Lump wave dynamics of saturated two-dimensional superfluid He-film with nontrivial bottom boundary conditions

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In this article, the free surface dynamics of saturated ($\sim 10^{-6}$ cm) superfluid $^4$He film is considered under the condition that there exists a very weak downward superfluid flow velocity into the substrate. For saturated film, the effect of surface tension plays a decisive role in the dynamics of the system. The free surface is shown to be governed by forced Kadomtsev Petviashvili-I equation, with the forcing function depending on downward superfluid velocity in the substrate. Exact and perturbative lump solutions of the (2+1) dimensional evolution equation are obtained and the effect of leakage velocity function on the lump solutions are shown.

I. INTRODUCTION

Detailed research on nonlinear dynamical systems has gained vast popularity because linear theory fails in explaining phenomena related to large amplitude waves, wave-wave interactions etc [1, 2]. But in majority of the cases, the analytical treatment becomes difficult because of unsolvability of the associated differential equations. Only rescue from this situation occurs when the differential equations become integrable leading to exact solvability.

Superfluidity is a state of matter when the matter behaves like a fluid with zero viscosity. Study of nonlinear waves in superfluid $^4$He films has started long ago. In superfluid $^4$He films, if there exist small, finite amplitude localized density fluctuations then that can lead to the existence of solitons. These solitons arises from the balance between dispersion and nonlinearity in which the nonlinearity emerges from Van der Waals potential of the substrate. Huberman[3] considered monolayer superfluid $^4$He films ($\sim 10^{-7}$cm) and showed that small amplitude localized perturbations in superfluid density would lead to the occurrence of gap-less solitons. He found this soliton as the solution of Korteweg de-Vries(KdV) equation which is a completely integrable system[4]. Nakajima et.al[5] considered thin $^4$He film and derived KdV equation from Landau-two fluid hydrodynamic approach. In their next paper[6] they considered saturated superfluid $^4$He film ($\sim 10^{-6}$ cm) where the surface tension plays a decisive role to the wave dynamics, at low temperature and solved the nonlinear surface wave from KdV equation. Biswas and Warke[7] also derived KdV equation from the phenomenological Hamiltonian given in [3] and predicted theoretically that superfluid solitons can exist. Condat and Guyer[9] considered mixture of $^4$He films and discussed the propagation of “troughlike” and “bumplike” solitons in such films. Johnson[10] on the other hand presented some linearized theory on $^4$He films and also discussed the far-field nonlinear problem. He showed that the relevant equation valid in the far-field region is the Burgers equation. In [11] Gopakumar et.al studied superfluid films with the solitons overtaking collisions and found detailed expression with appropriate correction for the amplitude dependence of the solitary wave on wave speed. These works are all on the (1+1) dimensional models.

Biswas and Warke[12] considered (2+1) dimensional model and derived Kadomtsev Petviashvili (KP) equation for surface density fluctuations in superfluid $^4$He film. Later Sreekumar and Nandakumaran[13] studied two-soliton resonances of the KP equation for the superfluid surface density. They also considered large amplitude density fluctuations in a thin superfluid film and discussed about existence of ”quasi-solitons” under collision [14]. In [15] they showed the free surface dynamics of saturated two-dimensional superfluid $^4$He film is governed by the KP equation. It is showed that soliton resonance could happen at lowest order nonlinearity, if two dimensional effects are considered. In an interesting recent work [16], the first experimental observation of bright solitons has been done in bulk superfluid $^4$He.

In all these previous works on (1+1) and (2+1) dimensions on $^4$He films where free surface evolution was considered, an assumption was made that the superfluid does not flow into the substrate. But there may be situations [17, 18] where this bottom boundary condition will change. In the derivations of long wave, small amplitude nonlinear integrable equations like KdV, KP equations etc the standard trivial bottom boundary conditions are used. But in case of present
problem, the downward superfluid flow into the substrate will change the dynamics of the wave.

In this work, we have considered saturated \((\sim 10^{-6} \text{ cm})\) two dimensional \(^4\text{He}\) film where the surface tension plays a decisive role in the dynamics of the wave, such that there is a very weak superfluid flow velocity into the substrate, which seems to be relevant with the actual physical situations. The lump wave dynamics has been discussed in presence of such nontrivial bottom boundary condition.

