THE DEFOCUSING ENERGY-CRITICAL KLEIN-GORDON-HARTREE EQUATION

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ABSTRACT. In this paper, we study the scattering theory for the defocusing energy-critical Klein-Gordon equation with a cubic convolution \( u_{tt} - \Delta u + u + (|x|^{-4} * |u|^2)u = 0 \) in the spatial dimension \( d \geq 5 \). We utilize the strategy in [10] derived from concentration compactness ideas to show that the proof of the global well-posedness and scattering is reduced to disprove the existence of the soliton-like solution. Employing technique from [28], we consider a virial-type identity in the direction orthogonal to the momentum vector so as to exclude such solution.

Key Words: Klein-Gordon-Hartree equation; Scattering theory; Strichartz estimate.

AMS Classification: Primary 35P25. Secondary 35B40, 35Q40, 81U99.

1. Introduction

This paper is devoted to the study of the Cauchy problem of the defocusing energy-critical Klein-Gordon-Hartree equation

\[
\begin{aligned}
\ddot{u} - \Delta u + u + f(u) &= 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^d, d \geq 5, \\
\dot{u}(0, x) &= u_0(x), \quad u_1(0, x) = u_1,
\end{aligned}
\] (1.1)

where \( f(u) = (V(x) * |u|^2)u \) with \( V(x) = |x|^{-4} \). Here \( u \) is a real-valued function defined in \( \mathbb{R}^{d+1} \), the dot denotes the time derivative, \( \Delta \) is the Laplacian in \( \mathbb{R}^d \), \( V(x) \) is called the potential, and \(*\) denotes the spatial convolution in \( \mathbb{R}^d \).

Formally, the solution \( u \) of (1.1) conserves the energy

\[
\begin{aligned}
E(u(t), \dot{u}(t)) &= \frac{1}{2} \int_{\mathbb{R}^d} \left( |\dot{u}(t, x)|^2 + |\nabla u(t, x)|^2 + |u(t, x)|^2 \right) dx \\
&\quad + \frac{1}{4} \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|u(t, x)|^2 |u(t, y)|^2}{|x-y|^4} dxdy \\
&= E(u_0, u_1),
\end{aligned}
\]

and the momentum

\[
P(u(t)) = \int_{\mathbb{R}^d} u_1(t, x) \nabla u(t, x) dx = P(u)(0). (1.2)
\]

For the equation (1.1) with nonlinearity \( f(u) = \mu(|x|^{-\gamma} * |u|^2)u, \mu = \pm 1 \), using the ideas of Strauss [30], [31] and Pecher [29], Mochizuki [24] showed that if \( d \geq 3 \), \( 2 \leq \gamma < \min(d, 4) \), then global well-posedness and scattering results with small data hold in the energy space \( H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d) \). For the general initial data, we refer to the authors [23] where we develop a complete scattering theory in the energy space for (1.1) with the subcritical nonlinearity (i.e. \( 2 < \gamma < \min(d, 4) \)) for both
defocusing \( (\mu = 1) \) and focusing \( (\mu = -1) \) in spatial dimension \( d \geq 3 \). In this paper, we will focus on the energy-critical case, i.e. \( \gamma = 4 \) and \( d \geq 5 \). We refer also to Miao-Zhang \([21]\) where the low regularity for the cubic convolution defocusing Klein-Gordon-Hartree equation is discussed.

Before stating our main results, we recall the scattering theory for the classical Klein-Gordon equation, i.e. \((1.1)\) with nonlinearity \( f(u) = \mu|u|^{p-1}u \). For \( \mu = 1 \) and

\[
1 + \frac{4}{d} < p < 1 + \frac{4\gamma_d}{d - 2}, \quad \gamma_d = \begin{cases} 1, & 3 \leq d \leq 9; \\ \frac{d}{d + 1}, & d \geq 10, \end{cases}
\]

Brenner \([5]\) established the scattering results in the energy space in dimension \( d \geq 10 \). Thereafter, Ginibre and Velo \([7]\) exploited the Birman-Solomjak space \( \ell^m(L^2, I, B) \) in \([8]\) and the delicate estimates to improve the results in \([5]\), which covered all subcritical cases. Finally K. Nakanishi \([25]\) obtained the scattering results for the energy-critical case by the strategy of induction on energy \([6]\) and a new Morawetz-type estimate. And recently, S. Ibrahim, N. Masmoudi and K. Nakanishi \([10, 11]\) utilized the concentration compactness ideas to give the scattering threshold for the focusing (i.e. \( \mu = -1 \)) nonlinear Klein-Gordon equation. We remark that their method also works for the defocusing case. We will utilize their argument to study the scattering theory for the defocusing energy-critical Klein-Gordon-Hartree equation.

On the other hand, the scattering theory for the Hartree equation

\[
i\dot{u} = -\Delta u + (|x|^{-\gamma} * |u|^2)u
\]

has been also studied by many authors (see \([9, 15, 16, 17, 18, 19]\)). For the energy-subcritical case, i.e. \( \gamma < 4 \), Ginibre and Velo \([9]\) obtained the asymptotic completeness in the energy space \( H^1(\mathbb{R}^d) \) by deriving the associated Morawetz inequality and extracting an useful Birman-Solomjak type estimate. Nakanishi \([20]\) improved the results by a new Morawetz estimate. For the energy-critical case \( (\gamma = 4 \text{ and } d \geq 5) \), Miao, Xu and Zhao \([16]\) took advantage of a new kind of the localized Morawetz estimate to rule out the possibility of the energy concentration at origin and established the scattering results in the energy space for the radial data. We refer also to \([17, 18, 19]\) for the general data and also mass-critical case.

Compared with the classical Klein-Gordon equation with the local nonlinearity \( f(u) = |u|^{p-1}u \), the nonlinearity \( f(u) = (V(\cdot) * |u|^2)u \) is nonlocal, which brings us many difficulties. The main difficulty is the absence of a Lorentz invariance which could be used to control the momentum efficiently. We will overcome this difficulty by considering a Virial-type identity in the direction orthogonal to the momentum vector following the technique in \([28]\).

Now we introduce the definition of the strong solution for \((1.1)\).

**Definition 1.1** (solution). A function \( u : I \times \mathbb{R}^d \to \mathbb{R} \) on a nonempty time interval \( 0 \in I \) is a strong solution to \((1.1)\) if for any compact \( J \subset I \), \( (u, u_t) \in C_0^1(J; H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)) \) and

\[
u \in W(J), \quad W(I) := L_t^{2(d+1)}(J; B_x^{2(d+1)-1}((\mathbb{R}^d))
\]

and for each \( t \in I \), \( (u(t), \dot{u}(t)) \) satisfies the following Duhamel’s formula:

\[
\begin{pmatrix} u(t) \\ \dot{u}(t) \end{pmatrix} = V_0(t) \begin{pmatrix} u_0(x) \\ u_1(x) \end{pmatrix} - \int_0^t V_0(t-s) \begin{pmatrix} 0 \\ f(u(s)) \end{pmatrix} ds,
\]

\((1.4)\)
where
\[ V_0(t) = \left( \dot{K}(t), \ddot{K}(t) \right), \quad K(t) = \frac{\sin(t\omega)}{\omega}, \quad \omega = (1 - \Delta)^{1/2}. \]

The interval \( I \) is called the lifespan of \( u \). Moreover, if the solution \( u \) cannot be extended to any strictly large interval, then we say that \( u \) is a maximal-lifespan solution. We say that \( u \) is a global solution if \( I = \mathbb{R} \).

**Remark 1.2.** From Remark 2.5 below, we obtain the solution \( u \) lies in the space \( W(I) \) locally in time. Also, the finiteness of the norm on maximal-lifespan implies the solution is global and scatters in both time directions by standard argument. In view of this, we define
\[ S_I(u) = \|u\|_{ST(I)} = \|u\|_{[W](I)} \quad (1.5) \]
as the scattering size of \( u \).

Our main result is the following global well-posedness and scattering result in the energy space.

**Theorem 1.3.** Assume that \( d \geq 5 \), and \((u_0, u_1) \in H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)\). Then there exists a unique global solution \( u(t) \) of (1.1) which scatters in the sense that there exist solutions \( v_\pm \) of the free Klein-Gordon equation
\[ \ddot{v} - \Delta v + v = 0 \quad (1.6) \]
with \((v_\pm(0), \dot{v}_\pm(0)) \in H^1 \times L^2\) such that
\[ \left\| \left( u(t), \dot{u}(t) \right) - \left( v_\pm(t), \dot{v}_\pm(t) \right) \right\|_{H^1 \times L^2} \to 0, \quad \text{as} \quad t \to \pm\infty. \quad (1.7) \]

The outline for the proof of Theorem 1.3 we define the function \( \Lambda \) by
\[ \Lambda(E) = \sup \{ \|u\|_{ST(I)} : E(u, u_t) \leq E \} \quad (1.8) \]
where the supremum is taken over all strong solutions \( u \) of (1.1) on any interval \( I \) with energy not greater than \( E \), and define
\[ E_{\text{max}} = \sup \{ E : \Lambda(E) < +\infty \}. \]
The small data scattering (Theorem 2.3 below) tells us \( E_{\text{max}} > 0 \). Our goal next is to prove that \( E_{\text{max}} = +\infty \). We argue by contradiction. We show that if \( E_{\text{max}} < +\infty \), then there exists a nonlinear solution of (1.1) with energy be exactly \( E_{\text{max}} \). Moreover, this solution satisfies some strong compactness properties. This is completed in Section 4 where we utilize the profile decomposition that was established in [10], and a strategy introduced by Kenig and Merle [13]. We consider a virial-type identity in the direction orthogonal to the momentum vector following the technique [28] to obtain a contradiction. We refer to Section 5 for more details.

