On the absolute $N_{q\alpha}$-summability of $r$th derived conjugate series

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Abstract. The object of the present paper is to study the absolute $N_{q\alpha}$-summability of $r$th derived conjugate series generalizing a known result.

Keywords. Fourier series; conjugate series; derived conjugate series; Nevanlinna summability; kernel.

1. Introduction

1.1.

In the year 1921, Nevanlinna [7] suggested and discussed an interesting method of summation called $N_q$-method. Moursund [5] applied this method for summation of Fourier series and its conjugate series. Later, Moursund [6] developed $N_{q\alpha}$-method (where $p$ is a positive integer) and applied it to $p$th derived Fourier series. Samal [9] discussed $N_{q\alpha}$-method ($0 \leq \alpha < 1$) and studied absolute $N_{q\alpha}$-summability of some series associated with Fourier series. In his Ph.D. thesis [10] he extended $N_{q\alpha}$-method of summation to $N_{q\alpha}$-method for any $\alpha \geq 0$ and studied absolute $N_{q\alpha}$-summability of Fourier series. Earlier we [8] have studied absolute $N_{q\alpha}$-summability of a series conjugate to a Fourier series. In the present paper we shall study the absolute $N_{q\alpha}$-summability of $r$th ($r < \alpha$) derived series of a conjugate series.

1.2.

DEFINITION 1. [6,10]

Let $F(w)$ be a function of a continuous parameter $w$ defined for all $w > 0$. The $N_{q\alpha}$-method consists in forming the $N_{q\alpha}$-transform or mean

$$N_{q\alpha}F(w) = \int_0^1 q\alpha(t)F(wt)\,dt$$

and then considering the limit

$$\lim_{w \to \infty} N_{q\alpha}F(w),$$
where the class of functions $q_{\alpha}(t)$ is such that

1. $q_{\alpha}(t) \geq 0$ for $0 \leq t \leq 1$,
2. $\int_0^1 q_{\alpha}(t) \, dt = 1$,
3. $\frac{d^\beta}{dt^\beta} q_{\alpha}(t)$ exists and is absolutely continuous for $0 \leq t \leq 1$, $\beta = 1, 2, \ldots, k-1$, where $[\alpha] = k$,
4. $\frac{d^\beta}{dt^\beta} q_{\alpha}(t) = 0$ for $t = 1$, $\beta = 0, 1, 2, \ldots, k-1$,
5. $\frac{d^k}{dt^k} q_{\alpha}(t)$ exists for $0 < t < 1$,
6. $(-1)^k \frac{d^k}{dt^k} q_{\alpha}(t) \geq 0$ and monotonic increasing for $0 < t < 1$,
7. $\int_0^t Q_k(u) \, u^{1+\alpha-k} \, du = O \left( \frac{Q_k(t)}{t^{\alpha-k}} \right)$,

where

$$Q_k(t) = \int_{1-t}^1 (-1)^k \frac{d^k}{du^k} q_{\alpha}(u) \, du.$$

Also we set

$$Q(t) = \int_{1-t}^1 q_{\alpha}(u) \, du.$$

If $\lim_{w \to \infty} N_{\alpha} F(w)$ exists, we say that $N_{\alpha}$-limit of $F(w)$ exists.

**DEFINITION 2.**

Let $\sum_{n=0}^\infty u_n$ be an infinite series with $S(w) = \sum_{n \leq w} u_n$. If $\lim_{w \to \infty} \{ \sum_{n \leq w} u_n Q(1 - (n/w)) \} = 1$, we say that $\sum u_n$ is summable by $N_{\alpha}$-method to the sum 1. In short we write that $\sum u_n = 1$ ($N_{\alpha}$). Further the series $\sum u_n$ is said to be $|N_{\alpha}|$-summable (absolute $N_{\alpha}$-summable) if

$$\int_A^\infty \frac{dw}{w^2} \left| \sum_{n \leq w} n u_n q_{\alpha} \left( \frac{n}{w} \right) \right| < \infty$$

for some positive constant $A$.

For $\alpha = 0$, the method reduces to the original $N_q$-method [7] and if $\alpha$ is any positive integer $p$, then the method reduces to $N_{qp}$-method of Moursund [6].

1.3.

Let $f(t)$ be a periodic function with period $2\pi$ and Lebesque integrable over $(-\pi, \pi)$.
On the absolute $N_{qa}$-summability of rth derived conjugate series

Let
\[ f(t) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) \equiv \sum_{n=0}^{\infty} A_n(t). \] (1.3.1)

The series conjugate to (1.3.1) at $t = x$ is given by
\[ \sum_{n=1}^{\infty} (b_n \cos nx - a_n \sin nx) \equiv \sum_{n=1}^{\infty} B_n(x), \] (1.3.2)
\[ P(u) = \sum_{i=0}^{n} \frac{\theta_i}{i!} u^i \quad \text{for} \quad -\pi \leq u \leq \pi, \]
where $\theta_i$s for $i = 0, 1, 2, \ldots, r - 1$ are arbitrary constants.
\[ h(u) = \frac{\{f(x + u) - P(u)\} - (-1)^r\{f(x - u) - P(-u)\}}{2u^r}, \]
\[ H_0(t) = h(t), \]
\[ H_\beta(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t - u)^{\beta - 1}h(u)du, \quad (\beta > 0), \]
\[ h_\beta(t) = \Gamma(1 + \beta)t^{-\beta}H_\beta(t), \quad (\beta \geq 0). \]

2. Purpose of the present paper

In the present paper we shall prove the following theorems:

**Theorem 1.** Let $\beta = \alpha - r$. If $H_\beta(+0) = 0$ and $\int_0^\pi t^{-\beta}|dH_\beta(t)| < \infty$, then the rth derived series of the conjugate series of $f(t)$ at $t = x$ is $|N_{qa}|$-summable.

