COMPLETE SOBOLEV TYPE INEQUALITIES

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ABSTRACT. We establish Sobolev type inequalities in the noncommutative settings by generalizing monotone metrics in the space of quantum states, such as matrix-valued Beckner inequalities. We also discuss examples such as random transpositions and Bernoulli-Laplace models.

1. INTRODUCTION

Poincaré inequalities (PIs) and log Sobolev inequalities (LSIs) have been well developed in the last few decades. See [Led99, GZ03] for properties, applications and criterion of PIs and LSIs. Gross showed that log Sobolev inequalities and hypercontractivity are equivalent for Dirichlet form operators, see [Gro75]. Beckner inequalities (BIs), as an interpolation between PIs and LSIs, were introduced by Beckner ([Bec89]) in 1989 for the canonical Gaussian measures on $\mathbb{R}^n$. Later Ledoux ([Led97]) introduced a family of inequalities with the same pattern of LSIs and PIs, which also solved the regularity issues of porous medium equations ([Dem05, Váz07, BGL13]).

Log Sobolev inequalities in the quantum (noncommutative) settings have been studied recently, see [CM17, CM20, LJL20, GJL18, DR20, BCR20]. The idea of characterizing matrix-valued Sobolev type inequalities is still absent from the literature. Surprisingly, we explore a vast variety of Sobolev type inequalities by introducing the generalized monotone metrics in the space of quantum states.

Let us first recall that an ergodic system $T_t = e^{-t\Delta}$ on a probability space $(\Omega, \mu)$ satisfies the $\lambda$-LSI if there exists $\lambda > 0$ such that

$$\int \rho^2 \ln(\rho^2) d\mu - \lambda \int \rho^2 d\mu \leq \frac{1}{\lambda} \mathcal{E}_\Delta(\rho, \rho)$$

for any function $\rho$, where $\mathcal{E}_\Delta(\rho, \sigma) = \int_\Omega \Delta(\rho) \sigma d\mu$ is the energy form. We now use the notation $\text{Ent}(\rho) = \int \rho \ln(\rho) d\mu - \int \rho d\mu \ln(\int \rho d\mu)$ for the relative entropy. An equivalent formulation of LSIs is the exponential decay of the relative entropy:

$$\text{Ent}(T_t(\rho)) \leq e^{-\lambda t} \text{Ent}(\rho)$$

for any positive $\rho$, see [BGL13, LJL20]. For a convex function $f$, let us consider the relative entropy functional

$$\text{Ent}_f(\rho) = \int f(\rho) - f(\sigma) - (\rho - \sigma) f'(\sigma) d\mu,$$

where $\sigma = \int \rho d\mu$. We observe that $\text{Ent}_f = \text{Ent}$ for $f(x) = x \ln(x)$. Then $T_t$ satisfies the generalized Sobolev inequality associated to $f$ if there exists $\lambda > 0$ such that

$$\text{Ent}_f(T_t(\rho)) \leq e^{-\lambda t} \text{Ent}_f(\rho)$$

for any positive $\rho$. Again $f(x) = x \ln(x)$ returns the classical LSIs. Let $p \in (1, 2)$ and $f(x) = x^p$, then (3) is equivalent to

$$\|\rho\|_p^p - \|\rho\|_1^p \leq \frac{p}{\lambda(p)} \mathcal{E}_\Delta(\rho, \rho^{p-1}),$$

for any positive $\rho$. Again $f(x) = x \ln(x)$ returns the classical LSIs.
where \( p\mathcal{E}_\Delta(\rho, \rho^{p^{-1}}) \) is the analogue of the Fisher information associated to \( \Delta \). The limiting cases \( p \to 1^+ \) and \( p \to 2^+ \) reduce to LSIs and PIs, respectively ([BT06]). Setting \( q = \frac{2}{p} \) and \( g = \rho^{1/q} \), we obtain BIs

\[
\|g\|^2_2 - \|g\|^2_q \leq \frac{4 - 2q}{\lambda(2/q)} \mathcal{E}_\Delta(g, g),
\]

which was first introduced Beckner ([Bec89]) in 1989 for the canonical Gaussian measure on \( \mathbb{R}^n \) with optimal constants \( \lambda(q) = 2 \).

We aim at extending [3] to a finite von Neumann algebra \((\mathcal{N}, \tau)\) equipped with a normal faithful tracial state \( \tau \). We consider the semigroup \( T_t = e^{-tA} : \mathcal{N} \to \mathcal{N} \) of completely positive self-adjoint unital maps. Let \( \mathcal{N}_{\text{fix}} = \{ \rho | T_t(\rho) = \rho \} \) be the fixed point algebra of \( T_t \), which admits the conditional expectation. Then the generator \( A \) is said to satisfy the \( \lambda \)-modified \( f \)-Sobolev inequality (MfSI) if

\[
d^f(T_t(\rho)\|E(\rho)) \leq e^{-\lambda t} d^f(\rho\|E(\rho)), \quad \forall \rho \in \mathcal{N}_+, \]

where \( d^f(\rho\|\sigma) = \tau(f(\rho) - f(\sigma) - (\rho - \sigma)f'(\sigma)) \) for \( \rho, \sigma \in \mathcal{N}_+ \). We say \( A \) satisfies \( \lambda \)-complete \( f \)-Sobolev inequality (CfSI) if the above inequality remains true for \( A \otimes id_{\mathcal{M}} \), where \( \mathcal{M} \) is any finite von Neumann algebra. The case \( f(x) = x\ln(x) \) has been studied in a series of paper, see [CJL18, BGJ20, LJL20]. By imposing more conditions on \( f \), we would recover most properties of CLSIs such as stability under tensorization and change of measure.

Intriguingly, the study of generalized monotone metrics in the space of quantum states sheds light on \( C_f \)SIs and the Bregman relative entropy. The monotone metric was anticipated by Morozova and Chenstov ([MC89]) to transfer the geometric techniques to the noncommutative settings. Motivated by Morozova and Chenstov, Petz ([Pet96]) introduced monotone metrics systematically using the relative modular operators and discovered the equivalent relation between operator monotone functions and the monotone metrics. Later on, Hiai and Petz ([HP12]) extended the monotone metrics to two parameters. Continuing Petz’ study, we define define the generalized monotone metrics associated to two-variable functions via the double operator integral. By this new definition of generalized monotone metrics, we explore a wide range of Sobolev type inequalities.

The paper is organized as follows. In section 2, we introduce the generalized monotone metrics. In section 3, we define \( C_f \)SIs and establish \( C_f \)SIs for derivation triples. In section 4, we discuss examples and applications such as complete Beckner inequalities and random transpositions and Bernoulli-Laplace models.

## 2. Generalized Monotone Metrics

### 2.1. Monotone metrics.

Let \( \mathcal{N} \) be a finite von Neumann algebra equipped with a normal faithful tracial state \( \tau \) and \( \beta : \mathcal{N} \to \mathcal{N} \) be a completely positive trace preserving (CPTP) map. The set of positive elements in \( \mathcal{N} \) is denoted by \( \mathcal{N}_+ \). Let \( L_p(\mathcal{N}, \tau) \) denote the noncommutative \( L_p \) space, written as \( L_p(\mathcal{N}) \) if the trace \( \tau \) is clear from the context. Let \( \mathbb{R}^+ = (0, \infty) \) in the sequel. The left and right multiplications by \( \rho \in \mathcal{N} \) are defined by

\[
L_\rho(a) = \rho a \quad \text{and} \quad R_\rho(a) = a\rho, \quad \forall a \in \mathcal{N}.
\]

Note \( L_\rho \) and \( R_\sigma \) commute for any \( \rho, \sigma \in \mathcal{N} \). For \( \rho, \sigma \in \mathcal{N}_+ \) and \( f : \mathbb{R}^+ \to \mathbb{R}^+ \), we define \( J_{f,\rho,\sigma} : \mathcal{N} \to \mathcal{N} \) by

\[
J_{f,\rho,\sigma} = f(L_\rho R_\sigma^{-1}) R_\sigma,
\]

where \( p \mathcal{E}_\Delta(\rho, \rho^{p^{-1}}) \) is the analogue of the Fisher information associated to \( \Delta \). The limiting cases \( p \to 1^+ \) and \( p \to 2^+ \) reduce to LSIs and PIs, respectively ([BT06]). Setting \( q = \frac{2}{p} \) and \( g = \rho^{1/q} \), we obtain BIs

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where \( d^f(\rho\|\sigma) = \tau(f(\rho) - f(\sigma) - (\rho - \sigma)f'(\sigma)) \) for \( \rho, \sigma \in \mathcal{N}_+ \). We say \( A \) satisfies \( \lambda \)-complete \( f \)-Sobolev inequality (CfSI) if the above inequality remains true for \( A \otimes id_{\mathcal{M}} \), where \( \mathcal{M} \) is any finite von Neumann algebra. The case \( f(x) = x\ln(x) \) has been studied in a series of paper, see [CJL18, BGJ20, LJL20]. By imposing more conditions on \( f \), we would recover most properties of CLSIs such as stability under tensorization and change of measure.

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where $L_\rho R_\sigma^{-1}$ is the relative modular operator, see [Pet07] for more information. We use $\mathbb{J}_\rho^f$ if $\rho = \sigma$. The inverse of $\mathbb{J}_{\rho,\sigma}^f$ is given by

$$\left(\mathbb{J}_{\rho,\sigma}^f\right)^{-1} = f^{-1}(L_\rho R_\sigma^{-1}) R_\sigma^{-1}.$$ 

Let $\rho, \sigma \in \mathcal{N}_+$ and $f : \mathbb{R}^+ \to \mathbb{R}^+$, then the following conditions are equivalent ([HPT12]):

\begin{align}
\beta^* (\mathbb{J}_{\beta(\rho),\beta(\sigma)}^f)^{-1} &\leq (\mathbb{J}_{\rho,\sigma}^f)^{-1}; \\
\beta \mathbb{J}_{\rho,\sigma}^f &\beta^* \leq \mathbb{J}_{\beta(\rho),\beta(\sigma)}^f.
\end{align}

Let us recall the following generalized Lieb’s concavity theorem ([Pet85, HP12, HP13]).

