Adding Matter to Poincare Invariant Branes

Benjamín Grinstein,∗ Detlef R. Nolte,† and Witold Skiba‡

Department of Physics, University of California at San Diego, La Jolla, CA 92093

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Abstract

A solution to the cosmological constant problem has been proposed in which our universe is a 3-brane in a 5-dimensional spacetime. With a bulk scalar, the field equations admit a Poincare invariant brane solution regardless of the value of the cosmological constant (tension) on the brane. However, the solution does not include matter in the brane. We find new exact static solutions with matter density and pressure in the brane. We study small perturbations about these solutions. None seem consistent with observational cosmology. As a byproduct we find a class of new matterless static solutions and a non-static solution which, curiously, requires the string value for the dilaton coupling.

I. INTRODUCTION

The cosmological constant problem has evaded solution since its inception [1]. If only, it has become more severe: the triumph of Quantum Field Theories as the correct description

∗e-mail address: bgrinstein@ucsd.edu
†e-mail address: dnolte@ucsd.edu
‡e-mail address: wskiba@ucsd.edu
of the fundamental interactions came at the cost of additive contributions to the cosmological constant at big and disparate scales. It is natural to expect that the cosmological constant gets contributions of order $M_P^4$, where $M_P$ is the Planck scale, from short distance gravitational dynamics; $M_W^4$, where $M_W$ is the mass of the $W$-boson, from the phase transition associated with electroweak symmetry breaking; $\Lambda_{QCD}^4$, where $\Lambda_{QCD}$ is the scale of Quantum Chromo Dynamics, from the chiral symmetry breaking phase transition; etc. A proper solution to the problem has to explain how all such contributions are cancelled to absurdly high precision.

An intriguing solution has been proposed \cite{ref2,ref3} in which spacetime is five dimensional, but the observable universe is constrained to a four-dimensional hypersurface, a “3-brane.” The authors exhibit solutions to the field equations which give a flat, Poincare invariant, brane regardless of the value of the cosmological constant. The geometry of space includes naked singularities which are four dimensional hypersurfaces on either side of the brane on which spacetime ends. The significance of and consistency of theories with these singularities remains unclear \cite{ref4}. It has been suggested \cite{ref5} that the singularities hide a fine tuning equivalent to that required to set the cosmological constant to zero. In Ref. \cite{ref6} the gravitational action is modified by including a Gauss-Bonnet term, of second order in the curvature tensor, and it is found that the singularity can be smoothed out but only at the price of a fine tuning. It has also been shown \cite{ref7} that in some cases the field equations admit solutions that correspond to an Einstein-de Sitter universe on the brane. It has been suggested that this type of models may be derived from string theory \cite{ref8}.

The solution is however incomplete. There is no matter in the toy model of \cite{ref2,ref3}. It is necessary to incorporate matter if the solution is to be relevant to cosmology. This is a non-trivial issue: the standard paradigm, namely the big-bang cosmology based on a Friedman-Robertson-Walker (FRW) metric, is observationally very successful, so one should aim at reproducing, or at least approximating this paradigm once matter is added.

We investigate inclusion of matter into these models. We discover new, exact solutions to the brane models with matter. They are static and therefore incompatible with observa-
tional cosmology. Since the solutions are not unique, it is possible that other solutions exist which appropriately describe the expansion of the universe. Therefore we look for time dependent solutions by linearizing the field equations about our new solutions. As will be seen, generically the small perturbations correspond to propagating modes or to non-expanding universes (static solutions). Perturbations about special, non-generic backgrounds yield expanding universes that seem, however, inconsistent with observational cosmology.

The paper is organized as follows. In section II we briefly review the basic equations and the models of [2,3]. In section III we present our new solutions with non-zero matter on the brane. In section IV we perform a small perturbation analysis about these new solutions. We present our conclusions in section V.

The cosmology of brane models has been investigated in a number of papers. A general formulation was given in Ref. [9]. The work in Refs. [10,11] is concerned with the cosmology of brane models of the Randall-Sundrum type [12]. In addition, Randall-Sundrum models to which scalars are added have been of interest [13]. A method to generate solutions to the non-linear field equations in classes of Randall-Sundrum models with scalars was given in Ref. [15]. However, there has been little, and only very recent, work on the cosmology of automatically Poincare invariant branes [16].

II. PRELIMINARIES

We denote the coordinates of spacetime by $x^A$, $A = 0, \ldots, 4$, and often use $t = x^0$ and $y = x^4$. The 3-brane is located at $y = 0$. The class of spherically symmetric metrics we study is parameterized by three functions of $t$ and $y$ only [3]

$$ds^2 = G_{AB}dx^A dx^B = n^2(t, y)dt^2 - a^2(t, y)dx^2 - b^2(t, y)dy^2. \quad (2.1)$$

Fixing $y = 0$ we see that the metric gives a flat FRW cosmology on the brane with scale factor $R(t') = a(t(t'), 0)$ where $dt' = n(t, 0)dt$. We will denote by $g_{\mu\nu}$, with $\mu, \nu = 0, \ldots, 3$, the induced metric on the brane.
The models of Refs. [2,3] include a scalar field $\phi$. The action is

$$ S = \int d^5x \sqrt{G} \left[ -R + \frac{4}{3}(\nabla \phi)^2 - \Lambda \bar{e}^{\phi} \right] + \int d^4x \sqrt{-g} \left[ -V e^{\bar{b} \phi} \right]. \quad (2.2) $$

We have adopted the notation of Ref. [3], save for the constants $\bar{a}$ and $\bar{b}$ which we have adorned with a bar to distinguish them from the metric components $a^2(t, y)$ and $b^2(t, y)$. $R$ denotes the Ricci scalar.

