TRACE ANOMALIES AND COCYCLES OF WEYL AND DIFFEOMORPHISMS GROUPS

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Abstract

The general structure of trace anomaly, suggested recently by Deser and Shwimmer, is argued to be the consequence of the Wess-Zumino consistency condition. The response of partition function on a finite Weyl transformation, which is connected with the cocycles of the Weyl group in \(d = 2k\) dimensions is considered, and explicit answers for \(d = 4, 6\) are obtained. Particularly, it is shown, that addition of the special combination of the local counterterms leads to the simple form of that cocycle, quadratic over Weyl field \(\sigma\), i.e. the form, similar to the two-dimensional Liouville action. This form also establishes the connection of the cocycles with conformal-invariant operators of order \(d\) and zero weight. Beside that, the general rule for transformation of that cocycles into the cocycles of diffeomorphisms group is presented.
1 Introduction.

The conformal, or Weyl, or trace anomaly, discovered more than 20 years ago \[1\] plays an important role in understanding of many phenomena. It helps in the calculation of the effective action of 2D gravity \[2\], in understanding of the properties of gauge and supersymmetric gauge theories, and in other places. Despite the much efforts, aimed on the understanding of the structure of gauge anomalies at the beginning of 80-s, the structure of the trace anomaly remains unclerified. The understanding of that structure appeared at 1993 due to the work of Deser and Schwimmer \[3\], where they investigated a structure of Weyl anomaly using a scaling property of effective action. Beside that in the ref.\[4\] the new approach of obtaining of sigma-model-like generating functional (like in ref.\[5\] for gravitational and gauge anomaly) for conformal anomaly in high dimension was elaborated. But both of these approaches don’t lead to manifest expressions in the dimension \(d > 4\). Our approach of investigation of conformal anomaly in higher even dimensions is based on the well-known criterion for definition of possible forms of anomalies - on the Wess-Zumino consistensy condition \[6\], which in this abelian case coincides with the property of symmetricity of the conformal variation of anomaly. In this article we consider the general structure of trace anomaly in all dimensions and argue, that the structure, suggested by Deser and Schwimmer \[3\], is actually the consequence of Wess-Zumino consistency condition, although we were not able to present the complete proof. Then we investigate the finite form of the trace anomaly - i.e the change of the measure in the functional integral under finite Weyl transformations. It is well-known \[7\], that corresponding factor has the properties of the 1-cocycle - of the Weyl group, in this case. The analysis of the situation in \(d=4\) (\[8\]) and \(d=6\) (present work, see next section) leads to the beautiful relation between structure of anomaly, 1-cocycles of Weyl group, Euler classes and zero weight conformal-invariant operators of \(d\)-th order in \(d = 2k\) dimensions. We found, that the non-local effective action, wich generates the trace anomaly, has a form of Polyakov’s non-local action in \(d=2\) with \(d\)-th order conformal invariant operator instead of Laplacian and (essentially) the density of Euler characteristik instead of scalar curvature. It means that corresponding nontrivial cocycle of Weyl group is a second-order local, Weyl-invariant functional on group parameter \(\sigma(x)\) (see Sect.3). It is shown below, that generally, all the cocycles are connected with Weyl-invariant local Lagrangians - polynomials over
derivatives of $\sigma(x)$, but all such Lagrangians of order higher than two give rise to the trivial cocycles, i.e. coboundaries. Beside that there are linear nontrivial cocycles connected with conformal-invariant scalar combination of curvature tensor and its derivativs.

We investigate also in general form the well-known possibility of transforming the Weyl anomaly through local counterterms in effective action into the anomaly of diffeomorphisms group, and obtain the general prescription for that mapping (the resulting diffeomorphisms anomaly actually violates diffeomorphisms group only partially, maintaining the volume-preserving part of that group\cite{footnote1}, \(2d\) case see in \(\text{[10]}\)). Difference with \(2d\) case lies in a fact, that at \(d > 2\) the Weyl anomaly is not enough for construction of full effective action \(W(g_{\mu\nu})\) of conformal matter field, because the parameters of local symmetry group:

\[
\text{Weyl} \otimes \text{Diff}(2k)
\]

do not cover all components of the metric. But one has a possibility of construction of the finite variation of effective action on Weyl rescaling of metric:

\[
S(\sigma, g_{\mu\nu}) = W(e^\sigma g_{\mu\nu}) - W(g_{\mu\nu})
\]

