Entropy-based random models for hypergraphs

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Network theory has primarily focused on pairwise relationships, disregarding many-body interactions: neglecting them, however, can lead to misleading representations of complex systems. Hypergraphs represent an increasingly popular alternative for describing polyadic interactions: our innovation lies in leveraging the representation of hypergraphs based on the incidence matrix for extending the entropy-based framework to higher-order structures. In analogy with the Exponential Random Graphs, we name the members of this novel class of models Exponential Random Hypergraphs. Here, we focus on two explicit examples, i.e. the generalisations of the Erdős-Rényi Model and of the Configuration Model. After discussing their asymptotic properties, we employ them to analyse real-world configurations: more specifically, i) we extend the definition of several network quantities to hypergraphs, ii) compute their expected value under each null model and iii) compare it with the empirical one, in order to detect deviations from random behaviours. Differently from currently available techniques, ours is analytically tractable, scalable and effective in singling out the structural patterns of real-world hypergraphs differing significantly from those emerging as a consequence of simpler, structural constraints.

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I. INTRODUCTION

Networks provide a powerful language to model interacting systems [1, 2]. Within a network framework, the basic unit of interaction, i.e. the edge, involves two nodes and the complexity of the structure as a whole arises from the combination of these units. Despite its many successes, network science disregards certain aspects of interacting systems, notably the possibility that more-than-two constituent units could interact at a time [3]. Yet, it has been increasingly shown that, for a variety of systems, interactions cannot be always decomposed into a pairwise fashion and that neglecting higher-order ones can lead to an incomplete, if not misleading, representation of them [3–5]. Examples include chemical reactions involving several compounds, coordination activities within small teams of co-working people, brain activities mediated by groups of neurons. Generally speaking, thus, modelling the joint coordination of multiple entities calls for a framework that generalises the traditional, edge-centered one. An increasingly popular alternative to support a science of many-body interactions is provided by hypergraphs as these versatile, mathematical objects naturally allow nodes to interact in groups [6].

Given the recent interest towards these objects, the definition of analytical tools to study them is still in its infancy [7–10]. This paper represents our contribution to fill the gap: hereby, we extend the class of entropy-based null models [11, 12] to hypergraphs. These models work by preserving a given set of quantities while randomising everything else, hence destroying all, possible correlations between structural properties except for those that are genuinely embodied by the constraints themselves [13–15]. The versatility of such an approach allows it to be employed either in presence of full information (to quantify the level of self-organisation of a given configuration by spotting out the patterns that are incompatible with simpler, structural constraints [16–20]) or in presence of partial information (to infer the missing portion of a given configuration [21]).

Our strategy for defining null models for hypergraphs is based on the randomisation of their incidence matrix. The (generally, rectangular) table contains information about the connectivity of nodes, i.e. the set of hyperedges they belong to, and the connectivity of hyperedges, i.e. the set of nodes they cluster. We will explicitly derive two members of the novel class of models hereby named Exponential Random Hypergraphs, i.e. the Random Hypergraph Model (RHM, generalising the Erdős-Rényi Model) and the Hypergraph Configuration Model (HCM, generalising the Configuration Model), and provide an analytical characterisation of their behaviour. To this aim, we will exploit the formal equivalence between the incidence matrix of a hypergraph and the biadjacency matrix of a bipartite graph [8, 10]. As a next step, we...
II. FORMALISM AND BASIC QUANTITIES

A hypergraph can be defined as a pair $H(\mathcal{V}, \mathcal{E}_H)$ where $\mathcal{V}$ is the set of vertices and $\mathcal{E}_H$ is the set of hyperedges. Moving from the observation that the edge set $\mathcal{E}_G$ of a traditional, binary, undirected graph $G(\mathcal{V}, \mathcal{E}_G)$ is a subset of the power set of $\mathcal{V}$, several definitions of the hyperedge set $\mathcal{E}_H$ have been provided: the two most popular ones are those proposed in [22, 23], where hyperedges ‘link’ one or more vertices, and in [24], where hyperedges are allowed to be empty sets as well. Hereby, we adopt the definition according to which $\mathcal{E}_H$ is a multiset of the power set of $\mathcal{V}$: in words, we consider non-simple hypergraphs, admitting loops and parallel edges (i.e. hyperedges involving exactly the same set nodes) of any size, including 0 (corresponding to empty hyperedges), and $|\mathcal{V}|$ (corresponding to hyperedges clustering all vertices together).

As for traditional graphs, an algebraic representation of hypergraphs can be devised as well. For a formal analogy with the traditional case, let us call the cardinality of the set of nodes $|\mathcal{V}| = N$ and the cardinality of the set of hyperedges $|\mathcal{E}_H| = L$: then, we consider the $N \times L$ table known as incidence matrix, each row of which corresponds to a node and each column of which corresponds to a hyperedge. If we indicate the incidence matrix with $I$, its generic entry $I_{i\alpha}$ will be 1 if vertex $i$ belongs to hyperedge $\alpha$ and 0 otherwise. Notice that the number of 1s along each column can vary between 0 and $N$, the former case indicating an empty hyperedge and the latter one indicating a hyperedge that includes all nodes. As explicitly noticed elsewhere [8, 10], representing a hypergraph via its incidence matrix is equivalent at considering the bipartite graph defined by the sets $\mathcal{V}$ and $\mathcal{E}_H$. Table I represents the incidence matrix describing the binary, undirected hypergraph shown in Fig. 1.

Once the incidence matrix has been defined, several quantities needed for the description of hypergraphs can be defined quite straightforwardly: for example, the ‘degree of node $i$’ (hereby, degree) reads

$$k_i = \sum_{\alpha=1}^{L} I_{i\alpha}$$

and counts the number of hyperedges that are incident to it; analogously, the ‘degree of hyperedge $\alpha$’ (hereby, hyperdegree) reads

$$h_\alpha = \sum_{i=1}^{N} I_{i\alpha}$$

and counts the number of nodes it clusters. Both the sum of degrees and that of hyperdegrees equal the total number of 1s, i.e. $\sum_{i=1}^{N} k_i = \sum_{\alpha=1}^{N} \sum_{i=1}^{L} I_{i\alpha} = \sum_{\alpha=1}^{L} \sum_{i=1}^{N} I_{i\alpha} = \sum_{\alpha=1}^{N} h_\alpha \equiv T$. Importantly, a node degree no longer coincides with the number of its neighbours: instead, it matches the number of hyperedges it belongs to; a hyperdegree, instead, provides information on the hyperedge size.

III. BINARY, UNDIRECTED HYPERGRAPHS RANDOMISATION

An early attempt to define randomisation models for hypergraphs can be found in [25]. Its authors, however, have just considered hyperedges that are incident to triples of nodes - a framework that has been, later, applied to the study of the World Trade Network [8]. Considering the incidence matrix has two, clear advantages over the tensor-based representation employed in [8, 25]: i) generality, because the incidence matrix
matrix allows hyperedges of any size to be handled at once; ii) compactness, because the order of the tensor $I$ never exceeds two, hence allowing any hypergraph to be represented as a traditional, bipartite graph.

In order to extend the rich set of null models induced by graph-specific, global and local constraints to hypergraphs, we, first, need to identify the quantities that can play this role within the novel setting. In what follows, we will consider the total number of 1s, i.e. $T$, the degree and the hyperdegree sequences, i.e. $\{k_i\}_{i=1}^N$ and $\{h_{i\alpha}\}_{i=1}^L$ - either separately or in a joint fashion; moreover, we will distinguish between microcanonical and canonical randomisation techniques.

A. Homogeneous benchmarks: the Random Hypergraph Model (RHM)

The model is defined by just one, global constraint, that, in our case, reads $T = \sum_{i=1}^N \sum_{\alpha=1}^L I_{i\alpha}$. Its microcanonical version extends the model by Erdős and Rényi [26] - also known as Random Graph Model (RGM) - to hypergraphs and prescribes to count the number of incidence matrices that are compatible with a given, total number of 1s, say $T^*$: they are

$$\Omega_{\text{RHM}} = \left( \begin{array}{c} V \\ T^* \end{array} \right)$$

with $V = NL$ being the total number of entries of the incidence matrix $I$. Once the total number of configurations composing the microcanonical ensemble is determined, a procedure to generate them is needed: in the case of the RHM, it simply boils down to reshuffling the entries of the incidence matrix, a procedure ensuring that the total number of 1s is kept fixed while any other correlation is destroyed.

The canonical version of the RHM, instead, extends the model by Gilbert [27] and rests upon the constrained maximisation of Shannon entropy, i.e.

$$\mathcal{L} \equiv S[P] - \sum_{i=0}^M \theta_i [P(I)C_i(I) - \langle C_i \rangle]$$

where $S[P] = -\sum_{I \in \mathcal{I}} P(I) \ln P(I)$, $C_0 \equiv \langle C_0 \rangle \equiv 1$ sums up the normalisation condition and the remaining $M - 1$ constraints represent proper, topological properties. The sum defining Shannon entropy runs over the set $\mathcal{I}$ of incidence matrices described in the introductory paragraph and known as canonical ensemble.

Such an optimisation procedure defines the Exponential Random Hypergraph (ERH) framework, described by the expression

$$P(I) = \frac{e^{-H(I)}}{Z} = \sum_{I \in \mathcal{I}} e^{-H(I)} = \sum_{I \in \mathcal{I}} e^{-\sum_{i=1}^M \theta_i C_i(I)}$$

(5)

In the simplest case, the only, global constraint is represented by $T$ and leads to the expression

$$P(I) = \frac{e^{-\theta T(I)}}{\sum_{I \in \mathcal{I}} e^{-\theta T(I)}}$$

$$= \frac{e^{-\sum_{i=1}^N \sum_{\alpha=1}^L \theta I_{i\alpha}}}{\sum_{I \in \mathcal{I}} e^{-\sum_{i=1}^N \sum_{\alpha=1}^L \theta I_{i\alpha}}}$$

$$= \prod_i \prod_{\alpha=1}^L x^{I_{i\alpha}} \prod_{i=1}^N \prod_{\alpha=1}^L (1+x)^{-1}$$

(6)

that can be re-written as

$$P(I) = \prod_{i=1}^N \prod_{\alpha=1}^L p^{I_{i\alpha}} (1-p)^{1-I_{i\alpha}} = p^{T(I)} (1-p)^{NL-T(I)}$$

(7)

with $e^{-\theta} \equiv x$ and $p \equiv x/(1+x)$. The canonical ensemble, now, includes all $N \times L$ rectangular matrices whose number of entries equating 1 ranges from 0 to $NL$. According to such a model, the entries of the incidence matrix are i.i.d. Bernoulli random variables, i.e. $I_{i\alpha} \sim \text{Ber}(p)$, $\forall i, \alpha$: as a consequence, the total number of 1s, the degrees and the hyperdegrees obey Binomial distributions - being all defined as sums of i.i.d. Bernoulli random variables. Specifically, $T \sim \text{Bin}(NL,p)$, $k_i \sim \text{Bin}(L,p)$, $\forall i$ and $h_{i\alpha} \sim \text{Bin}(N,p)$, $\forall \alpha$, in turn, implying that $\langle T \rangle_{\text{RHM}} = NLp$, $\langle k_i \rangle_{\text{RHM}} = Lp$, $\forall i$ and $\langle h_{i\alpha} \rangle_{\text{RHM}} = Np$, $\forall \alpha$.

