NEW FUETER VARIABLES ASSOCIATED TO THE GLOBAL OPERATOR IN THE QUATERNIONIC CASE

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Abstract. The purpose of this paper is to develop a new theory of three non-commuting quaternionic variables and its related Schur analysis theory for a modified version of the quaternionic global operator.

Please note that no datasets were generated or analyzed during the current study.

1. Introduction

In 2007 a new theory of regular functions of a quaternionic variable was introduced in [26]. These are the so-called slice hyperholomorphic functions on quaternions. For more details on this topic we refer the reader to the books [17, 18, 25] and the references therein. This theory generated several interesting applications in different areas of mathematics and physics due to the discovery of the notion of the S-spectrum for quaternionic operators. In particular, this function theory allowed to develop the quaternionic counterpart of operator theory, Schur analysis, and quantum mechanics, see [3, 14, 13, 18]. A fundamental technique that we use in this paper is based on the research developed in [16] where the authors discovered a new approach to the theory of slice hyperholomorphic functions using a special operator with non-constant coefficients called the quaternionic global operator. The purpose of this paper is to develop a theory of three non-commuting quaternionic variables and a related Schur analysis for the quaternionic global operator. We first set the framework for our work, and define the skew field of quaternions (denoted by $\mathbb{H}$) as the space of elements of the form:

$$q = x_0 + x_1 e_1 + x_2 e_2 + x_3 e_3,$$

where $x_0, x_1, x_2, x_3$ are real numbers and $e_1, e_2, e_3$ satisfy the Cayley multiplication table. A number of the form $q$ in (1.1) is called a quaternion, and

$$\bar{q} = x_1 e_1 + x_2 e_2 + x_3 e_3$$

is called its vector part (more information on quaternions is provided in Section ??).
As in Definition 2.4, the global operator with non-constant coefficients, which was introduced for the first time in [16], was initially written as:

(1.2) \[ G_q(f) := |\vec{q}|^2 \partial_{x_0} f(q) + \vec{q} \sum_{l=1}^{3} x_l \partial_{x_l} f(q). \]

The operator \( G_q \) acts on functions of a quaternionic variable which are differentiable with respect to the real variables and it was first used to develop a Cauchy formula in the case of slice hyperholomorphic functions, see [16]. For a distributional approach to the Cauchy problem in this context see [19]. In particular, in [16], it was proved that slice-hyperholomorphic functions are strictly included in the kernel of the global operator \( G_q \).

However, the global operator we work with, denoted by \( V_q \), is the normalized form of \( G_q \), namely:

(1.3) \[ V_q = \frac{G_q}{|\vec{q}|^2}, \]

The global operator \( V_q \) was used in [4] to develop the Fueter mapping theorem for poly slice monogenic functions. In [2] the global operator was used to study infinite-order differential operators acting on entire hyperholomorphic functions and it was used also in [27] to study global differential equations for slice regular functions. Recent extensions and new results on the global operator and related topics can be found in papers [28, 29].

**Definition 1.1.** A quaternionic-valued function \( f \) which is at least \( C^1 \) in the real variables in a domain \( \Omega_R \subset \mathbb{R}^4 \) is called \( V_q \)-regular if \( V_q f = 0 \).

In this paper we will focus on a family of \( V_q \)-regular functions which are real-analytic and we develop a Cauchy-Kovalevskaya product (denoted CK-product) for this space. We consider the version of Gleason problem associated to \( V_q \), which will allow us to introduce a new type of Fueter-like variables and develop a Schur analysis and rational functions for this operator. For consistency we now introduce the notations that will be used throughout the paper.

**Definition 1.2.** Given \( x = (x_0, x_1, x_2, x_3) \in \mathbb{R}^4 \) such that \( x_u \neq 0 \) for some \( u = 1, 2, 3 \). We define the \( V_q \)-Fueter variables:

(1.4) \[ \mu_u(x) = x_u \left( 1 + \frac{x_0}{\vec{q}} \right), \quad u = 1, 2, 3, \]

and, for \( \alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}_0^3 \), we write, with a slight abuse of notation:

\[ x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3}, \]

where \( x^\alpha \) does not depend on \( x_0 \).

The functions \( \mu_u, u = 1, 2, 3 \), are the counterpart of the classical Fueter variables (see Definition 2.1 for the latter) in the present setting. These new variables allow us to define and study a counterpart of Schur analysis in this setting. In Theorem 3.6 we prove a more general form of the following:
Theorem 1.3. The functions

\[ \mu_u(x) = x_u \left( 1 + \frac{x_0}{\vec{q}} \right), \quad u = 1, 2, 3, \]

are \( V_q \)-regular on \( \mathbb{H}^* := \{ q = x_0 + \vec{q}, \quad \vec{q} \neq 0 \} \).

Throughout the paper we will exchange the quadruple \( x = (x_0, x_1, x_2, x_3) \) with \( x_u \neq 0 \) for some \( u = 1, 2, 3 \) with the quaternionic variable \( q = x_0 + \vec{q} \in \mathbb{H}^* \) so that we can consider \( \mu_u \) as functions of both. For a given \( x \in \mathbb{H}^* \), \( \mu_u(x) \) commute pairwise and we recover Fueter-like monomials as in Definition 3.3:

\[ \mu^\alpha(x) = \mu_1^{\alpha_1}(x)\mu_2^{\alpha_2}(x)\mu_3^{\alpha_3}(x), \]

where \( \alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}_0^3 \).

The functions \( \mu^\alpha \) are \( V_q \)-regular on \( \mathbb{H}^* \) (see Section 3) and we will see that they can be rewritten as

\[ \mu^\alpha(x) = x^\alpha \left( 1 + \frac{x_0}{\vec{q}} \right)^{|\alpha|}. \]

As explained in the following remarks, and throughout the paper there are important and fundamental differences between the cases considered in [6, 8, 10] (and in particular the case of Fueter variables) and the present setting.

Remark 1.4. In the setting of Fueter variables, the Cauchy-Dirac operator and the operators of partial differentiation commute for smooth functions. This is not the case for the operator \( V_q \), which does not commute with differentiations with respect to \( x_u \), \( u = 1, 2, 3 \). So, for instance, \( \frac{\partial f}{\partial x_1} \) need not be \( V_q \)-regular when \( f \) is \( V_q \)-regular. Since \( V_q \) and \( \frac{\partial f}{\partial x_0} \) commute, \( \frac{\partial f}{\partial x_0} \) is \( V_q \)-regular when \( f \) is \( V_q \)-regular. As an illustration the case \( f(x) = \mu_1(x) \) is considered in Example 3.9. The function \( \frac{\partial \mu_1}{\partial x_2} \) is not \( V_q \)-regular, while the function \( \frac{\partial \mu_1}{\partial x_0} \) is \( V_q \)-regular but cannot be written as a convergent Fueter-like series of the \( \mu^\alpha \) monomials.

To present these ideas we adopt the following structure: in Section 2 we review basic notations and definitions of quaternions, slice hyperholomorphic functions, and the quaternionic global operator with nonconstant coefficients. In Section 3 we introduce the Fueter-like variables and prove a Gleason type theorem in this setting. Section 4 is devoted to the study of a Cauchy-Kovalevskaya type extension and CK-type product corresponding to these Fueter-like variables. In Section 5 and 6 we study some examples of reproducing kernel Hilbert spaces generated by the Fueter-like variables including a counterpart of Averson space and Schur multipliers. Finally, in Section 7 we obtain Blaschke factors in this framework followed by building a rational function theory in Section 8. We conclude the paper with a general description of the next steps we will undertake using these new Fueter-like variables.
2. Preliminary results

We recall that the non-commutative field of quaternions is defined and denoted by

\[ H = \{ q = x_0 + x_1i + x_2j + x_3k : x_0, x_1, x_2, x_3 \in \mathbb{R} \}, \]

where the imaginary units satisfy the multiplication rules

\[ i^2 = j^2 = k^2 = -1 \quad \text{and} \quad ij = -ji = k, jk = -kj = i, ki = -ik = j. \]

On \( H \) the conjugate and the modulus of \( q \) are defined respectively by

\[ \bar{q} = x_0 - \bar{q}, \quad \bar{q} = x_1i + x_2j + x_3k \]

and

\[ |q| = \sqrt{q\bar{q}} = \sqrt{x_0^2 + x_1^2 + x_2^2 + x_3^2}. \]

Throughout the paper we use the notations \( e_0 = 1, e_1 = i, e_2 = j \) and \( e_3 = k \) interchangeably for the imaginary units. It is important to note that the conjugation satisfies the property \( pq = q\bar{p} \) for any \( p, q \in H \).

An important tool in the quaternionic case is the symmetric product of \( n \) quaternionic numbers \( q_1, q_2, \ldots, q_n \):

\[ q_1 \times q_2 \times \cdots \times q_n = \frac{1}{n!} \sum_{\sigma \in S_n} q_{\sigma(1)} q_{\sigma(2)} \cdots q_{\sigma(n)}, \]

where the sum is over the set \( S_n \) of all permutations on \( n \) indices.

In the classical case, Fueter [23, 24] used the following variables to describe the kernel of the Cauchy-Fueter operator:

**Definition 2.1.** The classical Fueter variables are: \( \zeta_l = x_l - x_0 e_l \) and we denote by \( \zeta^\alpha \) the symmetric product: \( \zeta_1^{\alpha_1} \times \zeta_2^{\alpha_2} \times \zeta_3^{\alpha_3} \), where \( \alpha = (\alpha_1, \alpha_2, \alpha_3) \).

