Generalised coherent states for $SU(n)$ systems

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Generalised coherent states are developed for $SU(n)$ systems for arbitrary $n$. This is done by first iteratively determining explicit representations for the $SU(n)$ coherent states, and then determining parametric representations useful for applications. For $SU(n)$, the set of coherent states is isomorphic to a coset space $SU(n)/SU(n-1)$, and thus shows the geometrical structure of the coset space. These results provide a convenient $(2n-1)$-dimensional space for the description of arbitrary $SU(n)$ systems. We further obtain the metric and measure on the coset space, and show some properties of the $SU(n)$ coherent states.

I. INTRODUCTION

Coherent states were originally constructed and developed for the Heisenberg–Weyl group to investigate quantized electromagnetic radiation [1]. These coherent states were generated by the action of the Heisenberg–Weyl group operators on the vacuum state which led to group theoretic generalizations by Peleromov [2] and Gilmore [3]. These two mathematical frameworks differ in some points, such as the representations of groups and the reference states; these differences are summarized in [4]. Coherent states for $SU(2)$ have also been called atomic coherent states [4,5], and have been found useful for treating atom systems, and also for investigations of quantum optical models such as nonlinear rotators [6]. $SU(2)$ coherent states have been successfully applied to the analysis of the classical limit of quantum systems, and more recently, to the investigations of nonlinear quantum systems and quantum entanglement [7].

In spite of these many successful $SU(2)$ coherent state applications, not much work has been done towards generalising the analysis to other $SU(n)$ groups, although $SU(3)$ symmetries were employed to treat a schematic nuclear shell model [8]. More recently, this lack has been addressed for $SU(3)$ systems with the explicit construction of the $SU(3)$ coherent states [9,10], the calculation of Clebsch–Gordon coefficients [11,12], and the investigation of Wigner functions [13]. Further, geometrical phases for $SU(3)$ systems have been discussed in [14]. These developments for $SU(3)$ are technologically useful and allow treatment of more complex quantum systems such as coupled Bose–Einstein condensates [15].

In this paper, we construct a set of explicit coherent states for $SU(n)$, and apply group theoretic techniques to facilitate the investigation of nonlinear quantum systems and quantum entanglement. In order to construct explicit coherent states, we need to specify the group representation and the reference states. For the chosen group representation, it is necessary to show a useful decomposition and a parameterization giving usable expressions for the coherent states. Formal approaches to the definition of coherent states are often not readily applicable. For instance, while the Baker–Campbell–Hausdorff relation derived for $SU(2)$ can be used to define coherent states, this approach does not yield explicit formulae and parameterisations.

In this paper, we employ the decomposition for $SU(n)$ in [16] and exploit its symmetric parameterisation. A set of coherent states of $SU(n)$ is called an orbit, and is produced by the action of group elements on a reference state which here is chosen to be the highest weight state. For instance, for $SU(2)$ the highest weight state for a spin $1/2$ system is spin-up, and the orbit is the surface of a 3-sphere. For general $n$, this orbit corresponds to a $(2n-1)$–sphere, which is isomorphic to the coset space $SU(n)/SU(n-1)$. The geometrical properties of this coset space generalise the $SU(3)$ properties described in [12]. The coset space considered here, $SU(n)/SU(n-1)$, differs slightly from the coset space normally considered for coherent states, $SU(n)/U(n-1)$, by including an arbitrary phase. The coset space $SU(n)/SU(n-1)$ enables us to provide a more general method to construct coherent states. Developing the representations and decompositions of higher rank groups becomes rapidly messy, however the decomposition in [16] leads to a systematic procedure for the derivation of the coherent states on the coset space $SU(n)/SU(n-1)$ without additional complexity. Hence we can easily extract an arbitrary phase carrying no physical significance for application to physical systems.

In Section II, we obtain an iterative equation in $SU(n)$ coherent states for the simplest irreducible unitary representation of $SU(n)$. We also show the geometrical structure of the coset space $SU(n)/SU(n-1)$, and provide the metric and measure on the space. In Section III our analysis is generalised to the case of finding coherent states of irreducible unitary representations for arbitrarily large dimension, and parametric representations are derived. We also show some properties of the coherent states. Finally, we summarise our results in Section IV.
II. DECOMPOSITION AND COSET SPACES FOR FUNDAMENTAL REPRESENTATIONS OF SU(n)

In order to construct the SU(n) coherent states for the fundamental n × n matrix representation, we first specify the reference state |φ₀⟩ as (1, 0, ..., 0)^T, where T denotes transpose. This state is a highest weight state, in the sense that it is annihilated by each of the SU(n) raising operators. The raising (lowering) operators J⁺, J⁻ are equivalent to elementary matrices $e^{iλ h_j}$, $h < j$ ($h > j$) in the n × n matrix representation. Appendix A shows the commutation relations of these matrices. In this section we review the construction of SU(2) coherent states, which provides the origin of the recursive relation of the SU(n) coherent states. We then derive the displacement operators for SU(3) and SU(4), employing the n × n matrix representation of SU(n). Finally our results are extended to the SU(n) case.

