Simulating Dense Matter

Simon HANDS

Department of Physics, Swansea University,
Singleton Park, Swansea SA2 8PP, U.K.

I review the Sign Problem hindering lattice QCD simulations of dense baryonic matter, focussing where possible on its physical relevance. The possibility of avoiding the Sign Problem via a duality transformation is also briefly considered. Finally, I review evidence for deconfinement at non-zero quark density in recent simulations of Two Color QCD.

§1. Motivation

What is the nature of the QCD ground state in the limit $\mu_B/T \gg 1$, where $T$ is temperature and $\mu_B$ the baryon chemical potential? The insight that diquark Cooper pair condensation in the color anti-triplet channel is naturally promoted by one-gluon exchange suggests that in this régime QCD is a color superconductor. At asymptotic densities $\mu_B \to \infty$ where weak-coupling methods can be trusted, the favoured ground state of QCD with three light quark flavors $^\ast$ exhibits Color-Flavor Locking, with spontaneous symmetry breaking pattern

$$SU(3)_c \otimes SU(3)_L \otimes SU(3)_R \otimes U(1)_B \otimes U(1)_Q \to SU(3)_{\Delta} \otimes U(1)_{\tilde{Q}}.$$ (1.1)

Italicised and regular fonts denote respectively local and global symmetries of massless quarks: since both are spontaneously broken the CFL phase is simultaneously superconducting and superfluid.

At the densities available in stellar cores, the QCD coupling $g(\mu_B)$ is no longer small, making reliable calculation difficult. In matter with $\mu_B \sim O(m_s)$ pairing may only take place between $u$ and $d$ quarks, and further non-trivial constraints are imposed by requirements of charge- and color-neutrality. Model approaches have predicted many exotic scenarios, such as gapless superconductivity, mixed states of normal and superconducting matter, and crystalline LOFF phases.$^1$ The issue of which is the true ground state is ideally resolved, of course, by a systematic non-perturbative lattice QCD calculation, as suggested by the talk’s title. It is worth recalling, however, that the most urgent question about quark matter is whether it exists at all in our universe inside compact stars, or whether the star would have collapsed into a black hole before the required core density can be attained. To settle this theoretically we need to solve the Tolman-Oppenheimer-Volkoff equations for relativistic stellar structure, which requires quantitative knowledge of the equation of state, ie quark density $n_q$, pressure $p$ and energy density $\varepsilon$ as functions of $\mu_B$ for all $\mu_B > \mu_{B_0}$, where $\mu_{B_0} \approx 924 \text{MeV}$ is the onset value corresponding to self-bound nuclear matter. This issue, surely, is the first goal of lattice QCD with $\mu_B \neq 0$.

$^\ast$ Astrophysical arguments suggest that QCD matter with four or more quark flavors cannot form a stable gravitationally bound system.
§2. The Sign Problem, and why we need it

Let me remind you why this problem was not solved years ago. In Euclidean metric the QCD Lagrangian is written
\[ L_{\text{QCD}} = \bar{\psi} M \psi + \frac{1}{4} F_{\mu\nu} F_{\mu\nu} \] with
\[ M(\mu) = \mathcal{D} + \mu \gamma_0 + m, \] (2.1)
where \( \mu = \mu_B/N_c \) is the quark chemical potential. It is straightforward to show
\[ \gamma_5 M(\mu) \gamma_5 \equiv M^\dagger(-\mu) \] implying \( \det M(\mu) = \det M(-\mu)^* \), so that the path integral measure is not positive definite for \( \mu \neq 0 \). This is not solely an issue for fermions; it can be traced to the explicit breaking of time reversal symmetry by the term with \( \mu \neq 0 \), which in Euclidean metric corresponds to a breaking of the symmetry under \( i \mapsto -i \). The consequences are drastic; Monte Carlo importance sampling, the mainstay of lattice QCD, becomes ineffective.

