The Oberbeck–Boussinesq approximation as a constitutive limit

Abstract We derive the usual Oberbeck–Boussinesq approximation as a constitutive limit of the full system describing the motion of an compressible linearly viscous fluid. To this end, the starting system is written, using the Gibbs free energy, in the variables \(v, \theta\) and \(p\). The Oberbeck–Boussinesq system is then obtained as the thermal expansion coefficient \(\alpha\) and the isothermal compressibility coefficient \(\beta\) tend to zero.

Keywords Oberbeck–Boussinesq approximation · Constitutive limit · Gibbs free energy

1 Introduction

The well-known Oberbeck–Boussinesq \([3,11]\) approximation was designed as a simplified model for the thermo-mechanical response of linear viscous fluids undergoing isochoric motions in isothermal processes but not necessarily isochoric ones in non-isothermal processes. Its roots stem from the end of the nineteenth century. Nevertheless, its justification from the point of view of continuum mechanics was quite recently given in 1996 by \([12]\) (cf. \([7,13]\) and \([10]\) for earlier contributions).

From the mathematical point of view, the expansion used in \([12]\) is still formal. In \([8,9]\), a rigorous justification of simplified problems has been given. We refer the reader to \([5,6]\) for a completely different approach in which singular limits of the full system are discussed.

In this paper, we use a new approach. Motivated by the studies in \([1,2]\), we obtain the Oberbeck–Boussinesq approximation as a constitutive limit. Since this limit is singular, it depends on the way how it is achieved. To achieve our result, we rewrite the full thermo-mechanical system with the help of the Gibbs free energy in the variables \(v, \theta\) and \(p\). It is important that the Gibbs free energy depends on two (dimensional) parameters \(a, b\), which tend to zero. Then we non-dimensionalize the system and make assumptions on the behavior of the thermal expansion coefficient, the isothermal compressibility coefficient, the specific heat coefficient at constant pressure and the “density” on the non-dimensional versions of the parameters \(a, b\), which are denoted by \(A, B\). We show that these requirements can be fulfilled by an easy example, which assumes that the density is linear in the pressure and the temperature. Provided that weak solutions of the full thermo-mechanical system satisfying a uniform estimate exist, we show that the limits satisfy the Oberbeck–Boussinesq approximation. Moreover, if we assume that the approximation parameters are fixed and sufficiently small, we can also recover the results obtained by a power series expansion in \([12]\).

Communicated by Andreas Öchsner.

Y. Kagei
Faculty of Mathematics, Kyushu University, Fukuoka 819-0395, Japan
E-mail: kagei@math.kyushu-u.ac.jp

M. Růžička
Institute of Applied Mathematics, University of Freiburg, Eckerstr. 1, 79104 Freiburg, Germany
E-mail: rose@mathematik.uni-freiburg.de
Let us finally introduce some notation:

In what follows boldfaced minuscules always stand for vectors and vector-valued functions, whereas boldfaced capital letters represent tensor-valued functions, i.e., \( \mathbf{v} = (v_1, v_2, v_3)^\top \), \( \mathbf{T} = (T_{kl})_{k, l = 1}^3 \), \( \mathbf{T}^\top := (T_{kl})_{k, l = 1}^3 \). All quantities are considered at points \( \mathbf{x} = (x_1, x_2, x_3)^\top \in \mathbb{R}^3 \) and at a certain time \( t \). We use the abbreviations \( \partial_k := \frac{\partial}{\partial x_k} \), \( \partial_t := \frac{\partial}{\partial t} \) (analogously for \( \partial_\theta \ldots \)), \( \partial_\rho^2 := \partial_\rho \partial_\rho \), \( \text{div} \mathbf{v} := \partial_1 v_1 + \partial_2 v_2 + \partial_3 v_3 \) and \( \nabla \rho := (\partial_1 \rho, \partial_2 \rho, \partial_3 \rho)^\top \). The dot between two quantities denotes the corresponding scalar product, whereas the superposed dot is the usual material time derivative:

\[
\mathbf{v} \cdot \mathbf{w} := \sum_{k=1}^3 v_k w_k, \quad \dot{\rho} := \partial_t \rho + \mathbf{v} \cdot \nabla \rho = \partial_t \rho + \sum_{k=1}^3 v_k \partial_k \rho, \\
\mathbf{T} \cdot \mathbf{L} = \sum_{k, l = 1}^3 T_{kl} L_{kl}, \quad \mathbf{b} := \partial_t \mathbf{b} + |\nabla \mathbf{b}| \mathbf{v} = \partial_t \mathbf{b} + \sum_{k=1}^3 v_k \partial_k \mathbf{b}.
\]

The trace of some tensor \( \mathbf{D} \) is denoted by \( \text{tr} \mathbf{D} \) and \( ||\mathbf{D}||^2 := \mathbf{D} \cdot \mathbf{D} \). For the identity tensor, we write \( \mathbf{I} \).

