Low \( Rm \) MHD turbulence: the role of boundaries

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Abstract

In this short review, we present the main known features of MHD turbulence at Low Magnetic Reynolds number, for which the flow isn’t intense nor electrically conductive enough to disturb an externally applied magnetic field. The emphasis is deliberately placed on the very specific physical mechanisms of these flows, rather than their numerical modelling. We also focus on homogeneous magnetic fields which have received most attention. Since the basic properties of these flows have been thoroughly reviewed a number of times, this review is deliberately biased towards flows in bounded domains, in which the tendency to two-dimensionality observed in MHD flows casts the boundaries of the domain into a leading role.

1 Introduction

Magnetic fields are routinely used to attempt to control flows of electrically conducting fluids in metallurgical processes. The intensity of these flows however places them most often in a turbulent state, with direct impact on their dissipative, mixing and transport properties. These properties being key to these processes, there is a wide need to understand MHD turbulence in an externally applied magnetic field. In this class of flows, the magnetic Reynolds number is low and the flow cannot disturb an externally imposed magnetic field \( B_\text{e} \). The main effect of the Lorentz force that results from the induced electric currents is then to diffuse momentum along the magnetic field lines \([1]\). Flow structures tend to become elongated along this direction and even invariant if this effect isn’t effectively opposed by inertia or viscosity. In turbulent flows, 3D inertia precisely resists this phenomenon as it breaks up larger flows structures into smaller ones and promotes a return to isotropy. In an homogeneous magnetic field, turbulence is precisely determined by the antagonism between these two forces, and by the interaction parameter \( N = \sigma B^2 L/\rho U \) which represents their ratio (\( \sigma \) and \( \rho \) are the fluid’s electrical conductivity and density, and \( U \) and \( L \) are macroscopic...
velocity and length scales). Whether diffusion along magnetic field lines is effective enough to stretch all structures all the way between the boundaries of the domain or not decides whether the flow is 3D or in any form of 2D state.

The systematic experimental investigation of MHD turbulence in liquid metals started in the 1960’s, with the works of [2, 3, 4, 5]. Driven by applications, these pioneering experiments analysed turbulence in ducts and pipes, rather than "homogeneous" turbulence and were reviewed in detail by Lykoudis [6]. In this review, we focus on the physical properties of turbulence with no mean flow, as first investigated experimentally by [7]. We shall underline the influence of the walls not only on the 2D states of MHD turbulence but also on the crucial intermediate states that exist between two-dimensionality and full three-dimensionality. We shall first explore the consequences of the antagonism between inertia and the Lorentz force on turbulence far from walls (section 2). Section 3 will be dedicated to the mechanisms of the transition between 3D and strictly 2D turbulence. In section 4, we shall review the possible states of quasi-2D turbulence in the presence of Hartmann walls, while section 5 will underline some of the author’s recent results on the appearance of three-dimensionality in quasi-2D MHD flows.

2 MHD turbulence far from walls

The most generic mechanisms of MHD turbulence are best singled out in flow regions far from boundaries where the Lorentz force is at least strong enough to balance inertia, \( i.e. N \simeq 1 \) or \( N \gg 1 \). Despite the presence of a strong magnetic field, it is usually acknowledged that the three ranges of 3D homogeneous non-MHD turbulence still exist: the flow is usually assumed forced at some large scales, whose non-universal behaviour is dictated by the particular forcing and the boundary conditions. Energy is cascaded down along the inertial range, which by contrast with the large scales is believed to exhibit a somewhat universal behaviour. The inertial range stops at the "small scales" where viscous friction becomes dominant. Unlike in non-MHD turbulence at high Reynolds number, however, the Joule dissipation incurred by the Lorentz force extracts energy at all scales so that not all the energy pumped out of the large scales survives along the inertial range, which therefore exhibits a steeper energy spectrum than the \( E(k) \sim k^{-5/3} \) law of homogeneous hydrodynamic turbulence (\( k \) stands for the usual wavenumber used to identify vortex size) [8]. The dynamics of the inertial range of MHD turbulence are usually described using two assumptions [7]: 1) At each scale, inertial forces balance Lorentz forces and 2) Anisotropy remains the same at all scales, over the inertial range. These lead to the scalings for the power
spectral density, and for a “geometric anisotropy”:

\[ E(k⊥) \sim U_0^2 k_{⊥}^{-3} \]  
\[ \frac{k_z}{k_{⊥}} \sim \frac{Re^{1/2}}{Ha} = N^{-1/2} \]

