PROLATE SPHEROIDAL OPERATOR AND ZETA

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ABSTRACT. In this paper we describe a remarkable new property of the self-adjoint extension \( W_{sa} \) of the prolate spheroidal operator introduced in \[1,3\]. The restriction of this operator to the interval \( J \) whose characteristic function commutes with it is well known, has discrete positive spectrum and is well understood \[14,15,16,9\]. What we have discovered is that the restriction of \( W_{sa} \) to the complement of \( J \) admits (besides a replica of the above positive spectrum) negative eigenvalues whose ultraviolet behavior reproduce that of the squares of zeros of the Riemann zeta function. Furthermore, their corresponding eigenfunctions belong to the Sonin space. This feature fits with the proof \[4\] of Weil’s positivity at the archimedean place, which uses the compression of the scaling action to the Sonin space. As a byproduct we construct an isospectral family of Dirac operators whose spectra have the same ultraviolet behavior as the zeros of the Riemann zeta function.

1. Introduction

The prolate spheroidal wave functions play a key role in \[2,4,5\] in relation with the Riemann zeta function. In all these applications they appear as eigenfunctions of the angle operator between two orthogonal projections in the Hilbert space \( L^2(\mathbb{R})^{ev} \) of even square integrable function on \( \mathbb{R} \). These projections depend on a parameter \( \lambda > 0 \), the projection \( P_\lambda \) is given by the multiplication with the characteristic function of the interval \( [-\lambda, \lambda] \subset \mathbb{R} \). The projection \( \hat{P}_\lambda \) is its conjugate by the Fourier transform \( \mathcal{F}_{ev} \) which is the unitary operator in \( L^2(\mathbb{R})^{ev} \) defined by

\[
\mathcal{F}_{ev}(\xi)(y) = \int \xi(x) \exp(-2\pi i xy) dx.
\]

In all the above applications of prolate spheroidal wave functions the miraculous existence, discovered by the Bell Labs group \[14,15,16\], of a differential operator \( W_\lambda \) commuting with the angle operator, plays only an auxiliary role. In the present paper we uncover another “miracle”: a careful study of the natural self-adjoint extension of \( W_\lambda \) introduced in \[1, \text{Lemma 6}\] (see also \[9, \S 3.3\]) to \( L^2(\mathbb{R}) \) shows that it still has discrete spectrum and that its negative eigenvalues reproduce the ultraviolet behavior of the squares of zeros of the Riemann zeta function. In a similar way the positive spectrum corresponds, in the ultraviolet regime, to the trivial zeros. This coincidence holds for two values \( \lambda = 1 \) and \( \lambda = \sqrt{2} \). The conceptual reason for this coincidence is the link between the operator

\[
(W_\lambda \xi)(x) = -\partial_x (\lambda^2 - x^2) \partial_x \xi(x) + (2\pi \lambda)^2 x^2 \xi(x)
\]

and the square of the scaling operator \( S := x \partial_x \). In \[4\] the compression of \( f(S) \) to Sonin’s space (for \( \lambda = 1 \)) was shown to be the root of Weil’s positivity at the archimedean place on test functions with support in the interval \([2^{-1/2}, 2^{1/2}]\), but

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since Sonin’s space is not preserved by scaling, one could not restrict scaling to this space. It turns out that $W_\lambda$ commutes with the orthogonal projection on Sonin’s space. Thus one can restrict $W_\lambda$ to Sonin’s space and the ultraviolet spectral similarity with the squares of non-trivial zeros of zeta suggests that one has spectrally captured the contribution of the archimedean place to the mysterious zeta spectrum. In fact using the Darboux process we construct a Dirac square root of $W_\lambda$ depending on a deformation parameter, and whose spectrum has the same ultraviolet behavior as the zeros of the Riemann zeta function.

Our paper is organized as follows: In Section 2 we show that there exists a unique selfadjoint extension $W_{sa}$ of the symmetric operator $W_{min}$ defined on Schwartz space $S(\mathbb{R})$ by (1). Moreover $W_{sa}$ commutes with Fourier transform and has discrete spectrum unbounded in both directions. In Section 3 we show that the eigenvectors for negative eigenvalues of $W_{sa}$ belong to Sonin’s space. In Section 4 we compute the semiclassical approximation to the number of negative eigenvalues of $W_{sa}$ whose absolute value is less than $E^2$. In Section 5 we use the Darboux method combined with solutions of a Riccati equation to construct an isospectral family of Dirac operators $D$ whose squares are direct sums of two copies of $W_{sa}$. In Section 6 we specialize to the case $\lambda = \sqrt{2}$ and show that the operator $2D$ has discrete simple spectrum contained in $\mathbb{R} \cup i\mathbb{R}$ with imaginary eigenvalues symmetric under complex conjugation and counting function $N(E)$ (counting those of positive imaginary part less than $E$) fulfilling the same as the Riemann formula

\begin{equation}
N(E) \sim \frac{E}{2\pi} \left( \log \left( \frac{E}{2\pi} \right) - 1 \right) + O(1)
\end{equation}

We also show the numerical evidence for the ultraviolet spectral similarity between the eigenvalues of $2D$ and the zeros of the Riemann zeta function. Lastly, Section 7 contains more speculative final remarks, in particular on a natural two-dimensional black hole geometry intrinsically related to the operator $2D$.

2. The selfadjoint prolate wave operator

The prolate spheroidal operator (1) is an operator of Sturm-Liouville type,

\begin{equation}
(W_\lambda \xi)(x) = -\partial_x \left( p(x) \partial_x \xi(x) \right) + q(x) \xi(x), \quad x \in \mathbb{R}
\end{equation}

but having two interior singular points it is not directly treatable by the usual Sturm-Liouville theory. However its restrictions to each of the intervals $(-\infty, -\lambda)$, $(-\lambda, \lambda)$ and $(\lambda, \infty)$ are standard, in fact quasi-regular, Sturm-Liouville operators.