To see why consider the formal solution to the Sign Problem known as reweighting. Here the phase of \( \det M \) is treated as an observable, and expectation values defined by
\[ \langle O \rangle = \frac{\langle \langle \text{O} \rangle \rangle}{\langle \langle \text{arg} \rangle \rangle}, \] (2.2)
with \( \langle \langle \ldots \rangle \rangle \) defined using the positive measure \( |\det M| e^{-S_{\text{boson}}} \). Unfortunately, both numerator and denominator of (2.2) are exponentially suppressed as \( V \to \infty \), eg:
\[ \langle \langle \text{arg} \rangle \rangle = \frac{Z_{\text{true}}}{Z_{\text{false}}} = \exp(-\Delta F) \sim \exp(-\#V) \] (2.3)
where in the last step we assume the free energy \( F \) is extensive. On general grounds we expect any signal for \( \langle O \rangle \) to be overwhelmed by statistical noise in the thermodynamic limit.

It is instructive and surprisingly easy to introduce a Sign Problem into QCD at \( \mu = 0 \). Consider the polymer representation\(^2\) for the QCD partition function:
\[ Z_{\text{QCD}} = \int DU \det M[U; m] e^{-S_W[U]} \]
\[ \propto \int DU \sum_{\{C\}} (2m)^{N_m} (-1)^{N_R} \prod_{\Gamma \in \mathcal{C}} \left( \text{tr} \prod_{\ell \in \Gamma} (\gamma_5)(\text{tr} U_{\Gamma}) \right) e^{-S_W[U]}. \] (2.4)
Each non-vanishing term in the expansion of the determinant is represented by a partition \( \mathcal{C} \) of the lattice into \( N_m \) monomers, \( N_d \) dimers and \( N_T \) polymers, the latter

\[ \text{Fig. 1. Two polymers of opposite sign.} \]
defined as oriented closed paths of links. In the strong-coupling limit where only monomers and dimers contribute each term is positive. Polymers contribute not only a Wilson loop \( U_\Gamma \) to the effective action but also a shape-dependent sign factor 
\[-\text{tr} \prod_{\ell \in \Gamma} \gamma_\ell, \] 
where \( \gamma_\ell = \pm \gamma_\mu \) depending on whether the link \( \ell \) points along \( \pm \hat{\mu} \).

The resulting Sign Problem makes simulation of even the non-interacting system difficult. At weak gauge coupling it is tempting to interpret the polymers as quark worldlines, but note that the overall sign contributed by \( C \) can be changed by the innocent-looking flip shown in Fig. 1. It is difficult to believe that in this limit there can be any significant correlation between the sign of \( C \) and long-range physics.

The positive measure \( \det M^\dagger M \) used in practical fermion algorithms describes color triplet quarks \( q \) and color anti-triplet conjugate quarks \( q^c \). There are thus gauge-invariant \( q q^c \) bound states with baryon number \( B \neq 0 \). At \( \mu = 0 \) we are content to consider these states as extra “mesons” and move on. Once \( \mu \neq 0 \), however, this position is untenable. The lightest baryon in this model’s spectrum is degenerate with the pion, so that there is an unphysical onset transition between vacuum and baryonic matter at \( \mu_0 \approx \frac{1}{2} m_\pi \). Only calculations performed with the correct complex measure \( \det^2 M \) can yield cancellations among configurations with differing phases, which nullify the effect of \( q q^c \) states and postpone the onset transition to the phenomenologically-observed \( \mu_0 \approx \frac{1}{3} m_N \).

This cancellation has been numerically verified in simulations of Two Color QCD (QC\( _2 \)D) with a single staggered quark flavor in the adjoint representation, where it was found that the signal for a fake transition at \( \mu \simeq \frac{1}{2} m_\pi \), to a superfluid phase whose order parameter \( \langle \bar{\psi} \psi \rangle \) vanishes identically due to the Pauli principle, went away once the sign of the determinant was correctly taken into account. More recently a visualisation of the Sign Problem has emerged from an analytic solution of a random matrix model with the same global symmetries as QCD, corresponding to the so-called mesoscopic limit of \( V \to \infty \) with \( m_\pi^2 f_\pi^2 V \) fixed. The chiral condensate can be expressed in terms of the distribution \( \rho(z) \) of eigenvalues of \( M - m \) in the complex plane:
\[
\langle \bar{\psi} \psi \rangle = \lim_{m \to 0} \lim_{V \to \infty} V^{-1} \int dz dz' \rho(z,m;\mu) \frac{z}{z + m}. \tag{2.5}
\]

