DECAY ESTIMATES WITH SHARP RATES OF GLOBAL SOLUTIONS OF NONLINEAR SYSTEMS OF FLUID DYNAMICS EQUATIONS

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Dedicated to Professor Boling Guo on the occasion of his 80th birthday!

Abstract. Consider the Cauchy problems for the \( n \)-dimensional incompressible Navier-Stokes equations

\[
\frac{\partial u}{\partial t} - \alpha \Delta u + (u \cdot \nabla)u + \nabla p = f(x, t), \quad u(x, 0) = u_0(x).
\]

In this system, the dimension \( n \geq 3 \), \( u(x, t) = (u_1(x, t), u_2(x, t), \cdots, u_n(x, t)) \) and \( f(x, t) = (f_1(x, t), f_2(x, t), \cdots, f_n(x, t)) \) are real vector valued functions of \( x = (x_1, x_2, \cdots, x_n) \) and \( t \). Additionally, \( \alpha > 0 \) is a positive constant. Suppose that the initial function and the external force satisfy appropriate conditions.

The main purpose of this paper is to make complete use of the uniform energy estimates of the global smooth solutions and couple together a well known Gronwall’s inequality to improve the Fourier splitting method to accomplish the decay estimates with sharp rates. The decay estimates with sharp rates of the global smooth solutions of the Cauchy problems for the \( n \)-dimensional magnetohydrodynamics equations may be established very similarly.

1. Introduction.

1.1. The mathematical model equations. Consider the Cauchy problems for the \( n \)-dimensional incompressible Navier-Stokes equations

\[
\frac{\partial u}{\partial t} - \alpha \Delta u + (u \cdot \nabla)u + \nabla p = f(x, t), \quad \nabla \cdot u = 0, \quad \nabla \cdot f = 0,
\]

\[
u(x, 0) = u_0(x), \quad \nabla \cdot u_0 = 0.
\]

In this system, \( x = (x_1, x_2, \cdots, x_n) \) represents the spatial variable, the dimension \( n \geq 3 \). In this system, \( u(x, t) = (u_1(x, t), u_2(x, t), \cdots, u_n(x, t)) \) represents the velocity of the fluid at position \( x \) and time \( t \), the real scalar function \( p = p(x, t) \) represents the pressure of the fluid at \( x \) and \( t \), \( f(x, t) = (f_1(x, t), f_2(x, t), \cdots, f_n(x, t)) \) represents the external force, \( \alpha > 0 \) is a positive constant.

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Also consider the Cauchy problems for the $n$-dimensional magnetohydrodynamics equations

\[ \frac{∂u}{∂t} - \frac{1}{RE} \triangle u + (u \cdot ∇)u - (A \cdot ∇)A + ∇P = f(x, t), \quad (3) \]
\[ \frac{∂A}{∂t} - \frac{1}{RM} \triangle A + (u \cdot ∇)A - (A \cdot ∇)u = g(x, t), \quad (4) \]
\[ \nabla \cdot u = 0, \quad \nabla \cdot f = 0, \quad \nabla \cdot A = 0, \quad \nabla \cdot g = 0, \]
\[ u(x, 0) = u_0(x), A(x, 0) = A_0(x), \nabla \cdot u_0 = 0, \nabla \cdot A_0 = 0. \quad (6) \]

In this system, the real vector valued function $u = u(x, t)$ represents the velocity of the fluid at position $x$ and time $t$, the real vector valued function $A = A(x, t)$ represents the magnetic field at position $x$ and time $t$. The real scalar function $P(x, t) = p(x, t) + \frac{M^2}{\text{RE} \cdot \text{RM}} |A(x, t)|^2$ represents the total pressure, where the real scalar function $p = p(x, t)$ represents the pressure of the fluid and $\frac{1}{2} |A(x, t)|^2$ represents the magnetic pressure. Additionally, $M > 0$ represents the Hartman constant, $\text{RE}$ represents the Reynolds constant and $\text{RM}$ represents the magnetic Reynolds constant.

Many mathematicians have accomplished the existence of the global weak solutions of the Cauchy problems for the $n$-dimensional incompressible Navier-Stokes equations (1)-(2) and for the $n$-dimensional magnetohydrodynamics equations (3)-(6). The existence of the global smooth solutions with small initial functions of these equations have also been established. See Lin [7], Temam [11], [12], Tian and Xin [13]. However, the uniform energy estimates of all order derivatives of the global weak solutions have been open. Very recently, many mathematicians have obtained very exciting results for special large global smooth solutions of the Cauchy problems for the $n$-dimensional incompressible Navier-Stokes equations, see Chemin [1]-[2], Hou, Lei and Li [4], Lei and Lin [5], Lei, Lin and Zhou [6], Peng and Zhang [8].

1.2. The main purposes. The main purposes of this paper are to couple together uniform energy estimates, the Fourier transformation, the Plancherel’s identity and the Gronwall’s inequality to simplify the Fourier splitting method to accomplish the $L^2$ decay estimates with sharp rates for the global smooth solutions of the Cauchy problems for the $n$-dimensional incompressible Navier-Stokes equations (1)-(2).

The Fourier splitting method was developed by Maria Schonbek to accomplish the decay estimates with sharp rates for the global weak solutions of many nonlinear evolution equations with dissipations. The Fourier splitting method involves the splitting of the frequency space into two time-dependent subspaces (a small ball $Ω(t)$ with radius $r(t) = \sqrt{2n/\left[2(1 + t)\right]}$ and the complementary $Ω(t)^c$ of the small ball) and the estimate of the Fourier transformation $\hat{u}(ξ, t)$ of the global weak solution, see [9]-[10]. See also Guo and Zhang [3], Zhang [14], [15] and [16]. To obtain the optimal decay rate, one must iterate the above process for finitely many times. We will make complete use of the uniform energy estimates and the Gronwall’s inequality to avoid the iteration process for $n$-dimensional equations, where $n \geq 3$. The key point of the improvement is that for many nonlinear evolution equations with dissipations, such as the $n$-dimensional magnetohydrodynamics equations, we may apply the simplified version of the Fourier splitting method to accomplish the decay estimates with sharp rates for the global smooth solutions.
1.3. The main results - decay estimates with sharp rates. Here are the main assumptions on the initial function and the external force needed for the main results. For the decay estimate of the global weak solutions, let us make the following assumptions. Suppose that the initial function \( u_0 \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \) and the external force \( f \in L^1(\mathbb{R}^n \times \mathbb{R}^n) \cap L^1(\mathbb{R}, L^2(\mathbb{R}^n)) \). Suppose that the external force satisfies the condition

\[
\int_0^\infty (1 + t)^{2 + n/2} \int_{\mathbb{R}^n} |f(x, t)|^2 dx dt < \infty.
\]

Suppose that there exist real scalar functions \( \phi_{kl} \in C^1(\mathbb{R}^n) \cap L^1(\mathbb{R}^n) \) and \( \psi_{kl} \in C^1(\mathbb{R}^n \times \mathbb{R}^+) \cap L^1(\mathbb{R}^n \times \mathbb{R}^+) \), such that

\[
\begin{align*}
    u_0(x) &= \left( \sum_{l=1}^n \frac{\partial \phi_{1l}}{\partial x_l}(x), \sum_{l=1}^n \frac{\partial \phi_{2l}}{\partial x_l}(x), \ldots, \sum_{l=1}^n \frac{\partial \phi_{nl}}{\partial x_l}(x) \right), \\
    f(x, t) &= \left( \sum_{l=1}^n \frac{\partial \psi_{1l}}{\partial x_l}(x, t), \sum_{l=1}^n \frac{\partial \psi_{2l}}{\partial x_l}(x, t), \ldots, \sum_{l=1}^n \frac{\partial \psi_{nl}}{\partial x_l}(x, t) \right),
\end{align*}
\]

and

\[
\frac{\partial \phi_{kl}}{\partial x_l} \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n), \quad \frac{\partial \psi_{kl}}{\partial x_l} \in L^1(\mathbb{R}^n \times \mathbb{R}^+) \cap L^2(\mathbb{R}^n \times \mathbb{R}^+),
\]

for all \( k = 1, 2, \ldots, n \) and \( l = 1, 2, \ldots, n \).

There exists a global weak solution to (1)-(2):

\[
u \in L^\infty(\mathbb{R}^+, L^2(\mathbb{R}^n)), \quad \nabla u \in L^2(\mathbb{R}^+, L^2(\mathbb{R}^n)).
\]

For the decay estimate of the global smooth solutions of the Cauchy problems for the \( n \)-dimensional incompressible Navier-Stokes equations, let us make the following additional assumptions.

