A few aspects of analysis on metric spaces

Stephen Semmes

Contents

1 Metric spaces and Lipschitz functions 1
2 Lipschitz functions of order \( \alpha \) 4
3 Some functions on the real line 6
4 Sums on general metric spaces 10
5 The Zygmund class 11
6 Approximation operators, 1 13
7 Approximation operators, 2 14
8 A kind of Calderón–Zygmund decomposition related to Lipschitz functions 16

1 Metric spaces and Lipschitz functions

Let \((M, d(x, y))\) be a metric space. Thus \(M\) is a nonempty set, \(d(x, y)\) is a nonnegative real-valued function on \(M \times M\) which is equal to 0 exactly when \(x = y\), \(d(x, y)\) is symmetric in \(x\) and \(y\), so that \(d(x, y) = d(y, x)\) for all \(x, y \in M\), and \(d(x, y)\) satisfies the triangle inequality,

\[
(1.1) \quad d(x, z) \leq d(x, y) + d(y, z)
\]
for all \( x, y, \) and \( z \) in \( M \). If \( x \) is an element of \( M \) and \( r \) is a positive real number, then we write \( B(x, r) \) and \( \overline{B}(x, r) \) for the open and closed balls in \( M \) with center \( x \) and radius \( r \), i.e.,

\[
(1.2) \quad B(x, r) = \{ y \in M : d(x, y) < r \}, \quad \overline{B}(x, r) = \{ y \in M : d(x, y) \leq r \}.
\]

Suppose that \( f(x) \) is a real or complex-valued function on \( M \), and that \( L \) is a nonnegative real number. We say that \( f \) is \( L \)-Lipschitz if

\[
(1.3) \quad |f(x) - f(y)| \leq L d(x, y)
\]

for all \( x, y \in M \). We also simply say that \( f \) is Lipschitz if it is \( L \)-Lipschitz for some \( L \). If \( f \) is Lipschitz, then we define \( \|f\|_{\text{Lip}} \) to be the supremum of

\[
(1.4) \quad \frac{|f(x) - f(y)|}{d(x, y)}
\]

over all \( x, y \in M \), where this ratio is replaced with 0 when \( x = y \). Thus \( f \) is \( \|f\|_{\text{Lip}} \)-Lipschitz when \( f \) is Lipschitz, and this is the smallest choice of \( L \) for which \( f \) is \( L \)-Lipschitz. Note that \( \| \cdot \|_{\text{Lip}} \) is a seminorm, so that

\[
(1.5) \quad \|a f + b g\|_{\text{Lip}} \leq |a| \|f\|_{\text{Lip}} + |b| \|g\|_{\text{Lip}}
\]

for all constants \( a, b \) and Lipschitz functions \( f, g \) on \( M \). Also, \( \|f\|_{\text{Lip}} = 0 \) if and only if \( f \) is a constant function on \( M \).

If \( f \) and \( g \) are real-valued \( L \)-Lipschitz functions on \( M \), then \( \max(f, g) \) and \( \min(f, g) \) are \( L \)-Lipschitz functions too. Let us check this for \( \max(f, g) \). It is enough to show that

\[
(1.6) \quad \max(f, g)(x) - \max(f, g)(y) \leq L d(x, y)
\]

for all \( x, y \in M \), i.e., one can interchange the roles of \( x \) and \( y \) to get an inequality in the opposite direction. Assume, for the sake of definiteness, that \( \max(f, g)(x) = f(x) \). Then we have

\[
(1.7) \quad \max(f, g)(x) = f(x) \leq f(y) + L d(x, y) \leq \max(f, g)(y) + L d(x, y),
\]

which is what we wanted.

Here is a generalization of this fact.
Lemma 1.8 Let \( \{ f_\sigma \}_{\sigma \in A} \) be a family of real-valued functions on \( M \) which are all \( L \)-Lipschitz for some \( L \geq 0 \). Assume also that there is point \( p \) in \( M \) such that the set of real numbers \( \{ f_\sigma(p) : \sigma \in A \} \) is bounded from above. Then the set \( \{ f_\sigma(x) : \sigma \in A \} \) is bounded from above for every \( x \) in \( M \) (but not uniformly in \( x \) in general), and \( \sup \{ f_\sigma(x) : \sigma \in A \} \) is an \( L \)-Lipschitz function on \( M \).

Indeed, because \( f_\sigma \) is \( L \)-Lipschitz for all \( \sigma \) in \( A \), we have that

\[
f_\sigma(x) \leq f_\sigma(y) + L d(x,y)
\]

for all \( x, y \) in \( M \). Applying this to \( y = p \), we see that \( \{ f_\sigma(x) : \sigma \in A \} \) is bounded from above for every \( x \), because of the corresponding property for \( p \). If \( F(x) = \sup \{ f_\sigma(x) : \sigma \in A \} \), then

\[
F(x) \leq F(y) + L d(x,y)
\]

for all \( x, y \) in \( M \), so that \( F \) is \( L \)-Lipschitz on \( M \).

For the record, let us write down the analogous statement for infima of \( L \)-Lipschitz functions.

Lemma 1.11 Let \( \{ f_\sigma \}_{\sigma \in A} \) be a family of real-valued functions on \( M \) which are all \( L \)-Lipschitz for some \( L \geq 0 \). Assume also that there is point \( q \) in \( M \) such that the set of real numbers \( \{ f_\sigma(q) : \sigma \in A \} \) is bounded from below. Then the set \( \{ f_\sigma(x) : \sigma \in A \} \) is bounded from below for every \( x \) in \( M \), and \( \inf \{ f_\sigma(x) : \sigma \in A \} \) is an \( L \)-Lipschitz function on \( M \).

