Black hole remnant of black hole-neutron star coalescing binaries with arbitrary black hole spin

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A model for determining the dimensionless spin parameter and mass of the black hole remnant of black hole-neutron star mergers with arbitrary initial black hole spin angular momentum, binary mass ratio, and neutron star mass and cold equation of state is formulated. Tests against numerical-relativity results are carried out, showing that both the dimensionless spin parameter and the final mass are accurately reproduced. For the first time, the behavior of both quantities and of the \( l = 2, m = 2, n = 0 \) quasinormal mode frequency is inspected throughout the parameter space. Predictions of this frequency may be exploited to guide gravitational-wave modeling and detection efforts and to extract physical information from detected gravitational-wave signals that would help us break degeneracies between binary black hole and black hole-neutron star systems, improve our understanding of compact binary formation, and constrain the neutron star equation of state.

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I. INTRODUCTION

Following its formation, a black hole-neutron star (BH-NS) binary system loses orbital energy and angular momentum by emitting gravitational radiation. During the inspiral stage, it readjusts its orbital separation accordingly, until the energy reservoir is emptied and the two compact objects merge. The remnant of this coalescence consists of a black hole possibly surrounded by a hot, massive accretion torus [1]. BH-NS binaries have not been observed yet, but population synthesis studies suggest that BH-NS mergers are likely to occur frequently in the Hubble volume [2–5]. By detecting the radiation emitted during their inspiral and merger, gravitational-wave (GW) observatories — such as LIGO [6], Virgo [7], KAGRA [8], and the Einstein Telescope [9] — are therefore likely to be involved in the first direct observations of BH-NS systems. Advanced LIGO is expected to see between 0.2 and 300 BH-NS binaries per year [10]. Other than being GW sources [11–14], BH-NS binaries, or mixed binaries, are among promising progenitor candidates for a fraction of the short-hard gamma-ray bursts (SGRBs) we observe [15, 16]. Further, NSs in BH-NS systems may undergo tidal distortions and disruptions. This aspect has drawn considerable attention in relation to the possibility of constraining the equation of state (EOS) of matter at supranuclear densities [17–20] and in relation to the physical processes the merger ejecta may be involved in. Radioactive decays triggered in the outflows by the formation of heavy isotopes may power day-long optical transients (“kilonovae”), and shocks generated by the outflows themselves in an interstellar medium of sufficiently high density may also lead to visible electromagnetic counterparts [21, 22]. Additionally, the ejecta are believed to be linked to the abundances of heavy elements formed in \( r \)-processes, by rapid neutron capture [23].

Numerical-relativity dynamical simulations of BH-NS mergers will ultimately allow us to fully understand and explore the physics behind these events. Research with this intent was initiated fairly recently [24–54] and has benefitted from relativistic quasiequilibrium studies [55–61]. As fully relativistic dynamical simulations remain challenging and computationally intensive, pseudo-Newtonian calculations, e.g. [62], and analytical studies, e.g. [17–19, 63–71], have been carried out as well for BH-NS systems. These studies generally focus on a few selected physical aspects — possibly even a single one — of BH-NS mergers and provide insight on the problem by offering access to the large space of parameters of mixed binaries at low computational costs.

This paper belongs to this last category of approaches and extends the work presented in [71] (paper I, henceforth). Its focus is to predict the dimensionless spin parameter and the mass of the BH remnant of BH-NS coalescing binaries. As opposed to paper I, no restriction to cases with parallel BH spin angular momentum and orbital angular momentum is made: in this sense, this article generalizes the model of paper I. Our semianalytical approach is based on the model of Buonanno, Kidder, and Lehner (BKL) to estimate the final BH spin of a BH-BH coalescence which exploits angular momentum and energy conservation for the binary from its last stable orbit onwards [72]. In order to account for the fraction of orbital angular momentum and energy that may not accrete immediately onto the BH, our model requires us to know the baryonic mass \( M_{b,\text{torus}} \) surviving the BH-NS merger and surrounding the BH remnant. As in paper I, when making predictions for BH-NS systems for which no numerical-relativity simulations exist yet, we rely on the relativistic fitting formula of [70] to determine \( M_{b,\text{torus}} \). This fit was calibrated to numerical-relativity simulations of aligned BH-NS mergers, but, as recently pointed out [50, 73], it may also be used for mixed binaries with nonparallel BH spin angular momentum and orbital angular momentum. In terms of \( M_{b,\text{torus}} \), a BH-NS merger with misaligned BH spin is roughly equivalent to a BH-NS merger with an effective, lower, aligned BH spin angular momentum given by Eq. (13). This allows us to go beyond the formulation of paper I and to generalize the model formulated therein. The key equations of our approach are Eqs. (8)-(10), along with the ansatz of Eq. (5), which appeared in paper I and is left unchanged for simplicity. Our tests show that our simple method accurately reproduces the
results of numerical-relativity simulations of BH-NS mergers with tilted initial BH spin [41, 50], despite the mathematical complexity of the BH-NS coalescence problem. Further, by exploring the BH-NS space of parameters, we illustrate how our predictions may be used to guide GW detection efforts and to extract valuable astrophysical information from detected GW signals.

