A NOTE ON HIGHER REGULARITY BOUNDARY HARNACK INEQUALITY

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Abstract. We show that the quotient of two positive harmonic functions vanishing on the boundary of a $C^{k,\alpha}$ domain is of class $C^{k,\alpha}$ up to the boundary.

1. Introduction

In this note we obtain a higher order boundary Harnack inequality for harmonic functions, and more generally, for solutions to linear elliptic equations.

Let $\Omega$ be a $C^{k,\alpha}$ domain in $\mathbb{R}^n$, $k \geq 1$. Assume for simplicity that

$$\Omega := \{(x', x_n) \in \mathbb{R}^n \mid x_n > g(x')\}$$

with

$$g : \mathbb{R}^{n-1} \to \mathbb{R}, \quad g \in C^{k,\alpha}, \quad \|g\|_{C^{k,\alpha}} \leq 1, \quad g(0) = 0.$$

Our main result is the following.

Theorem 1.1. Let $u > 0$ and $v$ be two harmonic functions in $\Omega \cap B_1$ that vanish continuously on $\partial \Omega \cap B_1$. Assume $u$ is normalized so that $u(e_n/2) = 1$, then

$$\left\| \frac{v}{u} \right\|_{C^{k,\alpha}(\Omega \cap B_1/2)} \leq C\|v\|_{L^\infty},$$

with $C$ depending on $n, k, \alpha$.

We remark that if $v > 0$, then the right hand side of (1.1) can be replaced by $v(u(e_n/2)$ as in the classic boundary Harnack inequality.

For a more general statement for solutions to linear elliptic equations, we refer the reader to Section 3.

The classical Schauder estimates imply that $u, v$ are of class $C^{k,\alpha}$ up to the boundary. Using that on $\partial \Omega$ we have $u = v = 0$ and $u_\nu > 0$, one can easily conclude that $v/u$ is of class $C^{k-1,\alpha}$ up to the boundary.

Theorem 1.1 states that the quotient of two harmonic functions is in fact one derivative better than the quotient of two arbitrary $C^{k,\alpha}$ functions that vanish on the boundary. To the best of our knowledge the result of Theorem 1.1 is not known in the literature for $k \geq 1$. The case when $k = 0$ is well known as boundary Harnack inequality: the quotient of two positive harmonic functions as above must be $C^\alpha$ up to the boundary if $\partial \Omega$ is Lipschitz, or the graph of a Hölder function, see [HW, CPMS, JK, HW, F].

A direct application of Theorem 1.1 gives smoothness of $C^{1,\alpha}$ free boundaries in the classical obstacle problem without making use of a hodograph transformation, see [KNS, C].

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Corollary 1.2. Let $\partial \Omega \in C^{1,\alpha}$ and let $u$ solve
\[
\Delta u = 1 \quad \text{in } \Omega, \quad u = 0, \quad \nabla u = 0 \quad \text{on } \partial \Omega \cap B_1.
\]
Assume that $u$ is increasing in the $e_n$ direction. Then $\partial \Omega \in C^\infty$.

The corollary follows by repeatedly applying Theorem 1.1 to the quotient $u_i/u_n$.

Our motivation for the results of this paper comes from the question of higher regularity in thin free boundary problems, which we recently began investigating in [DS].

The idea of the proof of Theorem 1.1 is the following. Let $v$ be a harmonic function vanishing on $\partial \Omega$. The pointwise $C^{k+1,\alpha}$ estimate at $0 \in \partial \Omega$ is achieved by approximating $v$ with polynomials of the type $x_n P$ with $\deg P = k$. It turns out that we may use the same approximation if we replace $x_n$ by a given positive harmonic function $u \in C^{k,\alpha}$ that vanishes on $\partial \Omega$. Moreover, the regularity of $\partial \Omega$ does not play a role since the approximating functions $u P$ already vanish on $\partial \Omega$.

In order to fix ideas we treat the case $k = 1$ separately in Section 2, and we deal with the general case in Section 3.

2. The case $k = 1 - C^{1,\alpha}$ estimates.

In this section, we provide the proof of our main Theorem 1.1 in the case $k = 1$. We also extend the result to more general elliptic operators.