For any point \( w \) in \( M \), \( d(x,w) \) defines a 1-Lipschitz function of \( x \) on \( M \). This can be shown using the triangle inequality. Suppose now that \( f(x) \) is an \( L \)-Lipschitz function on \( M \). For each \( w \in M \), define \( f_w(x) = f(w) + L d(x,w) \). The fact that \( f \) is \( L \)-Lipschitz implies that

\[
f(x) \leq f_w(x) \quad \text{for all} \ x, w \in M.
\]

Of course \( f_x(x) = x \), and hence

\[
f(x) = \inf \{ f_w(x) : w \in M \}.
\]

Each function \( f_w(x) \) is \( L \)-Lipschitz in \( x \), since \( d(x,w) \) is 1-Lipschitz in \( w \).
Similarly, we can set \( \tilde{f}_w(x) = f(x) - Ld(x, w) \), and then we have that
\[
(1.14) \quad f(x) = \sup\{\tilde{f}_w(x) : w \in M\},
\]
and that \( \tilde{f}_w(x) \) is an \( L \)-Lipschitz function of \( x \) for every \( w \).

Here is a variant of these themes. Let \( E \) be a nonempty subset of \( M \), and suppose that \( f \) is a real-valued function on \( E \) which is \( L \)-Lipschitz, so that
\[
(1.15) \quad |f(x) - f(y)| \leq Ld(x, y)
\]
for all \( x, y \) in \( M \). For each \( w \) in \( E \), set \( f_w(x) = f(x) + Ld(x, w) \) and \( \tilde{f}_w(x) = f(x) - Ld(x, w) \). Consider
\[
(1.16) \quad F(x) = \inf\{f_w(x) : w \in E\}, \quad \tilde{F}(x) = \sup\{\tilde{f}_w(x) : w \in E\},
\]
for \( x \) in \( M \). For the same reasons as before, \( F(x) = \tilde{F}(x) = f(x) \) when \( x \) lies in \( E \). Using Lemmas 1.8 and 1.11, one can check that \( F \) and \( \tilde{F} \) are \( L \)-Lipschitz real-valued functions on all of \( M \), i.e., they are extensions of \( f \) from \( E \) to \( M \) with the same Lipschitz constant \( L \).

If \( H(x) \) is any other real-valued function on \( M \) which agrees with \( f \) on \( E \) and is \( L \)-Lipschitz, then
\[
(1.17) \quad \tilde{f}_w(x) \leq H(x) \leq f_w(x)
\]
for all \( w \) in \( E \) and \( x \) in \( M \), and hence
\[
(1.18) \quad \tilde{F}(x) \leq H(x) \leq F(x)
\]
for all \( x \) in \( M \).

**Remark 1.19** If \( S \) is any nonempty subset of \( M \), define \( \text{dist}(x, S) \) for \( x \) in \( M \) by
\[
(1.20) \quad \text{dist}(x, S) = \inf_{y \in S} d(x, y).
\]
This function is always 1-Lipschitz in \( x \), by Lemma 1.11.

## 2 Lipschitz functions of order \( \alpha \)

Let \((M, d(x, y))\) be a metric space, and let \( \alpha \) be a positive real number. A real or complex-valued function \( f \) on \( M \) is said to be *Lipschitz of order \( \alpha \)* if there is nonnegative real number \( L \) such that
\[
(2.1) \quad |f(x) - f(y)| \leq L d(x, y)^\alpha
\]
for all $x, y \in M$. This reduces to the Lipschitz condition discussed in Section 1 when $\alpha = 1$. We shall sometimes write $\text{Lip}_\alpha$ for the collection of Lipschitz functions of order $\alpha$, which might be real or complex valued, depending on the context. One also sometimes refers to these functions as being “Hölder continuous of order $\alpha$”.

If $f$ is Lipschitz of order $\alpha$, then we define $\|f\|_{\text{Lip}_\alpha}$ to be the supremum of
\begin{equation}
\frac{|f(x) - f(y)|}{d(x, y)^\alpha}
\end{equation}
over all $x, y \in M$, where this quantity is replaced with 0 when $x = y$. In other words, $\|f\|_{\text{Lip}_\alpha}$ is the smallest choice of $L$ so that (2.1) holds for all $x, y \in M$. This defines a seminorm on the space of Lipschitz functions of order $\alpha$, as before, with $\|f\|_{\text{Lip}_\alpha} = 0$ if and only if $f$ is constant. Of course $\|f\|_{\text{Lip}_1}$ is the same as $\|f\|_{\text{Lip}}$ from Section 1.

If $f$ and $g$ are real-valued functions on $M$ which are Lipschitz of order $\alpha$ with constant $L$, then $\max(f, g)$ and $\min(f, g)$ are also Lipschitz of order $\alpha$ with constant $L$. This can be shown in the same manner as for $\alpha = 1$. Similarly, the analogues of Lemmas 1.8 and 1.11 for Lipschitz functions of order $\alpha$ hold for essentially the same reasons as before.

However, if $\alpha > 1$, it may be that the only functions that are Lipschitz of order $\alpha$ are the constant functions. This is the case when $M = \mathbb{R}^n$, for instance, equipped with the standard Euclidean metric, because a function in $\text{Lip}_\alpha$ with $\alpha > 1$ has first derivatives equal to 0 everywhere. Instead of using derivatives, it is not hard to show that the function has to be constant through more direct calculation too.

