SOLUTION OF W-CONSTRAINTS FOR R-SPIN INTERSECTION NUMBERS

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Abstract. We present a solution to the W-constraints satisfied by the intersection numbers on the moduli spaces of r-spin curves. We make use of a grading suggested by the selection rule for the correlators determined by the geometry of the moduli space.

1. Introduction

The famous Witten Conjecture [25] proved by Kontsevich [19] relates intersection numbers on moduli spaces of stable curves to the KdV hierarchy. Witten [26] introduce r-spin curves, their moduli spaces and conjectured that the intersection numbers on them are related to generalized KdV hierarchies (Gelf’and-Dickey hierarchies). This conjecture has been proved by Faber-Shadrin-Zvokine [6]. More recently, such problems have been also been studied from the point of view of Givental quantization formalism [15, 16, 11] and also from the point of view of Fan-Jarvis-Ruan-Witten theory [7].

As pointed out by Witten [25], together with the string equation, the KdV hierarchy completely determines all the intersection numbers of psi-classes on $\overline{M}_{g,n}$; similarly, the generalized KdV hierarchy together with the string equation completely determine the r-spin intersection numbers [26]. Liu, Vakil and Xu [20] have developed an algorithm to compute the r-spin intersection numbers based on such these facts. Some explicit results can be found in loc. cit. for $r = 5$ and similar results for $r = 4$ and 5 can be found in an earlier paper by Liu and Xu [21]. Such results match with results by other methods obtained by e.g. Shadrin [24], Brezin and Hikami [4].

It is well-known that such intersection numbers also satisfy linear constraints called the Virasoro constraints in the $r = 2$ case and the $W$-constraints in the general case. For derivations of the equivalence of such linear constraints with the generalized KdV hierarchy together with the string equation, see Dijkgraaf-Verlinde-Verlinde [2], Fukuma-Kawai-Nakayama [13, 14], Goeree [17], and Kac-Schwarz [18]. For algebraic background on $W$-algebras, we refer to Fateev-Lukyanov [8], Feigin-Frenkel [9, 10] and Frenkel-Kac-Radul-Wang [12].
Bakalov and Milanov \cite{bakalov_milanov_1,bakalov_milanov_2} constructed W-constraints for Frobenius manifolds associated with simple singularities, and they conjectured their constraints uniquely determine the partition function up to a factor.

Recently, Liu, Yang and Zhang \cite{liu_yang_zhang} prove the conjecture of Bakalov and Milanov and extend their construction of to more general Frobenius manifolds. A key ingredient of this proof is a natural grading of the coupling constants. We will use a different grading which is compatible with selection rule for nonvanishing correlator as determined by the geometry of the moduli spaces. Then we obtain a solution of the W-constraints similar to the case of Witten-Kontsevich partition function as in Alexandrov \cite{alexandrov}.

We will focus on the type A case, i.e., the r-spin curve intersection numbers, in this paper. The method also works in the D and E cases, and the details will be presented in a separate paper.

The rest of the paper is arranged as follows. In Section 2 we recall some well-known backgrounds on KP hierarchy and its reductions into generalized KdV hierarchies. In Section 3 we recall the partition of Witten’s r-spin intersection number and the generalized KdV hierarchies satisfied by them. We specify the change of variables to make Witten’s original formulation compatible with the standard notations. The W-constraints for r-spin numbers will be given in Section 4 and their solutions will be presented in Section 5. In Section 6 we present some examples that verify our solution.

\section{KP Hierarchy and Its Reductions}

For reference, see Miwa-Jimbo-Date \cite{miwa_jimbo_date}.

\subsection{The KP hierarchy}

Consider the algebra of pseudodifferential operators of the form

\[ P = \sum_{j \leq n} f_j(x)D^j \]

for some \( n \in \mathbb{Z} \), where \( D = \frac{\partial}{\partial x} \). One can define the multiplications of such operators using:

\begin{equation}
D^n \circ f = \sum_{j \geq 0} \binom{n}{j} \cdot (D^j f) \cdot D^{n+j}.
\end{equation}
Define

\[ P_+ = \sum_{j \geq 0} f_j(x)D^j, \]

\[ P_- = \sum_{j < 0} f_j(x)D^j, \]

\[ P^* = \sum_{j \leq n} f_j(x)(-D)^j, \]

\[ \text{res}(P) = f_{-1}(x). \]

The KP hierarchy is a system of evolution equations on the space of pseudodifferential operators of the form:

\[ Q = D + \sum_{j=1}^{\infty} f_j(T)D^{-j}, \]

given by

\[ \frac{\partial}{\partial T_n}Q = [(Q^n)_+, Q] = [Q, (Q^n)_-]. \]

Here \( T = T_1, T_2, \ldots \) and \( T_1 = x. \)

2.2. The dressing operator and the wave functions. Write \( Q = WDW^{-1}, \) where

\[ W = 1 + \sum_{j=1}^{\infty} w_jD^{-j}. \]

If \( W \) satisfies

\[ \frac{\partial}{\partial T_n}W = -(Q^n)_-W, \]

then the KP hierarchy is satisfied by \( Q. \) The wave function of the KP hierarchy is defined by:

\[ w(T; z) = We^{\xi(T;z)} = (1 + \sum_{j=1}^{\infty} w_j(T)z^{-j}) \cdot e^{\xi(t;z)}, \]

where \( \xi(T; z) = \sum_{j=1}^{\infty} T_jz^j, \) \( t_1 = x. \) Applying \( Q \) to \( w \) one gets:

\[ Qw(T; z) = z \cdot w(T; z). \]

Applying \( \frac{\partial}{\partial T_n} \) to \( w \) one gets:

\[ \frac{\partial}{\partial T_n}w(T; z) = (Q^n)_+w(T; z). \]
Similarly, the adjoint wave function \( w^*(T; z) \) is defined by
\[
(13) \quad w^*(T; z) = (W^{-1})^* e^{-\xi(T;z)}.
\]