Henceforth $W_\lambda$ will be simply denoted $W$ whenever $\lambda$ is a general parameter. To begin with, we regard $W$ as an unbounded operator on $L^2(\mathbb{R})$ with core the Schwartz space $S(\mathbb{R})$. As such, $W$ is real, symmetric and invariant under the parity exchange $x \mapsto -x$. These features are inherited by its closure in the graph norm $W_{min}$, as well as by $W_{max} = W_{min}^*$, the latter having domain

\begin{equation}
\text{Dom}(W_{max}) = \{ \xi \in L^2(\mathbb{R}) \mid W\xi \in L^2(\mathbb{R}) \},
\end{equation}
with \( W\xi \) viewed as a tempered distribution. In addition \( W \) has the remarkable property of commuting with the Fourier transform

\[
\mathbb{F}_{\text{eu}}(f)(y) := \int_{-\infty}^{\infty} f(x) \exp(-2\pi i xy) dx.
\]

(5)

Since both the Schwartz space \( \mathcal{S}(\mathbb{R}) \) and its dual are globally invariant under the Fourier transform, the domains \( \text{Dom}W_{\text{min}} \) and \( \text{Dom}W_{\text{max}} \) are invariant too, therefore both \( W_{\text{min}} \) and \( W_{\text{max}} \) commute with \( \mathbb{F}_{\text{eu}} \).

**Lemma 2.1.** The deficiency indices of \( W_{\text{min}} \) are \((4,4)\).

**Proof.** Any \( \xi \in \text{Dom}(W_{\text{max}}) \) satisfying \( W\xi = \pm i\xi \) is a piecewise real analytic function and is uniquely specified by six parameters in the complement of the two regular singular points \( \pm \lambda \). The known form of the solutions (cf. [13]) together with the fact that \( W\xi \in L^2(\mathbb{R}) \) imply that the logarithmic singularities of \( \xi \) on the left and the right of \( \pm \lambda \) have to match. This reduces the number of parameters to 4. Conversely, since all 4 singular points are LC (limit circle case), any solution of \( W\xi = \pm i\xi \) belongs to \( \text{Dom}(W_{\text{max}}) \), hence \( \dim \ker(W_{\text{max}} \pm iI) = 4 \).

\( \square \)

**Lemma 2.2.** Let \( \xi \in \text{Dom}W_{\text{max}} \) and denote \( a = \pm \lambda \). The distribution \( p(x)\partial_x\xi \) coincides with a continuous function \( f \) in a neighborhood of \( a \) and the evaluation map \( L(\xi) := f(a) \) defines a non-zero continuous linear form on \( \text{Dom}W_{\text{max}} \) which vanishes on the closed subspace \( \text{Dom}W_{\text{min}} \).

**Proof.** Let \( V = [b,c] \) be a compact interval neighborhood of \( a = \pm \lambda \) where \( a \) is the only zero of \( p(x) \). Let \( \psi \) be the distribution \( \psi = p(x)\partial_x\xi(x) \), one has by definition,

\[
\langle \psi \mid \phi \rangle = -\int_{\mathbb{R}} \xi(x)\partial_x(p(x)\phi(x)) dx, \quad \forall \phi \in \mathcal{S}(\mathbb{R})
\]

Let \( \eta = W_{\text{max}}\xi \), one has by definition,

\[
\langle \xi \mid \phi \rangle = \langle \xi \mid W_{\text{min}}\phi \rangle = \int_{\mathbb{R}} \xi(x) (\partial_x(p(x)\partial_x\phi(x)) + q(x)\phi(x)) dx, \quad \forall \phi \in \mathcal{S}(\mathbb{R})
\]

Let \( \xi_1 \in L^2(\mathbb{R}) \) coincide with \( q(x)\xi(x) \) on \( V \). Then for any smooth function \( \phi \) with support in \( V \),

\[
\langle \psi \mid \partial_x\phi \rangle = -\int_{\mathbb{R}} \xi(x)\partial_x(p(x)\partial_x\phi(x)) dx = \langle \xi_1 - \eta \mid \phi \rangle
\]

The restriction of \( \xi_1 - \eta \) to \( V \) belongs to \( L^2(V) \subset L^1(V) \) and the function \( f_1(x) = -\int_b^x (\xi_1 - \eta)(t)dt \) is continuous and fulfills

\[
\int_V f_1(x)\partial_x\phi(x) dx = \langle \xi_1 - \eta \mid \phi \rangle
\]

It follows that \( \langle \psi - f_1 \mid \partial_x\phi \rangle = 0 \) for all smooth functions \( \phi \) with support in \( V \) and choosing a positive smooth function \( \phi_1 \) with support in \( V \) and integral 1, one obtains

\[
\langle \psi \mid \phi \rangle = \langle f_1 + s \mid \phi \rangle, \quad \forall \phi \in C^\infty_c(V), \quad s = \langle (\psi - f_1) \mid \phi_1 \rangle.
\]

Thus the distribution \( p(x)\partial_x\xi \) coincides with the function \( f(x) := f_1(x) + s \) on \( V \). One has

\[
f(a) = s + f_1(a) = \langle (\psi - f_1) \mid \phi_1 \rangle - \int_b^a (\xi_1 - \eta)(x) dx
\]
Moreover \( \langle \psi \mid \phi_1 \rangle = \int p(x) \partial_x \phi_1(x) dx = -\int \xi(x) \partial_x (p(x) \phi_1(x)) dx = \langle \xi \mid \eta_1 \rangle \) where \( \eta_1 \in C_c^\infty(V) \). One has also,

\[
-(f_1 \mid \phi_1) - \int_b^a (\xi_1 - \eta)(x) dx = \int_b^c \int_b^x (\xi_1 - \eta)(t) \phi_1(x) dt dx - \int_b^a (\xi_1 - \eta)(x) dx \equiv \langle \xi \mid \eta_2 \rangle + \langle \eta \mid \eta_3 \rangle
\]

where the vectors \( \eta_j \in L^2(\mathbb{R}) \). Thus the linear form \( L(\xi) := f(\xi) \) is continuous in the graph norm of \( \text{Dom}W_{\text{max}} \). For \( \xi \in \mathcal{S}(\mathbb{R}) \) the distribution \( \psi = p(x) \partial_x \xi(x) \) is a function vanishing at \( x = a \) and thus \( L(\xi) = 0 \). By the density of \( \mathcal{S}(\mathbb{R}) \) in \( \text{Dom}W_{\text{min}} \) for the graph norm, it follows that \( L \) vanishes on the closed subspace \( \text{Dom}W_{\text{min}} \).

