BIHARMONIC SURFACES OF $S^4$

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Abstract. In this note we prove that a constant mean curvature surface is proper-biharmonic in the unit Euclidean sphere $S^4$ if and only if it is minimal in a hypersphere $S^3(\sqrt{2})$.

1. Introduction

Biharmonic maps $\varphi : (M, g) \to (N, h)$ between Riemannian manifolds are critical points of the bienergy functional

$$E_2(\varphi) = \frac{1}{2} \int_M |\tau(\varphi)|^2 v_g,$$

where $\tau(\varphi) = \text{trace} \nabla d\varphi$ is the tension field of $\varphi$ that vanishes for harmonic maps (see [11]). The Euler-Lagrange equation corresponding to $E_2$ is given by the vanishing of the bitension field

$$\tau_2(\varphi) = -J^\varphi(\tau(\varphi)) = -\Delta \tau(\varphi) - \text{trace} R^N(d\varphi, \tau(\varphi))d\varphi,$$

where $J^\varphi$ is formally the Jacobi operator of $\varphi$ (see [15]). The operator $J^\varphi$ is linear, thus any harmonic map is biharmonic. We call proper-biharmonic the non-harmonic biharmonic maps.

The study of proper-biharmonic submanifolds, i.e. submanifolds such that the inclusion map is non-harmonic (non-minimal) biharmonic, constitutes an important research direction in the theory of biharmonic maps. The first ambient spaces taken under consideration for their proper-biharmonic submanifolds were the spaces of constant sectional curvature. Non-existence results were obtained for proper-biharmonic submanifolds in Euclidean and hyperbolic spaces (see [2, 5, 8, 10, 13]).

The case of the Euclidean sphere is different. Indeed, the hypersphere $S^{n-1}(\sqrt{2})$ and the generalized Clifford torus $S^{n_1}(\sqrt{2}) \times S^{n_2}(\sqrt{2})$, $n_1 + n_2 = n - 1$, $n_1 \neq n_2$, are the main examples of proper-biharmonic submanifolds in $S^n$ (see [4, 15]). Moreover, the following

Conjecture 1.1 (2). The only proper-biharmonic hypersurfaces in $S^n$ are the open parts of hyperspheres $S^{n-1}(\sqrt{2})$ or of generalized Clifford tori $S^{n_1}(\sqrt{2}) \times S^{n_2}(\sqrt{2})$, $n_1 + n_2 = n - 1$, $n_1 \neq n_2$.

was proposed. This proved to be true for certain classes of hypersurfaces with additional geometric properties (see [2, 13]).

In codimension greater than 1, the family of proper-biharmonic submanifolds is rather large. For example, any minimal submanifold in $S^{n-1}(\sqrt{2})$ is proper-biharmonic in $S^n$. In particular, any minimal surface in $S^3(\sqrt{2})$, see [16], provides a proper-biharmonic surface in $S^4$.

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All proper-biharmonic submanifolds of $S^2$ and $S^3$ were determined (see [6, 4]). The next step towards the classification of proper-biharmonic submanifolds in spheres is represented by the case of $S^4$, and the first achievement was the proof of Conjecture 1.1 for compact hypersurfaces in $S^4$ (see [3]). Since all proper-biharmonic curves in $S^n$, and therefore in $S^4$, were determined (see [3]), the aim of this paper is to give a partial answer to the

**Open problem** ([3]). Are there other proper-biharmonic surfaces in $S^4$, apart from the minimal surfaces of $S^3(\sqrt{2})$?

We show that the answer is negative in the case of proper-biharmonic surfaces with constant mean curvature in $S^4$ (Theorem 3.1).

For other results on proper-biharmonic submanifolds in spaces of non-constant sectional curvature see, for example, [12, 14, 17, 19].

2. Preliminaries

Let $\varphi : M \to S^n$ be the canonical inclusion of a submanifold $M$ in the $n$-dimensional unit Euclidean sphere. The expressions assumed by the tension and bitension fields are

$\tau(\varphi) = mH, \quad \tau_2(\varphi) = -m(\Delta H - mH),$

where $H$ denotes the mean curvature vector field of $M$ in $S^n$, while $\Delta$ is the rough Laplacian on $\varphi^{-1}TS^n$.

The following characterization result proved to be the main ingredient in the study of proper-biharmonic submanifolds in spheres.

