Matrix factorizations, Reality and Knörrer periodicity

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Abstract
Motivated by periodicity theorems for Real $K$-theory and Grothendieck–Witt theory and, separately, work of Hori and Walcher on the physics of Landau–Ginzburg orientifolds, we introduce and study categories of Real matrix factorizations. Our main results are generalizations of Knörrer periodicity to categories of Real matrix factorizations. These generalizations are structurally similar to (1,1)-periodicity for $KR$-theory and 4-periodicity for Grothendieck–Witt theory. We use techniques from Real categorical representation theory, which allow us to incorporate into our main results equivariance for a finite group and discrete torsion twists.

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1 | INTRODUCTION

The main results of this paper are generalizations of Knörrer periodicity to categories of matrix factorizations which are Real in the sense of Atiyah’s Real $K$-theory or symmetric in the sense of its algebraic counterpart, Grothendieck–Witt theory.

1.1 | Background and motivation

Let $R = \mathbb{C}[x_1, \ldots, x_n]$ and $w \in R$ a non-zero polynomial without a constant term. A matrix factorization of $w$ is a $\mathbb{Z}/2\mathbb{Z}$-graded finite rank-free $R$-module $M$ with an odd $R$-linear endomorphism $d_M$ which satisfies $d_M^2 = w \cdot \text{id}_M$. The 2-periodic differential graded (dg) category of matrix factorizations $MF(R, w)$ and its triangulated homotopy category $HMF(R, w)$ are categorical invariants of the singularity $(R, w)$ introduced by Eisenbud to study the homological algebra of $R/(w)$-modules [22]. Much recent work on matrix factorizations stems from their interpretation as $D$-branes in affine Landau–Ginzburg $B$-models [30, 31, 49] and their fundamental role in the theory of a categorified knot invariants [13, 33, 34].

A fundamental property of matrix factorization categories is Knörrer periodicity.

**Theorem 1.1** [36]. There is a quasi-equivalence of dg categories

$$MF(R, w) \sim MF(R[[y, z]], w + y^2 + z^2).$$

Knörrer periodicity plays an important role in the classification of hypersurface rings of finite maximal Cohen–Macaulay type [36, 38] and represents a basic quantum symmetry of Landau–Ginzburg models [26]. Knörrer periodicity has been generalized in a number of directions. Versions of Knörrer periodicity for global matrix factorizations were proved in [40, 46] and used to study derived categories of projective hypersurfaces [6, 41] and to prove instances of homological projective duality [5]. Inspired by Homological Mirror Symmetry, an $A$-model counterpart of global Knörrer periodicity was proved for Fukaya–Seidel categories of singular hypersurfaces [29]. A different generalization is to matrix factorizations which are equivariant for a group of $\mathbb{C}$-algebra automorphisms of $R$ which preserve $w$. Hirano proved a global version of equivariant Knörrer periodicity and used this to study equivariant-derived categories of projective hypersurfaces [27]. Finally, an 8-periodic version of Knörrer periodicity for matrix factorizations over $\mathbb{R}$ was proved by Brown using derived Morita theory [8].

1.2 | Main results

Recall that the fundamental geometric objects in $KR$-theory are Real vector bundles, that is, complex vector bundles $M$ over a topological space with involution $(X, \sigma)$ with a lift of $\sigma$ to $M$ which is fiberwise $\mathbb{C}$-antilinear [3]. In Grothendieck–Witt theory, the algebraic counterpart of $KR$-theory, one instead lifts $\sigma$ to a $\mathbb{C}$-linear isomorphism $M \to \sigma^* M^\vee$ which is symmetric or skew-symmetric [35]. According to [19, section 7], when $w$ is an isolated hypersurface singularity at the origin, $MF(R, w)$ can be regarded as the dg category of perfect complexes on a hypothetical
non-commutative space $\mathcal{X}$ which is dg affine, homologically smooth and proper. Correspondingly, we define two versions of Real matrix factorizations, one antilinear and one contravariant, which can be regarded as the geometric objects of the $KR$-theory and Grothendieck–Witt theory of $\mathcal{X}$. The main results of this paper are Real equivariant generalizations of Knörrer periodicity for both versions of Real matrix factorizations.

To state our results more precisely, let $C_2$ be the multiplicative group $\{1, -1\}$ and $\pi : \hat{G} \to C_2$ a $C_2$-graded finite group. In the antilinear setting, let $\hat{G}$ act on $R$ by ring automorphisms $\sigma : R \to R$, $\sigma \in \hat{G}$, which are $\mathbb{C}$-linear if $\pi(\sigma) = 1$ and $\mathbb{C}$-antilinear if $\pi(\sigma) = -1$ and let $w$ be an isolated hypersurface singularity at the origin which is $\hat{G}$-invariant:

$$\sigma(w) = w, \quad \sigma \in \hat{G}. \quad (1)$$

We stress that while the group $G = \ker \pi$ is a symmetry of $(R, w)$, in the sense of equivariant matrix factorizations discussed above, $\hat{G}$ itself is not because of its antilinear action on $R$. A Real $G$-equivariant structure on a matrix factorization $(M, d_M)$ is the data of coherent isomorphisms $u_{\sigma} : M \to M$, $\sigma \in \hat{G}$, of graded abelian groups which commute with $d_M$ and satisfy $u_{\sigma}(rm) = \sigma^{-1}(r)u_{\sigma}(m)$ for all $r \in R$, $m \in M$. Let $MF_{\hat{G}}(R, w)$ be the dg category of Real $G$-equivariant matrix factorizations. To state the first form of Real Knörrer periodicity, extend the $\mathbb{C}$-antilinear action of $\hat{G}$ on $R$ to $R[y, z]$ by

$$\sigma(y) = \pi(\sigma)y, \quad \sigma(z) = z, \quad \sigma \in \hat{G}. \quad (2)$$

Denote by $\text{Perf}(C)$ the triangulated hull of a dg category $C$.

**Theorem 1.2** (Theorem 3.9). There is a quasi-equivalence of $\mathbb{R}$-linear dg categories

$$\text{Perf}(MF_{\hat{G}}(R, w)) \cong \text{Perf}(MF_{\hat{G}}(R[y, z], w + y^2 + z^2)).$$

Consider now the contravariant setting, where our approach is motivated by the work of Hori and Walcher on the physics of Landau–Ginzburg orientifolds [28]. Let a $C_2$-graded finite group $\hat{G}$ act on $R$ by $\mathbb{C}$-algebra automorphisms such that the isolated hypersurface singularity $w$ is $\pi$-semi-invariant:

$$\sigma(w) = \pi(\sigma)w, \quad \sigma \in \hat{G}. \quad (3)$$

Again, only $G = \ker \pi$ is a symmetry of $(R, w)$. From these data, we construct a duality structure on the dg category $MF_{\hat{G}}(R, w)$ of $G$-equivariant matrix factorizations, that is, a dg functor $MF_{\hat{G}}(R, w)^{op} \to MF_{\hat{G}}(R, w)$ with coherence data asserting that it is an involution. Extend the $\mathbb{C}$-linear action of $\hat{G}$ on $R$ to $R[y, z]$ by requiring $G$ to act trivially and

$$\sigma(y) = -iz, \quad \sigma(z) = iy, \quad \sigma \in \hat{G} \setminus G. \quad (4)$$

The second form of Real Knörrer periodicity is as follows.
Theorem 1.3 (Corollaries 5.13 and 5.14). There is a quasi-equivalence of $\mathbb{C}$-linear dg categories with duality

$$\text{Perf}(\text{MF}_G(R, w)) \sim \text{Perf}(\text{MF}_G(R[y, z], w + y^2 + z^2)),$$

where the duality structure of the codomain is a shifted and signed version of that of the domain. In particular, there is a quasi-equivalence of dg categories with duality

$$\text{Perf}(\text{MF}_G(R, w)) \sim \text{Perf}(\text{MF}_G(R[y_1, z_1, y_2, z_2], w + y_1^2 + z_1^2 + y_2^2 + z_2^2)),$$

where both dg categories are given the same duality structure.

To put Theorems 1.2 and 1.3 in context, we recall standard periodicities for $KR$-theory and Grothendieck–Witt theory. Atiyah’s (1,1)-periodicity theorem states that for a compact Hausdorff space $X$ with $C_2$-action, there is an isomorphism

$$([H] - 1) : KR^{*,*}(X) \sim KR^{*+1,*+1}(X),$$

where $[H] - 1 \in \widetilde{KR}^0(\mathbb{C}P^1) \simeq KR^{1,1}(pt)$ is the reduced Hopf bundle with $C_2$-action given by complex conjugation [3]. As $C_2$-spaces, $\mathbb{C}P^1 \simeq D^{1,1}/S^{1,1}$, where $D^{1,1}$ is the closed unit ball in $\mathbb{R} \oplus \mathbb{R}$ with involution $(y, z) \mapsto (-y, z)$ and $S^{1,1}$ is its boundary. That the eigenspaces of the ambient involution with eigenvalue $+1$ and $-1$ each have dimension 1 is responsible for the (1,1)-terminology. Atiyah’s (1,1)-periodicity unifies many $K$-theoretic periodicities, including Bott’s 2-periodicity for complex $K$-theory and 8-periodicity for $KO$-theory. In the algebraic setting, associated to a complex affine variety, $X$ are its higher Grothendieck–Witt groups $GW^*(X)$. These groups are 4-periodic,

$$GW^*(X) \sim GW^{*+4}(X)$$

and enjoy natural isomorphisms $GW^0(X) \simeq GW^+(X)$ and $GW^2(X) \simeq GW^-(X)$, where $GW^\pm(X)$ is the Grothendieck–Witt group of vector bundles with orthogonal (+) or symplectic (−) forms.

We view Theorems 1.2 and 1.3 as structural analogues of the (1,1)-periodicity and 4-periodicity theorems, respectively, for matrix factorizations. The (1,1)-nature of Theorem 1.2 is apparent from the form of the extension of the $\hat{G}$-action from $R$ to $R[y, z]$; see also the connection with periodicity of categories of Clifford modules below. As for Theorem 1.3, because of the appearance of shifted and signed duality structures, the quasi-equivalence (5) relates bilinear forms of parity $\epsilon \in C_2$ on matrix factorizations of $w$ to bilinear forms of parity $-\epsilon$ on matrix factorizations of $w + y^2 + z^2$, and so reflects a shift by two in Grothendieck–Witt theory. The quasi-equivalence (6) then relates bilinear forms of the same parity on matrix factorizations of $w$ and $w + y_1^2 + z_1^2 + y_2^2 + z_2^2$, reflecting the 4-periodicity of Grothendieck–Witt theory.

Theorems 1.2 and 1.3 recover a number of known periodicities. Consider first the degenerate case $\hat{G} = C_2$, so that $G$ is trivial. When $C_2$ acts on $R$ by complex conjugation, we recover from Theorem 1.2 the 8-periodicity of matrix factorizations of non-degenerate quadratic forms over $\mathbb{R}$, as proved by Brown [8]; see Corollary 4.4. If instead $C_2$ acts on $R$ by a $\mathbb{C}$-algebra involution which negates $w$, the second part of Theorem 1.3 is a precise version of the extended Knörrer periodicity.
discovered by Hori and Walcher [28]. In the degenerate case of trivial $C_2$-grading, so that $\hat{G} = G$, Theorems 1.2 and 1.3 give an elementary proof of equivariant Knörrer periodicity for affine Landau–Ginzburg models, a special case of Hirano’s result [27]; see Corollary 3.11. In Section 4, motivated by work Buchweitz, Eisenbud and Herzog [12], we prove in the antilinear setting that the category of Real matrix factorizations of a non-degenerate quadratic form is equivalent to a category of Real graded Clifford modules; see Theorem 4.3. We use this result in Section 4.3 to relate Theorem 1.2 to classical periodicities of Clifford modules, including their (1,1)-periodicity [1, 37], thereby giving a precise sense in which Theorem 1.2 is a (1,1)-periodicity theorem.

A second, equally important, context for our results is Landau–Ginzburg orientifolds. Orientifolding is a physical construction which produces an unoriented string theory from an oriented one, in contrast to the standard orbifold construction, which preserves orientability. The mathematics of orientifolds is underdeveloped in comparison to orbifolds. The results of this paper are concrete mathematical consequences of the orientifold construction. In the physics literature, it is well-appreciated that orientifold data in string theory induce on the category of $D$-branes a contravariant involution [17, 28]. Hori and Walcher identify these involutions for Landau–Ginzburg models and, among other things, use them to formulate extended Knörrer periodicity. See [11] for physical applications of this periodicity. A mathematical approach to some of the ideas of Hori and Walcher was suggested by Bertin and Rosay [7] but, as far as the authors are aware, was not pursued. Our approach to Theorem 1.3, and therefore the mathematics of Landau–Ginzburg orientifolds, is different, combining techniques from categorical representation theory, equivariant Grothendieck–Witt theory and the original physical work of Hori and Walcher. While Hori and Walcher and Bertin and Rosay focus on the degenerate case $\hat{G} = C_2$, we work with general $C_2$-graded finite groups $\hat{G}$ and their discrete torsion twists. In this way, we obtain complete results for general affine Landau–Ginzburg orientifolds.

Our results suggest two natural $K$-theoretic problems. The first is to relate Theorem 1.2 to the (1,1)-periodicity of the $KR$-theory of the Milnor fiber of $w$. For matrix factorizations over $\mathbb{C}$ and $\mathbb{R}$, Brown found such a relation for $K$-theory and $KO$-theory, respectively, under mild assumptions on $w$ [8]. The second is to deduce from Theorem 1.3 periodicity for Grothendieck–Witt groups of $G$-equivariant matrix factorizations, as envisioned by Hori and Walcher. At present, the missing ingredient is a 2-periodic generalization of Schlichting’s theorem on the invariance of Grothendieck–Witt groups under dg form equivalence [45].

1.3 Strategy of proof

We work in the framework of Real 2-representation theory, as developed by the second author [44, 51], in which a $C_2$-graded finite group $\hat{G}$ acts coherently on a $\mathbb{C}$-linear category $C$ by functors $\rho(\sigma) : C \to C$, $\sigma \in \hat{G}$, which are linear and covariant if $\pi(\sigma) = 1$ and, depending on the setting, antilinear and covariant or linear and contravariant if $\pi(\sigma) = -1$. In the context of matrix factorizations, a (possibly antilinear) action of $\hat{G}$ on $R$ for which the potential satisfies condition (1) or (3) defines a Real 2-representation on $\text{MF}(R, w)$; see Lemmas 3.2 and 5.6.

The quasi-equivalence of Theorem 1.1 is the dg functor given by tensoring with the matrix factorization

$$K = \left( \begin{array}{c|c} C[[y, z]] & y + iz \\ \hline \end{array} \right) \in \text{MF}(C[[y, z]], y^2 + z^2).$$
Our strategy is to equip $K$, and hence Knörrer’s quasi-equivalence by Proposition 2.11, with a Real $G$-equivariant structure and then deduce, upon taking Real $G$-equivariant objects, Theorems 1.2 and 1.3. The details of precisely how this is carried out differ depending on the setting. In both settings, however, a key point is that $K$ admits a Real $G$-equivariant structure only for particular $\hat{G}$-actions on $\mathbb{C}[[y, z]]$, namely those given by Equations (2) and (4); see Propositions 3.8 and 5.11. This effectively determines the form of our periodicities. There are additional technical difficulties in the contravariant setting. This leads us to prove two general results of independent interest about Real 2-representations on dg categories (Theorems 5.3 and 5.5) which assert that one may take (Real) equivariant objects of Real 2-representations in stages and that this process is natural.

Without any assumptions on the potential $w$, the triangulated category $\text{HMF}_G(R, w)$ of $G$-equivariant matrix factorizations, and its Real generalizations, need not be idempotent complete. For this reason, our techniques lead to statements about $\text{Perf}(\text{MF}_G(R, w))$, whose homotopy category is idempotent complete, instead of $\text{MF}_G(R, w)$ itself. The assumption that $w$ be an isolated hypersurface singularity, the case of primary interest for Landau–Ginzburg models, is to ensure idempotent completeness of $\text{HMF}(R, w)$ [19], which allows us to use powerful equivariant results of Elagin [24]. We show in Proposition 3.6 that $\text{Perf}(\text{MF}_G(R, w))$ is a dg enhancement of the idempotent completion of $\text{HMF}(R, w)^G$. Idempotent completeness is crucial to many applications of matrix factorizations [14, 21] and we expect the same to be true equivariantly.