The paper is organized as follows. In Sec. II we review the approach developed in paper I to predict the spin parameter and mass of BH remnants of BH-NS binaries with aligned BH spin angular momentum and orbital angular momentum. In Sec. III we propose an extension of this method and successfully test it against available numerical-relativity data. In Sec. IV we gather and discuss the results obtained by systematically varying the binary mass ratio, the initial BH spin parameter and inclination, and the NS mass and EOS. Finally, in Sec. V, we draw our conclusions and collect our remarks.

II. NONPRECESSING BH-NS BINARIES

In paper I we formulated and validated an approach to estimate the final spin parameter and mass of BH remnants of BH-NS mergers in which the initial BH spin and the orbital angular momentum are aligned. Our method was based on the BKL approach to estimate the final spin of BH-BH mergers [72] and is reviewed in this section, prior to extending it to arbitrary initial BH spin orientations in the next section.

We begin by recalling that we set the initial spin angular momentum of the NS to zero, as this is believed to be a reliable approximation of astrophysically realistic systems [74, 75] and because the same assumption is made in building initial

\[
\tag{1}
\begin{align*}
\bar{r}_{\text{ISCO}} & = \left[3 + Z_2 \mp \sqrt{(3 - Z_1)(3 + Z_1 + 2Z_2)}\right] \\
Z_1 & = 1 + (1 - a^2)^{1/3} \left[(1 + a)^{1/3} + (1 - a)^{1/3}\right] \\
Z_2 & = \sqrt{3a^2 + Z_1^2},
\end{align*}
\]

where the upper(lower) sign holds for corotating(counterrotating) orbits, and where \(\bar{r} = r/m\) denotes the (dimensionless) Boyer-Lindquist radial coordinate. In this same notation, the orbital angular momentum per unit mass of a test particle orbiting the BH may be expressed as

\[
\tag{2}
l_z(\bar{r}, a) = \pm \frac{\bar{r}^2 \pm 2a\sqrt{\bar{r}} + a^2}{\bar{r}^2 - 3\bar{r} \pm 2a\sqrt{\bar{r}}}^{1/2},
\]

while its orbital energy reads

\[
\tag{3}
e(\bar{r}, a) = \frac{\bar{r}^2 - 2\bar{r} \pm a\sqrt{\bar{r}}}{\bar{r}^2 - 3\bar{r} \pm 2a\sqrt{\bar{r}}}^{1/2}. 
\]

Given a BH-NS binary formed by a BH of mass (at isolation) \(M_{\text{BH}}\) and initial dimensionless spin parameter \(a_i\), and a NS of mass (at isolation) \(M_{\text{NS}}\), the dimensionless spin parameter \(a_f\) of the BH remnant may be determined by solving (numerically) the following equation:

\[
\tag{4}
a_f = \frac{a_i M_{\text{BH}}^2 + l_z(\bar{r}_{\text{ISCO}, f}, a_i) M_{\text{BH}} \left[\left[ 1 - f(\nu) \right] M_{\text{NS}} + f(\nu) M_{\text{BH,NS}} - M_{\text{b,torus}} \right]}{M \left[ 1 - \left(1 - e(\bar{r}_{\text{ISCO}, f}, a_i)\right)\nu \right] - e(\bar{r}_{\text{ISCO}, f}, a_i) M_{\text{b,torus}}}^{1/2}. 
\]

In the previous expression, the notation \(M = M_{\text{BH}} + M_{\text{NS}}\) was used, \(\bar{r}_{\text{ISCO}}\) and \(\bar{r}_{\text{ISCO}, f}\) denote \(\bar{r}_{\text{ISCO}}\) calculated for the initial and final BH spin parameter, respectively, \(\nu = M_{\text{BH}} M_{\text{NS}}/M^2\) is the symmetric mass ratio, \(M_{\text{b,torus}}\) is the baryonic mass of the torus remnant, and

\[
\tag{5}
f(\nu) = \begin{cases} 0 & \nu \leq 0.16 \\ \frac{1}{2} \left[ 1 - \cos \left( \frac{\pi(\nu - 0.16)}{2/9 - 0.16} \right) \right] & 0.16 < \nu < 2/9 \\ 1 & 2/9 \leq \nu \leq 0.25 \end{cases}
\]

is a function that regulates the transition between nondisruptive and disruptive coalescences, which happen for low and high values of \(\nu\), respectively. We remind the reader that Eq. (4) follows from angular momentum and energy conservation considerations. The terms \(-l_z(\bar{r}_{\text{ISCO}, f}, a_i) M_{\text{BH}} M_{\text{b,torus}}\) and \(-e(\bar{r}_{\text{ISCO}, f}, a_i) M_{\text{b,torus}}\) that appear in the numerator and denominator, respectively, are expressions that approximate the angular momentum and energy of the NS material that survives the merger, remains outside the BH horizon, and is not accreted onto the BH. Let us also remark that the term \(-M [1 - e(\bar{r}_{\text{ISCO}, f}, a_i)] \nu \) in the denominator accounts for the orbital energy radiated away in GWs by the binary system during its inspiral [77].

