Chapter 1

MESOSCOPIC AHARONOV-BOHM OSCILLATIONS IN METALLIC RINGS

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Abstract We study the amplitude of mesoscopic Aharonov-Bohm oscillations in quasi-one-dimensional (Q1D) diffusive rings. We consider first the low-temperature limit of a fully coherent sample. The variance of oscillation harmonics is calculated as a function of the length of the leads attaching the ring to reservoirs. We further analyze the regime of relatively high temperatures, when the dephasing due to electron-electron interaction suppresses substantially the oscillations. We show that the dephasing length $L_{\phi}^{AB}$ governing the damping factor $\exp(-2\pi R/L_{\phi}^{AB})$ of the oscillations is parametrically different from the common dephasing length for the Q1D geometry. This is due to the fact that the dephasing is governed by energy transfers determined by the ring circumference $2\pi R$, making $L_{\phi}^{AB}$ $R$-dependent.

Keywords: mesoscopic fluctuations, Aharonov-Bohm effect, electron-electron interaction, dephasing

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1. Introduction

The Aharonov-Bohm (AB) oscillations of conductance are one of the most remarkable manifestations of electron phase coherence in mesoscopic samples. Quantum interference of contributions of different electron paths in a ring threaded by a magnetic flux $\Phi$ makes the conductance $g$ an oscillatory function of $\Phi$, with a period of the flux quantum $\Phi_0 = \hbar c/e$; see Refs. [1, 2, 3] for reviews. In a diffusive ring these $\Phi_0$-periodic conductance oscillations are sample-specific (and would vanish upon the ensemble averaging), due to a random phase associated with diffusive paths. In this respect, the $\Phi_0$-periodic AB effect is a close relative of mesoscopic conductance fluctuations.

Another type of the AB effect is induced by interference of time-reversed paths encircling the ring and is intimately related to the weak localization (WL) correction [2]. Its principal period is $\Phi_0/2$. It survives the ensemble averaging but is suppressed by a magnetic field penetrating the sample. Below we concentrate on the first (mesoscopic, or $\Phi_0$-periodic) AB effect. Our results for the dephasing are, however, applicable to the second (weak-localization, or $\Phi_0/2$-periodic) type of AB oscillations as well, as we discuss in the end of Section 3.

Interaction-induced inelastic processes lead to dephasing of electrons, and thus to a damping of interference phenomena, in particular of AB oscillations. The mesoscopic AB oscillations can thus serve as a “measuring device” for the electron decoherence. This idea was, in particular, implemented in recent experiments [4, 5], where the low-temperature behavior of the dephasing time $\tau_\phi$ was studied, and two mechanisms of decoherence were identified: scattering off magnetic impurities and electron-electron scattering.

In Section 2 we will calculate the variance of harmonics of mesoscopic Aharonov-Bohm oscillations in the low-temperature regime when the dephasing effects are negligible. In Section 3 we will then analyze the opposite limit of high temperatures when the dephasing due to electron-electron interaction strongly suppresses the amplitude of the AB oscillations.

2. Low-temperature limit: fully coherent sample

In this section we will study the mesoscopic AB oscillations in the low-temperature regime when both the thermal length $L_T$ and the dephasing length $L_\phi^{AB}$ (see Sec. 3) are much larger than the sample size. These conditions correspond to the regime of universal conductance fluctuations [6, 7, 8]. In this limit, the variance of conductance fluctuations of a Q1D wire takes a universal value $8/15$ (in units of $(e^2/h)^2$) in the absence of
spin-orbit interaction, or 2/15 for strong spin-orbit interaction. If the
time-reversal symmetry is broken by the magnetic field, these values are
reduced by a factor of 2.

A natural question is what is the counterpart of these universal values
for the AB oscillations in a mesoscopic ring. It turns out, however,
that the situation in this case is more delicate, and the amplitude of
the oscillations depends in a non-trivial way on the length of the wires
connecting the ring to the bulk electrodes.

We will assume that the wires forming the ring are thin (i.e. of Q1D
character), which allows us to solve the problem analytically. Although
the formalism we will use for this purpose is well-known [6, 7, 9], we
are not aware of such a calculation in the literature. In some earlier pa-
ners, the problem of mesoscopic AB oscillations was studied numerically
[10, 11]. In Ref. [12] some analytical calculations were performed, but
the role of leads connecting the ring to the reservoirs was completely
disregarded. In paper [13] the contacts were included, but only as an
escape probability at the junctions. The diffusive dynamics of electrons
in the leads was not taken into account. For this reason our results in
this section, although qualitatively similar to the results of [13] in the
no-dephasing regime, do differ quantitatively.