These variables are Fueter-regular with respect to the classical Cauchy-Fueter operator. We now define the setting for slice hyperholomorphic functions. The unit sphere

\[ \mathbb{S} = \{ q \in \mathbb{H} : q^2 = -1 \}. \]

**Definition 2.2.** Let \( f : \Omega \rightarrow \mathbb{H} \) be a \( C^1 \) function on a given domain \( \Omega \subset \mathbb{H} \). Then, \( f \) is said to be (left) slice hyperholomorphic function if, for every \( I \in \mathbb{S} \), the restriction \( f_I \) to \( \mathbb{C}_I = \mathbb{R} + I\mathbb{R} \), with variable \( q = x + Iy \), is holomorphic on \( \Omega_I := \Omega \cap \mathbb{C}_I \), that is it has continuous partial derivatives with respect to \( x \) and \( y \) and the function \( \partial_I f : \Omega_I \rightarrow \mathbb{H} \) defined by

\[ \partial_I f(x + Iy) := \frac{1}{2} \left( \frac{\partial}{\partial x} + I \frac{\partial}{\partial y} \right) f_I(x + yI) \]

vanishes identically on \( \Omega_I \). The set of slice hyperholomorphic functions will be denoted by \( \mathcal{SR}(\Omega) \).

The right quaternion vector space \( \mathcal{SR}(\Omega) \) is endowed with the natural topology of uniform convergence on compact sets. The characterization of such functions on a ball centered at the origin is given by:
Theorem 2.3 (Series expansion [26]). An $\mathbb{H}$-valued function $f$ is slice hyperholomorphic on $B(0, R)$ if and only if it has a series expansion of the form:

$$f(q) = \sum_{n=0}^{+\infty} q^n a_n$$

converging on $B(0, R) = \{q \in \mathbb{H}; |q| < R\}$.

Another interesting approach to define slice hyperholomorphic functions is to consider them as solutions of a special global operator with non-constant coefficients that was introduced and studied in [16, 19, 27]. This leads to the following definition

**Definition 2.4.** Let $\Omega$ be an open set in $\mathbb{H}$ and $f : \Omega \rightarrow \mathbb{H}$ a function of class $C^1$. We define the global operator $G_q(f)$ by

$$G_q(f) := |\vec{q}|^2 \partial_{x_0} f(q) + \vec{q} \sum_{l=1}^{3} x_l \partial_{x_l} f(q),$$

for any $q = x_0 + \vec{q} \in \Omega$.

It was proved in [16] that any slice hyperholomorphic function is in the kernel of $G_q$ on axially symmetric slice domains. We briefly recall the definition of such a domain:

**Definition 2.5.** A domain $\Omega \subset \mathbb{H}$ is said to be a slice domain (or just $s$-domain) if $\Omega \cap \mathbb{R}$ is nonempty and for all $I \in S$, the set $\Omega_I := \Omega \cap C_I$ is a domain of the complex plane $\mathbb{C}$. If moreover, for every $q = x + I y \in \Omega$, the whole sphere $x + y S := \{x + Jy; J \in \mathbb{S}\}$ is contained in $\Omega$, we say that $\Omega$ is an axially symmetric slice domain.

There are many other interesting properties of the global operator $G_q$ that were studied in the literature, in particular in [15]. We recall some here:

**Proposition 2.6.** Let $\Omega$ be an open set in $\mathbb{H}$ and $f, g : \Omega \rightarrow \mathbb{H}$ two functions of class $C^1$. Then, for $q = x_0 + \vec{q} \in \Omega$ we have

(1) $G(fg) = G(f)g + fG(g) + (\vec{q} f - f \vec{q}) \sum_{l=1}^{3} x_l \partial_{x_l} g$.

In particular, it holds:
(2) $G(f\lambda + g) = G(f)\lambda + G(g), \forall \lambda \in \mathbb{H}$.
(3) $G(x_0 f) = |\vec{q}|^2 f + x_0 G(f)$ and $G(\vec{q} f) = -|\vec{q}|^2 f + \vec{q} G(f)$.
(4) $G(q^k f) = q^k G(f), \forall k \in \mathbb{N}$.

3. Fueter-type variables

In this section we introduce the Gleason setting that will yield the new $V_q$-Fueter variables in the quaternionic case, in a natural way. Through this process, we find these variables which correspond to the operator $V_q$ when we use the same technique the authors applied in the real ternary case [10], in the split quaternionic case [9], and in the regular Fueter quaternionic case [8]. For completion, in the case of Grassmann variables, see also [7].

It is also worth pointing out that our work here is the first that finds a counterpart of the classical Fueter variables through the treatment of this case of a differential operator with non-constant coefficients. All the applications mentioned before were using this method for operators with constant coefficients.
The strategy involves a clever application of the chain rule, and, for a $V_q$-regular function $f$, we compute:

$$\int_a^b \frac{d}{dt} f(tx) dt.$$ 

In the previous examples of ternary and split quaternions, we can take $a = 0$ and $b = 1$. In the present case $a$ cannot be chosen to be 0 because of the singularity at the origin. We will see that we can apply the same technique to obtain the Fueter-like variables $\mu_u$ in the proof of the following theorem:

**Theorem 3.1.** Let $\Omega$ be an open domain of $\mathbb{H}^*$. For a $V_q$-regular function $f \in C^1(\Omega)$, let $a, b \in \Omega$ such that $[a, b] = \{(1 - t)a + tb, 0 \leq t \leq 1\} \subset \Omega$. Then

$$f(b) - f(a) = \sum_{u=1}^3 \mu_u(b - a) R_u^{ab} f,$$

where

$$R_u^{ab} f = \int_a^b \frac{\partial f}{\partial x_u} (a + t(b - a)) dt.$$

**Proof.** The chain rule gives

$$\frac{d}{dt} f(a + tx) = x_0 \frac{\partial f}{\partial x_0} (a + tx) + \sum_{u=1}^3 x_u \frac{\partial f}{\partial x_u} (a + tx),$$

$$= \frac{x_0}{\vec{q}} \sum_{u=1}^3 x_u \frac{\partial f}{\partial x_u} (a + tx) + \sum_{u=1}^3 x_u \frac{\partial f}{\partial x_u} (a + tx),$$

$$= \sum_{u=1}^3 \mu_u(x) \frac{\partial f}{\partial x_u} (a + tx),$$

and the result follows by integrating back and setting $x = b - a$. The reader should note that in the second step of the equality we use the fact that $f$ is in the kernel of $V_q$. □

**Remark 3.2.** Note that

$$\mu_u(b - a) \neq \mu_u(b) - \mu_u(a).$$

Furthermore, the operators $R_u^{ab}$ do not commute with $V_q$.

For $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}_0^3$ we use the multi-index notation and set

$$x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} \quad \text{and} \quad |\alpha| = \alpha_1 + \alpha_2 + \alpha_3.$$

**Definition 3.3.** For the Fueter variables $\mu_u$ and $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}_0^3$, we define the product:

$$\mu^\alpha(x) = \mu_1^{\alpha_1}(x) \mu_2^{\alpha_2}(x) \mu_3^{\alpha_3}(x),$$

for every $x \in \mathbb{H}^*$

**Remark 3.4.** Since the $V_q$-Fueter variables $\mu_u$ commute, in the above definition we do not need to use the symmetric product and, in fact, we can rewrite the product as:

$$\mu^\alpha(x) = x^\alpha \left(1 + \frac{x_0}{\vec{q}}\right)^{|\alpha|}.$$
Proposition 3.5. Let $\Omega$ be an open domain in $\mathbb{H}^*$. For every $x \in \Omega \subset \mathbb{H}^*$ it holds that

\[ |\mu^\alpha|^2 = |x|^{2\alpha} \left( 1 + \frac{x_0^2}{x_1^2 + x_2^2 + x_3^2} \right)^{|\alpha|}, \]

and, in particular, we have:

\[ |\mu_u(x)|^2 \leq |\zeta_u(x)|^2, \quad u = 1, 2, 3. \]

Proof. This follows from the fact that $\frac{x_0}{q}$ has no real part on $\Omega$ and:

\[ |\mu_u(x)|^2 = x_u^2 + \frac{x_u^2 x_0^2}{x_1^2 + x_2^2 + x_3^2} \leq x_u^2 + x_0^2 = |\zeta_u(x)|^2. \]

We can now prove that the $V_q$-Fueter products $\mu^\alpha$ are in the kernel of $V_q$ on $\mathbb{H}^*$:

Theorem 3.6. It holds that $\mu^\alpha$ are in the kernel of the operator $V_q$ on any open domain $\Omega \subset \mathbb{H}^*$. Moreover, we have:

\[ V_q \mu^\alpha(x) = 0, \]

for every $x \in \Omega$.

Proof. We divide the verification into a number of steps.

STEP 1: It holds that

\[ \frac{\partial}{\partial x_u} \frac{1}{q} = \frac{e_u}{q^2} + \frac{2x_u}{q^3}, \quad u = 1, 2, 3. \]

Indeed, we have

\[ \frac{1}{q} = \frac{\bar{q}}{q^2} = -\frac{\bar{q}}{x_1^2 + x_2^2 + x_3^2}. \]

Hence

\[ \frac{\partial}{\partial x_u} \frac{1}{q} = -\frac{e_u (x_1^2 + x_2^2 + x_3^2) + 2x_u \bar{q}}{(x_1^2 + x_2^2 + x_3^2)^2} \]

\[ = \frac{e_u \bar{q}^2}{q^4} + \frac{2x_u \bar{q}}{q^3} \]

and hence the result.