The closed expression in Eq. (4) may be solved numerically with root-finding techniques to determine the dimensionless spin parameter of the BH remnant. Once this is done, the denominator on the right-hand side of Eq. (4) automatically provides a prediction for the mass of the BH remnant itself:

\[
\tag{6}
M_f = M \left[ 1 - \left(1 - e(\bar{r}_{\text{ISCO}, f}, a_i)\right)\nu \right] - e(\bar{r}_{\text{ISCO}, f}, a_i) M_{\text{b,torus}}. 
\]
III. GENERIC INITIAL BH SPIN CONFIGURATIONS

Consider now the more general scenario in which the initial spin angular momentum of the BH has an arbitrary direction and is therefore not necessarily parallel to the orbital angular momentum of the binary: in order to predict the spin and mass of the BH remnant of the BH-NS coalescence, it is possible to track the BKL approach and to modify our model — Eqs. (4)-(6) — accordingly.

We stress that we do not drop the assumption that the NS initial spin angular momentum vanishes. Once again, this approximation is present in the numerical-relativity simulations against which our model is tested. The total spin of the system thus coincides with the BH spin angular momentum \( \vec{S}_{BH} \) and has magnitude \( S_{BH} \equiv |\vec{S}_{BH}| \) and direction \( \vec{S}_{BH} \equiv \vec{S}_{BH}/S_{BH} \). For an unequal mass binary with a single spinning component, i.e. the case we are treating, the magnitude of the total spin and the angle \( \theta_i \equiv \arccos(\bar{L}_{orb}, \vec{S}_{BH}) \) between the total spin and the orbital angular momentum, having direction \( \bar{L}_{orb} \) are constant up to the innermost stable spherical orbit (ISSO), to an excellent approximation [78].

\[
\bar{L}_{orb}(\bar{r}_{ISSO}, a_I, \theta_I) M_{BH}\left\{1 - f(\nu)\right\} M_{NS} + f(\nu) M_{b,NS} - M_{b,torus}\cos\theta_I + a_I M_{BH}^2 \cos(\theta_I - \theta) = a_I M_{BH}^2
\]

where \( l_{orb} = |\bar{L}_{orb}|, a_{i,f} = |\bar{a}_{i,f}| \), and the mass of the BH remnant \( M_f \) is

\[
M_f = M \left\{ 1 - \frac{c(\bar{r}_{ISSO}, a_I, \theta_I) \nu}{c(\bar{r}_{ISSO}, a_I, \theta_I) M_{b,torus}} \right\}
\]

a straightforward generalization of Eq. (6). In extending the equatorial case model of paper I, Eqs. (4)-(6), we decide not to modify the ansatz of Eq. (5) that defines \( f(\nu) \): this is done for the sake of simplicity and because this does not seem to be necessary in light of the validation tests performed (see Sec. IIIA). The formulation of the model is completed by discussing \( l_{orb}, c, \) and \( M_{b,torus} \). As in [72], we compute the orbital angular momentum \( \bar{L}_{orb} \) of the inclined orbit using the fitting formula of [79]; similarly, we choose to compute the orbital energy \( e \) at the ISSO via the equivalent fitting formula provided in the same reference. The two expressions are

\[
l_{orb}(\bar{r}_{ISSO}, a, \theta) = \frac{1}{2} \left( 1 + \cos \theta \right) e_{pro}^\prime(\bar{r}_{ISCO}^\prime, a) + \frac{1}{2} \left( 1 - \cos \theta \right) e_{ret}^\prime(\bar{r}_{ISCO}^\prime, a)
\]

\[
e(\bar{r}_{ISSO}, a, \theta) = \frac{1}{2} \left( 1 + \cos \theta \right) e_{pro}(\bar{r}_{ISCO}^\prime, a) + \frac{1}{2} \left( 1 - \cos \theta \right) e_{ret}(\bar{r}_{ISCO}^\prime, a)
\]

As in the BKL approach, we assume that \( M_{BH}, M_{NS}, \bar{S}_{BH}, \) and \( \bar{L}_{orb} \) are all known at some point of the inspiral prior to the ISSO. Under the assumption that the angular momentum of the system changes only by a small amount during the merger and ringdown stages (cf. paper I), the total angular momentum \( \bar{J}_f \equiv \bar{a}_I M_{BH}^2 \) of the BH remnant may be computed by adding the initial spin angular momentum of the system, \( \bar{S}_{BH} \), and the orbital angular momentum accreted onto the BH itself. As pointed out in [72], the key point in estimating the orbital angular momentum contribution is to allow for the last stable orbit to be inclined with respect to the total angular momentum of the BH remnant \( \bar{J}_f \). Bearing this in mind, we generalize the term in Eq. (4) for the orbital angular momentum accreted onto the BH as

\[
l_{orb}(\bar{r}_{ISCO}, a_I, \theta_I) M_{BH}\left\{1 - f(\nu)\right\} M_{NS} + f(\nu) M_{b,NS} - M_{b,torus}\cos\theta_I + a_I M_{BH}^2 \cos(\theta_I - \theta) = a_I M_{BH}^2
\]

where \( l_{orb} \) is the orbital angular momentum per unit mass of a test particle orbiting a BH at the ISSO and \( \theta_I \) labels the angle between \( \bar{L}_{orb} \) at the ISSO and \( \bar{J}_f \). The angular momentum of the BH remnant is therefore the sum of the last expression and \( \bar{S}_{BH} \equiv \bar{a}_I M_{BH}^2 \). We can project this sum along \( \bar{J}_f \) itself and orthogonally to it. This readily yields the two equations