**Theorem 2.1.** Let $\beta : \mathcal{N} \to \mathcal{N}$ be a CPTP map and $f : \mathbb{R}^+ \to \mathbb{R}^+$ be an operator monotone function. Assume that $\rho, \sigma \in \mathcal{N}_+$, then

$$\beta^* \left(\mathbb{J}_{\beta(\rho),\beta(\sigma)}^f\right)^{-1} \beta \leq \left(\mathbb{J}_{\rho,\sigma}^f\right)^{-1}.$$ 

Hiai and Petz usually require that $\rho, \sigma, \beta(\rho), \beta(\sigma)$ are invertible. As we pointed out in [LJL20] that it is enough to assume the positivity by perturbation argument $\rho + \epsilon I$ for $\epsilon \to 0^+$. Consequently Hiai and Petz defined the **monotone metrics with two parameters $\gamma_{f,\rho,\sigma}^f$** by

\begin{equation}
\gamma_{f,\rho,\sigma}^f(a,b) = \langle a, \left(\mathbb{J}_{\rho,\sigma}^f\right)^{-1}(b) \rangle, \quad \forall a, b \in \mathcal{N},
\end{equation}

where $\langle a, b \rangle = \tau(a^* b)$ is the Hilbert-Schmidt inner product. For $\rho, \sigma \in \mathcal{N}_+$ and an operator monotone function $f : \mathbb{R}^+ \to \mathbb{R}^+$, we have

$$\gamma_{\beta(\rho),\beta(\sigma)}^f(\beta(a),\beta(b)) \leq \gamma_{f,\rho,\sigma}^f(a,a), \quad a \in \mathcal{N}.$$ 

**Corollary 2.2.** For an operator monotone function $f$, the monotone metric $\gamma_{f,\rho,\sigma}^f(a,a)$ is a jointly convex function for $(\rho, \sigma, a)$ for $\rho, \sigma \in \mathcal{N}_+$ and $a \in \mathcal{N}$.

### 2.2. Generalized monotone metrics.

Let us recall that for $F : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ and $\rho, \sigma \in \mathcal{N}_+$ the double operator integral is defined by

$$Q_{F,\rho,\sigma}(a) = \int_0^\infty \int_0^\infty F(s,t)dE_\rho(s)dE_\sigma(t),$$

where $E_\rho((s,t]) = 1_{[s,t]}(\rho)$ is the spectral projection of $\rho$. We denote it by $Q_{\rho,\sigma}^F$ if $\rho = \sigma$. For a comprehensive account of the double operator integral, see [DK51, kre56, dPS04, dPS07, BS03, PST0]. For operators $\rho = \sum_{i=1}^k s_ip_i$ and $\sigma = \sum_{j=1}^l t_jq_j$ with discrete spectrum, this simplifies to a Schur multiplier

$$Q_{F,\rho,\sigma}^\rho(y) = \sum_{i=1}^k \sum_{j=1}^l F(s_i, t_j)p_i q_j, \quad \forall y \in \mathcal{N}.$$ 

Note that $(Q_{F,\rho,\sigma}^\rho)^{-1} = Q_{F,\rho,\sigma}^{\rho}$ . Let $f_{[0]}(x,y) = f(\frac{x}{y})$ for $f : \mathbb{R}^+ \to \mathbb{R}^+$, then $Q_{f_{[0]}}^\rho = \mathbb{J}_{\rho,\sigma}^f$. Let us introduce two families of functions:

\begin{align}
\mathcal{C}^- &= \{ F; \quad \beta Q_{F,\rho,\sigma}^\rho \beta^* \leq Q_{F,\beta(\rho),\beta(\sigma)}^\rho, \forall \rho, \sigma \in \mathcal{N}_+ \text{ and CPTP } \beta\}, \\
\mathcal{C}^+ &= \{ F; \quad \beta^* Q_{F,\rho,\sigma}^\rho \beta \leq Q_{F,\rho,\sigma}^\rho, \forall \rho, \sigma \in \mathcal{N}_+ \text{ and CPTP } \beta\}.
\end{align}

**Definition 2.3.** Let $F \in \mathcal{C}^+$ and $\rho, \sigma \in \mathcal{N}_+$. We define the (two-variable) generalized monotone metric $\gamma_{F,\rho,\sigma}^F : \mathcal{N} \to \mathcal{N}$ by

$$\gamma_{F,\rho,\sigma}^F(a,b) = \langle a, Q_{F,\rho,\sigma}^\rho(b) \rangle.$$
It follows from the definition that
\[
\gamma^F_{\beta(\rho),\beta(\sigma)}(\beta(a),\beta(a)) \leq \gamma^F_{\rho,\sigma}(a,a), \quad \forall a \in \mathcal{N}.
\]
We use the same notation as [8] defined by [HPT12], but we only refer to [8] if the superscript function \( f \) is one-variable. Let \( f \) be operator monotone, then we identify
\[
\gamma^f_{\rho,\sigma} = \gamma^F_{\rho,\sigma}
\]
with \( F = f^{-1}_{[0]} \).

**Theorem 2.4.** Let \( F \in \mathcal{C}^+ \) satisfying \( \lambda F(\lambda x, \lambda y) \leq F(x, y) \) for any \( \lambda \in [0, 1] \). Then the generalized monotone metric \( \gamma^F_{\rho,\sigma}(a,a) \) is a convex function for \( (\rho,\sigma,a) \) of \( \rho,\sigma \in \mathcal{N}^+ \) and \( a \in \mathcal{N} \).

**Proof.** We use the standard trick and consider \( \beta : M_2 \otimes \mathcal{N} \to \mathcal{N} \) defined by
\[
\left( \begin{array}{cc} x_1 & x_2 \\ x_3 & x_4 \end{array} \right) \mapsto x_1 + x_4.
\]
Then \( \beta \) is CPTP. Let \( \rho = (\lambda \rho_1 \ 0 \ 0 \ (1-\lambda) \rho_2), \sigma = (\lambda \sigma_1 \ 0 \ 0 \ (1-\lambda) \sigma_2), \) and \( a = (\lambda a_1 \ 0 \ 0 \ (1-\lambda) a_2) \) for some \( \lambda \in [0,1] \). By (12), we obtain that
\[
\gamma^F_{\lambda \rho_1+(1-\lambda) \rho_2,\lambda \sigma_1+(1-\lambda) \sigma_2}(\lambda a_1 + (1-\lambda)a_2,\lambda a_1 + (1-\lambda)a_2) 
\leq \gamma^F_{\lambda \rho_1,\lambda \sigma_1}(\lambda a_1,\lambda a_1) + \gamma^F_{(1-\lambda) \rho_2,(1-\lambda) \sigma_2}((1-\lambda)a_2,(1-\lambda)a_2).
\]
We further have
\[
\gamma^F_{\lambda \rho,\lambda \sigma}(\lambda a,\lambda a) \leq \lambda \gamma^F_{\rho,\sigma}(a,a).
\]
Indeed
\[
\gamma^F_{\lambda \rho,\lambda \sigma}(\lambda a,\lambda a) = \lambda^2 \langle a, \int_0^\infty \int_0^\infty F(x,y)dE_{\lambda \rho}(x)adE_{\lambda \sigma}(y) \rangle 
= \lambda^2 \langle a, \int_0^\infty \int_0^\infty F(\lambda x, \lambda y)dE_{\rho}(x)adE_{\sigma}(y) \rangle 
\leq \lambda \gamma^F_{\rho,\sigma}(a,a).
\]
Applying (13) completes the proof. \( \Box \)

The monotonicity of \( \gamma^F_{\rho,\sigma}(a,a) \) does not necessarily imply the joint convexity for \( F \in \mathcal{C}^+ \) since the condition \( \lambda F(\lambda x, \lambda y) \leq F(x, y) \) sometimes fails. Let \( f \) be operator monotone and \( F = f^{-1}_{[0]} \), we actually have the equality.

**Proposition 2.5.** We have the following properties.

1. The sets \( \mathcal{C}^+ \) and \( \mathcal{C}^- \) are positive cones.
2. Let \( F_1 \in \mathcal{C}^+ \) and \( F_2 \in \mathcal{C}^- \), let \( F'_1(x,y) = F_1(x+t,y+s) \) and \( F'_2(x,y) = F_2(x+t,y+s) \) for any fixed \( t, s \geq 0 \). Then \( F'_1 \in \mathcal{C}^+ \) and \( F'_2 \in \mathcal{C}^- \).
3. If \( F \in \mathcal{C}^+ \), then \( \frac{1}{F} \in \mathcal{C}^- \). Similarly if \( F \in \mathcal{C}^- \), then \( \frac{1}{F} \in \mathcal{C}^+ \).
4. Let \( f : \mathbb{R}^+ \to \mathbb{R}^+ \) be operator monotone, then \( f_{[0]} \in \mathcal{C}^- \) and \( f^{-1}_{[0]} \in \mathcal{C}^+ \).

**Proof.** We only give proofs for (3) and (4). The equivalence between
\[
\beta^* (Q_F^{\beta(\rho),\beta(\sigma)})^{-1} \beta \leq (Q_F^{\rho,\sigma})^{-1}
\]
and
\[
\beta Q_F^{\rho,\sigma} \beta^* \leq Q_F^{\beta(\rho),\beta(\sigma)}
\]
yields (3). (4) follows directly from Theorem 2.1 and (3). \( \Box \)
Example 2.6. Let $f(x) = x^{-1}$, then $f$ is operator monotone. Indeed, $f(x) = \int_0^1 x^r dr$ and $x^r$ is operator monotone for $r \in [0, 1]$. Then we have
\[
\begin{align*}
f_0(x, y) &= \frac{x - y}{\ln(x) - \ln(y)} \in \mathcal{C}^- \quad \text{and} \quad f_0^{-1}(x, y) = \frac{\ln(x) - \ln(y)}{x - y} \in \mathcal{C}^+.
\end{align*}
\]
Let $f^{[1]}(x, y) = \frac{f(x) - f(y)}{x - y}$ denote the difference quotient of $f$. We consider the following set
\[
(16) \quad \mathcal{C}^+ = \{ f' | f^{[1]} \in \mathcal{C}^+ \}.
\]

Proposition 2.7. We have the following properties.
(1) The set $\mathcal{C}^+$ is a positive cone.
(2) The set $\mathcal{C}^+$ is invariant under right translation.
(3) Let $f(x) = \frac{x^p - x^q}{x^r}$ with $k \geq 0$, then $f \in \mathcal{C}^+$.

**Proof.** We only give the proof of (3). Let $g(x) = \ln(x + k)$. By Example 2.6 and Proposition 2.5, we have $g^{[1]} \in \mathcal{C}^+$. If follows from the definition that $f = g' \in \mathcal{C}^+$.

