A CHARACTERIZATION OF ORTHOGONAL CONVERGENCE IN SIMPLY CONNECTED DOMAINS

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Abstract. Let $\mathbb{D}$ be the unit disc in $\mathbb{C}$ and let $f : \mathbb{D} \to \mathbb{C}$ be a Riemann map, $\Delta = f(\mathbb{D})$. We give a necessary and sufficient condition in terms of hyperbolic distance and horocycles which assures that a compactly divergent sequence $\{z_n\} \subset \Delta$ has the property that $\{f^{-1}(z_n)\}$ converges orthogonally to a point of $\partial \mathbb{D}$. We also give some applications of this to the slope problem for continuous semigroups of holomorphic self-maps of $\mathbb{D}$.

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1. Introduction

A sequence $\{\zeta_n\} \subset \mathbb{D} := \{ \zeta \in \mathbb{C} : |\zeta| < 1 \}$ is said to converge orthogonally to a point $\sigma \in \partial \mathbb{D}$ provided $\{\zeta_n\}$ converges to $\sigma$ and $\lim_{n \to \infty} \arg(1 - \overline{\sigma} \zeta_n) = 0$.

Let $\Delta \subseteq \mathbb{C}$ be a simply connected domain, and $\{z_n\} \subset \Delta$ a sequence with no accumulation points in $\Delta$. Let $f : \mathbb{D} \to \Delta$ be a Riemann map. The aim of this note is to give an answer to the following question:

**What are (useful) geometric conditions on $\Delta$ which detect whether $\{f^{-1}(z_n)\}$ converges orthogonally to a point $\sigma \in \partial \mathbb{D}$?**
The previous question, aside being interesting by itself, is particularly important in studying dynamics of continuous semigroups of holomorphic self-maps of the unit disc (or more generally of holomorphic self-maps of the unit disc). Indeed, every semigroup of holomorphic self-maps of the unit disc has an essentially unique holomorphic model where the dynamical properties of the original semigroup translates into geometrical properties of the base domain of the model.

A similar question for non-tangential convergence has been settled in the recent paper [8], using hyperbolic geometry.

A remarkable result in the direction of the previous question has been proved using harmonic measure theory by D. Betsakos [4, Theorem 2]:

**Theorem** (Betsakos). Let $\Delta \subset \mathbb{C}$ be a simply connected domain starlike at infinity (namely $\Delta + it \subset \Delta$ for all $t \geq 0$) and $f : \mathbb{D} \to \Delta$ a Riemann map. If $\partial \Delta \subset \{ z \in \mathbb{C} : a < \text{Re} z < b, \text{Im} z < c \}$ for some $-\infty < a < b < +\infty$ and $c \in \mathbb{R}$, then there exists $\tau \in \partial \mathbb{D}$ such that $f^{-1}(w + it)$ converges orthogonally to $\tau$ for all $w \in \Delta$.

In this paper we give a complete answer to the above question in terms of hyperbolic geometry and “horocycles” in the spirit of [7], explaining also more geometrically Betsakos’s theorem. For the sake of clearness, we give here a definition of horocycles, referring the reader to Section 3 for details. Let $\bar{y}$ be a prime end of the simply connected domain $U \subset \mathbb{C}$. Let $f : \mathbb{D} \to U$ be a Riemann map such that 1 corresponds to $\bar{y}$ under $f$. Let $z_0 := f(0)$. For $R > 0$, the horocycle $E^{\Delta}_{z_0}(\bar{y}, R)$ centered at $\bar{y}$, with base point $z_0$ and radius $R$, is given by $f(E(1, R))$, where $E(1, R) := \{ z \in \mathbb{D} : |1 - z|^2 < R(1 - |z|^2) \}$ is a classical horocycle in the unit disc.

If $U \subset \mathbb{C}$ is a simply connected domain, we denote by $\partial_C U$ the set of prime ends of $U$ and by $\tilde{U} := U \cup \partial_C U$, endowed with the Carathéodory prime ends topology, or Carathéodory topology of $U$ for short (see [9] for precise definitions).

The main result of this paper is the following (see Section 2 for definitions and properties of hyperbolic distance and geodesics):

**Theorem 1.1.** Let $\Delta \subset \mathbb{C}$ be a simply connected domain, $f : \mathbb{D} \to \Delta$ a Riemann map. Let $\{z_n\} \subset \Delta$ be a sequence with no accumulation points in $\Delta$. Then there exists $\sigma \in \partial \mathbb{D}$ such that $\{f^{-1}(z_n)\}$ converges orthogonally to $\sigma$ if and only if there exist a simply connected domain $U \subset \mathbb{C}$, $z_0 \in U$, $\bar{y} \in \partial_C U$ and $R > 0$ such that

1. $E^{\Delta}_{z_0}(\bar{y}, R) \subset \Delta \subset U$, 
2. $\lim_{n \to \infty} k_U(z_n, \gamma([0, +\infty))) = 0$, where $\gamma : [0, +\infty) \to U$ is any geodesic for the hyperbolic distance in $U$ such that $\lim_{t \to +\infty} \gamma(t) = \bar{y}$ in the Carathéodory topology of $U$.

In particular, $\gamma(t)$ is eventually contained in $\Delta$ and $f^{-1}(\gamma(t))$ converges orthogonally to $\sigma$. 
Betsakos’ Theorem can be seen then as a consequence of Theorem 1.1: the domain starlike at infinity $\Delta$ is contained in a Koebe domain $K := \mathbb{C} \setminus \{z : \text{Re } z = s_0, \text{Im } z \leq s_1\}$ for some $s_0, s_1 \in \mathbb{R}$, and the condition that $\partial \Delta$ is contained in a vertical semistrip implies that $\Delta$ contains a horocycle of $K$ centered at the prime end corresponding to “infinity”. The line $t \mapsto s_0 + it$ is a geodesic in $K$ (for $t >> 1$) which converges to “infinity”, and hence one gets the orthogonal convergence of $f^{-1}(s_0 + it)$ as $t \to +\infty$ (and of $f^{-1}(w + it)$ for all $w \in \Delta$ since $k_K(s_0 + it, w + it) \to 0$ as $t \to +\infty$).

Another immediate consequence of Theorem 1.1 is the following: if $\Delta \subset \mathbb{C}$ is a simply connected domain such that $(\mathbb{H} + a) \subset \Delta \subset \mathbb{H}$ where $\mathbb{H} := \{z \in \mathbb{C} : \text{Re } z > 0\}$ and $a > 0$, then $f^{-1}(t)$ converges orthogonally to a point $\tau \in \partial \mathbb{D}$ as $t \to +\infty$.

As we mentioned before, Theorem 1.1 has direct applications to the study of the so-called “slope problem” for continuous semigroups of holomorphic self-maps of the unit disc (or more generally, for discrete iteration of holomorphic self-maps of the unit disc). We discuss this in Section 5.

The proof of Theorem 1.1 is rather involved. Using invariance of hyperbolic objects by Riemann mappings, one can essentially reduce to the case $(\mathbb{H} + 1) \subset \Delta \subset \mathbb{H}$. In this case, the curve $\Gamma : (0, +\infty) \ni t \mapsto t \in \mathbb{C}$, for $t$ sufficiently large, is a uniform quasi-geodesic in $\Delta$ in the sense of Gromov. This implies easily that $f^{-1}(\Gamma)$ converges non-tangentially to a point $\tau \in \partial \mathbb{D}$. However, in order to show that the convergence is orthogonal, one has to make careful estimates of the hyperbolic distance, proving that the geodesic of $\Delta$ which “shadows” $\Gamma$ becomes closer and closer in the hyperbolic distance of $\Delta$ to $\Gamma$. This is the content of Section 4.