Let \( P_\lambda \) be the cutoff projection associated to the interval \([−λ, λ] \), i.e. the multiplication operator by the characteristic function \( 1_{[−λ, λ]} \), and let \( \tilde{P}_\lambda = \mathcal{F}_{\text{ex}} P_\lambda \mathcal{F}_{\text{ex}}^{-1} \) denote its conjugate by the Fourier transform.

**Lemma 2.3.** If \( \xi \in \text{Dom}W_{\text{min}} \) then \( P_\lambda \xi \in \text{Dom}W_{\text{max}} \) and \( WP_\lambda \xi = P_\lambda W \xi \). The same holds with respect to \( \tilde{P}_\lambda \).

**Proof.** Let \( f \in C^\infty(V) \) where \( V \) is a neighborhood of the interval \([−λ, λ] \). Then \( P_\lambda f \in \text{Dom}W_{\text{max}} \) and viewing \( W(P_\lambda f) \) as distribution one gets, for any \( \phi \in \mathcal{S}(\mathbb{R}) \)

\[
\langle W(P_\lambda f), \phi \rangle = \int_{−\lambda}^\lambda f(x)(W \phi)(x) dx = \int_{−\lambda}^\lambda -f(x) \partial_x(\lambda^2 - x^2) \partial_x \phi(x) dx
\]

Using twice integration by parts, together with the fact that \((\lambda^2 - x^2) \phi(x) \) and \((\lambda^2 - x^2) f'(x) \) vanish on the boundary, one obtains

\[
\langle W(P_\lambda f), \phi \rangle = \int_{−\lambda}^\lambda f'(x)((\lambda^2 - x^2) \phi')(x) dx + \int_{−\lambda}^\lambda f(x)(2\pi \lambda)^2 x^2 \phi(x) dx
\]

\[
= -\int_{−\lambda}^\lambda (\partial_x((\lambda^2 - x^2) f'(x))) \phi(x) dx + \int_{−\lambda}^\lambda f(x)(2\pi \lambda)^2 x^2 \phi(x) dx
\]

\[
= \int_{−\lambda}^\lambda W f(x) \phi(x) dx,
\]

which shows that \( W(P_\lambda f) = P_\lambda W f \). In particular the same is true for any \( f \in \mathcal{S}(\mathbb{R}) \), and by the density of \( \mathcal{S}(\mathbb{R}) \) in \( \text{Dom}W_{\text{min}} \) for the graph norm it follows that \( \xi \in \text{Dom}W_{\text{min}} \implies P_\lambda \xi \in \text{Dom}W_{\text{max}} \) and \( W_{\text{max}}P_\lambda \xi = P_\lambda W_{\text{max}} \xi \).

The claim now follows from the fact that \( W \) commutes with \( \mathcal{F}_{\text{ex}} \).

The selfadjoint extensions of \( W_{\text{min}} \) are parametrized by self-orthogonal subspaces of \( \mathcal{E} := \text{Dom}(W_{\text{max}})/\text{Dom}(W_{\text{min}}) \) with respect to the anti-symmetric sesquilinear form given by the pairing

\[
\Omega(\xi, \eta) := \frac{1}{4} \left( \langle W_{\text{max}} \xi \mid \eta \rangle - \langle \xi \mid W_{\text{max}} \eta \rangle \right), \quad \xi, \eta \in \text{Dom}(W_{\text{max}})
\]

which descends to a non-degenerate form on \( \mathcal{E} \).
Lemma 2.4. The following auxiliary lemma will be used in the ensuing discussion.

Let $\mathcal{E}$ also that corresponding splittings $W$.

Lemma 2.5. The quadruplet \( \{\alpha_+, \beta_+, \alpha_-, \beta_-\} \) forms a basis of \( \mathcal{E}_\pm \).
The Ω-pairings with the above basis elements yield boundary conditions of Sturm-Liouville type. Denoting, for \( \xi \in \text{Dom}(W^\pm_{\max}) \),

\[
L_{\alpha_\pm}(\xi) := i\Omega_\pm(\xi, \alpha_+) , \quad L_{\beta_\pm}(\xi) := i\Omega_\pm(\xi, \beta_+) ,
\]

the minimal domains are characterized in these terms as being the intersection

\[
\text{Dom}(W^\pm_{\min}) = \ker L_{\alpha_\pm} \cap \ker L_{\beta_\pm} \cap \ker L_{\alpha_\pm} \cap \ker L_{\beta_\pm}
\]

and the induced functionals on \( \mathcal{E}_\pm = \text{Dom}(W^+_{\max})/\text{Dom}(W^\pm_{\min}) \) form a basis of \( \mathcal{E}_\pm^* \). By straightforward calculation, using the fact that one can always restrict the computation to \( \mathbb{R}^+ \), one obtains explicit expressions for the boundary functionals. Up to a nonzero constant factor they are as follows. In the even case,

\[
L_{\alpha_\pm}(\xi) = \lim_{x \to \lambda^-} ((x - \lambda) \log(\lambda - x) \partial_x \xi(x) - \xi(x))
- \lim_{x \to \lambda^+} ((x - \lambda) \log(x - \lambda) \partial_x \xi(x) - \xi(x));
\]

\[
L_{\beta_\pm}(\xi) := \lim_{x \to \lambda^-} ((\lambda - x) \partial_x \xi(x)) = \lim_{x \to \lambda^+} ((\lambda - x) \partial_x \xi(x));
\]

\[
L_{\alpha_\pm}(\xi) := 2 \lim_{x \to +\infty} \left[ \frac{x \cos(2\pi \lambda x) \partial_x \xi(x) + (2\pi \lambda \sin(2\pi \lambda x) + \cos(2\pi \lambda x)) \xi(x)}{2\pi x} \right];
\]

\[
L_{\beta_\pm}(\xi) := -2 \lim_{x \to -\infty} \left[ \frac{\sin(2\pi \lambda x) \partial_x \xi(x) - (2\pi \lambda \cos(2\pi \lambda x) - \sin(2\pi \lambda x)) \xi(x)}{2\pi x} \right].
\]

We note that the existence of the limit defining \( L_{\beta_\pm}(\xi) \), i.e. the equality of the lateral limits, is ensured by Lemma 2.2. Similar formulas define the functionals \( L_{\alpha_-}, L_{\beta_-}, L_{\alpha_-}, L_{\beta_-} \) in the odd case.