We consider a thin (Q1D) ring coupled symmetrically by two leads to
the bulk electrodes. The only geometric parameter characterizing the
problem is then the ratio \( \gamma \) of the resistance of the ring itself to the
total resistance of the ring with the leads (see Fig. 1.1). By definition,
\( 0 < \gamma < 1 \).

\[ \gamma = \frac{R_1}{R_1 + R_2} \]

\textbf{Figure 1.1.} The sample geometry. The geometric parameter \( \gamma \) is defined as the ratio
of the resistance of the ring without the leads to the resistance of the total sample,
\( \gamma = R_1/(R_1 + R_2) \).
2.1 Real-space formalism

To calculate the conductance fluctuations of a ring with leads, we use the formalism developed in Refs. [8, 10, 14]. To make the present paper self-contained we include a brief exposition of the formalism.

The real-space DC conductivity $\sigma(r, r')$ in the linear response regime at zero temperature can be calculated by the Kubo formula:

$$\sigma_{\alpha\beta}(r, r') = \frac{e^2}{4\pi m^2} \nabla_\alpha \left[ G^R(r, r') - G^A(r, r') \right] \nabla'_\beta \left[ G^A(r', r) - G^R(r', r) \right]$$  \hspace{2cm} (1.1)

where $G^A$ and $G^R$ denote the advanced and retarded Green’s functions for electrons at the Fermi level. Note that by particle number conservation the conductivity tensor must be divergenceless.

In real space, the impurity-averaged conductivity tensor $\langle \sigma \rangle$ is a long-ranged object which in leading order can be expressed diagrammatically by the sum of a bare conductivity bubble and a ladder diagram [8]. This sum is conveniently represented by defining a “flow function” $\phi$ (See Refs. [8, 14] for details),

$$\phi_{\alpha\beta}(r, r') \equiv \delta_{\alpha\beta} \delta(r - r') - \nabla_\alpha \nabla'_\beta D(r, r')$$  \hspace{2cm} (1.2)

where $D$ satisfies $-\nabla^2 D(r, r') = -\delta(r, r')$, so that

$$\langle \sigma_{\alpha\beta}(r, r') \rangle = \sigma_0 \phi_{\alpha\beta}(r, r')$$  \hspace{2cm} (1.3)

where $\sigma_0 = e^2 \nu D$ is the Boltzmann conductivity. Calculating the mesoscopic fluctuations of the conductivity (1.1), one finds:

$$\langle \delta \sigma_{\alpha\beta}(r_1, r_2) \rangle \delta \sigma_{\gamma\delta}(r_3, r_4) = \int \int \int \int \int d^4 r'_1 d^4 r'_2 d^4 r'_3 d^4 r'_4 \phi_{\alpha\alpha'}(r_1, r'_1) \phi_{\beta\beta'}(r_2, r'_2) \phi_{\gamma\gamma'}(r_3, r'_3) \phi_{\delta\delta'}(r_4, r'_4) \times \Gamma_{\alpha'\beta'\gamma'\delta'}(r'_1, r'_2; r'_3, r'_4)$$  \hspace{2cm} (1.4)

Here $\Gamma$ is given by a set of two-diffuson and two-cooperon diagrams [8, 15], yielding

$$\Gamma_{xxxx}(r_1, r_2; r_3, r_4) = 24 \delta(r_1 - r_3) \delta(r_2 - r_4) \left| \tilde{P}_D(r_1, r_2) \right|^2$$  \hspace{2cm} (1.5)

where $\tilde{P}_D$ is a rescaled diffusion propagator satisfying $(-i\nabla - eA)^2 \tilde{P}_D(r, r') = -\delta(r, r')$. To evaluate the conductance, the conductivity is integrated over the cross sections of the leads. For a sample which consists of Q1D parts we can switch to a one-dimensional formulation. It is convenient
to absorb a cross-sectional factor $S$ into the propagator, so that the
one-dimensional propagator $\tilde{P}_D^{(1)}$ is defined as