. This local action corresponds in \(d = 2\) to Liouville action \(\text{[2]}\). This action \(S(\sigma, g_{\mu\nu})\) has a property of being 1-cocycle of Weyl group, and may be used for transition from \(W(g_{\mu\nu})\) to Weyl invariant effective action \(\tilde{W}(g_{\mu\nu})\), the finite variation of which under the diffeomorphism \(x^\mu \to f^\mu(x)\) gives us the 1-cocycle of diffeomorphisms group \(\tilde{S}(f^\alpha: g_{\mu\nu})\), \(2d\) case see in \(\text{[10]}\). All these statements we have proved for \(d = 4, 6\) by explicit calculations, and they are the basis of our hypothesis in higher dimensions, together with some additional arguments. The organization of the paper is as follows. In section 2 we consider the connection between Wess-Zumino consistency condition \(\text{[3]}\) and structure of trace anomaly. In section 3 we construct all 1-cocycles of Weyl group in \(d = 4, 6\) and show the connection between nontrivial cocycles and conformal-invariant operators. In section 4 we define the local counterterms for transition from Weyl to \(\text{Diff}(d)\) invariant effective action. In conclusion the results and perspectives are summarised.
2 Weyl anomaly and Wess-Zumino consistency condition

Let’s consider the effective action for conformal matter $\varphi$ in external gravitational field:

$$ W(g) = \ln \int D\varphi \exp\{-S_w(\varphi; g)\} $$

(2.1)

where $S_w(\varphi; g)$ is classical Weyl and diffeomorphism invariant action for matter fields and where Weyl transformation is defined as:

$$ g_{\alpha\beta} \rightarrow e^{\sigma(x)} g_{\alpha\beta}; \quad \varphi \rightarrow e^{\Delta(x)} \varphi $$

(2.2)

Where $\Delta$ is conformal weight of matter field.

Then, taking an infinitesimal $\sigma$, we can write down the equation for anomaly

$$ \delta_\sigma W(g) = \int T^{\mu}_\mu \sigma(x) \sqrt{g} d^k x $$

(2.3)

The Wess-Zumino consistency condition in the case of Weyl transformations is simply a statement of a symmetricity of second conformal variation of effective action:

$$ \frac{\delta^2 W(g)}{\delta \sigma(x) \sqrt{g} \delta \sigma(y)} = \frac{\delta^2 W(g)}{\delta \sigma(y) \sqrt{g} \delta \sigma(x)} $$

(2.4)

or, in other words

$$ \frac{\delta A(x)}{\delta \sigma(y)} = \frac{\delta A(y)}{\delta \sigma(x)} $$

(2.5)

where we denote $T^{\mu}_\mu = A(x)$

Then we can propose the following hypothesis about the structure of solution of Wess-Zumino consistency condition in all (even) dimensions. Let we have local function of metric $A(g)$ i.e. the anomaly. The requirement, that the conformal variation of that function is symmetric (WZ consistency condition) leads to the following statement, concerning the structure of $A(g)$:

1) $A(g)$ is the sum with an arbitrary coefficients of the following terms.
   a) Euler density.
   b) Weyl-invariant polynomials over Riemann tensor and it’s covariant derivatives.
   c) Covariant total derivatives of polynomials over Riemann tensor and it’s
covariant derivatives.
In addition, we are making also the following statement 2) Terms c) are the Weyl variations of local functionals of metric. Taking into account the fact, that the definition of the measure in the functional integral always can be changed by multiplying on the exponent of the local functionals (counterters) of metric, one deduce, that third type (i.e. c) type) solutions of WZ condition are in that sense inessential and will be called trivial below.

This hypothesis agree with results obtained earlier in $d = 2,4$ [1] and agree with our investigation in $d = 6$. This allow us to be suggest these statements also in $d > 6$.

Let’s consider the solution of equation (2.5), and, correspondingly, test the validity of the above hypothesis, in some special cases, when anomaly $A(x)$ satisfies some additional constraints.

