CANTOR VERSUS CANTOR

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ABSTRACT. This paper examines the possibilities of extending Cantor’s two arguments on the uncountable nature of the set of real numbers to one of its proper denumerable subsets: the set of rational numbers. The paper proves that, unless certain restrictive conditions are satisfied, both extensions are possible. It is therefore indispensable to prove that those conditions are in fact satisfied in Cantor’s theory of transfinite sets. Otherwise that theory would be inconsistent.

Part I: Cantor’s 1874 argument

Cantor’s first proof of the uncountability of the set \( \mathbb{R} \) of the real numbers was published in the year 1874 [5], (French edition [6], Spanish edition [10]), in a short paper that also included a proof of the countable nature of the set \( \mathbb{A} \) of algebraic numbers, and then of the set \( \mathbb{Q} \) of rational numbers. The discussion that follows examines Cantor’s original argument and then the possibilities of its application to the set of rational numbers.

1. Cantor’s proof

1-1. Assume the set \( \mathbb{R} \) is denumerable. In these conditions there would be a one to one correspondence \( f \) between the set \( \mathbb{N} \) of natural numbers and \( \mathbb{R} \). Consequently, the elements of \( \mathbb{R} \) could be \( \omega \)-ordered as:

\[
    r_1, r_2, r_3, \ldots
\]

being \( r_i = f(i), \forall i \in \mathbb{N} \). Obviously, the sequence \( \langle r_i \rangle_{i \in \mathbb{N}} \) defined by \( f \) would contain all real numbers.

1-2. Consider now any real interval \( (a, b) \). Cantor’s 1874-argument consists of proving the existence of a real number \( \eta \) in \( (a, b) \) which is not in \( \langle r_i \rangle_{i \in \mathbb{N}} \). The existence of \( \eta \) would, in fact, prove the falseness of the assumption on the countable nature of \( \mathbb{R} \). The proof goes as follows.
1-3. Starting from $r_1$, find the first two elements of $\langle r_i \rangle_{i \in \mathbb{N}}$ within $(a, b)$. Denote the smaller of them by $a_1$ and the larger by $b_1$. Define the real interval $(a_1, b_1)$.

1-4. Starting from $r_1$, find the first two elements of $\langle r_i \rangle_{i \in \mathbb{N}}$ within $(a_1, b_1)$. Denote the smaller of them by $a_2$ and the larger by $b_2$. Define the real interval $(a_2, b_2)$. Evidently it holds: $(a_1, b_1) \supset (a_2, b_2)$.

1-5. Starting from $r_1$, find the first two elements of $\langle r_i \rangle_{i \in \mathbb{N}}$ within $(a_2, b_2)$. Denote the smaller of them by $a_3$ and the larger by $b_3$. Define the real interval $(a_3, b_3)$. Evidently it holds: $(a_1, b_1) \supset (a_2, b_2) \supset (a_3, b_3)$.

1-6. The continuation of the above procedure (RP from now on) defines a sequence of nested real intervals (RP-intervals):

$$(a_1, b_1) \supset (a_2, b_2) \supset (a_3, b_3) \supset \ldots$$

(II)

whose left endpoints $a_1, a_2, a_3, \ldots$ form a strictly increasing sequence, and whose right endpoints $b_1, b_2, b_3, \ldots$ form a strictly decreasing sequence, being every element of the first sequence smaller than every element of the second one.

1-7. From the $\omega$-ordering of $\langle r_i \rangle_{i \in \mathbb{N}}$ and the ordered way R defines the successive RP-intervals, it immediately follows that:

- If $r_n$ defines an endpoint $a_i$ or $b_i$, then it must hold $i \leq n$.
- If $r_n$ defines an endpoint $a_i$ or $b_i$, then $r_n$ will not lie within the successive intervals $(a_i, b_i), (a_{i+1}, b_{i+1}), (a_{i+2}, b_{i+2}), \ldots$

In consequence, we can ensure that, being $r_v$ any element of $\langle r_i \rangle_{i \in \mathbb{N}}$, it will never belong to any of the successive intervals $(a_v, b_v), (a_{v+1}, b_{v+1}), (a_{v+2}, b_{v+2}), \ldots$

1-8. The number of RP-intervals can be finite or infinite, and both possibilities have to be considered.

