Operator algebras and conjugacy problem for the pseudo-Anosov automorphisms of a surface

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In memory of W. P. Thurston

Abstract

The conjugacy problem for the pseudo-Anosov automorphisms of a compact surface is studied. To each pseudo-Anosov automorphism $\phi$, we assign an AF $C^*$-algebra $A_\phi$ (an operator algebra). It is proved that the assignment is functorial, i.e., every $\phi'$, conjugate to $\phi$, maps to an AF $C^*$-algebra $A_{\phi'}$, which is stably isomorphic to $A_\phi$. The new invariants of the conjugacy of the pseudo-Anosov automorphisms are obtained from the known invariants of the stable isomorphisms of the AF $C^*$-algebras. Namely, the main invariant is a triple $(\Lambda, [I], K)$, where $\Lambda$ is an order in the ring of integers in a real algebraic number field $K$ and $[I]$ an equivalence class of the ideals in $\Lambda$. The numerical invariants include the determinant $\Delta$ and the signature $\Sigma$, which we compute for the case of the Anosov automorphisms. A question concerning the $p$-adic invariants of the pseudo-Anosov automorphism is formulated.

Key words and phrases: mapping class group, AF $C^*$-algebras

MSC: 46L85 (noncommutative topology), 57M27 (invariants of 3-manifolds)

*Partially supported by NSERC.
Introduction

A. Conjugacy problem. Let \( \text{Mod}(X) \) be the mapping class group of a compact surface \( X \), i.e. the group of orientation preserving automorphisms of \( X \) modulo the trivial ones. Recall that \( \phi, \phi' \in \text{Mod}(X) \) are conjugate automorphisms, whenever \( \phi' = h \circ \phi \circ h^{-1} \) for an \( h \in \text{Mod}(X) \). It is not hard to see that conjugation is an equivalence relation which splits the mapping class group into disjoint classes of conjugate automorphisms. The construction of invariants of the conjugacy classes in \( \text{Mod}(X) \) is an important and difficult problem studied by Hemion [9], Mosher [13], and others. Any knowledge of such invariants leads to a topological classification of three-dimensional manifolds, which fiber over the circle with monodromy \( \phi \in \text{Mod}(X) \) [17].

B. Pseudo-Anosov automorphisms. It is known that any \( \phi \in \text{Mod}(X) \) is isotopic to an automorphism \( \phi' \), such that either (i) \( \phi' \) has a finite order, or (ii) \( \phi' \) is a pseudo-Anosov (aperiodic) automorphism, or else (iii) \( \phi' \) is reducible by a system of curves \( \Gamma \) surrounded by the small tubular neighborhoods \( N(\Gamma) \), such that on \( X \setminus N(\Gamma) \) \( \phi' \) satisfies either (i) or (ii). Let \( \phi \) be a representative of the equivalence class of a pseudo-Anosov automorphism. Then there exist a pair consisting of the stable \( F_s \) and unstable \( F_u \) mutually orthogonal measured foliations on the surface \( X \), such that \( \phi(F_s) = \frac{1}{\lambda_\phi} F_s \) and \( \phi(F_u) = \lambda_\phi F_u \), where \( \lambda_\phi > 1 \) is called a dilatation of \( \phi \). The foliations \( F_s, F_u \) are minimal, uniquely ergodic and describe the automorphism \( \phi \) up to a power. In the sequel, we shall focus on the conjugacy problem for the pseudo-Anosov automorphisms of a surface \( X \).

C. AF C*-algebras. The C*-algebra is an algebra \( A \) over \( \mathbb{C} \) with a norm \( a \mapsto ||a|| \) and an involution \( a \mapsto a^* \) such that it is complete with respect to the norm and \( ||ab|| \leq ||a|| \cdot ||b|| \) and \( ||a^*a|| = ||a^2|| \) for all \( a, b \in A \). The C*-algebras have been introduced by Murray and von Neumann as rings of bounded operators on a Hilbert space and are strongly connected with the geometry and topology of manifolds [3], §24. Any simple finite-dimensional C*-algebra is isomorphic to the algebra \( M_n(\mathbb{C}) \) of the complex \( n \times n \) matrices. A natural completion of the finite-dimensional semisimple C*-algebras (as \( n \to \infty \)) is known as an AF C*-algebra [6]. AF C*-algebra is most conveniently given by an infinite graph, which records the inclusion of the finite-dimensional subalgebras into the AF C*-algebra. The graph is called a Bratteli diagram. When the diagram is periodic, the AF C*-algebra is
stationary; this is an important special case. In addition to the usual isomorphism $\cong$, the $C^*$-algebras $A, A'$ are called \textit{stably isomorphic} whenever $A \otimes \mathcal{K} \cong A' \otimes \mathcal{K}$, where $\mathcal{K}$ is the $C^*$-algebra of compact operators.

**D. Motivation.** Let $\phi \in \text{Mod} (X)$ be a pseudo-Anosov automorphism. The main idea of present paper is to assign to $\phi$ an AF $C^*$-algebra, $\mathbb{A}_\phi$, so that for every $h \in \text{Mod} (X)$ the following diagram commutes:

$$
\begin{array}{c}
\phi \\
\downarrow \\
\mathbb{A}_\phi
\end{array}
\quad \cong 
\quad
\begin{array}{c}
\phi' = h \circ \phi \circ h^{-1} \\
\downarrow \\
\mathbb{A}_{\phi'}
\end{array}
$$

(In other words, if $\phi$ and $\phi'$ are conjugate pseudo-Anosov automorphisms, then the AF $C^*$-algebras $\mathbb{A}_\phi$ and $\mathbb{A}_{\phi'}$ are stably isomorphic.) For the sake of clarity, we shall consider an example illustrating the idea in the case $X = T^2$ (a torus).

**E. Model example.** Let $\phi \in \text{Mod} (T^2)$ be the Anosov automorphism given by a non-negative matrix $A_\phi \in SL_2(\mathbb{Z})$. (The assumption is not restrictive; each $A_\phi$ with $\text{Tr} (A_\phi) > 0$ is similar to a non-negative matrix. The case $\text{Tr} (A_\phi) < 0$ is treated likewise – by reduction to a non-positive matrix; then the absolute value of all entries must be taken.) Consider a stationary AF $C^*$-algebra, $\mathbb{A}_\phi$, given by the following periodic Bratteli diagram:

$$
A_\phi = \begin{pmatrix}
a_{11} & a_{12} \\
 a_{21} & a_{22}
\end{pmatrix},
$$

where $a_{ij}$ indicate the multiplicity of the respective edges of the graph. We encourage the reader to verify that $F : \phi \mapsto \mathbb{A}_\phi$ is a well-defined function.
on the set of Anosov automorphisms given by the hyperbolic matrices with
the non-negative entries. Let us show that if \( \phi, \phi' \in \text{Mod} (T^2) \) are conjugate
Anosov automorphisms, then \( \mathcal{A}_\phi, \mathcal{A}_{\phi'} \) are stably isomorphic AF \( C^* \)-algebras.
Indeed, let \( \phi' = h \circ \phi \circ h^{-1} \) for an \( h \in \text{Mod} (X) \). Then \( A_{\phi'} = TA_\phi T^{-1} \)
for a matrix \( T \in \text{SL}_2(\mathbb{Z}) \). Note that \((A'_\phi)^n = (TA_\phi T^{-1})^n = TA_\phi^n T^{-1}, \) where
\( n \in \mathbb{N} \). We shall use the following criterion: the AF \( C^* \)-algebras
\( \mathcal{A}_\phi, \mathcal{A}_{\phi'} \) are stably isomorphic if and only if their Bratteli diagrams contain a common
block of an arbitrary length (compare with [6], Theorem 2.3; recall that an
order-isomorphism mentioned in the theorem is equivalent to the condition
that the corresponding Bratteli diagrams have the same infinite tails – i.e.
a common block of infinite length). Consider the following sequences of
matrices:
\[
\begin{align*}
A_\phi A_\phi \ldots A_\phi \\
T A_\phi A_\phi \ldots A_\phi T^{-1},
\end{align*}
\]
which mimic the Bratteli diagrams of \( \mathcal{A}_\phi \) and \( \mathcal{A}_{\phi'} \). Letting \( n \to \infty \), we
conclude that \( \mathcal{A}_\phi \otimes \mathbb{K} \cong \mathcal{A}_{\phi'} \otimes \mathbb{K} \).

