \times Z_2$ or $Z_8 \times Z_2$ (or even $Z_8$ as suggested in Ref. 81) is an interesting issue to be resolved.

I like to thank Zheng-Cheng Gu and Xie Chen for many helpful discussions. This research is supported by NSF Grant No. DMR-1005541, NSF 11074140, and NSF 11274192. Research at Perimeter Institute is supported by the Government of Canada through Industry Canada and by the Province of Ontario through the Ministry of Research.

### Appendix A: Group cohomology theory

#### 1. Homogeneous group cocycle

In this section, we will briefly introduce group cohomology. The group cohomology class $\mathcal{H}^d(G, \mathcal{M})$ is an Abelian group constructed from a group $G$ and an Abelian group $\mathcal{M}$. We will use "$+$" to represent the multiplication of the Abelian groups. Each elements of $G$ also induce a mapping $\mathcal{M} \rightarrow \mathcal{M}$, which is denoted as

$$g \cdot m = m', \quad g \in G, \; m, m' \in \mathcal{M}. \quad (A1)$$

The map $g \cdot$ is a group homomorphism:

$$g \cdot (m_1 + m_2) = g \cdot m_1 + g \cdot m_2. \quad (A2)$$

The Abelian group $\mathcal{M}$ with such a $G$-group homomorphism, is call a $G$-module.

A homogeneous $d$-cochain is a function $\nu_d : G^{d+1} \rightarrow \mathcal{M}$, that satisfies

$$\nu_d(g_0, \ldots, g_d) = g \cdot \nu_d(g_0, \ldots, g_d), \quad g, g_i \in G. \quad (A3)$$

We denote the set of $d$-cochains as $\mathcal{C}^d(G, \mathcal{M})$. Clearly $\mathcal{C}^d(G, \mathcal{M})$ is an Abelian group. homogeneous group cocycle.

Let us define a mapping $d$ (group homomorphism)
from $C^d(G, \mathbb{M})$ to $C^{d+1}(G, \mathbb{M})$:

$$(d\nu_d)(g_0, \cdots, g_{d+1}) = \sum_{i=0}^{d+1} (-1)^i \nu_d(g_0, \cdots, \hat{g}_i, \cdots, g_{d+1})$$

(A4)

where $g_0, \cdots, \hat{g}_i, \cdots, g_{d+1}$ is the sequence $g_0, \cdots, g_i, \cdots, g_{d+1}$ with $g_i$ removed. One can check that $d^2 = 0$. The homogeneous $d$-cocycles are then the homogeneous $d$-cochains that also satisfy the cocycle condition

$$d\nu_d = 0.$$  
(A5)

We denote the set of $d$-cocycles as $Z^d(G, \mathbb{M})$. Clearly $Z^d(G, \mathbb{M})$ is an Abelian subgroup of $C^d(G, \mathbb{M})$.

Let us denote $B^d(G, \mathbb{M})$ as the image of the map $d : C^{d-1}(G, \mathbb{M}) \rightarrow C^d(G, \mathbb{M})$ and $B^0(G, \mathbb{M}) = \{0\}$. The elements in $B^d(G, \mathbb{M})$ are called $d$-coboundaries. Since $d^2 = 0$, $B^d(G, \mathbb{M})$ is a subgroup of $Z^d(G, \mathbb{M})$:

$$B^d(G, \mathbb{M}) \subset Z^d(G, \mathbb{M}).$$  
(A6)

The group cohomology class $H^d(G, \mathbb{M})$ is then defined as

$$H^d(G, \mathbb{M}) = Z^d(G, \mathbb{M})/B^d(G, \mathbb{M}).$$  
(A7)

We note that the $d$ operator and the cochains $C^d(G, \mathbb{M})$ (for all values of $d$) form a so called cochain complex,

$$\cdots \rightarrow C^d(G, \mathbb{M}) \xrightarrow{d} C^{d+1}(G, \mathbb{M}) \xrightarrow{d} \cdots$$  
(A8)

which is denoted as $C(G, \mathbb{M})$. So we may also write the group cohomology $H^d(G, \mathbb{M})$ as the standard cohomology of the cochain complex $H^d[C(G, \mathbb{M})]$.

