STEADY PERIODIC EQUATORIAL WATER WAVES WITH VORTICITY

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ABSTRACT. Of concern are steady two-dimensional periodic geophysical water waves of small amplitude near the equator. The analysis presented here is based on the bifurcation theory due to Crandall-Rabinowitz. Dispersion relations for various choices of the vorticity distribution, including constant, affine, and some nonlinear vorticities are obtained.

1. Introduction. Geophysical fluid dynamics is the study of fluid motion where the Earth’s rotation plays a significant role. In particular, the Coriolis forces are incorporated into the governing Euler equations. Geophysical fluid dynamics applies to a wide range of oceanic and atmospheric flows (see the discussions in the monographs [21, 44]) and shows a high complexity, which leads to an inherent mathematical intractability [1]. In order to mitigate the level of complexity, it is natural and common to derive simpler approximate equations of Euler equations. One widely used approach is the \( f \)-plane approximation, which takes a constant Coriolis parameter into account and does not consider the latitudinal variations. This approximation has been applied to oceanic flows within a zonal region of approximately \( 2^\circ \) on each side of the Equator [3, 10, 21]. We refer the reader to [6, 11, 12, 23, 24, 29, 30] for recent progresses in the study of the exact solutions and dynamical behaviors of the equatorial water waves. Recently a unified approach towards exact solutions concerning geophysical flows was presented in [35].

Equatorial water waves exhibit particular dynamics due to the vanishing of the Coriolis parameter along the equator, and the vertical stratification of the ocean is greater in this region than anywhere else. Both factors facilitate the propagation of geophysical waves that either raise or lower the equatorial thermocline, which is the sharp boundary between warm and deeper cold waters. The inherent ocean adjustment depends crucially on the existence of geophysical waves that can alter the depth of the thermocline [21]. See [11, 40] for analytical results concerning the

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dynamics of the thermocline in the equatorial region. An advantage of equatorial gravity water waves is that we can consider the flows as being two-dimensional because the equator acts as a natural waveguide and as a result the equatorial waves are predominantly zonal. The rigorous mathematical study of equatorial water waves was initiated by Constantin who presented in [3] the model of wave-current interactions in the $f$-plane approximation for underlying currents of positive constant vorticity. Starting with this pioneering paper, recently some essential results have been achieved. We refer the reader to [10, 24, 28, 29, 30, 32, 33] for the study of exact solutions and instability, and [5, 31, 34, 41] for the related properties of the periodic geophysical water flows with vorticity. In this paper, we will prove the existence of steady two-dimensional periodic equatorial water waves in the $f$-plane approximation and the underlying current has a very general vorticity. The study of steady periodic water waves waves with vorticity has received much attention since the work [18] by Constantin and Strauss. We refer the reader to [7, 8, 9, 19, 25, 26, 27] for recent results. Vorticity plays the key role in describing oceanic flows, and this aspect was very recently emphasized in thorough analytical studies [13, 14, 15, 17, 42].

The dispersion relation is the necessary and sufficient condition for local bifurcation. Although explicit dispersion relation allows us to obtain some essential feature of water waves, it is very difficult to present the explicit expressions when the vorticity is not constant. For the case of constant vorticity, explicit dispersion relations were derived in [3, 5] for the equatorial waves and in [18] for the classical gravity water waves. In [36], Karageorgis provided a wide range of examples of non-constant vorticity functions for which the dispersion relation can be determined explicitly for the classical gravity water wave problem. We remark that some progress of the study of the dispersion equation for steady water waves with vorticity has been presented in [37, 38]. See [4, 34, 39] and the references therein for more results on dispersion relations. We will obtain several explicit dispersion functions for equatorial water waves, which involve constant vorticity, linear vorticity, and some possibly nonlinear vorticity distributions.

2. The governing equations and equivalent formulation. We approximate the shape of the Earth as a perfect sphere which rotates with a constant rotational speed $\Omega = 7.29 \times 10^{-5}$ rad/s. We then choose an orthogonal reference framework with origin at a point on the Earth’s surface such that the $x$-axis points horizontally due east, the $y$-axis horizontally due north and the $z$-axis orthogonally outward. The upper boundary of the centre layer is located at $z = -d$, while $z = -\eta(x, y, t)$ is the thermocline. In the region $-\eta(x, y, t) \leq z \leq -d$, the full governing equations in the $f$-plane approximation near the equator are the Euler equations [3]

$$
\begin{align*}
\frac{ud_t + uw_x + uv_y + wu_z + 2\Omega w}{\rho} &= -\frac{1}{\rho} P_x, \\
\frac{v_t + uw_x + vv_y + wv_z}{\rho} &= -\frac{1}{\rho} P_y, \\
w_t + uw_x + vw_y + wu_z - 2\Omega u &= -\frac{1}{\rho} P_z - g,
\end{align*}
$$

(1)

or together with the equation of mass conservation

$$u_x + v_y + w_z = 0.
$$

(2)

Here $t$ is the time, $(u, v, w)$ is the fluid’s velocity, $P$ is the pressure, $g$ is the standard gravitational acceleration at the Earth’s surface and $\rho$ is the water’s density. Beneath the thermocline $z = -\eta(x, y, t)$, equation (2) is the same and equation (1)
holds with a slightly higher density $\rho + \Delta \rho$ of the motionless colder water (for example, the typical value of $\Delta \rho/\rho$ for the equatorial Pacific is 0.006 [23]), and therefore $u = v = w = 0$ and

$$P = P_0 - g(\rho + \Delta \rho)z, \quad z \leq -\eta(x, y, t)$$

for some constant $P_0$. A peculiar feature of (1)-(2) can be inferred from the vorticity equation [44]: it is a two-dimensional flow, independent upon the $y$-coordinate and with $v = 0$. While the vorticity equation plays an appreciable role in proving the two-dimensionality of gravity wave trains over flow with constant vorticity vector, the boundary conditions are decisive in proving the two-dimensionality (see the rigorous analytical argument in [2]). Such a two-dimensional character holds also in the case of time dependent water flows [43] and the capillary water flows of constant non-zero vorticity [16]. For this reason, we are able to look for two-dimensional flows, moving in the zonal direction along the equator independent of $y$. Therefore the system governing the motion becomes

$$\begin{cases}
u + uu_x + wu_z + 2\Omega w = -\frac{1}{\rho}P_x, \\
w + uw_x + ww_z - 2\Omega u = -\frac{1}{\rho}P_z - g, \\
u_x + w_z = 0.
\end{cases}$$

(4)

We now present the boundary conditions. The kinematic boundary conditions are

$$w = -\eta_t - u\eta_x \quad \text{on} \quad z = -\eta(x, t),$$

(5)

and

$$w = 0 \quad \text{on} \quad z = -d.$$ 

(6)

The dynamic boundary condition is

$$P = P_0 - g(\rho + \Delta \rho)z \quad \text{on} \quad z = -\eta(x, t),$$

(7)

and ensures the continuity of the pressure across the thermocline in view of (3).