1. In the case when $A$ is local polynomial function of Riemann tensor $R_{\mu\nu}^{\alpha\beta}$, eq.(2.5) gives the condition

$$\nabla^\alpha \frac{\delta A(R)}{\delta R_{\mu\nu}^{\alpha\beta}} g_{\mu\nu} = 0 \quad (2.6)$$

Then we have two types of solutions of (2.6): First, Weyl invariant $k$-th degree polynomial of Weyl tensor $C_{\mu\nu}^{\alpha\beta}$, and, next, (Weyl noninvariant) Euler density:

$$E_{2k} = \frac{1}{2k!} \varepsilon^{\mu_1 \nu_1 \mu_2 \nu_2 \cdots \mu_k \nu_k} \varepsilon_{\alpha_1 \beta_1 \alpha_2 \beta_2 \cdots \alpha_k \beta_k} R_{\mu_1 \nu_1}^{\alpha_1 \beta_1} R_{\mu_2 \nu_2}^{\alpha_2 \beta_2} \cdots R_{\mu_k \nu_k}^{\alpha_k \beta_k} \quad (2.7)$$

These are the solutions of type a) and b).

2. In the case when $A$ is total derivative of the form:

$$A(x) = \nabla_\alpha \nabla_\beta V^{\alpha\beta}(R) \quad (2.8)$$

where $V^{\alpha\beta}(R)$ is local polynomial $k - 1$ degree on $R$, we obtain the following restriction:

$$\frac{\delta V^{\alpha\beta}(R)}{\delta R_{\mu\nu}^{\lambda}} \delta_{\rho}^{\lambda} = \frac{\delta V^{\mu\nu}(R)}{\delta R_{\alpha\beta}^{\lambda}} \delta_{\rho}^{\lambda} \quad (2.9)$$

The solutions of type (2.8), which satisfy (2.9), coincide with variations of all independent local counterterms to effective action. For example in $d = 4$ one
has only one appropriate counterterm, which is a second order in curvature, with independent conformal variation:

\[ W_0 = \int R^2 \sqrt{g} d^4x; \quad \frac{\delta S_0}{\delta \sigma} = -6 \sqrt{g} \Box R \quad (2.10) \]

Two other possible counterterms

\[ \int R_{\mu\alpha\nu\beta} R^{\mu\alpha\nu\beta} \sqrt{g} d^4x; \quad \int R_{\mu\nu} R^{\mu\nu} \sqrt{g} d^4x; \quad (2.11) \]

give the same (2.10) contribution to anomaly. That can be checked directly, but also follows from the fact, that there are two independent constraints on the variation of these three counterterms. These constraints are consequences of the Weyl-invariance of the actions with following densities:

1. \( C_{\mu\nu\rho} C^{\mu\nu\rho} \)

2. density of the Euler characteristic: \( E_4 \) Weyl variations of these actions give abovementioned constraints.

From the other hand, the terms of the type (2.8) are in this actually unique, and coincide with (2.10), so the statements 1) and 2) are proved in this (i.e. \( d = 4 \)) case.

The similar situation is in \( d = 6 \). Let’s consider all third-order combinations of \( C_{\mu\nu\rho}, R_{\mu\nu}, R \) and Laplace operator - i.e. all six-dimensional dimensionless actions.

\[ A_1 = C^{\alpha\beta\mu\nu} C_{\alpha\beta\lambda\rho} C^{\lambda\rho}_{\mu\nu} \quad (2.12) \]
\[ A_2 = C^{\alpha\beta\mu\nu} C_{\alpha \beta \lambda \rho} C^\rho_{\mu \nu} \quad (2.13) \]

\[ L_1 = R^3 \]
\[ L_2 = R^{\alpha\beta} R_{\alpha\beta} R \]
\[ L_3 = C^{\alpha\beta\mu\nu} R_{\alpha\beta} R_{\mu\nu} \quad (2.14) \]
\[ L_4 = C^{\mu\nu\rho} C_{\beta \mu \nu \rho} R^\beta_\alpha \]
\[ L_5 = C^{\alpha\beta\mu\nu} C_{\alpha\beta\mu\nu} R \]
\[ L_6 = R \Box R \]

