Bounds on the Maximum Number of Minimum Dominating Sets

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Abstract

Given a graph with domination number \( \gamma \), we find bounds on the maximum number of minimum dominating sets. First, for \( \gamma \geq 3 \), we obtain lower bounds on the number of \( \gamma \)-sets that do not dominate a graph on \( n \) vertices. Then, we show that \( \gamma \)-fold lexicographic product of the complete graph on \( n^{1/\gamma} \) vertices has domination number \( \gamma \) and \( \binom{n}{\gamma} - O(n^{\gamma-\frac{1}{2}}) \) dominating sets of size \( \gamma \). Finally, we see that a certain random graph has, with high probability, (i) domination number \( \gamma \); and (ii) all but \( o(n^\gamma) \) of its \( \gamma \)-sets being dominating.

1 Introduction

A set \( S \) of vertices in a graph \( G \) is a dominating set if each vertex in the complement of \( S \) is adjacent to at least one vertex in \( S \); the minimum cardinality \( \gamma = \gamma(G) \) of such an \( S \) is called the domination number. A graph \( G \) with domination number \( \gamma \) thus has at least one dominating set of size \( \gamma \) and no dominating set of size at most \( \gamma - 1 \). It is easy to give an example of a graph with only one dominating set of size \( \gamma \), but how abundant can the
number of minimum dominating sets be? Let $X_{\gamma} = X_{\gamma}(G)$ be the number of dominating sets of size $\gamma(G)$. Then we let

$$M_{n,\gamma} = \max \{ X_{\gamma}(G) : |V(G)| = n \text{ and } \gamma(G) = \gamma \}$$

be the maximum number of minimum dominating sets in a graph on $n$ vertices and domination number $\gamma$. In view of the nature of the results in this paper, we will often state these in terms of $m_{n,\gamma}$, the minimum number of non-dominating $\gamma$-sets, since

$$m_{n,\gamma} = \min \left\{ \binom{n}{\gamma} - X_{\gamma}(G) : |V(G)| = n \text{ and } \gamma(G) = \gamma \right\}.$$  

Godbole et al. [5] provided a construction that gives a lower bound for $M_{n,\gamma}$, $\gamma \geq 3$. In this paper we use both probabilistic and constructive approaches to improve the results in [5] as well as to study the random variable $X_{\gamma}(G)$. Specifically,

(i) We present results (Theorem 2.1 and Corollary 2.3) that show that for every $\gamma \geq 3$ and every $G$ with dominating number $\gamma$, there are $\Omega(n^{\gamma - 1 - \frac{1}{\gamma} + \epsilon})$ non-dominating sets of size $\gamma$, and thus $m_{n,\gamma} \geq \Omega(n^{\gamma - 1 - \frac{1}{\gamma} + \epsilon})$;

(ii) In Proposition 2.2, we improve the constructions in [5] to produce a graph $G$ with domination number $\gamma$ for which the number of dominating sets is equal to $\binom{n}{\gamma} - O(n^{\gamma - 1 - \frac{1}{\gamma}})$, thus showing that $m_{n,\gamma} = O(n^{\gamma - 1 - \frac{1}{\gamma}})$;

(iii) Finally, in Theorem 3.1, we show that for every $\gamma \geq 3$, there exists an edge probability $p = p_n$ so that the random graph $G(n,p)$ satisfies, with high probability, the conditions $\gamma(G(n,p)) = \gamma$ and

$$X_{\gamma}(G(n,p)) = \binom{n}{\gamma} - O(n^{\gamma - 1 - \frac{1}{\gamma}}).$$

The related algorithmic question of counting dominating sets using a “measure and conquer” approach together with linear programming is addressed in [2]. Counting dominating sets by taking advantage of the low treewidth of a graph is studied in [3].

Throughout the paper, $\gamma$ will be a constant that does not depend on $n$. We will refer to a fraction that is asymptotically $(1 - o(1))$ of all $\gamma$-sets as being “almost all” $\gamma$-sets. Moreover, it will be tacitly assumed in such situations that the domination number of the graph $G$ in question is $\gamma$.
2 Lower Bounds on the Number of Sets that do not Dominate

We let \( D_G := A + I_n \), where \( A \) is the adjacency matrix corresponding to some ordering of the vertices of the graph. Thus a set of vertices \( a_1, a_2, \ldots, a_k \) in a graph \( G \) is a dominating set if and only if there are no rows in \( D_G \) with zeroes in the \( a_i \)th column for all \( 1 \leq i \leq k \).