**Theorem 2.** Let $\rho = \alpha - r - 1$. If $\rho \geq 0$ and $\int_0^\pi t^{-1}|h_\rho(t)|dt < \infty$, then the rth derived series of the conjugate series of $f(t)$ at $t = x$ is $|N_{qa}|$-summable.

By taking $\beta = \rho + 1, \rho \geq 0$ in Theorem 1, we can obtain Theorem 2 at once as it is known [4] that
\[ H_{\rho+1}(+0) = 0 \quad \text{and} \quad \int_0^\pi t^{-\rho-1}|dH_{\rho+1}(t)| < \infty \]
\[ \iff h_{\rho+1}(t) \in BV(0, \pi) \quad \text{and} \quad \frac{h_{\rho+1}(t)}{t} \in L(0, \pi) \]
\[ \iff h_\rho(t) \in L(0, \pi). \]

By taking $q_\alpha(t) = (\alpha + \delta)(1-t)^{\alpha+\delta-1}$, where $\delta > 0$ and $\alpha + \delta < k + 1$ ($|\alpha| = k$) in Theorems 1 and 2, we obtain the following corollaries respectively.

**COROLLARY 1.** [3]
If $H_\beta(+0) = 0$ and $\int_0^\pi t^{-\beta}|dH_\beta(t)| < \infty$, then the rth derived series of the conjugate series of $f(t)$ at $t = x$ is summable $|C, \beta + r + \delta|$, where $\beta > 0$ and $\delta > 0$.

**COROLLARY 2.** [3]
If $\rho \geq 0$ and $\int_0^\pi t^{-1}|h_\rho(t)|dt < \infty$, then the rth derived series of the conjugate series of $f(t)$ at $t = x$ is summable $|C, \rho + r + 1 + \delta|$. 
3. Notations and lemmas

3.1. Notations

For our purpose we use the following notations throughout this paper.

\[ [\alpha] = k, \]
\[ m = \min (k - r, r), \]
\[ q^k(u) = (-1)^k \frac{d^k}{du^k} q_\alpha(u), \]
\[ (\cos nu)_j = \left( \frac{d}{du} \right)^j \cos nu, \]
\[ S^{(i)}(x,u) = \sum_{n \leq x} (x-n)^i (\cos nu)_j, \]
\[ G_i(w,u) = \sum_{n \leq w} q_\alpha \left( \frac{n}{w} \right) \left( \frac{d}{du} \right)^{k+1-i} \cos nu, \quad \text{for} \quad i = 0, 1, 2, \ldots, m, \]
\[ g_i(x,w,u) = \frac{1}{k!} (-1)^k \left( \frac{d}{dx} \right)^k q_\left( \frac{x}{w} \right) \left( \frac{d}{dx} \right)^{k+1-i} S^{(k+1-i)}(x,u) \]
\[ \quad \text{for} \quad i = 0, 1, 2, \ldots, m. \]

3.2. Lemmas

We need the following lemmas for the proof of our theorem.

Lemma 1 [1]. If \( \beta > \alpha > 0, H_\alpha(t) \) is of BV \((0, \pi)\) and \( H_\alpha(+0) = 0 \), then \( H_\beta(t) \) is an integral in \((0, \pi)\) and for almost all values of \( t \),
\[ H'_\beta(t) = \frac{1}{\Gamma(\beta - \alpha)} \int_0^t (t-u)^{\beta-\alpha-1} dH_\alpha(u). \]

Lemma 2 [6]. If \( \alpha \geq 1 \), the kernel \( q_\alpha(t) \) is monotonic decreasing, its derivatives of odd orders less than \( k \) are negative and monotonic increasing, its derivatives of even orders less than \( k \) are positive and monotonic decreasing and there exists a constant \( A_k \) such that
\[ \left| \frac{d^\beta}{dt^\beta} q_\alpha(t) \right| < A_k \quad (\beta = 0, 1, 2, \ldots, k-1) \]
and
\[ \int_0^t \left| \frac{d^k}{dt^k} q_\alpha(t) \right| dt < A_k. \]

Lemma 3 [10]. \( Q_k(t) \) is continuous and monotonic increasing function of \( t \), \( Q_k(t) \geq 0, Q(0) = 0 \) and \( Q(1) = 1 \).

This follows directly from the definition of \( Q(t) \) and \( Q_k(t) \).

Lemma 4 [10, 8]. \( \int_0^1 q_k(t)/((1-t)^{\alpha-k}) dt \) exists.
Lemma 5 [8]. Let \( x > 0 \).

(i) If \( 1/x < u \leq \pi \), then

\[
S^{i,j}(x,u) = \begin{cases} 
O(x^i u^{-j-1}) & \text{for } 0 \leq j \leq i, \\
O(x^i u^{-i-1}) & \text{for } j > j \geq 0.
\end{cases}
\]

(ii) If \( 1/x \geq u > 0 \), then

\[
S^{i,j}(x,u) = O(x^{i+j+1}).
\]

Lemma 6 [2]. Let \( \lambda = \{\lambda_n\} \) be a positive monotonic increasing sequence with \( \lambda_n \to \infty \) as \( n \to \infty \). Then

\[
A_\lambda(x) = A_\lambda^0(x) = \sum_{\lambda_n \leq x} a_n
\]

and

\[
A_\lambda^r(x) = \sum_{\lambda_n \leq x} (x - \lambda_n)^r a_n (r > 0).
\]

If \( k \) is a positive integer,

\[
A_\lambda(x) = \frac{1}{k!} \left( \frac{d}{dx} \right)^k A_\lambda^k(x).
\]

Lemma 7 [8,10]. For \( \alpha \geq 1 \),

\[
\sum_{n \leq w} (-1)^n n^k q_{\alpha} \left( \frac{n}{w} \right) = O \left\{ q^k \left( 1 - \frac{1}{w} \right) \right\} + O \left\{ w Q_{\alpha} \left( \frac{1}{w} \right) \right\}.
\]

