ON HARDY TYPE INEQUALITIES FOR WEIGHTED QUASIDEVIATION MEANS

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Abstract. Using recent results concerning the homogenization and the Hardy property of weighted means, we establish sharp Hardy constants for concave and monotone weighted quasideviation means and for a few particular subclasses of this broad family. More precisely, for a mean \( D \) like above and a sequence \( (\lambda_n) \) of positive weights such that \( \lambda_n/(\lambda_1 + \ldots + \lambda_n) \) is nondecreasing, we determine the smallest number \( H \in (1, +\infty] \) such that
\[
\sum_{n=1}^{\infty} \lambda_n D((x_1, \ldots, x_n), (\lambda_1, \ldots, \lambda_n)) \leq H \cdot \sum_{n=1}^{\infty} \lambda_n x_n \text{ for all } x \in \ell_1(\lambda).
\]

It turns out that \( H \) depends only on the limit of the sequence \( (\lambda_n/(\lambda_1 + \ldots + \lambda_n)) \) and the behaviour of the mean \( D \) near zero.

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