This problem does not occur when $\alpha < 1$.

**Lemma 2.3** If $0 < \alpha \leq 1$ and $a, b$ are nonnegative real numbers, then $(a + b)^\alpha \leq a^\alpha + b^\alpha$.

To see this, observe that
\begin{equation}
\max(a, b) \leq (a^\alpha + b^\alpha)^{1/\alpha},
\end{equation}
and hence
\begin{equation}
a + b \leq \max(a, b)^{1-\alpha} (a^\alpha + b^\alpha) \\
\leq (a^\alpha + b^\alpha)^{1+(1-\alpha)/\alpha} = (a^\alpha + b^\alpha)^{1/\alpha}.
\end{equation}
Corollary 2.6 If \((M, d(x, y))\) is a metric space and \(\alpha\) is a real number such that \(0 < \alpha \leq 1\), then \(d(x, y)^\alpha\) also defines a metric on \(M\).

This is easy to check. The main point is that \(d(x, y)^\alpha\) satisfies the triangle inequality, because of Lemma 2.3 and the triangle inequality for \(d(x, y)\).

A function \(f\) on \(M\) is Lipschitz of order \(\alpha\) with respect to the original metric \(d(x, y)\) if and only if it is Lipschitz of order 1 with respect to \(d(x, y)^\alpha\), and with the same norm. In particular, for each \(w\) in \(M\), \(d(x, w)^\alpha\) satisfies (2.1) with \(L = 1\) when \(0 < \alpha \leq 1\), because of the triangle inequality for \(d(u, v)^\alpha\).

3 Some functions on the real line

Fix \(\alpha, 0 < \alpha \leq 1\). For each nonnegative integer \(n\), consider the function

\[
2^{-n\alpha} \exp(2^n i x)
\]

on the real line \(\mathbb{R}\), where \(\exp u\) denotes the usual exponential \(e^u\). Let us estimate the Lip\(\alpha\) norm of this function.

Recall that

\[
|\exp(i u) - \exp(i v)| \leq |u - v|
\]

for all \(u, v \in \mathbb{R}\). Indeed, one can write \(\exp(i u) - \exp(i v)\) as the integral between \(u\) and \(v\) of the derivative of \(\exp(i t)\), and this derivative is \(i \exp(i t)\), which has modulus equal to 1 at every point.

Thus, for any \(x, y \in \mathbb{R}\), we have that

\[
|2^{-n\alpha} \exp(2^n i x) - 2^{-n\alpha} \exp(2^n i y)| \leq 2^{n(1-\alpha)} |x - y|.
\]

Of course

\[
|2^{-n\alpha} \exp(2^n i x) - 2^{-n\alpha} \exp(2^n i y)| \\
\leq 2^{-n\alpha} |\exp(2^n i x)| + 2^{-n\alpha} |\exp(2^n i y)| = 2^{-n\alpha + 1}
\]

as well. As a result,

\[
|2^{-n\alpha} \exp(2^n i x) - 2^{-n\alpha} \exp(2^n i y)| \\
\leq \left(2^{n(1-\alpha)} |x - y|\right)^\alpha \left(2^{-n\alpha + 1}\right)^{1-\alpha} = 2^{1-\alpha} |x - y|^\alpha.
\]
This shows that the function (3.1) has Lip $\alpha$ norm (with respect to the standard Euclidean metric on $\mathbb{R}$) which is at most $2^{1-\alpha}$. In the opposite direction, if $2^n(x - y) = \pi$, then

\[(3.6) \quad |2^{-n\alpha} \exp(2^n i x) - 2^{-n\alpha} \exp(2^n i y)| \]
\[= 2^{-n\alpha}|\exp(2^n i x)| + 2^{-n\alpha}|\exp(2^n i y)| \]
\[= 2^{-n\alpha + 1} = 2\pi^{-\alpha} |x - y|^{\alpha}, \]
so that the Lip $\alpha$ norm is at least $2\pi^{-\alpha}$.

Now suppose that $f(x)$ is a complex-valued function on $\mathbb{R}$ of the form

\[(3.7) \quad f(x) = \sum_{n=0}^{\infty} a_n 2^{-n\alpha} \exp(2^n i x), \]

where the $a_n$’s are complex numbers. We assume that the $a_n$’s are bounded, which implies that the series defining $f(x)$ converges absolutely for each $x$. Set

\[(3.8) \quad A = \sup_{n \geq 0} |a_n|. \]

Let $m$ be a nonnegative integer. For each $x$ in $\mathbb{R}$ we have that

\[(3.9) \quad \left| \sum_{n=m}^{\infty} a_n 2^{-n\alpha} \exp(2^n i x) \right| \leq \sum_{n=m}^{\infty} A 2^{-n\alpha} = A (1 - 2^{-\alpha})^{-1} 2^{-m\alpha}. \]

If $m \geq 1$ and $x, y \in \mathbb{R}$, then (3.2) yields

\[(3.10) \quad \left| \sum_{n=0}^{m-1} a_n 2^{-n\alpha} \exp(2^n i x) - \sum_{n=0}^{m-1} a_n 2^{-n\alpha} \exp(2^n i y) \right| \]
\[\leq \sum_{n=0}^{m-1} A 2^{n(1-\alpha)} |x - y| \leq A 2^{(m-1)(1-\alpha)} \left( \sum_{j=0}^{\infty} 2^{-j(1-\alpha)} \right) |x - y| \]
\[= A 2^{(m-1)(1-\alpha)} \left( 1 - 2^{-(1-\alpha)} \right)^{-1} |x - y|. \]

Here we should assume that $\alpha < 1$, to get the convergence of $\sum_{j=0}^{\infty} 2^{-j(1-\alpha)}$.