Lemma 8 [8,10]. For \( \alpha \geq 1 \) and \( r = 0, 1, 2, \ldots, k - 1 \),

\[
\sum_{n \leq w} (-1)^n n^r q_{\alpha} \left( \frac{n}{w} \right) = O(1).
\]

Lemma 9. For \( i = 0, 1, 2, \ldots, k - 1 \),

\[
\sum_{n \leq w} (-1)^n n^i \in |Nq_{\alpha}|.
\]

Proof. For \( i = 0, 1, 2, \ldots, k - 2 \),

\[
\int_1^w \frac{dw}{w^2} \sum_{n \leq w} n(-1)^n n^i q_{\alpha} \left( \frac{n}{w} \right) = \int_1^w O(1) \frac{dw}{w^2} \quad \text{by Lemma 8}
\]

\[
= O(1)
\]
and

\[
\int_1^\infty \frac{dw}{w^2} \left| \sum_{n \leq w} (-1)^n n^k q_\alpha \left( \frac{n}{w} \right) \right|
\]

\[
= \int_1^\infty O\left\{ q^k \left( 1 - \frac{1}{w} \right) \right\} \frac{dw}{w^2}
\]

\[
+ \int_1^\infty O\left\{ wQ_k \left( \frac{1}{w} \right) \right\} \frac{dw}{w^2} \quad \text{by Lemma 7}
\]

\[
= O\left( \int_0^1 q^k(u) du \right) + O\left( \int_0^1 \frac{Q_k(u)}{u} du \right)
\]

\[
= O(1)
\]

by Lemma 2 and the definitions of \( q^k(u) \) and \( Q_k(u) \). This completes the proof of Lemma 9.

**Lemma 10.** For \( i = 0, 1, 2, \ldots, m \),

\[
G_i(w, u) = \int_1^w g_i(x, w, u) dx.
\]

**Proof.**

\[
G_i(w, u) = \sum_{n \leq w} q_\alpha \left( \frac{n}{w} \right) \left( \frac{d}{du} \right)^{k+1-i} \cos nu
\]

\[
= q_\alpha(1) \sum_{n \leq w} \left( \frac{d}{du} \right)^{k+1-i} \cos nu
\]

\[
- \int_1^w \frac{d}{dx} q_\alpha \left( \frac{x}{w} \right) \left\{ \sum_{n \leq x} \left( \frac{d}{du} \right)^{k+1-i} \cos nu \right\} dx
\]

\[
= - \int_1^w \frac{d}{dx} q_\alpha \left( \frac{x}{w} \right) \frac{1}{k!} \left( \frac{d}{dx} \right)^k \left\{ \sum_{n \leq x} (x-n)^k \left( \frac{d}{du} \right)^{k+1-i} \cos nu \right\} dx
\]

by Lemma 6

\[
= \left[ \frac{1}{k!} \sum_{\rho=1}^{k-1} (-1)^\rho \left( \frac{d}{dx} \right)^\rho q_\alpha \left( \frac{x}{w} \right) \left( \frac{d}{dx} \right)^{k-\rho} S^{k,k+1-i}(x, u) \right]_{x=1}^w
\]

\[
+ \int_1^w \frac{(-1)^k}{k!} \left( \frac{d}{dx} \right)^k q_\alpha \left( \frac{x}{w} \right) \frac{d}{dx} S^{k,k+1-i}(x, u) dx
\]

(integrating by parts for \( k - 1 \) times)

\[
= \int_1^w g_i(x, w, u) dx
\]

as the integrated part vanishes for \( x = w \) and \( x = 1 \).
Lemma 11. For \( wt \leq \pi \) and \( i = 0, 1, 2, \ldots, m \),
\[
\int_{t}^{t+(1/w)} u^{r-i}(u-t)^{k-\alpha} G_i(w,u) du = O(w^{\alpha-r+1}).
\]

Proof. For \( i = 0, 1, 2, \ldots, m \),
\[
\int_{t}^{t+(1/w)} u^{r-i}(u-t)^{k-\alpha} G_i(w,u) du
= \int_{t}^{t+(1/w)} u^{r-i}(u-t)^{k-\alpha} \left( \sum_{n \leq w} q_{\alpha} \left( \frac{n}{w} \right) \left( \frac{d}{du} \right)^{k+1-i} \cos nu \right) du
= \int_{t}^{t+(1/w)} u^{r-i}(u-t)^{k-\alpha} O(w^{k+2-i}) du
= O \left\{ \left( t + \frac{1}{w} \right)^{r-i} w^{k+2-i} \int_{t}^{t+(1/w)} (u-t)^{k-\alpha} du \right\}
= O \left\{ \left( \frac{wt + 1}{w} \right)^{r-i} w^{k+2-i} \cdot \frac{1}{w^{k-\alpha+1}} \right\}
= O(w^{\alpha-r+1}) \text{ as } wt \leq \pi.
\]

Lemma 12. For \( i = 0, 1, 2, \ldots, m \) and \( wt \leq \pi \),
\[
\int_{t+(1/w)}^{\pi} u^{r-i}(u-t)^{k-\alpha} G_i(w,u) du = O(w^{\alpha-r+1}).
\]

Proof. By the use of Lemma 10,
\[
\int_{t+(1/w)}^{\pi} u^{r-i}(u-t)^{k-\alpha} G_i(w,u) du
= \int_{t+(1/w)}^{\pi} u^{r-i}(u-t)^{k-\alpha} du \int_{1}^{w} g_i(x,w,u) dx
= \int_{t+(1/w)}^{\pi} u^{r-i}(u-t)^{k-\alpha} du \frac{1}{k!}
\times \int_{1}^{w} (-1)^k \left( \frac{d}{dx} \right)^k q_{\alpha} \left( \frac{x}{w} \right) \frac{d}{dx} s^{k+1-i}(x,u) dx
= \frac{1}{(k-1)!} \int_{1}^{w} (-1)^k \left( \frac{d}{dx} \right)^k q_{\alpha} \left( \frac{x}{w} \right) dx
\times \int_{t+(1/w)}^{\pi} u^{r-i}(u-t)^{k-\alpha} s^{k-1,k+1-i}(x,u) du
= \frac{1}{(k-1)!} \int_{1}^{w} (-1)^k \left( \frac{d}{dx} \right)^k q_{\alpha} \left( \frac{x}{w} \right) dx \cdot w^{\alpha-k}
\times \int_{t+(1/w)}^{\xi} u^{r-i} s^{k-1,k+1-i}(x,u) du,
\]