Since both \( \text{Dom}(W^\pm_{\min}) \) and \( \text{Dom}(W^\pm_{\max}) \), as well as the symplectic form \( \Omega \), are globally invariant under the Fourier transform, the quotient inherits induced transformations \( f^\pm_{\mathbb{R}} : \mathcal{E}_\pm \to \mathcal{E}_\pm \) which relates the boundary functionals as follows:

\[
L_{\beta_\pm} = L_{\beta_\pm} \circ f_{\mathbb{R}} \quad \text{and} \quad L_{\alpha_\pm} = L_{\alpha_\pm} \circ f_{\mathbb{R}}.
\]

This association gives rise to two distinguished self-orthogonal subspaces, namely

\[
\mathcal{L}_\beta = \bigcap_{\pm} \ker L_{\beta_\pm} \cap \bigcap_{\pm} \ker L_{\beta_\pm} \quad \text{and} \quad \mathcal{L}_\alpha = \bigcap_{\pm} \ker L_{\alpha_\pm} \cap \bigcap_{\pm} \ker L_{\beta_\pm}.
\]

**Definition.** We denote by \( W_{sa} \) the restriction of the operator \( W_{\max} \) to the subspace \( \mathcal{L}_\beta = \bigcap_{\pm} \ker L_{\beta_\pm} \cap \bigcap_{\pm} \ker L_{\beta_\pm} \). Explicitly, its domain \( \text{Dom}(W_{sa}) \) consists of the
elements $\xi \in \text{Dom}(W_{\max})$ satisfying the following boundary conditions:

$$\lim_{x \to \pm \lambda} (\lambda^2 - x^2) \partial_x \xi(x) = 0,$$

and at $\pm \infty$, writing $\xi = \xi^+ + \xi^-$ with $\xi^\pm \in \text{Dom}(W_{\max}^\pm)$,

$$\lim_{x \to \pm \infty} \left( x \sin(2\pi \lambda x) \partial_x \xi^+(x) - (2\pi \lambda x \cos(2\pi \lambda x) - \sin(2\pi \lambda x)) \xi^+(x) \right) = 0,$$

$$\lim_{x \to \pm \infty} \left( x \cos(2\pi \lambda x) \partial_x \xi^-(x) + (2\pi \lambda x \sin(2\pi \lambda x) + \cos(2\pi \lambda x)) \xi^-(x) \right) = 0.$$

We are now in a position to establish the main result of this section.

**Theorem 2.6.** (i) $W_{sa}$ is selfadjoint and commutes with the Fourier transform.

(ii) $W_{sa}$ commutes with the projections $P_\lambda$ and $\hat{P}_\lambda$.

(iii) $W_{sa}$ is the only selfadjoint extension of $W_{\min}$ commuting with $P_\lambda$ and $\hat{P}_\lambda$.

(iv) The spectrum of $W_{sa}$ is discrete and unbounded on both sides.

**Proof.** (i) $W_{sa}$ is selfadjoint by construction, and its domain $\mathcal{L}_\beta$ is invariant under the Fourier transform also by construction.

(ii) Since $\text{Dom}W_{\min}$ is given by $\{12\}$, every element of $\mathcal{L}_\beta$ is a linear combination of an element $\xi \in \text{Dom}W_{\min}$ and the 4 vectors $\beta_\pm, \hat{\beta}_\pm$ of Lemma 2.1. Each $\beta_\pm$ is of the form $P_\lambda f_\pm$ with $f_\pm$ smooth with compact support and thus one has, using Lemma 2.3,

$$P_\lambda \beta_\pm = \beta_\pm \in \mathcal{S}, \quad W_{sa} P_\lambda \beta_\pm = W_{sa} P_\lambda f_\pm = P_\lambda W f_\pm,$$

which shows that $W_{sa} P_\lambda \beta_\pm = P_\lambda W_{sa} P_\lambda \beta_\pm = P_\lambda W_{sa} \beta_\pm$ giving the required commutation for the $\beta_\pm$.

(iii) The domain of a selfadjoint extension of $W_{\min}$ commuting with $P_\lambda$ and $\hat{P}_\lambda$ must be contained in $\text{Dom}W_{\max}$ and also contain both $P_\lambda \mathcal{S}(\mathbb{R})$ and $\hat{P}_\lambda \mathcal{S}(\mathbb{R})$. Thus it must contain $\mathcal{L}_\beta$, and cannot be larger due to self-adjointness.

(iv) The operators $P_\lambda W_{sa}$ and $(I - P_\lambda)W_{sa}$ are selfadjoint on $(-\lambda, \lambda)$, resp. on $(-\infty, \lambda) \cup (-\lambda, \infty)$, and thus covered by standard results in Sturm-Liouville theory (cf. [11], [17], [9]). Indeed all four endpoints are limit circle case in the Weyl classification (see e.g. [17] §§5-6) for relevant definitions and properties), which can be easily checked by using explicit bases of formal solutions for $W_\xi - \mu \xi = 0$, $\mu \in \mathbb{C}$, around each singular point (cf. e.g. [14] §2). The endpoints $\pm \lambda$ are LCNO (non-oscillatory limit circle), while $\pm \infty$ are LCO (oscillatory limit circle) since the prolate spheroidal wave functions (which provide principal solutions around $\pm \lambda$) have infinitely many zeros in the neighborhood of $\pm \infty$ (cf. [14], [15], [16]). By well-known results (cf. e.g. [11] page 90, [18]) it follows that both $P_\lambda W_{sa}$ and $(I - P_\lambda)W_{sa}$ have discrete spectrum and that the spectrum of the latter is unbounded on both sides.