The peculiar normalization of the scalar field is adopted from string theory: when $\phi$ is a string theory dilaton its couplings are fixed. In particular, in this normalization, $\bar{b} = 2/3$ at lowest order. We are not interested solely in this particular set of string theory inspired parameters, so we keep the values unspecified.

The constants $\Lambda$ and $V$ represent the cosmological constant in the bulk (5-dimensional space) and on the brane, respectively. For our analysis we set $\Lambda = 0$. As seen in Ref. [2] this simplifies the analysis without compromising the essential features of the model. Moreover, one could imagine that if the model is embedded in a supersymmetric setting, $\Lambda$ could naturally vanish. The cosmological constant problem is associated with standard model fields’ contributions to $V$, $V \sim M_P^4 + M_W^4 + \Lambda_{\text{QCD}}^4 + \cdots$.

Einstein’s equations are

$$ R^{AB} - \frac{1}{2} G^{AB} R = \kappa^2 T^{AB}. \quad (2.3) $$

Here $R^{AB}$ and $R$ are the Ricci tensor and scalar. The gravitational constant is $\kappa^2$ and from now on we work in units of $\kappa^2 = 1$. $T^{AB}$ is the stress-energy tensor, which has two components:

$$ T^{AB} = \tilde{T}^{AB} + \frac{S^{AB}}{\bar{b}} \delta(y), \quad (2.4) $$

where $\tilde{T}^{AB}$ is derived as usual by varying the action with respect to the metric, and $S^{AB}$ is a contribution from a perfect fluid of density $\rho$ and pressure $p$ on the brane,

$$ S^A_B = \text{diag} (\rho, -p, -p, -p, 0). \quad (2.5) $$

Alternatively one may take the matter fluid to couple to the scalar, thus
These two ways of writing \( S_B^A \) correspond to distinct physical models. However the distinction turns out to be irrelevant for the exact solutions that we present in Sec. III. The field equation for the scalar \( \phi \) is

\[
\frac{8}{3} \nabla^2 \phi - \frac{\sqrt{-g}}{\sqrt{G}} \delta(y) \ddot{b} V e^{\dot{b} \phi} = 0. \tag{2.7}
\]

For the particular metric (2.1) Einstein’s equations and the \( \phi \) field equation are

\[
3 \left[ \left( \frac{\dot{a}}{a} \right)^2 + \frac{\dot{a} \dot{b}}{a b} \right] + \frac{n^2}{b^2} \left( \frac{a''}{a} - \left( \frac{\dot{a}}{a} \right)^2 + \frac{a' b'}{a b} \right) = \frac{2}{3} n^2 \left( \frac{\dot{\phi}^2}{n^2} + \frac{\phi'^2}{b^2} \right) + \delta(y) \frac{n^2}{b} \left( \frac{1}{2} V e^{\dot{b} \phi} \right), \tag{2.8}
\]

\[
3 \left( \frac{\dot{a} n'}{a n} + \frac{a' \dot{b}}{a b} - \frac{\dot{a}'}{a} \right) = \frac{4}{3} \phi \phi', \tag{2.9}
\]

\[
3 \left( \frac{\dot{a}}{a} \right)^2 \left( \frac{\dot{a}}{a} - \frac{\ddot{a}}{a} + 2 \frac{\dot{a} \dot{n}}{a n} - 2 \frac{\dot{a} b}{a b} - \frac{\ddot{b}}{b} + \frac{\dot{b} \dot{n}}{n b} \right) + \frac{a^2}{b^2} \left( \frac{\dot{a}'}{a'} + 2 \frac{a'' n'}{a n} - 2 \frac{a' \dot{b}'}{a b} + \frac{a n'}{n b} \right) = \frac{2}{3} a^2 \left( \frac{\dot{\phi}^2}{n^2} - \frac{\phi'^2}{b^2} \right) - \delta(y) \frac{a^2}{b} \left( \frac{1}{2} V e^{\dot{b} \phi} - p \right), \tag{2.10}
\]

\[
3 \left[ \frac{\dot{b}^2}{b^2} \left( \frac{\ddot{a}}{a} - \frac{\dot{a}}{a} + \frac{\ddot{n}}{n} \right) + \left( \frac{a'}{a} \right)^2 + \frac{a' n'}{a n} \right] = \frac{2}{3} b^2 \left( \frac{\dot{\phi}^2}{n^2} + \frac{\phi'^2}{b^2} \right), \tag{2.11}
\]

\[
8 \frac{1}{3} b^2 \left[ \frac{\dot{\phi}^2}{n^2} - \frac{\dot{\phi}''}{n^2} \right] - \phi' \left( 3 \frac{a'}{a} - \frac{b'}{b} + \frac{n'}{n} \right) + \frac{\dot{b}^2}{b^2} \left( \frac{3 \dot{a}}{a} + \frac{\dot{b}}{b} - \frac{n}{n} \right) = -\delta(y) \frac{1}{b} V \ddot{b} e^{\dot{b} \phi}. \tag{2.12}
\]

Here a dot is a shorthand for \( \partial / \partial t \) and a prime for \( \partial / \partial y \). The first four equations correspond to the 00, 04, 11 and 44 components of Einstein’s equations.