Fix $x, y \in \mathbb{R}$. If $|x - y| > 1/2$, then we apply (3.9) with $m = 0$ to both $x$ and $y$ to get that

\[(3.11) \quad |f(x) - f(y)| \leq |f(x)| + |f(y)| \leq 2 A (1 - 2^{-\alpha})^{-1} \]
\[\leq 2^{1+\alpha} A (1 - 2^{-\alpha})^{-1} |x - y|^{\alpha}. \]
Assume now that $|x - y| \leq 1/2$, and choose $m \in \mathbb{Z}_+$ so that

$$2^{-m-1} < |x - y| \leq 2^{-m}. \tag{3.12}$$

Combining (3.9) and (3.10), with (3.9) applied to both $x$ and $y$, we obtain that

$$|f(x) - f(y)| \leq 2 A (1 - 2^{-\alpha})^{-1} 2^{-m\alpha} + A 2^{(m-1)(1-\alpha)} (1 - 2^{-(1-\alpha)})^{-1} |x - y|$$

$$\leq 2^{1+\alpha} A (1 - 2^{-\alpha})^{-1} |x - y|^\alpha + A 2^{-(1-\alpha)} (1 - 2^{-(1-\alpha)})^{-1} |x - y|^\alpha. \tag{3.13}$$

Therefore, for all $x, y \in \mathbb{R}$, we have that

$$|f(x) - f(y)| \leq A (2^{1+\alpha} (1 - 2^{-\alpha})^{-1} + 2^{-(1-\alpha)} (1 - 2^{-(1-\alpha)})^{-1}) |x - y|^\alpha \tag{3.14}$$

when $0 < \alpha < 1$. In other words, $f$ is Lipschitz of order $\alpha$, and

$$\|f\|_{\text{Lip}} \leq \left( \sup_{n \geq 0} |a_n| \right) (2^{1+\alpha} (1 - 2^{-\alpha})^{-1} + 2^{-(1-\alpha)} (1 - 2^{-(1-\alpha)})^{-1}). \tag{3.15}$$

To get an inequality going in the other direction we shall compute as follows. Let $\psi(x)$ be a function on $\mathbb{R}$ such that the Fourier transform $\hat{\psi}(\xi)$ of $\psi$,

$$\hat{\psi}(\xi) = \int_{\mathbb{R}} \exp(i \xi x) \psi(x) \, dx \tag{3.16}$$

is a smooth function which satisfies $\hat{\psi}(1) = 1$ and $\hat{\psi}(\xi) = 0$ when $0 \leq \xi \leq 1/2$ and when $\xi \geq 2$. One can do this with $\psi(x)$ in the Schwartz class of smooth functions such that $\psi(x)$ and all of its derivatives are bounded by constant multiples of $(1 + |x|)^{-k}$ for every positive integer $k$.

For each nonnegative integer $j$, let us write $\psi_{2^j}(x)$ for the function $2^j \psi(2^j x)$. Thus

$$\hat{\psi}_{2^j}(\xi) = \hat{\psi}(2^{-j} \xi). \tag{3.17}$$

In particular, $\psi_{2^j}(2^j) = 1$, and $\psi_{2^j}(2^l) = 0$ when $l$ is a nonnegative integer different from $j$. Hence

$$\int_{\mathbb{R}} f(x) \psi_{2^j}(x) \, dx = \sum_{n=0}^{\infty} a_n 2^{-n\alpha} \psi_{2^j}(2^n) = a_j 2^{-j\alpha}. \tag{3.18}$$
On the other hand,

\[ \int_{\mathbb{R}} \psi_2(x) \, dx = \hat{\psi}_2(0) = \hat{\psi}(0) = 0, \]

so that

\[ \int_{\mathbb{R}} f(x) \psi_2(x) \, dx = \int_{\mathbb{R}} (f(x) - f(0)) \psi_2(x) \, dx. \]

Therefore

\[ |\int_{\mathbb{R}} f(x) \psi_2(x) \, dx| \leq \int_{\mathbb{R}} |f(x) - f(0)| |\psi_2(x)| \, dx \]
\[ \leq \|f\|_{\text{Lip}} \int_{\mathbb{R}} |x|^\alpha |\psi_2(x)| \, dx \]
\[ = \|f\|_{\text{Lip}} \sum_{\alpha} 2^{-j\alpha} \int_{\mathbb{R}} |x|^\alpha |\psi(x)| \, dx. \]

Combining this with (3.18), we obtain that

\[ |a_j| \leq \|f\|_{\text{Lip}} \sum_{\alpha} \int_{\mathbb{R}} |x|^\alpha |\psi(x)| \, dx \]

for all nonnegative integers \( j \). The integral on the right side converges, because of the decay property of \( \psi \).

