An experiment for the direct determination of the g-factor of a single proton in a Penning trap

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Abstract. A new apparatus has been designed that aims at a direct precision measurement of the g-factor of a single isolated proton or antiproton in a Penning trap. We present a thorough discussion on the trap design and a method for the experimental trap optimization using a single stored proton. A first attempt at the g-factor determination has been made in a section of the trap with a magnetic bottle. The Larmor frequency of the proton has been measured with a relative uncertainty of 1.8 × 10⁻⁶ and the magnetic moment has been determined with a relative uncertainty of 8.9 × 10⁻⁶. A g-factor of 5.585 696(50) has been obtained, which is in excellent agreement with previous measurements and predictions. Future experiments shall drive the spin-flip transition in a section of the trap with a homogeneous magnetic field. This has the potential to improve the precision of the measured g-factor of the proton and the antiproton by several orders of magnitude.

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1. Introduction

Charged particles can be stored by a superposition of static magnetic and electric fields. In this so-called Penning trap system, a strong magnetic field confines the particle radially and an electrostatic field prevents it from following the magnetic field lines, thus providing axial confinement. A single particle confined in a Penning trap is a suitable system for high-precision measurements of fundamental particle properties and tests of theories, as in the pioneering experiments for the precise comparison of the electron and positron magnetic moments [1] or the measurement of the proton-to-electron mass ratio [2]. Based on a very accurately determined electron $g$-factor, together with quantum electrodynamic calculations, the fine structure constant $\alpha$ has been determined to be $3.7 \times 10^{-10}$ [3]. High-precision studies have also been carried out in more complex systems such as hydrogen-like ions, with the aim of testing bound-state quantum electrodynamics [4]. Furthermore, Penning traps are crucial in the field of high-accuracy mass spectrometry [5–8].

With our Penning-trap setup, we aim to achieve a relative uncertainty of $10^{-9}$ or better for the $g$-factor of a single (anti)proton [9]. The magnetic moment of the proton is currently known to be $10^{-8}$. It was determined based on the electron-to-proton magnetic moment ratio, measured by observing electron and proton transitions simultaneously with a hydrogen maser [10]. We intend to determine the $g$-factor directly, based on the non-destructive measurement of two frequencies of a single proton in a Penning trap [11]. The advantage is that no theoretical corrections due to atomic binding need to be considered as in the case of [10], and most importantly, the measurement can be carried out in the same way with the antiproton, whose $g$-factor is currently known to be only $10^{-3}$ [12]. The precision of the comparison between the proton and antiproton magnetic moments would be increased by six orders of magnitude, thus providing a very stringent test of charge, parity, time (CPT) symmetry in the baryon sector [13].
The developments, techniques and methods presented in this paper are of great interest to other high-precision Penning trap experiments, such as those for the determination of the \( g \)-factor of highly charged ions [14] and of the electron [3]. Also high-accuracy mass measurements [15, 16] can directly benefit from the acquired knowledge. The method for determining the magnetic moment of the proton using a double Penning trap is presented in section 2. In section 3, the experimental setup is described. In section 4, the details of the design of one of the Penning traps are discussed. This trap is a crucial element in the determination of the \( g \)-factor, since it provides the strong magnetic inhomogeneity by means of which single (anti)proton spin flips can be detected. Utilizing a magnetic bottle of 300 mT mm\(^{-2}\) we recently succeeded in observing spin flips with a single trapped proton for the very first time [17]. The key to this was the characterization and optimization of the trap electrostatic characteristics presented in section 5, together with the systematic reduction of external noise. Further improvements of measurement techniques allowed the determination of the proton \( g \)-factor to be 5.585 696(50) with a relative uncertainty of \( 8.9 \times 10^{-6} \) in the presence of the magnetic bottle, as described in section 6.

2. Fundamentals of the \( g \)-factor experiment

The \( g \)-factor is a dimensionless quantity which relates the proton magnetic moment \( \vec{\mu} \) to its spin angular momentum \( \vec{S} \) according to \( \vec{\mu} = g \frac{e}{2m_p} \vec{S} \). The determination of the \( g \)-factor of a single proton stored in a Penning trap relies on the measurement of two frequencies from which the \( g \)-factor can be calculated:

\[
g = \frac{2\omega_L}{\omega_c}.
\]

Here \( \omega_c = \frac{e}{m_p} B \) is the free cyclotron frequency, \( \frac{e}{m_p} \) the charge-to-mass ratio of the proton, \( B \) the magnitude of the magnetic field and \( \omega_L = g \frac{e}{2m_p} B \) the Larmor frequency, which is the precession frequency of the spin magnetic moment about the direction of the external magnetic field.

Two traps are used in our \( g \)-factor apparatus. The trap setup, which is shown in figure 1, consists of a stack of electrodes in which two cylindrical, five-electrode Penning traps are implemented: the precision trap and the analysis trap. The traps are spatially separated by transport electrodes.

In the ideal case, the trap geometry produces an electric quadrupole potential of the form

\[
\phi (\rho, z) = V_0 c_2 \left( z^2 - \frac{\rho^2}{2} \right),
\]

where \( V_0 \) is the trapping voltage applied to the ring electrode (shaded electrodes in figure 1), \( c_2 \) is a coefficient related to the trap geometry and \((z, \rho)\) are the cylindrical coordinates. In such a harmonic potential, the particle oscillation frequencies do not depend on the amplitude of its motion, which can be described as a superposition of three decoupled harmonic oscillations: the modified cyclotron motion at \( \omega_+ \), which is the free cyclotron motion modified by the electric potential, the magnetron motion, which is a drift motion at \( \omega_- \) due to the crossed magnetic and electric fields, and the axial motion at \( \omega_z \), which is the oscillation of the trapped particle in the electrostatic potential. The frequencies are

\[
\omega_+ = \frac{\omega_c}{2} \pm \sqrt{\left( \frac{\omega_c}{2} \right)^2 - \frac{\omega_z^2}{2}},
\]

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Figure 1. The double Penning trap setup consists of the analysis trap, in which the magnetic field is deliberately inhomogeneous, and the precision trap, located in the homogeneous field region. The magnitude of the magnetic field along the axial direction is shown at the bottom of the figure. For details, see text.

\[
\omega_+ - \frac{\omega_c}{2} = \sqrt{\left(\frac{\omega_c}{2}\right)^2 - \omega_z^2}, \tag{2.4}
\]

\[
\omega_z = \sqrt{\frac{e}{m_p}2c_2V_0}. \tag{2.5}
\]

Typical values for the proton eigenfrequencies in our experiment are \(\omega_+/2\pi = 28.97\) MHz, \(\omega_-/2\pi = 8.26\) kHz and \(\omega_z/2\pi = 674.4\) kHz in the precision trap (see figure 1), and \(\omega_+/2\pi = 17.93\) MHz, \(\omega_-/2\pi = 12.9\) kHz and \(\omega_z/2\pi = 674.4\) kHz in the analysis trap (see figure 1), respectively.

The oscillating motions of a trapped charged particle induce image currents in the trap electrodes, which can be measured using highly sensitive detection systems [4, 18–21]. The proton signal is amplified, down-converted and a fast Fourier transform gives the frequency. Eventually, the free cyclotron frequency is calculated using the ‘invariance theorem’ \(\omega_c^2 = \omega_+^2 + \omega_-^2 + \omega_z^2\) [22].

The Larmor frequency can be measured by making use of the fact that the energy splitting \(\Delta U\) between the two spin eigenstates \(S_z = \pm \frac{\hbar}{2}\) of a proton with magnetic moment \(\vec{\mu} = g_e e \hbar 2m_p\) in an external magnetic field is given by

\[
\Delta U = g_e \frac{e \hbar}{2m_p} B = \hbar \omega_L. \tag{2.6}
\]

This means that transitions between the spin states can be induced by a radio-frequency (rf) field close to the Larmor frequency whose typical values in our setup are \(\omega_L/2\pi = 80.92\) MHz in the precision trap and \(\omega_L/2\pi = 50.06\) MHz in the analysis trap. It is thus possible to determine the Larmor frequency by measuring the spin-flip probability as a function of the drive frequency, provided that the spin orientation can be monitored. The determination of the spin orientation is carried out by means of the coupling between the magnetic moment of a particle in a Penning trap and its axial frequency in the presence of an inhomogeneous magnetic field, the so-called continuous Stern–Gerlach effect [23]. The magnetic field inhomogeneity is produced by a ferromagnetic ring electrode in the analysis trap as illustrated in figure 1. The ring locally
distorts the magnetic field lines of the external homogeneous magnetic field, \( \vec{B} = B_0 \vec{e}_z \), into an inhomogeneous field of the form of a so-called ‘magnetic bottle’,

\[
\vec{B} = B_0 \vec{e}_z + \vec{B}_\text{bottle} = B_0 \vec{e}_z + B_2 \left( z^2 - \frac{\rho^2}{2} \right) \vec{e}_z - z \vec{e}_\rho .
\]  

(2.7)

The magnetic potential energy \( U_{\text{mag}} = -\vec{\mu} \cdot \vec{B} \) due to the interaction between the magnetic moment of the trapped proton and the external magnetic field thus takes the form

\[
U_{\text{mag}} = \mp \mu_z \left[ B_0 + B_2 \left( z^2 - \frac{\rho^2}{2} \right) \right] .
\]  

(2.8)

\( B_0 \) is the homogeneous component of the magnetic field, which does not affect the axial frequency of the particle. The second term of equation (2.8), however, leads to an additional force in the axial direction, \( F_\text{mag}^z = \pm 2 \mu_z B_2 z \), which adds to or subtracts from the restoring force due to the electric potential of equation (2.2). Consequently, the axial frequency in equation (2.5) becomes

\[
\omega_z = \sqrt{\frac{e^2 c V_0}{m_p} \pm \frac{2 \mu_z B_2}{m_p}} \approx \pm \frac{\mu_z B_2}{m_p} \omega_z ,
\]  

(2.9)

where \( \omega_z \) is given by equation (2.5). This means that the axial frequency of the proton in the magnetic bottle can take two different values depending on the spin orientation. Thus, a spin flip can be detected by measuring the axial frequency jump

\[
\delta \nu^\text{SF}_z = \frac{\omega_z|\uparrow\rangle - \omega_z|\downarrow\rangle}{2\pi} = \frac{1}{2\pi^2} \frac{\mu_z B_2}{m_p} \nu_z .
\]  

(2.10)

Spin-flip transitions are induced by irradiating a magnetic rf field with frequency \( \nu_{\text{rf}} \). The detection of the spin orientation after each spin-flip attempt is performed in the analysis trap. The drive frequency is varied around \( \nu_L \) and for each value of \( \nu_{\text{rf}} \) the transition probability is measured. The actual Larmor frequency can be determined from the spin-flip resonance obtained after a series of attempts [17].

