Polarization observables in $e^+e^-$ annihilation to a baryon-antibaryon pair

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(Dated: September 12, 2018)

Using the helicity formalism of Jacob and Wick we derive spin density matrices of baryon antitibaryon pairs produced in $e^+e^-$ annihilation. We consider the production of pairs with spins $1/2 + 1/2$, $1/2 + 3/2$ (+c.c.) and $3/2 + 3/2$. We provide modular expressions to include chains of weak hadronic two-body decays of the produced hyperons. The expressions are suitable for the analysis of high statistics data from $J/\psi$ and $\psi(2S)$ decays at $e^+e^-$ colliders, by fits to the fully differential angular distributions of the measured particles. We illustrate the method by examples, such as the inclusive measurement of the $e^+e^- \rightarrow \psi(2S) \rightarrow \Omega^-\Omega^+$ process where one decay chain $\Omega^- \rightarrow \Lambda K^-$ followed by $\Lambda \rightarrow p\pi^-$ is considered. Finally we show that the inclusive angular distributions can be used to test spin assignment of the produced baryons.

I. INTRODUCTION

Charmonia are excellent sources of spin entangled hyperon antihyperon pairs. In particular the states $J/\psi$ or $\psi(2S)$, which carry $J^{PC} = 1^{--}$, are directly produced at electron positron colliders. They are perfectly suited for precise determination of hyperon decay parameters and searches for $CP$ symmetry violation in the baryon sector.

With a large number of collected $J/\psi$, $(1310.6 \pm 7.0) \times 10^6$, and $\psi(2S)$, $(448.1 \pm 2.9) \times 10^6$, at the BESIII experiment [1-3] detailed studies of such systems are now possible. Examples of the available data samples from recent publications are given in Table I. The branching fractions, $B$, for the listed decay modes range between $10^{-4}$ and $10^{-3}$ and the reconstructed data samples are up to $10^6$ events. In addition, considering world averages of the $B$ values for other $B_1\bar{B}_2$ decays, one can anticipate that more modes are accessible with the collected data sets (Table II). However, all of the published results focus only on the determination of the branching fractions and the angular distributions of the produced hyperons. In our opinion there is much more potential in the data. It was already shown in Ref. [4] that past analyses have assumed simplified expressions for the amplitudes which precluded an extensive analysis. The first analysis of $e^+e^- \rightarrow J/\psi \rightarrow \Lambda\bar{\Lambda}$ at BESIII using the correct formalism is reported in Ref. [5].

The correct expressions for the production amplitudes of such processes are described by a limited set of form factors — complex numbers at fixed center-of-mass (CM) energy. For instance, in the case of a spin-1/2 hyperon pair there are just two such form factors. The angular distribution is described by two real numbers: one related to the ratio of the absolute values of the form factors and the other giving the relative phase. In this case, provided that there is a non-negligible phase between the form factors, one can determine the decay parameters of the produced hyperons and carry out $CP$ violation tests in the baryon sector. For the spin-1/2 hyperons with single-step decay modes (analogous to $\Lambda$), the formulas provided in Ref. [4] could be used directly. However, to include other interesting cases the formalism has to be extended for states where the hyperon antihyperon pair can have a combination of spins 1/2 and 3/2 and for multi-step decay chains.

Several approaches are suitable to provide the amplitude for a process where the final states carry nonzero spins. We use the helicity formalism originally developed by Jacob and Wick [10]. This formalism had been used in the past for several hyperon production reactions and decays [11-15]. However, we did not find a general and modular formulation which could be applied directly to describe high statistics exclusive data, i.e. data where momenta of all particles are measured for each event. For this purpose fully differential angular distributions are needed, to be used for event generators and for maximum likelihood fits. It is the purpose of the present paper to document the construction of such a framework.

We derive spin density matrices for $e^+e^- \rightarrow B_1\bar{B}_2$ processes where the baryon (antibaryon) can have spin 1/2 or 3/2. In practice we focus on the cases where all baryons have positive parity and all antibaryons have negative parity. This fits to the ground state baryons of spin 1/2 and spin 3/2 [9]. The presented formalism can be applied to study decays of $J^{PC} = 1^{--}$ vector mesons produced in electron positron colliders, such as $J/\psi$ or $\psi(2S)$, into $B_1\bar{B}_2$ pairs. We will also revise some misleading assumptions and formulas used in the analyses of weak decay chains within this framework.

In order to establish our notation we start with applying the helicity formalism to the well known case of $1/2 + 1/2$ baryons, then we proceed to the $1/2 + 3/2$ and $3/2 + 3/2$ cases. We present a general formalism together with detailed expressions for the spin density...
measured directly, this information is encoded in the angular distributions. The angles are measured with respect to some axes, which makes it necessary to define appropriate frames of reference and cartesian coordinate systems.

The production process defines the first coordinate system; see below. For the decays it is useful to boost to the rest frame of the mother particle. Yet it is helpful to perform rotations before this boost. We will be very explicit to motivate and define these rotations.

Following the ideas of [10, 11] we use the helicity formalism. Here the spin quantization axis is not chosen along a fixed axis but along the flight direction of the state. The advantage is that the helicity does not change when boosting to the rest frame of this state. On the other hand, the use of angular-momentum ($\mathbf{J}$) conservation for the production and for each decay process suggests to single out the $z$-axis, based on the convention to use $\mathbf{J}^2$ and $J_z$ for the characterization of states.

Following this spirit it is useful to spell out how helicity states are constructed. To motivate this construction we discuss first how one deals with changes of reference frames in experimental analyses. Afterwards we will describe how to mimic these changes on the theory side.

- **a. Experimental procedure:** Suppose one has produced a “mother” particle that decays further. One wants to change from the production frame of this state to its rest frame. Given the state’s three-momentum

$$\mathbf{p}_m = p_m (\cos \phi_m, \sin \theta_m, \sin \phi_m \cos \theta_m, \cos \theta_m) \quad (1)$$

and the $z$-axis in the production frame, one possibility would be to perform a single rotation that aligns $\mathbf{p}_m$ with the $z$-axis. Subsequently one then boosts to the rest frame of the mother particle. The single rotation would be around an axis perpendicular to $\mathbf{p}_m$ and $\hat{\mathbf{z}}$. Yet when viewed as rotations around the coordinate

### TABLE I. Available $B_1\bar{B}_2$ data samples from recent BESIII publications [6–8].

| decay mode       | $B(\times 10^{-3})$ |
|------------------|---------------------|
| $J/\psi \to \Lambda \bar{\Lambda}$ | 440675 ± 670 | 19.43 ± 0.03 ± 0.33 |
| $\psi(2S) \to \Lambda \bar{\Lambda}$ | 6012 ± 82 | 2.44 ± 0.03 ± 0.11 |
| $J/\psi \to \Sigma(1385)^0 \Sigma(1385)^0$ | 5337 | 11.64 ± 0.04 ± 0.23 |
| $J/\psi \to \Xi(1385)^0 \Xi(1385)^0$ | 111026 ± 335 | 3.97 ± 0.02 ± 0.12 |
| $\psi(2S) \to \Xi(1385)^0 \Xi(1385)^0$ | 107762 ± 582 | 10.71 ± 0.09 |
| $\psi(2S) \to \Sigma(1385)^0 \Sigma(1385)^0$ | 134846 ± 637 | 11.65 ± 0.04 |
| $\psi(2S) \to \Xi(1385)^0 \Xi(1385)^0$ | 2214 ± 148 | 0.69 ± 0.06 |
| $\psi(2S) \to \Xi(1385)^0 \Xi(1385)^0$ | 10839 ± 123 | 2.73 ± 0.03 |
| $\psi(2S) \to \Xi(1385)^0 \Xi(1385)^0$ | 42811 ± 231 | 10.40 ± 0.06 |
| $\psi(2S) \to \Xi(1385)^0 \Xi(1385)^0$ | 42595 ± 467 | 10.96 ± 0.12 |
| $\psi(2S) \to \Xi(1385)^0 \Xi(1385)^0$ | 52523 ± 596 | 12.58 ± 0.14 |
| $\psi(2S) \to \Xi(1385)^0 \Xi(1385)^0$ | 5337 ± 83 | 2.78 ± 0.05 |
| $\psi(2S) \to \Xi(1385)^0 \Xi(1385)^0$ | 1375 ± 98 | 0.85 ± 0.06 |
| $\psi(2S) \to \Xi(1385)^0 \Xi(1385)^0$ | 1470 ± 95 | 0.84 ± 0.05 |

### TABLE II. Possible other hyperon antihyperon final states which can be studied at BESIII. The quoted branching fractions are from the Particle Data Group [9].

| decay mode       | $B(\times 10^{-3})$ |
|------------------|---------------------|
| $J/\psi \to \Xi(1530)^+ \Xi^+$ | 5.9 ± 1.5 |
| $J/\psi \to \Xi(1530)^0 \Xi^+$ | 3.3 ± 1.4 |
| $J/\psi \to \Sigma(1385)^- \Sigma^+$ | 3.1 ± 0.5 |
| $\psi(2S) \to \Omega^- \Omega^+$ | 0.47 ± 0.10 |

As long as the momentum direction is not flipped, boosts do not change the helicity. Therefore in the helicity amplitude method one can disregard the boost part of the Lorentz group, which allows to obtain angular distributions without using full expressions for the spinors as required by the Feynman diagram technique. This is very convenient but comes with a disadvantage: the energy dependence of the contributing amplitudes cannot be determined and therefore not even their relative importance. Yet for fixed production energy of a two-particle system and for two-body decays of the produced states all kinematical variables, i.e. all angles, are fully covered by the helicity framework.