F. Invariants of torus automorphisms obtained from the operator algebras. Conjugacy problem for the Anosov automorphisms can
now be recast in terms of AF \( C^* \)-algebras: find invariants of stable isomorphism classes of the stationary AF \( C^* \)-algebras. One such invariant is due
to Handelman [7]. Consider an eigenvalue problem for the hyperbolic matrix
\( A_\phi \in \text{SL}_2(\mathbb{Z}) \): \( A_\phi v_A = \lambda_A v_A \), where \( \lambda_A > 1 \) is the Perron-Frobenius eigen-
value and \( v_A = (v_A^{(1)}, v_A^{(2)}) \) the corresponding eigenvector with the positive
entries normalized so that \( v_A^{(i)} \in K = \mathbb{Q}(\lambda_A) \). Denote by \( m = z v_A^{(1)} + z v_A^{(2)} \) the
\( \mathbb{Z} \)-module in the number field \( K \). Recall that the coefficient ring, \( \Lambda \), of mod-
ule \( m \) consists of the elements \( \alpha \in K \) such that \( \alpha m \subseteq m \). It is known that \( \Lambda \) is
an order in \( K \) (i.e. a subring of \( K \) containing 1) and, with no restriction, one
can assume that \( m \subseteq \Lambda \). It follows from the definition, that \( m \) coincides with
an ideal, \( I \), whose equivalence class in \( \Lambda \) we shall denote by \( [I] \). It has been
proved by Handelman, that the triple \((\Lambda, [I], K)\) is an arithmetic invariant
of the stable isomorphism class of \( \mathcal{A}_\phi \): the \( \mathcal{A}_\phi, \mathcal{A}_{\phi'} \) are stably isomorphic AF
\( C^* \)-algebras if and only if \( \Lambda = \Lambda', [I] = [I'] \) and \( K = K' \). It is interesting to
compare the operator algebra invariants with the matrix invariants obtained
in [12] and [19].
G. AF $C^*$-algebra $\mathcal{A}_\phi$ (pseudo-Anosov case). Denote by $F_\phi$ the stable foliation of a pseudo-Anosov automorphism $\phi \in \text{Mod}(X)$. For brevity, we assume that $F_\phi$ is an oriented foliation given by the trajectories of a closed 1-form $\omega \in H^1(X; \mathbb{R})$. Let $v^{(i)} = \int_{\gamma_i} \omega$, where $\{\gamma_1, \ldots, \gamma_n\}$ is a basis in the relative homology $H_1(X, \text{Sing} F_\phi; \mathbb{Z})$, such that $\theta = (\theta_1, \ldots, \theta_{n-1})$ is a vector with positive coordinates $\theta_i = v^{(i+1)}/v^{(1)}$. (Note that the $\theta_i$ depend on a basis in the homology group; but a $\mathbb{Z}$-module generated by the $\theta_i$ does not – see lemma 1.) Consider the (infinite) Jacobi-Perron continued fraction \[ \left( \frac{1}{\theta} \right) = \lim_{k \to \infty} \left( \begin{array}{cc} 0 & 1 \\ I & b_1 \end{array} \right) \cdots \left( \begin{array}{cc} 0 & 1 \\ I & b_k \end{array} \right) \left( \begin{array}{c} 0 \\ I \end{array} \right), \] where $b_i = (b^{(i)}_1, \ldots, b^{(i)}_{n-1})^T$ is a vector of the nonnegative integers, $I$ the unit matrix and $I = (0, \ldots, 0, 1)^T$. By the definition, $\mathcal{A}_\phi$ is an (isomorphism class of the) AF $C^*$-algebra given by the Bratteli diagram whose incidence matrices coincide with $B_k = \left( \begin{array}{cc} 0 & 1 \\ I & b_k \end{array} \right)$ for $k = 1, \ldots, \infty$. Note that this yields the Bratteli diagram derived in the model example (the Anosov case).

H. Main results. For a matrix $A \in GL_n(\mathbb{Z})$ with positive entries, we denote by $\lambda_A$ the Perron-Frobenius eigenvalue and let $(v_A^{(1)}, \ldots, v_A^{(n)})$ denote the corresponding normalized eigenvector with $v_A^{(i)} \in K = \mathbb{Q}(\lambda_A)$. The coefficient (endomorphism) ring of the module $m = \mathbb{Z}v_A^{(1)} + \ldots + \mathbb{Z}v_A^{(n)}$ will be denoted by $\Lambda$. The equivalence class of ideal $I$ in $\Lambda$ will be denoted $[I]$. Finally, we denote by $\Delta = \text{Det} (a_{ij})$ and $\Sigma$ the determinant and signature of the symmetric bilinear form $q(x, y) = \sum_{i,j} a_{ij} x_i x_j$, where $a_{ij} = \text{Tr} (v_A^{(i)} v_A^{(j)})$ and $\text{Tr} (\bullet)$ the trace function. Our main results can be expressed as follows.

**Theorem 1** $\mathcal{A}_\phi$ is a stationary AF $C^*$-algebra.

Let $\Phi$ be a category of all pseudo-Anosov (Anosov, resp.) automorphisms of a surface of the genus $g \geq 2$ ($g = 1$, resp.); the arrows (morphisms) are conjugations between the automorphisms. Likewise, let $\mathcal{A}$ be the category of all stationary AF $C^*$-algebras $\mathcal{A}_\phi$, where $\phi$ runs over the set $\Phi$; the arrows of $\mathcal{A}$ are stable isomorphisms among the algebras $\mathcal{A}_\phi$.