A. Review of SU(2)

Elements g ∈ SU(2) in the fundamental 2 × 2 matrix representation of SU(2) may be parameterized as

$$g(θ, φ_1, φ_2) = \begin{pmatrix} e^{iφ_1} \cos θ & -e^{-iφ_1} \sin θ \\ e^{iφ_2} \sin θ & e^{-iφ_2} \cos θ \end{pmatrix},$$

(1)

where angles are real and lie on 0 < θ ≤ π/2 and 0 ≤ φ₁, φ₂ ≤ 2π respectively. The standard approach to obtain a set of coherent states corresponding to SU(2)/SU(1) begins with the decomposition of this matrix as

$$g(θ', φ_1, φ_2) = e^{-i(φ_2-φ_1)^2} \begin{pmatrix} \cos θ' & -e^{-iφ_2} \sin θ' \\ -e^{-iφ_1} \sin θ' & \cos θ' \end{pmatrix} e^{iφ_1},$$

(2)

where the new variable θ' is introduced as θ' = 2θ. The action of the right matrix on the highest weight vector

$$|φ₀⟩ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

changes only a phase factor and does not otherwise change the highest weight vector. Absolute phases φ₁ and φ₂ by themselves do not carry physical significance, and only their difference is physically important. This allows the removal of an arbitrary phase usually done by setting $e^{iφ_1} = 1$. Now the action of the left matrix on the highest weight state gives us coherent states $|n'_2⟩ = g|φ₀⟩ = \begin{pmatrix} \cos \theta' \\ e^{iφ_2} \sin \theta' \end{pmatrix}$, which corresponds to the 2–sphere, i.e. the surface of three–dimensional ball, with the unit vector ($cos θ', e^{iφ_2} sin θ'$), or ($cos θ', sin θ' cos φ_2, sin θ' sin φ_2$) in real coordinates, and the measure $dμ'_2 = sin θ'dθ'dφ_2$. However this method is not very convenient to construct coherent states for general SU(n). The decomposition of SU(n) equivalent to (4) is not trivial especially for larger n, and is dependent on the choice of which arbitrary phase is extracted. In this paper, to avoid using the equivalent decomposition to (2), we begin more generally with the parameterisation (2), derive coherent states and then easily remove an arbitrary phase from our SU(n) coherent states.

We now apply $g(θ, φ_1, φ_2)$ of (1) to the highest weight state $|φ₀⟩$. The action yields the SU(2) coherent states

$$|n_2⟩ = g|φ₀⟩ = \begin{pmatrix} e^{iφ_1} \cos θ \\ e^{iφ_2} \sin θ \end{pmatrix},$$

(3)

which correspond to points on a 3–sphere with unit vector ($e^{iφ_1} \cos θ, e^{iφ_2} \sin θ$), from which we derive the expression for the metric on the sphere

$$|ds_2|^2 = dθ^2 + cos^2(θ) dφ_1^2 + sin^2(θ) dφ_2^2,$$

(4)

and the measure associated with this metric (4) as

$$dμ_2 = cos θ sin θ dθ dφ_1 dφ_2.$$  

(5)

From this set of coherent states, we now give the procedure to obtain the SU(2) coherent states $|n_2⟩$. This is done by setting $e^{iφ_1} = 1$ and introducing the variable θ’ so the coherent states now correspond to 2–sphere. This shows that one can readily remove an arbitrary phase from our SU(n) coherent states without changing the decomposition of the group representation.

For the convenience of the later use, we here introduce λ–matrices and their parameterisation of SU(2). g may also be parameterised with using λ–matrices (i.e. Pauli matrices) $λ_1 = σ_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $λ_2 = σ_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, and $λ_3 = σ_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, as
\[ g = e^{i\alpha \lambda_3} e^{i\beta \lambda_2} e^{i\gamma \lambda_3}, \]

where \( \varphi_1 = \alpha + \gamma \), \( \varphi_2 = -\alpha + \gamma \), and \( \theta = -\beta \), viz \( \alpha = \frac{1}{2}(\varphi_1 - \varphi_2) \), \( \beta = -\theta \), and \( \gamma = \frac{1}{2}(\varphi_1 + \varphi_2) \).

### B. Structure of SU(n) for arbitrary n

We here employ the symmetric parameterisation for the SU(n) matrices provided in [13] to obtain an iterative equation for the SU(n) coherent states. This matrix representation efficiently yields the orbit of the highest weight state \((1, 0, \cdots, 0)^T\), because of its symmetric decomposition. The parameterisation influences the structure of the iterative equation, which we demonstrate by example for small \( n \). We derive firstly the SU(3) coherent states (which may be compared with the simplest case of [12]), and secondly the SU(4) coherent states. For each example, the expression for the coherent states shows their geometrical structure and determines the metric and measure of the coset space isomorphic to the coherent states. These examples are then generalised to SU(n) by determining the iterative equation for the SU(n) coherent states. Lastly, we give the measure of the coset space \( SU(n)/SU(n - 1) \).

