4-Dimensional Power Geometry and its Application to $P_1 - P_5$

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Abstract

In the first section of this work we introduce 4-dimensional Power Geometry for second-order ODEs of a polynomial form. In the next five sections we apply this construction to the first five Painlevé equations.

Keywords: Painlevé equations, asymptotic expansions.

MSC classes: 34m25, 34m55.

1 4-dimensional Power Geometry

We introduce definitions and notations of 4-dimensional Power Geometry analogous to the way it has been done in two and three-dimensional cases in [1] and [2].

Let us be given a second order differential equation of the form

$$f(z, w, w', w'') = 0,$$  \hspace{1cm} (1)

where $f(z, w, w', w'')$ is a polynomial, $z$ is an independent, $w$ is a dependent variable, $w' = dw/dz$.

To each monomial $a(z, w)$ in polynomial (1) we put in correspondance its four-dimensional exponent $Q(a(z, w)) = (q_1, q_2, q_3, q_4)$ according to the following rule:

$$Q(c z^r w^s) = (r, s, 0, 0); \quad Q\left(\frac{dw}{dz}\right) = (0, 0, 1, 0); \quad Q\left(\frac{d^2w}{dz^2}\right) = (0, 0, 0, 1);$$

$$Q(a(z, w)b(z, w)) = Q(a(z, w)) + Q(b(z, q)).$$

The set of all exponents of the monomials in polynomial $f(z, w)$ is called a support of a differential sum $f(z, w)$ and is denoted as $\tilde{S}(f)$. The convex hull $\Gamma(f)$ of the support $\tilde{S}(f)$ is called polyhedron of a differential sum $f(z, w)$, the boundary $\partial \Gamma(f)$ consists of the vertices $\Gamma_j^{(0)}$, edges $\Gamma_j^{(1)}$, two-dimensional faces $\Gamma_j^{(2)}$ and three-dimensional faces $\Gamma_j^{(3)}$ (hyper faces).
We put in correspondence a truncated equation \( f_j^{(d)}(z, w, w', w'') = 0 \) to every edge \( \Gamma_j^{(d)} \), where

\[
 f_j^{(d)}(z, w, w', w'') = \sum a_s(z, w)
\]

\( a_s(z, w) : Q(a_s(z, w)) \in \Gamma_j^{(d)} \).

Let us compare a four-dimensional construction with three and two-dimensional ones.

In three-dimensional case we put in correspondence to each differential monomial \( a(z, w) \) its three-dimensional exponent \( Q(a(z, w)) = (q_1, q_2, q_3) \) according to the following rule:

\[
 Q(cz^r w^s) = (r, s, 0); \ Q \left( \frac{d^l w}{dz^l} \right) = (0, 1, l);
\]

\[
 Q(a(z, w) b(z, w)) = Q(a(z, w)) + Q(b(z, w)).
\]

In three-dimensional case we put in correspondence to each differential monomial \( a(z, w) \) its three-dimensional exponent \( Q(a(z, w)) = (q_1, q_2, q_3) \) according to the following rule:

\[
 Q(cz^r w^s) = (r, s, 0); \ Q \left( \frac{d^l w}{dz^l} \right) = (-l, 1);
\]

\[
 Q(a(z, w) b(z, w)) = Q(a(z, w)) + Q(b(z, w)).
\]

Let us redefine a notion of order of the function. Let us be given a function \( \psi(z) \), for which infinity is not an accumulation point of poles and zeroes, \( z = re^{i\varphi} \). We call

\[
 p_+(\psi(z), \varphi) = \lim_{r \to \infty} \frac{\ln|\psi(re^{i\varphi})|}{\ln|r|}
\]

an order of a function on a ray with direction \( \varphi \) \((z \to \infty)\) if this limit exists.

Let us be given a function \( \psi(z) \), for which zero is not an accumulation point of poles and zeroes of the function, \( z = re^{i\varphi} \). We call

\[
 p_-(\psi(z), \varphi) = \lim_{r \to 0} \frac{\ln|\psi(re^{i\varphi})|}{\ln|r|}
\]

an order of a function on a ray with direction \( \varphi \) \((z \to 0)\) if this limit exists.

These orders of a function \( f(z) \) are of special interest if they coincide for \( \varphi \in (\varphi_1, \varphi_2) \), i.e. for points in some sector on Riemann surface of the logarithm.