2. Nonhomogeneous group cocycle

The above definition of group cohomology class can be rewritten in terms of nonhomogeneous group cochains/cocycles. An nonhomogeneous group $d$-cochain is a function $\omega_d : G^d \rightarrow M$. All $\omega_d(g_1, \cdots, g_d)$ form $C^d(G, \mathbb{M})$. The nonhomogeneous group cochains and the homogeneous group cochains are related as

$$\nu_d(g_0, g_1, \cdots, g_d) = \omega_d(\hat{g}_1, \cdots, \hat{g}_d),$$  
(A9)

with

$$g_0 = 1, \ g_1 = g_0 \hat{g}_1, \ g_2 = g_1 \hat{g}_2, \ \cdots \ g_d = g_{d-1} \hat{g}_d.$$  
(A10)

Now the $d$ map has a form on $\omega_d$:

$$(d\omega_d)(\hat{g}_1, \cdots, \hat{g}_{d+1}) = \hat{g}_1 \cdot \omega_d(g_2, \cdots, \hat{g}_{d+1})$$

$$+ \sum_{i=1}^{d} (-1)^i \omega_d(\hat{g}_1, \cdots, \hat{g}_i \hat{g}_{i+1}, \cdots, \hat{g}_{d+1})$$

$$+ (-1)^{d+1} \omega_d(\hat{g}_1, \cdots, \hat{g}_d)$$  
(A11)

This allows us to define the nonhomogeneous group $d$-cocycles which satisfy $d\omega_d = 0$ and the nonhomogeneous group $d$-coboundaries which have a form $\omega_d = d\mu_{d-1}$. In the following, we are going to use nonhomogeneous group cocycles to study group cohomology.

3. “Normalized” cocycles

We know that each elements in $H^d(G, \mathbb{R}/\mathbb{Z})$ can be represented by many cocycles. In the following, we are going to find ways to simplify the cocycles, so that the simplified cocycles can still represent all the elements in $H^d(G, \mathbb{R}/\mathbb{Z})$.

One simplification can be obtained by considering “normalized” cochains, which satisfy

$$\omega_d(g_1, \cdots, g_d) = 0, \text{ if one of } g_i = 1.$$  
(A12)

One can check that the $d$-operator maps a “normalized” cochain to a “normalized” cochain. The group cohomology classes obtained from the ordinary cochains is isomorphic to the group cohomology classes obtained from the “normalized” cochains. Let us use $C^d(G, \mathbb{M})$, $Z^d(G, \mathbb{M})$, and $B^d(G, \mathbb{M})$ to denote the “normalized” cochains, cocycles, and coboundaries. We have

$$H^d(G, \mathbb{M}) = Z^d(G, \mathbb{M})/B^d(G, \mathbb{M}).$$

Appendix B: The Künneth formula

The Künneth formula is a very helpful formula that allows us to calculate the cohomology of chain complex $X \times X'$ in terms of the cohomology of chain complex $X$ and chain complex $X'$. The Künneth formula is given by (see Ref. 85 page 247)

$$H^d(X \times X', \mathbb{M} \otimes R \mathbb{M}') \cong \bigoplus_{k=0}^{d} H^k(X, \mathbb{M}) \otimes_R H^{d-k}(X', \mathbb{M}')$$

$$+ \bigoplus_{k=0}^{d+1} \text{Tor}_1^R(H^k(X, \mathbb{M}), H^{d-k+1}(X', \mathbb{M}')).$$  
(B1)

Here $R$ is a principle ideal domain and $\mathbb{M}, \mathbb{M}'$ are $R$-modules such that $\text{Tor}_1^R(\mathbb{M}, \mathbb{M}') = 0$. Note that $\mathbb{Z}$ and $R$ are principal ideal domains, while $\mathbb{R}/\mathbb{Z}$ is not. $R$-module is like a vector space over $R$ (i.e. we can “multiply” a vector by an element of $R$.) For more details on principal ideal domain and $R$-module, see the corresponding Wiki articles.

The tensor-product operation $\otimes_R$ and the torsion-product operation $\text{Tor}_1^R$ have the following properties:

$$A \otimes \mathbb{Z} B \simeq B \otimes A,$$
$$\mathbb{Z} \otimes \mathbb{M} \simeq \mathbb{M} \otimes \mathbb{Z} \simeq \mathbb{M},$$
$$\mathbb{Z}_n \otimes \mathbb{M} \simeq \mathbb{M} \otimes \mathbb{Z} = \mathbb{M}/n\mathbb{M},$$
$$\mathbb{Z}_m \otimes \mathbb{Z} \mathbb{N} = \mathbb{Z}_{(m,n)},$$
$$(A \otimes B) \otimes_R \mathbb{M} = (A \otimes_R \mathbb{M}) \otimes (B \otimes_R \mathbb{M}),$$
$$\mathbb{M} \otimes_R (A \oplus B) = (\mathbb{M} \otimes_R A) \oplus (\mathbb{M} \otimes_R B);$$  
(B2)
and
\[
\text{Tor}^R_1(A, B) \simeq \text{Tor}^R_1(B, A),
\]
\[
\text{Tor}^Z_1(Z, M) = \text{Tor}^Z_1(\mathbb{M}, Z) = 0,
\]
\[
\text{Tor}^Z_1(Z_n, \mathbb{M}) = \{ m \in \mathbb{M} | nm = 0 \},
\]
\[
\text{Tor}^Z_1(Z_m, Z_n) = Z_{(m,n)},
\]
\[
\text{Tor}^R_1(A \oplus B, \mathbb{M}) = \text{Tor}^R_1(A, \mathbb{M}) \oplus \text{Tor}^R_1(B, \mathbb{M}),
\]
\[
\text{Tor}^R_1(\mathbb{M}, A \oplus B) = \text{Tor}^R_1(\mathbb{M}, A) \oplus \text{Tor}^R_1(\mathbb{M}, B),
\]
where \((m, n)\) is the greatest common divisor of \(m\) and \(n\). These expressions allow us to compute the tensor-product \(\otimes_R\) and the torsion-product \(\text{Tor}^R_1\).

The Künneth formula works for topological cohomology where \(X\) and \(X'\) is treated as spaces. The Künneth formula also works for group cohomology where \(X\) and \(X'\) is treated as groups.

As the first application of Künneth formula, we like to use it to calculate \(H^*(X, \mathbb{M})\) from \(H^*(X, Z)\). By choosing \(R = Z = Z\). In this case, the condition \(\text{Tor}^R_1(\mathbb{M}, \mathbb{M}') = \text{Tor}^Z_1(Z, \mathbb{M}') = 0\) is always satisfied. So we have
\[
H^d(X \times X', \mathbb{M}') \simeq \bigoplus_{k=0}^d H^k(X, Z) \otimes Z H^{d-k}(X', \mathbb{M}') + \left[ \bigoplus_{k=0}^{d+1} \text{Tor}^Z_1(H^k(X, Z), H^{d-k+1}(X', \mathbb{M}')) \right].
\]

Now we can further choose \(X'\) to be the space of one point, and use
\[
H^d(X', \mathbb{M}') = \begin{cases} \mathbb{M}' , & \text{if } d = 0, \\ 0 , & \text{if } d > 0, \end{cases}
\]
to reduce eqn. (B4) to
\[
H^d(X, \mathbb{M}) \simeq H^d(X, Z) \otimes Z \mathbb{M} \oplus \text{Tor}^Z_1(H^{d+1}(X, Z), \mathbb{M}),
\]
where \(\mathbb{M}'\) is renamed as \(\mathbb{M}\). The above is a form of the universal coefficient theorem which can be used to calculate \(H^*(X, \mathbb{M})\) from \(H^*(X, Z)\) and the module \(\mathbb{M}\).

Using the universal coefficient theorem, we can rewrite eqn. (B4) as
\[
H^d(X \times X', \mathbb{M}) \simeq \bigoplus_{k=0}^d H^k[X, H^{d-k}(X', \mathbb{M})].
\]

**Appendix C: Lyndon-Hochschild-Serre spectral sequence**

The Lyndon-Hochschild-Serre spectral sequence\(^{84,86}\) allows us to understand the structure of \(H^d(GG \times

---

1. L. D. Landau, Phys. Z. Sowjetunion 11, 26 (1937)
2. V. L. Ginzburg and L. D. Landau, Zh. Ekaper. Teoret. Fiz.
80 D. A. Ivanov, Phys. Rev. Lett. 86, 268 (2001), arXiv:cond-mat/0005069
81 S. Ryu and S.-C. Zhang, Phys. Rev. 85, 245132 (2012)
82 X.-L. Qi (2012), arXiv:1202.3983
83 H. Yao and S. Ryu (2012), arXiv:1202.5805
84 G. Hochschild and J.-P. Serre, Transactions of the American Mathematical Society (American Mathematical Society) 74, 110 (1953)
85 E. H. Spanier, *Algebraic Topology* (McGraw-Hill, New York, 1966)
86 R. C. Lyndon, Duke Mathematical Journal 15, 271 (1948)