Can Everyone Benefit from Social Integration?*

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Abstract

We study the integration of stable marriage problems (SMPs) of equal sizes into an extended society. We show that it is impossible to make every agent weakly better off by merging all SMPs if the matching that occurs before and after integration is stable. We show that integration always weakly benefits at least one-half of the society, which implies that it can be implemented by majority voting.

A stronger pro-integration condition requires that no agent is hurt whenever any number of SMPs merge sequentially. This property, that we call integration monotonicity, is even incompatible with Pareto efficiency.

KEYWORDS: Social integration, integration monotonicity, matching schemes, two-sided matching.

JEL CLASSIFICATION: C78 (matching theory).

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1. Introduction

The stable marriage problem (SMP), proposed by Gale and Shapley (1962), combines a rich mathematical structure with important real-life applications. The SMP consists of two equally sized groups of agents, the men and the women. Each man wants to marry a woman, and each woman wants to marry a man. Both men and women have a strict preference ordering over the set of potential spouses. The marriage terminology is a metaphor for arbitrary two-sided matching problems, such as the assignment of workers to firms, students to universities, doctors to hospitals, and so on.

Gale and Shapley were interested in systematic rules to decide who should marry whom. We refer to such rules as matchings. They recognized that a matching should be “in accordance with agents’ preferences” and should satisfy “some agreed-upon criterion of fairness”. Many such criteria can be formulated. Two standard ones are core-like stability and Pareto optimality. We refer to these requirements as efficiency criteria.

Other desirable criteria require that matchings incentivise the integration of different SMPs of equal size (communities) into an extended SMP (a society). We refer to them as pro-integration criteria. Roughly, they impose that whenever disjoint communities of same size merge as one, every agent weakly prefers the outcome obtained in an integrated society to the one obtained in a segregated one. We introduce two pro-integration properties in this paper.

The first one is integration monotonicity. We illustrate it with an example. It requires that, given a society with three communities $A$, $B$, and $C$, the matching obtained in societies $A \cup B$, $A \cup C$, and $B \cup C$ is weakly preferred by every agent to the one obtained in each society $A$, $B$, or $C$ alone. Furthermore, it also imposes that the matching obtained in the completely integrated society $A \cup B \cup C$ is weakly preferred by everybody to the one obtained in societies $A \cup B$, $A \cup C$, and $B \cup C$.

The second pro-integration property we study is weak integration monotonicity. It only requires that the matching obtained in the completely integrated society $A \cup B \cup C$ is weakly preferred by everyone to the one produced in each isolated community. It imposes no comparisons between the matching obtained in $A \cup B \cup C$ and those obtained in $A \cup B$, $A \cup C$, or $B \cup C$. 
In the Gale-Shapley original definition, a matching is a rule that specifies who should marry whom in a two-sided community. To define pro-integration rules, we use the concept of a matching scheme, which defines a matching for every combination of disjoint communities. The concept of matching schemes is based on assignments schemes used in cooperative game theory (Dutta and Ray, 1989; Sprumont, 1990). Assignment schemes specify an allocation of jointly generated surplus within every subset of the grand coalition in a cooperative game. Similarly, a matching scheme specifies a matching for every combination of communities of equal size. A matching mechanism is a systematic procedure to assign a matching scheme to every extended SMP.

This paper aims at improving our understanding of social integration dynamics in the context of SMPs. Particularly, we are interested in understanding which efficiency criteria are compatible with pro-integration requirements. We are also interested in quantifying the welfare losses that some agents may experience after integration occurs. Societies could achieve integration easily if such welfare losses were small, even in the absence of integration monotonic matching mechanisms.

1.1 Overview of Results

We obtain two negative and two positive results. The main negative result states that there is no Pareto optimal and integration monotonic matching mechanism (Proposition 2). To prove this result, we provide an SMP in which integration monotonicity forces agents to marry the worst partner for them in every combination of communities. Thus, integration monotonic matchings mechanisms fare badly with respect to efficiency.

A second, less surprising negative result states that weak integration monotonicity cannot be combined with stability (Proposition 1). It is nevertheless compatible with Pareto optimality. Pareto optimal and weakly integration monotonic matching mechanisms resemble matching with veto power. In those, agents may block improving changes for their spouses whenever such exchange gives them a less desirable partner.