1-9. Assume the number of RP-intervals is finite. In this case there would be a last real interval $\langle a_n, b_n \rangle$ in the sequence. This last interval would contain, at best, one element $r_v$ of $\langle r_i \rangle_{i \in \mathbb{N}}$, otherwise it would be possible to define at least one new real interval $\langle a_{n+1}, b_{n+1} \rangle$. Let, therefore, $\eta$ be any element within $\langle a_n, b_n \rangle$ different from $r_v$ (in the case that $r_v$ exist). Evidently $\eta$ is a real number within $(a, b)$ which does not belong to the sequence $\langle r_i \rangle_{i \in \mathbb{N}}$. Thus, $\eta$ proves the falseness of Cantor’s initial assumption on the countable nature of $\mathbb{R}$: the sequence $\langle r_i \rangle_{i \in \mathbb{N}}$ does not contain all real numbers.

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1Including the case that RP defines no interval.

2Or the whole interval $(a, b)$ in the case that RP defines no interval.
Consider now the number of RP-intervals is infinite. Since the sequence \( \langle a_i \rangle_{i \in \mathbb{N}} \) is strictly increasing and upper bounded by every element of \( \langle b_i \rangle_{i \in \mathbb{N}} \), the limit \( L_a \) of \( \langle a_i \rangle_{i \in \mathbb{N}} \) does exist. On its part, the sequence \( \langle b_i \rangle_{i \in \mathbb{N}} \) is strictly decreasing and lower bounded by every element of \( \langle a_i \rangle_{i \in \mathbb{N}} \), in consequence the limit \( L_b \) of this sequence also exists. Taking into account that every \( a_i \) is less than every \( b_i \) it must hold: \( L_a \leq L_b \).

Assume that \( L_a < L_b \). In this case, any of the infinitely many elements within the real interval \((L_a, L_b)\) is a real number within \((a, b)\) which does not belong to the sequence \( \langle r_i \rangle_{i \in \mathbb{N}} \), and then a proof of the falseness of Cantor’s initial hypothesis on the countable nature of \( \mathbb{R} \).

Finally, assume that \( L_a = L_b = L \). It is immediate that \( L \) is a real number within \((a, b)\) which is not in \( \langle r_i \rangle_{i \in \mathbb{N}} \). In fact, assume that \( L \) is an element \( r_v \) of \( \langle r_i \rangle_{i \in \mathbb{N}} \). According to 1-7 \( r_v \) does not belong to any of the successive intervals \((a_v, b_v), (a_{v+1}, b_{v+1}), (a_{v+2}, b_{v+2}), \ldots \), while \( L \) belongs to all of them. Therefore, \( L \) is not \( r_v \). The limit \( L \) is a real number in \((a, b)\) which is not in \( \langle r_i \rangle_{i \in \mathbb{N}} \), and then a proof of the falseness of Cantor’s initial assumption on the countable nature of \( \mathbb{R} \).

2. Cantor’s 1874-argument applied to rational numbers

Next paragraphs 2-11/2-14 extend Cantor’s 1874-argument to the set \( \mathbb{Q} \) of rational numbers. Both arguments are identical except in the last step.

Assume the set \( \mathbb{Q} \) of rational numbers is denumerable. In these conditions there would be a one to one correspondence \( f \) between the set \( \mathbb{N} \) of natural numbers and \( \mathbb{Q} \) so that the elements of \( \mathbb{Q} \) could be \( \omega \)-ordered as:

\[
q_1, q_2, q_3, \ldots \quad (\text{III})
\]

being \( q_i = f(i), \forall i \in \mathbb{N} \). Obviously, the sequence \( \langle q_i \rangle_{i \in \mathbb{N}} \) defined by \( f \) would contain all rational numbers.

Consider now any rational interval \((a, b)\). We will try to prove the existence of a rational number \( \eta \) in \((a, b)\) which is not in \( \langle q_i \rangle_{i \in \mathbb{N}} \). Evidently, \( \eta \) would prove the falseness of our initial assumption on the countability of \( \mathbb{Q} \). The proof goes as follows.

Starting from \( q_1 \), find the first two elements of \( \langle q_i \rangle_{i \in \mathbb{N}} \) within \((a, b)\). Denote the smaller of them by \( a_1 \) and the larger by \( b_1 \). Define the rational interval \((a_1, b_1)\).
2-4. Starting from \( q_1 \), find the first two elements of \( \langle q_i \rangle_{i \in \mathbb{N}} \) within \((a_1, b_1)\). Denote the smaller of them by \( a_2 \) and the larger by \( b_2 \). Define the rational interval \((a_2, b_2)\). Evidently it holds: \((a_1, b_1) \supset (a_2, b_2)\).

