Dissipative Properties of Systems Composed of High-Loss and Lossless Components

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December 21, 2013

Abstract

We study here dissipative properties of systems composed of two components one of which is highly lossy and the other is lossless. A principal result of our studies is that all the eigenmodes of such a system split into two distinct classes characterized as high-loss and low-loss. Interestingly, this splitting is more pronounced the higher the loss of the lossy component. In addition, the real frequencies of the high-loss eigenmodes can become very small and even can vanish entirely, which is the case of overdamping.

1 Introduction

We introduce a general framework to study dissipative properties of two component systems composed of a high-loss and lossless components. This framework covers conceptually any dissipative physical system governed by a linear evolution equation. Such systems include in particular damped mechanical systems, electric networks or any general Lagrangian system with losses accounted by the Rayleigh dissipative function. [Pars, Sec. 10.11, 10.12], [Gant, Sec. 8, 9, 46].

An important motivation and guiding examples for our studies come from two-component dielectric media composed of a high-loss and lossless components. Any dielectric medium always absorbs a certain amount of electromagnetic energy, a phenomenon which is often referred to as loss. When it comes to the design of devices utilizing dielectric properties very often a component which carries a useful property, for instance, magnetism is also lossy. So the question stands: is it possible to design a composite material/system which would have a desired property comparable with a naturally occurring bulk substance but with significantly reduced losses. In a search of such a low-loss composite it is appropriate to assume that the lossy component, for instance magnetic, constitutes the significant fraction which carries the desired property. But then it is far from clear whether a significant loss reduction is achievable at all. It is quite remarkable that the answer to the above question is affirmative, and an example of a simple layered structure having magnetic properties comparable with a natural bulk material but with 100 times lesser losses in wide frequency range is constructed in [FigVit8]. The primary goal of this paper is to find out and explain when and how a two...
component system involving a high-loss component can have low loss for a wide frequency range. The next question of how this low loss performance for wide frequency range is combined with a useful property associated with the lossy component is left for another forthcoming paper.

A principal result of our studies here is that a two component system involving a high-loss component can be significantly low loss in a wide frequency range provided, to some surprise, that the lossy component is sufficiently lossy. An explanation for this phenomenon is that if the lossy part of the system has losses exceeding a critical value it goes into essentially an overdamping regime, that is a regime with no oscillatory motion. In fact, we think that for Lagrangian systems there will be exactly an overdamping regime but these studies will also be conducted in already mentioned forthcoming paper.

The rest of the paper is organized as follows. The model setup and discussion of main results and their physical significance is provided in Section 2. In the following Section 3 we apply the developed general approach to an electric circuit showing all key features of the method. Sections 4 and 5 is devoted to a precise formulation of all significant results in the form of Theorems, Propositions and so on. Finally, in the last Section 6 we provide the proofs of these results.

2 Model setup and discussion of main results

A framework to our studies of energy dissipation is as in \[\text{FigSch1}\] and \[\text{FigSch2}\]. This framework covers many physical systems including dielectric, elastic and acoustic media. Our primary subject is a linear system (medium) whose state is described by a time dependent \textit{generalized velocity} \(v(t)\) taking values in a Hilbert space \(H\) with scalar product \((\cdot, \cdot)\). The evolution of \(v\) is governed by a linear equation incorporating \textit{retarded friction}

\[
m\partial_tv(t) = -iAv(t) - \int_0^\infty a(\tau)v(t-\tau)\,d\tau + f(t),
\]

where \(m > 0\) is a positive mass operator in \(H\), \(A\) is a self-adjoint operator in \(H\), \(f(t)\) is a time dependent external \textit{generalized force}, and \(a(t), t \geq 0\), is an operator valued function which we call the \textit{operator valued friction retardation function} [KubTod2 Section 1.6], or just the \textit{friction function}. The names “generalized velocity” and “generalized force” are justified when we interpret the real part of the scalar product \(\text{Re}\{(v(t), f(t))\}\) as the work \(W\) done by \(f(t)\) per unit of time at instant \(t\), that is

\[
W = \int_{-\infty}^{\infty} \text{Re}\{(v(t), f(t))\} \, dt.
\]

The \textit{internal energy of the system} is naturally defined as

\[
\text{internal energy } U = \frac{1}{2} (v(t), mv(t)),
\]

and it readily follows from (1) that it satisfies the following energy balance equation

\[
\frac{dU}{dt} = \text{Re}\{(v(t), f(t))\} - \text{Re}\left\{(v(t), \int_0^\infty a(\tau)v(t-\tau)\,d\tau)\right\}.
\]
The second term of the right hand side of (4) is interpreted as the instantaneous rate of “work done by the system on itself,” or more properly the negative rate of energy dissipation due to friction.

If we rescale the variables according to the formulas

\[ \tilde{v} = \sqrt{m}v, \quad \Omega = \frac{1}{\sqrt{m}} A \frac{1}{\sqrt{m}}, \quad \tilde{a} = \frac{1}{\sqrt{m}} a \frac{1}{\sqrt{m}}, \quad \tilde{f} = \frac{1}{\sqrt{m}} f, \]

then the equation (4) reduces to the special form in which \( m \) in the new variables is the identity operator, i.e.

\[ \partial_t \tilde{v} (t) = -i\Omega \tilde{v} (t) - \int_0^\infty \tilde{a} (\tau) \tilde{v} (t - \tau) d\tau + \tilde{f} (t), \]

and in view of (3) the internal energy \( U \) associated with the state \( \tilde{v} \) turns into the scalar product, that is

\[ \text{internal energy } U = \frac{1}{2} (\tilde{v}(t), \tilde{v}(t)). \]

The system evolution equation of the special form (6) has two important characteristic properties: (i) the operator \( \Omega = \frac{1}{\sqrt{m}} A \frac{1}{\sqrt{m}} \) can be interpreted as the system frequency operator in \( H \); (ii) the system internal energy is simply the scalar product (7). We refer to this important form (6) as the system canonical evolution equation.

For the sake of simplicity we assume the friction function to be “instantaneous,” that is \( a (t) = \beta B \delta (t) \) with \( B \) self-adjoint and \( \beta \geq 0 \) is a dimensionless loss parameter which scales the intensity of dissipation. Of course, such an idealized friction function is a simplification that we take to avoid significant difficulties associated with more realistic friction functions as in [FigSch1], [FigSch2].

Now, assuming that the rescaling (5) was applied, the canonical evolution equation (6) takes the form

\[ \partial_t v (t) = -iA (\beta) v (t) + f (t), \quad \text{where } A (\beta) = \Omega - i\beta B, \ \beta \geq 0. \]

Importantly, for the system operator \( \Omega - i\beta B \) it is assumed the operator \( B \) satisfies the power dissipation condition

\[ B \geq 0. \]

The general energy balance equation (4) takes now a simpler form

\[ \frac{dU}{dt} = \text{Re} \{ (v (t), f (t)) \} - W_{\text{dis}}. \]

where

\[ U = \text{system energy} = \frac{1}{2} (v(t), v(t)), \]

\[ W_{\text{dis}} = \text{system dissipated power} = \beta (v (t), Bv (t)). \]

Most of the time it is assumed that the system governed by (8) is at rest for all negative times, i.e.

\[ v (t) = 0, \ f (t) = 0, \ t \leq 0. \]
To simplify further technical aspects of our studies and to avoid the nontrivial subtleties involved in considering unbounded operators in infinite-dimensional Hilbert spaces, we assume the Hilbert space $H$ to be of a finite dimension $N$. Keeping in mind our motivations we associate the operator $B$ with the lossy component of a composite and express the significance of the lossy component in terms of the rank $N_B$ of the operator $B$. Continuing this line of thought we introduce a space $H_B$, the range of the operator $B$, and the corresponding orthogonal projection $P_B$ on it, that is

$$H_B = \{Bu : u \in H\}, \quad N_B = \dim H_B.$$  \hspace{1cm} (13)

In what follows we refer to $H_B$ as a *subspace of degrees of freedom susceptible to losses or just loss subspace*. We also refer to the orthogonal complement of $H_B^\perp = H \ominus H_B$ as the *subspace of lossless degrees of freedom or no-loss subspace*. The number $N_B$ plays an important role in the Livsic theory of open systems where $N_B$ is called “the index of non-Hermiticity” of the operator $\Omega - i\beta B$, \cite[Liv, pp. 24, 27-28]{Liv}. The definition of $P_B$ readily implies the following identity

$$B = P_B BP_B.$$  \hspace{1cm} (14)

We suppose the dimension $N_B$ to satisfy the following *loss fraction condition*

$$0 < \delta_B = \frac{N_B}{N} < 1,$$  \hspace{1cm} (15)

which signifies in a rough form that only a fraction $\delta_B < 1$ of the degrees of freedom are susceptible to lossy behavior. As we will see later when the loss parameter $\beta \gg 1$, only a fraction of the system eigenmodes are associated with high losses and this fraction is exactly $\delta_B$. For this reason we may refer to $\delta_B$ as the *fraction of high-loss eigenmodes*.

It turns out that the system dissipative behavior is qualitatively different when the loss parameter $\beta$ is small or large. It seems that common intuition about losses is associated with the small values of $\beta$. The spectral analysis of the system operator $\Omega - i\beta B$ for small $\beta$ can be handled by the standard perturbation theory, \cite[Bau85]{Bau}. The results of our analysis are contained in Theorem 15 and may be summarized as follows: Let $\omega_j$, $1 \leq j \leq N$ denote the all eigenvalues of the operator $\Omega$ repeated according to their multiplicities then there exists a corresponding orthonormal basis of eigenvectors $u_j$, $1 \leq j \leq N$ such that if $0 \leq \beta \ll 1$ then the operator $\Omega - i\beta B$ is diagonalizable with a complete set of $\zeta_j(\beta)$ eigenvalues and eigenvectors $v_j(\beta)$ having the expansions

$$\zeta_j(\beta) = \omega_j - i\beta (u_j, Bu_j) + O(\beta), \quad \omega_j = (u_j, \Omega u_j), \quad v_j(\beta) = u_j + O(\beta), \quad 1 \leq j \leq N, \beta \ll 1.$$  \hspace{1cm} (16)

The effect of small losses described by the above formula is well known, of course, see, for instance, \cite[Gant, Sec. 46]{Gant}.

The perturbation analysis of the system operator $\Omega - i\beta B$ for large values of the loss parameter $\beta \gg 1$ requires more efforts and its results are quite surprising. It shows, in particular, that all the eigenmodes split into two distinct classes according to their dissipative behavior: high-loss and low-loss modes. We refer to such a splitting as modal dichotomy.

In view of the above discussion we decompose the Hilbert space $H$ into the direct sum of invariant subspaces of the operator $B \geq 0$, that is,

$$H = H_B \oplus H_B^\perp,$$  \hspace{1cm} (17)
where $H_B = \text{ran} B$ is the loss subspace of dimension $N_B$ with orthogonal projection $P_B$ and its orthogonal complement, $H^\perp_B = \ker B$, is the no-loss subspace of dimension $N - N_B$ with orthogonal projection $P^\perp_B$. Then the operators $\Omega$ and $B$, with respect to this direct sum, are $2 \times 2$ block operator matrices

$$
\Omega = \begin{bmatrix} \Omega_2 & \Theta \\ \Theta^* & \Omega_1 \end{bmatrix}, \quad B = \begin{bmatrix} B_2 & 0 \\ 0 & 0 \end{bmatrix},
$$

where $\Omega_2 := P_B \Omega P_B|_{H_B} : H_B \to H_B$ and $B_2 := P_B B P_B|_{H_B} : H_B \to H_B$ are restrictions of the operators $\Omega$ and $B$ respectively to loss subspace $H_B$ whereas $\Omega_1 := P^\perp_B \Omega P^\perp_B|_{H^\perp_B} : H^\perp_B \to H^\perp_B$ is the restriction of $\Omega$ to complementary subspace $H^\perp_B$. Also, $\Theta : H^\perp_B \to H_B$ is the operator $\Theta := P_B \Omega P_B^\perp|_{H^\perp_B}$, whose adjoint is given by $\Theta^* = P^\perp_B \Omega P_B|_{H_B} : H_B \to H^\perp_B$. The block representation (19) plays an important role in our analysis involving the perturbation theory as well as the \textit{Schur complement} concept described in Appendix 7.

### 2.1 Modal dichotomy for the high-loss regime

Notice first that in view of (8) the operator $(-i\beta)^{-1} A(\beta) = B + i\beta^{-1}\Omega$ is analytic in $\beta^{-1}$ in a vicinity of $\beta = \infty$. Let then $\zeta(\beta)$ be an analytic in $\beta^{-1}$ eigenvalue of $A(\beta)$ in the same vicinity, with the possible exception of a pole at $\beta = \infty$. Notice that if use the substitution $\varepsilon = (-i\beta)^{-1}$ the operator $\varepsilon A(i\varepsilon^{-1}) = B + \varepsilon\Omega$ is a self-adjoint for real $\varepsilon$ and consequently the eigenvalue $\lambda(\varepsilon) = \varepsilon\zeta(i\varepsilon^{-1})$ of the operator $B + \varepsilon\Omega$ must be an analytic function of $\varepsilon$ and real-valued for real $\varepsilon$. Hence it satisfies the identity $\lambda(\overline{\varepsilon}) = \lambda(\varepsilon)$ where $\overline{\varepsilon}$ is the complex conjugate to $\varepsilon$. The later in view of the identity $\zeta(\beta) = (-i\beta) \lambda((-i\beta)^{-1})$ readily implies the following identities for the eigenvalue $\zeta(\beta)$ for real $\beta$ in a vicinity of $\beta = \infty$

$$
\overline{\zeta(\beta)} = \zeta(-\beta), \text{ or } \text{Re} \zeta(-\beta) = \text{Re} \zeta(\beta), \quad \text{Im} \zeta(-\beta) = -\text{Im} \zeta(\beta).
$$

Consequently, $\text{Re} \zeta(\beta)$ and $\text{Im} \zeta(\beta)$ are respectively an even and an odd function for real $\beta$ in a vicinity of $\beta = \infty$ implying that their Laurent series in $\beta^{-1}$ have respectively only even and odd powers.

The perturbation analysis for $\beta \gg 1$ of the operator $A(\beta) = \Omega - i\beta B$ described in Section 4.1 introduces an orthonormal basis $\{\hat{w}_j\}_{j=1}^N$ diagonalizing the operators $\Omega_1$ and $B_2$ from the block form (19), that is

$$
B_2 \hat{w}_j = \zeta_j \hat{w}_j \quad \text{for } 1 \leq j \leq N_B; \quad \Omega_1 \hat{w}_j = \rho_j \hat{w}_j \quad \text{for } N_B + 1 \leq j \leq N,
$$

where

$$
\begin{align*}
\zeta_j &= \langle \hat{w}_j, B_2 \hat{w}_j \rangle = \langle \hat{w}_j, B \hat{w}_j \rangle \quad \text{for } 1 \leq j \leq N_B; \\
\rho_j &= \langle \hat{w}_j, \Omega_1 \hat{w}_j \rangle = \langle \hat{w}_j, \Omega \hat{w}_j \rangle \quad \text{for } N_B + 1 \leq j \leq N.
\end{align*}
$$

The summary of the perturbation analysis for the high-loss regime $\beta \gg 1$, as described in Theorem 3, is as follows. The system operator $A(\beta)$ is diagonalizable and there exists a complete set of eigenvalues $\zeta_j(\beta)$ and eigenvectors $w_j(\beta)$ satisfying

$$
A(\beta) w_j(\beta) = \zeta_j(\beta) w_j(\beta), \quad 1 \leq j \leq N, \quad \beta \gg 1
$$

(22)
which split into two distinct classes

\[ \text{high-loss: } \zeta_j(\beta), \ w_j(\beta), \ 1 \leq j \leq N_B; \]
\[ \text{low-loss: } \zeta_j(\beta), \ w_j(\beta), \ N_B + 1 \leq j \leq N, \]

with the following properties.

**In the high-loss case** the eigenvalues have poles at \( \beta = \infty \) whereas their eigenvectors are analytic at \( \beta = \infty \), having the asymptotic expansions

\[ \zeta_j(\beta) = -i\kappa_j \beta + \rho_j + O\left(\beta^{-1}\right), \ \zeta_j > 0, \ \rho_j \in \mathbb{R}, \ w_j(\beta) = \hat{w}_j + O\left(\beta^{-1}\right), \ 1 \leq j \leq N_B. \]  

(24)

The vectors \( \hat{w}_j, 1 \leq j \leq N_B \) form an orthonormal basis of the loss subspace \( H_B \) and

\[ B\hat{w}_j = \hat{\zeta}_j \hat{w}_j, \ \rho_j = (\hat{w}_j, \Omega \hat{w}_j), \text{ for } 1 \leq j \leq N_B. \]  

(25)

**In the low-loss case** the eigenvalues and eigenvectors are analytic at \( \beta = \infty \), having the asymptotic expansions

\[ \zeta_j(\beta) = \rho_j - id_j \beta^{-1} + O\left(\beta^{-2}\right), \ \rho_j \in \mathbb{R}, \ d_j \geq 0, \]  

(26)

\[ w_j(\beta) = \hat{w}_j + w_j^{(-1)} \beta^{-1} + O\left(\beta^{-2}\right), \ N_B + 1 \leq j \leq N. \]

The vectors \( \hat{w}_j, N_B + 1 \leq j \leq N \) form an orthonormal basis of the no-loss subspace \( H_B^\perp \) and

\[ B\hat{w}_j = 0, \ \rho_j = (\hat{w}_j, \Omega \hat{w}_j), \ d_j = \left(w_j^{(-1)}, Bw_j^{(-1)}\right) \text{ for } N_B + 1 \leq j \leq N. \]  

(27)

The expansions (24) and (26) together with (19) readily imply the following asymptotic formulas for the real and imaginary parts of the complex eigenvalues \( \zeta_j(\beta) \) for \( \beta \gg 1 \)

high-loss: \[ \text{Re } \zeta_j(\beta) = \rho_j + O\left(\beta^{-2}\right), \ \text{Im } \zeta_j(\beta) = -\check{\zeta}_j \beta + O\left(\beta^{-1}\right), \ 1 \leq j \leq N_B; \]  

(28)

low-loss: \[ \text{Re } \zeta_j(\beta) = \rho_j + O\left(\beta^{-2}\right), \ \text{Im } \zeta_j(\beta) = -d_j \beta^{-1} + O\left(\beta^{-3}\right), \ N_B + 1 \leq j \leq N. \]  

(29)

Observe that the expansions (28) and (29) readily yield

\[ \lim_{\beta \to \infty} \text{Im } \zeta_j(\beta) = -\infty \text{ for } 1 \leq j \leq N_B; \ \lim_{\beta \to \infty} \text{Im } \zeta_j(\beta) = 0 \text{ for } N_B + 1 \leq j \leq N, \]  

(30)

justifying the names high-loss and low-loss. Notice also that the relations (24)–(27) imply that the high-loss eigenmodes projection on the no-loss subspace \( H_B^\perp \) is of order \( \beta^{-1} \) in contrast to the low-loss eigenmodes for which the projection on the loss subspace \( H_B \) is of order \( \beta^{-1} \). In other words, for \( \beta \gg 1 \) the high-loss eigenmodes are essentially confined to the loss subspace \( H_B \) whereas the low-loss modes are essentially expelled from it.

### 2.2 Losses and the quality factor associated with the eigenmodes

Here we consider the energy dissipation associated with high-loss and low-loss eigenmodes. The power dissipation is commonly quantified by the so called *quality factor* \( Q \) that can naturally be introduced in a few not entirely equivalent ways, [Pain] pp. 47, 70, 71]. The most common way to define the quality factor is based on relative rate of the energy dissipation
per cycle when the system is in a state of damped harmonic oscillations \( v(t) \) with a given frequency \( \omega \), namely,

\[
Q = 2\pi \frac{\text{energy stored in system}}{\text{energy lost per cycle}} = |\omega| \frac{U}{W_{\text{dis}}} = |\omega| \frac{(v(t), v(t))}{2 \beta (v(t), Bv(t))},
\]

(31)

where we used for the system energy \( U \) and the dissipated power \( W_{\text{dis}} \) their expressions \[1\]. Notice also that in the above formula we use the absolute value \(|\omega|\) of the frequency \( \omega \) since in our settings the frequency \( \omega \) can be negative. The state of damped harmonic oscillations \( v(t) \) is defined by an eigenvector \( w \) of the system operator \( A(\beta) = \Omega - i\beta B \) with eigenvalue \( \zeta \), and it evolves as \( v(t) = w e^{-i \beta t} \) with the frequency \( \omega = \Re \zeta \) and the damping factor \(-\Im \zeta\).

