Numerical Analysis of Electroconvection Phenomena in Cross-flow

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Electroconvective phenomena in the presence of cross-flow between two parallel electrodes is investigated in a numerical study. The two-relaxation-time Lattice Boltzmann Method with fast Poisson solver solves for the spatiotemporal distribution of flow field, electric field, and charge density. Couette and Poiseuille cross-flows are applied to the solutions with electroconvective vortices. Increasing cross-flow velocity deforms the vortices and eventually suppresses them when threshold values of shear stress are reached. This behavior is parameterized by a non-dimensional parameter, $Y$, formulated as the ratio of the electrical force to the viscous force in the Navier-Stokes Equations. For high values of $Y$, the electric force dominates the flow, while for $Y$ values below the critical threshold, the electric force influence on the flow is negligible and the flow is dominated by the shear.

I. INTRODUCTION

Electrohydrodynamics (EHD) is an interdisciplinary field describing the interaction of fluids with an electric field. Insights into complex multiphysics interactions are essential for understanding EHD flows: (1) the electric field from the potential difference between the anode and cathode and its modifications by the space charge effects; (2) the ion motion in the electric field; (3) the interaction between the motion of ions and the neutral molecules; and (4) the inertial and viscous forces in the complex flow. As a subset of EHD, electroconvection (EC) is a phenomenon where convective transport is induced by unipolar discharge into a dielectric fluid [1-20]. The EC stability problem was first analyzed by a simplified non-linear hydraulic model [21, 22] and linear stability analysis without charge diffusion [23, 24]. Atten & Moreau [25] showed that in the weak-injection limit, $C \ll 1$, where $C$ is the charge injection level, the flow stability is determined by the criterion $T_c C^2$, where $T_c$ is the linear stability threshold for the electric Rayleigh number $T_c$ — a ratio between electric force to the viscous force. In the space-charge-limited (SCL) injection, $C \rightarrow \infty$, the flow stability is determined by $T_c$ only. The experimental observations [26, 27] have shown that, for the SCL scenario, $T_c = 100$, while linear stability analysis suggests $T_c = 160.45$ for the same conditions [25]. Atten suggested that the discrepancy is due to the omission of the charge diffusion term in the analysis [28]. The effect of charge diffusion was investigated by Zhang et al. by employing linear stability analysis with a Poiseuille flow [11] and by non-linear analysis using a multiscale method [16]. The authors found that the charge diffusion has a non-negligible effect on $T_c$ and the transient behavior depends on the Reynolds number ($Re$) [11, 16].

To gain insight into the complexity of the EC flow, the problem can be investigated by numerical simulations. The earlier finite difference model simulations have shown that the strong numerical diffusivity may contaminate the model [2]. Other numerical approaches include the particle-in-cell method [29], finite volume method with the flux-corrected transport scheme [30], total variation diminishing scheme [4, 7, 13-15], and the method of characteristics [3]. Recently, Luo et al. showed that a Lattice Boltzmann model (LBM) could predict the linear and finite amplitude stability criteria of the subcritical bifurcation in the EC flow [17-20] for

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both 2D and 3D flow scenarios. This unified LBM transforms the elliptic Poisson equation into a parabolic reaction-diffusion equation and introduces artificial coefficients to control the evolution of the electric potential.

The EC stability problem was shown to be analogous to Rayleigh-Bernard convection (RBC) [20, 31-36]. Of particular interest is the suppression of the RBC cells in the cross-flow [37]. A non-dimensional group \( Gr / Re^2 \), the ratio of buoyancy to the inertia force, was used to parametrize the effect of the applied shear, where \( Gr \) is the Grashof number. For \( Gr / Re^2 > 10 \), the effect of the cross-flow is insignificant, while for \( Gr / Re^2 < 0.1 \), the effect of the buoyancy can be neglected. In EC flow scenario, 2D finite volume simulations of Poiseuille flow show that the critical electric Rayleigh number, \( T_c \), depends on the \( Re \) and ion mobility parameter, \( M \) [12].