**Theorem 2** Let $F : \Phi \to \mathcal{A}$ be a map given by the formula $\phi \mapsto \mathcal{A}_\phi$. Then:

(i) $F$ is a functor; it maps conjugate pseudo-Anosov automorphisms to stably isomorphic AF $C^*$-algebras;
(ii) \( \text{Ker } F = [\phi], \) where \([\phi] = \{\phi' \in \Phi \mid (\phi')^m = \phi^n, \ m, n \in \mathbb{N}\}\) is the commensurability class of the pseudo-Anosov automorphism \( \phi. \)

**Corollary 1** The following are invariants of the conjugacy classes of the pseudo-Anosov automorphisms:

(i) triples \((\Lambda, [I], K)\);
(ii) integers \(\Delta\) and \(\Sigma.\)

I. **How to calculate invariants \((\Lambda, [I], K), \Delta\) and \(\Sigma?\)** There is no easy way; the problem is comparable to that of numerical invariants of the fundamental group of a knot. A step in this direction would be computation of the matrix \(A;\) the latter is similar to the matrix \(\rho(\phi),\) where \(\rho : \text{Mod} (X) \rightarrow PIL\) is a faithful representation of the mapping class group as a group of the piecewise-integral-linear (PIL) transformations [14], p.45. The entries of \(\rho(\phi)\) are the linear combinations of the Dehn twists along the \((3g – 1)\) (Lickorish) curves on the surface \(X.\) Then one can effectively determine whether the \(\rho(\phi)\) and \(A\) are similar matrices (over \(\mathbb{Z}\)) by bringing the polynomial matrices \(\rho(\phi) \circ xI\) and \(A \circ xI\) to the Smith normal form; when the similarity is established, the numerical invariants \(\Delta\) and \(\Sigma\) become the polynomials in the Dehn twists. A tabulation of the simplest elements of \(\text{Mod} (X)\) is possible in terms of \(\Delta\) and \(\Sigma\) (see §4.3); however, this task lies beyond the scope of present paper.

K. **Structure of the paper.** Proofs of the main results can be found in section 3. Sections 1 and 2 consist of lemmas used to prove the main results. Section 4 includes some examples, open problems and conjectures. Since the paper does not include a formal section on the preliminaries, we encourage the reader to consult [3], [6], [11] (operator algebras & dynamics), [10], [18] (measured foliations) and [2], [15] (Jacobi-Perron continued fractions).

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1 Jacobian of a measured foliation

1.1 Definition of the jacobian

Let $\mathcal{F}$ be a measured foliation on a compact surface $X$ [18]. For the sake of brevity, we shall always assume that $\mathcal{F}$ is an oriented foliation, i.e. given by the trajectories of a closed 1-form $\omega$ on $X$. (The assumption is not a restriction – each measured foliation is oriented on a surface $\widetilde{X}$, which is a double cover of $X$ ramified at the singular points of the half-integer index of the non-oriented foliation [10].) Let $\{\gamma_1, \ldots, \gamma_n\}$ be a basis in the relative homology group $H_1(X, \operatorname{Sing} \mathcal{F}; \mathbb{Z})$, where $\operatorname{Sing} \mathcal{F}$ is the set of singular points of the foliation $\mathcal{F}$. It is well known that $n = 2g + m - 1$, where $g$ is the genus of $X$ and $m = |\operatorname{Sing} (\mathcal{F})|$. The periods of $\omega$ in the above basis will be written

$$\lambda_i = \int_{\gamma_i} \omega.$$

The real numbers $\lambda_i$ are coordinates of $\mathcal{F}$ in the space of all measured foliations on $X$ (with the fixed set of singular points) [5].

**Definition 1** By a jacobian $\text{Jac} (\mathcal{F})$ of the measured foliation $\mathcal{F}$, we understand a $\mathbb{Z}$-module $m = \mathbb{Z}\lambda_1 + \ldots + \mathbb{Z}\lambda_n$ regarded as a subset of the real line $\mathbb{R}$. 
An importance of the jacobians stems from an observation that although the periods, $\lambda_i$, depend on the choice of basis in $H_1(X, \text{Sing } F; \mathbb{Z})$, the jacobian does not. Moreover, up to a scalar multiple, the jacobian is an invariant of the equivalence class of the foliation $F$. We formalize these observations in the following two sections.

1.2 Invariance of the jacobian

**Lemma 1** The $\mathbb{Z}$-module $m$ is independent of choice of basis in $H_1(X, \text{Sing } F; \mathbb{Z})$ and depends solely on the foliation $F$.

**Proof.** Indeed, let $A = (a_{ij}) \in GL_n(\mathbb{Z})$ and let

$$\gamma'_i = \sum_{j=1}^{n} a_{ij} \gamma_j$$

be a new basis in $H_1(X, \text{Sing } F; \mathbb{Z})$. Then using the integration rules:

$$\lambda'_i = \int_{\gamma'_i} \omega = \int \sum_{j=1}^{n} a_{ij} \gamma_j \omega = \sum_{j=1}^{n} \int_{\gamma_j} \omega = \sum_{j=1}^{n} a_{ij} \lambda_j.$$

To prove that $m = m'$, consider the following equations:

$$m' = \sum_{i=1}^{n} Z \lambda'_i = \sum_{i=1}^{n} Z \sum_{j=1}^{n} a_{ij} \lambda_j = \sum_{j=1}^{n} \left( \sum_{i=1}^{n} a_{ij} Z \right) \lambda_j \subseteq m.$$ 

Let $A^{-1} = (b_{ij}) \in GL_n(\mathbb{Z})$ be an inverse to the matrix $A$. Then $\lambda_i = \sum_{j=1}^{n} b_{ij} \lambda'_j$ and

$$m = \sum_{i=1}^{n} Z \lambda_i = \sum_{i=1}^{n} Z \sum_{j=1}^{n} b_{ij} \lambda'_j = \sum_{j=1}^{n} \left( \sum_{i=1}^{n} b_{ij} Z \right) \lambda'_j \subseteq m'.$$

Since both $m' \subseteq m$ and $m \subseteq m'$, we conclude that $m' = m$. Lemma 1 follows. $\square$
1.3 Projective invariance

Recall that the measured foliations $\mathcal{F}$ and $\mathcal{F}'$ are equivalent, if there exists an automorphism $h \in \text{Mod}(X)$, which sends the leaves of the foliation $\mathcal{F}$ to the leaves of the foliation $\mathcal{F}'$. This equivalence deals with topological foliations, i.e. projective classes of measured foliations, see [18] for an explanation.

**Lemma 2** Let $\mathcal{F}, \mathcal{F}'$ be the equivalent measured foliations on a surface $X$. Then

$$Jac(\mathcal{F}') = \mu \, Jac(\mathcal{F}),$$

where $\mu > 0$ is a real number.

**Proof.** Let $h : X \to X$ be an automorphism of the surface $X$. Denote by $h_*$ its action on $H_1(X, \text{Sing}(\mathcal{F}); \mathbb{Z})$ and by $h^*$ on $H^1(X; \mathbb{R})$ connected by the formula:

$$\int_{h_*(\gamma)} \omega = \int_{\gamma} h^*(\omega), \quad \forall \gamma \in H_1(X, \text{Sing}(\mathcal{F}); \mathbb{Z}), \quad \forall \omega \in H^1(X; \mathbb{R}).$$

Let $\omega, \omega' \in H^1(X; \mathbb{R})$ be the closed 1-forms whose trajectories define the foliations $\mathcal{F}$ and $\mathcal{F}'$, respectively. Since $\mathcal{F}, \mathcal{F}'$ are equivalent measured foliations, $\omega' = \mu \, h^*(\omega)$

for a $\mu > 0$.

