Integrable random matrix ensembles

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Received 19 April 2011, in final form 2 September 2011
Published 14 October 2011
Online at stacks.iop.org/Non/24/3179

Abstract

We propose new classes of random matrix ensembles whose statistical properties are intermediate between statistics of Wigner–Dyson random matrices and Poisson statistics. The construction is based on integrable \(N\)-body classical systems with a random distribution of momenta and coordinates of the particles. The Lax matrices of these systems yield random matrix ensembles whose joint distribution of eigenvalues can be calculated analytically thanks to the integrability of the underlying system. Formulae for spacing distributions and level compressibility are obtained for various instances of such ensembles.

Mathematics Subject Classification: 05.45.−a, 05.45.Mt, 02.30.Ik, 71.30.+h

(Some figures in this article are in colour only in the electronic version)

1. Introduction

The theory of random matrices, introduced by Wigner in the 1950s, has proved to be a very useful tool in many fields of physics, from localization theory to quantum transport (see e.g. [1] and references therein). In quantum chaos, a well accepted conjecture states that Wigner–Dyson random matrix ensembles describe statistical properties of spectra of quantum systems whose classical counterpart is chaotic [2], while statistics of integrable systems is best described by Poisson statistics of independent random variables [3]. The corresponding wave functions are extended in the chaotic case and localized in the integrable case. The choice of the random matrix ensemble suited to describe the statistical behaviour of a system depends on the symmetries of that system. In the usual setting [4], standard random matrix ensembles consist of matrices \(M\) with independent Gaussian random elements whose measure is invariant over conjugation

\[M \rightarrow U^{-1} MU,\]

where \(U\) is an arbitrary matrix belonging to one of the three following groups of matrices: unitary, orthogonal or symplectic. The unitary group defines the Gaussian unitary ensemble (GUE), which is supposed to describe statistical properties of energy levels of chaotic systems without time-reversal invariance. The orthogonal group corresponds to Gaussian orthogonal
ensemble (GOE), used for time-reversal invariant chaotic systems. The symplectic group gives rise to Gaussian symplectic ensemble (GSE), applicable to time-reversal chaotic systems with half-integer spin without rotational symmetry.

Although many extensions and generalizations of random matrices have been proposed in order to best describe various models [5], the existence of a large invariance group as in (1) remains their characteristic feature. Without such invariance it is very difficult to connect analytically simple properties of matrix elements with complex properties of matrix eigenvalues. For all random matrix ensembles with invariance group it is possible to integrate over unnecessary variables in order to obtain explicitly the joint distribution of eigenvalues under the form

\[ P(\lambda_1, \ldots, \lambda_N) \sim \prod_{i<j} |\lambda_j - \lambda_i|^{\beta} e^{-\sum V(\lambda_i)}, \]

with \( V(x) \) a system-dependent potential and \( \beta \) a parameter. For the Gaussian ensembles the potential is quadratic and the parameter \( \beta \) is equal to 1 for GOE, 2 for GUE and 4 for GSE.

All correlation functions for invariant ensembles can be calculated analytically [4]. However, the resulting formulae are cumbersome. For the nearest-neighbour distribution \( P(s) \), instead of the exact expression one often uses a simple surmise proposed by Wigner. This surmise has correct functional dependence at small and large argument and takes the form

\[ P(s) = a s^d e^{-b s^2} \]

with constants \( a \) and \( b \) determined from the normalization conditions

\[ \int_0^\infty P(s) \, ds = \int_0^\infty s P(s) \, ds = 1. \]

The Wigner-type surmise for the probability \( P(n, s) \) that between two eigenvalues separated by \( s \) there exist exactly \( n - 1 \) other levels (with \( P(1, s) \equiv P(s) \)) is [6]

\[ P(n, s) = a_n s^{dn} e^{-b_n s^2}, \quad d_n = n - 1 + \frac{1}{2} n(n + 1) \beta \]

and \( a_n, b_n \) are fixed by the normalizations

\[ \int_0^\infty P(n, s) \, ds = 1, \quad \int_0^\infty s P(n, s) \, ds = n. \]

While for chaotic systems it is possible to argue that eigenstates may statistically be invariant under rotations, this is not the case for more general models. In order to describe statistical properties of such systems one has to consider non-invariant ensembles of random matrices. One of the most investigated examples is the three-dimensional Anderson model [7], with on-site disorder and nearest-neighbour coupling. Depending on the strength of the disorder, it can display metallic behaviour well described by standard random matrix ensembles, or insulator behaviour with Poisson-like spectrum. However, at the metal–insulator transition, spectral statistics are of an intermediate type and are not described by invariant ensembles [8]. Similar behaviours have been observed in pseudo-integrable billiards [9], quantum maps corresponding to diffractive classical maps [10], or quantum Hall transitions [11]. Models have been proposed to describe such intermediate statistics [12], and random matrix ensembles which possess similar features have been constructed, such as e.g. power-law random banded matrix ensembles [13, 14] or critical ultrametric ensembles [15].

The main purpose of this paper is to construct random matrix ensembles which are not invariant over rotations of eigenstates, but whose joint distributions of eigenvalues can nevertheless be calculated analytically. A short version of the paper has been published in [16]. All of these ensembles have intermediate statistics, and for certain of them spectral correlation
functions, e.g. the nearest-neighbour distribution, are obtained explicitly. Eigenfunctions of these ensembles are neither localized (as for integrable systems) nor extended (as for chaotic models) but have fractal properties [17].

Random matrices of the proposed critical ensembles are constructed from the Lax matrices of classical integrable models. These models are systems of \( N \) classical particles labelled by an index \( i, 1 \leq i \leq N \) in a one-dimensional space. Each particle \( i \) is characterized by its position in space \( q_i \) and its momentum \( p_i \). The dynamics of the particles is entirely described by the Hamiltonian \( H(p, q) \), where \( p = (p_1, \ldots, p_N) \) and \( q = (q_1, \ldots, q_N) \). The characteristic property of these models is the existence of a pair of \( N \times N \) matrices \( L \) and \( M \), called the Lax pair of the system [18], such that the equations of motion (the Hamilton equations, derived from the system Hamiltonian) are equivalent to

\[
\frac{dL}{dt} = M L - LM. \tag{7}
\]

The Lax matrix \( L \) is a matrix depending on momenta \( p \) and coordinates \( q \). We propose to consider these Lax matrices as random matrices with a certain ‘natural’ measure of random variables \( p_j \) and \( q_j \)

\[
dL = P(p, q) d^N p d^N q. \tag{8}
\]

The explicit form of this measure depends on the system and will be discussed below. We do not impose any dynamics on variables \( p \) and \( q \). The only information we use from the integrability of the underlying classical system is the existence and explicit form of action-angle variables \( I_\alpha(p, q) \) and \( \phi_\alpha(p, q) \). In particular, it is well known that the transformation from momenta and coordinates to action-angle variables is canonical, so that

\[
\prod_j dp_j dq_j = \prod_\alpha dI_\alpha d\phi_\alpha. \tag{9}
\]

Direct proof that the transformation is canonical is difficult in general, and implicit methods have been used to establish it for specific systems [19–21]. In the models we consider here, action variables turn out to be the eigenvalues \( \lambda_\alpha \) of the Lax matrix, or a simple function of them. The canonical change of variables from momenta and coordinates to action-angle variables in (8) leads to a formal relation

\[
dL = P(\lambda, \phi) d^N \lambda d^N \phi, \tag{10}
\]

where \( P(\lambda, \phi) \equiv P(p(\lambda, \phi), q(\lambda, \phi)) \). The exact joint distribution of eigenvalues is then obtained by integration over angle variables, which can easily be performed in all cases considered, and yields

\[
P(\lambda) = \int P(\lambda, \phi) d^N \phi. \tag{11}
\]

This scheme is general and can be adapted to many different models.

In this paper we consider in detail four typical models of \( N \)-particle classical integrable systems. The three first, labelled CM\(_r\), CM\(_h\) and CM\(_t\), correspond to the rational, hyperbolic and trigonometric Calogero–Moser models [22, 23]. The fourth model, labelled RS, is a trigonometric variant of the Ruijsenaars–Schneider model [24].

The Calogero–Moser models are defined by the Hamiltonian

\[
H(p, q) = \frac{1}{2} \sum_j p_j^2 + g^2 \sum_{j<k} v(q_j - q_k). \tag{12}
\]
where \( v(\xi) \) is a potential depending on the distance between particles and \( g \) is a constant \([25]\). For the models considered here it has the form \( v(\xi) = x^2(\xi) \), where

\[
x(\xi) = \begin{cases} 
  \frac{1}{\xi} & \text{model CM}_r, \\
  \frac{\mu/2}{\sinh(\mu \xi/2)} & \text{model CM}_b, \\
  \frac{\mu/2}{\sin(\mu \xi/2)} & \text{model CM}_t.
\end{cases}
\]

The Hamiltonian of our fourth model, the trigonometric Ruijsenaars–Schneider model, is \([24]\)

\[
H(p, q) = \sum_{j=1}^{N} \cos(p_j) \prod_{k \neq j} \left(1 - \frac{\sin^2[\mu g/2]}{\sin^2[\mu(q_j - q_k)/2]}\right)^{1/2} \text{ model RS.}
\]

The plan of the paper is the following. Sections 2, 3, and 4 are devoted to the construction of critical ensembles related respectively with the rational, hyperbolic and trigonometric Calogero–Moser models. In each of these sections we briefly present the construction of the action-angle variables and choose a ‘natural’ measure of random momenta and coordinates which allows an easy change of variables as in (10). We then give explicit formulæ for the joint distribution of eigenvalues for the resulting critical ensembles of Lax matrices. In section 5 this scheme is applied to the Ruijsenaars–Schneider model. For this model the joint distribution of eigenvalues takes a form which makes it suitable for the application of the transfer operator formalism. This approach is detailed in section 6, and in section 7 it is applied to the analytic calculation of nearest-neighbour distributions for the RS model. The spectral compressibility for this model is obtained in section 8.

For clarity we state below the principal results for the four models considered in this paper.

**CM\(_{r}\) ensemble.** The CM\(_{r}\) ensemble is defined as the ensemble of \( N \times N \) Hermitian matrices of the form

\[
L_{kr} = p_r \delta_{kr} + ig \frac{1 - \delta_{kr}}{q_k - q_r},
\]

with \( g \) a real constant. Positions \( q \) and momenta \( p \) are random variables distributed according to the density

\[
P(p, q) \sim \exp \left[ -A \left( \sum_j p_j^2 + g^2 \sum_{j \neq k} \frac{1}{(q_j - q_k)^2} \right) - B \sum_j q_j^2 \right],
\]

with \( A \) and \( B \) arbitrary positive constants. The joint distribution of eigenvalues for this ensemble is then given by

\[
P(\lambda) \sim \exp \left[ -A \sum_a \lambda_a^2 - g^2 B \sum_{a \neq b} \frac{1}{(\lambda_a - \lambda_b)^2} \right].
\]

A characteristic property of this ensemble is the exponentially strong level repulsion: the nearest-neighbour spacing distribution \( P(s) \) is characterized by

\[
\ln P(s) \sim -\frac{b}{s^2} + O(1).
\]

We propose the following Wigner-type surmise for the next-to-nearest-neighbour spacing distributions \( P(n, s) \), depending on four parameters:

\[
P(n, s) = a s^n \exp \left( -\frac{b}{s^2} - cs \right).
\]
It contains two fitting constants depending on \( n \). The other two are fixed by the normalization (6).

**CM\( h \) ensemble.** The CM\( h \) ensemble is defined as the ensemble of \( N \times N \) Hermitian matrices of the form

\[
L_{kr} = p_r \delta_{kr} + i g \frac{\mu (1 - \delta_{kr})}{2 \sinh[\mu(q_k - q_r)/2]}
\]

with \( g \) and \( \mu \) real constants, and \( q \) and \( p \) distributed according to the density

\[
P(p, q) \sim \exp \left[ -A \left( \sum_j p_j^2 + g^2 \sum_{j \neq k} \frac{\mu^2}{4 \sinh^2[\mu(q_j - q_k)/2]} \right) - B \sum_j \cosh \mu q_j \right].
\]

The exact joint distribution for this model is

\[
P(\lambda) \sim \exp \left( -A \sum_\alpha \lambda_\alpha^2 \right) \prod_\alpha K_0 \left( B \prod_{\beta \neq \alpha} \left| 1 + \frac{ig\mu}{\lambda_\alpha - \lambda_\beta} \right| \right),
\]

where \( K_0(x) \) is the modified Bessel function of the second kind. The nearest-neighbour spacing distribution has an exponential asymptotic similar to (18) but with 1/s leading term instead of 1/s^2, namely

\[
\ln P(s) \sim -\frac{b}{s} + O(\ln s).
\]

The Wigner-type surmise for CM\( h \) is

\[
P(n, s) = a s^d \exp \left( -\frac{b}{s} - cs \right).
\]