2 Derivation of the approximation

2.1 Governing equations and assumptions

The starting point for our analysis is the balance of mass, linear momentum and energy and the second law of thermodynamics in the form of the Clausius–Duhem inequality:

\[
\dot{\rho} + \rho \text{div} \mathbf{v} = 0, \\
\rho \dot{\mathbf{v}} - \text{div} \mathbf{T} = \rho \mathbf{b}, \\
\rho \dot{\mathbf{e}} - \text{div} \mathbf{q} = \mathbf{T} : \mathbf{D} + \rho \mathbf{r},
\]

\[
\rho \dot{\theta} - \text{div} \left( \frac{\mathbf{q}}{\theta} \right) - \frac{\rho}{\theta} \mathbf{r} \geq 0, \tag{2.2}
\]

where \( \rho \) denotes the density, \( \mathbf{v} \) the velocity field, \( \mathbf{T} \) the symmetric Cauchy stress tensor, \( \mathbf{b} \) the density of external body forces, \( e \) the specific internal energy, \( \mathbf{L} \) the velocity gradient, \( \mathbf{r} \) the radiant heating, \( \theta \) the temperature, \( \eta \) the entropy and \( \mathbf{q} \) the heat flux vector.

In the following, we neglect radiant heating, i.e., \( \mathbf{r} = 0 \), and assume that the body forces have a potential, i.e., \( \mathbf{b} = \nabla f \). Moreover, we restrict ourselves to the case of a compressible linearly viscous fluid. Thus, we assume that

\[
\mathbf{T} = -\rho \mathbf{I} + \lambda (\text{tr} \mathbf{D}) \mathbf{I} + 2\mu \mathbf{D}, \\
\mathbf{q} = \kappa \nabla \theta, \tag{2.3}
\]

where \( p \) is the pressure, \( \lambda, \mu \) are the constant viscosities and \( \kappa \) is the constant thermal conductivity.

If \( \mathbf{v}, \theta \) and \( \rho \) are considered as independent variables in (2.1)–(2.3), it is useful to introduce the Helmholtz free energy \( \psi \) through

\[
\psi(\rho, \theta) := e(\rho, \eta) - \theta \eta. \tag{2.4}
\]

In this case, we obtain from (2.3) that

\[
\eta = -\partial_\theta \psi, \quad p = \rho \partial_\rho \psi, \\
\mu \geq 0, \quad 3\lambda + 2\mu \geq 0, \quad \kappa \geq 0, \tag{2.5}
\]

while (2.1) reads

\[
\dot{\rho} + \rho \text{div} \mathbf{v} = 0, \\
\rho \dot{\mathbf{v}} - 2\mu \text{div} \mathbf{D} - \lambda \nabla (\text{tr} \mathbf{D}) + \nabla (\rho^2 \partial_\rho \psi) = \rho \nabla f, \\
-\rho \theta \left( \partial_\rho^2 \psi \dot{\theta} + \partial_\rho^2 \psi \dot{\rho} \right) - \kappa \Delta \theta = 2\mu ||\mathbf{D}||^2 + \lambda ||\text{tr} \mathbf{D}||^2. \tag{2.6}
\]
However, for our purposes it is more convenient to view $v, \theta$ and $p$ as independent variables in (2.1)–(2.3). To this end, we introduce the Gibbs free energy $\phi$ through

$$
\phi(p, \theta) := \psi(\rho, \theta) + p \rho^{-1}.
$$

(2.7)

In this situation, we conclude from (2.3) that

$$
\eta = -\partial_\theta \phi, \quad \rho^{-1} = \partial_p \phi, \\
\mu \geq 0, \quad 3\lambda + 2\mu \geq 0, \quad \kappa \geq 0.
$$

(2.8)

The system (2.1) now reads

$$
\begin{align*}
\frac{1}{\partial_p \phi} \left( (\partial^2_{\rho\phi}) \dot{\theta} + (\partial^2_p \phi) \dot{p} \right) &= \text{div } v, \\
\frac{1}{\partial_p \phi} \dot{\psi} - 2\mu \text{ div } D - \lambda \nabla (\text{tr } D) + \nabla p &= (\partial_p \phi)^{-1} \nabla f, \\
-\frac{\theta}{\partial_p \phi} \left( (\partial^2_{\rho\phi}) \dot{\theta} + (\partial^2_p \phi) \dot{p} \right) - \kappa \frac{\Delta \theta}{\Delta} &= 2\mu |D|^2 + \lambda |\text{tr } D|^2.
\end{align*}
$$

(2.9)