Its system energy \( U \), dissipated power \( W_{\text{dis}} \), and quality factor \( Q \) satisfy

\[
U = U[w] e^{2 \Im \zeta t}, \quad W_{\text{dis}} = W_{\text{dis}}[w] e^{2 \Im \zeta t}, \quad Q = Q[w],
\]

(32)

where

\[
U[w] = \frac{1}{2} (w, w), \quad W_{\text{dis}}[w] = -2 \Im \zeta U[w],
\]

(33)

\[
\Re \zeta = \frac{(w, \Omega w)}{(w, w)}, \quad \Im \zeta = -\frac{(w, \beta B w)}{(w, w)},
\]

(34)

\[
Q[w] = -\frac{1}{2} \frac{|\Re \zeta|}{\Im \zeta}.
\]

(35)

For an eigenvector \( w \) we refer to the terms \( U[w] \), \( W_{\text{dis}}[w] \), and \( Q[w] \) as its energy, power of energy dissipation, and quality factor, respectively. Observe, that eigenvectors with the same eigenvalue have equal quality factors. Notice also that an eigenvector \( w \) with eigenvalue \( \zeta \) has power of energy dissipation \( W_{\text{dis}}[w] \) equal to the product \(- (w, w) \Im \zeta \) and quality factor \( Q[w] \) equal the ratio \( \frac{\Re \zeta}{2 \Im \zeta} \).

Consider now the high-loss regime \( \beta \gg 1 \). Let \( \zeta_j(\beta), 1 \leq j \leq N \) denote the high-loss and low-loss eigenvalues of the system operator \( A(\beta) \) which have the expansions \[28\], \[29\]. Then for any eigenvectors \( w_j(\beta), 1 \leq j \leq N \) with these eigenvalues, respectively, which are normalized in the sense

\[
(w_j(\beta), w_j(\beta)) = 1 + O(\beta^{-1}), \quad 1 \leq j \leq N
\]

(36)

as \( \beta \to \infty \), the following asymptotic formulas holds as \( \beta \to \infty \) for the energy and the power of energy dissipation of these modes

\[
U[w_j(\beta)] = \frac{1}{2} + O(\beta^{-1}), \quad 1 \leq j \leq N;
\]

(37)

high-loss: \( W_{\text{dis}}[w_j(\beta)] = \zeta_j \beta + O(1), \quad 1 \leq j \leq N_B; \)

(38)

low-loss: \( W_{\text{dis}}[w_j(\beta)] = d_j \beta^{-1} + O(\beta^{-2}), \quad N_B + 1 \leq j \leq N. \)

(39)

We see clearly now the modal dichotomy, i.e. eigenmode splitting according to their dissipative properties: high-loss modes \( w_j(\beta), 1 \leq j \leq N_B \) and low-loss modes \( w_j(\beta), N_B + 1 \leq j \leq N \). Indeed, these asymptotic formulas \[38\], \[39\] imply

\[
\text{high-loss modes: } \lim_{\beta \to \infty} W_{\text{dis}}[w_j(\beta)] = \infty; \quad \text{low-loss modes: } \lim_{\beta \to \infty} W_{\text{dis}}[w_j(\beta)] = 0.
\]

(40)
The quality factor \( Q[w_j(\beta)] \) for each high-loss eigenmode has a series expansion containing only odd powers of \( \beta^{-1} \) with the asymptotic formula as \( \beta \to \infty \)

\[
Q[w_j(\beta)] = \frac{1}{2} \frac{|\rho_j|}{\xi_j} \beta^{-1} + O(\beta^{-3}), \quad 1 \leq j \leq N_B. \tag{41}
\]

The quality factor \( Q[w_j(\beta)] \) for each low-loss eigenvectors has a series expansion containing only odd powers of \( \beta^{-1} \) as well provided \( \text{Im} \zeta_j(\beta) \neq 0 \) for \( \beta \gg 1 \). Moreover, it satisfies the following asymptotic formula as \( \beta \to \infty \)

\[
Q[w_j(\beta)] = \frac{1}{2} \frac{|\rho_j|}{d_j} \beta + O(\beta^{-1}), \quad N_B + 1 \leq j \leq N, \tag{42}
\]

provided \( d_j \neq 0 \). In fact, it is true under rather general conditions that \( d_j > 0 \), for \( j = N_B + 1, \ldots, N \) (see (26) and Remark 9). These asymptotic formulas (41), (42) readily imply that

\[
\text{high-loss modes: } \lim_{\beta \to \infty} Q[w_j(\beta)] = 0; \quad \text{low-loss modes: } \lim_{\beta \to \infty} Q[w_j(\beta)] = \begin{cases} \infty & \text{if } \rho_j \neq 0, \\ 0 & \text{if } \rho_j = 0. \end{cases} \tag{43}
\]

Observe, that the relations (39) and (42) clearly indicate that for the low-loss modes the larger values of \( \beta \) imply lesser losses and the possibility of a higher quality factor! In particular, the more lossy is the lossy component the less lossy are the low-loss modes. This somewhat paradoxical conclusion can be explained by the fact that the low-loss eigenmodes are being expelled from the loss subspace \( H_B \) in the sense that their projection onto this subspace satisfies asymptotically \( P_B w_j(\beta) = O(\beta^{-1}) \) as \( \beta \to \infty \).

### 2.3 Losses for external harmonic forces

Let us subject now our system to a harmonic external force \( \hat{f}(\omega) e^{-i\omega t} \) of a frequency \( \omega \) that will set the system into a stationary oscillatory motion of the form \( \hat{v}(\omega) e^{-i\omega t} \) of the same frequency \( \omega \) and amplitude \( \hat{v}(\omega) \) depending on the energy dissipation. Or more generally we can subject the system to an external force \( f(t) \) and observe its response \( v(t) \) governed by the evolution equation (8). The solution to this problem in view of the rest condition (12) can be obtained with the help of the Fourier-Laplace transform

\[
\hat{v}(\xi) = \int_0^\infty e^{i\xi t} v(t) dt, \quad \text{Im} \xi > 0, \tag{44}
\]

applied to the evolution equation (8) resulting in the following equation

\[
\xi \hat{v}(\xi) = [\Omega - iB] \hat{v}(\xi) + i \hat{f}(\xi), \quad \xi = \omega + i\eta, \quad \eta = \text{Im} \xi > 0. \tag{45}
\]

For \( \text{Im} \xi > 0 \) in view of \( B \geq 0 \) the operator \( \xi I - (\Omega - iB) \) is invertible, and hence

\[
\hat{v}(\xi) = \mathfrak{A}(\xi) \hat{f}(\xi), \quad \xi = \omega + i\eta, \quad \eta = \text{Im} \xi > 0, \tag{46}
\]

\[
\mathfrak{A}(\xi) = i [\xi I - (\Omega - iB)]^{-1}. \tag{47}
\]

For a harmonic force \( f(t) = f e^{-i\omega t} \) the corresponding harmonic solution is \( v(t) = v e^{-i\omega t} \) where

\[
v = \mathfrak{A}(\omega) f, \quad \mathfrak{A}(\omega) = i [\omega I - A(\beta)]^{-1}, \quad A(\beta) = \Omega - i\beta B, \quad \beta \geq 0. \tag{48}
\]
The operator $\mathfrak{A}(\omega)$ is called the \textit{admittance operator}.

For the stationary regime associated with a harmonic external force $f(t) = fe^{-i\omega t}$ the quality factor can be naturally defined by a formula analogous to (31), namely

$$Q = Q_{f,\omega} = 2\pi \frac{\text{energy stored in system}}{\text{energy lost per cycle}},$$

where the energy lost refers specifically to the energy loss due to friction in the system. By the expressions (11), the quality factor turns into

$$Q = \frac{|\omega|}{W_{\text{dis}}} = \frac{|\omega|}{\beta (v, Bv)},$$

(50)

where according to (10) $U = \frac{1}{2} (v, v)$ is the stored energy, $W_{\text{dis}} = \beta (v, Bv)$ is the power of dissipated energy.

In many cases of interest the external force $f$ is outside the loss subspace corresponding to a situation when the driving forces/sources are located outside the lossy component of the system. This important factor is described by the projection on the no-loss space $H_B^\perp = H \ominus H_B$, that is by $P_B^\perp f$. We may expect the effect of losses to depend significantly on whether $P_B^\perp f = 0$ or $P_B^\perp f \neq 0$. But even if $P_B^\perp f = f$, that is $f$ is outside the loss subspace $H_B$, there may still be losses since all system degrees of freedom can be coupled. The analysis of the stored and dissipated energies, in view of the relations (46)-(51), depends on the admittance operator $\mathfrak{A}(\omega)$. To study the properties of the admittance operator $\mathfrak{A}(\omega)$ defined by (48) we consider the block form (17) and (18) and represent $\omega I - A(\beta)$ as $2 \times 2$ block operator matrix

$$\omega I - A(\beta) = \begin{bmatrix} \Xi_2(\omega, \beta) & -\Theta \\ -\Theta^* & \Xi_1(\omega) \end{bmatrix},$$

(52)

$$\Xi_2(\omega, \beta) := \omega I_2 - (\Omega_2 - i\beta B_2), \quad \Xi_1(\omega) := \omega I_1 - \Omega_1$$

where $I_2$ and $I_1$ denote the identity operators on the spaces $H_B$ and $H_B^\perp$, respectively. With respect to this block representation, the \textit{Schur complement} of $\Xi_2(\omega, \beta)$ in $\omega I - A(\beta)$ is defined as the operator

$$S_2(\omega, \beta) = \Xi_1(\omega) - \Theta^* \Xi_2(\omega, \beta)^{-1} \Theta,$$

(53)

whenever $\Xi_2(\omega, \beta)$ is invertible.

In what follows we assume the frequency $\omega \neq 0$ is not one of the resonance frequencies, that is $\omega \neq \rho_j, \ N_B + 1 \leq j \leq N$. Then we know by Proposition 21 that the operators $\Xi_1(\omega), \ \Xi_2(\omega, \beta), \ S_2(\omega, \beta)$, and $\omega I - A(\beta)$ are invertible for $\beta \gg 1$. To simplify lengthy expressions we will suppress the symbols $\omega, \beta$ appearing as arguments in operators $\Xi_1(\omega), \ \Xi_2(\omega, \beta), \ S_2(\omega, \beta)$. Furthermore, the explicit formula based on the Schur complement is derived for the admittance operator

$$\mathfrak{A}(\omega) = i [\omega I - A(\beta)]^{-1} = i \begin{bmatrix} \Xi_2^{-1} + \Xi_2^{-1} \Theta S_2^{-1} \Theta^* \Xi_2^{-1} & \Xi_2^{-1} \Theta S_2^{-1} \\ S_2^{-1} \Theta^* \Xi_2^{-1} & S_2^{-1} \end{bmatrix}.$$  

(54)
A perturbation analysis at $\beta = \infty$ of the admittance operator $\mathfrak{A}(\omega)$, the results of which are summarized in Proposition 21, yields the following asymptotic expansion for $\beta \to \infty$

$$\mathfrak{A}(\omega) = \begin{bmatrix} 0 & 0 \\ 0 & i\Xi^{-1} \end{bmatrix} + W^{(-1)}\beta^{-1} + O(\beta^{-2}) , \quad W^{(-1)} \geq 0,$$

(55)

where

$$W^{(-1)} = \begin{bmatrix} B_2^{-1} \Xi_1^{-1} & B_2^{-1}\Theta\Xi_1^{-1} \\ (\Xi_1^{-1})^*\Theta*B_2^{-1} & (\Xi_1^{-1})^*\Theta*B_2^{-1}\Theta\Xi_1^{-1} \end{bmatrix} , \quad B_2^{-1} > 0.$$

(56)

These asymptotics for the admittance operator lead to asymptotic formulas as $\beta \to \infty$ for the energy $U$, the dissipation power $W_{\text{dis}}$, and the quality $Q$ factor which depend on whether $P_B^\perp f = 0$ or $P_B^\perp f \neq 0$. Namely, if $P_B^\perp f = 0$, that is if $f$ is inside the loss subspace $H_B$, then Theorem 23 tells us that

$$U = \frac{1}{2} \left( f, B_2^{-2} + B_2^{-1}\Theta(\Xi_1^{-1})^*\Xi_1^{-1}\Theta*B_2^{-1} \right) f^{\beta - 2} + O(\beta^{-3}),$$

$$W_{\text{dis}} = (f, B_2^{-1} f)\beta^{-1} + O(\beta^{-2}),$$

$$Q = |\omega| \frac{1}{2} \left( f, B_2^{-2} + B_2^{-1}\Theta(\Xi_1^{-1})^*\Xi_1^{-1}\Theta*B_2^{-1} \right) f \beta^{-1} + O(\beta^{-2}),$$

and the leading order terms of $U$, $W_{\text{dis}}$ and $Q$ are positive numbers. In particular, the quality factor $Q \to 0$ as $\beta \to \infty$.

If $P_B^\perp f \neq 0$ then Theorem 23 tells us that

$$U = \frac{1}{2} \left( \Xi_1^{-1} P_B^\perp f, \Xi_1^{-1} P_B^\perp f \right) + O(\beta^{-1}),$$

$$W_{\text{dis}} = (f, W^{(-1)} f)\beta^{-1} + O(\beta^{-2}),$$

and the leading order term of $U$ and $W_{\text{dis}}$ is a positive and a nonnegative number, respectively. Furthermore, the quality factor is either infinite for $\beta \gg 1$ (the case $W_{\text{dis}} \equiv 0$) or $Q \to \infty$ as $\beta \to \infty$. Moreover,

$$Q = |\omega| \frac{1}{2} \left( \Xi_1^{-1} P_B^\perp f, \Xi_1^{-1} P_B^\perp f \right) (f, W^{(-1)} f) \beta + O(1)$$

(59)

provided $(f, W^{(-1)} f) \neq 0$, in which case the leading order term for $Q$ is a positive number.

Therefore the quality factor $Q$ satisfies $\lim_{\beta \to \infty} Q = \infty$ provided $f$ has a non-zero projection on the no-loss subspace $H_B^\perp = H \ominus H_B$, whereas otherwise $\lim_{\beta \to \infty} Q = 0$.

3 An electric circuit example

One of the important applications of our methods described above is electric circuits and networks involving resistors representing losses. A general study of electric networks with losses can be carried out with the help of the Lagrangian approach, and that systematic study is left for another publication. For Lagrangian treatment of electric networks and circuits we refer to [Gant, Sec. 9], [Gold, Sec. 2.5], [Pars].

We illustrate the idea and give a flavor of the efficiency of our methods by considering below a rather simple example of an electric circuit as in Fig. 1. This example will show the essential features of two component systems incorporating high-loss and lossless components.
Figure 1: An electric circuit involving three capacitances $C_1$, $C_2$, $C_{12}$, two inductances $L_1$, $L_2$, a resistor $R_2$, and two sources $E_1$, $E_2$.

To derive evolution equations for the electric circuit in Fig. 1, we use a general method for constructing Lagrangians for circuits, [Gant, Sec. 9], that yields

$$T = \frac{1}{2} L_1 \dot{q}_1^2 + \frac{1}{2} L_2 \dot{q}_2^2,$$

$$U = \frac{1}{2} C_1 q_1^2 + \frac{1}{2} C_{12} (q_1 - q_2)^2 + \frac{1}{2} C_2 q_2^2,$$

$$R = \frac{R_2}{2} \dot{q}_2^2,$$

where $T$ and $U$ are respectively the kinetic and the potential energies, $T - U$ is the Lagrangian, and $R$ is the Rayleigh dissipative function. Notice that $I_1 = \dot{q}_1$ and $I_2 = \dot{q}_2$ are the currents. The general Euler-Lagrange equations are, [Gant, Sec. 8],

$$\frac{\partial}{\partial t} \frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} = -\frac{\partial U}{\partial q_j} - \frac{\partial R}{\partial \dot{q}_j},$$

and for (60) the general Euler-Lagrange equations therefore have the following form

$$\frac{\partial}{\partial t} L_1 \dot{q}_1 = -\frac{1}{C_1} q_1 - \frac{1}{C_{12}} (q_1 - q_2) + \frac{1}{C_1} \dot{q}_1 + f_1,$$

$$\frac{\partial}{\partial t} L_2 \dot{q}_2 = -\frac{1}{C_2} q_2 + \frac{1}{C_{12}} (q_1 - q_2) - R_2 \dot{q}_2 + f_2.$$

If we introduce

$$Q = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}, \quad F = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix},$$

$$L = \begin{bmatrix} L_1 & 0 \\ 0 & L_2 \end{bmatrix}, \quad G = \begin{bmatrix} C_1^{-1} + C_{12}^{-1} & -C_{12}^{-1} \\ -C_{12}^{-1} & C_2^{-1} + C_{12}^{-1} \end{bmatrix}, \quad R = \begin{bmatrix} 0 & 0 \\ 0 & R_2 \end{bmatrix},$$

the system (62) can be recast into

$$L \dot{\partial}_t^2 Q + R \partial_t Q + GQ = F.$$

To provide for an efficient spectral study of the vector equation (65) we notice that

$$L > 0, \quad G > 0,$$
and introduce
\[ \tilde{Q} = L^{-\frac{1}{2}} Q, \quad \tilde{F} = L^{-\frac{1}{2}} F, \quad \tilde{R} = L^{-\frac{1}{2}} R L^{-\frac{1}{2}}, \quad \tilde{G} = L^{-\frac{1}{2}} G L^{-\frac{1}{2}} = \Phi^2. \] (67)

The vector equation (65) is transformed then into
\[ \partial_t^2 \tilde{Q} + \tilde{R} \partial_t \tilde{Q} + \tilde{G} \tilde{Q} = \tilde{F}. \] (68)

To rewrite the above second-order ODE as the first-order ODE we set
\[ X = \begin{bmatrix} y \\ x \end{bmatrix} = \begin{bmatrix} \partial_t \tilde{Q} \\ \Phi \tilde{Q} \end{bmatrix}, \text{ that is } y = \partial_t \tilde{Q}, \quad x = \Phi \tilde{Q}, \] (69)
allowing one to recast the vector equation (68) as
\[ \partial_t X = \begin{bmatrix} -\tilde{R} & -\Phi \\ \Phi & 0 \end{bmatrix} X + \begin{bmatrix} \tilde{F} \\ 0 \end{bmatrix}, \] (70)
where
\[ \begin{bmatrix} -\tilde{R} & -\Phi \\ \Phi & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -\Phi_{11} & -\Phi_{12} \\ 0 & -R_2 L_2^{-1} & -\Phi_{12} & -\Phi_{22} \\ \Phi_{11} & \Phi_{12} & 0 & 0 \\ \Phi_{12} & \Phi_{22} & 0 & 0 \end{bmatrix}. \] (71)

Consequently, the general Euler-Lagrange equations (62) for the electric circuit in Fig. 1 are transformed into the canonical form (8), namely
\[ \partial_t X = -i (\Omega - i \beta B) X + \begin{bmatrix} \tilde{F} \\ 0 \end{bmatrix}, \] with the system operator
\[ A(\beta) = \Omega - i \beta B, \quad \beta \geq 0, \] (72)
\[ \Omega = \begin{bmatrix} 0 & -i \Phi \\ i \Phi & 0 \end{bmatrix}, \quad \Phi = \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{12} & \Phi_{22} \end{bmatrix} > 0, \] (73)
\[ B = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \tau^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \text{and } \beta = \frac{R_2 \tau}{L_2}, \text{ where } \tau > 0 \text{ is a unit of time.} \] (74)

As we can see this electric circuit example fits within the framework of our model. Indeed, since resistors represent losses, this two component system consists of a lossy component and a lossless component – the right and left circuits in Fig. 1 respectively.