Compared with experiments with electrons and positrons [1], the difficulty lies in the fact that the ratio \( \frac{\mu_z}{m_p} \) in the case of the proton is about one million times smaller, \( \frac{\mu_z}{m_p} \frac{m_e}{\mu_z} = 8 \times 10^{-7} \), so that a much stronger magnetic bottle is necessary to achieve a comparable \( \delta \nu^\text{SF}_z \) [24]. In our case, with an axial frequency of 674 kHz, a \( B_2 \) of 300 mT mm\(^{-2} \) is needed to provide a frequency jump of about 190 mHz.

A strong magnetic bottle increases \( \delta \nu^\text{SF}_z \) and thus the spin-flip detectability. On the other hand, however, in an inhomogeneous magnetic field the eigenfrequencies are no longer energy independent, but rather shift in proportion to the magnetic bottle strength. The perturbations due to the presence of the magnetic bottle are summarized in matrix form [25] as

\[
\begin{pmatrix}
\Delta \omega_+ / \omega_+ \\
\Delta \omega_+ / \omega_+ \\
\Delta \omega_- / \omega_- \\
\Delta \omega_- / \omega_- \\
\end{pmatrix} = \frac{B_2}{B_0 m_p \omega_z^2} \begin{pmatrix}
- \left( \frac{\omega_0}{\omega_z} \right)^2 & 1 & 2 \\
1 & 0 & -1 \\
2 & -1 & -2 \\
- \left( \frac{\omega_0}{\omega_z} \right)^2 & 1 & 2 \\
\end{pmatrix} \begin{pmatrix}
\Delta E_+ \\
\Delta E_z \\
\Delta E_- \\
\end{pmatrix} .
\]  

(2.11)
Using equation (2.11) to estimate axial frequency shifts in our magnetic bottle yields
\[
\frac{\Delta \omega_z}{(\Delta E_+ - \Delta E_-)} = \frac{B_2}{B_0 m_p \omega_z} \Rightarrow \approx \frac{1}{\mu eV}.
\] (2.12)

Obviously, fluctuations in the radial energy in the analysis trap are highly undesirable, since it might become impossible to resolve the small frequency jump due to a spin flip because of the instability of the axial frequency signal. The effects of an inhomogeneous magnetic field on the eigenfrequencies are the reason for a double Penning-trap setup, since the frequency shifts described by equation (2.11) limit the precision of the frequency measurements and consequently the precision of the determination of the $g$-factor. The double trap allows the measurement of the eigenfrequencies in a region where the magnetic field is highly homogeneous [21], namely in the precision trap in which $B_2 < 1 \mu T \text{mm}^{-2}$, in spite of the huge magnetic inhomogeneity produced by the analysis trap in close vicinity.

3. Experimental setup

A sketch of the experimental setup is shown in figure 2. A superconducting magnet provides an homogeneous magnetic field of $B_0 = 1.899 \text{T}$ for the Penning trap. The double-trap setup shown in figure 1 is mounted in a sealed ultra-high vacuum chamber, which is placed in the horizontal bore of a superconducting magnet. For details, see text.

Figure 2. Schematic diagram of the experimental setup. The double Penning trap setup is placed in the horizontal bore of a superconducting magnet. For details, see text.
Figure 3. Top: the trap chamber in cross-sectional view. The chamber is a sealed system in which a vacuum pressure below $10^{-14}$ mbar can be achieved due to cryopumping. Bottom: the proton signal can be detected as a minimum in the thermal noise spectrum of the axial detector, which is represented by a resonant circuit.

The trap chamber is shown in figure 3. The cylindrical chamber made of oxygen-free electrolytic (OFE) copper is closed on one side by a mounting flange with feedthroughs. On the other side, a copper tube allows the chamber to be evacuated to a pressure of $1 \times 10^{-6}$ mbar before being pinched off. When cooled down to cryogenic temperatures a vacuum pressure below $10^{-14}$ mbar can be achieved due to cryopumping [26, 27], which allows for storage times of the order of months. The electrode stack consists of the trap and transport electrodes, an electron beam source and a polyethylene (PE) target. The electrodes are mounted between precision sapphire spacers. To generate the magnetic rf field for the spin-flip drive a disc coil is used [28, 29], which is mounted outside the electrode stack but inside the trap chamber.

The feedthrough flange connects the dc-supply lines for the trap electrodes, the excitation lines and the detection lines to the outside. The trap electrodes are biased by an ultra-high

The 4 K pulse-tube cooler was recently replaced by a liquid-He bath cryostat, because the cooler vibrations with an amplitude of a few $\mu$m used to limit the relative uncertainty of the cyclotron frequency measurement to $\approx 5 \times 10^{-9}$.

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precision voltage source (Stahl Electronics UM1-14) placed at room temperature. Three low-pass filter stages are implemented between the power supply and the feedthrough flange at 5, 80 and 300 K. The location of the filter boards is indicated in figure 2.

The detection systems consist of a resonant circuit and a cryogenic low-noise amplifier. The resonant circuit is a solenoid coil mounted in a cylindrical housing. The inductance and self-capacitance of the coil together with the parasitic capacitance of the trap and attached electronics form a high-quality ($Q$) tuned circuit. Two such detectors are implemented, the superconducting Nb/Ti axial detector and the cyclotron detector made of OFE copper. The resonators with their respective amplifiers are thermally anchored to the cold stage (see figure 2). At the resonance frequency of $\nu_0 = 680$ kHz, the effective parallel resistance of the axial detector is $R_p = 36$ M$\Omega$ and the quality factor $Q = 5700$. The cyclotron detector operates at a frequency of $\nu_0 = 28.969$ MHz, with $R_p = 380$ k$\Omega$ and $Q = 1250$ [30, 31]. The cyclotron detector is placed inside the trap chamber, which aims at reducing the parasitic parallel capacitances and resistive losses due to the wiring and feedthrough. Since at the lower operating frequency of the axial resonator such parasitic effects are less critical, this resonator is placed outside the chamber.

The protons are produced inside the electrode stack. By applying a high voltage to the acceleration electrode of the electron source (see figure 3), electrons are emitted from a field emission point (FEP), thus establishing an electron beam that bombards the target. This sputters atoms and molecules from its surface. Charged particles, including protons, are produced by electron impact ionization and stored in the precision trap. The undesired species are expelled from the trap by means of broadband noise excitation and resonant rf excitation. The remaining proton cloud is then reduced to a single proton.

The image currents induced by the single oscillating proton in the trap electrodes can be detected as a voltage drop across the effective resistance $R_p$ of the resonant circuit. By means of this interaction with the external circuit the proton is resistively cooled. In thermal equilibrium, the particle shortens the voltage noise of the detector at its oscillation frequency, resulting in a minimum, the so-called ‘dip’, in the thermal noise spectrum of the axial detector [19], as shown in figure 3. The proton dip in the axial mode can be observed in the precision trap as well as in the analysis trap, since the axial detector is connected to both. The same principle is used to measure the radial frequencies in the precision trap by coupling the radial modes to the axial mode as discussed in [21].

4. The analysis trap design

The most important trap design goal for observing single-proton spin flips is the maximization of the magnetic bottle produced by a ferromagnetic ring electrode (see figure 1), while obeying certain design criteria for an optimal electrostatic trapping potential. Deviations from the ideal quadrupole potential in equation (2.2) in combination with the strong magnetic inhomogeneity can make it impossible to detect a proton signal. With the former design of the analysis trap, with a combination of a toroidal ring and cylindrical correction electrodes and endcaps [32], a roughly 30% stronger magnetic bottle was achieved compared to the current trap geometry. However, this strongest inhomogeneity was obtained at the expense of the electrostatic properties suited for the efficient detection of a single proton. For the new design both the maximization of the magnetic bottle and the optimization of the trapping potential were investigated, as will be discussed in sections 4.1 and 4.2, respectively.

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To calculate the static electric and magnetic fields generated by any arbitrary shape of the electrodes, a well-established numerical technique, the finite-element method (FEM) is used. The numerical calculations presented in the following sections were performed with an FEM-based software package, COMSOL Multiphysics. The precision of the electrode dimensions employed in the calculations was chosen to be 1 µm and the fields are calculated with a tolerance of $10^{-6}$. The uncertainties of the results presented in this section are dominated by the errors in the fit coefficients by the further analysis of the numerical data obtained from the FEM calculations.