Example 2.8. Let $F(x, y) = \frac{x^p - y^p}{x - y}$ for $p \in (0, 1)$. Let us recall that
\[
f(x) = x^p = \frac{\sin(p\pi) x}{\pi} \int_0^\infty \frac{t^{p-1}}{r + x} dr.
\]
By Proposition 2.7, we have $f' \in \mathcal{C}^+$ and $F = f^{[1]} \in \mathcal{C}^+$. We cannot find a function $f$ such that $F^{-1} = f_0$. Thus $\mathcal{C}^+$ is a strict extension of the operator monotone functions.

Remark 2.9. In a discussion, we noticed that Haonan Zhang also gave a proof of the example above separately. Haonan Zhang was trying to develop the matrix-valued Beckner inequalities using the geodesic convexity techniques in [CM20] and [CM17].

3. Complete Sobolev type inequality

3.1. Derivations. Let $\mathcal{N}$ be a finite von Neumann algebra equipped with a normal faithful tracial state $\tau$. Let $\mathcal{H} \mathcal{N}$ be a self-adjoint Hilbert $\mathcal{N}$-$\mathcal{N}$ bimodule with the antilinear form $J$. A derivation of a von Neumann algebra $\mathcal{N}$ is a densely defined linear operator $\delta : L_2(\mathcal{N}, \tau) \to \mathcal{H}$ such that
(1) $\text{dom}(\delta)$ is a weakly dense $\ast$-subalgebra in $\mathcal{N}$;
(2) the identity element $1 \in \text{dom}(\delta)$;
(3) $\delta(xy) = x\delta(y) + \delta(x)y$, for any $x, y \in \text{dom}(\delta)$.

We always work with a closable derivation and denote the closure by $\bar{\delta}$. A derivation $\delta$ is said to be $\ast$-preserving if $J(\delta(x)) = \delta(x^*)$. Every closable $\ast$-preserving derivation $\delta$ determines a positive operator $\delta^* \delta$ on $L_2(\mathcal{N}, \tau)$. It was shown in [Sau90] that $T_t = e^{-t\delta} : \mathcal{N} \to \mathcal{N}$ is a strongly continuous semigroup of CPTP maps. See [HN95], [IO80], [Pet09], [Kap53], and [BR76] for more details. The functional calculus of a derivation $\delta$ is given by
\[
(17) \quad \delta(f(\rho)) = Q^\rho_{f^{[1]}}(\delta(\rho)) = \int_0^\infty \int_0^\infty \frac{f(s) - f(t)}{s - t} dE_\rho(s) \delta(\rho) dE_\rho(t).
\]

Now let $T_t = e^{-tA} : \mathcal{N} \to \mathcal{N}$ be a strongly continuous semigroup of completely positive unital self-adjoint maps on $L_2(\mathcal{N}, \tau)$. The generator $A$ is a positive operator on $L_2(\mathcal{N}, \tau)$ given by
\[
A(x) = \lim_{t \to 0^+} \frac{1}{t} (T_t(x) - x), \forall x \in \text{dom}(A).
\]

It was pointed out in [Sau90] that $\text{dom}(\delta) = \{ x \in \mathcal{N} \parallel A^{1/2} x \parallel_2 < \infty \}$ is indeed a $\ast$-algebra and invariant under the semigroup. The weak gradient form of $A$ is defined by
\[
\Gamma_A(x, y)(z) = \frac{1}{2} [\tau(A(x)^* yz + \tau(x^* A(y)z) - \tau(x^* yA(z))].
\]
If the weak gradient form $\Gamma_A(x, y) \in L_1(\mathcal{N})$ for all $x, y \in \text{dom}(A^{1/2})$, we say the generator $A$ (or $T_t$) satisfies $\Gamma$-regularity.

**Theorem 3.1** (JRS14). If $A$ satisfies $\Gamma$-regularity, then there exists a finite von Neumann algebra $(\mathcal{M}, \tau)$ containing $\mathcal{N}$ and a $*$-preserving derivation $\delta_A : \text{dom}(A^{1/2}) \to L_2(\mathcal{M})$ such that

$$\tau(\Gamma_A(x, y)z) = \tau(\delta_A(x)^*\delta_A(y)z).$$

Equivalently $\Gamma_A(x, y) = E_\mathcal{N}(\delta_A(x)^*\delta_A(y))$, where $E_\mathcal{N} : \mathcal{M} \to \mathcal{N}$ is the conditional expectation.

Throughout the paper, we always work with a closable $*$-preserving derivation $\delta$ and a strongly continuous semigroup $T_t = e^{-tA}$ of completely positive unital self-adjoint maps on $L_2(\mathcal{N}, \tau)$ satisfying $\Gamma$-regularity.

### 3.2. Generalized Fisher information.

The $f$-Fisher information $I_A^{f,\tau}$ of $A$ is defined as

$$I_A^{f,\tau}(\rho) = \tau(A(\rho)f'(\rho)), \forall \rho \in \text{dom}(A^{1/2}) \cap L_2(\mathcal{N}) \text{ and } f'(\rho) \in L_\infty(\mathcal{N}).$$

Equivalently

$$I_A^{f,\tau}(\rho) = \lim_{\epsilon \to 0^+} \tau(A(\rho)(f'(\rho + \epsilon 1))).$$

For a derivation $\delta$, the Fisher information is defined as

$$I_{\delta}^{f,\tau}(\rho) = \tau\left(\delta(\rho)Q_{f|_{\mathbb{R}^+}}^\rho(\delta(\rho))\right), \forall \rho \in \text{dom}(\delta) \subset \mathcal{N},$$

where

$$f^{[2]}(x, y) = \frac{f'(x) - f'(y)}{x - y}.$$

Then $I_{\delta}^{f}(\rho) = I_{\delta_A}^{f}(\rho)$. We use $I_A^{f,\tau}$ or $I_{\delta}^{f,\tau}$ if the trace is clear from the context. In the rest of this section, we always consider convex and continuously differentiable $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that $f^{[2]} \in C^+$. By Theorem 3.1, for any $A$ satisfying $\Gamma$-regularity, there exists a closable $*$-preserving derivation $\delta_A : \text{dom}(A^{1/2}) \to L_2(\mathcal{M})$ such that $\Gamma_A(x, y) = E_\mathcal{N}(\delta_A(x)^*\delta_A(y))$ where $E_\mathcal{N} : \mathcal{M} \to \mathcal{N}$. Thus

$$I_A^{f}(\rho) = I_{\delta_A}^{f}(\rho).$$

The choice of $\delta_A$ is not necessarily unique, but $I_A^{f}$ is uniquely determined. We recapture the widely used Fisher information $I_A(\rho) = \tau(A(\rho)\ln(\rho))$ by choosing $f(x) = x \ln(x)$. We shall also observe the relation between the $f$-Fisher information and the generalized monotone metric

$$I_{\delta}^{f}(\rho) = \gamma_{\rho, f}^{[2]}(\delta(\rho), \delta(\rho)).$$

**Lemma 3.2** (non-negativity). The $f$-Fisher information is nonnegative.

**Proof.** The convexity of $f$ implies that $f^{[2]} \geq 0$. Set $w = (Q_{f|_{\mathbb{R}^+}}^\rho)^{1/2}(\delta_A(\rho))$ with

$$(Q_{f|_{\mathbb{R}^+}}^\rho)^{1/2}(y) = \int_0^\infty \int_0^\infty \left(\frac{f'(s) - f'(t)}{s - t}\right)^{1/2} dE_\rho(s)ydE_\rho(t).$$

Thus

$$I_A^{f}(\rho) = \tau(\delta_A(\rho)Q_{f|_{\mathbb{R}^+}}^\rho(\delta_A(\rho)))$$

$$= \tau(E_{\mathcal{N}}(ww)) \geq 0.$$

Similarly $I_{\delta}$ is also nonnegative. \qed

An important example is $f(x) = x^p$ for $p \in (1, 2)$, and we denote such $p$-Fisher information by $P_A^p$ or $I_{\delta}^p$. As an application of Theorem 2.4, we get the following result.

**Corollary 3.3.** The $p$-Fisher information is convex.
Recall that for any finite von Neumann algebra $\mathcal{N}$, there exists a $\sigma$-finite measure space $(X, \mu)$ such that $\mathcal{Z}(\mathcal{N}) \cong L_\infty(X, \mu)$ and $\mathcal{N} = \int_X \mathcal{N}_x d\mu(x)$, where $\mathcal{Z}(\mathcal{N})$ is the center of $\mathcal{N}$ and $\mathcal{N}_x$ is a factor for any $x \in X$. Now we rewrite the $f$-Fisher information by using the direct integral

$$I_\delta^f(\rho) = \int_X I_\delta^f(\rho_x) d\mu(x).$$

**Lemma 3.4.** Let $\tau_1$ and $\tau_2$ be normal faithful traces over $\mathcal{N}$ and $\frac{d\mu}{d\tau_2} \geq c$ for some $c > 0$. Then for any $\rho \in \mathcal{N}_+$,

$$cI_A^{\tau_2}(\rho) \leq I_A^{\tau_1}(\rho).$$

The result remains true for $I_\delta^f$.

**Proof.** Two traces only differ by two measures $\mu_1$ and $\mu_2$ over the center $L_\infty(X, \mu_1) \cong L_\infty(X, \mu_2) \cong \mathcal{Z}(\mathcal{N})$. Note that $\frac{d\mu_1}{d\mu_2} \geq c$ if and only $\frac{d\mu_2^*}{d\mu_1} \geq c$. Setting the pointwise differential form $w_x = (Q_{f[2]}^0)^{1/2}(\delta_A(\rho_x))$, we infer that

$$cI_A^{\tau_2}(\rho) = c \int_X \tau_2(E_N(w_x w_x)) d\mu_2(x) \leq \int_X \tau_1(E_N(w_x w_x)) d\mu_1(x) = I_A^{\tau_1}(\rho).$$

\[\square\]

### 3.3. Bregman relative entropy.

Let us recall the definition of $f$-Bregman relative entropy

$$d^f(\rho || \sigma) = \tau(f(\rho) - f(\sigma) - (\rho - \sigma)f'(\sigma)), $$

for $\rho, \sigma \in \mathcal{N}_+$. For simplicity, we would call it as $f$-relative entropy. Equivalently

$$d^f(\rho || \sigma) = \lim_{\epsilon \to 0^+} d^f(\rho || \sigma + \epsilon 1).$$

We write $d^f(\rho || \sigma)$ if the trace $\tau$ is clear from the context. For a comprehensive study of Bregman relative entropy, see [MPV16] [PV15] [Vir16]. It follows from the definition that $d^f(\rho || \sigma) \geq 0$ with the equality if and only if $\rho = \sigma$. Note that we identify the Lindblad relative entropy

$$d^f(\rho || \sigma) = \tau(\rho \ln \rho - \rho \ln \sigma - \rho + \sigma) = D_{\text{Lind}}(\rho || \sigma)$$

with the choice $f(x) = x \ln(x)$. The $f$-relative entropy admits an integral representation ([PV15])

$$d^f(\rho || \sigma) = \int_{s=0}^1 \tau \left( (\rho - \sigma) \frac{d}{dt} f(\sigma + (s+t)(\rho - \sigma)) |_{t=0} \right) ds.$$  

Let $\mathcal{K} \subset \mathcal{N}$ be a von Neumann subalgebra of $\mathcal{N}$ and $E_\mathcal{K}$ be the conditional expectation onto $\mathcal{K}$. The relative entropy with respect to $\mathcal{K}$ is defined by

$$d^f_\mathcal{K}(\rho) = d^f(\rho || E_\mathcal{K}(\rho)).$$

Noting $\tau((\rho - E_\mathcal{K}(\rho)) f'(E_\mathcal{K}(\rho))) = 0$, then

$$d^f_\mathcal{K}(\rho) = \tau(f(\rho) - f(E_\mathcal{K}(\rho))).$$

**Lemma 3.5.** The following equality remains true

$$d^f(\rho || \sigma) = d^f_\mathcal{K}(\rho) + d^f(E_\mathcal{K}(\rho) || \sigma)$$

for any $\sigma \in \mathcal{K}$.

