Edge Tunneling of Vortices in Superconducting Thin Films

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Abstract

We investigate the phenomenon of the decay of a supercurrent due to the zero-temperature quantum tunneling of vortices from the edge in a thin superconducting film in the absence of an external magnetic field. An explicit formula is derived for the tunneling rate of vortices, which are subject to the Magnus force induced by the supercurrent, through the Coulomb-like potential barrier binding them to the film’s edge. Our approach ensues from the non-relativistic version of a Schwinger-type calculation for the decay of the 2D vacuum previously employed for describing vortex-antivortex pair-nucleation in the bulk of the sample. In the dissipation-dominated limit, our explicit edge-tunneling formula yields numerical estimates which are compared with those obtained for bulk-nucleation to show that both mechanisms are possible for the decay of a supercurrent.

I. INTRODUCTION

Recently there has been much renewed interest in the physics of vortices in high-temperature superconductors in the presence of applied magnetic fields [1]. Phenomena involving vortex dynamics and their hindrance by means of added pinning centres pose well-defined physics problems of great relevance for the technological applications of the new high-$T_c$ materials.

Some of the lesser understood issues are related to the problem of the residual resistance due to the thermal or quantum motion of the vortices in the presence of an externally applied supercurrent, but in the absence of the applied magnetic field. Even in the absence of an external field, vortices can be dragged into the bulk under the action of the Lorentz-like Magnus force which they experience sitting at the edge of the sample. This phenomenon, leading to an electrical resistance in the presence of dissipation, has been investigated
recently by relatively few authors \cite{2,3} considering its importance from both the conceptual and the practical points of view. In addition, in a recent article \cite{4} we have pointed out how such a residual resistance may arise, in the absence of an applied field, also due to the spontaneous homogeneous nucleation of vortex-antivortex pairs in the bulk of the sample. These are created as fluctuations of the electromagnetic-like Magnus field acting on the quantised vortices and antivortices thought of as electron-positron-like pairs. By means of this analogy with quantum electrodynamics (QED), we have set up a powerful “relativistic” quantum field theory (QFT) approach to study vortex nucleation in the two-dimensional (2-D) geometry. This study has been conducted in the presence of quantum dissipation and for the cases of either an harmonic local pinning potential \cite{4} or, more recently \cite{5}, for a periodic-lattice distribution of pinning centers. Our analysis has produced an explicit analytic result, and an estimate of the magnitude of the vortex pair-nucleation rate has shown that the effect may become experimentally accessible at low temperatures and high enough current densities.

In this article we investigate, by means of our QFT approach, the original issue of tunneling of quantised vortices from the sample’s boundary as an alternative mechanism for the decay of an applied supercurrent in a thin-film geometry. The problem has already been introduced by Ao and Thouless \cite{2} and by Stephen \cite{3}, with and without the inclusion of quantum dissipation, and discussions that give full weight to the inertial mass of the moving vortex can be found in the literature \cite{4} alongside treatments \cite{3,6} in which the inertial mass is taken to be negligible. The physics of the problem has been captured mainly through semi-classical evaluations of the tunneling rate and discussion has centered on the effects of dissipation and pinning in opposing the stabilising influence of the Magnus force on the classical orbits of a vortex \cite{2,3}. Beside the issue of the residual superconductor’s resistance, the quantum tunneling of the magnetic flux lines into the bulk of the material has been investigated to understand theoretically \cite{6} the observed \cite{8} flux-creep phenomenon in the presence of a field.

In this work we will analyze further the problem of vortex tunneling from the point of view of the instability of the “vacuum” represented by a thin superconducting film in the presence of an externally-driven supercurrent. Our aim is in fact that of obtaining an explicit formula (complete of prefactors) for the tunneling rate. From the theoretical point of view, our approach differs from other semi-classical evaluations of the path integral in that use is made of the Schwinger formalism for pair creation in QFT. Since this fully-relativistic formulation provides for a description of vortex-antivortex pair nucleation, we will be concerned here with the completely non-relativistic limit of this approach, formally corresponding to the case of a negligible vortex inertial mass and to contributions to the path integral relating to quantum particles moving “forward” in the time coordinate. This non-relativistic version of the Schwinger formalism indeed appears as very convenient for carrying out explicit and straightforward calculations, especially in the dissipation-dominated
regime. We in fact work out the tunneling rate in the case when dissipation dominates over inertia, and for the case of a potential barrier made up of the Coulomb-like attraction to the edge and the electric-like potential extracting the vortex into the bulk. Our central result shows that the tunneling rate $\Gamma$ has a strong exponential dependence on the number current density $J$, much as in the case of vortex nucleation in the bulk of the sample. The numerical value of the tunneling rate per unit length, as obtained from our approach with material parameters typical of the high-temperature cuprate superconductors, compares favourably with analogous results obtained for the bulk nucleation rate. This qualifies the novel vortex nucleation mechanism by us proposed in recent articles as almost as likely as vortex tunneling from the sample’s edge for the decay of a supercurrent. In our estimates, edge tunneling appears to be more favourable than bulk nucleation for equal given extension of edge length and surface area.

