On Quantum Statistical Mechanics of a Schwarzschild Black Hole

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Abstract

Quantum theory of geometry, developed recently in the framework of non-perturbative quantum gravity, is used in an attempt to explain thermodynamics of Schwarzschild black holes on the basis of a microscopical (quantum) description of the system. We work with the formulation of thermodynamics in which the black hole is enclosed by a spherical surface $B$ and a macroscopic state of the system is specified by two parameters: the area of the boundary surface and a quasilocal energy contained within. To derive thermodynamical properties of the system from its microscopics we use the standard statistical mechanical method of Gibbs. Under a certain number of assumptions on the quantum behavior of the system, we find that its microscopic (quantum) states are described by states of quantum Chern-Simons theory defined by sets of points on $B$ labelled with spins. The level of the Chern-Simons theory turns out to be proportional to the horizon area of black hole measured in Planck units. The statistical mechanical analysis turns out to be especially simple in the case when the entire interior of $B$ is occupied by a black hole. We find in this case that the entropy contained within $B$, that is, the black hole entropy, is proportional to the horizon surface area.

Key words: black hole thermodynamics, quantum gravity

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I. INTRODUCTION

The statistical mechanical method of Gibbs, developed in his celebrated *Elementary Principles of Statistical Mechanics* (1902), turned out to be the most efficient tool for explaining the bulk properties of matter on the basis of its microscopics. So far, self-gravitating systems have resisted the application of this method. In this paper we apply the statistical mechanical method of Gibbs to such systems. The aim of the paper is to try to derive thermodynamics of gravitational systems using methods developed by Gibbs.

Our discussion is restricted to a simple gravitational system whose thermodynamics is widely studied: a spherically symmetric uncharged black hole. Although this paper deals only with a special example of gravitational system, there is a natural way to generalize the ideas presented to other types of black holes (see [1]).

As a basis of our discussion we take the thermodynamics of self-gravitating systems in the form it is formulated by Martinez [2]. In this formulation the gravitational system composed of a spherically symmetric black hole is characterized by the surface area $A = 4\pi R^2$ of a two-dimensional spherical boundary surface $B$ (located at $r = R$) that encloses the black hole, and a quasilocal energy $E$ contained within. Following Martinez [2], we use the quasilocal energy of Brown and York [3] as $E$. In equilibrium states the system is completely characterized macroscopically when these two variables are specified (see [2]).

The expression of the entropy function $S$ as a function $S(A, E)$ of extensive variables is called fundamental equation of a thermodynamical system. Once known, the fundamental equation contains all thermodynamical information about the system. For the system of our interest the fundamental equation is given by [2]

$$S(A, E) = 4\pi E^2 \left(1 - \frac{E}{2R}\right)^2,$$

which ‘radius’ $R$ is used as an extensive variable instead of the area $A$. Due to the spherical symmetry of the system one can use $R$ and $A$ interchangeably.

The fundamental equation (1) is derived from Hawking’s semiclassical expression [4] for the temperature of black hole radiation. It is our aim in this paper to try to explain the fundamental equation (1) on the basis of the microscopics of the system. To derive the fundamental equation (1) we apply the Gibbs’ method. The method of Gibbs, when adapted to our case, is to construct a function $Q(\alpha, \beta)$ called the statistical sum of the system, which is a function of intensive parameters of the system (here $\alpha$ stands for the product $\beta p$ of the inverse temperature $\beta$ and the quantity $p$ that plays a role of the ‘surface pressure’; see [2]).

The statistical sum is given by

$$Q(\alpha, \beta) = \text{Tr} \ e^{-\alpha \hat{A} - \beta \hat{E}},$$

where $\hat{A}$ and $\hat{E}$ are quantum mechanical operators corresponding to the classical quantities $A, E$. The statistical sum (2) contains all thermodynamical information about the system. In particular, the fundamental equation of the system can be obtained from $Q(\alpha, \beta)$ by means of well-known thermodynamical relations.

As a basis of microscopical description of our system we use the one given by non-perturbative quantum gravity based on the loop representation [5] (see also [6]). An important modification arises, however, because our system has a boundary. In this paper we
propose a construction of the space of quantum states which takes into account the presence of the boundary. Having found quantum states of the system, to calculate the statistical sum \( Q \) one needs to construct quantum operators corresponding to the variables \( A, E \), and find the corresponding spectra. The latter problem in the case of the area operator has been resolved successfully in [6]. However, there exists only rather tentative results concerning the quasilocal energy operator. Surprisingly, even without knowledge of the operator \( \hat{E} \) it turns out to be possible to analyze the thermodynamics of the system in a special case. Namely, we are able to complete the statistical mechanical analysis in the case when the boundary surface \( \mathcal{B} \) that encloses the system coincides with the horizon surface of black hole. The analysis of this case constitutes the main result of the paper.