4.1. Magnetic bottle design

The strength of the magnetic bottle term $B_2$, obtained by introducing a ferromagnetic CoFe ring electrode with a saturation magnetization of 2.35 T [33] into the external homogeneous magnetic field of 1.899 T, is investigated as a function of four parameters related to the geometry of the ring. The parameters $\theta$, $\rho_0$, $l_0$, and $l_n$ define the spatial distribution of ferromagnetic material around the trap center and are indicated in figure 4. The choice of such geometry of the electrode is discussed at the end of this section. The effect of the variation of each parameter on the magnetic bottle is investigated independently. The parameter $\rho_s = 5$ mm is defined by the inner radius of the sapphire spacers (not shown in the figure), which are mounted between the electrodes. The effect of the variation of $\rho_s$ was not investigated. The parameter $l_0$ is fixed by the electrostatic design discussed in section 4.2 and kept constant at $l_0 = 0.386$ mm for the magnetic field calculations. For each ring geometry, an FEM calculation is performed. In figure 5(a) an example of the magnetic field resulting from a two-dimensional FEM calculation is shown. Because of the axial symmetry, only one half of the $z\rho$-plane is considered, from which only a small region (compared to figure 1) around the ring electrode of the analysis trap is shown. The axial component of the field $B_z$ along the $\rho = 0$ axis is shown in figure 5(b). By means of a polynomial fit to the calculated data around the trap center, the $B_2$ term is obtained.

We first analyze the angular positioning of ferromagnetic material with respect to the trap center. The magnetic bottle term is calculated as described above for different ring profiles,
Figure 5. (a) Result of a finite-element calculation. The magnetic field lines for
the pictured electrode arrangement are shown. Because of the axial symmetry
only one half of the analysis trap in a cross-sectional view is shown. In (b) the
axial component of the magnetic field $B_z$ is plotted against $z$. The $B_z$-term can
be obtained from a polynomial fit to the calculated data.

represented by $\theta$ in figure 4. In this process $\theta$ is varied, while the other parameters are kept
constant.

In figure 6, an optimal angle that yields the highest $B_2$ term for the parameter $\theta$ can be
clearly identified, $= 21.3(1.1)^\circ$. This means that placing ferromagnetic material in the region
between $\theta \approx \pm 21^\circ$ (and naturally also in the equivalent axial symmetric region) will generate a
positive contribution to $B_2$. In regions where $\theta < -21^\circ$ or $\theta > 21^\circ$ the contribution to $B_2$ has the
opposite sign, thus diminishing the entire $B_2$. This result is consistent with a detailed analytical
calculation [25]. Consideration was given to the geometry of the adjacent electrodes when
choosing the most appropriate $\theta$ for the ring. In order to not compromise the mechanical stability
of the adjacent electrodes (see figure 5(a)), an angle $\theta = 18^\circ$ was chosen, which generates a $B_2$
only 1.6% smaller than the maximum in figure 6.

In a second step the inner radius of the ring $\rho_0$ and its radial extension $l_{\rho}$ are varied. The
magnetic bottle term as a function of the trap radius $\rho_0$ is shown in figure 7(a). The strength of
the magnetic bottle increases rapidly with decreasing trap radius. Nevertheless, the choice of $\rho_0$
has to be limited, since a small radius makes the whole trap difficult to machine. A trap radius
of 1.8 mm was chosen, which generates a $B_2$ of about 300 mT mm$^{-2}$. Next, keeping $\rho_0$ fixed at
1.8 mm, the radial extension $l_{\rho}$ is varied. The results in figure 7(b) show that the variation of
$B_2$ with $l_{\rho}$ is much smaller than its variation with $\rho_0$. Note that only the portion of magnetic
material placed close to the trap center contributes significantly to the magnetic bottle, which is
again consistent with the analytical calculation [25]. Thus, the radial extension of the ring $l_{\rho}$ is
chosen to be 5.7 mm, so as to match the outer diameter of the sapphire ring (see figure 9).

Finally, $\theta$, $\rho_0$ and $l_{\rho}$ are kept fixed and the influence of a fourth parameter is analyzed.
This is represented by $l_n$ (‘ring nose’) in figure 4. The length of the inner surface of the ring
electrode $l_n$ is uniquely determined for a given trap radius $\rho_0$ (see section 4.2); however, there
are different ways to combine it with the chosen $\theta$. In figure 8 the magnetic bottle is plotted as
a function of $l_n$. Although a stronger magnetic bottle is reached with a smaller $l_n$, too small a $l_n$
Figure 6. Effect of the angular positioning of ferromagnetic material on the magnetic bottle. Each angle $\theta$ corresponds to a different design for the ring electrode. The $B_2$-term with respective error is plotted as a function of $\theta$ for $\rho_0 = 1.8$, $l_\rho = 5.7$, $l_n = 0.35$ and $l_0 = 0.386$ mm.

Figure 7. Effect of the radial positioning of ferromagnetic material on $B_2$. (a) The magnetic bottle strength with respective error as a function of the trap radius for $\theta = 18^\circ$, $l_\rho = 5.7$, $l_n = 0.35$ and $l_0 = 0.386$ mm. The $B_2$-term grows strongly with decreasing $\rho_0$. (b) For a constant radius of $\rho_0 = 1.8$ mm, the radial extension $l_\rho$ is varied.

also implies that the adjacent electrodes have a pointed shape, which is difficult to manufacture with precision (see figure 9). Taking this into account, an $l_n$ of 0.35 mm was chosen.

Two criteria were considered for choosing the geometry of the ring electrode: the strength of the magnetic bottle and the shielding of the magnetic rf field for the spin-flip drive. As illustrated in figure 9, the drive is applied through a disc coil from the outside of the trap. Since the magnetic field generated by the coil is shielded by the electrodes and not by the sapphire spacers, the path for the signal is provided by the gap between the electrodes, which is shown in figure 9. The geometry chosen for the final design (figure 9(a)) is compared to a hypothetical design (figure 9(b)), in which the values of the parameters are the same as those in figure 9(a), but the ring is designed with right angles in the inner edges. This geometry, although easier to manufacture, would attenuate the spin-flip drive more strongly. In a separate measurement...
similar to that described in [28, 29], a field amplitude attenuation of 10 dB due to the trap electrodes in figure 9(a) was observed. For the trap in figure 9(b) the attenuation is expected to be comparable to that due to the precision trap in figure 1, which is about 15 dB. The alternative design in figure 9(b) also generates a weaker magnetic bottle, since less magnetic material is placed close to the trap center.

The magnetic field calculated for the optimized geometry has a magnetic bottle term $B_2 = 300.3807(63)$ mT mm$^{-2}$. This magnetic bottle leads to an axial frequency jump due to a proton spin-flip transition of $\delta\nu_{SF}^z = 188.1(2.2)$ mHz out of $\nu_z = 682(8)$ kHz. The calculated $B_2$ agrees with the measured value of $B_2 = 297(10)$ mT mm$^{-2}$ (see section 5.1).

4.2. Design and assembly of a five-electrode cylindrical Penning trap

The electrostatic potential generated by a stack of cylindrical electrodes can be expanded in a Taylor series at $(\rho = 0, z = 0)$

$$\phi (\rho, z) = \sum_{k=0}^{\infty} \left( \sum_{i=0}^{k} C_{i,k} \rho^i z^{k-i} \right) \quad \text{with} \quad C_{i,k} = \left( \frac{k}{i} \right) \frac{\partial^i \phi}{\partial \rho^i \partial z^{k-i}} \bigg|_{(0,0)} . \quad (4.1)$$

The potential along the axial direction is

$$\phi (z) = \sum_{k=0}^{\infty} C_{0,k} z^k \quad \text{with} \quad C_{0,k} = \frac{1}{k!} \frac{\partial^k \phi}{\partial z^k} \bigg|_{(0,0)} = c_{k,0} V_0 , \quad (4.2)$$

where the notation $c_{k,0} = C_{k,0}/V_0$ for the normalized coefficients was introduced. For simplicity, the coefficients $C_{0,k}(c_{0,k})$ will be designated as $C_k(c_k)$. The potential generated by real trap electrodes is thus no longer a pure quadrupole potential as in equation (2.2), but rather contains anharmonic terms. The electrostatic anharmonicity leads to energy-dependent shifts in the eigenfrequencies [25, 34, 35], which may limit the precision of the measurements.

In Penning traps used for high-precision experiments, the electrostatic potential should be as harmonic as possible, which means that the terms with order higher than $(k = 2)$ in equation (4.1) should vanish. For a particle confined within a small volume at the center of the trap, only the lower order terms, particularly $C_4$ and $C_6$, are of relevance. Anharmonicity compensation of the potential can be accomplished by using the five-electrode configuration.
Figure 9. (a) Final design of the ring electrode (in dark gray). A calculated magnetic bottle of $B_2 = 300.3807(63)$ mT mm$^{-2}$ is achieved with this geometry. The electrodes adjacent to the ring are shown in order to illustrate the gap through which the spin-flip drive leaks to the trap center. (b) Alternative design of the ring electrode with the same set of parameters ($\theta$, $\rho_0$, $l$, $l_n$, $l_0$) as in (a). The right angles in the ring shape, which are not appropriate due to the spin-flip drive method used, are the only difference between the designs. This geometry generates a magnetic bottle of $B_2 = 262.9182(12)$ mT mm$^{-2}$.

depicted in figure 10(a) [34, 35]. The five-electrode Penning trap consists of a ring electrode, two endcaps and two extra electrodes, the so-called correction electrodes. The five electrodes are arranged so as to ensure mirror symmetry about the ($z = 0$)-plane. Thus, the odd $k$ coefficients vanish. By applying a proper voltage $V_c$ to the correction electrodes, the $C_4$-coefficient can be eliminated. The ratio between $V_c$ and the voltage $V_0$ applied to the ring is called the tuning ratio, $T_R = V_c / V_0$. By means of a proper choice of the trap geometry and tuning ratio, $C_4$ and $C_6$ can be eliminated simultaneously, thus providing compensation of the potential in the lowest order [34, 35]. Elliptical corrections of the electrostatic potential due to harmonic imperfections, which are caused by deviations from the ideal geometry in the electrodes (including patch effects [36]), can be neglected in the scope of this work. This is because the invariance theorem used to determine the free cyclotron frequency (see section 2) is insensitive to those kinds of imperfections as well as to misalignments between the magnetic field axis and the trap axis as demonstrated in [22].