The paper is organized as follows. In Section 2, we deal with the local theory for the equation (1.1). In Section 3, we give the linear and nonlinear profile decomposition and show some properties of the profile. In Section 4, we extract a critical solution. Finally in Section 5, we preclude the critical solution, which completes the proof of Theorem 1.3.
2. Preliminaries

2.1. Notation. First, we give some notations which will be used throughout this paper. We always assume the spatial dimension $d \geq 5$ and let $2^* = \frac{2d}{d-2}$. For any $r : 1 \leq r \leq \infty$, we denote by $\| \cdot \|_r$ the norm in $L^r = L^r(\mathbb{R}^d)$ and by $r'$ the conjugate exponent defined by $\frac{1}{r} + \frac{1}{r'} = 1$. For any $s \in \mathbb{R}$, we denote by $H^s(\mathbb{R}^d)$ the usual Sobolev space. Let $\psi \in \mathcal{S}(\mathbb{R}^d)$ be such that $\text{supp} \ \hat{\psi} \subseteq \{ \xi : \frac{1}{2} \leq |\xi| \leq 2 \}$ and $\sum_{j \in \mathbb{Z}} \hat{\psi}(2^{-j}\xi) = 1$ for $\xi \neq 0$. Define $\psi_0$ by $\hat{\psi}_0 = 1 - \sum_{j \geq 1} \hat{\psi}(2^{-j}\xi)$. Thus $\text{supp} \ \hat{\psi}_0 \subseteq \{ \xi : |\xi| \leq 2 \}$ and $\hat{\psi}_0 = 1$ for $|\xi| \leq 1$. We denote by $\Delta_j$ and $P_0$ the convolution operators whose symbols are respectively given by $\hat{\psi}(\xi/2^j)$ and $\hat{\psi}_0(\xi)$.

For any $s \in \mathbb{R}$, $1 \leq r \leq \infty$, the inhomogeneous Besov space $B^s_{r,2}(\mathbb{R}^d)$ is defined by

$$B^s_{r,2}(\mathbb{R}^d) = \left\{ u \in \mathcal{S}'(\mathbb{R}^d), \| P_0 u \|_{L^r}^2 + \| 2^{js} \| \Delta_j u \|_{L^r}^2 \right\}_{j \in \mathbb{N}} < \infty \right\}.$$ 

For details of Besov space, we refer to [2]. For any interval $I \subset \mathbb{R}$ and any Banach space $X$ we denote by $\mathcal{C}(I; X)$ the space of strongly continuous functions from $I$ to $X$ and by $L^q(I; X)$ the space of strongly measurable functions from $I$ to $X$ with $\| u(\cdot); X \| \in L^q(I)$. Given $d$, we define, for $2 \leq r \leq \infty$,

$$\delta(r) = d\left( \frac{1}{2} - \frac{1}{r} \right).$$

Sometimes abbreviate $\delta(r)$, $\delta(r_1)$ to $\delta$, $\delta_1$ respectively. We denote by $\langle \cdot, \cdot \rangle$ the scalar product in $L^2$. We let $L^p_\ast$ denote the weak $L^p$ space.

2.2. Strichartz estimate. In this section, we consider the Cauchy problem for the equation (1.1)

$$\begin{cases}
\hat{u} - \Delta u + u + f(u) = 0, \\
u(0) = u_0, \ \dot{u}(0) = u_1.
\end{cases}$$

(2.1)

The integral equation for the Cauchy problem (2.1) can be written as

$$u(t) = \tilde{K}(t)u_0 + K(t)u_1 - \int_0^t K(t-s)f(u(s))ds, (2.2)$$

or

$$\begin{pmatrix} u(t) \\ \dot{u}(t) \end{pmatrix} = V_0(t) \begin{pmatrix} u_0(x) \\ u_1(x) \end{pmatrix} - \int_0^t V_0(t-s) \begin{pmatrix} 0 \\ f(u(s)) \end{pmatrix}ds, (2.3)$$

where

$$K(t) = \frac{\sin(t\omega)}{\omega}, \quad V_0(t) = \begin{pmatrix} \tilde{K}(t) \\ \tilde{K}(t) \end{pmatrix}, \quad \omega = (1 - \Delta)^{1/2}.$$ 

Let $U(t) = e^{it\omega}$, then

$$\tilde{K}(t) = \frac{U(t) + U(-t)}{2e^{it\omega}}, \quad K(t) = \frac{U(t) - U(-t)}{2i\omega}.$$

Now we recall the following dispersive estimate for the operator $U(t) = e^{it\omega}$.

Lemma 2.1 ([5, 7]). Let $2 \leq r \leq \infty$ and $0 \leq \theta \leq 1$. Then

$$\| e^{it\omega}f \|_{B^s_{r,2}} \leq \mu(t) \| f \|_{B^{(d+1+\theta)(\frac{1}{2} - \frac{1}{r})/2}_{r',2}}$$

where

$$\mu(t) = C \min \left\{ \| t \|^{-(d-1-\theta)(\frac{1}{2} - \frac{1}{r})}, \| t \|^{-(d-1+\theta)(\frac{1}{2} - \frac{1}{r})} \right\}.$$
Combining the above lemma, the abstract duality and interpolation argument (see \textbf{[8, 12]}), we have the following Strichartz estimates.

\textbf{Lemma 2.2 \textbf{[5, 7, 20]}.} Let $0 \leq \theta_i \leq 1$, $\rho_i \in \mathbb{R}$, $2 \leq q_i, r_i \leq \infty$, $i = 1, 2$. Assume that $(\theta_i, d, q_i, r_i) \neq (0, 3, 2, \infty)$ satisfy the following admissible conditions

\begin{equation}
\begin{aligned}
0 \leq \frac{2}{q_i} &\leq \min \left\{ (d - 1 + \theta_i)\left(\frac{1}{2} - \frac{1}{r_i}\right), 1 \right\}, \quad i = 1, 2 \\
\rho_1 + (d + \theta_1)\left(\frac{1}{2} - \frac{1}{r_1}\right) &- \frac{1}{q_1} = \mu, \quad (2.4) \\
\rho_2 + (d + \theta_2)\left(\frac{1}{2} - \frac{1}{r_2}\right) &- \frac{1}{q_2} = 1 - \mu.
\end{aligned}
\end{equation}

Then, for $f \in H^\mu$, we have

\begin{equation}
\begin{aligned}
\|U(\cdot)f\|_{L^{q_1}(\mathbb{R}; B_{r_1}^{\theta_1})} &\leq C\|f\|_{H^\mu}; \\
\|K * f\|_{L^{q_1}(I; B_{r_1}^{\theta_1})} &\leq C\|f\|_{L^{q_2}_t(I; B_{r_2}^{\theta_2})}; \\
\|K_R * f\|_{L^{q_1}(I; B_{r_1}^{\theta_1})} &\leq C\|f\|_{L^{q_2}_t(I; B_{r_2}^{\theta_2})}.
\end{aligned}
\end{equation}

where the subscript $R$ stands for retarded, and

\begin{align*}
K * f &= \int_\mathbb{R} K(t - s)f(u(s))ds, \\
K_R * f &= \int_0^t K(t - s)f(u(s))ds.
\end{align*}

In addition to the $W$-norm defined in (1.5), we also need the following space

\begin{equation}
[W]^*_I = L_t^{2(d+1)}(I; B_{r_1}^{\theta_1}(\mathbb{R}^d)).
\end{equation}

Now we give a nonlinear estimate which will be applied to show the small data scattering.

\textbf{Lemma 2.3.} We have

\begin{equation}
\left\|(V(\cdot) * |u|^2)v\right\||_{[W]^*_I} + \left\|(V(\cdot) * (uv))u\right\||_{[W]^*_I} \leq C\|u\|_{[W]_I}^{\frac{d+1}{2}} \|u\|_{L_t^{\frac{\alpha(\mu+1)}{\mu+1}}(I; L^{\mu})}^{\frac{\alpha}{\mu+1}} + C\|u\|_{[W]_I} \|u\|_{L_t^{\frac{\alpha}{\mu+1}}(I; L^{\mu})} \|v\|_{L_t^{\frac{4d-\sigma}{d-\sigma}}(I; L^{\frac{4d-3}{d-\sigma}})} \|v\|_{L_t^\infty(I; L^\infty)}.
\end{equation}

In particular,

\begin{equation}
\left\|(V(\cdot) * |u|^2)u\right\||_{[W]^*_I} \leq C\|u\|_{[W]_I}^{\frac{d+1}{2}} \|u\|_{L_t^{\frac{\alpha(\mu+1)}{\mu+1}}(I; L^{\mu})}^{\frac{\alpha}{\mu+1}}.
\end{equation}

\textbf{Proof.} We only need to prove the estimate $\|(V(\cdot) * |u|^2)v\|_{[W]^*_I}$, since the estimate $\|(V(\cdot) * (uv))u\|_{[W]^*_I}$ is similar. From the Sobolev embedding: $W^{s,p}(\mathbb{R}^d) \hookrightarrow B_{p,2}^s(\mathbb{R}^d)$, $p \leq 2; B_{p,2}^s(\mathbb{R}^d) \hookrightarrow W^{s,q}(\mathbb{R}^d)$, $q \geq 2$, the fractional Leibnitz rule \textbf{[14]}, and the H"older and the Young inequalities, we have

\begin{align*}
\left\|(V(\cdot) * |u|^2)v\right\|_{L_t^{\frac{\alpha}{\mu+1}}(I; B_{r_1}^{\theta_1})} \leq &\left\|V\right\|_{L_t^\infty(I; L^{\frac{\alpha}{\mu+1}})} \|u\|_{L_t^\infty(I; L^{\mu})} \left\|\left\|u\right\|_{L_t^{\frac{\alpha}{\mu+1}}(I; L^{\mu})}\right\|^2 + \left\|V\right\|_{L_t^\infty(I; L^{\mu})} \left\|\left\|u\right\|_{L_t^\infty(I; L^{\mu})}\right\| \left\|\right\|u\|_{L_t^{\frac{\alpha}{\mu+1}}(I; L^{\mu})}\right\| \left\|\right\|v\|_{L_t^{\frac{4d-\sigma}{d-\sigma}}(I; L^{\frac{4d-3}{d-\sigma}})} \|v\|_{L_t^\infty(I; L^\infty)}.
\end{align*}
where the exponents satisfy
\[
\begin{align*}
\frac{d}{p} &= 2\delta(r) + 2\delta(s), \\
\frac{2}{q} + \frac{2}{k} &= 1.
\end{align*}
\] (2.11)

Since \( V(x) = |x|^{-d} \in L^2_\mathbb{R} \), if we take admissible pair \( q = r = \frac{2(d+1)}{d+2} \) and \( \delta(s) = 1 + \frac{1}{k} \) (then \( \delta(r) = \frac{d}{d+2}, k = d + 1 \), then
\[
\|(V * |u|^2)v\|_{L^\infty(I; L^\frac{1}{q}, \mathbb{R}^d)} \lesssim \|v\|_{W(I)}\|u\|_{L^\infty(I; L^r)}^2 + \|u\|_{W(I)}\|v\|_{L^\infty(I; L^r)}\|v\|_{L^\infty(I; L^r)}.
\] (2.12)

The H\ölder inequality and the Sobolev embedding theorem yield that
\[
\|v\|_{L^\infty(I; L^r)} \lesssim \|v\|_{W(I)}\|u\|_{L^\infty(I; L^r)}^2 \|v\|_{L^\infty(I; L^r)} \lesssim \|v\|_{W(I)}\|v\|_{W(I)}.
\] (2.13)

Plugging (2.13) into (2.12), we get
\[
\|(V * |u|^2)v\|_{L^\infty(I; L^\frac{1}{q}, \mathbb{R}^d)} \lesssim \|v\|_{W(I)}\|u\|_{L^\infty(I; L^r)}^2 \|v\|_{L^\infty(I; L^r)} \|v\|_{W(I)} \|v\|_{W(I)}.
\]

Thus we complete the proof of Lemma 2.3. \( \square \)

Now, we can state the local well-posedness for (1.1) with large initial data and small data scattering in the energy space \( H^1 \times L^2 \).