We wish to note that our work uses some of the ideas presented in earlier works [7–9] on black hole thermodynamics performed within the loop approach to quantum gravity. The important difference between the present and earlier works is that we propose to use Chern-Simons theory to describe states of the black hole quantum mechanically.

The organization of this paper is as follows. Section II describes the space of quantum states of our system. Section III constitutes the core of the paper. It contains a statistical mechanical analysis of the case when the whole interior of \( \mathcal{B} \) is occupied by a black hole. We conclude with the discussion of the results obtained.

II. QUANTUM STATES

In this section we describe physical quantum states of our system. We give only main points of the construction. For details see [1].

Let us recall that our system consists of black hole of a certain mass \( M \) enclosed by a spherical boundary surface \( \mathcal{B} \) (located at \( r = R, R \geq 2M \)). Macroscopically the system is completely characterized by two variables, that is, the area \( A \) of \( \mathcal{B} \) and the quasilocal energy of Brown and York \( E \) contained within, that are both defined on \( \mathcal{B} \). The quantities \( A, E \) completely define (via vacuum Einstein’s equations) the state of the gravitational field within \( \mathcal{B} \). Therefore, one can think of a macroscopic (classical) state of our system as of a state of the gravitational field inside the surface \( \mathcal{B} \).

We shall describe microscopic (quantum) states of the system as states of the quantized gravitational field within \( \mathcal{B} \). To construct these states we need to solve two problems, in complete analogy with the classical case. The first problem is to find the space of kinematical states; these are analogs of ‘geometries’ of the classical case. The second problem consists in imposing ‘the quantum Einstein’s equations’. Solving these equations one finds the so-called physical states.

A major progress has been made on the first of these problems. Following the pioneering work [4] of Rovelli and Smolin, a mathematically well-defined theory of ‘quantum geometry’ has been constructed (see [6,9] for the most recent account). However, a general treatment of ‘the quantum Einstein equations’ is still a matter of much controversy, despite the recent progress on this front [10].

In this paper we propose a solution to the second problem for our special case, and find the physical states. To describe the main idea of our treatment we need a simple observation from the classical theory. Let us note that in the classical case the geometry
within $\mathcal{B}$ is completely determined (via vacuum Einstein’s equations) once the geometry on the surface $\mathcal{B}$ is specified. Indeed, for our special case of static Schwarzschild spacetime dynamics (in the sense of the Hamiltonian formulation) is trivial. Therefore, the problem of solution of Einstein’s equations reduces to the problem of solution of constraints, which is an elliptic problem. This means that, to find the geometry within $\mathcal{B}$, one should specify an (allowable) geometry on $\mathcal{B}$ (or, so to say, boundary conditions). This determines (via constraint equations) the geometry within the entire $\mathcal{B}$.

Following the spirit of the correspondence principle advocated by N.Bohr one can expect that, whatever precise form of equations of quantum dynamics is, these equations will keep some key properties of their classical analogs. For example, it is natural to expect that solutions to quantum dynamical equations describing static (in some appropriate sense) geometry within $\mathcal{B}$ should be completely determined by a state of geometry on the boundary $\mathcal{B}$, as this happens in the classical case. We assume that (still unknown) equations of quantum dynamics possess this property. The assumption implies that physical states of our system are labeled by quantum states of geometry on the boundary surface $\mathcal{B}$. Note that our assumption does not tell us anything about non-stationary cases for which the corresponding classical property does not hold.

Thus, the analogy with the classical case tells us that equations of quantum dynamics should be such that, once a state of quantized gravitational field on $\mathcal{B}$ is specified, one finds a state of quantized gravitational field within the entire $\mathcal{B}$ solving these equations. However, even in the classical case there does not exist a solution for any choice of ‘boundary data’. We expect, therefore, that in the quantum case not all states of quantum geometry on the surface $\mathcal{B}$ will give rise to physical states of the system. Moreover, in the classical case we need to know only certain minimal set of data to find the geometry within $\mathcal{B}$. We expect to have an analog of this in the quantum case.

Thus, to construct quantum states of the Schwarzschild black hole let us discuss the classical case in more details, and find a minimal set of surface data that completely specifies a macrostate of the system. We describe the gravitational field using Ashtekar variables [11]. The geometry of spatial hypersurfaces $\Sigma_t$ is described in this formalism by a pair of (canonically conjugated) variables $\tilde{E}^a, A_a$. Let us now introduce fields describing the geometry on $\mathcal{B}$. First, we introduce the pullback $a$ of the connection $A$ on $\mathcal{B}$. Next, let us denote by $e$ the two-form that is the pullback on $\mathcal{B}$ of the dual to the denseitized triad $\tilde{E}^a$ two-form $\epsilon_{abc} \tilde{E}^c$.