2-5. Starting from \( q_1 \), find the first two elements of \( \langle q_i \rangle_{i \in \mathbb{N}} \) within \((a_2, b_2)\). Denote the smaller of them by \( a_3 \) and the larger by \( b_3 \). Define the rational interval \((a_3, b_3)\). Evidently it holds: \((a_1, b_1) \supset (a_2, b_2) \supset (a_3, b_3)\).

2-6. The continuation of the above procedure (QP from now on) defines a sequence of nested rational intervals (QP-intervals):

\[
(a_1, b_1) \supset (a_2, b_2) \supset (a_3, b_3) \supset \ldots
\]

(IV) whose left endpoints \( a_1, a_2, a_3 \ldots \) form a strictly increasing sequence, and whose right endpoints \( b_1, b_2, b_3 \ldots \) form a strictly decreasing sequence, being every element of the first sequence smaller than every element of the second one.

2-7. From the \( \omega \)-ordering of \( \langle q_i \rangle_{i \in \mathbb{N}} \) and the ordered way QP defines the successive QP-intervals, it immediately follows that:

(1) If \( q_n \) defines an endpoint \( a_i \) or \( b_i \), then it must hold \( i \leq n \).

(2) If \( q_n \) defines an endpoint \( a_i \) or \( b_i \), then \( q_n \) will not lie within the successive intervals \((a_i, b_i), (a_{i+1}, b_{i+1}), (a_{i+2}, b_{i+2}), \ldots\)

In consequence, we can ensure that, being \( q_v \) any element of \( \langle q_i \rangle_{i \in \mathbb{N}} \), it will never belong to any of the successive intervals \((a_v, b_v), (a_{v+1}, b_{v+1}), (a_{v+2}, b_{v+2}), \ldots\)

2-8. The number of QP-intervals can be finite or infinite, and both possibilities have to be considered.

2-9. Assume the number of QP-intervals is finite\(^3\). In this case there would be a last rational interval\(^4\) \((a_n, b_n)\) in the sequence. This last interval would contain, at best, one element \( q_v \) of \( \langle q_i \rangle_{i \in \mathbb{N}} \), otherwise it would be possible to define at least one new rational interval \((a_{n+1}, b_{n+1})\). Let, therefore, \( \eta \) be any element within \((a_n, b_n)\) different from \( q_v \) (in the case that \( q_v \) exist). Evidently \( \eta \) is a rational number within \((a, b)\) which does not belong to the sequence \( \langle q_i \rangle_{i \in \mathbb{N}} \). Thus, \( \eta \) proves the falseness of the initial assumption on the countable nature of \( \mathbb{Q} \): the sequence \( \langle q_i \rangle_{i \in \mathbb{N}} \) does not contain all rational numbers.

\(^3\)Including the case that QP defines no interval.

\(^4\)Or the whole interval \((a, b)\) in the case that QP defines no interval.
2-10. Consider now the number of QP-intervals is infinite. Since the sequence \( \langle a_i \rangle_{i \in \mathbb{N}} \) is strictly increasing and upper bounded by every element of \( \langle b_i \rangle_{i \in \mathbb{N}} \), the real limit \( L_a \) of \( \langle a_i \rangle_{i \in \mathbb{N}} \) does exist. On its part, the sequence \( \langle b_i \rangle_{i \in \mathbb{N}} \) is strictly decreasing and lower bounded by every element of \( \langle a_i \rangle_{i \in \mathbb{N}} \), in consequence the real limit \( L_b \) of this sequence also exists. Taking into account that every \( a_i \) is less than every \( b_i \) it must hold: \( L_a \leq L_b \), being \( L_a \) and \( L_b \) two real (rational or irrational) numbers.

2-11. Assume that \( L_a < L_b \). In this case, any of the infinitely many rationals within the real interval \( (L_a, L_b) \) is a rational number within \( (a, b) \) which does not belong to the sequence \( \langle q_i \rangle_{i \in \mathbb{N}} \), and then a proof of the falseness of the initial hypothesis on the countable nature of \( \mathbb{Q} \).

