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Pseudogradient Flows of Geometric Energies

Abstract: For differentiable functions on Riemannian manifolds, the gradient vector field and its induced gradient flow are well-defined and well-understood concepts. Unfortunately, many nonquadratic, infinite-dimensional optimization problems cannot be formulated on Riemannian manifolds in a convenient way. We introduce a category of infinite-dimensional Banach manifolds that allows us to define a generalized gradient. Moreover, we show short-time existence of the induced flows and apply their discretizations to the numerical minimization of various geometric energies of immersed curves and surfaces.

Keywords: gradient flow, minimal surfaces, Euler elastica, Willmore energy

MSC: 49Q05, 49Q10

4.1 Introduction

Gradient flows are a popular tool for minimizing (or maximizing) continuously differentiable functions, both theoretically and numerically. Algorithms derived from gradient flows, often subsumed as methods of steepest descent, belong to virtually every library for (smooth) numerical optimization. The idea behind gradient flows is very intuitive and appealing:

Let \((M, g)\) be a Riemannian manifold, \(F: M \to \mathbb{R}\) be a function of class \(C^1\), and \(a \in M\) some point. A tangent vector \(u \in T_aM\) of length \(r > 0\) is called a direction of steepest descent of \(F\) at the point \(a \in M\) of length \(r\) if it minimizes the slope \(S(u) := R(u)^{-1}(dF|_a, u)\) among all tangent vectors of length \(r\), where \(R(u) := g(u, u)^{\frac{1}{2}}\). That means, \(u\) is a solution of the following constraint optimization problem:

\[
\begin{align*}
\text{Minimize } S(u) \text{ over } u \in T_aM \text{ subject to the constraint } R(u) = r.
\end{align*}
\]

By introducing a scalar Lagrange multiplier \(\lambda \in \mathbb{R}\), we have to solve the system

\[
\begin{cases}
\langle dS|_u, v \rangle + \lambda \langle dR|_u, v \rangle = 0 & \text{for all } v \in T_aM, \\
R(u) = r.
\end{cases}
\]

This system is equivalent to

\[
\begin{cases}
r^2 \langle dF|_a, v \rangle - \langle dF|_a, u \rangle g(u, v) + \lambda r^2 g(u, v) = 0 & \text{for all } v \in T_aM, \\
g(u, u) = r^2.
\end{cases}
\]

Testing with \(v = u\) leads to \(\lambda = 0\). Afterwards, testing with all \(v \in \ker(dF|_a)\) leads to \(u \in \ker(dF|_a)^\perp\). Thus, the system is solved by \(u = \sigma \text{ grad}(F)|_a\) with \(\sigma \in \mathbb{R}\).
\{-r |dF|_g^{-1}, r |dF|_g^{-1}\}. Here, the \textit{gradient} \(\text{grad}(F)|_a \in T_a M\) is the vector defined by the equation
\[
g(\text{grad}(F)|_a, v) := \langle dF|_a, v \rangle \quad \text{for all } v \in T_a M.
\] (4.1.2)
Moreover, \(|dF|_g\) is the Frobenius norm of the linear form \(dF|_a : T_a M \to \mathbb{R}\) with respect to the inner product \(g\). Summarized, the direction of steepest descent of length \(r\) is given by
\[
u = -r |dF|_g^{-1} \text{grad}(F)|_a.
\] Usually, one chooses \(r = |dF|_g\) and defines the \textit{direction of steepest descent} by
\[
u = -\text{grad}(F)|_a.
\]
We point out a simple but often forgotten fact: One needs a metric \(d\) on \(T_a M\) in order to be able to talk about the slope of a linear function \(\xi : T_a M \to \mathbb{R}\) in direction \(u\), the slope being given by
\[
\langle \xi, u \rangle - \langle \xi, 0 \rangle
d(u, 0).
\] In particular, the notion of steepest descent requires the specification of a metric, at least on tangent spaces; different metrics will lead to different directions of steepest descent. \textit{There is no such thing as a direction of steepest descent per se.}

A flow \(\Phi : U \times I \subset M \times \mathbb{R} \to M\) is called a \textit{downward gradient flow of} \(F\) if it is generated by the vector field \(-\text{grad}(F)\), i.e., if it satisfies
\[
\frac{d}{dt} \Phi(a, t) = -\text{grad}(F)_{\Phi(a, t)} \quad \text{and} \quad \Phi(a, 0) = a \quad \text{for all } (a, t) \in U \times I.
\] (4.1.3)
Analogously, an \textit{upward gradient flow of} \(F\) is generated by \(\text{grad}(F)\). Under mild regularity assumptions on \(F\) (e.g., \(dF\) is locally Lipschitz continuous), both the downward and upward flow exist at least locally and for short times. Moreover, each of the flows is unique.

This theory carries over to infinite-dimensional manifolds as far as only Riemannian/Hilbert manifolds are considered. Unfortunately, many interesting variational problems can only be formulated as \textit{differentiable} variational problems on Hilbert manifolds if one shrinks the feasible set to an extent that minimizers of the actual problem of interest might be excluded. Many geometric variational problems involving nonquadratic energies and nonlinear constraints belong to this class; we discuss several examples in Section 4.5.1 and Section 4.5.2. The essential reason for this phenomenon is that the Sobolev spaces \(W^{s,2}(\mathbb{R}^n; \mathbb{R})\) do not embed continuously into \(W^{1,\infty}(\mathbb{R}^n; \mathbb{R})\) for moderate \(n \geq 2\) and small \(s \in [0, 1 + \frac{n}{2}]\). Thus, a generic continuous functional on \(W^{1,\infty}(\mathbb{R}^n; \mathbb{R})\) can only be made continuous on \(W^{s,2}(\mathbb{R}^n; \mathbb{R})\) by choosing \(s\) overly large.

This is why the generalization of gradients and gradient flows to more general infinite-dimensional manifolds is a quite delicate business. As far as we know, there have been the following two main approaches so far:
1. One may generalize gradients as directions of steepest descent in the sense of (4.1.1), using $R(u) = \|u\|_{T_a M}$ and $r = \|dF|_a\|_{(T_a M)'}$. If the Banach space $T_a M$ is reflexive, this leads again to a well-defined gradient $\operatorname{grad}(F)|_a \in T_a M$. However, this gradient depends nonlinearly on $dF|_a$. In practice, this means that a costly minimization or root finding algorithm has to be used in order to determine $\operatorname{grad}(F)|_a$. Moreover, even more costly algorithms have to be applied if the norm $\|\cdot\|_{T_a M}$ is not differentiable on $T_a M \setminus \{0\}$. For more details on this kind of gradient flows and for generalizations to more general metric spaces, we refer the reader to [1].

2. In some cases, one may exploit a Gelfand triple $T_a M \hookrightarrow H_a \hookrightarrow (T_a M)'$, i.e., a sequence of continuous, linear, and dense embeddings, where $(H_a, g_a)$ is a Hilbert space. Then one may use (4.1.2) to define a “gradient” $\operatorname{grad}^H(F)|_a \in H_a$. Since $\operatorname{grad}^H(F)|_a \not\in T_a M$ is not a tangent vector, the mapping $a \mapsto \operatorname{grad}^H(F)|_a$ cannot be interpreted as a vector field so that (4.1.3) is no longer an ordinary differential equation. Under certain conditions on $dF$, (4.1.3) may still be formulated as a parabolic partial differential equation; existence and uniqueness of the “gradient flow” can be ensured for short positive times $t \in [0, \varepsilon]$. Prominent examples are the mean curvature flow (the $L^2$-gradient flow of the surface area functional, see [5]) and the Willmore flow (the $L^2$-gradient flow of the Willmore functional). An extensive overview and further literature on these flows can be found in [7].

In Section 4.4, we propose another way to define (pseudo)gradients for certain classes of Banach manifolds. In contrast to the first of the two approaches mentioned above, the pseudogradient $\operatorname{grad}(F)$ depends linearly on $dF$. In constrast to the second approach, these pseudogradients form indeed a vector field $\operatorname{grad}(F)$ and this vector field has the same regularity as $dF$. If $F$ is of class $C^{1,1}_{\text{loc}}$, then the pseudogradient flow defined by (4.1.3) exists locally and for small times, both in forward and in backward direction (see 4.4.2). Of course, one cannot expect to obtain such a strong result for arbitrary Banach manifolds and arbitrary functions. It turns out that the suitable category of manifolds to consider is the category of Riesz manifolds which we introduce in Section 4.3. We will also see there that the concept of Riesz manifolds is a bit richer than the concept of Riemannian manifolds.

We aim at treating pseudogradient flows for quite general classes of infinite-dimensional Banach manifolds—and not only for open sets in Banach spaces. Such situations arise, e.g., when treating optimization problems with nonlinear constraints. The class of examples we have primarily in mind is the minimization of knot energies among knots in parameterization by arclength. The treatment of infinite-dimensional Banach manifolds will inevitably involve certain infinite-dimensional vector bundles so that we require the notion of a Banach bundle. This is why we give a very brief introduction to fiber bundles in general and to Banach bundles in particular (see Section 4.2).

Section 4.5 is devoted to applications of the developed theory. We consider volume functionals (Section 4.5.1) and certain functionals depending on extrinsic cur-
vature (Section 4.5.2) and show that these functionals together with suitably chosen domains belong to the category of Riesz manifolds (4.5.1 and 4.5.5). As a side effect, we verify that the category of Riesz manifolds (i) is strictly larger than the category of Riemannian manifolds and (ii) contains many examples of practical relevance. Moreover, we give several numerical examples of discretized pseudogradient descent for the minimization of these energies subject to various constraints. In particular, these examples will demonstrate the efficiency and robustness of pseudogradient methods when applied to numerical minimization of geometric energies subject to nonlinear equality constraints.

4.2 Banach Bundles

Fiber bundles of various types and in particular vector bundles are typical working horses in algebraic topology, differential geometry, and global analysis. Since the reader might not be familiar with the notion of a fiber bundle, we give a brief introduction, being well aware that a thorough treatment of this matter may easily fill a full-semester master course. A beautiful and short introduction to Banach manifolds and vector bundles can be found in [13]. Moreover, we refer to [12], maybe the classical text on fiber bundles, and to [14], a rather modern exposition which also covers great parts of the theory of partial differential equations for fields with values in fiber bundles.

