Generation of the Trigonometric Cubic B-Spline Collocation Solutions for the Kuramoto-Sivashinsky (KS) Equation

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November 21, 2018

Abstract

A recent type of B-spline functions, namely trigonometric cubic B-splines, are adapted to the collocation method for the numerical solutions of the Kuramoto-Sivashinsky equation. Having only first and second order derivatives of the trigonometric cubic B-splines at the nodes forces us to convert the Kuramoto-Sivashinsky equation to a coupled system of equations by reducing the order of the higher order terms. Crank-Nicolson method is applied for the time integration of the space discretized system resulted by trigonometric cubic B-spline approach. Some initial boundary value problems are solved to show the validity of the proposed method.

Keywords: Kuramoto-Sivashinsky Equation; Trigonometric cubic B-spline; collocation.

1 Introduction

The original form of the Kuramoto-Sivashinsky was constructed to describe pattern formations and dissipation of them in reaction-diffusion system[1]. In that study, the reductive perturbation method was implemented for deriving a scale-invariant part from original macroscopic motion equations. It was also shown that the Ginzburg-Landau equation can govern the dynamics near an instability point in many cases. The origin of persistent wave propagation in reaction-diffusion medium was explored by the same equation[2]. It was also used to explain the chaotic behavior in a distributed chemical reaction due to the unstable growth of a spatial inhomogeneity taking place in an oscillating medium[3]. Small model thermal diffusive instabilities in laminar flame fronts can also be represented by the same equation[4, 5]. Nonlinear analysis of flame front stability assuming stioochiometric composition of the combustible mixture was also studied with constant-density
model of a premixed flame[6]. The one-dimensional form
\[ u_t + uu_x + \alpha u_{xx} + \vartheta u_{xxxx} = 0 \]  
(1)
of the equation appeared in the study [7]. Hyman and Nicolaenko characterized the transition to chaos of the solutions by numerical simulations [8]. The Weiss-Tabor-Carnevale technique applied to the generalised Kuramoto-Sivashinsky equation to extract some particular analytical solutions[9]. In the related literature, the methods covering simplest equation, homotopy analysis and tanh and extended tanh techniques derived to determine solitary wave, or multiple soliton solutions to the Kuramoto-Sivashinsky equation[10, 11, 12, 13]. Besides the analytical solutions, many numerical techniques including Chebyshev spectral collocation[14], finite difference and collocation [15], quintic B-spline [16], radial basis meshless method of lines [17], and exponential cubic B-spline method [18] have been applied to derive the numerical solutions to Kuramoto-Sivashinsky equation.

Different from the other B-splines techniques based on classical polynomial cubic, quartic and quintic B-splines[19, 20, 21] or exponential cubic B-splines [22], the trigonometric cubic B-spline functions have recently appeared. In this study, we construct a collocation method based on trigonometric cubic B-spline functions for some initial boundary value problems for the Kuramoto-Sivashinsky equation. After reducing the order of the term with the fourth order derivative to two, we discretize the resultant system by using Crank-Nicolson method in time. Performing the linearization of the nonlinear term lead us to discretize the system by trigonometric cubic B-spline functions. As a result of adapting the initial and boundary conditions, the iteration algorithm will be ready to run.

To solve the initial value (1) numerically we first replace it by a system which is first order in the time derivative
\[ u_t + uu_x + \alpha v + \vartheta v_{xx} = 0 \]
\[ v - u_{xx} = 0 \]  
(2)
To complete the usual classical mathematical statement of the problem, the initial and the boundary conditions are chosen as to be
\[ u(x, 0) = u_0 \]  
(3)
and
\[ u(x_0, t) = g_0, \]  
\[ u(x_N, t) = g_1, \]
\[ u_x(x_0, t) = 0, \]  
\[ u_x(x_N, t) = 0, \]
\[ u_{xx}(x_0, t) = 0, \]  
\[ u_{xx}(x_N, t) = 0. \]  
(4)