On solutions of Einstein’s equations the surface fields $a, e$ are not independent. In the Schwarzschild spacetime there exists a simple relation between the curvature two-form of the connection $a$ and the two-form $e$. Taking from [12] explicit expressions for $a, e$ in the Schwarzschild spacetime, one can find that

$$f = \frac{2M}{R} \frac{1}{R^2} e. \quad (3)$$

Note that throughout this paper we use the units in which $\hbar = G = c = 1$.

We take $f$ as the sought surface constraint equation. Thus, we view two fields $e, a$ on the surface $\mathcal{B}$, where the connection $a$ is such that its curvature two-form is related to $e$ via $f = \psi e$, $\psi$ being a number, as specifying a macroscopic state of our system. Indeed, it is easy to see that classically this set of data carries all information necessary to reconstruct
the geometry within the entire $\mathcal{B}$. Let us note that the two form $e$ carries information about areas of regions on $\mathcal{B}$, and, in particular, the area of the entire $\mathcal{B}$ can be determined once $e$ is known. If, moreover, one is given that the curvature two-form $f$ of the connection on $\mathcal{B}$ is related to $e$ via $f = \psi e$, one can find the mass of the corresponding Schwarzschild solution from (3). Thus, a set of fields $e, a$ together with the relation (3) between them carries all information about a macrostate.

Following our method of analogy with the classical case, we now simply have to find how to describe states of fields $e, a$ quantum mechanically, and how to impose the constraint (3) in the quantum case. The states we find this way will label the physical states of our system.

We are lucky, for there exists a complete description of states we look for. Quantum theory of geometry [4] provides us with a complete description of states of quantized $e$ field. Conformal field theory [14] tells us what are quantum states of connection subject to a quantum analog of constraint (3) on a two-dimensional surface. The corresponding quantum states, although in a different context, have already been used in non-perturbative quantum gravity by Smolin [15]. In this paper we simply give a description of these quantum states in the amount we need for our statistical mechanical treatment.

The states we are going to describe are invariant under diffeomorphism transformations that are tangent to the boundary surface $\mathcal{B}$, gauge invariant, and satisfy, in some precise sense, the quantum analog of the constraint (3). We give a description of a basis in the space of quantum states that is formed by eigenstates of operators measuring areas of regions on $\mathcal{B}$. This basis is especially convenient for our purposes. A basis quantum state is labeled by a set of points on the surface $\mathcal{B}$, which we, following [3], shall call vertices. It is important that two states that differ only in position of vertices on $\mathcal{B}$ should be considered as a single state if one deals with diffeomorphism invariant states. Each vertex $v$ is labeled by a set of quantum numbers. These include: (i) spin (half-integer) $j_v^d$ that enters the interior of $\mathcal{B}$ starting at the vertex $v$; (ii) spin $j_v^u$ that enters the surface $\mathcal{B}$ on an edge lying up the surface; (iii) spin $j_v^t$ that stays on the surface (see Fig. 1). Because of gauge invariance quantum number $j_v^t$ can take values only from the range $|j_v^u - j_v^d| \leq j_v^t \leq j_v^u + j_v^d$. Note also, that because of gauge invariance not all sets of spins are allowed [3]. Namely, in the case of a closed surface, which we consider here, the half-integers $j_v^u, j_v^d$ should satisfy the additional condition that the sums $\sum_{v \in \mathcal{B}} j_v^u$ and $\sum_{v \in \mathcal{B}} j_v^d$ over all vertices in $\mathcal{B}$ are integers.

The set of quantum numbers we have described can be thought of as specifying a state of $e$ field on $\mathcal{B}$. As it is described by conformal field theory, for each choice of these quantum numbers there exists a finite number of states of the quantized connection $a$ on $\mathcal{B}$ that satisfy the constraint (3). These states are the states of quantum Chern-Simons theories defined by a set of spins $\{j_v^t\}$. Different possible states of the quantized connection $a$ are labeled by additional quantum numbers.

It turns out (see, for instance, [16] for a discussion of this point) that the zero vector is the only vector in the physical Hilbert space of states of quantized connection $a$, unless the representations of $SU(2)$ labeled by spins $j_v^t$ all satisfy a certain condition of integrability. It is shown, for example, in [15] that for the case when the constraint equation is

$$e = \frac{k}{4\pi} f,$$  \hspace{1cm} (4)

the integrability condition is simply that all spins $j_v^t$ satisfy the inequality $j_v^t \leq k$. Here $k$ is
required to be an integer. Comparing (4) with the relation (3) we find that in our case

\[ k = A \frac{1}{(2M/R)}. \]  

(5)

It is known that representations satisfying the above integrability condition are, in fact, representations of the quantum group \( SU(2)_q \). The number \( k \) is known as level, or coupling constant of the corresponding Chern-Simons theory [15]. It is related in a simple way to the deformation parameter \( q \) (see, for example, [15]).