We expect the methods developed in this paper to lead to further mathematical applications of Landau–Ginzburg orientifolds. To mention one, recall that for $w$ an isolated hypersurface singularity at the origin, the dg category $\text{MF}(R, w)$ is Calabi–Yau and so defines a 2-dimensional non-semisimple oriented extended topological field theory [14, 20]. We expect that contravariant Real matrix factorizations are the correct framework to lift these theories to unoriented topological field theories. This would confirm the physical predictions of [28].

Note added

After this paper appeared, Brown and Walker proved that $\text{HMF}_G(R, w)$ is idempotent complete when $w$ is an isolated hypersurface singularity [9]. As discussed above, this, together with the results of this paper, leads to Real Knörrer periodicities for the categories $\text{MF}_G(R, w)$ themselves, as opposed to their perfect dg categories.

2  |  PRELIMINARY MATERIAL

Throughout the paper, $k$ denotes a ground field with $\text{char } k \neq 2$.

2.1  |  2-periodic dg categories

We recall standard material on dg categories, following [32] in the $\mathbb{Z}$-graded setting and [19, section 5] for 2-periodic modifications. All categories are assumed to be small by choosing small quasi-equivalent dg categories.

Let $k[u, u^{-1}]$ be the algebra of Laurent polynomials, viewed as a differential $\mathbb{Z}$-graded (dg) algebra with $u$ in degree 2 and trivial differential. Let $\text{Com}(k)$ be the dg category of complexes of $k$-modules. Let $\text{Com}(k[u, u^{-1}])$ be the symmetric monoidal dg category of dg functors $k[u, u^{-1}] \to \text{Com}(k)$, where $k[u, u^{-1}]$ is viewed as a dg category with a single object.
Definition 2.1. A 2-periodic dg category is a category enriched over \( \text{Com}(k[u, u^{-1}]) \).

We often cite results about differential \( \mathbb{Z} \)-graded categories and apply them in the 2-periodic setting if their proofs are effectively unchanged. Since we consider only 2-periodic dg categories in what follows, we abbreviate ‘2-periodic dg’ to ‘dg’.

Given a dg category \( \mathcal{C} \), let \( Z^0(\mathcal{C}) \) and \( H^0(\mathcal{C}) \) be the categories with the same objects as \( \mathcal{C} \) and \( \text{Hom}_{Z^0}(\mathcal{C}, \mathcal{C}) = Z^0(\text{Hom}(\mathcal{C}, \mathcal{C})) \) and \( \text{Hom}_{H^0}(\mathcal{C}, \mathcal{C}) = H^0(\text{Hom}(\mathcal{C}, \mathcal{C})) \).

**Definition 2.2.** A dg natural transformation \( \alpha : F \Rightarrow G \) of dg functors \( F, G : \mathcal{C} \to \mathcal{D} \) is a family of morphisms \( \{ \alpha_C \in \text{Hom}_{\mathcal{D}}(F(C), G(C)) \} \) such that \( \alpha_{C_2} \circ F(f) = G(f) \circ \alpha_{C_1} \) for each morphism \( f : C_1 \to C_2 \). A dg natural transformation whose components are dg isomorphisms, that is, closed degree 0 isomorphisms, is called a dg natural isomorphism.

We often refer to dg natural transformations simply as natural transformations.

Let \( \mathcal{C} \) be a dg category. Its opposite dg category \( \mathcal{C}^{\text{op}} \) has the same objects as \( \mathcal{C} \), morphism complexes \( \text{Hom}_{\mathcal{C}^{\text{op}}}(C_1, C_2) = \text{Hom}_{\mathcal{C}}(C_2, C_1) \) and composition \( f^{\text{op}} \circ g^{\text{op}} = (-1)^{|f||g|}(g \circ f)^{\text{op}} \). Let \( \mathcal{C}^{\text{op}}\text{-mod} \) be the dg category of dg functors \( \mathcal{C}^{\text{op}} \to \text{Com}(k[u, u^{-1}]) \). The Yoneda dg functor is \( \mathcal{C} \to \mathcal{C}^{\text{op}}\text{-mod}, C \mapsto \text{Hom}_C(\_ , C) \). The pretriangulated hull of \( \mathcal{C} \), denoted \( \mathcal{C}^{\text{ptr}} \), is the smallest dg subcategory of \( \mathcal{C}^{\text{op}}\text{-mod} \) which contains the Yoneda image and is closed under shifts and cones of closed morphisms. The triangulated hull of \( \mathcal{C} \), denoted \( \text{Perf}(\mathcal{C}) \), is the full dg subcategory of compact objects of the dg-derived category of \( \mathcal{C}^{\text{op}}\text{-mod} \).

**Definition 2.3.** A dg category \( \mathcal{C} \) is called pretriangulated (resp. triangulated) if the Yoneda dg functor \( \mathcal{C} \to \mathcal{C}^{\text{ptr}} \) (resp. \( \mathcal{C} \to \text{Perf}(\mathcal{C}) \)) is a quasi-equivalence.

Let \( F : C \to D \) be a dg functor with restriction dg functor \( F^* : D^{\text{op}}\text{-mod} \to \mathcal{C}^{\text{op}}\text{-mod} \). The derived left adjoint of \( F^* \) restricts to a dg functor \( \text{Ind} F : \text{Perf}(\mathcal{C}) \to \text{Perf}(D) \). See [18, section C] for properties of \( \text{Ind} F \).

2.2 Matrix factorizations

We recall background material on matrix factorizations, following [19, 42].

Let \( R = k[x_1, \ldots, x_n] \) with maximal ideal \( m \). A non-zero polynomial \( w \in m \) is called a potential. Further restrictions on the potential are common in the literature, depending on the applications in mind. See, for example, [19, section 3]. One particularly important class of potentials, especially with regard to the physics of affine Landau–Ginzburg models [31, 49], are the isolated hypersurface singularities.

**Definition 2.4.**

(i) The dg category of matrix factorizations \( \text{MF}(R, w) \) has

- objects **matrix factorizations** \( M = (M, d_M) \), which consist of a finite rank-free \( \mathbb{Z}/2\mathbb{Z} \)-graded \( R \)-module \( M \) and an odd \( R \)-linear map \( d_M : M \to M \), the twisted differential, which satisfies \( d_M^2 = w \cdot \text{id}_M \), and
- morphism complexes the \( \mathbb{Z}/2\mathbb{Z} \)-graded \( R \)-modules \( \text{Hom}_{\text{MF}}(M, N) = \text{Hom}_R(M, N) \) with differential \( D \) defined on homogeneous elements by \( D(f) = d_N \circ f - (-1)^{|f|} f \circ d_M \).
(ii) The homotopy category of matrix factorizations is $\text{HMF}(R, w) = H^0(\text{MF}(R, w))$.

Let $M \in \text{MF}(R, w)$. After fixing a homogeneous $R$-module decomposition $M = M_0 \oplus M_1$, we can write $d_M = \begin{pmatrix} 0 & d'_M \\ d_M & 0 \end{pmatrix}$ and depict $M$ as

$$M = \begin{pmatrix} M_0 & d'_M \\ d_M & M_1 \end{pmatrix}.$$ 

The shift dg functor $\Sigma : \text{MF}(R, w) \to \text{MF}(R, w)$ is defined on objects and morphisms by $\Sigma(M_0 \oplus M_1, d_M) = (M_1 \oplus M_0, -d_M)$ and $\Sigma f = (-1)^{|f|} f$. Note that $\Sigma \circ \Sigma = \text{id}_{\text{MF}(R, w)}$.

Let $w \in R$ and $w' \in R'$ be potentials. Given $M \in \text{MF}(R, w)$ and $N \in \text{MF}(R', w')$, the external tensor product $M \boxtimes N \in \text{MF}(R \otimes_k R', w \otimes 1 + 1 \otimes w')$ is

$$M \boxtimes N = \begin{pmatrix} M_0 \otimes N_0 \oplus M_1 \otimes N_1 & d_M \otimes \text{id}_{N_0} & -\text{id}_{M_0} \otimes d'_N \\ \text{id}_{M_0} \otimes d_N & d_M \otimes \text{id}_{N_1} & -\text{id}_{M_0} \otimes d'_N \end{pmatrix} \rightarrow M_1 \otimes N_0 \oplus M_0 \otimes N_1.$$ 

External tensor product extends to a dg functor

$$\boxtimes : \text{MF}(R, w) \otimes_k \text{MF}(R', w') \to \text{MF}(R \otimes_k R', w \otimes 1 + 1 \otimes w').$$ 

Under the swap isomorphism $R \otimes_k R' \simeq R' \otimes_k R$, there is a natural isomorphism

$$M \boxtimes N \simeq N \boxtimes M, \quad m \otimes n \mapsto (-1)^{|m||n|} n \otimes m$$

in $\text{MF}(R \otimes_k R', w \otimes 1 + 1 \otimes w')$. The shift functor and tensor product satisfy

$$\Sigma M \boxtimes N \simeq M \boxtimes \Sigma N \simeq \Sigma(M \boxtimes N).$$

See [50] for a detailed discussion of tensor products of matrix factorizations.

The category $\text{HMF}(R, w)$ admits a triangulated structure with shift functor induced by $\Sigma$ [39, Proposition 3.3]. If, moreover, $w$ is an isolated hypersurface singularity at the origin, then the dg category $\text{MF}(R, w)$ is triangulated [19, Lemma 5.6], so that $\text{HMF}(R, w)$ is idempotent complete.

A factorization $w = ab$ defines a rank 1 matrix factorization

$$\{a, b\} = \left( \begin{array}{c} a \\ b \end{array} \right) \in \text{MF}(R, w).$$

Particularly important to this paper are the cases $w = uv \in k[[u, v]]$ with associated matrix factorization $\{u, v\}$ and, if $k$ is algebraically closed, $w = y^2 + z^2$ with associated matrix factorization $\{y + iz, y - iz\}$. Knörrer periodicity can now be stated as follows.
Theorem 2.5 [36, Theorem 3.1]. Let $k$ be algebraically closed with char $k \neq 2$ and $w \in R = k[[x_1, \ldots, x_n]]$ a potential. Let $K = \{ y + iz, y - iz \}$. Then, the dg functor
\[
\mathcal{K} = - \boxtimes K : \text{MF}(R, w) \to \text{MF}(R[[y, z]], w + y^2 + z^2)
\]
is a quasi-equivalence.

2.3 Real 2-representation theory

In this section, we establish the representation theoretic framework of the paper, following [51].

Denote by $C_2$ the multiplicative group $\{ \pm 1 \}$. A $C_2$-graded group is a group homomorphism $\pi : \hat{G} \to C_2$. A morphism of $C_2$-graded groups is a group homomorphism which commutes with the $C_2$-gradings. The ungraded group of $\hat{G}$ is $G = \ker \pi$. The terminal $C_2$-graded group is the identity map $C_2 \to C_2$, often denoted simply by $C_2$.

Let $\hat{G}$ be a $C_2$-graded group with ungraded group $G$. We use two distinct forms of the Real 2-representation theory of $G$ (with respect to the fixed $\hat{G}$), an antilinear approach in Sections 3 and 4 and a contravariant approach in Section 5. The relevant form will always be clear from the context, so we do not distinguish terminology. When the $C_2$-grading of $\hat{G}$ is trivial, so that $G = \hat{G}$, both forms reduce to earlier notions of categorical representations [16, 24, 48]. We formulate definitions for dg categories, although we sometimes apply them to linear categories, viewed as dg categories concentrated in degree 0.

Given a group $G$, denote by $BG$ the locally discrete 2-category with a single object $\star$ and 1-morphisms $1\text{Hom}(\star, \star) = G$. A group homomorphism $f : G \to H$ induces a 2-functor $Bf : BG \to BH$.

2.3.1 The antilinear approach

Let $C$, $D$ be $\mathbb{C}$-linear dg categories. An $\mathbb{R}$-linear dg functor $F : C \to D$ is called antilinear if its maps on morphism complexes are $\mathbb{C}$-antilinear. Let $\text{dgCat}_{\mathbb{C}/\mathbb{R}}$ be the 2-category whose objects are $\mathbb{C}$-linear dg categories, 1-morphisms are dg functors which are linear or antilinear and 2-morphisms are dg natural transformations. There is a 2-functor $\text{dgCat}_{\mathbb{C}/\mathbb{R}} \to BC_2$ which records the linearity of the functors.

Fix a $C_2$-graded group $\hat{G}$ with ungraded group $G$.

Definition 2.6 [51, section 6.4]. A Real 2-representation of $G$ is a 2-functor $\rho : BG \to \text{dgCat}_{\mathbb{C}/\mathbb{R}}$ over $BC_2$.

Explicitly, a Real 2-representation $\rho$ is the data of

(i) a dg category $\rho(\star) = C \in \text{dgCat}_{\mathbb{C}/\mathbb{R}}$;
(ii) $\mathbb{R}$-linear dg functors $\rho(\sigma) : C \to C, \sigma \in \hat{G}$, which are $\mathbb{C}$-linear if $\pi(\sigma) = 1$ and $\mathbb{C}$-antilinear if $\pi(\sigma) = -1$;
(iii) natural isomorphisms $\theta_{\sigma_2, \sigma_1} : \rho(\sigma_2) \circ \rho(\sigma_1) \Rightarrow \rho(\sigma_2 \sigma_1), \sigma_1, \sigma_2 \in \hat{G}$; and
(iv) a natural isomorphism $\theta_e : \rho(e) \Rightarrow \text{id}_C$. 
These data are required to satisfy the following coherence conditions:

(a) For each \( \sigma_1, \sigma_2, \sigma_3 \in \hat{G} \), there is an equality of natural isomorphisms

\[
\theta_{\sigma_3 \sigma_2 \sigma_1} \circ \left( \theta_{\sigma_3 \sigma_2} \circ \text{id}_{\rho(\sigma_1)} \right) = \theta_{\sigma_3, \sigma_2 \sigma_1} \circ \left( \text{id}_{\rho(\sigma_3)} \circ \theta_{\sigma_2 \sigma_1} \right).
\]

(b) For each \( \sigma \in \hat{G} \), there are equalities \( \theta_{e, \sigma} = \theta_e \circ \text{id}_{\rho(\sigma)} \) and \( \theta_{\sigma, e} = \text{id}_{\rho(\sigma)} \circ \theta_e \) of natural isomorphisms.

We often denote a Real 2-representation by \( \rho_C \) or \((\rho, \theta)\).

**Definition 2.7.** A **Real \( G \)-equivariant structure** on a dg functor \( F : \mathcal{C} \to \mathcal{D} \) between Real 2-representations of \( G \) is a family of natural isomorphisms \( \{ \eta_\sigma : F \circ \rho_C(\sigma) \Rightarrow \rho_D(\sigma) \circ F \}_{\sigma \in \hat{G}} \) which makes the diagram

\[
\begin{array}{ccc}
F \circ \rho_C(\sigma_2) \circ \rho_C(\sigma_1) & \xrightarrow{\eta_{\sigma_2} \circ \text{id}_{\rho_C(\sigma_1)}} & \rho_D(\sigma_2) \circ F \circ \rho_C(\sigma_1) \\
\downarrow \quad \rho_C(\sigma_2) \circ \theta_{\sigma_1} & & \downarrow \quad \rho_D(\sigma_2) \circ \rho_C(\sigma_1) \circ F \\
F \circ \rho_C(\sigma_2 \sigma_1) & \xrightarrow{\eta_{\sigma_2 \sigma_1}} & \rho_D(\sigma_2 \sigma_1) \circ F
\end{array}
\]  

(9)

commute for each \( \sigma_1, \sigma_2 \in \hat{G} \).