The present preprint is meant as a reference for the ongoing analyses of high-statistics BESIII data. It will be extended to a full article in the near future.

## II. GENERAL FRAMEWORK

In general we look at the production of two unstable particles in an initial scattering reaction. Subsequently the produced particles decay in one or several steps. The general task is to deduce information about the spins and their correlations among the involved (unstable) particles. If none of the spins are

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axes this amounts to a succession of three rotations. Viewed as active rotations these are (a) a rotation around the z-axis by \(-\phi_m\); (b) a rotation around the y-axis by \(-\theta_m\); (c) a rotation around the z-axis by \(+\phi_m\); see also [10]. In principle, however, the first two rotations are sufficient to align \(p_m\) with the z-axis. In line with the present BESIII analyses we follow this two-rotation procedure in the present work. The rotation matrix for \(p_m\) is given by

\[
\begin{pmatrix}
\cos \theta_m \cos \phi_m & \cos \phi_m \sin \phi_m & -\sin \phi_m \\
-\sin \theta_m \cos \phi_m & \cos \theta_m \cos \phi_m & \sin \phi_m \\
\sin \phi_m & -\cos \phi_m & 0
\end{pmatrix}.
\] (2)

This rotation defines in a unique way the helicity reference frame for a daughter particle. In an experimental analysis the boosts and rotations in Eq (2) are applied recursively to all decay products of a decay chain, thus defining a set of helicity variables to describe an event.

b. Matching amplitude: To mimic this procedure on the theory side we construct helicity states by the inverse procedure, following essentially [11]. A one-particle state with helicity \(\lambda\) and momentum \(p = p (\cos \phi \sin \theta, \sin \phi \sin \theta, \cos \theta)\) is constructed from a state \(|p, \lambda\rangle\) that moves along the z-direction by

\[
|p, \theta, \phi, \lambda\rangle := R(\phi, \theta, 0)|p, \lambda\rangle
\] (3)

with [10]

\[
R(\alpha, \beta, \gamma) := e^{-i\alpha J_x} e^{-i\beta J_y} e^{-i\gamma J_z}.
\] (4)

Correspondingly a two-particle state in its CM frame is given by

\[
|p, \theta, \phi, \lambda_1, \lambda_2\rangle := R(\phi, \theta, 0)|p, \lambda_1, \lambda_2\rangle.
\] (5)

In practice we follow all the steps of [10] except for the fact that we use a two-angle rotation procedure as spelled out in Eq. (3). When constructing (5) the first particle has momentum \(p = p (\cos \phi \sin \theta, \sin \phi \sin \theta, \cos \theta)\) and helicity \(\lambda_1\) while the second has momentum \(-p\) and helicity \(\lambda_2\). The most important consequence of our construction of these two-particle states is their projection on angular-momentum eigenstates [11]:

\[
\langle J, M, \lambda_1', \lambda_2' | \theta, \phi, \lambda_1, \lambda_2\rangle
= \frac{\sqrt{2J+1}}{4\pi} D^J_{M, \lambda_1-\lambda_2} (\phi, \theta, 0) \delta_{\lambda_1', \lambda_1} \delta_{\lambda_2', \lambda_2}
\] (6)

where \(D^{(n)}_{m', m}(\alpha, \beta, \gamma) := \langle jm' | R(\alpha, \beta, \gamma) | jm\rangle\) is the Wigner D-matrix.\(^1\)

\(^1\) Note that our definition is in line with [10] but differs from the conventions used in Mathematica [16]. In particular, we have \(D^{(n)}_{m', m}(\alpha, \beta, \gamma) = e^{-im'\alpha-i(m'-m)\gamma} D^{(n)}_{m', m}(0, \beta, 0)\) while the built-in “WignerD” function of Mathematica satisfies \(D^{(n)}_{m', m}(\alpha, \beta, \gamma) = e^{im'\alpha+im\gamma} D^{(n)}_{m', m}(0, \beta, 0)\).
We have to evaluate three matrix elements. The first and the third bring in Wigner functions. The general formula is given in Eq. (6). For the transition amplitude one finds in the one-photon approximation

\[ \rho_{\lambda_1, \lambda_2}(J, M, \lambda_1, -\lambda_2) = \delta_{J,1} A_{\lambda_1, \lambda_2}^\text{in} A_{\lambda_1, -\lambda_2}^\text{out} \, . \]  

(10)

Here \( A_{\lambda_1, \lambda_2} \) denotes the transition amplitude between helicity states. Only transitions fulfilling the inequality

\[ |\lambda_1 - \lambda_2| \leq J = 1 \]  

(11)

are different from zero. For a parity conserving process the amplitudes between opposite helicity states are related:

\[ A_{\lambda_1, \lambda_2} = \eta_1 \eta_2 \theta_{-1} J_{-s_1} J_{-s_2} A_{-\lambda_1, -\lambda_2} \, , \]  

(12)

where \( \eta \) is the parity of the initial state, \( \eta_1 \) and \( \eta_2 \) are the parities of the final state particles. Moreover parity symmetry of QED implies \( A_{\lambda_1, \lambda_2} = A_{\lambda_2, \lambda_1} \) for the initial \( e^+ e^- \rightarrow \gamma^* \) production amplitude. Here we are not interested in the \( p_m \) dependence of the reaction and therefore we can drop \( A_{\text{in}} \).

We obtain for the production density matrix:

\[ \rho_{\lambda_1, \lambda_2}(J, M, \lambda_1, -\lambda_2) = \sqrt{\frac{1}{2}} \delta_{J,1} A_{\lambda_1, \lambda_2}^\text{in} A_{\lambda_1, -\lambda_2}^\text{out} \, . \]  

(13)

We we find in the following relativistic expression for the density matrix:

\[ \rho_{\lambda_1, \lambda_2}(J, M, \lambda_1, -\lambda_2) = \sqrt{\frac{1}{2}} A_{\lambda_1, \lambda_2} A_{\lambda_1, -\lambda_2} \, . \]  

(14)

We note in passing that here one also could rotate to a frame where the baryons do not lie in the \( x-z \) plane, i.e. where they have a non-vanishing value of \( \phi \). This would not change the density matrix because of the following relation:

\[ D_{k,i,j}^\text{in}(\theta, \phi, 0) = D_{k,j,i}^\text{in}(\theta, \phi, 0) \, . \]  

(16)

The explicit form of the reduced density matrix \( \rho_1 \) is given by

\[ \rho_1(\theta) = \begin{pmatrix} \frac{1 + \cos^2 \theta}{2} & -\cos \theta \sin \theta & \sin^2 \theta \\ -\cos \theta \sin \theta & \frac{\sqrt{2}}{2} & \cos \theta \sin \theta \\ \sin^2 \theta & \cos \theta \sin \theta & \frac{1 + \cos^2 \theta}{2} \end{pmatrix} \, . \]  

(17)

Finally we note that for the case where \( B_1 \) and \( B_2 \) are of the same type and in the one-photon approximation, charge conjugation provides the following (schematic) relation: \( \langle \gamma^* | B_1, B_2 \rangle = \langle \gamma^* | B_2, B_1 \rangle \). The minus sign emerging from the virtual photon is compensated by the reordering of the two (anti-commuting) fermions from \( |B_1, B_2\rangle \) to \( |B_2, B_1\rangle \).