### 2.3. Tau-function and the vertex operators.

It turns out that there is a function \( \tau(t) \) such that
\[
(14) \quad w(T; z) = \tau(T_1 - \frac{1}{2} T_2 - \frac{1}{2} \xi(T; z), T_2 + \frac{1}{2} \xi(T; z), \ldots) \tau(T),
\]
\[
(15) \quad w^*(t; z) = \tau(T_1 + \frac{1}{2} t T_2 + \frac{1}{2} \xi(T; z), T_2 - \frac{1}{2} \xi(T; z), \ldots) \tau(T).
\]

In terms of the vertex operators
\[
(16) \quad X(T; z) = e^{\xi(T;z)} e^{-\xi(\tilde{D}, 1/z)}, \quad \tilde{X}(T; z) = e^{-\xi(T;z)} e^{\xi(\tilde{D}, 1/z)},
\]
where \( \tilde{D} = (\frac{\partial}{\partial T_1}, \frac{1}{2} \frac{\partial}{\partial T_2}, \ldots) \), one has
\[
(17) \quad w(T; z) = \frac{X(T; z) \tau(T)}{\tau(T)}, \quad w^*(T; z) = \frac{\tilde{X}(T; z) \tau(T)}{\tau(T)}.
\]

The vertex operators can be rewritten in terms of the following field of operators:
\[
(18) \quad \phi(T; z) = \sum_{n \in \mathbb{Z}} \phi_n(T) z^{-n},
\]
where \( \phi_n \)'s are operators defined by:
\[
(19) \quad \phi_n(T) = \begin{cases} 
-\frac{1}{n} \frac{\partial}{\partial T_n}, & n > 0, \\
T_{-n}, & n < 0, \\
0, & \text{otherwise}.
\end{cases}
\]

For \( n > 0 \), the operators \( \phi_{-n} \)'s are creators, and the operators \( \phi_n \)'s are annihilators. In other words,
\[
(20) \quad \phi(T; z) = \sum_{n=1}^{\infty} z^{-n} T_n \cdot - \sum_{n=1}^{\infty} z^n \frac{1}{n} \frac{\partial}{\partial T_n}.
\]

Then
\[
(21) \quad X(T; z) = : e^{\phi(z)} :, \quad \tilde{X}(T; z) = : e^{-\phi(T;z)} :.
\]

Here \( : : \) means the normal ordering, i.e., the annihilators are always put on the right of the creators.
2.4. **The r-th reduction of KP hierarchy.** For a positive integer \( r \), the \( r \)-th generalized KdV hierarchy (also called the Gelf’and-Dikii hierarchy) is obtained from the KP hierarchy by imposing the condition that the tau-function \( \tau(T) \) does not depend on \( t_k \) when \( k \equiv 0 \pmod{r} \).

By (9), \( Q^r_- = 0 \), hence one can write \( Q^r \) as some operator

\[
L = D^r + u_{r-2}D^{r-2} + \cdots + u_1 D + u_0
\]

for some functions \( u_0, \ldots, u_{r-2} \). One can then rewrite the \( r \)-th reduced KP hierarchy in terms of the operator \( L \) as follows:

\[
\frac{\partial}{\partial t_n} L = [(L^{n/r} +), L] = [L, (L^{n/r})_-].
\]

### 3. Partition Function of Witten’s R-Spin Intersection Numbers

#### 3.1. Witten’s r-spin intersection numbers.** They are defined by Witten [26] as follows:

\[
\langle \tau_{m_1, a_1} \cdots \tau_{m_n, a_n} \rangle_g = \frac{1}{r^g} \int_{\mathcal{M}_{g,n}} \prod_{i=1}^{n} \psi(x_i)^{m_i} \cdot e(V).
\]

These are nonzero only when the following selection rule is satisfied:

\[
(r + 1)(2g - 2) + rn = r \sum_{i=1}^{n} m_i + \sum_{i=1}^{n} a_i,
\]

and

\[
a_i \neq r - 1, \quad i = 1, \ldots, n.
\]

These intersection numbers satisfy the string equation

\[
\langle \tau_{0,0} \prod_{i=1}^{n} \tau_{m_i, a_i} \rangle_g = \sum_{j=1}^{n} \langle \tau_{m_j-1, a_j} \cdot \prod_{1 \leq i \leq n}^{n} \tau_{m_i, a_i} \rangle_g,
\]

and the dilaton equation:

\[
\langle \tau_{1,0} \prod_{i=1}^{n} \tau_{m_i, a_i} \rangle_g = (2g - 2 + n) \cdot \prod_{i=1}^{n} \tau_{m_i, a_i} \rangle_g.
\]

In genus 0, the following was calculated in [26]:

\[
\langle \tau_{0,a_1} \tau_{0,a_2} \tau_{0,a_3} \rangle_0 = \delta_{a_1 + a_2 + a_3 + 2}.
\]
3.2. Generalized KdV hierarchies from $r$-spin curves. Let us recall now the generalized Witten Conjecture for $r$-spin intersection numbers [26, 6]. Introduce formal variables $t_{m,a}$ corresponding to $\tau_{m,k}$ ($m = 0, 1, 2, \ldots$, $a = 0, 1, \ldots, r - 2$), consider

$$ F_g(t) = \sum \langle \tau_{m_1,a_1} \cdots \tau_{m_n,a_n} \rangle_g \cdot \frac{1}{n!} \prod_{i=1}^{n} t_{m_i,a_i}, \quad (30) $$

$$ F(t; \lambda) = \sum_{g \geq 0} \lambda^{2g-2} F_g(t), \quad (31) $$

$$ Z(t; \lambda) = \exp F(t; \lambda). \quad (32) $$

The following are the first few terms of $F_0$ and $F_1$:

$$ F_0(t) = \frac{1}{3!} \sum_{a_1+a_2+a_3=r-2} t_{0,a_1} t_{0,a_2} t_{0,a_3} + \cdots, \quad (33) $$

$$ F_1(t) = \frac{r-1}{24} t_{1,0} + \cdots. \quad (34) $$

The string equation and the dilaton equation can be reformulated as the following two differential equations:

$$ L_{-1} Z = 0, \quad (35) $$

$$ L_0 Z = 0, \quad (36) $$

where $L_{-1}$ and $L_0$ are given by

$$ L_{-1} = -\frac{\partial}{\partial t_{0,0}} + \sum_{k=1}^{\infty} \sum_{a=1}^{2} t_{k,a} \frac{\partial}{\partial t_{k-1,a}} + \frac{1}{2\lambda^2} \sum_{a=0}^{r-2} t_{0,a} t_{0,r-2-a}, \quad (37) $$

$$ L_0 = -\frac{\partial}{\partial t_{1,0}} + \sum_{k=1}^{\infty} \sum_{a=1}^{2} \frac{r k + a + 1}{r + 1} t_{k,a} \frac{\partial}{\partial t_{k,a}} + \frac{r - 1}{24}, \quad (38) $$

respectively. The generalized Witten Conjecture can be stated as follows. There is a pseudodifferential operator

$$ L = D^r + \sum_{i=0}^{r-2} u_i(t) D^i, \quad D = \frac{\sqrt{-1}}{\sqrt{r}} \frac{\partial}{\partial x}, \quad x = t_{0,0}, \quad (39) $$

such that

$$ \frac{\partial^2 F}{\partial t_{0,0} \partial t_{n,a}} = -c_{n,a} \text{res}(L^{n+\frac{a+1}{r}}), \quad (40) $$

where

$$ c_{n,a} = \frac{(-1)^{n+1}}{(a+1)(a+1+r)\cdots(a+1+nr)}, \quad (41) $$
and
\[(42) \quad \sqrt{-1} \frac{\partial L}{\partial t_{n,a}} = \frac{c_{n,a}}{\sqrt{r}} \cdot [(L^{n+(a+1)/r})_+, L].\]

Comparing with the notations in last section, we set
\[(43) \quad t_{n,a} = T_{nr+a+1} \cdot \frac{\sqrt{-r}}{c_{n,a}} = (-1)^n \sqrt{-r} T_{nr+a+1} \cdot \prod_{j=0}^{n}(j + \frac{a + 1}{r}).\]

Then in the new coordinates \(\{T_1, \ldots, T_{r-1}, T_{r+1}, \ldots\}\), \(L\) satisfies
\[(44) \quad \frac{\partial L}{\partial T_k} = [(L^{k/r})_+, L],\]

and
\[(45) \quad \frac{\partial^2 F}{\partial T_1 \partial T_k} = \text{res}(L^{k/r}).\]

4. W-Constraints for Witten’s r-spin intersection numbers

4.1. String equation and dilaton equation in new coordinates.

The operators in string equation and the dilaton equation now become
\[(46) \quad L_{-1} = \sqrt{-r} \frac{\partial}{\partial T_1} - \sum_{k=r+1}^{\infty} \frac{k}{r} T_k \frac{\partial}{\partial T_{k-r}} - \frac{1}{2r \lambda^2} \sum_{b+c=r} bT_b \cdot cT_c,\]
\[(47) \quad L_0 = -\sqrt{-r} \frac{1}{1 + \frac{r}{2} \partial T_{r+1}} + \sum_{k=1}^{\infty} \frac{k}{r+1} T_k \frac{\partial}{\partial T_k} + \frac{r - 1}{24} r.\]

We change them by multiplications of some constants and take now:
\[(48) \quad \tilde{L}_{-1} = \sum_{k=r+1}^{\infty} \frac{k}{r} \tilde{T}_k \frac{\partial}{\partial T_{k-r}} + \frac{1}{2r \lambda^2} \sum_{b+c=r} bT_b \cdot cT_c,\]
\[(49) \quad \tilde{L}_0 = \sum_{k=1}^{\infty} \frac{k}{r} \tilde{T}_k \frac{\partial}{\partial T_k} + \frac{r^2 - 1}{24r},\]

where we have made the following dilaton shift:
\[(50) \quad \tilde{T}_k = T_k - \delta_{k,r+1} \cdot \frac{\sqrt{-r}}{1 + \frac{r}{2}}.\]
4.2. W-Constraints for Witten’s r-spin intersection numbers.

One can use the method Goeree [17] to derive the W-constraints for r-spin intersection numbers. For an integer \( r \geq 2 \), let

\[
\alpha(z) = \sum_{n \in \mathbb{Z}} \alpha_n z^{-\frac{n}{r} - 1},
\]

where

\[
\alpha_m = \alpha_{-m} = 0, \quad m \in \mathbb{Z},
\]

and for \( m \geq 0, 1 \leq j \leq r - 1 \),

\[
\begin{align*}
\alpha_{m+j} &= \frac{\lambda}{\partial T} \partial T_{rm+j}, \\
\alpha_{-m-j} &= \lambda^{-1}(rm + j) \tilde{T}_{rm+j},
\end{align*}
\]

It follows that one can rewrite \( \alpha(z) \) as follows:

\[
\alpha(z) = \alpha_1(z) + \cdots + \alpha_{r-1}(z),
\]

where

\[
\alpha_j(z) = \sum_{m \in \mathbb{Z}} \alpha_{m+\frac{j}{r}} z^{-\frac{m-j}{r} - 1},
\]