Equation (2.9)$_1$ nicely reflects the fact that changes in volume are induced by changes in temperature and changes in pressure. In fact, if we introduce the thermal expansion coefficient $\alpha$ and the isothermal compressibility coefficient $\beta$ through

$$
\alpha(p, \theta) := (\partial_p \phi)^{-1} \partial^2_{\rho\phi}, \\
\beta(p, \theta) := -\left( \frac{\partial_p \phi}{\partial_p \phi} \right)^{-1} \partial^2_p \phi,
$$

(2.10)

we can rewrite (2.9)$_1$ as

$$
\alpha \dot{\theta} - \beta \dot{p} = \text{div } v.
$$

(2.11)

It is well known that for many fluids, the thermal expansion coefficient $\alpha$ is small ($\alpha \approx (10^{-4} - 10^{-3}) \text{K}^{-1}$) and the isothermal compressibility coefficient $\beta$ is even smaller ($\beta \approx (10^{-11} - 10^{-10}) \text{Pa}^{-1}$). Finally, it is convenient to introduce the specific heat coefficient at constant pressure through

$$
c_p(p, \theta) := -\theta \partial^2_\theta \phi.
$$

(2.12)

With this notation, we can rewrite (2.9)$_3$ as

$$
c_p (\partial_p \phi)^{-1} \dot{\theta} - \alpha \theta \dot{p} - \kappa \Delta \theta = 2\mu |D|^2 + \lambda |\text{tr } D|^2.
$$

(2.13)

Remark 2.14 Note that in [12], special fluids that can sustain isochoric motions in isothermal processes have been considered. This formally corresponds to neglecting $\beta$ in (2.11). For such fluids, the Oberbeck–Boussinesq approximation has been formally derived in [12] with the help of a power series expansion. In [8,9] a rigorous mathematical justification of a simplified model has been carried out. However, the mathematical justification starting with the full system from [12] is still lacking. One of the difficulties is the lack of appropriate a priori estimates, which is related to the fact that $\beta$ has been neglected in (2.11).

In [1,2], thermal expansion models have been considered as a constitutive limit for free energies. In these papers, the predictions of the compressible theory have been compared to the prediction of different limiting theories. However, these limiting theories are from the mathematical point of view singular limits. Thus, the way how the limit is achieved is important and different ways can result in different limiting systems.

In the present paper, we combine ideas from [8,12] and [1,2] in order to derive the Oberbeck–Boussinesq approximation as a constitutive limit. To this end, we consider the system (2.9) for a family of Gibbs free energies depending on (dimensional) parameters $a, b > 0$, i.e.,

$$
\phi(p, \theta) = \phi^{a,b}(p, \theta).
$$
The dependence on the parameters $a, b$ will be suppressed in the notation in most cases. Under certain assumptions, we consider the limit as $a, b$ tend to zero. Before that, we render the system (2.9) non-dimensional by introducing dimensionless variables

$$\tilde{x} := \frac{x}{L}, \quad \tilde{t} := \frac{t}{T}, \quad \tilde{V} := \frac{V}{V}, \quad \tilde{p} := \frac{p}{\pi}, \quad \tilde{\theta} := \frac{\theta}{\theta_r}, \quad \tilde{f} := \frac{f}{gL},$$

where $L, T, V, \pi$ and $\theta_r, \vartheta$ are typical length, time, velocity, pressure and temperatures, while $g$ is the gravitational constant.\footnote{This form of the non-dimensionalization is motivated by a typical situation when the Oberbeck–Boussinesq approximation is used, namely a fluid layer in a gravitational field with a certain difference between the temperature $\theta^t$ at the top and $\theta^b$ at the bottom. In this case, we would choose $\vartheta = \theta^b - \theta^t$ and $\theta_r = 2^{-1} \vartheta^{-1}(\theta^t + \theta^b)$.}

Moreover, we assume that the mapping $\vartheta \mapsto A$ is invertible and that $\vartheta \mapsto B$ is invertible and that $(a, b) \mapsto (A, B)$ is invertible and that $(a, b) \rightarrow (0, 0)$ implies $(A, B) \rightarrow (0, 0)$. Finally, set

$$\tilde{\phi}^{A, B}(\tilde{p}, \tilde{\theta}) := \frac{\phi^{\delta(A, B), b(A, B)}(p, \theta)}{\phi_0^{a(A, B), b(A, B)}}.$$

where $\alpha_0$ and $\beta_0$ are typical values for the thermal expansion coefficient $\alpha$ and the isothermal compressibility coefficient $\beta$. The non-dimensional version of (2.11) then reads

$$\alpha_0 \vartheta \tilde{\alpha} \tilde{\vartheta} - \beta_0 \pi \tilde{\beta} \tilde{\vartheta} = \text{div} \tilde{\nu}, \quad (2.15)$$

where the superposed dot now stands for the non-dimensional material time derivative. In typical applications, we see that

$$\alpha_0 \vartheta \approx 10^{-3} - 10^{-2}, \quad \beta_0 \pi \approx 10^{-6} - 10^{-5}. \quad (2.16)$$