2. Geodesics and quasi-geodesics in simply connected domains

Let $\Delta \subset \mathbb{C}$ be a simply connected domain. We denote by $\kappa_\Delta$ the infinitesimal metric in $\Delta$, that is, for $z \in \Delta$, $v \in \mathbb{C}$, we let

$$\kappa_\Delta(z; v) := \frac{|v|}{f'(0)},$$

where $f : \mathbb{D} \to \Delta$ is the Riemann map such that $f(0) = z$, $f'(0) > 0$. The hyperbolic distance $k_\Delta$ in $\Delta$ is defined for $z, w \in \Delta$ as

$$k_\Delta(z, w) := \inf \int_0^1 \kappa_\Delta(\gamma(t); \gamma'(t))dt,$$

where the infimum is taken over all piecewise smooth curves $\gamma : [0, 1] \to \Delta$ such that $\gamma(0) = z, \gamma(1) = w$.

Let $-\infty < a < b < +\infty$ and let $\gamma : [a, b] \to \Delta$ be a piecewise $C^1$-smooth curve. For $a \leq s \leq t \leq b$, we define the hyperbolic length of $\gamma$ in $\Delta$ between $s$ and $t$ as

$$\ell_\Delta(\gamma; [s, t]) := \int_s^t \kappa_\Delta(\gamma(u); \gamma'(u))du.$$
In case the length is computed in all the interval \([a, b]\) of definition of the curve, we will simply write
\[
\ell_{\Delta}(\gamma) := \ell_{\Delta}(\gamma; [a, b]).
\]
In particular recall that, if \(H := \{z \in \mathbb{C} : \text{Re } z > 0\}\), then for every \(z, w \in H\),
\[
(2.1) \quad k_{\mathbb{H}}(z, w) := \frac{1}{2} \log \frac{1 + \frac{z-w}{z+w}}{1 - \frac{z-w}{z+w}}.
\]

**Definition 2.1.** Let \(\Delta \subsetneq \mathbb{C}\) be a simply connected domain. A \(C^1\)-smooth curve \(\gamma : (a, b) \to \Delta, -\infty \leq a < b \leq +\infty\), such that \(\gamma'(t) \neq 0\) for all \(t \in (a, b)\) is called a geodesic of \(\Delta\) if for every \(a < s \leq t < b\),
\[
\ell_{\Delta}(\gamma; [s, t]) = k_{\Delta}(\gamma(s), \gamma(t)).
\]
Moreover, if \(z, w \in \Delta\) and there exist \(a < s < t < b\) such that \(\gamma(s) = z\) and \(\gamma(t) = w\), we say that \(\gamma|_{[s, t]}\) is a geodesic which joins \(z\) and \(w\).

With a slight abuse of notation, we also call geodesic the image of \(\gamma\) in \(\Delta\).

Using Riemann maps and the invariance of hyperbolic metric and distance for biholomorphisms, we have the following result (see, e.g., [9] for the definition and properties of the Carathéodory’s prime ends topology):

**Proposition 2.2.** Let \(\Delta \subsetneq \mathbb{C}\) be a simply connected domain. Let \(-\infty \leq a < b \leq +\infty\).

1. If \(\eta : (a, b) \to \Delta\) is a geodesic, then
   \[
   \eta(a) := \lim_{t \to a^+} \eta(t), \quad \eta(b) := \lim_{t \to b^-} \eta(t)
   \]
   exist as limits in the Carathéodory topology of \(\Delta\). Moreover, if \(\eta(a), \eta(b) \in \Delta\) then
   \[
   k_{\Delta}(\eta(a), \eta(b)) = \lim_{\epsilon \to 0^+} \ell_{\Delta}(\eta; [a + \epsilon, b - \epsilon]).
   \]

2. If \(\eta : (a, b) \to \Delta\) is a geodesic such that \(\eta(a), \eta(b) \in \partial_C \Delta\), then \(\eta(a) \neq \eta(b)\).

3. For any \(z, w \in \Delta\), \(z \neq w\), there exists a real analytic geodesic \(\gamma : (a, b) \to \Delta\) such that \(\gamma(a) = z\) and \(\gamma(b) = w\). Moreover, such a geodesic is essentially unique, namely, if \(\eta : (\tilde{a}, \tilde{b}) \to \Delta\) is another geodesic joining \(z\) and \(w\), then \(\gamma([a, b]) = \eta([\tilde{a}, \tilde{b}])\) in \(\tilde{\Delta}\).

4. If \(\gamma : (a, b) \to \Delta\) is a geodesic such that either \(\gamma(a) \in \Delta\) or \(\gamma(b) \in \Delta\) (or both), then there exists a geodesic \(\eta : (\tilde{a}, \tilde{b}) \to \Delta\) such that \(\eta(\tilde{a}), \eta(\tilde{b}) \in \partial_C \Delta\) and such that \(\gamma([a, b]) \subset \eta([\tilde{a}, \tilde{b}])\) in \(\tilde{\Delta}\).

5. If \(\gamma : (a, b) \to \Delta\) is a geodesic such that \(\gamma(a) \in \partial_C \Delta\) then the cluster set \(\Gamma(\gamma, a) = \Pi(\gamma(a))\), the principal part of the prime end \(\gamma(a)\) (and similarly for \(b\) in case \(\gamma(b) \in \partial_C \Delta\)).
Given a simply connected domain, it is in general a hard task to find geodesics. The aim of this section is indeed to recall a powerful method due to Gromov to localize geodesics via simpler curves which are called quasi-geodesics.

**Definition 2.3.** Let $\Delta \subset \mathbb{C}$ be a simply connected domain. Let $A \geq 1$ and $B \geq 0$. A piecewise $C^1$-smooth curve $\gamma : [a, b] \rightarrow \Delta$, $-\infty < a < b < +\infty$, is a $(A, B)$-quasi-geodesic if for every $a \leq s \leq t \leq b$,

$$\ell_\Delta(\gamma; [s, t]) \leq Ak_\Delta(\gamma(s), \gamma(t)) + B.$$

The importance of quasi-geodesics is contained in the following result (see, e.g., [14]):

**Theorem 2.4** (Gromov’s shadowing lemma). For every $A \geq 1$ and $B \geq 0$ there exists $\delta = \delta(A, B) > 0$ with the following property. Let $\Delta \subset \mathbb{C}$ be any simply connected domain. If $\gamma : [a, b] \rightarrow \Delta$ is a $(A, B)$-quasi-geodesic, then there exists a geodesic $\tilde{\gamma} : [\tilde{a}, \tilde{b}] \rightarrow \Delta$ such that $\tilde{\gamma}(\tilde{a}) = \gamma(a), \tilde{\gamma}(\tilde{b}) = \gamma(b)$ and for every $u \in [a, b]$ and $v \in [\tilde{a}, \tilde{b}]$,

$$k_\Delta(\gamma(u), \tilde{\gamma}(\tilde{a})), \gamma([a, b])) < \delta, \quad k_\Delta(\tilde{\gamma}(v), \gamma([a, b])) < \delta.$$