In this paper, we parameterize the EC stability in the cross-flow between two parallel electrodes. The segregated solver used in the study combines a two-relaxation-time LBM modeling fluid and charged species transport and a Fast Fourier Transform Poisson solver to solve for the electric field directly [38]. Couette and Poiseuille cross-flow scenarios provide shear stress, the dominant terms are determined from non-dimensional analysis of the governing equations. A subcritical bifurcation is described by the ratio of the electrical force to the viscous force.

II. NON-DIMENSIONAL ANALYSIS

The governing equations for EHD flow include the Navier-Stokes equations (NSE) with the electric forcing term \( F_e = -\rho \nabla \varphi \) in the momentum equation, the charge transport equation, and the Poisson equation for electric potential.

\[
\nabla \cdot \mathbf{u} = 0, \\

\rho \frac{D \mathbf{u}}{Dt} = -\nabla P + \mu \nabla^2 \mathbf{u} - \rho_e \nabla \varphi, \\

\frac{\partial \rho_e}{\partial t} + \mathbf{u} \cdot \nabla \left( \rho_e \mathbf{u} - \rho_e D_c \nabla \varphi \right) = 0, \\

\nabla^2 \varphi = -\frac{\rho_e}{\varepsilon},
\]

where \( \rho \) is the density, \( \mu \) is the dynamic viscosity, \( \mathbf{u} = (u_x, u_y) \) is the velocity vector field, \( P \) is the static pressure, \( \mu_e \) is the ion mobility, \( D_c \) is the ion diffusivity, \( \rho_e \) is the charge density, \( \varepsilon \) is the electric permittivity, and \( \varphi \) is the electric potential. The electric force provides a source term in the momentum equation (Eq. 2) [11, 39-41].

Non-dimensional analysis of the governing equations (Eq. 1-4) yields:
\[ \nabla \cdot \mathbf{u}^* = 0 \]  
\[ \frac{D \mathbf{u}^*}{Dt} = -\nabla P^* + \frac{M^2}{T} \nabla^2 \mathbf{u}^* - CM^2 \rho^*_i \nabla \phi^* , \]  
\[ \frac{\partial \rho^*_i}{\partial t^*} + \nabla \cdot \left[ (\mathbf{u}^* - \nabla \phi^*) \rho^*_i - \frac{1}{F_e} \nabla \rho^*_i \right] = 0 . \]  
\[ \nabla^2 \phi^* = -C \rho^*_i , \]

where the asterisk denotes the non-dimensional variables. In the absence of cross-flow, non-dimensional governing equations yield four dimensionless parameters describing the system’s state \([4, 6, 7, 9, 11-20]\).

\[ M = \left( \varepsilon / \rho \right)^{1/2}, \quad T = \frac{\varepsilon \Delta \phi_0}{\mu b}, \quad C = \frac{\rho_b H^2}{\varepsilon \Delta \phi_0}, \quad F_e = \frac{\mu_b \Delta \phi_0}{D_e}, \]

where \( H \) is the distance between the electrodes (two infinite plates), \( \rho_b \) is the injected charge density at the anode, and \( \Delta \phi_0 \) is the voltage difference between the electrodes. The physical interpretations of these parameters are as follows: \( M \) is the ratio between hydrodynamic mobility and the ionic mobility; \( T \) is the ratio between electric force to the viscous force; \( C \) is the charge injection level \([11, 16]\); and \( F_e \) is the reciprocal of the charge diffusivity coefficient \([11, 16]\).

