Fuzzy Linear space using triangular fuzzy numbers

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Abstract. In this paper, we discussed the elementary structure of linear space for triangular fuzzy system, linear subspace of the triangular fuzzy numbers and linear transformations between two triangular fuzzy spaces. We introduce a study of new concept of a Hamel base for any linear space and the uniqueness of its cardinality of fuzzy numbers. We begin with some basic definitions and properties of linear space triangular fuzzy system which will be used throughout this paper. Also the axioms of the primary field and the addition and scalar multiplication operations for triangular fuzzy system have been included.

Keywords: Linear space, triangular fuzzy numbers, linear transformations, etc.

1. Introduction
A linear space, denoted by $F$, is a primary field of (real or complex) numbers. The elements of the linear space are called scalars which are denoted by lower case Greek letters, for example, $\lambda$ and $\mu$. $\text{Re} \lambda$ denotes real part and $\text{Im} \lambda$ denotes the imaginary part of a complex number. However, once in a while the symbols $m$ or $M$ may be used to denote a scalar which is used to represent a type of boundedness.

The upper case letter $T$ denotes the linear transformation between linear spaces. When the range and domain of the transformation are linear topological spaces, it can be referred to as a linear map.

In 1965, the concept of fuzzy set theory was introduced first by Zadeh [6] and thereafter, several authors have developed the concept through the contribution of the different articles and applied the same on several branches of pure and applied mathematics. Katsaras [4] introduced the concept of fuzzy norm in 1984 and in 1992. The idea of fuzzy norm linear space was introduced by Felbin [2]. A different idea of fuzzy norm on a linear space was introduced by Cheng - Moderson [1] whose related metric is same as defined by Katsaras [4]. Later on Bag and Samanta [3] customized the definition of fuzzy norm of Cheng – Moderson and there after they have studied finite dimensional fuzzy normed linear spaces and established the concept of continuity and boundedness of a function with respect to their fuzzy norm. Moreover, the definition of intuitionistic fuzzy n-normed linear space was introduced in the paper [5] and a sufficient condition for an intuitionistic fuzzy n-normed linear space to be complete was established.

In this paper, we use the above generalized notion of fuzzy linear space and triangular fuzzy numbers in order to introduce a new generalized triangular fuzzy linear space, triangular fuzzy Hamel space and also the triangular fuzzy linear transformation and the work is extended to obtain the important theorem results.

2. Preliminaries
In this section, some basic definitions of fuzzy numbers have been given.

2.1 Fuzzy number
Fuzzy numbers are of incredible significance in fuzzy frameworks. The fuzzy numbers that generally applied as a part of utilizations are the triangular (shaped) and the trapezoidal (shaped) fuzzy numbers [20].

2.2 Definition
A fuzzy number is defined as \( \tilde{U} : R \rightarrow I = [0, 1] \) satisfying [20],

i) \( \tilde{U} \) is upper semi continuous,

ii) \( \tilde{U}(y) = 0 \) outside some interval \([c, d]\),

iii) There are real numbers \( a, b \) in such a way that \( c \leq a \leq b \leq d \) and

- \( \tilde{U}(y) \) is monotonic increasing on \([c, a]\)
- \( \tilde{U}(y) \) is monotonic decreasing on \([b, d]\)
- \( \tilde{U}(y) = 1, \forall y \in [a, b] \)

\( F(\Re) \) denotes the set of all fuzzy numbers. An identical parametric is additionally given in [16]. Another definition or parametric form of a fuzzy number which gives the same \( F(\Re) \) is given by Kaleva[30].

Arithmetic operations between two triangular fuzzy numbers defined on universal set of real numbers \( \Re \) are investigated in[14].