**Theorem 2.4** (small data scattering). Assume \( d \geq 5 \), and \((u_0, u_1) \in H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)\). There exists a small constant \( \delta = \delta(E) \) such that if \( \|(u_0, u_1)\|_{H^1 \times L^2} \leq E \) and \( I \) is an interval such that
\[
\|\hat{K}(t)u_0 + K(t)u_1\|_{W(I)} \leq \delta,
\]
then there exists a unique strong solution \( u \) to (1.1) in \( I \times \mathbb{R}^d \), with \( u \in C(I; H^1) \cap C^1(I; L^2) \) and
\[
\|u\|_{W(I)} \leq 2C\delta.
\] (2.14)

Let \((T_-(u_0, u_1), T_+(u_0, u_1))\) be the maximal time interval on which \( u \) is well-defined.

**Remark 2.5.** (1) There exists \( \tilde{\delta} \) such that if \( \|(u_0, u_1)\|_{H^1 \times L^2} \leq \tilde{\delta} \), the conclusion of Theorem 2.4 applies to any interval \( I \). Indeed, by Strichartz estimates, \( \|\hat{K}(t)u_0 + K(t)u_1\|_{W(I)} \leq C\tilde{\delta} \) and the claim follows.

(2) Given \((u_0, u_1) \in H^1 \times L^2\), there exists \( 0 \in I \) such that the hypothesis of Theorem 2.4 is verified on \( I \). This is clear because, by Strichartz estimates, \( \|\hat{K}(t)u_0 + K(t)u_1\|_{W(\mathbb{R})} < \infty \).

Finally, we conclude this subsection by recalling the following standard finite blow-up criterion.

**Lemma 2.6** (Standard finite blow-up criterion). If \( T_+(u_0, u_1) < +\infty \), then
\[
\|u\|_{W([0, T_+(u_0, u_1))]} = +\infty.
\]

A corresponding result holds for \( T_-(u_0, u_1) \).

The proof is similar to the one in Lemma 2.11 of [13].
2.3. Perturbation lemma. In this part, we give the perturbation theory of the solution of (1.1) with the global space-time estimate. First we recall some notations in [10].

With any real-valued function \( u(t, x) \), we associate the complex-valued function \( \bar{u}(t, x) \) by
\[
\bar{u} = \langle \nabla \rangle u - i \dot{u}, \quad u = \text{Re}(\nabla)^{-1} \bar{u}.
\]
Then the free and nonlinear Klein-Gordon equations are given by
\[
(\Box + 1)u = 0 \iff (i \partial_t + \langle \nabla \rangle) \bar{u} = 0,
\]
\[
(\Box + 1)u = -f(u) \iff (i \partial_t + \langle \nabla \rangle) \bar{u} = -f(\langle \nabla \rangle^{-1} \text{Re} \bar{u}),
\]
and the energy are written as
\[
\dot{E}(\bar{u}) = E(u, \dot{u}) = \frac{1}{2} \int_{\mathbb{R}^d} (|\bar{u}|^2 + |\nabla u|^2 + |u|^2) \, dx + \frac{1}{4} \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|u(t, x)|^2|u(t, y)|^2}{|x - y|^4} \, dx \, dy.
\]

Lemma 2.7. Let \( I \) be a time interval, \( t_0 \in I \) and \( \bar{u}, \bar{w} \in C(I; L^2(\mathbb{R}^d)) \) satisfy
\[
(i \partial_t + \langle \nabla \rangle) \bar{u} = -f(u) + \text{eq}(u) \quad (i \partial_t + \langle \nabla \rangle) \bar{w} = -f(w) + \text{eq}(w).
\]
for some function \( \text{eq}(u) \), \( \text{eq}(w) \). Assume that for some constants \( M, E > 0 \), we have
\[
\|u\|_{ST(I)} \leq M, \quad \|\bar{u}\|_{L^p_I L^q_x(\mathbb{R}^d)} + \|\bar{w}\|_{L^p_I L^q_x(\mathbb{R}^d)} \leq E,
\]
Let \( t_0 \in I \), and let \((u(t_0), u_0(t_0))\) be close to \((w(t_0), w_0(t_0))\) in the sense that
\[
\left\| (u(t_0) - w(t_0), u_0(t_0) - w_0(t_0)) \right\|_{H^1 \times L^2} \leq \epsilon.
\]
Let \( \gamma_0 = e^{i(\langle \nabla \rangle(t - t_0))} (\bar{u} - \bar{w})(t_0) \) and assume also that we have smallness conditions
\[
\|\gamma_0\|_{ST(I)} + \|(\text{eq}(u), \text{eq}(w))\|_{ST'(I)} \leq \epsilon,
\]
where \( 0 < \epsilon < \epsilon_1 = \epsilon_1(M, E) \) is a small constant and
\[
ST^*(I) = [W]^*(I) + L^1_t(I; L^2_x(\mathbb{R}^d)).
\]
Then we conclude that
\[
\|u - w\|_{ST(I)} \leq C(M, E)\epsilon, \quad \|u\|_{ST(I)} \leq C(M, E, E').
\]

Proof. Since \( \|u\|_{ST(I)} \leq M \), there exists a partition of the right half of \( I \) at \( t_0 \):
\[
t_0 < t_1 < \cdots < t_N, \quad I_j = (t_j, t_{j+1}), \quad I \cap (t_0, \infty) = (t_0, t_N),
\]
such that \( N \leq C(L, \delta) \) and for any \( j = 0, 1, \cdots, N - 1 \), we have
\[
\|w\|_{ST(I_j)} \leq \delta \ll 1.
\]
The estimate on the left half of \( I \) at \( t_0 \) is analogue, we omit it.

Let \( \gamma(t) = u(t) - w(t), \quad \gamma_j(t) = e^{i(\langle \nabla \rangle(t - t_j))} \gamma_j(t_j), \quad 0 \leq j \leq N - 1, \)
then \( \gamma \) satisfies the following difference equation
\[
(i \partial_t + \langle \nabla \rangle) \gamma = (V * |w|^2) \gamma + 2[V * (\gamma w)] + 2[V * (\gamma w)] - (V * |\gamma|^2) \gamma + \text{eq}(u) - \text{eq}(w),
\]
\[
\gamma_j(t_j) = \gamma_j(t_j),
\]
which implies that
\[
\tilde{\gamma}(t) = \tilde{\gamma}_j(t) - i \int_{t_j}^{t} e^{i(\nabla)(t-s)} \left( (V * |w|^2)\gamma + 2[V * (\gamma w)] w + 2[V * (\gamma w)] \gamma \\
+ (V * |\gamma|^2) w + (V * |\gamma|^2) \gamma + eq(u) - eq(w) \right) ds,
\]
\[
\tilde{\gamma}_{j+1}(t) = \tilde{\gamma}_j(t) - i \int_{t_j}^{t_{j+1}} e^{i(\nabla)(t-s)} \left( (V * |w|^2)\gamma + 2[V * (\gamma w)] w + 2[V * (\gamma w)] \gamma \\
+ (V * |\gamma|^2) w + (V * |\gamma|^2) \gamma + eq(u) - eq(w) \right) ds.
\]

By Lemma 2.22 and Lemma 2.23, we have
\[
\|\gamma - \gamma_j\|_{ST(I_j)} + \|\gamma_{j+1} - \gamma_j\|_{ST(\mathbb{R})} \leq \|\gamma - \gamma_j\|_{W(I_j)} + \|\gamma_{j+1} - \gamma_j\|_{W(\mathbb{R})} \leq \|\gamma - \gamma_j\|_{ST^c(I_j)} + \|\gamma_{j+1} - \gamma_j\|_{ST^c(\mathbb{R})} \leq \|\gamma - \gamma_j\|_{ST(I_j)} + \|\gamma_{j+1} - \gamma_j\|_{ST(\mathbb{R})} \leq \|\gamma - \gamma_j\|_{W(I_j)} + \|\gamma_{j+1} - \gamma_j\|_{W(\mathbb{R})} \leq \|\gamma - \gamma_j\|_{ST(I_j)} + \|\gamma_{j+1} - \gamma_j\|_{ST(\mathbb{R})}.
\]

Therefore, assuming that
\[
\|\gamma\|_{ST(I_j)} \leq \delta \ll 1, \quad \forall \, j = 0, 1, \cdots, N - 1,
\]
then by (2.22) and (2.24), we have
\[
\|\gamma\|_{ST(I_j)} + \|\gamma_{j+1}\|_{ST(I_{j+1}, t_N)} \leq C\|\gamma\|_{ST(I_j, t_N)} + \epsilon,
\]
for some absolute constant $C > 0$. By (2.20) and iteration on $j$, we obtain
\[
\|\gamma\|_{ST(t)} \leq (2C)^N \epsilon \leq \frac{\delta}{2},
\]
if we choose $\epsilon_1$ sufficiently small. Hence the assumption (2.23) is justified by continuity in $t$ and induction on $j$. Then repeating the estimate (2.24) once again, we can get the ST-norm estimate on $\gamma$, which implies the Strichartz estimates on $u$.

3. Profile decomposition

In this section, we first recall the linear profile decomposition of the sequence of $H^1$-bounded solutions of (1.1) which was established in [10]. And then we utilize it to show the orthogonal analysis for the nonlinear energy and the nonlinear profile decomposition which will be used to construct the critical element and obtain its compactness properties.