\[
\frac{1}{2} \left( 1 + \cos \theta \right) e_{pro}(\bar{r}_{ISCO}^\prime, a) + \frac{1}{2} \left( 1 - \cos \theta \right) e_{ret}(\bar{r}_{ISCO}^\prime, a) = \frac{1}{2} \left( 1 + \cos \theta \right) e_{pro}(\bar{r}_{ISCO}^\prime, a)
\]

where \( r_{ISCO}^\prime, l_{ISCO}^\prime, \) and \( e_{ISCO} \) are given in Eqs. (1)-(3) for prograde (pro) and retrograde (ret) orbits. Finally, the baryonic mass surviving the merger, \( M_{b,torus} \), may be estimated by following the consideration made by Stone and collaborators in [73] and further validated by Foucart and collaborators in [50]: in terms of \( M_{b,torus} \), a BH-NS merger with misaligned BH spin is roughly equivalent to a BH-NS merger with lower, aligned BH spin angular momentum with dimensionless parameter \( a_{i,Eff} \). We determine this effective dimensionless spin parameter by solving

\[
r_{ISCO}(a_{i,Eff}, \theta) = \bar{r}_{ISCO}(a_I, \theta_I)
\]

where \( \bar{r}_{ISCO}(a_{i,Eff}) \) may be calculated via Eq. (1) and where the second equality expresses \( \bar{r}_{ISCO}(a_I, \theta_I) \) and is carried out by following the fitting formulas of [79], in accordance with Eqs. (11)-(12). \( M_{b,torus} \) is then computed using \( a_{i,Eff} \) in the relativistic fitting formula of [70] for aligned BH-NS mergers.

Summarizing, given a BH-NS binary formed by a NS with gravitational mass \( M_{NS} \), baryonic mass \( M_{b,NS} \), and radius \( R_{NS} \) (determined by its EOS), and a BH with gravitational mass \( M_{BH} \), and initial spin of (dimensionless) magnitude \( a_i \) and ini-
tial inclination $\theta_i$ with respect to the orbital angular momentum vector of the binary, the properties of the BH remnant may be calculated as follows:

1. solve Eq. (13) numerically for $a_{i,\text{EFF}}$;
2. use $a_{i,\text{EFF}}$ to calculate the baryonic mass of the matter that survives the merger, $M_{\text{bh,torus}}$, following [70];
3. solve Eqs. (8)-(9) numerically to determine (a) the (dimensionless) magnitude of the spin angular momentum of the BH remnant, $a_i$, and (b) its inclination angle $\theta_i$ with respect to the orbital angular momentum;
4. determine the gravitational mass of the BH remnant by plugging $M_{\text{bh,torus}}$ and $a_i$ in Eq. (10).

As a final remark, we stress that for $\theta_i = 0$, i.e. for nontilted BH spins, this procedure reduces to the model formulated and tested in paper I.

### A. Validation of the model

Numerical-relativity simulations of BH-NS coalescences with misaligned BH spin and orbital angular momentum vectors were reported in [41, 50], for a total of seven binaries. These form the test bed currently available for our model.¹ The tests are collected in Table I. The first column of the table numbers the tests, while the second column provides the reference in which the numerical-relativity simulation was performed by Foucart and collaborators find tilt angles of $\theta_i$ between the disk possibly formed in a BH-NS merger and the BH remnant spin angular momentum vector [73]. The interest in $\psi_d$ derives from its direct link to the possibility of having quasi-periodic signals of jet precession as an observational signature of SGRBs originating from BH-NS mergers, as opposed to ones originating from binary NS systems [73]. A unique $\psi_d$ angle is an idealized concept, as the postmerger dynamics of BH-NS systems are complex, but a disk tilt angle measurement is provided for the simulations reported in [41]. This is calculated where the surface density of the disk is maximum, at late times. The best we can do to link $\theta_i$ to numerical-relativity results is, therefore, to compare it to the quantity just discussed. For the first four cases discussed in Table I, Foucart and collaborators find tilt angles of $2^\circ$, $4^\circ$, $7^\circ$, and $8^\circ$ [41]. These are all a factor $\sim 3$–$4$ smaller than $\theta_i$. In light of these comparisons, we may thus conclude that $\theta_i$ may be used to constrain the disk tilt angle via

$$\theta_i/4 \lesssim \psi_d \lesssim \theta_i/3.$$  (14)

In Table II, we repeat the tests of Table I using the numerical-relativity values of $M_{\text{bh,torus}}$, instead of the ones obtained by exploiting Eq. (13) and the relativistic fit of [70]. As we see, this slightly improves the performance of our model.