Thus we see that in two-dimensional case construction of the support assumes automatically that formal asymptotics of the solutions to the equation satisfy the following condition: order of the derivative of the function is one less that an order of the function (for example, the function \( z^n \) satisfies this condition).
In the three-dimensional structure is assumed that at each differentiation order of the function is changed by $\gamma_1 \in \mathbb{R}$. Functions with $\gamma_1 \neq 1$ as in 2-D variant also exist: consider $p_-(\sin z, 0)$ and $p_-(\cos z, 0)$.

Introduction of the fourth coordinate differential monomials exponents included in the second order ODEs is due to the following condition on the asymptotic behavior of the possible solutions: the first differentiation changes order of a function by $\gamma_1 \in \mathbb{R}$, the second differentiation changes order of a function by $\gamma_2 \in \mathbb{R}$. In 2-D construction $\gamma_1 = \gamma_2 = 1$, in 3-D construction $\gamma_1 = \gamma_2$. Functions with $\gamma_1 \neq \gamma_2$ exist: consider $p_-(z + z^\alpha, 0)$, $p_-(1 + z^\alpha, 0)$ and $p_-(z^\alpha - 2, 0)$, $\alpha > 1$.

We formulate a necessary condition of the fact that a truncated equation corresponding to hyper-face in four-dimensional case can have a solution which is an asymptotic form of a solution to the initial equation.

**Assertion 1.** Let us be given a differential equation of the second order. We consider a truncated equation corresponding to a hyper face $\Gamma^{(3)}$ with an external normal $N = (n_1, n_2, n_3, n_4)$. If $n_1 = 0$, then a truncated equation has no solution of finite order which can be an asymptotic form of the solution to the initial equation.

**Proof.** Negative proof. Let the hyper face have an external normal $N = (0, n_2, n_3, n_4)$, hyperplane containing the face has an equation $n_2 q_2 + n_3 q_3 + n_4 q_4 + e = 0$. (2)

There exist 4 points $Q_i = (q_{1i}, q_{2i}, q_{3i}, q_{4i}), i = 1, \ldots, 4$ in the face, which lie on the hyperplane (2) but do not lie in a plane of smaller dimension. That means that the rank of the matrix $Q = (q_{ij}), i, j = 1, \ldots, 4$ is equal to three.

All the points $Q_i$ satisfy an equation (2). We subtract from each equation the first equation and obtain a system

$$\hat{Q} = (q_{ij} - q_{i1}), i = 1, \ldots, 4, j = 1, 2, 3.$$

As $\hat{Q} N^T = 0$, and $N = (0, n_2, n_3, n_4)$, we obtain that the matrix $\hat{Q} = (q_{ij} - q_{i1}), i = 2, \ldots, 4, j = 1, 2, 3$ has a rank less than three. If we assume that it is less than two we arrive at a contradiction, we obtain that the rank of $\hat{Q}$ is equal to two, i.e. its columns are linearly dependent.

If a truncated equation corresponding to a hyper face has a solution an order of which is equal to $\gamma$, an order of the first derivative is equal to $\gamma_1$, an order of the second derivative is equal to $\gamma_2$, then the points $Q_i, i = 1, \ldots, 4$ satisfy the system

$$q_{1i} + q_{2i} \gamma + q_{3i} \gamma_1 + q_{4i} \gamma_2 + f = 0, i = 1, \ldots, 4.$$

We subtract from the second, third and fourth equation the first one and obtain a system

$$\hat{Q} x = y,$$
where $x = (\gamma, \gamma_1, \gamma_2)^T$, $y = (q_{11} - q_{12}, q_{11} - q_{13}, q_{11} - q_{14})^T$. But the rank of the matrix $\hat{Q}$ is equal to 2, and a column $y$ is a linear combination of the columns of the matrix $\hat{Q}$. We obtain that rank of the matrix $\hat{Q}$ is equal to 2: this leads us to contradiction.

2 The first Painlevé equation

The first Painlevé equation

$$w'' = 6w^2 + z$$

has one singular point $z = \infty$.

All three points of the support, of course, lie in the plane of dimension 3, so the use of the methods of four-dimensional power geometry does not make sense.