STEP 2: It holds that

\[ \frac{\partial}{\partial x_u} \left( 1 + \frac{x_0}{q} \right)^{|\alpha|} = \sum_{t, s \in \mathbb{N}_0, t + s = |\alpha|} \left( 1 + \frac{x_0}{q} \right)^t \left( \frac{x_0 e_u}{q^2} + \frac{2x_u x_0}{q^3} \right) \left( 1 + \frac{x_0}{q} \right)^s, \]

where $u = 1, 2, 3$.

\[ \text{(3.12)} \]

is a direct consequence of \[ \text{(3.11)} \] and of the formula for the derivative of $f^n$ when $f$ is a matrix-valued function (and in particular quaternionic valued) of (say) a real variable.
(3.13) \[ \frac{df^n}{dw} = \sum_{t,s \in \mathbb{N}_0 \atop t+s=|\alpha|} \frac{f^t f^s}{t! s!}. \]

STEP 3: We have

(3.14) \[ \frac{\partial}{\partial x_0} \left( 1 + \frac{x_0}{q^\alpha} \right)^{|\alpha|} = \frac{|\alpha|}{q^\alpha} \left( 1 + \frac{x_0}{q^\alpha} \right)^{|\alpha|-1}. \]

This is because \( 1 + \frac{x_0}{q^\alpha} \) commutes with its derivative with respect to \( x_0 \), and formula (3.13) reduces then to the classical formula.

STEP 4: We now calculate

\[ \frac{1}{q} \sum_{u=1}^{3} x_u \frac{\partial}{\partial x_u} \mu^\alpha. \]

We have:

\[ \frac{1}{q} \sum_{u=1}^{3} x_u \frac{\partial}{\partial x_u} x^\alpha \left( 1 + \frac{x_0}{q^\alpha} \right)^{|\alpha|} = \]

\[ = \frac{1}{q} \left[ \sum_{u=1}^{3} x_u \left( \alpha_u x^\alpha - e_u \right) \left( 1 + \frac{x_0}{q^\alpha} \right)^{|\alpha|} + x^\alpha \sum_{t,s \in \mathbb{N}_0 \atop t+s=|\alpha|-1} \left( 1 + \frac{x_0}{q^\alpha} \right)^t \left( \frac{x_0 e_u}{q^2} + \frac{2 x_u x_0}{q^3} \right) \left( 1 + \frac{x_0}{q^\alpha} \right)^s \right] \]

\[ = \frac{1}{q} \left[ |\alpha| x^\alpha \left( 1 + \frac{x_0}{q^\alpha} \right)^{|\alpha|} + x^\alpha \sum_{t,s \in \mathbb{N}_0 \atop t+s=|\alpha|-1} \left( 1 + \frac{x_0}{q^\alpha} \right)^t \left( \frac{3}{q^2} \sum_{u=1}^{3} x_u x_0 e_u + \frac{2 x^2 x_0}{q^3} \right) \left( 1 + \frac{x_0}{q^\alpha} \right)^s \right] \]

\[ = \frac{x^\alpha}{q^\alpha} \left[ |\alpha| \left( 1 + \frac{x_0}{q^\alpha} \right)^{|\alpha|} + \sum_{t,s \in \mathbb{N}_0 \atop t+s=|\alpha|-1} \left( 1 + \frac{x_0}{q^\alpha} \right)^t \left( \frac{x_0 q^2 x_0}{q^2} - \frac{2 q^2 x_0}{q^3} \right) \left( 1 + \frac{x_0}{q^\alpha} \right)^s \right] \]

\[ = \frac{x^\alpha}{q^\alpha} \left[ |\alpha| \left( 1 + \frac{x_0}{q^\alpha} \right)^{|\alpha|} - \sum_{t,s \in \mathbb{N}_0 \atop t+s=|\alpha|-1} \left( 1 + \frac{x_0}{q^\alpha} \right)^t \frac{x_0}{q^2} \left( 1 + \frac{x_0}{q^\alpha} \right)^s \right] \]

\[ = \frac{x^\alpha}{q^\alpha} \left[ |\alpha| \left( 1 + \frac{x_0}{q^\alpha} \right)^{|\alpha|} - |\alpha| \left( 1 + \frac{x_0}{q^\alpha} \right)^{|\alpha|-1} \frac{x_0}{q^\alpha} \right]. \]

STEP 5: We can now compute \( V_q \mu^\alpha \).
Using (3.14) and the previous step we have:

\[
V_q \mu^\alpha = x^\alpha \frac{|\alpha|}{q} \left(1 + \frac{x_0}{q}\right)^{|\alpha|-1} - \frac{x^\alpha}{q} \left[|\alpha| \left(1 + \frac{x_0}{q}\right)^{|\alpha|} - |\alpha| \left(1 + \frac{x_0}{q}\right)^{|\alpha|-1} \frac{x_0}{q}\right] \\
= \frac{|\alpha|x^\alpha}{q} \left(1 + \frac{x_0}{q}\right)^{|\alpha|-1} \left(1 - \left(1 + \frac{x_0}{q}\right) + \frac{x_0}{q}\right) \\
= 0.
\]

This ends the proof.

Remark 3.7. We observe that both functions \(\mu^\alpha\) and \(\zeta^\alpha\) coincide with \(x^\alpha\) when \(x_0 = 0\). It is important to note that these are two different extensions of the same real function \(x^\alpha\) leading to two different regular function theories. In fact, \(\mu^\alpha\) is the \(V_q\)-regular extension of \(x^\alpha\) while \(\zeta^\alpha\) gives the classical Fueter extension. However, the classical Fueter variables \(\zeta^\alpha\) extend \(x^\alpha\) to the whole space of quaternions while \(\mu^\alpha\) extend \(x^\alpha\) to domains of \(\mathbb{H}^*\).

**Proposition 3.8.** We prove that \(q^n\) is in ker \(V_q\) and that, moreover,

\[
q^n = \sum_{|\alpha|=n} \mu^\alpha c_{\alpha,n}
\]

where, with \(\alpha = (\alpha_1, \alpha_2, \alpha_3)\),

\[
(3.15) \quad c_{\alpha,n} = \frac{n!}{\alpha!} e_1^{\alpha_1} \times e_2^{\alpha_2} \times e_3^{\alpha_3},
\]

where the symmetric product is taken among all the products of the units \(e_u\).

**Proof.** In \(\mathbb{H}^*\) we have

\[
q^n = (x_0 + \vec{q})^n \\
= \left(1 + \frac{x_0}{q}\right)^n (\vec{q})^n \\
= \left(1 + \frac{x_0}{q}\right)^n \left(\sum_{|\alpha|=n} x^\alpha c_{\alpha,n}\right) \\
= \sum_{|\alpha|=n} x^\alpha \left(1 + \frac{x_0}{q}\right)^n c_{\alpha,n},
\]

for some \(c_{\alpha,n} \in \mathbb{H}\) which can be expressed in term of symmetrized products as in (3.15) by known formulas.

We note that (3.15) does not take into account the Cayley table of multiplication for the quaternions.

**Example 3.9.** Let us examine \(\mu_1(x) = x_1 \left(1 + \frac{x_0}{q}\right) = x_1 + x_0 \frac{x_1}{q}\). The function \(\frac{\partial \mu_1}{\partial x_2} = x_1 x_0 \left(\frac{e_2}{q^2} + \frac{2x_2}{q^3}\right)\) is not \(V_q\)-regular, while the function \(\frac{\partial \mu_1}{\partial x_0} = \frac{x_1}{q}\) is \(V_q\)-regular but cannot be written as a convergent Fueter-like series of the \(\mu^\alpha\).
Discussion of Example 3.9. We have
\[
\frac{\partial \mu_1}{\partial x_2}(x) = x_1 x_0 \left( \frac{e_2}{q^2} + \frac{2x_2}{q^3} \right),
\]
and so
\[
\frac{\partial^2 \mu_1}{\partial x_0 \partial x_2}(x) = x_1 \left( \frac{e_2}{q^2} + \frac{2x_2}{q^3} \right)
\]
is independent of \(x_0\). On the other hand
\[
3 \sum_{u=1} x_u \frac{\partial}{\partial x_u} \left( \frac{\partial \mu_1}{\partial x_2} \right) = 3 \sum_{u=1} x_u \frac{\partial}{\partial x_u} \left( x_1 x_0 \left( \frac{e_2}{q^2} + \frac{2x_2}{q^3} \right) \right)
\]
\[
= x_0 \left[ 3 \sum_{u=1} x_u \frac{\partial}{\partial x_u} \left( x_1 \left( \frac{e_2}{q^2} + \frac{2x_2}{q^3} \right) \right) \right].
\]
Hence
\[
\left( V_q \frac{\partial \mu_1}{\partial x_2} \right)(x) = x_1 \left( \frac{e_2}{q^2} + \frac{2x_2}{q^3} \right) - x_0 \left[ 3 \sum_{u=1} x_u \frac{\partial}{\partial x_u} \left( x_1 \left( \frac{e_2}{q^2} + \frac{2x_2}{q^3} \right) \right) \right].
\]
Setting \(x_0 = 0\) shows that \(V_q \frac{\partial \mu_1}{\partial x_2} \neq 0\).