Denote by \( \hat{G} \)-dgCat\(_{\mathbb{C}/\mathbb{R}} \) the category of Real 2-representations of \( G \) and their Real \( G \)-equivariant dg functors.

**Definition 2.8.** Let \( C \) be a Real 2-representation of \( G \). The **homotopy fixed point dg category** \( C^{\hat{G}} \) has

- objects **homotopy fixed points** (or **Real \( G \)-equivariant objects**), which are pairs \((C, u)\) with \( C \in C \) and \( u = \{ u_\sigma : C \to \rho(\sigma)(C) \}_{\sigma \in \hat{G}} \) a family of dg isomorphisms such that

\[
u_{\sigma_2 \sigma_1} = \theta_{\sigma_2 \sigma_1} \circ \rho(\sigma_2)(u_{\sigma_1}) \circ u_{\sigma_2} \tag{10}
\]

for each \( \sigma_1, \sigma_2 \in G \), and

- morphisms \( f : (C, u) \to (C', u') \) given by a morphism \( f : C \to C' \) in \( C \) such that

\[
u_{\sigma}' \circ f = \rho(\sigma)(f) \circ u_{\sigma} \tag{11}
\]

for each \( \sigma \in \hat{G} \).

In Equation (10), we have written \( \theta_{\sigma_2 \sigma_1} \) for what is really its component \( \theta_{\sigma_2 \sigma_1, C'} \). Similar notational simplifications are made below without comment.

Given a Real \( G \)-equivariant dg functor \((F, \eta) : \mathcal{C} \to \mathcal{D}\), let \( F^{\hat{G}} : \hat{G} \to \hat{D} \) be the dg functor defined on objects by \( F^{\hat{G}}(X, u) = (F(X), \emptyset_{\sigma \in \hat{G}}(F(u))) \) and on morphisms by \( F^{\hat{G}}(f) = f \). This defines a functor \((\cdot)^{\hat{G}} : \hat{G} \text{-dgCat}_{\mathbb{C}/\mathbb{R}} \to \text{dgCat}_{\mathbb{R}} \). See [24, section 8], [48, section 2] in the \( \mathbb{C} \)-linear
case, of which the Real case is a direct modification. If the $C_2$-grading of $\hat{G}$ is trivial, then $(\cdot)^{\hat{G}} = (\cdot)^G$ factors through the forgetful functor $\text{dgCat}_C \to \text{dgCat}_R$.

**Proposition 2.9.** Let $(F, \eta) : C \to D$ be a Real $G$-equivariant dg functor between Real 2-representations of a finite group $G$ such that $F$ is a quasi-equivalence.

(i) If $C$ and $D$ are triangulated, then $\text{Ind}_{\hat{G}} : \text{Perf}(C^{\hat{G}}) \to \text{Perf}(D^{\hat{G}})$ is a quasi-equivalence.

(ii) If $C^{\hat{G}}$ and $D^{\hat{G}}$ are triangulated, then $F^{\hat{G}} : C^{\hat{G}} \to D^{\hat{G}}$ is a quasi-equivalence.

**Proof.**

(i) Since $C$ and $D$ are triangulated, $H^0(C)$ and $H^0(D)$ are triangulated and idempotent complete. Moreover, $H^0(C)$ and $H^0(D)$ inherit the structure of Real 2-representations of $G$, with $\hat{G}$ acting by triangle equivalences. It follows from [24, Corollary 6.10] that $H^0(C)^{\hat{G}}$ and $H^0(D)^{\hat{G}}$ are again triangulated. The dg functor $F^{\hat{G}} : C^{\hat{G}} \to D^{\hat{G}}$ induces a dg functor $\text{Ind}_{\hat{G}} : \text{Perf}(C^{\hat{G}}) \to \text{Perf}(D^{\hat{G}})$ which fits into the commutative diagram

$$
\begin{array}{ccc}
H^0(\text{Perf}(C^{\hat{G}})) & \longrightarrow & H^0(C)^{\hat{G}} \\
\downarrow & & \downarrow \\
H^0(\text{Perf}(D^{\hat{G}})) & \longrightarrow & H^0(D)^{\hat{G}}
\end{array}
$$

whose horizontal arrows are equivalences [24, Theorem 8.7]. Since $H^0(F)$ is an equivalence, the same is true of $H^0(F)^{\hat{G}}$ [47, Proposition 2.20]. Commutativity of the diagram therefore implies that $H^0(\text{Ind}_{\hat{G}})$ is an equivalence. Since $\text{Perf}(C^{\hat{G}})$ and $\text{Perf}(D^{\hat{G}})$ are pretriangulated, $\text{Ind}_{\hat{G}}$ is a quasi-equivalence.

(ii) If $C^{\hat{G}}$ and $D^{\hat{G}}$ are triangulated, then the Yoneda arrows in the commutative diagram

$$
\begin{array}{ccc}
C^{\hat{G}} & \longrightarrow & \text{Perf}(C^{\hat{G}}) \\
\downarrow F^{\hat{G}} & & \downarrow \text{Ind}_{\hat{G}} \\
D^{\hat{G}} & \longrightarrow & \text{Perf}(D^{\hat{G}})
\end{array}
$$

are quasi-equivalences. Part (i) then implies that $F^{\hat{G}}$ is a quasi-equivalence.

**Remark 2.10.** Proposition 2.9, and its proof, apply to $G$-equivariant dg functors between 2-representations of a finite group $G$ on $k$-linear dg categories, where $k$ is any field with $\text{char} k \nmid |G|$.

**Proposition 2.11.** Let $\rho_C, \rho_D, \rho_E$ be Real 2-representations of $G$ and $(F, \eta) : C \times D \to E$ a Real $G$-equivariant dg functor. Each object $(D, u) \in D^{\hat{G}}$ induces a Real $G$-equivariant structure on the dg functor $F(-, D)$. 

**Proof.** Define $\eta^D = \{ \eta^D_\sigma : F(-, D) \circ \rho_c(\sigma) \Rightarrow \rho_c(\sigma) \circ F(-, D) \}_{\sigma \in \hat{G}}$ so that the component of $\eta^D_\sigma$ at $C \in C$ is

$$
\eta^D_{\sigma, C} : F(\rho_c(\sigma)(C), D) \xrightarrow{F(id_C, u_\sigma)} F(\rho_c(\sigma)(C), \rho_D(\sigma)(D)) \xrightarrow{\eta_\sigma} \rho_c(\sigma)(F(C, D)).
$$

Naturality of $\eta_\sigma$ implies that of $\eta^D_\sigma$. That $\eta^D$ satisfies the coherence condition (9) follows from the fact that $\eta$ satisfies the same conditions and $u$ satisfies condition (10). \qed

2.3.2 The contravariant approach

Let $dgCat_k$ be the 2-category of $k$-linear dg categories, dg functors and dg natural transformations. Taking the opposite of a dg category extends to a duality involution of $dgCat_k$. In particular, if $\theta : F \Rightarrow G$ is a natural transformation, then its opposite is $\theta^{op} : G^{op} \Rightarrow F^{op}$. Given $\epsilon \in C_2$ and $C \in dgCat_k$, write

$$
\epsilon C = \begin{cases} 
C & \text{if } \epsilon = 1, \\
C^{op} & \text{if } \epsilon = -1
\end{cases}
$$

with similar notation for dg functors and natural transformations.

For a direct contravariant analogue of Definition 2.6, see [51, section 4.4]. We instead use an explicit unpacking of this definition. Fix a $C_2$-graded group $\hat{G}$ with ungraded group $G$.

**Definition 2.12.** A **Real 2-representation** of $G$ on a dg category $C$ is the data of

(i) dg functors $\rho(\sigma) : \pi(\sigma)C \to C$, $\sigma \in \hat{G}$,

(ii) natural isomorphisms $\theta_{\sigma_2, \sigma_1} : \rho(\sigma_2) \circ \rho(\sigma_1) \Rightarrow \rho(\sigma_2 \sigma_1)$, $\sigma_1, \sigma_2 \in \hat{G}$, and

(iii) a natural isomorphism $\theta_e : \rho(e) \Rightarrow id_C$.

These data are required to satisfy the following coherence conditions:

(a) For each $\sigma_1, \sigma_2, \sigma_3 \in \hat{G}$, there is an equality of natural isomorphisms

$$
\theta_{\sigma_3, \sigma_2, \sigma_1} \circ \left( \theta_{\sigma_3, \sigma_2} \circ id_{\rho(\sigma_3 \sigma_2) \circ \rho(\sigma_1)} \right) = \theta_{\sigma_3, \sigma_2, \sigma_1} \circ \left( id_{\rho(\sigma_3)} \circ \rho(\sigma_3) \circ \theta_{\sigma_3, \sigma_2} \right). \quad (12)
$$

(b) For each $\sigma \in \hat{G}$, there are equalities $\theta_{e, \sigma} = \theta_e \circ id_{\rho(\sigma)}$ and $\theta_{\sigma, 1} = id_{\rho(\sigma)} \circ \rho(\sigma) \circ \theta_e$ of natural isomorphisms.

A **Real $G$-equivariant structure** on a dg functor $F : C \to D$ between Real 2-representations is a family $\{ \eta_\sigma : F \circ \rho_c(\sigma) \Rightarrow \rho_D(\sigma) \circ F \}_{\sigma \in \hat{G}}$ of natural isomorphisms such that

$$
\theta_{D, \sigma_2, \sigma_1} \circ \left( id_{\rho_D(\sigma_2)} \circ \rho(\sigma_2) \circ \eta_{\sigma_1} \right) \circ \left( \eta_{\sigma_2} \circ id_{\rho(\sigma_2)} \circ \rho_c(\sigma_1) \right) = \eta_{\sigma_2 \sigma_1} \circ F(\theta_{C, \sigma_2, \sigma_1}). \quad (13)
$$
for each $\sigma_1, \sigma_2 \in \hat{G}$. A homotopy fixed point of $C$ is a pair $(C, u)$ consisting of an object $C \in C$ and dg isomorphisms $u = \{u_{\sigma} : C \to \rho(\sigma)(C)\}_{\sigma \in \hat{G}}$ such that

$$u_{\sigma_2 \sigma_1} = \theta_{\sigma_2, \sigma_1} \circ \rho(\sigma_2)(u_{\sigma_1})$$

for each $\sigma_1, \sigma_2 \in \hat{G}$.

**Proposition 2.13.** A Real 2-representation of $G$ on a dg category $C$ induces a Real 2-representation of $G$ on $\text{Perf}(C)$ which makes the Yoneda dg functor $C \to \text{Perf}(C)$ Real-$G$-equivariant. Moreover, a Real $G$-equivariant dg functor $(F, \eta) : C \to C'$ lifts to a Real $G$-equivariant dg functor $(\text{Ind} F, \text{Ind} \eta) : \text{Perf}(C) \to \text{Perf}(C')$.

**Proof.** Let $(\rho, \theta)$ be a Real 2-representation of $G$ on $C$. Define a Real 2-representation $(\rho', \theta')$ on $\text{Perf}(C)$ as follows. Set $\rho'(\sigma) = \text{Ind} \rho(\sigma)$, $\sigma \in \hat{G}$, where, for $\sigma \in \hat{G} \setminus G$, we implicitly use $\text{Perf}(C'^\text{op}) \simeq \text{Perf}(C)^\text{op}$. The natural isomorphism $\theta_{\sigma_2, \sigma_1}$ induces a natural isomorphism of pullback dg functors and hence also of their derived left adjoints,

$$\theta'_{\sigma_2, \sigma_1} : \text{Ind} \rho(\sigma_2) \circ \rho(\sigma_1) \Rightarrow \text{Ind} \rho(\sigma_2 \sigma_1).$$

Coherence of $\theta'$ follows from that of $\theta$. The identity natural transformations define a Real $G$-equivariant structure on the Yoneda dg functor; see [18, section C.10]. The proof of the final statement is similar and so is omitted. □

### 2.3.3 Twists by 2-cocycles

There is a simple way to construct new Real 2-representations of $G$ from a given Real 2-representation. To treat both approaches simultaneously, let $k_\pi^\times$ be the $\hat{G}$-module defined as follows:

- antilinear approach: $k_\pi^\times = \mathbb{C}^\times$ with $\hat{G}$ acting through $\pi$ by complex conjugation;
- contravariant approach: $k_\pi^\times = k^\times$ with $\hat{G}$ acting through $\pi$ by inversion.

Let $C^*(\hat{G}; k_\pi^\times)$ be the complex of normalized group cochains on $\hat{G}$ with coefficients in $k_\pi^\times$ and $Z^*(\hat{G}; k_\pi^\times)$ and $H^*(\hat{G}; k_\pi^\times)$ the groups of cocycles and cohomology classes, respectively.

Given $\mu \in Z^2(\hat{G}; k_\pi^\times)$ and $(\rho, \theta)$ a Real 2-representation of $G$ (either of the two approaches), define a Real 2-representation $(\rho', \theta')$ by $\rho' = \rho$ and $\theta'_{\sigma_2, \sigma_1} = \mu([\sigma_2 | \sigma_1]) \theta_{\sigma_2, \sigma_1}$, $\sigma_2, \sigma_1 \in \hat{G}$. Up to equivalence, $(\rho', \theta')$ depends on $\mu$ through $[\mu] \in H^2(\hat{G}; k_\pi^\times)$.

The terminal $C_2$-graded group satisfies $H^2(C_2; k_\pi^\times) \simeq C_2$. A representative of the non-trivial class is $\hat{c}([\sigma_2 | \sigma_1]) = (-1)^{\frac{\sigma_2 - 1}{2} \frac{\sigma_1 - 1}{2}}$. For any $C_2$-graded finite group $\pi : \hat{G} \to C_2$, pullback along the $C_2$-grading defines $\pi^\times : H^2(C_2; k_\pi^\times) \to H^2(\hat{G}; k_\pi^\times)$. In particular, associated to any Real 2-representation $(\rho, \theta)$ of $G$ is a second Real 2-representation $(\rho, \theta_-)$, where $\theta_- = \pi^\times \hat{c} \cdot \theta$. 
3 | KNÖRRER PERIODICITY FOR REAL EQUIVARIANT MATRIX FACTORIZATIONS

In this section, $k = \mathbb{C}$ and Real 2-representations are in the antilinear approach of Section 2.3.1. We introduce Real matrix factorizations, in the antilinear setting, and prove the first instance of Real Knörrer periodicity.

3.1 | Equivariant module categories

Let $R$ be a $\mathbb{C}$-algebra. The group $\text{Aut}_{\mathbb{C}/\mathbb{R}}(R)$ of generalized algebra automorphisms of $R$ is the group of ring automorphisms $R \to R$ which are either $\mathbb{C}$-linear or $\mathbb{C}$-antilinear. The map $\text{Aut}_{\mathbb{C}/\mathbb{R}}(R) \to C_2$ which records linearity makes $\text{Aut}_{\mathbb{C}/\mathbb{R}}(R)$ into a $C_2$-graded group with ungraded group $\text{Aut}_{\mathbb{C}}(R)$.

Given $\sigma \in \text{Aut}_{\mathbb{C}/\mathbb{R}}(R)$, let $(-)\sigma : \text{R-mod} \to \text{R-mod}$ be the functor given on a finitely generated $R$-module by $M \mapsto M^\sigma$, where $M^\sigma = M$ as abelian groups with $R$-module structure $r \cdot m = \sigma^{-1}(r)m$, and on an $R$-module homomorphism $f : M \to N$ by $f \mapsto f^\sigma$, where $f^\sigma(m) = f(m)$, $m \in M^\sigma$.

**Lemma 3.1.** The functors $\{(-)\sigma\}_{\sigma \in \text{Aut}_{\mathbb{C}/\mathbb{R}}(R)}$ extend to a Real 2-representation of $\text{Aut}_{\mathbb{C}}(R)$ on $\text{R-mod}$.