It is not hard to see, that the level \( k \) takes the minimal (for a fixed area \( A \) value \( A \) when the surface \( B \) coincides with the horizon surface of black hole. When the mass of black hole inside \( B \) decreases, the level increases, as it is obvious from (5). Thus, the level \( k \) is proportional to the area of the surface \( B \) (measured in Planckian units) when this surface coincides with the horizon surface of black hole, and this value is the minimal possible value once \( A \) is fixed. This has an important consequence to us, for this means that to large (in Planckian units) black holes correspond large levels \( k \). It is well-known that in the limit of large \( k \) the dimensions of the quantum group representations go into dimensions of their classical analogs. Also, there exists a simple formula for the number of different states of quantized connection in the limit of large \( k \). Roughly speaking, a basis in the space of states of quantum connection \( a \) is labeled by different ways that the spins \( j^t_v \) can be combined consistently according to the rules of addition of angular momentum of the quantum group \( SU(2)_q \). In the limit of large \( k \) this number is given by (see [15])

\[ \prod_v (2j^t_v + 1). \]  

(6)

One can expect that statistical mechanical methods can only be applied to macroscopical objects that are composed of many ‘elementary’ excitations. For our geometrical system this means that statistical mechanical description is presumably legitimate only when applied to systems large as compared with Planckian scales. As we have seen, for large black holes the level \( k \) of the quantum theory is large, and one can use the simple formulas that hold in the limit of a large \( k \). Thus, we shall not keep track of the fact that \( k \) is finite. This can be shown to be legitimate, for the corrections that appear when one takes the finiteness of \( k \) are negligible.
The states we have described (see Fig. 1 for a picture of a typical quantum state) are eigenstates of operators that measure areas of regions on $B$. In particular, the eigenvalues of operator $\hat{A}_R$ that measures the area of a region $R$ on $B$ are given by

$$A_R(S) = 16\pi \gamma \sum_{v \in R} \frac{1}{2} \sqrt{2j_v^u(j_v^u + 1) + 2j_v^d(j_v^d + 1) - j_v^t(j_v^t + 1)},$$

where we have introduced the notation $S$ for the basis quantum states. The sum here is taken over all vertices of $S$ that lie in the region $R$. We have taken into account in (7) the fact that states $S$ are gauge invariant. The quantity $\gamma$ in (7) is the parameter that appears in the loop quantization of general relativity, as it is discussed in [17]. In the case of closed surface, which we consider here, not all of the eigenvalues (7) are eigenvalues of the operator $\hat{A}$ that measures the area of the entire $B$. Namely, in the case when $B$ is closed gauge invariant quantum states are those that satisfy the additional condition that we mentioned above.

Unfortunately, there does not exist yet such a complete description of operator $\hat{E}$ that measures the quasilocal energy. However, as we shall see in the next section, a partial statistical mechanical analysis is possible even without knowledge of $\hat{E}$.

### III. STATISTICAL MECHANICAL ANALYSIS

The statistical mechanical method of Gibbs consists in constructing the statistical sum over all different quantum states of the system. Since a macroscopic state is specified by the two extensive parameters: the area $A$ and the quasilocal energy $E$, we have to introduce two conjugate intensive parameters that we shall denote by $\alpha, \beta$ correspondingly. The statistical sum is a function of the intensive parameters

$$Q(\alpha, \beta) = \text{Tr} e^{-\alpha \hat{A} - \beta \hat{E}}.$$
The statistical sum can not be calculated, unless the operator $\hat{E}$ is known. Let us note, however, that for two values of the parameter $\beta$, namely, for $\beta = 0, \infty$, the calculation is possible. Indeed, for $\beta = 0$ a precise form of the operator $\hat{E}$ simply does not matter. In the latter case only the states of zero energy survive in (8).

