SECOND ORDER RIESZ TRANSFORMS ON
MULTIPLY–CONNECTED LIE GROUPS AND PROCESSES
WITH JUMPS

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Abstract. We study a class of combinations of second order Riesz transforms on Lie groups \( \mathbb{G} = \mathbb{G}_x \times \mathbb{G}_y \) that are multiply connected, composed of a discrete abelian component \( \mathbb{G}_x \) and a compact connected component \( \mathbb{G}_y \). We prove sharp \( L^p \) estimates for these operators, therefore generalizing previous results \[31\][17][5].

We construct stochastic integrals with jump components adapted to functions defined on the semi-discrete set \( \mathbb{G}_x \times \mathbb{G}_y \). We show that these second order Riesz transforms applied to a function may be written as conditional expectation of a simple transformation of a stochastic integral associated with the function. The analysis shows that Itô integrals for the discrete component must be written in an augmented discrete tangent plane of dimension twice larger than expected, and in a suitably chosen discrete coordinate system. Those artifacts are related to the difficulties that arise due to the discrete component, where derivatives of functions are no longer local. Previous representations of Riesz transforms through stochastic integrals in this direction do not consider discrete components and jump processes.

1. Introduction

Sharp \( L^p \) inequalities for pairs of differentially subordinate martingales date back to the celebrated work of Burkholder [10] in 1984 where the optimal constant is exhibited. See also from the same author [12] [13]. The relation between differentially subordinate martingales and Caldéron-Zygmund operators is known since Gundy–Varopoulos [23]. Banuelos–Wang [8] were the first to exploit this connection to prove new sharp inequalities for singular integrals. A vast literature has since then been accumulating on this line of research, some of which will be discussed below.

In this article we bring for the first time this whole circle of ideas to a semi-discrete setting, applying it to a family of second order Riesz transforms on multiply–connected Lie groups. We prove optimal norm estimates in \( L^p \) for these operators as well as derive their representation through stochastic integrals using jump processes on multiply–connected Lie groups.

Before we state our results in a complete form, we present a sample case which requires little notation.

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Consider a Lie group $\mathbb{G} := \mathbb{G}_x \times \mathbb{G}_y$, where $\mathbb{G}_x$ is a discrete abelian group with a fixed finite set $G = \{g_i, g_i^{-1} : i = 1, \ldots, m\}$ of generators, and their reciprocals, and $\mathbb{G}_y$ is a connected, compact Lie group of dimension $n$ endowed with a biinvariant metric. Given a smooth function $f \in \mathbb{G}$, consider a fixed orthonormal basis $Y_1, \ldots, Y_n$ of the Lie algebra $\mathfrak{g}_y$ of $\mathbb{G}_y$. The gradient of a function and the divergence of a vector field with respect to the variable $y$, and the Laplace operator $\Delta_y$, can all be expressed in terms of the fixed basis.

Discrete partial derivatives are defined in analogy. Given any point $(x, y) \in \mathbb{G}_x \times \mathbb{G}_y$, and given a direction $i \in \{1, \ldots, m\}$, the right and the left derivative at $(x, y)$ in the direction $i$ are:

$$\partial_i^+ f(x, y) := f(x + g_i, y) - f(x, y) =: X_i^+ f(x, y),$$

$$\partial_i^- f(x, y) := f(x, y) - f(x - g_i, y) =: X_i^- f(x, y).$$

The discrete Laplace operator $\Delta_x$ is then defined as

$$\Delta_x f(x, y) = \sum_{i=1}^m X_i^- X_i^+ f(x, y).$$

Let $z = (x, y)$ and $\Delta_z = \Delta_x + \Delta_y$. For $1 < p < \infty$, let $p^* = \max\{p, \frac{p}{p-1}\}$.

The $L^p$-norm of a function is computed w.r.t. to the Haar measure on $\mathbb{G}$.

The inequalities in the Theorem below can be rephrased in terms of the Riesz transform estimates proved in Theorem 2.

**Theorem.** If $f : \mathbb{G} \to \mathbb{C}$ is smooth and $1 < p < \infty$, then

$$\|Y_j Y_l f\|_{L^p} \leq (p^* - 1)\|(-\Delta_z) f\|_{L^p}$$

and

$$\|X_i^+ X_i^- f\|_{L^p} \leq (p^* - 1)\|(-\Delta_z) f\|_{L^p}$$

for $1 \leq j, l \leq n$ and $1 \leq i \leq m$.

The inequalities in the Theorem can be rephrased in terms of Riesz transforms. These can be formally defined as $R_i^2 := R_i^+ \circ R_i^- := (X_i^+ \circ (-\Delta_z)^{-1/2}) \circ (X_i^- \circ (-\Delta_z)^{-1/2}) = X_i^+ \circ X_i^- \circ (-\Delta_z)^{-1}, 1 \leq i \leq m$, in the discrete directions, and $R_{jk} = R_j \circ R_k = \ldots = Y_j \circ Y_k \circ (-\Delta_z)^{-1}, 1 \leq j, k \leq n$ in the continuous directions.

The standard procedure for obtaining inequalities for singular integrals from inequalities for martingales can be described as follows. Starting with a test function $f$, martingales are built using Brownian motion and Poisson extensions in the upper half space $\mathbb{R}^+ \times \mathbb{R}^n$. It is shown that the martingale arising from $Rf$, where $R$ is a Riesz transform in $\mathbb{R}^n$, is a martingale transform of that arising from $f$. The two form a pair of martingales with differential subordination and orthogonality. Thus, in the case of Riesz tranforms, the optimal $L^p$ constants could be recovered using probabilistic methods. One derives martingale inequalities under hypotheses of strong differential subordination and orthogonality relations, see Banuelos–Wang [8].
In the case of second order Riesz transforms, the use of heat extensions in the upper half space instead of Poisson extensions originated in Petermichl-Volberg \cite{26} and was used to prove $L^p$ estimates for the second order Riesz transforms based on the results of Burkholder in Nazarov-Volberg \cite{31} as part of their best-at-time estimate for the Beurling-Ahlfors operator, whose real and imaginary parts themselves are second order Riesz transforms.

Here is how this idea takes shape in our case.

**Theorem.** The second order Riesz transforms $R_{ij}^2 f$, $1 \leq i \leq m$, and $R_{jk}^f$, $1 \leq j,k \leq n$, of a function $f \in L^2(\mathbb{G})$ as defined in \cite{7} can be written as the conditional expectations

$$R_{ij}^2 f(z) = \mathbb{E}(M_{ij}^{2,f} | Z_0 = z)$$

and

$$R_{jk}^f(z) = \mathbb{E}(M_{jk}^{f} | Z_0 = z).$$

Here $M_{ij}^{2,f}$ and $M_{jk}^{f}$ are suitable martingale transforms of the martingale $M_t^f$ associated to $f$, and $Z_t$ is a suitable random walk on $\mathbb{G}$ (see Section 2).

All these $L^p$ norm inequalities use special functions found in Pichorides \cite{27}, Essén \cite{21}, Banuelos-Wang \cite{8} when orthogonality is present in addition to differential subordination or Burkholder \cite{10,11,12}, Wang \cite{32} when differential subordination is the only hypothesis.