2.3 Triangular Fuzzy Number
It is a fuzzy number represented with three points as given: \( \tilde{U} = (\tilde{u}_1, \tilde{u}_2, \tilde{u}_3) \)

This is interpreted as membership functions and holds the accompanying conditions

i) \( \tilde{u}_1 \) to \( \tilde{u}_2 \) is increasing function

ii) \( \tilde{u}_2 \) to \( \tilde{u}_3 \) is decreasing function

iii) \( \tilde{u}_1 \leq \tilde{u}_2 \leq \tilde{u}_3 \)

\[ \mu_{\tilde{U}}(x) = \begin{cases} 0, & \text{for } y < \tilde{u}_1 \\ \frac{y - \tilde{u}_1}{\tilde{u}_2 - \tilde{u}_1}, & \text{for } \tilde{u}_1 \leq y \leq \tilde{u}_2 \\ \frac{\tilde{u}_3 - y}{\tilde{u}_3 - \tilde{u}_2}, & \text{for } \tilde{u}_2 \leq y \leq \tilde{u}_3 \\ 0, & \text{for } y > \tilde{u}_3 \end{cases} \]

2.4 Positive triangular fuzzy number
A fuzzy number \( \tilde{U} = (\tilde{u}_1, \tilde{u}_2, \tilde{u}_3) \) is said to be positive triangular if all \( \tilde{u}_j \)'s > 0 for \( j = 1, 2, 3 \).

2.5 Negative triangular fuzzy number
A fuzzy number \( \tilde{U} = (\tilde{u}_1, \tilde{u}_2, \tilde{u}_3) \) is said to be negative triangular if all \( \tilde{u}_j \)'s < 0 for \( j = 1, 2, 3 \).

2.5.1 Note.
A negative triangular fuzzy number can be composed as the negative multiplication of a positive triangular fuzzy number.
2.6 Equal triangular fuzzy numbers
Consider \( \tilde{U} = (\tilde{u}_1, \tilde{u}_2, \tilde{u}_3) \) and \( \tilde{V} = (\tilde{v}_1, \tilde{v}_2, \tilde{v}_3) \) as two triangular fuzzy numbers. If \( \tilde{U} \) is identically equal to \( \tilde{V} \) only if \( \tilde{u}_1 = \tilde{v}_1, \tilde{u}_2 = \tilde{v}_2 \) and \( \tilde{u}_3 = \tilde{v}_3 \).

### 3. Elementary Properties

Some elementary properties of linear space are discussed here.

#### 3.1. Definition of linearly independent

Let \( \tilde{Y} \) be a linear space. A subset \( \tilde{A} \) of it is called linearly independent given that for every finite subset \( \{ \tilde{U}_1, \tilde{U}_2, ..., \tilde{U}_n \} \) of \( \tilde{A} \), where \( \tilde{U}_1 = (u_1^1, u_1^m, u_1^n), \tilde{U}_2 = (u_2^1, u_2^m, u_2^n) \) and so on, the following conditions hold:

\[
\begin{align*}
\lambda_1 \tilde{U}_1 + \lambda_2 \tilde{U}_2 + \cdots + \lambda_n \tilde{U}_n &= 0 \\
\lambda_1(u_1^1, u_1^m, u_1^n) + \lambda_2(u_2^1, u_2^m, u_2^n) + \cdots + \lambda_n(u_n^1, u_n^m, u_n^n) &= 0 \\
\lambda_1u_1^1 + \lambda_2u_2^1 + \cdots + \lambda_nu_n^1 &= 0 \\
\lambda_1u_1^m + \lambda_2u_2^m + \lambda_nu_n^m &= 0 \\
\lambda_1u_1^n + \lambda_2u_2^n + \lambda_nu_n^n &= 0 \\
\text{if and only if in all the above equations } &\lambda_1 = \lambda_2 = \cdots = \lambda_n = 0.
\end{align*}
\]

#### 3.2. Definition of linear span

Let \( \tilde{A} \) be the subset of the linear space \( \tilde{Y} \). Then Span of \( \tilde{A} \), is the set of all finite linear combinations of vectors in \( \tilde{A} \), given by

\[
\text{Span}(\tilde{A}) = \{ \lambda_1 \tilde{U}_1 + \lambda_2 \tilde{U}_2 + \cdots + \lambda_n \tilde{U}_n ; n \in N, \tilde{U}_j \in \tilde{A}, \lambda_j \in F, 1 \leq j \leq n \}
\]

\[
\text{Span}(\tilde{A}) = \{ \lambda_1(u_1^1, u_1^m, u_1^n) + \lambda_2(u_2^1, u_2^m, u_2^n) + \cdots + \lambda_n(u_n^1, u_n^m, u_n^n) ; n \in N, \tilde{U}_j \in \tilde{A}, \lambda_j \in F, 1 \leq j \leq n \}
\]

