Geometric Drawing Method for Form-finding and Force-finding to Triangular Prism Tensegrity with End Surfaces not Paralleled

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Abstract. The triangular prism tensegrity with arbitrary triangle bottom and end surfaces not paralleled is the most common type tensegrity structure. A geometric drawing method with four steps’ process for form-finding was given based on the property of tensegrity structure, and the degree of freedom for form-finding was analysed. The force-finding method with ten steps’ process was given after form-finding, and the self-balancing internal forces was obtained by measuring the lengths of correspondent line segments. The geometric drawing method was carried out in CAD software by an example, and the example tensegrity structure obtained by the method was analysed by finite element method software, the correctness of the method was verified. The form-finding method is the geometric implementation of the force density method, and the process is intuitive and controllable, it is suitable and operational for triangular prism of tensegrity structures of any shape.

Keywords: Triangular prism of tensegrity; Form-finding; Force-finding; Geometric drawing method.

1. Introduction
Mechanics and geometry are inextricably linked [1]. Tensegrity structure is composed of discrete compression bars and continuous cables (tension bars), which is the perfect unity of force and shape. Since the tensegrity structures is put forward in the 1940s by famous American architect fuller (R.B.Fuller), scholars have conducted extensive studies on the structure from the aspects of structure form-finding [2]--[6], stability [7]--[9], self-balancing internal forces [10], geometric nonlinearity [11][12], etc. However, most of the tensegrity structures were only designed to present the artistic beauty in the form of architectural sculpture, tensegrity structures in engineering is rarely used. Recent years, tensegrity structures have been applied in spatial deployable structures [7][13].

The main characteristic of tensegrity structure is that internal forces only meet the balance under a specific shape, which determines the key problem of the structure system is the form-finding, it is to determine the shape of the tensegrity structure satisfying the self-balancing. The form-finding methods for tensegrity structure can be divided into two categories: kinematic method and static method. The typical kinematic method is dynamic relaxation method [2], while the typical static method is force density method [6].

The dynamic relaxation method is used to find the form of prestressed structure by solving the dynamic equation. When the prestress is applied, the structure nodes would move dynamically because of the structure is not in the self-equilibrium state, and the vibration of tensegrity structure stops gradually.
under the action of damping. When the structure is stable after dynamic relaxation, the shape is the result of forming-finding and internal forces are self-balanced. The disadvantage of dynamic relaxation method is that it often encounters the problem of slow convergence or divergency. As a form-finding method for cable structures, force density method has been applied to tensegrity structure recent years. The idea of force density method is to solve the self-balancing state of structure under the condition of given force density factor. The force density method is a method combining force and shape analysis, the length control of the bar requires some experience. The force density is not easy to determine, and the relationship of force and shape is not evident. Some methods of form-finding change the problem of tensegrity structure to the constrained optimization problem of maximum length of compression bars or minimum length of tension cables, and use the method of nonlinear programming to solve the problem. The existing form-finding methods for tensegrity structure are usually realized by computer software. As a nonlinear process, form-finding process by computer software is often disordered, and the process lacks of intuitiveness, controllability and initiative of designers, and often meets the difficulties of non-convergence. In many cases, structural form-finding is carried out according to the requirements of architects, designers should participate in the process of form-finding of tensegrity structure according to aesthetic requirements, and the current method of form-finding is difficult to meet the requirements of designers.

Tensegrity structure is a perfect combination of mechanics and geometry, and the process of form-finding and force-finding of tensegrity structure is a topological optimization process of structure [2][3]. Because of tensegrity structures is made up of axial force elements, and the axial force can be reflected by the length of line segment of the element, form-finding and force-finding can be realized by the geometric drawing method[14]. Method to find the form and force of triangular prism tensegrity structure with arbitrary triangle bottom and the end faces paralleled has been given[14]. The geometric drawing method to find the form and force of triangular tensegrity with arbitrary triangle bottom and two end faces with arbitrary angle would be given in the paper below.