We start with the observation that for \( a \leq n \), if we have \( n(a) \geq n(b) \), then it follows that

\[
\frac{n^{a^b}}{b!} = \frac{a^b}{a(a-1) \ldots (a-b+1)} \cdot n \cdot \binom{a}{b}
\]

\[
\geq \left( \prod_{i=1}^{b-1} \frac{1}{1 - i/\alpha} \right) \binom{n}{b}
\]

\[
= \left( \prod_{i=1}^{b-1} \frac{1 - i/n}{1 - i/\alpha} \right) \frac{n^b}{b!}
\]

\[
\geq \frac{n^b}{b!}.
\]

**Theorem 2.1.** For every \( \gamma \geq 3 \) and each graph \( G \) on \( n \) vertices with domination number \( \gamma \), let \( 3 \leq k \leq \gamma + 1 \). Then \( G \) must contain at least \( \left( \binom{n}{k-2}/(k-1)! \right) \) non-dominating sets of \( k \) vertices.

*Proof.* Let \( r \) be the largest integer such that \( \binom{n}{r-1} \leq n \binom{r+1}{k-1} \). This implies that \( \binom{n}{k-1} \leq n \binom{r+1}{k-1} \), which, by the above observation (with \( a = r + 1 \) and \( b = k - 1 \)) shows that

\[
\frac{n(r + 1)^{k-1}}{(k-1)!} \geq \frac{n^{k-1}}{(k-1)!},
\]

and thus that \( r + 1 \geq n^{(k-2)/(k-1)} \). We claim that there is a row in \( D_G \) with at least \( r + 1 \) zeros. Otherwise, each row induces at most \( \binom{r}{k-1} \) non-dominating sets of size \( k - 1 \), and the total number of non-dominating sets of size \( k - 1 \) is at most \( n \binom{r}{k-1} < \binom{n}{k-1} \), a contradiction to the fact that \( \gamma > k - 1 \).

Using the row of \( D_G \) with \( r + 1 \) zeros, we can construct \( \binom{r+1}{k} > \binom{n^{(k-2)/(k-1)}}{k} \) non-dominating sets of size \( k \). This proves Theorem 2.1. \( \square \)
We next give an explicit construction of a graph $G$ with $X_\gamma(G) = \binom{n}{\gamma} - O(n^{\gamma - \frac{1}{2}})$. For $n$ such that $n^{1/\gamma}$ is an integer, let $V(G) = \{1, 2, \ldots, n^{1/\gamma}\}^\gamma$. Let vertices $(u_1, \ldots, u_\gamma)$ and $(v_1, \ldots, v_\gamma)$ be adjacent if $u_i \neq v_i$ for each $i > 1$. $G$ is the complete graph for $\gamma = 1$, and for $\gamma \geq 2$, $G$ is the $\gamma$-fold lexicographic product of $K_{n^{1/\gamma}}$, in which the first coordinate is irrelevant as far as edges are concerned. For $\gamma = 2$, we consider the two-fold lexicographic product of $K_{\sqrt{n}}$, with $n = m^2$. A pair of vertices with the same second coordinate cannot dominate, because neither of the pair are adjacent to any other vertex with that second coordinate. Also, any pair of vertices with different second coordinate do dominate, since any other vertex has its second coordinate with that second coordinate. Thus no set of $\gamma = 2$ in which every pair of vertices dominates. In this case, we need (i) to verify that the domination number is in fact $\gamma$, and (ii) to identify the non-dominating sets of cardinality $\gamma$.

**Proposition 2.2.** For every $M \geq 1$ and every $\gamma \geq 2$, let $G$, $|V(G)| = n = M\gamma$, be the $\gamma$-fold lexicographic product of $K_M$. Then $\gamma(G) = \gamma$ and the number of non-dominating sets of size $\gamma$ is of order $n^{\gamma - \frac{1}{2}}$.

