**The πN Sigma term – a lattice investigation**

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**Abstract**

A lattice calculation of the πN sigma term is described using dynamical staggered fermions. Preliminary results give a sea term comparable in magnitude to the valence term.

**Introduction and Theoretical Discussion**

Hadrons appear to be far more complicated than the (rather successful) constituent quark model would suggest. For example a recent result from the EMC experiment, \(^1\), suggests that constituent quarks are responsible for very little if any of the nucleon spin. Another, older result, is from the πN sigma term, see e.g. \(^2\), which seems to give a large strange component to the nucleon mass. To explain theoretically these results and other experiments involves the computation of certain matrix elements – a non-perturbative calculation. This is thus an area where lattice calculations may be of some help.

The πN sigma term, \(\sigma_{\pi N}\), is defined\(^3\) as that part of the mass of the nucleon (for definiteness the proton) coming from the vacuum connected expectation value of the up (\(u\)) and down (\(d\)) quark mass terms in the QCD Hamiltonian,

\[
\sigma_{\pi N} = m \langle N | \bar{u} u + \bar{d} d | N \rangle,
\]

where we have taken these quarks to have equal current mass (\(= m\)). Other contributions to the nucleon mass come from the chromo-electric and chromo-magnetic gluon pieces and the sea terms due to the \(s\) quarks. Experimentally this matrix element has been measured from low energy π-N scattering. A delicate extrapolation to the chiral limit \(^4\) gives a result for the isospin even amplitude of \(\Sigma/f_\pi^2\) with \(\Sigma = \sigma_{\pi N}\), from which the πN sigma term may be found. The precise value obtained this way has been under discussion for many years. However within the limits of our lattice calculation, this will not concern us here and for orientation we shall just quote a range of results from later analyses of \(\sigma_{\pi N} \approx 56\text{MeV}, \) \(^5\) down to 45MeV, \(^6\).

To estimate valence and sea contributions to \(\sigma_{\pi N}\), classical current algebra analyses assume octet dominance and make first order perturbation theory about the \(SU_F(3)\) flavour symmetric Hamiltonian. We have

\[
\hat{H}_m = m(\bar{u} u + \bar{d} d) + m_s \bar{s} s \equiv m_{av} \bar{\psi} I \psi - \frac{m_s - m}{\sqrt{3}} \bar{\psi} \lambda^8 \psi.
\]

The first term in the second equation is the flavour symmetric Hamiltonian with mass \(m_{av} = (2m + m_s)/\sqrt{3}\), while the second term is the flavour breaking piece. (\(\psi\) is the column vector \((u, d, s)\).) We thus have from eq. \(\text{(2)}\)

\[
\sigma_{\pi N} \approx m \langle N | \bar{u} u + \bar{d} d - 2\bar{s} s | N \rangle_{\text{symm}} + 2m \langle N | \bar{s} s | N \rangle_{\text{symm}}
\]

\[\overset{\text{def}}{=} \sigma_{\pi N}^{\text{val}} + \sigma_{\pi N}^{\text{sea}},\]

where we have first assumed that the nucleon wavefunction does not change much around the symmetric point. We then subtract and add a strange component. At the symmetric point the \(u\) and \(d\) quarks each have equal valence and sea part, while the \(s\) quark matrix element only has a sea component. Thus in the first term the sea contribution cancels, justifying the definitions given in the last line of eq. \(\text{(3)}\). Also this means that in the following we shall interchangeably talk about ‘strange’ and ‘sea’ contributions to

\(^1\)The MT\(_2\) Collaboration.

\(^2\)Talk presented by R. Horsley at the Workshop on Field-Theoretical Aspects of Particle Physics, Kyffhäuser, Germany, September 13\(^{th}\) to 17\(^{th}\) 1993. To be published in the NTZ Series.

\(^3\)Actually, originally as the matrix element of the double commutator of the Hamiltonian with two axial charges. However this is equivalent to the definition given in eq. \(\text{(3)}\), see for example ref. \(\text{(4)}\).
the nucleon. At first order perturbation theory $\sigma_{\pi N}^{val}$ may easily be calculated to give $\sigma_{\pi N}^{val} \approx 25 \text{MeV}$ and so $\sigma_{\pi N}^{seq} \approx 31 \sim 20 \text{MeV}$. This in turn means that 
\[ m_s(N|\bar{s}s|N) \approx 400 \sim 250 \text{MeV}, \]
which would indicate a sizeable portion of the nucleon mass comes from the strange quark contribution. (Remember that the mass of the proton is about $938 \text{MeV}$.) This would not be expected from the constituent quark model, where the nucleon is made only of (dressed) $u$ and $d$ quarks.