Let $\Omega \subset \mathbb{R}^n$ with $\partial \Omega \in C^{1,\alpha}$. Precisely,
\[
\partial \Omega = \{(x', g(x')) | x' \in \mathbb{R}^{n-1}, \quad g(0) = 0, \quad \nabla_{x'} g(0) = 0, \quad \|g\|_{C^{1,\alpha}} \leq 1\}.
\]
Let $u$ be a positive harmonic function in $\Omega \cap B_1$, vanishing continuously on $\partial \Omega \cap B_1$. Normalize $u$ so that $u(e_n/2) = 1$. Throughout this section, we refer to positive constants depending only on $n, \alpha$ as universal.

Theorem 2.1. Let $v$ be a harmonic function in $\Omega \cap B_1$ vanishing continuously on $\partial \Omega \cap B_1$. Then,
\[
\|v - P\|_{C^{1,\alpha}(\Omega \cap B_{1/2})} \leq C\|v\|_{L^\infty(\Omega \cap B_1)}
\]
with $C$ universal.

First we remark that from the classical Schauder estimates and Hopf lemma, $u$ satisfies
\[
(2.1) \quad u \in C^{1,\alpha}, \quad \|u\|_{C^{1,\alpha}(\Omega \cap B_{1/2})} \leq C, \quad u \nu > c > 0 \quad \text{on } \partial \Omega \cap B_{1/2}.
\]

Thus, after a dilation and multiplication by a constant we may assume that
\[
(2.2) \quad \|g\|_{C^{1,\alpha}(B_1)} \leq \delta, \quad \nabla u(0) = e_n, \quad |\nabla u|_{C^{\alpha}} \leq \delta,
\]
where the constant $\delta$ will be specified later.

We claim that Theorem 2.1 will follow, if we show that there exists a linear function
\[
(2.3) \quad P(x) = a_0 + \sum_{i=1}^n a_i x_i, \quad a_n = 0
\]
such that
\[
(2.4) \quad \left| \frac{v}{u}(x) - P(x) \right| \leq C|x|^{1+\alpha}, \quad x \in \Omega \cap B_1
\]
for $C$ universal.

To obtain (2.3), we prove the next lemma.
Lemma 2.2. Assume that, for some \( r \leq 1 \) and \( P \) as in (2.3) with \( |a_i| \leq 1 \),
\[
\|v - uP\|_{L^\infty(\Omega \cap B_r)} \leq r^{2+\alpha}.
\]
Then, there exists a linear function
\[
P(x) = \bar{a}_0 + \sum_{i=1}^{n} \bar{a}_i x_i, \quad \bar{a}_n = 0
\]
such that
\[
\|v - u\bar{P}\|_{L^\infty(\Omega \cap B_{\rho r})} \leq (\rho r)^{2+\alpha},
\]
for some \( \rho > 0 \) universal, and
\[
\|P - \bar{P}\|_{L^\infty(B_r)} \leq C r^{1+\alpha},
\]
with \( C \) universal.

