The Gap Between Model-Based and Model-Free Methods on the Linear Quadratic Regulator: An Asymptotic Viewpoint

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Abstract

The effectiveness of model-based versus model-free methods is a long-standing question in reinforcement learning (RL). Motivated by recent empirical success of RL on continuous control tasks, we study the sample complexity of popular model-based and model-free algorithms on the Linear Quadratic Regulator (LQR). We show that for policy evaluation, a simple model-based plugin method requires asymptotically less samples than the classical least-squares temporal difference (LSTD) estimator to reach the same quality of solution; the sample complexity gap between the two methods can be at least a factor of state dimension. For policy evaluation, we study a simple family of problem instances and show that nominal (certainty equivalence principle) control also requires a factor of state dimension fewer samples than the policy gradient method to reach the same level of control performance on these instances. Furthermore, the gap persists even when employing baselines commonly used in practice. To the best of our knowledge, this is the first theoretical result which demonstrates a separation in the sample complexity between model-based and model-free methods on a continuous control task.

1 Introduction

The relative merits of model-based versus model-free methods in reinforcement learning (RL) is a decades old question. This debate has become reinvigorated in the last few years due to the impressive success of RL techniques in various domains such as game playing, robotic manipulation, and locomotion tasks. A common rule of thumb amongst RL practitioners is that model-free methods have worse sample complexity compared to model-based methods, but are generally able to achieve better performance asymptotically since they do not suffer from biases in the model that lead to sub-optimal behavior [10, 29, 33]. However, there is currently no general theory which rigorously explains the gap between performance of model-based versus model-free methods. While there has been theoretical work studying both model-based and model-free methods in RL, prior work has primarily shown specific upper bounds [5, 6, 17, 19, 41] which are not directly comparable, or information-theoretic lower bounds [17, 19] which are currently too coarse-grained to delineate between model-based and model-free methods. Furthermore, most of the prior work has focused primarily on the tabular Markov Decision Process (MDP) setting.

We take a first step towards a theoretical understanding of the differences between model-based and model-free methods for continuous control settings. While we are ultimately interested in comparing these methods for general MDPs with non-linear state transition dynamics, in this work we build upon recent progress in understanding the performance guarantees of data-driven
methods for the Linear Quadratic Regulator (LQR). We study the asymptotic behavior of both *policy evaluation* and *policy optimization* on LQR, comparing the performance of simple model-based methods which use empirical state transition data to fit a dynamics model versus the performance of popular model-free methods from RL: *temporal-difference learning* for policy evaluation and *policy gradient methods* for policy optimization.

Our analysis shows that in the policy evaluation setting, a simple model-based plugin estimator is always asymptotically more sample efficient than the classical least-squares temporal difference (LSTD) estimator; the gap between the two methods can be at least a factor of state-dimension. For policy optimization, we construct a simple family of instances for which nominal control (also known as the certainty equivalence principle in control theory) is also at least a factor of state-dimension more efficient than the widely used policy gradient method. Furthermore, the gap persists even when we employ commonly used baselines to reduce the variance of the policy gradient estimate. In both settings, we also show minimax lower bounds which highlight the near-optimality of model-free methods in certain regimes. To the best of our knowledge, our work is the first to rigorously show a setting where a strict separation between a model-based and model-free method solving the same continuous control task occurs.

2 Main Results

In this paper, we study the performance of model-based and model-free algorithms for the Linear Quadratic Regulator (LQR) via two fundamental primitives in reinforcement learning: *policy evaluation* and *policy optimization*. In both tasks we fix an unknown dynamical system

\[ x_{t+1} = A_x x_t + B_x u_t + w_t , \]

starting at \( x_0 = 0 \) (for simplicity) and driven by Gaussian white noise \( w_t \overset{i.i.d.}{\sim} N(0, \sigma_w^2 I_n) \). We let \( n \) denote the state dimension and \( d \) denote the input dimension, and assume the system is underactuated (i.e. \( d \leq n \)). We also fix two positive semi-definite cost matrices \((Q, R)\).

2.1 Policy Evaluation

Given a controller \( K \in \mathbb{R}^{d \times n} \) that stabilizes \((A_x, B_x)\), the policy evaluation task is to compute the (relative) value function \( V^K(x) \):

\[
V^K(x) := \lim_{T \to \infty} \mathbb{E} \left[ \sum_{t=0}^{T-1} (x_t^T Q x_t + u_t^T R u_t - \lambda_K) \bigg| x_0 = x \right] , \quad u_t = K x_t .
\] (2.1)

Above, \( \lambda_K \) is the infinite horizon average cost. It is well-known that \( V^K(x) \) can be written as:

\[
V^K(x) = \sigma_w^2 x^T P_* x ,
\] (2.2)

where \( P_* = \text{dlyap}(A_x + B_x K, Q + K^T R K) \) solves the discrete-time Lyapunov equation:

\[
(A_x + B_x K)^T P_* (A_x + B_x K) - P_* + Q + K^T R K = 0 .
\] (2.3)

From the Lyapunov equation, it is clear that given \((A_x, B_x)\), the solution to policy evaluation task is readily computable. In this paper, we study algorithms which only have input/output access to
Specifically, we study on-policy algorithms that operate on a single trajectory, where the input \( u_t \) is determined by \( u_t = K x_t \). The variable that controls the amount of information available to the algorithm is \( T \), the trajectory length. The trajectory will be denoted as \( \{x_t\}_{t=0}^T \). We are interested in the asymptotic behavior of algorithms as \( T \to \infty \).

**Model-based algorithm.** In light of Equation (2.3), the plugin estimator is a very natural model-based algorithm to use. Let \( L_* := A_* + B_*K \) denote the true closed-loop matrix. The plugin estimator uses the trajectory \( \{x_t\}_{t=0}^T \) to estimate \( L_* \) via least-squares; call this \( \hat{L}(T) \). The estimator then returns \( \hat{P}_{\text{plug}}(T) \) by using \( \hat{L}(T) \) in-place of \( L_* \) in (2.3). Algorithm 1 describes this estimator in more detail.

**Algorithm 1** Model-based algorithm for policy evaluation.

**Input:** Policy \( \pi(x) = K x \), rollout length \( T \), regularization \( \lambda > 0 \), thresholds \( \zeta \in (0, 1) \) and \( \psi > 0 \).

1. Collect trajectory \( \{x_t\}_{t=0}^T \) using the feedback \( u_t = \pi(x_t) = K x_t \).
2. Estimate the closed-loop matrix via least-squares:

   \[
   \hat{L}(T) = \left( \sum_{t=0}^{T-1} x_{t+1} x_t^T \right) \left( \sum_{t=0}^{T-1} x_t x_t^T + \lambda I_n \right)^{-1}.
   \]

3. **if** \( \rho(\hat{L}(T)) > \zeta \) or \( \|\hat{L}(T)\| > \psi \) **then**
4. **set** \( \hat{P}_{\text{plug}}(T) = 0 \).
5. **else**
6. **set** \( \hat{P}_{\text{plug}}(T) = \text{lyap}(\hat{L}(T), Q + K^T R K) \).
7. **end if**
8. **return** \( \hat{P}_{\text{plug}}(T) \).

**Model-free algorithm.** By observing that \( V^K(x) = \sigma^2_w x^T P_* x = \sigma^2_w \langle \text{svec}(P_*), \text{svec}(x x^T) \rangle \), one can apply Least-Squares Temporal Difference Learning (LSTD) \([8, 9]\) with the feature map \( \phi(x) := \text{svec}(x x^T) \) to estimate \( P_* \). Here, \( \text{svec}(\cdot) \) vectorizes the upper triangular part of a symmetric matrix, weighting the off-diagonal terms by \( \sqrt{2} \) to ensure consistency in the inner product. This is a classical algorithm in RL; the pseudocode is given in Algorithm 2.

We now proceed to compare the risk of Algorithm 1 versus Algorithm 2. Our notion of risk will be the expected squared error of the estimator: \( \mathbb{E}[\|\hat{P} - P_*\|_F^2] \). Our first result gives an upper bound on the asymptotic risk of the model-based plugin Algorithm 1.

**Theorem 2.1.** Let \( K \) stabilize \( (A_*, B_*) \). Define \( L_* \) to be the closed-loop matrix \( A_* + B_*K \) and let \( \rho(L_*) \in (0, 1) \) denote its stability radius. Recall that \( P_* \) is the solution to the discrete-time Lyapunov equation (2.3) that parameterizes the value function \( V^K(x) \). We have that Algorithm 1 with thresholds \( (\zeta, \psi) \) satisfying \( \zeta \in (\rho(L_*), 1) \) and \( \psi \in (\|L_*\|, \infty) \) and any fixed regularization parameter \( \lambda > 0 \) has the asymptotic risk upper bound:

\[
\lim_{T \to \infty} T \cdot \mathbb{E}[\|\hat{P}_{\text{plug}}(T) - P_*\|_F^2] \leq 4 \text{Tr}((I - L_*^T \otimes_s L_*^T)^{-1}(L_*^T P_*^2 L_* \otimes_s \sigma^2_w P_*^{-1})(I - L_*^T \otimes_s L_*^T)^{-T}).
\]
shows that the asymptotic risk of the model-free method always exceeds that of the model-based plugin method. We remark that we prove the theorem under an idealized setting (where the estimate \( \hat{\sigma}_w \) does not affect both higher order terms and the rate of convergence to the limiting risk). We include these thresholds as they simplify the proof. In practice, we find that thresholding or regularization is generally not needed, with the caveat that if the estimate \( \hat{\sigma}_w \) is set to the true cost \( \lambda \), then the solution to the discrete Lyapunov equation is not guaranteed to exist (and when it exists is not guaranteed to be positive semidefinite). Finally, we remark that a non-asymptotic high probability upper bound for the risk of Algorithm 2 is not available.

We now turn our attention to the model-free LSTD algorithm. Our next result gives a lower bound on the asymptotic risk of Algorithm 2.

**Theorem 2.2.** Let \( K \) stabilize \((A_*,B_*)\). Define \( L_* \) to be the closed-loop matrix \( A_* + B_*K \). Recall that \( P_* \) is the solution to the discrete-time Lyapunov equation (2.3) that parameterizes the value function \( V^K(x) \). We have that Algorithm 2 with the cost estimates \( \lambda_t \) set to the true cost \( \lambda_* := \sigma_w^2 \text{Tr}(P_*) \) satisfies the asymptotic risk lower bound:

\[
\lim inf_{T \rightarrow \infty} T \cdot \mathbb{E}[\|\hat{P}_{\text{std}}(T) - P_*\|^2_F] \geq 4R_{\text{plug}} + 8\sigma_w^2 \langle P_\infty, L_*^T P_*^2 L_* \rangle \text{Tr}((I - L_*^T \otimes_s L_*^T)^{-1}(P_\infty^{-1} \otimes_s P_\infty^{-1})(I - L_*^T \otimes_s L_*^T)^{-T})
\]

Here, \( R_{\text{plug}} := \lim_{T \rightarrow \infty} T \cdot \mathbb{E}[\|\hat{P}_{\text{plug}}(T) - P_*\|^2_F] \) is the asymptotic risk of the plugin estimator, \( P_\infty = \text{dlyap}(L_*^T, \sigma_w^2 I_n) \) is the stationary covariance matrix of the closed loop system \( x_{t+1} = L_* x_t + w_t \), and \( \otimes_s \) denotes the symmetric Kronecker product.

Theorem 2.2 shows that the asymptotic risk of the model-free method always exceeds that of the model-based plugin method. We remark that we prove the theorem under an idealized setting where the infinite horizon cost estimate \( \lambda_t \) is set to the true cost \( \lambda_* \). In practice, the true cost is not known and must instead be estimated from the data at hand. However, for the purposes of our comparison this is not an issue because using the true cost \( \lambda_* \) over an estimator of \( \lambda_* \) only reduces the variance of the risk.
To get a sense of how much excess risk is incurred by the model-free method over the model-based method, consider the following family of instances, defined for \( \rho \in (0, 1) \) and \( 1 \leq d \leq n \):

\[
\mathcal{F}(\rho, d, K) := \{(A_*, B_*): A_* + B_* K = \tau P_E + \gamma I_n, \ (\tau, \gamma) \in (0, 1), \ \tau + \gamma \leq \rho, \ \dim(E) \leq d\}.
\]  

(2.4)

With this family, one can show with elementary computations that under the simplifying assumptions that \( Q + K^T R K = I_n \) and \( d > n \), Theorem 2.1 and Theorem 2.2 state that:

\[
\lim_{T \to \infty} T \cdot \mathbb{E}[\|\hat{P}_\text{plug}(T) - P_\star\|^2_F] \leq O\left(\frac{\rho^2 n^2}{(1 - \rho^2)^3}\right),
\]

\[
\liminf_{T \to \infty} T \cdot \mathbb{E}[\|\hat{P}_\text{lstd}(T) - P_\star\|^2_F] \geq \Omega\left(\frac{\rho^2 n^3}{(1 - \rho^2)^3}\right).
\]

That is, for \( \mathcal{F}(\rho, d, K) \), the plugin risk is a factor of state-dimension \( n \) less than the LSTD risk. Moreover, the non-asymptotic result for LSTD from Lemma 4.1 of Abbasi-Yadkori [1] (which extends the non-asymptotic discounted LSTD result from Tu and Recht [44]) gives a bound of \( \|\hat{P}_\text{lstd}(T) - P_\star\|^2_F \leq \tilde{O}(n^3/T) \) w.h.p., which matches the asymptotic bound of Theorem 2.2 in terms of \( n \) up to logarithmic factors.

Our final result for policy evaluation is a minimax lower bound on the risk of any estimator over \( \mathcal{F}(\rho, d, K) \).

**Theorem 2.3.** Fix a \( \rho \in (0, 1) \) and suppose that \( K \) satisfies \( Q + K^T R K = I_n \). Suppose that \( n \) is greater than an absolute constant and \( T \gtrsim n(1 - \rho^2)/\rho^2 \). We have that:

\[
\inf_{\hat{P}} \sup_{(A_*, B_*) \in \mathcal{F}(\rho, d, K)} \mathbb{E}[\|\hat{P} - P_\star\|^2_F] \geq \frac{\rho^2 n^2}{(1 - \rho^2)^3 T},
\]

where the infimum is taken over all estimators \( \hat{P} \) taking input \( \{x_t\}_{t=0}^T \).

Theorem 2.3 states that the rate achieved by the model-based Algorithm 3 over the family \( \mathcal{F}(\rho, d, K) \) cannot be improved beyond constant factors, at least asymptotically; its dependence on both the state dimension \( n \) and stability radius \( \rho \) is optimal.

### 2.2 Policy Optimization

Given a finite horizon length \( T \), the policy optimization task is to solve the finite horizon optimal control problem:

\[
J_\star := \min_{u_t(\cdot)} \mathbb{E}\left[\sum_{t=0}^{T-1} (x_t^T Q x_t + u_t^T R u_t) + x_T^T Q x_T\right], \quad x_{t+1} = A_* x_t + B_* u_t + w_t.
\]  

(2.5)

We will focus on a special case of this problem when there is no penalty on the input: \( Q = I_n, \ R = 0, \) and \( \text{range}(A_*) \subseteq \text{range}(B_*) \). In this situation, the cost function reduces to \( \mathbb{E}[\sum_{t=0}^T \|x_t\|^2] \) and the optimal solution simply chooses a \( u_t \) that cancels out the state \( x_t \); that is \( u_t = -B_*^T A_* x_t \). We work with this simple class of instances so that we can ensure that policy gradient converges to the optimal solution; in general this is not guaranteed.
We consider a slightly different input/output oracle model in this setting than we did in Section 2.1. The horizon length \( T \) is now considered fixed, and \( N \) rounds are played. At each round \( i = 1, \ldots, N \), the algorithm chooses a feedback matrix \( K_i \in \mathbb{R}^{n \times d} \). The algorithm then observes the trajectory \( \{ x_t^{(i)} \}_{t=0}^T \) by playing the control input \( u_t^{(i)} = K_i^T x_t^{(i)} + \eta_t^{(i)} \), where \( \eta_t^{(i)} \sim \mathcal{N}(0, \sigma_i^2 I_d) \) is i.i.d. noise used for the policy. This process then repeats for \( N \) total rounds. After the \( N \) rounds, the algorithm is asked to output a \( \hat{K}(N) \) and is assigned the risk \( \mathbb{E}[J(\hat{K}(N)) - J_*] \), where \( J(\hat{K}(N)) \) denotes playing the feedback \( u_t = \hat{K}(N)^T x_t \) on the true system \((A_*, B_*)\). We will study the behavior of algorithms when \( N \to \infty \) (and \( T \) is held fixed).

**Model-based algorithm.** Under this oracle model, a natural model-based algorithm is to first use random open-loop feedback (i.e. \( K_i = 0 \)) to observe \( N \) independent trajectories (each of length \( T \)), and then use the trajectory data to fit the state transition matrices \((A_*, B_*)\); call this estimate \((\hat{A}(N), \hat{B}(N))\). After fitting the dynamics, the algorithm then returns the estimate of \( K_* \) by solving the finite horizon problem (2.5) with \((\hat{A}(N), \hat{B}(N))\) taking the place of \((A_*, B_*)\). In general, however, the assumption that \( \text{range}(\hat{A}(N)) \subseteq \text{range}(\hat{B}(N)) \) will not hold, and hence the optimal solution to (2.5) will not be time-invariant. Moreover, solving for the best time-invariant static feedback for the finite horizon problem in general is not tractable. In light of this, to provide the fairest comparison to the model-free policy gradient method, we use the time-invariant static feedback that arises from infinite horizon solution given by the discrete algebraic Riccati equation as a proxy. We note that under our range inclusion assumption, the infinite horizon solution is a consistent estimator of the optimal feedback. The pseudo-code for this model-based algorithm is described in Algorithm 3.

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**Algorithm 3** Model-based algorithm for policy optimization.

**Input:** Horizon length \( T \), rollouts \( N \), regularization \( \lambda > 0 \), thresholds \( \zeta > 0 \), \( \psi > 0 \), \( \gamma > 0 \).

1. Collect trajectories \( \{(x_t^{(i)}, u_t^{(i)})\}_{t=0}^T \) using the feedback \( K_i = 0 \) (open-loop).
2. Estimate the dynamics matrices \((A_*, B_*)\) via regularized least-squares:
   \[
   \hat{\Theta}(N) = \left( \sum_{i=1}^N \sum_{t=0}^{T-1} x_{t+1}^{(i)}(z_t^{(i)})^T \right)^{-1} \left( \sum_{i=1}^N \sum_{t=0}^{T-1} z_t^{(i)}(z_t^{(i)})^T + \lambda I_{n+d} \right), \quad z_t^{(i)} := (x_t^{(i)}, u_t^{(i)}).
   \]
3. Set \((\hat{A}, \hat{B}) = \hat{\Theta}(N)\).
4. if \( \|\hat{A}\| > \zeta \) or \( \|\hat{B}\| > \psi \) or \( \sigma_d(\hat{B}) < \gamma \) then
5. Set \( \hat{K}_{\text{plug}}(N) = 0 \).
6. else
7. Set \( \hat{P} = \text{dare}(\hat{A}, \hat{B}, I_n, 0) \) as the positive definite solution to\(^1\):
   \[
   P = \hat{A}^T \hat{P} \hat{A} - \hat{A}^T \hat{P} \hat{B} (\hat{B}^T \hat{P} \hat{B})^{-1} \hat{B}^T \hat{P} \hat{A} + I_n.
   \]
8. Set \( \hat{K}_{\text{plug}}(N) = -(\hat{B}^T \hat{P} \hat{B})^{-1} \hat{B}^T \hat{P} \hat{A} \).
9. end if
10. return \( \hat{K}_{\text{plug}}(N) \).

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\(^1\) A sufficient condition for the existence of a unique positive definite solution to the discrete algebraic Riccati equation when \( R = 0 \) is that \((A, B)\) is stabilizable and \( B \) has full column rank (Lemma 6.1).
Model-free algorithm. We study a model-free algorithm based on policy gradients (see e.g. [32, 45]). Here, we choose to parameterize the policy as a time-invariant linear feedback. The algorithm is described in Algorithm 4.

**Algorithm 4** Model-free algorithm for policy optimization (REINFORCE) [32, 45].

**Input:** Horizon length $T$, rollouts $N$, baseline functions $\{\Psi_t(\cdot; \cdot)\}$, step-sizes $\{\alpha_i\}$, initial $K_1$, threshold $\zeta$.

1: for $i = 1, \ldots, N$ do
2: Collect trajectory $T^{(i)} := \{(x_t^{(i)}, u_t^{(i)})\}_{t=0}^T$ using feedback $K_i$.
3: Compute policy gradient $g_i$ as:
   $$g_i = \frac{1}{\sigma_u^2} \sum_{t=0}^{T-1} x_t^{(i)} (\eta_t^{(i)})^T \Psi_t(T^{(i)}; K_i).$$
4: Take policy gradient step:
   $$K_{i+1} = \text{Proj}_{\|\cdot\| \leq \zeta}(K_i - \alpha_i g_i).$$
5: end for
6: Set $\hat{K}_{pg}(N) = K_N$.
7: return $\hat{K}_{pg}(N)$.