In case of hydrodynamic systems in shallow water in \((1+1)\) dimensions, we have considered such nontrivial bottom boundary conditions \([19, 20]\). Phase and amplitude modifications of solitary waves in such cases are obtained analytically. But in present problem, surface tension plays a decisive role in the dynamics which is neglected in long water wave systems.

The article is organized as follows. In section-II, the derivation of two dimensional nonlinear evolution equation has been done for saturated superfluid \(^4\text{He}\) film with nontrivial bottom boundary effect. Details of its lump wave solutions has been discussed in section-III. Conclusive remarks are given in section-IV followed by bibliography.

II. DERIVATION OF THE NONLINEAR \((2+1)\) DIMENSIONAL EVOLUTION EQUATION OF THE FREE SURFACE

![FIG. 1: Propagation of small but finite amplitude localized wave in saturated superfluid \(^4\text{He}\) film.](image)

In this section, we shall consider the propagation of small amplitude surface wave on saturated superfluid \(^4\text{He}\) film. In case of very thin \(^4\text{He}\) film \((\sim 10^{-7} \text{ cm})\) the effect of surface tension is neglected. Nakajima et al\([6]\) has considered saturated film of superfluid \(^4\text{He}\) whose thickness is of the order of \(10^{-6} \text{ cm}\). In such films, the surface tension plays a decisive role in the wave dynamics. In this work, we also consider saturated superfluid \(^4\text{He}\) film with nontrivial bottom boundary effects. In this case, the acceleration of the superfluid film due to finite temperature gradient gives a small correction factor\([8]\), hence it is neglected. The geometry of the system is shown in FIG.1. The depth of the Helium film is taken as \(d\).

In the previous scientific literatures of this subject, it was considered that there is no downward velocity of the superfluid into the substrate. Here we shall consider that effect by considering a very weak yet finite downward velocity. We shall see that due to that effect the dynamics of the surface wave will be changed. We follow the derivation of \([15]\) with nontrivial boundary conditions at the bottom.

Since the superflow is irrotational and we also consider incompressibility hence,

\[
\nabla^2 \phi(x, y, z, t) = 0,
\]

where \(\phi(x, y, z, t)\) is the velocity potential.

We consider in this work, that there is a small but finite flow of superfluid into the substrate. Hence, the bottom boundary condition emerges as

\[
\frac{\partial \phi}{\partial z}|_{z=0} = C(x, y, t),
\]
where $C(x, y, t)$ is the downward velocity dependent on both space and time coordinates.

On the film vapor interface, we have the nonlinear boundary condition as,
\[
\frac{\partial z_1}{\partial t} + \left( \frac{\partial \phi}{\partial x} \right)_1 \frac{\partial z_1}{\partial x} + \left( \frac{\partial \phi}{\partial y} \right)_1 \frac{\partial z_1}{\partial y} - \left( \frac{\partial \phi}{\partial z} \right)_1 = 0. \tag{3}
\]

The index 1 refers to the quantities in the film vapor interface and $z_1 = d + a(x, y, t)$ where $a(x, y, t)$ is the departure of the film surface from the equilibrium position.

The another nonlinear boundary condition is
\[
\left( \frac{\partial \phi}{\partial t} \right)_1 + \left( \frac{1}{2} \right) \left( \nabla \phi \right)^2 - \frac{\sigma}{\rho} \left( \frac{\partial^2 a}{\partial x^2} + \frac{\partial^2 a}{\partial y^2} \right) + g_1 a - (1/2) g_2 \frac{d}{d a} a^2 = 0. \tag{4}
\]

The last two terms in (4) comes the expansion of the Van der Waals force term and $g_1 = \frac{3\alpha}{d^4}$ and $g_2 = \frac{12\alpha}{d^6}$, where $\alpha$ is the Van der Waals constant. $\rho$ and $\sigma$ represent the density and surface tension respectively.