A consequence of Gromov’s shadowing lemma we will take advantage of is the following result, whose proof is based on standard arguments of normality:

**Corollary 2.5.** Let $\Delta \subset \mathbb{C}$ be a simply connected domain. Let $\gamma : [0, +\infty) \rightarrow \Delta$ be a piecewise $C^1$-smooth curve such that $\lim_{t \rightarrow +\infty} k_\Delta(\gamma(0), \gamma(t)) = +\infty$ and there exist $A \geq 1$, $B \geq 0$, such that for every fixed $T > 0$ the curve $[0, T) \ni t \mapsto \gamma(t)$ is a $(A, B)$-quasi-geodesic. Then there exists a prime end $x \in \partial \Delta$ such that $\gamma(t) \rightarrow x$ in the Carathéodory topology of $\Delta$ as $t \rightarrow +\infty$. Moreover, there exists $\epsilon > 0$ such that, if $\eta : [0, +\infty) \rightarrow \Delta$ is the geodesic of $\Delta$ parameterized by arc length such that $\eta(0) = \gamma(0)$ and $\lim_{t \rightarrow +\infty} \eta(t) = x$ in the Carathéodory topology of $\Delta$, then, for every $t \in [0, +\infty)$,

$$k_\Delta(\gamma(t), \eta([0, +\infty))) < \epsilon, \quad k_\Delta(\eta(t), \gamma([0, +\infty))) < \epsilon.$$

3. **Horocycles in simply connected domains**

In this section we define horocycles (also called “horospheres”) in simply connected domains, using in this context an abstract approach introduced in [7], and inspired on the general definition of horospheres in several complex variables introduced by M. Abate [1, 2].

Recall (see e.g., [2] Prop. 1.2.2) that for every $\sigma \in \partial \mathbb{D}$,

$$\lim_{w \rightarrow \sigma}[k_{\mathbb{D}}(z, w) - k_{\mathbb{D}}(0, w)] = \frac{1}{2} \log \frac{1}{1 - |z|^2},$$

so that, given $R > 0$,

$$E(\sigma, R) := \{z \in \mathbb{D} : \frac{1}{1 - |z|^2} < R\} = \{z \in \mathbb{D} : \lim_{w \rightarrow \sigma}[k_{\mathbb{D}}(z, w) - k_{\mathbb{D}}(0, w)] < \frac{1}{2} \log R\}$$
is a horocycle of \( D \) of center \( \sigma \) and radius \( R \).

Let \( \Delta \subset \mathbb{C} \) be a simply connected domain, \( z_0 \in \Delta \) and let \( f : D \to \Delta \) be a Riemann map such that \( f(0) = z_0 \). Let \( y \in \partial_C \Delta \) be a prime end of \( \Delta \). There exists exactly one \( \sigma \in \partial D \) which corresponds to a prime end \( x_\sigma \in \partial_C D \) such that \( \hat{f}(x_\sigma) = y \). Moreover, a sequence \( \{w_n\} \subset \Delta \) converges to \( y \) in the Carathéodory topology of \( \Delta \) if and only if \( \{f^{-1}(w_n)\} \) converges to \( \sigma \) (in the Euclidean topology). Therefore, if \( \{z_n\} \) and \( \{w_n\} \) are two sequences in \( \Delta \) which converge to \( y \) in the Carathéodory topology of \( \Delta \), taking into account that \( f \) is an isometry for the hyperbolic distance, by (3.1), we have for every \( z \in \Delta \):

\[
\lim_{n \to \infty} [k_\Delta(z, w_n) - k_\Delta(z_0, w_n)] = \lim_{n \to \infty} [k_D(f^{-1}(z), f^{-1}(w_n)) - k_D(0, f^{-1}(w_n))] = \lim_{n \to \infty} [k_D(f^{-1}(z), f^{-1}(z_n)) - k_D(0, f^{-1}(z_n))] = \lim_{n \to \infty} [k_\Delta(z, z_n) - k_\Delta(z_0, z_n)].
\]

Moreover, by the same equation (3.1),

\[
\lim_{n \to \infty} [k_D(f^{-1}(z), f^{-1}(w_n)) - k_D(0, f^{-1}(w_n))] \in (-\infty, +\infty),
\]

hence \( \lim_{n \to \infty} [k_\Delta(z, w_n) - k_\Delta(z_0, w_n)] \in (-\infty, +\infty) \).

By the previous remark, the following definition is well posed (that is, it is independent of the sequence \( \{w_n\} \) chosen):

**Definition 3.1.** Let \( \Delta \subset \mathbb{C} \) be a simply connected domain and \( z_0 \in \Delta \). Let \( y \in \partial_C \Delta \) be a prime end of \( \Delta \). Let \( R > 0 \). The horocycle \( E^\Delta_{z_0}(y, R) \) of center \( y \), base point \( z_0 \) and hyperbolic (radius) \( R > 0 \) is

\[
E^\Delta_{z_0}(y, R) := \{ z \in \Delta : \lim_{n \to \infty} [k_\Delta(z, w_n) - k_\Delta(z_0, w_n)] < \frac{1}{2} \log R \},
\]

where \( \{w_n\} \subset \Delta \) is any sequence which converges to \( y \) in the Carathéodory topology of \( \Delta \).

Note that, by (3.1), \( E(\sigma, R) = E^D_{\sigma}(y, R) \) for every \( \sigma \in \partial D \) and \( R > 0 \).

The base point \( z_0 \) in the definition of horocycles is essentially irrelevant. Indeed, as proven by a direct computation, for every \( R > 0 \), \( z_0, z_1 \in \Delta \) and \( y \in \partial_C \Delta \)

\[
E^\Delta_{z_0}(y, R) = E^\Delta_{z_1}(y, AR),
\]

where \( A \in (0, +\infty) \) and, in fact,

\[
-k_\Delta(z_0, z_1) \leq \frac{1}{2} \log A := \lim_{n \to \infty} [k_\Delta(z_0, w_n) - k_\Delta(z_1, w_n)] \leq k_\Delta(z_0, z_1).
\]

If \( \Delta, \bar{\Delta} \subset \mathbb{C} \) are simply connected domain and \( f : \Delta \to \bar{\Delta} \) is a biholomorphism, then we denote by \( \hat{f} \) the homeomorphism induced by \( f \) in the Carathéodory topology. Since biholomorphisms are isometries for the hyperbolic distance, we immediately get:
Proposition 3.2. Let $\Delta, \tilde{\Delta} \subseteq \mathbb{C}$ be two simply connected domains. Let $f : \Delta \to \tilde{\Delta}$ be a biholomorphism, $y \in \partial_C \Delta$ and $z_0 \in \Delta$. Then, for every $R > 0$

$$f(E^\Delta_{z_0}(y, R)) = E^\tilde{\Delta}_{f(z_0)}(f(y), R).$$

Now we need a “localization lemma”, comparing hyperbolic distance in horospheres with hyperbolic distance in the domain. We start with a simple lemma whose proof is a direct computation based on (2.1):

Lemma 3.3. Let $\beta \in (-\frac{\pi}{2}, \frac{\pi}{2})$.