We now turn to \(\frac{\partial \mu_1}{\partial x_0}\), which is \(V_q\)-regular since differentiation with respect to \(x_0\) commutes with \(V_q\). Assume now that
\[
\frac{\partial \mu_1}{\partial x_0} = \sum_{\alpha \in \mathbb{N}_0^3} \mu^\alpha c_\alpha
\]
for some quaternionic coefficients \(c_\alpha\), and where the convergence is assumed in some set \(\Omega \subset \mathbb{H}^*\). In other words
\[
\frac{x_1}{q} = \sum_{\alpha \in \mathbb{N}_0^3} x^\alpha \left( 1 + \frac{x_0}{q} \right)^{|\alpha|} c_\alpha.
\]
Setting \(x_0 = 0\) in this equation, and multiplying both sides by \(q\) leads to
\[
x_1 = \sum_{\alpha \in \mathbb{N}_0^3} x_1^\alpha x_2^{a_2} x_3^{a_3} e_1 c_\alpha + \sum_{\alpha \in \mathbb{N}_0^3} x_1^\alpha x_2^{a_2} x_3^{a_3} e_2 c_\alpha + \sum_{\alpha \in \mathbb{N}_0^3} x_1^\alpha x_2^{a_2} x_3^{a_3} e_3 c_\alpha,
\]
where this equality is valid \textit{a priori} only for \((x_1, x_2, x_3)\) such that \(x_u \neq 0\) for some \(u = 1, 2, 3\). By classical results on series and summable families (3.17) can be extended to \(x_u = 0\). Comparing the linear terms on both sides of (3.17) leads to
\[
x_1 = x_1 e_1 c_{0,0,0} + x_2 e_2 c_{0,0,0} + x_3 e_3 c_{0,0,0},
\]
which is impossible, leading to a contradiction.

Just as in the classical case, let us now define the conjugate operator of \(V_q\), denoted by \(\overline{V_q}\), to be the operator defined by:
\[
\overline{V_q} := \frac{\partial}{\partial x_0} + \frac{1}{q} \sum_{u=1}^3 e_u \frac{\partial}{\partial x_u}.
\]
We can prove the following:

**Proposition 3.10.** Let $\Omega$ be an open domain in $\mathbb{H}^*$. If $f$ is $V_q$-regular on $\Omega$, then $V_q(f)$ is also $V_q$-regular on $\Omega$. Moreover, in this case we have

\[(3.19) \quad \frac{1}{2} V_q f(q) = \frac{\partial}{\partial x_0} f(q), \quad \forall q = x_0 + \vec{q}.\]

**Proof.** We first observe that

\[(V_q + \overline{V}_q)f(q) = 2 \frac{\partial}{\partial x_0} f(q).\]

However, since $f$ is $V_q$-regular, then it belongs to $\ker(V_q)$ so that $V_q(f) = 0$, leading to

\[\overline{V}_q f(q) = 2 \frac{\partial}{\partial x_0} f(q).\]

Thus, applying the operator $V_q$ and, using the fact that it commutes with $\frac{\partial}{\partial x_0}$, we obtain

\[V_q \overline{V}_q f(q) = 2 V_q \frac{\partial}{\partial x_0} f(q) = 2 \frac{\partial}{\partial x_0} V_q f(q) = 0.\]

So, the function $\overline{V}_q(f)$ is $V_q$-regular, which ends the proof. \(\square\)

**Proposition 3.11.** It holds that for every $q \in \mathbb{H}^*$ we have

\[\frac{1}{2} \overline{V}_q \mu^\alpha(q) = \frac{|\alpha|}{q} \mu^\alpha(q).\]

**Proof.** We use the previous computations of $\frac{\partial}{\partial x_0} \mu^\alpha$ and $\frac{1}{q} \sum_{u=1}^3 x_u \frac{\partial}{\partial x_u} \mu^\alpha$ to obtain

\[\overline{V}_q \mu^\alpha(q) = 2|\alpha| \frac{x^\alpha}{q} \left(1 + \frac{x_0}{q}\right)^{|\alpha|-1}.\]

Hence, we get

\[\overline{V}_q \mu^\alpha(q) = 2|\alpha| q^{-1} \mu^\alpha(x),\]

which is the desired equality. \(\square\)

**Remark 3.12.** It is easy to see that a consequence of this proposition yields the counterpart of the properties of the number operators and we have

\[M_q \overline{V}_q(\mu^\alpha) = 2|\alpha| \mu^\alpha,\]

where $M_q$ represents the left quaternionic multiplication.
4. Cauchy-Kovalevskaia product

In this section we build a Cauchy-Kovalevskaia product using the new variables $\mu^\alpha$. We start by building a CK-extension of a function in the kernel of $V_q$, $f = f_0 + e_1 f_1 + e_2 f_2 + e_3 f_3$, where $f_0, f_1, f_2, f_3$ are real valued differentiable functions on an open subset $\Omega \subset \mathbb{H}^*$. The equation $V_q f = 0$ can be rewritten as

$$\frac{\partial f_j}{\partial x_0} = G_j, \quad j = 0, 1, 2, 3,$$

where $G_j$ is real analytic in the variables $x_0, x_1, x_2, x_3$. The right hand side can be written in terms of the partial derivatives with respect to the other three variables, i.e. $\frac{\partial f_j}{\partial x_k}$, $j = 0, 1, 2, 3$, $k = 1, 2, 3$, and they are given by the following system on an open subset $\Omega \subset \mathbb{H}^*$:

$$\begin{align*}
\frac{\partial f_0}{\partial x_0} &= \frac{1}{x_1^2 + x_2^2 + x_3^2} (x_1 E f_1 + x_2 E f_2 + x_3 E f_3) \\
\frac{\partial f_1}{\partial x_0} &= -\frac{1}{x_1^2 + x_2^2 + x_3^2} (-x_1 E f_0 - x_2 E f_3 + x_3 E f_2) \\
\frac{\partial f_2}{\partial x_0} &= -\frac{1}{x_1^2 + x_2^2 + x_3^2} (-x_2 E f_0 - x_1 E f_3 + x_3 E f_1) \\
\frac{\partial f_3}{\partial x_0} &= -\frac{1}{x_1^2 + x_2^2 + x_3^2} (-x_3 E f_0 - x_1 E f_2 + x_2 E f_1)
\end{align*}$$

(4.2)

where $E$ denotes the Euler operator:

$$E = \sum_{u=1}^{3} x_u \frac{\partial}{\partial x_u}.$$ 

We can therefore apply the Cauchy-Kovalevskaia theorem (see e.g. [20] §7 p. 39], which asserts that the system (4.1) has a unique solution near a real point possibly different from the origin) for given initial real analytic values $f_j(0, x_1, x_2, x_3)$ on an open domain $\Omega \subset (\mathbb{H}^* \cap \{x_0 = 0\})$. This solution, $F$, is defined on an open set $\Omega \subset \mathbb{H}^*$, where $\tilde{\Omega} = \Omega \cap \{x_0 = 0\}$ and we call it the CK-extension of $f$ with respect to $V_q$.

**Definition 4.1.** The CK-extension to $\Omega$ with respect to the operator $V_q$ found above is denoted by $F = CKV_q(f)$.

**Remark 4.2.** The choice

$$f_0(0, x_1, x_2, x_3) = x_1^{a_1} x_2^{a_2} x_3^{a_3} \quad \text{and} \quad f_1(0, x_1, x_2, x_3) = f_2(0, x_1, x_2, x_3) = f_3(0, x_1, x_2, x_3) = 0$$

leads, in the case of the variables $x^\alpha$ to:

$$\mu^\alpha = CKV_q(x^\alpha).$$

**Definition 4.3.** We can now define the CK-product of two functions $f, g$ in the kernel of $V_q$ to be:

$$f *_{V_q} g = CKV_q(f(0, x_1, x_2, x_3)g(0, x_1, x_2, x_3)),$$

(4.4)

for every $x \in \Omega$.

It is easy to see that, for any two quaternions $c, d$ we have:
**Proposition 4.4.** On the entire domain $\mathbb{H}^*$ we have that

\begin{equation}
\mu^\alpha c \ast_{V_q} \mu^\beta d = \mu^{\alpha+\beta}cd.
\end{equation}

*Proof.* The proof is left to the reader. $\square$

**Theorem 4.5.** For every $x \in \mathbb{H}^*$ it holds that:

\begin{equation}
\left( \exp \left( x_0 \frac{1}{q} E \right) \right) (x^\alpha) = \mu^\alpha.
\end{equation}

*Proof.* We proceed in a number of steps.

**STEP 1:** It holds that

\begin{equation}
E \left( \frac{1}{q} \right) = -\frac{1}{q}.
\end{equation}

Indeed, using (3.11) we can write

\begin{align*}
E \left( \frac{1}{q} \right) &= \sum_{u=1}^{3} x_u \left( \frac{e_u}{q^2} + \frac{2x_u}{q^3} \right) \\
&= \frac{\vec{q}}{q^2} + 2 \frac{x_1^2 + x_2^2 + x_3^2}{q^3} \\
&= \frac{\vec{q}}{q^2} - 2 \frac{\vec{q}^2}{q^3} \\
&= -\frac{1}{q}.
\end{align*}

**STEP 2:** For non-commuting functions of a real variable it holds that

\begin{equation}
(fg)' = f'g + fg'.
\end{equation}

This follows from

\begin{align*}
f(t)g(t) - f(t_0)g(t_0) &= (f(t) - f(t_0))g(t) + f(t_0)(g(t) - g(t_0)).
\end{align*}

Note that $f$ and $g$ may commute at a joint value $t$ but we do not assume that $f(t)g(s) = g(s)f(t)$ for $t \neq s$.

