THE ELLIPTIC HYPERGEOMETRIC FUNCTION
AND 6j-SYMBOLS FOR THE SL(2,C) GROUP

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Abstract. We show that the complex hypergeometric function describing 6j-symbols for SL(2,C) group is a special degeneration of the V-function — an elliptic analogue of the Euler-Gauss 2F1 hypergeometric function. For this function, we derive mixed difference-recurrence relations as limiting forms of the elliptic hypergeometric equation and some symmetry transformations. At the intermediate steps of computations, there emerge a function describing the 6j-symbols for the Faddeev modular double and the corresponding difference equations and symmetry transformations.

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1. Introduction

Classical special functions [1] have natural interpretation as matrix elements of the operators realizing representations of standard Lie groups. For example, matrix elements of the SL(2,R) group representations are described by the Euler-Gauss 2F1 hypergeometric function [32]. One of the problems in the representation theory of Lie groups is the decomposition of tensor products of irreducible representations into a direct sum of irreducible representations. In the simplest case of the product of two representations the expansion coefficients are called Clebsch-Gordan coefficients, or 3j-symbols, which are again related to classical special functions. For triple tensor products, there are two different ways of sequential pairwise tensoring of representations leading to two different expressions for the expansion coefficients in terms of 3j-symbols. Matrix elements of the map between these two possible expansions are called 6j-symbols [31] and play a very important role both in physics and mathematics. In particular, they have found applications in quantum

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mechanics, two-dimensional conformal field theory, three-dimensional gravity, solutions of the Yang-Baxter equation and related integrable systems, knot theory, topology, and so on.

The well-known Racah polynomials were introduced as $6j$-symbols of the $SU(2)$ group \[20\]. Their explicit expression uses a special (Saalschützian) terminating hypergeometric $4F_3$-series. Similar situation holds for the $q$-Racah and Askey-Wilson polynomials \[2\]: they are related to $6j$-symbols of the quantum algebra $sl_q(2, \mathbb{R})$ \[13,16\] and are explicitly described by terminating $q$-hypergeometric $4\phi_3$-series.

The most complicated form of $6j$-symbols emerges for non-compact groups and their principal-series representations. Our key object here is the $6j$-symbols for the group $SL(2, \mathbb{C})$ (or proper Lorentz group $SO(3, 1)^+$ isomorphic to it), constructed in \[14\] and \[7\]. They are expressed in terms of a three complex-dimensional integral (six real dimensions), or, equivalently, as an infinite bilateral sum of the Mellin-Barnes type integrals. Another group-theoretical object that we are interested in the present paper is the Faddeev modular double $sl_q(2, \mathbb{R}) \times \tilde{sl}_{\tilde{q}}(2, \mathbb{R})$ \[10\]. Its continuous-series $6j$-symbols are built in \[19\] and have the form of a contour integral of a combination of Faddeev’s modular quantum dilogarithms \[8,9\]. This function is directly related to the hyperbolic hypergeometric function introduced by Ruijsenaars in \[22\], which can be considered a hyperbolic analogue of the Askey-Wilson function since both satisfy one finite-difference equation.

The next level of generalization of the $6j$-symbols is associated with the elliptic hypergeometric functions. Such objects appeared first in the context of elliptic solutions of the IRF (interaction round a face) type Yang-Baxter equation, which combine into an elliptic function taking the form of a terminating elliptic hypergeometric series \[11\]. The general $V$-function — a genuine elliptic hypergeometric function that is transcendental over the field of elliptic functions and which absorbs the elliptic $6j$-symbols of \[11\] — was constructed in \[28\]. This $V$-function represents an elliptic analogue of the Euler-Gauss hypergeometric function satisfying a second order difference equation with elliptic coefficients, which is called the elliptic hypergeometric equation \[29,30\].

As shown in \[24\], the elliptic beta integral \[27\] and $V$-function can be reduced to rather general complex hypergeometric functions in the Mellin-Barnes representation (with the hyperbolic hypergeometric functions obtained at the intermediate steps). In this paper, we show that the Mellin-Barnes form of $6j$-symbols obtained in \[7,14\] is a special subcase of a complex hypergeometric function constructed independently in \[6\] and \[24\]. We give also a detailed comparison of these $6j$-symbols with the functions emerging from $b \to i$ limit of the $6j$-symbols for the Faddeev modular double \[19\].

In addition to the $b \to i$ limit for the hyperbolic integral of Ruijsenaars \[22\] associated with the $6j$-symbols of \[19\], we describe its $b \to 0$ limit in some detail and derive corresponding identities. In particular, we derive difference equations satisfied by the limiting hypergeometric functions in different normalizations. In \[25\], a new singular limit $b \to 1$ was found for the Faddeev modular dilogarithm. It can be applied to the same Ruijsenaars integral \[22\] in order to derive symmetry relations for a particular rational hypergeometric function emerging in this limit. However, we skip consideration of the corresponding identities in this paper. Our general conclusion here is that the complex
6j-symbols, together with many other generalized hypergeometric functions, are degenerations of the elliptic hypergeometric $V$-function, which confirms its status of a universal special function of hypergeometric type.

2. Complex 6j-Symbols

The well known Euler's beta integral has the form

$$\int_0^1 x^{\alpha-1}(1-x)^{\beta-1} dx = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)}\text{, \quad Re}(\alpha), \text{Re}(\beta) > 0,$$

where $\Gamma(x)$ is the Euler gamma function. Its generalization to the field of complex numbers $x \to z \in \mathbb{C}$ was suggested in [12]. Its description requires the complex gamma function

$$\Gamma(x,n) = \Gamma(\alpha|\alpha') := \frac{\Gamma(\alpha)}{\Gamma(1-\alpha')} = \frac{\Gamma(\frac{n + ix}{2})}{\Gamma(1 + \frac{n - ix}{2})}, \quad \alpha = \frac{n + ix}{2}, \quad \alpha' = -\frac{n + ix}{2},$$

where $x \in \mathbb{C}$ and $n \in \mathbb{Z}$. For two complex numbers $\alpha, \alpha' \in \mathbb{C}$ such that $\alpha - \alpha' = n \in \mathbb{Z}$, we use the notation

$$[z]^{\alpha} := z^{\alpha}z^{\alpha'} = |z|^{2\alpha'}z^n, \quad \int_{\mathbb{C}} d^2z := \int_{\mathbb{R}^2} d(\text{Re} z)\, d(\text{Im} z),$$

where $\bar{z}$ is the complex conjugate of $z$. Then the complex beta integral in [12] can be represented, after a linear fractional transformation of the integration variable, in the form of the star-triangle relation

$$\int_{\mathbb{C}} [z_1 - w]^{\alpha-1}[z_2 - w]^{\beta-1}[z_3 - w]^{\gamma-1} d^2w = \frac{\Gamma(\alpha,\beta,\gamma)}{[z_3 - z_2]^{\alpha}[z_1 - z_3]^{\beta}[z_2 - z_1]^{\gamma}},$$

where $\Gamma(\alpha_1,\ldots,\alpha_k) := \prod_{j=1}^k \Gamma(\alpha_j|\alpha'_j)$ and $\alpha + \beta + \gamma = \alpha' + \beta' + \gamma' = 1$.

The right-hand side expression in (2.3) can take different forms due to the reflection equations

$$\Gamma(\alpha|\alpha') = (-1)^{\alpha-\alpha'}\Gamma(\alpha'|\alpha), \quad \Gamma(x,-n) = (-1)^n\Gamma(x,n),$$

and

$$\Gamma(\alpha|\alpha')\Gamma(1-\alpha|1-\alpha') = (-1)^{\alpha-\alpha'}, \quad \Gamma(x,n)\Gamma(-x-2i,n) = 1.$$
and
\[ \Psi_2(a_1, a_2, a_3 | \ell, c, z) = \frac{1}{2 \pi i} \oint \frac{d^2 z_0}{(1 - z_0) \Gamma(\alpha)} \int_c \frac{dz}{z - z_0} \frac{\Gamma(\alpha)}{\Gamma(\alpha + a_1 + a_2 + \ell + \bar{c})}. \]

The principal series representation parameters \(a_1, a_2, a_3, l, c, \bar{c}\) are of the form \(\alpha = (n_\alpha + ix_\alpha)/2\), where \(n_\alpha \in \mathbb{Z}, x_\alpha \in \mathbb{R}\) are such that linear combinations of integers \(n_\alpha\) appearing in the powers of square bracket expressions \((2.2)\) take integer values only, so that all resulting functions are single-valued.