In the presence of cross-flow, the velocity term in the non-dimensional analysis of the momentum equation is modified to account for external flow, \( \mathbf{u}_{ext} \), while in the previous definitions (Eq. 9), the velocity term was non-dimensionalized by the drift velocity of charges. Here, we consider the velocity of the upper wall in Couette flow or the centerline velocity for Poiseuille flow as \( \mathbf{u}_{ext} \).

\[ \nabla \cdot \mathbf{u}^* = 0 \]  
\[ \frac{D \mathbf{u}^*}{Dt} = -\nabla P^* + \frac{1}{Re} \nabla^2 \mathbf{u}^* - \left[ \frac{\rho_0 \phi_0}{\rho \mathbf{u}_{ext}^2} \right] \rho^*_i \nabla \phi^* , \]  
\[ \frac{\partial \rho^*_i}{\partial t^*} + \nabla \cdot \left[ (\mathbf{u}^* - \nabla \phi^*) \rho^*_i - \frac{1}{F_e} \nabla \rho^*_i \right] = 0 . \]  
\[ \nabla^2 \phi^* = -C \rho^*_i , \]

where \( Re = \frac{\rho \mathbf{u}_{ext} H}{\mu} \) and \( X = \frac{\rho_0 \phi_0}{\rho \mathbf{u}_{ext}^2} \) as proposed by Guan et al. \([40]\). Since \( Re \) is essentially the ratio of inertia to viscous force and \( X \) is the ratio of electric force to inertia, the product of these (denoted as \( Y \)) is the ratio of electric force to viscous force:

\[ Y = X \times Re = \frac{\rho_0 \phi_0 H}{\mu \mathbf{u}_{ext}} = \frac{\rho_0 \phi_0}{\tau} , \]

where \( \tau \) is the shear stress. In Couette flow, \( \tau = \text{constant} \); in Poiseuille flow, the average value for the channel flow is used.

**III. RESULT AND DISCUSSION**

To model EC vortices, the hydrostatic base-state is perturbed using wave-form functions with a small amplitude that satisfies the boundary conditions and continuity equation:
\[
    u_x = L_x \sin\left(2\pi y / L_y \right) \sin(2\pi x / L_x) \times 10^{-3},
\]
\[
    u_y = L_y \left[ \cos\left(2\pi y / L_y \right) - 1 \right] \cos(2\pi x / L_x) \times 10^{-3}.
\]

The physical domain size \( L_x = 1.22 \text{m} \) and \( L_y = 1 \text{m} \) limits the perturbation wavenumber to \( \lambda_x = 2\pi / L_x \approx 5.15(1/\text{m}) \), yielding the most unstable mode under the conditions \( C = 10, M = 10 \) and \( Fe = 4000 \) [18]. The electric Nusselt number, \( Ne = I / I_o \), serves as a flow stability criteria, where \( I \) is the cathode current for a given solution and \( I_o \) is the cathode current for the hydrostatic solution [4, 18]. For cases where EC vortices exist, \( Ne > 1 \). For a strong ion injection, the EC stability largely depends on \( T \), so, in this analysis, \( T \) is varied, while other non-dimensional parameters are held constant at \( C = 10, M = 10 \), and \( Fe = 4000 \).

The Couette cross-flow is added to the simulation with EC vortices by assigning constant velocity of the upper wall. To model the Poiseuille flow, a body force in the \( x \)-direction is added. FIG. 1 shows the charge density and \( x \)-direction velocity for Couette cross-flow \( (u_{wall} = 0.5m / s) \) and Poiseuille cross-flow \( (u_{center} = 0.5m / s) \). The Couette cross-flow stretches the vortices in the direction of the bulk flow and may eliminate one of the two vortices. In a Poiseuille cross-flow, the vortex pair becomes separated; the vortices are pushed toward the opposite walls. For strong cross-flow, both vortices in the pair are eliminated, and \( I = I_o \), \( Ne = 1 \) (see FIG. 4). The EC contribution to the flow field is negligible at higher values of shear stress (higher velocity), and the flow field is exactly the same as the applied cross-flow.