**CM\( t \) ensemble.** Matrices from this ensemble correspond to a situation where \( \mu \) in equation (20) is allowed to take pure imaginary values. They are of the form

\[
L_{kr} = p_r \delta_{kr} + i g \frac{\mu (1 - \delta_{kr})}{2 \sin[\mu(q_k - q_r)/2]}
\]

with \( g \) and \( \mu \) real constants, and \( q \) and \( p \) distributed according to the density

\[
P(p, q) \sim \exp \left[ -A \left( \sum_j p_j^2 + g^2 \sum_{j \neq k} \frac{\mu^2}{4 \sin^2[\mu(q_j - q_k)/2]} \right) \right]
\]

with the restrictions that all \( q_j \) are between 0 and \( 2\pi/\mu \). The exact joint distribution of eigenvalues for this ensemble is

\[
P(\lambda) \sim \exp \left( -A \sum_\alpha \lambda_\alpha^2 \right) \chi(\lambda),
\]

where the function \( \chi(\lambda) \) is equal to 1 if \( \lambda_1 < \lambda_2 < \cdots < \lambda_N \) and \( \lambda_{\alpha+1} - \lambda_\alpha > \mu g \) for all \( \alpha \). The nearest-neighbour spacing distribution is given by a shifted Poisson distribution of the form

\[
P(s) = \begin{cases} 
0, & 0 < s < b, \\
\frac{1}{1-b} \exp \left( -\frac{s-b}{1-b} \right), & s > b
\end{cases}
\]

with \( b \) some fitting constant.
RS ensemble. The RS ensemble is defined as the ensemble of $N \times N$ matrices of the form [24]

$$L_{kr} = e^{i\sigma p_{r}/2} \tilde{W}_{k}^{1/2} \frac{\sin[\mu g \sigma/2]}{\sin[\mu(q_{k} - q_{r} + g \sigma)/2]} \tilde{V}_{r}^{1/2} e^{i\sigma p_{r}/2}$$

(29)

with

$$\tilde{V}_{k} = \prod_{j \neq k} \frac{\sin[\mu(q_{k} - q_{j} - g \sigma)/2]}{\sin[\mu(q_{k} - q_{j} + g \sigma)/2]}, \quad \tilde{W}_{k} = \prod_{j \neq k} \frac{\sin[\mu(q_{k} - q_{j} - g \sigma)/2]}{\sin[\mu(q_{k} - q_{j} + g \sigma)/2]}.$$

(30)

Let $\tilde{\Omega}$ be the set of $q$ such that for all $k$ the sign of both $\tilde{V}_{k}$ and $\tilde{W}_{k}$ is the same as the sign of $\sin(N \mu g \sigma/2)/\sin(\mu g \sigma/2)$. The matrix $L$ is unitary if and only if $q \in \tilde{\Omega}$. The variables $q$ and $p$ are chosen to be distributed according to the uniform density in the region where $L$ is unitary. That is, we choose momentum variables $p_{j}$ independent and uniformly distributed between 0 and $2\pi/\sigma$ and coordinate variables $q$ uniformly distributed over $\tilde{\Omega}$. In this case eigenvalues of the Lax matrices (29) are also uniformly distributed over $\tilde{\Omega}$. Choosing $\mu = 2\pi/N$, $\sigma = 1$ and $g = a$, we compute correlation functions of eigenvalues of matrix (29) for fixed $a$ and $N \to \infty$. The results strongly depend on the integer part of $a$. For $0 < a < 1$ the nearest-neighbour spacing distribution is similar to (28) with constant $b$ now equal to $a$. For $1 < a < 2$ the nearest-neighbour distribution takes the form

$$P(s) = \begin{cases} A^{2} \sin^{2}(\rho s) & \text{when } 1 < g < 4/3, \\ \frac{81}{\pi^{2}} s^{2} & \text{when } g = 4/3, \\ A^{2} \sin^{2}(\rho s) & \text{when } 4/3 < g < 2. \end{cases}$$

(31)

Constants $A$ and $\rho$ are determined from the normalization conditions (4). Other correlation functions are also obtained in section 7.

Numerical implementation. The results presented above are quite robust with respect to alterations in the distribution of $q$ and $p$. In all models considered we chose (as explained above) a distribution of coordinates such that the $q_{j}$ are confined to a finite interval while having a strong repulsion between each other. As may be expected physically (though we do not have a rigorous proof for this), numerical evidence shows that, if we keep these two characteristic features, spectral properties for $N \to \infty$ depend only weakly on the precise choice for the distribution of $q$ and $p$.

From these considerations it is thus natural to use, rather than the exact complicated distribution of $q$, the picket-fence configuration when all coordinates are just fixed and equally spaced. As all definitions of our ensembles involve only differences $q_{j} - q_{k}$ multiplied by a parameter ($\mu$ or $g$, depending on the model), we can without loss of generality choose to take $q_{j} = j = 1, \ldots, N$.

For numerical implementation we chose $q_{j} = j$, and $p_{j}$ as independent Gaussian variables with zero mean and with variance equal 1 (CM ensembles) or independent variables uniformly distributed between 0 and $2\pi$ (RS ensemble). For concreteness, we fixed $\mu = 4\pi/N$ for CM$_{b}$ and CM$_{t}$, and $\mu = 2\pi/N$ for RS. For such a choice, the $N \times N$ Lax matrices take the form

$$L_{kr} = p_{k} \delta_{kr} + ig \frac{1 - \delta_{kr}}{k - r}, \quad \text{model CM}_{r},$$

$$L_{kr} = p_{k} \delta_{kr} + \frac{g}{N} \frac{2\pi(1 - \delta_{kr})}{\sin[2\pi(k - r)/N]}, \quad \text{model CM}_{b},$$

$$L_{kr} = p_{k} \delta_{kr} + \frac{g}{N} \frac{2\pi(1 - \delta_{kr})}{\sin[2\pi(k - r)/N]}, \quad \text{model CM}_{t},$$

$$L_{kr} = \frac{e^{i\rho_{r}}}{N(1 - e^{2\pi i(k - r + \rho)/N})}, \quad \text{model RS}.$$
For CM\(_t\) matrices with even \(N\), to avoid the singularity we changed \(N \rightarrow N + 1\) in the above formula. As the figures in the next sections show, despite this particular choice for the distribution of \(q\) and \(p\), the agreement between the computed spectral statistics and analytical formul\(\alpha\)e is remarkable.

### 2. Rational Calogero–Moser model

The first model we consider is the rational Calogero–Moser model CM\(_r\) [25], characterized by the Lax matrix

\[
L_{kr} = p_r \delta_{kr} + ig \frac{1 - \delta_{kr}}{q_k - q_r}.
\]

(33)

It depends on a real constant \(g\) and on a set of \(2N\) random variables \(p_k\) and \(q_k\) whose distribution will be specified later on. We are interested in eigenvalues \(\lambda_\alpha\) and eigenfunctions \(u_k(\alpha)\) of this matrix (here and below we will use the Greek letters to label eigenvalues and corresponding eigenfunctions)

\[
\sum_{r=1}^{N} L_{kr} u_r(\alpha) = \lambda_\alpha u_k(\alpha).
\]

(34)

To construct angle-action variables let us define the new quantities

\[
Q_{\alpha\beta} = \sum_k u_k^*(\alpha) q_k u_k(\beta).
\]

(35)

From equation (33) one obtains

\[
L_{kr} q_r - q_k L_{kr} = -ig(1 - \delta_{kr}).
\]

(36)

Multiplying both sides by \(u_k^*(\alpha) u_r(\beta)\) and summing over \(k\) and \(r\) one obtains

\[
Q_{\alpha\beta}(\lambda_\alpha - \lambda_\beta) = -ig(e_\alpha^* e_\beta - \delta_{\alpha\beta}),
\]

(37)

where

\[
e_\alpha = \sum_k u_k(\alpha).
\]

(38)

For \(\alpha = \beta\), equation (37) implies that \(|e_\alpha|^2 = 1\), and one can choose the overall phase of the eigenvector \(u_k(\alpha)\) in such a way that \(e_\alpha = 1\). Let \(\phi_\alpha\) be new variables defined by

\[
Q_{\alpha\alpha} = \phi_\alpha.
\]

(39)

Then from equation (37) we have

\[
Q_{\alpha\beta} = \phi_\alpha \delta_{\alpha\beta} - ig \frac{1 - \delta_{\alpha\beta}}{\lambda_\alpha - \lambda_\beta}.
\]

(40)

The matrix \(Q\) can be seen as the dual matrix of \(L\), with \(\phi_\alpha\) playing the role of momenta and \(\lambda_\alpha\) the role of positions. In [19] it was proved that there is a canonical transformation from position and momentum variables \((q_k, p_k)\) to action and angle variables \((\lambda_\alpha, \phi_\alpha)\). Showing that the transformation is canonical is a rather technical mathematical result. However, one can easily check that the new variables \(\lambda_\alpha\) and \(\phi_\alpha\) verify Hamilton–Jacobi equations (see appendix A).

We now consider an ensemble of Hermitian matrices of the form (33) with random variables \(p_k\) and \(q_k\) drawn according to the measure

\[
P(L) dL = N \exp \left[ -A Tr L^2 - B \sum_k q_k^2 \right] \prod_k dp_k dq_k.
\]

(41)
where $A$ and $B$ are given constants and $N$ a normalization factor. The first term in equation (41) is the analogue of the usual Gaussian weight of RMT; the second term is a quadratic confinement potential. Since the action-angle transformation is canonical one has

$$
\prod_k dp_k \, dq_k = \prod_a d\lambda_a \, d\phi_a.
$$

(42)

From equation (35), using orthogonality of eigenvectors one obtains

$$
\text{Tr} Q^2 = \sum_j q_j^2.
$$

(43)

Using these relations one can rewrite the distribution (41) in action-angle variables $\lambda_a$ and $\phi_a$ as

$$
P(L) \, dL = N \exp \left[ -A \sum_a \lambda_a^2 - B \left( \sum_a \phi_a^2 + g^2 \sum_{a \neq \beta} \frac{1}{(\lambda_a - \lambda_\beta)^2} \right) \right] \prod_a d\lambda_a \, d\phi_a.
$$

(44)

Integration over the $\phi_a$ gives a constant. We thus obtain the joint distribution of eigenvalues for the ensemble of random matrices $L$ with the measure (41) as

$$
P(\lambda_1, \ldots, \lambda_N) \sim \exp \left[ -A \sum_a \lambda_a^2 - B g^2 \sum_{a \neq \beta} \frac{1}{(\lambda_a - \lambda_\beta)^2} \right].
$$

(45)

Note that, similarly as in the standard RMT case, this joint eigenvalue distribution can be interpreted via the Coulomb gas model as the partition function of an ensemble of particles on a line, here with inverse square repulsion. After rescaling $x_k = \lambda_k(Bg^2/A)^{-1/4}$, equilibria positions of the particles at positions $\lambda_a$ are given by

$$
x_k = 2 \sum_{j \neq k} \frac{1}{(x_j - x_k)^3}, \quad 1 \leq k \leq N.
$$

(46)

Such a relation characterizes the zeros of Hermite polynomials of degree $N$ (see also equation (10.3) of [25]). It is known from RMT [4] that the distribution of eigenvalues of Gaussian random ensembles has a similar property, which implies that the asymptotic density of eigenvalues is given by Wigner’s semi-circle law.

An immediate consequence of the distribution (45) is the unusual very strong level repulsion at small distances. For all standard random matrix ensembles the nearest-neighbour distribution $P(s)$ behaves as $s^\beta$ at small $s$. By contrast, in our case it follows from (45) that

$$
P(s) \sim ae^{-b/s^2}.
$$

(47)

As the potential between eigenvalues decreases as the inverse square of the distance between them, the probability of having a gap of size $s$ for large $s$ is exponentially small. We could not calculate exactly the correlation functions for the distribution (45). However, combining the two asymptotics above, we build a Wigner-type surmise for the nearest-neighbour spacing distribution of the form

$$
P(s) = ae^{-b/s^2 - cs},
$$

(48)

where $b$ is a fitting constant, and constants $a$ and $c$ are determined from the normalization conditions (4). For the $n$th nearest-neighbour spacing distributions $P(n, s)$, with $n \geq 2$, we conjecture, by analogy with the Wigner surmise (5) for standard random matrices, the form

$$
P(n, s) = as^d e^{-b/s^2 - cs}
$$

(49)

with two fitting constants $b$ and $d$. 
To assess this conjecture we compare the analytical expressions (48)–(49) with numerical results, with the choice of parameters detailed in section 1. Results displayed in figure 1 show that the agreement is remarkable.

