A Highly Predictive Ansatz for Leptonic Mixing and CP Violation

G.C. Branco ∗ and J.I. Silva-Marcos †

CFTP, Departamento de Física
Instituto Superior Técnico, Avenida Rovisco Pais, 1
1049-001 Lisboa, Portugal

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Abstract

We suggest a simple highly predictive ansatz for charged lepton and light neutrino mass matrices, based on the assumption of universality of Yukawa couplings. Using as input the charged lepton masses and light neutrino masses, the six parameters characterizing the leptonic mixing matrix $V_{PMNS}$, are predicted in terms of a single phase $\phi$, which takes a value around $\phi = \frac{\pi}{2}$. Correlations among various physical quantities are obtained, in particular $V_{13}^{PMNS}$ is predicted as a function of $\Delta m^2_{21}$, $\Delta m^2_{31}$ and $\sin^2(\theta_{sol})$, and restricted to the range $0.167 < |V_{13}^{PMNS}| < 0.179$.

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∗e-mail: gbranco@ist.utl.pt
†e-mail: juca@cftp.ist.utl.pt
1 Introduction

Understanding the pattern of fermion masses and flavour mixing is still one of the open fundamental questions in particle physics. The discovery of large leptonic mixing, in contrast to small quark mixing, has rendered the flavour puzzle even more intriguing.

In the Standard Model (SM) and in most of its extensions, the arbitrariness of fermion masses and mixing stems from the fact that the gauge invariance does not constrain the flavour structure of the Yukawa couplings. The fact that, in the SM, only Yukawa couplings can be complex, has motivated the hypothesis of universality of strength of Yukawa couplings (USY) [1], which would have all the same strength, with flavour-dependent phases. The consequences of USY have been analyzed in various works, both for the quark [2] and lepton sectors [3]. Such an USY structure implies interesting correlations among various physical quantities.

The size of $V_{ij}^{PMNS}$ is predicted as a function of $\tan(\theta_{sol})$ and the neutrino mass differences $\Delta m^2_{31}, \Delta m^2_{21}$. For central values of $\sin^2(\theta_{sol})$ and $\Delta m^2_{21}$, one obtains $|V_{13}^{PMNS}| = 0.178$, clearly at the reach of the next round of experiments [5]. The ansatz also predicts the strength of Dirac type CP violation, measured by the invariant quartet $I_{CP} = \text{Im}[V_{12} V_{23} V_{13} V_{21}]^{PMNS}$. For central values of $\sin^2(\theta_{sol}), \sin^2(\theta_{sol})$ and $\Delta m^2_{21}$, one obtains $I_{CP} = 0.00906$, which can be measured in neutrino oscillation experiments [5].

This paper is organized as follows. In the next section, we describe the Ansatz and its parameter space, both in the charged lepton and neutrino sectors. In section 3, we evaluate the lepton mixing and derive some predictions of the ansatz for various physical quantities, including $|V_{13}^{PMNS}|$, double beta decay and the strength of the Dirac-type CP violation. Section 4 contains some numerical results and figures illustrating correlations among various physical quantities. Finally, our conclusions are contained in section 5.

2 The Ansatz and its Parameter Space

2.1 The charged lepton sector

We propose the following USY structure for the charged lepton mass matrix

\[
M_l = \frac{c_l}{\sqrt{3}} \cdot K_\phi^+ \cdot \begin{pmatrix} 1 & 1 & 1 \\ 1 & e^{ia_1} & 1 \\ 1 & 1 & e^{ib_1} \end{pmatrix} \quad ; \quad K_\phi = \text{diag}(1, 1, e^{i\phi})
\]

The phase $\phi$ does not affect the charged lepton mass spectrum but contributes to the leptonic mixing. Using the trace, determinant and second invariant of $H_l = M_l M_\ell^\dagger$, one can derive exact expressions for the phases $a_1, b_1$ and the parameter $c_l$ in terms of the masses:

\[
c_l = \frac{1}{\sqrt{3}} \sqrt{m_\mu^2 + m_\tau^2 + m_e^2}
\]

\[
3 \sin^2 \left( \frac{a_1}{2} \right) + 3 \sin^2 \left( \frac{b_1}{2} \right) + \sin^2 \left( \frac{a_1 + b_1}{2} \right) = \frac{81}{4} \frac{m_e^2 m_\mu^2 + m_e^2 m_\tau^2 + m_\mu^2 m_\tau^2}{(m_\mu^2 + m_\tau^2 + m_e^2)}
\]

\[
\left| \sin \left( \frac{a_1}{2} \right) \sin \left( \frac{b_1}{2} \right) \right| = \frac{27}{4} \frac{m_e m_\mu m_\tau}{\sqrt{(m_\mu^2 + m_\tau^2 + m_e^2)}}
\]