$$\tilde{P}_D^{(1)}(x, x') \equiv S \tilde{P}_D^{(3)}(r, r'), \quad (1.6)$$

where $S$ is the cross-section at the coordinate $x$. The one-dimensional
diffusion propagator satisfies the diffusion equation $-\nabla^2 \tilde{P}_D^{(1)}(x, x') = \delta(x - x')$, and the conductance fluctuations are given by

$$\langle \delta g \delta g \rangle = 24 \int dx_1' \phi^2 \left(-\frac{L}{2}, x_1'\right) \int dx_2' \phi^2 \left(\frac{L}{2}, x_2'\right) \tilde{P}_D(x_1', x_2') \tilde{P}_D(x_2', x_1'), \quad (1.7)$$

where $\pm \frac{L}{2}$ denote the ends of the leads at the reservoirs and the system
size $L$ is given by $L = \pi R/\gamma$, where $R$ is the radius of the ring.

### 2.2 Zero-temperature Aharonov-Bohm oscillations

It is convenient to work in a gauge where the vector potential gives
just a phase shift in the boundary conditions for the diffusion propa-
gator at the junctions. Then the Q1D diffusion propagator is a l inear
function of each of the coordinates in each segment of the sample. The
coefficients are determined by the boundary conditions (unit jump in the
slope for equal coordinates, vanishing propagators at the reservoirs, and
particle number conservation – modified by the phase shift due to the
vector potential – at the junctions) and can be evaluated by solving a
set of linear equations. After a lengthy but straightforward calculation
[20], integrating both coordinates of Eq. (1.7) over the entire sample
(including the leads) yields the correlator

$$\langle \delta g(\Phi) \delta g(\Phi + \delta \Phi) \rangle = D(\delta \Phi) + C(2\Phi + \delta \Phi), \quad (1.8)$$

where the diffusion contribution has the form

$$D(\delta \Phi) = \frac{1}{30} \left[ (1 - \gamma)^4 + \frac{320 \gamma^2 (1 + \gamma)^4}{\left(1 + 6\gamma + \gamma^2 - (1 - \gamma)^2 \cos (2\pi \delta \Phi / \Phi_0)\right)^2} + \frac{16 \gamma (1 + \gamma)^2 (1 - 10\gamma + \gamma^2)}{1 + 6\gamma + \gamma^2 - (1 - \gamma)^2 \cos (2\pi \delta \Phi / \Phi_0)} \right]. \quad (1.9)$$

The second term in (1.9) is the cooperon contribution which has the same
form as the diffusion one with $\delta \Phi$ replaced by $2\Phi + \delta \Phi$ (i.e. $C(x) = D(x)$)
if the magnetic flux threading the material of the ring is much less than
one flux quantum. In the opposite limit the cooperon contribution is
negligibly small. Equation 1.9 corresponds to spinful electrons in the absence of spin-orbit interaction; for strong spin-orbit interaction the result is reduced by a factor of 4. Note that in Eq. 1.9 the limits of \( \gamma \to 0 \) (small ring) and \( \delta \Phi \to 0 \) do not commute. If one first sets \( \gamma = 0 \) and then takes the limit \( \delta \Phi \to 0 \), one gets for the correlation function the value \( 1/30 \), whereas the opposite order of limits yields the value \( 4/15 \), as expected for a plain wire.

We define Fourier components of the conductance oscillations in the following way:

\[
\delta g(\Phi) = \delta g_0 + 2 \sum_{n=1}^{\infty} \delta g_n \cos \left( 2\pi n \Phi / \Phi_0 + \theta_n \right).
\]

(1.10)

The variance of the amplitude \( \delta g_n \) of the \( n \)-th harmonic of the oscillations is then found as the Fourier transform of (1.9),

\[
\langle \delta g_n^2 \rangle = \frac{1}{30} \left( \gamma^{1/2} - 1 \right)^{2n} \left( \gamma^{1/2} + 1 \right)^{-2n} \gamma^{1/2} (\gamma + 1) 
\times \left[ 9 - 10\gamma + 9\gamma^2 + 20n \gamma^{1/2} (\gamma + 1) \right]
\]

(1.11)

for \( n \geq 1 \), and

\[
\langle \delta g_0^2 \rangle = \frac{1}{30} \left( 1 + 9\gamma^{1/2} - 4\gamma - \gamma^{3/2} + 6\gamma^2 - \gamma^{5/2} - 4\gamma^3 + 9\gamma^{7/2} + \gamma^4 \right).
\]