3. Hyperbolic Calogero–Moser model

The Lax matrix for the hyperbolic Calogero–Moser model CM$_h$ reads [25]

$$L_{kr} = p_r \delta_{kr} + ig(1 - \delta_{kr}) \frac{\mu}{2 \sinh(\mu(q_k - q_r)/2)}.$$  (50)

Let us define two matrices $Q$ and $R$ by

$$Q_{\alpha\beta} = \sum_k u_k^*(\alpha)e^{\mu q_k}u_k(\beta), \quad R_{\alpha\beta} = \sum_k u_k^*(\alpha)e^{-\mu q_k}u_k(\beta),$$  (51)

and two vectors

$$e_\alpha = \sum_k u_k(\alpha)e^{\mu q_k/2}, \quad f_\alpha = \sum_k u_k(\alpha)e^{-\mu q_k/2}.$$  (52)

From (50) one can obtain the two equivalent equations

$$e^{\mu q_k} L_{kr} - L_{kr} e^{\mu q_r} = ig\mu (1 - \delta_{kr}) e^{\mu(q_k + q_r)/2},$$  (53)

$$e^{-\mu q_k} L_{kr} - L_{kr} e^{-\mu q_r} = ig\mu (1 - \delta_{kr}) e^{-\mu(q_k + q_r)/2}.$$  (54)

Multiplying both sides by $u_k(\alpha)^*u_r(\beta)$ and summing over all $k$ and $r$ one obtains

$$Q_{\alpha\beta}(\lambda_\alpha - \lambda_\beta) = -ig\mu (e_\alpha^*e_\beta - Q_{\alpha\beta}),$$  (55)

$$R_{\alpha\beta}(\lambda_\alpha - \lambda_\beta) = ig\mu (f_\alpha^*f_\beta - R_{\alpha\beta}),$$  (56)

which implies that matrices $Q$ and $R$ take the form

$$Q_{\alpha\beta} = e_\alpha^* \frac{ig\mu}{\lambda_\beta - \lambda_\alpha + ig\mu e_\beta}, \quad R_{\alpha\beta} = f_\alpha^* \frac{ig\mu}{\lambda_\beta - \lambda_\alpha - ig\mu f_\beta}.$$  (57)
By their definition (51), matrices $Q$ and $R$ are inverse of each other, so that $\sum_\gamma Q_{\alpha\gamma} R_{\gamma\beta} = \delta_{\alpha\beta}$ for all $\alpha, \beta$. For $\alpha = \beta$ this condition implies that

$$- g^2 \mu^2 e^*_\alpha f_\alpha \sum_\gamma \frac{e_\gamma f_\gamma^*}{(\lambda_\gamma - \lambda_\alpha + ig\mu)^2} = 1$$  \hspace{1cm} (58)$$

(in particular, it follows that all $e_\alpha$ and $f_\alpha$ are non-zero). For $\alpha \neq \beta$ one obtains

$$\sum_\gamma \frac{e_\gamma f_\gamma^*}{(\lambda_\beta - \lambda_\alpha + ig\mu)(\lambda_\gamma - \lambda_\beta - ig\mu)} = 0.$$ \hspace{1cm} (59)$$

Using the identity

$$\frac{1}{(\lambda_\beta - \lambda_\alpha + ig\mu)(\lambda_\gamma - \lambda_\beta - ig\mu)} = \frac{1}{\lambda_\beta - \lambda_\gamma} \frac{1}{\lambda_\gamma - \lambda_\beta} - \frac{1}{\lambda_\beta - \lambda_\alpha + ig\mu} \frac{1}{\lambda_\alpha - \lambda_\gamma},$$ \hspace{1cm} (60)$

valid for $\alpha \neq \beta$, one concludes that

$$\sum_\gamma \frac{e_\gamma f_\gamma^*}{\lambda_\gamma - \lambda_\alpha + ig\mu} = c,$$ \hspace{1cm} (61)$$

where $c$ is a certain constant independent on $\alpha$. According to this equation the quantities $b_\gamma = e_\gamma f_\gamma^*/c$ obey a system of linear equations of the form

$$\sum_\gamma \frac{b_\gamma}{x_\gamma - y_\alpha} = 1,$$ \hspace{1cm} (62)$$

with $x_\gamma = \lambda_\gamma$ and $y_\alpha = \lambda_\alpha - ig\mu$. This equation coincides with equation (B1) in appendix B. From (B2) it follows that

$$e_\alpha f_\alpha^* = ig\mu c V_\alpha,$$ \hspace{1cm} (63)$$

where

$$V_\alpha = \prod_{\beta \neq \alpha} \left(1 + \frac{ig\mu}{\lambda_\alpha - \lambda_\beta} \right),$$ \hspace{1cm} (64)$$

while (B4) implies that

$$\sum_\alpha e_\alpha f_\alpha^* = ig\mu c N.$$ \hspace{1cm} (65)$$

It readily follows from the definition (52) of $e_\alpha$ and $f_\alpha$ that $\sum_\alpha e_\alpha f_\alpha^* = N$, thus the value of $c$ is fixed by $ig\mu c = 1$. Equation (58) is then fulfilled as a direct consequence of (B3). Finally, we have

$$e_\alpha f_\alpha^* = V_\alpha.$$ \hspace{1cm} (66)$$

Let $\phi_\alpha$ be new variables defined from diagonal elements of matrix $Q$ by

$$Q_{aa} = |V_\alpha| e^{i\phi_\alpha}.$$ \hspace{1cm} (67)$$

Then from (57) and (66) it follows that

$$R_{aa} = |V_\alpha| e^{-\mu\phi_\alpha}.$$ \hspace{1cm} (68)$$

In the definition (52) of $e_\alpha$ it is convenient to choose the overall phase of the eigenvector $u_\alpha(\alpha)$ in such a way that $e_\alpha$ be real. As $Q_{aa} = |e_\alpha|^2$ one has

$$e_\alpha = |V_\alpha|^{1/2} e^{i\phi_\alpha/2}.$$ \hspace{1cm} (69)$$
Using equation (57), the matrix $Q$ can now be expressed in terms of the new variables $\lambda_\alpha$ and $\phi_\alpha$, as

$$Q_{\alpha\beta} = |V_\alpha|^{1/2} e^{i\phi_\alpha/2} \frac{i g \mu}{\lambda_\beta - \lambda_\alpha + i g \mu} e^{i\phi_\beta/2} |V_\beta|^{1/2}. \quad (70)$$

As in the case of model CM$_r$, the matrix $Q$ can be seen as the dual matrix of $L$. Indeed, $Q$ coincides with the Lax matrix of the rational Ruijsenaars–Schneider model with coordinates $\lambda_\alpha$ and momenta $\phi_\alpha$ [19]. In [19] it has been proved that the transformation from position and momentum variables $(q_k, p_k)$ to action-angle variables $(\lambda_\alpha, \phi_\alpha)$ is canonical. Again one can check that the new variables $\lambda_\alpha$ and $\phi_\alpha$ verify Hamilton–Jacobi equations.

We now consider an ensemble of Hermitian matrices of the form (50) with random variables $p_k$ and $q_k$ drawn according to the measure $P(L) dL = N \exp\left[-A \text{Tr} L^2 - B \sum_\alpha |V_\alpha| \cosh \mu \phi_\alpha \right] \prod_k p_k \, d q_k. \quad (71)$

As in the case of model CM$_r$, equation (71) contains a standard RMT Gaussian weight and a confinement potential which can be rewritten as $\text{Tr} Q + \text{Tr} R$. Using (67), (68), and the fact that the transformation is canonical, we obtain the distribution in terms of the new variables $\lambda_\alpha$ and $\phi_\alpha$ as

$$P(L) dL = N \exp\left[-A \sum_\alpha \lambda_\alpha^2 - B \sum_\alpha |V_\alpha| \cosh \mu \phi_\alpha \right] \prod_\alpha d\lambda_\alpha d\phi_\alpha. \quad (72)$$

The joint distribution of eigenvalues is then obtained by integrating over the angle variables, using

$$\int_{-\infty}^{\infty} \exp\left[-B |V_\alpha| \cosh \mu \phi_\alpha \right] d\phi_\alpha = \frac{1}{\mu} K_0 \left(B |V_\alpha| \right) \quad (73)$$

where $K_0$ is the modified Bessel function of the second kind. This yields the joint distribution of eigenvalues for model CM$_h$ as

$$P(\lambda_1, \ldots, \lambda_N) \sim \exp\left(-A \sum_\alpha \lambda_\alpha^2 \right) \prod_\alpha K_0 \left(B \prod_\beta \frac{1 + i g \mu}{\lambda_\alpha - \lambda_\beta} \right). \quad (74)$$

This expression is exact but difficult to handle. In order to find a Wigner-type surmise for the nearest-neighbour distributions we consider the limiting behaviour $P(\lambda)$ when two nearby eigenvalues $\lambda_1$ and $\lambda_2$ get close to each other. Setting $s = \lambda_1 - \lambda_2$ we see that the factor $\exp(-b/s^2)$ in the case of model CM$_r$ is replaced by a factor

$$K_0 \left(B \sqrt{1 + \frac{g^2 \mu^2}{s^2}} \right)^2 \sim s \exp\left(-\frac{2Bg\mu}{s} \right). \quad (75)$$

We therefore expect the nearest-neighbour spacing distribution to behave as

$$P(n, s) = as^d \exp(-b/s - cs). \quad (76)$$

In figure 2 we show the results of numerical computations of the nearest-neighbour spacing distributions for matrices of the form (50) with the choice of parameters and variables detailed in section 1. The surmise (76) perfectly reproduces numerical results.

In order to assess better the validity of the exponentially strong level repulsion for models CM$_r$ and CM$_h$, we compare in figure 3 the beginning of the distributions $P(s)$ for these models. Clearly the $1/s^2$ repulsion for CM$_r$ and the $1/s$ repulsion for CM$_h$ fit numerical curves very well. However, the precision of our numerical results does not permit to confirm or reject the presence of the logarithmic term $d \ln s$ in $P(s)$ for CM$_h$ model.
Figure 2. Nearest-neighbour spacing distributions $P(n, s)$ for the random matrices of model CM$_h$ with $\mu = 4\pi/N$ and $g = 0.05$ (top left), 0.25 (top right), 0.5 (bottom left), and 1 (bottom right), averaged over the central quarter of the spectrum for 32000 realizations of matrices of size $N = 256$. Solid lines are numerical results, dashed lines indicate the fit (76) for $P(n, s)$ with (in each panel from left to right) $P(s) = P(1, s)$ (red), $P(2, s)$ (green) and $P(3, s)$ (blue).

Figure 3. Nearest-neighbour spacing distributions $P(s)$ for the random matrices CM$_r$ ($g = 0.5$, black circles) and $g = 1$, red squares) and CM$_h$ ($g = .5$, green triangles up, and $g = 1$, blue triangles down), with $\mu = 4\pi/N$. Symbols are numerical results, solid lines indicate the fit $b/s^2 + cs - \ln a$ (model CM$_r$, equation (48)) and $b/s + cs - \ln a - d\ln s$ (model CM$_h$, equation (76)). Logarithm is natural.

4. Trigonometric Calogero–Moser model

For the trigonometric Calogero–Moser model CM$_t$ the Lax matrix is [25]

$$L_{kr} = p_r \delta_{kr} + ig(1 - \delta_{kr}) \frac{\mu}{2\sin(\mu(q_k - q_r)/2)}.$$

(77)

The only difference with the previous model, (50), is the sin function which replaces the sinh. Matrix (77) can be obtained from (50) by the substitution $\mu \rightarrow i\mu$. However, the fact that positions of the particles are now defined on a circle (because of the sin function) makes the resulting spectral statistics entirely different from the previous models CM$_r$ and CM$_h$. 
To construct action-angle variable we introduce, as in the previous section, two matrices
\[ Q_{\alpha\beta} = \sum_k u_k(\alpha) e^{i\mu q_k/2}, \quad R_{\alpha\beta} = \sum_k u_k^*(\alpha) e^{-i\mu q_k/2}, \] (78)
and two vectors
\[ e_{\alpha} = \sum_k u_k(\alpha) e^{i\mu q_k/2}, \quad f_{\alpha} = \sum_k u_k^*(\alpha) e^{-i\mu q_k/2}. \] (79)

Following the same steps as above with \( \mu \) replaced by \( i\mu \), one obtains
\[ Q_{\alpha\beta} = f_{\alpha}^* g_{\lambda\beta} - \lambda_{\alpha} + g_{\mu} e_{\beta}, \quad R_{\alpha\beta} = e_{\alpha}^* g_{\lambda\beta} - \lambda_{\alpha} + g_{\mu} f_{\beta}. \] (80)

Again, using the fact that \( Q \) is the inverse of \( R \) we obtain that
\[ \sum_{\gamma} |e_{\gamma}|^2 \lambda_{\gamma} - \lambda_{\alpha} = c_1, \quad \sum_{\gamma} |f_{\gamma}|^2 \lambda_{\gamma} - \lambda_{\alpha} + g_{\mu} = c_2 \] (81)
with certain constants \( c_1 \) and \( c_2 \) independent of \( \alpha \). Repeating the same arguments as in the previous section and using results of appendix B one concludes that \( c_2 = -c_1 = 1/(\mu g) \) and
\[ |e_{\alpha}|^2 = V_{\alpha}, \quad |f_{\alpha}|^2 = W_{\alpha}. \] (82)

The new variables \( \phi_{\alpha} \) are defined as above from the diagonal elements of matrix \( Q \) as follows:
\[ Q_{\alpha\alpha} = V_{\alpha}^{1/2} W_{\alpha}^{1/2} e^{i\mu \phi_{\alpha}}. \] (84)

In definition (79) of \( e_{\alpha} \) one has a freedom to choose the overall phase of the eigenvector \( u_k(\alpha) \). Since from (80) one must have \( Q_{\alpha\alpha} = f_{\alpha}^* e_{\alpha} \), one can choose phases, for example, as follows:
\[ e_{\alpha} = V_{\alpha}^{1/2} e^{i\mu \phi_{\alpha}/2}, \quad f_{\alpha} = W_{\alpha}^{1/2} e^{-i\mu \phi_{\alpha}/2}. \] (85)

Then the matrix \( Q \) can be expressed in terms of new variables \( \lambda_{\alpha} \) and \( \phi_{\alpha} \) as
\[ Q_{\alpha\beta} = e^{i\mu \phi_{\alpha}/2} W_{\alpha}^{1/2} g_{\mu} \lambda_{\alpha} - \lambda_{\beta} + g_{\mu} V_{\beta}^{1/2} e^{i\mu \phi_{\beta}/2}. \] (86)

The inverse matrix \( R \) plays a symmetric role, as it is obtained from \( Q \) by exchanging \( \mu \) to \( -\mu \). Again, there is a canonical transformation from position and momentum variables \( (q_k, p_k) \) to action and angle variables \( (\lambda_{\alpha}, \phi_{\alpha}) \) [19].