\( Re \) is a Reynolds number based on the large scale velocity \( U \) and length \( L \), the ratio \( \frac{Ha^2}{Re} = N \) is the corresponding interaction parameter, and the square of the Hartmann number \( Ha = LB(\sigma/\rho \nu)^{1/2} \) represents the large scale ratio of the Lorentz to viscous forces. The subscript \( ⊥ \) stands for components of vectors orthogonal to \( B \), assumed aligned with \( e_z \). Unlike the Lorentz force, viscous friction is only effective at very small scale. When active, it stops the energy cascade so that the smallest scales are heuristically defined as the smallest possible structures of the inertial range which are not destroyed by viscosity. This leads to

\[ k_{⊥,\text{max}} \sim Re^{\frac{1}{2}}, \quad k_{z,\text{max}} \sim \frac{Re}{Ha}. \]  

The corresponding number of degrees of freedom of the flow scales as \( N_f \sim k_{⊥,\text{max}}^2 k_{z,\text{max}} \sim \frac{Re^2}{Ha} \). These scalings reflect that Joule dissipation strongly reduces the number of degree of freedom, since \( N_f \sim Re^{3/4} \) in non-MHD turbulence. In other words, the lesser amount of energy that survives the journey down the inertial range is dissipated by viscous friction at much larger “small scales” than in non-MHD turbulence.

The validity of these simple scalings has been tested experimentally in several occasions, in particular the \( k^{-3} \) law. \cite{9} and \cite{7} measured turbulent spectra in a turbulent flow of liquid metal, either subjected to an homogeneous magnetic field or to no field. They were able to convincingly recover both the \( E(k) \sim k^{-5/3} \) law in non-MHD case, and the \( E(k) \sim k^{-3} \) law with an applied magnetic field corresponding to an interaction parameter of at least a few units. In both cases, the experiments were designed in such a way as to eliminate any significant influence from the walls. More recently, \cite{10} measured turbulent spectra in a rectangular channel, to find that the spectral exponent in the inertial range depended non-monotonously on the value of \( N \), with values in the range \([-5, -5/3]\) (see figure\cite{1}). Two main explanations were put forward: firstly, it was deemed possible that for \( N > 1 \), the larger structures, at least, could extend across the whole channel. Their behaviour would then be governed by 2D turbulence leaving the flow in a partly 2D, partly 3D state. For exponents lower than \(-3\), \cite{11, 12}’s theory was invoked, as it suggests that the presence of helicity in the flow can lead to such steep spectra. Although no definite evidence for this explanation could be put forward, it does find support in the fact that the presence of large 2D structures in the vicinity of walls can indeed generate helicity by
Variations with $N$ of the exponent in the scaling law for the inertial range of the energy spectrum found by [10] in the turbulent flow in the wake of a fixed grid in a rectangular channel.

Ekman pumping (see section 4).

On the more theoretical side, attempts were made to justify these heuristic scalings either numerically or by analysing the dynamical system associated to a generic turbulent flow. Rigorous estimates of the dimension of its attractor were derived by [13, 14] that indeed confirmed the exponent of $Ha$ in these scalings, without the need for any additional assumption than those the Navier Stokes equations rely on.

Since the 80’s, the properties of Low-Rm MHD turbulence have been to a large part analysed numerically, in periodic domains, with various large scale forcings. These aspects are reviewed in detail in [15]. Among these studies, [16] raised the question of the validity of the assumption that anisotropy is constant across the inertial range. They stressed that anisotropy could be defined in a number of ways, and were able to show that for $N > 1$, both the anisotropy of the velocity gradients and the kinematic anisotropy (defined as the ratio of energies along and across the field direction) were reasonably scale-independent in the inertial range, for several types of forcing.

