LOW-DEGREE RATIONAL CURVES ON HYPERSURFACES
IN PROJECTIVE SPACES AND THEIR FAN DEGENERATIONS

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ABSTRACT. We study rational curves on general Fano hypersurfaces in projective space, mostly by
degenerating the hypersurface along with its ambient projective space to reducible varieties. We
prove results on existence of low-degree rational curves with balanced normal bundle, and reprove
some results on irreducibility of spaces of rational curves of low degree.

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INTRODUCTION

The celebrated work of Mori starting with [5] had brought out the importance of rational
curves in higher-dimensional geometry, and in the early 1990s Campana and Kollár-Miyaoka-
Mori applied Mori’s bend-and-break method to prove that the family of rational curves on any
Fano manifold X is a large enough to connect a pair of general points (i.e. X is ‘rationally con-
nected’). This work and much more is exposed in Kollár’s book [4]. Since then there has been
considerable interest, especially by Joe Harris and his school (e.g. [3]), in studying the family of
rational curves on a general Fano hypersurface in projective space, especially as to dimension

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and irreducibility. The expected dimension of the family of rational curves of degree $e$ in a hypersurface $X$ of degree $d$ in $\mathbb{P}^n$ is $e(n-d+1)+n-4$, and it has been conjectured that when $X$ is general, this family is irreducible and of the expected dimension provided $d \leq n-1$ and $n > 3$. This conjecture has been proven for $d \leq n-2$ by Riedl and Yang [10] which also contains extensive references. See also [11] for a partial extension to the case $d = n-1$.

The purpose of this paper is to bring to bear on these questions a technique used previously [6], [7] to study curves (of any geometric genus) in the projective plane. This technique is based on degenerating the ambient projective space itself to a reducible variety called a fan and simultaneously degenerate a general hypersurface in projective space to a general hypersurface of suitable type on the fan. In a sense, one is degenerating the entire family of projective hypersurfaces of given degree to an analogous family of the same dimension on a fan. This has the advantage that the components of the limit are simpler, consisting, e.g. in the case of a 2-component fan, of a hypersurface of degree $d-1$ and a blowup of a (rational) hypersurface of degree $d$ with a point of multiplicity $d-1$, called a quasi-cone. Moreover the union has double points only. We call this a 2-fan of type $(d, d-1)$. We will focus mainly on the case $d_2 = d-1$, where $X_1$ is a blowup of a special rational hypersurface called a quasi-cone and is in turn the blowup of $\mathbb{P}^{n-1}$ in a $(d, d-1)$ complete intersection.

Now the limit on a 2-component fan $X_1 \cup X_2$ of a rational curve of degree $e$ on a general hypersurface of degree $d$ essentially takes the form $C_1 \cup C_2$ where $C_1$ is a birational transform of a rational curve of degree $e_1$ in $\mathbb{P}^{n-1}$ that is $(e_1(d-1) - e_2)$-secant to a certain $(d, d-1)$ complete intersection, while $C_2$ is the rational curve of degree $e_2 \geq 0$ on $X_2$, a general hypersurface of degree $d-1$ and $e_1 + e_2 = e$. This leads us to study secant rational curves to complete intersections. We will show in Thm 21 that for fixed $d, e, a$ in a suitable range, the locus of rational curves of degree $e$ that are $a$-secant to a general $(d, d-1)$ complete intersection is irreducible and of the expected dimension. This will be used to reprove a result (Thm 26) on irreducibility of families of rational curves of low degree on general hypersurfaces of degree $d < n$ in $\mathbb{P}^n$. This result is mostly subsumed by the theorem of Riedl-Yang and, as noted by the referee, can also be deduced from the results of Harris-Roth-Starr [3]. Those proofs are different. It is interesting to note that for the bend-and-break method as used in [10] and elsewhere, the case of low-degree curves is hardest, but for our method the opposite is true. Also, in §8, we prove some results on irreducibility of the family of rational curves going through a fixed, general point.

Our main new result (Theorem 24) concerns the existence of rational curves of low degree with balanced normal bundle, and extends a result of Coskun-Riedl [1]:

**Theorem.** For $d \leq n, n \geq 4, e \leq 2n - 2$ there exists on a general hypersurface of degree $d$ in $\mathbb{P}^n$ a rational curve of degree $e$ with balanced normal bundle.

The detailed contents of the paper are as follows.

- We begin in §1 by studying certain (decomposable) bundles on rational chains and combs which smooth out to balanced bundles on rational curves.
- This study is applied in §2 to polygons on a union of hyperplanes to give a preliminary construction for rational curves of low degree with balanced normal bundles.
• In §3 we study certain rational projective hypersurfaces called quasi cones which arise, essentially as componenets, in the study of hypersurfaces on fans.
• Fans themselves along with their deformations and hypersurfaces on them are introduced in §4.
• In §5 we prove a technical result concerning what happens to a curve on a general hypersurface in projective space as the latter degenerates to a fan.
• In §6 we study families of curves in a given homology class on a quasi-cone. Using the identification of the quasi-cone with a blowup of projective space, these can be identified with families of curves in projective space that are multisecant to a given codimension-2 complete intersection.
• The results on balanced normal bundles and irreducibility of families are then proved in §7 and §8 using fans.

Throughout the paper, we work over \( \mathbb{C} \).

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1. Balanced bundles on chains and combs

Recall that a vector bundle \( E \) on a rational curve \( C \) is said to be balanced if
\[
E \cong r_+ \mathcal{O}_C(a + 1) \oplus (r - r_+) \mathcal{O}_C(a), r_+ > 0
\]
where \( \mathcal{O}_C(*) \) denotes the unique line bundle of degree \( * \). Balancedness is an open property on vector bundles. The subbundle \( E_+ = r_+ \mathcal{O}_C(a + 1) \) is uniquely determined and called the upper subbundle. We can write \( E_+ = V_+(E) \otimes \mathcal{O}_C(a + 1) \) where the vector space \( V_+(E) \) is canonical up to scalar multiplication and for any point \( p \in C \), \( V_+(E) \) may be identified, again canonically up to scalar, with a uniquely determined subspace of the fibre \( E(p) \), called the upper subspace at \( p \).

Now let
\[
C = C_1 \cup ... \cup C_e
\]
be a (connected, nodal) tree of smooth rational curves. We denote by \( \mathcal{O}_C(a_1, ..., a_e) \) any line bundle on \( C \) whose restriction on each \( C_j \) is \( \mathcal{O}_{C_j}(a_j) \). A balanced decomposition for a vector bundle \( E \) on \( C \) is an isomorphism to a direct sum of line bundles whose total degrees differ by at most 1, i.e.
\[
E \cong \bigoplus_{i=1}^{r_+} \mathcal{O}_C(a_{i1}, ..., a_{ie}) \oplus \bigoplus_{i=1}^{r-r_+} \mathcal{O}_C(b_{i1}, ..., b_{ie})
\]
where
\[
\sum_{j=1}^e a_{ij} = \deg_+(E), i = 1, ..., r_+, \sum_{j=1}^e b_{ij} = \deg_+(E) - 1, i = 1, ..., r - r_+.
\]
The first sum above is called an upper subbundle of \( E \). Thus \( r(\deg_+(E) - 1) + r_+ = \deg(E) \). Then \( \deg_+(E) \) is called the upper degree and \( r_+ = r_+(E) \) is called the upper rank. A vector bundle \( E \) on a rational tree \( C \) is said to be balanced if its restriction on any connected subtree admits a balanced decomposition. A bundle \( E \) on a rational tree is said to be strongly balanced
if it is balanced and there is a subbundle \( E_+ \subset E \)- necessarily unique- that restricts on each component \( C_i \) to the upper subbundle of \( E_i \).

**Remarks 1.** (i) Clearly every strongly balanced bundle is balanced with upper degree and rank being the degree and rank of \( E_+ \). Indeed strongly balanced bundles behave just like balanced bundles on \( \mathbb{P}^1 \) and in particular there is a well-defined upper subspace \( V_+(E) \).

(ii) Unlike in the irreducible case, not every balanced bundle is strongly balanced because an ‘upper’ subbundle is in general not unique when some of the \( a_{ij} \) may not equal the upper degree of \( E_j \).

(iii) Another subtlety of weakly (non-strongly) balanced bundles is that ‘upper’ line subbundles may be rigid and it is not in general possible to form linear combinations of them.

(iv) The condition that just \( E \) itself, rather than all its restrictions, admit a balanced decomposition seems too weak. In particular Corollary 4 below seems unlikely to hold in that generality.

(v) Balancedness is defined only for vector bundles which split as direct sums of line bundles and it is not claimed that any vector bundle on a rational chain splits thusly.

We now consider a connected chain of rational trees

\[
C = C_1 \cup \ldots \cup C_e
\]

where each link \( C_i \) is itself a rational tree (rather than a \( \mathbb{P}^1 \)) and where each \( E_i = E|_{C_i} \) is strongly balanced. Set \( p_i = C_i \cap C_{i+1} \). Note that if each \( E_i \) is strongly balanced then \( V_+(E_i) \), being a subspace of both \( E_i(p_{i-1}) \) and \( E_i(p_i) \), may be intersected with both \( V_+(E_{i-1}) \) and \( V_+(E_{i+1}) \). Repeating this operation we obtain a pair of descending flags, ‘forward’ and ‘backward’ on \( V_+(E_i) \). We will say that \( E \) is in balanced general position if these flags each have the generic dimension (e.g. the first member of the forward flag is of dimension \( \max(r_+(E_i) + r_+(E_{i+1}) - r, 0) \) etc.) and the pair is in mutual general position, \( \forall i \).

**Lemma 2.** Let \( C \) be a rational tree of the form \( C = C_1 \cup \ldots \cup C_e \) where each \( C_i \) is itself a rational tree. Let \( E = E_1 \cup \ldots \cup E_e \) be a vector bundle \( C \) such that each restriction \( E_i = E|_{C_i} \) is strongly balanced and such that \( E \) is in balanced general position. Then \( E \) is balanced.