We expand $\phi(x, y, z, t)$ in a series as
\[
\phi(x, y, z, t) = \sum_{n=0}^{\infty} \phi_n(x, y, t) z^n \tag{5}
\]

Using (2) in (5) we get
\[
\phi_1(x, y, t) = C(x, y, t). \tag{6}
\]

Ultimately we can write $\phi(x, y, z, t)$ as
\[
\phi(x, y, z, t) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n!} \phi_{(02n)} z^{2n} + \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} C_{2n} z^{2n+1}, \tag{7}
\]

where $\phi_{(02n)} = \frac{\partial^{2n} \phi_0}{\partial^2 x^{2n}} + \frac{\partial^{2n} \phi_0}{\partial^2 y^{2n}}$ and $C_{2n} = \frac{\partial^{2n} C}{\partial^2 x^{2n}} + \frac{\partial^{2n} C}{\partial^2 y^{2n}}$.

We study the dynamics of localized disturbances of long wavelength and small but finite amplitude in the superfluid film thickness. We chose $x$ axis as the principal direction of propagation of the wave.

We expand $\phi_0$, $a$ and $C$ in powers of $\epsilon$ as
\[
\phi_0(x, y, t) = \epsilon^0 a_0(x, y, t) + \epsilon^2 a_2(x, y, t) + \ldots, \tag{8}
\]

The series of the velocity term $C(x, y, t)$ is different in terms of smallness parameters because we have assumed that $C(x, y, t)$ is very weak, which is reflected in the expansion.

Using the moving frame transformation
\[
X = x + C_3 t, t \to t, \tag{9}
\]

where $C_3$ is the frame velocity and the asymmetric scaling on the space and time
\[
\bar{x} = \epsilon^{1/2} X, \bar{y} = \epsilon y, \bar{t} = \epsilon^{3/2} t \tag{10}
\]

where $\epsilon$ is an infinitesimal small constant, on equations (3) and (4) and comparing coefficients of
\(e^{3/2}\) and \(e^{5/2}\) we get,

\[
C_3 \frac{\partial^2 \phi_0^{(2)}}{\partial x^2} - (1/2)C_3 d^2 \frac{\partial^4 \phi_0^{(1)}}{\partial x^4} + \frac{\partial}{\partial t} \frac{\partial \phi_0^{(1)}}{\partial x} + (1/2) \frac{\partial}{\partial t} \left( \frac{\partial \phi_0^{(1)}}{\partial x} \right)^2 - \frac{\sigma}{\rho} \frac{\partial^3 a_1}{\partial x^3} + g_1 \frac{\partial a_2}{\partial x} - \frac{\sigma d a_1}{\partial x} = 0,
\]

\[
C_3 \frac{\partial^2 \phi_0^{(2)}}{\partial x^2} - (1/2)C_3 d^2 \frac{\partial^4 \phi_0^{(1)}}{\partial x^4} + \frac{\partial}{\partial t} \frac{\partial \phi_0^{(1)}}{\partial x} + (1/2) \frac{\partial}{\partial t} \left( \frac{\partial \phi_0^{(1)}}{\partial x} \right)^2 - \frac{\sigma}{\rho} \frac{\partial^3 a_1}{\partial x^3} + g_1 \frac{\partial a_2}{\partial x} - \frac{\sigma d a_1}{\partial x} = 0,
\]

From \(e^{3/2}\) order calculations we get

\[
C_3^2 = g_1 d\tag{12}
\]

Finally at \(e^{5/2}\) order, we get the forced Kadomtsev Petviashvili (KP) equation as

\[
A_1 \frac{\partial a_1}{\partial t} + A_2 a_1 \frac{\partial a_1}{\partial x} + A_3 \frac{\partial^3 a_1}{\partial x^3} - A_4 \frac{\partial^2 a_1}{\partial y^2} \int a_1 dx = A_5 C^{(1)}, \tag{13}
\]

where the coefficients are given by \(A_1 = \frac{2}{7}, A_2 = (\frac{2}{6} - \frac{3\sigma}{\rho} \frac{d^2}{dx^2}), A_3 = (\frac{2}{6} - \frac{3\sigma}{\rho} \frac{d^2}{dx^2}), A_4 = \frac{2}{6}, A_5 = \frac{1}{2}\).

There are two types of KP equations namely KP-I and KP-II depending the sign of the coefficients.