3 The second Painlevé equation

We consider the second Painlevé equation

$$w'' = 2w^3 + zw + \alpha,$$

where $\alpha$ is a complex parameter. The equation has one singular point $z = \infty$.

The support of the equation consists of four points: $(0, 0, 0, 1)$, $(0, 3, 0, 0)$, $(1, 1, 0, 0)$ and $(0, 0, 0, 0)$, all these points lie in a hyperplane $q_3 = 0$, an external normal to it is equal either to $(0, 0, 1, 0)$ or to $(0, 0, -1, 0)$. According to the assertion 1 we obtain that the truncated solution has no solutions the leading term of which with its derivatives have a finite order.

4 The third Painlevé equation $\alpha \beta \gamma \delta \neq 0$

We consider the third Painlevé equation

$$w'' = \frac{(w')^2}{w} - \frac{w'}{z} + \frac{\alpha w^2 + \beta}{z} + \gamma w^3 + \frac{\delta}{w},$$

where $\alpha, \beta, \gamma, \delta$ are complex parameters, the equation (3) has two singular points: $z = 0$ and $z = \infty$.

We rewrite the equation (3) in a form of a differential sum:

$$f_3(z, w) \overset{def}{=} -zw'' + z (w')^2 - w w' + \alpha w^3 + \beta w + \gamma z w^4 + \delta z = 0. \quad (4)$$
The four-dimensional support of the equation (4) consists of the points

\[ Q_1 = (1, 0, 0, 0), \ Q_2 = (1, 4, 0, 0), \ Q_3 = (0, 1, 0, 0), \ Q_4 = (0, 3, 0, 0), \]
\[ Q_5 = (0, 1, 1, 0), \ Q_6 = (1, 0, 2, 0), \ Q_7 = (1, 1, 0, 1). \]

A convex hull of the equation (4) is a polygon with vertices \( Q_1, \ldots, Q_7 \).

Its hyper faces are \( \Gamma_1^{(3)} = \text{conv}(Q_1, \ldots, Q_6), \Gamma_2^{(3)} = \text{conv}(Q_1, Q_3, Q_5, Q_6, Q_7), \)
\[ \Gamma_3^{(3)} = \text{conv}(Q_1, Q_2, Q_6, Q_7), \Gamma_4^{(3)} = \text{conv}(Q_1, Q_2, Q_3, Q_4, Q_7), \]
\[ \Gamma_5^{(3)} = \text{conv}(Q_2, Q_3, \ldots, Q_7), \Gamma_6^{(3)} = \text{conv}(Q_3, Q_4, Q_5, Q_7). \]

According to the Assertion 1 we consider only the following 3D faces (i.e. the faces the first coordinate of the external normal to which is not equal to zero):

1. \( \Gamma_2^{(3)} \) lies in the plane \( q_1 + q_2 - q_4 - 1 = 0 \) with an external normal \( N_2 = (-1, -1, 0, 1) \). The truncated equation corresponding to the face

\[ -zw'' + z(w')^2 - w w' + \beta w + \delta z = 0 \]

has already been considered as a truncated equation corresponding to a 2D face in 3D case and as a truncated equation corresponding to an edge in 2D case.

2. \( \Gamma_3^{(3)} \) lies in the plane \( q_1 = 1 \) with an external normal \( N_3 = (-1, 0, 0, 0) \). The truncated equation corresponding to the face

\[ -zw'' + z(w')^2 + \gamma zw^4 + \delta z = 0 \]

has already been considered as a truncated equation corresponding to a 2D face in 3D case.

3. \( \Gamma_5^{(3)} \) lies in the plane \( q_1 - q_2 - 2q_3 - 3q_4 + 3 = 0 \) with an external normal \( N_5 = (-1, 1, 2, -3) \). The truncated equation corresponding to the face

\[ -zw'' + z(w')^2 - w w' + \alpha w^3 + \gamma zw^4 = 0 \]

has already been considered as a truncated equation corresponding to an edge in 2D case.

4. \( \Gamma_6^{(3)} \) lies in the plane \( q_1 - q_4 = 0 \) with an external normal \( N_6 = (-1, 0, 0, 1) \). The truncated equation corresponding to the face

\[ -zw'' - w w' + \alpha w^3 + \beta w = 0. \]

This equation has not been considered neither in 2D nor in 3D case.

