Local Covering Optimality of Lattices: 
Leech Lattice versus Root Lattice $E_8$

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Abstract

We show that the Leech lattice gives a sphere covering which is locally least dense among lattice coverings. We show that a similar result is false for the root lattice $E_8$. For this we construct a less dense covering lattice whose Delone subdivision has a common refinement with the Delone subdivision of $E_8$. The new lattice yields a sphere covering which is more than 12% less dense than the formerly best known given by the lattice $A_8^*$. Currently, the Leech lattice is the first and only known example of a locally optimal lattice covering having a non-simplicial Delone subdivision. We hereby in particular answer a question of Dickson posed in 1968. By showing that the Leech lattice is rigid our answer is even strongest possible in a sense.

1 Introduction

The Leech lattice is the exceptional lattice in dimension 24. Soon after its discovery by Leech [Lee67] it was conjectured that it is extremal for several geometric problems in $\mathbb{R}^{24}$: the kissing number problem, the sphere packing problem, and the sphere covering problem.

In 1979, Odlyzko and Sloane and independently Levenshtein solved the kissing number problem in dimension 24 by showing that the Leech lattice gives an optimal solution. Two years later, Bannai and Sloane showed that it gives the unique solution up to isometries (see [CS88], Ch. 13, 14). Unlike the kissing number problem, the other two problems are still open.

Recently, Cohn and Kumar [CK04] showed that the Leech lattice gives the unique densest lattice sphere packing in $\mathbb{R}^{24}$. Furthermore they showed, that the density of any sphere packing (without restriction to lattices) in $\mathbb{R}^{24}$ cannot exceed the one given by the Leech lattice by a factor of more than $1 + 1.65 \cdot 10^{-30}$.

At the moment it is not clear how one can prove a corresponding result for the sphere covering problem. In this paper we take a first step into this direction by showing

**Theorem 1.1.** The Leech lattice gives a sphere covering which is locally optimal among lattices.

In Section 2 we give precise definitions of “locally optimality” and of all other terms needed.

Surprisingly, a result similar to Theorem 1.1 does not hold for the root lattice $E_8$, which is the exceptional lattice in dimension 8. As the Leech lattice, $E_8$ gives the unique solution to the kissing number problem (see [CS88], Ch. 13, 14) and is conjectured to be extremal for the sphere packing problem. Blichfeldt [Bli34] showed that it gives an optimal lattice sphere packing. Later, Vetchinkin [Vet82] showed that

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sphere packings of all other $8$-dimensional lattices are less dense. Besides giving another proof of this, Cohn and Kumar [CK04] demonstrated that the density of a sphere packing in $\mathbb{R}^8$ cannot exceed the one of $E_8$ by a factor of more than $1 + 10^{-14}$. In contrast to the Leech lattice, $E_8$ cannot be an extremal sphere covering in its dimension though. Conway and Sloane note in [CS88], Ch. 2: "It is surprising that $A_8^*$, with [covering density] $\Theta = 3.6658\ldots$, is better than $E_8$, which has $\Theta = 4.0587\ldots$". Here, we show that $E_8$ does not even give a locally optimal lattice sphere covering, by constructing a new $8$-dimensional lattice, which yields a less dense sphere covering and whose Delone subdivision has a common refinement with the one of $E_8$.

**Theorem 1.2.** There exists a lattice with covering density $\Theta < 3.2013$ whose Delone subdivision has a common refinement with the one of $E_8$.

Note that this new sphere covering beats the old record holder $A_8^*$ in dimension $8$. The proof of Theorem 1.2 relies on computational methods we developed in [SV04] and up to now we can only give an approximation of the best “known” covering lattice. In any case, by Proposition 2.1 due to Barnes and Dickson [BD84] yields:

**Corollary 1.3.** The root lattice $E_8$ does not give a locally optimal lattice sphere covering.

By Theorem 1.2 we give a first example of a locally optimal covering lattice whose Delone subdivision is not simplicial. By this we give an affirmative answer to a question of Dickson posed in [Dic68]. The Leech lattice gives even a strongest possible example, in the sense that it is rigid (see Section 2 for details). Our proof in Section 3 immediately applies to $E_8$, giving a new proof of $E_8$’s rigidity, first observed by Baranovskii and Grishukhin [BG01].

**Theorem 1.4.** The Leech lattice and $E_8$ are rigid.

## 2 Lattices, Positive Quadratic Forms and Delone Subdivisions

In this section we briefly review some concepts and results about lattices and their relation to positive definite quadratic forms (PQFs from now on). For further reading we refer to [CS88], Ch. 2 §2.2 and [SV04].

Let $\mathbb{R}^d$ denote a $d$-dimensional Euclidean space with unit ball $B^d$. Its volume is $\kappa_d = \pi^{d/2}/\Gamma(d/2 + 1)$. A (full rank) lattice $L$ is a discrete subgroup in $\mathbb{R}^d$, that is, there exists a regular matrix $A \in \text{GL}_d(\mathbb{R})$ with $L = A\mathbb{Z}^d$. The determinant $\det(L) = |\det(A)|$ of $L$ is independent of the chosen basis $A$. The Minkowski sum $L + \alpha B^d = \{v + \alpha x : v \in L, x \in B^d\}$, $\alpha \in \mathbb{R}_{>0}$, is called a lattice packing if the translates of $\alpha B^d$ have mutually disjoint interiors and a lattice covering if $\mathbb{R}^d = L + \alpha B^d$. The packing radius

$$\lambda(L) = \max\{\lambda : L + \lambda B^d \text{ is a lattice packing}\},$$

and the covering radius

$$\mu(L) = \min\{\mu : L + \mu B^d \text{ is a lattice covering}\},$$

are both attained. They are homogeneous and therefore, the covering density $\Theta(L) = \frac{\mu(L)^d \kappa_d}{\det(L)}$ and the packing density $\delta(L) = \frac{\lambda(L)^d \kappa_d}{\det(L)}$ are invariant with respect to scaling of $L$.