3.1. Linear profile decomposition. First, we give some notation as introduced in [10]. For any triple $(t_n^j, x_n^j, h_n^j) \in \mathbb{R} \times \mathbb{R}^d \times (0, \infty)$ with arbitrary suffix $n$ and $j$, let $\tau_n^j$, $T_n^j$, and $(\nabla)^j_n$ respectively denote the scaled time shift, the unitary and the self-adjoint operators in $L^2(\mathbb{R}^d)$, defined by
\[
\tau_n^j = -\frac{t_n^j}{h_n^j}, \quad T_n^j \varphi(x) = (h_n^j)^{-\frac{d}{2}} \varphi \left( \frac{x - x_n^j}{h_n^j} \right), \quad (\nabla)^j_n = \sqrt{-\Delta + (h_n^j)^2}.
\]


We denote the set of Fourier multipliers on
\[ \mathcal{MC} = \{ \mu = \mathcal{F}^{-1} \hat{\mu} \mid \hat{\mu} \in C(\mathbb{R}^d), \exists \lim_{|x| \to \infty} \hat{\mu}(x) \in \mathbb{R} \}. \]

Now we can state the linear profile decomposition as follows

**Lemma 3.1** (Linear profile decomposition, [10]). Let \( \vec{v}_n(t) = e^{i(t^j)} \vec{v}_n(0) \) be a sequence of free Klein-Gordon solutions with uniformly bounded \( L_2^2 \) norm. Then after replacing it with some subsequence, there exist \( K \in \{0, 1, 2, \ldots, \infty\} \) and, for each integer \( j \in [0, K) \), \( \varphi^j \in L^2(\mathbb{R}^d) \) and \( \{(t_i^n, x_i^n, h_i^n)\}_{n \in \mathbb{N}} \subset \mathbb{R} \times \mathbb{R}^d \times (0, 1] \) satisfying the following. Define \( \vec{v}_n \) and \( \omega^k_n \) for each \( j < k \leq K \) by
\[
\vec{v}_n(t, x) = \sum_{j=0}^{k-1} \vec{v}_j^n(t, x) + \omega^k_n(t, x),
\]
then we have
\[
\lim_{k \to K} \lim_{n \to \infty} \|\omega^k_n\|_{L^\infty(\mathbb{R}; B_{2, \infty}(\mathbb{R}^d))} = 0,
\]
and for any \( \mu \in \mathcal{MC} \), any \( l < j < k \leq K \) and any \( t \in \mathbb{R} \),
\[
\lim_{n \to \infty} \langle \mu \vec{v}_j^n, \mu \vec{v}_k^n \rangle_{L^2} = 0 = \lim_{n \to \infty} \langle \mu \vec{v}_j^n, \omega^k_n \rangle_{L^2},
\]
\[
\lim_{n \to \infty} \frac{h_j^n}{h_k^n} + \frac{h_j^n}{h_k^n} + \frac{|t_j^n - t_k^n| + |x_j^n - x_k^n|}{h_k^n} = +\infty.
\]
Moreover, each sequence \( \{h_j^n\}_{n \in \mathbb{N}} \) is either going to 0 or identically 1 for all \( n \).

**Remark 3.2.** We call \( \{\vec{v}_j^n\}_{n \in \mathbb{N}} \) a free concentrating wave for each \( j \), and \( \omega^k_n \) the remainder. From (3.3), we have the following asymptotic orthogonality
\[
\lim_{n \to +\infty} \left( \|\mu \vec{v}_n(t)\|_{L^2}^2 - \sum_{j=0}^{k-1} \|\mu \vec{v}_j^n(t)\|_{L^2}^2 - \|\mu \omega^k_n(t)\|_{L^2}^2 \right) = 0, \quad \forall \mu \in \mathcal{MC}.
\]

Next we begin with the orthogonal analysis for the nonlinear energy. It follows from Mikhlin’s theorem that the following estimates for \( 1 < p < \infty \),
\[
\|\|\nabla - \langle \nabla \rangle\|_p \varphi \|_p \leq \|\nabla/h_n\|_p \|\langle \nabla/h_n\rangle^{-1} \varphi\|_p,
\]
\[
\|\|\nabla^{-1} - \langle \nabla \rangle^{-1}\|_p \varphi \|_p \leq \|\langle \nabla/h_n\rangle^{-2} |\nabla|^{-1} \varphi\|_p,
\]
hold uniformly for \( 0 < h_n \leq 1 \).

**Lemma 3.3.** Let \( \vec{v}_n \) be a sequence of free Klein-Gordon solutions satisfying \( \vec{v}_n(0) \in L^2_x \). Let \( \vec{v}_n = \sum_{j=0}^{k-1} \vec{v}_j^n + \omega^k_n \) be the linear profile decomposition given by Lemma 3.1.

If \( \lim_{n \to \infty} \hat{E}(\vec{v}_n(0)) < +\infty \), then we have \( \vec{v}_n(0) \in L^2_x \) for large \( n \), and
\[
\lim_{k \to K} \lim_{n \to \infty} \left| \hat{E}(\vec{v}_n(0)) - \sum_{j=0}^{k-1} \hat{E}(\vec{v}_j^n(0)) - \hat{E}(\omega^k_n(0)) \right| = 0.
\]
Moreover, we have for all \( j < k \)
\[
0 \leq \lim_{n \to \infty} \hat{E}(\vec{v}_j^n(0)) \leq \lim_{n \to \infty} \hat{E}(\vec{v}_j^n(0)) \leq \lim_{n \to \infty} \hat{E}(\vec{v}_n(0)),
\]
where the last inequality becomes equality only if \( K = 1 \) and \( \omega^1_n \to 0 \) in \( L^\infty_t L^2_x \).
Thus, we can discard those

In fact, on one hand, by the Hölder and Bernstein equalities, we have

On the other hand, from the sharp interpolation [1], we know

which concludes the claim.

Proof. First, we claim that

\[ \|u\|_{L^2_x} \lesssim \|u\|_{H^1}^{d-2} \|u\|_{B_{1,\infty}^{d-4}}^2, \quad 2^* = \frac{2d}{d-2}. \]  

(3.11)

In fact, on one hand, by the Hölder and Bernstein equalities, we have

\[ \|P_{\leq 1} u\|_{L^2_x} \lesssim \|P_{\leq 1} u\|_{L^2_x}^{d-2} \|P_{\leq 1} u\|_{L^2_x}^2 \lesssim \|u\|_{H^1}^{d-2} \|u\|_{B_{1,\infty}^{d-4}}^2, \]

(3.15)

On the other hand, from the sharp interpolation [1], we know

\[ \|P_{> 1} u\|_{L^2_x} \lesssim \|P_{> 1} u\|_{L^2_x}^{d-2} \|P_{> 1} u\|_{B_{1,\infty}^{d-4}} \lesssim \|u\|_{H^1}^{d-2} \|u\|_{B_{1,\infty}^{d-4}}, \]

which concludes the claim.

Thus, by (3.11) and (3.15), we obtain

\[ \lim_{k \to K} \lim_{n \to \infty} \|\omega_n^k\|_{L^2_x} \leq \lim_{k \to K} \lim_{n \to \infty} \|\omega_n\|_{H^1}^{d-2} \|\omega_n^k\|_{B_{1,\infty}^{d-4}} = 0, \]

where \( \omega_n^k = \text{Re}(\nabla)^{-1}\omega_n^k \). This implies that, if there exists \( i \in \{1, 2, 3, 4\} \) such that \( u_i = \omega_n^k \), then by the Hölder and the Hardy-Littlewood-sobolev inequalities, we get

\[ \lim_{k \to K} \lim_{n \to \infty} \|(V(x) * (u_1 u_2))(u_3 u_4)\|_{L^2_x} \leq \lim_{k \to K} \lim_{n \to \infty} 4 \|u_i\|_{L^2_x} = 0. \]

This together with (3.6) reduces us to prove

\[ \lim_{k \to K} \lim_{n \to \infty} \left| F\left( \sum_{j<k} v_n^j(0) \right) - \sum_{j<k} F\left( v_n^j(0) \right) \right| = 0, \]

(3.12)

where \( F(u) = \|(V(x) * |u|^2)|u|^2\|_{L^1_x} \).

Moreover, using the decay of \( e^{it\nabla} \) in \( S \to L^2_x \) uniform w.r.t. \( n \) and the Sobolev embedding \( H^1(\mathbb{R}^d) \subset L^2(\mathbb{R}^d) \), we have

\[ \|v_n^j\|_{L^2_x} \leq \| (\nabla)^{-1} e^{-it\nabla} \tau_n^j \varphi^j(\tau_n^j) \|_{L^2_x} \to 0, \text{ as } n \to \infty. \]

Thus, we can discard those \( j \) where \( \tau_n^j = -\frac{\tau_n^j}{\tau_n^j} \to +\infty \).

Hence, up to subsequence, we may assume that \( \tau_n^j \to \exists \tau_n^j \in \mathbb{R} \) for all \( j \). Let

\[ \psi^j := \text{Re} e^{-it\nabla} \tau_n^j \varphi^j \in L^2_x(\mathbb{R}^d), \]

(3.13)

we have

\[ \left| F\left( \sum_{j<k} v_n^j(0) \right) - \sum_{j<k} F\left( v_n^j(0) \right) \right| \]

\[ \leq \left| F\left( \sum_{j<k} v_n^j(0) \right) - F\left( \sum_{j<k} (\nabla)^{-1} T_n^j \psi^j \right) \right| \]

\[ + \left| \sum_{j<k} F\left( v_n^j(0) \right) - \sum_{j<k} F\left( (\nabla)^{-1} T_n^j \psi^j \right) \right| \]

\[ + \left| F\left( \sum_{j<k} (\nabla)^{-1} T_n^j \psi^j \right) - \sum_{j<k} F\left( (\nabla)^{-1} T_n^j \psi^j \right) \right|. \]

(3.14)

(3.15)

(3.16)

By the continuity of the operator \( e^{it\nabla} \) in \( t \) in \( H^1 \), we have

\[ v_n^j(0) - (\nabla)^{-1} T_n^j \psi^j \to 0 \text{ in } H^1(\mathbb{R}^d), \text{ as } n \to \infty. \]
This together with the following nonlinear estimate
\[ \| (V(\cdot) * (g_1 g_2)) g_3 g_4 \|_{L^1_T} \lesssim \prod_{j=1}^4 \| g_j \|_{L^2_T} \tag{3.17} \]
show that as \( n \to \infty \),
\[ |F(\sum_{j<k} \langle \nabla \rangle^{-1} T_n^j \psi^j) - \sum_{j<k} F(\langle \nabla \rangle^{-1} T_n^j \psi^j)| \to 0, \]
\[ |\sum_{j<k} F(\psi^j) - \sum_{j<k} F(\langle \nabla \rangle^{-1} T_n^j \psi^j)| \to 0. \]

