Monogamy inequality of entanglement of pure tripartite qudit states

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We analytically establish an inequality analogous to the Coffman-Kundu-Wootters inequality, which succinctly describes monogamy of entanglement in $\mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathbb{C}^d$ dimensional pure states. The derivation of this inequality is based on the G-concurrence [27] measure of entanglement. It is shown that the shared entanglement of the subsystems of a pure tripartite qudit state always satisfy a monogamy constraint.

I. INTRODUCTION

Entanglement manifests several counter-intuitive phenomenon of quantum mechanics making it remarkably distinguished from its classical counterpart. Unlike classical correlation, entanglement cannot be shared freely among the involved parties. Nature imposes severe restriction on sharing entanglement arbitrarily between the subsystems of a given multipartite state, and this is referred to as monogamy of entanglement. For example, given a tripartite state shared by three parties namely, A, B and C, if A and B are maximally entangled between themselves, there cannot be any entanglement between A (B) and C. The restriction is strong, since it excludes the possibility of sharing even any kind of classical correlation in the mentioned scenario [1].

From an operational perspective, the concept of monogamy of entanglement is also significant as it lies at the heart of many information processing tasks such as the security of entanglement based quantum key distribution protocols [2, 3]. Monogamy was used to show that quantum cloning and state estimation are equivalent in the asymptotic regime [4]. Monogamy of quantum correlations is also found to be important in other branches of physics such as condensed-matter physics [5], frustrated spin systems [6], statistical physics [7], black-hole physics [8] etc.

The notion of monogamy of entanglement was first formulated in a form of inequality known as the Coffman-Kundu-Wootters (CKW) inequality [11]. The narrative nicely demonstrates a trade-off between shared entanglement of A with B and C. The authors used concurrence [9, 10] as a measure of entanglement to describe monogamy in a tripartite state in $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$. Then it was generalised for arbitrary multi-qubit states in [12]. The interesting feature of monogamy has also been studied with other measures of entanglement. The nature of monogamy based on concurrence of assistance [13] was shown for tripartite states [14] and subsequently it was generalized for multi-partite states [15], and for $2 \otimes 2 \otimes n$ [16] systems, as well. A monogamy inequality in terms of negativity in tripartite system was derived in [17]. Monogamous nature of squashed entanglement and Tsallis-q entanglement were described in [18] and [19], respectively.

Realizing its indispensability in a large number of applications, monogamy has been extended to more general scenarios, such as for multiqubit mixed states [21]. The relevant inequality in this case may be derived using the measure of entanglement of formation. Another interesting direction is to explore monogamy of entanglement involving higher dimensional systems. In [22] a monogamy inequality for tripartite state in dimension 2, 3 and 4 was derived using negativity as the measure of entanglement. A monogamy inequality was presented for a specific class of multi-qudit systems by convex-roof extended negativity [23]. Recently, monogamy of entanglement was shown taking Rényi-α entanglement with respect to any partition for generalized W-class states [24]. However, a general monogamy relation for qudit systems has hitherto not been established even for pure states. In this work, our motivation is to address this fundamental issue. Here we provide proof of a monogamy inequality for an arbitrary dimensional tripartite system.

Towards this goal it is relevant to note that quantification of entanglement in higher dimensional systems is yet to be understood completely. In the context of pure bipartite states it is known that the Schmidt coefficients play a pivotal role in capturing all entanglement properties. A finite set of entanglement monotones was introduced in [26] to quantify entanglement of a pure bipartite qudit state. These monotones are functions of the Schmidt coefficients of the given state. Subsequently, in [27], another set of monotones, namely, concurrence monotones were introduced, the last member of which is known as the G-concurrence. It measures to which extent the maximum Schmidt rank (for Schmidt number for mixed states, see [28]) is contained in a state. Although G-concurrence is easily computable for pure states, the mixed state generalisation is a considerably difficult task. However, a lower bound of G-concurrence for arbitrary mixed bipartite state has been provided in [29]. From a practical point of view, this particular monotone of entanglement has been found important in remote entanglement distribution protocols such as entanglement swapping and remote preparation of bipartite entangled states [27].