Since $\alpha_0 \vartheta$ and $\beta_0 \pi$ are non-dimensional numbers (depending on the parameters $a, b$), we roughly want that the non-dimensional analog $A$ of $a$ behaves as $\alpha_0 \vartheta$ and the non-dimensional analog $B$ of $b$ behaves as $\beta_0 \pi$. This is made precise in the following way: we set

$$h^{a, b}(\tilde{p}, \tilde{\theta}) := \frac{\phi^{\delta(a, b)}(p, \theta)}{\phi_0^{a, b}}, \quad (2.17)$$

and define the non-dimensional parameters $A, B$ through

$$A = A(a, b) := (\partial_{\tilde{p}} h^{a, b}(1, 1))^{-1} \partial_{\tilde{\theta}}^2 h^{a, b}(1, 1),$$

$$B = B(a, b) := - (\partial_{\tilde{p}} h^{a, b}(1, 1))^{-1} \partial_{\tilde{\theta}}^2 h^{a, b}(1, 1). \quad (2.18)$$

Moreover, we assume that the mapping $(a, b) \mapsto (A, B)$ is invertible and that $(a, b) \rightarrow (0, 0)$ implies $(A, B) \rightarrow (0, 0)$. Finally, set

$$\phi^{A, B}(\tilde{p}, \tilde{\theta}) := \frac{\phi^{\delta(A, B), b(A, B)}(p, \theta)}{\phi_0^{a(A, B), b(A, B)}}.$$

\footnote{This form of the non-dimensionalization is motivated by a typical situation when the Oberbeck–Boussinesq approximation is used, namely a fluid layer in a gravitational field with a certain difference between the temperature $\theta^t$ at the top and $\theta^b$ at the bottom. In this case, we would choose $\vartheta = \theta^b - \theta^t$ and $\theta_r = 2^{-1} \vartheta^{-1}(\theta^t + \theta^b)$.}
The system (2.9) for the non-dimensional quantities and differential operators then becomes (we skip all bars for convenience):

\[
(\partial_p \phi^{A,B})^{-1} \left( (\partial^2_{p p} \phi^{A,B}) \dot{\theta} + (\partial^2_{p p} \phi^{A,B}) \dot{p} \right) = \text{div} \, \mathbf{v},
\]

\[
(\partial_p \phi^{A,B})^{-1} \mathbf{v} - \frac{\phi_0}{\pi L} \left( 2 \mu \text{div} \, \mathbf{D} + \lambda \nabla (\text{tr} \, \mathbf{D}) \right) + \frac{\phi_0}{\gamma L^2} \nabla p
\]

\[
= \frac{\gamma L}{\lambda} (\partial_p \phi^{A,B})^{-1} \nabla f,
\]

\[
- \left( \theta + \theta_i \right) (\partial_p \phi^{A,B})^{-1} \left( (\partial^2_{p p} \phi^{A,B}) \dot{\theta} + (\partial^2_{p p} \phi^{A,B}) \dot{p} \right) - \frac{\kappa}{\gamma L} \frac{\partial}{\partial \Delta \theta}
\]

\[
= \frac{V}{\lambda} \left( 2 \mu |\mathbf{D}|^2 + \lambda |\text{tr} \, \mathbf{D}|^2 \right). \tag{2.20}
\]

where we used the notation \( \phi_0 := \phi_0^{a(A,B),b(A,B)} \). Of course, for the quantities \( V, T, L, \vartheta \) and \( \pi \), the same convention is used, e.g., \( V = V^{a(A,B),b(A,B)} \).

A priori there is no obvious representative velocity \( V \) in natural convection processes. As already observed in [4], such processes are reflected by the assumption \( V^2 \approx g L \alpha_0 \vartheta \). This is translated in our situation by

\[
V^2 := A g L. \tag{2.21}
\]

From (2.20)_2, it follows that non-trivial body forces are only possible if \( \phi_0 \approx g L \). Consequently, we set

\[
\gamma := \frac{\phi_0}{g L}, \tag{2.22}
\]

and require that \( \gamma = O(1) \) as \( A, B \) tend to zero. We also introduce the Reynolds numbers \( Re_\mu \) and \( Re_\lambda \) as well as the Prandtl number \( Pr \) by setting

\[
Re_\mu := \frac{\pi V L}{2 \mu \phi_0}, \quad Re_\lambda := \frac{\pi V L}{\lambda \phi_0}, \quad Pr := \frac{\phi_0 \mu}{\vartheta \kappa}, \tag{2.23}
\]

and require that \( Re_\mu = O(1), Re_\lambda = O(1), Pr = O(1) \) as \( A, B \) tend to zero. Note that all these requirements can be fulfilled simultaneously, e.g., if

\[
V \approx A^{\frac{1}{2}}, \quad L \approx A^{\frac{1}{3}}, \quad \phi_0 \approx A^{\frac{1}{4}}, \quad \pi \approx A^{\frac{1}{2}}, \quad \vartheta \approx A^{\frac{1}{3}}. \tag{2.24}
\]

then all above requirements are satisfied.