If \( \alpha = 1 \), then let us pass to the derivative and write

\[ f'(x) = \sum_{n=0}^{\infty} a_n i \exp 2^n i x \]

(where one should be careful about the meaning of \( f' \) and of this series). This leads to

\[ \frac{1}{2\pi} \int_{0}^{2\pi} |f'(x)|^2 \, dx = \sum_{n=0}^{\infty} |a_n|^2. \]

The main idea is that

\[ \sum_{n=0}^{\infty} |a_n|^2 \leq \|f\|_{\text{Lip}}^2 \]

if \( f \) is Lipschitz. Conversely, if \( \sum_{n=0}^{\infty} |a_n|^2 < \infty \), then the derivative of \( f \) exists in an \( L^2 \) sense, and in fact one can show that \( f' \) has “vanishing mean oscillation”.
4 Sums on general metric spaces

Let \((M, d(x, y))\) be a metric space. For each integer \(n\), suppose that we have chosen a complex-valued Lipschitz function \(\beta_n(x)\) such that

\[
\sup_{x \in M} |\beta_n(x)| \leq 1 \quad \text{and} \quad \|\beta\|_{\text{Lip}} \leq 2^n. \tag{4.1}
\]

Fix a real number \(\alpha, 0 < \alpha < 1\).

Let \(a_n, n \in \mathbb{Z}\) be a family (or doubly-infinite sequence) of complex numbers which is bounded, and set

\[
A = \sup_{n \in \mathbb{Z}} |a_n|. \tag{4.2}
\]

Consider

\[
f(x) = \sum_{n \in \mathbb{Z}} a_n 2^{-n\alpha} \beta_n(x). \tag{4.3}
\]

The sum on the right side does not really converge in general, although it would if we restricted ourselves to \(n\) greater than any fixed number, because of the bound on \(\beta_n(x)\). However, this sum does converge “modulo constants”, in the sense that the sum in

\[
f(x) - f(y) = \sum_{n \in \mathbb{Z}} a_n 2^{-n\alpha} (\beta_n(x) - \beta_n(y)), \tag{4.4}
\]

converges absolutely for all \(x, y\) in \(M\).

To see this, suppose that \(k\) is any integer. For \(n \geq k\) we have that

\[
\sum_{n=k}^{\infty} |a_n| 2^{-n\alpha} |\beta_n(x)| \leq A (1 - 2^{-\alpha})^{-1} 2^{-k\alpha}, \tag{4.5}
\]

and similarly for \(y\) instead of \(x\). For \(n \leq k - 1\) we have that

\[
\sum_{n=-\infty}^{-k-1} |a_n| 2^{-n\alpha} |\beta_n(x) - \beta_n(y)| \leq A \sum_{n=-\infty}^{-1} 2^{n(1-\alpha)} d(x, y)
= A 2^{(k-1)(1-\alpha)} (1 - 2^{-(1-\alpha)})^{-1} d(x, y). \tag{4.6}
\]

Thus

\[
\sum_{n \in \mathbb{Z}} |a_n| 2^{-n\alpha} |\beta_n(x) - \beta_n(y)| \leq A (1 - 2^{-\alpha})^{-1} 2^{-k\alpha} + A 2^{(k-1)(1-\alpha)} (1 - 2^{-(1-\alpha)})^{-1} d(x, y) \tag{4.7}
\]

for all \(x, y \in M\) and \(k \in \mathbb{Z}\).
5 The Zygmund class

Let \( f(x) \) be a real or complex-valued function on the real line. We say that \( f \) lies in the \textit{Zygmund class} \( Z \) if \( f \) is continuous and there is a nonnegative real number \( L \) such that

\[
|f(x + h) + f(x - h) - 2 f(x)| \leq L |h|
\]

for all \( x, y \in \mathbb{R} \). In this case, the seminorm \( \| f \|_Z \) is defined to be the supremum of

\[
\frac{|f(x + h) + f(x - h) - 2 f(x)|}{|h|}
\]

over all \( x, h \in \mathbb{R} \) with \( h \neq 0 \). This is the same as the smallest \( L \) so that (5.1) holds. Clearly \( f \) is in the Zygmund class when \( f \) is Lipschitz (of order 1), with \( \| f \|_Z \leq 2 \| f \|_{\text{Lip}} \).

Suppose that \( \{a_n\}_{n=0}^{\infty} \) is a bounded sequence of complex numbers, and consider the function \( f(x) \) on \( \mathbb{R} \) defined by

\[
f(x) = \sum_{n=0}^{\infty} a_n 2^{-n} \exp(2^n i x).
\]

Let us check that \( f \) lies in the Zygmund class, with \( \| f \|_Z \) bounded in terms of

\[
A = \sup_{n \geq 0} |a_n|.
\]

Note that \( f \) is continuous.

Observe that

\[
| \exp(i(u + v)) + \exp(i(u - v)) - 2 \exp(i u) |
\]

\[
= | \exp(i v) + \exp(-i v) - 2 |
\]

for all real numbers \( u, v \), and that

\[
\exp(i v) + \exp(-i v) - 2 = \int_0^v i(\exp(i t) - \exp(-i t)) \, dt
\]

when \( v \geq 0 \). Since \( | \exp(i t) - \exp(-i t) | \leq 2 t \) for \( t \geq 0 \), we obtain that

\[
| \exp(i v) + \exp(-i v) - 2 | \leq \int_0^v 2 t \, dt = v^2.
\]
Hence
\[
| \exp(i(u + v)) + \exp(i(u - v)) - 2 \exp(iu) | \leq v^2,
\]
and this works for all real numbers \( u, v \), since there is no real difference between \( v \geq 0 \) and \( v \leq 0 \).

