Poincare’-Birkhoff-Witt property for bicovariant differential algebras on simple quantum groups.

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Abstract

We investigate the possibility to construct bicovariant differential calculi on quantum groups $SO_q(N)$ and $Sp_q(N)$ as a quantization of an underlying bicovariant bracket. We show that, opposite to $GL(N)$ and $SL(N)$-cases, neither of possible graded $SO$- and $Sp$- bicovariant brackets (associated with a quasitriangular $r$-matrices) obey the Jacobi identity when the differential forms are Lie algebra-valued. The absence of a classical Poisson structure gives an indication that differential algebras describing bicovariant differential calculi on quantum orthogonal and symplectic groups are not of Poincaré-Birkhoff-Witt type.

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Bicovariant differential calculus (BDC) on quantum groups initiated by Woronowicz’s \cite{1} provides a meaningful example of the noncommutative differential geometry \cite{1}. On the other hand, it serves as the starting point to formulate a new class of gauge theories with a simple quantum group playing the role of a gauge group \cite{3},\cite{4}. Many of the phenomena, which one can encounter studying these theories, are not privileges of a model but have their origin in the theory of BDC. Thus, it is extremely important to investigate the general properties of BDC on simple quantum groups.

In this letter we are aimed to find out whether external algebras on quantum groups $SO_q(N)$ and $Sp_q(N)$ are of Poincare’-Birkhoff-Witt (PBW) type, i.e. they possess a unique basis of lexicographically ordered monomials. This is not an academic question, since it has a strong influence on all differential geometry associated with these groups. In particular, the missing of PBW property under quantization means that the classical and corresponding to it quantum system has different number of observables.

Our consideration is based on the $R$-matrix approach of \cite{5}, which is very useful in dealing with BDC’s.

Recall that the central point of Woronowicz’s theory is the constructing of bicovariant bimodules $\Gamma$ over a Hopf algebra $\mathcal{A}$ (the algebra of functions on a quantum group). The bimodules over $\mathcal{A}$ supplied with two coactions:

$$\Delta_R : \Gamma \rightarrow \Gamma \otimes \mathcal{A} \quad \text{and} \quad \Delta_L : \Gamma \rightarrow \mathcal{A} \otimes \Gamma$$ \hspace{1cm} (1)

satisfying the set of axioms \cite{1}. Bicovariant bimodules are interpreted as noncommutative analogues of tensor bundles $\Gamma_{cl}$ over Lie groups. For the case of simple quantum groups the classification of bicovariant bimodules was obtained in \cite{6} and confirmed in \cite{7}.

A first order differential calculus is defined as a pair $(\Gamma, d)$, where differential $d : \mathcal{A} \rightarrow \Gamma$ is a nilpotent mapping obeying the Leibnitz rule.

The bicovariant wedge product of two left-invariant 1-forms $\Omega_i$ is defined via tensor algebra construction:

$$\Omega_i \wedge \Omega_j = \Omega_i \otimes_{\mathcal{A}} \Omega_j - \sigma_{ij}^{lk} \Omega_l \otimes_{\mathcal{A}} \Omega_k$$ \hspace{1cm} (2)

or in the concise matrix notation:

$$\Omega_1 \wedge \Omega_2 = (I_{12} - \sigma_{12})\Omega_1 \otimes_{\mathcal{A}} \Omega_2.$$ \hspace{1cm} (3)

Here matrix $\sigma_{12}$ satisfies the Yang-Baxter equation (YBE) \cite{1}:

$$\sigma_{23}\sigma_{12}\sigma_{23} = \sigma_{12}\sigma_{23}\sigma_{12},$$ \hspace{1cm} (4)

which provides the assosiativity of the wedge product:

$$(\Omega_1 \wedge \Omega_2) \wedge \Omega_3 = \Omega_1 \wedge (\Omega_2 \wedge \Omega_3).$$ \hspace{1cm} (5)

Therefore, adopting (2),(3) one can construct, starting from $\Gamma$, an associative external algebra $\Gamma^\wedge = \sum \Gamma^{(n)}$, where $\Gamma^{(0)} = \mathcal{A}$, $\Gamma^{(1)} = \Gamma$ and $\Gamma^{(n)}$ is the space of $n$-forms. It is proved \cite{1} that a first order differential calculus can be lifted to higher order differential forms via extending $\Gamma$ by an additional bi-invariant 1-form $X$ generating $d : d\Omega = [X, \Omega]_{\pm}$, $\Omega \in \Gamma^\wedge$. 

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However, let us stress that YBE (4) does not guarantee that \( \Gamma^\wedge \) constructed in such a way is the algebra of PBW type. From the point of view of general BDC theory the fulfillment of PBW property for quantum external algebras is an additional physical requirement.