From the charged lepton hierarchy, one obtains to an excellent approximation

\[
|a_1| \simeq 6 \frac{m_e}{m_\tau}, \quad |b_1| \simeq 9 \frac{m_\mu}{2 m_\tau},
\]
Obviously, in Eq. (2) $a_l$ and $b_l$ enter in a symmetric way. The choice of Eq. (3) is required in order to obtain the right eigenvalue ordering.

### 2.2 The effective neutrino mass matrix

We assume that lepton number is violated at a high energy scale, leading at low energies to the following effective neutrino mass matrix,

$$ M_\nu = \frac{c_\nu}{\sqrt{3}} \begin{bmatrix} e^{ia_\nu} & 1 & 1 \\ 1 & e^{-ia_\nu} & 1 \\ 1 & 1 & e^{ib_\nu} \end{bmatrix} $$  \hspace{1cm} (4)

The three parameters, $c_\nu$, $b_\nu$, and $a_\nu$, of the neutrino mass matrix ansatz in Eq. (4) are entirely determined by the three neutrino masses. We find from the trace, second invariant and determinant of $H_\nu \equiv M_\nu M_\nu^\dagger$:

$$ 3c_\nu^2 = m_3^2 + m_2^2 + m_1^2 $$

$$ \cos(a_\nu) = 1 - \frac{27}{2} d_\nu $$

$$ \cos(b_\nu) = \frac{1 + \frac{27}{4} d_\nu - \frac{1}{4} \chi_\nu}{1 - \frac{27}{4} d_\nu} $$  \hspace{1cm} (5)

where

$$ d_\nu = \frac{m_1 m_2 m_3}{(m_3^2 + m_2^2 + m_1^2)^2} ; \quad \chi_\nu = \frac{m_1^2 m_2^2 + m_1^2 m_3^2 + m_2^2 m_3^2}{(m_3^2 + m_2^2 + m_1^2)^2} $$  \hspace{1cm} (6)

At this stage, it is worth emphasizing the predictive power of the Ansatz. From Eqs. (2, 5), it is clear that once the parameters ($c_l, b_l, a_l$), and ($c_\nu, b_\nu, a_\nu$) are fixed by the charged lepton and neutrino masses, the six parameters of the Pontecorvo–Maki–Nakagawa–Sakata matrix, $V_{PMNS}$, are completely determined in terms of a single parameter, the phase $\phi$.

### 3 Evaluation of Lepton Mixing

#### 3.1 Diagonalization and parametrization of the lepton mass matrices

The diagonalization of the Hermitian charged lepton mass matrix $H_l \equiv M_l M_l^\dagger$ is carried out through

$$ V_l^\dagger \cdot H_l \cdot V_l = diag(m_e^2, m_\mu^2, m_\tau^2) $$

with the unitary matrix $V_l$ given by

$$ V_l = K_\phi^\dagger \cdot F \cdot W_l $$

where $F$

$$ F = \begin{pmatrix} \frac{\sqrt{2}}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & \frac{\sqrt{3}}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \sqrt{\frac{3}{2}} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix} $$  \hspace{1cm} (7)

and $W_l$ is a unitary matrix close to the identity. Given the strong hierarchy of the charged lepton masses, to an excellent approximation, one obtains for $W_l$

$$ W_l \simeq \begin{pmatrix} 1 & \frac{m_\mu}{\sqrt{3} m_\tau} & -i \sqrt{\frac{3}{2} m_\tau} \\ -\frac{m_e}{\sqrt{3} m_\tau} & 1 - \frac{m_\mu}{m_\tau} \left( \frac{m_\mu}{m_\tau} \right)^2 & i \frac{m_\mu}{\sqrt{2} m_\tau} \\ -i \sqrt{\frac{3}{2} m_\mu} & i \frac{m_\mu}{\sqrt{2} m_\tau} & 1 - \frac{m_\mu}{m_\tau} \left( \frac{m_\mu}{m_\tau} \right)^2 \end{pmatrix} $$  \hspace{1cm} (8)