Given $c > 0$, we are looking for periodic waves traveling at speed $c$, that is, $u, w, P, \eta$ have the form $(x - ct)$ and all of them are periodic with period $L$. In the new reference frame $(x - ct, z) \mapsto (x, z)$, we assume that there are no stagnation points of the flow, that is,

$$u < c \quad \text{for} \quad -\eta(x) \leq z \leq -d,$$

(8)

throughout the fluid.

We define, up to an additive constant, the stream function $\psi(x, z)$ by

$$\psi_x = -w, \quad \psi_z = u - c, \quad \text{for} \quad -\eta(x) < z < -d.$$

The relative mass flux is given by

$$m = \int_{-\eta(x)}^{-d} \left(u(x, z) - c\right)dz < 0,$$

which is independent of $x$ by the kinematic boundary conditions (5) and (6). Moreover, $\psi(x, z)$ is constant on $z = -d$ and on $z = -\eta(x)$. We normalize $\psi$ by choosing $\psi = 0$ on $z = -\eta(x)$, and then $\psi = m$ on $z = -d$. From the equation of mass conservation, we can deduce that $\psi$ is $L$-periodic in $x$. Now the equations of motion
and the boundary conditions (5)-(7) are expressed as
\[
\begin{cases}
\psi_z \psi_{zz} - \psi_x \psi_{xz} - 2\Omega \psi_x = -\frac{1}{\rho} P_x, & \text{for } -\eta(x) < z < -d, \\
-\psi_z \psi_{xx} + \psi_x \psi_{xz} - 2\Omega (\psi_z + c) = -\frac{1}{\rho} P_z - g, & \text{for } -\eta(x) < z < -d, \\
\psi_x = \psi_z \eta_x, & \text{on } z = -\eta(x), \\
P = P_0 - g(\rho + \Delta \rho) z, & \text{on } z = -\eta(x), \\
\psi = 0, & \text{on } z = -d.
\end{cases}
\]

Let \(\omega = u_z - w_x = \Delta \psi\) be the vorticity of the flow. The validity of (8) guarantees the existence of a function \(\gamma\), called the vorticity function, such that \(\omega = \gamma(\psi)\) throughout the fluid (see [18]). Thus
\[
\Delta \psi = \gamma(\psi).
\]

Let
\[
\Gamma(p) = \int_0^p \gamma(-s) ds, \quad 0 \leq p \leq -m,
\]
attain its maximum value \(\Gamma_{\text{max}}\) at some \(p \in [0, -m]\). From the first two equations in (9) we obtain in analogy with Bernoulli’s law for gravity water waves [1], that the expression
\[
E = \frac{\psi_x^2 + \psi_z^2}{2} - 2\Omega \psi + (g - 2\Omega c) z + \frac{P}{\rho} + \Gamma(-\psi)
\]
is constant throughout the layer \(-\eta(x) < z < -d\). Therefore the dynamic boundary condition (7) is equivalent to
\[
\frac{\psi_x^2 + \psi_z^2}{2} - (\tilde{g} + 2\Omega c) z = \frac{Q}{2}, \quad \text{on } z = -\eta(x),
\]
where \(Q = 2(E - \frac{P_0}{\rho})\) and
\[
\tilde{g} = g - \frac{\Delta \rho}{\rho}
\]
is the reduced gravity [23].

Summarizing the above considerations, we can reformulate the problem as the following free boundary problem
\[
\begin{cases}
\Delta \psi = \gamma(\psi), & \text{for } -\eta(x) < z < -d, \\
|\nabla \psi|^2 - 2(\tilde{g} + 2\Omega c) z = Q, & \text{on } z = -\eta(x), \\
\psi = 0, & \text{on } z = -\eta(x), \\
\psi = m, & \text{on } z = -d,
\end{cases}
\]
which is to be solved for functions that are of period \(L\) in the \(x\)-variable. For simplicity, we set \(L = 2\pi\) in the following.

Since \(\psi\) is constant on both boundaries, with \(z \mapsto \psi(x, z)\) strictly decreasing in view of (8), we are able to use the hodograph transform [22]
\[
q = x, \quad p = -\psi(x, z)
\]
to convert the unknown domain
\[
D_\eta = \{(x, z) : x \in (-\pi, \pi), -\eta(x) < z < -d\}
\]
of one wavelength into the fixed rectangular domain
\[
R = (-\pi, \pi) \times (0, -m).
\]
For every fixed \(x\), in the new \((q,p)\)-variables, the height function
\[
h(q,p) = z + d
\]
represents the level beneath the upper flat boundary of the centre layer, and it is a single-valued function of \(\psi\) in the fixed strip \(0 < p < -m\). Since
\[
h_q = \frac{w}{u - c}, \quad h_p = -\frac{1}{c - u},
\]
we have
\[
w = -\frac{h_q}{h_p}, \quad u = c - \frac{1}{h_p}, \quad \partial_x = \partial_q - \frac{h_q}{h_p} \partial_p, \quad \partial_z = \frac{1}{h_p} \partial_p.
\]
Therefore,
\[
\partial_q \omega = \left(\partial_x - \frac{w}{c - u} \partial_z\right) \omega.
\]
It follows from the equation (4) that
\[
(u - c)\omega_x + w \omega_z = 0.
\]
Consequently, \(\partial_q \omega = 0\) and then \(\omega\) is a function of \(p\) only throughout the rectangle \(R\), \(\omega = \gamma(-p)\), with
\[
\gamma(-p) = \gamma(\psi) = \omega = u_z - w_x.
\]
Using the above facts, problem (10) becomes
\[
\begin{cases}
(1 + h_p^2)h_{pp} - 2h_q h_p h_{qp} + h_q^2 h_{qq} = \gamma(-p)h_p^3, & \text{in } 0 < p < -m, \\
1 + h_p^2 = \left[2(\hat{g} + 2\Omega c)(h - d) + Q\right]h_p^2, & \text{on } p = 0, \\
h = 0, & \text{on } p = -m,
\end{cases}
\]
with \(h\) even and of period \(2\pi\) in the \(q\)-variable. Condition (8) is replaced by
\[
h_p > 0 \quad \text{throughout the closed rectangle } \overline{R}.
\]
Using similar arguments as in [1, 18], we conclude that the governing equations (4)-(7) are equivalent to problem (11). We will construct a unique solution of (11) in the space \(C^{3,\alpha}_{\text{per}}(\overline{R})\) for some \(\alpha \in (0, 1)\). As a consequence the vorticity function \(\gamma\) then belongs to \(C^{1,\alpha}[0, -m]\) and \((u, w, \eta) \in C^{2,\alpha}_{\text{per}}(D_\eta) \times C^{2,\alpha}_{\text{per}}(\overline{D_\eta}) \times C^{3,\alpha}_{\text{per}}(\overline{R})\) with \(D_\eta\) being the closure of the fluid domain \(D_\eta = \{(x, z) : x \in \mathbb{R}, -\eta(x) < z < -d\}\).