2 Cubic Trigonometric B-spline Collocation Method

Consider a uniform partition of the problem domain \([a = x_0, b = x_N]\) at the knots \(x_i, i = 0, ..., N\) with mesh spacing \(h = (b - a)/N\). On this partition together with additional knots \(x_{-2}, x_{-1}, x_{N+1}, x_{N+2}\) outside the problem domain, \(T_i(x)\) can be defined as
\[ T_i(x) = \frac{1}{\gamma} \begin{cases} 
W^3(x_{i-2}), & x \in [x_{i-2}, x_{i-1}] \\
W(x_{i-2})(W(x_{i-2})Y(x_i) + Y(x_{i+1})W(x_{i-1})) + Y(x_{i+2})W^2(x_{i-1}), & x \in [x_{i-1}, x_i] \\
W(x_{i-2})Y^2(x_{i+1}) + Y(x_{i+2})(W(x_{i-1})Y(x_{i+1}) + Y(x_{i+2})W(x_i)), & x \in [x_{i}, x_{i+1}] \\
Y^3(x_{i+2}), & x \in [x_{i+1}, x_{i+2}] \\
0, & \text{otherwise} \end{cases} \]

(5)

where \( W(x_i) = \sin\left(\frac{x-x_i}{2}\right), \), \( \hat{Y}(x_i) = \sin\left(\frac{x-x_i}{2}\right), \gamma = \sin\left(\frac{h}{2}\right)\sin\left(\frac{2\pi}{2}\right). \) The twice continuously differentiable piecewise trigonometric B-spline function set \( \{T_i(x)\}_{i=-1}^{N+1} \), forms a basis for the functions defined in the same interval [24, 25].

\( T_i(x) \) are twice continuously differentiable piecewise trigonometric cubic B-spline on the interval \([a, b]\). The iterative formula

\[ T^k_i(x) = \frac{\sin\left(\frac{x-x_i}{2}\right)}{\sin\left(\frac{x_{i+k-1}-x}{2}\right)} T^{k-1}_i(x) + \frac{\sin\left(\frac{x_{i+k}-x}{2}\right)}{\sin\left(\frac{x_{i+k-1}-x_{i+1}}{2}\right)} T^{k-1}_{i+1}(x), \quad k = 2, 3, 4, ... \]  

(6)

gives the cubic B-spline trigonometric functions starting with the CTB-splines of order 1:

\[ T^1_i(x) = \begin{cases} 
1, & x \in [x_i, x_{i+1}) \\
0, & \text{otherwise.} \end{cases} \]

The graph of the trigonometric cubic B-splines over the interval [0, 1] is depicted in Fig. 1.

Fig.1: Trigonometric cubic B-splines over the interval [0, 1]
The nonzero functional and derivative values of trigonometric cubic B-spline functions at the grids are given in Table 1.

Table 1: Values of $T_i(x)$ and its principle two derivatives at the knot points

| $x_i$  | $T_i(x_k)$ | $T'_i(x_k)$ | $T''_i(x_k)$ |
|--------|------------|-------------|--------------|
| $x_{i-2}$ | 0          | 0           | 0            |
| $x_{i-1}$ | $\sin^2\left(\frac{h}{2}\right) \csc(h) \csc\left(\frac{3h}{2}\right)$ | $\frac{3(1 + 3\cos(h)) \csc^2\left(\frac{h}{2}\right)}{16 \left[2 \cos\left(\frac{h}{2}\right) + \cos\left(\frac{3h}{2}\right)\right]}$ | 
| $x_i$ | $\frac{2}{1 + 2\cos(h)}$ | 0 | $-\frac{3\cot^2\left(\frac{h}{2}\right)}{2 + 4\cos(h)}$ |
| $x_{i+1}$ | $\sin^2\left(\frac{h}{2}\right) \csc(h) \csc\left(\frac{3h}{2}\right)$ | $-\frac{3}{4} \csc\left(\frac{3h}{2}\right)$ | $\frac{3(1 + 3\cos(h)) \csc^2\left(\frac{h}{2}\right)}{16 \left[2 \cos\left(\frac{h}{2}\right) + \cos\left(\frac{3h}{2}\right)\right]}$ |
| $x_{i+2}$ | 0          | 0           | 0            |