**Proof.** Let $\sigma_1, \sigma_2 \in \text{Aut}_{\mathbb{C}/\mathbb{R}}(R)$.

Identity maps of underlying abelian groups therefore define the components of the required 2-isomorphisms $\{f_{\sigma_2, \sigma_1}\}_{\sigma_2, \sigma_1 \in \text{Aut}_{\mathbb{C}/\mathbb{R}}(R)}$ and $\theta_e$. \hfill □

Let $\hat{G}$ be a $C_2$-graded finite group. A $C_2$-graded group homomorphism $f : \hat{G} \to \text{Aut}_{\mathbb{C}/\mathbb{R}}(R)$ is called an action of $\hat{G}$ on $R$ by generalized algebra automorphisms. Pullback of the Real 2-representation of Lemma 3.1 along $Bf : B\hat{G} \to B\text{Aut}_{\mathbb{C}/\mathbb{R}}(R)$ defines a Real 2-representation of $G$ on $\text{R-mod}$. The category of Real $G$-equivariant $R$-modules is the homotopy fixed point category $R$-$\text{mod}^\hat{G}$. An object $(M, u) \in R$-$\text{mod}^\hat{G}$ is an $R$-module $M$ and a family of $R$-module isomorphisms $\{u_\sigma : M \to M^\sigma\}_{\sigma \in \hat{G}}$ which satisfy $u_{\sigma_2 \sigma_1} = (u_{\sigma_1})^{\sigma_2} \circ u_{\sigma_2}$. More generally, we can twist this Real 2-representation by $\mu \in Z^2(\hat{G}; C_\mathbb{R}^\times)$, as in Section 2.3.3, in which case an object $(M, u) \in R$-$\text{mod}^{\hat{G}}$ is a $\mu$-twisted Real $G$-equivariant $R$-module: the family of $R$-module isomorphisms $\{u_\sigma\}_{\sigma \in \hat{G}}$ satisfy $u_{\sigma_2 \sigma_1} = \mu([\sigma_2 | \sigma_1])(u_{\sigma_1})^{\sigma_2} \circ u_{\sigma_2}$.

The case of primary interest for this paper is $R = \mathbb{C}[x_1, \ldots, x_n]$. Using Reynolds operators, we may assume that $\hat{G}$ acts on $R$ by degree preserving generalized algebra automorphisms. In more invariant terms, there exists a finite-dimensional Real representation $V$ of $G$, in the sense of [51, section 3.1], so that $\hat{G}$ acts on $V$ linearly or antilinearly according to the $C_2$-grading, such that $R$ is Real $G$-equivariantly isomorphic to the completed symmetric algebra $\hat{\text{Sym}}V^\vee$. We henceforth work in such coordinates.

3.2 | Equivariant matrix factorization categories

Let $R = \mathbb{C}[x_1, \ldots, x_n]$ with potential $w \in m$. For each $\sigma \in \text{Aut}_{\mathbb{C}/\mathbb{R}}(R)$, let $(-)^\sigma : \text{MF}(R, w) \to \text{MF}(R, \sigma(w))$ be the dg functor defined on objects and morphisms by $(M, d_M) \mapsto (M^\sigma, d_M^\sigma)$ and $f \mapsto f^\sigma$, respectively.
Let $\hat{G}$ be a $C_2$-graded finite group acting on $R$ by generalized algebra automorphisms which preserve the potential:

$$\sigma(w) = w, \quad \sigma \in \hat{G}.$$ 

Each $(-)\sigma$ is then a dg endofunctor of $MF(R, w)$ and we obtain the following result.

**Lemma 3.2.** The dg functors $\{(-)\sigma\}_{\sigma \in \hat{G}}$ extend to a Real 2-representation of $G$ on $MF(R, w)$.

**Definition 3.3.**

(i) The category of Real $G$-equivariant matrix factorizations $MF_G(R, w)$ is the homotopy fixed point category $MF(R, w)^G$.

(ii) The homotopy category of Real $G$-equivariant matrix factorizations $HMF_G(R, w)$ is $H^0(MF^G(R, w))$.

**Example 3.4.**

(i) As a degenerate case, let a finite group $G$ (with trivial $C_2$-grading) act on $\mathbb{C}[x_1, \ldots, x_n]$ by $\mathbb{C}$-algebra automorphisms. Any $G$-invariant potential $w \in \mathbb{C}[x_1, \ldots, x_n]$ is then invariant in the above sense. In this example, Definition 3.3 reduces to the $\mathbb{C}$-linear dg category $MF_G(R, w)$ of $G$-equivariant matrix factorizations, a model for the the category of $D$-branes in the Landau–Ginzburg orbifold associated to the symmetry group $G$ of $(R, w)$ [2, section 2.2].

(ii) Let a finite group $G$ act on $\mathbb{R}[x_1, \ldots, x_n]$ by $\mathbb{R}$-algebra automorphisms and let $w \in \mathbb{R}[x_1, \ldots, x_n]$ be a $G$-invariant potential. Then, $w$ is also invariant with respect to the generalized action of $\hat{G} = G \times C_2$ on $\mathbb{R}[x_1, \ldots, x_n] \otimes_{\mathbb{R}} \mathbb{C} \simeq \mathbb{C}[x_1, \ldots, x_n]$ in which $C_2$ acts on $\mathbb{C}$ by complex conjugation. In particular, when $G$ is the trivial group, taking $C_2$-fixed points defines a dg isomorphism

$$MF_{C_2}(\mathbb{C}[x_1, \ldots, x_n], w) \simto MF(\mathbb{R}[x_1, \ldots, x_n], w).$$

In this way, matrix factorizations over $\mathbb{R}$ appear as a special case of Real equivariant matrix factorizations.

(iii) Let $w = x^m + y^m \in \mathbb{C}[x, y], m \in \mathbb{Z}_{\geq 2}$. Let $\hat{G} = D_{2m}$ be the dihedral group of order $2m$ with $\pi$ the projection with kernel $G$ the cyclic group of order $m$. Define a generalized $\hat{G}$-action on $\mathbb{C}[x, y]$ by letting a generator of $G$ act by $x \mapsto \zeta_m x$ and $y \mapsto \zeta_m y$, with $\zeta_m$ a primitive $m$th root of unity, and a fixed element of the non-identity coset of $\hat{G}$ act by complex conjugation. Then, $w$ is invariant.

(iv) Let $w \in \mathbb{C}[x_1, \ldots, x_n]$ be homogeneous of degree divisible by 4. Then, $w$ is invariant with respect to the generalized action of the terminal $C_2$-graded group determined by $x_j \mapsto ix_j, j = 1, \ldots, n$. 

Lemma 3.5.

(i) For each \( \sigma \in \text{Aut}_{\mathbb{C}/\mathbb{R}}(R) \) and \( \sigma' \in \text{Aut}_{\mathbb{C}/\mathbb{R}}(R') \), the diagram

\[
\begin{array}{ccc}
MF(R, w) \otimes_{\mathbb{C}} MF(R', w') & \xrightarrow{\otimes} & MF(R \otimes_{\mathbb{C}} R', w \otimes 1 + 1 \otimes w') \\
\downarrow (-)^{\sigma} \otimes (-)^{\sigma'} & & \downarrow (-)^{\sigma \otimes \sigma'} \\
MF(R, \sigma(w)) \otimes_{\mathbb{C}} MF(R', \sigma'(w')) & \xrightarrow{\otimes} & MF(R \otimes_{\mathbb{C}} R', \sigma(w) \otimes 1 + 1 \otimes \sigma'(w'))
\end{array}
\]

commutes up to natural isomorphism.

(ii) With respect to the Real 2-representations of \( G \) of Lemma 3.2, the dg functor

\[
\otimes : MF(R, w) \otimes_{\mathbb{C}} MF(R', w') \to MF(R \otimes_{\mathbb{C}} R', w \otimes 1 + 1 \otimes w')
\]

admits a Real \( G \)-equivariant structure.

Proof.

(i) Let \( M \in MF(R, w) \) and \( N \in MF(R', w') \). Direct sums of identity maps of underlying abelian groups give isomorphisms

\[
(M_0 \otimes_{\mathbb{C}} N_0)^{\sigma \otimes \sigma'} \oplus (M_1 \otimes_{\mathbb{C}} N_1)^{\sigma \otimes \sigma'} \to (M_0^\sigma \otimes_{\mathbb{C}} N_0^{\sigma'}) \oplus (M_1^\sigma \otimes_{\mathbb{C}} N_1^{\sigma'})
\]

and

\[
(M_1 \otimes_{\mathbb{C}} N_0)^{\sigma \otimes \sigma'} \oplus (M_0 \otimes_{\mathbb{C}} N_1)^{\sigma \otimes \sigma'} \to (M_1^\sigma \otimes_{\mathbb{C}} N_0^{\sigma'}) \oplus (M_0^\sigma \otimes_{\mathbb{C}} N_1^{\sigma'}).
\]

These are the components of an isomorphism \((M \boxtimes N)^{\sigma \otimes \sigma'} \to M^\sigma \boxtimes N^{\sigma'}\) that fulfill the coherence condition (9).

(ii) The identity natural isomorphisms from part (i) are the data for the required Real \( G \)-equivariant structure on \( \otimes \).

Proposition 3.6. Let \( w \) be an isolated hypersurface singularity at the origin. The canonical functor \( HMF_G(R, w) \to HMF(R, w)^G \) is an idempotent completion. In particular, \( \text{Perf} \ (MF_G(R, w)) \) is a dg enhancement of \( HMF(R, w)^G \).

Proof. The Yoneda dg functor \( MF_G(R, w) \to \text{Perf} \ (MF_G(R, w)) \) induces on homotopy categories an idempotent completion \( HMG_G(R, w) \to H^0(\text{Perf} \ (MF_G(R, w))) \). The dg category \( MF_G(R, w) \) is pretriangulated, as can be verified directly. Since \( MF(R, w) \) is triangulated [19, Lemma 5.6], we can apply [24, Theorem 8.7] to conclude that the canonical functor \( H^0(\text{Perf} \ (MF_G(R, w))) \to HMF(R, w)^G \) is an equivalence.

Remark 3.7. The definition of \( HMF_G(R, w) \) given in Definition 3.3 agrees with [2, section 2.2], [43, section 6], [28, section 3.1], [42, section 2.1]. On the other hand, Carqueville–Runkel, for example, define \( HMF_G(R, w) \) to be \( HMF(R, w)^G \) [15, section 7.1]. In view of Proposition 3.6, these definitions agree for \( w \) an isolated hypersurface singularity at the origin precisely if \( HMF_G(R, w) \) is
idempotent complete, which does not seem to be known. If $\text{HMF}_G(R, w)$ (or its Real generalization) is idempotent complete, then some of the proofs below can be simplified by using part (ii) of Proposition 2.9 in place of part (i).

For comparison, if $X$ is a quasi-projective $G$-variety over a field $k$ with $\text{char } k \nmid |G|$, then $D^b(\text{Coh}(X)^G) \to D^b(\text{Coh}(X))^G$ is a triangle equivalence [23, Theorem 9.6], [24, Theorem 7.3]. A key ingredient of the proof is idempotent completeness of the bounded derived category of an abelian category [4, Corollary 2.10].

### 3.3 Real Knörrer periodicity

To formulate a Real generalization of Knörrer periodicity (Theorem 2.5), we study Real $G$-equivariant structures on $K = \{y + iz, y - iz\}$.

**Proposition 3.8.** The matrix factorization $\{u, v\} \in \text{MF}(\mathbb{C}[u, v], uv)$ admits a Real $G$-equivariant structure if and only if there exists $\chi \in Z^1(\hat{G}; \mathbb{C}\langle u, v \rangle)$ such that $\sigma(u) = \chi(\sigma)u$ and $\sigma(v) = \chi(\sigma)^{-1}v$ for each $\sigma \in \hat{G}$, in which case the set of dg isomorphism classes of Real $G$-equivariant structures on $\{u, v\}$ is in bijection with $H^1(\hat{G}; \mathbb{C}[u, v]_\chi)$.

**Proof.** A Real $G$-equivariant structure on $\{u, v\}$ is the data of commutative diagrams

$$
\begin{align*}
\mathbb{C}[u, v] & \xrightarrow{u} \mathbb{C}[u, v] \xrightarrow{v} \mathbb{C}[u, v] \\
\mathbb{C}[u, v]^\sigma & \xrightarrow{u^\sigma} \mathbb{C}[u, v]^\sigma \xrightarrow{v^\sigma} \mathbb{C}[u, v],
\end{align*}
$$

where $\chi \in Z^1(\hat{G}; \mathbb{C}[u, v]_\chi^\sigma)$ and $u^\sigma(r) = \chi_0(\sigma^{-1})\sigma^{-1}(r)$ for each $\sigma \in \hat{G}$ and $r \in \mathbb{C}[u, v]$. Set $\chi = \chi_0\chi_1^{-1}$. Commutativity of the diagram is then equivalent to the equations $\sigma(u) = \chi(\sigma)u$ and $\sigma(v) = \chi(\sigma)^{-1}v$. Since $\hat{G}$ acts on $\mathbb{C}[u, v]$ by degree preserving maps, we find the restriction $\chi \in Z^1(\hat{G}; \mathbb{C}\langle u, v \rangle)$. Direct calculation then shows that isomorphism classes of Real $G$-equivariant structures are in bijection with

$$
\{(\chi_0, \chi_1) \in Z^1(\hat{G}; \mathbb{C}[u, v]_\chi^\sigma)^2 \mid \chi = \chi_0\chi_1^{-1}\}/B^1(\hat{G}; \mathbb{C}[u, v]_\chi^\sigma).
$$

Projection to the first factor gives the desired bijection with $H^1(\hat{G}; \mathbb{C}[u, v]_\chi^\sigma)$. □

By Proposition 3.8, $\{u, v\}$ admits a Real $G$-equivariant structure precisely when $\mathbb{C}[u, v] \simeq \text{Sym} V^\vee$ with $V \simeq C_{\chi^{-1}} \oplus C_{\chi}$ for some $\chi \in Z^1(\hat{G}; \mathbb{C}\langle u, v \rangle)$, where $C_{\chi}$ is the one-dimensional Real representation of $G$ determined by $\chi$ and $u$ (resp. $v$) is the coordinate dual to $C_{\chi^{-1}}$ (resp. $C_{\chi}$). Up to isomorphism, $C_{\chi}$ depends on $\chi$ through $[\chi] \in H^1(\hat{G}; \mathbb{C}\langle u, v \rangle)$. In the terminal case, $H^1(C_2; \mathbb{C}_\chi)$ is trivial so that, without loss of generality, we may take $\chi(\sigma) = -1$ for the generator $\sigma \in C_2$. Pull back along the $C_2$-grading then gives a universal choice of $\chi$ for all $C_2$-graded finite groups $\hat{G}$. In what follows, fix a Real $G$-equivariant structure on $\{u, v\}$ by taking $\chi_0 = \chi$ and $\chi_1$ to be trivial.

---

1 See “Note added” in the Introduction.
Proposition 3.8 applies to \( \{y + iz, y - iz\} \in MF(C[y, z], y^2 + z^2) \) via the coordinate change \( y = \frac{u + v}{2} \) and \( z = \frac{u - v}{2i} \). We conclude that \( \{y + iz, y - iz\} \) admits a Real structure precisely when, up to Real equivariant isomorphism, \( C_2 \) acts on \( C[y, z] \) by \( (y, z) \mapsto (-y, z) \).

We can now state the first form of Real Knörrer periodicity.

**Theorem 3.9.** Let a \( C_2 \)-graded finite group \( \hat{G} \) act on \( R = C[[x_1, ..., x_n]] \) by generalized algebra automorphisms which preserve the potential \( w \). Extend the generalized \( \hat{G} \)-action to \( R[[y, z]] \) by \( \sigma(y) = \pi(\sigma)y \) and \( \sigma(z) = z, \sigma \in \hat{G} \). Let \( K = \{y + iz, y - iz\} \), considered as a Real \( G \)-equivariant matrix factorization as above. If \( w \) is an isolated hypersurface singularity at the origin, then the dg functor

\[
\text{Ind} K^{\hat{G}} : \text{Perf}(MF_{\hat{G}}(R, w)) \to \text{Perf}(MF_{\hat{G}}(R[[y, z]], w + y^2 + z^2))
\]

is a quasi-equivalence.