3. Local bifurcation. The laminar flow solutions (independent of \(q\)) to (11) are given explicitly by
\[
H(p, \lambda) = \int_0^p \frac{1}{\sqrt{\lambda - 2\Gamma(s)}} ds + d + \frac{\lambda - Q}{2(\hat{g} + 2\Omega c)}, \quad 0 < p < -m
\]
and the parameters \(\lambda\) and \(Q\) are related by
\[
0 < \int_0^{-m} \frac{1}{\sqrt{\lambda - 2\Gamma(s)}} ds = \frac{Q - \lambda}{2(\hat{g} + 2\Omega c)} - d,
\]
in view of the boundary condition \(H(-m) = 0\). Therefore,
\[
H(p, \lambda) = \int_{-m}^{p} \frac{1}{\sqrt{\lambda - 2\Gamma(s)}} ds.
\]
The function \(\lambda \mapsto Q(\lambda)\) is strictly convex for \(\lambda > 0\), and it attains its minimum \(Q_0\) at the unique point \(\lambda_0\) with
\[
\int_0^{-m} \left(\lambda_0 - 2\Gamma(s)\right)^{-\frac{1}{2}} ds = \frac{1}{\hat{g} + 2\Omega c}.
\]
Now we linearize the problem (11) about a laminar solution $H$. We look for solutions $h \in C^{3,0}_{\text{per}}(\overline{\mathcal{R}})$, even in the $q$-variable and zero on $p = -m$, of the form

$$h = H + \varepsilon f.$$ 

Let

$$a(\lambda, p) = \sqrt{\lambda - 2\Gamma(p)}.$$ 

Then we obtain the following boundary problem for the function $f$,

$$\begin{cases}
(a^3 f_p)_p + af_{qq} = 0, & \text{in } \mathcal{R}, \\
a^3 f_p = - (\check{g} + 2\Omega c) f, & \text{on } p = 0, \\
f = 0, & \text{on } p = -m.
\end{cases} \tag{14}$$

Keeping in mind that the even function $f$ has the Fourier series representation

$$f(q, p) = \sum_{k=0}^{\infty} f_k(p) \cos(kq) \quad \text{in } C^2_{\text{per}}(\mathcal{R}).$$

It is easy to see that $f$ solves (14) if and only if each $f_k$ solves the Sturm-Liouville problem

$$\begin{cases}
(a^3 M_p)_p = k^2 aM, & \text{in } (0, -m), \\
a^3 M_p = - (\check{g} + 2\Omega c) M, & \text{on } p = 0, \\
M = 0, & \text{on } p = -m.
\end{cases} \tag{15}$$

Since we are looking for $2\pi$-periodic solutions of (15), we consider the case $k = 1$ and analyse the related minimization problem

$$\mu(\lambda) = \inf_{\phi \in \mathcal{V}} \frac{-(\check{g} + 2\Omega c)\phi^2(0) + \int_{0}^{-m} a^3 \phi_p^2 dp}{\int_{0}^{-m} a\phi^2 dp}, \tag{16}$$

where $\mathcal{V} := \{ \phi \in H^1(0, -m), \phi(-m) = 0, \phi \neq 0 \}$. Using the same arguments as in [1], we know that a minimizer $M \in C^{3,\alpha}[0, -m]$ of (16) is a classical solution of the weighted Sturm-Liouville problem

$$\begin{cases}
(a^3 M_p)_p = - \mu(\lambda) aM, & \text{in } (0, -m), \\
a^3 M_p = - (\check{g} + 2\Omega c) M, & \text{on } p = 0, \\
M = 0, & \text{on } p = -m.
\end{cases} \tag{17}$$

For the existence of solutions to the linear problem (15), it is necessary that $\mu(\lambda) = -1$ for some $\lambda > 2\Gamma_{\text{max}}$. First, for $\lambda > (\check{g} + 2\Omega c) + 2\Gamma_{\text{max}}$, we have

$$a(\lambda, p) = \sqrt{\lambda - 2\Gamma(p)} \geq \sqrt{\check{g} + 2\Omega c}, \quad p \in [0, -m].$$

Therefore

$$\int_{0}^{-m} (a^3 \phi_p^2 + a\phi^2) dp \geq \sqrt{\check{g} + 2\Omega c} \int_{0}^{-m} (\check{g} + 2\Omega c) \phi_p^2 dp + \int_{0}^{-m} \phi^2 dp$$

$$\geq 2(\check{g} + 2\Omega c) \int_{0}^{-m} |\phi| dp$$

$$\geq 2(\check{g} + 2\Omega c) \left| \int_{0}^{-m} \phi dp \right|$$

$$= (\check{g} + 2\Omega c) \phi^2(0), \quad \text{for } \phi \in H^1((0, -m)) \text{ with } \phi(-m) = 0,$$

which implies that $\mu(\lambda) > -1$ for $\lambda > (\check{g} + 2\Omega c) + 2\Gamma_{\text{max}}$. Moreover, using the similar method in [18], we can show that $\mu(\lambda)$ is continuous and a strictly increasing function of $\lambda$ in any interval where it is negative and the solution $\lambda^*$ to $\mu(\lambda) = -1$
is unique if exists. Therefore, if we are able to show that $\mu(\lambda) \leq -1$ for some $\lambda > 2\Gamma_{\text{max}}$, then we know that there exists some $\lambda > 2\Gamma_{\text{max}}$ such that $\mu(\lambda) = -1$.

Now we prove the following sufficient condition for the vorticity function to ensure that $\mu(\lambda) \leq -1$ for some $\lambda > 2\Gamma_{\text{max}}$.