In general for problems with a continuous action space, when applying policy gradient one has many degrees of freedom in choosing how to represent the policy $\pi$. Some of these degrees of freedom include whether or not the policy should be time-invariant and how much of the history before time $t$ should be used to compute the action at time $t$. More broadly, the question is what function class should be used to model the policy. Ideally, one chooses a function class which is both capable of expressing the optimal solution and is easy to optimize over.

Another issue that significantly impacts the performance of policy gradient in practice is choosing a baseline which effectively reduces the variance of the policy gradient estimate. What makes computing a baseline challenging is that good baselines (such as value or advantage functions) require knowledge of the unknown MDP transition dynamics in order to compute. Therefore, one has to estimate the baseline from the empirical trajectories, adding another layer of complexity to the policy gradient algorithm.

In general, these issues are still an active area of research in RL and present many hurdles to a general theory for policy optimization. However, by restriction our attention to LQR, we can sidestep these issues which enables our analysis. In particular, by studying problems with no penalty on the input and where the state can be cancelled at every step, we know that the optimal control is a static time-invariant linear feedback. Therefore, we can restrict our policy representation to static linear feedback controllers without introducing any approximation error. Furthermore, it turns out that we can further parameterize instances so that the optimization landscape satisfies a standard notion of restricted strong convexity. This allows us to study policy gradient by leveraging the existing theory on the asymptotic distribution of stochastic gradient descent for strongly convex objectives. Finally, we can compute many of the standard baselines used in closed form, which further enables our analysis.
We note that in the literature, the model-based method is often called nominal control or the certainty equivalence principle. As noted in Dean et al. [11], one issue with this approach is that on an infinite horizon, there is no guarantee of robust stability with nominal control. However, as we are dealing with only finite horizon problems, the notion of stability is irrelevant.

We will consider a restricted family of instances \((A_*, B_*)\) to obtain a sharp asymptotic analysis. For a \(\rho \in (0, 1)\) and \(1 \leq d \leq n\), we define the family \(\mathcal{G}(\rho, d)\) over \((A_*, B_*)\) as:

\[
\mathcal{G}(\rho, d) := \{(\rho U_*^T, \rho U_*): U_* \in \mathbb{R}^{n \times d}, U_*^T U_* = I_d\}.
\]

This is a simple family where the \(A_*\) matrix is stable, contractive, and symmetric. Observe that for \((A_*, B_*) \in \mathcal{G}(\rho, d)\) we have \(\text{range}(A_*) = \text{range}(B_*)\). Furthermore, the optimal feedback \(K_* = -U_*\) for each of these instances. Our first result for policy optimization gives the asymptotic risk of the model-based Algorithm 3.

**Theorem 2.4.** Fix a \(\rho \in (0, 1)\). For any \((A_*, B_*) \in \mathcal{G}(\rho, d)\), we have that the model-based plugin Algorithm 3 with thresholds \((\zeta, \psi, \gamma)\) such that \(\zeta \in (\rho, 1), \psi \in (\rho, \infty), \text{and } \gamma \in (0, \rho)\) satisfies the asymptotic risk bound:

\[
\lim_{N \to \infty} N \cdot \mathbb{E}[J(\hat{\mathcal{R}}_{\text{plug}}(N)) - J_*] = O(d^2 + (n - d)d) + o_T(1).
\]

Here, \(O(\cdot)\) hides constants depending only on \(\rho, \sigma_w^2, \sigma_u^2\).

Theorem 2.4 states that when \(d \asymp n\), the RHS of the risk bound for the model-based case is \(O(n^2)\). It will turn out that the \(O(n^2)\) dependence on \(n\) is optimal for the family \(\mathcal{G}(\rho, d)\). Similar to Theorem 2.1, Theorem 2.4 requires the setting of thresholds \((\zeta, \psi, \gamma)\). These thresholds serve two purposes. First, they ensure the existence of a unique positive definite solution to the discrete algebraic Riccati solution with the input penalty \(R = 0\) (the details of this are worked out in Section 6.2). Second, they simplify various technical aspects of the proof related to uniform integrability. In practice, such strong thresholds are not needed, and we leave either removing them or relaxing their requirements to future work.

Next, we look at the model-free case. As mentioned previously, baselines are very influential on the behavior of policy gradient. In our analysis, we consider three different baselines:

\[
\Psi_t(T; K) = \sum_{t=1}^{T} ||x_t||_2^2, \quad \text{ (Simple baseline } b_t(x_t; K) = ||x_t||_2^2).\]

\[
\Psi_t(T; K) = \sum_{t=1}^{T} ||x_t||_2^2 - V_t^K(x_t), \quad \text{ (Value function baseline } b_t(x_t; K) = V_t^K(x_t)).\]

\[
\Psi_t(T; K) = A_t^K(x_t, u_t). \quad \text{ (Advantage baseline } A_t^K(x_t, u_t) = Q_t^K(x_t, u_t) - V_t^K(x_t)).\]

Above, the simple baseline should be interpreted as having effectively no baseline; it turns out to simplify the variance calculations. On the other hand, the value function baseline \(V_t^K\) is a very popular heuristic used in practice [32]. Typically one has to actually estimate the value function for a given policy, since computing it requires knowledge of the model dynamics. In our analysis however, we simply assume the true value function is known. While this is an unrealistic assumption in practice, we note that this assumption substantially reduce the variance of policy gradient, and hence only serves to reduce the asymptotic risk. The last baseline we consider is to use the advantage
function \( A^K \). Using advantage functions has been shown to be quite effective in practice \cite{38}. It has the same issue as the value function baseline in that it needs to be estimated from the data; once again in our analysis we simply assume we have access to the true advantage function.

Our main result for model-free policy optimization is the following asymptotic risk lower bound on Algorithm 4.

**Theorem 2.5.** Fix a \( \rho \in (0, 1) \). For any \((A_*, B_*) \in \mathcal{A}(\rho, d)\) consider Algorithm 4 with \( K_1 = 0_{n \times d} \), step-sizes \( \alpha_i = \frac{1}{2(T-1)\rho^2 \sigma^2_{\text{pg} i}} \), and threshold \( \zeta \in (1, \infty) \). We have that:

\[
\liminf_{N \to \infty} N \cdot \mathbb{E}[J(\hat{K}_{\text{pgi}}(N)) - J_*] \geq \left\{ \begin{array}{ll}
\Omega(T^2 \cdot (d^4 + n^3d)) + o_T(T^2) & \text{(Simple baseline)} \\
\Omega(T \cdot dn^2) + o_T(T) & \text{(Value function baseline)} \\
\Omega(d^3 + nd^2) & \text{(Advantage baseline)}
\end{array} \right.
\]

Here, \( \Omega(\cdot) \) hides constants depending only on \( \rho, \sigma^2_{\text{w}}, \sigma^2_{\text{a}} \).

Theorem 2.5 states that when \( d \asymp n \), for the simple baseline the RHS is \( \Omega(T^2 n^4) \), for the value function baseline the RHS is \( \Omega(Tn^3) \), and finally for the advantage baseline the RHS is \( \Omega(n^3) \). In all cases (even with the advantage baseline), the dependence on \( n \) is at least one factor more than in the model-free case, which is \( O(n^2) \) (Theorem 2.4). Furthermore, for the simple and value function baseline, we even see the RHS depending on the horizon length \( T^2 \) and \( T \), respectively. The extra factors of the horizon length appear due to the large variance of the policy gradient estimator without the variance reduction effects of the advantage baseline. Finally, we note that we prove Theorem 2.5 with a specific choice of step size \( \alpha_i \). This step size corresponds to the standard \( 1/(mt) \) step sizes commonly found in proofs for SGD on strongly convex functions (see e.g. Rakhlin et al. \cite{34}), where \( m \) is the strong convexity parameter. We leave to future work extending our results to support Polyak-Ruppert averaging, which would yield asymptotic results that are more robust to specific step size choices.

Finally, we turn to our information-theoretic lower bound for any (possibly adaptive) method over the family \( \mathcal{A}(\rho, d) \).

**Theorem 2.6.** Fix a \( d \leq n/2 \) and suppose \( d(n - d) \) is greater than an absolute constant. Consider the family \( \mathcal{A}(\rho, d) \) as described above. Fix a time horizon \( T \) and number of rollouts \( N \). The risk over any algorithm \( A \) which plays (possibly adaptive) feedbacks of the form \( u_t = K_t^T x_t + \eta_t \) with \( \|K_t\| \leq 1 \) and \( \eta_t \sim \mathcal{N}(0, \sigma^2_{\text{w}} I_d) \) is lower bounded by:

\[
\inf_A \sup_{\rho \in (0, 1/4), (A_*, B_*) \in \mathcal{A}(d, \rho)} \mathbb{E}[J(A) - J_*] \geq \frac{\sigma^4_{\text{w}}}{\sigma^2_{\text{w}} + \sigma^2_{\text{a}}} \frac{d^2(n - d)}{nN}.
\]

When \( d \asymp n \), the RHS of the risk bound is \( \Omega(n^2/N) \). Therefore, Theorem 2.6 tells us that asymptotically, the model-based method in Algorithm 3 is nearly optimal in terms of its dependence on the state dimension \( n \).

### 3 Related Work

For general Markov Decision Processes (MDPs), the setting which is the best understood theoretically is the finite-horizon episodic case with discrete state and action spaces, often referred to
as the “tabular” setting. Jin et al. [19] provide an excellent overview of the known regret bounds in the tabular setting; here we give a brief summary of the highlights. We focus only on regret bounds for simplicity, but note that many results have also been established in the PAC setting (see e.g. [23, 40, 41]). For tabular MDPs, a model-based method is one which stores the entire state-transition matrix, which takes \( O(S^2AH) \) space where \( S \) is the number of states, \( A \) is the number of actions, and \( H \) is the horizon length. The best known regret bound in the model-free case is \( \tilde{O}(\sqrt{H^2SAT}) \) [6], which matches the known lower bound of \( \Omega(\sqrt{H^2SAT}) \) [17, 19] up to log factors. On the other hand, a model-free method is one which only stores the \( Q \)-function and hence requires only \( O(SAH) \) space. The best known regret bound in the model-free case is \( \tilde{O}(\sqrt{H^3SAT}) \), which is worse than the model-based case by a factor of the horizon length \( H \). Interestingly, there is no gap in terms of the number of states \( S \) and actions \( A \). It is open whether or not the gap in \( H \) is fundamental or can be closed. Sun et al. [42] present an information-theoretic definition of model-free algorithms. Under their definition, they construct a family of factored MDPs with horizon length \( H \) where any model-free algorithm incurs sample complexity \( \Omega(2^H) \), whereas there exists a model-based algorithm that has sample complexity polynomial in \( H \) and other relevant quantities. We leave proving lower bounds for LQR under their more general definition of model-free algorithms to future work.

For LQR, the story is less complete. Unlike the tabular setting, the storage requirements of a model-based method are comparable to a model-free method. For instance, it takes \( O(n(n+d)) \) space to store the state transition model and \( O((n+d)^2) \) space to store the \( Q \)-function. In presenting the known results of LQR, we will delineate between offline (one-shot) methods versus online (adaptive) methods.

In the offline setting, the first non-asymptotic result is from Fiechter [15], who studied the sample complexity of the discounted infinite horizon LQR problem. Later, Dean et al. [11] study the average cost infinite horizon problem, using tools from robust control to quantify how the uncertainty in the model affects control performance in an interpretable way. Both works fall under model-based methods, since they both propose to first estimate the state transition matrices from sample trajectories using least-squares and then use the estimated dynamics in a control synthesis procedure.

For model-free methods for LQR, Tu and Recht [44] study the performance of least-squares temporal difference learning (LSTD) [8, 9], which is a classic policy evaluation algorithm in RL. They focus on the discounted cost LQR setting and provide a non-asymptotic high probability bound on the risk of LSTD. Later, Abbasi-Yadkori et al. [1] extend this result to the average cost LQR setting. Most related to our analysis for policy gradient is Fazel et al. [14], who study the performance of model-free policy gradient related methods on LQR. Unfortunately, their bounds do not give explicit dependence on the problem instance parameters and are therefore difficult to compare to. Furthermore, Fazel et al. study a simplified version of the problem where the problem is a infinite horizon problem (as opposed to finite horizon in this work) and the only noise is in the initial state; all subsequence state transitions have no process noise. Other than our current work, we are unaware of any analysis (asymptotic or non-asymptotic) which explicitly studies the behavior of policy gradient on the finite horizon LQR problem. We also note that Fazel et al. analyze a policy optimization method which is more akin to random search (e.g. [25, 36]) than REINFORCE. Finally, note that all the results mentioned for LQR are only upper bounds; we are unaware of any lower bounds in the literature for LQR which give explicit dependence on the problem instance.

We now discuss known results for the online (adaptive) setting for LQR. For model-based algo-
rithms, both \textit{optimism in the face of uncertainty} (OFU) \cite{kearns2008near, abbasiyadkori2015online, d временем} and \textit{Thompson sampling} \cite{thompson1933Bayes, abbasiyadkori2015online, d временем} have been analyzed in the online learning literature. In both cases, the algorithms have been shown to achieve $\tilde{O}(\sqrt{T})$ regret, which is known to be nearly optimal in the dependence on $T$. However, in nearly all the bounds the dependence on the problem instance parameters is hidden. Furthermore, it is currently unclear how to solve the OFU subproblem in polynomial time for LQR. In response to the computational issues with OFU, Dean et al. \cite{dean2016adaptivity} propose a polynomial time adaptive algorithm with sub-linear regret $\tilde{O}(T^{2/3})$; their bounds also make the dependence on the problem instance parameters explicit, but are quite conservative in this regard.

For model-free algorithms, Abbasi-Yadkori et al. \cite{abbasi2010online} study the regret of a model-free algorithm similar in spirit to least-squares policy iteration (LSPI) \cite{ross2009value}. They prove that their algorithm has regret $\tilde{O}(T^{2/3+\varepsilon})$ for any $\varepsilon > 0$, nearly matching the bound given by Dean et al. in terms of the dependence on $T$. In terms of the dependence on the problem specific parameters, however, their bound is not directly comparable to that of Dean et al. Experimentally, Abbasi-Yadkori et al. observe that their model-free algorithm performs quite sub-optimally compared to model-based methods; these empirical observations are also consistent with similar experiments conducted in \cite{kearns2008near, d временем, d временем}.

4 Asymptotic Toolbox

Our analysis relies heavily on computing limiting distributions for the various estimators we study. A crucial fact we use is that if the matrix $L_s$ is stable, then the Markov chain $\{x_t\}$ given by $x_{t+1} = L_s x_t + w_t$ with $w_t \sim \mathcal{N}(0, \sigma_w^2 I_n)$ is geometrically ergodic. This allows us to apply well known limit theorems for ergodic Markov chains.

In what follows, we let $\overset{a.s.}{\to}$ denote almost sure convergence and $\overset{D}{\to}$ denote convergence in distribution. We also let $\otimes$ denote the standard Kronecker product and $\otimes_s$ denote the \textit{symmetric} Kronecker product; see e.g. \cite{zheng2019kronecker} for a review of the basic properties of the Kronecker and symmetric Kronecker product which we will use extensively throughout the sequel. For a matrix $M$, the notation $\operatorname{vec}(M)$ denotes the vectorized version of $M$ by stacking the columns. We will also let $\operatorname{svec}(\cdot)$ denote the operator that satisfies $\langle \operatorname{svec}(M_1), \operatorname{svec}(M_2) \rangle = \langle M_1, M_2 \rangle$ for all symmetric matrices $M_1, M_2 \in \mathbb{R}^{n \times n}$, where the first inner product is with respect to $\mathbb{R}^{n(n+1)/2}$ and the second is with respect to $\mathbb{R}^{n \times n}$. Finally, we let $\operatorname{mat}(\cdot)$ and $\operatorname{smat}(\cdot)$ denote the functional inverses of $\operatorname{vec}(\cdot)$ and $\operatorname{svec}(\cdot)$. The proofs of the results presented in this section are deferred to the appendix.

We first state a well-known result that concerns the least-squares estimator of a stable dynamical system. In the scalar case, this result dates back to Mann and Wald \cite{mann1947mean}.

\textbf{Lemma 4.1.} Let $x_{t+1} = L_s x_t + w_t$ be a dynamical system with $L_s$ stable and $w_t \sim \mathcal{N}(0, \sigma_w^2 I)$. Given a trajectory $\{x_t\}_{t=0}^T$, let $\hat{L}(T)$ denote the least-squares estimator of $L_s$ with regularization $\lambda \geq 0$:

$$
\hat{L}(T) = \arg \min_{L \in \mathbb{R}^{n \times n}} \left\{ \frac{1}{2} \sum_{t=0}^{T-1} \| x_{t+1} - L x_t \|_2^2 + \frac{\lambda}{2} \| L \|_F^2 \right\}.
$$

Let $P_\infty$ denote the stationary covariance matrix of the process $\{x_t\}_{t=0}^\infty$, i.e. $L_s P_\infty L_s^T = P_\infty + \sigma_w^2 I_n = 0$. We have that $\hat{L}(T) \overset{a.s.}{\to} L_s$ and furthermore:

$$
\sqrt{T} \operatorname{vec}(\hat{L}(T) - L_s) \overset{D}{\to} \mathcal{N}(0, \sigma_w^2 (P_\infty^{-1} \otimes I_n)) .
$$
We now consider a slightly altered process where the system is no longer autonomous, and instead will be driven by white noise.

**Lemma 4.2.** Let \( x_{t+1} = A_*x_t + B_*u_t + w_t \) be a stable dynamical system driven by \( u_t \sim N(0, \sigma_u^2 I_d) \) and \( w_t \sim N(0, \sigma_w^2 I_n) \). Consider a least-squares estimator \( \hat{\Theta} \) of \( \Theta_* := (A_*, B_*) \in \mathbb{R}^{n \times (n+d)} \) based off of \( N \) independent trajectories of length \( T \), i.e. given \( \{(z_t^{(i)}, w_t^{(i)})\}_{t=0}^T \) \( i = 1, \ldots, N \),

\[
\hat{\Theta}(N) = \min_{(A,B) \in \mathbb{R}^{n \times (n+d)}} \frac{1}{2} \sum_{i=1}^N \sum_{t=0}^{T-1} \|x_t^{(i)} - Ax_t^{(i)} - Bu_t^{(i)}\|_2^2 + \frac{\lambda}{2} \|A \ B\|_F^2.
\]

Let \( P_\infty \) denote the stationary covariance of the process \( \{x_t\}_{t=0}^\infty \), i.e. \( P_\infty \) solves

\[
A_*P_\infty A_*^T - P_\infty + \sigma_u^2 B_*B_*^T + \sigma_w^2 I_n = 0.
\]

We have that \( \hat{\Theta}(N) \xrightarrow{a.s.} \Theta_* \) and furthermore:

\[
\sqrt{N}\text{vec}(\hat{\Theta}(N) - \Theta_*) \xrightarrow{D} \mathcal{N}\left(0, \frac{\sigma_u^2}{T} \begin{bmatrix} P_\infty^{-1} & 0 \\ \frac{1}{\sigma_u^2}I_d \end{bmatrix} \otimes I_n + o(1/T) \right).
\]

Next, we consider the asymptotic distribution of Least-Squares Temporal Difference Learning for LQR.