(1.12)

The cooperon contribution does not change the oscillation amplitudes Eq. 1.11, but only affects the statistics of the phases \( \theta_n \). Specifically, if the magnetic flux through the material of the ring is small and the cooperon contribution is present, \( C(x) = D(x) \), the phases \( \theta_n \) are equal to 0 or \( \pi \). In the opposite limit, when the cooperon contribution is suppressed, the phases \( \theta_n \) are randomized. As to the aperiodic fluctuations, Eq. (1.12), the variance becomes larger by a factor of 2 is the presence of the cooperon term.

In Figure 1.2 we plot \( \langle \delta g_n^2 \rangle \) for \( n = 1, \ldots, 4 \) as a function of the geometric parameter \( \gamma \). It is seen that the oscillation amplitude depends on \( \gamma \) in a non-monotonous way, vanishing in both limits of long (\( \gamma \to 0 \)) and short (\( \gamma \to 1 \)) leads.

3. Dephasing by electron-electron interaction

We now turn to the high-temperature regime and analyze how the oscillations are suppressed due to dephasing induced by electron-electron scattering processes. We will be mainly interested in the exponential suppression factor and will not calculate the (\( \gamma \)-dependent) numerical prefactor of order unity.
Figure 1.2. Variance $\langle \delta g_n^2 \rangle$ of the first four harmonics of the oscillations in the absence of spin-orbit interaction. With spin-orbit interaction, $\langle \delta g_n^2 \rangle$ is reduced by a factor of 4.

Within the conventional approach, when the dephasing time $1/\tau_\phi$ is introduced as a mass of the diffuson and cooperon propagators, $P_{D,C}(q, \omega) \sim 1/(Dq^2 - i\omega + 1/\tau_\phi)$, the variance of the $n$-th harmonic of the oscillations is suppressed by the factor [2]

$$\langle \delta g_n^2 \rangle \sim L_T^2 L_\phi L^2 e^{-2\pi Rn/L_\phi}, \quad (1.13)$$

where $L_\phi = (D\tau_\phi)^{1/2}$ is the dephasing length, $L_T = (D/T)^{1/2}$ the thermal length, $D$ the diffusion constant, $T$ the temperature, $R$ the radius of the ring, and we set $\hbar = 1$. (It is assumed in Eq. (1.13) that $L_\phi, L_T \ll 2\pi R$.) For a thin ring, $L_\phi$ is then identified with the dephasing length governing the WL correction in the quasi-one-dimensional (Q1D) geometry, which was found by Altshuler, Aronov and Khmelnit-
ski [16, 17] to be

\[ L_\phi = (D \tau_\phi)^{1/2}, \quad \tau_\phi^{-1} \sim \left( \frac{T}{\nu D^{1/2}} \right)^{2/3}. \]  (1.14)

In fact, Aleiner and Blanter showed recently [18] that \( \tau_\phi \) relevant to the mesoscopic conductance fluctuations in wires has indeed the same form, Eq. (1.14), as the WL dephasing time. This seems to support the assumption that the dephasing times governing different mesoscopic phenomena are identical. Equations (1.13), (1.14) are commonly used for the analysis of experimental data.

We will show below, however, that contrary to the naive expectations the formulas (1.13) and (1.14) do not describe correctly the dephasing of AB oscillations. Specifically, if the interaction-induced exponential damping factor of AB oscillations is presented in the form \( \langle \delta g^2 \rangle \sim \exp(-2\pi Rn/L_{AB}^\phi) \), the corresponding length \( L_{AB}^\phi \) is parametrically different from Eq. (1.14). Moreover, \( L_{AB}^\phi \) depends on the system size \( R \).