The general properties of \( \sigma \) for quantum simple Lie groups (in particular, the projector expansion) were studied in [8], [9]. In [8] the authors modified the definition (2) by imposing the additional quadratic relations on the generators \( \Omega_i \) (we will comment this later). The direct investigation of the PBW property for (2) is rather involved and to our knowledge it has been tackled only for the external algebras on \( GL_q(N) \) and \( SL_q(N) \) \([10]\), [11], [12]. Fortunately, for quantum groups \( SO_q(N) \) and \( Sp_q(N) \) \((N = 2n)\), one can exploit ideas of [14], [15] that the quantum groups can be obtained by quantizing of classical Poisson structures and try to answer the question, about PBW property, just on the semiclassical level. This is due to the existence of the infinitesimal version of bicovariant differential calculi on the quantum groups provided by graded bicovariant brackets \([16]\) which can be deduced directly as a semiclassical limit of the PBW algebras presented in [11], [12], [13]. In this approach, it is assumed that the anticommutator in \( \Gamma \) is determined, in the semiclassical approximation, by a graded bracket on \( \Gamma_{cl}^\wedge \):

\[
\Omega \wedge \Omega' + \Omega' \wedge \Omega = \hbar \{ \Omega, \Omega' \} + \hbar^2(\ldots),
\]

satisfying the condition of bicovariance:

\[
\Delta_{L,R}(\{ \Omega, \Omega' \}) = \{ \Delta_{L,R}(\Omega), \Delta_{L,R}(\Omega') \}, \quad \Omega, \Omega' \in \Gamma_{cl}^1.
\]

The coactions \( \Delta_L : \Gamma \to A \otimes \Gamma \) and \( \Delta_R : \Gamma \to \Gamma \otimes A \) coming in (7) are the classical analogues of (2). On the matrix elements \( \Omega_{ij}^i \) \((i, j = 1, 2, \ldots N)\) of the Lie-valued left-invariant Cartan’s form \( \Omega \) \((\Omega_{ij}^i \text{ generates } \Gamma_{cl}^1)\) \(\Delta_{L,R} \) are defined as

\[
\Delta_L(\Omega_{ij}^i) = I \otimes \Omega_{ij}^i \equiv \Omega_{ij}^i, \quad \Delta_R(\Omega_{ij}^i) = \Omega_{ij}^k \otimes (T^{-1})_{kj}^l \equiv (T^{-1} \Omega T)^i_j,
\]

and to arbitrary element of \( \Gamma_{cl}^i \) they are extended as homomorphisms with respect to the wedge product. We introduce the matrix element \( T^i_j \) of the groups in (8) and, for the sake of simplicity, we omit the signs of the tensor products in the last parts of (8). In the following we will refer to bracket satisfying (7) as to bicovariant with respect to the \( L, R \)-coactions of \( G \).

To turn \( \Gamma_{cl}^\wedge \) into a classical Poisson system we also impose on the graded bracket in (6) the following conventional requirements:

i) the symmetry condition:

\[
\{ \rho, \rho' \} = (-1)^{\deg(\rho) \deg(\rho')} \{ \rho', \rho \},
\]

ii) the graded Jacobi identity:

\[
(-1)^{\deg \rho_1 \deg \rho_2} \{ \{ \rho_1, \rho_2 \}, \rho_3 \} + (-1)^{\deg \rho_2 \deg \rho_3} \{ \{ \rho_3, \rho_1 \}, \rho_2 \} + (-1)^{\deg \rho_1 \deg \rho_3} \{ \{ \rho_2, \rho_3 \}, \rho_1 \} = 0,
\]

and

\[
(\cdot, \cdot) = \frac{1}{2} (\{ \cdot, \cdot \} + \{ \cdot, \cdot \}),
\]

\[
\{ \cdot, \cdot \} = \{ \cdot, \cdot \},
\]

\[
\{ \Gamma_{cl}^\wedge, \Gamma_{cl}^\wedge \} = \Gamma_{cl}^\wedge. 
\]
In addition we demand (as usual) this bracket to be a graded differentiation:

\[ \{ \rho_1 \otimes \rho_2, \rho_3 \otimes \rho_4 \} = \{ \rho_1, \rho_3 \} \otimes \rho_2 \rho_4 + \rho_1 \rho_3 \otimes \{ \rho_2, \rho_4 \} . \]

If a bracket satisfying the requirements above exists, then \( \Gamma^3 \) is said to be equipped with a graded Poisson-Lie (PL) structure \([16]\) and one can consider \( \Gamma^3 \) as a phase space for some graded dynamical system.