2. Form-finding to the Triangular Prism Tensegrity

2.1. Property of Triangular Prism Tensegrity with End Surfaces not Paralleled

The bottom surface of the triangular prism is ABC, the top surface is abc, and the shape of the triangle ABC is arbitrary. How to find three points a, b and c of top surface with self-balancing forces, and keep the top surface abc passing through point o and forming the specified angle and direction with the bottom surface ABC. There are two forms of triangular prismatic tensegrity: right-handed and left-handed, as shown in Fig.1. The paper takes right-handed as an example to introduce the method of form-finding and force-finding, which is also suitable for left-handed form.

![Figure 1. Right-handed and left-handed triangular prism tensegrity.](image1)

![Figure 2. Property of triangular prism.](image2)

For node a, there are 2 bars connected to the bottom surface, one is the tension bar Aa, another one is the compression bar Ca. Two tension bars ac and ab connected node a in the top surface. The resultant force of compression bar Ca and tension bar Aa must be in the plane of triangle CAa. The resultant force of the tension bar ab and the tension bar ac must be in the plane of triangle abc. At point a, the resultant force of bar Ca and Aa must be balanced with the resultant force of bar ac and ab. Therefore the resultant force must be in the intersecting line of plane CAa and abc, oa shown in Fig.2. The intersecting line oa
can be find by geometric drawing method. In a similar way, intersecting line $ob$ for plane $ABb$ and $abc$, intersecting line $oc$ for plane $BCc$ and $abc$. With these three intersecting lines, we can find the form of the triangular prism of tenseginty.

2.2. Geometric Drawing Method for Form-finding

Based on the above analysis of the properties of triangular prism of tensegrity structure with two end surfaces not paralleled, the methods and steps for form-finding can be obtained below:

Step 1: The top surface through the specified point $o$ must be intersect to the plane of $ABC$, the intersecting line is $LL$, as shown in Fig.3.

Step 2: The line $CA$ intersects to line $LL$ at point $E$, the line $AB$ intersects to line $LL$ at point $G$, and the line $BC$ intersects to line $LL$ at point $H$.

Step 3: Select point $a$ on line $oE$ so that $l_{oa}=\alpha l_{CA}$, select point $b$ on line $oG$ so that $l_{ob}=\beta l_{AB}$, and select point $c$ on line $oH$ so that $l_{oc}=\gamma l_{BC}$. To satisfy the right-handed form structure, point $b$ is in the opposite direction of ray $oG$, as shown in Fig.4.

Step 4: Connect $ab$, $bc$, $ca$, $Ca$, $Ab$, $Bc$, $Aa$, $Bb$, $Cc$.

A triangular prism tensegrity structure satisfying self-equilibrium of prestress is obtained. in the tensegrity structure, $Ca$, $Ab$ and $Bc$ are compression bars and the rest bars are tension bars.

Figure 3. Initial condition for form-finding.  Figure 4. Form-finding process of triangular prism.

2.3. Degree of Freedom of Form-finding

The height of point $o$ can be arbitrary, and the position of point $o$ in the plane can also be arbitrarily determined. That is to say, point $o$ is determined by three degrees of freedom, which can be determined according to the requirements of the architect. Once the position of point $o$ is determined, there are two degrees of freedom left to determine the direction of the plane in which $abc$ is located. After the direction of the plane $abc$ is determined, there are three degrees of freedom $a, b$ and $c$ to determine the positions of points $a, b$ and $c$. Therefore, after the determination of the bottom surface $ABC$, there are eight degrees of freedom in the form-finding, three degrees of freedom to determine the spatial position of the resultant point $o$ $(x, y, z)$, two degrees of freedom to determine the direction of the top surface, and three degrees of freedom to determine the positions of $a, b$ and $c$ on the line $oE$, $oG$ and $oH$ respectively.

3. Force-finding

3.1. Force Density

For any bar connecting nodes $i$ and $j$ with internal force $f_{ij}$ and length $l_{ij}$. The ratio of internal force to the length of the bar is expressed as $q_{ij}=f_{ij}/l_{ij}$, and $q_{ij}$ is the force density (or tensioning coefficient, prestress coefficient).

