ON THE NUMBER OF PRIMES UP TO THE $n$TH RAMANUJAN PRIME

CHRISTIAN AXLER

Abstract. The $n$th Ramanujan prime is the smallest positive integer $R_n$ such that for all $x \geq R_n$ the interval $(x/2, x]$ contains at least $n$ primes. In this paper we undertake a study of the sequence $(\pi(R_n))_{n \in \mathbb{N}}$, which tells us where the $n$th Ramanujan prime appears in the sequence of all primes. In the first part we establish new explicit upper and lower bounds for the number of primes up to the $n$th Ramanujan prime, which imply an asymptotic formula for $\pi(R_n)$ conjectured by Yang and Togbé. In the second part of this paper, we use these explicit estimates to derive a result concerning an inequality involving $\pi(R_n)$ conjectured by of Sondow, Nicholson and Noe.

1. Introduction

Let $\pi(x)$ denotes the number of primes not exceeding $x$. In 1896, Hadamard [6] and de la Vallée-Poussin [15] proved, independently, the asymptotic formula $\pi(x) \sim x/\log x$ as $x \to \infty$, which is known as the Prime Number Theorem. Here, $\log x$ is the natural logarithm of $x$. In his later paper [16], where he proved the existence of a zero-free region for the Riemann zeta-function $\zeta(s)$ to the left of the line $\text{Re}(s) = 1$, de la Vallée-Poussin also estimated the error term in the Prime Number Theorem by showing

$$
\pi(x) = \frac{x}{\log x} + O\left(\frac{x}{\log^2 x}\right).
$$

The prime counting function and the asymptotic formula (1.1) play an important role in the definition of Ramanujan primes, which have their origin in Bertrand’s postulate.

Bertrand’s Postulate. For each $n \in \mathbb{N}$ there is a prime number $p$ with $n < p \leq 2n$.

In terms of the prime counting function, Bertrand’s postulate states that $\pi(2n) - \pi(n) \geq 1$ for every $n \in \mathbb{N}$. Bertrand’s postulate was first proved by Chebyshev [4] in 1850. In 1919, Ramanujan [8] proved an extension of Bertrand’s postulate by investigating inequalities of the form $\pi(x) - \pi(x/2) \geq n$ for $n \in \mathbb{N}$. In particular, he found that

$$
\pi(x) - \pi\left(\frac{x}{2}\right) \geq 1 \quad \text{for every } x \geq 2 \quad \text{(respectively } 2, 3, 4, 5, \ldots \text{).}
$$

Using the fact that $\pi(x) - \pi(x/2) \to \infty$ as $x \to \infty$, which follows from (1.1), Sondow [10] introduced the notation $R_n$ to represent the smallest positive integer for which the inequality $\pi(x) - \pi(x/2) \geq n$ holds for every $x \geq R_n$. In [11], Ramanujan calculated the numbers $R_1 = 2, R_2 = 11, R_3 = 17, R_4 = 29$, and $R_5 = 41$. All these numbers are prime, and it can easily be shown that $R_n$ is actually prime for every $n \in \mathbb{N}$. In honor of Ramanujan’s proof, Sondow [10] called the number $R_n$ the $n$th Ramanujan prime.

A legitimate question is, where the $n$th Ramanujan prime appears in the sequence of all primes. Letting $p_k$ denotes the $k$th prime number, we have $R_n = p_{\pi(R_n)}$, and it seems natural to study the sequence $(\pi(R_n))_{n \in \mathbb{N}}$. The first few values of $\pi(R_n)$ for $n = 1, 2, 3, \ldots$ are

$$
\pi(R_n) = 1, 5, 7, 10, 13, 15, 17, 19, 20, 25, 26, 28, 31, 35, 36, 39, 41, 42, 49, 50, 51, 52, 53, \ldots.
$$

For further values of $\pi(R_n)$, see [9]. Since both $R_n$ for large $n$ and $\pi(x)$ for large $x$ are hard to compute, we are interested in explicit upper and lower bounds for $\pi(R_n)$. Sondow [10] Theorem 2] found a first lower bound for $\pi(R_n)$ by showing that the inequality

$$
\pi(R_n) > 2n
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1
holds for every positive integer \( n \geq 2 \). Combined with [10] Theorem 3 and the Prime Number Theorem, we get the asymptotic relation
\[
(1.4) \quad \pi(R_n) \sim 2n \quad (n \to \infty).
\]
This, together with [13], means, roughly speaking, that the probability of a randomly chosen prime being a Ramanujan prime is slight less than 1/2. The first upper bound for \( \pi(R_n) \) is due to Sondow [10, Theorem 2]. He found that the upper bound \( \pi(R_n) < 4n \) holds for every positive integer \( n \), and conjectured [10, Conjecture 1] that the inequality \( \pi(R_n) < 3n \) holds for every positive integer \( n \). This conjecture was proved by Laishram [7, Theorem 2] in 2010. Applying Theorem 4 from the paper of Sondow, Nicholson and Noe [11], we get a refined upper bound for the number of primes less or equal to \( \pi(R_n) \), namely that the inequality \( \pi(R_n) \leq \pi(41p_{3n}/47) \) holds for every positive integer \( n \) with equality at \( n = 5 \). Srinivasan [12, Theorem 1.1] proved that for every \( \varepsilon > 0 \) there exists a positive integer \( N = N(\varepsilon) \) such that
\[
(1.5) \quad \pi(R_n) < [2n(1 + \varepsilon)]
\]
for every positive integer \( n \geq N \) and conclude [12, Corollary 2.1] that \( \pi(R_n) \leq 2.6n \) for every positive integer \( n \). The present author [1, Theorem 3.22] showed independently that for each \( \varepsilon > 0 \) there is a computable positive integer \( N = N(\varepsilon) \) so that \( \pi(R_n) \leq [2n(1 + \varepsilon)] \) for every positive integer \( n \geq N \) and conclude that
\[
(1.6) \quad \pi(R_n) \leq \lfloor tn \rfloor
\]
for every positive integer \( n \), where \( t \) is an arbitrary real number satisfying \( t > 48/19 \). The inequality (1.6) was improved by Srinivasan and Nicholson [13, Theorem 1]. They proved that
\[
\pi(R_n) \leq 2n \left(1 + \frac{3}{\log n + \log \log n - 4}\right)
\]
for every positive integer \( n \geq 242 \). Later, Srinivasan and Arés [13, Theorem 1.1] found a more precise result by showing that for every \( \varepsilon > 0 \) there exists a positive integer \( N = N(\varepsilon) \) such that
\[
(1.7) \quad \pi(R_n) < 2n \left(1 + \frac{\log 2 + \varepsilon}{\log n + j(n)}\right)
\]
for every positive integer \( n \geq N \), where \( j \) is any positive function satisfying \( j(n) \to \infty \) and \( nj'(n) \to 0 \) as \( n \to \infty \). Setting \( \varepsilon = 0.5 \) and \( j(n) = \log \log n - \log 2 - 0.5 \), they found [13, Corollary] that the inequality (1.7) holds for every positive integer \( n \geq 44 \). In 2016, Yang and Togbê [17, Theorem 1.2] established the following current best upper and lower bound for \( \pi(R_n) \) when \( n \) satisfies \( n > 10^{300} \).