Deterministic proofs of sharp $L^p$ estimates of Caldéron-Zygmund operators that use Burkholder’s theorems are available in the literature. The technique of Bellman functions was used in Nazarov-Volberg \cite{31} for an $L^p$ estimate for certain second order Riesz transforms in the Euclidean plane as well as in the recent version on discrete abelian groups Domelevo-Petermichl \cite{17}.

The aim of the present work is two-fold. On the one hand, we want to generalize the estimate to second order Riesz transforms acting on multiply connected Lie groups, built as the cartesian product of a discrete abelian group with a connected compact Lie group. Previous works based on stochastic methods for the analysis of Riesz transforms on connected compact Lie groups are in Arcozzi \cite{1,2}, and sharp $L^p$ estimates were proved in this setting in Banuelos-Baudoin \cite{5} for second order Riesz transforms. The novelty of this text is the generalization to the multiply connected setting. In this sense, it is also a generalization of \cite{17}, by regarding each point in the discrete abelian group as a Lie group of dimension zero.

On the other hand we want to derive a probabilistic proof through the use of an identity formula involving stochastic integration. We make use of stochastic integrals with jump components adapted to functions defined on the semi-discrete set $\mathbb{G}_x \times \mathbb{G}_y$. We show that corresponding second order Riesz transforms applied to a function may be written as conditional expectation of a simple transformation of a stochastic integral associated with the function. The desired $L^p$ estimates then follow from the work by Wang \cite{32}, who had identified the correct requirements on differential subordination in the presence of discontinuity in space.
Such probabilistic representations can be very advantageous. For example, the conjectured estimate for the $L^p$ norms of the Beurling-Ahlfors transform is a fascinating, famous open question. The argument in [31] is deterministic in spirit and gave a new best estimate at its time, twice larger than expected. Subsequent improvements profited substantially from the finer structure that remains intact through the use of stochastic integration and a probabilistic representation formula of second order Riesz transforms due to Bañuelos–Méndez–Hernandez [3]. This representation was subsequently used on many occasions, notably in the improvement for the norm estimate of the Beurling-Ahlfors operator by Bañuelos–Janakiraman [6]. Their result was a major advance in this direction, since it was the first that dropped the constant below twice larger than expected, thus confirming the suspicion that the conjectured estimate should hold true. Further improvements were made by Borichev–Janakiraman–Volberg [9], also through the refined use of stochastic integration formulae and a deep, new martingale estimate.

The sharpness of the constant for the real part of the Beurling-Ahlfors operator, a combination of perfect squares of Riesz transforms, was proved using probabilistic methods in conjunction with a modification of a technique by Bourgain in Geiss–Montgomery–Saksman [22]. It turns out the real part of the Beurling-Ahlfors operator alone already attains the conjectured norm estimate. See also applications in Bañuelos–Baudoin [5].

We expect our representation of semi–discrete second order Riesz transforms to have further applications, such as to UMD spaces (see [22]), as well as logarithmic and weak–type estimates (see [24][25]), currently under investigation with Osekowski.

Bañuelos–Wang [8] addressed càdlàg processes in the presence of orthogonality relations, such as one sees in the continuous setting when considering martingales that represent the Hilbert transform through stochastic integrals. It is remarkable that the $L^p$ estimates of the discrete Hilbert transform on the integers are still unknown. It is a famous conjecture that this operator has the same norm as its continuous counterpart. We hope advances made in this paper give new ideas on how to address this question.

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1.1. Differential operators and Riesz transforms.

First order derivatives and tangent planes. We will consider Lie groups $G := G_x \times G_y$, where $G_x$ is a discrete abelian group with a fixed set $G$ of $m$ generators, and their reciprocals, and $G_y$ is a connected, compact Lie group of dimension $n$ endowed with a biinvariant metric. The choice of the set $G$ of generators in $G_x$ corresponds to the choice of a bi-invariant metric structure on $G_x$. We will use on $G_x$ the multiplicative notation
for the group operation. We will define a product metric structure on $G$, which agrees with the Riemannian structure on the first factor, and with the discrete “word distance” on the second. We will at the same time define a “tangent space” $T_xG$ for $G$ at a point $z = (x, y) \in (G_x \times G_y) = G$. We will do this in three steps.

First, since $G_y$ is an $n$-dimensional connected Lie group with Lie algebra $\mathfrak{g}_y$. We can identify each left-invariant vector field $y$ in $\mathfrak{g}_y$ with its value at the identity $e$, $\mathfrak{g}_y \equiv T_eG_y$. Since $G$ is compact, it admits a bi-invariant Riemannian metric, which is unique up to a multiplicative factor. We normalize it so that the measure $\mu_y$ associated with the metric satisfies $\mu_y(G_y) = 1$. The measure $\mu_y$ is also the normalized Haar measure of the group. We denote by $\langle \cdot, \cdot \rangle$ the corresponding inner product on $T_yG_y$ and by $\nabla_y f(y)$ the gradient at $y \in G_y$ of a smooth function $f : G_y \to \mathbb{R}$. Let $Y_1, \ldots, Y_n$ be an orthonormal basis for $\mathfrak{g}_y$. The gradient of $f$ can be written $\nabla_y f = Y_1(f)Y_1 + \ldots + Y_n(f)Y_n$.

Second, in the discrete component $G_x$, let $\mathfrak{g}_x = (g_i)_{i=1,...,m}$ be a set of generators for $G_x$, such that for $i \neq j$ and $\sigma = \pm 1$ we have $g_i \neq g_j^\sigma$. The choice of a particular set of generators induces a word metric, hence, a geometry, on $G_x$. Any two sets of generators induce bi-Lipschitz equivalent metrics.

At any point $x \in G_x$, and given a direction $i \in \{1, \ldots, m\}$, we can define the right and the left derivative at $x$ in the direction $i$:

$$(\partial^+ f/\partial x_i)(x, y) := f(x + g_i, y) - f(x, y) := (\partial_i^+ f)(x, y)$$

$$(\partial^- f/\partial x_i)(x, y) := f(x, y) - f(x - g_i, y) := (\partial_i^- f)(x, y).$$

Comparing with the continuous component, this suggests that the tangent plane $T_xG_x$ at a point $x$ of the discrete group $G_x$ might actually be split into a “right” tangent plane $T_x^+G_x$ and a “left” tangent plane $T_x^-G_x$, according to the direction with respect to which discrete differences are computed. We consequently define the augmented discrete gradient $\nabla_x f(x)$, with a hat, as the $2m$-vector of $\hat{T}_xG_x := T_x^+G_x \oplus T_x^-G_x$ accounting for all the local variations of the function $f$ in the direct vicinity of $x$; that is, the $2m$-column–vector

$$\hat{\nabla}_x f(x) := (X_1^+ f, X_2^+ f, \ldots, X_m^+ f, X_m^- f, \ldots)(x) = \sum_{i=1}^{m} \sum_{\tau=\pm} X_i^\tau f(x)$$

with $X_i^\tau \in \hat{T}_xG_x$, where we noted the discrete derivatives $X_i^\tau f := \partial_i^\tau f$ and introduced the discrete $2m$–vectors $X_i^\pm$ as the column vectors of $\mathbb{Z}^{2m}$

$$X_i^+ = (0, \ldots, 1, \ldots, 0) \times 0_m, \quad X_i^- = 0_m \times (0, \ldots, 1, \ldots, 0).$$

Here the 1’s in $X_i^\pm$ are located at the $i$–th position of respectively the first or the second $m$–tuple. Notice that those vectors are independent of the
point \( x \). The scalar product on \( T_xG_x := T_x^+G_x \oplus T_x^-G_x \) is defined as

\[
(U, V)_{T_xG_x} := \frac{1}{2} \sum_{i=1}^{m} \sum_{\tau=\pm} U_i^\tau V_i^\tau.
\]

We chose to put a factor \( \frac{1}{2} \) in front of the scalar product to compensate for the fact that we consider both left and right differences.