**Proof.** It suffices to prove \( E \) itself admits a balanced decomposition as the hypotheses descend to subchains. We use induction on \( e \) and on the rank \( r \). The assertion is trivial if \( r = 1 \), vacuous if \( e = 1 \) and easy if \( e = 2 \). In fact if \( e = 2 \) and \( r_+(E_1) + r_+(E_2) > r \), then \( E \) is clearly strongly balanced.

Now we will assume \( e \geq 3 \) and use induction on \( e \). By the last remark, we may assume

\[
(2) \quad r_+(E_i) + r_+(E_{i+1}) \leq r, \forall i,
\]
or else \( E|_{C_i \cup C_{i+1}} \) is strongly balanced and \( C_i \cup C_{i+1} \) may be treated as a single link of the chain. This together with general position imply that the upper subspaces of \( E_i \) at \( p_i = C_i \cap C_{i+1} \) have trivial intersection, \( \forall i \). We switch notation, writing \( C \) in the form \( C_1^* \cup C_2 \cup C_3 \) where \( C_1^* \) is itself a chain on \( e - 2 \) links. By induction, we may assume the assertion holds for \( C_1^* \) and \( C_1^* \cup C_2 \), i.e. \( E_1^* := E_{C_1^*} \) and \( E_{12} = E_{C_1^* \cup C_2} \) are both admit balanced decompositions. We will say that a line subbundle of \( E_{12} \) is of type (11) if its restrictions on \( C_1^* \) and \( C_2 \) are both of the upper
degree, and likewise for (10), (111) etc. Assume to begin with that \( r_+(E_1^1) + r_+(E_2^1) > r \). Then by induction, \( E_{12} \) is a sum of line subbundles of types (11), (01) and (10), where the first two make up the upper subspace at \( p_2 \). By (2) (together with general position), the first two types may be glued to a direct summand line subbundle of type 0 on \( E_3 \), while the last type may be glued to summands of type 0 or 1. In total this gives a decomposition of \( E \) with summands of types (110), (010), (101), (100), which is a balanced decomposition.

Now suppose \( r_+(E_1^1) + r_+(E_2^1) \leq r \). Then \( E_{12} \) is a sum of bundles of types (01), (10), (00). Again by (2), the first type glues to a summand of type 0 on \( E_3 \) to yield a summand of type (010). At this point we could use induction on \( r \) to conclude. Alternatively, suppose now \( r_+(E_1^1) + r_+(E_3^1) > r - r_+(E_2^1) \). Then summands of type 0 on \( E_2 \) can be extended to types (101, (001), (100), so we get a balanced decomposition for \( E \). Finally suppose \( r_+(E_1^1) + r_+(E_3^1) \leq r - r_+(E_2^1) \). Then summands of type 0 on \( E_2 \) can be extended to types (100), (001), (000), so again we get a balanced decomposition.

\[ \square \]

**Remark 3.** The Lemma says in particular that \( E \) is a direct sum of line bundles. It does not say \( E \) is strongly balanced. Also, the Lemma will mainly be used in the case where all the \( C_i \) are irreducible, but the more general statement is convenient for the induction.

**Corollary 4.** If \( E, C \) are as in Lemma 2 then a general smoothing of \( (E, C) \) is balanced.

**Proof.** Note that \( \tilde{E} \otimes E \) is a direct sum of line bundles whose total degree on any subchain (a fortiori, on any component) is \( \geq -1 \). It is easy to see by induction on chain length that such a line bundle has \( H^1 = 0 \). Now consider a deformation \( (\tilde{E}, \tilde{C}) \) of \( (E, C) \) where \( C \) smooths. Let \( (\tilde{E}_1, \tilde{C}) \) be an analogous deformation where \( \tilde{E}_1 \) is a direct sum of line bundles deforming the RHS of (1). It is easy to check that

\[ H^1(C, (\tilde{E}_1 \otimes \tilde{E}) \otimes O_C) = H^1(C, \tilde{E} \otimes E) = 0 \]

and consequently the isomorphism (1) extends to a map \( \tilde{E}_1 \to \tilde{E} \), which must be an isomorphism on the general fibre. \[ \square \]

**Remark 5.** The line bundle \( O(1, -1, -1) \) has \( H^1 \neq 0 \).

We conclude with some remarks on bundles on combs. These will be used in the proof of Theorem 24 below. By definition a comb is a rational tree of the form \( B \cup \bigcup_{i=1}^{l} T_i \) where \( B \), the base, and \( T_1, ..., T_l \), the teeth, are \( \mathbb{P}^1 \) and each tooth meets the base. For ‘generalized comb’, replace \( \mathbb{P}^1 \) by ‘rational tree’ (thus every rational tree is a generalized comb but it is understood that a particular pettinal presentation is specified). In general, the behavior of bundles on combs is complicated and we will just consider a special, dentally challenged case used below (see Theorem 24).

**Lemma 6.** Let \( E \) be a rank-\( r \) bundle on a generalized comb \( B \cup \bigcup_{i=1}^{l} T_i \). Assume
(i) $E_B, E_{T_1}, \ldots, E_{T_t}$ are strongly balanced;
(ii) $r_+(E_{T_i}) = r - 1, i = 1, \ldots, t$ and the upper hyperplanes in $V_+(E_{T_i}) \subset E(p_i), i = 1, \ldots, t$ are general, where $p_i = B \cap T_i$;
(iii) $t \leq r_+(E_B)$ (resp. $t < r_+(E_B)$).

Then $E$ is balanced (resp. strongly balanced).

Proof. We assume $t < r_+(E_B)$ as the equality case is similar. In this case we can see easily as above that $E$ is strongly balanced with unique upper subbundle or rank $r_+(E_B) - t$ and degree $d_+ = \deg_+(E_B) + \sum_{i=1}^{t} \deg_+(E_{T_i})$ corresponding to $V_+(E_B) \cap \bigcap_{i=1}^{t} V_+(E_{T_i})$, complemented by $t$ many general line subbundles of degree $d_+ - 1$ corresponding to $V_+(E_B) \cap \bigcap_{i=1}^{t} V_+(E_{T_i}), j = 1, \ldots, t$

(mod $V_+(E_B) \cap \bigcap_{i=1}^{t} V_+(E_{T_i})$), glued to a general, non-upper subbundle of $E_{T_j}$, plus $r - r_+ < r - t$

many general line subbundles corresponding to linearly independent elements of $\bigcap_{i=1}^{t} V_+(E_{T_i})$
glued to general, non-upper, line subbundles on $E_B$. □

2. RATIONAL CURVES WITH BALANCED NORMAL BUNDLE

The normal bundle $N_{C/X}$ of a rational curve $C$ on $X$ is related to the movement of $C$ on $X$: $N_{C/X}$ being semipositive, i.e. globally generated, means that the curve moves freely, filling $X$; when that is so, $N_{C/X}$ being balanced means $C$ can be ‘pinned down’ at the maximum number of general points on $X$, i.e. that $C$ has ‘maximal momentum’ (or energy).

Our purpose in this section is to prove, by a suitable degeneration, the existence of rational curves $C$ of low degree on general Fano hypersurfaces $X$ such that the normal bundle $N_{C/X}$ is balanced. The idea is to work with a polygon, i.e. a chain of lines, on a union of hyperplanes, with at most one line per hyperplane, and suitably modify its normal bundle at non-lci points, which are the smooth points on the polygon that are singular on the union of hyperplanes. This result, which will be used in the proof of our irreducibility result, will later be generalized to higher-degree curves using fans (see Theorem 24). No fans will be used here.

We begin with a local construction. In $C \times C^n, n \geq 3$ with coordinates $t, x, y, z, \ldots$ consider a hypersurface $U$ with equation $tz = xy$. We view $U$ as a family of hypersurfaces $U_t \subset C^n$ over $C$. Then $U_t$ has a normal-crossing double point for $t = 0$ and is smooth for $t \neq 0$. Now blow up $U$ in the locus $t = y = 0$, i.e. one component $U_{0,1}$ of the special fibre $U_0$, which has coordinates $x, z, \ldots$. The blowup $\bar{U}$ is smooth and the map $\bar{U} \to U$ is small. In the relevant open set we can write

$t = uy, x = uz$,

so the special fibre $\bar{U}_0$ has two components $\bar{U}_{0,1}, \bar{U}_{0,2}$, with respective equations $y, u$, and $\bar{U}_{0,2} \to U_{0,2}$ is an isomorphism while $\bar{U}_{0,1} \to U_{0,1}$ is the blow-up of the smooth codimension-2 locus $x = z = 0$. 

6
Now working globally, let $H_i = (x_i)$ be the $i$-th coordinate hyperplane in $\mathbb{P}^n$ and let $X_0 = \bigcup_{i=1}^d H_i = (x_1...x_d)$, $d \leq n$. Consider an $e$-gon, $e \leq d$, $C_0 \subset X_0$ of the form

$$C_0 = \bigcup_{i=1}^e L_{i,i} \subset H_i, L_i \cap L_{i+1} = L_i \cap H_{i+1} =: p_{i,i+1}, i = 1, ..., e-1,$$

$L_i \subset H_i$ being general lines. Set $p_{i,j} = L_i \cap H_j, K_{i,j} = H_i \cap H_j, i \neq j$. Thus each point $p_{i,j}, j \neq i + 1$, is a smooth, non-lci point on $C_0$ relative to $X_0$, while each $p_{i,i+1}$ is an lci point relative to $X_0$. Note that $K_{i,j}$ can be identified with the fibre at $p_{i,j}$ of the projectivized normal bundle $\mathbb{P}(N_{L_i/H_i})$ which is a trivial projective bundle $L_i \times K_i$, so we identify $K_{i,j} \simeq K_i, \forall j \neq i$.