Thus, in general, an algebra of quantum external forms is expected to be a graded bicovariant algebra with the graded commutator that produces in the semiclassical limit a graded bicovariant bracket. Now the fact that this algebra is of PBW type leads, in semiclassics, to the requirement on the corresponding bicovariant bracket to be Poisson, i.e., to satisfy the Jacobi identity (here and below we confine ourselves only with a consideration of the exterior algebras \([2]\) having usual classical limit).

In the cases of \( GL(N) \) and \( SL(N) \), graded PL structures exist \([14], [17]\) and the corresponding algebras of quantum external forms are of PBW type \([11], [12]\). Moreover, if a graded PL structure exists, then bicovariance and PBW property can be considered as main quantization principles.

Let \( G \) be \( SO(N) \) or \( Sp(N) \) groups and \( G \) be the corresponding Lie algebra. The following terminology (see \([18]\)) will be useful. A skewsymmetric solution \( r (r \in G \wedge G) \) of the classical YBE (CYBE) will be refered as a triangular \( r \)-matrix and a skewsymmetric \( r \) obeying the modified YBE (mYBE) will be refered as a quasitriangular one.

Our strategy is as follows. It is natural to consider a graded PL structure on \( \Gamma^3 \) generated by the brackets between the components of \( G \)-valued Cartan’s form \( \Omega \), i.e., when \( \Omega^i_j = \omega_\alpha(t^\alpha)^i_j \) where \( t^\alpha \in G \). It is shown below (see also \([19]\)) that these brackets are defined via a triangular \( r \)-matrix. However, this is not for the case of the standard \( r \)-matrix \([13]\) associated with simple Lie algebras. In other words, if we employ a quasitriangular \( r \) (that is relevant for subsequent quantization) and require \( G \)-covariant Poisson bracket \( \{ \Omega_1, \Omega_2 \} \) to be an element of \( G \otimes G \) (as a matrix), then we get a unique solution \( \{ \Omega_1, \Omega_2 \} = 0 \) (see below). One may hope that by discarding the requirement \( \Omega \in G \) it would be possible to obtain a graded bicovariant bracket with a quasitriangular \( r \). Below we analyze this possibility and, hence, assume the general situation when \( \Omega \in Mat(N, C) \sim gl(N, C) \). In this case, a covariancy group \( G \) of graded brackets on \( gl(N) \) is a subgroup of \( GL(N) \). Let us stress that this is in agreement with the quantum external algebra construction \([1]\) where \( \dim \Gamma^{(1)} = N^2 \).

One remark is in order. We will not consider in this letter the graded bicovariant brackets which are covariant under the groups isomorphic with the linear groups of \( A_{n-1} \) series (e.g. \( SO(3) \sim Sp(2) \sim SL(2) \), see \([19]\) for discussion).

Now we recall briefly the basic facts about Lie groups \( G \) corresponding to \( so(N) \) or \( sp(N) \) (\( N = 2n \)) Lie algebras. The fundamental representation of \( G \) is given by

\[ T C^t T C^{-1} = C T C^{-1} T = I , \]

where \( N \times N \) metric \( C \) is \( C^{ij} = \delta^{ij} \) for \( SO(N) \) and \( C^{ij} = \epsilon_i \delta^{ij} \) for \( Sp(N) \), \( i' = N + 1 - i \), \( \epsilon_i = 1 \) (\( i = 1, \ldots, n \)), and \( \epsilon_i = -1 \) if \( i = n, \ldots, 2n \). We denote by \( C^{ij} (C_{ij}) \) matrix elements of \( C (C^{-1}) \).
The fundamental representations of the corresponding Lie algebras are defined as follows

\[ \mathcal{G} = \{ X \in \text{Mat}(N, \mathbb{C}) | X^t = -CX^{-1} \} . \]

To simplify the calculations we introduce an operation \( \tilde{\Omega} \) acting on \( \Omega, \Omega^2, \) etc., in the following way

\[ \tilde{\Omega} = C\Omega^tC^{-1}, \quad \tilde{\Omega}^2 = C(\Omega^2)^tC^{-1} = -(\tilde{\Omega})^2. \]

Clearly, \( \tilde{\Omega} = \Omega. \) Using this operation we split matrix-valued forms as \( \Omega^\pm = \Omega \pm \tilde{\Omega}. \)

Note that the form \( \Omega^- \) belongs to \( \mathcal{G} \) in the fundamental representation.