The diagonalization of the neutrino mass matrix is achieved through

$$ V_\nu^\dagger \cdot M_\nu \cdot V_\nu^* = D_\nu = diag(m_1, m_2, m_3) $$  \hspace{1cm} (9)
where $m_i$ denote the neutrino masses. In order to understand the main features of $V_\nu$ in the framework of our Ansatz, it is useful to introduce a convenient parametrization. Let us now introduce the dimensionless parameters $\varepsilon, \delta$ defined by

$$\varepsilon = \frac{m_2}{\sqrt{m_3^2 + m_2^2 + m_1^2}} ; \quad \delta = \frac{m_1}{m_2}$$

The neutrino masses can then be written:

$$m_1 = \sqrt{3} \ c_\nu \ \varepsilon \ \delta$$
$$m_2 = \sqrt{3} \ c_\nu \ \varepsilon$$
$$m_3 = \sqrt{3} \ c_\nu \sqrt{1 - \varepsilon^2 - \delta^2 \varepsilon^2}$$

By substituting $m_i$ as functions of $\varepsilon, \delta$ in Eqs. (5, 6), we obtain $d_\nu, \chi_\nu$ as well as $a_\nu, b_\nu$ as functions of $\varepsilon$ and $\delta$:

$$d_\nu = \delta \ \varepsilon^2 \ \sqrt{1 - \varepsilon^2 (1 + \delta^2)} ; \quad \chi_\nu = \varepsilon^2 \ [1 + \delta^2 - \varepsilon^2 \ (1 + \delta^2 + \delta^4)]$$

The matrix $V_\nu$ is then entirely given as a function of these two parameters ($\varepsilon, \delta$), which are fixed by neutrino mass ratios. Furthermore, for our ansatz, $V_\nu$ is exactly factorizable in the following way:

$$V_\nu = F \cdot K_\gamma \cdot O_\nu \cdot K_M$$

where $F$ was given in Eq. (7) and $K_\gamma, K_M$ are diagonal unitary matrices containing phases, which will contribute to the Dirac and Majorana type phases of the lepton mixing, $K_\gamma = \text{diag}(1, e^{i\gamma}, -i)$ and $K_M = \text{diag}(e^{i\alpha M}, e^{i\beta M}, e^{i\gamma M})$. As mentioned, all these phases and the angles of orthogonal matrix $O_\nu$ can be expressed as functions of $\delta$ and $\varepsilon$.

So far all our results are exact. Our numerical results for $V_{PMNS}$ will be obtained through exact numerical diagonalization of $H_\nu$ and $H_\nu$. However, in order to get an overview of the physical implications of this USY ansatz, it is useful to derive some analytical expressions which hold to a good approximation. Let us parametrize $O_\nu$ in the following way:

$$O_\nu = O_{23} \cdot O_{13} \cdot O_{12}$$

with

$$O_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\hat{\theta}_{23}) & \sin(\hat{\theta}_{23}) \\ 0 & -\sin(\hat{\theta}_{23}) & \cos(\hat{\theta}_{23}) \end{bmatrix} ; \quad O_{13} = \begin{bmatrix} \cos(\hat{\theta}_{13}) & 0 & \sin(\hat{\theta}_{13}) \\ 0 & 1 & 0 \\ -\sin(\hat{\theta}_{13}) & 0 & \cos(\hat{\theta}_{13}) \end{bmatrix}$$

$$O_{12} = \begin{bmatrix} \cos(\hat{\theta}_{12}) & \sin(\hat{\theta}_{12}) & 0 \\ -\sin(\hat{\theta}_{12}) & \cos(\hat{\theta}_{12}) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