4.2.1 General Fiber Bundles

The central idea of a fiber bundle is that it looks locally like a product of a base space and a fiber. The following definitions make this more precise.

**Definition 4.2.1.** Let $M, E$ and $X$ be topological spaces, let $\pi: E \to M$ be a continuous, surjective map, and let $U \subset M$ be an open set. We write $E|U := \pi^{-1}(U) \subset E$. A local trivialization of $\pi: E \to M$ on $U$ with typical fiber $X$ is a homeomorphism $\varphi: E|U \to U \times X$ such that the following diagram commutes

$$
\begin{array}{c}
E|U \\
\downarrow \varphi
\end{array}
\begin{array}{c}
U \\
\downarrow \pi_{|E|U}
\end{array}
\begin{array}{c}
U \times X \\
\downarrow \mathrm{pr}
\end{array}

$$

Here, $\mathrm{pr}: U \times X \to U, (a, x) \mapsto a$ is the projection onto the first factor.

**Definition 4.2.2.** Let $M, E$ and $X$ be topological spaces and let $\pi: E \to M$ be a continuous, surjective map. A fiber bundle atlas on $\pi: E \to M$ with typical fiber $X$ is a
covering family of local trivializations \((\varphi_i: E|U_i \to U_i \times X)_{i \in I}\) in the sense that \((U_i)_{i \in I}\) is an open covering of \(M\).

Two fiber bundle atlantes with typical fiber \(X\) are called \textit{compatible} if their union is a fiber bundle atlas. A fiber bundle atlas is called \textit{maximal} if it contains each compatible fiber bundle atlas.

A \textit{fiber bundle with typical fiber} \(X\) is a continuous and surjective mapping \(\pi: E \to M\) together with a maximal fiber bundle atlas with typical fiber \(X\). With a slight abuse of notation, one usually omits mentioning the precise atlas and simply refers to \(\pi: E \to M\) or even to \(E\) as the fiber bundle. One refers to \(E\) as the \textit{total space}, to \(M\) as the \textit{base space}, and to \(\pi\) as the \textit{bundle projection} of the fiber bundle \(\pi: E \to M\). For \(a \in M\), one calls \(E_a := \pi^{-1}(\{a\})\) the \textit{fiber over} \(a\). Note that \(E_a\) is homeomorphic to \(X\).

For each pair \(i, j \in I\) with \(U_{ij} := U_i \cap U_j \neq \emptyset\), the following commutative diagram defines a homeomorphism \(\varphi_{ij}: U_{ij} \times X \to U_{ij} \times X\), the so-called \textit{transition map}:

\[
\begin{array}{c}
U_{ij} \times X \\
\downarrow \varphi_{ij} \\
E|U_{ij} \\
\downarrow \pi \\
U_{ij} \times X
\end{array}
\]

(4.2.1)

Note that one has \(\varphi_{ij} = (\varphi_{ji})^{-1}\) and \(\varphi_{ij} \circ \varphi_{jk} = \varphi_{ik}\).

Let \(R \in \{ C_{k,\alpha}^l \mid k \in \mathbb{N} \cup \{0, \infty\}, \alpha \in [0, 1] \text{ with } k + \alpha \geq 1 \}\). If \(M \times X\) is a Banach manifold of class \(R\) and if each transition \(\varphi_{ji}\) is of class \(R\), we say that \(\pi: E \to M\) is a \textit{fiber bundle of class} \(R\). Note that in this case, the total space \(E\) is a manifold of class \(R\) and \(\pi\) is a submersion of class \(R\).

\textbf{Example 4.2.3.} Let \(M\) and \(X\) be topological spaces. Then \(E := M \times X\) together with the projection \(\pi: E \to M\), \((a, f) \mapsto a\) is a fiber bundle with typical fiber \(X\). Such fiber bundles are called \textit{trivial}, since there exists a global trivialization. By definition, every fiber bundle restricted to a sufficiently small set is a trivial fiber bundle.

\textbf{Example 4.2.4.} Let \(E \subset \mathbb{R}^3\) be the embedded Möbius strip depicted in Figure 4.1. Let \(M \subset E\) be the centerline of the Möbius strip and \(\pi: E \to M\) be the shortest distance projection with respect to the intrinsic distance of \(E\). Then \(\pi: E \to M\) is a \textit{nontrivial} smooth fiber bundle with a closed interval as typical fiber.

\textbf{Example 4.2.5.} Let \(\pi_1: E_1 \to M\) and \(\pi_2: E_2 \to M\) be fiber bundles with typical fibers \(X_1\) and \(X_2\) respectively. Then the \textit{product bundle} \(\pi: E_1 \times_M E_2 \to M\) defined by

\[
E_1 \times_M E_2 := \{ (e_1, e_2) \in E_1 \times E_2 \mid \pi_1(e_1) = \pi_2(e_2) \}
\quad \text{and}
\]

\[
\pi: E_1 \times_M E_2 \to M, \quad \pi(e_1, e_2) := \pi_1(e_1) = \pi_2(e_2).
\]
is a fiber bundle with typical fiber $X_1 \times X_2$.

**Definition 4.2.6.** Let $\pi: E \to M$ be a fiber bundle with typical fiber $X$. A mapping $\sigma: M \to E$ is called a *section* of the bundle $E$ if it satisfies $\pi \circ \sigma = \text{id}_M$, i.e., $\sigma$ maps each point $a \in M$ into its fiber $E_a$. If both $\pi: E \to M$ and $\sigma: M \to E$ are of class $R$, we say that $\sigma$ is a *section of class $R$*.

**Definition 4.2.7.** Let $\pi_1: E_1 \to M_1$ and $\pi_1: E_2 \to M_2$ be fiber bundles and let $F: E_1 \to E_2$ and $f: M_1 \to M_2$ be continuous maps. We say that $F$ is a *fiber bundle morphism from $E_1$ to $E_2$ over $f$* if the following diagram is commutative:

$$
\begin{array}{ccc}
E_1 & \xrightarrow{F} & E_1 \\
\downarrow{\pi_1} & & \downarrow{\pi_2} \\
M_1 & \xrightarrow{f} & M_2
\end{array}
$$

Thus one has well-defined and continuous *fiber mappings* $F_a := F_{(E_1)_a}: (E_1)_a \to (E_2)_{f(a)}$ for each $a \in M_1$. We say that $F: E_1 \to E_2$ is an *isomorphism of fiber bundles over $f$*, if both $f$ and $F$ are homeomorphisms. In that case, the inverse $F^{-1}: E_2 \to E_1$ is a fiber bundle morphism over $f^{-1}$. 

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**Fig. 4.1:** An embedded Möbius strip as nontrivial fiber bundle over its centerline (white). Several fibers are depicted in orange.
4.2.2 Banach Bundles and Hilbert Bundles

Note that the transition maps $\varphi_{ij} = U_{ij} \times X \to U_{ij} \times X$ of a fiber bundle $\pi: E \to M$ are of the special form

$$\varphi_{ij}(a, x) = (a, \varphi_{ij}(a)(x))$$

with some transition function $\varphi_{ij}: U_{ij} \to \text{Homeo}(X)$ into the group of homeomorphisms of $X$. Moreover, one has $(\varphi_{ij}(a))^{-1} = \varphi_{ji}(a)$ and $\varphi_{ij}(a) \varphi_{jk}(a) = \varphi_{ik}(a)$.

One may define various special types of fiber bundles in the following way: Let $X$ be a space with an additional structure that is compatible with the topology of $X$. For example, $X$ can be a topological vector space, a topological group, or a metric space. Denote by $G \subset \text{Homeo}(X)$ the group of continuous isomorphisms of $X$, i.e., those homeomorphisms of $X$ that also respect the additional structure. By adding the further requirement to a bundle atlas that its transition functions take only values in this group $G$, one may define fiber bundles whose fibers exhibit the same structure as its typical fiber $X$. In the following, we outline this construction for the special case that the typical fiber $X$ is a Banach space (or Hilbert space), leading to the notion of a Banach bundle (or Hilbert bundle, respectively).

**Definition 4.2.8.** Let $X$ be a Banach space and let $\pi: E \to M$ be a fiber bundle with typical fiber $X$. A **Banach bundle atlas on** $\pi: E \to M$ with typical fiber $X$ is a covering family of local trivializations $(\varphi_i: E|U_i \to U_i \times X)_{i \in I}$ such that each transition function is a continuous map $\varphi_{ij}: U_{ij} \to L(X; X)$, where $L(X; X)$ denotes the Banach algebra of continuous linear operators from $X$ to $X$.

A fiber bundle $\pi: E \to M$ with typical fiber $X$ is called a **Banach bundle with typical fiber $X$** if its maximal bundle atlas of $\pi: E \to M$ contains a (maximal) Banach bundle atlas.

Analogously, one defines the notion of a Hilbert bundle. The definitions are such that each Hilbert bundle is also a Banach bundle and that each Banach bundle is also a fiber bundle.

**Proposition 4.2.9.** Let $\pi: E \to M$ be a Banach bundle. Then each $E_a, a \in M$ carries the structure of a topological vector space in the following way: Choose a local trivialization $\varphi: E|U \to U \times X$ from a Banach bundle atlas of $\pi$ such that $a \in U$. Then define for $\nu, \nu_1, \nu_2 \in E_a$ and $\lambda \in \mathbb{R}$:

$$\nu_1 (+)_a \nu_2 := \varphi^{-1}(a, \text{pr}_X(\varphi(\nu))) + \text{pr}_X(\varphi(\nu_2)) \quad \text{and} \quad \lambda \cdot (\cdot)_a \nu := \varphi^{-1}(a, \lambda \cdot \text{pr}_X(\varphi(\nu))).$$
By the special structure of Banach bundle atlantes (see 4.2.8), this is independent of the choice of \( \varphi \) and the induced mappings\(^{4.1}\)

\[
\begin{align*}
+ : E \times_M E &\to E, & v_1 + v_2 &:= v_1 \langle + \rangle_{\pi(v_1, v_2)} v_2 \quad \text{and} \\
\cdot : \mathbb{R} \times E &\to E, & \lambda \cdot v &:= \lambda \langle \cdot \rangle_{\pi(v)} v
\end{align*}
\]

are continuous.