Proof. We write
\[
v(x) = u(x)P(x) + r^{2+\alpha} \tilde{v}\left(\frac{x}{r}\right), \quad x \in \Omega \cap B_r,
\]
with
\[
\|\tilde{v}\|_{L^\infty(\tilde{\Omega} \cap B_1)} \leq 1, \quad \tilde{\Omega} := \frac{1}{r} \Omega.
\]
Define also,
\[
\tilde{u}(x) := u(rx), \quad x \in \tilde{\Omega} \cap B_1.
\]
We have,
\[
0 = \Delta v = \Delta (uP) + r^\alpha \Delta \tilde{v}\left(\frac{x}{r}\right), \quad x \in \Omega \cap B_r,
\]
and
\[
\Delta (uP) = 2\nabla u \cdot \nabla P = 2 \sum_{i=1}^{n-1} a_i u_i, \quad x \in \Omega \cap B_r.
\]
Moreover, from (2.2) we have
\[
\|\nabla u - e_n\|_{L^\infty(\Omega \cap B_r)} \leq \delta r^\alpha.
\]
Thus, \( \tilde{v} \) solves
\[
|\Delta \tilde{v}| \leq 2\delta \quad \text{in } \tilde{\Omega} \cap B_1, \quad \tilde{v} = 0 \quad \text{on } \partial \tilde{\Omega} \cap B_1,
\]
and
\[
\|\tilde{v}\|_{L^\infty(\tilde{\Omega} \cap B_1)} \leq 1.
\]
Hence, as \( \delta \to 0 \) (using also (2.2)) \( \tilde{v} \) must converge (up to a subsequence) uniformly to a solution \( v_0 \) of
\[
\Delta v_0 = 0 \quad \text{in } B_1^+, \quad v_0 = 0 \quad \text{on } \{x_n = 0\} \cap \bar{B}_1^+
\]
and
\[
|v_0| \leq 1 \quad \text{in } B_1^+.
\]
Such a \( v_0 \) satisfies,
\[
\|v_0 - x_n Q\|_{L^\infty(B_1^+)} \leq C \rho^3 \leq \frac{1}{4} \rho^{2+\alpha},
\]
for some \( \rho \) universal and \( Q = b_0 + \sum_{i=1}^{n} b_i x_i, |b_i| \leq C \). Notice that \( b_n = 0 \) since \( x_n Q \) is harmonic.
By compactness, if \( \delta \) is chosen sufficiently small, then

\[
\| \tilde{v} - x_n Q \|_{L^\infty(\tilde{\Omega} \cap B_\rho)} \leq \frac{1}{2} \rho^{2+\alpha}.
\]

From (2.2),

\[
| \tilde{u} - x_n | \leq \delta
\]

thus

\[
\| \tilde{v} - \tilde{u} Q \|_{L^\infty(\tilde{\Omega} \cap B_\rho)} \leq \rho^{2+\alpha}
\]

from which the desired conclusion follows by choosing

\[
\bar{P}(x) = P(x) + r^{1+\alpha} Q \left( \frac{x}{r} \right).
\]

\[ \Box \]

Remark 2.3. Notice that, from boundary Harnack inequality, \( \tilde{v} \) satisfies (see (2.5) and recall that \( u(\frac{1}{2}e_n) = 1 \))

\[
| \tilde{v} | \leq C \tilde{u} \text{ in } \tilde{\Omega} \cap B_{1/2},
\]

with \( C \) universal. Thus our assumption can be improved in \( B_{r/2} \) to

\[
| v(x) - uP(x) | \leq Cu(x)r^{1+\alpha} \text{ in } \Omega \cap B_{r/2}.
\]

Moreover,

\[
\left[ \frac{\tilde{v}}{\tilde{u}} \right]_{C^{1,\alpha}(\tilde{\Omega} \cap B_{r/4}(\frac{1}{2}e_n))} \leq C
\]

since \( \tilde{u} \) is bounded below in such region. This, together with the identity

\[
\frac{v}{u} = P + r^{1+\alpha} \frac{\tilde{v}}{\tilde{u}} \left( \frac{x}{r} \right) \quad x \in \Omega \cap B_r
\]

implies

\[
(2.6) \quad \left[ \nabla \left( \frac{v}{u} \right) \right]_{C^{\alpha}(\Omega \cap B_{r/4}(\frac{1}{2}e_n))} = \left[ \frac{\tilde{v}}{\tilde{u}} \right]_{C^{1,\alpha}(\tilde{\Omega} \cap B_{r/4}(\frac{1}{2}e_n))} \leq C.
\]

Proof of Theorem 2.1 After multiplying \( v \) by a small constant, the assumptions of the lemma are satisfied with \( P = 0 \) and \( r = r_0 \) small. Thus, if we choose \( r_0 \) small universal, we can apply the lemma indefinitely and obtain a limiting linear function \( P_0 \) such that

\[
| v - uP_0 | \leq Cr^{2+\alpha}, \quad r \leq r_0.
\]

In fact, from Remark 2.3 we obtain

\[
| \frac{v}{u} - P_0 | \leq C|x|^{1+\alpha}
\]

which together with (2.6) gives the desired conclusion. \[ \Box \]