Suppose that the initial function \( u_0 \in L^1(\mathbb{R}^n) \cap H^{2m+1}(\mathbb{R}^n) \) and the external force \( f \in L^1(\mathbb{R}^n \times \mathbb{R}^+) \cap L^1(\mathbb{R}^n \times \mathbb{R}^+) \cap L^2(\mathbb{R}^n \times \mathbb{R}^+, H^{2m}(\mathbb{R}^n)) \). Suppose that there exists a global smooth solution \( u \in L^\infty(\mathbb{R}^+, H^{2m+1}(\mathbb{R}^n)) \), such that \( \nabla u \in L^2(\mathbb{R}^+, H^{2m+1}(\mathbb{R}^n)) \), where \( m \geq 1 \) is a positive integer.

This assumption is valid if the initial function and the external force are reasonably small or if there exist positive constants \( p > n \geq 3 \) and \( q > 2 \), with \( \frac{n}{p} + \frac{2}{q} = 1 \), such that

\[
\int_0^\infty \left( \int_{\mathbb{R}^n} |u(x, t)|^p dx \right)^{q/p} dt < \infty.
\]

Suppose that

\[
\int_0^\infty (1 + t)^{2m+2+n/2} \int_{\mathbb{R}^n} |\nabla^m f(x, t)|^2 dx dt < \infty.
\]

**Theorem 1.1.** (1) For the global weak solutions of the Cauchy problems for the \( n \)-dimensional incompressible Navier-Stokes equations (1)-(2), there holds the following decay estimate with sharp rate

\[
(1 + t)^{1+n/2} \int_{\mathbb{R}^n} |u(x, t)|^2 dx \leq C,
\]

for all time \( t > 0 \), where \( C > 0 \) is a positive constant, independent of \( u \) and \( (x, t) \).
For the global smooth solution of the Cauchy problems for the $n$-dimensional incompressible Navier-Stokes equations (1)-(2), there hold the following decay estimates with sharp rates

$$(1 + t)^{1+n/2} \int_{\mathbb{R}^n} |u(x, t)|^2 dx \leq C,$$

$$(1 + t)^{2+n/2} \int_{\mathbb{R}^n} |\nabla u(x, t)|^2 dx \leq C,$$

$$(1 + t)^{2m+1+n/2} \int_{\mathbb{R}^n} |\Delta^m u(x, t)|^2 dx \leq C,$$

$$(1 + t)^{2m+2+n/2} \int_{\mathbb{R}^n} |\nabla \Delta^m u(x, t)|^2 dx \leq C,$$

and

$$(1 + t)^{1/2+n/2} \|u(\cdot, t)\|_{L^\infty} \leq C,$$

$$(1 + t)^{1+n/2} \|\nabla u(\cdot, t)\|_{L^\infty} \leq C,$$

$$(1 + t)^{m+1+n/2} \|\Delta^m u(\cdot, t)\|_{L^\infty} \leq C,$$

$$(1 + t)^{m+2+n/2} \|\nabla \Delta^m u(\cdot, t)\|_{L^\infty} \leq C,$$

for all positive integers $m \geq 1$ and for all time $t > 0$, where $C > 0$ is a positive constant, independent of $u$ and $(x, t)$.

2. The mathematical analysis and the proofs of the main results. The improved Fourier splitting method is a very valuable tool to accomplish the decay estimates with sharp rates for the global smooth or weak solutions of many nonlinear differential equations with dissipations.

2.1. The uniform energy estimates. The main purpose is to use traditional ideas, methods and techniques to establish some uniform energy estimates.

**Lemma 2.1.** Suppose that the initial function $u_0 \in L^2(\mathbb{R}^n)$ and the external force $f \in L^1(\mathbb{R}^+, L^2(\mathbb{R}^n))$. Then there holds the following uniform energy estimate

$$\left[ \int_{\mathbb{R}^n} |u(x, t)|^2 dx + 2\alpha \int_0^t \int_{\mathbb{R}^n} |\nabla u(x, \tau)|^2 dxd\tau \right]^{1/2} \leq \left[ \int_{\mathbb{R}^n} |u_0(x)|^2 dx \right]^{1/2} + \int_{\mathbb{R}^n} \left[ \int_{\mathbb{R}^n} |f(x, t)|^2 dx \right]^{1/2} dt.$$

**Proof.** Multiplying system (1) by $2u$ and integrating the result with respect to $x$ over $\mathbb{R}^n$ yield the following energy equation

$$\frac{d}{dt} \int_{\mathbb{R}^n} |u(x, t)|^2 dx + 2\alpha \int_{\mathbb{R}^n} |\nabla u(x, t)|^2 dx = 2 \int_{\mathbb{R}^n} u(x, t) \cdot f(x, t) dx,$$

where

$$\int_{\mathbb{R}^n} u(x, t) \cdot [(u(x, t) \cdot \nabla)u(x, t)] dx = 0.$$

By using Cauchy-Schwartz’s inequality, there hold the following estimates

$$\left[ \int_{\mathbb{R}^n} u(x, t) \cdot f(x, t) dx \right] \leq \left[ \int_{\mathbb{R}^n} |u(x, t)|^2 dx \right]^{1/2} \left[ \int_{\mathbb{R}^n} |f(x, t)|^2 dx \right]^{1/2}.$$
Lemma 2.2. (I) There holds the following Fourier representation
\[ \hat{u}(\xi, t) = \exp(-\alpha|\xi|^2 t)\hat{u}_0(\xi) + \int_0^t \exp[-\alpha|\xi|^2 (t - \tau)]\hat{f}(\xi, \tau) d\tau - \int_0^t \exp[-\alpha|\xi|^2 (t - \tau)] \left[ (\hat{u} \cdot \nabla)\hat{u}(\xi, \tau) + \hat{v}_p(\xi, \tau) \right] d\tau, \]
for all \((\xi, t) \in \mathbb{R}^n \times \mathbb{R}^+\).

(II) There holds the following estimate
\[ |\hat{u}(\xi, t)| \leq |\hat{u}_0(\xi)| + \int_0^t |\hat{f}(\xi, \tau)| d\tau + C \left[ \int_0^t \int_{\mathbb{R}^n} |u(x, \tau)|^2 dx d\tau \right]^{1/2} \left[ \int_0^t \int_{\mathbb{R}^n} |\nabla u(x, \tau)|^2 dx d\tau \right]^{1/2}, \]
for all \((\xi, t) \in \mathbb{R}^n \times \mathbb{R}^+\), where \(C > 0\) is a positive constant, independent of \(\hat{u}(\xi, t)\) and \((\xi, t)\).

(III) There holds the following estimate
\[ |\hat{u}(\xi, t)| \leq C|\xi|, \]
for all \((\xi, t) \in \mathbb{R}^n \times \mathbb{R}^+\), where \(C > 0\) is a positive constant, independent of \(\hat{u}(\xi, t)\) and \((\xi, t)\), if
\[ (1 + t)^{n/2} \int_{\mathbb{R}^n} |u(x, t)|^2 dx \leq C, \]
for all time \(t > 0\) and for another positive constant \(C > 0\), independent of \(u\) and \((x, t)\).
Proof. (I) Performing the Fourier transformation to (1) leads to
\[
\frac{d}{dt} \hat{u}(\xi, t) + \alpha |\xi|^2 \hat{u}(\xi, t) + (\hat{u} \cdot \nabla) \hat{u}(\xi, t) + \hat{\nabla} \hat{\rho}(\xi, t) = \hat{f}(\xi, t).
\]
Multiplying this equation by the integrating factor \(\exp(\alpha |\xi|^2 t)\) gives
\[
\frac{d}{dt}[\exp(\alpha |\xi|^2 t) \hat{u}(\xi, t)] + \exp(\alpha |\xi|^2 t) \left[(\hat{u} \cdot \nabla) \hat{u}(\xi, t) + \hat{\nabla} \hat{\rho}(\xi, t)\right] = \exp(\alpha |\xi|^2 t) \hat{f}(\xi, t).
\]
Integrating with respect to time \(t\) yields
\[
\exp(\alpha |\xi|^2 t) \hat{u}(\xi, t) + \int_0^t \exp(\alpha |\xi|^2 \tau) \left[(\hat{u} \cdot \nabla) \hat{u}(\xi, \tau) + \hat{\nabla} \hat{\rho}(\xi, \tau)\right] d\tau = \hat{u}_0(\xi) + \int_0^t \exp(\alpha |\xi|^2 \tau) \hat{f}(\xi, \tau) d\tau.
\]
Finally, we obtain the Fourier representation.