Now let $X_1 = (f_d)$ be a general degree-$d$ hypersurface through all the non-lci points $p_{1,j}, j \neq i + 1$. By Lemma 7 below, $X_1$ is smooth at all the non-lci points. Consider a family $\mathcal{X}$ in $C \times \mathbb{P}^n$ with equation $\prod_{i=1}^d x_i + tf_d$. Construct the modification $\tilde{\mathcal{X}}$ as above where at each non-lci point $p_{i,j}$, only $H_i$ is blown up (i.e. $H_i$ plays the role of $U_0,1$ and $f_d$ plays the role of the $z$ coordinate). Then $C_0$ lifts to a polygon $\tilde{C}_0$ in $\tilde{\mathcal{X}}$ and $\tilde{\mathcal{X}}$ is smooth along $\tilde{C}_0$ and $\tilde{X}_0$ is smooth at the points corresponding to the $p_{i,j}, j \neq i + 1$ because the proper transform of $L_i$ is disjoint from the inverse image of $H_j$. Thus $\tilde{C}_0$ is everywhere lci relative to $\tilde{X}_0$.

**Lemma 7.** A general $X_1$ as above is smooth in all $p_{i,j}, j \neq i + 1$ and transverse to $H_j$. Moreover, for each fixed $i$, if we identify $K_{i,j} \simeq K_i, \forall j \neq i + 1$, then the collection of tangent planes to $X_1 \cap K_{i,j}$ at the $p_{i,j}, j \neq i + 1$ become identified with a general collection of hyperplanes in $K_i$.

**Proof.** Use induction on $d$ and $e$, using the exact sequence

$$0 \to \mathcal{O}(d-1)(-\sum_{j \geq 2 \atop j \neq i+1} p_{i,j}) \to \mathcal{O}(d)(-\sum_{j \geq 2 \atop j \neq i+1} p_{i,j}) \to \mathcal{O}(H_i)(-\sum_{j \geq 2 \atop j \neq i+1} p_{i,j}) \to 0.$$

By induction, the first term has vanishing $H^1$, hence so does the middle term and the map on $H^0$ is surjective. \hfill $\Box$

At a point of $\tilde{C}_0$ coming from a non-lci point $p_{i,j}$, the normal bundle $N_{\tilde{C}_0/X_0}$ is an elementary ‘down’ modification (i.e. subsheaf of colength 1) of $N_{L_i/H_i}$ corresponding to the hyperplane $T_{i,j} = T_{p_{i,j}}X_1 \cap T_{p_{i,j}}(K_{i,j}) \subset K_{i,j} \simeq K_i = N_{L_i/H_i}(p_{i,j})$, i.e, the kernel of the surjection $N_{L_i/H_i} \to N_{L_i/H_i}(p_{i,j})/T_{i,j}$. From the Lemma and the results of the previous section it then follows easily that the normal bundle $N_{\tilde{C}_0/X_0}$ is balanced. In particular, it is unobstructed and $\tilde{C}_0$ smooths out in the family $\mathcal{X}$ to a smooth curve on $X_i$ with balanced normal bundle. Since a balanced bundle of nonnegative degree is semipositive, hence has vanishing $H^1$, a rational curve with balanced normal bundle on a smooth Fano hypersurface automatically deforms with the hypersurface, we conclude:
Proposition 8. For all \( e \leq d \leq n \), a general hypersurface \( X \) of degree \( d \) in \( \mathbb{P}^n \) contains a rational curve \( C \) of degree \( e \) such that the normal bundle \( N_{C/X} \) is balanced:

\[
N_{C/X} \cong r_+ \mathcal{O}_C(a_+ + 1) \oplus (n - 2 - r_+) \mathcal{O}_C(a_+)
\]

where

\[
a_+ = \left\lfloor \frac{(n + 1 - d)e - 2}{n - 2} \right\rfloor, \quad r_+ = (n + 1 - d)e - 2 - (n - 2)a_+.
\]

This result will later be reproved and generalized to higher-degree curves (see Thm 24). The lower-degree case will play a role in the proof of our main result, through the following consequence:

Corollary 9. Notations as above, if \( a_+ \geq 0 \) and \( q_1, ..., q_{a_++1} \) are general points on \( X \), there is a rational curve \( C \) of degree \( e \) on \( X \) through them and the family of such curves near \( C \) passing through \( q_1, ..., q_{a_++1} \) is \( r_+ \)-dimensional and fills up a subvariety whose proper transform on the blowup of \( X \) in \( q_1, ..., q_{a_++1} \) is locally smooth and \( (r_+ + 1) \)-dimensional at its intersection with each exceptional divisor.

Proof. For \( C \) with balanced normal bundle we have \( H^1(N_{C/X}(-p_1...-p_{a_++1})) = 0 \). Therefore by deforming \( C \) and the points we get a general \( (a_+ + 1) \)-tuple on \( X \) and moreover for any fixed collection of points \( q_i \) the curves sweep up a subvariety whose tangent spaces on the blowup come from \( H^0(N_{C/X}(-p_1...-p_{a_++1})) = H^0(N_{C/X,+}(-p_1...-p_{a_++1})) \), where \( N_{C/X,+}(-p_1...-p_{a_++1}) \) is a trivial bundle. \( \square \)

Remark 10. The variety filled up by the curves through the \( (q_\bullet) \) points will naturally be singular at those points, so its dimension cannot be computed by looking at tangent spaces.; what is important for our arguments below is its local dimension at the points.

Remark 11. In general, there exists rational curves with positive, hence unobstructed, but unbalanced normal bundle on general hypersurfaces. For example, a quadric \( X \) of large enough dimension is ruled by planes, and conics in those planes- which are special among conics on \( X \)- have positive, unbalanced normal bundle.

Remark 12. After this was written, we found that closely related results on balanced normal bundles has been obtained earlier by Coskun and Riedl\(^{11}\) using different, degeneration-free methods. Their results apply in a larger range of degrees compared to Lemma\(^{8}\) and for complete intersections as well, rather than just hypersurfaces. Anyhow a more general result for hypersurfaces will be given below in Proposition\(^{24}\)

Next we briefly review with proof, in a form convenient for our purposes, some facts on unfolding and unkinking rational curves, which can be found in Kollár’s book \(^{4}\), especially Sect. II.3, Thm. 3.14 on free rational curves. Let

\[
f : \mathbb{P}^1 \to X
\]

be a morphism to a smooth \( n \)-fold, which we assume is a general member of a maximal family filling up \( X \).
Lemma 13. Notation as above, $f$ is an immersion of $n \geq 2$ and an embedding if $n \geq 3$.

Proof. Recall that a bundle on $\mathbb{P}^1$ is semipositive when it is a direct sum of line bundles of non-negative degrees. Note that our filling hypothesis implies - indeed is equivalent to - the property that $f^*(T_X)$ is semipositive. We show $f$ is injective if $n \geq 3$. Suppose for contradiction $p_1 \neq p_2 \in \mathbb{P}^1$ are such that $f(p_1) = f(p_2) = x$. Then, for any first-order deformation of $f$, the associated section $s$ of $f^*(T_X)$ has the property that $s(p_1) = s(p_2) \in T_X(x) = T_{X,x} \otimes \mathbb{C}(x)$. Let $L$ be any hyperplane in $T_X(x)$ containing $df(T_{p_1}(p_1)) + df(T_{p_1}(p_1))$. There is an associated colength-1 subsheaf $M \subset f^*(T_X)$ such that $f^*(T_X)/M$ is the skyscraper sheaf $T_X(x)/L$ at $x$, and then $s \in H^0(M)$. But since $M$ is a direct sum of line bundles of degree $\geq -1$, we have

$$h^1(M) = 0 = h^1(f^*(T_X)), \quad h^0(M) < h^0(f^*(T_X)).$$

This contradicts the fact that $f$ is general in an unobstructed family of dimension $h^0(f^*(T_X))$.

The argument that $f$ is unramified in $n \geq 2$ proceeds similarly: if $f$ is ramified at $p \in \mathbb{P}^1$, there exists $k \geq 2$ and an injection $\text{Sym}^k(T_{p_1}(p)) \rightarrow T_X(f(p))$ such that the value at $p$ of any first-order deformation of $f$ is contained in the image. Then we can take for $L$ any hyperplane containing the image and argue as above. \hfill \square

3. QUASI-CONES

A quasi-cone in $\mathbb{P}^n$ is by definition a hypersurface with equation

$$F = Q(F_d, F_{d-1}) = F_d(x_1, ..., x_n) + x_n F_{d-1}(x_1, ..., x_n)$$

where $x_0, ..., x_n$ are homogeneous coordinates and $F_d, F_{d-1}$ are homogeneous polynomials of degree $d, d - 1$ respectively. This is equivalent to saying $F$ has degree $d$ and multiplicity $d - 1$ at $p = [1, 0, ..., 0]$. Note that $F_{d-1}$ is uniquely determined by $F$ while $F_d$ is uniquely determined mod $F_{d-1}$. Let

$$b : \tilde{P} \rightarrow \mathbb{P}^n$$

denote the blowup of $p$, with exceptional divisor $E \simeq \mathbb{P}^{n-1}$, and let

$$\pi : \tilde{P} \rightarrow \mathbb{P}^{n-1}$$

denote the natural projection, which is an isomorphism on $E$.

Now let $\tilde{X} \subset \mathbb{P}^n$ be a quasi-cone with equation $Q(F_d, F_{d-1})$ as above, and let

$$b_X : X \rightarrow \tilde{X}$$

be the proper transform of $\tilde{X}$ in $\tilde{P}$, with exceptional divisor $E_X = X \cap E \simeq F_{d-1}$. Via $\pi_X$, $X$ can be realized as the blowup of $\mathbb{P}^{n-1}$ in $Y = V(F_{d-1}) \cap V(F_d)$. In fact the rational map $b_X \circ \pi_X^{-1} : \mathbb{P}^{n-1} \cdots \rightarrow \mathbb{P}^n$ corresponds to the linear system on $\mathbb{P}^{n-1}$ with base locus $Y$ generated by $x_1 F_{d-1}, ..., x_n F_{d-1}, F_d$. $\tilde{X}$ is said to be quasi-smooth if $Y$ is smooth. In this case it is easy to see that $X$ is smooth.