It is easy to see that our proof holds in greater generality. For example, if \( v \) solves \( \Delta v = f \in C^\alpha \) in \( \Omega \cap B_1 \) and vanishes continuously on \( \partial \Omega \cap B_1 \), then we get

\[
\| \frac{v}{u} \|_{C^{1,\alpha}(\Omega \cap B_{r/2})} \leq C(\| v \|_{L^\infty} + \| f \|_{C^\alpha}).
\]

To obtain this estimate it suffices to take in Lemma 2.2 linear functions \( P(x) = a_0 + \sum_{i=1}^n a_i x_i \) satisfying \( 2a_n u_n(0) = f(0) \). In fact, the following more general Theorem holds.
Theorem 2.4. Let
\[ L u := \text{Tr}(A D^2 u) + b \cdot \nabla u + c u, \]
with \( A \in C^\alpha, b, c \in L^\infty \) and
\[ \lambda I \leq A \leq \Lambda I, \quad \|A\|_{C^\alpha}, \|b\|_{L^\infty}, \|c\|_{L^\infty} \leq \Lambda. \]
Assume
\[ L u = 0, \quad u > 0 \quad \text{in} \quad \Omega \cap B_1, \quad u = 0 \quad \text{on} \quad \partial \Omega \cap B_1 \]
and
\[ L v = f \in C^\alpha \quad \text{in} \quad \Omega \cap B_1, \quad v = 0 \quad \text{on} \quad \partial \Omega \cap B_1. \]
Then, if \( u \) is normalized so that \( \| u \|_{C^{1,\alpha}(\Omega \cap B_{1/2})} = 1 \)
\[ \| u \|_{C^{1,\alpha}(\Omega \cap B_{1/2})} \leq C(\|v\|_{L^\infty} + \|f\|_{C^\alpha}) \]
with \( C \) depending on \( \alpha, \lambda, \Lambda \) and \( n \).

Remark 2.5. We emphasize that the conditions on the matrix \( A \) and the right hand side \( f \) are those that guarantee interior \( C^{2,\alpha} \) Schauder estimates. However the conditions on the domain \( \Omega \) and the lower order coefficients \( b, c \) are those that guarantee interior \( C^{1,\alpha} \) Schauder estimates.

Remark 2.6. The theorem holds also for divergence type operators
\[ Lu = \text{div}(A \nabla u + bu), \quad A \in C^\alpha, \quad b \in C^\alpha. \]

The proof of Theorem 2.4 follows the same argument of Theorem 2.1. For convenience of the reader, we give a sketch of the proof.

Sketch of the proof of Theorem 2.4. After a dilation we may assume that (2.2) holds and also
\[ A(0) = I, \quad \max\{[A]_{C^\alpha}, \|b\|_{L^\infty}, \|c\|_{L^\infty}, [f]_{C^\alpha}\} \leq \delta \]
with \( \delta \) to be chosen later. Again, it suffices to show the analogue of Lemma 2.2 in this context, with the \( x_n \) coefficient of \( P \) and \( \bar{P} \) satisfying
\[ 2a_n = 2\bar{a}_n = f(0). \]
Define \( \tilde{v} \) as before. Then
\[ f = L v = L(uP) + r^\alpha \tilde{L} \tilde{v} \left( \frac{x}{r} \right) \quad x \in \Omega \cap B_r \]
with
\[ \tilde{L} \tilde{v} := \text{Tr}(\tilde{A} D^2 \tilde{v}) + \tilde{b} \cdot \nabla \tilde{v} + r^2 \tilde{c} \tilde{v}, \]
\[ \tilde{A}(x) = A(rx), \quad \tilde{b}(x) = b(rx), \quad \tilde{c}(x) = c(rx), \quad x \in \tilde{\Omega} \cap B_1. \]

On the other hand,
\[ \tilde{L} (uP) = (Lu)P + 2(\nabla u)^T \tilde{A} \nabla P + u b \cdot \nabla P \]
thus, using (2.2)-(2.7) and the fact that \( 2a_n = f(0) \)
\[ |L(uP) - f| \leq C \delta r^\alpha, \quad x \in \Omega \cap B_r. \]
From this we conclude that
\[ |\tilde{L} \tilde{v}| \leq C \delta \quad \text{in} \quad \tilde{\Omega} \cap B_1 \]
and we can argue by compactness exactly as before. \( \square \)
3. THE GENERAL CASE, $k \geq 2$.

