SPECTRAL STRUCTURE OF TRANSFER OPERATORS FOR EXPANDING CIRCLE MAPS

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ABSTRACT. We explicitly determine the spectrum of transfer operators (acting on spaces of holomorphic functions) associated to analytic expanding circle maps arising from finite Blaschke products. This is achieved by deriving a convenient natural representation of the respective adjoint operators.

1. INTRODUCTION

One of the major strands of modern ergodic theory is to exploit the rich links between dynamical systems theory and functional analysis, making the powerful tools of the latter available for the benefit of understanding complex dynamical behaviour. In classical ergodic theory, composition operators occur naturally as basic objects for formulating concepts such as ergodicity or mixing [24]. These operators, known in this context as Koopman operators, are the formal adjoints of transfer operators, the spectral data of which provide insight into fine statistical properties of the underlying dynamical systems, such as rates of mixing (see, for example, [5]).

In the literature, the term ‘composition operator’ is mostly used to refer to compositions with analytic functions mapping a disk into itself, a setting in which operator-theoretic properties such as boundedness, compactness, and most importantly explicit spectral information are well-established (good references are [20] or the encyclopedia on the subject [7]). The purpose of the present article is to demonstrate that in a particular analytic setting, the spectra of transfer operators can be deduced directly from certain composition operators.

Let $\tau$ be a real analytic expanding map on the circle and $\{\phi_k: k = 1, \ldots, K\}$ the set of local inverse branches of $\tau$. Then the associated transfer operator $L$ (also referred to as Ruelle-Perron-Frobenius or simply Ruelle operator), defined by

$$(Lf)(z) = \sum_{k=1}^{K} \phi_k'(z)(f \circ \phi_k)(z),$$

preserves and acts compactly on certain spaces of holomorphic functions [17, 18, 21].

The spectrum of $L$ can be understood by passing to its (Banach space) adjoint operator $L^*$. This strategy has been explored in the context of Ruelle operators acting on the space of functions locally analytic on the Julia set of a rational function $R$, see [1, 10, 11, 23]; in particular, explicit expressions for Fredholm determinants of certain Ruelle operators have been derived. In our setting of analytic expanding circle maps, we adopt a similar approach, that is, we analyse the spectrum of $L$ by deriving a natural explicit representation of $L^*$ (Proposition 4.5). As a by-product we obtain a more conceptual proof of results in [21], where the spectrum of $L$ for
a certain family of analytic circle maps was determined using a block-triangular
matrix representation.

In the spirit of approaching problems in the world of real numbers by making
recourse to complex numbers, we consider finite Blaschke products, a class of ra-
tional maps on the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ preserving the unit circle. One
of the striking features of Blaschke products is that they partition the Riemann
sphere into simple dynamically invariant regions: the unit circle, the unit disk and
the exterior disk in $\hat{\mathbb{C}}$. As a consequence, the spectrum of $L^*$ can be determined by
studying the spectrum of composition operators on spaces of holomorphic functions
on these dynamically invariant regions.

Our main result (Theorem 5.4) can be summarized as follows. Let $B$ be a finite
Blaschke product such that its restriction $\tau$ to the unit circle $T = \{z : |z| = 1\}$ is expanding. Denote by $H^2(A)$ the Hardy-Hilbert space of functions which are
holomorphic on some suitable annulus $A$ (containing $T$) and square integrable on
its boundary $\partial A$ (see Definition 2.1). Then the transfer operator $L$ associated to $\tau$
is compact on $H^2(A)$, with spectrum

$$\sigma(L) = \{\lambda(z_0)^n : n \in \mathbb{N}_0\} \cup \{\overline{\lambda(z_0)}^n : n \in \mathbb{N}\} \cup \{0\},$$

where $\lambda(z_0)$ is the multiplier of the unique attracting fixed point $z_0$ of $B$ in the
unit disk. This implies that for finite Blaschke products which give rise to analytic
expanding circle maps, the derivative of the fixed point in the unit disk completely
determines the spectrum of $L$.

The paper is organized as follows. In Section 2, we review basic definitions and
facts about Hardy-Hilbert spaces on annuli. The following Section 3 is devoted
to analytic expanding circle maps and their corresponding transfer operators. In
Section 4, we explicitly derive the structure of the corresponding adjoint operators
after having established a suitable representation of the dual space. This structure
is then used in Section 5 in order to obtain the spectrum of transfer operators
associated to analytic expanding circle maps arising from finite Blaschke products,
thus proving our main result.

2. HARDY-HILBERT SPACES

Throughout this article $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ denotes the one point compactification of
$\mathbb{C}$. For $U \subset \hat{\mathbb{C}}$, we write $\text{cl}(U)$ to denote the closure of $U$ in $\hat{\mathbb{C}}$. For $r > 0$ we use $T_r = \{z \in \mathbb{C} : |z| = r\}$, $T = T_1$ to denote circles centred at 0, and

$$D_r = \{z \in \mathbb{C} : |z| < r\},$$

$$D_r^\infty = \{z \in \hat{\mathbb{C}} : |z| > r\},$$

$D = D_1$ to denote disks centred at 0 and $\infty$. Given $r < 1 < R$ the symbol

$$A_{r,R} = \{z \in \mathbb{C} : r < |z| < R\}$$

will denote an open annulus containing the unit circle $T$. 
We write $L^p(\mathbb{T}_r) = L^p(\mathbb{T}_r, d\theta/2\pi)$ with $1 \leq p < \infty$ for the Banach space of $p$-integrable functions with respect to normalized one-dimensional Lebesgue measure on $\mathbb{T}_r$. Finally, for $U$ an open subset of $\hat{\mathbb{C}}$ we use $\text{Hol}(U)$ for the space of holomorphic functions on $U$.

Hardy-Hilbert spaces on disks and annuli will provide a convenient setting for our analysis. We briefly recall their properties in the following.