(3.2.1)
for some $t + (1/w) < \xi < \pi$, by an application of the mean value theorem. For $i \geq 2$, using Lemma 5(i) in (3.2.1), we get

$$ \int_{t + (1/w)}^\pi u^{r-i}(u-t)^{k-a}G_i(w,u)du $$

$$ = \frac{1}{(k-1)!} \int_1^w (-1)^k \left( \frac{d}{dx} \right)^k q_a \left( \frac{x}{w} \right) w^{a-k}dx $$

$$ \times \int_{t + (1/w)}^\xi u^{r-i}O(x^{k-1}u^{-k-2+i})du $$

$$ = \frac{1}{(k-1)!} \int_1^w (-1)^k \left( \frac{d}{dx} \right)^k q_a \left( \frac{x}{w} \right) w^{a-k}O \left( \frac{x^{k-1}}{(t + \frac{1}{w})^{k-r+i}} \right) dx $$

$$ = \frac{1}{w^{a-r+1}} \int_1^w (-1)^k \left( \frac{d}{dx} \right)^k q_a \left( \frac{x}{w} \right) dx $$

$$ = O \left( \frac{1}{w^{a-r}} \right)^i $$

by Lemma 2.

For $i = 1$, we have

$$ \int_{t + (1/w)}^\xi u^{r-i} S^{k-1,k+1-i}(x,u)du $$

$$ = \int_{t + (1/w)}^\xi u^{r-i} S^{k-1,k}(x,u)du $$

$$ = \left[ u^{r-1} S^{k-1,k}(x,u) \right]_{t + (1/w)}^\xi $$

$$ - (r-1) \int_{t + (1/w)}^\xi u^{r-2} S^{k-1,k-1}(x,u)du $$

$$ = O \left( \frac{x^{k-1}}{(t + \frac{1}{w})^{k-r+1}} \right) + \int_{t + (1/w)}^\xi u^{r-2}O(x^{k-1}u^{-k})du $$

by Lemma 5(i)

$$ = O(w^{2k-r}). $$

Similarly, for $i = 0$, integrating by parts twice and using Lemma 5(i), it follows that

$$ \int_{t + (1/w)}^\xi u^{r-i} S^{k-1,k+1-i}(x,u)du = O(w^{2k-r}). $$
Proof. For some \( 1 \leq i \leq 13 \), using the above estimation in (3.2.1)

\[
\int_{t+1/\omega}^{0} \frac{\omega^{k-i}(u-t)^{k-\alpha}G_i(w,u)\,du}{w^{k+1-i}} = O \left( \int_{0}^{1} (-1)^{k} \left( \frac{d}{dx} \right)^{k} q_{\alpha} \left( \frac{x}{w} \right) w^{\alpha+k-\alpha} \,dx \right)
\]

\[
= O \left( \int_{0}^{1} \left( \frac{w^{k-i+1}}{w^{k+1-i}} \right) q_{\alpha}(\theta) \,d\theta \right)
\]

\[
= O(w^{\alpha-k+1}) \quad \text{by Lemma 2.}
\]

This completes the proof of Lemma 12.

Lemma 13.

\[
\int_{t}^{t+1/\omega} \frac{u^{k-i}(u-t)^{k-\alpha}du}{w^{k+1-i}} \int_{1}^{w-(\pi/i)} u^{k-\alpha} \,dx = O \left( \frac{w^{\alpha-k}}{k!} q_{\alpha} \left( \frac{1}{\omega} \right) \right).
\]

Proof. For some \( 1 < \xi < w-(\pi/i) \), by an application of the mean value theorem,

\[
\int_{t}^{t+1/\omega} \frac{u^{k-i}(u-t)^{k-\alpha}du}{w^{k+1-i}} \int_{1}^{w-(\pi/i)} u^{k-\alpha} \,dx
\]

\[
= \int_{t}^{t+1/\omega} \frac{u^{k-i}(u-t)^{k-\alpha}du}{w^{k+1-i}}
\]

\[
\times \int_{1}^{w-(\pi/i)} (-1)^{k} \left( \frac{d}{dx} \right)^{k} q_{\alpha} \left( \frac{x}{w} \right) \frac{dx}{w^{\alpha+k-\alpha}} \,dx
\]

\[
= \int_{t}^{t+1/\omega} \frac{1}{k!} u^{k-i}(u-t)^{k-\alpha} \left[ (-1)^{k} \left( \frac{d}{dx} \right)^{k} q_{\alpha} \left( \frac{x}{w} \right) \right]_{x=\omega-(\pi/i)}
\]

\[
\times \int_{\xi}^{w-(\pi/i)} \frac{dx}{w^{\alpha+k-\alpha}} \left[ q_{\alpha}^{k+1-i}(x,u) \right]_{x=\xi} \,dx.
\]

(3.2.2)

For \( i = 0 \), using Lemma 5(i) in (3.2.2), we get

\[
\int_{t}^{t+1/\omega} \frac{u^{k-i}(u-t)^{k-\alpha}du}{w^{k+1-i}} \int_{1}^{w-(\pi/i)} u^{k-\alpha} \,dx
\]

\[
= \frac{1}{k!} \int_{t}^{t+1/\omega} \frac{u^{k-i}(u-t)^{k-\alpha}du}{w^{k+1-i}} q_{\alpha} \left( 1 - \frac{\pi}{\omega} \right) O \left( \frac{(w-\frac{\pi}{\omega})^{k+1}}{w^{k+1}} \right) \,du
\]
This completes the proof of Lemma 13.