Using the above notation, we can rewrite (2.20) as

\[
(\partial_p \phi^{A,B})^{-1} \left( (\partial^2_{p p} \phi^{A,B}) \dot{\theta} + (\partial^2_{p p} \phi^{A,B}) \dot{p} \right) = \text{div} \, \mathbf{v},
\]

\[
(\partial_p \phi^{A,B})^{-1} \mathbf{v} - \frac{1}{Re_\mu} \text{div} \, \mathbf{D} - \frac{1}{Re_\lambda} \nabla (\text{tr} \, \mathbf{D}) + \frac{\gamma}{A} \nabla p
\]

\[
= \frac{1}{A} (\partial_p \phi^{A,B})^{-1} \nabla f,
\]

\[
- \left( \theta + \theta_i \right) (\partial_p \phi^{A,B})^{-1} \left( (\partial^2_{p p} \phi^{A,B}) \dot{\theta} + (\partial^2_{p p} \phi^{A,B}) \dot{p} \right) - \frac{1}{Pr Re_\mu} \Delta \theta
\]

\[
= A \left( \frac{1}{\gamma Re_\mu} |\mathbf{D}|^2 - \frac{1}{\gamma Re_\lambda} |\text{tr} \, \mathbf{D}|^2 \right). \tag{2.25}
\]

Concerning the behavior with respect to \( A, B \) of the remaining quantities “thermal expansion coefficient,” “isothermal compressibility coefficient,” “specific heat at constant pressure” and “density” in (2.25), we make
the following assumptions: There exist constants \( c_0 > 0, k_i^{A,B}, i = 1, 2 \), and functions \( \alpha_i^{A,B}(p, \theta), \beta_i^{A,B}(p, \theta), c_i^{A,B}(p, \theta), \rho_i^{A,B}(p, \theta), \) such that

\[
\begin{align*}
\frac{\partial^2 \phi^{A,B}(p, \theta)}{\partial \rho \partial \phi^{A,B}(p, \theta)} &= A(1 + A \alpha_1^{A,B}(p, \theta) + B \alpha_2^{A,B}(p, \theta)), \\
\frac{\partial^2 \phi^{A,B}(p, \theta)}{\partial \phi \partial \phi^{A,B}(p, \theta)} &= -B(1 + A \beta_1^{A,B}(p, \theta) + B \beta_2^{A,B}(p, \theta)) , \\
\frac{1}{\partial \rho \phi^{A,B}(p, \theta)} &= (1 - A(\theta + \theta_r) + Bp + \rho_i^{A,B}(p, \theta))k_1^{A,B}, \\
-(\theta + \theta_r)\frac{\partial^2 \phi^{A,B}(p, \theta)}{\partial \phi^2} &= (c_0 + A^2 c_1^{A,B}(p, \theta) + B^2 c_2^{A,B}(p, \theta))k_2^{A,B},
\end{align*}
\]

where we require that

\[
\lim_{(A,B) \to (0,0)} k_i^{A,B} = 1,
\]

and locally uniformly in \( p, \theta \)

\[
\begin{align*}
\alpha_i^{A,B}(p, \theta) &= O(1), & \beta_i^{A,B}(p, \theta) &= O(1), & c_i^{A,B}(p, \theta) &= O(1), \\
\rho_i^{A,B}(p, \theta) &= O(A^2) + O(B^2) + O(AB)
\end{align*}
\]

with \( i = 1, 2 \), as \( A, B \) tend to zero.

Let us illustrate the above procedure and assumptions by the following example:

**Example 2.29** We assume that

\[
\rho^{a,b}(p, \theta) = (\partial \rho \phi^{a,b}(p, \theta))^{-1} = \rho_0(1 + b p - \theta),
\]

i.e., the density depends linearly on the pressure and the temperature. This constitutive equation is in dimensional form, and the constants \( \rho_0, a, b \) are assumed to be positive. A possible Gibbs free energy compatible with such a behavior is given by

\[
\phi^{a,b}(p, \theta) = b^{-1}\rho_0^{-1} \left( \ln(1 + b p - \theta) - \ln(1 - \theta) \right) - c_0 \theta (\ln \theta - 1),
\]

where \( c_0 > 0 \). From the definition in (2.18), we easily compute

\[
\begin{align*}
A &= A(a, b) = a \vartheta \left( 1 + b \pi - a \vartheta (1 + \theta_r) \right)^{-1}, \\
B &= B(a, b) = b \pi \left( 1 + b \pi - a \vartheta (1 + \theta_r) \right)^{-1}.
\end{align*}
\]