This 6-dimensional integral can be rewritten as an infinite bilateral sum of integrals corresponding to the Mellin-Barnes type representation of the function of interest. Such an expression was obtained in \([14]\) and slightly corrected in \([7]\):

\[ R_t(c, \bar{c}) = (-1)^{\bar{c}-c} \pi^2 a \left( \frac{1-a_1-l+\bar{c}}{2} \right) a \left( \frac{1+a_1+l+\bar{c}}{2} \right) \]
\[ \times \sum_{n \in \mathbb{Z}} \int_L a \left( \frac{1+a_1-a_2+\bar{c}}{2} + s \right) a \left( \frac{1-a_1-a_2+\bar{c}}{2} + s \right) a \left( \frac{1+a_1+l+\bar{c}}{2} + s \right) a \left( \frac{1-a_1+l+\bar{c}}{2} + s \right) \frac{du}{a(s) a(\bar{c} + s) a(\bar{c} + s) a(\bar{c} + s)}. \]

where \(s = (n + iu)/2\) and
\[ a(\alpha) = \frac{\Gamma(1 - \alpha')}{\Gamma(\alpha)}. \]

The integration contour \(L\) may be any contour lying in the strip \(\text{Im}(u) \in ]-1, 0[\) (singularities of the integrand lie on the real axis due to the unitarity condition \(x_\alpha \in \mathbb{R}\)). Expression \((2.7)\) differs from the one in \([14]\) by the sign of the parameter \(\bar{c}\), corresponding to the transition to an equivalent representation, which we believe is the correct prescription.

Now we rewrite the complex 6j-symbols in terms of the complex gamma function \((2.1)\). If we set \(\alpha = N/2 + i\sigma/2\) and \(\alpha' = -N/2 + i\sigma/2\) in \((2.8)\), we obtain
\[ a(\alpha) = \frac{1}{\Gamma(\sigma, N)} = \Gamma(-\sigma - 2i, N). \]

Using relation \((2.9)\) and setting
\[ a_1 = N_1/2 + i\sigma_1, \quad a_3 = N_3/2 + i\sigma_3, \quad c = M_1/2 + i\rho_1, \quad s = N/2 + iu/2, \]
\[ a_2 = N_2/2 + i\sigma_2, \quad l = N_4/2 + i\sigma_4, \quad \bar{c} = M_2/2 + i\rho_2, \]

we rewrite expression \((2.7)\) as
\[ \left\{ \begin{array}{c} \sigma_1, N_1 \sigma_2, N_2 | \rho_1, M_1 \\ \sigma_3, N_3 \sigma_4, N_4 | \rho_2, M_2 \end{array} \right\} = \frac{\pi^2}{4} \frac{\Gamma(\sigma_1 - \sigma_2 + \rho_2 - i, A_1)}{\Gamma(-\sigma_3 - \sigma_4 + \rho_2 - i, A_3)} \frac{\Gamma(\sigma_2 - \sigma_3 + \rho_1 - i, A_2)}{\Gamma(\sigma_1 + \sigma_4 + \rho_1 - i, A_4)} \]
\[ \times (-1)^{M_2-N_3+N_4} \sum_{N \in \mathbb{Z}} \int_{u \in L} \prod_{j=1}^4 \Gamma(R_j - u, S_j - N) \Gamma(U_j + u, T_j + N) \frac{du}{u}, \]
where
\[
R_1 = -\sigma_1 + \sigma_2 - \rho_2 - i, \quad U_1 = -\rho_1 - \sigma_2 + \sigma_4 + \rho_2, \quad S_1 = (-N_1 + N_2 - M_2)/2,
\]
\[
R_2 = \sigma_1 + \sigma_2 - \rho_2 - i, \quad U_2 = \rho_1 - \sigma_2 + \sigma_4 + \rho_2, \quad S_2 = (N_1 + N_2 - M_2)/2,
\]
\[
R_3 = -\sigma_3 - \sigma_4 - \rho_2 - i, \quad U_3 = 0, \quad S_3 = -(N_3 + N_4 + M_2)/2,
\]
\[
R_4 = \sigma_3 - \sigma_4 - \rho_2 - i, \quad U_4 = 2\rho_2, \quad S_4 = (N_3 - N_4 - M_2)/2,
\]
and
\[
T_1 = (-M_1 - N_2 + N_4 + M_2)/2, \quad T_2 = (M_1 - N_2 + N_4 + M_2)/2, \quad T_3 = 0, \quad T_4 = M_2.
\]
(2.13)

and, finally,
\[
A_1 = \frac{N_1 - N_2 + M_2}{2}, \quad A_2 = \frac{N_2 - N_3 + M_1}{2},
\]
\[
A_3 = \frac{-N_3 - N_4 + M_2}{2}, \quad A_4 = \frac{N_1 + N_4 + M_1}{2}.
\]

We note that
\[
\sum_{a=1}^{4} (R_a + U_a) = -4i \quad \text{and} \quad \sum_{a=1}^{4} (S_a + T_a) = 0 \quad (2.14)
\]

and
\[
A_1 + A_2 = A_3 + A_4. \quad (2.15)
\]

Racah coefficients (2.7) depend on 6 complex parameters (pairs of continuous and discrete variables), whereas function (2.11) formally depends on 8 such parameters \((R_j, S_j), (U_j, T_j), j = 1, \ldots, 4\), and the balancing condition leaves 7 independent quantities. However, the structure of integration and summation in (2.11) allows shifting the integration and summation variables by arbitrary constants (with an appropriate shift of the integration contour). Therefore, one of the pairs of variables can be set equal to zero, which was done in (2.12) and (2.13) by the choice \(U_3 = 0\) and \(T_3 = 0\).

3. Relation to 6j-symbols for the Faddeev modular double

We consider Faddeev’s quantum modular dilogarithm \(\gamma^{(2)}(y; \omega_1, \omega_2)\) [8][9] also called the hyperbolic gamma function [23],
\[
\gamma^{(2)}(u; \omega) = \gamma^{(2)}(u; \omega_1, \omega_2) := e^{-\frac{\omega}{2}B_{2,2}(u; \omega)}\gamma(u; \omega), \quad (3.1)
\]
where \(B_{2,2}\) is the second-order multiple Bernoulli polynomial
\[
B_{2,2}(u; \omega) = \frac{1}{\omega_1 \omega_2} \left( (u - \omega_1 + \omega_2)^2 - \frac{\omega_1^2 + \omega_2^2}{12} \right)
\]
and
\[
\gamma(u; \omega) := \frac{qe^{2\pi i \frac{u}{\omega_1} \frac{1}{\omega}}}{(e^{2\pi i \frac{u}{\omega} \frac{1}{\omega}}; q)_{\infty}} = \exp \left( -\int_{\mathbb{R}+i0} \frac{e^{ux}}{(1-e^{\omega_1 x})(1-e^{\omega_2 x})} \frac{dx}{x} \right). \quad (3.2)
\]