**Lemma 4.3.** Let \( x_{t+1} = A_*x_t + B_*u_t + w_t \) be a linear system driven by \( u_t = Kx_t \) and \( w_t \sim N(0, \sigma_w^2 I_n) \). Suppose the closed-loop matrix \( A_* + B_*K \) is stable. Let \( \nu_\infty \) denote the stationary distribution of the Markov chain \( \{x_t\}_{t=0}^\infty \). Define the two matrices \( A_\infty, B_\infty \), the mapping \( \psi(x) \), and the vector \( w_* \) as

\[
A_\infty := \mathbb{E}_{x \sim \nu_\infty, x' \sim \rho(\cdot|x, \pi(x))}[\phi(x)(\phi(x) - \phi(x'))^T],
\]

\[
B_\infty := \mathbb{E}_{x \sim \nu_\infty, x' \sim \rho(\cdot|x, \pi(x))}[(\phi(x') - \psi(x))^T w_*]w_*^T \phi(x)\phi(x)^T,
\]

\[
\psi(x) := \mathbb{E}_{x \sim \nu_\infty, x' \sim \rho(\cdot|x, \pi(x))}[\phi(x')],
\]

\[
w_* := \text{svec}(P_\infty).
\]

Let \( \hat{w}_{\text{lstd}}(T) \) denote the LSTD estimator given by:

\[
\hat{w}_{\text{lstd}}(T) = \left( \sum_{t=0}^{T-1} \phi(x_t)(\phi(x_t) - \phi(x_{t+1}))^T \right)^{-1} \left( \sum_{t=0}^{T-1} (c_t - \lambda_t) \phi(x_t) \right).
\]

Suppose that LSTD is run with the true \( \lambda_t = \lambda_* := \sigma_w^2 \text{Tr}(P_\ast) \) and that the matrix \( A_\infty \) is invertible. We have that \( \hat{w}_{\text{lstd}}(T) \xrightarrow{a.s.} w_* \) and furthermore:

\[
\sqrt{T}(\hat{w}_{\text{lstd}}(T) - w_*) \xrightarrow{D} \mathcal{N}(0, A_\infty^{-1}B_\infty A_\infty^{-T}).
\]

As a corollary to Lemma 4.3, we work out the formulas for \( A_\infty \) and \( B_\infty \) and a useful lower bound.
Corollary 4.4. In the setting of Lemma 4.3, with $L_* = A_* + B_* K$, we have that the matrix $A_\infty$ is invertible, and:

$$A_\infty = (P_\infty \otimes_s P_\infty) - (P_\infty L_*^T \otimes_s P_\infty L_*^T),$$

$$B_\infty = (\sigma_w^2 (P_\infty, L_*^T P_*^2 L_*) + 2 \sigma_w^4 \|P_\|_2^4) (2(P_\infty \otimes_s P_\infty) + \text{svec}(P_\infty) \text{svec}(P_\infty)^T)$$

$$+ 2 \sigma_w^2 (\text{svec}(P_\infty) \text{svec}(P_\infty L_*^T P_*^2 L_* P_\infty)^T + \text{svec}(P_\infty L_*^T L_*) \text{svec}(P_\infty)^T)$$

$$+ 8 \sigma_w^2 (P_\infty L_*^T P_*^2 L_* P_\infty \otimes_s P_\infty).$$

Furthermore, we can lower bound the matrix $A_\infty^{-1} B_\infty A_\infty^{-T}$ by:

$$A_\infty^{-1} B_\infty A_\infty^{-T} \geq 8 \sigma_w^2 (P_\infty, L_*^T P_*^2 L_*) (I - L_*^T \otimes_s L_*^T)^{-1} (P_\infty^{-1} \otimes_s P_\infty^{-1}) (I - L_*^T \otimes_s L_*^T)^{-T}$$

$$+ 16 \sigma_w^2 (I - L_*^T \otimes_s L_*^T)^{-1} (L_*^T P_*^2 L_* \otimes_s P_\infty^{-1}) (I - L_*^T \otimes_s L_*^T)^{-T}. \quad (4.1)$$

Next, we state a standard lemma which we will use to convert convergence in distribution guarantees to guarantees regarding the convergence of risk.

Lemma 4.5. Suppose that $\{X_n\}$ is a sequence of random vectors and $X_n \overset{D}{\rightarrow} X$. Suppose that $f$ is a non-negative continuous real-valued function such that $\mathbb{E}[f(X)] < \infty$. We have that:

$$\lim_{n \to \infty} \inf \mathbb{E}[f(X_n)] \geq \mathbb{E}[f(X)].$$

If additionally we have $\sup_{n \geq 1} \mathbb{E}[f(X_n)^{1+\varepsilon}] < \infty$ holds for some $\varepsilon > 0$, then the limit $\lim_{n \to \infty} \mathbb{E}[f(X_n)]$ exists and

$$\lim_{n \to \infty} \mathbb{E}[f(X_n)] = \mathbb{E}[f(X)].$$

Proof. Both facts are standard consequences of weak convergence of probability measures; see e.g. Chapter 5 of Billingsley [7] for more details.

The next claim uniformly controls the $p$-th moments of the regularized least-squares estimate when $T$ is large enough. This technical result will allow us to invoke Lemma 4.5 to obtain convergence in $L^p$.

Lemma 4.6. Let $x_{t+1} = L_* x_t + w_t$ with $w_t \sim N(0, \sigma_w^2 I_n)$ and $L_*$ stable. Fix a regularization parameter $\lambda > 0$ and let $\hat{L}(T)$ denote the LS estimator:

$$\hat{L}(T) = \arg \min_{L \in \mathbb{R}^{n \times n}} \frac{1}{2} \sum_{t=0}^{T-1} \|x_{t+1} - L x_t\|^2 + \frac{\lambda}{2} \|L\|^2_F.$$

Fix a finite $p \geq 1$. Let $C_{L_*,\lambda,n}$ and $C_{L_*,\lambda,n,p}$ denote constants that depend only on $L_*,\lambda,n$ (resp. $L_*,\lambda,n,p$) and not on $T,\delta$. Fix $a \delta \in (0,1)$. With probability at least $1 - \delta$, as long as $T \geq C_{L_*,\lambda,n} \log(1/\delta)$ we have:

$$\|\hat{L}(T) - L_*\| \leq C'_{L_*,\lambda,n} \sqrt{\frac{\log(1/\delta)}{T}}.$$

Furthermore, as long as $T \geq C_{L_*,\lambda,n,p}$, then:

$$\mathbb{E}[\|\hat{L}(T) - L_*\|^p] \leq C'_{L_*,\lambda,n,p} \frac{1}{T^{p/2}}.$$
The next result is the analogue of Lemma 4.6 for the non-autonomous system driven by white noise.

**Lemma 4.7.** Let \( x_{t+1} = A_x x_t + B_x u_t + w_t \) with \( w_t \sim \mathcal{N}(0, \sigma_w^2 I_n) \), \( u_t \sim \mathcal{N}(0, \sigma_u^2 I_d) \), and \( A_x \) stable. Fix a regularization parameter \( \lambda > 0 \) and let \( \hat{\Theta}(N) \) denote the LS estimator:

\[
\hat{\Theta}(N) = \arg \min_{(A, B) \in \mathbb{R}^{n \times (n+d)}} \frac{1}{2} \sum_{t=0}^{T-1} \sum_{i=1}^N \| x_{t+1}^{(i)} - A x_t^{(i)} - B u_t^{(i)} \|^2 + \frac{\lambda}{2} \| [A \quad B] \|_F^2.
\]

Fix a finite \( p \geq 1 \). Let \( C_{\Theta, T, \lambda, n, d} \) and \( C_{\Theta, T, \lambda, n, d, p} \) denote constants that depend only on \( \Theta_* = T, \lambda, n, d \) (resp. \( \Theta, T, \lambda, n, d, p \)) and not on \( N, \delta \). Fix a \( \delta \in (0, 1) \). With probability at least \( 1 - \delta \), as long as \( N \geq C_{\Theta, T, \lambda, n, d} \log(1/\delta) \) we have:

\[
\| \hat{\Theta}(N) - \Theta_* \| \leq C_{\Theta, T, \lambda, n, d}' \sqrt{\log(1/\delta)}.
\]

Furthermore, as long as \( N \geq C_{\Theta, T, \lambda, n, d, p} \), then:

\[
\mathbb{E}[\| \hat{\Theta}(N) - \Theta_* \|^p] \leq C_{\Theta, T, \lambda, n, d, p}' \frac{1}{N^{p/2}}.
\]

**Proof.** The proof is nearly identical to that of Lemma 4.6, except we use the concentration result of Proposition 1.1 from Dean et al. [11] instead of Theorem 2.4 of Simchowitz et al. [39] to establish concentration over multiple independent rollouts. We omit the details as they very closely mimic that of Lemma 4.6.

We note that in doing this we obtain a sub-optimal dependence on the horizon length \( T \). This can be remedied by a more careful argument combining the concentration along each trajectory from Simchowitz et al. with the concentration across independent trajectories from Dean et al. However, as in our limit theorems only \( N \) the rollout length is being sent to infinity (e.g. \( T \) is considered a constant), a sub-optimal bound in \( T \) will suffice for our purpose.

Our final asymptotic result deals with the performance of stochastic gradient descent (SGD) with projection. This will be our key ingredient in analyzing policy gradient (Algorithm 4). While the asymptotic performance of SGD (and more generally stochastic approximation) is well-established (see e.g. [21]), we consider a slight modification where the iterates are projected back into a compact convex set at every iteration. As long as the optimal solution is not on the boundary of the projection set, then one intuitively does not expect the asymptotic distribution to be affected by this projection, since eventually as SGD converges towards the optimal solution the projection step will effectively be inactive. Our result here makes this intuition rigorous. It follows by combining the asymptotic analysis of Toulis and Airoldi [43] with the high probability bounds for SGD from Rakhlin et al. [34].

To state the result, we need a few definitions. First, we say a differentiable function \( F: \mathbb{R}^d \to \mathbb{R} \) satisfies **restricted strong convexity** (RSC) on a compact convex set \( \Theta \subseteq \mathbb{R}^d \) if it has a unique minimizer \( \theta_* \in \text{int}(\Theta) \) and for some \( m > 0 \), we have \( \langle \nabla F(\theta), \theta - \theta_* \rangle \geq m \| \theta - \theta_* \|^2 \) for all \( \theta \in \Theta \). We denote this by RSC\((m, \Theta)\).

**Lemma 4.8.** Let \( F \in C^3(\Theta) \) and suppose \( F \) satisfies RSC\((m, \Theta)\). Let \( \theta_* \in \Theta \) denote the unique minimizer of \( F \) in \( \Theta \). Suppose we have a stochastic gradient oracle \( g(\theta; \xi) \) such that \( g \) is continuous in
Given a sequence \( \{ \xi \} \subseteq \mathbb{R} \) drawn i.i.d. from the law of \( \xi \), consider the sequence of iterates \( \{ \theta_t \}_{t=1}^{\infty} \), starting with \( \theta_1 \in \Theta \) and defined as:

\[
\theta_{t+1} = \text{Proj}_\Theta(\theta_t - \alpha_t g(\theta_t; \xi_t)), \quad \alpha_t = \frac{1}{mt}.
\]

We have that:

\[
\lim_{T \to \infty} mT \cdot \text{Var}(\theta_T) = \Xi,
\]

where \( \Xi = \text{lyap}(\frac{m}{2} I_d - \nabla^2 F(\theta_*), \mathbb{E}_\xi [g(\theta_*; \xi)g(\theta_*; \xi)^T]) \) solves the continuous-time Lyapunov equation:

\[
\left( \frac{m}{2} I_d - \nabla^2 F(\theta_*) \right) \Xi + \Xi \left( \frac{m}{2} I_d - \nabla^2 F(\theta_*) \right) + \mathbb{E}_\xi [g(\theta_*; \xi)g(\theta_*; \xi)^T] = 0.
\]

We also have that for any \( G \in C^2(\Theta) \) with \( \nabla G(\theta_*) = 0 \) and \( \nabla^2 G(\theta_*) \succ 0 \),

\[
\liminf_{T \to \infty} T \cdot \mathbb{E} [G(\theta_T) - G(\theta_*)] \geq \frac{1}{2m} \text{Tr}(\nabla^2 G(\theta_*) \cdot \Xi).
\]

We defer the proof of this lemma to Section B of the Appendix. We quickly comment on how the last inequality can be used. Taking trace of both sides from Equation 4.6, we obtain:

\[
\text{Tr}(\Xi \cdot (\nabla^2 F(\theta_*) - \frac{m}{2} I_d)) = \frac{1}{2} \mathbb{E}_\xi [\|g(\theta_*; \xi)\|_2^2].
\]

We now upper bound the LHS as:

\[
\text{Tr}(\Xi \cdot (\nabla^2 F(\theta_*) - \frac{m}{2} I_d)) = \text{Tr}(\Xi \cdot \nabla^2 G(\theta_*)^{-1/2} \cdot \nabla^2 G(\theta_*)^{-1/2} (\nabla^2 F(\theta_*) - \frac{m}{2} I_d) \nabla^2 G(\theta_*)^{-1/2} \cdot \nabla^2 G(\theta_*)^{1/2})
\leq \text{Tr}(\Xi \cdot \nabla^2 G(\theta_*)) \lambda_{\max}(\nabla^2 G(\theta_*)^{-1/2} (\nabla^2 F(\theta_*) - \frac{m}{2} I_d) \nabla^2 G(\theta_*)^{-1/2})
= \text{Tr}(\Xi \cdot \nabla^2 G(\theta_*)) \lambda_{\max}(\nabla^2 G(\theta_*)^{-1} (\nabla^2 F(\theta_*) - \frac{m}{2} I_d)).
\]

Combining the last two equations we obtain that:

\[
\liminf_{T \to \infty} T \cdot \mathbb{E} [G(\theta_T) - G(\theta_*)] \geq \frac{1}{2m} \text{Tr}(\Xi \cdot \nabla^2 G(\theta_*)) \geq \frac{1}{4m \lambda_{\max}(\nabla^2 G(\theta_*)^{-1} (\nabla^2 F(\theta_*) - \frac{m}{2} I_d))} \mathbb{E}_\xi [\|g(\theta_*; \xi)\|_2^2].
\]

We will use this last estimate in our analysis.
5 Analysis of Policy Evaluation Methods

In this section, recall that $Q, R, K$ are fixed, and furthermore define $M := Q + K^T R K$.

5.1 Proof of Theorem 2.1

The strategy is as follows. Recall that Lemma 4.1 gives us the asymptotic distribution of the (regularized) least-squares estimator $\hat{L}(T)$ of the true closed-loop matrix $L_*$. For a stable matrix $L$, let $P(L) = \text{dlyap}(L, M)$. Since the map $L \mapsto P(L)$ is differentiable, using the delta method we can recover the asymptotic distribution of $\sqrt{T} \text{vec}(P(\hat{L}(T)) - P_*)$. Upper bounding the trace of the covariance matrix for this asymptotic distribution then yields Theorem 2.1.

Let $[DP(L)]$ denote the Fréchet derivative of the map $P(\cdot)$ evaluated at $L$, and let $[DP(L)](X)$ denote the action of the linear operator $[DP(L)]$ on $X$. By a straightforward application of the implicit function theorem, we have that:

$$[DP(L_*)](X) = \text{dlyap}(L_*, X^T P_* L_* + L^T_* P_* X).$$

Before we proceed, we introduce some notation surrounding Kronecker products. Let $\Gamma$ denote the matrix such that $\Gamma \text{vec}(S) = \text{vec}(S)$ for any symmetric matrix $S$. Also let $\Pi$ be the orthonormal matrix such that $\Pi \text{vec}(X) = \text{vec}(X^T)$ for all square matrices $X$. It is not hard to verify that $\Pi^T (A \otimes B) \Pi = (B \otimes A)$, a fact we will use later. With this notation, we proceed as follows:

\[
\text{svec}([DP(L_*)](X)) = (I - L^T_* \otimes s L^*_T)^{-1} \text{svec}(X^T P_* L_* + L^T_* P_* X) \\
= (I - L^T_* \otimes s L^*_T)^{-1} \Gamma \text{vec}(X^T P_* L_* + L^T_* P_* X) \\
= (I - L^T_* \otimes s L^*_T)^{-1} \Gamma X^T P_* (I \otimes L^*_T) \Pi + (I \otimes L^T_* P_*)) \text{vec}(X) .
\]

Applying Lemma 4.1 in conjunction with the delta method, we obtain:

$$\sqrt{T} \text{vec}(P(\hat{L}(T)) - P_*) \overset{D}{\sim} N(0, \sigma^2 w (I - L^T_* \otimes s L^*_T)^{-1} V (I - L^T_* \otimes s L^*_T)^{-T}) ,$$

where,

$$V := \Gamma [((L^T_* P_* \otimes I_*) \Pi + (I_*) \otimes L^T_* P_* (P^{-1}_* \otimes I_*)) ((L^T_* P_* \otimes I_*) \Pi + (I_*) \otimes L^T_* P_* (P^{-1}_* \otimes I_*)) ] \Gamma^T$$

$$\leq 2 \Gamma [((L^T_* P_* \otimes I_*) \Pi (P^{-1}_* \otimes I_*) \Pi^T (P_* L_* \otimes I_*) + (I_*) \otimes L^T_* P_* (P^{-1}_* \otimes I_*) (I_*) \otimes P_* L_*) ] \Gamma^T$$

$$= 2 \Gamma [(L^T_* P_*^2 L_* \otimes P^{-1}_*) + (P^{-1}_* \otimes L^T_* P_*^2 L_*)] \Gamma^T$$

$$= 4 (L^T_* P_*^2 L_* \otimes P^{-1}_*).$$

In (a), we used the inequality for any matrices $X, Y$ and positive definite matrices $F, G$, (see e.g. Chapter 3, page 94 of [46]):

$$(X + Y)(F + G)^{-1} (X + Y)^T \leq X F^{-1} X^T + Y G^{-1} Y^T.$$ 

Suppose that the sequence $\{\|Z_T\|^2_F\}$ is uniformly integrable, where $Z_T := \sqrt{T} \text{vec}(P(\hat{L}(T)) - P_*)$. Then:

$$\lim_{T \to \infty} T \cdot E[\|P(\hat{L}(T)) - P_*\|^2_F] \leq 4 \text{Tr}((I - L^T_* \otimes s L^*_T)^{-1} (L^T_* P^2_* L_* \otimes s \sigma^2 w P^{-1}_*) (I - L^T_* \otimes s L^*_T)^{-T}) .$$

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which is the desired bound on the asymptotic risk.

We now show that the sequence \( \{ \| Z_T \|^2_F \} \) is uniformly integrable. To do this, we need a simple matrix stability perturbation bound.

**Lemma 5.1.** Let \( A \) be a stable matrix that satisfies \( \| A^k \| \leq C \rho^k \) for all \( k \geq 0 \) and \( \rho \in (0, 1) \). Fix a \( \gamma \in (\rho, 1) \). Suppose that \( \Delta \) is a perturbation that satisfies: \[
\| \Delta \| \leq \frac{\gamma - \rho}{C}.
\]

Then we have that (a) \( A + \Delta \) is a stable matrix with \( \rho(A + \Delta) \leq \gamma \) and (b) \( \| (A + \Delta)^k \| \leq C \gamma^k \) for all \( k \geq 0 \).

**Proof.** We start by proving (b). Fix an integer \( k \geq 1 \). Consider the expansion of \( (A + \Delta)^k \) into \( 2^k \) terms. Label all these terms as \( T_{i,j} \) for \( i = 0, \ldots, k \) and \( j = 1, \ldots, \binom{k}{i} \) where \( i \) denotes the degree of \( \Delta \) in the term (hence there are \( \binom{k}{i} \) terms with a degree of \( i \) for \( \Delta \)). Using the fact that \( \| A^k \| \leq C \rho^k \) for all \( k \geq 0 \), we can bound \( \| T_{i,j} \| \leq C^{i+1} \rho^{k-i} \| \Delta \|^i \). Hence by triangle inequality:

\[
\| (A + \Delta)^k \| \leq \sum_{i=0}^{k} \sum_{j=0}^{k} \| T_{i,j} \|
\leq \sum_{i=0}^{k} \binom{k}{i} C^{i+1} \rho^{k-i} \| \Delta \|^i
= C \sum_{i=0}^{k} \binom{k}{i} (C \| \Delta \|)^i \rho^{k-i}
= C(C \| \Delta \| + \rho)^k
\leq C \gamma^k,
\]

where the last inequality uses the assumption \( \| \Delta \| \leq \frac{\gamma - \rho}{C} \). This gives the claim (b).

To derive the claim (a), we use the inequality that \( \rho(A + \Delta) \leq \| (A + \Delta)^k \|^{1/k} \leq C^{1/k} \gamma^k \) for any \( k \geq 1 \). Since this holds for any \( k \geq 1 \), we can take the infimum over all \( k \geq 1 \) on the RHS, which yields the desired claim. \( \square \)

Fix a finite \( p \geq 1 \). Since \( L_* \) is stable and \( \zeta \in (\rho(L_*), 1) \), there exists a \( C_* \) such that \( \| L_*^k \| \leq C_* \zeta^k \) for all \( k \geq 0 \). For the rest of the proof, \( O(\cdot), \Omega(\cdot) \) will hide constants that depend on \( L_*, C_*, n, p, \lambda, \zeta, \psi \), but not on \( T \). Set \( \delta_T = O(1/T^{p/2}) \) and let \( T \) be large enough so that there exists an event \( \mathcal{E}_{Bdd} \) promised by Lemma 4.6 such that \( \mathbb{P}(\mathcal{E}_{Bdd}) \geq 1 - \delta_T \) and on \( \mathcal{E}_{Bdd} \) we have \( \| \hat{L}(T) - L_* \| \leq O(\sqrt{\log(1/\delta_T)}/T) \). Let \( T \) also be large enough so that on \( \mathcal{E}_{Bdd} \), we have \( \| \hat{L}(T) - L_* \| \leq \min((\gamma - \rho_*)/C_*, \psi - \| L_* \|) \). With this setting, we have that on \( \mathcal{E}_{Bdd} \), for any \( \alpha \in (0, 1) \),

\[
\hat{L}(\alpha) := \alpha \hat{L}(T) + (1 - \alpha)L_* \in \left\{ L \in \mathbb{R}^{n \times n} : \rho(L) \leq \zeta, \| L \| \leq \min \left( \| L_* \| + \frac{\gamma - \rho_*}{C_*}, \psi \right) \right\} =: \mathcal{G}.
\]

Therefore on \( \mathcal{E}_{Bdd} \), for some \( \alpha \in (0, 1) \),

\[
\| P(\hat{L}(T)) - P_* \| = \| [DP(\hat{L}(\alpha))](\hat{L}(T) - L_*) \| \leq \sup_{L \in \mathcal{G}} \| [DP(\hat{L})](\hat{L}(T) - L_*) \| := S\| \hat{L}(T) - L_* \|.
\]
Here the norm $\|H\| := \sup_{\|X\| \leq 1} \|H(X)\|$. We have that $S$ is finite since $G$ is a compact set.