(II) Now let us make estimates about the Fourier transformation \(\hat{u}(\xi, t)\). First of all, there hold the following estimates
\[
\left| (\hat{u} \cdot \nabla) \hat{u}(\xi, t) \right| \leq |\xi| \int_{\mathbb{R}^n} |u(\mathbf{x}, t)|^2 d\mathbf{x},
\]
\[
\left| (\hat{u} \cdot \nabla) \hat{u}(\xi, t) \right| \leq \left\{ \int_{\mathbb{R}^n} |u(\mathbf{x}, t)|^2 d\mathbf{x} \right\}^{1/2} \left\{ \int_{\mathbb{R}^n} \left| \nabla u(\mathbf{x}, t) \right|^2 d\mathbf{x} \right\}^{1/2},
\]
for all \((\xi, t) \in \mathbb{R}^n \times \mathbb{R}^+\).

Secondly, taking the divergence of the Navier-Stokes equations yields
\[
\triangle \hat{p} + \sum_{i=1}^n \sum_{j=1}^n \frac{\partial^2}{\partial x_i \partial x_j} (u_i u_j) = 0.
\]
Performing Fourier transformation to this equation leads to
\[
|\xi|^2 \hat{\rho}(\xi, t) + \sum_{i=1}^n \sum_{j=1}^n \xi_i \xi_j \hat{u}_i \hat{u}_j(\xi, t) = 0.
\]
By using Cauchy-Schwartz’s inequality, we get the estimates
\[
|\xi|^4 |\hat{\rho}(\xi, t)|^2 = \left| \sum_{i=1}^n \sum_{j=1}^n \xi_i \xi_j \hat{u}_i \hat{u}_j(\xi, t) \right|^2 \\
\leq \sum_{i=1}^n \sum_{j=1}^n |\xi_i \xi_j|^2 \sum_{i=1}^n \sum_{j=1}^n |\hat{u}_i \hat{u}_j(\xi, t)|^2 \\
= \sum_{i=1}^n \sum_{j=1}^n |\hat{u}_i \hat{u}_j(\xi, t)|^2.
\]
For all \(\xi \neq 0\), we see that
\[
|\hat{\rho}(\xi, t)|^2 \leq \sum_{i=1}^n \sum_{j=1}^n |\hat{u}_i \hat{u}_j(\xi, t)|^2.
\]
Therefore, we obtain the estimates
\[
|\hat{\rho}(\xi, t)|^2 \leq \sum_{i=1}^n \sum_{j=1}^n |u_i u_j(\xi, t)|^2 \leq \sum_{i=1}^n \sum_{j=1}^n \left[ \int_{\mathbb{R}^n} |u_i(\mathbf{x}, t) u_j(\mathbf{x}, t)| d\mathbf{x} \right]^2.
\]
\[
\leq \left\{ \int_{\mathbb{R}^n} |u(x,t)|^2 \, dx \right\}^2,
\]
and
\[
|\nabla p(\xi,t)|^2 \leq \sum_{i=1}^{n} \sum_{j=1}^{n} |\nabla (u_i u_j)(\xi,t)|^2
\leq 4 \left\{ \int_{\mathbb{R}^n} |u(x,t)|^2 \, dx \right\} \left\{ \int_{\mathbb{R}^n} |\nabla u(x,t)|^2 \, dx \right\}.
\]

Therefore, from the representation for the Fourier transformation
\[
\hat{u}(\xi,t) = \exp(-\alpha|\xi|^2 t)\hat{u}_0(\xi) + \int_0^t \exp[-\alpha|\xi|^2 (t - \tau)]\hat{f}(\xi,\tau)d\tau
\]
\[
- \int_0^t \exp[-\alpha|\xi|^2 (t - \tau)] \left[ (u \cdot \nabla) u(\xi,\tau) + \nabla \hat{p}(\xi,\tau) \right] d\tau,
\]
there hold the following estimates
\[
|\hat{u}(\xi,t)| \leq |\hat{u}_0(\xi)| + \int_0^t |\hat{f}(\xi,\tau)| d\tau
+ 3 \int_0^t \left[ \int_{\mathbb{R}^n} |u(x,\tau)|^2 \, dx \right]^{1/2} \left[ \int_{\mathbb{R}^n} |\nabla u(x,\tau)|^2 \, dx \right]^{1/2} d\tau
\leq |\hat{u}_0(\xi)| + \int_0^t |\hat{f}(\xi,\tau)| d\tau
+ 3 \left[ \int_0^t \int_{\mathbb{R}^n} |u(x,\tau)|^2 \, dx \, d\tau \right]^{1/2} \left[ \int_0^t \int_{\mathbb{R}^n} |\nabla u(x,\tau)|^2 \, dx \, d\tau \right]^{1/2}.
\]

(III) By using the assumptions on \(u_0\) and \(f\), we have
\[
\hat{u}_0(\xi) = i \left( \sum_{l=1}^{n} \xi_l \hat{\phi}_1(\xi), \sum_{l=1}^{n} \xi_l \hat{\phi}_2(\xi), \cdots, \sum_{l=1}^{n} \xi_l \hat{\phi}_n(\xi) \right),
\]
\[
\hat{f}(\xi, t) = i \left( \sum_{l=1}^{n} \xi_l \hat{\psi}_1(\xi, t), \sum_{l=1}^{n} \xi_l \hat{\psi}_2(\xi, t), \cdots, \sum_{l=1}^{n} \xi_l \hat{\psi}_n(\xi, t) \right).
\]

By applying Cauchy-Schwartz’s inequality to the Fourier transformations, we get the following estimates
\[
|\hat{u}_0(\xi)|^2 = \sum_{k=1}^{n} \left| \sum_{l=1}^{n} \xi_l \hat{\phi}_{kl}(\xi) \right|^2
\leq \sum_{k=1}^{n} \sum_{l=1}^{n} \xi_l^2 \sum_{l=1}^{n} |\hat{\phi}_{kl}(\xi)|^2 = |\xi|^2 \sum_{k=1}^{n} \sum_{l=1}^{n} |\hat{\phi}_{kl}(\xi)|^2,
\]
\[
|\hat{f}(\xi, t)|^2 = \sum_{k=1}^{n} \left| \sum_{l=1}^{n} \xi_l \hat{\psi}_{kl}(\xi, t) \right|^2
\leq \sum_{k=1}^{n} \sum_{l=1}^{n} \xi_l^2 \sum_{l=1}^{n} |\hat{\psi}_{kl}(\xi, t)|^2 = |\xi|^2 \sum_{k=1}^{n} \sum_{l=1}^{n} |\hat{\psi}_{kl}(\xi, t)|^2.
\]
Lemma 2.3. There holds the following decay estimate
\[ |(\mathbf{u} - \nabla)\mathbf{u}(\xi, t)| \leq |\xi| \int_{\mathbb{R}^n} |\mathbf{u}(x, t)|^2dx, \]
for all \((\xi, t) \in \mathbb{R}^n \times \mathbb{R}^+\). Now we obtain the following estimates
\[
|\hat{u}(\xi, t)| \leq |\hat{u}_0(\xi)| + \int_0^t \hat{f}(\xi, \tau) d\tau + 2|\xi| \int_0^t \int_{\mathbb{R}^n} |\mathbf{u}(x, \tau)|^2 dx d\tau \\
\leq |\xi| \left[ \sum_{k=1}^n \sum_{l=1}^n |\hat{\phi}_{kl}(\xi)|^2 \right]^{1/2} + |\xi| \int_0^t \left[ \sum_{k=1}^n \sum_{l=1}^n |\hat{\psi}_{kl}(\xi, \tau)|^2 \right]^{1/2} d\tau \\
+ C|\xi| \int_0^\infty \frac{1}{(1+t)^{n/2}} dt \\
\leq C|\xi|,
\]
for all \((\xi, t) \in \mathbb{R}^n \times \mathbb{R}^+\). The proof of Lemma 2.2 is finished. \(\square\)

2.3. The improved Fourier splitting method.

Lemma 2.3. There holds the following decay estimate
\[ (1 + t)^{n/2} \int_{\mathbb{R}^n} |\mathbf{u}(x, t)|^2 dx \leq C, \]
for all time \(t > 0\), where \(C > 0\) is a positive constant, independent of \(\mathbf{u}\) and \((x, t)\).