Our impossibility results show that stable matching mechanisms must always hurt someone. Yet, our first positive result establishes that stable matching mechanisms never hurt more than one-half of the society after complete integration occurs (Proposition 3). This implies that integration can be approved by majority voting. Simulation
results using 1,000 random SMPs suggest that the actual fraction of agents who suffer welfare losses after integration remains relatively constant around 25%.

In our second result we quantify the expected welfare changes after integration occurs in random SMPS. We measure welfare using the expected ranking of an agent’s spouse. Using these computations, we show that in expectation both men and women obtain a better spouse after social integration occurs. We use the seminal work of Pittel (1989) to obtain this result.

Restricting our attention to those hurt by integration, we find very asymmetric welfare changes between men and women. The side who proposes in the well-known deferred acceptance algorithm, say women, have minor losses. On the contrary, men suffer large welfare reductions. Yet, 1,000 simulations of random SMPs suggest that the welfare losses of both sides become smaller with respect to the size of the SMP as \( n \) grows. This finding suggests that social integration is easier to achieve in large societies.

Simulations have been used in the past to give a better understanding of impossibility results in matching theory. For example, a well-known theorem states that there is no matching rule that neither men nor woman can manipulate by misrepresenting their preferences. At best, a matching rule can only be immune to manipulation by one side of the society, say women, but there are always men who benefit from lying (Gale and Sotomayor, 1985).

Although this impossibility theorem may seem very strong, Teo et al. (2001) use simulations of SMPs to show that the number of men who benefit from lying is very small. They simulate 1,000 SMPs with random and independent preferences. They find that in 74% of the cases, there is no profitable manipulation for any man. In their simulations, the average percentage of men who benefit from misrepresenting their preferences is barely over 5%.

Their results show that simulations are a helpful tool to understand the scope of impossibility theorems in two-sided matching.
1.2 Related Literature

There is a well-known monotonicity result in the matching literature. It establishes that, whenever a woman is added to an SMP, every man becomes weakly better off. Similarly, adding an extra woman makes every existing woman weakly worse off. This result extends to many-to-one SMPs. The references include Theorem 5 in Kelso and Crawford (1982), Theorems 1 and 2 in Crawford (1991), and Theorems 2.25 and 2.26 in Roth and Sotomayor (1992).

However, it was until very recently that we were able to quantify the expected welfare gains and losses of adding an extra agent to an SMP. To describe these welfare changes, we need to present a seminal result by Pittel (1989). He studies an SMP with \( n \) men and \( n \) women in which agents’ preferences are drawn independently at random. He analyses the expected ranking of an agent’s spouse. A ranking of 1 means the best possible spouse, whereas a ranking of \( n \) means the worst possible one. He shows that in the women-optimal stable matching, women get a partner ranked \( \log(n) \) with high probability (whp). Men get a partner ranked \( n/\log(n) \) whp. In an SMP with 100 men and 100 women, women would get their 5th best choice whp (\( \log(100)=4.6 \)). Men would be in a much worse situation. They would obtain their 22nd best choice whp (\( 100/\log(100)=21.7 \)).

If we add an extra woman to those 100 existing ones, keeping the population of men unchanged, we observe that the asymmetric ranking of an agent’s spouse reverses. Now men obtain a partner ranked \( \log(n) \) whp (the 5th best in our example), whereas women obtain a spouse ranked \( n/\log(n) \) whp (the 22nd best). This interesting result is due to Ashlagi et al. (2017).

The results we have summarized consider what occurs when one agent is added to the problem. We analyse instead what occurs when communities of equal sizes merge. These communities contain both men and women.

A more general monotonicity property allows groups of arbitrary size to integrate. This notion has not been studied in matching problems. It was first proposed by Sprumont (1990) for cooperative games with transferable utility. He calls this property population monotonicity. It requires that whenever two communities of arbitrary size join as one, there exists a way to share the surplus generated by its members so that every agent becomes weakly better off. He shows that the extended Shapley value of
every convex cooperative game is population monotonic. He provides a characterization of population monotonicity using monotonic games with veto players.\footnote{A game is monotonic whenever adding any agent to a group always increases the surplus generated by the group. A game has a veto player if any coalition without such player produces no surplus.} Sprumont’s work only applies to games with transferable utility. This large class of games does not include the SMP.