Now we consider the term (3.16). Let
\[ \hat{\psi}^j = \begin{cases} |\nabla|^{-1} \psi^j, & \text{if } h_n^j \to 0 \\ \langle \nabla \rangle^{-1} \psi^j, & \text{if } h_n^j \equiv 1, \end{cases} \]
then we have \( \hat{\psi}^j \in L^2_T \), and
\[ \| \langle \nabla \rangle^{-1} T_n^j \psi^j - h_n^j T_n^j \hat{\psi}^j \|_{L^2_T} \lesssim \begin{cases} \| \langle \nabla \rangle^{-1} T_n^j \psi^j - h_n^j T_n^j \langle \nabla \rangle^{-1} \psi^j \|_{L^2_T} & \text{if } h_n^j \to 0 \\ \| \langle \nabla \rangle^{-1} T_n^j \psi^j - h_n^j T_n^j \langle \nabla \rangle^{-1} \psi^j \|_{L^2_T} & \text{if } h_n^j \equiv 1 \end{cases} \tag{3.18} \]
\[ = \begin{cases} 0 & \text{if } h_n^j \equiv 1 \\ \| \langle \nabla \rangle^{-1} \psi^j - |\nabla|^{-1} \psi^j \|_{L^2_T} & \text{if } h_n^j \to 0 \end{cases} \to 0, \text{ as } n \to \infty. \]

Combining this with (3.17), we obtain that as \( n \to \infty \),
\[ F(\sum_{j<k} \langle \nabla \rangle^{-1} T_n^j \psi^j) - \sum_{j<k} F(h_n^j T_n^j \hat{\psi}^j) \to 0, \]
\[ \sum_{j<k} F(\langle \nabla \rangle^{-1} T_n^j \psi^j) - \sum_{j<k} F(h_n^j T_n^j \hat{\psi}^j) \to 0. \]
Thus it suffices to show that as \( n \to \infty \)
\[ |F(\sum_{j<k} h_n^j T_n^j \hat{\psi}^j) - \sum_{j<k} F(h_n^j T_n^j \hat{\psi}^j)| \to 0. \tag{3.21} \]

Now we define \( \hat{\psi}_{n,R}^j \) for any \( R \gg 1 \) by
\[ \hat{\psi}_{n,R}^j(x) = \chi_R(x) \hat{\psi}^j \prod \{(1 - \chi_{h_n^j R})(x - x_n^{j,l}) \mid 1 \leq l < k; h_n^j R < h_n^j \}, \]
where \( (h_n^{j,l}, x_n^{j,l}) = (h_n^j, x_n^j - x_n^l)/h_n^j \), and \( \chi_R(x) = \chi(x/R) \) with \( \chi(x) \in C_c^\infty(\mathbb{R}^d) \) satisfying \( \chi(x) = 1 \) for \( |x| \leq 1 \) and \( \chi(x) = 0 \) for \( |x| \geq 2 \). Then \( \psi_n^j \to \chi_R \psi^j \) in \( L^2_x \) as \( n \to \infty \), since either \( h_n^{j,l} \to 0 \) or \( |x_n^{j,l}| \to \infty \) by (3.5). Moreover, we have \( \chi_R \psi^j \to \psi^j \) in \( L^2_x \) as \( R \to \infty \).

Hence we may replace \( \psi^j \) by \( \hat{\psi}_n^j R \) in (3.21). Since \( \{ \text{supp}(t,x)h_n^l \psi_n^j R \psi_n^j \} \) are mutually disjoint for large \( n \), and so for large \( n 

\[ | \sum_{j<k} h_n^j T_n^j \hat{\psi}_n^j R^j |^2 = \sum_{j<k} | h_n^j T_n^j \hat{\psi}_n^j R^j |^2. \] (3.22)

Then

\[
| F( \sum_{j<k} h_n^j T_n^j \hat{\psi}_n^j R^j ) - \sum_{j<k} F( h_n^j T_n^j \hat{\psi}_n^j R^j ) | \leq \sum_{j \neq l} \left\| (V(\cdot) * |h_n^j T_n^j \hat{\psi}_n^j R^j|^2) |h_n^j T_n^j \hat{\psi}_n^j R^j| \right\|_{L^1(\mathbb{R}^d)} \\
= \sum_{j \neq l} (h_n^j)^{2-d} \left\| (V(\cdot) * |\hat{\psi}_n^j R^j|^2) |\hat{\psi}_n^j R^j| \right\|_{L^1_1(\mathbb{R}^d)} \\
\to 0, \text{ as } n \to \infty,
\]

by Lebesgue dominated convergence theorem, since either \( h_n^j \to 0 \) or \( |x_n^{j,l}| \to \infty \) by (3.5). This concludes the proof of Lemma 3.3. \( \square \)

3.2. Nonlinear profile decomposition. After the linear profile decomposition of a sequence of initial data in the last subsection, we now show the nonlinear profile decomposition of a sequence of the solutions of (1.1) with the same intial data in the last subsection, we now show the non linear profile decomposition of (1.1). After the linear profile decomposition of (1.1), we undo the group action \( T_n^j \) to look for the linear profile \( \tilde{V}^j \). Let

\[
\tilde{V}_n^j(t,x) = T_n^j \tilde{V}^j \left( (t-t_n^j)/h_n^j \right)
\]

then we have

\[
\tilde{V}_n^j(t,x) = e^{it(\cdot)} \varphi^j
\]
Now let $\tilde{v}_n^j$ be the nonlinear solution with the same initial data $\tilde{v}_n^j(0)$

$$
\begin{cases}
(i\partial_t + \langle \nabla \rangle)\tilde{v}_n^j = -f(\text{Re}(\langle \nabla \rangle^{-1}\tilde{v}_n^j)), \\
\tilde{v}_n^j(0) = \tilde{v}_n^j(0) = T_n^j\tilde{V}_n^j(\tau_n^j),
\end{cases}
$$

where $\tau_n^j = -t_n^j/h_n^j$. In order to look for the nonlinear profile $\tilde{U}_n^j$ associated with the free concentrating wave $\tilde{v}_n^j$, we also need undo the group action. Define

$$
\tilde{v}_n^j(t, x) = T_n^j\tilde{U}_n^j((t - t_n^j)/h_n^j),
$$

then $\tilde{U}_n^j$ satisfies the rescaled equation

$$
\begin{cases}
(i\partial_t + \langle \nabla \rangle^j)\tilde{U}_n^j = -f(\text{Re}(\langle \nabla \rangle^j_n)^{-1}\tilde{U}_n^j), \\
\tilde{U}_n^j(\tau_n^j) = \tilde{V}_n^j(\tau_n^j).
\end{cases}
$$

Up to subsequence, we may assume that there exist $h_n^\infty \in \{0, 1\}$ and $\tau_n^\infty \in [-\infty, \infty]$ for every $j$, such that as $n \to \infty$

$$
h_n^j \to h_n^\infty, \text{ and } \tau_n^j \to \tau_n^\infty.
$$

And then the limit equations are given by

$$
\tilde{V}_n^j = e^{it\langle \nabla \rangle^j_n}\varphi^j, \quad \begin{cases}
(i\partial_t + \langle \nabla \rangle^j_\infty)\tilde{U}_n^j = -f(\text{Re}(\langle \nabla \rangle^j_\infty)^{-1}\tilde{U}_n^j), \\
\tilde{U}_n^j(\tau_n^\infty) = \tilde{V}_n^j(\tau_n^\infty),
\end{cases}
$$

where $\tilde{U}_n^j$ is defined by

$$
\tilde{U}_n^j := \text{Re}(\langle \nabla \rangle^j_\infty)^{-1}\tilde{U}_n^j = \begin{cases}
\text{Re}(\langle \nabla \rangle^j)^{-1}\tilde{U}_n^j & \text{if } h_n^j = 1, \\
\text{Re}|\langle \nabla \rangle|^{-1}\tilde{U}_n^j & \text{if } h_n^j = 0.
\end{cases}
$$

We remark that by using the standard iteration with the Strichartz estimate, we can obtain the unique existence of a local solution $\tilde{U}_n^j$ around $t = \tau_n^\infty$ in all cases, including $h_n^\infty = 0$ and $\tau_n^\infty = \pm \infty$ (the later corresponding to the existence of the wave operators). We denote $\tilde{U}_n^j$ on the maximal existence interval to be the nonlinear profile associated with the free concentrating wave $(\tilde{v}_n^j; t_n^j, x_n^j, h_n^j)$.

The nonlinear concentrating wave $\tilde{u}_n^j$ associated with $\tilde{v}_n^j$ is defined by

$$
\tilde{u}_n^j(t, x) := T_n^j\tilde{U}_n^j((t - t_n^j)/h_n^j).
$$

It is easy to see that $u_n^j$ solves (1.1) when $h_n^\infty = 1$. When $h_n^\infty = 0$, $u_n^j$ solves

$$
\begin{cases}
(\partial_t - \Delta + 1)u_n^j = (i\partial_t + \langle \nabla \rangle)\tilde{u}_n^j = (\langle \nabla \rangle - |\nabla|)\tilde{u}_n^j - f(|\nabla|^{-1}\langle \nabla \rangle u_n^j), \\
\tilde{u}_n^j(0) = T_n^j\tilde{U}_n^j(\tau_n^j).
\end{cases}
$$

The existence time interval of $u_n^j$ may be finite and even go to 0, but at least we have

$$
\|\tilde{u}_n(0) - \tilde{v}_n^j(0)\|_{L^2} = \|T_n^j\tilde{U}_n(\tau_n^j) - T_n^j\tilde{V}_n^j(\tau_n^j)\|_{L^2}
\leq\|\tilde{V}_n^j(\tau_n^j) - \tilde{V}_n^j(\tau_n^j)\|_{L^2} + \|\tilde{V}_n^j(\tau_n^j) - \tilde{U}_n^j(\tau_n^j)\|_{L^2} \to 0,
$$

as $n \to \infty$. 