**Proposition 1.1** (Yang, Togbê). Let \( n \) be a positive integer with \( n > 10^{300} \). Then
\[
\beta < \pi(R_n) < \alpha,
\]
where
\[
\alpha = 2n \left(1 + \frac{\log 2}{\log n}\right) \frac{\log 2 \log \log n - \log 2 - 2 - 0.13}{\log^2 n},
\]
\[
\beta = 2n \left(1 + \frac{\log 2}{\log n}\right) \frac{-\log 2 \log \log n - \log 2 - 2 + 0.11}{\log^2 n}.
\]

The proof of Proposition 1.1 is based on explicit estimates for the \( k \)th prime number \( p_k \) obtained by Dusart [5] Proposition 6.6 and Proposition 6.7] and on Srinivasan’s lemma [12, Lemma 2.1] concerning Ramanujan primes. Instead of using Dusart’s estimates, we use the estimates obtained in [3, Corollary 1.2 and Corollary 1.4] to get the following improved upper bound for \( \pi(R_n) \).

**Theorem 1.2.** Let \( n \) be a positive integer satisfying \( n \geq 5 \times 225 \) and let
\[
(1.8) \quad U(x) = \frac{\log 2 \log x (\log \log x)^2 - c_1 \log x \log \log x + c_2 \log x - \log 2 \log \log x + \log 2 + 0.565}{\log^4 x + \log^3 x \log x - \log^2 x \log 2 - \log x \log 2},
\]
where \( c_1 = 2 \log^2 2 + \log 2 \) and \( c_2 = \log^3 2 + 2 \log^2 2 + 0.565 \). Then
\[
\pi(R_n) < 2n \left(1 + \frac{\log 2}{\log n} - \frac{\log 2 \log \log n - \log 2 - 2}{\log^2 n} + U(n)\right).
\]

With the same method, we used for the proof of Theorem 1.2, we get the following more precised lower bound for the number of primes not exceeding the \( n \)th Ramanujan prime.
Theorem 1.3. Let $n$ be a positive integer satisfying $n \geq 1245$ and let

\begin{equation}
L(x) = \frac{\log 2 \log x (\log \log x)^2 - d_1 \log x \log \log x + d_2 \log x}{\log^3 x + \log^2 x \log \log x - \log^2 2 \log \log x + \log^3 2 + \log^2 2},
\end{equation}

where $d_1 = 2 \log^2 2 + \log 2 + 1.472$ and $d_2 = \log^3 2 + 2 \log^2 2 - 2.51$. Then

\[\pi(R_n) > 2n \left(1 + \frac{\log 2}{\log n} - \frac{\log 2 \log \log n - \log^2 2 - \log 2}{\log^2 n} + L(n)\right).\]

A direct consequence of Theorem 1.2 and Theorem 1.3 is the following result, which implies the correctness of a conjecture stated by Yang and Togbé [14, Conjecture 5.1] in 2015.

Corollary 1.4. Let $n \geq 2$ be a positive integer. Then

\[\pi(R_n) = 2n \left(1 + \frac{\log 2}{\log n} - \frac{\log 2 \log \log n - \log^2 2 - \log 2}{\log^2 n} + \frac{\log 2 (\log \log n)^2}{\log^4 n} + O\left(\frac{\log \log n}{\log^4 n}\right)\right).\]

The initial motivation for writing this paper, was the following conjecture stated by Sondow, Nicholson and Noe [11, Conjecture 1] involving $\pi(R_n)$.

Conjecture 1.5 (Sondow, Nicholson, Noe). For $m = 1, 2, 3, \ldots$, let $N(m)$ be given by the following table:

| $m$ | 1 | 2 | 3 | 4 | 5 | 6 | 7, 8, \ldots, 19 | 20, 21, \ldots |
|-----|---|---|---|---|---|---|-----------------|----------------|
| $N(m)$ | 1 | 1245 | 189 | 189 | 85 | 85 | 10 | 2 |

Then we have

\begin{equation}
\pi(R_{mn}) \leq m \pi(R_n) \quad \forall n \geq N(m).
\end{equation}

Note that the inequality (1.10) clearly holds for $m = 1$ and every positive integer $n$. In the cases $m = 2, 3, \ldots, 20$, the inequality (1.10) has been verified for every positive integer $n$ with $R_{mn} < 10^5$. For any fixed positive integer $m$, we have, by (1.10), $\pi(R_{mn}) \sim 2mn \sim m \pi(R_n)$ as $n \to \infty$. A first result in the direction of Conjecture 1.5 is due to Yang and Togbé [17, Theorem 1.3]. They used Proposition 1.1 to find the following result, which proves Conjecture 1.5 when $n$ satisfies $n > 10^{300}$.

Proposition 1.6 (Yang, Togbé). For $m = 1, 2, 3, \ldots$, and $n > 10^{300}$, we have

\[\pi(R_{mn}) \leq m \pi(R_n).\]

Using the same method, we apply Theorem 1.2 and Theorem 1.3 to get the following result.

Theorem 1.7. The Conjecture 1.5 of Sondow, Nicholson and Noe holds except for $(m, n) = (38, 9)$.

2. Preliminaries

Let $n$ be a positive integer. For the proof of Theorem 1.2 and Theorem 1.3, we need sharp estimates for the $n$th prime number. The current best upper and lower bound for the $n$th prime number were obtained in [3, Corollary 1.2 and Corollary 1.4] and are given as follows.

Lemma 2.1. For every positive integers $n \geq 46254381$, we have

\[p_n < n \left(\log n + \log \log n - 1 + \frac{\log \log n - 2}{\log n} - \frac{(\log \log n)^2 - 6 \log \log n + 10.667}{2 \log^2 n}\right).\]

Lemma 2.2. For every positive integer $n \geq 2$, we have

\[p_n > n \left(\log n + \log \log n - 1 + \frac{\log \log n - 2}{\log n} - \frac{(\log \log n)^2 - 6 \log \log n + 11.508}{2 \log^2 n}\right).\]

3. Proof of Theorem 1.2

To prove Theorem 1.2 we use the method investigated by Yang and Togbé [17] for the proof of the upper bound for $\pi(R_n)$ given in Proposition 1.1. First, we note following result, which was obtained by Srinivasan [12, Lemma 2.1]. Although it is a direct consequence of the definition of a Ramanujan prime, it plays an important role in the proof of the upper bound for $\pi(R_n)$ in Proposition 1.1.

Lemma 3.1 (Srinivasan). Let $R_n = p_s$ be the $n$th Ramanujan prime. Then we have $2p_{s-n} < p_s$ for every positive integer $n \geq 2$. 