Given a $d$-dimensional lattice $L = A\mathbb{Z}^d$ with basis $A$ we associate a $d$-dimensional PQF $Q[x] = x^t A^t A x = x^t G x$, where the Gram matrix $G = A^t A$ is symmetric and positive definite. We will carelessly identify quadratic forms with symmetric matrices by saying $Q = G$ and $Q[x] = x^t Q x$. The set of quadratic forms is a $\binom{d+1}{2}$-dimensional real vector space $S^d$, in which the set of PQFs forms an open,
convex cone $S_{>0}^d$. The PQF $Q$ depends on the chosen basis $A$ of $L$. For two arbitrary bases $A$ and $B$ of $L$ there exists a $U \in \text{GL}_d(\mathbb{Z})$ with $A = BU$. Thus, $\text{GL}_d(\mathbb{Z})$ acts on $S_{>0}^d$ by $Q \mapsto U'QU$. A PQF $Q$ can be associated to different lattices $L = A\mathbb{Z}^d$ and $L' = A'\mathbb{Z}^d$. In this case there exists an orthogonal transformation $O$ with $A = OA'$. Note that the packing and covering density are invariant with respect to orthogonal transformations.

The determinant (or discriminant) of a PQF is defined by $\det(Q)$. The homogeneous minimum $\lambda(Q)$ and the inhomogeneous minimum $\mu(Q)$ are given by

$$
\lambda(Q) = \min_{v \in \mathbb{Z}^d \setminus \{0\}} Q[v], \quad \mu(Q) = \max_{x \in \mathbb{R}^d} \min_{v \in \mathbb{Z}^d} Q[x - v].
$$

If $Q$ is associated to $L$, then $\det(L) = \sqrt{\det(Q)}$, $\mu(L) = \sqrt{\mu(Q)}$, $\lambda(L) = \sqrt{\lambda(Q)/2}$.

We say that a lattice $L$ with associated PQF $Q$ gives a locally optimal lattice packing or a locally optimal lattice packing, if there is a neighborhood of $Q$ in $S_{>0}^d$, so that for all $Q'$ in the neighborhood we have $\Theta(Q) \leq \Theta(Q')$, respectively $\delta(Q) \geq \delta(Q')$.

A polytope $P = \text{conv}\{v_1, \ldots, v_n\}$, with $v_1, \ldots, v_n \in \mathbb{Z}^d$, is called a Delone polytope of $Q$ if there exists a $c \in \mathbb{R}^d$ and a real number $r \in \mathbb{R}$ with $Q[v_i - c] = r^2$ for all $i = 1, \ldots, n$, and for all other lattice points $v \in \mathbb{Z}^d \setminus \{v_1, \ldots, v_n\}$ we have strict inequality $Q[v - c] > r^2$. The set of all Delone polytopes is called the Delone subdivision of $Q$. Note that the inhomogeneous minimum of $Q$ is at the same time the maximum squared circumradius of its Delone polytopes. We say that the Delone subdivision of a PQF $Q'$ is a refinement of the Delone subdivision of $Q$, if every Delone polytope of $Q'$ is contained in a Delone polytope of $Q$.

By a theory of Voronoi [Vor08] (see also [SV04]), the set of PQFs with a fixed Delone subdivision is an open polyhedral cone in $S_{>0}^d$ — the secondary cone of the subdivision. In the literature the secondary cone is sometimes called $L$-type domain of the subdivision. The topological closures of these secondary cones give a face-to-face tessellation of $S_{>0}^d$, the set of all positive semi-definite quadratic forms. The relative interior of a face in this tessellation contains PQFs that have the same Delone subdivision. If a face is contained in the boundary of a second face, then the corresponding Delone subdivision of the first is a true refinement of the second one. The relative interior of faces of minimal dimension 1 contain rigid PQFs. They have the special property that every PQF $Q'$ in a sufficiently small neighborhood and not being a multiple of $Q$, has a Delone subdivision which is a true refinement of $Q$’s subdivision. The relative interior of faces of maximal dimension $\binom{d+1}{2}$ contain PQFs whose Delone subdivision is a triangulation, that is, it consists of simplices only. We refer to such a subdivision as a simplicial Delone subdivision or Delone triangulation.

We transfer the terminology of Delone subdivisions from PQFs to lattices by saying that the Delone subdivision of the lattice $L'$ is a refinement of the Delone subdivision of the lattice $L$, if there are associated PQFs $Q'$ and $Q$ so that the Delone subdivision of $Q'$ is a refinement of the Delone subdivision of $Q$. A lattice is called rigid if an associated PQF is rigid.

In [Dic68], Dickson states: “Whether it is possible for $f_0$ [a PQF giving a locally optimal lattice covering] to occur on the boundary of a cone [...] is still a matter of conjecture.” Hence to answer his question affirmatively, one has to find a PQF giving a locally optimal lattice covering with non-simplicial Delone subdivision. The following proposition by Barnes and Dickson (see [BD67], [Dic68] §5) shows that there is essentially at most one such lattice covering for each Delone subdivision:

**Proposition 2.1.** A lattice $L$ (or an associated PQF) gives a locally optimal lattice sphere covering iff it minimizes the covering density among all lattices whose Delone subdivisions have a common refinement with $L$’s Delone subdivision. Moreover, such a lattice $L$ is determined uniquely up to dilations and orthogonal transformations.
3 The Leech Lattice, the Root Lattice $E_8$, and their Rigidity

Let us introduce the two lattices. We gathered the information mostly from [CS88], Ch. 4 §8, §11.

The Leech lattice $\Lambda$ with associated PQF $Q_\Lambda$ satisfies $\det(Q_\Lambda) = 1$, $\lambda(Q_\Lambda) = 4$ and $\mu(Q_\Lambda) = 2$. Thus, its packing density, already given by Leech [Lee67], is $\delta(Q_\Lambda) = \kappa_{24}$ and best possible among all lattices in $\mathbb{R}^{24}$ ([CK02], Th. 9.3). Its covering density is $\Theta(Q_\Lambda) = 4096 \cdot \kappa_{24}$. The first proof of this fact is due to Conway, Parker and Sloane [CS88], Ch. 23. There they also classified the 23 different (up to congruences) Delone polytopes of $Q_\Lambda$ attaining the maximum circumradius $\sqrt{2}$.

From their list, we will consider Delone polytopes of type $A_{24}^2$ to prove the rigidity in this section and those of type $A_{24}$ to prove the local optimality in Section 5. To describe them we define $Q_\Lambda(n) = \{ v \in \mathbb{Z}^{24} : Q_\Lambda[v] = 2n \}$. The Delone polytopes of type $A_{24}^2$ are 24-dimensional regular cross polytopes with respect to the metric induced by $Q_\Lambda$. They are of the form $v + \text{conv}\{v_0, \ldots, v_{17}\}$, $v \in \mathbb{Z}^{24}$, where $v_0 = 0$, $v_{24} \in Q_\Lambda(4)$ and all the other $v_i \in Q_\Lambda(2)$ satisfy $v_j + v_{j+24} = v_{24}$, $j = 0, \ldots, 47$ (indices computed modulo 48). The Delone polytopes of type $A_{24}$ are 24-dimensional simplices having 275 edge vectors in $Q_\Lambda(2)$ and 25 in $Q_\Lambda(3)$. This is the only information about $A_{24}$ we will need in Section 5.