**STEP 3:** It holds that

\begin{equation}
E \left( \frac{x^\alpha}{q^n} \right) = \begin{cases} 
(\alpha - n)\frac{x^\alpha}{q^n}, & 0 \leq n \leq |\alpha|, \\
0, & \text{otherwise}.
\end{cases}
\end{equation}

We proceed by induction. The case $n = 0$ corresponds to the formula

\begin{align*}
E(x^\alpha) &= |\alpha|x^\alpha.
\end{align*}

We then write

\begin{align*}
\frac{x^\alpha}{q^{n+1}} &= \frac{x^\alpha}{q^n} \frac{1}{q}.
\end{align*}
and apply (4.8) with $f(x) = \frac{x^\alpha}{q^n}$ and $g(x) = \frac{1}{q}$ to get:

$$E\left(\frac{x^\alpha}{q^{n+1}}\right) = \sum_{u=1}^{3} x_u \frac{\partial}{\partial x_u} \left(\frac{x^\alpha}{q^{n+1}}\right)$$

$$= \left(\sum_{u=1}^{3} x_u \frac{\partial}{\partial x_u} \left(\frac{x^\alpha}{q^n}\right)\right) \frac{1}{q} + \frac{x^\alpha}{q^n} \left(\sum_{u=1}^{3} x_u \frac{\partial}{\partial x_u} \frac{1}{q}\right)$$

$$= (|\alpha| - n) \frac{x^\alpha}{q^n} \frac{1}{q} + \frac{x^\alpha}{q^n} \frac{-1}{q}$$

induction at rank $n$ by (4.7)

$$= (|\alpha| - n - 1) \frac{x^\alpha}{q^{n+1}}.$$ 

**STEP 4:** We prove (4.10)

$$\left(E\frac{1}{q}\right)^n (x^\alpha) = (|\alpha| - 1)(|\alpha| - 2) \cdots (|\alpha| - n + 1) \frac{x^\alpha}{q^n}, \quad n = 1, 2, \ldots$$

We proceed by induction. The case $n = 1$ corresponds to the previous step.

$$\left(E\frac{1}{q}\right)^{n+1} (x^\alpha) = E\left(\frac{1}{q}\right) \left(E\frac{1}{q}\right)^n (x^\alpha) = E\frac{1}{q} (|\alpha| - 1)(|\alpha| - 2) \cdots (|\alpha| - n + 1) E\frac{1}{q^n}$$

$$= (|\alpha| - 1)(|\alpha| - 2) \cdots (|\alpha| - n + 1) E\left(E\frac{1}{q}\right)^n - 1 \left(E\frac{1}{q}\right)^n (x^\alpha)$$

where we have used the induction hypothesis to go from the first to the second line, and formula (4.9) to go from the third line to the fourth.

**STEP 5:** We prove (4.6).

We first note that, in view of (4.10)

$$\left(E\frac{1}{q}\right)^n (x^\alpha) = 0$$

for $n \geq |\alpha|$. Furthermore, since

$$\left(E\frac{1}{q}\right)^n = \frac{1}{q} \left(E\frac{1}{q}\right)^{n-1} E, \quad n = 1, 2, \ldots,$$

we have that the series

$$\left(\exp\left(x_0 \frac{1}{q} E\right)\right) (x^\alpha) = x^\alpha + \sum_{n=1}^{\infty} \frac{x^n_0}{n!} \left(E\frac{1}{q}\right)^n (x^\alpha)$$

$$= x^\alpha + \sum_{n=1}^{\infty} \frac{x^n_0}{n! q^n} \left(\frac{1}{q}\right)^n E(x^\alpha)$$
has only a finite number of non-zero terms, and is therefore equal to
\[
\left( \exp \left( x_0 \frac{1}{q} E \right) \right) (x^\alpha) = x^\alpha + \sum_{n=1}^{\infty} \frac{x_0^n}{n!} \left( \frac{1}{q} \right)^n E(x^\alpha)
\]
\[
= x^\alpha \left( \sum_{n=0}^{\lvert \alpha \rvert - 1} \frac{\lvert \alpha \rvert (\lvert \alpha \rvert - 1) \cdots (\lvert \alpha \rvert - n + 1)}{n!} \left( \frac{x_0}{q} \right)^n \right)
\]
\[
= x^\alpha \left( 1 + \frac{x_0}{q} \right)^{\lvert \alpha \rvert}.
\]

The theorem is now proven. \( \square \)

Let \( b \) and \( c \) be two quaternions and writing
\[
q^n b = \sum_{\alpha \in \mathbb{N}_0^3} \mu^\alpha c_{\alpha,n} b \quad \text{and} \quad q^m c = \sum_{\alpha \in \mathbb{N}_0^3} \mu^\alpha c_{\alpha,m} c,
\]
we have the \( \star_{V_q} \) product of the two:
\[
q^n b \star_{V_q} q^m c = \sum_{\alpha,\beta \in \mathbb{N}_0^3} \mu^{\alpha+\beta} c_{\alpha,n} b c_{\alpha,m} c.
\]

In the expression above, since \( b \) and \( c \) are quaternions they do not commute with \( c_{\alpha,n} \) and \( c_{\beta,m} \), however, when \( b \in \mathbb{R} \) this expression reduces to
\[
q^n b \star_{V_q} q^m c = \sum_{\alpha,\beta \in \mathbb{N}_0^3} \mu^{\alpha+\beta} c_{\alpha,n} c_{\beta,m} b c
\]
(4.11)
\[
= \sum_{\gamma \in \mathbb{N}_0^3} \left( \sum_{\alpha+\beta = \gamma} c_{\alpha,n} c_{\beta,m} \right) b c, \quad b \in \mathbb{R}, \ c \in \mathbb{H}.
\]

When \( x_0 = 0 \) we have \( q = \bar{q} \) and the \( \star_{V_q} \) product reduces to the pointwise product. We have
\[
\bar{q}^n = \sum_{\alpha \in \mathbb{N}_0} x^\alpha c_{\alpha,n}
\]
and
\[
\bar{q}^n \bar{q}^m = \bar{q}^{n+m}
\]
so that
\[
c_{\gamma,n+m} = \sum_{\alpha+\beta = \gamma} c_{\alpha,n} c_{\beta,m}.
\]

\[
q^n b \star_{\text{slice}} q^m c = q^{n+m} b c
\]
(4.12)
\[
= \sum_{\gamma \in \mathbb{N}_0^3} \mu^\gamma c_{\gamma,n+m} b c.
\]

We then have:
Proposition 4.6. Let \( b \in \mathbb{R} \) and \( c \in \mathbb{H} \). On the entire domain \( \mathbb{H}^* \) it holds that

\[
q^n b \ast_{V_q} q^m c = q^n b \ast_{\text{slice}} q^m c = q^{n+m} b c
\]

and, more generally, for \( f \) intrinsic

\[
f \ast_{V_q} g = f \ast_{\text{slice}} g = fg.
\]

The term intrinsic used in slice quaternionic analysis means that the function preserves all slices. This equality does not hold for more general functions in the kernel of \( V_q \).

5. Reproducing kernel Hilbert spaces

We now set the stage for the Schur analysis theory in this case and we begin by defining a reproducing kernel for the Hilbert space of power series in \( \mu^\alpha \).

For \( r, R \) and \( \rho \) strictly positive, let us first define the domain:

\[
\Omega_{r,R,\rho} = \{ x \in \mathbb{R}^4 : r < |x_u| < R, \ u = 1, 2, 3, \text{ and, } |x_0| < \rho \}.
\]

First we see that for any \( x \in \Omega_{r,R,\rho} \subset \mathbb{H}^* \), we have:

\[
|\mu^\alpha(x)| \leq L_{r,R,\rho}^{2|\alpha|} \rho / \sqrt{3r},
\]

where \( L_{r,R,\rho} = R \left( 1 + \frac{\rho}{\sqrt{3r}} \right) \).

Let \( \Omega_{r,R,\rho} \) and \( L \) be defined as above.

Proposition 5.1. Let \( c_\alpha \) be a family of positive numbers for \( \alpha \) in a subset \( S \) of \( \mathbb{N}_0^3 \). Assume that

\[
\sum_{\alpha \in S} L_{r_1,R_1,\rho_1}^{2|\alpha|} c_\alpha < \infty
\]

for all \( r_1, R_1, \rho_1 \) such that

\[
r < r_1 < R_1 < R \quad \text{and} \quad 0 < \rho_0 < \rho.
\]

Then the function

\[
K_c(x, y) = \sum_{\alpha \in S} \frac{\mu^\alpha(x) \overline{\mu^\alpha(y)}}{c_\alpha},
\]

is positive definite in \( \Omega_{r,R,\rho} \) and the associated reproducing kernel Hilbert space consists of the Fueter-like series with quaternionic coefficients \( f_\alpha \):

\[
\mathcal{H}(K_c) = \{ f = \sum_{\alpha \in \mathbb{N}_0^3} \mu^\alpha f_\alpha \mid f \text{ abs. conv. in } \Omega_{r,R,\rho}, \sum_{\alpha \in \mathbb{N}_0^3} c_\alpha |f_\alpha|^2 < \infty \}.
\]

Proof. The proof follows from the usual arguments as in [6, 10]. \( \square \)

Proposition 5.2. Elements of \( \mathcal{H}(K_c) \) are \( V_q \)-regular on

\[
\Omega(\mathcal{H}(K_c)) = \left\{ x \in \mathbb{H}^* \mid \sum_{\alpha \in \mathbb{N}_0^3} \frac{|\mu^\alpha(x)|^2}{c_\alpha} < \infty \right\}.
\]
Proof. We proceed in a number of steps.