Note

\[
\begin{align*}
\tilde{L}_{-1} &= \frac{1}{2r} \text{res}_z : \alpha(z) \alpha(z) ;, \\
\tilde{L}_0 &= \frac{1}{2r} \text{res}_z z : \alpha(z) \alpha(z) ; + \frac{r^2 - 1}{24r}.
\end{align*}
\]

Consider the mode expansion of the following \( r-1 \) W-fields:

\[
\begin{align*}
W^{(2)}(z) &= \frac{1}{2!} : \alpha(z)^2 : + \frac{r^2 - 1}{24 z^2} ;, \\
W^{(3)}(z) &= \frac{1}{3!} : \alpha(z)^3 ;, \\
& \quad \vdots \\
W^{(r)}(z) &= \frac{1}{r!} : \alpha(z)^r ;.
\end{align*}
\]

I.e., write them as follows:

\[
W^{(k)}(z) = \sum_{m \in \mathbb{Z}} \sum_{j=0}^{r-1} W^{(k)}_{m+j\frac{r}{r}} z^{-\frac{m-j}{r} - k}.
\]

Then the W-constraints satisfied by the partition function of Witten’s r-spin intersection numbers are:

\[
W^{(k)}_m \tau = 0, \quad 2 \leq k \leq r, \quad m \geq -k + 1.
\]
Note

\[ W^{(k)}_m = \frac{1}{k!} \sum_{i_1, \ldots, i_k \in 1/r \cdot \mathbb{Z}, \ i_1 + \cdots + i_k = m} : \alpha_{i_1} \cdots \alpha_{i_k} : \]

5. **Solution of the W-Constraints for R-Spin Intersection Numbers**

5.1. **Grading.** In Liu-Yang-Zhang [22], the following grading for the r-spin case is used:

\[ \text{deg} t_{n,a} = n + \frac{a + 1}{r}. \]

For our purpose, we define

\[ \text{deg} T_n = \frac{n}{r + 1}. \]

The motivation for this definition is as follows. First introduce some operators:

\[ \mathcal{O}_{nr+a+1} = (-1)^n \sqrt{-r} \cdot \prod_{j=0}^{n} (j + \frac{a + 1}{r}) \cdot \tau_{n,a}. \]

They satisfy:

\[ t_{n,a} \tau_{n,a} = T_{nr+a+1} \cdot \mathcal{O}_{nr+a+1}. \]

By the selection rule [25] for \( \langle \tau_{m_1,k_1} \cdots \tau_{m_n,k_n} \rangle_g \), a correlator of the form \( \langle \mathcal{O}_{a_1} \cdots \mathcal{O}_{a_n} \rangle_g \) is nonvanishing only if

\[ \frac{a_1}{r+1} + \cdots + \frac{a_n}{r+1} = 2g - 2 + n. \]

Hence one has

\[ F(T, \lambda) = \sum_{k=1}^{\infty} F^{(k)}(T, \lambda), \]

where

\[ F^{(k)}(T, \lambda) = \sum_{a_1 + \cdots + a_n = (r+1)k} \lambda^{2g-2} \langle \mathcal{O}_{a_1} \cdots \mathcal{O}_{a_n} \rangle_g \prod_{j=1}^{n} T_{a_j}. \]

Furthermore, one can write

\[ \tau(T) = \sum_{k \geq 0} \tau^{(k)}(T), \]

where \( \tau^{(k)}(T) \) has degree \( k \), and clearly \( \tau^{(0)} = 1 \).
Define the Euler operator

\[ E = \frac{1}{r+1} \sum_{n=1}^{\infty} nT_n \cdot \partial T_n. \]

Then one has

\[ E \tau^{(k)} = k \cdot \tau^{(k)}. \]

5.2. **Gradings of the Virasoro operators.** Based on (67) we also define:

\[ \text{deg} \frac{\partial}{\partial T_n} = -\frac{n}{r+1}. \]

Using these gradings one can examine the gradings of the \( W \)-operators.

Let us first look at \( W^{(2)}(z) \) first. Its mode expansion is given by:

\[ W^{(2)}(z) = \frac{1}{2} \sum_{i_1, i_2 = 1}^{r-1} \sum_{m_1, m_2 \in \mathbb{Z}} :\alpha_{m_1+i_1} \alpha_{m_2+i_2} : z^{-(m_1+m_2+i_1+i_2)/r^2 - 2} + \frac{r^2 - 1}{24z^2}. \]

The relevant modes are:

\[ W_{-1}^{(2)} = \frac{1}{2} \sum_{m_1 \in \mathbb{Z}} \sum_{i_1 = 1}^{r-1} :\alpha_{m_1+i_1} \alpha_{-m_1-2+i_1} : ; \]

\[ W_{0}^{(2)} = \frac{1}{2} \sum_{m_1 \in \mathbb{Z}} \sum_{i_1 = 0}^{r-1} :\alpha_{m_1+i_1} \alpha_{-m_1-1+i_1} : + \frac{r^2 - 1}{24} ; \]

\[ W_{m}^{(2)} = \frac{1}{2} \sum_{m_1 \in \mathbb{Z}} \sum_{i_1 = 1}^{r-1} :\alpha_{m_1+i_1} \alpha_{m-m_1-1+i_1} : ; m > 0. \]

Written explicitly as differential operators they are:

\[ W_{-1}^{(2)} = -r \sqrt{-r} \frac{\partial}{\partial T_1} + \sum_{k=r+1}^{\infty} kT_k \frac{\partial}{\partial T_{k-r}} + \frac{1}{2\lambda^2} \sum_{b+c=r} bT_b \cdot cT_c, \]

\[ W_{0}^{(2)} = -r \sqrt{-r} \frac{\partial}{\partial T_{r+1}} + \sum_{k=1}^{\infty} kT_k \frac{\partial}{\partial T_k} + \frac{r^2 - 1}{24}, \]

\[ W_{m}^{(2)} = -r \sqrt{-r} \frac{\partial}{\partial T_{(m+1)r+1}} + \sum_{k=1}^{\infty} kT_k \frac{\partial}{\partial T_{k+rm}} + \frac{\lambda^2}{2} \sum_{b+c=rm} \frac{\partial}{\partial T_b} \cdot \frac{\partial}{\partial T_c}. \]