For the proof of rigidity it is convenient to work with the following coordinates with respect to the standard basis of $\mathbb{R}^{24}$. The vectors of squared length 4 of $\Lambda$ are of shape $\frac{1}{\sqrt{8}}((\pm 4)^20^{22})$, $\frac{1}{\sqrt{8}}((\pm 2)^80^{16})$ and $\frac{1}{\sqrt{8}}((\pm 3(\pm 1)^{23})$, where permitted permutations of coordinates and permitted positions of minus signs are explained in [CS88], Ch. 10 §3.2. Here, we only need the $2^2(23)$ vectors of the first type, where all permutations of coordinates and all positions of minus signs are allowed.

The root lattice $E_8$ with associated PQF $Q_{E_8}$ satisfies $\det(Q_{E_8}) = 1$, $\lambda(Q_{E_8}) = 2$ and $\mu(Q_{E_8}) = 1$. Thus, its packing density is $\delta(Q_{E_8}) = \frac{1}{16} \cdot \kappa_8$ and best possible among all lattices in $\mathbb{R}^8$ ([Bli34]). Its covering density is $\Theta(Q_{E_8}) = \kappa_8$. The first proof of this fact is due to Coxeter [Cox46], who gave a complete description of the Delone subdivision of $Q_{E_8}$. It is a tiling of $\mathbb{R}^8$ into regular simplices and regular cross polytopes with respect to the metric induced by $Q_{E_8}$. Define $Q_{E_8}(n) = \{ v \in \mathbb{Z}^8 : Q_{E_8}[v] = n \}$. Note that this slightly differs from the definition of $Q_\Lambda(n)$ above. Then the cross polytopes are of the form $v + \text{conv}\{v_0, \ldots, v_{15}\}$, $v \in \mathbb{Z}^8$, where $v_0 = 0$, $v_8 \in Q_{E_8}(4)$ and all the other $v_i \in Q_{E_8}(2)$ satisfy $v_j + v_{j+8} = v_8$, $j = 0, \ldots, 15$ (indices computed modulo 16). The regular simplices have 36 edge vectors in $Q_{E_8}(2)$. We shall give more information about this Delone subdivision in Section 6.

For the proof of $E_8$’s rigidity and for the construction of a new sphere covering in Section 6 it is again convenient to work with explicit coordinates with respect to the standard basis of $\mathbb{R}^8$. Set

$$E_8 = \{ x \in \mathbb{R}^8 : x \in \mathbb{Z}^8 \cup (\frac{1}{2} + \mathbb{Z})^8 \text{ and } \sum_{i=1}^{8} x_i \in 2\mathbb{Z} \}. \tag{1}$$

The automorphism group of $E_8$ is generated by all permutations of the 8 coordinates, by all even sign changes and by the matrix $H = \text{diag}(H_4, H_4)$ where

$$H_4 = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}.$$  

There are 240 vectors of squared length 2 in $E_8$: $2^3(5)$ of shape $((\pm 1)^20^6)$ and $2^7$ of shape $((\pm \frac{1}{2})^8)$ where the number of minus signs is even. The 2160 vectors of squared length 4 are: $2 \cdot 8$ of shape $((\pm 2)^07)$, $2^4(8)$ of shape $((\pm 1)^40^4)$ and $2^7 \cdot 8$ of shape $(\pm \frac{1}{2}(\pm \frac{1}{2})^7)$ where the number of minus signs is odd.
Now we proceed to the proof of Theorem 1.4. We will handle both cases simultaneously. For this, denote by \( L \) the Leech lattice \( \Lambda \) or the root lattice \( \mathbb{E}_8 \) and write \( d \) for the rank of \( L \). We shall show that every PQF \( Q \), whose Delone subdivision contains the above mentioned cross polytopes, is a multiple of \( Q_L \).

By \( \langle \cdot, \cdot \rangle \) we denote the inner product given by \( Q_L \), i.e. \( \langle x, y \rangle = x^t Q_L y \), and by \( \langle \cdot, \cdot \rangle \) we denote the inner product given by \( Q \).

Let \( v, w \in Q_L(2) \) with \( \langle v, w \rangle = 0 \). So, \( v + w \in Q_L(4) \). Therefore, \( 0, v + w, v, w \) are vertices of a Delone cross polytope, as considered above. Let \( e \) be the center of its circumsphere, hence \( Q[0 - e] = Q[v + w - e] = Q[v - e] = Q[w - e] \). Then a straightforward calculation reveals \( \langle v, w \rangle = 0 \).

Now we switch to coordinates with respect to the standard basis. Choosing a basis \( A \) of \( L \) gives the associated PQF \( Q_L = A^t A \). Obviously, \( Q = A^t (A^t)^{-1} Q A^{-1} A \). We denote the entries of \( C = (A^t)^{-1} Q A^{-1} \) by \( (c_{ij}) \) and by \( e_i \) we denote the \( i \)-th canonical basis vector of \( \mathbb{R}^d \). Let \( v_i \in \mathbb{R}^d \) be the coordinate vector of \( e_i \) with respect to the basis \( A \), that is \( A v_i = e_i \). Then by our choice of coordinates and by the argument above we have for \( i \neq j \) the orthogonality

\[
0 = (e_i + e_j)^t (e_i - e_j) = (v_i + v_j)^t Q (v_i - v_j) = (v_i + v_j)^t Q (v_i - v_j) = (e_i + e_j)^t C (e_i - e_j) = c_{ii} - c_{jj}.
\]

Moreover, for pairwise different indices \( i, j, k, l \), the orthogonality yields

\[
0 = (\pm e_i \pm e_j)^t (\pm e_k \pm e_l) = \pm c_{ik} \pm c_{il} \pm c_{jk} \pm c_{jl}.
\]

Hence, the matrix \( C \) is a multiple of the identity matrix and so \( Q \) is a multiple of \( Q_L \), which proves Theorem 1.4.

4 Local Lower Bounds for the Covering Density

In this section we briefly describe a variant of a method due to Ryshkov and Delone which enables us to compute local lower bounds for the covering density. This is a slight variation of the method described in [SV04] and [Val03].