**Definition 2.1** (Hardy-Hilbert spaces). For $\rho > 0$ and $f : \mathbb{T}_\rho \to \mathbb{C}$ write

$$M_\rho(f) = \int_0^{2\pi} |f(\rho e^{i\theta})|^2 \frac{d\theta}{2\pi}.$$  

Then the Hardy-Hilbert spaces on $D_r$ and $A_{\rho,R}$ are given by

$$H^2(D_r) = \left\{ f \in \text{Hol}(D_r) : \sup_{\rho < r} M_\rho(f) < \infty \right\},$$

and

$$H^2(A_{\rho,R}) = \left\{ f \in \text{Hol}(A_{\rho,R}) : \sup_{\rho < R} M_\rho(f) + \sup_{\rho > \rho} M_\rho(f) < \infty \right\}.$$

The Hardy-Hilbert space on the exterior disk $D_R^\infty$ is defined accordingly, that is $f \in H^2(D_R^\infty)$ if $f \in \text{Hol}(D_R^\infty)$ (or, equivalently, $f \circ \varsigma$ holomorphic on $D_{1/R}$ with $\varsigma(z) = 1/z$) and $\sup_{\rho > R} M_\rho(f) < \infty$. Finally, $H^2_0(D_{1/R}^\infty) \subset H^2(D_R^\infty)$ denotes the subspace of functions vanishing at infinity.

A comprehensive account of Hardy spaces over general domains is given in the classic text [8]. A crisp treatment of Hardy spaces on the unit disk can be found in [16, Chapter 17]), while a good reference for Hardy spaces on annuli is [19]. We shall now collect a number of results which will be useful in what follows.

Any function in $H^2(U)$, where $U$ is a disk or an annulus, can be extended to the boundary in the following sense. For $f \in H^2(D_r)$ there is an $f^* \in L^2(\mathbb{T}_r)$ such that

$$\lim_{\rho \to r} f(\rho e^{i\theta}) = f^*(re^{i\theta}) \quad \text{for a.e.} \ \theta,$$

and analogously for $f \in H^2(D_R^\infty)$. Similarly, for $f \in H^2(A_{\rho,R})$ there are $f^*_1 \in L^2(\mathbb{T}_r)$ and $f^*_2 \in L^2(\mathbb{T}_R)$, with $\lim_{\rho \to r} f(\rho e^{i\theta}) = f^*_1(re^{i\theta})$ and $\lim_{\rho \to R} f(\rho e^{i\theta}) = f^*_2(Re^{i\theta})$ for a.e. $\theta$. It turns out that the spaces $H^2(U)$ are Hilbert spaces with inner products

$$(f,g)_{H^2(D_r)} = \int_0^{2\pi} f^*(re^{i\theta}) g^*(re^{i\theta}) \frac{d\theta}{2\pi},$$

and

$$(f,g)_{H^2(A_{\rho,R})} = \int_0^{2\pi} f^*_1(Re^{i\theta}) g^*_1(Re^{i\theta}) \frac{d\theta}{2\pi} + \int_0^{2\pi} f^*_2(Re^{i\theta}) g^*_2(Re^{i\theta}) \frac{d\theta}{2\pi}.$$  

Similarly, for $H^2(D_{1/R}^\infty)$.

**Notation 2.2.** In order to avoid cumbersome notation, we shall write $f(z)$ instead of $f^*(z)$ for $z$ on the boundary of the domain.
Remark 2.3. It is not difficult to see that \( E = \{ e_n : n \in \mathbb{Z} \} \) where
\[
e_n(z) = \frac{z^n}{d_n}, \quad \text{with} \quad d_n = \sqrt{r^{2n} + R^{2n}} \tag{2.1}
\]
is an orthonormal basis for \( H^2(A_r,R) \). In particular, it follows that \( f \in \text{Hol}(A_r,R) \) is in \( H^2(A_r,R) \) if and only if \( f(z) = \sum_{n=-\infty}^{\infty} c_n e_n(z) \) with 
\[
\sum_{n=-\infty}^{\infty} |c_n|^2 < \infty,
\]
where the coefficients are given by \( c_n = c_n(f) = (f, e_n)_{H^2(A_r,R)} \). Note also that 
\[
\|f\|_{H^2(A_r,R)}^2 = \sum_{n=-\infty}^{\infty} |c_n|^2.
\]

3. Circle maps and transfer operators

The purpose of this section is to establish compactness of transfer operators on Hardy-Hilbert spaces for analytic expanding circle maps, defined as follows.

Definition 3.1. We say \( \tau : T \to T \) is an analytic expanding circle map if
(i) \( \tau \) is analytic on \( T \);
(ii) \( |\tau'(z)| > 1 \) for all \( z \in T \).

In particular, \( \tau \) is a \( K \)-fold covering for some \( K > 1 \). With a slight abuse of notation we continue to write \( \tau \) for its holomorphic extension to an open annulus \( A_r,R \) for \( r < 1 < R \) and let 
\[
A = \{ A_{r,R} : \tau \text{ and } 1/\tau \text{ holomorphic on } A_{r,R} \}.
\]

Before proceeding to the definition of transfer operator associated to \( \tau \) we require some more notation. Given two open subsets \( U \) and \( V \) of \( \mathbb{C} \) we write
\[
U \subset \subset V
\]
if \( \text{cl}(U) \) is a compact subset of \( V \).

By the expansivity of \( \tau \) and [21, Lemma 2.2], we can choose \( A_0, A' \) and \( A \) in \( A \) with 
\[
A_0 \subset A' \subset A \text{ and } \tau(\partial A_0) \cap \text{cl}(A) = \emptyset. \tag{3.1}
\]

Given an analytic expanding circle map \( \tau \), we associate with it a transfer operator \( L \) by setting 
\[
(Lf)(z) = \sum_{k=1}^{K} \phi_k^i(z)(f \circ \phi_k)(z), \tag{3.2}
\]
where \( \phi_k \) denotes the \( k \)-th local inverse of \( \tau \).