An approximate solution $U$ and $V$ to the unknown $u$ and $v$ is written in terms of the expansion of the CTB as

$$U(x, t) = \sum_{i=-1}^{N+1} \delta_i T_i(x), \quad V(x, t) = \sum_{i=-1}^{N+1} \phi_i T_i(x).$$  \hspace{0.5cm} (7)

where $\delta_i$ and $\phi_i$ are time dependent parameters to be determined from the collocation points $x_i, i = 0, \ldots, N$ and the boundary and initial conditions. The nodal values $U$ and its first and second derivatives at the knots can be found from the (7) as

$$U_i = \alpha_1 \delta_{i-1} + \alpha_2 \delta_i + \alpha_1 \delta_{i+1}, \quad V_i = \alpha_1 \phi_{i-1} + \alpha_2 \phi_i + \alpha_1 \phi_{i+1},$$

$$U'_i = \beta_1 \delta_{i-1} - \beta_1 \delta_{i+1}, \quad V'_i = \beta_1 \phi_{i-1} - \beta_1 \phi_{i+1},$$

$$U''_i = \gamma_1 \delta_{i-1} + \gamma_2 \delta_i + \gamma_1 \delta_{i+1}, \quad V''_i = \gamma_1 \phi_{i-1} + \gamma_2 \phi_i + \gamma_1 \phi_{i+1}.$$  \hspace{0.5cm} (8)

$$\alpha_1 = \sin^2\left(\frac{h}{2}\right) \csc(h) \csc\left(\frac{3h}{2}\right), \quad \alpha_2 = \frac{2}{1 + 2\cos(h)}, \quad \beta_1 = -\frac{3}{4} \csc\left(\frac{3h}{2}\right)$$

$$\gamma_1 = \frac{3(1 + 3\cos(h)) \csc^2\left(\frac{h}{2}\right)}{16 \left[2 \cos\left(\frac{h}{2}\right) + \cos\left(\frac{3h}{2}\right)\right]}, \quad \gamma_2 = -\frac{3\cot^2\left(\frac{h}{2}\right)}{2 + 4\cos(h)}.$$  \hspace{0.5cm} (9)

When KS equation is space-split as (2), the system includes the second-order derivatives so that smooth approximation can constructed with the combination of the trigonometric cubic B-splines. The time integration of the space-split system (2) is performed by the Crank-Nicolson method as

$$\frac{U^{n+1} - U^n}{\Delta t} + (UU_x)^{n+1} + (UU_x)^n = \alpha \frac{V^{n+1} + V^n}{2} + \beta \frac{V_x^{n+1} + V_x^n}{2} = 0$$

$$\frac{V^{n+1} + V^n}{2} - \frac{U^{n+1} + U^n}{2} = 0.$$  \hspace{0.5cm} (10)
where \( U^{n+1} = U(x, (n+1)\Delta t) \) represent the solution at the \((n+1)\)th time level. Here \( t^{n+1} = t^n + \Delta t \), \( \Delta t \) is the time step, superscripts denote \( n \) th time level, \( t^n = n\Delta t \).

One linearize terms \((UU_x)^{n+1}\) and \((UU_x)^n\) in (10) as [23]

\[
(UU_x)^{n+1} = U^{n+1}U_x^n + U^nU_{x+1} - U^nU_x
\]

\[
(UU_x)^n = U^nU_x^n
\]
to obtain the time-integrated linearized the KS Equation:

\[
\frac{2}{\Delta t}U^{n+1} - \frac{2}{\Delta t}U^n + U^{n+1}U_x^n + U^nU_{x+1} + \alpha (V^{n+1} + V^n) + \vartheta (V^{n+1} + V^n) = 0
\]

\[
\frac{V^{n+1} + V^n}{2} - \frac{U^{n+1} + U^n}{2} = 0
\]