Next, define the set $\mathcal{G}_{\text{Alg}}$ as:

$$\mathcal{G}_{\text{Alg}} := \{ L \in \mathbb{R}^{n \times n} : \rho(L) \leq \zeta, \|L\| \leq \psi \},$$

and define the event $\mathcal{E}_{\text{Alg}}$ as $\mathcal{E}_{\text{Alg}} := \{ \hat{L}(T) \in \mathcal{G}_{\text{Alg}} \}$. Consider the decomposition:

$$\mathbb{E}[\|\hat{P}_{\text{plug}}(T) - P_*\|^p] = \mathbb{E}[\|\hat{P}_{\text{plug}}(T) - P_*\|^p 1_{\mathcal{E}_{\text{Alg}}}] + \mathbb{E}[\|\hat{P}_{\text{plug}}(T) - P_*\|^p 1_{\mathcal{E}_{\text{Alg}}}^c] + \mathbb{E}[\|\hat{P}_{\text{plug}}(T) - P_*\|^p 1_{\mathcal{E}_{\text{Alg}}}^c \cap \mathcal{E}_{\text{Alg}}].$$

In what follows we will assume that $T$ is sufficiently large.

**On $\mathcal{E}_{\text{Bdd}}$.** On this event, since we have $\mathcal{E}_{\text{Bdd}} \subseteq \mathcal{E}_{\text{Alg}}$, we can bound by Lemma 4.6:

$$\mathbb{E}[\|\hat{P}_{\text{plug}}(T) - P_*\|^p 1_{\mathcal{E}_{\text{Alg}}}^c] \leq \mathbb{E}[\|\hat{P}_{\text{plug}}(T) - P_*\|^p 1_{\mathcal{E}_{\text{Alg}}}^c \cap \mathcal{E}_{\text{Alg}}] \leq \mathbb{E}[\|\hat{P}_{\text{plug}}(T) - P_*\|^p 1_{\mathcal{E}_{\text{Alg}}}^c \cap \mathcal{E}_{\text{Alg}}] \leq S^p \mathbb{E}[\|\hat{L}(T) - L_*\|^p] \leq O(1/T^{p/2}).$$

**On $\mathcal{E}_{\text{Bdd}}^c \cap \mathcal{E}_{\text{Alg}}$.** On this event, we use the fact that $\mathcal{G}_{\text{Alg}}$ is compact to bound:

$$\mathbb{E}[\|\hat{P}_{\text{plug}}(T) - P_*\|^p 1_{\mathcal{E}_{\text{Alg}}}^c \cap \mathcal{E}_{\text{Alg}}] \leq \sup_{\hat{L} \in \mathcal{G}_{\text{Alg}}} \|\text{dlyap}(\hat{L}, Q + K^T R K) - P_*\|^p \mathbb{P}(\mathcal{E}_{\text{Bdd}}^c \cap \mathcal{E}_{\text{Alg}}) \leq \sup_{\hat{L} \in \mathcal{G}_{\text{Alg}}} \|\text{dlyap}(\hat{L}, Q + K^T R K) - P_*\|^p \delta_T \leq O(1/T^{p/2}).$$

**On $\mathcal{E}_{\text{Bdd}}^c \cap \mathcal{E}_{\text{Alg}}$.** On this event, we simply have:

$$\mathbb{E}[\|\hat{P}_{\text{plug}}(T) - P_*\|^p 1_{\mathcal{E}_{\text{Alg}}}^c \cap \mathcal{E}_{\text{Alg}}] = \|P_*\|^p \mathbb{P}(\mathcal{E}_{\text{Bdd}}^c \cap \mathcal{E}_{\text{Alg}}) \leq \|P_*\|^p \delta_T \leq O(1/T^{p/2}).$$

**Putting it together.** Combining these bounds we obtain that $\mathbb{E}[\|\hat{P}_{\text{plug}}(T) - P_*\|^p] \leq O(1/T^{p/2})$. Recall that $Z_T = \text{svect}(P(\hat{L}(T)) - P_*$. We have that for any finite $\gamma > 0$ and $T \geq \Omega(1)$:

$$\mathbb{E}[\|Z_T\|^{2+\gamma}_F] = T^{(2+\gamma)/2} \mathbb{E}[\|P(\hat{L}(T)) - P_*\|^{2+\gamma}_F] \leq n^{(2+\gamma)/2} T^{(2+\gamma)/2} \mathbb{E}[\|P(\hat{L}(T)) - P_*\|^{2+\gamma}] \leq n^{(2+\gamma)/2} T^{(2+\gamma)/2} O(1/T^{(2+\gamma)/2}) \leq n^{(2+\gamma)/2} O(1).$$

On the other hand, when $T \leq O(1)$ it is easy to see that $\mathbb{E}[\|Z_T\|^{2+\gamma}_F]$ is finite. Hence we have $\sup_{T \geq 1} \mathbb{E}[\|Z_T\|^{2+\gamma}_F] < \infty$ which shows the desired uniformly integrable condition. This concludes the proof of Theorem 2.1.
5.2 Proof of Theorem 2.2

Lemma 4.3 (specifically (4.1)) combined with Lemma 4.5 tells us that:

\[
\liminf_{T \to \infty} T \cdot \mathbb{E} \left[ \left\| \hat{P}_{\text{std}}(T) - P_* \right\|_F^2 \right] \geq \text{Tr}(A_{\infty}^{-1} B_{\infty} \mathbf{A}_{\infty}^{-T})
\]
\[
\geq 8\sigma_{\text{w}}^2 \text{Tr}((P_{\infty}, L_*^T P_*^2 L_*)^{-1}(I - L_*^T \otimes_s L_*^T)^{-1}(P_{\infty}^{-1} \otimes_s P_{\infty}^{-1})(I - L_*^T \otimes_s L_*^T)^{-T})
\]
\[
+ 16\sigma_{\text{w}}^2 \text{Tr}((I - L_*^T \otimes_s L_*^T)^{-1}(L_*^T P_*^2 L_* \otimes_s P_{\infty}^{-1})(I - L_*^T \otimes_s L_*^T)^{-T})
\]

The claim now follows by using the risk bound from Theorem 2.1.

5.3 Proof of Theorem 2.3

Let \( E_1, \ldots, E_N \) be \( d \)-dimensional subspaces of \( \mathbb{R}^n \) with \( d \leq n/2 \) such that \( \|P_{E_i} - P_{E_j}\|_F \geq \sqrt{d} \). By Proposition 8 of Pajor [31], we can take \( N \geq e^{n(n-d)} \). Now consider instances \( A_i \) with \( A_i = \tau P_{E_i} + \gamma I_n \) for a \( \tau, \gamma \) \((0, 1)\) to be determined. We will set \( \tau + \gamma = \rho \) so that each \( A_i \) is contractive (i.e. \( \|A_i\| < 1 \)) and hence stable. This means implicitly that we will require \( \tau < \rho \). Let \( \mathbb{P}_i \) denote the distribution over \((x_1, \ldots, x_T)\) induced by instance \( A_i \). We have that:

\[
\text{KL}(\mathbb{P}_i, \mathbb{P}_j) = \sum_{t=1}^T \mathbb{E}_{x_t \sim \mathbb{P}_i} [\text{KL}(\mathcal{N}(A_i x_t, \sigma^2 I), \mathcal{N}(A_j x_t, \sigma^2 I))]
\]
\[
= \frac{1}{2\sigma^2} \sum_{t=1}^T \mathbb{E}_{x_t \sim \mathbb{P}_i} [\|A_i - A_j\|_F x_t x_t^T]
\]
\[
\leq \frac{\|A_i - A_j\|_F^2}{2\sigma^2} \sum_{t=1}^T \text{Tr}((\mathbb{E}_{x_t \sim \mathbb{P}_i} [x_t x_t^T])
\]
\[
\leq \frac{\tau^2}{\sigma^2} T \text{Tr}(P_{\infty})
\]
\[
= \tau^2 T \left( \frac{d}{1 - \rho^2} + \frac{n - d}{1 - \gamma^2} \right)
\]
\[
\leq \tau^2 T \frac{n}{1 - \rho^2}.
\]

Now if we choose \( n(n-d) \geq 4 \log 2 \) and \( T \gtrsim n(1 - \rho^2)/\rho^2 \), we can set \( \tau^2 \asymp \frac{n(1-\rho^2)}{\rho^2} \) and obtain that

\[
\frac{I(V;X) + \log 2}{\log |\mathcal{V}|} \leq 1/2.
\]

On the other hand, let \( P_t = \text{dlyap}(A_i, I_n) \). We have that for any integer \( k \geq 0 \):

\[
(\tau P_{E_i} + \gamma I_n)^k - (\tau P_{E_j} + \gamma I_n)^k = \sum_{\ell=0}^k \binom{k}{\ell} \gamma^{k-\ell} \tau^\ell (P_{E_i}^\ell - P_{E_j}^\ell)
\]
\[
= k \gamma^{k-1} \tau (P_{E_i} - P_{E_j}) + \sum_{\ell=2}^k \binom{k}{\ell} \gamma^{k-\ell} \tau^\ell (P_{E_i}^\ell - P_{E_j}^\ell).
\]
Hence,

\[ P_i - P_j = \sum_{k=1}^{\infty} ((A^k_i)^T A^k_i - (A^k_j)^T A^k_j) \]

\[ = \sum_{k=1}^{\infty} (A^2k_i - A^2k_j) \]

\[ = \left( \sum_{k=1}^{\infty} 2k\gamma^{2k-1}\tau + \sum_{k=2}^{\infty} \sum_{\ell=2}^{k} \left( \frac{2k}{\ell} \gamma^{2k-\ell}\tau^\ell \right) \right) (P_{Ei} - P_{Ej}) \]

\[ = \left( \frac{2\gamma \tau}{(1 - \gamma^2)^2} + \sum_{k=2}^{\infty} \sum_{\ell=2}^{k} \left( \frac{k}{\ell} \gamma^{k-\ell}\tau^\ell \right) \right) (P_{Ei} - P_{Ej}). \]

Therefore,

\[ \|P_i - P_j\|_F \geq \frac{2\gamma \tau}{(1 - \gamma^2)^2}\|P_{Ei} - P_{Ej}\|_F \gtrsim \frac{\gamma \tau}{(1 - \gamma^2)^2} \sqrt{d}. \]

The claim now follows by Fano’s inequality and setting \( d = n/4 \).

6 Analysis of Policy Optimization Methods

6.1 Preliminary Calculations

Given \((\rho U_\ast U_\ast^T, \rho U_\ast) \in \mathcal{G}(\rho, d)\), let \( J_\Sigma(K) \) for a \( K \in \mathbb{R}^{n \times d} \) denote the following cost:

\[ J_\Sigma(K) := \mathbb{E} \left[ \sum_{t=1}^{T} \|x_t\|_2^2 \right], \quad x_{t+1} = \rho U_\ast U_\ast^T x_t + \rho U_\ast u_t + w_t, \quad u_t = K^T x_t, \quad w_t \sim \mathcal{N}(0, \Sigma). \]

Here we assume \( T \geq 2 \) and \( \Sigma \) is positive definite. We write \( J(K) = J_{\sigma^2_{\ast \ast} I_{\ast \ast}}(K) \) as shorthand. Under this feedback law, we have \( x_t \sim \mathcal{N}(0, \sum_{\ell=0}^{T-1} L^\ell \Sigma (L^\ell)^T) \) with \( L = \rho U_\ast (U_\ast + K)^T \). Hence the cost can be written as:

\[ J_\Sigma(K) = \sum_{t=1}^{T} \sum_{\ell=0}^{t-1} \text{Tr}(L^\ell \Sigma (L^\ell)^T) \]

\[ = T \text{Tr}(\Sigma) + \sum_{t=1}^{T} \sum_{\ell=1}^{t-1} \rho^{2\ell} \text{Tr}\left( (U_\ast(U_\ast + K)^T)^\ell \Sigma ((U_\ast + K)U_\ast^T)^\ell \right). \]

Let \( K_\ast \) denote the minimizer of \( J_\Sigma(K) \); it is evident that \( K_\ast = -U_\ast \). While the function \( J_\Sigma(K) \) is not convex, it has many nice properties. First, the Hessian evaluated at \( K_\ast \) is positive definite:

\[ \text{Hess} J_\Sigma(K_\ast)[H, H] = 2(T - 1)\rho^2 \text{Tr}(H^T \Sigma H). \]
The next nice property is a quadratic growth condition:

\[ J_\Sigma(K) - J_\Sigma(K_\star) = \sum_{t=1}^{T} \sum_{\ell=1}^{t-1} \rho^{2\ell} \text{Tr}((U_\star(U_\star + K)^T\Sigma((U_\star + K)U_\star^T)^\ell) \geq (T-1)\rho^2 \text{Tr}(U_\star(U_\star + K)^T\Sigma(U_\star + K)U_\star^T) \]

\[ = (T-1)\rho^2 \text{Tr}((U_\star + K)^T\Sigma(U_\star + K)) \]

\[ \geq (T-1)\rho^2 \lambda_{\min}(\Sigma)\|U_\star + K\|_F^2 \]

\[ = (T-1)\rho^2 \lambda_{\min}(\Sigma)\|K - K_\star\|_F^2. \]

We now compute the gradient \( \nabla J_\Sigma(K) \). First, we consider the function \( M \mapsto M^\ell \) for any integer \( \ell \geq 2 \). We have that the derivatives are:

\[ [DM^\ell](\Delta) = \sum_{k=0}^{\ell-1} M^k \Delta M^{\ell-k-1}, \quad [D(M^\ell)^T](\Delta) = \sum_{k=0}^{\ell-1} (M^k)^T \Delta^T (M^{\ell-k-1})^T. \]

Let \( L(K) = \rho U_\star(U_\star + K)^T \). By the chain rule,

\[ [DL(K)^\ell](\Delta) = \sum_{k=0}^{\ell-1} L(K)^k B_\star \Delta^T L(K)^{\ell-k-1}, \quad [D(L(K)^\ell)^T](\Delta) = \sum_{k=0}^{\ell-1} (L(K)^k)^T \Delta B_\star^T (L(K)^{\ell-k-1})^T. \]

Hence by the chain rule again,

\[ [D \text{Tr}(L(K)^\ell \Sigma(L(K)^\ell)^T)](\Delta) = \text{Tr} \left( L(K)^\ell \Sigma \sum_{k=0}^{\ell-1} (L(K)^k)^T \Delta B_\star^T (L(K)^{\ell-k-1})^T \right) \]

\[ + \text{Tr} \left( \sum_{k=0}^{\ell-1} L(K)^k B_\star \Delta^T L(K)^{\ell-k-1} \Sigma(L(K)^\ell)^T \right) \]

\[ = 2 \left( \sum_{k=0}^{\ell-1} L^k \Sigma(L^\ell)^T L^{\ell-k-1} B_\star, \Delta \right). \]

Above, we use \( L = L(K) \) as shorthand. We have shown that:

\[ \nabla_K \text{Tr}(L(K)^\ell \Sigma(L(K)^\ell)^T) = 2 \sum_{k=0}^{\ell-1} L^k \Sigma(L^\ell)^T L^{\ell-k-1} B_\star. \]

Therefore we can compute the gradient of \( J_\Sigma(K) \) as:

\[ \nabla J_\Sigma(K) = 2(T-1)\Sigma L^T B_\star + 2 \sum_{\ell=2}^{T-1} \sum_{k=0}^{\ell-1} (T-\ell) L^k \Sigma(L^\ell)^T L^{\ell-k-1} B_\star. \]
Now observe that $B_*(U_* + K)^T = L$ and therefore:
\[
\langle \nabla J_\Sigma(K), K - K_* \rangle = \text{Tr}(\nabla J_\Sigma(K)(K + U_*)^T)
\]
\[
= 2(T - 1) \text{Tr}(\Sigma L^T L) + 2 \sum_{\ell=2}^{T-1} \sum_{k=0}^{T-\ell-1} (T - \ell) \text{Tr}(L^k \Sigma(L^\ell)^T L^{\ell-k-1} L)
\]
\[
= 2(T - 1) \text{Tr}(\Sigma L^T L) + 2 \sum_{\ell=2}^{T-1} \sum_{k=0}^{T-\ell-1} (T - \ell) \text{Tr}(\Sigma(L^\ell)^T L^\ell)
\]
\[
\geq 2(T - 1) \rho^2 \text{Tr}(\Sigma(U_* + K)U_*^T(U_* + K)^T)
\]
\[
= 2(T - 1) \rho^2 \text{Tr}(\Sigma(U_* + K)(U_* + K)^T)
\]
\[
\geq 2(T - 1) \rho^2 \lambda_{\text{min}}(\Sigma) \text{Tr}((U_* + K)(U_* + K)^T)
\]
\[
= 2(T - 1) \rho^2 \lambda_{\text{min}}(\Sigma) \|U_* + K\|_F^2
\]
\[
= 2(T - 1) \rho^2 \lambda_{\text{min}}(\Sigma) \|K - K_*\|_F^2.
\]

Above, (a) follows since $\text{Tr}(AB) \geq 0$ for positive semi-definite matrices $A, B$. This condition proves that $K = K_*$ is the unique stationary point, and establishes the restricted strong convexity $\text{RSC}(m, \mathbb{R}^{n \times d})$ condition for $J_\Sigma(K)$ with constant $m = 2(T - 1) \rho^2 \lambda_{\text{min}}(\Sigma)$.

### 6.2 Proof of Theorem 2.4

We define the pair $(A, B)$ is stabilizable if there exists a feedback matrix $K$ such that $\rho(A + BK) < 1$. We first state a result which gives a sufficient condition for the existence of a unique positive definite solution to the discrete algebraic Riccati equation.

**Lemma 6.1** (Theorem 2, [28]). Suppose that $Q > 0$, $(A, B)$ is stabilizable, and $B$ has full column rank. Then there exists a unique positive definite solution $P$ to the DARE:
\[
P = A^T PA - A^T PB(B^T PB)^{-1} B^T PA + Q.
\] (6.1)

This $P$ satisfies the lower bound $P \succeq Q$, and if $A$ is contractive (i.e. $\|A\| < 1$) satisfies the upper bound $\|P\| \leq \frac{\|Q\|}{1 - \|A\|^2}$.

**Proof.** Define the map $\Psi(z; A) := B^T(z^{-1}I_n - A)^{-T}Q(zI_n - A)^{-1}B$. Let $K$ be such that $A + BK$ is stable. We observe that for $|z| = 1$, we have that:
\[
\Psi(z; A + BK) = B^*(zI_n - (A + BK))^{-*}Q(zI_n - (A + BK))^{-1}B > 0.
\]

This is because $Q > 0$, $B^*B > 0$, and the matrix $zI_n - (A + BK)$ does not drop rank since $A + BK$ has no eigenvalues on the unit circle. Therefore by Theorem 2 of [28], there exists a unique symmetric solution $P$ that satisfies (6.1) with the additional constraint that $B^T PB > 0$ and that
\[ \rho(A_c) < 1 \] with \( A_c := A - B(B^T PB)^{-1}B^T PA \). But (6.1) means that:

\[
A_c^T PA_c = (A - B(B^T PB)^{-1}B^T PA)^T P(A - B(B^T PB)^{-1}B^T PA) = A^T PA - A^T PB(B^T PB)^{-1}B^T PA - A^T PB(B^T PB)^{-1}B^T PA \\
+ A^T PB(B^T PB)^{-1}B^T PB(B^T PB)^{-1}B^T PA = A^T PA - A^T PB(B^T PB)^{-1}B^T PA = P - Q.
\]

Hence, we have \( A_c^T PA_c - P + Q = 0 \), and since \( A_c \) is stable by Lyapunov theory we know that \( P \succeq Q \). Furthermore, since \( P \succeq 0 \), (6.1) implies that \( P \preceq A^T PA + Q \) from which the upper bound on \( \|P\| \) follows under the contractivity assumptions.

Next, we state a result which gives the derivative of the discrete algebraic Riccati equation.

**Lemma 6.2** (Section A.2 of [3]). Let \((Q, R)\) be positive semidefinite matrices. Suppose that \((A, B)\) are such that there exists a unique positive definite solution \( P(A, B) \) to \( \text{dare}(A, B, Q, R) \). For a perturbation \( [\Delta A \ \Delta B] \in \mathbb{R}^{n \times (n+d)} \), we have that the Fréchet derivative \( [D_{(A,B)}P(A,B)] \) evaluated at the perturbation \( [\Delta A \ \Delta B] \) is given by:

\[
[D_{(A,B)}P(A,B)]([\Delta A \ \Delta B]) = \text{dlyap} \left( A_c, A_c^T P [\Delta A \ \Delta B] [I_n \ K] + [I_n]^T [\Delta A \ \Delta B]^T PA_c \right),
\]

where \( P = P(A, B) \), \( K = -(B^T PB + R)^{-1}B^T PA \), and \( A_c = A + BK \).