Now it is clear that one can write:

\[ W_{m}^{(2)} = W_{m}^{(2,0)} + W_{m}^{(2,1)}, \]
with

\[ \deg W^{(2,0)}_m = -\frac{rm}{r+1}, \quad \deg W^{(2,1)} = -1 - \frac{rm}{r+1}. \]

5.3. **Gradings of higher W-operators.** We will use the following notation: \( \sum' \) means a summation over some indices in \( \frac{1}{r} \mathbb{Z} \) non of which is \( -\frac{r+1}{r} \) or integral. We will also use operators

\[ \beta^\pm = \alpha^\pm + r\sqrt{-r}\delta_{n, -(r+1)}\lambda^{-1}, \]

i.e.,

\[ \beta^{m+\pm} = \lambda \frac{\partial}{\partial T_{rm+j}}, \]

\[ \beta^{m-\pm} = \lambda^{-1}(rm+j)T_{rm+j}. \]

It is clear that

\[ \deg \beta_i = \frac{ri}{r+1}. \]

Since one has

\[
W^{(k)}_m = \frac{1}{k!} \left( k(\alpha-(r+1)/r)^{k-1}\alpha_{m+(k-1)(r+1)/r} \right.
+ \binom{k}{2}(\alpha-(r+1)/r)^{k-2} \sum' : \alpha_{i_1} \alpha_{i_2} : \\
+ \cdots \\
+ \left. \sum' : \alpha_{i_1} \cdots \alpha_{i_k} : + \delta_{k,2}\delta_{m,0} \frac{r^2 - 1}{24} \right),
\]
after using \( \alpha_{-(r+1)/r} = \beta_{-(r+1)/r} - r\sqrt{-1}\lambda^{-1} \) and \( \alpha_i = \beta_i \) for \( i \neq -(r+1)/r \), one can rewrite \( W_m^{(k)} \) in terms of the \( \beta \)-operators as follows:

\[
W_m^{(k)} = \frac{1}{k!} \sum_{i_1,\ldots,i_k \in 1/rZ \atop i_1 + \cdots + i_k = m} \beta_{i_1} \cdots \beta_{i_k} : \\
- \frac{r\sqrt{-r\lambda^{-1}}}{(k-1)!} \sum_{i_1,\ldots,i_{k-1} \in 1/rZ \atop i_1 + \cdots + i_{k-1} = m+(r+1)/r} \beta_{i_1} \cdots \beta_{i_{k-1}} : \\
+ \frac{(r\sqrt{-r\lambda^{-1}})^2}{2!(k-2)!} \sum_{i_1,\ldots,i_{k-2} \in 1/rZ \atop i_1 + \cdots + i_{k-2} = m+2(r+1)/r} \beta_{i_1} \cdots \beta_{i_{k-2}} : \\
+ \cdots \cdots \cdots \cdots \\
+ \frac{(-r\sqrt{-r\lambda^{-1}})^{k-2}}{(k-2)!2!} \sum_{i_1,i_2 \in 1/rZ \atop i_1+i_2 = m+(k-2)(r+1)/r} \beta_{i_1} \beta_{i_2} : \\
+ \frac{(-r\sqrt{-r\lambda^{-1}})^{k-1}}{(k-1)!1!} \beta_{m+(k-1)(r+1)/r} \delta_{k,2}\delta_{m,0} \delta_{j,0} \frac{r^2 - 1}{24}.
\]

It follows that

\[ W_m^{(k)} = W_m^{(k,0)} + \cdots + W_m^{(k,k-1)}, \]

where

\[
W_m^{(k,j)} = \frac{(r\sqrt{-r\lambda^{-1}})^j}{j!(k-j)!} \sum_{i_1,\ldots,i_{k-j} \in 1/rZ \atop i_1 + \cdots + i_{k-j} = m+j(r+1)/r} \beta_{i_1} \cdots \beta_{i_{k-j}} : \\
+ \delta_{k,2}\delta_{m,0}\delta_{j,0} \frac{r^2 - 1}{24}.
\]

In particular,

\[
W_m^{(k,k-1)} = \frac{(r\sqrt{-r\lambda^{-1}})^{k-1}}{(k-1)!} \beta_{m+(k-1)(r+1)/r} \\
= \frac{(r\sqrt{-r\lambda^{-1}})^{k-1}}{(k-1)!} r\lambda \frac{\partial}{\partial T_{m+(k-1)(r+1)}}.
\]

Note

\[ \deg W_m^{(k,j)} = -\frac{rm}{r+1} - j. \]
5.4. **Solution of the W-constraints.** Now we use the grading introduced above to rewrite the W-constraint equations as follows:

\begin{align}
(W_m^{(k,0)}) + \cdots + W_m^{(k,k-1)}(\tau^{(0)} + \tau^{(1)} + \cdots) &= 0.
\end{align}

For each \( j \geq 0 \), one then has:

\begin{align}
W_m^{(k,0)}\tau^{(j-k+1)} + W_m^{(k,1)}\tau^{(j-k+2)} + \cdots + W_m^{(k,k-1)}\tau^{(j)} &= 0.
\end{align}

Or equivalently,

\begin{align}
\frac{\partial \tau^{(j)}}{\partial T_{rm+(k-1)(r+1)}} &= -\frac{(k-1)!\lambda^{k-2}}{(-r\sqrt{-r})^{k-1}} \sum_{l=1}^{k-1} W_m^{(k,k-1-l)}\tau^{(j-l)}.
\end{align}