**Theorem 2.1** ([7, 18]). The canonical inclusion $\varphi : M^m \to S^n$ of a submanifold $M$ in the $n$-dimensional unit Euclidean sphere $S^n$ is biharmonic if and only if

\[
\begin{align*}
\Delta H + \text{trace } B(\cdot, A_H \cdot) - mH &= 0, \\
4 \text{trace } A_{\nabla^H} H(\cdot) + m \text{grad}(|H|^2) &= 0,
\end{align*}
\]

where $A$ denotes the Weingarten operator, $B$ the second fundamental form, $H$ the mean curvature vector field, $\nabla^H$ and $\Delta^H$ the connection and the Laplacian in the normal bundle of $M$ in $S^n$.

Using the main examples, two methods of construction for proper-biharmonic submanifolds in spheres were given.

**Theorem 2.2** (Composition property, [5]). Let $M$ be a minimal submanifold of $S^{n-1}(a) \subset S^n$. Then $M$ is proper-biharmonic in $S^n$ if and only if $a = \sqrt{2}$.

We note that such submanifolds are pseudo-umbilical, i.e. $A_H = |H|^2 \text{Id}$, have parallel mean curvature vector field and mean curvature $|H| = 1$.

**Theorem 2.3** (Product composition property, [5]). Let $M_1^{m_1}$ and $M_2^{m_2}$ be two minimal submanifolds of $S^{n_1}(r_1)$ and $S^{n_2}(r_2)$, respectively, where $n_1 + n_2 = n - 1$, $r_1^2 + r_2^2 = 1$. Then $M_1 \times M_2$ is proper-biharmonic in $S^n$ if and only if $r_1 = r_2 = \frac{1}{\sqrt{2}}$ and $m_1 \neq m_2$.

The proper-biharmonic submanifolds obtained in this way are no longer pseudo-umbilical, but still have parallel mean curvature vector field and their mean curvature is bounded, $|H| \in (0, 1)$. For dimension reasons, this second method cannot be applied in order to produce proper-biharmonic surfaces in $S^4$. 
In [11, 20] the authors obtained explicit examples of proper-biharmonic submanifolds in $S^5$ with constant mean curvature, which are neither pseudo-umbilical nor of parallel mean curvature vector field.

We note that all known examples of proper-biharmonic submanifolds in $S^n$ have constant mean curvature.

We end by recalling here the following results which are needed in the next section.

**Theorem 2.4** ([2]). Let $M^m$ be a pseudo-umbilical submanifold in $S^{m+2}$, $m \neq 4$. Then $M$ is proper-biharmonic in $S^{m+2}$ if and only if it is minimal in $S^{m+1} (\frac{1}{\sqrt{2}})$.

**Theorem 2.5** ([2]). Let $M^2$ be a surface with parallel mean curvature vector field in $S^n$. Then $M$ is proper-biharmonic in $S^n$ if and only if it is minimal in $S^{n-1} (\frac{1}{\sqrt{2}})$.

### 3. Biharmonic surfaces with constant mean curvature in $S^4$

We shall prove the following

**Theorem 3.1.** Let $M^2$ be a proper-biharmonic constant mean curvature surface in $S^4$. Then $M^2$ is minimal in $S^3 (\frac{1}{\sqrt{2}})$.

**Proof.** Following [9], we shall first prove that any proper-biharmonic constant mean curvature surface in $S^4$ has parallel mean curvature vector field. Then we shall conclude by using Theorem 2.5.

Denote by $H$ the mean curvature vector field of $M^2$ in $S^4$. Since $M$ is proper-biharmonic with constant mean curvature, its mean curvature does not vanish at any point and we denote by

\[(3.1)\]

\[E_4 = \frac{H}{|H|} \in C(NM).\]

Consider $\{E_1, E_2\}$ to be a local orthonormal frame field on $M$ around an arbitrary fixed point $p \in M$ and let $E_4$ be a local unit section in the normal bundle, orthogonal to $E_3$. We can assume that $\{E_1, E_2, E_3, E_4\}$ is the restriction of a local orthonormal frame field around $p$ on $S^4$, also denoted by $\{E_1, E_2, E_3, E_4\}$.

Denote by $B$ the second fundamental form of $M$ in $S^4$ and by $A_3$ and $A_4$ the Weingarten operators associated to $E_3$ and $E_4$, respectively.