#### 3.3. Definition of Hamel base

Let \( \tilde{Y} \) be the linear space. A subset \( \tilde{H} \) of it is said to be a Hamel base if and only if for every non zero vector \( \tilde{y} \in \tilde{Y} \), there exists a unique set \( \{ \tilde{h}_1, \tilde{h}_2, ..., \tilde{h}_n \} \) of vectors in \( \tilde{H} \) and a unique set \( \{ \lambda_1, \lambda_2, ..., \lambda_n \} \) of scalars so that

\[
\tilde{y} = \lambda_1 \tilde{h}_1 + \lambda_2 \tilde{h}_2 + \cdots + \lambda_n \tilde{h}_n
\]

#### 3.4. Definitions

Consider two subsets \( \tilde{U} \) and \( \tilde{V} \) of a linear space \( \tilde{Y} \). Then the algebraic sum \( \tilde{U} + \tilde{V} \) is the set consisting of all sums \( (u_1^1 + v_1^1, u_1^m + v_1^m, u_1^n + v_1^n) \) where \( (u_1^1, u_1^m, u_1^n) \in \tilde{U}_1 \) and \( (v_1^1, v_1^m, v_1^n) \in \tilde{V}_1 \).

Also, if \( \tilde{y} \in \tilde{Y} \), then the \( \tilde{y} \)-translate of \( \tilde{U} \) is given by

\[
\tilde{y} + \tilde{U} = \{ (y_1^1 + u_1^1, y_1^m + u_1^m, y_1^n + u_1^n) ; (u_1^1, u_1^m, u_1^n) \in \tilde{U}_1 \}
\]

Consider a scalar \( \lambda \in F \) and \( \tilde{U} \) be the subset of \( \tilde{Y} \), then \( \lambda \tilde{U} \) is the set of all scalar multiples \( \lambda(u_1^1, u_1^m, u_1^n) \) with \( (u_1^1, u_1^m, u_1^n) \in \tilde{U} \).
4. Linear transformation and theorem on fuzzy linear space

In the following theorem, Hamel base of linear system and the uniqueness of its cardinality of triangular fuzzy numbers have been proved.

4.1. Definition of linear transformation

A linear transformation \( \tilde{T} \) from a linear space \( \tilde{Y}_1 \) to a linear space \( \tilde{Y}_2 \) (on the same scalar field \( F \)) is a function which satisfies

i) \( \tilde{T}(\tilde{y}_1 + \tilde{y}_2) = \tilde{T}(\tilde{y}_1) + \tilde{T}(\tilde{y}_2), \forall \tilde{y}_1, \tilde{y}_2 \in \tilde{Y} \) and

ii) \( \tilde{T}(\lambda \tilde{y}_1) = \lambda \tilde{T}(\tilde{y}_1), \forall \tilde{y}_1 \in \tilde{Y} \) and for all \( \lambda \in F \).

4.2. Theorem

Let \( \tilde{T} \) be the linear transformation from the linear space \( \tilde{Y}_1 \) to the linear space \( \tilde{Y}_2 \).

a) If \( \tilde{U}, \tilde{V} \subset \tilde{Y}_1 \), then \( \tilde{T}(\tilde{U} + \tilde{V}) = \tilde{T}(\tilde{U}) + \tilde{T}(\tilde{V}) \)

b) If \( \tilde{U} \subset \tilde{Y}_1 \) and \( \lambda \in F \), then \( \tilde{T}(\lambda \tilde{U}) = \lambda \tilde{T}(\tilde{U}) \)

Proof

a) If \( \tilde{y}_2 \in \tilde{T}(\tilde{U} + \tilde{V}) \), then there exists a vector \( y \in \tilde{U} + \tilde{V} \), such that \( y_2 = \tilde{T}(y_1) \).