\[ L'_2 = R^{\alpha\mu} R_{\mu\beta} R^{\beta}_\alpha \quad (2.15) \]
\[ L'_6 = R_{\alpha\beta} \Box R^{\alpha\beta} \quad (2.16) \]
\[ L''_6 = C^{\alpha\beta\mu\nu} \Box C_{\alpha\beta\mu\nu} \quad (2.17) \]
All other non-full-derivative combinations coincide with the combination of this Lagrangians due to the Bianchi identities. It’s easy to see that first two densities are Weyl-invariant and can be considered as a contribution in anomaly from nonlocal part of effective action. Beside that, one of these densities (say, \( L'_2 \)), has dependent conformal variation due to the Gauss-Bonnet theorem. The counterterms constructed from last two densities have dependent conformal variation, since there are two additional conformal-invariant actions with following densities:

\[
A_3 = C^{\alpha\beta\mu\nu} \nabla C_{\alpha\beta\mu\nu} + 2C^{\alpha\mu\rho\nu}C_{\beta\mu\nu\rho}R_{\alpha}^\beta \\
-3C^{\alpha\mu\beta\nu}R_{\alpha\beta}R_{\mu\nu} - \frac{3}{2}R^{\alpha\mu}R_{\mu\beta}R_{\alpha}^\beta \\
+ \frac{27}{20}R^{\alpha\beta}R_{\alpha\beta}R - \frac{21}{100}R^3
\]

\[
A_4 = R_{\alpha\beta} \Box R^{\alpha\beta} - \frac{3}{10}R \Box R \\
-2C^{\alpha\mu\beta\nu}R_{\alpha\beta}R_{\mu\nu} + R^{\alpha\mu}R_{\mu\beta}R_{\alpha}^\beta \\
-\frac{1}{10}R^{\alpha\beta}R_{\alpha\beta}R - \frac{1}{50}R^3
\]

They differ from \( A_{1,2} \) in that they have nonzero Weyl variations (\( \delta_\sigma A_{3,4} = \) full derivatives), therefore we can use them only as constraint on local counterterms, but not as an independent contributions into the anomaly.

Therefore in \( d = 6 \) there are only 6 independent local counterterms

\[
W_i^o = \int L_i \sqrt{g} d^6 x, \quad i = 1, 2, \ldots 6
\]

and 3 nontrivial contributions in anomaly \( E_6, A_1, A_2 \).

Again, it is possible to show that all possible solutions of the WZ condition (2.9) of the type (2.8) coincide with the combinations of the variations of the actions \( W_i \).

These considerations give the complete proof of statements 1) and 2) at \( d=6 \).
Cocycles and conformal-invariant operators

Let’s consider the problem of the change of the measure in the functional integral for conformal matter field $\varphi$ in external gravitational field under the finite Weyl transformation (2.2).

The measure in the functional integral changes in the following way:

$$D_{e^{\sigma(x)}g}\varphi = D_g\varphi \exp S(\sigma; g_{\alpha\beta})$$

This type of relations are very important, being the starting point of DDK calculation of the critical exponent of 2d gravity [11].

The action $S(\sigma; g)$ in (2.2) has to satisfy some conditions. First, in the case of infinitesimal transformation $\delta \sigma(x)$ it has to reproduce the trace anomaly:

$$S(\delta \sigma(x); g_{\alpha\beta}) = \int T^\mu_\mu \delta \sigma(x) \sqrt{g} d^2x$$

Second, $S(\sigma; g)$ has to satisfy the following property, which follows from the application of (2.2) to the composition of two Weyl transformations $\sigma_1$ and $\sigma_2$:

$$S(\sigma_1 + \sigma_2; g) = S(\sigma_1; e^{\sigma_2}g) + S(\sigma_2; g)$$

which means that $S(\sigma; g)$ is the 1-cocycle of the group of Weyl transformations [7].