An important consequence of (82) is that for all \( \alpha \) we should have
\[ V_{\alpha} > 0, \quad W_{\alpha} > 0. \] (87)

These inequalities impose non-trivial restrictions on eigenvalues \( \lambda_{\alpha} \), as we will see now. We label eigenvalues so that \( \lambda_1 < \lambda_2 < \cdots < \lambda_N \), and we consider the function
\[ h(x) = \sum_{\gamma} \frac{V_{\gamma}}{\lambda_{\gamma} - x} + \frac{1}{\mu g}. \] (88)

It has \( N \) poles at \( x = \lambda_{\gamma} + \mu g \), and according to equations (81) and (85) it has \( N \) zeros at \( x = \lambda_{\alpha} + \mu g \). If all numerators \( V_{\alpha} \) are positive then the derivative of \( h \) is positive, and it is easy to check from the graph of the function that between two consecutive poles there is one and only one zero. Suppose \( \mu g > 0 \). Then for \( x \to -\infty \) the function \( h(x) \) has a strictly positive limit. The lowest zero \( \lambda_1 + \mu g \) must thus lie in the interval \( [\lambda_1, \lambda_2] \). More generally one must have
\[ \lambda_\alpha + \mu g \in ]\lambda_\alpha, \lambda_{\alpha+1}[ \text{ for } 1 \leq \alpha \leq N - 1, \text{ while the largest zero } \lambda_N + \mu g \text{ lies in the interval } ]\lambda_N, \infty[. \] Thus eigenvalues fulfil the inequalities
\[ \lambda_{\alpha+1} - \lambda_\alpha > g \mu. \tag{89} \]

Conversely, if these inequalities are fulfilled then trivially all \( V_\alpha \) are positive. Therefore, (89) are the necessary and sufficient conditions for the positivity of all \( V_\alpha \). In particular, eigenvalues of the Lax matrix (77) obey inequalities (89) for any choice of the \( q \). These results adapt straightforwardly to the case where \( \mu g \) is negative.

Since in (77) the \( q_k \) only appear as an argument in the sin function, there is no need to choose a confining potential for the particle distribution as in models CM\(_r\) and CM\(_h\). We consider the probability distribution of \( p \) and \( q \) in the form
\[ P(p, q) \sim \exp \left[ -A \left( \sum_j p_j^2 + g^2 \sum_{j \neq k} \frac{\mu^2}{4 \sin^2[\mu(q_k - q_r)/2]} \right) \right] \] (90)
with the restrictions that all \( q_j \) are between 0 and \( 2\pi/\mu \). Since the change of variables from \( p \) and \( q \) to \( \lambda_\alpha \) and \( \phi_\alpha \) is canonical and the restrictions (89) do not depend on phase variables the joint distribution of eigenvalues is
\[ P(\lambda_1, \ldots, \lambda_N) \sim \exp \left[ -A \sum_\alpha \lambda_\alpha^2 \right] \chi(\lambda_1, \ldots, \lambda_N), \tag{91} \]
where the function \( \chi(\lambda) \) is equal to 1 if (89) is fulfilled for all \( \alpha \), and 0 otherwise.

It turns out that model CM\(_t\) is very similar to a fourth model, the Ruijsenaars–Schneider model, that we will consider in the next section. Therefore we postpone analytical calculations of the nearest-neighbour spacing distributions to section 6. The nearest-neighbour spacing distributions \( P(n, s) \) are shifted Poisson distributions of the form
\[ P(n, s) = \begin{cases} 0, & 0 < s < nb, \\ \frac{(s - nb)^{n-1}}{(n-1)!(1-b)^n} e^{-(s-nb)/(1-b)}, & s > nb \end{cases} \] (92)
with some numerical constant \( b \). In figure 4 we show the results of numerical computations for matrices of the form (77) with the choice of parameters and variables detailed in section 1.

5. Ruijsenaars–Schneider model

The Calogero–Moser models considered in the previous sections are such that there exists a matrix \( Q \) which is, in a certain sense, dual to the Lax matrix \( L \). Namely, the canonical transformation from variables \((p_k, q_k)\) to action-angle variables \((\lambda_\alpha, \phi_\alpha)\) is such that the action variables \( \lambda_\alpha \) are eigenvalues of \( L \) and angle variables \( \phi_\alpha \) are related to \( Q_\alpha \) in a simple way.

In fact, the matrix \( Q \) is also a Lax matrix, corresponding to a possibly different Hamiltonian, and \( L \) plays the role of a matrix dual to \( Q \) for the inverse of the canonical transformation [19]. The rational Calogero–Moser system is self-dual since matrices \( L \) and \( Q \), given by (33) and (40), are equal up to labelling of the variables.

The model we consider in this section is related to the above models in that its Lax matrix \( L \) and the dual matrix \( Q \) are a kind of generalization of those of model CM\(_t\) (86). The treatment of this model closely follows the previous section.

The Lax matrix for Ruijsenaars–Schneider model is the \( N \times N \) unitary matrix given by [21]
\[ L_{kr} = e^{i\sigma p_r/2} \tilde{W}_k^{1/2} \frac{\sin[\mu \sigma/2]}{\sin[\mu(q_k - q_r + g \sigma)/2]} \tilde{V}_r^{1/2} e^{i\sigma p_r/2}. \tag{93} \]
This Lax matrix is related with the Hamiltonian (14) by
\[ H(p, q) = \frac{1}{2} \text{Tr}(L + L^\dagger). \] (95)

It is convenient to introduce the vectors
\[ \tilde{e}_k = \tilde{V}_k^{1/2} e^{i\sigma \mu q_k/2}, \quad \tilde{f}_k = \tilde{W}_k^{1/2} e^{-i\mu q_k/2}, \] (96)
so that matrix \( L \) can be rewritten as
\[ L_{kr} = \tilde{f}_k^\ast \tilde{e}_r \sin[\mu g / 2] \sin[\mu (q_k - q_r + g \sigma) / 2] e^{-i\mu q_k/2} \tilde{e}_r. \] (97)

As in the previous section, the condition that the matrix (93) is unitary imposes certain restrictions on the coordinates \( q \), which we will discuss later. Assuming that the Lax matrix is unitary, we choose to denote its eigenvalues by \( e^{i\sigma \lambda \omega} \). The dual matrices for the Ruijsenaars–Schneider model are defined by [21]
\[ Q_{\alpha \beta} = \sum_k u_k^\ast(\alpha) e^{i\mu q_k/2} u_k(\beta), \quad R_{\alpha \beta} = Q_{\beta \alpha}^* = \sum_k u_k^\ast(\alpha) e^{-i\mu q_k/2} u_k(\beta), \] (98)
and the vectors \( e_\alpha \) and \( f_\alpha \) by
\[ e_\alpha = \sum_k u_k(\alpha) \tilde{e}_k, \quad f_\alpha = \sum_k u_k(\alpha) \tilde{f}_k. \] (99)

From equation (97) one has
\[ e^{i\mu (q_k + q_r)/2} L_{kr} \sin[\mu (q_k - q_r + g \sigma) / 2] = \tilde{f}_k^\ast \sin[\mu g / 2] \tilde{e}_r. \] (100)

Multiplying this expression by \( u_k^\ast(\alpha) u_r(\beta) \) and summing both sides over \( k \) and \( r \) leads to
\[ \frac{1}{2i} Q_{\alpha \beta} \left( e^{i\sigma \lambda \omega} - e^{-i\sigma \lambda \omega} \right) = f_\alpha^* \sin[\mu \sigma / 2] e_\beta, \] (101)
which yields the analogue of (80),
\[ Q_{\alpha \beta} = f_\alpha^* e^{-i\sigma \lambda \omega/2} \frac{\sin[\sigma g \mu / 2]}{\sin[\sigma (\lambda_\beta - \lambda_\alpha + g \mu) / 2]} e^{-i\sigma \lambda / 2} e_\beta. \] (102)
Let us rewrite

$$Q_{\alpha \beta} = f_{\alpha}^* \frac{e^{-i\tau} - 1}{e^{i(\sigma_{\alpha} - \tau)} - e^{i\sigma_{\beta}}} e_{\beta},$$

(103)

where we have set $\tau = \sigma_{\mu}$. From its definition (98) it is clear that $Q$ has to be an unitary matrix, i.e. $\sum_{\gamma} Q_{\alpha \gamma} Q_{\gamma \beta}^* = \delta_{\alpha \beta}$. Selecting terms with $\alpha = \beta$ and $\alpha \neq \beta$ yields the two equations

$$|\rho|^2 f_{\alpha}^2 \sum_{\gamma} \frac{|e_{\gamma}|^2}{|e^{i(\sigma_{\gamma} - \tau)} - e^{i\sigma_{\beta}}|^2} = 1,$$

(104)

$$\sum_{\gamma} \frac{|e_{\gamma}|^2}{(e^{i(\sigma_{\gamma} - \tau)} - e^{i\sigma_{\beta}})(e^{-i(\sigma_{\gamma} - \tau)} - e^{-i\sigma_{\beta}})} = 0,$$

(105)

where we have set $\rho = e^{-i\tau} - 1$. Using the identity

$$\frac{1}{(e^{i(\sigma_{\gamma} - \tau)} - e^{i\sigma_{\beta}})(e^{-i(\sigma_{\gamma} - \tau)} - e^{-i\sigma_{\beta}})} \quad \frac{e^{i\sigma_{\gamma}}}{e^{i(\sigma_{\gamma} - \tau)} - e^{i\sigma_{\beta}}} \quad \frac{e^{i\sigma_{\gamma}}}{e^{i(\sigma_{\gamma} - \tau)} - e^{i\sigma_{\beta}}},$$

(106)

it follows from (105) that there exists a constant $c$ such that

$$\sum_{\gamma} |e_{\gamma}|^2 e^{i\sigma_{\gamma}} = c.$$

(107)

One has $c \neq 0$. Indeed, if $c = 0$ then inverting the system of equations (107) leads to $e_{\gamma} = 0$ for all $\gamma$, which contradicts the fact that $L$ is unitary. Equations (B1)–(B2) then allow us to obtain

$$|e_{\alpha}|^2 = -c\rho V_{\alpha} e^{-i\tau(N-1)/2},$$

(108)

while from equation (104) one obtains, using equation (B3)

$$|f_{\alpha}|^2 = -\frac{1}{c\rho} W_{\alpha} e^{i\tau(N-1)/2},$$

(109)

with new vectors $V_{\alpha}$ and $W_{\alpha}$ defined by

$$V_{\alpha} = \prod_{\beta \neq \alpha} \frac{\sin[\sigma(\lambda_{\alpha} - \lambda_{\beta} + g\mu)/2]}{\sin[\sigma(\lambda_{\alpha} - \lambda_{\beta})/2]}, \quad W_{\alpha} = \prod_{\beta \neq \alpha} \frac{\sin[\sigma(\lambda_{\alpha} - \lambda_{\beta} - g\mu)/2]}{\sin[\sigma(\lambda_{\alpha} - \lambda_{\beta})/2]}.$$

(110)

Using (B5) we obtain

$$\sum_{\alpha} |e_{\alpha}|^2 = c(1 - e^{-i\tau N}).$$

(111)

There is some overall freedom in the definition of vectors $\tilde{e}_{\alpha}$ and $\tilde{f}_{\alpha}$ in equation (96), as one could multiply $\tilde{e}_{\alpha}$ by some constant factor and divide $\tilde{f}_{\alpha}$ by the same factor. This in turn entails the same freedom for vectors $e_{\alpha}$ and $f_{\alpha}$ in (99). If one chooses for instance $\sum_{j} |\tilde{e}_{j}|^2 = t$ then from unitarity of the transformation in (99) one has $\sum_{\alpha} |e_{\alpha}|^2 = t$, which fixes the value

$$c = \frac{t}{1 - e^{-i\tau N}},$$

(112)

and thus

$$|e_{\alpha}|^2 = \frac{t \sin(\tau/2)}{\sin(N\tau/2)} V_{\alpha}, \quad |f_{\alpha}|^2 = \frac{\sin(N\tau/2)}{t \sin(\tau/2)} W_{\alpha}.$$
By definition $t = \sum_j |\tilde{e}_j|^2$ is positive. A convenient choice is to take
\[ t = \frac{|\sin(N\tau/2)|}{|\sin(\tau/2)|}. \tag{114} \]
Since $|e_\alpha|^2$ and $|f_\alpha|^2$ are non-negative one concludes that $V_\alpha$ and $W_\alpha$ have the same sign as $\sin(N\tau/2)/\sin(\tau/2)$ for all $\alpha$. The choice (114) implies that
\[ |e_\alpha|^2 = |V_\alpha|, \quad |f_\alpha|^2 = |W_\alpha|. \tag{115} \]

The new action variables are the $\lambda_\alpha$, and angle variables $\phi_\alpha$ are defined by
\[ Q_{\alpha\beta} = (V_\alpha W_\beta)^{1/2} e^{i\mu\phi_\alpha}. \tag{116} \]

Equation (102) implies that $Q_{\alpha\alpha} = f_\alpha^* e_\alpha e^{-i\lambda_\alpha}$. In view of (115) and (116) one can choose the phases of the eigenvectors $u_\alpha(\alpha)$ such that $e_\alpha$ and $f_\alpha$ defined by (99) can be expressed as
\[ e_\alpha = V^{1/2}_\alpha e^{i\mu\phi_\alpha/2} e^{i\lambda_\alpha/2}, \quad f_\alpha = W^{1/2}_\alpha e^{-i\mu\phi_\alpha/2} e^{-i\lambda_\alpha/2}. \tag{117} \]

In terms of the new variables, $Q$ thus reads
\[ Q_{\alpha\beta} = e^{i\mu\phi_\alpha/2} W^{1/2}_\alpha \frac{|\sin(\sigma g\mu/2)|}{\sin[\sigma (\lambda_\beta - \lambda_\alpha + g\mu)/2]} V^{1/2}_\beta e^{i\mu\phi_\beta/2}. \tag{118} \]

Comparing (93) and (118) one sees that for model RS the dual matrix matrix $Q$ is obtained from $L$ by changing $\sigma \leftrightarrow \mu$, $g \leftrightarrow -g$, $p \leftrightarrow \phi$ and $q \leftrightarrow \lambda$. It means that this model is self-dual (the matrices $L$ and $Q$ are the same up to a change of notation). One can show that the transformation from $(q_k, p_k)$ to action and angle variables $(\lambda_\alpha, \phi_\alpha)$ is canonical [21]. As mentioned, a consequence of the unitarity of the matrix $Q$ is that $V_\alpha$ and $W_\alpha$ have the same sign as $\sin(N\tau/2)/\sin(\tau/2)$ for all $\alpha$. This implies that certain inequalities have to be verified by the $\lambda_\alpha$, which we now derive in a way similar as in the previous section.