The picture can be refined by inspecting how the energy is distributed in the $(k_\perp, k_z)$ spectral space. In this regard, it was early recognised that the selective nature of Joule dissipation severely damped modes within the ”Joule cone”, defined as the region of spectral space such that $k_z/k_\perp < N^{-1/2}$ [17]. Consequently, hardly any energy remains there unless it is maintained by external forcing. Experiments by [18] showed that because of their finite number, energy containing modes were in fact localised in a torus in $(k_\perp, k_z)$, whose pointy inward edge coincided with the Joule Cone. This was later confirmed by the numerical simulations of [19, 20], who showed that the radial section of the torus was shaped as a cardioid curve, and that spectral energy transfer essentially occurred through surfaces of the same family. [20]
found that the lines of constant energy tended to coincide with those of the decay rate due to combined viscous and Joule friction \( \lambda_k = k^2 + Ha^2 k_z / k^2 \). They noted that the anisotropy of low-\( Rm \) MHD turbulence was naturally rendered by the sequence of scalar decay rates \( \lambda_k \). This lifted the need for separate laws along and across the field \( \| \lambda \| \), which could be replaced by a single one involving the forcing scale \( k_f \):\[
\frac{\sqrt{\| \lambda \| \text{max}}} {2\pi k_f} \approx 0.5 \text{Re}^{1/2}.
\]

3 Transition to and from strict two-dimensionality

One of the most distinctive features of MHD turbulence is its tendency to become 2D, highlighted in introduction. But the question of just how close to two-dimensionality MHD turbulence can become cannot find an answer without specifying the boundary conditions of the problem, at least along those boundaries that intercept the magnetic field lines. In his seminal 1967 paper \[17\], Moffatt showed that in a freely decaying unbounded flow, the anisotropy of structures increased indefinitely, and that the flow tended towards a limit state where the ratio of kinetic energies along to across the magnetic field direction would be of 1/2. This spread a vision of 2D MHD turbulence as a limit state rather than as an achievable flow. Later, numerical simulations of freely decaying MHD turbulence in a spatially periodic domain by \[21\] indeed exhibited strictly 2D structures when the initial interaction parameter was above 50. Here, strict two-dimensionality was only made possible by the finite length imposed by periodic boundary conditions chosen along the magnetic field lines. Similar observations were more recently made in forced MHD flows \[22\], still in 3D periodic domains.
The reverse transition that leads three-dimensionality to appear in an initially strictly 2D flow has received less attention. [22] mention the existence of intermittent regimes in forced flows. More recently, [23] and [24] argued that such intermittency could appear in regimes where diffusion of momentum along the field direction by the Lorentz force was sufficiently strong to create 2D structures but not sufficiently dominant to prevent the development of 3D instabilities that disrupted these structures. Indeed, without dissipation at the boundaries, the flow can become strictly 2D, a state in which the Lorentz force vanishes completely, leaving the growth of 3D perturbations unimpeded. These perturbations break down 2D structures to restore a 3D state in which the Lorentz force starts acting again. This dynamical instability can produce an intermittent behaviour in domains with periodic or no-slip boundaries, but this effect has never been observed in domains bounded by no-slip walls, where strong dissipation is always present in wall boundary layers.

When strict two-dimensionality is achieved, the electric current density and the Lorentz force vanish entirely so the flow strictly recovers the properties of non-MHD 2D turbulence: above the injection scale, energy is cascaded upwards up to large coherent structures whose dynamics is dictated by the forcing and the conditions along boundaries parallel to \( B \). The corresponding power density spectrum exhibit a \( k^{-5/3} \) slope. Below this scale, enstrophy is cascaded along a \( k^{-3} \) energy spectrum, down to the Kraichnan scales \( k_{\perp,\max} \sim Re^{1/2} \) (reviews of 2D turbulence can be found in [25] and [26]).