**Lemma 3.1.** Assume that there exists $\beta \in (3, \infty]$ such that

$$
\tilde{g} + 2\Omega c > \frac{\sqrt{2}}{3} \beta^*\|\gamma\|^3/2 p_1^{3/2 - 1} + \frac{2\sqrt{2} \beta^*}{1 + 4\beta^*} \|\gamma\|_2^{1/2} p_1^{1 + \frac{1}{2\beta^*}},
$$

(18)

where $\beta^* = \frac{\beta}{\pi - 1}$ is the conjugate number of $\beta$ (if $\beta = \infty$ then $\beta^* = 1$),

$$p_1 = \text{min}\{p \in [0, -m] : \Gamma(p) = \Gamma_{\text{max}}\},$$

and

$$
\|\gamma\|_\beta = \begin{cases} 
\left( \int_0^m |\gamma(s)|^\beta ds \right)^{1/\beta}, & \text{if } 3 < \beta < \infty, \\
\max_{p \in [0, -m]} |\gamma(p)|, & \text{if } \beta = \infty.
\end{cases}
$$

Then there exists $\lambda > 2\Gamma_{\text{max}}$ such that $\mu(\lambda) \leq -1$.

**Proof.** Consider the expression

$$E(k) = \left(2\|\gamma\|_\beta\right)^{1/2} \frac{k^2}{2k - 1 + \frac{2}{2\beta^*}} p_1^{3/2 - 1} + \left(2\|\gamma\|_\beta\right)^{1/2} \frac{1}{2k + 1 + \frac{1}{2\beta^*}} p_1^{1 + \frac{1}{2\beta^*}}$$

for $k > 1/2$. Because the limit of $E(k)$ as $k \to 1/2$ equals the right-hand side of (18), we can find a $k > 1/2$ such that $\tilde{g} + 2\Omega c > E(k)$. Let $n \geq 2$ and the function

$$\phi_n(p) = \begin{cases} 
0, & p_n \leq p \leq -m, \\
(p_n - p)^k, & 0 \leq p \leq p_n,
\end{cases}
$$

where

$$p_n = \left(1 - \frac{1}{n}\right)p_1 - \frac{m}{n} > 0.$$

Clearly $\phi_n \in H^{1}(0, -m)$ and $\phi_n(0) = 0, \phi_n \neq 0$. Using Hölder’s inequality, we obtain

$$a(p, 2\Gamma_{\text{max}}) \leq \sqrt{2\Gamma_{\text{max}} - 2\Gamma(p)} = \sqrt{2\Gamma(p_1) - 2\Gamma(p)} \leq \left(2\|\gamma\|_\beta\right)^{1/2} |p_1 - p|^{1/(2\beta^*)}.$$

Thus, using the fact $\phi_n^2(0) = p_n^{2k}$, we obtain

$$\int_0^{-m} a^2(p, 2\Gamma_{\text{max}}) \left(\phi_n^2(p)\right)^2 dp + \int_0^{-m} a(p, 2\Gamma_{\text{max}}) \phi_n^2(p) dp$$

$$\leq \left(2\|\gamma\|_\beta\right)^{2/3} k^3 \int_0^{p_1} |p_1 - p|^{2k} (p_n - p)^{2k-2} dp + \left(2\|\gamma\|_\beta\right)^{2/3} \int_0^{p_1} |p_1 - p|^{2k} (p_n - p)^{2k-2} dp$$

$$= \left(2\|\gamma\|_\beta\right)^{2/3} k^3 \left\{ \int_0^{p_1} |p_1 - p|^{2k} (p_n - p)^{2k-2} dp + \int_0^{p_1} |p_1 - p|^{2k} (p_n - p)^{2k-2} dp \right\}$$

$$+ \left(2\|\gamma\|_\beta\right)^{2/3} k^3 \left\{ \int_0^{p_1} |p_1 - p|^{2k} (p_n - p)^{2k-2} dp + \int_0^{p_1} |p_1 - p|^{2k} (p_n - p)^{2k-2} dp \right\}$$

$$\leq \left(2\|\gamma\|_\beta\right)^{2/3} k^3 \left\{ \int_0^{p_1} (p_n - p)^{2k-2} dp + (p_n - p_1)^{2k} \int_0^{p_1} (p_n - p)^{2k-2} dp \right\}$$

$$+ \left(2\|\gamma\|_\beta\right)^{2/3} k^3 \left\{ \int_0^{p_1} (p_n - p)^{2k-2} dp + (p_n - p_1)^{2k} \int_0^{p_1} (p_n - p)^{2k-2} dp \right\}$$

$$\leq \phi_n^2(0)(A_n + B_n).$$
By direct computations,
\[
A_n = \left(2\|\gamma\|_\beta\right)^{\frac{1}{2}} \frac{k^2}{2k - 1 + \frac{3}{2}\rho} p_n^{-\rho - 1} + \left(2\|\gamma\|_\beta\right)^{\frac{1}{2}} \frac{1}{2k + 1 + \frac{1}{2}\rho} p_n^{1 + \frac{1}{\rho}},
\]
and
\[
B_n = \left(\frac{1}{2k - 1} - \frac{1}{2k + 1 + \frac{1}{2}\rho}\right)(2\|\gamma\|_\beta)^{\frac{1}{2}} k^2 \left(\frac{p_n - p_1}{p_n}\right)^{2k-1+\frac{3}{2}\rho} p_n^{2k} \\
+ \left(\frac{1}{2k + 1} - \frac{1}{2k + 1 + \frac{1}{2}\rho}\right)(2\|\gamma\|_\beta)^{\frac{1}{2}} \left(\frac{p_n - p_1}{p_n}\right)^{2k+1+\frac{1}{2}\rho} p_n^{2k}.
\]
Using the facts $|p_n - p_1| < 1$ and $\lim_{n \to \infty} p_n = p_1$, given $\varepsilon \in (0, \hat{g} + 2\Omega c)$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$,
\[
A_n < \hat{g} + 2\Omega c - \varepsilon \quad \text{and} \quad B_n < \varepsilon.
\]
Therefore we show the existence of $\phi_n \in H^1((0, -m))$ with $\phi_n(-m) = 0, \phi_n \neq 0$ and satisfies
\[
\int_0^{-m} a^3(p, 2\Gamma_{\text{max}})\left(\phi_n'(p)\right)^2 dp + \int_0^{-m} a(p, 2\Gamma_{\text{max}})\phi_n^2(p) dp < (\hat{g} + 2\Omega c)\phi_n^2(0),
\]
which implies that $\mu(2\Gamma_{\text{max}}) < -1$. Using the continuity of the function $\lambda \mapsto \mu(\lambda)$, we know the existence of some $\lambda > 2\Gamma_{\text{max}}$ with $\mu(\lambda) \leq -1$. \hfill \Box