B. Baryon spin density matrices

The general spin-density matrix for a spin-1/2 particle has the following form:

\[ \rho_{1/2} = \frac{1}{2} \begin{pmatrix} I_0 + I_z & I_x - iI_y \\ I_x + iI_y & I_0 - I_z \end{pmatrix} \]  

(18)

or expressed in a compact way:

\[ \rho_{1/2} = \frac{1}{2} \sum_{\mu} I_\mu \sigma_\mu \, , \]  

(19)

where \( \mu = 0, x, y, z \); \( \sigma_x, \sigma_y, \sigma_z \) are the Pauli matrices and \( \sigma_0 \) is the identity 2 \( \times \) 2 matrix. \( I_0 \) is the cross section term and \( I \) is a three vector \( I = I_0 \cdot P \), where \( P \) is the polarization vector for the fermion. For some formulas we also use notation with a numeric index: \( \mu = 0, 1, 2, 3 \).

The density matrix of a spin-3/2 particle can be written in terms of sixteen Hermitian 4 \( \times \) 4 matrices \( Q_\mu \) with \( \mu = 0, ..., 15 \) as described in Ref. [17]. The explicit expression for these matrices is given in Appendix A. The general density matrix for a single spin-3/2 particle can be expressed as

\[ \rho_{3/2} = \sum_{\mu=0}^{15} r_\mu Q_\mu \, , \]  

(20)

where \( r_0 \) is the cross section term, \( Q_0 = \frac{1}{4} I \) where \( I \) is the 4 \( \times \) 4 identity matrix and \( r_\mu \) are real numbers.

III. SPECIFIC PRODUCTION PROCESSES

A. Two spin-\( \frac{1}{2} \) baryons

It is well known how the spin density matrices look like for a reaction \( e^+ e^- \rightarrow B_1 B_2 \) where both produced particles have spin 1/2. The results were obtained using different approaches [4, 18–22]. Here we reproduce the result using the helicity method. We focus on the case where the baryon has positive parity \( \eta_1 = 1 \) and the antibaryon negative parity \( \eta_2 = -1 \). This fits to the production of a pair of ground-state hyperons. In general only two out of four possible helicity transitions are independent. Using \( \eta_1 \eta_2 = -1 \) for the baryon antibaryon pair one can set \( A_{1/2,1/2} = -A_{-1/2,-1/2} =: h_1 \) and \( A_{1/2,-1/2} = A_{-1/2,1/2} =: h_2 \). The transition amplitude matrix is

\[ \begin{pmatrix} h_1 & h_2 \\ h_2 & h_1 \end{pmatrix} \, . \]  

(21)

The spin density matrix for a two-particle 1/2 + 1/2 system can be expressed in terms of a set of 4 \( \times \) 4
matrices obtained from the outer product, \( \otimes \), of \( \sigma_\mu \) and \( \sigma_\nu \) [13]:

\[
\rho_{1/2,1/2} = \frac{1}{4} \sum_{\mu,\nu} C_{\mu\nu} \sigma_\mu \otimes \sigma_\nu. \quad (22)
\]

Here the spin matrices are expressed in the helicity frames denoted by \( \mu = 0, x, y, z \) and \( \nu = 0, x, y, z \) for the baryon \( B_1 \) and the antibaryon \( \bar{B}_2 \), respectively. The real coefficients \( C_{\mu\nu} \) are functions of the scattering angle \( \theta_1 \) of \( B_1 \).

Suppose one is not interested in the absolute size of the cross section but only in the (not normalized) angular distributions. For their description we do not need all information contained in the two complex form factors \( h_1 \) and \( h_2 \). Instead we can use just two real parameters: First, \( \alpha_0 \) as defined below and, second, the relative phase between the form factors \( \Delta \Phi = \arg(h_1/h_2) \), i.e. we disregard the normalization and the overall phase. More specifically without any loss of generality we take \( h_1 \) as real and set \( h_1 = \sqrt{1+\alpha_0}/\sqrt{2} \) and \( h_2 = \sqrt{1-\alpha_0} \exp(-i \Delta \Phi) \).

Only 8 coefficients \( C_{\mu\nu} \) are non-zero and they are given by

\[
\begin{align*}
C_{00} &= 2(1+\alpha_0 \cos^2 \theta_1), \\
C_{0y} &= 2\sqrt{1-\alpha_0^2} \sin \theta_1 \cos \theta_1 \sin(\Delta \Phi), \\
C_{x\bar{x}} &= 2 \sin^2 \theta_1, \\
C_{y\bar{y}} &= 2\sqrt{1-\alpha_0^2} \cos \theta_1 \sin(\Delta \Phi), \\
C_{xy} &= -C_{0y}, \\
C_{y\bar{y}} &= \alpha_0 C_{x\bar{x}}, \\
C_{z\bar{z}} &= -C_{x\bar{x}}, \\
C_{zz} &= -2(\alpha_0 + \cos^2 \theta_1). 
\end{align*}
\]

For the case when the antibaryon \( \bar{B}_2 \) is not measured (the decay products are not registered), the corresponding **inclusive** density matrix can be obtained by taking the trace of the formula in Eq. (22) with respect to the spin variables of \( \bar{B}_2 \). The result is

\[
\rho_{1/2} = \frac{1}{2} \sum_{\mu} C_{\mu 0} \sigma_\mu, \quad (24)
\]

where

\[
\begin{align*}
C_{00} &= I_0 = 2(1+\alpha_0 \cos^2 \theta_1), \\
C_{y0} &= -2\sqrt{1-\alpha_0^2} \sin \theta_1 \cos \theta_1 \sin(\Delta \Phi). 
\end{align*}
\]

If the produced spin-1/2 baryon is a hyperon decaying weakly, one can determine the polarization of \( B_1 \) in the \( e^+e^- \rightarrow B_1 \bar{B}_2 \) production process from the angular distributions of the decay products. The most common case is a weak decay into a spin-1/2 fermion and a pseudoscalar (\( e.g. \, \Lambda \rightarrow p\pi^- \)). For the case of a one-step process, when the decay product is stable and its polarization is not measured, the final angular distribution is given by:

\[
d\sigma \propto (I_0 + \alpha_1 I_y \sin \theta_1 \sin \phi_\pi) d\Omega_\pi, \quad (26)
\]

where \( \alpha_1 \) is the decay asymmetry parameter for the corresponding weak decay mode of \( B_1 \).

**B. Spin 1/2 and spin 3/2 baryon**

To be specific we consider \( e^+e^- \rightarrow B_1 \bar{B}_2 \) where \( B_1 \) has spin 1/2 and \( \bar{B}_2 \) spin 3/2. We focus on the case where the baryon has positive parity, \( \eta_1 = 1 \), and the antibaryon negative parity, \( \eta_2 = -1 \). This fits to the production of ground-state hyperons with the respective spins. In general only three out of eight transition amplitudes are independent: Parity symmetry of the production process relates the amplitudes pairwise. In addition, the one-photon approximation does not allow for the helicity combination where \( |\lambda_1 - \lambda_2| = 2 \) on account of Eq. (11).

Again we have \( \eta_1 \eta_2 = -1 \) for the baryon antibaryon pair so that

\[
A_{\lambda_1,\lambda_2} = -A_{-\lambda_1,-\lambda_2,} \text{ follows from Eq. (12). For simplicity we introduce } A_{1/2,1/2} = -A_{-1/2,-1/2} =: h_1, \quad A_{1/2,1/2} = -A_{-1/2,1/2} =: h_2 \text{ and } A_{1/2,3/2} = -A_{-1/2,3/2} =: h_3.
\]

In the one-photon approximation the remaining amplitudes vanish: \( A_{-1/2,3/2} = A_{1/2,-3/2} = 0 \). Therefore the transition amplitude can be expressed as:

\[
\begin{pmatrix}
2h_3 & h_1 & h_2 & 0 \\
0 & h_2 & -h_1 & -h_3
\end{pmatrix}
\]

The density matrix for the \( 1/2 + 3/2 \) system can be expressed in terms of a set of \( 8 \times 8 \) matrices obtained from the outer product of \( \sigma_\mu \) and \( Q_\nu \):

\[
\rho_{1/2,3/2} = \frac{1}{8} \sum_{\mu=0}^{3} \sum_{\nu=0}^{15} C_{\mu\nu} \sigma_\mu \otimes Q_\nu, \quad (28)
\]

where \( C_{\mu\nu} \) are real coefficients. Only 30 coefficients are non-zero. Here we just give the expressions for the inclusive spin density matrices for the 1/2 and the 3/2 baryon, respectively.