As we shall see presently, the above choices of the annuli guarantee that \( L \) is a well-defined linear operator which maps \( H^2(A) \) compactly to itself. In order to show this, we shall employ a factorization argument, similar to the ones used in [4, 21]. Let \( H^\infty(A') \) be the Banach space of bounded holomorphic functions on \( A' \) equipped with the supremum norm. We can write \( L = \hat{\varphi}_L \hat{J} \), where \( \varphi_L : H^\infty(A') \to H^2(A) \) is a lifted transfer operator given by the same functional expression (3.2) and \( J : H^2(A) \to H^\infty(A') \) is the canonical embedding
\[
\begin{align*}
H^\infty(A') \xrightarrow{J} H^2(A) \xrightarrow{\varphi_L} \varphi_L \xrightarrow{\hat{J}} H^2(A)
\end{align*}
\]
Lemma 3.3. Let \( \mathcal{J} \) be a finite rank operator on \( H^\infty(A') \). For \( f \in H^\infty(A') \), let the transfer operator \( \mathcal{J} \) be continuous by [21, Lemma 2.5], and let \( \mathcal{J} : H^\infty(A') \to H^2(A) \) be the canonical embedding.

Proof. We can factorize \( \mathcal{L} \) as \( \mathcal{L} = \mathcal{J} \tilde{\mathcal{L}} \), where \( \tilde{\mathcal{L}} : H^\infty(A') \to H^\infty(A) \), given by the functional expression (3.2), is continuous by [21, Lemma 2.5], and \( \tilde{\mathcal{L}} : H^\infty(A') \to H^2(A) \) is the canonical embedding.

Next, we establish compactness of \( \mathcal{J} : H^2(A) \hookrightarrow H^\infty(A') \) given by

\[
(\mathcal{J} f)(z) = f(z) \quad \text{for } z \in A'.
\]

Proof. We can factorize \( \mathcal{L} \) as \( \mathcal{L} = \mathcal{J} \tilde{\mathcal{L}} \), where \( \tilde{\mathcal{L}} : H^\infty(A') \to H^\infty(A) \), given by the functional expression (3.2), is continuous by [21, Lemma 2.5], and \( \tilde{\mathcal{L}} : H^\infty(A') \to H^2(A) \) is the canonical embedding.

Lemma 3.2. The transfer operator \( \mathcal{L} \) given by (3.2) maps \( H^\infty(A') \) continuously to \( H^2(A) \).

Proof. We can factorize \( \mathcal{L} \) as \( \mathcal{L} = \mathcal{J} \tilde{\mathcal{L}} \), where \( \tilde{\mathcal{L}} : H^\infty(A') \to H^\infty(A) \), given by the functional expression (3.2), is continuous by [21, Lemma 2.5], and \( \tilde{\mathcal{L}} : H^\infty(A') \to H^2(A) \) is the canonical embedding.

Lemma 3.3. Let \( \mathcal{J} \) and \( \mathcal{J}_N \) be as above. Then

\[
\lim_{N \to \infty} \| \mathcal{J} - \mathcal{J}_N \|_{H^2(A) \to H^\infty(A')} = 0.
\]

In particular, the embedding \( \mathcal{J} \) is compact.

Proof. For \( z \in A' \), it follows by the Cauchy-Schwarz inequality that

\[
| (\mathcal{J} f)(z) - (\mathcal{J}_N f)(z) | \leq \left( \sum_{|n| \geq N} |c_n(f)|^2 \right)^{1/2} \left( \sum_{|n| \geq N} |c_n(z)|^2 \right)^{1/2}
\]

\[
\leq \| f \|_{H^2(A)} \left( \sum_{|n| \geq N} \frac{|z^n|^2}{r^{2n} + R^{2n}} \right)^{1/2}
\]

\[
\leq \| f \|_{H^2(A)} \left( \sum_{n \geq N} \frac{z^n}{R} \right)^{2n} + \sum_{n \geq N} \left( \frac{r^n}{R^n} \right)^2 \right)^{1/2}.
\]

Thus,

\[
\| \mathcal{J} f - \mathcal{J}_N f \|_{H^\infty(A')} \leq \| f \|_{H^2(A)} \left( \left( \frac{R}{R'} \right)^{2N} \frac{1}{1 - (\frac{R}{R'})^2} + \left( \frac{r}{r'} \right)^{2N} \frac{1}{1 - (\frac{r}{r'})^2} \right)^{1/2},
\]

and the assertion follows.

The factorization \( \mathcal{L} = \mathcal{L}_N \) together with Lemmas 3.2 and 3.3 now imply the following result.

Proposition 3.4. The transfer operator \( \mathcal{L} : H^2(A) \to H^2(A) \) in (3.2) is compact.
Remark 3.5. Closer inspection of Lemma 3.3 reveals that the singular values of $J$ decay at an exponential rate. Thus $J$ and hence $L$ are trace-class. In fact, using results from [2] it is possible to show that both the singular values and the eigenvalues of $L$ decay at an exponential rate, a property that $L$ shares with other transfer operators arising from analytic maps (see, for example, [9, 3]).

4. Adjoint Operator

A central step in showing our main result is to find an appropriate representation of the dual space on which the adjoint of the transfer operator has a simple structure.

For the remainder of this section we set $A = A_{r,R}$ and denote by $H^2(A)^*$ the strong dual of $H^2(A)$, that is, the space of continuous linear functionals on $H^2(A)$ equipped with the topology of uniform convergence on the unit ball. We will show that $H^2(A)^*$ is isomorphic to the topological direct sum $H^2(D_r) \oplus H^2_0(D_R^\infty)$, equipped with the norm $\| (h_1, h_2) \|^2 = \| h_1 \|^2_{H^2(D_r)} + \| h_2 \|^2_{H^2_0(D_R^\infty)}$. Similar representations of the duals of Hardy spaces for multiply connected regions can be found in [15, Proposition 3]. The present set-up is sufficiently simple to allow for a short proof of the representation.