### 3.1 Effective electron-electron interaction

Following Refs. [16, 17, 18, 19], the electron-electron interaction can be represented by external time-dependent random fields \( \varphi^\alpha(\mathbf{r}, t) \), with the correlation function \( \langle \varphi^\alpha(\mathbf{r}, t) \varphi^\beta(\mathbf{r}', t') \rangle \) determined from the fluctuation-dissipation theorem,

\[ \langle \varphi^\alpha(\mathbf{r}) \varphi^\beta(\mathbf{r}') \rangle_\omega = -\text{Im} \, U(\mathbf{r}, \mathbf{r}'; \omega) \delta_{\alpha\beta} \coth \frac{\omega}{2T}. \]  (1.15)

The conventional form for the dynamically screened Coulomb interaction in a diffusive system is [17]

\[ U(q, \omega) = \frac{1}{U_0^{-1}(q) + \Pi(q, \omega)} \simeq \Pi^{-1}(q, \omega), \]  (1.16)

where \( U_0(q) \) is the bare Coulomb interaction, \( \Pi(q, \omega) = \nu D q^2 / (D q^2 - i\omega) \) is the polarization operator, and \( \nu \) is the density of states. As we will see below, the characteristic momenta \( q \) for our problem are of the order of the inverse system size \( R^{-1} \). In view of the non-trivial geometry of our system, it is thus more appropriate to work in the coordinate representation. A corresponding generalization of Eq. (1.16) can be readily obtained, yielding

\[ \text{Im} \, U(\mathbf{r}, \mathbf{r}'; \omega) \simeq \text{Im} \, \Pi^{-1}(\mathbf{r}, \mathbf{r}'; \omega) = -\frac{\omega}{\nu D} D(\mathbf{r}, \mathbf{r}'), \]  (1.17)
where $D$ is the propagator for the Laplace equation, $-\nabla^2 D(r, r') = \delta(r-r')$ with zero boundary conditions at the contacts with bulk electrodes. Substituting Eq. (1.17) in Eq. (1.15), we get, for relevant frequencies $\omega \ll T$,

$$
\left\langle \varphi^\alpha(r, t) \varphi^\beta(r', t') \right\rangle = \frac{2T}{\nu D} D(r, r') \delta_{\alpha\beta} \delta(t-t') .
$$

(1.18)

### 3.2 Aharonov-Bohm oscillations in presence of interaction

Since the ring we are considering consists of Q1D wires, the corresponding diffusion propagator satisfies the one-dimensional diffusion equation

$$
\left\{ \partial_t - D \partial_x^2 + i \left[ \varphi^\alpha(x, t) - \varphi^\beta(x, t) \right] \right\} P^\alpha\beta_{\delta\Phi}(x, t; x', t') = \delta(x-x') \delta(t-t')
$$

(1.19)

supplemented by appropriate matching conditions at junctions of the ring and leads. Here $\delta \Phi = \Phi_1 - \Phi_2$ is difference in the AB flux between the two measurements, which is incorporated in the matching conditions. The conductance correlation function in the high-$T$ limit is again given by the conventional two-diffuson diagrams [15, 8] (again we drop the cooperon contribution which affects only the phases but not the amplitudes of the oscillations), yielding (we drop the prefactor of order unity)

$$
\left\langle \delta g(\Phi_1) \delta g(\Phi_2) \right\rangle \sim \frac{D^2}{TRt^4} \int dx_1 dx_2 \int dt dt' \times \tilde{\delta}(t-t') \left\langle P^{12}_{\delta\Phi}(x_1, x_2, t) P^{21}_{\delta\Phi}(x_2, x_1, t') \right\rangle
$$

(1.20)

where angular brackets denote averaging over the external fields, $\tilde{\delta}(t)$ is given by

$$
\tilde{\delta}(t) = 12\pi T \int \frac{d\epsilon_1}{2\pi} \frac{d\epsilon_2}{2\pi} f'(\epsilon_1) f'(\epsilon_2) e^{i(\epsilon_1-\epsilon_2)t}
$$

$$
= 3\pi T^3 t^2 \sinh^{-2}(\pi T t) ,
$$

(1.21)

and $f(\epsilon)$ is the Fermi distribution function. The function $\tilde{\delta}(t)$ is peaked at $t = 0$ with a width $T^{-1}$. We will replace it below by the delta-function $\delta(t)$. This is justified if the dephasing effect during the time $t-t' \sim T^{-1}$ is negligible, i.e. $\left\langle \phi^\alpha(x, t) \phi^\alpha(x, t) \right\rangle T^{-1} \ll 1$. Using Eq. (1.18), we find that the latter condition is equivalent to the requirement that the conductance of the sample is much larger than the conductance quantum
\[ e^2/h \simeq (25 \text{k}\Omega)^{-1}. \] This condition is well satisfied in typical experiments with metallic rings, thus justifying the replacement \( \delta(t) \rightarrow \delta(t) \).