3.2. Geometric Drawing Method for Force-finding

After form-finding is completed, the triangular prism tensegrity structure is obtained shown in Fig4. Since the self-balancing internal forces of the triangular prism tensegrity structure can be changed linear, when the internal force of one element specified first, the internal forces of othermembers can be solved uniquely. The force-finding steps are provided below:
Step 1: For the convenience of drawing method for force-finding, the force density of the member $Ab$ is specified as $q_{Ab}=-1 \text{ N/mm}$ first, then $f_{Ab}=-l_{Ab}$, and the self-balancing internal forces can be determined by measuring the length of the corresponding line segments.

Step 2: Select the plane $BbA$ with point $b$ from Fig.4, which contains the line $bo$, and obtain the spatial relationship during form-finding, as shown in Fig.5 (a). Since $f_{Ab}=l_{Ab}$, the length of $Ab$ is decomposed to the line $bB$ and line $bo$ to obtain the force $f_{bB}$ of the bar $bB$ and the force $f_{bo}$ which is the resultant force of bar $ba$ and $bc$ in the top surface.

Step 3: In the plane $abc$, the tension $f_{ba}$ is decomposed to two sides $ba$ and $bc$ at point $b$, the tension $f_{ba}$ of the bar $ba$ and the tension $f_{bc}$ of the bar $bc$ are obtained, as shown in Fig.5 (b).

Step 4: In the plane $abc$, the direction of $ao$ has been determined in the form-finding in section 2. At point $a$, the tension $f_{ba}$ obtained in step 3 is decomposed to two sides $ao$ and $ac$, to obtain the tension $f_{ac}$ of bar $ac$, and $f_{ao}$ which is the resultant force of bar $ab$ and $ac$, as shown in Fig.5 (c).

Step 5: In the plane $abc$, the direction of $co$ has been determined in the form-finding in section 2. According to the direction of $co$, the tension $f_{bc}$ obtained in step 3 is decomposed into two sides $co$ and $ca$ to obtain $f_{co}$ which is the resultant force of bar $cb$ and $ca$, and the tension $f_{ca}$ of bar $ca$, as shown in Fig.5(d).

**Figure 5.** Force-finding of end surface $abc$.

Step 6: Select the plane $AaC$ passing point $a$ in Fig.4, which contains the line $ao$, as shown in Fig.6(a). The force $f_{ao}$ has been obtained in step 4, according to three directions $ao$, $aA$, $aC$ and according to parallelogram decomposition methods, get the force $f_{aA}$ and the force $f_{ao}$.

**Figure 6.** Force-finding of bars connecting the top and bottom nodes.

Step 7: Select the plane $CcB$ passing through point $c$ in Fig.4, which contains the line $co$, as shown in Fig.6(b). The force $f_{co}$ has been obtained in Step 5, according to three directions $co$, $cC$, $cB$ and according to parallelogram decomposition methods, get the force $f_{cC}$ and the force $f_{cb}$.

**Figure 7.** Finding point $O$.

**Figure 8.** Force-finding of bottom surface.
Step 8: Straight lines $ab$, $bc$, $ca$ intersect to line $LL$ at point $g$, $h$ and $e$ respectively, let's connect $gA$, $hB$ and $eC$. The three lines $gA$, $hB$ and $eC$ intersect at point $O$, as shown in Fig.7.

Step 9: The lines $AO$, $Ab$ and $Aa$ are in the same plane. Based on the value of $f_{Ab}$ in step 1, according to parallelogram decomposition methods, obtain $f_{AO}$ which is the resultant force of bar $AB$ and $AC$, as shown in Fig.8 (a).

Step 10: Decompose force $f_{AO}$ to line $AB$ and $AC$, obtain forces $f_{AB}$ and $f_{AC}$, as shown in Fig.8 (b). $f_{Ab}$ is used to obtain $f_{Bc}$ at point $B$, as shown in Fig.8 (c).

Through the above geometric drawing method, the internal forces of 12 bars are obtained, and each force can be directly measured from the length of the corresponding line segment in the graph. Since the balance of forces at each node is considered in the process of form-finding and force-finding, the internal forces obtained by geometric drawing naturally meet the self-balance condition. In addition, internal forces can be scaled up or down linearly.