2-12. Finally, assume that \( L_a = L_b = L \). It is immediate that \( L \) is a real number within the real interval \( (a, b) \) which is not in \( \langle q_i \rangle_{i \in \mathbb{N}} \). In fact, If \( L \) is irrational then it is clear that it will not in \( \langle q_i \rangle_{i \in \mathbb{N}} \); assume then \( L \) is rational, and assume also it is an element \( q_v \) of \( \langle q_i \rangle_{i \in \mathbb{N}} \). According to 2-7, \( q_v \) does not belong to any of the successive intervals \( (a_v, b_v), (a_{v+1}, b_{v+1}), (a_{v+2}, b_{v+2}), \ldots \), while \( L \) belongs to all of them. Therefore, \( L \) is not \( q_v \). The limit \( L \) is a real number (rational or irrational) in the real interval \( (a, b) \) which is not in \( \langle q_i \rangle_{i \in \mathbb{N}} \). Thus, if \( L \) were rational then our initial assumption on the countable nature of \( \mathbb{Q} \) would be false.

2-13. Under the assumption 2-12 let \( \langle c_i \rangle_{i \in \mathbb{N}} \) be the sequence of elements of \( \langle q_i \rangle_{i \in \mathbb{N}} \) within \( (a, b) \) which are not endpoints of the QP-intervals. If there would exist an \( v \) such that:

\[
(a_v, a_{v+1}) \cap \langle c_i \rangle_{i \in \mathbb{N}} \neq \emptyset
\]

then there would exist at least a rational \( \eta \) within \( (a_v, a_{v+1}) \) which is not an element of \( \langle c_i \rangle_{i \in \mathbb{N}} \) and such that \( a_v < \eta < a_{v+1} \). Now then, being \( a_v \) the immediate predecessor of \( a_{v+1} \) in \( \langle a_i \rangle_{i \in \mathbb{N}} \), and being this sequence strictly increasing it would be impossible for \( \eta \) to be a left endpoint of any QP-interval. And being all right points \( \langle b_i \rangle_{i \in \mathbb{N}} \) greater than \( a_{v+1} \), and then than \( \eta \), this last number could not be an element of \( \langle b_i \rangle_{i \in \mathbb{N}} \) either. Thus, \( \eta \) would be a rational number within \( (a, b) \) which is not in \( \langle q_i \rangle_{i \in \mathbb{N}} \), and then a proof of the falseness of our initial assumption on the countable nature of \( \mathbb{Q} \).

2-14. The same argument 2-13 applies to each of the successive intervals \( (b_{i+1}, b_i) \) defined by the successive \( \langle b_i \rangle_{i \in \mathbb{N}} \), the right endpoints of the QP-intervals.

\[\text{\textsuperscript{5}}\text{That any real interval contains infinitely many rationals is an elementary result that can easily be proved.}\]
3. Conclusion on Cantor’s 1874 argument

We have just proved in 3-1/3-14 the alternatives of Cantor 1874-argument on the cardinality of the real numbers can be applied to the set \( \mathbb{Q} \) of rational numbers, except the last one, that applies only if the common limit of the sequences of left and right endpoints of the QP-intervals is rational. Evidently, if Cantor’s 1874-argument could be extended to the rational numbers we would have a contradiction: the set \( \mathbb{Q} \) would and would not be denumerable. Accordingly, in order to ensure the impossibility of that contradiction, each of the following points have to be proved:

3-1. Whatsoever be the rational interval \((a, b)\) and whatsoever be the reordering of \(\langle q_i \rangle_{i \in \mathbb{N}}\), it must holds:

1. The number of QP-intervals can never be finite.
2. The sequences of endpoints \(\langle a_i \rangle_{i \in \mathbb{N}}\) and \(\langle b_i \rangle_{i \in \mathbb{N}}\) can never have different limits.
3. The common limit of \(\langle a_i \rangle_{i \in \mathbb{N}}\) and \(\langle b_i \rangle_{i \in \mathbb{N}}\) can never be rational.
4. Being \(\langle c_i \rangle_{i \in \mathbb{I}}\) the sequence defined in 2-13 it must hold:

\[ (a, a_1) \cap \langle c_i \rangle_{i \in \mathbb{I}} = \emptyset; \quad (b_1, b) \cap \langle c_i \rangle_{i \in \mathbb{I}} = \emptyset \quad \text{(VI)} \]

\[ \forall i \in \mathbb{N} : \begin{cases} (a_i, a_{i+1}) \cap \langle c_i \rangle_{i \in \mathbb{I}} = \emptyset \\ (b_{i+1}, b_i) \cap \langle c_i \rangle_{i \in \mathbb{I}} = \emptyset \end{cases} \quad \text{(VII)} \]

Until those proofs be given, Cantor’s 1874-argument should be suspended, and the possibility of a contradiction involving the foundation of (infinitist) set theory should be considered.