STEP 1: Let \( a \in (0,1) \). Then
\[
\sum_{\alpha \in \mathbb{N}_0^3} a^{2|\alpha|} |\alpha|^2 < \infty,
\]
and in particular \( a^{|\alpha|}|\alpha| \) are uniformly bounded. By the Cauchy-Schwarz inequality
\[
(\alpha_1 + \alpha_2 + \alpha_3)^2 \leq 3(\alpha_1^2 + \alpha_2^2 + \alpha_3^2).
\]
Hence
\[
\sum_{\alpha \in \mathbb{N}_0^3} a^{2|\alpha|} |\alpha|^2 \leq 3 \sum_{\alpha \in \mathbb{N}_0^3} a^{2|\alpha|} (\alpha_1^2 + \alpha_2^2 + \alpha_3^2)
\]
\[
= 3 \sum_{u=1}^3 \left( \sum_{\alpha \in \mathbb{N}_0^3} a^{2\alpha_1} a^{2\alpha_2} a^{2\alpha_3} \alpha_u^2 \right).
\]
But
\[
\left( \sum_{\alpha \in \mathbb{N}_0^3} a^{2\alpha_1} a^{2\alpha_2} a^{2\alpha_3} \alpha_1^2 \right) = \left( \sum_{\alpha_1=0}^\infty a^{2\alpha_1} \right) \left( \sum_{\alpha_2=0}^\infty a^{2\alpha_2} \right) \left( \sum_{\alpha_3=0}^\infty a^{2\alpha_3} \right) < \infty
\]
and similarly for \( u = 2 \) and \( u = 3 \).

STEP 2: For \( r_1 \) and \( R_1 \) as in (5.3)
\[
(5.7) \quad \sum_{\alpha \in \mathbb{N}_0^3} \left| \frac{\partial \mu^\alpha}{\partial x_1} \right| c_\alpha^2 < \infty
\]
for \( r_1 \leq |x_1| \leq R_1 \), and similarly for \( x_2 \) and \( x_3 \).
We have
\[
\frac{\partial \mu^\alpha}{\partial x_1} = \alpha_1 x^{\alpha-e_1} \left( 1 + \frac{x_0}{q} \right)^{|\alpha|} +
\]
\[
x^\alpha \left[ \sum_{t+s=|\alpha|-1 \atop \in \mathbb{N}_0} \left( 1 + \frac{x_0}{q} \right)^t \left( \frac{x_0 e_u}{q^2} + \frac{2 x_u x_0}{q^3} \right) \left( 1 + \frac{x_0}{q} \right)^s \right]
\]
and so
\[
\left| \frac{\partial \mu^\alpha}{\partial x_1} \right| \leq \alpha_1 R_1^{1|\alpha|-1} + R_1^{|\alpha|} \left( 1 + \frac{\rho_1}{\sqrt{3} r_1} \right)^{|\alpha|-1} \left( \frac{\rho_1}{3 r_1^2} + \frac{2 R_1 \rho_1}{3^{3/2} r_1^{3/2}} \right)
\]
\[
\leq R_1^{|\alpha|} \left( 1 + \frac{\rho_1}{\sqrt{3} r_1} \right)^{|\alpha|} \left[ 1 + \frac{1}{1 + \frac{\rho_1}{\sqrt{3} r_1}} \left( \frac{\rho_1}{3 r_1^2} + \frac{2 R_1 \rho_1}{3^{3/2} r_1^{3/2}} \right) \right].
\]
\[
=M, \text{ independent of } \alpha
\]
Let $R_2$ be such that $R_1 < R_2 < R$. We can write
\[
\sum_{\alpha \in \mathbb{N}_0^3} \frac{\left| \partial \mu^\alpha \right|^2}{c_\alpha} \leq M \sum_{\alpha \in \mathbb{N}_0^3} \frac{R_1^{2|\alpha|} \left( 1 + \frac{\rho_1}{\sqrt{3}r_1} \right)^{2|\alpha|} |\alpha|^2}{c_\alpha} 
\]
\[
= M \sum_{\alpha \in \mathbb{N}_0^3} \frac{R_2^{2|\alpha|} \left( 1 + \frac{\rho_1}{\sqrt{3}r_1} \right)^{2|\alpha|} |\alpha|^2}{c_\alpha} \left[ \left( \frac{R_1}{R_2} \right)^{2|\alpha|} |\alpha|^2 \right] 
\]
\[
\leq M \sum_{\alpha \in \mathbb{N}_0^3} \frac{R_2^{2|\alpha|} \left( 1 + \frac{\rho_1}{\sqrt{3}r_1} \right)^{2|\alpha|} |\alpha|^2}{c_\alpha} 
\]
\[
< \infty 
\]
using Step 1, for some constant $M$.

STEP 3: For $r_1$ and $R_1$ as in (5.3) we have:

(5.8) \[
\sum_{\alpha \in \mathbb{N}_0^3} \frac{\left| \partial_0 \mu^\alpha \right|^2}{c_\alpha} < \infty 
\]
for $|x_0| \leq \rho_1$.

This follows from

\[
\frac{\partial}{\partial x_0} \mu^\alpha = x_0^\alpha \left[ \sum_{t,s \in \mathbb{N}_0 \, \, t+s=|\alpha|-1} \left( 1 + \frac{x_0}{q} \right)^t \left( \frac{1}{q} \right) \left( 1 + \frac{x_0}{q} \right)^s \right] 
\]
and the corresponding bound

\[
\left| \frac{\partial}{\partial x_0} \mu^\alpha \right| \leq |\alpha|R_1^{|\alpha|} \left( 1 + \frac{\rho_1}{\sqrt{3}r_1} \right)^{|\alpha|-1} \frac{1}{\sqrt{3}r_1}. 
\]

This step can be proven directly using the Appell-type property, i.e. $V_T$ acts on powers of $\mu$ the same as $\frac{\partial}{\partial x_0}$.

STEP 4: We prove that, pointwise, for $f = \sum_{\alpha \in \mathbb{N}_0^3} \mu^\alpha f_\alpha \in \mathcal{H}(K_c)$,

\[
\frac{\partial}{\partial x_u} \sum_{\alpha \in \mathbb{N}_0^3} \mu^\alpha f_\alpha = \sum_{\alpha \in \mathbb{N}_0^3} \frac{\partial}{\partial x_u} \mu^\alpha f_\alpha. 
\]

Using the Cauchy-Schwarz inequality and the previous lemma we see that the series of derivatives

\[
\sum_{\alpha \in \mathbb{N}_0^3} \frac{\partial}{\partial x_u} \mu^\alpha f_\alpha = \sum_{\alpha \in \mathbb{N}_0^3} \frac{\partial}{\partial x_u} \mu^\alpha \sqrt{c_\alpha} f_\alpha 
\]
converges uniformly in intervals \([r_1, R_1]\). Since the series \(\sum_{\alpha \in \mathbb{N}_0^3} \mu^\alpha f_\alpha\) converges in \(\mathcal{F}(K_c)\) then it converges pointwise and a classical calculus theorem allows us to conclude that we have convergence in Step 4. This classical theorem speaks of sequences and not of summable families, but we can reduce the latter to the case of sequences by identifying \(\mathbb{N}_0^3\) and \(\mathbb{N}_0^3\) via a bijection.

This solves the cases of \(u = 1, 2, 3\). The case \(u = 0\) is treated in a similar way and we leave the details to the reader.

We now apply Step 4 four times in the definition of \(V_q\). More precisely,

\[
V_q \left( \sum_{\alpha \in \mathbb{N}_0^3} \mu^\alpha f_\alpha \right) = \frac{\partial}{\partial x_0} \left( \sum_{\alpha \in \mathbb{N}_0^3} \mu^\alpha f_\alpha \right) - \frac{1}{q} \sum_{u=1}^{3} x_u \frac{\partial}{\partial x_u} \left( \sum_{\alpha \in \mathbb{N}_0^3} \mu^\alpha f_\alpha \right)
\]

This concludes the proof. \(\square\)

Let us now turn to a definition of a backward-shift and multiplication operator in this case and we set

\[
(5.9) \quad e_1 = (1, 0, 0), \quad e_2 = (0, 1, 0) \quad \text{and} \quad e_3 = (0, 0, 1).
\]

**Definition 5.3.** The multiplication and backward shift operators on \(\mathbb{H}^*\) are:

\[
(5.10) \quad M_u \mu^\alpha = \mu^{\alpha + e_u}
\]

\[
(5.11) \quad B_u \mu^\alpha = \frac{\alpha_u}{|\alpha|} \mu^{\alpha - e_u}, \quad u = 1, 2, 3
\]

with the understanding that \(B_u \mu^\alpha = 0\) if \(\alpha_u = 0\).