Lemma 3.3. Let 

\[ G(x) = x \left( \log x + \log \log x - 1 + \frac{\log \log x - 2}{\log x} - \frac{(\log \log x)^2 - 6 \log \log x + 10.667}{2 \log^2 x} \right) \tag{3.1} \]

and

\[ H(x) = x \left( \log x + \log \log x - 1 + \frac{\log \log x - 2}{\log x} - \frac{(\log \log x)^2 - 6 \log \log x + 11.508}{2 \log^2 x} \right), \tag{3.2} \]

and consider the function \( F_1 : (2n, 2.6n) \to \mathbb{R} \) defined by

\[ F_1(x) = G(x) - 2H(x - n). \tag{3.3} \]

In the following proposition, we note a first property of the function \( F_1(x) \) concerning its derivative.

**Proposition 3.2.** Let \( n \) be a positive integer with \( n \geq 16 \). Then \( F_1(x) \) is a strictly decreasing function on the interval \((2n, 2.6n)\).

**Proof.** Setting

\[ q_1(x) = \frac{\log x - 2}{\log x} - \frac{(\log \log x)^2 - 4 \log \log x + 4.667}{2 \log^2 x} + \frac{(\log \log x)^2 - 7 \log \log x + 13.667}{\log^2 x} \]

and

\[ r_1(x) = -\frac{2(\log \log (x - n) - 1)}{\log (x - n)} + \frac{(\log \log (x - n))^2 - 4 \log \log (x - n) + 5.508}{2 \log^2 (x - n)} \]

\[ -\frac{2(\log \log (x - n))^2 - 14 \log \log (x - n) + 29.016}{\log^2 (x - n)}, \]

a straightforward calculation shows that the derivative of \( F_1(x) \) is given by

\[ F_1'(x) = \log x - 2 \log(x - n) + \log x - 2 \log \log(x - n) + \frac{1}{\log x} + q_1(x) + r_1(x). \]

Note that \( \log \log(x - n) \geq 1, t^2 - 4t + 4.667 > 0 \) and the fact that the functions \( t \mapsto (\log \log t)^2 - 7 \log \log t + 13.667 \) and \( t \mapsto (\log \log t)^2 - 4 \log \log t + 5.508 \) are monotonic decreasing for every \( t > 1 \), we obtain that

\[ F_1'(x) < 1.772 \log n + \log \log (2.6n) - \log \log^2 n. \]

Finally, we use the fact that \( t \log t > e^{1.772} \log (2.6t) \) for every \( t \geq 6 \) to get \( F_1(x) < 0 \) for every \( x \in (2n, 2.6n) \), which means that \( F_1(x) \) is a strictly decreasing function on the interval \((2n, 2.6n)\). \( \square \)

Next, we define the function \( \gamma : \mathbb{R}_{\geq 4} \to \mathbb{R} \) by

\[ \gamma(x) = \frac{\log 2 + 2 \log x + 0.565}{\log x + \log 2 \log x - 2} \tag{3.4} \]

A simple calculation shows that

\[ \gamma(x) = \frac{\log 2}{\log x} - \frac{2 \log 2 \log x - \log^2 2 - \log 2}{\log^2 x} + U(x), \tag{3.5} \]

where \( U(x) \) is defined as in \((1.8)\). In the following lemma, we note some useful properties of \( \gamma(x) \).

**Lemma 3.3.** Let \( \gamma(x) \) be defined as in \((3.4)\). Then the following hold:

(a) \( \gamma(x) > 0 \) for every \( x \geq 8 \),

(b) \( \gamma(x) < \log 2 / \log x \) for every \( x \geq 10734 \),

(c) \( \gamma(x) < 1/4 \) for every \( x \geq 10734 \).

**Proof.** The statement in (a) is clear. To prove (b), we first note that \( U(x) < \log 2(\log \log x)^2 / \log^3 x \) for every \( x \geq 230 \geq \exp(\exp(1 + \log 2)) \). Now we use \((3.5)\) and the fact that \( (\log \log x - \log 2 - 1) \log x \geq (\log \log x)^2 \) for every \( x \geq 10734 \), to conclude (b). Finally, (c) is a direct consequence of (b). \( \square \)
Now, we give a proof of Theorem 1.2.

Proof of Theorem 1.2. First, we consider the case where $n$ is a positive integer with $n \geq 528,491,312 \geq \exp(\exp(3))$. By (1.3) and (1.6), we have $2n < \pi(R_n) < 2.6n$. Hence $\pi(R_n) \geq 2n \geq 1056,982,624$ and $\pi(R_n) - n \geq 528,491,312$. Now we apply Lemma 2.1 and Lemma 2.2 to get that $F_1(\pi(R_n)) > p_{\pi(R_n)} - 2p_{\pi(R_n)-n}$, where $F_1$ is defined as in (3.3). Since $R_n = p_{\pi(R_n)}$, Srinivasan’s Lemma 5.1 yields

\begin{equation}
F_1(\pi(R_n)) > 0.
\end{equation}

For convenience, we write in the following and

\begin{equation}
\pi(3.7)
\end{equation}

$p\gamma(3.8)$

For convenience, we write in the following and

\begin{equation}
\pi(3.9)
\end{equation}

$A$ and

\begin{equation}
\pi(3.10)
\end{equation}

$B$

Next, we estimate $\xi$. Since

\begin{equation}
\pi(3.11)
\end{equation}

$D$

Using the right-hand side inequality of (3.8), we easily get

\begin{equation}
\pi(3.12)
\end{equation}

$E$

In the following, we give upper bounds for the quantities $A_1$, $B_1$, $C_1$ and $D_1$. We start with $A_1$. We use the inequalities

\begin{equation}
\pi(3.13)
\end{equation}

$F$

which hold for every real $t > 0$, and Lemma 3.3(c) to get

\begin{equation}
\pi(3.14)
\end{equation}

$G$

Next, we estimate $B_1$. Using the right-hand side inequality of (3.8), we easily get

\begin{equation}
\pi(3.15)
\end{equation}

$H$

To find an upper bound for $C_1$, we note that $t = (\log \log t - 2)/\log t$ is a decreasing function on the interval $(\exp(\exp(3)), \infty)$. Together with Lemma 3.3(a), we obtain that the inequality

\begin{equation}
\pi(3.16)
\end{equation}

$I$

holds. Finally, we estimate $D_1$. For this purpose, we consider the function $f : (1, \infty) \to \mathbb{R}$ defined by

\begin{equation}
\pi(3.17)
\end{equation}

$J$

By the mean value theorem, there exists a real number $\xi \in (n + 2n\gamma, 2n + 2n\gamma)$ such that $f(2n + 2n\gamma) - f(n + 2n\gamma) = nf'(\xi)$). Since $f''(x) > 0$ for every $x > 1$, we get $f'(\xi) = f'(n + 2n\gamma) \geq f'(n)$. Hence we get

\begin{equation}
\pi(3.18)
\end{equation}

$K$

Therefore

\begin{equation}
\pi(3.19)
\end{equation}

$L$

Since $\sigma(x)$ is a strictly decreasing function on the interval $(1, \infty)$, it follows that the inequality

\begin{equation}
\pi(3.20)
\end{equation}

$M$
Corollary 3.5. For every positive integer \( n \in (2^n, 2^{n+1}) \), we have
\[
\frac{F_1(n)}{2n} < (1 + \gamma) \log 2 - \gamma \log n + \frac{(1 + \gamma) \log 2}{\log n} - \gamma \log \log n + \frac{(1 + \gamma) r_1(\log \log n)}{\log^2 n} + \gamma \frac{r_2(\log \log n)}{2 \log^2 n} + (1 + 2\gamma) \frac{0.841}{2 \log^2 n},
\]
where \( r_1(t) = t^2 - 7t + 13.667 \) and \( r_2(t) = t^2 - 6t + 10.667 \). The functions \( t \mapsto r_1(\log \log t)/\log t \), \( t \mapsto r_1(\log \log t)/\log^2 t \), and \( t \mapsto r_2(\log \log t)/\log t \) are decreasing on the interval \((1, \infty)\). Hence \( r_1(\log \log n) \leq r_1(3) \) and \( r_2(\log \log n) \leq r_2(3) \). Together with Lemma 3.3(a), Lemma 3.3(b) and Proposition 3.2, we get \( \pi(R_n) < \alpha \). We conclude by direct computation.

