An Algebraic-Geometry Approach to Prime Factorization

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New algorithms for prime factorization that outperform the existing ones or take advantage of particular properties of the prime factors can have a practical impact on present implementations of cryptographic algorithms that rely on the complexity of factorization. Currently used keys are chosen on the basis of the present algorithmic knowledge and, thus, can potentially be subject to future breaches. For this reason, it is worth to investigate new approaches which have the potentiality of giving a computational advantage. The problem has also relevance in quantum computation, as an efficient quantum algorithm for prime factorization already exists. Thus, better classical asymptotic complexity can provide a better understanding of the advantages offered by quantum computers. In this paper, we reduce the factorization problem to the search of points of parametrizable varieties, in particular curves, over finite fields. The varieties are required to have an arbitrarily large number of intersection points with some hypersurface over the base field. For a subexponential or polynomial factoring complexity, the number of parameters have to scale sublinearly in the space dimension $n$ and the complexity of computing a point given the parameters has to be subexponential or polynomial, respectively. We outline a procedure for building these varieties, which is illustrated with two constructions. In one case, we show that there are varieties whose points can be evaluated efficiently given a number of parameters not greater than $n/2$. In the other case, the bound is dropped to $n/3$. Incidentally, the first construction resembles a kind of retro-causal model. Retro-causality is considered one possible explanation of quantum weirdness.

I. INTRODUCTION

Prime factorization is a problem in the complexity class NP of problems that can be solved in polynomial time by a nondeterministic machine. Indeed, the prime factors can be verified efficiently by multiplication. At present, it is not known if the problem has polynomial computational complexity and, thus, is in the complexity class P. Nonetheless, the most common cryptographic algorithms rely on the assumption of hardness of factorization. Website certificates and bitcoin wallets are examples of resources depending on that assumption. Since no strict lower bound on the computational complexity is actually known, many critical services are potentially subject to future security breaches. Consequently, cryptographic keys have gradually increased their length
to adapt to new findings. For example, the general number-field sieve [1] can break keys that would have been considered secure against previous factoring methods.

Prime factorization is also important for its relation with quantum computing, since an efficient quantum algorithm for factorization is known. This algorithm is considered a main argument supporting the supremacy of quantum over classical computing. Thus, the search for faster classical algorithms is relevant for better understanding the actual gap between classical and quantum realm.

Many of the known factoring methods use the ring of integers modulo $c$ as a common feature, where $c$ is the number to be factorized. Examples are Pollard’s $\rho$ and $p-1$ algorithms [2, 3], Williams’ $p+1$ algorithm [4], their generalization with cyclotomic polynomials [5], Lenstra elliptic curve factorization [6], and quadratic sieve [7]. These methods end up to generate a number, say $m$, having a factor of $c$. Once $m$ is obtained, the common factor can be efficiently computed by the Euclidean algorithm. Some of these methods use only operations defined in the ring. Others, such as the elliptic-curve method, perform also division operations by pretending that $c$ is prime. If this operation fails at some point, the divisor is taken as outcome $m$. In other words, the purpose of these methods is to compute a zero over the field of integers modulo some prime factor of $c$ by possibly starting from some random initial state. Thus, the general scheme is summarized by a map $X \mapsto m$ from a random state $X$ in some set $\Omega$ to an integer $m$ modulo $c$. Different states may be tried until $m$ is equal to zero modulo some prime of $c$. The complexity of the algorithm depends on the computational complexity of generating $X$ in $\Omega$, the computational complexity of evaluating the map, and the average number of trials required to find a zero.

In this paper, we employ this general scheme by focusing on a class $\Theta$ of maps defined as multivariate rational functions over prime fields $\mathbb{Z}_p$ of order $p$ and, more generally, over a finite field $\mathbb{GF}(q)$ of order $q = p^k$ and degree $k$. The set $\Omega$ of inputs is taken equal to the domain of definition of the maps. More precisely, the maps are first defined over some algebraic number field $\mathbb{Q}(\alpha)$ of degree $k_0$, where $\alpha$ is an algebraic number, that is, solution of some irreducible polynomial $P_I$ of degree $k_0$. Then, the maps are reinterpreted over a finite field. Using the general scheme of the other methods, the class $\Theta$ takes to a factoring algorithm with polynomial complexity if

(a) the number of distinct zeros, say $N_P$, of the maps in $\Theta$ is arbitrarily large over $\mathbb{Q}(\alpha)$;

(b) a large fraction of the zeros remain distinct when reinterpreted over a finite field whose order is greater than about $N_P^{1/M}$;

(c) the product of the number of parameters by the field degree is upper-bounded by a sublinear power function of $\log N_P$;
(d) the computational complexity of evaluating the map given any input is upper-bounded by a polynomial function in $\log NP$.

A subexponential factoring complexity is achieved with weaker scaling conditions on the map complexity, as discussed later in Sec. III-B. Later, this approach to factorization will be reduced to the search of rational points of a variety having an arbitrarily large number of rational intersection points with a hypersurface.

The scheme employing rational functions resembles some existing methods, such as Pollard’s $p - 1$ algorithm. The main difference is that these algorithms generally rely on algebraic properties over finite fields, whereas the present scheme relies on algebraic properties over the field $\mathbb{Q}(\alpha)$. For example, Pollard’s method ends up to build a polynomial $x^n - 1$ with $p - 1$ roots over a finite $\mathbb{Z}_p$, where $p$ is some prime factor of $c$. This feature of the polynomial comes from Fermat’s little theorem and is satisfied if the integer $n$ has $p - 1$ as factor. Thus, the existence of a large number of zeros of $x^n - 1$ strictly depends on the field. Indeed, the polynomial does not have more than 2 roots over the rationals. In our scheme, the main task is to find rational functions having a sufficiently large number of zeros over an algebraic number field. This feature is then inherited by the functions over finite fields. Some specific properties of finite fields can eventually be useful, such as the reducibility of $P_1$ over $\mathbb{Z}_p$. This will be mentioned later.

Let us illustrate the general idea with an example. Suppose that the input of the factorization problem is $c = pp'$, with $p$ and $p'$ prime numbers and $p < p'$. Let the map be a univariate polynomial of the form

$$P(x) = \prod_{i=1}^{NP} (x - x_i),$$

where $x_i$ are integer numbers somehow randomly distributed in an interval between 1 and $i_{\text{max}} \gg p'$. More generally, $x_i$ can be rational numbers $n_i/m_i$ with $n_i$ and/or $m_i$ in $\{1, \ldots, i_{\text{max}}\}$. When reinterpreted modulo $p$ or modulo $p'$, the numbers $x_i$ take random values over the finite fields. If $NP < p$, we expect that the polynomial has about $NP$ distinct roots over the finite fields. Thus, the probability that $P(x) \mod p = 0$ or $P(x) \mod p' = 0$ is about $NP/p$ or $NP/p'$, respectively, which are the ratio between the number of zeros and the size of the input space $\Omega$ over the finite fields. The probability that $P(x) \mod c$ contains a nontrivial factor of $c$ is about $\frac{NP}{p} \left(1 - \frac{NP}{p}\right) + \frac{NP}{p'} \left(1 - \frac{NP}{p'}\right)$. Thus, if $NP$ is of the order of $p$, we can get a nontrivial factor by the Euclidean algorithm in few trials. More specifically, if $p \approx p'$ and $NP \approx \sqrt{c}/2$, then the probability of getting a nontrivial factor is roughly $1/2$. It is clear that a computational complexity of the map scaling
subexponentially or polynomially in $\log N_P$ leads to a subexponential or polynomial complexity of the factoring algorithm. Thus, the central problem is to build a polynomial $P(x)$ or a rational function with friendly computational properties with respect to the number of zeros. The scheme can be generalized by taking multivariate maps with $M$ input parameters. In this case, the number of zeros needs to be of the order of $p^M$, which is the size of the input space over the field $\mathbb{Z}_p$. As a further generalization, the rational field can be replaced by an algebraic number field $\mathbb{Q}(\alpha)$ of degree $k_0$. A number in this field is represented as a $k_0$-dimensional vector over the rationals. Reinterpreting the components of the vector over a finite field $\mathbb{Z}_p$, the size of the sampling space is $p^{k_0M}$, so that we should have $N_P \sim p^{k_0M}$ in order to get nontrivial factors in few trials. Actually, this is the worst-case scenario since the reinterpretation of $\mathbb{Q}(\alpha)$ modulo $p$ can lead to a degree of the finite field much smaller than $k_0$. For example, if $\alpha$ is the root $e^{2\pi i/n}$ of the polynomial $x^n - 1$, the degree of the corresponding finite field with characteristic $p$ collapses to 1 if $n$ is a divisor of $p - 1$.

On one hand, it is trivial to build polynomials with an arbitrarily large number $N_P$ of roots over the rationals as long as the computational cost grows linearly in $N_P$. On the other hand, it is also simple to build polynomials with friendly computational complexity with respect to $\log N_P$ if the roots are taken over algebraically closed fields. The simplest example is the previously mentioned polynomial $P(x) = x^n - 1$, which has $n$ distinct complex roots and a computational complexity scaling as $\log n$. However, over the rationals, this polynomial has at most 2 roots. We can include other roots by extending the rational field to an algebraic number field, but the extension would have a degree proportional to the number of roots, so that the computational complexity of evaluating $P(x)$ would grow polynomially in the number of roots over the extension.

**A. Algebraic-geometry rephrasing of the problem**

It is clear that an explicit definition of each root of polynomial (1) leads to an amount of memory allocation growing exponentially in $\log c$, so that the resulting factoring algorithm is exponential in time and space. Thus, the roots has to be defined implicitly by some simple rules. Considering a purely algebraic definition, we associates the roots to rational solutions of a set of $n$ non-linear polynomial equations in $n$ variables $x = (x_1, \ldots, x_n)$,

$$P_k(x) = 0, \quad k \in \{0, \ldots, n - 1\}. \quad (2)$$
The solutions are intersection points of \( n \) hypersurfaces. The roots of \( P(x) \) are defined as the values of some coordinate, say \( x_n \), at the intersection points. By eliminating the \( n - 1 \) variables \( x_1, \ldots, x_{n-1} \), we end up with a polynomial \( P(x_n) \) with a number of roots generally growing exponentially in \( n \). This solves the problem of space complexity in the definition of \( P(x) \). There are two remaining problems. First, we have to choose the polynomials \( P_0, \ldots, P_{n-1} \) such that an exponentially large fraction of the intersection points are rational. Second, the variable elimination, given a value of \( x_n \), has to be performed as efficiently as possible over a finite field. If the elimination has polynomial complexity, then factorization turns out to have polynomial complexity. Note that the elimination of \( n - 1 \) variables given \( x_n \) is equivalent to a consistency test of the \( n \) polynomials.

The first problem can be solved in a simple way by defining the polynomials as elements of the ideal generated by products of linear polynomials. Let us denote a linear polynomial with a symbol with a hat. For example, the quadratic polynomials

\[
G_i = \hat{a}_i \hat{b}_i, \quad i \in \{1, \ldots, n\}
\]  

have generally \( 2^n \) rational common zeros, provided that the coefficients of \( \hat{a}_i \) and \( \hat{b}_i \) are rational. Identifying the polynomials \( P_0, \ldots, P_{n-1} \) with elements of the ideal generated \( G_1, \ldots, G_n \), we have

\[
P_k = \sum_i c_{k,i}(x)\hat{a}_i \hat{b}_i, \quad i \in \{0, \ldots, n-1\},
\]

whose set of common zeros contains the \( 2^n \) rational points of the generators \( G_i \). In particular, if the polynomials \( c_{k,i} \) are set equal to constants, then the system \( P_0 = \ldots P_{n-1} = 0 \) is equivalent to the system \( G_1 = \ldots G_n = 0 \).

At this point, the variable elimination is the final problem. A working method is to compute a Gröbner basis. For the purpose of factorizing \( c = pp' \), the task is to evaluate a Gröbner basis to check if the \( n \) polynomials given \( x_n \) are consistent modulo \( c \). If they are consistent modulo some non-trivial factor \( p \) of \( c \), we end up at some point with some integer equal to zero modulo \( p \). However, the complexity of this computation is doubly exponential in the worst case. Thus, we have to search for a suitable set of polynomials with a large set of rational zeros such that there is an efficient algorithm for eliminating \( n - 1 \) variables. The variable elimination is efficient if \( n - 1 \) out of the \( n \) polynomial equations \( P_k = 0 \) form a suitable triangular system for some set of low-degree polynomials \( c_{k,i} \). Let us assume that the last \( n - 1 \) polynomials have the triangular
form

\[
\begin{align*}
P_{n-1}(x_{n-1}, x_n) \\
P_{n-2}(x_{n-2}, x_{n-1}, x_n) \\
\vdots \\
P_1(x_1, \ldots, x_{n-1}, x_{n-2}, x_n)
\end{align*}
\]

such that the \(k\)-th polynomial is linear in \(x_k\). Thus, the corresponding polynomial equations can be sequentially solved in the first \(n-1\) variables through the system

\[
\begin{align*}
x_{n-1} &= \frac{N_{n-1}(x_n)}{D_{n-1}(x)} \\
x_{n-2} &= \frac{N_{n-2}(x_n, x_{n-1})}{D_{n-2}(x_{n-1})} \\
\vdots \\
x_1 &= \frac{N_1(x_n, x_{n-1}, \ldots, x_2)}{D_1(x_n, x_{n-1}, \ldots, x_2)},
\end{align*}
\]

where \(D_k \equiv \partial P_k / \partial x_k\) and \(N_k \equiv P_k |_{x_k=0}\). This system defines a parametrizable curve, say \(V\), in the algebraic set defined by the polynomials \(P_1, \ldots, P_{n-1}\), the variable \(x_n\) being the parameter. Let us remind that a curve is parametrizable if and only if its geometric genus is equal to zero. The overall set of variables can be efficiently computed over a finite field, provided that the polynomial coefficients \(c_{k,i}\) are not too complex. Once determined the variables, the remaining polynomial \(P_0\) turns out to be equal to zero if \(x\) is an intersection point. Provided that the rational intersection points have distinct values of \(x_n\) (which is essentially equivalent to state that the points are distinct and are in the variety \(V\)), then the procedure generates a value \(P_0(x)\) which is zero modulo \(p\) with high probability if \(p\) is of the order of the number of rational intersection points. For this inference, it is pivotal to assume that a large fraction of rational points remain distinct when reinterpreted over the finite field.

This algebraic-geometry rephrasing of the problem can be stated in a more general form. Let \(V\) and \(H\) be some irreducible curve in an \(n\)-dimensional space and a hypersurface, respectively. The curve \(V\) is not necessarily parametrizable, thus its genus may take strictly positive values. The points in \(H\) are the zero locus of some polynomial \(P_0\). Let \(N_P\) be the number of distinct intersection points between \(V\) and \(H\) over the rational field \(\mathbb{Q}\). Over a finite field \(\text{GF}(q)\), Weil’s theorem states that the number of rational points, says \(N_1\), of a smooth curve is bounded by the inequalities

\[
-2g\sqrt{q} \leq N_1 - (q + 1) \leq 2g\sqrt{q},
\]

where \(g\) is the geometric genus of the curve. Generalizing to singular curves \(8\), we have

\[
-2g\sqrt{q} - \delta \leq N_1 - (q + 1) \leq 2g\sqrt{q} + \delta,
\]

(7)
where \( \delta \) is the number of singularities, properly counted. These inequalities have the following geometric interpretation. For the sake of simplicity, let us assume that the singularities are ordinary double points. A singular curve, says \( \mathcal{S} \), with genus \( g \) is birationally equivalent to a smooth curve, say \( \mathcal{R} \), with same genus, for which Wiel’s theorem holds. That is, the rational points of \( \mathcal{R} \) are bijectively mapped to the non-singular rational points of \( \mathcal{S} \), apart from a possible finite set \( \Omega \) of \( 2m \) points mapping to \( m \) singular points of \( \mathcal{S} \). The cardinality of \( \Omega \) is at most \( 2\delta \) (attained when the \( \delta \) singularities have tangent vectors over the finite field). We have two extremal cases. In one case, \( \#\Omega = 2\delta \), so that, \( \mathcal{S} \) has \( \delta \) points less than \( \mathcal{R} \) (two points in \( \Omega \) are merged into a singularity of \( \mathcal{S} \)). This gives the lower bound in (7). In the second case, \( \Omega \) is empty and the singular points of \( \mathcal{S} \) are rational points. Thus, \( \mathcal{S} \) has \( \delta \) rational points more. Given this interpretation, Weil’s upper bound still holds for the number of non-singular rational points, say \( N_1' \),

\[
N_1' \leq (q + 1) + 2g\sqrt{q}.
\]

Thus, if the genus is much smaller than \( \sqrt{q} \), \( N_1' \) is upper-bounded by a number close to the order of the field.