Proof. Multiplying system (1) by \(2\mathbf{u}\) and integrating the result with respect to \(x\) over \(\mathbb{R}^n\) yield
\[
\frac{d}{dt} \int_{\mathbb{R}^n} |\mathbf{u}(x, t)|^2 dx + 2\alpha \int_{\mathbb{R}^n} |\nabla \mathbf{u}(x, t)|^2 dx = 2 \int_{\mathbb{R}^n} \mathbf{u}(x, t) \cdot \mathbf{f}(x, t) dx,
\]
where
\[
\int_{\mathbb{R}^n} \mathbf{u}(x, t) \cdot (\mathbf{u}(x, t) \cdot \nabla) \mathbf{u}(x, t) dx = 0.
\]
Applying the Plancherel’s identity to this equation gives
\[
\frac{d}{dt} \int_{\mathbb{R}^n} |\hat{\mathbf{u}}(\xi, t)|^2 d\xi + 2\alpha \int_{\mathbb{R}^n} |\xi|^2 |\hat{\mathbf{u}}(\xi, t)|^2 d\xi = 2 \int_{\mathbb{R}^n} \hat{\mathbf{u}}(\xi, t) \cdot \hat{\mathbf{f}}(\xi, t) d\xi.
\]
Multiplying it by \((1 + t)^{2n}\) to get the energy equation
\[
\frac{d}{dt} \left( (1 + t)^{2n} \int_{\mathbb{R}^n} |\hat{\mathbf{u}}(\xi, t)|^2 d\xi \right) + 2\alpha (1 + t)^{2n} \int_{\mathbb{R}^n} |\xi|^2 |\hat{\mathbf{u}}(\xi, t)|^2 d\xi \\
= 2n(1 + t)^{2n-1} \int_{\mathbb{R}^n} |\hat{\mathbf{u}}(\xi, t)|^2 d\xi + 2(1 + t)^{2n} \int_{\mathbb{R}^n} \hat{\mathbf{u}}(\xi, t) \cdot \hat{\mathbf{f}}(\xi, t) d\xi.
\]
By applying Cauchy-Schwartz’s inequality, we have
\[
2(1 + t)^{2n} \left| \int_{\mathbb{R}^n} \hat{\mathbf{u}}(\xi, t) \cdot \hat{\mathbf{f}}(\xi, t) d\xi \right| \\
\leq 2n(1 + t)^{2n-1} \int_{\mathbb{R}^n} |\hat{\mathbf{u}}(\xi, t)|^2 d\xi + \frac{1}{2n} (1 + t)^{2n+1} \int_{\mathbb{R}^n} |\hat{\mathbf{f}}(\xi, t)|^2 d\xi.
\]
Now the above energy equation becomes the inequality

\[
\frac{d}{dt} \left\{ (1 + t)^{2n} \int_{\mathbb{R}^n} |\hat{\mathbf{u}}(\xi, t)|^2 d\xi \right\} + 2\alpha(1 + t)^{2n} \int_{\mathbb{R}^n} |\xi|^2 |\hat{\mathbf{u}}(\xi, t)|^2 d\xi \\
\leq 4n(1 + t)^{2n-1} \int_{\mathbb{R}^n} |\hat{\mathbf{u}}(\xi, t)|^2 d\xi + \frac{1}{2n}(1 + t)^{2n+1} \int_{\mathbb{R}^n} |\hat{\mathbf{f}}(\xi, t)|^2 d\xi.
\]

Let \( t > 0 \) and define the small ball whose radius depends on time:

\[
\Omega(t) = \{ \xi \in \mathbb{R}^n : \alpha|\xi|^2(1 + t) \leq 2n \}.
\]

Then we have the following estimates

\[
2\alpha(1 + t)^{2n} \int_{\mathbb{R}^n} |\xi|^2 |\hat{\mathbf{u}}(\xi, t)|^2 d\xi \\
= 2\alpha(1 + t)^{2n} \int_{\Omega(t)} |\xi|^2 |\hat{\mathbf{u}}(\xi, t)|^2 d\xi \\
+ 2\alpha(1 + t)^{2n} \int_{\Omega(t)^c} |\xi|^2 |\hat{\mathbf{u}}(\xi, t)|^2 d\xi \\
\geq 2\alpha(1 + t)^{2n} \int_{\Omega(t)} |\xi|^2 |\hat{\mathbf{u}}(\xi, t)|^2 d\xi \\
\geq 4n(1 + t)^{2n-1} \int_{\Omega(t)} |\hat{\mathbf{u}}(\xi, t)|^2 d\xi \\
= 4n(1 + t)^{2n-1} \int_{\Omega(t)} |\hat{\mathbf{u}}(\xi, t)|^2 d\xi \\
- 4n(1 + t)^{2n-1} \int_{\Omega(t)} |\hat{\mathbf{u}}(\xi, t)|^2 d\xi.
\]

Therefore, we get

\[
\frac{d}{dt} \left\{ (1 + t)^{2n} \int_{\mathbb{R}^n} |\hat{\mathbf{u}}(\xi, t)|^2 d\xi \right\} \\
\leq 4n(1 + t)^{2n-1} \int_{\Omega(t)} |\hat{\mathbf{u}}(\xi, t)|^2 d\xi + \frac{1}{2n}(1 + t)^{2n+1} \int_{\mathbb{R}^n} |\hat{\mathbf{f}}(\xi, t)|^2 d\xi.
\]

There hold the following estimates

\[
(1 + t)^{n/2} \int_{\Omega(t)} |\hat{\mathbf{u}}(\xi, t)|^2 d\xi \\
\leq (1 + t)^{n/2} \int_{\Omega(t)} \left\{ |\hat{\mathbf{u}}_0(\xi)| + \int_0^t |\hat{\mathbf{f}}(\xi, \tau)| d\tau \right\} d\xi \\
+ C \left[ \int_0^t \int_{\mathbb{R}^n} |\mathbf{u}(x, \tau)|^2 dx d\tau \right]^{1/2} \left[ \int_0^t \int_{\mathbb{R}^n} |\nabla \mathbf{u}(x, \tau)|^2 dx d\tau \right]^{1/2} \\
\leq C \left\{ \int_{\mathbb{R}^n} |\mathbf{u}_0(x)| dx + \int_0^t \int_{\mathbb{R}^n} |\mathbf{f}(x, \tau)| dx d\tau \right\}^2 \\
+ C \int_0^t \int_{\mathbb{R}^n} |\mathbf{u}(x, \tau)|^2 dx d\tau \int_0^t \int_{\mathbb{R}^n} |\nabla \mathbf{u}(x, \tau)|^2 dx d\tau.
\]
Lemma 2.4. There holds the decay estimate with sharp rate
\[
(1 + t)^{1 + n/2} \int_{\mathbb{R}^n} |u(x, t)|^2 dx \leq C,
\]
for all \( t > 0 \), where \( C > 0 \) is a positive constant, independent of \( u(x, t) \) and \( (x, t) \).

Proof. Recall that from Lemma 2.2
\[
|\hat{u}(\xi, t)| \leq C|\xi|,
\]
Lemma 2.5. For all $(\xi, t) \in \mathbb{R}^n \times \mathbb{R}^+$. Now we have the following estimates

$$
\frac{d}{dt} \left\{ (1 + t)^{2n} \int_{\mathbb{R}^n} |\hat{u}(\xi, t)|^2 d\xi \right\}
\leq 4n(1 + t)^{2n-1} \int_{\Omega(t)} |\hat{u}_0(\xi)|^2 d\xi + \frac{1}{2n} (1 + t)^{2n+1} \int_{\mathbb{R}^n} |\hat{f}(\xi, t)|^2 d\xi
\leq 4n(1 + t)^{2n-1} \int_{\Omega(t)} C^2 |\xi|^2 d\xi + \frac{1}{2n} (1 + t)^{2n+1} \int_{\mathbb{R}^n} |\hat{f}(\xi, t)|^2 d\xi
\leq C(1 + t)^{3n/2 - 2} + \frac{1}{2n} (1 + t)^{2n+1} \int_{\mathbb{R}^n} |\hat{f}(\xi, t)|^2 d\xi.
$$

Integrating in time $t$ yields the estimate

$$(1 + t)^{2n} \int_{\mathbb{R}^n} |\hat{u}(\xi, t)|^2 d\xi \leq \int_{\mathbb{R}^n} |\hat{u}_0(\xi)|^2 d\xi
+ C(1 + t)^{3n/2 - 1} + \frac{1}{2n} (1 + t)^{3n/2 - 1} \int_0^\infty (1 + t)^{2n+2} \int_{\mathbb{R}^n} |\hat{f}(\xi, t)|^2 d\xi dt.
$$