It can be shown [20] that the general form of a \( Z \)-graded bicovariant bracket \( \{ \Omega_1, \Omega_2 \} \) is

\[ \{ \Omega_1, \Omega_2 \} = [\Omega_1, [\Omega_2, r_{12}]] + Tr_{34}(W_{1234}\Omega_3\Omega_4), \]

where \( r_{12} \) is the quasitriangular \( r \)-matrix and \( W_{1234} \) is a \( G \)-invariant tensor:

\[ W_{1234} = T_1 T_2 T_3 T_4 W_{1234} T_1^{-1} T_2^{-1} T_3^{-1} T_4^{-1}, \]

with symmetry properties:

\[ W_{1234} = W_{2134} = -W_{1243}, \]

Here indices 1, 2, 3, 4 denote the numbers of the matrix spaces. Thus, to construct a general \( SO-(Sp) \)-bicovariant bracket we have to enumerate all tensors \( [13] \) invariant under \( G \)-action \( [12] \). Classification of all possible \( W_{1234} \) leads to the following explicit form of bracket \( [11] \) (the detailed proof of this statement will be published elsewhere):

\[ \{ \Omega_1, \Omega_2 \} = [\Omega_1[\Omega_2, r_{12}]] + X^{(1)}_{12}(\Omega_1^2 + \Omega_2^2) + (\tilde{\Omega}_1^2 + \tilde{\Omega}_2^2)X^{(2)}_{12} + \]

\[ (\tilde{\Omega}_1 X^{(3)}_{12} \Omega_1 + \tilde{\Omega}_2 X^{(3)}_{12} \Omega_2) + (\tilde{\Omega}_1 X^{(4)}_{12} \Omega_2 + \tilde{\Omega}_2 X^{(4)}_{12} \Omega_1) + X^{(5)}_{12}(\Omega_1 \tilde{\Omega}_1 + \Omega_2 \tilde{\Omega}_2) + \]

\[ (X^{(6)}_{12}(\tilde{\Omega}_1 + \tilde{\Omega}_2) + (\Omega_1 + \Omega_2)X^{(7)}_{12})\text{tr}\Omega, \]

where all \( X^{(i)} \) are symmetric \( G \)-invariant matrices in \( \text{Mat}(N, \mathbb{C}) \otimes \text{Mat}(N, \mathbb{C}) \):

\[ X^{(i)} = a_i I + b_i P + c_i K^0. \]

and \( a_i, b_i, c_i \) are complex numbers, \( I \) is the identity matrix, \( P \) is a permutation matrix and \( K^0: (K^0)_{ij}^{kl} = C_{kl}P_{ij}. \)

Due to the identities

\[ K^0_{12} \Omega_1 = K^0_{12} \tilde{\Omega}_2, \quad K^0_{12} \Omega_2 = K^0_{12} \tilde{\Omega}_1, \]

we find that \( K^0_{12}(\Omega_1 \tilde{\Omega}_1 + \Omega_2 \tilde{\Omega}_2) = 0 \), i.e., we can put \( c_5 = 0 \) and, therefore, the bracket \( [14] \) depends on twenty arbitrary parameters \( a_i, b_i, c_i \). In fact this number coincides with dimension of the cohomology group \( H^0(\mathcal{G}, SV \otimes \wedge V) \), where \( V = \text{Mat}(N, \mathbb{C}) \) and \( SV \) (\( \wedge V \)) stands for the symmetric (antisymmetric) part of \( V \otimes V \). We note that operators \( X^{(i)} \) have the matrix structure of Yangian \( R \)-matrices.

