A CHARACTERIZATION OF WHITNEY FORMS

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Abstract. We give a characterization of Whitney forms on an $n$-simplex $\sigma$ and prove that for every real valued simplicial $k$-cochain $c$ on $\sigma$, the form $Wc$ is the unique differential $k$-form $\varphi$ on $\sigma$ with affine coefficients that pulls back to a constant form of degree $k$ on every $k$-face $\tau$ of $\sigma$ and satisfies $\int_\tau \varphi = \langle c, \tau \rangle$.

1. Introduction

Whitney forms have been extraordinarily useful in several areas of mathematics: algebraic topology [8], [6]; global analysis and spectral geometry [4], [3]; numerical electromagnetism [1], [2]; vibrations of thin plates [7]. Their definition in Whitney’s book [9, p. 140] appears somewhat mysterious. Attempts to gain a better insight into the definition have continued up to now. For example, the recent paper of Lohi and Kettunen [5] contains three different equivalent definitions. In this note we give a conceptual, easily stated characterization of Whitney forms.

On a triangulated differentiable manifold $M$ of $n$ dimensions with a triangulation $h : K \to M$, cf. [9, p. 124], the Whitney form $Wc$ corresponding to the cochain $C^k(K)$ is a family $\omega_\sigma$ of smooth $k$-forms, satisfying certain compatibility conditions, on each closed $n$-simplex $\sigma$. Namely, if $\tau$ is a common face of two top dimensional faces $\sigma_1$ and $\sigma_2$, than the pull-backs to $\tau$ of $\omega_{\sigma_1}$ and $\omega_{\sigma_2}$ coincide. Thus to describe the Whitney form $Wc$ it suffices to give a description of $Wc|_\sigma = \omega_\sigma$ for every simplex $\sigma$ of top dimension. Note that the homeomorphism $h$ defines an affine structure on $\sigma$ and the induced affine structures on common faces of two $n$-simplexes agree. Thus the concept of an affine function on a simplex is well-defined and so is a notion of a “constant” form of degree $k$ on a $k$-simplex.

From now on we work on a fixed $n$-simplex $\sigma$. Our characterization of $Wc$ is stated precisely in the Theorem below. It asserts that $Wc$ restricted to $\sigma$ is the unique $k$-form on $\sigma$ with affine coefficients and constant pull-backs to $k$-faces whose integrals over $k$-faces $\tau$ are prescribed by the values $\langle c, \tau \rangle$ of $c$ on $\tau$.

2. Proof of the Theorem

A simplex $\tau = [p_0, p_1, \ldots, p_k]$ of $k$ dimensions is a convex hull of $k + 1$ points in general position in $\mathbb{R}^n$. In particular, every simplex is closed. We will consider a fixed $n$-simplex $\sigma$ together with all its $k$-faces $\tau$ with $0 \leq k \leq n$. Thus a point $q \in \sigma$
is a convex linear combination

\[ q = m_0p_0 + m_1p_1 + \ldots + m_np_n \]

\[ m_i \geq 0 \quad \text{for} \quad i = 0, 1, \ldots, n \]

\[ m_0 + m_1 + \ldots + m_n = 1 \]

and the barycentric coordinate functions \( v_i(q) \) are defined by

\[ v_i(q) = m_i. \]

We observe that, if \( q = (x^1, x^2, \ldots, x^d) \) the barycentric coordinates are affine functions of \( x^1, x^2, \ldots, x^d \) i.e. are of the form \( a_1x^1 + a_2x^2 + \ldots + a_nx^n + b \). We regard all simplices as oriented with the orientation determined by the order of vertices with the usual convention that \( -\tau \) is \( \tau \) with the opposite orientation and that under a permutation of vertices the orientation changes by the sign of the permutation. A cochain \( c \) of degree \( k \) is then defined as a formal linear combination with real coefficients the \( \tau \)-faces of \( k \)-faces \( \tau \) of \( \sigma \) and we denote by \( C^k(\sigma) = C^k \) the space of all such cochains. If \( c = \sum \tau a_\tau \tau \) we will write \( a_\tau = \langle c, \tau \rangle \). Finally, we will denote by \( \Lambda^k(\sigma) = \Lambda^k \) the space of all smooth exterior differential forms of degree \( k \) on the simplex \( \sigma \). With this notation, one defines the Whitney mapping

\[ W : C^k \to \Lambda^k \]

for all \( k = 0, 1, \ldots, n \), cf. [9] or [3] for a detailed discussion. We will call forms in the image of \( W \) the Whitney forms. It follows immediately from the definition that the Whitney forms when expressed in terms of the coordinates of \( \mathbb{R}^n \) have affine coefficients. We abuse the language and say that a form \( \eta \in \Lambda^k(\tau) \) is constant if it is a constant multiple of the Euclidean volume element on \( \tau \). After these preliminaries we state our theorem.