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¹ Sixteen BH-BH tests were carried out in the BKL paper [72].
TABLE II. Tests against results of numerical-relativity simulations of BH-NS coalescences with misaligned spin reported in the literature. The first five columns are organized as in Table I. The sixth column provides the difference between our estimate $a_{\text{f}}$ and $a_{\text{NR}}$, as opposed to Table I, where the fit of [70] and the approximation outlined in Eq. (13) were used, $a_{\text{f}}$ is computed using the numerical-relativity result for $M_{\text{beyond}}$. In the seventh column, we report the difference between the BKL prediction for the final spin parameter and $a_{\text{NR}}$. The last two columns are the same as in Table I (the values are once more obtained using the numerical-relativity result for $M_{\text{beyond}}$).

| Ref. | $Q$ | $a_i$ | $\theta_i$ | $a_i - a_{\text{NR}}$ | $a_{\text{f}}^{\text{BKL}} - a_{\text{NR}}$ | $\epsilon(M_f/M)$ | $\theta_f$ |
|------|-----|-------|------------|------------------------|--------------------------------|------------------|-----------|
| 1    | [41]| 3    | 0.5       | 20°                    | 0.01                           | 0.02              | 0.00      |
| 2    | [41]| 3    | 0.5       | 40°                    | 0.01                           | 0.02              | 0.00      |
| 3    | [41]| 3    | 0.5       | 60°                    | 0.01                           | 0.03              | 0.01      |
| 4    | [41]| 3    | 0.5       | 80°                    | 0.02                           | 0.03              | 0.02      |
| 5    | [50]| 7    | 0.9       | 20°                    | 0.01                           | 0.00              | 0.00      |
| 6    | [50]| 7    | 0.9       | 40°                    | 0.01                           | 0.01              | 0.01      |
| 7    | [50]| 7    | 0.9       | 60°                    | 0.01                           | 0.01              | 0.00      |

In order to gauge how well our approach performs, and in addition to the comparisons of Table II, in concluding this section we would like to draw a comparison with a model formulated in [73] to predict the spin of the BH remnant of BH-NS mergers. We wish to remark that the scope of the work presented by Stone and collaborators was not to build the most accurate model to calculate $a_{\text{f}}$, but to study the origin of SGRBs. The method of [73] is based on a post-Newtonian fit, and it, too, uses the results of [70] to predict $M_{\text{beyond}}$, albeit via the Newtonian fit, and not the relativistic one used in this paper and in paper I. The predictions for the magnitude of the dimensionless spin angular momentum vector of the BH remnant yielded by the model ($a_{\text{PN}}^\text{f}$) are tested in Table II of [73] against the numerical-relativity results of [41, 44, 50]. In Table III, we report these same comparisons with the addition of our predictions for $a_{\text{f}}$. The models seem to perform equivalently for high BH masses, i.e. for $Q = 7$, regardless of the initial spin configuration of the BH and the NS compactness. We observe that, for these binaries, our prediction almost systematically slightly exceeds the numerical-relativity result, while the post-Newtonian model slightly underestimates it. For $Q = 5$, that is, for a moderate BH mass, the post-Newtonian model underestimates $a_{\text{f}}^{\text{NR}}$ by 0.03, while our model happens to reproduce it exactly. The trend continues to progress this way as the mass ratio is lowered: with the exception of cases 11 and 17, for which $\{a_i = 0.0, \theta_i = 0°\}$ and $\{a_i = 0.5, \theta_i = 80°\}$, our model performs significantly better. The most notable case is the one with $\{a_i = 0.9, \theta_i = 0°\}$, for which $a_{\text{f}}^{\text{NR}} - a_{\text{f}}^{\text{PN}} = 0.10$, while $a_{\text{f}}^{\text{NR}} - a_{\text{f}} = -0.01$. From these comparisons, we therefore conclude that, overall, our phenomenological model is better at reproducing the numerical-relativity results.

IV. RESULTS

We may now use our model to explore the space of parameters of BH-NS systems systematically and to study the properties of the BH remnant ($a_{\text{f}}, \theta_{\text{f}}$, and its $l = m = 2, n = 0$ quasinormal mode frequency). We do so by varying

(i) the binary mass ratio $Q$ between 2 and 10;
(ii) the initial dimensionless spin parameter of the BH, $a_i$, between 0 to 0.99;
(iii) the angle the BH spin initially forms with respect to the orbital angular momentum, $\theta_i$, from 0° to 180°;

2 We remind the reader that $M_f = M$ in the BKL approach.
(iv) the NS mass, between $1.2 M_\odot$ and $2.0 M_\odot$, compatibly with the measurement reported in [80];

(v) the NS EOS. More specifically, we quote most results for the APR2 EOS [81, 82], since this is the most complete nuclear many-body study to date; additionally, when considering EOS-dependent effects (see Fig. 2), we use the WFF1 EOS [83] and the PS EOS [84] as representatives of the softest and stiffest possible EOS, yielding the most and least compact NSs, respectively.

We wish to remark that the model is untested in the $a_i \sim 0.99$ regime, where studies have shown that BH-NS systems behave very nonlinearly [53] and where a basic assumption behind the $M_{\text{torus}}$ fit of [70] may break down.

A first noteworthy result is that, when spanning the space of parameters just outlined, our model does not produce over-spinning BHs ($a_i > 1$), which were instead encountered, and discarded, in [73].