We set

\[
\vartheta := \vartheta_0(a \vartheta_0)^{-1}, \quad \pi := \pi_0(a \vartheta_0)^{-1}, \quad \phi_0^{a,b} := \phi^{a,b}(\pi, \vartheta \frac{b \pi_0}{a \vartheta_0}),
\]

where \( \vartheta_0 \) and \( \pi_0 \) are positive constants independent of \( a, b \). Moreover, we require that

\[
\lim_{(a,b) \to (0,0)} \frac{b \pi_0}{(a \vartheta_0)^{5/4}} = 0.
\]

Note that in typical situations, this requirement is fulfilled [cf. (2.16)]. Moreover, this requirement ensures that \( (A, B) \to (0,0) \) if \( (a, b) \to (0,0) \). The system of equations (2.30) for \( a, b \) is solvable and we compute

\[
\begin{align*}
a &= \frac{A}{\vartheta_0} \left( \frac{\vartheta}{1 + A(1 + \theta_r) - B} \right)^{4/3} = \frac{1}{\vartheta_0} \left( \frac{A}{\chi_{A,B}} \right)^{4/3}, \\
b &= \frac{B A^{1/3}}{\chi_{A,B}^{4/3}}.
\end{align*}
\]
Moreover, straightforward manipulations show (with skipped bars)

\[
\frac{\partial^2 \phi^{A,B}(p, \theta)}{\partial p \partial \phi^{A,B}(p, \theta)} = A \left( 1 - \frac{B (p - 1) - A (\theta - 1)}{1 + B (p - 1) - A (\theta - 1)} \right),
\]

\[
\frac{\partial^2 \phi^{A,B}(p, \theta)}{\partial \theta \partial \phi^{A,B}(p, \theta)} = -B \left( 1 - \frac{B (p - 1) - A (\theta - 1)}{1 + B (p - 1) - A (\theta - 1)} \right),
\]

\[
\frac{1}{\partial p \partial \phi^{A,B}(p, \theta)} = \left( 1 - A (\theta + \theta_r) + B p + \rho_r^{A,B}(p, \theta) \right) k_{1}^{A,B},
\]

\[
-(\theta + \theta_r) \partial^2 \phi^{A,B}(p, \theta) = \left( \frac{c_0 \theta_0 \rho_0}{\pi_0} + \frac{A^2 c_1^{A,B}(p, \theta)}{k_2^{A,B}} \right) k_{2}^{A,B},
\]

where

\[
\rho_r^{A,B}(p, \theta) = -AB \left( (p - 1) \frac{1 + \theta_r}{\chi_{A,B}} + \frac{\theta - 1}{\chi_{A,B}} \right) + A^2 (\theta - 1) \frac{1 + \theta_r}{\chi_{A,B}} + B^2 p - 1 \frac{1}{\chi_{A,B}},
\]

\[
k_{1}^{A,B} = \frac{\chi_{A,B}}{B} \left( \ln \left( 1 + \frac{B}{\chi_{A,B}} - \frac{B}{\chi_{A,B}} \right) - \ln \left( 1 - \frac{B}{\chi_{A,B}} \right) \right)
\]

\[-A^{1/3} \frac{c_0 \rho_0 \theta_0}{\pi_0} \frac{1}{\chi_{A,B}} \left( \ln \left( \frac{\theta_0^{1/3} B}{A^{4/3}} \right) - 1 \right),
\]

\[
c_1^{A,B}(p, \theta) = \frac{-x_{A,B} p (\theta + \theta_r) \left( 2 + B (p - 2) - 2A (\theta - 1) \right)}{\left( 1 + B (p - 1) - A (\theta - 1) \right)^2 \left( 1 - B - A (\theta - 1) \right)^2},
\]

\[
k_{2}^{A,B} = \left( k_{1}^{A,B} \right)^{-1}.
\]

Note that due to (2.32), we have \( B A^{-4/3} \to 0 \) if \( A, B \) tend to zero. Thus, our example fulfills the requirements (2.26)–(2.28) as well as the requirements for (2.22) and (2.23).