Let $Jac(\mathcal{F}) = \mathbb{Z}\lambda_1 + \ldots + \mathbb{Z}\lambda_n$ and $Jac(\mathcal{F}') = \mathbb{Z}\lambda'_1 + \ldots + \mathbb{Z}\lambda'_n$. Then:

$$\lambda'_i = \int_{\gamma_i} \omega' = \mu \int_{\gamma_i} h^*(\omega) = \mu \int_{h_*(\gamma_i)} \omega, \quad 1 \leq i \leq n.$$

By lemma 1, it holds:

$$Jac(\mathcal{F}) = \sum_{i=1}^{n} \mathbb{Z} \int_{\gamma_i} \omega = \sum_{i=1}^{n} \mathbb{Z} \int_{h_*(\gamma_i)} \omega.$$

Therefore:

$$Jac(\mathcal{F}') = \sum_{i=1}^{n} \mathbb{Z} \int_{\gamma_i} \omega' = \mu \sum_{i=1}^{n} \mathbb{Z} \int_{h_*(\gamma_i)} \omega = \mu \, Jac(\mathcal{F}).$$

Lemma 2 follows. □
2 Equivalent foliations are stably isomorphic

Let $\mathcal{F}$ be a measured foliation on the surface $X$. We introduce an AF $C^*$-algebra, $\mathbb{A}_\mathcal{F}$, corresponding to the foliation $\mathcal{F}$ as explained in item G of introduction (for the foliation $\mathcal{F}_\phi$). The goal of present section is the commutativity of the following diagram:

\[
\begin{array}{ccc}
\mathcal{F} & \xrightarrow{\text{equivalent}} & \mathcal{F}' \\
\downarrow & & \downarrow \\
\mathbb{A}_\mathcal{F} & \xrightarrow{\text{stably isomorphic}} & \mathbb{A}_{\mathcal{F}'}
\end{array}
\]

2.1 Modules and continued fractions

The following lemma is a simple property of the Jacobi-Perron fractions [2].

**Lemma 3** Let $m = \mathbb{Z}\lambda_1 + \ldots + \mathbb{Z}\lambda_n$ and $m' = \mathbb{Z}\lambda'_1 + \ldots + \mathbb{Z}\lambda'_n$ be two $\mathbb{Z}$-modules, such that $m' = \mu m$ for $\mu > 0$. Then the Jacobi-Perron continued fractions of the vectors $\lambda$ and $\lambda'$ coincide except, possibly, at a finite number of terms.

**Proof.** Let $m = \mathbb{Z}\lambda_1 + \ldots + \mathbb{Z}\lambda_n$ and $m' = \mathbb{Z}\lambda'_1 + \ldots + \mathbb{Z}\lambda'_n$. Since $m' = \mu m$, where $\mu$ is a positive real, one gets the following identity of the $\mathbb{Z}$-modules:

\[
\mathbb{Z}\lambda'_1 + \ldots + \mathbb{Z}\lambda'_n = \mathbb{Z}(\mu\lambda_1) + \ldots + \mathbb{Z}(\mu\lambda_n).
\]

One can always assume that $\lambda_i$ and $\lambda'_i$ are positive reals. For obvious reasons, there exists a basis $\{\lambda''_1, \ldots, \lambda''_n\}$ of the module $m'$, such that:

\[
\begin{aligned}
\lambda''_i &= A(\mu\lambda) \\
\lambda''_i &= A'\lambda',
\end{aligned}
\]

where $A, A' \in GL_n^+(\mathbb{Z})$ are the matrices, whose entries are non-negative integers. In view of the Proposition 3 of [1]:

\[
\begin{aligned}
A &= \begin{pmatrix} 0 & 1 \\ I & b_1 \end{pmatrix} \ldots \begin{pmatrix} 0 & 1 \\ I & b_k \end{pmatrix} \\
A' &= \begin{pmatrix} 0 & 1 \\ I & b'_1 \end{pmatrix} \ldots \begin{pmatrix} 0 & 1 \\ I & b'_l \end{pmatrix},
\end{aligned}
\]

10
where \( b_i, b'_i \) are non-negative integer vectors. Since the (Jacobi-Perron) continued fraction for the vectors \( \lambda \) and \( \mu \lambda \) coincide for any \( \mu > 0 \) [2], we conclude that:

\[
\begin{align*}
\begin{pmatrix} 1 \\ \theta \end{pmatrix} &= \begin{pmatrix} 0 & 1 \\ I & b_1 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ I & b_k \end{pmatrix} \begin{pmatrix} 0 & 1 \\ I & a_1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ I & a_2 \end{pmatrix} \cdots \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \\
\begin{pmatrix} 1 \\ \theta' \end{pmatrix} &= \begin{pmatrix} 0 & 1 \\ I & b'_1 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ I & b'_l \end{pmatrix} \begin{pmatrix} 0 & 1 \\ I & a_1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ I & a_2 \end{pmatrix} \cdots \begin{pmatrix} 0 \\ 1 \end{pmatrix},
\end{align*}
\]

where

\[
\begin{pmatrix} 1 \\ \theta'' \end{pmatrix} = \lim_{i \to \infty} \begin{pmatrix} 0 & 1 \\ I & a_1 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ I & a_i \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}.
\]

In other words, the continued fractions of the vectors \( \lambda \) and \( \lambda' \) coincide except at a finite number of terms. \( \square \)

### 2.2 Main lemma

**Lemma 4** Let \( \mathcal{F} \) and \( \mathcal{F}' \) be equivalent measured foliations on a surface \( X \). Then the AF \( C^* \)-algebras \( \mathcal{A}_F \) and \( \mathcal{A}_{F'} \) are stably isomorphic.

**Proof.** Notice that lemma 2 implies that equivalent measured foliations \( \mathcal{F}, \mathcal{F}' \) have proportional jacobians, i.e. \( m' = \mu m \) for a \( \mu > 0 \). On the other hand, by lemma 3 the continued fraction expansion of the basis vectors of the proportional jacobians must coincide, except a finite number of terms. Thus, the AF \( C^* \)-algebras \( \mathcal{A}_F \) and \( \mathcal{A}_{F'} \) are given by the Bratteli diagrams, which are identical, except a finite part of the diagram. It is well known ([6], Theorem 2.3) that the AF \( C^* \)-algebras, which have such a property, are stably isomorphic. \( \square \)

### 3 Proofs

#### 3.1 Proof of theorem 1

Let \( \phi \in \text{Mod} (X) \) be a pseudo-Anosov automorphism of the surface \( X \). Denote by \( \mathcal{F}_\phi \) the invariant foliation of \( \phi \). By definition of such a foliation, \( \phi (\mathcal{F}_\phi) = \lambda_\phi \mathcal{F}_\phi \), where \( \lambda_\phi > 1 \) is the dilatation of \( \phi \).
Consider the jacobian $Jac\,(\mathcal{F}_\phi) = m_\phi$ of foliation $\mathcal{F}_\phi$. Since $\mathcal{F}_\phi$ is an invariant foliation of the pseudo-Anosov automorphism $\phi$, one gets the following equality of the $\mathbb{Z}$-modules:
\[ m_\phi = \lambda_\phi m_\phi, \quad \lambda_\phi \neq \pm 1. \] (1)

Let $\{v^{(1)}, \ldots, v^{(n)}\}$ be a basis in module $m_\phi$, such that $v^{(i)} > 0$. In view of (1), one obtains the following system of linear equations:
\[
\begin{align*}
\lambda_\phi v^{(1)} &= a_{11} v^{(1)} + a_{12} v^{(2)} + \ldots + a_{1n} v^{(n)} \\
\lambda_\phi v^{(2)} &= a_{21} v^{(1)} + a_{22} v^{(2)} + \ldots + a_{2n} v^{(n)} \\
\vdots \\
\lambda_\phi v^{(n)} &= a_{n1} v^{(1)} + a_{n2} v^{(2)} + \ldots + a_{nn} v^{(n)},
\end{align*}
\] (2)

where $a_{ij} \in \mathbb{Z}$. The matrix $A = (a_{ij})$ is invertible. Indeed, since foliation $\mathcal{F}_\phi$ is minimal, real numbers $v^{(1)}, \ldots, v^{(n)}$ are linearly independent over $\mathbb{Q}$. So do numbers $\lambda_\phi v^{(1)}, \ldots, \lambda_\phi v^{(n)}$, which therefore can be taken for a basis of the module $m_\phi$. Thus, there exists an integer matrix $B = (b_{ij})$, such that $v^{(j)} = \sum_{i,j} w^{(i)}$, where $w^{(i)} = \lambda_\phi v^{(i)}$. Clearly, $B$ is an inverse to matrix $A$. Therefore, $A \in GL_n(\mathbb{Z})$.

