ON UNITARY REPRESENTATIONS OF ALGEBRAIC GROUPS
OVER LOCAL FIELDS

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Abstract. Let $G$ be an algebraic group over a local field $k$ of characteristic zero. We show that the locally compact group $G(k)$ consisting of the $k$-rational points of $G$ is of type I. Moreover, we complete Lipsman’s characterization of the groups $G$ for which every irreducible unitary representation of $G(k)$ is a CCR representation and show at the same time that such groups belong to the family of trace class groups, recently studied by Deitmar and van Dijk.

1. Introduction

Given a second countable locally compact group $G$, a problem of major interest is to determine the unitary dual space $\hat{G}$ of $G$, that is, the set of equivalence classes of irreducible unitary representations of $G$. The space $\hat{G}$ carries a natural Borel structure, called the Mackey Borel structure, which is defined as follows (see [Mac57, §6] or [Dix77, §18.6]). For every $n \in \{1, 2, \ldots, \infty\}$, let $\text{Irr}_n(G)$ denote the space of all irreducible unitary representations of $G$ in a fixed Hilbert space $H_n$ of dimension $n$. The set $\text{Irr}_n(G)$ is equipped with the weakest Borel structure for which the functions $\pi \mapsto \langle \pi(g)\xi, \eta \rangle$ are measurable for all $\xi, \eta \in H_n$; the disjoint union $\text{Irr}(G) = \bigcup_n \text{Irr}_n(G)$ is endowed with the sum Borel structure of the Borel structures of the $\text{Irr}_n(G)$’s and $\hat{G}$ with the quotient structure for the canonical surjective map $\text{Irr}(G) \to \hat{G}$.

A classification of $\hat{G}$ is only possible if $\hat{G}$ is countably separated, that is, if there exists a sequence of Borel subsets of $\hat{G}$ which separates the points of $\hat{G}$. By a result of Glimm (Gli61), a group $G$ as above has this last property if and only if $G$ is of type I in the sense of the following definition.

Recall that a von Neumann algebra is a selfadjoint subalgebra of $L(H)$ which is closed for the weak operator topology of $L(H)$, where $H$ is a Hilbert space. A von Neumann algebra $M$ is a factor if the center of $M$ consists of the scalar operators. A unitary representation $\pi$ of $G$ in a Hilbert space $H$ is a factor representation if the von Neumann subalgebra $W^*_\pi$ of $L(H)$ generated by $\pi(G)$ in $L(H)$ is a factor.

Definition 1. A locally compact group $G$ is of type I if, for every factor representation $\pi$ of $G$, the factor $W^*_\pi$ is of type I, that is, $W^*_\pi$ is isomorphic to the
von Neumann algebra $\mathcal{L}(\mathcal{K})$ for some Hilbert space $\mathcal{K}$; equivalently, if $\pi$ is equivalent to a multiple $n\sigma$ of an irreducible unitary representation $\sigma$ of $G$ for some $n \in \{1, 2, \ldots, \infty\}$.

The main result of this note is to establish the type I property for algebraic groups. Let $K$ be an algebraically closed field. A linear algebraic group over $K$ is a Zariski closed subgroup of $GL_n(K)$ for some $n \geq 1$. If $k$ is a subfield of $K$ and if $G$ is defined over $k$, the group of $k$-rational points of $G$ is the subgroup $G(k) = G \cap GL_n(k)$. By a local field, we mean a non discrete locally compact field.

**Theorem 2.** Let $k$ be a local field of characteristic 0 and let $G$ be a linear algebraic group defined over $k$. The locally compact group $G(k)$ is of type I.

Some comments on Theorem 2 are in order.

**Remark 3.**

(i) The case where $G$ is reductive is due to Harish-Chandra ([HC53]) for $k = \mathbb{R}$ and to Bernstein ([Ber74]) for non archimedean $k$.

(ii) In the case $k = \mathbb{R}$, Theorem 2 is a result due to Dixmier ([Dix57]).

(iii) Duflo ([Duf82]) describes a general procedure to determine the unitary dual for a group $G(k)$ as in Theorem 2. As is known to experts, one could derive the fact that $G(k)$ is type I from Duflo’s results, provided Bernstein’s result is extended to certain finite central extensions of reductive groups. This note offers a direct approach, independent of [Duf82]. Indeed, our proof of Theorem 2 closely follows the main strategy in the case $k = \mathbb{R}$ from [Dix57] (see also [Puk99]). In particular, we obtain a new proof of the known fact ([Dix59], [Kir62], [Moo65]) that $G(k)$ is of type I when $G$ is unipotent. As a necessary preliminary step, we establish the extension mentioned above of Bernstein’s result to certain finite central extensions of reductive groups (see Subsection 2.6).

(iv) Assume that $k$ is a local field of positive characteristic. Then $G(k)$ is still of type I when $G$ is reductive ([Ber74]). To extend the result to an arbitrary $G$, the major difficulty to overcome is to show that $G(k)$ is of type I when $G$ is unipotent, a fact which – to our knowledge – is unknown.

Let $G$ be a locally compact group. Recall that, if $\pi$ is a unitary representation of $G$ on a Hilbert space $\mathcal{H}$, then $\pi$ extends to a representation by bounded operators on $\mathcal{H}$ of the convolution algebra $C_c(G)$ of continuous functions with compact support on $G$, defined by

$$\pi(f) = \int_G f(x)\pi(x)dx \quad \text{for all } f \in C_c(G),$$

where $dx$ denotes a Haar measure on $G$. The representation $\pi$ is said to be a CCR representation if $\pi(f)$ is a compact operator for every $f \in C_c(G)$. The group $G$ is a **CCR group** if every $\pi \in \hat{G}$ is a CCR representation. It follows from [Dix77, Theorem 9.1] that every CCR group is type I.

Let $C_c^\infty(G)$ be the space of test functions on $G$ as defined in [Bru61]. Following [DvD16], we say that the representation $\pi$ of $G$ is of trace class if $\pi(f)$ is a trace class operator for every $f \in C_c^\infty(G)$. The group $G$ is called a **trace class group**, if every irreducible unitary representation of $G$ is trace class. Since $C_c^\infty(G)$ is dense in $L^1(G, dx)$, it is clear that a trace class group is CCR, and hence of type I.
Let $G$ and $k$ be as in Theorem 2. It is known that $G(k)$ is a CCR group, if $G$ is reductive (see [HC53 and Ber74]) or if $G$ is unipotent (see [Dix59, Kir62, Moo65]). This result has been strengthened in [DvD16] by showing that $G(k)$ is a trace class group in these cases.

Let $U$ be the unipotent radical of $G$ and $L$ a Levi subgroup defined over $k$; so, we have a semi-direct product decomposition $G = LU$. Let

$$M = \{g \in L \mid gu = ug \text{ for all } u \in U\}.$$ 

Then $M$ is an algebraic normal subgroup of $L$ defined over $k$. Lipsman ([Lip75]) showed that, if $G(k)$ is CCR, then $L(k)/M(k)$ is compact. Our second result concerns the converse statement (see also the comment immediately after Theorem 3.1 in [Lip75]).

**Theorem 4.** Let $G$ and $k$ be as in Theorem 2. Let $L$ and $M$ be as above. The following properties are equivalent:

(i) $G(k)$ is a CCR group;

(ii) $L(k)/M(k)$ is compact;

(iii) $G(k)$ is trace class.

Theorem 4 generalizes several results from [vD19] and provides a positive solution to a much stronger version of Conjecture 14.1 stated there.

### 2. Some preliminary results

#### 2.1. Projective representations

Let $G$ be a second countable locally compact group. We will need the notion of a projective representation of $G$. For all what follows, we refer to [Mac58].

Let $H$ be a Hilbert space. Recall that a map $\pi : G \to U(H)$ from $G$ to the unitary group of $H$ is a projective representation of $G$ if the following holds:

(i) $\pi(e) = I$,

(ii) for all $x, y \in G$, there exists $\omega(x, y) \in T$ such that

$$\pi(xy) = \omega(x, y)\pi(x)\pi(y),$$

(iii) the map $g \mapsto \langle \pi(y)\xi, \eta \rangle$ is Borel for all $\xi, \eta \in H$.

The map $\omega : G \times G \to T$ has the following properties:

(iv) $\omega(x, e) = \omega(e, x)$ for all $x \in G$,

(v) $\omega(xy, z)\omega(x, y) = \omega(x, yz)\omega(y, z)$ for all $x, y, z \in G$.