In order to obtain the coefficients in equation (4.2) in terms of the trap dimensions, the potential $\phi(\rho, z)$ has to be calculated. The solution of the Laplace equation in cylindrical coordinates can be expanded in a series of modified Bessel functions and trigonometric functions [37]. Considering the boundary-value problem illustrated in figure 10(a), in which axial and mirror symmetry are assumed, the potential at any point inside the cylinder is
Figure 10. (a) Idealized representation of a five-electrode cylindrical Penning trap. The boundary-value problem illustrated in the figure can be solved analytically. (b) Real geometry of the analysis trap in our experiment. The inner surfaces of the real electrodes have the same cylindrical geometry as that assumed for the analytical calculation.

given by [38]

$$\phi (\rho, z) = V_0 \sum_{m=1}^{\infty} [A_m + T R B_m] I_0 (\kappa_m \rho) \cos (\kappa_m z)$$

(4.3)

with

$$A_m = \frac{8}{L l_s \kappa_m^2 I_0 (\kappa_m \rho_0)} \sin \left(\frac{\kappa_m l_s}{2}\right) \sin \left(\frac{\kappa_m (l_s + l_0)}{2}\right), \quad \text{and}$$

$$B_m = \frac{16}{L l_s \kappa_m^2 I_0 (\kappa_m \rho_0)} \sin \left(\frac{\kappa_m l_s}{2}\right) \sin \left(\frac{\kappa_m (l_s + l_0)}{2}\right) \cos \left(\frac{\kappa_m (2 l_s + l_0 + l_c)}{2}\right),$$

(4.4)

where $\kappa_m = \frac{m \pi}{L}$ with $m = 2n + 1$, $n \in \mathbb{N}$. $I_0$ is the zeroth-order modified Bessel function of the first kind and $L = 4l_s + 2l_c + 2l_e + l_0$ is the total length of the trap. The coefficients in equation (4.2) are given by

$$c_k = \frac{1}{V_0 k!} \left. \frac{\partial^k \phi}{\partial z^k} \right|_{(0,0)} = \frac{1}{k!} \sum_{m=1}^{\infty} [A_m + T R B_m] \left[ \left. \frac{\partial^k \cos (\kappa_m z)}{\partial z^k} \right|_{z=0} \right],$$

(4.5)

where $I_0(0) = 1$. The derivative on the right side for $z = 0$ is nonzero only for $k$ even. Thus, the odd terms in $k$ vanish, as expected due to the mirror symmetry. The $c_k$-coefficients in equation (4.5) can be written as the sum:

$$c_k = e_k + T R d_k$$

(4.6)
with
\[
e_k = \frac{1}{k!} \sum_{m=1}^{\infty} A_m (-1)^{\frac{k}{2}} \kappa_m^k,
\]
\[
d_k = \frac{1}{k!} \sum_{m=1}^{\infty} B_m (-1)^{\frac{k}{2}} \kappa_m^k.
\] (4.7)

The axial frequency of the particle is given by equations (2.5) and (4.6)
\[
\omega_z = \sqrt{\frac{q}{m} 2V_0 (e_2 + T_R d_2)}.
\] (4.8)

Changes in the tuning ratio thus shift the axial frequency, unless \(d_2\) vanishes. In this case the trap is said to be orthogonal \([34, 35]\).

Equations (4.3)–(4.7) establish an analytical model, by means of which the electrostatic potential of a five-electrode cylindrical Penning trap can be calculated for different sets of trap dimensions \((\rho_0, l_0, l_c, l_s)\). The precision trap of our experiment was designed on the basis of this analytical calculation \([28, 39]\). In the case of the analysis trap, however, the analytical model is not accurate enough for the calculation of the electrostatic potential. Although the inner surfaces of the real electrodes form the same hollow cylinder as in the geometry considered for the analytical calculation (figure 10(a)), the real electrodes shape in the region where \(\rho > \rho_0\) (see figure 10(b)) deviates from the idealized geometry. The effect on the electric potential, which arises due to the existence of gaps between the trap electrodes, is stronger the smaller the trap radius. If the same idealized trap geometry is assumed (figure 10(a)), then the results of the FEM calculation and those of the analytical calculation agree within a tolerance of \(\pm 1 \mu m\). In this section the importance of the exact geometry of the trap electrodes for the calculation of the electrostatic potential is demonstrated for a trap radius of \(\rho_0 = 1.8\) mm.

In the numerical method the electrostatic potential produced by the ring \((A_3)\) and the correction electrodes \((A_2/A_4)\) in figure 10(b), when a voltage of \(-1 V\) is applied to one of them while all the others are grounded, is calculated separately. These ‘normalized’ potentials represent the individual contributions of the electrodes, from which the total potential along the \(z\)-axis resulting from any desired voltage scheme can be calculated:
\[
\phi(z) = T_R V_0 \phi_{A_2}(z) + V_0 \phi_0(z) + T_R V_0 \phi_{A_4}(z),
\] (4.9)

where \(\phi_0(z)\) and \(\phi_{A_4}(z)\) are the contributions of the ring and correction electrodes, respectively. By fitting a polynomial function of the form
\[
\phi(z) = \sum_{n=0}^{\infty} C_{2n} z^{2n}
\] (4.10)

to the data resulting from equation (4.9) near the center of the trap, the coefficients \(C_2, C_4\) and \(C_6\) can be extracted. The compensation of the potential and the orthogonality of the trap geometry defined by the set of dimensions \((\rho_0, l_0, l_c)\) are analyzed. The ideal tuning ratio for which \(C_4\) \((C_6)\) vanishes can be extracted from the \(C_i\) versus \(T_R\) curves as illustrated in figure 11(a). The difference between the tuning ratios that satisfy the conditions \(C_4 = 0\) and \(C_6 = 0\), \(\Delta_{TR} = T_R^{C_4} - T_R^{C_6}\), is an indicator of the quality of the anharmonicity compensation of the potential. \(\Delta_{TR} = 0\) implies that \(C_4\) and \(C_6\) can be simultaneously eliminated, which is desirable.
Figure 11. (a) $C_4 (C_6)$ as a function of the tuning ratio. For a given trap geometry ($\rho_0 = 1.8\text{ mm, } l_0 = 0.37\text{ mm, } l_c = 1.385\text{ mm}$), the electric potential is calculated for different values of $T_R$ and the coefficients are extracted from the polynomial fit to the numerical data. The $T_R$ that tunes out the respective coefficient is indicated in each case. (b) Axial frequency as a function of the tuning ratio for $\rho_0 = 1.8\text{, } l_0 = 0.38\text{, and } l_c = 1.35\text{ mm}$. The slope of the curve $D_{\nu z}^v$ characterizes the orthogonality of the trap geometry under investigation (see text for details).

The orthogonality of the trap is examined by calculating $D_2$. In terms of the axial frequency in equation (4.8) it can be written as

$$D_2^v \left[ \text{Hz/Unit}_T \right] = \frac{\partial \nu_z}{\partial T_R} = \frac{1}{2\pi} \sqrt{\frac{q}{m}} \frac{2V_0}{2 \sqrt{\varepsilon_2 + d_2 T_R}},$$

(4.11)

where $\text{Unit}_T$ is a change in $T_R = V_c/V_0$ of one. The axial frequency is calculated from the $C_2$-coefficient of the electric potential for different values of $T_R$. The result is the linear relation in figure 11(b), from which the slope of the curve representing $D_2^v$ can be calculated. This parameter quantifies the orthogonality of the trap. $D_2^v = 0$ means that the trap is orthogonal, i.e. the axial frequency is independent of the tuning ratio. The ideal set of dimensions ($\rho_0, l_0, l_c$) for the trap geometry is that with which both requirements, $\Delta_{TR} = 0$ and $D_2^v = 0$, can be fulfilled. In the example of figure 11, $\Delta_{TR} = -0.0096$ in (a) and $D_2^v = 8\text{ kHz/Unit}_T$ in (b) and hence in both cases far from the ideal geometry.

We have adjusted two parameters, $l_0$ and $l_c$, in order to tune the trap properties. The slit between the electrodes (see figure 10) is chosen to be $l_s = 0.14\text{ mm}$. We start the optimization procedure with roughly estimated values for $l_0$ and $l_c$. For this geometry $\Delta_{TR}$ and $D_2^v$ are calculated. For the same $l_0$, a second geometry with a different $l_c$ is analyzed. Again, $\Delta_{TR}$ and $D_2^v$ are calculated. Comparing the two tested configurations, $l_0$ with $l_0^1$ and $l_0$ with $l_0^2$, the right direction for $l_c$ that minimizes $\Delta_{TR}$ can be identified. It is demonstrated that $\Delta_{TR}$ scales linearly with $l_c$ for a fixed $l_0$, as shown in the summary of the results in figure 12(a). Therefore, only three such points are needed to uniquely determine the $l_c$ that makes $\Delta_{TR} = 0$ for a given $l_0$. The $D_2^v$, however, is not necessarily optimized yet.