To proceed with space integration of the (11), an approximation of \( U^n \) and \( V^n \) in terms of the unknown element parameters and trigonometric cubic B-splines separately can be written as (7). Substitute Eqs (8) into (11) and collocate the resulting equation at the knots \( x_i, i = 0, \ldots, N \) yields a linear algebraic system of equations:

\[
\left[ \left( \frac{2}{\Delta t} + K_2 \right) \alpha_1 + K_1 \beta_1 \right] \delta_{m-1}^{n+1} + (\alpha \alpha_1 + \vartheta \gamma_1) \phi_{m-1}^{n+1} + \left[ \left( \frac{2}{\Delta t} + K_2 \right) \alpha_2 \right] \delta_{m-1}^{n+1} + (\alpha \alpha_2 + \vartheta \gamma_2) \phi_{m-1}^{n+1} + \left[ \left( \frac{2}{\Delta t} + K_2 \right) \alpha_1 - K_1 \beta_1 \right] \delta_{m+1}^{n+1} + (\alpha \alpha_1 + \vartheta \gamma_1) \phi_{m+1}^{n+1} = \frac{2}{\Delta t} \alpha_1 \delta_{m-1}^{n+1} - (\alpha \alpha_1 + \vartheta \gamma_1) \phi_{m-1}^{n+1} + \frac{2}{\Delta t} \alpha_2 \delta_{m+1}^{n+1} - (\alpha \alpha_2 + \vartheta \gamma_2) \phi_{m+1}^{n+1}
\]

\[
= \frac{1}{\Delta t} \gamma_1 \delta_{m-1}^{n+1} + \alpha_1 \phi_{m-1}^{n+1} + \gamma_2 \delta_{m+1}^{n+1} + \alpha_2 \phi_{m+1}^{n+1} + \gamma_1 \delta_{m+1}^{n+1} + \alpha_1 \phi_{m+1}^{n+1} - \gamma_1 \delta_{m-1}^{n+1} - \alpha_1 \phi_{m-1}^{n+1} + \gamma_2 \delta_{m+1}^{n+1} - \alpha_2 \phi_{m+1}^{n+1} + \gamma_1 \delta_{m+1}^{n+1} - \alpha_1 \phi_{m+1}^{n+1}, \quad m = 0 \ldots N, \ n = 0, 1, \ldots
\]

where

\[
K_1 = \alpha_1 \delta_{i-1}^{n+1} + \alpha_2 \delta_{i}^{n+1} + \alpha_1 \delta_{i+1}^{n+1}
\]

\[
K_2 = \beta_1 \delta_{i-1}^{n+1} - \beta_1 \delta_{i+1}^{n+1}.
\]

The system (12) can be converted the following matrices system:

\[
A\textbf{x}^{n+1} = \textbf{b}\textbf{x}^n
\]

where

\[
A = \begin{bmatrix}
\nu_{m1} & \nu_{m2} & \nu_{m3} & \nu_{m4} & \nu_{m5} & \nu_{m6} \\
-\gamma_1 & \alpha_1 & -\gamma_2 & \alpha_2 & -\gamma_1 & \alpha_1 \\
\nu_{m1} & \nu_{m2} & \nu_{m3} & \nu_{m4} & \nu_{m5} & \nu_{m6} \\
-\gamma_1 & \alpha_1 & -\gamma_2 & \alpha_2 & -\gamma_1 & \alpha_1 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\nu_{m1} & \nu_{m2} & \nu_{m3} & \nu_{m4} & \nu_{m5} & \nu_{m6} \\
-\gamma_1 & \alpha_1 & -\gamma_2 & \alpha_2 & -\gamma_1 & \alpha_1
\end{bmatrix}
\]
Thus the following requirements help to determine initial parameters:

\[
\begin{bmatrix}
\nu_{m6} & \nu_{m7} & \nu_{m8} & \nu_{m9} & \nu_{m6} & \nu_{m7} \\
\gamma_1 & -\alpha_1 & \gamma_2 & -\alpha_2 & \gamma_1 & -\alpha_1 \\
\nu_{m6} & \nu_{m7} & \nu_{m8} & \nu_{m9} & \nu_{m6} & \nu_{m7} \\
\gamma_1 & -\alpha_1 & \gamma_2 & -\alpha_2 & \gamma_1 & -\alpha_1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\nu_{m6} & \nu_{m7} & \nu_{m8} & \nu_{m9} & \nu_{m6} & \nu_{m7} \\
\gamma_1 & -\alpha_1 & \gamma_2 & -\alpha_2 & \gamma_1 & -\alpha_1 \\
\end{bmatrix}
\]

