A NOTE ON SOME PARTITIONS RELATED TO TERNARY QUADRATIC FORMS

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Abstract. We offer some partition functions related to ternary quadratic forms, and note on its asymptotic behavior. We offer these results as an application of a simple method related to conjugate Bailey pairs.

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1. Introduction

We write a positive ternary quadratic form as

\[ f(x, y, z) = ax^2 + by^2 + cz^2 + ryz + szx + txy, \]

and say it is primitive if \( \gcd(a, b, c, r, s, t) = 1 \). We recall some elementary facts about \( f(x, y, z) \). We first consider \((a, b, c, r, s, t) = (1, 1, 1, 0, 0, 0)\). The number \( S(X) \) of integers less than or equal to \( X \) which are representable as a sum of three squares is \( \sim \frac{5}{6}X \). (See [7] for an interesting discussion on this result and related material.) In general, it is known that \( (X \to \infty) \)

\[ \#\{n \leq X : n = f(x, y, z)\} \sim \alpha X, \]

where \( \alpha \in (0, 1] \). Fomenko [3] gives a nice introduction on primitive quadratic forms in \( k \geq 2 \) variables. In particular, we recommend the discussion mentioned there on ternary quadratic forms in relation to (1.2). See Blomer and Granville [2] for a discussion on estimates for quadratic forms and a discussion on Bernay’s result that \( (X \to \infty) \)

\[ \#\{n \leq X : n = ax^2 + by^2 + cxy\} \sim \beta \frac{X}{\sqrt{\log X}}, \]

for a positive constant \( \beta \).

In [5] we find partition functions related to (1.3), offering some interesting applications of \( q \)-series arising from Bailey pairs associated with indefinite binary quadratic
forms. A simple method used in [6], which uses a definition of a Bailey pair in conjunction with conjugate Bailey pairs, produced the tools needed to obtain $q$-series related to ternary quadratic forms. The motivation of this paper is to obtain partition functions with asymptotic behavior similar to (1.2), similar in spirit to $S(X)$.

2. Bailey’s Lemma and identities

Here we discuss the analytic tools used to obtain our generating functions. Put [4]

$$ (Z; q)_n = (Z)_n = (1 - Z)(1 - Zq) \cdots (1 - Zq^{n-1}) \). 

Bailey [1] introduced the idea of a pair of sequences $(\alpha_n, \beta_n)$, relative to $a$, which satisfy

$$ \beta_n(a, q) = \sum_{r \geq 0} \frac{\alpha_r(a, q)}{(aq)_{n+r}(q)_{n-r}}. $$

Further formulas needed for our study are a specialization of Bailey’s lemma [1]

$$ \sum_{n \geq 0} (q)_n (-1)^n \beta_n q^{n(n+1)/2} = \sum_{n \geq 0} (-1)^n q^{n(n+1)/2} \alpha_n, $$

and the pairs [6, eq(3.5)] $(\alpha_n(q^2, q^2), \beta_n(q^2, q^2))$,

$$ \beta_n(q^2, q^2) = \frac{q^n}{(-q)_{2n}}, $$

$$ \alpha_n(q^2, q^2) = (-1)^n q^{n(n-1)} \frac{(1 - q^{4n+2})}{1 - q^2} \sum_{i \geq 0} q^{i(i+1)/2} \sum_{2|j| \leq i} (-1)^j q^{-j(j-1)+2nj}, $$

and [6, eq(3.6)] $(\alpha_n, \beta_n)$,

$$ \beta_n = \frac{(q)_n (-1)^n q^{n(n-1)/2}}{(q)_{2n}}, $$

$$ \alpha_n = (-1)^n q^{n(n-1)/2} \frac{(1 - q^{2n+1})}{1 - q} \sum_{i \geq 0} (-1)^i q^{i(i+1)/2} (1 - q^{2i+1}) \sum_{|j| \leq i} (-1)^j q^{-j(j-1)/2+2nj}. $$

Inserting (2.3)–(2.4) and (2.5)–(2.6) into (2.2) gives us the following lemma containing our needed identities, which will be used in the next section.

**Lemma 2.1.** We have,

$$ \sum_{n \geq 0} \frac{(q^2; q^2)_n (-1)^n q^{n^2}}{(-q)_{2n}} = \sum_{n \geq 0} q^{2n^2} (1 - q^{4n+2}) \sum_{i \geq 0} q^{i(i+1)/2} \sum_{2|j| \leq i} (-1)^j q^{-j(j-1)+2nj}, $$
and

\begin{equation}
\sum_{n \geq 0} \frac{(q)_{n} q^{n^2}}{(q^{n+1})_{n}} = \sum_{n \geq 0} q^{n^2} (1-q^{2n+1}) \sum_{i \geq 0} (-1)^i q^{i(3i+1)/2} (1-q^{2i+1}) \sum_{|j| \leq i} (-1)^j q^{-j(j-1)/2+nj}.
\end{equation}

3. Partitions

We are concerned with one main generating function from which our partition theorems will follow.