After dividing equation \([13]\) by \(A_1\) and simplifying other coefficients using \([12]\), we get

\[
\frac{A_4}{A_1} = \frac{C_3^2}{2} = \tilde{A}, \quad \frac{A_2}{A_1} = \frac{1}{2} \left( \frac{2\sigma}{\rho} - \frac{\sigma d^2}{dx^2} \right) = \tilde{B}, \quad \frac{A_3}{A_1} = \frac{g_1 d}{\sigma}, \quad \frac{A_4}{A_1} = \frac{1}{2} = \tilde{D}.
\]

For the sake of definiteness, if we take the positive value of \(C_3\), then equation \([13]\) describes the motion of the wave in the positive velocity \(C_3\) with reference to the x axis. The sign of \(\tilde{B}\), depends on the size of the thickness \(d\), and changes sign at the critical thickness \(d_c = \sqrt{\frac{2\sigma}{\rho}}\), which in this case is of the order of \(10^{-7}\) cm.

Hence, surface tension is completely neglected in case of a very thin film consisting of a few atomic layers. In the present work, we consider saturated film whose thickness \(d\) is larger than the critical value \(d_c\), hence the surface tension plays a role.

This means that coefficients \(\tilde{A}, \tilde{B}\) and \(\tilde{C}\) are all positive.

Hence we can reduce equation \([13]\) as

\[
\frac{\partial a_1}{\partial t} + \tilde{A} a_1 \frac{\partial a_1}{\partial x} + \tilde{B} \frac{\partial^3 a_1}{\partial x^3} - \tilde{C} \frac{\partial^2 a_1}{\partial y^2} \int a_1 dx = \frac{1}{2} \tilde{C}^{(1)}, \tag{14}
\]

Rescaling variables as \(a_1 = \frac{\sqrt{\rho}}{A} U, \bar{t} = \frac{X}{B}, \tilde{y} = \sqrt{\frac{2\sigma}{B}} Y\) and \(C^{(1)} = \frac{12\bar{B}^2}{A} \tilde{C}\) we get from \([14]\)

\[
\frac{\partial U}{\partial \bar{t}} + 6 \frac{\partial U}{\partial X} + \frac{\partial^3 U}{\partial X^3} - 3 \frac{\partial^2 U}{\partial \tilde{y}^2} \int U dX = -\tilde{C}, \tag{15}
\]

where we have replaced \(X\) in place of \(\bar{x}\) for notational convenience and negative sign in \(\tilde{C}\) arises due to the fact that the leakage velocity is downwards i.e in -\(z\) direction.

**III. NATURE OF SOLUTIONS**

From \([15]\), we see that ultimately we have got forced KP-I equation as the (2+1) dimensional evolution equation of free surface. It is known that all integrable equations possess soliton solutions.
which are exponentially localized solutions in certain directions. Where as, lump solutions are a kind of rational function solutions, localized in all directions in the space.

Lump solutions are found in many systems such as the KP-I equation [21], the three-dimensional three-wave resonant interaction [22], the B-KP equation [23], the DaveyStewartson-II equation [21] and the Ishimori-I equation [24]. A new integrable nonlinear equation possessing lump solution in (2+1) dimension has been developed recently [25, 26] which is known as Kundu-Mukherjee-Naskar(KMN) equation [27–30]. Since, lump solutions play an important role in describing localized phenomena in various physical systems, we will consider the dynamics of lump solutions of equation [15] which is KP-I equation with a forcing function. Whereas for KP-I equation, the N-line soliton phenomena in various physical systems, we will consider the dynamics of lump solutions of equation [15] which is KP-I equation with a forcing function. Whereas for KP-I equation, the N-line soliton states are unstable but the lump solutions are stable. Details about the rational lump solutions of equation [15] are given in next two subsections.