More recently, Chambers and Hayashi (2017) use the population monotonicity axiom (which they call integration monotonicity) to study exchange economies. Their definition of integration monotonicity corresponds to the one we use, although they allow communities of arbitrary size to merge. They find that an allocation rule satisfies Pareto optimality and integration monotonicity only if the order in which communities integrate matters. Interestingly, they show that whenever integration occurs it must hurt at least one-third of the society if equals are treated equally.

The term population monotonicity is sometimes used to refer to the solidarity axiom, introduced by Thomson (1983). This axiom is suitable for scenarios in which a society produces a fixed amount of welfare, no matter how many agents belong to it. Examples of those include bargaining and fair division. The solidarity axiom requires that if an agent joins a society, the new assignment does not increase the utility received by any existing member. In other words, the burden imposed by a new agent should be shared by all existing members of a society. Population monotonicity and solidarity are similar concepts but have very different interpretations.

Another similar property is resource monotonicity. It was introduced by Moulin and Thomson (1988) in the context of fair division of a growing set of resources. It imposes that an increase in the society’s endowment weakly benefits everyone. They show that resource monotonicity is incompatible with Pareto optimality if agents are guaranteed an equal split of the society’s endowment. Earlier, Aumann and Peleg (1974) provided an example showing that the competitive equilibrium is not resource monotonic.

### 2. Model

A **community** $C$ consists of $n$ men and $n$ women. There are $\kappa$ disjoint communities. A **society** $S$ is the set of all communities. A **population** $P$ is a set of communities in
The power set $2^S$ is the set of all populations in $S$. $M_P$ and $W_P$ denote the men and women in $P$.

Each man $m$ (resp. woman $w$) has strict preferences over the set of all women in the society $W^S$ (resp. men $M^S$). We write $w \succ_m w'$ to denote that $m$ prefers $w$ to $w'$. Similarly, $w \succ_m w'$ if either $w \succ_m w'$ or $w = w'$. We represent women’s preferences using the same notation. We call $\succ = (\succ_x)_{x \in S}$ a preference profile.

An extended SMP is a pair $(S, \succ)$. An extended SMP consists of $\kappa$ disjoint SMPs $(C, (\succ_x)_{x \in C})$.

Given a population $P$, a matching $\mu : P \to P$ is a function that assigns a man to every woman and a woman to every man. It satisfies $\mu(\mu(x)) = x$ for every agent (man or woman) $x \in P$. A matching scheme $\sigma : P \times 2^S \to P$ is a function that specifies a matching $\sigma(\cdot, P)$ for every $P \in 2^S$.

A matching mechanism $\Gamma$ is a function $\succ \to \Gamma(\succ)$ from the set of all preference profiles to the set of all matching schemes.

To give an example of an extended SMP and a matching scheme, consider a society with $n = 1$ and two communities, $A$ and $B$. Agents’ preferences are given by

\[
\begin{align*}
w^A & \succ_{m^A} w^B \\
w^A & \succ_{m^B} w^B \\
m^B & \succ_{w^A} m^A \\
m^B & \succ_{w^B} m^A
\end{align*}
\]

With these preferences, the man from community $A$ prefers the woman from community $A$ over the woman of community $B$, and so on. A possible matching scheme $\sigma$ is $\sigma(m^A, A) = w^A$, $\sigma(m^B, B) = w^B$, $\sigma(m^A, S) = w^B$ and $\sigma(m^B, S) = w^A$.

2.1 Efficient Matching Schemes

We consider two well-known efficiency properties. The first one is stability. Besides its intuitive appeal and close relationship to the core of a cooperative game, the concept of stability is a good predictor of the success of several real-life matching mechanisms (Roth, 2002).
Definition 1 (Stability). A matching $\mu$ is stable if there is no man $m$ and woman $w$ that are not married to each other such that $w \succ_m \mu(m)$ and $m \succ_w \mu(w)$. Any such pair $(m, w)$ is called a blocking pair.

A weaker efficiency property is Pareto optimality. It is arguably the most basic fairness consideration in economics. It only requires that there is no way to make one agent better off without hurting any other agent.

Definition 2 (Pareto optimality). A matching $\mu$ is Pareto optimal if there is no other matching $\mu'$ such that $\mu'(x) \succeq_x \mu(x)$ for every $x \in P$, and $\mu'(y) \succ_y \mu(y)$ for some $y \in P$.