**Example 4.2.10.** Let \( M \) be a Banach manifold of class \( C^{k, \alpha}_{\text{loc}} \), \( k + \alpha \geq 2 \). Then the tangent bundle \( \pi_{TM} : TM \to M \) is a Banach bundle of class \( C^{k-1, \alpha}_{\text{loc}} \). Sections of \( TM \) are precisely the vector fields on \( M \).

**Example 4.2.11.** Let \( \pi_X : X \to M \) and \( \pi_Y : Y \to M \) be Banach bundles over \( M \). Then the space \( L(X; Y) := \bigsqcup_{a \in M} \{a\} \times L(X_a; Y_a) \) together with \( \pi_{L(X; Y)} : L(X; Y) \to M \), \( (a, A) \mapsto a \) is a Banach bundle in a canonical way; a Banach bundle atlas for \( L(X; Y) \) can be constructed from bundle atlantes for \( X \) and \( Y \).

In particular, the dual bundle \( X' := L(X; M \times \mathbb{R}) \) is also a Banach bundle with fibers \( (X')_a \cong (X_a)' \) for all \( a \in M \).

**Definition 4.2.12.** Let \( \pi_1 : E_1 \to M_1 \) and \( \pi_2 : E_2 \to M_2 \) be Banach bundles, and let \( F : E_1 \to E_2 \) be a fiber bundle morphism over \( f : M_1 \to M_2 \) (see 4.2.7). We say that \( F \) is a Banach bundle morphism from \( E_1 \) to \( E_2 \) over \( f \) if for each \( a \in M_1 \) the fiber mapping \( F_a := F|_{(E_1)_a} : (E_1)_a \to (E_2)_{f(a)} \) induced by \( F \) is linear. We say that \( F : E_1 \to E_2 \) is an isomorphism of Banach bundles over \( f \), if it is also a fiber bundle isomorphism. Note that in this case, the inverse \( F^{-1} : E_2 \to E_1 \) is automatically a Banach bundle morphism over \( f^{-1} \).

### 4.3 Riesz Structures

First, we introduce Riesz structures on Banach spaces (Section 4.3.1). To some extent, Banach spaces with Riesz structures look and feel quite much like Hilbert spaces. For example, 4.3.5 introduces the concept of a pseudoadjoint \( A^* \in L(X_2; X_1) \) of a Riesz morphism \( A \in L(X_1; X_2) \) (a concept which we utilize to introduce pseudogadients in Section 4.4). The generalization of Riesz structures to Banach bundles is straightforward. Hence, a great deal of Section 4.3.2 will occupy us with producing a practically relevant example. Finally, we introduce the category of Riesz manifolds (Section 4.3.3), a category that carries many desirable properties of the category of Riemannian manifolds.

\(^{4.1}\) see 4.2.5 for the definition of the product bundle \( \pi : E \times_M E \to M \)
4.3.1 Riesz Structures

**Definition 4.3.1.** Let $X$ be a Banach space. A *Riesz structure* on $X$ consists of
1. a continuous, linear and dense injection $i : X \hookrightarrow H$ into a Hilbert space $H$;
2. a continuous, linear and dense injection $j : H \hookrightarrow Y$ into a Banach space $Y$;
3. a fixed Hilbert norm $\| \cdot \|_H$ or equivalently, its Riesz isomorphism $I : H \to H'$;
4. and of an isomorphism of Banach spaces $J : X \to Y'$ such that the following diagram commutes:

$$
\begin{array}{ccc}
X & \xrightarrow{J} & Y' \\
\downarrow{i} & & \downarrow{j} \\
H & \xrightarrow{I} & H'
\end{array}
$$

We may call $J$ a *pseudo-Riesz isomorphism* of $X$. Moreover, every Banach space $X$ with Riesz structure $(i, j, I, J)$ exhibits a pre-Hilbert metric, which can be expressed by the Riesz structure in the following ways

$$b(x_1, x_2) := \langle i x_1, i x_2 \rangle_H = \langle I i x_1, i x_2 \rangle = \langle J x_1, j i x_2 \rangle, \quad \text{for all } x_1, x_2 \in X.$$

**Remark 1.** At first glance, the notion of a Riesz structure seems to boil down to the notion of a Gelfand triple, i.e., a topological vector space $X$, a Hilbert space $H$ together with linear, dense embeddings $X \hookrightarrow H \hookrightarrow X'$. However, this is not true, since Riesz structures involve a third Banach space $Y$ which need not coincide with $X'$, in particular if $X$ is nonreflexive.

The prototypical Riesz structure is given by the following example.

**Example 4.3.2.** Let $(\Omega, \mathcal{A}, \mu)$ be a finite measure space, let $p \in [2, \infty]$, and let $q \in [1, 2]$ be the Hölder conjugate of $p$. Put $X := L^p(\Omega, \mu), H := L^2(\Omega, \mu), Y := L^q(\Omega; \mu)$, and denote by $i : L^p(\Omega, \mu) \hookrightarrow L^2(\Omega, \mu), j : L^2(\Omega, \mu) \to L^q(\Omega, \mu)$ the canonical embeddings. The Riesz isomorphism $I : L^2(\Omega, \mu) \to (L^2(\Omega, \mu))'$ is given by $\langle I v_1, v_2 \rangle := \int_{\Omega} v_1 v_2 \, d\mu$ for $v_1, v_2 \in H$. Analogously, one may consider the operator $J : L^p(\Omega, \mu) \to (L^q(\Omega, \mu))'$ defined by $\langle J u, w \rangle := \int_{\Omega} u \, w \, d\mu$ for $u \in X$ and $w \in Y$. Observe that $I \circ i = J' \circ J$. By the Radon-Nikodym theorem, $J$ is an isomorphism of Banach spaces, hence $(i, j, I, J)$ is a Riesz structure on $X = L^p(\Omega, \mu)$.

It is essential for our applications that Riesz structures are able to encode elliptic operators.

**Example 4.3.3.** Let $(\Sigma, g)$ be a compact Riemannian manifold all whose connected components have nontrivial boundary, let $p \in [2, \infty[, \text{ and let } q \in ]1, 2]$ be the...
H"older conjugate of $p$. Let $X := W^{2,p}(\Sigma; \mathbb{R}^m) \cap W^{1,p}_0(\Sigma; \mathbb{R}^m)$, $H := W^{1,2}_0(\Sigma; \mathbb{R}^m)$, $Y := L^q(\Sigma; \mathbb{R}^m)$ and let $i: X \hookrightarrow H$ and $j: H \hookrightarrow Y$ be the canonical embeddings. Let $0 < c \leq C < \infty$ be constants and let $b \in W^{1,r}(\Sigma; \text{Sym}^2(T\Sigma))$ with $r > \dim(\Sigma)$ be a tensor field of symmetric, bilinear forms with $c g|_a \leq b|_a \leq C g|_a$ for all $a \in \Sigma$. Thus, the Laplace-Beltrami operator $-\Delta_b : W^{2,p}(\Sigma; \mathbb{R}^m) \to L^p(\Sigma; \mathbb{R}^m)$ with respect to the metric $b$ is well-defined and continuous (see 4.5.3). This allows us to define the operator $J_b : X \to Y'$ by

$$\langle J_b u, w \rangle := \int_{\Sigma} g_0(-\Delta_b u, w) \text{vol}_b, \quad u \in X, \ w \in Y',$$

where $\text{vol}_b$ denotes the Riemannian volume density induced by $b$. Elliptic regularity for the Dirichlet problem (see Theorem 9.15 in [10]) implies that $J_b$ is an isomorphism provided that $r \geq p$. Integration by parts leads to the weak formulation $I_b : W^{1,2}_0(\Sigma; \mathbb{R}^m) \to (W^{1,2}_0(\Sigma; \mathbb{R}^m))^\prime$ of the Laplace-Beltrami operator:

$$\langle I_b v_1, v_2 \rangle := \int_{\Sigma} \langle dv_1, dv_2 \rangle_b \text{vol}_b \quad \text{for} \quad v_1, v_2 \in W^{1,2}_0(\Sigma; \mathbb{R}^m).$$

The condition $c g \leq b \leq C g$ guarantees that $I_b$ is both continuous and continuously invertible. The fact that $I_b$ and $J_b$ are connected via integration by parts is reflected in the equality $I_b \circ i = j^\prime \circ J_b$, showing that $(i, j, I_b, J_b)$ constitutes a Riesz structure on $X$.

**Definition 4.3.4.** Let $(X_1, i_1, j_1, I_1, J_1)$ and $(X_2, i_2, j_2, I_2, J_2)$ be Banach spaces with Riesz structure. We call a triple $(A, B, C)$ with $A \in L(X_1; X_2)$, $B \in L(H_1; H_2)$, and $C \in L(Y_1; Y_2)$ a morphism of Riesz structures or shorter, a Riesz morphism, if the following diagram commutes:

$$\begin{array}{ccc}
X_1 & \xrightarrow{A} & X_2 \\
\downarrow{i_1} & & \downarrow{i_2} \\
H_1 & \xrightarrow{B} & H_2 \\
\downarrow{j_1} & & \downarrow{j_2} \\
Y_1 & \xrightarrow{C} & Y_2.
\end{array}$$

**Proposition 4.3.5.** Let $(X_1, i_1, j_1, I_1, J_1)$ and $(X_2, i_2, j_2, I_2, J_2)$ be Banach spaces with Riesz structure and let $(A, B, C)$ be a Riesz morphism. We define the pseudoadjoint $A^*$ of $A$ by $A^* := J_1^{-1} C^\prime J_2$, where $C^\prime \in L(Y_2'; Y_1')$ denotes the dual map given by $C^\prime \xi := \xi \circ C$. The pseudoadjoint has the following properties:

1. $A^* \in L(X_2; X_1)$.
2. $A^*$ is an adjoint with respect to the induced pre-Hilbert metrics $b_1$ and $b_2$, i.e., one has

$$b_2(A x_1, x_2) = b_1(x_1, A^* x_2) \quad \text{for all} \ x_1 \in X_1 \text{ and } x_2 \in X_2.$$
3. \( A^* A \) is symmetric and positive semi-definite with respect to \( b_1 \).
4. \( AA^* \) is symmetric and positive semi-definite with respect to \( b_2 \).