For \( i \geq 1 \), using Lemma 5(i) in (3.2.2), we obtain
\[
\int_{t}^{t+(1/w)} u^{-i}(u-t)^{k-\alpha} du \int_{1}^{w-(\pi/i)} g_i(x,w)dx
= \frac{1}{k!} \int_{t}^{t+(1/w)} u^{-i}(u-t)^{k-\alpha} \left(1 - \frac{\pi}{wt}\right) O\left(\frac{w^k}{u^{k+2-i}}\right) du
= O\left(\frac{w^{\alpha-k+1}}{t^{k+2-r}} \int_{t}^{t+(1/w)} (u-t)^{k-\alpha} du\right)
= O\left(\frac{w^{\alpha-k}}{t^{k+1-r}} q^k \left(1 - \frac{\pi}{wt}\right)\right).
\]

As \( wt > \pi \),

This completes the proof of Lemma 13.

**Lemma 14.** For \( i = 0,1,2,\ldots,m \),
\[
\int_{t+(1/w)}^{\pi} u^{-i}(u-t)^{k-\alpha} S^{k,k+1-i} \left(w - \frac{\pi}{t},u\right) du = O\left(\frac{w^\alpha}{t^{k+1-r}}\right).
\]

**Proof.** By an application of the mean value theorem for some \( t + (1/w) < \xi < \pi \),
\[
\int_{t+(1/w)}^{\pi} u^{-i}(u-t)^{k-\alpha} S^{k,k+1-i} \left(w - \frac{\pi}{t},u\right) du
= w^{\alpha-k} \int_{t+(1/w)}^{\pi} u^{-i} S^{k,k+1-i} \left(w - \frac{\pi}{t},u\right) du
= w^{\alpha-k} \left[ u^{\alpha-k} S^{k,k-i} \left(w - \frac{\pi}{t},u\right) \right]_{u=t+(1/w)}^{\xi} + (r-i)w^{\alpha-k} \int_{t+(1/w)}^{\xi} u^{-i-1} S^{k,k-i} \left(w - \frac{\pi}{t},u\right) du
= w^{\alpha-k} O\left(\frac{w^k}{(t + \frac{1}{w})^{k+1-r}}\right) + w^{\alpha-k} \int_{t+(1/w)}^{\xi} u^{-i-1} O\left(\frac{w^k}{u^{k+2-i}}\right) du
\]
by Lemma 5(i)
\[
= O\left(\frac{w^{\alpha}}{t^{k+1-r}}\right) + O\left(\frac{w^\alpha}{t^{k+1-r}} \int_{t+(1/w)}^{\xi} \frac{1}{u^{k+2-i}} du\right)
= O\left(\frac{w^{\alpha}}{t^{k+1-r}}\right).
\]
Lemma 15. For the absolute \( N_{q\alpha}^{\infty} \)-summability of \( r \)-th derived conjugate series

\[
\int_{t+1/w}^{\pi} u^{r-i} (u-t)^{k-q} \, du \int_{t}^{w-(\pi/t)} g_i(x, w, u) \, dx
\]

\[
= O\left(\frac{u^{\alpha-k} \left(1 - \frac{\pi}{w}\right)}{t^{k+1-r}}\right).
\]

Proof. For some \( 1 < \eta < w - (\pi/t) \), by an application of the mean value theorem

\[
\int_{t+1/w}^{\pi} u^{r-i} (u-t)^{k-q} \, du \int_{t}^{w-(\pi/t)} g_i(x, w, u) \, dx
\]

\[
= \frac{1}{k!} \int_{t-1/w}^{\pi} u^{r-i} (u-t)^{k-q} \, du \left[ (-1)^k \left(\frac{d}{dx}\right)^k q\alpha\left(\frac{x}{w}\right) \right]_{x=w-(\pi/t)}
\]

\[\times \int_{\eta}^{w-(\pi/t)} \frac{d}{dx} s^{k+1-i}(x, u) \, dx\]

\[
= q^k \left(1 - \frac{\pi}{w}\right) \left\{ \int_{t+1/w}^{\pi} u^{r-i} (u-t)^{k-q} s^{k+1-i}(w - \frac{\pi}{t}, u) \, du
\]

\[\quad - \int_{t+1/w}^{\pi} u^{r-i} (u-t)^{k-q} s^{k+1-i}(\eta, u) \, du \right\}
\]

\[= O\left(\frac{q^k \left(1 - \frac{\pi}{w}\right) w^{\alpha}}{w^k t^{k+1-r}}\right)
\]

since by Lemma 14, the first integral is \( O(w^{\alpha}/(t^{k+1-r})) \) and the second integral is dominated by the first integral.

Lemma 16. For \( i = 0, 1, 2, \ldots, m \) and \( w > \pi \),

\[
\int_{w-(\pi/t)}^{w} g_i(x, w, u) \, dx = O\left( w^{2} u^{-k+i} Q_k\left(\frac{\pi}{w^t}\right) \right).
\]