FIG. 1. Charge density and \( x \)-direction velocity contour of the EC with cross-flow. Top: Couette flow with \( u_{wall} = 0.5m / s \); one of the two vortices is suppressed. Bottom: Poiseuille flow with \( u_{center} = 0.5m / s \); two vortices are suppressed and pushed towards the walls.

FIG. 2 shows the extended stability analysis of EC without cross-flow[38] by introducing (a) finite velocity of the upper wall (cathode) and (b) a uniform body force for pressure driven flow \( dp / dx \). For a constant \( T \), \( Ne \) decreases as \( U_{wall} \) or \( dp / dx \) increases. The applied shear stress stabilizes the EC flow.
FIG. 2. Electric Nusselt number depends on the electric Rayleigh number $T$ and applied velocity of the upper wall $U_{wall}$ for Couette type cross-flow or applied body force $dp/dx$ for Poiseuille type cross-flow.

FIG. 3 shows the dependency of $Ne$ on non-dimensional parameter $Y$. For varying values of $T$, the solutions lie on the same curve of $Ne$ normalized by $Ne_{\infty}$ ($Ne$ at $Y \to \infty$), solutions without cross-flow [38]) suggesting that the EC stability with cross-flow can be characterized by a single non-dimensional parameter $Y$, which is inversely proportional to $\tau$.

FIG. 4 shows $Ne = f(Y)$ for $C = 10$, $M = 10$, $T = 170.07$, and $Fe = 4000$ for Couette and Poiseuille cross-flow. A hysteresis loop with subcritical bifurcation is observed; the bifurcation thresholds are $Y_c = 625.25$, $Y_f = 297.32$ for Couette flow and $Y_c = 218.58$, $Y_f = 159.36$ for Poiseuille flow. Similar to stability parameter $T$ for the hydrostatic case (FIG. 2), for $Y < Y_c$, the system does not yield the EC instability, returning to the unperturbed state ($I = I_0$ and $Ne = 1$). If $Y$ decreases after the EC vortices are formed, $Ne$ decreases nonlinearly, until $Y = Y_f$, then the EC vortices are suppressed; the flow is not influenced by the Coulombic forces.
FIG. 4. Electrical Nusselt number $Ne$ versus $Y$. Bifurcation thresholds are: (a) Couette cross-flow $Y_c = 625.25$ and $Y_f = 297.32$; (b) Poiseuille cross-flow $Y_c = 159.36$ and $Y_f = 218.58$.

IV. CONCLUSION

The 2D numerical study extends the EC stability analysis to Couette and Poiseuille flows between two infinitely long parallel electrodes. The numerical approach utilizes the two-relaxation-time LBM to solve the flow and charge transport equations and a Fast Poisson Solver to solve the Poisson equation. Shear stress from applied cross-flow deforms the EC vortices and leads to their suppression. The non-dimensional analysis of the governing equations is used to derive parameter $Y$, a ratio of electric force to viscous force, in the presence of cross-flow. The non-dimensional parameter $Y$ accounts for the effect of the shear stress, analogous to a non-dimensional group $Gr/Re^2$ (ratio of buoyancy to the inertial forces) used to parametrize the effect of the applied shear in RBC. The electric $Ne$, defined as a current ratio, is used as stability criteria. Similar to stability parameter $T$ for the hydrostatic case, a hysteresis loop with subcritical bifurcation $Ne=f(Y)$ is observed. The bifurcation thresholds are $Y_c = 625.25, Y_f = 297.32$ for Couette flow and $Y_c = 218.58, Y_f = 159.36$ for Poiseuille flow.

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VI. REFERENCES

[1] B. Malraison and P. Atten, Chaotic behavior of instability due to unipolar ion injection in a dielectric liquid, Physical Review Letters 49, 723 (1982).
[2] A. Castellanos and P. Atten, Numerical modeling of finite amplitude convection of liquids subjected to unipolar injection, IEEE transactions on industry applications, 825 (1987).
[3] K. Adamiak and P. Atten, Simulation of corona discharge in point–plane configuration, Journal of electrostatics 61, 85 (2004).
[4] P. Traoré and A. Pérez, Two-dimensional numerical analysis of electroconvection in a dielectric liquid subjected to strong unipolar injection, Physics of Fluids 24, 037102 (2012).
[5] R. Kwak, V. S. Pham, K. M. Lim, and J. Han, Shear flow of an electrically charged fluid by ion concentration polarization: scaling laws for electroconvective vortices, Physical review letters 110, 114501 (2013).