Let $\nabla^{S^4}$ and $\nabla$ be the Levi-Civita connections on $S^4$ and on $M$, respectively, and denote by $\omega^B_A$ the connection 1-forms of $S^4$ with respect to $\{E_1, E_2, E_3, E_4\}$, i.e.

\[(3.2)\]

\[\nabla^{S^4} E_A = \omega^B_A E_B, \quad A, B = 1, \ldots, 4.\]

From (3.1) we have $H = |H| E_3$ and, since $2H = B(E_1, E_1) + B(E_2, E_2)$, we obtain that

\[(3.3)\]

\[0 = 2\langle H, E_4 \rangle = \langle B(E_1, E_1), E_4 \rangle + \langle B(E_2, E_2), E_4 \rangle = \langle A_4(E_1), E_1 \rangle + \langle A_4(E_2), E_2 \rangle,
\]

i.e. trace $A_4 = 0$. As a consequence, we have

\[(3.4)\]

\[|A_4|^2 = |A_4(E_1)|^2 + |A_4(E_2)|^2 = \langle A_4(E_1), E_1 \rangle^2 + 2\langle A_4(E_1), E_2 \rangle^2 + \langle A_4(E_2), E_2 \rangle^2 = 2\langle A_4(E_1), E_1 \rangle^2 + \langle A_4(E_1), E_2 \rangle^2.
\]

The tangent part of the biharmonic equation (2.2) now writes

\[(3.5)\]

\[A_{\nabla_{E_1} E_3} E_1 + A_{\nabla_{E_2} E_3} E_2 = 0.
\]
Equations (3.7) can be thought of as a linear homogeneous system in \( \omega \).

Case II.

Since \( \omega_3(E_1)A_4(E_1) + \omega_3(E_2)A_4(E_2) = 0 \),

we get

\begin{align*}
\omega_3(E_1)A_4(E_1) + \omega_3(E_2)A_4(E_2) &= 0, \\
(3.6) &\quad \omega_3(E_1)A_4(E_1) + \omega_3(E_2)A_4(E_2) = 0.
\end{align*}

Considering now the scalar product by \( E_1 \) and \( E_2 \) in (3.6), we obtain

\begin{align*}
\begin{cases}
\langle A_4(E_1), E_1 \rangle \omega_3(E_1) + \langle A_4(E_2), E_1 \rangle \omega_3(E_2) = 0, \\
\langle A_4(E_1), E_2 \rangle \omega_3(E_1) + \langle A_4(E_2), E_2 \rangle \omega_3(E_2) = 0.
\end{cases}
\end{align*}

Equations (3.7) can be thought of as a linear homogeneous system in \( \omega_3(E_1) \) and \( \omega_3(E_2) \). By using (3.3) and (3.4), the determinant of this system is equal to \(-\frac{1}{2}|A_4|^2\).

Suppose now that \((\nabla^\perp H)(p) \neq 0\). Then there exists a neighborhood \( U \) of \( p \) in \( M \) such that \( \nabla^\perp H \neq 0 \), at any point of \( U \). Since

\[ \nabla^\perp H = |H|\nabla^\perp E_3 = |H|\{\omega_3(E_1)E_1^\otimes E_4 + \omega_3(E_2)E_2^\otimes E_4\}, \]

the hypothesis \( \nabla^\perp H \neq 0 \) on \( U \) implies that (3.7) admits non-trivial solutions at any point of \( U \). Therefore, the determinant of (3.7) is zero, which means that \( |A_4|^2 = 0 \), i.e. \( A_4 = 0 \) on \( U \).

We have two cases.

Case I. If \( U \) is pseudo-umbilical in \( S^4 \), i.e. \( A_3 = |H| \text{Id} \), from Theorem 2.4 we get that \( U \) is minimal in \( S^3(\frac{1}{|H|}) \) and we have a contradiction, since any minimal surface in \( S^3(\frac{1}{|H|}) \) has parallel mean curvature vector field in \( S^4 \).

Case II. Suppose that there exists \( q \in U \) such that \( A_3(q) \neq |H| \text{Id} \). Then, eventually by restricting \( U \), we can suppose that \( A_3 \neq |H| \text{Id} \) on \( U \). Since the principal curvatures of \( A_3 \) have constant multiplicity 1, we can suppose that \( E_1 \) and \( E_2 \) are such that

\[ A_3(E_1) = k_1E_1, \quad A_3(E_2) = k_2E_2, \]

where \( k_1 \neq k_2 \) at any point of \( U \). As \( A_4 = 0 \), we obtain

\begin{align*}
(3.8) &\quad B(E_1, E_1) = k_1E_3, \quad B(E_1, E_2) = 0, \quad B(E_2, E_2) = k_2E_3,
\end{align*}

on \( U \).