The set $Z^2(G, T)$ of all Borel maps $\omega : G \times G \to T$ with properties (iv) and (v) is an abelian group for the pointwise product.

For a given $\omega \in Z^2(G, T)$, a map $\tau : G \to U(H)$ with properties (i), (ii) and (iii) as above is called an $\omega$-representation of $G$.

For $i = 1, 2$, let $\omega_i \in Z^2(G, T)$ and $\pi_i$ an $\omega_i$-representation of $G$ on a Hilbert space $H_i$. Then $\pi_1 \otimes \pi_2$ is an $\omega_1\omega_2$-representation of $G$ on the Hilbert space $H_1 \otimes H_2$. In particular, if $\omega_2 = \omega_1^{-1}$, then $\pi_1 \otimes \pi_2$ is an ordinary representation of $G$.

Every projective unitary representation of $G$ can be lifted to an ordinary unitary representation of a central extension of $G$. More precisely, for $\omega \in Z^2(G, T)$, let $S$ be the closure in $T$ of the subgroup generated by the image of $\omega$; define a group $G^\omega$ with underlying set $S \times G$ and multiplication $(s, x)(t, y) = (s\omega(x, y), xy)$. Equipped with a suitable topology, $G^\omega$ is a locally compact group. Let $\pi : G \to U(H)$ be an $\omega$-representation of $G$. Then $\pi^0 : G^\omega \to U(H)$, defined by $\pi^0(s, x) = s\pi(x)$ is an
ordinary representation of $G^\omega$; moreover, the map $\pi \to \pi^0$ is a bijection between the $\omega$-representations of $G$ and the representations of $G^\omega$ for which the restriction to the central subgroup $S \times \{e\}$ is a multiple of the one dimensional representation $(s,e) \mapsto s$.

A projective representation $\pi$ of $G$ on $\mathcal{H}$ is irreducible (respectively, a factor representation) if the only closed $\pi(G)$-invariant subspaces of $\mathcal{H}$ are $\{0\}$ and $\mathcal{H}$ (respectively, if the von Neumann algebra generated by $\pi(G)$ is a factor). A factor projective representation $\pi$ is of type $I$, if the factor generated by $\pi(G)$ is of type $I$.

2.2. Factor representations. Let $G$ be a second countable locally compact group. Let $N$ be a type $I$ closed normal subgroup of $G$. Then $G$ acts by conjugation on $\hat{N}$: for $\pi \in \hat{N}$, the conjugate representation $\pi^g \in \hat{N}$ is defined by $\pi^g(n) = \pi(gng^{-1})$, for $g \in G, n \in N$.

Let $\pi \in \hat{N}$. The stabilizer

$$G_\pi = \{g \in G : \pi^g \text{ is equivalent to } \pi\}$$

of $\pi$ is a closed subgroup of $G$ containing $N$. There exists a projective unitary representation $\bar{\pi}$ of $G_\pi$ on $\mathcal{H}$ which extends $\pi$. Indeed, for every $g \in G_\pi$, there exists a unitary operator $\pi(g)$ on $\mathcal{H}$ such that

$$\pi^g(n) = \pi(g)\pi(n)\pi(g)^{-1} \quad \text{for all } n \in N.$$ 

One can choose $\pi(g)$ such that $g \mapsto \pi(g)$ is an $\omega \circ (p \times p)$-representation of $G_\pi$ which extends $\pi$ for some $\omega \in Z^2(G/N, T)$, where $p : G \to G/N$ is the canonical projection (see Theorem 8.2 in [Mac58]).

The normal subgroup $N$ is said to be regularly embedded in $G$ if the quotient space $\hat{N}/G$, equipped with the quotient Borel structure inherited from $\hat{N}$, is a countably separated Borel space.

The following result is part of the so-called Mackey machine; it is a basic tool which reduces the determination of the factor (or irreducible) representations of $G$ to the determination of the factor (or irreducible) representations of subgroups of $G$. The proof is a direct consequence of Theorems 8.1, 8.4 and 9.1 in [Mac58].

**Theorem 5.** Let $N$ be a closed normal subgroup of $G$ and denote by $p : G \to G/N$ the canonical projection. Assume that $N$ is of type $I$ and is regularly embedded in $G$.

(i) Let $\pi \in \hat{N}$ and $\pi'$ a factor representation of $G_\pi$ such that the restriction of $\pi'$ to $N$ is a multiple of $\pi$. Then the induced representation $\text{Ind}^G_{G_\pi} \pi'$ is a factor representation. Moreover, $\text{Ind}^G_{G_\pi} \pi'$ is of type $I$ if and only if $\pi'$ is of type $I$.

(ii) Every factor representation of $G$ is equivalent to a representation of the form $\text{Ind}^G_{G_\pi} \pi'$ as in (i).

(iii) Let $\pi \in \hat{N}$ and assume that $G = G_\pi$. Let $\omega \in Z^2(G/N, T)$ and $\bar{\pi}$ an $\omega \circ (p \times p)$-representation of $G$ which extends $\pi$. Then every factor representation $\pi'$ of $G$ such that the restriction of $\pi'$ to $N$ is a multiple of $\pi$ is equivalent to a representation of the form $\sigma \otimes \bar{\pi}$, where $\sigma$ is a factor $\omega^{-1}$-representation of $G/N$ lifted to $G$. Moreover, $\pi'$ is of type $I$ if and only if $\sigma$ is of type $I$. 
We will need the well-known fact that being of type I is inherited from cocompact normal subgroups; for the sake of completeness, we reproduce a short proof modeled after [Puk99].

**Proposition 6.** Let $N$ be a normal subgroup of $G$. Assume that $N$ is of type I and that $G/N$ is compact. Then $G$ is of type I.

**Proof.** Since $N$ is of type I, the Borel space $\hat{N}$ is countably separated, by Glimm’s result (mentioned in the introduction). The action of $G$ on $\hat{N}$ factorizes to an action of the compact group $K := G/N$.

We claim that $\hat{N}/K$ is countably separated, that is, $N$ is regularly embedded in $G$. Indeed, by a theorem of Varadarajan (see [Zim84, 2.1.19]), there exists a compact metric space $X$ on which $K$ acts by homeomorphisms and an injective $K$-equivariant Borel map $\hat{N} \to X$. Since $K$ is compact, $X/K$ is easily seen to be countably separated and the claim follows.

Let $\pi \in \hat{N}$ and $\sigma$ a projective factor representation of $G\pi/N$. Then $\sigma$ lifts to an ordinary representation $\tilde{\sigma}$ of a central extension $\tilde{G}_{\pi}$ of $G_{\pi}/N$ by a closed subgroup of $T$. Since $G_{\pi}/N$ is compact, $\tilde{G}_{\pi}$ is compact and therefore of type I. So, $\tilde{\sigma}$ and hence $\sigma$ is of type I. Theorem 5 shows that $G$ is of type I. □

We will use the following consequence of Proposition 6 in the proof of Theorem 4.

**Corollary 7.** Let $G$ and $N$ be as in Proposition 6. For every $\rho \in \hat{G}$, there exists a representation $\pi \in \hat{N}$ such that $\rho$ is a subrepresentation of $\text{Ind}^G_N \pi$.

**Proof.** Let $\rho \in \hat{G}$. By Proposition 6, $N$ is regularly embedded. Hence, by Theorem 5, there exist $\pi \in \hat{N}$ and $\sigma \in \hat{G}_{\pi}$ such that $\sigma|_N$ is a multiple of $\pi$ and such that $\rho \simeq \text{Ind}^G_{G_{\pi}} \sigma$.

