On the properties of Laplace transform originating from one-sided Lévy stable laws

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Abstract
We consider the conventional Laplace transform of \( f(x) \), denoted by
\[
\mathcal{L}[f(x); p] \equiv F(p) = \int_0^\infty e^{-px}f(x)dx \quad \text{with} \quad \Re(p) > 0.
\]
For \( 0 < \alpha < 1 \) we furnish the closed form expressions for the inverse Laplace transforms \( \mathcal{L}^{-1}[F(p^\alpha); x] \) and \( \mathcal{L}^{-1}[p^{\alpha-1}F(p^\alpha); x] \). In both cases they involve definite integration with kernels which are appropriately rescaled one-sided Lévy stable probability distribution functions \( g_\alpha(x) \), \( 0 < \alpha < 1, x > 0 \). Since \( g_\alpha(x) \) are exactly and explicitly known for rational \( \alpha \), i.e. for \( \alpha = l/k \) with \( l, k = 1, 2, \ldots, l < k \), our results extend the known and tabulated case of \( \alpha = 1/2 \) to any rational \( 0 < \alpha < 1 \). We examine the integral kernels of this procedure as well as the resulting two kinds of Lévy integral transformations.

Keywords: one-sided Lévy stable distribution, Laplace transform, special functions

1. Introduction

The (direct) Laplace transform of the function \( f(x), x > 0 \), is defined as
\[
\mathcal{L}[f(x); p] \equiv F(p) = \int_0^\infty \exp(-px)f(x)dx, \quad \Re(p) > 0,
\]
and its inverse \( L^{-1} \) can be formally written as

\[
L^{-1}[F(p); x] = f(x). \tag{2}
\]

(The condition \( \Re(p) > 0 \) is assumed to hold throughout the paper.) The usefulness of the Laplace transform in all fields of science and technology does not need to be emphasized. Numerous physical and engineering applications are presented from the computational viewpoint in [2]. The practical use of the Laplace transform is enhanced by the existence of voluminous tables:

see [3] for the direct Laplace transform and [4] for the inverse Laplace transform.

We shall be concerned in this note with the extensions of two integral formulas for the inverse Laplace transform:

\[
L^{-1}\left[ F\left(p^{1/2}\right); x \right] = \frac{1}{2\sqrt{\pi x^3}} \int_0^\infty t \exp\left(-\frac{t^2}{4x}\right)f(t)dt, \tag{3}
\]

which is formula 1.1.1.26, p 4 of [4], and

\[
L^{-1}\left[ p^{-1/2}F\left(p^{1/2}\right); x \right] = \frac{1}{\sqrt{\pi x}} \int_0^\infty \exp\left(-\frac{t^2}{4x}\right)f(t)dt, \tag{4}
\]

which is formula 1.1.1.31, p 5 of [4]. Compare also the closely related formulas 1.1.5.22 and 1.1.5.23, p 8 of [3]. For the derivation of equations (3) and (4), see equations (16) and (22) respectively, below.

The purpose of this work is to place equations (3) and (4) in a wider context in order to derive more general Laplace inversion formulas which are not tabulated so far. The new ingredient here are well-defined integrable probability distribution functions called Lévy stable functions, and denoted by \( g_\alpha(x) \), \( x > 0 \). They are defined through their Laplace transform as follows [5, 6]:

\[
L[g_\alpha(x); p] = \exp(-p^\alpha), \quad 0 < \alpha < 1. \tag{5}
\]

See [6–8] for more properties of \( g_\alpha(x) \). The right-hand side of equation (5) is the so-called stretched exponential, also sometimes referred to as the Kohlrausch–Watts–Williams function, see [9] and references therein. The simplest case of \( g_\alpha(x) \) is \( g_{1/2}(x) \), a historically first known Lévy stable function [10], which reads:

\[
g_{1/2}(x) = \left(2\sqrt{\pi x^{3/2}}\right)^{-1} \exp[-1/(4x)], \quad x > 0, \tag{6}
\]

also known as Lévy–Smirnov function.

The paper is structured as follows: in section 2 we present our main results, in section 3 we discuss further properties of integral kernels of section 2 and in section 4 we present conclusions and discussion.