Now, let us assume that most of the \( N_P \) rational points in \( \mathcal{V} \cap \mathcal{H} \) over \( \mathbb{Q} \) remain distinct when reinterpreted over \( \mathbb{Z}_p \), with \( p \approx aN_P \), where \( a \) is a number slightly greater than 1, say \( a = 2 \). We also assume that these points are not singularities of \( \mathcal{V} \). Weil’s inequality (8) implies that the curve does not have more than about \( p \) points over \( \mathbb{Z}_p \). Since \( p \gtrsim N_1' \gtrsim N_P \sim p/2 \), we have that the number of non-singular rational points of the curve is about the number of intersection points over \( \mathbb{Z}_p \). This implies that a large fraction of points \( x \in \mathcal{V} \) over the finite field are also points of \( \mathcal{H} \). We have the following.

**Claim 1.** Let \( \mathcal{V} \) and \( \mathcal{H} \) be an algebraic curve with genus \( g \) and a hypersurface, respectively. The hypersurface is the zero locus of the polynomial \( P_0 \). Their intersection has \( N_P \) distinct points over the rationals, which are not singularities of \( \mathcal{V} \). Let us also assume that \( g \ll \sqrt{N_P} \) and that most of the \( N_P \) rational points remain distinct over \( \mathbb{Z}_p \) with \( p \gtrsim 2N_P \). If we pick up at random a point in \( \mathcal{V} \), then

\[
P_0(x) = 0 \mod p
\]

with probability equal to about ratio \( N_P/p \).

If there are pairs \( (\mathcal{V}, \mathcal{H}) \) for every \( N_P \) that satisfy the premises of this claim, then prime factorization is reduced to the search of rational points of a curve. Actually, these pairs always exist, as
shown later in the section. Assuming that \( c = pp' \) with \( p \sim p' \), the procedure for factorizing \( c \) is as follows.

1. Take a pair \((V, H)\) with \( N_P \sim c^{1/2} \) such that the premises of Claim \( \square \) hold.

2. Search for a rational point \( x \in V \) over \( Z_p \).

3. Compute \( \text{GCD}[P_0(x), c] \), the greatest common divisor of \( P_0(x) \) and \( c \).

The last step gives \( \text{GCD}[P(x), c] \) equal to 1, \( c \) or one of the factors of \( c \). The probability of getting a nontrivial factor can be made close to \( 1/2 \) with a suitable tuning of \( N_P \) (as shown later in Sec. \( \Pi \)).

Finding rational points of a general curve with genus greater than 2 is an exceptionally complex problem. For example, just to prove that the plane curve \( x^h + y^h = 1 \) with \( h > 2 \) does not have zeros over the rationals took more than three centuries since Fermat stated it. Curves with genus 1 are elliptic curves, which have an important role in prime factorization (See Lenstra algorithm \( \square \)). Here, we will focus on parametrizable curves, which have genus 0. In particular, we will consider parametrizations generated by the sequential equations \( \{5\} \). It is interesting to note that it is always possible to find a curve \( V \) with parametrization \( \{5\} \) and a hypersurface \( H \) such that their intersection contains a given set of rational points. In particular, there is a set of polynomials \( P_0, \ldots, P_{n-1} \) of the form \( \{4\} \), such that the zero locus of the last \( n-1 \) polynomials contains a parametrizable curve with parametrization \( \{5\} \), whose intersection with the hypersurface \( P_0 = 0 \) contains \( 2^n \) distinct rational points. Let the intersection points be defined by the polynomials \( \{3\} \). Provided that \( x_n \) is a separating variable, the set of intersection points admits the rational univariate representation \( \{9\} \)

\[
\begin{align*}
x_{n-1} &= \frac{\tilde{N}_{n-1}(x_n)}{\tilde{D}_{n-1}(x_n)} \\
x_{n-2} &= \frac{\tilde{N}_{n-2}(x_n)}{\tilde{D}_{n-2}(x_n)} \\
&\vdots \\
x_1 &= \frac{\tilde{N}_1(x_n)}{\tilde{D}_1(x_n)} \\
\tilde{N}_0(x_n) &= 0
\end{align*}
\]

The first \( n-1 \) equations are a particular form of equation \( \{5\} \) and define a parametrizable curve with \( x_n \) as parameter. The last equation can be replaced by some linear combination of the polynomials \( G_i \). It is also interesting to note that the rational univariate representation is unique once the separating variable is chosen. This means that the parametrizable curve is uniquely determined by the set of intersection points and the variable that is chosen as parameter.

It is clear that the curve and hypersurface obtained through this construction with a general set of polynomials \( G_i \) satisfy the premises of Claim \( \square \). Indeed, a large part of the common zeros
of the polynomials $G_i$ are generally distinct over a finite field $\mathbb{Z}_p$ with $p \simeq N_P$. For example, the point $\hat{a}_1 = \cdots = \hat{a}_n = 0$ is distinct from the other points if and only if $\hat{b}_i \neq 0$ at that point for every $i \in \{1, \ldots, n\}$. Thus, the probability that a given point is not distinct over $\mathbb{Z}_p$ is of the order of $p^{-1} \sim N_P^{-1}$, hence a large part of the points are distinct over the finite field. There is an apparent paradox. With a suitable choice of the linear functions $\hat{a}_i$ and $\hat{b}_i$, the intersection points can be made distinct over a field $\mathbb{Z}_p$ with $p \ll N_P$, which contradicts Weil’s inequality (8). The contradiction is explained by the fact that the curve is broken into the union of reducible curves over the finite field. In other words, some denominator $D_k$ turns out to be equal to zero modulo $p$ at some intersection points. This may happen also with $p \sim N_P$, which is not a concern. Indeed, possible zero denominators can be used to find factors of $c$. 

B. Contents

In Section II, we introduce the general scheme of the factoring algorithm based on rational maps and discuss its computational complexity in terms of the complexity of the maps, the number of parameters and the field degree. In Section III, the factorization problem is reduced to the search of rational points of parametrizable algebraic varieties $\mathcal{V}$ having an arbitrarily large number $N_P$ of rational intersection points with a hypersurface $\mathcal{H}$. Provided that the $N_P$ grows exponentially in the space dimension, the factorization algorithm has polynomial complexity if the number of parameters and the complexity of evaluating a point in $\mathcal{V}$ over a finite field grow sublinearly and polynomially in the space dimension, respectively. Thus, the varieties $\mathcal{V}$ and $\mathcal{H}$ have to satisfy two requirements. On one side, their intersection has to contain a large set of rational points. On the other side, $\mathcal{V}$ has to be parametrizable and its points have to be computed efficiently given the values of the parameters. The first requirement is fulfilled with a generalization of the construction given by Eq. (4). First, we define an ideal $I$ generated by products of linear polynomials such that the associated algebraic set contains $N_P$ rational points. The relevant information on this ideal is encoded in a satisfiability formula (SAT) in conjunctive normal form (CNF) and a linear matroid. Then, we define $\mathcal{V}$ and $\mathcal{H}$ as elements of the ideal. By construction, $\mathcal{V} \cap \mathcal{H}$ contains the $N_P$ rational points. The ideal $I$ and the polynomials defining the varieties contain some coefficients. The second requirement is tackled in Sec. IV. By imposing the parametrization of $\mathcal{V}$, we get a set of polynomial equations for the coefficients. These equations always admit a solution, provided that the only constraint on $\mathcal{V}$ and $\mathcal{H}$ is being an element of $I$. The task is to find an ideal $I$ and a set of coefficients such that the computation of points in $\mathcal{V}$ is as efficient as possible, given a number
of parameters scaling sublinearly in the space dimension.

In this general form, the problem of building the varieties \( V \) and \( H \) is quite intricate. A good strategy is to start with simple ideals and search for varieties in a subset of these ideals, so that the polynomial constraints on the unknown coefficients can be handled with little efforts. With these restrictions, it is not guaranteed that the required varieties exist, but we can have hints on how to proceed. This strategy is employed in Sec. [V] where we consider an ideal generated by the polynomials \( P_0, P_1, \ldots, P_{n-1} \). The varieties are defined by linear combinations of these generators with constant coefficients, that is, \( H \) and \( V \) are in the zero locus of \( P_0 \) and \( P_1, \ldots, P_{n-1} \), respectively, defined by Eq. (4). The \( 2^n \) rational points associated with the ideal are taken distinct in \( \mathbb{Q} \). First, we prove that there is no solution with one parameter \( (M = 1) \), for a dimension greater than 4. We give an explicit numerical example of a curve and hypersurface in dimension 4. The intersection has 16 rational points. We also give a solution with about \( n/2 \) parameters. Suggestively, this solution resembles a kind of retro-causal model. Retro-causality is considered one possible explanation of some strange aspects of quantum theory, such as non-locality and wave-function collapse after a measurement. Finally, we close the section by proving that there is a solution with \( 2 \leq M \leq (n-1)/3 \). This is shown by explicitly building a variety \( V \) with \( (n-1)/3 \) parameters. Whether it is possible to drop the number of parameters below this upper bound is left as an open problem. If \( M \) grows sublinearly in \( n \), then there is automatically a factoring algorithm with polynomial complexity, provided that the coefficients defining the polynomials \( P_k \) are in \( \mathbb{Q} \) and can be computed efficiently over a finite field. The conclusions and perspectives are drawn in Sec. [VI].

II. GENERAL SCHEME AND COMPLEXITY ANALYSIS

At a low level, the central object of the factoring algorithm under study is a class \( \Theta \) of maps \( \bar{\tau} \mapsto R(\bar{\tau}) \) from a set \( \bar{\tau} \equiv (\tau_1, \ldots, \tau_M) \) of \( M \) parameters over the field \( \mathbb{Q}(\alpha) \) to a number in the same field, where \( R \) is a rational function, that is, the algebraic fraction of two polynomials. Let us write it as

\[
R(\bar{\tau}) \equiv \frac{N(\bar{\tau})}{D(\bar{\tau})}.
\]

This function may be indirectly defined by applying consecutively simpler rational functions, as done in Sec. [III]. Note that the computational complexity of evaluating \( R(\bar{\tau}) \) can be lower than the complexity of evaluating the numerator \( N(\bar{\tau}) \). For this reason we consider more general rational functions rather than polynomials. Both \( M \) and \( \alpha \) are not necessarily fixed in the class \( \Theta \). We
denote by $N_P$ the number of zeros of the polynomial $N$ over $\mathbb{Q}(\alpha)$. The number $N_P$ is supposed to be finite, we will come back to this assumption later in Sec. II C. For the sake of simplicity, first we introduce the general scheme of the algorithm over the rational field. Then, we outline its extension to algebraic number fields. We mainly consider the case of semiprime input, that is, $c$ is taken equal to the product of two prime numbers $p$ and $p'$. This case is the most relevant in cryptography. If the rational points are somehow randomly distributed when reinterpreted over $\mathbb{Z}_p$, then the polynomial $N$ has at least about $N_P$ distinct zeros over the finite field, provided that $N_P$ is sufficiently smaller than the size $p^M$ of the input space $\Omega$. We could have additional zeros in the finite field, but we conservatively assume that $N_P$ is a good estimate for the total number. For $N_P$ close to $p^M$, two different roots in the rational field may collapse to the same number in the finite field. We will account for that later in Sec. II B. Given the class $\Theta$, the factorization procedure has the same general scheme as other methods using finite fields. Again, the value $m = R(\tau_1, \tau_2, \ldots)$ is computed by pretending that $c$ is prime and $\mathbb{Z}/c\mathbb{Z}$ is a field. If an algebraic division takes to a contradiction during the computation of $R$, the divisor is taken as outcome $m$. For the sake of simplicity, we neglect the zeros of $D$ and consider only the zeros of $N$. In Section III, we will see that this simplification is irrelevant for a complexity analysis. It is clear that the outcome $m$ is zero in $\mathbb{Z}/c\mathbb{Z}$ with high probability for some divisor $p$ of $c$ if the number of zeros is about or greater than the number of inputs $p^M$. Furthermore, if $p' > p$ and $N_P$ is sufficiently smaller than $(p')^M$, then the outcome $m$ contains the nontrivial factor $p$ of $c$ with high probability. This is guaranteed if $N_P$ is taken equal to about $c^{M/2}$, which is almost optimal if $p \simeq p'$, as we will see later in Sec. II B. Thus, we have the following.

**Algorithm 1.** Factoring algorithm with input $c = pp'$, $p$ and $p'$ being prime numbers.

1. Choose a map in $\Theta$ with $M$ input parameters and $N_P$ zeros over the rationals such that $N_P \simeq c^{M/2}$ (see Sec. III B for an optimal choice of $N_P$);
2. generate a set of $M$ random numbers $\tau_1, \ldots, \tau_M$ over $\mathbb{Z}/c\mathbb{Z}$.
3. compute the value $m = R(\tau_1, \ldots, \tau_M)$ over $\mathbb{Z}/c\mathbb{Z}$ (by pretending that $c$ is prime).
4. compute the greatest common divisor between $m$ and $c$.
5. if a nontrivial factor of $c$ is not obtained, repeat from point (2).

The number $M$ of parameters may depend on the map picked up in $\Theta$. Let $M_{\min}(N_P)$ be the minimum of $M$ in $\Theta$ for given $N_P$. The setting at point (1) is possible only if $M_{\min}$ grows less than linearly in $\log N_P$, which is condition (c) enumerated in the introduction. A tighter condition is
necessary if the computational complexity of evaluating the map scales subexponentially, but not polynomially. This will be discussed with more details in Sec. II B.

If \( c \) has more than two prime factors, \( N_P \) must be chosen about equal to about \( p^M \), where \( p \) is an estimate of one prime factor. If there is no knowledge about the factors, the algorithm can be executed by trying different orders of magnitude of \( N_P \) from 2 to \( c^{1/2} \). For example, we can increase the guessed \( N_P \) by a factor 2, so that the overall number of executions grows polynomially in \( \log_2 p \). However, better strategies are available. A map with a too great \( N_P \) ends up to produce zero modulo \( p \) for every factor \( p \) of \( c \) and, thus, the algorithm always generates the trivial factor \( c \). Conversely, a too small \( N_P \) gives a too small probability of getting a factor. Thus, we can employ a kind of bisection search. A sketch of the search algorithm is as follows.

1. set \( a_d = 1 \) and \( a_u = c^M \);
2. set \( N_P = \sqrt{a_d a_u} \) and choose a map in \( \Theta \) with \( N_P \) zeros;
3. execute Algorithm 1 from point (2) and break the loop after a certain number of iterations;
4. if a nontrivial factor is found, return it as outcome;
5. if the algorithm found only the trivial divisor \( c \), set \( a_u = N_P \), otherwise set \( a_d = N_P \);
6. go back to point (2).

This kind of search can reduce the number of executions of Algorithm 1. In the following, we will not discuss these optimizations for multiple prime factors, we will consider mainly semiprime integer numbers \( c = pp' \).