**Staggered Fermions**

As we see from above, we need to calculate certain matrix elements. One of the most promising methods at present to calculate them comes from the lattice, where Euclidean space-time is discretised as part of the sea contribution.) As we shall be interested in the questions of sigma/nucleon and baryon octet mass splittings are given by

\[ M_B - M_{\text{symm}} = -(m_s - m)(B|\bar{\psi}\lambda^8\psi|B)_{\text{symm}}/\sqrt{3}, \]

where $B$ is the $\frac{1}{2}^+$ nucleon multiplet. From the Wigner-Eckart theorem for $SU(3)$ we have

\[ (B|\bar{\psi}\lambda^8\psi|B)_{\text{symm}} = F \text{tr}(B^\dagger \{\lambda^8, B\}) + D \text{tr}(B^\dagger \lambda^8, B)). \]

From the known baryon masses $F, D$ can then be estimated. (We use the numbers given in Table 3 as $(m_s - m)F \approx 190, (m_s - m)D \approx 61$.) Furthermore

\[ \sigma_{\pi N}^{val} = m\sqrt{3}(N|\bar{\psi}\lambda^8\psi|N)_{\text{symm}} = m(3F - D). \]

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\[ \langle 0 | \hat{B} \hat{S}_4^1 \hat{B}_W | 0 \rangle \approx A_{NN} \mu_N^1 + A_{LL} \mu_L^1, \quad \frac{1}{2} T \gg t \gg 0. \] (6)

In the second line we have gone from the (numerically) calculable 2-point correlation function to the equivalent expression in the operator formalism. The transfer matrix is denoted by \( \hat{T} = \hat{S}_4^1 = \exp (-2\hat{H}) \). A shift of one unit still lies in the hypercube, which then involves a flavour transformation. Thus we can write \( \hat{T}_4 = \hat{S}_4^1 \hat{T}_4^1 \) where \( \hat{S}_4^1 \) is a non-local flavour transformation. (No explicit representation of \( \hat{S}_4^1 \) is at present known.) Eigenstates of \( \hat{S}_4^1 \) are defined by

\[ \hat{S}_4^1 | \alpha \rangle = \mu_\alpha | \alpha \rangle \quad \mu_\alpha = \xi_\alpha e^{-M_\alpha}, \] (7)

where \( \langle \alpha | \alpha \rangle = 1 \) and \( \xi_\alpha = \pm 1 \) is the eigenvalue of \( \hat{S}_4^1 \). Inserting complete sets of states in the second line of eq. (6) gives the third line where the amplitude \( A_{\alpha \beta} = \langle 0 | \hat{B} | \alpha \rangle \langle \beta | \hat{B}_W | 0 \rangle \), and \( \mu_N = \exp (-M_N), \) \( \mu_L = -\exp (-M_L) \). (We shall call \( \Lambda \), a little artificially, the parity partner to the nucleon.) The time box length (\( T \), here = 24) has been assumed to be so large that there are no finite time-size affects (although these can be easily taken into account). The parity partner arises because on the lattice inversion \( I_s : x_i \to -x_i \) is not parity (\( P \)) as it effects the flavour degrees of freedom. (One can show that \( \hat{P} = \hat{I}_s \hat{S}_4 \).) As a consequence hadron operators in general do not have a definite parity.

As an example of a correlation function we show in Fig. 1 a baryon correlation function. The typical zigzagging of the results (and fit) is caused by the parity partner. To pick out the ground states we must take \( t \to \infty \) and/or choose an operator with a good overlap with this state. Experience has shown that a non-local operator (the ‘wall’ and hence the subscript ‘\( W \)’) has rather good overlap properties with the ground state for very moderate \( t \), so this has been used as the source; the sink is taken as a local baryon operator. The mass obtained this way agrees with that which comes from using a local baryon operator for both sink and source.

**Measuring \( \sigma_{\pi N} \)**

Practically there are several possibilities open to us for the evaluation of the matrix element. The easiest is simply to differentiate eq. (6). As

\[ \frac{\partial \hat{S}_4^1}{\partial m} = -\bar{\chi} \hat{S}_4 \]

we have

\[ m \frac{\partial M_N}{\partial m} |_\beta = \langle N | m \bar{\chi} \chi | N \rangle = \sigma_{\pi N}, \] (8)

an application of the Feynman-Hellmann theorem. Thus all we need to do is to measure \( M_N \) for different masses \( m \) (at constant \( \beta \)) and numerically estimate the gradient. In Fig. 2 we show such a procedure. (At present we have simply fitted a straight line through all points.) We find

\[ \sigma_{\pi N} \approx 11.9(8) m |_{m=0.01} \approx 0.12(1), \] (9)

In the staggered formalism the operator expression for \( m \bar{\chi} \chi \) has a more complicated appearance than that given here.
which gives
\[
\frac{\sigma_{\pi N}}{M_N} \approx \frac{11.9m}{11.9m + 0.64}|_{m=0.01} \approx 0.15,
\] (10)
which is to be compared with the experimental result of \(\sigma_{\pi N}/M_N \approx 0.06 \sim 0.05\). The numerical result is much larger, but presumably this simply indicates that we have used much too large a quark mass in our simulation (which at present we, and everybody else, cannot avoid doing due to computer limitations).