Proof. For \( 0 \leq 1 \), by use of Lemma 5(i),

\[
\int_{w-(\pi/t)}^{w} g_i(x, w, u) \, dx
\]

\[= \int_{w-(\pi/t)}^{w} (-1)^k (k-1)! \left(\frac{d}{dx}\right)^k q\alpha\left(\frac{x}{w}\right) s^{k+1-i}(x, u) \, dx
\]

\[= \frac{1}{(k-1)!} \int_{w-(\pi/t)}^{w} (-1)^k \left(\frac{d}{dx}\right)^k q\alpha\left(\frac{x}{w}\right) O\left(\frac{x^{k+1-i}}{u^k}\right) \, dx
\]

\[= O\left(\frac{w^{2-i}}{u^k} \int_{1-(\pi/w)}^{1} q^k(\theta) \, d\theta\right)
\]

\[= O\left( w^{2} u^{-k+i} Q_k\left(\frac{\pi}{w^t}\right) \right) \text{ as } w > \pi.
\]
For \( i \geq 2 \), by use of Lemma 5(i)

\[
\int_{w-(\pi/i)}^{w} g_i(x, w, u) dx \\
= \int_{w-(\pi/i)}^{w} \left( \frac{-1}{(k-1)!} \left( \frac{d}{dx} \right)^k q_a \left( \frac{x}{w} \right) \right) S^{k-1, k+1-i}(x, u) dx \\
= \frac{1}{(k-1)!} \int_{w-(\pi/i)}^{w} (-1)^k \left( \frac{d}{dx} \right)^k q_a \left( \frac{x}{w} \right) O \left( \frac{x^{k-1}}{u^{k+2-i}} \right) dx \\
= O \left( u^{-k-2+i} \int_{1-(\pi/wt)}^{1} q^k(\theta) d\theta \right) \\
= O \left( w^{2} u^{-k+i} Q_k \left( \frac{\pi}{wt} \right) \right) \quad \text{as} \quad wu > \pi.
\]

Hence

\[
\int_{w-(\pi/i)}^{w} g_i(x, w, u) dx = O \left( w^{2} u^{-k+i} Q_k \left( \frac{\pi}{wt} \right) \right).
\]

**Lemma 17.** For \( i = 0, 1, 2, \ldots, m \) and \( wt > \pi \),

\[
\int_{t+\xi/(1/w)}^{\pi} u^{-i}(u-t)^{k-\alpha} \frac{d}{dx} S^{k, k+1-i}(x, u) du = O \left( \frac{w^{\alpha}}{t^{k-\tau}} \right).
\]

**Proof.** Let \( i = 0 \). By mean value theorem for some \( t + (1/w) < \xi < \pi \),

\[
\int_{t+\xi/(1/w)}^{\pi} u^{-i}(u-t)^{k-\alpha} \frac{d}{dx} S^{k, k+1-i}(x, u) du \\
= k \int_{t+\xi/(1/w)}^{\pi} u^r(u-t)^{k-\alpha} S^{k-1, k+1}(x, u) du \\
= kw^{\alpha-k} \int_{t+\xi/(1/w)}^{\pi} u^r S^{k-1, k+1}(x, u) du \\
= kw^{\alpha-k} \left[ u^r S^{k-1, k}(x, u) \right]_{u=t+\xi/(1/w)}^{\xi} \\
- kw^{\alpha-k} \int_{t+\xi/(1/w)}^{t} u^{-i} S^{k-1, k}(x, u) du \\
= O \left( \frac{w^{\alpha}}{t^{k-\tau}} \right).
\]
On the absolute $N_{q\alpha}$-summability of $r$th derived conjugate series

\[ k\omega_{\alpha}^{-k}O\left\{ \frac{\alpha^k}{(t+1/w)^{k-r}} \right\} \]

\[ -kr\omega_{\alpha}^{-k}[u^{-1}s^{k-1,k-1}(x,u)]_{u=t+(1/w)} \]

\[ + kr(r-1)\omega_{\alpha}^{-k}\int_{t+(1/w)}^{\xi} u^{r-2}s^{k-1,k-1}(x,u)\,du \quad \text{by Lemma 5(i)} \]

\[ = O\left(\frac{\omega_{\alpha}}{t-r} \right) + kr\omega_{\alpha}^{-k}O\left\{ \frac{\alpha^{-1}}{(t+1/w)^{k-r+1}} \right\} \]

\[ + kr(r-1)\omega_{\alpha}^{-k}\int_{t+(1/w)}^{\xi} u^{r-2}O\left(\frac{\alpha^{-1}}{u^k} \right)\,du \]

\[ = O\left(\frac{\omega_{\alpha}}{t-r} \right) + O\left(\frac{\omega_{\alpha}^{-1}}{t-r+1} \right) \]

\[ = O\left(\frac{\omega_{\alpha}}{t-r} \right) \quad \text{as} \quad wt > \pi. \]

For $i > 1$, using the technique used in the proof of Lemma 14, it can be proved that

\[ \int_{\frac{\pi}{t+1/w}}^{\pi} u^{r-i}(u-t)^{k-\alpha}G_i(w,u)\,du \]

\[ = O\left(\frac{\omega_{\alpha}^{-1}}{t-k-r+1} \right) \]

\[ = O\left(\frac{\omega_{\alpha}}{t-r} \right) \quad \text{as} \quad wt > \pi. \]

This completes the proof of Lemma 17.

**Lemma 18.** For $i = 0, 1, 2, \ldots, m$,

\[ \int_{\frac{\pi}{w^2}}^{\pi} dw \left| \int_{\frac{\pi}{w}}^{\pi} u^{r-i}(u-t)^{k-\alpha}G_i(w,u)\,du \right| = O\left(\frac{1}{t^{\alpha-r}} \right). \]

**Proof.**

\[ \int_{\frac{\pi}{w}}^{\pi} dw \left| \int_{\frac{\pi}{w}}^{\pi} u^{r-i}(u-t)^{k-\alpha}G_i(w,u)\,du \right| \]

\[ \leq \int_{\frac{\pi}{w^2}}^{\pi} dw \left| \int_{\frac{\pi}{w}}^{\pi} u^{r-i}(u-t)^{k-\alpha}G_i(w,u)\,du \right| \]

\[ + \int_{\frac{\pi}{w^2}}^{\pi} dw \left| \int_{\frac{\pi}{w}}^{\pi} u^{r-i}(u-t)^{k-\alpha}G_i(w,u)\,du \right| \]

\[ = \int_{\frac{\pi}{w^2}}^{\pi} dw O(w^{\alpha-r+1}), \quad \text{by Lemmas 11 and 12} \]

\[ = O\left(\frac{1}{t^{\alpha-r}} \right). \]
Lemma 19. For \( i = 0, 1, 2, \ldots, m \) and \( \omega r > \pi \),
\[
\int_{\pi}^{\pi} u^{-i}(u-t)^{k-\alpha} G_i(w, u) \, du
= O\left(\frac{w^{\alpha-k}q^k(1 - \frac{\pi}{\omega})}{t^{k-r+1}}\right) + O\left(\frac{w^{\alpha-k+1}Q_k\left(\frac{\pi}{\omega}\right)}{t^{k-r}}\right).
\]