While generalizing the concept of monogamy to higher dimensional systems, it was observed that the CKW in-
equality fails to comply with the result of the qubit scenario. A counter example was produced to show that the CKW inequality does not hold for a tripartite qutrit state [30]. Being an intrinsic property of nature, formulation of monogamy of entanglement for higher dimensional systems is indeed required. Though investigation of monogamy needs a suitable measure of entanglement, closed form expressions of mixed bipartite qudit state entanglement are hard to find. This is the first difficulty one usually faces to deal with this problem. Nonetheless, a CKW type inequality involving the G-concurrence measure of entanglement has been conjectured in [31]. In this work we provide a detailed proof of the conjecture analytically, by adopting the framework for formalizing G-concurrence as prescribed in [1]. To illustrate the applicability of our derived inequality, we further provide examples of some renowned classes of states in $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$ and arbitrary $\mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathbb{C}^d$, as well.

We have organised this paper as follows. In section (II) we briefly revisit the G-concurrence measure of entanglement. In section (III) we present the proof of the monogamy inequality. Examples are discussed in section (IV). Finally, we provide some concluding remarks in section (V).

II. CONCURRENCE MONOTONES AND G-CONCURRENCE

Concurrence monotones [27] were presented in an attempt to characterize entanglement properties of a given pure qudit entangled state. These monotones can be obtained as a function of Schmidt coefficients of the given state. In this way, one can perceive all non-local aspects of a pure state by knowing all of its monotones. For a given pure state $|\psi\rangle$ in $\mathbb{C}^d \otimes \mathbb{C}^d$ with the Schmidt coefficients $\lambda_0, \ldots, \lambda_{d-1}$ respectively, we can define its $d$ concurrence monotones up to normalization as follows:

$$C_1(|\psi\rangle) = \sum_{i=0}^{d-1} \lambda_i$$

$$C_2(|\psi\rangle) = \left(\sum_{i<j} \lambda_i \lambda_j\right)^{\frac{1}{2}}$$

$$\vdots$$

$$C_d(|\psi\rangle) = \left(\prod_{i=0}^{d-1} \lambda_i\right)^{\frac{1}{2}}.$$  \hspace{1cm} (1)

The first monotone is trivial and sums up to one due to the normalization constraint. Amongst the others, the last monotone, known as G-concurrence, is of particular interest for us. Henceforth, we shall denote it as $G(|\psi\rangle)$ for convenience. For a $2 \otimes 2$ pure state, it agrees with the concurrence presented in [9]. G-concurrence of a pure state can be easily obtained by evaluating the determinant of one of its reduced density matrix.

Thus, G-concurrence is easily computable and can be extended for mixed states by convex roof extension. Consider, a $d \otimes d$-dimensional bipartite mixed state $\rho$. Now G-concurrence of $\rho$ is the minimized average G-concurrence of any ensemble of pure states $|\psi_i\rangle$:

$$G(\rho) = \inf \sum_i G(|\psi_i\rangle\langle\psi_i|),$$  \hspace{1cm} (2)

where the infimum is taken over all pure state decompositions realising $\rho = \sum_i |\psi_i\rangle\langle\psi_i|$, and $G(|\psi_i\rangle\langle\psi_i|)$ is the G-concurrence of $|\psi_i\rangle$, which can be easily determined.

III. PROOF OF MONOGAMY INEQUALITY

Consider, a tripartite state $|\Psi\rangle_{123}$ of Hilbert space dimension $\mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathbb{C}^d$ ($d > 2$) which is shared between three observers 1, 2 and 3. We denote the subsystem of 12 as $\rho_{12}$, 13 as $\rho_{13}$, and the single particle subsystem of 1 shared with 23 as $\rho_{1|23}$, respectively. We then proceed to prove one of the conjectures given in [31] which describes an entanglement trade-off between three of these subsystems $\rho_{12}$, $\rho_{13}$ and $\rho_{1|23}$, respectively.