We now wish to estimate the valence and sea contributions. This is technically more complicated and involves the evaluation of a 3-point function, \[C(t; \tau) = \langle B(t)m\bar{\chi}\chi B(0)\rangle, \] (11)
This may be diagrammatically sketched as the sum of two terms:

Diagrams (a) and (b) are the connected (or valence) piece; the nucleon quark lines are connected with the mass insertion operator. Diagram (b) represents the disconnected (or sea) piece.

For the connected piece we have fixed \(t\) to be 8 or 9 and then evaluated \(C(t; \tau)\) as a function of \(\tau\). The appropriate fit function is
\[
C(t; \tau) \approx \sum_{\alpha, \beta=N, \Lambda} A_{\alpha\beta} \langle \alpha|m\bar{\chi}\chi|\beta\rangle \mu_\alpha^{t-\tau} \mu_\beta \quad \frac{1}{2}T \gg t \gg \tau \gg 0.
\] (12)

The \(A_{\alpha\beta}\) and \(M_\alpha\) are known from 2-point correlation functions. We see that when \(\alpha = \beta = N\) we have the matrix element that we require: \(\langle N|m\bar{\chi}\chi|N\rangle\). However there are other terms which complicate the fit: \(\langle \Lambda|m\bar{\chi}\chi|\Lambda\rangle\) and the cross terms \(\langle \Lambda|m\bar{\chi}\chi|N\rangle \equiv \mu_\Lambda \mu_\Lambda^{-1} \langle N|m\bar{\chi}\chi|\Lambda\rangle\). As can be seen from eq. (12) these cross terms are responsible for oscillations in the result. To disentangle the wanted result from \(\langle \Lambda|m\bar{\chi}\chi|\Lambda\rangle\) we need to make a joint fit to the \(t = 8\) and \(t = 9\) results. At present we have not done this,
but just checked that separate fits give consistent results. In Fig. 3 we show preliminary results for \( t = 8 \).

Up until now we have only evaluated about a quarter of our available configurations, so the final results should have reduced error bars. We find

\[
\sigma^\text{val}_\pi N \approx 0.08(2),
\]

with about the same result for the \( \Lambda \) matrix element. The cross term is smaller, roughly 0.02.

Finally we have attempted to estimate the disconnected term using a stochastic estimator, \[12\]. This is costly in CPU time. One can improve the statistics by summing the 3-point correlation function over \( \tau \); this is then equivalent to a differentiation of the 2-point function with respect to \( m \). This gives

\[
\sum_\tau C(t; \tau) \approx t \sum_{\alpha=N,\Lambda} A_{\alpha\alpha} \langle \alpha \mid m \bar{\chi} \mid \alpha \rangle \mu^\dagger_\alpha \quad \frac{1}{2} T \gg t \gg 0.
\]

In Fig. 4 we show the disconnected part of the 3-point correlation function. As expected the quality of

\[
\sigma^\text{sea}_\pi N \approx 0.05(4).
\]
Discussion

We see that (roughly at least) eq. (9) is consistent with eqs. (13, 15) and that
\[ \frac{\sigma_{\pi N}}{\sigma_{\pi N}^{val}} \approx 1.5, \]
which is to be compared with the experimental value of about 2.2 ~ 1.8. Although we can draw no firm conclusions at present our result tentatively indicates that the valence part of the \( \pi N \) sigma term is slightly larger than the sea term.

Comparing our results with other work obtained using dynamical fermions, Gupta et. al., [7] who use 2 flavours of Wilson fermions, find \( \sigma_{\pi N}/\sigma_{\pi N}^{val} \approx 2 \sim 3 \), which indicates a somewhat larger sea component in the \( \pi N \) sigma term. On the other hand Patel, [13], using results from [14] for 2 staggered flavours finds for the ratio \( 1.5 \sim 2.0 \), while Bernard et. al., [15], have \( \sim 2.0 \). These, like our result, seems to be lower than for the Wilson fermion case.

We would also like to emphasise that although lattices can give a first principle calculation, at present one is not able to do this. Technically the fit formulae that we employ, eqs. (12, 14) are true only for the complete correlation function. However we have used them for either the connected or the disconnected part separately. The best way to circumvent this problem is to make simulations at different strange quark masses; differentiation as in eq. (8) would then give directly \( m_s\langle N|\bar{s}s|N\rangle \), the strange content of the nucleon. However this calculation is not feasible at the present time. This should be the ultimate goal of lattice simulations, as other recent (non-lattice) theoretical results, [6], have hinted that perhaps the strange quark content of the nucleon is not as large as supposed, previous results being explained by a combination of factors, such as \( \Sigma \neq \sigma_{\pi N} \) and higher order corrections to first order perturbation theory. (Indeed there are already tantalising lattice indications that this may be so, [3, 7].)

In conclusion we would just like to say that lattice results at present are generally in qualitative agreement with other theoretical and experimental results. However much improvement in the calculations is required to be able to make quantitative predictions.

Acknowledgements

This work was supported in part by the Deutsche Forschungsgemeinschaft. The numerical computations were performed on the Cray Y-MP in Jülich with time granted by the Scientific Council of the HLRZ. We wish to thank both institutions for their support.

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