Proof. Using Lemma 10,
\[
\int_{\pi}^{\pi} u^{-i}(u-t)^{k-\alpha} G_i(w, u) \, du
= \int_{\pi}^{\pi} u^{-i}(u-t)^{k-\alpha} \, du \int_{1}^{w} g_i(x, w, u) \, dx
= \int_{\pi}^{\pi} u^{-i}(u-t)^{k-\alpha} \, du \int_{1}^{w} g_i(x, w, u) \, dx
\]
Using Lemmas 13 and 16,
\[
J_1 = \int_{\pi}^{\pi} u^{-i}(u-t)^{k-\alpha} \, du \int_{1}^{w} g_i(x, w, u) \, dx
= O\left(\frac{w^{\alpha-k}q^k(1 - \frac{\pi}{\omega})}{t^{k-r+1}}\right)
+ O\left(\int_{\pi}^{\pi} u^{-k}(u-t)^{k-\alpha} w^2 Q_k\left(\frac{\pi}{\omega}\right) \, du\right)
= O\left(\frac{w^{\alpha-k}q^k(1 - \frac{\pi}{\omega})}{t^{k-r+1}}\right) + O\left(\frac{w^{\alpha-k+1}Q_k\left(\frac{\pi}{\omega}\right)}{t^{k-r}}\right) \text{ as } k \geq r
\]

and
\[
J_2 = \int_{\pi}^{\pi} u^{-i}(u-t)^{k-\alpha} \, du \int_{1}^{w} g_i(x, w, u) \, dx
= O\left(\frac{w^{\alpha-k}q^k(1 - \frac{\pi}{\omega})}{t^{k-r+1}}\right)
+ \int_{1}^{w} \int_{\pi/(\omega-\omega)} \, dx
\]
\[
\times \int_{\pi}^{\pi} u^{-i}(u-t)^{k-\alpha} g_i(x, w, u) \, dx \quad \text{by Lemma 15},
\]
Proof.

Lemma

This completes the proof of Lemma 19.

Proof of Theorem

4. Proof of the theorem

Proof of Theorem 1. We have for \( r \geq 1 \),

\[
\left( \frac{d}{dx} \right)^r B_u(x) = \frac{2}{\pi} \int_0^\pi \frac{(-1)^r}{2} \left\{ f(x+u) - (-1)^r f(x-w) \right\} \times \left( \frac{d}{du} \right)^r \sin nu \, du
\]
\[\begin{align*}
&= (-1)^r \frac{2}{\pi} \int_0^\pi h(u) u' \left( \frac{d}{du} \right)^r \sin nu \, du \\
&\quad + (-1)^r \frac{2}{\pi} \int_0^\pi \frac{1}{2} \{ P(u) - (-1)^r P(-u) \} \left( \frac{d}{du} \right)^r \sin nu \, du \\
&= \alpha_n + \beta_n, \quad \text{say.}
\end{align*}\]

For the proof of our theorem it is enough to show that

\[\sum \alpha_n \in |N_{q\alpha}|\]

and

\[\sum \beta_n \in |N_{q\alpha}|.\]

Now

\[\begin{align*}
n\alpha_n &= (-1)^r \frac{2}{\pi} \int_0^\pi nh(u) u' \left( \frac{d}{du} \right)^r \sin nu \, du \\
&= (-1)^{r+1} \frac{2}{\pi} \int_0^\pi h(u) u' \left( \frac{d}{du} \right)^{r+1} \cos nu \, du \\
&= (-1)^{r+1} \frac{2}{\pi} \sum_{j=1}^{k-r} (-1)^{j-1} H_j(u) \left( \frac{d}{du} \right)^{j-1} \\
&\quad \times \left\{ u' \left( \frac{d}{du} \right)^{r+1} \cos nu \right\}_u^\pi \\
&\quad + (-1)^{k+1} \frac{2}{\pi} \int_0^\pi H_{k-r}(u) \left( \frac{d}{du} \right)^{k-r} \left\{ u' \left( \frac{d}{du} \right)^{r+1} \cos nu \right\} \, du \\
&= J_1(n) + J_2(n), \quad \text{say. (4.1)}
\end{align*}\]

Since for \(j = 1, 2, \ldots, k - r, H_j(+0) = O\) it is clear that \(J_1(n)\) is the sum of the terms containing \((-1)^n n^p\), where \(p\) is even and \(r + 1 \leq p \leq k\).

By the use of Lemma 9, for \(p = 1, 2, \ldots, k\),

\[\int_1^\infty \frac{dw}{w^2} \left| \sum_{n \geq w} n^p (-1)^n q\alpha \left( \frac{n}{w} \right) \right| < \infty.\]