Conservation of the stress-energy tensor would be automatic were it derived from a local action. However, since a fluid component has been added on the brane, the equation \( T^{AB} \mid_B = 0 \) contains additional non-trivial information. Conservation of energy, \( T^{0B} \mid_B = 0 \), gives

\[
8 \frac{1}{3} b^2 \left[ \frac{\dot{\phi}^2}{n^2} - \frac{\dot{\phi}''}{n^2} \right] - \phi' \left( 3 \frac{a'}{a} - \frac{b'}{b} + \frac{n'}{n} \right) + \frac{\dot{b}^2}{b^2} \left( \frac{3 \dot{a}}{a} + \frac{\dot{b}}{b} - \frac{n}{n} \right) = -\delta(y) \frac{1}{b} V \ddot{b} e^{\dot{b} \phi} + 2 \left( \dot{\rho} + 3 \frac{\dot{a}}{a} (\rho + p) \right) \tag{2.13}
\]
while $T_{4B:B} = 0$ yields

$$
\frac{8}{3} \frac{1}{b^2} \phi' \left[ \left( \frac{\phi' b^2}{n^2} - \phi'' \right) - \phi' \left( \frac{3\phi'}{a} - \frac{b'}{b} + \frac{n'}{n} \right) + \phi' \left( \frac{3\phi'}{a} + \frac{b'}{b} - \frac{n'}{n} \right) \right] = \delta(y) \frac{1}{b} \left( \frac{n'}{n} (Ve^{b\phi} + 2\rho) + \frac{\rho'}{a} (Ve^{b\phi} - 2\rho) \right).$$

(2.14)

Using the field equation for the scalar, Eq. (2.12), in the conservation of energy equation gives

$$
\dot{\rho} + 3 \frac{\dot{a}}{a} (\rho + p) = 0
$$

(2.15)
on the brane. The second conservation equation, $T_{4B:B} = 0$, is always satisfied as a consequence of the $\phi$ equation of motion and Einstein’s equations. Only the brane part of $T_{4B:B} = 0$ does not follow immediately from Eq. (2.12). The identity for $y = 0$ can be confirmed by taking the $y$ derivative of Eq. (2.11) and then using Eqs. (2.8) and (2.10).

It is stated in Ref. [9] that for brane geometries of the form given by Eq. (2.1) the equation of conservation of transverse momentum, $T_{4A:A} = 0$, reduces, on the brane, to

$$
\frac{n'}{n} \rho = \frac{3}{a} \rho
$$

(2.16)

It is understood here that when discontinuous quantities are evaluated on the brane, like $n'$ and $a'$, they are given by their average, i.e.,

$$
n'(y = 0) = \frac{1}{2} (n'(y = 0+) + n'(y = 0-)).
$$

(2.17)
The equation is seldom considered any further because, for $Z_2$ symmetric brane-spaces, that is for metrics with $y \rightarrow -y$ symmetry, which are overwhelmingly most common, the averages both vanish separately, $a' = 0 = n'$, and the equation is trivially satisfied.

However the solutions we consider are not $Z_2$ symmetric. This is also true of the solutions in Refs. [4] but there Eq. (2.16) is still trivially satisfied since the matter density and pressure both vanish. It is easy to see that our solutions below, Eqs. (3.8)–(3.10), do not satisfy Eq. (2.16). The reason is, in fact, that Eq. (2.16) does not apply to the case in which there are bulk scalars. The conservation of $y$-momentum is automatically satisfied in the bulk because
the action is translation invariant in the bulk. So at issue here is only the conservation equation on the brane. Retaining only the bulk terms involving second derivatives, the conservation Eq. \((2.14)\) gives

\[
- \frac{4}{3} \phi' \phi'' + b \delta(y) \left[ -\frac{n'}{n} \rho + 3 \frac{a'}{a} p - \frac{1}{2} V e^{b \phi} \left( \frac{n'}{n} + 3 \frac{a'}{a} \right) \right] = 0. \tag{2.18}
\]

Following Ref. \[9\] we interpret the discontinuous derivatives on the brane as averages, and this will remain implicit in what follows. Integrating gives a jump equation,

\[
- \frac{2}{3b} \Delta (\phi')^2 = \frac{n'}{n} \rho - 3 \frac{a'}{a} p + \frac{1}{2} V e^{b \phi} \left( \frac{n'}{n} + 3 \frac{a'}{a} \right) \tag{2.19}
\]

We have verified that our solutions, Eqs. \((3.8)\)–\((3.10)\), and also the perturbations, Eqs. \((4.7)\)–\((4.9)\) and \((4.11)\)–\((4.13)\) satisfy this equation. In fact one can prove this without reference to the explicit form of the solution.