We begin our analysis by discussing the results collected in the four panels of Fig. 1, where $M_{\text{NS}} = 1.4 M_\odot$ and the APR2 EOS is used. In the top panels, we show $a_i$ and $\Delta a \equiv a_i - a_i$ as a function of the symmetric mass ratio of the binary, $\nu$, and $a_i$ for specific values of the initial tilt angle $\theta_i$; in the bottom panels, we show $\theta_i$ and $\Delta \theta \equiv \theta_i - \theta_i$ as a function of $\nu$ and $\theta_i$ for specific values of $a_i$. In the top left panel we consider $a_i$ for $\theta_i = 0^\circ$, $120^\circ$, and $180^\circ$. Notice that the three surfaces all intersect at $a_i = 0$, as expected. With the exception of this line, for any combination of $a_i$ and $\nu$, $a_i$ is highest if $\theta_i = 0^\circ$, i.e. if the BH spin angular momentum and the orbital angular momenta are aligned. As $\theta_i$ is increased, the values of $a_i$ decrease progressively. In particular, the conditions for having $\theta_i = 0^\circ$, as it is evident for the $\theta_i = 180^\circ$ surface. Recalling that $a_i$ and $a_i$ are positive, the shape of the $\theta_i = 180^\circ$ surface tells us that, under certain combinations of the initial physical parameters, a spin flip may take place in BH-NS binaries. This means that the initial and final BH spin angular momenta may be antiparallel. For high values of $a_i$, this happens for high symmetric mass ratios, that is, when the values of the NS and BH mass are close, so that the angular momentum accreted onto the BH is more “effective” in increasing the BH spin angular momentum. Notice that for $a_i \lesssim 0.4$ a spin flip always occurs in the range of $\nu$ we consider.

In the top right panel of Fig. 1 we look at our results in terms of $\Delta a$. This time, we show the surface for $\theta_i = 100^\circ$ instead of $120^\circ$, so that the results for $\theta_i = 180^\circ$ are visible more clearly. Again, we see that the more the BH spin angular momentum is initially aligned to the orbital angular momentum, the more this is transferred to the BH, as expected, so that the highest values of $\Delta a$ are obtained when $\theta_i = 0^\circ$. The highest of these, in particular, is encountered when the BH is initially nonspinning and $\nu = 2/9$, which corresponds to the lowest value of $Q$ that we use. Notice, once more, the effects of spin flip on the shape of the $\theta_i = 180^\circ$ surface.

In the remaining two panels of Fig. 1 we show $\theta_i$ and $\Delta \theta = \theta_i - \theta_i$ as a function of $\nu$ and $\theta_i$ for $a_i = 0.15, 0.30, 0.45, 0.75$, and $0.99$. For $a_i < 0.30$, $\theta_i \lesssim 50^\circ$, no matter what $\theta_i$ is considered. When $a_i$ is increased, however, higher and higher angles $\theta_i$ are achieved for any given $\nu$ and $\theta_i$. At $\theta_i = 180^\circ$, this behavior culminates in spin flip and, with the right combinations of $a_i$ and $\nu$, $\theta_i$ can be $0^\circ$. As the plots show, the critical value of the mass ratio for the onset of spin flip increases with $a_i$. This means that as $a_i$ grows it is increasingly harder to flip the spin of the BH. On the other hand, $\theta_i = 0^\circ$ for any value of the symmetric mass ratio when $a_i$ is low.

Knowing the mass and spin of the BH remnant, we may determine its $l = 2, m = 2, n = 0$ quasinormal mode (QNM) frequency $f_{220}^{\text{QNM}}$. This is done via the fit provided in [85]. Our results for $f_{220}^{\text{QNM}}$ are reported in the panels of Fig. 2. The mass of the NS in the BH-NS binary system is taken to be $M_{\text{NS}} = 1.4 M_\odot$ in all panels. The dimensionless BH spin parameter is progressively increased from top to bottom. The WFF1 and PS EOS are used in the left and right panels, respectively. As in paper I, this choice is motivated by the fact that these EOSs yield the most and least compact NSs, respectively. In each panel, we report $f_{220}^{\text{QNM}}$ as a function of $\theta_i$ for $q \equiv Q^{-1} = M_{\text{NS}}/M_{\text{BH}} = 0.1, 0.2, 0.3, 0.4$, and $0.5$. Physically, a decrease in the BH mass and an increase in the BH spin both cause the QNM frequency value to increase. It is therefore not surprising to observe that, overall, the QNM frequency increases as $q$ and/or $a_i$ grow. A second characteristic all panels share, is that, all else being fixed, increasing the initial tilt angle $\theta_i$ causes $f_{220}^{\text{QNM}}$ to decrease. Once again, this is due to the fact that the spin of the BH remnant diminishes as $\theta_i$ grows. A third general behavior we observe is that, given an NS EOS and an initial BH spin parameter, the range of values spanned by $f_{220}^{\text{QNM}}$ as $q$ runs from 0.1 to 0.5 narrows as $\theta_i$ increases.

Within the space of parameters we consider, $f_{220}^{\text{QNM}}$ varies between $1 \text{kHz}$ and $6.5 \text{kHz}$. A ringdown occurring at the lower end of this range may be detected by second generation gravitational-wave detectors. On the other hand, third generation detectors with enhanced sensitivity in the high-frequency regime, like the Einstein Telescope, will be necessary to access the possible ringdown of systems yielding $f_{220}^{\text{QNM}} \gtrsim 2 \text{Hz}$, unless these are particularly close to Earth, obviously.