Theorem 1 For any pure state $|\Psi\rangle_{123}$, the entanglement quantified by G-concurrence between the subsystem labelled by 1 and 2, between 1 and 3, and between 1 and subsystem 23 satisfy the following monogamy inequality

$$G^d(\rho_{1|23}) \geq G^d(\rho_{12}) + G^d(\rho_{13})$$  \hspace{1cm} (3)

Proof: Let us denote $|\Psi\rangle_{123}$ in computational basis $|ijk\rangle_{i,j,k=0}^{d-1}$,

$$|\Psi\rangle_{123} = \sum_{i,j,k=0}^{d-1} a_{ijk} |ijk\rangle,$$  \hspace{1cm} (4)

where $a_{ijk} \in \mathbb{C}$ and satisfies the normalisation constant

$$\sum_{i,j,k} |a_{ijk}|^2 = 1.$$  \hspace{1cm}

We obtain the subsystem shared by the observers 12 denoted as $\rho_{12}$ by tracing over the third subsystem

$$\rho_{12} = \sum_{i,j,k} a_{ijk}^* a_{ijk} |i\rangle\langle j|.$$  \hspace{1cm} (5)

Although the state space of $\rho_{12}$ is nine dimensional, it would be sufficient to consider only three of them which are necessary to be entangled with the single qutrit subsystem of the observer 3 [11]. Henceforth, we will consider the rank of $r(\rho_{123}) = 3$, and the same for the other subsystem $\rho_{13}$ shared by the observers 1 and 3. Now, the mixed state $\rho_{12}$ can be realised by the following pure state decomposition

$$\rho_{12} = \sum_k |\phi_k\rangle\langle\phi_k|,$$  \hspace{1cm} (6)

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$$\rho_{12} = \sum_k |\phi_k\rangle\langle\phi_k|,$$  \hspace{1cm} (6)
where \(|\phi_k\rangle = \sum_{ij} a_{ijk} |ij\rangle\) are sub-normalized orthogonal state vectors with norm no greater than one. The range of \(\rho_{12}\) is spanned by these three vectors. At this point, we face a difficulty as the closed expression of G-concurrence of a mixed state is not known. However, we can circumvent our problem by invoking the theorem presented in [1] which states that \(\rho_{12}\) written in the above form must contain at least one such pure state decomposition in which the G-concurrence of one \(|\phi_k\rangle\) is non-vanishing, vanishes for the other pure states. Without loss of generality, we assume

\[ G(|\phi_0\rangle) \neq 0 \quad \text{and} \quad G(|\phi_k\rangle) = 0, \quad \forall \ k = 1 \ldots \text{rank}(d-1). \]

(7)

So, we can evaluate G-concurrence of the mixed state \(\rho_{12}\) by calculating only \(G(|\phi_0\rangle)\), and hence, we can write

\[ G(\rho_{12}) = G(|\phi_0\rangle \langle \phi_0|). \]

(8)

Let us evaluate G-concurrence of \(|\phi_0\rangle\) by computing its Schmidt coefficients. To do this, we write \(|j\rangle = \sum_t U_{ji} |i\rangle\) and obtain the following

\[ |\phi_0\rangle = \sum_{ij} a_{ij0} U_{ji}|i\rangle \]

(9)

\[ = \sum_{i} \lambda_i^0 |i\rangle, \]

(10)

where \(\lambda_i^0 = \sum_j a_{ij0} U_{ji}\). Immediately, we obtain the Schmidt coefficients of \(|\phi_0\rangle\) as \(\lambda_i^0|^2\). So, the G-concurrence of \(|\phi_0\rangle\) is given by

\[ G^d(|\phi_0\rangle) = \prod_i |\lambda_i^0|^2. \]