In the following we shall use the Codazzi and Gauss equations in order to get to a contradiction.

The Codazzi equation is given in this setting by

\begin{align*}
(3.9) &\quad 0 = (\nabla^g_XB)(Y, Z, \eta) - (\nabla^g_YB)(X, Z, \eta), \quad \forall X, Y, Z \in C(TM), \forall \eta \in C(NM),
\end{align*}

where \( \nabla^g_XB \) is defined by

\[ (\nabla^g_XB)(Y, Z, \eta) = X\langle B(Y, Z), \eta \rangle - \langle B(\nabla_XY, Z), \eta \rangle - \langle B(Y, \nabla_XZ), \eta \rangle - \langle B(Y, \nabla^\perp_X\eta) \rangle. \]
For $X = Z = E_1$, $Y = E_2$ and $\eta = E_3$, equation (3.9) leads to

$$0 = E_1 \langle B(E_2, E_1), E_3 \rangle - E_2 \langle B(E_1, E_1), E_3 \rangle$$

$$- \langle B(\nabla_{E_1} E_2, E_1), E_3 \rangle + \langle B(\nabla_{E_2} E_1, E_1), E_3 \rangle$$

$$- \langle B(E_2, \nabla_{E_1} E_1), E_3 \rangle + \langle B(E_1, \nabla_{E_2} E_1), E_3 \rangle$$

(3.10)

$$- \langle B(E_2, E_1), \nabla_{E_1} E_3 \rangle + \langle B(E_1, E_1), \nabla_{E_2} E_3 \rangle.$$ 

Now, from (3.8) we have

$$B(\nabla_{E_1} E_2, E_1) = k_1 \omega_3^1(E_1) E_3, \quad B(E_2, \nabla_{E_1} E_1) = -k_2 \omega_3^1(E_1) E_3,$$

$$B(\nabla_{E_2} E_1, E_1) = 0, \quad \langle B(E_1, E_1), \nabla_{E_2} E_3 \rangle = 0,$$

thus (3.10) implies

(3.11) \quad $E_2(k_1) = (k_2 - k_1) \omega_3^1(E_1)$.

Analogously, for $X = Z = E_2$, $Y = E_1$ and $\eta = E_3$ in (3.9), we obtain

(3.12) \quad $E_1(k_2) = (k_2 - k_1) \omega_3^1(E_2)$.

For $X = Z = E_1$, $Y = E_2$ and $\eta = E_4$ in (3.9), we obtain

$$0 = \langle B(E_2, E_1), \nabla_{E_1} E_4 \rangle - \langle B(E_1, E_1), \nabla_{E_2} E_4 \rangle$$

$$= -k_1 \langle E_3, \nabla_{E_2} E_4 \rangle,$$

which implies

(3.13) \quad $k_1 \omega_3^1(E_2) = 0$.

Analogously, for $X = Z = E_2$, $Y = E_1$ and $\eta = E_4$ in (3.9), we obtain

(3.14) \quad $k_2 \omega_3^1(E_1) = 0$.

Since $\nabla^{\perp} H \neq 0$ on $U$, we can suppose that $\omega_3^1(E_1) \neq 0$ on $U$. This, together with (3.14), leads to $k_2 = 0$. From here we get $|k_1| = 2|H| \neq 0$, and consequently $k_1$ is a non-zero constant. As $k_1 \neq k_2$, from (3.11) and (3.12) we obtain

(3.15) \quad $\omega_3^1(E_1) = \omega_3^1(E_2) = 0$.

thus $M$ is flat.

Consider now the Gauss equation,

$$\langle R^{S^4}(X, Y)Z, W \rangle = \langle R(X, Y)Z, W \rangle$$

$$+ \langle B(X, Z), B(Y, W) \rangle - \langle B(X, W), B(Y, Z) \rangle.$$

(3.16)

As $M$ is flat, for $X = W = E_1$ and $Y = Z = E_2$, equations (3.16) and (3.8) lead to

$$1 = \langle B(E_1, E_2), B(E_2, E_1) \rangle - \langle B(E_1, E_1), B(E_2, E_2) \rangle = -k_1k_2$$

(3.17) \quad $= 0$,

and we have a contradiction.

Therefore, $\nabla^{\perp} H = 0$ and we conclude. \quad \Box
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