Having the general form \( [14] \) one can calculate the bracket between the variables \( \Omega^\pm \). For this purpose one needs explicit expressions for \( \{ \tilde{\Omega}_1, \Omega_2 \} \) and \( \{ \Omega_1, \tilde{\Omega}_2 \} \) that are
obtained from (14) by acting with \( \tilde{\cdot} \) in the first or in the second matrix spaces. Now if we take into account that
\[
\tilde{X}^{(i)}(a, b, c) = X^{(i)}_1(a, \epsilon c, \epsilon b),
\]
then we get
\[
\{\Omega^+_1, \Omega^+_2\} = [\Omega^+_1[\Omega^-_2, r_{12}]]_+ + Z^\pm_{12}(\Omega^+_1 + \Omega^+_2) - (\tilde{\Omega}^2_1 + \tilde{\Omega}^2_2)Z^\pm_{12} + (V^\pm_{12}(\Omega_1 + \Omega_2) + (\tilde{\Omega}_1 + \tilde{\Omega}_2)V^\pm_{12})tr\Omega,
\]
where
\[
Z^\pm_{12} = (X^1_{12} - X^2_{12}) \pm (\tilde{X}^1 - \tilde{X}^2) = \alpha^\pm Y^\pm_{12} + 2\delta^\pm(a_1 - a_2),
\]
\[
V^\pm_{12} = (X^6_{12} + X^7_{12}) \pm (\tilde{X}^6 + \tilde{X}^7) = \beta^\pm Y^\pm_{12} + 2\delta^\pm(a_6 + a_7),
\]
and \( Y^\pm_{12} = P_{12} \pm \epsilon K^0_{12} \). Thus, we see that Lie-valued generators \( \Omega^- \) form the closed algebra
\[
\{\Omega^-_1, \Omega^-_2\} = [\Omega^-_1[\Omega^-_2, r_{12}]]_+ + Z^\pm_{12}(\Omega^-_1 + \Omega^-_2)Z^\pm_{12} + (V^\pm_{12}(\Omega_1 + \Omega_2) + (\tilde{\Omega}_1 + \tilde{\Omega}_2)V^\pm_{12})tr\Omega,
\]
also only if \( Z^{-12} = V^{-12} = 0 \) or \( \alpha^- = \beta^- = 0 \). Then the calculation of the Jacobi identity (14) gives:
\[
\{\{\Omega^-_1, \Omega^-_2\}, \Omega^-_3\} + (\text{cycle } 1, 2, 3) = -[\Omega^-_1, [\Omega^-_2, [\Omega^-_3, C(r)]]_+]_+,
\]
where
\[
C(r) = [r_{12}, r_{23} + r_{13}] + [r_{13}, r_{23}].
\]
If \( r_{12} \) is a quasitriangular \( r \)-matrix \( (C(r) \neq 0 \) is \( ad \)-invariant tensor), i.e., \( r_{12} \) is a solution of the mYBE (20), then the related bracket (18) is non Poisson (see (13)). Correspondingly, if \( r_{12} \) is a triangular \( r \)-matrix \( (C(r) = 0 \) in eq. (20)), then the bracket (18) is Poisson. These statements agree with the results of [14].

Before considering the general bracket (14) we recall how the exterior derivative \( d \) comes into this scheme. If we relate in the quantum case the co-invariant element \( X \) of Woronowicz with the quantum trace \( tr_q\Omega \) (the definition of \( q \)-trace see in [3], [21], [13], [4], [22]), then semiclassically it means that the ordinary exterior derivative \( d \) is expressed via the corresponding bicovariant bracket:
\[
d = \frac{1}{\kappa}\{tr\Omega, \ldots\},
\]
where \( \kappa \) is some numerical parameter depending on a bracket under consideration. The fulfillment of the nilpotency condition: \( d^2 = 0 \) is equivalent to the identity:
\[
\{\{\Omega, tr\Omega\}, tr\Omega\} = 0;
\]
and the Leibnitz rule is guaranteed by:
\[
\{\{\Omega_1, \Omega_2\}, tr\Omega\} + \{\{\Omega_1, tr\Omega\}, \Omega_2\} - \{\Omega_1, \{\Omega_2, tr\Omega\}\} = 0.
\]
Bracket (14) satisfying (22) and (23) will be referred as differential. If (14) satisfy the Jacobi identity (9) then (22) and (23) are fulfilled automatically. It was done by using the symbolic manipulation program REDUCE. The next step is to impose identity (23) proving the Leibnitz rule for the differential resulting differential bicovariant brackets are presented in Appendix.

Substitution of (26) in (22) gives four solutions for coefficients $\mu$:
- $\mu_1 = \mu$, $\mu_2 = \mu_3 = \mu_4 = \nu$, $\mu_5 = \mu_6 = 0$,
- $\mu_2 = -\mu_3 = -\mu_4 \neq 0$, $\mu_5 = \mu_6 = 0$;
- $\mu_5 = -\mu_6 = a_4 \neq 0$, $\mu_i = 0$ ($i = 1, \ldots, 4$);
- $\mu_i = 0$ ($i = 1, \ldots, 6$).

Thus, for bracket (24) (for (23) respectively) we have four possibilities

1) $\{\Omega, \text{tr}\Omega\} = \mu \Omega^2 + \nu (\tilde{\Omega}^2 + \tilde{\Omega} \Omega + \Omega \tilde{\Omega})$, (for all $\mu, \nu$ except $\mu = \nu = 0$),
2) $\{\Omega, \text{tr}\Omega\} = \mu (\Omega^2 + \tilde{\Omega}^2 - \tilde{\Omega} \Omega - \Omega \tilde{\Omega})$, ($\mu \neq 0$),
3) $\{\Omega, \text{tr}\Omega\} = \mu (\tilde{\Omega} - \Omega) \text{tr}\Omega$, ($\mu = a_4 \neq 0$),
4) $\{\Omega, \text{tr}\Omega\} = 0$.

The next step is to impose identity (23) proving the Leibnitz rule for the differential d (21). It was done by using the symbolic manipulation program REDUCE. The resulting differential bicovariant brackets are presented in Appendix.