We get the following weaker but more compact upper bounds for the parameter \( s \).

**Corollary 3.4.** For every positive integer \( n \geq 2 \), we have
\[
\pi(R_n) < 2n \left( 1 + \frac{\log 2}{\log n} - \frac{\log 2 \log \log n - \log^2 2 - \log 2}{\log^2 n} + \frac{\log 2(\log \log n)^2}{\log^3 n} \right).
\]

**Proof.** If \( n \geq 5 \, 225 \), the corollary follows directly from Theorem 1.2, since \( U(x) \leq \log 2(\log \log x)^2/\log^3 x \) for every \( x \geq 230 \). For the remaining cases of \( n \), we use a computer.

In the next corollary, we reduce the number \( 10^{300} \) in Proposition 1.1 as follows.

**Corollary 3.5.** For every positive integer \( n \) satisfying \( n \geq 4 \, 842 \, 763 \, 560 \, 306 \), we have
\[
\pi(R_n) < 2n \left( 1 + \frac{\log 2}{\log n} - \frac{\log 2 \log \log n - \log^2 2 - \log 2 - 0.13}{\log^2 n} \right),
\]

**Proof.** Note that \( U(x) \leq 0.13/\log^2 x \) for every \( x \geq 4 \, 842 \, 763 \, 560 \, 306 \). Now we can use Theorem 1.2.

**Corollary 3.6.** Let \( n \) be a positive integer satisfying \( n \geq 640 \). Then
\[
\pi(R_n) < 2n \left( 1 + \frac{\log 2}{\log n} \right).
\]

**Proof.** For every positive integer \( n \geq 10 \, 734 \), we have
\[
\frac{\log 2 \log \log n - \log^2 2 - \log 2}{\log^2 n} + \frac{\log 2(\log \log n)^2}{\log^3 n} > 0,
\]

an it suffices to apply Corollary 3.4. We conclude by direct computation.

**4. PROOF OF THEOREM 1.3**

Using a similar argument as in the proof of Lemma 3.1, Yang and Togbé [17, p. 248] derived the following result.

**Lemma 4.1** (Yang, Togbé). Let \( R_n = p_s \) be the \( s \)th Ramanujan prime. Then we have \( p_s < 2p_{s-n+1} \) for every positive integer \( n \).

Next, we define for each positive integer \( n \) the function \( F_2 : (2n, 2.6n) \to \mathbb{R} \) by
\[
F_2(x) = H(x) - 2G(x - n + 1),
\]
where the functions \( G(x) \) and \( H(x) \) are given by 3.1 and 3.2, respectively. In Proposition 3.2 we showed that for every positive integer \( n \geq 16 \), the function \( F_1(x) \) is decreasing on the interval \((2n, 2.6n)\). In the following proposition, we get a similar result for the function \( F_2(x) \).

**Proposition 4.2.** Let \( n \) be a positive integer with \( n \geq 15 \). Then \( F_2(x) \) is a strictly decreasing function on the interval \((2n, 2.6n)\).
Proof. A straightforward calculation shows that the derivative of $F_2(x)$ is given by

$$F'_2(x) = \log x - 2 + \frac{2 \log \log x - 2 \log \log (x-n+1) + \frac{1}{\log x} + \frac{\log \log x - 2}{\log x}}{2 \log^2 x}$$

$$- \frac{(\log \log x)^2 - 4 \log \log x + 5.508 + (\log \log x)^2 - 7 \log \log x + 14.508}{2 \log^2 x}$$

$$= \frac{2(\log \log (x-n+1) - 1)}{\log (x-n+1)} + \frac{(\log \log (x-n+1))^2 - 4 \log \log (x-n+1) + 4.667}{2 \log^2 (x-n+1)}$$

$$- \frac{2(\log \log (x-n+1))^2 - 14 \log \log (x-n+1) + 27.334}{\log^2 (x-n+1)}.$$

Now we argue as in the proof of Proposition 4.2 to obtain that the inequality $F_2(x) < 1.717 - \log n + \log \log (2.6n) - \log (\log^2 n)$ holds for every real $x$ such that $2n < x < 2.6n$. Since $t \log^2 t > e^{1.717} \log (2.6t)$ for every $t \geq 6$, we get that $F_2(x)$ is a strictly decreasing function on the interval $(2n, 2.6n)$. □

Now, we define the function $\delta : R_{\geq 4} \to R$ by

$$\delta(x) = \frac{\log 2 + \log 2/\log x - (1.472 \log \log x + 2.51)/\log^2 x}{\log x + \log \log x - \log 2 - \log 2/\log x}.$$

A simple calculation shows that

$$\delta(x) = \frac{\log 2 - \log 2/\log x - \log 2 \log x - \log 2 + \log x}{\log^2 x} + L(x),$$

where $L(x)$ is given by (4.2). In the following lemma, we note two properties of the function $\delta(x)$, which will be useful in the proof of Theorem 1.3.

Lemma 4.3. Let $\delta(x)$ be defined as in (4.2). Then the following two inequalities hold:

(a) $\delta(x) > 0.638/\log x$ for every $x \geq \exp(\exp(3))$,

(b) $\delta(x) < \log 2/\log x$ for every $x \geq 230$.

