On the Complexity of the Singly Connected Vertex Deletion.

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Singly Connected Vertex Deletion problem
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Singly Connected Digraphs

- Singly Connected Digraphs: For every pair $u$ and $v$ of digraph $D$, at most one (directed) path from $u$ to $v$. 
Singly Connected Vertex Deletion problem

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- Singly Connected Digraphs: For every pair \( u \) and \( v \) of digraph \( D \), at most one (directed) path from \( u \) to \( v \).

Figure: Examples of Singly Connected Digraphs.
Lemma 1

Digraph $D$ is not singly connected $\iff \exists$ two vertices $u, v$ with two internally vertex disjoint paths from $u$ to $v$. 
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Figure: Obstruction of SCVD
Singly Connected Vertex Deletion problem

Problem statement

- **Input**

  - digraph $D$
  - positive integer $k$

  Does there exist $S \subseteq V(D)$ such that $|S| \leq k$ and $D \setminus S$ is singly connected?
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  ▶ digraph $D$
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Main Results

- SCVD in $\alpha$-bounded digraphs
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- SCVD in DAG local tournaments
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- SCVD in in-tournaments
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- Linear kernel for SCVD in local tournaments
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Motivation
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- Singly connectivity in undirected graphs means forests
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- 2 versions of FVS in undirected graphs:
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- 2 versions of FVS in undirected graphs:
  - Deletion to cycle free graphs: DFVS
  - Deletion to singly connected graphs: SCVD
Motivation

- Similar Obstruction Structure.
Motivation

- Similar Obstruction Structure.

(a) Obstruction of SCVD

(b) Obstruction of DFVS
Singly Connected Vertex Deletion problem

Related Work
Singly Connected Vertex Deletion problem

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- Recognition of Singly Connected digraphs: $O(s.t + m)$ (Dietzfelbinger and Jaberi).
Singly Connected Vertex Deletion problem

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- Recognition of Singly Connected digraphs: $O(s.t + m)$ (Dietzfelbinger and Jaberi).
- NP Completeness of SCVD in digraphs: Dietzfelbinger and Jaberi.
SCVD in $\alpha$-bounded digraphs
$\alpha$-bounded digraphs

$\alpha$-bounded digraphs are digraphs whose maximum independent set size is bounded by $\alpha$. 

**SCVD in $\alpha$-bounded digraphs**

$\alpha$-bounded digraphs
SCVD in $\alpha$-bounded digraphs

Tournaments

Lemma 2

*Tournament on four or more vertices is not singly connected.*
SCVD in $\alpha$-bounded digraphs

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- Algorithm for SCVD in tournaments is implied trivially by the lemma.
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- Check if $n - k \leq 3$, otherwise output no.
SCVD in \( \alpha \)-bounded digraphs

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- Algorithm for SCVD in tournaments is implied trivially by the lemma.
- Check if \( n \leq k \leq 3 \), otherwise output no.
- Iterate over all three subset induced digraphs.

The runtime of the algorithm is \( O(n^3) \).
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SCVD in $\alpha$-bounded digraphs.

Definitions

**Forward arcs.** For a path $P = v_1 \ldots v_l$ in digraph $D$, $(v_i, v_j)$ is a forward arc with respect to $P$ if $j > i + 1$. 
SCVD in \( \alpha \)-bounded digraphs.

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![Figure: Forward arcs in a path.](image-url)
SCVD in \( \alpha \)-bounded digraphs.

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**Figure:** Backward arcs in a path.
Observations

Observation 1

If a digraph $D$ has a path $P$ such that there is a forward arc with respect to $P$, then $D$ is not singly connected.
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*If a digraph $D$ has a path $P$ such that there is a forward arc with respect to $P$, then $D$ is not singly connected.*
Lemma 3

A tournament on four or more vertices is not singly connected.
SCVD in $\alpha$-bounded digraphs

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A tournament on four or more vertices is not singly connected.
- Long paths and bounded independence number
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- FVS in tournaments: NP hard
SCVD in $\alpha$-bounded digraphs

Algo Lemma

Lemma (Algo Lemma)

For every $\alpha \in \mathbb{N}$, every $\alpha$-bounded digraph with at least $\alpha(2\alpha + 4)$ vertices is not singly connected.