One can multiply both sides by \( \frac{1}{r+1}(rm + (k-1)(r+1))T_{rm+(k-1)(r+1)} \) then take summation \( \sum_{k=2}^{r} \sum_{m=-(k-1)}^{\infty} \) to get:

\begin{align*}
E_{\tau^{(j)}} &= -\sum_{k=2}^{r} \sum_{m=-(k-1)}^{\infty} \frac{(k-1)!\lambda^{k-2}}{(-r\sqrt{-r})^{k-1}} \sum_{l=1}^{k-1} \left( \frac{rm}{r+1} + k-1 \right)
\cdot T_{rm+(k-1)(r+1)} W_m^{(k,k-1-l)}\tau^{(j-l)}
\end{align*}

\begin{align*}
&= -\sum_{l=1}^{r-1} \sum_{k=l+1}^{r} \sum_{m=-(k-1)}^{\infty} \frac{(k-1)!\lambda^{k-2}}{(-r\sqrt{-r})^{k-1}} \left( \frac{rm}{r+1} + k-1 \right)
\cdot T_{rm+(k-1)(r+1)} W_m^{(k,k-1-l)}\tau^{(j-l)}.
\end{align*}

One can rewrite it also as follows

\begin{align*}
E_{\tau^{(j)}} &= -\frac{1}{r+1} \sum_{l=1}^{r-1} \sum_{k=l+1}^{r} \sum_{m=-(k-1)}^{\infty} \frac{(k-1)!\lambda^{k-1}}{(-r\sqrt{-r})^{k-1}}
\end{align*}

\begin{align*}
&\quad \cdot \beta_{-(m+k-1+(k-1)/r)} W_m^{(k,k-1-l)}\tau^{(j-l)}.
\end{align*}

After introducing the following operators for \( l = 1, \ldots, r-1 \):

\begin{align}
A_l &= -\frac{1}{r+1} \sum_{k=l+1}^{r} \sum_{m=0}^{\infty} \frac{(k-1)!\lambda^{k-1}}{(-r\sqrt{-r})^{k-1}} \beta_{-(m+k-1+(k-1)/r)} W_m^{(k,k-1-l)}
\end{align}

one gets

**Theorem 5.1.** The partition function of \( r \)-spin intersection numbers can be computed recursively as follows:

\begin{align}
\tau^{(j)} &= \sum_{l=1}^{r-1} A_l \tau^{(j-l)}.
\end{align}
Together with the initial value $\tau^{(0)} = 1$, this then provides a solution of the W-constraints for the r-spin intersection numbers. We conjecture that

\begin{equation}
[A_i, A_j] = 0
\end{equation}

for $i, j = 1, \ldots, r - 1$. If this is true, then we have

\begin{equation}
\tau = \exp\left(\sum_{j=1}^{r-1} \frac{1}{j} A_j\right).1.
\end{equation}

6. Examples

6.1. The $r = 2$ case. In this case,

\begin{equation}
A_1 = \frac{\lambda}{6\sqrt{-2}} \left( \sum_{a,b \in \frac{1}{2} \mathbb{Z}^+} \beta_{-a+b} \beta_{a+b-\frac{3}{2}} + \frac{1}{2} \sum_{a,b \in \frac{1}{2} \mathbb{Z}^+} \beta_{-a-b+\frac{3}{2}} \beta_a \beta_b \right.
\end{equation}

\begin{equation}
+ \frac{1}{2} \beta^3_{-\frac{1}{2}} + \frac{1}{8} \beta_{-\frac{3}{2}}^3 \right).
\end{equation}

Our result gives:

\begin{equation}
\tau = e^{A_1}1.
\end{equation}

This matches with Alexandrov [1].

6.2. The $r = 3$ case. We use a Maple program by Hao Xu to compute the 3-spin intersection numbers. For $g = 0$ and $n = 3$,

$\langle \tau_{0,0} \tau_{0,0} \tau_{0,1} \rangle_0 = 1$;

for $g = 0$ and $n = 4$,

$\langle \tau_{0,1}^4 \rangle_0 = \frac{1}{3}$, $\langle \tau_{1,1} \tau_{0,0}^3 \rangle_0 = 1$, $\langle \tau_{1,0} \tau_{0,1}^2 \rangle_0 = 1$;

for $g = 0$ and $n = 5$,

$\langle \tau_{2,1} \tau_{0,0}^4 \rangle_0 = 1$, $\langle \tau_{0,0} \tau_{0,1}^3 \rangle_0 = 1$,

$\langle \tau_{1,0} \tau_{0,0}^2 \tau_{0,0}^3 \rangle_0 = 2$, $\langle \tau_{1,0}^2 \tau_{0,1} \tau_{0,0}^2 \rangle_0 = 2$;

hence one has

\begin{equation}
F_0 = \frac{1}{2} t_{0,0} t_{0,1} + \left( \frac{1}{12} t_{0,1}^4 + \frac{1}{6} t_{1,1} t_{0,0}^3 + \frac{1}{2} t_{1,0} t_{0,1} t_{0,0} \right)
\end{equation}

\begin{equation}
+ \left( \frac{1}{24} t_{2,1} t_{0,0}^4 + \frac{1}{6} t_{2,0} t_{0,1} t_{0,0}^3 + \frac{1}{3} t_{1,1} t_{0,1} t_{0,0} + \frac{1}{2} t_{1,0} t_{0,1} t_{0,0}^2 \right) + \cdots
\end{equation}
For $g = 1$ and $n = 1$,

$$\langle \tau_{1,0} \rangle_1 = \frac{1}{12};$$

for $g = 1$ and $n = 2$,

$$\langle \tau_{2,0} \rangle_1 = \frac{1}{12}, \quad \langle \tau_{1,0} \rangle_1 = \frac{1}{12};$$