Moreover, without loss of the generality one can assume that $a_{ij} \geq 0$. Indeed, if it is not yet the case, consider the conjugacy class $[A]$ of the matrix $A$. Since $v^{(i)} > 0$, there exists a matrix $A^+ \in [A]$ whose entries are non-negative integers. One has to replace basis $v = (v^{(1)}, \ldots, v^{(n)})$ in the module $m_\phi$ by a basis $Tv$, where $A^+ = TAT^{-1}$. It will be further assumed that $A = A^+$.

**Lemma 5** Vector $(v^{(1)}, \ldots, v^{(n)})$ is the limit of a periodic Jacobi-Perron continued fraction.

**Proof.** It follows from the discussion above, that there exists a non-negative integer matrix $A$, such that $Av = \lambda_\phi v$. In view of [1], Proposition 3, matrix $A$ admits a unique factorization:
\[ A = \left( \begin{array}{cccc} 0 & 1 \\ I & b_1 \end{array} \right) \ldots \left( \begin{array}{cccc} 0 & 1 \\ I & b_k \end{array} \right), \] (3)

where $b_i = (b_{i1}, \ldots, b_{in})^T$ are vectors of the non-negative integers. Let us consider the periodic Jacobi-Perron continued fraction:
\[ Per\left( \begin{array}{cccc} 0 & 1 \\ I & b_1 \end{array} \right) \ldots \left( \begin{array}{cccc} 0 & 1 \\ I & b_k \end{array} \right) \left( \begin{array}{c} 0 \\ 1 \end{array} \right). \] (4)
According to [15], Satz XII, the above fraction converges to vector \( w = (w(1), \ldots, w(n)) \), such that \( w \) satisfies equation \((B_1B_2\ldots B_k)w = Aw = \lambda_\phi w\). In view of equation \( Av = \lambda_\phi v \), we conclude that vectors \( v \) and \( w \) are collinear. Therefore, the Jacobi-Perron continued fractions of \( v \) and \( w \) must coincide.

\[ \square \]

It is now straightforward to prove, that the AF \( C^* \)-algebra attached to foliation \( \mathcal{F}_\phi \) is stationary. Indeed, by lemma 5, the vector of periods \( v(i) = \int_{\gamma_i} \omega \) unfolds into a periodic Jacobi-Perron continued fraction. By the definition, the Bratteli diagram of the AF \( C^* \)-algebra \( \mathcal{A}_\phi \) is periodic as well. In other words, the AF \( C^* \)-algebra \( \mathcal{A}_\phi \) is stationary. \( \square \)

### 3.2 Proof of theorem 2

(i) Let us prove the first statement. For the sake of completeness, let us give a proof of the following (well-known) lemma.

**Lemma 6** Let \( \phi \) and \( \phi' \) be conjugate pseudo-Anosov automorphisms of a surface \( X \). Then their invariant foliations \( \mathcal{F}_\phi \) and \( \mathcal{F}_{\phi'} \) are equivalent as measured foliations.

**Proof.** Let \( \phi, \phi' \in \text{Mod} \ (X) \) be conjugate, i.e \( \phi' = h \circ \phi \circ h^{-1} \) for an automorphism \( h \in \text{Mod} \ (X) \). Since \( \phi \) is the pseudo-Anosov automorphism, there exists a measured foliation \( \mathcal{F}_\phi \), such that \( \phi(\mathcal{F}_\phi) = \lambda_\phi \mathcal{F}_\phi \). Let us evaluate the automorphism \( \phi' \) on the foliation \( h(\mathcal{F}_\phi) \):

\[
\phi'(h(\mathcal{F}_\phi)) = h\phi h^{-1}(h(\mathcal{F}_\phi)) = h\phi(\mathcal{F}_\phi) = \lambda_\phi h(\mathcal{F}_\phi).
\]

Thus, \( \mathcal{F}_{\phi'} = h(\mathcal{F}_\phi) \) is the invariant foliation for the pseudo-Anosov automorphism \( \phi' \) and \( \mathcal{F}_\phi, \mathcal{F}_{\phi'} \) are equivalent foliations. Note also that the pseudo-Anosov automorphism \( \phi' \) has the same dilatation as the automorphism \( \phi \).

\[ \square \]

Suppose that \( \phi \) and \( \phi' \) are conjugate pseudo-Anosov automorphisms. Functor \( F \) acts by the formulas \( \phi \mapsto \mathcal{A}_\phi \) and \( \phi' \mapsto \mathcal{A}_{\phi'} \), where \( \mathcal{A}_\phi, \mathcal{A}_{\phi'} \) are the AF \( C^* \)-algebras corresponding to invariant foliations \( \mathcal{F}_\phi, \mathcal{F}_{\phi'} \). In view of lemma 6, \( \mathcal{F}_\phi \) and \( \mathcal{F}_{\phi'} \) are equivalent measured foliations. Then, by lemma 4, the
AF $C^*$-algebras $A_\phi$ and $A_{\phi'}$ are stably isomorphic AF $C^*$-algebras. Item (i) follows.

(ii) Let us prove the second statement. We start with an elementary observation. Let $\phi \in \text{Mod} (X)$ be a pseudo-Anosov automorphism. Then there exists a unique measured foliation, $\mathcal{F}_\phi$, such that $\phi(\mathcal{F}_\phi) = \lambda_\phi \mathcal{F}_\phi$, where $\lambda_\phi > 1$ is an algebraic integer. Let us evaluate automorphism $\phi^2 \in \text{Mod} (X)$ on the foliation $\mathcal{F}_\phi$:

$$
\phi^2(\mathcal{F}_\phi) = \phi(\phi(\mathcal{F}_\phi)) = \phi(\lambda_\phi \mathcal{F}_\phi) = \\
= \lambda_\phi \phi(\mathcal{F}_\phi) = \lambda_\phi^2 \mathcal{F}_\phi = \lambda_{\phi^2} \mathcal{F}_\phi,
$$

(6)

where $\lambda_{\phi^2} := \lambda_\phi^2$. Thus, foliation $\mathcal{F}_\phi$ is an invariant foliation for the automorphism $\phi^2$ as well. By induction, one concludes that $\mathcal{F}_\phi$ is an invariant foliation of the automorphism $\phi^n$ for any $n \geq 1$.