It turns out that in the relevant region of parameter space, $\varepsilon$ is relatively small, $\varepsilon \approx 0.2$. Therefore, we make an expansion in powers of $\varepsilon$ which yield:

$$\tan(\hat{\theta}_{12}) = -\sqrt{3} \left( 1 + \frac{8\delta^2 - 3\delta^4 - 3}{4(1-\delta^2)} \varepsilon^2 + O(\varepsilon^4) \right)$$
$$\tan(\hat{\theta}_{23}) = \varepsilon \ \frac{(1-\delta)}{\sqrt{2}} \left( 1 + \frac{37\delta^2 - 2\delta^4 - 2}{8} \varepsilon^2 + O(\varepsilon^4) \right)$$
$$\tan(\hat{\theta}_{13}) = \varepsilon \ \sqrt{2\delta} \left( 1 + \frac{4\delta^2 + 8\delta^4}{3} \varepsilon^2 + O(\varepsilon^4) \right)$$
$$\tan(\gamma) = -\varepsilon \ \frac{(10\delta^2 - 3\delta^4)}{4(1-\delta^2)} + O(\varepsilon^3)$$

The leptonic mixing matrix is given by:

$$V_{PMNS} = V_\nu^T \cdot V_\nu = \left( W_\nu^T F^T \ K_\phi \right) \cdot (F \ K_\gamma \ O_\nu \ K_M)$$
This formula is exact and it will be used in the numerical computation of $V_{PMNS}$. However, it is useful to obtain analytical approximate expressions for $V_{PMNS}$. Using Eqs. (16, 14), and neglecting the small contribution from $W_{i}$ given by Eq. (8), one obtains

$$\left| V_{11}^{PMNS} \right| = \tan(\theta_{sol}) = \tan(\theta_{12})$$

$$\left| V_{13}^{PMNS} \right| = \sin(\theta_{13})$$

which identifies these two lepton mixing angles in terms of our parametrization. Up to second order in $\varepsilon$, from Eq. (15), one obtains $\tan^{2}(\theta_{sol})$ and $|V_{13}^{PMNS}|$ expressed in terms of the measured $\Delta m_{31}^{2}$, $\Delta m_{21}^{2}$ and the lightest neutrino mass, $m_{1}$:

$$\tan^{2}(\theta_{sol}) = \frac{m_{1}}{\Delta m_{31}^{2} + m_{1}^{2}}$$

$$\left| V_{13}^{PMNS} \right|^{2} = \frac{2m_{1}\sqrt{\Delta m_{31}^{2} + m_{1}^{2}}}{\Delta m_{31}^{2} + \Delta m_{21}^{2} + \Delta m_{1}^{2}}$$

Eliminating $m_{1}$ from Eq. (18), one obtains the interesting sum rule expressing $|V_{13}^{PMNS}|$ in terms of measured quantities

$$\left| V_{13}^{PMNS} \right| = \sqrt{2} \tan(\theta_{sol}) \sqrt{\frac{\Delta m_{21}^{2}}{\Delta m_{31}^{2}}} \frac{1}{\sqrt{1 - \tan^{4}(\theta_{sol}) + (1 + 2\tan^{4}(\theta_{sol})) \frac{\Delta m_{1}^{2}}{\Delta m_{31}^{2}}}}$$

For central values of $\sin^{2}(\theta_{sol})$ and $\Delta m_{ij}^{2}$ one finds

$$\left| V_{13}^{PMNS} \right| = 0.178$$

For $\theta_{atm}$ one obtains:

$$\sin^{2}(\theta_{atm}) = \frac{4}{9} \left[ 1 - \cos(\phi) + \frac{3}{2} \varepsilon (1 - \delta) \sin(\phi) + O(\varepsilon^{2}) \right]$$

It is clear that $\theta_{atm}$ crucially depends on $\phi$, the phase defined in Eq. (1). It is interesting to note that a good fit of $\theta_{atm}$ is obtained for $\phi = \frac{\pi}{2}$.

### 3.2 Double Beta Decay

We evaluate now $M_{ee}$, which controls the strength of double beta decay and is given by:

$$M_{ee} \equiv \left| m_{1} \left( V_{11}^{PMNS} \right)^{2} + m_{2} \left( V_{12}^{PMNS} \right)^{2} + m_{3} \left( V_{13}^{PMNS} \right)^{2} \right|$$