▶ The algorithm for SCVD trivially follows from Algo Lemma. It is exactly like SCVD in tournaments.

▶ Runtime: $O((n^{\alpha(2\alpha + 4)} - 1))$. 
SCVD in $\alpha$-bounded digraphs

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- The algorithm for SCVD trivially follows from Algo Lemma. It is exactly like SCVD in tournaments.
- Runtime: $O(n^{\alpha(2\alpha+4)-1})$. 
SCVD in $\alpha$-bounded digraphs.

Lemmas

**Theorem 4 (Gallai Milgram)**

*Every directed graph $D$ has a path cover $\mathcal{P}$ and a maximal independent set $\{v_P | P \in \mathcal{P}\}$ such that $v_P \in P$ for every $P \in \mathcal{P}$.*
SCVD in \( \alpha \)-bounded digraphs.

Lemmas

**Theorem 4 (Gallai Milgram)**

*Every directed graph \( D \) has a path cover \( \mathcal{P} \) and a maximal independent set \( \{ v_P \mid P \in \mathcal{P} \} \) such that \( v_P \in P \) for every \( P \in \mathcal{P} \).*
Observation 2

For a path $P$ in a singly connected digraph $D$, $v_P \in V(P)$ can have at most two backward arcs incident on it.
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*For a path $P$ in a singly connected digraph $D$, $v_P \in V(P)$ can have at most two backward arcs incident on it.*

- Assume not. Let there be three backward arcs incident in $v_P$. There are four possibilities.
SCVD in $\alpha$-bounded digraphs.

Observations

(a) Possibility 1
SCVD in $\alpha$-bounded digraphs.

Observations

(a) Possibility 1

(b) Possibility 2
SCVD in $\alpha$-bounded digraphs.

Observations

(a) Possibility 1

(b) Possibility 2

(c) Possibility 3
SCVD in $\alpha$-bounded digraphs.

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(a) Possibility 1

(b) Possibility 2

(c) Possibility 3

(d) Possibility 4
SCVD in $\alpha$-bounded digraphs.

Observations

(a) 2 possible paths from $v_j$ to $v_P$

(b) 2 possible paths from $v_P$ to $v_j$

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Observations

**Figure**: Only possible configuration for 2 backward arcs incident on $v$
Observation 3

For a vertex $v \notin V(P)$, there can be at most two edges between $v$ and $V(P)$. 
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*For a vertex* $v \notin V(P)$, *there can be at most two edges between* $v$ *and* $V(P)$. 
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![Diagram showing Observation 3](attachment:observation_3_diagram.png)
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> Assume not. Let $D$ be $\alpha$-bounded singly connected digraph with $|V(D)| \geq \alpha(2\alpha + 4)$. 
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- By Gallai Milgram, there is a path cover $\mathcal{P}$, and maximal independent set $I = \{v_P | P \in \mathcal{P}\}$.
SCVD in $\alpha$-bounded digraphs

Proof Outline of algo lemma

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- By Gallai Milgram, there is a path cover $\mathcal{P}$, and maximal independent set $I = \{v_P | P \in \mathcal{P}\}$.
- By averaging argument, there is a path $P$ such that $|V(P)| \geq 2\alpha + 4$. 
SCVD in $\alpha$-bounded digraphs

Proof Outline of algo lemma

$V_P$
SCVD in $\alpha$-bounded digraphs

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▶ $|N[I] \cap V(P)| \leq 2(\alpha - 1) + 4 + 1 = 2\alpha + 3$ (in the underlying undirected graph).

▶ There exists a vertex $v \in V(P)$ which is not adjacent to $I$.

▶ This contradicts our assumption that $I$ is a maximal independent set.
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Open Problems

▶ Is there a polynomial time algorithm for SCVD in local tournaments?

▶ Is there a $o(2^n)$ exact algorithm for SCVD in general digraphs?

▶ Is SCVD problem in general digraphs parameterized by the solution set size FPT?
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Thank You.