Finally, for a function \( f \) defined on the cartesian product \( G := G_x \times G_y \), the (augmented) gradient \( \hat{\nabla}_z f(z) \) at the point \( z = (x, y) \) is an element of the tangent plane \( T_zG := T_xG_x \oplus T_yG_y \), that is a \((2m+n)\)-column–vector

\[
\hat{\nabla}_z f(z) := \sum_{i=1}^{m} \sum_{\tau=\pm} X_i^\tau f(z) \hat{X}_i^\tau + \sum_{j=1}^{n} Y_j f(z) \hat{Y}_j(z)
\]

where \( \hat{X}_i^\tau \) and \( \hat{Y}_j(z) \) can be identified with column vectors of size \((2m+n)\) with obvious definitions and scalar product \((\cdot, \cdot)_{\hat{T}_zG} \).

Let \( d\mu := d\mu_x d\mu_y \), \( d\mu_x \) being the counting measure on \( G_x \) and \( d\mu_y \) being the Haar measure on \( G_y \). The inner product of \( \varphi, \psi \) in \( L^2(G) \) is

\[
(\varphi, \psi)_{L^2(G)} := \int_G \varphi(z) \psi(z) d\mu_z(z).
\]

**Riesz transforms.** Following [1][2], recall first that for a compact Riemannian manifold \( M \) without boundary, one denotes by \( \nabla_M \), \( \text{div}_M \) and \( \Delta_M := \text{div}_M \nabla_M \) respectively the gradient, the divergence and the Laplacian associated with \( M \). Then \( -\Delta_M \) is a positive operator and the vector Riesz transform is defined as the linear operator

\[
R_M := \nabla_M \circ (-\Delta_M)^{-1/2}
\]

acting on \( L^2(M) \) \((L^2 \text{ functions with vanishing mean})\). It follows that if \( f \) is a function defined on \( M \) and \( y \in M \) then \( R_M f(y) \) is a vector of the tangent plane \( T_yM \).

Similarly on \( M = G \), we define \( \nabla_G := \hat{\nabla}_z \) as before, and then we define the divergence operator as its formal adjoint, that is \(-\text{div}_G = -\hat{\text{div}}_z := \hat{\nabla}^* \), with respect to the natural \( L^2 \) inner product of vector fields:

\[
(U, V)_{L^2(\hat{T}_G)} := \int_G (U(z), V(z))_{\hat{T}_zG} d\mu_z(z)
\]

We have the \( L^2 \)-adjoints \((X_i^\pm)^* = -X_i^\mp \) and \( Y_j^* = -Y_j \). If \( U \in \hat{T}_G \) is defined by

\[
U(z) = \sum_{i=1}^{m} \sum_{\tau=\pm} U_i^\tau(z) \hat{X}_i^\tau + \sum_{j=1}^{n} U_j(z) \hat{Y}_j,
\]
we define its divergence $\nabla_*^z U$ as

$$\nabla_*^z U(z) := -\frac{1}{2} \sum_{i=1}^m \sum_{\tau = \pm} X_{\tau}^i U_{\tau}^i(z) - \sum_{j=1}^n Y_j U_j(z).$$

The Laplacian $\Delta_G$ is as one might expect:

$$\Delta_z f(z) := -\nabla_*^z \nabla_x f(z) - \nabla_*^z \nabla_y f(z) = -\sum_{i=1}^m X_i^+ X_i^- f(z) + \sum_{j=1}^n Y_j^2 f(z) = \sum_{i=1}^m X_i^2 f(z) + \sum_{j=1}^n Y_j^2 f(z) = \Delta_x f(z) + \Delta_y f(z)$$

where we denoted $X_i^2 := X_i^+ X_i^- = X_i^- X_i^+$. We have chosen signs so that $-\Delta_G \geq 0$ as an operator. The Riesz vector $(\hat{R}_z f)(z)$ is the $(2m + n)$–column–vector of the tangent plane $\hat{T}_G \mathcal{G}$ defined as the linear operator

$$\hat{R}_z f := (\hat{\nabla}_z f) \circ (-\Delta_z f)^{-1/2}$$

We also define transforms along the coordinate directions:

$$R_i^\pm = X_i^\pm \circ (-\Delta_z)^{-1/2} \quad \text{and} \quad R_j = Y_j \circ (-\Delta_z)^{-1/2}.$$ 

If $\mathcal{G}_x$ is a finite group, then the transforms $R_j$ apply to

$$L^p_0(\mathcal{G}) := \left\{ f \in L^p(\mathcal{G}) \text{ such that for } x \in \mathcal{G}_x \text{ one has } \sum_{x \in \mathcal{G}_x} \int_{\mathcal{G}_y} f(x, \cdot) d\mu_y = 0 \right\}$$

1.2. **Main results.** In this text, we are concerned with second order Riesz transforms and combinations thereof. We first define the square Riesz transform in the (discrete) direction $i$ to be

$$R_i^2 := R_i^+ R_i^- = R_i^- R_i^+.$$ 

Then, given $\alpha := ((\alpha_i^x)_{i=1\ldots m}, (\alpha_j^y)_{j,k=1\ldots n}) \in \mathbb{C}^m \times \mathbb{C}^{n \times n}$, we define $R_\alpha^2$ to be the following combination of second order Riesz transforms:

$$R_\alpha^2 := \sum_{i=1}^m \alpha_i^x R_i^2 + \sum_{j,k=1}^n \alpha_j^y R_j R_k,$$

where the first sum involves squares of discrete Riesz transforms as defined above, and the second sum involves products of continuous Riesz transforms. This combination is written in a condensed manner as the quadratic form

$$R_\alpha^2 = (\hat{R}_z, A_\alpha \hat{R}_z)$$
where $A_{\alpha}$ is the $(2m + n) \times (2m + n)$ block matrix
\begin{equation}
A_{\alpha} := \begin{pmatrix}
A^x_{\alpha} & 0 \\
0 & A^y_{\alpha}
\end{pmatrix}
\end{equation}
with
\begin{align*}
A^x_{\alpha} &= \text{diag}(\alpha_1^x, \ldots, \alpha_m^x, \alpha_1^x, \ldots, \alpha_m^x) \in \mathbb{C}^{2m \times 2m}, \\
A^y_{\alpha} &= (\alpha_{jk})_{j,k=1 \ldots n} \in \mathbb{C}^{n \times n}.
\end{align*}

The first result is a representation formula of second order Riesz transforms $R^2_{\alpha}$ à la Gundy–Varopoulos (see [23]).