In Ref. \[17\] it is advocated that the correct form of the conservation of transverse momentum equation is

\[
T^{4\mu}_{\mu} = 0, \tag{2.20}
\]

where the index \(\mu\) runs from 0 to 3 only. However, the first term in Eq. \((2.18)\), involving the all important second derivative term \(\phi'' \phi'\), arises from the derivative \(T^{44}A,4\), which is ommitted from Eq. \((2.20)\).

Let us now describe the models studied in Refs. \[2,3\]. They take \(\rho = p = 0\). The solutions all have \(n(t, y) = a(t, y)\) and \(b(t, y) = 1\), and are static, \(\dot{a} = \dot{\phi} = 0\). For example, case I studied in Ref. \[2\] has \(\Lambda = 0\) and solutions

\[
n = a = \begin{cases} (1 - y/y_+) \gamma^+, & \text{for } y > 0 \\ (1 - y/y_-) \gamma^-, & \text{for } y < 0 \end{cases} \tag{2.21}
\]

\[
\phi = \begin{cases} \varphi_+ \log(1 - y/y_+) + c, & y > 0 \\ \varphi_- \log(1 - y/y_-) + c, & y < 0 \end{cases}. \tag{2.22}
\]

The constants \(\gamma_{\pm} = 1/4\) and \(|\varphi_{\pm}| = 3/4\) are fixed by the field equations in the bulk. Case I has, in particular, \(\varphi_{\pm} = \mp 3/4\). The constants \(y_{\pm}\) are determined by “jump” conditions, that
is, by requiring that the second derivatives of fields in the field equations correctly reproduce the $\delta$-function terms from the brane. The constant $c$ is an irrelevant constant shift of the scalar field.

III. SOLUTIONS WITH MATTER

The models of Refs. [2,3] provide a solution to the cosmological constant problem which is deficient in several ways: (1) There are naked singularities. Whether these are problematic remains an open question; see, for example, Refs. [4,5]. (2) There is a massless scalar which interacts with all matter with a universal, gravity-like coupling strength. This is ruled out [18] unless the coupling is made sufficiently weak. It can be arranged, however, by choosing the parameter $\bar{b}$ small enough. (3) It describes a static cosmology, in conflict with observation (see, however, Ref. [19]). (4) It does not include matter density (and pressure) on the brane.

Here we address the last two of these problems. One hopes the two are connected: when matter is included in the model the universe will evolve in time. Of course, not only should the universe evolve, but the rate of expansion should be adequate.

However, even after the introduction of matter the model admits static solutions. This is somewhat surprising, particularly if it is contrasted with the FRW cosmology which admits a static solution only if the matter density is precisely balanced by a cosmological term giving the Einstein universe. Since we are ultimately interested in time dependent solutions, we will look at time dependent small fluctuations about these solutions in the next section.

We look for solutions to the field equations in the bulk with the ansatz

$$a = y^\alpha,$$  \hspace{1cm} (3.1)

$$n = y^\nu,$$  \hspace{1cm} (3.2)

$$b = 1,$$  \hspace{1cm} (3.3)

$$\phi = \varphi \log y.$$  \hspace{1cm} (3.4)

The $\phi$ field equation gives
\[ 3\alpha + \nu = 1. \]  

(3.5)

All Einstein field equations give then

\[ 2\alpha^2 - \alpha + \frac{2}{9}\varphi^2 = 0. \]  

(3.6)

For definiteness we consider solutions akin to case I of Ref. [2], with

\[ \varphi = \mp 3\sqrt{\alpha/2 - \alpha^2}, \]  

(3.7)

where the upper and lower signs correspond to the regions \( y > 0 \) and \( y < 0 \), respectively.

The full solution is found by shifting \( y \) by \( y_+ (y_-) \) on \( y > 0 (y < 0) \), and pasting these using the jump equations. We have

\[ a = A \left( 1 - \frac{y}{y_\pm} \right)^{\alpha_\pm}, \]  

(3.8)

\[ n = N \left( 1 - \frac{y}{y_\pm} \right)^{\nu_\pm}, \]  

(3.9)

\[ \phi = \varphi_\pm \log (1 - y/y_\pm) + c. \]  

(3.10)

Here \( A \) and \( N \) are arbitrary constants that can be set to unity by a coordinate rescaling.