**Corollary 2.7.** If $\phi$ is an eigenfunction of $W_{sa}^\pm$ then

(i) $\phi$ is regular on $[\lambda, \lambda + \epsilon)$ and on $(\lambda - \epsilon, \lambda]$ for some $\epsilon > 0$, with a possible discontinuity at $\lambda$;

(ii) the leading term of the asymptotic expansion of $\phi$ at $\infty$ is proportional to $\frac{\sin(2\pi \lambda x)}{x}$ if $\phi$ is even and to $\frac{\cos(2\pi \lambda x)}{x}$ if $\phi$ is odd.
Proof. This follows from the above characterization \cite{16, 17, 18} of the domain of \( W_{sa} \) combined with the known bases of formal solutions for the equation \( W\xi = \mu\xi \), \( \mu \in \mathbb{R} \), around \( \pm \lambda \) and \( \pm \infty \) (cf. \cite{13}). \qed

3. Sonin space and negative eigenvalues

We translate the requirement that the Fourier transform \( \mathbb{F}_{eq}f \) of an \( f \in \text{Dom}W_{\text{max}} \) has no logarithmic singularity at the singular points into a condition on the asymptotic behavior of \( f \) at \( \infty \). For simplicity we only deal with even functions, and for notational convenience take \( \lambda = 1 \).

We can then find the asymptotic expansion at \( \infty \) using the boundary condition that the leading term there is \( \sin(2\pi \lambda y) \). We take for simplicity \( \lambda = 1 \) and use \cite{13} to get for the tentative eigenvector for eigenvalue \( \mu \) the expansion at \( \infty \)

\[ \xi_\mu(x) \sim \frac{\sin(2\pi x)}{x} + \frac{(\mu - 4\pi^2)\cos(2\pi x)}{4\pi x^2} + \frac{-\mu^2 + 8\pi^2\mu + 2\mu - 16\pi^4 + 8\pi^2}{32\pi^2 x^3}L(2\pi x) + (x^{-4}) \]

In fact as shown in Proposition 14 of \cite{13}, the coefficients of this expansion are directly related to the coefficients of the expansion of the finite solution at \( \lambda \) and taking for simplicity \( \lambda = 1 \), if the latter is of the form

\[ f_\mu(x) = \sum U_n(\mu)(x - 1)^n, \quad U_0(\mu) = 1, \quad U_1(\mu) = \frac{\mu - 4\pi^2}{2}, \quad U_2(\mu) = \frac{\mu^2 - 8\pi^2\mu - 2\mu + 16\pi^4 - 8\pi^2}{16}, \quad \ldots \]

then the asymptotic series at infinity which governs the solution which has leading term in \( \exp(-2\pi ix)/x \) is equal to \( v(x) \exp(-2\pi ix)/x \) where

\[ v(x) \sim \sum n!U_n(\mu)(2\pi ix)^{-n} \]

When one applies the Borel summation to this series the first step is to replace it by its Borel transform which is, up to normalization,

\[ B(y) := \sum U_n(\mu)y^n \]

and is related to \( v(x) \) by \( \int_0^\infty t^n \exp(-zt) \, dt = z^{-n-1}\Gamma(n+1) \) i.e. the Laplace transform

\[ \frac{v(x)}{2\pi ix} = \int_0^\infty \exp(-2\pi ixt)B(t) \, dt \]

Lemma 3.1. For any \( \mu \in \mathbb{R} \) the asymptotic expansion of the unique solution \( \xi_\mu \) which at \( \infty \) is asymptotically \( -\frac{\sin(2\pi x)}{2\pi x} \) is Borel summable and is equal to the Fourier transform of the unique even solution \( \phi_\mu \) which is zero on \([-1, 1]\) and agrees with \( f_\mu(x) \) for \( x > 1 \).

Proof. One has the equality

\[ \frac{v(x)}{2\pi ix} = \int_0^\infty \exp(-2\pi ixt)B(t) \, dt = \int_0^\infty \exp(-2\pi ixt)f_\mu(t + 1) \, dt = \]

\[ = \int_1^\infty \exp(-2\pi ixy - 1)f_\mu(y) \, dy \]

Thus one gets

\[ v(x) \exp(-2\pi ix)/(2\pi ix) = \int_1^\infty \exp(-2\pi ixy)f_\mu(y) \, dy \]
The function \( \phi_\mu \) is even and vanishes on \([-1,1]\) so
\[
\int_{-\infty}^{\infty} \exp(-2\pi i xy)\phi_\mu(y)dy = \int_{-1}^{1} \exp(-2\pi i xy)f_\mu(y)dy + \int_{1}^{\infty} \exp(-2\pi i xy)f_\mu(y)dy = v(x)\exp(-2\pi i x)/(2\pi i x) + \overline{v(x)\exp(-2\pi i x)/(2\pi i x)}
\]
Now these two terms are asymptotic solutions since \( \mu \) is real and \( v(x)\exp(-2\pi i x)/(2\pi i x) \) is an asymptotic solution. Moreover the leading behavior at \( \infty \) is in \( \exp(-2\pi i x)/(2\pi i x) - \exp(2\pi i x)/(2\pi i x) = -\frac{\sin(2\pi x)}{\pi x} \)
Thus it follows that the Fourier transform \( F_{ex}\phi_\mu = \xi_\mu \).

**Corollary 3.2.** With the above notation, assume \( \mu \) is a negative eigenvalue. Then \( \phi_\mu \) belongs to the Sonin space.

**Proof.** In fact Sonin’s space is the orthogonal of the eigenspaces of \( W_{sa} \) associated to the classical prolate functions and their Fourier transforms.