Even more is true. Suppose that $\psi \in \text{Mod} (X)$ is a pseudo-Anosov automorphism, such that $\psi^m = \phi^n$ for some $m \geq 1$ and $\psi \neq \phi$. Then $\mathcal{F}_\phi$ is an invariant foliation for the automorphism $\psi$. Indeed, $\mathcal{F}_\phi$ is invariant foliation of the automorphism $\psi^m$. If there exists $\mathcal{F}' \neq \mathcal{F}_\phi$ such that the foliation $\mathcal{F}'$ is an invariant foliation of $\psi$, then the foliation $\mathcal{F}'$ is also an invariant foliation of the pseudo-Anosov automorphism $\psi^m$. Thus, by the uniqueness, $\mathcal{F}' = \mathcal{F}_\phi$. We have just proved the following lemma.

**Lemma 7** Let $\phi$ be the pseudo-Anosov automorphism of a surface $X$. Denote by $[\phi]$ a set of the pseudo-Anosov automorphisms $\psi$ of $X$, such that $\psi^m = \phi^n$ for some positive integers $m$ and $n$. Then the pseudo-Anosov foliation $\mathcal{F}_\phi$ is an invariant foliation for every pseudo-Anosov automorphism $\psi \in [\phi]$.

In view of lemma 7, one arrives at the following identities among the AF $C^*$-algebras:

$$
A_\phi = A_{\phi^2} = \ldots = A_{\phi^n} = A_{\psi^m} = \ldots = A_{\psi^2} = A_{\psi}.
$$

(7)

Thus, functor $F$ is not an injective functor: the preimage, $\text{Ker} F$, of algebras $A_\phi$ consists of a countable set of the pseudo-Anosov automorphisms $\psi \in [\phi]$, commensurable with the automorphism $\phi$.

Theorem 2 is proved. □
3.3 Proof of corollary 1

(i) It follows from theorem 1, that $A_\phi$ is a stationary AF $C^*$-algebra. An arithmetic invariant of the stable isomorphism classes of the stationary AF $C^*$-algebras has been found by D. Handelman in [7]. Summing up his results, the invariant is as follows.

Let $A \in GL_n(\mathbb{Z})$ be a matrix with the strictly positive entries, such that $A$ is equal to the minimal period of the Bratteli diagram of the stationary AF $C^*$-algebra. (In case the matrix $A$ has zero entries, it is necessary to take a proper minimal power of the matrix $A$.) By the Perron-Frobenius theory, matrix $A$ has a real eigenvalue $\lambda_A > 1$, which exceeds the absolute values of other roots of the characteristic polynomial of $A$. Note that $\lambda_A$ is an invertible algebraic integer (the unit). Consider the real algebraic number field $K = \mathbb{Q}(\lambda_A)$ obtained as an extension of the field of the rational numbers by the algebraic number $\lambda_A$. Let $(v_A^{(1)}, \ldots, v_A^{(n)})$ be the eigenvector corresponding to the eigenvalue $\lambda_A$. One can normalize the eigenvector so that $v_A^{(i)} \in K$.

The departure point of Handelman’s invariant is the $\mathbb{Z}$-module $m = \mathbb{Z}v_A^{(1)} + \ldots + \mathbb{Z}v_A^{(n)}$. The module $m$ brings in two new arithmetic objects: (i) the ring $\Lambda$ of the endomorphisms of $m$ and (ii) an ideal $I$ in the ring $\Lambda$, such that $I = m$ after a scaling ([4], Lemma 1, p.88). The ring $\Lambda$ is an order in the algebraic number field $K$ and therefore one can talk about the ideal classes in $\Lambda$. The ideal class of $I$ is denoted by $[I]$. Omitting the embedding question for the field $K$, the triple $(\Lambda, [I], K)$ is an invariant of the stable isomorphism class of the stationary AF $C^*$-algebra $A_\phi$ (§5 of [7]). Item (i) follows.

(ii) Numerical invariants of the stable isomorphism classes of the stationary AF $C^*$-algebras can be derived from the triple $(\Lambda, [I], K)$. These invariants are the rational integers – called the determinant and signature – can be obtained as follows.

Let $m, m'$ be the full $\mathbb{Z}$-modules in an algebraic number field $K$. It follows from (i), that if $m \neq m'$ are distinct as the $\mathbb{Z}$-modules, then the corresponding AF $C^*$-algebras cannot be stably isomorphic. We wish to find the numerical invariants, which discern the case $m \neq m'$. It is assumed that a $\mathbb{Z}$-module is given by the set of generators $\{\lambda_1, \ldots, \lambda_n\}$. Therefore, the problem can be formulated as follows: find a number attached to the set of generators $\{\lambda_1, \ldots, \lambda_n\}$, which does not change on the set of generators $\{\lambda'_1, \ldots, \lambda'_n\}$ of the same $\mathbb{Z}$-module.
One such invariant is associated with the trace function on the algebraic number field \( K \). Recall that \( \text{Tr} : K \to \mathbb{Q} \) is a linear function on \( K \) such that 
\[
\text{Tr} (\alpha + \beta) = \text{Tr} (\alpha) + \text{Tr} (\beta) \quad \text{and} \quad \text{Tr} (a\alpha) = a \text{Tr} (\alpha) \quad \forall \alpha, \beta \in K \quad \text{and} \quad \forall a \in \mathbb{Q}.
\]

Let \( m \) be a full \( \mathbb{Z} \)-module in the field \( K \). The trace function defines a symmetric bilinear form \( q(x, y) : m \times m \to \mathbb{Q} \) by the formula:
\[
(x, y) \mapsto \text{Tr} (xy), \quad \forall x, y \in m.
\]

The form \( q(x, y) \) depends on the basis \( \{\lambda_1, \ldots, \lambda_n\} \) in the module \( m \):
\[
q(x, y) = \sum_{j=1}^{n} \sum_{i=1}^{n} a_{ij} x_i y_j, \quad \text{where} \quad a_{ij} = \text{Tr} (\lambda_i \lambda_j).
\]

However, the general theory of the bilinear forms (over the fields \( \mathbb{Q}, \mathbb{R}, \mathbb{C} \) or the ring of rational integers \( \mathbb{Z} \)) tells us that certain numerical quantities will not depend on the choice of such a basis.

Namely, one such invariant is as follows. Consider a symmetric matrix \( A \) corresponding to the bilinear form \( q(x, y) \):
\[
A = \begin{pmatrix}
  a_{11} & a_{12} & \cdots & a_{1n} \\
  a_{12} & a_{22} & \cdots & a_{2n} \\
  \vdots & \vdots & & \vdots \\
  a_{1n} & a_{2n} & \cdots & a_{nn}
\end{pmatrix}.
\]

It is known that the matrix \( A \), written in a new basis, will take the form \( A' = U^T A U \), where \( U \in GL_n(\mathbb{Z}) \). Then \( \text{Det} (A') = \text{Det} (U^T A U) = \text{Det} (U^T) \text{Det} (A) \text{Det} (U) = \text{Det} (A) \). Therefore, the rational integer number:
\[
\Delta = \text{Det} (\text{Tr} (\lambda_i \lambda_j)),
\]
called a determinant of the bilinear form \( q(x, y) \), does not depend on the choice of the basis \( \{\lambda_1, \ldots, \lambda_n\} \) in the module \( m \). We conclude that the determinant \( \Delta \) discerns \( m \neq m' \).