**Proof.** **Claim 1.** Since \( C', J_1^{-1} \) and \( J_2 \) are continuous, \( A^* \) is clearly continuous.

**Claim 2.** Observe
\[
i_1 A^* = (i_1 J_1^{-1}) C J_2 = I_1^{-1} (j_1 C) J_2 = I_1^{-1} B' (j_2 J_2) = I_1^{-1} B' I_2 i_2 = B^* i_2.
\]
With \( x_1 \in X_1 \) and \( x_2 \in X_2 \), the second statement follows from
\[
b_1(x_1, A^* x_2) = \langle i_1 x_1, B^* i_2 x_2 \rangle_{H_1} = \langle B i_1 x_1, i_2 x_2 \rangle_{H_1} = \langle i_2 A x_1, i_2 x_2 \rangle_{H_2} = b_2(A x_1, x_2).
\]

**Claim 3.** This follows from
\[
b_1(u, A^* A v) = b_2(A u, A v)
\]
and
\[
b_1(u, A^* A u) = b_2(A u, A u) = \| i_2 A u \|_{H_2}^2 \geq 0
\]
for all \( u, v \in X_1 \).

**Claim 4.** One has \( b_2(AA^* u, v) = b_1(A^* u, A^* v) \) and
\[
b_2(AA^* u, u) = b_1(A^* u, A^* u) = \langle i_1 A^* u, i_1 A^* u \rangle_{H_1} = \| B^* i_2 u \|_{H_1}^2 \geq 0
\]
for all \( u, v \in X_2 \). □

### 4.3.2 Riesz Bundle Structures

Analogously to 4.3.1 and 4.3.4, we define Riesz structures on Banach bundles and Riesz bundle morphisms as follows.

**Definition 4.3.6.** Let \( \pi_X : X \to M \) be a Banach bundle. A *Riesz structure on \( X \) consists of
1. a continuous, linear, and dense bundle injection \( i : X \hookrightarrow H \) over \( \text{id}_M \) into a Hilbert bundle \( \pi_H : H \to M \);
2. a continuous, linear, and dense bundle injection \( j : H \hookrightarrow Y \) over \( \text{id}_M \) into a further Banach bundle \( \pi_Y : Y \to M \);
3. a fixed Hilbert bundle norm \( \| \cdot \|_H \) or equivalently, its Riesz isomorphism \( I : H \to H' \);
4. and an isomorphism \( J : X \to Y' \) of Banach bundles over \( \text{id}_M \),
such that the following diagram commutes.

\[
\begin{array}{ccc}
X & \xrightarrow{J} & Y' \\
\downarrow{\pi_x} & \cong & \downarrow{\pi_{y'}} \\
H & \xrightarrow{\pi_H} & H'.
\end{array}
\]

Let \( R \in \{ C_{\text{loc}}^{k,\alpha} \mid k \in \mathbb{N} \cup \{0, \infty\}, \alpha \in [0,1] \) with \( k + \alpha \geq 1 \) and suppose that \( \pi: X \to M \) is a Banach bundle of class \( R \). We say that a Riesz structure \((i, j, I, J)\) is of class \( R \), if

1. \( H \) and \( Y \) are Banach bundles of class \( R \) and
2. \( i, j, I, \) and \( J \) are Banach bundle morphisms of class \( R \).

**Example 4.3.7.** Let \( \Sigma \) be a connected, compact, \( n \)-dimensional manifold with non-trivial boundary. Denote by \( g_0 \) the Euclidean metric on \( \mathbb{R}^m \). For \( f \in W^{1,\infty}(\Sigma; \mathbb{R}^m) \), denote by \( f^# g_0 := g_0(\text{d}f \cdot, \text{d}f \cdot) \) the pullback of the bilinear form \( g_0 \) and define the space of Lipschitz immersions as

\[
\text{Imm}^{1,\infty}(\Sigma; \mathbb{R}^m) := \{ f \in W^{1,\infty}(\Sigma; \mathbb{R}^m) \mid \exists C > 0: C^{-1} g \leq f^# g_0 \leq C g \},
\]

where \( g \) may be any smooth Riemannian metric on \( \Sigma \). Note that \( \text{Imm}^{1,\infty}(\Sigma; \mathbb{R}^m) \subset W^{1,\infty}(\Sigma; \mathbb{R}^m) \) is an open set. For \( p > n \), the Sobolev embedding theorem shows that \( W^{2,p}(\Sigma; \mathbb{R}^m) \) embeds continuously into \( W^{1,\infty}(\Sigma; \mathbb{R}^m) \) which enables us to define

\[
\text{Imm}^{2,p}(\Sigma; \mathbb{R}^m) := \text{Imm}^{1,\infty}(\Sigma; \mathbb{R}^m) \cap W^{2,p}(\Sigma; \mathbb{R}^m).
\]

The trace theorem for Sobolev spaces states that the so-called trace operator 
\( \text{res}: C^\infty(\Sigma; \mathbb{R}^m) \to C^\infty(\partial \Sigma; \mathbb{R}^m) \), \( \text{res}(f) = f|_{\partial \Sigma} \) can be continuously extended to 
\( \text{res}: W^{2,p}(\Sigma; \mathbb{R}^m) \to W^{2-\frac{1}{p},p}(\partial \Sigma; \mathbb{R}^m) \). We fix an immersion \( \gamma \in \text{Imm}^{2-\frac{1}{p},p}(\partial \Sigma; \mathbb{R}^m) \) and consider the configuration space of immersed surfaces

\[
\mathcal{C} := \text{Imm}^{2,p}(\Sigma; \mathbb{R}^m) := \{ f \in \text{Imm}^{2,p}(\Sigma; \mathbb{R}^m) \mid \text{res}(f) = \gamma \}.
\]

Analogously to 4.3.3, we define the (trivial) Banach bundles

\[
\mathcal{X} := \mathcal{C} \times (W^{2,p}(\Sigma; \mathbb{R}^m) \cap W^{1,p}_0(\Sigma; \mathbb{R}^m)), \quad \mathcal{H} := \mathcal{C} \times W^{1,2}_0(\Sigma; \mathbb{R}^m), \quad \mathcal{Y} := \mathcal{C} \times L^q(\Sigma; \mathbb{R}^m)
\]

along with canonical injections \( i: \mathcal{X} \hookrightarrow \mathcal{H} \) and \( j: \mathcal{H} \hookrightarrow \mathcal{Y} \). These mappings are fiber-wise continuous and dense injections. We summarize the setting in the commutative diagram

\[
\begin{array}{cccc}
\mathcal{X} & \xrightarrow{i} & \mathcal{H} & \xrightarrow{j} \mathcal{Y} \\
\downarrow{\text{pr}} & & \downarrow{\text{pr}} & \\
\mathcal{C} & \xrightarrow{id_C} & \mathcal{C} & \xrightarrow{id_C} \mathcal{C}.
\end{array}
\]
Each immersion \( f \in \mathcal{C} \) induces a Riemannian metric \( b := f^\# g_0 \) of Sobolev class \( W^{1, p} \). Thus, the Laplace-Betrami operator \( \Delta_f := \Delta_b : W^{2, p}(\Sigma; \mathbb{R}^m) \to L^p(\Sigma; \mathbb{R}^m) \) is well-defined. With the definitions and results of 4.3.3, we observe that \( I_f := I_b : \mathcal{X}_f \to (\mathcal{X}_f)' \) and \( I_f := I_b : \mathcal{X}_f \to (\mathcal{X}_f)' \) are isomorphisms satisfying \( I_f \circ i_f = i_f' \circ f \). Hence the data \((i, j, I, J)\) constitutes a Riesz structure on the Banach bundle \( \mathcal{X} \). Without going into detail, we mention that \((i, j, I, J)\) is even a Riesz structure of class \( C^\infty \).

**Definition 4.3.8.** Let \((X_1, i_1, j_1, I_1, J_1)\) and \((X_2, i_2, j_2, I_2, J_2)\) be Banach bundles with Riesz bundle structures. We call a triple \((A, B, C)\) of Banach bundle morphisms \( A : X_1 \to X_2, B : H_1 \to H_2, \) and \( C : Y_1 \to Y_2 \) a Riesz bundle morphism, if \((A_a, B_a, C_a)\) is a Riesz morphism for each \( a \in M \), i.e., if the following diagram commutes:

\[
\begin{array}{ccc}
X_1 & \xrightarrow{A} & X_2 \\
\downarrow{i_1} & & \downarrow{i_2} \\
H_1 & \xrightarrow{B} & H_2 \\
\downarrow{j_1} & & \downarrow{j_2} \\
Y_1 & \xrightarrow{C} & Y_2
\end{array}
\]

### 4.3.3 Riesz Manifolds

**Definition 4.3.9.** Let \( k \in \mathbb{N} \cup \{ \infty \} \) and \( \alpha \in [0, 1] \) with \( k + \alpha \geq 2 \). A Riesz manifold of class \( C_{\text{loc}}^{k, \alpha} \) is a Banach manifold \( M \) of class \( C_{\text{loc}}^{k, \alpha} \) together with a Riesz structure \((i, j, I, J)\) of class \( C_{\text{loc}}^{k-1, \alpha} \) on \( TM \).

**Definition 4.3.10.** Let \((M_1, i_1, j_1, I_1, J_1)\) and \((M_2, i_2, j_2, I_2, J_2)\) be Riesz manifolds of class \( C_{\text{loc}}^{k, \alpha} \) and let \( F : M_1 \to M_2 \) be a mapping of class \( C_{\text{loc}}^{k, \alpha} \). We say that \( F \) is a Riesz morphism of class \( C_{\text{loc}}^{k, \alpha} \) if the tangent map \( TF : TM_1 \to TM_2 \) induces a morphism of Riesz bundles of class \( C_{\text{loc}}^{k-1, \alpha} \). In particular, for a Riesz morphism \( F : M \to \mathbb{R} \) into the real numbers, we also say that \( F \) is a Riesz function.