That is

$$(1 + t)^{1+n/2} \int_{\mathbb{R}^n} |\hat{u}(\xi, t)|^2 d\xi \leq \int_{\mathbb{R}^n} |\hat{u}_0(\xi)|^2 d\xi
+ C + \int_0^\infty (1 + t)^{n/2} \int_{\mathbb{R}^n} |\hat{f}(\xi, t)|^2 d\xi dt,
$$

The proof of Lemma 2.4 is finished. $\square$

Lemma 2.5. Let the initial function $u_0 \in L^1(\mathbb{R}^n) \cap H^{2m+1}(\mathbb{R}^n)$ and the external force $f \in L^1(\mathbb{R}^n \times \mathbb{R}^+ \cap L^1(\mathbb{R}^+L^2(\mathbb{R}^n)) \cap L^2(\mathbb{R}^+,H^{2m}(\mathbb{R}^n))$. Suppose that there exists a global smooth solution to the Cauchy problems for the $n$-dimensional incompressible Navier-Stokes equations (1)-(2): $u \in L^\infty(\mathbb{R}^+,H^{2m+1}(\mathbb{R}^n))$, such that $\nabla u \in L^2(\mathbb{R}^+,H^{2m+1}(\mathbb{R}^n))$. There hold the following decay estimates with sharp rates

$$(1 + t)^{1+n/2} \int_{\mathbb{R}^n} |u(\mathbf{x}, t)|^2 d\mathbf{x} \leq C,$$

$$(1 + t)^{2+n/2} \int_{\mathbb{R}^n} |\nabla u(\mathbf{x}, t)|^2 d\mathbf{x} \leq C,$$

$$(1 + t)^{2m+1+n/2} \int_{\mathbb{R}^n} |\Delta u(\mathbf{x}, t)|^2 d\mathbf{x} \leq C,$$

$$(1 + t)^{2m+2+n/2} \int_{\mathbb{R}^n} |\nabla \Delta u(\mathbf{x}, t)|^2 d\mathbf{x} \leq C,$$

and

$$(1 + t)^{1/2+n/2} \|u(\cdot, t)\|_{L^\infty} \leq C,$$

$$(1 + t)^{1+n/2} \|\nabla u(\cdot, t)\|_{L^\infty} \leq C,$$

$$(1 + t)^{m+1/2+n/2} \|\Delta u(\cdot, t)\|_{L^\infty} \leq C,$$

$$(1 + t)^{m+1+n/2} \|\nabla \Delta u(\cdot, t)\|_{L^\infty} \leq C,$$

for all positive integers $m \geq 1$ and for all time $t > 0$, where $C > 0$ is a positive constant, independent of $u$ and $(\mathbf{x}, t)$. 
Proof. Applying the differential operator $\Delta^m$ to the Navier-Stokes equations gives

$$\frac{\partial}{\partial t} \Delta^m u - \alpha \Delta^{m+1} u + \Delta^m (u \cdot \nabla) u + \nabla \Delta^m p = \Delta^m f(x, t).$$

Multiplying this equation by $2\Delta^m u$ and integrating the result with respect to $x$ over $\mathbb{R}^n$ yields

$$\frac{d}{dt} \int_{\mathbb{R}^n} |\Delta^m u(x, t)|^2 dx + 2\alpha \int_{\mathbb{R}^n} |\nabla \Delta^m u(x, t)|^2 dx$$

$$+ \int_{\mathbb{R}^n} \Delta^m u \cdot \nabla^2 \Delta^m u \cdot u dx = 2 \int_{\mathbb{R}^n} \Delta^m u \cdot \nabla^2 \Delta^m f(x, t) dx,$$

where

$$\int_{\mathbb{R}^n} \Delta^m u \cdot \nabla \Delta^m p(x, t) dx = \int_{\mathbb{R}^n} (\nabla \cdot \Delta^m u) \Delta^m p(x, t) dx = 0.$$

Applying the Plancherel’s identity leads to

$$\frac{d}{dt} \int_{\mathbb{R}^n} |\xi|^{4m} \tilde{u}(\xi, t)|^2 d\xi + 2\alpha \int_{\mathbb{R}^n} |\xi|^{4m+2} \tilde{u}(\xi, t)|^2 d\xi$$

$$+ \int_{\mathbb{R}^n} |\xi|^{4m} \tilde{u}(\xi, t) \cdot \overline{\tilde{f}(\xi, t)} d\xi = 2 \int_{\mathbb{R}^n} |\xi|^{4m} \tilde{u}(\xi, t) \cdot \tilde{f}(\xi, t) d\xi.$$

Multiplying it by $(1 + t)^{2m+2n}$ leads to

$$\frac{d}{dt} \left\{ (1 + t)^{2m+2n} \int_{\mathbb{R}^n} |\xi|^{4m} \tilde{u}(\xi, t)|^2 d\xi \right\}$$

$$+ 2\alpha (1 + t)^{2m+2n} \int_{\mathbb{R}^n} |\xi|^{4m+2} \tilde{u}(\xi, t)|^2 d\xi$$

$$+ 2(1 + t)^{2m+2n} \int_{\mathbb{R}^n} |\xi|^{4m} \tilde{u}(\xi, t) \cdot \overline{\tilde{f}(\xi, t)} d\xi$$

$$= (2m + 2n)(1 + t)^{2m+2n-1} \int_{\mathbb{R}^n} |\xi|^{4m} \tilde{u}(\xi, t)|^2 d\xi$$

$$+ 2(1 + t)^{2m+2n} \int_{\mathbb{R}^n} |\xi|^{4m} \tilde{u}(\xi, t) \cdot \tilde{f}(\xi, t) d\xi.$$

By using Cauchy-Schwartz’s inequality, we have the following estimates

$$\left| (1 + t)^{2m+2n} \int_{\mathbb{R}^n} |\xi|^{4m} \tilde{u}(\xi, t) \cdot \tilde{f}(\xi, t) d\xi \right|$$

$$\leq (2m + 2n)(1 + t)^{2m+2n-1} \int_{\mathbb{R}^n} |\xi|^{4m} \tilde{u}(\xi, t)|^2 d\xi$$

$$+ \frac{1}{2m + 2n} (1 + t)^{2m+2n+1} \int_{\mathbb{R}^n} |\xi|^{4m} \tilde{f}(\xi, t)|^2 d\xi,$$

$$\left| (1 + t)^{2m+2n} \int_{\mathbb{R}^n} |\xi|^{4m} \tilde{u}(\xi, t) \cdot \overline{\tilde{f}(\xi, t)} d\xi \right|$$

$$\leq \alpha (1 + t)^{2m+2n} \int_{\mathbb{R}^n} |\xi|^{4m+2} \tilde{u}(\xi, t)|^2 d\xi$$

$$+ C_0 (1 + t)^{2m+2n} \left\{ \int_{\mathbb{R}^n} |\tilde{u}(\xi, t)|^2 d\xi \right\} \int_{\mathbb{R}^n} |\xi|^{4m} \tilde{u}(\xi, t)|^2 d\xi.$$. 
Now the above energy equation becomes the differential inequality

$$
\frac{d}{dt} \left\{ (1 + t)^{2m+2n} \int_{\mathbb{R}^n} |\xi|^{4m} |\hat{u}(\xi, t)|^2 d\xi \right\}
+ \alpha (1 + t)^{2m+2n} \int_{\mathbb{R}^n} |\xi|^{4m+2} |\hat{u}(\xi, t)|^2 d\xi
\leq (4m + 4n)(1 + t)^{2m+2n-1} \int_{\mathbb{R}^n} |\xi|^{4m} |\hat{u}(\xi, t)|^2 d\xi
+ \frac{1}{2m + 2n} (1 + t)^{2m+2n+1} \int_{\mathbb{R}^n} |\xi|^{4m} |\hat{f}(\xi, t)|^2 d\xi
+ C_0 (1 + t)^{2m+2n-1} \int_{\Omega(t)} |\xi|^{4m} |\hat{u}(\xi, t)|^2 d\xi.
$$

Let

$$
\Omega(t) = \{ \xi \in \mathbb{R}^n : \alpha |\xi|^2 (1 + t) \leq 4m + 4n + C_0 \}.
$$