On the other hand, the action $S(\sigma; g)$ coincides with the finite variation of anomalous effective action, due to the property (2.1) and (3.1). In other words

$$S(\sigma, g) = W(e^\sigma g) - W(g)$$

and non-triviality of the cocycle $S(\sigma; g)$ follows from the fact that $W(g)$ is non-local, $Diff(2k)$-invariant functional of $g_{\alpha\beta}$. Thus, we can easily calculate the trivial cocycles as a coboundary of local counterterms $W^i_0(g)$ which we shall call from now on 0-cochains:

$$S^i_0(\sigma, g) = \Delta W^i_0(g)$$

where we have defined coboundary operator $\Delta$ on 0-cochains as finite Weyl variation (3.4). Then we can define 1-cochains as a local functions $W_1(\sigma, g)$ of group parameter and metric with coboundary operator:

$$\Delta W_1(\sigma_1, \sigma_2, g) = W_1(\sigma_1 + \sigma_2; g) - W_1(\sigma_1; e^{\sigma_2}g) - W_1(\sigma_2; g)$$
It is easy to see that $\Delta^2 = 0$ which is exactly the cocyclic property (3.3). One can generalize this construction on higher cohomologies of Weyl group [7].

The nontrivial cocycles can be obtained from the solution of eq. (3.3). If we are looking for the general solution of equation (3.3) with condition (3.2) we have to take $\sigma_2 = \sigma$ and $\sigma_1 = \delta \sigma$ and get the differential form of (3.3):

$$\delta S(\sigma; g) = S(\delta \sigma; e^\sigma g) = \int A(R(e^\sigma g)) \delta \sigma \sqrt{g} d^2k x$$ (3.7)

The explicit form of solution for two-dimensional case is the famous Liouville action [2]

$$S_{d=2}(\sigma, g) = \frac{c}{48\pi} \int d^2x \sqrt{g} \left( \frac{1}{2} g^{\alpha\beta} \partial_\alpha \sigma \partial_\beta \sigma + R \sigma \right)$$ (3.8)

In four dimensions explicit form of cocycle, corresponding to $E_4$, firstly has been found in [8]

$$S_{E}(\sigma, g) = \int d^4x \sqrt{g} \left( \frac{1}{2} \left( \nabla_\alpha \sigma \nabla^\alpha \sigma \right)^2 + \frac{1}{2} \nabla_\alpha \sigma \nabla^\alpha \sigma \nabla^2 \sigma - \left( R^{\alpha\beta} - \frac{1}{2} g^{\alpha\beta} R \right) \nabla_\alpha \sigma \nabla_\beta \sigma \right) + \sigma E_4$$ (3.9)

In $d = 4$ there is linear nontrivial cocycle corresponding to single invariant density $C_{\mu\alpha\beta} C^{\mu\alpha\beta}$.

$$S_{E}(\sigma, g) = \int d^4x \sqrt{g} \left( \frac{1}{2} \left( \nabla_\alpha \sigma \nabla^\alpha \sigma \right)^2 + \frac{1}{2} \nabla_\alpha \sigma \nabla^\alpha \sigma \nabla^2 \sigma - \left( R^{\alpha\beta} - \frac{1}{2} g^{\alpha\beta} R \right) \nabla_\alpha \sigma \nabla_\beta \sigma \right) + \sigma E_4$$ (3.10)

This expression satisfies the cocyclic property (3.3) and can appear in the Weyl transformation of the measure (3.1).

Then in $d=4$ we can obtain only one trivial cocycle:

$$S_0(\sigma, g) = \Delta \int R^2 \sqrt{g} d^4x$$ (3.11)

On the other side, one can check that

$$S(\sigma, g) = S(\sigma, g) + \frac{2}{3} S_0(\sigma, g) = \int \left[ \frac{1}{2} \Delta_4 \sigma \right] + (E_4 + \frac{2}{3} \Box R) \sigma \sqrt{g} d^4x$$ (3.12)

where

$$\sqrt{g} \Delta_4 = \sqrt{g} \left( \Box^2 - 2 R^{\mu\nu} \nabla_\mu \nabla_\nu + \frac{2}{3} \Box - \frac{1}{3} (\nabla^\mu R) \nabla_\mu \right)$$ (3.13)
is the fourth-order conformal-invariant differential operator acting on a scalar field of zero conformal weight \[8\] (exact analogy of \[\sqrt{g} \Box \] in \(d = 2\)).