A. Extension to algebraic number fields

Before outlining how the algorithm can be extended to algebraic number fields, let us briefly remind what a number field is. The number field \( \mathbb{Q}(\alpha) \) is a rational field extension obtained by adding an algebraic number \( \alpha \) to the field \( \mathbb{Q} \). The number \( \alpha \) is solution of some irreducible polynomial \( P_I \) of degree \( k_0 \), which is also called the degree of \( \mathbb{Q}(\alpha) \). The extension field includes all the elements of the form \( \sum_{i=0}^{k_0-1} r_i \alpha^i \), where \( r_i \) are rational numbers. Every power \( \alpha^h \) with \( h \geq k_0 \) can be reduced to that form through the equation \( P_I(\alpha) = 0 \). Thus, an element of \( \mathbb{Q}(\alpha) \) can be represented as a \( k_0 \)-dimensional vector over \( \mathbb{Q} \). Formally, the extension field is defined as the quotient ring \( \mathbb{Q}[X]/P_I \), the polynomial ring over \( \mathbb{Q} \) modulo \( P_I \). The quotient ring is also a
field as long as $P_I$ is irreducible. Reinterpreting the rational function $\mathcal{R}$ over a finite field $\text{GF}(p^k)$ means to reinterpret $r_i$ and the coefficients of $P_I$ as integers modulo a prime number $p$. Since the polynomial $P_I$ may be reducible over $\mathbb{Z}_p$, the degree $k$ of the finite field is some value between 1 and $k_0$ and equal to the degree of one the irreducible factors of $P_I$. Let $D_1, \ldots, D_f$ be the factors of $P_I$. Each $D_i$ is associated with a finite field $\mathbb{Z}_p[X]/D_i \cong \text{GF}(p^{k_i})$, where $k_i$ is the degree of $D_i$. Smaller values of $k$ take to a computational advantage, as the size $p^{kM}$ of the input space $\Omega$ is smaller and the probability, about $N_P/p^{kM}$, of getting the factor $p$ is higher. For example, the cyclotomic number field with $\alpha = e^{2\pi i/n}$ has a degree equal to $\phi(n)$, where $\phi$ is the Euler totient function, which is asymptotically lower-bounded by $Kn/\log \log n$, for every constant $K < e^{-\gamma}$, $\gamma$ being the Euler constant. In other words, the highest degree of the polynomial prime factors of $x^n - 1$ is equal to $\phi(n)$. Let $P_I$ be equal to the factor with $e^{2\pi i/n}$ as root. If $n$ is a divisor of $p - 1$ for some prime number $p$, then $P_I$ turns out to have linear factors over $\mathbb{Z}_p$. Thus, the degree of the number field collapses to 1 when mapped to a finite field with characteristic $p$. Thus, the bound $k_0$ sets a worst case.

For general number fields, the equality $k = k_0$ is more an exception than a rule, apart from the case of the rational field, for which $k = k_0 = 1$. For the sake of simplicity, let us assume for the moment that $k = k_0$ for one of the two factors of $c$, say $p'$. Algorithm II is modified as follows. The map is chosen at point (1) of Algorithm II such that $N_P \simeq c^{k_0M/2}$; the value $m$ computed at point (3) is a polynomial over $\mathbb{Z}/c\mathbb{Z}$ of degree $k_0 - 1$; the greatest common divisor at point (4) is computed between one of the coefficients of the polynomial $m$ and $c$. If the degree $k$ of the finite field of characteristic $p$ turns out to be smaller than $k_0$, we have to compute the polynomial greatest common divisor between $m$ and $P_I$ by pretending again that $\mathbb{Z}/c\mathbb{Z}$ is a field. If $m$ is zero over $\text{GF}(p^k)$, then the Euclidean algorithm generates at some point a residual polynomial with the leading coefficient having $p$ as a factor (generally, all the coefficients turn out to have $p$ as a factor). If $k \neq k_0$ for both factors and most of the maps, then the algorithm ends up to generate the trivial factor $c$, so that we need to decrease $N_P$ until a non-trivial factor is found.

B. Complexity analysis

The computational cost of the algorithm grows linearly with the product between the computational cost of the map, say $C_0(\mathcal{R})$, and the average number of trials, which is roughly $p^{k_0M}/N_P$ provided that $N_P \ll p^{k_0M}$ and $P_I$ is irreducible over $\mathbb{Z}_p$. The class $\Theta$ may contain many maps with a given number $N_P$ of zeros over some number field. We can choose the optimal map for each
$N_P$, so that we express $k_0$, $M$ and $R$ as functions of $\log N_P \equiv \xi$. The computational cost $C_0(R)$ is written as a function of $\xi$, $C_0(\xi)$.

Let us evaluate the computational complexity of the algorithm in terms of the scaling properties of $k_0(\xi)$, $M(\xi)$ and $C_0(\xi)$ as functions of $\xi = \log N_P$. The complexity $C_0(\xi)$ is expected to be a monotonically increasing function. If the functions $k_0(\xi)$ and $M(\xi)$ were decreasing, then they would asymptotically tend to a constant, since they are not less than 1. Thus, we assume that these two functions are monotonically increasing or constant.

As previously said, the polynomial $N$ has typically about $N_P$ distinct roots over GF($p^{k_0}$), provided that $N_P$ is sufficiently smaller than $p^{k_0}M$. If $N_P$ is greater than $p^{k_0}M$, then almost every value of $\vec{\tau}$ is a zero of the polynomial. Assuming that the zeros are somehow randomly distributed, the probability that a number picked up at random is different from any zero over GF($p^{k_0}$) is equal to $(1 - p^{-k_0M})^{N_P}$. Thus, the number of roots over GF($p^{k_0}$) is expected to be of the order of $p^{k_0M}[1 - (1 - p^{-k_0M})^{N_P}]$, which is about $N_P$ for $N_P \ll p^{k_0M}$. Thus, the average number of trials required for getting a zero is

$$N_{\text{trials}} \equiv \frac{1}{1 - (1 - p^{-k_0M})^{N_P}}, \quad (11)$$

A trial is successful if it gives zero modulo some nontrivial factor of $c$, thus the number of required trials can be greater than $N_{\text{trials}}$ if some factors are close each other. Let us consider the worst case with $c = pp'$, where $p$ and $p'$ are two primes with $p' \simeq p$ such that $(p')^{k_0M} \simeq p^{k_0M}$. Assuming again that the roots are randomly distributed, the probability of a successful trial is $Pr_{\text{succ}} \equiv 2[1 - (1 - p^{-k_0M})^{N_P}](1 - p^{-k_0M})^{N_P}$. The probability has a maximum equal to $1/2$ for $\xi$ equal to the value

$$\xi_0 \equiv \log \left[ \frac{\log 2}{\log(1 - p^{-k_0M})} \right]. \quad (12)$$

Evaluating the Taylor series at $p = \infty$, we have that

$$\xi_0 = k_0M \log p + \log \log 2 - \frac{1}{2p^{k_0M}} + O(p^{-2k_0M}). \quad (13)$$

The first two terms give a very good approximation of $\xi_0$. At the maximum, the ratio between the number of zeros and the number of states $p^{k_0M}$ of the sampling space is about $\log 2$. It is worth to note that, for the same value of $\xi$, the probability of getting an isolated factor with $p \ll p'$ is again exactly $1/2$. Thus, we have in general

$$N_P \simeq 0.69p^{k_0M} \Rightarrow Pr_{\text{succ}} = 1/2. \quad (14)$$
Since the maximal probability is independent of $k_0$ and $M$, this value is also maximal if $k_0$ and $M$ are taken as functions of $\xi$. The maximal value $\xi_0$ is solution of the equation

$$\xi_0 = \log \left( \frac{\log 2}{\log (1 - p^{-f(\xi_0)})} \right).$$

(15)

where $f(\xi) \equiv k_0(\xi)M(\xi)$. If the equation has no positive solution, then the probability is maximal for $\xi = 0$. That is, the optimal map in the considered class is the one with $NP = 1$. This means that the number of states $p^{k_0M}$ of the sampling space grows faster than the number of zeros. In particular, there is no solution for $\log p$ sufficiently large if $f(\xi)$ grows at least linearly (keep in mind that $f(\xi) \geq 1$). Thus, the function $f(\xi)$ has to be a sublinear power function, as previously said.

The computational cost of the algorithm for a given map $\mathcal{R}(\xi)$ is

$$C(p, \xi) \equiv \frac{C_0(\xi)}{2[1 - (1 - p^{-f(\xi)})^\exp \xi](1 - p^{-f(\xi)})^\exp \xi}.\quad (16)$$

The optimal map for given $p$ is obtained by minimizing $C(p, \xi)$ with respect to $\xi$. The computational complexity of the algorithm is

$$C(p) = \min_{\xi > 0} C(p, \xi) \equiv C(p, \xi_m),\quad (17)$$

which satisfies the bounds

$$C_0(\xi_m) \leq C(p) \leq 2C_0(\xi_0).\quad (18)$$

The upper bound in Eq. (15) is the value of $C(p, \xi)$ at $\xi = \xi_0$, whereas the lower bound is the computational complexity of the map at the minimum $\xi_m$.

It is intuitive that the complexity $C_0(\xi)$ must be subexponential in order to have $C(p)$ subexponential in $\log p$. This can be shown by contradiction. Suppose that the complexity $C(p)$ is subexponential in $\log p$ and $C_0(\xi) = \exp(a\xi)$ for some positive $a$. The lower bound in Eq. (18) implies that the optimal $\xi_m$ grows less than $\log p$. Asymptotically,

$$\left. \frac{p^{k_0M}}{NP} \right|_{\xi = \xi_m} \sim e^{f(\xi_m) \log p - \xi_m} \geq Kp^{1/2},\quad (19)$$

for some constant $K$. Thus, the average number of trials grows exponentially in $\log p$, implying that the computational complexity is exponential, in contradiction with the premise. Since $f(\xi)$ and $\log C_0(\xi)$ must grow less than linearly, we may assume that they are concave.

**Property 1.** The functions $f(\xi)$ and $\log C_0(\xi)$ are concave, that is,

$$\frac{d^2}{d\xi^2} f(\xi) \leq 0,\quad \frac{d^2}{d\xi^2} \log C_0(\xi) \leq 0.$$

(20)
The lower bound in Eq. (18) depends on \( \xi_m \), which depends on the function \( C_0(\xi) \). A tighter bound which is also simpler to evaluate can be derived from Property 1 and the inequality

\[
C(p, \xi) \geq \frac{1}{2} e^{f(\xi) \log p - \xi} C_0(\xi). \tag{21}
\]

**Lemma 1.** If Property 1 holds and \( C(p) \) is asymptotically sublinear in \( p \), then there is an integer \( \bar{p} \) such that the complexity \( C(p) \) is bounded from below by \( \frac{C_0(\xi_0)}{2 \log 2} \) for \( p > \bar{p} \).

**Proof.** The minimum \( \xi_m \) is smaller than \( \xi_0 \), since the function \( C_0(\xi) \) is monotonically increasing. Thus, we have

\[
C(p) = \min_{\xi \in \{0, \xi_0\}} C(p, \xi) \geq \min_{\xi \in \{0, \xi_0\}} e^{f(\xi) \log p - \xi + \log C_0(\xi) / 2}. \tag{22}
\]

Since the exponential is monotonic and the exponent is concave, the objective function has a maximum and two local minima at the \( \xi = 0 \) and \( \xi = \xi_0 \). Keeping in mind that \( f(\xi) \geq 1 \), The first local minimum is not less than \( pC_0(0)/2 \). The second minimum is \( e^{f(\xi_0) \log p - \xi_0} C_0(\xi_0)/2 \), which is greater than or equal to \( C_0(\xi_0)/(2 \log 2) \). This can be proved by eliminating \( p \) through Eq. (15) and minimizing in \( \xi_0 \). Since \( C(p) \) is sublinear in \( p \), there is an integer \( \bar{p} \) such that the second minimum is global for \( p > \bar{p} \). \( \square \)

Summarizing, we have

\[
0.72 C_0(\xi_0) \leq C(p) \leq 2 C_0(\xi_0) \tag{23}
\]

for \( p \) greater than some integer. Thus, the complexity analysis of the algorithm is reduced to study the asymptotic behavior of \( C_0(\xi_0) \). The upper bound is asymptotically tight, that is, \( \xi = \xi_0 \) is asymptotically optimal. Taking

\[
f(\xi) = b \xi^\beta \text{ with } \beta \in [0 : 1),
\]

the optimal value of \( \xi \) is

\[
\xi_0 = (b \log p)^{1/\beta} + O(1).
\]

The function \( f(\xi) \) cannot be linear, but we can take it very close to a linear function,

\[
f(\xi) = b \frac{\xi}{(\log \xi)^\gamma}, \quad \gamma > 1. \tag{24}
\]

In this case, the optimal \( \xi \) is

\[
\xi_0 = e^{(b \log p)^{1/\beta}} + O \left[ (\log p)^{1/\beta} \right].
\]

There are three scenarios taking to subexponential or polynomial complexity.
(a) The functions $C_0(\xi)$ and $f(\xi)$ scale polynomially as $\xi^\alpha$ and $\xi^\beta$, respectively, with $\beta \in [0 : 1)$. Then, the computational complexity $C(p)$ scales polynomially in $\log p$ as $(\log p)^{\frac{\alpha}{1-\beta}}$.

(b) The function $C_0(\xi)$ is polynomial and $f(\xi) \sim \xi/(\log \xi)^\beta$ with $\beta > 1$. Then the computational complexity $C(p)$ scales subexponentially in $\log p$ as $[b(\log p)^{1/\beta}]$.

(c) The function $C_0(\xi)$ and $f(\xi)$ are superpolynomial and polynomial respectively, with $C_0(\xi) \sim \exp[b\xi^\alpha]$ and $f(\xi) \sim \xi^\beta$. If $\alpha + \beta < 1$, then the complexity $C(p)$ is subexponential in $\log p$ and scales as $[b(\log p)^{\frac{\alpha}{1-\beta}}]$.

The algorithm has polynomial complexity in the first scenario. The other cases are subexponential. This is also implied by the following.

**Lemma 2.** The computational complexity $C(p)$ is subexponential or polynomial in $\log p$ if the function $C_0(\xi)f(\xi)$ grows less than exponentially, that is, if

$$\lim_{\xi \to \infty} \frac{f(\xi) \log C_0(\xi)}{\xi} = 0.$$ 

In particular, the complexity is polynomial if $C_0(\xi)$ is polynomial and $f(\xi)$ scales sublinearly.

This lemma can be easily proved directly from Eq. (15) and the upper bound in Eq. (18), the former implying the inequality $\xi_0 \leq f(\xi_0) \log p + \log \log 2$. Let us prove the first statement.

$$\lim_{p \to \infty} \frac{\log C(p)}{\log p} \leq \lim_{\xi \to \infty} \frac{f(\xi) \log 2C_0(\xi)}{\xi - \log log 2} = \lim_{\xi \to \infty} \frac{f(\xi) \log C_0(\xi)}{\xi} = 0.$$ 

Using the lower bound in Eq. (23), the lemma can be strengthened by adding the inferences in the other directions (if replaced by if and only if).

Summarizing, we have the following.

**Claim 2.** The factoring algorithm has subexponential (polynomial) complexity if, for every $\xi = \log N_P > 0$ with $N_P$ positive integer, there are rational univariate functions $R(\bar{\tau}) = \frac{N(\bar{\tau})}{D(\bar{\tau})}$ of the parameters $\bar{\tau} = (\tau_1, \ldots, \tau_{M(\xi)})$ over an algebraic number field $\mathbb{Q}(\alpha)$ of degree $k_0(\xi)$ with polynomials $N$ and $D$ coprime, such that

1. the number of distinct roots of $N$ in $\mathbb{Q}(\alpha)$ is equal to about $N_P$. Most of the roots remain distinct when interpreted over finite fields of order equal to about $N_P^{1/M}$;

2. given any value $\bar{\tau}$, the computation of $R(\bar{\tau})$ takes a number $C_0(\xi)$ of arithmetic operations growing less than exponentially (polynomially) in $\xi$. 