4. A short epilog to Cantor’s 1874-argument

The following short argument proves the existence of rational numbers within any rational interval \((a, b)\) which are not in the sequence \(\langle q_i \rangle_{i \in \mathbb{N}}\) of the above extension of Cantor’s 1874-argument.

4-1. Let \(x\) be a rational variable initially defined as \(b\), the right endpoint of any rational interval \((a, b)\), and \(\langle q_i \rangle_{i \in \mathbb{N}}\) the sequence of rational numbers defined in 2-11. Then, consider the following procedure P:

\[ q_i \in (a, b) \land q_i < x \Rightarrow x = q_i; \quad i = 1, 2, 3, \ldots \quad \text{(VIII)} \]

that compares \(x\) with the successive elements of \(\langle q_i \rangle_{i \in \mathbb{N}}\) that belong to \((a, b)\), and defines \(x\) as the compared element each time the compared element is less than \(x\).
4-2. Whatsoever be the finite or infinite number of times that P defines \( x \), and whatsoever be the current value of \( x \) once performed P, \( x \) will be a rational number within \((a, b)\) because, by definition, it is always defined as a rational number within \((a, b)\), and only as a rational number within \((a, b)\). Consider then the rational interval \((a, x)\) and any element \( \eta \) within \((a, x)\), as for instance \( \frac{1}{2}(a + x) \). It evidently holds \( \eta \in (a, b) \) and \( \eta < x \). But \( \eta \) does not belong to \( \langle q_i \rangle_{i \in \mathbb{N}} \) for if that were the case there would exist a \( q_v \) such that \( \eta = q_v \), and then we would have \( q_v < x \), which is impossible because according to P it must be \( x \leq q_v, \forall v \in \mathbb{N} \). The rational \( \eta \) proves, therefore, the existence of rational numbers within \((a, b)\) which are not in \( \langle q_i \rangle_{i \in \mathbb{N}} \), and then the falseness of our initial assumption on the countable nature of \( Q \).

4-3. Infinitist mathematics assumes the performance of procedures of infinitely many steps, as the above Cantor’s argument, extension and epilog; and many, many others. Mathematics is not usually concerned with the way those procedures could be, in fact, carried out; it is only concerned with the consistency of the involved arguments. When the result of an infinite procedure is an infinite set (or sequence), then the set (or sequence) is assumed to be a complete infinite totality, which implies the completion of the infinitely many steps of the defining procedure. That said, it could be appropriate to recall we dispose of a formal theory whose main objective is just the analysis of the performance of those infinite procedures in a finite or infinite interval of time (supertask and bifurcated supertask respectively).

4-4. Notice that conclusion 4-2 is not derived from the successive performed operations (Benacerraf criticism of supertasks) but from the own definition of P: \( x \) can only be defined as a rational number in \((a, b)\). And notice also that if we assume the actual infinity then we can also assume the existence of a conceptual universe in which supertasks and hypercomputations are possible.

**Part II: Cantor’s diagonal method**

Cantor’s diagonal argument makes use of a hypothetical table \( T \) containing all real numbers within the real interval \((0, 1)\). That table can be easily redefined in order to ensure it contains at least all rational numbers within \((0, 1)\). In these conditions, could the rows of \( T \) be reordered so that the resulting diagonal and antidiagonal were rational numbers? In that case not only the set of real

\[6\text{A procedure of countably many steps could be accomplished by performing each step } s_i \text{ at the precise instant } t_i \text{ of an } \omega \text{-ordered sequence of instants } \langle t_i \rangle_{i \in \mathbb{N}} \text{ within a finite interval of time } (t_a, t_b) \text{ whose limit is } t_b. \text{ See for instance } [3, 11, 17, 12, 18, 1, 16].\]
numbers but also, and for the same reason, the set of rational numbers would be non denumerable. And then we would have a contradiction since Cantor also proved the set of rational numbers is denumerable. Should, therefore, Cantor’s diagonal argument be suspended until it be proved the impossibility of such a reordering? Is that reordering possible? The discussion that follows addresses both questions.