It is useful to describe the problem at hand by means of the classical equation of motion for a single vortex moving at relatively low velocity in a supercurrent having density $J$

$$m\ddot{q} = -\nabla U(q) + eE - e\dot{q}\times B - \eta\dot{q}$$

Here $m$ is the (negligible) inertial mass of the vortex carrying topological charge $e = \pm 2\pi$ and treated as a single, point-like particle of 2-D coordinate $q(t)$. Also, $U(q)$ is the phenomenological potential acting on the vortex, in the present case the “Coulomb-like” interaction binding it to the sample’s edge. The supercurrent $J$ gives rise to an electric-like field $E = \times J$ (a notation implying $E \cdot J = 0$) superposed to a magnetic-like field $B = \hat{z}d\rho_s^{(3)}$ for a thin film orthogonal to the vector $\hat{z}$ and having thickness $d$ with a superfluid component characterised by a 3-D number density $\rho_s^{(3)}$. Finally, $\eta$ is a phenomenological friction coefficient taking dissipation into account. A derivation of this electromagnetic analogy for the Magnus force was given by us in Ref. (but see also [9–11]). Further contributions to $B$ arising from other quantum effects are possible and have been recently debated, however in the dissipation-dominated regime the actual explicit dependence of $B$ on the material’s parameters will be irrelevant for our study. The quantum-mechanical counterpart of Eq. (1.1) is constructed through the Feynman path-integral transposition in which the dissipation is treated quantistically through the formulation due to Caldeira and Leggett. This approach views quantum dissipation as described by the linear coupling of the vortex coordinate to the coordinates of a bath of harmonic oscillators of prescribed dynamics. By means of this description of quantum dissipation, we formulate our own approach to the dissipative tunneling of vortices subject to the fields $E$ and $B$ (the magnetic-like field being treated in an effective way).

The organisation of our article is as follows. In Section II we sketch some basic facts about our Schwinger method for the relativistic vortex dynamics and show how the completely non-relativistic limit yields a convenient formulation of the vortex tunneling problem. This is expanded in Section III, where
we treat in detail the situation in which dissipation and a Coulomb interaction are present and dominate over vortex inertia. In Section IV we estimate the tunneling rate for a range of reasonable values of the material parameters and compare the results with previous estimates for the bulk nucleation rate. This Section contains also our conclusions. We work in the units system for which $\hbar = 1$.

II. VORTEX QUANTUM DYNAMICS: A UNIFIED APPROACH TO VACUUM DECAY THROUGH NUCLEATION AND TUNNELING

The calculation that follows is the entirely non-relativistic version of our published QFT formulation [4] for the decay, via vortex-antivortex pair nucleation, of the “vacuum” represented by a supercurrent having number density $\mathbf{J}$ flowing in a superconducting thin film. Since the would-be particles should experience a force entirely analogous to the Lorentz force from a uniform electromagnetic field, we expect this vacuum to be unstable and decay via vortex pair creation. We evaluate the probability amplitude for the vacuum decay in time $T$,

$$Z = \langle 0| e^{-i\hat{H}T} |0\rangle \equiv e^{-iT W_0}$$ (2.1)

where $W_0 = \mathcal{E}(\text{vac}) - i\frac{\rho}{2}$ gives the energy of this vacuum, $\mathcal{E}(\text{vac})$, and its decay rate, $\Gamma$. With a suitable normalization factor, the probability amplitude is given by a functional integral over field configurations. In the presence of a gauge field $A_\mu(r, t)$ and external potential $V(r)$, we have

$$Z = \mathcal{N} \int \mathcal{D}\phi \exp \left\{ -i \int d^2 r \ dt \phi^* \left( -D_0^2 + \frac{1}{\gamma} D^2 - \mathcal{E}_0^2 - V(r) \right) \phi \right\}$$

$$= \exp \left\{ -Tr \ln \left( -\frac{1}{\gamma} D^2 - D_3^2 + \mathcal{E}_0^2 + V(r) \right) \right\}$$ (2.2)

which can be evaluated, in the Euclidean metric, by means of the identity

$$Tr \ln \frac{-D_3^2 + \mathcal{E}_0^2 + V(r)}{\Lambda^2}$$

$$= -\lim_{\epsilon \to 0} \int_0^\infty \frac{d\tau}{\tau} Tr \left\{ e^{-(\mathcal{E}_0^2 + V)r} - e^{-\Lambda^2 \tau} \right\}$$ (2.3)

where (with $x_3 \equiv it$) $D_3^2 = D_3^2 + \frac{1}{4}(D_1^2 + D_2^2)$ and where $D_\mu = \partial_\mu - iA_\mu$ is the usual covariant derivative. Notice that $\gamma = m/\mathcal{E}_0$, the ratio between the inertial mass $m$ and the activation energy $\mathcal{E}_0$, plays the role of the inverse square of the velocity of light. Thus the non-relativistic limit is implicitly taken in the dissipation-dominated case, where $m \to 0$.

The evaluation of the trace is straightforward by means of the Feynman path-integral. We have, with a suitable normalization factor $\mathcal{N}(\tau)$

$$Tr \left\{ e^{-(\mathcal{E}_0^2 + V)r} \right\} = \mathcal{N}(\tau) \int dq_0 \int_{q(0)=q(\tau)=q_0} \mathcal{D}q(s) e^{-\int_0^\tau ds \mathcal{L}E}$$ (2.4)
where the Euclidean version of the relativistic Lagrangian reads

$$L_E = \frac{1}{4} \dot{q}^2_t + \frac{\gamma}{4} \dot{q}^2 - i \dot{q}_\mu A_\mu(q) + \xi^2_0 + V(q)$$  \hspace{1cm} (2.5)