3. the function $C_0(\xi)^{k_0(\xi)M(\xi)}$ is subexponential (the function $k_0(\xi)M(\xi)$ scales sublinearly).

Let us stress that the asymptotic complexity is less than exponential if and only if $C_0(\xi)^{f(\xi)}$ is less than exponential. Thus, the latter condition is a litmus test for a given class of rational functions. However, the function $C_0(\xi)^{f(\xi)}$ does not provide sufficient information on the asymptotic computational complexity of the factoring algorithm. The general number-field sieve is the algorithm with the best asymptotic complexity, which scales as $e^{a \log p)^{1/3}}$. Thus, algorithm \[ \text{is asymptotically more efficient than the general number-field sieve if } C_0(\xi) \text{ and } f(\xi) \text{ are asymptotically upper-bounded by a subexponential function } e^{b(\log p)^{\alpha}} \text{ and a power function } c\xi^\beta, \text{ respectively, such that } \alpha < (1 - \beta)/3. \text{ In the limit case of } \beta \to 1 \text{ and polynomial complexity of the map, the function } f(\xi) \text{ must be asymptotically upper-bounded by } b\xi/(\log \xi)^3. \]

C. Number of rational zeros versus polynomial degree

Previously we have set upper bounds on the required computational complexity of the rational function $R = N/D$ in terms of the number of its rational zeros. For a polynomial (subexponential) complexity of prime factorization, the computational complexity $C_0$ of $R$ must scale polynomially (subexponentially) in the logarithm in the number of rational zeros. Thus, for a univariate rational function, it is clear that $C_0$ has to scale polynomially (subexponentially) in the logarithm of the degree $d$ of $N$, since the number of rational zeros is upper-bounded by the degree (fundamental theorem of algebra). An extension of this inference to multivariate functions is more elaborate, as upper bounds on the number of rational zeros are unknown. However, we are interested more properly to a set of $N_P$ rational zeros that remain in great part distinct when reinterpreted over a finite field whose order is greater than about $N_P^{1/M}$. Under this restriction, let us show that the number of rational zeros of a polynomial of degree $d$ and with $M$ variables is upper-bounded by $Kd^{2M}$ with some constant $K > 0$. This bound allows us to extend the previous inference on $C_0$ to the case of multivariate functions.

Assuming that the $N_P$ rational zeros over $\mathbb{Q}$ are randomly distributed when reinterpreted over $\text{GF}(q)$, their number over the finite field is about $q^M \left[ 1 - (1 - q^{-M})^{N_P} \right]$, as shown previously. Since an upper bound on the number of zeros $N(q)$ of a smooth hypersurface over a finite field of order $q$ is known, we can evaluate an upper bound on $N_P$. Given the inequality \[ \text{[10]} \]

$$N(q) \leq \frac{q^M - 1}{q - 1} + [(d - 1)^M - (-1)^M] (1 - d^{-1}) q^{(M-1)/2} \quad (25)$$
and

\[ q^M [1 - (1 - q^{-M})^{N_P}] \leq N(q), \]  

(26)

we get an upper bound on \( N_P \) for each \( q \). Requiring that Eq. (26) is satisfied for every \( q > N_P^{1/M} \), we get

\[ N_P < K d^{2M^2/M} < Kd^{2M} \]  

(27)

for some constant \( K \) (the same result is obtained by assuming that Eq. (26) holds for every \( q \)). Note that a slight break of bound (27) with \( N_P \) growing as \( d_0^{M^a} \) in \( M \) for some particular \( d = d_0 \) and \( a > 1 \) would make the complexity of prime factorization polynomial, provided the computational complexity of evaluating the function \( R \) is polynomial in \( M \). This latter condition can be actually fulfilled, as shown with an example later. Ineq. (25) holds for smooth irreducible hypersurfaces. However, dropping these conditions are not expected to affect the bound (27). For example, if \( M = 2 \), then Ineq. (25) gives

\[ N_P \leq q + 1 + (d - 1)(d - 2)\sqrt{q} \]  

(28)

which is the Weil’s upper bound (6) for a smooth plane curve, whose geometric genus \( g \) is equal to \((d - 1)(d - 2)/2\). This inequality holds also for singular curves [8]. Indeed, this comes from the upper bound (17) and the equality \( g = (d - 1)(d - 2)/2 - \delta \). Also reducibility does not affect Ineq (27).

It is simple to find examples of multivariate functions with a number of rational zero quite close to the bound \( Kd^{2M} \). Trivially, there are polynomials \( \mathcal{N}(\tau_1, \ldots, \tau_M) \) of degree \( d \) with a number of rational zeros at least equal to the number of coefficients minus 1, that is, equal to

\( \tilde{N}_P \equiv M!^{-1} \prod_{k=1}^{M} (d + k) - 1 \sim d^M/M! + O(d^{M-1}) \). For \( M = 1 \), this corresponds to take the univariate polynomial

\[ \mathcal{N}(\tau) = (\tau - x_1)(\tau - x_2) \cdots (\tau - x_d). \]  

(29)

A better construction of a multivariate polynomial is a generalization of the univariate polynomial in Eq. (29). Given linear functions \( L_{i,s}(\vec{\tau}) \), the polynomial

\[ \tilde{P} = \sum_{i=1}^{M} \prod_{s=1}^{d} L_{i,s}(\vec{\tau}) \]

has generally a number of rational points \( N_P \) at least equal to \( d^M \), which is the square root of the upper bound, up to a constant. For \( d < 4 \) and \( M = 2 \), the number of rational zeros turns out to be
infinite, since the genus is smaller than 2 (see Sec. II C 1 for the case of infinite rational points). A naive computation of $\tilde{P}(\tau)$ takes $dM^2$ arithmetic operations, that is, its complexity is polynomial in $M$. This example provides an illustration of the complexity test described previously in Claim 2.

Expressing $d$ in terms of $M$ and $\xi = \log N_P$ and assuming that $C_0 \sim dM^2$, we have that

$$C_0(\xi) = M^2 e^{M^{-1}\xi},$$

which is subexponential in $\xi$ (provided that $M$ is a growing function of $\xi$), which is a necessary condition for a subexponential algorithm. However, the polynomial does not pass the litmus test, as $C_0(\xi)^M$ grows exponentially.

1. The case of infinite rational zeros

Until now, we have assumed that the rational function has a finite number of rational zeros over the rationals. However, in the multivariate case, it is possible to have non-zero functions with an infinite number of zeros. For example, this is the case of a bivariate polynomials with genus equal to zero and one, which correspond to parametrizable curves and elliptic curves, respectively. We can also have functions whose zero locus contains linear subspaces with positive dimension, which can have infinite rational points. Since the probability of having $R$ equal to zero modulo $p$ increases with the number of zeros over the rationals, this would imply that the probability is equal to 1 if the number of zeros is infinite. This is not evidently the case. For example, if $R$ is zero for $x_1 = 0$ and $M > 1$, evidently the function has infinite rational points over $\mathbb{Q}$, but the number of points with $x_1 = 0$ over $\mathbb{Z}_p$ is $p^{M-1}$, which is $p$ times less than the number of points in the space. Once again, we are interested more properly to sets of $N_P$ rational zeros over $\mathbb{Q}$ such that a large fraction of them remain distinct over finite fields whose order is greater than about $N_P^{1/M}$. Under this condition, $N_P$ cannot be infinite and is constrained by Ineq. (27). If there are linear subspaces with dimension $h > 0$ in the zero locus of $R$, we may fix some of the parameters $\tau$, so that these spaces become points. In the next sections, we will build rational functions having isolated rational points and possible linear subspaces in the zero locus. If there are subspaces with dimension $k > 0$ giving a dominant contribution to factorization, we can transform them to isolated rational points by fixing some parameters without changing the asymptotic complexity of the algorithm. Isolated rational points are the only relevant points for an asymptotic study of the complexity of the factoring algorithm, up to a dimension reduction. Thus, we will consider only them and will not care of the other linear subspaces.
III. SETTING THE PROBLEM IN THE FRAMEWORK OF ALGEBRAIC GEOMETRY

Since the number of zeros \( N_P \) is constrained by Ineq. \((27)\), the complexity of computing the rational function \( R(\vec{\tau}) \) must be subexponential or polynomial in \( \log d \) in order to have \( C_0(\xi) \) subexponential or polynomial. This complexity scaling is attained if, for example, \( R \) is a polynomial with few monomials. The univariate polynomial \( P = \tau^d - 1 \), which is pivotal in Pollard’s \( p - 1 \) algorithm, can be evaluated with a number of arithmetic operations scaling polynomially in \( \log d \). This is achieved by consecutively applying polynomial maps. For example, if \( d = 2^g \), then \( \tau^d \) is computed through \( g \) applications of the map \( x \rightarrow x^2 \) by starting with \( x = \tau \). However, polynomials with few terms have generally few zeros over \( \mathbb{Q} \). More general polynomials and rational functions with friendly computational complexity are obviously available and are obtained by consecutive applications of simple functions, as done for \( \tau^d - 1 \). This leads us to formulate the factorization problem in the framework of algebraic geometry as an intersection problem.

A. Intersection points between a parametrizable variety and a hypersurface

Considering only the operations defined in the field, the most general rational functions \( R(\vec{\tau}) \) with low complexity can be evaluated through the consecutive application of a small set of simple rational equations of the form

\[
\begin{align*}
  x_{n-M} &= \frac{N_{n-M}(x_{n-M+1}, \ldots, x_n)}{D_{n-M}(x_{n-M+1}, \ldots, x_n)} \\
  x_{n-M-1} &= \frac{N_{n-M-1}(x_{n-M}, \ldots, x_n)}{D_{n-M-1}(x_{n-M}, \ldots, x_n)} \\
  &\quad \vdots \\
  x_1 &= \frac{N_1(x_2, \ldots, x_n)}{D_1(x_2, \ldots, x_n)} \\
  R &= P_0(x_1, \ldots, x_n),
\end{align*}
\]

where \( P_0 \) is a polynomial. If the numerators and denominators \( N_k \) and \( D_k \) do not contain too many monomials, then the computation of \( N_k/D_k \) can be performed efficiently. Assuming that the computational complexity of these rational functions is polynomial in \( n \), the complexity of \( R \) is polynomial in \( n \). The computation of \( R(\vec{\tau}) \) is performed by setting the last \( M \) components \( x_{n-M+1}, \ldots, x_n \) equal to \( \tau_1, \ldots, \tau_M \) and generating the sequence \( x_{n-M}, x_{n-M-1}, \ldots, x_1, R \) according to Eqs. \((30)\), which ends up with the value of \( R \). The procedure may fail to compute the right value of \( R(\vec{\tau}) \) if some denominator \( D_k(\vec{\tau}) \equiv D_k[x_{k+1}(\vec{\tau}), \ldots, x_n(\vec{\tau})] \) turns out to be equal to zero during the computation. However, since our only purpose is to generate the zero of the field, we can take a zero divisor as outcome and stop the computation of the sequence. In this way, the
algorithm generates a modified function $R'(\tau)$. Defining $N_k(\tau)$ as the numerator of the rational function $D_k(\tau)$, we have

$$R'(\tau) = \begin{cases} R(\tau) & \text{if } N_1(\tau) \ldots N_{n-M}(\tau) \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

The function $R'(\tau)$ has the zeros of $N(\tau)N_1(\tau) \ldots N_{n-M}(\tau)$. For later reference, let us define the following.

**Algorithm 2.** Computation of $R'(\tau)$.

1. set $(x_{n-M+1}, \ldots, x_n) = (\tau_1, \ldots, \tau_M)$;
2. set $k = n - M > 0$;
3. set $x_k = \frac{N_k(x_{k+1}, \ldots, x_n)}{D_k(x_{k+1}, \ldots, x_n)}$. If the division fails, return the denominator as outcome;
4. set $k = k - 1$;
5. if $k = 0$, return $P_0(x_1, \ldots, x_n)$, otherwise go back to 3.

The zeros of the denominators are not expected to give an effective contribution on the asymptotic complexity of the factoring algorithm, otherwise it would be more convenient to reduce the number of steps of the sequence by one and replace the last function $P_0$ with the denominator $D_1$. Let us show that. Let us denote by $N_1$ the number of rational zeros of $R'$ over some finite field with all the denominators different from zeros. They are the zeros returned at step 5 of Algorithm 2. Let $N_T$ be the total number of zeros. We remind that the factoring complexity is about $p^{kM}$ times the ratio between the complexity of $R$ and the number of zeros. If the algorithm is more effective than the one with one step less and $P_0$ replaced with $D_1$, then $\frac{C_1}{N_T} < \frac{C_T - C_1}{N_T - N_T}$, where $C_1$ and $C_T$ are the number of arithmetic operations of the last step and of the whole algorithm, respectively. Since $C_1 \geq 1$, we have

$$N_1 > N_T C_T^{-1}$$

In order to have a subexponential factoring algorithm, $C_T$ must scale subexponentially in $\log N_T$. Thus,

$$N_1 > N_T e^{-\alpha (\log N_T)^\beta}$$

for some positive $\alpha$ and $0 < \beta < 1$. That is,

$$\log N_1 > \log N_T - \alpha (\log N_T)^\beta.$$
If we assume polynomial complexity, we get the tighter bound

$$\log N_1 > \log N_T - \alpha \log \log N_T.$$  

These inequalities imply that the asymptotic complexity of the factoring algorithm does not change if we discard the zero divisors at Step 3 in Algorithm 2. Thus, for a complexity analysis, we can consider only the zeros of $R' (\mathbf{\tau})$ with all the denominators $D_k (\mathbf{\tau})$ different from zero. This will simplify the subsequent discussion. Each of these zeros are associated with an $n$-tuple $(x_1, \ldots, x_n)$, generated by Algorithm 2 and solutions of Eqs. (30). Let us denote by $Z_P$ the set of these $n$-tuples.

By definition, an element in $Z_P$ is a zero of the set of $M - n + 1$ polynomials

$$P_0 (x_1, \ldots, x_n),$$

$$P_k (x_1, \ldots, x_n) = x_k D_k (x_{k+1}, \ldots, x_n) - N_k (x_{k+1}, \ldots, x_n), \quad k \in \{1, \ldots, n - M\}.$$  

(32)

The last $n - M$ polynomials define an algebraic set of points, say $A$, having one irreducible branch parametrizable by Eqs. (30). This branch defines an algebraic variety which we denote by $V$. The algebraic set may have other irreducible components which do not care about. The polynomial $P_0$ defines a hypersurface, say $H$. Thus, the set $Z_P$ is contained in the intersection between $V$ and $H$. This intersection may contain singular points of $V$ with $D_k (x_{k+1}, \ldots, x_n) = 0$ for some $k$, which are not relevant for a complexity analysis, as shown previously. Thus, the factorization problem is reduced to search for non-singular rational points of a parametrizable variety $V$, whose intersection with a hypersurface $H$ contains an arbitrarily large number $N_P$ of rational points. If $N_P$ and $C_0$ scale exponentially and polynomially in the space dimension $n$, respectively, then the complexity of factorization is polynomial, provided that the number of parameters $M$ scales sublinearly as a power of $n$. In the limit case of

$$M \sim n/(\log n)^{\beta}$$  

(33)

with $\beta > 1$, the complexity scales subexponentially as $e^{\beta (\log n)^{1/\beta}}$. Thus, if $M$ has the scaling property (33) with $\beta > 3$, then there is an algorithm outperforming asymptotically the general number field sieve. A subexponential computational complexity is also obtained if the complexity of evaluating a point in $V$ is subexponential.

The parametrization of $V$ is a particular case of rational parametrization of a variety. We call it Gaussian parametrization since the triangular form of the polynomials $P_1, \ldots, P_{n-M}$ resembles Gaussian elimination. Note that this form is invariant under the transformation

$$P_k \rightarrow P_k + \sum_{k' = k+1}^{n-M} \omega_{k,k'} P_{k'}.$$  

(34)
The form is also invariant under the variable transformation

\[ x_k \rightarrow x_k + \sum_{k'=k+1}^{n+1} \eta_{k,k'} x_{k'} \]  

(35)

with \( x_{n+1} = 1 \).