**Proof.** Note $\tau(\rho f'(\sigma)) = \tau(E_\mathcal{K}(\rho) f'(\sigma))$. Then we have

$$d^f(\rho || \sigma) = \text{tr} \left( f(\rho) - f(E_\mathcal{K}(\rho)) + f(E_\mathcal{K}(\rho)) - f(\sigma) + (\rho - \sigma)f'(\sigma) \right)$$

$$= d^f_\mathcal{K}(\rho) + \tau \left( f(E_\mathcal{K}(\rho)) - f(\sigma) + (E_\mathcal{K}(\rho) - \sigma)f'(\sigma) \right)$$

$$= d^f_\mathcal{K}(\rho) + d^f(E_\mathcal{K}(\rho) || \sigma).$$

\[\square\]
Together with nonnegativity of $f$-relative entropy, Lemma 3.5 implies that
\begin{equation}
    d\ell^K_\rho = \inf_{\sigma \in \mathcal{K}} d\ell(f(\rho\|\sigma)).
\end{equation}

Let $\mathcal{N}_{\text{fix}} \subset \mathcal{N}$ be the fixed point algebra of the semigroup $T_t = e^{-tA}$ and $E$ be the conditional expectation onto $\mathcal{N}_{\text{fix}}$, then $ET_t = T_tE = E$.

**Lemma 3.6** (gradient form). The semigroup $T_t$ relates the $f$-relative entropy and the $f$-Fisher information. The $f$-Fisher information is the negative derivative of $d\ell^K_\rho (T_t(\rho))$,
\begin{equation}
    \frac{d}{dt} d\ell^K_\rho (T_t(\rho)) = -I_A^f(T_t(\rho)).
\end{equation}

**Proof.** Let $g(t) = d\ell^K_\rho (\rho_t) = \tau(f(\rho_t) - f(E(\rho_t)))$. By the chain rule, we obtain that
\begin{equation}
    g'(t) = \frac{d}{dt} \tau(f(\rho_t) - f(E(\rho))) = \tau(-A(T_t(\rho)))f'(T_t(\rho)) = -I_A^f(T_t(\rho)).
\end{equation}

\hfill \Box

**Lemma 3.7.** Let $\tau_1$ and $\tau_2$ be two normal faithful traces over a finite von Neumann algebra $\mathcal{N}$ such that $\frac{d\tau_1}{d\tau_2} \leq c$ for $c > 0$. For any $\rho, \sigma \in \mathcal{N}_+$,
\begin{equation}
    d\ell_\tau^f(\rho\|\sigma) \leq c d\ell_\tau^f(\rho\|\sigma).
\end{equation}

In particular, we have $d\ell_\tau^f(\rho) \leq c d\ell_\tau^f(\rho)$.

**Proof.** We follow the notations and idea in the proof of Lemma 3.4. Also note that $\frac{d\tau_1}{d\tau_2} \leq c$ if and only if $\frac{d\mu_1}{d\mu_2} \leq c$. Again by the non-negativity of the $f$-relative entropy, we have
\begin{equation}
    d\ell_\tau^f(\rho\|\sigma) = \int_X d\ell_\tau^f(\rho_x\|\sigma_x)d\mu_1(x) 
\end{equation}
\begin{equation}
    \leq c \int_X d\ell_\tau^f(\rho_x\|\sigma_x)d\mu_2(x) = c d\ell_\tau^f(\rho\|\sigma).
\end{equation}

The second assertion follows from (23). \hfill \Box

**Theorem 3.8** (Data Processing Inequality). Let $\Phi : \mathcal{N} \to \mathcal{N}$ a quantum channel (CPTP), then
\begin{equation}
    d\ell(\Phi(\rho\|\Phi(\sigma)) \leq d\ell(f(\rho\|\sigma)) \quad \forall \rho \in \mathcal{N}_+, \sigma \in \mathcal{N}_{\text{fix}}.
\end{equation}

**Proof.** Let $a(t) = (1-t)\sigma + t\rho$ for $t \in [0,1]$, and we consider the function
\begin{equation}
    H_{\rho,\sigma}(t) = \tau(f(a(t))).
\end{equation}

By chain rule $H'(t) = \tau(f'(a(t)|(\rho - \sigma))$. It follows from the integration by parts that
\begin{equation}
    \int_0^1 (1-t)H''_{\rho,\sigma}(t)dt = (1-t)H''_{\rho,\sigma}(0)_t |_0 = \int_0^1 (1-t)H''_{\rho,\sigma}(t)dt
\end{equation}
\begin{equation}
    = -H'_{\rho,\sigma}(0) + H_{\rho,\sigma}(1) - H_{\rho,\sigma}(0) = d\ell(\rho\|\sigma).
\end{equation}

Recall that $\lim_{t \to 0^+} \frac{a(\rho + t\sigma) - a(\rho)}{t} = Q_{\gamma_{\rho1}^0}(\sigma)$, then we have
\begin{equation}
    H''_{\rho,\sigma}(t) = \tau\left( \lim_{\epsilon \to 0^+} \frac{f'(a(t + \epsilon)) - f'(a(t))}{\epsilon}(\rho - \sigma) \right)
\end{equation}
\begin{equation}
    = \tau\left( \lim_{\epsilon \to 0^+} \frac{f'(a(t) + \epsilon(\rho - \sigma)) - f'(a(t))}{\epsilon}(\rho - \sigma) \right)
\end{equation}
\begin{equation}
    = \tau((\rho - \sigma)Q_{f^t_{\rho2}}(\rho - \sigma)) = \gamma_{\alpha(t),\alpha(t)}^t(\rho - \sigma, \rho - \sigma).
Then $f^{[2]} \in \mathcal{C}^+$ implies that
\[ H''_{\Phi(\rho), \Phi(\sigma)}(t) \leq H''_{\rho, \sigma}(t). \]
Together with (24) it yields the assertion. \hfill \Box

Let $f(x) = x^p$ for $p \in (1, 2)$. We obtain the $p$-relative entropy
\[ d^p(\rho \| \sigma) = \tau(\rho^p - \sigma^p - p(\rho - \sigma)\sigma^{p-1}). \]
Thus
\[ d^p_\rho(\rho) = \tau(\rho^p - (E_K(\rho))^p). \]

It shall be noted that $p$-relative entropy is different from the (sandwiched) Rényi entropy. The $p$-relative entropy with respect the conditional expectation (27) appeared in [BT06], where they studied the classical (commutative) situations and ergodic systems. The general properties of Bregman relative entropy are systematically studied in [PV15, Vir16, MPV16]. Again we use the standard argument as in Theorem 2.4 and obtain the joint convexity.

**Corollary 3.9.** The $f$-relative entropy $d^p(\rho \| \sigma)$ is a jointly convex function for $(\rho, \sigma)$ for $\rho, \sigma \in \mathcal{N}_+$ if $d^f(\lambda \rho \| \lambda \sigma) \leq \lambda d^f(\rho \| \sigma)$ for any $\lambda \in [0, 1]$. Thus $D_{\text{Lind}}$ and $d^p$ are jointly convex.

3.4. $C_t$SI.

**Definition 3.10.** The semigroup $T_t = e^{-tA}$ or the generator $A$ with the fixed-point algebra $\mathcal{N}_{\text{fix}}$ is said to satisfy:

1. the modified $f$-Sobolev inequality $\lambda$-$\text{M}_f\text{SI}$ (with respect to the trace $\tau$) if there exists a constant $\lambda > 0$ such that
   \[ \lambda d^f_{\tau}(\rho) \leq I^f_A(\rho), \quad \forall \rho \in \text{dom}(\delta) \cap \mathcal{N}_+. \]
   (or equivalently $d^f_{\tau}(T_t(\rho)) \leq e^{-\lambda t}d^f_{\tau}(\rho)$, $\forall \rho \in \mathcal{N}_+$.)

2. the complete $f$-Sobolev inequality $\lambda$-$\text{M}_f\text{SI}$ (with respect to the trace $\tau$) if $A \otimes \text{id}_{\mathcal{F}}$ satisfies $\lambda$-$\text{M}_f\text{SI}$ for any finite von Neumann algebra $\mathcal{F}$.

Let $C_{t}\text{SI}(A, \tau)$ be the supremum of $\lambda$ such that $A$ satisfies $\lambda$-$C_{t}\text{SI}$, or denoted by $C_{t}\text{SI}(A)$ if there is no ambiguity. Sometimes we use $C_{t}\text{SI}(T_t)$ for convenience.

$C_{t}\text{SI}$ is a generalization of the complete logarithmic Sobolev inequality. An important example is $f(x) = x^p$ for $p \in (1, 2)$, which induces the so-called $C_p\text{SI}$ and $\text{CLSI}^+$ [LJL20].

**Lemma 3.11.** Let $E_K : \mathcal{N} \to \mathcal{K}$ be a conditional expectation. Then
\[ I^f_{I-E_K}(\rho) = d^f(\rho \| E_K(\rho)) + d^f(E_K(\rho)\| \rho) \]
and hence $C_{t}\text{SI}(I - E_K) \geq 1$.