Proof. Since $0.055 \log x + 0.812 > 0.638 \log \log x$ for every $x \geq 4.71 \cdot 10^8$, it follows that the inequality

$$(\log 2 - 0.638) \log x + (1 + 0.638) \log 2 - \frac{1.472 \cdot 3 + 2.51 - 0.638 \log 2}{e^\delta} > 0.638 \log \log x$$

holds for every $x \geq 4.71 \cdot 10^8$. The function $t \mapsto \log \log t/\log t$ is decreasing for $x \geq e^\delta$. Hence

$$(\log 2 - 0.638) \log x + (1 + 0.638) \log 2 - \frac{1.472 \log \log x + 2.51 - 0.638 \log 2}{\log x} > 0.638 \log \log x$$

for every $x \geq \exp(\exp(3))$. Now it suffices to note that the last inequality is equivalent to $\delta(x) > 0.638/\log x$. This proves (a). Next, we prove (b). Since $\log 2 \log \log x > \log 2 + \log 2$ for every $x \geq 230 \geq \exp(\exp(1 + 2 \log 2))$, we obtain that the inequality

$$\log 2 + \log^2 2 < \log \log x + \frac{1.472 + 2.51 - \log^2 2}{\log x}$$

holds for $x \geq 230$. Again, it suffices to note that the last inequality is equivalent to $\delta(x) < \log 2/\log x$. □

Finally, we give the proof of Theorem 1.3. First, we consider the case where $n$ is a positive integer with $n \geq 528491312 \geq \exp(\exp(3))$. By (1) and (1.9), we have $2n < \pi(R_n) < 2.6n$. Further, $\pi(R_n) > 2n \geq 1056982624$ and $\pi(R_n) - n > 528491312$. Applying Lemma 2.1 and Lemma 2.2, we get $F_2(\pi(R_n)) < p_\pi(R_n) - 2p_{\pi(R_n)-n+1}$, where $F_2$ is defined as in (4.1). Note that $R_n = p_{\pi(R_n)}$. Hence, by Lemma 4.3, we get

$$F_2(\pi(R_n)) < p_{\pi(R_n)} - 2p_{\pi(R_n)-n+1} < 0.$$

In the following, we use, for convenience, the notation $\delta = \delta(n)$ and write $\beta = 2n(1 + \delta)$. So, by (4.3), we need to prove that $\beta < \pi(R_n)$. For this purpose, we first show that $F_2(\beta) > 0$. From Lemma 4.3 it follows that $2n < \beta < 2.6n$. Furthermore, we have

$$\frac{F_2(\beta)}{2n} = (1 + \delta) \log 2 - \delta \log n - \frac{\log n}{n} + \delta + \frac{1}{n} + A_2 + B_2 + C_2 + D_2 - \frac{0.841(1 + \delta)}{2 \log^2 (2n + 2\delta)},$$

where
where the quantities $A_2$, $B_2$, $C_2$ and $D_2$ are given by

\[
A_2 = (1 + \delta) \log(1 + \delta) - \left(1 + 2\delta + \frac{1}{n}\right) \log \left(1 + 2\delta + \frac{1}{n}\right),
\]

\[
B_2 = (1 + \delta) \log \log(2n + 2\delta) - \left(1 + 2\delta + \frac{1}{n}\right) \log \log(n + 2n\delta + 1),
\]

\[
C_2 = (1 + \delta) \log \frac{\log(2n + 2\delta)}{\log(2n + 2\delta)} - \left(1 + 2\delta + \frac{1}{n}\right) \log \log(n + 2n\delta + 1) - 2
\]

\[
D_2 = -(1 + \delta) \frac{\log \log(2n + 2\delta) - 2}{\log(2n + 2\delta)} \log \log(n + 2n\delta + 1) + 10.667
\]

\[
+ \left(1 + 2\delta + \frac{1}{n}\right) \frac{\log \log(n + 2n\delta + 1) - 6 \log \log(n + 2n\delta + 1) + 10.667}{2 \log^2(n + 2n\delta + 1)}.
\]

To show that $F_2(\beta) > 0$, we give in the following some lower bounds for the quantities $A_2$, $B_2$, $C_2$ and $D_2$. To find a lower bound for $A_2$, we consider the function $f : (0, \infty) \to \mathbb{R}$ defined by $f(x) = x \log x$. Then $A_2 = f(1 + \delta) - f(1 + 2\delta + 1/n)$. By the mean value theorem, there exists $\xi \in (1 + \delta, 1 + 2\delta + 1/n)$, so that $A_2 = -(\delta + 1/n) (\log \xi + 1)$. Since $\log \xi \leq \log(1 + 2\delta + 1/n) \leq 2\delta + 1/n$, we get

\[
A_2 \geq -\delta - 2\delta^2 - \frac{1}{n} \left(1 + 3\delta + \frac{1}{n}\right).
\]

Applying Lemma 4.3(b) to the last inequality, we obtain that

\[
A_2 \geq -\delta - 2 \frac{\log^2 2}{\log^2 n} - \frac{1}{\log n} \frac{(1 + 3\delta + 1/n)^2 \log^2 n}{n} \geq -\delta - 0.961 \frac{\log^2 n}{\log^2 n}.
\]

Our next goal is to estimate $B_2$. For this purpose, we use the right-hand side inequality of (3.8), Lemma 4.3(b) and the inequality $1/(x \log x) < 0.0037 / \log^2 x$, which holds for every $x \geq 2036$, to get

\[
\log \log(n + 2n\delta + 1) < \log \log n + \frac{2 \log 2}{\log^2 n} + \frac{1}{n \log n} < \log \log n + \frac{1.39}{\log^2 n}.
\]

On the other hand, we have

\[
\log \log(2n + 2n\delta) = \log \log n + \log \left(1 + \frac{\log 2 + \log(1 + \delta)}{\log n}\right).
\]

Applying the left-hand side inequality of (3.8), we obtain that

\[
\log \log(2n + 2n\delta) \geq \log \log n + \frac{2 \log 2}{\log^2 n} + \frac{\log(1 + \delta)}{\log n} + \frac{(\log 2 + \log(1 + \delta))^2}{2 \log^2 n}.
\]

Combined with

\[
(\log 2 + \log(1 + \delta))^2 \leq (\log 2 + \delta)^2 \leq \left(\log 2 + \frac{\log 2}{\log n}\right)^2 \leq 0.53,
\]

it follows that the inequality

\[
\log \log(2n + 2n\delta) \geq \log \log n + \frac{2 \log 2}{\log^2 n} + \frac{\log(1 + \delta)}{\log n} - \frac{0.265}{\log^2 n}
\]

holds. Again, we use the left-hand side inequality of (3.8) to establish

\[
\log \log(2n + 2n\delta) \geq \log \log n + \frac{2 \log 2}{\log^2 n} + \frac{\delta - \delta^2/2}{\log n} - \frac{0.265}{\log^2 n}
\]

Now we apply Lemma 4.3(a) and Lemma 4.3(b) to obtain that

\[
\log \log(2n + 2n\delta) \geq \log \log n + \frac{2 \log 2}{\log^2 n} + \frac{0.361}{\log^2 n}.
\]

Together with the definition of $B_2$ and (4.3), we get

\[
B_2 \geq -\delta \log \log n + \frac{(1 + \delta) \log 2}{\log n} - \frac{1.029 + 2.419\delta}{\log^2 n} - \frac{1.39}{\log^2 n}.
\]