2. Derivation of main results

As a first step we shall rewrite equations (3) and (4) using the function \( g_{1/2}(x) \) of equation (6). It is easy to see that
\[ \mathcal{L}^{-1}[F(p^{1/2}); x] = \int_0^\infty \frac{1}{t^{1/\alpha}} \mathfrak{g}_\alpha \left( \frac{x}{t^{1/\alpha}} \right) f(t) dt, \]  
(7)

as well as
\[ \mathcal{L}^{-1}[p^{-1/2}F(p^{1/2}); x] = 2x \int_0^\infty \frac{1}{t^{1/\alpha}} \mathfrak{g}_\alpha \left( \frac{x}{t^{1/\alpha}} \right) f(t) dt. \]  
(8)

Anticipating the sought for generalization for \( \alpha \neq 1/2 \) we introduce the following two positive two-variable kernels for \( t, x > 0 \):
\[ M_\alpha(t, x) = \frac{1}{t^{1/\alpha}} \mathfrak{g}_\alpha \left( \frac{x}{t^{1/\alpha}} \right), \]  
(9)
\[ N_\alpha(t, x) = \frac{x}{\alpha t} M_\alpha(t, x) = \frac{x}{\alpha t^{1+1/\alpha}} \mathfrak{g}_\alpha \left( \frac{x}{t^{1/\alpha}} \right), \]  
(10)

which in turn define the following two Lévy integral transformations:
\[ \tilde{f}_\alpha(x) = \int_0^\infty M_\alpha(t, x)f(t) dt = \int_0^\infty \frac{1}{t^{1/\alpha}} \mathfrak{g}_\alpha \left( \frac{x}{t^{1/\alpha}} \right) f(t) dt \]  
(11)
\[ f_\alpha(x) = \int_0^\infty N_\alpha(t, x)f(t) dt = \frac{x}{\alpha} \int_0^\infty \frac{1}{t^{1+1/\alpha}} \mathfrak{g}_\alpha \left( \frac{x}{t^{1/\alpha}} \right) f(t) dt. \]  
(12)

We tacitly assume that the integrals in equations (11) and (12) are convergent. Observe also that \( \tilde{f}_\alpha/2(x) = \mathcal{L}^{-1}[F(p^{1/2}); x] \) and \( f_\alpha(x) = \mathcal{L}^{-1}[p^{-1/2}F(p^{1/2}); x] \). The forms of equations (9) and (10) may look ad hoc at this stage but their relevance will become evident later on.

In the following, assuming \( \mathcal{L} \left[ f(x); p \right] = F(p) \), we calculate the Laplace transforms of \( \tilde{f}_\alpha(x) \) and of \( f_\alpha(x) \):
\[ \mathcal{L} \left[ \tilde{f}_\alpha(x); p \right] = \int_0^\infty e^{-px} \left[ \int_0^\infty \frac{1}{t^{1/\alpha}} \mathfrak{g}_\alpha \left( \frac{x}{t^{1/\alpha}} \right) f(t) dt \right] dx, \]  
(13)
\[ = \int_0^\infty f(t) \left[ \int_0^\infty e^{-px} \frac{1}{t^{1/\alpha}} \mathfrak{g}_\alpha \left( \frac{x}{t^{1/\alpha}} \right) dx \right] dt. \]  
(14)
\[ = \int_0^\infty f(t) \left[ \int_0^\infty e^{-pt^{1/\alpha}} \mathfrak{g}_\alpha(y) dy \right] dt, \]  
(15)
\[ = \int_0^\infty e^{-pt} f(t) dt = F(p), \]  
(16)

compare [11]. Analogously:
\[ \mathcal{L} \left[ f_\alpha(x); p \right] = \int_0^\infty e^{-px} \left[ \frac{x}{\alpha} \int_0^\infty \frac{1}{t^{1+1/\alpha}} \mathfrak{g}_\alpha \left( \frac{x}{t^{1/\alpha}} \right) f(t) dt \right] dx, \]  
(17)
\[ = \int_0^\infty \frac{1}{\alpha t^{1+1/\alpha}} f(t) \left[ \int_0^\infty xe^{-px} \frac{1}{t^{1/\alpha}} \mathfrak{g}_\alpha \left( \frac{x}{t^{1/\alpha}} \right) dx \right] dt, \]  
(18)
In equation (14) we have applied a simple change of variable and in equation (16) we have used equation (5), see [11]. Similarly, the change of variable was applied in equation (19) and equation (5) was used in equation (21). The above results can be summarized in the following statement:

**Theorem 1.** If \( L[f(x); p] = F(p) \) and \( 0 < \alpha < 1 \), then with equation (11)

\[
L\left[\tilde{f}_\alpha(x); p\right] = F(p^\alpha)
\]

and with equation (12)

\[
L\left[\tilde{f}_\alpha(x); p\right] = p^{\alpha-1}F(p^\alpha).
\]