1. Let $0 < \rho_0 < \rho_1$ and let $\Gamma := \{pe^{i\beta} : \rho_0 \leq \rho \leq \rho_1\}$. Then, $\ell_H(\Gamma) = \frac{1}{2\cos \beta} \log \frac{\rho_1}{\rho_0}$.
2. Let $\rho_0, \rho_1 > 0$. Then, $k_H(\rho_0, \rho_1e^{i\beta}) - k_H(\rho_0, \rho_1) \geq \frac{1}{2} \log \frac{1}{\cos \beta}$.
3. Let $\rho_0 > 0$ and $\alpha \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Then, $(0, +\infty) \ni \rho \mapsto k_H(\rho e^{i\alpha}, \rho_0 e^{i\beta})$ has a minimum at $\rho = \rho_0$, it is increasing for $\rho > \rho_0$ and decreasing for $\rho < \rho_0$.
4. Let $\theta_0, \theta_1 \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and $\rho > 0$. Then $k_H(\rho e^{i\theta_0}, \rho e^{i\theta_1}) = k_H(e^{i\theta_0}, e^{i\theta_1})$. Moreover, $k_H(1, e^{i\theta}) = k_H(1, e^{-i\theta})$ for all $\theta \in [0, \pi/2)$ and $[0, \pi/2) \ni \theta \mapsto k_H(1, e^{i\theta})$ is strictly increasing.
5. Let $\beta_0, \beta_1 \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and $0 < \rho_0 < \rho_1$. Then $k_H(\rho_0 e^{i\beta_0}, \rho_1 e^{i\beta_1}) \geq k_H(\rho_0, \rho_1)$.

Let $\infty \in \partial_C \mathbb{H}$ denote the prime end corresponding to $1 \in \partial \mathbb{D}$, via the Cayley transform $C : \mathbb{D} \to \mathbb{H}$, $C(z) = \frac{1+z}{1-z}$. For $R > 0$ a direct computation shows $E^H_1(\infty, R) = \{w \in \mathbb{C} : \text{Re } w > R\}$. In particular, if $t_0 > 0$ and $\gamma : [1, +\infty) \to \mathbb{H}$ is the geodesic defined by $\gamma(t) = t$, it follows that $\gamma(t) \to \infty$ in the Carathéodory topology of $\mathbb{H}$ and $\gamma(t)$ is eventually contained in $E^H_1(\infty, R)$ for every $R > 0$.

Also, if $\gamma : [0, +\infty)$ is a geodesic in a simply connected domain $\Delta \subseteq \mathbb{C}$ such that $\lim_{t \to +\infty} k_\Delta(\gamma(0), \gamma(t)) = +\infty$ and $R > 0$ we define

$$S_\Delta(\gamma, R) := \{z \in \Delta : k_\Delta(z, \gamma) < R\},$$

a hyperbolic sector around $\gamma$.

Lemma 3.4. Let $U \subseteq \mathbb{C}$ be a simply connected domain, $z_0 \in U$, $y \in \partial_C U$ and $\gamma : [0, +\infty) \to U$ a geodesic which converges to $y$ in the Carathéodory topology of $U$. Let $\{z_n\}$ be a sequence with no accumulation points in $U$. If $\lim_{n \to \infty} k_U(z_n, \gamma([0, +\infty))) = 0$ then $\{z_n\}$ is eventually contained in $E^{U'_{z_0}}(y, R)$ and $\lim_{n \to \infty} k_{E^{U'_{y,R}}}(z_n, \gamma([0, +\infty))) = 0$, for all $R > 0$.

Proof. Using Proposition 3.2 and the fact that horospheres do not depend essentially on the base point, we can assume $U = \mathbb{H}$, $z_0 = 1$, $y = \infty \in \partial_C \mathbb{H}$ and $\gamma(t) = t, t \geq 1$.

Write $z_n = \rho_n e^{i\theta_n}$, with $\rho_n > 0$ and $\theta_n \in (-\pi/2, \pi/2)$. Hence, $k_H(\rho_n e^{i\theta_n}, [1, +\infty)) = k_H(\rho_n e^{i\theta_n}, \rho_n)$ by Lemma 3.3. Moreover, by assumption, $\lim_{n \to \infty} k_H(\rho_n e^{i\theta_n}, \rho_n) = 0$. 


Hence, by Lemma 3.3(4), \{z_n\} is eventually contained in \(V(r, M) := \{\rho e^{i\theta} : \rho > M, |\theta| < r\}\) for all \(r > 0, M > 0\).

Note that, given \(R > 0\), then for \(M\) sufficiently large, \(V(r, M) \subset E^H_1(\infty, R)\)—and, in particular, \(\{z_n\} \subset E^H_1(\infty, R)\) eventually.

Let \(R > 0\). By [8, Lemma 4.4] and the previous consideration, it follows that for all \(N > 0\), there exists \(t_0 \geq 1\) such that \(S^H(\gamma|_{[t_0, \infty)}, N) \subset E^H_1(\infty, R)\) and \(\{z_n\} \subset S^H(\gamma|_{[t_0, \infty)}, N)\) eventually. Hence, by [8, Lemma 4.6], once fixed \(t_0\) and \(N\), and setting \(S := S^H(\gamma|_{[t_0, \infty)}, N)\) for short, there exists a constant \(C > 0\) such that for all \(n\) such that \(z_n \in S\),

\[
k_S(\rho_n, z_n) \leq C k_H(\rho_n, z_n).
\]

Therefore, \(\lim_{n \to \infty} k_S(\rho_n, z_n) = 0\). Now, given \(R > 0\), we can choose \(N > 0\) and \(t_0 \geq 1\) such that \(S := S^H(\gamma|_{[t_0, \infty)}, N) \subset E^H_1(\infty, R)\). Hence, for \(n\) sufficiently large,

\[
k_{E^H_1(\infty, R)}(\rho_n, z_n) \leq k_S(\rho_n, z_n) \to 0.
\]

This proves Lemma 3.4. \(\square\)

4. Proof of Theorem 1.1

For \(\beta \in (0, \pi)\), we denote

\[
V(\beta) := \{\rho e^{i\theta} : \rho > 0, |\theta| < \beta\}.
\]

By Lemma 3.3, it follows immediately that \(V(\beta)\) is a hyperbolic sector around the geodesic \((0, +\infty)\) of \(\mathbb{H}\).

The main result we need is the following:

**Proposition 4.1.** Let \(\Delta \subset \subset \mathbb{C}\) be a simply connected domain and let \(f : \mathbb{D} \to \Delta\) be a Riemann map. Suppose that \(\mathbb{H} + a \subset \Delta \subset \mathbb{H}\) for some \(a > 0\). Then there exists \(\xi \in \partial \mathbb{D}\) such that \(f^{-1}(t)\) converges orthogonally to \(\xi\) as \(t \to +\infty\).

**Proof.** We can assume without loss of generality that \(a = 1\) and let \(U := \mathbb{H} + 1\). Note that, for every \(z, w \in U\),

\[
(4.1) \quad k_U(z, w) = k_H(z - 1, w - 1).
\]

We divide the proof in several steps:

**Step 1.** Let \(\beta \in (0, \pi/4)\). Then there exists a constant \(K(\beta) > 0\) such that for every \(\theta_0, \theta_1 \in [-\beta, \beta]\) and for every \(\rho \geq 2\), \(k_U(\rho e^{i\theta_0}, \rho e^{i\theta_1}) < K(\beta)\). Moreover, \(\lim_{\beta \to 0} K(\beta) = 0\).