Here \( q \) is taken to be a \((d + 1)\)-dimensional coordinate describing the closed trajectories in the Euclidean space-time as a function of the Schwinger proper time \( s \). We denote \( q = (q_t(s), q(s)) \) in terms of its time- and space-like components, respectively, the dot representing the derivative \( \dot{q} = \frac{dq}{ds} \). The part of the trajectory moving backward in time \( q_t \) is therefore interpreted as describing the antivortex. The expression for the vacuum decay rate is therefore, quite generally

$$\frac{\Gamma}{2} = \text{Im} \int_0^\infty d\tau \int dq_0 N(\tau) \int_{q(0)=q_0}^{q(\tau)=q_0} Dq(s)e^{-\int_0^\tau ds L_E}$$  \hspace{1cm} (2.6)

where the integral over \( q_0 \), the initial point in space of the closed paths, runs over the region of particle nucleation. In the case of vortex-antivortex pair creation in the bulk of the thin film, this region is the surface of area \( L^2 \) of the film.

In the problem considered in the present paper, vortices are already present in a narrow strip of width \( a \approx \xi \) (with \( \xi \) the coherence length of the superconducting order parameter) along the edge of the sample where \( \nabla \times J \) is different from zero. Thus the “vacuum” in this case corresponds to a uniform distribution of vortices along the edge, whilst none of them is present in the bulk in the absence of an external magnetic field. The “vacuum decay” thus corresponds to the possibility that some of the vortices are dragged away from the edge and enter the bulk. We can still take Eq. (2.6) as the starting point for determining the decay rate, provided we factor out the contributions coming from all paths describing particles propagating backwards in time \( q_t \) and replace them with a suitable normalization factor. Also, since vortices are already present, a chemical potential \( \mu^* \) will be introduced to cancel out their nucleation energy. We will take the non-relativistic limit explicitly, since now the advantage of the relativistic formulation (which in our case provided for a simultaneous description of both particles and antiparticles) is no longer useful.

We find that this way of approaching the tunneling problem, based on Eq. (2.6), has in our opinion a number of advantages over the standard instanton-type calculation \cite{7,14}. The instanton (or WKB for point-like particles) calculation, although well established, nevertheless calls for the determination of the “bounce” solution of the path integral’s saddle point equation and the functional integration of the fluctuations around it, a task often too hard to carry out explicitly. The present formulation based on an entirely non-relativistic Schwinger approach leads to a promising viable alternative, as we now show. The integration over the time paths \( q_t(s) \) in Eq. (2.6) is carried out by means of a saddle-point approximation which fixes the correct non-relativistic form of the Lagrangian. Our uniform-field situation corresponds
to $A_\mu(q) = \frac{1}{2} F_{\mu\nu} q^\nu$, with the (2+1)-dimensional field tensor given by, after the analytic continuation to Euclidean time

$$F_{\mu\nu} = \begin{pmatrix} 0 & B & iE_x \\ -B & 0 & iE_y \\ -iE_x & -iE_y & 0 \end{pmatrix}$$

(2.7)

In the present treatment, the vortex chemical potential $\mu^*$ is also added to the scalar potential, $A_0 \to A_0 + \mu^*$, so that with a suitable choice the forward-moving vortices can be selected from the backward-moving antivortices. In our previous work [4] we have shown that the main role of the magnetic-like field in the nucleation of vortex-antivortex pairs is to renormalise the friction coefficient (mimicked by the bath of harmonic oscillators included in the potential $V(q)$),

$$_{\eta \to \eta_R = (\eta^2 + B^2)/\eta}$$

(2.8)

(denoted simply by $\eta$ in what follows), as well as the nucleation energy

$$_{\mathcal{E}_0^2 \to \mathcal{E}_{0R}^2 = \mathcal{E}_0^2 + \Delta \mathcal{E}(B)^2}$$

(2.9)

where $\Delta \mathcal{E}(B)$ is a $B$- and $\eta$-dependent renormalization. However, the friction coefficient $\eta$ is a rather uncertain normal-metal parameter and, moreover, we assume that there is an infinite reservoir of vortices at the border of the film, their $\mu^*$ cancelling the value of the effective nucleation energy as well as all its renormalizations. Thus, we may safely neglect the effects of the magnetic-like component of the Magnus force and write the relativistic action in the form

$$S_E(B = 0) = \int_0^\tau ds \left\{ \frac{\gamma}{4} \dot{q}^2 + \frac{1}{4} \dot{q}_t^2 \dot{q}_t (\mathbf{E} \cdot \mathbf{q} + \mu^*) + \mathcal{E}_{0R}^2 + V(q) \right\}$$

(2.10)

The saddle point approximation fixes the function $\dot{q}_t = dq_t/ds$ so as to have an extremum for the action: namely, we have the saddle-point condition

$$\frac{\delta S_E(0)}{\delta \dot{q}_t} = \frac{1}{2} (\dot{q}_t - 2(\mathbf{E} \cdot \mathbf{q} + \mu^*)) = 0$$

(2.11)

from which we obtain

$$\frac{dq_t}{ds} = 2(\mathbf{E} \cdot \mathbf{q} + \mu^*) \to 2\mu^*$$

(2.12)

where we take $\mu^* = \pm \mathcal{E}_{0R}$ to be the dominant energy scale. Here, the positive sign applies to particles moving forward in the time $q_t$ and the negative one to backward-moving antiparticles. The resulting saddle-point action for those trajectories moving forward up to the standard Euclidean time $T = 2\mathcal{E}_{0R}\tau$ can therefore be taken as (indicating for short with $t$ the time component $q_t$)
\[ S_E^+(0) \approx \int_0^T dt \left\{ \frac{1}{2} m \left( \frac{dq}{dt} \right)^2 - \mathbf{E} \cdot \mathbf{q} + U(\mathbf{q}) \right\} \equiv S_{NR} \]  