The properties of matchings trivially extend to matching schemes and mechanisms. A matching scheme $\sigma$ is stable (resp. Pareto optimal) if the matching $\sigma(\cdot, P)$ is stable (resp. Pareto optimal) in every population $P \in 2^S$. A matching mechanism $\Gamma$ is stable (resp. Pareto optimal) if the matching scheme $\Gamma(\succ)$ is stable (resp. Pareto optimal) with respect to the preference profile $\succ$.

### 2.2 Pro-Integration Matching Schemes

We introduce two pro-integration properties. The first one is integration monotonicity. It requires that no agent is hurt whenever any two disjoint populations integrate.

Definition 3 (Integration Monotonicity). A matching scheme $\sigma$ is integration monotonic if $\forall P, P' \subseteq 2^S$ such that $P \cap P' = \emptyset$ and $\forall x \in P$, $\sigma(x, P \cup P') \succeq_x \sigma(x, P)$.

If an extended SMP admits an integration monotonic matching scheme, then we can guarantee that the integration of any number of communities would be weakly beneficial for every agent. It is natural to expect that social integration would occur in its corresponding society.

Unfortunately, we will show in the next Section that integration monotonicity is a very strict requirement. That is why we consider a milder criterion called weak integration monotonicity. This property requires that no agent is hurt after all communities integrate, compared to his position in his segregated community.

Definition 4 (Weak Integration Monotonicity). A matching scheme $\sigma$ is weakly integration monotonic if $\forall C \in S$ and $\forall x \in C$, $\sigma(x, S) \succeq_x \sigma(x, C)$. 


Weak integration monotonicity compares the spouse that an agent obtains when she is only allowed to marry someone from her community against the one she obtains when agents are allowed to marry any potential partner in the full society.

If a matching scheme is weakly integration monotonic, then agents know that complete social integration will be beneficial for them. However, it may be that some agents obtain a worst match if only partial integration occurs, i.e. if only some but not all communities integrate.

A matching mechanism \( \Gamma \) is (weakly) integration monotonic if the matching scheme \( \Gamma(\succ) \) is (weakly) integration monotonic with respect to the preference profile \( \succ \).

### 3. Results

It would be ideal that we can find a stable and integration matching scheme for every extended SMP. But unfortunately, extended SMPs may even lack a stable and weakly integration monotonic matching scheme.

**Proposition 1.** For each society with at least two communities, no stable matching mechanism is weakly integration monotonic.

**Proof.** Let \( A \) and \( B \) be the two communities, each with one man and one woman. Consider the following preference profile \( \succ \)

\[
\begin{align*}
 w^A &\succ_{m^A} w^B & m^B &\succ_{w^A} m^A \\
 w^A &\succ_{m^B} w^B & m^B &\succ_{w^B} m^A
\end{align*}
\]

All women prefer \( m^B \) and all men prefer \( w^A \). Stability of a matching scheme requires that \( \sigma(w^A, A) = m^A \), \( \sigma(w^B, B) = m^B \), and \( \sigma(w^A, A \cup B) = m^B \). However, both man \( m^A \) and woman \( w^B \) obtain a worse partner when communities \( A \) and \( B \) merge.

Therefore, for the preference profile \( \succ \) any stable matching mechanism \( \Gamma \) produces a matching scheme \( \Gamma(\succ) \) that is not weakly integration monotonic. \( \square \)

Proposition 1 is an expected result. As we mentioned earlier, a well-known result in two-sided matching establishes that adding a man to an SMP makes every existing men
weakly worse off. Similarly, adding a new woman makes every existing women weakly worse off. Therefore, it is not surprising that adding a man-woman pair can generate welfare losses for some agents.

However, Proposition 2 shows that even a considerably weaker efficiency property, such as Pareto optimality, is also at odds with a strong pro-integration requirement.

**Proposition 2.** For each society with at least three communities, no Pareto optimal matching mechanism is integration monotonic.