With these two lemmas, we are ready to proceed. We differentiate the map \( h(A, B) := -(B^T P(A, B)B + R)^{-1}B^T P(A, B)A \). By the chain rule:

\[
[D_{(A,B)}h(A, B)](\Delta) = -(B^T PB + R)^{-1}(B^T P \Delta A + \Delta B^T PA + B^T [D_{(A,B)}P](\Delta)A) + (B^T PB + R)^{-1}(\Delta B^T PB + B^T P \Delta B + B^T [D_{(A,B)}P](\Delta)B)(B^T PB + R)^{-1}B^T PA.
\]

We now evaluate this derivative with:

\[
A = \rho U^*_T U^*_T, \ B = \rho U^*_T, Q = I_n, R = 0.
\]

Observe that by Lemma 6.2, we have that \( [D_{(A,B)}P(A,B)] = 0 \), since \( A_c = 0 \). Therefore the derivative \( [D_{(A,B)}h(A, B)](\Delta) \) simplifies to:

\[
[D_{(A,B)}h(A, B)](\Delta) = -\rho^{-2}(\rho U^*_T \Delta A + \rho \Delta B^T U^*_T U^*_T) + \rho^{-4}(\Delta B^T \rho U^*_T + \rho U^*_T \Delta B)\rho^2 U^*_T \\
= -\rho^{-1} U^*_T \Delta A - \rho^{-1} \Delta B^T U^*_T + \rho^{-1} \Delta B \rho U^*_T + \rho^{-1} U^*_T \Delta B U^*_T \\
= -\rho^{-1} U^*_T \Delta A + \rho^{-1} U^*_T \Delta B U^*_T.
\]

Hence we have:

\[
\text{vec}(D_{(A,B)}h(A, B)](\Delta)) = \text{vec}( -\rho^{-1} U^*_T \Delta A + \rho^{-1} U^*_T \Delta B U^*_T ) \\
= -\rho^{-1}(I_n \otimes U^*_T) \text{vec}(\Delta A) + \rho^{-1}(U^*_T \otimes U^*_T) \text{vec}(\Delta B) \\
= \rho^{-1} \left[ -(I_n \otimes U^*_T) \ (U^*_T \otimes U^*_T) \right] \text{vec}(\Delta).
\]

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Now from Lemma 4.2, we have that by the delta method:
\[
\sqrt{N}\text{vec}(h(\hat{A}(N), \hat{B}(N)) - K^T)
\]
\[
\overset{D}{\sim} N\left(0, \frac{\sigma_w^2}{T \rho^2} \begin{bmatrix} -(I_n \otimes U_n^T) & (U_n \otimes U_n^T) \end{bmatrix} \left( \begin{bmatrix} P_{\infty}^{-1} & 0 \\ 0 & (1/\sigma_u^2)I_d \end{bmatrix} \otimes I_n \right) \begin{bmatrix} -(I_n \otimes U_n^T) \\ (U_n^T \otimes U_n^T) \end{bmatrix} + o(1/T) \right) =: \varphi.
\]

We now make use of the second order delta method. Recall that the Hessian of \( J \) at \( K_* \) is \( \text{Hess}(J(K_*))[H,H] = 2(T-1)\rho^2\sigma_u^2\|H\|_F^2 \). If \( \sqrt{N}\text{vec}(\hat{K}(N) - K_*) \overset{D}{\sim} \varphi \), then by the second order delta method:
\[
N \cdot (J(\hat{K}(N)) - J_*) \overset{D}{\sim} (T-1)\rho^2\sigma_u^2\|\varphi\|_2^2.
\]
Let \( Z_N := N \cdot (J(\hat{K}(N)) - J_*) \). To conclude the proof, we show that the sequence \( \{Z_N\} \) is uniformly integrable. Once we have the uniform integrability in place, then by Lemma 4.5:
\[
\lim_{N \to \infty} N \cdot (J(\hat{K}(N)) - J_*) = (T-1)\rho^2\sigma_u^2\frac{\sigma_w^2}{T \rho^2} \text{Tr}\left( \left( \begin{bmatrix} P_{\infty}^{-1} & 0 \\ 0 & (1/\sigma_u^2)I_d \end{bmatrix} \otimes I_n \right) \begin{bmatrix} -(I_n \otimes U_n^T) \\ (U_n^T \otimes U_n^T) \end{bmatrix} + \sigma_T(1) \right)
\]
\[
= \sigma_w^4 \frac{T-1}{T} \text{Tr}(P_{\infty}^{-1} + \frac{d}{\sigma_u^2}) d + \sigma_T(1).
\]

We readily compute \( P_{\infty} \) as:
\[
P_{\infty} = \sum_{\ell=0}^{\infty} A_\ell \left( \sigma_u^2 B_\ell B_\ell^T + \sigma_w^2 I_n \right) A_\ell^T.
\]
Plugging in \( A_* = \rho U_* U_*^T \) and \( B_* = \rho U_* \), we have:
\[
P_{\infty} = \frac{\sigma_w^2 \rho^2 + \sigma_w^2}{1 - \rho^2} P_{U_*} + \sigma_w^2\sigma_w^2 P_{U_*}^T \Rightarrow P_{\infty}^{-1} = \frac{1 - \rho^2}{\sigma_u^2 \rho^2 + \sigma_w^2} P_{U_*} + \frac{1}{\sigma_w^2} P_{U_*}^T.
\]
Hence we have:
\[
\lim_{N \to \infty} N \cdot (J(\hat{K}(N)) - J_*) = \sigma_w^4 \frac{T-1}{T} \left( \frac{1 - \rho^2}{\sigma_u^2 \rho^2 + \sigma_w^2} + \frac{n - d}{\sigma_u^2} + \frac{d}{\sigma_w^2} \right) \text{d} + \sigma_T(1),
\]
from which Theorem 2.4 follows.

To conclude the proof, define the events:
\[
\mathcal{E}_{\text{Alg}} := \{ \|\hat{A}(N)\| \leq \zeta, \|\hat{B}(N)\| \leq \psi, \sigma_d(\hat{B}(N)) \geq \gamma \},
\]
\[
\mathcal{E}_{\text{Bdd}} := \{ \|\hat{A}(N) - A_*\| \leq \min\{\zeta - \rho, (1 - \rho)/2\}, \|\hat{B}(N) - B_*\| \leq \min\{\gamma - \rho, \rho/2\} \}.
\]
Observe that $\mathcal{E}_{\text{Bdd}} \subseteq \mathcal{E}_{\text{Alg}}$. Fix a finite $p \geq 1$. We write:

\[
\mathbb{E}[Z^p_N] = N^p \mathbb{E}[(J(\tilde{K}(N)) - J_*)^p 1_{\mathcal{E}_{\text{Bdd}}}] + N^p \mathbb{E}[(J(\tilde{K}(N)) - J_*)^p 1_{\mathcal{E}_{\text{Alg}}}] \\
= N^p \mathbb{E}[(J(\tilde{K}(N)) - J_*)^p 1_{\mathcal{E}_{\text{Bdd}}}] + N^p \mathbb{E}[(J(\tilde{K}(N)) - J_*)^p 1_{\mathcal{E}_{\text{Bdd}} \cap \mathcal{E}_{\text{Alg}}}] \\
+ N^p \mathbb{E}[(J(\tilde{K}(N)) - J_*)^p 1_{\mathcal{E}_{\text{Bdd}} \cap \mathcal{E}_{\text{Alg}}}] \\
= N^p \mathbb{E}[(J(\tilde{K}(N)) - J_*)^p 1_{\mathcal{E}_{\text{Bdd}}}] + N^p \mathbb{E}[(J(\tilde{K}(N)) - J_*)^p 1_{\mathcal{E}_{\text{Bdd}} \cap \mathcal{E}_{\text{Alg}}}] \\
+ N^p (0 - J_*)^p \mathbb{P}(\mathcal{E}_{\text{Bdd}} \cap \mathcal{E}_{\text{Alg}}) \\
\leq N^p \mathbb{E}[(J(\tilde{K}(N)) - J_*)^p 1_{\mathcal{E}_{\text{Bdd}}}] + N^p \mathbb{E}[(J(\tilde{K}(N)) - J_*)^p 1_{\mathcal{E}_{\text{Bdd}} \cap \mathcal{E}_{\text{Alg}}}] + N^p (J(0) - J_*)^p \mathbb{P}(\mathcal{E}_{\text{Bdd}}).
\]

We now consider what happens on these three events. For the remainder of the proof, we let $C$ denote a constant that depends on $n, d, p, \rho, \zeta, \gamma, A_*, B_*, T, \varepsilon, \sigma_{u_*}^2, \sigma_{d_*}^2$ but not on $N$, whose value can change from line to line.

**On the event $\mathcal{E}_{\text{Bdd}}$.** By a Taylor expansion we write:

\[
h(A(N), \tilde{B}(N)) = h_{\lambda_N}(A, B) = [D_{(A, B)}h(A, \tilde{B})]\left(\left[\tilde{A}(N) - A_* \quad \tilde{B}(N) - B_*\right]\right),
\]

where $\tilde{A} = tA_* + (1 - t)\tilde{A}(N)$ and $\tilde{B} = tB_* + (1 - t)\tilde{B}(N)$ for some $t \in [0, 1]$. Observe that on $\mathcal{E}_{\text{Bdd}}$, we have that

\[
\tilde{A}, \tilde{B} \in \mathcal{G} := \{(A, B) : \|A\| \leq (\rho + 1)/2, \|B\| \leq 3\rho/2, \sigma_d(B) \geq \rho/2\}.
\]

By Lemma 6.1, for any $(A, B) \in \mathcal{G}$ we have that $\text{dare}(A, B, I_n, 0)$ has a unique positive definite solution and its derivative is well defined. By the compactness of $\mathcal{G}$ and the continuity of $h$ and its derivative, we define the finite constants

\[
C_K := \sup_{A, B \in \mathcal{G}} \|h(A, B)\|, \quad C_{\text{deriv}} := \sup_{A, B \in \mathcal{G}} \|[D_{(A, B)}h(A, B)]\|.
\]

We can now Taylor expand $J(K)$ around $K_*$ and obtain:

\[
J(\tilde{K}(N)) - J_* = \frac{1}{2} \text{Hess} J(\tilde{K})[\tilde{K}(N) - K_*, \tilde{K}(N) - K_*] \\
\leq \frac{1}{2} \left(\sup_{\|	ilde{K}\| \leq C_K + \|K_*\|} \|\text{Hess} J(\tilde{K})\|\right) \|\tilde{K}(N) - K_*\|^2_F \\
\leq \frac{1}{2} \left(\sup_{\|	ilde{K}\| \leq C_K + \|K_*\|} \|\text{Hess} J(\tilde{K})\|\right) C_{\text{deriv}}^2 (\|\tilde{A}(N) - A_*\|^2 + \|\tilde{B}(N) - B_*\|^2).
\]

Hence for $N$ sufficiently large, by Lemma 4.7 we have

\[
N^p \cdot \mathbb{E}[(J(\tilde{K}(N)) - J_*)^p 1_{\mathcal{E}_{\text{Bdd}}}] \leq C N^p (\mathbb{E}[\|\tilde{A}(N) - A_*\|^{2p}] + \mathbb{E}[\|\tilde{B}(N) - B_*\|^{2p}]) \\
\leq C N^p \left(\frac{1}{N^p}\right) = C.
\]
On the event $\mathcal{E}_\text{Bdd}^c \cap \mathcal{E}_\text{Alg}$. In this case, we use the bounds given by $\mathcal{E}_\text{Alg}$ to bound the controller $\hat{K}(N)$. Lemma 6.1 ensures that the solution $\hat{P} = \text{dare}(\hat{A}(N), \hat{B}(N), I_n, 0)$ exists and satisfies $\|\hat{P}\| \leq \frac{1}{1-\zeta}$ and $\hat{P} \succeq I_n$. We can then bound $\|\hat{K}(N)\|$ as follows. Dropping the indexing of $N$,

$$
\|\hat{K}\| = \| (\hat{B}^T \hat{P} \hat{B})^{-1} \hat{B}^T \hat{P} \hat{A} \| \leq \frac{1}{\sigma_{\text{min}}(\hat{B}^T \hat{P} \hat{B})} \|\hat{B}^T \hat{P} \hat{A}\| \leq \frac{1}{\gamma^2} \frac{\psi \zeta}{1-\zeta^2}.
$$

Therefore:

$$
N^p \cdot \mathbb{E}[(J(\hat{K}(N)) - J_*)^p 1_{\mathcal{E}_\text{Bdd}^c \cap \mathcal{E}_\text{Alg}}] \leq N^p \cdot \left( \sup_{\|K\| \leq \frac{\psi \zeta}{1-\zeta^2}} (J(K) - J_*)^p \right) \mathbb{P}(\mathcal{E}_\text{Bdd}^c) \leq C N^p \mathbb{P}(\mathcal{E}_\text{Bdd}^c).
$$

By Lemma 4.7, we can choose $N$ large enough such that $\mathbb{P}(\mathcal{E}_\text{Bdd}^c) \leq 1/N^p$ so that $N^p \cdot \mathbb{E}[(J(\hat{K}(N)) - J_*)^p 1_{\mathcal{E}_\text{Bdd}^c \cap \mathcal{E}_\text{Alg}}] \leq C$.

On the event $\mathcal{E}_\text{Bdd}^c \cap \mathcal{E}_\text{Alg}$. This case is simple. We simply invoke Lemma 4.7 to choose an $N$ large enough such that $\mathbb{P}(\mathcal{E}_\text{Bdd}^c) \leq 1/(N(J(0) - J_*)^p)$.

Putting it together. If we take $N$ as the maximum over the three cases described above, we have hence shown that for all $N$ greater than this constant:

$$
\mathbb{E}[Z_N^p] \leq C.
$$

This shows the desired uniform integrability condition for $Z_N$. The asymptotic bound now follows from Lemma 4.5.

6.3 Proof of Theorem 2.5

The proof works by applying Lemma 4.8 with the function $F(\theta) = J_\mathcal{C}(K)$ with $\Sigma = \sigma_u^2 \rho^2 P_{\mathcal{U}} + \sigma_w^2 I_n$ and $G(\theta) = J(K)$. We first need to verify the hypothesis of the lemma. We define the convex domain $\Theta$ as $\Theta = \{K \in \mathbb{R}^{n \times d} : \|K\| \leq \zeta\}$. Note that $K_*$ is in the interior of $\Theta$, since $\|K_*\| = 1 < \zeta$. Recall that the policy gradient $g(K; \xi)$ is:

$$
g(K; \xi) = \frac{1}{\sigma_u^2} \sum_{t=1}^{T-1} x_t \eta_t^T \Psi_t, \quad \xi = (\eta_0, w_0, \eta_1, w_1, \ldots, \eta_{T-1}, w_{T-1}).
$$

It is clear that $x_t$ is a polynomial in $(K, \xi)$. Furthermore, all three of the $\Psi_t$’s we study are also polynomials in $(K, \xi)$. Hence $[D_K g(K; \xi)]$ is a matrix with entries that are polynomial in $(K, \xi)$. Therefore, for every $\xi$, for all fixed $K_1, K_2 \in \Theta$,

$$
\|g(K_1; \xi) - g(K_2; \xi)\|_F \leq \sup_{K \in \Theta} \|D_K g(K; \xi)\|_F \|K_1 - K_2\|_F.
$$

Hence squaring and taking expectations,

$$
\mathbb{E}_\xi[\|g(K_1; \xi) - g(K_2; \xi)\|^2_F] \leq \mathbb{E}_\xi \left[ \sup_{K \in \Theta} \|D_K g(K; \xi)\|^2_F \right] \|K_1 - K_2\|^2_F.
$$

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We can now define the constant $L := \mathbb{E}_\xi \left[ \sup_{K \in \Theta} \left\| D_K g(K; \xi) \right\|_F^2 \right]$. To see that this quantity $L$ is finite, observe that $\left\| \left\| D_K g(K; \xi) \right\|_F^2 \right\|_F^2$ is a polynomial of $\xi$ with coefficients given by $K$ (and $A_*, B_*$). Since $K$ lives in a compact set $\Theta$, these coefficients are uniformly bounded and hence the their moments are bounded. In Section 6.1, we showed that the function $J(K)$ satisfies the RSC $(m, \Theta)$ condition with $m = 2(T-1)\rho^2 \sigma_w^2$. Also it is clear that the high probability bound on $\left\| g(K; \xi) \right\|_F$ can be achieved by standard Gaussian concentration results. Hence by Lemma 4.8, and in particular Equation 4.8,

$$\liminf_{N \to \infty} N \cdot \mathbb{E}[J(\hat{K}) - J_*] \geq \frac{1}{8(T-1)\rho^2 \sigma_u^2 \lambda_{\max}((\nabla^2 J(K_*))^{-1}(\nabla^2 J_\Sigma(K_*)) - \frac{m}{2} I_{nd})} \mathbb{E}_\xi [\left\| g(K_*; \xi) \right\|_F^2] = \frac{1}{4(T-1)\rho^2 (\sigma_w^2 + 2\rho^2 \sigma_u^2)} \mathbb{E}_\xi [\left\| g(K_*; \xi) \right\|_F^2].$$

(6.2)

Above, the equality holds since we have that,

$$\nabla^2 J(K_*) = 2(T-1)\rho^2 \sigma_u^2 I_{nd} = mI_{nd},$$

$$\nabla^2 J_\Sigma(K_*) = 2(T-1)\rho^2 (I_d \otimes \Sigma) = mI_{nd} + m\rho^2 \sigma_u^2 (I_d \otimes P_{U_*}),$$

and therefore,

$$\lambda_{\max}((\nabla^2 J(K_*))^{-1}(\nabla^2 J_\Sigma(K_*)) - \frac{m}{2} I_{nd}) = \lambda_{\max} \left( \frac{1}{2} I_{nd} + \frac{\rho^2 \sigma_u^2}{\sigma_w^2} (I_d \otimes P_{U_*}) \right) = \frac{1}{2} + \frac{\rho^2 \sigma_u^2}{\sigma_w^2} = \frac{\sigma_w^2 + 2\rho^2 \sigma_u^2}{2\sigma_w^2}.$$

The remainder of the proof is to estimate the quantity $\mathbb{E}_\xi [\left\| g(K_*; \xi) \right\|_F^2]$. Note that at $K = K_*$, $x_t = \rho U_* \eta_{t-1} + w_{t-1}$ since the dynamics are cancelled out. Define $c_{t \to T} := \sum_{t=1}^{T} \left\| x_t \right\|_2^2$. At $K = K_*$, we have $c_{t \to T} = \sum_{t=1}^{T-1} \left\| \rho U_* \eta_t + w_t \right\|_2^2$. Observe that we have for $t_2 > t_1$, for any $h$ that depends on only $(\eta_{t_1}, \eta_{t_1+1}, w_{t_1+1}, \ldots)$:

$$\mathbb{E}[\langle \eta_{t_1}, \eta_{t_2} \rangle \langle x_{t_1}, x_{t_2} \rangle h] = \mathbb{E}[\langle \eta_{t_1}, \eta_{t_2} \rangle \left\langle \rho^2 \langle \eta_{t_1-1}, \eta_{t_2-1} \rangle + \langle w_{t_1-1}, w_{t_2-1} \rangle + \rho \langle U_* \eta_{t_1-1}, w_{t_1-1} \rangle + \rho \langle U_* \eta_{t_2-1}, w_{t_1-1} \rangle \right\rangle h] = 0.$$

As a consequence, we have that as long as $\Psi_t$ only depends on $(\eta_t, w_t, \eta_{t+1}, w_{t+1}, \ldots)$:

$$\mathbb{E}[\left\| g(K; \xi) \right\|_F^2] = \frac{1}{\sigma_w^2} \sum_{t=1}^{T-1} \mathbb{E}[\left\| \eta_t \right\|_2^2 \left\| x_t \right\|_2^2 \Psi_t] + \frac{2}{\sigma_u^4} \sum_{t_2 > t_1=1}^{T-1} \mathbb{E}[\langle \eta_{t_1}, \eta_{t_2} \rangle \langle x_{t_1}, x_{t_2} \rangle \Psi_{t_1} \Psi_{t_2}] = \frac{1}{\sigma_w^2} \sum_{t=1}^{T-1} \mathbb{E}[\left\| \eta_t \right\|_2^2 \left\| x_t \right\|_2^2 \Psi_t] .$$

6.3.1 Simple baseline

Recall that the simple baseline is to set $b_t(x_t; K) = \left\| x_t \right\|_2^2$. Hence, the policy gradient estimate simplifies to:

$$g(K; \xi) = \frac{1}{\sigma_w^2} \sum_{t=1}^{T-1} x_t \eta_t^T c_{t+1 \to T} .$$