(2.13)

where the residual contribution in terms of the nucleation energy is cancelled by our choice for the chemical potential and where a term \((\mathbf{E} \cdot \mathbf{q})^2/2\mathcal{E}_{0R}\) has been dropped. Also, we have denoted by \(U(\mathbf{q}) = V(\mathbf{q})/2\mathcal{E}_{0R}\) the non-relativistic potential energy. We now carry out the functional integral over the trajectories moving forward in time, replacing the contribution of the integral over the antidvortex trajectories with some factor \(\Phi\). This factor will also include the contribution arising from the fluctuations around the saddle point. The resulting expression for the tunneling rate through the barrier represented by the overall potential \(U(\mathbf{q}) - \mathbf{E} \cdot \mathbf{q}\) becomes, from Eq. (2.6)

\[
\frac{\Gamma}{2} \approx \text{Im} \int_0^\infty \frac{dT}{T} \Phi \mathcal{N}_{NR}(T) \int_{\mathbf{q}(T)=\mathbf{q}(0)} \mathcal{D}\mathbf{q}(t) e^{-S_{NR}}
\]

(2.14)

This expression contains only closed paths forward-moving in time \(t\) and weighted by the non-relativistic action. The standard normalization factor \(\mathcal{N}_{NR}(T)\) corresponds to the path integral for the non-relativistic free particle in \(D = 2\) dimensions, in the absence of dissipation

\[
\mathcal{N}_{NR}(T) \int_{\mathbf{q}(0)=\mathbf{q}(T)} \mathcal{D}\mathbf{q}(t) e^{-\int_0^T dt \frac{1}{2} m q^2} = \frac{m}{2\pi T}
\]

(2.15)

To fix the overall factor \(\Phi\), we proceed in the following manner. The expression for the non-relativistic vacuum decay amplitude is as follows

\[
\langle 0 | e^{-H_{NR}T} | 0 \rangle = \mathcal{N}_{NR}(T) \int \mathcal{D}\mathbf{q} e^{-S_{NR}}
\]

\[
= \exp \left( - \mathcal{E}(\text{vac}) - i \frac{\Gamma}{2} T \right) e^{-\mathcal{E}(\text{vac})T} \left( \cos \frac{\Gamma T}{2} + i \sin \frac{\Gamma T}{2} \right)
\]

(2.16)

Since we are dealing with a highly-stable vacuum, we take the lowest order term in the expansion in powers of \(\Gamma T\) for \(\Gamma \ll \mathcal{E}(\text{vac})\) to obtain

\[
\mathcal{N}_{NR}(T) \int \mathcal{D}\mathbf{q} e^{-S_{NR}} = i \frac{\Gamma T}{2} e^{-\mathcal{E}(\text{vac})T} + \ldots
\]

(2.17)

which determines, inserted in Eq. (2.14), the following expression for \(\Phi\)

\[
\Phi = \mathcal{E}(\text{vac}) = \left\{ \int_0^\infty dT e^{-\mathcal{E}(\text{vac})T} \right\}^{-1}
\]

(2.18)

In deriving this expression we have taken into account the fact that the calculation for the rate \(\Gamma\) is based on a saddle-point approximation where \(\Phi\) is a function of the material’s parameters only. The exponential \(e^{-\mathcal{E}(\text{vac})T}\) represents the vacuum probability amplitude in the absence of the electric field instability, thus we finally write

\[
\frac{\Gamma}{2} = \text{Im} \frac{\int_0^\infty dT \mathcal{N}_{NR}(T) \int \mathcal{D}\mathbf{q} e^{-S_{NR}}}{\int_0^\infty dT \mathcal{N}_{NR}(T) \int \mathcal{D}\mathbf{q} e^{-S_{NR}^0}}
\]

(2.19)
where the action $S_{NR}^{(0)}$ is the non-relativistic action in the absence of the electric field. We have in this way reached a formulation for the tunneling rate which leads to an expression in agreement with the standard WKB method. This can be checked by examples insofar as the exponential term $\beta$ is concerned, in the characteristic expression for the tunneling rate $\Gamma = \alpha e^{-\beta}$.

### III. EVALUATION OF THE TUNNELING RATE THROUGH THE EDGE POTENTIAL.

In this Section we will apply the above general formalism to the situation at hand, where the vortices tunnel from the edge strip under the combined effect of the “electric potential” $-E \cdot \mathbf{q}$ and the 2-D “Coulomb electrostatic potential”. The latter is due to the attraction of the vortex by the edge, which is equivalent to the attraction by a virtual antivortex, like in the familiar virtual-charge method in standard electrostatics. This Coulomb potential takes one of the equivalent forms $U_C(y) = K \ln(1+y/a)$ or $U_C(y) = \frac{1}{\pi} K \ln \left(1 + \left(\frac{y}{a}\right)^2\right)$, where $y$ is the distance from the edge, both acceptable extrapolations of the known large-distance behavior. The coupling constant $K$ depends on the superfluid density $\rho_s^{(3)}$ and carrier’s mass $m_0$ through the relationship $K = 2\pi \rho_s^{(3)} d/m_0$. However, its precise value in real materials is unknown and this parameter will be treated, like many others in this calculation, phenomenologically.