As is well-known (see for instance [BdlHV08, Proposition E.2.5]),

$$\text{Ind}^{G_{\pi}}_N (\sigma)|_N \simeq \sigma \otimes \text{Ind}^{G_{\pi}}_N 1_N,$$

that is,

$$\text{Ind}^{G_{\pi}}_N (\sigma)|_N \simeq \sigma \otimes \lambda_{G_{\pi}/N},$$

where $\lambda_{G_{\pi}/N}$ is the regular representation of $G_{\pi}/N$ lifted to $G_{\pi}$. Since $G_{\pi}/N$ is compact, $\lambda_{G_{\pi}/N}$ contains the trivial representation $1_{G_{\pi}}$. It follows that $\text{Ind}^{G_{\pi}}_N (\sigma)|_N$ contains $\sigma$. Since $\sigma|_N$ is multiple of $\pi$ and since $\sigma$ is irreducible, we see that $\sigma$ is contained in $\text{Ind}^{G_{\pi}}_N \pi$. Induction by stages shows then that $\rho \simeq \text{Ind}^{G_{\pi}}_{G_{\pi}} \sigma$ is contained in $\text{Ind}^G_{G_{\pi}} (\text{Ind}^{G_{\pi}}_N \pi) \simeq \text{Ind}^G_N \pi$. □

### 2.3. Actions of algebraic groups

Assume now that $k$ is a local field of characteristic 0 and $G$ a linear algebraic group defined over $k$. If $V$ is an algebraic variety defined over $k$, then the set $V(k)$ of $k$-rational points in $V$ has also a locally compact topology, which we call the Hausdorff topology.

The following well-known result (see [Zim84, 3.1.3]) is a crucial tool for the sequel. We indicate briefly the main steps in its proof.

**Theorem 8.** Let $G \times V \to V$ be a $k$-rational action of $G$ on an algebraic variety $V$ defined over $k$. Then every $G(k)$-orbit in $V(k)$ is open in its closure for the Hausdorff topology.
Proof. Let \( W \) be a \( G \)-orbit in \( V \). Then \( W \) is open in its closure for the Zariski topology (see [Bor91, 1.8]). This implies that \( W(\mathbf{k}) \) is open in its closure for the Hausdorff topology. By [BS64, 6.4], there are only finitely many \( G(\mathbf{k}) \)-orbits contained in \( W(\mathbf{k}) \) and the claim follows. \( \square \)

We will use the previous theorem in the case of a linear representation of \( G \). More precisely, let \( K \) be an algebraic closure of \( \mathbf{k} \) and let \( V \) be a finite dimensional vector space over \( K \), equipped with a \( \mathbf{k} \)-structure \( V_k \). Let \( \rho : G \to GL(V) \) be a \( \mathbf{k} \)-rational representation of \( G \). Consider the dual adjoint \( \rho^* \) of \( G \) on the dual vector space \( V^* = \text{Hom}(V, K) \), equipped with the \( \mathbf{k} \)-structure \( V^*_k = \text{Hom}(V_k, \mathbf{k}) \). Then \( \rho^* : G \to GL(V^*) \) is a \( \mathbf{k} \)-rational representation of \( G \). Fix a non-trivial unitary character \( \varepsilon \in \hat{\mathbf{k}} \) of the additive group of \( \mathbf{k} \). The map \( \Phi : V_k^* \to \hat{V_k} \), given by

\[
\Phi(f)(v) = \varepsilon(f(v)) \quad \text{for all } f \in V^* \text{ and } v \in V_k,
\]

is an isomorphism of topological groups (see [Wei64, Chap.II, §5]). The group \( G(\mathbf{k}) \) acts on \( V_k, V_k^* \) and \( \hat{V_k} \); the map \( \Phi \) is \( G(\mathbf{k}) \)-equivariant. Therefore, the following corollary is a direct consequence of Theorem 8.

Corollary 9. Let \( \rho : G \to GL(V) \) be a \( \mathbf{k} \)-rational representation of \( G \) as above. Every \( G(\mathbf{k}) \)-orbit in \( \hat{V_k} \) is open in its closure.

Let \( U \) be the unipotent radical of \( G \) and \( u \) its Lie algebra. Then \( U \) is an algebraic subgroup of \( G \) defined over \( \mathbf{k} \) and the exponential map \( \exp : u \to U \) is an isomorphism of \( \mathbf{k} \)-varieties; moreover, \( m \to \exp m \) is a bijection between Lie subalgebras of \( u \) and (connected) algebraic subgroups of \( U \).

The Lie algebra \( u \) has a \( \mathbf{k} \)-structure \( u_k \) which is the Lie \( \mathbf{k} \)-subalgebra of \( u \) for which \( \exp : u_k \to U(\mathbf{k}) \) is a bijection.

The action of \( G \) by conjugation on \( U \) induces an action, called the adjoint representation \( \text{Ad} : G \to GL(u) \), which is a \( \mathbf{k} \)-rational representation by automorphisms of the Lie algebra \( u \) for which \( \exp : u \to U \) is \( G \)-equivariant. The map \( m \to \exp m \) is a bijection between \( \text{Ad}(G) \)-invariant ideals of \( u \) and algebraic normal subgroups of \( G \) contained in \( U \).

Lemma 10. Let \( M \) be an abelian algebraic normal subgroup of \( G \) contained in \( U \). Then \( M(\mathbf{k}) \) is regularly embedded in \( G(\mathbf{k}) \).

Proof. By an elementary argument (see [Zim84, 2.1.12]), it suffices to show that every \( G(\mathbf{k}) \)-orbit in \( \hat{M(\mathbf{k})} \) is open in its closure. Let \( m \) be the \( \text{Ad}(G) \)-invariant ideal of \( u \) corresponding to \( M \) and set \( m_k = m \cap u_k \). Since \( M \) is abelian, \( \exp : m_k \to M(\mathbf{k}) \) is an isomorphism of topological groups. So, it suffices to show that every \( G(\mathbf{k}) \)-orbit in \( \hat{m_k} \) is open in its closure. This is indeed the case by Corollary 9. \( \square \)

2.4. Heisenberg groups and Weil representation. Let \( \mathbf{k} \) be a field of characteristic 0 and let \( n \geq 1 \) be an integer. The Heisenberg group \( H_{2n+1}(\mathbf{k}) \) is the nilpotent group with underlying set \( \mathbf{k}^{2n} \times \mathbf{k} \) and product

\[
((x, y), (x', y')) = (x + x', y + y', s + t + \frac{1}{2} \beta((x, y), (x', y'))),
\]

for \((x, y), (x', y') \in \mathbf{k}^{2n}, s, t \in \mathbf{k}, \) where \( \beta \) is the standard symplectic form on \( \mathbf{k}^{2n} \).

The group \( H_{2n+1}(\mathbf{k}) \) is the group of \( \mathbf{k} \)-rational points of a unipotent algebraic group \( H_{2n+1} \) defined over \( \mathbf{k} \). Its Lie algebra \( \mathfrak{h}_{2n+1} \) has a basis \( \{X_1, \ldots, X_n, Y_1, \ldots, Y_n, Z\} \) with non trivial commutators \([X_i, Y_i] = Z\) for all \( i = 1, \ldots, n \). The center of \( \mathfrak{h}_{2n+1} \)
is equal to the commutator subalgebra $[h_{2n+1}, h_{2n+1}]$ and is spanned by $Z$. This last property characterizes $H_{2n+1}$; for the proof, see [Dix57, Lemme 4].

**Lemma 11.** Let $U$ be a unipotent algebraic group defined over $k$ with Lie algebra $u$. Assume that the center $z$ of $u$ equals $[u,u]$ and that $\dim z = \dim([u,u]) = 1$. Then $U$ is isomorphic to $H_{2n+1}$ over $k$ for some $n \geq 1$.

The symplectic group $Sp_{2n}$, which is the isometry group of $\beta$, acts by rational automorphisms of $H_{2n+1}$, given by

$$\varphi_g((x,y),t) = (g(x,y),t) \quad \text{for all} \quad g \in Sp_{2n}, \quad ((x,y),t) \in H_{2n+1}.$$

Let $\text{Aut}_c(H_{2n+1})$ be the group of automorphisms of $H_{2n+1}$ which acts trivially on the center of $H_{2n+1}$. Observe that $\varphi_l \in \text{Aut}_c(H_{2n+1})$ for every $g \in Sp_{2n}$ and that $I_h \in \text{Aut}_c(H_{2n+1})$ for every $h \in H_{2n+1}$, where $I_h$ is the inner automorphism given by $h$. The following proposition is proved in [Fol89, 1.22] in the case $k = \mathbb{R}$; however, its proof is valid in our setting.

**Proposition 12.** Every automorphism in $\text{Aut}_c(H_{2n+1})$ can be uniquely written as the product $\varphi_l \circ I_h$ for $g \in Sp_{2n}$ and $h \in H_{2n+1}$. So, $\text{Aut}_c(H_{2n+1})$ can be identified with the semi-direct product $Sp_{2n} \ltimes H_{2n+1}$.