Proposition 4.1. The dual space $H^2(A)^*$ is isomorphic to $H^2(D_r) \oplus H^2_0(D_R^\infty)$ with the isomorphism given by

$$J : H^2(D_r) \oplus H^2_0(D_R^\infty) \rightarrow H^2(A)^*$$

$$(h_1, h_2) \mapsto l,$$

where

$$l(f) = \frac{1}{2\pi i} \int_{T_r} f(z) h_1(z) \, dz + \frac{1}{2\pi i} \int_{T_R} f(z) h_2(z) \, dz \quad (f \in H^2(A)).$$

Proof. We will first show that (4.1) defines a continuous functional $l \in H^2(A)^*$ and that $J$ is a bounded linear operator. In order to see this note that for any $(h_1, h_2) \in H^2(D_r) \oplus H^2_0(D_R^\infty)$ the linear functional $l = J(h_1, h_2)$ is bounded, since for any $f \in H^2(A)$ with $\| f \|_{H^2(A)} \leq 1$

$$|l(f)| \leq \left( r \| h_1 \|_{H^2(D_r)} + R \| h_2 \|_{H^2_0(D_R^\infty)} \right).$$

It follows that

$$\| J(h_1, h_2) \|_{H^2(A)^*} \leq \sqrt{r^2 + R^2} \sqrt{\| h_1 \|^2_{H^2(D_r)} + \| h_2 \|^2_{H^2_0(D_R^\infty)}}$$

and $\| J \|_{H^2(D_r) \oplus H^2_0(D_R^\infty) \rightarrow H^2(A)^*} \leq \sqrt{r^2 + R^2}$. Hence, $J$ is well defined and bounded.

For injectivity, we suppose that $l = J(h_1, h_2) = 0$ and show that $h_1 = 0$ and $h_2 = 0$. In order to see this note that any $(h_1, h_2) \in H^2(D_r) \oplus H^2_0(D_R^\infty)$ can be written $h_1(z) = \sum_{n=0}^{\infty} a_n z^n$ and $h_2(z) = \sum_{n=1}^{\infty} a_n z^{-n}$ with suitable coefficients $a_n \in \mathbb{C}$. Now let $\mathcal{E} = \{ e_n : n \in \mathbb{Z} \}$ with $e_n(z) = \frac{z^n}{d_n}$ denote the orthonormal basis of $H^2(A)$ given in Remark 2.3. A short calculation using Lebesgue dominated convergence shows that

$$0 = (J(h_1, h_2))(e_n) = \frac{a_{-n-1}}{d_n} \quad \text{for all } n \in \mathbb{Z},$$

(4.2)
which implies \( h_1 = 0 \) and \( h_2 = 0 \). Thus \( J \) is injective.

Finally, in order to show that \( J \) is surjective, fix \( l \in H^2(A)^* \). We will construct \((h_1, h_2) \in H^2(D_r) \oplus H^2_0(\mathbb{R}^\infty)\) such that \( J(h_1, h_2) = l \).

By the Riesz representation theorem there is a unique \( g \in H^2(A) \) such that \( l(f) = (f, g)_{H^2(A)} \) for all \( f \in H^2(A) \). Moreover, \( g \) can be uniquely expressed as 
\[
g = \sum_{n \in \mathbb{Z}} c_n g_n.
\]

Now define
\[
h_1(z) = \sum_{n=0}^{\infty} c_{-n-1}(g) d_{-n-1} z^n \quad \text{for } z \in D_r,
\]
\[
h_2(z) = \sum_{n=1}^{\infty} c_{n-1}(g) d_{n-1} z^{-n} \quad \text{for } z \in D^\infty_R.
\]

Using \( \|g\|_{H^2(A)}^2 = \sum_{n \in \mathbb{Z}} |c_n(g)|^2 < \infty \), it follows that \( h_1 \in H^2(D_r) \) and \( h_2 \in H^2_0(\mathbb{R}^\infty) \). Combining (4.2) and (4.3) we obtain
\[
(J(h_1, h_2))(c_n) = \frac{a_{-n-1}}{d_n} c_{n-1}(g) d_{-n} = \frac{c_n(g) d_n}{d_n} = c_n(g) = (c_n, g)_{H^2(A)}
\]
for every \( n \in \mathbb{Z} \). Since the above equality also holds for all finite linear combinations of elements in \( E \) the continuity of \( J \) implies 
\[
(J(h_1, h_2))(f) = (f, g)_{H^2(A)} = l(f)
\]
for all \( f \in H^2(A) \). Thus \( J \) is surjective. \( \square \)

**Remark 4.2.** The inverse \( J^{-1} \) of \( J \) can be obtained using the kernel \( K_z \in H^2(A) \) defined by \( K_z(w) = 1/(z - w) \) for \( z \in \hat{C} \setminus \text{cl}(A) \). More precisely, \( J^{-1} \) is given by \( l \mapsto (h_1, h_2) \), where \( h_1(z) = l(-K_z) \) for \( z \in D_r \) and \( h_2(z) = l(K_z) \) for \( z \in D^\infty_R \).

Returning to the setting of Section 3, let \( \tau \) be an analytic expanding circle map and \( A = A_{r, R} \subset \mathcal{A} \) an annulus satisfying (3.1) such that the associated transfer operator \( \mathcal{L}: H^2(A) \to H^2(A) \) is well defined and compact. Using the representation of the dual space \( H^2(A)^* \) obtained in the previous lemma, we shall shortly derive an explicit form for the adjoint operator of \( \mathcal{L} \).