In our opinion equation (23) is a far-reaching generalization of formula 1.1.1.26, p 4 of [4], see our equation (3), and equation (24) similarly generalizes formula 1.1.1.31, p 5 of [4], see our equation (4). Since \( g_\alpha(x) \) are exactly and explicitly known for any rational \( 0 < \alpha < 1 \), see [6], equations (23) and (24) are explicit for this case. For reader’s convenience we reproduce below the Maple® procedure LevyDist which generates \( g_\alpha(x) \) for any rational \( \alpha \) and \( k \), compare [6]:

\[
\text{LevyDist} := \text{proc}(k, l, x) \text{ simplify} (\text{convert}((\sqrt{k*l}) \ast \text{MeijerG}([[],[seq(j1/l, j1 = 0..l-1)]],[[seq(j2/k, j2 = 0..k-1)],[[]], l^1/(k^1*l^1)/(x*(2*Pi)^1*(k-1)/2)), \text{StandardFunctions})); \text{end};
\]

3. Further properties of integral kernels

In addition to properties displayed in equations (23) and (24) the integral kernels \( M_\alpha(t, x) \) and \( N_\alpha(t, x) \) possess further characteristics connected with integration over the positive half-axis. We set out to evaluate the following integral of convolution type:

\[
\int_0^\infty M_\alpha(t, x)M_\beta(y, t)\,dt = \int_0^\infty \frac{1}{t^{1/\alpha}} g_\alpha\left(\frac{x}{t^{1/\alpha}}\right) \frac{1}{y^{1/\beta}} g_\beta\left(\frac{t}{y^{1/\beta}}\right)\,dt
\]

\[
= \int_0^\infty \frac{1}{y^{1/\beta}} g_\alpha\left(\frac{x}{y^{1/\beta}}\right) g_\beta(u)\,du
\]

\[
= \frac{1}{1-y^{1/\beta}t^{1/\alpha}} g_\beta(u)\,du
\]
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\[ \begin{align*}
\int_0^\infty \frac{1}{y^{\alpha \gamma}} g_{\alpha \beta} \left( \frac{x}{y} \frac{1}{u^{\alpha \gamma}} \right) g_{\gamma}(u) du &= \frac{1}{y^{\alpha \gamma}} g_{\alpha \beta} \left( \frac{x}{y} \right) \equiv M_{\alpha \beta}(y, x). \\
\int_0^\infty N_\alpha(t, x) N_\beta(y, t) dt &= \int_0^\infty \frac{x}{\alpha t^{1+\frac{1}{\alpha \beta}}} g_{\alpha \beta} \left( \frac{x}{t} \right) \frac{1}{y^{\beta \gamma}} g_{\beta}(u) du \\
&= \frac{x}{\alpha \beta} \int_0^\infty \frac{1}{t^{1+\frac{1}{\alpha \beta}}} g_{\alpha \beta} \left( \frac{x}{t} \frac{1}{y^{\beta \gamma}} \right) g_{\beta}(u) du \\
&= \frac{1}{\alpha \beta} \frac{x}{y^{1+\frac{1}{\alpha \beta}}} g_{\alpha \beta} \left( \frac{x}{y} \right) \equiv N_{\alpha \beta}(y, x).
\end{align*} \]

In equation (26) we have performed a change of variable \( u^{1/\beta} = t \), and in equation (27) we have used equation (29) of [11]. Very similarly, the following convolution-type integral reads:

\[ \int_0^\infty N_\alpha(t, x) N_\beta(y, t) dt = \int_0^\infty \frac{x}{t^{1+\frac{1}{\alpha \beta}}} g_{\alpha \beta} \left( \frac{x}{t} \right) \frac{1}{y^{\beta \gamma}} g_{\beta}(u) du \]

In equation (30) we have used the change of variable \( u^{1/\beta} = t \), followed by employing again equation (29) of [11] in equation (31). Thus equations (28) and (32) express the transitivity of kernels \( M_\alpha \) and \( N_\beta \) with respect to the index \( \alpha \), via convolution-type integration of type equations (25) and (29). They generalize equation (29) of [11] and are valid for any real \( \alpha, \beta \) such that \( 0 < \alpha, \beta < 1 \). We remark that neither \( M_\gamma(y, x) \) nor \( N_\gamma(y, x), 0 < \gamma < 1 \), are symmetric with respect to \( x \) and \( y \).