This function obeys the first order difference equations
\[
\frac{\gamma^{(2)}(y + \omega_1; \omega_1, \omega_2)}{\gamma^{(2)}(y; \omega_1, \omega_2)} = 2 \sin \frac{\pi y}{\omega_2}, \quad \frac{\gamma^{(2)}(y + \omega_2; \omega_1, \omega_2)}{\gamma^{(2)}(y; \omega_1, \omega_2)} = 2 \sin \frac{\pi y}{\omega_1} \quad (3.3)
\]
and has the asymptotics \[ \text{[15]} \]

\[
\begin{align*}
\text{I} & : \quad \lim_{y \to \infty} e^{\pm \frac{\pi}{2} i B_{2,2}(y, \omega_1, \omega_2)} \gamma^{(2)}(y; \omega_1, \omega_2) = 1, \quad \arg \omega_1 < \arg y < \arg \omega_2 + \pi, \\
\text{II} & : \quad \lim_{y \to \infty} e^{\mp \frac{\pi}{2} i B_{2,2}(y, \omega_1, \omega_2)} \gamma^{(2)}(y; \omega_1, \omega_2) = 1, \quad \arg \omega_1 - \pi < \arg y < \arg \omega_2. 
\end{align*}
\]

In what follows, we shall a special degeneration limit

\[
b := \sqrt{\frac{\omega_1}{\omega_2}} = i + \delta, \quad \delta \to 0^+, 
\]

in which case

\[
Q = \omega_1 + \omega_2 = 2\delta \sqrt{\omega_1 \omega_2} + O(\delta^2).
\]

In [24], it was rigorously shown that in the limit (3.6), the estimate

\[
\gamma^{(2)}(i \sqrt{\omega_1 \omega_2}(n + x\delta); \omega_1, \omega_2) = e^{\pm \frac{\pi}{2} n^2 (4\pi \delta)^{\frac{1}{2}} \Gamma(x, n, \sqrt{\omega_1 \omega_2})} \delta \to 0^+, 
\]

holds uniformly on compacta, where \( n \in \mathbb{Z}, x \in \mathbb{C} \), and complex gamma function is defined in [21]. Such a limit was qualitatively considered first in [3].

The \( 6j \)-symbols corresponding to the principal unitary series representations of the Faddeev modular double \( sl_q(2, \mathbb{R}) \times sl_q(2, \mathbb{R}) \) [10] were constructed by Ponsot and Teschner in [19] in the form of an explicit expression

\[
\begin{align*}
\{ \alpha_1 & \alpha_2 \mid \alpha_3 \alpha_4 \} \big|_b = \frac{S_b(\alpha_s + \alpha_2 - \alpha_1)S_b(\alpha_1 + \alpha_t - \alpha_4)}{S_b(\alpha_t + \alpha_2 - \alpha_3)S_b(\alpha_3 + \alpha_4 - \alpha_4)} |S_b(2\alpha_t)|^2 J_b(\mu, \nu), 
\end{align*}
\]

where \( S_b(z) = \gamma^{(2)}(z; b, b^{-1}) \),

\[
J_b(\mu, \nu) = \int_{-i\infty}^{i\infty} \prod_{a=1}^{4} S_b(\mu_a - z)S_b(\nu_a + z)dz, 
\]

and

\[
\begin{align*}
\nu_1 & = \alpha_s + \alpha_1 - \alpha_2, \quad \mu_1 = -Q - \alpha_s + \alpha_t + \alpha_4 + \alpha_2, \\
\nu_2 & = Q + \alpha_s - \alpha_1 - \alpha_2, \quad \mu_2 = -\alpha_s - \alpha_t + \alpha_4 + \alpha_2, \\
\nu_3 & = \alpha_s + \alpha_3 - \alpha_4, \quad \mu_3 = Q - 2\alpha_s, \\
\nu_4 & = Q + \alpha_s - \alpha_3 - \alpha_4, \quad \mu_4 = 0, 
\end{align*}
\]

with \( Q = b + b^{-1} \). We note that the parameters \( \mu_a \) and \( \nu_a \) satisfy the balancing condition

\[
\sum_{a=1}^{4} (\nu_a + \mu_a) = 2Q. 
\]

Formally, function (3.10) depends on 7 independent complex variables, but one of the parameters can be set equal to zero by shifting the integration variable, which was done in (3.11) by the choice \( \mu_4 = 0 \).

The factor \( |S_b(2\alpha_t)|^2 \) corresponds to the measure in the orthogonality relation for \( 6j \)-symbols. In principle, we can remove it from the definition (3.9) and lift the unitary principal series restrictions imposed on the parameters of this function \( \alpha_1, 2, 3, 4, \alpha_s, \alpha_t \in Q/2 + i\mathbb{R} \) (see the definition of the hyperbolic hypergeometric function \( J_h(\mu, \nu) \) in (4.28) below, where we do not assume such a constraint).
Now we intend to show that formula (2.11) can be derived from expression (3.9) in the limit $b \to i$. First, we rewrite the last function in a slightly different notation. We shift the integration variable $z \to z + Q - 2\alpha_s$ and afterwards define new parameters

$$\alpha_s = \alpha'_s + Q/2, \quad \alpha_1 = -\alpha'_1 + Q/2, \quad \alpha_3 = -\alpha'_3 + Q/2,$$
$$\alpha_t = -\alpha'_t + Q/2, \quad \alpha_2 = -\alpha'_2 + Q/2, \quad \alpha_4 = \alpha'_4 + Q/2. \quad (3.13)$$

After the shifts, parameters $\nu_a$ and $\mu_a$ take the following form in terms of the $\alpha'$-variables

$$\nu_1 = Q/2 - \alpha'_s - \alpha'_1 + \alpha'_2, \quad \mu_1 = \alpha'_s - \alpha'_1 + \alpha'_4 - \alpha'_2,$$
$$\nu_2 = Q/2 - \alpha'_s + \alpha'_1 + \alpha'_2, \quad \mu_2 = \alpha'_s + \alpha'_4 - \alpha'_2,$$
$$\nu_3 = Q/2 - \alpha'_s - \alpha'_3 - \alpha'_4, \quad \mu_3 = 0,$$
$$\nu_4 = Q/2 - \alpha'_s + \alpha'_3 - \alpha'_4, \quad \mu_4 = 2\alpha'_s. \quad (3.14)$$

For $\omega_1 = b$, $\omega_2 = b^{-1}$, relation (3.8) takes the form:

$$S_{1+\delta}(i(n + x\delta)) = e^{\frac{x\mu^2}{2\delta}}(4\pi\delta)^{i\nu-1}\Gamma(x, n). \quad (3.15)$$

We parametrize our new $\alpha'$ variables and the integration variable in accordance with the form of the argument of the $S_b$-function in formula (3.15):

$$\alpha'_1 = i(N_1/2 + \sigma_1\delta), \quad \alpha'_3 = i(N_3/2 + \sigma_3\delta), \quad \alpha'_4 = i(M_1/2 + \rho_1\delta),$$
$$\alpha'_2 = i(N_2/2 + \sigma_2\delta), \quad \alpha'_4 = i(N_4/2 + \sigma_4\delta), \quad \alpha'_s = i(M_2/2 + \rho_2\delta),$$
$$z = i(-N - u\delta). \quad (3.16)$$

In this parametrization, $N_k$ and $M_k$ are integers chosen such that their linear combinations appearing in the expressions for $\mu_j$ and $\nu_j$ (3.14) take even values in order to be divisible by 2.