Then, $l_0$ is varied and the above compensation procedure is repeated, in order to find the $l_0$ for which the trap becomes orthogonal. For each $l_0$ there is a $l_c$ for which $\Delta_{TR} = 0$. The compensated pairs ($l_0, l_c$) are linearly related to $D_2^v$. The stars in figure 12(b) represent this line, along which the trapping potential is compensated, as clearly depicted in figure 12(a). Following the line of stars across the different $l_0$s (see figure 12(b)), one can find a unique point for which
Figure 12. (a) $\Delta_{TR}$ as a function of $l_c$ for different $l_0$s. All trap geometries represented by a pair $(l_0, l_c)$ along the line of stars satisfy the compensation condition $\Delta_{TR} \approx 0$. (b) $D_{Vz}^2 (V_0 = -1 V)$ is no longer minimized for all points on the compensation line, but rather varies linearly along it. The crossed circle represents the unique pair $(l_0, l_c)$ with which both optimization conditions can be satisfied. For comparison, the results obtained for an idealized cylindrical geometry with the same set $(l_0, l_c)$ optimized for the real geometry are also plotted. These are represented by the crossed triangle.

The crossed triangles in both graphs of figure 12 represent the results for the same set of dimensions of the inner surfaces of the electrodes calculated using the corresponding idealized cylindrical geometry in figure 10(a). The results obtained using the real geometry (row 1 in table 1) and those obtained using the idealized cylindrical geometry (row 2 in table 1) differ significantly. For comparison, the optimization procedure was repeated for the idealized cylindrical geometry in figure 10(a) with $\rho_0 = 1.8$ mm. The ideal dimensions $(l_0^\gamma, l_c^\gamma)$ obtained in this case are also shown in table 1 (row 4). Considering the real trap geometry with the dimensions $(l_0^\gamma, l_c^\gamma)$ the results in row 3 of table 1 are obtained. With the former design of the analysis trap [32], $\Delta_{TR} = -1.13(1) \times 10^{-1}$ and $D_{Vz}^2 (V_{res} = -0.940 V) = -616.82(32)$ kHZ/Unit$_{TR}$ are obtained from the numerical calculations.

The analysis trap electrodes are machined from high-purity OFE copper, with a manufacturing tolerance of less than 3 $\mu$m. The inner surfaces of the electrodes are polished. In order to avoid oxidation, these are treated with electroless nickel plating, which serves as a diffusion layer, followed by galvanic gold plating. Each layer is 5 $\mu$m thick, which is taken into account by dimensioning the electrodes, so as to ensure that the real trap conforms to the design. A curvature radius of at most 5 $\mu$m at the sharp edge of the correction electrodes is estimated. The polishing prior to plating aims to ensure the uniformity of the galvanic coating thickness and the homogeneity of the surfaces. The machining grooves on the inner surfaces were entirely removed, since otherwise these could lead to suppression of the gold deposition at
The axial frequency asymmetry compensation for a given configuration is evaluated by comparing the variation that better compensates for the asymmetry of the potential is determined. The quality of the electrodes, the offset on the ring parameters are available to manipulate the electrostatic properties of the trap: the offset on the endcaps ($O_{\text{e}}$, $O_{\text{r}}$, $O_{\text{c}}$) when the frequency variation is the same for both correction electrodes, $\Delta_{\nu}^{A2} = \Delta_{\nu}^{A4}$. The sought-after configuration corresponds to the pair of offsets for which $\Delta_{\nu}^{A2} = \Delta_{\nu}^{A4} = 0$. The offset on the ring $O_{\text{fA3}}$ can be varied to optimize the orthogonality by minimizing the absolute value of $D_{\zeta}^{\nu}$.

Table 1. Comparison of the results of the design optimization procedure obtained for the real trap geometry and for an idealized cylindrical geometry. In both cases the calculation is performed using the numerical method.

| Trap geometry | $l_0$ (mm) | $l_c$ (mm) | $V_{\text{res}}$ (V) | $D_{2}^{\nu}$ (Hz/Unit$T_R$) | $\Delta_{\text{TR}}$ |
|---------------|------------|------------|----------------------|-----------------------------|---------------------|
| 1 | Real | 0.386 | 1.359 | -0.814 | 65.117(9) | -3.7(2) x 10^{-4} |
| 2 | Cylindrical | 0.386 | 1.359 | -0.823 | 29 259.9(7) | -2.85(4) x 10^{-3} |
| 3 | Real | 0.443 | 1.33 | -0.806 | -26 704.5(6) | 2.54(4) x 10^{-3} |
| 4 | Cylindrical | 0.443 | 1.33 | -0.814 | -316.96(1) | 6(2) x 10^{-5} |

certain areas, thus resulting in a non-uniform variation of the work function due to the presence of different materials on the surfaces [36]. By the low operating voltage of the analysis trap, such non-equipotential surfaces could make controlled particle trapping and manipulation extremely difficult.

5. Experimental trap tune

Although the trap is designed to be orthogonal and to allow the compensation of the electrostatic potential up to the sixth order, imperfections may arise in the trap assembled in reality. The voltages applied to the trap electrodes are subject to uncontrolled offsets, which arise from contact potentials and voltage drops due to leakage currents. Such kinds of offsets can reach a few hundreds of millivolts, thus affecting the electrostatic properties of our small Penning trap.

A method to tune the trap using the stored proton was developed that allows for the compensation of asymmetries in the trapping potential due to unknown offsets and even to minor assembly imperfections. It consists in deliberate offset voltages to the trap electrodes as indicated in figure 13. Due to the mirror symmetry with respect to the $z = 0$ plane, only three parameters are available to manipulate the electrostatic properties of the trap: the offset on the ring $O_{\text{fA3}}$, the offset on the endcaps ($O_{\text{fA1}}$, $O_{\text{fA5}}$) and the offset on the correction electrodes ($O_{\text{fA2}}$, $O_{\text{fA4}}$).

For a given offset on the endcaps $O_{\text{fA1,A5}}$, the offset on the correction electrodes $O_{\text{fA2,A4}}$ that better compensates for the asymmetry of the potential is determined. The quality of the asymmetry compensation for a given configuration is evaluated by comparing the variation of the axial frequency $\Delta_{\nu}^{A} = \frac{\Delta\nu}{\Delta V_{AI}}$ when the voltage applied to each correction electrode ($V_{AI}$, $i = 2, 4$, in figure 13) is varied. The asymmetry compensation is partially optimized for a given set ($O_{\text{fA1,A5}}$, $O_{\text{fA2,A4}}$) when the frequency variation is the same for both correction electrodes, $\Delta_{\nu}^{A2} = \Delta_{\nu}^{A4}$. The sought-after configuration corresponds to the pair of offsets for which $\Delta_{\nu}^{A2} = \Delta_{\nu}^{A4} = 0$. The offset on the ring $O_{\text{fA3}}$ can be varied to optimize the orthogonality by minimizing the absolute value of $D_{\zeta}^{\nu}$.

To carry out the optimization process, only the proton signal in the thermal noise spectrum of the axial detector is needed (see section 3). To begin with, the orthogonality of the trap and the symmetry of the potential are investigated for the case when no offsets are applied to the electrodes, so $O_{\text{fA3}} = 0$ V, $O_{\text{fA1,A5}} = 0$ V and $O_{\text{fA2,A4}} = 0$ V with $V_0 = -0.93$ V (see table 2). The axial frequency is measured for different values of the tuning ratio, as illustrated in figure 14. Then, $D_{\zeta}^{\nu}$ is determined, which is the slope of $\nu_z$ versus $T_R$ in figure 14(b).
Asymmetries in the trapping potential due to offset voltages on the electrodes can be compensated systematically by properly adjusting the values of $O_{\text{IfA1,A5}}$, $O_{\text{IfA2,A4}}$, and $O_{\text{IfA3}}$.

The value obtained from the fit, $D_z^{v_c} = -6.43(10) \times 10^4$ Hz/Unit $T_R$, clearly differs from the calculated $D_z^{v_c} = 6.5117(9) \times 10^4$ Hz/Unit $T_R$. According to the results presented in figure 12, this discrepancy in $D_z^{v_c}$ would correspond to a deviation of $\approx 110 \mu m$ (29%) in the length of the ring electrode $l_0$, which is completely outside the manufacturing tolerance range of 3 $\mu m$. This indicates the existence of offset potentials or assembly imperfections.

The best tuning ratio for this configuration ($T_R = 0.8002$ in figure 14(a)) is chosen. This is determined by the narrowest dip with the largest depth, which indicates the most harmonic potential ($C_4 = 0$) [19, 40]. The symmetry of the trapping potential is analyzed by varying $C_4$ leads to a dependence between the axial frequency and the energy, thus broadening the line shape of the axial signal. For the trap tune, the observation of the dip line shape suffices for determining the ideal $T_R$.
In practice, correction electrodes are applied. In figure 15(a), this configuration is compared to the case when no offsets are applied. In figure 15(b), this configuration is compared to the case when no offsets are applied. In practice, the axial frequency variation to that obtained by repeating the same procedure with the other correction electrodes in addition to $V_{\text{off}}$, is best achievable using only the correction electrodes. The expected response of the axial frequency in a trap free of offsets is that $\nu_{A} = \nu_{0}$. With $O_{\text{fA2,A4}} = 35 \text{ mV}$, the asymmetry is partially compensated as represented by the circles with $(\Delta_{\nu}^{A2} = 338.1 (7.3) \times 10^{-4} \text{ Hz} \mu \text{V}^{-1}$ and $\Delta_{\nu}^{A4} = 326.7 (5.2) \times 10^{-4} \text{ Hz} \mu \text{V}^{-1}$). The total frequency shift is up to 1000 times bigger than in the ideal case. The axial frequency scales linearly with $\nu = A_{1} V_{A}$. This asymmetric behavior can be compensated by applying an extra voltage $V_{\text{off}}$ to the correction electrodes in addition to $V_{A}$. This offset is equally distributed between the correction electrodes, since the effects of $O_{\text{fA2,A4}} = V_{\text{off}}$ and $O_{\text{fA4}} = -V_{\text{off}}/m$, $O_{\text{fA4}} = V_{\text{off}}/m$ (with $\frac{1}{2} + \frac{1}{m} = 1$) on the potential concerning $\nu_{A}^{i}$ and $D_{v}^{c}$ cannot be distinguished experimentally.