\(^1\text{Note that if} \Delta = \Delta' \text{for the modules} m, m', \text{one cannot conclude that} m = m'. \text{The problem of equivalence of the symmetric bilinear forms over} \mathbb{Q} \text{(i.e. the existence of a linear substitution over} \mathbb{Q}, \text{which transforms one form to the other), is a fundamental question of number theory. The Minkowski-Hasse theorem says that two such forms are}
\]
Finally, recall that the form \( q(x, y) \) can be brought by an integer linear transformation to the diagonal form:

\[
a_1x_1^2 + a_2x_2^2 + \ldots + a_nx_n^2,
\]

where \( a_i \in \mathbb{Z} \setminus \{0\} \). We let \( a_i^+ \) be the positive and \( a_i^- \) the negative entries in the diagonal form. In view of the law of inertia for the bilinear forms, the integer number \( \Sigma = (#a_i^+) - (#a_i^-) \), called a signature, does not depend on a particular choice of the basis in the module \( m \). Thus, \( \Sigma \) discerns the modules \( m \neq m' \). Corollary 1 follows. □

4 Examples, open problems and conjectures

In the present section we shall calculate invariants \( \Delta \) and \( \Sigma \) for the Anosov automorphisms of the two-dimensional torus. Examples of two non-conjugate Anosov automorphisms with the same Alexander polynomial, but different determinants \( \Delta \) are constructed. Recall that isotopy classes of the orientation-preserving diffeomorphisms of the torus \( T^2 \) are bijective with the \( 2 \times 2 \) matrices with integer entries and determinant +1, i.e. \( \text{Mod} \ (T^2) \cong SL(2, \mathbb{Z}) \). Under the identification, the non-periodic automorphisms correspond to the matrices \( A \in SL(2, \mathbb{Z}) \) with \( |\text{Tr} A| > 2 \).

4.1 Full modules and orders in the quadratic field

Let \( K = \mathbb{Q}(\sqrt{d}) \) be a quadratic extension of the field of rational numbers \( \mathbb{Q} \). Further we suppose that \( d \) is a positive square free integer. Let

\[
\omega = \begin{cases} 
\frac{1+\sqrt{d}}{2} & \text{if } d \equiv 1 \mod 4, \\
\sqrt{d} & \text{if } d \equiv 2, 3 \mod 4.
\end{cases}
\]

(13)

Proposition 1 Let \( f \) be a positive integer. Every order in \( K \) has form \( \Lambda_f = \mathbb{Z} + (f \omega)\mathbb{Z} \), where \( f \) is the conductor of \( \Lambda_f \).

equivalent if and only if they are equivalent over the field \( \mathbb{Q}_p \) for every prime number \( p \) and over the field \( \mathbb{R} \). Clearly, the resulting \( p \)-adic quantities will give new invariants of the stable isomorphism classes of the AF \( C^* \)-algebras. The question is much similar to the Minkowski units attached to knots, see e.g. Reidemeister [16]. We will not pursue this topic here and refer the reader to the problem part of present article.
Proof. See [4] pp. 130-132. □

The proposition 1 allows to classify the similarity classes of the full modules in the field $K$. Indeed, there exists a finite number of $m_f^{(1)}, \ldots, m_f^{(s)}$ of the non-similar full modules in the field $K$, whose coefficient ring is the order $\Lambda_f$, cf Theorem 3, Ch 2.7 of [4]. Thus, proposition 1 gives a finite-to-one classification of the similarity classes of full modules in the field $K$.

4.2 Numerical invariants of the Anosov automorphisms

Let $\Lambda_f$ be an order in $K$ with the conductor $f$. Under the addition operation, the order $\Lambda_f$ is a full module, which we denote by $m_f$. Let us evaluate the invariants $q(x, y), \Delta$ and $\Sigma$ on the module $m_f$. To calculate $(a_{ij}) = \text{Tr} (\lambda_i \lambda_j)$, we let $\lambda_1 = 1, \lambda_2 = f\omega$. Then:

\[
a_{11} = 2, \quad a_{12} = a_{21} = f, \quad a_{22} = \frac{1}{2}f^2(d+1) \quad \text{if} \quad d \equiv 1 \mod 4
\]
\[
a_{11} = 2, \quad a_{12} = a_{21} = 0, \quad a_{22} = 2f^2d \quad \text{if} \quad d \equiv 2, 3 \mod 4, \quad (14)
\]

and

\[
q(x, y) = 2x^2 + 2fxy + \frac{1}{2}f^2(d+1)y^2 \quad \text{if} \quad d \equiv 1 \mod 4
\]
\[
q(x, y) = 2x^2 + 2f^2dy^2 \quad \text{if} \quad d \equiv 2, 3 \mod 4. \quad (15)
\]

Therefore

\[
\Delta = \begin{cases} 
  f^2d & \text{if } d \equiv 1 \mod 4, \\
  4f^2d & \text{if } d \equiv 2, 3 \mod 4,
\end{cases} \quad (16)
\]

and $\Sigma = +2$ in the both cases, where $\Sigma = \#(\text{positive}) - \#(\text{negative})$ entries in the diagonal normal form of $q(x, y)$.

4.3 Examples

Let us consider some numerical examples, which illustrate advantages of our invariants in comparison to the classical Alexander polynomials.

Example 1 Denote by $M_A$ and $M_B$ the hyperbolic 3-dimensional manifolds obtained as a torus bundle over the circle with the monodromies

\[
A = \begin{pmatrix} 5 & 2 \\
2 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 5 & 1 \\
4 & 1 \end{pmatrix}, \quad (17)
\]
respectively. The Alexander polynomials of $M_A$ and $M_B$ are identical $\Delta_A(t) = \Delta_B(t) = t^2 - 6t + 1$. However, the manifolds $M_A$ and $M_B$ are not homotopy equivalent. Indeed, the Perron-Frobenius eigenvector of matrix $A$ is $v_A = (1, \sqrt{2} - 1)$ while of the matrix $B$ is $v_B = (1, 2\sqrt{2} - 2)$. The bilinear forms for the modules $m_A = Z + (\sqrt{2} - 1)\mathbb{Z}$ and $m_B = Z + (2\sqrt{2} - 2)\mathbb{Z}$ can be written as

$$q_A(x, y) = 2x^2 - 4xy + 6y^2, \quad q_B(x, y) = 2x^2 - 8xy + 24y^2,$$  \qquad (18)

respectively. The modules $m_A, m_B$ are not similar in the number field $K = \mathbb{Q}(\sqrt{2})$, since their determinants $\Delta(m_A) = 8$ and $\Delta(m_B) = 32$ are not equal. Therefore, matrices $A$ and $B$ are not conjugate\(^2\) in the group $SL(2, \mathbb{Z})$. Note that the class number $h_K = 1$ for the field $K$.

**Example 2 ([8], p.12)** Let $M_A$ and $M_B$ be 3-dimensional manifolds corresponding to matrices

$$A = \begin{pmatrix} 4 & 3 \\ 5 & 4 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 4 & 15 \\ 1 & 4 \end{pmatrix},$$  \qquad (19)

respectively. The Alexander polynomials of $M_A$ and $M_B$ are identical $\Delta_A(t) = \Delta_B(t) = t^2 - 8t + 1$. Yet the manifolds $M_A$ and $M_B$ are not homotopy equivalent. Indeed, the Perron-Frobenius eigenvector of matrix $A$ is $v_A = (1, 1\sqrt{\frac{1}{2}}\sqrt{15})$ while of the matrix $B$ is $v_B = (1, 1\sqrt{\frac{1}{2}}\sqrt{15})$. The corresponding modules are $m_A = Z + (\frac{1}{2}\sqrt{15})\mathbb{Z}$ and $m_B = Z + (\frac{1}{10}\sqrt{15})\mathbb{Z}$; note that $d = 15 \equiv 3 \mod 4$ in both cases, but the corresponding conductors are $f_A = 3$ and $f_B = 15$. Using formulas (15) one finds

$$q_A(x, y) = 2x^2 + 18y^2, \quad q_B(x, y) = 2x^2 + 450y^2,$$  \qquad (20)

respectively. The modules $m_A, m_B$ are not similar in the number field $K = \mathbb{Q}(\sqrt{15})$, since formulas (16) imply that their determinants $\Delta(m_A) = 36$ and $\Delta(m_B) = 900$ are not equal. Therefore, matrices $A$ and $B$ are not conjugate in the group $SL(2, \mathbb{Z})$.