Then

$$
\alpha (1 + t)^{2m+2n} \int_{\mathbb{R}^n} |\xi|^{4m+2} |\hat{u}(\xi, t)|^2 d\xi
= \alpha (1 + t)^{2m+2n} \int_{\Omega(t)} |\xi|^{4m+2} |\hat{u}(\xi, t)|^2 d\xi
+ \alpha (1 + t)^{2m+2n} \int_{\Omega(t)^c} |\xi|^{4m+2} |\hat{u}(\xi, t)|^2 d\xi
\geq \alpha (1 + t)^{2m+2n} \int_{\Omega(t)^c} |\xi|^{4m+2} |\hat{u}(\xi, t)|^2 d\xi
\geq (4m + 4n + C_0)(1 + t)^{2m+2n-1} \int_{\Omega(t)^c} |\xi|^{4m} |\hat{u}(\xi, t)|^2 d\xi
= (4m + 4n + C_0)(1 + t)^{2m+2n-1} \int_{\mathbb{R}^n} |\xi|^{4m} |\hat{u}(\xi, t)|^2 d\xi
- (4m + 4n + C_0)(1 + t)^{2m+2n-1} \int_{\Omega(t)} |\xi|^{4m} |\hat{u}(\xi, t)|^2 d\xi.
$$

Now

$$
\frac{d}{dt} \left\{ (1 + t)^{2m+2n} \int_{\mathbb{R}^n} |\xi|^{4m} |\hat{u}(\xi, t)|^2 d\xi \right\}
\leq (4m + 4n)(1 + t)^{2m+2n-1} \int_{\Omega(t)} |\xi|^{4m} |\hat{u}(\xi, t)|^2 d\xi
+ \frac{1}{2m + 2n} (1 + t)^{2m+2n+1} \int_{\mathbb{R}^n} |\xi|^{4m} |\hat{f}(\xi, t)|^2 d\xi
+ C_0 (1 + t)^{2m+2n-1} \int_{\Omega(t)} |\xi|^{4m} |\hat{u}(\xi, t)|^2 d\xi
\leq C(1 + t)^{2m+2n-1} \int_{\Omega(t)} |\xi|^{4m+2} d\xi
+ \frac{1}{2m + 2n} (1 + t)^{2m+2n+1} \int_{\mathbb{R}^n} |\xi|^{4m} |\hat{f}(\xi, t)|^2 d\xi
$$
\[ \leq C(1 + t)^{3n/2 - 2} + \frac{1}{2m + 2n} (1 + t)^{2m+2n+1} \int_{\mathbb{R}^n} |\xi|^{4m} |\hat{f}(\xi, t)|^2 d\xi. \]

Integrating in time \( t \) yields
\[ (1 + t)^{2m+2n} \int_{\mathbb{R}^n} |\xi|^{4m} |\hat{u}(\xi, t)|^2 d\xi \]
\[ \leq \int_{\mathbb{R}^n} |\xi|^{4m} |\hat{u}_0(\xi)|^2 d\xi + C(1 + t)^{3n/2 - 1} \]
\[ + \frac{1}{2m + 2n} (1 + t)^{3n/2 - 1} \int_0^\infty (1 + t)^{2m+2n+1/2} \int_{\mathbb{R}^n} |\xi|^{4m} |\hat{f}(\xi, t)|^2 d\xi dt. \]

Therefore, we obtain the decay estimate with sharp rate
\[ (1 + t)^{2m+1+n/2} \int_{\mathbb{R}^n} |\xi|^{4m} |\hat{u}(\xi, t)|^2 d\xi \]
\[ \leq \int_{\mathbb{R}^n} |\xi|^{4m} |\hat{u}_0(\xi)|^2 d\xi + C \]
\[ + \frac{1}{2m + 2n} \int_0^\infty (1 + t)^{2m+2n+1/2} \int_{\mathbb{R}^n} |\xi|^{4m} |\hat{f}(\xi, t)|^2 d\xi dt. \]

By using the Gagliardo-Nirenberg’s interpolation inequality, we have the estimate
\[ \|\Delta^m u(\cdot, t)\|_{L^\infty}^8 \leq C \left\{ \int_{\mathbb{R}^n} |\Delta^m u(x, t)|^2 dx \right\}^3 \cdot \left\{ \int_{\mathbb{R}^n} |\Delta^{m+n} u(x, t)|^2 dx \right\}. \]

Recall that there hold the decay estimates
\[ (1 + t)^{2m+1+n/2} \int_{\mathbb{R}^n} |\Delta^m u(x, t)|^2 dx \leq C, \]
\[ (1 + t)^{2m+2n+1+n/2} \int_{\mathbb{R}^n} |\Delta^{m+n} u(x, t)|^2 dx \leq C, \]

where the positive constant \( C > 0 \) is independent of \( u \) and \( (x, t) \). Therefore, there hold the following estimates
\[ (1 + t)^{8m+4n} \|\Delta^m u(\cdot, t)\|_{L^\infty}^8 \leq C \left\{ (1 + t)^{2m+1+n/2} \int_{\mathbb{R}^n} |\Delta^m u(x, t)|^2 dx \right\}^3 \cdot \left\{ (1 + t)^{2m+2n+1+n/2} \int_{\mathbb{R}^n} |\Delta^{m+n} u(x, t)|^2 dx \right\} \leq C. \]

The proofs of Lemma 2.5 is finished. \( \square \)

3. Conclusions and remarks.

3.1. Summary. In this paper, we made complete use of the uniform energy estimates to improve the Fourier splitting method to accomplish decay estimates with sharp rates of the global smooth solutions of the Cauchy problems for the \( n \)-dimensional incompressible Navier-Stokes equations and for the \( n \)-dimensional magnetohydrodynamics equations.
3.2. Summary about the $n$-dimensional incompressible Navier-Stokes equations. Consider the Cauchy problems for the $n$-dimensional incompressible Navier-Stokes equations

$$\frac{\partial u}{\partial t} - \alpha \Delta u + (u \cdot \nabla)u + \nabla p = f(x, t), \quad \nabla \cdot u = 0, \nabla \cdot f = 0,$$

$$u(x, 0) = u_0(x), \quad \nabla \cdot u_0 = 0.$$

Suppose that the initial function $u_0 \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ and the external force $f \in L^1(\mathbb{R}^n \times \mathbb{R}^+ \cap L^1(\mathbb{R}^+L^2(\mathbb{R}^n))$. There exists a global weak solution $u \in L^\infty(\mathbb{R}^+, L^2(\mathbb{R}^n))$, such that $\nabla u \in L^2(\mathbb{R}^+, L^2(\mathbb{R}^n))$.

There holds the following uniform energy estimate

$$\left\{ \int_{\mathbb{R}^n} |u(x, t)|^2 dx + 2 \alpha \int_0^t \int_{\mathbb{R}^n} |\nabla u(x, \tau)|^2 dx d\tau \right\}^{1/2} \leq \left\{ \int_{\mathbb{R}^n} |u_0(x)|^2 dx \right\}^{1/2} + \int_0^t \left\{ \int_{\mathbb{R}^n} |f(x, \tau)|^2 dx \right\}^{1/2} d\tau.$$

Suppose that there exist real scalar functions $\phi_{kl} \in C^1(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ and $\psi_{kl} \in C^1(\mathbb{R}^n \times \mathbb{R}^+) \cap L^1(\mathbb{R}^n \times \mathbb{R}^+)$, such that the initial function $u_0$ and the external force $f$ satisfy

$$u_0(x) = \left( \sum_{l=1}^n \frac{\partial \phi_{1l}(x)}{\partial x_l}, \sum_{l=1}^n \frac{\partial \phi_{2l}(x)}{\partial x_l}, \cdots, \sum_{l=1}^n \frac{\partial \phi_{nl}(x)}{\partial x_l} \right),$$

$$f(x, t) = \left( \sum_{l=1}^n \frac{\partial \psi_{1l}(x, t)}{\partial x_l}, \sum_{l=1}^n \frac{\partial \psi_{2l}(x, t)}{\partial x_l}, \cdots, \sum_{l=1}^n \frac{\partial \psi_{nl}(x, t)}{\partial x_l} \right),$$

and

$$\frac{\partial \phi_{kl}}{\partial x_l} \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n), \quad \frac{\partial \psi_{kl}}{\partial x_l} \in L^1(\mathbb{R}^n \times \mathbb{R}^+) \cap L^1(\mathbb{R}^+, \mathbb{R}^n),$$

for all $k = 1, 2, \cdots, n$ and $l = 1, 2, \cdots, n$.