An arbitrary element \( g \in SU(n) \) in the \( n \times n \) matrix representation [13] may be parameterised as

\[ g = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & e^{i\varphi_1} \cos\xi_1 & e^{i\varphi_2} \sin\xi_1 & 1 \\ \vdots & e^{i\varphi_2} \sin\xi_1 & e^{i\varphi_1} \cos\xi_1 & 0 \\ 0 & e^{i\varphi_3} \cos\xi_2 & e^{i\varphi_2} \sin\xi_2 & 0 \\ & 0 & \vdots & \vdots \\ & 0 & e^{i\varphi_4} \sin\xi_2 & e^{i\varphi_3} \cos\xi_2 \\ & & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & e^{i\varphi} \cos\theta & -\sin\theta & 0 \\ \sin\theta & e^{-i\varphi} \cos\theta & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = L_{n-1} M(\theta, \varphi) R_{n-1} \]

where \( X_{n-1}, Y_{n-1} \) are the appropriate \((n - 1) \times (n - 1)\) matrices representing elements of \( SU(n - 1) \), and \( I_k \) is the \( k \times k \) identity matrix, and we have defined three matrices \( L_{n-1}, M, R_{n-1} \) for convenience. (See Appendix B for a justification of this parameterisation.)

#### 1. Structure of SU(3)

For SU(3), since the matrices \( X_{n-1}, Y_{n-1} \) may be parameterised as [14], [8] gives

\[ g(\varphi_1, \xi_1, \varphi_2, \varphi, \theta, \varphi_3, \xi_2, \varphi_4) \]

\[ = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\varphi_1} \cos\xi_1 & e^{i\varphi_2} \sin\xi_1 \\ 0 & e^{i\varphi_2} \sin\xi_1 & e^{i\varphi_1} \cos\xi_1 \end{pmatrix} \begin{pmatrix} e^{i\varphi} \cos\theta & -\sin\theta & 0 \\ \sin\theta & e^{-i\varphi} \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\varphi_3} \cos\xi_2 & e^{i\varphi_2} \sin\xi_2 \\ 0 & e^{i\varphi_2} \sin\xi_2 & e^{i\varphi_3} \cos\xi_2 \end{pmatrix} = L_2 M(\theta, \varphi) R_2. \]

We take the highest weight state \((1, 0, 0)^T\) as the reference state and now obtain the expression of the orbit. Noting that the right matrix \( R_2 \) does not change the reference state, the displacement operator for the SU(3) coherent states is the product of the left and middle matrices, \( L_2 M(\theta, \varphi) \). The left matrix \( L_2 \) corresponds to SU(2), hence the orbit of the reference state is the coset space \( SU(3)/SU(2) \). The first column of the middle matrix \( M(\theta, \varphi) \) and the first and second columns of the left matrix \( L_2 \) in Eq. (14) can change the reference state, giving

\[ |n_3\rangle \equiv g|\phi_0\rangle = L_2 \begin{pmatrix} e^{i\varphi} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix} \]

\[ = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} e^{i\varphi} \cos\theta \\ 0 \\ 0 \end{pmatrix} + \sin\theta \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\varphi_1} \cos\xi_1 & e^{i\varphi_2} \sin\xi_1 \\ 0 & e^{i\varphi_2} \sin\xi_1 & e^{i\varphi_1} \cos\xi_1 \end{pmatrix} \begin{pmatrix} 0 \\ \sin\theta \\ 0 \end{pmatrix} \]

\[ = e^{i\varphi} \cos\theta \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \sin\theta |n_2\rangle. \]

\[ \text{(10)} \]
The state $|n_3\rangle$ is the $SU(2)$ coherent state given in (3), hence the coset space is isomorphic to a 5–sphere which has unit normal

$$n_3 = (e^{i\varphi_1} \cos \theta, e^{i\varphi_2} \sin \theta \cos \xi_1, e^{i\varphi_2} \sin \theta \sin \xi_1),$$  \hspace{1cm} (11)

metric

$$|ds_3|^2 = d\theta^2 + \cos^2 \theta d\varphi_1^2 + \sin^2 \theta (d\xi_1^2 + \cos^2 \xi_1 d\varphi_2^2 + \sin^2 \xi_1 d\varphi_2^2),$$  \hspace{1cm} (12)

and measure

$$d\mu_3 = \cos \theta \sin^3 \theta \cos \xi_1 \sin \xi_1 d\theta d\xi_1 d\varphi_1 d\varphi_2.$$  \hspace{1cm} (13)

We note here that an arbitrary phase in these coherent states for $SU(3)$ can be easily removed as discussed in Subsection (II A), and this process is also applicable to general $SU(n)$ cases. $SU(3)$ may also be decomposed using $\lambda$–matrices (4), which yields a slightly different parameterisation from (3). The $SU(2)$ $\lambda$–matrix decomposition for each matrix in (9) gives the $\sigma$–matrices [16], which yields a slightly different parameterisation from (9).