**Theorem 1.** The second order Riesz transform $R^2_{\alpha} f$ of a function $f \in L^2(\mathbb{G})$ as defined in (7) can be written as the conditional expectation
\[ \mathbb{E}(M^\alpha_{0}\, | \, Z_0 = z). \]

Here $M^\alpha_{t}$ is a suitable martingale transform of a martingale $M^f_{t}$ associated to $f$, and $Z_t$ is a suitable random walk on $\mathbb{G}$ (see Section 2).

When $p$ and $q$ are conjugate exponents, let $p^* = \max \left\{ p, \frac{p}{p-1} \right\}$. We have the estimate

**Theorem 2.** Let $\mathbb{G}$ be a Lie group and $R^2_{\alpha} : L^p_0(\mathbb{G}, \mathbb{C}) \to L^p(\mathbb{G}, \mathbb{C})$ be a combination of second order Riesz transforms as defined above. This operator satisfies the estimate
\[ \|R^2_{\alpha}\| \leq \|A_{\alpha}\|_2 \ (p^* - 1). \]

The estimate above is sharp when the group $\mathbb{G} = \mathbb{G}_x \times \mathbb{G}_y$ and $\text{dim}(\mathbb{G}_y) + \text{dim}^\infty(\mathbb{G}_x) \geq 2$, where $\text{dim}^\infty(\mathbb{G}_x)$ denotes the number of infinite components of $\mathbb{G}_x$.

Above, we have set:
\[ \|A_{\alpha}\|_2 = \max \left\{ \|A^x_{\alpha}\|_2, \|A^y_{\alpha}\|_2 \right\} = \max \left\{ |\alpha_1^x|, \ldots, |\alpha_m^x|, \|A^y_{\alpha}\|_2 \right\}. \]

In the case where $\mathbb{G} = \mathbb{G}_x$ only consists of the discrete component, this was proved in [18,17] using the deterministic Bellman function technique. In the case where $\mathbb{G} = \mathbb{G}_y$ is a connected compact Lie group, this was proved in [7] using Brownian motions defined on manifolds and projections of martingale transforms.

In the case where the function $f$ is real valued, we obtain better estimates involving the Choi constants (see Choi [14]). Compare with [7,17].

**Theorem 3.** Assume that $a I \preceq A_{\alpha} \preceq b I$ in the sense of quadratic forms, where $a, b$ are real numbers. Then $R^2_{\alpha} : L^p(\mathbb{G}, \mathbb{R}) \to L^p(\mathbb{G}, \mathbb{R})$ enjoys the norm estimate $\|R^2_{\alpha}\|_p \leq C_{a,b,p}$, where these are the Choi constants.

The Choi constants (see [14]) are not explicit, except $C_{-1,1,p} = p^* - 1$. An approximation of $C_{0,1,p}$ is known and writes as
\[ C_{0,1,p} = \frac{p}{2} + \frac{1}{2} \log \left( \frac{1+e^{-2}}{2} \right) + \frac{\beta_2}{p} + \ldots, \]
with $\beta_2 = \log^2 \left( \frac{1+e^{-2}}{2} \right) + \frac{1}{2} \log \left( \frac{1+e^{-2}}{2} \right) - 2 \left( \frac{e^{-2}}{1+e^{-2}} \right)^2$.  

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1.3. **Weak formulations.** Let $f : \mathbb{G} \to \mathbb{C}$ be given. The heat extension $\tilde{f}(t)$ of $f$ is defined as $\tilde{f}(t) := e^{t\Delta_x}f =: P_tf$. We have therefore $\tilde{f}(0) = f$.

The aim of this section is to derive weak formulations for second order Riesz transforms. We start with the weak formulation of the identity operator $I$.

**Lemma 1.** Assume $f$ and $g$ in $L^2_0(\mathbb{G})$, then

$$
(\mathcal{I}f,g) = (f,g)_{L^2(\mathbb{G})} = 2\int_0^\infty \left( \nabla_z P_tf, \nabla_z P_tg \right)_{L^2(\mathcal{I}G)} dt
$$

$$
= 2\int_0^\infty \int_{z \in \mathbb{G}} \left\{ \frac{1}{2} \sum_{i=1}^m \sum_{\tau = \pm} (X_\tau^i P_tf)(z)(X_\tau^i P_tg)(z) + \sum_{j=1}^n (Y_j P_tf)(z)(Y_j P_tg)(z) \right\} d\mu_z(z) dt
$$

and the sums and integrals that arise converge absolutely.

**Proof.** This classical formula can be obtained by observing that $d_t P_t = \Delta_x P_t$ and writing the ODE satisfied by $\phi(t) := (P_t f, P_t g)_{L^2(\mathbb{G})}$. \qed

In order to pass to the weak formulation for the squares of Riesz transforms, we need the following hypothesis and commutation properties.

**Hypothesis.** We assume everywhere in the sequel:

1. The discrete component $\mathbb{G}_x$ of the Lie group $\mathbb{G}$ is an abelian group
2. The connected component $\mathbb{G}_y$ of the Lie group $\mathbb{G}$ is a compact Lie group endowed with a biinvariant Riemannian metric, so that the family $(Y_j)_{j=1,...,n}$ commutes with $\Delta_y$.

**Lemma 2. (Commutation relations)** Assuming the Hypothesis above, we have

$$
Y_j \circ \Delta_z = \Delta_z \circ Y_j \quad \text{and} \quad X_\tau^i \circ \Delta_z = \Delta_z \circ X_\tau^i, \quad \tau \in \{+, -\}
$$

**Proof.** Since $\mathbb{G} = \mathbb{G}_x \times \mathbb{G}_y$ is a cartesian product, we have the commutator $[Y_j, X_\tau^i] = 0$ and as a consequence $[Y_j, \Delta_z] = 0$ and $[X_\tau^i, \Delta_y] = 0$. Then $[Y_j, \Delta_y] = 0$ yields $[Y_j, \Delta_z] = 0$, and since $\mathbb{G}_x$ is abelian we have successively $[X_\tau^i, \Delta_x] = 0$ and $[X_\tau^i, \Delta_z] = 0$. \qed
Lemma 3. Assume the Hypothesis and the Commutation lemma above. Assume \( f \) and \( g \) in \( L^2(G) \), then

\[
(R^2_\alpha f, g)_{L^2(G)} = -2 \int_0^{\infty} (A_\alpha \nabla_z P_t f, \nabla_z P_t g)_{L^2(I \backslash G)} \, dt
\]

\[
= -2 \int_0^{\infty} \int_{z \in G} \left\{ \frac{1}{2} \sum_{i=1}^m \sum_{\tau = \pm} \alpha_i^\tau (X_i^\tau P_t f)(z)(X_i^\tau P_t g)(z)
\right. \\
+ \sum_{j,k=1}^n \alpha_{jk}^y (Y_j P_t f)(z)(Y_k P_t g)(z) \right\} \, d\mu_z(z) \, dt
\]

and the sums and integrals that arise converge absolutely.