In order to ensure that $\langle T \rangle_{\text{RHM}} = T^*$, parameters have to be tuned opportune. To this aim, the likelihood maximisation principle can be invoked [28]: it prescribes to maximise the function $\mathcal{L}(\theta) \equiv \ln P(I^*|\theta)$ with respect to the unknown parameter(s) that define it. Such a recipe leads us to find $p = T^*/NL$, with $T^* = T(I^*)$ indicating the empirical value of the constraint defining the RHM.

The RHM (also considered in [10], although a derivation from first principles is missing, there) is formally equivalent to the Bipartite Random Graph Model (BiRGM) [29]. Such an identification is guaranteed by our focus on non-simple hypergraphs.

1. Estimation of the number of empty hyperedges

As non-simple hypergraphs admit the presence of empty as well as parallel hyperedges, let us evaluate how
frequently they appear in the ensembles induced by our benchmarks.

To this aim, let us start by considering the probability for the generic hyperedge $\alpha$ to be empty or, in other terms, that its hyperdegree $h_\alpha$ is zero. Upon remembering that $h_\alpha \sim \text{Bin}(N, p)$, one finds

$$p_0 \equiv (1 - p)^N. \quad (8)$$

Such a quantity leads us to find the probability of observing at least one, empty hyperedge (or, in other terms, the complementary of the probability that no hyperedge is empty), i.e.

$$p_1 \equiv 1 - (1 - p)^L = 1 - [1 - (1 - p)^N]^L, \quad (9)$$

and the expected number of empty hyperedges: upon remembering that they are i.i.d. Binomial random variables, one finds

$$\langle N_0 \rangle \equiv Lp_0 = L(1 - p)^N. \quad (10)$$

Let us, now, inspect the behaviour of the aforementioned quantities on the ensemble induced by the RHM, as the density of 1s in the incidence matrix, i.e. $p = T/NL$, varies. The two regimes of interest are the dense one, defined by $T \to NL$, and the sparse one, defined by $T \to 0$. In the dense case, one finds

$$\lim_{T \to NL} p_0 = \lim_{T \to NL} \left(1 - \frac{T}{NL}\right)^N = 0, \quad (11)$$

a relationship inducing $p_1 \xrightarrow{T \to NL} 0$ and $\langle N_0 \rangle \xrightarrow{T \to NL} 0$; in words, the probability of observing empty hyperedges progressively vanishes as the density of 1s increases. Consistently, in the sparse case one finds

$$\lim_{T \to 0} p_0 = \lim_{T \to 0} \left(1 - \frac{T}{NL}\right)^N = 1, \quad (12)$$

a relationship inducing $p_1 \xrightarrow{T \to 0} 1$ and $\langle N_0 \rangle \xrightarrow{T \to 0} L$: in words, the probability of observing empty hyperedges progressively rises as the density of 1s decreases.

To evaluate the density of 1s in the incidence matrix in correspondence of which the transition from the sparse to the dense regime happens, let us consider the case $N \gg 1$: more formally, this amounts to consider the asymptotic framework defined by letting $N \to +\infty$ while posing $T = O(L)$, i.e. $p = O(1/N)$. Since $p = T/NL$ and $h \equiv T/L$ remains finite, the probability for a generic hyperedge to be empty remains finite as well: consistently, the expected number of empty hyperedges becomes

$$\lim_{N \to +\infty} \langle N_0 \rangle = \lim_{N \to +\infty} L\left(1 - \frac{h}{N}\right)^N = Le^{-h}; \quad (13)$$

upon imposing $Le^{-h} \leq 1$, i.e. that the expected number of empty hyperedges is at most 1, one derives what may be called filling threshold, corresponding to $h^\text{RHM} = \ln L$. In words, a value $p > p^\text{RHM} \equiv h^\text{RHM}/N = \ln L/N$ ensures that the expected number of empty hyperedges in our random hypergraph is strictly less than one. The calculation above complements the result derived in [10] about the percolation threshold for hypergraphs (see also Appendix A).

2. Estimation of the number of parallel hyperedges

Let us, now, move to considering the issue of parallel hyperedges. By definition, two, parallel hyperedges $\alpha$ and $\beta$ are characterised by identical columns: hence, their Hamming distance, defined as the number of positions at which the corresponding symbols are different, is zero. More formally,

$$d_{\alpha \beta} \equiv \sum_{i=1}^N [I_{i\alpha}(1 - I_{i\beta}) + I_{i\beta}(1 - I_{i\alpha})], \quad (14)$$

a sum whose generic addendum is 1 in just two cases: either $I_{i\alpha} = 1$ and $I_{i\beta} = 0$ or $I_{i\alpha} = 0$ and $I_{i\beta} = 1$. Since $d_{\alpha \beta} \sim \text{Bin}(N, 2p(1 - p))$, one finds that $P(d_{\alpha \beta} = 0) = [1 - 2p(1 - p)]^N$ and $\langle d_{\alpha \beta} \rangle = 2p(1 - p)N, \forall \alpha \neq \beta$. Notice that

$$\lim_{T \to NL} P(d_{\alpha \beta} = 0) = \lim_{T \to 0} P(d_{\alpha \beta} = 0) = 1 \quad (15)$$

and

$$\lim_{T \to NL} \langle d_{\alpha \beta} \rangle = \lim_{T \to 0} \langle d_{\alpha \beta} \rangle = 0 \quad (16)$$

as $(2T/NL)(1 - T/NL) \to 0$ in both regimes: in words, the probability of observing parallel hyperedges progressively rises both as a consequence of having many 1s and as a consequence of having few 1s. Analogously for the expected Hamming distance.

Let us, now, evaluate the expected Hamming distance between any, two hyperedges $\alpha$ and $\beta$, within the asymptotic framework defined by letting $N \to +\infty$ while posing $T = O(L)$, i.e. $p = O(1/N)$. Since $p = T/NL$ and $h \equiv T/L$ remains finite, one finds that
\[
\lim_{N \to \infty} \langle d_{\alpha \beta} \rangle = \lim_{N \to \infty} \frac{2h}{N} \left(1 - \frac{h}{N}\right) N = 2h; \quad (17)
\]
upon imposing \(2h \geq 1\), i.e. that the expected Hamming distance between any, two hyperedges \(\alpha\) and \(\beta\) is at least 1, one derives what may be called \textit{resolution threshold}, corresponding to \(h_{\text{RHM}}^r = 1/2\). In words, a value \(p > p_{r_{\text{RHM}}}^f \equiv h_{\text{RHM}}^r / N = 1/2N\) ensures that, on average, any, two hyperedges \(\alpha\) and \(\beta\) differ by at least one element (see also Appendix A).

3. Estimation of the overlap between hyperedges

Finally, let us move to considering the issue of overlapping hyperedges. To this aim, let us require that the probability for any, two hyperedges to overlap amounts to 1. In formulas,
\[
1 - (1 - p^2)^N = 1, \quad (18)
\]
a relationship leading to the equation
\[
(1 - p^2)^N \simeq 1 - Np^2 = 0 \quad (19)
\]
that holds true in the sparse regime and further implies
\[
h^2 = N; \quad (20)
\]
such a value, firstly derived in [10], individuates the so-called \textit{percolation threshold}, corresponding to \(h_{\text{RHM}}^f = \sqrt{N}\). In words, a value \(p > p_{f_{\text{RHM}}}^f \equiv h_{\text{RHM}}^f / N = 1/\sqrt{N}\) ensures that any, two hyperedges in our random hypergraph share at least one node.

Since \(h_{\text{RHM}}^r \leq h_{\text{RHM}}^f \leq h_{\text{RHM}}^L\), such a result contributes to describe the following situation: by progressively rising the parameter \(p\), three, different thresholds are met, i.e. \(i\) the resolution threshold \(p_{r_{\text{RHM}}} \equiv 1/2N\), ensuring that, on average, any, two hyperedges differ by at least one element; \(ii\) the filling threshold \(p_{f_{\text{RHM}}} \equiv \ln L/N\), ensuring that the expected number of empty hyperedges is strictly less than one; \(iii\) the percolation threshold \(p_{f_{\text{RHM}}}^f = 1/\sqrt{N}\), ensuring that any, two hyperedges overlap.