Fix \(\beta \in (0, \pi/4)\) and \(\rho \geq 2\). Let \(\tilde{\beta} = \beta(\rho) \in (0, \pi/2)\) be such that \(e^{i\tilde{\beta}} = \frac{\rho e^{i\beta} - 1}{|\rho e^{i\beta} - 1|}\). Hence,

\[
\sin \tilde{\beta} = \frac{\sin \beta}{|e^{i\beta} - 1/\rho|} \leq \frac{2 \sin \beta}{|2e^{i\beta} - 1|},
\]

which shows that \(\lim_{\beta \to 0} \sup_{\rho \geq 2} \tilde{\beta}(\rho) = 0\).
Let $A_1 := \{ e^{i\beta} \rho - 1 : \rho \leq \beta \}$. Note that $A_1$ is the arc of the circle with center 1, radius $|e^{i\beta} - 1|$ and end points $p_0 := e^{i\beta} \rho = |e^{i\beta} - 1|e^{i\beta} + 1$ and $p_1 := e^{-i\beta} \rho = |e^{i\beta} - 1|e^{-i\beta} + 1$.

Let $A_2 := \{ (\rho - 1)e^{i\beta} + 1 : |\theta| \leq \beta \}$. Note that $A_2$ is the arc of the circle with center 1 and radius $\rho - 1$ with end points $q_0 := (\rho - 1)e^{i\beta} + 1$ and $q_1 := (\rho - 1)e^{-i\beta} + 1$.

Note that by construction, $A_1, A_2$ are arcs of circles that intersect $\partial U$ orthogonally, hence they are geodesics for the hyperbolic distance $k_U$.

Let $B_1 := \{ re^{i\beta} + 1 : \rho - 1 \leq r \leq |e^{i\beta} - 1| \}$ and let $B_2 := \{ re^{-i\beta} + 1 : \rho - 1 \leq r \leq |e^{-i\beta} - 1| \}$.

By construction, $A_1 \cup B_1 \cup A_2 \cup B_2$ is a Jordan curve which bounds a simply connected domain $Q \subset \mathbb{C}$. Moreover, by simple geometric considerations, the curve $\{ re^{i\beta} : |\theta| \leq \beta \}$ is contained in $Q$. Hence,

$$k_U(p_{e^{i\beta}}G_{e^{i\beta}}) \leq \text{diam}_U(Q),$$

where $\text{diam}_U(Q) := \sup_{z, w \in Q} k_U(z, w)$ is the hyperbolic diameter of $Q$. Clearly,

$$\text{diam}_U(Q) \leq \ell_U(A_1) + \ell_U(A_2) + \ell_U(B_1) + \ell_U(B_2).$$

Now, since $A_1, A_2$ are geodesics for $U$, it follows that $\ell_U(A_1) = k_U(p_0, p_1)$ and $\ell_U(A_2) = k_U(q_0, q_1)$. Hence, by (1.1)

$$\ell_U(A_1) = k_U(p_0, p_1) = k_H(p_0 - 1, p_1 - 1) = k_H(|e^{i\beta} - 1|e^{i\beta}, |e^{i\beta} - 1|e^{-i\beta}),$$

and, by Lemma 3.3(4), $k_H(|e^{i\beta} - 1|e^{i\beta}, |e^{i\beta} - 1|e^{-i\beta})$ depends on $\beta$ and goes to 0 as $\beta$ goes to 0. Similarly, $\ell_U(A_2)$ goes to 0 as $\beta$ goes to 0.

On the other hand, by (1.1) and Lemma 3.3(1),

$$\ell_U(B_1) = \ell_H(B_1 - 1) = \ell_H(|e^{i\beta} - 1|e^{i\beta}, |e^{i\beta} - 1|e^{-i\beta}) = \frac{1}{2 \cos \beta} \log \frac{|e^{i\beta} - 1|}{\rho - 1}.$$

Since $\sin(\beta) \leq \frac{2 \sin(\pi/4)}{|e^{i\beta} - 1|} = \sqrt{\frac{2}{5 - 2\sqrt{2}}}$, we have that $\cos(\beta) \geq \sqrt{\frac{3 - 2\sqrt{2}}{5 - 2\sqrt{2}}}$ and

$$\ell_U(B_1) \leq \sqrt{\frac{5 - 2\sqrt{2}}{3 - 2\sqrt{2}}} \log \frac{|e^{i\beta} - 1|}{\rho - 1} \leq \sqrt{\frac{5 - 2\sqrt{2}}{3 - 2\sqrt{2}}} \log |2e^{i\beta} - 1|,$$

for every $\rho \geq 2$, which shows that $\ell_U(B_1)$ goes to 0 as $\beta$ goes to 0. A similar argument shows that also $\ell_U(B_2)$ goes to 0 as $\beta$ goes to 0 and Step 1 follows.

**Step 2.** Let $\beta \in (0, \pi/2)$. Let $\alpha_\beta := (1 - \cos^2 \beta)^{-1}$. Then for every $x_1 > x_0 \geq \alpha_\beta$ the geodesic in $\Delta$ joining $x_0$ and $x_1$ is contained in $V(\beta)$.

Fix $x_0 \geq \alpha_\beta$ and $x_1 > x_0$. Let $\sigma : [0, 1] \to \Delta$ be the geodesic for $\Delta$ such that $\sigma(0) = x_0$ and $\sigma(1) = x_1$. Assume that $\sigma([0, 1])$ is not contained in $V(\beta)$. Hence, there exist $0 < t_1 < t_2 < 1$ such that $\sigma(t_1) \in \partial V(\beta)$, $j = 1, 2$ and $\{ \sigma(t) : t_1 \leq t \leq t_2 \} \cap V(\beta) = \emptyset$. Since $V(\beta)$ disconnects $\mathbb{H}$ in two connected components, we can assume without loss of generality that $\sigma(t_1) = y_1 e^{i\beta}$ and $\sigma(t_2) = y_2 e^{i\beta}$ for some $y_1, y_2 > 0$ (possibly $y_1 = y_2$).
Denote $R := \{r : r > 0\}$. Let $\gamma_1$ be the segment in $R$ joining $x_0$ and $y_1$, namely, if $y_1 \geq x_0$, let $\gamma_1 := \{r : x_0 \leq r \leq y_1\}$, while, if $y_1 < x_0$, let $\gamma_1 := \{r : y_1 \leq r \leq x_0\}$. Let $\gamma_2 = \{ye^{i\theta} : \theta \in [0, \beta]\}$. Let $\gamma_3$ be the segment on $\partial V(\beta)$ joining $\sigma(t_1) = ye^{i\beta}$ with $\sigma(t_2) = ye^{i\beta}$, i.e., if for instance $y_1 \leq y_2$, $\gamma_3 := \{re^{i\beta} : y_1 \leq r \leq y_2\}$. Then, let $\gamma_4 := \{ye^{i\theta} : \theta \in [0, \beta]\}$. Finally, let $\gamma_5$ be the segment on $R$ joining $y_2$ with $x_1$.