Note that

\[
(5.12) \quad \left( \sum_{u=1}^{3} B_u M_u \right) \mu^\alpha = \mu^\alpha.
\]

As expected, we have the following result:

**Theorem 5.4.** Assume that the domain of definition of \(M_u\) in \(\mathcal{F}(K_c)\) contains the linear span of the Fueter polynomials \(\mu^\alpha\). Then,

\[
(5.13) \quad M_u^*(\mu^\alpha) = \frac{c_\alpha}{c_{\alpha - e_u}} \mu^{\alpha - e_u},
\]
with the understanding that the right hand side of \((5.13)\) equals 0 if \(\alpha_u = 0\).

Proof. It is easy to check that:
\[
\langle M_\mu^\alpha, \mu^\beta \rangle = \langle \mu^\alpha, \mu^{\beta + e_u} \rangle = c_\alpha \delta_{\alpha, \beta + e_u} = c_\alpha \delta_{\alpha - e_u, \beta} = \frac{c_\alpha}{c_{\alpha - e_u}} c_{\alpha - e_u} \delta_{\alpha - e_u, \beta} = \frac{c_\alpha}{c_{\alpha - e_u}} \langle \mu^{\alpha - e_u}, \mu^\beta \rangle.
\]
This concludes the proof. \(\square\)

6. Arveson space and Schur multipliers

For properties of the classical Arveson space, as well as a motivation for its definition, the reader can turn to [11, 21]. We now turn to the definition of an Arveson space in our case, which we will denote by \(A\), in the present setting. We start with the definition of the counterpart of the unit ball in our case:

\[
\Omega_A = \Omega(\mathfrak{A}) = \left\{ x \in \mathbb{H}^*, \sum_{\alpha \in \mathbb{N}_0^3} |\mu^\alpha(x)|^2 \frac{\alpha!}{\alpha!} < \infty \right\}.
\]

Definition 6.1. The \(V_q\)–Arveson space is the Hilbert space of absolutely convergent \(V_q\)-series on the unit ball

\[
\sum_{\alpha \in \mathbb{N}_0^3} \mu^\alpha(x) c_\alpha
\]

with

\[
\langle f, f \rangle = \sum_{\alpha \in \mathbb{N}_0^3} \frac{\alpha!}{|\alpha|!} |c_\alpha|^2 < \infty,
\]

i.e.:

\[
\mathfrak{A} = \{ f(x) = \sum_{\alpha \in \mathbb{N}_0^3} \mu^\alpha(x) c_\alpha | f \text{ abs. conv. with } \langle f, f \rangle = \sum_{\alpha \in \mathbb{N}_0^3} \frac{\alpha!}{|\alpha|!} |c_\alpha|^2 < \infty \}.
\]

Here the notations are the ones of Definition 3.3.

Definition 6.2. The reproducing kernel for this Arveson space is:

\[
K_\mathfrak{A}(x, y) = \sum_{\alpha \in \mathbb{N}_0^3} \mu^\alpha(x) \overline{\mu^\alpha(y)} \frac{|\alpha|!}{\alpha!}.
\]

The domain of this kernel is defined by the following:

Proposition 6.3. Elements of the Arveson space are \(V_q\)-regular in

\[
\Omega_A = \Omega(\mathfrak{A}) = \left\{ x \in \mathbb{H}^*, \sum_{\alpha \in \mathbb{N}_0^3} |\mu^\alpha(x)|^2 \frac{\alpha!}{\alpha!} < \infty \right\}.
\]
Proof. This follows from the reproducing kernel property, using the same type of arguments as in Section 5. \( \square \)

Evaluation at the origin does not exist, however, we can now define its counterpart by setting, for \( f \in \mathfrak{A} \):

\[
C(f) = c_{0,0,0}.
\]

**Theorem 6.4.** The Arveson space is the unique reproducing kernel Hilbert space (up to a multiplicative constant) of \( V_q \) power series such that \( M_1, M_2 \) and \( M_3 \) are bounded operators there and satisfy

\[
I - \sum_{u=1}^{3} M_u M_u^* = CC^*.
\]

Furthermore,

\[
M_u^* = B_u, \quad u = 1, 2, 3,
\]

in the Arveson space.

**Proof.** We first assume that (6.7) (sometimes called the structural identity) is in force and compute the associated inner product. We proceed in a number of steps.

**STEP 1:** We have that

\[
C^*1 = 1_{\mathbb{H}},
\]

(the constant function equal identically to 1).

Indeed,

\[
(C^*1)(y) = \langle C^*1, K_{\mathfrak{A}}(\cdot, y) \rangle_{\mathfrak{A}} = \langle 1_H, CK_{\mathfrak{A}}(\cdot, y) \rangle_{\mathbb{H}} = 1.
\]

Here we used the fact that \( CK_{\mathfrak{A}}(\cdot, y) = 1_H \).

**STEP 2:** \( M_u^*1 = 0 \)

This is (5.13) with \( \alpha = (0, 0, 0) \).

**STEP 3:** Let \( \alpha \) and \( \beta \) different from \( (0, 0, 0) \), and assume that (6.7) is in force. We have

\[
1 = \sum_{\alpha=1}^{3} \frac{c_{\alpha}}{c_{\alpha-e_u}}, \quad \alpha \in \mathbb{N}^3.
\]

Indeed,

\[
\langle \mu^\alpha, \mu^\beta \rangle = \sum_{u=1}^{3} \langle M_u^\alpha \mu^\alpha, M_u^\beta \mu^\beta \rangle
\]

that is:

\[
c_{\alpha} \delta_{\alpha, \beta} = \sum_{u=1}^{3} \frac{c_{\alpha}c_{\beta}}{c_{\alpha-e_u}c_{\beta-e_u}} \langle \mu^\alpha-e_u \mu^\beta-e_u \rangle
\]

or, equivalently (6.9) holds.
STEP 4: The inner product is that of the Arveson space.

From (6.9) we have that the sequence \( d_\alpha = \frac{1}{c_\alpha} \) satisfies

\[
d_\alpha = \sum_{\mu=1}^{3} d_{\alpha - e_\mu}, \quad \alpha \in \mathbb{N}^3.
\]

Together with \( d_{0,0,0} = 1 \) we get \( c_\alpha = c_{0,0,0} \frac{\alpha!}{|\alpha|!} \).

The converse is proven by reading these arguments backwards, with \( c_\alpha = \frac{\alpha!}{|\alpha|!} \).

\[
\square
\]

**Proposition 6.5.** Assume that \( S \) is a \( \mathbb{H}^{n \times m} \)-valued multiplier defined on \( \Omega_A \). Then,

\[
(6.10) \quad (M^*_S(K_A(\cdot, y)) \xi)(x) = \sum_{\alpha \in \mathbb{N}^3} \frac{|\alpha|!}{\alpha!} \mu^\alpha(x) (S *_{V_q} \mu^\alpha)(y)^* \xi, \quad \forall \xi \in \mathbb{H}^n.
\]

**Proof.** Since \( M^*_S \) is continuous we obtain:

\[
\langle M^*_S(K_A(\cdot, y)) \xi, K_A(\cdot, x) \eta \rangle_\alpha = \langle K_A(\cdot, y) \xi, S *_{V_q} K_A(\cdot, x) \eta \rangle_\alpha
\]

\[
= \sum_{\alpha \in \mathbb{N}^3} \frac{|\alpha|!}{\alpha!} \mu^\alpha(x) \langle K_A(\cdot, y) \xi, S \eta *_{V_q} \mu^\alpha(\cdot) \rangle_\alpha
\]

\[
= \sum_{\alpha \in \mathbb{N}^3} \frac{|\alpha|!}{\alpha!} \mu^\alpha(x) \xi^* S \eta *_{V_q} \mu^\alpha(y)
\]

\[
= \eta^* \sum_{\alpha \in \mathbb{N}^3} \frac{|\alpha|!}{\alpha!} \mu^\alpha(x) (S \xi *_{V_q} \mu^\alpha(y))^*
\]

\[
= \eta^* \sum_{\alpha \in \mathbb{N}^3} \frac{|\alpha|!}{\alpha!} \mu^\alpha(x) (S *_{V_q} \mu^\alpha(\cdot))(y)^* \xi.
\]

\[
\square
\]

**Theorem 6.6.** A \( \mathbb{H}^{n \times m} \)-valued function \( S \) is a contractive multiplier from \( \mathfrak{A}^m \) into \( \mathfrak{A}^n \) if and only if the \( \mathbb{H}^{n \times n} \)-valued kernel

\[
(6.11) \quad K(x, y) = \sum_{\alpha \in \mathbb{N}^3} \frac{|\alpha|!}{\alpha!} \left\{ \mu^\alpha(x) \overline{\mu^\alpha(y)} I_n - (\mu^\alpha *_{V_q} S)(x) (\mu^\alpha *_{V_q} S)(y)^* \right\}
\]

is positive definite in \( \Omega_A = \Omega(\mathfrak{A}) \).