B. Heterogeneous benchmarks: the Hypergraph Configuration Model (HCM)

The number of constraints can be enlarged to include the degrees, i.e. the sequence \(\{k_i\}_{i=1}^N\), and the hyperdegrees, i.e. the sequence \(\{h_{\alpha}\}_{\alpha=1}^L\). Counting the number of configurations on which both sequences match their empirical values is a hard task - although numerical recipes to shuffle the entries of a rectangular matrix, while preserving its marginals, exist [7, 30, 31]. Solving the corresponding problem in the canonical framework is, instead, straightforward. Indeed, Shannon entropy maximisation leads to
\[
P(I) = \frac{e^{-\sum_{i=1}^N \alpha_i k_i(I) - \sum_{\alpha=1}^L \beta_\alpha h_{\alpha}(I)}}{\sum_{I \in \mathcal{I}} e^{-\sum_{i=1}^N \alpha_i k_i(I) - \sum_{\alpha=1}^L \beta_\alpha h_{\alpha}(I)}}
\]
\[
= \frac{\prod_{i=1}^N x_{i,(I)} \prod_{\alpha=1}^L y_{\alpha,(I)} \prod_{i=1}^N \prod_{\alpha=1}^L (1 + x_i y_{\alpha})^{-1}}, \quad (21)
\]
an expression that can be re-written as
\[
P(I) = \prod_{i=1}^N \prod_{\alpha=1}^L p_{i\alpha} (1 - p_{i\alpha})^{1 - I_{i\alpha}} \quad (22)
\]
with \(e^{-\alpha_i} \equiv x_i, \forall i, e^{-\beta_\alpha} \equiv y_{\alpha}, \forall \alpha\) and \(p_{i\alpha} \equiv x_i y_{\alpha}/(1 + x_i y_{\alpha}), \forall i, \alpha\). According to such a model, the entries of the incidence matrix of a hypergraph are independent random variables that obey different Bernoulli distributions, i.e. \(I_{i\alpha} \sim \text{Ber}(p_{i\alpha}), \forall i, \alpha\). As a consequence, both degrees and hyperdegrees obey Poisson-Binomial distributions, i.e. \(k_i \sim \text{PoissBin}(L, \{p_{i\alpha}\}_{i=1}^L), \forall i, \alpha\) and \(h_{\alpha} \sim \text{PoissBin}(N, \{p_{i\alpha}\}_{i=1}^N), \forall \alpha\) [17].

In this case, solving the likelihood maximisation problem amounts to solve the system of coupled equations
\[
k_i^* = \sum_{\alpha=1}^L \frac{x_i y_{\alpha}}{1 + x_i y_{\alpha}} = \sum_{\alpha=1}^L p_{i\alpha} = \langle k_i \rangle, \forall i \quad (23)
\]
\[
h_{\alpha}^* = \sum_{i=1}^N \frac{x_i y_{\alpha}}{1 + x_i y_{\alpha}} = \sum_{i=1}^N p_{i\alpha} = \langle h_{\alpha} \rangle, \forall \alpha \quad (24)
\]
ensuring that \(\langle k_i \rangle = k_i^*, \forall i, \langle h_{\alpha} \rangle = h_{\alpha}^*, \forall \alpha\) (and, as a consequence, \(\langle T \rangle = T^*\)). In case hypergraphs are sparse and in absence of hubs
\[
p_{i\alpha} \simeq x_i y_{\alpha} = \frac{k_i^* h_{\alpha}^*}{L}, \forall i, \alpha. \quad (25)
\]
The HCM reduces to a ‘partial’ Configuration Model [17] when either the degree or the hyperdegree sequence is left unconstrained (see also Appendix B). The canonical ensemble of each randomisation model (Table II in Appendix B sums up the set of constraints
The HCM is formally equivalent to the Bipartite Configuration Model (BiCM) [29]. Such an identification is guaranteed by our focus on non-simple hypergraphs.

1. Estimation of the number of empty hyperedges

Let us, now, consider the probability for the generic hyperedge $\alpha$ to be empty or, in other terms, that its hyperdegree $h_{\alpha}$ is zero. Upon remembering that $h_{\alpha} \sim \text{PoissBin}(N, \{p_{i\alpha}\}_{i=1}^{N})$, one finds

$$p_{0}^{\alpha} = \prod_{i=1}^{N} (1 - p_{i\alpha}).$$

(26)

Such a quantity leads to the probability of observing at least one, empty hyperedge, now reading

$$p_{1} = 1 - \prod_{\alpha=1}^{L} [1 - p_{0}^{\alpha}] = 1 - \prod_{\alpha=1}^{L} \left[ 1 - \prod_{i=1}^{N} (1 - p_{i\alpha}) \right],$$

(27)

and to the expected number of empty hyperedges

$$\langle N_{0} \rangle = \sum_{\alpha=1}^{L} p_{0}^{\alpha} = \sum_{\alpha=1}^{L} \prod_{i=1}^{N} (1 - p_{i\alpha}).$$

(28)

As in the previous section, let us, now, inspect the behaviour of the aforementioned quantities on the ensemble induced by the HCM, as the density of 1s in the incidence matrix varies. Although it depends on (the heterogeneity of) the sets of coefficients $\{x_{i}\}_{i=1}^{N}$ and $\{y_{\alpha}\}_{\alpha=1}^{L}$, general conclusions can still be drawn within a simpler framework. To this aim, let us consider the functional form reading

$$p_{i\alpha} = \frac{zf_{i}g_{\alpha}}{1 + zf_{i}g_{\alpha}}, \ \forall \ i, \alpha$$

(29)

where the vector of fitnesses $\{f_{i}\}_{i=1}^{N}$ accounts for the heterogeneity of nodes, the vector of fitnesses $\{g_{\alpha}\}_{\alpha=1}^{L}$ accounts for the heterogeneity of hyperedges and $z$ tunes the density of 1s in the incidence matrix. Within such a framework, the fitnesses of the nodes and the fitnesses of the hyperedges can be drawn from any distribution.

The dense and sparse regimes are, now, defined by the positions $z \to +\infty$ and $z \to 0$, respectively. In the dense case, one finds

$$\lim_{z \to +\infty} p_{0}^{\alpha} = \lim_{z \to +\infty} \prod_{i=1}^{N} (1 - p_{i\alpha})$$

$$= \lim_{z \to +\infty} \prod_{i=1}^{N} \left( 1 - \frac{zf_{i}g_{\alpha}}{1 + zf_{i}g_{\alpha}} \right) = 0,$$

(30)

a relationship inducing $p_{1} \stackrel{z \to +\infty}{\to} 1$ and $\langle N_{0} \rangle \stackrel{z \to +\infty}{\to} 0$: in words, the probability of observing empty hyperedges progressively vanishes as the density of 1s increases. Consistently, in the sparse case one finds

$$\lim_{z \to 0} p_{0}^{\alpha} = \lim_{z \to 0} \prod_{i=1}^{N} (1 - p_{i\alpha})$$

$$= \lim_{z \to 0} \prod_{i=1}^{N} \left( 1 - \frac{zf_{i}g_{\alpha}}{1 + zf_{i}g_{\alpha}} \right) = 1,$$

(31)

a relationship inducing $p_{1} \stackrel{z \to 0}{\to} 1$ and $\langle N_{0} \rangle \stackrel{z \to 0}{\to} L$. In words, the probability of observing empty hyperedges progressively rises as the density of 1s decreases.

2. Estimation of the number of parallel hyperedges

Let us, now, focus on the issue of parallel hyperedges. As in the case of the RHM, we consider the Hamming distance between the columns representing the two hyperedges $\alpha$ and $\beta$. Since, now, $d_{\alpha\beta} \sim \text{PoissBin}(N, \{q_{i}^{\alpha\beta}\}_{i=1}^{N})$, where $q_{i}^{\alpha\beta} = p_{i\alpha}(1 - p_{i\beta}) + p_{i\beta}(1 - p_{i\alpha})$ with $p_{i\alpha} = zf_{i}g_{\alpha}/(1 + zf_{i}g_{\alpha})$ and $p_{i\beta} = zf_{i}g_{\beta}/(1 + zf_{i}g_{\beta})$, one finds that $P(d_{\alpha\beta} = 0) = \prod_{i=1}^{N} (1 - q_{i}^{\alpha\beta})$ and $\langle d_{\alpha\beta} \rangle = \sum_{i=1}^{N} q_{i}^{\alpha\beta}$, $\forall \ \alpha \neq \beta$. Notice that

$$\lim_{z \to +\infty} P(d_{\alpha\beta} = 0) = \lim_{z \to 0} P(d_{\alpha\beta} = 0) = 1$$

(32)

and

$$\lim_{z \to 0} (d_{\alpha\beta}) = \lim_{z \to 0} \langle d_{\alpha\beta} \rangle = 0$$

(33)

as $q_{i}^{\alpha\beta} \to 0$ in both regimes. As in the case of the RHM, the probability of observing parallel hyperedges progressively rises both as a consequence of having many 1s and as a consequence of having few 1s. Analogously for the expected Hamming distance.

When considering heterogeneous benchmarks, closed-form expressions for the asymptotic estimates can be provided only in specific cases (see also Appendix B).

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1 Partial Configuration Models are recovered upon posing either $f_{i} = 1$, $\forall \ i$ or $g_{\alpha} = 1$, $\forall \ \alpha$. 
FIG. 2: Left panel: trend of \( p_1 \xrightarrow{N \to +\infty} 1 - (1 - e^{-h})^L \), i.e. the probability of observing at least one empty hyperedge as a function of \( T/NL \); evaluating it in correspondence of \( h_f^{RHM} = \ln L \simeq 6.907 \) (whose related, critical density value \( p_f^{RHM} = h_f^{RHM}/N = \ln L/N \simeq 0.023 \) individuates a steep transition separating the sparse regime from the dense one) returns the value \( 1 - (1 - 1/L)^L \approx 0.6323 \). Right panel: trend of \( p_1 \) as defined by Eq. \((27)\). Notice that the trend induced by the HCM is shifted on the right with respect to the trend induced by the RHM: this is probably due to the presence of small fitnesses increasing the probability of observing at least one, empty hyperedge, hence requiring a larger value of \( z \) to let \( p_1 \) vanish. The dense (sparse) regime is recovered for large (small) values of \( T/NL \). Each dot represents an average taken over an ensemble of 1000 configurations (explicitly sampled from either the RHM or the HCM) and is accompanied by the corresponding 95% confidence interval.

IV. RESULTS

A. Hypergraphs in the dense and sparse regime

Let us start by verifying the correctness of the estimations of the number of empty and parallel hyperedges provided by our benchmarks: to this aim, we have considered the values \( N = 300 \) and \( L = 1000 \) (for other values, see Appendix A and Appendix B).

1. The Random Hypergraph Model

Let us start focusing on our homogeneous benchmark. The left panel of Fig. 2 depicts the trend of \( p_1 = 1 - \left( 1 - \frac{T}{NL} \right)^L \). The left panel of Fig. 3 depicts the trend of \( \langle N_\alpha \rangle_L = \left( 1 - \frac{T}{NL} \right)^N \). Each quantity has been plotted as a function of \( T/NL \in [10^{-6}, 1] \). The dense (sparse) regime is recovered for large (small) values of \( T/NL \). Each dot of Figs. 2 and 3 represents an average taken over an ensemble of 1000 configurations explicitly sampled from the RHM and is accompanied by the corresponding 95% confidence interval. A first, general comment concerns the agreement between the analytical trends (solid lines) and the numerical estimations (dots), confirming the correctness of our formulas.