A. $\bar{C}$ is dependent only on time

For the sake of simplicity if we consider the velocity function $\bar{C}$ is dependent on time only i.e, $\bar{C} = C(T)$ then equation [15] becomes exactly solvable. We use a shift $u = U + \int CdT$, and translate to a moving frame $\bar{x} = X + a(T), T = T$, where $a(T)$ is an arbitrary function of time. If we choose $a(T)$ such that

$$\frac{da}{dT} = 6 \int CdT$$

then equation [15] becomes

$$\frac{\partial u}{\partial T} + 6u \frac{\partial u}{\partial \bar{x}} + \frac{\partial^3 u}{\partial \bar{x}^3} - 3 \frac{\partial^2 u}{\partial Y^2} \int ud\bar{x} = 0.$$  \hspace{1cm} (17)

which is nothing but KP-I equation.

The standard one lump solution of (17) is given by

$$u = 4 \frac{-\{\bar{x} + a_1 Y + 3(a_1^2 - b_1^2)T\}^2 + b_1^2(Y + 6a_1 T)^2 + \frac{1}{b_1^2}}{\{\{\bar{x} + a_1 Y + 3(a_1^2 - b_1^2)T\}^2 + b_1^2(Y + 6a_1 T)^2 + \frac{1}{b_1^2}\}^2},$$  \hspace{1cm} (18)

where $a_1, b_1$ are real parameters.

Replacing to the old variables we get,

$$U = 4 \frac{-\{(X + a(T)) + a_1 Y + 3(a_1^2 - b_1^2)T\}^2 + b_1^2(Y + 6a_1 T)^2 + \frac{1}{b_1^2}}{\{(X + a(T)) + a_1 Y + 3(a_1^2 - b_1^2)T\}^2 + b_1^2(Y + 6a_1 T)^2 + \frac{1}{b_1^2}\}^2} - \int \bar{C}dT,$$  \hspace{1cm} (19)

where $a(T)$ is given by [16]. We can see from (19) that the structure of $U$ will change due to the function $\bar{C}$. In order to understand it more clearly, we plot both $X - T$ dependence and $Y - T$ dependence of $U$ in the following figures and see the dynamics.

B. $\bar{C}$ is dependent only on both space and time

In this section, we will consider more general case, i.e, when the velocity function $\bar{C}$ is dependent on both space and time coordinates i.e, $\bar{C} = \bar{C}(X, Y, T)$. But for this case the exact solution of equation [15] becomes very difficult. But when the forcing function $\bar{C}$, is assumed to be fast compared to the evolution of the unforced equation it can be solved by perturbation technique. This suggests the introduction of two time scales. The effects of rapid forcing on some evolution equations has been investigated in [31–33]. In [33], two-dimensional integrable KP equation has been considered and solved for general initial conditions and forcing functions.

We follow the method from [33] and proceed to find the solution of (15). In the first place, we consider $\bar{C}$ to be a rapidly varying function i.e, $\bar{C} = f(X, Y, T/\epsilon_1)$, where $0 < \epsilon_1 << 1$. Following[32],
we introduce, $\tau = T/\epsilon_1$ so that time derivatives transform according to the rule $\frac{\partial}{\partial T} \rightarrow (1/\epsilon_1) \frac{\partial}{\partial \tau}$ + $\frac{\partial}{\partial T}$.

Applying these, we get from (15),
\[
(U_T + 6U_{XX} + U_{XXX})X - 3U_{YY} + \frac{1}{\epsilon_1} U_{X\tau} = f(X,Y,\tau),
\]
where subscripts denote partial derivatives.

Introducing the series expansion,
\[
U(X,Y,T,\tau) = \sum_{n=0}^{\infty} U_n(X,Y,T,\tau)\epsilon_1^n
\]
on (20) and equating powers of $\epsilon_1$ to zero we get,

$O(1/\epsilon_1)$:
\[
U_0X\tau = 0
\]
\[ O(1) : \]
\[ U_{0XT} + 6(U_0 U_{0X})_X + U_{0XXX} - 3U_{0YY} + U_{1X\tau} = f \]  \hspace{1cm} (23)

\[ O(\epsilon_1) : \]
\[ U_{1XT} + 6(U_0 U_{1X})_X + 6(U_1 U_{0X})_X + U_{1XXX} - 3U_{1YY} + U_{2X\tau} = 0 \]  \hspace{1cm} (24)

From (22) integrating twice we get,
\[ U_0 = V_0(X, Y, T) + W(Y, T, \tau) \]  \hspace{1cm} (25)

For lump solutions with fixed \( Y \), we have \( U_0 \to 0 \), as \( X \to \pm \infty \). Hence, \( W(Y, T, \tau) = 0 \), that’s why we can write
\[ U_0(X, Y, T, \tau) = V_0(X, Y, T) \]  \hspace{1cm} (26)

Now, \( V_0(X, Y, T) \) will be determined in next order.