4 Quasi-2D MHD turbulence

Strictly and intermittently 2D flows have only ever been achieved in numerical simulations with either periodic or free-slip boundary conditions, but never in a laboratory where the influence of walls and wall friction can never be completely avoided. As such, no-slip walls are an intrinsic part of MHD turbulence and become especially important in flow regimes that approach the 2D state.

The most obvious feature of wall-bounded flows is the presence of Hartmann boundary layers along boundaries that intercept the magnetic field lines. In these layers, of thickness \( \sim Ha^{-1} \), viscous friction opposes the Lorentz force to maintain a velocity gradient along \( B \) [27]. Strict two-dimensionality is thus only possible outside of these layers and the corresponding flows are only quasi-2D, rather than strictly 2D. The influence of the walls in quasi-2D MHD was first analysed by [1], who showed that in the limit of large \( N \) and \( Ha \), inertia was negligible in the Hartmann layers and the flow in the core was not only dynamically 2D (i.e. \( u \cdot B = 0 \)), but also kinematically 2D (i.e. \( \partial_B u = 0 \)). The absence of inertia in the layers doesn’t prevent the development of 2D turbulence in the core, but the Hartmann layers exert
linear friction on it so that a flow confined within a channel orthogonal to the field is described by a shallow water equation of the (dimensional) form:

\[ \frac{d\mathbf{u}_\perp}{dt} + \nabla \cdot p = \nu \nabla^2 \mathbf{u}_\perp - \frac{\mathbf{u}_\perp}{t_H} + f, \]  

(5)

where \( f \) represent an externally applied force density. For a given fluid, the linear damping time \( t_H = \frac{H^2}{\nu H a} \) is controlled by the intensity of the magnetic field and the channel depth \( H \). In this same channel configuration, \([9]\) and \([28]\) experimentally showed that the dynamics of such flows was essentially that of 2D turbulence and the latter work presents an evidence of an inverse energy cascade (see figure 3). Unlike in 2D turbulence however, linear friction introduces an energy sink at scale \( k_{\text{min}}^\perp \sim \frac{Ha}{Re} \), as structures larger than the corresponding size cannot survive the action of friction. If this largest possible scale is smaller than the typical dimension of the domain, Hartmann friction prevents the condensation of energy in modes dictated by the boundary conditions and stops the possible energy pile-up in the large scales associated with this phenomenon \([25]\).

In the regimes of moderately high inertia where \( N \sim 1 \), rotation at the scale of individual vortices becomes strong enough to drive local poloidal recirculations, through a local *Ekman pumping* mechanism \([29]\), an effect that is well known in rotating flows \([30]\). These secondary flows were shown to alter the properties of quasi-2D turbulence, by introducing non-linear anisotropic diffusion along the flow streamlines, that tends to damp small scale fluctuations \([31, 32]\). Local Ekman pumping is also a source of helicity, which, according to \([11]\) induces steep turbulent spectra. This too, is an expression of the damping of small scales.

The more recent experiments of \([33]\) exhibited a further interesting regime of MHD turbulence in a channel where the core remained 2D, as in \([28]\)’s experiments, but where the Hartmann boundary layers were turbulent. This happened whenever \( Re/Ha > 380 \) and \( N \gg 1 \). Unlike the friction exerted by laminar Hartmann layers, that due to turbulent Hartmann layers varies non-linearly with the core velocity (see figure 3). It incurs a much higher global dissipation, alters the scaling of the large scales \([34]\), and most likely the rest of the spectrum, all the way down to the size of the small scales.