Let \( x \) and \( h \) be real numbers, and let \( m \) be a nonnegative integer. From (5.8) we get that
\[
\left| \sum_{n=0}^{m} a_n 2^{-n} (\exp(2^n i(x + h)) + \exp(2^n i(x - h)) - 2 \exp(2^n i x)) \right| \leq A \sum_{n=0}^{m} 2^{-n} 2^{2n} |h|^2 \leq A 2^{m+1} |h|^2.
\]

If \( |h| \geq 1/2 \), then
\[
\left| f(x + h) + f(x - h) - 2 f(x) \right| \leq |f(x + h)| + |f(x - h)| + 2 |f(x)| \leq 4 A \leq 8 A |h|.
\]

If \( |h| \leq 1/2 \), then choose a positive integer \( m \) such that \( 2^{-m-1} \leq |h| \leq 2^{-m} \).

We can write \( f(x + h) + f(x - h) - 2 f(x) \) as
\[
\sum_{n=0}^{m} a_n 2^{-n} (\exp(2^n i(x + h)) + \exp(2^n i(x - h)) - 2 \exp(2^n i x)) + \sum_{n=0}^{\infty} a_n 2^{-n} (\exp(2^n i(x + h)) + \exp(2^n i(x - h)) - 2 \exp(2^n i x))
\]
This leads to
\[
\left| f(x + h) + f(x - h) - 2 f(x) \right| \leq |f(x + h)| + |f(x - h)| + 2 |f(x)| \leq A 2^{m+1} |h|^2 + 4 A 2^{-m} \leq A \cdot 2 \cdot |h| + 4 \cdot A \cdot 2 \cdot |h| = 10 A |h|.
\]

This shows that \( f \) lies in the Zygmund class, with constant less than or equal to \( 10 A \).
6 Approximation operators, 1

Let \((M, d(x, y))\) be a metric space. Fix a real number \(\alpha, 0 < \alpha < 1\), and let \(f\) be a real-valued function on \(M\) which is Lipschitz of order \(\alpha\). For each positive real number \(L\), define \(A_L(f)\) by

\[
A_L(f)(x) = \inf \{ f(w) + L \, d(x, w) : w \in M \}
\]

for all \(x\) in \(M\).

For arbitrary \(x, w\) in \(M\) we have that

\[
f(w) \geq f(x) - \|f\|_{\text{Lip} \alpha} \, d(x, w)^\alpha.
\]

As a result,

\[
f(w) + L \, d(x, w) \geq f(x)
\]

when \(L \, d(x, w)^{1-\alpha} \geq \|f\|_{\text{Lip} \alpha}\). Thus we can rewrite (6.1) as

\[
A_L(f)(x) = \inf \{ f(w) + L \, d(x, w) : w \in M, \ L \, d(x, w)^{1-\alpha} \leq \|f\|_{\text{Lip} \alpha} \},
\]

i.e., one gets the same infimum over this smaller range of \(w\)’s. In particular, the set of numbers whose infimum is under consideration is bounded from below, so that the infimum is finite.

Because we can take \(w = x\) in the infimum, we automatically have that

\[
A_L(f)(x) \leq f(x)
\]

for all \(x\) in \(M\). In the other direction, (6.2) and (6.4) lead to

\[
A_L(f)(x) \geq f(x) - \|f\|_{\text{Lip} \alpha} \left(\frac{\|f\|_{\text{Lip} \alpha}}{L}\right)^{\alpha/(1-\alpha)}
\]

\[
= f(x) - \|f\|_{\text{Lip} \alpha}^{1/(1-\alpha)} \, L^{-\alpha/(1-\alpha)}.
\]

We also have that \(A_L(f)\) is \(L\)-Lipschitz on \(M\), as in Lemma 1.11.

Suppose that \(h(x)\) is a real-valued function on \(M\) which is \(L\)-Lipschitz and satisfies \(h(x) \leq f(x)\) for all \(x\) in \(M\). Then

\[
h(x) \leq h(w) + L \, d(x, w) \leq f(w) + L \, d(x, w)
\]

for all \(x, w\) in \(M\). Hence

\[
h(x) \leq A_L(f)(x)
\]
for all $x$ in $M$.

Similarly, one can consider

$$B_L(f)(x) = \sup \{ f(w) - L d(x, w) : w \in M \},$$

and show that

$$B_L(f)(x) = \sup \{ f(w) - L d(x, w) : w \in M, \ L d(x, w)^{1-\alpha} \leq \|f\|_{\text{Lip}} \alpha \}. \tag{6.10}$$

This makes it clear that the supremum is finite. As before,

$$f(x) \leq B_L(f)(x) \leq f(x) + \|f\|_{\text{Lip}}^{1/(1-\alpha)} L^{-\alpha/(1-\alpha)}, \tag{6.11}$$

and $B_L(f)$ is $L$-Lipschitz. If $h(x)$ is a real-valued function on $M$ which is $L$-Lipschitz and satisfies $f(x) \leq h(x)$ for all $x$ in $M$, then

$$B_L(f)(x) \leq h(x) \tag{6.12}$$

for all $x$ in $M$.

7 Approximation operators, 2

Let $(M,d(x,y))$ be a metric space, and let $\mu$ be a positive Borel measure on $M$. We shall assume that $\mu$ is a doubling measure, which means that there is a positive real number $C$ such that

$$\mu(B(x,2r)) \leq C \mu(B(x,r)) \tag{7.1}$$

for all $x$ in $M$ and positive real numbers $r$, and that the $\mu$-measure of any open ball is positive and finite.