The inclusive density matrix for the spin-1/2 baryon \( B_1 \) is obtained by taking the trace of the formula in Eq. (28) with respect to the spin variables of the antibaryon \( \bar{B}_2 \). One obtains the general form (19) with entries

\[
\begin{align*}
I_0 &= 2|h_1|^2 \sin^2 \theta_1 + (1 + \cos^2 \theta_1)(|h_2|^2 + |h_3|^2), \\
I_y &= 2\sqrt{2} \Im(h_1 h_2^* \sin \theta_1 \cos \theta_1), \\
I_x &= I_z = 0.
\end{align*}
\]

The corresponding inclusive spin density matrix obtained for the baryon \( B_2 \) can be expressed as

\[
\rho_{3/2}(\theta_1) = \begin{pmatrix}
m_{11} & c_{12} & c_{13} & 0 \\
c_{12}^* & m_{22} & m_{23} & c_{13}^* \\
c_{13}^* & -m_{23}^* & m_{22}^* & -c_{12}^* \\
0 & c_{13} & -c_{12} & m_{11}
\end{pmatrix}, \quad (30)
\]
where $m_{11}$, $m_{22}$ and $m_{23}$ are real while $c_{12}$ and $c_{13}$ are complex functions of the scattering angle $\theta_1$. These elements of the spin density matrix are

\[
m_{11} = \frac{1 + \cos^2\theta_1}{2} |h_3|^2, \\
m_{22} = |h_1|^2 \sin^2\theta_1 + \frac{1 + \cos^2\theta_1}{2} |h_2|^2, \\
m_{23} = \sqrt{2} \Im(h_3 h_1^\dagger) \cos\theta_1 \sin\theta_1, \\
c_{12} = \frac{h_3 h_1^\dagger \cos\theta_1 \sin\theta_1}{\sqrt{2}}, \\
c_{13} = \frac{1}{2} h_3 h_2^* \sin^2\theta_1.
\]

The density matrix $\rho_{3/2}$ can be also written in terms of the polarization parameters introduced in Eq. (20). Since we are considering a parity conserving process it turns out that only seven parameters are non-zero: $r_0$, $r_1$, $r_6$, $r_7$, $r_{10}$, $r_{11}$ and $r_{12}$. This fits to the production of ground-state hyperons with spin 3/2. Actually all elements of the spin density matrix are

\[
r_0 = (\cos^2\theta_1 + 1)(|h_3|^2 + |h_1|^2) + 2|h_1|^2 \sin^2\theta_1, \\
r_1 = 2 \sin 2\theta_1 \frac{2 \Im(h_3 h_1^\dagger) + \sqrt{2} \Im(h_1 h_3^\dagger)}{3\sqrt{3}}, \\
r_6 = 2 \sin^2\theta_1 |h_1|^2 + (|h_2|^2 - |h_3|^2)(\cos^2\theta_1 + 1), \\
r_7 = \sqrt{2} \sin 2\theta_1 \frac{\Re(h_1 h_3^\dagger)}{\sqrt{3}}, \\
r_8 = 2 \sin^2\theta_1 \frac{\Re(h_2 h_1^\dagger)}{\sqrt{3}}, \\
r_{10} = 2 \sin^2\theta_1 \frac{3 \Im(h_2 h_3^\dagger)}{\sqrt{3}}, \\
r_{11} = 2 \sin 2\theta_1 \frac{3 (\sqrt{2} h_2 h_1^\dagger + h_1 h_3^\dagger)}{\sqrt{15}}.
\]

\[C. \ \text{Two spin-}\frac{3}{2} \text{ baryons}\]

We focus again on the case where the baryon has positive parity $\eta_1 = 1$ and the antibaryon negative parity $\eta_2 = -1$. This fits to the production of ground-state hyperons with spin 3/2. Actually all such ground-state hyperons are distinct from each other by strangeness or electric charge. Thus we focus on the case where the produced antibaryon is the antiparticle of the produced baryon (and not an arbitrary spin-3/2 state). This allows to involve arguments from charge conjugation invariance.

For $e^+e^- \rightarrow B_1 \bar{B}_2$, where both $B_1$ and $\bar{B}_2$ are spin-3/2 particles, only 4 out of 16 amplitudes are independent. From Eq. (12) it follows that $A_{\lambda_1, \lambda_2} = A_{-\lambda_1, -\lambda_2}$. We only need to consider $A_{1/2, -1/2} = A_{-1/2, 1/2} =: h_2$, $A_{1/2, 1/2} = A_{-1/2, -1/2} =: h_1$, $A_{-3/2, -1/2} = A_{1/2, 1/2} =: h_3$, and $A_{3/2, -1/2} = A_{3/2, 3/2} =: h_4$. Due to Eq. (11) expressing the constraint for the spin projection values of the initial state (one-photon approximation) the following amplitudes vanish: $A_{-1/2, 3/2} = A_{1/2, -3/2} = 0$. Moreover $A_{-1/2, -3/2} = A_{1/2, 3/2} = h_3$ due to charge conjugation invariance. Thus the transition amplitude is given by

\[
\begin{pmatrix}
  h_4 & h_3 & 0 & 0 \\
  h_3 & h_1 & h_2 & 0 \\
  0 & h_2 & h_1 & h_3 \\
  0 & 0 & h_3 & h_4
\end{pmatrix}.
\]

The density matrix for the 3/2 + 3/2 system can be expressed in terms of a set of $16 \times 16$ matrices constructed from the outer product of $Q_\mu$ and $Q_\nu$:

\[
\rho_{3/2, 3/2} = \sum_{\mu=0}^{15} \sum_{\nu=0}^{15} C_{\mu\nu} Q_\mu \otimes Q_\nu,
\]

where $C_{\mu\nu}$ is a set of 256 real coefficients of which 140 are zero.

If the antibaryon is not registered the inclusive density matrix of the spin-3/2 baryon $B_1$ is again given by Eq. (30). In this case, the elements are

\[
m_{11} = \frac{1 + \cos^2\theta_1}{2} |h_3|^2 + |h_4|^2 \sin^2\theta_1, \\
m_{22} = \frac{1 + \cos^2\theta_1}{2} (|h_2|^2 + |h_3|^2) + |h_1|^2 \sin^2\theta_1, \\
m_{23} = \sqrt{2} \Im(h_3 h_1^\dagger) \cos\theta_1 \sin\theta_1, \\
c_{12} = \frac{1}{\sqrt{2}} (h_4 h_3^\dagger - h_3 h_4^\dagger) \cos\theta_1 \sin\theta_1, \\
c_{13} = \frac{1}{2} h_3 h_2^* \sin^2\theta_1.
\]

The angular distribution is given by the trace of the density matrix:

\[
\frac{d\sigma}{d\cos\theta_1} \propto 2(m_{11} + m_{22}).
\]

Defining

\[
\alpha_\psi = \frac{|h_2|^2 - 2(|h_1|^2 - |h_3|^2 + |h_4|^2)}{|h_2|^2 + 2(|h_1|^2 - |h_3|^2 - |h_4|^2)},
\]

it can be written as $1 + \alpha_\psi \cos^2\theta_1$. Using Eq. (20) an alternative representation for the inclusive density matrix for the spin-3/2 baryon is given by the following seven real $r_\mu = C_{\mu0}$ coefficients (the remaining
nine are zero):

\[ r_0 = \left[ |h_2|^2 + 2(|h_1|^2 + |h_4|^2) \right] (1 + \alpha \psi \cos^2 \theta_1), \]
\[ r_1 = 2 \sin 2 \theta_1 \frac{3 \Im(h_2 h_1^* \bar{h}_1) + \sqrt{3} \Im(h_3(h_1^* + h_4^*))}{\sqrt{3}}, \]
\[ r_6 = -2 \sin^2 \theta_1 \left( |h_1|^2 - |h_4|^2 \right) + |h_2|^2 \left( \cos^2 \theta_1 + 1 \right), \]
\[ r_7 = \sqrt{2} \sin 2 \theta_1 \frac{\Re(h_2 \bar{h}_4)}{\sqrt{3}}, \]
\[ r_8 = \sqrt{2} \sin 2 \theta_1 \frac{\Re(h_2 h_4^*)}{\sqrt{3}}, \]
\[ r_{10} = 2 \sin 2 \theta_1 \frac{\Im(h_2 h_4^*)}{\sqrt{3}}, \]
\[ r_{11} = 2 \sin 2 \theta_1 \frac{\Im \left( \sqrt{3} h_1 h_4 + h_3 (h_1^* + h_4^*) \right)}{\sqrt{15}}. \]

The corresponding coefficients for the inclusive density matrix of the antibaryon are the same, provided one uses the scattering angle of the antibaryon, i.e. \( \theta_1 \rightarrow \pi - \theta_1 \).