Let $u_n$ be a sequence of (local) solutions of \((1.1)\) around $t = 0$, and let $v_n$ be the sequence of the free solutions with the same initial data. We consider the linear profile decomposition of $\{v_n\}$ given by Lemma 3.1.

$$
\bar{v}_n = \sum_{j=0}^{k-1} v^{(j)}_n + \bar{\omega}_n,
$$

where the implicit constants do not depend on the interval $I$. Assume that in the nonlinear profile decomposition

$$
\hat{u}^j \in L^\infty_t L^2_x(\mathbb{R}^d),
$$

\(j < k\).

**Definition 3.4** (Nonlinear profile decomposition). Let $\{\bar{u}^{(j)}_n\}_{n \in \mathbb{N}}$ be the free concentrating wave, and $\{\hat{u}^j\}_{n \in \mathbb{N}}$ be the sequence of the nonlinear concentrating wave associated with $\{\bar{u}^{(j)}_n\}_{n \in \mathbb{N}}$. Then we define the nonlinear profile decomposition of $u_n$ by

$$
\bar{u}^{(n)} < k := \sum_{j=0}^{k-1} \bar{u}^{(j)}_n + \bar{\omega}^k_n,
\hat{u}^j = e^{i(\nabla)(t - t^j_n)} T_n^j \varphi^j.
$$

**Proof.**

One can refer to [10] for the proof of (3.31) and (3.32). Now we turn to prove (3.33). By the definition of $u^{(j)}_n$ and $\hat{U}_\infty$, we know that

$$
u^{(j)}_n(x, t) = Re\langle \nabla \rangle^{-1} u^{(j)}_n(t, x) = Re\langle \nabla \rangle^{-1} T_n^j \hat{U}^j \left( \frac{t - t^j_n}{h_n^j} \right) = h_n^j T_n^j \langle \nabla \rangle^{-1} \hat{U}^j \left( \frac{t - t^j_n}{h_n^j} \right).
$$

The following lemmas derive from Lemma 3.1 and the perturbation lemma. The first lemma concerns the orthogonality in the Strichartz norms.

**Lemma 3.5.** Assume that in the nonlinear profile decomposition \((3.28)\), we have

$$
\|\hat{U}^j_{\infty}\|_{ST^j_{\infty}(\mathbb{R})} + \|\hat{U}^j_{\infty}\|_{L^\infty_t L^2_x(\mathbb{R})} < \infty, \forall j < k.
$$

Then, for any finite interval $I$, $j < k$, we have

$$
\lim_{n \to \infty} \|u^{(j)}_n\|_{ST(I)} \lesssim \|\hat{U}^j_{\infty}\|_{ST^j_{\infty}(\mathbb{R})},
$$

$$
\lim_{n \to \infty} \|u^{<k}_n\|^2_{ST(I)} \lesssim \lim_{n \to \infty} \sum_{j=0}^{k-1} \|u^{(j)}_n\|^2_{ST(I)}.
$$

where the implicit constants do not depend on the interval $I$ or $j$. We also have

$$
\lim_{n \to \infty} \left\| f(u^{<k}_n) - \sum_{j=0}^{k-1} f(\langle \nabla \rangle^{-1} \langle \nabla \rangle u^{(j)}_n) \right\|_{ST^*_{\infty}(I)} = 0,
$$

where $f(u) = (V(x) * |u|^2)u$.

**Proof.**

One can refer to [10] for the proof of (3.31) and (3.32). Now we turn to prove (3.33). By the definition of $u^{(j)}_n$ and $\hat{U}_\infty$, we know that
Let \( u_{(n)}^{<k}(t, x) = \sum_{j<k} w_{(n)}^j(x, t) \), where \( w_{(n)}^j(x, t) \) is defined by
\[
u_{(n)}^j(x, t) = \frac{\langle \nabla \rangle}{\langle \nabla \rangle_{\infty}} u_{(n)}^j = h_n^2 T_n^j \hat{U}_n^j (t - t_n^j / h_n^j).
\]
Then we have
\[
\|f(u_{(n)}^{<k}) - \sum_{j<k} f(\langle \nabla \rangle_{\infty}^{-1} \langle \nabla \rangle u_{(n)}^j)\|_{ST^*(I)} \leq \|f(u_{(n)}^{<k}) - \sum_{j<k} f(u_{(n)}^j)\|_{ST^*(I)} + \|f(u_{(n)}^{<k}) - \sum_{j<k} f(u_{(n)}^j)\|_{ST^*(I)}^2.
\]
First, we estimate (3.34). Let \( G(I) = L^{3(d+1)}_{t,x} \cap L^{6d/(d+1)}_{x} \). It follows from (2.9) and the Hölder inequality that \( q = \frac{2(d+1)}{d-1} \)
\[
\|f(u_{(n)}^{<k}) - f(u_{(n)}^{<k})\|_{L^q(I; B^s_{2,2})} + \|f(u_{(n)}^{<k}) - f(u_{(n)}^{<k})\|_{L^q(I; B^s_{2,2})} \leq \|u_{(n)}^{<k} - u_{(n)}^{<k}\|_{ST^*_2(I)} \|\|u_{(n)}^{<k}, u_{(n)}^{<k}\|_{L^\infty \cap ST^*_2(I)} \|_{G(I)} \|_{G(I)}^2 + \|u_{(n)}^{<k} - u_{(n)}^{<k}\|_{ST^*_2(I)} \|\|u_{(n)}^{<k}, u_{(n)}^{<k}\|_{L^\infty \cap ST^*_2(I)} \|_{G(I)} \|_{G(I)}^2 \to 0, \text{ as } n \to \infty,
\]
where we utilize (3.8) in the last second inequality and the fact \( \hat{U}_\infty^j \in L^\infty \cap ST^*_2(I) \subset [G(I)]. \)

Next we estimate (3.35). For \( R \gg 1 \), we define \( \hat{U}_{n,R}^j \) by
\[
\hat{U}_{n,R}^j(t, x) = \chi R(t, x) \hat{U}_\infty^j(t, x) \prod_{l<k} \{ 1 - \chi_{h_l^j}(R) (t - t_l^j, x - x_l^j) \} \leq h_l^j, \quad R^{-1} \leq h_l^j, R \to \infty
\]
where \( (h_l^j, t_l^j, x_l^j) = (h_l^j, t_l^j, x_l^j, x_l^j, x_l^j + x_l^j) / h_l^j \), and \( \chi R(t, x) = \chi \left( \frac{t}{R}, \frac{x}{R} \right) \) with \( \chi \in C^\infty_c(\mathbb{R}^{d+1}) \) satisfying \( \chi(t, x) = 1 \) for \( |(t, x)| \leq 1 \) and \( \chi(t, x) = 0 \) for \( |(t, x)| \geq 2 \). Then, noting that either \( h_l^j, R \to 0 \) or \( |t_l^j| + |x_l^j| \to \infty \) by (3.33), we obtain \( \hat{U}_{n,R}^j \to \chi R \hat{U}_\infty^j \) in \( ST^*_2(I) \) and \( [G(I)] \) as \( n \to \infty \). Furthermore, we get \( \chi R \hat{U}_\infty^j \to \hat{U}_\infty^j \) in the same spaces.

Therefore, we may replace \( u_{(n)}^j \) by \( u_{(n),R}^j := h_n^2 T_n^j \hat{U}_{n,R}^j \). By the support property of \( u_{(n),R}^j \), we have for large \( n \)
\[
\left( \sum_{j<k} u_{(n),R}^j \right)^2 = \sum_{j<k} |u_{(n),R}^j|^2.
\]
Thus, we obtain
\[
\left\| f(u_{(n)}^{<k}, R) - \sum_{j<k} f(u_{(n)}^{j}, R) \right\|_{ST^{*}(I)}
\]
\[\leq \sum_{j \neq l} \left\| (V(\cdot) \ast |u_{(n)}^{j}, R|)^2 u_{(n)}^{l} \right\|_{ST^{*}(I)}
\]
\[= \sum_{j \neq l} (h_n^{j,l})^{1-\frac{2}{s}} \left\| (V(\cdot) \ast |\hat{U}_{(n)}^{j}(R)|^2) (t, x)\hat{U}_{(n)}^{l}(R) \left( \frac{t - t_n^{j,l}}{h_n^{j,l}}, x - x_n^{j,l} \right) \right\|_{ST^{*}(I)}
\]
\[\to 0, \quad \text{as } n \to \infty,
\]
by Lebesgue dominated convergence theorem, since either $h_n^{j,l} \to 0$ or $|t_n^{j,l}| + |x_n^{j,l}| \to \infty$ by (3.35). Thus we concludes the proof of Lemma 3.5.

After this preliminaries, we now show that $\tilde{u}_{(n)}^{<k} + \tilde{\omega}_n^{k}$ is a good approximation for $\tilde{u}_{(n)}$ provided that each nonlinear profile has finite global Strichartz norm.

**Lemma 3.6.** Let $u_n$ be a sequence of local solutions of (1.1) around $t = 0$ satisfying $\lim_{n \to \infty} E(u_n, \tilde{u}_n) < +\infty$. Assume that in its nonlinear profile decomposition (3.28), we have for any $j$

\[
\left\| \tilde{U}_{(n)}^{j} \right\|_{ST^{*}\left((\mathbb{R}) \right)} + \left\| \tilde{U}_{(n)}^{j} \right\|_{L_t^{\infty}L_x^{s}\left(\mathbb{R}^s\right)} < \infty.
\]

Then, for large $n$, $u_n$ is bounded in the Strichartz and the energy norms, that is

\[
\lim_{n \to \infty} \left( \left\| u_n \right\|_{ST^{*}(\mathbb{R})} + \left\| \tilde{u}_n \right\|_{L_t^{\infty}L_x^{s}(\mathbb{R} \times \mathbb{R}^s)} \right) < +\infty.
\]

**Proof.** We only need to verify the conditions of Lemma 2.7. For this purpose, we always use the fact that $u_{(n)}^{<k} + \omega_n^{k}$ satisfies that

\[
(i\partial_t + \langle \nabla \rangle)(\tilde{u}_{(n)}^{<k} + \tilde{\omega}_n^{k}) = -f(u_{(n)}^{<k} + \omega_n^{k}) + eq(u_{(n)}^{<k}, \omega_n^{k}),
\]

where the error term $eq(u_{(n)}^{<k}, \omega_n^{k})$ is

\[
eq \sum_{j<k} (\langle \nabla \rangle - \langle \nabla \rangle_{\infty}) \tilde{u}_{(n)}^{j} + f(u_{(n)}^{<k} - \sum_{j<k} f(u_{(n)}^{j})
\]

\[+ f(u_{(n)}^{<k} + \omega_n^{k}) - f(u_{(n)}^{<k}),
\]

and $u_{(n)}^{j} = (\langle \nabla \rangle_{\infty})^{-1}(\langle \nabla \rangle)_{\infty}^{j}$ is as before.