Finally, we use a computer and Lemma 4.3(b) to get

\[
B_2 \geq -\delta \log \log n + \frac{(1 + \delta) \log 2}{\log n} - \frac{1.113}{\log^2 n}.
\]
Next, we find an lower bound for $C_2$. For this, we apply the inequality
\[
\frac{2(1 + 2\delta + 1/n)}{\log(n + 2n\delta + 1)} \geq \frac{2(1 + \delta)}{\log(2n + 2n\delta)}
\]
to the definition of $C_2$ to get
\[
C_2 \geq (1 + \delta) \frac{\log\log(2n + 2n\delta)}{\log(2n + 2n\delta)} - \left(1 + 2\delta + \frac{1}{n}\right) \frac{\log\log(n + 2n\delta + 1)}{\log(n + 2n\delta + 1)}.
\]
We use $2n + 2n\delta \geq n + 2n\delta + 1 \geq n$ to obtain that the inequality
\[
C_2 \geq -\log\log(n + 2n\delta + 1) \log n \geq (1 + 2\delta + 1/n)\log 2 + \log(1 + \delta)
\]
holds. Applying the right-hand side inequality of (3.8) and Lemma 4.3(b) to the last inequality, we get
\[
C_2 \geq -\log\log(n + 2n\delta + 1) \frac{\log 2/\log n + 1/n}{\log(2n + 2n\delta)\log(n + 2n\delta + 1)}.
\]
A computation shows that
\[
\left(1 + \frac{2\log 2}{\log n} + \frac{1}{n}\right) \frac{\log 2}{\log n} \leq 0.778.
\]
Hence
\[
C_2 \geq -\frac{(2 + 0.778)\log\log(n + 2n\delta + 1)}{\log(2n + 2n\delta)\log(n + 2n\delta + 1)} - \frac{\log n \log\log(n + 2n\delta + 1)}{n \log(2n + 2n\delta)\log(n + 2n\delta + 1)}.
\]
Note that the function $t \mapsto \log\log t / \log t$ is a decreasing function for every $t > e^e$, we obtain that
\[
C_2 \geq -\frac{(2 + 0.778)\log\log n}{\log^2 n} - \frac{\log n}{n \log n} \geq -\frac{1.472 \log\log n}{\log^2 n}.
\]
Finally, we estimate $D_2$. For this purpose, we consider the function $f : (1, \infty) \to \mathbb{R}$ defined by
\[
f(x) = \frac{\log\log x - 6 \log\log n + 10.667}{2 \log^2 x}.
\]
Note that $f(x)$ is a strictly decreasing function on the interval $(1, \infty)$ and the numerator of $f(x)$ is positive for every real $x > 1$. Together with $2n + 2n\delta \geq n + 2n\delta + 1 \geq n$, we get
\[
D_2 \geq \left(\delta + \frac{1}{n}\right) \frac{(\log\log n)^2 - 6 \log\log n + 10.667}{2 \log^2 n} > 0.
\]
Finally, we combine (4.9) with (4.6) and (4.8)-(4.10) to get that the inequality
\[
\frac{F_2(\beta)}{2n} > \left(\delta + \frac{1}{n}\right) \frac{(\log 2 + 2 \log 2/\log n) - \log n - 1}{\log^2 n} - \frac{1.472 \log\log n + 2.4945}{\log^2 n} - \frac{\log n - 0.841\delta}{2 \log^2 n}
\]
holds. Now it suffices to use (4.12) to get that the right-hand side of the last inequality is equal to $0$ and it follows that $F_2(\beta) > 0$. Together with $2n < \pi(R_n)$, $\beta < 2.6n$, the inequality (4.14) and Proposition 4.2 we obtain that $\pi(R_n) > \beta$ for every positive integer $n \geq 528,491,312$. We conclude by direct computation. □

Since $L(x) \geq 0$ for every $x \geq 10^{57}$, we use Theorem 1.3 to get the following weaker but more compact lower bound for $\pi(R_n)$.

**Corollary 4.4.** Let $n$ be a positive integer satisfying $n \geq 10^{57}$. Then
\[
\pi(R_n) > 2n \left(1 + \frac{\log 2}{\log n} - \frac{\log 2 \log\log n - \log^2 2 - \log 2}{\log^2 n}\right).
\]

In the next corollary, we use Theorem 1.3 to find that the lower bound for $\pi(R_n)$ given in Proposition 1.1 also holds for every positive integer $n$ satisfying $51,396,214,158,824 \leq n \leq 10^{400}$.

**Corollary 4.5.** Let $n$ be a positive integer satisfying $n \geq 51,396,214,158,824$. Then
\[
\pi(R_n) > 2n \left(1 + \frac{\log 2}{\log n} - \frac{\log 2 \log\log n - \log^2 2 - \log 2 + 0.11}{\log^2 n}\right).
\]

**Proof.** The claim follows directly by Theorem 1.3 and the fact that $L(x) \geq -0.11/\log^2 x$ for every $x \geq 51,396,214,158,824$. □
Finally, we give the following result concerning a lower bound for $\pi(R_n)$.

**Corollary 4.6.** Let $n$ be a positive integer satisfying $n \geq 85$. Then

$$\pi(R_n) > 2n \left(1 + \frac{\log 2}{\log n} - \frac{\log 2 \log \log n}{\log^2 n}\right).$$

**Proof.** Since $L(x) + (\log^2 2 + \log 2)/\log^2 x \geq 0$ for every $x \geq 20$, we apply Theorem 1.3 to get the correctness of the corollary for every positive integer $n \geq 1245$. We conclude by direct computation. \[\square\]

5. **Proof of Theorem 1.7**

In this section we give a proof of Theorem 1.7 by using Theorem 3.22 of [1]. For this, we need to introduce the following notations. By [2] Corollary 3.4 and Corollary 3.5, we have

$$\frac{x}{\log x - 1 - \frac{1}{\log x}} < \pi(x) < \frac{x}{\log x - 1 - \frac{1}{\log x}},$$

where the left-hand side inequality is valid for every $x \geq 468049$ and the right-hand side inequality holds for every $x \geq 5.43$. Using the right-hand side inequality of (5.1), we get $p_0 > n(\log p_n - 1 - 1.17/\log p_n)$ for every positive integer $n$. In addition, we set $\varepsilon > 0$ and $\lambda = \varepsilon/2$. Let $S = S(\varepsilon)$ be defined by

$$S = \exp \left(\sqrt{1.17 + \frac{2(1 + \varepsilon)}{\varepsilon}} \left(0.17 + \frac{2}{\log(2 \cdot 5.43)}\right) + \frac{1}{2} \left(\frac{1 + \varepsilon}{\varepsilon} \log 2\right)^2 + \frac{1}{2} \left(\frac{1 + \varepsilon}{\varepsilon} \log 2\right)\right)$$

and let $T = T(\varepsilon)$ be defined by $T = \exp(1/2 + \sqrt{1.17 + 0.17/\lambda + 1/4})$. By setting $X_0 = X_0(\varepsilon) = \max\{468049, 25, T\}$, we get the following result.