Let $\Gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4 \cup \gamma_5$. Hence, $\Gamma$ is a piecewise smooth curve in $\mathbb{H}$ which joins $x_0$ and $x_1$.

Now, since $\sigma$ is a geodesic in $\Delta$, $\Delta \subset \mathbb{H}$ and by Lemma 3.3(5),

$$\ell_\Delta(\sigma) = k_\Delta(x_0, ye^{i\beta}) + k_\Delta(ye^{i\beta}, ye^{i\beta}) + k_\Delta(ye^{i\beta}, x_1)$$

$$\geq k_\mathbb{H}(x_0, ye^{i\beta}) + k_\mathbb{H}(ye^{i\beta}, ye^{i\beta}) + k_\mathbb{H}(ye^{i\beta}, x_1)$$

$$\geq k_\mathbb{H}(x_0, ye^{i\beta}) + k_\mathbb{H}(ye^{i\beta}, ye^{i\beta}) + k_\mathbb{H}(ye^{i\beta}, x_1)$$

$$= k_\mathbb{H}(x_0, y_1) + k_\mathbb{H}(y_1, y_2) + k_\mathbb{H}(y_2, x_1)$$

$$+ [k_\mathbb{H}(x_0, ye^{i\beta}) - k_\mathbb{H}(x_0, y_1)] + [k_\mathbb{H}(ye^{i\beta}, x_1) - k_\mathbb{H}(y_2, x_1)]$$

$$\geq k_\mathbb{H}(x_0, x_1) + [k_\mathbb{H}(x_0, ye^{i\beta}) - k_\mathbb{H}(x_0, y_1)] + [k_\mathbb{H}(ye^{i\beta}, x_1) - k_\mathbb{H}(y_2, x_1)]$$

$$\geq k_\mathbb{H}(x_0, x_1) + \log \frac{1}{\cos \beta},$$

where the penultimate inequality follows from the triangle inequality, and the last inequality follows from Lemma 3.3(2). Moreover, a direct computation shows that

$$k_\mathbb{H}(x_0, x_1) = k_U(x_0, x_1) - \frac{1}{2} \log \frac{1 - \frac{1}{x_1}}{1 - \frac{1}{x_0}}.$$

Therefore, taking into account that $U \subset \Delta$, from the previous inequality we have

$$k_\Delta(x_0, x_1) = \ell_\Delta(\sigma) \geq k_U(x_0, x_1) - \frac{1}{2} \log \frac{1 - \frac{1}{x_1}}{1 - \frac{1}{x_0}} + \log \frac{1}{\cos \beta}$$

$$\geq k_\Delta(x_0, x_1) - \frac{1}{2} \log \frac{1 - \frac{1}{x_0}}{1 - \frac{1}{x_0}} + \log \frac{1}{\cos \beta},$$

which forces

$$\log \frac{1}{\cos^2 \beta} \leq \log \frac{1 - \frac{1}{x_0}}{1 - \frac{1}{x_0}}.$$

However, if $x_0 \geq \alpha \beta$,

$$\log \frac{1}{\cos^2 \beta} \leq \log \frac{1 - \frac{1}{x_0}}{1 - \frac{1}{x_0}} \leq \log \frac{1 - \frac{1}{x_0}}{\cos^2 \beta},$$

getting a contradiction, and Step 2 follows.
Step 3. Let \( \beta \in (0, \pi/4) \) and let \( 2 \leq x_0 < x_1 \). Let \( \sigma : [0,1] \to \Delta \) be a geodesic for \( \Delta \) such that \( \sigma(0) = x_0 \) and \( \sigma(1) = x_1 \). Let \( K(\beta) \) be the constant defined in Step 1 and let \( c \in (0, x_0 e^{-K(\beta)}) \). Suppose \( \sigma([0,1]) \subset V(\beta) \). Then \( |\sigma(t)| > c \) for all \( t \in [0,1] \).

Assume by contradiction that there exists \( t_1 \in (0,1) \) such that \( |\sigma(t_1)| = c \). Then \( \sigma(t_1) = ce^{i\theta_1} \) for some \( \theta_1 \in (-\beta, \beta) \). Moreover, by continuity of \( \sigma \), there exist \( \tilde{t}_1 \in [0,t_1) \) and \( \tilde{t}_2 \in (t_1,1) \) such that \( \sigma(\tilde{t}_1) = x_0 e^{i\theta_1} \), \( \sigma(\tilde{t}_2) = x_0 e^{i\theta_2} \) for some \( \tilde{\theta}_1, \tilde{\theta}_2 \in (-\beta, \beta) \) and \( |\sigma(t)| \leq x_0 \) for all \( t \in [\tilde{t}_1, \tilde{t}_2] \).

Now, since \( \sigma \) is a geodesic in \( \Delta \), and \( \Delta \subset \mathbb{H} \),

\[
k_\Delta(x_0 e^{i\tilde{\theta}_1}, x_0 e^{i\tilde{\theta}_2}) = \ell_\Delta(\sigma; [\tilde{t}_1, \tilde{t}_2]) = \ell_\Delta(\sigma; [\tilde{t}_1, t_1]) + \ell_\Delta(\sigma; [t_1, \tilde{t}_2])
\]

\[
= k_\Delta(x_0 e^{i\tilde{\theta}_1}, e^{i\tilde{\theta}_1}) + k_\Delta(e^{i\tilde{\theta}_1}, x_0 e^{i\tilde{\theta}_2})
\]

\[
\geq k_\mathbb{H}(x_0, c) + k_\mathbb{H}(c, x_0) = 2k_\mathbb{H}(x_0, c) = \log \frac{x_0}{c},
\]

where the last inequality follows from Lemma 3.3(5).

On the other hand, since \( U \subset \Delta \), and by Step 1,

\[
k_\Delta(x_0 e^{i\tilde{\theta}_1}, x_0 e^{i\tilde{\theta}_2}) \leq k_U(x_0 e^{i\tilde{\theta}_1}, x_0 e^{i\tilde{\theta}_2}) \leq K(\beta).
\]

Hence, \( \log \frac{x_0}{c} \leq K(\beta) \), which contradicts the choice of \( c \) and Step 3 follows.

Step 4. For every \( \delta > 0 \) there exists \( \mu_\delta \geq 2 \) such that for every \( x_1 > x_0 \geq \mu_\delta \), if \( \sigma : [0,1] \to \Delta \) is a geodesic of \( \Delta \) such that \( \sigma(0) = x_0 \) and \( \sigma(1) = x_1 \), then for every \( x \in [x_0, x_1] \) there exists \( t_x \in [0,1] \) such that \( k_\Delta(x, \sigma(t_x)) < \delta \).

In order to prove Step 4, we first claim that for every \( \nu \in (0, \frac{\pi}{4}) \) there exists \( \mu_\nu \geq 2 \) such that for every \( x_1 > x_0 \geq \mu_\nu \), \( \sigma([0,1]) \subset V(\nu) + 1 \).