Before doing so we require some more notation. Define \( C(r): H^2(D_r) \to L^2(\mathbb{T}_r) \) by
\[
(C(r)h)(z) = h(\tau(z)) \quad \text{for } z \in \mathbb{T}_r,
\]
and \( C(R): H^2_0(\mathbb{R}^\infty) \to L^2(\mathbb{T}_R) \) by
\[
(C(R)h)(z) = h(\tau(z)) \quad \text{for } z \in \mathbb{T}_R.
\]

It turns out that \( C(r) \) and \( C(R) \) are compact, the proof of which relies on the following fact.

**Lemma 4.3.** Let \( K \) be a compact subset of a disk \( D \) in \( \mathbb{C} \). Then there exists a constant \( c_K \) depending on \( K \) only such that for any \( f \in H^2(D) \)
\[
\sup_{z \in K} |f(z)| \leq c_K \|f\|_{H^2(D)}.
\]

**Proof.** This follows, for example, from [4, Lem. 2.9], or by a calculation using the Cauchy-Schwarz inequality similar to the proof of Lemma 3.3. \( \square \)

We now have the following.

**Lemma 4.4.** The operators \( C(r) \) and \( C(R) \) are compact.
Proof. The choice of $A = A_{r,R}$ in (3.1) implies that $r_0 = \sup_{z \in \mathbb{T}}, |\tau(z)| < r$, and we can choose a disk $D_{r'}$ with $D_{r_0} \subseteq D_{r'} \subseteq D_r$.

Let $\tilde{C}^{(r)} : H^2(D_{r'}) \to L^2(\mathbb{T}_r)$ be defined by the functional expression as in (4.4), but now considered on $H^2(D_{r'})$. The operator is continuous since

$$\left\|\tilde{C}^{(r)} h\right\|_{L^2(\mathbb{T}_r)} \leq \sup_{z \in \mathbb{T}_r} |h(z)| \leq \sup_{z \in \text{cl}(D_{r_0})} |h(z)| \leq c_K \|h\|_{H^2(D_{r'})},$$

where we have used Lemma 4.3 with $K = \text{cl}(D_{r_0})$. The lemma follows since we can write $C^{(r)} = \tilde{C}^{(r)} \tilde{J}$ with $\tilde{J} : H^2(D_{r}) \to H^2(D_{r'})$ denoting the canonical embedding, which is compact (see, for example, [4, Lemma 2.9]). The argument for $C^{(R)}$ is similar. \hfill $\square$

Next, we need to define certain projection operators on $L^2(\mathbb{T}_\rho)$. For any $g \in L^2(\mathbb{T}_\rho)$ we can write $g(z) = \sum_{n \in \mathbb{Z}} g_n z^n$, so that $g = g_+ + g_-$ with $g_+(z) = \sum_{n=0}^{\infty} g_n z^n$ and $g_-(z) = \sum_{n=1}^{\infty} g_n z^{-n}$. Since $\|g\|_{L^2(\mathbb{T}_\rho)}^2 = \sum_{n=\infty}^{\infty} |a_n|^2 < \infty$, the functions $g_+$ and $g_-$ can be viewed as functions in $H^2(D_\rho)$ and $H^2(D_\rho)$, respectively. Then we define the bounded projection operators $\Pi_+^{(\rho)} : L^2(\mathbb{T}_\rho) \to H^2(D_\rho)$ and $\Pi_-^{(\rho)} : L^2(\mathbb{T}_\rho) \to H^2(D_\rho)$ by

$$\Pi_+^{(\rho)}(g) = g_+ \quad \text{and} \quad \Pi_-^{(\rho)}(g) = g_-.$$

(4.6)

Finally, let $\mathcal{L}^* : H^2(A)^* \to H^2(A)^*$ denote the adjoint operator of $\mathcal{L}$ in the Banach space sense, that is, $(\mathcal{L}^* l)(f) = l(\mathcal{L} f)$ for all $l \in H^2(A)^*$ and $f \in H^2(A)$. The following proposition provides an explicit representation $\mathcal{L}'$ of $\mathcal{L}^*$ via

$$\mathcal{L}' = J^{-1} \mathcal{L}^* J,$$

as an operator on $H^2(D_r) \oplus H^2(D_R)$ given by compositions of $C^{(\rho)}$, $\Pi_-^{(\rho)}$ and $\Pi_+^{(\rho)}$ for $\rho = r, R$.

**Proposition 4.5.** Let $\mathcal{L} : H^2(A) \to H^2(A)$ be the transfer operator associated to an analytic expanding circle map $\tau$, with $A \in \mathcal{A}$ as in (3.1). Then the isomorphism $J$ conjugates the adjoint $\mathcal{L}^*$ of $\mathcal{L}$ to

$$\mathcal{L}' : H^2(D_r) \oplus H^2(D_R) \to H^2(D_r) \oplus H^2(D_R)$$

given by

$$\mathcal{L}' = \begin{pmatrix} \Pi_+^{(\rho)} C^{(r)} & \Pi_-^{(\rho)} C^{(r)} \\ \Pi_+^{(R)} C^{(r)} & \Pi_-^{(R)} C^{(r)} \end{pmatrix},$$

(4.7)

that is $\mathcal{L}' = J^{-1} \mathcal{L}^* J$.