We should note that at this point we do not claim (although this is supported by numerical evidence) that all eigenvalues of the restriction of \( W_{sa} \) to Sonin’s space are negative, however there could be only finitely many exceptions.

4. **Semiclassical approximation and counting function**

In this section we use the semiclassical estimate for the function counting the number of eigenvalues and investigate the negative eigenvalues of the operator \( W_{sa} \). We consider the classical Hamiltonian
\[
H_\lambda(p,q) = (p^2 - \lambda^2)(q^2 - \lambda^2)
\]
and use it as a semiclassical approximation of \( W_{sa} \) via the formal relation
\[
W_\lambda \sim -4\pi^2 H_\lambda + 4\pi^2 \lambda^4
\]
using the correspondence \( q \to x \) and \( p \to \frac{1}{2\pi i} \partial_x \) associated to the choice of the Fourier transform \( F_{ex} \). Sonin’s space corresponds to the conditions \( p^2 - \lambda^2 \geq 0 \) and \( q^2 - \lambda^2 \geq 0 \) and the region of interest for the counting of eigenvalues is thus
\[
\Omega_\lambda(E) := \{(q,p) \mid q \geq \lambda, p \geq \lambda, H_\lambda(p,q) \leq \left( \frac{E}{2\pi} \right)^2 + \lambda^4 \}
\]
The area of \( \Omega_\lambda(E) \) is given, with \( a = \left( \frac{E}{2\pi} \right)^2 + \lambda^4 \), by the convergent integral
\[
I_\lambda(a) = \int_{\lambda}^{\infty} \left( \frac{\sqrt{a + \lambda^4 x^2} - \lambda^4}{\sqrt{x^2 - \lambda^2}} - \lambda \right) dx
\]
One has, with \( x = \lambda y \), the equality
\[
I_\lambda(a) = \lambda \int_{1}^{\infty} \left( \frac{\sqrt{a + \lambda^4 y^2} - \lambda^4}{\sqrt{\lambda^4 y^2 - \lambda^2}} - 1 \right) dy = \lambda^2 \int_{1}^{\infty} \left( \frac{\sqrt{a \lambda^4 + y^2 - 1}}{\sqrt{y^2 - 1}} - 1 \right) dy
\]
Thus one obtains the equality
\[
I_\lambda(a) = \lambda^2 I_1(a \lambda^{-4})
\]
We recall that the elliptic integrals $E(m)$ and $K(m)$ are defined by

$$E(m) := \int_0^{\pi/2} \sqrt{1 - m \sin^2 \theta} \, d\theta, \quad K(m) := \int_0^{\pi/2} \frac{1}{\sqrt{1 - m \sin^2 \theta}} \, d\theta$$

**Lemma 4.1.** The integral $I(a) = I_1(a)$ is given by the sum of elliptic integrals

$$I(a) = aK(1-a) - E(1-a) + 1 \quad (22)$$

**Proof.** One has, with $m = 1 - a$, $x = 1/t$

$$I(a) = \int_0^1 \left( \frac{\sqrt{1 - mt^2}}{\sqrt{1 - t^2}} - 1 \right) \frac{dt}{t^2}$$

Let

$$g(t) := -\frac{\sqrt{1 - t^2} \left( m^2 t^2 + \sqrt{1 - t^2} \sqrt{1 - mt^2} - 1 \right)}{t\sqrt{1 - mt^2}}$$

One has $g(0) = 0$, $g(1) = 0$ and the derivative of $g$ is equal to

$$g'(t) = -\left( \frac{\sqrt{1 - mt^2}}{\sqrt{1 - t^2}} - 1 \right) t^{-2} + \frac{1 - m}{\sqrt{1 - t^2} \sqrt{1 - mt^2}} - \frac{\sqrt{1 - mt^2}}{\sqrt{1 - t^2}} + 1$$

so that the equality $\int_0^1 g'(t) \, dt = 0$ gives (22). \qed
We thus get

**Proposition 4.2.** The semiclassical approximation to the number of negative eigenvalues $\xi$ of $W_{sa}$ with $-\xi \leq E^2$ on even functions is the same as on odd functions and is equal to $2\sigma(E, \lambda)$ where

$$\sigma(E, \lambda) \sim \frac{E}{2\pi} \left( \log \left( \frac{E}{2\pi} \right) - 1 + \log(4) - 2\log(\lambda) \right) + \lambda^2 + o(1) \quad (23)$$

**Proof.** The semiclassical approximation corresponds, for the restriction to even functions (or to odd functions), to twice the area of $\Omega(E)$ and hence to $I_\lambda(a) = \lambda^2 I(a \lambda^{-4})$, for $a = \left( \frac{E}{2\pi} \right)^2 + \lambda^4$. One has the asymptotic expansion for $a \to \infty$

$$I(a) \sim \frac{1}{2} \sqrt{a} \left( \log(a) - 2 + 4 \log(2) \right) + 1 + \frac{1}{8} \sqrt{\frac{1}{a}} (-\log(a) - 4 \log(2)) + o \left( \frac{1}{a} \right) \quad (24)$$

so that

$$I_\lambda(a) \sim \frac{1}{2} \sqrt{a} \left( \log(a) - 2 + 4 \log(2) - 4 \log \lambda \right) + \lambda^2 + o(1) \quad (25)$$

We then use the expansions

$$\sqrt{a} = \frac{E}{2\pi} + O(1/E), \quad \log(a) = 2 \log \left( \frac{E}{2\pi} \right) + O(1/E^2)$$

and obtain \[23].

---

5. **Dirac operators**

The results of Section 4 show that for suitable values of $\lambda$ the negative spectrum of $W_{sa}$ has the same ultraviolet behavior as the squares of zeros of the Riemann zeta function. Since $W_{sa}$ is a differential operator of second order we liken it to the Klein-Gordon operator and construct the analogue of the Dirac operator. We first use the Darboux process (see [7], [8]) to factorize $W_{sa}$ as a product of two first order differential operators.