We compute $M_{ee}$ in two steps. First, we evaluate the contribution to Majorana phases from $K_{M} = diag(e^{i\hat{\alpha}_{M}}, e^{i\hat{\beta}_{M}}, e^{i\hat{\gamma}_{M}})$. This can be done by focussing only on the diagonalization of the neutrino mass matrix: $V_{i} \cdot M_{\nu} \cdot V_{i}^{\dagger} = diag(m_{1}, m_{2}, m_{3})$. It is clear that these phases appear when diagonalizing $M_{\nu}$ only with $F K_{\gamma} O_{\nu}$, without $K_{M}$:

$$(F K_{\gamma} O_{\nu})^{\dagger} \cdot M_{\nu} \cdot (F K_{\gamma} O_{\nu})^{*} = diag(m_{1} e^{2i\hat{\alpha}_{M}}, m_{2} e^{2i\hat{\beta}_{M}}, m_{3} e^{2i\hat{\gamma}_{M}})$$

In leading order, we find:

$$2\hat{\alpha}_{M} = -\frac{\pi}{2} - \frac{9 - 12\delta - 4\delta^{2}}{4(1 - \delta)} \varepsilon$$

$$2\hat{\beta}_{M} = \frac{\pi}{2} + \frac{1 + 12\delta - 9\delta^{2}}{4(1 - \delta)} \varepsilon$$

$$2\hat{\gamma}_{M} = \frac{\pi}{2} + \frac{1 - 2\delta}{2} \varepsilon$$

We can then write

$$M_{ee} = \left| m_{1} e^{2i\hat{\alpha}_{M}} (V_{11})^{2} + m_{2} e^{2i\hat{\beta}_{M}} (V_{12})^{2} + m_{3} e^{2i\hat{\gamma}_{M}} (V_{13})^{2} \right|$$

\(^1\) Obviously, the phases $\hat{\alpha}_{M}$, $\hat{\beta}_{M}$, $\hat{\gamma}_{M}$ are defined modulo $\pi$. 
where here $V$ is the lepton mixing matrix $V_{PMNS}$ but without the last $K_M$ phases, i.e. $V = V_{PMNS} \cdot K_M = W_{1}^{T} F T K_{\phi} F K_{\gamma} O_{\nu}$.

Since, the matrix $F T K_{\phi} F$ in $V$ only gives a contribution in the 2–3 plane, and $W_{12}^{l}$ and $W_{13}^{l}$ are all of the order of $\varepsilon^{5}$ or smaller, we may read the expressions for $V_{11}, V_{12}$ and $V_{13}$ directly from the leading order expressions for $V_{PMNS}$.

Using Eqs. (24, 25, 26) together with the leading order expressions for $V_{PMNS}$, we have:

$$
\tan(\hat{\theta}_{12}) = -\sqrt{\delta} \left( 1 + \frac{8\delta + 8\delta^{2} - 3\delta^{3} - 3}{4(1 - \delta)} \varepsilon^{2} \right)
$$

$$
\tan(\hat{\theta}_{13}) = \varepsilon \sqrt{2\delta} \left( 1 + \frac{4\delta + 4\delta^{2}}{8} \varepsilon^{2} \right)
$$

$$
\tan(\gamma) = -\varepsilon \frac{(10\delta - 3\delta^{2} - 3)}{4(1 - \delta)}
$$

Using Eqs. (24, 25, 26) together with $m_1, m_2$ and $m_3$ expressed in terms of $c_{\nu}, \delta$ and $\varepsilon$ (as in Eq. (11)), we find the following leading order expression

$$
M_{ee} = \frac{9\sqrt{3}}{2} \delta \varepsilon^{2} \left( 1 - \frac{(1 + \delta^{2})}{2} \varepsilon^{2} - \frac{(1 + \delta^{2})^{2}}{8} \varepsilon^{4} \right) c_{\nu}
$$

### 3.3 Dirac-type CP Violation

The strength of the Dirac-type CP violation is given by the imaginary part of any rephasing invariant quartet of $V_{PMNS}$, e.g.

$$
I_{CP} = \left| \text{Im} [V_{12} V_{23} V_{23}^{*} V_{13}^{*}]^{PMNS} \right|
$$

Using Eqs. (1, 7, 8, 14), we can evaluate $I_{CP}$ in terms of $\varepsilon, \delta$ and $\phi$, obtaining in second order of $\varepsilon$:

$$
I_{CP} = \frac{2\delta \varepsilon}{9(1 + \delta)} \left[ 1 - \cos(\phi) - \varepsilon \frac{3(10\delta - 3\delta^{2} - 3)}{4(1 - \delta)} \sin(\phi) \right]
$$

From Eq. (21), it is clear that the phase $\phi$ is strongly correlated with $\sin(\theta_{atm})$. Then, for the central value of $\sin^{2}(\theta_{atm})$, which is obtained with $\phi = \frac{\pi}{4}$ and central values of $\sin^{2}(\theta_{sol})$ and $\Delta m_{32}^{2}$, one gets $|I_{CP}| = 0.0105$, a value obtained neglecting the charged lepton contribution, which is small. Further on, in Section 4, we shall give an exact numerical example.