Now we can apply asymptotic formula (3.15) and find that in the limit $\delta \to 0$, the $6j$-symbols for Faddeev’s modular double are converted to the complex $6j$-symbols for the $SL(2, \mathbb{C})$ group,

$$\{ \alpha_1 \alpha_2 | \alpha_3 \alpha_4 \alpha_5 \} \mid_{\delta \to 0} = e^{\frac{\pi i F}{2}} \frac{M_1^2 + 4\mu_1^2}{16\pi^3 i\delta} \{ \sigma_1, N_1 \sigma_2, N_2 \mid \rho_1, M_1 \}, \quad (3.17)$$

where

$$F = A_1^2 - A_2^2 - A_3^2 + A_4^2 + \sum_{a=1}^4 (S_a^2 + T_a^2) + 2(A_4 - A_2). \quad (3.18)$$

Conditions (2.14) and (2.15) imply that $F$ is an even integer. Therefore $e^{\frac{\pi i F}{2}}$ is a plain sign factor. The details of the computations can be found in Appendix A. The proportionality coefficient in the right-hand side of relation (3.17) without the diverging factor $e^{\frac{\pi i F}{2}}/i\delta$ coincides with the $\rho$-function defined in equation (28) of [7], which describes the measure weight function. It emerged from the multiplier $|S_b(2\alpha_t)|^2$ in definition (3.9).
4. Difference equations

An elliptic analogue of the Euler-Gauss hypergeometric function \( V(t_1, \ldots, t_8; p, q) \) was introduced in [28],
\[
V(t_1, \ldots, t_8; p, q) = \frac{(p; p)_\infty (q; q)_\infty \Gamma(z)}{4\pi i} \int_T \frac{\prod_{a=1}^{8} \Gamma(t_a z; p, q) \Gamma(t_a z^{-1}; p, q) \, dz}{\Gamma(z^2; p, q) \Gamma(z^{-2}; p, q) / z},
\]
with the parameters \( t_a \) satisfying the balancing condition
\[
\prod_{a=1}^{8} t_a = p^2 q^2.
\]
Here, \( \Gamma(z; p, q) \) is the elliptic gamma function defined as a double infinite product
\[
\Gamma(z; p, q) = \prod_{j,k=0}^{\infty} \frac{1 - z^{-1} p^{j+1} q^{k+1}}{1 - z p^j q^k}, \quad |p|, |q| < 1, \quad z \in \mathbb{C}.
\]
It satisfies the equations
\[
\begin{align*}
\Gamma(z; p, q) &= \Gamma(z; q, p), \\
\Gamma(qz; p, q) &= \theta(z; p) \Gamma(z; q, p), \quad \Gamma(pz; p, q) = \theta(z; q) \Gamma(z; q, p),
\end{align*}
\]
where \( \theta(z; p) \) is a short Jacobi theta-function
\[
\theta(z; q) = (z; q)_\infty (qz^{-1}; q)_\infty, \quad (z; q)_\infty := \prod_{j=0}^{\infty} (1 - z q^j),
\]
related to Jacobi’s \( \theta_1 \)-function as follows
\[
\theta_1(u|\tau) = -\theta_{11}(u) = - \sum_{\ell \in \mathbb{Z} + 1/2} e^{\pi i \ell^2} e^{2\pi i \ell (u + 1/2)} = i q^{1/8} e^{-\pi i u} (q; q)_\infty \theta(e^{2\pi i u}; q).
\]
As shown in [29, 30], the \( V \)-function satisfies a finite-difference equation, called the elliptic hypergeometric equation,
\[
\mathcal{L}(t) (U(qt_6, q^{-1} t_7) - U(t)) + (t_6 \leftrightarrow t_7) + U(t) = 0,
\]
where
\[
U(t) = \frac{V(t_1, \ldots, t_8; p, q)}{\Gamma(t_6 t_8^{1/2}; p, q) \Gamma(t_7 t_8^{1/2}; p, q)},
\]
the first \( U \)-function containing parameters \( qt_6, q^{-1} t_7 \) instead of \( t_6, t_7 \), and
\[
\mathcal{L}(t) = \frac{\theta \left( \frac{t_6}{qt_7}; p \right) \theta \left( \frac{t_6 t_8}{p}; p \right) \theta \left( \frac{t_6 t_k}{p}; p \right)}{\theta \left( \frac{t_6}{q t_7}; p \right) \theta \left( \frac{t_6 t_8}{q}; p \right) \theta \left( \frac{t_6 t_k}{q}; p \right)} \prod_{k=1}^{5} \theta \left( \frac{t_k t_8}{q}; p \right).
\]
We recall the asymptotic relation [23]
\[
\Gamma(e^{-2\pi y}; e^{-2\pi \omega_1}, e^{-2\pi \omega_2}) \bigg|_{y \to 0} = e^{-\pi (2y - \omega_1 - \omega_2)/12 \omega_1 \omega_2 \gamma(2)(y; \omega_1, \omega_2)},
\]
\[
(4.11)
\]
which was shown in [21] to be uniform on compacta, and the asymptotics
\[
\theta(e^{-2\pi v y}; e^{-2\pi r \omega_1}) = e^{-\frac{\pi}{\omega_1}2 \sin \frac{\pi y}{\omega_1}}.
\] (4.12)

With their help, we can derive difference equations for the corresponding hyperbolic hypergeometric functions.

A hyperbolic analogue of the \( V \)-function (4.11) appearing from it in the limit (4.11) is
\[
I_h(u) = \int_{-\infty}^{\infty} \prod_{a=1}^{8} \frac{\gamma(2)(u_a \pm z; \omega_1, \omega_2)}{\gamma(2)(\pm 2z)} \frac{dz}{2i\sqrt{\omega_1\omega_2}},
\] (4.13)

with \( u_a \) satisfying the condition:
\[
\sum_{a=1}^{8} u_a = 2(\omega_1 + \omega_2).
\] (4.14)

It can be shown [4] that in the limit (4.11), elliptic hypergeometric equation (4.8) is converted into a difference equation for \( I_h(u) \),
\[
\mathcal{A}(u; \omega_1, \omega_2)(Y(u_6 + \omega_2, u_7 - \omega_2) - Y(u)) + (u_6 \leftrightarrow u_7) + Y(u) = 0,
\] (4.15)

where
\[
\mathcal{A}(u; \omega_1, \omega_2) = \frac{\sin \frac{\pi}{\omega_1}(u_6 - u_8 - \omega_2) \sin \frac{\pi}{\omega_1}(u_6 + u_8) \sin \frac{\pi}{\omega_1}(u_8 - u_6)}{\sin \frac{\pi}{\omega_1}(u_6 - u_7) \sin \frac{\pi}{\omega_1}(u_7 - u_6 - \omega_2) \sin \frac{\pi}{\omega_1}(u_7 + u_6 - \omega_2)} \times \prod_{k=1}^{5} \frac{\sin \frac{\pi}{\omega_1}(u_7 + u_k - \omega_2)}{\sin \frac{\pi}{\omega_1}(u_8 + u_k)}
\] (4.16)

and
\[
Y(u) = \frac{I_h(u)}{\gamma(2)(u_6 \pm u_8, u_7 \pm u_8; \omega_1, \omega_2)}.
\] (4.17)

The function \( I_h(u) \) is symmetric in the quasiperiods \( \omega_1 \) and \( \omega_2 \). Therefore, we have a second difference equation
\[
\mathcal{A}(u; \omega_2, \omega_1)(Y(u_6 + \omega_1, u_7 - \omega_1) - Y(u)) + (u_6 \leftrightarrow u_7) + Y(u) = 0.
\] (4.18)

A detailed consideration of the limiting transitions \( \omega_1 \to 0 \) (for a fixed \( \omega_2 \)) and \( \omega_1 \to \pm \omega_2 \) for the function \( Y(u) \) and for the corresponding difference equations was given in [20]. Here, we investigate a special degenerate case of this function: our key object is now the integral
\[
E_h(u) = \int_{-\infty}^{\infty} \prod_{a=1}^{6} \frac{\gamma(2)(u_a \pm z; \omega_1, \omega_2)}{\gamma(2)(\pm 2z; \omega_1, \omega_2)} \frac{dz}{2i\sqrt{\omega_1\omega_2}}
\] (4.19)

without any balancing condition. For \( \Re(u_a) > 0 \), we can fix the integration contour as the imaginary axis. For other values of parameters, the integration contour is of the Mellin-Barnes type, i.e., it should separate sequences of poles going to infinity to the left and right of the imaginary axis.