**Proof.** To prove the domination number is at least $\gamma$, take a set of vertices $\{v^1, \ldots, v^{\gamma-1}\}$. Each $v^i$ is a $\gamma$-tuple $(v^i_1, \ldots, v^i_\gamma)$. Consider the vertex $u = (1, v^1_2, v^2_3, \ldots, v^{\gamma-1}_\gamma)$ obtained by a Cantor-diagonal-like process. For every $v^i$, $u_{i+1} = v^i_{i+1}$, so $u$ is not adjacent to any vertex $v^1, \ldots, v^{\gamma-1}$ since it coincides with each $v^i$ in at least one coordinate other than the first. Thus no set of size $\gamma - 1$ can dominate.

We claim that a set $S = \{v^1, \ldots, v^\gamma\}$ dominates if for all $i > 1$, $v^k_i \neq v^j_i$, provided that $j \neq k$. To prove the claim, assume that $S$ satisfies this property. Let $u$ be a vertex of $G$ and for each $2 \leq i \leq \gamma$, define $A_i = \{j : v^j_i = u_i\}$. By the properties of the set $S$, we have $|A_i| \leq 1$. By a pigeonhole argument, there exists $1 \leq k \leq \gamma$ such that $k \notin \bigcup_{i=2}^{\gamma} A_i$. This implies that $u_i \neq v^k_i$ for every $2 \leq i \leq \gamma$ and therefore $v^k$ dominates $u$.

We now enumerate the number of dominating sets. These can be thought upon as being the rows of a $\gamma \times \gamma$ matrix, with entries from the set $[n^{1/\gamma}]$ and satisfying the condition that each column except for the first has distinct entries. There are thus $(n^{1/\gamma})^\gamma = n$ choices for the first column, and $(n^{1/\gamma})_{\gamma - 1}$ choices for each of the others, where $(a)_b$ denotes the descending factorial of $a$. For $\gamma = 2$, we consider the two-fold lexicographic product of $K_{\sqrt{n}}$, with $n = m^2$. A pair of vertices with the same second coordinate cannot dominate, because neither of the pair are adjacent to any other vertex with that second coordinate. Also, any pair of vertices with different second coordinate do dominate, since any other vertex has its second coordinate with that second coordinate. Thus no set of $\gamma = 2$ in which every pair of vertices dominates. In this case, we need (i) to verify that the domination number is in fact $\gamma$, and (ii) to identify the non-dominating sets of cardinality $\gamma$. 

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length \( b \) starting at \( a \). This yields \( \binom{n}{\gamma} \cdot ((n^{1/\gamma})^{\gamma - 1}) \) for the total number of matrices, but, since a dominating set is unordered, we divide by \( \gamma! \) to get the number of non-dominating sets as being equal to

\[
\binom{n}{\gamma} \cdot \frac{n \cdot ((n^{1/\gamma})^{\gamma - 1})}{\gamma!}.
\] (1)

We now analyze the asymptotic nature of the two terms in (1). First we calculate that

\[
\binom{n}{\gamma} = \left(1 - \frac{1}{n}\right) \left(1 - \frac{2}{n}\right) \ldots \left(1 - \frac{\gamma - 1}{n}\right) \frac{n^\gamma}{\gamma!}.
\] (2)

Likewise we have that

\[
\frac{n \cdot ((n^{1/\gamma})^{\gamma - 1})}{\gamma!} = \left\{ \left(1 - \frac{1}{n^{1/\gamma}}\right) \left(1 - \frac{2}{n^{1/\gamma}}\right) \ldots \left(1 - \frac{\gamma - 1}{n^{1/\gamma}}\right) \right\}^{\gamma - 1} \frac{n^\gamma}{\gamma!}.
\] (3)

Substituting (2) and (3) into (1) and using the fact that \( \lim_{x \to 0} (1 - e^{-x})/x = 1 \), we get the number of non-dominating sets as being asymptotic to

\[
\frac{n^\gamma}{\gamma!} \left( \frac{\gamma(\gamma - 1)^2}{2n^{1/\gamma}} - \frac{\gamma(\gamma - 1)}{2n} \right) \sim \frac{n^\gamma}{\gamma!} \left( \frac{\gamma(\gamma - 1)^2}{2n^{1/\gamma}} \right),
\]

as claimed. \( \square \)