### IV. DECAY CHAINS

The density matrices of the produced hyperons can be used to derive the angular distributions of the particles produced in the subsequent decays. When considering multi-step decay processes, also the density matrices of the intermediate states are needed. Moreover one should keep track of the spin correlations for the initial \( B_1 \bar{B}_2 \) pair. We propose a general modular method to obtain the distributions in a systematic way. Since the joined production density matrices of Eqs. (22), (28) and (34) are expressed as outer products of the basis matrices \( \sigma_\mu \) and \( Q_\mu \), it is enough to know how the latter individually transform under a decay process.

We consider two weak decay modes, which cover most of the relevant cases2: 1) spin-3/2 hyperon decaying into spin-1/2 hyperon and pseudoscalar, 2) spin-1/2 hyperon decaying into spin-1/2 hyperon and pseudoscalar. If we neglect the widths of the initial and final particles, the CM momentum of the decay particles is fixed. The angular distribution is specified by two spherical angles \( \theta \) and \( \phi \), which give the direction of the final baryon in the helicity frame of the initial hyperon. The spin configuration of the final system is fully specified by the spin density matrix of the final baryon, which has spin 1/2 in both cases, since the accompanying particle is a pseudoscalar meson. Let us start considering a decay of type 1).

The aim is to relate the basis matrices of the mother hyperon \( Q_\mu \) to those of the daughter baryon \( \sigma^d_\nu \). In other words one has to find the transition matrix \( b_{\mu \nu} \) such that:

\[ Q_\mu \rightarrow \sum_{\nu=0}^3 b_{\mu \nu} \sigma^d_\nu. \quad (39) \]

The \( 16 \times 4 \) \( b_{\mu \nu} \) matrix depends only on the final baryon \( \theta \) and \( \phi \) angles, and on the decay parameters of the considered decay mode. If the initial particle density matrix is given by Eq. (20) then the final baryon density matrix is:

\[ \rho^{d}_{1/2} = \sum_{\mu=0}^3 \sum_{\nu=0}^3 c_{\mu \nu} \sigma^d_\nu. \quad (40) \]

The differential cross section is simply obtained by taking the trace of \( \rho^{d}_{1/2} \):

\[ \text{Tr} \rho^{d}_{1/2} = \sum_{\mu=0}^3 r_{\mu} b_{\mu,0}. \quad (41) \]

Let us now consider a decay of type 2). Similarly we introduce a \( 4 \times 4 \) matrix \( a_{\mu \nu} \) which allows us to express the \( \sigma^d_\nu \) matrices in the mother helicity frame in terms of \( \sigma^d_\nu \) matrices in the daughter helicity frame:

\[ \sigma_\mu \rightarrow \sum_{\nu=0}^3 a_{\mu \nu} \sigma^d_\nu. \quad (42) \]

The decay matrices \( a_{\mu \nu} \) and \( b_{\mu \nu} \) introduced above allow to keep track of the spin correlation between the decay products of the \( B_1 \) and \( B_2 \) decays chains.

In the following example we start from the two-particle \( 1/2 + 3/2 \) density matrix given by Eq. (28). After the \( B_1 \) decay (1/2 \( \rightarrow \) 1/2 + 0) the density matrix is transformed into

\[ \rho^{(f)}_{1/2,3/2} = \frac{1}{2} \sum_{\mu=0}^3 \sum_{\nu=0}^3 C_{\mu \nu} \left( \sum_{\kappa=0}^{3} a_{\mu \kappa} \sigma^d_\kappa \right) \otimes Q_\nu, \quad (43) \]

where the \( \sigma^d_\nu \) matrices act in the daughter helicity frame. Correspondingly after the \( B_2 \) decay (3/2 \( \rightarrow \) 1/2 + 0) the density matrix would read:

\[ \rho^{(f)}_{1/2,3/2} = \frac{1}{2} \sum_{\mu=0}^3 \sum_{\nu=0}^3 C_{\mu \nu} \sigma_\mu \otimes \left( \sum_{\kappa=0}^{3} b_{\nu \kappa} \sigma^d_\kappa \right). \quad (44) \]

Below we provide the explicit expression for the decay matrices \( a_{\mu \nu} \) and \( b_{\mu \nu} \). Consider a \( J = 1/2 \) or \( J = 3/2 \) hyperon (with initial helicity \( \kappa \)) decaying into a \( J = 1/2 \) baryon (with helicity \( \lambda_1 = \lambda \)) and a pseudoscalar particle (\( \lambda_2 = 0 \)). By evaluating the transition operator between the initial hyperon and the daughter baryon state one gets:

\[ \langle \theta, \phi, \lambda | S | 0, 0, \kappa \rangle_m = a_{\theta, \phi, \lambda} \langle \theta, \phi, \lambda | J, \lambda \rangle_d \]
\[ \times \langle J, \lambda | S | J, \kappa \rangle_m \times \langle J, \kappa | 0, 0, \kappa \rangle_m \]
where the angles \( \theta \) and \( \phi \) are given with respect to the helicity frame of the mother hyperon \( \eta \). The amplitude \( B_\lambda = d(\lambda, \lambda | S, J, \kappa)_m \) depends only on the helicity of the daughter baryon and it is therefore called helicity amplitude. Recalling also Eq. (6) the transition amplitude becomes:

\[
d(\theta, \phi, \lambda | S, 0, \kappa)_m \propto D^{1/2}_{\kappa, \lambda}(\Omega) B_\lambda,
\]

where \( D^{1/2}_{\kappa, \lambda}(\Omega) = D^{1/2}_{\kappa, \lambda}(\phi, \theta, 0) \). The coefficients \( a_{\mu\nu} \) are then obtained by multiplying the amplitude above by its conjugate and inserting basis \( \sigma \) matrices for the mother and the daughter baryon:

\[
a_{\mu\nu} = \frac{1}{4\pi} \sum_{\lambda, \lambda' = -1/2}^{1/2} B_\lambda B_{\lambda'}^* \times 
\sum_{\kappa, \kappa' = -1/2}^{1/2} (\sigma_\mu)^{\kappa, \kappa'} (\sigma_\nu)^{\lambda', \lambda} D^{1/2}_{\kappa, \lambda}(\Omega) D^{1/2}_{\kappa', \lambda'}(\Omega).
\]

These coefficients can be rewritten in terms of the decay parameters \( \alpha_D \) and \( \phi_D \) defined in Ref. [9]. For completeness we first relate the helicity amplitudes to the \( S \) and \( P \) wave amplitudes \( A_S \) and \( A_P \), corresponding respectively to the parity violating and parity conserving transitions. If a hyperon of spin \( J \) decays (weakly) into a hyperon of spin \( S \) and a (pseudo)scalar state, then the relation between helicity amplitudes and canonical amplitudes is given by [10]

\[
B_\lambda = \sum_L \left( \frac{2L + 1}{2J + 1} \right)^{1/2} (L, 0; S, \lambda | J, \lambda) A_L
\]

where \( (s_1, m_1, s_2, m_2 | s, m) \) is a Clebsch-Gordan coefficient. For \( J = S = 1/2 \) the helicity amplitudes are\(^3\)

\[
B_{-1/2} = \frac{A_S + A_P}{\sqrt{2}},
B_{1/2} = \frac{A_S - A_P}{\sqrt{2}}.
\]