**Lemma 5.1.** Let $\varepsilon > 0$. For every positive integer $n$ satisfying $n \geq (\pi(X_0) + 1)/(2(1 + \varepsilon))$, we have

$$R_n \leq p_{[2(1+\varepsilon)n]}.$$

**Proof.** This follows from Theorem 3.22 and Lemma 3.23 of [1]. \[\square\]

The following proof of Theorem 1.7 consists of three steps. In the first step, we apply Theorem 1.2 and Theorem 1.3 to derive a lower bound for the quantity $m\pi(R_m) - \pi(R_{mn})$, which holds for every positive integers $m$ and $n$ satisfying $m \geq 2$ and $n \geq \max\{5225/m, 1245\}$. Then, in the second step, we use this lower bound and a computer to establish Theorem 1.7 for the cases $m = 2$ and $m \in \{3, 4, \ldots, 19\}$. Finally, we consider the case where $m \geq 20$. In this case, we first show that the inequality $\pi(R_{mn}) \leq m\pi(R_n)$ holds for every positive integer $n \geq 1245$. So it suffices to show that the required inequality also holds for every positive integers $m$ and $n$ with $m \geq 20$ and $N(m) \leq n \leq 1244$, where $N(m)$ is defined as in Theorem 1.7, with the only exception $(n, m) = (38, 9)$. For this purpose, note that

$$\pi(R_{mn}) \leq m\pi(R_n) \iff R_{mn} \leq p_{m\pi(R_n)}.$$

Now, for each $n \in \{2, \ldots, 1244\}$ we use (5.2) and Lemma 5.1 with $\varepsilon = \pi(R_n)/2n - 1$ (note that $\varepsilon > 0$ by (1.3)) to find a positive integer $M(n)$, so that $R_{mn} \leq p_{m\pi(R_n)}$ for every positive integer $m \geq M(n)$. Finally we check with a computer for which $m < M(n)$ the inequality $R_{mn} \leq p_{m\pi(R_n)}$ holds.

**Proof of Theorem 1.7.** First, we note that the inequality (1.10) holds for $m = 1$. So, we can assume that $m \geq 2$. Let $n$ be a positive integer with $n \geq \max\{5225/m, 1245\}$. By (5.3), (5.5) and Theorem 1.2 we have

$$\pi(R_{mn}) < 2mn \left(1 + \frac{\log 2 + \log 2/\log(mn) + 0.565/\log^2(mn)}{\log(mn) + \log\log(mn) - \log 2 - \log 2/\log(mn)}\right)$$

and, by (4.12), (4.3) and Theorem 4.3 we have

$$\pi(R_n) > 2n \left(1 + \frac{\log 2 + \log 2/\log n - (1.472\log\log n + 2.51)/\log^2 n}{\log n + \log\log n - \log 2 - \log 2/\log n}\right).$$

We set $\lambda(x) = \log x + \log\log x - \log 2 - \log 2/\log x$ and $\phi(x) = 1.472\log\log x + 2.51$. Then, by (5.3) and (5.4), we get

$$\frac{m\pi(R_n) - \pi(R_{mn})}{2mn} > \frac{W_m(n)}{X(n)\lambda(mn)},$$

where $X(n)\lambda(mn)$ is defined as in Theorem 1.7.
where
\[ W_m(n) = \log 2 \log m + \log 2(\log(\log(mn)) - \log \log n) + \log 2 \left( \frac{\log(mn)}{\log n} - \frac{\log n}{\log(mn)} \right) \]

Clearly, it suffices to show that \( W_m(n) \geq 0 \). Setting \( g(x) = \log \log x \), we get, by the mean value theorem, that there exists a real number \( \xi \in (n, mn) \) such that \( g(mn) - g(n) = (m - 1)g'(\xi) \).

Further, we have
\[ \log \log(mn) - \log \log n = \frac{(m - 1)n}{\xi \log \xi} \geq \frac{m - 1}{m \log(mn)} \geq \frac{1}{2 \log(mn)}. \]

Further, we have
\[ \frac{\log(mn)}{\log n} - \frac{\log n}{\log(mn)} = \frac{m - 1}{m \log(mn)}, \]

as well as
\[ \frac{\log(mn)}{\log n} - \frac{\log n}{\log(mn)} > \frac{m \log \log n}{\log^2(mn)}. \]

Combining (5.6) with the definition of \( W_m(n) \), we obtain that the inequality
\[ W_m(n) > \log m \left( \log 2 + \frac{\log 2}{\log n} + \log 2 \left( \frac{\log m + 1/2}{\log(mn)} + \frac{\log m \log \log n}{\log^2(mn)} \right) - \frac{\phi(n) \lambda(mn)}{\log^2 n} - \frac{0.565 \lambda(n)}{\log^2(mn)} \right), \]

Since \( \lambda(x) < \log x + \log 2 < \log x + \log x \), we get
\[ W_m(n) > \log m \left( \log 2 + \frac{\log 2}{\log n} - \frac{\phi(n)}{\log^2 n} \right) + \log 2 \left( \frac{\log m + 1/2}{\log(mn)} + \frac{\log m \log \log n}{\log^2(mn)} \right)
- \frac{\phi(n)}{\log n} - \frac{\phi(n) \log \log(mn)}{\log^2 n} + \frac{\phi(n) \log 2}{\log^2 n} - \frac{0.565 \log n}{\log^2(mn)} - \frac{0.565 \log \log n}{\log^2(mn)} \right). \]

Now, we use the right-hand side inequality of (3.8) to get \( \log \log(mn) \leq \log \log n + \log m/\log n \). Finally, we have
\[ W_m(n) > \log m \left( \log 2 + \frac{\log 2}{\log n} - \frac{\phi(n)}{\log^2 n} \right) - \frac{\phi(n)}{\log n} - \frac{\phi(n) \log \log n}{\log^2 n} + \frac{\phi(n) \log 2}{\log^2 n} - \frac{0.565 \log n}{\log^2(mn)} \right)
+ \frac{(\log m + 1/2) \log 2 - 0.565}{\log^2(mn)} \right) + \frac{\log(m \log 2 - 0.565) \log \log n}{\log^2(mn)} \right), \]

for every positive integers \( m \) and \( n \) satisfying \( m \geq 2 \) and \( n \geq \max(\lfloor 5225/m \rfloor, 1.245) \). Next, we use this inequality to prove the theorem. For this purpose, we consider the following three cases:

(i) **Case 1**: \( n = 2 \).

First, let \( n = 4903689 \). In this case, we have \((\log m + 1/2) \log 2 - 0.565 \geq 0.262 \) and \( \log m \log 2 - 0.565 > -0.085 \). Hence
\[ \frac{(\log m + 1/2) \log 2 - 0.565}{\log^2(mn)} \right) + \frac{(\log m \log 2 - 0.565) \log \log n}{\log^2(mn)} > 0. \]

Applying this inequality to (5.9), we get
\[ W_2(n) > \log 2 \left( \log 2 + \frac{\log 2}{\log n} - \frac{\phi(n)}{\log^2 n} \right) - \frac{\phi(n)}{\log n} - \frac{\phi(n) \log \log n}{\log^2 n} \right) + \frac{\phi(n) \log 2}{\log^2 n} - \frac{0.565 \log n}{\log^2(mn)} \right) \]

Since \( \log 2 - \phi(x)/\log x - \phi(x)/\log^2 x > 0 \) for every real \( x \geq 10.877 \), we get
\[ W_2(n) > \log^2 2 - \frac{\phi(n)}{\log n} - \frac{\phi(n) \log \log n - \log 2}{\log^2 n}. \]