Now substituting the calculated coefficients in (14) and analyzing the identity (4) with the help of the REDUCE program, we arrive at the conclusion that neither of the nontrivial differential brackets is Poisson. Thus, among the family (14) of bicovariant brackets there are differential brackets but no Poisson brackets. Note that we essentially use the requirement that $\Omega$'s lie in the algebras 1.) so($N$), sp(2$n$) or in 2.) gl($N$) = Mat($N$). In the first case we have an additional relation on the generators $\Omega^+ = 0$. We stress that if we consider some other relations (cubic relations or $\Omega_1 \Omega_2 K_{12}^0 = -K_{12}^0 \Omega_1 \Omega_2$), then, the Poisson structure can exist.

Now we analyze the external bicovariant algebra (9) on quantum groups $SO_q(N)$ and $Sp_q(N)$ directly in quantum case. For these $q$-groups the $R$-matrix satisfies the cubic characteristic equation (5):

$$ R = R^{-1} + \lambda - \lambda K, \quad K \equiv -\frac{1}{\lambda \nu} (R^2 - \lambda R - 1), $$

(28)
where \( \nu = \epsilon q^{-N} \), \( R = \hat{R}_{12} = P_{12}R_{12} \) and the matrix \( K = K_{12} = K^{i_1i_2}_{j_1j_2} = C^{i_1i_2}_{j_1j_2} \) is proportional to the singlet projector \( P^{(0)} \):

\[
P^{(0)} = \mu^{-1}K, \quad \mu = (1 + \epsilon[N - \epsilon]q).
\]  

(29)

Note that this time \( C \) is a quantum metric \([4]\). Below we also use the projectors:

\[
P^{(\pm)} = \frac{1}{q + q^{-1}}(\pm R + q^{\pm 1}I + \mu_{\pm}K), \quad \mu_{\pm} = -\frac{q^{\mp 1} \pm \nu}{\mu}.
\]  

(30)

It has been shown in \([8], [9]\) that for differential 1-forms one has the following relations coming from definition (3):

\[
X^{(\pm\pm)} = P^{(\pm)}\Omega' R \Omega' P^{(\pm)} = 0, \quad X^{(00)} = P^{(0)}\Omega' R \Omega' P^{(0)} = 0.
\]  

(31)

Here \( \Omega' = I \otimes \Omega = \Omega_2 \) and the signs of the wedge products are omitted. Taking the following sum

\[
q X^{(++)} + \frac{1}{q} X^{(--)} - \frac{q\mu_{+}^2 + q^{-1}\mu_{-}^2}{(q + q^{-1})^2} X^{(00)} = 0
\]  

(32)

and using the identities

\[
\frac{\mu_{+} + \mu_{-}}{q + q^{-1}} = -\frac{1}{\mu}, \quad \frac{q\mu_{+} - q^{-1}\mu_{-}}{q + q^{-1}} = -\frac{\nu}{\mu},
\]  

(33)

one can show that relations (31) are equivalent to the unique relation:

\[
(R \Omega' R \Omega' R + \Omega' R \Omega') - \frac{1}{\mu}(K \Omega' R \Omega' K + \Omega' R \Omega' K) - \frac{\nu}{\mu}(K \Omega' R \Omega' R + R \Omega' R \Omega' K) = 0.
\]  

(34)

This form of the defining relations for \( \Gamma^\wedge \) is suitable to produce a graded bicovariant bracket on \( \Gamma^\wedge_0 \).

The semiclassical expansions of projector \( P^{(0)}_{12} \) and \( R \)-matrix are:

\[
P^{(0)}_{12} = \hat{P}^{(0)}_{12} + \hbar \epsilon N K_{12} + O(\hbar^2), \quad R_{12} = P_{12} + \hbar P_{12}\tilde{r}_{12} + O(\hbar^2),
\]

where \( \hat{P}^{(0)}_{12} = \frac{1}{N}K_{12}^0 \) and \( \tilde{r} \) satisfies CYBE. It follows from \( KR = RK = \nu K \) that in the first order in \( \hbar \):

\[
K_{12}^1 - \epsilon K_{12}^1 P_{12} = K_{12}^0 \tilde{r}_{12} - \epsilon(1 - \epsilon N)K_{12}^0,
K_{12}^1 - \epsilon P_{12} K_{12}^1 = \tilde{r}_{21}K_{12}^0 - \epsilon(1 - \epsilon N)K_{12}^0.
\]  

(35)

Then, by expanding (34) in powers of \( \hbar \), taking into account (35) and the correspondence principle (4), we obtain:

\[
(I - \hat{P}^{(0)}_{12})(\{\Omega_1, \Omega_2\} + G_{12}) - (\{\Omega_1, \Omega_2\} + G_{12})\hat{P}^{(0)}_{12} = 0,
\]  

(36)
where
\[ G_{12} = -[\Omega_1, [\Omega_2, r_{12}]] + P_{12}(\Omega_1^2 + \Omega_2^2) - \epsilon (K_{12} \Omega_1 \Omega_2 + \Omega_1 \Omega_2 K_{12} + \Omega_1 K_{12} \Omega_2 + \Omega_2 K_{12} \Omega_1) \] (37)
and we made use of the quasitriangular \( r \)-matrix: \( r = \tilde{r} - (P - \epsilon K) \).