**Example 4.3.11.** Let \((M, g)\) be a Riemannian manifold, i.e., a (smooth) Banach manifold together with a (smooth) section \( g \) of \( \text{Sym}^2(TM) \) such that for each \( a \in M \), the symmetric bilinear form \( g_a : T_a M \times T_a M \to \mathbb{R} \) is positive-definite and generates the topology on \( T_a M \). In particular, \((T_a M, g_a)\) is a Hilbert space and \( I_a : T_a M \to T_a M, \langle I_a u, v \rangle := g_a(u, v) \) is a Riesz isomorphism. With \( TM = H = Y \), and \( f = I, i = j = id_{TM} \), we observe that \((M, i, j, I, J)\) is a Riesz manifold in a natural way. Note that a smooth map between Riemannian manifolds induces also a (unique) Riesz morphism between the induced Riemannian manifolds.

---

4.2 Since \( f^\# g_0 \) depends quadratically on \( df \), the condition \( p > n \) in necessary here.
**Example 4.3.12.** Let \((M, i, j, I, J)\) be a smooth Riesz manifold. Then one may define a section \(g\) of \(\text{Sym}^2(TM)\) by

\[
g_a(u, v) := \langle J_a u, j i v \rangle = \langle I_a i u, i v \rangle \quad \text{for } u, v \in T_a M.
\]

Since \(I_a\) is a Riesz isomorphism, \(g_a\) is a symmetric and positive-definite bilinear form. Thus, \((T_a M, g_a)\) is a pre-Hilbert space with completion \((H_a, \langle I_a \cdot, \cdot \rangle)\). If \(T_a M\) is not isomorphic to a Hilbert space, \((TM, g)\) is not a Riemannian manifold. In general, there is no chance to complete \((M, g)\) to a Hilbert manifold \((\tilde{M}, \tilde{g})\). Moreover, in the rare cases where this is possible, many continuous (or even differentiable) functions on \(M\) cannot be extended continuously (differentiably) to \(\tilde{M}\).

These examples show that Riemannian manifolds form a (proper) full subcategory of the category of Riesz manifolds in the sense that the functor described in 4.3.11 is injective (but not surjective) on objects and fully faithful. Still, the latter category shares many useful properties with the former (see Section 4.4 below for a few examples). Forgetting the underlying Riesz structure induces a forgetful functor from the category of Riesz manifolds to the category of Banach manifolds. In this sense, Riesz manifolds may be considered as lying somewhere between Riemannian manifolds and arbitrary Banach manifolds. More precisely, we have the following commutative diagram of functors

\[
\begin{array}{c}
\{ \text{Riemannian manifolds} \} \\
\downarrow \quad (M,g) \rightarrow M \\
\{ \text{Riesz manifolds} \} \\
\downarrow \quad (M,i,j,I,J) \rightarrow M \\
\{ \text{Banach manifolds} \}
\end{array}
\]

### 4.4 Pseudogradient Flow

Now we have all necessary ingredients for defining the pseudogradient of a Riesz function.

**Definition 4.4.1.** Let \((M, i, j, I, J)\) be a Riesz manifold of class \(C^{k, \alpha}_{\text{loc}}\) and let \(F: M \rightarrow \mathbb{R}\) be a Riesz mapping of class \(C^{k, \alpha}_{\text{loc}}\). We define the *pseudogradient field* \(\text{grad}(F): M \rightarrow TM\) by setting \(\text{grad}(F) := (TF)^*\).

Compared to the elaborate theory that is needed for the treatment of nonlinear flows of parabolic type, the existence theory of pseudogradient flows relies on quite elemen-
tary principles. We gather the central properties of pseudogradient fields and their flows in the following theorem.

**Theorem 4.4.2.** Let \( k \in \mathbb{N} \cup \{ \infty \} \) and \( \alpha \in [0, 1] \) with \( k + \alpha \geq 2 \).

Let \( M \) be a Riesz manifold of class \( C_{\text{loc}}^{k,\alpha} \) and let \( F: M \to \mathbb{R} \) be a Riesz function of class \( C_{\text{loc}}^{k,\alpha} \). Then one has

1. For all \( a \in M \) : \( \langle dF|_a, \text{grad}(F)|_a \rangle \geq 0. \)
2. Let \( a \in M \). Then \( \text{grad}(F)|_a = 0 \) if and only if \( dF|_a = 0 \).
3. The pseudogradient field \( \text{grad}(F) \) is a vector field of class \( C_{\text{loc}}^{k-1,\alpha} \).
4. For each interior point \( a_0 \in M \setminus \partial M \) there is a neighborhood \( U \) of \( a_0 \) and an \( \varepsilon > 0 \) such that there is a unique flow \( \Phi: U \times ]-\varepsilon, \varepsilon[ \to M \) with
   \[
   \frac{d}{dt} \Phi(a, t) = -\text{grad}(F)|_{\Phi(a,t)} \quad \text{and} \quad \Phi(a, 0) = a \quad \text{for all} \quad (a, t) \in U \times ]-\varepsilon, \varepsilon[.
   \]
5. For each interior point \( a \in M \setminus \partial M \), the function \( t \mapsto F \circ \Phi(a, t) \) is monotonically decreasing.

**Proof.** Denote the Riesz structure on \( M \) with

\[
i: TM \to H, \quad j: H \to Y, \quad I: H \to H' \quad \text{and} \quad J: TM \to Y'
\]

and let \( B: H \to T \mathbb{R} \) and \( C: Y \to T \mathbb{R} \) be the Banach bundle morphisms over \( F \) of class \( C_{\text{loc}}^{k-1,\alpha} \) with \( i \circ B = B \circ i \) and \( j \circ C = C \circ j \).

**Claim 1.** Denote by \( b_\mathbb{R} \) the Euclidean inner product on \( T \mathbb{R} \). By 4.3.5, we have for each \( a \in M \) that

\[
\langle dF|_a, \text{grad}(F)|_a \rangle = b_\mathbb{R}(1, T_a F \text{grad}(F)|_a \cdot 1) = \| B_a^* \cdot 1 \|^2_{H_a} \geq 0.
\]

**Claim 2.** Note that \( dF|_a = 0 \) is equivalent to \( T_a F = 0 \). On the one hand, the definition of \( \text{grad}(F) \) implies immediately that \( \text{grad}(F)|_a \) vanishes if \( T_a F \) vanishes (see 4.4.1). On the other hand, the inequality above shows that \( \text{grad}(F)|_a = 0 \) implies \( B_a^* = 0 \) and thus \( B_a = 0 \). We have \( (i_\mathbb{R})_a T_a F = B_a i_a = 0 \) and since \( (i_\mathbb{R})_a \) is an isomorphism, we obtain \( T_a F = 0 \) and hence \( dF|_a = 0 \).

**Claim 3.** With \( J \) also \( J^{-1} \) is of class \( C_{\text{loc}}^{k-1,\alpha} \). Moreover, continuous bilinear combinations of Banach-space valued mappings of class \( C_{\text{loc}}^{k-1,\alpha} \) are again of class \( C_{\text{loc}}^{k-1,\alpha} \). Hence \( \text{grad}(F) = J^{-1} C \) is a section of class \( C_{\text{loc}}^{k-1,\alpha} \) in the Banach bundle \( TM \), hence a vector field of class \( C_{\text{loc}}^{k-1,\alpha} \).

**Claim 4.** Note that \( a \mapsto -\text{grad}(F)|_a \) is a vector field at least of class \( C_{\text{loc}}^0 \). Hence the statement follows from the Picard-Lindelöf theorem.

**Claim 5.** This follows from Claim 2 and from the chain rule:

\[
\frac{d}{dt} F(\Phi(a, t)) = -\langle dF|_{\Phi(a,t)}, \text{grad } F|_{\Phi(a,t)} \rangle \leq 0.
\]

If the initial condition is not a critical point, the function \( t \mapsto F(\Phi(a, t)) \) is even strictly monotonically decreasing, since stationary points cannot be reached within finite time. \( \square \)
4.5 Applications

Geometric functionals for curves and surfaces, such as those we discuss in this section, play a major role in the field of geometry processing and are applied to a variety of tasks. Since reducing computational time is a specific objective in this field, it is no wonder that the advantages of $H^1$- and $H^2$-gradient flows have already been observed and utilized within this community (see [9]). Our aim here is to give some more theoretical justification on these flows, even for the pre-discretized, infinite-dimensional setting, which is often not covered appropriately.

4.5.1 Minimal Surfaces

Let us return to the setting of 4.3.7. Observe that $X = TC$ is precisely the tangent bundle of $C = \text{Imm}^2_{\gamma}(\Sigma; \mathbb{R}^m)$, the configuration space of immersions subject to Dirichlet boundary conditions. As we have seen in 4.3.7, the data $(i, j, I, J)$ represents a smooth Riesz structure on the Banach bundle $TC$, hence $(C, i, j, I, J)$ is a smooth Riesz manifold. Now consider the volume functional

$$F: C \rightarrow \mathbb{R}, \quad F(f) := \int_{\Sigma} \text{vol}_{f^*g_0}.$$ 

**Theorem 4.5.1.** The volume functional $F$ is a smooth Riesz function on the smooth Riesz manifold $\text{Imm}^2_{\gamma}(\Sigma; \mathbb{R}^m)$.