There holds the following decay estimate

$$(1 + t)^{1+n/2} \int_{\mathbb{R}^n} |u(x, t)|^2 dx \leq C,$$

for all time $t > 0$, where $C > 0$ is a positive constant, independent of $u$ and $(x, t)$.

Suppose that the initial function $u_0 \in L^1(\mathbb{R}^n) \cap H^{2m+1}(\mathbb{R}^n)$ and the external force $f \in L^1(\mathbb{R}^n \times \mathbb{R}^+) \cap L^1(\mathbb{R}^+, L^2(\mathbb{R}^n)) \cap L^2(\mathbb{R}^+, H^{2m}(\mathbb{R}^n))$. Suppose that there exists a global smooth solution $u \in L^\infty(\mathbb{R}^+, H^{2m+1}(\mathbb{R}^n))$, such that $\nabla u \in L^2(\mathbb{R}^+, H^{2m+1}(\mathbb{R}^n))$, where $m \geq 1$ is a positive integer.

This assumption is true if there exist positive constants $p > n \geq 3$ and $q > 2$, with $\frac{2}{p} + \frac{2}{q} = 1$, such that

$$\int_0^\infty \left[ \int_{\mathbb{R}^n} |u(x, t)|^p dx \right]^{q/p} dt < \infty.$$ 

We can prove that $u \in L^\infty(\mathbb{R}^+, H^{2m+1}(\mathbb{R}^n))$ and that $\nabla u \in L^2(\mathbb{R}^+, H^{2m+1}(\mathbb{R}^n))$.

There hold the following decay estimates with sharp rates for the global smooth solution

$$(1 + t)^{1+n/2} \int_{\mathbb{R}^n} |u(x, t)|^2 dx \leq C,$$
\[(1 + t)^{2+n/2} \int_{\mathbb{R}^n} |\nabla u(x, t)|^2 \, dx \leq C,\]
\[(1 + t)^{2m+1+n/2} \int_{\mathbb{R}^n} |\Delta^m u(x, t)|^2 \, dx \leq C,\]
\[(1 + t)^{2m+2+n/2} \int_{\mathbb{R}^n} |\nabla \Delta^m u(x, t)|^2 \, dx \leq C,\]

and
\[(1 + t)^{1/2+n/2} \|u(\cdot, t)\|_{L^\infty} \leq C,\]
\[(1 + t)^{1+n/2} \|
abla u(\cdot, t)\|_{L^\infty} \leq C,\]
\[(1 + t)^{m+1+2+n/2} \|\Delta^m u(\cdot, t)\|_{L^\infty} \leq C,\]
\[(1 + t)^{m+1+2+n/2} \|\nabla \Delta^m u(\cdot, t)\|_{L^\infty} \leq C,\]

for all positive integers \(m \geq 1\) and for all time \(t > 0\), where \(C > 0\) is a positive constant, independent of \(u\) and \((x, t)\).

**Remark 1.** The decay estimates with sharp rates stated in Theorem 1.1 will have positive influence on how to compute the following exact limits
\[
\lim_{t \to \infty} \left\{ (1 + t)^{1+n/2} \int_{\mathbb{R}^n} |u(x, t)|^2 \, dx \right\},
\lim_{t \to \infty} \left\{ (1 + t)^{2+n/2} \int_{\mathbb{R}^n} |\nabla u(x, t)|^2 \, dx \right\},
\lim_{t \to \infty} \left\{ (1 + t)^{2m+1+n/2} \int_{\mathbb{R}^n} |\Delta^m u(x, t)|^2 \, dx \right\},
\lim_{t \to \infty} \left\{ (1 + t)^{2m+2+n/2} \int_{\mathbb{R}^n} |\nabla \Delta^m u(x, t)|^2 \, dx \right\},
\]
for the global smooth solutions of the \(n\)-dimensional incompressible Navier-Stokes equations in terms of the integrals of the initial function and the external force.

**Remark 2.** Due to the divergence free conditions \(\nabla \cdot u_0 = 0\) and \(\nabla \cdot f = 0\) in the \(n\)-dimensional incompressible Navier-Stokes equations (1)-(2), it is necessarily true that \(\int_{\mathbb{R}^n} u_0(x) \, dx = 0\) and \(\int_{\mathbb{R}^n} f(x, t) \, dx = 0\), for all \(t > 0\).

**Open problem:** The existence and uniqueness of the global smooth solution of the Cauchy problems for the \(n\)-dimensional incompressible Navier-Stokes equations (1)-(2) have been open for a long time, where \(n \geq 3\). Suppose that the initial function \(u_0 \in H^{2m+1} (\mathbb{R}^n)\) and the external force \(f \in L^1 (\mathbb{R}^+, L^2 (\mathbb{R}^n)) \cap L^1 (\mathbb{R}^+, H^{2m} (\mathbb{R}^n))\). There exist special structures in the \(n\)-dimensional incompressible Navier-Stokes equations (1)-(2). By making complete use of the special structures of (1)-(2), it is possible to accomplish the uniform energy estimates for all order derivatives to accomplish the existence of the global smooth solution: \(u \in L^\infty (\mathbb{R}^+, H^{2m+1} (\mathbb{R}^n))\), such that \(\nabla u \in L^2 (\mathbb{R}^+, H^{2m+1} (\mathbb{R}^n))\), where \(m \geq 1\) is a positive integer.

### 3.3. Summary about the \(n\)-dimensional magnetohydrodynamics equations
Consider the Cauchy problems for the \(n\)-dimensional magnetohydrodynamics equations
\[
\frac{\partial u}{\partial t} - \frac{1}{\text{RE}} \Delta u + (u \cdot \nabla)u - (A \cdot \nabla)A + \nabla P = f(x, t), \\
\frac{\partial A}{\partial t} - \frac{1}{\text{RM}} \Delta A + (u \cdot \nabla)A - (A \cdot \nabla)u = g(x, t), \\
\nabla \cdot u = 0, \quad \nabla \cdot f = 0, \quad \nabla \cdot A = 0, \quad \nabla \cdot g = 0,
\]
\[
u(x, 0) = u_0(x), \quad A(x, 0) = A_0(x), \quad \nabla \cdot u_0 = 0, \quad \nabla \cdot A_0 = 0.
\]

Here are the main assumptions on the initial functions and the external forces needed for the main results.

Suppose that the initial functions
\[ u_0 \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n), \quad A_0 \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n). \]

Suppose that the external forces
\[ f \in L^1(\mathbb{R}^n \times \mathbb{R}^+) \cap L^1(\mathbb{R}^+, L^2(\mathbb{R}^n)), \]
\[ g \in L^1(\mathbb{R}^n \times \mathbb{R}^+) \cap L^1(\mathbb{R}^+, L^2(\mathbb{R}^n)). \]

There exists a global weak solution
\[ u \in L^\infty(\mathbb{R}^+, L^2(\mathbb{R}^n)), \quad A \in L^\infty(\mathbb{R}^+, L^2(\mathbb{R}^n)), \]
such that
\[ \nabla u \in L^2(\mathbb{R}^+, L^2(\mathbb{R}^n)), \quad \nabla A \in L^2(\mathbb{R}^+, L^2(\mathbb{R}^n)). \]

There holds the following uniform energy estimate
\[
\left\{ \int_{\mathbb{R}^n} \left[ |u(x, t)|^2 + |A(x, t)|^2 \right] \, dx \right\}^{1/2} \\
+ \int_0^t \int_{\mathbb{R}^n} \left[ \frac{2}{\text{RE}} |\nabla u(x, \tau)|^2 + \frac{2}{\text{RM}} |\nabla A(x, \tau)|^2 \right] \, dx \, d\tau \right\}^{1/2} \\
\leq \left\{ \int_{\mathbb{R}^n} \left[ |u_0(x)|^2 + |A_0(x)|^2 \right] \, dx \right\}^{1/2} \\
+ \int_0^t \left\{ \int_{\mathbb{R}^n} \left[ |f(x, \tau)|^2 + |g(x, \tau)|^2 \right] \, dx \right\}^{1/2} \, d\tau.
\]