Since it is easy to check that the lower bound of Theorem 2.1 with \( m = \gamma \) is asymptotic to \( n^{\gamma - 1 - \frac{1}{\gamma - 1}} \), Theorem 2.1 and Proposition 2.2 lead to the following corollary:

**Corollary 2.3.** For \( \gamma \geq 3 \),

\[
\Omega \left( n^{\gamma - 1 - \frac{1}{\gamma - 1}} \right) \leq m_{n, \gamma} = O(n^{\gamma - \frac{1}{\gamma}}).
\]
3 Domination by a Large Fraction of $\gamma$-sets in Highly Dense Random Graphs

In this section, we show that for every $\gamma \geq 1$ and large enough $n$, there exists a $p = p_{n,\gamma}$ very close to 1, such that the Erdős-Rényi random graph $G(n, p)$ has domination number $\gamma$ and is dominated by almost all $\gamma$-sets. This provides a probabilistic counterpart to Proposition 2.2.

**Theorem 3.1.** Given any $\gamma \in \mathbb{N}$, there exists a sequence $\epsilon = \epsilon_{n,\gamma} \to 0$ (as $n \to \infty$), such that, w.h.p., the Erdős-Rényi random graph $G(n, 1-\epsilon_{n,\gamma})$ has fixed domination number $\gamma$ and is dominated by almost all the $\gamma$-sets ($n \to \infty$).

**Proof.** Set $\epsilon = \epsilon_{n,\gamma} = n^{-2/(2\gamma-1)}$, and let $X_{\gamma-1}$ be the number of dominating sets, in $G(n, 1-\epsilon)$, of size one less than the desired domination number. By Markov’s inequality and linearity of expectation,

$$P(X_{\gamma-1} \geq 1) \leq E(X_{\gamma-1}) = \binom{n}{\gamma-1} (1-%2d n^{-\gamma+1})$$

$$\leq \frac{n^{-\gamma-1}}{(\gamma-1)!} \exp\{-n(\gamma+1)n^{-\gamma-1}\} \to 0,$$ (4)

since with the choice of $\epsilon = n^{-2/(2\gamma-1)}$, $(\gamma-1) \log n - n\epsilon^{-1} \to -\infty$. So with high probability there exists no dominating set of size $\gamma - 1$. By a similar argument, and using the inequality $1-x \geq \exp\{-x/(1-x)\}$, we see that

$$E(X_{\gamma}) = \binom{n}{\gamma} (1-\epsilon)^{n-\gamma} \geq \binom{n}{\gamma} \exp\{-\frac{(n-\gamma)\epsilon^{-\gamma}}{1-\epsilon^{-\gamma}}\}.$$  

Thus as $n \to \infty$, $E(X_{\gamma})/\binom{n}{\gamma} \to 1$ since $\epsilon \ll \frac{1}{\sqrt{n}}$.

There is an easy way to show that this implies that $X_{\gamma} = \binom{n}{\gamma} (1-o(1))$ with high probability: Assume that $X_{\gamma} \leq q(n)$ with probability $p > 0$ for some $q < 1$. Then $E(X_{\gamma}) \leq pq \binom{n}{\gamma} + (1-p) \binom{n}{\gamma} = (1-p+pq) \binom{n}{\gamma} \neq \binom{n}{\gamma} (1-o(1))$, a contradiction. Thus, for $n$ large, it is improbable that the value of $X_{\gamma}$ falls further than an asymptotically negligible fraction below its expected value, and so this process, with high probability, constructs a graph of domination number $\gamma$ in which almost all sets of size $\gamma$ dominate. However, we want to quantify the above behavior in terms of a concentration result. We can write

$$X_{\gamma} = \sum_{S \in \binom{V}{\gamma}} I_S,$$
where $I_S$ is the indicator variable of the event that $S$ is a dominating set, and thus

$$\text{Var}(X_\gamma) = \sum_{S \in \binom{V}{\gamma}} \mathbb{E}(I_S)(1 - \mathbb{E}(I_S)) + \sum_{(S,T): S \neq T} (\mathbb{E}(I_SI_T) - \mathbb{E}(I_S)\mathbb{E}(I_T))$$