Next we obtain the $SU(3)$ coherent states for arbitrary $n$. For convenience, we define a matrix $L_2$ as $(\sqrt{3} \lambda_8 - \lambda_3)/2$, which describes the $SU(2)$ diagonal generator $\sigma_3$ in the bottom right corner of the $SU(3)$ matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & \sigma_3 \end{pmatrix}.$$  \hspace{1cm} (14)

The left matrix of (9) may be expressed by $\lambda_7, \lambda_8$, and $\lambda_3$. For convenience, we define a matrix $L_2$ as $(\sqrt{3} \lambda_8 - \lambda_3)/2$, which describes the $SU(2)$ diagonal generator $\sigma_3$ in the bottom right corner of the $SU(3)$ matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & \sigma_3 \end{pmatrix}.$$  \hspace{1cm} (15)

where $\varphi_1 = \alpha + \gamma, \varphi_2 = -\alpha + \gamma,$ and $\xi_1 = -\beta$. These two expressions (14) and (15) give

$$L_2 M(\theta, \varphi) = e^{i\alpha \lambda_8} e^{i\beta \lambda_7} e^{i\gamma \lambda_8} e^{i\varphi/2\lambda_3} e^{-i\theta \lambda_2} e^{i\varphi/2\lambda_3}$$

$$= e^{i\sqrt{3} \alpha/2 \lambda_8} e^{-i\alpha/2 \lambda_3} e^{i\beta \lambda_7} e^{i\gamma \lambda_8} e^{i\varphi/2\lambda_3} e^{-i\theta \lambda_2} e^{i\varphi/2\lambda_3} e^{-i\beta \lambda_7} e^{i\varphi/2\lambda_3}.$$  \hspace{1cm} (16)

The coherent states $|n_3\rangle$ in this representation are thus

$$|n_3\rangle = e^{i\alpha \lambda_8} e^{i\beta \lambda_7} e^{i\gamma \lambda_8} e^{i\varphi/2\lambda_3} e^{-i\theta \lambda_2} e^{i\varphi/2\lambda_3} \langle 0 \rangle$$

$$= e^{i\sqrt{3} \alpha/2 \lambda_8} e^{-i\alpha/2 \lambda_3} e^{i\beta \lambda_7} e^{i\gamma \lambda_8} e^{i\varphi/2\lambda_3} e^{-i\theta \lambda_2} e^{i\varphi/2\lambda_3} e^{-i\beta \lambda_7} e^{i\varphi/2\lambda_3} \langle 0 \rangle.$$  \hspace{1cm} (17)

2. $SU(4)$ and $SU(n)$ for arbitrary $n$

Next we obtain the $SU(4)$ coherent states by applying the above procedure to the $SU(3)$ coherent states. This process shows the iterative structure of the $SU(n)$ coherent states, which allows us to define generalised coherent states for arbitrary $n$. An arbitrary element $g \in SU(4)$ can be factored by (3) as

$$g = L_3 M(\theta, \varphi) R_3$$

$$= \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} e^{i\varphi} \cos \theta & -\sin \theta \\ \sin \theta & e^{-i\varphi} \cos \theta \end{pmatrix} \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & I_2 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & X_2 \\ 0 & 0 & X_3 \end{pmatrix} \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & e^{i\varphi_1} \cos \xi_1 & \cdots & 0 \\ 0 & \sin \xi_1 & e^{-i\varphi_1} \cos \xi_1 & \cdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & Y_2 \\ 0 & 0 & Y_3 \end{pmatrix} M(\theta, \varphi) R_3,$$  \hspace{1cm} (18)
where (8) has been iteratively applied twice. Here $X_3$, $Y_3$ are $SU(3)$ matrices and $X_2$ may be parameterised as

$$X_2 = \begin{pmatrix} e^{i\varphi_2} \cos \xi_2 & -e^{-i\varphi_2} \sin \xi_2 \\ e^{i\varphi_2} \sin \xi_2 & e^{-i\varphi_2} \cos \xi_2 \end{pmatrix}. \quad (20)$$

Taking the highest weight state $|\phi_0\rangle = (1, 0, 0)^T$ and evaluating $g|\phi_0\rangle$ as before, we observe that only two columns, the first column of $X_3$ and the first column of the matrix $M(\theta, \varphi)$, are important. The $SU(4)$ coherent states are

$$|n_4\rangle = g|\phi_0\rangle = e^{i\varphi} \cos \theta \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \sin \theta \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} (X_2) + \sin \theta \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} |n_3\rangle. \quad (21)$$

We note that the matrix including $Y_2$ in (13) commutes with the matrix to its right, that is $[I_2 \otimes Y_2, M(\theta, \varphi)] = 0$, and does not change the state $|\phi_0\rangle$. The expression of the metric for this coset space is

$$|ds_4|^2 = d\theta^2 + \cos^2(\theta) d\varphi^2 + \sin^2(\theta) \{d\xi_2^2 + \cos^2(\xi_1) d\varphi_1^2 + \sin^2(\xi_1) (d\xi_2^2 + \cos^2(\xi_2) d\varphi_2^2 + \sin(\xi_2) d\varphi_3)\}, \quad (22)$$

and the measure is

$$d\mu_4 = \cos(\theta) \sin^3(\theta) \cos(\xi_1) \sin^3(\xi_2) \cos(\xi_2) \sin(\xi_2) \ d\theta \ d\xi_1 d\xi_2 \ d\varphi_1 d\varphi_2 d\varphi_3. \quad (23)$$

We note that the total volume is $(2\pi)^4/(6 \cdot 4 \cdot 2)$.