Let $\tilde{Y} \subset X$ be the exceptional divisor of $\phi_X$, which is isomorphic to the projectivized conormal bundle of $Y$ in $\mathbb{P}^{n-1}$, and maps to the ruled subvariety $\tilde{Y} \subset \tilde{X}$ defined by $F_{d-1}, F_d$, contracting the ‘infinity section’ $\tilde{Y} \cap E_X$. 


Now let $L, H$ be the line bundles on $X$ pulled back respectively from the hyperplane bundles on $\mathbb{P}^n, \mathbb{P}^{n-1}$. Then we have

$$L = dH - \tilde{Y},$$
$$\tilde{F} = (d - 1)H - \tilde{Y},$$
and consequently

$$L = \tilde{F} + H.$$

The following is worth noting

**Lemma 14.** If $d \leq n - 1$ and $\tilde{X}$ is quasi-smooth then $X$ is Fano; in fact, $-K_X$ is very ample.

**Proof.** $-K_X = nH - \tilde{Y} = (n - d)H + L$. The map

$$b \times \pi : X \to \mathbb{P}^n \times \mathbb{P}^{n-1}$$

is clearly an embedding, hence $-K_X = (b \times \pi)^*(\mathcal{O}(n - d, 1))$ is very ample. □

Now let $C$ be an irreducible curve in $X$ corresponding to a curve in $\tilde{X}$ of degree $e$ and multiplicity $m$ at $p$. Then

$$C.L = e, C.\tilde{F} = m.$$

Consequently,

$$C.H = e - m, C.E = (d - 1)(e - m) - m.$$

Thus, $C$ maps to a curve $C_s$ in $\mathbb{P}^{n-1}$ that has degree $e - m$ and meets $Y$ in $(d - 1)(e - m) - m$ points (among the $(d - 1)(e - m)$ intersections of $C_s$ and $F_{d-1}$).

For instance, when $m = e - 1$, $C_s$ is a $(d - e)$-secant line of $Y$. Next, we make an elementary remark about curves in $\tilde{Y}$, considered as ‘infinitely-near secants’ to $Y$. This result will not be needed later, but it is psychologically important. To begin with note that

$$\tilde{Y} = \mathbb{P}(\tilde{N}_Y) = \mathbb{P}(\mathcal{O}_Y((-d + 1)H, -dH))$$

and that

$$\mathcal{O}_{\tilde{Y}}(\tilde{Y}) = \mathcal{O}_{\tilde{Y}}(-1) = \mathcal{O}_{\mathbb{P}(\tilde{N}_Y)}(-1).$$

Now let $C \to \tilde{Y}$ be an irreducible curve. Such a curve corresponds to a map $f : C \to Y$ together with an invertible quotient $f^*(\tilde{N}_Y) \to A$, and clearly

$$A = \mathcal{O}_{\tilde{Y}}(1).C.$$

Now clearly the smallest-degree invertible quotient of $\tilde{N}_Y$ is $\mathcal{O}(-dH)$ which corresponds to the ‘infinity section’ $\tilde{Y} \cap \tilde{F}$. For any curve not contained in the infinity section, the degree of the quotient $A$ is at least $(-d + 1) \deg(H)$. Thus we conclude

**Lemma 15** (Infinitely-near secants). For an irreducible curve $C \to \tilde{Y}$, not contained in $\tilde{F}$, we have

$$C.\tilde{Y} \leq (d - 1)C.H.$$

If $C$ is contained in $\tilde{F}$, we have

$$C.\tilde{Y} = dC.H.$$
4. Fans and their Hypersurfaces

In this section we fix a dimension $n$. Let $\tilde{\mathbb{P}}^n$ denote the blowup of $\mathbb{P}^n$ at a point.

**Definition 16.** A fan (of fibre length $m$ or $m$-fan) is a normal-crossing variety of the form

$$P_1 \cup ... \cup P_m$$

where $P_1 \simeq ... \simeq P_{m-1} \simeq \tilde{\mathbb{P}}^n$, $P_m \simeq \mathbb{P}^n$ and for $i = 1, ..., m - 1$, $P_i \cap P_{i+1}$ is the exceptional divisor in $P_i$ and a hyperplane in $P_{i+1}$ not meeting its exceptional divisor if any.

A relative fan is a flat proper morphism $\mathcal{P} \to T$ whose fibres are fans (NB the fibre length of these fans may vary).

Our interest in fans stems from the fact that they are degenerate forms of $\mathbb{P}^n$, as illustrated by the following construction of a relative fan with general fibre a 1-fan (i.e. $\mathbb{P}^n$) and special fibre a 2-fan. Let $(B, 0_B)$ be a smooth affine pointed curve (e.g. $\mathbb{A}^1$) with coordinate $t$ and let $\mathcal{P}(1)/B$ denote the blow-up of $\mathbb{P}^n \times B$ in $(0, 0_B)$. The fibre of $\mathcal{P}(1)$ over $0_B \in B$, is a 2-fan $P_1 \cup P_2$ where $P_1$ is the blowup of $\mathbb{P}^n \times 0_B$ in $(0, 0_B)$, with exceptional divisor $E_1 \simeq \mathbb{P}^{n-1}$ and $P_2$ is the exceptional divisor, which is a copy of $\mathbb{P}^n$ containing $E_1 = P_1 \cap P_2$ as a hyperplane. Thus $\mathcal{P}(1)/B$ itself is a relative fan (of max fibre length 2). We will also denote by $E_0 \subset P_1$ the pullback of a general hyperplane from $\mathbb{P}^n$.

The above construction can be iterated to yield for all $M \geq 1$ a relative fan $\mathcal{P}(M)/T = B^M$ called a standard fan with smooth total space and fibres $m$-fans for $1 \leq m \leq M + 1$. Inductively, $\mathcal{P}(M + 1)$ is the blowup of $\mathcal{P}(M) \times B$ in $0_M \times B^M \times 0_B$ where $0_M$ is a point in the 'last component' $P_{M+1}$ of $\mathcal{P}(M)$ over $0_B$ which is not in $E_M$. An $m$-fan fibre of $\mathcal{P}(M)$ is determined up to locally trivial deformation (and even up to isomorphism) by a sequence $(i_1 < ... < i_m) \subset [1, M + 1]$ and has the form $P'_1 \cup ... \cup P'_m$ where each $P'_j$ is a smoothing of $P_{i_j} \cup ... \cup P_{i_{j+1}-1}$ for $j < m$ or $P_i \cup ... \cup P_{M+1}$ for $j = m$. We call this a fibre of type $(i_\bullet)$.

Note that the base $B^M$ of $\mathcal{P}(M)$ is endowed with a natural coordinate stratification so that over the $i$th coordinate hyperplane $t_i = 0$, the $i$-th 'joint' $E_i$ forms a flat family whose general fibre is a 2-fan with double locus $E_i$. The pullback of $t_i = 0$ is a sum of 2 divisors $D^-_i, D^+_i$ whose fibres over a given point consist of all fan components 'below' or 'above' $E_i$ respectively. The fact that $E_i$ has opposite normal bundles in $P_i$ and $P_{i+1}$ is a direct consequence of the smoothness of the total space of the relative fan $\mathcal{P}$: indeed we have

$$N_{E_i/P_i} = N_{P_{i+1}/P_i}|_{E_i}, N_{E_i/P_{i+1}} = N_{P_i/P}|_{E_i},$$

however, if $F$ is a complete fibre, we have

$$N_{P_i/P}|_{E_i} \otimes N_{P_{i+1}/P}|_{E_i} = N_{F/P}|_{E_i} = \mathcal{O}_{E_i}.$$

There is a useful operation on relative fans called stretching. Given a relative fan $\mathcal{P}/T$ with smooth total space, plus a component $E_i$ of the relative double locus, living over a divisor $(t_i) \subset T$, the stretch $\mathcal{P}_i^+ / T^+$ is defined as follows. Consider a base change $T^+ \to T$ given locally by $s_i^2 = t_i$. The pullback of $E_i$ becomes singular, with local equation $s_i^2 = xy$. Blowing this up, the total space $\mathcal{P}_i^+$ becomes smooth and the pullback of $E_i$ is a conic bundle $P_i^+$ meeting the proper
transforms of $P_i$ and $P_{i+1}$ in disjoint sections denoted $E'_i, E''_i$ whose respective normal bundles in $P'_i$ are $O_{E'_i}(1), O_{E''_i}(-1)$. This easily implies that

\[ P'_i = P_{E'_i}(O(1) \oplus O) = P_{E''_i}(O \oplus O(-1)) \]

so again $P'_i$ is a $\mathbb{P}^n$ blown up at a point and $P'_i + T^+$ is a relative fan, with max fibre chain length increased by 1.

The stretch $P_i^{++}/T^+$ just constructed is a subvariety of a relative fan $P/T$ constructed as follows. Consider the base change $T^{++} \to T, t_i = t'_i t''_i$, which has, locally over $E_i$ in the above setup, the equation $t'_i t''_i = xy$.

Then blow up the Weil divisor defined locally $(t'_i, x)$ (geometrically and globally this consists of the sum of the components of the pullback family over $(t'_i)$ lying to one side of $E_i$). This yields a new relative fan $P_i^{++}/T^{++}$ with smooth total space which contains the regular stretch $P_i^+/T^+$ as the divisor with equation $t'_i = t''_i$. The same relative fan is obtained by blowing up $(t''_i, y)$.

The parameter $t'_i$ (resp. $t''_i$) cuts out on $P^{++}/T^{++}$ the divisor $E'_i$ (resp. $E''_i$). It is easy to check that a big stretch of a standard relative fan $P(m)/B^m$ is $P(m + 1)/B^{m + 1}$.

The following result shows that every relative fan is essentially standard:

**Proposition 17.** Locally at each point of $B^m$, $P(m)$ is a versal deformation of its fibres. At $0^m_B$, $P(m)/B^m$ is miniversal.