We search for the offset $V_{\text{off}}$ that makes the two frequency slopes $\Delta_{\nu}^{A2}$ and $\Delta_{\nu}^{A4}$ equal for $O_{\text{fA1,A5}} = 0$. This offset also minimizes $\Delta_{\nu}^{A1}$ at least partially. In figure 16(a) the results for different values of $V_{\text{off}}$ are summarized. As one can see, a $V_{\text{off}} = 70 \text{ mV}$ ($O_{\text{fA2}} = -35 \text{ mV}$ and $O_{\text{fA4}} = 35 \text{ mV}$) makes $\Delta_{\nu}^{A2} = 338.1 (7.3) \times 10^{-4} \text{ Hz} \mu \text{V}^{-1}$ and $\Delta_{\nu}^{A4} = 326.7 (5.2) \times 10^{-4} \text{ Hz} \mu \text{V}^{-1}$, which agree within error bars. Although this still represents a corresponding frequency shift higher than expected, it is best achievable using only the correction electrodes. In figure 15(b), this configuration is compared to the case when no offsets are applied. In practice, $\Delta_{\nu}^{A1}$ must vanish, so that the ideal behavior of figure 15(a) is obtained.
Figure 16. (a) Partial asymmetry compensation of the trapping potential by applying offset voltages to the correction electrodes with $O_{\text{ffA1,A5}} = 0$. The offset configuration that equalizes the frequency variations $\Delta_{\text{Ai}}^v$ is sought. This corresponds to the point at which the curves intersect, $V_{\text{off}} = 70$ mV ($O_{\text{ffA2}} = -35$ mV, $O_{\text{ffA4}} = 35$ mV). Since $\Delta_{\text{Ai}}^v$ at the intersection point does not vanish, further adjustment is necessary. (b) The orthogonality, quantified by $D_{\nu z}^2$, is measured for each offset configuration. It is improved with the asymmetry compensation.

This result indicates that the offsets on the correction electrodes are not the only ones that should be compensated for. The $D_{\nu z}^2$ for each offset configuration is also measured as shown in figure 16(b). The orthogonality is improved with the asymmetry compensation.

The absolute value of $D_{\nu z}^2$ can be minimized partially by applying the adequate offset on the ring electrode $O_{\text{ffA3}}$. In figure 17, $D_{\nu z}^2$ is plotted as a function of $O_{\text{ffA3}}$ for fixed arbitrary $O_{\text{ffA1,A5}}$ and $O_{\text{ffA2,A4}}$. The ideal offset for the ring is independent of the offsets applied to the correction electrodes and endcaps, so that this procedure does not have to be repeated for each new partial asymmetry compensation.

To complete the experimental trap tuning procedure, offsets are now applied to the endcap $A_1$, $O_{\text{ffA1}}$. Due to experimental constraints, no offsets could be applied to the endcap $A_5$. The values for $O_{\text{ffA2}}$ and $O_{\text{ffA4}}$, as well as the tuning ratio and the ring voltage $V_0$, have to be re-adjusted for each different offset applied to $A_1$. In figure 18 a summary of the experimental results is presented. In figure 18(a), the asymmetry, quantified by $\Delta_{\nu A2}^4$ and $\Delta_{\nu A4}^4$, is shown as a function of the offset applied to $A_1$. Each point corresponds to a different trapping potential, represented by a set ($O_{\text{ffA1}}, O_{\text{ffA2}}, O_{\text{ffA4}}$), which has a partially compensated asymmetry. Two points can be found, $O_{\text{ffA1}} = -448.56$ mV and $O_{\text{ffA1}} = 750$ mV, at which the asymmetry is compensated to the absolute minimum. According to figure 18(b), at these points $D_{\nu z}^2$ is also consistent with zero, so that both optimization conditions are satisfied. In table 2 the offsets and relevant operating parameters for both candidates and for the starting configuration are explicitly given. The negative offset on $A_1$ is preferred, because of its more convenient operating voltage.

In figure 18(b) the experimental and simulated data are compared. Based on the extensive data set, the experimental data can be reproduced (dashed curve in figure 18(b)). The numerical method based on the FEM calculation of the potential allows the simulation of any geometric deviations, misalignments or assembly imperfections. The slope of the experimental data curve
Figure 17. Partial orthogonality optimization. The offset voltage on the ring that minimizes the absolute value of $D_{2z}^\nu$ for fixed offsets on the endcaps and correction electrodes is sought. Based on the data, an offset $O_{\text{IfA3}} = -90 \text{ mV}$ was chosen, which roughly corresponds to the minimal absolute value of $D_{2z}^\nu$. This value of $O_{\text{IfA3}}$ is supposed to fulfil the requirement $D_{2z}^\nu = 0$ at the end of the procedure with the total asymmetry compensation of the potential.

Table 2. Operating parameters at the starting point (row 1) and at the points at which both optimization criteria are met (rows 2 and 3).

| $O_{\text{IfA1}}$ (mV) | $O_{\text{IfA3}}$ (mV) | $O_{\text{IfA2},A4}$ (mV) | $V_0$ (V) | $T_R$ | $D_{2z}^\nu$ (Hz/Unit$_{TR}$) |
|-----------------------|-----------------------|-----------------------------|-----------|-------|-----------------------------|
| 1                     | 0                     | 0                           | 0         | -0.93 | 0.8002                      |
| 2                     | -448.56               | -90                         | ±21.977   | -1.17 | 0.8961                      |
| 3                     | 750                   | -90                         | ±53.517   | -0.55 | 0.7631                      |

in figure 18(b) can be explained simply by the presence of offset potentials. The vertical shift of the curve, however, cannot be exactly reproduced by adjusting the voltages on the electrodes or considering a manufacturing tolerance up to $\pm 10 \mu m$. The results of the numerical simulations indicate an assembly imperfection of a larger gap of about 20–40 $\mu m$ between the ring and the correction electrodes due to a sapphire ring which was not seated properly. Recently, the sapphire rings between the ring and correction electrodes were replaced by a new pair, whose dimensions are identical to $\pm 2 \mu m$. Preliminary results show a reduction of the vertical shift, with a $D_{2z}^\nu = -120(533) \text{ Hz/Unit}_{TR}$ for $O_{\text{IfA1}} = 0$, $O_{\text{IfA2}} = -50 \text{ mV}$, $O_{\text{IfA3}} = -90 \text{ mV}$ and $O_{\text{IfA4}} = 50 \text{ mV}$.

5.1. Magnetic bottle measurement

The magnetic bottle strength can be experimentally determined by measuring the magnetic field at different positions in the trap. To this end, the proton is moved along the $z$-axis by applying a specific asymmetric offset configuration to the trap electrodes. At each different position, which can be calculated numerically, the modified cyclotron frequency of the proton is measured. Using the invariance theorem [22] (see section 2) the free cyclotron frequency is determined. The magnetic field is thus calculated using the relation $B = \frac{m_e}{e} \omega_c$ and the coefficient $B_2$ is obtained from a polynomial fit to the data.
Figure 18. Summary of the experimental results. (a) Demonstration of the total asymmetry compensation of the trapping potential. For each set of offsets ($O_{\text{IA1}}$, $O_{\text{IA2,AA}}$) along the curves, the asymmetry is partially compensated ($\Delta_1^{A2} \approx \Delta_1^{A4}$). Two points can be identified, at which the potential is symmetric ($\Delta_1^{A_i} \approx 0$). (b) The $D_{2z}^{\nu}$ measurement for each set of compensating offsets confirms that these two points in fact correspond to a fully optimized Penning trap. Based on the data set, the result could be reproduced by a numerical simulation (dashed curve). The crossed triangle indicates the value calculated for the trap without offsets.

To determine the modified cyclotron frequency $\omega_+$ at a given position, the evolution of the axial frequency is observed when the cyclotron motion is excited by an external rf signal (a burst of 1000 cycles at $\approx 18$ MHz). The excitation frequency is varied in steps of 3 kHz, while the axial frequency is continuously measured and recorded 60 s after each excitation pulse (see figure 19(a)). At resonance, the proton absorbs energy, which, in the presence of the magnetic bottle, leads to a shift of the axial frequency (see equation (2.11)). In figure 19(a), this is indicated by an abrupt jump in the axial frequency between two measurements. To determine $\omega_+$, the absolute value of the difference between two adjacent axial frequencies, $|\nu_z^i - \nu_z^{i-1}|$, is plotted against the excitation frequency. These data are fitted with a Gaussian, whose maximum gives the modified cyclotron frequency, whereas the standard deviation gives the measurement error, as shown in figure 19(b). Although this evaluation does not make use of the Boltzmann convoluted line shape of the cyclotron frequency, which actually describes the cyclotron resonance [27, 41, 42] (see also section 6.3), it is accurate enough to determine the magnetic bottle. To reduce the linewidth of the cyclotron resonance, the particle temperature is lowered electronically during the measurement by applying active feedback to the axial detection system [43, 44].

Using the same method described above, the magnetron frequency $\omega_-$ was determined at only one position in the trap. The change in $\omega_-$ due to the magnetic field variation can be neglected in comparison to that of $\omega_+$ (see equations (2.3) and (2.4)). The value obtained for the magnetron frequency, $\nu_- = 12.90(33)$ kHz, is thus used to calculate the free cyclotron frequency at different positions along the magnetic field axis, which are shown in figure 20. A magnetic bottle term $B_2 = 297(10)$ mT mm$^{-2}$ is obtained by fitting a fourth-order polynomial to the data, which perfectly agrees with the expected value, $B_2 = 300.3807(63)$ mT mm$^{-2}$, numerically calculated as described in section 4.1.
Figure 19. Determination of the modified cyclotron frequency. (a) In the presence of a magnetic bottle, the axial frequency depends on the radial energy. In resonance with the excitation frequency, the proton absorbs energy in the cyclotron mode, which can be detected as an abrupt shift in the axial frequency. (b) The modified cyclotron frequency is determined from the Gaussian fit to the frequency difference between two consecutive measurements as a function of the excitation frequency.