The following facts about the irreducible representations of $H_{2n+1}(k)$ and associated metaplectic (or oscillator) representations of $Sp_{2n}(k)$ will play a crucial role in the sequel; for more details, see [Wei64].

**Theorem 13.** Denote by $Z$ the center of $H_{2n+1}(k)$ and let $\chi \in \hat{Z}$ be a non trivial character of $Z$.

(i) **(Stone-von Neumann)** There exists, up to equivalence, a unique irreducible unitary representation $\pi_\chi$ of $H_{2n+1}(k)$ such that $\pi_\chi|_Z$ is a multiple of $\chi$.

(ii) **(Segal-Shale-Weil)** The representation $\pi_\chi$ extends to a unitary representation of $\overline{Sp_{2n}(k)} \times H_{2n+1}(k)$, where $\overline{Sp_{2n}(k)}$ is a twofold cover of $Sp_{2n}(k)$, called the metaplectic group.

2.5. **Large compact subgroups and admissible representations.** Let $G$ be a locally compact group. We introduce a few notions from [War72, 7.5].

**Definition 14.** Let $K$ be a compact subgroup of $G$.

(i) $K$ is said to be **large** in $G$ if, for every $\sigma \in \hat{K}$,

$$\sup_{\pi \in \hat{G}} m(\sigma, \pi|_K) < +\infty,$$

where $m(\sigma, \pi|_K)$ is the multiplicity of $\sigma$ in $\pi|_K$.

(ii) $K$ is said to be **uniformly large** in $G$, if there exists an integer $M$ such that, for every $\sigma \in \hat{K}$,

$$\sup_{\pi \in \hat{G}} m(\sigma, \pi|_K) \leq M \dim \sigma,$$

**Proposition 15.** Let $G$ be a second countable locally compact group.

(i) Assume that $G$ contains a large compact subgroup. Then $G$ is CCR.

(ii) Assume that $G$ contains a uniformly large compact subgroup. Then $G$ is trace class.
Proof. Item (i) is well known (see [War72, Theorem 4.5.7.1]). Assume that $G$ contains a uniformly large compact subgroup. Let $f \in C_c(G)$ and $\pi \in \hat{G}$. Then $\pi(f)$ is a Hilbert-Schmidt operator ([War72, Theorem 4.5.7.4]). It follows from [DvD16, Proposition 1.6] that $\pi(f)$ is a trace class operator if $f \in C_\infty^c(G)$.

We introduce further notions in the context of totally disconnected groups.

**Definition 16.**

(i) A representation $\pi : G \to GL(V)$ in a complex vector space $V$ is said to be **smooth** if the stabilizer in $G$ of every $v \in V$ is open.

(ii) A smooth representation or unitary representation $(\pi, V)$ of $G$ is **admissible** if the space $V^K$ of $K$-fixed vectors in $V$ is finite dimensional, for every compact open subgroup $K$ of $G$.

(iii) $G$ is said to have a **uniformly admissible** smooth dual (respectively, unitary dual) if, for every compact open subgroup $K$ of $G$, there exists a constant $N(K)$ such that $\dim V^K \leq N(K)$, for every irreducible smooth (respectively, irreducible unitary) representation $(\pi, V)$ of $G$.

Let $G$ be a totally disconnected locally compact group. For a compact open subgroup $K$ of $G$, let $\mathcal{H}(G, K)$ be the convolution algebra of continuous functions on $G$ which are bi-invariant under $K$. The algebra $\mathcal{H}(G, K)$ is a $\ast$-algebra, for the involution given by

$$f^\ast(g) = \Delta(g)f(g^{-1}) \quad \text{for all } f \in \mathcal{H}(G, K), g \in G,$$

where $\Delta$ is the modular function of $G$. Observe that

$$\mathcal{H}(G, K) = e_K \ast C_c(G) \ast e_K,$$

where $e_K = \frac{1}{\mu(K)}1_K$ and $\mu$ is a Haar measure on $G$.

Let $(\pi, V)$ be a smooth (respectively, unitary) representation of $G$ and let $K$ be a compact open subgroup of $G$. A representation (respectively, a $\ast$-representation) $\pi_K$ of $\mathcal{H}(G, K)$ is defined on $V^K$ by

$$\pi_K(f)v := \int_G f(g)\pi(g)v\,d\mu(g) \quad \text{for all } f \in \mathcal{H}(G, K), v \in V^K.$$ If $\pi$ is irreducible, then $\pi_K$ is an algebraically (respectively, topologically) irreducible representation of $\mathcal{H}(G, K)$. Moreover, every algebraically (respectively, topologically) irreducible representation of $\mathcal{H}(G, K)$ is of the form $\pi_K$ for some irreducible smooth (respectively, unitary) representation $\pi$ of $G$.

**Proposition 17.** Let $G$ be a totally disconnected locally compact group.

(i) Assume that every irreducible unitary representation of $G$ is admissible. Then $G$ is trace class.

(ii) Assume that $G$ has a uniformly admissible smooth dual. Then $G$ has a uniformly admissible unitary dual.

(iii) Assume that $G$ has a uniformly admissible unitary dual. Then every compact open subgroup of $G$ is large.

Proof. To show item (i), observe that the algebra $C_c^\infty(G)$ of test functions on $G$ mentioned in the introduction is the union $\bigcup_K \mathcal{H}(G, K)$, where $K$ runs over the compact open subgroups of $G$. Let $K$ be such a subgroup and $(\pi, V)$ an irreducible
unitary representation of $G$. Since $\pi(e_K)$ is the orthogonal projection on $V^K$ and since $V^K$ is finite dimensional, $\pi(f) = \pi(e_K)\pi(f)\pi(e_K)$ has finite rank, for every $f \in \mathcal{H}(G, K)$; see also [DvD16, Theorem 2.3].

Item (ii) is proved in [FR19, Theorem B].

To show item (iii), let $K$ be a compact open subgroup of $G$ and let $\sigma \in \hat{K}$. Let $(\pi, \mathcal{H})$ be an irreducible representation of $G$. Since $K$ is a totally disconnected compact group, there exists a normal open subgroup $L = L(\sigma)$ of $K$ such that $\sigma|_L$ is the identity representation. By assumption, there exists an integer $N(L)$ such that $\dim \mathcal{H}_L \leq N(L)$ for every $(\pi, \mathcal{H}) \in \hat{G}$. Since

$$m(\sigma, \pi|_K) \dim \sigma \leq \dim \mathcal{H}_L,$$

the claim follows. $\Box$

The following immediate corollary is a useful criterion for the trace class property in the totally disconnected case.

Corollary 18. A totally disconnected locally compact group $G$ is trace class if and only if $G$ is CCR.

Proof. As already mentioned, $G$ being trace class implies that $G$ is CCR. Conversely, assume that $G$ is CCR. Then, for every compact open subgroup $K$ and every irreducible representation $\pi$ of $G$, the projection $\pi(e_K)$ is a compact operator and has therefore finite rank. Hence, every irreducible unitary representation of $G$ is admissible and $G$ is trace class by Proposition 17(i). $\Box$

As mentioned in the introduction, it is known that reductive algebraic groups over local fields are of type I. In fact, the following stronger result holds.

Theorem 19.

(i) (*Harish-Chandra*) Let $G$ be a connected reductive Lie group with finite center. Then every maximal compact subgroup of $G$ is uniformly large in $G$.

(ii) (*Bernstein*) Let $G = G(k)$, where $G$ is a reductive linear algebraic group over a non archimedean local field $k$. Then $G$ has a uniformly admissible smooth dual and hence a uniformly admissible unitary dual.

Every group $G$ as in (i) or (ii) is trace class.

Proof. For (i), we refer to [HC53]. Item (ii) follows from [Ber74] in combination with results from [HC70]. Alternatively, Item (ii) is a direct consequence of [Ber74], of the fact that $G$ has an admissible smooth dual ([Ber92]), and of Proposition 17(ii).

The last statement follows from Propositions 15 and 17. $\Box$

2.6. Covering groups of reductive groups. Let $k$ be a local field and $G$ a reductive linear algebraic group defined over $k$. When $k$ is archimedean, every finite covering group of $G = G(k)$ is of type I (see Theorem 19(i)). We need to show that this result also holds in the non archimedean case, at least for certain finite covers of $G$.