for $g = 1$ and $n = 3$,

$$\langle \tau_{3,0} \rangle_1 = \frac{1}{12}, \quad \langle \tau_{2,0} \rangle_1 = \frac{1}{6}, \quad \langle \tau_{1,0} \rangle_1 = \frac{1}{6};$$

$$\langle \tau_{2,1} \rangle_1 = \frac{1}{36}, \quad \langle \tau_{1,1} \rangle_1 = \frac{1}{36};$$

hence

$$F_1 = \frac{1}{12} t_{1,0} + \left( \frac{1}{12} t_{2,0} t_{0,0} + \frac{1}{24} t_{1,0}^2 \right)$$

$$+ \left( \frac{1}{24} t_{3,0} t_{0,0} + \frac{1}{6} t_{2,0} t_{1,0} t_{0,0} + \frac{1}{36} t_{1,0}^3 + \frac{1}{72} t_{2,1} t_{0,1}^2 + \frac{1}{72} t_{1,1} t_{0,1} \right) + \cdots.$$
one then has

\[
\begin{align*}
F_0 &= -\frac{\sqrt{-3}}{9} T_2 T_1^2 + \left( \frac{2}{81} T_2^4 - \frac{5}{81} T_5 T_1^3 - \frac{4}{27} T_4 T_2 T_1^2 \right) + \sqrt{-3} \left( \frac{2}{243} T_5 T_1^4 + \frac{28}{729} T_7 T_2 T_1^3 - \frac{20}{243} T_5 T_2 T_1^3 + \frac{16}{243} T_4^2 T_2 T_1^2 \right) + \cdots , \\
F_1 &= -\frac{\sqrt{-3}}{27} T_4 - \left( \frac{7}{81} T_7 T_1 + \frac{2}{81} T_4^2 \right) + \sqrt{-3} \left( \frac{56}{729} T_7 T_4 T_1 + \frac{16}{2187} T_4^3 - \frac{40}{729} T_8 T_2 - \frac{25}{729} T_5 T_2^2 \right) + \cdots , \\
F_2 &= -\frac{770}{6561} T_4 T_2 - \frac{605}{6561} T_1 T_5 - \frac{340}{6561} T_8^2 + \cdots .
\end{align*}
\]

One can see that

\[
\begin{align*}
F^{(1)} &= -\frac{\sqrt{-3}}{9} T_2 T_1^2 \lambda^{-2} - \frac{\sqrt{-3}}{27} T_4 , \\
F^{(2)} &= \left( \frac{2}{81} T_2^4 - \frac{5}{81} T_5 T_1^3 - \frac{4}{27} T_4 T_2 T_1^2 \right) \lambda^{-2} - \left( \frac{7}{81} T_7 T_1 + \frac{2}{81} T_4^2 \right) , \\
F^{(3)} &= \sqrt{-3} \left( \frac{2}{243} T_5 T_1^4 + \frac{28}{729} T_7 T_2 T_1^3 - \frac{20}{243} T_5 T_2 T_1^3 + \frac{16}{243} T_4^2 T_2 T_1^2 \right) \lambda^{-2} \\
&\quad + \sqrt{-3} \left( \frac{56}{729} T_7 T_4 T_1 + \frac{16}{2187} T_4^3 - \frac{40}{729} T_8 T_2 - \frac{25}{729} T_5 T_2^2 \right) .
\end{align*}
\]

It follows that

\[
\begin{align*}
\tau^{(1)} &= -\frac{\sqrt{-3}}{9} T_2 T_1^2 \lambda^{-2} - \frac{\sqrt{-3}}{27} T_4 , \\
\tau^{(2)} &= -\frac{\lambda^{-4}}{54} T_2 T_1^4 \lambda^{-2} \left( \frac{13}{81} T_4 T_2 T_1^2 - \frac{2}{81} T_2^4 + \frac{5}{81} T_5 T_1^3 \right) - \frac{13}{486} T_4^2 - \frac{7}{81} T_7 T_1 , \\
\tau^{(3)} &= \sqrt{-3} \left( \frac{\lambda^{-6}}{1458} T_2^3 T_1^6 + \lambda^{-4} \left( \frac{25}{1458} T_2^2 T_4 T_1^4 T_4 / x^4 - \frac{2}{729} T_2^5 T_1^2 + \frac{5}{729} T_2 T_1^5 T_5 \right) \right. \\
&\quad + \lambda^{-2} \left( \frac{2}{243} T_5 T_1^4 - \frac{2}{2187} T_4 T_2^3 + \frac{325}{4374} T_4^2 T_2 T_1^2 + \frac{35}{729} T_7 T_2 T_1^3 - \frac{20}{243} T_5 T_2 T_1^3 + \frac{5}{2187} T_4 T_5 T_1^3 \right) \\
&\quad - \frac{40}{729} T_8 T_2^2 - \frac{25}{729} T_5 T_2^2 + \frac{325}{39366} T_4^3 + \frac{175}{2187} T_7 T_4 T_1 .
\end{align*}
\]
On the other hand, in this case

\[ A_1 = \frac{\lambda}{12\sqrt{-3}} \sum_{m=0}^{\infty} \beta_{-(m+1/3)} W_{m-1}^{(2,0)} + \frac{\lambda^2}{54} \sum_{m=0}^{\infty} \beta_{-(m+2/3)} W_{m-2}^{(3,1)}, \]

\[ A_2 = \frac{\lambda^2}{54} \sum_{m=0}^{\infty} \beta_{-(m+4/3)} W_{m-2}^{(3,0)}, \]

where the components of the \( W \)-operators are given by:

\[ W_{m}^{(2,0)} = \frac{1}{2} \sum_{i_1, i_2 \in 1/3 \mathbb{Z}, i_1 + i_2 = m} : \beta_{i_1} \beta_{i_2} : + \frac{\delta_{m,0}}{3}, \]

\[ W_{m}^{(3,0)} = \frac{1}{6} \sum_{i_1, \ldots, i_3 \in 1/3 \mathbb{Z}, i_1 + \cdots + i_3 = m} : \beta_{i_1} \beta_{i_2} \beta_{i_3} :, \]

\[ W_{m}^{(3,1)} = \frac{3\sqrt{-3}\lambda^{-1}}{2} \sum_{i_1, i_2 \in 1/3 \mathbb{Z}, i_1 + i_2 = m+4/3} : \beta_{i_1} \beta_{i_2} :. \]