Using the normalization \( |A_S|^2 + |A_P|^2 = |B_{-1/2}|^2 + |B_{1/2}|^2 = 1 \), the relation between helicity amplitudes and the decay parameters is:

\[
\begin{align*}
\alpha_D &= -2 \Re (A_S^* A_P) = |B_{1/2}|^2 - |B_{-1/2}|^2, \\
\beta_D &= -2 \Im (A_S^* A_P) = 2 \Im (B_{1/2} B_{-1/2}^*) , \\
\gamma_D &= |A_S|^2 - |A_P|^2 = 2 \Re (B_{1/2} B_{-1/2}^*).
\end{align*}
\]

where \( \beta_D = \frac{\sqrt{1 - \alpha_D^2}}{\alpha_D} \sin \phi_D \) and \( \gamma_D = \frac{\sqrt{1 - \alpha_D^2}}{\alpha_D} \cos \phi_D \). The non-zero elements of the decay matrix \( a_{\mu\nu} \) are (where an overall \( \frac{1}{12} \) factor is omitted):

\[
\begin{align*}
a_{00} &= 1, \\
a_{03} &= \alpha_D, \\
a_{10} &= \alpha_D \cos \phi \sin \theta, \\
a_{11} &= \gamma_D \cos \theta \cos \phi - \beta_D \sin \phi, \\
a_{12} &= -\beta_D \cos \theta \cos \phi - \gamma_D \sin \phi, \\
a_{13} &= \sin \theta \cos \phi, \\
a_{20} &= \alpha_D \sin \theta \sin \phi, \\
a_{21} &= \beta_D \cos \phi + \gamma_D \cos \theta \sin \phi, \\
a_{22} &= \gamma_D \cos \phi - \beta_D \cos \theta \sin \phi, \\
a_{23} &= \sin \theta \sin \phi, \\
a_{30} &= \alpha_D \cos \theta, \\
a_{31} &= -\gamma_D \sin \theta, \\
a_{32} &= \beta_D \sin \theta, \\
a_{33} &= \cos \theta.
\end{align*}
\]

Analogously, the elements of the \( b_{\mu\nu} \) matrix are given by

\[
\begin{align*}
b_{\mu\nu} &= \frac{1}{2} \sum_{\lambda, \lambda' = -1/2}^{1/2} B_\lambda B_{\lambda'}^* 
\times \sum_{\kappa, \kappa' = -3/2}^{3/2} (Q_\mu)^{\kappa, \kappa'} (\sigma_\nu)^{\lambda', \lambda} D^{3/2}_{\kappa, \lambda}(\Omega) D^{3/2}_{\kappa', \lambda'}(\Omega).
\end{align*}
\]

Out of 64 \( b_{\mu\nu} \) coefficients 12 are zero. The coefficients relevant for the inclusive distributions are presented in Eq. (54) as a part of an example in section V. The remaining coefficients are straightforward to obtain. As before, we first rewrite the helicity amplitudes in terms of the canonical amplitudes using Eq. (47):

\[
\begin{align*}
B_{-1/2} &= \frac{A_P + A_D}{\sqrt{2}}, \\
B_{1/2} &= \frac{A_P - A_D}{\sqrt{2}}.
\end{align*}
\]

In this case the \( P \) and \( D \) amplitudes \( A_P \) and \( A_D \) are the contributing ones. The definition of the decay parameters \( \alpha_D, \beta_D \) and \( \gamma_D \) is analogous to that of Eq. (49):

\[
\begin{align*}
\alpha_D &= -2 \Re (A_P^* A_D) = |B_{1/2}|^2 - |B_{-1/2}|^2, \\
\beta_D &= -2 \Im (A_P^* A_D) = 2 \Im (B_{1/2} B_{-1/2}^*), \\
\gamma_D &= |A_P|^2 - |A_D|^2 = 2 \Re (B_{1/2} B_{-1/2}^*).
\end{align*}
\]

Again, they can be expressed in terms of the parameters \( \alpha_D \) and \( \phi_D \).

**V. EXAMPLES**

We provide two examples with the aim to clarify how to apply our method: 1) inclusive reaction

---

\(^3\) Note that the Particle Data Group [9] uses \( -A_P = A_P^{\text{PDG}} \).
\( e^+ e^- \rightarrow \Omega^- \Omega^+ \), where only the decay products of the \( \Omega^- \) are measured, 2) exclusive reaction \( e^+ e^- \rightarrow \Lambda \bar{\Lambda} \), where the two spin-1/2 hyperons decay in a one-step process and all decay products are measured.

In the first example the \( \Omega^- \) produced in the \( e^+ e^- \rightarrow \Omega^- \bar{\Omega}^+ \) reaction is identified using the following sequence of decays: (a) \( \Omega^- \rightarrow \Lambda K^- \) and (b) \( \Lambda \rightarrow p \pi^- \).

To describe the decay chain we introduce helicity reference frames and the spherical coordinates \((\theta_\Lambda, \phi_\Lambda)\) and \((\theta_p, \phi_p)\) for the \( \Lambda \) and \( p \) directions, respectively. The scattering angle of \( \Omega \) in the overall CM system is denoted as \( \theta_\Omega \). The density matrix of \( \Omega^- \) is given by Eq. (20):

\[
\rho_\Omega = \sum_{\mu=0}^{15} r_\mu (\theta_\Omega; h_1, h_2, h_3, h_4) Q_\mu,
\]

where only seven real coefficients \( r_\mu \) are non-zero and are given by Eq. (38). The \( r_\mu \) parameters depend on the scattering angle \( \theta_\Omega \) and on four complex form factors. If we are not interested in the overall normalization then only six real parameters are enough to describe the \( \Omega \) production process. They have to be determined by fitting to the experimental data. The density matrix of the \( \Lambda \) coming from the \( \Omega^- \rightarrow \Lambda K^- \) decay can be obtained from Eq. (40):

\[
\rho_\Lambda = \sum_{\mu=0}^{15} \sum_{\nu=0}^{3} r_\mu \cdot b_\mu^\Omega \cdot (\theta_\Lambda, \phi_\Lambda; \alpha_\Omega, \beta_\Omega, \gamma_\Omega) \sigma_\nu^\Lambda,
\]

by (where an overall \( \frac{1}{8\pi} \) factor is omitted):

\[
\begin{align*}
\b_{0,0}^\Omega & = 1, \\
\b_{1,3}^\Omega & = \sqrt{\frac{3}{5}} \sin \theta_\Lambda \sin \phi_\Lambda, \\
\b_{6,0}^\Omega & = -\frac{\sqrt{3}}{4} (3 \cos 2\theta_\Lambda + 1), \\
\b_{7,0}^\Omega & = -3 \sin \theta_\Lambda \cos \theta_\Lambda \cos \phi_\Lambda, \\
\b_{8,0}^\Omega & = -\frac{3}{2} \sin^2 \theta_\Lambda \cos 2\phi_\Lambda, \\
\b_{10,3}^\Omega & = -9 \sin^2 \theta_\Lambda \cos \phi_\Lambda \sin \phi_\Lambda, \\
\b_{11,3}^\Omega & = -\frac{9}{4\sqrt{10}} (5 \cos 2\theta_\Lambda + 3) \sin \phi_\Lambda \sin \theta_\Lambda,
\end{align*}
\]

Finally including also the last decay of the chain, \( \Lambda \rightarrow p \pi^- \), the proton density matrix in the proton helicity frame can be obtained:

\[
\rho_p = \sum_{\mu=0}^{15} \sum_{\nu=0}^{3} r_\mu \cdot b_\mu^\Omega \cdot a_\nu^\Lambda (\theta_p, \phi_p; \alpha_\Lambda, \beta_\Lambda, \gamma_\Lambda) \sigma_\nu^p.
\]

Since the proton polarization is not measured, we are only interested in the trace of the density matrix \( \text{Tr} \rho_p \), which gives the differential distribution of the final state specified by the five kinematic variables \( \xi := (\cos \theta_\Omega, \cos \theta_\Lambda, \cos \theta_p, \phi_\Lambda, \phi_p) \):

\[
\frac{d\Gamma}{d\xi} = 2 \sum_{\mu=0}^{15} \sum_{\nu=0}^{3} r_\mu b_\mu^\Omega \phi_\nu^\Lambda,
\]
where the relevant \( a_{\nu_0}^{\Lambda,0} \) can be directly taken from Eq. (50).

Moving to the second example, we start from the two-particle density matrix for the \( \Lambda-\bar{\Lambda} \) pair coming from the \( e^+e^- \rightarrow \Lambda\bar{\Lambda} \) reaction, which is given by Eq. (22). After considering the subsequent two-body weak decays into \( p\pi^-/\bar{p}\pi^+ \), the joint angular distribution of the \( p/\bar{p} \) pair is given within the present formalism as:

\[
\text{Tr}[\rho_{p\bar{p}}] \propto \sum_{\mu,\nu=0}^3 C_{\mu\nu} a_{\mu0}^{\Lambda,0} \bar{a}_0^{\Lambda,0},
\]  

(55)

with the \( a_{\mu0} \) matrices given by Eq. (50), where only decay asymmetries \( a^{\Lambda}/\bar{a}^{\Lambda} \) for \( \Lambda/\bar{\Lambda} \) enter.