Proof. We apply Lemma (1) to \( R^2_\alpha f \) instead of \( f \) and we are left with integrands of the form

\[
\left( \nabla_z P_t R^2_\alpha f, \nabla_z P_t g \right)_{L^2(I \backslash G)}
\]

\[
= \left( (-\Delta_z) P_t P_t R^2_\alpha f, P_t g \right)_{L^2(G)}
\]

\[
= \sum_i \alpha_i^y (X_i^- P_t f, P_t g) + \sum_{j,k} \alpha_{jk}^y (X_j^- P_t f, X_k^- P_t g)
\]

\[
= \frac{1}{2} \sum_i \sum_{\tau = \pm} \alpha_i^\tau (X_i^\tau P_t f, X_i^\tau P_t g) + \sum_{j,k} \alpha_{jk}^y (X_j P_t f, X_k P_t g)
\]

where we used successively the commutation properties of the Laplacian \( \Delta_z \) with the vector fields and the commutation properties of the vector fields with \( P_t = e^{t\Delta_z} \). This yields the desired result. \( \square \)

2. Stochastic Integrals and Martingale Transforms

In what follows, we assume that we have a complete probability space \((\Omega, \mathcal{F}, \mathbb{P})\) with a cadlag (i.e., right continuous left limit) filtration \((\mathcal{F}_t)_{t \geq 0}\) of sub-\(\sigma\)–algebras of \(\mathcal{F}\). We assume as usual that \(\mathcal{F}_0\) contains all events of probability zero. All random walks and martingales are adapted to this filtration.

We define below a continuous-time random process \(Z\) with values in \(G\), \(Z_t := (X_t, Y_t) \in G_x \times G_y\), having infinitesimal generator \(L = \Delta_z\). The pure-jump component \(X_t\) is a compound Poisson jump process on the discrete set \(G_x\), whereas the continuous component \(Y_t\) is a standard brownian motion on the manifold \(G_y\). Then, Itô’s formula ensures that semi-discrete “harmonic”
functions \( f : \mathbb{R}^+ \times G \to \mathbb{C} \) solving the backward heat equation \((\partial_t + \Delta_z)f = 0\) give rise to martingales \( M^f_t := f(t, Z_t) \) for which we define a class of martingale transforms.

**Stochastic integrals on Riemannian manifolds and Itô integral.** Following EMERY [19, 20], see also ARCOZZI [1, 2], we define the Brownian motion \( Y_t \) on \( \mathbb{B}Z_y \), a compact Riemannian manifold, as the process \( Y_t : \Omega \to (0, T) \times \mathbb{B}Z_y \) such that for all smooth functions \( f : \mathbb{B}Z_y \to \mathbb{C} \), the quantity

\[
\left( Y_t - Y_0 \right) - \frac{1}{2} \int_0^t (\Delta_y f)(Y_s) \, ds =: (I_{d_y f})_t
\]

is an \( \mathbb{R} \)-valued continuous martingale. For any adapted continuous process \( \Psi \) with values in the cotangent space \( T^* \mathbb{B}Z_y \) of \( \mathbb{B}Z_y \), if \( \Psi_t(\omega) \in T^*_Y(\omega) \mathbb{B}Z_y \) for all \( t \geq 0 \) and \( \omega \in \Omega \), then one can define the continuous Itô integral \( I_{\Psi} \) of \( \Psi \) as

\[
(I_{\Psi})_t := \int_0^t \langle \Psi_s, dY_s \rangle
\]

so that in particular

\[
(I_{d_y f})_t := \int_0^t \langle d_y f(Y_s), dY_s \rangle
\]

The integrand above involves the 1–form of \( T^*_y \mathbb{B}Z_y \)

\[
d_y f(y) := \sum_j (Y_j f)(y) \, Y_j^*
\]

**A pure jump process on \( \mathbb{B}Z_x \).** We will now define the discrete \( m \)-dimensional process \( \mathcal{N}_t \) on the discrete abelian group \( \mathbb{B}Z_x \) as a generalized compound Poisson process. In order to do this we need a number of independent variables and processes:

First, for any given \( 1 \leq i \leq m \), let \( N^i_t \) be a càdlàg Poisson process of parameter \( \lambda \), that is

\[
\forall t, \quad \mathbb{P}(N^i_t = n) = \frac{(\lambda t)^n}{n!} e^{-\lambda t}.
\]

The sequence of instants where the jumps of the \( N^i_t \) occur is noted \( (T^i_k)_{k \in \mathbb{N}} \), with the convention \( T^i_0 = 0 \).

Second, we set

\[
\mathcal{N}_t = \sum_{i=1}^m N^i_t
\]

Almost surely, for any two distinct \( i \) and \( j \), we have \( \{T^i_k\}_{k \in \mathbb{N}} \cap \{T^j_k\}_{k \in \mathbb{N}} = \emptyset \). Let therefore \( \{T^i_k\}_{k \in \mathbb{N}} = \bigcup_{i=1}^m \{T^i_k\}_{k \in \mathbb{N}} \) be the ordered sequence of instants of jumps of \( \mathcal{N}_t \) and let \( i_t \equiv i_t(\omega) \) be the index of the coordinate where the jump occurs at time \( t \). We set \( i_t = 0 \) if no jump occurs. The random variables \( i_t \)
are measurable: \( i_t = (\mathcal{N}_1^t - \mathcal{N}_{i-1}^t, \mathcal{N}_2^t - \mathcal{N}_{i-2}^t, \ldots, \mathcal{N}_m^t - \mathcal{N}_{i-m}^t) \cdot (1, 2, \ldots, m) \).

In differential form,

\[
\text{d}\mathcal{N}_t = \sum_{i=1}^{m} \text{d}\mathcal{N}_i^i = \text{d}\mathcal{N}_i^i.
\]

Third, we denote by \((\tau_k)_{k \in \mathbb{N}}\) a sequence of independent Bernoulli variables \(\forall k, \quad \mathbb{P}(\tau_k = 1) = \mathbb{P}(\tau_k = -1) = 1/2\).

Finally, the random walk \(\mathcal{X}_t\) started at \(\mathcal{X}_0 \in \mathbb{G}_x\) is the càdlàg compound Poisson process (see e.g. Protter [30], Privault [28, 29]) defined as

\[
\mathcal{X}_t := \mathcal{X}_0 + \sum_{k=1}^{\mathcal{N}_t} G^\tau_{i_k},
\]

where \(G_i^\tau = (0, \ldots, 0, \tau g_i, 0, \ldots, 0)\) when \(i \neq 0\) and \((0, \ldots, 0)\) when \(i = 0\).