4. Example

It is known that the bar lengths of the bottom surface $ABC$ of the triangular prism tensegrity are $l_{AB}=1000$ mm, $l_{BC}=800$ mm, and $l_{AC}=700$ mm. The spatial coordinates are $A(0, 0, 0)$, $B(984.808, -173.648, 0)$, and $C(515.129, 473.964, 0)$, as shown in Fig.3. Choose any position of point $o(500,150,545.955)$. The top end surface and the bottom surface form a angle of 20°, and intersect at line $LL$. The equation of the line $LL$ is: $x=-1000$ mm, $z=0$.

Find the form and get the three points of the top surface: $a(117.5378, -122.8456, 406.7505)$, $b(1063.7389, 140.1056, 751.1395)$, $c(292.6227, 483.6129, 470.4762)$.

Set the self-balancing internal forces: $f_{Ab} = -l_{Ab}=-1309.725$ N. According to the 10 steps of force-finding, the forces of the tensegrity can be finded as below:

$f_{ab} = 666.233$ N, $f_{bc} = 482.494$ N, $f_{ca} = 347.541$ N, $f_{Ab} = 814.118$ N, $f_{Bb} = 412.056$ N, $f_{Cc} = 548.502$ N, $f_{Ab} = 381.624$, $f_{bc} = 375.960$ N, $f_{Ab} = -1309.725$, $f_{Bc} = -856.004$ N, $f_{Ca} = -1004.856$ N.

Figure 9. Internal forces of pressure bars (N). Figure 10. Internal forces of tension bars (N).

The tensegrity structure obtained by form-finding can be analysed by ANSYS. The section of compression bar $Ab$, $Bc$ and $Ca$ is $\varnothing 40 \times 3$ with a section area of $348$ mm$^2$, and the others are tension bars of $\varnothing 8$ with a section area of $50$ mm$^2$. The elastic modulus of the material is adjusted to $2 \times 10^8$ MPa, and the initial tensile strain $=7.0428 \times 10^{-8}$ was applied to the tension bars, so as to obtain the self-balancing prestress value of the bars without much deformation. The internal forces of all the bars were calculated. The force of the bar $Ab$ is the maximum compression, it is 1309.8N as shown in Fig.9. The force of the bar $Aa$ is the maximum tension, it is 814.137N as shown in Fig.10. The maximum displacement of the structure node is only 0.0007mm.

The self-balancing forces of 12 members obtained by FEM is $(f_{AB}, f_{BC}, f_{CA}, f_{ab}, f_{bc}, f_{ca}, f_{Ab}, f_{Bb}, f_{Cc}, f_{Aa}, f_{Bb}, f_{Cc})=(381.64,375.97,639.86,666.25,482.50,347.55,-1309.8,-856.03,-1004.9,814.14,412.07,548.52)$.

Based on the example we can find, the form and the internal forces can be obtained simultaneously by the geometric drawing method. The internal forces of finite element method are the same as the first four significant digits of the results of drawing method in this paper, which proves that the geometric drawing method for form-finding and force-finding in this paper is correct.
5. Conclusion

Based on the relationship between the self-balancing forces of the triangular prism tensegrity structure and its geometric dimension when the two end surfaces are unparallel, the method and steps of geometric drawing for form-finding and force-finding are given.

(1) Geometric drawing method does not need formula calculation, it can be realized by drawing in 3d CAD, which is simple and intuitive, and can be freely realized according to designers’ requirements for structure shape.

(2) In the case that $ABC$ is fixed at three points on the bottom surface, the triangular prism tensegrity structure has eight degrees of freedom to find the shape, three degrees of freedom to determine the position of the resultant point $o$ $(x, y, z)$, two degrees of freedom to determine the direction of the plane of $abc$, and three degrees of freedom to determine points $a$, $b$ and $c$ on $oa$, $ob$ and $oc$.

(3) The method and steps to determine the self-balancing internal forces of the bars are given.

(4) The correctness of the form-finding and force-finding method is verified by an example, and the internal forces naturally meet the self-balance.

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