**Theorem.** Let \( \sigma \) be a simplex of \( n \) dimensions and \( c \) a cochain of degree \( k \) on \( \sigma \). We is the unique \( k \)-form \( \omega \) on \( \sigma \) satisfying the following conditions.

1. \( \omega \) has affine coefficients.
2. The pull-back \( \iota_\tau \omega \) is constant for every \( k \)-dimensional face \( \tau \) of \( \sigma \), where \( \iota_\tau : \tau \hookrightarrow \sigma \) denotes the inclusion map.
3. \( \int_\tau \omega = \langle c, \tau \rangle \) for every \( k \)-face \( \tau \) of \( \sigma \).

**Proof.** We first observe that without any loss of generality we can assume that \( \sigma \) is the standard simplex in \( \mathbb{R}^n \) i.e. is given by

\[ \sigma = \left\{ (x^1, x^2, \ldots, x^n) \in \mathbb{R}^n \mid x^i \geq 0 \quad \text{for} \quad i = 1, 2, \ldots, n; \quad \sum_{i=0}^n x^i \leq 1 \right\}. \]

Thus \( \sigma = [0, e_1, e_2, \ldots, e_n] \) where \( e_i \) is the point on the \( i \)-th coordinate axis with \( x^i = 1 \). The barycentric coordinate functions restricted to \( \sigma \) are then given by

\[ v_0 = 1 - (x^1 + x^2 + \ldots + x^n) \quad \text{and} \quad v_i = x^i \quad \text{for} \quad i = 1, 2, \ldots, n. \]

We first do a quick dimension count that makes the theorem plausible. The dimension of the space of \( k \)-forms with affine coefficients on \( \sigma \) is \( \binom{n}{k}(n+1) \). Requiring that \( \iota_\tau \omega \) is constant on a \( k \)-simplex \( \tau \) imposes \( k \) conditions and the number of \( k \)-faces of an \( n \)-simplex is \( \binom{n+1}{k+1} \). Thus, the dimension of the space of \( k \)-forms

satisfying (1) and (2) above ought to be
\[
\binom{n}{k}(n + 1) - \binom{n + 1}{k + 1}k = \binom{n + 1}{k + 1}.
\]
This last integer is the number of \(k\)-faces of \(\sigma\), i.e. the dimension of the space \(C^k(\sigma)\) of \(k\)-cochains.

It is instructive to consider the simplest cases \(k = 0\) and \(k = n\) of the theorem. A 0-cochain is a sum \(c = \sum a_i p_i\) and
\[
Wc = a_0 v_0 + a_1 v_1 + \ldots + a_n v_n
\]
\[
= a_0 \left(1 - \sum_{i=1}^{n} x_i^j \right) + \sum_{i=1}^{n} a_i x_i^j
\]
\[
= a_0 + \sum_{i=1}^{n} (a_i - a_0)x_i^j
\]
is the unique affine function \(f\) taking prescribed values \(f(p_i) = \int_{p_i} f = \langle c, p_i \rangle\), where the integration of a form of degree 0 over a vertex is just the evaluation.

If \(k = n\), \(\sigma\) is the only face of dimension \(n\) so every cochain is a multiple of \(\sigma^*\). For \(c = \sigma^*\), we have
\[
Wc = W\sigma^*
\]
\[
= \left(n! \sum_{j=0}^{n} (-1)^j v_j \wedge \ldots \wedge \hat{v_j} \wedge \ldots \wedge v_n \right)
\]
\[
= n!dx^1 \wedge \ldots \wedge dx^n
\]
where we used the explicit expressions of the barycentric coordinates (1) in terms of the coordinates \(x^1, \ldots, x^n\) and the hat over a factor means that the factor is omitted. Since the volume of the standard \(n\)-simplex in \(\mathbb{R}^n\) is equal to \(1/n!\), \(\int_\sigma W(\sigma^*) = \langle \sigma^*, \sigma \rangle = 1\), \(W\sigma^*\) is the unique constant form with prescribed integral equal to one.