Let \( L = \text{conv} \{ v_1, \ldots, v_{d+1} \} \subset \mathbb{R}^d \) be a simplex in the \( d \)-dimensional Euclidean space with inner product given by the PQF \( Q \). Its centroid is \( m = \frac{1}{d+1} \sum_i v_i \). Let \( e \) be the center of its circumsphere and let \( r \) be its circumradius. Using Apollonius’ formula (see [Ber87] §9.7.6) we get

\[
r^2 = Q[c - m] + \frac{1}{(d+1)^2} \sum_{k \neq l} Q[v_k - v_l].
\]

Proposition 4.1. Let \( L_1 = \text{conv} \{ v_{1,1}, \ldots, v_{1,d+1} \}, \ldots, L_n = \text{conv} \{ v_{n,1}, \ldots, v_{n,d+1} \} \) be a collection of Delone simplices of \( Q \) with radii \( r_1, \ldots, r_n \). Then, the inhomogeneous minimum is bounded by

\[
\mu(Q) \geq \max_i r_i^2 \geq \frac{1}{n(d+1)^2} \sum_i \sum_{k \neq l} Q[v_{i,k} - v_{i,l}]. \tag{2}
\]

The proof is straightforward. We can use the foregoing proposition to get local lower bounds for the covering density of PQFs having \( L_1, \ldots, L_n \) as Delone simplices. We fix the determinant of \( Q \) and minimize the right hand side of (2), which is a linear function:

Proposition 4.2. Let \( D > 0 \). A linear function \( f(Q') = \text{trace}(F Q') \) with a PQF \( F \) has a unique minimum on the determinant \( D \) surface \( \{ Q \in S^d_{>0} : \det(Q) = D \} \). Its value is \( d \sqrt{D \det F} \) and the minimum is attained by the PQF \( \sqrt{D \det F} \).
For a proof of Proposition 4.2 we refer to [Val03], Proposition 8.2.2. Together, Proposition 4.1 and Proposition 4.2 yield

\[ \Theta(Q) \geq \sqrt{\left( \frac{d}{d+1} \right)^d \det F \cdot \kappa_d}, \]

with \( F = \frac{1}{n(d+1)} \sum_i \sum_{k \neq l} (v_{i,k} - v_{i,l})(v_{i,k} - v_{i,l})^t \) which is a PQF.

5 Local Optimality of the Leech Lattice

For the proof of Theorem 1.1 we use the fact that any non-empty set \( \frac{1}{\sqrt{4n}} Q_A(n), n > 0 \), forms a spherical 11-design ([CS88], Ch. 7, Th. 23) in the Euclidean space with inner product \( \langle \cdot, \cdot \rangle \). Generally, a spherical \( t \)-design \( X \) is a non-empty finite subset of the unit sphere \( S^{d-1} = \{ x \in \mathbb{R}^d : \langle x, x \rangle = 1 \} \) satisfying

\[ \frac{1}{\operatorname{vol}S^{d-1}} \int_{S^{d-1}} f(x)dx = \frac{1}{|X|} \sum_{x \in X} f(x) \]

for every polynomial \( f: \mathbb{R}^d \to \mathbb{R} \) of degree at most \( t \). Here, \( \operatorname{vol}S^{d-1} \) denotes the surface volume of \( S^{d-1} \), not the volume of the enclosed ball. Equivalently, \( X \) is a spherical \( t \)-design if it satisfies the equalities (see [Ven01], Th. 3.2):

\[ \sum_{x \in X} \langle x, y \rangle^k = 0, \quad \sum_{x \in X} \langle x, y \rangle^k = \frac{1}{d(d+1)\cdots(d+k-2)} |X| \langle y, y \rangle^k/2, \quad \text{for all } y \in \mathbb{R}^d \text{ and all odd } k \leq t, \]

\[ \sum_{x \in X} \langle x, y \rangle^{k+1} = 0, \quad \sum_{x \in X} \langle x, y \rangle^{k+1} = \frac{1}{d(d+1)\cdots(d+k-1)} |X| \langle y, y \rangle^{k+1/2}, \quad \text{for all } y \in \mathbb{R}^d \text{ and all even } k \leq t. \]

For the proof of Theorem 1.1 the following spherical 2-design property is even sufficient:

Lemma 5.1. Let \( Q \in S^d_{>0} \) and let \( X \subseteq \mathbb{R}^d \) denote a spherical 2-design with respect to the inner product given by \( Q \). Then

\[ \sum_{x \in X} xx^t = \frac{|X|}{d} Q^{-1}. \]

Proof. Since \( X \) forms a spherical 2-design, we have \( \sum_{x \in X} (x^t Q y)^2 = \frac{|X|}{d} (y^t Q y) \). On the other hand

\[ \sum_{x \in X} (x^t Q y)^2 = \sum_{y \in X} y^t Q (x x^t) Q y = y^t Q \left( \sum_{x \in X} xx^t \right) Q y. \]

Thus because both identities are valid for all \( y \in \mathbb{R}^d \) we derive the equality stated in the lemma.

Now we finish the proof of Theorem 1.1. Let \( L \) be a Delone simplex of \( Q_A \) of type \( A_{24} \). We apply Corollary 4.3 to the orbit of \( L \) under the automorphism group \( C_0 = \{ T \in \mathbb{GL}_{24}(\mathbb{Z}) : T^t Q A T = Q_A \} \) of \( Q_A \).

For every PQF \( Q \) for which the simplices \( TL, T \in C_0 \), are Delone simplices, we have \( \Theta(Q) \geq \sqrt{(\frac{24}{25})^{24} \det F \cdot \kappa_{24}} \) with \( F = \frac{1}{25|C_0|} \sum_{T \in C_0} \sum_{e} ee^t \), where \( e \) runs through all the edge vectors of \( TL \). Since \( L \) has 275 edges in \( Q_A(2) \) and 25 edges in \( Q_A(3) \) and because of the transitivity of \( C_0 \) on \( Q_A(2) \) and \( Q_A(3) \) ([CS88], Ch. 10, Th. 27) we get

\[ F = \frac{1}{25|C_0|} \left( \frac{275|C_0|}{|Q_A(2)|} \sum_{e \in Q_A(2)} ee^t + \frac{25|C_0|}{|Q_A(3)|} \sum_{e \in Q_A(3)} ee^t \right). \]
By Lemma 5.1 (applied to $Q_8/4$ and $Q_8/6$) this yields

$$F' = \frac{1}{25} \left( \frac{275}{|Q_8(2)|} \cdot \frac{|Q_8(2)|}{6} \cdot Q_8^{-1} + \frac{25}{|Q_8(3)|} \cdot \frac{|Q_8(3)|}{4} \cdot Q_8^{-1} \right) = \frac{5^2}{2^2 \cdot 3} Q_8^{-1}$$

Since $\det Q_8^{-1} = 1$, it follows $F = \frac{5^8}{2^8 \cdot 3}$ and finally, by Corollary 4.4 we derive $\Theta(Q) \geq 4096 \cdot \kappa_{24} = \Theta(Q_8)$.