For \( \mu = 0 \) or the quenched limit \( N_f = 0 \), \( \rho \) is real, but in general it is a complex-valued function. The explicit solution for \( \mu > m_\pi/2 \) shows that for \( \text{Re} \ z > m \), \( \rho \) develops an oscillatory structure, with a characteristic wavelength of \( O(V^{-1}) \) and amplitude \( O(e^V) \). Any function calculated using a formula such as (2.5) must receive wildly fluctuating contributions from different regions of the plane, but remarkably, it can be shown that the result behaves entirely in accord with physical expectations, namely that \( \langle \bar{\psi} \psi(m) \rangle \) changes sign at \( m = 0 \), but exhibits no sign of discontinuous behaviour as \( \mu \) passes through \( m_\pi/2 \).

§3. High Density Effective Theory and an Optimistic Conjecture

For \( \mu \gg T, \mu \gg \Lambda_{\text{QCD}} \), QCD is supposed to exist in a deconfined phase of degenerate weakly interacting quarks. Is it possible to construct an effective theory
in terms of quasiparticle degrees of freedom at the Fermi surface, with momenta $p, k \ll \mu$. Define “fast” (−) and “slow” (+) degrees of freedom via the decomposition:

$$\psi(x) = \exp(i\mu x \cdot \hat{p})[\psi_+(x) + \psi_-(x)] \text{ with } \psi_\pm(p) = \frac{1}{2}(1 \pm \alpha \cdot \hat{p})\psi(p).$$

The phase and projection factors ensure that the $\psi_\pm$ fields scatter off gluons with physical momenta $q$, according to the kinematics shown in Fig. 2. The quark Lagrangian in Minkowski metric now reads

$$\bar{\psi}(\gamma_0 + (1, -\hat{p})\gamma_\nu i\tilde{D}_\nu \psi + \gamma_\nu (\gamma_0 \tilde{A}_\nu \psi + h.c.) + \frac{g^2}{2\mu} \psi_+ (\gamma^\nu \tilde{A}_\nu \psi_+ + O(D^3\mu^{-2}))$$

with $\gamma^\nu = (\gamma_0, \hat{p} \gamma_\nu \hat{p})\gamma^\nu$, $\gamma^\nu_\perp = \gamma^\nu - \gamma^\nu_\parallel$.

The resulting theory has been used with some success to analyse the color superconducting phase; however, for our purposes it is more interesting to continue to Euclidean metric. Since $\gamma_\parallel \tilde{D}_\nu$ is anti-hermitian and satisfies $\{\gamma_\parallel \tilde{D}_\nu, \gamma_5 \} = 0$, it is straightforward to show $\text{det}(\gamma_\parallel \tilde{D}_\nu)$ is positive definite. There is therefore no Sign Problem in the limit $\mu \to \infty$.

I have argued the Sign Problem is intractable almost everywhere in the $(\mu, T)$ plane as $V \to \infty$; however, it is perhaps possible to distinguish between regions such as $\mu \in (\frac{1}{2}m_\pi, \frac{1}{3}m_N)$ where we know from Sec. 2 that sign cancellations are both subtle and crucial to obtaining physically sensible predictions, from regions where the sign fluctuations are not so strongly correlated with long range physics. One such region appears to be the upper left-hand corner of the QCD phase diagram $\mu/T \lesssim 1$ where RHIC physics takes place: on finite volumes there is a pleasing consistency between approaches based on reweighting (which must inevitably fail in the thermodynamic limit) and alternative methods based on analytic continuation.

While a systematic numerical treatment of HDET is yet to emerge, the previous paragraph at least suggests that in the cold dense régime $\mu/T \gg 1$ a “solution” of the Sign Problem may not be a crucial component of the physics, and that it may be possible to perform controlled calculations on reasonable volumes. Another

---

* An exception appears to be exactly at $T = 0$ below the fake onset at $\mu = m_\pi/2$. 

---

Fig. 2. Kinematics of quasiquark-gluon scattering at the Fermi surface.
approach, which I will pursue in Sec. 5 below, is to argue that a theory with no Sign Problem such as QC2D still models much relevant physics in this régime.

§4. Are we using the Right Basis?