This transitivity appears to break down for other types of convolution integrals which mix \( M_\gamma \) and \( N_\gamma \), namely for \( \int_0^\infty M_\gamma(t, x) N_\gamma(y, t) dt \). In fact the evaluation of this last integral for arbitrary \( \alpha \) and \( \beta \) does not appear to be possible. However, we shall demonstrate that for specific forms of \( \alpha \) and \( \beta \), namely for \( \alpha \) and \( \beta \) rational, this evaluation is possible in terms of special functions. As a first step we define the Laplace transform of the aforementioned integral through

\[ \mathcal{F}_{\alpha, \beta}(p, y) = \mathcal{L} \left[ \int_0^\infty M_\alpha(t, x) N_\beta(y, t) dt; p \right], \]

where, by convention, the integration variable for the Laplace transform is fixed to be \( x \), and \( y \) becomes a parameter. (We remind the reader that in equation (33) all the variables \( t, x, y \) and \( p \) are positive.) For the moment no further assumptions about \( \alpha \) and \( \beta \) are made, except \( 0 < \alpha, \beta < 1 \). Quite surprisingly, \( \mathcal{F}_{\alpha, \beta}(p, y) \) can be calculated exactly:

\[ \mathcal{F}_{\alpha, \beta}(p, y) = \mathcal{L} \left[ \int_0^\infty M_\alpha(t, x) N_\beta(y, t) dt; p \right]. \]
\[ F_{\alpha,\beta}(p, y) = \int_0^\infty e^{-px} \left[ \int_0^\infty \frac{x}{t^{1+\beta}} g_{\alpha,\beta,\gamma} \left( \frac{x}{t} \right) \frac{t^{1+\beta}}{y^{1+\beta}} g_{\alpha,\beta,\gamma} \left( \frac{t}{y} \right) \, dt \right] \, dx \]  

(34)

\[ = \int_0^\infty \frac{t}{\beta y^{1+\beta}} g_{\beta} \left( \frac{t}{y} \right) \left[ \int_0^\infty e^{-px} \frac{1}{t^{1+\beta}} g_{\alpha,\beta,\gamma} \left( \frac{x}{t} \right) \, dx \right] \, dt. \]  

(35)

The inner integral in equation (35), via the change of variable \( ut^{1/\beta} = x \) has the value \( e^{pt} \), see equation (5), and therefore

\[ F_{\alpha,\beta}(p, y) = \int_0^\infty \frac{t}{\beta y^{1+\beta}} e^{-p\gamma t} g_{\beta} \left( \frac{t}{y} \right) \, dt \]  

(36)

\[ = \frac{1}{\beta y^{1+\beta}} \int_0^\infty e^{-\gamma t^{1+\beta}} g_{\beta} \left( \frac{t}{y} \right) \, dz \]  

(37)

\[ = \frac{1}{\beta y^{1+\beta}} \left[ - \frac{d}{dz} \int_0^\infty e^{-sz} g_{\beta} \left( \frac{z}{y} \right) \, dz \right]_{s=p^{-1/\beta}} \]  

(38)

\[ = - \frac{1}{\beta y^{1+\beta}} \left[ \frac{d}{dz} \left( e^{-sz} \right) \right]_{s=p^{-1/\beta}} \]  

(39)

\[ = p^{\beta(\beta-1)} e^{-y^{1/\beta}} \]  

(40)

valid for any real \( 0 < \alpha, \beta < 1 \). In equation (36) we used the change of variable \( y t^{1/\beta} = z \) and in equation (36) we employed again equation (5). Equation (40) immediately implies the equality

\[ F_{\alpha,\beta}(p, y) = p^{\alpha-\beta} F_{\alpha,\beta}(p, y). \]  

(41)

Consequently, equation (40) implies

\[ J_{\alpha,\beta}(x, y) \equiv \int_0^\infty M_{\alpha}(t, x) N_{\beta}(y, t) \, dt = \mathcal{L}^{-1} \left[ F_{\alpha,\beta}(p, y); x \right] \]  

(42)

\[ = \mathcal{L}^{-1} \left[ p^{\beta(\beta-1)} e^{-y^{1/\beta}}; x \right]. \]  

(43)

According to the Laplace inversion formula 2.2.1.19 on p 53 of [4] equation (43) can be inverted if \( \alpha \beta < 1 \) is rational, in terms of Meijer G functions [12]. Therefore for \( \alpha \) and \( \beta \) rational the integral \( \int_0^\infty M_{\alpha}(t, x) N_{\beta}(y, t) \, dt \) can be evaluated exactly.