This establishes that the $SU(n)$ coherent states $|n_n\rangle$ in this representation may be obtained from the iterative relation

$$|n_n\rangle = g|\phi_0\rangle = e^{i\varphi} \cos \theta \begin{pmatrix} 1 \\ \vdots \\ 0 \end{pmatrix} + \sin \theta \begin{pmatrix} 1 \\ \vdots \\ 0 \end{pmatrix} (X_{n-1}) + \sin \theta \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} |n_{n-1}\rangle, \quad (24)$$

where $X_{n-1}$ are $SU(n-1)$ matrices, and $|n_{n-1}\rangle$ is an $SU(n-1)$ coherent state. Since $|n_n\rangle$ is the unit vector of the $(2n-1)$–sphere, the measure on the hypersphere is

$$d\mu_n = \cos \theta \sin^{2n-3}(\theta) \cos(\xi_1) \sin^{2n-5}(\xi_1) \cdots \cos(\xi_{n-2}) \sin(\xi_{n-2}) \ d\theta \ d\xi_1 \cdots d\xi_{n-2} \ d\varphi_1 \cdots d\varphi_{n-1}. \quad (25)$$

### III. ARBITRARY $SU(n)$ REPRESENTATIONS

We extend the results of the previous section to irreducible unitary representations of arbitrarily large dimension for $SU(n)$. We define infinitesimal operators and the basis of the group representation. Using the decomposition of $SU(n)$, we derive an iterative equation for the $SU(n)$ coherent states and further obtain its recurrence equation. For the purpose of applications, some properties of the $SU(n)$ coherent states are given.
A. Infinitesimal operators

We denote $T_N^n$ as a representation of $SU(n)$ where the size number $N$ determines the dimension of the representation. A set of simultaneous normalised eigenstates of the Cartan operators $J_h^h (1 < h \leq n-1)$ is employed as an appropriate basis to describe the set of coherent states. This basis will be denoted as $|m_1, \ldots , m_n\rangle$ where the $m_j$ satisfy $N = \sum_{j=1}^n m_j$. The basis elements are also simultaneous eigenstates of the size operator $\hat{N}$ such that $\hat{N}|m_1, \ldots , m_n\rangle = N|m_1, \ldots , m_n\rangle$. For $SU(2)$, these are equivalent to the angular momentum eigenstates. The $n^2 - n$ raising operators $J_h^h, 1 \leq h < j \leq n$, and the same number of lowering operators $J_j^h, 1 \leq j < h \leq n$, of $SU(n)$ satisfy the relations

$$J_h^h|m_1, \ldots , m_h, \ldots , m_j, \ldots , m_n\rangle = \sqrt{(m_h+1)m_j} |m_1, \ldots , m_h+1, \ldots , m_j-1, \ldots , m_n\rangle,$$

[raising operators, $(h < j)$]

$$J_j^h|m_1, \ldots , m_h, \ldots , m_j, \ldots , m_n\rangle = \sqrt{m_h(m_j+1)} |m_1, \ldots , m_h-1, \ldots , m_j+1, \ldots , m_n\rangle.$$

[lowering operators, $(h > j)$]

For Cartan operators $J_h^h, 1 \leq h \leq n-1$ we have

$$J_h^h|m_1, \ldots , m_h, \ldots , m_j, \ldots , m_n\rangle = \sqrt{\frac{2}{h(h+1)}} \left( \sum_{k=1}^n m_k - h m_{h+1} \right) |m_1, \ldots , m_h, \ldots , m_j, \ldots , m_n\rangle.$$

B. $SU(n)$ coherent states

It is appropriate to choose $|\phi_0\rangle = |N, 0, \ldots , 0\rangle$ as the highest weight state, since the action of any raising operator on this state gives zero. The parameterization shows that the representation $T_N^n(g)$ may be decomposed as

$$T_N^n(g) = T_N^n(L_{n-1})T_N^n(M)T_N^n(R_{n-1}).$$

The action of the representation $T_N^n(R_{n-1})$ does not change the highest weight state as we have seen in the examples in the previous section, hence the coherent state is determined as

$$|n_N^n\rangle = T_N^n(L_{n-1})T_N^n(M(\theta, \varphi))|N, 0, \ldots , 0\rangle.$$ (30)

The right element $T_N^n(M(\theta, \varphi))$ in (30) acts as an $SU(2)$ operator on the subspace $|m_1, m_2\rangle$, a cross section obtained by taking the first two elements of $|m_1, m_2, \ldots , m_n\rangle$. It is well-known that an arbitrary $g \in SU(2)$ may be decomposed as

$$g = e^{-\zeta^* J_2^*} e^{-\nu J_1^*} e^{\zeta J_2} e^{\varphi J_1} = e^{\zeta J_2} e^{\varphi J_1} e^{-\zeta^* J_2^*} e^{-\nu J_1^*}.$$ (31)