Let us define the function
\[ h(x) = \sum_\gamma V_\gamma\cot[(x - \sigma \lambda_\gamma)/2] - \frac{\cos(N\tau/2)}{\sin(\tau/2)}, \tag{119} \]
which is periodic with period $2\pi$ and can be considered as a function on the unit circle. It has $N$ poles at $x = \sigma \lambda_\gamma$. Taking the imaginary part of equation (107), with $c$ given by (112) and $e_\alpha$ given by (117), one obtains that $h(x)$ has $N$ zeros at $x = \sigma \lambda_\gamma - \tau$. When all $V_\gamma$ are positive, the same arguments as in the previous section imply that between two consecutive poles of $h(x)$ there must be exactly one zero. It means that between two nearby eigenvalues $\sigma \lambda_\gamma$ there is one and only one number of the form $\sigma \lambda_\gamma - \tau$. A similar reasoning starting from matrix $R = Q^\dagger$ leads to the conclusion that between two consecutive $\sigma \lambda_\gamma$ there must also be one and only one number of the form $\sigma \lambda_\gamma + \tau$. These two conditions (shift by $+\tau$ or $-\tau$) are equivalent, thus we can restrict ourselves to a shift by $+\tau$, i.e. the condition that the sets $[\sigma \lambda_\gamma, 1 \leq \gamma \leq N]$ and $[\sigma \lambda_\gamma + \tau, 1 \leq \gamma \leq N]$ intertwine on the unit circle. It is a necessary and sufficient condition for the matrix (102) to be an unitary matrix. A similar conclusion is readily obtained when all $V_\gamma$ are negative.

The conditions implied by this type of intertwining have been discussed in [28, 29]. There is a fundamental difference between these eigenvalue conditions in the RS model and in the CMt model discussed in the previous section. In the CMt case the Lax matrix is Hermitian, and the $\lambda_\alpha$ can take values on the whole real axis. In the RS case, as the Lax matrix is unitary, the $\lambda_\alpha$ lie on the unit circle. Therefore poles and zeros cannot be ordered in a simple way as in the previous case, and the analysis of the previous section does not apply.

For completeness we shortly repeat the arguments of the papers [28, 29]. Let us put all eigenvalues $\sigma \lambda_\alpha$ of the Lax matrix (93) on the unit circle and divide the circle into sectors.
with angle $\tau$. Denote the (positive) angular distance from the boundaries of the $k$th sector in counter clockwise direction by $x_k$ and in clockwise direction by $y_k$, as in figure 5. After a shift by $\tau$, the intertwining relations imply that only one of the two points corresponding to $x_k$ and $y_k$ will fall in-between points corresponding to $x_{k+1}$ and $y_{k+1}$. The first case corresponds to $x_k > x_{k+1}$ and $y_{k+1} > y_k$. In the second case the inequalities are reversed and $x_k < x_{k+1}$ and $y_{k+1} < y_k$. In both cases the inequality

$$(y_{k+1} - y_k)(x_{k+1} - x_k) < 0$$

is fulfilled.

Let us consider consecutive sectors of angle $\tau$ as in figure 5. Denote the number of eigenvalues in each sector by $n_k$. After a shift by $\tau$, eigenphases from the $k$th sector will move into the $(k+1)$th sector. The $n_k$ shifted points divide this sector into $n_k + 1$ intervals. As was proved above, eigenphases in the $(k+1)$th sector have to intertwine with these shifted eigenphases. Therefore all $n_k + 1$ intervals except the first and the last will be occupied. The first will be occupied if $x_{k+1} < x_k$ and the last interval will be occupied provided $y_{k+1} > y_{k+2}$. These statements can be rewritten in the form of the recurrence relation

$$n_{k+1} = n_k - 1 + \Theta(x_k - x_{k+1}) + \Theta(y_{k+1} - y_{k+2}),$$

(121)

where $\Theta(t)$ is the Heaviside step function, $\Theta(t) = 1$ when $t > 0$ and $\Theta(t) = 0$ for $t < 0$. From (120) it follows that this relation can be rewritten in the form

$$n_{k+1} = n_k - 1 + \Theta(x_k - x_{k+1}) + \Theta(x_{k+2} - x_{k+1}).$$

(122)

We now specialize to the case where $\tau$ depends on the size $N$ of the Lax matrix. We set

$$\tau = \frac{2\pi}{N} a$$

(123)

with fixed $a$. The case where $a$ is an integer corresponds to a situation where $L$ is not unitary (equation (111) implies in this case that $e^{i \gamma} = 0$ for all $\gamma$). We thus assume that $a$ is not an integer. The total number of sectors of angle $\tau$ in the unit circle is

$$K = \left\lfloor \frac{N}{a} \right\rfloor.$$
Integrable random matrix ensembles

where \([t]\) denotes the integer part of \(t\). Suppose the beginning of the first sector lies at position \(\sigma \lambda_1\). We choose to consider that this eigenvalue does not belong to the first sector, i.e. \(y_1 = 0\). Applying equation (120) for \(k = 0\) implies that \(x_1 > x_0\). Thus necessarily \(n_2 = n_1 + 1\) and for \(k \geq 3\) one easily obtains from equation (122) that

\[
n_k = n_1 + \Theta(x_{k+1} - x_k).
\]

The total number of eigenvalues lying into all \(K\) sectors obeys the inequalities

\[
N - n_1 - 1 \leq \sum_{k=1}^{K} n_k \leq N - 1.
\]

The right-hand side inequality comes from the fact that when \(a\) is not an integer the union of all \(K\) intervals does not overlap the whole circle: in particular, it does not contain the first eigenvalue \(\sigma \lambda_1\) from which we start our sectors. The left-hand side inequality is a consequence of the fact that if eigenvalues were shifted by \(-\tau\), the first sector in the opposite direction would have exactly the same number of eigenvalues as the second sector, i.e. \(n_1 + 1\) eigenvalues: as all \(K\) sectors does not cover the whole circle and the uncovering region is smaller than the sector of angle \(\tau\), it follows that the number of eigenvalues in all \(K\) sectors is larger than \(N - (n_1 + 1)\).

From (125) one easily obtains a second inequality

\[
Kn_1 + 1 \leq \sum_{k=1}^{K} n_k \leq K(n_1 + 1).
\]

From these two inequalities it follows that

\[
\frac{N}{K + 1} - 1 < n_1 < \frac{N - 2}{K}.
\]

By definition of \(K\) we have

\[
\frac{N}{a} - 1 < K < \frac{N}{a}.
\]

Substituting in the right-hand side of (128) the minimum of \(K\) and in the left-hand side the maximum value of \(K\) one obtains

\[
a - 1 - \frac{a^2}{N + a} < n_1 < a + \frac{a(a - 2)}{N - a},
\]

which entails that for \(N\) large enough \(a - 1 < n_1 < a\). Since \(n_1\) has to be an integer, \(n_1 = [a]\). This means that for sufficiently large \(N\), within an interval of length \(\tau\) from any eigenvalue there are always exactly \([a]\) eigenvalues.

The inverse statement is also true. If within an interval of length \(\tau\) from any eigenvalue there exist exactly \([a]\) other eigenvalues then all \(V_{\alpha}\) and \(W_{\alpha}\) have the sign of \(\sin(\tau/2)/\sin(N\tau/2)\). To see this let us consider e.g. \(V_{\alpha}\) given by equation (110). It is a product of terms \(\sin x / \sin(x - \tau)\) where \(x = \sigma(\lambda_\alpha - \lambda_\beta)/2\). As the distance between two eigenvalues may be restricted, \(0 < \sigma \lambda_\alpha - \sigma \lambda_\beta < 2\pi\), \(x\) obeys inequality \(0 < x < \pi\). Therefore \(\sin(x) > 0\), and \(\sin(x - \tau)\) is negative when \(0 < x < \tau\) and positive when \(\tau < x < \pi\). If within an interval of length \(\tau\) from \(\sigma \lambda_\alpha\) there are exactly \([a]\) eigenvalues, then in the product formula for \(V_{\alpha}\) there are exactly \([a]\) negative terms, so that its total sign is \((-1)^{[a]}\). For \(N\) large enough the sign of \(\sin(\tau/2)/\sin(N\tau/2)\) with \(\tau = 2\pi a/N\) is the sign of \(\sin \pi a\), which is precisely \((-1)^{[a]}\).

The above arguments prove that for sufficiently large \(N\) (whose value depends only on \(a\)) the necessary and sufficient condition for the unitarity of matrix \(Q\) is that at distance \(|\tau|\) from any eigenvalue there exist \([a]\) other eigenvalues.
As matrices $L$ and $Q$ are dual, the unitarity condition for $L$ can be readily deduced: it is that at distance $|\tau|$ from one coordinate $\mu q_k$ there are exactly $[a]$ other coordinates. Note that in [21] only the case $0 < a < 1$ had been considered.

These restrictions determine the allowed region in coordinate space. We choose as the ‘natural’ measure of momenta and coordinates the uniform distribution for momenta (between 0 and $2\pi/\sigma$) and coordinates uniformly distributed in the allowed region as explained above. After the change of variables from coordinates and momentum to action-angle variables it follows that the resulting distribution of eigenvalues will be also uniform but in the allowed region of eigenvalues with the only restriction that any interval $[\sigma \lambda_a, \sigma \lambda_a + 2\pi a/N]$ contains exactly $[a]$ eigenvalues. In next section we will show that it is possible to calculate asymptotic expressions for the joint distribution of eigenvalues using a transfer operator technique.

For numerical investigations, we consider an ensemble of unitary matrices of the form (97), with $p_k$ chosen as independent random variables uniformly distributed between 0 and $2\pi$ and the picket-fence distribution of coordinates $q_j = j, 1 \leq j \leq N$ (see section 1). Choosing constants such that $\mu = 2\pi/N, \sigma = 1$ and $g = a$, so that $\tau = 2\pi a/N$, direct calculations yield

$$\tilde{V}_k = \tilde{W}_k = \left| \frac{\sin N\tau/2}{N \sin \tau/2} \right|. \quad (131)$$

With a slightly different choice of phases in (93), matrix $L$ simplifies to

$$L_{kr} = \frac{e^{i\Phi_k}}{N} \frac{1 - e^{2i\pi a}}{1 - e^{2i\pi (k-r+a)/N}}, \quad (132)$$

where we denote $p_k = \Phi_k$. This is a particular specialization of the Lax matrix for the Ruijsenaars–Schneider model. In the form (132) but with $a = bN$ with fixed $b$ it first appeared in [28] as a result of the quantization of a classical parabolic map on the torus proposed in [10]. When $b$ is a rational number the map considered in [10] corresponds to a pseudo-integrable map of exchange of two intervals. For this particular case where $a$ depends on $N$, the spectral statistics of the unitary matrix (132) has been obtained analytically in [28, 29] without knowledge of the relation with the Lax matrix of the Ruijsenaars–Schneider model. The nearest-neighbour spacing distributions in that case are given by polynomials multiplied by decreasing exponentials. In the present case, where $a$ is a fixed parameter independent on $N$, results are quite different. In particular, level repulsion at small arguments of the nearest-neighbour spacing distribution is much stronger. In the case where $0 < a < 1$ the distribution even vanishes in the range $[0, a]$. Analytical calculations of spectral correlation functions for this model are performed in the next sections.

To illustrate the accuracy of the analytical results that we derive in the next sections, we show in figure 6 results of numerical computations for matrices of the form (132) for different values of the parameter $a$. Agreement is remarkable for all parameter values.