*Proof.* Let $A$, $B$ and $C$ be the three communities, each with one man and one woman. Consider the following preference profile $\succ$

\[
\begin{align*}
   w_B &\succ_{m_A} w_C \succ_{m_A} w_A & m_B &\succ_{w_A} m_C \succ_{w_A} m_A \\
   w_C &\succ_{m_B} w_A \succ_{m_B} w_B & m_C &\succ_{w_B} m_A \succ_{w_B} m_B \\
   w_A &\succ_{m_C} w_B \succ_{m_C} w_C & m_A &\succ_{w_C} m_B \succ_{w_C} m_C
\end{align*}
\]

Agents’ preferences are such that agents of community $A$ prefer those from $B$, agents from $B$ prefer those from $C$, and agents from $C$ prefer to those from $A$. There are only two Pareto optimal matching schemes. Both require $\sigma(w^A, A \cup B) = m^B$, $\sigma(w^B, B \cup C) = m^C$, and $\sigma(w^C, C \cup A) = m^A$. Note that in any Pareto optimal matching scheme there is always a community that gets her first choice whenever we merge only two communities.

Integration monotonicity implies that when we aggregate all communities, they should all do at least as good as when only two societies merge. This is clearly impossible, because some agent would not be able to obtain their first choice any more. The only matching scheme that satisfies integration monotonicity is the segregated one, in which each man always marries the woman from his own race. This matching is clearly not Pareto optimal, because every agent gets her worst possible partner after integration occurs.

Therefore, for the preference profile $\succ$ any integration monotonic matching mechanism $\Gamma$ produces a matching scheme $\Gamma(\succ)$ that is not Pareto optimal. □

Propositions 1 and 2 show that efficiency and pro-integration properties are at odds with each other. They can only be satisfied together in their weak versions.
Matching schemes that are Pareto optimal and weakly integration monotonic always exist. We can construct them as follows. Let \(\sigma(\cdot, P)\) be an arbitrary Pareto optimal matching for every \(P \in 2^S\), except for \(S\). For \(S\), let \(\sigma(\cdot, S)\) be an arbitrary Pareto optimal matching unless there exists an agent that vetoes it, i.e. an agent that prefers \(\sigma(x, C) \succ_x \sigma(x, S)\). If every such matching is blocked by vetoes, assign \(\sigma(x, C) = \sigma(x, S)\) \(\forall C \in S, \forall x \in C\).

4. Integration in Stable Matching Schemes

In the previous Section we insisted in matching mechanisms that were at least weakly integration monotonic. We found that those cannot be stable. Alternatively, we may focus on stable matching mechanisms and analyse how close they are to being weakly integration monotonic. We follow this route in the remainder of the paper. We focus on how many agents get hurt by complete integration and in the magnitude of their welfare losses in stable matching mechanisms. Henceforth we only focus on complete integration, i.e. all communities merging at once.

4.1 How Many People Dislike Integration?

In the proof of Proposition 1, half of the society becomes worse off after complete integration takes place in a stable matching scheme. Our next Proposition shows that this fraction can never be larger than one-half.

**Proposition 3.** In any stable matching scheme \(\sigma^*\),

\[
|\{x \in S \mid \sigma^*(x, C) \succ_x \sigma^*(x, S)\}| \leq \kappa n
\]

*The bound is tight.*

**Proof.** As a reminder, we have \(\kappa\) communities, each with \(n\) men and \(n\) women. Hence, \(\kappa n\) is exactly half the number of agents in \(S\). Let us partition \(S\) into three sets \(B^0\), \(B^+\) and \(B^-\), defined as

\[
B^0 = \{x \in S \mid \sigma^*(x, S) = \sigma^*(x, C)\}
\]

\[
B^+ = \{x \in S \mid \sigma^*(x, S) \succ_x \sigma^*(x, C)\}
\]

\[
B^- = \{x \in S \mid \sigma^*(x, C) \succ_x \sigma^*(x, S)\}
\]

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So $B^0$ is the set of people who keep the same partner after integration, $B^+$ are those who prefer their “integrated” partner, and $B^-$ are those who prefer the “segregated” partner.

Consider an arbitrary couple $(x, \sigma^*(x, C))$. If $x \in B^-$, $\sigma^*(x, C) \in B^+$ because otherwise $(x, \sigma^*(x, C))$ constitutes a blocking pair to the matching $\sigma^*(\cdot, S)$, contradicting the fact that $\sigma^*$ is a stable matching scheme. It follows that $|B^+| \geq |B^-|$, and thus $|B^0| + |B^+| \geq |B^-|$, completing the proof.