(11)

Thus, the G-concurrence for the subsystem \(\rho_{12}\) is

\[ G^d(\rho_{12}) = \prod_i |\lambda_i^0|^2. \]

(12)

Now we proceed to find the G-concurrence of the other subsystem \(\rho_{13}\). In a similar fashion, we obtain

\[ G^d(\rho_{13}) = \prod_i |\lambda_i^0|^2, \]

(13)

where \(\Gamma_i^0 = \sum_k a_{i0k} U_{ki}\). At this point, we use the hermiticity of the elements \(a_{ijk}\), namely, \(a_{ijk} = a_{ikj}^*\). Thus, we can write the following:

\[ \Gamma_i^0 = \sum_k a_{i0k} U_{ki} = \sum_j a_{ij0}^* U_{ji}. \]

(14)

Now we rewrite

\[ G^d(\rho_{12}) + G^d(\rho_{13}) = 2 \prod_i |\lambda_i^0|^2 \]

\[ \leq 2 \prod_i \sum_j |a_{ij0}|^2 \]

\[ = 2 \left( \sum_j |a_{0j0}|^2 \cdots \sum_j |a_{d'd'j0}|^2 \right) \]

\[ = 2 \left( |a_{000}|^2 + |a_{010}|^2 + \cdots + |a_{0(d-1)0}|^2 \right) \cdots \]

\[ \left( |a_{d'd'0}|^2 + |a_{d'd'10}|^2 + \cdots + |a_{d'd'(d-1)0}|^2 \right) \]

\[ = 2\alpha \beta \]

(15)

where \(d' = d - 1\), \(\alpha = |a_{000}|^2 + |a_{010}|^2 + \cdots + |a_{0(d-1)0}|^2\) and \(\beta = |a_{(d-1)00}|^2 + |a_{(d-1)10}|^2 + \cdots + |a_{(d-1)(d-1)0}|^2\). The implication of the above equation may be further elucidated if we consider a special instance of qudit system. We substitute \(d = 3\) and obtain

\[ G^3(\rho_{12}) + G^3(\rho_{13}) \leq 2 \left( |a_{000}|^2 + |a_{010}|^2 + |a_{020}|^2 \right) \]

\[ \left( |a_{100}|^2 + |a_{110}|^2 + |a_{120}|^2 \right) \]

\[ \left( |a_{200}|^2 + |a_{210}|^2 + |a_{220}|^2 \right) \]

\[ \equiv \delta \]

(16)

where \(\delta\) is a positive quantity. It is to be noted that our theorem (3) aims to provide an algebraic constraint between G-concurrence of the subsystems \(\rho_{12}, \rho_{13}\) and the single particle subsystem \(\rho_{123}\). Now, we evaluate the G-concurrence of \(\rho_{123}\), which is obtained by tracing over the subsystems 2 and 3,

\[ \rho_{123} = \sum_{i,i',j,k} a_{i'jk}^* a_{ijk} |i'\rangle \langle i| \]

\[ = \sum_{i,j,k} a_{ij0}^* a_{ij0} |i\rangle \langle i| + \sum_{i,j,k} a_{ij0}^* a_{ij0} |i\rangle \langle i| \]

(17)

In order to calculate G-concurrence of \(\rho_{123}\), we treat the compound system of 2 and 3 as a single entity which is entangled with the single particle subsystem. Thus following [11], we obtain G-concurrence of \(\rho_{123}\) as follows,

\[ G^d(\rho_{123}) = d^d det \rho_{123} \]

(18)

Using Eq.(17) it follows that

\[ G^d(\rho_{123}) = d^d det \left( \sum_{i,j,k} a_{i'jk}^* a_{ijk} |i\rangle \langle i| + \sum_{i,j,k} a_{ij0}^* a_{ij0} |i\rangle \langle i| \right) \]