As the following example shows, a central finite extension of a locally compact group of type I is in general not of type I.
Example 20. Let $F_p$ be the field of order $p$ for a prime $p \geq 3$. Let $V$ be the vector space over $F_p$ of sequences $(x_n)_{n \in \mathbb{N}}$ with $x_n \in F_p$ and $x_n = 0$ for almost all $n \in \mathbb{N}$, and let $\omega$ be the symplectic form on $V \oplus V$, defined by

$$\omega((x, y), (x', y')) = \sum_{n \in \mathbb{N}} (x_n y'_n - y_n x'_n) \text{ for } (x, y), (x', y') \in V \oplus V.$$ 

Let $H_\infty(F_p)$ be the discrete group with underlying set $V \oplus V \oplus F_p$ and with multiplication defined by

$$(x, y, z)(x', y', z') = (x + x', y + y', z + z' + \omega((x, y), (x', y'))).$$

for all $(x, y, z), (x', y', z') \in H_\infty(F_p)$.

The "Heisenberg group" $H_\infty(F_p)$ is a finite central cover of the abelian group $V \oplus V$. It is easily checked that $H_\infty(F_p)$ is not virtually abelian. So, $H_\infty(F_p)$ is not of type I, by Thoma’s characterization of discrete groups of type I ([Tho68]).

Let $k$ be a non archimedean local field, $G$ a reductive linear algebraic group defined over $k$, and $G = G(k)$.

Let $\tilde{G}$ be a finite central extension of $G$, that is, there exist a finite normal subgroup $F$ contained in the center of $\tilde{G}$ and a continuous surjective homomorphism $p : \tilde{G} \to G$ with ker $p = F$.

Let $P$ be a parabolic subgroup of $G$. The subgroup $\tilde{P} = p^{-1}(P)$ is called a parabolic subgroup of $\tilde{G}$. Let $N$ be the unipotent radical of $P$ and $P = MN$ a Levi decomposition. There exists a unique closed subgroup $\tilde{N}$ of $\tilde{G}$ such that $p|_N : N \to \tilde{N}$ is an isomorphism (see [Duf82] 11 Lemme). By its uniqueness property, $\tilde{N}$ is a normal subgroup of $\tilde{P}$ called the unipotent radical of $\tilde{P}$, and we have a Levi decomposition $\tilde{P} = \tilde{M}\tilde{N}$, where $\tilde{M} = p^{-1}(M)$.

We recall a few facts about the structure of $G = G(k)$ from [BT72].

Let $P_0$ be a fixed minimal parabolic subgroup of $G$ with unipotent radical $N_0$ and Levi decomposition $P_0 = M_0N_0$, where $N_0$ is the unipotent radical of $P_0$. Let $A_0$ be a maximal split torus contained in $P_0$. For any root $\alpha$ of $A_0$, let $\xi_\alpha$ be the corresponding character of $A_0$ and let $A_0^+$ be the set of all $a \in A_0$ with $|\xi_\alpha(a)| \geq 1$. Let $Z$ be a maximal split torus contained in the center of $G$. Let $P_0^-$ be the opposite parabolic subgroup to $P_0$ and $N_0^-$ the unipotent radical of $P_0^-$. There exists a compact maximal subgroup $K_0$ of $G$ such that $G = K_0 P_0$ (Iwasawa decomposition). We have a Cartan decomposition

$$G = K_0SZ\Omega K_0,$$

where $S$ is a finitely generated semigroup contained in $A_0^+$ and $\Omega$ is a finite subset of $G$; moreover, the following holds: every neighbourhood of $e$ contains a compact open subgroup $K$ with the following properties:

(i) $K \subset K_0$ and $K_0$ normalizes $K$;

(ii) We have $K = K^+K^-$, where $K^+ = K \cap P_0$ and $K^- = K \cap N_0^-$;

(iii) $a^{-1}K^+a \subset K^+$ and $aK^-a^{-1} \subset K^-$ for every $a \in S$.

For all this, see [BT72].

We assume from now on that $\tilde{G}$ satisfies the following Condition (*):

(*) For every $g, h \in \tilde{G}$, we have

$$gh = hg \iff p(g)p(h) = p(h)p(g).$$
Set
\[ \widetilde{K}_0 = p^{-1}(K_0), \quad \widetilde{S} = p^{-1}(S), \quad \widetilde{Z} = p^{-1}(Z), \quad \text{and} \quad \widetilde{\Omega} = p^{-1}(\Omega). \]

Let \( N_0 \) and \( N_0^- \) be the unique subgroups of \( \widetilde{G} \) corresponding to \( N_0 \) and \( N_0^- \).

**Proposition 21.** We have
\begin{enumerate}
\item \( \widetilde{G} = \widetilde{K}_0 \widetilde{S} \widetilde{Z} \widetilde{\Omega} \widetilde{K}_0; \)
\item \( \widetilde{Z} \) is contained in the center of \( G; \)
\item \( \widetilde{S} \) is a finitely generated commutative semigroup of \( \widetilde{G}; \)
\item every neighbourhood of \( e \) in \( \widetilde{G} \) contains a compact open subgroup \( L \) with the following properties:
  \begin{enumerate}
  \item \( L \subset \widetilde{K}_0 \) and \( \widetilde{K}_0 \) normalizes \( L; \)
  \item \( L = L^+L^- \), where \( L^+ = L \cap \widetilde{P}_0 \) and \( L^- = L \cap \widetilde{N}_0^-; \)
  \item \( \bar{a}^{-1}L^+\bar{a} \subset L^+ \) and \( \bar{a}L^-\bar{a}^{-1} \subset L^- \) for every \( \bar{a} \in \widetilde{S}. \)
  \end{enumerate}
\end{enumerate}

**Proof.** It is clear that property (1) holds. It follows from Condition (*) that \( \widetilde{Z} \)


is contained in the center of \( \widetilde{G} \) and that \( \widetilde{S} \) is commutative. Moreover, \( \widetilde{S} \) is a finitely generated commutative semigroup of \( \widetilde{G} \), since \( \widetilde{G} \) is a finite covering of \( G \).

So, properties (2) and (3) are satisfied.

Let \( \widetilde{U} \) be an open neighbourhood of \( e \) in \( \widetilde{G} \) and set \( U := p(\widetilde{U}) \). We may assume that \( \widetilde{U} \cap F = \{ e \} \), where \( F = \ker p; \) so,

\[ p|_{\widetilde{U}} : \widetilde{U} \to U \]

is a homeomorphism and we can therefore identify \( p^{-1}(U) \) with \( U \times F \) and \( \widetilde{U} \) with the open subset \( U \times \{ e \} \) of \( U \times F \).

Let \( \bar{a}_1, \ldots, \bar{a}_n \) be a generating set of \( \widetilde{S} \). Choose a neighborhood \( V \) of \( e \) contained in \( U \) such that \( V^{-1} = V, V^2 \subset U \), and

\[ \bar{a}_i(V \times \{ e \})\bar{a}_i^{-1} \subset U \times \{ e \} \]

for every \( i \in \{ 1, \ldots, n \} \).

Fix a compact open subgroup \( K_1 \) contained in \( V \). We can identify \( p^{-1}(K_1) \) with \( K_1 \times F \), as topological groups. Observe that \( K_1 \times \{ e \} \) is an open subgroup of the compact group \( \widetilde{K}_0 \) and has therefore finite index in \( \widetilde{K}_0 \). Hence, there exists a subgroup of finite index \( K_2 \) of \( K_1 \) such that \( K_2 \times \{ e \} \) is normal in \( \widetilde{K}_0 \).

Let \( K \) be a compact open subgroup of \( K_0 \) contained in \( K_2 \) and with the properties (i), (ii), and (iii) as above. Set

\[ L := K \times \{ e \} \quad \text{and} \quad L^+ := L \cap \widetilde{P}_0, \quad L^- := L \cap \widetilde{N}_0^- . \]

Since \( K \) is normal in \( K_0 \) and since \( L \) is contained in the normal subgroup \( K_2 \times \{ e \} \) of \( \widetilde{K}_0 = p^{-1}(K_0) \), it is clear that \( L \) is normal in \( \widetilde{K}_0 \).