5. An elementary previous result

5-5. Let \( M \) be the set of all real numbers in the real interval \((0, 1)\) expressed in decimal notation and completed, in the cases of finitely many decimal digits, with infinitely many 0’s in the right side of their decimal expansions. The subset of all rational numbers in \( M \) will be denoted by \( M_Q \).

5-6. For every natural number \( n \) there are infinitely many elements in \( M_Q \) with the same decimal digit \( d_n \) in the same \( n \)-th position of its decimal expansion. In fact, consider the following element \( r_0 \) of \( M_Q \): \( r_0 = 0.d_1d_2\ldots d_n000\ldots \) where \( d_1, d_2, \ldots d_n \) are any decimal digits. From \( r_0 \) we define the sequence:

\[
\begin{align*}
 r_1 &= 0.d_1d_2\ldots d_n1000\ldots \\
 r_2 &= 0.d_1d_2\ldots d_n11000\ldots \\
 r_3 &= 0.d_1d_2\ldots d_n111000\ldots \\
 \ldots \\
 r_i &= 0.d_1d_2\ldots d_n(i).1000\ldots \\
 \ldots
\end{align*}
\]

The function \( f \) from \( \mathbb{N} \) (the set of natural numbers) to \( M_Q \) defined according to:

\[
f(i) = r_i, \quad \forall i \in \mathbb{N}
\]

proves the existence of a denumerable subset \( f(\mathbb{N}) \) of \( M_Q \), all of whose elements have the same decimal digit \( d_n \) in the same \( n \)-th position of its decimal expansion.

6. Discussion

6-1. Next paragraph 6-2 defines a table \( T \) of the real numbers within \((0, 1)\) which, according to Cantor [4], contains at least all rational numbers in \((0, 1)\). Subsequent paragraphs 6-3/6-5 consider the consequences of a rational diagonal and antidiagonal in \( T \).
Cantor versus Cantor

6-2. Cantor’s set \( M \) is the union of two disjoint sets: the denumerable set \( M_Q \) of all rational numbers in \((0, 1)\) and the set \( M_I \) of all irrationals in the same interval \((0, 1)\). Assume, as Cantor did in 1891 [7], \( M \) is denumerable. In these conditions, it is evident that \( M_I \) would also be denumerable. Let then \( g \) be a bijection between \( \mathbb{N} \) and \( M_Q \), and \( h \) a bijection between \( \mathbb{N} \) and \( M_I \). From \( g \) and \( h \) we define a one to one correspondence \( f \) between \( \mathbb{N} \) and \( M \) according to:

\[
\begin{align*}
    f(2n - 1) &= g(n) \\
    f(2n) &= h(n)
\end{align*}
\]

\( \forall n \in \mathbb{N} \) (XIV)

We can therefore consider the \( \omega \)-ordered table \( T \) whose successive rows \( r_1, r_2, r_3 \ldots \) are just \( f(1), f(2), f(3) \ldots \). By construction, and being \( M_Q \) denumerable, \( T \) contains a denumerable subtable with all rational numbers in \((0, 1)\).

6-3. The diagonal of \( T \) is a real number \( D = 0.d_1d_2d_3 \ldots \) whose \( n \)-th decimal digit \( d_{nn} \) is the \( n \)-th decimal digit of the \( n \)-th row \( r_n \) of \( T \). Cantor successfully proved [7] the existence of another real number in \( M \) derived from \( D \), the antidiagonal \( D^- \), which by construction cannot be in \( T \). In consequence, \( M \) cannot be denumerable as was assumed to be (Cantor’s diagonal argument, an impeccable Modus Tollens[7]).

6-4. Since \( D^- \) is a real number in \((0, 1)\), it will be either rational or irrational. But if it were rational then, and for the same reason as in the case of \( M \), the subset \( M_Q \) of all rational numbers in \( M \) would also be non denumerable. The problem here is that Cantor proved the set \( \mathbb{Q} \) of all rational numbers, and therefore \( M_Q \), is denumerable [5].

6-5. According to 6-4, if it were possible to reorder the rows of \( T \) in such a way that a rational antidiagonal could be defined, then we would have two contradictory results: the set \( \mathbb{Q} \) of rational numbers would and would not be denumerable. Both results can be considered as proved by Cantor, although the second one only as an unexpected (and so far unknown) consequence of his famous diagonal method. Accordingly, we can state our first conclusion:

\[
\begin{itemize}
    \item Cantor’s diagonal argument and all its formal consequences should be suspended until it be proved the impossibility of defining a rational antidiagonal in all possible reorderings of \( T \)’s rows.
\end{itemize}
\]

6-6. Next paragraphs 6-7/6-17 examines the possibilities and consequences of reordering the rows of \( T \) in the sense indicated in 6-5.