6 A New Sphere Covering in Dimension 8

If we apply the method of Theorem 1.1 to $E_8$ we get a local lower bound of $\sqrt{(8/9)^5} \cdot \kappa_8 \approx 0.6243 \cdot \kappa_8$. But $\Theta(E_8) = \kappa_8$, since the circumsphere of the regular cross polytopes in $E_8$’s Delone subdivision is 1. Despite this gap, $E_8$ could be a locally optimal covering lattice. The following proof of Theorem 1.2 shows that this is not the case. By Proposition 2.1 we have to find a PQF $Q$ with $\Theta(Q) < \Theta(Q_{E_8})$ so that $\text{Del}(Q)$ and $\text{Del}(Q_{E_8})$ have a common refinement.

Below, we describe a systematic way to attain a refining Delone triangulation of $\text{Del}(Q_{E_8})$. Given such a triangulation we can find an approximation of the unique PQF minimizing $\Theta$ among all PQFs in the closure of its secondary cone by solving a convex programming problem on a computer. We give such an approximation in Appendix A and verify its properties with a simple computer program. Since we carry out the verification using exact arithmetic only, the proof of Theorem 1.2 is rigorous.

First, we describe all $Z^8$-periodic triangulations refining the Delone subdivision of $Q_{E_8}$, that is, all sets $P$ of simplices satisfying the following conditions:

**no additional vertices:** all vertices of simplices $L \in P$ lie in $Z^8$.

**periodicity:** $\forall L \in P, v \in Z^8 : v + L \in P$.

**face-to-face tiling:** $\forall L, L' \in P : L \cap L' \in P$.

**refinement:** $\forall L \in P \exists L' \in \text{Del}(Q_{E_8}) : L \subseteq L'$.

**covering:** $\forall x \in R^8 \exists L \in P : x \in L$.

Recall from Section 5 that $\text{Del}(Q_{E_8})$ consists of simplices and cross polytopes only. Thus for a $Z^8$-periodic triangulation refining $\text{Del}(Q_{E_8})$ we have to specify how to split the 8-dimensional cross polytopes into simplices.

We say that two polytopes $P$ and $P'$ are $Z^8$-equivalent if $P' = v + P$ for some $v \in Z^8$. Every $w \in Q_{E_8}(4)$ defines a Delone cross polytope $P_w = \text{conv} \{v_0, \ldots, v_{15}\}$ with $v_0 = 0, v_8 = w$ and all other $v_j \in Q_{E_8}(2)$ with $v_j + v_{j+8} = v_8$ (indices computed modulo 16).

Two $w, w' \in Q_{E_8}(4)$ define $Z^8$-equivalent cross polytopes iff $w' \in w + 2Z^8$, because then the difference $\frac{1}{2}w' - \frac{1}{2}w$ of their centers is in $Z^8$. Under this equivalence relation the set $Q_{E_8}(4)$ splits into 135 classes, containing 8 pairs of mutually orthogonal vectors $\pm w_1, \ldots, \pm w_8$. Each of the $w_i$ equivalent to $w$ is a diagonal of $P_w$, e.g. $w_i = v_i - v_{i+8}$, $i = 0, \ldots, 7$. In the coordinate system introduced in 11 the 135 classes are (see CS88, Ch. 6, §3):

1 class: $\pm 2e_1, \ldots, \pm 2e_8$. 

7
70 classes: 8 elements $\pm e_a \pm e_b \pm e_c \pm e_d$ with an even number of minus signs and 8 elements $\pm e_e \pm e_f \pm e_g \pm e_h$ with an even number of minus signs and with $\{a, \ldots, h\} = \{1, \ldots, 8\}$; or the same with an odd number of minus signs.

64 classes: 8 pairs of vectors of shape $(\pm \frac{3}{2} (\pm \frac{1}{2})^7)$ with odd number of minus signs, where the position of $\pm \frac{3}{2}$ is permuted to all 8 coordinates.

We can split each cross polytopes $P_w$ into simplices in eight different ways by adding a diagonal. Without loss of generality we add the diagonal $\text{conv} \{v_0, v_8\}$ and split the cross polytope $P_w$ into the 128 simplices $\text{conv} \{0, v_8, v_{j_1}, \ldots, v_{j_7}\}$, where $j_k \in \{k, k + 8\}$. Thus altogether we get $8^{135}$ different $Z^8$-periodic triangulations refining $\text{Del}(Q_{E_8})$.

Now, which of these periodic triangulations are Delone triangulation for some PQF? To decide this, we take a closer look at the tiling $\text{Del}(Q_{E_8})$ and at secondary cones $\Delta(T)$ of Delone triangulations $T$ refining $\text{Del}(Q_{E_8})$.

We already described the cross polytopes of $\text{Del}(Q_{E_8})$. Centers of simplices of $\text{Del}(Q_{E_8})$ containing the origin are the 17280 vectors $\frac{1}{2} \nu$, where $\nu$ is a vector of $Q_{E_8}(8)$ not in $2Q_{E_8}(2)$. We say two polytopes are adjacent in the tiling, if they share a facet. Each simplex is adjacent to 9 cross polytopes and each cross polytope is adjacent to 128 simplices and 128 cross polytopes. A simplex and a cross polytope both containing the origin are adjacent iff the inner product, with respect to $Q_{E_8}$, of their centers equals $\frac{5}{6}$. Two cross polytopes both containing the origin are adjacent iff the inner product of their centers equals $\frac{1}{2}$.