$$= \binom{n}{\gamma}(1 - \epsilon^\gamma)^{n-\gamma} + \binom{n}{\gamma}(n-k)\left(\frac{\gamma}{\gamma-k}\right)\mathbb{E}(I_{S'}I_{T'})$$

$$- \binom{n}{\gamma}^2(1 - \epsilon^\gamma)^{2n-2\gamma},$$

where the sets denoted by $S'$ and $T'$ are any two such that $|S' \cap T'| = k$. Now,

$$\mathbb{E}(I_{S'}I_{T'}) = \mathbb{P}(S', T' \text{ both dominate})$$

$$\leq \mathbb{P}(S' \text{ dominates } (S' \cup T')^C; T' \text{ dominates } (S' \cup T')^C)$$

$$= (1 - 2\epsilon^\gamma + \epsilon^{2\gamma-k})^{n-2\gamma+k}, \quad (5)$$

where $A^C$ denotes the complement of $A$, since each of the $n - 2\gamma + k$ vertices in $(S' \cup T')^C$ must have a neighbour in $S'$ and in $T'$. For each such vertex, this equals one minus the probability that it has no neighbour in $S'$ or in $T'$. Inequality (5) shows that the variance is bounded by

$$\text{Var}(X_\gamma) \leq \binom{n}{\gamma}(1 - \epsilon^\gamma)^{n-\gamma} - \binom{n}{\gamma}^2(1 - \epsilon^\gamma)^{2n-2\gamma}$$

$$+ \binom{n}{\gamma}(n-k)\left(\frac{\gamma}{\gamma-k}\right)\left(1 - 2\epsilon^\gamma + \epsilon^{2\gamma-k}\right)^{n-2\gamma+k}. \quad (6)$$

The goal is to show that the $k = 0$ term is asymptotically larger than the other ones in (6). Towards this end, observe that the difference between the $k = 0$ term and $\binom{n}{\gamma}^2(1 - \epsilon^\gamma)^{2n-2\gamma}$ can be bounded from above as follows:

$$\binom{n}{\gamma}\left(\frac{n-\gamma}{\gamma}\right)(1 - \epsilon^\gamma)^{2(n-2\gamma)} - \binom{n}{\gamma}^2(1 - \epsilon^\gamma)^{2n-2\gamma}$$

$$= \binom{n}{\gamma}\left(1 - \epsilon^\gamma\right)^{2n-2\gamma}\left\{\left(\frac{n-\gamma}{\gamma}\right)(1 - \epsilon^\gamma)^{-2\gamma} - 1\right\}$$

$$\leq \binom{n}{\gamma}^2(1 - \epsilon^\gamma)^{2n-2\gamma}(1 - \epsilon^\gamma)^{-2\gamma} - 1$$

$$\leq (\mathbb{E}(X_\gamma))^2 \cdot 4\gamma \epsilon^3. \quad (7)$$
Next we bound the intricate summand \( \binom{\gamma}{k} \binom{n-\gamma}{\gamma-k} (1 - 2\epsilon^\gamma + \epsilon^{2\gamma-k})^{n-2\gamma+k} \) as follows:

\[
f(k) := \binom{\gamma}{k} \binom{n-\gamma}{\gamma-k} (1 - 2\epsilon^\gamma + \epsilon^{2\gamma-k})^{n-2\gamma+k}
\]
\[
\leq 2 \binom{\gamma}{k} \frac{n^{\gamma-k}}{(\gamma-k)!} (1 - 2\epsilon^\gamma + \epsilon^{2\gamma-k})^n
\]
\[
\leq 2 \frac{n^{\gamma-k}}{(\gamma-k)!} \exp\{n(\epsilon^{2\gamma-k} - 2\epsilon^\gamma)\} := g(k). \tag{8}
\]

The reasoning behind the first inequality in (8) is due to the following: First we have \( \binom{n-\gamma}{\gamma-k} \leq \binom{n}{\gamma-k} \leq n^{\gamma-k}/(\gamma-k)! \), and, secondly, we used the fact that \((1 - 2\epsilon^\gamma + \epsilon^{2\gamma-k})^{n-2\gamma+k} \leq 2(1 - 2\epsilon^\gamma + \epsilon^{2\gamma-k})^n\), which is valid for large \( n \) and fixed \( \gamma \). The ratio of consecutive terms in \( g \) is given by

\[
h(k) := \frac{g(k+1)}{g(k)} = \frac{(\gamma-k)^2}{n(k+1)} \exp\{n(1 - \epsilon)\epsilon^{2\gamma-k-1}\}.
\]