We shall now exemplify the Laplace inversion of equation (43) with three pairs of rational parameters \( \alpha \) and \( \beta \). First we reproduce the inversion formula 2.2.1.19 of [4] using our parametrization. It reads then, for \( \alpha \beta \equiv l/k \), with \( l \) and \( k \) relatively prime positive integers, as

\[ \mathcal{L}^{-1} \left[ \frac{1}{p^{\alpha-1} k^{\gamma}} e^{-p^{\gamma/\beta}}; x \right] = \frac{\sqrt{k}}{(2\pi)^{1/2}} \int_0^x t^{-\alpha+1/2} \alpha^{-1} k^{-1} G_{\alpha,\gamma} \left( \frac{y}{k} \right) \left( \frac{l}{k} \right) \Delta(l, \alpha - l/k, 0) \Delta(k, 0) \]  

(44)
\[
\sqrt[k]{\frac{1}{(2\pi)^{d}}} \Gamma^{d-\alpha-\frac{1}{k}} \alpha \cdot \frac{1}{k} G \left( \left\lceil \frac{1}{k} \right\rceil, \left\lceil \Delta \left( I, \alpha - \frac{1}{k} \right) \right\rceil, \left\lceil \left( \Delta(k, 0) \right), \left\lfloor \frac{1}{k} \right\rfloor \right\rceil, \left( \frac{y^k}{k} \right) \left( \frac{l}{k} \right) \right).
\]

where in equations (44) and (45) for the Meijer G function, denoted by G, we employed a traditional notation, see [12], and a simplified notation inspired by the computer algebra systems\(^4\), respectively. Above, \(\Delta(k, a) = \frac{a}{k}, \frac{a + 1}{k}, \ldots, \frac{a + k - 1}{k} \), \(k \neq 0\), is a special sequence. For a precise definition of Meijer G function as a Mellin transform see [12] and [13].

(a) Case \(\alpha = 2/3\) and \(\beta = 1/2\) 
Here \(\alpha \beta = 1/3\), \(l = 1\), \(k = 3\), \(\alpha - 1/3 = 1/3\).

\[
\mathcal{J}_{\frac{1}{3} \frac{1}{2}}(x, y) = \mathcal{L}^{-1} \left[ p^\frac{1}{3} \exp \left( -yp^\frac{1}{3} \right) ; x \right]
\]

\[
= \sqrt[3]{3} \frac{2\pi x^{2/3}}{27x} G_{1,3}^{1,0} \left( \left\lceil \frac{y}{27x} \right\rceil, \left\lceil 0, \frac{1}{3}, \frac{2}{3} \right\rceil \right)
\]

\[
= \sqrt[3]{3} \frac{2\pi x^{2/3}}{27x} G_{0,2}^{1,0} \left( \left\lceil \frac{y}{27x} \right\rceil, \left\lceil 0, \frac{2}{3} \right\rceil \right)
\]

\[
= \sqrt[3]{3} \frac{2\pi x^{2/3}}{27x} G_{0,2}^{1,0} \left( \left\lceil \left\lceil 0, \frac{2}{3} \right\rceil \right\rceil, \left\lceil \frac{y}{27x} \right\rceil \right)
\]

\[
= \frac{1}{3} \sqrt[3]{\pi x} y K_{2} \left( \frac{2\pi y^{3}}{3\sqrt[3]{x}} \right)
\]

where in equation (47) the simplification property of Meijer G function was used and in equation (49) we applied formula 8.4.23.1 of [12], where \(K_{\nu}(z)\) is the modified Bessel function of the second kind of order \(\nu\).

(b) Case \(\alpha = 1/2\) and \(\beta = 2/3\) 
Here \(\alpha \beta = 1/3\), \(l = 1\), \(k = 3\), \(\alpha - 1/3 = 1/6\).

\[
\mathcal{J}_{\frac{1}{3} \frac{1}{2}}(x, y) = \mathcal{L}^{-1} \left[ p^\frac{1}{6} \exp \left( -yp^{\frac{1}{6}} \right) ; x \right] = \sqrt[3]{3} \frac{2\pi x^{2/3}}{27x} G_{1,3}^{1,0} \left( \left\lceil \frac{y}{27x} \right\rceil, \left\lceil 0, \frac{1}{3}, \frac{2}{3} \right\rceil \right)
\]

\[
= \sqrt[3]{3} \frac{2\pi x^{2/3}}{27x} G_{0,2}^{1,0} \left( \left\lceil \left\lceil 0, \frac{1}{3}, \frac{2}{3} \right\rceil \right\rceil, \left\lceil \frac{y}{27x} \right\rceil \right)
\]