It is obvious that the latter case describes the zero energy state of the system (it corresponds to the zero temperature, that, as we know, is proportional to the inverse $\beta$). In other words, in the case $\beta = \infty$ our system is simply a region of flat spacetime enclosed by the surface $\mathcal{B}$. When one decreases $\beta$ (increases the temperature) the mean value of energy $E$ will increase. One finds that the value $\beta = 0$ (infinite temperature) corresponds to the largest possible value of energy for fixed $A$. But, we know that the maximal (for a fixed area $A$) possible ‘amount’ of quasilocal energy is contained within $\mathcal{B}$ when all the interior of $\mathcal{B}$ is occupied by a black hole. Thus, the case of $\beta = 0$ will describe the situation when $\mathcal{B}$ coincides with the horizon surface of black hole. This is in agreement with the thermodynamics described in [2]. Indeed, $\beta$ becomes zero when the surface $\mathcal{B}$ coincides with the horizon surface of black hole (see [2]). Thus, we find that some thermodynamical functions describing black hole can be calculates even without knowledge of $\hat{E}$. To calculate thermodynamical functions in this case, we simply have to put $\beta = 0$ in (8).

To find $Q(\alpha, \beta)$ in the case $\beta = 0$ let us use the basis of eigenstates of $\hat{A}$ described in the previous section. It is convenient to think about vertices on the surface $\mathcal{B}$ as about imaginary particles. Different particles correspond to different values of quantum numbers $j^u, j^d, j^t$. There can be any number of particles of each sort in a state. It is important to take into account the fact that, according to formula (6), each particle ‘carries’ the degeneracy $(2j^t + 1)$ (that is, the addition to a state of a particle increases the dimension of space of states of quantized connection). The sum over all states becomes the sum over numbers of particles

$$Q(\alpha) = \prod_{\Gamma} \sum_{n_{\Gamma}} (d(\Gamma))^{n_{\Gamma}} \exp \left(-\alpha n_{\Gamma} A(\Gamma)\right) = \prod_{\Gamma} \frac{1}{1 - d(\Gamma) \exp \left(-\alpha A(\Gamma)\right)}, \quad (9)$$

where we have denoted by $\Gamma$ a sort of particles ($\Gamma$ simply labels a set of spins $j^u, j^d, j^t$), by $d(\Gamma) := (2j^t + 1)$ the degeneracy carried by a particle of sort $\Gamma$, and by $n_{\Gamma}$ the number of particles of sort $\Gamma$ in a state. The summations over $n_{\Gamma}$ run from zero to infinity, $A(\Gamma)$ denotes the contribution to the area of $\mathcal{B}$ from a particle of sort $\Gamma$.

For simplicity, we have not taken into account in (9) the fact that not all eigenvalues (10) are eigenvalues of area operator for a closed surface. In our language of particles this means that not all combinations of particles can be realized. Thus, strictly speaking, the summations over numbers $n_{\Gamma}$ of particles in (9) are not independent. Let us forget about this for a moment and proceed with our analysis. We shall discuss the consequences of this restriction later on.

We have found that the state of the system when the entire interior of $\mathcal{B}$ is occupied by a black hole corresponds to $\beta = 0$. Thus, such a state of the system is described by a single thermodynamical parameter $\alpha$. The mean values of the area $A$ and the energy $E$, that can be expressed as derivatives of $\ln Q(\alpha, \beta)$ at $\beta = 0$, become functions of this single parameter. One could exclude $\alpha$, and express all thermodynamical functions in terms of, for example, $A$. In other words, a macroscopic state of a Schwarzschild black hole is specified by a single
parameter. We are particularly interested to find the entropy function of the system as a function of \( A \). Clearly, it may serve as a first test to our method whether it predicts the Bekenstein-Hawking formula for the entropy of black hole. Namely, the thermodynamics described in [2] tells us that, when the whole of \( B \) is occupied by a black hole, the entropy ‘contained’ within \( B \) is simply the Bekenstein-Hawking entropy of black hole

\[
S_{BH} = \frac{1}{4} A
\]  

(10)

To find the entropy as a function of \( A \) one should, in principle, find \( S(\alpha) \) and \( A(\alpha) \). Then, one would exclude \( \alpha \) and find \( S \) as a function of \( A \). This problem can, in principle, be solved; however, there exists a simpler way to get the dependence \( S(A) \). We note that we expect the Bekenstein-Hawking formula (10) to hold only for large (as compared with Planckian area) areas \( A \). Indeed, from the point of view of quantum geometry only large as compared with the Planckian scale black holes can be considered macroscopical, and thus, the thermodynamical description can be applied only to such black holes. Thus, we are interested in the dependence \( S(A) \) predicted by the statistical mechanics only for large values of \( A \). One can change the mean value of area changing the intensive parameter \( \alpha \). Some values of \( \alpha \) correspond to large mean values of \( A \), and we are to find the thermodynamical functions for these \( \alpha \).