**Stochastic integrals on discrete groups.** We recall for the convenience of the reader the derivation of stochastic integrals for jump processes. We will emphasize the fact that the corresponding Itô’s formula involves the action of a discrete 1–form written in a well-chosen local coordinate system of the discrete augmented cotangent plane (see details below). Let \(1 \leq k \leq \mathcal{N}_t\) and let \((T_k, i_k, \tau_k)\) be respectively the instant, the axis and the direction of the \(k\)-th jump. We set \(T_0 = 0\). Let \(f := f(t, x), t \in \mathbb{R}^+, x \in \mathbb{G}_x\) a function defined on \(\mathbb{R}^+ \times \mathbb{G}_x\). Then

\[
f(t, \mathcal{X}_t) - f(0, \mathcal{X}_0)
= f(t, \mathcal{X}_t) - f\left(T_{\mathcal{N}_t}, \mathcal{X}_{T_{\mathcal{N}_t}}\right) + \sum_{k=1}^{\mathcal{N}_t} \left\{f(T_k, \mathcal{X}_{T_k}) - f(T_{k-1}, \mathcal{X}_{T_{k-1}}) \right\}
= f(t, \mathcal{X}_t) - f\left(T_{\mathcal{N}_t}, \mathcal{X}_{T_{\mathcal{N}_t}}\right)
+ \sum_{k=1}^{\mathcal{N}_t} \left\{f(T_k, \mathcal{X}_{T_k}) - f(T_{k-1}, \mathcal{X}_{T_{k-1}}) \right\}
= \int_{T_{\mathcal{N}_t}}^{t} (\partial_t f)(s, \mathcal{X}_s)ds
+ \sum_{k=1}^{\mathcal{N}_t} \left\{f(T_k, \mathcal{X}_{T_k}) - f(T_{k-1}, \mathcal{X}_{T_{k-1}}) + \int_{T_{k-1}}^{T_k} (\partial_t f)(s, \mathcal{X}_s)ds \right\}
= \int_{0}^{t} (\partial_t f)(s, \mathcal{X}_s)ds + \int_{0}^{t} (f(s, \mathcal{X}_s) - f(s, \mathcal{X}_{s-})) \, d\mathcal{N}_s
= \int_{0}^{t} (\partial_t f)(s, \mathcal{X}_s)ds + \sum_{i=1}^{m} \int_{0}^{t} (f(s, \mathcal{X}_s) - f(s, \mathcal{X}_{s-})) \, d\mathcal{N}_i^i.
\]
At an instant $s$, the integrand in the last term writes as
\[(f(s, x_s) - f(s, x_{s-})) dN_s^i = (f(s, x_s + G^T_N s) - f(s, x_{s-})) dN_s^i = (X^T_N s f)(s, x_{s-}) \tau_{N_i} dN_s^i = \frac{1}{2} \{(X^2 f)(s, x_{s-}) + \tau_{N_s} (X^0_i f)(s, x_{s-})\} dN_s^i\]
where we introduced, for all $1 \leq i \leq m$,
\[\begin{align*}
X^0_i &= X^+ - X^- \\
X^2_i &= X^+ - X^-.
\end{align*}\]
Notice that, for any given $1 \leq i \leq m$, up to a normalisation factor, the system of coordinate $(X^2_i, X^0_i)$ is obtained thanks to a rotation of $\pi/4$ of the canonical system of coordinate $(X^+_i, X^-_i)$. Finally,
\[f(t, x_t) - f(0, x_0) = \int_0^t (\partial_t f)(s, x_s) ds + \frac{1}{2} \sum_{i=1}^m \int_0^t \{(X^2_i f)(s, x_{s-}) + \tau_{N_i} (X^0_i f)(s, x_{s-})\} dN_s^i + \frac{1}{2} \sum_{i=1}^m \int_0^t (X^2_i f)(s, x_{s-}) ds + (X^0_i f)(s, x_{s-}) dX_s^i \]
having set $dX_s^i = \tau_{N_i} dN_s^i$
\[= \int_0^t \left\{ (\partial_t f)(s, x_s) + \frac{\lambda}{2} (\Delta x f)(s, x_s) \right\} ds + \int_0^t \left\{ \hat{d} f(s, x_{s-}), d\hat{W}_s \right\} =: \int_0^t \left\{ (\partial_t f)(s, x_s) + \frac{\lambda}{2} (\Delta x f)(s, x_s) \right\} ds + \left( I_{\hat{d} x f} \right)_t .\]
It is easy to see that $dX_s^i$ is the stochastic differential of a martingale. Here and in the sequel, we take $\lambda = 2$.

**Discrete Itô integral.** The stochastic integral above shows that Itô formula (3) for continuous processes has a discrete counterpart involving stochastic integrals for jump processes, namely we have the *discrete* Itô integral
\[(I_{\hat{d} x f})_t := \frac{1}{2} \sum_{i=1}^m \int_0^t (X^2_i f)(s, x_{s-}) d(N_s^i - \lambda s) + (X^0_i f)(s, x_{s-}) dX_s^i \]
This has a more intrinsic expression similar to the continuous Itô integral (3). If we regard the discrete component $G_x$ as a “discrete Riemannian”
manifold, then this discrete Itô integral involves discrete vectors (resp. 1–forms) defined on the augmented discrete tangent (resp. cotangent) space $\hat{T}_x^*\mathcal{G}_x$ of dimension $2m$ defined as

$$
\hat{T}_x\mathcal{G}_x = \text{span}\{X_1^+, X_2^+, \ldots, X_1^-, X_2^-, \ldots\}
$$

$$
\hat{T}_x^*\mathcal{G}_x = \text{span}\{(X_1^+)^*, (X_2^+)^*, \ldots, (X_1^-)^*, (X_2^-)^*, \ldots\}
$$

Let $\text{d}\hat{W}_x \in \hat{T}_x\mathcal{G}_x$ be the vector and $\text{d}\hat{f} \in \hat{T}_x^*\mathcal{G}_x$ be the 1–form respectively defined as:

$$
d\hat{W}_s = d(N^1_t - \lambda s)X_1^2 + \ldots + d(N^m_t - \lambda s)X_m^2 + d\lambda_t^1 X_1^2 + \ldots + d\lambda_t^m X_m^2
$$

$$
d\hat{f}_x f = X_1^2 f(X_1^2)^* + \ldots + X_m^2 f(X_m^2)^* + X_1^0 f(X_1^0)^* + \ldots + X_m^0 f(X_m^0)^*
$$

We have with these notations

$$
\left( I_{\hat{d}_x f} \right)_t := \langle \hat{d}_x f, d\hat{W}_s \rangle_{\hat{T}_s^*\mathcal{G}_x \times \hat{T}_s\mathcal{G}_x}
$$

where the factor $1/2$ is included in the pairing $\langle \cdot, \cdot \rangle_{\hat{T}_s^*\mathcal{G}_x \times \hat{T}_s\mathcal{G}_x}$.