**Proof.** Clearly locally trivial deformations of a fan $P$ are trivial, i.e. $H^1(P, T^0_P) = 0$ and likewise for $H^2$. On the other hand the $T^1$ sheaf is clearly $\bigoplus O_{E_i}$ because the singularity of $P$ along each $E_i$ is just an ordinary curve double points times $E_i$. This shows that the natural ‘Kodaira-Spencer’ map $T_1 T \to H^0(P, T^1_P)$ is surjective and even an isomorphism if $t = 0^m_B$. This proves our assertion. □

Next we discuss hypersurfaces on fans. Let $(d_\bullet) = (d_1 \geq d_2 \geq \ldots \geq d_m \geq 0)$ be a decreasing sequence of integers. Consider the line bundle on $P(m)$:

\[ O(d_1, ..., d_m) = O(d_1 D_1^+ - \sum_{i=2}^{m} d_i D_i^+). \]

This is a line bundle whose restriction on each fibre component $P_i$ is the line bundle $O(d_i E_i - d_{i+1} E_{i+1})$ or $O(d_m)$ if $i = m$. On the generic fibre, it is just $O(d_1)$. Note that this bundle has no higher cohomology on any fibre, hence locally constant sections: in fact it suffices to prove this for the most special fibre, an $m$-fan, and the latter can be easily done by induction on $m$, writing the $m$-fan as $P_1 \cup (P_2 \cup \ldots \cup P_m)$ and using a Mayer-Vietoris sequence. Also this bundle is clearly very ample if the sequence $(d_\bullet)$ is strictly decreasing. Finally, note that the restriction of $O(d_\bullet)$ to a fibre of type $(i_\bullet)$ can be identified with $O(d_0, d_\bullet)$.
5. Curves on fan hypersurfaces

Let $\pi : \mathcal{P} \to B$ be a relative fan and let $\mathcal{X} \subset \mathcal{P}/B$ be a relative hypersurface not containing any fibre component of $\pi$. We denote by $\mathcal{M}_0(\mathcal{X}, e)$ the Kontsevich space of stable genus-0 maps to fibres of $\mathcal{X}/B$ having the homology class of a degree-$e$ curve on a general fibre (which equals the homology class of a degree-$e$ curve on the bottom component of any fibre that is disjoint from all other components, that is, $e(D_0^+)\sim 1$).

**Proposition 18.** Assume the fibre $X_0$ of $\mathcal{X}/B$ over $0 \in B$ is a general hypersurface of type $(d, d)$ of the fan $P\cdot = \pi^{-1}(0)$, let $\mathcal{M}_*$ be a component of $\mathcal{M}_0(\mathcal{X}, e)$ that surjects to $B$, and let $f : C \to X_0$ correspond to a general member of the fibre of $\mathcal{M}_*$ over 0. Then no component of $C$ maps with nonzero degree to the singular locus of $X_0$.

**Proof.** We may assume $P\cdot$ is a 2-fan as the general case is similar. If $C_1$ is a component of $C$ mapping nonconstantly to the singular locus $E$ of $X_0$, then the local ideal of $E$ vanishes identically to some order $m$ on $C$. Then by a stretch operation we may reduce $m$ successively until $m = 0$. Note that in the stretch operation the limit curve may acquire new components, but these do not map to the singular locus of the new fan: indeed if $xy$ is a local equation of the special fibre, where $(x, y)$ are equations of the singular locus, extra components arise at points where the function $x/y$ is indeterminate on the curve family, so after blowing up this function becomes locally regular and nonconstant, which means new components do not map into the singular locus.

Thus, after sufficient stretching of the relative fan, $C_1$ maps nonconstantly into some component of a fan fibre other than the bottom one. Then by going to a versal deformation of the special fan, we may consider a general 2-fan fibre and there $C_1$ still maps nonconstantly. This implies $f$ could not be constant to begin with. □

6. Multisecant rational curves of complete intersections

Motivated by the construction of the preceding section, we now consider families of multisecant rational curves of general complete intersections. For simplicity, we have stated only a particular codimension-2 case, namely that of a $(d, d-1)$ complete intersection, which is the one we need, but the result and the proof are valid in greater generality. We begin with the case of lines.

**Proposition 19.** Let $Y$ be a general $(d, d-1)$ complete intersection in $\mathbb{P}^{n-1}, n \geq 4$ and let $a \leq d-1 \leq n-2$. Then

(i) the locus $\text{Sec}_a(Y)$ of $a$-secant lines to $Y$ is irreducible reduced $(2n-4-a)$-dimensional, has a smooth normalization, and its general member meets $Y$ transversely in exactly $a$ points.

(ii) For a general point $q \in \mathbb{P}^{n-1}$, the locus $\text{Sec}_a(Y, q)$ of $a$-secant lines to $Y$ through $q$ is $n-2-a$-dimensional and nonempty.

**Proof.** Let $U_d$ be the space of pairs of hypersurfaces $(F_{d-1}, F_d)$ such that $Y = V(F_{d-1}, F_d)$ has at most one singular point at which $F_{d-1}$ and $F_d$ are smooth and simply tangent (i.e. have same linear terms and transverse quadratic terms). This is a ‘big’ open subset of $\mathbb{P}^{Nd-1} \times \mathbb{P}^{Nd}$, i.e. the
complement of $U_d$ has codimension $> 1$, and hence $U_d$ is simply connected. Consider the locus $W_d$ of quadruples $(L, A, F_{d-1}, F_d)$ such that $(F_{d-1}, F_d) \in U_d$, $L$ is a line, and $A$ is a divisor of degree $a$ on $L$ contained in $Y = V(F_{d-1}, F_d)$. This is clearly an irreducible nonsingular variety of dimension $2n - 4 + N_{d-1} + N_d - a$, and the projection $\pi_d : W_d \to U_d$ is proper. Moreover by working over a fixed $L$ note that the subset of $W_d$ where $Y$ is singular along $L$ has codimension $> 1$. It follows from the Lemma below that $\pi_d$ is also surjective; alternatively, since the tangent spaces of $Y$ can be specified generically mod $L$ at a general element of $W_d$ (see Lemma 22), it follows that $\pi_d$ is generically smooth, hence surjective.

Our claim is that a general fibre of $\pi_d$ is irreducible, and we argue by contradiction. To begin with, note that by irreducibility of $W_d$, the components of such a general fibre are monodromy-interchangeable. Also, again by irreducibility of $W_d$, if $Z$ is any codimension-1 subvariety of $U_d$ then the dimension of the fibre of $\pi_d$ over any general point $E \in Z$ is still $n - 2 - a$, same as the general fibre dimension. Now if a general fibre of $\pi_d$ is reducible, consider a Stein factorization of $\pi_d$. Then by simple connectivity of $U_d$, there would be a codimension-1 locus $Z$ in $U_d$ over which the fibre of $\pi_d$ has a nonreduced component, whose support is algebraically equivalent to a component of a general fibre over $U_d$, and hence to a fraction of an entire general fibre. By dimension counting, for a general point of such a fibre component, $Y$ is smooth at its intersection with $L$. Moreover a general line $L$ in the fibre goes through a general point of $\mathbb{P}^n$ relative to $Y$. Consequently by [9], $L$ is a good (unobstructed) secant to $V(F_{d-1}, F_d)$, i.e. a smooth point of the fibre over $U_d$. This contradiction proves assertion (i).

Assertion (ii) now follows from the fact that for $a \leq d - 1$ the family of $a$-secant lines to $Y$ fills the ambient projective space, which is a consequence of the following easy result:

\[ \boxdot \]

**Lemma 20.** Let $Y \subset \mathbb{P}^n$ be a codimension-2 complete intersection of type $(c, d)$ and let $a \leq \min(c, d) < n$. Then there is an $a$-secant line to $Y$ through a general point of $\mathbb{P}^n$.

*Proof.* (One of many). Working by induction on $a$, it suffices to prove that in the indicated range, the family of $(a + 1)$-secants is nonempty and has codimension 1 in that of $a$-secants, which would follow if the family of $(a + 1)$-secants has nonempty intersection with a general curve section of the family of $a$-secants. It thus suffices to prove that for $a < \min(c, d) < n$ any 1-parameter family of $a$-secants contains an $(a + 1)$-secant. To that end, note the tautological $\mathbb{P}^1$-bundle over such a 1-parameter family maps to a cone in $\mathbb{P}^n$ whose desingularization we denote by $K$. If there is no $(a + 1)$-secant in our family, then the intersection points of the rulings with $Y$ trace out a divisor $R$ disjoint from the section that contracts to the vertex of the cone, hence $R$ is linearly equivalent to $ah$. Now $Y$ corresponds to a section of $O(c, d)$ whose pullback minus $R$ yields a nowhere vanishing section of $O(c - a, d - a)$ on $K$. But this clearly cannot exist. \[ \boxdot \]

Our aim now is to extend the Proposition from the case of lines to that of rational curves and some of their limits. By a *simply covered rational curve* we mean a curve $C$ admitting a surjective map from a rational tree $\tilde{C}$, which has fibre degree 1 over a general point of each component of $C$. Thus, no component of $\tilde{C}$ may map with degree $> 1$, though some may map to a point, and
Theorem 21. Let $Y$ be a general $(d, d-1)$ complete intersection in $\mathbb{P}^{n-1}$ and let $a \leq e(d-1)$, $d \leq n-1$ and $e \leq d-1$. Then the locus $\text{Sec}_a^e(Y)$ of simply covered rational curves of degree $e$ that are $a$-secant to $Y$ is an irreducible reduced $((e+1)n-4-a)$-dimensional and has a smooth normalization, and its general member is an irreducible nonsingular curve that meets $Y$ transversely in exactly $a$ points and passes through a general point of $\mathbb{P}^{n-1}$.