From the formula

\[
A(\alpha) = - \frac{d \ln Q(\alpha)}{d \alpha}
\]  

(11)

we find for the mean value \( A(\alpha) \) of the area

\[
A(\alpha) = \sum_{\Gamma} A(\Gamma) f(\Gamma),
\]  

(12)

where we have introduced the function

\[
f(\Gamma) = \frac{d(\Gamma) \exp (-\alpha A(\Gamma))}{1 - d(\Gamma) \exp (-\alpha A(\Gamma))}.
\]  

(13)

It is easy to see from (12) that \( f(\Gamma) \) plays the role of the mean number of particles in the ‘state’ \( \Gamma \) for fixed value of parameter \( \alpha \). Large values of \( A(\alpha) \) correspond to a case when at least for some \( \Gamma \) the function \( f(\Gamma) \) takes large values.

It can happen that the second term in the denominator of (13) for some \( \Gamma \) is close to unity. This will correspond to a large number of particles of the sort \( \Gamma \), and, therefore, to a large mean area. Let us see to what values of \( \alpha \) this corresponds.

As it was noted in [4], there exists only few possibilities. First, it can happen that the thermodynamical functions such as \( Q(\alpha), A(\alpha), S(\alpha) \) get large values only when \( \alpha \to 0 \). It follows then from the Euler relation that in the thermodynamical limit (large areas) the dependence \( S(A) \) of the entropy on the area is different from the linear one (in fact, \( S(A) \) grows slower than \( A \)). The second case is when the thermodynamical functions diverge as \( \alpha \) goes to some ‘critical’ value \( \alpha_{cr} \). Then, as it is discussed in [7], the Euler relation implies the linear dependence \( S(A) = \alpha_{cr} A \). Finally, there exists the possibility that the statistical
sum $Q$ diverges for all $\alpha$, which means that the density of states of the system grows faster than $\exp A$.

It is not hard to show, that in our case the second possibility is realized, and, therefore, the dependence $S(A)$ of the entropy on the area (for $\beta = 0$) is linear. Indeed, let us consider the function $d(\Gamma) \exp -\alpha A(\Gamma)$ as a function of $\Gamma$ and $\alpha$. Let us recall that $\Gamma$ stands for three quantum numbers $j^u, j^d, j^t$. It turns out to be convenient to use instead of $j^u, j^d, j^t$ the following linear combinations of them

$$I := j^u + j^d,$$
$$J := j^u - j^d,$$
$$K := j^u + j^d - j^u + j^d. \tag{14}$$

Then, as it is not hard to see, $I, J$ take all positive integer and half-integer values, and $K$ runs from zero to $2\min(j^u, j^d)$ ($K$ also takes both integer and half-integer values). The contribution to the area from a particle of sort $\Gamma$ written as a function of this new quantum numbers becomes

$$A(\Gamma) = A(I, J, K) = 8\pi \gamma \sqrt{J^2 + I + 2IK + K - K^2} \tag{15}$$

The degeneracy that is ‘carried’ by a particle of sort $\Gamma$ becomes $2(I - K) + 1$. As a straightforward analysis shows, for all values of $\alpha < \alpha_{cr} = 1.138/8\pi \gamma$ the second term in the denominator of (13) is less than unity for all types of particles $\Gamma$ (for all non-zero values of quantum numbers $I, J, K$). When $\alpha \approx \alpha_{cr}$, the maximal value of the function $d(\Gamma) \exp -\alpha A(\Gamma)$ in the allowable range of the quantum numbers $I, J, K$ is just the unity. This is realized for the following values of quantum numbers: $j^u = j^d; j^u + j^d = j^t = 2$.

Thus, what we find is that for the value of $\alpha$ close to $\alpha_{cr}$ the number of particles $f(\Gamma)$ in one of the ‘states’ $\Gamma = \Gamma_{cr}$ becomes large. This means that the thermodynamical functions diverge for $\alpha = \alpha_{cr}$. As we have said above, this implies that for large values of $A$ the dependence of the entropy on the area is linear

$$S = \alpha_{cr} A,$$
$$\alpha_{cr} = 1.138/8\pi \gamma \tag{16}$$

Note that we have obtained this result analyzing the behavior of $f(\Gamma)$ as a function of $\Gamma$ and $\alpha$. It is crucial for the result that our particles ‘carry’ the degeneracy $d(\Gamma)$, for it is this degeneracy in (13) that makes the number of particles in one of the states diverge for a finite (non-zero) value of $\alpha$.