**Proof.** We have
\[ d^f(E_K(\rho)\| \rho) = \tau (f(E_K(\rho)) - f(\rho) - (E_K(\rho) - \rho)f'(\rho)) = \tau ((I - E_K)(\rho)f'(\rho)) = -d^f(\rho \| E_K(\rho)) + I^f_{I-E_K}(\rho). \]
Thus $C_{t}\text{SI}(I - E_K)$ follows from the nonnegativity of $f$-relative entropy. \hfill \Box

This result for $\text{CLSI}$ was given by [DPR17]. We can obtain a better constant for $C_p\text{SI}$, see [LJL20]. Applying Lemma 3.6 we have the following equivalence:

**Proposition 3.12.** The following conditions are equivalent:

1. $\lambda d^f_{\tau}(\rho) \leq I^f_A(\rho)$ for any $\rho \in \mathcal{N}_+$;
2. $d^f_{\tau}(T_t(\rho)) \leq e^{-\lambda t}d^f_{\tau}(\rho)$ for any $\rho \in \mathcal{N}_+$.
Combining Lemma 3.7 and Lemma 3.4 we obtain the following change of measure principle.

**Theorem 3.13** (change of measure principle). Let $\tau_1$ and $\tau_2$ be normal faithful traces over $\mathcal{N}$ and $c_2 \leq \frac{dN_1}{d\tau_2} \leq c_1$ for some $c_1, c_2 > 0$. Then $C_t\text{SI}(A, \tau_1) \geq \frac{c_2}{c_1} C_t\text{SI}(A, \tau_2)$.

The change of measure principle is often referred to as Holley and Stroock (HSS86) argument. They proved that LSIs are stable under change of measures. This remains true for CLSIs [LJL20].

**Theorem 3.14** (tensorization stability). Let $T^j_t : \mathcal{N}_j \to \mathcal{N}_j$ be a family of semigroups with fixed-point algebras $\mathcal{N}_j = \mathcal{N}_j$ for $1 \leq j \leq k$. Then the tensor semigroup $T_t = \otimes_{j=1}^k T^j_t$ has the fixed-point algebra $\mathcal{N}_f = \otimes_{j=1}^k \mathcal{N}_f$. Moreover, we have

$$C_t\text{SI}(T_t) \geq \inf_{1 \leq j \leq k} C_t\text{SI}(T^j_t).$$

**Proof.** It suffices to prove for the 2-fold tensor product. For the $n$-fold tensor product, we may use the standard induction argument. Let $E_1, E_2, E_3$ be the conditional expectations onto $\mathcal{N}_f, \mathcal{N}_f$ and $\mathcal{N}_f$, respectively. Applying (3.3) and Theorem 3.8 gives

$$d^f_{\mathcal{N}_f} = d^f(\mathcal{N}_f) + d^f(\mathcal{N}_f) + d^f(\mathcal{N}_f) \leq d^f(\mathcal{N}_f) + d^f(\mathcal{N}_f) + d^f(\mathcal{N}_f).$$

For the $n$-fold tensor product, we may use the standard induction argument. □

The following result is motivated by [Spo78] and [GJL18] (lemma 2.6).

**Theorem 3.15.** Let $T_t = e^{-tA}$ be a semigroup of completely positive self-adjoint unital maps on $L_2(\mathcal{N}, \tau)$. Suppose there exists some positive constant $\lambda$ such that

$$I^f_{\mathcal{N}}(T_t(\rho)) \leq e^{-\lambda t} I^f_{\mathcal{N}}(\rho), \forall \rho \in \mathcal{N}_+,$$

then $C_t\text{SI}(A) \geq \lambda$.

**Proof.** We use Lemma 3.6 again. Define $g(t) = d^f_{\mathcal{N}_f} (T_t(\rho))$, then

$$-g(t) \leq -e^{-\lambda t} g(0).$$

Integrating both sides from $[0, \infty)$ yields $C_t\text{SI}(A) \geq \lambda$. □

**Remark 3.16.** In many situations we do not need the condition $f^{[2]} \in \mathbb{C}^+$, such as Lemma 3.2, 3.4, 3.6, 3.7, 3.11, Prop 3.12, Theorem 3.13 and 3.15. However, this condition is necessary to obtain the data processing inequality.

3.5. $C_t\text{SI}$ of derivation triple. Let us recall the definition of a derivation triple in [LJL20]. Let $\mathcal{N}$ be a finite von Neumann algebra equipped with a normal faithful tracial state $\tau$, and $\delta$ be a closable $\tau$-preserving derivation on $\mathcal{N}$. Suppose there exists a larger finite von Neumann algebra $(\mathcal{M}, \tau)$ containing $\mathcal{N}$ and a weakly dense $\tau$-subalgebra $\mathcal{A} \subset \mathcal{N}$ such that

1. $\mathcal{A} \subset \text{dom}(\delta)$;
2. $\delta^\tau : \mathcal{A} \to \mathcal{A}$;
3. $\delta : \mathcal{A} \to L_2(\mathcal{M}, \tau)$.

We define $\pi_{\delta} : \Omega^1(\mathcal{A}) \to \mathcal{M}$ by

$$\pi_{\delta}(a \otimes b - 1 \otimes ab) = \delta(a)b,$$

where $\Omega^1(\mathcal{A}) = \{ \sum_j [a_j \otimes b_j - 1 \otimes a_j b_j] | a_j, b_j \otimes \mathcal{A} \} \subset \mathcal{A} \otimes \mathcal{A}$. Thus $\Omega(\mathcal{A})$ is Hilbert $\mathcal{A}$-bimodule with inner product

$$(\delta(a_1)b_1, \delta(a_2)b_2)_\mathcal{A} = b_1^* E_{\mathcal{N}}(\delta(a_1))\delta(a_2)b_2,$$

where $E_{\mathcal{N}} : \mathcal{M} \to \mathcal{N}$ is the conditional expectation and $(\cdot, \cdot)_\mathcal{A}$ is the $\mathcal{N}$-valued inner product. A linear operator $Rc : \Omega(\mathcal{A}) \to \mathcal{M}$ is called the Ricci operator of $(\mathcal{N} \subset \mathcal{M}, \delta, \tau)$ provided that
(1) $\text{Rc}(a \rho b) = a \text{Rc}(\rho) b$, \quad \forall a, b \in \mathcal{A}, \rho \in \Omega_{\delta}(\mathcal{A})$;

(2) there exists a strongly continuous semigroup $\hat{T}_t = e^{-tL} : \mathcal{M} \to \mathcal{M}$ of completely positive trace preserving maps such that $\Gamma_L(a, b) = E_N(\delta(a^* \delta b))$ and $\delta(\delta^* \delta a) - L(\delta(a)) = \text{Rc}(\delta(a))$ for any $a, b \in \mathcal{A}$.

The derivation $\delta$ is said to admit a Ricci curvature $\text{Rc} \geq \lambda$ bounded below by a constant $\lambda$, if $(\text{Rc}(\rho), \rho)_A \geq \lambda E_N(\rho^* \rho)$ for any $\rho \in \Omega_{\delta}(\mathcal{A})$. We say the generator $A$ of $T_t = e^{-tA}$ admits $\text{Rc} \geq \lambda$ if there exists a derivation triple $(\mathcal{N} \subset \mathcal{M}, \delta, \tau)$ such that

$$\Gamma_A(a, b) = E_N(\delta(a^* \delta b)), \quad \forall a, b \in \mathcal{A}$$

and $\delta$ admits $\text{Rc} \geq \lambda$. It shall be noted that the choice of $\delta$ is not unique, thus we may find a larger Ricci lower bound of $A$ by choosing a good $\delta$.

**Lemma 3.17.** Let $(\mathcal{N} \subset \mathcal{M}, \delta, \tau)$ be a derivation triple with a Ricci curvature $\text{Rc} \geq \lambda > 0$. Then

$$I_\delta^f(e^{-t \delta^* \delta}(\rho)) \leq e^{-2\lambda t} I_\delta^f(\rho), \quad \forall \rho \in \mathcal{N}_+.$$  

**Proof.** In the proof, we use the following notations $A = \delta^* \delta$, $T_t = e^{-tL}$, and $T_t(\rho) = \rho_t$. Let $\hat{T}_t = e^{-tL}$ be the semigroup given in the definition of the Ricci curvature. Let us consider two functions

$$h(t) = I_\delta^f(\rho_t) = \tau \left( \int_0^\infty \int_0^\infty \delta(\rho_t) \frac{f'(s)}{s-l} dE_{\rho_t}(s) dE_{\rho_t}(l) \right)$$

and

$$k(t) = \gamma_{\rho_t, \rho_t}^{f[2]}(\hat{T}_t(\delta(\rho)), \hat{T}_t(\delta(\rho)))$$

$$= \tau \left( \int_0^\infty \int_0^\infty \hat{T}_t(\delta(\rho)) \frac{f'(s)}{s-l} dE_{\rho_t}(s) dE_{\rho_t}(l) \right).$$

By $\Gamma_L(a, b) = E_N(\delta(a^* \delta b))$, then

$$k(t) = \gamma_{\rho_t, \rho_t}^{f[2]}(\hat{T}_t(\delta(\rho)), \hat{T}_t(\delta(\rho))).$$

Noting $f^{[2]} \in \mathcal{C}^+$, we deduce that $k(t) \leq k(0)$ and

$$k'(0) \leq 0.$$

By the product rule, we have

$$h'(t) = -2\tau \left( \int_0^\infty \int_0^\infty \delta(A(\rho_t)) \frac{f'(s)}{s-l} dE_{\rho_t}(s) dE_{\rho_t}(l) \right) + h'_r(t)$$

and

$$k'(t) = -2\tau \left( \int_0^\infty \int_0^\infty L(\hat{T}_t(\delta(\rho))) \frac{f'(s)}{s-l} dE_{\rho_t}(s) dE_{\rho_t}(l) \right) + k'_r(t),$$

where $h'_r$ and $k'_r$ are the derivatives corresponding to $dE_{\rho_t}$. We make an important observation

$$h'_r(0) = k'_r(0).$$

Together with $\delta(\delta^* \delta a) - L(\delta(a)) = \text{Rc}(\delta(a))$, then

$$h'(0) - k'(0) = -2\tau \left( \int_0^\infty \int_0^\infty \text{Rc}(\delta(\rho)) \frac{f'(s)}{s-l} dE_{\rho}(s) dE_{\rho}(l) \right)$$

$$= -2\tau \left( \text{Rc} \left( Q_{f^{[2]}}^{1/2}(\delta(\rho)) \right) \left( Q_{f^{[2]}}^{1/2}(\delta(\rho)) \right) \right).$$

Using that $\text{Rc} \geq \lambda$, we infer that

$$h'(0) - k'(0) \leq -2\lambda h(0).$$
By (28), then
\[ h'(t) \leq -2\lambda h(t). \]

Setting \( h_s(t) = I^t_h(\rho_{t,s}) \), then \( h'_s(0) = h'(s) \). Inequality (30) remains true for \( h_s \). Hence
\[ h'(s) = h'_s(0) \leq -2\lambda h_s(0) = -2\lambda h(s) \]
completes the proof. \( \square \)

Applying Theorem 3.15, we get the following complete Sobolev inequality.