It is interesting to note that if \( N'_P \) out of the \( N_P \) points in \( Z_P \) are collinear, then it is possible to build another variety with Gaussian parametrization and a hypersurface over a \((n-1)\)-dimensional subspace such that their intersection contains the \( N'_P \) points. For later reference, let us state the following.

**Lemma 3.** Let \( Z_P \) be the set of common zeros of the polynomials (32) with \( D_k(x_{k+1}, \ldots, x_n) \neq 0 \) over \( Z_P \) for \( k \in \{1, \ldots, n-M\} \). If \( N'_P \) points in \( Z_P \) are solutions of the linear equation \( L(x_1, \ldots, x_n) = 0 \), then there is a variety with Gaussian parametrization and a hypersurface over the \((n-1)\)-dimensional subspace defined by \( L(x_1, \ldots, x_n) = 0 \) such that their intersection contains the \( N'_P \) points.

**Proof.** Given the linear function \( L(x_1, \ldots, x_n) \equiv l_{n+1} + \sum_{k=1}^{n} l_k x_k \), let us first consider the case with \( l_k = 0 \) for \( k \in \{1, \ldots, n-M\} \). Using the constraint \( L = 0 \), we can set one of the \( M \) variables \( x_{n-M+1}, \ldots, x_n \) as a linear function of the remaining \( M-1 \) variables. Thus, we get a new set of polynomials retaining the original triangular form. The new parametrizable variety, say \( V' \), has \( M-1 \) parameters. The intersection of \( V' \) with the new hypersurface over the \((n-1)\)-dimensional space contains the \( N'_P \) points. Let us now consider the case with \( l_k = 0 \) for \( k \in \{1, \ldots, \hat{k}\} \), where \( \hat{k} \) is some integer between 0 and \( n-M-1 \), such that \( l_{\hat{k}+1} \neq 0 \). We can use the constraint \( L = 0 \) to set \( x_{\hat{k}+1} \) as a linear function of \( x_{\hat{k}+2}, \ldots, x_n \). We discard the polynomial \( P_{\hat{k}+1} \) and eliminate the \((\hat{k}+1)\)-th variable from the remaining polynomials. We get \( n-1 \) polynomials retaining the original triangular form in \( n-1 \) variables \( x_1, \ldots, x_{\hat{k}}, x_{\hat{k}+2}, \ldots, x_n \). The intersection between the new parametrizable variety and the new hypersurface contain the \( N'_P \) points. □

This simple lemma will turn out to be a useful tool in different parts of the paper.

In Section IIIB, we show how to build a set of polynomials \( P_k \) with a given number \( N_P \) of common rational zeros by using some tools of algebraic geometry described in Appendix A. In Sec. IV, we close the circle by imposing the form (32) for the polynomials \( P_k \) with the constraint that \( D_k(x_{k+1}, \ldots, x_n) \neq 0 \) for \( k \in \{1, \ldots, n-M\} \) over the set of \( N_P \) points.
B. Sets of polynomials with a given number of zeros over a number field

In this subsection, we build polynomials with given \( N_P \) common rational zeros as elements of an ideal \( I \) generated by products of linear functions. This construction is the most general. The relevant information on the ideal \( I \) is summarized by a satisfiability formula in conjunctive normal form without negations and a linear matroid. The formula and the matroid uniquely determine the number \( N_P \) of rational common zeros of the ideal. Incidentally, we also show that the information can be encoded in a more general formula with negations by a suitable choice of the matroid.

Every finite set of points in an \( n \)-dimensional space is an algebraic set, that is, they are all the zeros of some set of polynomials. More generally, the union of every finite set of linear subspaces is an algebraic set. In the following, we will denote linear polynomials by a symbol with a hat; namely, \( \hat{a} \) is meant as \( a_{n+1} + \sum_{i=1}^{n} a_i x_i \). Let us denote by \( \vec{x} \) the \((n+1)\)-dimensional vector \( (x_1, \ldots, x_n, x_{n+1}) \), where \( x_{n+1} \) is an extra-component that is set equal to 1. A linear polynomial \( \hat{a} \) is written in the form \( \vec{a} \cdot \vec{x} \). Let \( V_1, \ldots, V_L \) be a set of linear subspaces and \( I_1, \ldots, I_L \) their associated radical ideals. The codimension of the \( k \)-th subspace is denoted by \( n_k \). The minimal set of generators of the \( k \)-th ideal contains \( n_k \) independent linear polynomials, say \( \hat{a}_{k,1}, \ldots, \hat{a}_{k,n_k} \), so that

\[ \vec{x} \in V_k \Leftrightarrow \hat{a}_{k,i} \cdot \vec{x} = 0 \quad \forall i \in \{1, \ldots, n_k\}. \]  
(36)

If the codimension \( n_k \) is equal to \( n \), then \( V_k \) contains one point. We are mainly interested to these points, whose number is taken equal to \( N_P \). The contribution of higher dimensional subspaces to the asymptotic complexity of the factoring algorithm is irrelevant up to a dimension reduction (see also Sec. [II C 1] and the remark in the end of the section). Since only isolated points are relevant, we could just consider ideals whose zero loci contain only isolated points. However, we allow for the possible presence of subspaces with positive dimension since they may simplify the set of the generators or the form of the polynomials \( P_k \) that eventually we want to build.

Let \( \mathcal{Z} \) be the union of the subspaces \( V_k \). The product \( I_1 \cdot I_2 \cdot \ldots \cdot I_L \equiv \tilde{I} \) is associated with \( \mathcal{Z} \), that is, \( \mathcal{Z} = \mathbf{V}(\tilde{I}) \). A set of generators of the ideal \( \tilde{I} \) is

\[ \prod_{k=1}^{L} \hat{a}_{k,i_k} \equiv G_{i_1, \ldots, i_L}(\vec{x}) \quad i_r \in \{1, \ldots, n_r\}, r \in \{1, \ldots, L\}. \]  
(37)

Thus, we have that

\[ \vec{x} \in \mathcal{Z} \Leftrightarrow G_{i_1, \ldots, i_L}(\vec{x}) = 0 \quad i_r \in \{1, \ldots, n_r\}, r \in \{1, \ldots, L\}. \]  
(38)

Polynomials in the ideal \( \tilde{I} \) are zero in the set \( \mathcal{Z} \). This construction is not the most general, as \( \tilde{I} \) is not radical. Thus, there are polynomials that are not in \( \tilde{I} \), but their zero locus contains \( \mathcal{Z} \).
Furthermore, the number of generators and the number of their factors grow polynomially and linearly in $N_P$, respectively. This makes it hard to build polynomials in $\tilde{I}$ whose complexity is polynomial in $\log N_P$. The radicalization of the ideal and the assumption of special arrangements of the subspaces in $\mathcal{Z}$ can reduce drastically both the degree of the generators and their number.

For example, let us assume that $V_1$ and $V_2$ are two isolated points in the $n$-dimensional space and, thus, $n_1 = n_2 = n$. The overall number of generators $a_{1,1}, \ldots, a_{1,n}$ and $a_{2,1}, \ldots, a_{2,n}$ is equal to $2n$. Thus, there are $n - 1$ linear constraints among the generators. Using linear transformations, we can write these constraints as

$$\hat{a}_{1,i} = \hat{a}_{2,i} \equiv \hat{a}_i \quad \forall i \in \{2, \ldots, n\}.$$ 

Every generator $G_{i,i_3,\ldots,i_L}$ with $i \neq 1$ is equal to $\tilde{G}_{i,i_3,\ldots,i_L} = a_i^2 \prod_{k=3}^L \hat{a}_{k,i_k}$. The polynomial $\tilde{G}_{i,i_3,\ldots,i_L} \equiv a_i \prod_{k=3}^L \hat{a}_{k,i_k}$ is not an element of the ideal $\tilde{I}$, but it is an element of its radical. Indeed, it is zero in the algebraic set $\mathcal{Z}$. Thus, we extend the ideal by adding these new elements. This extension allows us to eliminate all the generators $G_{i_1,i_2,\ldots,i_L}$ with $i_1 = 1$ or $i_2 = 1$, since they are generated by $\tilde{G}_{i,i_3,\ldots,i_L}$. Thus, the new ideal has the generators,

$$\begin{cases} 
\hat{a}_{1,1} \hat{a}_{2,1} \prod_{k=3}^L a_{k,i_k} \\
\hat{a}_i \prod_{k=3}^L a_{k,i_k}, & i \in \{2, \ldots, n\}, r \in \{3, \ldots, L\} 
\end{cases}$$

Initially, we had $n^2 \prod_{k=3}^L n_k$ generators. Now, their number is $n \prod_{k=3}^L n_k$. A large fraction of them has the degree reduced by one. We can proceed with the other points and further reduce both degrees and number of generators. Evidently, this procedure cannot take to a drastic simplification of the generators if the points in $\mathcal{Z}$ are in general position, since the generators must contain the information about these positions. A simplification is possible if the points have special arrangements taking to contraction of a large number of factors in the generators. Namely, coplanarity of points is the key feature that can take to a drastic simplification of the generators.

In a $n$-dimensional space, there are at most $n$ coplanar points in general position. Let us consider algebraic sets containing larger groups of coplanar points. For example, let us assume that the first $m$ sets $V_1, \ldots, V_m$ are distinct coplanar points, with $m \gg n$. Then, there is a vector $\vec{a}_1$ such that $\vec{a}_1 \cdot \vec{x} = 0$ for every $\vec{x}$ in the union of the first $m$ linear spaces. It is convenient to choose the linear polynomial $\vec{a}_1 \cdot \vec{x}$ as common generator of the first $m$ ideals $I_1, \ldots, I_m$. Let us set $\hat{a}_{k,1} = \hat{a}_1$ for $k \in \{1, \ldots, m\}$. Every generator $G_{i_1,\ldots,i_L}$ with $i_k = 1$ for some $k \in \{1, \ldots, m\}$ is contracted to a generator of the form $\hat{a} \prod_{k=m+1}^L \hat{a}_{k,i_k}$. If there are other groups of coplanar points, we can perform other contractions.
Definition 1. Given an integer $\bar{n} > n$, we define $\Gamma_s$ with $s \in \{1, \ldots, \bar{n}\}$ as a set of $s$-tuples $(i_1, \ldots, i_s) \in \{1, \ldots, \bar{n}\}^s$ with $i_k < i_{k'}$ for $k < k'$. That is,

$$\forall s \in \{1, \ldots, \bar{n}\}, \quad \Gamma_s \subseteq \{(i_1, \ldots, i_s) \in \{1, \ldots, \bar{n}\}^s | i_k < i_{k'}, \forall k, k' \text{ s.t. } k < k'\}. \quad (40)$$

The final result of the inclusion of elements of the radical ideal is another ideal, say $I$, with generators of the form

$$\hat{a}_{i_1} \quad \forall i_1 \in \Gamma_1$$
$$\hat{a}_{i_1} \hat{a}_{i_2} \quad \forall (i_1, i_2) \in \Gamma_2$$
$$\quad \cdots$$
$$\hat{a}_{i_1} \hat{a}_{i_2} \cdots \hat{a}_{i_{\bar{n}}} \quad \forall (i_1, i_2, \ldots, i_{\bar{n}}) \in \Gamma_{\bar{n}}, \quad (41)$$

where $\{\hat{a}_1, \ldots, \hat{a}_{\bar{n}}\} \equiv \Phi$ is a set of $\bar{n}$ linear polynomials. Polynomials in this form generate the most general ideals whose zero loci contain a given finite set of points. This is formalized by the following.

Lemma 4. Every radical ideal associated with a finite set $Z_P$ of points is generated by a set of polynomials of form (41) for some $\bar{n}$.

Proof. This can be shown with a naive construction. Given a set $\bar{S}$ of $N_P$ points associated with the ideals $I_1, \ldots, I_{N_P}$, the product $I_1 \cdots I_{N_P}$ is an ideal associated with the set $\bar{S}$, which can be radicalized by adding a certain number of univariate square-free polynomials as generators. The resulting ideal is generated by a set of polynomials of form (41). \(\square\)

With the construction used in the proof, $\bar{n}$ ends up to be equal to the number of points in $Z_P$, which is not optimal for our purposes. We are interested to keep $\bar{n}$ sufficiently small, possibly scaling polynomially in the dimension $n$. This is possible only if the points in the zero locus have a ‘high degree’ of collinearity. Thus, a bound on $\bar{n}$ sets a restriction on $Z$.

The minimal information on $\Phi$ that is relevant for determining the number $N_P$ of points in $Z$ is encoded in a linear matroid, of which $\Phi$ is one linear representation. Thus, the sets $\Gamma_s$ and the matroid determine $N_P$. Note that the last set $\Gamma_{\bar{n}}$ has at most one element. The linear generators can be eliminated by reducing the dimension of the affine space, see Lemma 3. Thus, we can set $\Gamma_1 = \emptyset$. Every subset $\Phi_{\text{sub}}$ of $\Phi$ is associated with a linear space $V_{\text{sub}}$ whose points are the common zeros of the linear functions in $\Phi_{\text{sub}}$. That is, $V_{\text{sub}} = V(I_{\text{sub}})$, where $I_{\text{sub}}$ is the ideal generated by $\Phi_{\text{sub}}$. Let us denote briefly by $V(\Phi_{\text{sub}})$ the linear space $V(I_{\text{sub}})$. This mapping from subsets of $\Phi$ to subspaces is not generally injective. Let $\hat{a}' \in \Phi \setminus \Phi_{\text{sub}}$ be a linear superposition of the functions in $\Phi_{\text{sub}}$, then $\Phi_{\text{sub}}$ and $\Phi_{\text{sub}} \cup \{\hat{a}'\}$ represent the same linear space. An injective mapping is obtained
by considering only the maximal subset associated with a linear subspace. These maximal subsets are called\,flats\,in matroid theory.

**Definition 2.** Flats of the linear matroid $\Phi$ are defined as subsets $\Phi_{\text{sub}} \subseteq \Phi$ such that no function in $\Phi \setminus \Phi_{\text{sub}}$ is linearly dependent on the functions in $\Phi_{\text{sub}}$.

Let us also define the closure of a subset of $\Phi$.

**Definition 3.** Given a subset $\Phi_{\text{sub}}$ of $\Phi$ associated with subspace $V$, its closure \(\text{cl}(\Phi_{\text{sub}})\) is the flat associated with $V$.

The number of independent functions in a flat is called rank of the flat. The whole set $\Phi$ is a flat of rank $n+1$, which is associated with an empty space. Flats of rank $n$ define points of the $n$-dimensional affine space (with $x_{n+1} = 1$). More generally, flats of rank $k$ define linear spaces of dimension $n-k$. The dimension of a flat $\Phi_{\text{flat}}$ is meant as the dimension of the space $V(\Phi_{\text{flat}})$.

The structure of the generators (41) resembles a Boolean satisfiability problem (SAT) in conjunctive normal form without negations. Let us interpret $\hat{a}_i$ as a logical variable $a_i$ which is true or false if the function is zero or different from zero, respectively. Every subset $\Phi_{\text{sub}} \subseteq \Phi$ is identified with a string $(a_1, \ldots, a_n)$ such that $a_i = \text{true}$ if and only if $\hat{a}_i \in \Phi_{\text{sub}}$.