The next two sections are devoted to the analysis of this electric circuit both theoretically and numerically in the high-loss regime $\beta \gg 1$ using the methods developed in this paper. For this purpose the following properties of the matrix $\Phi$ in (73) are useful:
\[ \Phi = \left( \sqrt{\text{Tr} (\Phi^2)} + 2 \sqrt{\det (\Phi^2)} \right)^{-1} \left( \sqrt{\det (\Phi^2)} I_2 + \Phi^2 \right), \] (75)
\[ \Phi^2 = \begin{bmatrix} \frac{1}{L_1} \left( \frac{1}{c_1} + \frac{1}{c_{12}} \right) & -\frac{1}{\sqrt{L_1 L_2} c_{12}} \\ -\frac{1}{\sqrt{L_1 L_2} c_{12}} & \frac{1}{L_2} \left( \frac{1}{c_2} + \frac{1}{c_{12}} \right) \end{bmatrix}, \quad \Phi_{11}, \Phi_{22} > 0, \quad \Phi_{12} \in \mathbb{R} \setminus \{0\}, \] where $\sqrt{\cdot}$ denotes the positive square root.
3.1 Spectral analysis in the high-loss regime

In this section, a spectral analysis of the electric circuit example in Fig. 1 in the high-loss regime is given using the main results of this paper.

3.1.1 Perturbation analysis

The finite dimensional Hilbert space is $H = \mathbb{C}^4$ under the standard inner product $(\cdot, \cdot)$. It is decomposed into the direct sum of invariant subspace of the operator $B \geq 0$ in (74),

$$H = H_B \oplus H_B^\perp,$$

where $H_B = \text{ran } B$, $H_B^\perp = \ker B$ are the loss subspace and no-loss subspace with dimensions $N_B = 1$, $N - N_B = 3$ and orthogonal projections

$$P_B = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad P_B^\perp = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

respectively. The operators $\Omega$ and $B$, with respect to this direct sum, are $2 \times 2$ block operator matrices

$$\Omega = \begin{bmatrix} \Omega_2 & \Theta \\ \Theta^* & \Omega_1 \end{bmatrix}, \quad B = \begin{bmatrix} B_2 & 0 \\ 0 & 0 \end{bmatrix},$$

where $\Omega_2 := P_B \Omega P_B|_{H_B} : H_B \to H_B$ and $B_2 := P_B B P_B|_{H_B} : H_B \to H_B$ are restrictions of the operators $\Omega$ and $B$, respectively, to loss subspace $H_B$ whereas $\Omega_1 := P_B^\perp \Omega P_B^\perp|_{H_B^\perp} : H_B^\perp \to H_B^\perp$ is the restriction of $\Omega$ to complementary subspace $H_B^\perp$. Also, $\Theta : H_B^\perp \to H_B^\perp$ is the operator whose adjoint is given by $\Theta^* = P_B^\perp \Omega P_B|_{H_B}$, Moreover, according to our perturbation theory the operator $\Theta^* B_2^{-1} \Theta : H_B^\perp \to H_B^\perp$ plays a key role in the analysis. These operators act on the $4 \times 1$ column vectors in their respective domains as matrix multiplication by the $4 \times 4$ matrices

$$\Omega_2 = 0, \quad B_2 = B, \quad \Omega_1 = \begin{bmatrix} 0 & 0 & -i\Phi_{11} & -i\Phi_{12} \\ 0 & 0 & 0 & 0 \\ i\Phi_{11} & 0 & 0 & 0 \\ i\Phi_{12} & 0 & 0 & 0 \end{bmatrix},$$

$$\Theta = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i\Phi_{12} & -i\Phi_{22} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \Theta^* = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i\Phi_{12} & 0 & 0 & 0 \\ i\Phi_{22} & 0 & 0 & 0 \end{bmatrix},$$

$$B_2^{-1} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \tau & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \Theta^* B_2^{-1} \Theta = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \tau \Phi_{12}^2 & \tau \Phi_{12} \Phi_{22} & \tau \Phi_{22}^2 \\ 0 & 0 & \tau \Phi_{12} \Phi_{22} & \tau \Phi_{22}^2 \end{bmatrix},$$

where by "=" we mean equality as functions from the domain of the operator on the LHS of the equal sign.
The operators $\Omega_1$ and $B_2$ in this example have only simple eigenvalues and so we will use Corollary 12. We introduce below a fixed orthonormal basis $\{\hat{w}_j\}_{j=1}^4$ diagonalizing the operators $\Omega_1$ and $B_2$ and then determine the values $\hat{\zeta}_j$, $\rho_j$, $d_j$ from the relations

$$B_2 \hat{w}_j = \hat{\zeta}_j \hat{w}_j, \quad \rho_j = (\hat{w}_j, \Omega \hat{w}_j) \quad \text{for} \ j = 1;$$

$$\Omega_1 \hat{w}_j = \rho_j \hat{w}_j, \quad d_j = (\hat{w}_j, \Theta^* B_2^{-1} \Theta \hat{w}_j) \quad \text{for} \ 2 \leq j \leq 4.$$

In particular,

$$\hat{w}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \hat{\zeta}_1 = \tau^{-1}, \quad \rho_1 = 0; \quad (76)$$

$$\hat{w}_2 = \frac{1}{\sqrt{\Phi_{12}^2 + \Phi_{12}^2}} \begin{bmatrix} 0 \\ 0 \\ -\Phi_{12} \\ \Phi_{11} \end{bmatrix}, \quad \rho_2 = 0, \quad d_2 = \frac{\tau (\Phi_{12}^2 - \Phi_{11} \Phi_{22})^2}{\Phi_{11}^2 + \Phi_{12}^2} > 0, \quad (77)$$

$$\hat{w}_3 = \frac{1}{\sqrt{2}} \frac{1}{\sqrt{\Phi_{11}^2 + \Phi_{12}^2}} \begin{bmatrix} -i \sqrt{\Phi_{11}^2 + \Phi_{12}^2} \\ 0 \\ \Phi_{11} \\ \Phi_{12} \end{bmatrix}, \quad \rho_3 = \sqrt{\Phi_{11}^2 + \Phi_{12}^2}, \quad d_3 = \frac{1}{2} \tau \Phi_{12}^2 (\Phi_{11} + \Phi_{22})^2}{\Phi_{11}^2 + \Phi_{12}^2},$$

$$\hat{w}_4 = \hat{w}_3, \quad \rho_4 = -\rho_3 < 0, \quad d_4 = d_3 > 0.$$

By Theorem 3 and Corollary 12 of this paper it follows that in the high-loss regime $\beta \gg 1$, the system operator $A(\beta) = \Omega - i\beta B$ is diagonalizable and there exists a complete set of eigenvalues and eigenvectors satisfying

$$A(\beta) w_j(\beta) = \zeta_j(\beta) w_j(\beta), \quad 1 \leq j \leq 4,$$

which splits into two classes

$$\text{high-loss:} \quad \zeta_j(\beta), \ w_j(\beta), \quad j = 1;$$

$$\text{low-loss:} \quad \zeta_j(\beta), \ w_j(\beta), \quad 2 \leq j \leq 4,$$

having the following properties.

**High-loss modes.** The high-loss eigenvalue has a pole at $\beta = \infty$ whereas its eigenvector is analytic at $\beta = \infty$, having the asymptotic expansion

$$\zeta_1(\beta) = -i \hat{\zeta}_1 \beta + \rho_1 + O(\beta^{-1}), \quad \hat{\zeta}_1 > 0, \quad \rho_1 \in \mathbb{R}; \quad (79)$$

$$w_1(\beta) = \hat{w}_1 + O(\beta^{-1}),$$

as $\beta \to \infty$. The vector $\hat{w}_1$ is an orthonormal basis of the loss subspace $H_B$ and

$$B \hat{w}_1 = \hat{\zeta}_1 \hat{w}_1, \quad \rho_1 = (\hat{w}_1, \Omega \hat{w}_1).$$

**Low-loss modes.** The low-loss eigenvalues and eigenvectors are analytic at $\beta = \infty$, having the asymptotic expansions

$$\zeta_j(\beta) = \rho_j - id_j \beta^{-1} + O(\beta^{-2}), \quad \rho_j \in \mathbb{R}, \quad d_j > 0, \quad (80)$$

$$w_j(\beta) = \hat{w}_j + w_j^{(-1)} \beta^{-1} + O(\beta^{-2}), \quad 2 \leq j \leq 4,$$
as $\beta \to \infty$. The vectors $\hat{w}_j, 2 \leq j \leq 4$ form an orthonormal basis of the no-loss subspace $H^1_B$ and

$$B\hat{w}_j = 0, \quad \rho_j = (\hat{w}_j, \Omega \hat{w}_j), \quad d_j = \left(w_j^{(-1)}, Bw_j^{(-1)}\right), \quad 2 \leq j \leq 4.$$ 

### 3.1.2 Overdamping and symmetries of the spectrum

The phenomenon of overdamping (also called heavy damping) is best known for a simple damped oscillator. Namely, when the damping exceeds certain critical value all oscillations cease entirely, see, for instance, [Pain, Sec. 2]. In other words, if the damped oscillations are described by the exponential function $e^{-i\zeta t}$ with a complex constant $\zeta$ then in the case of overdamping (heavy damping) $\Re \zeta = 0$. Our interest in overdamping is motivated by the fact that if an eigenmode becomes overdamped then it will not resonate at any frequency. Consequently, the contribution of such a mode to losses becomes minimal, and that provides a mechanism for the absorption suppression for systems composed of lossy and lossless components.

The treatment of overdamping for systems with many degrees of freedom involves a number of subtleties particularly in our case when the both lossy and lossless degrees of freedom are present. We have reasons to believe though that any Lagrangian system with losses accounted by the Rayleigh dissipation function can have all high-loss eigenmodes overdamped for a sufficiently large value of the loss parameter $\beta$. In order to give valuable insights into far more general systems, we focus on the electric circuit example in Fig. 1 giving statements and providing arguments on the spectral symmetry and overdamping for the circuit. This analysis is used in the next section to interpret the behavior of the eigenvalues of the circuit operator $A(\beta)$.

Our first principal statement is on a symmetry of the spectrum of the system operator $A(\beta)$ with respect to the imaginary axis.

**Proposition 1 (spectral symmetry)** Let $A(\beta)$ denote the system operator (72) for the electric circuit given in Fig. 1. Then for each $\beta \geq 0$, its spectrum $\sigma (A(\beta))$ lies in the lower half of the complex plane and is symmetric with respect to the imaginary axis, that is

$$\sigma (A(\beta)) = -\overline{\sigma(A(\beta))}. \quad (81)$$

Moreover, except for a finite set of values of $\beta$, the system operator $A(\beta)$ is diagonalizable with four nondegenerate eigenvalues.

**Proof.** From the asymptotic analysis in (79)–(77) it follows that all the eigenvalues of $A(\beta)$ must be distinct for $\beta \gg 1$. Now the operator $\Omega - i\beta B$, $\beta \in \mathbb{C}$ is analytic on $\mathbb{C}$ and so, by a well-known fact from perturbation theory [Ban85, p. 25, Theorem 3; p. 225, Theorem 1], its Jordan structure is invariant except on a set $S \subseteq \mathbb{C}$ which is closed and isolated. These facts imply the system operator $A(\beta)$ is diagonalizable with distinct eigenvalues except on the closed and isolated set $S \cap [0, \infty)$ which must be bounded since the eigenvalues of $A(\beta)$ are distinct for $\beta \gg 1$. In particular, this implies $S \cap [0, \infty)$ is a finite set. This proves that except for a finite set of values of $\beta$, the system operator $A(\beta)$ is diagonalizable with four nondegenerate eigenvalues.

Next, since $A(\beta) = \Omega - i\beta B$ in (72) is a system operator satisfying the power dissipation condition $B \geq 0$ then it follows from Lemma 27 in Appendix 8 that if $\beta \geq 0$ then $\Im \zeta \leq 0$ if $\zeta$ is an eigenvalue of $A(\beta)$, i.e., the spectrum $\sigma (A(\beta))$ lies in the lower half of the complex
plane. Finally, one can show that \( \det (\zeta I - A(\beta)) = \det L^{-1} \det (\zeta^2 L + \zeta i R - G) \) for all \( \zeta \in \mathbb{C}, \beta \geq 0 \). Moreover, from our assumptions \( \beta \geq 0, \tau > 0 \) and \( L, G > 0 \) it follows that the \( 2 \times 2 \) matrices \( L, R, G \) must have real entries. Using these two facts we conclude
\[
\det (\zeta I - A(\beta)) = \det L^{-1} \det (\zeta^2 L + \zeta i R - G) = \det L^{-1} \det \left( (-\zeta)^2 L + (-\zeta)iR - G \right) = \det \left( (-\zeta) I - A(\beta) \right),
\]
and, hence, (81) holds. ■

**Corollary 2 (eigenvalue symmetry)** Let \( \mathcal{I} \) be any open interval in \((0, \infty)\) with the property that the eigenvalues of system operator \( A(\beta) \) are nondegenerate for every \( \beta \in \mathcal{I} \). Then there exists a unique set of functions \( \zeta_j : \mathcal{I} \to \mathbb{C}, j = 1, 2, 3, 4 \) which are analytic at each \( \beta \in \mathcal{I} \) and whose values \( \zeta_1(\beta), \zeta_2(\beta), \zeta_3(\beta), \zeta_4(\beta) \) are the eigenvalues of the system operator \( A(\beta) \). Moreover, there exists a unique permutation \( \kappa : \{1, 2, 3, 4\} \mapsto \{1, 2, 3, 4\} \) depending only on the interval \( \mathcal{I} \) such that for each \( j = 1, 2, 3, 4 \),
\[
\zeta_j(\beta) = -\zeta_{\kappa(j)}(\beta) \quad \text{for every} \ \beta \in \mathcal{I}.
\]

**Proof.** It is a well-known fact from perturbation theory for matrices depending analytically on a parameter \([\text{Bau85}]\), that simple eigenvalues can be chosen to be analytic locally in the perturbation parameter and analytically continued as eigenvalues along any path in the domain of analyticity of the matrix function which does not intersect a closed and isolated set of singularities. These singularities are necessarily contained in the set of parameters in the domain where the value of matrix function has repeated eigenvalues. The proof of the first part of this corollary now follows immediately from this fact and the fact the high-loss and low-loss eigenvalues of \( A(\beta) \) are meromorphic and analytic at \( \beta = \infty \), respectively, with distinct values for \( \beta \gg 1 \). The existence and uniqueness of the permutation is now obvious from this and symmetry of the spectrum described in the previous proposition. This completes the proof. ■

**Corollary 3 (overdamping)** Let \( \zeta_j(\beta), j = 1, 2, 3, 4 \) be the high-loss and low-loss eigenvalues of the system operator \( A(\beta) \) given by (78)-(77). Then, in the high-loss regime \( \beta \gg 1 \), these eigenvalues lie in the lower open half-plane and, moreover, the eigenvalues \( \zeta_j(\beta), j = 1, 2 \) are on the imaginary axis whereas the eigenvalues \( \zeta_j(\beta), j = 3, 4 \) lie off this axis and symmetric to it, i.e., \( \zeta_4(\beta) = -\zeta_3(\beta) \).

**Proof.** First, it follows the asymptotic analysis in (79)-(77) that there exists a \( \beta_0 > 0 \) such that \( \zeta_j(\beta), j = 1, 2, 3, 4 \) are all the eigenvalues of system operator \( A(\beta) \) and are distinct for every \( \beta \in (\beta_0, \infty) \). By the previous corollary there exists a unique permutation \( \kappa : \{1, 2, 3, 4\} \mapsto \{1, 2, 3, 4\} \) depending only on the interval \((\beta_0, \infty)\) such that for each \( j = 1, 2, 3, 4 \), the identity (83) for every \( \beta \in (\beta_0, \infty) \).

Next, we will now show that for this permutation we have \( \kappa(1) = 1, \kappa(2) = 2, \kappa(3) = 4, \kappa(4) = 3 \). Well, consider the the asymptotic expansions of the imaginary and real parts of high-loss and low-loss eigenvalues. First, \( \lim_{\beta \to \infty} \text{Im} \zeta_1(\beta) = -\infty, \lim_{\beta \to \infty} \text{Im} \zeta_j(\beta) = 0, j = 2, 3, 4 \) and since \( \zeta_1(\beta) = -\zeta_{\kappa(1)}(\beta) \) these properties imply \( \kappa(1) = 1 \). Second, \( \lim_{\beta \to \infty} \text{Re} \zeta_2(\beta) = 0, \lim_{\beta \to \infty} \text{Re} \zeta_4(\beta) = \rho_4 = -\rho_3 = -\lim_{\beta \to \infty} \text{Re} \zeta_3(\beta) \) with \( \rho_3 > 0 \) and since \( \zeta_j(\beta) = -\zeta_{\kappa(j)}(\beta) \) with \( \kappa(j) \neq 1 \) for \( j = 2, 3, 4 \) these properties imply \( \kappa(2) = 2, \kappa(3) = 4, \kappa(4) = 3 \).
To complete the proof we notice that since $\kappa(1) = 1, \kappa(2) = 2, \kappa(3) = 4, \kappa(4) = 3$ then for $\beta \gg 1$ we have $\zeta_j(\beta) = -\zeta_j(\beta)$, $j = 1,2$ and $\zeta_3(\beta) = -\zeta_3(\beta)$. The proof now follows immediately from this and the facts $-\text{Re}\zeta_4(\beta) = \text{Re}\zeta_3(\beta) = \rho_3 + O(\beta^{-2})$, $\text{Im}\zeta_4(\beta) = -d_3\beta^{-1} + O(\beta^{-3})$ as $\beta \to \infty$ where $\rho_3, d_3 > 0$. \hfill \blacksquare

Remark 4 The boundary of the overdamping regime known as critical damping, corresponds to a value of the loss parameter $\beta = \beta_0 > 0$ at which the system operator $A(\beta)$ develops a purely imaginary but degenerate eigenvalue $\zeta_0$. The spectral perturbation analysis of $A(\beta)$ in a neighborhood of the point $\beta = \beta_0$ is theoretically and computationally a difficult problem since it is a perturbation of the non-self-adjoint operator $A(\beta_0)$ with a degenerate eigenvalue. This type of perturbation problem was considered in [Welt] where asymptotic expansions of the perturbed eigenvalues and eigenvectors were given and explicit recursive formulas to compute their series expansions were found [Welt, Theorem 3.1], under a generic condition [Welt, p. 2, (1.1)]. In particular, this condition is satisfied for the system operator $A(\beta)$ at the point $\beta = \beta_0$ for the degenerate eigenvalue $\zeta_0$ since

$$\frac{\partial}{\partial \beta} \det (\zeta I - A(\beta))|_{(\zeta,\beta)=(\zeta_0,\beta_0)} = i\tau^{-1}\zeta_0^{-3} - i\tau^{-1}(\Phi_{11} + \Phi_{12}^2) \zeta_0 \neq 0.$$

3.2 Numerical analysis

In order to illustrate the behavior of the eigenvalues of the system operator for the circuit in Fig. 1 we fix positive values for capacitance $C_1$, $C_2$, $C_{12}$, inductances $L_1$, $L_2$, and the unit of time $\tau$. Once these are fixed, the system operator $A(\beta)$ is computed using (64), (67), and (72)–(74). These values constrain the magnitude of the resistance $R_2$ of the corresponding circuit in Fig. 1 to be proportional to the dimensionless loss parameter $\beta$ since it follows from (74) that

$$R_2 = \frac{L_2}{\tau \beta}. \quad (84)$$

The high-loss regime $\beta \gg 1$ is associated with the right circuit in Fig. 1 experiencing huge losses due to the resistance $R_2 \gg 1$ while the left circuit remains lossless. In particular, each choice of these values provides a numerical example of a physical model with a two component system composed of a high-loss and lossless components.

For the numerical analysis in this section we chose

$$C_1 := 2, \quad C_2 := 3, \quad C_{12} := 4, \quad L_1 := 5, \quad L_2 := 6, \quad \tau := 1. \quad (85)$$

All graphs were plotted in Maple® using these fixed values and with the loss parameter in the domain $0 \leq \beta \leq 1$.