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Since we have that $c_{t+1} \rightarrow T$ at optimality only depends only on $(\eta_t, w_t, \eta_{t+1}, w_{t+1}, \ldots)$, we compute $\mathbb{E}[\|g(K_*; \xi)\|^2_F]$ as follows:

\[
\mathbb{E}[\|g(K_*; \xi)\|^2_F] = \frac{1}{\sigma_u^2} \sum_{t=1}^{T-1} \mathbb{E}[\|\eta_t\|^2 \|x_t\|^2 \|c_{t+1} \rightarrow T\] 

\[
= \frac{1}{\sigma_u^2} \sum_{t=1}^{T-1} \mathbb{E}\left[\|\eta_t\|^2 \|\rho U_* \eta_t - w_t\|^2 + 2 \sum_{\ell=1}^{T-1} \|\rho U_* \eta_\ell + w_\ell\|^2\right] 

= \frac{1}{\sigma_u^2} \sum_{t=1}^{T-1} \mathbb{E}\left[\|\rho U_* \eta_t - w_t\|^2 \|\rho U_* \eta_t + w_t\|^2\right] 

+ \frac{1}{\sigma_u^2} \sum_{t=1}^{T-1} \sum_{\ell=t+1}^{T-1} \mathbb{E}[\|\rho U_* \eta_t - w_t\|^2 \|\rho U_* \eta_\ell + w_\ell\|^2] 

+ \frac{2}{\sigma_u^2} \sum_{t=1}^{T-1} \sum_{\ell_2 > t+1}^{T-1} \mathbb{E}[\|\rho U_* \eta_t - w_t\|^2 \|\rho U_* \eta_\ell + w_\ell\|^2] 

+ \frac{2}{\sigma_u^2} \sum_{t=1}^{T-1} \sum_{\ell_2 > \ell_1 + 1}^{T-1} \mathbb{E}[\|\rho U_* \eta_t - w_t\|^2 \|\rho U_* \eta_\ell + w_\ell\|^2] 

= \frac{2}{\sigma_u^2} \sum_{t=1}^{T-1} \sum_{\ell_2 > t+1}^{T-1} \mathbb{E}[\|\rho U_* \eta_t - w_t\|^2 \|\rho U_* \eta_\ell + w_\ell\|^2] 

\leq T^3 \frac{1}{\sigma_u^2} d(\rho^2 \sigma_u^2 d + \sigma_u^2 n)^3 + o(T^3) 

6.3.2 Value function baseline

Recall that the value function at time $t$ for a particular policy $K$ is defined as:

\[V^K_t(x) = \mathbb{E}\left[\sum_{t=1}^{T} \|x_t\|^2 \|x_t = x\right].\]

We now consider policy gradient with the value function baseline $b_t(x_t; K) = V^K_t(x_t)$:

\[g(K; \xi) = \frac{1}{\sigma_u^2} \sum_{t=1}^{T-1} x_t \eta_t^T (c_t \rightarrow T - V^K_t(x_t))\]

Recalling that under $K_*$ the dynamics are cancelled out, we readily compute:

\[V^K_t(x) = \|x\|^2 + (T - t)(\rho^2 \sigma_u^2 d + \sigma_u^2 n)\]

Therefore:

\[g(K_*; \xi) = \frac{1}{\sigma_u^2} \sum_{t=1}^{T-1} x_t \eta_t^T (c_{t+1} \rightarrow T - (T - t)(\rho^2 \sigma_u^2 d + \sigma_u^2 n))\].

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Define $\beta := (\rho^2 \sigma_u^2 d + \sigma_w^2 n)$. We compute the variance as:

$$
\mathbb{E}[\|g(K_\star; \xi)\|^2_F] = \frac{1}{\sigma_u^4} \sum_{t=1}^{T-1} \mathbb{E} \left[ \| \eta_t \|^2_2 \| \rho U_\star \eta_{t-1} + w_{t-1} \|^2_2 \right] \\
\times \left( \sum_{t=1}^{T-1} \| \rho U_\star \eta_t + w_t \|^2_2 - \beta \right)^2 + 2 \sum_{t=1}^{T-1} \| \rho U_\star \eta_t + w_t \|^2_2 - \beta \right) (\| \rho U_\star \eta_{t-1} + w_{t-1} \|^2_2 - \beta) \right) \\
= \frac{1}{\sigma_u^4} \sum_{t=1}^{T-1} \mathbb{E}[\| \eta_t \|^2_2 \| \rho U_\star \eta_{t-1} + w_{t-1} \|^2_2] \\
\times \left( 1 \sigma_u^4 \sum_{t=1}^{T-1} \sum_{t=t+1}^{T-1} \mathbb{E}[\| \eta_t \|^2_2 \| \rho U_\star \eta_{t-1} + w_{t-1} \|^2_2] + o(T^2) \\
\times T^2 \frac{1}{\sigma_u^2} d (\rho^2 \sigma_u^2 d + \sigma_w^2 n) (\mathbb{E}[\| \rho U_\star \eta_t + w_t \|^2_2] - \beta^2) + o(T^2) \\
\times T^2 \frac{1}{\sigma_u^2} d (\rho^2 \sigma_u^2 d + \sigma_w^2 n) (\rho^4 \sigma_u^4 d + \sigma_w^2 n + 2 \rho^2 \sigma_u^2 d) + (\rho^2 \sigma_u^2 d + \sigma_w^2 n)^2 .
$$

Above, (a) follows because:

$$
\mathbb{E}[\| \rho U_\star \eta_t + w_t \|^2_2] = 2 \| \rho^2 \sigma_u^2 P \rho U_\star \eta_t + \rho^2 \sigma_u^2 I_n \|^2_F + (\rho^2 \sigma_u^2 d + \sigma_w^2 n)^2 \\
= 2 (\rho^4 \sigma_u^4 d + \sigma_w^2 n + 2 \rho^2 \sigma_u^2 d) + (\rho^2 \sigma_u^2 d + \sigma_w^2 n)^2 .
$$

### 6.3.3 Ideal advantage baseline

Let us first compute $Q_t^{K_\star}(x_t, u_t)$. Under $K_\star$, $x_{t+1} = \rho U_\star \eta_t + w_t$. So we have:

$$
Q_t^{K_\star}(x_t, u_t) = \| x_t \|^2_2 + \mathbb{E}_{w_t} [\| \rho P \rho U_\star x_t + \rho U_\star u_t + w_t \|^2_2] + (T - t - 1) (\rho^2 \sigma_u^2 d + \sigma_w^2 n) \\
= \| x_t \|^2_2 + \| \rho P \rho U_\star x_t + \rho U_\star u_t \|^2_2 + \sigma_w^2 n + (T - t - 1) (\rho^2 \sigma_u^2 d + \sigma_w^2 n) .
$$

Recalling that $V_t^{K_\star}(x) = \| x \|^2_2 + (T - t) (\rho^2 \sigma_u^2 d + \sigma_w^2 n)$,

$$
A_t^{K_\star}(x_t, u_t) = Q_t^{K_\star}(x_t, u_t) - V_t^{K_\star}(x_t) = \| \rho P \rho U_\star x_t + \rho U_\star u_t \|^2_2 - \rho^2 \sigma_u^2 d .
$$

Therefore, if $u_t = K_t x_t + \eta_t$, we have $A_t^{K_\star}(x_t, u_t) = \| \rho U_\star \eta_t \|^2_2 - \rho^2 \sigma_u^2 d$. Since $A_t^{K_\star}(x_t, u_t)$ depends only on $\eta_t$,

$$
\mathbb{E}[\| g(K_\star; \xi) \|^2_F] = \frac{1}{\sigma_u^4} \sum_{t=1}^{T-1} \mathbb{E}[\| \eta_t \|^2_2 \| x_t \|^2_2 (\| \rho U_\star \eta_t \|^2_2 - \rho^2 \sigma_u^2 d)^2] \\
= \frac{1}{\sigma_u^4} (T - 1) (\rho^2 \sigma_u^2 d + \sigma_w^2 n) \rho^4 \mathbb{E}[\| \eta_t \|^2_2 (\| \eta_t \|^2_2 - \sigma_u^2 d)^2] .
$$

We have that:

$$
\mathbb{E}[\| \eta_t \|^2_2] = \sigma_u^2 d , \mathbb{E}[\| \eta_t \|^2] = \sigma_u^4 (d^2 + 2d) , \mathbb{E}[\| \eta_t \|^6] = \sigma_u^6 (d^3 + 6d^2 + 8d) .
$$

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Hence,
\[
\mathbb{E}[||\eta_1||^2_2(||\eta_1||^2_2 - \sigma^2_d d)^2] = \mathbb{E}[||\eta_1||^6_2 + ||\eta_1||^2_2 \sigma^4_d d^2 - 2||\eta_1||^4_2 \sigma^2_d d] = \sigma^6_n (2d^2 + 8d) .
\]

Therefore,
\[
\mathbb{E}[||g(K_*; \xi)||^2_F] \approx \rho^4 T (\rho^2 \sigma^2_u d + \sigma^2_w n)^{1/2} d^2 .
\]

### 6.3.4 Putting it together

Combining Equation (6.2) with the calculations for \( \mathbb{E}_\xi[||g(K_*; \xi)||^2_F] \), we obtain:

\[
\lim_{N \to \infty} \inf N \cdot \mathbb{E}[J(\hat{K}_p(N)) - J_*] \geq \begin{cases} 
T^2 \cdot \left( d^1 \rho^2 \sigma^4_u + n^3 d \rho^2 \sigma^6_u (\sigma^2_u + \rho^2 \sigma^2_u) \right) + o(T^2) & \text{(Simple baseline)} \\
T \cdot \left( d(\rho^2 \sigma^2_u)^2 + \sigma^2_u n \right)^2 \rho^2 \sigma^2_u (\sigma^2_u + \rho^2 \sigma^2_u) + o(T) & \text{(Value function baseline)} \\
d^1 \rho^4 \sigma^4_u / (\sigma^2_u + \rho^2 \sigma^2_u) + nd^2 \sigma^2_u / (\sigma^2_u + \rho^2 \sigma^2_u) & \text{(Advantage baseline)}
\end{cases}
\]

from which Theorem 2.5 follows.

### 6.4 Proof of Theorem 2.6

Our proof is inspired by lower bounds for the query complexity of derivative-free optimization of stochastic optimization (see e.g. [18]).

Recall that the function \( J(\hat{K}) \) satisfies the quadratic growth condition \( J(K) - J_* \geq (T - 1) \rho^2 \sigma^2_w \|K - K_*\|^2_F \). Therefore for any \( \vartheta > 0 \),

\[
\inf_{K} \sup_{(A_*, B_*) \in \mathcal{U}(\rho, d)} \mathbb{E}[J(\hat{K}) - J_*) \geq \inf_{K} \sup_{(A_*, B_*) \in \mathcal{U}(\rho, d)} (T - 1) \rho^2 \sigma^2_w \vartheta^2 \cdot \mathbb{P}(J(\hat{K}) - J_* \geq (T - 1) \rho^2 \sigma^2_w \vartheta^2) \\
\geq \inf_{K} \sup_{(A_*, B_*) \in \mathcal{U}(\rho, d)} (T - 1) \rho^2 \sigma^2_w \vartheta^2 \cdot \mathbb{P}((-U_* \geq \hat{K})^2 \geq (T - 1) \rho^2 \sigma^2_w \vartheta^2) \\
= \inf_{K} \sup_{(A_*, B_*) \in \mathcal{U}(\rho, d)} (T - 1) \rho^2 \sigma^2_w \vartheta^2 \cdot \mathbb{P}((-U_* \geq \hat{K})^2 \geq \vartheta) .
\]

Above, the first inequality is Markov’s inequality and the second is the quadratic growth condition.

We first state a result regarding the packing number of \( O(n, d) \), which we define as:

\[
O(n, d) := \{ U \in \mathbb{R}^{n \times d} : U^T U = I_d \}.
\]

**Lemma 6.3.** Let \( \delta > 0 \), and suppose that \( d \leq n/2 \). We have that the packing number \( M \) of \( O(n, d) \) in the Frobenius norm \( \|\cdot\|_F \) satisfies

\[
M(O(n, d), \|\cdot\|_F, \delta d^{1/2}) \geq \left( \frac{c}{\delta} \right)^{d(n-d)} ,
\]

where \( c > 0 \) is a universal constant.

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Proof. Let $G_{n,d}$ denote the Grassman manifold of $d$-dimensional subspaces of $\mathbb{R}^n$. For two subspaces $E, F \in G_{n,d}$, equip $G_{n,d}$ with the metric $\rho(E, F) = \|P_E - P_F\|_F$, where $P_E, P_F$ are the projection matrices onto $E, F$ respectively. Proposition 8 of Pajor [31] tells us that the covering number $N(G_{n,d}, \rho, \delta d^{1/2}) \geq \left(\frac{s}{\delta}\right)^{d(n-d)}$. But since $M(G_{n,d}, \rho, \delta d^{1/2}) \geq N(G_{n,d}, \rho, \delta d^{1/2})$, this gives us a lower bound on the packing number of $G_{n,d}$. Now for every $E \in G_{n,d}$ we can associate a matrix $E_1 \in O(n, d)$ such that span($E_1$) = $E$. The projector $P_E$ is simply $P_E = E_1E_1^T$. Now let $E, F \in G_{n,d}$ and observe the inequality,

$$\|P_E - P_F\|_F = \|E_1E_1^T - F_1F_1^T\|_F \leq 2\|E_1 - F_1\|_F.$$ 

Hence a packing of $G_{n,d}$ also yields a packing of $O(n, d)$ up to constant factors.

Now letting $U_1, \ldots, U_M$ be a $2\theta$-separated set we have by the standard reduction to multiple hypothesis testing that that the risk is lower bounded by:

$$(T - 1)\rho^2\sigma_w^2\hat{g}^2 \cdot \inf_{\bar{V}} P(\bar{V} \neq V) \geq (T - 1)\rho^2\sigma_w^2\hat{g}^2 \cdot \left(1 - \frac{I(V; Z) + \log 2}{\log M}\right). \quad (6.3)$$

where $V$ is a uniform index over $\{1, \ldots, M\}$ and the inequality is Fano’s inequality.

Now we can proceed as follows. First, we let $U_1, \ldots, U_M$ be elements of $O(n,d)$ that form a $2\theta \times \sqrt{d}$ packing in the $\|\cdot\|_F$ norm. We know we can let $M \geq e^{d(n-d)}$ by Lemma 6.3. Each $U_i$ induces a covariance $\Sigma_i = \sigma_w^2I_n + \rho^2\sigma_u^2U_iU_i^T \leq (\sigma_w^2 + \rho^2\sigma_u^2)I_n$. Furthermore, the closed-loop $L_i$ given by playing a feedback matrix $K$ that satisfies $\|K\| \leq 1$ is:

$$L_i = \rho U_i(U_i + K)^T.$$ 

It is clear that $\|L_i\| \leq 2\rho$ and hence if $\rho < 1/2$ then this system is stable. With this, we can control:

$$\mathbb{E}[x_t x_t^T] = \sum_{\ell=0}^{t-1} L_i^\ell \Sigma_i (L_i^\ell)^T \leq (\sigma_w^2 + \rho^2\sigma_u^2) \sum_{\ell=0}^{t-1} L_i^\ell (L_i^\ell)^T \leq \frac{\sigma_w^2 + \rho^2\sigma_u^2}{1 - (2\rho)^2} I_n.$$ 

Hence for one trajectory $Z = (x_0, u_0, x_1, u_1, \ldots, x_{T-1}, u_{T-1}, x_T)$,

$$\text{KL}(\mathbb{P}_i, \mathbb{P}_j) \leq \sum_{t=0}^{T-1} \frac{1}{2\sigma_w^2} \mathbb{E}[x_t x_t^T][\|L_i(L_j - L_j)x_t\|^2]$$ 

$$\leq \frac{8\rho^2}{\sigma_w^2} \sum_{t=0}^{T-1} \text{Tr}(\mathbb{E}[x_t x_t^T])$$ 

$$\leq \frac{8(\sigma_w^2 + \rho^2\sigma_u^2)\rho^2 Tn}{\sigma_w^2(1 - (2\rho)^2)}.$$ 

Assuming $d(n-d)$ is greater than an absolute constant, we can set $\rho$ to be (recall we have $N$ different rollouts):

$$\rho^2 \propto \frac{\sigma_w^2}{\sigma_w^2 + \sigma_u^2} \frac{d(n-d)}{nTN},$$ 

and bound $\frac{I(V; Z) + \log 2}{\log M} \leq 1/2$. The result now follows from plugging in our choice of $\rho$ into (6.3).
7 Conclusion

We compared the asymptotic performance of both model-based and model-free methods for LQR. We showed that for policy evaluation, a simple plugin estimator is always more asymptotically sample efficient than the classical LSTD estimator. For policy optimization, we studied a family of instances where the convergence of policy gradient to the optimal solution is guaranteed, and showed that in this setting a simple plugin estimator is asymptotically at least a factor of state-dimension more efficient than policy gradient, depending on what specific baseline is used.

This work opens a variety of new directions for future research. The first is to broaden our results for policy gradient and analyze a larger family of instances. As mentioned earlier, this would require expanding our current understanding for under what conditions policy gradient on a finite horizon objective converges to an optimal solution. Another interesting direction is to use our framework to analyze the effect of various baseline estimators in policy gradient. Designing efficient baseline estimators is still an open problem in RL, and using asymptotic analysis to more carefully understand the various estimators could be very insightful. Finally, extending the asymptotic analysis to the online learning setting may help further our understanding of the effects of optimistic exploration versus $\varepsilon$-greedy exploration for LQR.

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A Deferred Proofs for Asymptotic Toolbox

Our main limit theorem is the following CLT for ergodic Markov chains.

**Theorem A.1** (Corollary 2 of [20]). Suppose that \( \{x_t\}_{t=0}^{\infty} \subseteq X \) is a geometrically ergodic (Harris) Markov chain with stationary distribution \( \pi \). Let \( f : X \to \mathbb{R} \) be a Borel function. Suppose that \( \mathbb{E}_\pi[|f|^{2+\delta}] < \infty \) for some \( \delta > 0 \). Then for any initial distribution, we have:

\[
\sqrt{n} \left( \frac{1}{n} \sum_{i=1}^{n} f(x_i) - \mathbb{E}_\pi[f(x)] \right) \overset{D}{\to} \mathcal{N}(0, \sigma_f^2),
\]

where

\[
\sigma_f^2 := \text{Var}_\pi(f(x_0)) + 2 \sum_{i=1}^{\infty} \text{Cov}_\pi(f(x_0), f(x_i)).
\]

### A.1 Proof of Lemma 4.1

**Proof.** Let \( X \in \mathbb{R}^{T \times n} \) be the data matrix with rows \( (x_0, \ldots, x_{T-1}) \) and \( W \in \mathbb{R}^{T \times n} \) be the noise matrix with rows \( (w_0, \ldots, w_{T-1}) \). We write:

\[
\tilde{L}(T) - L_* = -\lambda L_*(X^T X + \lambda I_n)^{-1} + W^T X^T X + \lambda I_n)^{-1}.
\]

Using the fact that \( \text{vec}(AXB) = (B^T \otimes A)\text{vec}(X) \),

\[
\sqrt{T} \text{vec}(\tilde{L}(T) - L_*) = -\sqrt{T} \text{vec}(\lambda L_*(X^T X + \lambda I_n)^{-1}) + ((T^{-1} X^T X)^{-1} \otimes I_n) \text{vec}(T^{-1/2} W^T X).
\]

It is well-known that \( \{x_t\} \) is geometrically ergodic (see e.g. [27]), and therefore the augmented Markov chain \( \{(x_t, w_t)\} \) is geometrically ergodic as well. By Theorem A.1 combined with the Cramér-Wold theorem we conclude:

\[
\text{vec}(T^{-1/2} W^T X) = T^{-1/2} \sum_{t=1}^{T} \text{vec}(w_t x_t^T) \overset{D}{\to} \mathcal{N}(0, \mathbb{E}_{x \sim \nu_\infty, w}[\text{vec}(w x^T) \text{vec}(w x^T)^T]).
\]

Above, we let \( \nu_\infty \) denote the stationary distribution of \( \{x_t\} \). We note that the cross-correlation terms disappear in the asymptotic covariance due to the martingale difference property of \( \sum_{t=0}^{T-1} w_t x_t^T \).