The parameters in the above expressions correspond to the angle parameters in (1) as $\zeta = e^{i(\varphi_2 - \varphi_1)} \tan \theta$, and $\nu = \ln \cos \theta$. Setting $\varphi_1 = \varphi$ and $\varphi_2 = 0$, $T_N^n(M(\theta, \varphi))$ is decomposed as (31), and acts on the subspace $|N, 0, \ldots , 0\rangle$ as

$$T_N^n(M(\theta, \varphi))|N, 0, \ldots , 0\rangle = \sum_{j=0}^N e^{i\varphi(N-j)} \sin^j(\theta) \cos^{N-j}(\theta) \left( \begin{array}{c} N \\ j \end{array} \right) 1/2 |N - j, j, 0, \ldots , 0\rangle.$$ (32)

The left element of the decomposition, $T_N^n(L_{n-1})$, does not change the first element of the state $|N - j, j, 0, \ldots , 0\rangle$, and acts on the subspace $|j, 0, \ldots , 0\rangle$ in the state $|N - j\rangle \otimes |j, 0, \ldots , 0\rangle$. This element acts as an $SU(n-1)$ operator on the subspace $|j, 0, \ldots , 0\rangle$, which generates the $SU(n-1)$ coherent states, giving

$$|n_N^n\rangle = \sum_{j=0}^N e^{i\varphi (N-j)} \sin^j(\theta) \cos^{N-j}(\theta) \left( \begin{array}{c} N \\ j \end{array} \right) 1/2 |N - j\rangle \otimes |n_{n-1}^j\rangle.$$ (33)
In order to obtain a more convenient expression of the SU(n) coherent states, we derive a recurrence relation from the above iterative equation (33). The last decomposition in (31) gives the SU(2) coherent states

\[ |n_2^N \rangle = \sum_{j=0}^{N} e^{ij\varphi_2} e^{i(N-j)\varphi_1} \sin^j(\theta) \cos^{N-j}(\theta) \left( \begin{array}{c} N \\ j \end{array} \right)^{1/2} |N-j,j\rangle \]

\[ = \sum_{j=0}^{N} \eta_j^N(\varphi_1, \varphi_2, \theta) |N-j,j\rangle, \quad (34) \]

where we define

\[ \eta_j^N(\varphi_1, \varphi_2, \theta) \equiv e^{i\varphi_2} e^{i(N-j)\varphi_1} \sin^j(\theta) \cos^{(N-j)}(\theta) \left( \begin{array}{c} N \\ j \end{array} \right)^{1/2}. \quad (35) \]

The SU(3) coherent states are constructed using the SU(2) coherent states, and the relations (33) and (35) give

\[ |n_3^N \rangle = \sum_{j_1=0}^{N} e^{i\varphi(N-j_1)\sin^{j_1}\cos^{(N-j_1)}(\theta)} \left( \begin{array}{c} N \\ j_1 \end{array} \right)^{1/2} |N, j_1\rangle \otimes |n_2^{j_1}\rangle \]

\[ = \sum_{j_1=0}^{N} \eta_{j_1}^N(\varphi, 0, \theta) \sum_{j_2=0}^{j_1} \eta_{j_2}^{j_1}(\varphi_1, \varphi_2, \xi_1) |N-j_1, j_1-j_2, j_2\rangle, \quad (36) \]

in agreement with the SU(3) coherent states developed in [12].

Recursive, the SU(n) coherent states may be expressed by the function \( \eta_n^N(\alpha, \beta, \gamma) \) of (33),

\[ |n_n^N \rangle = \sum_{j_1=0}^{N} \eta_{j_1}^N(\varphi, 0, \theta) \sum_{j_2=0}^{j_1} \eta_{j_2}^{j_1}(\varphi_1, 0, \xi_1) \cdots \sum_{j_{n-3}=0}^{j_{n-2}} \eta_{j_{n-3}}^{j_{n-2}}(\varphi_{n-3}, 0, \xi_{n-3}) \]

\[ \sum_{j_{n-2}=0}^{j_{n-3}} \eta_{j_{n-2}}^{j_{n-3}}(\varphi_{n-2}, \varphi_{n-1}, \xi_{n-2}) |N-j_1, j_1-j_2, \cdots, j_{n-1}\rangle. \quad (37) \]

**C. Properties of the SU(n) coherent states**

For the purpose of applications, here we describe some fundamental properties of the SU(n) coherent states.