On Figure 2 In Fig. 2 is a series of plots which compares the imaginary part of each of the eigenvalues $\zeta_j(\beta)$, $1 \leq j \leq 4$ of the system operator $A(\beta)$ for the electric circuit in Fig. 1 to the quality factor $Q$ of their corresponding eigenmodes. To plot the quality factor as a function of the loss parameter $\beta$ we have used formula (85). As is evident by this figure, there is clearly modal dichotomy caused by dissipation.

Indeed, for the high-loss eigenpair $\zeta_1(\beta), w_1(\beta)$ we can see in plots (a)–(b) that as the loss parameter $\beta$ grows large so too does $\text{Im}\zeta_1(\beta)$ whereas the quality factor $Q[w_1(\beta)] = -\frac{1}{2} \frac{\text{Re}\zeta_1(\beta)}{\text{Im}\zeta_1(\beta)}$ of the mode goes to zero as predicted by our general theory (cf. (99) of Proposition.
Figure 2: (a)-(d) For each high-loss eigenvalue $\zeta_j(\beta)$, $j = 1$ and low-loss eigenvalue $\zeta_j(\beta)$, $2 \leq j \leq 4$ of the system operator $A(\beta)$ for the electric circuit in Fig. 1 with the values (84) and (85), comparing its imaginary part $\text{Im} \, \zeta_j(\beta)$ to the quality factor $Q_j = -\frac{1}{2} \frac{|\text{Re} \, \zeta_j(\beta)|}{\text{Im} \, \zeta_j(\beta)}$ of any one of its eigenmodes (not shown is $Q_j \to \infty$ as $\beta \to 0$). For $\beta \geq \beta_0 \approx 0.57282$ (critical damping) and $j = 1, 2$, the eigenmodes with eigenvalue $\zeta_j(\beta)$ are overdamped since $\text{Re} \, \zeta_j(\beta) = 0$. The overdamping phenomenon is manifested graphically in (a), (c) with the cusps in the curves at the intersection of vertical dotted line $\beta = \beta_0$ and (b), (d) with the curves on the line $Q = 0$ for $\beta \geq \beta_0$. (a) $\text{Im} \, \zeta_1(\beta)$ vs. $\beta$. (b) $Q_1$ vs. $\beta$. Due to overdamping, $Q_1 = 0$ for $\beta \geq \beta_0$. (c) $\text{Im} \, \zeta_j(\beta)$ vs. $\beta$, $j = 2, 3, 4$. The curves $\text{Im} \, \zeta_2(\beta)$ and $\text{Im} \, \zeta_4(\beta)$ are shown in blue and green, respectively. Due to eigenvalue symmetry, the curves $\text{Im} \, \zeta_3(\beta)$ and $\text{Im} \, \zeta_4(\beta)$ cannot be distinguished in this plot and $\text{Im} \, \zeta_2(\beta) = \text{Im} \, \zeta_1(\beta)$ for $\beta \leq \beta_0$. (d) $Q_j$ vs. $\beta$, $j = 2, 3, 4$. The curves $Q = Q_2$ and $Q = Q_4$ are shown in blue and green, respectively, and because of overdamping $Q_2 = 0$ for $\beta \geq \beta_0$. Due to eigenvalue symmetry, the curves $Q = Q_3$ and $Q = Q_4$ cannot be distinguished in this plot.
Figure 3: (a)-(d) A comparison of the real and imaginary parts of the low-loss eigenvalue $\zeta_3(\beta)$ to its truncated asymptotic expansion $\tilde{\zeta}_3(\beta) = \rho_3 - i d_3 \beta^{-1}$ for the values $\rho_3, d_3$ predicted by our theory. As evident from these plots, $\zeta_3(\beta) \approx \tilde{\zeta}_3(\beta)$ for $\beta$ large. (a) Re $\zeta_3(\beta)$ vs. $\beta$. (b) Re $\tilde{\zeta}_3(\beta)$ vs. $\beta$ (the solid line) and Re $\zeta_3(\beta)$ vs. $\beta$ (the diagonal crosses). (c) Im $\zeta_3(\beta)$ vs. $\beta$. (d) Im $\zeta_3(\beta)$ vs. $\beta$ (the solid line) and Im $\tilde{\zeta}_3(\beta)$ vs. $\beta$ (the diagonal crosses).
and (110) of Proposition 14). In fact, by our results on the overdamping phenomenon described in Corollary 3, we know that it is exactly zero for all $\beta \geq \beta_0$, where $\beta_0$ denotes the boundary of the overdamped regime as discussed in Remark 4. For the fixed values in (83), $\beta_0 \approx 0.57282$ and we have place a vertical dotted line in each of the plots in Fig. 2 to indicate this boundary.

The behavior is quite different for the low-loss eigenpairs $\zeta_j(\beta)$, $w_j(\beta)$, $2 \leq j \leq 4$. The plots (c)–(d) show that as the loss parameter $\beta$ grows large, the values $\operatorname{Im} \zeta_j(\beta)$, $j = 2, 3, 4$ all become small with the quality factors $Q[w_j(\beta)] = -\frac{1}{2} \frac{|\operatorname{Re} \zeta_j(\beta)|}{\operatorname{Im} \zeta_j(\beta)}$ of the eigenmodes $w_3(\beta)$ and $w_4(\beta)$ becoming large as predicted by our theory (cf. (99) of Proposition 7, (112) of Proposition 14 and formulas (77)). The quality factor $Q[w_2(\beta)] = -\frac{1}{2} \frac{|\operatorname{Re} \zeta_2(\beta)|}{\operatorname{Im} \zeta_2(\beta)}$ of the low-loss mode $w_2(\beta)$ becomes zero for $\beta \geq \beta_0$, again a fact which is predicted for this electric circuit from the overdamping phenomenon described in Corollary 3.

**On Figure 3.** Figure 3 compares the low-loss eigenvalue $\zeta_3(\beta)$ of the system operator $A(\beta)$ to the truncation $\tilde{\zeta}_3(\beta)$ of its asymptotic expansion as predicted by our theory in (80) and (77), namely,

$$\zeta_3(\beta) \approx \tilde{\zeta}_3(\beta) = \rho_3 - i d_3 \beta^{-1} = \sqrt{\Phi_{11}^2 + \Phi_{12}^2} - i \frac{1}{2} \frac{\Phi_{12}^2 (\Phi_{11} + \Phi_{22})^2}{\Phi_{11}^2 + \Phi_{12}^2} \beta^{-1}, \quad \beta \gg 1.$$  

The plots (a) and (c) in the figure are the real and imaginary parts, respectively, of the eigenvalue $\zeta_3(\beta)$ and plots (b) and (d) are the real and imaginary parts, respectively, of both the eigenvalue $\zeta_3(\beta)$ and the truncation of its asymptotic expansion $\tilde{\zeta}_3(\beta)$.

**On Figure 4** In Fig. 4 we have the images in the complex plane of the eigenvalues $\zeta_j(\beta)$, $1 \leq j \leq 4$. This figure displays the spectral symmetry and overdamping phenomena as predicted in Proposition 1 and Corollaries 2, 3. According to Lemma 27 and as evident in the figure, the eigenvalues lie in the lower half-plane. By Theorem 13 these eigenvalues converge to the real axis as $\beta \to 0$ and, in particular, to the eigenvalues of the frequency operator $\Omega$.

In plot (a) we see the images of the eigenvalues $\zeta_j(\beta)$, $j = 1, 2$ in the complex plane. We observe that overdamping does occur but only for these two eigenvalues. Indeed, as the loss parameter $\beta$ increase from zero these two eigenvalues eventually merge on the negative imaginary axis when $\beta = \beta_0 \approx 0.57282$ and stay on this axis for all $\beta \geq \beta_0$ with $\operatorname{Im} \zeta_1(\beta) \to -\infty$ and $\operatorname{Im} \zeta_2(\beta) \to 0$ as $\beta \to \infty$.

In plot (b) we see the images of the eigenvalues $\zeta_j(\beta)$, $j = 3, 4$ in the complex plane. This plot shows the eigenvalue symmetry for the system operator $A(\beta)$ for the electric circuit which forces these eigenvalues to satisfy $\zeta_4(\beta) = -\zeta_3(\beta)$, to lie off the imaginary axis, and to be in the lower open half-plane for all $\beta > 0$ with $\operatorname{Im} \zeta_j(\beta) \to 0$ as $\beta \to \infty$ for $j = 3, 4$. 

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Figure 4: (a)-(b) Image in the complex plane of the eigenvalues $\zeta_j(\beta)$, $1 \leq j \leq 4$ of the system operator $A(\beta)$ for the electric circuit in Fig. 1 with the values (84) and (85). The power dissipation condition implies $\text{Im} \ zeta_j(\beta) \leq 0$ for $\beta \geq 0$, as evident in the figure. (a) Image of the high-loss eigenvalue $\zeta_1(\beta)$ and low-loss eigenvalue $\zeta_2(\beta)$ displayed in red and blue, respectively. For the purpose of comparison, the view is restricted to a box around the image of $\zeta_2(\beta)$. Off the imaginary axis the blue curve is symmetric about this axis due to the eigenvalue symmetry $\zeta_2(\beta) = -\zeta_1(\beta)$ for $\beta \leq \beta_0 \approx 0.57282$. The two curves intersect, for $\beta = \beta_0$, on the negative imaginary axis and due to overdamping stay there for all $\beta \geq \beta_0$. Moreover, $\text{Im} \ zeta_1(\beta) \to -\infty$ and $\text{Im} \ zeta_2(\beta) \to 0$ as $\beta \to \infty$. (b) Image of the low-loss eigenvalues $\zeta_3(\beta)$ and $\zeta_4(\beta)$ displayed in orange and green, respectively. The green curve is symmetric about the imaginary axis to the orange curve due to the eigenvalue symmetry $\zeta_4(\beta) = -\zeta_3(\beta)$ for all $\beta \geq 0$. Moreover, $\text{Im} \ zeta_4(\beta) = \text{Im} \ zeta_3(\beta) \to 0$ and $-\text{Re} \ zeta_4(\beta) = \text{Re} \ zeta_3(\beta) \to \rho_3$ as $\beta \to \infty$, where $\rho_3 \approx 0.40825$. 

4 Perturbation analysis of the system operator

This and the following sections are devoted to a rigorous perturbation analysis of the eigenvalues and eigenvectors of the system operator from (8),

\[ A(\beta) := \Omega - i\beta B, \quad \beta \geq 0, \quad (86) \]

in both the high-loss regime, \( \beta \gg 1 \), and the low-loss regime, \( \beta \ll 1 \). We mainly focus on the high-loss regime and our goal is to develop a mathematical framework based on perturbation theory for an asymptotic analytic description of the effects dissipation have on the system (8) including modal dichotomy, i.e., splitting of eigenmodes into two distinct classes according to their dissipative properties: high-loss and low-loss modes. This framework and its rigorous analysis provides for insights into the mechanism of losses in composite systems and in ways to achieve significant absorption suppression.

The rest of this section is organized as follows. We first recall the basic assumptions, definitions, and notations from earlier in this paper regarding the system operator (86), the quality factor \( Q \) and power of energy dissipation \( W_{\text{dis}} \) associated with its modes. In the next section we state our main results on the perturbation analysis of the eigenvalues and eigenvectors for this operator \( A(\beta) \). The result for the high-loss regime, \( \beta \gg 1 \), and the low-loss regime, \( \beta \ll 1 \), are placed in separate sections. Finally, we prove the statement of our main results in Section 6.

The system operator \( A(\beta) \) in (86), for each value of the loss parameter \( \beta \), is a linear operator on the finite dimensional Hilbert space \( H \) with \( N := \dim H \) and scalar product \( (\cdot, \cdot) \). The frequency operator \( \Omega \) and the operator associated with dissipation \( B \) are self-adjoint operators on \( H \). The operator \( B \) satisfies the power dissipation condition (9) and the loss fraction condition (15), namely,

\[ B \geq 0, \quad 0 < \delta_B < 1 \quad (87) \]

where \( N_B := \text{rank } B \) denotes the rank of the operator \( B \) and \( \delta_B := \frac{N_B}{N} \) is referred to as the fraction of high-loss modes.

The range of the operator \( B \), i.e., the loss subspace, is denoted by \( H_B \) and the orthogonal projection onto this space is denoted by \( P_B \). It follows immediately from these definitions and the fact \( B \) is self-adjoint that

\[ H = H_B \oplus H_B^\perp \quad (88) \]

where \( H_B^\perp \), i.e., the no-loss subspace, is the orthogonal complement of \( H_B \) in \( H \) and is the kernel of \( B \) with the orthogonal projection onto this space given by \( P_B^\perp := I - P_B \). In particular,

\[ H_B = \text{ran } B, \quad N_B = \dim H_B, \quad H_B^\perp = \ker B, \quad N - N_B = \dim H_B^\perp. \quad (89) \]

The energy \( U[w] \), power of energy dissipation \( W_{\text{dis}}[w] \), and the quality factor \( Q[w] \) of an eigenvector \( w \) of the system operator \( A(\beta) \) with eigenvalue \( \zeta \) is

\[ U[w] = \frac{1}{2} (w, w), \quad W_{\text{dis}}[w] = (w, \beta B w), \quad Q[w] = |\text{Re } \zeta| \frac{(w, w)}{(w, \beta B w)}, \quad (90) \]
where $Q[w]$ is said to be finite if $W_{\text{dis}}[w] \neq 0$. In Appendix we show that
\[
\text{Im} \zeta = -\frac{(w, \beta Bw)}{(w, w)}, \quad W_{\text{dis}}[w] = -2 \text{Im} \zeta U[w], \quad Q[w] = -\frac{1}{2} \frac{|\text{Re} \zeta|}{\text{Im} \zeta},
\]
where $Q[w]$ is finite if and only if $\text{Im} \zeta \neq 0$.

### 4.1 The high-loss regime

We begin this section with our results on the perturbation analysis of the eigenvalues and eigenvectors for this operator $A(\beta)$ for the high-loss regime in which $\beta \gg 1$.

**Theorem 5 (eigenmodes dichotomy)** Let $\hat{\zeta}_j$, $1 \leq j \leq N_B$ be an indexing of all the nonzero eigenvalues of $B$ (counting multiplicities) where $N_B = \text{rank} B$. Then for the high-loss regime $\beta \gg 1$, the system operator $A(\beta)$ is diagonalizable and there exists a complete set of eigenvalues $\zeta_j(\beta)$ and eigenvectors $w_j(\beta)$ satisfying
\[
A(\beta) w_j(\beta) = \zeta_j(\beta) w_j(\beta), \quad 1 \leq j \leq N,
\]
which split into two distinct classes of eigenpairs
\[
\begin{align*}
\text{high-loss:} & \quad \zeta_j(\beta), \ w_j(\beta), \quad 1 \leq j \leq N_B; \\
\text{low-loss:} & \quad \zeta_j(\beta), \ w_j(\beta), \quad N_B + 1 \leq j \leq N,
\end{align*}
\]
having the following properties:

(i) The high-loss eigenvalues have poles at $\beta = \infty$ whereas their eigenvectors are analytic at $\beta = \infty$. These eigenpairs have the asymptotic expansions
\[
\zeta_j(\beta) = -i\hat{\zeta}_j \beta + O(\beta^{-1}), \quad \hat{\zeta}_j > 0, \quad \rho_j \in \mathbb{R},
\]
\[
w_j(\beta) = \hat{w}_j + O(\beta^{-1}), \quad 1 \leq j \leq N_B
\]
as $\beta \to \infty$. The vectors $\hat{w}_j$, $1 \leq j \leq N_B$ form an orthonormal basis of the loss subspace $H_B$ and
\[
B\hat{w}_j = \hat{\zeta}_j \hat{w}_j, \quad \rho_j = (\hat{w}_j, \Omega \hat{w}_j),
\]
for $1 \leq j \leq N_B$.

(ii) The low-loss eigenpairs are analytic at $\beta = \infty$ and have the asymptotic expansions
\[
\zeta_j(\beta) = \rho_j - id_j \beta^{-1} + O(\beta^{-2}), \quad \rho_j \in \mathbb{R}, \quad d_j \geq 0,
\]
\[
w_j(\beta) = \hat{w}_j + w_j^{(-1)} \beta^{-1} + O(\beta^{-2}), \quad N_B + 1 \leq j \leq N
\]
as $\beta \to \infty$. The vectors $\hat{w}_j$, $N_B + 1 \leq j \leq N$ form an orthonormal basis of the no-loss subspace $H_B^\perp$ and
\[
B\hat{w}_j = 0, \quad \rho_j = (\hat{w}_j, \Omega \hat{w}_j), \quad d_j = \left( w_j^{(-1)}, Bw_j^{(-1)} \right)
\]
for $N_B + 1 \leq j \leq N$. 

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Corollary 6 (eigenmode expulsion) The projections of the eigenvectors \( w_j (\beta) \), \( 1 \leq j \leq N \) onto the loss subspace \( H_B \) and the no-loss subspace \( H_B^\perp \) have the asymptotic expansions
\[
P_B w_j (\beta) = \hat{w}_j + O (\beta^{-1}) , \quad P_B^\perp w_j (\beta) = O (\beta^{-1}) , \quad 1 \leq j \leq N_B; \quad (98)
P_B w_j (\beta) = \hat{w}_j + O (\beta^{-1}) , \quad P_B w_j (\beta) = O (\beta^{-1}) , \quad N_B + 1 \leq j \leq N
\]
as \( \beta \to \infty \).

Proposition 7 (eigenfrequency expansions) The functions \( \text{Re} \zeta_j (\beta) \), \( 1 \leq j \leq N \) at \( \beta = \infty \) are analytic and their series expansions contain only even powers of \( \beta^{-1} \). The functions \( \text{Im} \zeta_j (\beta) \), \( 1 \leq j \leq N \) at \( \beta = \infty \) have poles for \( 1 \leq j \leq N_B \), are analytic for \( N_B + 1 \leq j \leq N \), and their series expansions contain only odd powers of \( \beta^{-1} \). Moreover, they have the asymptotic expansions
\[
\text{Re} \zeta_j (\beta) = \rho_j + O (\beta^{-2}) , \quad \text{Im} \zeta_j (\beta) = -\hat{\zeta}_j \beta + O (\beta^{-1}) , \quad 1 \leq j \leq N_B; \quad (99)
\text{Re} \zeta_j (\beta) = \rho_j + O (\beta^{-2}) , \quad \text{Im} \zeta_j (\beta) = -d_j \beta^{-1} + O (\beta^{-3}) , \quad N_B + 1 \leq j \leq N
\]
as \( \beta \to \infty \).

Proposition 8 For each \( j = 1, \ldots, N \) and in the high-loss regime \( \beta \gg 1 \), the following statements are true:

1. If \( 1 \leq j \leq N_B \) then \( \text{Im} \zeta_j (\beta) < 0 \).
2. If \( N_B + 1 \leq j \leq N \) then either \( \text{Im} \zeta_j (\beta) \equiv 0 \) or \( \text{Im} \zeta_j (\beta) < 0 \). Moreover, \( \text{Im} \zeta_j (\beta) \equiv 0 \) if and if \( \zeta_j (\beta) \equiv \rho_j \).
3. If \( N_B + 1 \leq j \leq N \) then \( \hat{w}_j \not\in \ker (\rho_j I - \Omega) \) if and only if \( d_j \neq 0 \).
4. If \( N_B + 1 \leq j \leq N \) then \( \hat{w}_j \not\in \ker \Omega \) if and only if \( \rho_j \neq 0 \) or \( d_j \neq 0 \).

Remark 9 Typically, one can expect that the asymptotic expansion of the low-loss eigenvalues have \( d_j \neq 0 \), for \( j = N_B + 1, \ldots, N \). Indeed, if this were not the case then Theorem 2 and the previous proposition tell us that the intersection of one of the eigenspaces of the operator \( \Omega \) with the kernel of the operator \( B \) would contain a nonzero vector. And this is obviously atypical behavior.