The SAT formula associated with the generators (41) is

$$
\bigwedge_{k=2}^{n} \left( \bigvee_{i \in \Gamma_k} a_i \right).
$$

Given a flat $\Phi_{\text{flat}}$, the linear space $V(\Phi_{\text{flat}})$ is a subset of $\mathcal{Z}$ if and only if $\Phi_{\text{flat}}$ is solution of the SAT formula. If a set $\Phi_{\text{sub}} \subseteq \Phi$ is solution of the SAT formula, then the flat $\text{cl}(\Phi_{\text{sub}})$ is also solution of the formula. Thus, satisfiability implies that there are flats as solutions of the formula. This does not mean that satisfiability implies that $\mathcal{Z}$ is non-empty. Indeed, if the dimension of $V(\Phi_{\text{sub}})$ is negative for every solution $\Phi_{\text{sub}}$, then the set $\Phi$ is the only flat solution of the formula. We are interested to the isolated points in $\mathcal{Z}$. A point $p \in \mathcal{Z}$ is isolated if there is a SAT solution $\Phi_{\text{flat}}$ with zero dimension such that $p \in V(\Phi_{\text{flat}})$ and no flat $\Phi'_{\text{flat}} \subset \Phi_{\text{flat}}$ is solution of the Boolean formula. We denote by $\mathcal{Z}_p$ the subset in $\mathcal{Z}$ containing the isolated points. Since the number $N_P$ of isolated points is completely determined by the SAT formula and the linear matroid, the information on these two latter objects is the most relevant. Given them, the linear functions $\hat{a}_i$ have some free coefficients.

**Remark.** In general, we do not rule out sets $\mathcal{Z}$ containing subspaces with positive dimension, however these subspaces are irrelevant for the complexity analysis of the factoring algorithm. For
example, if subspaces of dimension $d_s < M$ give a dominant contribution to factorization, then we can generally eliminate $d_s$ out of the $M$ parameters by setting them equal to constants, so that the subspaces are reduced to points. Furthermore, subspaces with dimension greater than $M - 1$ are not in the parametrizable variety $\mathcal{V}$, whose dimension is $M$. Neither the overall contribution of all the subspaces with positive dimension can provide a significant change in the asymptotic complexity up to parameter deletions. Thus, only isolated points of $\mathcal{Z}$ are counted without loss of generality.

C. Boolean satisfiability and algebraic-set membership

As we said previously, the Boolean formula does not encode all the information about the number of isolated points in $\mathcal{Z}$, which also depends on the independence relations among the vectors $\vec{a}_i$, specified by the matroid. A better link between the SAT problem and the membership to $\mathcal{Z}$ can be obtained if we consider sets $\Phi$ with cardinality equal to $2^n$ and interpret half of the functions in $\Phi$ as negations of the others. Let us denote by $\hat{a}_{0,1}, \ldots, \hat{a}_{0,n}$ and $\hat{a}_{1,1}, \ldots, \hat{a}_{1,n}$ the $2n$ linear functions of $\Phi$. For general functions, we have the following.

**Property 2.** The set of vectors $\{\vec{a}_{s_1,1}, \ldots, \vec{a}_{s_n,n}, \vec{a}_{1-s_k,k}\}$ is independent for every string $\vec{s} = (s_1, \ldots, s_n) \in \{0, 1\}^n$ and every $k \in \{1, \ldots, n\}$.

This property generally holds if the functions are picked up at random. Let us assume that $\Phi$ satisfies Property 2. This implies that $\{\hat{a}_{s_1,1}, \ldots, \hat{a}_{s_n,n}\}$ are linearly independent and equal to zero at one point $\vec{x}_{\vec{s}}$. Furthermore, Property 2 also implies that different strings $\vec{s}$ are associated with different points $\vec{x}_{\vec{s}}$.

**Lemma 5.** Let $\{\vec{a}_{0,1}, \ldots, \vec{a}_{0,n}, \vec{a}_{1,1}, \ldots, \vec{a}_{1,n}\}$ be a set of $2n$ vectors satisfying Property 2. Let $\vec{x}_{\vec{s}}$ be the solution of the equations $\hat{a}_{s_1,1} = \cdots = \hat{a}_{s_n,n} = 0$. If $\vec{s} \neq \vec{r}$, then $\vec{x}_{\vec{s}} \neq \vec{x}_{\vec{r}}$.

**Proof.** Let us assume that $\vec{x}_{\vec{s}} = \vec{x}_{\vec{r}}$ with $\vec{s} \neq \vec{r}$. There is an integer $k \in \{1, \ldots, n\}$ such that $s_k \neq r_k$. Thus, the set of vectors $\{\vec{a}_{s_1,1}, \ldots, \vec{a}_{s_n,n}, \vec{a}_{1-s_k,k}\}$ are orthogonal to $\vec{x}_{\vec{s}}$. Since the dimension of the vector space is $n + 1$, the set of $n + 1$ vectors are linearly dependent, in contradiction with the hypotheses. □
Now, let us define the set $\mathcal{Z}$ as the zero locus of the ideal generators

$$
\hat{a}_{0,i} \hat{a}_{1,i} \quad \forall i \in \{1, \ldots, n\}
$$

$$
\hat{a}_{s_1,i_1} \hat{a}_{s_2,i_2} \quad \forall (s_1,i_1; s_2,i_2) \in \Gamma_2
$$

$$
dots
$$

$$
\hat{a}_{s_1,i_1} \hat{a}_{s_2,i_2} \ldots \hat{a}_{s_n,i_n} \quad \forall (s_1,i_1; s_2,i_2, \ldots, s_n,i_n) \in \Gamma_n.
$$

(43)

The first $n$ generators provide an interpretation of $\hat{a}_{1,i}$ as negation of $\hat{a}_{0,i}$, as consequence of Property 2. The $i$-th generator implies that $(a_{0,i}, a_{1,i})$ is equal to $(true, false)$, $(false, true)$ or $(true, true)$. However, the last case is forbidden by Property 2. Assume that $(a_{0,i}, a_{1,i})$ is equal to $(true, true)$ for some $i$. Then, there would be $n + 1$ functions $\hat{a}_{s_1,1}, \ldots, \hat{a}_{s_n,n}, \hat{a}_{1-s_i}$ equal to zero, which is impossible since they are independent. Thus, the algebraic set defined by the first $n$ generators contains $2^n$ distinct points, as implied by Lemma 5 which are associated with all the possible states taken by the logical variables. The remaining generators set further constraints on these variables and define a Boolean formula in conjunctive normal form. With this construction there is a one-to-one correspondence between the points of the algebraic set $\mathcal{Z}$ and the solutions of the Boolean formula.

There is a generalization of the generators (43) that allows us to weaken Property 2 while retaining the one-to-one correspondence. Let $R_1, \ldots, R_m$ be $m$ disjoint non-empty sets such that $\bigcup_{k=1}^{m} R_k = \{1, \ldots, n\}$.

**Property 3.** The set of vectors $\bigcup_{k=1}^{m} \{\vec{a}_{s,k}|i \in R_k\} \equiv A_{\vec{s}}$ is independent for every $\vec{s} = (s_1, \ldots, s_m) \in \{0,1\}^m$. Furthermore, every vector $\vec{a}_{s,i} \notin A_{\vec{s}}$ is not in span($A_{\vec{s}}$), with $s \in \{0,1\}$ and $i \in \{1, \ldots, n\}$.

**Lemma 6.** Let $\{\vec{a}_{0,1}, \ldots, \vec{a}_{0,n}, \vec{a}_{1,1}, \ldots, \vec{a}_{1,n}\}$ be a set of $2n$ vectors satisfying Property 3. Let $\vec{x}_{\vec{s}}$ be the solution of the equations

$$
\hat{a}_{s_k,i} = 0 \quad i \in R_k, k \in \{1, \ldots, m\}
$$

for every $\vec{s} \in \{0,1\}^m$. If $\vec{s} \neq \vec{r}$, then $\vec{x}_{\vec{s}} \neq \vec{x}_{\vec{r}}$.

The generators (43) are generalized by replacing the first line with

$$
\hat{a}_{0,i} \hat{a}_{1,j}, \quad (i, j) \in \bigcup_{k=1}^{m} (R_k \times R_k).
$$

(44)

Provided that Property 3 holds there is a one-to-one correspondence between the points in the algebraic set and the solutions of a SAT formula built according to the following interpretation.
Each set of functions \( \{ \hat{a}_{0,i} | i \in R_k \} \equiv a_k \) is interpreted as a Boolean variable, which is true if the functions in there are equal to zero. The set \( \{ \hat{a}_{1,i} | i \in R_k \} \) is interpreted as negation of \( \{ \hat{a}_{0,i} | i \in R_k \} \). The SAT formula is built in obvious way from the set of generators. For example, the generator \( \hat{a}_{0,i} \hat{a}_{0,j} \) with \( i \in R_1 \) and \( j \in R_2 \) induces the clause \( a_1 a_2 \). Different generators can induce the same clause. Since the total number of solutions depends only on the SAT formula, it is convenient to take the maximal set of generators compatible with a given formula. That is, if \( a_1 a_2 \) is a clause, then \( \hat{a}_{0,i} \hat{a}_{0,j} \) is taken as a generator for every \( i \in R_1 \) and \( j \in R_2 \).

### D. 3SAT-like generators

SAT problems have clauses with an arbitrarily large number of literals. Special cases are 2SAT and 3SAT, in which clauses have at most 2 or 3 literals. It is known that every SAT problem can be converted to a 3SAT one by increasing the number of variables and replacing a clause with a certain number of smaller clauses containing the new variables. For example, the clause \( a \lor b \lor c \lor d \) can be replaced by \( a \lor b \lor x \) and \( c \lor d \lor (\neg x) \). An assignment satisfies the first clause if and only if the other two clauses are satisfied for some \( x \). An identical reduction can be performed also on the generators \( \hat{a}_i \). For example, a generator in \( I \) of the form \( \hat{a}_1 \hat{a}_2 \hat{a}_3 \hat{a}_4 \equiv G_0 \) can be replaced by \( \hat{a}_1 \hat{a}_2 y \equiv G_1, \hat{a}_1 \hat{a}_2 (1 - y) \equiv G_2 \) and \( y (1 - y) \equiv G_3 \), where \( y \) is an additional variable. Also in this case, \( G_0 \) is equal to zero if and only if \( G_1 \) and \( G_2 \) are zero for \( y = 0,1 \). Furthermore, the new extended ideal contains the old one. Indeed, we have that \( G_0 = \hat{a}_3 \hat{a}_4 G_1 + \hat{a}_1 \hat{a}_2 G_2 \).

Note that all the polynomials in the ideal \( I \) are independent of the additional variables used in the reduction. Thus, if we build the polynomials \( P_k \) by using 3SAT-like generators, then all these polynomials may be independent of some variables. Thus, we can consider generators in a 3SAT form,

\[
\hat{a}_{i_1} \hat{a}_{i_2} \quad \forall (i_1, i_2) \in \Gamma_2 \\
\hat{a}_{i_1} \hat{a}_{i_2} \hat{a}_{i_3} \quad \forall (i_1, i_2, i_3) \in \Gamma_3.
\]

There is no loss of generality, provided that all the polynomials \( P_k \) are possibly independent from \( n_I \) variables.

The number of isolated points satisfies the inequality

\[
N_P \leq 3^n.
\]

The actual number can be considerably smaller, depending on the matroid and the number of clauses defining the Boolean formula. The bound is attained if \( n_c = 3n \), the generators have the
form \( a_i b_i c_i \) with \( i \in \{1, \ldots, n\} \), and the independent sets of the matroid contain \( n + 1 \) elements. If there are only clauses with 2 literals, then the bound is

\[
N_P \leq 2^n, \tag{47}
\]

which is strict if the generators have the form \( a_i b_i \) with \( i \in \{1, \cdots, n\} \). A consequence of these constraints is that the number \( M \) of parameters must scale sublinearly in \( n \),

\[
M \leq K n^\beta, \quad 0 \leq \beta < 1 \tag{48}
\]

for some \( K > 0 \).

**IV. BUILDING UP THE PARAMETRIZABLE VARIETY AND THE HYPERPLANE**

In this section, we put together the tools introduced previously to tackle our problem of building the rational function \( R \) with the desired properties of being computationally simple and having a sufficiently large set of zeros. This problem has being reduced to the search of computationally simple polynomials \( P_k \) of the form (32) with a number of common rational zeros growing sufficiently fast with the space dimension. To build these polynomials, we first choose a set of generators of the form (41) such that the associated algebraic set \( Z \) has a set of \( N_P \) points. Then, we write the polynomials \( P_k \) as elements of the ideal associated with \( Z \). Finally, we impose that the polynomials \( P_k \) have the form of Eqs. (32).

**Procedure 1.** Building up of a parametrizable variety \( V \) with \( M \) parameters and \( N_P \) intersection points.

1. Take a set of \( \bar{n} \) unknown non-homogeneous linear functions in \( n \) variables with \( \bar{n} > n \), say \( \hat{a}_1, \ldots, \hat{a}_{\bar{n}} \). Additionally, specify which set of vectors are linearly independent. In other words, a linear matroid with \( \bar{n} \) elements is defined.

2. Choose and ideal \( I \) with generators of the form (45) such that the associated algebraic set \( Z \) contains a subset \( Z_P \) of \( N_P \) isolated points over some given number field.

3. Set the polynomials \( P_s \) equal to elements of the ideal \( I \) with \( s \in \{0, \ldots, n - M\} \). That is,

\[
P_s(\vec{x}) = \sum_{(i,j) \in \Gamma_2} C_{s,i,j}(\vec{x})\hat{a}_i\hat{a}_j + \sum_{(i,j,k) \in \Gamma_3} D_{s,i,j,k}(\vec{x})\hat{a}_i\hat{a}_j\hat{a}_k, \tag{49}
\]

The polynomials \( P_s \) with \( s \in \{1, \ldots, n - M\} \) define an algebraic set \( A \). The polynomial \( P_0 \) defines a hyperplane \( H \). The number of parameters \( M \) and the polynomial coefficients \( C_{s,i,j}(\vec{x}) \) and \( D_{s,i,j,k}(\vec{x}) \) are also unknown.
4. Search for values of the coefficients such that there is a parametrizable branch \( V \) in \( \mathcal{A} \) with a number of parameters as small as possible. All the polynomials \( P_s \) with \( s \in \{0, \ldots, n - M\} \) are possibly independent of some subset of variables (see Sec. III D). The polynomials \( D_k \), as defined in Eq. (32) must be different from zero in the set \( \mathcal{Z}_P \).

More explicitly, the last step leads us to the following.

**Problem 1.** Given the sets \( \Gamma_2 \) and \( \Gamma_3 \), and polynomials of the form (49), find linear functions \( \hat{a}_1, \ldots, \hat{a}_{\bar{n}} \) and coefficients \( C_{s,i,j}(\bar{x}), D_{s,i,j,k}(\bar{x}) \) such that

\[
\frac{\partial P_s}{\partial x_k} = 0, \quad 1 \leq k < s \leq n - M, \\
\frac{\partial^2 P_s}{\partial x_s^2} = 0, \quad 1 \leq s \leq n - M, \\
\bar{x} \in \mathcal{Z}_P \Rightarrow \frac{\partial P_s}{\partial x_s} \equiv D_s(x_{s+1}, \ldots, x_n) \neq 0,
\]

under the constraint that \( (\hat{a}_1, \ldots, \hat{a}_{\bar{n}}) \) is the representation of a given matroid.

**Remark.** If the algebraic set associated with the ideal \( I \) is zero-dimensional, this problem has always a solution for any \( M \), since a rational univariate representation always exists (see introduction). Essentially, the task is to find ideals such that there is a solution with the coefficients \( C_{s,i,j}(\bar{x}) \) and \( D_{s,i,j,k}(\bar{x}) \) as simple as possible, so that their computation is efficient, given \( \bar{x} \).

Let us remind that the constraints (50) are invariant under transformations (34, 35). All the polynomials are possibly independent of a subset of \( n_I \) variables, say \( \{x_{n-n_I+1}, \ldots, x_n\} \),

\[
\frac{\partial P_s}{\partial x_k} = 0, \quad \begin{cases} 
 s \in \{0, \ldots, n - M\} \\
 k \in \{n - n_I + 1, \ldots, n\}
\end{cases} \quad (51)
\]

These \( n_I \) variables can be set equal to constants, so that the actual number of significant parameters is \( M - n_I \). The input of Problem 1 is given by a 3SAT formula of form (45) and a linear matroid.