We now use the identity \( \text{vec}(w x^T) = (x \otimes I_n)w \) and compute

\[
\mathbb{E}_{x \sim \nu_\infty,w}[\text{vec}(w x^T) \text{vec}(w x^T)^T] = \mathbb{E}_{x \sim \nu_\infty,w}[(x \otimes I_n)ww^T(x^T \otimes I_n)]
\]

\[
= \sigma_w^2 \mathbb{E}_{x \sim \nu_\infty}[(x \otimes I_n)(x^T \otimes I_n)]
\]

\[
= \sigma_w^2 \mathbb{E}_{x \sim \nu_\infty}[(xx^T \otimes I_n)]
\]

\[
= \sigma_w^2 (P_\infty \otimes I_n).
\]

We have that \( T^{-1} X^T X \overset{a.s.}{\to} P_\infty \) by the ergodic theorem. Therefore by the continuous mapping theorem followed by Slutsky’s theorem, we have that

\[
(T^{-1} X^T X)^{-1} \otimes I_n) \text{vec}(T^{-1/2} W^T X) \overset{D}{\to} \mathcal{N}(0, \sigma_w^2 (P_\infty^{-1} \otimes I_n)).
\]

On the other hand, we have:

\[
\sqrt{T} \text{vec}(\lambda L_*(X^T X + \lambda I_n)^{-1}) = \frac{1}{\sqrt{T}} \text{vec}(\lambda L_*(T^{-1} X^T X + T^{-1} \lambda I_n)^{-1}) \overset{a.s.}{\to} 0.
\]

The claim now follows by another application of Slutsky’s theorem. \( \Box \)
A.2 Proof of Lemma 4.2

Proof. Let $Z(i) \in \mathbb{R}^{T \times (n+d)}$ be a data matrix with the rows $(z_0^{(i)}, \ldots, z_{T-1}^{(i)})$, and let $W(i) \in \mathbb{R}^{T \times n}$ be the noise matrix with the rows $(w_0^{(i)}, \ldots, w_{T-1}^{(i)})$. With this notation we write:

$$
\hat{\Theta}(N) - \Theta_* = \left( \sum_{i=1}^N \frac{1}{T} \sum_{t=0}^{T-1} z_{t+1}^{(i)} (z_t^{(i)})^T \right) \left( \sum_{i=1}^N \frac{1}{T} \sum_{t=0}^{T-1} z_{t}^{(i)} (z_t^{(i)})^T + \lambda I_{n+d} \right)^{-1} - \Theta_*
$$

$$
= \Theta_* \left( \sum_{i=1}^N \frac{1}{T} (Z^{(i)})^T Z^{(i)} \right) \left( \sum_{i=1}^N \frac{1}{T} (Z^{(i)})^T Z^{(i)} + \lambda I_{n+d} \right)^{-1} - \Theta_*
$$

$$
+ \left( \sum_{i=1}^N \frac{1}{T} (W^{(i)})^T Z^{(i)} \right) \left( \sum_{i=1}^N \frac{1}{T} (Z^{(i)})^T Z^{(i)} + \lambda I_{n+d} \right)^{-1}
$$

$$
= -\lambda \Theta_* \left( \sum_{i=1}^N \frac{1}{T} (Z^{(i)})^T Z^{(i)} + \lambda I_{n+d} \right)^{-1}
$$

$$
+ \left( \sum_{i=1}^N \frac{1}{T} (W^{(i)})^T Z^{(i)} \right) \left( \sum_{i=1}^N \frac{1}{T} (Z^{(i)})^T Z^{(i)} + \lambda I_{n+d} \right)^{-1}
$$

$$
=: G_1(N) + G_2(N).
$$

Taking vec of $G_2(N)$:

$$
\text{vec}(G_2(N)) = \left( \frac{1}{N} \sum_{i=1}^N \frac{1}{T} (Z^{(i)})^T Z^{(i)} + \frac{\lambda}{N} I_{n+d} \right)^{-1} \otimes I_n \left( \frac{1}{N} \sum_{i=1}^N \frac{1}{T} \sum_{t=0}^{T-1} w_t^{(i)} (z_t^{(i)})^T \right).
$$

Now we write vec$(w_t z_t^T) = (z_t \otimes I_n) w_t$ and hence

$$
\mathbb{E} \left[ \text{vec} \left( \frac{1}{T} \sum_{t=0}^{T-1} w_t z_t^T \right) \text{vec} \left( \frac{1}{T} \sum_{t=0}^{T-1} w_t z_t^T \right)^T \right] = \frac{1}{T^2} \sum_{t_1, t_2=0}^{T-1} \mathbb{E}[(z_{t_1} \otimes I_n) w_{t_1} w_{t_2}^T (z_{t_2} \otimes I_n)]
$$

$$
= \frac{\sigma_w^2}{T^2} \sum_{t=0}^{T-1} \mathbb{E}[z_t z_t^T] \otimes I_n.
$$

We have that:

$$
\frac{1}{N} \sum_{i=1}^N \frac{1}{T} (Z^{(i)})^T Z^{(i)} + \frac{\lambda}{N} I_{n+d} \xrightarrow{\text{a.s.}} \frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E}[z_t z_t^T].
$$

Hence by the central limit theorem combined with the continuous mapping theorem and Slutsky’s theorem,

$$
\sqrt{N} \text{vec}(G_1(N)) \xrightarrow{\text{a.s.}} 0,
$$

$$
\sqrt{N} \text{vec}(G_2(N)) \xrightarrow{D} \mathcal{N} \left( 0, \frac{\sigma_w^2}{T} \left[ \frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E}[z_t z_t^T] \right]^{-1} \otimes I_n \right)
$$

$$
= \mathcal{N} \left( 0, \frac{\sigma_w^2}{T} \left[ \frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E}[z_t z_t^T] \right]^{-1} \otimes I_n \right).
$$
To finish the proof, we note that $\mathbb{E}[x_t x_t^T] = \sum_{t=0}^{T-1} \phi_t^T M(A_t^k)^T := P_t$ with $M := \sigma_n^2 B \sigma_n^T + \sigma_n^2 I_n$ and $P_0 = 0$ (since $x_0 = 0$). Since $A_k$ is stable, there exists a $\rho \in (0, 1)$ and $C > 0$ such that $\|A_k^k\| \leq C \rho^k$ for all $k \geq 0$. Hence,

$$
\|P_\infty - P_t\| = \left\| \sum_{t=t}^{\infty} A_k^k M(A_k^k)^T \right\| \leq C^2\|M\| \sum_{t=t}^{\infty} \rho^{2t} = C^2\|M\| \frac{\rho^{2t}}{1 - \rho^2}.
$$

Therefore,

$$
\left\| \frac{1}{T} \sum_{t=0}^{T-1} P_t - P_\infty \right\| = \left\| \frac{1}{T} \sum_{t=1}^{T-1} (P_t - P_\infty) + \frac{1}{T} P_\infty \right\|
\leq \frac{C^2\|M\|}{T(1 - \rho^2)} \sum_{t=1}^{T-1} \rho^{2t} + \frac{1}{T} \|P_\infty\|
\leq \frac{C^2\|M\|}{T(1 - \rho^2)^2} + \frac{1}{T} \|P_\infty\| = O(1/T).
$$

Therefore, $[\frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E}[x_t x_t^T]]^{-1} = P_\infty^{-1} + O(1/T)$ from which the claim follows.

\begin{proof}
Proof. Let $c_t = x_t^T (Q + K^T R K) x_t$. From Bellman’s equation, we have $c_t - \lambda = (\phi(x_t) - \psi(x_t))^T w_*$. We write:

$$
\tilde{w}_{\text{std}}(T) - w_* = \left( \sum_{t=0}^{T-1} \phi(x_t)(\phi(x_t) - \phi(x_{t+1}))^T \right)^{-1} \left( \sum_{t=0}^{T-1} (c_t - \lambda) \phi(x_t) \right) - w_*
= \left( \sum_{t=0}^{T-1} \phi(x_t)(\phi(x_t) - \phi(x_{t+1}))^T \right)^{-1} \left( \sum_{t=0}^{T-1} \phi(x_t)(\phi(x_t) - \psi(x_t))^T \right) w_* - w_*
= \left( \sum_{t=0}^{T-1} \phi(x_t)(\phi(x_t) - \phi(x_{t+1}))^T \right)^{-1} \left( \sum_{t=0}^{T-1} \phi(x_t)(\phi(x_{t+1}) - \psi(x_t))^T w_* \right)
= \left( \frac{1}{T} \sum_{t=0}^{T-1} \phi(x_t)(\phi(x_t) - \phi(x_{t+1}))^T \right)^{-1} \left( \frac{1}{T} \sum_{t=0}^{T-1} \phi(x_t)(\phi(x_{t+1}) - \psi(x_t))^T w_* \right).
$$

We now proceed by considering the Markov chain $\{z_t := (x_t, w_t)\}$. Observe that $x_{t+1}$ is $z_t$-measurable, and furthermore the stationary distribution of this chain is $\nu_\infty \times \mathcal{N}(0, \sigma_w^2 I_n)$. From this we conclude two things. First, we conclude by the ergodic theorem that the term inside the inverse converges a.s. to $A_\infty$ and hence the inverse converges a.s. to $A_\infty^{-1}$ by the continuous mapping theorem. Next, Theorem A.1 combined with the Cramér-Wold theorem allows us to conclude that

$$
\frac{1}{\sqrt{T}} \sum_{t=1}^{T} \phi(x_t)(\phi(x_{t+1}) - \psi(x_t))^T w_* \xrightarrow{D} \mathcal{N}(0, B_\infty).
$$

The final claim now follows by Slutsky’s theorem.
\end{proof}
A.4 Proof of Corollary 4.4

Proof. In the proof we write $\Sigma = \sigma_w^2 I_n$. First, we note that a quick computation shows that $\psi(x) = \text{svec}(Lxx^T L^T + \Sigma)$.

Matrix $A_\infty$. We have

$$\phi(x) - \phi(x') = \text{svec}(xx^T - (Lx + w)(Lx + w)^T)$$

$$= \text{svec}(xx^T - Lxx^T L^T - Lxw^T - wx^T L^T - ww^T).$$

Hence, conditioning on $x$ and iterating expectations, we have

$$A_\infty = \mathbb{E}_{x \sim \nu_\infty}[\phi(x)\text{svec}(xx^T - Lxx^T L^T - \Sigma)^T].$$

Now let $m,n$ be two test vectors and $M = \text{smat}(m), N = \text{smat}(n)$. We have that,

$$m^T A_\infty n = \mathbb{E}_{x \sim \nu_\infty}[x^T M x (x^T - Lxx^T L^T - \Sigma, N)]$$

$$= \mathbb{E}_{x \sim \nu_\infty}[x^T M x (N - L^T NL)x - (\Sigma, N)]$$

$$= \mathbb{E}_{x \sim \nu_\infty}[x^T M x (N - L^T NL)x - (\Sigma, N)] \mathbb{E}_{x \sim \nu_\infty}[x^T M x]$$

$$= \mathbb{E}_g [g^T P_\infty^{1/2} MP_\infty^{1/2} g^T P_\infty^{1/2} (N - L^T NL) P_\infty^{1/2} g] - (\Sigma, N) \langle M, P_\infty \rangle$$

$$= 2 \langle P_\infty^{1/2} MP_\infty^{1/2}, P_\infty^{1/2} (N - L^T NL) P_\infty^{1/2} \rangle + \langle M, P_\infty \rangle \langle N - L^T NL, P_\infty \rangle - (\Sigma, N) \langle M, P_\infty \rangle$$

$$= 2 \langle P_\infty^{1/2} MP_\infty^{1/2}, P_\infty^{1/2} (N - L^T NL) P_\infty^{1/2} \rangle,$$

where the last identity follows since $LP_\infty L^T - P_\infty + \Sigma = 0$. We therefore have:

$$A_\infty = (P_\infty \otimes_s P_\infty) - (P_\infty L^T \otimes_s P_\infty L^T)$$

$$= (P_\infty \otimes_s P_\infty)(I - L^T \otimes_s L^T) .$$

Note that this writes $A_\infty$ as the product of two invertible matrices and hence $A_\infty$ is invertible.

Matrix $B_\infty$. We have

$$\langle \phi(x') - \psi(x), w_\star \rangle = \text{svec}(Lxw^T + wx^T L^T + ww^T - \Sigma)^T w_\star$$

$$= 2x^T L^T P_\star w + \langle ww^T - \Sigma, P_\star \rangle.$$

Hence,

$$\langle \phi(x') - \psi(x), w_\star \rangle^2 = 4(x^T L^T P_\star w)^2 + \langle ww^T - \Sigma, P_\star \rangle^2 + 4x^T L^T P_\star w \langle ww^T - \Sigma, P_\star \rangle$$

$$=: T_1 + T_2 + T_3.$$

Now we have that $m^T B_\infty n$ is

$$m^T B_\infty n = \mathbb{E}[T_1 x^T Mxx^T N x] + \mathbb{E}[T_2 x^T Mxx^T N x] + \mathbb{E}[T_3 x^T Mxx^T N x].$$

(A.1)
First, we have
\[
\mathbb{E}[T_1 x^T M x x^T N x] = 4 \mathbb{E}[(x^T L^T P \Sigma P L P^T x)^2 x^T M x x^T N x]
\]
\[
= 4 \mathbb{E}[x^T L^T P \Sigma P L x x^T M x x^T N x]
\]
\[
= 4 \mathbb{E}[x^T L^T P \Sigma P L x x^T M x x^T N x]
\]
\[
= 4 \mathbb{E}_{g}[g^T (P_{1/2}^T L^T P \Sigma P L P_{1/2}^T) g g^T (P_{1/2}^T M P_{1/2}^T) g]
\]

Now we state a result from Magnus to compute the expectation of the product of three quadratic forms of Gaussians.

**Lemma A.2** (See e.g. Magnus [24]). Let \( g \sim \mathcal{N}(0, I) \) and \( A_1, A_2, A_3 \) be symmetric matrices. Then,
\[
\mathbb{E}[g^T A_1 g g^T A_2 g g^T A_3 g] = \text{Tr}(A_1) \text{Tr}(A_2) \text{Tr}(A_3)
\]
\[
+ 2(\text{Tr}(A_1) \text{Tr}(A_2 A_3) + \text{Tr}(A_2) \text{Tr}(A_1 A_3) + \text{Tr}(A_3) \text{Tr}(A_1 A_2))
\]
\[
+ 8 \text{Tr}(A_1 A_2 A_3).
\]

Now by setting
\[
A_1 = P_{1/2}^T L^T P \Sigma P L P_{1/2}^T,
\]
\[
A_2 = P_{1/2}^T M P_{1/2}^T,
\]
\[
A_3 = P_{1/2}^T N P_{1/2}^T,
\]
we can compute the expectation \( \mathbb{E}[T_1 x^T M x x^T N x] \) using Lemma A.2. In particular,
\[
\text{Tr}(A_1) \text{Tr}(A_2) \text{Tr}(A_3) = \langle P_{\infty}, L^T P \Sigma P L \rangle m^T \text{svec}(P_{\infty}) \text{svec}(P_{\infty})^T n,
\]
\[
\text{Tr}(A_1) \text{Tr}(A_2 A_3) = \langle P_{\infty}, L^T P \Sigma P L \rangle m^T \text{svec}(P_{\infty}) \text{svec}(P_{\infty}) n,
\]
\[
\text{Tr}(A_2) \text{Tr}(A_1 A_3) = m^T \text{svec}(P_{\infty}) \text{svec}(P_{\infty} L^T P \Sigma P L P_{\infty})^T n,
\]
\[
\text{Tr}(A_3) \text{Tr}(A_1 A_2) = m^T \text{svec}(P_{\infty} L^T P \Sigma P L P_{\infty}) \text{svec}(P_{\infty})^T n,
\]
\[
\text{Tr}(A_1 A_2 A_3) = m^T (P_{\infty} L^T P \Sigma P L P_{\infty} \otimes_s P_{\infty}) n.
\]

Hence,
\[
\mathbb{E}[g^T A_1 g g^T A_2 g g^T A_3 g] = m^T \langle P_{\infty}, L^T P \Sigma P L \rangle (2 \text{svec}(P_{\infty}) \text{svec}(P_{\infty})^T)
\]
\[
+ 2 \text{svec}(P_{\infty}) \text{svec}(P_{\infty} L^T P \Sigma P L P_{\infty})^T + 2 \text{svec}(P_{\infty} L^T P \Sigma P L P_{\infty}) \text{svec}(P_{\infty})^T
\]
\[
+ 8 \text{svec}(P_{\infty} L^T P \Sigma P L P_{\infty} \otimes_s P_{\infty}) n
\]

Next, we compute
\[
\mathbb{E}[T_2 x^T M x x^T N x] = \mathbb{E}[(w w^T - \Sigma, P_x)^2 x^T M x x^T N x]
\]
\[
= \mathbb{E}[(w w^T - \Sigma, P_x)^2] \mathbb{E}[x^T M x x^T N x].
\]

First, we have
\[
\mathbb{E}[(w w^T - \Sigma, P_x)^2] = \mathbb{E}[(w^T P_x w)^2] - 2 \langle \Sigma, P_x \rangle \mathbb{E}[w^T P_x w] + \langle \Sigma, P_x \rangle^2
\]
\[
= 2 \| P_x \Sigma P_x \|_F^2 + 2 \langle \Sigma, P_x \rangle^2 - 2 \langle \Sigma, P_x \rangle^2 + \langle \Sigma, P_x \rangle^2
\]
\[
= 2 \| P_x \Sigma P_x \|_F^2.
\]
On the other hand,
\[ \mathbb{E}[x^TMxx^TNx] = 2\langle P_\infty^{1/2}MP_\infty^{1/2}, P_\infty^{1/2}NP_\infty^{1/2}\rangle + \langle M, P_\infty\rangle \langle N, P_\infty\rangle. \]

Combining these calculations,
\[
\mathbb{E}[T_2x^TMxx^TNx] = 2\|\Sigma_1^{1/2}P_1\Sigma_1^{1/2}\|_F^2(2\langle P_\infty^{1/2}MP_\infty^{1/2}, P_\infty^{1/2}NP_\infty^{1/2}\rangle + \langle M, P_\infty\rangle \langle N, P_\infty\rangle)
\]
\[
= 2\|\Sigma_1^{1/2}P_1\Sigma_1^{1/2}\|_F^2m^T(2(P_\infty \otimes s P_\infty) + \text{svec}(P_\infty)\text{svec}(P_\infty)^T)n
\]

Finally, we have \( \mathbb{E}[T_3x^TMxx^TNx] = 0 \), which is easy to see because it involves odd powers of \( w \). This gives us that \( B_\infty \) is:
\[
B_\infty = (\langle P_\infty, L^TP_1\Sigma P_1L \rangle + 2\|\Sigma_1^{1/2}P_1\Sigma_1^{1/2}\|_F^2)(2(P_\infty \otimes s P_\infty) + \text{svec}(P_\infty)\text{svec}(P_\infty)^T)
\]
\[
+ 2\text{svec}(P_\infty)\text{svec}(P_\infty)L^TP_1\Sigma P_1LP_\infty)\text{svec}(P_\infty)^T + 2\text{svec}(P_\infty)L^TP_1\Sigma P_1LP_\infty)\text{svec}(P_\infty)^T + 8(P_\infty L^TP_1\Sigma P_1LP_\infty \otimes s P_\infty).
\]

This completes the proof of the formulas for \( A_\infty \) and \( B_\infty \).

To obtain the lower bound, we need the following lemma which gives a useful lower bound to Lemma A.2.

**Lemma A.3.** Let \( A_1 \) be positive semi-definite and let \( A_2 \) be symmetric. Let \( g \sim \mathcal{N}(0, I) \). We have that:
\[
\mathbb{E}[g^T A_1 g (g^T A_2 g)] \geq 2 \text{Tr}(A_1) \text{Tr}(A_2^2) + 4 \text{Tr}(A_1 A_2^2).
\]

**Proof.** Suppose that \( A_1 \neq 0 \), otherwise the bound holds vacuously. From Lemma A.2,
\[
\mathbb{E}[g^T A_1 g (g^T A_2 g)] = \text{Tr}(A_1) \text{Tr}(A_2^2) + 2 \text{Tr}(A_1) \text{Tr}(A_2^2) + 4 \text{Tr}(A_2) \text{Tr}(A_1 A_2) + 8 \text{Tr}(A_1 A_2^2).
\]

Since \( A_1 \) is PSD and non-zero, this means that \( \text{Tr}(A_1) > 0 \). We proceed as follows:
\[
4|\text{Tr}(A_2)\text{Tr}(A_1 A_2)| = 2|\text{Tr}(A_2)\text{Tr}(A_1)^{1/2}|2\frac{\text{Tr}(A_1 A_2)}{\text{Tr}(A_1)^{1/2}}
\]
\[
\leq \text{Tr}(A_1)\text{Tr}(A_2)^2 + 4\frac{\text{Tr}(A_1 A_2)^2}{\text{Tr}(A_1)}
\]
\[
= \text{Tr}(A_1)\text{Tr}(A_2)^2 + 4\frac{\text{Tr}(A_1^{1/2} A_2^{1/2} A_2^2)}{\text{Tr}(A_1)}
\]
\[
\leq \text{Tr}(A_1)\text{Tr}(A_2)^2 + 4\frac{\|A_1^{1/2}\|_F^2 \|A_1^{1/2} A_2\|_F^2}{\text{Tr}(A_1)}
\]
\[
= \text{Tr}(A_1)\text{Tr}(A_2)^2 + 4\text{Tr}(A_1 A_2^2),
\]

where in (a) we used Young’s inequality and in (b) we used Cauchy-Schwarz. The claim now follows.