Remark 10 An important subspace which arises in studies of open systems in [Liu, pp. 27-28] as well as in [FigSha, Sec. 4.1] is
\[
\mathcal{O}_\Omega (\text{ran} B) = \text{Span} \{ \Omega^n B u : u \in H, \ n = 0, 1, \ldots \}, \quad (100)
\]
where it is called the orbit and is the smallest subspace of \( H \) containing \( \text{ran} B \) that is invariant under \( \Omega \). It is shown there that the orthogonal complement \( \mathcal{O}_\Omega (\text{ran} B)^\perp = H \ominus \mathcal{O}_\Omega (\text{ran} B) = \cap_{n \geq 0} \ker (B \Omega^n) \) is the subspace which is invariant with respect to the operator \( \Omega - iB \) and the restriction \( \Omega - iB |_{H_{\Omega}^\perp} \) is self-adjoint. Consequently, the evolution over this subspace in entirely decoupled from the operator \( B \) and there is no energy dissipation there. Moreover, the importance of the orbit to the perturbation analysis is that in applications it’s often not hard to see that
\[
\mathcal{O}_\Omega (\text{ran} B) = H. \quad (101)
\]
In this case, by Remark 9 we know that the asymptotic expansion of the low-loss eigenvalues have \( d_j \neq 0 \), for \( j = N_B + 1, \ldots, N \).
For computational purposes the next proposition and its corollary are important results. We first recall some notation. The orthogonal projection onto the loss subspace \( \ker B = H_B \) is \( P_B \) and \( P_B^\perp \) is the orthogonal projection onto the no-loss subspace \( \text{ran} B = H_B^\perp \). Then from (18) we will need the operators \( B_2 = P_B P_B^\perp |_{H_B} : H_B \to H_B, \) \( \Omega_1 = P_B^\perp \Omega P_B^\perp |_{H_B^\perp} : H_B^\perp \to H_B^\perp, \) and \( \Theta = P_B \Omega P_B^\perp |_{H_B} : H_B \to H_B^\perp \) whose adjoint is \( \Theta^* = P_B^\perp \Omega P_B |_{H_B^\perp} : H_B^\perp \to H_B \).

**Proposition 11 (asymptotic spectrum)** The following statements are true:

1. In the asymptotic expansions (94) for the high-loss eigenpairs, the coefficients \( \hat{\zeta}_j, \hat{w}_j, 1 \leq j \leq N_B \) form a complete set of eigenvalues and orthonormal eigenvectors for the operator \( B_2 \) with

\[
B_2 \hat{w}_j = \hat{\zeta}_j \hat{w}_j, \quad 1 \leq j \leq N_B. \tag{102}
\]

In particular, \( B_2 \) is a positive definite operator as is its inverse \( B_2^{-1} \), i.e.,

\[
B_2 > 0, \quad B_2^{-1} > 0. \tag{103}
\]

2. In the asymptotic expansions (96) for the low-loss eigenpairs, the coefficients \( \rho_j, \hat{w}_j, N_B + 1 \leq j \leq N \) form a complete set of eigenvalues and orthonormal eigenvectors for the self-adjoint operator \( \Omega_1 \) with

\[
\Omega_1 \hat{w}_j = \rho_j \hat{w}_j, \quad N_B + 1 \leq j \leq N. \tag{104}
\]

3. The coefficients \( d_j, N_B + 1 \leq j \leq N \) in the asymptotic expansions of the low-loss eigenvalues are given by the formulas

\[
d_j = (\hat{w}_j, \Theta^* B_2^{-1} \Theta \hat{w}_j), \quad N_B + 1 \leq j \leq N. \tag{105}
\]

**Corollary 12 (computing expansions)** The following statements give sufficient conditions that allow computation of \( \hat{w}_j, \hat{\zeta}_j, \rho_j, \) and \( d_j \) in the asymptotic expansions of the eigenpairs:

1. If the eigenvalues of \( \rho_j = \rho_j, 1 \leq j \leq N_B \) are distinct and \( \zeta_j, 1 \leq j \leq N_B \) is any indexing of these eigenvalues then in Theorem 8 after a possible reordering of the high-loss eigenpairs in (94), the coefficients in the asymptotic expansions (94) are uniquely determined by the relations

\[
\hat{\zeta}_j = \zeta_j, \quad B_2 \hat{w}_j = \hat{\zeta}_j \hat{w}_j, \quad ||\hat{w}_j|| = 1, \quad \rho_j = (\hat{w}_j, \Omega \hat{w}_j), \quad 1 \leq j \leq N_B. \]

2. If the eigenvalues of \( \rho_j = \rho_j, N_B + 1 \leq j \leq N \) are distinct and \( \zeta_j, N_B + 1 \leq j \leq N \) is any indexing of these eigenvalues then in Theorem 8 after a possible reordering of the low-loss eigenpairs in (96), the coefficients in the asymptotic expansions (96) are uniquely determined by the relations

\[
\rho_j = \zeta_j, \quad \Omega_1 \hat{w}_j = \rho_j \hat{w}_j, \quad ||\hat{w}_j|| = 1, \quad d_j = (\hat{w}_j, \Theta^* B_2^{-1} \Theta \hat{w}_j), \quad N_B + 1 \leq j \leq N. \]

The next two propositions give the asymptotic expansions as \( \beta \to \infty \) of the energy, power of energy dissipation, and quality factor for the high-loss and low-loss eigenvectors \( w_j(\beta), 1 \leq j \leq N \).
Proposition 13 (energy and dissipation) The energy for each of the high-loss and low-loss eigenvectors have the asymptotic expansions

\[ U[w_j(\beta)] = \frac{1}{2} + O(\beta^{-1}), \quad 1 \leq j \leq N \]  

as \( \beta \to \infty \). The power of energy dissipation for the high-loss and low-loss eigenvectors have the asymptotic expansions

\[ W_{\text{dis}}[w_j(\beta)] = \tilde{\zeta}_j \beta + O(1), \quad 1 \leq j \leq N_B; \]  

\[ W_{\text{dis}}[w_j(\beta)] = d_j \beta^{-1} + O(\beta^{-2}), \quad N_B + 1 \leq j \leq N \]  

as \( \beta \to \infty \). In particular,

\[ \lim_{\beta \to \infty} W_{\text{dis}}[w_j(\beta)] = \begin{cases} \infty & \text{if } 1 \leq j \leq N_B, \\ 0 & \text{if } N_B + 1 \leq j \leq N. \end{cases} \]  

Proposition 14 (quality factor) For each \( j = 1, \ldots, N \), the following statements are true regarding the quality factor of the high-loss and low-loss eigenvectors:

1. For \( \beta \gg 1 \), the quality factor \( Q[w_j(\beta)] \) is finite if and only if \( \text{Im} \zeta_j(\beta) \neq 0 \).

2. If the quality factor \( Q[w_j(\beta)] \) is finite for \( \beta \gg 1 \) then it is either analytic at \( \beta = \infty \) or has a pole, in either case its series expansion contains only odd powers of \( \beta^{-1} \) and, in particular,

\[ \lim_{\beta \to \infty} Q[w_j(\beta)] = 0 \text{ or } \infty. \]

3. The quality factor of each high-loss eigenvector is finite for \( \beta \gg 1 \) and has the asymptotic expansion

\[ Q[w_j(\beta)] = \frac{1}{2} \frac{|\rho_j|}{\zeta_j} \beta^{-1} + O(\beta^{-3}), \quad 1 \leq j \leq N_B \]  

as \( \beta \to \infty \). In particular,

\[ \lim_{\beta \to \infty} Q[w_j(\beta)] = 0, \quad 1 \leq j \leq N_B. \]

4. If \( j \in \{N_B + 1, \ldots, N\} \) and \( d_j \neq 0 \) then the quality factor of the low-loss eigenvector \( w_j(\beta) \) is finite for \( \beta \gg 1 \) and has the asymptotic expansion

\[ Q[w_j(\beta)] = \frac{1}{2} \frac{|\rho_j|}{d_j} \beta + O(\beta^{-1}) \]  

as \( \beta \to \infty \). In particular,

\[ \lim_{\beta \to \infty} Q[w_j(\beta)] = \begin{cases} \infty & \text{if } \rho_j \neq 0, \\ 0 & \text{if } \rho_j = 0. \end{cases} \]
4.2 The low-loss regime

We now give our results on the perturbation analysis of the eigenvalues and eigenvectors for the system operator $A(\beta)$ in the low-loss regime $0 \leq \beta \ll 1$. The focus of this paper is on the high-loss regime and so we do not try to give results as general as those in previous section. Instead, the goal of this section is to show the fundamentally different asymptotic behavior in the low-loss regime compared to that of the high-loss regime.

Theorem 15 (low-loss asymptotics) Let $\omega_j$, $1 \leq j \leq N$ be an indexing of all the eigenvalues of $\Omega$ (counting multiplicities). Then for $0 \leq \beta \ll 1$, the system operator $A(\beta) = \Omega - i\beta B$ is diagonalizable and there exists a complete set of eigenvalues $\zeta_j(\beta)$ and eigenvectors $v_j(\beta)$ of $A(\beta)$ satisfying

$$A(\beta) v_j(\beta) = \zeta_j(\beta) v_j(\beta), \quad 1 \leq j \leq N$$

with the following properties:

(i) The eigenvalues and eigenvectors are analytic at $\beta = 0$ and have the asymptotic expansions

$$\zeta_j(\beta) = \omega_j - i\sigma_j \beta + O(\beta^2), \quad \omega_j \in \mathbb{R}, \quad \sigma_j \geq 0, \quad 1 \leq j \leq N$$

$$v_j(\beta) = u_j + O(\beta), \quad 1 \leq j \leq N$$

as $\beta \to 0$. The vectors $u_j$, $1 \leq j \leq N$ form an orthonormal basis of eigenvectors of $\Omega$ and

$$\Omega u_j = \omega_j u_j, \quad \sigma_j = (u_j, Bu_j), \quad 1 \leq j \leq N.$$ 

Corollary 16 (energy and dissipation) The energy and power of energy dissipation of these eigenvectors have the asymptotic expansions

$$U[v_j(\beta)] = \frac{1}{2} + O(\beta^{-1}), \quad W_{\text{dis}}[v_j(\beta)] = \sigma_j \beta + O(\beta^2), \quad 1 \leq j \leq N$$

as $\beta \to 0$. In particular,

$$\lim_{\beta \to 0} W_{\text{dis}}[v_j(\beta)] = 0, \quad 1 \leq j \leq N.$$ 

Corollary 17 (quality factor) The quality factor of each of these eigenvectors has the asymptotic expansion

$$Q[v_j(\beta)] = \frac{1}{2} \left| \frac{\omega_j}{\sigma_j} \right| \beta^{-1} + O(\beta),$$

as $\beta \to 0$, provided $\sigma_j \neq 0$, in which case it has the limiting behavior

$$\lim_{\beta \to 0} Q[v_j(\beta)] = \begin{cases} \infty & \text{if } \omega_j \neq 0, \\ 0 & \text{if } \omega_j = 0. \end{cases}$$

Remark 18 Typically, one can expect that the asymptotic expansion of these eigenvalues have $\sigma_j \neq 0$, for $1 \leq j \leq N$. Indeed, if this were not the case then it would follow from the assumption $B \geq 0$ and (115) of Theorem 15 that the intersection of one of the eigenspaces of the operator $\Omega$ with the kernel of the operator $B$ would contain a nonzero vector. And this is obviously atypical behavior as mentioned previously in Remark 9.
These results show that in the low-loss regime $0 \leq \beta \ll 1$, all the modes behave as low-loss modes since the power of energy dissipation is small and typically the quality factor is very high. In contrast, the high-loss regime $\beta \gg 1$ has both a fraction $0 < \delta_B < 1$ of high-loss modes and a fraction $0 < 1 - \delta_B < 1$ of low-loss modes. The behaviour of the low-loss modes in either regime is similar whereas the behavior of the high-loss modes has the opposite behavior with power of energy dissipation large and quality factor always small.

5 Perturbation analysis of a system subjected to harmonic forces

In this section we give an asymptotic description of the stored energy, power of dissipated energy, and quality factor for a harmonic solution $\nu(t) = \nu e^{-i\omega t}$ of the system (8) in the high-loss regime $\beta \gg 1$ subjected to a harmonic external force $f(t) = fe^{-i\omega t}$ with nonzero amplitude $f \in H$ and frequency $\omega \in \mathbb{R}$. We will state our main results in this section but hold off on their proofs until Section 6.

To begin we recall that according to (46)–(48), assuming $\omega$ is not in the resolvent set of the system operator $A(\beta) = \Omega - i\beta B$, there is a unique harmonic solution $\nu(t) = \nu e^{-i\omega t}$ to the system (8) with the harmonic force $f(t) = fe^{-i\omega t}$ whose amplitude $\nu$ is given by

$$\nu = \mathfrak{A}(\omega) f = i [\omega I - (\Omega - i\beta B)]^{-1} f,$$

(120)

where $\mathfrak{A}(\omega)$ is the admittance operator.

As was introduced in Section 2.3, the stored energy $U$, power of dissipated energy $W_{\text{dis}}$, and quality factor $Q = Q_{f,\omega}$ associated with the harmonic external force $f(t) = fe^{-i\omega t}$ is given by the quantities

$$U = \frac{1}{2} (\nu, \nu), \quad W_{\text{dis}} = \beta (\nu, B\nu), \quad Q = |\omega| \frac{U}{W_{\text{dis}}} = |\omega| \frac{1}{\beta} (\nu, \nu),$$

(121)

where $Q$ is said to be finite if $W_{\text{dis}} \neq 0$.

For the results in the rest of this section we assume that $f$, $\omega$ are independent of the loss parameter $\beta$.

The techniques of analysis in the high-loss regime $\beta \gg 1$ differ significantly depending on whether the frequency $\omega$ is an asymptotic resonance frequency or not.

Definition 19 (nonresonance frequency) A real number $\omega$ is called an asymptotic nonresonance frequency of the system (8) provided $\omega \neq \rho_j$, $N_B + 1 \leq j \leq N$, otherwise it is an asymptotic resonance frequency.

The usage of this terminology is justified by the following proposition:

Proposition 20 Let $\omega \in \mathbb{R}$. Then the admittance operator $\mathfrak{A}(\omega)$ is analytic at $\beta = \infty$ if and only if $\omega$ is an asymptotic nonresonance frequency.

In this paper we will only consider the nonresonance frequencies.
5.1 Nonresonance frequencies

In this section we state the results of our analysis of losses for external harmonic forces with asymptotic nonresonance frequencies in the high-loss regime.

Recall from \([17]\) the Hilbert space \(H\) decomposes into the direct sum of orthogonal subspaces invariant with respect to the operator \(B \geq 0\), namely,

\[
H = H \oplus H_B^\perp
\]

where \(H_B = \text{ran} B\), \(H_B^\perp = \text{ker} B\) are the loss and no-loss subspaces with orthogonal projections \(P_B\), \(P_B^\perp\), respectively. It follows from this and the block representation of \(\Omega\) and \(B\) in \([18]\) that \(\xi I - A(\beta)\), with respect to this decomposition, is the \(2 \times 2\) block operator matrix in \([52]\), namely,

\[
\omega I - A(\beta) = \begin{bmatrix}
\Xi_2(\omega, \beta) & -\Theta \\
-\Theta^* & \Xi_1(\omega)
\end{bmatrix},
\]

\[
\Xi_2(\omega, \beta) := \omega I_2 - (\Omega_2 - i\beta B_2), \quad \Xi_1(\omega) := \omega I_1 - \Omega_1,
\]

where \(\Omega_1, \Omega_2,\) and \(B_2\) are self-adjoint operators, the latter of which has an inverse satisfying \(B_2^{-1} > 0\). With respect to this block representation, the Schur complement of \(\Xi_2(\omega, \beta)\) in \(\omega I - A(\beta)\) is the operator in \([53]\), namely,

\[
S_2(\omega, \beta) = \Xi_1(\omega) - \Theta^* \Xi_2(\omega, \beta)^{-1} \Theta,
\]

whenever \(\Xi_2(\omega, \beta)\) is invertible.

To simplify lengthy expressions we will often suppress the symbols \(\omega, \beta\) appearing as arguments in the operators \(\Xi_1(\omega), \Xi_2(\omega, \beta), S_2(\omega, \beta)\). We now give the main results of this section for an asymptotic nonresonance frequency \(\omega\).

**Proposition 21 (admittance asymptotics)** For \(\beta \gg 1\), each of the operators \(\Xi_1(\omega), \Xi_2(\omega, \beta), S_2(\omega, \beta)\), and \(\omega I - A(\beta)\) are invertible and the admittance operator \(\mathfrak{A}(\omega) = i(\omega I - A(\beta))^{-1}\) is given by the formula

\[
\mathfrak{A}(\omega) = i \begin{bmatrix}
I_2 & \Xi_2^{-1} \Theta \\
0 & I_1
\end{bmatrix} \begin{bmatrix}
\Xi_2^{-1} & 0 \\
0 & S_2^{-1}
\end{bmatrix} \begin{bmatrix}
I_2 & 0 \\
\Theta^* \Xi_2^{-1} & I_1
\end{bmatrix}
\]

\[
= i \begin{bmatrix}
\Xi_2^{-1} + \Xi_2^{-1} \Theta S_2^{-1} \Theta^* \Xi_2^{-1} & \Xi_2^{-1} \Theta S_2^{-1} \\
S_2^{-1} \Theta^* \Xi_2^{-1} & S_2^{-1}
\end{bmatrix}.
\]

Moreover, \(\mathfrak{A}(\omega)\) is analytic at \(\beta = \infty\) and has the asymptotic expansion

\[
\mathfrak{A}(\omega) = \begin{bmatrix}
0 & 0 \\
0 & i\Xi_1^{-1}
\end{bmatrix} + W^{(-1)} \beta^{-1} + O(\beta^{-2}),
\]

as \(\beta \to \infty\), where

\[
W^{(-1)} = \begin{bmatrix}
B_2^{-1} & B_2^{-1} \Theta \Xi_1^{-1} \\
(\Xi_1^{-1})^* \Theta^* B_2^{-1} & (\Xi_1^{-1})^* \Theta^* B_2^{-1} \Theta \Xi_1^{-1}
\end{bmatrix}
\]

\[
= \begin{bmatrix}
I_2 & 0 \\
(\Xi_1^{-1})^* \Theta^* & I_1
\end{bmatrix} \begin{bmatrix}
B_2^{-1} & 0 \\
0 & 0
\end{bmatrix} \begin{bmatrix}
I_2 & \Theta \Xi_1^{-1} \\
0 & I_1
\end{bmatrix}.
\]

In particular, this is a positive semidefinite operator, i.e.,

\[
W^{(-1)} \geq 0.
\]
Corollary 22 The operators $\mathcal{A}(\omega)^* \mathcal{A}(\omega)$, $P_B \mathcal{A}(\omega)^* \mathcal{A}(\omega) P_B$, and $\mathcal{A}(\omega)^* \beta B \mathcal{A}(\omega)$ are analytic at $\beta = \infty$. Moreover, they have the asymptotic expansions

$$\mathcal{A}(\omega)^* \mathcal{A}(\omega) = \begin{bmatrix} 0 & 0 \\ 0 & (\Xi_1^{-1})^* \Xi_1^{-1} \end{bmatrix}$$

$$P_B \mathcal{A}(\omega)^* \mathcal{A}(\omega) P_B = \begin{bmatrix} B_2^{-2} + B_2^{-4} \Theta (\Xi_1^{-1})^* \Xi_1^{-1} \Theta^* B_2^{-1} & 0 \\ 0 & 0 \end{bmatrix} \beta^{-2} + O(\beta^{-3}),$$

$$\mathcal{A}(\omega)^* \beta B \mathcal{A}(\omega) = W(-1) \beta^{-1} + O(\beta^{-2})$$

as $\beta \to \infty$.

The following statements give our main results regarding the stored energy $U$, power of dissipated energy $W_{\text{dis}}$, and quality factor $Q = Q_{f,\omega}$ associated a harmonic external force $f(t) = fe^{-i \omega t}$, where $f$, $\omega$ are independent of $\beta$, $f \neq 0$, and $\omega$ an asymptotic nonresonance frequency of the system $[8]$. As we shall see the behaviour in the high-loss regime $\beta \gg 1$ of these quantities is drastically different depending on whether the amplitude $f$ has a component in the no-loss subspace $H^1_B$ or not, i.e., $P_B^\perp f = 0$ or $P_B f = 0$.