For some computations it is useful to have coordinates of vertices, with respect to the coordinate system [1]: The vertices of the cross polytope $P$ defined by the center $e_1$ are $0, 2e_1, e_1 \pm e_i, i = 2, \ldots, 8$. An adjacent Delone cross polytope is defined by the center $e = (\frac{3}{4}, -\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$. Its vertices are $0, 2e, e_1 - e_2, 2e_1 - (e_1 - e_2), e_1 + e_i, 2e - (e_1 + e_i), i = 3, \ldots, 8$. A Delone simplex adjacent to $P$ is defined by the center $e' = (\frac{5}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6})$. Its vertices are $0, (\frac{1}{2}, \ldots, \frac{1}{2}), e_1 + e_i, i = 2, \ldots, 8$.

Since the automorphism group of $E_8$ acts transitively on vectors of squared length 4 and since the stabilizer of $\pm 2e_1$ in $E_8$’s automorphism group is the group generated by even sign changes and by permutations of the last 7 coordinates, the knowledge of the coordinates given is enough to describe the whole Delone subdivision.

The secondary cones $\Delta(T)$ are open polyhedral cones and by the theory of Voronoi, they are given by linear forms on $S^8$ called regulators. Each pair of adjacent simplices $L = \text{conv} \{v_0, \ldots, v_8\}$, $L' = \text{conv} \{v_1, \ldots, v_9\}$ gives a regulator $\theta(L, L')$. If $\alpha_0, \ldots, \alpha_9 \in \mathbb{Q}$ are the uniquely determined numbers with $\alpha_0 = 1$, $\sum_{i=0}^9 \alpha_i = 0$ and $\sum_{i=0}^9 \alpha_i v_i = 0$, then $\theta(L, L')(Q) = \sum_{i=0}^9 \alpha_i Q[v_i]$ for $Q \in S^8$ and

$$\Delta(T) = \{ Q \in S^8 : \theta(L, L')(Q) > 0, (L, L') \text{ pair of adjacent simplices of } T \}.$$ 

Note that $\theta(L+v, L'+v) = \theta(L, L')$ for all $v \in Z^8$.

For a triangulation $T$ which is a refinement of $\text{Del}(Q_{E_8})$ we distinguish between three types of pairs of adjacent simplices $(L, L')$. In the first case one of the simplices is a simplex of $\text{Del}(Q_{E_8})$ and the other one is not. In the two other cases both simplices are not simplices of $\text{Del}(Q_{E_8})$. In the second case they refine adjacent cross polytopes, in the third case they refine the same cross polytope. In the first two cases we have $\theta(L, L')(Q_{E_8}) > 0$ and in the last case $\theta(L, L')(Q_{E_8}) = 0$. Since $E_8$ is rigid, the closures of secondary cones of Delone triangulations refining $\text{Del}(Q_{E_8})$ cover a sufficient small neighborhood of the ray containing multiples of $Q_{E_8}$. Thus, $T$ is a Delone triangulation for some PQF refining $\text{Del}(Q_{E_8})$ iff

$$\{ Q \in S^8 : \theta(L, L')(Q) > 0, (L, L') \text{ refining the same cross polytope of } \text{Del}(Q_{E_8}) \}$$

is not empty.
The three types of regulators are easily computed, e.g. with help of the coordinates given above. In the first case, let \( P = \text{conv}\{v_0, \ldots, v_{15}\} \) be a cross polytope of \( \text{Del}(Q_{E_8}) \) with the notational convention: \( v_8 \in Q_{E_8}(4), v_i + v_{i+8} = v_8 \). Let \( L' = \text{conv}\{v_0', \ldots, v_8'\} \) be a simplex of \( \text{Del}(Q_{E_8}) \) with \( v_i' = v_i, i = 0, \ldots, 7 \). Let \( c \) be the centroid of \( P \) and \( c' \) be the centroid of \( L' \). Then \( c' = \frac{1}{8}(v_0 + \ldots + v_8) \) and \( \frac{1}{c} = \frac{1}{2}(v_0 + \ldots + v_7) \). Suppose the edge \( \text{conv}\{v_k, v_{k+8}\}, k \in \{0, \ldots, 7\}, \) belongs to \( T \). We have \( c = \frac{1}{2}(v_k + v_{k+8}) \) and we derive \( \frac{1}{8} (v_k + v_{k+8}) + \frac{1}{12} (v_8' + v_0 + \ldots + v_7) = \frac{1}{8} (v_0 + \ldots + v_7) \). Therefore we get the regulator

\[
\psi(L,L')(Q) = Q[v_k] + Q[v_{k+8}] + \frac{2}{3}Q[v_8'] - \frac{1}{3}Q[v_0] - \cdots - \frac{1}{3}Q[v_7].
\]

In the second case, let \( P_1 = \text{conv}\{v_0, \ldots, v_{15}\} \) and \( P_2 = \text{conv}\{v_0', \ldots, v_{15}'\} \) be two adjacent cross polytopes with the usual notational convention and with \( v_i = v_i' \) for \( i = 0, \ldots, 7 \). Then the centers \( c = \frac{1}{2}v_8 \) and \( c' = \frac{1}{2}v_8' \) of the cross polytopes satisfy the relation \( \frac{1}{2}(c + c') = \frac{1}{8}(v_0 + \cdots + v_7) \). Let us assume that the diagonals \( \text{conv}\{v_k, v_{k+8}\} \) and \( \text{conv}\{v_k', v_{k+8}'\} \) with \( k, k' \in \{0, \ldots, 7\} \) belong to \( T \). Then, since \( c, c' \) are the centers of these diagonals, we derive \( v_k + v_{k+8} + v_k' + v_{k+8}' = \frac{1}{2}v_0 + \cdots + \frac{1}{2}v_7 \). Therefore we get the regulator

\[
\psi(L,L')(Q) = Q[v_k] + Q[v_{k+8}] + Q[v_k'] + Q[v_{k+8}'] - \frac{1}{2}Q[v_0] - \cdots - \frac{1}{2}Q[v_7].
\]

In the third case, let \( P = \text{conv}\{v_0, \ldots, v_{15}\} \) be a cross polytope with the usual notational convention. Then adjacent simplices are of the form \( L = \text{conv}\{v_0, v_8, v_{j}, \ldots, v_{j'k}\}, L' = \text{conv}\{v_0, v_8, v_{j'}, \ldots, v_{j'k}\}, \) where \( j, j_k \in \{k, k+8\} \) and \( j' = j + 8 \) only for one \( k \in \{1, \ldots, 7\} \). Because of \( v_{jk} + v_{j+k+8} = v_0 + v_8 \) we get seven regulators

\[
\psi(L,L')(Q) = Q[v_{jk}] + Q[v_{j+k+8}] - Q[v_0] - Q[v_8], \quad k = 1, \ldots, 7.
\]

Note that these conditions are equivalent to \( Q[v_8] < Q[v_{jk} - v_{j+k+8}], k = 1, \ldots, 7 \), which means that the chosen diagonal \( v_8 \) is shorter than the other seven with respect to the metric induced by \( Q \).