Let \( \Omega \subset \mathbb{R}^n \) with \( \partial \Omega \in C^{k, \alpha} \). Precisely,
\[
\partial \Omega = \{(x', g(x')) \mid x' \in \mathbb{R}^{n-1}\}, \quad g(0) = 0, \quad \nabla x'g(0) = 0, \quad \|g\|_{C^{k, \alpha}} \leq 1.
\]

**Theorem 3.1.** Let
\[
Lu := Tr(AD^2u) + b \cdot \nabla u + cu
\]
with
\[
\lambda I \leq A \leq \Lambda I,
\]
and
\[
\max\{\|A\|_{C^{k-1, \alpha}}, \|b\|_{C^{k-2, \alpha}}, \|c\|_{C^{k-2, \alpha}}\} \leq \Lambda.
\]
Assume
\[
Lu = 0, u > 0 \quad \text{in} \quad \Omega \cap B_1, \quad u = 0 \quad \text{on} \quad \partial \Omega \cap B_1
\]
and
\[
Lv = f \in C^{k-1, \alpha} \quad \text{in} \quad \Omega \cap B_1, \quad v = 0 \quad \text{on} \quad \partial \Omega \cap B_1.
\]
Then, if \( u \) is normalized so that \( u(\frac{1}{2}e_n) = 1 \)
\[
\left\| \frac{\partial}{\partial u} \right\|_{C^{k, \alpha}(\Omega \cap B_{1/2})} \leq C(\|v\|_{L^\infty} + \|f\|_{C^{k-1, \alpha}})
\]
with \( C \) depending on \( k, \alpha, \lambda, \Lambda \) and \( n \).

From now on, a positive constant depending on \( n, k, \alpha, \lambda, \Lambda \) is called universal.

**Remark 3.2.** If we are interested only in \( C^{k, \alpha} \) estimates for \( \frac{\partial}{\partial u} \) on \( \partial \Omega \cap B_{1/2} \), then the regularity assumption on \( c \) can be weakened to \( \|c\|_{C^{k-3, \alpha}} \leq \Lambda \).

If \( u \) and \( v \) solve (3.1) and (3.2) respectively, the rescalings
\[
\hat{u}(x) = \frac{1}{r_0}u(r_0x), \quad \hat{v}(x) = \frac{1}{r_0}v(r_0x)
\]
satisfy the same problems with \( \tilde{\Omega}, \tilde{A}, \tilde{b}, \tilde{c} \) and \( f \) replaced by
\[
\tilde{\Omega} = \frac{1}{r_0} \Omega, \quad \tilde{A}(x) = A(r_0x), \quad \tilde{b}(x) = r_0b(r_0x), \quad \tilde{c}(x) = r_0^2c(r_0x), \quad \tilde{f}(x) = r_0f(r_0x).
\]

Thus, as in the case \( k = 1 \), we may assume that
\[
\nabla u(0) = e_n, \quad A(0) = I
\]
and that the following norms are sufficiently small:
\[
\max\{\|g\|_{C^{k, \alpha}}, \|A - I\|_{C^{k-1, \alpha}}, \|b\|_{C^{k-2, \alpha}}, \|c\|_{C^{k-2, \alpha}}, \|f\|_{C^{k-1, \alpha}}, \|u - x_n\|_{C^{k, \alpha}}\} \leq \delta,
\]
with \( \delta \) to be specified later.

The proof of Theorem 3.1 is essentially the same as in the case \( k = 1 \). However, we now need to work with polynomials of degree \( k \) rather than linear functions.

We introduce some notation. A polynomial \( P \) of degree \( k \) is denoted by
\[
P(x) = a_m x^m, \quad m = (m_1, m_2, \ldots, m_n), |m| = m_1 + \ldots + m_n,
\]
with the \( a_m \) non-zero only if \( m \geq 0 \) and \( |m| \leq k \). We use here the summation convention over repeated indices and the notation
\[
x^m = x_1^{m_1} \ldots x_n^{m_n}.
\]
Also, in what follows, \( \tilde{i} \) denotes the multi-index with 1 on the \( i \)th position and zeros elsewhere and \( \| P \| = \max |a_m| \).