The external normals to the faces \( \Gamma_1^{(3)} \) (\( q_4 = 0, N_1 = (0, 0, 0, -1) \)) and \( \Gamma_4^{(3)} \) (\( q_3 = 0, N_4 = (0, 0, -1, 0) \)) do not satisfy Assertion 1.
5 The fourth Painlevé equation $\alpha \beta \neq 0$

We consider the fourth Painlevé equation
\[ w'' = \frac{(w')^2}{w} - \frac{3}{2} w^3 + 4z w^2 + 2(z^2 - \alpha)w + \frac{\beta}{w}, \tag{5} \]
where $\alpha, \beta$, are complex parameters.

We rewrite the equation (5) in a form of a differential sum:
\[ f_4(z, w) \triangleq -2ww'' + (w')^2 + 3w^4 + 8zy^4 + 4(z^2 - \alpha)w^2 + 2\beta = 0. \tag{6} \]

The 4D support of the equation (6) consists of the points
\[ Q_1 = (0, 0, 0, 0), \ Q_2 = (0, 2, 0, 0), \ Q_3 = (2, 2, 0, 0), \ Q_4 = (1, 3, 0, 0), \ Q_5 = (0, 4, 0, 0), \ Q_6 = (0, 0, 2, 0), \ Q_7 = (0, 1, 0, 1). \tag{7} \]

A convex hull of the equation (4) is a polygon with vertices $Q_1, Q_3, Q_5, Q_6, Q_7$ with hyper faces $\Gamma_1^{(3)} = \text{conv}(Q_1, \ldots, Q_6), \Gamma_2^{(3)} = \text{conv}(Q_3, \ldots, Q_7), \Gamma_3^{(3)} = \text{conv}(Q_1, Q_3, Q_6, Q_7), \Gamma_4^{(3)} = \text{conv}(Q_1, \ldots, Q_5, Q_7), \Gamma_5^{(3)} = \text{conv}(Q_1, Q_2, Q_5, Q_6, Q_7).

According to the Assertion 1 we consider only the following 3D faces:

1. $\Gamma_2^{(3)}$ lies in the plane $q_1 + q_2 + 2q_3 + 3q_4 = 0$ with an external normal $N_2 = (1, 1, 2, 3)$. The truncated equation corresponding to the face
\[-2ww'' + (w')^2 + 3w^4 + 8zy^4 + 4z^2w^2 = 0\]
has already been considered as a truncated equation corresponding to a 2D face in 3D case.

2. $\Gamma_3^{(3)}$ lies in the plane $q_1 - q_2 + q_4 = 0$ with an external normal $N_3 = (1, -1, 0, 1)$. The truncated equation corresponding to the face
\[-2ww'' + (w')^2 + 4z^2w^2 + 2\beta = 0\]
has already been considered as a truncated equation corresponding to a 2D face in 3D case.

3. $\Gamma_5^{(3)}$ lies in the plane $q_4 = 0$ with an external normal $N_5 = (-1, 0, 0, 0)$. The truncated equation corresponding to the face
\[-2ww'' + (w')^2 + 3w^4 - 4\alpha w^2 + 2\beta = 0\]
has already been considered as a truncated equation corresponding to a 2D face in 3D case.

The external normals to the faces $\Gamma_1^{(3)}$ with equation $q_4 = 0$ and $\Gamma_4^{(3)}$ with equation $q_3 = 0$ ($N_1 = (0, 0, 0, -1)$ and $N_4 = (0, 0, -1, 0)$) do not satisfy the above conditions.
6 The fifth Painlevé equation (the case $\delta \neq 0$)

We consider the fifth Painlevé equation

$$w'' = \left( \frac{1}{2w} + \frac{1}{w-1} \right) (w')^2 - \frac{w'}{z} + \frac{(w-1)^2}{z^2} \left( \alpha w + \frac{\beta}{w} \right) + \gamma w + \frac{\delta w(w+1)}{w-1},$$

where $\alpha, \beta, \gamma, \delta$ are complex parameters. The equation has two singular points $z = 0$ and $z = \infty$.