The components \( \tilde{P}_{12}(0) \{ \Omega_1, \Omega_2 \} \) are not defined by (36). Thus, we see that relations (31) are unsufficient to generate, in the limit \( \hbar \to 0 \), a genuine bicovariant bracket. In the quantum case it means that the number of defining relations (31) is not enough to reorder lexicographically arbitrary monom in \( \Omega \)'s. Therefore, if we confine ourselves only with (31), then we can not conclude that \( \dim \Gamma^\wedge \) is equal to \( \dim \Gamma^\wedge_{\Omega} \).

On the other hand, we can not assume the solution of (36) as
\[ \{\Omega_1, \Omega_2\} = -G_{12} \] (38)
since \( G_{12} \) is symmetric under \( 1 \leftrightarrow 2 \) only if the following relation hold:
\[ K_{12} \Omega_1 \Omega_2 + \Omega_1 \Omega_2 K_{12} = 0 \] (39)
But this relation contradicts to the requirement that \( \Omega \in \mathcal{G} \) or that the number of \( \Omega \)s are \( N^2 \) (\( \Omega \in \text{Mat}(N) \)). Note, however, that the bracket (38) is Poisson for \( \Omega \)'s restricted by constraint (39).

To improve the situation the authors of [8], in addition to (31), have assumed the relations:
\[ X^{(0+)} = P^{(0)} \Omega' R \Omega' P^{(+)} = 0, \quad X^{(+0)} = P^{(+)} \Omega' R \Omega' P^{(0)} = 0. \] (40)
One can obtain without problems that (34) and (40) are equivalent to the relation:
\[ R \Omega' R \Omega' R + \Omega' R \Omega' + \frac{1}{\mu} (\nu q^{-1} - 1) (K \Omega' R \Omega' + \Omega' R \Omega' K) = 0. \] (41)
By expanding (41) in \( \hbar \), as it was done for the general relation (34), we get the following bicovariant bracket:
\[ \{\Omega_1, \Omega_2\} = \{\Omega_1, [\Omega_2, r_{12}]\} + P_{12}(\Omega_1^2 + \Omega_2^2) + (\Omega_1 K_{12} \Omega_2 + \Omega_2 K_{12} \Omega_1). \] (42)
being a particular case of (14). Now we see that according to our classification this bracket neither Poisson nor differential. It means in the quantum case that the requirement \( d^2 = 0 \) (22) implies some additional cubic relations on generators \( \Omega^j \), which were not assumed from the beginning. The situation is somewhat improved when we require \( d^2 = 0 \) only on "physical" components \( \Omega = \Omega^- \). This requirement is consistent with (23), since \( \{\Omega^-, \text{tr} \Omega\} = -2(\Omega^-)^2 \). However, the substitution \( \Omega \to \Omega^- \) in (23) leads to the conclusion:
\[ \{\{\Omega^-, \Omega^-, \text{tr} \Omega\} \} \neq 0. \]
Thus, one can not assume the Leibnitz rule for \( d \) on the "physical" subalgebra generated by \( \Omega^- \) without imposing new cubic relations on \( \Omega \)'s. Note that if we impose the unacceptable relations (39), then, the bracket (42) coincides with (38) and, therefore, is Poisson.
Seemingly, the absence of bicovariant Poisson structure for $SO(N)$ and $Sp(N)$ ($N$ is generic) reflects the fact that we cannot confine ourselves by considering only $G$-invariant tensors $W$ in (1). Considering in (1) tensor $W$ which is not $G$-invariant, we disturb, of course, bicovariance but may hope to keep the Jacobi identity. Then we expect that the bicovariance will be restored on the surface $\Omega^+ = 0$ if we treat $\Omega^+ = 0$ as the first order constraint (in the Dirac sense).

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**APPENDIX**

A differential $SO$- and $Sp$- covariant brackets on $Mat(N)$.