2.2 Constitutive limit

Now we want to show that we are able to obtain the Oberbeck–Boussinesq approximation as a constitutive limit of (2.25) as \( A, B \) tend to zero. We assume that the requirements (2.26)–(2.28) as well as the requirements for (2.22) and (2.23) are fulfilled. We consider solutions \( \mathbf{v} = \mathbf{v}^{A,B}, \theta = \theta^{A,B} \) and \( p = p^{A,B} \) of (2.25) in the following sense:

Let \( \Omega \subset \mathbb{R}^3 \) be a given sufficiently smooth bounded domain and \( I = (0, T), T > 0 \), be a given time interval. We set \( Q = I \times \Omega \) and assume that \( \nabla f \in L^2(Q) \) and appropriate boundary and initial conditions for \( \mathbf{v} \) and \( \theta \) are given. The system (2.25) is satisfied in the sense of distributions over \( Q \) and \( \mathbf{v}, \theta \) are uniformly bounded in \( W^{1,2}(Q) \cap L^\infty(Q) \) with respect to \( A, B \). Finally, we assume that \( B = o(A) \).

From our assumptions, it follows that there exists \( \mathbf{v}, \theta, p \in W^{1,2}(Q) \) such that

\[
\mathbf{v}^{A,B} \to \mathbf{v} \text{ weakly in } W^{1,2}(Q),
\]

\[
\theta^{A,B} \to \theta \text{ weakly in } W^{1,2}(Q),
\]

\[
p^{A,B} \to p \text{ weakly in } W^{1,2}(Q)
\]

and for all \( q \in [1, \infty) \)

\[
\mathbf{v}^{A,B} \to \mathbf{v} \text{ strongly in } L^q(Q),
\]

\[
\theta^{A,B} \to \theta \text{ strongly in } L^q(Q),
\]

\[
p^{A,B} \to p \text{ strongly in } L^q(Q)
\]
These convergences and the assumptions (2.26)–(2.28) as well as the requirements for (2.22) and (2.23) immediately imply that the limit as \( A, B \) tend to zero in (2.25) yield for all \( \varphi, \psi \in C_0^\infty(Q) \)

\[
\int_Q \operatorname{div} \varphi \cdot \varphi \, dx \, dt = 0,
\]

\[
\int_Q c_0 \partial_t \varphi + c_0 \nabla \varphi \cdot \nabla \psi + \frac{1}{PrRe_\mu} \nabla \varphi \cdot \nabla \psi \, dx \, dt = 0.
\]

(2.35)

To treat (2.25)\(_2\), we add \( k_i^{A,B} \nabla f \) on both sides and obtain for all \( \varphi \in C_0^\infty(Q) \)

\[
\int_Q k_i^{A,B} (1 - A (\theta^{A,B} + \theta_0) + B p^{A,B} + \rho_p^{A,B} (p^{A,B}, \theta^{A,B}))
\]

\[
\times (\partial_t \varphi^{A,B} + [\nabla \varphi^{A,B}]\varphi^{A,B}) \cdot \varphi \, dx \, dt
\]

\[
+ \int_Q \frac{1}{Re_\mu} \operatorname{Dv}^{A,B} \cdot \varphi + \frac{1}{Re_\mu} \operatorname{div} \varphi^{A,B} \operatorname{div} \varphi \, dx \, dt
\]

\[
- \int_Q \left( \frac{c}{A} p^{A,B} + k_i^{A,B} f (1 - \theta_t) - \frac{k_i^{A,B}}{A} \right) \operatorname{div} \varphi \, dx \, dt
\]

\[
= \int_Q k_i^{A,B} \left( (1 - \theta^{A,B}) + B p^{A,B} + \frac{1}{A} \rho_p^{A,B} (p^{A,B}, \theta^{A,B}) \right) \nabla f \cdot \varphi \, dx \, dt.
\]

From this, we deduce for \( A, B \) tending to zero for all \( \varphi \in C_0^\infty(Q) \) with \( \operatorname{div} \varphi = 0 \)

\[
\int_Q (\partial_t \varphi + [\nabla \varphi] \varphi) \cdot \varphi \, dx \, dt + \frac{1}{Re_\mu} \int_Q \operatorname{Dv} \cdot \varphi \, dx \, dt
\]

\[
= \int_Q (1 - \theta) \nabla f \cdot \varphi \, dx \, dt.
\]

(2.36)

The system (2.35) and system (2.36) are the weak formulation of the celebrated Oberbeck–Boussinesq approximation.

2.3 Formal expansion

Our approach enables us also to recover the results in [12] and [8] for dissipation number \( \mathrm{Di} = 0 \). To this end, we assume that the parameters \( A, B \) are fixed but sufficiently small. Moreover, we assume that \( A \) is smaller than \( B \) in the sense that \( A^2 \leq B \leq A \). This requirement is fulfilled in typical applications. In this situation, we formally expand the non-dimensional quantities \( \vartheta, \beta \) and \( p \) in (2.25) into power series with respect to the perturbation parameters \( A, B \) in the form

\[
\vartheta = \sum_{m,k=0}^\infty A^m B^k \vartheta_{m,k}, \quad \beta = \sum_{m,k=0}^\infty A^m B^k \beta_{m,k}, \quad p = \sum_{m,k=0}^\infty A^m B^k p_{m,k}.
\]