Proof. We want to show that $\mathcal{L}^* J = J \mathcal{L}'$, that is,

$$(\mathcal{L}^* J(h_1, h_2))(f) = (J \mathcal{L}'(h_1, h_2))(f)$$

(4.8)

for all $(h_1, h_2) \in H^2(D_r) \oplus H^2(D_R)$ and $f \in H^2(A)$. For any such $(h_1, h_2)$ and $f$, the adjoint property yields

$$(\mathcal{L}^* J(h_1, h_2))(f) = (J(h_1, h_2))(\mathcal{L} f)$$

$$= \frac{1}{2\pi i} \int_{\mathbb{T}_r} (\mathcal{L} f)(z) h_1(z) dz + \frac{1}{2\pi i} \int_{\mathbb{T}_R} (\mathcal{L} f)(z) h_2(z) dz.$$
Now let a basis for $H^2(D_r) \oplus H^2_0(D^\infty_R)$ be given by $\mathcal{P} = \{(p_n, 0) : n \in \mathbb{N}_0\} \cup \{(0, \cdot (-n) : n \in \mathbb{N}\}$ with $p_n(z) = z^n$, where $p_n \in H^2(D_r)$ if $n \geq 0$ and $p_n \in H^2_0(D^\infty_R)$ if $n < 0$.

Take $f \in \mathcal{E}$, where $\mathcal{E}$ is the basis for $H^2(A)$ given by (2.1). For $n \in \mathbb{N}_0$ and $(h_1, h_2) = (p_n, 0) \in \mathcal{P}$ we get

$$\begin{align*}
(\mathcal{L}^* J(h_1, 0))(f) &= \frac{1}{2\pi i} \int_{\mathbb{T}_r} (\mathcal{L} f)(z) h_1(z) \, dz \\
&\overset{(a)}{=} \frac{1}{2\pi i} \int_{\mathbb{T}} (\mathcal{L} f)(z) h_1(z) \, dz \\
&= \frac{1}{2\pi i} \sum_{k=1}^{\infty} \int_{\mathbb{T}} \phi_k(z) (f \circ \phi_k)(z) h_1(z) \, dz \\
&\overset{(b)}{=} \frac{1}{2\pi i} \sum_{k=1}^{\infty} \int_{\phi_k(\mathbb{T})} f(w) (h_1 \circ \tau)(w) \, dw \\
&\overset{(c)}{=} \frac{1}{2\pi i} \int_{\mathbb{T}} f(w) (h_1 \circ \tau)(w) \, dw \\
&\overset{(d)}{=} \frac{1}{2\pi i} \int_{\mathbb{T}_r} f(w) (h_1 \circ \tau)(w) \, dw.
\end{align*}$$

Here, equalities $(a)$ and $(d)$ follow since the integrands are analytic on $A$, equality $(b)$ follows by change of variables with $w = \phi_k(z)$ and $\tau(w) = z$, and equality $(c)$ is a consequence of the fact that $\bigcup_k \phi_k(\mathbb{T}) = \mathbb{T}$ up to measure zero. Then, by the definition of $\Pi^r_+$ and $\Pi^r_-$,

$$\begin{align*}
(\mathcal{L}^* J(h_1, 0))(f) &= \frac{1}{2\pi i} \int_{\mathbb{T}_r} f(w) (\Pi^r_+(h_1 \circ \tau))(w) \, dw + \frac{1}{2\pi i} \int_{\mathbb{T}_r} f(w) (\Pi^r_-(h_1 \circ \tau))(w) \, dw \\
&= \frac{1}{2\pi i} \int_{\mathbb{T}_r} f(w) (\Pi^r_+(h_1 \circ \tau))(w) \, dw + \frac{1}{2\pi i} \int_{\mathbb{T}_r} f(w) (\Pi^r_-(h_1 \circ \tau))(w) \, dw \\
&= (\mathcal{L}' h_1, 0)(f).
\end{align*}$$

The penultimate equality follows from the fact that $\Pi^r_-(h_1 \circ \tau) \in H^2_0(D^\infty_R)$.

Analogously, for $n \in \mathbb{N}$ and $(h_1, h_2) = (0, \cdot (-n)) \in \mathcal{P}$, the same argument shows

$$(\mathcal{L}^* J(0, h_2))(f) = (\mathcal{L}'(0, h_2))(f).$$

Hence, for $f \in \mathcal{E}$, by linearity (4.8) holds for all finite linear combinations of basis elements $(h_1, h_2)$ in $\mathcal{P}$. Since these form a dense subspace of $H^2(D_r) \oplus H^2_0(D^\infty_R)$, and $\mathcal{L}^*$, $\mathcal{L}'$ and $J$ are continuous operators, equality (4.8) holds for all $(h_1, h_2) \in H^2(D_r) \oplus H^2_0(D^\infty_R)$ and $f \in \mathcal{E}$. By continuity, this extends to all $f \in H^2(A)$, which completes the proof. \qed

**Remark 4.6.** Lemma 4.4 and continuity of the projection operators in (4.6) imply that $\mathcal{L}'$ is compact. Note, however, that this also follows from compactness of $\mathcal{L}$ guaranteed by the choice of $A$ in (3.1).
5. Spectrum for Blaschke products

Having discussed transfer operators $L$ associated with analytic expanding circle maps and a convenient representation of the corresponding adjoint operators (Proposition 4.5), we shall now use this representation to obtain the full spectrum of $L$ for finite Blaschke products, a class of circle maps defined as follows.

**Definition 5.1.** For $n \geq 2$, let $\{a_1, \ldots, a_n\}$ be a finite set of complex numbers in the open unit disk $D$. A finite Blaschke product is a map of the form

$$B(z) = C \prod_{i=1}^{n} \frac{z - a_i}{1 - \overline{a_i}z},$$

where $|C| = 1$.

It follows from the definition that

(i) $B$ is a meromorphic function on $\hat{\mathbb{C}}$ with zeros $a_i$ and poles $1/\overline{a_i}$;

(ii) $B$ is holomorphic on a neighbourhood of $\overline{D}$ with $B(D) = D$ and $B(\mathbb{T}) = \mathbb{T}$.

Note also that a function $f$ is holomorphic on an open neighbourhood of $D$ with $f(\mathbb{T}) = \mathbb{T}$ if and only if $f$ is a finite Blaschke product (see, for example, [6, Exercise 6.12]).