The filling threshold. Although the value of the filling threshold has been determined by inspecting the asymptotic behaviour of \( \langle N_\alpha \rangle \), the quantity showing the neatest transition from the sparse to the dense regime is \( p_1 \xrightarrow{N \to +\infty} 1 - (1 - e^{-h})^L \); evaluating it in correspondence of \( h_f^{RHM} = \ln L \simeq 6.907 \) returns the value \( 1 - (1 - 1/L)^L \approx 0.6323 \). Interestingly enough, letting \( L \) grow asymptotically as well, the previous value tends to \( 1 - e^{-1} \approx 0.6321 \).

Upon considering that \( p_0 \xrightarrow{N \to +\infty} e^{-h} \), evaluating it in correspondence of \( h_f^{RHM} = \ln L \simeq 6.907 \) returns the value \( 1/L = 0.001 \): in words, although the filling threshold ensures that each, single hyperedge is empty with an overall small probability, the likelihood of observing at least one, empty hyperedge is still large (i.e. \( \approx 2/3 \)).

The steepness of the trend of \( p_1 \), however, suggests it to quickly vanish as the density of 1s in the incidence matrix crosses the value \( h_f^{RHM} = h_f^{RHM}/N = \ln L/N \approx 0.023 \).

The resolution threshold. The value of the resolution threshold has been determined by inspecting the asymptotic behaviour of \( \langle d_{\alpha\beta} \rangle \): evaluating \( P(d_{\alpha\beta} = 0) \)
ensuring that each, expected hyperdegree amounts to 1. Right panel: trends of $p$ hyperedges are no longer observed, parallel hyperedges a limited amount of parallel hyperedges, when empty portion of the incidence matrix is not required to observe hyperedge to be empty: in words, while filling a large decreases faster than the probability of each, single probability for any, two hyperedges to be parallel

Crossing the thresholds. As Fig. 3 confirms, the probability for any, two hyperedges to be parallel decreases faster than the probability of each, single hyperedge to be empty: in words, while filling a large portion of the incidence matrix is not required to observe a limited amount of parallel hyperedges, when empty hyperedges are no longer observed, parallel hyperedges are no longer observed, too. More formally, $h_{rRHM} = 1/2 < h_{rHCM} = \ln L$: hence, progressively rising the density of 1s in the incidence matrix leads us crossing the resolution threshold before crossing the filling threshold. In more quantitative terms, evaluating $\langle N_0 \rangle \xrightarrow{N \to +\infty} L e^{-h}$ in correspondence of $h_{rRHM} = 1/2$ returns the value $L/\sqrt{e} \approx 606$ which is larger than $\langle d_{\alpha \beta} \rangle \xrightarrow{N \to +\infty} 2h$ evaluated in correspondence of $h_{rRHM} = 1/2$ - returning the value 1 by definition.

2. The Hypergraph Configuration Model

Let us, now, focus on the trends induced by our heterogeneous benchmarks. Following the procedure described in the previous sections, both the fitnesses of nodes and those of hyperedges were drawn from

FIG. 3: Left panel: trends of $p_0 \xrightarrow{N \to +\infty} e^{-h}$ (aquamarine line) and $P(d_{\alpha \beta} = 0) \xrightarrow{N \to +\infty} e^{-2h}$ (dark-blue line) as a function of $T/NL$; evaluating them in correspondence of $h_{rRHM} = 1/2$ (the related, critical density value is indicated in light-blu) returns the values $e^{-h_{rRHM}} = 1/\sqrt{e} \approx 0.606$ and $e^{-2h_{rRHM}} = e^{-1} \approx 0.367$; evaluating them in correspondence of $h_{rHCM} = \ln L \approx 6.907$ (the related, critical density value is indicated in magenta) returns the values $e^{-h_{rHCM}} = 1/L = 0.001$ and $e^{-2h_{rHCM}} = 1/L^2 = 10^{-6}$. The red line indicates the density value corresponding to $h = 1$, ensuring that each, expected hyperdegree amounts to 1. Right panel: trends of $p_0$, as defined by Eq. (20), and $P_0$, i.e. the percentage of pairs of parallel hyperedges. As already noticed, the shift on the right of the trend induced by the HCM is probably due to the presence of small fitnesses, decreasing the magnitude of the set of probability coefficients. The dense (sparse) regime is recovered for large (small) values of $T/NL$. Each dot represents an average taken over an ensemble of 1000 configurations (explicitly sampled from either the RHM or the HCM) and is accompanied by the corresponding 95% confidence interval. The yellow line indicates the density value in correspondence of which the minimum, expected hyperdegree amounts to 1.
a Pareto distribution\(^2\) with \(\alpha = 2\). The right panel of Fig. 2 depicts the trend of \(p_1\), as defined by Eq. (27). The right panel of Fig. 3 depicts the trend of \(p_0\), as defined by Eq. (26), and the arithmetic mean \(P_0 = \frac{\sum_{i=0}^{N} P(d_{\alpha \beta} = 0) / L(L - 1)}{\sum_{i=0}^{N} P(d_{\alpha \beta} = 0)} = \prod_{i=1}^{N-1} (1 - q_i)\), i.e. the average of the value \(P(d_{\alpha \beta} = 0)\) over all pairs of hyperedges - a choice dictated by the will of accounting for the heterogeneity of different pairs of hyperedges. Both quantities have been plotted as a function of \((T) / NL \in [10^{-6}, 1]\) where \((T) = \sum_{i=1}^{N} \sum_{\alpha=1}^{L} z_i g_{\alpha} / (1 + z_i g_{\alpha})\). The dense (sparse) regime is recovered for large (small) values of \(z\). Each dot of Figs. 2 and 3 represents an average taken over an ensemble of 1000 configurations explicitly sampled from the HCM and is accompanied by the corresponding 95\% confidence interval. As in the case of the RHM, the agreement between the analytical trends (solid lines) and the numerical estimations (dots) confirms the correctness of our formulas.

Comparing the analytical estimation of (the trend of) the percentage of pairs of parallel hyperedges with its sample counterpart reveals the former to provide a quite accurate estimation of the latter. Besides, we explicitly notice that the trends characterising the HCM are systematically shifted on the right: this is probably due to the presence of small fitnesses that, at least in case of Fig. 2, increase the probability of observing at least one, empty hyperedge, hence requiring a larger value of \(z\) to let \(p_1\) vanish. Such a result seems to be confirmed by the trends depicted in Fig. 3.

Deriving an explicit expression for the filling threshold in the case of the HCM is a rather difficult task; still, we can proceed in a purely numerical fashion and individuate the value of the density of 1s in the incidence matrix guaranteeing that the minimum, expected hyperdegree amounts to 1: as evident from Fig. 3, such a threshold still lies in the right tail of the trends induced by the HCM - although its, precise, numerical value depends on those of the fitnesses.

B. Solving the HCM on real-world hypergraphs

In order to test our benchmarks on real-world configurations, we have focused on a number of data sets taken from Austin R. Benson’s website\(^3\), i.e. the contact-primary-school, the email-Enron and the NDC-classes ones.

To have an explicit expression for the expected value of any quantity of interest, let us comment on the goodness of the approximation

\[
p_{i\alpha} \simeq x_i y_{\alpha} = \frac{k_i^* h_{\alpha}^*}{T^*}, \quad \forall i, \alpha. \tag{34}
\]

As Fig. 7 in Appendix A shows, it is is quite accurate for each data set considered here - in fact, one can safely assume that \(x_i \simeq k_i^* / \sqrt{T^*}, \forall i\) and \(h_{\alpha}^* / \sqrt{T^*}, \forall \alpha\).

C. ‘Hypergraph to graph’ projection

The canonical formalism that we have adopted leads to factorisable distributions, i.e. distributions that can be written as a product of pair-wise probability distributions; this allows the expectation of several quantities of interest to be evaluated analytically. To this aim, let us consider the matrix introduced in [4], reading

\[
W = \mathbf{I} \cdot \mathbf{I}^T - K \tag{35}
\]

with \(K\) being the diagonal matrix whose \(i\)-th entry reads \(k_i\); according to the definition above

\[
w_{ij} = \sum_{\alpha=1}^{L} I_{i\alpha} I_{j\alpha} - \delta_{ij} k_i \tag{36}
\]

counts the number of hyperedges nodes \(i\) and \(j\) are connected by - more explicitly, \(w_{ij} = \sum_{\alpha=1}^{L} I_{i\alpha} I_{j\alpha}, i \neq j\) and \(w_{ii} = \sum_{\alpha=1}^{L} I_{i\alpha} I_{i\alpha} = k_i = \sum_{\alpha=1}^{L} I_{i\alpha} - k_i = k_i - k_i = 0\). The null models discussed so far can be employed to calculate \(\langle w_{ij} \rangle, i \neq j\) that, in a perfectly general fashion, reads

\[
\langle w_{ij} \rangle = \sum_{\alpha=1}^{L} \langle I_{i\alpha} I_{j\alpha} \rangle = \sum_{\alpha=1}^{L} \langle I_{i\alpha} \rangle \langle I_{j\alpha} \rangle = \sum_{\alpha=1}^{L} p_{i\alpha} p_{j\alpha}; \tag{37}
\]

the total number of hyperedges shared by node \(i\) with any other node in the hypergraph (in a sense, its ‘strength’ - see also Fig. 4) can be computed as

\[
\sigma_i = \sum_{j=1}^{N} \sum_{(j \neq i)} w_{ij} = \sum_{j=1}^{N} \sum_{(j \neq i)} I_{i\alpha} I_{j\alpha} \tag{38}
\]

whose expected value reads

\[
\langle \sigma_i \rangle = \sum_{j=1}^{N} \langle w_{ij} \rangle = \sum_{\alpha=1}^{L} p_{i\alpha} [h_{\alpha} - p_{i\alpha}]. \tag{39}
\]

\(^2\) Other fat-tailed distributions were considered: qualitatively, results do not change.