As we have said, when $\alpha$ approaches $\alpha_{cr}$ the number of particles in one of the states $\Gamma = \Gamma_{cr}$ goes to infinity. It is interesting to compare this with the phenomenon of Bose-Einstein condensation. In the case of an ideal Bose gas a macroscopic fraction of particles that compose the gas end up on the ground energy level when the temperature of the gas reaches certain critical temperature. Let us note that the number of particles in the case of a Bose gas is fixed. In our case particles are imaginary creatures, and their number is not fixed. As we have found, when the intensive parameter $\alpha$ approaches the critical value $\alpha_{cr}$, much larger number of particles end up in the state $\Gamma_{cr}$ than in all other states. Thus, what we find is very similar to Bose-Einstein condensation; the important difference is, however,
that the number of particles in our ‘gas’ is not fixed. Indeed, the fact that the number of particles is not fixed can be shown to be crucial for our result of linear dependence of the entropy on the area to hold.

When $\alpha$ is close to $\alpha_{cr}$ only a negligible fraction of particles is in states different from $\Gamma_{cr}$. Thus, one can write

$$A = A(\Gamma_{cr}) N,$$

(17)

where we have introduced the number $N$ of particles in the state $\Gamma_{cr}$.

Let us now recall that all results of this section were obtained neglecting the fact that, in the case of a closed surface $B$, not all sets of spins on $B$ correspond to physical states. As we have mentioned above, there exists a simple condition on the total sums of spins that enter and leave the surface. The way to take into account this condition is to exclude from the statistical sum all terms with spins not satisfying the conditions. Generally, such an operation may significantly change the behavior of all thermodynamical functions that are derived from the statistical sum. It is not hard to see, however, that our condition is very weak in the sense that the results we have obtained for the case of an open surface continue to hold when one considers closed surfaces $B$. Here we present only a heuristic argument in the support of this, leaving a more rigorous treatment for another occasion.

It is not hard to see that the results (16), (17) stem from one and the same fact that we have discovered about the microscopic state describing black hole. We saw that a state describing a large black hole is very special in the sense that a macroscopic fraction of ‘particles’ resides in one and the same ‘state’. Then the total area is given simply by the number of particles times the contribution from an individual particle, as it is described by (17). The entropy, on the other hand, is given by the logarithm of the number of states of quantized connection

$$S = \ln d(\Gamma_{cr})^N = N \ln d(\Gamma_{cr}).$$

(18)

Combining this equation with (17) we find that

$$S = \frac{\ln d(\Gamma_{cr})}{A(\Gamma_{cr})} A = \frac{\ln 5}{8\pi\gamma\sqrt{2}} A = \alpha_{cr} A.$$

(19)

Since $j^u, j^d$ in this state happen to have one and the same spin 1, the conditions we have mentioned above are satisfied in any state describing a large black hole. Thus, in the thermodynamical limit (large black holes) the results we have obtained continue to hold even when one imposes the conditions arising in the case of a closed $B$.

1A similar phenomenon, when the entropy of the system grows linearly as a function of energy, is known to occur in ordinary thermodynamics for systems for which the number of particles is not fixed. The author is grateful to P. Aichelburg from whom he learned about this fact.
IV. DISCUSSION

We have found that the microscopic states of Schwarzschild black hole can be described by states of SU(2) Chern-Simons theory, which are defined by choices of vertices and spins on $\mathcal{B}$. We have found that the integer level of Chern-Simons theory is proportional to the horizon area of black hole (see (5)). Thus, large black holes correspond to large levels $k$. We have used this description as the basis of our statistical mechanical analysis.

Although we have not reached our goal of explaining the fundamental equation (1) on the basis of quantum microscopics of the system, we were able to complete the statistical mechanical analysis of the case when the entire interior of the system is occupied by a black hole. We have found that in this case the entropy contained within $\mathcal{B}$ is proportional to the area of the boundary $\mathcal{B}$, with the proportionality coefficient given by (16). The proportionality coefficient between black hole entropy $S$ and the horizon area $A$ turns out to be a function of the parameter $\gamma$.

As it is discussed in [17], the parameter $\gamma$ is a free parameter that arises in the loop quantization of general relativity. Unless a value of the parameter $\gamma$ is fixed by some independent considerations, a comparison of the dependence $S(A)$ predicted by our analysis with the Bekenstein-Hawking formula (10) is not possible. Thus, the predicted dependence (16) by itself does not provide us with a test of our approach. Note, however, that the approach presented gives us not just the dependence (16). The considerations of section III led us to the conclusion that the statistical state of a black hole should be described by a density matrix

$$\hat{\rho} = \frac{1}{Q(\alpha)} e^{-\alpha \hat{A}},$$

where $Q(\alpha)$ is the statistical sum (8). The entropy was then defined as $S = \text{Tr}(\hat{\rho} \ln \hat{\rho})$. However, with the density matrix $\hat{\rho}$ in one's disposal one knows much more than simply the entropy. For example, the knowledge of the density matrix allows one to analyze properties of the black hole radiation spectrum [18]. Results of such an analysis, together with the result (16), can serve as a test of the validity of the approach presented.