From (23) we can separate out two equations as \( \tau \) independent and \( \tau \) dependent parts respectively as,
\[ V_{0XT} + 6(V_0 V_{0X})_X + V_{0XXX} - 3V_{0YY} = 0, \]  \hspace{1cm} (27)

\[ U_{1X\tau} = f(X, Y, \tau) \]  \hspace{1cm} (28)

Thus we see that \( V_0 \) satisfies unforced KP-I equation. Now, solving (28) we get,
\[ U_1(X, Y, T, \tau) = V_1(X, Y, T) + G_1(X, Y, T, \tau), \quad G_1(X, Y, T, \tau) = \int \int f(X, Y, \tau)dXd\tau \]  \hspace{1cm} (29)

and \( V_1(X, Y, T) \) will be determined in next order.

From (24) we again separate out two equations as \( \tau \) independent and \( \tau \) dependent parts respectively as,
\[ V_{1XT} + 6(V_0 V_{1X})_X + 6(V_1 V_{0X})_X + V_{1XXX} - 3V_{1YY} = 0, \]  \hspace{1cm} (30)

\[ G_{1XT} + 6(V_0 G_{1X})_X + 6(G_1 V_{0X})_X + G_{1XXX} - 3G_{1YY} + U_{2X\tau} = 0 \]  \hspace{1cm} (31)

Differentiating (27) w.r.to \( X \), we can easily see that (30) is identical if we take \( V_1 = V_0X \).

Integrating (31) we get,
\[ U_2(X, Y, T, \tau) = V_2(X, Y, T) + G_2(X, Y, T, \tau), \]  \hspace{1cm} (32)

where
\[ G_2(X, Y, T, \tau) = -\int (G_{1T} + 6(V_0 G_{1X}) + 6(G_1 V_{0X}) + G_{1XXX} - 3G_{1YY} dX)d\tau \]  \hspace{1cm} (33)

Again in the same procedure we can get \( V_2(X, Y, T) \) in the next order as \( V_2 = \frac{1}{2}V_0XX \). Thus we now have the solution of the forced KP-I equation (20) as
\[ U(X, Y, T, \tau) = V_0(X, Y, T) + \epsilon_1 \{ V_0X(X, Y, T) + G_1(X, Y, T, \tau) \} + \epsilon_2 \{ \frac{1}{2}V_0XX(X, Y, T) + G_2(X, Y, T, \tau) \} + \ldots, \]  \hspace{1cm} (34)

where the functions \( G_1, G_2 \) are given by (29) and (33) respectively.


1. Role of initial conditions

We consider, (20) is to be solved with the initial condition \( U(X, Y, 0, 0) = g(X, Y) \). In \( \text{33} \), authors have discussed in detail how the initial conditions and the forcing function must be related. In that paper, they have deduced that when the initial data enter at leading order only, inconsistencies can arise in the higher-order problems. On the other hand when the initial data influences
FIG. 3: We see the dynamics of the lump solution $U$ from (38) with the downward space-time dependent superfluid leakage velocity $f(X, Y, \tau) = V_{0 XX}(X, Y, 0) \sin \tau$ with $a_1 = 1, b_1 = 2$. Since $V_0$ satisfies unforced KP-I equation (27), hence we take the function $V_0$ from (18). We can compare these figures with FIG. 2 ((a) and (b)) to see the effect of space-time dependent leakage velocity on the lump wave structure.

all terms in the perturbation expansion, such inconsistencies can be avoided by an appropriate choice of initial conditions at each order.