Lemma 3.1. Let \( A_{k,m}(n) \) be the number of partitions of \( n \) where: (i) \( k \) appears at most twice. (ii) all parts \( < k \) appear at least twice and at most thrice. (iii) \( m \) is the number of parts \( > k \) and \( \leq 2k \). (iv) parts \( \geq k+1 \) or \( \leq 2k \) may appear any number of times. Further, let \( \bar{A}_{k,m}(n) \) be those partitions counted by \( A_{k,m}(n) \) with number of parts that are \( \leq k \) odd minus those with number of parts that are \( \leq k \) even. Then,

\begin{equation}
\sum_{n, m \geq 0} \bar{A}_{k,m}(n) a^m q^n = \frac{(q)_{k} q^{1+1+2+2+\cdots+(k-1)+(k-1)+k}}{(aq^{k+1})_k} = \frac{(q)_{k} q^{k(k+1)/2+k(k-1)/2}}{(aq^{k+1})_k}.
\end{equation}

We write out the numerator of the right side of (3.1) for the reader to show the weight associated with parts that are \( \leq k \) for the sake of clarity. Define the polynomial in \( x \) by \( f_k(x) := (x; x)_k x^{1+1+2+2+\cdots+(k-1)+(k-1)+k}. \) Then

\begin{equation}
f_1(x) = x^1 - x^{1+1}.
\end{equation}

\begin{equation}
f_2(x) = x^{1+1+2} - x^{1+1+1+2} - x^{1+1+2+2} + x^{1+1+1+2+2}.
\end{equation}

\begin{equation}
f_3(x) = x^{3+2+2+1+1} + x^{1+1+1+2+2+2+3} + x^{1+1+1+2+2+3+3} + x^{1+1+2+2+3+3} - x^{1+1+1+2+2+2+3} - x^{1+1+2+2+3+3} - x^{1+1+1+2+2+2+3} - x^{1+1+2+2+3+3}.
\end{equation}

It is seen from (3.2)–(3.4) that the weight is +1 when the number of parts is odd and −1 if the number of parts is even.

Put

\begin{equation}
\sum_{m, k \geq 0} \bar{A}_{k,m}(n) = B(n),
\end{equation}
and
\[ \sum_{m, k \geq 0} (-1)^{m+k} \tilde{A}_{k,m}(n) = \tilde{B}(n). \]

**Theorem 3.2.** We have,
\[ \sum_{n \geq 0} \frac{(q^2; q^2)_n (-1)^n q^{n^2}}{(-q)^{2n}} = \sum_{n \geq 0} \frac{(q)_n (-1)^n q^{n(n+1)/2 + n(n-1)/2}}{(-q^{n+1})_{2n}} = \sum_{n \geq 0} \tilde{B}(n) q^n, \]
\[ \sum_{n \geq 0} (q)_n q^{n^2} = \sum_{n \geq 0} B(n) q^n. \]

4. FURTHER THEOREMS

We state our main results (or corollaries) concerning \( B(n) \) and \( \tilde{B}(n) \). First we note that the second sum in (2.7) may be written in the form
\[ \sum_{n \geq 0} q^{2n^2} (1 - q^{4n+2}) \sum_{i \geq 0, j \in \mathbb{Z}} (-1)^i q^{(i+1)/2 + j(j+1) + (2i+1)|j| + 2jn}. \]

Secondly, the second sum in (2.8) may similarly be rewritten as a \( q \)-series related to a positive ternary quadratic form with \( \gcd = 1 \). Identities (2.7) (respectively (2.8)) tell us that \( \tilde{B}(n) \) (respectively \( B(n) \)) are nonzero only when \( n \) is represented by a ternary quadratic form. Consequently it is observed from (1.2) that the following two results follow.

**Theorem 4.1.** \# \{ \( n \leq X : B(n) \neq 0 \) \} \( \sim \alpha_1 X, \ (X \to \infty) \) where \( \alpha_1 \in (0, 1] \).

**Theorem 4.2.** \# \{ \( n \leq X : \tilde{B}(n) \neq 0 \) \} \( \sim \alpha_2 X, \ (X \to \infty) \) where \( \alpha_2 \in (0, 1] \).

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