Let $\tau: \mathbb{T} \to \mathbb{T}$ denote the restriction of a finite Blaschke product $B$ to $\mathbb{T}$. A short calculation shows that $\tau$ is expanding if \(\sum_{i=1}^{n} (1 - |a_i|)/(1 + |a_i|) > 1\) (see [12, Corollary to Prop. 1] for details). Expansiveness of $\tau$ can also be expressed in terms of the nature of the fixed points of $B$, as the following result shows.

**Proposition 5.2.** Let $B$ and $\tau$ be as above. Then the following conditions are equivalent.

(a) $|\tau'(z)| > 1$ for all $z \in \mathbb{T}$.

(b) $B$ has exactly $n - 1$ fixed points on $\mathbb{T}$, which are repelling, and two fixed points $z_0 \in D$ and $\hat{z}_0 = 1/\overline{z}_0 \in \hat{\mathbb{C}} \setminus \overline{D}$, which are attracting.

**Proof.** See [14, Prop. 2.1] and [22].

Crucial for the proof of our main theorem is the notion of a composition operator, which we briefly recall.

**Definition 5.3.** Let $U$ be an open region in $\hat{\mathbb{C}}$. If $\psi: U \to U$ is holomorphic, then $C_\psi: \text{Hol}(U) \to \text{Hol}(U)$ defined by $C_\psi f = f \circ \psi$ is called a composition operator (with symbol $\psi$).

Note that in the literature the term ‘composition operator’ is mostly used in the context of holomorphic functions. The operators in (4.4) considered on $L^2(\mathbb{T}_\tau)$ do not formally fall into this category, but will turn out to be composition operators for Blaschke product symbols.

We are now able to state our main result.

**Theorem 5.4.** Let $B$ be a finite Blaschke product such that $\tau = B|_\mathbb{T}$ is an analytic expanding circle map. Then

(a) the transfer operator $L: H^2(A) \to H^2(A)$ associated with $\tau$ is well defined and compact for some annulus $A \in \mathcal{A}$, and
(b) the spectrum of $\mathcal{L} : H^2(A) \to H^2(A)$ is given by

$$\sigma(\mathcal{L}) = \{\lambda(z_0)^n : n \in \mathbb{N}_0\} \cup \{\lambda(\hat{z}_0)^n : n \in \mathbb{N}\} \cup \{0\},$$  \hspace{1cm} (5.1)

where $\lambda(z_0)$ and $\lambda(\hat{z}_0) = \overline{\lambda(z_0)}$ are the multipliers\(^1\) of the unique fixed points $z_0$ and $\hat{z}_0$ of $B$ in $\mathbb{D}$ and $\mathbb{C} \setminus \overline{\mathbb{D}}$, respectively.

**Proof.** The first assertion is obvious, as $\tau$ is an analytic expanding circle map and we can choose $A = A_{r,R} \in \mathcal{A}$ as in (3.1) such that $\mathcal{L}$ is well defined and compact by the results in Section 3.

For the second claim, we will use the fact that the spectrum of $\mathcal{L}$ coincides with that of its adjoint $\mathcal{L}^*$, which together with the structure of the representation $\mathcal{L}'$ of $\mathcal{L}^*$ will allow us to deduce (5.1).

We start by observing that for the chosen $A$ we have $B(\partial A) \cap \text{cl}(A) = \emptyset$, as well as $B(D_r) \subseteq D_r$ and $B(D^\infty_R) \subseteq D^\infty_R$. It follows that $f \circ B \in H^2(D_r)$ for any $f \in H^2(D_r)$, and $f \circ B \in H^2(D^\infty_R)$ for any $f \in H^2(D^\infty_R)$, so that $C_B^{(r)} f = f \circ B$ and $C_B^{(R)} f = f \circ B$ define composition operators on $H^2(D_r)$ and $H^2(D^\infty_R)$, respectively. It is a standard fact that $B(D_r) \subseteq D_r$ guarantees compactness of $C_B^{(r)}$ (see, for example, [7, pp. 128-129]). Similarly for $C_B^{(R)}$. It is also well known (see [13, Lem. 7.10] or [7, Thm. 7.20]) that all eigenvalues of a compact composition operator $C_\psi$ are simple and are given by the non-negative integer powers of the multiplier of the unique attracting fixed point of $\psi$. Hence,

$$\sigma(C_B^{(r)}) = \{\lambda(z_0)^n : n \in \mathbb{N}_0\} \cup \{0\}$$

and

$$\sigma(C_B^{(R)}) = \{\lambda(\hat{z}_0)^n : n \in \mathbb{N}_0\} \cup \{0\},$$

where $z_0$ and $\hat{z}_0$ are the unique attracting fixed points of $B$ in $D_r$ and $D^\infty_R$, respectively (see Proposition 5.2).

We now explain how to use these observations to determine the spectrum of $\mathcal{L}'$ given in (4.7). Note that $\Pi_+^{(r)} C_B^{(r)} = C_B^{(r)}$ and $\Pi_-^{(r)} C_B^{(r)} = 0$, where $\Pi_+^{(r)}$ and $\Pi_-^{(r)}$ are the projection operators in (4.6). Thus the operator $\mathcal{L}'$ is given by

$$\mathcal{L}' = \begin{pmatrix}
C_B^{(r)} & \Pi^{(r)}_+ C_B^{(R)} \\
0 & \Pi^{(r)}_- C_B^{(R)}
\end{pmatrix}. \hspace{1cm} (5.2)$$

In particular, $\mathcal{L}'$ leaves $H^2(D_r) \oplus \{0\}$ invariant. The operator $\Pi_-^{(R)} C_B^{(R)}$ is not a composition operator on $H^2(D^\infty_R)$, but we can relate its spectrum to the spectrum of $C_B^{(R)}$ on $H^2(D^\infty_R)$. More precisely,

$$\sigma(\Pi_-^{(R)} C_B^{(R)}) = \sigma(C_B^{(R)}) \setminus \{1\},$$  \hspace{1cm} (5.3)

as we shall see below. Then, using (5.3) the assertion of the theorem follows, since

$$\sigma(\mathcal{L}') = \sigma(C_B^{(r)}) \cup \sigma(\Pi_-^{(R)} C_B^{(R)})$$
\hspace{1cm}
$$= \{\lambda(z_0)^n : n \in \mathbb{N}_0\} \cup \{\lambda(\hat{z}_0)^n : n \in \mathbb{N}\} \cup \{0\},$$

and $\sigma(\mathcal{L}) = \sigma(\mathcal{L}^*) = \sigma(\mathcal{L}')$.