Proof. To begin with, we use the same construction as in the case of lines above and let $W_d^e$ denote the set of triples $(C, A, F)$ such that $F \in U_d$, $C$ is a smooth rational curve of degree $e$ and $A$ is a subscheme of length $a$ of the schematic intersection $C \cap Y$ (notations as above). Note that a general $C$ will be a rational normal curve in its linear span which is a $\mathbb{P}^e$. Let $\bar{W}_d^e$ denote the closure of $W_d^e$ in the product $\mathcal{H} \times U_d$, where $\mathcal{H}$ is the relative Hilbert scheme (of zero-dimensional subschemes of length $a$) of the universal (stable) curve over $\mathcal{M}$, the Kontsevich space of stable genus-0 unpointed maps of degree $e$ to $\mathbb{P}^{n-1}$. Points of $\bar{W}_d^e$ have the form $(f, C, A, F'_{d-1}, F_d)$ such that $f : C \to \mathbb{P}^{n-1}$ is a stable map and $A$ is a length-$a$ subscheme of $f^* (F'_{d-1}, F_d)$, and the whole tuple is a limit of similar ones where $f$ is an embedding of a smooth rational (normal) curve. By a result of Gruson et al. [2], $W_d^e$ is irreducible of dimension $(e+1)n-4 + N_{d-1} + N_d - a$. For the general point $(C, A, F) \in W_d^e$, we can choose $F'_{d-1}, F_d$ to have general enough tangent spaces at $A$ (this is already true when $C$ is a chain of lines), by Lemma 22. Consequently, using Lemma 23 below, it follows that the secant bundle $N^e_C$ has vanishing $H^1$. This shows that the projection to $U_d$ is a smooth morphism at that point (hence it is dominant and has a smooth general fibre of the expected dimension).

Note that $\mathcal{H}$ is smooth [8] and there is an open subset $\mathcal{H}' \subset \mathcal{H}$ whose complement has codimension 3, such that for $(f, C, A) \in \mathcal{H}'$, $A$ has length 1 or 0 at any singular point of $C$ (if $C$ has any) and at any point lying in a non-singleton fibre of $f$. Over $\mathcal{H}'$ the projection from $\bar{W}$ is an affine bundle. Note that $\bar{W}_d^e$ contains points where $C$ is an embedded curve of the form $L \cup C_{e-1}$, where $L$ is a line and $C_{e-1}$ has degree $e-1$, both going through a general point $q \in \mathbb{P}^{n-1}$ with respect to $Y$, and $A$ consists of $a_1 \leq d-1$ points on $L \setminus q$ plus $a_2 \leq (e-1)(d-1)$ points on $C_{e-1} \setminus q$. Indeed such secant curves $L, C_{e-1}$ exist by induction and by construction the secant sheaves $N^e_{C_{e-1}}, N^e_L$ are generically globally generated, hence semipositive, where $N^e_C$ denotes the secant sheaf, which may be identified with the normal bundle to the proper transform in the blowup of $\mathbb{P}^{n-1}$ in $Y$. Now $N^e_L(q)$ and $N^e_{C_{e-1}}(q)$ admit a common quotient $Q$ such that the kernel $K$ of the natural map $N^e_L \oplus N^e_{C_{e-1}} \to Q$ corresponds to connected, locally trivial deformations of $L \cup C_{e-1}$ (compare the proof of Lemma 13). Since $K$ contains $N^e_L(-q) \oplus N^e_{C_{e-1}}(-q)$, it has $H^1 = 0$. Therefore $L \cup C_{e-1}$ is unobstructed and smoothable as secant. It follows that for a general point in $W_d^e$, $C$ also goes through a general point of the ambient space. This also gives another, inductive proof that $H^1(N^e_C) = 0$ for $C$ general.

Let $\bar{W}$ denote the normalization of $\bar{W}_d^e$. If a general fibre of $\bar{W}_d^e$ over $U_d$ is reducible, then so is a general fibre of $\bar{W}$ over $U_d$. But as $\bar{W}$ is normal, so is its general fibre hence that fibre is locally irreducible, and therefore it must be disconnected. Using as above a Stein factorization

no two components of $\bar{C}$ may map to the same curve. Clearly, any simply covered rational curve is smoothable in $\mathbb{P}^n$.

Theorem 21. Let $Y$ be a general $(d, d-1)$ complete intersection in $\mathbb{P}^{n-1}$ and let $a \leq e(d-1)$, $d \leq n-1$ and $e \leq d-1$. Then the locus $\text{Sec}_a^e(Y)$ of simply covered rational curves of degree $e$ that are $a$-secant to $Y$ is an irreducible reduced $((e+1)n-4-a)$-dimensional and has a smooth normalization, and its general member is an irreducible nonsingular curve that meets $Y$ transversely in exactly $a$ points and passes through a general point of $\mathbb{P}^{n-1}$.

Proof. To begin with, we use the same construction as in the case of lines above and let $W_d^e$ denote the set of triples $(C, A, F)$ such that $F \in U_d$, $C$ is a smooth rational curve of degree $e$ and $A$ is a subscheme of length $a$ of the schematic intersection $C \cap Y$ (notations as above). Note that a general $C$ will be a rational normal curve in its linear span which is a $\mathbb{P}^e$. Let $\bar{W}_d^e$ denote the closure of $W_d^e$ in the product $\mathcal{H} \times U_d$, where $\mathcal{H}$ is the relative Hilbert scheme (of zero-dimensional subschemes of length $a$) of the universal (stable) curve over $\mathcal{M}$, the Kontsevich space of stable genus-0 unpointed maps of degree $e$ to $\mathbb{P}^{n-1}$. Points of $\bar{W}_d^e$ have the form $(f, C, A, F'_{d-1}, F_d)$ such that $f : C \to \mathbb{P}^{n-1}$ is a stable map and $A$ is a length-$a$ subscheme of $f^* (F'_{d-1}, F_d)$, and the whole tuple is a limit of similar ones where $f$ is an embedding of a smooth rational (normal) curve. By a result of Gruson et al. [2], $W_d^e$ is irreducible of dimension $(e+1)n-4 + N_{d-1} + N_d - a$. For the general point $(C, A, F) \in W_d^e$, we can choose $F'_{d-1}, F_d$ to have general enough tangent spaces at $A$ (this is already true when $C$ is a chain of lines), by Lemma 22. Consequently, using Lemma 23 below, it follows that the secant bundle $N^e_C$ has vanishing $H^1$. This shows that the projection to $U_d$ is a smooth morphism at that point (hence it is dominant and has a smooth general fibre of the expected dimension).

Note that $\mathcal{H}$ is smooth [8] and there is an open subset $\mathcal{H}' \subset \mathcal{H}$ whose complement has codimension 3, such that for $(f, C, A) \in \mathcal{H}'$, $A$ has length 1 or 0 at any singular point of $C$ (if $C$ has any) and at any point lying in a non-singleton fibre of $f$. Over $\mathcal{H}'$ the projection from $\bar{W}$ is an affine bundle. Note that $\bar{W}_d^e$ contains points where $C$ is an embedded curve of the form $L \cup C_{e-1}$, where $L$ is a line and $C_{e-1}$ has degree $e-1$, both going through a general point $q \in \mathbb{P}^{n-1}$ with respect to $Y$, and $A$ consists of $a_1 \leq d-1$ points on $L \setminus q$ plus $a_2 \leq (e-1)(d-1)$ points on $C_{e-1} \setminus q$. Indeed such secant curves $L, C_{e-1}$ exist by induction and by construction the secant sheaves $N^e_{C_{e-1}}, N^e_L$ are generically globally generated, hence semipositive, where $N^e_C$ denotes the secant sheaf, which may be identified with the normal bundle to the proper transform in the blowup of $\mathbb{P}^{n-1}$ in $Y$. Now $N^e_L(q)$ and $N^e_{C_{e-1}}(q)$ admit a common quotient $Q$ such that the kernel $K$ of the natural map $N^e_L \oplus N^e_{C_{e-1}} \to Q$ corresponds to connected, locally trivial deformations of $L \cup C_{e-1}$ (compare the proof of Lemma 13). Since $K$ contains $N^e_L(-q) \oplus N^e_{C_{e-1}}(-q)$, it has $H^1 = 0$. Therefore $L \cup C_{e-1}$ is unobstructed and smoothable as secant. It follows that for a general point in $W_d^e$, $C$ also goes through a general point of the ambient space. This also gives another, inductive proof that $H^1(N^e_C) = 0$ for $C$ general.

Let $\bar{W}$ denote the normalization of $\bar{W}_d^e$. If a general fibre of $\bar{W}_d^e$ over $U_d$ is reducible, then so is a general fibre of $\bar{W}$ over $U_d$. But as $\bar{W}$ is normal, so is its general fibre hence that fibre is locally irreducible, and therefore it must be disconnected. Using as above a Stein factorization
of $\bar{W} \to U_d$, it follows again that there is a codimension-1 locus $Z \subset U_d$ whose inverse image in $\bar{W}$ has a multiple component $\hat{Z}$, i.e. such that the general fibre of $\bar{W}$, hence of $W^e_d$ has a multiple component $B$ whose support is a limit of a general fibre component. Note that because a multiple of $B$ moves, i.e. is in a 1-parameter family filling up $\bar{W}$, $B$ cannot be contractible, i.e. cannot map to a subvariety of codimension $> 1$ in $\mathcal{M}$, because that is a topological property. Hence $\hat{Z}$ must map to a locus of codimension 1 or less in $\mathcal{M}$.

Now if $(C, f)$ is a general element of a codimension $\leq 1$ locus in $\mathcal{M}$ then either

(i) $C$ is smooth and $f$ is an embedding, or

(ii) $C$ is smooth, $n - 1 = 3$ and $f$ is an embedding except of one transverse double point, or

(iii) $C$ has exactly 2 smooth components and $f$ is an embedding.

Now case (ii) is excluded by the assumption $d \leq n - 1$, $e \leq d - 1$, so $C$ would map to a line or conic so we are in fact in case (i) or (iii). In case (i), $C$ being irreducible and containing a general point of $\mathbb{P}^{n-1}$, it is an unobstructed secant to the corresponding $Y$ so the fibre over $U_d$ cannot be multiple. In case (iii), the locus of curves $(C, f)$ already has codimension 1 so the $Y$ is general for such $C$; however it is easy to see that for such $(C, f)$ and $Y$ general, $C$ is an unobstructed secant.