We now express the diffusion propagators in Eq. (1.19) as path integrals. We are interested in the regime of strong dephasing, when the relevant paths propagate only inside the ring and do not extend into the leads (see below). It is convenient to introduce the angular coordinate \( \theta \) on the ring \((-\pi \leq \theta \leq \pi)\), with \( \theta = \pm \pi/2 \) corresponding to the junctions with the leads.

Expanding the conductance fluctuations in Fourier harmonics with respect to the flux, \( \delta g(\Phi) \rightarrow \delta g_n \), we then get

\[
\langle \delta g_n^2 \rangle \sim \frac{D^2}{TR^4} \int d\Theta_1 d\Theta_2 \int dt \frac{\Theta_1}{\Theta_2} \frac{\Theta_2}{\Theta_1} \int D\theta_1(t) \int D\theta_2(t) \times \exp \left\{ -\int_0^t dt' \left[ \frac{R^2\theta_1^2}{4D} + \frac{R^2\theta_2^2}{4D} + V(\theta_1, \theta_2) \right] \right\},
\]

where the path integral goes over pairs of paths \( \theta_1(t), \theta_2(t) \) which have a relative winding number \( n \). The “potential” \( V(\theta_1, \theta_2) \) in Eq. (1.22) is given by \( V(\theta_1, \theta_2) = \langle (\phi^{\alpha}(\theta_1) - \phi^{\alpha}(\theta_2))^2 \rangle \); its explicit form can be straightforwardly obtained according to Eq. (1.18) by solving the diffusion equation in the ring with leads as in Section 2.2 and presented in detail in [20]. We will only need below the form of \( V(\theta_1, \theta_2) \) for both
coordinates being in the same arm of the ring. For $|\theta_i| \leq \pi/2$ we find

$$V(\theta_1, \theta_2) = \frac{2TR}{\nu D} \left[ |\theta_1 - \theta_2| - \frac{\gamma + 1}{2\pi} (\theta_1 - \theta_2)^2 \right];$$

(1.23)

the expression for $|\theta_i| > \pi/2$ follows from symmetry considerations.

### 3.3 Strong-dephasing limit

We consider first the fundamental harmonic ($n = 1$) of the AB oscillations; a generalization to higher harmonics, $n = 2, 3, \ldots$ will be done in the end. For $n = 1$ the relevant pairs of paths interfere after half-encircling the ring in the opposite directions. We are interested in the regime of a relatively high temperature, when the dephasing effect is strong. In this case, the path integral in Eq. (1.22) can be evaluated via the saddle-point method. As has been mentioned above, the paths representing the saddle-point solution (instanton) do not extend into the leads. Indeed, exploring a part of a lead and returning back into the ring would only increase the action of the path. It is clear from the symmetry considerations that the optimal paths satisfy $\theta_1(t) = -\theta_2(t)$. Furthermore, it is easy to see that the optimal initial and final points are located at maximum distance from the leads, i.e. $\Theta_1 = 0$ and $\Theta_2 = \pi$ (or vice versa). To within exponential accuracy, the problem is then reduced to that of a particle of mass $R^2/D$ tunneling with zero energy in the potential

$$V(\theta) = \frac{4TR}{\nu D} \times \left\{ \begin{array}{ll}
\left[ \theta - \frac{\gamma + 1}{\pi} \theta^2 \right], & 0 \leq \theta \leq \frac{\pi}{2} \\
\left[ (\pi - \theta) - \frac{\gamma + 1}{\pi} (\pi - \theta)^2 \right], & \frac{\pi}{2} \leq \theta \leq \pi 
\end{array} \right.$$  

(1.24)

from $\theta = 0$ to $\theta = \pi$. Since the potential is composed of quadratic parts, the corresponding instanton action is easily calculated, yielding

$$\langle \delta g_1^2 \rangle \propto e^{-S}$$

with

$$S = C_\gamma \frac{T^{1/2} R^{3/2}}{\nu^{1/2} D}.$$  

(1.25)

Here $C_\gamma$ is a coefficient of order unity depending on the geometrical factor $\gamma$,

$$C_\gamma = \left[ \frac{\pi}{2(\gamma + 1)} \right]^{3/2} \left[ 2\gamma (1 - \gamma^2)^{1/2} + \pi + 2 \arcsin \gamma \right];$$  