Moreover, as \( K = K^+K^- \) for \( K^+ = K \cap P_0 \) and \( K^- = K \cap N_0^- \), we have \( L = L^+L^- \). So, properties (4.1) and (4.2) are satisfied. Property (4.3) is also satisfied. Indeed, let \( i \in \{ 1, \ldots, n \} \) and \( a_i = p(\bar{a}_i) \). On the one hand, we have

\[ p(\bar{a}_i^{-1}L^+\bar{a}_i) = a_i^{-1}K^+a_i \subset K^+ = p(L^+) \]

and

\[ p(\bar{a}_iL^-\bar{a}_i^{-1}) = a_iK^-a_i^{-1} \subset K^- = p(L^-) . \]
On the other hand, since $L$ is contained in $V \times \{e\}$, we have
\[ \widetilde{a}_i^{-1}L^+\widetilde{a}_i \subset U \times \{e\} \quad \text{and} \quad \widetilde{a}_iL^{-}\widetilde{a}_i^{-1} \subset U \times \{e\} \]
and the claim follows. \(\square\)

It is known (see [Ber92, Theorem 15]) that every irreducible smooth representation of $G$ is admissible; the following proposition extends this result to the central cover $\widetilde{G}$. The proof depends on an analysis of cuspidal representations of $\widetilde{G}$.

Let $\tilde{P}$ be a parabolic subgroup of $\tilde{G}$ with Levi decomposition $\tilde{P} = \tilde{M}\tilde{N}$, as defined above.

Let $(\sigma, W)$ be a smooth representation of $\tilde{M}$. Since $\tilde{P}/\tilde{N} = \tilde{M}$, the representation $\sigma$ extends to a unique representation of $\tilde{P}$ which is trivial on $\tilde{N}$. One defines the induced representation $\text{Ind}_{\tilde{N}}^{\tilde{P}} \sigma$ to be the right regular representation of $\tilde{G}$ on the vector space $V$ of all locally constant functions $\tilde{G} \to W$ with $f(pg) = \sigma(p)f(g)$ for all $g \in \tilde{G}$ and $p \in \tilde{P}$.

Let $(\pi, V)$ be a smooth representation of $\tilde{G}$. Let $V(\tilde{N})$ be the $\pi(\tilde{M})$-invariant subspace generated by $\{\pi(n)v - v \mid n \in \tilde{N}, v \in V\}$; so, a smooth representation $\pi_{\tilde{N}}$ of $\tilde{M}$ is defined on $V_{\tilde{N}} := V/V(\tilde{N})$.

An irreducible smooth representation $(\pi, V)$ of $\tilde{G}$ is called \textit{cuspidal} if $V_{\tilde{N}} = \{0\}$, for all unipotent radicals $\tilde{N}$ of proper parabolic subgroups of $\tilde{G}$.

**Proposition 22.** Every irreducible smooth representation of $\tilde{G}$ is admissible.

**Proof.** The proof is along the lines given for $G = G(k)$ in [Ber92] or [Ren10].

- **First step:** Let $(\pi, V)$ be a smooth cuspidal representation of $G$. Then the matrix coefficients of $\pi$ (that is, the functions $g \mapsto v^*(\pi(g)v)$, for $v \in V$ and $v^* \in V^*$) are compactly supported modulo the center of $\tilde{G}$.

  This follows by an adaptation of the proof of the corresponding result for $G$ as in [Ber92, Theorem 14] or [Ren10, §VI.2].

- **Second step:** Let $(\pi, V)$ be a smooth cuspidal representation of $\tilde{G}$. Then $\pi$ is admissible.

  This is the same proof as in the case of $G$ (see [Ber92, Corollary p.53] or [Ren10, §VI.2]).

- **Third step:** Let $(\pi, V)$ be a smooth irreducible representation of $\tilde{G}$. Then there exists a parabolic subgroup $\tilde{P} = \tilde{M}\tilde{N}$ and a cuspidal representation $\sigma$ of $\tilde{M}$ such that $\pi$ is isomorphic to a subrepresentation of $\text{Ind}_{\tilde{P}}^{\tilde{G}} \sigma$.

  The proof is identical to the proof in the case of $G$ as in [Ber92, Lemma 17] or [Ren10, Corollaire p.205].

- **Fourth step:** Let $(\pi, V)$ be a smooth irreducible representation of $\tilde{G}$. Then $\pi$ is admissible.

  By the third step, we may assume that $\pi$ is a subrepresentation of $\text{Ind}_{\tilde{P}}^{\tilde{G}} \sigma$ for some parabolic subgroup $\tilde{P} = \tilde{M}\tilde{N}$ and a cuspidal representation $\sigma$ of $\tilde{M}$. By the second step, $\sigma$ is admissible. Since $\tilde{G}/\tilde{P}$ is compact, it is easily seen that $\text{Ind}_{\tilde{P}}^{\tilde{G}} \sigma$ is admissible (see [Ren10, Lemma III.2.3]). It follows that $\pi$ is admissible. \(\square\)

**Corollary 23.** Let $G$ be a reductive linear algebraic group over a non archimedean local field $k$. Let $\tilde{G}$ be a central finite cover of $G = G(k)$ satisfying Condition (*) as above. Then $\tilde{G}$ is trace class.
Proof. Let $L$ be a compact open subgroup of $\tilde{G}$. Proposition 21 shows that Assertion A from [Ber74] is satisfied for the totally disconnected locally compact group $\tilde{G}$. Hence, there exists an integer $N(L) \geq 1$ such that $\dim V^L \leq N(L)$ for every admissible smooth representation $(\pi, V)$ of $\tilde{G}$. Since, by Proposition 22 every smooth representation of $\tilde{G}$ is admissible, it follows that $\tilde{G}$ has a uniformly admissible smooth dual. Proposition 17(ii) shows that $\tilde{G}$ is therefore trace class. □

The metaplectic group $Sp_{2n}(k)$ satisfies Condition (*)& as above (see [MVW87, Corollaire p. 38]). The following result is therefore a direct consequence of Corollary 23.

**Corollary 24.** Let $G$ be a reductive algebraic subgroup of $Sp_{2n}$ and let $G = G(k)$. Let $\tilde{G}$ be the inverse image of $G$ in $Sp_{2n}(k)$. Then $\tilde{G}$ is a trace class group.

3. Proofs

3.1. **Proof of Theorem 2.** Let $G^0$ be the connected component of $G$. Since $G^0(k)$ is a normal subgroup of finite index in $G(k)$, in view of Proposition 6 it suffices to prove the claim when $G$ is connected.

Let $U$ be the unipotent radical of $G$ and $u$ its Lie algebra. Through a series of reduction steps, we will be lead eventually to the case where $U$ is a Heisenberg group.

If $u = \{0\}$, then $G$ is reductive and the claim follows from [Dix57] and [Ber74].

- **First reduction step:** we assume from now on that $G$ is connected and that $u \neq \{0\}$.

We proceed by induction on $\dim G$. So, assume that $\dim G > 0$ and the claim is proved for every connected algebraic group defined over $k$ with dimension strictly smaller than $\dim G$.

Let $\rho$ be a factor representation of $G := G(k)$, fixed in the sequel. We have to show that $\rho$ is of type $I$.

Let $m$ be a non-zero $G$-invariant abelian ideal of $u$. (An example of such an ideal is the center of $u$.) Then $M = \exp(m)$ is a non-trivial abelian normal subgroup of $G$ contained in $U$. By Lemma 10, $M := M(k)$ is regularly embedded in $G$. Hence, by Theorem 5(ii), there exists $\chi \in \hat{M}$ and a factor representation $\pi$ of $G_\chi$ such that $\pi|_M$ is a multiple of $\chi$ and such that $\rho = \Ind_{G_\chi}^G \pi$.

Recall that $G$ acts $k$-rationally by the co-adjoint action of $G$ on $m^*$ and that $\chi$ is defined by a unique linear functional $f_m \in m^*$ (see Subsection 2.3).

Assume that either $\dim m \geq 2$ or $\dim m = 1$ and $f_m = 0$. In both cases, there exists a non-zero subspace $\ell$ of $m$ contained in $\ker(f_m)$. Let $K$ be the corresponding algebraic normal subgroup of $G$. Then $\rho$ factorizes through the group of $k$-points of the connected algebraic group $G/K$ and $\dim G/K < \dim G$. Hence, $\rho$ is of type $I$, by the induction hypothesis.

- **Second reduction step:** we assume from now on that $u$ contains no $G$-invariant abelian ideal $m$ with $\dim m \geq 2$ or with $\dim m = 1$ and such that $f_m = 0$. In particular, we have $\dim z = 1$ and $f_z \neq 0$, where $z$ is the center of $u$.