**Semi-discrete stochastic integrals.** Let finally $Z_t = (X_t, Y_t)$ be a semi-discrete random walk on the cartesian product $\mathcal{G} = \mathcal{G}_x \times \mathcal{G}_y$, where $X_t$ is the random walk above defined on $\mathcal{G}_x$ with generator $\Delta_x$ and where $Y_t$ is the Brownian motion defined on $\mathcal{G}_y$ with generator $\Delta_y$. For $f := f(t, z) = f(t, x, y)$ defined from $\mathbb{R}^+ \times \mathcal{G}$ onto $\mathcal{C}$, we have easily the stochastic integral involving both discrete and continuous parts:

$$
f(t, Z_t) = \int_0^t \{(\partial_t f)(s, Z_s) + (\Delta_x f)(s, Z_s)\} \, ds + \left( I_{\hat{d}_x f} \right)_t
$$

where the semi-discrete Itô integral writes as

$$
\left( I_{\hat{d}_x f} \right)_t := \left( I_{d_x f} \right)_t + (I_{d_y f})_t
$$

$$
:= \int_0^t \langle \hat{d}_x f(s, Z_{s-}), d\hat{W}_s \rangle_{\hat{T}_s^*\mathcal{G}_x \times \hat{T}_s\mathcal{G}_x}
$$

$$
+ \int_0^t \langle d_y f(s, Z_{s-}), dY_{s} \rangle_{\hat{T}_s^*\mathcal{G}_y \times \hat{T}_s\mathcal{G}_y}
$$

2.1. Martingale transforms and quadratic covariations.

**Martingale transforms.** We are interested in martingale transforms allowing us to represent second order Riesz transforms. Let $f(t, z)$ be a solution to the heat equation $\partial_t - \Delta_z = 0$. Fix $T > 0$ and $Z_0 \in \mathcal{G}$. Then define

$$
\forall 0 \leq t \leq T, \quad M_t^{f, T, Z_0} = f(T - t, Z_t).
$$
This is a martingale since $f(T - t)$ solves the backward heat equation $\partial_t + \Delta z = 0$, and we have in terms of stochastic integrals

$$M_t^{f,T,Z_0} = f(T - t, Z_t) = f(T, Z_0) + \int_0^t \left< \tilde{d}_z f(T - s, Z_{s-}), dZ_s \right>$$

Given $A_\alpha$ the $\mathbb{C}^{(2m+n) \times (2m+n)}$ matrix defined earlier, we note $M_t^{\alpha,f,T,Z_0}$ the martingale transform $A_\alpha * M_t^{f,T,Z_0}$ defined as

$$M_t^{\alpha,f,T,Z_0} := f(T, Z_0) + \int_0^t \left( A_\alpha \tilde{\nabla}_z f(s, Z_{s-}), dZ_s \right)$$

where the first integral involves the $L^2$ scalar product on $\hat{T}_z \mathbb{G} \times \hat{T}_z \mathbb{G}$ and the second integral involves the duality $\hat{T}_z^* \mathbb{G} \times \hat{T}_z \mathbb{G}$. In differential form:

$$dM_t^{\alpha,f,T,Z_0} = \left( A_\alpha \tilde{\nabla}_z f(s, Z_{s-}), dZ_s \right)$$

$$= \sum_{i=1}^m \alpha_i^x \left( (X_i^2 f)(T - t, Z_{t-}) d(N_i^2 t - \lambda t) + (X_i^0 f)(t, Z_{t-}) d\lambda_i^t \right)$$

$$+ \sum_{j=1}^n \alpha_{j,k}^y (X_j f)(T - t, Z_{t-}) d\gamma_i^k$$

**Quadratic covariation and subordination.** We have the quadratic covariations (see Protter [30], Dellacherie–Meyer [15], or Privault [28, 29])

$$d[N^i - \lambda t, N^j - \lambda t]_t = dN^i_t$$

$$d[N^i - \lambda t, X^i]_t = \tau_{N_i} dN^i_t$$

$$d[X^i, X^j]_t = dN^i_t$$

$$d[Y^j, Y^j]_t = dt,$$

the other quadratic covariations being zero. For any two martingales $M_t^f$ and $M_t^g$ defined as above thanks to their respective heat extensions $P_t f$ et
\( P_t g \), we have the quadratic covariations
\[
\begin{align*}
d[M^f, M^g]_t & = \sum_{i=1}^{m} (X_i^2 f)(T - t, Z_{t-}) (X_i^2 g)(T - t, Z_{t-}) d[N^i - \lambda t, N^i - \lambda t]_t \\
 & + \sum_{i=1}^{m} (X_i^0 f)(T - t, Z_{t-}) (X_i^0 g)(T - t, Z_{t-}) d[\mathcal{X}^i, \mathcal{X}^i]_t \\
 & + \sum_{i=1}^{m} (X_i^2 f)(T - t, Z_{t-}) (X_i^0 g)(T - t, Z_{t-}) d[N^i - \lambda t, \mathcal{X}^i]_t \\
 & + \sum_{i=1}^{m} (X_i^0 f)(T - t, Z_{t-}) (X_i^2 g)(T - t, Z_{t-}) d[\mathcal{X}^i, N^i - \lambda t]_t \\
 & + \sum_{i=1}^{m} (X_j f)(T - t, Z_{t-}) (X_j g)(T - t, Z_{t-}) d[\mathcal{Y}^j, \mathcal{Y}^j]_t \\
 & = \sum_{i=1}^{m} \sum_{\tau = \pm 1} (X_i^* f)(X_i^* g)(T - t, Z_{t-}) 1(\tau N_i = \tau 1) dN^i_t \\
 & + (\nabla y f, \nabla y g)(T - t, Z_{t-}) dt
\end{align*}
\]
so that finally
\[
(4) \quad d[M^f, M^g]_t = \sum_{i=1}^{m} \sum_{\tau = \pm 1} (X_i^* f)(X_i^* g)(T - t, Z_{t-}) 1(\tau N_i = \tau 1) dN^i_t \\
 & + (\nabla y f, \nabla y g)(T - t, Z_{t-}) dt
\]

**Differential subordination.** Following Wang [32], given two adapted càdlàг Hilbert space valued martingales \( X_t \) and \( Y_t \), we say that \( Y_t \) is differentially subordinate by quadratic variation to \( X_t \) if \( |Y_0|_H \leq |X_0|_H \) and \( [Y, Y]_t - [X, X]_t \) is nondecreasing nonnegative for all \( t \). In our case, we observe that
\[
d[M^{\alpha f}, M^{\alpha f}]_t = \sum_{i=1}^{m} |\alpha_i|^2 \left\{ (X_i^+ f)^2(T - t, Z_{t-}) 1(\tau N_i = 1) \\
 & + (X_i^- f)^2(T - t, Z_{t-}) 1(\tau N_i = -1) \right\} dN^i_t \\
 & + (A^{\alpha f}_g, A^{\alpha f}_g)(T - t, Z_{t-}) dt
\]
Hence
\[
(5) \quad d[M^{\alpha f}, M^{\alpha f}]_t \leq \|A_{\alpha}\|_2^2 d[M^f, M^f]_t
\]
This means that \( M^{\alpha f}_t \) is differentially subordinate to \( \|A_{\alpha}\|_2 M^f_t \).
3. Proofs of the main results

3.1. Proof of Theorem 1

To establish the estimate itself, we adapt the well-known connection between martingale transforms and classical singular integral operators, through the use of projection operators. We refer to Gundy–Varopoulos [23] as well as Banuelos [4] and Banuelos–Baudoin [5]. Following the same strategy, the random trajectories we use \( \mathcal{B}_t := (t, Z_t) \) defined on the band \([-T, 0] \times \mathbb{G} \) by

\[
\mathcal{B}_t := (-t, Z_t), \quad -T \leq t \leq 0, \quad Z_{-T} \in \mathbb{G}
\]

are replaced by random trajectories \( \mathcal{B}_t := (-t, Z_t) \) defined on the upper half space \( \mathbb{R}^+ \times \mathbb{G} \) after exhaustion of the upper half space.