Figure 20. Magnetic bottle measurement. The magnetic field is determined at different positions along the $z$-axis of the trap. A polynomial fit to the data gives the magnetic bottle term $B_z = 297(10) \text{ mT mm}^{-2}$.

6. Determination of the $g$-factor of the proton

Achieving the required axial frequency stability in the magnetic bottle in order to detect a single proton spin-flip is a challenging task. To illustrate this, the drift of the axial frequency, monitored during a period of 11 h before and after diverse optimization measures, is compared in figure 21(a). The difference between two consecutively measured frequencies, $\alpha = \nu_z(t) - \nu_z(t - \Delta t)$, is plotted as a histogram in figure 21(b). The standard deviation of $\alpha$, defined as $\Sigma = \sqrt{\frac{1}{N-1} \sum_{i=1}^{N}(\alpha_i - \bar{\alpha})^2}$, quantifies the frequency stability (see also [17]).
Figure 21. Improvement of the axial frequency stability. (a) Comparison between the frequency drift measured before and after the systematic reduction of external noise on the trap electrodes and the optimization of the trapping potential. The total drift within the same time interval was reduced from 60 Hz to 1.4 Hz. (b) Comparison of the differences $\alpha = v_z(t) - v_z(t - \Delta t)$ before and after optimization. The standard deviation of $\alpha$ quantifies the frequency stability. For details, see text.

The systematic reduction of external noise on the trap electrodes by shielding and filtering all connecting channels to the external environment, together with the trapping potential tune presented in the previous sections, led to a drastic improvement of the frequency stability $\Xi$. The initial value of $\Xi = 1.55$ Hz was reduced to the current best value of $\Xi = 113$ mHz (see figure 21(b)), measured with the optimal fast Fourier transformation (FFT) averaging time of 90 s.

Although the stability is still not good enough to allow the detection of a discrete spin flip, we recently demonstrated the detection of spin flips with a statistical method [17]. The implementation of active feedback to reduce the particle temperature [43, 44] allowed the measurement of a narrower Larmor frequency resonance than in [17]. The modified cyclotron frequency was also measured in the magnetic bottle, so that a preliminary value for the $g$-factor of the proton could be determined.

6.1. Particle temperature measurement

Exact knowledge of the axial temperature of the proton is fundamental for the analysis of the Larmor frequency measurement, presented in the next section. In order to achieve the best possible relative uncertainty for the $g$-factor determined in the presence of the magnetic bottle, the axial temperature was lowered to the minimum value attainable with our detection system when negatively phased feedback is applied.

For the determination of the temperature of the axial mode of the proton, we make use of the concept of sideband coupling [20, 25, 45, 46]. Two eigenmodes of a particle in a Penning trap can be coupled by means of an rf signal with frequency $\omega_{12} = \omega_1 \pm \omega_2$, where $\omega_1$ and $\omega_2$ are the respective oscillation frequencies in each mode. The consequence is that energy is exchanged between the modes. The equilibrium state is defined by the equality of the quantum...
Figure 22. Determination of the axial temperature. The coupling of the axial and magnetron modes leads to an energy exchange between them. (a) In the presence of the magnetic bottle, the axial frequency shift $\Delta \nu_z$ after each coupling pulse can be expressed in terms of the magnetron energy, whose distribution, in turn, reflects the thermal Boltzmann distribution of the axial mode shown in (b).

numbers of the modes, such that

$$n_1 = n_2 \Rightarrow \frac{E_1}{h \omega_1} = \frac{E_2}{h \omega_2} \Rightarrow \langle E_1 \rangle = \frac{E_2 \omega_1}{\omega_2} \Rightarrow T_1 = T_2 \frac{\omega_1}{\omega_2}. \quad (6.1)$$

By applying an excitation with frequency $\nu_+ + \nu_-$, the magnetron mode is coupled to the axial mode, which is in contact with the thermal bath of the axial detection system, whose effective temperature we want to determine. The energy exchange between the modes causes the measured axial frequency to change in the magnetic bottle after the application of sideband coupling pulses, as shown in figure 22(a). The frequency shift $\Delta \nu_z^i = \nu_z^i - \nu_z^{i=0}$, defined as the difference between the frequency measured after each coupling $\nu_z^i$ and the lowest frequency value $\nu_z^{(0)}$, is related to the magnetron energy by $\Delta E_\pm = 4\pi^2 m \nu_z^{(0)} \frac{\hbar^2}{2\mu} \Delta \nu_z$, according to equation (2.11). By displaying the data in a histogram, the thermal distribution of the axial mode is obtained, as shown in figure 22(b). This result can be fitted by a Boltzmann distribution of the form

$$f_B(E) = \frac{1}{k_B T} e^{-E/k_B T},$$

from which the average energy can be extracted as

$$\langle E \rangle = \frac{1}{k_B T} \int_0^\infty E e^{-E/k_B T} dE = k_B T.$$

In practice, the average energy is calculated from the slope of the natural logarithm of the histogram as a function of $\Delta E_\pm$. Either way, $\langle E \rangle = k_B T$, where $k_B T$ is the fit parameter. Finally, the axial temperature can be calculated using the relations given in equation (6.1):

$$\langle E \rangle = \langle E_\pm \rangle \frac{\nu_z}{\nu_+} = k_B T_z,$$

where $k_B$ is the Boltzmann constant.

Using this method, the temperature of the proton axial mode was determined. With negative feedback, the axial temperature is $T_z = 2.22(0.19)$ K, which is measured at a physical temperature of the detection electronics of $T = 9.5$ K, achieved using the pulse-tube cooler.

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6.2. Determination of the Larmor frequency

The Larmor frequency is determined from a spin-flip resonance, which is measured in the magnetic bottle with the method described in [17]. Spin flips are induced by applying an rf magnetic field, whose frequency is varied in discrete steps over a range around \( \omega_L \). The corresponding spin-flip probabilities are determined by comparing the frequency fluctuations \( \Xi_{\alpha} \) and \( \Xi_{\chi} \), which are statistically increased due to spin flips, with the background frequency fluctuation \( \Xi_{\text{back}} \). The difference in comparison with the measurement in [17] is that negatively phased feedback is applied to the axial detection system to reduce the particle temperature from 9.5 to 2.2 K, thus improving the precision of the measurement.

The spin-flip probability is evaluated for 15 different excitation frequencies around \( \omega_L \), a reference frequency far from the resonance, and also for the case when no excitation is applied. The last two serve as a check to ensure that the observed effects are in fact due to spin flips. The procedure is carried out in a sequence of 16 consecutive excitations. The sequence starts with a measurement of the axial frequency (\( v_z^i=0 \)). After 90 s of FFT averaging time, the axial frequency (\( v_z^i=1 \)) is measured again. Then, feedback is turned on and an excitation with the reference frequency \( v_{\text{ref}} \) is applied. Feedback is turned off and after 90 s the axial frequency (\( v_z^i=2 \)) is recorded. The excitation with the Larmor frequency \( v_{\text{L}} \) is applied, the axial frequency (\( v_z^i=3 \)) is measured and so on until the whole frequency range has been scanned. The order in which the frequencies are irradiated is chosen to be random. This sequence is repeated several hundreds of times.

From each sequence, the frequency differences \( \alpha_{\text{back}} = v_z^i=1 - v_z^i=0 \), \( \alpha_{\text{ref}} = v_z^i=2 - v_z^i=1 \), \( \alpha_{\chi}^{i=1} = v_z^i=3 - v_z^i=2 \), \ldots, \( \alpha_{\chi}^{i=15} = v_z^i=17 - v_z^i=16 \) are calculated. From the whole group of sequences, the standard deviations of the differences \( \alpha_{\text{back}} \), \( \alpha_{\text{ref}} \) and the \( \alpha_{\chi} \) (\( \Xi_{\text{back}}, \Xi_{\text{ref}} \) and \( \Xi_{\chi} \), respectively) are calculated. The sequence is repeated until the fluctuations \( \Xi_{\text{back}}, \Xi_{\text{ref}} \) and \( \Xi_{\chi} \) converge to nearly constant values. As explained in [17], comparing \( \Xi_{\text{back}} \) and \( \Xi_{\chi} \), the respective spin-flip probability \( P_{\text{SF}} \) can be determined, given that for \( M \) spin flips in \( N \) trials, the frequency fluctuation is

\[
\Xi_{\chi} = \sqrt{\frac{\sum_{i=1}^{M} (\alpha_i + \delta v_z^{\text{SF}} - \bar{\alpha})^2}{N-1} + \sum_{i=M}^{N} (\alpha_i - \bar{\alpha})^2} \approx \sqrt{\Xi_{\text{back}}^2 + P_{\text{SF}} (\delta v_z^{\text{SF}})^2}, \quad (6.2)
\]

where \( \delta v_z^{\text{SF}} = 188 \) mHz is the frequency jump due to a spin flip (see equations (2.10) and (6.6)). The resolution limit is reached when the error \( [\Xi_{\text{back}}/\sqrt{2(N-2)}] + [\Xi_{\chi}/\sqrt{2(N-2)}] \) is of the same size as the increase in the frequency fluctuation \( \Delta \Xi = \Xi_{\chi} - \Xi_{\text{back}} \). The probability \( P_{\text{SF}} \) is given by [41]

\[
P_{\text{SF}} = \frac{1}{2} \left( 1 - \exp\left( -\frac{1}{2} \Omega_{\text{R}}^2 t_0 \chi (2\pi v_{\text{rf}}, B_2, T_z) \right) \right). \quad (6.3)
\]