Hence

\[\int_1^\infty \frac{dw}{w^2} \left| \sum_{n \geq w} J_1(n) q\alpha \left( \frac{n}{w} \right) \right| < \infty.\]
Now

\[ J_2(n) = (-1)^{k+1} \frac{2}{\pi} \int_0^\pi H_{k-r}(u) \left( \frac{d}{du} \right)^{k-r} \left\{ u' \left( \frac{d}{du} \right)^{r+1} \cos nu \right\} du \]

\[ = \frac{2}{\pi \Gamma(k - \alpha + 1)} \int_0^\pi dH_\beta(t) \int_0^\pi (u-t)^{k-\alpha} \left( \frac{d}{du} \right)^{k-r} \left\{ u' \left( \frac{d}{du} \right)^{r+1} \cos nu \right\} du \]

\[ \times \int_0^u (u-t)^{k-\alpha} dH_\beta(t) \] by Lemma 1 as \( \beta = \alpha - r \) and \( |\alpha| = k \)

\[ = \frac{2(-1)^{k+1}}{\pi \Gamma(k - \alpha + 1)} \int_0^\pi dH_\beta(t) \int_0^\pi (u-t)^{k-\alpha} \left( \frac{d}{du} \right)^{k-r} \left\{ u' \left( \frac{d}{du} \right)^{r+1} \cos nu \right\} du \]

\[ \times \left\{ \sum_{i=0}^m \left( \begin{array}{c} k-r \\ i \end{array} \right) \left( \frac{d}{du} \right)^i u' \left( \frac{d}{du} \right)^{k+1-i} \cos nu \right\} du \]

where \( m = \min(k-r,r) \)

\[ = \frac{2(-1)^{k+1}}{\pi \Gamma(k - \alpha + 1)} \sum_{i=0}^m \left( \begin{array}{c} k-r \\ i \end{array} \right) \frac{r!}{(r-i)!} \int_0^\pi dH_\beta(t) \]

\[ \times \int_0^\pi (u-t)^{k-\alpha} u'^{-i} \left( \frac{d}{du} \right)^{k+1-i} \cos nu du. \]

By the use of Lemmas 20 and 18,

\[ \int_1^\infty \frac{dw}{w^2} \left| \sum_{n \leq w} J_2(n) q_n \left( \frac{n}{w} \right) \right| \]

\[ \leq \frac{2}{\pi \Gamma(k - \alpha + 1)} \sum_{i=0}^m \left( \begin{array}{c} k-r \\ i \end{array} \right) \frac{r!}{(r-i)!} \int_0^\pi \left| dH_\beta(t) \right| \int_1^\infty \frac{dw}{w^2} \]

\[ \times \left| \int_0^\pi u'^{-i} (u-t)^{k-\alpha} G_i(w,u) du \right| \]

\[ = \frac{2}{\pi \Gamma(k - \alpha + 1)} \sum_{i=0}^m \left( \begin{array}{c} k-r \\ i \end{array} \right) \frac{r!}{(r-i)!} \int_0^\pi \left| dH_\beta(t) \right| \]

\[ \times \left\{ \int_1^{\pi/2} \frac{dw}{w^2} \left| \int_0^\pi u'^{-i} (u-t)^{k-\alpha} G_i(w,u) du \right| \right. \]

\[ \left. + \int_{\pi/2}^\infty \frac{dw}{w^2} \left| \int_0^\pi u'^{-i} (u-t)^{k-\alpha} G_i(w,u) du \right| \right\} \]
From (4.1), (4.2) and (4.3) it is clear that

\[ \sum \alpha_n \in |d_{n0}|. \]

Let \( r \) be an odd number, i.e. \( r = 2p + 1 \), where \( p = 0, 1, 2, \ldots \). Then

\[
\beta_n = -\frac{2}{\pi} \int_0^\pi \frac{1}{2} \{ P(u) + P(-u) \} \left( \frac{d}{du} \right)^{2p+1} \sin nu du
\]

\[ = (-1)^{p+1} \frac{2}{\pi} n^{2p+1} \int_0^\pi \left( \sum_{j=0}^p \frac{\theta_{2j} \mu_j}{(2j)!} \right) \cos nu du \]

\[ = (-1)^{p+1} \frac{2}{\pi} n^{2p+1} \int_0^\pi \left( \sum_{j=0}^p \frac{\theta_{2j} \mu_j}{(2j)!} \right) \cos nu du \]

\[ = (-1)^{p+1} \frac{2}{\pi} n^{2p+1} \sum_{j=0}^p \frac{\theta_{2j} \mu_j}{(2j)!} \left( -1 \right)^n \]

\[ \times \left( \sum_{j=1}^p \left( -1 \right)^{\mu+1} n^{-2\mu} \pi^{2j-2\mu+1} \frac{(2j)!}{(2j-2\mu)!} \right) \]

\[ = 2(-1)^n \sum_{\mu=1}^p \left( -1 \right)^{p+\mu} n^{2p-2\mu+1} \sum_{j=\mu}^p \frac{\theta_{2j-1}}{(2j-2\mu)!} \pi^{2j-2\mu}. \]

Let \( r \) be an even number, i.e. \( r = 2p \), where \( p = 1, 2, \ldots \). Then

\[
\beta_n = \frac{2}{\pi} \int_0^\pi \frac{1}{2} \{ P(u) - P(-u) \} \left( \frac{d}{du} \right)^{2p} \sin nu du
\]

\[ = (-1)^p \frac{2}{\pi} n^{2p} \sum_{j=1}^p \frac{\theta_{2j-1}}{(2j-1)!} \int_0^\pi u^{2j-1} \sin nu du \]

\[ = (-1)^p \frac{2}{\pi} n^{2p} \sum_{j=1}^p \frac{\theta_{2j-1}}{(2j-1)!} (-1)^n \]

\[ \times \left( \sum_{\mu=1}^j \left( -1 \right)^{\mu-1} n^{-2\mu+1} \pi^{2j-2\mu+1} \frac{(2j-1)!}{(2j-2\mu)!} \right) \]

\[ = 2(-1)^n \sum_{\mu=1}^p \left( -1 \right)^{p+\mu-1} n^{2p-2\mu+1} \sum_{j=\mu}^p \frac{\theta_{2j-1}}{(2j-2\mu)!} \pi^{2j-2\mu}. \]
So by the use of Lemma 9,
\[
\int_1^\infty \frac{dw}{w^2} \left| \sum_{n \leq w} n \beta_n q^\alpha \left( \frac{n}{w} \right) \right| < \infty,
\]
i.e. \( \sum \beta_n \in |N_{q\alpha}| \). This terminates the proof of Theorem 1.

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