(1.26)

$C_\gamma$ is equal to $\frac{\pi^{5/2}}{2\gamma^{3/2}}$ in the limit of long leads ($\gamma \to 0$) and to $\frac{\pi^{5/2}}{4}$ in the limit of short leads ($\gamma \to 1$).
The above calculation can be straightforwardly generalized to higher harmonics of the AB oscillations, \( n = 2, 3, \ldots \). The optimal paths still begin at \( \Theta = 0 \) or \( \pi \) but now perform \( n/2 \) windings in the opposite directions. Therefore, the corresponding action is \( S_n = nS \).

To calculate the preexponential factor, we have to take into account small fluctuations of the initial and final points \( \Theta_1, \Theta_2 \) around their optimal values, as well as fluctuations of the paths \( \theta_1(t), \theta_2(t) \) around the instanton solution. We will only calculate the parametric dependence of the prefactor, neglecting numerical factors of order unity. First, let us consider small offsets of the initial and final points of the paths from their optimal position. The second order variation of the action \( \delta^2 S \) will be a quadratic form of the offsets, \( \delta^2 S = u_{ij} \delta \Theta_i \delta \Theta_j \), where \( i = 1, 2 \). Using that \( \delta^2 S \sim 1 \) for \( \delta \Theta_i \sim 1 \), we get

\[
(\det u_{ij})^{-1/2} \sim S^{-1}.
\] (1.27)

Second, we have to account for small deviations of the paths from the instanton solution. The corresponding factor can be identified as the propagator for a harmonic oscillator with the parameters \( m \sim R^2/D \) and \( m\omega^2 \sim RT/D\nu \). There are two such factors (one for each of the
paths), yielding together
\[ \left[ (m\omega)^{1/2} \right]^2 \sim \frac{T^{1/2} R^{3/2}}{\nu^{1/2} D}. \] (1.28)

Finally, there is a Gaussian integration over the deviations of the time \( t \) spent on the path from its optimal value \( t_{\text{opt}} \sim (\nu R/T)^{1/2} \). The corresponding factor can be estimated as
\[ \left( \frac{\partial^2 S}{\partial t^2} \right)_{t=t_{\text{opt}}} \sim \left( \frac{S}{t_{\text{opt}}^2} \right) \sim \nu^{3/4} D^{1/2} \sim \frac{T^{3/4} R^{1/4}}{1}. \] (1.29)

Combining Eqs. (1.22), (1.27), (1.28) and (1.29), we obtain the final result for the variance of the harmonics of the mesoscopic AB oscillations
\[ \langle \delta g_n^2 \rangle \sim \left( \frac{L_T}{R} \right)^{7/2} \left( \frac{L^{\text{AB}} \phi}{R} \right)^{3/2} e^{-2\pi n R/L^{\text{AB}} \phi}, \] (1.30)
where \( n = 1, 2, \ldots \), and the action \( S \) is given by Eq. (1.25).

### 3.4 Aharonov-Bohm dephasing time

Let us discuss the obtained result (1.30), (1.25). First of all, it is essentially different from what one would obtain by using the formulas (1.13), (1.14). Indeed, the exponent in Eq. (1.30) scales in a different way with the temperature and with the system size, as compared to Eqs. (1.13), (1.14). It is instructive to rewrite Eq. (1.30) in a form analogous to Eq. (1.13),
\[ \langle \delta g_n^2 \rangle \sim \left( \frac{L_T}{R} \right)^{7/2} \left( \frac{L^{\text{AB}} \phi}{R} \right)^{3/2} e^{-2\pi n R/L^{\text{AB}} \phi}, \] (1.31)
thus defining the Aharonov-Bohm dephasing length \( L^{\text{AB}} \phi \),
\[ L^{\text{AB}} \phi = \frac{2\pi}{C_\gamma} \frac{\nu^{1/2} D}{T^{1/2} R^{1/2}}. \] (1.32)

The corresponding dephasing rate \( 1/\tau_{\phi}^{\text{AB}} = D/(L^{\text{AB}} \phi)^2 \) is thus given by
\[ 1/\tau_{\phi}^{\text{AB}} = \left( \frac{C_\gamma}{2\pi} \right)^2 \frac{TR}{\nu D}. \] (1.33)