**Theorem 3.18.** Let \( (N \subset M, \delta, \tau) \) be a derivation triple with a Ricci curvature \( Rc \geq \lambda > 0 \). Let \( f : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) be continuously differentiable and \( f'' \in C^+ \). Then we have
\[ C_{p,SI}(N \subset M, \delta, \tau) \geq 2\lambda. \]

4. Applications

4.1. \( p \) norm estimate.

**Theorem 4.1.** Let \( (N \subset M, \delta, \tau) \) be a derivation triple with a Ricci curvature \( Rc \geq \lambda > 0 \) and \( T_\delta = e^{-\delta^2} \). Then we have
\[ \| T_\delta (\rho) - E(\rho) \|_p \leq e^{-\lambda t} \sqrt{\frac{2}{p(p-1)}} \| \rho \|_p^{1-p/2} (\| \rho \|_p^p - \| E(\rho) \|_p^p)^{1/2}, \forall \rho \in N. \]

**Proof.** This proof is inspired by [RX16]. For self-adjoint \( a, b \in N \), we define
\[ G_{a,b}(s) = \| a + sb \|_p^p - \frac{p(p-1)}{2} s^2 \| a + sb \|_p^{p-2} \| b \|_p^2. \]

It follows from the definition that \( G_{a,b} \) is convex over \( \mathbb{R} \) if \( G''_{x,y}(0) \geq 0 \) for any self-adjoint \( x, y \in N \). In [RX16], they considered the following function
\[ \psi(s) = \| a + sb \|_p^p \]
with an additional condition that \( a \) is invertible. They proved that
\[ \psi''(0) \geq p(p-1) \| a \|_p^{p-2} \| b \|_p^2. \]

It implies that \( G''_{a,b}(0) \geq 0 \). This remains true if \( a \) is not invertible, see the proof of Theorem 2 in [RX16]. Let \( a = E(\rho_t) = E(\rho) \) and \( b = \rho_t - E(\rho) \), then
\[ \| a + sb \|_p^p \geq \| E(a + sb) \|_p^p = \| a \|_p^p. \]

Hence the right derivative of \( G_{a,b} \) at 0 is nonnegative, and convexity implies \( G'_{a,b}(s) \geq 0 \) for any \( s \geq 0 \). In particular \( G(1) \geq G(0) \), then
\[ \| E(\rho) \|_p^p + \frac{p(p-1)}{2} \| \rho_t \|_p^{p-2} \| \rho_t - E(\rho) \|_p^2 \leq \| \rho_t \|_p^p. \]

By \( C_{p,SI} \), we have
\[ \| \rho_t \|_p^p \leq \| E(\rho) \|_p^p + e^{-2\lambda t} \left( \| \rho \|_p^p - \| E(\rho) \|_p^p \right). \]

Chaining the two inequalities gives
\[ \| \rho_t - E(\rho) \|_p^2 \leq e^{-2\lambda t} \frac{2}{p(p-1)} \| \rho_t \|_p^{2-p} \left( \| \rho \|_p^p - \| E(\rho) \|_p^p \right). \]

Noting \( \| \rho_t \|_p \leq \| \rho \|_p \) and taking the square root of the inequality above complete the proof. \( \square \)
4.2. Bakry-Émery criterion. Let \((M, g)\) be a smooth \(n\)-dimensional Riemannian manifold without boundary. For a smooth function \(U \in C^\infty(M)\), we define a probability measure \(\mu\) by

\[
\frac{1}{Z_v} e^{-U} \, dv
\]

with the normalization factor \(Z_v = \int_M e^{-U} \, dv\) and a Bakry-Émery Ricci curvature \(Rc_v\) by

\[
Rc_v = Rc + \text{Hess}(U).
\]

Applying Theorem 3.18, we obtain the modified Laplace operator

\[
\Delta_v = \Delta + \nabla U \cdot \nabla,
\]

where \(\Delta\) is the Laplace-Beltrami operator. See [LJL20] for the definition of a derivation triple of a Riemannian manifold.

**Theorem 4.2.** Let \((M, g, \mu)\) be a smooth Riemannian manifold with the measure \(\mu\) defined by

\[
\frac{1}{Z_v} e^{-U} \, dv
\]

with \(Z_v = \int_M e^{-U} \, dv\) for \(U \in C^\infty(M)\). Given that \(Rc_v \geq \kappa > 0\) and \(f[2] \in \mathcal{C}^+\), then

\[
C_f \text{SI}(\Delta_v) \geq 2\kappa.
\]

Let \(f(x) = x^p\), we obtain the complete Beckner inequalities.

**Corollary 4.3** (complete Beckner Inequalities). Let \((M, g, \mu)\) be a smooth Riemannian manifold with the measure \(\mu\) defined by

\[
\frac{1}{Z_v} e^{-U} \, dv
\]

with \(Z_v = \int_M e^{-U} \, dv\) for \(U \in C^\infty(M)\). Given that \(Rc_v \geq \kappa > 0\), then

\[
C_p \text{SI}(\Delta_v) \geq 2\kappa.
\]

By Theorem 3.13, we have the following result.

**Theorem 4.4.** Let \(\nu\) be the probability measure defined by

\[
\frac{1}{Z_v} e^{-V} \, dv
\]

with \(Z_v = \int_M e^{-V} \, dv\), where \(Z_v\) is the normalization factor. If \(\|U - V\|_\infty \leq C\) and \(f[2] \in \mathcal{C}^+\), then

\[
e^{2C} C_f \text{SI}(\Delta_v) \geq C_f \text{SI}(\Delta_v).
\]

4.3. Random Transpositions. Let \(S_n\) be the permutation group on \(\{1, \ldots, n\}\), and we consider the Laplace operator \(\Delta_n\) given by

\[
(\Delta_n f)(\sigma) = \frac{1}{n} \sum_{i,j=1}^{n} \left[ f(\sigma) - f(\sigma^{ij}) \right],
\]

where \(\sigma^{ij}\) denotes the configuration of \(\sigma\) after swapping the elements on \(i\)-th site and \(j\)-th site. For example let \(\sigma = (1 \ 3 \ 2)\) and \(i = 1\) and \(j = 2\), then \(\sigma^{ij} = (3 \ 1 \ 2)\). It is well-known that \(\Delta_n\) is ergodic, and thus \(E_n(f) = \frac{1}{n} \sum_{\sigma \in S_n} f(\sigma)\). The lower bound of Ricci curvature \(Rc\) of the random transposition on the Symmetric group \(S_n\) is defined using the geodesic convexity, and \(Rc \geq \frac{4}{n}\) ([EMT15] and [FM16]). MLSIs were studied in [Goe04], [BT06], and [GQ03] using the martingale methods in [LY98], and they proved that

\[
1 \leq \text{MLSI}(\Delta_n) \leq 4.
\]

The upper was given by the spectral gap \(\lambda_2(\Delta_n) = 2\) ([DS87]). We also apply the martingale methods and establish a similar relation between \(C_p \text{SI}(\Delta_{n+1})\) and \(C_p \text{SI}(\Delta_n)\):

**Theorem 4.5.** Let \(p \in (1, 2)\), then

\[
p \leq C_p \text{SI}(\Delta_n) \leq 4 \quad \text{and} \quad 1 \leq \text{CLSI}(\Delta_n) \leq 4
\]

for any \(n \geq 2\).
As pointed out by [BT06] (Section 4), the upper bound of $M_{p}\text{SI}(\Delta)$ is also given by the spectral gap. It is sufficient to give the lower bound. Let $\mathcal{M}$ be a finite von Neumann algebra equipped with a normal faithful trace $\tau$, and we consider $\mathcal{M}$-valued function. For $f \in \ell_{\infty}^m$, let $\tau(f) = \frac{1}{m} \sum_{j=1}^{m} \tau(f(j))$.

**Lemma 4.6.** For any $\mathcal{M}$-valued function $f$ defined over $\{1, \ldots, n\}$ and $n \geq 2$, we have

$$\tau(f^p - E(f)) \leq \frac{1}{2n^2} \sum_{i,j=1}^{n} \tau \left[ (f(i) - f(j)) \left( f(i)^{p-1} - f(j)^{p-1} \right) \right],$$

where $E(f) = \frac{1}{n} \sum_{i=1}^{n} f(i)$.

This lemma is an immediate application of $C_{p}\text{SI}(I - E) \geq p$ ([LJL20]). The scalar case of the following lemma was proven in [BT06], and here we give an operator valued version.

**Lemma 4.7.** Let $p \in (1, 2)$ and $F(\rho, \sigma) = \tau \left[ (\rho - \sigma)(\rho^{p-1} - \sigma^{p-1}) \right]$, then $F$ is jointly convex for $x, y \in \mathcal{M}_+$.