Note that the right-hand side of the last inequality is positive. Combined with (5.5), we get that \( \pi(R_{2n}) \leq 2\pi(R_n) \) holds for every positive integer \( n \geq 4903689 \). A direct computation shows that the inequality \( \pi(R_{2n}) \leq 2\pi(R_n) \) also holds for every positive integer \( n \) so that \( 1 \leq n \leq 4903689 \).
(ii) Case 2: $m \in \{3, 4, \ldots, 19\}$.
First, we consider the case where $n \geq 6675$. By (5.9), we have

$$W_m(n) > \log m \left( \log 2 + \frac{\log 2}{\log n} \frac{\phi(n)}{\log^2 n} - \frac{\phi(n)}{\log n} - \frac{\phi(n)(\log \log n - \log 2)}{\log^2 n} \right).$$

We set $\delta_2 = 0.003314$ to obtain that the inequality

$$\delta_2 + \frac{\log 2}{\log x} \frac{\phi(x)}{\log^2 x} - \frac{\phi(x)}{\log^3 x} > 0$$

holds for every real $x \geq 6675$. So we see that

$$W_m(n) > (\log 2 - \delta_2) \log 3 - \frac{\phi(n)}{\log n} - \frac{\phi(n)(\log \log n - \log 2)}{\log^2 n}$$

and since the right-hand side of the last inequality is positive, we use (5.10) to conclude that $\pi(R_{nm}) \leq m\pi(R_n)$ holds for every positive integer $n \geq 6675$. For $m \in \{3, 4\}$, we verify with a direct computation that the inequality $\pi(R_{nm}) \leq m\pi(R_n)$ also holds for every positive integer $n$ so that $189 \leq n \leq 6674$. For $m \in \{5, 6\}$, we use a computer to check that the inequality $\pi(R_{nm}) \leq m\pi(R_n)$ is also valid for every positive integer $n$ satisfying $85 \leq n \leq 6674$. Finally, if $m \in \{7, 8, \ldots, 19\}$, a computer check shows that the required inequality also holds for every positive integer $n$ with $10 \leq n \leq 6674$.

(iii) Case 3: $m \geq 20$.
First, let $n \geq 1245$. Setting $\delta_3 = 0.03$, we obtain, similar to Case 2, that

$$W_m(n) > (\log 2 - \delta_3) \log 20 - \frac{\phi(n)}{\log n} - \frac{\phi(n)(\log \log n - \log 2)}{\log^2 n}.$$

Note that the right-hand side of the last inequality is positive. Together with (5.5), we get that $\pi(R_{nm}) \leq m\pi(R_n)$ holds for all positive integers $m$ and $n$ satisfying $m \geq 20$ and $n \geq 1245$. Now, for each $n \in \{2, \ldots, 1244\}$, we use (5.2), Lemma 5.1 with $\varepsilon = \pi(R_n)/2n - 1$ and a C++ version of the following MAPLE code to find positive integer $M(n) \geq 20$, so that $R_{mn} \leq p_{m\pi(R_n)}$ for every positive integer $m \geq M(n)$ and then we check for which $m$ with $20 \leq m < M(n)$ the inequality $R_{mn} \leq p_{m\pi(R_n)}$ holds:

```maple
> restart: with(numtheory): Digits := 100:
> for n from 1244 by -1 to 2 do
> ep := pi(R[n])/(2*n)-1: # R[n] denotes the nth Ramanujan prime
> lambda := ep^2:
> S := ceil(evalf(exp(sqrt(1.17+2*(1+ep)/ep)*(0.17+log(2)/log(2*5.43)))+
> (1/2+(1+ep)*log(2)/ep)^2+1/2+log(2/ep))/ep): # Hence pi(R[mm]) <= m*pi(R[n]) for all m >= M by Lemma 5.1
> T := ceil(evalf(exp(sqrt(1.17+0.17/lambda+1/4)+1/2))):
> X9 := max(468049,2*S,T): M := ceil((1+pi(X9))/(2*(1+ep))):
> while M*pi(R[n]) - pi(R[n*M]) > 0 and M >= 20 do
> M := M-1:
> end do:
> L[n] := M+1:
> end do:
```

Since $L[i] = 20$ for every $i \in \{2, \ldots, 1244\} \setminus \{9\}$ and $L[9] = 39$, we get that $\pi(R_{nm}) \leq m\pi(R_n)$ for every positive integers $n, m$ with $n \in \{2, \ldots, 1244\} \setminus \{9\}$ and $m \geq 20$ and for every positive integers $n, m$ with $n = 9$ and $m \geq 39$. A direct computation shows that the inequality $\pi(R_{nm}) \leq m\pi(R_n)$ holds for every $m$ with $20 \leq m \leq 37$ as well and that $38\pi(R_9) - \pi(R_{9,38}) = -2$.

So, we showed that the inequality $\pi(R_{nm}) \leq m\pi(R_n)$ holds for every $m \in \mathbb{N}$ and every positive integer $n \geq N(m)$ with the only exception $(m, n) = (38, 9)$, as desired. \qed

We use Theorem [1.7] and a computer to get the following remark.
Remark. The inequality $\pi(R_{mn}) \leq m\pi(R_n)$ fails if and only if $(m, n) \in \mathbb{N}_{>2} \times \{1\}$ (see [13]) or $(m, n) \in \{(2, 3), (2, 7), (2, 8), (2, 9), (2, 22), (2, 23), (2, 25), (2, 37), (2, 38), (2, 49), (2, 53), (2, 54), (2, 55), (2, 66), (2, 82), (2, 83), (2, 84), (2, 85), (2, 86), (2, 87), (2, 101), (2, 102), (2, 113), (2, 114), (2, 115), (2, 160), (2, 161), (2, 162), (2, 179), (2, 180), (2, 184), (2, 185), (2, 186), (2, 232), (2, 240), (2, 241), (2, 246), (2, 247), (2, 376), (2, 377), (2, 378), (2, 380), (2, 381), (2, 386), (2, 387), (2, 388), (2, 412), (2, 531), (2, 532), (2, 537), (2, 538), (2, 547), (2, 548), (2, 549), (2, 550), (2, 551), (2, 552), (2, 554), (2, 555), (2, 556), (2, 557), (2, 558), (2, 792), (2, 793), (2, 794), (2, 795), (2, 796), (2, 797), (2, 798), (2, 799), (2, 800), (2, 801), (2, 802), (2, 803), (2, 804), (2, 1140), (2, 1141), (2, 1142), (2, 1146), (2, 1147), (2, 1202), (2, 1241), (2, 1242), (2, 1243), (2, 1244), (3, 9), (3, 11), (3, 23), (3, 25), (3, 49), (3, 54), (3, 55), (3, 56), (3, 57), (3, 66), (3, 67), (3, 83), (3, 84), (3, 114), (3, 115), (3, 160), (3, 187), (3, 188), (4, 9), (4, 11), (4, 37), (4, 38), (4, 42), (4, 54), (4, 55), (4, 82), (4, 83), (4, 84), (4, 114), (4, 115), (4, 188), (5, 3), (5, 9), (5, 84), (6, 28), (6, 54), (6, 55), (6, 84), (7, 3), (7, 9), (8, 9), (9, 9), (10, 9), (11, 3), (11, 9), (12, 9), (13, 9), (14, 9), (15, 3), (15, 9), (16, 9), (17, 9), (18, 9), (19, 9), (38, 9)).

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