If the claim is true, since \( \sigma \) is continuous, for every \( x \in [x_0, x_1] \) there exists \( |\theta_x| < \nu \) and \( t_x \in [0,1] \) such that \( \sigma(t_x) = (x - 1)e^{i\theta_x} + 1 \). Hence, by (4.1) and Lemma 3.3(4), and recalling that \( U \subset \Delta \),

\[
k_\Delta(\sigma(t_x), x) \leq k_U((x - 1)e^{i\theta_x} + 1, (x - 1) + 1) = k_\mathbb{H}((x - 1)e^{i\theta_x}, 1) = k_\mathbb{H}(e^{i\theta_x}, 1).
\]

Since \( k_\mathbb{H}(e^{i\nu}, 1) \to 0 \) as \( \nu \to 0 \), Step 4 follows.

In order to prove the claim, given \( \nu \in (0, \frac{\pi}{4}) \), let \( \beta \in (0, \nu) \). Hence, there exists \( \alpha > 1 \) such that \( V(\beta) \cap \{w \in U : |w| > \alpha\} \subset V(\nu) + 1 \). Let \( \alpha_\beta \) be given by Step 2. Let \( \mu_\nu > e^{K(\beta)} \max\{\alpha_\beta, \alpha\} \), where \( K(\beta) \) is given by Step 1. Hence, by Step 2, for every \( x_1 > x_0 \geq \mu_\nu \) the geodesic \( \sigma \) for \( k_\Delta \) joining \( x_0, x_1 \) is contained in \( V(\beta) \). By Step 3, \( \sigma([0,1]) \subset \{w : |w| > \alpha\} \), hence, \( \sigma \) is contained in \( V(\nu) + 1 \).

Let \( \xi \in \partial \mathbb{D} \) be such that \( \lim_{t \to +\infty} f^{-1}(t) = \xi \) (see [17, Page 162]). Let \( \gamma : [0,1] \to \mathbb{D} \) be the geodesic of \( \mathbb{D} \) defined by \( \gamma(t) = t\xi \).

Step 5. For every \( \epsilon > 0 \) there exists \( t_\epsilon > 0 \) such that \( f^{-1}(t) \in S_D(\gamma, \epsilon) \) for all \( t \geq t_\epsilon \).
Fix $\epsilon > 0$. Let $\delta = \frac{\epsilon}{3}$ and let $\mu_\delta \geq 2$ be the point defined in Step 4. Let $\{x_n\}$ be an increasing sequence of positive real numbers converging to $+\infty$. Let $\sigma_n : [0, R_n] \to \Delta$ be the geodesic in $\Delta$ parameterized by arc length such that $\sigma_n(0) = \mu_\delta$ and $\sigma_n(R_n) = x_n$. Using a normality argument, up to extracting a subsequence, we can assume that $\{\sigma_n\}$ converges uniformly on compacta of $[0, +\infty)$ to a geodesic $\sigma : [0, +\infty) \to \Delta$, parameterized by arc length such that $\sigma(0) = \mu_\delta$ and $\lim_{s \to +\infty} \sigma(s) = y \in \partial C\Delta$ in the Carathéodory topology of $\Delta$.

In particular, for every fixed $T > 0$ there exists $n_T \in \mathbb{N}$ such that for every $n \geq n_T$ we have $R_n \geq T$ and for every $s \in [0, T]$, 
\begin{equation}
(4.2) \quad k_\Delta(\sigma_n(s), \sigma(s)) < \delta.
\end{equation}

By Step 4, for every $t \in [\mu_\delta, x_n]$ there exists $s^n_t \in [0, R_n]$ such that $k_\Delta(\sigma_n(s^n_t), t) < \delta$.

We claim that, for every fixed $x_1 > \mu_\delta$ there exists $C_{x_1} > 0$ such that for all $n \in \mathbb{N}$ and all $t \in [\mu_\delta, x_1]$, we have $s^n_t \leq C_{x_1}$. Indeed, since $[\mu_\delta, x_1]$ is compact in $\Delta$, $C_0 := \max_{x \in [\mu_\delta, x_1]} k_\Delta(x, \mu_\delta) < +\infty$. Hence, recalling that $\sigma_n$ is parameterized by arc length, for all $t \in [\mu_\delta, x_1]$, we have
\[ s^n_t = k_\Delta(\sigma_n(s^n_t), \sigma_n(0)) = k_\Delta(\sigma_n(s^n_t), \mu_\delta) \leq k_\Delta(\sigma_n(s^n_t), t) + k_\Delta(t, \mu_\delta) \leq \delta + C_0 =: C_{x_1}. \]

Therefore, fix $x_1 > \mu_\delta$, and set $T := C_{x_1}$. By (4.2), for all $t \in [\mu_\delta, x_1]$ we have
\[ k_\Delta(\sigma(s^n_t), t) \leq k_\Delta(\sigma(s^n_t), \sigma_n(\mu_\delta)) + k_\Delta(\sigma_n(\mu_\delta), t) < 2\delta. \]

By the arbitrariness of $x_1$, this proves that $t \in S_\Delta(\sigma, 2\delta)$ for all $t \geq \mu_\delta$.

Since $f$ is an isometry for the hyperbolic distance, $f^{-1} \circ \sigma$ is a geodesic in $\mathbb{D}$ parameterized by arc length and $f^{-1}(t) \in S_\mathbb{D}(f^{-1} \circ \sigma; 2\delta)$ for all $t \geq \mu_\delta$. In particular, for every $t \geq \mu_\delta$ there exists $s_t \in [0, +\infty)$ such that
\begin{equation}
(4.3) \quad k_\mathbb{D}(f^{-1}(t), f^{-1}(\sigma(s_t))) < 2\delta.
\end{equation}

Note that this implies in particular that $s_t \to +\infty$ as $t \to +\infty$. Hence, $\lim_{t \to +\infty} f^{-1}(\sigma(s_t)) = \xi$ and $\lim_{t \to +\infty} f^{-1}(\sigma(t)) = \xi$.

Using a Cayley transform from $\mathbb{D}$ to $\mathbb{H}$ which maps $\xi$ to $\infty$, it follows that there exists $s_1 \geq 0$ such that $f^{-1}(\sigma(s)) \in S_\mathbb{D}(\gamma; \delta)$ for all $s \geq s_1$. In particular, for every $s \geq s_1$, there exists $r_s \in [0, 1)$ such that
\begin{equation}
(4.4) \quad k_\mathbb{D}(f^{-1}(\sigma(s)), \gamma(r_s)) < \delta.
\end{equation}

Let $t_\epsilon \geq \mu_\delta$ be such that $s_t \geq s_1$ for all $t \geq t_\epsilon$. Then by (4.3) and (4.4), for all $t \geq t_\epsilon$,
\[ k_\mathbb{D}(\gamma(r_{s_t}), f^{-1}(t)) \leq k_\mathbb{D}(f^{-1}(\sigma(s_t)), \gamma(r_{s_t})) + k_\mathbb{D}(f^{-1}(t), f^{-1}(\sigma(s_t))) < 3\delta = \epsilon, \]
and Step 5 follows.

From Step 5 we see that $\limsup_{t \to +\infty} k_\mathbb{D}(f^{-1}(t), [0, 1]) = 0$, hence, $f^{-1}(t)$ converges to $\xi$ orthogonally.

Now, we may prove Theorem 1.1. \qed
Proof of Theorem 1.1. If \( \{z_n\} \subset \Delta \) is such that \( \{f^{-1}(z_n)\} \) converges orthogonally to some \( \sigma \in \partial \mathbb{D} \), then the result follows at once taking \( U = \Delta \) and any \( R > 0 \). Indeed, by Proposition 3.2 we can assume \( \Delta = U = \mathbb{D} \), and the statement is immediate.