**Proof.** Let $f = (\rho, \sigma)$ and $E(f) = \frac{1}{p}(\rho + \sigma)$, then $\delta(f) = \frac{1}{p}(f(2) - f(1), f(1) - f(2))$ and $\delta^*\delta = I - E$. We can rewrite $F$ as $F(\rho, \sigma) = \frac{2}{p} \delta^p(f)$. By Corollary 3.3 $F$ is jointly convex. $\square$

Now we prove Theorem 4.5

**Proof.** Let $\mathcal{N}_i \subset L_\infty(S_{n+1}, \mathcal{M})$ be a von Neumann subalgebra generated by $\{e_i^j\}_{j=1}^{n+1}$ satisfying

$$e_i^j(\sigma) = \begin{cases} 1, & \text{if } \sigma_i = j; \\ 0, & \text{otherwise}. \end{cases}$$

We denote the corresponding conditional expectation by $E_{\mathcal{N}_i}$. By martingale equality, we have

$$d^p(f\|E(f)) = d^p(f\|E_{\mathcal{N}_i}(f)) + d^p(E_{\mathcal{N}_i}(f)\|E(f)).$$

Since $i$ is also uniformly chosen from from the $n + 1$ sites, then

$$d^p(f\|E(f)) = \frac{1}{n+1} \sum_{i=1}^{n+1} \left[ d^p(f\|E_{\mathcal{N}_i}(f)) + d^p(E_{\mathcal{N}_i}(f)\|E(f)) \right].$$

(32)

For any fixed $i$, we define

$$f_i(j) = \sum_{\sigma_i=j} f(\sigma).$$

Let

$$E_{ij}(f)(\sigma) = \frac{1}{n!} \begin{cases} f_i(j), & \text{if } \sigma_i = j; \\ 0, & \text{otherwise}, \end{cases}$$

then

$$E_{\mathcal{N}_i}(f) = \frac{1}{n+1} \sum_{j=1}^{n+1} E_{ij}(f).$$

Let us define a projection map

$$P_{ij}(f)(\sigma) = \begin{cases} f(\sigma), & \text{if } \sigma_i = j; \\ 0, & \text{otherwise}, \end{cases}$$

then

$$d^p(f\|E_{\mathcal{N}_i}(f)) = \frac{1}{(n+1)!} \sum_{j=1}^{n+1} \tau (P_{ij}(f)^p - E_{ij}(f)^p).$$
Also by our assumption
\[
\frac{C_p \text{SI}(\Delta_n)}{(n+1)!} \tau \left( P_{ij}(f)^p - E_{ij}(f)^p \right)
\]
\[
\leq \frac{p}{(n+1)!2n} \sum_{\{(\sigma,k,l)|\sigma_i = \sigma_i^{kl} = j\}} \tau \left[ \left( f(\sigma^{kl}) - f(\sigma) \right) \left( f(\sigma^{kl})^{p-1} - f(\sigma)^{p-1} \right) \right].
\]

It is important to observe that
\[
\sum_{i=1}^{n+1} \sum_{j=1}^{n+1} \tau \left[ \left( f(\sigma^{kl}) - f(\sigma) \right) \left( f(\sigma^{kl})^{p-1} - f(\sigma)^{p-1} \right) \right] = (n-1) \sum_{\sigma,k,l} \tau \left[ \left( f(\sigma^{kl}) - f(\sigma) \right) \left( f(\sigma^{kl})^{p-1} - f(\sigma)^{p-1} \right) \right].
\]

Thus
\[
(33) \quad \frac{1}{n+1} \sum_{i=1}^{n+1} d^p(f\|E_{Ni}(f)) \leq \frac{n-1}{n C_p \text{SI}(\Delta_n)} I_{\Delta_{n+1}}^p(f).
\]

Applying Lemma 4.6, we obtain that
\[
d^p(E_{Ni}(f)\|E(f)) \leq \frac{1}{2(n+1)^2} \sum_{k,l=1}^{n+1} \tau \left[ (E_{ik}(f) - E_{il}(f)) (E_{ij}(f)^{p-1} - E_{il}(f)^{p-1}) \right].
\]

By the definition of $E_{ik}$, we have
\[
(E_{ik}(f) - E_{il}(f)) (E_{ij}(f)^{p-1} - E_{il}(f)^{p-1})
\]
\[
= \left( \frac{1}{n!} \sum_{\sigma_i = l} f(\sigma^{kl}) - \frac{1}{n!} \sum_{\sigma_i = l} f(\sigma) \right) \left( \left( \frac{1}{n!} \sum_{\sigma_i = l} f(\sigma^{kl}) \right)^{p-1} - \left( \frac{1}{n!} \sum_{\sigma_i = l} f(\sigma) \right)^{p-1} \right).
\]

Together with Lemma 4.7, it implies that
\[
\tau \left[ (E_{ik}(f) - E_{il}(f)) (E_{ij}(f)^{p-1} - E_{il}(f)^{p-1}) \right]
\]
\[
\leq \frac{1}{n!} \sum_{\sigma_i = l} \tau \left[ (f(\sigma^{kl}) - f(\sigma)) (f(\sigma^{kl})^{p-1} - f(\sigma)^{p-1}) \right].
\]

Then we have
\[
d^p(E_{Ni}(f)\|E(f)) \leq \frac{1}{2(n+1)^2n} \sum_{k,l=1}^{n+1} \sum_{\sigma_i = l} \tau \left[ (f(\sigma^{kl}) - f(\sigma)) (f(\sigma^{kl})^{p-1} - f(\sigma)^{p-1}) \right].
\]

Thus
\[
(34) \quad \frac{1}{n+1} \sum_{i=1}^{n+1} d^p(E_{Ni}(f)\|E(f)) \leq \frac{1}{(n+1)p} I_{\Delta_{n+1}}^p(f).
\]

Combining (33) and (34), we obtain
\[
\frac{1}{C_p \text{SI}(\Delta_{n+1})} \leq \frac{n-1}{n C_p \text{SI}(\Delta_n)} + \frac{1}{p(n+1)}.
\]

Lemma 4.6 implies that $C_p \text{SI}(\Delta_2) \geq 2p$. By induction method, we have
\[
C_p \text{SI}(\Delta_{n+1}) \geq p.
\]
Indeed, by assuming that \( C_{p} SI(\Delta_n) \geq p \) we obtain that

\[
\frac{1}{C_{p} SI(\Delta_{n+1})} \leq \frac{1}{p} \left( \frac{n-1}{n} + \frac{1}{n+1} \right) \leq \frac{1}{p}.
\]

We only prove the estimate for \( C_{p} SI \) and the argument remains true for CLSI.

\[\square\]

4.4. Bernoulli-Laplace Model. We consider the Bernoulli-Laplace model with \( n \) distinct sites \( \{1, \ldots, n\} \) and \( r \) identical particles, where \( n \geq 2 \) and \( 1 \leq r \leq n - 1 \). Each site can be occupied by at most 1 particle. Let \( C_{n,r} \) be the state space of the configurations of \( r \) elements occupying \( n \) sites. The Laplace operator \( \Delta_{n,r} : L_{\infty}(C_{n,r}) \rightarrow L_{\infty} \) is defined by

\[
(\Delta_{n,r} f)(\sigma) = \frac{1}{n} \sum_{i<j} [f(\sigma) - f(\sigma^{ij})].
\]

Again \( \sigma^{ij} \) is the configuration of \( \sigma \) after we swap the \( i \)-th site and the \( j \)-th site. Let \( \sigma_{i} \) denote the number of particles occupying the \( i \)-th site. The lower bound of the Ricci curvature of the BL model was also studied in [EMT15] and [FM16]. [Goe04], [BT06], and [GQ03] proved that

\[1/2 \leq MLSI(\Delta_{n,r}) \leq 2\]

Again we use the noncommutative martingale method and obtain a similar estimate.

**Theorem 4.8.** Let \( p \in (1,2) \), then

\[p/2 \leq C_{p} SI(\Delta_{n,r}) \leq 2 \quad \text{and} \quad 1/2 \leq CLSI(\Delta_{n,r}) \leq 2\]

for any \( n \geq 2 \) and \( 1 \leq r \leq n - 1 \).

Again the upper bound of \( M_{p} SI(\Delta) \) is also given by the spectral gap, see [DS87]. Let \( \mathcal{M} \) be a finite von Neumann algebra equipped with a normal faithful trace \( \tau \), and we consider \( \mathcal{M} \)-valued function. For \( f \in \ell_{\infty}^{m} \), let \( \tau(f) = \frac{1}{m} \sum_{j=1}^{m} \tau(f(j)) \). The proof of 4.8 is quite similar to the proof of 4.5.

**Proof.** Let \( \mathcal{N}_{i} \subset L_{\infty}(C_{n+1,r}, \mathcal{M}) \) be a von Neumann subalgebra generated by \( \{e_{i}^{0}, e_{i}^{1}\} \) defined by

\[
e_{i}^{j}(\sigma) = \begin{cases} 1, & \text{if } \sigma_{i} = j; \\ 0, & \text{otherwise}, \end{cases} \quad j = 0, 1.
\]

Let \( E_{\mathcal{N}_{i}} \) be the corresponding conditional expectation onto \( \mathcal{N}_{i} \). By martingale equality, we have that

\[
d^p(f \| E(f)) = d^p(f \| E_{\mathcal{N}_{i}}(f)) + d^p(E_{\mathcal{N}_{i}}(f) \| E(f)).
\]

Since \( i \) is uniformly chosen from from the \( n+1 \) sites, then

\[
d^p(f \| E(f)) = \frac{1}{n+1} \sum_{i=1}^{n+1} \left[ d^p(f \| E_{\mathcal{N}_{i}}(f)) + d^p(E_{\mathcal{N}_{i}}(f) \| E(f)) \right].
\]

(35)

For any fixed \( i \), we define

\[
f_{i}(j) = \sum_{\sigma_{i} = j} f(\sigma), \quad j = 0, 1.
\]

Let

\[
E_{i,j}(\sigma) = \begin{cases} f_{i}(j), & \text{if } \sigma_{i} = j; \\ 0, & \text{otherwise}, \end{cases}
\]

where \( a_{0} = \binom{n-1}{r} \) and \( a_{1} = \binom{n-1}{r-1} \). Let us define projections

\[
P_{i,j}(\sigma) = \begin{cases} f(\sigma), & \text{if } \sigma_{i} = j; \\ 0, & \text{otherwise}, \end{cases} \quad j = 0, 1.
\]
Then
\[ d^p(f\|E_{N_0}(f)) = \frac{1}{n+1} \sum_{j=0,1} \tau [P_{i,j}(f)^p - E_{i,j}(f)^p]. \]

By the definition of \( C_{p\SI} \), then we have
\[ C_{p\SI}(\Delta_{n,r-\bar{j}}) \tau [P_{i,j}(f)^p - E_{i,j}(f)^p] \]
\[ \leq \frac{p^p}{2n} \sum_{\{\sigma, k, l|\sigma_i = \sigma_i^{kl} = j\}} \tau \left[ (f(\sigma^{kl}) - f(\sigma)) \left( f(\sigma^{kl})^{p-1} - f(\sigma)^{p-1} \right) \right]. \]

An important observation is that
\[ \sum_{i=1}^{n+1} \sum_{j=0,1} \sum_{\{\sigma, k, l|\sigma_i = \sigma_i^{kl} = j\}} \tau \left[ (f(\sigma^{kl}) - f(\sigma)) \left( f(\sigma^{kl})^{p-1} - f(\sigma)^{p-1} \right) \right] \]
\[ = (n-1) \sum_{\sigma, k, l} \tau \left[ (f(\sigma^{kl}) - f(\sigma)) \left( f(\sigma^{kl})^{p-1} - f(\sigma)^{p-1} \right) \right]. \]