\(^3\) https://www.cs.cornell.edu/~arb/data/
average incident hyperedges degree

Let us, now, extend the concept of assortativity to hypergraphs. To this aim, we consider the quantity named average incident hyperedges degree, defined as

\[ k_i^{an} = \frac{\sum_{\alpha=1}^{L} I_{i\alpha} h_{\alpha}}{k_i} = \frac{\sigma_i + k_i}{k_i} + 1 \approx \frac{\sigma_i}{k_i} \]  

and representing the arithmetic mean of the degrees of the hyperedges including node \( i \). An analytical approximation of its expected value can be provided as well:

\[ \langle k_i^{an} \rangle \approx \frac{\sum_{\alpha=1}^{L} p_{i\alpha} [\langle h_{\alpha} \rangle + 1 - p_{i\alpha}]}{\langle k_i \rangle} = \frac{\langle \sigma_i \rangle + \langle k_i \rangle}{\langle k_i \rangle} = \frac{\langle \sigma_i \rangle}{\langle k_i \rangle} + 1 \approx \frac{\langle \sigma_i \rangle}{\langle k_i \rangle}. \]  

D. Disparity ratio and degree in the projection

More information about the patterns shaping real-world hypergraphs can be obtained upon defining the ratio \( f_{ij} = w_{ij}/\sigma_i, \ i \neq j \) that induces the quantity

\[ Y_i = \sum_{j=1}^{N} \left( \frac{f_{ij}^{2}}{\sigma_i^2} \right) \]  

known as disparity ratio and quantifying the (un)evenness of the distribution of the weights constituting the strength of node \( i \) over the \( \kappa_i = \sum_{j=1}^{N} \Theta[w_{ij}] \equiv \sum_{j=1}^{N} a_{ij} \) links characterising its connectivity - since \( a_{ij} = 1 \) if nodes \( i \) and \( j \) share, at least, one hyperedge, \( \kappa_i \) is the degree of node \( i \) in the projection of the hypergraph (see also Fig. 4 and Fig. 5b). Since, under the RHM, \( w_{ij} \sim \text{Bin}(L, p^2) \), we find that \( \langle a_{ij} \rangle = 1 - (1 - p^2)^L \), i.e. the expected value of \( a_{ij} \) coincides with the probability of observing a non-zero overlap. Under the HCM, instead, \( w_{ij} \sim \text{PoisBin}(L, \{p_{i\alpha}p_{j\alpha}\}_{\alpha=1}^{L}) \), hence

\[ \langle a_{ij} \rangle = 1 - \prod_{\alpha=1}^{L} (1 - p_{i\alpha}p_{j\alpha}). \]  

Let us also notice that

\[ Y_i = \frac{1}{\kappa_i} \]  

in case weights are equally distributed among the connections established by node \( i \), i.e. \( w_{ij} = a_{ij}\sigma_i/\kappa_i, \ i \neq j \). Any larger value signals an excess concentration of weight in one or more links. An analytical approximation of the expected value of the disparity ratio of node \( i \) can be provided as well:

\[ \langle Y_i \rangle \approx \sum_{j=1}^{N} \frac{\langle w_{ij}^2 \rangle}{\langle w_{ij} \rangle^2}. \]  

Contrarily to what has been previously observed, the expected value of the disparity ratio cannot be always safely decomposed as a ratio of expected values - not even if the 'full' HCM is employed. In fact, while this approximation works relatively well for the contact-primary-school data set, it does not for the email-Enron and the NDC-classes ones (see also Fig. 10 in Appendix C). For this reason, the expected value of the disparity ratio has been evaluated by explicitly sampling the ensemble of incidence matrices induced by the 'full' HCM. In any case, as Fig. 5c shows, such a null...
Observed $\sigma_i$, $\kappa_i$, $Y_i$, CEC$_i$ and the expected values $\langle \sigma_i \rangle_{\text{HCM}}$, $\langle \kappa_i \rangle_{\text{HCM}}$, $\langle Y_i \rangle_{\text{HCM}}$, $\langle \text{CEC}_i \rangle_{\text{HCM}}$. The HCM overestimates the extent to which any two nodes overlap, as well as the CEC; the disparity ratio, instead, is underestimated by it. These results can be understood by considering that the HCM just constrains the degree sequences, hence inducing an ensemble where connections are ‘distributed’ more evenly than observed.

F. Eigenvector centrality

Centrality measures for hypergraphs have been defined as well. An example is provided by the *clique motif eigenvector centrality* (CEC), defined in [34] (see also Appendix D): CEC$_i$ corresponds to the $i$-th entry of the Perron-Frobenius eigenvector of $W$. As Fig. 5d shows, the HCM underestimates the CEC as well: such a result can be understood by considering that the HCM constrains only the degree sequences, hence inducing an ensemble where connections are ‘distributed’ more evenly than observed, an evidence letting the nodes overlap more, thus causing the entries of $\langle W \rangle$ to be overall larger and less dissimilar, as well as those of its Perron-Frobenius eigenvector.

G. Confusion matrix

Let us, now, consider the set of properties composing the so-called *confusion matrix* (see also Appendix E). They are known as the *true positive rate* (TPR), i.e. the percentage of 1s correctly recovered by a given method and whose expected value reads

$$\langle \text{TPR} \rangle = \sum_{i=1}^{N} \sum_{\alpha=1}^{L} \frac{I_{i\alpha} p_{i\alpha}}{T};$$

the *specificity* (SPC), i.e. the percentage of 0s correctly recovered by a given method and whose expected value reads

$$\langle \text{SPC} \rangle = \sum_{i=1}^{N} \sum_{\alpha=1}^{L} \frac{(1 - I_{i\alpha})(1 - p_{i\alpha})}{NL - T};$$

the *positive predictive value* (PPV), i.e. the percentage of 1s correctly recovered by a given method with respect to the total number of 1s predicted by it and whose expected value reads

$$\langle \text{PPV} \rangle = \sum_{i=1}^{N} \sum_{\alpha=1}^{L} \frac{I_{i\alpha} p_{i\alpha}}{(T)};$$

and the *accuracy* (ACC), measuring the overall performance of a given method in correctly placing both 1s and 0s and whose expected value reads

$$\langle \text{ACC} \rangle = \frac{\langle \text{TP} \rangle + \langle \text{TN} \rangle}{NL}.$$
FIG. 6: Top panels: projections of the contact-primary-school, email-Enron and NDC-classes data sets onto the layer of nodes. Bottom panels: validated counterparts of the aforementioned projections: any two nodes are linked if they share a significantly large number of hyperedges. Communities have been detected by running the Louvain algorithm.

approximated version) to reproduce the density of 1s - and, as a consequence, the density of 0s - ensures the SPC to be recovered quite precisely, in turn ensuring the overall accuracy of the model to be large (for an overall evaluation of the performance of the HCM in reproducing real-world hypergraphs, see also Table IV in Appendix F).

H. Community detection

Communities are commonly understood as densely connected groups of nodes. Representing an hypergraph via its incidence matrix allows this statement to be made more precise from a statistical perspective: in fact, the null models discussed so far can be employed to test if any two nodes share a significantly large number of hyperedges - hence can be clustered together, should this be the case.

To this aim, we can adapt the recipe proposed in [17] to project bipartite networks and summarised in the following. One, first, computes

\[ p\text{-value}(w_{ij}^*) = \sum_{x \geq w_{ij}^*} f(x) \]  

(50)

for each pair of nodes; \( f(x) \) depends on the chosen null model: in case the RHM is employed, it coincides with the Binomial distribution Bin\( (x|L, p) \); in case the HCM is employed, it coincides with the Poisson-Binomial distribution PoissBin\( (x|L, \{p_{\alpha} p_{j\alpha}\}_{\alpha=1}^L) \). Second, one implements the FDR procedure, designed to handle multiple tests of hypothesis [35]: in practice, after ranking the p-values in increasing order, i.e. \( p\text{-value}_1 \leq p\text{-value}_2 \leq \cdots \leq p\text{-value}_n \), one individuates the largest integer \( i \) satisfying the condition

\[ p\text{-value}_i \leq \frac{it}{n} \]  

(51)

where \( n = N(N-1)/2 \) and \( t \) is the single-test significance level, set to 0.01 in the present analysis. Third, one links the (pairs of) nodes whose related p-value is smaller than the aforementioned threshold.
V. CONCLUSIONS

Our paper contributes to current research on hypergraphs by extending the constrained entropy-maximisation framework to incidence matrices, i.e. their simplest, tabular representation. Differently from the currently-available techniques [7], our methodology has the advantage of being analytically tractable, scalable and versatile enough to be straightforwardly extensible to directed and/or weighted hypergraphs.

Beside leading to results whose relevance is mostly theoretical (i.e. the individuation of different regimes for higher-order structures and the estimation of the actual impact of empty and parallel hyperedges on the analysis of empirical systems), our models prove to be particularly useful when employed as benchmarks for real-world systems, i.e. for detecting patterns that are not imputable to purely random effects. Specifically, our results suggest that real-world hypergraphs are characterised by a degree of self-organisation that is absolutely non-trivial (see also Appendix F).

This is even more surprising when considering that our results are obtained under a benchmark such as the HCM, i.e. a null model constraining both the degree and the hyperdegree sequences: since it overestimates the extent to which any two nodes overlap - a result whose relevance becomes evident as soon as one considers the effects that higher-order structures have on spreading and cooperation processes [38–40] - our future efforts will be directed towards the analysis of benchmarks constraining non-linear quantities such as the co-occurrences between nodes and/or hyperedges.