**Proof.** One has

$$D(f \mapsto \text{vol}_{f^*g_0}) u = \langle df, du \rangle_{f^*g_0} \text{vol}_{f^*g_0}$$

for each Lipschitz immersion $f \in \text{Imm}^{1,\infty}(\Sigma; \mathbb{R}^m)$ and each $u \in T_f \text{Imm}(\Sigma; \mathbb{R}^m) = W^{1,\infty}(\Sigma; \mathbb{R}^m)$. Hence, the volume functional is differentiable and its differential $dF: C \rightarrow T' C$ is given by

$$\langle dF_f, u \rangle := \int_{\Sigma} \langle df, du \rangle_{f^*g_0} \text{vol}_{f^*g_0} \quad \text{for } u \in T_f C = W^{2,p}(\Sigma; \mathbb{R}^m) \cap W^{1,p}_0(\Sigma; \mathbb{R}^m).$$

Extending $dF_f$ continuously to $H_f$ leads to the smooth section $B: C \rightarrow H'$ given by

$$\langle B_f, v \rangle := \int_{\Sigma} \langle df, dv \rangle_{f^*g_0} \text{vol}_{f^*g_0} \quad \text{for } v \in H_f = W^{1,2}(\Sigma; \mathbb{R}^m).$$

Via integration by parts, we obtain

$$\langle dF_f, u \rangle = \int_{\Sigma} g_0(-\Delta f, u) \text{vol}_{f^*g_0}.$$
Since $\Delta f \in L^p(\Sigma; \mathbb{R}^m)$, this leads to the smooth section $C: \mathcal{C} \to \mathcal{Y}$ given by

$$\langle C_f, w \rangle := \int_{\Sigma} g_0(-\Delta f, w) \text{vol}_{f^*g_0} \quad \text{for } w \in \mathcal{Y}_f = L^q(\Sigma; \mathbb{R}^m).$$

Moreover, the family $(C_f)_{f \in \mathcal{C}}$ induces a smooth Banach bundle morphism $C: \mathcal{Y} \to \mathcal{C} \times \mathbb{R}$ and one has $j^*_R i^*_R T^*_F = C j^*_R i^*_R$, hence $F$ is a Riesz function on $\mathcal{C}$.

Combining 4.4.2 and 4.5.1 allows us to discuss the pseudogradient flow of the volume function $F$. For an immersion $f \in \mathcal{C}$, the defining equation for the pseudogradient $u = \text{grad} \, F|_f$ can be written as

$$\int_{\Sigma} \langle du, dv \rangle_{f^*g_0} \text{vol}_{f^*g_0} = \int_{\Sigma} \langle df, dv \rangle_{f^*g_0} \text{vol}_{f^*g_0}, \quad \text{for all } v \in \mathcal{H}_f = W^{1,2}_0(\Sigma; \mathbb{R}^m).$$

(4.5.2)

Note that the weak formulation of the Laplace-Beltrami operator occurs on both sides of the equation. For numerical optimization, we have to discretize the configuration space $\mathcal{C}$ and the objective function. To this end, we may fix a triangulation $\mathcal{T}$ of $\Sigma$ and define the discrete configuration space $\mathcal{C}_\mathcal{T}$ as the set of all immersions that are piecewise-linear with respect to $\mathcal{T}$ and that restrict to $\gamma$ on all boundary vertices of $\mathcal{T}$. This way, the image $f(\Sigma)$ of an element $f \in \mathcal{C}_\mathcal{T}$ is an immersed simplicial mesh whose boundary complex is inscribed into $\gamma(\partial \Sigma)$. A discrete volume function $\mathcal{F}_\mathcal{T}$ can be defined straight-forwardly as the restriction of the volume function to $\mathcal{C}_\mathcal{T}$. Since $\mathcal{C}_\mathcal{T}$ is finite dimensional, it is a Riemannian manifold when equipped with the discretization of the weak formulation of the Laplace-Beltrami operator and the discretized pseudogradient $u = \text{grad} \, \mathcal{F}_\mathcal{T}|_f$ can be defined by

$$\int_{\Sigma} \langle du, dv \rangle_{f^*g_0} \text{vol}_{f^*g_0} = \langle d\mathcal{F}_\mathcal{T}|_f, v \rangle = \int_{\Sigma} \langle df, dv \rangle_{f^*g_0} \text{vol}_{f^*g_0}, \quad \text{for all } v \in T_f \mathcal{C}_\mathcal{T}.$$  

(4.5.3)

The pseudogradient descent algorithm consists now in choosing an initial guess $f_0 \in \mathcal{C}_\mathcal{T}$ and by computing recursively

$$f_{n+1} = f_n - \tau_n \, \text{grad} \, \mathcal{F}_\mathcal{T}|_{f_n}, \quad \text{for } n \in \mathbb{N} \cup \{0\},$$

(4.5.4)

where $\text{grad} \, \mathcal{F}_\mathcal{T}|_{f_n}$ has to be computed from (4.5.3) and $\tau_n \geq 0$ is a step size parameter that has to be chosen appropriately. Again, the (discretized) Laplace-Beltrami operator occurs on both sides of the equation. This comes in handy, as it has to be reassembled in each iteration and this way, it can be used at least twice: once for computing $d\mathcal{F}_\mathcal{T}|_f$ and once for solving for the pseudogradient. Since assembling the discrete Laplace-Beltrami operator is a standard task, the implementation is pleasantly easy. We point out that this algorithm has already been introduced in [15], however, from a different
Fig. 4.2: From left to right: initial surface (512 faces); surface after a single explicit Euler step in negative discrete mean curvature direction; surface after a single explicit Euler step in negative discrete pseudogradient direction.

Fig. 4.3: The same as in Figure 4.2 with the same step sizes, but with refined mesh (32 768 faces).

perspective. The mentioned article is also a good reference for the assembly details of discrete Laplace-Beltrami operators on discrete surfaces.

When $\tau_n$ is chosen to be independent of $n$, (4.5.4) boils down to an explicit Euler scheme for the (discrete) pseudogradient flow. In principle, one may apply also more sophisticated step size rules for line search, such as the Armijo algorithm or the Wolfe-Powell algorithm.

In the case of the area functional, it turns out that one can go quite well with constant step size of magnitude $\tau = 1$ and this is essentially independent of the mesh resolution. In our experiments, we rarely needed more than two dozen pseudogradient steps in order to reach a configuration from which Newton’s algorithm converged within several steps to a critical point—no matter if the discrete surface consisted of a few thousand or a million triangles. As Figure 4.4 suggest, the discrete pseudogradient flow performs well and is very robust even in cases where minimizers do not exist.

These findings are in strong contrast to the step size rules for discrete mean curvature flow, the $L^2$-gradient flow of the volume functional: As discrete mean curvature flow is a parabolic partial differential equation (in time and space), the step size has to be decreasing along with the mesh size, even if more sophisticated integration schemes such as (semi-)implicit methods are employed. As a comparison, we include Figure 4.2 and Figure 4.3, each depicting a step of discrete $L^2$-gradient descent and discrete pseudogradient descent at different mesh resolutions. We can also see there
that the $L^2$-gradient is increasingly localized with raising $L^\infty$-norm as the mesh size is decreased. Of course, the reason is that the initial surface is not twice differentiable so that its mean curvature is a vector-valued distribution with support concentrated on the creases. Hence the discrete mean curvature, trying to approximate this distribution, blows up under mesh refinement. In order to obtain a stable algorithm, the step size has to be decreased accordingly.

On the other hand, the information contained in the mean curvature distribution is spread out over the whole surface when solving for the pseudogradient.\textsuperscript{4.3} In particular, points in medium distance to an “incident” get informed immediately during the next pseudogradient step so that they have the opportunity to “react” just-in-time. See also Figure 4.5 for a typical example of an “incident” that occurs quite frequently with discrete mean curvature flows but considerably less often with discrete pseudogradient flows. We point out that this behavior is primarily an artifact of the time discretization: Mean curvature flows with infinitesimal step size have infinite propagation speed and this infinite speed is hard to capture by a discrete time stepping algorithm.

\textbf{4.3} Note that the pseudo-Riesz isomorphism is still an elliptic operator such its inverse is smoothing.
Fig. 4.5: $L^2$-gradient descent (first column) and pseudogradient descent (second column), both for the area functional, with explicit Euler scheme and constant step sizes. 

First row: initial surfaces; second row: after first step; third row: after nine steps.
4.5.2 Elasticae

Curvature dependent energies such as the elastica functional can also be well-formulated on the space $\text{Imm}^{2,p}(\Sigma; \mathbb{R}^m)$, when $p$ is suitably chosen. While the discussion of the volume functional was dominated by the Laplacian operator, it turns out that a suitable operator for defining a Riesz structure for curvature energies is the bi-Laplacian. If $p > \dim(\Sigma)$, the discussion is even simplified as we do not need to resort to the space $\text{Imm}^{4,p}(\Sigma; \mathbb{R}^m)$, where the strong formulation of the bi-Laplacian resides.

Let $\Sigma$ be an $n$-dimensional, compact, and connected smooth manifold. In order to keep the exposition as brief as possible, we focus our attention to manifolds with nontrivial smooth boundary and to the space

$$C := \text{Imm}^{2,p}_{\gamma}(\Sigma; \mathbb{R}^m), \quad p \in [2, \infty[ \cap \mathbb{N}, \infty[$$

for a given $\gamma \in \text{Imm}^{2-\frac{1}{p}}(\Sigma; \mathbb{R}^m)$.\(^{4,5}\) From now on we use the Banach bundles

$$X := T\mathcal{C} = \mathcal{C} \times (\text{Imm}^{2,p}(\Sigma; \mathbb{R}^m) \cap W^{1,p}_{0}(\Sigma; \mathbb{R}^m)),$$

$$\mathcal{H} := \mathcal{C} \times (\text{Imm}^{2,2}(\Sigma; \mathbb{R}^m) \cap W^{1,2}_{0}(\Sigma; \mathbb{R}^m)),$$

$$\mathcal{Y} := \mathcal{C} \times (\text{Imm}^{2,q}(\Sigma; \mathbb{R}^m) \cap W^{1,q}_{0}(\Sigma; \mathbb{R}^m)),$$

where $q$ is the Hölder conjugate of $p$.\(^{4,5}\) Note that $p$ is chosen such that the canonical injections $i: X \hookrightarrow \mathcal{H}$ and $j: \mathcal{H} \hookrightarrow \mathcal{Y}$ are dense in each fiber. From 4.3.3, we recall that the Laplace-Beltrami operator $\Delta_f$ maps $W^{2,r}(\Sigma; \mathbb{R}^m) \cap W^{1,r}_{0}(\Sigma; \mathbb{R}^m)$ isomorphically to $L^{r}(\Sigma; \mathbb{R}^m)$ for $r \in [1, p]$ such that we obtain a Riesz structure $(i, j, I, J)$ with the isomorphisms

$$I: \mathcal{H} \to \mathcal{H}^{'}, \quad \langle I_f v_1, v_2 \rangle := \int_\Sigma g_0(\Delta_f v_1, \Delta_f v_2) \text{ vol}\_f^g, \quad (4.5.5)$$

$$J: \mathcal{X} \to \mathcal{Y}^{'}, \quad \langle J_f u, w \rangle := \int_\Sigma g_0(\Delta_f u, \Delta_f w) \text{ vol}\_f^g. \quad (4.5.6)$$

In order to define the elastica functional, we have to introduce the second fundamental form of an immersion. Therefore, we make a short excursion to the Hessian of a function on a Riemannian manifold.