**Definition 4.** A 3SAT formula of form (45) and a linear matroid with \( \bar{n} \) elements is called a model.

In literature, the term ‘model’ is occasionally used with a different meaning and refers to a solution of a SAT formula.

Problem 1 in its general form is quite intricate. First, it requires the definition of a linear matroid and a SAT formula with an exponentially large number of solutions associated with isolated points. Whereas it is easy to find examples of matroids and Boolean formulas with this feature, it is not generally simple to characterize models with an exponentially large number of isolated points.
Second, Eqs. (50) take to a large number of polynomial equations in the unknown coefficients. Lemma 3 can help to reduce the search space by dimension reduction. This will be shown in Sec. IV B with a simple example. A good strategy is to start with simple models and low-degree coefficients in Eq. (49). In particular, we can take the coefficients constant, as done later in Sec. V.

This restriction does not guarantees that Problem 1 has a solution for a sufficiently small number of parameters $M$, but we can have some hints on how to proceed.

### A. Required number of rational points vs space dimension

Let assume that the computational complexity $C_0$ of $R$ is polynomial in the space dimension $n$, that is,

$$C_0 \sim n^{\alpha_0}. \quad (52)$$

The factoring algorithm has polynomial complexity if

$$\begin{align*}
K_1 n^\alpha &\leq \log N_P \leq K_2 n \quad 0 < \alpha \leq 1 \\
M &\leq (\log N_P)^\beta \quad \beta < 1
\end{align*}$$

(53)

for $n$ sufficiently great, where $K_1$ is some positive constant and $K_2 = \log 3$. The upper bound is given by Eq. (46). The algorithm has subexponential complexity $C \sim e^{b(\log N_P)^\alpha}$ with $0 < \alpha < 1$ if

$$\begin{align*}
\log N_P &\sim (\log n)^{1/\alpha} \quad 0 < \alpha < 1, \\
M &\sim (\log n)^{\beta} \quad 0 < \beta < 1 - \alpha.
\end{align*}$$

(54)

The upper bound on $\beta$ comes from Lemma 2. Thus, the number of rational points is required to scale much less than exponentially for getting polynomial or subexponential factoring complexity. Note that a slower increase of $N_P$ induces stricter bounds on $M$ in terms of $n$.

### B. Reduction of models

In this subsection, we describe an example of model reduction. The model reduction is based on Lemma 3 and can be useful for simplifying Problem 1. The task is to reduce a class of models associated with an efficient factoring algorithm to a class of simpler models taking to another efficient algorithm, so that it is sufficient to search for solutions of Problem 1 over the latter smaller class.

In our example, the matroid contains $2n$ elements and is represented by the functions $\hat{a}_{0,1}, \ldots, \hat{a}_{0,n}, \hat{a}_{1,1}, \ldots, \hat{a}_{1,n}$ satisfying Property 2.
**Model A.**
Matroid with representation \((\hat{a}_{0,1}, \ldots, \hat{a}_{0,n}, \hat{a}_{1,1}, \ldots, \hat{a}_{1,n})\) satisfying Property 2.
Generators:
\[
\hat{a}_{0,i} \hat{a}_{1,i} \quad i \in \{1, \ldots, n\}
\]
\[
\hat{a}_{0,i} \hat{a}_{0,j} \quad (i, j) \in \Gamma.
\]

**Definition 5.** A diagonal model is defined as Model A with \(\Gamma = \emptyset\).

Clearly, a diagonal model defines an algebraic set with \(2^n\) isolated points. Each point satisfies the linear equations
\[
\hat{a}_{s,i} = 0, \quad i \in \{1, \ldots, n\}
\]
for some \((s_1, \ldots, s_n) \in \{0, 1\}^n\).

If there is an algorithm with polynomial complexity and associated with Model A, then it is possible to prove that there is another algorithm with polynomial complexity and associated with a diagonal model. More generally, this formula reduction takes to a subexponential factoring algorithm, provided that the parent algorithm outperforms the quadratic sieve algorithm. If the parent algorithm outperforms the general number field sieve, then the reduced algorithm outperforms the quadratic sieve. Thus, if we are interested to find a competitive algorithm from Model A, we need to search only the space of reduced formulas. If there is no algorithm outperforming the quadratic sieve with \(\Gamma = \emptyset\), then there is no algorithm outperforming the general number field for \(\Gamma \neq \emptyset\).

**Theorem 1.** If there is a factoring algorithm with subexponential asymptotic complexity \(e^{a(\log p)^\gamma}\) and associated with Model A, then there is another algorithm associated with the diagonal model with computational complexity upper-bounded by the function \(e^{\bar{a}(\log p)^{1-\gamma}}\) for some \(\bar{a} > 0\). In particular, if the first algorithm has polynomial complexity, also the latter has polynomial complexity.

**Proof.** Let us assume that the asymptotic computational complexity of the parent algorithm is \(e^{a(\log p)^\gamma}\). For every \(N_P\), there is a Model A with \(N_P\) isolated points and generating a rational function \(R\) with complexity \(C_0(\xi)\) scaling as \(e^{\alpha(\log N_P)\beta}\) and a number of parameters \(M\) scaling as \((\log N_P)\gamma\), where \(\gamma = \alpha/(1 - \beta)\) and \(0 \leq \beta < 1\) (See [11]). We denote by \(Z\) the set of isolated points. Since the complexity \(C_0\) is lower-bounded by a linear function of the dimension \(n\), we have
\[
\log n \leq a(\log N_P)^\alpha + O(1).
\]
Let \(\hat{a}_{0,1}, \ldots, \hat{a}_{0,n}, \hat{a}_{1,1}, \ldots, \hat{a}_{1,n}\) be the set of linear functions representing the matroid and satisfying Property 2. The ideal generators are given by Eq. (55).
Let \( m \) be the maximum number of functions in \( \{ \hat{a}_0, \ldots, \hat{a}_0, \ldots, \hat{a}_0, n \} \) which are simultaneously different from zero for \( \vec{x} \in \mathbb{Z} \). Thus, we have that

\[
N_P \leq \sum_{j=0}^{m} \frac{n!}{(n-j)!j!} \leq (1 + n^{-1}) n^m. \tag{58}
\]

There is a point \( \vec{x}_2 \) in \( \mathbb{Z} \) such that \( \hat{a}_{0,1}, \ldots, \hat{a}_{0,m} \) are different from zero and \( \hat{a}_{0,m+1} = \hat{a}_{0,m+1} = \cdots = \hat{a}_{0,n} = 0 \), up to permutations of the indices.

Let us set these last \( n - m \) functions equal to zero by dimension reduction. The new set of generators is associated with another factoring algorithm (Lemma 3) and contains the clauses of the form

\[
\hat{a}_{0,i} \hat{a}_{1,i} \quad i \in \{1, \ldots, m\}
\]

\[
\hat{a}_{0,j} \hat{a}_{0,j} \quad (i, j) \in \bar{\Gamma} \subseteq \{1, \ldots, m\} \times \{1, \ldots, m\}. \tag{59}
\]

Since there is a point \( \vec{x}_2 \) such that \( \hat{a}_{0,i} \neq 0 \) for \( i \in \{1, \ldots, m\} \), the set \( \bar{\Gamma} \) turns out to be empty, so that the reduced model is diagonal. The number of common zeros of the generators, say \( N_1 \), is equal to \( 2^m \). Using Ineq. (58), we have that

\[
(\log_2 N_1)(\log n) + \log (1 + n^{-1}) \geq \log N_P. \tag{60}
\]

Ineq. (57) and this last inequality implies that

\[
\log N_P \leq K (\log N_1)^{\frac{1}{1-\alpha}} \tag{61}
\]

for some constant \( K \). Since the computational complexity, say \( \bar{C}_0 \) of the rational function \( \mathcal{R} \) associated with the reduced model is not greater than \( C_0 \), which scales as \( e^{\bar{a}(\log N_P)^{\bar{\alpha}}} \), we have that

\[
\bar{C}_0 \leq e^{\bar{a}(\log N_1)^{\frac{\bar{\alpha}}{1-\alpha}}}, \tag{62}
\]

for some constant \( \bar{\alpha} \). Similarly, since the number of parameters, say \( \bar{M} \), of the reduced rational function is not greater than \( M \), we have that

\[
\bar{M} \leq \bar{K} (\log \bar{N}_1)^{\frac{\bar{\alpha}}{1-\alpha}} \tag{63}
\]

for some constant \( \bar{K} \). Thus, the resulting factoring algorithm has a computational complexity upper-bounded by

\[
e^{\bar{a}(\log p)^{\frac{\bar{\alpha}}{1-\alpha-\beta}}} = e^{\bar{a}(\log p)^{\frac{\bar{\gamma}}{\bar{\gamma}}}}
\]

up to a constant factor. The last statement of the theorem is proved in a similar fashion. \( \square \)
The diagonal model with generators

\[ G_i = \hat{a}_{0,i} \hat{a}_{1,i}, \quad \forall i \in \{1, \ldots, n\} \]  

provides the simplest example of polynomials with an exponentially large number of common zeros. The algebraic set \( Z = Z_P \) contains \( 2^n \) points, which are distinct because of Property 2. This guarantees that the generated ideal is radical. Thus, Hilbert’s Nullstellensatz implies that every polynomial which is zero in \( Z \) can be written as \( \sum_i F_i(\bar{x})G_i(\bar{x}) \), where \( F_1, \ldots, F_n \) are polynomials (let us remind that \( x_{n+1} = 1 \)).

We impose that the polynomials \( P_0, \ldots, P_{n-M} \) are in the ideal generated by \( G_1, \ldots, G_n \), that is,

\[ P_k(\bar{x}) = \sum_i C_{k,i}(\bar{x})\hat{a}_{0,i} \hat{a}_{1,i} \quad \forall k \in \{0, \ldots, n-M\}. \]  

As there is no particular requirement on \( P_0 \), we can just set \( C_{0,i}(\bar{x}) \) equal to constants. In particular, we can take \( P_0 = \hat{a}_{0,1} \hat{a}_{1,1} \). In this case, the unknown variables of Problem 1 are the polynomials \( C_{k,i}(\bar{x}) \) and the linear equations \( \hat{a}_{s,k} \) under the constraints of Property 2. In the following section we tackle this problem with \( C_{k,i}(\bar{x}) \) constant.

V. QUADRATIC POLYNOMIALS

In this section, we illustrate the procedure described previously by considering the special case of \( n-M+1 \) quadratic polynomials in the ideal \( I \) generated by the polynomials \( G_1, \ldots, G_n \). Namely, we take the polynomials \( P_l \) of the form

\[ P_l(\bar{x}) = \sum_{i=1}^{n} c_{l,i} \hat{a}_{0,i} \hat{a}_{1,i}, \quad l \in \{0, \ldots, n-M\}, \]  

where \( c_{l,i} \) are rational numbers and the linear functions \( \hat{a}_{s,i} \) satisfy Property 2. Thus, there are \( 2^n \) common rational zeros of the \( n-M+1 \) polynomials, which are also the zeros of the generators. Each rational point is associated with a vector \( \bar{s} \in \{0, 1\}^n \) so that the linear equations \( \bar{a}_{s,1} \cdot \bar{x} = 0, \ldots, \bar{a}_{s,n} \cdot \bar{x} = 0 \) are satisfied.

First, we consider the case with one parameter \((M = 1)\). We also assume that all the \( 2^n \) rational points are in the parametrizable variety. Starting from these assumptions, we end up to build a variety \( V \) with a number \( M \) of parameters equal to \( n/2 - 1 \) for \( n \) even and \( n \geq 4 \). Furthermore, we prove that there is no solution with \( M = 1 \) if \( n > 4 \). We give a numerical example for \( n = 4 \), which takes to a rational function \( R \) with 16 zeros. Then we build a parametrizable variety with a
number of parameters equal to \((n - 1)/3\). Thus, the minimal number of parameters is some value between 2 and \((n - 1)/3\) for the considered model with the polynomials of the form \((66)\).

A. One parameter? \((M = 1)\)

Given polynomials \((66)\) and vectors \(\vec{a}_{s,i}\) satisfying Property \((2)\), we search for a solution of Problem 1 under the assumption \(M = 1\). Let us first introduce some notations and definitions.

We define the \((n - 1) \times n\) matrices

\[
\mathbf{M}^s \equiv \begin{pmatrix}
A_{1,1}^{(s_1)} & \cdots & A_{1,n}^{(s_n)} \\
\vdots & \ddots & \vdots \\
A_{n-1,1}^{(s_1)} & \cdots & A_{n-1,n}^{(s_n)}
\end{pmatrix},
\]

(67)

where

\[
A_{k,i}^{(s)} \equiv \frac{\partial \vec{a}_{s,i}}{\partial x_k},
\]

(68)

The square submatrix of \(\mathbf{M}^s\) obtained by deleting the \(j\)-th column is denoted by \(\mathbf{M}^s_j\), that is,

\[
\mathbf{M}^s_j = \begin{pmatrix}
A_{1,1}^{(s_1)} & \cdots & A_{1,j-1}^{(s_{j-1})} & A_{1,j}^{(s_{j+1})} & \cdots & A_{1,n}^{(s_n)} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
A_{n-1,1}^{(s_1)} & \cdots & A_{n-1,j-1}^{(s_{j-1})} & A_{n-1,j}^{(s_{j+1})} & \cdots & A_{n-1,n}^{(s_n)}
\end{pmatrix},
\]

(69)

The vectors \(\vec{a}_{0,i}\) and \(\vec{a}_{1,i}\) are also briefly denoted by \(\vec{a}_i\) and \(\vec{b}_i\), respectively. Similarly, we also use the symbols \(A_{k,i}\) and \(B_{k,i}\) for the derivatives \(A_{k,i}^{(0)}\) and \(A_{k,i}^{(1)}\).

Problem 1 takes the specific form

**Problem 2.** Find coefficients \(c_{l,i}\) and vectors \(\vec{a}_{s,i}\) satisfying Property \((2)\) such that

\[
\sum_{i=1}^{n} c_{l,i} \left( A_{k,i} \vec{a}_i + B_{k,i} \vec{b}_i \right) = 0 \quad 1 \leq k < l \leq n - 1, \quad (70)
\]

\[
\sum_{i=1}^{n} c_{l,i} A_{l,i} B_{l,i} = 0 \quad 1 \leq l \leq n - 1, \quad (71)
\]

\[
\vec{x} \in \mathbb{Z}_P \Rightarrow \sum_{i=1}^{n} c_{l,i} \left( A_{l,i} \vec{a}_i + B_{l,i} \vec{b}_i \right) \neq 0 \quad 1 \leq l \leq n - 1. \quad (72)
\]

Let us stress again that the problem is invariant with respect to the transformations \((32,35)\), the latter taking to the transformation

\[
A_{k,i}^{(s)} \to A_{k,i}^{(s)} + \sum_{l=1}^{k-1} \bar{\eta}_{k,l} A_{l,i}^{(s)} \quad (73)
\]
of the derivatives.

We have the following.