Thus
\[ (36) \quad \frac{1}{n+1} \sum_{i=1}^{n+1} d^p(f\|E_{N_0}(f)) \leq \frac{n-1}{n} \frac{1}{\min_{j=0,1}\{C_{p\SI}(\Delta_{n,r-\bar{j}})\}} I_{\Delta_{n+1,r}}(f). \]

Applying Lemma 4.6, we obtain that
\[ d^p(E_{N_0}(f)\|E(f)) \]
\[ \leq \frac{a_0a_1}{(a_0 + a_1)^2} \tau \left[ \left( \frac{f_i(1)}{a_1} - \frac{f_i(0)}{a_0} \right) \left( \left( \frac{f_i(1)}{a_1} \right)^{p-1} - \left( \frac{f_i(0)}{a_0} \right)^{p-1} \right) \right]. \]

The definition of \( f_i(j) \) infers that
\[ f_i(1) = \sum_{\sigma_i = 1} f(\sigma) = \frac{1}{(n-r+1)} \sum_{k=1}^{n+1} \sum_{\sigma_k = 1, \sigma_i = 0} f(\sigma^{ki}), \]
\[ f_i(0) = \sum_{\sigma_i = 0} f(\sigma) = \frac{1}{r} \sum_{k=1}^{n+1} \sum_{\sigma_k = 1, \sigma_i = 0} f(\sigma). \]

Together with Lemma 4.7, it implies that
\[ \tau \left[ (E_{i,1}(f) - E_{i,0}(f)) (E_{i,1}(f)^{p-1} - E_{i,0}(f)^{p-1}) \right] \]
\[ \leq \frac{(n-r)!(r-1)!}{n!} \sum_{k=1}^{n+1} \sum_{\sigma_k = 1, \sigma_i = 0} \tau \left[ (f(\sigma^{ik}) - f(\sigma)) \left( f(\sigma^{ik})^{p-1} - f(\sigma)^{p-1} \right) \right]. \]

Similarly
\[ \tau \left[ (E_{i,1}(f) - E_{i,0}(f)) (E_{i,1}(f)^{p-1} - E_{i,0}(f)^{p-1}) \right] \]
\[ \leq \frac{(n-r)!(r-1)!}{n!} \sum_{k=1}^{n+1} \sum_{\sigma_k = 1, \sigma_i = 0} \tau \left[ (f(\sigma^{ik}) - f(\sigma)) \left( f(\sigma^{ik})^{p-1} - f(\sigma)^{p-1} \right) \right]. \]
Thus
\[ d^p(E_{X_n}(f)\|E(f)) \leq \frac{a_0a_1}{2(a_0 + a_1)^2} \frac{(n-r)!(r-1)!}{n!} \sum_{k=1}^{n+1} \sum_{\sigma_i \neq \sigma_k} \tau \left[ (f(\sigma^k) - f(\sigma)) \left( f(\sigma^k)^{p-1} - f(\sigma)^{p-1} \right) \right]. \]

Then we have
\[ \sum_{i=1}^{n+1} d^p(E_{X_n}(f)\|E(f)) \leq \frac{2}{p} I_{n+1,r}(f). \]

Together with \([33]\), it implies that
\[ \frac{1}{C_p\text{SI}_{\Delta_{n+1,r}}} \leq \frac{n-1}{n} \frac{1}{\text{min}_{j=0,1}\{C_p\text{SI}(\Delta_{n,r-j})\}} + \frac{2}{(n+1)p}. \]

Noting \( \Delta_{n,1} = \Delta_{n,n-1} = I - E \), we obtain that \( C_p\text{SI}(\Delta_{n,1}) \geq p \) and \( C_p\text{SI}(\Delta_{n,n-1}) \geq p \). By induction method, we have \( C_p\text{SI}(\Delta_{n+1,r}) \geq \frac{2}{p} \). Indeed, let us assume that \( C_p\text{SI}(\Delta_{n,r}) \geq \frac{2}{p} \), then
\[ \frac{1}{C_p\text{SI}(\Delta_{n+1,r})} \leq \frac{2}{p} - \left( \frac{1}{n} - \frac{1}{n+1} \right) \frac{2}{p} \leq \frac{2}{p}. \]

The argument remains true for CLSI.

\[ \square \]

**References**

[BCR20] Ivan Bardet, Angela Capel, and Cambyse Rouzé. Approximate tensorization of the relative entropy for noncommuting conditional expectations. *arXiv preprint arXiv:2001.07981*, 2020.

[Bec89] William Beckner. A generalized poincaré inequality for gaussian measures. *Proceedings of the American Mathematical Society*, pages 397–400, 1989.

[BR76] Ola Bratteli and Derek W. Robinson. Unbounded derivations of von neumann algebras. 25(2):139, 1976.

[BS03] Mikhail Sh Birman and Michael Solomyak. Double operator integrals in a hilbert space.

[BR76] Ola Bratteli and Derek W. Robinson. Unbounded derivations of von neumann algebras. 25(2):139, 1976.

[BS03] Mikhail Sh Birman and Michael Solomyak. Double operator integrals in a hilbert space. *Integral equations and operator theory*, 47(2):131–168, 2003.

[BT06] Sergey G Bobkov and Prasad Tetali. Modified logarithmic sobolev inequalities in discrete settings. *Journal of Theoretical Probability*, 19(2):289–336, 2006.

[CM17] Eric A Carlen and Jan Maas. Gradient flow and entropy inequalities for quantum markov semigroups with detailed balance. *Journal of Functional Analysis*, 273(5):1810–1869, 2017.

[CM20] Eric A Carlen and Jan Maas. Non-commutative calculus, optimal transport and functional inequalities in dissipative quantum systems. *Journal of Statistical Physics*, 178(2):319–378, 2020.

[Dem05] Jérôme Demange. Porous media equation and sobolev inequalities under negative curvature. *Bulletin des sciences mathématiques*, 129(10):804–830, 2005.

[DK51] Yu. L. Daleckii and S. G. Krein. Formulas of differentiation according to a parameter of functions of hermitian operators. 76:13, 1951.

[DPR17] Nilanjana Datta, Yan Pautrat, and Cambyse Rouzé. Contractivity properties of a quantum diffusion semigroup. *Journal of Mathematical Analysis*, 58(1):012205, 2017.

[dPS04] B. de Pagter and F. A. Sukochev. Differentiation of operator functions in non-commutative \( l_p \)-spaces. *Journal of Functional Analysis*, 212(1):28, 2004.

[dPS07] Ben de Pagter and Fyodor Sukochev. Commutator estimates and \( r \)-flows in non-commutative operator spaces. *Proceedings of the Edinburgh Mathematical Society, Series II*, 50(2):293, 2007.

[DR20] Nilanjana Datta and Cambyse Rouzé. Relating relative entropy, optimal transport and fisher information: A quantum lwi inequality. *Annales Henri Poincaré*, pages 1–36, Springer, 2020.

[DS87] Persi Diaconis and Mehrdad Shahshahani. Time to reach stationarity in the bernoulli–laplace diffusion model. *SIAM Journal on Mathematical Analysis*, 18(1):208–218, 1987.
[EMT15] Matthias Erbar, Jan Maas, and Prasad Tetali. Discrete ricci curvature bounds for bernoulli-laplace and random transposition models. In *Annales de la Faculté des sciences de Toulouse: Mathématiques*, volume 24, pages 781–800, 2015.

[FM16] Max Fathi and Jan Maas. Entropic ricci curvature bounds for discrete interacting systems. *The Annals of Applied Probability*, 26(3):1774–1806, 2016.

[GJL18] Li Gao, Marius Junge, and Nicolas LaRacuente. Fisher information and logarithmic sobolev inequality for matrix valued functions. *arXiv preprint arXiv:1807.08838*, 2018.

[Goe04] Sharad Goel. Modified logarithmic sobolev inequalities for some models of random walk. *Stochastic processes and their applications*, 114(1):51–79, 2004.

[GQ03] Fuqing Gao and Jeremy Quastel. Exponential decay of entropy in the random transposition and bernoulli-laplace models. *The Annals of Applied Probability*, 13(4):1591–1600, 2003.

[Gl75] Leonard Gross. Logarithmic sobolev inequalities. *American Journal of Mathematics*, 97(4):1061–1083, 1975.

[GZ03] Alice Guionnet and B Zegarlinski. Lectures on logarithmic sobolev inequalities. In *Séminaire de probabilités XXXVI*, pages 1–134. Springer, 2003.

[HN95] S. Hejazian and A. Niknam. Derivations of operator algebras, automatic closability. In *Different aspects of differentiability*, 1995.

[HP12] Fumio Hiai and D´enes Petz. From quasi-entropy to various quantum information quantities. *Publications of the Research Institute for Mathematical Sciences*, 48(3):525–542, 2012.

[HP13] Fumio Hiai and D´enes Petz. Convexity of quasi-entropy type functions: Lieb's and ando's convexity theorems revisited. *Journal of Mathematical Physics*, 54(6):062201, 2013.

[HS86] Richard Holley and Daniel W Stroock. Logarithmic sobolev inequalities and stochastic ising models. 1986.

[IO80] Atsushi Inoue and Shôichi Ota. Derivations on algebras of unbounded operators. *Transactions of the American Mathematical Society*, 261(2):567–577, 1980.

[JS14] M. Junge, E. Ricard, and D. Shlyakhtenko. Noncommutative diffusion semigroups and free probability. *Preprint*, 3, 2014.

[Kap53] Irving Kaplansky. Modules over operator algebras. *American Journal of Mathematics*, 75(4):839–858, 1953.

[KS85] Elena Aleksandrovna Morozova and Nikolai Nikolaevich Chentsov. Markov invariant geometry on state manifolds. *Itogi Nauki i Tekhniki. Seriya “ Sovremennye Problemy Matematiki. Noveishie Dostizheniya”*, 36:69–102, 1989.

[LP07] Jesse Peterson. A 1-cohomology characterization of property (t) in von neumann algebras. *Pacific journal of mathematics*, 243(1):181–199, 2009.

[PS08] D Potapov and F Sukochev. Double operator integrals and submajorization. *Mathematical Modelling of Natural Phenomena*, 5(4):317–339, 2010.

[PV15] József Pitrik and Dániel Virosztek. On the joint convexity of the bregman divergence of matrices. *Letters in Mathematical Physics*, 105(5):675–692, 2015.
[Vir16] Dániel Virosztek. Maps on quantum states preserving bregman and jensen divergences. Letters in Mathematical Physics, 106(9):1217–1234, 2016.

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