VI. FORM FACTORS AND HELICITY AMPLITUDES

We follow the definitions of [24] for constraint-free form factors. When relating them to the helicity amplitudes we use the conventions of [10]. This makes our helicity amplitudes \( A_{\lambda_1,\lambda_2} \) somewhat different from the expressions \( \Gamma^{\lambda,\lambda} \) of [24].

The form factors for a particle-antiparticle pair of spin 1/2 and mass \( m \) are introduced by

\[
\langle B(p_2, \lambda_2) \bar{B}(p_1, \lambda_1) \mid j_\mu(0) \rangle = \bar{u}(p_2, \lambda_2) \Gamma_\mu v(p_1, \lambda_1)
\]

(56)

with the electromagnetic current

\[
j_\mu = \frac{2}{3} u_\gamma \gamma u - \frac{1}{3} \bar{u}_\gamma \gamma \bar{d} - \frac{1}{3} s_\gamma \gamma s + \ldots
\]

(57)

and [24]

\[
\Gamma_\mu := F_1(q^2) \gamma_\mu + F_2(q^2) \frac{i \sigma_{\mu\nu} q^\nu}{2m}
\]

(58)

where \( q = p_1 + p_2 \) denotes the momentum of the virtual photon.

These form factors are related to the helicity amplitudes by

\[
A_{+1/2, +1/2} = 2m(F_1 + \tau F_2),
\]

\[
A_{+1/2, -1/2} = \sqrt{2}q^2(F_1 + F_2),
\]

(59)

where \( \tau = \frac{q^2}{2m^2} \). We have defined

\[
A_{\lambda_1,\lambda_2} := \sqrt{\frac{3}{4\pi}} (J = 1, M; \lambda_1, \lambda_2) \mid j(M) \rangle \mid 0\rangle
\]

(60)

with

\[
j(M = +1) := -\frac{1}{\sqrt{2}} (j^1 + ij^2),
\]

\[
j(M = 0) := j^3,
\]

(61)

\[
j(M = -1) := \frac{1}{\sqrt{2}} (j^1 - ij^2).
\]

In the following we will stick to the more compact notation for the helicity form factors from Sect. III: \( A_{1/2,1/2} = h_1, A_{1/2, -1/2} = h_2 \) etc. Close to threshold \( \tau \approx 1 \) one finds

\[
h_1 \approx \frac{1}{\sqrt{2}} h_2.
\]

(62)

The form factors for a particle-antiparticle pair of spin 3/2 and mass \( m \) are given by

\[
\langle B'(p_2, \lambda_2) \bar{B}'(p_1, \lambda_1) \mid j_\mu(0) \rangle = \bar{u}^\alpha(p_2, \lambda_2) \Gamma_{\alpha\beta} v^\beta(p_1, \lambda_1)
\]

(63)

with [24]

\[
\Gamma_{\alpha\beta} := g_{\alpha\beta} \left( F_1(q^2) \gamma_\mu + F_2(q^2) \frac{i \sigma_{\mu\nu} q^\nu}{2m} \right) + \frac{g_{\alpha\beta}}{m^2} \left( F_3(q^2) \gamma_\mu + F_4(q^2) \frac{i \sigma_{\mu\nu} q^\nu}{2m} \right).
\]

(64)

These form factors are related to the helicity amplitudes by

\[
h_1 = 2m \left( 1 - \frac{4}{3} \tau \right) (F_1 + \tau F_2)
\]

\[
+ 2m \frac{4}{3} \tau (1 - \tau) (F_3 + \tau F_4),
\]

\[
h_2 = -\frac{2}{3} \sqrt{2q^2} \left[-(1 - 2\tau) (F_1 + F_2)
\]

\[
- 2\tau (1 - \tau) (F_3 + F_4) \right],
\]

\[
h_3 = \sqrt{\frac{2}{3}} \sqrt{2q^2} (F_1 + F_2),
\]

\[
h_4 = 2m (F_1 + \tau F_2).
\]

(65)

Close to threshold \( \tau \approx 1 \) one finds

\[
h_4 \approx -3h_1 \approx \frac{\sqrt{3}}{2} h_3 \approx -\frac{3}{\sqrt{8}} h_2.
\]

(66)

Transition form factors for a particle with \( J^P = \frac{3}{2}^+ \), mass \( M \) and an antiparticle with \( J^P = \frac{1}{2}^- \), mass \( m \) are encoded in

\[
\langle B'(p_2, \lambda_2) \bar{B}(p_1, \lambda_1) \mid j_\mu(0) \rangle = \bar{u}^\nu(p_2, \lambda_2) \Gamma_{\nu\mu} v(p_1, \lambda_1)
\]

(67)

with [24]

\[
\Gamma_{\nu\mu} := G_1(q^2) (q^\nu \gamma^\mu - q^\mu \gamma^\nu) \gamma_5
\]

\[
+ G_2(q^2) (q^\nu p_2^\mu - (q \cdot p_2) g^\nu \mu) \gamma_5
\]

\[
+ G_3(q^2) (q^\nu q^\mu - q^2 g^\nu \mu) \gamma_5.
\]

(68)

These form factors are related to the helicity ampli-
tudes by

\[ h_1 = \sqrt{\frac{2}{3}} N \sqrt{q^2} \left( G_1 + MG_2 + \frac{q^2 + M^2 - m^2}{2M} G_3 \right), \]
\[ h_2 = \frac{1}{\sqrt{3}} N \left( \frac{q^2 - m(m + M)}{M} G_1 + \frac{q^2 + M^2 - m^2}{2} G_2 + q^2 G_3 \right), \]
\[ h_3 = N \left( (m + M) G_1 + \frac{q^2 + M^2 - m^2}{2} G_2 + q^2 G_3 \right), \]

with a “normalization factor”

\[ N(q^2) := \sqrt{q^2 - (M - m)^2}. \]

Close to threshold, \( q^2 \approx (m + M)^2 \), one finds

\[ h_1 \approx \sqrt{2} h_2 \approx \sqrt{\frac{2}{3}} h_3. \] (71)

To facilitate the matching between Feynman matrix elements and expressions in the helicity framework of Jacob and Wick [10] we provide in appendix B some explicit formulas for the particle and antiparticle spinors.

VII. DISCUSSION

We would like to draw attention to some interesting properties of the derived angular distributions close to threshold. For the production of two spin-1/2 baryons the parameters \( \alpha_\phi \) and \( \Delta \Phi \) are zero at threshold. Therefore, there is no spin polarization implying the inclusive distributions of the decay products are isotropic. For the spin 3/2 + 3/2 production the baryons are polarized even at threshold. The inclusive distributions of the decay products would be isotropic if \( r_0 = 1 \) (assuming normalization \( |h_2|^2 + 2(|h_1|^2 + |h_3|^2 + |h_4|^2) = 1 \)) and all other \( r_i \) terms were zero in Eq. (38). Using the close-to-threshold relation between the form factors from Eq. (66) one sees that three additional terms are not zero:

\[ r_6 = \frac{1}{5 \sqrt{3}} (1 - 3 \cos^2 \theta_1), \]
\[ r_7 = \frac{1}{5} \sin 2\theta_1, \]
\[ r_8 = -\frac{1}{5} \sin^2 \theta_1. \] (72)

An inclusive distribution that is only differential in the production angle is not sensitive to these parameters. Indeed, \( \alpha_\phi \) as introduced in (37) vanishes at threshold. However, distributions differential in the angles of decay products are sensitive. It is not even necessary that the decay is parity violating. If one assumed that the decay \( 3/2 \rightarrow 1/2 + 0 \) would be parity conserving, implying \( \gamma_D = 1 \), then the angular distribution of the decay products is already not isotropic:

\[ \frac{d\Gamma}{d\cos \theta_1 d\cos \theta_D} \propto 1 + \frac{(1 - 3 \cos^2 \theta_1)(1 - 3 \cos^2 \theta_D)}{10}. \]

This property of the reaction close to threshold could be used to establish spin assignment of the produced baryons by studying inclusive angular distributions. One possible test is to calculate the moment \((1 - 3 \cos^2 \theta_D) \), where \( \theta_D \) is the helicity angle of the daughter baryon. For the spin 1/2 + 1/2 reaction this quantity is zero.