**Remark 1.** If we take $\beta = \infty$ in Lemma 3.1, then (18) becomes the following condition
\[
\hat{g} + 2\Omega c > \frac{\sqrt{3}}{3} \|\gamma\|_\infty^{3/2} p_1^{\frac{3}{4}} + \frac{2\sqrt{2}}{5} \|\gamma\|_1^{3/2} p_1^{\frac{3}{4}},
\]
which is in analogy with that given by Constantin in [1] for the classical gravity water waves. Note however, that the class of vorticity functions satisfying the latter condition are a true subclass of those satisfying (18). If the vorticity $\gamma \leq 0$, then $\Gamma_{\text{max}} = 0$ and $p_1 = 0$, and the linearized problem has always solutions since (18) is true. Moreover (18) holds also true for vorticities with positive values, provided $\|\gamma\|_\beta$ is small enough. See the next Section for detailed analysis of the dispersion relation.

Before we prove our main result, let us cite the following bifurcation theorem due to Crandall and Rabinowitz.

**Theorem 3.2.** [20] Let $X, Y$ be real Banach spaces and let $F \in C^k(\mathcal{O}, Y)$ with $k \geq 2$, where $\mathcal{O} \subset \mathbb{R} \times X$ is an open set containing the point $(\lambda^*, 0)$ satisfy
(i) $F(\lambda, 0) = 0$ for all $(\lambda, 0) \in \mathcal{O}$;
(ii) $\mathcal{L} = \partial_f F(\lambda^*, 0) \in \mathcal{L}(X, Y)$ is a linear Fredholm operator of index zero and one-dimensional kernel $N(\mathcal{L})$ generated by some $f^* \in X \setminus \{0\}$;
(iii) the transversality condition $[\partial_{\lambda, f} F(\lambda^*, 0)](1, f^*) \notin \mathcal{R}(\mathcal{L})$ holds, where $\mathcal{R}(\mathcal{L})$ is the range of the linear operator $\mathcal{L}$ and
\[
\partial_{\lambda, f} F(\lambda^*, 0) = \partial_\lambda [\partial_f F(\lambda, 0)]\bigg|_{\lambda = \lambda^*} \in \mathcal{L}(\mathbb{R}, \mathcal{L}(X, Y)) = \mathcal{L}(\mathbb{R} \times X, Y).
\]
Then there exists $\varepsilon > 0$ and a branch of solutions
\[
\{(\lambda, f) = (\lambda(s), s\chi(s)) : s \in \mathbb{R}, |s| < \varepsilon \} \subset \mathbb{R} \times X
\]
of $F(\lambda, f) = 0$ with $\lambda(0) = \lambda^*, \chi(0) = f^*$ and such that $s \mapsto \lambda(s) \in \mathbb{R}, s \mapsto s\chi(s) \in X$ are of class $C^{k-1}$ on $(-\varepsilon, \varepsilon)$. Furthermore, there exists an open set $O_0 \subset O$ with $(\lambda^*, 0) \in O_0$ and
\[
\left\{ (\lambda, f) \in O : F(\lambda, f) = 0, f \neq 0 \right\} = \left\{ (\lambda(s), s\chi(s)) : \lambda < |s| < \varepsilon \right\}
\]
If $F$ is real analytic, then $\lambda$ and $\chi$ are real analytic on $(-\varepsilon, \varepsilon)$.

To apply Theorem 3.2 above to our problem, we rewrite the problem (11) in the following operator form
\[
F(\lambda, f) = 0 \quad \text{with} \quad f \in X,
\]
where
\[
X = \{ f \in C^{3, \alpha}_{\text{per}}(R) : f = 0 \text{ on } p = -m \},
\]
and
\[
F = (F_1(\lambda, f), F_2(\lambda, f)) : (2\Gamma_{\text{max}}, \infty) \times X \rightarrow Y
\]
for
\[
Y = Y_1 \times Y_2 = C^{1, \alpha}_{\text{per}}(R) \times C^{2, \alpha}_{\text{per}}(T)
\]
with
\[
T = \{ (q, p) : q \in [-\pi, \pi], p = 0 \},
\]
is given by
\[
F_1(\lambda, f) = (1 + f_q^2)(H_{pp} + f_{pp}) - 2f_q(H_p + f_p)f_{pq} + (H_p + f_p)^2f_{qq} - \gamma(-p)(H_p + f_p)^3,
\]
and
\[
F_2(\lambda, f) = 1 + f_q^2 - [2(\tilde{g} + 2\Omega c)(H + f - d) + Q](H_p + f_p)^2 \big|_T.
\]
Since $H$ is the laminar solution, we know that
\[
F(\lambda, 0) = 0 \quad \text{for all } \lambda > 2\Gamma_{\text{max}}.
\]
The linearized operator $F_f = (F_{1f}, F_{2f})$ at $f = 0$ is given by
\[
F_{1f}(\lambda, 0) = \partial_p^2 + H_p^2\partial_q^2 - 3\gamma(-p)H_p^2\partial_p, \quad \text{in } R,
\]
and
\[
F_{2f}(\lambda, 0) = -2\left( \frac{\tilde{g} + 2\Omega c}{\lambda} + \lambda^{1/2}\partial_p \right) \big|_T.
\]

**Lemma 3.3.** The null space $\ker\{F_f(\lambda^*, 0)\}$ is one-dimensional.

**Proof.** Under the condition (18), we know that there exists a unique $\lambda^* > 2\Gamma_{\text{max}}$ with $\mu(\lambda^*) = -1$ such that the null space $\ker\{F_f(\lambda^*, 0)\}$ contains at least one element $M(p)\cos q$, where $M \in C^{3, \alpha}([0, -m])$ is the unique eigenfunction of (17) corresponding to the eigenvalue $\mu(\lambda^*) = -1$, normalized by $M(0) = 1$.