Suppose that there exist real scalar functions
\[
\phi_{kl} \in C^1(\mathbb{R}^n) \cap L^1(\mathbb{R}^n), \quad \psi_{kl} \in C^1(\mathbb{R}^n \times \mathbb{R}^+) \cap L^1(\mathbb{R}^n \times \mathbb{R}^+), \\
\kappa_{kl} \in C^1(\mathbb{R}^n) \cap L^1(\mathbb{R}^n), \quad \omega_{kl} \in C^1(\mathbb{R}^n \times \mathbb{R}^+) \cap L^1(\mathbb{R}^n \times \mathbb{R}^+),
\]
such that
\[
u(x) = \left( \sum_{l=1}^n \frac{\partial \phi_{1l}(x)}{\partial x_l} + \sum_{k=1}^n \frac{\partial \phi_{2l}(x)}{\partial x_l}, \ldots, \sum_{l=1}^n \frac{\partial \phi_{nl}(x)}{\partial x_l} \right), \\
f(x, t) = \left( \sum_{l=1}^n \frac{\partial \psi_{1l}(x, t)}{\partial x_l} + \sum_{k=1}^n \frac{\partial \psi_{2l}(x, t)}{\partial x_l}, \ldots, \sum_{l=1}^n \frac{\partial \psi_{nl}(x, t)}{\partial x_l} \right), \\
A_0(x) = \left( \sum_{l=1}^n \frac{\partial \kappa_{1l}(x)}{\partial x_l} + \sum_{k=1}^n \frac{\partial \kappa_{2l}(x)}{\partial x_l}, \ldots, \sum_{l=1}^n \frac{\partial \kappa_{nl}(x)}{\partial x_l} \right), \\
g(x, t) = \left( \sum_{l=1}^n \frac{\partial \omega_{1l}(x, t)}{\partial x_l} + \sum_{k=1}^n \frac{\partial \omega_{2l}(x, t)}{\partial x_l}, \ldots, \sum_{l=1}^n \frac{\partial \omega_{nl}(x, t)}{\partial x_l} \right),
\]
and that
\[
\frac{\partial \phi_{kl}}{\partial x_l} \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n), \quad \frac{\partial \omega_{kl}}{\partial x_l} \in L^1(\mathbb{R}^n \times \mathbb{R}^+) \cap L^1(\mathbb{R}^+, L^2(\mathbb{R}^n)),
\]
\[
\frac{\partial \omega_{kl}}{\partial x_l} \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n), \quad \frac{\partial \omega_{kl}}{\partial x_l} \in L^1(\mathbb{R}^n \times \mathbb{R}^+) \cap L^1(\mathbb{R}^+, L^2(\mathbb{R}^n)),
\]
for all \(k = 1, 2, \ldots, n\) and \(l = 1, 2, \ldots, n\).

There holds the following decay estimate with sharp rate
\[
(1 + t)^{1+n/2} \int_{\mathbb{R}^n} |u(x, t)|^2 + |A(x, t)|^2 dx \leq C,
\]
for all \(t > 0\), where \(C > 0\) is a positive constant, independent of \((u, A)\) and \((x, t)\).

**Theorem 3.1.** Suppose that the initial functions
\[
u_0 \in L^1(\mathbb{R}^n) \cap H^{2m+1}(\mathbb{R}^n), \quad A_0 \in L^1(\mathbb{R}^n) \cap H^{2m+1}(\mathbb{R}^n).
\]
Suppose that the external forces
\[
f \in L^1(\mathbb{R}^n \times \mathbb{R}^+) \cap L^1(\mathbb{R}^+, L^2(\mathbb{R}^n)) \cap L^2(\mathbb{R}^+, H^{2m}(\mathbb{R}^n)),
\]
\[
g \in L^1(\mathbb{R}^n \times \mathbb{R}^+) \cap L^1(\mathbb{R}^+, L^2(\mathbb{R}^n)) \cap L^2(\mathbb{R}^+, H^{2m}(\mathbb{R}^n)).
\]
Suppose that there exists a global smooth solution
\[
u \in L^\infty(\mathbb{R}^+, H^{2m+1}(\mathbb{R}^n)), \quad A \in L^\infty(\mathbb{R}^+, H^{2m+1}(\mathbb{R}^n)),
\]
such that
\[
\nabla \nu \in L^2(\mathbb{R}^+, H^{2m+1}(\mathbb{R}^n)), \quad \nabla A \in L^2(\mathbb{R}^+, H^{2m+1}(\mathbb{R}^n)).
\]
For the global smooth solution of the \(n\)-dimensional magnetohydrodynamics equations, there hold the following decay estimates with sharp rates
\[
(1 + t)^{1+n/2} \int_{\mathbb{R}^n} |u(x, t)|^2 + |A(x, t)|^2 dx \leq C,
\]
\[
(1 + t)^{2+n/2} \int_{\mathbb{R}^n} |\nabla u(x, t)|^2 + |\nabla A(x, t)|^2 dx \leq C,
\]
\[
(1 + t)^{2m+1+n/2} \int_{\mathbb{R}^n} |\Delta^m u(x, t)|^2 + |\Delta^m A(x, t)|^2 dx \leq C,
\]
\[
(1 + t)^{2m+2+n/2} \int_{\mathbb{R}^n} |\nabla \Delta^m u(x, t)|^2 + |\nabla \Delta^m A(x, t)|^2 dx \leq C,
\]
and
\[
(1 + t)^{1/2+n/2} \|u(\cdot, t)\|_{L^\infty} + \|A(\cdot, t)\|_{L^\infty} \leq C,
\]
\[
(1 + t)^{1+n/2} \|\nabla u(\cdot, t)\|_{L^\infty} + \|\nabla A(\cdot, t)\|_{L^\infty} \leq C,
\]
\[
(1 + t)^{m+1+2+n/2} \|\Delta^m u(\cdot, t)\|_{L^\infty} + \|\Delta^m A(\cdot, t)\|_{L^\infty} \leq C,
\]
\[
(1 + t)^{m+1+n/2} \|\nabla \Delta^m u(\cdot, t)\|_{L^\infty} + \|\nabla \Delta^m A(\cdot, t)\|_{L^\infty} \leq C,
\]
for all positive integers \(m \geq 1\) and for all time \(t > 0\), where \(C > 0\) is a positive constant, independent of \((u, A)\) and \((x, t)\).

**Proof.** The proof of Theorem 3.1 is very similar to that of Theorem 1.1 and the details are omitted. \(\Box\)
3.4. Other remarks. In one-dimensional space $\mathbb{R}$, if the integral of a continuous and integrable function satisfies $\int_{\mathbb{R}} u_0(x) dx = 0$, then we may define the function $\phi(x) = \int_{-\infty}^{x} u_0(y) dy$. It is easy to see that $\phi \in C^{1}(\mathbb{R})$ and $u_0(x) = \phi'(x)$. In $n$-dimensional space $\mathbb{R}^n$, if the integral of a vector valued function $u_0$ satisfies $\int_{\mathbb{R}^n} u_0(x) dx = 0$, then it is reasonable to let every component be equal to the linear combination of the partial derivatives of finitely many functions. That is why we make the assumptions for the initial function and the external force for the main results of this paper.

3.5. Some technical lemmas.

**Lemma 3.2. (The Cauchy-Schwartz’s inequality)** Let the functions $f \in L^2(\mathbb{R}^n)$ and $g \in L^2(\mathbb{R}^n)$. There holds the following Cauchy-Schwartz’s inequality

$$\left[ \int_{\mathbb{R}^n} f(x) g(x) dx \right]^2 \leq \int_{\mathbb{R}^n} |f(x)|^2 dx \int_{\mathbb{R}^n} |g(x)|^2 dx.$$ 

**Lemma 3.3. (The Hölder’s inequality)** Let $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$, where $p \geq 1$ and $q \geq 1$ are positive constants, such that $\frac{1}{p} + \frac{1}{q} = 1$. There holds the following estimate

$$\left| \int_{\mathbb{R}^n} f(x) g(x) dx \right| \leq \left[ \int_{\mathbb{R}^n} |f(x)|^p dx \right]^{1/p} \left[ \int_{\mathbb{R}^n} |g(x)|^q dx \right]^{1/q}.$$ 

**Lemma 3.4. (The Gronwall’s inequality)** Suppose that the nonnegative continuous functions $f \geq 0$, $g \geq 0$ and $h \geq 0$ satisfy the inequality

$$g(t) \leq f(t) + \int_0^t g(\tau) h(\tau) d\tau,$$

for all $t > 0$, where the derivative $f' \geq 0$. Then

$$g(t) \leq f(t) \exp \left\{ \int_0^t h(\tau) d\tau \right\},$$

for all $t > 0$.

**Lemma 3.5. (The Plancherel’s identity)** There holds the following Plancherel’s identity

$$\int_{\mathbb{R}^n} |f(x)|^2 dx = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 d\xi,$$

for all real vector valued functions $f \in L^2(\mathbb{R}^n)$.

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