Given \( u \) a solution to (3.1), we will approximate a solution \( v \) to (3.2) with polynomials \( P \) such that \( \mathcal{L}(uP) \) and \( f \) are tangent at 0 of order \( k - 1 \).

Below we show that the coefficients of such polynomials must satisfy a certain linear system.

Indeed,

\[
\mathcal{L}(uP) = (\mathcal{L}u)P + 2(\nabla u)^T A \nabla P + u \text{tr}(A D^2 P) + u b \cdot \nabla P.
\]

Since \( \mathcal{L}u = 0 \), we find

\[
\mathcal{L}(uP) = g^i P_i + g^{ij} P_{ij}, \quad g^i \in C^{k-2, \alpha}, \quad g^{ij} \in C^{k-1, \alpha}.
\]

Using the first order in the expansions below (l.o.t = lower order terms),

\[
A = I + \text{l.o.t.}, \quad u = x_n + \text{l.o.t.}, \quad \nabla u = e_n + \text{l.o.t.},
\]

we write each \( g^i, g^{ij} \) as a sum of a polynomial of degree \( k - 1 \) and a reminder of order \( O(|x|^{k-1+\alpha}) \). We find

\[
g^i = 2\delta^i_n + \text{l.o.t.}, \quad g^{ij} = \delta_{ij} x_n + \text{l.o.t.}.
\]

In the case \( P = x^m \) we obtain

\[
\mathcal{L}(u x^m) = m_n(m_n + 1)x^{m-\tilde{n}} + \sum_{\tilde{i} \neq \tilde{n}} m_{\tilde{i}}(m_{\tilde{i}} - 1)x^{m-2\tilde{i}+\tilde{n}} + c_{\tilde{i}}^m x^l + w_m(x),
\]

with

\[
c_{\tilde{i}}^m \neq 0 \quad \text{only if } |m| \leq |l| \leq k - 1, \quad \text{and} \quad w_m = O(|x|^{k-1+\alpha}).
\]

Also in view of (3.3)

\[
|c_{\tilde{i}}^m| \leq C\delta, \quad |w_m| \leq C\delta|x|^{k-1+\alpha}, \quad \|w_m\|_{C^{k-2, \alpha}(B_r)} \leq C\delta r.
\]

Thus, if \( P = a_m x^m \), with \( \| P \| \leq 1 \) then

\[
\mathcal{L}(uP) = R(x) + w(x), \quad R(x) = d_l x^l, \quad \text{deg } R = k - 1,
\]

with \( w \) as above and the coefficients of \( R \) satisfying

\[
d_l = (l_n + 1)(l_n + 2)a_{l+\tilde{n}} + \sum_{\tilde{i} \neq \tilde{n}} (l_{\tilde{i}} + 1)(l_{\tilde{i}} + 2)a_{l+2\tilde{i} - \tilde{n}} + c_{\tilde{i}}^m a_m.
\]

**Definition 3.3.** We say that \( P \) is an approximating polynomial for \( v/u \) at 0 if the coefficients \( d_l \) of \( R(x) \) coincide with the coefficients of the Taylor polynomial of order \( k - 1 \) for \( f \) at 0.

We think of (3.6) as an equation for \( a_{l+\tilde{n}} \) in terms of \( d_l \) and a linear combination of \( a_m \)'s with either \( |m| < |l| + 1 \) or when \( |m| = l + 1 \) with \( m_n < l_n + 1 \). Thus the \( a_m \)'s are uniquely determined from the system (3.6) once \( d_l \) and \( a_m \) with \( m_n = 0 \) are given.

The proof of Theorem 3.1 now follows as in the case \( k = 1 \), once we establish the next lemma.
Lemma 3.4. Assume that for some $r \leq 1$ and an approximating polynomial $P$ for $v/u$ at 0, with $\|P\| \leq 1$, we have

$$\|v - uP\|_{L^\infty(\Omega \cap B_r)} \leq r^{k+1+\alpha}.$$ 

Then, there exists an approximating polynomial $\bar{P}$ for $v/u$ at 0, such that

$$\|v - u\bar{P}\|_{L^\infty(\Omega \cap B_{\rho r})} \leq (\rho r)^{k+1+\alpha}$$ 

for $\rho > 0$ universal, and

$$\|P - \bar{P}\|_{L^\infty(B_r)} \leq C r^{k+\alpha},$$

with $C$ universal.