\(^2\)The reader may verify this fact using the method of periods, which dates back to Gauss. First we have to find the fixed points $Ax = x$ and $Bx = x$, which gives us $x_A = 1 + \sqrt{2}$ and $x_B = \frac{1 + \sqrt{2}}{10}$, respectively. Then one unfolds the fixed points into a periodic continued fraction, which gives us $x_A = [2, 2, 2, \ldots]$ and $x_B = [1, 4, 1, 4, \ldots]$. Since the period (2) of $x_A$ differs from the period (1, 4) of $B$, the matrices $A$ and $B$ belong to different conjugacy classes in $SL(2, \mathbb{Z})$.  

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Example 3 ([8], p.12) Let \(a, b\) be a pair of positive integers satisfying the Pell equation \(a^2 - 8b^2 = 1\); the latter has infinitely many solutions, e.g. \(a = 3, b = 1\), etc. Denote by \(M_A\) and \(M_B\) the 3-dimensional manifolds corresponding to matrices

\[
A = \begin{pmatrix} a & 4b \\ 2b & a \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} a & 8b \\ b & a \end{pmatrix},
\]

(21) respectively. The Alexander polynomials of \(M_A\) and \(M_B\) are identical \(\Delta_A(t) = \Delta_B(t) = t^2 - 2at + 1\). Yet manifolds \(M_A\) and \(M_B\) are not homotopy equivalent. Indeed, the Perron-Frobenius eigenvector of matrix \(A\) is \(v_A = (1, \frac{1}{4b} \sqrt{a^2 - 1})\) while of the matrix \(B\) is \(v_B = (1, \frac{1}{8b} \sqrt{a^2 - 1})\). The corresponding modules are \(m_A = \mathbb{Z} + (\frac{1}{4b} \sqrt{a^2 - 1})\mathbb{Z}\) and \(m_B = \mathbb{Z} + (\frac{1}{8b} \sqrt{a^2 - 1})\mathbb{Z}\). It is easy to see that the discriminant \(d = a^2 - 1 \equiv 3 \mod 4\) for all \(a \geq 2\). Indeed, \(d = (a-1)(a+1)\) and, therefore, integer \(a \not\equiv 1; 3 \mod 4\); hence \(a \equiv 2 \mod 4\) so that \(a - 1 \equiv 1 \mod 4\) and \(a + 1 \equiv 3 \mod 4\) and, thus, \(d = a^2 - 1 \equiv 3 \mod 4\). Therefore the corresponding conductors are \(f_A = 4b\) and \(f_B = 8b\), and

\[
q_A(x, y) = 2x^2 + 32b^2(a^2 - 1)y^2, \quad q_B(x, y) = 2x^2 + 128b^2(a^2 - 1)y^2,
\]

(22) respectively. The modules \(m_A, m_B\) are not similar in the number field \(K = \mathbb{Q}(\sqrt{a^2 - 1})\), since their determinants \(\Delta(m_A) = 64b^2(a^2 - 1)\) and \(\Delta(m_B) = 256b^2(a^2 - 1)\) are not equal. Therefore, matrices \(A\) and \(B\) are not conjugate in the group \(SL(2, \mathbb{Z})\).

4.4 Open problems and conjectures

This section is reserved for some questions and conjectures, which arise in connection with the invariants \((\Lambda, [I], K), q(x, y), \Delta\) and \(\Sigma\).

1. \(P\)-adic invariants of the pseudo-Anosov automorphisms

A. Let \(\phi \in Mod(X)\) be pseudo-Anosov automorphism of a surface \(X\). If \(\lambda_{\phi}\) is the dilatation of \(\phi\), then one can consider a \(\mathbb{Z}\)-module \(m = \mathbb{Z}v^{(1)} + \ldots + \mathbb{Z}v^{(n)}\) in the number field \(K = \mathbb{Q}(\lambda_{\phi})\) generated by the normalized eigenvector \((v^{(1)}, \ldots, v^{(n)})\) corresponding to the eigenvalue \(\lambda_{\phi}\). The trace function on the number field \(K\) gives rise to a symmetric bilinear form \(q(x, y)\) on the module \(m\). The form is defined over the field \(\mathbb{Q}\). It has been shown that a pseudo-Anosov automorphism \(\phi'\), conjugate to \(\phi\), yields a form \(q'(x, y)\), equivalent
to \( q(x, y) \), i.e. \( q(x, y) \) can be transformed to \( q'(x, y) \) by an invertible linear substitution with the coefficients in \( \mathbb{Z} \).

**B.** Recall that two rational bilinear forms \( q(x, y) \) and \( q'(x, y) \) are equivalent whenever the following conditions are met:

(i) \( \Delta = \Delta' \), where \( \Delta \) is the determinant of the form;

(ii) for each prime number \( p \) (including \( p = \infty \)) certain \( p \)-adic equation between the coefficients of forms \( q, q' \) must be satisfied, see e.g. [4], Ch.1, §7.5. (In fact, only a finite number of such equations have to be verified.)

Condition (i) has been already used to discern between the conjugacy classes of the pseudo-Anosov automorphisms. One can use condition (ii) to discern between the pseudo-Anosov automorphisms with \( \Delta = \Delta' \). The following question can be posed: *Find the \( p \)-adic invariants of the pseudo-Anosov automorphisms.*

2. **Signature of the pseudo-Anosov automorphism**

The signature is an important and well-known invariant connected to the chirality and knotting number of knots and links [16]. It will be interesting to find a geometric interpretation of the signature \( \Sigma \) for the pseudo-Anosov automorphisms. One can ask the following question: *Find a geometric meaning of the invariant \( \Sigma \).*

3. **Number of conjugacy classes of the pseudo-Anosov automorphisms with the same dilatation**

The dilatation \( \lambda_\phi \) is an invariant of the conjugacy class of the pseudo-Anosov automorphism \( \phi \in \text{Mod} \( X \). On the other hand, it is known that there exist non-conjugate pseudo-Anosov’s with the same dilatation and the number of such classes is finite [18]. It is natural to expect that the invariants of operator algebras can be used to evaluate the number. We conclude with the following conjecture.

**Conjecture 1** Let \((\Lambda, [I], K)\) be the triple corresponding to a pseudo-Anosov automorphism \( \phi \in \text{Mod} \( X \). Then the number of the conjugacy classes of the pseudo-Anosov automorphisms with the dilatation \( \lambda_\phi \) is equal to the class number \( h_\Lambda = |\Lambda/[I]| \) of the integral order \( \Lambda \).

**Acknowledgment.** I thank the referee for helpful comments and Daniel Silver and Susan Williams for their interest and hospitality.
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