Proposition 3 is interesting because it tells us that integration is always approved by a weak majority. This result and its proof can be extended almost verbatim to the case when some communities are larger than others, or when communities have more men than women and vice versa.

A natural conjecture related to Proposition 3 is that the integration would be approved by an arbitrarily large super-majority when $n$ grows. The reasoning behind is that agents gain access to a large set of potential spouses when integration occurs, and thus are likely to be better off after integration occurs.

However, simulation results suggest this conjecture is wrong. We simulate an extended SMP with $n$ up to 500 and $\kappa$ up to 5. We assign agents with random and independent preferences. We use the women-optimal stable matching (WOSM) mechanism in the simulations.\(^2\) Table 1 suggests that the fraction of agents hurt by integration remains around 25% when $n$ grows.\(^3\)

4.2 Gains from Integration

In this Subsection, we compare agents’ welfare before and after integration occurs. We evaluate welfare using the expected ranking of an agent’s spouse. We continue to assume that agents are endowed with random and independent preferences. We relax the assumption of independent preferences in Section 5.

We use a seminal result of Pittel (1989) for an SMP with $n$ men and $n$ women. Pittel shows that in the WOSM women get a partner ranked $\log(n)$ whp, whereas men

\(^2\)We could have similarly selected the men-optimal stable matching. We pick the women-optimal stable matching to have a consistent selection among the set of stable matchings.

\(^3\)The code is available at www.josueortega.com. For $\kappa = 5, n = 500$ it took 2 days to run using the high performance facilities at the University of Glasgow.
Table 1: How many people (in percentage) prefer segregation?
Average over a thousand simulations with preferences drawn uniformly at random. Standard errors in parenthesis.

| κ \ n | 50   | 100  | 500  |
|-------|------|------|------|
| 2     | 25.4 | 25.8 | 25.5 |
|       | (0.04) | (0.02) | (0.01) |
| 3     | 25.4 | 25.8 | 26   |
|       | (0.03) | (0.01) | (0.01) |
| 4     | 24.8 | 25.1 | 25.6 |
|       | (0.02) | (0.01) | (0.01) |
| 5     | 24.3 | 24.6 | 25.1 |
|       | (0.02) | (0.01) | (0.01) |

get a partner ranked \( n/\log(n) \) whp. A partner ranked 1st is the best possible match, whereas a partner ranked \( n \) means is the worst possible one.

It is easy to adapt Pittel’s result to extended SMPs. His result directly implies that women get a partner ranked \( \log(\kappa n) \) whp after integration occurs. Men get someone ranked \( \kappa n/\log(\kappa n) \) whp. Pittel’s result also imply that before integration occurs women and men obtain an expected spouse ranked \( \log(n) \) and \( n/\log(n) \) whp within their own community. We need to know where such agent is in the complete ranking of all possible partners.

To answer this question, suppose that a man is ranked \( q \) among men in his community. That means that a random agent from another community has \( q/(n+1) \) chances of being higher ranked than him. There are \( n(\kappa - 1) \) men from other races. On average, \( \frac{n(\kappa - 1)}{n+1} \) men will be ranked higher than him. Remember that there were already \( q \) men in his own community better ranked than him. This implies that his expected ranking is \( q + \frac{n(\kappa - 1)}{n+1} = \frac{q(n+1)}{n+1} \).

Substituting \( q \) for \( \log(n) \) and \( n/\log(n) \) respectively, we obtain the expected welfare changes for women and men. These appear in expressions (1) and (2), respectively.

\[
\begin{align*}
(1) \quad & \log(n) \left( \frac{\kappa n + 1}{n+1} \right) - \log(\kappa n) \\
& \quad \text{ranking w. segregation} - \text{ranking w. integration} \\
& = \frac{n(\kappa - 1)}{n+1} \log(n) - \log(\kappa) \\
\end{align*}
\]

\[
\begin{align*}
(2) \quad & \frac{n}{\log(n)} \left( \frac{\kappa n + 1}{n+1} \right) - \frac{\kappa n}{\log(\kappa n)} \\
& \quad \text{ranking w. segregation} - \text{ranking w. integration} \\
& = H[(\kappa n + 1) \log(\kappa n) - \kappa(n + 1) \log(n)]
\end{align*}
\]
where $H = n/(n + 1) \log(n) \log(\kappa n)$]. Expressions (1) and (2) are positive for sensible\(^4\) values of $\kappa$ and $n$, and thus both men and women benefit from social integration in expectation. The normalized gains from integration by gender appear in Figure 2.