Conversely, assume that Conditions (1) and (2) of Theorem 1.1 are satisfied. By Proposition 3.2 we can assume \( U = \mathbb{H} \) and \( \gamma(t) = t \), for every \( t \geq 1 \). Since \( E^u_R(\infty, R) = \{z \in \mathbb{C} : \Re w > R\} = \mathbb{H} + R \), then \( \mathbb{H} + R \subset \Delta \subset \mathbb{H} \) by Condition (1) and it follows from Proposition 4.1 that there exists \( \sigma \in \partial \mathbb{D} \) such that \( f^{-1}(\gamma(t)) \to \sigma \) orthogonally as \( t \to +\infty \). In particular, this implies that, if \( \eta : [0, 1) \to \mathbb{D} \) is the geodesic \( \eta(t) = t\sigma \), then for every \( \epsilon > 0 \) there exists \( t_0 \geq 1 \) such that for all \( t \geq t_0 \),

\[
(4.5) \quad k_{\mathbb{D}}(f^{-1}(\gamma(t)), \eta([0, 1))) < \epsilon.
\]

On the other hand, by Condition (2) of Theorem 1.1 and by Lemma 3.4 for every \( \epsilon > 0 \) there exists \( n_0 \) such that for all \( n \geq n_0 \)

\[
(4.6) \quad k_{\mathbb{D}}(f^{-1}(z_n), f^{-1}(\gamma([0, +\infty)))) = k_{\Delta}(z_n, \gamma([0, +\infty))) \leq k_{E^u_R(\infty, R)}(z_n, \gamma([0, +\infty))) < \epsilon.
\]

Note that, since \( \{z_n\} \) is compactly divergent in \( \Delta \) — and so is \( \{f^{-1}(z_n)\} \) in \( \mathbb{D} \) — this implies in particular that \( \{f^{-1}(z_n)\} \) converges to \( \sigma \).

By the triangle inequality, (4.5) and (4.6) imply that for every \( \epsilon > 0 \) there exists \( n_1 \) such that \( k_{\mathbb{D}}(f^{-1}(z_n), \eta([0, 1])) < \epsilon \) for all \( n \geq n_1 \). Therefore, \( \{f^{-1}(z_n)\} \) converges orthogonally to \( \sigma \). \( \square \)

5. Applications to semigroups

In this section we state some direct but interesting consequences of Theorem 1.1 for the dynamics of semigroups in \( \mathbb{D} \). Similar results hold for discrete dynamics of holomorphic self-maps of \( \mathbb{D} \). We leave details to the interested reader.

A one-parameter continuous semigroup of holomorphic self-maps of \( \mathbb{D} \) — or, for short, semigroup in \( \mathbb{D} \) — is a continuous homomorphism of the real semigroup \([0, +\infty)\) endowed with the Euclidean topology to the semigroup under composition of holomorphic self-maps of \( \mathbb{D} \) endowed with the topology of uniform convergence on compacta. Semigroups in \( \mathbb{D} \) have been largely studied (see, e.g., \([2, 5, 18]\)). It is known that, if \( (\phi_t) \) is a semigroup in \( \mathbb{D} \), which is not a group of hyperbolic rotations, then there exists \( \tau \in \mathbb{D} \), the Denjoy-Wolff point of \( (\phi_t) \), such that \( \lim_{t \to +\infty} \phi_t(z) = \tau \), and the convergence is uniform on compacta. In case \( \tau \in \mathbb{D} \), the semigroup is called elliptic. Non-elliptic semigroups can be divided into three types: hyperbolic, parabolic of positive hyperbolic step and parabolic of zero hyperbolic step. It is known (see \([10, 11]\)) that if \( (\phi_t) \) is a hyperbolic semigroup then \( \{\phi_t(z)\} \) always converges non-tangentially to its Denjoy-Wolff point as \( t \to +\infty \) for every \( z \in \mathbb{D} \), while, if it is parabolic of positive hyperbolic step then \( \{\phi_t(z)\} \) always converges tangentially to its Denjoy-Wolff point as \( t \to +\infty \) for every \( z \in \mathbb{D} \).

In case of parabolic semigroups of zero hyperbolic step, the behavior of trajectories can be rather wild. All the trajectories have the same slope, that is the cluster set of
arg(1 − τφ_t(z)) as \( t \to +\infty \)—which is a compact subset of \([-π/2, π/2]\)—does not depend on \( z \) (see [10, 11]). In most cases this slope is just a point, but in [4, 12] examples are constructed such that the slope is the full interval \([-π/2, π/2]\) and in [4, 8] examples are constructed where the slope is a closed subset of \((-π/2, π/2]\) not reduced to a single point.

Recall that (see, e.g., [2, 3, 13, 6]) \((φ_t)\) is a parabolic semigroup in \(D\) of zero hyperbolic step if and only if there exists a univalent function \(h\), the Königs function of \((φ_t)\) such that \(h(D)\) is starlike at infinity, \(h(φ_t(z)) = h(z) + it\) for all \(t \geq 0\) and \(z \in D\), and for every \(w \in \mathbb{C}\) there exists \(t_0 \geq 0\) such that \(w + it_0 \in h(D)\). The triple \((\mathbb{C}, h, z + it)\) is called a canonical model for \((φ_t)\) and it is essentially unique.

Using Theorem 1.1 applied to \(h(D)\), we can find geometric conditions that guarantee that the slope reduces to \(\{0\}\).

We recall that for \(β \in (0, π)\), \(V(β) := \{ρe^{iθ} : ρ > 0, |θ| < β\}\).

**Corollary 5.1.** Let \((φ_t)\) be a parabolic semigroup in \(D\) of zero hyperbolic step and with canonical model \((\mathbb{C}, h, z + it)\) and Denjoy-Wolff point \(τ \in ∂D\). If

1. either there exists \(a > 0\) such that \((iH + it) \subset Δ \subset iH\),
2. or, there exist \(β \in (0, π)\), \(a > 0\), such that \((iV(β) + it) \subset Δ \subset iV(β)\),
3. or, there exist \(-∞ < a < b < +∞\) and \(c \in \mathbb{R}\) such that \(∂Δ\) is contained in the semistrip \(\{ζ ∈ \mathbb{C} : a < \Re ζ < b, \Im ζ < c\}\),

then \(φ_t(z) → τ\) orthogonally as \(t \to +∞\), for every \(z \in D\).

Statement (3) in Corollary 5.1 was proved by D. Betsakos in [4], as a consequence of Theorem 2 in [4].

**Proof.** Statements (1) and (2) follow straightforwardly from Theorem 1.1 considering, respectively, \(U = H\) and \(U = iV(β)\) and, for (2), computing horocycles in \(iV(β)\) using a Riemann map from \(H\) to \(iV(β)\).

As for (3), since \(Δ\) is starlike at infinity, if \(p \in ∂Δ\), then

\[Δ \subset \mathbb{C} \setminus \{ζ ∈ \mathbb{C} : \Re ζ = \Re p, \Im ζ ≤ \Im p\} =: U.\]

Therefore, using the Koebe map from \(D\) onto \(U\) in order to compute horocycles in \(U\), one obtains (3) directly from Theorem 1.1. □

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