The latter are therefore trajectories starting at time \( t = -\infty \) from a chosen point \( Z_{-\infty} \in \mathbb{G} \), and stopping at time \( t = 0 \), when hitting the bottom boundary \( \mathbb{G} \) of the upper half space. If \( f(t) = \hat{P}_t f \) is as in the previous section, then \( M_t^f = f(\mathcal{B}_t) \) for \( -\infty \leq t \leq 0 \) is a martingale and \( \mathcal{M}_t^{\alpha, f} \) its martingale transform as defined previously. Let the projection operator \( \mathcal{T}^\alpha \) be defined as the following conditional expectation of the martingale transform \( \mathcal{M}_t^{\alpha, f} \) of the stochastic integral \( \mathcal{M}_t^f \):

\[
\forall z \in \mathbb{G}, \quad (\mathcal{T}^\alpha f)(z) := \mathbb{E} \left( \mathcal{M}_0^{\alpha, f} \mid Z_0 = z \right).
\]

Following Gundy–Varopoulos [23] (see also [5]) this operator is the second order Riesz transform we are interested in. Indeed, recalling the expression (4) of the quadratic covariations of two martingale increments, it is not difficult to calculate

\[
\forall g, \quad (\mathcal{T}^\alpha f, g) = \int_{\mathbb{G}} (\mathcal{T}^\alpha f)(z) g(z) \, d\mu_z(z)
\]

\[
= 2 \int_0^{\infty} \left( A_\alpha \hat{\nabla}_z P_t f, \hat{\nabla}_z P_t g \right)_{L^2(\mathbb{T})} \, dt.
\]

This means thanks to Lemma 3 that \( \mathcal{T}^\alpha = R_\alpha^2 \). This concludes the proof of Theorem 1. \( \square \)

3.2. Proof of Theorem 2

Recall that the subordination estimate (5) shows that the martingale transform \( Y_t := M_t^\alpha \) is differentially subordinate to the martingale \( X_t := \| A_\alpha \|_2 M_t^f \). Following this result of Wang [32]:

**Theorem 4.** (Wang, 1995) Let \( X_t \) and \( Y_t \) be two adapted càdlàg Hilbert–valued martingales such that \( Y_t \) is differentially subordinate by quadratic covariation to \( X_t \). For \( 1 < p < \infty \),

\[
\| Y_t \|_p \leq (p^* - 1) \| X_t \|_p
\]
and the constant \( p^* - 1 \) is best possible. Strict inequality holds when \( 0 < \|X\|_p < \infty \) and \( p \neq 2 \),

we get immediately

**Lemma 4.** Let \( M_t^f \) and \( M_t^{\alpha,f} \) as defined above. We have

\[
\forall t, \quad \|M_t^{\alpha,f}\|_p \leq \|A_\alpha\|_2 (p^* - 1) \|M_t^f\|_p.
\]

Finally, the operator \( T^\alpha \) being a conditional expectation of \( M_t^{\alpha,f} \), this proves the estimate \( \|T^\alpha\|_p \leq \|A_\alpha\|_2 (p^* - 1) \).

**Sharpness.** The sharpness in Lie groups with at least two infinite directions is inherited from the continuous case, where it is seen that two continuous directions are enough for sharpness and optimality of the estimate is seen when considering the operator \( R_1^2 - R_2^2 \). If one or both continuous directions are replaced by \( \mathbb{Z} \) just consider the isomorphic groups \((t\mathbb{Z})^N\) for \( 0 < t \leq 1 \) in conjunction with the Lax equivalence theorem [1]. By the same argument, sharpness for a uniform estimate in \( m \) for the cyclic case \((\mathbb{Z}/m\mathbb{Z})^N\) is inherited from that on the torus \( \mathbb{T}^N \).

3.3. **Proof of Theorem 3.** The proof of Theorem 3 follows exactly the same procedure. Recall Choi’s result [14] for discrete martingales.

**Theorem 5.** (Choi, 1992) Let \((\Omega, (\mathcal{F})_{n\in\mathbb{N}}, \mathbb{P})\) a probability space and \( X_n \) an adapted real valued martingale. Let \((\alpha_n)_{n\in\mathbb{N}}\) be a predictable sequence taking values in \([0,1]\). Let \( Y := \alpha \ast X \) be the martingale transform of \( X \) defined for almost all \( \omega \in \Omega \) as

\[
Y_0(\omega) = X_0(\omega), \quad (Y_{n+1} - Y_n)(\omega) = \alpha_n (X_{n+1} - X_n)(\omega).
\]

Then there exists a constant \( C_p \) depending only on \( p \) such that \( \|Y\|_p \leq C_p \|X\|_p \) and the estimate is best possible.

The previous result from Choi is only for discrete martingales. For continuous-in-time martingales, we invoke Theorem 1.6 from the result of Bañuelos and Osekowski [7], namely

**Lemma 5.** (Bañuelos–Osekowski, 2012) Let \( X_t \) and \( Y_t \) be two real-valued martingales satisfying

\[
\mathbb{d} \left[ Y - \frac{a+b}{2} X, Y - \frac{a+b}{2} X \right]_t \leq \mathbb{d} \left[ \frac{b-a}{2} X, \frac{b-a}{2} X \right]_t
\]

for all \( t > 0 \). Then for all \( 1 < p < \infty \), we have \( \|Y\|_p \leq C_p \|X\|_p \).

The result is now a corollary of Lemma 5 above with \( X_t = M_t^f \) and \( Y_t = M_t^{\alpha,f} \). It is not difficult to prove that the difference of quadratic
variations above writes in terms of a jump part and a continuous part as
\[
\left[ Y - \frac{a+b}{2}X, Y - \frac{a+b}{2}X \right]_t - d \left[ \frac{b-a}{2}X, \frac{b-a}{2}X \right]_t
\]
\[
= \sum_{i=1}^{m} \sum_{\pm} (\alpha_x^i - a)(\alpha_x^i - b)(X_{\pm}^i f)^2(\mathcal{B}_t) \mathbb{1}(\tau_{N_i} = \pm 1) dN_i^x
\]
\[
+ \langle (A^y_I - bI)(A^y_I - bI) \nabla_y f(\mathcal{B}_t), \nabla_y f(\mathcal{B}_t) \rangle dt,
\]
which is nonpositive since we assumed precisely \( aI \leq A_\alpha \leq bI \). This proves Theorem 3. \( \square \)

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