Let $t$ be a positive real number. Define a function $p_t(x,y)$ on $M \times M$ by

$$p_t(x,y) = \begin{cases} 1 - t^{-1}d(x,y) & \text{when } d(x,y) \leq t \\ 0 & \text{when } d(x,y) > t, \end{cases} \tag{7.2}$$

and put

$$\rho_t(x) = \int_M p_t(x,y) \, d\mu(y). \tag{7.3}$$
This is positive for every \( x \) in \( M \), because of the properties of \( \mu \). Also put
\[
\phi_t(x, y) = \rho_t(x)^{-1} p_t(x, y),
\]
so that
\[
\int_M \phi_t(x, y) \, d\mu(y) = 1
\]
for all \( x \) in \( M \) by construction.

Fix a real number \( \alpha, 0 < \alpha \leq 1 \), and let \( f \) be a complex-valued function on \( M \) which is Lipschitz of order \( \alpha \). Define \( P_t(f) \) on \( M \) by
\[
P_t(f)(x) = \int_M \phi_t(x, y) \, f(y) \, d\mu(y).
\]
Because of (7.5),
\[
P_t(f)(x) - f(x) = \int_M \phi_t(x, y) (f(y) - f(x)) \, d\mu(y),
\]
and hence
\[
|P_t(f)(x) - f(x)| \leq \int_M \phi_t(x, y) |f(y) - f(x)| \, d\mu(y)
\leq \int_M \phi_t(x, y) \|f\|_{\text{Lip}, \alpha} \, t^\alpha \, d\mu(y) = \|f\|_{\text{Lip}, \alpha} \, t^\alpha.
\]
In the second step we employ the fact that \( \phi_t(x, y) = 0 \) when \( d(x, y) \geq t \).

Suppose that \( x \) and \( z \) are elements of \( M \), and consider
\[
|P_t(f)(x) - P_t(f)(z)|.
\]
If \( d(x, z) \geq t \), then
\[
|P_t(f)(x) - P_t(f)(z)|
\leq |P_t(f)(x) - f(x)| + |f(x) - f(z)| + |P_t(f)(z) - f(z)|
\leq \|f\|_{\text{Lip}, \alpha} (2t^\alpha + d(x, z)^\alpha) \leq 3t^{\alpha-1} \|f\|_{\text{Lip}, \alpha} \, d(x, z).
\]
Assume instead that \( d(x, z) \leq t \). In this case we write \( P_t(f)(x) - P_t(f)(z) \) as
\[
\int_M (\phi_t(x, y) - \phi_t(z, y)) \, f(y) \, d\mu(y)
= \int_M (\phi_t(x, y) - \phi_t(z, y)) (f(y) - f(x)) \, d\mu(y),
\]
using (7.5). This yields

\[ |P_t(f)(x) - P_t(f)(z)| \]
\[ \leq \int_M |\phi_t(x, y) - \phi_t(z, y)| |f(y) - f(x)| \, d\mu(y) \]
\[ \leq (2t)^\alpha \|f\|_{\text{Lip } \alpha} \int_{B(x, 2t)} |\phi_t(x, y) - \phi_t(z, y)| \, d\mu(y), \]

where the second step relies on the observation that \( \phi_t(x, y) - \phi_t(z, y) \) is supported, as a function of \( y \), in the set

\[ \overline{B}(x, t) \cup \overline{B}(z, t) \subseteq \overline{B}(x, 2t). \]

Of course

\[ \phi_t(x, y) - \phi_t(z, y) = (\rho_t(x)^{-1} - \rho_t(z)^{-1}) p_t(x, y) + \rho_t(z)^{-1} (p_t(x, y) - p_t(z, y)). \]

Notice that

\[ |p_t(x, y) - p_t(z, y)| \leq t^{-1} d(x, z) \]

for all \( y \) in \( M \). To see this, it is convenient to write \( p_t(u, v) \) as \( \lambda_t(d(u, v)) \), where \( \lambda_t(r) \) is defined for \( r \geq 0 \) by \( \lambda_t(r) = 1 - t^{-1} r \) when \( 0 \leq r \leq t \), and \( \lambda_t(r) = 0 \) when \( r \geq t \). It is easy to check that \( \lambda_t \) is \( t^{-1} \)-Lipschitz, and hence \( \lambda_t(d(u, v)) \) is \( t^{-1} \)-Lipschitz on \( M \) as a function of \( u \) for each fixed \( v \), since \( d(u, v) \) is \( 1 \)-Lipschitz as a function of \( u \) for each fixed \( v \). These computations and the doubling condition for \( \mu \) permit one to show that

\[ \int_{B(x, 2t)} |\phi_t(x, y) - \phi_t(z, y)| \, d\mu(y) \leq C_1 t^{-1} d(x, z) \]

for some positive real number \( C_1 \) which does not depend on \( x, z \), or \( t \). (Exercise.) Altogether, we obtain that

\[ \|P_t(f)\|_{\text{Lip } 1} \leq \max(3, 2^\alpha C_1) t^{\alpha - 1} \|f\|_{\text{Lip } \alpha}. \]

8 A kind of Calderón–Zygmund decomposition related to Lipschitz functions

Let \( (M, d(x, y)) \) be a metric space, and let \( f \) be a real-valued function on \( M \). Consider the associated maximal function

\[ N(f)(x) = \sup_{y \in M \atop y \neq x} \frac{|f(y) - f(x)|}{d(y, x)}. \]
where this supremum may be $+\infty$.