\( \Omega_{\text{R}} = 2\pi v_{\text{rf}} \frac{b_{\text{rf}}}{R_0} \) is the Rabi frequency with which the proton spin precesses around the axis of the magnetic excitation field \( \vec{b}_{\text{rf}} \). This transverse rf magnetic field with frequency \( v_{\text{rf}} \) is generated by the spin-flip coil shown in figures 3 and 9. In equation (6.3), \( T_z \) and \( t_0 \) are the axial temperature and irradiation time, respectively, while \( \chi (2\pi v_{\text{rf}}, B_2, T_z) \) is the transition line shape, which can
Figure 23. The Larmor frequency resonance. The solid line represents the fit of the resonance line shape to the data, from which the Larmor frequency $\nu_L = 50.064\,971(91)$ MHz could be extracted. The linewidth corresponds to an axial temperature of $T_z = 2.5(1.2)$ K.

be written in a simplified form (for details see [41])

$$\chi (\omega_\ell) = \frac{1}{\pi \delta \omega_L} \left[ \arctan \left( \frac{\gamma_z \left( \omega_\ell - \left( \omega_0 + \frac{\nu_T}{2\text{Re}\gamma} \delta \omega_L \right) \right)}{2\text{Re}\gamma} \right) + \frac{\pi}{2} \right]$$

$$\times \exp \left( - \frac{\omega_\ell - \left( \omega_0 + \frac{\nu_T}{2\text{Re}\gamma} \delta \omega_L \right)}{\delta \omega_L} \right),$$

with $\gamma = \sqrt{\gamma_z^2 + 4\gamma_z \delta \omega_L}$, where $\delta \omega_L = \omega_0 \frac{B_0}{\gamma_L} z^2 = \omega_0 \frac{B_0}{\gamma_L} \frac{k_B T_z}{m_p \omega_Z^2}$ is a correction parameter that defines the linewidth. This arises from the axial oscillation of the particle in the magnetic bottle. The oscillatory motion is coupled to the axial detector with the coupling constant $\gamma_z = \frac{z^2}{2} \frac{B_p}{\gamma_D}$, where $R_p$ is the effective parallel resistance at resonance (see figure 3) and $D$ is a characteristic length related to the trap geometry and the detection circuit design [31]. The resonance frequency is given by $\omega_0 = \omega_L$.

In figure 23, the probabilities $P_{SF}'$ are plotted against different excitation frequencies. The solid line is the result of a fit of equation (6.3) to the experimental data.

The parameters $t_0 = 1$ s and $B_2 = 297$ mT mm$^{-2}$ are fixed, while $\nu_0$, $b_\ell$ and $T_z$ are free fit parameters. The values for the axial temperature obtained from the fit, $T_z = 2.5(1.2)$ K, and that measured as described in section 6.1, $T_z = 2.2(0.2)$ K, agree well. A Larmor frequency of $\nu_L = 50.064\,971(91)$ MHz is obtained from the fit, which corresponds to a relative uncertainty of $1.8 \times 10^{-6}$.

6.3. Determination of the modified cyclotron frequency

Immediately after completing the measurement of the Larmor resonance, the modified cyclotron frequency was determined. The measuring method was modified in comparison with that
used for the determination of the magnetic bottle described in section 5.1, which allowed the achievement of a relative uncertainty of only $1.6 \times 10^{-4}$.

The measurement principle is based on the sideband coupling method, briefly explained in section 6.1. The cyclotron motion is coupled to the axial motion by means of an rf signal of frequency $\nu_{\text{coup}} = \nu_+ - \nu_z$. In the analysis trap, no cyclotron detection system is attached to the trap electrodes. The proton is previously cooled in the precision trap by means of the coupling to the cyclotron resonator with negative feedback and then transported to the analysis trap. A cyclotron temperature below 100 mK can be achieved [31], which corresponds to a cyclotron quantum number $n_z \approx 730$. The axial mode is kept weakly coupled to the detector at a temperature of $T_z = 2.2$ K, which corresponds to $n_z \approx 4 \times 10^5$. Thus, according to equation (6.1) the coupling of the modes will result in a heating of the cyclotron motion.

When the coupling field is applied, energy is exchanged between the axial and cyclotron modes. The resulting energy fluctuation in the cyclotron mode gives rise to axial frequency shifts, according to equation (2.9). Reducing the amplitude of the rf signal to the limit below which no measurable effect can be observed, the axial frequency shifts turn into a small fluctuation. The magnitude of this fluctuation depends on the coupling frequency, reaching its maximum value when the coupling is resonant, $\nu_{\text{coup}} - (\nu_+ - \nu_z) = 0$.

The measurement method is very similar to those used for determining the Larmor frequency. A significant difference is that the excitation frequency is not varied in discrete steps but continuously swept within a given interval. The whole frequency range to be scanned is divided into eight parts, each of which is represented by a center frequency $\nu_i$ and a sweep interval. Within a sequence, each frequency $\nu_i$ is irradiated a few dozen times. After each excitation pulse, the axial frequency is recorded twice: 90 s after the excitation pulse and 90 s after shorting the excitation line to the ground, which serves as a reference measurement. The eight frequencies of a sequence are irradiated in random order. The standard deviations $\xi_i^r$ and $\xi_i^c$ are calculated for each frequency interval centered at $\nu_i$. The sequence is repeated until the fluctuations $\xi^r$ and $\xi^c$ converge to nearly constant values.

In figure 24, the difference $\xi_i^c - \xi_i^r$ is plotted against $\nu_i$. The horizontal bars represent the swept intervals.

To interpret the experimental data, it is necessary to relate the measured quantity $\Delta \xi_i^c = \xi_i^c - \xi_i^r$ to the cyclotron energy. Given that not only the spin but also the cyclotron and magnetron motions are associated with a magnetic moment, transitions between the energy levels in the radial modes will lead to axial frequency shifts as well. Thus, the effect in equation (2.9) can be generalized as [25]

$$\nu_z = \nu_{z0} + \delta \nu_z(n_+, n_-, n_s),$$

with

$$\delta \nu_z(n_+, n_-, n_s) = \frac{h\nu_+}{2\pi m_p \nu_z} B_0 \left( n_+ + \frac{1}{2} + \frac{\nu_-}{\nu_+} \left( n_- + \frac{1}{2} \right) + \frac{gn_s}{2} \right),$$

where $n_+, n_-$ and $n_s$ are the cyclotron, magnetron and spin quantum numbers, respectively. So, a change of $\Delta n_+ = \pm 1$ in the cyclotron quantum number shifts the axial frequency by $\delta \nu_z = \pm 68$ mHz. The corresponding energy shift $\Delta E_z = \pm 74$ neV can be calculated using equation (2.11). The difference $\Delta \xi_i^c = \xi_i^c - \xi_i^r$ plotted in figure 24 can be expressed...
Figure 24. Determination of the modified cyclotron frequency. The axial and modified cyclotron modes are coupled. The coupling frequency is swept within the intervals represented by the horizontal bars centered at \( \nu_{\text{coup}} \). The resonant coupling frequency is extracted from the fit (solid line). The corresponding cyclotron frequency \( \nu_c = 17.926 \pm 13(16) \text{ MHz} \) is obtained.

in terms of the cyclotron energy

\[
\Delta \Xi (\nu_{\text{coup}}) = \frac{1}{4\pi^2 m_p \nu_c} \frac{B_2}{B_0} \Delta E_+ (\nu_{\text{coup}}),
\]

(6.7)

On the other hand, the response to a cyclotron excitation can be expressed as the convolution of the line shape of a damped cyclotron resonance with the transition line shape \( \chi(\omega) \) of equation (6.4), as demonstrated in [25]. According to that, the excitation energy of the cyclotron motion can be written as \( E_+ = \frac{q^2 e^2}{m_p \gamma_c} \pi \chi(\omega) \), where \( \gamma_c \) is the damping constant and \( e \) the amplitude of the excitation field. Assuming that \( \gamma_c = |\gamma_{\text{coup}}| \), where \( \gamma_{\text{coup}} = \frac{q^2 e^2}{2m_p \nu_{\text{coup}}^2 \omega_{\text{bott}}} \) represents the transfer rate for the sideband coupling [45], a line shape for \( \Delta \Xi (\nu_{\text{coup}}) \) to fit the experimental data can be derived. The solid line in figure 24 represents the best fit to the data points, from which the resonant \( \nu_{\text{coup}} \) can be extracted. The offset of the curve arises from the background noise of the frequency generator, when this is turned off but still connected to the excitation line.

From \( \nu_{\text{coup}} \) the modified cyclotron frequency and thus the free-cyclotron frequency can be calculated, yielding \( \nu_c = 17.926 \pm 13(16) \text{ MHz} \).

The \( g \)-factor is determined from \( \nu_c \) and \( \nu_L \) to be \( g = 5.585696(50) \), in agreement with the value obtained in [10] and with the most recent value obtained in [46] by using the technique developed by us and described in this paper.

7. Conclusions

In this paper, we presented the Penning trap, which is the core of the experiment for the direct determination of the magnetic moment of a single proton. A complete overview including the design, characterization and an optimization method using the stored proton was given. The detection of spin flips of a single proton was demonstrated for the very first time in a magnetic bottle of \( B_2 = 300 \text{ mT mm}^{-2} \) [17]. Further improvements allowed the \( g \)-factor to be determined in this work with a relative uncertainty of \( 8.9 \times 10^{-6} \). This represents a milestone in the direct
determination of the proton magnetic moment with extremely high precision. This breakthrough brings us closer to the possibility of testing matter–antimatter symmetry at a high precision level in the baryon sector, since the measuring method presented in this paper is also suitable for the determination of the $g$-factor of the antiproton.

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