When the Caldeira-Leggett quantum dissipation is taken into account, we end up with an overall non-relativistic potential

$$U(\mathbf{q}) = \frac{1}{2\varepsilon_0 R} V(\mathbf{q}) = U_C(y) + U_D(\mathbf{q})$$

$$U_D(\mathbf{q}) = \sum_k \left\{ \frac{1}{2} m_k x_k^2 + \frac{1}{2} m_k \omega_k^2 \left(x_k + \frac{c_k}{2m_k \omega_k^2}\right)^2 \right\}$$

with the oscillators’ masses $m_k$ and frequencies $\omega_k$ constrained in such a way that the classical equation of motion for the resulting action reproduces the form (1.1). This requires the $c_k$ to satisfy the constraint (for the so-called ohmic case, which we consider)

$$\pi \sum_k \frac{c_k^2}{m_k \omega_k} \delta(\omega - \omega_k) \equiv J(\omega) = \eta \omega$$

$\eta$ being the phenomenological friction coefficient of Eq. (1.2) (renormalised by $B$). Notice that this coefficient is unaffected by the non-relativistic limit.

After a Fourier transformation in which the $x_k$ modes can be integrated out, we are lead to an effective non-relativistic action which in the dissipation-dominated ($m \to 0$) regime reads, with $\mathbf{q} = (x,y)$

$$S_{NR} = 2\pi \eta \sum_{n>0} n \mathbf{q}_n \cdot \dot{\mathbf{q}}_n^* - ET \dot{y} + K \int_0^T dt \ln \left(1 + \frac{y(t)}{a}\right)$$

(3.3)
Here \( q(t) = \tilde{q} + \sum_{n<0} a_n e^{i\omega_n t} \), where \( \omega_n = 2\pi n/T \), and furthermore \( \bar{y} = \int_0^T dt y(t)/T \) stands for the \( n=0 \) mode in the \( y \)-direction. In order to render the problem tractable, at this point, we introduce a further approximation for the interaction term of the action in Eq. (3.3), by writing

\[
\int_0^T dt \ln \left( 1 + \frac{y}{a} \right) \approx T \ln \left( 1 + \frac{\bar{y}}{a} \right)
\]

which is justified in the limit of large exit times \( T \) (see Appendix). We are therefore in a position to evaluate the numerator of the formula, Eq. (2.19), for the tunneling rate, which we write schematically as \( \Gamma/2 = N/D \).

We define, for the numerator

\[
N = \text{Im} \int_0^\infty \frac{dT}{T} N_{NR}(T) \int Dq e^{-S_{NR}} = \text{Im} \int_0^\infty \frac{dT}{T} N_{NR}(T) I_x I_y
\]

where the path integral over \( q(t) \) factorises into \( I_x = L \prod_{n=1}^\infty (1/2\pi \eta n) \), representing the contribution of the free dissipative motion along the \( x \)-axis of the edge having length \( L \), and in

\[
I_y = \int d\bar{y} I_G(\bar{y}, T) e^{ET\bar{y} - K T \ln(1+\bar{y}/a)}
\]

the factor containing the dynamical effects, which we now evaluate. This we do most conveniently by integrating out first the real and imaginary parts, \( \text{Re} \ y_n = \psi_n \) and \( \text{Im} \ y_n = \xi_n \), of the modes \( y_n \) for \( n > 0 \). Since the path integral is restricted to loops which have their origin at \( y(0) = y(T) \equiv y_0 = 0 \), the constraint \( y_0 = \bar{y} + \sum_{n<0} \psi_n = \bar{y} + 2 \sum_{n>0} \psi_n = 0 \) has to be imposed, corresponding to the vacuum wave-function \( |\Psi_0(y_0)|^2 \sim \delta(y_0) \). This leads to the constrained Gaussian integrals

\[
I_G(\bar{y}, T) = \int \prod_{n=1}^\infty d\xi_n d\psi_n \delta \left( \bar{y} + 2 \sum_{n=1}^\infty \psi_n \right) e^{-2\pi \eta \sum_{n=1}^\infty n(\xi_n^2 + \psi_n^2) + \frac{1}{2} e^{-\frac{1}{2} \pi \eta \bar{y}^2} \times \\
\times \prod_{n=1}^\infty \left( \frac{1}{2\pi n} \right)^{1/2} \int \prod_{n=2}^\infty d\psi_n \exp \left\{ -2\pi \eta \sum_{n,m=2}^\infty \psi_n(n\delta_{nm} + 1)\psi_m - 2\pi \eta \bar{y} \sum_{n=2}^\infty \psi_n \right\}
\]

(3.7)

where the \( \delta \)-function constraint has been taken care of through, e.g., the \( \psi_1 \)-integration. Introducing the matrix \( (n, m = 2, 3, 4 \ldots) \)

\[
M_{nm} = 2\pi \eta (n\delta_{nm} + 1)
\]

(3.8)

the integrals can be evaluated, formally, to give

\[
I_G(\bar{y}, T) = \frac{1}{2\sqrt{\pi}} \prod_{n=1}^\infty \left( \frac{\pi}{2\eta n} \right)^{1/2} \det M^{-1/2} e^{-\frac{1}{2} \pi \eta \bar{y}^2 + \pi \eta^2 \bar{y}^2 \sum_{n,m=1}^\infty M_{nm}^{-1}}
\]

(3.9)

Notice that the resulting expression involves infinite products and sums that would lead to divergencies. However, since the dissipation is believed to be suppressed above a characteristic material-dependent frequency \( \omega_c \), these
products and sums are to be cut at a characteristic integer $n^* = [\omega_c T/2\pi]$.