[6] P. Traoré and J. Wu, On the limitation of imposed velocity field strategy for Coulomb-driven electroconvection flow simulations, Journal of Fluid Mechanics 727 (2013).

[7] J. Wu, P. Traoré, P. A. Vázquez, and A. T. Pérez, Onset of convection in a finite two-dimensional container due to unipolar injection of ions, Physical Review E 88, 053018 (2013).

[8] S. M. Davidson, M. B. Andersen, and A. Mani, Chaotic induced-charge electro-osmosis, Physical review letters 112, 128302 (2014).

[9] A. Pérez, P. Vázquez, J. Wu, and P. Traoré, Electrohydrodynamic linear stability analysis of dielectric liquids subjected to unipolar injection in a rectangular enclosure with rigid sidewalls, Journal of Fluid Mechanics 758, 586 (2014).

[10] I. Rubinstein and B. Zaltzman, Equilibrium electroconvective instability, Physical review letters 114, 114502 (2015).

[11] M. Zhang, F. Martinelli, J. Wu, P. J. Schmid, and M. Quadrio, Modal and non-modal stability analysis of electrohydrodynamic flow with and without cross-flow, Journal of Fluid Mechanics 770, 319 (2015).

[12] P. Traore, J. Wu, C. Louste, P. A. Vazquez, and A. T. Perez, Numerical study of a plane poiseuille channel flow of a dielectric liquid subjected to unipolar injection, IEEE Transactions on Dielectrics and Electrical Insulation 22, 2779 (2015).

[13] J. Wu and P. Traoré, A finite-volume method for electro-thermoconvective phenomena in a plane layer of dielectric liquid, Numerical Heat Transfer, Part A: Applications 68, 471 (2015).

[14] J. Wu, A. T. Perez, P. Traore, and P. A. Vazquez, Complex flow patterns at the onset of annular electroconvection in a dielectric liquid subjected to an arbitrary unipolar injection, IEEE Transactions on Dielectrics and Electrical Insulation 22, 2637 (2015).

[15] J. Wu, P. Traoré, A. T. Pérez, and P. A. Vázquez, On two-dimensional finite amplitude electro-convection in a dielectric liquid induced by a strong unipolar injection, Journal of Electrostatics 74, 85 (2015).

[16] M. Zhang, Weakly nonlinear stability analysis of subcritical electrohydrodynamic flow subject to strong unipolar injection, Journal of Fluid Mechanics 792, 328 (2016).

[17] K. Luo, J. Wu, H.-L. Yi, and H.-P. Tan, Lattice Boltzmann model for Coulomb-driven flows in dielectric liquids, Physical Review E 93, 023309 (2016).

[18] K. Luo, J. Wu, H.-L. Yi, and H.-P. Tan, Three-dimensional finite amplitude electroconvection in dielectric liquids, Physics of Fluids 30, 023602 (2018).

[19] K. Luo, J. Wu, H.-L. Yi, L.-H. Liu, and H.-P. Tan, Hexagonal convection patterns and their evolutionary scenarios in electroconvection induced by a strong unipolar injection, Physical Review Fluids 3, 053702 (2018).

[20] K. Luo, T.-F. Li, J. Wu, H.-L. Yi, and H.-P. Tan, Mesoscopic simulation of electrohydrodynamic effects on laminar natural convection of a dielectric liquid in a cubic cavity, Physics of Fluids 30, 103601 (2018).