**Proof.** By Proposition 2.11 and Lemma 3.5, the Real \( G \)-equivariant structure on \( K \) induces a Real \( G \)-equivariant structure on the Knörrer dg functor \( K \). The theorem therefore follows from Theorem 2.5 and Proposition 2.9(i). Note that the idempotent completeness required for the application of Proposition 2.9(i) follows from [19, Lemma 5.6].

**Corollary 3.10.** Let \( w \in \mathbb{R}[[x_1, ..., x_n]] \) be an isolated hypersurface singularity at the origin. Then, there is a triangle equivalence

\[
\text{HMF}(\mathbb{R}[[x_1, ..., x_n]], w) \simeq \text{HMF}(\mathbb{R}[[x_1, ..., x_n, y, z]], w - y^2 + z^2).
\]

**Proof.** Let the terminal \( C_2 \)-graded group act on \( C[[x_1, ..., x_n]] \) by complex conjugation and on \( C[[y, z]] \) by the antilinear extension of \( y \mapsto -y \) and \( z \mapsto z \). The \( C_2 \)-fixed point subalgebra of \( C[[x_1, ..., x_n, y, z]] \) is \( \mathbb{R}[[x_1, ..., x_n, iy, z]] \), which is isomorphic to \( \mathbb{R}[[x_1, ..., x_n, y, z]] \). Under this isomorphism, the potential \( w + y^2 + z^2 \) is identified with \( w - y^2 + z^2 \). In view of Example 3.4(ii) and the fact that both triangulated categories under consideration are idempotent complete, Theorem 3.9 reduces to the desired statement.

When the \( C_2 \)-grading of \( \hat{G} \) is trivial, Theorem 3.9 gives a simple proof of the affine case of Hirano’s equivariant Knörrer periodicity [27, Theorem 1.2].

**Corollary 3.11.** Let a finite group \( G \) act on \( C[[x_1, ..., x_n]] \) by \( C \)-algebra automorphisms which preserve the isolated hypersurface singularity \( w \). There is a quasi-equivalence

\[
\text{Perf}(MF_{\hat{G}}(C[[x_1, ..., x_n]], w)) \simeq \text{Perf}(MF_{\hat{G}}(C[[x_1, ..., x_n, y, z]], w + y^2 + z^2)),
\]

where \( G \) acts trivially on \( C[[y, z]] \).

**Remark 3.12.** There is a straightforward generalization of Theorem 3.9, with the same proof, in which a Real character \( \chi \in Z^1(\hat{G}; C_{\pi}^\chi) \) is used to extend the generalized \( \hat{G} \)-action from \( R \) to \( R[[y, z]] \), as in Proposition 3.8. When the \( C_2 \)-grading of \( \hat{G} \) is trivial, this generalization recovers the extra character theoretic data in Hirano’s equivariant Knörrer periodicity.
4 | CLIFFORD MODULES AND REAL MATRIX FACTORIZATION

In this section, we study Real matrix factorizations of quadratic hypersurfaces using Clifford modules. In this way, we obtain Real generalizations of results of Buchweitz–Eisenbud–Herzog [12] and connect Real Knörrer periodicity to classical periodicities of categories of Clifford modules [1, 37].

4.1 Buchweitz–Eisenbud–Herzog equivalence

For background on Clifford algebras, see [1, section 1] and [37, section I]; our conventions match the former.

Let $V$ be a finite-dimensional vector space over a field $k$ with $\text{char} k \neq 2$. Let $R = \widehat{\text{Sym}} V^\vee$ be the completed symmetric algebra on $V^\vee$ and $T(V)$ the tensor algebra on $V$. The Clifford algebra of a non-degenerate quadratic form $q \in \text{Sym}^2 V^\vee$ is the $\mathbb{Z}/2\mathbb{Z}$-graded algebra

$$\text{Cl}(V, q) = T(V)/(v \otimes v - q(v) | v \in V).$$

Let $\text{Cl}(V, q)$-$\text{grmod}$ be the category of finitely generated $\mathbb{Z}/2\mathbb{Z}$-graded left $\text{Cl}(V, q)$-modules. Objects are therefore $\mathbb{Z}/2\mathbb{Z}$-graded vector spaces $A = A_0 \oplus A_1$ with an action of $\text{Cl}(V, q)$ such that homogeneous elements $c \in \text{Cl}(V, q)$ act by linear maps $c : A_i \to A_{i+|c|}$. Morphisms are $\text{Cl}(V, q)$-linear maps of degree 0.

Define a functor $\Phi : \text{Cl}(V, q)$-$\text{grmod} \to \text{HMF}(\widehat{\text{Sym}} V^\vee, q)$ as follows [12, section 2]. For $A \in \text{Cl}(V, q)$-$\text{grmod}$, let $\Phi(A) = A \otimes_k R$ as $\mathbb{Z}/2\mathbb{Z}$-graded $R$-modules with twisted differential

$$d^\Phi_{(\Phi(A))} \in \text{Hom}_k(A_i \otimes_k R, A_{i+1} \otimes_k R) \cong \text{Hom}_k(A_i, A_{i+1}) \otimes_k R$$

defined to be the image of the identity map under the composition

$$\text{End}_k(V) \hookrightarrow V \otimes_k V^\vee \to \text{Hom}_k(A_i, A_{i+1}) \otimes_k R,$$

where the second map uses the action of $V \subset \text{Cl}(V, q)$ on $A$ and the canonical inclusion $V^\vee \hookrightarrow R$.

Given a morphism $f : A \to A'$ in $\text{Cl}(V, q)$-$\text{grmod}$, set $\Phi(f) : \Phi(A) \to \Phi(A')$.

Theorem 4.1 [12, section 2]. The functor $\Phi$ induces a $k$-linear equivalence

$$\text{Cl}(V, q)$-$\text{grmod} \cong \text{HMF}(\widehat{\text{Sym}} V^\vee, q).$$

4.2 Buchweitz–Eisenbud–Herzog equivalence and reality

Suppose now that $\hat{G}$ is a $\mathbb{Z}_2$-graded finite group, $V$ is a finite-dimensional Real representation of $G$ over $k = \mathbb{C}$ and the non-degenerate quadratic form $q \in \text{Sym}^2 V^\vee$ is $\hat{G}$-invariant. Then, $\hat{G}$
acts on $\text{Cl}(V, q)$ by generalized algebra automorphisms so that, by Lemma 3.1, $\text{Cl}(V, q)\text{grmod}$ is a Real 2-representation of $G$. Objects of $\text{Cl}(V, q)\text{grmod}_G^\hat{G}$ are called Real $G$-equivariant Clifford modules. When $\hat{G} = C_2$, this recovers the Real Clifford modules of [3, section 4], [37, section I.10]. By Lemma 3.2, the dg category $\text{MF}(R, q)$ is a Real 2-representation of $G$, as is its subcategory $Z^0(\text{MF}(R, w))$.

**Proposition 4.2.** The functor $\Phi : \text{Cl}(V, q)\text{grmod} \to Z^0(\text{MF}(R, q))$ admits a Real $G$-equivariant structure.

**Proof.** Let $A \in \text{Cl}(V, q)\text{grmod}$ and $\sigma \in \hat{G}$. Define an $R$-linear isomorphism

$$
\eta_{\sigma, A} : \Phi(A^\sigma) \to \Phi(A)^\sigma, \quad a \otimes r \mapsto a \otimes \sigma^{-1}(r).
$$

That $\eta_{\sigma, A}$ is closed of degree zero is the statement that the diagrams

$$
\begin{array}{ccc}
A_i^\sigma \otimes R & \xrightarrow{d^i} & A_{i+1}^\sigma \otimes R \\
\eta_i^\sigma \downarrow & & \downarrow \eta_{i+1}^\sigma \\
A_i \otimes R^\sigma & \xrightarrow{d^\sigma} & A_{i+1} \otimes R^\sigma
\end{array}
$$

commute, $i = 0, 1$. The clockwise and counterclockwise compositions send $a \otimes r$ to $\sum_j \sigma^{-1}(v_j)a \otimes \sigma^{-1}(v_j^\vee r)$ and $\sum_j v'_ja \otimes v_j^\vee \sigma^{-1}(r)$, respectively, where $\{v_j\}$ and $\{v'_j\}$ are arbitrary bases of $V$ with dual bases $\{v_j^\vee\}$ and $\{v'_j^\vee\}$. Taking $v'_j = \sigma^{-1}(v_j)$ yields the required commutativity. The equalities

$$
\eta_{\sigma_2}^i \circ \eta_{\sigma_1}^i(a \otimes r) = \eta_{\sigma_1}^i(a \otimes \sigma^{-1}_2(r)) = a \otimes \sigma_1^{-1}(\sigma^{-1}_2(r)) = \eta_{\sigma_2 \sigma_1}^i(a \otimes r)
$$

verify the coherence condition (9). □

**Theorem 4.3.** Let $V$ be a Real representation of a finite $C_2$-graded group $\hat{G}$ and $q \in \text{Sym}^2 V^\vee$ a non-degenerate $\hat{G}$-invariant quadratic form. There is an $\mathbb{R}$-linear equivalence

$$
\text{Cl}(V, q)\text{grmod}_G^\hat{G} \simeq \text{HMF}(\text{Sym} V^\vee, q)^{\hat{G}}.
$$

**Proof.** Write $R = \text{Sym} V^\vee$. The Real 2-representation of $G$ on $\text{MF}(R, w)$ induces one on $\text{HMF}(R, w)$, for which the canonical functor $Z^0(\text{MF}(R, w)) \to \text{HMF}(R, w)$ admits an obvious Real $G$-equivariant structure. In view of Proposition 4.2, the following composition

$$
\Phi : \text{Cl}(V, q)\text{grmod} \xrightarrow{\Phi} Z^0(\text{MF}(R, w)) \to \text{HMF}(R, w)
$$

is Real $G$-equivariant. Moreover, $\Phi$ is an equivalence by Theorem 4.1. Applying [47, Proposition 2.20], we conclude that $\Phi^\hat{G} : \text{Cl}(V, q)\text{grmod}_G^\hat{G} \to \text{HMF}(R, q)^{\hat{G}}$ is an equivalence. □
4.3 Periodicities of Clifford modules

In this section, we connect Theorems 3.9 and 4.3 to classical periodicities of categories of Clifford modules over $\mathbb{R}$.

Let $V = \bigoplus_{j=1}^{n} \mathbb{C} \cdot e_j$ with coordinates $x_j = e_j^\vee$ and $R = \text{Sym} \ V^\vee$. Define a Real representation of the trivial group on $V$ by

$$\sigma(e_j) = e_j, \quad j = 1, \ldots, r, \quad \sigma(e_j) = -e_j, \quad j = r+1, \ldots, n,$$

where $\sigma \in C_2$ is the generator. The quadratic form $q = \sum_{j=1}^{n} x_j^2$ is $C_2$-invariant. Setting $s = n - r$, there is an equivalence $\text{Cl}(V, q)$-grmod$^{C_2} \simeq \text{Cl}_{r,s}$-grmod, the right-hand side being the category of graded modules over the $\mathbb{R}$-algebra $\text{Cl}_{r,s} = \text{Cl}(\mathbb{R}^n, \sum_{j=1}^{r} x_j^2 - \sum_{j=r+1}^{n} x_j^2)$. See [37, Proposition I.10.9].

The categories $\text{Cl}_{r,s}$-grmod, $r, s \in \mathbb{Z}_{\geq 0}$, enjoy a number of periodicities [1, section 4], [37, section I.4] which, in view of Theorem 4.3, correspond to periodicities of Real matrix factorization categories. In particular, the $(1,1)$-periodicity theorem

$$\text{Cl}_{r,s} \text{-grmod} \simeq \text{Cl}_{r+1,s+1} \text{-grmod}$$

corresponds to an equivalence

$$\text{HMF}(R, q)^{C_2} \simeq \text{HMF}(R[[y, z]], q + y^2 + z^2)^{C_2},$$

where the generalized $C_2$-action is extended from $R$ to $R[[y, z]]$ by $\sigma(y) = -y$ and $\sigma(z) = z$. This is precisely the form of Theorem 3.9, thereby making precise its $(1,1)$-nature. As a second example, recall that there are graded algebra isomorphisms

$$\text{Cl}_{1,1}^{\otimes 4} \simeq \text{Cl}_{4,4} \simeq \text{Cl}_{0,8} \simeq \text{Cl}_{8,0}.$$ 

In view of Example 3.4(ii), repeated application of Theorem 3.9, combined with Theorem 4.3, recovers a special case of Brown’s 8-periodicity.

**Corollary 4.4** [8, Theorem 1.2]. *Let $q$ be a non-degenerate quadratic form on $\mathbb{R}^n$. There is an $\mathbb{R}$-linear triangle equivalence*

$$\text{HMF}(\mathbb{R}[[x_1, \ldots, x_n]], q) \simeq \text{HMF} \left( \mathbb{R}[[x_1, \ldots, x_{n+8}]], q + \sum_{i=1}^{8} x_{n+i}^2 \right).$$

Note that Brown’s theorem applies to arbitrary isolated hypersurface singularities, not only quadratics, but relies on Dyckerhoff’s 2-periodic derived Morita theory [19, section 6]. On the other hand, our methods are elementary. It would be interesting to generalize Dyckerhoff’s results to the Real equivariant setting and use them to deduce Theorem 3.9 from the Morita triviality of $\text{Cl}_{1,1}$. 


Motivated by Grothendieck–Witt theory and the physics of Landau–Ginzburg orientifolds, as studied by Hori–Walcher [28], we give a second formulation of Real matrix factorizations and their periodicity, independent from those of Sections 3 and 4. All constructions are now linear over a ground field $k$ with char $k \neq 2$ and Real 2-representations are in the contravariant approach of Section 2.3.2.

5.1 2-periodic dg categories with duality

To begin this section, we define duality structures on (2-periodic) dg categories, parallel to the $\mathbb{Z}$-graded setting of [45, section 1].

**Definition 5.1.**

(i) A **dg category with duality** $(\mathcal{C}, P, \Theta)$ is a dg category $\mathcal{C}$, a dg functor $P : \mathcal{C}^{op} \to \mathcal{C}$ and a dg natural isomorphism $\Theta : id_{\mathcal{C}} \Rightarrow P \circ P^{op}$ such that

$$P(\Theta_C) \circ \Theta_{P(C)} = id_{P(C)}$$

for each $C \in \mathcal{C}$. The pair $(P, \Theta)$ is called a **dg duality structure** on $\mathcal{C}$.

(ii) A **dg form functor** $(T, \varphi) : (\mathcal{C}, P, \Theta) \to (\mathcal{D}, Q, \Xi)$ between dg categories with duality is a dg functor $T : \mathcal{C} \to \mathcal{D}$ and a dg natural transformation $\varphi : T \circ P \Rightarrow Q \circ T^{op}$ such that

$$Q(\varphi_C) \circ \Xi_{T(C)} = \varphi_{P(C)} \circ T(\Theta_C)$$

for each $C \in \mathcal{C}$.

Given dg form functors $(T, \varphi) : (\mathcal{C}, P, \Theta) \to (\mathcal{D}, Q, \Xi)$ and $(S, \psi) : (\mathcal{D}, Q, \Xi) \to (\mathcal{E}, R, \Lambda)$, their composition is the dg form functor $(S \circ T, \psi \circ \varphi) : (\mathcal{C}, P, \Theta) \to (\mathcal{E}, R, \Lambda)$, where $\psi \circ \varphi$ is defined so that its component at $C \in \mathcal{C}$ is $\psi_{T(C)} \circ S(\varphi_C)$. Denote by $\text{dgCat}_{D_k}$ the category with objects dg categories with duality and morphisms their dg form functors.

**Example 5.2.** Let $(\rho, \theta)$ be a Real 2-representation of the trivial group (with respect to the terminal $\mathbb{Z}$-graded group) on a dg category $\mathcal{C}$. Let $\sigma$ be the generator of $C_2$. Define a dg duality structure on $\mathcal{C}$ by $P = \rho(\sigma)$ and $\Theta = \theta_{\sigma, \sigma}^{-1} \circ \theta_{e}^{-1}$. Similarly, a Real equivariant functor between Real 2-representations induces a dg form functor. These assignments extend to an equivalence $\text{C}_2\text{-dgCat}_{D_k} \sim \text{dgCat}_{D_k}$.