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APPENDIX A.
THE RANDOM HYPERGRAPH MODEL

1. Asymptotic results: estimation of the number of empty hyperedges

In the main text, we derived the expression

\[ p_0 = (1 - p)^N = \left(1 - \frac{T}{NL}\right)^N = \left(1 - \frac{h}{N}\right)^N \]  

(52)

describing the probability that a generic hyperedge is empty. It can be manipulated more easily upon considering the case \( N \gg 1 \). In fact,

\[ p_0 = \left(1 - \frac{h}{N}\right)^N \rightarrow e^{-h} \]  

(53)

where the notation is meant to indicate \( \lim_{N \to +\infty} p_0 \). Such a result affects the asymptotic behaviour of quantities like \( p_1 \) and \( \langle N_0 \rangle \) since

\[ p_1 = 1 - [1 - (1 - p)^N]^L = 1 - [1 - p_0]^L \rightarrow 1 - [1 - e^{-h}]^L \]  

(54)

and

\[ \langle N_0 \rangle = L(1 - p)^N = Lp_0 \rightarrow Le^{-h}. \]  

(55)

Since \( p = T/NL \) and \( h = T/L \) remains finite, we are implicitly posing \( T = O(L) \), i.e. \( p = O(1/N) \). In words, all quantities of interest remains finite, allowing us to derive an explicit expression for the filling threshold. Requiring \( \langle N_0 \rangle = 1 \) leads to

\[ Le^{-h} = 1 \]  

(56)

which is solved by \( h = \ln L \), further inducing \( p = h/N = \ln L/N \). Numerical simulations confirm the correctness of the aforementioned estimations for a wide range of different combinations of \( N \) and \( L \) (see Fig. 6).


2. Asymptotic results: estimation of the number of parallel hyperedges

Let us, now, focus on the issue of parallel hyperedges. The position \( h = T/L \) leads to the results

\[ P(d_{\alpha\beta} = 0) = [1 - 2p(1 - p)]^N = \left[1 - \frac{2T}{NL} \left(1 - \frac{T}{NL}\right)\right]^N = \left[1 - \frac{2h}{N} (1 - \frac{h}{N})\right]^N \rightarrow e^{-2h} \]  

(57)

and

\[ \langle d_{\alpha\beta} \rangle = 2p(1 - p)^N = \frac{2T}{NL} \left(1 - \frac{T}{NL}\right)^N = \frac{2h}{N} \left(1 - \frac{h}{N}\right)^N \rightarrow 2h; \]  

(58)

all quantities of interest remain finite, allowing us to derive an explicit expression for the resolution threshold. Requiring \( \langle d_{\alpha\beta} \rangle = 1 \) leads to

\[ 2h = 1 \]  

(59)

which is solved by \( h = 1/2 \), further inducing \( p = h/N = 1/2N \). Numerical simulations confirm the correctness of the aforementioned estimations for a wide range of different combinations of \( N \) and \( L \) (see Fig. 7).
FIG. 7: Trends of $\langle N_0 \rangle = p_0 \to e^{-h}$ and $P_0 \equiv 2 \sum_{\alpha<\beta} P(d_{\alpha\beta} = 0)/L(L-1)$ as a function of $T/NL$ under the RHM (left panels) and the HCM (right panels). The dense (sparse) regime is recovered for large (small) values of $T/NL$. Each dot represents an average taken over an ensemble of 1000 configurations (explicitly sampled from either the RHM or the HCM) and is accompanied by the corresponding 95% confidence interval.
APPENDIX B.
THE HYPERGRAPH CONFIGURATION MODEL

1. Approximating the Hypergraph Configuration Model

Let us, first, consider that the generic probability coefficient induced by the HCM can be Taylor-expanded as

\[ p_{i\alpha}^{HCM} = \frac{x_i y_\alpha}{1 + x_i y_\alpha} = x_i y_\alpha - (x_i y_\alpha)^2 + (x_i y_\alpha)^3 \ldots \] (60)

one may, thus, wonder at which order the expansion can be safely truncated. In case hypergraphs are sparse, i.e. \( T/NL \ll 1 \), and in absence of hubs it turns out that just considering the first order is enough to obtain predictions as accurate as those achievable under the ‘full’ HCM: in other words, one can simply put \( p_{i\alpha}^{FOA} \simeq x_i y_\alpha \). Upon doing so, the system of equations defining the HCM simplifies to

\[ k_i^* = \sum_{\alpha=1}^{L} x_i y_\alpha, \ \forall \ i \] (61)

\[ h_\alpha^* = \sum_{i=1}^{N} x_i y_\alpha, \ \forall \ \alpha \] (62)

expressions leading us to find \( x_i = k_i^*/\sqrt{T^*}, \ \forall \ i \) and \( y_\alpha = h_\alpha^*/\sqrt{T^*}, \ \forall \ \alpha \). As a consequence, \( p_{i\alpha}^{FOA} = x_i y_\alpha = k_i^* h_\alpha^*/T^* \) - a position that is commonly named Chung-Lu approximation (CLA). Figure 8 shows the goodness of the latter one in approximating \( \langle \sigma_{i} \rangle_{HCM} \) for the contact-primary-school, the email-Enron and the congress-bills data sets, i.e. the accuracy of the following chain of equalities

\[ \langle \sigma_{i} \rangle_{HCM} = \sum_{\alpha=1}^{L} p_{i\alpha}^{HCM} \left[ h_\alpha - x_i y_\alpha \right] \simeq \sum_{\alpha=1}^{L} x_i y_\alpha \left[ h_\alpha - x_i y_\alpha \right] \overset{CLA}{\simeq} k_i^* \left( 1 - \frac{k_i^*}{T^*} \right) \sum_{\alpha=1}^{L} (h_\alpha^*)^2 \simeq k_i^* \sum_{\alpha=1}^{L} \frac{(h_\alpha^*)^2}{T^*}, \ \forall \ i. \] (63)

2. Partial Configuration Models for binary, undirected hypergraphs

The HCM reduces to a ‘partial’ Configuration Model in case one of the two degree sequences is left unconstrained.

Hypergraph Partial Configuration Model - layer of nodes. For instance, let us constrain only the degree sequence \( \{k_i\}_{i=1}^{N} \), by posing \( \beta_\alpha = 0, \ \forall \ \alpha \) (or, equivalently, \( y_\alpha = 1, \ \forall \ \alpha \)). Our canonical probability distribution becomes

\[ P(\mathbf{I}) = \prod_{i=1}^{N} \prod_{\alpha=1}^{L} \frac{x_i^{L_{i\alpha}}}{1 + x_i} = \prod_{i=1}^{N} p_i^k(1 - p_i)^{L - k_i} \] (64)

where \( e^{-\alpha} \equiv x_i \) and \( p_i \equiv x_i/(1 + x_i) \). The entries of the incidence matrix are, now, independent random variables obeying the Bernoulli distributions reading

\[ I_{i\alpha} \sim \text{Ber}(p_i), \ \forall \ i, \alpha; \] (65)

in words, the entries along the same row obey the same distribution while the ones along the same column obey different distributions. As a consequence, the \( i \)-th node degree (a sum of i.i.d. Bernoulli random variables), obeys the Binomial distribution

\[ k_i \sim \text{Bin}(L, p_i) \] (66)
FIG. 8: Left panels: numerical values of the node-specific Lagrange multipliers obtained by solving the ‘full’ HCM scattered versus the ones derived by adopting the CLA. Middle panels: numerical values of the hyperedge-specific Lagrange multipliers obtained by solving the ‘full’ HCM scattered versus the ones derived by adopting the CLA. Right panels: expected values \{⟨σ_i⟩_{HCM}\} scattered versus the expected values \{⟨σ_i⟩_{CLA}\}.

while the α-th hyperedge degree (a sum of independent random variables that obey different Bernoulli distributions)
obeys the Poisson-Binomial distribution

\[ h_\alpha \sim \text{PoissBin}(N, p_1 \ldots p_N). \] (67)

Interestingly, while the degrees of the nodes obey different Binomial distributions, the degrees of the hyperedges obey the same Poisson-Binomial distribution.

The resolution of the likelihood maximisation problem leads us to find the values \( p_i = k_i^*/L, \forall i \) which, in turn, ensure that \( \langle k_i \rangle_{\text{HPCM}} = \sum_{\alpha=1}^{L} p_i = k_i^*, \forall i \) and that \( \langle T \rangle_{\text{HPCM}} = T^* \); instead, \( \langle h_\alpha \rangle_{\text{HPCM}} \) will, in general, differ from \( h_\alpha^* - \) in fact, \( \langle h_\alpha \rangle_{\text{HPCM}} = T^*/L, \forall \alpha. \)

From a microcanonical perspective, the number of configurations satisfying the requirement that the degrees of the nodes match their empirical values amounts to

\[
\Omega_{\text{HPCM}} = \prod_{i=1}^{N} \binom{L}{k_i^*};
\] (68)

reshuffling the 1s along each row of the incidence matrix separately ensures the degrees of the nodes to be preserved, while destroying any other correlation.

Hypergraph Partial Configuration Model - layer of hyperedges. Analogously, the canonical probability distribution describing the case in which only the hyperdegree sequence \( \{h_\alpha\}_{\alpha=1}^{L} \) is constrained reads

\[
P(I) = \prod_{i=1}^{N} \prod_{\alpha=1}^{L} \frac{y_\alpha^{I_{i\alpha}}}{1 + y_\alpha} = \prod_{\alpha=1}^{L} p_\alpha^{h_\alpha}(1 - p_\alpha)^{N - h_\alpha}
\] (69)

where \( e^{-\beta_\alpha} \equiv y_\alpha \) and \( p_\alpha \equiv y_\alpha/(1 + y_\alpha) \). As for the previous null model, the entries of the incidence matrix are independent random variables obeying different Bernoulli distributions, i.e.

\[ I_{i\alpha} \sim \text{Ber}(p_\alpha), \forall i, \alpha; \] (70)

in words, the entries along the same column obey the same distribution while the ones along the same row obey different distributions. As a consequence, the \( i \)-th node degree (a sum of independent random variables that obey different Bernoulli distributions) obeys the Poisson-Binomial distribution

\[ k_i \sim \text{PoissBin}(L, p_1 \ldots p_L) \] (71)

while the \( \alpha \)-th hyperedge degree (a sum of i.i.d. Bernoulli random variables) obeys the Binomial distribution

\[ h_\alpha \sim \text{Bin}(N, p_\alpha). \] (72)

Interestingly, while the degrees of the nodes obey the same Poisson-Binomial distribution, the degrees of the hyperedges obey different Binomial distributions.