\[\begin{align*}
\text{(a) } \kappa &= 2 \\
\text{(b) } \kappa &= 5
\end{align*}\]

Figure 1: Individual expected gains from integration divided by $\kappa n$, by gender

4.3 Who Prefers Segregation?

Another natural conjecture is that people who oppose social integration have a lower expected desirability than those who do not. In other words, they are usually ranked lower in the preference lists of their potential spouses. This new conjecture is false too.\(^5\) Table 2 describes the expected rank of people who prefer segregation. It is immediate that they have the same expected ranking as a random agent, suggesting that people who prefer segregation are not particularly undesirable agents.

4.4 Quantifying Welfare Losses

Finally, we look at the welfare losses suffered by those who prefer segregation when integration realizes, in terms of ranking of their current partner. If their loss was relatively small it would be a strong argument for saying that the impossibilities described in Section 3 can be circumvented in real life. Table 3 presents an interesting observation: the side of the society who received the proposals in the deferred acceptance algorithm (in this case men) get severely hurt by integration. Conversely, women suffer a moderate hurt at most.

\[\text{For example, expressions (1) and (2) are positive for } k \geq 2 \text{ and } n \geq 2k.\]

\[\text{With correlated preferences, women who prefer segregation tend to be lower ranked than their peers who prefer integration. We discuss this finding further in Section 5.}\]
Table 2: Average rank of people who prefer segregation, by gender.
Average over a thousand simulations with preferences drawn uniformly at random. Standard errors in parenthesis.

| κ \ n | 50  | 100  | 500  |
|-------|-----|------|------|
|       | women | men | women | men | women | men |
| 2     | 50.7 | 50.5 | 100.6 | 100.5 | 500.5 | 500.5 |
|       | (0.3) | (0.07) | (0.22) | (0.1) | (0.21) | (0.09) |
| 3     | 75.7 | 75.5 | 150.7 | 150.5 | 750.6 | 750.5 |
|       | (0.25) | (0.09) | (0.24) | (0.08) | (0.25) | (0.08) |
| 4     | 100.6 | 100.5 | 200.6 | 200.5 | 1000.7 | 1000.5 |
|       | (0.25) | (0.09) | (0.27) | (0.07) | (0.24) | (0.08) |
| 5     | 125.6 | 125.5 | 250.6 | 250.5 | 1250.6 | 1250.5 |
|       | (0.29) | (0.08) | (0.29) | (0.08) | (0.28) | (0.08) |

Table 3: Average welfare loss by people who prefer segregation, by gender.
Average over a thousand simulations. Welfare loss measured in difference in ranking of partners. Standard errors in parenthesis.

| r \ n | 50  | 100  | 500  |
|-------|-----|------|------|
|       | women | men | women | men | women | men |
| 2     | 4.9 | 19.7 | 5.7 | 34.9 | 7.4 | 136.6 |
|       | (0.91) | (9.47) | (0.95) | (22.91) | (1.04) | (246.83) |
| 3     | 5.4 | 27.4 | 6.2 | 49.2 | 7.9 | 193.8 |
|       | (1.01) | (14.57) | (0.96) | (39.22) | (0.83) | (409.86) |
| 4     | 5.7 | 35.0 | 6.5 | 62.1 | 8.1 | 250.8 |
|       | (1.08) | (21.67) | (1.03) | (60.48) | (0.84) | (623.95) |
| 5     | 6.0 | 41.8 | 6.8 | 74.9 | 8.4 | 303.3 |
|       | (1.07) | (28.26) | (1.14) | (84.86) | (0.88) | (709.15) |

Interestingly, the welfare losses of both sides become smaller with respect to the size of the grand society as n grows. This finding suggests that integration could be more easily implemented in large societies.

5. Two Concluding Remarks

1. Although we have focused on one-to-one SMPs, our results extend to the many-to-one case. The extension of Propositions 1 and 2 is trivial. Proposition 3, which establishes that at most one-half of the society gets hurt by integration, can also be extended when agents’ preferences are responsive, defined below.