Assume now that $\phi \in C^{3, \alpha}([0, -m]) \cap \ker\{F_f(\lambda^*, 0)\}$, then its Fourier coefficients $\phi_k$ satisfy (15). If for some $k \geq 2$, $\phi_k \neq 0$ , then we have
\[
-\tilde{g} + 2\Omega c \phi_k^2(0) + \int_0^{-m} a^3(\partial_p \phi_k)^2 dp = -k^2 < -1,
\]
which contradicts the minimizing property of $\mu(\lambda^*) = -1$. Thus for $k \geq 2$, $\phi_k \equiv 0$, and then
\[
\phi(q, p) = \phi_0(p) + \phi_1(p) \cos q.
\]
Next we show that $\phi_0 \equiv 0$. In fact, it follows from (15) that $\phi_0$ satisfies

$$
\begin{cases}
(a^3(\lambda^*, p)\partial_p(\phi_0))_p = 0, & \text{in } R, \\
a^3(\lambda^*, p)\partial_p(\phi_0) = - (\tilde{g} + 2\Omega c)\phi_0, & \text{on } p = 0, \\
\phi_0(-m) = 0.
\end{cases}
$$

(21)

By the first equation, we know that

$$
\phi_0(p) = A_0 \int_{-m}^p \frac{1}{a^3(\lambda^*, s)} ds, \quad p \in [-m, 0],
$$

for some constant $A_0 \in \mathbb{R}$. Then using the second equation of (21), we obtain

$$
A_0 = (\tilde{g} + 2\Omega c)A_0 \int_{-m}^0 \frac{1}{a^3(\lambda^*, s)} ds,
$$

which implies either $A_0 = 0$ or by the boundary condition at $p = 0$,

$$
\int_{-m}^0 \frac{1}{a^3(\lambda^*, s)} ds = \frac{1}{\tilde{g} + 2\Omega c}.
$$

The latter relation is impossible because the monotonicity of $\mu(\lambda)$ and the fact that $\mu(\lambda^*) = -1$, $\mu(\lambda_0) = 0$ imply that $\lambda^* < \lambda_0$. However, we have

$$
\int_{-m}^0 \frac{1}{a^3(\lambda_0, s)} ds = \frac{1}{\tilde{g} + 2\Omega c}
$$

by the property of $\lambda_0$ as the point where $Q(\lambda)$ attains its minimum. \(\square\)

The following result can be proved by the same method as in [1] and we omit the proof here.

**Lemma 3.4.** The derivative $F_{\lambda} : X \rightarrow Y$ is a Fredholm operator of index zero for all $\lambda > 2\Gamma_{\max}$.

**Lemma 3.5.** The following transversality condition holds

$$
[F_{\lambda}(\lambda^*, 0)](1, \varphi^*) \notin R(F_{\lambda}(\lambda^*, 0)).
$$

**Proof.** It is well-known that the pair $(A, B) \in Y \cap R(F_{\lambda}(\lambda^*, 0))$ if and only if it satisfies the orthogonality condition

$$
\int_R A(q, p)a^3(p, \lambda^*)\varphi(q, p)dqdp + \frac{1}{2}\int_T B(q)a^2(0, \lambda^*)\varphi(q, 0)dq = 0,
$$

where $\varphi(q, p) = M(p)\cos q$ generates the null space $\ker \{F_{\lambda}(\lambda^*, 0)\}$. Let $a = a(\cdot, \lambda^*)$, then

$$
a_p = -\frac{\gamma(-p)}{a}.
$$

It follows from (19)-(20) that

$$
F_{\lambda}(\lambda^*, 0) = -\left( a^{-4}\partial^2_q + 3a_pa^{-3}\partial_p, \left\{ -2(\tilde{g} + 2\Omega c)\left(\frac{1}{\lambda^*}\right)^2 + \frac{1}{\sqrt{\lambda^*}}\partial_p \right\}_T \right).
$$

Thus, to prove the result, it is sufficient to check that

$$
\int_R a^3\varphi \left\{ a^{-4}\partial^2_q + 3a_pa^{-3}\partial_p \right\} dqdp \\
+ \int_T \left\{ -\frac{\tilde{g} + 2\Omega c}{\lambda^*}\varphi^2 + \frac{\sqrt{\lambda^*}}{2}\varphi_p \right\} dq \neq 0.
$$

(22)
Obviously, \( \varphi \) satisfies the equation
\[
\begin{aligned}
& (a^3 \varphi_p)_p = a \varphi, & \text{in } (0, -m), \\
& (\lambda^*)^{3/2} \varphi_p = -(\tilde{g} + 2\Omega_c) \varphi, & \text{on } p = 0, \\
& \varphi = 0, & \text{on } p = -m, \\
\end{aligned}
\]
from which we obtain that
\[
\int_R a_p \varphi \varphi_p d\eta dp = -\frac{1}{2} \int_T a \varphi \varphi_p dq + \frac{1}{2} \int \int_R \left( \frac{\varphi^2}{a} + a \varphi_p^2 \right) d\eta dp.
\]
Moreover, since \( a(0) = \sqrt{\lambda^*} \), it follows from the boundary condition at \( p = 0 \) that
\[
\int_T \left\{ \frac{\tilde{g} + 2\Omega_c}{\lambda^*} \varphi^2 + \sqrt{\lambda^*} \varphi_p \right\} dq = 0.
\]
Therefore, we can express and compute (22) as
\[
\int_R a^{-1} \varphi \varphi_p dqdp + \frac{3}{2} \int \int_R a^{-1} \varphi^2 dqdp + \frac{3}{2} \int \int_R a \varphi_p^2 dqdp
= \frac{1}{2} \int \int_R a^{-1} \varphi^2 dqdp + \frac{3}{2} \int \int_R a \varphi_p^2 dqdp > 0,
\]
since \( \varphi_{qq} = -\varphi \).

Summarizing the above considerations, we have proved the following result.

**Theorem 3.6.** Assume that condition (18) is satisfied. Then there exists a \( C^1 \)-curve \( \mathcal{C}_{loc} \) of solutions \( h \in C^{3,\alpha}(\mathbb{R}) \) to the system (11) and satisfying (12) and the solution curve \( \mathcal{C}_{loc} \) contains precisely one function that is independent of \( q \).

4. **Dispersion relation.** The dispersion relation provides a formula describing how the phase-speed of the wave varies with respect to certain physical parameters, such as the fixed mean-depth of the flow, the wavelength, and the vorticity. If \( k > 0 \) is the given wave number, then it follows from Theorem 3.6 that there exists a curve of small-amplitude \( \frac{2\pi}{k} \) periodic solutions of (11) bifurcating from the trivial flows if and only if the Sturm-Liouville problem
\[
\begin{aligned}
\vartheta(p)M''(p) + \frac{3}{2} \vartheta'(p)M'(p) &= k^2 M(p), & 0 < p < -m, \\
\lambda^{3/2} M'(0) &= -(\tilde{g} + 2\Omega_c) M(0), \\
M(-m) &= 0,
\end{aligned}
\]
has a nontrivial solution \( M(p) \) for some \( \lambda > 2\Gamma_{\max} \), here
\[
\vartheta(p) = a^2(\lambda, p) = \lambda - 2\Gamma(p).
\]

In general it is difficult to solve (23) and to derive explicit dispersion relations. However for particular classes of vorticity functions we provide explicit solutions to problem (23). Our idea is inspired by the work of [36] and [3].

