PERIODIC AND FALLING-FREE MOTION OF INVERTED SPHERICAL PENDULUM WITH MOVING PIVOT POINT

IVAN POLEKHIN

ABSTRACT. For the system of inverted spherical pendulum with friction and periodically moving pivot point we prove existence of at least one periodic solution with additional property of being falling-free. The last means that pendulum never becomes horizontal along considered periodic solution. Presented proof is an application of some recent results in the fixed point theory.

1. Introduction

One of the problems originally presented in What is mathematics? book by Courant and Robbins [CR] is stated as follows:

Suppose a train travels from station A to station B along a straight section of track. The journey need not be of uniform speed or acceleration. The train may act in any manner, speeding up, slowing down, coming to a halt, or even backing up for a while, before reaching B. But the exact motion of the train is supposed to be known in advance; that is, the function \( s = f(t) \) is given, where \( s \) is the distance of the train from station A, and \( t \) is the time, measured from the instant of departure. On the floor of one of the cars a rod is pivoted so that it may move without friction either forward or backward until it touches the floor. If it does touch the floor, we assume that it remains on the floor henceforth; this will be the case if the rod does not bounce. Is it possible to place the rod in such a position that, if it is released at the instant when the train starts and allowed to move solely under the influence of gravity and the motion of the train, it will not fall to the floor during the entire journey from A to B?

As an exercise, authors suggest to prove a more general result where spherical inverted pendulum is considered instead of planar pendulum. We consider slightly different problem introducing viscous friction force acting on the mass point of the pendulum and prove that for an arbitrary small friction coefficient periodic solution always exists. Moreover, we show that considered periodic solution never approaches horizontal plane.

In the first section, we present the system of governing dynamical equations for the system and prove some properties which we are going to use in the second section where we apply some recent results by Srzednicki, Wójcik and Zgliczynski [SWZ] to our system.

2. Governing equations

Let \( Oxyz \) be an orthogonal moving reference frame such that \( O \) coincide with the pivot point of the pendulum, \( Ox \) and \( Oy \) axis are in the horizontal plane and always remain parallel

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to themselves at the initial moment of time; $Oz$ is vertical and oriented in an opposite way to the gravitational force. By $r_{moving}$ we denote radius vector of the mass point in $Oxyz$. Let $x$, $y$, and $z$ be its components. Since pivot point moves periodically in the horizontal plane, then for the radius vector $r_{fixed}$ of the mass point in some fixed reference frame with axis parallel to the axis of $Oxyz$ we have

$$r_{fixed} = r_{moving} + \rho,$$

where $\rho = \xi e_x + \eta e_y$, where we assume $\xi, \eta$ to be $2\pi$-periodic smooth functions. Obviously, we have similar relations for the velocities and accelerations

$$\dot{r}_{fixed} = \dot{r}_{moving} + \dot{\rho}, \quad \ddot{r}_{fixed} = \ddot{r}_{moving} + \ddot{\rho}.$$ 

Mass point moves under the action of gravitational force, viscous friction force, and constraint force

$$m\ddot{r}_{fixed} = F_{grav} + F_{friction} + N. \quad (1)$$

Here $N$ is the constraint force which is parallel to the radius vector $r_{moving}$.

$$N = |N| \frac{r_{moving}}{|r_{moving}|} = |N| e_n. \quad (2)$$

2.1. Remark. Forces of constraint are the forces which allows one to consider system with constraints as constraint-free with additional unknown *apriori* forces. More on constraint forces and their use in mechanics one can find in [GPS].

We assume that friction force is determined by the following model

$$F_{friction} = -\gamma(\dot{r}_{fixed} - \dot{\rho}) = -\gamma \dot{r}_{moving}, \quad \gamma > 0. \quad (3)$$

We can now rewrite (1) as follows

$$m\ddot{r}_{moving} = |N| e_n - mge_z - m\ddot{\rho} - \gamma \dot{r}_{moving}. \quad (4)$$

2.2. Remark. For the above equation (3), the solution function $r_{moving}(\cdot, r_0, \dot{r}_0) : \mathbb{R} \to S^2$ is defined by the initial conditions $r_0$ and $\dot{r}_0$. 