Let $G$ be a $C_2$-graded finite group. We formulate two general results relating Real 2-representations of $G$ and duality structures. In view of Example 5.2, the first is an instance of the expectation that, given an algebraic or geometric object with an action of a group $G$, the fixed points of a normal subgroup $N \trianglelefteq G$ inherit an action of $G/N$. Our setting differs from standard ones (for example, [25, Exercise 4.15.3]), since our actions are not covariant. Some calculations in the proof are relegated to Appendix A.
Theorem 5.3. Let $(\rho, \Theta)$ be a Real 2-representation of $G$ on a dg category $C$:

(i) Each $\sigma \in \hat{G} \setminus G$ induces a dg duality structure $(\rho(\sigma), \Theta(\sigma))$ on $C^G$.

(ii) For each $\sigma_1, \sigma_2 \in \hat{G} \setminus G$, there exist dg form equivalences

$$(id_{C^G}, \varphi^{\sigma_1, \sigma_2}) : (C^G, \rho(\sigma_1), \Theta(\sigma_1)) \to (C^G, \rho(\sigma_2), \Theta(\sigma_2))$$

which satisfy the coherence condition

$$(id_{C^G}, \varphi^{\sigma_1, \sigma_3}) = (id_{C^G}, \varphi^{\sigma_3, \sigma_2}) \circ (id_{C^G}, \varphi^{\sigma_1, \sigma_2}).$$

In particular, $C^G$ inherits from $(\rho, \Theta)$ a canonical equivalence class of dg duality structures.

Proof.

(i) Lift $\rho(\sigma) : C^{op} \to C$ to a dg functor $(C^G)^{op} \to C^G$, again denoted by $\rho(\sigma)$, as follows. Given $(C, u) \in C^G$, let $v = \{v_g\}_{g \in G}$, where

$v_g : \rho(\sigma)(C) \xrightarrow{\rho(\sigma)(u^{-1}g_\sigma^{-1}g_\sigma)} \rho(\sigma)(\rho(\sigma^{-1}g_\sigma)(C)) \xrightarrow{\Theta(\sigma)(\rho(\sigma^{-1}g_\sigma)(C))} \rho(\sigma)(\rho(\sigma^{-1}g_\sigma)(C)) \xrightarrow{\Theta(\sigma)(\rho(\sigma^{-1}g_\sigma)(C))} \rho(\sigma)(\rho(\sigma^{-1}g_\sigma)(C))$.

Lemma A.1 verifies that $(\rho(\sigma)(C), v)$ is an object of $C^G$. Setting $\rho(\sigma)(C, v) = (\rho(\sigma)(C), v)$ and defining $\rho(\sigma)$ on morphisms in $C^G$ as for $C$ gives a dg functor $\rho(\sigma) : (C^G)^{op} \to C^G$. Let $\Theta(\sigma)$ be the natural isomorphism whose component at $(C, u) \in C^G$ is

$\Theta(\sigma)(C, u) : C \xrightarrow{\rho(\sigma)(u^{-1})} \rho(\sigma)^2(C) \xrightarrow{\Theta(\sigma)(\rho(\sigma)(u^{-1}))} \rho(\sigma)(\rho(\sigma)(u^{-1}))$.

Lemma A.2 verifies that $\Theta(\sigma)(C, u)$ is a morphism in $C^G$. We have

$\sigma(\Theta(\sigma)(C, u)) \circ \Theta(\sigma)(C, u) = \sigma(u_{\sigma\sigma}) \circ \Theta(\sigma)(\rho(\sigma^{-1}g_\sigma)(C)) \circ \Theta(\sigma)(\rho(\sigma^{-1}g_\sigma)(C)) \circ \Theta(\sigma)(\rho(\sigma^{-1}g_\sigma)(C))$,

which is the identity by Equation (12). Hence, $(\rho(\sigma), \Theta(\sigma))$ satisfies the coherence condition (14) and defines a dg duality structure on $C^G$.

(ii) Let $\varphi^{\sigma_1, \sigma_2} : \rho(\sigma_1) \Rightarrow \rho(\sigma_2)$ be the natural isomorphism whose component at $(C, u) \in C^G$ is

$\varphi^{\sigma_1, \sigma_2} : \rho(\sigma_1)(C) \xrightarrow{\rho(\sigma_2)(u_{\sigma_2}^{-1}g_\sigma^{-1}g_\sigma)} \rho(\sigma_2)(\rho(\sigma_2^{-1}g_\sigma)(C)) \xrightarrow{\rho(\sigma_2)(\rho(\sigma_2^{-1}g_\sigma)(C))} \rho(\sigma_2)(\rho(\sigma_2^{-1}g_\sigma)(C))$.

Lemma A.3 verifies that $\varphi^{\sigma_1, \sigma_2}$ is a morphism in $C^G$. By Lemma A.4, the pair $(id_{C^G}, \varphi^{\sigma_1, \sigma_2})$ is a dg form equivalence. The coherence condition involving $\sigma_1, \sigma_2, \sigma_3 \in \hat{G} \setminus G$ is proved in Lemma A.5.

\[\square\]

Corollary 5.4. A Real 2-representation of $G$ on $C$ induces on Perf $(C^G)$ a canonical equivalence class of dg duality structures.
Proof. By Theorem 5.3, the dg category $C^G$ inherits a dg duality structure, unique up to equivalence. Interpreting this dg duality structure as a Real 2-representation of the trivial group on $C^G$, as in Example 5.2, we conclude from Proposition 2.13 that Perf ($C^G$) inherits a dg duality structure which is unique up to equivalence. □

The second general result asserts naturality of the dg duality structures constructed in Theorem 5.3, and therefore also in Corollary 5.4.

**Theorem 5.5.** Let $(F, \eta): C \to D$ be a Real $G$-equivariant dg functor.

(i) For each $\sigma \in \hat{G} \setminus G$, there is an induced dg form functor

$$(F^G, \psi^G): (C^G, \rho_C(\sigma), \Theta(\sigma)) \to (D^G, \rho_D(\sigma), \Theta(\sigma))$$

whose second component $\psi^G$ is a natural isomorphism.

(ii) For each $\sigma_1, \sigma_2 \in \hat{G} \setminus G$, there is a commutative diagram of dg form functors

\[
\begin{array}{ccc}
(C^G, \rho_C(\sigma_1), \Theta(\sigma_1)) & \xrightarrow{(F^G, \psi^G)} & (D^G, \rho_D(\sigma_1), \Theta(\sigma_1)) \\
(id_{C^G}, \varphi^{G}_{\sigma_1}) \downarrow & & \downarrow (id_{D^G}, \varphi^{G}_{\sigma_1}) \\
(C^G, \rho_C(\sigma_2), \Theta(\sigma_2)) & \xrightarrow{(F^G, \psi^G)} & (D^G, \rho_D(\sigma_2), \Theta(\sigma_2)).
\end{array}
\]

Proof.

(i) Define $\psi^G: F^G \circ \rho_C(\sigma) \Rightarrow \rho_D(\sigma) \circ (F^G)^{op}$ so that its component at $(C, u) \in C^G$ is $\eta_{\sigma}: F(\rho_C(\sigma)C) \to \rho_D(\sigma)F(C)$. That $\psi^G$ is a morphism in $D^G$ amounts to commutativity of the square

\[
\begin{array}{ccc}
F(\rho_C(\sigma)C) & \xrightarrow{\eta_C} & \rho_D(\sigma)F(C) \\
\eta_C \circ F^G(\sigma) \downarrow & & \downarrow (\rho_D^{op} \circ \varphi^{G}_{\sigma}) \circ F^G(\sigma) \\
\rho_D(g)F(\rho_C(\sigma)C) & \xrightarrow{\rho_D(\eta_C)} & \rho_C(g)\rho_D(\sigma)F(C)).
\end{array}
\]

for each $g \in G$. The clockwise composition is

\[
\begin{align*}
\partial^{-1}_{\sigma \sigma^{-1} g \sigma} & \circ \partial_{\sigma^{-1} g \sigma} \circ \rho_D(\sigma)(\eta^{-1}_{\sigma^{-1} g \sigma}) \circ \rho_D(\sigma)(F(u^{-1}_{\sigma^{-1} g \sigma})) \\
& = \partial^{-1}_{\sigma \sigma^{-1} g \sigma} \circ \partial_{\sigma^{-1} g \sigma} \circ \rho_D(\sigma)(\eta^{-1}_{\sigma^{-1} g \sigma}) \circ \rho_D(\sigma)(F(\rho_C(\sigma)(u^{-1}_{\sigma^{-1} g \sigma}))) \\
& = \partial^{-1}_{\sigma \sigma^{-1} g \sigma} \circ \eta_{\sigma^{-1} g \sigma} \circ F(\partial_{\sigma^{-1} g \sigma}) \circ F(F(\rho_C(\sigma)(u^{-1}_{\sigma^{-1} g \sigma}))) \\
& = \partial^{-1}_{\sigma \sigma^{-1} g \sigma} \circ \eta_{\sigma^{-1} g \sigma} \circ F(\partial(\sigma^{-1} g \sigma)) \circ F(\rho_C(\sigma)(u^{-1}_{\sigma^{-1} g \sigma})).
\end{align*}
\]

The first equality follows from naturality of $\eta_{\sigma}$ and the second and third from Equation (13). The final expression is the counterclockwise composition.
To check that \((F^G, \psi^G)\) is a dg form functor, we verify Equation (15), which reads
\[
\rho_D(\sigma)(\psi^G_{(C,u)}) \circ \Theta(\sigma)_{F^G(C,u)} = \psi^G_{\rho_C(\sigma)(C,u)} \circ \Theta(\sigma)(C,u)
\]
for each \((C,u) \in \mathcal{C}^G\). We compute
\[
\rho_D(\sigma)(\psi^G_{(C,u)}) \circ \Theta(\sigma)_{F^G(C,u)} = \rho_D(\sigma)(\eta_{\sigma}) \circ \theta_{\sigma,\sigma}^{-1} \circ \eta_{\sigma^2} \circ F(u_{\sigma^2})
\]
\[
= \eta_{\sigma} \circ F(\theta_{\sigma,\sigma}^{-1}) \circ F(u_{\sigma^2})
\]
\[
= \psi^G_{\rho_C(\sigma)(C,u)} \circ \Theta(\sigma)(C,u),
\]
the second equality following from Equation (13) and the others by definition.

(ii) The component at \((C,u) \in \mathcal{C}^G\) of the counterclockwise composition is
\[
\eta_{\sigma^2} \circ F(\rho_C(\sigma^2)(u_{\sigma^{-1} \sigma_1})) \circ F(\theta_{\sigma,\sigma}^{-1}) = \rho_D(\sigma^2)(F(u_{\sigma^{-1} \sigma_1})) \circ \eta_{\sigma^2} \circ F(\theta_{\sigma,\sigma}^{-1}),
\]
the equality following by naturality of \(\eta_{\sigma^2}\). Using the definition of \(F(C,u)\), the previous expression is seen to equal
\[
\rho_D(\sigma^2)(\eta_{\sigma_2^{-1} \sigma_1} \circ F(u_{\sigma^{-1} \sigma_1})) \circ \theta_{\sigma,\sigma}^{-1} \circ \eta_{\sigma_1}.
\]
Contravariance of \(\rho_D(\sigma^2)\) and Equation (13) with \(\sigma_1 \mapsto \sigma_2^{-1} \sigma_1\) and \(\sigma_2 \mapsto \sigma_2\) then gives equality with the clockwise composition. \(\square\)

### 5.2 Duality for matrix factorizations

In this section, we record standard material about duality for matrix factorizations, following [42, 50].

Fix a potential \(w \in R = k[x_1, \ldots, x_n]\). Given \(M \in \text{MF}(R, w)\), put \(M_- = (M, -d_M) \in \text{MF}(R, w)\).

The grading morphism \(J_M = \text{id}_{M_0} \Theta (-\text{id}_{M_1}) : M \to M_-\) is a dg isomorphism which satisfies
\[
J_{\Sigma M} = -\Sigma J_M.
\]

Let \((-)^{\vee} : \text{MF}(R, w)^{\text{op}} \to \text{MF}(R, -w)\) be the dg functor defined on objects by \((M, d_M)^{\vee} = (M^\vee, d_M^\vee)\), where \(M^\vee = \text{Hom}_R(M, R)\) and \(d_M^\vee(f) = -(-1)^{|f|} f \circ d_M\), and on morphisms by \(\text{Hom}_R(-, R)\). In diagrammatic form, we have
\[
\left( \begin{array}{ccc}
M_0 & \xrightarrow{d_M^0} & M_1 \\
\xleftarrow{d_M^0} & \xrightarrow{d_M^1} & \end{array} \right)^\vee = \left( \begin{array}{ccc}
M_0^\vee & \xrightarrow{-d_M^{1\vee}} & M_1^\vee \\
\xleftarrow{-d_M^{1\vee}} & \xrightarrow{d_M^{0\vee}} & \end{array} \right).
\]
The dg isomorphism \(\Theta_M = J_{M^{\vee}} \circ \text{ev}_M : M \to M^{\vee\vee}\), where \(\text{ev}_M\) is the canonical evaluation \(R\)-module isomorphism \(M \to M^{\vee\vee}\), makes \((\text{MF}(R, w), (-)^{\vee}, \Theta)\) a dg category with duality. The grading morphisms are the components of a natural isomorphism
\[
J : \Sigma \circ (-)^{\vee} \Rightarrow (-)^{\vee} \circ \Sigma.
\] (16)
Let $M \in \text{MF}(R, w)$ and $N \in \text{MF}(R', w')$. Under the swap identification $R \otimes_k R' \simeq R' \otimes_k R$, there is an isomorphism
\begin{equation}
(M \boxtimes N)^\vee \simeq N^\vee \boxtimes M^\vee
\end{equation}
in $\text{MF}(R \otimes_k R', -w \otimes 1 - 1 \otimes w')$ given by the pairing $(n^\vee \otimes m^\vee, m \otimes n) = n^\vee(n)m^\vee(m)$.

5.3 Orientifold data for $G$-equivariant matrix factorizations

Let $\hat{G}$ be a $C_2$-graded finite group acting on $R$ by $k$-algebra automorphisms. We stress that $\hat{G}$ acts by $k$-linear automorphisms, unlike the $\hat{G}$-actions of Sections 3 and 4. Assume that the potential $w$ is $\pi$-semi-invariant:
\[ \sigma(w) = \pi(\sigma)w, \quad \sigma \in \hat{G}. \]

The ungraded group $G$ is therefore a group of symmetries of $w$, while $\hat{G}$ itself is not. The results of Section 3.2, applied to trivially $C_2$-graded groups, imply that each $\sigma \in \hat{G}$ defines a dg functor $(-)\sigma : \text{MF}(R, w) \to \text{MF}(R, \pi(\sigma)w)$. The identity maps are the components of a natural isomorphism
\begin{equation}
(-)^\sigma \circ (-)^\vee \simeq (-)^\vee \circ ((-)^\sigma)^{\text{op}} : \text{MF}(R, w)^{\text{op}} \to \text{MF}(R, \pi(\sigma)w).
\end{equation}

Given $g \in G$ and $\sigma \in \hat{G} \setminus G$, define dg functors $\rho(g) : \text{MF}(R, w) \xrightarrow{(-)^g} \text{MF}(R, w)$ and $\rho(\sigma) : \text{MF}(R, w)^{\text{op}} \xrightarrow{(-)^\vee} \text{MF}(R, -w) \xrightarrow{(-)^\sigma} \text{MF}(R, w)$.

**Lemma 5.6.** The dg functors $\{\rho(\sigma)\}_{\sigma \in \hat{G}}$ extend to a Real 2-representation on $\text{MF}(R, w)$.