Set $Z = \exp(z)$ and set $Z = Z(k)$. Let $\chi \in \hat{Z}$ be the character corresponding to $f_z$ and let $\pi$ be a factor representation of $G_\chi$ such that $\pi|_Z$ is a multiple of $\chi$ and such that $\rho = \Ind_{G_\chi}^G \pi$. 


Assume that \( \dim \mathbf{G}_X < \dim \mathbf{G} \). Then, by induction hypothesis, the group \( \mathbf{G}_X^0(\mathbf{k}) \) of the \( \mathbf{k} \)-points of the connected component of \( \mathbf{G}_X \) is of type I. Since \( \mathbf{G}_X^0(\mathbf{k}) \) has finite index in \( \mathbf{G}_X \), it follows from Proposition \( \ref{prop:finite-index} \) that \( \mathbf{G}_X \) is of type I. Hence, \( \rho \) is of type I by Theorem \( \ref{thm:unitary-1} \)(i).

- **Third reduction step**: we assume from now on that \( \dim \mathbf{G}_X = \dim \mathbf{G} \) for \( \chi \in \hat{\mathbb{Z}} \) as above. Then \( \mathbf{G}_X \) has finite index in \( \mathbf{G} \) and hence \( \mathbf{G}_X = \mathbf{G} \), since \( \mathbf{G} \) is connected. So, \( \mathbf{G}_X = \mathbf{G} \) and in particular \( \rho = \pi \).

Observe that, since \( \chi \neq 1_{\mathbb{Z}} \) and since \( \dim \mathbb{Z} = 1 \), it follows that \( \mathbf{G} \) acts trivially on \( \mathbb{Z} \). We claim that \( \mathfrak{u} \) is not abelian. Indeed, otherwise, we would have \( \mathfrak{u} = \mathfrak{z} \) and \( \mathbf{G} \) would act trivially on \( \mathfrak{u} \); so, \( \mathbf{G} \) would be reductive, in contradiction to the first reduction step.

We claim that \( \dim [\mathfrak{u}, \mathfrak{u}] = 1 \). Indeed, assume, by contradiction, that \( \dim [\mathfrak{u}, \mathfrak{u}] \geq 2 \). Then, setting \( \mathfrak{u}' := [\mathfrak{u}, \mathfrak{u}] \), there exist ideals \( \mathfrak{u}_1, \mathfrak{u}_2 \) in \( \mathfrak{u} \) of dimension 1 and 2 such that \( \mathfrak{u}_1 \subset \mathfrak{u}_2 \subset \mathfrak{u}' \) and \( [\mathfrak{u}, \mathfrak{u}_2] \subset \mathfrak{u}_1 \). Since the center \( Z' \) of \( \mathfrak{u}' \) is of dimension \( \geq 2 \). As \( Z' \) is a characteristic ideal, \( Z' \) is \( \mathbf{G} \)-invariant and this is a contradiction to the second reduction step.

In view of Lemma \( \ref{lem:central-extension} \) we can state our last reduction step.

- **Fourth reduction step**: we assume from now on that \( \mathbf{U} \) is the Heisenberg group \( H_{2n+1} \) for some \( n \geq 1 \) and that \( \chi \neq 1_{\mathbb{Z}} \).

By Theorem \( \ref{thm:unitary-2} \)(i), there exists a unique irreducible representation \( \pi_{\chi} \) of \( H_{2n+1}(\mathbf{k}) \) such that \( \pi_{\chi}|_{\mathbb{Z}} \) is a multiple of \( \chi \).

Let \( \mathbf{L} \) be a Levi subgroup of \( \mathbf{G} \) defined over \( \mathbf{k} \). So, \( \mathbf{L} \) is reductive and \( \mathbf{G} = \mathbf{L} \rtimes \mathbf{U} \). By Proposition \( \ref{prop:central-extension} \) the action of \( \mathbf{G} \) on \( H_{2n+1}(\mathbf{k}) \) by conjugation gives rise to a continuous homomorphism \( \varphi : \mathbf{G} \to Sp_{2n}(\mathbf{k}) \rtimes H_{2n+1}(\mathbf{k}) \).

The representation \( \pi_{\chi} \) of \( H_{2n+1}(\mathbf{k}) \) extends to an \( \omega \circ (p \times p) \)-representation \( \tilde{\pi}_{\chi} \) of \( Sp_{2n}(\mathbf{k}) \rtimes H_{2n+1}(\mathbf{k}) \) for some \( \omega \in Z^2(\mathbf{Sp}_{2n}(\mathbf{k}), \mathbf{T}) \), where \( p : \mathbf{Sp}_{2n}(\mathbf{k}) \rtimes H_{2n+1}(\mathbf{k}) \to \mathbf{Sp}_{2n}(\mathbf{k}) \) is the canonical projection. By Theorem \( \ref{thm:unitary-2} \)(ii), one can choose \( \omega \) so that the image of \( \omega \) is \( \{ \pm 1 \} \) and the corresponding central extension of \( \mathbf{Sp}_{2n}(\mathbf{k}) \) is the metaplectic group \( \widetilde{Sp}_{2n}(\mathbf{k}) \). Setting \( \omega' := \omega \circ (p \times p) \circ (\varphi \times \varphi) \in Z^2(\mathbf{G}, \mathbf{T}) \) and \( \tilde{\omega}' := \tilde{\pi}_{\chi} \circ \varphi \), we have that \( \tilde{\omega}' \) is an \( \omega' \)-representation of \( \mathbf{G} \) which extends \( \pi_{\chi} \).

Let \( \sigma \) be a factor \( \omega' \)-representation of \( \mathbf{L} = \mathbf{L}(\mathbf{k}) \) lifted to \( \mathbf{G} \). (Observe that \( \omega'^{-1} = \omega' \).) Let \( \tilde{\mathbf{L}} = \{ \pm 1 \} \times \mathbf{L} \) be the central extension of \( \mathbf{L} \) defined by \( \omega' \).

We claim that \( \tilde{\mathbf{L}} \) satisfies Condition \((*)\) as before Proposition \( \ref{prop:central-extension} \). Indeed, let \( x, y \in \mathbf{L} \) be such that \( xy = yx \); then \( \varphi(x)\varphi(y) = \varphi(y)\varphi(x) \) and hence \( \omega'(x, y) = \omega(\varphi(x), \varphi(y)) = \omega(\varphi(y), \varphi(x)) = \omega'(y, x) \), since Condition \((*)\) holds for \( \mathbf{Sp}_{2n}(\mathbf{k}) \). This proves the claim.

It follows from Corollary \( \ref{cor:central-extension} \) that \( \tilde{\mathbf{L}} \) is of type I. Theorem \( \ref{thm:unitary-2} \)(iii) then implies that the factor representation \( \rho \) of \( \mathbf{G} \) is of type I.

### 3.2. Proof of Theorem \( \ref{thm:unitary} \)

As mentioned before, the fact that (i) implies (ii) was already shown by Lipsman in [Lip75] Theorem 3.1 and Lemma 4.1.

We claim that (ii) implies (i). Indeed, let \( \mathbf{M} \) be as in Theorem \( \ref{thm:unitary} \). Then \( \mathbf{M} \) is an algebraic normal subgroup of the reductive group \( \mathbf{L} \) and is therefore reductive as
well (see \cite{Bor91} Corollary in \S14.2). Since $M(k)$ commutes with all elements in $U(k)$, it follows that

$$N := M(k)U(k) \cong M(k) \times U(k)$$

is a direct product of groups. Now, $M(k)$ and $U(k)$ are CCR (even trace class) by the results mentioned in the introduction. Hence, $N$ is CCR as well (and by \cite{DyD16} Proposition 1.9) even trace class). Since $G(k)/N \cong L(k)/M(k)$ is compact, it follows from a general result (see \cite{Sch70} Proposition 4.3) that $G(k)$ is CCR.

It remains to show that (ii) implies (iii).

If $k$ is non-archimedean then $G = G(k)$ is totally disconnected and CCR and it follows from Corollary \cite{BachirEchterh} that $G$ is trace class. The proof in the archimedean case is more involved.

Assume that $k = R$ and that (ii) holds. Let

$$N := M(R)U(R) \cong M(R) \times U(R)$$

be as above. We already observed above that $N$ is trace class and that $G/N$ is compact, where $G = G(R)$. Note that $N$ is unimodular, since $M(R)$ and $U(R)$ are unimodular. Since, moreover, $G/N$ is compact, it follows from Weil’s formula

$$\int_G f(g) \, dg = \int_{G/N} \int_N f(gn) \, dn \, dg, \quad f \in C_c(G)$$

that $G$ is unimodular as well.