First we rewrite (23) to a simpler equivalent form. Recall that \( H(p) = H(\lambda, p) \) are given as (13). Then
\[
\frac{dp}{dH} = \frac{1}{H_p} = a(\lambda, p) = c - u(H), \quad \frac{du}{dH} = \gamma(-p(H)),
\]
and
\[
\frac{d^2u}{dH^2} = -\gamma'(-p(H))[c - u(H)].
\]
Using the change of variable
\[
\varphi(H) = [c - u(H)] M(p(H)),
\]
then
\[ M'(p(H)) = \frac{\varphi'(H)}{[c-u(H)]^2} + \frac{\varphi(H)u'(H)}{[c-u(H)]^3}, \]
we get
\[ M''(p(H)) = \frac{\varphi''(H)}{[c-u(H)]^3} + \frac{3\varphi'(H)u'(H) + \varphi(H)u''(H)}{[c-u(H)]^4} + \frac{3\varphi(H)[u'(H)]^2}{[c-u(H)]^5}. \]
Substituting the above results into the first equation of (23), we obtain
\[ \vartheta(p)M''(p(H)) + \frac{3}{2}\vartheta'(H)M'(H) = \frac{\varphi''(H)}{c-u(H)} - \frac{\varphi(H)\gamma'(-p(H))}{c-u(H)} = k^2 \frac{\varphi(H)}{c-u(H)}. \]
Therefore we find the following ordinary differential equation
\[ \varphi''(H) = [k^2 + \gamma'(p(H))]\varphi(H). \]
Now the bottom boundary conditions in (23) become
\[ \varphi(0) = 0, \]
and
\[ \varphi'(H_0) + \left[ \frac{\gamma(0)}{\sqrt{\lambda}} + \frac{\hat{g} + 2\Omega c}{\lambda} \right] \varphi(H_0) = 0, \]
where
\[ H_0 = H(0) = \int_m^0 \frac{1}{\sqrt{\lambda - 2\Gamma(s)}} ds = \frac{\lambda - Q}{2(\hat{g} + 2\Omega c)} \]
is the thermocline. Summarizing we come to the conclusion that problem (23) is equivalent to the following boundary value problem:
\[ \begin{cases} \varphi''(H) = [k^2 + \gamma'(p(H))]\varphi(H), & H_0 < H < 0, \\ \varphi'(H_0) + \left[ \frac{\gamma(0)}{\sqrt{\lambda}} + \frac{\hat{g} + 2\Omega c}{\lambda} \right] \varphi(H_0) = 0, \\ \varphi(0) = 0. \end{cases} \]
(24)
Still (24) - in full generality - seems to be difficult to be solve explicitly. We therefore discuss some special cases of \( \gamma \), which allow us in turn to find explicit dispersion relations.

4.1. Constant vorticity. Let us first consider the case \( \gamma(\psi) \equiv \gamma \) for some \( \gamma \in \mathbb{R} \). Then (24) becomes
\[ \begin{cases} \varphi''(H) = k^2 \varphi(H), & H_0 < H < 0, \\ \varphi'(H_0) + \left[ \frac{\gamma(0)}{\sqrt{\lambda}} + \frac{\hat{g} + 2\Omega c}{\lambda} \right] \varphi(H_0) = 0, \\ \varphi(0) = 0. \end{cases} \]
(25)
The general solution of the first equation and the third condition in (25) is
\[ \varphi(H) = \alpha \sinh(kH) \]
for \( \alpha \in \mathbb{R} \). To satisfy the second condition in (25) we must have
\[ \tanh(kH_0) = -\frac{k\lambda}{\gamma \sqrt{\lambda + (\hat{g} + 2\Omega c)}}. \]
We can compute \( H_0 \) as
\[ H_0 = \int_{-m}^{0} \frac{1}{\sqrt{\lambda - 2\gamma s}} ds = \frac{\sqrt{\lambda + 2\gamma m} - \sqrt{\lambda}}{2\gamma}. \]
Therefore,
\[
\tanh \left( k \sqrt{\lambda} - \sqrt{\lambda + 2\gamma m} \right) = \frac{k\lambda}{\gamma \sqrt{\lambda} + \bar{g} + 2\Omega c}.
\] (26)
Consider a laminar flow with constant vorticity \( \gamma \). Then one has \( v = 0, w = 0, \) and
\[
u(z) - c = -\sqrt{\lambda^{*}} + \gamma(z + D),
\]
where \( z = -D \) is the thermocline. Moreover, the corresponding relative mass flux is
\[
m = \int_{-D}^{-d} (-\sqrt{\lambda^{*}} + \gamma(z + D)) dz = -\sqrt{\lambda^{*}}(D - d) + \frac{\gamma}{2}(D - d)^{2}.
\]
Therefore,
\[
D - d = \frac{\sqrt{\lambda^{*}}}{\gamma} - \sqrt{\lambda^{*}} + 2\gamma m,
\]
and we know
\[
\tanh \left( k(D - d) \right) = \frac{k\lambda^{*}}{\gamma \sqrt{\lambda^{*}} + \bar{g} + 2\Omega c}.
\]
Therefore we have recovered the relation provided in [3], in which the case of a positive constant vorticity was considered. But from (26) we can see that local bifurcation always occurs when the vorticity is negative.

4.2. **Non-constant vorticity.** Observe that in case of a linear vorticity function (24) is autonomous, while for general vorticity distributions (24) is non-autonomous. We next present a structural condition which allows to discuss multiplicative perturbations of the autonomous version of (24).