### 3.4 Majorana-type CP Violation

It is well known that in the case of Majorana neutrinos, the basic rephasing invariants, in the leptonic sector, are bilinears of the type $V_{PMNS}^{PMNS}$ with $k \neq l$. In fact, in the case of three leptonic flavours, it has recently been shown that there are six rephasing invariant independent "Majorana-type" phases from which one can reconstruct the full $V_{PMNS}$ matrix using $3 \times 3$ unitarity [6]. One can choose as basic Majorana phases

$$
\gamma_{1} = \text{Arg} [V_{11} (V_{13})^{*}] \quad \beta_{1} = \text{Arg} [V_{12} (V_{13})^{*}]
$$

$$
\gamma_{2} = \text{Arg} [V_{21} (V_{23})^{*}] \quad \beta_{2} = \text{Arg} [V_{22} (V_{23})^{*}]
$$

$$
\gamma_{3} = \text{Arg} [V_{31} (V_{33})^{*}] \quad \beta_{3} = \text{Arg} [V_{32} (V_{33})^{*}]
$$

where we have dropped the PMNS superscript in the $V_{ij}$’s. The $\gamma_i, \beta_i$ can be evaluated in the present USY ansatz and we obtain in leading order,

$$
\gamma_{1} = -\frac{3\pi}{4} - \frac{11 - 16\delta + 4\delta^{2}}{8(1 - \delta)} \varepsilon \quad \beta_{1} = -\frac{\pi}{4} + \frac{16\delta - 11\delta^{2} + 1}{8(1 - \delta)} \varepsilon
$$

$$
\gamma_{2} = -\frac{\pi}{4} - \arctan \left( \frac{3 \sin(\phi)}{1 - \cos(\phi)} \right) \quad \beta_{2} = \frac{3\pi}{4} - \arctan \left( \frac{3 \sin(\phi)}{1 - \cos(\phi)} \right)
$$

$$
\gamma_{3} = \frac{3\pi}{4} - \arctan \left( \frac{3 \sin(\phi)}{1 - \cos(\phi)} \right) \quad \beta_{3} = \frac{3\pi}{4} - \arctan \left( \frac{3 \sin(\phi)}{1 - \cos(\phi)} \right)
$$
4 Numerical Results

The predictive power of our Ansatz is best shown with a set of figures. Fig. 1 demonstrates the dependence of the solar mixing angle on the value of $m$. We allow for explicit experimental uncertainties of $\Delta m_{21}^2 = 7.65^{+22}_{-20} \times 10^{-5} \, eV^2$ and $\Delta m_{31}^2 = 2.40^{+12}_{-11} \times 10^{-3} \, eV^2$. It is clear, that a central value for $\sin^2(\theta_{sol}) = 0.3$ implies a prediction for the value of $m$: $0.0316 \, eV < m < 0.0345 \, eV$. From Fig. 2 and 3, it also follows that $0.167 < |V_{13}^{PMNS}| < 0.179$ and that $0.00315 < M_{ee} < 0.00345$. Notice that, choosing the neutrino mass differences and $\sin^2(\theta_{sol}) = 0.304^{+16}_{-16}$ within these experimental constraints, our model accommodates the upper limit for $|V_{13}^{PMNS}|^2 < 0.004$. The mixing angle $\sin^2(\theta_{atm})$ and the experimental observable measuring CP violation $I_{CP}$ depend crucially on the angle $\phi$ and thus we may plot the two experimental observables against each other. From Fig. 4 we find, that for a central value of $\sin^2(\theta_{sol}) = 0.5$, that $0.0090 < I_{CP} < 0.0098$.