Writing $M = 2\pi \eta M_0 (1 + M_0^{-1} V)$, with $M_{0nm} = n\delta_{nm}$ and $V_{nm} = 1$, we get

$$\text{det} \, M = \prod_{n=2}^{n^*} (2\pi \eta n) \exp(Tr \ln(1 + M_0^{-1} V))$$

(3.10)

The trace can be evaluated now by formally expanding $\ln(1 + M_0^{-1} V)$, to get

$$\text{det}(1 + M_0^{-1} V) = 1 + \sum_{n=2}^{n^*} \frac{1}{n} \equiv Q$$

(3.11)

In a similar way, we evaluate $M_{nm}^{-1} = \frac{1}{2\pi \eta} \left( \frac{1}{n} \delta_{nm} - \frac{1}{Q nm} \right)$, yielding

$$\sum_{n,m=2}^{n^*} M_{nm}^{-1} = \frac{1}{2\pi \eta} (1 - \frac{1}{Q})$$

(3.12)

from which we obtain

$$I_G(\bar{y}, T) = \frac{1}{2} \left( \frac{2\eta}{Q} \right)^{1/2} \prod_{n>0} \left( \frac{1}{2\eta n} \right) e^{-\frac{\pi n}{2Q} \bar{y}^2}$$

(3.13)

The numerator of our formula for the tunneling rate $\Gamma$ is therefore the double integral

$$\mathcal{N} = \text{Im} \frac{1}{2} L \left( \frac{2\eta}{Q} \right)^{1/2} \int_0^{\infty} \frac{dT}{T} \mathcal{N}_{NR}(T) \prod_{n>0} \left( \frac{1}{2\eta n} \right)^2 \int_0^{\infty} d\bar{y} e^{-\frac{\pi n}{2Q} \bar{y}^2 + ET \bar{y} - KT \ln(1 + \bar{y}/a)}$$

(3.14)

We remark that in the above formula the following formal expression appears:

$$\hat{N} \equiv \mathcal{N}_{NR}(T) \prod_{n>0} \left( \frac{\frac{1}{2\eta n}}{\eta m} \right)^2$$. We interpret this expression as an $m \to 0$ limit:

$$\hat{N} = \mathcal{N}_{NR}(T) \prod_{n>0} \left( \frac{\pi T^{-1}}{m \omega_n^2 + \eta \omega_n} \right)^2 = \frac{m}{2\pi T} \prod_{n=1}^{n^*} \left( 1 + \frac{\eta T}{2\pi m} \frac{1}{n} \right)^2$$

$$\to \frac{\eta}{2\pi} \exp(-T \Delta \mathcal{E})$$

(3.15)

where $\Delta \mathcal{E} = \frac{\omega_c}{\pi m} (1 + \ln(m \omega_c/\eta))$ is a further renormalization of the activation energy which is also cancelled by tuning the vortex chemical potential $\mu^*$, so that, effectively, $\hat{N} \to \frac{\eta}{2\pi}$.

The integral is now evaluated by means of the steepest descent approximation in both $\bar{y}$ and $T$; the saddle-point equations are, with

$$S = \frac{\pi \eta}{2Q} \bar{y}^2 - ET \bar{y} + KT \ln(1 + \frac{\bar{y}}{a})$$

(3.16)

as the reduced action,

$$\frac{\partial S}{\partial T} = -E \bar{y} + K \ln(1 + \frac{\bar{y}}{a}) = 0$$

$$\frac{\partial S}{\partial \bar{y}} = \frac{\pi \eta}{Q} \bar{y} - ET + KT \frac{1}{a + \bar{y}} = 0$$

(3.17)
the first of which yields $\bar{y}$ and the second $\bar{T}$, the mean exit distance and time, respectively (since $n^*$ is a large cutoff integer, $Q$ is kept fixed in the variation of the effective action). This leads to the expression $S_0 = \pi \eta \bar{y}^2 / 2Q$ for the saddle-point effective action. One has, furthermore, to integrate over the Gaussian fluctuations around $\bar{y}$ and $\bar{T}$. One can easily verify that there is a negative mode, thus giving an imaginary contribution, coming from the integration over the fluctuations in $T$. We in fact expand

$$S = S_0 + \alpha \delta y^2 + \beta \delta y \delta T = S_0 + \alpha (\delta y + \frac{\beta}{2\alpha} \delta T)^2 - \frac{\beta^2}{4\alpha} \delta T^2$$  \hspace{1cm} (3.18)$$

with $\alpha = \frac{1}{2} (\frac{\pi \eta}{Q} - \frac{K \bar{T}}{(a + \bar{y})^2})$ and $\beta = -E + \frac{K}{a + \bar{y}}$. This leads to the following evaluation of the numerator of Eq. (2.19)

$$N = \text{Im} \int_0^\infty dT N_{NR}(T) \int \mathcal{D}q e^{-S_{NR}} = \eta L \sqrt{\frac{\eta}{2Q \bar{T}}} \frac{e^{-S_0}}{(E - K/(a + \bar{y}))}$$  \hspace{1cm} (3.19)$$

where $Q = \ln[\omega_c \bar{T} / 2\pi]$ for large enough $\bar{T}$.