This contradiction completes the proof.

Lemma 22. Let $C$ be a general chain of $e$ lines in $\mathbb{P}^n, e \leq n$. Let $Y$ be a general $(d - 1, d)$ complete intersection, $d \leq n$, meeting $C$ in $a \leq e(d - 1)$ points with at most $d - 1$ on each line. Then the tangent spaces to $Y$ at $Y \cap C$ can be specified generally modulo the tangents to $C$.

Proof. There is evidently no loss of generality in assuming $C$ is a chain of $n$ lines in $\mathbb{P}^n$ and the intersection $Z = Y \cap C$ comprises $d - 1$ points on each line. We then specialize the chain to a pencil $C = \bigcup L_i$ of $n$ general lines through a point. For each $i$, let

$$G_i = \bigcup_{j=1}^{d-1} H_{ij} \cup H_i$$

where $H_{ij}$ is a general hyperplane through the $j$th point of $Z$ on $L_i$ while $H_i$ is a hyperplane containing all the lines $L_j, j \neq i$. Then $G_i$ contains $Z$ and has general tangents on $Z \cap L_i$. Then a general linear combination $\sum t_i G_i$ also contains $Z$ and has general tangents at all of $Z$.

Using Lemma 2 and, if appropriate, smoothing the chain, we can conclude:

Lemma 23. Let $C$ be a general rational curve of degree $e$ (resp. general chain of $e$ lines) in $\mathbb{P}^n, e \leq n$. Let $Y$ be a general $(d - 1, d)$ complete intersection meeting $C$ in $a \leq e(d - 1)$ points (resp. with at most $d - 1$ on each line). Then the associated secant bundle to $C$ relative to $Y$, viz. $N^e_{C/P^n}$, is balanced.

Note that by taking $a = e(d - 1)$ and putting the curve or chain on a component $X_1$ of a 2-fan hypersurface $X_1 \cup X_2$ of type $(d, d - 1)$ (so it is disjoint from $X_2$), and smoothing the fan hypersurface, we obtain rational curves of degree $e \leq n$ on a general hypersurface of degree $d \leq n$ in $\mathbb{P}^n$. However, we can do somewhat better:
Theorem 24. For \( d \leq n \) and \( e \leq 2n - 2 \), there exists a rational curve of degree \( e \) on a general hypersurface \( X \) of degree \( d \) in \( \mathbb{P}^n \) having balanced normal bundle.

Proof. Note that the case \( e \leq n \) is handled by Lemma 23 (alternatively, by the case \( C_2 = \emptyset, a = e_1(d - 1) \) of the constructions below); in some cases Proposition 8 applies also, as well as the results of [11]. So we may assume \( e > n \). We use an induction on \( d \) based on the following 2 constructions of curves on a general hypersurface \( X_1 \cup X_2 \) of bidegree \( (d, d - 1) \) on a 2-fan.

Construction 1: \( C_2 \subset X_2 \) is a general rational curve of degree \( e_2 = n - 2 \). \( R_1, \ldots, R_{e_2} \subset \tilde{Y} \subset X_1 \) are rulings connected to \( C_2 \), i.e. \((\bigcup R_i) \cap E = C_2 \cap E\). \( C_1 \subset X_1 \) is a general curve of degree \( e_1 \leq n \) meeting \( R_1 \) and no other \( R_i \), and meeting \( \tilde{Y} \) in \( a - 1 = e_1(d - 1) - 1 \) additional points (hence \( C_1 \) is disjoint from \( E \)). We let \( C = C_1 \cup \bigcup_{i=1}^{e_2} R_i \cup C_2 \). Note that this has degree \( e_1 + e_2 \) with respect to the line bundle \( \mathcal{O}(1, 0) \).

Construction 2: \( C_2, R_2, \ldots, R_{e_2} \) as above, \( C_1 \) meeting \( \tilde{Y} \) in \( e_1(d - 1) \) points not on any \( R_i \), and meeting \( C_2 \cap E \) in 1 point. We let \( C = C_1 \cup \bigcup_{i=2}^{e_2} R_i \cup C_2 \). It has \( \mathcal{O}(1, 0) \)-degree equal to \( e_1 + e_2 - 1 \).

Regarding Construction 1, given \( C_2 \subset X_2 \), a general degree-\( e_2 \) rational curve on a general degree-(\( d - 1 \)) hypersurface, we choose \( f_d \) generally through \( C_2 \cap E \), which then determines the quasi-cone \( Q(f_d, d_{d-1}) \). Note that the normal bundle to the rulings is of the form \((n - 2)\mathcal{O} \oplus \mathcal{O}(-1)\) where the first summand is the normal bundle in \( \tilde{Y} \). It follows easily that

\[
N_{C/X}|_{C_2} = N_{C_2/X_2}, N_{C/X}|_{R_i} = (n - 2)\mathcal{O} \oplus \mathcal{O}(-1), i \geq 2, N_{C/X}|_{R_1} = (n - 1)\mathcal{O}.
\]

Now the degree of \( N_{C_2/X_2} \) is \( e_2(n + 2 - d) - 2 = (n - 2)(n + 2 - d) - 2 \) and its rank is \( n - 2 \). Therefore, it has upper rank \( r_+ = n - 4 \) (or \( n - 2 = 2 \) if \( n = 4 \)). Moreover \( N_{C/X}|_{C_1} \) is an elementary ‘up’ modification (length-1 enlargement) of the secant sheaf \( N_{C_1}^s \) in the point \( C_1 \cap R_1 \) (and thus locally equal to \( N_{C_1/P^{n-1}} \)) where \( C_1 \) denotes the image of \( C_1 \) in \( \mathbb{P}^{n-1} \). By general choice of \( f_d \), the tangent space involved can be chosen generally and consequently by Lemma 9 and induction, \( N_{C/X} \) is balanced, hence likewise for the general smoothing \((C', X')\) of \((C, X)\) where \( C' \) has degree \( e_1 + e_2 \). The case of Construction 2 is similar.

Now for \( d = 1 \), it is well known that there exist rational curves of any degree with balanced normal bundle. Then using either of the above constructions plus induction and smoothing, we construct rational curves of all degrees \( e \leq 2n \) on \( X \) of any degree \( d \leq n \) with balanced normal bundles.

\[\square\]

Remark 25. The foregoing constructions require the existence of a hypersurface \( f_d \) in \( \mathbb{P}^{n-1} \) with given tangent hyperplanes at \( e_2 \) points. Counting dimensions, the latter ‘should’ be possible when \((n - 1)e_2 \leq \binom{n + d - 1}{n - 1} \). Thus, this construction could in principle produce curves with balanced normal bundle and degree \( e \) up to, roughly, \( \frac{1}{n - 1} \binom{n + d - 1}{n - 1} \) on a general hypersurface of degree \( d \) in \( \mathbb{P}^n \). However, I have no reason to believe this would be sharp or for that matter that there is any upper bound on the degrees of rational curves with balanced normal bundle on a general Fano hypersurface.
7. Irreducible families

Here we use out techniques to re-prove a result on irreducibility of families of rational curves degree $e < d$ on hypersurfaces of degree $d < n$ in $\mathbb{P}^n$. The idea is to specialize to a 2-fan $X_1 \cup X_2$ and try to get the curve to specialize to $X_1$ by imposing the maximum number of points to go into $X_1$. This 'maximum number' is known thanks to balancedness of the normal bundle (cf. Cor. 9). Then the part of the curve in $X_1$ is a suitable multisection curve amenable to the results of the last section.

**Theorem 26.** The family of rational curves of degree $e < d$ on a general hypersurface of degree $d$ in $\mathbb{P}^n$ for $d < n, n \geq 4$ is irreducible, generically unobstructed and of dimension $e(n + 1 - d) + n - 4$.

**Remark 27.** For $d \leq n - 2$ and $e$ arbitrary, this has been proven by Riedl-Yang [10] based on bend-and-break. For $d = n - 1$ some results for low $e$ were obtained by Tseng [11]. In fact, the referee notes that the Theorem also follows fully from the older work of Harris-Roth-Starr [3]. The following argument is due to him: [3] shows in the given range that the locus of $2$-points instead, leading to a 1-dimensional variety, i.e. the curve $C'$ itself.

Consider a degeneration of $X$ to a general hypersurface $X_1 \cup X_2$ of bidegree $(d, d - 1)$ on a 2-fan and with it a degeneration of $q_\bullet$ to general points $(q_\bullet)$ on $X_1$ (only!). Here $X_1$ is the blowup of a general quasi-cone of degree $d$ at its vertex, $X_2$ is a general degree-$(d - 1)$ hypersurface and $E = X_1 \cap X_2$ is the exceptional divisor on $X_1$ and a hyperplane section of $X_2$, hence a degree-$(d - 1)$ hypersurface in $\mathbb{P}^{n-1}$. As we have seen, $X_1$ can also be represented as the blowup of a general $(d, d - 1)$ complete intersection $Y = F_{d-1} \cap F_d$ in $\mathbb{P}^{n-1}$ with exceptional divisor $Y$ and $E$ being the proper transform of $F_{d-1}$.

Let $(C, f)$ be the Kontsevich limit on $X_1 \cup X_2$ of a general curve in $S$, considered as a stable (unpointed) map. This is the special fibre of a 1-parameter family

$$C \to X', \quad B$$

As we have seen in Prop. 18, no component of $C$ can map non-constantly to $E = X_1 \cap X_2$. Let $C_i$ be the sum of the components of $C$ mapping non-constantly to $X_i$ (but not to $E$), $i = 1, 2$ and let $m$ be the degree of $f_*(C_2)$, i.e.

$$m = f_*(C_2).E = C_2.f^*(X_1) = -C_2.f^*(X_2).$$
Likewise,
\[ m = f_\ast (C_1).E = C_1.f_\ast (X_2) = -C_1.f_\ast (X_1) \]
(note that \( f_\ast (X_1), f_\ast (X_2) \) have degree 0 on any fibre component mapping to a point). We call \((e, m)\) the type of \( f_\ast (C_1) \) (or, abusively, of \( C_1 \)). Then \( f_\ast (C_1) \) has \( H \)-degree \( e - m \) and intersection number \( a = (e - m)(d - 1) - m \) with \( \overline{Y} \) and by upper semicontinuity, the family of \( C_1 \)'s going through \((q_i)\) fills up a variety that has dimension \( r_+ + 1 \) or more- though possibly singular or non-reduced- locally at each \( q_i \), because this is true for the curves in \( X \) through \((q_i)\) (see Corollary 9).