A key point is, naturally, whether the ringdown occurs or not, that is, whether an oscillation at $f_{220}^{\text{QNM}}$ is excited or not. A simple criterion to estimate whether this is the case or not may be established by considering the mass $M_{\text{torus}}$ of the material that is not accreted onto the BH. If $M_{\text{torus}} > 0$, the NS is tidally disrupted, (a part of) its matter accretes incoherently onto the BH, and the ringdown of the BH is unlikely to be excited. When $M_{\text{torus}} = 0$ the accretion is more coherent — the majority of these mergers end with a plunge — and therefore the BH ringdown may be excited.\(^4\) In the panels

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\(^3\) Further details on the EOS are discussed in the Appendix of paper I.

\(^4\) This different definition of the mass ratio is used in order to obtain evenly spaced curves in the panels of Fig. 2 when varying the mass ratio by a fix amount.

\(^5\) See [86] for a more detailed discussion.
FIG. 1. (Color online) APR2 EOS and $M_{\text{NS}} = 1.4 M_\odot$. Top left: $a_f$ as a function of $a_i$ and $\nu$ for $\theta_i = \{0^\circ, 120^\circ, 180^\circ\}$ (as indicated by the labels on the surfaces). Top right: $\Delta a = a_f - a_i$ as a function of $a_i$ and $\nu$ for $\theta_i = \{0^\circ, 100^\circ, 180^\circ\}$ (as indicated by the labels on the surfaces). Bottom left: $\theta_f$ as a function of $\theta_i$ and $\nu$ for $a_f = \{0.15, 0.3, 0.45, 0.75, 0.99\}$ (as indicated by the labels on the surfaces). Bottom right: $\Delta \theta = \theta_f - \theta_i$ as a function of $\theta_i$ and $\nu$ for $a_i = \{0.15, 0.3, 0.45, 0.75, 0.99\}$ (as indicated by the labels on the surfaces).

of Fig. 2, a black, thick, continuous line divides systems with $M_{\text{bh,torus}} > 0$ from ones with $M_{\text{bh,torus}} = 0$: the former are above the line, and the latter are below. The $f_{220}^{\text{QNM}}$ curves are colored when $M_{\text{bh,torus}} = 0$, that is, when the BH ringdown is expected to be excited and, therefore, imprinted in the emitted gravitational radiation. Notice that the EOS affects the location of the boundary between systems such that $M_{\text{bh,torus}} = 0$ and ones that yield $M_{\text{bh,torus}} > 0$, and thus the portion of the parameter space in which a BH ringdown may take place.
FIG. 2. (Color online) $l = 2, m = 2, n = 0$ quasinormal mode frequency, $f_{220}^{\text{QNM}}$, as a function of the initial tilt angle $\theta_i$. The NS mass is taken to be $1.4M_\odot$. Left and right panels refer to the WFF1 EOS and PS EOS, respectively. The initial BH spin parameter $a_i$ increases from top to bottom, as indicated by the title of each panel. Different colors and combinations of dots and dashes refer to different values of the mass ratio $q = M_{\text{NS}}/M_{\text{BH}}$, as indicated in the legend of the bottom right panel. The thick, black, continuous line marks the boundary between systems with $M_{\text{torus}} = 0$ and ones with $M_{\text{torus}} > 0$: these are located below and above the line, respectively. The $f_{220}^{\text{QNM}}$ curves are gray above this line to indicate that the excitation of the quasinormal mode is unlikely.
In this paper we presented a model that allows us for the first time to accurately determine the final spin parameter, $a_f$, and mass, $M_f$, of the BH remnant of BH-NS coalescences, with arbitrary mass ratio, NS mass, NS cold, barotropic EOS, and BH spin magnitude and orientation. This work generalizes the model presented in paper I, which was limited to mixed binaries with aligned BH spin and angular momentum. It is based on the Buonanno, Kidder, and Lehner model for the final spin of binary BH mergers [72]. Our approach relies on a fit to the mass of the torus remnant of aligned BH-NS mergers [70] and on its use for BH-NS mergers with nonparallel BH spin and orbital angular momentum, a possibility just recently pointed out [50, 73].

Exploring the BH-NS parameter space via numerical-relativity simulations is desirable but in practice very time and resource consuming. Further, results for misaligned initial BH spin configurations were so far presented by the Caltech-Cornell-CITA Collaboration only [41, 50]. While this means there is just a handful of possible tests that allow us to validate the model discussed in this paper for binaries with inclined initial BH spin angular momentum, it also means that we are now able to study certain aspects of BH-NS coalescing systems for the first time.

The model was tested by comparing its predictions to the recent seven numerical-relativity simulations of BH-NS mergers with inclined initial BH spin angular momentum. The agreement between our predictions and the numerical results is at the level of the tests performed in paper I for systems with aligned BH spin: the maximum absolute error on the final BH spin parameter is 0.02, while the maximum relative error on the final BH gravitational mass is 1% when using the numerical-relativity result for the mass of the torus remnant and 2% when using the prediction of [70] for the same quantity. Additionally, in Eq. (14) we were able to establish how the final inclination angle $\theta_f$, which is another natural output of the model [see Eqs. (7)-(9)], may be used to constrain the misalignment angle between the disk possibly formed in the merger and the final BH spin angular momentum vector. The importance of this angle lies in the fact that it is directly linked to observational signatures that may help us distinguish between SGRBs of BH-NS origin and ones of NS-NS origin [73]. Finally, we compared our model and a PN-based model that predicts the dimensionless spin parameter of the BH remnant [73]. The former was seen to be more accurate, especially when low BH masses are involved, particularly in combination with high initial spin magnitudes. Further, and as opposed to the PN-based model, our model is well behaved, in the sense that it does not produce overspinning BH remnants.