and

\[
\begin{align*}
\nu_{m1} &= \left(\frac{2}{N} + K_2\right)\alpha_1 + K_1\beta_1 \\
\nu_{m2} &= (\alpha\alpha_1 + \vartheta\gamma_1) \\
\nu_{m3} &= \left(\frac{2}{N} + K_2\right)\alpha_2 \\
\nu_{m4} &= (\alpha\alpha_2 + \vartheta\gamma_2) \\
\nu_{m5} &= \left(\frac{2}{N} + K_2\right)\alpha_1 - K_1\beta_1 \\
\nu_{m6} &= \frac{2}{N}\alpha_2 \\
\nu_{m7} &= -\left(\alpha\alpha_1 + \vartheta\gamma_1\right) \\
\nu_{m8} &= \frac{2}{N}\alpha_2 \\
\nu_{m9} &= -\left(\alpha\alpha_2 + \vartheta\gamma_2\right)
\end{align*}
\]

The system (13) consist of 2N + 2 linear equation in 2N + 6 unknown parameters

\[
x^{n+1} = (\delta^{n+1}_-, \phi^{n+1}_-, \delta^{n+1}_0, \phi^{n+1}_0, \ldots, \delta^{n+1}_{n+1}, \phi^{n+1}_{n+1}).
\]

To obtain a unique solution, an additional four constraints are needed. These are obtained from the imposition of the Robin boundary conditions so that \(U_{xx}(a, t) = 0, V(a, t) = 0\) and \(U_{xx}(b, t) = 0, V(b, t) = 0\) gives the following equations:

\[
\begin{align*}
\gamma_1\delta_- + \gamma_2\delta_0 + \gamma_1\delta_1 &= 0 \\
\alpha_1\phi_- + \alpha_2\phi_0 + \alpha_1\phi_1 &= 0 \\
\gamma_1\delta_{N-1} + \gamma_2\delta_N + \gamma_1\delta_{N+1} &= 0 \\
\alpha_1\phi_{N-1} + \alpha_2\phi_N + \alpha_1\phi_{N+1} &= 0
\end{align*}
\]

Elimination of the parameters \(\delta_-, \phi_-, \delta_{N+1}, \phi_{N+1}\), from the Eq.(12), using the above equations gives a solvable system of \(2N + 2\) linear equations including \(2N + 2\) unknown parameters. After finding the unknown parameters via the application of a variant of Thomas algorithm, approximate solutions at the knots can be obtained by placing successive three parameters in the Eq.(8).

Initial parameters \(\delta^0_i, \phi^0_i, i = -1, \ldots, N + 1\) are needed to start the iteration procedure (13). Thus the following requirements help to determine initial parameters:

\[
\begin{align*}
U_{xx}(a, 0) &= 0 = \gamma_1\delta^0_- + \gamma_2\delta^0_0 + \gamma_1\delta^0_1, \\
U_{xx}(x_i, 0) &= \gamma_1\delta^0_{i-1} + \gamma_2\delta^0_i + \gamma_1\delta^0_{i+1} = u_{xx}(x_i, 0), i = 1, \ldots, N - 1 \\
U_{xx}(b, 0) &= 0 = \gamma_1\delta^0_{N-1} + \gamma_2\delta^0_N + \gamma_1\delta^0_{N+1}, \\
V(a, 0) &= 0 = \alpha_1\phi^0_- + \alpha_2\phi^0_0 + \alpha_1\phi^0_1, \\
V(x_i, 0) &= \alpha_1\phi^0_{i-1} + \alpha_2\phi^0_i + \alpha_1\phi^0_{i+1} = v(x_i, 0), i = 1, \ldots, N - 1 \\
V(a, 0) &= \alpha_1\phi^0_{N-1} + \alpha_2\phi^0_N + \alpha_1\phi^0_{N+1}
\end{align*}
\]
3 Numerical tests

To see versatility of the present method, three numerical examples are studied in this section. The efficiency and accuracy of the solutions will be determined by using the global relative error using formula

\[
\text{GRE} = \frac{\sum_{j=1}^{N} |U_j^n - u_j^n|}{\sum_{j=1}^{N} |u_j^n|} \quad (14)
\]

where \( U \) denotes numerical solution and \( u \) denotes analytical solution.