We now come to the evaluation of the normalization denominator in Eq. (2.19). We remark that, the numerator being already of the form $\alpha e^{-\beta}$, this denominator contributes only to the specification of the prefactor $\alpha$ in the expression for the tunneling rate $\Gamma$. To begin with, we carry out the Gaussian integrals over $\xi_n$ and $\psi_n$. Here we use the alternative form of the Coulomb interaction, $U_C = \frac{1}{2} K \ln(1 + (y(t)/a)^2)$, and we expand the logarithm around the minimum (now at $y = 0$) to lowest order (thus keeping all the integrals Gaussian). We define

$$D = \int_0^\infty dT N_{NR}(T) \int \mathcal{D}q e^{-S_{NR}^{(0)}} = \int_0^\infty dT N_{NR}(T) I_x I_y^{(0)}$$  \hspace{1cm} (3.20)$$

where $I_x$ is the same as for the numerator above, while

$$I_y^{(0)} = \prod_{n=1}^{\infty} d\xi_n d\psi_n \int d\bar{y} \delta(\bar{y} + 2 \sum_{n=1}^{\infty} \psi_n) \times$$

$$\times \exp \left\{ -2 \pi \eta \sum_{n=1}^{\infty} n (\xi_n^2 + \psi_n^2) - \frac{K T}{2a^2} [2 \sum_{n=1}^{\infty} (\xi_n^2 + \psi_n^2) + \bar{y}^2] \right\}$$  \hspace{1cm} (3.21)$$

Finally, repeating many of the steps of the calculation done for the Gaussian integrals of the numerator $N$, we end up with

$$D = \frac{a^2 \eta^2}{K} L \int_0^{\infty} du \left( 1 + 2aQ(u) \right)^{-1/2} \prod_{n=1}^{n^*} \left( 1 + u/n \right)^{-1} \equiv \frac{a^2 \eta^2}{K} L \varphi(\frac{\omega_c}{K} a^2 \eta)$$  \hspace{1cm} (3.22)$$

where $u = K T / 2a^2 \eta$, $Q(u) = \sum_{n=1}^{n^*} 1/(n + u)$ and $\varphi$ is a dimensionless function of order unity. The evaluation of the tunneling rate is now formally
completed, and an explicit analytic formula ensues from our (albeit approximate) treatment leading, from Eqs. (3.19) and (3.22), to the tunneling rate per vortex:

\[ \frac{\Gamma}{2} = \frac{N}{D} = \frac{K}{\varphi a^2 \sqrt{2Q\eta}} \frac{e^{-S_0}}{T (E - K/(a + \bar{y}))} \]  

(3.23)

This expression is readily evaluated, once the parameters \(a, \eta, \omega_c, K\) and \(J\) are fixed, by evaluating numerically the solutions of Eq. (3.17) and the integral in Eq. (3.22) by means of \(n^* = \frac{\omega_c T}{2\pi} = \frac{(\omega_c a^2\eta/K)u}{[(\text{integer part)}]}\). To obtain the tunneling rate per unit length, we must compute the density of vortices along the edge. Since \(\rho_v = \frac{1}{2\pi} \nabla \times \nabla \theta\) is the bulk density in terms of the condensate phase \(\theta\), and since the number current is \(J = \frac{\rho_v}{m_0} \int dy \) \((\nabla \theta - A)\), we obtain

\[ \int dy \rho_v = \frac{m_0}{2\pi \rho_s^{(3)}} \int dy \partial_y J_x = J/K \]  

(3.24)

The final expression for \(R\), the tunneling rate per unit length, is thus

\[ R = \frac{J}{K} \Gamma \]  

(3.25)

We conclude this Section by noticing that, since \(E = 2\pi J\) in terms of the supercurrent density, and since the saddle-point equation (3.17) has iterative solution \(\bar{y} = \frac{E}{K} \ln(1+\bar{y}/a) = \frac{K}{2\pi} (\ln(1+K/2\pi Ja) + \cdots)\), we can cast the dominant exponential factor determining \(\Gamma\) in the form

\[ S_0 = \frac{\eta \{ K \ln(1+K/2\pi Ja) + \cdots \}^2}{8\pi J^2 Q} \]  

(3.26)

This yields almost the same leading dependence (logarithmic corrections apart) \(\Gamma \approx e^{-(J_0/J)^2}\) on the external current that was also obtained for the homogeneous bulk-nucleation phenomenon \([4]\) (in terms of the tunneling length \(\ell_T \sim K/E\) we could also rewrite \(\Gamma \sim e^{-\eta \ell_T^2}\)). Indeed, the effective Coulomb energy \(K \ln(1+K/2\pi Ja)\) can be interpreted as playing the role of the activation energy \(\mathcal{E}_0\) renormalised by the vortex-antivortex interactions.