Theorem 23 (quality factor) If $P_B^\perp f = 0$ then the stored energy $U$, power of dissipated energy $W_{\text{dis}}$, and quality factor $Q$ are analytic at $\beta = \infty$ and have the asymptotic expansions

$$U = \frac{1}{2} \left( f, B_2^{-2} + B_2^{-4} \Theta (\Xi_1(\omega)^{-1})^* \Xi_1(\omega)^{-1} \Theta^* B_2^{-1} \right) f \beta^{-2} + O(\beta^{-3}),$$

$$W_{\text{dis}} = (f, B_2^{-1} f) \beta^{-1} + O(\beta^{-2}),$$

$$Q = |\omega| \frac{1}{2} \left( f, B_2^{-2} + B_2^{-4} \Theta (\Xi_1(\omega)^{-1})^* \Xi_1(\omega)^{-1} \Theta^* B_2^{-1} \right) f \beta^{-1} + O(\beta^{-2})$$

as $\beta \to \infty$. Moreover, the leading order terms of $U$ and $W_{\text{dis}}$ in these expansions are positive numbers, similarly for $Q$ provided $\omega \neq 0$, and satisfy the inequalities

$$\frac{1}{2} \left( f, B_2^{-2} + B_2^{-4} \Theta (\Xi_1(\omega)^{-1})^* \Xi_1(\omega)^{-1} \Theta^* B_2^{-1} \right) f \geq \frac{1}{2} \left( \sup_{1 \leq j \leq N_B} \left\{ \hat{\zeta}_j \right\} \right)^{-1} (f, B_2^{-1} f) \geq \frac{1}{2} \left( \sup_{1 \leq j \leq N_B} \left\{ \hat{\zeta}_j \right\} \right)^{-2} (f, f) > 0.$$

Theorem 24 (quality factor) If $P_B^\perp f \neq 0$ then the stored energy $U$ and power of dissipated energy $W_{\text{dis}}$ are analytic at $\beta = \infty$ and have the asymptotic expansions

$$U = \frac{1}{2} \left( \Xi_1(\omega)^{-1} P_B^\perp f, \Xi_1(\omega)^{-1} P_B^\perp f \right) + O(\beta^{-1})$$

$$W_{\text{dis}} = (f, W(-1) f) \beta^{-1} + O(\beta^{-2})$$

as $\beta \to \infty$. In particular, the leading order terms in the expansions of $U$ and $W_{\text{dis}}$ are positive and nonnegative numbers, respectively.
Moreover, if \( \text{W}_{\text{dis}} \neq 0 \) for \( \beta \gg 1 \) then the quality factor \( Q \) has a pole at \( \beta = \infty \) provided \( \omega \neq 0 \). In particular, if \( f \notin \text{ker} \text{W}^{-1} \) then it has the asymptotic expansion

\[
Q = |\omega| \frac{1}{2} \frac{(\Xi_1(\omega)^{-1} P_B^f, \Xi_1(\omega)^{-1} P_B^f)}{(f, \text{W}^{-1}(f))} \beta + O(1)
\]

as \( \beta \to \infty \), whose leading order term is a positive number provided \( \omega \neq 0 \).

**Corollary 25** The operator \( \text{W}^{-1} \) defined in (127) has the \( N - N_B \) dimensional kernel

\[
\text{ker} \text{W}^{-1} = \text{ker} \left( P_B + \Theta \Xi_1(\omega)^{-1} P_B^+ \right) = \{ f_1 + f_2 \in H : f_1 \in H_B^\perp \text{ and } f_2 = -\Theta \Xi_1(\omega)^{-1} f_1 \}.
\]

**Corollary 26 (quality factor)** The stored energy \( U \), power of dissipated energy \( \text{W}_{\text{dis}} \), and quality factor \( Q \) have the following limits as \( \beta \to \infty \):

\[
\lim_{\beta \to \infty} U = \begin{cases} \frac{1}{2} (\Xi_1(\omega)^{-1} P_B^f, \Xi_1(\omega)^{-1} P_B^f) > 0 & \text{if } P_B^f \neq 0, \\ 0 & \text{if } P_B^f = 0, \end{cases}
\]

\[
\lim_{\beta \to \infty} \text{W}_{\text{dis}} = 0,
\]

\[
\lim_{\beta \to \infty} Q = \begin{cases} \infty & \text{if } P_B^f \neq 0 \text{ and } \omega \neq 0, \\ 0 & \text{if } P_B^f = 0 \text{ or } \omega = 0, \end{cases}
\]

where we assume in the statement regarding quality factor for the case \( P_B^f \neq 0 \) that \( \text{W}_{\text{dis}} \neq 0 \) for \( \beta \gg 1 \). In particular, if \( P_B^f = 0 \) or \( f \notin \text{ker} \text{W}^{-1} \) then \( Q \) is finite for \( \beta \gg 1 \) and we have the above limits for \( U, \text{W}_{\text{dis}}, \text{and Q as } \beta \to \infty \).

**6 Proof of results**

This section contains the proofs of the results of this paper. We split these proofs into two subsections. In the first subsection we prove the statements given in Section 4 on the spectral perturbation analysis in the high-loss and low-loss regime for the system operator. In the second subsection we prove the statements in Section 5 on the perturbation analysis of losses for external harmonic forces. All assumptions, notation, and convention used here will adhere to that previously introduced in those two sections.

**6.1 Perturbation analysis of the system operator**

This purpose of this section is to prove the results given in Section 4. We do so by considering in separate subsections the high-loss regime \( \beta \gg 1 \) and the low-loss regime \( 0 \leq \beta \ll 1 \).

**6.1.1 The high-loss regime**

**Proof of Theorem 5.** Let \( \hat{\zeta}_j, 1 \leq j \leq N_B \) be an indexing of all the nonzero eigenvalues of \( B \) (counting multiplicities) where \( N_B = \text{rank} B \). These eigenvalues are all positive real numbers since by assumption \( B \geq 0 \).

We begin by extending the domain of the system operator by \( A(\beta) = \Omega - i\beta B, \beta \in \mathbb{C} \). Recall by our assumption \( \Omega \) is self-adjoint. Hence the operator \( (-i\beta)^{-1} A(\beta) = B + (-i\beta)^{-1} \Omega \)
is analytic in $\beta^{-1}$ in a complex neighborhood of $\beta = \infty$ and if we use the substitution $\varepsilon = (-i\beta)^{-1}$ then the operator $\varepsilon A (i\varepsilon^{-1}) = B + \varepsilon \Omega$ is an analytic operator which is self-adjoint for real $\varepsilon$. Thus by a theorem of Rellich [Ban85, p. 21, Theorem 1] we know for $\varepsilon \in \mathbb{C}$ with $|\varepsilon| \ll 1$, the operator $B + \varepsilon \Omega$ is diagonalizable and there exists a complete set of analytic eigenvalues $\lambda_j (\varepsilon)$ and eigenvectors $x_j (\varepsilon)$ satisfying

\begin{align*}
(B + \varepsilon \Omega) x_j (\varepsilon) &= \lambda_j (\varepsilon) x_j (\varepsilon), \quad \lambda_j (\varepsilon) = \lambda_j (\varepsilon), \quad 1 \leq j \leq N; \\
(x_j (\varepsilon), x_k (\varepsilon)) &= \delta_{jk}, \quad \text{for } \varepsilon \in \mathbb{R}, \quad 1 \leq j, k \leq N,
\end{align*}

where $\delta_{jk}$ denotes the Kronecker delta symbol. In particular, the vectors $\tilde{w}_j := x_j (0)$, $j = 1, \ldots, N$ form an orthonormal basis of eigenvectors for the operator $B$. Thus, after a possible reindexing of these analytic eigenpairs, we may assume without loss of generality that

\begin{align*}
B \tilde{w}_j &= \zeta_j \tilde{w}_j, \quad \lambda_j (0) = \zeta_j > 0, \quad 1 \leq j \leq N_B; \\
B \tilde{w}_j &= 0, \quad \lambda_j (0) = 0, \quad N_B + 1 \leq j \leq N.
\end{align*}

Denote the derivatives of these eigenvalues at $\varepsilon = 0$ by $\rho_j := \lambda'_j (0)$, $1 \leq j \leq N$. Then it follows that they satisfy

$$\rho_j = (\tilde{w}_j, \Omega \tilde{w}_j) \in \mathbb{R}, \quad 1 \leq j \leq N.$$

Indeed, for $\varepsilon$ real with $|\varepsilon| \ll 1$ by (139) we have

$$0 = (x_j (\varepsilon), (B + \varepsilon \Omega - \lambda_j (\varepsilon) I) x_j (\varepsilon)), \quad 1 \leq j \leq N$$

and the result follows immediately by taking the derivative on both sides and evaluating at $\varepsilon = 0$.

Now from these facts and recalling the substitution $\varepsilon = (-i\beta)^{-1}$ that was made, we conclude that if $\beta \in \mathbb{C}$ with $|\beta| \gg 1$ then the operator $A (\beta) = (i\beta)(B + (-i\beta)^{-1} \Omega)$ is diagonalizable with a complete set of eigenvalues $\zeta_j (\beta)$ and eigenvectors $w_j (\beta)$ satisfying

\begin{align*}
\zeta_j (\beta) &= (-i\beta) \lambda_j ((-i\beta)^{-1}), \quad w_j (\beta) = x_j ((-i\beta)^{-1}), \\
A (\beta) w_j (\beta) &= \zeta_j (\beta) w_j (\beta), \quad 1 \leq j \leq N.
\end{align*}

We will now show that the eigenpairs

- high-loss: $\zeta_j (\beta), w_j (\beta), \quad 1 \leq j \leq N_B$;
- low-loss: $\zeta_j (\beta), w_j (\beta), \quad N_B + 1 \leq j \leq N$,

have the properties described in Theorem 5.

We start with the high-loss eigenpairs. First, it follows from our results above that the eigenpairs $\lambda_j (\varepsilon), x_j (\varepsilon), 1 \leq j \leq N_B$ are analytic for $\varepsilon \in \mathbb{C}$ with $|\varepsilon| \ll 1$ and have the asymptotic expansions

$$\lambda_j (\varepsilon) = \zeta_j + \rho_j \varepsilon + O (\varepsilon^2), \quad \zeta_j > 0, \quad \rho_j \in \mathbb{R},$$

$$x_j (\varepsilon) = \tilde{w}_j + O (\varepsilon), \quad 1 \leq j \leq N_B$$

as $\varepsilon \to 0$. This implies by (141) the high-loss eigenvalues have poles at $\beta = \infty$ whereas their eigenvectors are analytic at $\beta = \infty$ and they have the asymptotic expansions

\begin{align*}
\zeta_j (\beta) &= (-i\beta) \lambda_j ((-i\beta)^{-1}) = -i\beta \zeta_j + \rho_j + O (\beta^{-1}), \quad \zeta_j > 0, \quad \rho_j \in \mathbb{R}, \\
w_j (\beta) &= x_j ((-i\beta)^{-1}) = \tilde{w}_j + O (\beta^{-1}), \quad 1 \leq j \leq N_B
\end{align*}
as $\beta \to \infty$, where the vectors $\hat{w}_j, 1 \leq j \leq N_B$ form an orthonormal basis of the loss subspace $H_B = \text{ran} B$ and

$$B\hat{w}_j = \zeta_j \hat{w}_j, \quad \rho_j = (\hat{w}_j, \Omega \hat{w}_j), \quad 1 \leq j \leq N_B.$$ 

This proves statement (i) of this theorem.

We now consider the low-loss eigenpairs. By the results above we know the eigenpairs $\lambda_j(\varepsilon), x_j(\varepsilon), N_B + 1 \leq j \leq N$ are analytic for $\varepsilon \in \mathbb{C}$ with $|\varepsilon| \ll 1$ and have the asymptotic expansions

$$\lambda_j(\varepsilon) = \rho_j \varepsilon + \frac{1}{2} \lambda_j''(0) \varepsilon^2 + O(\varepsilon^3),$$

$$x_j(\varepsilon) = \hat{w}_j + x_j'(0) \varepsilon + O(\varepsilon), \quad N_B + 1 \leq j \leq N$$

as $\varepsilon \to 0$. Moreover, there is an explicit formula for the second derivative of these eigenvalues

$$\lambda_j''(0) = 2 \left( x_j'(0), B x_j'(0) \right) \geq 0, \quad N_B + 1 \leq j \leq N.$$ 

Indeed, this follows from (139) and (140) since

$$\rho_j \varepsilon + \frac{1}{2} \lambda_j''(0) \varepsilon^2 + O(\varepsilon^3) = \lambda_j(\varepsilon) = (x_j(\varepsilon), \Omega x_j(\varepsilon)) + (x_j(\varepsilon), B x_j(\varepsilon))$$

$$= (\hat{w}_j, \Omega \hat{w}_j) \varepsilon + (x_j'(0), B x_j'(0)) \varepsilon^2 + O(\varepsilon^3)$$

for real $\varepsilon$ as $\varepsilon \to 0$. Thus we can conclude from this and (141) that the low-loss eigenpairs are analytic at $\beta = \infty$ and they have the asymptotic expansions

$$\zeta_j(\beta) = (i \beta) \lambda_j \left( (i \beta)^{-1} \right) = \rho_j - i d_j \beta^{-1} + O(\beta^{-2}), \quad \rho_j \in \mathbb{R}, \quad d_j \geq 0$$

$$w_j(\beta) = x_j \left( (i \beta)^{-1} \right) = \hat{w}_j + w_j^{(1)} \beta^{-1} + O(\beta^{-2}), \quad N_B + 1 \leq j \leq N$$

as $\beta \to \infty$, where the vectors $\hat{w}_j, N_B + 1 \leq j \leq N$ form an orthonormal basis of the no-loss subspace $H_B^\perp = \text{ker} B$ and

$$B\hat{w}_j = 0, \quad \rho_j = (\hat{w}_j, \Omega \hat{w}_j), \quad w_j^{(1)} = i x_j'(0),$$

$$-i d_j = -i \frac{1}{2} \lambda_j''(0) = -i \left( x_j'(0), B x_j'(0) \right) = -i \left( w_j^{(1)}, B w_j^{(1)} \right)$$

for $N_B + 1 \leq j \leq N$. This completes the proof. ■

**Proof of Corollary 6.** Let $w_j(\beta), 1 \leq j \leq N_B$ and $w_j(\beta), N_B + 1 \leq j \leq N$ denote the high-loss and low-loss eigenvectors, respectively, given in the previous theorem. Then by our results we know that the zeroth order terms in their asymptotic expansions must satisfy $P_B \hat{w}_j = \hat{w}_j, P_B^\perp \hat{w}_j = 0$ for $1 \leq j \leq N_B$ and $P_B \hat{w}_j = 0, P_B^\perp \hat{w}_j = \hat{w}_j$ for $N_B + 1 \leq j \leq N$ since $H = H_B \oplus H_B^\perp$ and $P_B, P_B^\perp$ are the orthogonal projections onto $H_B, H_B^\perp$, respectively. The proof of this corollary now follows. ■

**Proof of Proposition 7.** Let $\zeta_j(\beta), 1 \leq j \leq N$ be the high-loss and low-loss eigenvalues given in Theorem 5. Then as described in its proof we can extend the domain of the system operator $A(\beta) = \Omega - i \beta B, \beta \in \mathbb{C}$ and these eigenvalues can be extended uniquely to meromorphic functions in a neighborhood of $\beta = \infty$ whose values are eigenvalues of $A(\beta)$. Moreover, using the same notation to denote their extensions, it follows from (139) and (141) that these functions satisfy

$$\overline{\zeta_j(\beta)} = \zeta_j(-\beta), \quad 1 \leq j \leq N.$$
for all $\beta \in \mathbb{C}$ with $|\beta| \gg 1$. This implies $\frac{1}{2} (\zeta_j (\beta) + \zeta_j (-\beta))$ and $\frac{1}{2} (\zeta_j (\beta) - \zeta_j (-\beta))$ are even and odd functions, respectively, and meromorphic at $\beta = \infty$ and equal $\text{Re} \zeta_j (\beta)$ and $\text{Im} \zeta_j (\beta)$, respectively, for real $\beta$. In particular, this implies their Laurent series in $\beta^{-1}$ have only even and odd powers, respectively. The rest of the proof of this proposition now follows immediately by considering the real and imaginary part of the asymptotic expansions of the high-loss and low-loss eigenvalues given in Theorem 5.

**Proof of Proposition 8.** 1. & 2. Here we will prove just the first two statements of the proposition. We come back to the proof of the third and fourth statements after we have proved Proposition 11.

Let $j \in \{1, \ldots, N\}$ and $\beta \gg 1$. Well, since $\zeta_j (\beta), w_j (\beta)$ is an eigenpair of $A (\beta)$ then it follows from (91) and the fact $B \geq 0$ that $\text{Im} \zeta_j (\beta) \leq 0$. From this and Proposition 7 since $\text{Im} \zeta_j (\beta)$ either as a pole or is analytic at $\beta = \infty$, then either $\text{Im} \zeta_j (\beta) \equiv 0$ or $\text{Im} \zeta_j (\beta) \leq 0$. If $\text{Im} \zeta_j (\beta) \equiv 0$ then it follows by (91) that $Bw_j (\beta) \equiv 0$ and hence $\zeta_j (\beta) w_j (\beta) \equiv A (\beta) w_j (\beta) \equiv \Omega w_j (\beta)$ which implies that $\zeta_j (\beta) \equiv \rho_j$. Furthermore, if $1 \leq j \leq N_B$ then we know that $\text{Im} \zeta_j (\beta) = -\zeta_j \beta \mp O (\beta^{-1})$ as $\beta \to \infty$ with $\zeta_j > 0$ and so $\text{Im} \zeta_j (\beta) < 0$ for $\beta \gg 1$. This completes the proof of the first two statements.

**Proof of Proposition 11.** We begin by proving the first statement of this proposition. Recall that $B_2 = P_BBP_B \big|_{H_B} : H_B \to H_B$ where $P_B$ is the orthogonal projection onto $H_B$. By Theorem 5 it follows that $\zeta_j, \check{w}_j, 1 \leq j \leq N_B$ are a complete set of eigenvalues and eigenvectors for $B_2$ with these eigenvectors forming an orthonormal basis for $H_B$. As these eigenvalues are all positive this implies $B_2 > 0$ and, in particular, it is invertible and its inverse satisfies $B_2^{-1} > 0$. This completes the proof of the first statement.

Next, we prove the second statement of this proposition. Well, by Theorem 5 we know that the vectors $\check{w}_j, N_B + 1 \leq j \leq N$ form an orthonormal basis for the no-loss subspace $H_\beta$. By definition $P_B$ is the orthogonal projection onto $H_\beta$ implying $P_B \check{w}_j = \check{w}_j$, for $j = N_B + 1, \ldots, N$. And hence, since $\Omega_1 = P_B \Omega P_B \big|_{H_\beta} : H_\beta \to H_\beta$ which is obviously a self-adjoint operator because $P_B$ and $\Omega$ are, we have $\Omega_1 \check{w}_j = P_B \Omega \check{w}_j, j = N_B + 1, \ldots, N$. Therefore this, the fact $P_B B = 0$, and Theorem 5 imply

$$
\Omega_1 \check{w}_j = P_B \Omega \check{w}_j = \lim_{\beta \to \infty} P_B (\Omega - \beta B) \check{w}_j (\beta) = \lim_{\beta \to \infty} P_B \zeta_j (\beta) \check{w}_j (\beta) = \rho_j \check{w}_j
$$

for $j = N_B + 1, \ldots, N$. This completes the proof of the second statement.