6. Joint distribution of eigenvalue spacings for model RS

We now calculate asymptotic expressions for the joint distribution of eigenvalue spacings for the Lax matrix ensemble corresponding to the RS model. As discussed in the previous section, eigenvalues are such that within an interval of length $a$ from any eigenvalue there are exactly $[a]$ other eigenvalues. We introduce the rescaled nearest-neighbour spacings

$$\xi_k = \frac{N}{2\pi} (\lambda_{a+1} - \lambda_a). \quad (133)$$

The answer strongly depends on the integer part of $a$. Therefore we consider different cases separately.
6.1. $0 < a < 1$

The simplest case corresponds to $0 < a < 1$. In this case the only restriction is that the distance between the nearest eigenvalues is larger than $a$, namely $\xi_k > a$. It is convenient to rewrite this restriction as follows. The joint probability density of having $N + 1$ eigenvalues inside an interval of length $L$ is given by

$$p(\xi_1, \xi_2, \ldots, \xi_N) = \frac{1}{Z_N(L)} \prod_{j=1}^{N} g(\xi_j) \delta\left(L - \sum_{k=1}^{N} \xi_k\right),$$  \hspace{1cm} (134)$$

where

$$g(x) = \begin{cases} 0 & \text{when } x < a, \\ 1 & \text{otherwise,} \end{cases}$$ \hspace{1cm} (135)$$

and $Z_N(L)$ is the normalization constant

$$Z_N(L) = \int_{0}^{\infty} \int_{0}^{\infty} \cdots \int_{0}^{\infty} \prod_{j=1}^{N} g(\xi_j) \delta\left(L - \sum_{k=1}^{N} \xi_k\right).$$ \hspace{1cm} (136)$$

We are interested in the joint probability distribution of $n$ consecutive spacings

$$p(\xi_1, \ldots, \xi_n) = \int_{0}^{\infty} \int_{0}^{\infty} \cdots \int_{0}^{\infty} \prod_{j=1}^{N} g(\xi_j) \delta\left(L - \sum_{k=1}^{N} \xi_k\right),$$ \hspace{1cm} (137)$$

when $L, N \to \infty$ with mean level spacing $\Delta = L/N$ remaining constant. In the following we set $\Delta = 1$. We shall proceed as it was done in [30]. The multiple integrals in (137) are easily calculated by introducing the function

$$h_{n, N}(L) = \int_{0}^{\infty} \int_{0}^{\infty} \cdots \int_{0}^{\infty} \prod_{j=1}^{N} g(\xi_j) \delta\left(L - \sum_{k=1}^{N} \xi_k\right),$$ \hspace{1cm} (138)$$
whose Laplace transform reads
\[ g_{n,N}(t) = \lambda(t)^{N-n} \prod_{k=1}^{n} g(\xi_k) e^{-t\xi_k}. \] (139)

Here
\[ \lambda(t) = \int_{0}^{\infty} g(x) e^{-tx} \, dx = \frac{e^{-ta}}{t} \] (140)
is the Laplace transform of \( g(x) \). The inverse Laplace transform of \( \lambda(t)^{N-n} \) is then
\[ \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \lambda(t)^{N-n} \exp(Lt) \, dt = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{1}{\lambda(t)^n} \exp (N(\ln \lambda(t) + \Delta t)) \, dt. \] (141)
The large-\( N \) behaviour of (141) is obtained by saddle-point approximation. The value \( c \) corresponds to the solution of the saddle-point equation
\[ 1 + \frac{\lambda'(c)}{\lambda(c)} = 0 \] (142)
(recall that \( \Delta = 1 \)). The solution reads
\[ c = \frac{1}{1-a}. \] (143)
Similarly, the inverse Laplace transform of \( g_{n,N}(t) \) can be calculated by saddle-point approximation. Rather than calculating explicitly all prefactors coming from the integration, it is easier to observe that the large-\( N \) behaviour of the normalization factor \( Z_N = h_{0,N}(L) \) is obtained similarly. One finally obtains
\[ p(\xi_1, \ldots, \xi_N) = \frac{e^{na/(1-a)}}{(1-a)^n} \prod_{j=1}^{n} g(\xi_j) e^{-\xi_j/(1-a)}. \] (144)

6.2. \( 1 < a < 2 \)

We now consider the case \( 1 < a < 2 \). According to previous section for sufficiently large matrix size, any eigenvalue \( x_k \) is such that there exists exactly 1 eigenvalue in the interval \([x_k, x_k + 2\pi a/N]\). In other words, the constraints on \( x_k \) are that \( x_k + 1 \in [x_k, x_k + 2\pi a/N] \) and \( x_{k+2} > x_k + 2\pi a/N \). In terms of the differences (133) between consecutive eigenvalues, the above restriction are equivalent to
\[ 0 < \xi_k+1 < a, \quad a < \xi_k+1 + \xi_k. \] (145)
Introduce the function \( f(x) \) by
\[ f(x) = \begin{cases} 1 & \text{when } 0 < x < a, \\ 0 & \text{otherwise} \end{cases} \] (146)
and \( g(x) \) as in (135). Then the joint probability density of \( N + 1 \) eigenvalues inside an interval of length \( L \) is given by the following expression:
\[ p(\xi_1, \xi_2, \ldots, \xi_N) = \frac{1}{Z_N(L)} \prod_{j=1}^{N} f(\xi_j) g(\xi_j + \xi_{j+1}) \delta \left( L - \sum_{k=1}^{N} \xi_k \right). \] (147)
where \( Z_N(L) \) is the normalization constant
\[ Z_N(L) = \int_{0}^{\infty} d\xi_1 \ldots \int_{0}^{\infty} d\xi_N \prod_{j=1}^{N} f(\xi_j) g(\xi_j + \xi_{j+1}) \delta \left( L - \sum_{k=1}^{N} \xi_k \right). \] (148)
The large-$N$ behaviour of the joint probability distribution of $n$ consecutive spacings (137) can be then obtained as above, following [30]. We review the main steps of the procedure. Introducing the function

$$h_{n,N}(L) = \int_0^\infty d\xi_{n+1} \cdots \int_0^\infty d\xi_N \prod_{j=1}^N f(\xi_j)g(\xi_j + \xi_{j+1}) \delta \left( L - \sum_{k=1}^N \xi_k \right).$$  \hspace{1cm} (149)

its Laplace transform reads

$$g_{n,N}(t) = \int_0^\infty d\xi_{n+1} \cdots \int_0^\infty d\xi_N \prod_{j=1}^N e^{-t\xi_j} f(\xi_j)g(\xi_j + \xi_{j+1}).$$  \hspace{1cm} (150)

This quantity can be seen as a product of transfer operators

$$g_{n,N}(t) = K_t(\xi_1, \xi_2)K_t(\xi_2, \xi_3)\cdots K_t(\xi_{n-1}, \xi_n) \times \int_0^\infty d\xi_{n+1} \cdots \int_0^\infty d\xi_N K_t(\xi_n, \xi_{n+1}) \cdots K_t(\xi_N, \xi_1),$$  \hspace{1cm} (151)

where the transfer operator, $K_t(\xi, \xi')$ is defined as

$$K_t(\xi, \xi') = f(\xi)g(\xi + \xi')f(\xi') e^{-t(\xi + \xi')/2}.$$  \hspace{1cm} (153)

This is a real symmetric operator. Its real eigenvalues, $\lambda_j(t)$, and eigenfunctions, $\phi_j(t; \xi)$, verify

$$\int_0^\infty K_t(\xi, \xi')\phi_j(t; \xi')d\xi' = \lambda_j(t)\phi_j(t; \xi).$$  \hspace{1cm} (154)

As this operator is real symmetric its eigenfunctions can be chosen to be orthonormal

$$\int_0^\infty \phi_j(t; \xi)\phi_k(t; \xi)d\xi = \delta_{jk}.$$  \hspace{1cm} (155)

The transfer operator can be expanded over the basis of eigenfunctions as

$$K_t(\xi, \xi') = \sum_j \lambda_j(t)\phi_j(t; \xi)\phi_j(t; \xi').$$  \hspace{1cm} (156)

In the large-$N$ limit the dominant contribution to (151) is given by the largest eigenvalue $\lambda_0(t)$. Then using the orthogonality property (155) of the $\phi_j$ we obtain

$$g_{n,N}(t) \sim \sum_{\xi_1, \xi_{n+1}} \prod_{k=1}^{n-1} K_t(\xi_k, \xi_{k+1})\lambda_0(t)^n \phi_0(t; \xi_n)\phi_0(t; \xi_1).$$  \hspace{1cm} (157)

The large-$N$ behaviour of $h_{n,N}(L)$ is obtained by performing the inverse Laplace transform of $\lambda_0(t)^n$. As in (141) the leading term is obtained by saddle-point approximation; here the saddle-point is such that

$$1 + \frac{\lambda_0'(c)}{\lambda_0(c)} = 0$$  \hspace{1cm} (158)

(again we take $\Delta = 1$). Calculating in a similar way the large-$N$ behaviour of the normalization factor $Z_N = h_{0,N}(L)$, one finally obtains

$$p(\xi_1, \xi_2, \ldots, \xi_n) = \frac{1}{\lambda_0^{-1}(c)} \phi_0(c; \xi_1)K_c(\xi_1, \xi_2)K_c(\xi_2, \xi_3)\cdots K_c(\xi_{n-1}, \xi_n)\phi_0(c; \xi_n).$$  \hspace{1cm} (159)
6.3. \( m < a < m + 1 \)

The general case \( m < a < m + 1 \) can be treated in a similar way. In this case each interval \([x_k, x_{k+2}^N]e^{ia/N}\) contains exactly \( m \) eigenvalues \( x_{k+1}, \ldots, x_{k+m} \). In terms of the rescaled differences (133) between consecutive eigenvalues this condition is equivalent to the following two inequalities

\[
\begin{align*}
0 < \xi_k + \xi_{k+1} + \cdots + \xi_{k+m-1} < a, & \quad (160) \\
0 < \xi_k + \xi_{k+1} + \cdots + \xi_{k+m}. & \quad (161)
\end{align*}
\]

In analogy with equation (147), the joint probability of eigenvalues spacings thus reads

\[
p(\xi_1, \xi_2, \ldots, \xi_N) = \frac{1}{Z_N(L)} \prod_{j=1}^{N} f(\xi_j + \cdots + \xi_{j+m-1}) g(\xi_j + \cdots + \xi_{j+m}) \delta(L - \sum_{k=1}^{N} \xi_k)
\]  

(162)

with \( f \) and \( g \) defined by (146) and (135). The large-\( N \) behaviour is calculated as above by introducing a transfer operator, which in this case depends on two sets of variables \( \xi = (\xi_1, \ldots, \xi_m) \) and \( \xi' = (\xi'_1, \ldots, \xi'_m) \) shifted by one unit i.e. \( \xi_2 = \xi'_1, \xi_3 = \xi'_2, \ldots, \xi_m = \xi'_{m-1} \) (see e.g. [30]). The explicit form of the transfer operator is the following

\[
K(\xi, \xi') = \delta(\xi_2 - \xi'_1) \cdots \delta(\xi_m - \xi'_{m-1}) \times e^{-t\xi_1/2} f(\xi_1 + \cdots + \xi_m) g(\xi_1 + \cdots + \xi_m + \xi'_m) f(\xi'_1 + \cdots + \xi'_m) e^{-t\xi'_m/2}.
\]  

(163)

The eigenvalue equation

\[
\int K(\xi, \xi') \phi(\xi') d\xi' = \lambda \phi(\xi)
\]

(164)

reduces to a one-dimensional equation because of the \( \delta \)-functions appearing in the definition of the transfer operator. This equation can be written in the form

\[
e^{-t\xi_1/2} \int_0^{\infty} e^{-tz/2} g(\xi_1 + \xi_2 + \cdots + \xi_m + z) \phi(t; \xi, \xi', \xi, \xi, \ldots, \xi) dz = \lambda(t) \phi(t; \xi_1, \ldots, \xi_m).
\]  

(165)

Here it is implicitly assumed that all variables \( \xi_j > 0 \) and

\[
\phi(t; \xi, \xi, \ldots, \xi) = 0 \quad \text{when } \xi_1 + \cdots + \xi_m > a.
\]  

(166)

As above the largest eigenvalue, \( \lambda_0(t) \) as well as the corresponding eigenfunction \( \phi_0(t; \xi) \), calculated at point \( t = c \) obeying the same saddle-point condition (equation (158)), determine all correlation functions in the limit of large \( N \). The joint probability of \( n \) consecutive spacings takes a different form for \( n \leq m \) and \( n > m \). For \( n \leq m \)

\[
p(\xi_1, \ldots, \xi_n) = \int_0^a d\xi_{m+1} \ldots \int_0^a d\xi_m \phi_0(c; \xi_1, \ldots, \xi_m) \phi_0(c; \xi_m, \ldots, \xi_1),
\]  

(167)

while for \( n > m \)

\[
p(\xi_1, \ldots, \xi_n) = \lambda_0(c)^{-m} \phi_0(c; \xi_n, \ldots, \xi_{n-m}) \phi_0(c; \xi_1, \ldots, \xi_m) e^{-c \sum_{s=1}^{n-m} \xi_s}
\times \prod_{j=1}^{m} f(\xi_j + \cdots + \xi_{j+m-1}) \prod_{j=1}^{m} g(\xi_j + \cdots + \xi_{j+m}).
\]  

(168)
7. Nearest-neighbour spacing distributions for model RS

In the previous section we have derived expressions for the joint distribution of eigenvalue spacings \( p(\xi_1, \ldots, \xi_n) \). From these expressions the \( n \)th nearest-neighbour spacing distribution can be calculated as

\[
P(n, s) = \int_0^{\infty} d\xi_1 \ldots \int_0^{\infty} d\xi_n \ p(\xi_1, \ldots, \xi_n) \delta \left( s - \sum_{i=1}^{n} \xi_i \right). \tag{169}
\]

7.1. \( 0 < a < 1 \)

For \( 0 < a < 1 \) these integrals are easily calculable (e.g. by Laplace transform) and from the joint distribution equation (144) we obtain

\[
P(n, s) = \begin{cases} e^{(na - s)/(1 - s)}(s - na)^{n-1} & s \geq na, \\ 0 & 0 < s < na. \end{cases} \tag{170}
\]

Comparison with numerical simulations is displayed at figure 6.

7.2. \( 1 < a < 2 \)

When \( 1 < a < 2 \) the joint distribution is given by equation (159). What remains is to calculate the largest eigenvalue of the transfer operator \( K_t \), as well as its associated eigenfunction. As \( K_t \) is a positive operator, the analogue of the Perron–Frobenius theorem states that the eigenvector corresponding to the largest eigenvalue is positive. Orthogonality of the eigenfunctions implies that the converse is also true. Thus if one finds a positive eigenfunction then the corresponding eigenvalue is the largest one. The eigenvalue equation (154) is equivalent to

\[
e^{-t\xi/2} \int_{a-\xi}^{a} e^{-t\xi'/2} \phi(\xi') d\xi' = \lambda \phi(\xi). \tag{171}
\]

Let us look for solutions of equation (171) positive on \([0, a]\) under the form \( \phi(\xi) = \sinh \rho \xi \), with \( \rho \) some unknown complex parameter. Since equation (171) should hold for all \( \xi \in [0, a] \) we obtain the necessary condition

\[
t = -2 \rho \coth(\rho a). \tag{172}
\]

When \( t < -2/a \), equation (172) admits two real solutions \( \rho = \pm \rho_0 \). Thus \( \phi(\xi) = \sinh \rho_0 \xi \) with \( \rho_0 > 0 \) is a solution of equation (171). If \( t > -2/a \), equation (172) admits two pure imaginary solutions \( \rho = \pm i \rho_0 \), thus \( \phi(\xi) = \sin \rho_0 \xi \) with \( \rho_0 > 0 \) and \( i \rho_0 \) solution of equation (172) is a positive solution of equation (171). Finally if \( t = -2/a \), \( \rho = 0 \) is the unique solution to equation (172). In that case \( \phi(\xi) = \xi \) is a solution of equation (171) which is positive on \([0, a]\). Thus for all \( t \) we have a positive solution to equation (171). Properly normalized, this solution gives the eigenvector \( \phi_0(t; \xi) \).