Large cancellations between either Feynman diagrams or gauge configurations hint at low calculational efficiency. Maybe gauge covariant quarks and gluons are not the best degrees of freedom at high density? It is possible in some cases to effect a transformation to another basis in which the Sign Problem is absent. An intriguing example illustrating this comes from 3d scalar QED,11) with Lagrangian

\[ L_{SQED} = \frac{1}{4} F^2 + |D\phi|^2 + m^2|\phi|^2 + \lambda|\phi|^4 - \frac{1}{2} \varepsilon_{ijk} H_i F_{jk}, \tag{4.1} \]

where \( H \) is a real vector source coupled to a generalised \( B \)-field. The model has a transition separating Coulomb and Higgs phases which is second order for sufficiently large \( \lambda/e^2 \). There is a conjectured duality at this critical point with a complex scalar field theory described by

\[ L_{SFT} = [(\partial - \tilde{e} H) \bar{\phi} \phi^*] [(\partial + \tilde{e} H) \bar{\phi} \phi] + \tilde{m}^2|\bar{\phi}|^2 + \tilde{\lambda}|\bar{\phi}|^4 + \cdots \tag{4.2} \]

The point is that \( \tilde{e} H_3 \) with \( \tilde{e} = 2\pi/e \) is a real chemical potential associated with the conserved charge density \( 2\text{Im}(\bar{\phi} \partial_3 \phi) \). The Lagrangian (4-2) is in general complex, describing planar Bose-Einstein condensation (BEC) of \( \bar{\phi} \)-quanta for \( \tilde{e} H_3 \approx \tilde{m} \). In principle, however, it could be studied via simulations of the real action (4-1).

A more recent example due to Endres12) exploits an exact duality between complex scalar field theory in \( d \) dimensions and a loop gas:

\[ Z(\mu) = \int D\phi D\phi^* \exp \left[ -\sum_{x\nu} \left( 2\phi_x^* \phi_x - \phi_x^* e^{\mu \delta_{x,0}} \phi_{x+\nu} - \phi_x^* e^{-\mu \delta_{x,0}} \bar{\phi}_x \right) - m^2 \sum_x \phi_x^* \phi_x - \sum_x V(\phi_x^* \phi_x) \right] \]

\[ \alpha \sum_{\{\ell\}} \int_0^\infty \rho \rho e^\mu \sum_{x,0} \prod_x \left[ e^{-(2d+2+m^2)\rho_x^2 - V(\rho_x^2)} \prod_\nu I_{\ell_x,\nu}(2\rho_x \rho_{x+\nu}) \right], \tag{4.3} \]

where \( \rho_x = |\phi_x| \), and we have introduced integer-valued link variables \( \ell_{x,\nu} \) governed by the constraint \( \partial_{\ell_{\nu}} \ell_{\nu} = 0 \). Once again \( Z \) is recast in terms of a functional integral over real variables, and the Sign Problem averted. Remarkably, it has proved possible to simulate the action (4-3) efficiently, yielding non-trivial results of physical interest.12)

§5. Two Color Matters

QCD with gauge group SU(2) and an even number of fundamental quark flavors has a real functional measure even once \( \mu \neq 0 \), and remains the only dense matter system with long-range fundamental interactions amenable to study with orthodox
lattice methods. Since $q$ and $\bar{q}$ live in equivalent representations of the color group, hadron multiplets contain both $qq$ mesons and $q\bar{q}$, $\bar{q}q$ baryons. Near the chiral limit at $\mu = 0$ we expect spontaneous chiral symmetry breaking implying $m_\pi \ll m_\rho$, where $\rho$ denotes the next lightest hadron. The theory’s $\mu$-dependence for $\mu < m_\rho$ can be analysed using chiral perturbation theory ($\chi$PT). The key result is that a second order onset transition occurs at $\mu_o = \frac{1}{2} m_\pi$ to a phase with quark charge density $n_q > 0$. For $N_f = 2$ the matter which forms is composed of tightly-bound scalar diquarks, which Bose condense to form a gauge-invariant superfluid BEC $\langle \bar{q}q \rangle \equiv \langle \psi^T C\gamma_5\sigma_2\tau_2\psi \rangle \neq 0$, where the matrices act on spinor, flavor and color indices respectively. Since $n_q \to 0$ as $\mu \to \mu_o$ the matter can be arbitrarily dilute.