Let $L$ be a positive real number, and put

\[(8.2) \quad F_L = \{x \in M : N(f)(x) \leq L\}.\]

We shall assume for the rest of this section that

\[(8.3) \quad F_L \neq \emptyset.\]

As in Section 6, define $A_L(f)$ by

\[(8.4) \quad A_L(f)(x) = \inf \{f(w) + L d(x, w) : w \in M\}.\]

We shall address the finiteness of this infimum in a moment. As before,

\[(8.5) \quad A_L(f)(x) \leq f(x)\]

for all $x$ in $M$.

If $u$ is any element of $F_L$, then

\[(8.6) \quad |f(y) - f(u)| \leq L d(y, u)\]

for all $y$ in $M$. Let $x$ and $w$ be arbitrary points in $M$. The preceding inequality implies that

\[(8.7) \quad f(u) \leq f(w) + L d(u, w),\]

and hence

\[(8.8) \quad f(u) - L d(x, u) \leq f(w) + L (d(u, w) - d(x, u)) \leq f(w) + L d(x, w),\]

by the triangle inequality. This yields

\[(8.9) \quad f(u) - L d(x, u) \leq A_L(f)(x),\]

which includes the finiteness of $A_L(f)(x)$. If we take $x = u$, then we get $f(u) \leq A_L(f)(u)$, so that

\[(8.10) \quad f(u) = A_L(f)(u) \quad \text{for all } u \in F_L.\]
For $x \notin F_L$, we obtain

\[(8.11)\]
\[f(x) - 2L d(x, u) \leq A_L(f)(x)\]

for all $u$ in $F_L$, by combining (8.9) and (8.6) with $y = x$. In other words,

\[(8.12)\]
\[f(x) - A_L(f)(x) \leq 2L \text{dist}(x, F_L).\]

Note that $A_L(f)$ is $L$-Lipschitz on $M$, by Lemma [1.11].

In the same way, if

\[(8.13)\]
\[B_L(f)(x) = \sup \{ f(w) - L d(x, w) : w \in M \},\]

then

\[(8.14)\]
\[f(x) \leq B_L(f)(x) \leq f(x) + 2L \text{dist}(x, F_L)\]

for all $x$ in $M$, and $B_L(f)$ is $L$-Lipschitz.

References

[AmbT] L. Ambrosio and P. Tilli, Selected Topics on “Analysis on Metric Spaces”, Scuola Normale Superiore, Pisa, 2000.

[As1] P. Assouad, Espaces Métriques, Plongements, Facteurs, Thèse de Doctorat (January, 1977), Université de Paris XI, 91405 Orsay, France.

[As2] P. Assouad, Étude d’une dimension métrique liée à la possibilité de plongement dans $\mathbb{R}^n$, Comptes Rendus de l’Académie des Sciences Paris, Sér. A 288 (1979), 731–734.

[As3] P. Assouad, Plongements Lipschitziens dans $\mathbb{R}^n$, Bulletin de la Société Mathématique de France 111 (1983), 429–448.

[CoiW1] R. Coifman and G. Weiss, Analyse Harmonique Non-Commutative sur Certains Espaces Homogènes, Lecture Notes in Mathematics 242, Springer-Verlag, 1971.

[CoiW2] R. Coifman and G. Weiss, Extensions of Hardy spaces and their use in analysis, Bulletin of the American Mathematical Society 83 (1977), 569–645.
[DeVS] R. DeVore and R. Sharpley, *Maximal Functions Measuring Smoothness*, Memoirs of the American Mathematical Society 293, 1984.

[Dur] P. Duren, *Theory of $H^p$ Spaces*, Academic Press, 1970.

[FolS] G. Folland and E. Stein, *Hardy Spaces on Homogeneous Groups*, Princeton University Press, 1982.

[Hei] J. Heinonen, *Lectures on Analysis on Metric Spaces*, Springer-Verlag, 2001.

[Kig] J. Kigami, *Analysis on Fractals*, Cambridge University Press, 2001.

[Kra] S. Krantz, *Lipschitz spaces, smoothness of functions, and approximation theory*, Expositiones Mathematicae 3 (1983), 193–260.

[MacS] R. Macias and C. Segovia, *Lipschitz functions on spaces of homogeneous type*, Advances in Mathematics 33 (1979), 257–270.

[Rud] W. Rudin, *Principles of Mathematical Analysis*, McGraw-Hill, 1976.

[Sar] D. Sarason, *Function Theory on the Unit Circle*, Virginia Polytechnic Institute and State University, 1978.

[Sem1] S. Semmes, *Metric spaces and mappings seen at many scales*, appendix in *Metric Structures for Riemannian and Non-Riemannian Spaces*, M. Gromov et al., Birkhäuser, 1999.

[Sem2] S. Semmes, *Derivatives and difference quotients for Lipschitz or Sobolev functions on various spaces*, in *Lecture Notes on Analysis in Metric Spaces*, edited by L. Ambrosio and F. Serra Cassano, 71–103, Scuola Normale Superiore, Pisa, 2000.

[Ste1] E. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, 1970.

[Ste2] E. Stein, *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton University Press, 1993.

[SteW] E. Stein and G. Weiss, *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton University Press, 1971.
[TaiW] M. Taibleson and G. Weiss, *The molecular characterization of certain Hardy spaces*, Astérisque 77, 67–149, Société Mathématique de France, 1980.

[Zyg] A. Zygmund, *Trigonometric Series*, Volumes I and II, Cambridge University Press, 1979.