The resolution of the likelihood maximisation problem leads us to find the values \( p_\alpha = h_\alpha^*/N, \forall \alpha \) which, in turn, ensure that \( \langle h_\alpha \rangle_{\text{HPCM}} = \sum_{i=1}^{N} p_\alpha = h_\alpha^*, \forall \alpha \) and that \( \langle T \rangle_{\text{HPCM}} = T^* \); instead, \( \langle h_i \rangle_{\text{HPCM}} \) will, in general, differ from \( k_i^* - \) in fact, \( \langle h_i \rangle_{\text{HPCM}} = T^*/N, \forall i. \)

From a microcanonical perspective, the number of configurations satisfying the requirement that the degrees of the hyperedges match their empirical values amounts to
resuffling the 1s along each column of the incidence matrix separately ensures the degrees of the hyperedges to be preserved, while destroying any other correlation.

\[
\Omega_{\text{HPCM_hyperedges}} = \prod_{\alpha=1}^{L} \left( \frac{N}{h_{\alpha}^*} \right); \quad (73)
\]

| Table II: Summary of the constraints defining the canonical version of each benchmark considered here: while the total number of 1s is preserved by each model, the degrees are separately preserved by the two, partial HCMs and jointly preserved only by the HCM. Stars indicate the empirical values of the constraints.

| Random Hypergraph Model | Total number of 1s, \((T)\) | Degrees, \((k_{i\alpha})\) | Hyperdegrees, \((h_{\alpha})\) |
|-------------------------|--------------------------|---------------------|---------------------|
| Hypergraph Partial Configuration Model (node layer) | \(T^*\) | \(T^*/N\) | \(T^*/L\) |
| Hypergraph Partial Configuration Model (hyperedge layer) | \(T^*\) | \(k_{\alpha}^*\) | \(h_{\alpha}^*\) |
| Hypergraph Configuration Model | \(T^*\) | \(k_{\alpha}^*\) | \(h_{\alpha}^*\) |

3. Asymptotic results: estimation of the number of empty hyperedges

As noticed in the main text, the asymptotic behaviour of the HCM depends on the whole set of coefficients \(\{p_{i\alpha}\}_{i,\alpha}\): closed form expressions can be, thus, obtained only after having specified the functional dependence of each of them on \(N\). General conclusions can be drawn more easily for the two, ‘partial’ Configuration Models. Let us start by considering the behaviour of HPCM\(_{\text{hyperedges}}\), defined by the coefficients \(p_{i\alpha} = h_{\alpha}/N, \forall i, \alpha\). Such a position leads to

\[
p_{0}^\alpha = \prod_{i=1}^{N} (1 - p_{i\alpha}) = \left(1 - \frac{h_{\alpha}}{N}\right)^{N} \to e^{-h_{\alpha}}, \quad (74)
\]

that, in turn, induces the expression

\[
p_{1} = 1 - \prod_{\alpha=1}^{L} \left[ 1 - \frac{N}{i=1} (1 - p_{i\alpha}) \right] = 1 - \prod_{\alpha=1}^{L} \left[ 1 - \left(1 - \frac{h_{\alpha}}{N}\right)^{N} \right] \to 1 - \prod_{\alpha=1}^{L} (1 - e^{-h_{\alpha}}), \quad (75)
\]

for the probability of observing at least one, empty hyperedge, and the expression

\[
\langle N_{0} \rangle = \sum_{\alpha=1}^{L} N \prod_{i=1}^{N} (1 - p_{i\alpha}) = \sum_{\alpha=1}^{L} \left(1 - \frac{h_{\alpha}}{N}\right)^{N} \to \sum_{\alpha=1}^{L} e^{-h_{\alpha}}, \quad (76)
\]

for the expected number of empty hyperedges. It is interesting to consider the ‘full’ HCM and calculate \(p_{0}^\alpha\) in the sparse regime. We get

\[
p_{0}^\alpha = \prod_{i=1}^{N} (1 - p_{i\alpha}) \simeq 1 - \sum_{i=1}^{N} p_{i\alpha} = 1 - h_{\alpha} \simeq e^{-h_{\alpha}}, \quad (77)
\]

an expression proving that the trends returned by HPCM\(_{\text{hyperedges}}\) are compatible with those returned by the ‘full’ HCM (see also Fig. 9).

Let us, now, inspect the behaviour of HPCM\(_{\text{nodes}}\), defined by the coefficients \(p_{i\alpha} = k_{i}/L, \forall i, \alpha\). Such a position, in the sparse regime, leads to

\[
p_{0} = \prod_{i=1}^{N} (1 - p_{i\alpha}) = \prod_{i=1}^{N} \left(1 - \frac{k_{i}}{L}\right) \simeq 1 - \sum_{i=1}^{N} \frac{k_{i}}{L} = 1 - \frac{T}{L} = 1 - h \simeq e^{-h} \quad (78)
\]
4. Asymptotic estimation of the number of parallel hyperedges

Let us, now, focus on the issue of parallel hyperedges. Let us start by considering the behaviour of HPCM\textsubscript{hyperedges}, defined by the coefficients $p_{i\alpha} = h_\alpha/N$, $\forall i, \alpha$. Such a position leads to

$$P(d_{\alpha\beta} = 0) = \prod_{i=1}^{N} (1 - q_i^{\alpha\beta}) = \prod_{i=1}^{N} [1 - p_{i\alpha}(1-p_{i\beta}) - p_{i\beta}(1-p_{i\alpha})] = \prod_{i=1}^{N} \left(1 - \frac{h_\alpha + h_\beta}{N} + \frac{2h_\alpha h_\beta}{N^2}\right)$$

$$= \left(1 - \frac{h_\alpha + h_\beta}{N} + \frac{2h_\alpha h_\beta}{N^2}\right)^N \to e^{-(h_\alpha + h_\beta)} (79)$$

and to
\[
\langle d_{\alpha\beta} \rangle = \sum_{i=1}^{N} q_i^{\alpha\beta} = \sum_{i=1}^{N} \left[ p_{i\alpha} (1 - p_{i\beta}) + p_{i\beta} (1 - p_{i\alpha}) \right] = \left( \frac{h_\alpha + h_\beta}{N} - \frac{2h_\alpha h_\beta}{N^2} \right) N \to h_\alpha + h_\beta. \quad (80)
\]

It is interesting to consider the ‘full’ HCM and calculate \( q_i^{\alpha\beta} = p_{i\alpha} (1 - p_{i\beta}) + p_{i\beta} (1 - p_{i\alpha}) \) in the sparse regime, i.e. by truncating it to the first addendum: in words, considering \( q_i^{\alpha\beta} \simeq p_{i\alpha} + p_{i\beta} \) allows us to recover the trends returned by HPCM_{hyperedges} (see also Fig. 9).

Let us, now, inspect the behaviour of HPCM_{nodes}, defined by the coefficients \( p_{i\alpha} = k_i / L, \forall i, \alpha \). Such a position, in the sparse regime, leads to

\[
P(d_{\alpha\beta} = 0) = \prod_{i=1}^{N} (1 - q_i^{\alpha\beta}) = \prod_{i=1}^{N} \left[ 1 - p_{i\alpha} (1 - p_{i\beta}) + p_{i\beta} (1 - p_{i\alpha}) \right] = \prod_{i=1}^{N} \left[ 1 - \frac{2k_i}{L} \left( 1 - \frac{k_i}{L} \right) \right]
\]
\[
\simeq 1 - \sum_{i=1}^{N} \frac{2k_i}{L} = 1 - \frac{2T}{L} = 1 - 2h \simeq e^{-2h}, \quad (81)
\]

(where the fourth passage comes from the choice of considering only the leading order of the product, in the sparse regime, and the last one comes from the choice of truncating the exponential function to the first order of its Taylor expansion) and to

\[
\langle d_{\alpha\beta} \rangle = \sum_{i=1}^{N} q_i^{\alpha\beta} = \sum_{i=1}^{N} \left[ p_{i\alpha} (1 - p_{i\beta}) + p_{i\beta} (1 - p_{i\alpha}) \right] = \sum_{i=1}^{N} \frac{2k_i}{L} \left( 1 - \frac{k_i}{L} \right) \simeq \sum_{i=1}^{N} \frac{2k_i}{L} = \frac{2T}{L} = 2h \quad (82)
\]

(where the fourth passage comes from the choice of considering only the leading order of the sum, in the sparse regime, and the last one comes from the choice of truncating the exponential function to the first order of its Taylor expansion). The equation above allows us to recover the threshold value \( h = 1/2 \) and \( p_{i\beta}^\text{RHM} = h/N = 1/2N \) (see also Fig. 9).
APPENDIX C.
EXPECTED VALUE OF TOPOLOGICAL QUANTITIES

Let us, now, provide the explicit expression of the expected value of some topological properties of interest. Let us start with the entries of the matrix \( W \), for which the following results hold true

\[
\langle w_{ij} \rangle = \sum_{\alpha=1}^{L} \langle I_{i\alpha} I_{j\alpha} \rangle = \sum_{\alpha=1}^{L} p_{i\alpha} p_{j\alpha},
\]

\[
\text{Var}[w_{ij}] = \sum_{\alpha=1}^{L} \text{Var}[I_{i\alpha} I_{j\alpha}] + 2 \sum_{\beta<\gamma} \text{Cov}[I_{i\beta} I_{j\beta}, I_{i\gamma} I_{j\gamma}] = \sum_{\alpha=1}^{L} p_{i\alpha} p_{j\alpha} (1 - p_{i\alpha} p_{j\alpha}).
\]

Therefore,

\[
\sum_{j=1}^{N} \langle w_{ij} \rangle^2 = \sum_{\alpha=1}^{L} p_{i\alpha}^2 \left[ \sum_{j=1}^{N} p_{j\alpha}^2 - p_{i\alpha}^2 \right] + 2 \sum_{\beta<\gamma} p_{i\beta} p_{i\gamma} \left[ \sum_{j=1}^{N} p_{j\beta} p_{j\gamma} - p_{i\beta} p_{i\gamma} \right],
\]

\[
\sum_{j=1}^{N} \text{Var}[w_{ij}] = \sum_{\alpha=1}^{L} p_{i\alpha} (\langle h_{\alpha} \rangle - p_{i\alpha}) - \sum_{\alpha=1}^{L} p_{i\alpha}^2 \left[ \sum_{j=1}^{N} p_{j\alpha}^2 - p_{i\alpha}^2 \right].
\]