The $\chi$PT predictions for $\langle \bar{\psi}\psi(\mu) \rangle$ and $\langle q\bar{q}(\mu) \rangle$ have been confirmed by lattice simulations with staggered fermions.\(^4\), \(^1\)\(^4\) For our purposes the most relevant prediction is the equation of state for $T = 0$, $\mu > \mu_o$:

$$n_{\chi PT} = 8N_f f_\pi^2 \mu \left(1 - \frac{\mu_o^4}{\mu^4}\right), \quad (5.1)$$

leading to the pressure $p_{\chi PT} = \int_{\mu_o}^{\mu} n_q d\mu$ and energy density $\varepsilon_{\chi PT} = -p + \mu n_q$:

$$p_{\chi PT} = 4N_f f_\pi^2 \left(\mu^2 + \frac{\mu_o^4}{\mu^2} - 2\mu_o^2\right); \quad \varepsilon_{\chi PT} = 4N_f f_\pi^2 \left(\mu^2 - \frac{3\mu_o^4}{\mu^2} + 2\mu_o^2\right). \quad (5.2)$$

This is to be contrasted with another paradigm for cold dense matter, namely a degenerate system of weakly interacting quarks populating a Fermi sphere up to some maximum momentum $k_F \approx E_F = \mu$, obeying Stefan-Boltzmann (SB) scaling:

$$n_{SB} = \frac{N_f N_c}{3\pi^2} \mu^3; \quad \varepsilon_{SB} = 3p_{SB} = \frac{N_f N_c}{4\pi^2} \mu^4. \quad (5.3)$$

The appearance of $N_c$ underlines that (5.3) describes a deconfined phase. Superfluidity in this scenario arises from condensation of diquark Cooper pairs within a layer of thickness $\Delta$ centred on the Fermi surface, implying $\langle q\bar{q} \rangle \propto \Delta \mu^2$. Fig. 3 shows the ratio of $\chi$PT to SB predictions as functions of $\mu/\mu_o$ for the choice $f_\pi^2 = N_c/6\pi^2$. By equating pressures, we naively predict a first-order deconfining transition from BEC
Simulating Dense Matter

To quark matter at $\mu_d/\mu_o \approx 2.3^\star$.

To test whether this prediction is robust we have performed lattice simulations of QC$_2$D using $N_f = 2$ Wilson quarks.$^{15}$ Initial runs have been on a $8^3 \times 16$ lattice with lattice spacing (defined via the string tension) $a \approx 0.22$ fm, corresponding to $T \approx 60$ MeV. We have not attempted to get particularly close to the chiral limit; $m_\pi a = 0.79(1)$ and $m_\rho a = 0.80(1)$. The code’s only novelty is the inclusion of a diquark source term of the form $j(qq + \bar{q}\bar{q})$, with $ja = 0.04$ for the most part, which both ensures ergodicity and regularises IR fluctuations in the superfluid phase. Fig. 4 shows the resulting curves for $n_q$, $p$ and $\varepsilon_q$ in the same format as Fig. 3 (open symbols denote the $j \to 0$ extrapolation). There appears to be a transition from confined bosonic “nuclear matter” to deconfined quark matter at $\mu_d a \approx 0.65$. For large $\mu$, $n_q/n_{SB} \approx p/p_{SB} \approx 2$, consistent with a bound system having $E_F < k_F$. The claim is supported by Fig. 5, which shows the Polyakov line rise from zero at $\mu_d$, coincident with the superfluid order parameter assuming the scaling $\langle qq \rangle \propto \mu^2$, consistent with Cooper pairing at a Fermi surface. A condensed matter physicist would refer to this as a BEC/BCS crossover. Similar conclusions have been reached in a study of topological charge susceptibility using staggered fermions with $N_f = 8$.$^{16}$

A major motivation for studying QC$_2$D is to understand how deconfined quarks affect gluodynamics; as argued above, it is at least plausible that the lessons learned

$^\star$ The apparent transition at $\mu_d/\mu_o \approx 1.4$ can be eliminated by introducing a bag constant to stabilise the confined phase at low density.
may apply to physical QCD. In any medium with a preferred rest frame, the gluon propagator can be decomposed as follows:

\[
D_{\mu\nu}(q) = P^{M}_{\mu\nu}D_{M}(q^2_0, \vec{q}.\vec{q}) + P^{E}_{\mu\nu}D_{E}(q^2_0, \vec{q}.\vec{q}) + P^{L}_{\mu\nu}D_{L}(q^2_0, \vec{q}.\vec{q})
\]  