(19)

Now, using an elementary property of matrices namely,

\[ det(X + Y) \geq det(X) + det(Y) \]

for any given matrices X and Y, we can write

\[ G^d(\rho_{123}) \geq d^d \sum_{i,j,k} a_{ij0}^* a_{ij0} |i\rangle \langle i| \]

(20)
Let us now consider the particular case of $d = 3$. We have

$$G^3(\rho_{1|23}) \geq 27 \left( \sum_{j,k} |a_{0jk}|^2 \sum_{j,k} |a_{1jk}|^2 \sum_{j,k} |a_{2jk}|^2 \right)$$

$$= 27 \left( \sum_j (|a_{0j0}|^2 + |a_{0j1}|^2 + |a_{0j2}|^2) \right)$$

$$= 27 \left( \sum_j (|a_{1j0}|^2 + |a_{1j1}|^2 + |a_{1j2}|^2) \right)$$

$$= 27 \left( \sum_j (|a_{2j0}|^2 + |a_{2j1}|^2 + |a_{2j2}|^2) \right) \quad (21)$$

Comparing the above with Eq. (16), we observe that

$$G^3(\rho_{1|23}) \geq G^3(\rho_{12}) + G^3(\rho_{13}) \quad (23)$$

Thus we obtain a monogamy inequality for a tripartite pure qutrit state.

We provide the remaining part of the proof of the theorem (3) for arbitrary $d$, in Appendix (A). Note that we have derived (3) without taking into account the dimensionality of the state describing the physical system. This is the most striking feature of our derivation. The CKW inequality was obtained by taking concurrence as entanglement measure. As prescribed in [9], concurrence is strictly defined for $d = 2$ systems. On the other hand, the derivation of G-concurrence for arbitrary states does not rely on such limitation. So, our analysis can be straightforwardly generalised for any arbitrary $d \geq 2$.

IV. EXAMPLES

In this section we will present some examples to assert our theorem obtained in the last section. We start with the pure tripartite qutrit state introduced in [30] to show the violation of CKW inequality. The state is given as follows:

$$|\chi\rangle = \frac{1}{\sqrt{6}} (|012\rangle - |021\rangle + |120\rangle - |102\rangle + |201\rangle - |210\rangle) \quad (24)$$

The reduced density matrices for the subsystem $\rho_{12}$, $\rho_{13}$ and $\rho_{1|23}$ are

$$\rho_{12} = \frac{1}{3}(|x\rangle\langle x| + |y\rangle\langle y| + |z\rangle\langle z|) \quad (25)$$

$$\rho_{13} = \frac{1}{3}(|x\rangle\langle x| + |y\rangle\langle y| + |z\rangle\langle z|) \quad (26)$$

$$\rho_{1|23} = \frac{1}{3}(|0\rangle\langle 0| + |1\rangle\langle 1| + |2\rangle\langle 2|) \quad (27)$$

where $|x\rangle$, $|y\rangle$ and $|z\rangle$ in the first two expressions are given below:

$$|x\rangle = \frac{1}{\sqrt{2}} (|01\rangle - |10\rangle) \quad (28)$$

$$|y\rangle = \frac{1}{\sqrt{2}} (|02\rangle - |20\rangle) \quad (29)$$

$$|z\rangle = \frac{1}{\sqrt{2}} (|12\rangle - |21\rangle) \quad (30)$$

It was shown in [30], that the G-concurrence of $\rho_{12}$ and $\rho_{13}$ vanish. We also obtain readily $G(\rho_{1|23}) = 1$ using (18). Thus, we find $G^3(\rho_{1|23}) > G^3(\rho_{12}) + G^3(\rho_{13})$.