The functions of equations (51) and (52) can still be represented by better known special functions using the representation of Meijer G function as a finite sum of generalized hypergeometric functions \(pF_{0}^{\left( a_{p} \right)} \left( \frac{b_{q}}{c_{r}} ; x \right)\), see formula 8.2.2.3 of [12], where \(\left( a_{p} \right)\) is a list

\(^4\) We have extensively used Maple® and Mathematica® in this work.
of ‘upper’ parameters and \( (b_q) \) is a list of ‘lower’ parameters of \( \mu F_q \). An application of formula 8.2.2.3 of [12] to equation (52) gives directly

\[
\mathcal{J}_{\frac{1}{2}} \left( x, y \right) = \frac{\Gamma \left( \frac{3}{4} \right)}{2 \pi x^{3/4}} \left( F_2 \left( \frac{1}{2}, \frac{3}{4} ; \frac{y^3}{27x} \right) + \frac{y^4}{6 \pi x^{3/4}} F_2 \left( \frac{7}{6}, \frac{5}{6} ; \frac{y^7}{27x^3} \right) - \frac{y^2}{4 \pi x^{3/4}} F_2 \left( \frac{3}{4}, \frac{7}{8} \frac{y^4}{256x} \right) \right) \]

which also agrees with formula 2.2.1.6 of [4] for the parameter \( \nu = 1/6 \).

(c) Case \( \alpha = \beta = 1/2 \)

Here \( \alpha, \beta = 1/4, \Gamma = 1, k = 4, \alpha - 1/4 = 1/4 \).

\[
\mathcal{J}_{\frac{1}{2}} \left( x, y \right) = L^{-1} \left[ p^{-1/4} \exp \left( -y p^{1/4} \right) ; x \right] = \frac{2}{(2\pi)^{3/2} x^{3/4}} G_{4,4} \left( \frac{y^4}{256} \right) \]

5 We would like to correct a misprint in formula 2.2.1.6 p 51 of [4]: the third hypergeometric function in this

form should be \( F_2 \left( \frac{5}{3}, \frac{4}{4} ; \frac{y^3}{27x} \right) \).
allowed further extension for arbitrary $\alpha$ by choosing appropriate kernels $M_\alpha(t,x)$ and $N_\alpha(t,x)$ of equations (9) and (10), respectively, thereby permitting us to prove two Laplace transform formulas, equations (16) and (22). Since $g_\alpha(x)$ are explicitly known for $\alpha$ rational [6] that gives a wealth of novel Laplace inversion formulas. Subsequently the integration kernels $M_\alpha(t,x)$ and $N_\alpha(t,x)$ were shown to possess the transitivity (or a kind of a reproducing property) with respect to the index $\alpha$, under integration of convolution type. This can be clearly seen from equations (27) and (31) which, in the limit $\alpha \to 1$, become proportional to Mellin convolution of two Lévy stable distributions (with the caveat that the value $\alpha = 1$ is outside the range of our interest here).

It turns out that an even more general kernel correlation function, defined by equations (33) and (42), can be exactly evaluated but for rational $\alpha$ and $\beta$ only, again using the Laplace transform technique. Observe in passing that equation (40) as a function of $p$ can be cast in the form of the Weibull distribution [14], signaling a link between Lévy stable and extreme values distributions, which is a subject of recent publications, see [15] and [16]. Note that the calculation of closed form of the Laplace transform in equation (33), but with $y$ as integration variable, does not appear to be possible.

The kernels appearing in equations (9) and (10) are intimately related to the fractional Fokker–Planck equation. In fact the kernel $N_\alpha(t,x)$ appears in the work of Barkai, see equation (16) in [17], whereas the kernel $M_\alpha(t,x)$ was used in equation (25) of reference [18], in connection with partial differential equations in which fractional derivatives are acting on the spatial coordinates.

The field of applications of Lévy stable laws is huge. It ranges from theories of random matrices [19], descriptions of optical properties of nanocrystals [9] to predator search behavior in marine biology [20]. Many other examples could be quoted. Various calculational aspects of Lévy stable laws are critically reviewed in [21]. Theoretical and experimental aspects are reviewed in [22] and [23], respectively.

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