First, by the definition of the nonlinear concentrating wave $u_{(n)}^{j}$ and (3.27), we have

\[
\left\| (\tilde{u}_{(n)}^{<k}(0) + \tilde{\omega}_n^{k}(0)) - \tilde{u}_n(0) \right\|_{L_x^{2}} \leq \sum_{j=0}^{k-1} \left\| \tilde{u}_{(n)}^{j}(0) - \tilde{u}_n(0) \right\|_{L_x^{2}} \to 0,
\]
as $n \to +\infty$.

Next, by the linear profile decomposition in Lemma 3.1, we get

\[
\left\| \tilde{u}_n(0) \right\|_{L_x^{2}}^2 = \left\| \tilde{v}_n(0) \right\|_{L_x^{2}}^2 \geq \sum_{j=0}^{k-1} \left\| \tilde{v}_{n}^{j}(0) \right\|_{L_x^{2}}^2 + o_n(1) = \sum_{j=0}^{k-1} \left\| \tilde{v}_{n}^{j}(0) \right\|_{L_x^{2}}^2 + o_n(1).
\]

(3.40)
Thus, using the small data scattering (Lemma 3.1), we obtain that except for a finite set \( J \subset \mathbb{N} \), the energy of \( u_j^{(n)} \) with \( j \notin J \) is smaller than the iteration threshold. Hence

\[
\| u_j^{(n)} \|_{ST(I)} \lesssim \| u_j^{(n)}(0) \|_{L^2}, \quad j \notin J.
\]

This together with (3.31), (3.32), (3.38) and (3.40) yields that for any finite interval \( I \)

\[
\sup_k \lim_{n \to \infty} \| u_{<k}^{(n)} \|_{ST(I)}^2 \lesssim \sum_{j \in J} \| u_j^{(n)} \|_{ST(I)}^2 + \sum_{j \notin J} \| u_j^{(n)} \|_{ST(I)}^2
\]

\[
\lesssim \sum_{j \in J} \| \tilde{U}^{(j)} \|_{ST^*(I)}^2 + \lim_{n \to \infty} \| \tilde{u}_n(0) \|_{L^2}^2 < +\infty. \tag{3.41}
\]

Combining this with the Strichartz estimate for \( \omega_n^k \), we get

\[
\sup_k \lim_{n \to \infty} \| u_{<k}^{(n)} + \omega_n^k \|_{ST(I)} < +\infty.
\]

By Lemma 3.1 and Lemma 3.5, we have

\[
\| f(u_{<k}^{(n)}) + \omega_n^k - f(u_{<k}^{(n)}) \|_{ST^*(I)} \to 0,
\]

and

\[
\| f(u_{<k}^{(n)}) - \sum_{j=0}^{k-1} f(u_j^{(n)}) \|_{ST^*(I)} \to 0,
\]

as \( n \to +\infty \). On the other hand, the linear part in \( eq(u_{<k}^{(n)}, \omega_n^k) \) vanishes when \( h_{\infty}^j = 1 \), and is controlled when \( h_{\infty}^j = 0 \) by

\[
\| \langle \nabla \rangle - |\nabla| \| u_j^{(n)} \|_{L^1(I; L^2)} \lesssim \| \langle \nabla \rangle^{-1} u_j^{(n)} \|_{L^\infty(I; L^2)}
\]

\[
\simeq \| \langle \nabla / h_{\infty}^j \rangle^{-1} \tilde{U}^{(j)} \|_{L^\infty(I; L^2)}
\]

\[
\lesssim \| P_{\leq (h_{\infty}^j)^{1/2}} \tilde{U}^{(j)} \|_{L^\infty(I; L^2)} + (h_{\infty}^j)^{1/4} \| P_{> (h_{\infty}^j)^{1/2}} \tilde{U}^{(j)} \|_{L^\infty(I; L^2)}
\]

\[
\to 0, \quad \text{as} \quad n \to \infty.
\]

Thus, \( \| eq(u_{<k}^{(n)}, \omega_n^k) \|_{ST^*(I)} \to 0 \), as \( n \to \infty \).

Therefore, for \( k \) sufficiently close to \( K \) and \( n \) large enough, the true solution \( u_n \) and the approximate solution \( u_{<k}^{(n)} + \omega_n^k \) satisfy all the assumptions of the perturbation Lemma 2.7. Hence we can obtain the desired result.

4. Concentration Compactness

By the profile decomposition in the previous section and the perturbation theory, we argue in this section that if the scattering result does not hold, then there must exist a minimal energy solution with some good compactness properties. This is the object of the following proposition.

**Proposition 4.1.** Suppose that \( E_{\text{max}} < +\infty \). Then there exists a global solution \( u_c \) of (1.1) satisfying

\[
E(u_c) = E_{\text{max}}, \quad \| u_c \|_{ST(\mathbb{R})} = +\infty. \tag{4.1}
\]
Moreover, there exists \( c(t) : \mathbb{R}^+ \to \mathbb{R}^d \), such that \( K = \{(u_c, \dot{u}_c)(t, x - c(t)) \mid t \in \mathbb{R}^+\} \) is precompact in \( H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d) \). Besides, one can assume that \( c(t) \) is \( C^1 \) and satisfies
\[
|\dot{c}(t)| \lesssim_{u_c} 1 \tag{4.2}
\]
uniformly in \( t \).

**Proof.** The proof of [10] can be adapted verbatim, but we give a sketch for the sake of completeness. By the definition of \( E_{\text{max}} \), we can choose a sequence \( \{u_n(t)\} \) such that
\[
E(u_n, \dot{u}_n) \to E_{\text{max}}, \quad \text{and} \quad \|u_n\|_{ST(I_n)} \to \infty, \quad \text{as} \quad n \to \infty. \tag{4.3}
\]
Now we consider the linear and nonlinear profile decompositions of \( u_n \), using Lemma 3.1
\[
e^{it(\nabla)} \bar{u}_n(0) = \sum_{j=0}^{k-1} \bar{v}_n^j + \bar{\omega}_n^k, \quad \bar{v}_n = e^{i(t-t_0)}T_n^J \varphi^j(x),
\]
\[
u_{\beta}^k(n) = \sum_{j=0}^{k-1} \nu_{\beta}^j(n), \quad u_n^j(t, x) = T_n^J \bar{U}_n^j((t-t_0)/h_n^j),
\]
\[
\|\bar{v}_n^j(0) - \bar{v}_n^j(0)\|_{L_x^2} \to 0, \quad \text{as} \quad n \to \infty.
\]

Lemma 3.6 precludes that all the nonlinear profiles \( \bar{U}_n^j \) have finite global Strichartz norm. On the other hand, every solution of (1.1) with energy less than \( E_{\text{max}} \) has global finite Strichartz norm by the definition of \( E_{\text{max}} \). Hence by (4.3), we deduce that there is only one profile, i.e. \( K = 1 \), and so for large \( n \)
\[
\bar{E}(\bar{u}_n^0) = E_{\text{max}}, \quad \|\bar{U}_n^0\|_{ST(\mathbb{R})} = \infty, \quad \lim_{n \to \infty} \|\bar{v}^1_n\|_{L_t^\infty L_x^2} = 0. \tag{4.5}
\]
If \( h_n^0 \to 0 \), then \( \bar{U}_n^0 = \text{Re} |\nabla|^{-1}\bar{U}_n^0 \) solves the \( \dot{H}^1 \)-critical wave-Hartree equation
\[
\partial_t u - \Delta u + (|x|^{-2} * |u|^2)u = 0
\]
and satisfies
\[
E(\bar{U}_n^0(x, u_{\text{max}})) = E_{\text{max}} < +\infty, \quad \|\bar{U}_n^0\|_{L_t^2(\mathbb{R}; B_{\dot{y}_2}^{1/2})} = \infty, \quad q = \frac{2(d+1)}{d-1}.
\]
But Miao-Zhang-Zheng [22] has proven that there is no such solution. Hence \( h_n^0 = 1 \). And so there exist a sequence \( (t_n, x_n) \in \mathbb{R} \times \mathbb{R}^d \) and \( \phi \in L^2(\mathbb{R}^d) \) such that along some subsequence,
\[
\|\bar{u}_n(0, x) - e^{-it_n(\nabla)} \phi(x - x_n)\|_{L^2_x} \to 0, \quad n \to \infty. \tag{4.6}
\]

Now we show that \( \bar{U}_n^0 = \langle \nabla \rangle^{-1} \bar{U}_n^j \) is a global solution. Assume not, then we can choose a sequence \( t_n \in \mathbb{R} \) which approaches the maximal existence time. Since \( \bar{U}_n^0(t + t_n) \) satisfies (4.3), then applying the above argument to it, we obtain by (4.6) that for some \( \psi \in L^2 \) and another sequence \( (t'_n, x'_n) \in \mathbb{R} \times \mathbb{R}^d \) such that
\[
\|T_n^J \bar{U}_n^j(t_n) - e^{-it_n(\nabla)} \psi(x - x'_n)\|_{L^2_x} \to 0, \tag{4.7}
\]
as \( n \to \infty \). Let \( \bar{v} := e^{i(t(\nabla)} \psi \). For any \( \varepsilon > 0 \), there exist \( \delta > 0 \) with \( I = [-\delta, \delta] \) such that
\[
\|\langle \nabla \rangle^{-1} \bar{v}(t - t_n)\|_{ST(I)} \leq \varepsilon,
\]
which together with (4.7) shows that for sufficiently large \( n \)
\[
\left\| (\nabla)^{-1} e^{i t (\nabla)} \tilde{U}_\infty(t_n) \right\|_{ST(t)} \leq \epsilon.
\]
If \( \epsilon \) is small enough, this implies that the solution \( \tilde{U}_\infty \) exists on \([t_n - \delta, t_n + \delta]\) for large \( n \) by the small data theory (Lemma 2.4). This contradicts the choice of \( t_n \).