**Lemma 5.1.** Let $p(x) = x^2 - \lambda^2$, $V(x) = 4\pi^2 \lambda^2 x^2$, $L = \partial(p(x)\partial) + V(x)$, $(\nabla f)(x) := p(x)^{1/2} \partial f(x)$ and $U$ the unitary operator

$$U : L^2([\lambda, \infty), dx) \to L^2([\lambda, \infty), p(x)^{-1/2} dx), \quad U(\xi)(x) := p(x)^{1/4} \xi(x).$$

Let $w(x)$ be a solution of the equation

$$\nabla w(x) + w(x)^2 = -V(x) + \left( \frac{p''(x)}{4} - \frac{p'(x)^2}{16p(x)} \right), \quad \forall x \in [\lambda, \infty) \quad (26)$$

then one has $L = U^* (\nabla + w)(\nabla - w) U$.

**Proof.** Let $f$ be a smooth function on $\mathbb{R}$ and consider the differential operators $T_1 := f \partial_x f$ and $T_2 := \partial_x f^4 \partial_x$. Let us show that $T_1^2 - T_2$ is an operator of order zero: one has

$$T_1^2 = f \partial_x f^2 \partial_x f = -f' f^2 \partial_x f + \partial_x f^3 \partial_x f$$

$$-f' f^2 \partial_x f = -f' f^2 - f' f^3 \partial_x f, \quad \partial_x f^3 \partial_x f = \partial_x f^4 \partial_x + \partial_x f^3 f'$$

so that $T_1^2 - T_2$ is the multiplication by $2 f'' f^2 + f^3 f''$. Applying this for $f(x) = p(x)^{1/4}$ gives

$$(U^* \nabla U)^2 = \partial_x p(x) \partial_x + \frac{p''(x)}{4} - \frac{p'(x)^2}{16p(x)}$$

from which the conclusion follows using \[26\].
We now determine all solutions of the Riccati equation (26) which gives
\[ \sqrt{x^2 - \lambda^2} w'(x) + w(x)^2 = -4\pi^2 \lambda^2 x^2 - \frac{1}{4} \frac{x^2}{x^2 - \lambda^2} + \frac{1}{2} \] (27)
The next Lemma is standard using the reduction of a Riccati equation to a Bernoulli equation.

**Lemma 5.2.** Let \( u_j \) be two real valued solutions of \( Lu = 0 \) which generate the linear space of solutions in \( (\lambda, \infty) \).

(i) For \( z \in \mathbb{C} \) and \( u = u_1 + zu_2 \) the solution \( u \) has no zero in \( (\lambda, \infty) \) if \( z \notin \mathbb{R} \) and an infinity of zeros otherwise.

(ii) All solutions of the Riccati equation (27) are given by
\[ w_z(x) = \frac{(x^2 - \lambda^2)^{1/4} \partial \left( (x^2 - \lambda^2)^{1/4} u(x) \right)}{u(x)} \] (28)
where \( u = u_1 + zu_2 \) and \( z \in \mathbb{C} \setminus \mathbb{R} \).

(iii) The map \( z \mapsto w_z \) from \( \mathbb{C} \setminus \mathbb{R} \) to the space of solutions of (27) is a homeomorphism.

**Proof.** (i) Let \( x \in (\lambda, \infty) \) and \( z \in \mathbb{C} \setminus \mathbb{R} \), \( u = u_1 + zu_2 \). Assume \( u(x) = 0 \). Then \( u_1(x) + zu_2(x) = 0 \) and since \( z \in \mathbb{C} \setminus \mathbb{R} \) this implies \( u_1(x) = 0 \) and \( u_2(x) = 0 \). The Wronskian \( p(x)(u_1'(x)u_2(x) - u_1(x)u_2'(x)) \) is constant and non-zero since the \( u_j \) are independent solutions. Thus we get a contradiction and \( u \) has no zero. Moreover since the equation \( Lu = 0 \) is in the LCO case any real valued solution has infinitely many zeros.

(ii) The standard solution of the Riccati equation (26) is of the form
\[ w(x) = \frac{p(x)^{1/4} \partial \left( p(x)^{1/4} u(x) \right)}{u(x)} \] (29)
which gives
\[ \nabla w(x) + w(x)^2 = \frac{p''(x)}{4} + \frac{p'(x)u'(x)}{u(x)} - \frac{p'(x)^2}{16p(x)} + \frac{p(x)u''(x)}{u(x)} \]
so that
\[ Lu = 0 \Rightarrow \nabla w(x) + w(x)^2 = -V(x) + \left( \frac{p''(x)}{4} - \frac{p'(x)^2}{16p(x)} \right) \]
Thus by (i) any \( w_z, z \in \mathbb{C} \setminus \mathbb{R} \) is a solution of (27). Using the three values \( \{i, -i, j\} \) for \( z \) and the reduction to a Bernoulli equation one can express the general solution of (27) in the form
\[ w = w_i + \frac{(w_{-i} - w_i)(w_j - w_i)}{(1 - t)(w_j - w_i) + t(w_{-i} - w_i)} = w_z(t) \] (30)
where
\[ z(t) = \frac{i(i(t - 1) + j(t + 1))}{i(t + 1) + j(t - 1)} \in \mathbb{C} \setminus \mathbb{R} \]
(iii) The formula (30) establishes an homeomorphism between the space of solutions of the Riccati equation and the complement in \( \mathbb{P}^1(\mathbb{C}) \) of the circle
\[ \{ t \in \mathbb{P}^1(\mathbb{C}) \mid z(t) \in \mathbb{P}^1(\mathbb{R}) \} \]
and thus the map \( z \mapsto w_z \) from \( \mathbb{C} \setminus \mathbb{R} \) to the space of solutions of (27) is a homeomorphism. \( \square \)
Proposition 5.3. Let \( w \) be a solution of the Riccati equation (27) and \( \mathcal{D} \) be the matrix of order one operators
\[
\mathcal{D} = \begin{pmatrix} 0 & \nabla + w(x) \\ \nabla - w(x) & 0 \end{pmatrix}
\]
(31)

Then the square of \( \mathcal{D} \) is diagonal with each diagonal term spectrally equivalent to \( L \),
\[
U^* \mathcal{D}^2 U = \begin{pmatrix} L & 0 \\ 0 & L + 2\nabla w(x) \end{pmatrix}
\]
The proof is straightforward. The use of the Darboux process in this construction is related to the theory of isospectral deformations [7, 8].