We now start from the decomposition (A.1) for \( B_\infty \), with \( m = n \) and noting that \( \mathbb{E}[T_2(x^TMx)^2] \geq 0 \) and \( \mathbb{E}[T_3(x^TMx)^3] = 0 \):
\[
m^TB_\infty m \geq \mathbb{E}[T_1(x^TMx)^2]
\]
\[
\geq 8\langle P_\infty, L^TP_1\Sigma P_1L \rangle m^T(P_\infty \otimes s P_\infty)m + 16m^T(P_\infty L^TP_1\Sigma P_1LP_\infty \otimes s P_\infty)m.
\]
Above in (a) we applied the lower bound from Lemma A.3. Hence since $m$ is arbitrary,

$$B_{\infty} \geq 8(P_{\infty}, L^T P_s \Sigma P_s L)(P_{\infty} \otimes_s P_{\infty}) + 16(P_{\infty} L^T P_s \Sigma P_s L P_{\infty} \otimes_s P_{\infty}).$$

We also have that $A_{\infty} = (P_{\infty} \otimes_s P_{\infty})(I - L^T \otimes L^T)$, and hence $A_{\infty}^{-1} = (I - L^T \otimes L^T)^{-1}(P_{\infty}^{-1} \otimes_s P_{\infty}^{-1})$. Therefore,

$$A_{\infty}^{-1} B_{\infty} A_{\infty}^{-T} \geq 8(P_{\infty}, L^T P_s \Sigma P_s L)(I - L^T \otimes_s L^T)^{-1}(P_{\infty}^{-1} \otimes_s P_{\infty}^{-1})(I - L^T \otimes_s L^T)^{-T} + 16(I - L^T \otimes_s L^T)^{-1}(L^T P_s \Sigma P_s L \otimes_s P_{\infty}^{-1})(I - L^T \otimes_s L^T)^{-T}.$$

\[ \square \]

### A.5 Proof of Lemma 4.6

**Proof.** Recall in the notation of the proof of Lemma 4.1,

$$\tilde{L}(T) - L^* = -\lambda L^*(X^T X + \lambda I_n)^{-1} + W^T X(X^T X + \lambda I_n)^{-1}.$$

Now let us suppose that we are on an event where $X^T X$ is invertible. Let $X = U\Sigma V^T$ denote the compact SVD of $X$. We have:

$$\|\tilde{L}(T) - L^*\| \leq \lambda \frac{\|L^*\|}{\lambda_{\min}(X^T X + \lambda I_n)} + \|W^T X(X^T X + \lambda I_n)^{-1}\|$$

$$(a) \leq \lambda \frac{\|L^*\|}{\lambda_{\min}(X^T X + \lambda I_n)} + \|W^T X(X^T X)^{-1}\|.$$

The inequality (a) holds due to the following. First observe that $(X^T X + \lambda I_n)^{-2} \leq (X^T X)^{-2}$. Therefore with $M = W^T X$, conjugating both sides by $M$, we have $M(X^T X + \lambda I_n)^{-2} M^T \leq M(X^T X)^{-2} M^T$. Hence,

$$\|M(X^T X + \lambda I_n)^{-1}\| = \sqrt{\lambda_{\max}(M(X^T X + \lambda I_n)^{-2} M^T)}$$

$$\leq \sqrt{\lambda_{\max}(M(X^T X)^{-2} M^T)}$$

$$= \|M(X^T X)^{-1}\|.$$

By Theorem 2.4 of Simchowitz et al. [39] for $T \geq C_{L^*, n} \log(1/\delta)$, there exists an event $\mathcal{E}$ with $\mathbb{P}(\mathcal{E}) \geq 1 - \delta$ such that on $\mathcal{E}$ we have:

$$\|\tilde{L}_{\text{obs}}(T) - L^*\| \leq C_{L^*, n} \sqrt{\log(1/\delta)/T}, \quad X^T X \succeq C''_{L^*, n} T \cdot I_n.$$

Hence on this event we have $\|\tilde{L}(T) - L^*\| \leq C''_{L^*, n, \lambda} \sqrt{\log(1/\delta)/T}$.

For the remainder of the proof, $O(\cdot)$ will hide constants that depend on $L^*$, $n, p, \lambda$ but not on $T$ or $\delta$. We bound the $p$-th moment as follows. We decompose:

$$\mathbb{E}[\|\tilde{L}(T) - L^*\|^p] = \mathbb{E}[\|\tilde{L}(T) - L^*\|^p 1_{\mathcal{E}}] + \mathbb{E}[\|\tilde{L}(T) - L^*\|^p 1_{\overline{\mathcal{E}}}].$$

On $\mathcal{E}$ we have by the inequality $(a + b)^p \leq 2^{p-1}(a^p + b^p)$ for non-negative $a, b$,

$$\|\tilde{L}(T) - L^*\|^p \leq 2^{p-1}(O(X^T /T^p) + O((\log(1/\delta)/T)^{p/2})).$$

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On the other hand, we always have:

$$
\| \hat{L}(T) - L_* \|^p \leq 2^{p-1}(\| L_* \|^p + (\| W^T X \| / \lambda^p))^p.
$$

Hence:

$$
\mathbb{E}[\| \hat{L}(T) - L_* \|^p 1_{\mathcal{E}^c}] \leq 2^{p-1}\| L_* \|^p \mathbb{P}(\mathcal{E}^c) + \frac{2^{p-1}}{\lambda^p} \mathbb{E}[\| W^T X \|^p 1_{\mathcal{E}^c}]
\leq 2^{p-1}\| L_* \|^p \delta + \frac{2^{p-1}}{\lambda^p} \sqrt{\mathbb{E}[\| W^T X \|^2] \delta}.
$$

We will now compute a very crude bound on $\mathbb{E}[\| W^T X \|^2]$, which will suffice. For non-negative $a_t$, we have $(a_1 + \ldots + a_T)^2 \leq T^{2p-1}(\sum_{t=1}^T a_t^{2})$ by Hölder’s inequality. Hence

$$
\mathbb{E}[\| W^T X \|^2] = \mathbb{E}
\left[ \left\| \sum_{t=0}^{T-1} w_t x_t^T \right\|^{2p} \right]
\leq T^{2p-1} \mathbb{E}
\left[ \sum_{t=1}^{T} \| w_t \|^{2p} \| x_t \|^{2p} \right]
= T^{2p-1} \mathbb{E}[\| w_1 \|^{2p}] \sum_{t=1}^{T} \mathbb{E}[\| x_t \|^{2p}]
\leq T^{2p} \mathbb{E}[\| w_1 \|^{2p}] P_{\infty}^p \mathbb{E}[g \sim \mathcal{N}(0, I)][\| g \|^{2p}]
= O(T^{2p}).
$$

Above, $P_{\infty}$ denotes the covariance of the stationary distribution of $\{ x_t \}$. Continuing from above:

$$
\mathbb{E}[\| \hat{L}(T) - L_* \|^p 1_{\mathcal{E}^c}] = 2^{p-1}\| L_* \|^p \delta + \frac{2^{p-1}}{\lambda^p} \sqrt{O(T^{2p})} \delta.
$$

We now set $\delta = O(1/T^{3p})$ so that the term above is $O(1/T^{p/2})$. Doing this we obtain that for $T$ sufficiently large (as a function of only $L_*, p, \lambda$),

$$
\mathbb{E}[\| \hat{L}(T) - L_* \|^p] \leq O(1/T^{p/2})
$$

\[\square\]

## B Proof of Lemma 4.8

We now state a high probability bound for SGD. This is a straightforward modification of Lemma 6 from Rakhlin et al. [34] (modifications are needed to deal with the lack of almost surely bounded gradients), and hence we omit the proof.

**Lemma B.1** (Lemma 6, Rakhlin et al. [34]). *Let the assumptions of Lemma 4.8 hold. Define two constants:

$$
M := \sup_{\theta \in \Theta} \| \theta \|_2, \quad G_3 := \sup_{\theta \in \Theta} \| \nabla F(\theta) \|_2.
$$

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Note that since $\Theta$ is compact, both $M$ and $G_3$ are finite. Fix a $T \geq 4$ and $\delta \in (0, 1/e)$. We have that with probability at least $1 - \delta$, for all $t \leq T$, 

$$
\|\theta_t - \theta_*\|_2^2 \lesssim \frac{\text{polylog}(T/\delta)}{t} \left( \frac{G_1^2 + G_2^2}{m^2} \frac{M(G_2 + G_3)}{m} \right).
$$

We are now in a position to analyze the asymptotic variance of SGD with projection. As mentioned previously, our argument follows closely that of Toulis and Airoldi [43]. For the remainder of the proof, $O(\cdot)$ and $\Omega(\cdot)$ will hide all constants except those depending on $t$ and $\delta$. Introduce the notation:

$$
\tilde{\theta}_{t+1} = \theta_t - \alpha_t g_t(\theta_t; \xi_t),
\quad \theta_{t+1} = \text{Proj}_\Theta(\tilde{\theta}_{t+1}).
$$

Let $E_t := \{ \tilde{\theta}_t = \theta_t \}$ be the event that the projection step is inactive at time $t$. Recall that we assumed that $\theta_*$ is in the interior of $\Theta$. This means there exists a radius $R > 0$ such that $\{ \theta : \|\theta - \theta_*\|_2 \leq R \} \subseteq \Theta$. Therefore, the event $\{ \|\tilde{\theta}_t - \theta_*\|_2 \leq R \} \subseteq E_t$. We now decompose,

$$
\text{Var}(\theta_{t+1}) = \text{Var}(\theta_{t+1} - \tilde{\theta}_{t+1} + \tilde{\theta}_{t+1})
= \text{Var}(\tilde{\theta}_{t+1}) + \text{Var}(\theta_{t+1} - \tilde{\theta}_{t+1}) + \text{Cov}(\theta_{t+1} - \tilde{\theta}_{t+1}, \tilde{\theta}_{t+1}) + \text{Cov}(\tilde{\theta}_{t+1}, \theta_{t+1} - \tilde{\theta}_{t+1}).
$$

We have that,

$$
\theta_{t+1} - \tilde{\theta}_{t+1} = (\theta_{t+1} - \tilde{\theta}_{t+1})1_{E_{t+1}^c}.
$$

Hence,

$$
\|\text{Var}(\theta_{t+1} - \tilde{\theta}_{t+1})\| \leq \mathbb{E}[\|\tilde{\theta}_{t+1}1_{E_{t+1}^c} - \theta_{t+1}1_{E_{t+1}^c}\|_2^2]
\leq 2(\mathbb{E}[\|\tilde{\theta}_{t+1}\|_2^21_{E_{t+1}^c}] + \mathbb{E}[\|\theta_{t+1}\|_2^21_{E_{t+1}^c}])
\leq 2\left(\sqrt{\mathbb{E}[\|\tilde{\theta}_{t+1}\|_2^2]}\mathbb{E}[1_{E_{t+1}^c}] + M^2\mathbb{E}[1_{E_{t+1}^c}]\right).
$$

We can bound $\mathbb{E}[\|\tilde{\theta}_{t+1}\|_2^2]$ by a constant for all $t$ using our assumption (4.2). On the other hand,

$$
\mathbb{E}[1_{E_{t+1}^c}] \leq \mathbb{P}(\|\tilde{\theta}_{t+1} - \theta_*\|_2 > R).
$$

By triangle inequality,

$$
\|\tilde{\theta}_{t+1} - \theta_*\|_2 \leq \|\tilde{\theta}_t - \theta_*\|_2 + \alpha_t\|g_t\|_2.
$$

By Lemma B.1 and the concentration bound on $\|g_t\|_2$ from our assumption (4.3), with probability at least $1 - \delta$,

$$
\|\tilde{\theta}_{t+1} - \theta_*\|_2 \leq O(\text{polylog}(t/\delta)/\sqrt{t}).
$$

Hence for $t$ large enough, $\mathbb{E}[1_{E_{t+1}^c}] \leq O(\exp(-t^\alpha))$ for some $\alpha > 0$. This shows that $\|\text{Var}(\theta_{t+1} - \tilde{\theta}_{t+1})\| \leq O(\exp(-t^\alpha))$. Similar arguments show that $\max \{ \|\text{Cov}(\theta_{t+1} - \tilde{\theta}_{t+1}, \tilde{\theta}_{t+1})\|, \|\text{Cov}(\tilde{\theta}_{t+1}, \theta_{t+1} - \tilde{\theta}_{t+1})\| \} \leq O(\exp(-t^\alpha))$. Hence:

$$
\text{Var}(\theta_{t+1}) = \text{Var}(\tilde{\theta}_{t+1}) + O(\exp(-t^\alpha)).
$$
Therefore,

\[ \text{Var}(\theta_{t+1}) = \text{Var}(\tilde{\theta}_{t+1}) + O(\exp(-t^\alpha)) \]
\[ = \text{Var}(\theta_t - \alpha_t g(\theta_t; \xi_t)) + O(\exp(-t^\alpha)) \]
\[ = \text{Var}(\theta_t) + \alpha_t^2 \text{Var}(g(\theta_t; \xi_t)) - \alpha_t \text{Cov}(g(\theta_t; \xi_t), \theta_t) + O(\exp(-t^\alpha)) \]
\[ = \text{Var}(\theta_t) + \alpha_t^2 \text{Var}(g(\theta_t; \xi_t)) - \alpha_t \text{Cov}(\theta_t, \nabla F(\theta_t)) - \alpha_t \text{Cov}(\nabla F(\theta_t), \theta_t) + O(\exp(-t^\alpha)). \]  

(B.1)

Now we write:

\[ \text{Var}(g(\theta_t; \xi_t)) = \text{Var}(g(\theta_t; \xi_t) + (g(\theta_t; \xi_t) - g(\theta_\ast; \xi_t))) \]
\[ = \text{Var}(g(\theta_\ast; \xi_t)) + \text{Var}(g(\theta_t; \xi_t) - g(\theta_\ast; \xi_t)) \]
\[ + \text{Cov}(g(\theta_\ast; \xi_t), g(\theta_t; \xi_t) - g(\theta_\ast; \xi_t)) + \text{Cov}(g(\theta_t; \xi_t) - g(\theta_\ast; \xi_t), g(\theta_\ast; \xi_t)) \]

We have by our assumption (4.4),

\[ \|\text{Var}(g(\theta_t; \xi_t) - g(\theta_\ast; \xi_t))\| \leq \mathbb{E}[\|g(\theta_t; \xi_t) - g(\theta_\ast; \xi_t)\|_2^2] \]
\[ = \mathbb{E}_t \mathbb{E}_\xi [\|g(\theta_t; \xi_t) - g(\theta_\ast; \xi_t)\|_2^2] \]
\[ \leq L \mathbb{E}[\|\theta_t - \theta_\ast\|_2^2]. \]

On the other hand,

\[ \|\text{Cov}(g(\theta_\ast; \xi_t), g(\theta_t; \xi_t) - g(\theta_\ast; \xi_t))\| \leq 2\mathbb{E}[\|g(\theta_\ast; \xi_t)\|_2 \|g(\theta_t; \xi_t) - g(\theta_\ast; \xi_t)\|_2] \]
\[ \leq 2\sqrt{\mathbb{E}[\|g(\theta_\ast; \xi_t)\|_2^2] \mathbb{E}[\|g(\theta_t; \xi_t) - g(\theta_\ast; \xi_t)\|_2^2]} \]
\[ \leq 2\sqrt{LG^2 \mathbb{E}[\|\theta_t - \theta_\ast\|_2^2]}. \]

The same bound also holds for \( \|\text{Cov}(g(\theta_t; \xi_t) - g(\theta_\ast; \xi_t), g(\theta_\ast; \xi_t))\| \). Since we know that \( \mathbb{E}[\|\theta_t - \theta_\ast\|_2^2] \leq O(1/t) \), this shows that:

\[ \text{Var}(g(\theta_t; \xi_t)) = \text{Var}(g(\theta_\ast; \xi_t)) + o_t(1). \]

Next, by a Taylor expansion of \( \nabla F(\theta_t) \) around \( \theta_\ast \), we have that:

\[ \nabla F(\theta_t) = \nabla^2 F(\theta_\ast)(\theta_t - \theta_\ast) + \text{Rem}(\theta_t - \theta_\ast), \]

where \( \|\text{Rem}(\theta_t - \theta_\ast)\| \leq O(\|\theta_t - \theta_\ast\|_2^3) \). Therefore, utilizing the fact that adding a non-random vector does not change the covariance,

\[ \text{Cov}(\theta_t, \nabla F(\theta_t)) = \text{Cov}(\theta_t, \nabla^2 F(\theta_\ast)(\theta_t - \theta_\ast) + \text{Rem}(\theta_t - \theta_\ast)) \]
\[ = \text{Cov}(\theta_t, \nabla^2 F(\theta_\ast)(\theta_t - \theta_\ast)) + \text{Cov}(\theta_t, \text{Rem}(\theta_t - \theta_\ast)) \]
\[ = \text{Cov}(\theta_t, \nabla^2 F(\theta_\ast)\theta_t) + \text{Cov}(\theta_t - \theta_\ast, \text{Rem}(\theta_t - \theta_\ast)) \]
\[ = \text{Var}(\theta_t) \nabla^2 F(\theta_\ast) + \text{Cov}(\theta_t - \theta_\ast, \text{Rem}(\theta_t - \theta_\ast)). \]

We now bound \( \text{Cov}(\theta_t - \theta_\ast, \text{Rem}(\theta_t - \theta_\ast)) \) as:

\[ \|\text{Cov}(\theta_t - \theta_\ast, \text{Rem}(\theta_t - \theta_\ast))\| \leq O(\mathbb{E}[\|\theta_t - \theta_\ast\|_2^3]) \leq O(\text{polylog}(t)/t^{3/2}). \]
We now make two observations. Recall that \( \alpha \)

This matrix recursion can be solved by Corollary C.1 of Toulis and Airoldi [43],

Therefore,

\[
\text{Var}(\theta_{t+1}) = \text{Var}(\theta_t) + \alpha_t^2 (\text{Var}(g(\theta_*; \xi)) + o_t(1)) - \alpha_t (\text{Var}(\theta_t) \nabla^2 F(\theta_*) + \nabla^2 F(\theta_*) \text{Var}(\theta_t))
\]

\[+ \alpha_t O(\text{polylog}(t)/t^{3/2}) + O(\exp(-t^\alpha)) . \]

We now make two observations. Recall that \( \alpha_t = 1/(mt) \). Hence we have

\[O(\exp(-t^\alpha)) = \alpha_t^2 O(t^2 \exp(-t^\alpha)) = \alpha_t^2 o_t(1). \]

Similarly, \( \alpha_t O(\text{polylog}(t)/t^{3/2}) = \alpha_t^2 O(\text{polylog}(t)/t^{1/2}) = \alpha_t^2 o_t(1). \)

Therefore,

\[
\text{Var}(\theta_{t+1}) = \text{Var}(\theta_t) - \alpha_t (\text{Var}(\theta_t) \nabla^2 F(\theta_*) + \nabla^2 F(\theta_*) \text{Var}(\theta_t)) + \alpha_t^2 (\text{Var}(g(\theta_*; \xi)) + o_t(1)).
\]

This matrix recursion can be solved by Corollary C.1 of Toulis and Airoldi [43], yielding (4.5).

To complete the proof, by a Taylor expansion we have:

\[
T \cdot \mathbb{E}[F(\theta_T) - F(\theta_*/t)] = \frac{T}{2} \text{Tr}(\nabla^2 F(\theta_*) \mathbb{E}[(\theta_T - \theta_*) (\theta_T - \theta_*)^T]) + \frac{T}{6} \mathbb{E}[\nabla^3 f(\hat{\theta})(\theta_T - \theta_*)^3].
\]

As above, we can bound \( |\mathbb{E}[\nabla^3 f(\hat{\theta})(\Theta_T - \theta_*)^3]| \leq O(\mathbb{E}[\|\Theta_T - \theta_*\|^3]) \leq O(\text{polylog}(T)/T^{3/2}) \), and hence \( T \cdot |\mathbb{E}[\nabla^3 f(\hat{\theta})(\Theta_T - \theta_*)^3]| \rightarrow 0 \). On the other hand, letting \( \mu_T := \mathbb{E}[\theta_T] \), by a bias-variance decomposition,

\[
\mathbb{E}[(\theta_T - \theta_*) (\theta_T - \theta_*)^T] = \mathbb{E}[(\theta_T - \mu_T)(\theta_T - \mu_T)^T] + (\mu_T - \theta_*)(\mu_T - \theta_*)^T
\]

\[\geq \mathbb{E}[(\theta_T - \mu_T)(\theta_T - \mu_T)^T] = \text{Var}(\theta_T). \]

Therefore,

\[
T \cdot \mathbb{E}[F(\theta_T) - F(\theta_*)] \geq \frac{1}{2m} \text{Tr}(\nabla^2 F(\theta_*)(mT) \text{Var}(\theta_T)) - \frac{T}{6} |\mathbb{E}[\nabla^3 f(\hat{\theta})(\theta_T - \theta_*)^3]|.
\]

Taking limits on both sides yields (4.7). This concludes the proof of Lemma 4.8.