**Proof.** Define coherence data $\{\theta_{\sigma_2, \sigma_1}\}_{\sigma_2, \sigma_1 \in \hat{G}}$ as follows. If $\pi(\sigma_2) = -1$ and $\pi(\sigma_1) = 1$, let
\[ \theta_{\sigma_2, \sigma_1} : \rho(\sigma_2) \circ \pi(\sigma_2) \circ \rho(\sigma_1) = (-)^{\sigma_2} \circ (-)^\vee \circ ((-)^{\sigma_1})^{\text{op}} \simeq \]
\[ (-)^{\sigma_2} \circ (-)^\sigma \circ (-)^\vee \simeq (-)^{\sigma_2 \sigma_1} \circ (-)^\vee = \rho(\sigma_2 \sigma_1), \]
where the first and second natural isomorphisms are those of Equation (18) and the proof of Lemma 3.1 (restricted to ungraded groups), respectively. If $\pi(\sigma_2) = \pi(\sigma_1) = -1$, let
\[ \theta_{\sigma_2, \sigma_1} : \rho(\sigma_2) \circ \pi(\sigma_2) \circ \rho(\sigma_1) = (-)^{\sigma_2} \circ (-)^\vee \circ ((-)^{\sigma_1})^{\text{op}} \simeq \]
\[ (-)^{\sigma_2} \circ (-)^{\sigma_1} \circ (-)^\vee \circ ((-)^{\sigma_1})^{\text{op}} \simeq (-)^{\sigma_2 \sigma_1} = \rho(\sigma_2 \sigma_1), \]
where now the second natural isomorphism uses in addition the double dual isomorphism $\Theta$ for $\text{MF}(R, w)$. The remaining $\theta_{\sigma_2 \sigma_1}$ are defined similarly. The coherence of $\theta$ follows from the coherence of Lemma 3.1 and the coherence isomorphisms (14) for $\Theta$.

**Theorem 5.7.** The Real 2-representation of $G$ on $\text{MF}(R, w)$ from Lemma 5.6 induces a canonical equivalence classes of dg duality structures on $\text{MF}_G(R, w)$.

**Proof.** It follows immediately from Theorem 5.3.

**Example 5.8.** Assume that the ground field $k$ is algebraically closed. For $m \in \mathbb{Z}_{\geq 1}$, denote by $\zeta_m \in k^\times$ a primitive $m$th root of unity and $C_m$ the cyclic multiplicative group of order $m$.

(i) Let $w = x^m \in k[[x]]$, $m \geq 2$ and $\hat{G} = C_{2m}$ with $\pi$ the projection with kernel $G = C_m$. Then $w$ is $\pi$-semi-invariant with respect to the $\hat{G}$-action on $k[[x]]$ in which a generator acts by $x \mapsto \zeta_{2m} x$.

(ii) Let $w = x y^{2m} + x^{2n+1} \in k[[x, y]]$, $m, n \in \mathbb{Z}_{\geq 1}$, and $\hat{G} = C_{2m}$ with $\pi$ the projection with kernel $G = C_m$. Then, $w$ is $\pi$-semi-invariant with respect to the $\hat{G}$-action on $k[[x, y]]$ in which a generator acts by $x \mapsto -x$ and $y \mapsto \zeta_{2m} y$.

(iii) Let $w = x^m - y^m \in k[[x, y]]$, $m \in \mathbb{Z}_{\geq 2}$. As a first example, let $\hat{G} = C_m \times C_2$ with $\pi$ the projection to the second factor. Define a $\hat{G}$-action on $k[[x, y]]$ by letting a generator of $G$ act by $x \mapsto \zeta_m x$ and $y \mapsto \zeta_m y$ and a generator of $C_2$ act by swapping $x$ and $y$. Then, $w$ is $\pi$-semi-invariant. As a second example, let $\hat{G} = D_{2m}$ be the dihedral group of order $2m$ with $\pi$ the projection with kernel $G = C_m$. Define a $\hat{G}$-action on $k[[x, y]]$ by letting a generator of $G$ act by $x \mapsto \zeta_m x$ and $y \mapsto \zeta_m^{-1} y$ and a fixed element of the non-identity coset of $\hat{G}$ act by swapping $x$ and $y$. Then $w$ is $\pi$-semi-invariant. Finally, let a generator of $\hat{G} = C_{2m}$ act on $k[[x, y]]$ by $x \mapsto \zeta_{2m} x$ and $y \mapsto \zeta_{2m} y$. Then, $w$ is again $\pi$-semi-invariant. The final example generalizes to a homogeneous potential in any number of variables in an obvious way.

There is a shifted variant of Lemma 5.6, and so also Theorem 5.7, in which $\hat{G}$ acts on $\text{MF}(R, w)$ by the dg functors $\hat{\rho}(g) = \rho(g)$, $g \in G$ and $\hat{\rho}(\sigma) = \rho(\sigma) \circ \Sigma$, $\sigma \in \hat{G} \setminus G$. The shifted coherence data $\hat{\theta}$ are defined similarly to $\theta$, using in addition that $\Sigma$ commutes with $(-)^{\sigma}$, the natural isomorphism (16) and $\Sigma^2 \simeq \text{id}_{\text{MF}}$. For example, when $\pi(\sigma_2) = \pi(\sigma_1) = -1$, let

$$\hat{\theta}_{\sigma_2, \sigma_1} : \hat{\rho}(\sigma_2) \circ \rho(\sigma_2) \circ \rho(\sigma_1) = (-)^{\sigma_2} \circ (-)^{\sigma_1} \circ (-)^{\sigma_1} \circ (-)^{\sigma_1} \circ (-)^{\sigma_1} \circ (-)^{\sigma_1} \circ (\Sigma)^{\text{op}} \Rightarrow$$

$$(-)^{\sigma_2} \circ (-)^{\sigma_1} \circ (-)^{\sigma_1} \circ (-)^{\sigma_1} \circ (\Sigma)^{\text{op}} \circ \Sigma \circ \Sigma \Rightarrow (-)^{\sigma_2 \sigma_1} = \hat{\rho}(\sigma_2 \sigma_1),$$

where now the grading isomorphism $J$ is used in the first natural isomorphism.

**Example 5.9.** As discussed in Section 2.3.3, a 2-cocycle $\hat{\mu} \in Z^2(\hat{G}; k^\times)$ can be used to twist the Real 2-representation $(\rho, \theta)$, or $(\hat{\rho}, \hat{\theta})$, of $G$ on $\text{MF}(R, w)$. This construction admits the following physical interpretation. Let $\mu \in Z^2(G; k^\times)$ be the underlying untwisted 2-cocycle. As explained in [10, section 2.3], the homotopy fixed point category $\text{MF}_{G, \mu}(R, w)$ of the $\mu$-twisted 2-representation of $G$ consists of $D$-branes in the Landau–Ginzburg $G$-orbifold of $(R, w)$ with discrete torsion $\mu$. 

By Theorem 5.3, the data \((\hat{G}, \hat{\mu})\) define on \(\text{MF}_{G,\hat{\mu}}(R, w)\) a dg duality structure which, by generalizing the discussion in [28, section 3], is data required to orientifold the above Landau–Ginzburg \(G\)-orbifold. In particular, objects of the \(C_2\)-homotopy fixed point category \(\text{MF}_{G,\hat{\mu}}(R, w)^{C_2} \simeq \text{MF}_{\hat{G},\hat{\mu}}(R, w)\) are \(D\)-branes in the Landau–Ginzburg \(\hat{G}\)-orientifold of \((R, w)\) with discrete torsion \(\hat{\mu}\).

**Remark 5.10.** An analogue of Theorem 4.3 holds in the contravariant setting. Since it is not used in what follows, we restrict ourselves to a brief discussion of the terminal case \(\hat{G} = C_2\). In the present setting, the generator \(\sigma \in C_2\) is required to negate the non-degenerate quadratic form, \(\sigma(q) = -q\), and so yields a graded algebra involution

\[
\tilde{\sigma} : \text{Cl}(V, q)^{\text{op}} \xrightarrow{\sim} \text{Cl}(V, -q) \xrightarrow{\sigma} \text{Cl}(V, q),
\]

where \(\text{Cl}(V, q)^{\text{op}}\) denotes the superalgebra opposite of \(\text{Cl}(V, q)\). The \(k\)-linear dual of a graded \(\text{Cl}(V, q)\)-module is naturally a right \(\text{Cl}(V, q)\)-module, which we view as a left \(\text{Cl}(V, q)\)-module via \(\tilde{\sigma}\). In this way, we obtain a duality structure on \(\text{Cl}(V, q)\text{-grmod}\) with respect to which we verify that \(\tilde{\Phi}\) lifts to a form equivalence.

### 5.4 | Real Knörrer periodicity

Let \(k\) be algebraically closed with \(\text{char } k \nmid 2\) and \(\text{char } k \nmid |G|\). We begin this section with a contravariant analogue of Proposition 3.8.

**Proposition 5.11.** The matrix factorization \(\{u, v\} \in \text{MF}(k[\{u, v\}], uv)\) admits a Real \(G\)-equivariant structure

(i) with respect to the Real 2-representation \((\tilde{\rho}, \tilde{\theta})\) if and only if there exists \(\chi \in Z^1(\hat{G}; k_\pi^\times)\) such that \(\sigma(u) = \pi(\sigma) \chi(\sigma) u\) and \(\sigma(v) = \chi(\sigma)^{-1} v\) for each \(\sigma \in \hat{G}\), and

(ii) with respect to the Real 2-representation \((\rho, \theta)\) if and only if there exists \(\chi \in Z^1(\hat{G}; k_\pi^\times)\) such that

\[
\sigma \begin{pmatrix} u \\ v \end{pmatrix} = \begin{cases} \begin{pmatrix} \chi(\sigma)u \\ \chi(\sigma)^{-1} v \end{pmatrix} & \text{if } \pi(\sigma) = 1, \\ \begin{pmatrix} -\chi(\sigma)v \\ \chi(\sigma)^{-1} u \end{pmatrix} & \text{if } \pi(\sigma) = -1. \end{cases}
\]

In both cases, if non-empty, the set of dg isomorphism classes of Real \(G\)-equivariant structures on \(\{u, v\}\) is in bijection with \(H^1(\hat{G}; k[\{u, v\}]_\pi^\times)\).

**Proof.**

(i) A Real \(G\)-equivariant structure on \(\{u, v\}\) is the data of commutative diagrams

\[
\begin{array}{ccc}
R & \xrightarrow{u} & R & \xrightarrow{v} & R \\
\downarrow{u_\pi} & & \downarrow{u_\pi^1} & & \downarrow{v_\pi^0} \\
R^g & \xrightarrow{u_\pi} & R^g & \xrightarrow{v_\pi} & R^g
\end{array}
\]
and

\[
\begin{array}{c}
R \xrightarrow{u} R \xrightarrow{v} R \\
\downarrow^{u_0} \downarrow^{u_1} \downarrow^{u_0} \\
R^\sigma \xrightarrow{v_0} R^\sigma \xrightarrow{v_0} R^\sigma
\end{array}
\]

where \( R = k[[u, v]], \ g \in G \) and \( \sigma \in \hat{G} \setminus G \). Here, \( \chi_i \in Z^1(\hat{G}; k[[u, v]]^\times) \). Setting \( \chi = \chi_0^{-1} \), commutativity of the diagrams becomes the stated conditions and we conclude that \( \chi \in Z^1(\hat{G}; k^\times) \). The remainder of the proof is as for Proposition 3.8.

(ii) The proof is a minor variation of the previous part and so is omitted.

Proposition 5.11 distinguishes the Real 2-representation \((\hat{\rho}, \hat{\theta})\): For any \( C_2 \)-graded finite group \( \hat{G} \), there exists a Real \( G \)-equivariant structure on \( \{u, v\} \) for some action of \( \hat{G} \), say, by taking \( \chi \) trivial. This is in contrast to \((\rho, \theta)\), where there exists no Real structure in the terminal case \( \hat{G} = C_2 \).

To formulate a universal form of Real Knörrer periodicity, we therefore work in the setting of Proposition 5.11(i) and take \( k[[u, v]] \simeq \text{Sym}(V) \), where \( V \simeq k_{\chi^{-1}} \oplus k_{\chi} \) for some \( \chi \in Z^1(\hat{G}; k^\times) \) with \( u \) (resp. \( v \)) the coordinate dual to \( k_{\chi^{-1}} \) (resp. \( k_{\chi} \)). In the terminal case, the group \( H^1(C_2; k^\times) \) is trivial so that we may assume \( \chi \) to be trivial. This gives a universal choice of \( \chi \) for all \( C_2 \)-graded finite groups \( \hat{G} \). Fix a Real \( G \)-equivariant structure on \( \{u, v\} \), say, by taking \( \chi_0 = \chi_1 \) trivial.

The next result uses the universal 2-cocycle twist of a Real 2-representation \((\rho, \theta)\), denoted by \((\hat{\rho}, \hat{\theta}_-)\) in Section 2.3.3, and views Knörrer periodicity (Theorem 2.5) as a quasi-equivalence \( F = - \boxtimes \{u, v\} : MF(R, w) \to MF(R[[u, v]], w + uv) \).

**Theorem 5.12.** Let a \( C_2 \)-graded finite group \( \hat{G} \) act on \( R = \mathbb{C}[[x_1, \ldots, x_n]] \) by \( \mathbb{C} \)-algebra automorphisms which leave the potential \( w \) \( \pi \)-semi-invariant. Extend the \( \hat{G} \)-action to \( R[[u, v]] \) by \( \sigma(u) = \pi(\sigma)u \) and \( \sigma(v) = v \), \( \sigma \in \hat{G} \). The Knörrer dg functor \( F \) admits a Real \( G \)-equivariant structure when \( MF(R, w) \) and \( MF(R[[u, v]], w + uv) \) are viewed as Real 2-representations by \((\rho, \theta)\) and \((\hat{\rho}, \hat{\theta}_-)\), respectively.

**Proof.** Define natural isomorphisms \( \{\eta_\sigma : F_\rho(\sigma) \Rightarrow F_{\hat{\rho}(\sigma)} (F_{\pi(\sigma)} F)\}_{\sigma \in \hat{G}} \) as follows. Let \( M \in MF(R, w) \). For \( g \in G \), define the component of \( \eta_g \) at \( M \) to be

\[
F(\rho(g)(M)) = \rho(g)(M) \boxtimes K \xrightarrow{\text{id} \boxtimes \text{id}} \rho(g)(M) \boxtimes \rho(g)(K) \xrightarrow{\sim} \rho(g)(M \boxtimes K),
\]

where the final map is the monoidal coherence data for \((-)^g \). For \( \sigma \in \hat{G} \setminus G \), define the component of \( \eta_\sigma \) at \( M \) to be

\[
F(\rho(\sigma)(M)) = \rho(\sigma)(M) \boxtimes K \xrightarrow{\sim} K \boxtimes \rho(\sigma)(M) \xrightarrow{\text{id} \boxtimes \text{id}} \rho(\sigma)(K) \boxtimes \rho(\sigma)(M) \xrightarrow{\sim} \rho(\sigma)(M \boxtimes K) = \rho(\sigma)(F(M)).
\]
The first and third isomorphisms use the symmetry (7) and the isomorphisms (8) and (17), respectively. Explicitly, \( \eta_{\sigma} \) is the morphism

\[
\begin{array}{c}
M_0^{\sigma} \oplus M_1^{\sigma} \\
\begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}
\end{array}
\quad \begin{array}{c}
M_1^{\sigma} \oplus M_0^{\sigma} \\
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}
\end{array}
\quad \begin{array}{c}
M_0^{\sigma} \oplus M_1^{\sigma} \\
\begin{pmatrix}
-d_M^{\sigma} & v \\
u & -d_M^{\sigma}
\end{pmatrix}
\end{array}
\quad \begin{array}{c}
M_1^{\sigma} \oplus M_0^{\sigma} \\
\begin{pmatrix}
d_M^{\sigma} & v \\
u & -d_M^{\sigma}
\end{pmatrix}
\end{array}
\end{array}
\quad (19)

It is immediate that each \( \eta_{\sigma} \) is a natural isomorphism. Given the explicit description of \( \eta_{\sigma} \) (for example, in the above matrix form), verification of the coherence condition (13), with \( \theta_C \mapsto \theta \) and \( \theta_D \mapsto \tilde{\theta}_- \), is a direct calculation. For example, when \( \pi(\sigma_2) = \pi(\sigma_1) = -1 \), the coherence condition requires the equality

\[
\tilde{\theta}_{-\sigma_2 \sigma_1} \circ \left( \text{id}_{\hat{G}(\sigma_2)} \circ \eta_{\sigma_1}^{-1} \right) \circ \left( \eta_{\sigma_2} \circ \text{id}_{\hat{G}(\sigma_2)} \circ \rho(\sigma_1) \right) = \eta_{\sigma_2 \sigma_1} \circ \mathcal{K}(\tilde{\theta}_{\sigma_2 \sigma_1}).
\]

The components of the left-hand side of the desired equality in degrees 0 and 1 are represented by the matrices \(- \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \) and \( -1 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \), respectively, where we work in a basis as in diagram (19) and have suppressed from the notation all canonical evaluation isomorphisms. The extra factor of \(-1\) in the second equality is from the grading isomorphism \( J \), which appears in \( \tilde{\theta} \), and hence also \( \tilde{\theta}_- \). On the other hand, since \( \sigma_2 \sigma_1 \in G \), the right-hand side of the desired equality is also the identity. Calculations for the remaining cases of \( \pi(\sigma_1) \) and \( \pi(\sigma_2) \) are similar.