To shed more light on the physical reason for the difference between the conventional Q1D formula (1.14) and our result (1.32), (1.33), the
following qualitative argument is instructive. Calculating perturbatively the dephasing rate using the formula (1.16) for the screened Coulomb interaction in a diffusive system, one gets

\[
\tau_\phi^{-1} = \int \frac{dq}{2\pi} \frac{T}{\nu D q^2}.
\] 

(1.34)

In the calculation of the dephasing rate in a wire [17, 19, 18], the arising infrared divergence is cut off self-consistently, since only processes with energy transfers \( \omega > \tau_\phi^{-1} \) contribute. As a result, the lower limit of integration in Eq. (1.34) is \( \sim L_\phi^{-1} \), yielding the result (1.14). On the other hand, in the case of the Aharonov-Bohm dephasing rate, the relevant paths have to encircle the ring. Therefore, despite the fact that \( L_{AB} \ll 2\pi R \), the low-momentum cutoff in Eq. (1.34) is set by the inverse system size \( (2\pi R)^{-1} \). This yields \( 1/\tau_{AB}^\phi \sim TR/\nu D \), reproducing (up to a numerical coefficient) the result (1.33).

It is worth emphasizing that our result Eq. (1.33) for the dephasing rate depends not only on the ring radius, but also on the geometry of the leads through the coefficient \( C_\gamma \). We note a certain similarity between this result and the dependence of the dephasing rate in a ballistic AB-ring on the probe configuration recently found in [21].

As has been mentioned in the introduction, our results are also applicable to the WL \((h/2e\)-periodic\) AB-oscillations. Their \( n \)-th harmonic \( \delta g_{n, WL} \) is determined by cooperon paths with winding number \( n \). Assuming that the magnetic flux penetrating the sample is negligible and comparing the path-integral representations for \( \langle \delta g_{n, WL}^2 \rangle \) and \( \delta g_{n, WL} \) we find

\[
\langle \delta g_{n}^2 \rangle = \frac{e^2 D}{3\nu L^2} \delta g_{n, WL}^{WL}
\] 

(1.35)

where \( L = \pi R/\gamma \). This implies, in particular, that the dephasing length \( L_{AB}^\phi \), Eq. (1.32), is the same for both types of the AB effect. Equation (1.35) is a generalization of the relation [18] between the WL correction and conductance fluctuations for single-connected geometries.

4. Summary

In Section 2 we have studied the amplitude of mesoscopic Aharonov-Bohm oscillations in the low-temperature (phase-coherent) regime. We have shown that, in contrast to fluctuations in a wire, the oscillation amplitude does not take a universal value, but depends on the geometric factor \( \gamma \) characterizing the ratio of resistances of the ring and the leads. The \( \gamma \)-dependence of the variance of the oscillation harmonics is non-monotonous as shown in Figure 1.2. For a typical experiment [22, 23, 4],
where the resistances of the ring and the leads are comparable (so that \(\gamma \approx 0.5\)), our results predict the r.m.s. amplitude of the first harmonic \(\text{rms}(2\delta g_1) \approx 0.16\). This value compares well with the result of [23] at low bias voltage, as well as with the low-temperature results of [4] in strong magnetic fields, when the magnetic impurities are frozen.

In Section 3 we have studied how the Aharonov-Bohm oscillations are suppressed by dephasing caused by the electron-electron interaction. Using the path integral formalism and the instanton method, we have obtained the result (1.30), (1.25) which is parametrically different from the naive expectation (1.13), (1.14). This demonstrates that the Aharonov-Bohm dephasing rate \(1/\tau_{\phi}^{AB}\), Eq. (1.33), is parametrically different from the dephasing rate \(1/\tau_{\phi}\), Eq. (1.14), corresponding to a single-connected geometry. Physically, the difference can be traced back to the fact that \(1/\tau_{\phi}\) is determined self-consistently by the processes with energy transfers of the order of \(1/\tau_{\phi}\) itself (or equivalently with momentum transfers \(\sim 1/L_{\phi}\)), while the characteristic energy and momentum transfers governing \(1/\tau_{\phi}^{AB}\) are determined by the system size. For this reason, the Aharonov-Bohm dephasing rate \(1/\tau_{\phi}^{AB}\) depends on the ring radius \(R\), diverging in the limit \(R \to \infty\).

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