Thus nontrivial cocycle in \(d = 4\) can be cut up to the second order on \(\sigma\) by adding a trivial cocycle with appropriate coefficient. The reason of this reduction comes from the following observation, which also explains the Weyl-invariance property of the operator \(3.13\). It is easy to see that from cocyclic property \(3.3\) follows that the highest-order over \(\nabla_{\sigma}\) term in cocycle has to be Weyl-invariant. There is only one fourth-order invariant action in \(d = 4\)

\[I_4 = \int d^4x \sqrt{g} (\nabla_{\mu}\sigma \nabla^\mu \sigma)^2\] (3.14)

and there is no third-order invariant action in \(d = 4\), consequently we can use trivial cocycle \(3.11\) for the reduction of nontrivial one to the second-order cocycle \(3.12\). The corresponding highest order part involves Weyl-invariant operator \(3.13\). These considerations leads to the correspondence between cocycles and Weyl-invariant actions, moreover, nontrivial cocycle is connected to the second-order invariant action and conformal invariant fourth-order differential operator.

Let’s now consider the case \(d = 6\). Firstly we can easily construct two linear cocycles using invariant part of anomaly:

\[S_C^1 = \int \sqrt{g} d^6 x C^{\alpha\beta\mu\nu} C_{\alpha\beta\lambda\delta} C_{\mu\nu}^{\lambda\rho} \sigma(x)\] (3.15)

\[S_C^1 = \int \sqrt{g} d^6 x C^{\alpha\mu} C_{\alpha\rho\lambda} C_\mu^{\lambda\nu} \sigma(x)\] (3.16)

Secondly let’s obtain the solution of equation (27) with \(A(g) = E_6\):

\[S_E = \int \sqrt{g} d^6 x \{-6(\nabla^\mu \sigma \nabla_\mu \sigma)^3 - 18 \Box \sigma (\nabla^\mu \sigma \nabla_\mu \sigma)^2\]

\[+ 24 \nabla^\lambda \sigma \nabla_\lambda \sigma (\nabla^\mu \nabla^\nu \sigma \nabla_\mu \nabla_\nu \sigma - (\Box \sigma)^2) - 6 R(\nabla^\mu \sigma \nabla_\mu \sigma)^2\] (3.17)

\[-4 G^\mu_{\mu\nu} \nabla_\lambda \sigma \nabla^\nu \nabla_\mu \sigma - 3 G^\nu_{\mu
u} \nabla_\nu \sigma + 6! E_6 \sigma\}

where

\[G^\lambda_{\mu\nu} = \varepsilon^{\mu_1 \nu_1 \mu_2 \nu_2 \lambda \rho} \varepsilon_{\alpha_1 \beta_1 \mu_2 \nu_2 \mu \nu} R^{\alpha_1 \beta_1}_{\mu_1 \nu_1},\]

\[G^\nu_{\mu} = \varepsilon^{\mu_1 \nu_1 \mu_2 \nu_2 \lambda \nu} \varepsilon_{\alpha_1 \beta_1 \alpha_2 \beta_2 \lambda \mu} R^{\alpha_1 \beta_1}_{\mu_1 \nu_1} R^{\alpha_2 \beta_2}_{\mu_2 \nu_2}\] (3.18)
Then we can calculate 6 trivial cocycles $S_0^6$ using (2.14), (2.20) and (3.3). These cocycles correspond to all Weyl-invariant actions of order 3-6 over $\nabla_\mu \sigma$. In other words, we can easily check that:

\[
S_6 = \frac{1}{125} S_0^1 = I_6 + ...
\]
\[
S_4 = \frac{1}{125} S_0^1 + \frac{6}{25} S_6^6 = I_4 + ...
\]
\[
S_4' = \frac{1}{25} S_0^2 - \frac{1}{125} S_6^6 = I_4' + ...
\]
\[
S_3 = S_0^3 = I_3 + ...
\]
\[
S_2 = S_0^4 = I_2 + ...
\]
\[
S_2' = S_0^5 = I_2' + ...
\]