Next, we consider the tripartite generalized W-class state in $C^3 \otimes C^3 \otimes C^3$ as follows

$$|\mathcal{W}\rangle = \frac{1}{\sqrt{6}} (a_{11} |100\rangle + a_{12} |200\rangle + a_{21} |010\rangle + a_{22} |020\rangle + a_{31} |001\rangle + a_{32} |002\rangle) \quad (31)$$

where $|a_{ij}|^2 = 1$. It is to be noted that for $a_{ij} = \frac{1}{\sqrt{6}}$, we retain the familiar W-state. The subsystem $\rho_{12}$ is obtained by tracing over the third subsystem, and given by

$$\rho_{12} = |\tilde{x}\rangle \langle \tilde{x}| + |\tilde{y}\rangle \langle \tilde{y}| \quad (32)$$

where we write the un-normalized kets $|\tilde{x}\rangle$ and $|\tilde{y}\rangle$ as

$$|\tilde{x}\rangle = a_{11} |10\rangle + a_{12} |20\rangle + a_{21} |01\rangle + a_{22} |02\rangle \quad (33)$$

$$|\tilde{y}\rangle = \sqrt{|a_{31}|^2 + |a_{32}|^2} \langle 00| \quad (34)$$

At this point we shall follow the argument presented in [33] to evaluate the G-concurrence of the mixed state $\rho_{12}$. One can find a pure state decomposition of $\rho_{12}$ by suitable unitary transformation. We write one such pure state decomposition as $\rho_{12} = |\tilde{\phi}_1\rangle \langle \tilde{\phi}_1| + |\tilde{\phi}_2\rangle \langle \tilde{\phi}_2|$, where

$$|\tilde{\phi}_1\rangle = u_{11} |x\rangle + u_{12} |y\rangle \quad (35)$$

$$|\tilde{\phi}_2\rangle = u_{21} |x\rangle + u_{22} |y\rangle \quad (36)$$

Here $u_{hi}$ are elements of the $r \times r$ unitary matrix, $r$ being the rank of $\rho_{12}$. For this particular case, $r(\rho_{12}) = 2$. Thus we rewrite $\rho_{12}$ as

$$\rho_{12} = p_1 |\tilde{\phi}_1\rangle \langle \tilde{\phi}_1| + p_2 |\tilde{\phi}_2\rangle \langle \tilde{\phi}_2| \quad (37)$$

$$= \sum_{h=1}^{r} p_h |\phi_h\rangle \langle \phi_h| \quad (38)$$

where we have defined $p_i = \langle \tilde{\phi}_i| \phi_i\rangle$ and $|\phi_i\rangle = |\tilde{\phi}_i\rangle / \sqrt{p_i}$. Now the G-concurrence of $\rho_{12}$ is

$$G(\rho_{12}) = \min_h \sum_{h=1}^{r} p_h G(|\phi_h\rangle)$$

$$= p_1 G(|\tilde{\phi}_1\rangle) + p_2 G(|\tilde{\phi}_2\rangle)$$

$$= p_1 G\left( |\tilde{\phi}_1\rangle \langle \tilde{\phi}_1| \right) + p_2 G\left( |\tilde{\phi}_2\rangle \langle \tilde{\phi}_2| \right) \quad (39)$$
It can be verified that $G_i(\frac{\rho_{i1}}{\sqrt{d}})$ vanishes for $i = 1, 2$. So we obtain

$$G(\rho_{12}) = 0$$

(40)

Since, the G-concurrence vanishes for this particular decomposition, it is indeed the infimum of all pure state decompositions. In a similar fashion one can obtain $G(\rho_{13}) = 0$. Following (18), we evaluate $G(\rho_{1|23}) = 0$ . Thus, we can write for the $|\psi\rangle$ state,

$$G^3(\rho_{1|23}) = G^3(\rho_{12}) + G^3(\rho_{13}).$$

(41)

We conclude this section by showing that the monogamy inequality is indeed satisfied by the GHZ state in arbitrary dimension. In computational basis, it can be written as

$$|GHZ\rangle = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |iii\rangle.$$  