(1) Stereographic coordinates

The decomposition (31) implies that the SU(n) coherent states may be represented in the complex numbers \( \zeta_k \) such that \( \zeta_k = e^{i(\varphi_k+1-\varphi_k)} \tan(\xi_k) \). Routine change of variables gives the SU(n) coherent states in this stereographic coordinates as

\[ |n_n^N \rangle = e^{i\varphi N} \left( \frac{1}{1+|\xi|^2} \right)^N \sum_{j_1=0}^{N} (\zeta_1)^{j_1} \left( \begin{array}{c} N \\ j_1 \end{array} \right)^{1/2} \left( \frac{1}{1+|\zeta|^2} \right)^{j_1} \]

\[ \cdots \]

\[ \sum_{j_{n-2}=0}^{j_{n-3}} (\zeta_{n-2})^{j_{n-2}} \left( \begin{array}{c} j_{n-2} \\ j_{n-1} \end{array} \right)^{1/2} \left( \frac{1}{1+|\zeta|^2} \right)^{j_{n-1}} |N-j_1, \cdots, j_{n-2}-j_{n-1}, j_{n-1}\rangle. \quad (38) \]

(2) Resolution of unity

The set of coherent states provides a resolution of unity in the coset space as

\[ \frac{(N+n-1)!}{2\pi^n N!} \int d\mu_n |n_n^N \rangle \langle n_n^N | = \hat{1}. \quad (39) \]
The diagonal elements of the commutation relations of these non-diagonal elements are given by change of integral variables \( x = \cos^2(\xi_k) \) allowing
\[
\int_{0}^{\pi/2} d\xi_k \cos^{2(m-n)+1}(\xi_k) \sin^{2(n+1)}(\xi_k) = \frac{\eta^m(m-n)!}{(m+1)!},
\]
while the integrals in terms of \( \varphi_j \) produce delta functions. The result of the all integrals cancels with the normalization factor, and gives us
\[
\sum_{j_1=0}^{N} \cdots \sum_{j_n-1=0}^{j_n-1} |N-j_1, \ldots, j_n-1\rangle \langle N-j_1, \ldots, j_n-1| = I.
\]

(3) Overlap of two coherent states
The overlap of two coherent states may be calculated from (33), as
\[
\langle n_n^{N} | n_n^{N} \rangle = \left( e^{i(\varphi_{n-1}-\varphi'_{n-1})} \prod_{k=0}^{n-2} \sin \xi_k \sin \xi'_k + \sum_{m=0}^{n-2} \left[ e^{i(\varphi_m-\varphi'_m)} \cos \xi_m \cos \xi'_m \prod_{k=0}^{m-1} \sin \xi_k \sin \xi'_k \right] \right)^N,
\]
where we have changed the notation of angles, replacing \( \theta \) with \( \xi_0 \), and \( \varphi \) with \( \varphi_0 \), and where we have defined \( \prod_{k=0}^{n-2} \sin \xi_k \sin \xi'_k = 1 \).

IV. SUMMARY

In conclusion, we have described \( SU(n) \) coherent states for irreducible unitary representations for arbitrarily large dimension and some examples for small \( n \) demonstrated. The geometric structure of the \( SU(n) \) coherent states has been represented using spherical coordinates. We also gave expressions for the resolution of unity, and the non-orthogonality of the coherent states. It was shown the \( SU(n) \) coherent states may be recursively derived from \( SU(2) \) coherent states.

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APPENDIX A: \( \lambda \)-MATRICES

In general \( SU(n) \) generators can be represented by \( n^2 - n \) off–diagonal matrices and \( n - 1 \) diagonal matrices. For example, \( SU(4) \) has fifteen generators which can be constructed using twelve off–diagonal matrices and three diagonal matrices [20]. We take \( \{ e_j^h \} \) as a basis for the group \( SU(n) \), where \( e_j^h \) are elementary matrices. We also define \( e_j^h \) (\( h < j \)) as raising operators, and \( e_j^h \) (\( h > j \)) as lowering operators respectively. Non-diagonal elements of this basis are
\[
\{ \beta_j^h = -i(e_j^h - e_j^h), \Theta_j^k = e_j^h + e_j^h, 1 \leq h < j \leq n \}.
\]
The commutation relations of these non–diagonal elements are
\[
[\beta_j^h, \Theta_k^e] = -i\delta_j^k \Theta_j^h + i\delta_j^h \Theta_j^k + i\delta_j^k \Theta_j^e - i\delta_j^e \Theta_j^k.
\]
The diagonal elements \( \{ n_m^m \} \) are
\[
n_m^m = \sqrt{\frac{2}{m(m+1)} \left( \sum_{j=1}^{m} e_j^m - m \, e_m^{m+1} \right)}.
\]
For instance in \( SU(4) \) the fifteen \( \lambda \)-matrices are numbered as
These $\lambda$-matrices are the generators of the representation $T_1$.

These $SU(4)$ generators allow another expression for the coherent states. Defining a matrix $\lambda_1'$ as $(\sqrt{6}\lambda_1 - \sqrt{3}\lambda_3)/3$, the decomposition of $SU(4)$ using $\lambda$-matrices gives expressions for the coherent states

$$ |n_4\rangle = e^{i\alpha\lambda_1' e^{i\beta}} |\lambda_1' e^{i\beta} e^{i\gamma} e^{i\delta} e^{i\varepsilon} 2\lambda_3 e^{i\theta} e^{i\varphi}/2\lambda_1 e^{i\theta} e^{i\varphi}/2\lambda_3 |\phi_0\rangle $$

where $\varphi_2 = 2\alpha + 2\gamma$, $\varphi_3 = -\alpha + \gamma$, and $\xi_2 = -\beta$. These expressions have been obtained directly from (16) and (19), and show the displacement operator for the $SU(4)$ coherent states.