Consider an extended SMP with colleges G and students T instead of men and women. A matching pairs students with colleges, such that a college can be assigned
to many students but a student can only be assigned to one college. Each college \( g \) can accept a fixed number of students \( q_g \), and has a strict preference over all subsets of students up to size \( q_g \). A community is a pair of \( n \) students and a set of colleges that can accept \( n \) students in total. A society is a set of \( \kappa \) disjoint communities.

The preferences of a college \( g \) are responsive if, \( \forall T' \subset T \) such that \( |T'| < q_g \), and \( \forall t,t' \in T, T' \cup \{t\} \succ_g T' \cup \{t'\} \) if and only if \( \{t\} \succ_g \{t'\} \). If all colleges have responsive preferences, the many-to-one SMP can be rephrased as an equivalent one-to-one matching problem of assigning university’s desks to students (Roth and Sotomayor, 1992). Our Proposition 3 directly implies that at most one-half of the agents of the corresponding one-to-one problem get hurt by complete integration.

2. The simulation results were obtained using random and independent preferences, following the literature on random SMPs. However, correlated preferences are evident in some matching environments like school choice. We perform a robustness check of our simulation results introducing correlation in preferences as follows.

We define a status quo in preferences for both men and women. The status quo is a random order over all possible partners. Each agent’s preferences is identical to the status quo, except perhaps in at most \( c \) positions. For example, if \( c = 2 \) and the status quo over six partners is \( 1 \succ 2 \succ 3 \succ 4 \succ 5 \succ 6 \), an agent’s preferences could be \( 1 \succ 2 \succ 6 \succ 4 \succ 5 \succ 3 \) but not \( 2 \succ 3 \succ 1 \succ 4 \succ 5 \succ 6 \). The swaps in agents’ preferences are chosen randomly. The expected correlation coefficient between each agents’ preferences and the status quo equals \( \rho = 1 - \frac{c}{kn} \). The simulation results using correlated preferences appear in Table 4.\(^6\)

Firstly, we find that the fraction of people against integration still remains around 25%. Secondly, the expected ranking of women hurt by integration increases, whereas the one of men decreases. This finding suggests that the proposer’s advantage becomes larger with correlated preferences. This advantage achieves its peak with \( \rho = 0.7 \), but declines when preferences become more correlated. This is due to the fact that the size of the set of stable matching becomes smaller when preferences are highly correlated (Holzman and Samet, 2014).\(^7\) For the same reason, women’s welfare losses become larger and comparable to those suffered by men as \( \rho \) increases.

\(^6\)Alternatively, one could model correlation in preferences by defining a general ranking subject to idiosyncratic shocks, as in Lee (2017) and Che and Tercieux (2017).

\(^7\)The set of stable matchings even becomes a singleton for a variety of highly correlated preferences (Eeckhout, 2000; Clark, 2006; Ortega and Hergovich, 2017).
Table 4: Statistics for correlated preferences, $n = 100$, $r = 2$.
Average over a thousand simulations. Standard errors in parenthesis.

| $\rho$ | % worse | Exp. ranking women | Exp. ranking men | Welfare loss women | Welfare loss men |
|--------|---------|---------------------|------------------|-------------------|-----------------|
| 0.9    | 24.6    | 105.6               | 93.9             | 30.5              | 31.9            |
|        | (0.02)  | (59.22)             | (50.78)          | (17.75)           | (24.36)         |
| 0.7    | 26.1    | 114.3               | 89.2             | 17.2              | 34.7            |
|        | (0.02)  | (25.51)             | (20.07)          | (8.45)            | (25.87)         |
| 0.5    | 26.1    | 109.7               | 93.4             | 10.9              | 34.8            |
|        | (0.02)  | (15.81)             | (11.85)          | (3.46)            | (22.99)         |
| 0.3    | 25.9    | 104.4               | 97.5             | 7.8               | 34.1            |
|        | (0.02)  | (6.46)              | (4.43)           | (1.83)            | (21.68)         |
| 0.1    | 25.8    | 101.1               | 100.1            | 6.1               | 34.1            |
|        | (0.02)  | (1.05)              | (0.62)           | (1.24)            | (22.35)         |
| 0      | 25.8    | 100.6               | 100.5            | 5.7               | 34.9            |
|        | (0.02)  | (0.22)              | (0.1)            | (0.95)            | (22.91)         |

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