The approach presented and tested in the first part of this work was used to span the space of parameters consisting of the binary mass ratio, the NS mass, the NS EOS, and the BH initial spin magnitude and inclination. The model and results presented here improve our understanding of BH-NS systems throughout their space of parameters and may in principle be used in a number of ways. Prior to detection, they may be exploited to determine the best strategy to adopt when tar-

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**V. CONCLUSIONS AND REMARKS**

Results like the ones reported in Fig. 2 lead to a number of considerations regarding GW detection and astrophysics. In summary, our expectations in terms of the absence or presence of BH remnant ringdown in the GW signal emitted by a coalescing compact binary (i) may be used in guiding GW modeling and detection efforts and (ii) are a valuable source of astrophysical information when combined with detected GW signals. Let us illustrate these two statements by means of some examples.

- **Point-particle GW templates are sufficient to detect BH-NS inspirals** [19]. Figure 2 now tells us more: it shows where in the parameter space of BH-NS binaries a search may benefit from the use of BH-BH GW templates beyond the inspiral regime, and where it would not. For example, and with the exception of highly spinning BH cases, BH-NS systems with mass ratios of $1:10$ are expected to behave like BH-BH binaries and to exhibit the ringdown of the BH remnant, so that the use of complete inspiral-merger-ringdown (IMR) binary BH waveforms would be favorable. On the other hand, searches for BH-NS systems with moderate tilt angle and low BH mass may be limited to the inspiral stage and employ point-particle templates; alternatively, they could profit from the construction of BH-NS IMR templates.

- **Suppose the GW emission of a canonical BH-NS system with mass ratio $1:7$ is detected.** If, say, the initial BH spin and the orbital angular momentum are roughly aligned ($\theta_i \lesssim 25^{\circ}$) and $a_i \sim 0.8$, the lack of BH remnant ringdown in the detected GW may rule out a BH binary with equivalent physical parameters as a possible source, and it would rule out soft EOSs. On the other hand, the repeated observation of systems of this kind exhibiting ringing BH remnants would allow us to place upper bounds on the NS EOS stiffness.

- **Suppose, instead, that the GW emission of a compact binary system with mass ratio $1:4$ to $1:2$ is detected.** If, say, the initial spin angular momentum of the secondary is negligible, and the initial spin angular momentum of the primary is roughly aligned with the orbital angular momentum ($\lesssim 25^{\circ}$) and is such that the dimensionless spin parameter is $\sim 0.8$, then the absence of a BH remnant ringdown would tell us that this is a BH-NS system with a particularly low total mass, and not a BH-BH system. This, in turn, would have implications in terms of population synthesis, as would the repeated observations of binaries with these properties and manifest BH remnant ringdown.

In order to be feasible, the scenarios discussed necessitate further investigation in terms of data analysis, of course. In particular, determining whether the ringdown of the BH remnant is present in a detected GW requires a rigorous formulation. Our discussion, however, illustrates that our model may be used to break degeneracies between BH-BH and BH-NS systems. This in turn could improve our understanding of compact binary formation and help us constrain the NS EOS.
getting GWs emitted by BH-NS systems in a specific portion of the space of parameters (utilizing point-particle inspirals vs BH-BH IMR templates vs BH-NS IMR templates). Additionally, they show us where in the space of parameters the development of BH-NS IMR templates is especially important (roughly speaking, low total masses, moderate to high BH spins, and low to moderate, or even high if considering stiff NS EOSs, initial BH spin tilt angle): this is where improving the physics and the number of available numerical-relativity simulations would be particularly beneficial. At the same time, this also indicates the direction in which recent work on the interface between numerical and analytical modeling of GWs should be extended [20, 86], possibly using the model presented in this paper. The results also, inform us of where BH-NS GW detection directly benefits from the development of full BH-BH templates (roughly speaking, this is the rest of the parameter space, in line with the conclusions of [52]). Naturally, the comparison between BH-NS and BH-BH merger simulations and GW templates is important throughout the space of parameters, but it is particularly so on the border between the two areas of the parameter space that we identified. Strong efforts in this direction recently appeared in [52]. Following detection, knowledge of the properties of the BH remnants of BH-NS systems may allow us to break degeneracies and aid us in deciding whether a GW from a coalescing binary was emitted by a BH-NS system or not, in placing constraints on the NS EOS, and in constraining compact binary formation scenarios. Improving our understanding and predictions of the GW cutoff frequency, as recently done for nonspinning BH-NS systems [86], is especially important in the context of pinning down the NS EOS.

By allowing for generic initial BH spin configurations, the formulation of the model presented in this paper completes one of the possible extensions outlined in paper I. The original version of the model has proven to be useful in developing phenomenological waveforms for nonspinning BH-NS systems [86]. In the future, we plan to first extend the waveforms of [86] to cases with a spinning BH with aligned spin, and then to consider initially misaligned BH spin configurations [12, 13, 87]. As for paper I, we reiterate our intention to test and upgrade the model as more numerical-relativity simulation results become available.

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