APPENDIX B: THE SYMMETRIC PARAMETERIZATION FOR $SU(n)$

We here show a brief proof of the parameterization [8]. A proof of this parameterization for $n = 3$ was given in [17]. Showing any element of $g \in SU(n)$ can be transformed into $R_{n-1}$, we give the parameterization [8] as an inverse equation in terms of the element $g$. We first review the proof for $n = 3$, and prove [8] for arbitrary $n$ inductively.

(i) For $n=3$, an arbitrary element $g = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix}$ can be transformed as

$$ \begin{pmatrix} 1 & 0 & 0 \\ 0 & a & b \\ 0 & -b & a \end{pmatrix} \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix} = \begin{pmatrix} x_{11} \\ r_2^3 \sum_{k=2}^{3} x_{k1} x_{k2} \\ 0 \end{pmatrix} = \begin{pmatrix} x_{13} \\ r_2^3 \sum_{k=2}^{3} x_{k1} x_{k3} \\ 0 \end{pmatrix}, $$

where $a = x_{21}/r_2^3$, $b = x_{31}/r_2^3$ and $r_2^3 = \sqrt{\sum_{k=2}^{3} |x_{k1}|^2}$. Applying a matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix}$ from the left on the above matrix [B1] gives $\begin{pmatrix} 1 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix}$, where we used the constraints on $g$, which are $\sum_{j=1}^{3} x_{jk} x_{jl} = \delta_{kl}$. With suitably chosen parameters, the inversion of the above relation gives the devised $SU(3)$ parameterization [8].

Now we extend this procedure to the general result, and prove it inductively.

(ii) We assume the result in the case (i), that is, for $n = m$ an arbitrary element $g \in SU(m)$ can be parametrized as [8], and for any $g$ a matrix $X_{m-1} \in SU(m-1)$ exists such that

$$ \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & X_{m-1} \end{pmatrix} \begin{pmatrix} x_{11} & x_{12} \cdots & x_{1m} \\ x_{21} & x_{22} \cdots & x_{2m} \\ \vdots & \vdots & \vdots \\ x_{m1} & x_{m2} \cdots & x_{mm} \end{pmatrix} = \begin{pmatrix} x_{11} \\ r_2^m \sum_{k=2}^{m} x_{k1} x_{k2} \\ \vdots \\ 0 \end{pmatrix}. $$

(iii) For $n=m+1$, using [B2], an arbitrary matrix $g \in SU(m+1)$ can be transformed as
where \( I_k \) are \( k \times k \) identity matrices. Using the constraints for \( SU(m+1) \) matrices, \( \sum_{j=1}^{m+1} x^*_j x_{jl} = \delta_{kl} \), the matrix on the right hand side can be transformed to contain an \( SU(m) \) matrix as

\[
\begin{pmatrix}
  x_{11} & x_{12} & \cdots & x_{1,m+1} \\
  x_{21} & x_{22} & \cdots & x_{2,m+1} \\
  \vdots & \vdots & \ddots & \vdots \\
  x_{m+1} & x_{m+2} & \cdots & x_{m+1,m+1}
\end{pmatrix}
\times
\begin{pmatrix}
  \frac{x_{11}^*}{r_1} & \cdots & \frac{x_{1,m+1}^*}{r_1} \\
  \cdots & \ddots & \cdots \\
  \frac{x_{m+1}^*}{r_1} & \cdots & \frac{x_{m+1,m+1}^*}{r_1}
\end{pmatrix}
\]

\[= \begin{pmatrix}
  1 & 0 & \cdots & 0 \\
  0 & I_{m-1} \\
  \vdots & \ddots & \vdots & \vdots \\
  0 & \cdots & 1 & 0
\end{pmatrix}
\]

Since the second matrix can be parametrized equivalently to the \( n = m \) case, the product of the three matrices constructs an \( SU(n) \) matrix. The inversion of this relation gives

\[
\begin{pmatrix}
  x_{11} & x_{12} & \cdots & x_{1,m+1} \\
  x_{21} & x_{22} & \cdots & x_{2,m+1} \\
  \vdots & \vdots & \ddots & \vdots \\
  x_{m+1} & x_{m+2} & \cdots & x_{m+1,m+1}
\end{pmatrix}
\times
\begin{pmatrix}
  \frac{x_{11}}{r_2} & \cdots & \frac{x_{1,m+1}}{r_2} \\
  \cdots & \ddots & \cdots \\
  \frac{x_{m+1}}{r_2} & \cdots & \frac{x_{m+1,m+1}}{r_2}
\end{pmatrix}
\]

\[= \begin{pmatrix}
  1 & 0 & \cdots & 0 \\
  0 & I_{m-1} \\
  \vdots & \ddots & \vdots & \vdots \\
  0 & \cdots & 1 & 0
\end{pmatrix}
\]

The elements, \( x_{11} \) and \( \sqrt{1 - |x_{11}|^2} \), can be parametrized as

\[x_{11} = e^{i \varphi} \cos \theta, \quad \sqrt{1 - |x_{11}|^2} = \sin \theta. \quad \text{(B6)}\]
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