We represent the fifth Painlevé equation in a form of a differential sum:

$$f(z, w) \overset{def}{=} -z^2w(w-1)w'' + z^2 \left( \frac{3}{2}w - \frac{1}{2} \right) (w')^2 - zw(w-1)w' + (w-1)^3(\alpha w^2 + \beta) + \gamma z w^2(w-1) + \delta z^2 w^2(w+1) = 0. \quad (8)$$

The 4D support of the equation (6) consists of the points

\begin{align*}
Q_1 & = (2,3,0,1), \ Q_2 = (2,2,0,1), \ Q_3 = (2,3,2,0), \ Q_4 = (2,2,2,0), \\
Q_5 & = (1,2,1,0), \ Q_6 = (1,3,1,0), \ Q_7 = (0,0,0,0), \ Q_8 = (0,5,0,0), \\
Q_9 & = (1,2,0,0), \ Q_{10} = (1,3,0,0), \\
Q_{11} & = (2,2,0,0), \ Q_{12} = (2,3,0,0), \\
Q_{13} & = (0,1,0,0), \ Q_{14} = (0,2,0,0), \ Q_{15} = (0,3,0,0), \ Q_{16} = (0,4,0,0).
\end{align*}

A convex hull of the equation (6) is a polygon with vertices $Q_1$, $Q_2$, $Q_3$, $Q_4$, $Q_7$, $Q_8$, $Q_{11}$, $Q_{12}$ with hyper faces

$$\Gamma_1^{(3)} = \text{conv}(Q_1, Q_2, Q_3, Q_4, Q_{11}, Q_{12}), \quad \Gamma_2^{(3)} = \text{conv}(Q_2, Q_3, Q_7, Q_{11}),$$

$$\Gamma_3^{(3)} = \text{conv}(Q_1, Q_2, Q_7, Q_{12}, Q_{13}, \ldots, Q_{16}), \quad \Gamma_4^{(3)} = \text{conv}(Q_1, Q_3, Q_8, \ldots, Q_{12}),$$

$$\Gamma_5^{(3)} = \text{conv}(Q_3, \ldots, Q_{12}, Q_{13}, \ldots, Q_{16}), \quad \Gamma_6^{(3)} = \text{conv}(Q_1, \ldots, Q_8, Q_{13}, \ldots, Q_{16}).$$

According to the Assertion 1 we consider only the following 3D faces:

1. $\Gamma_1^{(3)}$ lies in the plane $q_1 = 2$ with an external normal $N_1 = (1,0,0,0)$. The truncated equation corresponding to the face

$$-z^2w(w-1)w'' + z^2 \left( \frac{3}{2}w - \frac{1}{2} \right) (w')^2 + \delta z^2 w^2(w+1) = 0$$

has already been considered as a truncated equation corresponding to a 2D face in 3D case.
2. $\Gamma^{(3)}_2$ lies in the plane $q_1 - q_2 - q_3 - q_4 = 0$ with an external normal $N_2 = (1, -1, -1, -1)$. The truncated equation corresponding to the face

$$z^2 w'' - \frac{1}{2} z^2 (w')^2 - \beta + \delta z^2 w = 0$$

has already been considered as a truncated equation corresponding to a 2D face in 3D case.

3. $\Gamma^{(3)}_4$ lies in the plane $q_1 + q_2 + q_3 + q_4 - 5 = 0$ with an external normal $N_4 = (1, 1, 1, 1)$. The truncated equation corresponding to the face

$$-z^2 w^2 w'' + \frac{3}{2} z^2 w (w')^2 + \alpha w^5 + \delta z^2 w^3 = 0$$

has already been considered as a truncated equation corresponding to a 2D face in 3D case.

4. $\Gamma^{(3)}_6$ lies in the plane $q_1 - q_3 - 2q_4 = 0$ with an external normal $N_6 = (1, 0, -1, -2)$. The truncated equation corresponding to the face

$$-z^2 w(w - 1)w'' + z^2 \left( \frac{3}{2} w - \frac{1}{2} \right) (w')^2 - zw(w - 1)w' + (w - 1)^3 (\alpha w^2 + \beta) = 0$$

has already been considered as a truncated equation corresponding to a 2D face in 3D case, this equation can be solved directly.

The external normals to the faces $\Gamma^{(3)}_3$ with equation $q_3 = 0$ and $\Gamma^{(3)}_5$ with equation $q_4 = 0 (N_3 = (0, 0, -1, 0)$ and $N_5 = (0, 0, 0, -1))$ do not satisfy the above conditions.

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