We tried to generate all Delone triangulations refining \( \text{Del}(Q_{E_8}) \) by an exhaustive computer search. But this seems to be hopeless since they are far too many. So we decided to generate a Delone triangulation which has a fairly large symmetry group. For this we choose a subgroup \( G \) of \( Q_{E_8} \)'s automorphism group which, in the coordinate system \( \mathbb{Z}^8 \), is generated by permutations of the last 7 coordinates and by the involution \( x \mapsto -x \).

Proposition 6.1. There are exactly four \( \mathbb{Z}^8 \)-periodic triangulations refining \( \text{Del}(Q_{E_8}) \) invariant under the group \( G \). Exactly two of them are Delone triangulations and both are equivalent under the action of \( Q_{E_8} \)'s full automorphism group.

Proof. To show that there is essentially one Delone triangulation with the prescribed symmetries, we will again work with the coordinate system \( \mathbb{Z}^8 \). In Table 1 we list the orbits of squared length 4 vectors under the action of \( G \).

| # | representative | orbit size |
|----|---------------|-----------|
| 1. | 20000000 | 1 |
| 2. | 02000000 | 7 |
| 3. | 11110000 | \( \frac{1}{3} \) |
| 4. | 11110000 | \( \frac{1}{2} \) |
| 5. | 11110000 | \( \frac{1}{3} \) |
| 6. | 11110000 | \( \frac{1}{2} \) |
| 7. | 01111100 | \( \frac{1}{4} \) |
| 8. | 01111100 | \( \frac{1}{6} \) |
| 9. | 01111100 | \( \frac{1}{2} \) |
| 10. | \( \frac{1}{3} \) | 31111111 |
| 11. | \( \frac{1}{3} \) | 31111111 |
| 12. | \( \frac{1}{3} \) | 31111111 |
| 13. | \( \frac{1}{3} \) | 31111111 |
| 14. | \( \frac{1}{7} \) | 31111111 |
| 15. | \( \frac{1}{7} \) | 31111111 |
| 16. | \( \frac{1}{7} \) | 31111111 |
| 17. | \( \frac{1}{7} \) | 31111111 |
| 18. | \( \frac{1}{7} \) | 31111111 |
| 19. | \( \frac{1}{7} \) | 31111111 |
| 20. | \( \frac{1}{7} \) | 31111111 |
Table 1. Orbits of squared length 4 vectors in $E_8$. Minus signs are given by bars.

To define a $\mathbb{Z}^8$-periodic triangulation refining $\text{Del}(Q_{E_8})$ we have to choose a collection of orbits $(O_i)_{i \in I}$, $I \subseteq \{1, \ldots, 20\}$, so that for every of the 135 classes of possible diagonals $C$ we have $|\bigcup_{i \in I} O_i \cap C| = 2$. This restriction immediately gives $\{2, 4, 5, 8, 9, 14, 15, 16, 17, 18, 19\} \cap I = \emptyset$. For example, the four vectors $\pm 02000000, \pm 00200000$ are in $O_2 \cap C$. On the other hand we have to have $\{1, 6, 11, 12, 13\} \subseteq I$. Now there are two binary choices left: either we have $3 \in I$ or $7 \in I$ and either we have $10 \in I$ or $20 \in I$. From these four triangulations only those given by $I_1 = \{1, 3, 6, 10, 11, 12, 13\}$ and $I_2 = \{1, 6, 7, 11, 12, 13, 20\}$ are Delone triangulations: Under the prescribed symmetry we can assume that there are numbers $\alpha, \beta, \gamma, \delta$ with

$$\alpha = (e_1, e_1), \quad \beta = (e_1, e_i), \quad \gamma = (e_i, e_i), \quad \delta = (e_i, e_j), \quad i = 2, \ldots, 8, j = i + 1, \ldots, 8.$$  

Suppose we choose orbit 3. By (5) this implies the inequality

$$\left(\frac{1}{2}(11111111), \frac{1}{2}(1111\overline{1111})\right) = \frac{1}{4}(\alpha + 6\beta - \gamma - 6\delta) < 0,$$

then choosing orbit 20 implies

$$\left(\frac{1}{2}(1\overline{0}000000), \frac{1}{2}(\overline{1}11111111)\right) = \frac{1}{4}(\alpha - 6\beta + \gamma + 6\delta) < 0,$$

yielding a contradiction. Hence we have to choose orbit 10 instead. A similar calculation shows that if we choose orbit 7, then we have to choose orbit 20. To see that these triangulations are Delone triangulations we still have to give a PQF satisfying all regulators in (5). We postpone this to Appendix A.

By applying the transformation $HAH$, where $A$ exchanges the first and fifth coordinate and their signs, and $H$ is the transformation $H = \text{diag}(H_4, H_4)$ we see that both Delone triangulations are equivalent. The transformation $HAH$ is an element of $E_8$’s automorphism group exchanging the relevant orbits of vectors of size 2 by $O_1 \leftrightarrow O_{13}, O_3 \leftrightarrow O_7, O_6 \leftrightarrow O_{11}, O_{10} \leftrightarrow O_{20}, O_{12} \leftrightarrow O_{12}$. $\square$

Given the Delone triangulation $T$ refining $\text{Del}(Q_{E_8})$, attained in this way or another, we can compute an approximation of the unique PQF minimizing the covering density $\Theta$ among all PQFs in the closure of its secondary cone. For details we refer to [SV04] and give only a brief sketch. We have to solve the optimization problem: maximize $\det Q$ where $Q$ lies in the closure of the secondary cone $\Delta(T)$ and the circumradius of every simplex in $T$ with respect to $Q$ is bounded by 1. This is a convex optimization problem and we can approximate the solution using a computer. With help of the software MAXDET written by Wu, Vandenberghe and Boyd (see [VBW98]) we found a PQF $\tilde{Q}$ with covering density $\Theta \approx 3.2012$. But MAXDET uses floating point arithmetic. So we have to verify that the Delone subdivision of the new PQF has a common refinement with $\text{Del}(Q_{E_8})$ and we have to verify $\tilde{Q}$’s covering density. We did this by writing a program which uses only rational arithmetic. We give more details and the PQF in Appendix A. The successful verification proves in particular Theorem 1.2.