The corresponding eigenvalue is given by

\[
\lambda_0(t) = \frac{e^{(t-1/2)a}}{\rho - t/2}, \tag{173}
\]

with \( \rho \) an implicit function of \( t \).

The saddle-point \( c \) is a solution of equation (158). For \( \lambda_0 \) given by equation (173) the condition becomes

\[
1 + 2a(2 - a)\rho^2 - \cosh 2\rho a + 2\rho(a - 1) \sinh 2\rho a = 0 \tag{174}
\]
and the saddle-point \( c \) is obtained from \( \rho \) through equation (172). Equivalently, this condition can be expressed as

\[
a = \frac{2z^2 - z \sinh 2z}{z^2 + \sinh^2 z - z \sinh 2z}, \quad z = \rho a. \tag{175}
\]

In figure 7 we plot \( a \) as a function of \( z = \rho a \). For \( 1 < a < 4/3 \) equation (175) has a unique real solution \( \rho_0 > 0 \), and \( \phi_0(\xi) = \sin \rho_0 \xi \) is a positive eigenfunction of the transfer operator. For \( 4/3 < a < 2 \) equation (175) has a unique pure imaginary solution \( i\rho_0 \) with \( \rho_0 > 0 \). Furthermore in that latter case \( \rho_0 a \in [0, \pi] \), so that \( \phi_0(\xi) = \sin \rho_0 \xi \) is an eigenfunction of the transfer operator which is positive on \([0, a]\). At \( a = 4/3 \) the unique solution is \( \rho = 0 \) and \( \phi_0(\xi) = \xi \) is a positive eigenfunction of the transfer operator. The \( n \)th nearest-neighbour spacing distribution can now be calculated from equation (159).

In the case \( n = 1 \) it directly gives us the nearest-neighbour spacing distribution \( P(s) = A^2 \phi_0(s)^2 \), where \( A \) is the normalization constant. It is nonzero only for \( s \in [0, a] \), where it takes the following form

\[
P(s) = \begin{cases} A^2 \sinh^2(\rho s) & \text{when } 1 < a < 4/3, \\ \frac{81}{64} s^2 & \text{when } a = 4/3, \\ A^2 \sin^2(\rho s) & \text{when } 4/3 < a < 2. \end{cases} \tag{176}
\]

Constants \( A \) and \( \rho \) can be determined either by solving equation (175) and normalizing the eigenfunction \( \phi_0(\xi) \), or equivalently by imposing the normalization conditions (4). The next-to-nearest distribution, \( P(2, s) \) is non-zero only when \( a < s < 2a \) and within this interval it is given by

\[
P(2, s) = \frac{A^2}{\lambda_0(c)} e^{-cs/2} \int_{s-a}^{a} \phi_0(\xi) \phi_0(s - \xi) \, d\xi. \tag{177}
\]

In particular for \( a = 4/3 \) all integrals can be calculated analytically and \( P(2, s) \) has the form

\[
P(2, s) = (-\frac{3}{2} + \frac{27}{16} s - \frac{81}{512} s^3) e^{3s/4 - 1}. \tag{178}
\]

In a similar manner one can obtain the higher nearest-neighbour functions. For example, \( P(3, s) \) is given by the formula

\[
P(3, s) = \frac{A^2 e^{-cs/2}}{\lambda_0(c)^2} \int_{0}^{a} d\xi_1 \phi_0(\xi_1) \int_{a-\xi_1}^{a} d\xi_2 e^{-c\xi_2/2} \int_{a-\xi_2}^{a} d\xi_3 \phi_0(\xi_3) \delta \left( s - \sum_{i=1}^{n} \xi_i \right). \tag{179}
\]
It is non-zero only when \( a < s < 3a \). In particular for \( a = 4/3 \) we obtain

\[
p(3, s) = \begin{cases} 
\left( \frac{3}{2} - \frac{s}{48} + \frac{24}{27} s^3 \right) e^{2s^2/3} + \frac{81/4}{27} s^4 & \text{when } 4/3 < s < 8/3, \\
\left( \frac{9}{4} + \frac{27}{32} s - \frac{81}{512} s^3 \right) e^{2s^2/3} + 9 e^{3s^2/2} & \text{when } 8/3 < s < 4.
\end{cases}
\]  

(180)

In figure 6 these formulae are compared with numerical simulations and show a remarkable agreement.

7.3. \( 2 < a < 3 \)

For \( 2 < a < 3 \) the joint distribution is given by (168). The largest eigenvalue and corresponding eigenfunction of the transfer operator (163) are solution of the eigenvalue equation (165). For \( m = 2 \) it takes the form

\[
e^{-\xi_1/2} \int_{\mu - \xi_2}^{\mu - \xi_1} e^{-\xi_2/2} \phi(\xi_2, \xi_3) d\xi_3 = \lambda \phi(\xi_1, \xi_2).
\]  

(181)

Let us look for solutions of the form similar to Bethe Ansatz

\[
\phi(\xi_1, \xi_2) = e^{\alpha \xi_1 + \beta \xi_2} + e^{-\beta \xi_1 + \alpha \xi_2} + e^{\alpha - \beta \xi_1 - \xi_3} - e^{-\beta \xi_1 - \xi_3} - e^{\alpha - \beta \xi_1 - \xi_3} - e^{\alpha + \beta \xi_1 + \xi_3},
\]  

(182)

where we have set \( \mu = -t/2 \). As equation (181) has to be fulfilled for all \( \xi_1, \xi_2 \), this function is a solution of (181) if and only if the following conditions are valid

\[
\frac{e^{\alpha(\mu + \beta)}}{\mu + \beta} = \frac{e^{\alpha(\mu - \beta)}}{\mu - \beta} = \frac{e^{\alpha(\mu - \beta)}}{\mu - \beta} = -\frac{\lambda}{\alpha}.
\]  

(183)

From the first equality in equation (183) one can express \( \alpha \) as a function of \( \alpha \) and \( \beta \). After inspection we found that the solutions of the above equations have the following form

\[
\alpha a = \frac{1}{2} x_1 + i x_2, \quad \beta a = -\frac{1}{2} x_1 + i x_2, \quad \mu a = \frac{1}{2} x_1 + x_3
\]  

(184)

with real parameters \( x_1, x_2 \) and \( x_3 \). Under this substitution the eigenfunction (182) is transformed to

\[
\phi(\xi_1, \xi_2) = e^{s_1(\xi_1 - \xi_2) / 2a} \left( \sin \frac{x_2(\xi_1 + \xi_2)}{a} - e^{(x_1 - x_3) \xi_1 / a} \sin \frac{x_2 \xi_1}{a} - e^{-(x_1 - x_3) \xi_1 / a} \sin \frac{x_2 \xi_3}{a} \right),
\]  

(185)

where from (183) \( x_1, x_2 \) and \( x_3 \) must fulfill the following equalities:

\[
\frac{e^{s_1}}{x_1} = e^{s_1 + i s_2} = \frac{e^{s_1 - i s_2}}{x_3 + i x_2} = \frac{e^{s_1 - i s_2}}{x_3 - i x_2} = -\frac{\lambda}{\alpha}
\]  

(186)

and depend on time \( t \) through the relation

\[
\frac{t a}{2} = \frac{x_1}{2} + x_3.
\]  

(187)

This implies that

\[
x_3 = \frac{x_2}{\tan x_2}
\]  

(188)

and, consequently, \( x_1 \) is related with \( x_2 \) as follows

\[
\frac{e^{s_1}}{x_1} = \frac{\sin x_2}{x_2} e^{s_2 / \tan x_2}.
\]  

(189)

The eigenfunction corresponding to the largest eigenvalue of the transfer operator is thus given by (185) with \( x_1, x_2 \) and \( x_3 \) real parameters depending on \( t \), which must verify (188) and (189) and be such that \( \phi(\xi_1, \xi_2) \) is a positive function over \([0, a]^2\).
The saddle-point condition is again given by equation (158). Using (186)–(189) we obtain a second relation between $x_1$ and $x_2$, namely

$$a = \frac{1}{1 - 1/x_1} + \frac{2 - \sin(2x_2)/x_2}{1 + \sin^2(x_2)/x_2^2 - \sin(2x_2)/x_2}.$$  \hspace{1cm} (190)$$

Equations (189) and (190) determine parameters $x_1$ and $x_2$ at a given $a$. To obtain a positive eigenfunction $\phi$ it is necessary to obtain the solutions in the intervals

$$x_1 < 0, \quad \pi < x_2 < 2\pi.$$ \hspace{1cm} (191)$$

The knowledge of these parameters allows us to calculate the eigenfunction (185), from which the nearest-neighbour distributions can be deduced through equation (168). The first distributions read

$$P(s) = A \int_0^{a-s} \phi(s, y)\phi(y, s) \, dy,$$ \hspace{1cm} (192)$$

$$P(2, s) = A \int_0^{s} \phi(s - y, y)\phi(y, s - y) \, dy \hspace{1cm} \text{(193)}$$

and

$$P(3, s) = \frac{A}{\lambda} \int_{a-a}^{a} dx e^{\mu x} \int_{a-a}^{s-x} dy e^{\mu y} \phi(s - x - y, x)\phi(s - x - y, y) \, dy.$$ \hspace{1cm} (194)$$

with $A$ the normalization constant

$$A = \left( \int_0^a dx \left( \int_0^{a-x} dy \phi(x, y)\phi(y, x) \right) \right)^{-1}. \hspace{1cm} (195)$$

These analytical expressions perfectly agree with numerical simulations, as shown in figure 6.

8. Level compressibility for model RS

The expressions for the joint distribution of eigenvalue spacings $p(\xi_1, \ldots, \xi_n)$ obtained in section 6 allow us to derive formulae for the level compressibility $\chi$, which characterizes the asymptotic behaviour of the number variance.

The number variance $\Sigma^2(L)$ is the average variance of the number of energy levels in an interval of length $L$. It is defined from the two-point correlation function $R_2(s) = \sum_{n=1}^{\infty} P(n, s)$ as

$$\Sigma^2(L) = L - 2 \int_0^{L} ds \, (L - s)(1 - R_2(s)). \hspace{1cm} (196)$$

For systems with intermediate spectral statistics, $\Sigma^2(L) \sim \chi L$ for large $L$. In order to obtain the large-$N$ behaviour of the level compressibility we calculate the Laplace transform of the two-point correlation function. It has a series expansion of the form

$$g_2(t) = \int_0^{L} ds \, R_2(s)e^{-ts} = \frac{1}{t} + \frac{\chi - 1}{2} + O(t) \hspace{1cm} (197)$$

which allows us to obtain $\chi$ (see [30] for more detail).
8.1. Case $0 < a < 1$

The $n$th nearest-neighbour spacing distributions are given by equation (170). Summation over $n$ gives the two-point correlation function, and its Laplace transform is readily obtained, yielding

$$g_2(t) = \frac{1}{\text{e}^{at} (1 + t - at) - 1}. \quad (198)$$

Small-$t$ expansion of $g_2(t)$ gives

$$\chi = (1 - a)^2. \quad (199)$$

8.2. Case $1 < a < 2$

The functions $P(n, s)$ are given by equations (159) and (169). Their Laplace transform reads

$$g(n, t) = \frac{1}{\lambda_0^{-1}(c)} \int_0^\infty d\xi_1 \cdots \int_0^\infty d\xi_n \phi_0(c; \xi_1) K_c(\xi_1, \xi_2) \cdots K_c(\xi_{n-1}, \xi_n) \phi_0(c; \xi_n) \text{e}^{-t(\xi_1 + \cdots + \xi_n)}, \quad (200)$$

where as in the previous sections $\lambda_0(c)$ is the largest eigenvalue of the transfer operator (153) and $\phi_0(c; \xi)$ its associated eigenfunction, both taken at the saddle-point $c$. Using the definition (153) of the transfer operator, we see that $g(n, t)$ can be rewritten

$$g(n, t) = \frac{1}{\lambda_0^{-1}(c)} \int_0^\infty d\xi \int_0^\infty d\xi' \phi_0(c; \xi) e^{-t\xi/2} K_{c+t}(\xi, \xi') \phi_0(c; \xi') e^{-t\xi'/2}. \quad (201)$$

Replacing the transfer operator by its expansion (156) and summing over $n$ we obtain

$$g_2(t) = \sum_j \frac{\lambda_0(c)}{\lambda_0(c) - \lambda_j(c + t)} \left( \int_0^\infty d\xi \phi_0(c; \xi) \phi_j(c + t; \xi) e^{-t\xi/2} \right)^2. \quad (202)$$

One can check, using normalization (155) of the eigenfunctions and the saddle-point condition (158), that the leading-order term is given by $g_2(t) \sim 1/t$. The next-order term can be simplified using the normalization of $\phi_0$. It yields

$$g_2(t) = \frac{1}{t} - \frac{\lambda_0''(c)}{2\lambda_0'(c)} - 1 + o(t^2), \quad (203)$$

from which one obtains

$$\chi = -1 - \frac{\lambda_0''(c)}{\lambda_0'(c)}. \quad (204)$$

Here $\lambda_0(t)$ is given by (173) (with $\rho$ depending on $t$ through (172)), and $c$ is given by condition (158). After calculation, $\chi$ can be expressed as a function of $\rho$ at the saddle-point. We obtain

$$\chi = \left( \frac{a^2}{4} - \frac{4a(1 - a) \zeta^2 + a^2 \sinh^2 \zeta}{(2\zeta - \sinh 2\zeta)^2} \right) \frac{\sinh^2 \zeta}{\zeta^2}, \quad z = \rho \alpha, \quad (205)$$

with $\rho$ the real positive solution $\rho_0$ of (175) for $1 < a < 4/3$ or the pure imaginary solution $i\rho_0$ of (175) for $4/3 < a < 2$. For $a = 4/3$, the limit $\rho \to 0$ in (205) gives $\chi = 4/9$. 