The jump equations are

\[ \frac{\Delta a'}{a} = -\frac{b}{3} \left( \frac{1}{2}V e^{b\phi} + \rho \right), \]  

(3.11)

\[ 2\frac{\Delta a'}{a} + \frac{\Delta n'}{n} = b \left( -\frac{1}{2}V e^{b\phi} + p \right), \]  

(3.12)

\[ \Delta \phi' = \frac{3}{8} b\bar{b}Ve^{b\phi}, \]  

(3.13)

where \( \Delta a' = a' (y = 0^+) - a' (y = 0^-) \), etc. These give three equations for five unknowns,

\[ \frac{\alpha_+}{y_+} - \frac{\alpha_-}{y_-} = \frac{1}{3} \left( \frac{1}{2}V e^{b\phi} + \rho \right), \]  

(3.14)

\[ \frac{1}{y_+} - \frac{1}{y_-} = -p + \frac{1}{3}\rho + \frac{2}{3}Ve^{b\phi}, \]  

(3.15)

\[ \frac{\sqrt{\alpha_+/2 - \alpha_+^2}}{y_+} + \frac{\sqrt{\alpha_-/2 - \alpha_-^2}}{y_-} = \frac{1}{8}bVe^{b\phi}. \]  

(3.16)

These can always be solved for three unknowns (say \( \alpha_+ \) and \( y_+ \)) in terms of \( \rho, p \) and two other unknowns (say \( \alpha_- \) and \( c \)). The reason not all unknowns are determined is twofold. First,
gauge (diffeomorphism) invariance allows us to make unphysical changes to our solutions. This will be explained below in detail, but for now it suffices to know that one may fix the gauge freedom by setting, say, \( c = 0 \). And secondly, even for vanishing \( \rho \) and \( p \) one can find a class of solutions parameterized by one parameter. These are new solutions to the model considered in [2], which could not be discovered there because it was assumed that \( n = a \). Indeed, imposing this one has \( \nu = \alpha \) which together with Eqs. (3.5) and (3.7) imply

\[
\alpha = \nu = \frac{1}{4} \tag{3.17}
\]

and

\[
\varphi = \pm \frac{3}{4} \tag{3.18}
\]

as found in case I of [4].

We have searched for time dependent solutions. We found one special solution, valid only for \( \rho = 0 \) and provided \( \bar{b} = 2/3 \). It is given by

\[
\begin{align*}
    ds^2 &= (1 - y/y_\pm) dt^2 - (1 - y/y_\pm)^2 dx^2 - t^2 dy^2, \\
    \phi &= -\frac{3}{2} \ln[t(1 - y/y_\pm)] + c, \\
    \frac{1}{y_+} - \frac{1}{y_-} &= \frac{1}{6} V e^{2c/3}.
\end{align*} \tag{3.19}
\]

Perplexingly, the special value \( \bar{b} = 2/3 \) corresponds to the string theory value of this parameter. Other non-static solutions were given in Ref. [7]. In this paper we will not consider these type of non-static matter-free solutions further.

**IV. SMALL PERTURBATIONS ABOUT LARGE MATTER DENSITY**

Armed with the new solutions with static matter density, we proceed to investigate the time dependence of small matter perturbations. Let us denote the static solution of the previous section by \( n_0, a_0, b_0, \phi_0, \rho_0 \) and \( p_0 \). We look for solutions to the field equations, Eqs. (2.8)–(2.12), of the form
\[ n = n_0(1 + \delta n), \]
\[ a = a_0(1 + \delta a), \]
\[ b = b_0(1 + \delta b), \]
\[ \phi = \phi_0 + \delta \phi, \]
\[ \rho = \rho_0 + \delta \rho, \]
\[ p = p_0 + \delta p. \] (4.1)

We count orders of the perturbative expansion parametrically in \( \delta \rho \) and \( \delta p \). That is, we re-scale \( \delta \rho \rightarrow \epsilon \delta \rho \), count powers of \( \epsilon \) and set \( \epsilon = 1 \) at the end of the calculation. In particular this implies that we make no assumption as to the relative importance of temporal or spatial derivatives \([11]\).

To derive the linearized equations in the bulk, we use again this parameterization and the explicit form of the zeroth order solutions, Eq. (3.1). The 00, 04, 11 and 44 components of Einstein’s equations and the \( \phi \) field equation give

\[ \delta a'' + 4\frac{\alpha}{y} \delta a' - \frac{\alpha}{y} \delta b' + \frac{4 \varphi}{9 y} \delta \phi' = 0, \] (4.2)

\[ \frac{\partial}{\partial t} \left( \nu \frac{\delta a}{y} + \frac{\alpha}{y} \frac{\delta b}{y} - \frac{\delta a}{y} - \frac{\delta a'}{y} - 4 \frac{\varphi}{9 y} \delta \phi \right) = 0, \] (4.3)

\[ 2 \delta a'' + 2 \frac{\alpha + \nu}{y} \delta a' + \frac{\alpha}{y} \delta n' + \delta n'' - \frac{2 \alpha + \nu}{y} \delta b' + \frac{4 \varphi}{3 y} \delta \phi' = \frac{1}{n_0^2} (2 \delta \ddot{a} + \dot{\delta} b), \] (4.4)

\[ \frac{2 \alpha + \nu}{y} \delta a' + \frac{\alpha}{y} \delta n' - \frac{4 \varphi}{9 y} \delta \phi' = \frac{1}{n_0^2} \delta \ddot{a}, \] (4.5)

\[ \frac{1}{y} \frac{\partial}{\partial y} [y \delta \phi' + \varphi (3 \delta a + \delta n - \delta b)] = \frac{1}{n_0^2} \delta \ddot{\phi}. \] (4.6)