Thus \( \hat{U}_\infty \) is a global solution and it is just the desired critical element \( u_c \). Moreover, since (1.1) is symmetric in \( t \), we may assume that
\[
\| u_c \|_{ST(0, +\infty)} = +\infty.
\]
(4.8)
We call such \( u \) a forward critical element.

One can refer to [23] for the choice of \( c(t) \). Thus we concludes the proof of Proposition 4.1. \( \square \)

As a consequence of the above proposition and the Hardy-Littlewood-Sobolev inequality, we have

**Corollary 4.2.** Let \( u \) be a forward critical element. And we denote
\[
E_{R,c} = - \int_{|x-c| \geq R} (|u|^2 + |\nabla u|^2 + |\dot{u}|^2) dx + \int_{\mathbb{R}^d} \int_{|x-c| \geq R, y \in \mathbb{R}^d} \frac{|u(t,x)|^2|u(t,y)|^2}{|x-y|^4} dxdy,
\]
then for any \( \eta > 0 \), there exists \( R(\eta) > 0 \) such that
\[
E_{R(\eta),c(t)} \leq \eta E(u, \dot{u}), \text{ for any } t > 0.
\]

The next corollary is the conclusion of this section.

**Corollary 4.3.** Let \( u \) be a nonlinear strong solution of (1.1) such that the set \( K \) defined in Proposition 4.1 is precompact in \( H^1 \times L^2 \), and \( E(u, \dot{u}) \neq 0 \). Then there exists a constant \( \beta = \beta(\tau) > 0 \) such that, for all time \( t > 0 \), there holds that
\[
\int_{t}^{t+\tau} \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|x_2 - y_2|^2}{|x-y|^6} |u(s,x)|^2|u(s,y)|^2 dxdyds \geq \beta,
\]
(4.9)
where \( x_2 \) denotes the second component of \( x \in \mathbb{R}^d \). In particular, there holds that
\[
\int_{0}^{t} \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|x_2 - y_2|^2}{|x-y|^6} |u(t,x)|^2|u(t,y)|^2 dxdyds \geq t.
\]
(4.10)

**Proof.** One can refer to [23] for the detail proof. \( \square \)

5. Extinction of the critical element

In this section, we utilize the technique in [28] to prove that the critical solution constructed in Section 4 does not exist, thus ensuring that \( E_{\max} = +\infty \). This implies Theorem 1.3.

**Proposition 5.1.** Assume that \( d \geq 5 \), then \( E_{\max} = +\infty \).

**Proof.** We use a Virial-type estimate in a direction orthogonal to the momentum vector. Up to relabeling the coordinates, we might assume that \( \text{Mom}(u) \) is parallel to the first coordinate. Thus we have
\[
\int_{\mathbb{R}^d} u(t,x)\partial_j u(t,x) dx = 0, \quad \forall j \geq 2.
\]
(5.1)
Let \( \phi_R(x) = \phi(x/R) \) where \( \phi(x) \) is a nonnegative smooth radial function such that \( \text{supp } \phi \subseteq B(0, 2) \) and \( \phi \equiv 1 \) in \( B(0, 1) \). We define the Virial action
\[
I(t) = \int_{\mathbb{R}^d} z_2 \phi_R(z) \partial_2 u(t,x) u_t(t,x) dx,
\]
where \( z = x - c(t) \) and \( z_2 \) denotes the second component of \( z \in \mathbb{R}^d \). Integrating by parts we get by \( (5.1) \)
\[
\partial_t I(t) = \int_{\mathbb{R}^d} \partial_t(z_2 \phi_R(z)) \partial_2 u(t,x) u_t(t,x) dx + \frac{1}{2} \int_{\mathbb{R}^d} z_2 \phi_R(z) \partial_2(u_t(x,t))^2 dx
\]
\[
+ \int_{\mathbb{R}^d} z_2 \phi_R(z) \partial_2 u(t,x) (\Delta u - u - (V(\cdot) * |u|^2) u) dx
\]
\[
= \frac{1}{2} \int_{\mathbb{R}^d} \left( - |u_t|^2 + |u|^2 + |\nabla u|^2 + (V(\cdot) * |u|^2)|u|^2 \right) dx - \int_{\mathbb{R}^d} |\partial_2 u|^2 dx
\]
\[
+ \frac{z_2}{2} \int_{\mathbb{R}^d} u_t |\partial_2 u| dx - 2 \int_{\mathbb{R}^d} z_2 \phi_R(z) |u|^2 \left( \frac{x_2}{|x|^6} * |u|^2 \right) dx
\]
\[
+ \int_{|z| \geq R} O_1(u) dx,
\]
where
\[
O_1(u) = \frac{1}{2} \left( z_2 \phi_R - (1 - \phi_R(x)) \right) \left[ - |u_t|^2 + |u|^2 + |\nabla u|^2 + (V(\cdot) * |u|^2)|u|^2 \right]
\]
\[
- (c'(t) \cdot \nabla \phi_R) z_2 \frac{\phi_R}{R} \partial_2 u u_t - c_2'(t)(1 - \phi_R(z)) \partial_2 u u_t - (\nabla \phi_R \cdot \nabla u) z_2 \partial_2 u,
\]
is supported on the set \( |z| \geq R \) and satisfies
\[
\left| \int_{|z| \geq R} O_1(u) dx \right| \lesssim \int_{|z| \geq R} (|u|^2 + |\nabla u|^2 + |u_t|^2) dx.
\]
Besides, we define the equi partition of energy action
\[
J(t) = \int_{\mathbb{R}^d} \phi_R(z) u(t,x) u_t(t,x) dx.
\]
Then
\[
\partial_t J(t) = \int_{\mathbb{R}^d} \left( |u_t|^2 - |u|^2 - |\nabla u|^2 - (V(\cdot) * |u|^2)|u|^2 \right) dx + \int_{|z| \geq R} O_2(u) dx,
\]
where
\[
O_2(u) = (1 - \phi_R(z)) \left[ |u_t|^2 - |u|^2 - |\nabla u|^2 - (V(\cdot) * |u|^2)|u|^2 \right] + (c'(t) \cdot \nabla u) \frac{u_t}{R} - \frac{u}{R} \nabla u \cdot \nabla u,
\]
has the same properties as \( O_1(u) \).
Considering \( A(t) = I(t) + \frac{1}{2} J(t) \), we get
\[
|A(t)| \lesssim RE(u, u_t), \quad \text{for all time } t,
\]
and
\[
\partial_t A(t) = - \int_{\mathbb{R}^d} |\partial_2 u|^2 dx - \int_{\mathbb{R}^d \times \mathbb{R}^d} \phi_R(x - c(t)) (x_2 - c_2(t)) \frac{x_2 - y_2}{|x - y|^6} |u(t,x)|^2 |u(t,y)|^2 dx dy
\]
\[
- \int_{|z| \geq R} (O_1(u) + \frac{1}{2} O_2(u)) dx.
\]
And so by symmetrization, \( \partial_t A(t) \) can be rewritten as

\[
-\partial_t A(t) = \int_{\mathbb{R}^d} |\partial_2 u|^2 dx + \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|x_2 - y_2|^2}{|x - y|^6} |u(t, x)|^2 |u(t, y)|^2 dxdy

+ I_2 + \int |z| \geq R (O_1(u) + O_2(u)) dx,
\]

where

\[
I_2 = \int_{\mathbb{R}^d \times \mathbb{R}^d} [(x_2 - c_2(t))\phi_R(x - c(t)) - (y_2 - c_2(t))\phi_R(y - c(t)) - (x_2 - y_2)]

\times \frac{x_2 - y_2}{|x - y|^6} u(t, x)^2 |u(t, y)|^2 dxdy.
\]

We will show that \( I_2 \) constitute only a small fraction of \( E(u, u_t) \). First, by Corollary 4.2, we know that if \( R \) is sufficient large depending on \( u \) and \( \eta \), then

\[
E_{R,c(t)}(u, u_t) \leq \eta E(u, u_t).
\]

Let \( \chi \) denote a smooth cutoff to the region \(|x - c(t)| \geq \frac{R}{2}\) such that \( \nabla \chi \) is bounded by \( R^{-1} \) and supported where \(|x - c(t)| \sim R\). In the region where \(|x - c(t)| \sim |y - c(t)|\), we have

\[
|x - c(t)| \sim |y - c(t)| \geq R,
\]

since otherwise \( I_2 \) vanish. Moreover, note that

\[
|(x_2 - c_2(t))\phi(x - c(t)) - (y_2 - c_2(t))\phi(y - c(t))| \leq |x - y|,
\]

we use the Hardy-Littlewood-Sobolev inequality and Sobolev embedding theorem to control the contribution to \( I_2 \) from this regime by

\[
\int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{\chi u(t, x)^2 |u(t, y)|^2}{|x - y|^4} dxdy \lesssim \|\nabla (\chi u)\|^4_2 \lesssim \eta^2.
\]

In the region where \(|x - c(t)| \ll |y - c(t)|\), we use the fact that

\[
|x - c(t)| \ll |y - c(t)| \sim |x - y| \text{ and } |y - c(t)| \gtrsim R
\]

to estimate the contribution from this regime by

\[
\int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{1}{|x - y|^4} |u(t, y)|^2 |u(t, x)|^2 dxdy \lesssim \|\nabla (\chi u)\|^4_{L^2} \|\nabla u\|^4_{L^2} \lesssim \eta.
\]

The last line follows from the same computation as the first case. Finally, since the remaining region \(|y - c(t)| \ll |x - c(t)|\) can be estimated in the same way, we conclude that

\[
I_2 \lesssim \eta.
\]

Chosen \( \eta \) sufficiently small depending on \( u \) and \( R \) sufficiently large depending on \( u \) and \( \eta \), we obtain

\[
-\partial_t A(t) \geq \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|x_2 - y_2|^2}{|x - y|^6} |u(t, x)|^2 |u(t, y)|^2 dxdy - \eta E(u, u_t).
\]

If \( E_{\max} < \infty \), integrating from 0 to \( T > 0 \) and using Corollary 4.2, we get that there exists \( \alpha = \alpha(1, u) > 0 \) such that

\[
\int_0^T \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|x_2 - y_2|^2}{|x - y|^6} |u(s, x)|^2 |u(s, y)|^2 dxdyds \geq \alpha T,
\]
for all $T > 1$. Thus $-A(t) \gtrsim T$ for large $T$, which contradicts with (5.3). Hence we have $E_{\text{max}} = +\infty$, this concludes the proof of Proposition 5.1. \hfill \Box

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