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8.3. Case $2 < a < 3$

As in the previous case $\chi$ is given by (204) with $\lambda_0(t)$ given by (186), with $x_1, x_2, x_3$ and $t$ related through (187)–(189). From (187)–(188), parameter $x_1$ can be expressed as

$$x_1 = -2x_2 \tan x_2 - at.$$  \hfill (206)

Differentiating both (189) and (206) with respect to time we obtain $dx_1/dt$ and $dx_2/dt$ as a function of $x_1$ and $x_2$, and then similarly $d^2x_1/dt^2$ and $d^2x_2/dt^2$. Using (186), the saddle-point condition (158) can be rewritten as

$$1 + \frac{dx_1}{dt} \left(1 - \frac{1}{x_1}\right) = 0,$$  \hfill (207)

and from (204) $\chi$ can then be expressed as

$$\chi = \frac{1}{(1-x_1)^2} + \frac{d^2x_1}{dt^2} \left(1 - \frac{1}{x_1}\right).$$  \hfill (208)

Using the expression obtained $d^2x_1/dt^2$ we finally obtain $\chi$ as a function of $x_1$ and $x_2$, with $x_1, x_2$ obtained as solution of (189)–(190). Inverting (190) we obtain

$$x_1 = \frac{a \sin^2 x_2 + (a-2)x_2^2 + (1-a)x_2 \sin 2x_2}{(a-1) \sin^2 x_2 + (a-3)x_2^2 + (2-a)x_2 \sin 2x_2}.$$  \hfill (209)

After some manipulation $\chi$ simplifies to

$$\chi = \frac{1}{a(\sin^2 x_2 + x_2^2 - x_2 \sin 2x_2)^2} \left[(a-3)^2(a-2)x_2^4 - (a-3)(a-1)(2a-5)x_2^2 \sin 2x_2 + 2(a-2)((\cos 2x_2 + 2)(a-1)(a-2) - 3)x_2^2 \sin^2 x_2 - 2a(a-2)(2a-3)x_2 \cos x_2 \sin^3 x_2 + a(a-1)^2 \sin^4 x_2\right].$$  \hfill (210)

Figure 8 is a plot of the level compressibility $\chi$. The theoretical prediction obtained from (199), (205) and (210) agrees with numerical data.
8.4. Asymptotics in the vicinity of integer \( a \)

For integer \( a \) the spectrum is rigid and thus the level compressibility is expected to take the value 0. Here we consider the first-order expansion of \( \chi \) in the vicinity of integer \( a \). We will show that at lowest order the expansion of \( \chi \) around \( a = n \) is given by \( \chi \simeq (1 - a)^2/n^3 \).

We first consider the expansion around \( a = 1 \). Let \( a = 1 + \epsilon \). For \( a < 1 \) we have \( \chi = (1 - a)^2 = \epsilon^2 \), thus expansion is trivial. For \( a > 1 \) \( \chi \) is given by (205) with \( a \) and \( z \) related by (175). At \( a = 1 \) the solution of (175) is \( z = \infty \). An asymptotic expansion of (205) and (175) yields

\[
a = \frac{2z}{2z - 1} + F(z)e^{-2z} + o(e^{-2z}),
\]

where \( F(z) \) is some rational fraction in \( z \). Thus at first order

\[
z = \frac{1}{2} \left( 1 + \frac{1}{\epsilon} \right).
\]

Expanding \( \chi \) for large \( z \) to lowest order gives

\[
\chi = a(a - 1) + a^2 \left( \frac{1}{4z^2} - \frac{1}{2z} \right) + G(z)e^{-2z} + o(e^{-2z})
\]

with \( G(z) \) is some rational fraction in \( z \). Using (212) one obtains \( \chi \simeq \epsilon^2 = (1 - a)^2 \).

Suppose now that \( a = 2 + \epsilon \). For \( a < 2 \) \( \chi \) is given by (205), with \( \rho = i\phi_0 \) solution of (175). Equivalently, \( \chi \) is given by

\[
\chi = \left( \frac{a^2}{4} + 4a(1 - a)z^2 + a^2 \sin^2 z \right) \left( 2z - \sin 2z \right) \sin^2 z, \tag{214}
\]

with \( z \) the real positive solution of

\[
a = \frac{2z^2 - z \sin 2z}{z^2 - 2z \sin 2z + \sin^2 z}. \tag{215}
\]

At point \( a = 2 \) the solution is \( z = \pi \). Expanding both sides of (215) at lowest order in \( \epsilon \) with \( z = \pi + \zeta \epsilon \) we obtain \( \zeta = \pi/2 \). Inserting this expansion for \( z \) in (214) we obtain

\[
\chi = \epsilon^2/4 + o(\epsilon^3).
\]

For \( a > 2 \) \( \chi \) is given by (210), with \( x_1 \) and \( x_2 \) specified by (189)–(190). At \( a = 2 \), we have

\[
\chi = \frac{2 \sin^4 x_2 - 3 \sin 2x_2}{2(x_2^2 + \sin^2 x_2 - x_2 \sin 2x_2)} \tag{216}
\]

which vanishes for \( x_2 = \pi \). For \( x_2 = \pi + t_1 \epsilon + t_2 \epsilon^2 \) we have the expansion

\[
\chi = \left( \frac{1}{2} - \frac{t_1}{\pi} \right) \epsilon + \left( \frac{5}{4} + \frac{9t_1}{2\pi} - \frac{3t_1^2}{\pi^2} - \frac{t_2}{\pi} \right) \epsilon^2 + o(\epsilon^3). \tag{217}
\]

Equation (189) is equivalent to

\[
\exp \left( x_1 - \frac{x_2}{\tan x_2} \right) = \frac{x_1 \sin x_2}{x_2} \tag{218}
\]

and the small-\( \epsilon \) expansion of both members of this equation reads

\[
x_1 - \frac{x_2}{\tan x_2} = -\frac{\pi}{\epsilon t_1} + \frac{\pi t_2}{t_1} - 1 + o(1), \tag{219}
\]

\[
\frac{x_1 \sin x_2}{x_2} = -\frac{\pi t_1}{\pi^2} \epsilon^2 + \frac{\pi^2 t_1 - 5\pi t_1^2 + 6t_1^3 + \pi^2 t_2 - 4\pi t_1 t_2}{\pi^3} \epsilon^3 + o(\epsilon^4) \tag{220}
\]
(we use (209) to obtain the expansion of $x_1$). This implies that the two first terms in the expansion (220) must vanish, thus $t_1 = \pi/2$ and $t_2 = 0$. Putting these values into (217) gives $\chi = \epsilon^2/4 + o(\epsilon^3)$.

The same result can be obtained from (210) and (189)–(190) for $a < 3$. At $a = 3$ again $x_2$ takes the value $\pi$, and an expansion $x_2 = \pi + t_1\epsilon$ gives $t_1 = \pi/3$, whence $\chi = \epsilon^3/9 + o(\epsilon^3)$.

9. Conclusion

In this paper we construct new random matrix ensembles with unusual properties. Random matrices from these ensembles are Lax matrices of $N$-body integrable classical systems with a certain measure of momenta and coordinates. Although such matrices are not invariant over rotation of the basis (as usual random matrix ensembles) the joint distribution of their eigenvalues can be calculated analytically. Four different models are considered in detail. Three of them correspond to rational, hyperbolic and trigonometric Calogero–Moser models. The fourth is related to the trigonometric Ruijsenaars–Schneider model. For the trigonometric Calogero–Moser model and the Ruijsenaars–Schneider model spectral correlation functions are calculated explicitly. For rational and hyperbolic Calogero–Moser models Wigner-type surmises are proposed. Our formulae are in good agreement with results of direct numerical calculations.

Appendix A. Hamilton–Jacobi equations

In this appendix we check that the action-angle variables $\lambda_\alpha$ and $\phi_\alpha$ for model CM$_r$ verify Hamilton–Jacobi equations by calculating their time derivative. We use the fact that for the Lax pair $(L, M)$ the matrix $M$ can be seen as a time derivative operator for the eigenfunctions of the matrix $L$. Namely, if $(u_k)_{1 \leq k \leq N}$ is a normalized eigenvector of $L$, then

$$
\dot{u}_k = \sum_r M_{kr} u_r \quad \text{and} \quad \dot{u}_k^* = -\sum_r u_r^* M_{rk}
$$

(A1)

(here $*$ denotes complex conjugation). For model CM$_r$, the Lax matrix $M$ is given by

$$
M_{kr} = -ig \delta_{kr} \sum_{j \neq k} \frac{1}{(q_k - q_j)^2} + ig(1 - \delta_{kr}) \frac{1}{(q_k - q_r)^2}.
$$

(A2)

One can easily check that for $1 \leq k, r \leq N$ one has

$$
p_r \delta_{kr} + M_{kr} (q_k - q_r) = L_{kr}.
$$

(A3)

Deriving (39) with respect to time, using the definition of $Q$, yields

$$
\dot{\phi}_\alpha = \sum_k [\dot{u}_k^*(\alpha) q_k u_k(\alpha) + u_k^*(\alpha) \dot{q}_k u_k(\alpha) + u_k^*(\alpha) q_k \dot{u}_k(\alpha)].
$$

(A4)

From Hamilton–Jacobi equations $\dot{q}_k = p_k$. Using (A1), we obtain that the time derivative of $\phi_\alpha$ is given by

$$
\dot{\phi}_\alpha = \sum_{k,r} u_k^*(\alpha) [p_k \delta_{kr} + M_{kr} (q_k - q_r)] u_r(\alpha) = \sum_{k,r} u_k^*(\alpha) L_{kr} u_r(\alpha) = \lambda_\alpha.
$$

(A5)

The time derivative of $\lambda_\alpha$ is easily obtained from (7),(A1) and (34), yielding $\dot{\lambda}_\alpha = 0$. This shows that the $\lambda_\alpha$ and $\phi_\alpha$ verify Hamilton–Jacobi equations.
Appendix B. Identities

The purpose of the appendix is to give, for completeness, the proofs of certain often used formulae.

Let coefficients $b_m$ obey the following system of linear equations for all $n = 1, \ldots, N$ with known $x_m$ and $y_m$

$$\sum_{m=1}^{N} \frac{b_m}{x_m - y_n} = 1. \quad (B1)$$

Then $b_m$ for all $m = 1, \ldots, N$ are expressed through $x_m$ and $y_m$ using e.g. the Cauchy determinants

$$b_m = \frac{\prod_n (x_m - y_n)}{\prod_{s \neq m} (x_m - x_s)}. \quad (B2)$$

The following identities are also useful. For all $l = 1, \ldots, N$ one has

$$\sum_{m=1}^{N} \frac{b_m}{(x_m - y_l)^2} = -\frac{\prod_{s \neq l} (y_l - y_s)}{\prod_l (y_l - x_s)}, \quad (B3)$$

$$\sum_{m=1}^{N} b_m = \sum_{m=1}^{N} (x_m - y_m) \quad (B4)$$

and

$$\sum_{m=1}^{N} \frac{b_m}{x_m} = 1 - \frac{1}{n} \frac{y_n}{x_n}. \quad (B5)$$

A simple way to check (B2) is to consider the function

$$f_n(x) = \frac{\prod_{r \neq n} (x - y_r)}{\prod_r (x - x_r)} \frac{(x - y_n)}{\prod_r (x - x_r)}. \quad (B6)$$

Asymptotically this function decreases as $1/x$ when $x \to \infty$, so that the integral over a large contour encircling all poles equals 1. Rewriting this integral as the sum over all finite poles gives

$$1 = \sum_m \frac{\prod_r (x_m - y_r)}{(x_m - y_n) \prod_{s \neq m} (x_m - x_s)}, \quad (B7)$$

which proves (B2).

Equality (B3) can be obtained by the integration of the function

$$\tilde{f}_n(x) = \frac{\prod_{r \neq n} (x - y_n)}{(x - y_l)^2 \prod_r (x - x_n)} \quad (B8)$$

over a contour which includes all poles. As this function decreases as $1/x^2$ when $x \to \infty$ the integral equals zero. Taking the sum over poles at $x = x_n$ with all $n = 1, \ldots, N$ and at $x = y_l$ one verifies (B3).

To obtain (B4) one has to integrate the function

$$\tilde{f}(x) = \prod_{n=1}^{N} \frac{x - y_n}{x - x_n} \quad (B9)$$

over the large contour and compare the residues at infinity and at $x = x_n$. 


Let us now consider the function

\[ \hat{f}(x) = \prod_n \frac{x - y_n}{x - x_n} \]

(B10)

It decreases as \(1/x\) when \(x \to \infty\) and has poles at \(x = 0\) and \(x = x_m\) with \(m = 1, \ldots, N\). Integrating it over a contour encircling all poles one obtains (B5).

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