(42)

It is a well known fact that the GHZ state is a genuinely entangled state. If we trace one of its subsystems, the resultant reduced density matrix becomes a maximally mixed state. Any entanglement measure yields zero for a maximally mixed state. Thus, we can infer that $G^d(\rho_{12}) = G^d(\rho_{13}) = 0$. We also obtain $\rho_{1|23} = \frac{1}{d} I_{d \times d}$ which is readily followed by $det(\rho_{1|23}) = \frac{1}{d}$, and using (18), $G^3(\rho_{1|23}) = 1$. Thus, the arbitrary dimensional GHZ state satisfies the monogamy inequality (3) \[ G^3(\rho_{1|23}) > G^3(\rho_{12}) + G^3(\rho_{13}). \]

V. CONCLUSIONS

In this work we have derived an inequality which expresses a quantitative constraint of sharing entanglement in a pure tripartite qudit state. Our inequality involves G-concurrence as a measure of entanglement. It is worth noting that one can reproduce the monogamy inequality akin to [11] from (3). For qubit systems, G-concurrence and concurrence are synonymous, hence we trace back the CKW inequality. However, the inequality based on G-concurrence is valid for arbitrary dimensional systems. Intuitively, it seems that other monotones also satisfy certain algebraic constraints, and it would be an interesting line of study to further explore the monogamous nature of entanglement using other monotones. On the other hand, a universal monogamy inequality must involve arbitrary dimensional multipartite systems. Although the CKW inequality can be generalised for arbitrary N-partite states, it fails to work for certain $d > 2$ systems. So, it would be worthwhile to check whether the inequality (3) can be extended for the multipartite scenario.

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Appendix A: Completion of proof of Theorem 1

We rewrite Eq. (20) as

\[ G_d(\rho_{1|23}) \geq d^d \prod \sum \sum |a_{ijk}|^2 \]  \hspace{1cm} (A1)

\[ = d^d \sum |a_{0jk}|^2 \ldots \sum |a_{(d-1)jk}|^2 \]  \hspace{1cm} (A2)

\[ = d^d \left( \sum |a_{0j0}|^2 + \ldots + |a_{0j(d-1)}|^2 \right) \ldots \]

\[ \left( \sum |a_{(d-1)j0}|^2 + \ldots + |a_{(d-1)j(d-1)}|^2 \right) \]

\[ = d^d \left( |a_{000}|^2 + |a_{010}|^2 + \ldots + |a_{0(d-1)0}|^2 + \sum |a_{0j(d-1)}|^2 \right) \]

\[ \ldots \left( |a_{(d-1)00}|^2 + |a_{(d-1)10}|^2 + \ldots + |a_{(d-1)(d-1)0}|^2 + \sum |a_{(d-1)j(d-1)}|^2 \right) \]  \hspace{1cm} (A3)

Now, recall that \( \alpha = |a_{000}|^2 + |a_{010}|^2 + \ldots + |a_{0(d-1)0}|^2 \) and \( \beta = |a_{(d-1)00}|^2 + |a_{(d-1)10}|^2 + \ldots + |a_{(d-1)(d-1)0}|^2 \). So, we can simplify further as follows:

\[ G_d(\rho_{1|23}) \geq d^d \left( \alpha + \sum |a_{0j(d-1)}|^2 \right) \ldots \left( \beta + \sum |a_{(d-1)j(d-1)}|^2 \right) \]  \hspace{1cm} (A4)

\[ = d^d \alpha \beta + \ldots \]  \hspace{1cm} (A5)

\[ \geq d^d \alpha \beta \]  \hspace{1cm} (A6)

\[ \geq 2\alpha \beta \quad \forall \ d \geq 2 \]  \hspace{1cm} (A7)

\[ \geq G_d(\rho_{12}) + G_d(\rho_{13}) \]  \hspace{1cm} (A8)

This completes the proof of our Theorem (3).