(5.4)

with

\[
P^{M}_{ij} = \delta_{ij} - \frac{q_iq_j}{\vec{q}.\vec{q}}, \quad P^{M}_{00} = P^{M}_{0i} = 0; \quad P^{E}_{\mu\nu} = \delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2_0 + \vec{q}.\vec{q}} - P^{M}_{\mu\nu}, \quad P^{L}_{\mu\nu} = \frac{q_\mu q_\nu}{q^2_0 + \vec{q}.\vec{q}}.
\]

We have used Landau gauge in which \(D_L = 0\). Figs. 6 and 7 plot \(D_E\) and \(D_M\) in the static limit \(q_0 = 0\) as functions of \(|\vec{q}|\), for various \(\mu\) on either side of the deconfinement transition. In the electric sector for \(\mu a \geq 0.9\) Fig. 6 shows evidence for some Debye screening as \(|\vec{q}| \to 0\). Deconfinement has a much more dramatic effect in the magnetic sector shown in Fig. 7, where in the same limit the propagator is screened by \(O(50\%)\).

This is significant because in perturbation theory magnetic gluons are not screened in the static limit; indeed, this is at the origin of the celebrated scaling \(\Delta \propto e^{-#/g}\) of the color superconducting gap predicted by weak-coupling methods.\(^{17}\)

In summary, even models with no Sign Problem may hold surprises for us at large \(\mu\). It would, of course, be nice to compare lattice results for QC2D with other non-perturbative approaches such as Schwinger-Dyson equations. My overall feeling, though, is that a radical reformulation of non-perturbative QCD is needed before numerical approaches can make further headway.

Acknowledgements

It is a pleasure to thank my hosts for organising such an excellent meeting. My trip to YKIS2006 was supported by the Royal Society.

References

1) For a recent review see I.A. Shovkovy, in Extreme QCD (Swansea, 2005) p.37.
2) M. Karowski, R. Schrader and H.J. Thun, Commun. Math. Phys. 97 (1985), 5
3) I. Montvay, Phys. Lett. B 227 (1989), 260
4) S.J. Hands, I. Montvay, L. Scorzato and J.I. Skullerud, Eur. Phys. J. C22 (2001), 451
5) J.C. Osborn, Phys. Rev. Lett. 93 (2004), 222001;
   G. Akemann, J.C. Osborn, K. Splittorff and J.J.M. Verbaarschot, Nucl. Phys. B 712 (2005), 287.
6) J.C. Osborn, K. Splittorff and J.J.M. Verbaarschot, Phys. Rev. Lett. 94 (2005), 202001.
7) D.K. Hong, Phys. Lett. B 473 (2000), 118.
8) D.K. Hong and S.D. Hsu, Phys. Rev. D 66 (2002), 071501.
9) K. Splittorff and J.J.M. Verbaarschot, hep-lat/0702011.
10) E. Laermann and O. Philipsen, Ann. Rev. Nucl. Part. Sci.53 (2003), 163.
11) K. Kajantie, M. Laine, T. Neuhaus, A. Rajantie and K. Rummukainen, Nucl. Phys. B 699 (2004), 632.
12) M.G. Endres, hep-lat/0610029.
13) J.B. Kogut, M.A. Stephanov, D. Toublan, J.J.M. Verbaarschot and A. Zhitnitsky, Nucl. Phys. B 582 (2000), 477.
14) S.J. Hands, I. Montvay, S.E. Morrison, M. Oevers, L. Scorzato and J.I. Skullerud, Eur. Phys. J. C10 (2000), 285;
   J.B. Kogut, D.K. Sinclair, J.S. Hands and S.E. Morrison, Phys. Rev. D 64 (2001), 094505.
15) S.J. Hands, S. Kim and J.I. Skullerud, Eur. Phys. J. C48 (2006), 193.
16) B. Alles, M. D’Elia and M.P. Lombardo, Nucl. Phys. B 752 (2006), 124.
17) D.T. Son, Phys. Rev. D 59 (1999), 094019.