Finally, we prove the third and final statement of this proposition. Recall the operator $\Theta = P_B \Omega P_B \big|_{H_B} : H_B \to H_B$. We now show that

$$
\Theta \check{w}_j = iB_2 P_B \check{w}_j^{(-1)}, \quad N_B + 1 \leq j \leq N.
$$

(142)

Well, by (92), (96), and (97) of Theorem 5 it follows that if $N_B + 1 \leq j \leq N$ then

$$
\rho_j \check{w}_j + O (\beta^{-1}) = \zeta_j (\beta) \check{w}_j (\beta) = A (\beta) \check{w}_j (\beta) = (\Omega - \beta B) \check{w}_j (\beta) + \beta^{-1} (\Omega - \beta B) \check{w}_j^{(-1)} + O (\beta^{-1}) = \Omega \check{w}_j - iB \check{w}_j^{(-1)} + O (\beta^{-1})
$$

as $\beta \to \infty$. Equating the zeroth order terms we conclude that

$$
\rho_j \check{w}_j = \Omega \check{w}_j - iB \check{w}_j^{(-1)}.
$$

Applying $P_B$ to both sides of this equation we find that

$$
0 = P_B \check{w}_j - iP_B B \check{w}_j^{(-1)} = (P_B \Omega P_B) \check{w}_j - i (P_B B P_B) \check{w}_j^{(-1)} = \Theta \check{w}_j - iB_2 P_B \check{w}_j^{(-1)}
$$
which proves the identity \((142)\).

Therefore by \((97)\) of Theorem \(5\) \((142)\), the facts \(P_B\) is the orthogonal projection onto 
\(H_B\), \(P_BBP_B = B\), \(B_2 = P_BBP_B|_{H_B}\), and since \(B_2^{-1}\) is self-adjoint we conclude that

\[
- id_j = -i \left( w_j^{(-1)}, B w_j^{(-1)} \right) = \left( P_B w_j^{(-1)}, iB_2 P_B w_j^{(-1)} \right) = \left( P_B w_j^{(-1)}, \Theta w_j \right)
\]

\[
= - \left( B_2^{-1} B_2 P_B w_j^{(-1)}, \Theta w_j \right) = -i \left( iB_2 P_B w_j^{(-1)}, B_2^{-1} \Theta w_j \right) = -i \left( \Theta w_j, B_2^{-1} \Theta w_j \right)
\]

This proves the final statement and hence the proof of the proposition is complete. ■

**Proof of Proposition \(8\)**: 3. & 4. We now complete the proof of Proposition \(8\) by proving the last two statements. Let \(j \in \{1, \ldots, N\}\). We begin by proving the third statement. Suppose that \(d_j = 0\). Then by Proposition \(11\) we know that \(0 = -i \left( \hat{w}_j, \Theta^* B_2^{-1} \Theta w_j \right)\) and 
\(B_2^{-1} > 0\). This implies \(\Theta \hat{w}_j = 0\). By Theorem \(5\) we have \(\hat{w}_j \in H_B^1\) so that \(P_B^1 \hat{w}_j = \hat{w}_j\) and hence

\[
0 = \Theta \hat{w}_j = P_B \Omega P_B^1 \hat{w}_j = P_B \Omega \hat{w}_j.
\]

From this, the fact \(P_B + P_B^1 = I\), and by the first statement in Proposition \(11\) it follows that

\[
\Omega \hat{w}_j = P_B^1 \Omega \hat{w}_j = P_B^1 \Omega P_B^1 \hat{w}_j = \Omega_1 \hat{w}_j = \rho_j \hat{w}_j.
\]

Thus we have shown if \(d_j = 0\) then \(\hat{w}_j \in \ker (\rho_j I - \Omega)\). We now prove the converse. Suppose \(\hat{w}_j \in \ker (\rho_j I - \Omega)\). This hypothesis and the fact \(\hat{w}_j \in H_B^1\) imply

\[
\Theta \hat{w}_j = P_B \Omega P_B^1 \hat{w}_j = P_B \Omega \hat{w}_j = \rho_j P_B \hat{w}_j = 0.
\]

Therefore by Proposition \(11\) we conclude \(d_j = \left( \hat{w}_j, \Theta^* B_2^{-1} \Theta \hat{w}_j \right) = 0\). This proves the third statement of Proposition \(8\).

Finally, we will complete the proof of Proposition \(8\) by proving the fourth statement. Suppose \(\hat{w}_j \notin \ker (\Omega)\) but \(\rho_j = 0\) and \(d_j = 0\). Then it would follow from the third statement of this proposition that \(\hat{w}_j \in \ker (\Omega)\), a contradiction. Thus we have shown if \(\hat{w}_j \notin \ker (\Omega)\) then \(\rho_j \neq 0\) or \(d_j \neq 0\). We now prove the converse. Suppose \(\rho_j \neq 0\) or \(d_j \neq 0\). If \(\hat{w}_j \in \ker (\Omega)\) then

\[
\rho_j \hat{w}_j = \Omega_1 \hat{w}_j = P_B^1 \Omega \hat{w}_j = 0.
\]

This implies \(\rho_j = 0\) and hence \(\hat{w}_j \in \ker (\rho_j I - \Omega)\). By the previous statement in this proposition this implies \(d_j = 0\). This yields a contradiction of our hypothesis. Therefore we have shown if \(\rho_j \neq 0\) or \(d_j \neq 0\) then \(\hat{w}_j \notin \ker (\Omega)\). This proves the fourth statement and hence completes the proof of this proposition. ■

**Proof of Corollary \(12\)** The proof of this corollary is straightforward and follows from Theorem \(5\) and Proposition \(8\). ■

**Proof of Proposition \(13\)** For high-loss and low-loss eigenvectors \(w_j (\beta), 1 \leq j \leq N\) we have \((w_j (\beta), w_j (\beta)) = 1 + O (\beta^{-1})\) as \(\beta \to \infty\). From this it follows that the energy has the asymptotic expansions

\[
U [w_j (\beta)] = \frac{1}{2} (w_j (\beta), w_j (\beta)) = \frac{1}{2} + O (\beta^{-1}), \quad 1 \leq j \leq N
\]

as \(\beta \to \infty\). Now by Proposition \(17\) we know the imaginary parts of high-loss and low-loss eigenvalues \(\zeta_j (\beta), 1 \leq j \leq N\) have the asymptotic expansions

\[
\text{Im } \zeta_j (\beta) = -\bar{\zeta}_j \beta + O (\beta^{-1}), \quad \bar{\zeta}_j > 0, \quad 1 \leq j \leq N_B;
\]

\[
\text{Im } \zeta_j (\beta) = -d_j \beta^{-1} + O (\beta^{-3}), \quad d_j \geq 0, \quad N_B + 1 \leq j \leq N
\]
as $\beta \to \infty$. Thus by these asymptotic expansions we conclude the power of energy dissipation of these eigenvectors by the formula (91) have the asymptotic expansions

$$W_{\text{dis}}[w_j(\beta)] = -2\text{Im} \, \zeta_j(\beta) U [w_j(\beta)] = \begin{cases} \hat{\zeta}_j \beta + O(1) & \text{if } 1 \leq j \leq N_B, \\ d_j \beta^{-1} + O(\beta^{-2}) & \text{if } N_B + 1 \leq j \leq N \end{cases}$$

as $\beta \to \infty$. The proof of the proposition now follows from these asymptotics. ■

**Proof of Proposition 14.** By the formula (91) for the quality factor of an eigenvector of the system operator and by Proposition 7 it follows for $\beta \gg 1$ we have $Q[w_j(\beta)]$ is finite if and only if $\text{Im} \, \zeta_j(\beta) \neq 0$ in which case

$$Q[w_j(\beta)] = \frac{-1}{2} \frac{|\text{Re} \, \zeta_j(\beta)|}{\text{Im} \, \zeta_j(\beta)}. $$

By Proposition 7 we know that $\text{Re} \, \zeta_j(\beta)$ and $\text{Im} \, \zeta_j(\beta)$ are either analytic or have poles at $\beta = \infty$ whose Laurent series expansions contain only even and odd powers of $\beta^{-1}$, respectively. In particular, this implies either $\text{Re} \, \zeta_j(\beta) \equiv 0$ or the sign of $\text{Re} \, \zeta_j(\beta)$ does not change for $\beta \gg 1$ and $|\text{Re} \, \zeta_j(\beta)| = \text{Re} \, \zeta_j(\beta)$, for $\beta \gg 1$ if this sign is positive or if this sign is negative $|\text{Re} \, \zeta_j(\beta)| = -\text{Re} \, \zeta_j(\beta)$, for $\beta \gg 1$. It follows from this and Proposition 7 that we have the asymptotic expansions

$$|\text{Re} \, \zeta_j(\beta)| = |\rho_j| + O(\beta^{-2}), \quad \text{Im} \, \zeta_j(\beta) = -\hat{\zeta}_j \beta + O(\beta^{-1}), \quad 1 \leq j \leq N_B;$$

$$|\text{Re} \, \zeta_j(\beta)| = |\rho_j| + O(\beta^{-2}), \quad \text{Im} \, \zeta_j(\beta) = -d_j \beta^{-1} + O(\beta^{-3}), \quad N_B + 1 \leq j \leq N$$

as $\beta \to \infty$. It also follows that if $\text{Im} \, \zeta_j(\beta) \neq 0$ for $\beta \gg 1$ then $Q[w_j(\beta)]$ is the product of two meromorphic functions one of which is even and one of which is odd in a neighborhood of $\beta = \infty$ which implies it has a Laurent series expansion at $\beta = \infty$ containing only odd powers of $\beta^{-1}$ and, in particular, $\lim_{\beta \to \infty} Q[w_j(\beta)] = 0$ or $\infty$. These facts and the fact $\hat{\zeta}_j > 0$ for $1 \leq j \leq N_B$ implies we have the asymptotic expansions

$$Q[w_j(\beta)] = \frac{-1}{2} \frac{|\text{Re} \, \zeta_j(\beta)|}{\text{Im} \, \zeta_j(\beta)} = \begin{cases} \frac{|\rho_j|}{\zeta_j} \beta^{-1} + O(\beta^{-3}) & \text{if } 1 \leq j \leq N_B, \\ \frac{|\rho_j|}{d_j} - \beta + O(\beta^{-1}) & \text{if } d_j \neq 0 \end{cases}$$

as $\beta \to \infty$. The proof of this propositions now follows from this. ■

6.1.2 The low-loss regime

**Proof of Theorem 15.** Let $\omega_j$, $1 \leq j \leq N$ be an indexing of all the eigenvalues of $\Omega$ (counting multiplicities). The proof of this theorem is similar in essence to the proof of Theorem 5. We begin by extending the domain of the system operator by $A(\beta) = \Omega - i\beta B$, $\beta \in \mathbb{C}$. Using the substitution $\varepsilon = -i\beta$ then the operator $A(i\varepsilon) = \Omega + \varepsilon B$ is an analytic operator which is self-adjoint for real $\varepsilon$. Thus by a theorem of Rellich [Bau85, p. 21, Theorem 1] we know for $\varepsilon \in \mathbb{C}$ with $|\varepsilon| \ll 1$, the operator $\Omega + \varepsilon B$ is diagonalizable and there exists a complete set of analytic eigenvalues $\lambda_j(\varepsilon)$ and eigenvectors $x_j(\varepsilon)$ satisfying

$$(\Omega + \varepsilon B)x_j(\varepsilon) = \lambda_j(\varepsilon)x_j(\varepsilon), \quad \overline{\lambda_j(\varepsilon)} = \lambda_j(\varepsilon), \quad 1 \leq j \leq N;$$

$$(x_j(\varepsilon), x_k(\varepsilon)) = 0 \quad \text{for } \varepsilon \in \mathbb{R}, \quad 1 \leq j, k \leq N,$$  

(143)
where $\delta_{jk}$ denotes the Kronecker delta symbol. In particular, the vectors $u_j := x_j(0)$, $j = 1, \ldots, N$ form an orthonormal basis of eigenvectors for the operator $\Omega$. Thus, after a possible reindexing of these analytic eigenpairs, we may assume without loss of generality that

$$\Omega u_j = \omega_j u_j, \quad \lambda_j(0) = \omega_j, \quad 1 \leq j \leq N.$$ 

Denote the derivatives of these eigenvalues at $\varepsilon = 0$ by $\sigma_j := \lambda_j'(0)$, $1 \leq j \leq N$. Then it follows that they satisfy

$$\sigma_j = (u_j, Bu_j) \in \mathbb{R}, \quad 1 \leq j \leq N.$$ 

Indeed, for $\varepsilon$ real with $|\varepsilon| \ll 1$ by (143) we have

$$0 = (x_j(\varepsilon), (\Omega + \varepsilon B - \lambda_j(\varepsilon) I) x_j(\varepsilon)), \quad 1 \leq j \leq N$$

and the result follows immediately by taking the derivative on both sides and evaluating at $\varepsilon = 0$.

Now from these facts and recalling the substitution $\varepsilon = -i\beta$ that was made, we conclude that if $\beta \in \mathbb{C}$ with $|\beta| \ll 1$ then the system operator $A(\beta) = \Omega + (-i\beta) B$ is diagonalizable with a complete set of eigenvalues $\zeta_j(\beta)$ and eigenvectors $v_j(\beta)$ satisfying

$$\zeta_j(\beta) = \lambda_j(-i\beta), \quad v_j(\beta) = x_j(-i\beta), \quad 1 \leq j \leq N.$$ 

(144)

We will now show that these eigenpairs have the properties described in Theorem 15. First, it follows from our results above that the eigenpairs $\lambda_j(\varepsilon), x_j(\varepsilon), 1 \leq j \leq N$ are analytic for $\varepsilon \in \mathbb{C}$ with $|\varepsilon| \ll 1$ and have the asymptotic expansions

$$\lambda_j(\varepsilon) = \omega_j + \sigma_j \varepsilon + O(|\varepsilon|^2), \quad \omega_j \in \mathbb{R}, \quad \sigma_j \geq 0,$$

$$x_j(\varepsilon) = u_j + O(\varepsilon), \quad 1 \leq j \leq N$$

as $\varepsilon \to 0$. This implies by (144) these eigenvalues and eigenvectors are analytic at $\beta = 0$ and they have the asymptotic expansions

$$\zeta_j(\beta) = \lambda_j(-i\beta) = \omega_j - i\sigma_j \beta + O(|\beta|^2), \quad \omega_j \in \mathbb{R}, \quad \sigma_j \geq 0,$$

$$v_j(\beta) = x_j(-i\beta) = u_j + O(\beta), \quad 1 \leq j \leq N$$

as $\beta \to 0$, where the vectors $u_j, 1 \leq j \leq N_B$ form an orthonormal basis of eigenvectors of $\Omega$ and

$$\Omega u_j = \omega_j u_j, \quad \sigma_j = (u_j, \Omega u_j), \quad 1 \leq j \leq N.$$ 

This completes the proof. ■

**Proof of Corollary 16.** This corollary follows immediately from the asymptotic expansions (144) of Theorem 15 and the formulas from (91). ■

**Proof of Corollary 17.** The proof of this corollary is similar in essence to the proof of Proposition 14. As in the proof of Theorem 15 we can extend the domain of the system operator $A(\beta) = \Omega - i\beta B$, $\beta \in \mathbb{C}$ and the functions $\zeta_j(\beta)$, $1 \leq j \leq N$ can be extended uniquely to analytic functions in a neighborhood of $\beta = 0$ whose values are the eigenvalues of $A(\beta)$. Moreover, using the same notation to denote their extensions, it follows from (143) and (144) that these functions satisfy

$$\overline{\zeta_j(\beta)} = \zeta_j(-\beta), \quad 1 \leq j \leq N.$$
for all $\beta \in \mathbb{C}$ with $|\beta| \ll 1$. This implies $\frac{1}{2} (\zeta_j (\beta) + \zeta_j (-\beta))$ and $\frac{1}{2} (\zeta_j (\beta) - \zeta_j (-\beta))$ are even and odd functions, respectively, are analytic at $\beta = 0$, and equal $\Re \zeta_j (\beta)$ and $\Im \zeta_j (\beta)$, respectively, for real $\beta$. In particular, this implies their Taylor series in $\beta$ have only even and odd powers, respectively. And this implies as in the proof of Proposition 13 that $|\Re \zeta_j (\beta)|$ is analytic at $\beta = 0$ and its Taylor series in $\beta$ has only even powers. Now by the formula in (91) we know that if $\Im \zeta_j (\beta) \equiv 0$ for $\beta \in \mathbb{R}$ with $|\beta| \ll 1$ then the quality factor is given by

$$Q [v_j (\beta)] = \frac{1}{2} \frac{|\Re \zeta_j (\beta)|}{\Im \zeta_j (\beta)},$$

and hence can be extended to a function meromorphic at $\beta = 0$ whose Laurent series in $\beta$ contains only odd powers. Using these facts and the asymptotic expansions in (114) of Theorem 13 we conclude

$$Q [v_j (\beta)] = -\frac{1}{2} \frac{|\Re \zeta_j (\beta)|}{\Im \zeta_j (\beta)} = -\frac{1}{2} \frac{|\omega_j| + O (\beta^2)}{-\sigma_j \beta + O (\beta^3)} = \frac{1}{2} \frac{|\omega_j|}{\sigma_j} \beta^{-1} + O (\beta)$$

as $\beta \to \infty$, provided $\sigma_j \neq 0$, in which case $\lim_{\beta \to 0} Q [v_j (\beta)] = 0$ or $\infty$ depending on whether $\omega_j = 0$ or $\omega_j \neq 0$, respectively. This completes the proof. ■

6.2 Perturbation analysis of a system subjected to harmonic forces

This purpose of this section is to prove the results given in Section 4.

Proof of Proposition 20. Let $\omega \in \mathbb{R}$. If $\omega$ is an asymptotic nonresonance frequency of the system (8) then Proposition 21 which we have proved in this section below, tells us the admittance operator $A (\omega)$ is analytic at $\beta = \infty$. So in order to complete the proof of this proposition we need only prove the admittance operator $A (\omega)$ cannot be analytic at $\beta = \infty$ if $\omega$ is an asymptotic resonance frequency of the system (8). Thus suppose $\omega = \rho_k$ for some $k \in \{N_B + 1, \ldots, N\}$ but $A (\omega)$ was analytic at $\beta = \infty$. Then it is continuous and so by (92), (96) this implies

$$0 = \lim_{\beta \to \infty} [-i (\omega I - \zeta_k (\beta)) A (\omega) w_k (\beta)] = \lim_{\beta \to \infty} w_k (\beta) = \hat{w}_k \neq 0,$$

a contraction. This contradiction proves the proposition. ■

For the rest of the proofs in this section the symbol $\omega$ whenever it appears will mean an asymptotic nonresonance frequency of the system (8), i.e., $\omega \in \mathbb{R}$ and $\omega \neq \rho_j$, $N_B + 1 \leq j \leq N$. Also, without loss of generality we may extend the domain of the system operator $A (\beta) = \Omega - i \beta B$, $\beta \in \mathbb{C}$. Similarly we extend the domains of the operators $\Xi_2 (\omega, \beta) = \omega I_2 - (\Omega_2 - i \beta B_2)$, $\beta \in \mathbb{C}$ and $S_2 (\omega, \beta) = \Xi_1 (\omega) - \Theta^* \Xi_2 (\omega, \beta)^{-1} \Theta$, $\beta \in \mathbb{C}$ provided $\Xi_2 (\omega, \beta)$ is invertible. For the rest of the proofs in this section we use these extensions. Also whenever it is convenient we will suppress the dependency of these operators on the symbols $\omega, \beta$.