**Corollary 5.13.** The quasi-equivalence

\[
\mathcal{K} \circ \mathcal{K} : MF(R, w) \to MF(\mathcal{M}[u_1, v_1, u_2, v_2], w + u_1 v_1 + u_2 v_2)
\]

admits a Real \( G \)-equivariant structure, where \( \hat{G} \) is extended to act on \( \mathcal{M}[u_1, v_1, u_2, v_2] \) by \( \sigma(u_i) = \pi(\sigma)u_i \) and \( \sigma(v_i) = v_i \), \( \sigma \in \hat{G} \), and both categories are viewed as Real 2-representations of \( G \) by \( (\rho, \theta) \).

**Proof.** This follows by applying Theorem 5.12 twice and noting that the universal twisting \( \pi^* \hat{c} \in Z^2(\hat{G}; k^G) \) has order 2.

**Corollary 5.14.** If \( w \) is an isolated hypersurface singularity at the origin, then there is a dg form quasi-equivalence

\[
\text{Perf} (MF_G(R, w)) \to \text{Perf} (\mathcal{M}[u_1, v_1, u_2, v_2], w + u_1 v_1 + u_2 v_2),
\]

where both dg categories are given the dg duality structure of Corollary 5.4.
Proof. Applying Theorem 5.5 to the quasi-equivalence \( \mathcal{K} \circ \mathcal{K} \) of Corollary 5.13 gives a dg form functor

\[
((\mathcal{K} \circ \mathcal{K})^G, \varphi) : \text{MF}_G(R, w) \to \text{MF}_G(R[[u_1, v_1, u_2, v_2]], w + u_1v_1 + u_2v_2))
\]

whose second component is a natural isomorphism. In view of Example 5.2, Proposition 2.13 applied to the terminal \( C_2 \)-graded group gives a dg form functor

\[
(\text{Ind} (\mathcal{K} \circ \mathcal{K})^G, \text{Ind} \varphi) : \text{Perf} (\text{MF}_G(R, w)) \to \text{Perf} (\text{MF}_G(R[[u_1, v_1, u_2, v_2]], w + u_1v_1 + u_2v_2))
\]

whose first component is a quasi-equivalence by Proposition 2.9(i) and Remark 2.10. □

In particular, when applied to the terminal \( C_2 \)-graded group, Corollary 5.14 implies an equivalence

\[
\text{HMF}(R, w) \simeq \text{HMF}(R[[u_1, v_1, u_2, v_2]], w + u_1v_1 + u_2v_2)
\]

of triangulated categories with duality. This is precisely the extended Knörrer periodicity of Hori–Walcher [28, section 4.5]. Corollary 5.14 therefore provides a precise mathematical formulation and generalization of Hori–Walcher’s extended Knörrer periodicity.

Theorem 5.12 and its corollaries generalize to the twisted setting of Example 5.9. The key observation is that the coherence condition (13) in the twisted case follows from the untwisted case, since the twisted coherence isomorphisms \( \theta_{\sigma_2, \sigma_1} \) and \( \hat{\theta}_{\sigma_2, \sigma_1} \) are the same scalar multiple of their untwisted counterparts, namely \( \hat{\mu}([\sigma_2 | \sigma_1]) \). For brevity, we state only the twisted analogue of Corollary 5.14.

**Corollary 5.15.** Let \( \hat{\mu} \in Z^2(\hat{G}; k^\times) \) with restriction \( \mu \in Z^2(G; k^\times) \). There is a dg form quasi-equivalence

\[
\text{Perf} (\text{MF}_{G, \hat{\mu}}(R, w)) \to \text{Perf} (\text{MF}_{G, \mu}(R[[u_1, v_1, u_2, v_2]], w + u_1v_1 + u_2v_2)),
\]

where both dg categories are given the dg duality structure of Theorem 5.7, but with \( \theta_{\sigma, \sigma} \) replaced with \( \hat{\mu}([\sigma | \sigma]) \theta_{\sigma, \sigma} \).

In the context of Example 5.9, Corollary 5.15 is an equivalence of categories of D-branes, with orientifold data determined by \( (\hat{G}, \hat{\mu}) \), for the Landau–Ginzburg \( G \)-orbifolds of \( (R, w) \) and \( (R[[u_1, v_1, u_2, v_2]], w + u_1v_1 + u_2v_2) \), both with discrete torsion \( \mu \).

**APPENDIX A: CALCULATIONS FOR THE PROOF OF THEOREM 5.3**

This appendix contains verifications of claims made in the proof of Theorem 5.3, whose notation we keep. For simplicity, we sometimes denote \( \rho(\sigma) \) by \( \sigma \).

**Lemma A.1.** The pair \( (\rho(\sigma)(C), \{v_g\}_{g \in G}) \) defines an object of \( CG \).

**Proof.** We need to verify the equality (10), which states \( v_{g_2g_1} = g_2 \circ g_2(v_{g_1}) \circ v_{g_2} \) for each \( g_1, g_2 \in G \). The right-hand side is

\[
\theta_{g_2, g_1} \circ g_2(v_{g_1}) \circ v_{g_2} = \theta_{g_2, g_1} \circ g_2 \left( \theta^{-1}_{g_1, \sigma} \circ \theta_{\sigma, \sigma^{-1} g_1 \sigma} \circ \sigma(u_{g_1 \sigma}^{-1}) \right) \circ \theta^{-1}_{g_2, \sigma} \circ \theta_{\sigma, \sigma^{-1} g_2 \sigma} \circ \sigma(u_{g_2 \sigma}^{-1}),
\]
which by Equation (12) and naturality of \( \theta_{g_2, \sigma} \) is equal to
\[
\theta_{g_2, g_1}^{-1} \circ \theta_{g_2, g_1} \circ g_2 (\theta_{\sigma, \sigma^{-1} g_1} \circ \theta_{g_1, \sigma^{-1} g_1} \circ g_1) \circ \theta_{\sigma, \sigma^{-1} g_2} \circ \sigma (u_{\sigma^{-1} g_2}).
\]

Equation (10) with \( g_i \mapsto \sigma^{-1} g_i, i = 1, 2 \), and naturality of \( \theta_{\sigma, \sigma^{-1} g_2} \) then gives
\[
\theta_{g_2, g_1}^{-1} \circ \theta_{\sigma, \sigma^{-1} g_2} \circ \sigma (\theta_{g_2, \sigma^{-1} g_1} \circ \theta_{g_2, \sigma^{-1} g_2} \circ \sigma (u_{\sigma^{-1} g_2})).
\]

By repeated application of Equation (12), this is equal to \( v_{g_2, g_1} \).

**Lemma A.2.** The map \( \Theta(\sigma) \) is a morphism in \( C^G \).

**Proof.** The statement amounts to the equality (11), which reads \( w_g \circ \Theta(\sigma) = g(\Theta(\sigma)) \circ u_g \) for each \( g \in G \). We compute
\[
w_g \circ \Theta(\sigma) = \theta_{g_2, \sigma} \circ \theta_{g_2, \sigma^{-1} g_2} \circ \sigma (\theta_{g_2, \sigma} \circ \theta_{g_2, \sigma^{-1} g_2} \circ \sigma (\theta_{g_2, \sigma^{-1} g_1} \circ \theta_{g_2, \sigma^{-1} g_2} \circ \sigma (u_{\sigma^{-1} g_2}))).
\]

The first equality follows from the definitions, the second from naturality of \( \theta_{\sigma, \sigma^{-1} g_2} \) and Equation (10) with \( g_1 \mapsto \sigma^{-2} g_2 \) and \( g_2 \mapsto \sigma^2 \) and the third from Equation (12).

**Lemma A.3.** The map \( \varphi_{\sigma_1, \sigma_2} : \rho(\sigma_1)(C, u) \to \rho(\sigma_2)(C, u) \) is a morphism in \( C^G \).

**Proof.** Write \( \{v_g^{\sigma_1}\}_{g \in G} \) and \( \{v_g^{\sigma_2}\}_{g \in G} \) for the homotopy fixed point data of \( \sigma_1 C \) and \( \sigma_2 C \), respectively. The present lemma amounts to the equality (11), which reads \( v_g^{\sigma_2} \circ \varphi_{\sigma_1, \sigma_2} = g(\varphi_{\sigma_1, \sigma_2}) \circ v_g^{\sigma_1} \) for each \( g \in G \). The definitions give
\[
v_g^{\sigma_2} \circ \varphi_{\sigma_1, \sigma_2} = g(\sigma_2(u_{\sigma_2^{-1} \sigma_1} \circ \theta_{\sigma_2^{-1} \sigma_1} \circ \theta_{\sigma_2^{-1} \sigma_1} \circ \sigma_1 \circ u_{\sigma_1^{-1} g_2} \circ \sigma_1)).
\]

Pre-applying \( \sigma_2^{-1} \) to \( \sigma_1(u_{\sigma_1^{-1} g_2}) \) gives
\[
\sigma_1(u_{\sigma_1^{-1} g_2}) = \theta_{\sigma_2, \sigma_1^{-1} \sigma_2} \circ \sigma_2(\theta_{\sigma_2^{-1} \sigma_1} \circ \sigma_2^{-1} \sigma_1 \circ \sigma_1 \circ u_{\sigma_1^{-1} g_2} \circ \sigma_1) \circ \theta_{\sigma_2^{-1} \sigma_1} \circ \sigma_1 \circ u_{\sigma_1^{-1} g_2} \circ \sigma_1 \circ \theta_{\sigma_2, \sigma_1^{-1} \sigma_2}^{-1}.
\]

Using Equation (10) to rewrite \( \sigma_2^{-1} \sigma_1(u_{\sigma_1^{-1} g_2}) \) gives
\[
\sigma_2^{-1} \sigma_1(u_{\sigma_1^{-1} g_2}) = \theta_{\sigma_2, \sigma_1^{-1} \sigma_2} \circ \sigma_2(\theta_{\sigma_2^{-1} \sigma_1} \circ \sigma_2^{-1} \sigma_1 \circ \sigma_1 \circ u_{\sigma_1^{-1} g_2} \circ \sigma_1 \circ \theta_{\sigma_2^{-1} \sigma_1} \circ \sigma_1 \circ u_{\sigma_1^{-1} g_2} \circ \sigma_1 \circ \theta_{\sigma_2, \sigma_1^{-1} \sigma_2}^{-1}).
\]
Similarly, we may pre-apply $\sigma_2\sigma_2^{-1}$ to $g(\sigma_2(u_{\sigma_2^{-1}\sigma_1}))$ to get a new expression for the latter. Plugging these expressions into $\varphi_{(C,u)}^{\sigma_1\sigma_2}$ and repeatedly applying Equation (12) gives

$$g(\varphi_{(C,u)}^{\sigma_1\sigma_2}) \circ \varphi_{(C,u)}^{\sigma_1} = \varphi_{(C,u)}^{\sigma_1\sigma_2} \circ g(\sigma_2(u_{\sigma_2^{-1}\sigma_1})) \circ \varphi_{(C,u)}^{\sigma_1\sigma_2} \circ \varphi_{(C,u)}^{\sigma_1},$$

as required.

$\square$

Lemma A.4. The pair $(\text{id}_{C^G}, \varphi^{\sigma_1\sigma_2})$ is a dg form functor.

Proof. It remains to verify the coherence condition (15), which states

$$\sigma_2(\varphi_{(C,u)}^{\sigma_1\sigma_2}) \circ \Theta(\sigma_2)(C,u) = \varphi_{(C,u)}^{\sigma_1\sigma_2} \circ \Theta(\sigma_1)(C,u)$$

for each $(C, u) \in C^G$. We have

$$\sigma_2(\varphi_{(C,u)}^{\sigma_1\sigma_2}) \circ \Theta(\sigma_2)(C,u) = \sigma_2(\sigma_2(u_{\sigma_2^{-1}\sigma_1})) \circ \varphi_{(C,u)}^{\sigma_1\sigma_2} \circ \sigma_2(\Theta_{\sigma_1\sigma_2})(C,u) = \sigma_2(\Theta_{\sigma_1\sigma_2})(C,u) \circ \varphi_{(C,u)}^{\sigma_1\sigma_2}.$$

The first through fourth equalities follow by definition, naturality of $\Theta_{\sigma_2\sigma_3}$, Equation (10) and Equation (12), respectively. Similarly,

$$\varphi_{(C,u)}^{\sigma_1\sigma_2} \circ \Theta(\sigma_1)(C,u) = \varphi_{(C,u)}^{\sigma_1\sigma_2} \circ \Theta(\sigma_1)(C,u) = \varphi_{(C,u)}^{\sigma_1\sigma_2} \circ \Theta(\sigma_1)(C,u) = \varphi_{(C,u)}^{\sigma_1\sigma_2} \circ \Theta(\sigma_1)(C,u).$$

The first, second and third equalities follow by definition, the explicit expression for $\varphi_{(C,u)}^{\sigma_1\sigma_2}$ and naturality of $\Theta_{\sigma_2\sigma_3}$, and Equation (12), respectively. Finally, Equation (10) with $g_1 \mapsto \sigma_2^{-1}\sigma_1\sigma_2^{-1}$ and $g_2 \mapsto \sigma_2\sigma_1$ gives $\varphi_{(C,u)}^{\sigma_1\sigma_2} \circ \Theta(\sigma_1)(C,u) = \Theta_{\sigma_1\sigma_2}\circ \varphi_{(C,u)}^{\sigma_1\sigma_2}.$

$\square$

Lemma A.5. For each $\sigma_1, \sigma_2, \sigma_3 \in \hat{G} \setminus G$, there are equalities of dg form functors

$$(\text{id}_{C^G}, \varphi^{\sigma_1\sigma_3}) = (\text{id}_{C^G}, \varphi^{\sigma_2\sigma_3}) \circ (\text{id}_{C^G}, \varphi^{\sigma_1\sigma_2}).$$
Proof. We need to verify that $\varphi^{\sigma_1,\sigma_3}_{(C,u)} = \varphi^{\sigma_2,\sigma_3}_{(C,u)} \circ \varphi^{\sigma_1,\sigma_2}_{(C,u)}$ for each $(C,u) \in \mathcal{C}$. We have

$$\varphi^{\sigma_1,\sigma_3}_{(C,u)} = \sigma_3(u^{-1}_{\sigma_1}) \circ \varphi^{\sigma_1,\sigma_1}_{(C,u)} \circ \sigma_1 \circ \sigma_3.$$  

The first equality follows by definition and the second by Equation (10) with $g_1 \mapsto \sigma_2^{-1}\sigma_1$ and $g_2 \mapsto \sigma_3^{-1}\sigma_2$. Applying Equation (12) to the fourth and fifth terms on the right-hand side then gives $\varphi^{\sigma_2,\sigma_3}_{(C,u)} \circ \varphi^{\sigma_1,\sigma_2}_{(C,u)}$. □

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