Finally, we report on some numerical evidences. The PQF $\tilde{Q}$ lies on the boundary of the secondary cone $\Delta(T)$. The closure of $\Delta(T)$ has 428 facets and only one of these facets does not contain $Q_{E_8}$. We applied the optimization to the Delone triangulation which belongs to the secondary cone adjacent to this facet. There we found a PQF with covering density $\Theta \approx 3.1423$, which is the best known covering density in dimension 8 so far. It seems that the PQF we approximated by $\tilde{Q}$ is not locally optimal. This would follow by Proposition 2.1 if we knew that the by $\tilde{Q}$ approximated PQF lies on the boundary of $\Delta(T)$. In any case we are left with the open problem to find a globally best covering lattice in dimension 8. Currently, we do not know where to search for such a lattice.
7 Remarks on the Packing-Covering Problem

Together with the local optimality of the Leech lattice with respect to the packing problem, we immediately derive the local optimality of the Leech lattice with respect to another problem (see [SV04]): The lattice packing-covering problem asks to minimize the packing-covering constant $\gamma(L) = \mu(L)/\lambda(L) = (\Theta(L)/\delta(L))^{1/d}$ among all $d$-dimensional lattices $L$. For the Leech lattice we derive $\gamma(\Lambda) = \sqrt{2}$ which again is a strict local minimum. We can offer two different proofs for this: either by using the global optimality of the Leech lattice with respect to the packing problem or by deriving a local lower bound along the lines of the proof of Theorem 1.1. Here, analogous tools to Proposition 4.2 and Corollary 4.3 (see [SV04], Sec. 10) are needed.

For $E_8$ the situation seems to differ from the covering case: Given a fixed Delone triangulation, the problem of finding the minimum $\gamma$ among all lattices with the same Delone triangulation can also be formulated as a convex optimization problem (see [SV04]). In contrast to the covering case, the application of MAXDET to the triangulation constructed in Section 6 indicates that $\gamma(E_8) = \sqrt{2}$ may in fact be a local optimum, as conjectured by Zong [Zon02]. He conjectured that $E_8$ gives the global optimum.

Note the remarkable fact that $d = 1, 2$ are the only known cases where the minimum covering density and the maximum packing density — and therefore the minimum packing-covering constant $\gamma_d = \min_L \gamma(L)$ — are known to be attained by the same lattice. Maybe yet another exceptional property of the beautiful Leech lattice...

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A Verification of Numerical Results

The PQF $\tilde{Q}$ of Section 5 is $\tilde{Q} = 34229189769Q_1 - 17121746137Q_2$ with

\[
    Q_1 = \begin{pmatrix}
        1 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\
        0 & 4/7 & -2/3 & 0 & 0 & 0 & 0 & 0 \\
        0 & -2/3 & 4/3 & -2/3 & 0 & 0 & 0 & 0 \\
        0 & 0 & -2/3 & 4/3 & -2/3 & 0 & 0 & 0 \\
        0 & 0 & 0 & -2/3 & 4/3 & -2/3 & 0 & 0 \\
        0 & 0 & 0 & 0 & -2/3 & 4/3 & -2/3 & 0 \\
        2 & 0 & 0 & 0 & 0 & 0 & 0 & 4 \\
    \end{pmatrix}
\]

and

\[
    Q_2 = \begin{pmatrix}
        0 & 1 & 0 & 0 & 0 & 0 & 0 & 7/2 \\
        1 & 0 & -2/3 & 0 & 0 & 0 & 0 & 0 \\
        0 & -2/3 & 4/3 & -2/3 & 0 & 0 & 0 & 0 \\
        0 & 0 & -2/3 & 4/3 & -2/3 & 0 & 0 & 0 \\
        0 & 0 & 0 & -2/3 & 4/3 & -2/3 & 0 & 0 \\
        0 & 0 & 0 & 0 & -2/3 & 4/3 & -2/3 & 0 \\
        7/2 & 0 & 0 & 0 & 0 & -2/3 & 4/3 & 0 \\
    \end{pmatrix}.
\]
Here, $Q_1$ and $Q_2$ form a basis of the subspace of $\mathcal{S}^8$ invariant under the group $G$ intersected with the subspace given by regulators $\theta_{(L,L')}\cdot (\ref{3}), \ref{4}, \ref{5}$ with $\theta_{(L,L')}\cdot (\tilde{Q}) = 0$. To give a rigorous proof of Theorem 1.2 we have to verify that one of the two triangulations we constructed in Section 6 is a refining Delone triangulation of $\tilde{Q}$’s Delone subdivision and that its covering density is at most $3.2013$. To do this we supply the MAGMA program newcovering8.m available from the arXiv.org e-print archive. To access it, download the source files for the paper math.MG/0405441. There we also included the Gram matrix of the “best known” covering lattice in the additional file currentbest8.txt.

The program newcovering8.m first verifies that there is a PQF, called $Q_{\text{interior}}$, which strictly satisfies the inequalities (5) for the triangulation given by $I_1$ (see proof of Proposition 6.1). By this we know that the triangulation is a Delone triangulation. Then the program checks that $\tilde{Q}$ satisfies all inequalities (3), (4), (5) given by regulators. This verifies that the Delone triangulation is in fact a refinement of $\tilde{Q}$’s Delone subdivision. Finally we compute the circumradii of a representative system of simplices with respect to the inner product $\langle \cdot, \cdot \rangle$ induced by $\tilde{Q}$. The squared circumradius $r^2$ of the simplex $L = \text{conv} \{0, v_1, \ldots, v_8\}$ is

$$r^2 = \frac{1}{4} \left| \begin{array}{cccc}
0 & (v_1, v_1) & (v_2, v_2) & \cdots (v_8, v_8) \\
(v_1, v_1) & (v_1, v_1) & (v_1, v_2) & \cdots (v_1, v_8) \\
& & & \cdots \\
(v_8, v_8) & (v_8, v_1) & (v_8, v_2) & \cdots (v_8, v_8) \\
\end{array} \right| \det \left( \begin{array}{cccc}
\langle v_i, v_j \rangle \\
\langle v_i, v_j \rangle \\
& & & \cdots \\
\langle v_i, v_j \rangle \\
\end{array} \right)_{1 \leq i, j \leq 8},$$

see e.g. [SV04]. All these evaluations involve only rational arithmetic and they can be carried out on a usual personal computer in less than 15 minutes.

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