We attach the boundary conditions of the quantities to the zero order term and set the boundary conditions to be zero for the higher-order terms. Similarly, we expand the quantities \( c_0, k_i^{A,B}, i = 1, 2, \) and functions \( \alpha_i^{A,B} (p, \theta), \beta_i^{A,B} (p, \theta), \gamma_i^{A,B} (p, \theta), \rho_i^{A,B} (p, \theta), i = 1, 2 \), where we also insert the series for \( p \) and \( \theta \) in the arguments. We replace all these quantities in the system (2.25) by the corresponding power series. At level (0, 0), we see from (2.25) that\(^2\)

\[
\operatorname{div} \vartheta_{0,0} = 0,
\]

\[
\gamma \nabla p_{0,0} = \nabla f,
\]

\[
c_0 \beta_{0,0} - \frac{1}{PrRe_\mu} \Delta \theta_{0,0} = 0.
\]

\(^2\) To simplify the notation, we use the superposed dot only at the level (0, 0).
At level (1, 0), we see from (2.25) that
\[
\dot{v}_{0,0} - \frac{1}{Re_\mu} \text{div} Dv_{0,0} + \gamma \nabla p_{1,0} = -(\theta_{0,0} - \theta_r) \nabla f + k_{1,1,0} \nabla f.
\]
Thus, setting \( v := v_{0,0}, \theta := \theta_{0,0} \) and \( p := \gamma p_{0,0} + A(\gamma p_{1,0} - \theta_r f - k_{1,1,0} f) \), we obtain
\[
\text{div } v = 0, \quad A \left( \dot{\theta} - \frac{1}{Re_\mu} \text{div} Dv \right) + \nabla p = (1 - A \theta) \nabla f, \quad c_0 \dot{\theta} - \frac{1}{Pr Re_\mu} \Delta \theta = 0.
\]

This system is the same as the one obtained in [12] and [8] for \( Di = 0 \). The approximation combines different levels in the temperature and the velocity equation and the velocity has zero divergence.

Acknowledgments Y.K. was partly supported by JSPS KAKENHI Grant Numbers 24340028, 22244009, 24224003, 15K13449. M.R. would like to thank for the financial support and the hospitality during several visits at the Kyushu University.

References

1. Bechtel, S.E., Forest, M.G., Rooney, F.J., Wang, Q.: Thermal expansion models of viscous fluids based on limits of free energy. Phys. Fluids 15, 2681 (2003)
2. Bechtel, S.E., Cai, M., Rooney, F.J., Wang, Q.: Investigation of simplified thermal expansion models for compressible Newtonian fluids applied to nonisothermal plane Couette and Poiseuille flows. Phys. Fluids 16, 3955 (2004)
3. Boussinesq, J.: Théorie Analytique de la Chaleur. Gauthier-Villars, Paris (1903)
4. Chandrasekhar, S.: The International Series of Monographs on Physics. Clarendon Press, Oxford (1961)
5. Feireisl, E., Novotný, A.: The Oberbeck–Boussinesq approximation as a singular limit of the full Navier–Stokes–Fourier system. J. Math. Fluid Mech. 11, 274 (2009)
6. Feireisl, E., Novotný, A.: Advances in Mathematical Fluid Mechanics. Birkhäuser Verlag, Basel (2009)
7. Hills, R.N., Roberts, P.H.: On the motion of a fluid that is incompressible in a generalized sense and its relationship to the Boussinesq approximation. Stab. Appl. Anal. Contin. Media 1, 205 (1991)
8. Kagei, Y., Růžička, M., Thäter, G.: Natural convection with dissipative heating. Commun. Math. Phys. 214, 287 (2000)
9. Kagei, Y., Růžička, M., Thäter, G.: A limit problem in natural convection. NoDEA 13, 447 (2006)
10. Milhaljan, J.M.: A rigorous exposition of the Boussinesq approximations applicable to a thin layer of fluid. Astrophys. J. 136, 1126 (1962)
11. Oberbeck, A.: Über die Wärmeleitung der Flüssigkeiten bei der Berücksichtigung der Strömungen infolge von Temperaturdifferenzen. Annalen der Physik und Chemie. 7, 271 (1879) Über die Bewegungsserscheinungen der Atmosphäre. Sitz. Ber. K. Preuss. Akad. Miss. 383 and 1120 (1888)
12. Rajagopal, K.R., Růžička, M., Srinivasa, A.R.: On the Oberbeck–Boussinesq approximation. Math. Models Methods Appl. Sci. 6, 1157–1167 (1996)
13. Spiegel, E.A., Veronis, G.: On the Boussinesq approximation for a compressible fluid. Astrophys. J. 131, 442 (1960)