Since subrepresentations of trace class representations are trace class and in view of Corollary \cite{BachirEchterh} it suffices to show that, for every $\pi \in \hat{N}$, the induced representation $\rho = \text{Ind}_N^G \pi$ is a (not necessarily irreducible) trace class representation of $G$, i.e., the operator $\rho(f)$ is a trace-class operator for all $f \in C_c^\infty(G)$.

Recall that, since $G/N$ is compact and both $G$ and $N$ are unimodular, the Hilbert space $H_\rho$ for the induced representation $\rho = \text{Ind}_N^G \pi$ is a completion of the vector space

$$(1) \quad \mathcal{F}_\rho = \{ \xi \in C(G, H_\pi) : \xi(gn) = \pi(n^{-1})\xi(g) \quad \forall g \in G \forall n \in N \},$$

with inner product given by $\langle \xi, \eta \rangle = \int_{G/N} \langle \xi(g), \eta(g) \rangle \, dg$. The induced representation $\rho = \text{Ind}_N^G \pi$ acts on the dense subspace $\mathcal{F}_\rho$ by the formula $(\rho(s)\xi)(g) = \xi(s^{-1}g)$. Now let $f \in C_c^\infty(G)$. Following some ideas in the proof of \cite{Sch70} Proposition 4.2, we compute for $\xi \in \mathcal{F}_\rho$:

$$(\rho(f)\xi)(g) = \int_G f(s)\xi(s^{-1}g) \, ds \overset{s \mapsto gs^{-1}}{=} \int_G f(gs^{-1})\xi(s) \, ds$$

$$= \int_{G/N} \int_N f(gn^{-1}s^{-1})\xi(sn) \, dn \, ds$$

$$\overset{n \mapsto n^{-1}}{=} \int_{G/N} \int_N f(gns^{-1})\xi(sn^{-1}) \, dn \, ds$$

$$= \int_{G/N} \int_N f(gns^{-1})\pi(n)\xi(s) \, dn \, ds$$

$$= \int_{G/N} k_f(g, s)\xi(s) \, ds,$$

where $k_f : G \times G \to B(H_\pi)$ is given by $k_f(g, s) = \pi(\varphi_f(g, s))$ and $\varphi_f : G \times G \to C_c^\infty(N)$ is given by $\varphi_f(g, s)(n) := f(gns^{-1})$. 


Since $\pi$ is trace class, the map $k_f = \pi \circ \varphi_f$ takes its values in the space of trace-class operators. As trace class operators are Hilbert-Schmidt operators, we may regard $k_f$ as a map into the set $\mathcal{HS}(\mathcal{H}_\pi)$ of Hilbert-Schmidt operators on $\mathcal{H}_\pi$.

We claim that the map $G \times G \to \|k_f(g, s)\|_{\text{HS}}$ is continuous, where $\|\cdot\|_{\text{HS}}$ denotes the Hilbert-Schmidt norm. This will follow from [DvD16, Proposition 1.4] as soon as we have shown that the map

$$
\varphi_f : G \times G \to C_c^\infty(N)
$$

is continuous, where $C_c^\infty(N)$ is equipped with the Fréchet space topology as introduced in [DvD16, Definition 1.1]. But this is a consequence of the continuity of the map $C_c^\infty(G) \to C_c^\infty(N)$, $\varphi \mapsto \varphi|_N$ and of the map $G \times G \to C_c^\infty(G)$, $(g, s) \mapsto L_s R_g f$, where $L_s$ (resp. $R_g$) denote left (resp. right) translation by $s$ (resp. $g$).

For $k, m \in N$, we have

$$
k_f(gk, sm) = \int_N f(gk n m^{-1} s^{-1}) \pi(n) \, dn \quad \overset{n \to k^{-1} n m}{=} \quad \int_N f(g s n^{-1}) \pi(k^{-1} n m) \, dn \quad = \quad \pi(k^{-1}) k_f(g, s) \pi(m).
$$

It follows that the Hilbert-Schmidt norm of $k_f(g, s)$ is constant on $N$-cosets in both variables. As $G/N$ is compact, this implies that the continuous map $(g, s) \mapsto \|k_f(g, s)\|_{\text{HS}}$ is bounded.

We want to conclude from this that $\rho(f)$ is a Hilbert-Schmidt operator. For this, let $c : G/N \to G$ be a Borel section for the quotient map $p : G \to G/N$. Let $V : \mathcal{H}_\rho \to L^2(G/N, \mathcal{H}_\pi)$ be defined on $\mathcal{F}_\rho$ by $\xi \mapsto V \xi := \xi \circ c$. Then it is straightforward to check that $V$ is a unitary operator which intertwines $\rho(f)$ with the integral operator $K_f : L^2(G/N, \mathcal{H}_\pi) \to L^2(G/N, \mathcal{H}_\pi)$ given by

$$
(K_f \xi)(\hat{g}) = \int_{G/N} \tilde{k}_f(\hat{g}, \hat{s}) \xi(\hat{s}) \, d\hat{s},
$$

where $\tilde{k}_f(\hat{g}, \hat{s}) := k_f(c(\hat{g}), c(\hat{s}))$ for $\hat{g}, \hat{s} \in G/N$.

It suffices to prove that $K_f$ is Hilbert-Schmidt. For this, let $\{e_i : i \in I\}$ and $\{v_j : j \in J\}$ be orthonormal bases of $L^2(G/N)$ and $\mathcal{H}_\pi$, respectively. Observe that $I$ and $J$ are at most countable, since $L^2(G/N)$ and $\mathcal{H}_\pi$ are separable, by the second countability of $G$.

For a pair $(j, l) \in J \times J$, denote by

$$
k_{jl}(\cdot, \cdot) := \langle \tilde{k}_f(\cdot, \cdot) v_j, v_l \rangle
$$

the $(j, l)$-th matrix coefficient of $\tilde{k}_f$ with respect to $\{v_j : j \in J\}$. Identifying elements $g \in G$ with their images in $G/N$ in the following formulas, we then get

$$
(\langle \tilde{k}_f(g, s) v_j, \tilde{k}_f(g, t) v_j \rangle) = \sum_l k_{jl}(g, s) k_{jl}(g, t)
$$

(2)
for $g, s, t \in G$. Using this and the fact that $\{\bar{e}_i : i \in I\}$ is also an orthonormal basis of $L^2(G/N)$, we compute

$$
\|K_f\|_{HS}^2 = \sum_{i,j} \langle K_f(e_i \otimes v_j), K_f(e_i \otimes v_j) \rangle = \sum_{i,j} \int_{G/N \times G/N \times G/N} \left( \tilde{k}_f(g,s) e_i(s) v_j \bar{\tilde{k}}_f(g,t) e_i(t) v_j \right) \ d(g,s,t)
$$

$$
= \sum_{i,j,l} \int_{G/N \times G/N \times G/N} e_i(s) e_i(t) \sum_l k_{jl}(g,s) \bar{k}_{jl}(g,t) \ d(g,s,t)
$$

$$
= \sum_{i,j,l} \int_{G/N} \langle k_{jl}(g, \cdot), \bar{e}_i \rangle \langle \bar{e}_i, k_{jl}(g, \cdot) \rangle \ dg
$$

$$
= \sum_{j,l} \int_{G/N} \|k_{jl}(g, \cdot)\|_2^2 \ dg
$$

$$
= \sum_{j,l} \|k_{jl}\|_2^2 = \sum_{j,l} \int_{G/N \times G/N} |k_{jl}(g,s)|^2 \ d(g,s)
$$

$$
= \int_{G/N \times G/N} \left\langle \tilde{k}_f(g,s) v_j, \bar{\tilde{k}}_f(g,s) v_j \right\rangle \ d(g,s)
$$

$$
= \int_{G/N \times G/N} \|\tilde{k}_f(g,s)\|_{HS}^2 \ d(g,s) < \infty.
$$

It follows that $\rho(f)$ is a Hilbert-Schmidt operator for all $f \in C_c^\infty(G)$, and since $C_c^\infty(G) = C_c^\infty(G) \ast C_c^\infty(G)$ by [DM78, Theorem 3.1], this implies that $\rho(f)$ is trace class for all $f \in C_c^\infty(G)$. This finishes the proof.

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