\[
(y_1^l, y_1^r, y_1^n) \in (u^l + v^l, u^r + v^r, u^n + v^n)
\]

\[
(y_2^l, y_2^r, y_2^n) = \tilde{T}(y_1^l, y_1^r, y_1^n)
\]

\[
\Rightarrow y_2^l = \tilde{T}(y_1^l)
\]

\[
y_2^r = \tilde{T}(y_1^r)
\]

\[
y_2^n = \tilde{T}(y_1^n)
\]

Because \((y_1^l, y_1^r, y_1^n) \in (u^l + v^l, u^r + v^r, u^n + v^n)\), we can write

\[
y_1^l = u_1^l + v_1^l, y_1^r = u_1^r + v_1^r, y_1^n = u_1^n + v_1^n
\]

where \((u_1^l, u_1^r, u_1^n) \in \tilde{U} \) and \((v_1^l, v_1^r, v_1^n) \in \tilde{V} \).

Consequently,

\[
y_2 = \tilde{T}(y_1)
\]

\[
(y_2^l, y_2^r, y_2^n) = \tilde{T}(y_1^l, y_1^r, y_1^n)
\]

\[
= \tilde{T}(u_1^l + v_1^l, u_1^r + v_1^r, u_1^n + v_1^n)
\]

\[
= \tilde{T}(u_1^l, u_1^r, u_1^n) + \tilde{T}(v_1^l, v_1^r, v_1^n)
\]

\[
\in \tilde{T}(\tilde{U} + \tilde{V})
\]

Therefore,

\[
\tilde{T}(\tilde{U} + \tilde{V}) = \tilde{T}(\tilde{U}) + \tilde{T}(\tilde{V})
\]

b) If \( y_2 \in \tilde{T}(\lambda \tilde{U}) \), then there exists a point \( y_1 \in \lambda \tilde{U} \) such that

\[
y_2 = \tilde{T}(y_1) \Rightarrow (y_2^l, y_2^r, y_2^n) = \tilde{T}(y_1^l, y_1^r, y_1^n)
\]

\[
\Rightarrow y_2^l = \tilde{T}(y_1^l)
\]

\[
y_2^r = \tilde{T}(y_1^r)
\]

\[
y_2^n = \tilde{T}(y_1^n)
\]
Because \( y_i^l \in \lambda u^l, y_i^m \in \lambda u^m, y_i^n \in \lambda u^n \), we can write
\[
y_i = \lambda(u^l, u^m, u^n).
\]

As a result,
\[
y_2 = T(y_1)
= T(y_i^l, y_i^m, y_i^n)
= T(\lambda u^l, \lambda u^m, \lambda u^n)
= \lambda T(u^l, u^m, u^n)
= \lambda T(\lambda U).
\]

Conversely, if \( y_1 \in \lambda T(\lambda U) \), then we can write \( y_i = \lambda y_2 \), where \( y_2 \in \lambda T(\lambda U) \).

Here \( y_2 = (y_2^l, y_2^m, y_2^n) \) and \( \lambda U = (u^l, u^m, u^n) \).

So there exists \( (u^l, u^m, u^n) \in \lambda U \), such that
\[
(y_2^l, y_2^m, y_2^n) = \lambda T(u^l, u^m, u^n)
= T(\lambda u^l, \lambda u^m, \lambda u^n).
\]

Therefore,
\[
(y_i^l, y_i^m, y_i^n) = \lambda (y_2^l, y_2^m, y_2^n) = \lambda T(u^l, u^m, u^n)
= T(\lambda u^l, \lambda u^m, \lambda u^n)
\in \lambda T(\lambda U).
\]

Thus, \( T(\lambda U) = \lambda T(\lambda U) \).

5. Conclusion
The current work is by constructing the elementary structure of the linear space and the linearly independent sets using triangular fuzzy numbers, we have shown Hamel base of linear system and the uniqueness of its cardinality of triangular fuzzy numbers. Also we have established linear transformation of a linear space using triangular fuzzy numbers. Finally the existence of a linear transformation for any linear space is proved.

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