Note that \( h(k) \geq 1 \) if and only if

\[
\exp\{n(1 - \epsilon)\epsilon^{2\gamma-k-1}\} \geq \frac{n(k+1)}{(\gamma-k)^2}
\]

if and only if

\[
n\epsilon^{2\gamma-k-1} \geq \log n + A \tag{9}
\]

for some constant \( A = A(\gamma) \). Furthermore, since \( n\epsilon^{2\gamma-k-1} \leq 1 \), (9) never holds. Thus \( g \) is decreasing. We conclude from (6), (7), and (8) that for some \( B = B(\gamma) > 0 \),

\[
\Var(X_\gamma) \leq \mathbb{E}(X_\gamma) + (\mathbb{E}(X_\gamma))^2 \left( -\frac{\gamma^2}{n} + 3\gamma\epsilon^\gamma \right) + \gamma \binom{n}{\gamma} g(1)
\]
\[
\leq \mathbb{E}(X_\gamma) + (\mathbb{E}(X_\gamma))^2 \left( -\frac{\gamma^2}{n} + 3\gamma\epsilon^\gamma \right) + Bn^{2\gamma-1}. \tag{10}
\]

since the term \( n(\epsilon^{2\gamma-1} - 2\epsilon^\gamma) \) of \( g(1) \) tends to zero.

Consequently, by Chebychev’s inequality [1],

\[
\Pr(|X_\gamma - \mathbb{E}(X_\gamma)| \geq \log n \cdot n^{\gamma-1/2}) \leq \frac{\Var(X_\gamma)}{\log^2(n)n^{2\gamma-1}}. \tag{11}
\]
The third term of the variance in (10) clearly causes no problems in (11). Since $\mathbb{E}(X_\gamma) \sim \binom{n}{\gamma}$, the first term of (10) gives a $o(1)$ contribution to (11) as well. Finally, the quantity $\left(-\frac{2^2}{n} + 3\gamma e^\gamma\right)$ is of order $O(1/n)$ and thus the probability in (11) is bounded above by $O(1/\log^2(n))$. We conclude that

$$\mathbb{P}(X_\gamma \leq \mathbb{E}(X_\gamma) - \log n \cdot n^{\gamma - \frac{1}{2}}) \to 0 \quad (n \to \infty).$$

Thus in this random graph, with high probability, we have

$$X_\gamma \geq \mathbb{E}(X_\gamma) - \log n \cdot n^{\gamma - \frac{1}{2}}$$

$$\geq \binom{n}{\gamma} \left(1 - e^\gamma\right)^n - \log n \cdot n^{\gamma - \frac{1}{2}}$$

$$\geq \binom{n}{\gamma} - \binom{n}{\gamma} ne^\gamma - \log n \cdot n^{\gamma - \frac{1}{2}}$$

$$= \binom{n}{\gamma} - O(n^{\gamma - \frac{1}{2}}).$$

This completes the proof. \qed

Remark 3.2. As pointed out by one of the referees, we are really studying the number of minimum dominating sets of a random graph $G(n, p)$, adjusting the edge probability to be close to one but not too close to one. In the literature, elementary as well as sophisticated methods have been used to study the sharp two-point concentration of the domination number; see, e.g. Wieland and Godbole [6] where the second moment method was used, or the improvements of Glebov et al. [4] to a larger range of edge probabilities using the more sophisticated Talagrand inequality [1]. In a similar fashion, it is conceivable that one can study the domination number of dense random graphs, as we do here, getting either a one or two point concentration for $\gamma$.

4 Open Question

We believe that the bounds in our results can be tightened, particularly by increasing the lower bound on number of non-dominating sets. This intuition arises from the fact that the lower bound is derived by considering the number of 0s in a single row of $D_G$. It is likely, however, that there are many more non-overlapping $\gamma$-sets of 0s among the $n - 1$ other rows. Further investigations along these lines would be quite revealing.
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