For a more general treatment, we consider the second case i.e., when the initial data influences all terms in the perturbation expansion such that

$$U_j(X, Y, 0, 0) = g_j(X, Y), j \geq 0.$$  

(35)

Let the forcing function be of the form $f(X, Y, \tau) = R(X, Y)S(\tau)$ hence, from (29) we get

$$\int R(X, Y)dX = -\frac{V_{0 XX}(X, Y, 0)}{f S(\tau)d\tau|_{\tau=0}}$$  

(36)

If we chose $S(\tau) = \sin \tau$, then we can get ultimately,

$$R(X, Y) = V_{0 XX}(X, Y, 0),$$  

(37)

hence we can get $f(X, Y, \tau) = V_{0 XX}(X, Y, 0) \sin \tau$. If we evaluate $G_2$ from (33) using (27), we can show that $G_2(X, Y, T, \tau) = 0$. Hence, we can write the solution up to $O(\epsilon_1^2)$ as

$$U(X, Y, T, \tau) = V_0(X, Y, T) + \epsilon_1 \{V_{0 XX}(1 - \cos \tau)\} + \epsilon_1^2 \{\frac{1}{2} V_{0 XX}(X, Y, T)\} + \ldots,$$  

(38)

Thus we can identify that in present case, $g_1 = 0, g_2 = \frac{V_{0 XX}(X, Y, 0)}{2}$.

The $X - T$ dependence and $Y - T$ dependence of the solution $U$ (38) in the present case is shown in FIG. 3. It can be compared with FIG. 2 ((a) and (b)) to see the effect of nontrivial bottom boundary effects on the lump wave solution.

In previous scientific literatures, the surface wave dynamics of saturated superfluid film has been considered in very few cases [6, 15]. In [6] KdV equation was derived as surface wave evolution equation where thermomechanical force was neglected. The resulting solitary waves derived from KdV equation were "cold", in contrast to the solitary wave of very thin film. Where as in [15], KP equation was derived in case of saturated superfluid film with the inclusion of weak transverse effects.

But in all such cases trivial bottom boundary conditions has been used such that there is no superflow into the substrate. But in case of porous substrates [17, 18] such conditions must change.
In shallow water long wave systems in (1+1) dimensions, we have considered such nontrivial bottom boundary conditions in [19, 20]. Phase and amplitude modifications of solitary waves in such cases as the solutions of perturbed KdV equation has been obtained analytically. But in present problem where two dimensional effects are considered where surface tension plays a effective role in the dynamics which is neglected in long water wave systems. So, the effects of nontrivial bottom boundary conditions on the dynamics of two dimensional rational lump solutions of superfluid $^4$He film has not been evaluated analytically so far as our knowledge goes. Hence, those obtained lump solutions in this work can be useful in relevant practical situations.

IV. CONCLUSIVE REMARKS:

In conclusion we can say that, we have considered saturated superfluid $^4$He film, with nontrivial bottom boundary conditions. In such situations, there is a very weak yet finite superfluid velocity into the substrate. For example, Nuclepore is a polycarbonate sheet which is around 10 micron thick, can be used as porous substrate For $^3$He film. In such cases, due to a very weak superflow into the substrate the bottom boundary condition will change, which will affect the dynamics of the surface waves. In case of saturated films, the effect of surface tension plays a decisive role in the surface wave propagation. We have shown that, in presence of very weak bottom boundary conditions, the dynamics of the free surface is governed by forced Kadomtsev Petviashvili-I equation, with the forcing function depending on downward superfluid velocity in the substrate.

Since, for KP-I equation, line solitons are unstable where as rational lump solutions are stable. So we have tried to concentrate on the dynamics of the simple one lump solution of obtained evolution equation. If the downward superfluid velocity depends only on time, then the system becomes exactly solvable. We have shown how such weak superfluid velocity affects the structure of lump solutions. If the leakage velocity depends on both space and time co ordinates, then exact solution becomes difficult. It is solved by pertubative technique, when there exist two times scales. If the leakage velocity function varies rapidly compared to the time evolution scale of unperturbed equation then it can be solved by perturbative method. In such case the forcing functions become related self consistently with the initial conditions. Plots of one lump solution in presence of such kind of leakage are shown. Since such kind of analytical treatment with nontrivial bottom boundary effects on superfluid $^4$He film is new as far as our knowledge goes it may have its useful applications and may pave new direction of research.

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