**Proof.** To simplify notation we assume \( n = m = 1 \). Assume first that \( M_S \) is a contraction. Then, \( I_A - M_SM^*_S \) is a positive operator from \( \mathfrak{A} \) into itself. The positivity of the kernel (6.11) follows then from the formula

\[
\langle ((I_A - M_SM^*_S)K_A(\cdot, y), K_A(\cdot, x))_\alpha \rangle = \sum_{\alpha \in \mathbb{N}^3} \frac{|\alpha|!}{\alpha!} \left\{ \mu^\alpha(x) \overline{\mu^\alpha(y)} I_n - (\mu^\alpha *_{V_q} S)(x) (\mu^\alpha *_{V_q} S)(y)^* \right\}
\]

which in turn is obtained from (6.10).
Conversely, if the kernel (6.11) is positive definite in $\Omega(\mathfrak{A})$, the right linear span of the pairs of functions
\[
(K_\mathfrak{A}(\cdot, y))(x), \sum_{\alpha \in \mathbb{N}_0^3} \frac{|\alpha|!}{\alpha!} (\mu^\alpha(x) (S * V_q \mu^\alpha)(y))^* \in \mathfrak{A} \times \mathfrak{A}
\]
defines a densely defined contractive relation, which extends therefore to the graph of an everywhere defined contraction; the adjoint of this contraction is $M_S$.

In view of the structural identity (6.9) the Fueter-like variables are Schur multipliers. In the next section we present another important Schur multiplier, which is the counterpart here of an elementary Blaschke factor.

7. $V_q$ Blaschke-type factors

We denote by $\Omega_1(\mathfrak{A})$ the set of $x \in \mathbb{H}^*$ such that
\[
\sum_{u=1}^{3} |\mu_u(x)|^2 < 1.
\]

By inequalities (3.8) we have
\[
\{ x \in \mathbb{H}^* ; 3x_0^2 + x_1^2 + x_2^2 + x_3^2 < 1 \} \subset \Omega_1(\mathfrak{A}).
\]

**Remark 7.1.** It is essential to consider elements of $\mathbb{H}^*$, for example the element $x = (\frac{1}{3}, 0, 0, 0) \in \{ x \in \mathbb{R}^4; 3x_0^2 + x_1^2 + x_2^2 + x_3^2 < 1 \}$. However, $\sum_{u=1}^{3} |\mu_u(x)|^2$ is not defined. It is also interesting to note that the above ellipsoid also appears in [8].

**Theorem 7.2.** Let $a \in \Omega_1(\mathfrak{A})$ and set $\mu(a) = (\mu_1(a) \mu_2(a) \mu_3(a))$. Then the multiplication operator by $\mu(a)$ on the left is a strict contraction from $\mathfrak{A}^3$ into $\mathfrak{A}$ and the map
\[
B_a(x) = (1 - \mu(a) \mu(a)^*)^{-1/2} (1 - \mu(x) \mu(a)^*)^{-*} V_q (\mu(x) - \mu(a)) (I_3 - \mu(a)^* \mu(a))^{-1/2}
\]
is a Schur multiplier from $\mathfrak{A}^3$ into $\mathfrak{A}$.

**Proof.** The proof follows the proofs in [5] Proposition 4.1 p. 11 and [8] Theorem 4.7 p. 146, and is briefly outlined. In the complex setting we also refer to [32], where a different, but equivalent expression is given for $B_a$. We set
\[
J = \begin{pmatrix}
I_3 & 0 \\
0 & -I_{3^3}
\end{pmatrix}.
\]

**STEP 1: The operator-matrix**
\[
H(a) = \begin{pmatrix}
(I_3 - M_{\mu(a)} M_{\mu(a)}^*)^{-1/2} & -M_{\mu(a)} (I_3 - M_{\mu(a)}^* M_{\mu(a)})^{-1/2} \\
-M_{\mu(a)}^* (I_3 - M_{\mu(a)} M_{\mu(a)}^*)^{-1/2} & (I_3 - M_{\mu(a)}^* M_{\mu(a)})^{-1/2}
\end{pmatrix}
\]
is $J$-unitary, i.e.
\[
H(a) J H(a)^* = H(a)^* J H(a) = J.
\]

See [22] for the general case where $M_{\mu(a)}$ is replaced by an arbitrary strict contraction between Hilbert spaces. The operator-matrix $H(a)$ is called the Halmos extension of $M_{\mu(a)}$. 
STEP 2: The operator
\[
(I_\mathcal{A} M_\mu) H(a) J H(a)^* \left( I_\mathcal{A} M_\mu^* \right)
\]

is non-negative.

Indeed, \(H(a) J H(a)^* = J\) by the previous step, and \(I_\mathcal{A} - M_\mu M_\mu^* \geq 0\) by (6.9).

STEP 3: \(B_a\) is a Schur multiplier.

It suffices to write
\[
(I_\mathcal{A} M_\mu) H(a) = (I_\mathcal{A} - M_\mu M_\mu^* \mu(a)) (I_\mathcal{A} - M_\mu M_\mu^*)^{-1} (I_\mathcal{A} - M_\mu M_\mu^* \mu(a))^{-1/2} (I_\mathcal{A} - M_\mu M_\mu^*) (I_\mathcal{A} M_\mu^* B_a) .
\]

This concludes the proof. \(\Box\)

8. \(V_q\) Rational Functions

The notion of rational function is important in Schur analysis, and leads to state space representations of linear systems. The study of hypercomplex rational functions using the Cauchy-Kowaleski extension theorem originates with the work of Laville; see [31].

Definition 8.1. The \(\mathbb{H}^{n \times m}\)-valued \(V_q\)-regular function on an open domain \(\Omega \subset \mathbb{H}^*\) is called \(V_q\)-rational if its restriction to \(x_0 = 0\) can be written as

\[
R(0, x_1, x_2, x_3) = D + C (I_N - \sum_{u=1}^{3} x_u A_u)^{-1} (\sum_{u=1}^{3} x_u B_u)
\]

where \(D \in \mathbb{H}^{n \times m}\), \(C \in \mathbb{H}^{n \times N}\), \(A_1, A_2, A_3 \in \mathbb{H}^{N \times N}\) and \(B_1, \ldots, B_3 \in \mathbb{H}^{N \times m}\).

Equivalently, taking the \(V_q\)-extension, we can write

\[
R(x) = D + C (I - \mu(x) A)^{-*_{V_q}} \mu(x) B
\]

where

\[
A = \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix}.
\]

Proposition 8.2. The Blaschke factor (7.1) is a \(V_q\)-rational function on \(\Omega_1(\mathcal{A})\).

Proof. We follow the proof of [5] Proposition 4.1.p. 12] and first recall that for a contraction \(K \in \mathbb{H}^{n \times r}\) it holds that

\[
K(I_r - K^* K)^{1/2} = (I_s - KK^*)^{1/2} K \quad \text{and} \quad K^* (I_s - KK^*)^{1/2} = (I_r - K^* K)^{1/2} K^*.
\]

These equalities are used in the computations below.
To see that $B_\alpha$ is $V_q$-rational we write (with $\mu(x) = (x_1 \ x_2 \ x_3)$)

$$B_\alpha(0, x_1, x_2, x_3) =$$

$$= B_\alpha(0, 0, 0, 0) + B_\alpha(0, x_1, x_2, x_3) - B_\alpha(0, 0, 0, 0) + (1 - \mu(a)\mu(a)^*)^{1/2}(1 - \mu(x)\mu(a)^*)^{-1} (\mu(x) - \mu(a))(I_3 - \mu(a)^*\mu(a))^{-1/2} +$$

$$= -\mu(a) + (1 - \mu(a)\mu(a)^*)^{1/2}(1 - \mu(x)\mu(a)^*)^{-1}x$$

which is of the form (8.1) with

$$T = \begin{pmatrix} A_1 & B_1 \\ A_2 & B_2 \\ A_3 & B_3 \\ C & D \end{pmatrix} = \begin{pmatrix} \mu(a)^* & (I_3 - \mu(a)^*\mu(a))^{1/2} \\ (1 - \mu(a)\mu(a)^*)^{1/2} & -\mu(a) \end{pmatrix}.$$ 

\[ \square \]

Remark 8.3. We note that (8.3) is co-isometric. Existence of a co-isometric realization is a general property of Schur multipliers, and will be considered elsewhere. Here, $T$ is in fact unitary.

Rational functions were defined in the setting of the Cauchy operator in [8]. When restricted to $x_0 = 0$, one obtains the same class of functions, namely functions of the of form (8.1). $V_q$-rational functions are stable under sum, $V_q$-product and $V_q$-inversion when sizes are compatible. The arguments are the same as in the paper [8], to which we refer the reader.

9. Conclusions and Future Endeavors

We are now extending this work to analyze a theory of Schur-Agler functions (see [1] and see [1] for the Fueter case) and Schur multipliers as well as expand the rational function theory in this context.

Consider a system of linear ordinary differential equations with variables coefficients of the form

$$V_j(f) = \frac{\partial f_j}{\partial x_0} - \sum_{u=1}^{3} \sum_{k=0}^{3} a_{k,u}(x) \frac{\partial f_k}{\partial x_u} = 0, \quad j = 0, 1, 2, 3,$$

where the $a_{k,u}$ are real analytic on some open subset of the real line, and the system of equations is denoted by $V$. Thanks to the Cauchy-Kovalevskaia theorem one can define $V$-Fueter variables as the CK extensions of the functions $x_u, \ u = 1, 2, 3$ and monomials as CK-extensions of the monomials $x^\alpha$. It seems difficult though to develop a general theory
which goes beyond some trivial facts. The cases considered in [8] (for the Cauchy-Fueter operator, factorizing the $\mathbb{R}^4$ Laplacian) and in [10] associated to the operator
\[
\frac{\partial^2}{\partial x_0^2} - \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}
\]
studied in the setting of split quaternion, together with the present analysis for the $V_q$-operator exhibit how different each specific case can be.

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