7The critiques of Cantor’s diagonal argument are invariably related to constructionist aspects which are not pertinent with the formal structure of Cantor’s demonstration.
6-7. A formal consequence of the existence of complete infinite totalities (actual infinity) is the existence of $\omega$-ordered sequences \[8\], \[9\]. In an $\omega$-ordered sequence, as our table $T$, every element -whatever it be- will always be preceded by a finite number of elements and succeeded by an infinite number of such elements. We will see now a conflictive consequence of this immense and suspicious asymmetry.

6-8. A row $r_i$ of $T$ will be said $n$-modular if its $n$-th decimal digit is $(n \mod 10)$. This means that a row is, for instance, 2348-modular if its 2348-th decimal digit is 8; or that it is 453-modular if its 453-th decimal digit is 3. If a row $r_n$ is $n$-modular (being $n$ in $n$-modular the same as $n$ in $r_n$) it will be said D-modular. For instance, the rows:

$$r_1 = 0.1007647464749943400034577774413\ldots \quad (XV)$$
$$r_2 = 0.2200045667778943000000000000000\ldots \quad (XVI)$$
$$r_3 = 0.0030000000000000000000000000000\ldots \quad (XVII)$$
$$r_4 = 0.100400000111111111111000000000000\ldots \quad (XVIII)$$
$$r_9 = 0.12345678999999966666666666333\ldots \quad (XIX)$$

are all of them D-modular.

6-9. Consider now the following permutation $P$ of the rows $\langle r_i \rangle_{i \in \mathbb{N}}$ of $T$:

- For each successive row $r_i$ in $T$:
  - If $r_i$ is D-modular then let it unchanged.
  - If $r_i$ is not D-modular then exchange it with any succeeding i-modular row $r_{j,j>i}$, provided that at least one of its succeeding rows be i-modular.

Notice that, thanks to the condition $j > i$ (in $r_{j,j>i}$), once a row $r_i$ has been exchanged with a succeeding i-modular row $r_{j,j>i}$, it remains D-modular and unaffected by the subsequent exchanges. And notice also the exchanges do not alter the nature $\omega$-ordered of the table: $P$ does not modify the $\omega$-ordered set $\mathbb{N}$ of indexes but, in any case, the real numbers indexed by its elements.

6-10. It is immediate to prove that each and every row of $T$ becomes D-modular as a consequence of permutation $P$. In fact, let us assume that a

\[8\]Infinitist mathematics assumes that procedures of infinitely many successive steps, as, for instance, $\omega$-recursive definitions, can be performed without worrying about the way they could be carried out. Supertask theory provides, on the other hand, a way by which any $\omega$-ordered sequence of theoretical actions could be accomplished in a finite interval of time, or even in an infinite interval of time in the case of bifurcated supertasks (hypercomputation) \[14\], \[15\], \[16\]. But this is not a problem infinitist mathematics is usually concerned with.
row \( r_n \) does not become \( D \)-modular as a consequence of \( P \). This means both that \( r_n \) was not \( D \)-modular in \( T \) and that it could not be exchanged with a succeeding \( n \)-modular row. Now then, all \( n \)-modular rows have the same digit \((n \mod 10)\) in the same \( n \)-th position of its decimal expansion, and according to 6-6 there are infinitely many rational numbers with the same digit in the same position of its decimal expansion, whatsoever be the digit and the position. Accordingly, since \( n \) is finite, the row \( r_n \) is preceded by a finite number and succeeded by an infinite number of \( n \)-modular rows. Any of these infinitely many succeeding \( n \)-modular rows had to be exchanged with \( r_n \). It is therefore impossible that \( r_n \) be not \( D \)-modular. In consequence (Modus Tollens), each and every row \( r_i \) becomes \( D \)-modular as a consequence of \( P \).

6-11. It is worth noting the result proved in 6-10 is a formal consequence of the fact that every row \( r_n \) of \( T \) is always preceded by a finite number of \( n \)-modular rows and succeeded by an infinite number of such \( n \)-modular rows. This monstrous asymmetry is an inevitable side effect of \( \omega \)-order which in turn is a consequence of assuming the existence of complete infinite totalities. Notice also 6-10 is not a constructive argument but a simple Modus Tollens.