### 5 Appearance of three-dimensionality

Although dominant in the quasi-2D state of MHD turbulence, the role played by boundaries isn’t confined to this regime. \([1]\) argued that since the anisotropy of a given structure of size \((l_\perp, l_z)\) resulted from a competition between diffusion of momentum along the magnetic field and return to isotropy driven by inertia, then in a channel of width \( H \), a critical size \( l_\perp^{1D} \) existed above which structures were quasi-2D and below which they were...
Figure 3: Left: Kinetic energy spectra measured by [28] in a quasi-2D turbulent layer of mercury pervaded by a transverse magnetic field. The inverse energy cascade is identified through the $k^{-5/3}$ slope. Right: variation of total angular momentum against $R = Re/Ha$, for the quasi-2D annular flow electrically driven in the MATUR experiment (experiments from [33] referred to as "MATUR", numerical simulations from [34], based on [35]’s model for the turbulent Hartmann layer). Values are normalised by the value of the angular momentum predicted on the assumption of a laminar Hartmann layer: the change in slope reflects the intensification of friction when the layers become turbulent.

3D:

$$\frac{L_{2D}}{H} < \left( \frac{\rho U}{\sigma B^2 H} \right)^{1/2}. \tag{6}$$

This remarkable property was verified experimentally only recently when [36] forced MHD turbulence in a cubic, insulating container placed in an homogeneous magnetic field. By comparing the frequency spectra derived from the electric potential gradients measured near both Hartmann walls, at opposite locations, they found a cutoff frequency $f_c$ separating 2D from 3D fluctuations:

$$f_c \simeq 1.7 \tau_{u'}^{-1} N_t^{0.67}. \tag{7}$$

The true interaction parameter $N_t = N(H/L_f)^2$ was based on the scale at which the flow was forced $L_f$ and the turnover time $\tau_{u'}$ was associated to RMS of velocity fluctuations. This experiment also singled out further mechanisms at play when the Lorentz force wasn’t strong enough to achieve quasi-two dimensionality in forced, established flows: at high $Ha$, the flow was quasi-2D as in [28]’s experiment. For slightly lower values of $Ha$, a form of three-dimensionality, called weak, was observed where flows in planes orthogonal to the field were topologically identical but of intensity decreasing with the distance to the Hartmann wall where the forcing was applied. This observation recovers the theoretical and numerical predictions of [29, 37].
These authors proved indeed that 2D inertia induced electric eddy currents between Hartmann layers and the bulk, that caused differential rotation and led columnar vortices to assume a 3D barrel-like shape. Weak three-dimensionality is therefore a direct consequence of the presence of Hartmann walls.

At moderate values of $Ha$, partial vortex merging was observed where vortices generated near one Hartmann wall were elongated along the magnetic field and merged near the opposite Hartmann wall, leading to Y-shaped vortices. At moderately high $Ha$, vortex pairing was unsteady, but most remarkably, for the lowest values of $Ha$, this phenomenon could lead to a re-stabilisation of the flow with steady Y-shaped vortices. Although the flow clearly wasn’t turbulent in these regimes, these findings single out some of the mechanisms of the antagonism between momentum diffusion along the field lines and inertia, that give birth to remarkable flow structures directly relevant to dynamics of MHD turbulence around the 2D-3D transition.

6 Conclusion

To conclude this short review, understanding Low-$Rm$ MHD turbulence is still very much a task in progress, particularly in wall-bounded configurations or more realistic ones. If some of the basic mechanisms are now well understood, at least heuristically, hardly any exact result is available for this rather specific type of turbulence (such as as 4/5th law in homogeneous turbulence). The most distinctive feature of MHD turbulence is possibly its tendency to two-dimensionality, which, unlike in turbulent flows in rotation, incurs strong dissipation. In this regard, the conditions of the transition
between quasi-2D and 3D turbulence are still very poorly understood. Recent progress indicate that the nature of the boundaries play a lead role in it: while strict two-dimensionality is only possible in domains bounded by non-dissipative boundaries, the presence of no-slip walls implies that three-dimensionality appears progressively in the flow, rather than because of the instability of quasi-2D structures. Three-dimensional instabilities still probably occur, but develop in states where turbulence may already exhibit several possible forms of three-dimensionality. Transition to three-dimensionality along such a route is to this day unexplored, and is likely to differ significantly from that found in simulations without dissipative boundaries.
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