**IV. DISCUSSION AND ESTIMATE OF THE TUNNELING RATE.**

We have presented a new treatment for the quantum tunneling of vortices from the boundary of a thin superconducting film. The formulation makes use of the idea that tunneling can be viewed as a pair-creation process in which only forward-moving time trajectories contribute to the Schwinger path integral. Backward-moving antivortices represent contributions to the path integral that are factorised out, and we have made use of the fact that an infinite reservoir of vortices is present at the edge. This leads to a mathematically convenient formulation of the evaluation of the tunneling rate, which
we have worked out in detail by assuming that the vortices experience an attractive 2D Coulomb-like potential confining them to the edge. By means of some approximate treatment of the logarithmic Coulomb interaction, and of the saddle-point evaluation of the path integral, we have reduced all integrations to Gaussian integrals which afford a closed analytical expression for the tunneling rate. This can then be estimated by choosing material parameters suitable for typical cuprate superconducting systems, e.g. YBCO films, that afford some of the highest critical current densities ($J_c \approx 10^7 \text{ A cm}^{-2}$ at 77 K [15]). To fix the indicative value of the friction coefficient $\eta$, we make use of the Bardeen-Stephen formula [16] linking the upper critical field $B_{c2}$ to the normal metal resistivity $\rho_n$: $\eta = \Phi_0 B_{c2}/\rho_n$ ($\Phi_0$ being the flux quantum). This yields $\eta \approx 10^{-2} \text{ A}^{-2}$ for YBCO. A rather uncertain parameter is the cutoff frequency $\omega_c$, although the dependence on it of the tunneling rate is rather weak. We therefore take indicatively $\omega_c \approx 80$ K, just under the expected value of the Kosterlitz-Thouless transition temperature in these films [1]. As for the edge thickness, we take $a \approx \xi \approx 10\AA$ of the order of magnitude of the coherence length. Finally, there is the value of the Coulomb interaction strength, $K$. Since we must have $K < \omega_c$, we present in Fig. 1 the dependence of the tunneling rate on the current density for some indicative values of $K$. The effect of temperature is taken into account phenomenologically [4], by introducing thermal current fluctuations in the exponential of the formula for the rate, $J^2 \rightarrow J^2 + \Delta J(T)^2$. The fluctuation term $\Delta J(T)$ is fixed by requiring that for $J = 0$ an Arrhenius form $\Gamma \propto e^{-U_{max}/T}$ is obtained for the rate, with the barrier’s height $U_{max} = K \ln(K/2\pi Ja) - K + 2\pi Ja$ as the activation energy. Notice that the barrier is infinite for $J \rightarrow 0$, so that there is no zero-current tunneling of vortices even for non-zero temperature. It can be seen from Fig. 1 that the effect of thermal fluctuations, as estimated from the above interpolative assumption, is indicatively to raise the tunneling rate by some orders of magnitude.

The results are summarised in Fig. 1, where it is shown that tunneling of vortices from the film’s boundary can become a highly likely phenomenon for sufficiently large current densities $J$ and relatively low Coulomb couplings $K$. The process is aided by the presence of thermal current fluctuations. For even larger currents than those considered here, the process could be described by means of a classical treatment based on the time-dependent Ginzburg-Landau equation [17]. It is interesting to compare the results obtained in Fig. 1 with the estimates carried out for the bulk nucleation process [4]. There, the maximum nucleation rates observed (also taking temperature into account) were in the region of $\approx 10^{13} \mu \text{m}^{-2}\text{s}^{-1}$ for a current $J = 10^7$ A cm$^{-2}$ and assuming that pinning centers act as nucleation seeds. We therefore conclude that edge tunneling appears to be a more likely supercurrent decay mechanism for normal superconducting sample geometries. Yet, the bulk-nucleation of vortex-antivortex pairs is also a competitive parallel mechanism that could be singled-out by means of an appropriate choice of sample geometry.

Although the measurements of the residual resistance in a superconductor
may not allow to distinguish between these two competing mechanisms, we
stress that from the point of view of vortex-tunneling microscopy the two pro-
cesses remain entirely separate observable phenomena. Bulk pair-nucleation
remains, in particular, a completely open challenge for observation of a phe-
nomenon that even in the context of standard QED has remained, to our
knowledge, so far elusive for static fields.

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APPENDIX

In this Appendix, we justify the use of the approximation, Eq. (3.4), used
in our calculation for handling the “Coulomb” interaction in the path integral
of the numerator of our formula for the nucleation rate. For large $\bar{y}$, and with
$\omega_n = 2\pi n/T$:

$$\int_0^T dt \ln \left(1 + \frac{y(t)}{a}\right) \approx T \ln \left(1 + \frac{\bar{y}}{a}\right) + \sum_{l=1}^{\infty} \frac{(-1)^{l+1}}{l} \int_0^T dt \left(\sum_{n \neq 0} \frac{y_n e^{i\omega_n t}}{\bar{y}}\right)^l$$

(A.1)

At the saddle point, $y_n = -\bar{y}/2nQ$ (as can be easily verified by varying Eq.
(3.3)) and therefore we have, for example

$$\int_0^T dt \left(\sum_{n \neq 0} \frac{y_n e^{i\omega_n t}}{\bar{y}}\right)^2 = \frac{T}{4Q^2} \sum_{n=1}^{\infty} \frac{1}{n^2}$$

(A.2)

Since $Q = \ln[\omega c T/\pi]$ and $T \sim \frac{\hbar K}{E^2}$, we get the estimate

$$\int_0^T dt \ln \left(1 + \frac{y(t)}{a}\right) \approx T \left[\ln \left(1 + \frac{\bar{y}}{a}\right) + O \left(\frac{\eta \omega c K}{E^2}\right)^{-2}\right]$$

(A.3)

Thus the approximation of taking the “Coulomb” potential of the form
$K \ln(\bar{y}/a)$ is justified for very low currents ($E = 2\pi J \to 0$). We still keep this
form, when the current is not so low, as we believe it captures the essential
point of the dynamical picture.
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FIGURE CAPTIONS

Figure 1. Plot of the tunneling rate per unit length, $R$, as function of the supercurrent density, $J$, for the values $K = 30$ K and $K = 50$ K. Dashed lines, $T = 0$ K; full lines, $T = 2.5$ K.