It follows that

\[ A_1 = \frac{\lambda}{24\sqrt{-3}} \sum_{m=0}^{\infty} \beta_{-(m+1/3)} \sum_{i_1, i_2 \in 1/3 \mathbb{Z}, i_1 + i_2 = m-1} : \beta_{i_1} \beta_{i_2} : + \frac{\lambda}{36\sqrt{-3}} \beta^{-4/3} \]

\[ + \frac{\lambda}{12\sqrt{-3}} \sum_{m=0}^{\infty} \beta_{-(m+2/3)} \sum_{i_1, i_2 \in 1/3 \mathbb{Z}, i_1 + i_2 = m-2/3} : \beta_{i_1} \beta_{i_2} :. \]

\[ A_2 = \frac{\lambda^2}{324} \sum_{m=0}^{\infty} \beta_{-(m+2/3)} \sum_{i_1, \ldots, i_3 \in 1/3 \mathbb{Z}, i_1 + \cdots + i_3 = m-2} : \beta_{i_1} \beta_{i_2} \beta_{i_3} :. \]

They can be written more explicitly as follows:

\[ A_1 = \frac{\lambda}{36\sqrt{-3}} \beta^{-4/3} + \frac{\lambda}{12\sqrt{-3}} \left( 2\beta^{-2/3} \beta_{-1/3} \beta_{-1/3} \right) \]

\[ + \sum_{m=0}^{\infty} \beta_{b+c=m+2} \beta_{-b} \beta_{-c} \beta_{m+\frac{7}{3}} + \frac{3}{2} \sum_{m=0}^{\infty} \sum_{b+c=m+2} \beta_{-b} \beta_{-c} \beta_{m+\frac{4}{3}} \]

\[ + \frac{1}{2} \sum_{m=0}^{\infty} \beta_{-(m+\frac{5}{3})} \sum_{b+c=m+1} \beta_{b} \beta_{c} + \sum_{m=0}^{\infty} \beta_{-(m+\frac{5}{3})} \sum_{b+c=m+\frac{4}{3}} \beta_{b} \beta_{c}, \]
and
\[
A_2 = \frac{\lambda}{324} \left\{ \beta_{-2/3} + 3\beta_{-4/3}\beta_{-2/3}\beta_{-1/3} + \beta_{-5/3}\beta_{-1/3}^3 \\
+ 9\beta_{-2/3}\beta_{-2/3}\beta_{-1/3} \\
+ (3\beta_{-2/3}\beta_{-2/3} + 6\beta_{-2/3}\beta_{-1/3} + 6\beta_{-2/3}\beta_{-1/3}^2 + 3\beta_{-2/3}\beta_{-1/3})\beta_{-2/3}^3 \right\} \\
= \lambda \left( \frac{1}{9\sqrt{-3}}T_4 + \frac{1}{3\sqrt{-3\lambda^2}}T_2T_1^2 \right).
\]

It follows that:
\[
A_{11} = \frac{\lambda}{36\sqrt{-3}}\beta_{-1/3} + \frac{\lambda}{6\sqrt{-3}}2\beta_{-2/3}\beta_{-1/3}^2 \\
= \frac{1}{9\sqrt{-3}}T_1 + \frac{1}{3\sqrt{-3\lambda^2}}T_2T_1^2,
\]

\[
A_{11}^2 = \left( \frac{1}{9\sqrt{-3}}T_4 + \frac{1}{3\sqrt{-3\lambda^2}}T_2T_1^2 \right)^2 \\
+ \frac{\lambda}{12\sqrt{-3}} \left( \frac{1}{9\sqrt{-3}} \sum_{b+c=1} \beta_{-b}\beta_{-c}\lambda + \frac{1}{3\sqrt{-3\lambda^2}} \sum_{b+c=\frac{3}{2}} \beta_{-b}\beta_{-c}2\lambda T_2T_1 \\
+ \frac{3}{2} \sum_{b+c=2} \beta_{-b}\beta_{-c}T_1^2 \right) \\
+ \frac{\lambda}{12\sqrt{-3}} \left( \frac{1}{2} \beta_{-\frac{3}{2}}2\lambda^2 \partial_t \partial_{\tau_2} \left( \frac{1}{3\sqrt{-3\lambda^2}}T_2T_1^2 \right) \right) \\
= \frac{-13}{243}T_4^2 - \frac{14}{81}T_7T_1^2 - \frac{5}{36}T_5T_1^3\lambda^{-2} - \frac{32}{81}T_4T_2T_1^2\lambda^{-2} - \frac{1}{27}T_2^2T_1^4\lambda^{-4}.
\]

\[
A_{21} = \frac{\lambda^2}{324} \left\{ \beta_{-2/3} + 3\beta_{-4/3}\beta_{-2/3}\beta_{-1/3} + \beta_{-5/3}\beta_{-1/3}^3 \right\} \\
= \lambda^{-2} \left( 16T_4^2 + 24T_4T_2T_1^2 + 5T_5T_1^3 \right).
\]

It can be checked that
\[
\tau^{(1)} = A_{11}, \\
\tau^{(2)} = \frac{1}{2}(A_{11}^2 + A_{21}).
\]

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