The above observation could be also expressed using the degree of polarization, which is defined for a spin 3/2 particle as [17]:

\[ d(3/2) = \sqrt{\frac{1}{15} \sum_{\mu=1}^{15} \left( \frac{r_\mu}{r_0} \right)^2}. \] (73)

It is easy to check that at threshold \( d(3/2) = \frac{2}{\sqrt{3}} \approx 23\% \), if the baryon-antibaryon pair is produced in an \( e^+e^- \) process.

ACKNOWLEDGMENTS

We would like to thank Changzheng Yuan for initiating this project and for the support. We are grateful to Patrik Adlarson for useful discussions. AK would like to thank Shuangshi Fang for support for the visit at IHEP and acknowledges grant of Chinese Academy of Science President’s International Fellowship Initiative (PIFI) for Visiting Scientist.

Appendix A: Spin \( \frac{1}{2} \) basis matrices

To describe a spin-3/2 particle density matrix the following set of \( Q_L^L \) matrices with \( 0 \leq L \leq 3 \) and \(-3 \leq M \leq L \) is needed, in total 16 \( 4 \times 4 \) matrices. The matrices are introduced in Ref. [17]. \( Q_0^0 = \frac{1}{4} \mathbb{I} \) where \( \mathbb{I} \) is the identity \( 4 \times 4 \) matrix. We use the following notation with only one index to denote the matrices:

\[ Q_L (L+1) + M := \frac{3}{4} Q_L^L. \] (A1)

Given the index \( \mu \) belonging to the matrix \( Q_{\mu} \), the corresponding values of \( M \) and \( L \) can be easily retrieved:

\[ \begin{align*}
\mu = 0 & : L = 0, \quad M = 0, \\
1 \leq \mu \leq 3 & : L = 1, \quad -1 \leq M \leq 1, \\
4 \leq \mu \leq 8 & : L = 2, \quad -2 \leq M \leq 2, \\
9 \leq \mu \leq 15 & : L = 3, \quad -3 \leq M \leq 3.
\end{align*} \] (A2)
Below the explicit expressions for the $Q_{ij}^L$ matrices are provided:

\[ Q_{1-1}^1 = \frac{i}{\sqrt{3}} \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & -\frac{1}{\sqrt{3}} & 0 \\ 0 & \frac{2}{\sqrt{3}} & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \]

\[ Q_{0}^1 = \sqrt{\frac{2}{3}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 & -\frac{1}{\sqrt{3}} \\ 0 & 0 & -\frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \]

\[ Q_{1}^1 = \frac{1}{\sqrt{3}} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & \frac{2}{\sqrt{3}} & 0 \\ 0 & \frac{2}{\sqrt{3}} & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \]

\[ Q_{2-2}^1 = \frac{i}{\sqrt{3}} \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \]

\[ Q_{2-1}^1 = \frac{i}{\sqrt{3}} \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \]

\[ Q_{0}^2 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \]

\[ Q_{1}^2 = \frac{1}{\sqrt{3}} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \]

\[ Q_{2}^2 = \frac{1}{\sqrt{3}} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \]

\[ Q_{3}^2 = i\sqrt{\frac{2}{3}} \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \]

\[ Q_{3}^3 = i\sqrt{\frac{2}{3}} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \]

Appendix B: Conventions for spin-1/2, spin-1 and spin-3/2 spinors for particles and antiparticles

Various conventions for spinors are used in the literature. Not all of them fit to the helicity framework of Jacob and Wick [10]. Therefore we provide here some explicit formulas for the spinors. To this end one has to be careful in the construction of the states denoted by type 2 in [10] as they are not obtained by just a rotation. As spelled out in [10], two-particle states flying in an arbitrary direction are obtained by two-particle states where state 1 flies in the (+z) direction and state 2 in the (−z) direction. In the following we present explicitly the spinors for the states 1 and 2 with which one starts. We use the Pauli-Dirac representation for the gamma matrices. For the spin-1/2 states with helicity $\lambda_{1,2}$, mass $m$, energy $E$, and mo-
menta $p_z$ or $-p_z$ ($p_z \geq 0$) one finds
\[
u(p_z, \pm 1/2) = \begin{pmatrix} \sqrt{E+m} \chi_+ \\ \pm \sqrt{E-m} \chi_+ \end{pmatrix},
\]
\[
u(-p_z, \pm 1/2) = \begin{pmatrix} \sqrt{E+m} \chi_- \\ \mp \sqrt{E-m} \chi_- \end{pmatrix},
\]
\[
\nu(p_z, \pm 1/2) = \begin{pmatrix} \sqrt{E-m} \chi_+ \\ \pm \sqrt{E+m} \chi_+ \end{pmatrix},
\]
\[
\nu(-p_z, \pm 1/2) = \begin{pmatrix} -\sqrt{E-m} \chi_- \\ \mp \sqrt{E+m} \chi_- \end{pmatrix}
\]
with the two-component spinors
\[
\chi_+ := \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \chi_- := \begin{pmatrix} 0 \\ 1 \end{pmatrix}.
\]

For the spin-1 states with helicity $\lambda_{1,2}$, mass $m$, energy $E$, and momenta $p_z$ or $-p_z$ ($p_z \geq 0$) we use
\[
\nu(p_z, +1) = \frac{1}{\sqrt{2}} (0, -1, -i, 0),
\]
\[
\nu(p_z, 0) = \frac{1}{m} (p_z, 0, 0, E),
\]
\[
\nu(p_z, -1) = \frac{1}{\sqrt{2}} (0, 1, i, 0),
\]
\[
\nu(-p_z, +1) = \frac{1}{\sqrt{2}} (0, 1, -i, 0),
\]
\[
\nu(-p_z, 0) = \frac{1}{m} (-p_z, 0, 0, E),
\]
\[
\nu(-p_z, -1) = \frac{1}{\sqrt{2}} (0, -1, i, 0).
\]

Finally, we present explicit expressions for the spin-3/2 states with helicity $\lambda_{1,2}$, mass $m$, energy $E$, and momenta $p_z$ or $-p_z$ ($p_z \geq 0$):
\[
\nu^\mu(p_z, \pm 3/2) = \epsilon^\mu(p_z, \pm 1) u(p_z, \pm 1/2),
\]
\[
\nu^\mu(p_z, \pm 1/2) = \frac{1}{\sqrt{3}} \epsilon^\mu(p_z, \pm 1) u(p_z, \mp 1/2)
\]
\[
\quad + \frac{\sqrt{2}}{3} \epsilon^\mu(-p_z, 0) u(p_z, \pm 1/2),
\]
\[
\quad + \frac{\sqrt{2}}{3} \epsilon^\mu(-p_z, 0) u(p_z, \pm 1/2),
\]
\[
\nu^\mu(-p_z, \pm 3/2) = \epsilon^\mu(-p_z, \pm 1) u(-p_z, \pm 1/2),
\]
\[
\nu^\mu(-p_z, \pm 1/2) = \frac{1}{\sqrt{3}} \epsilon^\mu(-p_z, \pm 1) u(-p_z, \mp 1/2)
\]
\[
\quad + \frac{\sqrt{2}}{3} \epsilon^\mu(-p_z, 0) u(-p_z, \pm 1/2),
\]
\[
\nu^\mu(p_z, \pm 3/2) = \epsilon^\mu(p_z, \pm 1) v(p_z, \pm 1/2),
\]
\[
\nu^\mu(p_z, \pm 1/2) = \frac{1}{\sqrt{3}} \epsilon^\mu(p_z, \pm 1) v(p_z, \mp 1/2)
\]
\[
\quad + \frac{\sqrt{2}}{3} \epsilon^\mu(p_z, 0) v(p_z, \pm 1/2),
\]
\[
\nu^\mu(-p_z, \pm 3/2) = \epsilon^\mu(-p_z, \pm 1) v(-p_z, \pm 1/2),
\]
\[
\nu^\mu(-p_z, \pm 1/2) = \frac{1}{\sqrt{3}} \epsilon^\mu(-p_z, \pm 1) v(-p_z, \mp 1/2)
\]
\[
\quad + \frac{\sqrt{2}}{3} \epsilon^\mu(-p_z, 0) v(-p_z, \pm 1/2).
\]

In general, if one takes a state flying in the $(+z)$ direction and applies to it just a rotation by $\pi$ around the $y$ axis, then the result differs by the Jacob/Wick construction by a factor $(-1)^{s_2 - \lambda_2}$. Thus for $s_2 = 1/2$ one picks up a minus sign for $\lambda_2 = -1/2$ while for $s_2 = 3/2$ one picks up a minus sign for $\lambda_2 = +1/2, -3/2$.

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