Numerical solution of KS equation (1) is obtained for \( \alpha = 1 \) and \( \vartheta = 1 \) with the exact solution given by

\[
u(x, t) = b + \frac{15}{19} d \left[ e \tanh \left( k (x - bt - x_0) \right) + f \tanh^3 \left( k (x - bt - x_0) \right) \right]
\]

the initial condition is taken from the exact solution together with boundary conditions given by (4). This example is studied in [26, 16, 27]. The above solution models the shock wave propagation with the speed \( b \) and initial position \( x_0 \).

We have considered domain as \([x_0, x_N] = [-30, 30]\) with time step \( \Delta t = 0.01 \) and number of partitions as 150. In order to compare the solutions with [16] and [27] we have taken \( b = 5, k = \frac{1}{2} \sqrt{119/19}, x_0 = -12, d = \sqrt{119/19}, e = -9, f = 11 \). Table 2 gives a comparison between the global relative error found by our method and by Quintic B-spline collocation method [16] and by Lattice Boltzmann method [27].

The numerical results are plotted at different time step for \( \Delta t = 0.005 \) and \( N = 400 \) in Fig. 2 and Fig. 3 shows projection of the solution on the x-t plane. Solution obtained by trigonometric cubic B-spline collocation method is very close to the exact solutions due to the global relative error obtained in Table 2.

Table 2: Comparison of global relative error for Example a at different time \( t \), \( N = 150 \)

| Time(t) | Present Method | [16] | [27] |
|---------|----------------|------|------|
| 1       | 2.98416 \times 10^{-5} | 3.81725 \times 10^{-4} | 6.7923 \times 10^{-4} |
| 2       | 7.00758 \times 10^{-5} | 5.51142 \times 10^{-4} | 1.1503 \times 10^{-3} |
| 3       | 9.51142 \times 10^{-5} | 7.03980 \times 10^{-4} | 1.5941 \times 10^{-3} |
| 4       | 1.79237 \times 10^{-4} | 8.63662 \times 10^{-4} | 2.0075 \times 10^{-3} |
This example represents chaotic behaviors with the initial condition,

\[ u(x, 0) = \cos\left(\frac{x}{2}\right) \sin\left(\frac{x}{2}\right) \]

with the boundary condition

\[ u_{xx}(0, t) = 0, \quad u_{xx}(4\pi, t) = 0 \]

The computational domain \([x_0, x_N] = [0, 4\pi]\) is used with \(N = 512, \Delta t = 0.001, \alpha = 1\). It is shown that KS-Equation is highly sensitive for choice of the parameter \(\vartheta\). In Figs. 4-7, we can observe the solution pattern exhibiting complete chaotic behaviors on the \(xt\)-plane, respectively. Figures illustrate that for the smaller value of \(\vartheta\), chaotic behavior
starts to evolve earlier and seen more complex instabilities.

Figure 4: Solutions on $xt$ – plane for $\vartheta = 0.05$

Figure 5: Solutions on $xt$ – plane for $\vartheta$
(c) The KS equation (1) is obtained for $\alpha = 1$ and $\vartheta = 1$. This example represents the simplest nonlinear partial differential equation showing chaotic behavior when spatial domain is finite, with the Gaussian initial condition,

$$u(x, 0) = -\exp(-x^2)$$

with the boundary condition

$$u(x_0, t) = 0, \ u(x_N, t) = 0$$

The computational domain $[x_0, x_N] = [-30, 30]$ with $N = 120$, $\Delta t = 0.001$. In Figs. 8 and 9, we can observe the convergent numerical results by our trigonometric cubic B-Spline method of lines with complete chaotic behavior at $t = 5$ and $t = 20$, respectively. It is
observed that the result shows same characteristics as in [16].

Figure 8: The Chaotic Solution of the KSE \( t = 5 \)  
Figure 9: The Chaotic Solution of the KSE \( t = 20 \)

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