We now consider the case when \(1 \leq k \leq n - 1\). We will write \(\Lambda^k_c\) for the space of \(k\)-forms on \(\sigma\) with affine coefficients and with constant pull-backs to \(k\)-faces of \(\sigma\). It is obvious from the definition of \(Wc\) and from (1) that \(Wc\) has affine coefficients on \(\sigma\) for every \(c \in C^k(\sigma)\). Similarly, since \(i^*_c W(c)\) is a form of maximal degree on \(\tau\), the calculation above, with \(k\) replacing \(n\), shows that \(i^*_c W(c)\) is constant on \(\tau\) for every \(k\)-face \(\tau\) of \(\sigma\). It follows that \(WC^k \subset \Lambda^k_c\). Now let \(\varphi \in \Lambda^k_c\). We use the restriction of the de Rham map \(R: \Lambda^k(\sigma) \rightarrow C^k(\sigma)\),
\[
\langle R\varphi, \tau \rangle = \int_\tau \varphi,
\]
to \(\Lambda^k_c\) and consider the difference \(\eta = \varphi - WR\varphi\). Clearly, \(\eta \in \Lambda^k_c\). Moreover basic properties of the Whitney mapping (cf. [9, 3]) imply that \(R\eta = R\varphi - WR\varphi = R\varphi - R\varphi = 0\), i.e. \(\eta\) integrates to zero on every \(k\)-face of \(\sigma\). Since the pull-back \(i^*\eta\) is constant on every such face \(\tau\), \(i^*\eta\) vanishes identically on every \(k\)-face \(\tau\). Thus to show that \(\varphi = WR\varphi\) (which would prove our theorem) it suffices to show that every form \(\eta \in \Lambda^k_c\), whose pull-backs to all \(k\)-faces vanish, is itself identically zero on \(\sigma\). Let \(\eta\) be such a form. We express it in the standard coordinates of \(\mathbb{R}^n\) as follows.
\[
\eta = \sum_I (b_I + a_{I,1}x^1 + \ldots + a_{I,n}x^n)dx^I
\]
Here $I$ is a multi-index $I = (i_1 < i_2 < \ldots < i_k)$, $1 \leq i_j \leq n$ for every $j$ and $dx^I = dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_k}$. We will abuse the notation at times and think of $I$ as a set. Fix a multi-index $J$ and consider the coordinate plane of the variables $x^{i_1}, x^{i_2}, \ldots, x^{i_k}$.

Let $\tau_J$ denote the $k$-face of $\sigma$ contained in that plane. By assumption $\iota^*_J \eta$ is identically zero. The variables $x_t$ for $t \notin J$ vanish in this plane so that

$$\iota^*_J \eta = \sum_{t \in J} (a_{J,t} x^t + b_J) dx^J \equiv 0.$$  

Since $J$ was arbitrary, $b_J = 0$ and $a_{J,t} = 0$ for all $J$ and all $t \in J$. It follows that we can rewrite (2) on $\sigma$ as follows.

$$\eta = \sum_I \sum_{j \in I} a_{I,j} x^j dx^I.$$  

Again, fix the multi-index $L$, an integer $m \notin L$, $1 \leq m \leq n - 1$, and the simplex $\tau = [e_m, e_{l_1}, \ldots, e_{l_k}]$. $\tau$ is a $k$-simplex in the $(k+1)$-plane $P$ with coordinates $x^m, x^{l_1}, \ldots, x^{l_k}$ as in the figure below. Recall that on $\tau$, $x^{l_1}, \ldots, x^{l_k}$ can be taken as local coordinates since

$$x^m = 1 - (x^{l_1} + \ldots + x^{l_k})$$

Moreover

$$dx^m = -(dx^{l_1} + \ldots + dx^{l_k})$$

We express the pull-back $\iota^*_J \eta$ in terms these coordinates using (5) and (6). Observe

that if $I \cup \{j\} \neq L \cup \{m\}$ one of the indices in $I \cup \{j\}$ is not in $L \cup \{m\}$. The corresponding variable is identically zero on the plane $P$ so that the summand $a_{I,j} x^j dx^I$ vanishes on $P$ and is therefore equal to zero when pulled back to $\tau$. Therefore

$$\iota^*_J \eta = \sum_{I \cup \{j\} \neq L \cup \{m\}} a_{I,j} x^j dx^I.$$  

Now consider the summand with $I = L$ and $j = m$. The coefficient of $dx^L$ in this term is

$$a_{L,m} x^m + a_{L,l_1} x^{l_1} + \ldots + a_{L,l_k} x^{l_k}$$

and we use (5) to eliminate $x^m$.

Thus, on $\tau$, the coefficient in question can be written as

$$a_{L,m} - a_{L,m} \sum_{s=1}^k x^{l_s} + a_{L,l_1} x^{l_1} + \ldots + a_{L,l_k} x^{l_k}.$$
Remaining terms in the sum (7) have \( j \neq m \). It follows that, for those terms, \( x^j \) is one of \( x^1, \ldots, x^k \) and \( x^m \) enters only into the differential monomial \( dx^j \) from which it can be eliminated using (6). It follows that
\[
\iota_\gamma^* \eta = (a_{L,m} + \text{linear terms}) \, dx^j.
\]
Since \( \iota_\gamma^* \eta \) is assumed to be identically zero, \( a_{L,m} = 0 \). \( L \) was fixed but arbitrary so that \( \eta \equiv 0 \).

\[\Box\]

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