6-12. Let \( T_p \) be the table resulting from permutation \( P \). Being all its rows \( D \)-modular, its diagonal \( D \) will be the rational number \( 0.\hat{1}234567890 \). It is now immediate to define infinitely many rational antidiagonals. In fact, let \( p_0 \) be the period 1234567890 of \( D \). We are interested in periods of ten digits none of which coincide in position with the digits of \( p_0 \), as is the case, for instance, of 0123456789 or 4545454545 (= 45). The number of those periods is \( 9^{10} \). Let us select the above two examples and denote them by \( p_1 \) and \( p_2 \) respectively \((p_1 = 0123456789; p_2 = 45)\). Now we define the \( \omega \)-ordered sequence of rational antidiagonals \( \langle A_i \rangle_{i \in \mathbb{N}} \) by:

\[
\forall n \in \mathbb{N} : A_n = 0.p_1p_1 \ldots p_1p_2
\]

whose elements cannot be in \( T_p \) for the same constructive reasons as in the case of Cantor diagonal method. And being all of them rational numbers, we must conclude that \( \mathbb{M}_\mathbb{Q} \) and its superset \( \mathbb{Q} \) are both non denumerable.

6-13. Permutation \( P \) allows to develop other arguments whose conclusions also suggest the inconsistency of the actual infinity. For instance, it is clear that rows as 0.21, 0.01\( \hat{2} \), 0.0001\( \hat{2} \), \ldots and infinitely many others, can never become \( D \)-modular, and then we would have to admit the absurdity that \( P \) makes all of them disappear from the table. In fact, let \( n \) be any natural number and assumes that, for instance, 0.21 is the \( n \)-th row of \( T_p \). Since \( n \) is finite, 0.21 will be preceded by a finite number of rows and succeeded by an infinite number of rows, infinitely many of which will be just \( n \)-modular according to 6-6. In
consequence $0.2\bar{1}$ was exchanged with any of those $n$-modular rows, and then it cannot be the $n$-th row of $T_p$. Thus, and being $n$ any natural number, we must conclude $0.2\bar{1}$ has disappeared from the table!

6-14. The above absurdity is the sort of things one can expect from a list in which each and every element has finitely many predecessors and infinitely many successors. A list in which, in spite of having infinitely many successive elements, it is impossible to reach an element with infinitely many predecessors (what, evidently, makes the above arguments possible). A list, in short, that is simultaneously complete (as the actual infinity requires) and uncompletable (because no last element completes it).

6-15. Permutation $P$ can even be considered as a case of supertask (hyper-computation). In fact, let $\langle t_i \rangle_{i \in \mathbb{N}}$ be an $\omega$-ordered sequence of instants within a finite interval of time $(t_a, t_b)$, being $t_b$ the limit of the sequence. Assume that $P$ is applied to each row $r_i$ just at the precise instant $t_i$. Consequently, $r_i$ will remain unchanged if it is D-modular (or if it is not D-modular and no i-modular row succeeds it) or it will be exchanged with any succeeding i-modular row. At $t_b$ permutation $P$ will have been applied to every row of $T$ as the bijection $f(t_i) = r_i$ proves.

6-16. Assume that at $t_b$, once accomplished the hypercomputation $P$, $T_p$ contains a row $r_n$ which is not D-modular. This row, whatsoever it be, will be preceded by a finite number of rows and succeeded by an infinite number of rows, infinitely many of which are $n$-modular, and then interchangeable with $r_n$. Thus, either $r_n$ is D-modular in $T_p$, or it has magically disappeared from the table!

6-17. To be simultaneously complete and uncompletable, as is the case of any $\omega$-ordered object, could be, after all, contradictory.

7. A final remark

7-1. Unnecessary as it may seem, let me end by recalling that an argument cannot be refuted by another different argument. In W. Hodges words: [15, p. 4]

How does anybody get into a state of mind where they persuade themselves that you can criticize an argument by suggesting a different argument which doesn’t reach the same conclusion?
This inadmissible strategy is frequently used in discussions related to infinity, for instance to refute Cantor’s arguments on the uncountability of the real numbers. However, to refute an argument means to indicate where and why that argument fails. If two arguments lead to contradictory conclusions, they simply prove the existence of a contradiction.

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