Let us, now, consider the strength sequence of matrix \( W \), for which the following results hold true

\[
\langle \sigma_i \rangle = \sum_{\alpha=1}^{L} p_{i\alpha} (\langle h_{\alpha} \rangle - p_{i\alpha}),
\]

\[
\text{Var}[\sigma_i] = \sum_{j=1}^{N} \text{Var}[w_{ij}] + 2 \sum_{j<k \atop (j,k\neq i)} \text{Cov}[w_{ij}, w_{ik}] = \sum_{j=1}^{N} \text{Var}[w_{ij}] + 2 \sum_{j<k \atop (j,k\neq i)} (\langle w_{ij} w_{ik} \rangle - \langle w_{ij} \rangle \langle w_{ik} \rangle).
\]

As a consequence, the expected value of the disparity ratio reads

\[
\langle Y_i \rangle \simeq \sum_{j=1}^{N} \frac{\langle w_{ij}^2 \rangle}{\langle \sigma_i^2 \rangle} = \sum_{j=1}^{N} \frac{\langle w_{ij} \rangle^2 + \text{Var}[w_{ij}]}{\langle \sigma_i \rangle^2 + \text{Var}[\sigma_i]},
\]

where the \( \simeq \) symbol is understood to approximate the expected value of a ratio as the ratio of expected values. This approximation just represents the first order of the Taylor expansion

\[
\mathbb{E} \left[ \frac{X}{Y} \right] \simeq \frac{\mathbb{E}[X]}{\mathbb{E}[Y]} - \frac{\mathbb{Cov}[X,Y]}{\mathbb{E}[Y]^2} + \frac{\mathbb{E}[X]}{\mathbb{E}[Y]^3} \text{Var}[X];
\]

due to the correlation between numerator and denominator, the second and the third terms of Eq. (90) may not be negligible: as Fig. 10 shows, while Eq. (89) holds true for the contact-primary-school data set, it does not for the others.
FIG. 10: Left panels: scatter plots between the analytical and the sample estimates of the numerator of the disparity ratio, under the ‘full’ HCM. Middle panels: scatter plots between the analytical and the sample estimates of the denominator of the disparity ratio, under the ‘full’ HCM. Right panels: scatter plots between the expected values \{\langle Y_i \rangle_{\text{HCM}} \}, computed as in Eq. (89), and the ones obtained by explicitly sampling the ensemble induced by the ‘full’ HCM. As analytical and sampled values of numerators and denominators nicely agree, the result concerning \{\langle Y_i \rangle \} must be imputed to the correlation between the two, disregarded by Eq. (89).
FIG. 11: Scatter plot between the ensemble average (of each component) of the Perron-Frobenius eigenvector of a given, empirical matrix $W^*$ and the numerical value (of each component) of the Perron-Frobenius eigenvector of the corresponding, expected matrix $⟨W⟩_{HCM}$. The extremely good agreement suggests a faster way to calculate the CEC that avoids to explicitly sample the ensemble induced by a null model.

APPENDIX D.
EXPECTED VALUE OF THE CLIQUE MOTIF EIGENVECTOR CENTRALITY

The clique motif eigenvector centrality (CEC) has been proposed in [34]: its $i$-th entry, $CEC_i$, is nothing else that the corresponding entry of the Perron-Frobenius eigenvector of $W$.

In order to evaluate the expected value of the CEC, we have explicitly sampled the ensemble of incidence matrices induced by the HCM; then, we have calculated the matrix $\tilde{W}$ induced by each $\tilde{I}$, according to Eq. (35). Afterwards, we have calculated the Perron-Frobenius eigenvector of each ‘projected’ matrix and taken their average in an entry-wise fashion.

Remarkably, we found that the aforementioned ensemble average basically coincides with the Perron-Frobenius eigenvector of the ensemble average of $W$ itself, i.e. $⟨W⟩_{HCM}$, as Fig. 11 shows.
APPENDIX E.
EXPECTED VALUE OF THE ENTRIES OF THE CONFUSION MATRIX

Before defining the entries of the confusion matrix, let us call $\tilde{I}$ the generic member of the ensemble of incidence matrices induced by one of the benchmarks considered in the present paper: the expected value of any quantity which is a function of $\tilde{I}$ is readily calculated by averaging it over the ensemble itself.

The true positive rate (TPR) is defined as the percentage of 1s correctly recovered by a given model and reads

$$\text{TPR} = \frac{TP}{T} = \sum_{i=1}^{N} \sum_{\alpha=1}^{L} \frac{I_{i\alpha} \tilde{I}_{i\alpha}}{T} \quad \implies \langle \text{TPR} \rangle = \frac{\langle TP \rangle}{T} = \sum_{i=1}^{N} \sum_{\alpha=1}^{L} \frac{I_{i\alpha} p_{i\alpha}}{T}; \quad (91)$$

the specificity (SPC) is defined as the percentage of 0s correctly recovered by a given model and reads

$$\text{SPC} = \frac{TN}{NL-T} = \sum_{i=1}^{N} \sum_{\alpha=1}^{L} \frac{(1-I_{i\alpha})(1-\tilde{I}_{i\alpha})}{NL-T} \quad \implies \langle \text{SPC} \rangle = \frac{\langle TN \rangle}{NL-T} = \sum_{i=1}^{N} \sum_{\alpha=1}^{L} \frac{(1-I_{i\alpha})(1-p_{i\alpha})}{NL-T} \quad (92)$$

the positive predictive value (PPV) is defined as the percentage of 1s correctly recovered by a given model with respect to the total number of predicted of 1s and reads

$$\text{PPV} = \sum_{i=1}^{N} \sum_{\alpha=1}^{L} \frac{I_{i\alpha} \tilde{I}_{i\alpha}}{T} \quad \implies \langle \text{PPV} \rangle = \frac{\langle TP \rangle}{\langle T \rangle} = \sum_{i=1}^{N} \sum_{\alpha=1}^{L} \frac{I_{i\alpha} p_{i\alpha}}{\langle T \rangle}; \quad (93)$$

lastly, the accuracy (ACC) measures the overall performance of a given reconstruction method in correctly placing both 0s and 1s and reads

$$\text{ACC} = \frac{TP + TN}{NL} \quad \implies \langle \text{ACC} \rangle = \frac{\langle TP \rangle + \langle TN \rangle}{NL}. \quad (94)$$

As stressed in the main text, results on the confusion matrix of a number of real-world hypergraphs reveal that the large sparsity of the latter ones makes it difficult to reproduce the TPR and the PPV (see Table III); on the other hand, the capability of the HCM to reproduce the density of 1s makes it capable of recovering the density of 0s as well, thus ensuring the overall accuracy of the model to be quite large.

| Data set           | $\langle \text{TPR} \rangle$ | $\langle \text{SPC} \rangle$ | $\langle \text{PPV} \rangle$ | $\langle \text{ACC} \rangle$ | $T/NL$       |
|-------------------|-----------------------------|-----------------------------|-----------------------------|-----------------------------|--------------|
| contact–primary-school | $3.57 \times 10^{-8}$       | $0.991$                     | $3.57 \times 10^{-8}$       | $0.983$                     | $8.66 \times 10^{-3}$ |
| email–Enron       | $8.15 \times 10^{-8}$       | $0.982$                     | $8.15 \times 10^{-8}$       | $0.966$                     | $1.72 \times 10^{-2}$ |
| NDC–classes       | $2.40 \times 10^{-10}$      | $0.997$                     | $2.40 \times 10^{-10}$      | $0.995$                     | $2.71 \times 10^{-3}$ |

TABLE III: Confusion matrix and density of 1s for the three real-world hypergraphs considered in the present paper.
APPENDIX F.
COMPARING THE RHM AND THE HCM PERFORMANCE

Let us, now, inspect the performance of the RHM and that of the HCM in reproducing the patterns of our real-world hypergraphs. To this aim, let us consider the coefficient of determination that, in our case, can be re-defined as

$$R^2_{\text{null}}(X) = 1 - \frac{\sum_i (X_i - (X_i)_{\text{null}})^2}{\sum_i (X_i - \bar{X})^2};$$  \hspace{1cm} (95)

it measures the goodness of a null model in reproducing empirical data when compared to a baseline model whose only prediction reads $\bar{X}$, i.e. the arithmetic mean of the empirical data themselves. While such a model is characterised by $R^2 = 0$, a model that is capable of matching the observed values exactly is characterised by $R^2 = 1$; models returning predictions whose discrepancies are larger than the discrepancies accompanying the predictions of the baseline model are characterised by $R^2 < 0$.

As Table IV shows, the HCM is often more accurate than the RHM in reproducing the quantities we have considered. More precisely, the RHM is found to steadily perform worse than it when the CEC is considered and to perform quite similar to it when the $Y$ is considered; the opposite tendency is observed when the $\sigma$ is considered. Overall, these results confirm that just constraining the total number of 1s of a given incidence matrix is not enough to achieve an accurate description of the corresponding hypergraph but also suggest that constraining the degree and the hyperdegree sequences may be not enough as well - as we have already noticed when commenting the highly non-trivial degree of self-organisation of the real-world hypergraphs (whose trends are) depicted in Fig. 4.

| Data set           | $R^2_{\text{HCM}}(\sigma)$ | $R^2_{\text{HCM}}(Y)$ | $R^2_{\text{HCM}}(\text{CEC})$ | $R^2_{\text{RHM}}(\sigma)$ | $R^2_{\text{RHM}}(Y)$ | $R^2_{\text{RHM}}(\text{CEC})$ |
|--------------------|-----------------------------|------------------------|-----------------------------|-----------------------------|------------------------|-----------------------------|
| contact-primary-school | -2.9697                     | -1.7248                | -0.1489                     | -2.7349                     | -1.8189                | -0.8478                     |
| email-Enron        | -0.1344                     | -0.2591                | -0.0519                     | -0.0131                     | -0.3462                | -0.1705                     |
| NDC-classes        | -0.0044                     | -0.1107                | -0.0482                     | -0.0008                     | -0.1222                | -0.0183                     |

TABLE IV: The $R^2$ index allows us to compare the performance of the RHM with that of the HCM in reproducing the patterns of our real-world hypergraphs. While both models perform worse than the baseline model, the HCM is, overall, more accurate than the RHM in reproducing all quantities we have considered, the only exception being $\sigma$: therefore, just constraining the total number of 1s of a given incidence matrix is, in general, not enough to achieve an accurate description of the hypergraph it represents - although constraining the degree and the hyperdegree sequences may be not enough as well.