We now derive the difference equation for this function following from hyperbolic hypergeometric equation (4.15). For this, we set \( u_1 = 2(\omega_1 + \omega_2) - \sum_{k=2}^{8} u_k \) and take the limit \( u_8 \to \infty \) inside cone I (3.4) (such that \( u_1 \to \infty \) inside cone II (3.5)). Then using
the corresponding asymptotics and renumbering the remaining parameters \( u_k \rightarrow u_{k-1} \), we obtain

\[
\mathcal{B}(u; \omega_1, \omega_2)(E_h(u_5 + \omega_2, u_6 - \omega_2) - E_h(u)) + (u_5 \leftrightarrow u_6) + E_h(u) = 0, \tag{4.20}
\]

\[
\mathcal{B}(u; \omega_1, \omega_2) = \prod_{k=1}^{4} \frac{1}{\sin \frac{\pi}{\omega_1}(u_6 + u_k - \omega_2)} \frac{\sin \frac{\pi}{\omega_1}(u_5 - u_6)}{\sin \frac{\pi}{\omega_1}(u_6 - u_5 - \omega_2)} \frac{\sin \frac{\pi}{\omega_1}(u_6 + u_5 - \omega_2)}{\sin \frac{\pi}{\omega_1}(2Q - \sum_{i=1}^{6} u_i)}.
\]

This equation was also derived in [4]. Despite the absence of the balancing condition, we can convert equation (4.20) into a finite-difference equation of the second order in the variable \( x \) by setting \( u_5 = c + x \) and \( u_6 = c - x \) for an arbitrary constant \( c \). Evidently, we also have the second equation

\[
\mathcal{B}(u; \omega_1, \omega_1)(E_h(u_5 + \omega_1, u_6 - \omega_1) - E_h(u)) + (u_5 \leftrightarrow u_6) + E_h(u) = 0. \tag{4.21}
\]

We can consider a different degeneration of equation (4.15). We reparametrize the variables \( u_a \) in identity (4.15) in the following asymmetric way:

\[
u_a = \nu_a + i\xi, \quad u_{a+4} = \mu_a - i\xi, \quad a = 1, 2, 3, 4. \tag{4.22}
\]

Then the balancing condition takes the form

\[
\sum_{a=1}^{4} (\nu_a + \mu_a) = 2(\omega_1 + \omega_2). \tag{4.23}
\]

In definition (4.13), we now shift the integration variable \( z \rightarrow z - i\xi \) and take the limit \( \xi \rightarrow -\infty \). From the above equations, we then obtain

\[
\mathcal{D}(\mu, \nu; \omega_1, \omega_2)(U(\mu_2 + \omega_2, \mu_3 - \omega_2) - U(\mu, \nu)) + (\mu_2 \leftrightarrow \mu_3) + U(\mu, \nu) = 0 \tag{4.24}
\]

and

\[
\mathcal{D}(\mu, \nu; \omega_2, \omega_1)(U(\mu_2 + \omega_1, \mu_3 - \omega_1) - U(\mu, \nu)) + (\mu_2 \leftrightarrow \mu_3) + U(\mu, \nu) = 0, \tag{4.25}
\]

where

\[
\mathcal{D}(\mu, \nu; \omega_1, \omega_2) = \frac{\sin \frac{\pi}{\omega_1}(\mu_2 - \mu_4 - \omega_2)}{\sin \frac{\pi}{\omega_1}(\mu_2 - \mu_3)} \frac{\sin \frac{\pi}{\omega_1}(\mu_4 - \omega_2)}{\sin \frac{\pi}{\omega_1}(\mu_3 - \omega_2)} \prod_{k=1}^{4} \frac{\sin \frac{\pi}{\omega_1}(\mu_3 + \nu_k - \omega_2)}{\sin \frac{\pi}{\omega_1}(\mu_4 + \nu_k)}, \tag{4.26}
\]

\[
U(\mu, \nu) = \frac{J_h(\mu, \nu)}{\gamma(2)(\mu_2 - \mu_4, \mu_3 - \mu_4; \omega_1, \omega_2)}, \tag{4.27}
\]

\[
J_h(\mu, \nu) = \int_{-i\infty}^{i\infty} \prod_{a=1}^{4} \gamma(2)(\mu_a - z; \omega_1, \omega_2) \gamma(2)(\nu_a + z; \omega_1, \omega_2) \frac{dz}{i\sqrt{\omega_1 \omega_2}}, \tag{4.28}
\]

with the parameters \( \mu_a \) and \( \nu_a \) satisfying the condition (4.23).

In fact, equality (4.24) is another form of equation (4.20) because, as established in [4], the integrals (4.19) and (4.28) are connected by the relation

\[
J_h(\mu, \nu) = \prod_{a=1}^{3} \gamma(2)(\mu_a + \nu_4; \omega_1, \omega_2) \gamma(2)(\nu_a + \mu_4; \omega_1, \omega_2)
\times \tilde{E}_h(\mu_1 + \eta, \mu_2 + \eta, \mu_3 + \eta, \nu_1 - \eta, \nu_2 - \eta, \nu_3 - \eta), \tag{4.29}
\]
where

\[ 2\eta = Q - \nu_1 - \sum_{a=1}^{3} \mu_a. \] (4.30)

5. The \( b \to 0 \) Reduction of Differential Equations

We now investigate various degenerations of hyperbolic integrals (4.19) and (4.28) corresponding to the limits \( b \to 0 \) and \( b \to i \). For the general difference equations (4.15) and (4.18), such degenerations are considered in [26]. We start from the standard \( \omega_1 \to 0 \) limit applied to equations (4.24) and (4.25). Recall the following uniform asymptotics computed in [23] (see also [26]):

\[ \gamma^{(2)}(\omega_1 x; \omega_1, \omega_2) \to \frac{\Gamma(x)}{\sqrt{2\pi}} \left( \frac{\omega_2}{2\pi \omega_1} \right)^{\frac{1}{2} - x} \] (5.1)

We plan to scale all the parameters according to the rule

\[ \mu_k = \omega_1 \beta_k, \quad \nu_k = \omega_1 \gamma_k, \quad k = 1, 2, 3, 4, \] and \( z = \omega_1 u \). (5.2)

However, we must first eliminate the \( \omega_2 \) term in the right-hand side of the balancing condition (4.23). For this, we shift the parameters \( \nu_{1,2} \to \nu_{1,2} + \omega_2 \) (such a shift can be made differently, say, \( \nu_1 \to \nu_1 + \omega_2, \mu_1 \to \mu_1 + \omega_2 \), but we limit ourselves to one example). Then, using the reflection rule

\[ \gamma^{(2)}(x; \omega_1, \omega_2) \gamma^{(2)}(x + \omega_2 - x; \omega_1, \omega_2) = 1, \] (5.3)

we rewrite integral (4.28) in the form:

\[ J_h(\mu, \nu) = \int_{-\infty}^{\infty} \prod_{i=1}^{4} \frac{\gamma^{(2)}(\mu_i - z; \omega_1, \omega_2) \prod_{i=1}^{4} \gamma^{(2)}(\nu_i + z; \omega_1, \omega_2)}{\prod_{i=1}^{4} \gamma^{(2)}(\mu_i - \nu_i - z; \omega_1, \omega_2)} \frac{dz}{i\sqrt{\omega_1 \omega_2}}. \] (5.4)

Now we apply the scaling (5.2), which leads to the balancing condition

\[ \sum_{k=1}^{4} (\beta_k + \gamma_k) = 2 \] (5.5)

and in the limit \( \omega_1 \to 0 \) obtain the relation

\[ J_h(\mu, \nu) \to \frac{1}{(2\pi)^4} \frac{\omega_2}{\omega_1} J_r(\beta, \gamma), \] (5.6)

where

\[ J_r(\beta, \gamma) = \int_{-\infty}^{\infty} \prod_{i=1}^{4} \frac{\Gamma(\beta_i - u) \prod_{i=1}^{4} \Gamma(\gamma_i + u)}{\prod_{i=1}^{4} \Gamma(1 - \gamma_i - u)} \] (5.7)