where

\[
I_6 = \int \sqrt{g}(\nabla^\mu \sigma \nabla_\mu \sigma)^3
\]
\[
I_4 = \int \sqrt{g}(\nabla^\mu \sigma \nabla_\mu \sigma)[\Box + \frac{1}{5} R](\nabla^\mu \sigma \nabla_\mu \sigma)
\]
\[
I_4' = \int \sqrt{g}[\nabla^\mu \nabla^\nu \sigma \nabla_\mu \nabla_\nu \sigma \nabla^\alpha \sigma \nabla_\alpha \sigma
\]
\[-\frac{1}{2} \nabla^\mu (\nabla^\alpha \sigma \nabla_\alpha \sigma) \nabla_\mu (\nabla^\beta \sigma \nabla_\beta \sigma) - \frac{1}{4}(\Box \sigma)^2 (\nabla^\alpha \sigma \nabla_\alpha \sigma)]
\]
\[
I_3 = \int \sqrt{g} C^{\alpha \beta \mu \nu} \nabla_\alpha \nabla_\beta \sigma \nabla_\mu \nabla_\nu \sigma
\]
\[
I_2 = \int \sqrt{g} C^{\alpha \mu \nu \rho} C_{\beta \mu \nu \rho} \nabla^\beta \sigma \nabla_\alpha \sigma
\]
\[
I_2' = \int \sqrt{g} C^{\alpha \beta \mu \nu} C_{\alpha \beta \mu \nu} \nabla_\rho \sigma \nabla^\rho \sigma
\]

are conformal-invariant actions. Then we can obtain the main result of this section. The following nontrivial cocycle:

\[
S_2 = S_E + 6S_6 + 12S_4 - 60S_4' - 8S_3
\]

is second order on $\sigma$ and has a form

\[
S_2 \sim \int \sqrt{g} \sigma \Delta_6 \sigma + ....
\]
where $\Delta_6$ is Weyl-invariant sixth-order differential operator in $d = 6$. Differently from $d = 4$ case this operator is not unique, since we can add to the action $S$ the second-order cocycles $S_2$ and $S_2'$ with arbitrary coefficients, which leads to the arbitrariness in definition of $\Delta_6$.

$$
\Delta_6 = \Box^3 - \frac{16}{3} \nabla_\alpha \nabla_\beta C^{\alpha\mu\beta\nu} \nabla_\mu \nabla_\nu - 2 \nabla_\alpha \nabla_\beta R^{\alpha\mu} \nabla^\beta \nabla_\mu \\
-2 \nabla_\alpha \nabla_\beta R^{\beta\mu} \nabla^\alpha \nabla_\mu + \frac{2}{3} \nabla_\alpha \nabla_\beta R \nabla^\alpha \nabla^\beta + \frac{3}{5} \Box R \Box \\
-\frac{8}{3} \nabla_\alpha C^{\alpha\mu\beta\nu} R_{\mu\nu} \nabla_\beta - \nabla_\alpha R^{\alpha\mu} R_{\mu\beta} \nabla^\beta + \frac{1}{2} \nabla^\alpha R^{\mu\nu} R_{\mu\nu} \nabla_\alpha \\
+ \frac{1}{5} \nabla_\alpha R R^{\alpha\beta} \nabla_\beta - \frac{6}{25} \nabla^\alpha R^2 \nabla_\alpha - \frac{2}{5} \nabla^\mu (\Box R) \nabla_\mu \\
+ \varepsilon^{\mu_1\nu_1\mu_2\nu_2\lambda\nu} \epsilon_{\alpha_1\beta_1\alpha_2\beta_2\lambda\nu} P_{\mu_1\nu_1}^\alpha \xi_{\rho_{\mu_2\nu_2}}^{\beta_1} \nabla_\mu \nabla_\nu \\
+ \theta \nabla_\alpha C^{\alpha\mu\rho} C_{\beta\mu\rho} \nabla_\beta + \tau \nabla_\alpha C^{\beta\mu\rho} C_{\beta\mu\rho} \nabla_\alpha
$$

where $\tau$ and $\theta$ arbitrary parameters.

The technical difficulties do not permit us to curry on the similar explicit calculations in higher dimensions, but these results convinced us, that in the same way one can cut the non-trivial cocycle, corresponding to the Euler characteristic term in the anomaly to the second order over $\nabla \sigma$ in all higher dimensions.