The solution to these equations gives \( \delta \phi, \delta b \) and \( \delta n' \) in terms of \( \delta a \):

\[ \delta \phi = \frac{1}{\alpha} (\varphi \delta a - F), \] (4.7)

\[ \delta b = \frac{1}{\alpha} \left[ (y \delta a)' - \frac{4 \varphi}{9 \alpha} F - \xi \right], \] (4.8)

\[ \delta n' = \frac{1}{\alpha} \left[ \frac{y}{n_0^2} \delta \ddot{a} - \frac{\partial}{\partial y} \left( (3 \alpha - 1) \delta a + \frac{4 \varphi}{9 \alpha} F \right) \right]. \] (4.9)

Here \( \xi \) is an arbitrary constant and \( F \) is a function of \( t \) and \( y \) satisfying
\[
\frac{1}{y} \frac{\partial}{\partial y} \left( y \frac{\partial F}{\partial y} \right) - \frac{1}{n_0^2} \frac{\partial^2 F}{\partial t^2} = 0. \tag{4.10}
\]

We connect the bulk solutions for \( y > 0 \) and \( y < 0 \) demanding continuity of the fields at the brane, \( y = 0 \), and using the jump equations (3.11) for the discontinuous derivatives at \( y = 0 \). The latter give jump conditions for the perturbations:

\[
\begin{align*}
\Delta \delta a' &= -\frac{1}{6} V e^{\phi_0}(\delta b + \bar{b} \delta \phi) - \frac{1}{3} \delta \rho - \frac{1}{3} \rho_0 \delta b, \\
\Delta \delta n' &= -\frac{1}{6} V e^{\phi_0}(\delta b + \bar{b} \delta \phi) + \delta p + \frac{2}{3} \delta \rho + (p_0 + \frac{2}{3} \rho_0) \delta b, \\
\Delta \delta \phi' &= \frac{3}{8} \bar{b} V e^{\phi_0}(\delta b + \bar{b} \delta \phi). 
\end{align*} \tag{4.11-4.13}
\]

It must be observed that both \( \xi \) and \( F \) can have different values on either side of the brane and, moreover, that \( F \) may be discontinuous at \( y = 0 \). In addition, conservation of energy gives, on the brane,

\[
\delta \rho + 3(\rho_0 + p_0) \delta a = \delta \rho_0, \tag{4.14}
\]

where \( \delta \rho_0 \) is a constant of integration.

There is a gauge freedom, that is, reparameterization invariance consistent with the form of our metric. Starting from the metric

\[
ds^2 = n^2(t', y') dt'^2 - a^2(t', y') dx^2 - b^2(t', y') dy'^2, \tag{4.15}
\]

we look for infinitesimal transformations

\[
\begin{align*}
t' &= t + T(t, y) \tag{4.16} \\
y' &= y + Y(t, y) \tag{4.17}
\end{align*}
\]

that leave the form of the metric invariant. Here \( T \) and \( Y \) are infinitesimal. The only constraint on these functions comes from the absence of off-diagonal terms in the metric:

\[
n^2 T' - b^2 \dot{Y} = 0. \tag{4.18}
\]

Under the gauge transformation the metric variations are
\[ \delta n = \frac{n'_0}{n_0} Y + \dot{T}, \quad (4.19) \]
\[ \delta a = \frac{a'_0}{a_0} Y, \quad (4.20) \]
\[ \delta b = Y'', \quad (4.21) \]
\[ \delta \phi = \phi'_0 Y. \quad (4.22) \]

For simplicity we have indicated the variation about a static solution with \( b_0 = 1 \) and \( a_0 = \dot{n}_0 = \dot{\phi}_0 = 0 \). It is instructive to check that our solutions of the field equations for the perturbations are invariant under these transformations, that is, that the perturbations (4.19)-(4.22) satisfy the field equations automatically. Indeed, the solution for \( \delta \phi \), Eq. (4.7), is satisfied provided one takes \( F = 0 \). Then, the solution for \( \delta b \), Eq. (4.8), requires \( \xi = 0 \). Finally the solution for \( \delta n' \), Eq. (4.9), is satisfied provided
\[ \dot{T}' - \frac{1}{n_0^2} \ddot{Y} = 0, \]
which is a consequence of the condition (4.18).

One may fix the gauge by imposing, for example,
\[ \delta b(y, t) = 0 \quad \text{and} \quad \delta \phi(y = 0, t) = 0. \quad (4.23) \]

There is some residual gauge freedom: there are further transformations with \( Y = 0 \) and \( T = T(t) \). These are uninteresting time reparameterizations. Were we to impose a gauge condition on any of the fields that have a discontinuous derivative on the brane, we could only require that it vanishes for either \( y > 0 \) or \( y < 0 \), but not both, since the reparameterization functions are smooth.