Now it is straightforward to show that equation (4.25) in this limit is converted to

\[ D(\beta, \gamma)(J_r(\beta_2 + 1, \beta_3 - 1, \gamma) - J_r(\beta, \gamma)) + (\beta_2 \leftrightarrow \beta_3) + J_r(\beta, \gamma) = 0, \] (5.8)

where

\[ D(\beta, \gamma) = \frac{(\beta_2 - \beta_4 - 1)(\beta_4 - \beta_2)}{(\beta_2 - \beta_3)(\beta_3 - \beta_2 - 1)} \prod_{k=1}^{4} \frac{(\beta_3 + \gamma_k - 1)}{(\beta_4 + \gamma_k)} \] (5.9)
and
\[ J_r(\beta, \gamma) = \frac{J_r(\beta, \gamma)}{\Gamma(\beta_2 - \beta_4)\Gamma(\beta_3 - \beta_4)}. \] (5.10)

In turn, equation (4.21) yields the following integral identity
\[ \frac{e^{\pi(\beta_2 - \beta_2)} \sin \pi(\beta_4 - \beta_2) \prod \sin \pi(\beta_3 + \gamma_k) \prod \sin \pi(\beta_4 + \gamma_k)}{\sin \pi(\beta_2 - \beta_3)} (\tilde{J}_r(\beta, \gamma) - J_r(\beta, \gamma)) + (\beta_2 \leftrightarrow \beta_3) + J_r(\beta, \gamma) = 0, \] where
\[ \tilde{J}_r(\beta, \gamma) = \Gamma(1 - \beta_2 + \beta_4) \Gamma(\beta_3 - \beta_4) \int_{-\infty}^{\infty} e^{\pi(\beta_4 - u)} \prod \Gamma(\beta_i - u) \prod \Gamma(\gamma_i + u) \Gamma(1 - \beta_2 + u) \prod \Gamma(1 - \gamma_i - u) \, du. \] (5.11)

The integral \( \tilde{J}_r \) converges in a rather cute way: the integrand falls off exponentially fast \( \propto u^{-2}e^{-2\pi i u} \) for \( u \to -\infty \), but it has only a power suppression \( \propto u^{-2} \) for \( u \to +i\infty \).

We can similarly simplify equation (4.21) for \( \omega_1 \to 0 \). Here, we have no balancing condition, and after scaling the parameters \( u_k \) and \( z \) as \( u_k = \omega_1 \alpha_k, z = \omega_1 u \), integral (4.19) in the limit \( \omega_1 \to 0 \) takes the form
\[ E_k(u) \to \left( \frac{1}{2\pi} \sqrt{\frac{\omega_2}{\omega_1}} \right)^9 \left( \frac{2\pi \omega_1}{\omega_2} \right)^{2\sum \alpha_i} \frac{E_r(\alpha)}{E_r(\alpha)}, \]
\[ E_r(\alpha) = \int_{-\infty}^{\infty} \prod \frac{\Gamma(\alpha_i + u)}{\Gamma(\pm 2u)} \frac{du}{4\pi i}, \]
and correspondingly equation (4.21) becomes
\[ C(\alpha)(E_r(\alpha_5 + 1, \alpha_6 - 1) - E_r(\alpha)) + (\alpha_5 \leftrightarrow \alpha_6) + E_r(\alpha) = 0, \] (5.13)
where
\[ C(\alpha) = \frac{\prod \Gamma(\alpha_6 + \alpha_k - 1)}{(\alpha_5 - \alpha_6)(\alpha_6 - \alpha_5 - 1)(\alpha_6 + \alpha_5 - 1)(2 - \sum \alpha_i)}. \] (5.14)

6. The \( b \to i \) reduction

In this section, we consider the \( b = i + \delta, \delta \to 0^+ \) limit (3.6) using asymptotic relation (3.8). Namely, we apply this limit to difference equation (4.24) with the following parametrization
\[ z = i\sqrt{\omega_1 \omega_2}(N + y\delta), \quad \mu_a = i\sqrt{\omega_1 \omega_2}(n_a + s_a \delta), \quad \nu_a = i\sqrt{\omega_1 \omega_2}(m_a + t_a \delta), \] (6.1)
where \( y, s_a, t_a \in \mathbb{C} \) and \( N, n_a, m_a \in \mathbb{Z} + \varepsilon, \varepsilon = 0, \frac{1}{2} \). Evidently, one has the relations
\[ i\sqrt{\omega_1 \omega_2}(N + y\delta) + \omega_2 = i\sqrt{\omega_1 \omega_2}(N - 1 + (y - i)\delta) + O(\delta^2), \]
\[ \frac{i}{\omega_1} \sqrt{\omega_1 \omega_2}(N + y\delta) = N + \delta(iN + y) + O(\delta^2). \]
Balancing condition (4.23) together with relation (3.7) imply that the parameters \( n_a, m_a, s_a, \) and \( t_a \) satisfy the constraints
\[ \sum_{a=1}^{4} (n_a + m_a) = 0, \quad \sum_{a=1}^{4} (s_a + t_a) = -4i. \] (6.2)
Now using the uniform estimate (3.8), we compute the asymptotics of function (4.27),
\[ U(\mu, \nu) \to (-1)^{2\varepsilon+(s^4+4s)} \frac{e^{\frac{i\pi}{2}(s^2-n^2+\sum_{m=1}^{n}m^2)}}{(4\pi \delta)^{1+i(s_2+s_3-2s_4)}} U(s, n; t, m), \tag{6.3} \]
where the subscript “cr” means “complex rational” and
\[ U(s, n; t, m) = \frac{J_{\text{cr}}(s, n; t, m)}{\Gamma(s_2-s_4, n_2-n_4)\Gamma(s_3-s_4, n_3-n_4)}, \tag{6.4} \]
where we have now a different parametrization \( u \rightarrow \frac{1}{4\pi} \sum_{N \in \mathbb{Z}+\varepsilon} \int_{-\infty}^{\infty} \prod_{a=1}^{4} \Gamma(s_a-y, n_a-N)\Gamma(t_a+y, m_a+N)dy. \) (6.5)
It turns out that the choice \( \varepsilon = 1/2 \) reduces to the case \( \varepsilon = 0 \) after the replacement of \( n_a \to n_a + \varepsilon, \) or, equivalently, \( m_a \to m_a - \varepsilon. \) Therefore, we can drop \( \varepsilon \) from the definition of \( J_{\text{cr}}. \) Evidently, function (6.5) coincides with the key element defining the 6j-symbols for the \( SL(2, \mathbb{C}) \) group (2.11).

We note that the diverging prefactor in (6.3) contains only the sum of \( s_2 \) and \( s_3, \) and does not contain \( n_2 \) and \( n_3. \) This implies the following limit form of equation (4.24):
\[ D(\beta, \gamma)(U(s_2-i, n_2-1, s_3+i, n_3+1) - U(s, n; t, m)) + (s_2, n_2 \leftrightarrow s_3, n_3) + U(s, n; t, m) = 0, \tag{6.6} \]
where the function \( D(\beta, \gamma) \) is fixed in (5.9) with the parametrization
\[ \beta_k = \frac{1}{2}(is_k - n_k), \quad \gamma_k = \frac{1}{2}(it_k - m_k). \tag{6.7} \]
In Appendix B, we consider a simple particular case of equation (6.6) when the function \( U \) is reduced to a computable beta integral, which explicitly demonstrates essential properties of this equation.