**Figure 1.** Fixed and moving reference frames
Since functions $\xi$ and $\eta$ are $2\pi$-periodic, then the extended phase space of our system can be considered as $\mathbb{R}/2\pi\mathbb{Z} \times TS^2$. Let $F: \mathbb{R}/2\pi\mathbb{Z} \times TS^2 \to \mathbb{R}$ be a function defined by the following equation

$$F = \frac{m}{2}(\dot{r}_{\text{moving}}, \dot{r}_{\text{moving}}).$$

(4)

Now consider submanifold $F = c$ of the extended phase space and show that if $c > 0$ is large, then along the solutions starting at $F = c$ function $F$ is locally decreasing. More specifically,

2.3. Lemma. There exists $c > 0$ such that

$$\dot{F} \bigg|_{F=c} < 0.$$

Proof. From the definition of $F$ and (1) we easily have

$$\dot{F} = m(\dot{r}_{\text{moving}} - \dot{\rho}, \dot{r}_{\text{moving}})$$

$$= (F_{\text{grav}} + F_{\text{friction}} + N, \dot{r}_{\text{moving}}) - m(\dot{\rho}, \dot{r}_{\text{moving}}).$$

Therefore, taking into account (2) and that $|\dot{r}_{\text{moving}}| = (2c/m)^{1/2}$, we obtain

$$\dot{F} = (F_{\text{grav}} - \gamma|\dot{r}_{\text{moving}}|^2 - m(\dot{\rho}, \dot{r}_{\text{moving}}$$

$$\leq |F_{\text{grav}}| - \gamma|\dot{r}_{\text{moving}}|^2 + m|\dot{\rho}||\dot{r}_{\text{moving}}|$$

$$= (2c/m)^{1/2}|F_{\text{grav}}| - 2c\gamma/m + (2cm)^{1/2}|\dot{\rho}|.$$

Since $|\dot{\rho}|$ is bounded, then for $c > 0$ sufficiently large we have $\dot{F} < 0$. 

Now we prove that if pendulum as well as its velocity vector is in the horizontal plane, then at least locally mass point is falling down, i.e. the following lemma is true.

2.4. Lemma. If solution $r_{\text{moving}}$ of (4) at the time $t$ satisfies $(r_{\text{moving}}(t), e_z) = z(t) = 0$ and $(\dot{r}_{\text{moving}}(t), e_z) = \dot{z}(t) = 0$, then $(\dot{r}_{\text{moving}}(t), e_z) = \ddot{z}(t) < 0$.

Proof. For the pendulum being in the horizontal position and moving horizontally as well, we have the following:

1. friction force is always parallel to the radius vector $\dot{r}_{\text{moving}}$, therefore at the given conditions friction force is directed horizontally;
2. constraint force is in the horizontal plane if $z = 0$;
3. $Oxyz$ is moving along the horizontal plane, therefore $(\dot{\rho}, e_z) = 0$.

Figure 2. Forces acting on the mass point when pendulum is horizontal.
Taking into account the above, from \([\mathbf{1}]\) we obtain
\[
(m\ddot{r}_{\text{moving}}, e_z) = (F_{\text{grav}} + F_{\text{friction}} + N - \dot{\rho}, e_z)
\]
\[
= -mg - \gamma(\dot{r}_{\text{fixed}} - \dot{\rho}, e_z) + (N, e_z) - (\dot{\rho}, e_z) = -mg < 0.
\]
\[
\square
\]

3. Main result

In this section we are going to apply some recent developments in fixed point theory by Srzednicki, Wójcik and Zgliczynski [SWZ] to our system and prove existence of falling-free periodic solutions. First, following [SWZ] we introduce some definitions slightly make up for our use.

From now on, we assume that \(v: \mathbb{R} \times M \to TM\) is a smooth time-dependent vector-field on a manifold \(M\).

3.1. Definition. For \(t_0 \in \mathbb{R}\) and \(x_0 \in M\), the map \(t \mapsto x(t, t_0, x_0)\) is the solution for the initial value problem for the system \(\dot{x} = v(t, x)\), such that \(x(0, t_0, x_0) = x_0\).

3.2. Definition. Let \(W \subset \mathbb{R} \times M\). Define the exit set \(W^-\) as follows. A point \((t, x)\) is in \(W^-\) if there exists \(\delta > 0\) such that \((t + t_0, x(t, t_0, x_0)) \notin W\) for all \(t \in (0, \delta)\).

3.3. Definition. We call \(W \subset \mathbb{R} \times M\) a Ważewski block for the system \(\dot{x} = v(t, x)\) if \(W\) and \(W^-\) are compact.

Now introduce some notations. By \(\pi_1\) and \(\pi_2\) we denote the projections of \(\mathbb{R} \times M\) onto \(\mathbb{R}\) and \(M\) respectively. If \(Z \subset \mathbb{R} \times M, t \in \mathbb{R}\), then we denote \(Z_t = \{z \in M: (t, z) \in Z\}\).