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\(^1\)Recall that the multiplier $\lambda(z^*)$ of a fixed point $z^*$ of a rational map $R$ is given by $R'(z^*)$ if $z^* \in \mathbb{C}$ and $1/R'(z^*)$ if $z^* = \infty$. For Blaschke products the equality $\lambda(\hat{z}_0) = X(\mathbb{D})$ follows from a straightforward calculation.
It remains to prove (5.3). For brevity, we drop the superscript $(R)$ from $\Pi_{-}^{(R)}$, $\Pi_{+}^{(R)}$ and $C_{B}^{(R)}$ since we only consider functions in $H^{2}(D_{R}^{\infty})$ in what follows. Observe that for $f \in H^{2}(D_{R}^{\infty})$, we have $(\Pi_{+}f)(z) = f(\infty)$, which implies

$$C_{B}\Pi_{+} = \Pi_{+} \quad \text{and} \quad \Pi_{-}C_{B} = \Pi_{-}C_{B}\Pi_{-}. \quad (5.4)$$

Note that 1 is an eigenvalue of $C_{B}$ if and only if the corresponding eigenfunction is constant. Take $\mu \in \sigma(C_{B})$ with $\mu(1 - \mu) \neq 1$. Since $C_{B}$ is compact, there is a non-zero $f \in H^{2}(D_{R}^{\infty})$ with $C_{B}f = \mu f$. The second equality in (5.4) now implies $\Pi_{-}C_{B}\Pi_{-}f = \mu\Pi_{-}f$. But since $\mu \neq 1$ the eigenvector $f$ is non-constant, so we have $0 \neq \Pi_{-}f \in H^{2}(D_{R}^{\infty})$ and thus $\mu \in \sigma(\Pi_{-}C_{B})$.

To show the converse inclusion, take $\mu \in \sigma(\Pi_{-}C_{B})$ with $\mu \neq 0$. Since $\Pi_{-}C_{B}$ is compact, there is a non-zero $f \in H^{2}(D_{R}^{\infty})$ with $\Pi_{-}C_{B}f = \mu f$. First observe that $2 \mu \neq 1$. Next we note that if $\mu(\mu - 1) \neq 0$, then $(1 - \mu)f - \Pi_{+}C_{B}f \neq 0$ (for otherwise $f$ would be zero). Finally, we use (5.4) to show that $(1 - \mu)f - \Pi_{+}C_{B}f$ is an eigenfunction of $C_{B}$ with eigenvalue $\mu$:

$$C_{B}((1 - \mu)f - \Pi_{+}C_{B}f) = (1 - \mu)(C_{B}f + (\mu f - \Pi_{-}C_{B}f)) - C_{B}\Pi_{+}C_{B}f$$

$$= \mu(1 - \mu)f + (1 - \mu)(I - \Pi_{-})C_{B}f - \Pi_{+}C_{B}f$$

$$= \mu((1 - \mu)f - \Pi_{+}C_{B}f).$$

Thus $\sigma(\Pi_{-}C_{B}) = \sigma(C_{B}) \setminus \{1\}$ as claimed. \hfill \Box

The following examples illustrate our main result.

**Example 5.5.** The map $B(z) = z^{n}$ for $n \geq 2$ has two attracting fixed points $z_{0} = 0$ and $\bar{z}_{0} = \infty$ with $\lambda(z_{0}) = \lambda(\bar{z}_{0}) = 0$. Thus $\sigma(\mathcal{L}) = \{0, 1\}$.

Curiously enough, these are not the only examples for which $\sigma(\mathcal{L}) = \{0, 1\}$, as the next example shows.

**Example 5.6.** Let $B(z) = z^{2}(z - b)/(1 - bz)$ for $b \in (-1, 1)$. Then $B|_{\bar{\mathbb{D}}}$ is an expanding 3-to-1 circle map. As the multipliers of the attracting fixed points of $B$ are vanishing, just as in Example 5.5, we get $\sigma(\mathcal{L}) = \{0, 1\}$.

**Example 5.7.** For the family of maps $B(z) = z(\mu - z)/(1 - \mu z)$ considered in [21], the restriction $B|_{\mathbb{D}}$ is an expanding circle map for any $\mu \in \mathbb{D}$. The attracting fixed points are $z_{0} = 0$ and $\bar{z}_{0} = \infty$ with $\lambda(z_{0}) = \mu$ and $\lambda(\bar{z}_{0}) = \bar{\mu}$. Thus

$$\sigma(\mathcal{L}) = \{\mu^{n} : n \in \mathbb{N}_{0}\} \cup \{\bar{\mu}^{n} : n \in \mathbb{N}\} \cup \{0\}.$$ 

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2In order to see this, note that otherwise $\Pi_{-}C_{B}f = f$, which implies $f \circ B - f = \text{const}$. However $f(B(z_{0})) - f(\bar{z}_{0}) = 0$, which implies $f = f \circ B$. Thus $f = \text{const}$, so $\Pi_{-}C_{B}f = 0$, contradicting the fact that $\mu \neq 0$. 
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