We can compute the \( \delta \to 0^+ \) form of equation (4.25) similarly. Using the relations
\[ i\sqrt{\omega_1\omega_2}(N+y\delta) + \omega_1 = i\sqrt{\omega_1\omega_2}(N+1+(y-i)\delta) + O(\delta^2), \]
\[ \frac{i}{\omega_2}\sqrt{\omega_1\omega_2}(N+y\delta) = -N - \delta(y - iN) + O(\delta^2), \]
it can be shown that in the limit \( \delta \to 0, \) difference equation (4.25) becomes
\[ D(\beta, \gamma)(U(s_2-i, n_2+1, s_3+i, n_3-1) - U(s, n; t, m)) + (s_2, n_2 \leftrightarrow s_3, N_3) + U(s, n; t, m) = 0, \tag{6.8} \]
where we have now a different parametrization
\[ \beta_k = \frac{1}{2}(is_k + n_k), \quad \gamma_k = \frac{1}{2}(it_k + m_k). \tag{6.9} \]
In a similar way, we can compute the \( b \to i \) limit forms of equations (4.20) and (4.21). We parametrize \( u_a \) and \( z \) as
\[ z = i\sqrt{\omega_1\omega_2}(N+y\delta), \quad u_a = i\sqrt{\omega_1\omega_2}(l_a + p_a\delta), \]
where \( y, p_a \in \mathbb{C} \) and \( N, l_a \in \mathbb{Z} + \varepsilon, \varepsilon = 0, \frac{1}{2}. \) Then, in the limit \( \delta \to 0 \) integral (4.19) has the asymptotics
\[ E_h(u) \to (-1)^{2\varepsilon + (2\varepsilon - 1)\sum_{k=1}^{n}l_k} (4\pi \delta)^{2i\sum_{k=1}^{n}p_k - 9} E_{\text{cr}}(p, l), \tag{6.10} \]
where
\[ E_{cr}(p, l) = \frac{1}{8\pi} \sum_{N \in \mathbb{Z} + \epsilon} \int_{-\infty}^{\infty} (y^2 + N^2) \prod_{k=1}^{6} \Gamma(p_k \pm y, l_k \pm N) dy. \] (6.11)

We note that in contrast to function (6.5), the variable \( \epsilon \) cannot be removed from the definition of \( E_{cr} \), and it plays a substantial role. The complex hypergeometric analogue of relation (4.29) connecting \( E_{cr} \) and \( J_{cr} \) functions is considered in the next section.

It is now straightforward to see that in the limit \( b \to i \), equation (4.20) reduces to the equality
\[ C(\alpha)(E_{cr}(p_5 - i, l_5 - 1, p_6 + i, l_6 + 1) - E_{cr}(p_5, l_5)) + (p_5, l_5 \leftrightarrow p_6, l_6) + E_{cr}(p, l) = 0, \] (6.12)
where the potential \( C(\alpha) \) is defined in (5.14) with the parametrization \( \alpha_k = \frac{1}{2}(-l_k + ip_k) \).

Equation (4.21) takes the form
\[ C(\alpha)(E_{cr}(p_5 - i, l_5 + 1, p_6 + i, l_6 - 1) - E_{cr}(p_5, l_5)) + (p_5, l_5 \leftrightarrow p_6, l_6) + E_{cr}(p, l) = 0, \] (6.13)
with \( \alpha_k = \frac{1}{2}(l_k + ip_k) \).

7. Additional symmetry transformation for complex 6j-symbols

The key symmetry transformation for the elliptic hypergeometric function \( V(t) \) established in [28] generates a rich family of nontrivial relations between the \( V \)-functions with different parametrizations of its arguments, as well as of its different limiting forms. The generic hyperbolic degeneration preserves the structure of the initial relations. However, if we take some of the hyperbolic level parameters to infinity, some of the permutational symmetries are lost, and the number of different symmetry transformations increases substantially. One such relations was described earlier in (4.29). By taking parameters to infinity in a slightly different way, another symmetry transformation for function \( J_h(\mu, \nu) \) was derived in [4],
\[ J_h(\mu, \nu) = \prod_{j,k=1}^{2} \gamma^{(2)}(\mu_j + \nu_k; \omega_1, \omega_2) \prod_{j,k=3}^{4} \gamma^{(2)}(\mu_j + \nu_k; \omega_1, \omega_2) \times J_h(\mu_1 + \eta, \mu_2 + \eta, \mu_3 - \eta, \mu_4 - \eta, \nu_1 + \eta, \nu_2 + \eta, \nu_3 - \eta, \nu_4 - \eta), \] (7.1)
where
\[ \eta = \frac{1}{2}(\omega_1 + \omega_2 - \mu_1 - \mu_2 - \nu_1 - \nu_2). \] (7.2)

We now consider what kind of identity emerges from this relation in the limit \( b \to i \). Using parametrization (6.11) and taking the limit \( \delta \to 0^+ \), we obtain the following
symmetry transformation:

\[ \mathcal{J}_{cr}(s, n; t, m) = e^{\pi i A} \prod_{j,k=1}^{2} \Gamma(s_j + t_k, n_j + m_k) \prod_{j,k=3}^{4} \Gamma(s_j + t_k, n_j + m_k) \]

\[ \times \mathcal{J}_{cr}(s_1 + Y, n_1 + K, s_2 + Y, n_2 + K, s_3 - Y, n_3 - K, s_4 - Y, n_4 - K; \]

\[ t_1 + Y, m_1 + K, t_2 + Y, m_2 + K, t_3 - Y, m_3 - K, t_4 - Y, m_4 - K), \]

\[ K = -\frac{n_1 + n_2 + m_1 + m_2}{2}, \quad Y = -\frac{s_1 + s_2 + t_1 + t_2 + 2i}{2}, \]

\[ A = (n_1 + n_2)(m_1 + m_2) + (n_3 + n_4)(m_3 + m_4) + 2(\varepsilon + \lambda) \left( 1 + \sum_{a=1}^{4} m_a \right). \]  

In this relation, we have two discrete parameters \( \varepsilon, \lambda = 0, \frac{1}{2} \) in the values of summation variables in \( \mathcal{J}_{cr} \) in the respective left- and right-hand sides. If \( K \) is an integer, then \( \varepsilon = \lambda \), and if \( K \) is a half-integer, then \( \varepsilon \neq \lambda \). However, as we have mentioned already, the variable \( \varepsilon \) can be removed from the \( \mathcal{J}_{cr} \) function in the left-hand side by the shifts \( n_k \rightarrow n_k + \varepsilon, m_k \rightarrow m_k - \varepsilon \). Therefore we can assume that \( \varepsilon = 0 \), and then in the right-hand side we have \( \lambda = 0 \) for integer \( K \) and \( \lambda = \frac{1}{2} \) for half-integer \( K \).

We now consider similar consequences for identity (6.1). Using again parametrization (6.1) and taking the limit \( \delta \rightarrow 0^+ \), we find a connection between the \( \mathcal{J}_{cr} \) and \( \mathcal{E}_{cr} \) functions

\[ \mathcal{J}_{cr}(s, n; t, m) = e^{\pi i A} \prod_{a=1}^{3} \Gamma(s_a + t_4, n_a + m_4) \Gamma(t_a + s_4, m_a + n_4) \mathcal{E}_{cr}(s + Z, n + L, t - Z, m - L), \]

where

\[ \mathcal{E}_{cr}(s + Z, n + L, t - Z, m - L) = \sum_{N \in \mathbb{Z} + \lambda} \int_{y \in \mathbb{R}} (y^2 + N^2) \]

\[ \times \prod_{a=1}^{3} \Gamma(s_a + Z \pm y, n_a + L \pm N) \Gamma(t_a - Z \pm y, m_a - L \pm N) dy, \]

\[ L = -\frac{1}{2} (m_4 + \sum_{a=1}^{3} n_a), \quad Z = -\frac{1}{2} (t_4 + 2i + \sum_{a=1}^{3} s_a), \]

\[ A = 2L^2 - \left( \sum_{a=1}^{4} n_a \right) - 2n_4 m_4 - \lambda + 2\varepsilon \left( 1 + \sum_{a=1}^{4} m_a \right). \]

Since \( \mathcal{E}_{cr}(s) \) evidently depends on 6 independent complex parameters, the derived identity shows that this is true for the \( \mathcal{J}_{cr} \) function as well.