Proof. We write

$$v(x) = u(x)P(x) + r^{k+1+\alpha} \tilde{v} \left( \frac{x}{r} \right), \quad x \in \Omega \cap B_r,$$

with

$$\|\tilde{v}\|_{L^\infty(\tilde{\Omega} \cap B_1)} \leq 1, \quad \tilde{\Omega} := \frac{1}{r} \Omega.$$

Define also,

$$\tilde{u}(x) := \frac{u(rx)}{r}, \quad x \in \tilde{\Omega} \cap B_1.$$

Then

$$f = \mathcal{L}v = \mathcal{L}(uP) + r^{k+\alpha-1} \tilde{\mathcal{L}}\tilde{v} \left( \frac{x}{r} \right) \quad x \in \Omega \cap B_r$$

with

$$\tilde{\mathcal{L}}\tilde{v} := Tr(\tilde{A} D^2 \tilde{v}) + r \tilde{b} \cdot \nabla \tilde{v} + r^2 \tilde{c} \tilde{v},$$

$$\tilde{A}(x) = A(rx), \quad \tilde{b}(x) = b(rx), \quad \tilde{c}(x) = c(rx), \quad x \in \tilde{\Omega} \cap B_1.$$

Using that $P$ is approximating, we conclude that

(3.7) \quad $\tilde{\mathcal{L}}\tilde{v} = \tilde{w}$ in $\tilde{\Omega} \cap B_1$, \quad $\tilde{v} = 0$ on $\partial \tilde{\Omega} \cap B_1$,

with

$$\|\tilde{v}\|_{L^\infty(\tilde{\Omega} \cap B_1)} \leq 1, \quad \|\tilde{w}\|_{C^{k+2,\alpha}} \leq C \delta.$$

By compactness $\tilde{v} \to v_0$ with $v_0$ harmonic. Thus we find,

$$\|v - x_n Q\|_{L^\infty(\tilde{\Omega} \cap B_1)} \leq C r^{k+2} \leq \frac{1}{2} r^{k+\alpha-1}, \quad \deg Q = k, \quad \|Q\| \leq C,$$

with $x_n Q$ a harmonic polynomial and $\rho$ universal. Thus,

$$\|v - u(P + r^{k+\alpha} Q(\frac{x}{r}))\|_{L^\infty(\Omega \cap B_{\rho r})} \leq \frac{1}{2} (\rho r)^{k+\alpha-1}.$$ 

However $P + r^{k+\alpha} Q(\frac{x}{r})$ is not approximating for $v/u$ at 0, and we need to modify $Q$ into a slightly different polynomial $\bar{Q}$.

We want the coefficients $\bar{q}$ of $\bar{Q}$ to satisfy (see (3.6))

(3.8) \quad $0 = (l + 1)(l + 2)\bar{q}_{l+2} + \sum_{i \neq n} (l_i + 1)(l_i + 2)\bar{q}_{l_i+2i-n} + \bar{c}_l^m \bar{q}_m$,

with (see (3.4)-(3.5))

$$\bar{c}_l^m = r^{l_l+1-m} c_l^m, \quad |c_l^m| \leq C \delta.$$
Moreover, since in the flat case i.e. $A = I$, $u = x_n$ and $g, b, c, f$ all vanishing, $Q$ is approximating for $v_0/x_n$ at 0, the coefficients of $Q$ satisfy the system \([3.6]\) with $c_1^n = 0$ and $d_l = 0$, i.e.

$$0 = (l_n + 1)(l_n + 2)q_l + n + \sum_{l \neq n} (l_i + 1)(l_i + 2)q_{l+2i} - n.$$ 

Thus, by subtracting the last two equations, the coefficients of $Q - \bar{Q}$ solve the system \([3.8]\) with left hand side bounded by $C\delta$, and we can find $\bar{Q}$ such that

$$\|Q - \bar{Q}\|_{L^\infty(B_1)} \leq C\delta.$$ 

\[\square\]

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