[21] N. Felici, Phénonomes hydro et aérodynamiques dans la conduction des diélectriques fluides, Rev. Gén. Electr. 78, 717 (1969).

[22] N. Felici and J. Lacroix, Electroconvection in insulating liquids with special reference to uni-and bi-polar injection: a review of the research work at the CNRS Laboratory for Electrostatics, Grenoble 1969–1976, Journal of Electrostatics 5, 135 (1978).

[23] J. Schneider and P. Watson, Electrohydrodynamic Stability of Space - Charge - Limited Currents in Dielectric Liquids. I. Theoretical Study, The Physics of Fluids 13, 1948 (1970).
[24] P. Watson, J. Schneider, and H. Till, Electrohydrodynamic Stability of Space Charge Limited Currents in Dielectric Liquids. II. Experimental Study, The Physics of Fluids 13, 1955 (1970).
[25] P. Atten and R. Moreau, Stabilité électrohydrodynamique des liquides isolants soumis à une injection unipolaire, J. Mécanique 11, 471 (1972).
[26] J. Lacroix, P. Atten, and E. Hopfinger, Electro-convection in a dielectric liquid layer subjected to unipolar injection, Journal of Fluid Mechanics 69, 539 (1975).
[27] P. Atten and J. Lacroix, Non-linear hydrodynamic stability of liquids subjected to unipolar injection, Journal de Mécanique 18, 469 (1979).
[28] P. Atten, Rôle de la diffusion dans le problème de la stabilité hydrodynamique d’un liquide diélectrique soumis à une injection unipolaire forte, CR Acad. Sci. Paris 283, 29 (1976).
[29] R. Chicón, A. Castellanos, and E. Martin, Numerical modelling of Coulomb-driven convection in insulating liquids, Journal of Fluid Mechanics 344, 43 (1997).
[30] P. Vazquez, G. Georghiou, and A. Castellanos, Characterization of injection instabilities in electrohydrodynamics by numerical modelling: comparison of particle in cell and flux corrected transport methods for electroconvection between two plates, Journal of Physics D: Applied Physics 39, 2754 (2006).
[31] S. Chandrasekhar, Hydrodynamic and hydromagnetic stability ( Courier Corporation, 2013).
[32] P. G. Drazin and W. H. Reid, Hydrodynamic stability (Cambridge university press, 2004).
[33] E. L. Koschmieder, Bénard cells and Taylor vortices (Cambridge University Press, 1993).
[34] P. Bergé and M. Dubois, Rayleigh-bénard convection, Contemporary Physics 25, 535 (1984).
[35] M. Krishnan, V. M. Ugaz, and M. A. Burns, PCR in a Rayleigh-Benard convection cell, Science 298, 793 (2002).
[36] A. V. Getling, Rayleigh-Bénard Convection: Structures and Dynamics (World Scientific, 1998), Vol. 11.
[37] A. Mohamad and R. Viskanta, Laminar flow and heat transfer in Rayleigh–Bénard convection with shear, Physics of Fluids A: Fluid Dynamics 4, 2131 (1992).
[38] Y. Guan and I. Novosselov, Two Relaxation Time Lattice Boltzmann Method Coupled to Fast Fourier Transform Poisson Solver: Application to Electroconvective Flow, arXiv preprint arXiv:1812.05711 (2018).
[39] Y. Zhang, L. Liu, Y. Chen, and J. Ouyang, Characteristics of ionic wind in needle-to-ring corona discharge, Journal of Electrostatics 74, 15 (2015).
[40] Y. Guan, R. S. Vaddi, A. Aliseda, and I. Novosselov, Experimental and Numerical Investigation of Electro-Hydrodynamic Flow in a Point-to-Ring Corona Discharge Physical Review Fluids 3, 14 (2018).
[41] Y. Guan, R. S. Vaddi, A. Aliseda, and I. Novosselov, Analytical model of electrohydrodynamic flow in corona discharge, Physics of plasmas 25, 083507 (2018).