3.4. Definition. A set \(W \subset [a, b] \times M\) is called a segment over \([a, b]\) if it is a block with respect to the system \(\dot{x} = v(t, x)\) and the following conditions hold:

- there exists a compact subset \(W^{--}\) of \(W^-\) called the essential exit set such that \(W^- = W^{--} \cup \{b\} \times W_b\), \(W^- \cap ([a, b) \times M) \subset W^{--}\),
- there exists a homeomorphism \(h: [a, b] \times W_a \to W\) such that \(\pi_1 \circ h = \pi_1\) and \(h([a, b] \times W^{--}_a) = W^{--}\). (5)

3.5. Definition. Let \(W\) be a segment over \([a, b]\). It is called periodic if \((W_a, W^{--}_a) = (W_b, W^{--}_b)\).

3.6. Definition. For periodic segment \(W\), we define the corresponding monodromy map \(m\) as follows

\[
m: (W_a, W^{--}_a) \to (W_a, W^{--}_a), \quad m(x) = \pi_2 h(a, \pi_2 h^{-1}(a, x)).
\]

3.7. Remark. Monodromy map \(m\) is a homeomorphism. Moreover, it can be proved that a different choice of \(h\) satisfying \([5]\) leads to the monodromy map homotopic to \(m\). It follows that the isomorphism in homologies

\[
\mu_W = H(m): H(W_a, W^{--}_a) \to H(W_a, W^{--}_a)
\]

is an invariant of \(W\).
3.8. Theorem. [SWZ] Let $W$ be a periodic segment over $[a, b]$. Then the set
\[ U = \{ x_0 \in W_a : x(t - a, a, x_0) \in W_t \setminus W_t^- \text{ for all } t \in [a, b] \} \]
is open in $W_a$ and the set of fixed point of the restriction $x(b - a, a, \cdot)|_U : U \to W_a$ is compact. Moreover, if $W$ and $W^-$ are ANRs then
\[ \text{ind}(x(b - a, a, \cdot)|_U) = \Lambda(\mu_W). \]

Where by $\Lambda(\mu_W)$ we denote the Lefschetz number of $\mu_W$. In particular, if $\Lambda(\mu_W) \neq 0$ then $x(b - a, a, \cdot)$ has a fixed point in $W_a$.

Using the above theorem and lemmas from the previous part, we can now prove that there exists periodic falling-free solution of (3).

3.9. Theorem. For any $\gamma > 0$ for the system (3), there exist $r_0$ and $\dot{r}_0$ such that solution $r_{\text{moving}}(t, r_0, \dot{r}_0)$ with the initial conditions $r_{\text{moving}}(0, r_0, \dot{r}_0) = r_0$ and $\dot{r}_{\text{moving}}(0, r_0, \dot{r}_0) = \dot{r}_0$ is periodic and $(r_{\text{moving}}(t, r_0, \dot{r}_0), e_z) > 0$ for all $t$.

Proof. Let $W \subset \mathbb{R}/2\pi \mathbb{Z} \times TS^2$ be a manifold with boundary defined by the inequalities $F \leq c$, $z \geq 0$, where $c$ is obtained from lemma 2.3. One can easily prove that $W$ is diffeomorphic to $\mathbb{R}/2\pi \mathbb{Z} \times D^2 \times D^2$, where $D^2$ is a 2-dimensional disk (with boundary). Let us now prove that $W$ is a periodic segment over $[0, 2\pi]$ for (1).

Indeed, from lemma 2.3 we see that the essential exit set is entirely on the boundary $z = 0$. Moreover, from lemma 2.4 we obtain that the points which satisfies $z = 0$ and $\dot{z} > 0$ do not belong to the essential exit set, at the same time, if for some point we have $z = 0$, $\dot{z} \leq 0$ then this point is in the essential exit set. Therefore, the essential exit set is compact. Homeomorphism $h$ in our case is an identity map, i.e. $[0, 2\pi] \times W_0 = W$.

![Figure 3](image_url)

Figure 3. Main features of behaviour of solutions at the boundary (in red) of $W$: a) trajectories can leave $W$ only through the $z = 0$ part of the boundary b) if for a point we have $z = 0$ then the solution starting from it, leaves $W$ iff $\dot{z} \leq 0$.

Finally, one can easily see that $W_0$ is homotopic to $D^2$ and $W_0^-$ is homotopic to $S^1$. Therefore, taking into account that $\mu_W = \text{id}$, we get
\[ \Lambda(\mu_W) = \Lambda(\text{id}_{W_0}) - \Lambda(\text{id}_{W_0^-}) = \chi(D^2) - \chi(S^1) = 1 - 0 \neq 0 \]
and theorem 3.8 can be applied. \qed
REFERENCES

[CR] R. Courant, H. Robbins, *What is mathematics?: an elementary approach to ideas and methods*, Oxford University Press (1996).

[SWZ] R. Srzednicki, K. Wójcik, and P. Zgliczynski, *Fixed point results based on Ważewski method* in *Handbook of topological fixed point theory*, Ed: R. Brown, M. Furi, L. Górniewicz, B. Jiang (2005), 903–941.

[GPS] H. Goldstein, C. Poole, J. Safko, *Classical Mechanics* (2002).

E-mail address: ivanpolekhin@gmail.com