### 4 Cocycles of diffeomorphism group from the cocycles of the Weyl group

Let’s now consider the new non-local effective action:

$$
\tilde{W}(g) = W(g) + S(\sigma; g) \bigg|_{\sigma = -\frac{1}{k} \ln \sqrt{g}} 
$$

where dimension of space-time is $d = 2k$. It’s ease to see that due to the relation \(3.3\) (cocyclic property) this action is Weyl-invariant, but not diffeomorphism-invariant. Let’s now in analogy with \(3.4\) calculate the finite variation of $\tilde{W}(g)$ under diffeomorphism transformation:

$$
\tilde{W}(f^* g) - \tilde{W}(g) = \tilde{S}(f; g)
$$

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It’s easy to see that from (3.3), (4.1) and from
\[ \ln \sqrt{f^* g(x)} = \ln \sqrt{g(f)} + \ln \Delta_f^x \]  \hfill (4.3)
where
\[ \Delta_f^x = \det \left[ \frac{\partial f^\alpha(x)}{\partial x^\beta} \right] \]  \hfill (4.4)
we obtain
\[ \tilde{S}(f; g) = S(\sigma; g_{\alpha \beta}/(\sqrt{g})^{1/k}) \bigg|_{\sigma = \frac{1}{k} \ln \Delta_f^x} \]  \hfill (4.5)
This action has a cocyclic property for the $Diff(2k)$ group
\[ \tilde{S}(f \circ h; g) = \tilde{S}(f; h^* g) + \tilde{S}(h; g) \]  \hfill (4.6)
Therefore we can define the change of the measure for $Diff(2k)$ transformation in the case when we are using Weyl-invariant regularization:
\[ D_{f^* g} \varphi = D_g \varphi \exp \tilde{S}(f; g_{\alpha \beta}) \]  \hfill (4.7)
where $\tilde{S}(f; g)$ is defined in (4.5) and has the property of being 1-cocycle of $Diff(2k)$ group. Finally we can substitute (3.12) and (3.21) in (4.5) and obtain the $Diff(4), Diff(6)$ cocycles
\[ \tilde{S}_2(f; g) = S_2(\sigma; g_{\alpha \beta}/(\sqrt{g})^{1/2}) \bigg|_{\sigma = \frac{1}{k} \ln \Delta_f^x} \quad k = 2, 3 \]  \hfill (4.8)
The corresponding $Diff(2)$ cocycle has been obtained in Ref. [10].

5 Conclusions and outlook

In this work we discussed the consequences of the Wess-Zumino consistency condition [3] on the structure of trace anomaly in any even dimensions. The Wess-Zumino condition follows from the fact that the second Weyl variation of the effective action $W(g)$ has to be symmetric, that means, that the first Weyl variation of the anomaly also has to be symmetric. The variation of the Euler density satisfies this condition in any even dimensions due to Bianchi identity. Beside that Weyl invariant terms (like $C^2$ in $d = 4$) [3] also, evidently, satisfy the WZ consistency condition. The other terms, with
lower order on $R$ can be obtained as variations of local functionals and can be removed by adding local counterterms to an effective action. This last statement, which is proved exactly at $d=2,4,6$, and remains a hypothesis at higher $d$, completes the classification of the structure of trace anomaly in all dimensions, following from the WZ consistency condition, and that classification coincides with that of Deser and Schwimmer [3].

Next, we have considered the response of the functional integral on the finite Weyl transformation, which, as is well-known for all the anomalous symmetries, is connected to the cocycles of the corresponding group. The corresponding cocycle of the Weyl group is calculated explicitly for $d=6$ and it is shown, that, surprisingly, it can be brought to the form, which is only of a second order over the Weyl parameter $\sigma$. This form contains a Weyl-invariant differential operators of the sixth order. These results give rise to the general hypothesis, concerning the cocycles of the Weyl group in an arbitrary dimensions - it appear to be the exact connection between Euler class, cocycle of the Weyl group, and the Weyl-invariant differential operator of order $d$ (dimensionality of the space-time) and weight zero.

Finally, it is shown, how one can use the cocycle of the Weyl group for obtaining the cocycle of the diffeomorphisms group in all dimensions, and the exact expressions for this transformation are obtained.

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