Proof of Proposition 21. We begin by proving for $|\beta| \gg 1$, the operators $\Xi_1 (\omega), \Xi_2 (\omega, \beta), S_2 (\omega, \beta)$ are invertible with $\Xi_2 (\omega, \beta)^{-1}, S_2 (\omega, \beta)$, and $S_2 (\omega, \beta)^{-1}$ analytic at $\beta = \infty$. By Proposition 11 it follows that the spectrum of the operator $\Omega_1$ is the set $\{\rho_j : N_B + 1 \leq j \leq N\}$ and since $\omega \not\in \{\rho_j : N_B + 1 \leq j \leq N\}$ then the operator $\Xi_1 (\omega) = \omega I_1 - \Omega_1$ is invertible. Now recall the well-known fact from perturbation theory that if
$T$ is a linear operator on $H$ such that in the operator norm $\|T\| < 1$ then $I - T$ is invertible and the series $\sum_{n=0}^{\infty} T^n$ converges absolutely and uniformly to $(I - T)^{-1}$. This implies if $T(\beta)$ is an operator-valued function which is analytic at $\beta = \infty$ then it has an asymptotic expansion $T(\beta) = T_0 + T_1 \beta^{-1} + O(\beta^{-2})$ as $\beta \to \infty$ and if $T_0$ is invertible then $T(\beta) = T_0 (I - T_0^{-1} [T(\beta) - T_0])$ is invertible for $|\beta| \gg 1$ and its inverse $T(\beta)^{-1}$ is analytic at $\beta = \infty$ with the asymptotic expansion $T(\beta)^{-1} = T_0^{-1} - T_0^{-1} T_1 T_0^{-1} \beta^{-1} + O(\beta^{-2})$ as $\beta \to \infty$. We will use some of these facts now to prove $\Xi_2(\omega, \beta)$, $S_2(\omega, \beta)$ are invertible for $|\beta| \gg 1$ and $\Xi_2(\omega, \beta)^{-1}$, $S_2(\omega, \beta)$, and $S_2(\omega, \beta)^{-1}$ are analytic at $\beta = \infty$. Well, the function $T(\beta) = i \beta^{-1} \Xi_2(\omega, \beta)$ is analytic at $\beta = \infty$ and its limit as $\beta \to \infty$ is an invertible operator since

$$\lim_{\beta \to \infty} i \beta^{-1} \Xi_2(\zeta, \beta) = \lim_{\beta \to \infty} i \beta^{-1} [\zeta I_2 - (\Omega_2 - i \beta B_2)] = B_2 > 0.$$ 

This implies $\Xi_2(\omega, \beta) = -i \beta T(\beta)$ is invertible for $|\beta| \gg 1$ with the inverse $\Xi_2(\omega, \beta)^{-1} = i \beta^{-1} T(\beta)^{-1}$ analytic at $\beta = \infty$ having the asymptotics

$$\Xi_2(\omega, \beta)^{-1} = -i B_2^{-1} \beta^{-1} + B_2^{-1} (\zeta I_2 - \Omega_2) B_2^{-1} \beta^{-2} + O(\beta^{-3}) \quad (145)$$

as $\beta \to \infty$. From which it follows that the operator

$$S_2(\omega, \beta) = \Xi_1(\omega) - \Theta^* \Xi_2(\omega, \beta)^{-1} \Theta = \Xi_1(\omega) (I_1 - \Xi_1(\omega)^{-1} \Theta^* \Xi_2(\omega, \beta)^{-1} \Theta)$$

is well-defined and invertible for $|\beta| \gg 1$ as well as it and its inverse, $S_2(\omega, \beta)^{-1}$, are analytic at $\beta = \infty$.

Now we prove $\omega I - A(\beta)$ is invertible for $|\beta| \gg 1$ and $\mathfrak{A}(\omega) = i (\omega I - A(\beta))^{-1}$ is analytic at $\beta = \infty$. First, by the $2 \times 2$ block operator matrix representation of $\omega I - A(\beta)$ from (122) and since $\Xi_2(\omega, \beta)$ is invertible for $|\beta| \gg 1$ then, as discussion in Appendix 7 on the Atiken block diagonalization formula (149)-(151), the operator admits for $|\beta| \gg 1$ the Frobenius-Schur factorization

$$\omega I - A(\beta) = \begin{bmatrix} \Xi_2 & -\Theta \\ -\Theta^* \Xi_1 \end{bmatrix} = \begin{bmatrix} I_2 & 0 \\ -\Theta^* \Xi_2^{-1} & I_1 \end{bmatrix} \begin{bmatrix} \Xi_2^{-1} & 0 \\ 0 & S_2 \end{bmatrix} \begin{bmatrix} I_2 & -\Xi_2^{-1} \Theta \\ 0 & I_1 \end{bmatrix}.$$ 

Furthermore, this implies for $\beta \gg 1$ that since $S_2(\omega, \beta)$ is invertible then $\omega I - A(\beta)$ is invertible and

$$\mathfrak{A}(\omega) = i \left[ \begin{bmatrix} I_2 & \Xi_2^{-1} \Theta \\ 0 & I_1 \end{bmatrix} \begin{bmatrix} \Xi_2^{-1} & 0 \\ 0 & S_2^{-1} \end{bmatrix} \begin{bmatrix} I_2 & 0 \\ \Theta^* \Xi_2^{-1} & I_1 \end{bmatrix} \right] \quad (146)$$

which proves formula (124). From this formula and the fact both $\Xi_2(\omega, \beta)^{-1}$ and $S_2(\omega, \beta)^{-1}$ are analytic at $\beta = \infty$ we conclude that $\mathfrak{A}(\omega)$ is analytic at $\beta = \infty$. Moreover, this formula, (151)-(154) in Appendix 7 that fact $(\Xi_1(\omega)^{-1})^* = \Xi_1(\omega)^{-1}$, and (145) imply we have the
This page contains a complex mathematical derivation involving block operators and asymptotic expansions. The text is structured around the asymptotic expansion of a particular operator and follows a proof that demonstrates the analytic properties of this operator. The proof involves several steps, including the use of asymptotic expansions and the application of specific operators to derive the required results. The operators are defined and their properties are explored, leading to conclusions about their analytic behavior as certain parameters approach infinity. The proof culminates in the demonstration of the analyticity of the operators in question, which is a key result in the context of the theorems and corollaries being discussed. The textbook page is part of a larger mathematical discussion that would typically appear in a course on advanced operator theory or mathematical analysis.
Proof of Corollary 25. From the block operator factorization of the operator $W^{(-1)}$ in (126) with respect to the direct sum $H = H_B \oplus H_B^\perp$ it follows immediately that $W^{(-1)} f = 0$ if and only if $P_B f + \Theta \Xi_1 (\omega)^{-1} P_B^\perp f = 0$, i.e., $f \in \ker (P_B + \Theta \Xi_1 (\omega)^{-1} P_B^\perp)$. But from this direct sum and the fact $P_B, P_B^\perp$ are the orthogonal projections onto $H_B, H_B^\perp$, respectively, we can compute this kernel to conclude

$$\ker W^{(-1)} = \ker (P_B + \Theta \Xi_1 (\omega)^{-1} P_B^\perp) = \{ f_1 + f_2 \in H : f_1 \in H_B^\perp \text{ and } f_2 = -\Theta \Xi_1 (\omega)^{-1} f_1 \}.$$ 

This representation of the kernel, the direct sum, and the fact that $\dim H_B^\perp = N - N_B$ implies the kernel has dimension $N - N_B$. This completes the proof. ■

Proof of Theorems 23 & 24. Let $f \in H, f \neq 0$. The stored energy $U$ and power of dissipated energy $W_{\text{dis}}$ associated with the harmonic external force $f(t) = e^{-i \omega t}$ by (120) and (121) are given by the formulas

$$U = \frac{1}{2} (f, \mathfrak{A}(\omega)^* \mathfrak{A}(\omega) f), \quad W_{\text{dis}} = (f, \mathfrak{A}(\omega)^* \beta B \mathfrak{A}(\omega) f)$$

for $\beta \gg 1$. In particular, by these formulas and Corollary 22 it follows that $U$ and $W_{\text{dis}}$ are analytic at $\beta = \infty$ and by (127), (129) have the asymptotic expansions

$$U = \frac{1}{2} (\Xi_1 (\omega)^{-1} P_B f, \Xi_1 (\omega)^{-1} P_B^\perp f) + O (\beta^{-1}), \quad W_{\text{dis}} = (f, W^{(-1)} f) \beta^{-1} + O (\beta^{-2})$$

as $\beta \to \infty$. In particular, by the fact $W^{(-1)} \geq 0$, the leading order term for $W_{\text{dis}}$ is a nonnegative number and if $P_B^\perp f \neq 0$ then the leading order term for $U$ is a positive number.

Now since $W_{\text{dis}}$ is analytic at $\beta = \infty$ then either $W_{\text{dis}} \equiv 0$ for $\beta \gg 0$ or $W_{\text{dis}} \neq 0$ for $\beta \gg 1$. Hence by the definition in (120) and (121) of the quality factor $Q = Q_{f, \omega}$ it will be finite for $\beta \gg 1$ if and only if $W_{\text{dis}} \neq 0$ for $\beta \gg 0$, in which case it is given by the formula $Q = |\omega| U / W_{\text{dis}}$ implying it is a meromorphic function at $\beta = \infty$. For example, by the asymptotic expansions just derived it follows that if $P_B^\perp f \neq 0$ then it must have a pole and if $P_B^\perp f = 0$ then it must be analytic. In particular, if we have $(f, W^{(-1)} f) \neq 0$, which is equivalent to $f \in \ker W^{(-1)}$, then it has the asymptotic expansion

$$Q = |\omega| \frac{1}{2} \left( \frac{\Xi_1 (\omega)^{-1} P_B f, \Xi_1 (\omega)^{-1} P_B^\perp f}{(f, W^{(-1)} f)} \right) \beta + O (1)$$

as $\beta \to \infty$ whose leading order term is nonnegative and if $P_B^\perp f \neq 0$ then it is positive.

Now we complete the proof of Theorem 23. Suppose $P_B^\perp f = 0$. Then we have $P_B f = f$. Then it follows from this and the block operator representation for $W^{(-1)}$ and $P_B$ in (126) and (148), respectively, that $P_B W^{(-1)} P_B f = B_2^{-1} f$. In particular, since $B_2^{-1} > 0$ then $(f, W^{(-1)} f) = (f, B_2^{-1} f) > 0$ and hence from the statements in this proof above the quality factor $Q$ is finite for $\beta \gg 1$ and is analytic at $\beta = \infty$. From our discussion above and (128)
we have the asymptotic expansions

\[ W_{\text{dis}} = (f, B^{-1}_2 f) \beta^{-1} + O(\beta^{-2}), \]

\[ U = \frac{1}{2} (f, \mathfrak{A}(\omega)^* \mathfrak{A}(\omega) f) = \frac{1}{2} (f, P_B \mathfrak{A}(\omega)^* \mathfrak{A}(\omega) P_B f) \]
\[ = \frac{1}{2} (f, [B^{-2}_2 + B^{-1}_2 \Theta (\Xi^{-1}_1)^* \Xi^{-1}_1 \Theta^* B^{-1}_2] f) \beta^{-2} + O(\beta^{-3}) \]

\[ Q = |\omega| \frac{1}{2} (f, [B^{-2}_2 + B^{-1}_2 \Theta (\Xi^{-1}_1)^* \Xi^{-1}_1 \Theta^* B^{-1}_2] f) \beta^{-1} + O(\beta^{-2}) \]

as \( \beta \to \infty \). Thus to complete the proof of Theorem 23 we need only prove the inequalities described in that theorem. First, since \( B^{-1}_2 > 0 \) then it has a positive square root \( B^{-1/2}_2 > 0 \). Second, it follows from Theorem 5 and (103) that for any \( u \in H_B \) we have

\[ \inf_{u \in H_B, u \neq 0} \frac{(u, B^{-1/2}_2 u)}{(u, u)} = \inf_{u \in H_B, u \neq 0} \frac{\sum_{1 \leq j \leq N_B} \hat{\varphi}_j^{-1} |(\hat{w}_j, u)|^2}{\sum_{1 \leq j \leq N_B} |(\hat{w}_j, u)|^2} \geq \min_{1 \leq j \leq N_B} \hat{\varphi}_j^{-1} = \left( \max_{1 \leq j \leq N_B} \hat{\varphi}_j \right)^{-1}. \]

Thus imply, with \( u = f, B^{-1/2}_2 f \), the inequalities

\[ \frac{1}{2} (f, B^{-1}_2 f) \geq \left( \max_{1 \leq j \leq N_B} \hat{\varphi}_j \right)^{-1} (f, f), \]
\[ \frac{1}{2} (f, B^{-2}_2 f) = \left( \max_{1 \leq j \leq N_B} \hat{\varphi}_j \right)^{-1} (f, B^{-1}_2 f) \]

These facts imply, since \( B^{-1}_2 \Theta (\Xi^{-1}_1)^* \Xi^{-1}_1 \Theta^* B^{-1}_2 \geq 0 \), that

\[ \frac{1}{2} (f, [B^{-2}_2 + B^{-1}_2 \Theta (\Xi^{-1}_1)^* \Xi^{-1}_1 \Theta^* B^{-1}_2] f) \geq \frac{1}{2} \left( \max_{1 \leq j \leq N_B} \hat{\varphi}_j \right)^{-1} (f, B^{-1}_2 f) \]
\[ \geq \frac{1}{2} \left( \max_{1 \leq j \leq N_B} \hat{\varphi}_j \right)^{-2} (f, f) > 0. \]

This completes the proof of the theorems. ■

**Proof of Corollary 26.** The proof of this corollary follows immediately from Theorem 23 and Theorem 24. ■

7 Appendix: Schur complement and the Aitken formula

Let \( M \)

\[ M = \begin{bmatrix} P & Q \\ R & S \end{bmatrix} \]  

(149)
be a square matrix represented in block form where $P$ and $S$ are square matrices with the former invertible, that is, $\|P^{-1}\| < \infty$. Then the following Aitken block-diagonalization formula holds [Zhang, Sec. 0.9, 1.1], [Aitken, p. 67 (4)],

$$ M = \begin{bmatrix} P & Q \\ R & S \end{bmatrix} = \begin{bmatrix} I & 0 \\ RP^{-1} & I \end{bmatrix} \begin{bmatrix} P & 0 \\ 0 & S \end{bmatrix} \begin{bmatrix} I & P^{-1}Q \\ 0 & I \end{bmatrix}, $$

(150)
i.e., the Frobenius-Schur factorization of the block matrix $M$ [Tret08, p. xiv], where the matrix

$$ S_P = M/P = S - RP^{-1}Q $$

is known as the Schur complement of $P$ in $M$. The Aitken formula (150) readily implies

$$ M^{-1} = \begin{bmatrix} I & -P^{-1}Q \\ 0 & I \end{bmatrix} \begin{bmatrix} P^{-1} & 0 \\ 0 & S^{-1}_P \end{bmatrix} \begin{bmatrix} I & 0 \\ -RP^{-1} & I \end{bmatrix} = \begin{bmatrix} P^{-1} + P^{-1}QS^{-1}_P & 0 \\ -S^{-1}_P & -P^{-1}QS^{-1}_P \end{bmatrix} $$

(152)

In particular, for $\|P^{-1}\| \ll 1$ and under the assumption of invertibility of the matrix $S$, formulas (151) and (152) imply

$$ S^{-1}_P = [S - RP^{-1}Q]^{-1} = S^{-1} [I - RP^{-1}QS^{-1}]^{-1} $$

$$ = S^{-1} + S^{-1}RP^{-1}QS^{-1} + O(\|P^{-2}\|), $$

(153)

$$ M^{-1} = \begin{bmatrix} 0 & 0 \\ 0 & S^{-1} \end{bmatrix} + O(\|P^{-1}\|), $$

(154)

$$ [M^{-1}]_1 = \begin{bmatrix} P^{-1} & -P^{-1}QS^{-1} \\ -S^{-1}_P & S^{-1} + S^{-1}RP^{-1}QS^{-1} \end{bmatrix} = \begin{bmatrix} P^{-1} & 0 \\ 0 & S^{-1} + S^{-1}RP^{-1}QS^{-1} \end{bmatrix} + \begin{bmatrix} P^{-1} & 0 \\ 0 & S^{-1} \end{bmatrix} \begin{bmatrix} 0 & -Q \\ -R & 0 \end{bmatrix} \begin{bmatrix} P^{-1} & 0 \\ 0 & S^{-1} \end{bmatrix}. $$

8 Appendix: Quality factor for eigenmodes

In this appendix we derive a simple and descriptive formula for the energy $U[w]$, power of energy dissipation $W_{\text{dis}}[w]$, and quality factor $Q[w]$, for any eigenmode $w$ of the system operator $A(\beta)$ with eigenvalue $\zeta$ which are the quantities

$$ U[w] = \frac{1}{2} (w, w) $$

(155)

$$ W_{\text{dis}}[w] = (w, \beta Bw) $$

(156)

$$ Q[w] = 2\pi \frac{\text{energy stored in the system}}{\text{energy lost per cycle}} $$

(157)

$$ = |\text{Re} \zeta| \frac{U[w]}{W_{\text{dis}}[w]} = |\text{Re} \zeta| \frac{\frac{1}{2} (w, w)}{\beta (w, Bw)}. $$

where $\text{Re} \zeta$ denotes the real part of the eigenvalue $\zeta$ and $Q[w]$ is finite if $W_{\text{dis}}[w] \neq 0$. 

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Lemma 27 If $w$ is an eigenvector of the system operator $A(\beta) = \Omega - i\beta B$ with eigenvalue $\zeta$ then

$$
\text{Re} \zeta = \frac{(w, \Omega w)}{(w, w)}, \quad \text{Im} \zeta = -\frac{(w, \beta B w)}{(w, w)}.
$$

(158)

In particular, if $\beta \geq 0$ then $\text{Im} \zeta \leq 0$.

**Proof.** By assumption $B \geq 0$ and by hypothesis we have $A(\beta)w = \zeta w, w \neq 0$ so that $(w, w) \neq 0$ and

$$
\frac{\text{Re} (w, A(\beta)w)}{(w, w)} = \text{Re} \zeta, \quad \frac{\text{Im} (w, A(\beta)w)}{(w, w)} = \text{Im} \zeta.
$$

On the other hand, since

$$
\frac{1}{2} [A(\beta) + A(\beta)^*] = \Omega, \quad \frac{1}{2i} [A(\beta) - A(\beta)^*] = -\beta B,
$$

this implies

$$
\text{Re} \zeta = \frac{\text{Re} (w, A(\beta)w)}{(w, w)} = \frac{(w, \Omega w)}{(w, w)},
$$

$$
\text{Im} \zeta = \frac{\text{Im} (w, A(\beta)w)}{(w, w)} = -\frac{(w, \beta B w)}{(w, w)} \leq 0.
$$

This completes the proof. ■

Proposition 28 (quality factor) If $w$ is an eigenvector of the system operator $A(\beta) = \Omega - i\beta B$ with eigenvalue $\zeta$ then the energy $U[w], \text{power of energy dissipation } W_{\text{dis}}[w],$ and quality factor $Q[w]$ satisfy

$$
\text{Re} \zeta U[w] = \frac{1}{2} (w, \Omega w), \quad W_{\text{dis}}[w] = (w, \beta B w) = -2 \text{Im} \zeta U[w],
$$

(159)

$$
Q[w] = \frac{1}{2} \frac{|(w, \Omega w)|}{(w, \beta B w)} = -\frac{1}{2} \frac{|\text{Re} \zeta|}{\text{Im} \zeta},
$$

(160)

and $Q[w]$ is finite if and only if $\text{Im} \zeta \neq 0$. In particular, the quality factor $Q[w] = -\frac{1}{2} \frac{|\text{Re} \zeta|}{\text{Im} \zeta}$ is independent of the choice of eigenvector $w$ since it depends only on its corresponding eigenvalue $\zeta$.

**Proof.** By the lemma it follows that $W_{\text{dis}}[w] = (w, \beta B w) = -(w, w) \text{Im} \zeta = -2 \text{Im} \zeta U[w]$ and $\text{Re} \zeta U[w] = \frac{1}{2} (w, w) \text{Re} \zeta = \frac{1}{2} (w, \Omega w)$. This implies that $Q[w]$ is finite if and only if $\text{Im} \zeta \neq 0$, in which case

$$
Q[w] = \frac{|\text{Re} \zeta|}{W_{\text{dis}}[w]} = \frac{1}{2} \frac{|(w, \Omega w)|}{(w, \beta B w)} = \frac{1}{2} \frac{|\text{Re} \zeta (w, w)|}{2 \text{Im} \zeta (w, w)} = -\frac{1}{2} \frac{|\text{Re} \zeta|}{\text{Im} \zeta}.
$$

This completes the proof. ■

**Acknowledgment:** The research of A. Figotin was supported through Dr. A. Nachman of the U.S. Air Force Office of Scientific Research (AFOSR), under grant number FA9550-11-1-0163. Both authors are indebted to the referees for their valuable comments on our original manuscript.
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