We are ready to present our solution to the linearized field equations. The solutions to the bulk equations express \( \delta \phi, \delta b \) and \( \delta n' \) in terms of \( \delta a \) and \( F \). Once \( F \) is determined in terms of \( \delta a \) our task is to determine \( \delta a \) only. We will proceed as follows. First we use the gauge conditions and our solutions in the bulk, Eqs. (4.7)-(4.9), to eliminate the function \( F \) everywhere, and to relate \( \delta a' \) to \( \delta a \) on the brane. Then we consider the jump equations which impose further restrictions on \( \delta a \) and its derivatives on the brane. These constraints
on the brane are used, finally, to determine $\delta a$ in the bulk. The cosmlogy depends only on
the fields on the brane. Therefore, we concentrate on determining as fully as possible the
fields on the brane.

We eliminate the function $F$ by fixing the gauge as in Eq. (4.23) and using our solution
of the field equations in the bulk, Eq. (4.8), thus

\[ F = \frac{9\alpha}{4\varphi} [(y\delta a)' - \xi]. \]  

(4.24)

We can now fix $\delta a'$ on the brane in terms of $\delta a$. Consider the constraints from requiring
continuity of $\delta \phi$ at $y = 0$. We have in addition the gauge choice in Eq. (4.23), $\delta \phi(y = 0, t) = 0$, so we obtain

\[ (4\alpha_+ - 1)\delta a = (y_+ \delta a_+' + \xi_+), \]

(4.25)

\[ (4\alpha_- - 1)\delta a = (y_- \delta a_- + \xi_-). \]

(4.26)

Here and below we denote the limiting values of fields as $y \to 0^\pm$ with a corresponding
subscript, eg, $\delta a'_\pm \equiv \lim_{y \to 0^\pm} \delta a'$.

The first jump equation, Eq. (4.11), gives a constraint between the constants of integration. From the jump equations for the exact solution, Eqs. (3.14) and (3.15), one has

\[ \rho_0 + p_0 = \frac{4\alpha_+ - 1}{y_+} + \frac{4\alpha_- - 1}{y_-}. \]  

(4.27)

Using this and Eqs. (4.25) and (4.26) in Eq. (4.11) we obtain

\[ \frac{1}{3} \delta \rho_0 = \frac{\xi_+}{y_+} + \frac{\xi_-}{y_-}. \]

(4.28)

The remaining two jump equations, (4.12) and (4.13), involve $\delta a''_\pm$ in addition to $\delta a'_\pm, \delta \bar{a}$
and $\delta a$. They can be solved to give, on the brane, $\delta a''_\pm$ in terms of $\delta \bar{a}$ and $\delta a$. However,
the resulting expressions are long and not terribly illuminating so we refrain from presenting
them here.

The jump equations do not fix the time behavior of $\delta a$ on the brane. We now describe
a procedure that determines the time behavior and the bulk dependence of $\delta a$. Note that
since $F$ is given in terms of $\delta a$ through Eq. (4.24), one must now impose that $\delta a$ satisfy the wave-like equation (4.10). The solution to Eq. (4.10) is straightforward,

$$F_{\pm}(y, t) = \int d\omega e^{i\omega t} \tilde{F}_{\pm}(\omega) \frac{J_0\left(\frac{\omega}{3a_{\pm}}|y - y_{\pm}|^{3a_{\pm}}\right)}{J_0\left(\frac{\omega}{3a_{\pm}}|y_{\pm}|^{3a_{\pm}}\right)},$$

(4.29)

where $J_0$ is a Bessel function. There is a second solution involving the Neuman function. However, we have dismissed it since it is arbitrarily large as $|y - y_{\pm}| \to 0$, which is outside the validity of perturbation theory. The Fourier coefficient $\tilde{F}_{\pm}(\omega)$ is determined by the function on the brane, $F(0, t)$. Using the conditions on derivatives of $\delta a$ on the brane one may express this solely in terms of $\delta a$:

$$\frac{9}{2} a_{\pm}(1 - 2a_{\pm}) \delta a(0, t) = \int d\omega e^{i\omega t} \tilde{F}_{\pm}(\omega).$$

(4.30)

Thus, knowledge of $\delta a$ on the brane determines all fields everywhere. The time-dependence of $\delta a$, however, is not arbitrary. Consistency of Eq. (4.29) and the jump equations gives an integro-differential equation for $\delta a$ on the brane. To see this, differentiate Eq. (4.29) and evaluate at $y = 0$:

$$- y_{\pm} \delta a''_{\pm} + 2\delta a'_{\pm} = \int d\omega e^{i\omega t}(4a_{\pm} - 2)\delta \tilde{a}(\omega)\omega|y_{\pm}|^{3a_{\pm}-1} \frac{J_1\left(\frac{\omega}{3a_{\pm}}|y_{\pm}|^{3a_{\pm}}\right)}{J_0\left(\frac{\omega}{3a_{\pm}}|y_{\pm}|^{3a_{\pm}}\right)},$$

(4.31)

where $(1 - 2a_{\pm})\delta \tilde{a}(\omega) = 2\varphi_{\pm}/9a_{\pm}\tilde{F}_{\pm}(\omega)$. Now the left hand side of this equation can be expressed in terms of $\delta \tilde{a}$ and $\delta a$ through the jump equations. One obtains

$$- y_{\pm} \delta a''_{\pm} + 2\delta a'_{\pm} = A_{\pm} \delta \tilde{a} + B_{\pm} \delta a + C_{\pm},$$