Next, we give an explicit numerical example, where six of the input parameters of the Ansatz are fixed by the known charged lepton masses, two neutrino mass differences $\Delta m_{21}^2, \Delta m_{31}^2$, together with a chosen value for the lightest neutrino mass $m_{1}$. Then, the six parameters of $V_{PMNS}$ are all predicted with a single free parameter, namely the phase $\phi$, which is taken to be $\phi = \frac{\pi}{2}$.

**INPUT:**

\[
\begin{align*}
  c_l &= 1023.72 \, eV \\
  a_l &= 1.729 \times 10^{-3} \\
  a_\nu &= 0.66 \\
  b_l &= 0.2677 \\
  b_\nu &= 0.5077 \\
  c_\nu &= 0.0290352 \, eV \\
  \phi &= \frac{\pi}{2}
\end{align*}
\]

where, for this particular example we have $\delta = 0.4286$ and $\varepsilon = 0.1927$. We then find

**OUTPUT**

\[
|V_{PMNS}| = \begin{pmatrix}
0.81573 & 0.55015 & 0.17867 \\
0.30173 & 0.66298 & 0.68514 \\
0.49350 & 0.50773 & 0.70616
\end{pmatrix};
\]

with

\[
\sin^2(\theta_{sol}) = 0.313 \quad ; \quad \sin^2(\theta_{atm}) = 0.485 \quad ; \quad |V_{13}^{PMNS}|^2 = 0.0319
\]

and

\[
\begin{align*}
  m_e &= 0.51 \, MeV \\
  m_1 &= 4.15 \times 10^{-3} \, eV \\
  m_\mu &= 105.5 \, MeV \\
  m_2 &= 9.69 \times 10^{-3} \, eV \\
  m_\tau &= 1770 \, MeV \\
  m_3 &= 0.04917 \, eV
\end{align*}
\]

We obtain for the Majorana observables

\[
\text{Arg} \begin{pmatrix}
V_{11}^{PMNS} & V_{12}^{PMNS} & V_{13}^{PMNS} \\
V_{21}^{PMNS} & V_{22}^{PMNS} & V_{23}^{PMNS} \\
V_{31}^{PMNS} & V_{32}^{PMNS} & V_{33}^{PMNS}
\end{pmatrix}^{PMNS} = \begin{pmatrix}
-2.535 & -0.6145 \\
-2.230 & 2.731 \\
0.7857 & -0.3545
\end{pmatrix}
\]

and for the strength of the Dirac type CP violation and double beta decay

\[
M_{ee} = 3.53 \times 10^{-3} \, eV \quad ; \quad I_{CP} = 0.00906
\]

5 Conclusions

We have pointed out that a simple ansatz, inspired by the hypothesis of universality of Yukawa couplings, leads to a highly predictive scheme for leptonic mixing. If one uses as input the charged lepton and neutrino masses, then the three mixing angles and the three CP violating phases entering in $V_{PMNS}$ are all predicted in terms of a single phase which takes the value $\phi \approx \frac{\pi}{2}$. The Ansatz predicts a relatively large value of $|V_{13}^{PMNS}|$ and of $I_{CP}$, clearly at the reach of the next round of experiments [5].

At this stage, it is worth recalling that when applied to the quark sector, the USY hypothesis can accommodate the main features of the CKM matrix, but cannot account for the observed strength of CP violation in the quark sector, measured by the rephasing invariant $|\text{Im}[V_{ub}V_{cb}V_{ub}^{\dagger}V_{cb}^{\dagger}]|$. However, it has been recently pointed out [7] that sufficient CP violation can be obtained in extensions of the
Standard Model where a USY structure is assumed but extra down singlet quarks are introduced and mix with the standard quarks. The USY ansatz has clearly a great appeal. A crucial open question is finding a symmetry principle, eventually implemented in a framework with extra dimensions [4], which can naturally lead to the universality of the strength of Yukawa couplings.

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Figure 2: $|V_{13}|$ as a function of $\sin^2 \theta_{sol}$, assuming 1\(\sigma\) uncertainties in neutrino mass differences

Figure 3: $M_{ee}$ as a function of $\sin^2 \theta_{sol}$, assuming 1\(\sigma\) uncertainties in neutrino mass differences
Figure 4: $I_{CP}$ as a function of $\sin^2 \theta_{atm}$, assuming $1\sigma$ uncertainties in neutrino mass differences

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