**Theorem 4.1.** Assume that \( \lambda > 2\Gamma_{\text{max}} \) and there exist \( G \in C^{2}[0, -m] \) and some constant \( \varsigma \in \mathbb{R} \) such that
\[
G''(p(H))[p'(H)]^{2} + G'(p(H))p''(H) - \gamma'(p)G(p(H)) = k^{2}G(p(H)) - \frac{\varsigma}{G(p(H))^{3}}.
\] (27)
Define the bifurcation parameter \( \lambda \) and the function \( \vartheta \) by
\[
\vartheta(p) = \lambda - 2 \int_{0}^{p} \gamma(-s) ds, \quad \lambda = \vartheta(0).
\]
Then (24) is solvable if and only if the dispersion relation
\[
- \left[ \bar{g} + 2\Omega c \right] \chi^{3/2} + \frac{\gamma(0)}{\lambda} = \frac{G'(0)}{G(0)} + \frac{1}{\chi^{3/2}G^{2}(0)F(m)}
\] holds, with \( F(m) \) being given by
\[
F(m) = \begin{cases} \frac{1}{\sqrt{\varsigma}} \tanh \int_{0}^{-m} \frac{\sqrt{\varsigma}}{G^{2}(s)\sqrt{\sigma(s)}} ds, & \varsigma > 0, \\ \int_{0}^{-m} \frac{1}{G^{2}(s)\sqrt{\sigma(s)}} ds, & \varsigma = 0, \\ \frac{1}{\sqrt{\varsigma}} \tan \int_{0}^{-m} \frac{\sqrt{\varsigma}}{G^{2}(s)\sqrt{\sigma(s)}} ds, & \varsigma < 0. \end{cases}
\]
**Proof.** Recall that we have the relation (13). Now let us suppose that the solution has the form
\[
\varphi(H) = G(p(H))\Psi(p(H)),
\] (29)
where \( \Psi \) will be determined in the following. Taking (29) into the first equation of (24), we obtain
\[
I + II = [k^{2} + \gamma'(p(H))]G(p(H))\Psi(p(H)),
\]
in which
\[ I = G''(p(H)) (p'(H))^2 \Psi(p(H)) + G'(p(H)) p''(H) \Psi(p(H)), \]
and
\[ II = 2G'(p(H)) (p'(H))^2 \Psi'(p(H)) + G(p(H)) \Psi''(p(H)) (p'(H))^2 \]
\[ + G(p(H)) \Psi'(p(H)) p''(H). \]
Using the condition (27), we obtain that \( \Psi \) has to solve
\[ G'(p') \Psi'' + [2G'(p')^2 + Gp''] \Psi' - \varsigma \Psi = 0. \] (30)
Suppose first that \( \varsigma > 0 \), then the general solution of (30) with the boundary condition \( \Psi(-m) = 0 \) is given by
\[ \Psi(p) = \alpha \sinh \left( \int_{-m}^{p} \frac{\sqrt{\varsigma}}{G^2(s) \sqrt{\vartheta(s)}} ds \right) \]
for some constant \( \alpha \). Therefore,
\[ \phi(H) = \alpha G(p(H)) \sinh \left( \int_{-m}^{p(H)} \frac{\sqrt{\varsigma}}{G^2(s) \sqrt{\vartheta(s)}} ds \right). \]
We know that \( p(H_0) = 0 \) and \( p'(H_0) = \sqrt{\lambda} \). It is easy to compute that
\[ \phi(H_0) = \alpha G(0) \sinh \int_{-m}^{0} \frac{\sqrt{\varsigma}}{G^2(s) \sqrt{\vartheta(s)}} ds, \]
and
\[ \phi'(H_0) = \alpha G'(0) \sqrt{\lambda} \sin \int_{-m}^{0} \frac{\sqrt{\varsigma}}{G^2(s) \sqrt{\vartheta(s)}} ds + \frac{\alpha \sqrt{\varsigma}}{G(0)} \cosh \int_{-m}^{0} \frac{\sqrt{\varsigma}}{G^2(s) \sqrt{\vartheta(s)}} ds. \]
Now the second condition
\[ \frac{\phi''(H_0)}{\phi(H_0)} = \left[ -\frac{b}{\sqrt{\lambda}} + \frac{\tilde{g} + 2\Omega c}{\lambda} \right] \]
in (24) reads
\[ \frac{G''(0)}{G(0)} + \frac{\sqrt{\varsigma}}{\lambda^{1/2} G^2(0) \tan h} \int_{-m}^{0} \frac{\sqrt{\varsigma}}{G^2(s) \sqrt{\vartheta(s)}} ds = \left[ \frac{\tilde{g} + 2\Omega c}{\lambda^{3/2}} + \frac{\gamma(0)}{\lambda} \right], \]
which is just the dispersion relation (28).
If we consider the case \( \varsigma < 0 \), the previous approach applies with minor changes to give
\[ \phi(H) = \alpha G(p(H)) \sin \left( \int_{-m}^{p(H)} \frac{\sqrt{-\varsigma}}{G^2(s) \sqrt{\vartheta(s)}} ds \right). \]
While \( \varsigma = 0 \), the solution is given as
\[ \phi(H) = \alpha G(p(H)) \int_{-m}^{p(H)} \frac{1}{G^2(s) \sqrt{\vartheta(s)}} ds. \]
In these both cases, we have the dispersion relation as given in (28). \( \square \)
Example 4.2. Let us consider the case where the vorticity function is linear, that is, 
\[ \gamma(\psi) = a\psi + b, \quad a, b \in \mathbb{R}. \]
It is easy to see that (27) holds with \( G \equiv 1 \) and \( \zeta = k^2 + a \). Note that
\[ \int_0^{-m} \frac{1}{G^2(s)\sqrt{\vartheta(s)}} ds = \int_0^{-m} \frac{1}{\sqrt{\lambda - 2\Gamma(s)}} ds = D - d. \]
Thus the dispersion relation reduces to
\[ \lambda + \left[ b\sqrt{\lambda} + (\tilde{g} + 2\Omega c) \right] F(D - d) = 0, \]
where \( F(D - d) \) is given by
\[ F(D - d) = \begin{cases} \text{tanh}\left(\sqrt{k^2 + a}(D - d)\right), & k^2 + a > 0, \\ D - d, & k^2 + a = 0, \\ \text{tan}\left(\sqrt{-k^2 - a}(D - d)\right), & k^2 + a < 0. \end{cases} \]
For a given function \( G \), we can produce vorticity functions obeying non-trivial dispersion relations. A simple case is the following

Example 4.3. Let \( G(p) = ap + b \) with \( a, b > 0 \). Then (27) becomes
\[ -a\gamma(-p) - \gamma'(p)(ap + b) = k^2(ap + b) - \frac{\zeta}{(ap + b)^3}, \]
which can be rewritten as its equivalent form
\[ \gamma'(p) + \frac{a}{ap - b}\gamma(p) = k^2 - \frac{\zeta}{(ap - b)^3}, \]
and the solution is given as
\[ \gamma(p) = \frac{cb}{ap - b} + \frac{b}{ap - b} \left[ \frac{ak^2}{2b} p^2 - k^2 p - \frac{\zeta}{2ab^2} \right] + \frac{\zeta}{2a(ap - b)^3} \]
for some \( c \in \mathbb{R} \).

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