(4.32)

where $A_{\pm}$ and $B_{\pm}$ are constants given in terms of the parameters of the background solution, and $C_{\pm}$ also depends on these parameters but is, in addition, linear in the small constants $\xi_{\pm}$. We have omitted the complicated and long expressions for these. Thus, we obtain an equation that determines the time dependence of $\delta a$

$$A_{\pm} \delta \tilde{a} + B_{\pm} \delta a + C_{\pm} = \int d\omega e^{i\omega t}(4a_{\pm} - 2)\delta \tilde{a}(\omega)\omega|y_{\pm}|^{3a_{\pm}-1} \frac{J_1\left(\frac{\omega}{3a_{\pm}}|y_{\pm}|^{3a_{\pm}}\right)}{J_0\left(\frac{\omega}{3a_{\pm}}|y_{\pm}|^{3a_{\pm}}\right)}.$$ 

(4.33)

A particular solution to this equation is easily found by Fourier transform:
\[
\delta a(0, t) = -C_\pm/B_\pm. \tag{4.34}
\]

To this solution one may add arbitrary linear combinations of solutions to the associated homogeneous equation (obtained by setting \(C_\pm = 0\)). These solutions have time dependence of the form of simple exponentials, \(\exp i\omega_0 t\), with characteristic frequencies \(\omega_0\) that are solutions to

\[
-\omega_0^2A_\pm + B_\pm - (2 - 4\alpha_\pm)\omega_0|y_\pm|^{3\alpha_\pm - 1} \frac{J_1\left(\frac{\omega_0}{3\alpha_\pm}|y_\pm|^{3\alpha_\pm}\right)}{J_0\left(\frac{\omega_0}{3\alpha_\pm}|y_\pm|^{3\alpha_\pm}\right)} = 0. \tag{4.35}
\]

There is an infinite number of solutions to this equation. The function \(J_0\) has an infinite number of simple zeros. Between two successive zeros the last term takes on any value. There may be additional complex solutions when \(A_\pm B_\pm < 0\).

The physical interpretation of our solution is straightforward. The infinite set of oscillatory modes simply correspond to field excitations in the bulk. Neither their amplitudes nor their frequencies depend on the additional density perturbation \(\delta \rho_0\). This is to be expected because even in the absence of additional matter there can be propagating gravitational and scalar field waves. It is only the particular solution that actually depends on the added matter. Ignoring the possible excitation of propagating modes, we see that the new solution is static.

There are some caveats. This conclusion does not hold if the parameters of the background are such that \(B_\pm = 0\) or if there are complex solutions \(\omega_0\). In the former case, if in addition \(C_\pm = 0\), there are further solutions of the form

\[
\delta a \propto t. \tag{4.36}
\]

This is of particular interest because the perturbations may now be time dependent. In fact, such time dependent perturbations are expected since they must correspond to the linearization of the exact solutions of Ref. [1] for which \(a \propto \exp(f(y) + \sqrt{\Lambda}t) \simeq 1 + f(y) + \sqrt{\Lambda}t\), where \(\Lambda\) is an arbitrary constant. We have verified that \(B_\pm = 0\) and \(A_\pm \neq 0\) for parameters corresponding to the background of Ref. [1] \((\alpha_\pm = 1/4, \varphi_\pm = \mp 3/4, \rho_0 = \delta \rho_0 = 0\). There are also solutions of the form \(\delta a \propto t^2\). If there are modes of complex \(\omega_0\), the solutions are not
simply oscillatory, but will in addition have exponential dependence. None of these solutions are suitable for observational cosmology.

V. CONCLUSIONS

The proposed solution to the cosmological constant problem of Refs. [2,3] is incomplete in that it does not include matter on the brane, i.e., in our universe. We have found new exact solutions including matter on the brane. These solutions are static and therefore describe an unacceptable cosmology.

However, the equations admit other solutions even under the same assumptions on the symmetry of the metric. This had been recognized in Ref. [7] which found curved brane solutions to the model dubbed case I in Ref. [4]. Here we have studied new solutions obtained as small perturbations about our new static solutions with matter. The small perturbations that are not simply propagating waves are generally static. In special cases the non-propagating, small perturbations grow linearly with time, but we have identified these as the De-Sitter-like solutions of Ref. [7]. On this basis it is tempting to rule out these as viable cosmologies.

In standard FRW cosmology the evolution of the scale factor is completely determined once the equation of state is fixed. Once spherical symmetry is chosen and matter is specified, Einstein’s equations determine the cosmology. However, this is not the case in the peculiar brane cosmologies of Refs. [2,3]. To understand what is happening one could consider this as an initial value problem. In standard cosmology if the metric and matter content were specified at an initial time (in a fixed gauge), one could evolve forward using the field equations. Thus one would recover the standard picture. Clearly this is not the case of the brane models. What else must be specified and why? This is an interesting question that we hope to explore further. Our guess is that the naked singularities introduce additional information that has been implicitly specified. In the absence of a new general principle that specifies these additional data one would have to give up the notion of causality (at least
on a global scale). This may be the price one must pay in order to solve the cosmological constant problem.

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