**Proposition 4.5.2.** Let $f \in \mathcal{C}$, $r \in [1, \infty]$, and $u \in W^{2,r}(\Sigma; \mathbb{R}^m)$. Define the Hessian $\text{Hess}_f(u)$ of $u$ with respect to $f$ by

$$\text{Hess}_f(u)(X, Y) = (d(du df^i) X) \cdot (df Y), \quad \text{for all } X, Y \in T_f \Sigma.$$

---

\(^{4,4}\) The case of manifolds without boundary can be treated, e.g., by imposing the nonlinear barycenter constraint $\int_\Sigma f \text{ vol}\_f^g = 0$.

\(^{4,5}\) We point out that the analogous Banach bundles for treating the case $\partial \Sigma = \emptyset$ would not be mere products anymore.
One has \( \text{Hess}_f(u) \in L^{\min(p,r)}(\Sigma; \text{Sym}^2(T\Sigma; \mathbb{R}^m)) \) and \( \text{Hess}_f \) is a continuous linear operator.\(^4,\text{6}\)

\[
\text{Hess}_f : W^{2,r}(\Sigma; \mathbb{R}^m) \to L^{\min(p,r)}(\Sigma; \text{Sym}^2(T\Sigma; \mathbb{R}^m)).
\]

**Proof.** Let \( u \in W^{2,r}(\Sigma; \mathbb{R}^m) \) so that \( du \) is an element of \( W^{1,r}(\Sigma; \text{Hom}(T\Sigma; \mathbb{R}^m)) \). The Moore-Penrose pseudoinverse restricted to linear maps of fixed rank is a smooth transformation. A concise treatment can be found, e.g., in [11] which allows us to deduce

\[
D(f \mapsto df^\dagger) u = -df^\dagger du df^\dagger + df^\dagger (du df^\dagger)^\dagger (\text{id}_{\mathbb{R}} - df df^\dagger). \tag{4.5.7}
\]

Hence one has \( df^\dagger \in W^{1,p}(\Sigma; \text{Hom}(\mathbb{R}^m; T\Sigma)) \) and 4.5.3 below completes the proof. \( \square \)

**Lemma 4.5.3.** Let \( E_1, E_2, \text{ and } E_3 \) be smooth Banach bundles over the compact, smooth manifold \( \Sigma \), let \( \mu : E_1 \times_M E_2 \to E_3 \) be a locally Lipschitz continuous bilinear bundle map and let \( p > \dim(\Sigma) \).

Then \( \mu \circ (\sigma_1, \sigma_2) \in W^{1,\min(p,r)}(\Sigma; E_3) \) holds for all sections \( \sigma_1 \in W^{1,p}(\Sigma; E_1) \) and \( \sigma_2 \in W^{1,r}(\Sigma; E_2) \). Moreover, the induced bilinear map

\[
A : W^{1,p}(\Sigma; E_1) \times W^{1,r}(\Sigma; E_2) \to W^{1,\min(p,r)}(\Sigma; E_1)
\]

is continuous.

**Proof.** It suffices to perform the regularity analysis locally. Thus, we may focus our attention to an open set \( U \subset \Sigma \) and we may assume that \( E_i|U \cong U \times \mathbb{R}^m_i, i \in \{1, 2, 3\} \) are trivial vector bundles. Moreover, we may write \( \sigma_1(x) = (x, f_1(x)), \sigma_2(x) = (x, f_2(x)) \), and \( \mu_x = B_x \) for all \( x \in U \) with \( f_1 \in W^{1,p}(\Sigma; \mathbb{R}^{m_1}), f_2 \in W^{1,r}(\Sigma ; \mathbb{R}^{m_2}) \), and \( B \in W^{1,\infty}(U; \text{Bil}(\mathbb{R}^{m_1} \times \mathbb{R}^{m_2}; \mathbb{R}^{m_1})) \).

The Sobolev embedding \( W^{1,p}(\Sigma; \mathbb{R}^{m_1}) \hookrightarrow L^{\infty}(\Sigma; \mathbb{R}^{m_1}) \) shows that \( B(f_1, f_2) \in L^{1}(\Sigma; \mathbb{R}^{m_2}) \). With \( n := \dim(\Sigma) \), one has the Sobolev embedding \( W^{1,\tilde{r}}(\Sigma; \mathbb{R}^{m_2}) \hookrightarrow L^{\tilde{r}}(\Sigma; \mathbb{R}^{m_2}) \) where

\[
\tilde{r} \in \begin{cases} [1, \frac{n\tilde{r}}{n-r}], & r < n, \\ [1, \infty], & r = n, \\ [1, \infty], & r > n. \end{cases}
\]

For each smooth vector field \( X \) on \( U \), we obtain

\[
d(B(f_1, f_2)) X = (dB X)(f_1, f_2) + B(df_1 X, f_2(x)) + B(f_1, df_2 X)
\]

4.6 We point out for geometers that the section \( \text{Hess}_f(u) \) coincides with the Hessian \( \nabla^{f^g_0} du \), provided that \( f \) and \( u \) are sufficiently smooth. Here, \( \nabla^{f^g_0} \) denotes the Levi-Civita connection of the Riemannian metric \( f^g_0 \).
and this together with the Hölder inequality implies $d(B(f_1, f_2)) \in L^{\min(s, r)}(\Sigma; \text{Hom}(T^* \Sigma; \mathbb{R}^m))$, hence $B(f_1, f_2) \in W^{1, \min(s, r)}(\Sigma; \mathbb{R}^m)$, where $s = (1/p + 1)\min(1, 1/r)$. We analyse the following three cases:

**Case 1.** $r < n$. Because of $p > n > r$, we have

$$s = \frac{1}{p + 1} > \frac{1}{n + 1} = r,$$

so that $\min(s, r) = r = \min(p, r)$.

**Case 2.** $r = n$. One may write $r = n = (1 + \varepsilon)^{-1} p < p$ with some $\varepsilon > 0$. Choosing $\tilde{r} = \frac{p}{1 + \varepsilon} < \infty$, we obtain

$$s = \frac{1}{p + \varepsilon} = \frac{p}{1 + \varepsilon} = r.$$

This shows $\min(s, r) = r = \min(p, r)$.

**Case 3.** $r > n$. Then one has $\tilde{r} = \infty$ and $s = p$, leading directly to $\min(s, r) = \min(p, r)$. The continuity of $A$ follows from the already mentioned Hölder and Sobolev inequalities. \(\square\)

**Definition 4.5.4.** The second fundamental form $\Pi(f)$ of $f$ can be written as $\Pi(f) := \text{Hess}_f(r)$ so that one has

$$\Pi(f) \in L^p(\Sigma; \text{Bil}(T^* \Sigma \times_T T \Sigma; \mathbb{R}^m)) \subset L^2(\Sigma; \text{Bil}(T^* \Sigma \times_T T \Sigma; \mathbb{R}^m)).$$

We define the elastica functional $\mathcal{F} : \mathcal{C} \to \mathbb{R}$ by

$$\mathcal{F}(f) = \frac{1}{2} \int_{\Sigma} |\Pi(f)|^2 \, \text{vol}_g \, r_g^s.$$

**Theorem 4.5.5.** For $p \in [2, \infty \cap \dim(\Sigma), \infty \cup \int, \text{the elastica functional } \mathcal{F} \text{ and the Willmore energy } \mathcal{W} \text{ are smooth Riesz functions on the smooth Riesz manifold } (\mathcal{C}, i, j, I, f) \text{ from (4.5.5) and (4.5.6)}.\]

**Proof.** Fix $f \in \mathcal{C}$ and let $u \in T_f \mathcal{C} \subset W^{2, p}(\Sigma; \mathbb{R}^m)$. From (4.5.7), we may deduce

$$D \Pi(f) u = (i_{\mathbb{R}^m} - df \cdot df^t) \text{Hess}_f(u) + (du \cdot df^t) \Pi(f).$$

Equation (4.5.1) provides us with a formula for the derivative of $f \mapsto \text{vol}_{r_g}$. Moreover, we have $|S|_{r_g} = \text{vol}_{r_g}$ for $S \in L^p(\Sigma; \text{Bil}(T^* \Sigma \times_T T \Sigma; \mathbb{R}^m))$. This would allow us to compute a precise expression for $\langle d\mathcal{F}(f), u \rangle$, but it already suffices for our considerations to observe that $d\mathcal{F}$ is of the form

$$\langle d\mathcal{F}(f), u \rangle = \frac{1}{2} \int_{\Sigma} \left( \langle \Pi(f), \text{Hess}_f(u) \rangle_{r_g} + \mu(\Pi(f), df, df^t, du) \right) \text{vol}_{r_g},$$
where \( \mu(\Pi(f), df, df^\dagger, du) \) is a polynomial expression in \( \Pi(f), df, df^\dagger, du \) with constant coefficients in which \( \Pi(f) \) occurs with order two and \( du \) occurs with order one.

Now let \( w \in \mathcal{W}_f = W^{2,q}(\Sigma; \mathbb{R}^m) \). Note that we have \( \Pi(f) \in L^p \), \( df \in L^\infty \), and \( df^\dagger \in L^\infty \), hence \( \mu(\Pi(f), df, df^\dagger, \gamma) \in L^{p/2} \). In the case \( n = 1 \), we have \( dw \in W^{1,q} \rightarrow L^\infty \). For \( n \geq 2 \), we have \( dw \in W^{1,q} \rightarrow L^r \) with \( r \geq \frac{p}{p-1 - \frac{p}{n}} \). Because of \( p > n \), we obtain

\[
0 \geq \frac{p}{p - 1 - \frac{p}{n}} > \frac{p}{p - 2} = \frac{(p/2)}{2},
\]

thus \( dw \in L^{(p/2)^r} \). In any case, we obtain \( \mu(\Pi(f), df, df^\dagger, dw) \in L^1 \). This shows that \( df\mathcal{F}|_{\mathcal{W}} \) can be continuously extended to \( \mathcal{W} : \mathcal{W}_f \rightarrow \mathbb{R} \) so that \( j_{\mathcal{W} i\mathbb{R}} T\mathcal{F} = C j i \) holds. Hence, \( \mathcal{F} \) is a Riesz function. The statement for \( \mathcal{W} \) follows from the identity

\[
H_f = \frac{1}{\dim(\Sigma)} \sum_{i=0}^m \Pi_f(df^i e_i, df^j e_j) \text{ for any } g_0\text{-orthonormal basis } e_1, \ldots, e_m \text{ of } \mathbb{R}^m
\]

and from the above discussion.