The other result of this paper is the discovery of the fact that the state of the system that corresponds to a macroscopically large black hole is realized for values of the intensive parameter $\alpha$ that are close to the critical point $\alpha_c$. This is quite similar to what one finds, for example, in the sum over lattices approach to quantum gravity in two dimensions [19]. One finds that, in order to go into a macroscopical limit, one needs to tune values of the parameters to a critical point. It is interesting that in our case, when the parameter $\alpha$ goes to a critical point, the system undergoes a phase transition similar to the phenomenon of Bose-Einstein condensation. We find that a macroscopical fraction of elementary excitations of geometry is in one and the same quantum state when we are dealing with large black holes. The state where most of the particles ‘condense’ to happen to have the following quantum numbers: $j^u = j^d = 1$.

Let us conclude summarizing the open problems of our approach. One of the most important of such problems is to develop a theory of quantum dynamics of geometry, and to answer the question whether the assumptions about dynamics made in this paper are true. This is, however, a difficult problem that may require mutual efforts of a large number of researchers in the field.
Probably a simpler, but not less important problem is to construct the operator \( \hat{E} \) corresponding to the quasilocal energy. This would allow one to analyze the case of an arbitrary mass contained within \( B \) and find the fundamental equation of the system. Work is in progress in this direction.

Another important open problem, related to the problem of construction of the energy operator, is to understand how the different mathematical techniques used in conformal field theory and in non-perturbative quantum gravity can be reconciled in a mathematically rigorous construction of quantum states. Work is in progress also in this direction [1].

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REFERENCES

[1] A. Ashtekar, J. Baez, A. Corichi, K. Krasnov, Quantum Geometry and Black Hole Entropy, in preparation.
[2] E. Martinez, The postulates of gravitational thermodynamics, Phys. Rev. D 54, 6302 (1996).
[3] D. Brown and J. York, Quasilocal energy and conserved charges derived from the gravitational action, Phys. Rev. D 47, 1407 (1993).
[4] S. Hawking, Particle Creation by Black Holes, Commun. Math. Phys. 43, 199 (1975).
[5] C. Rovelli, L. Smolin, Loop representation for quantum general relativity, Nucl. Phys. B133, 80 (1990).
[6] C. Rovelli, L. Smolin, Discreteness of Area and Volume in quantum gravity, Nucl. Phys. B 442, 593 (1995); A. Ashtekar, J. Lewandowski, Quantum theory of geometry I: area operators, Class. Quant. Grav. 14, 55 (1997); S. Fritelli, L. Lehner, C. Rovelli, The complete spectrum of the area from recoupling theory in loop quantum gravity, Class. Quant. Grav. 13, 2921 (1996).
[7] K. Krasnov, Geometrical Entropy from Loop Quantum Gravity, Phys. Rev. D55, 3505 (1997).
[8] C. Rovelli, Black Hole Entropy from Loop Quantum Gravity, Phys. Rev. Lett. 77, 3288 (1996).
[9] C. Rovelli, Loop Quantum Gravity and Black Hole Physics, Helv. Phys. Acta. 69, 583 (1996).
[10] T. Thiemann, Anomaly-free formulation of non-perturbative, four dimensional Lorentzian quantum gravity, Phys. Lett. B 380, 257 (1996).
[11] A. Ashtekar, New variables for classical and quantum gravity, Phys. Rev. Lett. 57, 2244 (1986); New Hamiltonian formulation of general relativity, Phys. Rev. D 36, 1587 (1987).
[12] I. Bengtsson, A new phase for general relativity? Class. Quant. Grav. 7, 27 (1990).
[13] M. Reisenberger, New constraints for canonical general relativity, Nucl. Phys. B457, 643 (1995).
[14] G. Segal, Conformal field theory, Oxford preprint.
[15] L. Smolin, Linking topological quantum field theory and non-perturbative quantum gravity, J. Math. Phys. 36, 6417 (1995).
[16] E. Witten, Quantum Field Theory and Jones Polynomial, Commun. Math. Phys. 121, 351 (1989).
[17] G. Immirzi, Quantum Gravity and Regge Calculus, available as preprint gr-qc/9701052; C. Rovelli, T. Thiemann, The Immirzi parameter in Quantum General Relativity, preprint available as gr-qc/9705059.
[18] K. Krasnov, Quantum Geometry and Thermal Radiation from Black Holes, in preparation.
[19] P. Ginsparg, G. Moore, Lectures on 2D Gravity and String Theory, Lectures given June 11-19, 1992 at TASI summer school, Boulder, CO, available as preprint hep-th/9304011.