6. Ultraviolet behavior of spectrum of Dirac, case \( \lambda = \sqrt{2} \)

In this section we take \( \lambda = \sqrt{2} \), and consider the operator \( 2\mathcal{D} \) where \( \mathcal{D} \) is as defined in Proposition 5.3.

Theorem 6.1. The operator \( 2\mathcal{D} \) has discrete simple spectrum contained in \( \mathbb{R} \cup i\mathbb{R} \). Its imaginary eigenvalues are symmetric under complex conjugation and the counting function \( N(E) \) counting those of positive imaginary part less than \( E \) fulfills
\[
N(E) \sim \frac{E}{2\pi} \left( \log \left( \frac{E}{2\pi} \right) - 1 \right) + O(1) \quad (32)
\]

Proof. By Proposition 5.3 the spectrum of \( 2\mathcal{D} \) consists of the complex numbers of the form \( \xi = \pm 2\sqrt{\alpha} \) where \( \alpha \) varies in the spectrum of \( L \). The latter is real and the number of negative eigenvalues \( \alpha \geq -E^2 \) is given by Proposition 4.2 as \( 2\sigma(E, \lambda) \), thus selecting the root with positive imaginary part one gets
\[
0 < \Im(\xi) \leq E \iff \alpha \geq -(E/2)^2
\]
and the number \( N(E) \) of such \( \xi \) is thus
\[
2\sigma(E/2, \sqrt{2}) = \frac{E}{2\pi} \left( \log \left( \frac{E/2}{2\pi} \right) - 1 + \log(4) - 2\log(\sqrt{2}) \right) + O(1) =
\]
\[
= \frac{E}{2\pi} \left( \log \left( \frac{E}{2\pi} \right) - 1 \right) + O(1)
\]
which gives the required estimate. \( \square \)
Figure 2. The spectrum (in blue) compared to zeros of zeta (red)

Figure 3. The spectrum compared to zeros of zeta
7. Final remarks

We gather in this final section a number of more speculative remarks.

7.1. Geometric meaning of Theorem 6.1. The operator $2\mathcal{D}$ of Theorem 6.1 together with the action by multiplication of smooth functions on the interval $[\sqrt{2}, \infty)$ would define a spectral triple if $2\mathcal{D}$ were self-adjoint (or skew adjoint) but its spectrum contains both real and imaginary pieces. The leading term $\nabla = (2\sqrt{x^2 - 2}) \partial_x$ shows that the corresponding classical metric is (with $\lambda = \sqrt{2}$ from now on)

$$ds^2 = -\frac{1}{\alpha(x)} dx^2, \quad \alpha(x) = -4(x^2 - 2)$$

This $ds^2$ changes sign when crossing the boundary $x = \sqrt{2}$ and this suggests, in order to handle all even functions on $\mathbb{R}$ and to take into account the real and imaginary eigenvalues of the square of $2\mathcal{D}$, to look for a two dimensional metric with signature $(-1, 1)$ of the form

$$ds^2 = -\alpha(x) dt^2 + \frac{1}{\alpha(x)} dx^2$$

This geometry corresponds to a black hole in two space-time dimensions with horizon at $x = \pm \sqrt{2}$. It fulfils the 2-dimensional analogue of Einstein’s equation with a cosmological constant $= 8$ and no source [10]. One can look at the null curves and this means

$$\frac{dx}{dt} = \alpha(x) = -4(x^2 - 2) \Rightarrow t(x) = \frac{1}{8\sqrt{2}} \log \left( (\sqrt{2} + x)/(x - \sqrt{2}) \right) + c$$

one then passes to the new coordinates $v = t - t(x)$, and $x$ unchanged. In these new coordinates one re-expresses the metric in the smooth form

$$ds^2 = 4 \left( x^2 - 2 \right) dv^2 - 2vdx$$
Figure 5. Light rays in the two dimensional geometry, and in black the original curve. The vertical lines are the horizons at \( x = \pm \sqrt{2} \).

In this metric the light rays are given by \( v = v_0 \) \( i.e. \) the horizontal lines of Figure 5) and by the curves

\[
v(x) = \frac{1}{4\sqrt{2}} \log \left| \frac{x - \sqrt{2}}{x + \sqrt{2}} \right| + c
\]

i.e. the solutions of the equation \( dv = \frac{dx}{2(x^2-2)} \). The original curve given by \( t = 0 \) corresponds to the graph of \( v = -t(x) \) as shown in black in Figure 5.

7.2. Positive eigenvalues of \( W_{sa} \) and trivial zeros of Zeta. The eigenvalues \( \chi(n) \) of the restriction of \( W_{sa} \) to even functions in the interval \([-\lambda, \lambda]\) have a well understood asymptotic form which by [12] Theorem 3.11 implies that, independently of the value of \( \lambda \), (note that we only consider even functions so that the index \( n \) of \( op.cit. \) is replaced by \( 2n \))

\[
\chi(n) = \left( 2n + \frac{1}{2} \right)^2 + O(1), \ n \to \infty
\]

This behavior is the same as that of the squares of the trivial zeros of the Riemann zeta function with the same shift of \( \frac{1}{2} \) as for the critical line. To obtain a convincing relation one would need to analyze the extension of \( 2D \) to (two copies of) the even functions on \( \mathbb{R} \) as well as the conditioning of the Hilbert space needed to eliminate the positive square roots of the \( \chi(n) \).

7.3. Spectral truncation. In order to eliminate the real eigenvalues of \( 2D \) coming from the positive eigenvalues of \( W_{sa} \) one can effect a spectral truncation [6], the algebra of functions acting by multiplication is then replaced by the operator system
obtained by compression on Sonin’s space. In a similar manner one can use spectral truncation to eliminate the positive square roots with the notations of §7.2.

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