4.5.3 Euler-Bernoulli Energy and Euler Elastica

The elastica functional is well-known under different names in dimensions one and two. For \( \dim(\Sigma) = 1 \), the functional \( F \) is identical to the Euler-Bernoulli bending energy

\[
\mathcal{F}(f) = \frac{1}{2} \int_{\Sigma} |\kappa_f|^2 g_0 \text{ vol}_{f^*g_0},
\]

where \( \kappa_f \) is the curvature of the regular curve \( f : \Sigma \rightarrow \mathbb{R}^m \). Let \( \Sigma \) be a compact interval equipped with the Euclidean metric \( g \), let \( \gamma : \partial \Sigma \rightarrow \mathbb{R}^m \), and \( \nu : \partial \Sigma \rightarrow S^{m-1} \subset \mathbb{R}^m \) be given.

The critical points of the Euler-Bernoulli bending energy \( \mathcal{F} \) on the set

\[
\{ f \in \mathcal{C} \mid \text{vol}_{f^*g_0} = \text{vol}_g, \ f|_{\partial \Sigma} = \gamma, \ n(f) = \nu \}
\]

are called Euler elasticae (with clamped ends) where the outward unit normals are denoted with \( n(f) : \partial \Sigma \rightarrow S^{m-1} \). While \( f|_{\partial \Sigma} = \gamma \) is clearly an inhomogeneous Dirichlet boundary condition, one may call \( n(f) = \nu \) a Neumann-type boundary condition. Note that \( n(f) \) depends nonlinearly on \( f \). The constraint \( \text{vol}_{f^*g_0} = \text{vol}_g \) amounts to the requirement that all curves \( f \) in the feasible set shall be in arclength parameterization (with respect to the given metric \( g \)).

With Figure 4.6 and Figure 4.7, we provide two numerical examples of discretized pseudogradient flows for the Euler-Bernoulli bending energy subject to arclength parameterization constraints. We discretize the space \( \mathcal{C} \) by polygonal lines. Since polygonal lines are almost never elements of \( W^{2,2}(\Sigma; \mathbb{R}^m) \) (unless they are straight lines), we have to discretize the energy \( \mathcal{F} \) as well. Our choice is

\[
\mathcal{F}(f) = \frac{1}{2} \sum_{i=1}^n \left( \frac{2 \arctan(\varphi_i/2)}{\ell_i} \right)^2 \ell_i,
\]
where \( \varphi_i \) denotes the turning angle of the polygonal line \( f \) at vertex \( i \) (i.e., \( \pi \) minus the angle enclosed by the neighboring edges) and where \( \ell_i \) is the average of the lengths of contiguous edges. As discrete pseudo-Riesz isomorphism, we use (4.5.5) and (4.5.6), where \( \Delta_f \) is replaced by the discrete Laplace-Beltrami operator. Strictly speaking, this is not justified by the smooth theory of pseudogradient flows as the discretization is nonconforming. However, pseudogradient search using this discretization performs surprisingly well as the figures attest. For comparison, we refer the reader to [3] and [4], where the \( L^2 \)-gradient flow of the Euler-Bernoulli energy is treated.

We point out that the presented flows preserve the parameterization by arclength. This was achieved by

- projecting the pseudogradient to the tangent space of the constraint manifold;
- performing an Armijo line search on the osculating circle of the constraint manifold in pseudogradient direction;
- and projecting the result onto the constraint manifold via a Newton-type algorithm involving a pseudoinverse of the linearized constraint map.

Of course, each of these steps involves operations that are usually only available for Riemannian manifolds. The methods go through numerically as (i) the discretized pseudo-Riesz isomorphism actually induces a Riemannian structure on the discrete configurations space \( C_T \) and (ii) the Euler-Bernoulli energy can be formulated on the Riemannian manifold \( \text{Imm}^2 \cdot 2(\Sigma; \mathbb{R}^m) \). However, we believe that at least some of these operations can also be established for general Riesz manifolds.

### 4.5.4 Willmore Energy

As mentioned before, the case \( \dim(\Sigma) = 1 \) is very special in that the elastica functional can be well-formulated on the Riemannian manifold \( \text{Imm}^2 \cdot 2(\Sigma; \mathbb{R}^m) \). Therefore, we also consider the case \( \dim(\Sigma) = 2 \). One has the relation \( |\Pi(f)|^2 = \kappa_1^2 + \kappa_2^2 \), \( K_f = \kappa_1 \kappa_2 \), and \( |H_f|^2 = \frac{1}{4}(\kappa_1 + \kappa_2)^2 \) for the the principle curvatures \( \kappa_1 \) and \( \kappa_2 \), the Gauss curvature \( K_f \), and the mean curvature vector \( H_f \) of the immersed surface \( f \). This leads to the identity

\[
\mathcal{F}(f) = \int_\Sigma (2 |H_f|^2 - K_f) \, \text{vol}_{f^*g_0}.
\]

In the case that \( \partial \Sigma = \emptyset \), the Gauss-Bonnet theorem and denseness considerations imply that

\[
\int_\Sigma K_f \, \text{vol}_{f^*g_0} = 2 \pi \chi(E) \tag{4.5.9}
\]
Fig. 4.6: Discrete elastic figure eight knot (2000 edges): (a) initial condition; (b)–(e) the first four iterations of pseudogradient flow with Armijo line search; (f) ultimate minimizer (the round circle), obtained after 6 pseudogradient steps in total and 3 Newton iterations.
Fig. 4.7: Three discrete elastic threads (6000 edges in total) joint together with clamped boundary conditions imposed at the free ends: (a) initial condition; (b)–(e) the first four iterations of pseudogradient flow with Armijo line search; (f) ultimate minimizer, obtained after 4 pseudogradient steps in total and 4 Newton iterations.
with the Euler characteristic of \( \Sigma \). Hence, \( \mathcal{F} \) is essentially identical to the Willmore energy

\[
\mathcal{W}(f) := \frac{1}{2} \int_{\Sigma} |H_f|^2 \, \text{vol}_{f^*g_0}.
\]

Different scaling conventions for \( H_f \) and equation (4.5.9) lead to a confusing variety of definitions for the Willmore energy in the literature, another favorite variant being \( \int_{\Sigma} |H_f|^2 \, \text{vol}_{f^*g_0} = \int_{\Sigma} (\kappa_1 - \kappa_2)^2 \, \text{vol}_{f^*g_0} \). In general, all these energies differ by a total Gauss curvature expressions and by boundary integrals involving Dirichlet and Neumann boundary data and thus lead to essentially the same optimization problems when subject to both Dirichlet and Neumann-type boundary conditions.

With Figure 4.8 and Figure 4.9, we provide two numerical examples for discretized pseudogradient flows for the Willmore functional subject to constrained total area. We use triangle meshes as discrete surfaces, so that the Willmore energy has also to be discretized (see [8] for details on the discretized Willmore energy that we used).

The numerical experiments we have conducted so far indicate that pseudogradient search provides a quite efficient and robust method for minimizing curvature dependent energies—at least if compared to the Willmore flow, the \( L^2 \)-gradient flow of the Willmore energy. The latter is a forth-order parabolic flow and it suffers even more severely from the time discretization issues discussed at the end of Section 4.5.1. More details on the Willmore flow along with numerical examples for comparison can be found, e.g., in [2] and [8].

Concerning the minimization of the Willmore energy of surfaces, we point out that there is a further very efficient method which involves an \( L^2 \)-gradient descent in “curvature space” (see [6]). However, this method is heavily based on the very special relationship between mean curvature and conformal geometry and it is not clear if and how this method can be carried over to other geometric energies.

### 4.6 Final Remarks

Another interesting feature of pseudogradient flows is that all points of a flow trajectory have regularity not below the initial condition, provided that the used Riesz structure is defined by an elliptic operator. This does not directly imply that critical points are arbitrarily smooth (since they will usually be obtained only as limit of the trajectory for \( t \to \infty \)), but, maybe, this can exploited for regularity theory in cases where long time existence and a priori bounds for the behavior along the flow (e.g., in the spirit of Grönwall’s inequalities) can be provided.

We plan to apply the presented techniques to knot energies such as the Möbius energy and integral Menger curvatures. Their \( L^2 \)-gradient flows (if existent) would be parabolic of fractional order somewhere between 2 and 4. The Möbius energy is best
Fig. 4.8: Discrete surface (≈ 93k faces) relaxing under the pseudogradient flow of the Willmore energy subject to an equality constraint on total area: (a) initial condition; (b)–(e) iterations 6, 12, 18, 24, and 30 of the flow. Initial model kindly provided to the public by Keenan Crane.
Fig. 4.9: Discrete surface (∼ 42k faces) relaxing under the pseudogradient flow of the Willmore energy subject to an equality constraint on total area: (a) initial condition; (b)–(e) iterations 6, 12, 18, 24, and 30 of the flow. Initial model kindly provided to the public by Keenan Crane.
described on the constraint manifold of an arclength parametrization constraint.\footnote{Indeed, this is one of the reasons why we also discussed arbitrary Riesz manifolds.} Moreover, integral Menger curvatures become arbitrarily small when scaling the knot with large factors (the same occurs for the Euler-Bernoulli energy), and constraints such as total length or parameterization by arclength have to be imposed in order to obtain a well-posed optimization problem. Thus, both energies require, in one way or the other, a nonlinear constraint for using smooth optimization techniques. We expect that their analysis might benefit significantly from pseudogradient descent with respect to suitable (yet to be found) Riesz structures.

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