Now I claim that \( C_1 \) is irreducible except for contracted components. Let \( C_{1,1}, ..., C_{1,k} \) be the connected components of \( C_1 \), let \((e_i, m_i)\) be the type of \( C_{1,i} \), and let \( b_i \) be the number of points \( q_i \) on \( C_{1,i} \). Thus,
\[
\sum b_i = a_+ + 1 = \left\lceil \frac{e(n-d+1)-2}{n-2} \right\rceil, e(n-d+1) - 2 = a_+(n-2) + r_+.
\]

By Proposition 18, no \( C_{1,i} \) can map nonconstantly to \( E \) and by Lemma 15 we have \( m_i \geq 0, \forall i \) (even if \( C_{1,i} \) maps to \( \overline{Y} \)).

Note that the degree of the secant bundle \( \mathcal{N}^s_{C_{1,i}} = \mathcal{N}_{C_{1,i}/X} \) is \( c_i = e_i(n-d+1) + m_i - 2 \) which we write as above in the form
\[ a_{+i}(n-2) + r_{+i}, \quad 0 < r_{+i} \leq n-2, \]

Thus if \( \mathcal{N}^s_{C_{1,i}} \) is balanced then it has the form \( r_{+i} \mathcal{O}(a_{+i} + 1) \oplus (n-2-r_{+i}) \mathcal{O}(a_{+i}) \) but in any case any family of curves of type \((e_i, m_i)\) can go through at most \( a_{+i} + 1 \) general points. Note that \( c_i > 0 \): indeed the only case where \( c_i = 0 \) is where \( e_i = 1, d = n-1, m = 0 \) and this would violate connectedness of \( C \). Also,
\[ \sum c_i \leq e(n-d+1) - 2. \]

Now we assume \( r_+ < n-2 \) as the case \( r_+ = n-2 \) is similar and simpler. Then since \( \sum b_i = a_+ + 1 \), we have
\[ (n-2) \sum b_i = e(n-d+1) - 2. \]

Therefore we may assume \( b_1 \geq a_{+1} + 1 \). But since \( C_{1,1} \) goes through \( b_1 \) general points this forces \( b_1 = a_{+1} + 1 \). Now suppose \( k > 1 \). If \( b_1 = a_+ \) (i.e. all the \( q \) points go to \( C_{1,1} \)), then we get \( r_{+1} < r_+ \) which is a contradiction since as noted above \( r_+ + 1 \) is the local dimension of the variety filled up by the curves going through the \( q \) points. Therefore we have \( b_1 < a_+ \) so we may assume \( b_2 > 0 \). But then freeing up a \( q \) point on \( C_{1,2} \) has no (dimension-raising) effect on the variety locally filled up by the curves at some \( q \) point on \( C_{1,1} \), unlike the situation on the general fibre, which is a contradiction.

Now a similar dimension argument shows that we must have \( m = 0 \). Then by Lemma 15 \( f \) is an embedding if \( n \geq 4 \). Then we can invoke Theorem 21 to conclude that the limit family is unique. Since the limit occurs with multiplicity 1, it follows that the family of curves of degree \( e \) on \( X \) is irreducible.

\( \square \)
Remark 28 (speculative). In the range $e \geq d$ not covered by our argument, The dimension of any component $S$ of the family of rational curves is at least $d(n + 1 - d) + n - 4$, which is $> 2(n - 1)$ provided $d$ is at least, roughly, $3n/4$ (the most interesting part of the $d$ range is near $n - 1$). Thus in this range bend-and-break is applicable to conclude that $S$ contains some reducible curves. Then, some of the simpler arguments from [10], avoiding their 'borrowing' argument, might conceivably be used to conclude irreducibility, but we have no proof.

8. CURVES THROUGH A GENERAL POINT

Let us denote by $R_e(X)$ (resp. $R_e(X, p)$) the set of rational curves of degree $e$ in $X$ (resp. in $X$ and through the point $p \in X$). Here we show that under some circumstances it is possible to deduce irreducibility of $R_e(X, p)$ for general $p$ from that of $R_e(X)$. As the referee notes, these results also follow from [3] as in Remark 27.

Proposition 29. Notations as above, assume $n \geq 4$ and $d \leq n - 2$. Then $R_e(X, p)$ is irreducible for general $p$.

Proof. By the theorem of Riedl-Yang [10], $R_e(X)$ is irreducible, hence the components of $R_e(X, p)$ for general $p$ are monodromy-interchangeable as $p$ ranges over $X$. Now we degenerate $X$ to a general $d$-fan of type $(d, d - 1, ..., 1)$, which is of the form $X(0) = X_d \cup X_{d-1} \cup ... \cup X_1$ where each $X_k$ is a blown up $k$-quasi-cone. Pick a general point $p \in X_1$ and consider the family of $e$-gons in $X_1$ of the form $C = L_1 \cup ... \cup L_e$, where $p \in L_1$, where the $L_i$ are lines and $L_i$ meets $L_{i+1}, i < e$. We may consider $Y = X_1 \cap X_2 \cap ... \cap X_d \cap E$ as a general $(d, d - 1, ..., 1)$ complete intersection in $E = \mathbb{P}^{n-1}$. When $d \leq n - 2$, $Y$ is irreducible. Choosing $L_i$ so that $L_i \cap E \in Y$, $C$ may be extended 'trivially' by attaching rulings, to a smoothable curve in $X(0)$. This leads to an irreducible unique family of $e$-gons contained in a unique component of $E_r(X, p)$. By the monodromy-interchangeability, it follows that $R_e(X, p)$ itself is irreducible. \qed

The argument is partly extendable to the case $d = n - 1$:

Proposition 30. Notations as above, assume $d = n - 1 \geq 3$ and $e < d$. Then $R_e(X, p)$ is irreducible for general $p$.

Proof. We argue similarly, now using Theorem 26, in lieu of [10] initially to conclude that the components of $R_e(X, p)$ are monodromy-interchangeable (this is the only place where the assumption $e < d$ is used). Only now we degenerate $X$ to $X_d \cup ... \cup X_3 \cup X'_2$ where $X'_2$ is a quadric. We will prove irreducibility for $e = 2, 3$ using conics and twisted cubics in $X'_2$ extended 'trivially' via rulings in $\tilde{Y}$. Then the general case follows similarly by using chains of conics or chains consisting of conics plus one twisted cubic in $X'_2$ extended via rulings. Note that in this case $Y = E \cap X_d \cap ... \cap X'_2$ is an irreducible curve. The conics in question are of the form $M \cap X'_2$ where $M$ is a $\mathbb{P}^2$ spanned by $q_1, q_2 \in Y$ and $p \in X'_2 \setminus E$ and the set of such, for fixed $p$, is obviously irreducible. This settles the case $e = 2$.

The case of twisted cubics, i.e. $e = 3$, is harder because for fixed, general $q_1, q_2, q_3 \in Y$ and $M = \langle q_1, q_2, q_3, p \rangle$, $M \cap X'_2$ is a smooth quadric surface which carries \textit{two} 1-parameter families of twisted cubics through the 4 points, each complementary or 'linked' to a family of lines (this
already shows that $R_3(X, p)$ has at most 2 components and then using chains with one cubic and some conics it follows that the same is true for $R_e(X, p)$ for all $e \geq 3$ odd. We are however claiming that $R_3(X, p)$ and then $R_e(X, p)$ has just one component. To this end, specialize $q_\bullet$ to $q_0^\bullet$, where $q_0^\bullet \in T \cap Y$ where $T$ is the tangent hyperplane to $X'_2$ at $p$. In this special case $M^0 \cap X'_2 = \langle q_0^\bullet \rangle \cap X'_2$ is a quadric cone with vertex $p$, which carries a unique family of twisted cubics $C$ through $p$, $q_0^\bullet$ linked to lines through the vertex $p$. If the locus of twisted cubics in $X'_2$ trisecant to $Y$ is reducible, it would be singular at $[C]$. The Zariski tangent space at this locus is $H^0$ of the secant bundle $N_C^e(-p)$, where $N_C$ is the normal bundle in $X'_2$, which has degree $3n - 5$, and $N_C^e$ is the elementary modification of $N_C$ corresponding to the tangent spaces $T_{q_0^\bullet}Y, i = 1, 2, 3$.

Now it is easy to check that for a twisted cubic $C$ contained in a smooth quadric surface $Q$ inside $X'_2$ the normal bundle $N_{C/X'_2}$ is $O(4, 3^{n-3})$ with $O(4) = N_{C/Q}$, $O(3^{n-3}) = N_{Q,X'_2}|C$, while for $C$ in a quadric cone surface the normal bundle becomes $O(5, 3^{n-4}, 2)$ (which shows that $C$ is a singular point in the space of twisted cubics through a fixed quadruple $(q_0^\bullet, p)$, because twisting down by these points yields a (-2) quotient). Now because the tangent $T_{q_0^\bullet}$ can be chosen generally, the secant bundle $N_C^e$ is a general elementary modification of $N_{C/X'_2}$ of colength $n - 2$ locally at each of the $q_0^\bullet, i = 1, 2, 3$ so $N_C^e$ is of type $O(2, 1, 0^{n-5})$, $N_C^e(-p)$ is of type $O(1, 0, 0, (-1)^{n-5})$, and consequently $H^1(N_C^e(-p)) = 0$, which makes $C$ a nonsingular point in the space of twisted cubics trough $p$ trisecant to $Y$ (allowing $q_0^\bullet$ to move on $Y$), contradiction. □

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