1. First solution: $\{\Omega, \text{tr}\Omega\} = \mu \Omega^2 + \nu(\tilde{\Omega}^2 + \tilde{\Omega}\Omega + \Omega\tilde{\Omega})$
   i) $\{\Omega_1, \Omega_2\} = [\Omega_1, [\Omega_2, r_{12}]] + \frac{1}{N}(2b_2 + \epsilon c_3 + Na_2)(\Omega_1^2 + \Omega_2^2)$
      
      
      $+ c_3 \Omega_1^+ K_{12} \Omega_2^+ \mu P_{12}(\Omega_1 + \Omega_2)$
      
      $+ b_2 P_{12}(\Omega_1^+)^2 + (\Omega_2^+)^2) + c_6(K_{12} \Omega_1^+ - \Omega_1^+ K_{12})\text{tr}\Omega;$
   
   ii) $\{\Omega_1, \Omega_2\} = [\Omega_1, [\Omega_2, r_{12}]] + \frac{1}{N}(2b_2 + \epsilon c_3 + Na_2)(\Omega_1^2 + \Omega_2^2)$
      
      $+ a_2((\Omega_1^+)^2 + (\Omega_2^+)^2) + c_3 \Omega_1^+ K_{12} \Omega_2^+ (2b_1 + \nu) P_{12}((\Omega_1^+)^2 + (\Omega_2^+)^2)$
      
      $- (b_1 + \nu)(P_{12}(\Omega_1^2 + \Omega_2^2) + (\tilde{\Omega}_1^2 + \tilde{\Omega}_2^2) P_{12}) - \frac{1}{2} \epsilon \nu(K_{12} \Omega_1^2 + \Omega_2^2 + (\Omega_1^2 + \Omega_2^2) K_{12}) +$
      
      $c_6(K_{12} \Omega_1^+ - \Omega_1^+ K_{12})\text{tr}\Omega;$
   
   iii) $\{\Omega_1, \Omega_2\} = [\Omega_1, [\Omega_2, r_{12}]] + \frac{1}{N}(2b_2 + \epsilon c_3 - \nu)((\Omega_1^+)^2 + (\Omega_2^+)^2)$
      
      $+ c_3 \Omega_1^+ K_{12} \Omega_2^+ + b_1 P_{12}((\Omega_1^+)^2 + (\Omega_2^+)^2) +$
\[(a_6(\tilde{\Omega}_1 + \tilde{\Omega}_2) - 3a_6(\Omega_1 + \Omega_2)) - \frac{1}{2}\epsilon(c_6 + c_7)P_{12}(\Omega_1^+ + \Omega_2^+) + c_6K_{12}\Omega_1^+ + c_7\Omega_1^+K_{12})\text{tr}\Omega;\]

2. Second solution: \(\{\Omega, \text{tr}\Omega\} = \mu(\Omega^2 + \tilde{\Omega}^2 - \tilde{\Omega}\Omega - \Omega\tilde{\Omega})\).

\[\{\Omega_1, \Omega_2\} = [\Omega_1, [\Omega_2, r_{12}]]_+ + a_1((\Omega_1^-)^2 + (\Omega_2^-)^2) + c_3\Omega_1^-K_{12}\Omega_2^- - \frac{1}{2}b_3P_{12}(\Omega_1^+\Omega_1^- + \Omega_2^+\Omega_2^-) + \]

\[(-\epsilon c_1P_{12} + c_1K_{12})(\Omega_1^2 + \Omega_2^2) + (\tilde{\Omega}_1^2 + \tilde{\Omega}_2^2)(\epsilon c_1P_{12} + c_2K_{12})\]

\[((a_6 + \epsilon c_0P_{12} + \epsilon c_0K_{12})\Omega_1^+ + \Omega_1^+(a_6 + \epsilon c_0P_{12} + \epsilon c_0K_{12}))\text{tr}\Omega,\]

where \(b_3 = -\epsilon(c_1 + c_2)\).

3. Third solution: \(\{\Omega, \text{tr}\Omega\} = \mu(\tilde{\Omega} - \Omega)\text{tr}\Omega\).

\[\{\Omega_1, \Omega_2\} = [\Omega_1, [\Omega_2, r_{12}]]_+ + (-a_3 - b_4 + c_1K_{12})(\Omega_1^2 + \Omega_2^2) + (\tilde{\Omega}_1^2 + \tilde{\Omega}_2^2)(a_3 + b_4 + c_1K_{12})\]

\[(a_3 + b_4P_{12})(\Omega_1^+\Omega_1^- + \Omega_2^+\Omega_2^-) + (\mu + b_3P_{12})(\tilde{\Omega}_1\Omega_1 + \tilde{\Omega}_1\Omega_2) +\]

\[(X_{12}^{(6)}(\tilde{\Omega}_1 + \tilde{\Omega}_2) + (\Omega_1 + \Omega_2)X_{12}^{(7)})\text{tr}\Omega,\]

where \(b_3 = -(Na_3 + 2b_4)\) and coefficients in \(X_{12}^{(6)}\) and in \(X_{12}^{(7)}\) remain to be arbitrary.