In relation (7.4) we have two discrete parameters \( \varepsilon, \lambda = 0, \frac{1}{2} \). If \( L \) is an integer, then \( \varepsilon = \lambda \), and if \( L \) is a half-integer, then \( \varepsilon \neq \lambda \). Again, without loss of generality, we can assume that \( n_k, m_k \in \mathbb{Z} \), i.e. \( \varepsilon = 0 \), and then \( \lambda = 0 \) for integer \( L \), whereas \( \lambda = \frac{1}{2} \) for half-integer \( L \). We conclude that both branches of the function \( \mathcal{E}_{cr} \) are related to \( \mathcal{J}_{cr} \) and thus together they provide, in a rather intricate way, a new unexpected integral representation of the 6j-symbols for the \( SL(2, \mathbb{C}) \) group, a representation structurally different from the Mellin-Barnes representation derived in [7, 14].
Appendix A. Details of the $b \to i$ asymptotics computations

In this appendix, we give asymptotic estimates for hyperbolic gamma functions in the limit $\delta \to 0^+$, which in combination reduce the 6j-symbols for the Faddeev modular double (3.9) to 6j-symbols for the $SL(2,\mathbb{C})$ group (2.11). The constraints on parameters were indicated in the main body of the paper:

\[
S_{i+\delta}(\mu_a - z) \to e^{\frac{\pi i}{2}(T_a + N)^2} (4\pi \delta)^{i(U_a + u) - 1} \Gamma(U_a + u, T_a + N), \quad (A.1)
\]

\[
S_{i+\delta}(\nu_a + z) \to e^{\frac{\pi i}{2}(S_a - N)^2} (4\pi \delta)^{i(R_a - u) - 1} \Gamma(R_a - u, S_a - N), \quad (A.2)
\]

\[
S_{i+\delta}(\alpha_s + \alpha_2 - \alpha_1) \to e^{\frac{\pi i}{2} A^2_3 (4\pi \delta)^{i(\sigma_1 - \sigma_2 + \rho_2)}} \Gamma(\sigma_1 - \sigma_2 + \rho_2 - i, A_1), \quad (A.3)
\]

\[
S_{i+\delta}(\alpha_1 + \alpha_t - \alpha_4) \to e^{\frac{\pi i}{2} A^2_4 (4\pi \delta)^{i(-\sigma_1 + \rho_2)}} \Gamma(\sigma_1 + \sigma_2 + \rho_2 - i, A_1), \quad (A.4)
\]

\[
S_{i+\delta}(\alpha_2 + \alpha_t - \alpha_3) \to e^{\frac{\pi i}{2} A^2_2 (4\pi \delta)^{i(-\sigma_2 + \rho_2)}} \Gamma(\sigma_2 + \sigma_3 - \rho_2 - i, A_2), \quad (A.5)
\]

\[
S_{i+\delta}(\alpha_3 + \alpha_s - \alpha_4) \to e^{\frac{\pi i}{2} A^2_1 (4\pi \delta)^{i(-\sigma_3 + \rho_2)}} \Gamma(-\sigma_3 - \sigma_4 + \rho_2 - i, A_3), \quad (A.6)
\]

\[
|S_{i+\delta}(2a_t)|^2 \to (4\pi \delta)^{3} \frac{M_t^2 + 4\rho_1^2}{4}. \quad (A.7)
\]

Appendix B. A check of equation (6.6)

We verify difference equation (6.6) for some particular choices of the parameters. Namely, we remark that integral (1.28) can be explicitly computed if, say, $\mu_4 + \nu_4 = Q$ and $\sum_3 (\mu_a + \nu_a) = Q$. In this case, $\gamma_3(\mu_4 - z; \omega_1, \omega_2) \gamma_3(\nu_4 + z; \omega_1, \omega_2) = 1$ and

\[
J_h(\mu, \nu) = \int_{i\infty}^{i\infty} \prod_{a=1}^{3} \gamma_3(\mu_a - z; \omega_1, \omega_2) \gamma_3(\mu_a + z; \omega_1, \omega_2) \frac{dz}{1/\omega_1 \omega_2} = \prod_{a,b=1}^{3} \gamma_3(\mu_a + \nu_b; \omega_1, \omega_2). \quad (B.1)
\]

Similarly, if $n_4 + m_4 = 0$ and $s_4 + t_4 = -2i$, then $\Gamma(s_4 - y, n_4 - N) \Gamma(t_4 + y, m_4 + N) = (-1)^{N-n_4}$. As a result, for

\[
\sum_{a=1}^{3} (n_a + m_a) = 0, \quad \sum_{a=1}^{3} (s_a + t_a) = -2i
\]

we have [24]

\[
\mathcal{J}_{cr}(s, n; t, m) = \frac{1}{4\pi} \sum_{N \in \mathbb{Z}} (-1)^{N-n_4} \int_{i\infty}^{i\infty} \prod_{a=1}^{3} \Gamma(s_a - y, n_a - N) \Gamma(t_a + y, m_a + N) dy
\]

\[
= (-1)^{\sum_{a=1}^{4} n_a} F(s, n; t, m), \quad (B.2)
\]

where

\[
F(s, n; t, m) = \prod_{a,b=1}^{3} \Gamma(s_a + t_a, n_a + m_a). \quad (B.3)
\]
From the equations
\[
\Gamma(x - i, n - 1) = \frac{\Gamma(x, n)(n - i x)}{2}, \quad \Gamma(x + i, n + 1) = \frac{\Gamma(x, n)}{n/2 - i x/2 + 1}
\] (B.4)
and parametrization (6.7), we find
\[
F(s_2 - i, n_2 - 1, s_3 + i, n_3 + 1) = \prod_{a=1}^{3} \frac{\beta_a + \gamma_a}{\beta_a + \gamma_a - 1} F(s, n; t, m), \quad (B.5)
\]
\[
\frac{\Gamma(s_2 - s_4, n_2 - n_4)\Gamma(s_3 - s_4, n_3 - n_4)}{\Gamma(s_2 - i - s_4, n_2 - 1 - n_4)\Gamma(s_3 + i - s_4, n_3 + 1 - n_4)} = \frac{\beta_3 - \beta_4 - 1}{\beta_2 - \beta_4}. \quad (B.6)
\]
As a result, imposing the above constraints on the parameters and relations (B.5), (B.6), we obtain the algebraic equation
\[
(\beta_2 - \beta_4 - 1)(\beta_3 - \beta_4)(\beta_3 - \beta_2 + 1) \left[ \prod_{k=1}^{3} (\beta_2 + \gamma_k)(\beta_3 - \beta_4 - 1) + \prod_{k=1}^{3} (\beta_3 + \gamma_k - 1)(\beta_4 - \beta_2) \right]
+ (\beta_3 - \beta_4 - 1)(\beta_2 - \beta_4)(\beta_3 - \beta_2 - 1) \left[ \prod_{k=1}^{3} (\beta_3 + \gamma_k)(\beta_2 - \beta_4 - 1) + \prod_{k=1}^{3} (\beta_2 + \gamma_k - 1)(\beta_4 - \beta_3) \right]
- (\beta_2 - \beta_3)(\beta_3 - \beta_2 + 1)(\beta_3 - \beta_2 - 1) \prod_{k=1}^{3} (\beta_4 + \gamma_k) = 0,
\] (B.7)
which is identically satisfied. Taking the limit \(\beta_4 \to \infty\) in (B.7), we obtain
\[
(\beta_3 - \beta_2 + 1) \left( \prod_{k=1}^{3} (\beta_2 + \gamma_k) - \prod_{k=1}^{3} (\beta_3 + \gamma_k - 1) \right) + (\beta_3 - \beta_2 - 1)
\times \left( \prod_{k=1}^{3} (\beta_3 + \gamma_k) - \prod_{k=1}^{3} (\beta_2 + \gamma_k - 1) \right) + (\beta_3 - \beta_2 + 1)(\beta_3 - \beta_2 - 1)(\beta_2 - \beta_3) = 0,
\] (B.8)
which leads to a difference equation for function (B.3),
\[
\mathcal{V}(s, n; t, m)(F(s_2 - i, n_2 - 1, s_3 + i, n_3 + 1) - F(s_2, n_2, s_3, n_3)) + (2 \leftrightarrow 3) + F(s_2, n_2, s_3, n_3) = 0,
\] (B.9)
where
\[
\mathcal{V}(s, n; t, m) = \frac{\prod_{k=1}^{3} (\beta_3 + \gamma_k - 1)}{(\beta_3 - \beta_2 - 1)(\beta_2 - \beta_3)}.
\] (B.10)

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