**Lemma 7.** For every $\vec{s} \in \{0, 1\}^n$ and $j \in \{1, \ldots, n\}$, the matrix $M_j^{\vec{s}}$ has maximal rank, that is,

$$\det M_j^{\vec{s}} \neq 0. \quad (74)$$

**Proof.** Let us prove the lemma by contradiction. There is a $j \in \{1, \ldots, n\}$, $l \in \{1, \ldots, n-1\}$, and an $\vec{s} \in \{0, 1\}^n$ such that the $l$-th row of $M_j^{\vec{s}}$ is linearly dependent on the first $l-1$ rows. Thus, there are coefficients $\lambda_1, \ldots, \lambda_{l-1}$ such that

$$A^{(s_l)}_{l,i} + \sum_{k=1}^{l-1} \lambda_k A^{(s_k)}_{k,i} = 0 \quad \forall i \neq j. \quad (75)$$

With a change of variables of the form of Eq. (35), this equation can be rewritten in the form

$$A^{(s_l)}_{l,i} = 0 \quad \forall i \neq j. \quad (76)$$

Up to permutations $\vec{a}_i \leftrightarrow \vec{b}_i$, we have

$$B_{l,i} = 0 \quad \forall i \neq j. \quad (77)$$

From Eq. (71), we have

$$\sum_{i=1}^{n} c_{l,i} A_{l,i} B_{l,i} = 0. \quad (78)$$

From this equation and Eq. (77), we get the equation $c_{l,j} A_{l,j} B_{l,j} = 0$, implying that $c_{l,j} A_{l,j} = 0$ or $c_{l,j} B_{l,j} = 0$. Without loss of generality, let us take

$$c_{l,j} B_{l,j} = 0. \quad (79)$$

Let $\vec{x}_0 \in \mathbb{Z}_P$ be the vector orthogonal to $\vec{b}_1, \ldots, \vec{b}_n$. From Eq. (72) we have that

$$c_{l,j} B_{l,j}(\vec{a}_j \cdot \vec{x}_0) \neq 0, \quad (80)$$

which is in contradiction with Ineq. (78). $\square$

**Corollary 1.** The coefficients $c_{n-1,i}$ are different from zero for every $i \in \{1, \ldots, n\}$.

**Proof.** Let us assume that the statement is false. Up to permutations, we have that $c_{n-1,1} = 0$. Lemma 7 implies that there is an integer $i_0 \in \{2, \ldots, n\}$ such that $B_{n-1,i} = 0$ for $i \notin \{1, i_0\}$, up to a transformation of the form of Eq. (55). Thus,

$$0 = \sum_{i=1}^{n} c_{n-1,i} A_{n-1,i} B_{n-1,i} = c_{n-1,i_0} A_{n-1,i_0} B_{n-1,i_0}, \quad (81)$$

where $A = \sum_{i=1}^{n} c_{n-1,i} A_{n-1,i} B_{n-1,i} = c_{n-1,i_0} A_{n-1,i_0} B_{n-1,i_0}$.
the first equality coming from Eq. (71). Lemma 7 also implies that $A_{n-1,i_0}B_{n-1,i_0} \neq 0$ Thus, on one hand, we have that $c_{n-1,i_0} = 0$. On the other hand, we have

$$c_{n-1,i_0}B_{n-1,i_0}(\hat{a}_{i_0} \cdot \bar{x}_0) \neq 0$$

(81)

from Eqs. (72), where $\bar{x}_0$ is the vector orthogonal to $\vec{b}_1, \ldots, \vec{b}_n$. Thus, we have a contradiction. □

Let us denote by $\bar{M}^{s}_{j_1,\ldots,j_m}$ the submatrix of $\bar{M}^s$ obtained by deleting the last $m-1$ rows and the columns $j_1, \ldots, j_m$. Given the coefficient matrix

$$c \equiv \begin{pmatrix} c_{0,1} & \cdots & c_{0,n} \\ \vdots & \ddots & \vdots \\ c_{n-1,1} & \cdots & c_{n-1,n} \end{pmatrix},$$

(82)

let us define $c_{j_1,\ldots,j_m}$ as the $m \times m$ submatrix of $c$ obtained by keeping the last $m$ rows and the columns $j_1, \ldots, j_m$.

Lemma 7 and Corollary 1 are generalized by the following.

**Theorem 2.** For every $m \in \{1, \ldots, n-1\}$, $s \in \{0,1\}^n$, and $m$ distinct integers $j_1, \ldots, j_m \in \{1, \ldots, n\}$, the matrices $\bar{M}^{s}_{j_1,\ldots,j_m}$ and $c_{j_1,\ldots,j_m}$ have maximal rank, that is,

$$\det \bar{M}^{s}_{j_1,\ldots,j_m} \neq 0,$$

$$\det c_{j_1,\ldots,j_m} \neq 0.$$  

(83)  

(84)

**Proof.** The proof is by recursion. For $m = 1$, the theorem comes from Lemma 7 and Corollary 1. Thus, we just need to prove Eqs. (83,84) by assuming that

$$\det \bar{M}^{s}_{j_1,\ldots,j_{m-1}} \neq 0,$$

$$\det c_{j_1,\ldots,j_{m-1}} \neq 0.$$  

(85)  

(86)

Let us first prove Eq. (83) by contradiction. If the equation is false, then there is an $\hat{s}_0 \in \{0,1\}^n$ and $m$ distinct integers $i_1, \ldots, i_m$ in $\{1, \ldots, n\}$ such that $\det \bar{M}^{\hat{s}_0}_{i_1,\ldots,i_m} = 0$. By permutations, we can set $i_h = h$. By suitable exchanges of $\hat{a}_i$ and $\hat{b}_i$, we can set $s_i = 1$ for every $i \in \{1, \ldots, n\}$. There is an integer $l \in \{1, \ldots, n-m\}$ such that $B_{l,i} = 0$ for $i \in \{m+1, \ldots, n\}$ up to a transformation of
the form of Eq. (35). From Eqs. (70,71), we have the \( m \) equations

\[
\begin{align*}
\sum_{i=1}^{m} c_{l,i} A_{i,i} B_{l,i} &= 0 \\
\sum_{i=1}^{m} c_{n+1-m,i} A_{i,i} B_{l,i} &= 0 \\
\sum_{i=1}^{m} c_{n+2-m,i} A_{i,i} B_{l,i} &= 0 \\
\vdots \\
\sum_{i=1}^{m} c_{n-2,i} A_{i,i} B_{l,i} &= 0 \\
\sum_{i=1}^{m} c_{n-1,i} A_{i,i} B_{l,i} &= 0.
\end{align*}
\tag{87}
\]

From Eq. (35), we have that \( A_{i,i} \neq 0 \) and \( B_{l,i} \neq 0 \) for some \( i \in \{1, \ldots, m\} \), so that

\[
\begin{pmatrix}
c_{l,1} & \cdots & c_{l,m} \\
c_{n+1-m,1} & \cdots & c_{n+1-m,m} \\
c_{n+2-m,1} & \cdots & c_{n+2-m,m} \\
\vdots & \ddots & \vdots \\
c_{n-2,1} & \cdots & c_{n-2,m} \\
c_{n-1,1} & \cdots & c_{n-1,m}
\end{pmatrix}
\] \( \text{rank} \) \( < m. \) \tag{88}

Up to a transformation of the form of Eq. (34), there is an integer \( l_0 \in \{n+1-m, \ldots, n-1\} \cup \{l\} \) such that \( c_{l_0,i} = 0 \) for \( i \in \{1, \ldots, m\} \). Eq. (36) implies that \( l_0 = l \). Thus, \( c_{l,1} = \cdots = c_{l,m} = 0 \), but this contradicts Eq. (72) with \( \bar{x} \in \mathbb{Z}_P \) orthogonal to \( \bar{b}_1, \ldots, \bar{b}_n \).

Let us now prove Eq. (34) by contradiction. If the equation is false, then there are \( m \) distinct integers \( i_1, \ldots, i_m \) in \( \{1, \ldots, n\} \) such that \( \det c_{i_1, \ldots, i_m} = 0 \). Without loss of generality, let us take \( i_h = h \). Up to the transformation (34), there is an integer \( l \in \{n-m, \ldots, n-1\} \) such that \( c_{l,i} = 0 \) for \( i \in \{1, \ldots, m\} \). Eq. (36) implies that \( l = n - m \). Thus,

\[
c_{n-m,1} = \cdots = c_{n-m,m} = 0. \tag{89}
\]

Eq. (33) implies that there is an integer \( i_0 \in \{m+1, \ldots, n\} \) such that \( A_{n-m,i} = 0 \) for \( i \in \{m+1, \ldots, n\} \setminus \{i_0\} \) up to transformation (35). Thus, we have from Eq. (71) that

\[
0 = \sum_{i=1}^{n} c_{n-m,i} A_{n-m,i} B_{n-m,i} = c_{n-m,i_0} A_{n-m,i_0} B_{n-m,i_0}. \tag{90}
\]

Eq. (33) also implies that \( A_{n-m,i_0} B_{n-m,i_0} \neq 0 \), so that \( c_{n-m,i_0} = 0 \), which is in contradiction with Eq. (72) for \( \bar{x} \) orthogonal to \( \bar{a}_1, \ldots, \bar{a}_n \). \( \square \)

In the following, this theorem will be used with \( m \in \{1, 2\} \).

Since all the coefficients \( c_{n-1,i} \) are different from zero, we can set them equal to 1 by rescaling the vectors \( \bar{a}_i \) or \( \bar{b}_i \). Let us denote by \( c_i \) the coefficients \( c_{n-2,i} \). Theorem 2 with \( m = 2 \) implies that
\( c_i \neq c_j \) for \( i \neq j \). Eq. (70) with \( l = n - 1 \) and \( l = n - 2 \) takes the form
\[
\frac{\partial}{\partial x_k} P_{n-1} = \sum_{i=1}^{n} \left( A_{k, i} \vec{a}_i + B_{k, i} \vec{b}_i \right) = 0 \quad 1 \leq k \leq n - 2,
\]
\[
\frac{\partial}{\partial x_k} P_{n-2} = \sum_{i=1}^{n} c_i \left( A_{k, i} \vec{a}_i + B_{k, i} \vec{b}_i \right) = 0 \quad 1 \leq k \leq n - 3.
\]
These equations impose the form (32) to the last two polynomials, \( P_{n-1} \) and \( P_{n-2} \), which must be independent from \( n - 2 \) and \( n - 3 \) variables, respectively. The first \( n - 2 \) vector equations are linearly independent. Let us assume the opposite. Then, there is a set of coefficients \( \lambda_1, \ldots, \lambda_{n-2} \) such that \( \sum_{k=1}^{n-2} \lambda_k (A_{k,i}, B_{k,i}) = 0 \). But this is impossible because of Property 2. It also contradicts Theorem 2. The theorem also implies that Eqs. (92) are linearly independent. Since the vector space is \( n + 1 \)-dimensional, the vectors \( \vec{a}_i \) and \( \vec{b}_i \) must have \( n - 1 \) vector constraints. Thus, at least \( n - 4 \) out of Eqs. (92) are linearly dependent on Eqs. (91). First, let us show that \( n - 4 \) is the maximal number of dependent equations. Assuming the converse, we have
\[
c_i (A_{k,i}, B_{k,i}) = \sum_{l=1}^{n-2} \lambda_{k,l} (A_{l,i}, B_{l,i}) \quad \forall k \in \{1, \ldots, n-3\}.
\]
for suitable coefficients \( \lambda_{k,l} \). Let us define the linear superposition
\[
(A_i, B_i) \equiv \sum_{k=1}^{n-3} v_k (A_{k,i}, B_{k,i})
\]
with the coefficients \( v_k \). Let \( \Lambda \) be the \( (n-2) \times (n-2) \) matrix with \( \Lambda_{k,n-2} = 0 \) and \( \Lambda_{k,l} = \lambda_{l,k} \) for \( k \in \{1, \ldots, n-2\} \) and \( l \in \{1, \ldots, n-3\} \). The coefficients \( (v_1, \ldots, v_{n-3}) \equiv \vec{v} \) are defined by imposing the \( n - 4 \) constraints
\[
(\Lambda^s \vec{v})_{n-2} = 0 \quad s \in \{1, \ldots, n-4\}.
\]
By construction, the pairs
\[
c_i^{k-1}(A_i, B_i) \quad k \in \{1, \ldots, n-2\}
\]
are linear superpositions of the derivatives \( (A_{k,i}, B_{k,i}) \) with \( k \in \{1, \ldots, n-2\} \). Furthermore, the first \( n - 3 \) pairs are linear superpositions of \( (A_{k,i}, B_{k,i}) \) with \( k \in \{1, \ldots, n-3\} \). That is,
\[
c_i^{k-1}(A_i, B_i) = \sum_{l=1}^{n-3} \tilde{\lambda}_{k,l}(A_{l,i}, B_{l,i}) \quad k \in \{1, \ldots, n-3\}
\]
\[
c_i^{n-3}(A_i, B_i) = \sum_{l=1}^{n-2} \tilde{\lambda}_{n-2,l}(A_{l,i}, B_{l,i})
\]
for some coefficients \( \tilde{\lambda}_{k,l} \). From Lemma 7 and Corollary 1 we have that the \( n - 2 \) pairs \( (96) \) are linearly independent. Indeed, Corollary 1 implies that \( A_i \neq 0 \) and \( B_i \neq 0 \) for every \( i \in \{1, \ldots, n\} \).
Lemma 7 implies that \( c_i k - 1 \) are linearly independent for \( k \in \{1, \ldots, n - 2\} \). Equations (96,97) can be also derived from Jordan’s theorem, Lemma 7 and Corollary 1. See Appendix B.

Thus, by a variable transformation, Eqs. (91,92) take the form

\[
\sum_{i=1}^{n} c_i k - 1 \left( A_i \tilde{a}_i + B_i \tilde{b}_i \right) = 0 \quad k \in \{1, \ldots, n - 2\}.
\]

(98)

and

\[
\frac{\partial (\hat{a}_i, \hat{b}_i)}{\partial x_k} = c_i k - 1 (A_i, B_i) \quad k \in \{1, \ldots, n - 2\}.
\]

(99)

These equations imply that \( \sum_{i=1}^{n} c_i k - 1 A_i B_i = 0 \) for \( k \in \{1, \ldots, 2n - 5\} \). For \( n > 4 \), we have in particular that

\[
\sum_{i=1}^{n} c_i k - 1 A_i B_i = 0 \quad k \in \{1, \ldots, n\}.
\]

(100)

Since

\[
det \begin{pmatrix}
1 & \ldots & 1 \\
c_1 & \ldots & c_n \\
\vdots & \ddots & \vdots \\
c_n^{n-1} & \ldots & c_n^{n-1}
\end{pmatrix} = \prod_{j>i}(c_j - c_i)
\]

(101)

and \( c_i \neq c_j \) for \( i \neq j \), Eq. (100) implies that \( A_i B_i = 0 \) for every \( i \in \{1, \ldots, n\} \). But this is in contradiction with Theorem 2.

Thus, let us take exactly \( n - 4 \) out of Eqs. (92) linearly dependent on Eqs (91). Let \( \bar{k} \) be an integer in \( \{1, \ldots, n - 3\} \) such that Eq. (92) with \( k = \bar{k} \) is linearly independent of Eqs. (91). Thus,

\[
c_i(A_{k,i}, B_{k,i}) = \lambda_k c_i(A_{\bar{k},i}, B_{\bar{k},i}) + \sum_{l=1}^{n-2} \lambda_{k,l} (A_{l,i}, B_{l,i}) \quad k \in \{1, \ldots, n - 3\} \setminus \{\bar{k}\}.
\]

By a transformation of the first \( n - 3 \) variables, we can rewrite this equation in the form

\[
c_i(A_{k,i}, B_{k,i}) = \sum_{l=1}^{n-2} \lambda_{k,l} (A_{l,i}, B_{l,i}) \quad k \in \{1, \ldots, n - 4\}.
\]

(102)

By a suitable transformation of the first \( n - 2 \) variables, the \( n - 2 \) pairs \( (A_{k,i}, B_{k,i}) \) can be split in two groups (see Appendix B), say,

\[
\frac{\partial}{\partial x_k}(\hat{a}_i, \hat{b}_i) \equiv (A'_{k,i}, B'_{k,i}) = c_i k - 1 (A'_i, B'_i) \quad k \in \{1, \ldots, n_1\}
\]

\[
\frac{\partial}{\partial x_k}(\hat{a}_i, \hat{b}_i) \equiv (A''_{k,i}, B''_{k,i}) = c_i k - 1 (A''_i, B''_i) \quad k \in \{1, \ldots, n_2\}
\]

\[
\begin{cases}
n_1 + n_2 = n - 2.
\end{cases}
\]

(103)
Equations (91) become

\[ \sum_{i=1}^n c_i^{k-1} \left( A_i' \vec{b}_i + B_i' \vec{a}_i \right) = 0 \quad k \in \{1, \ldots, n_1\} \]

\[ \sum_{i=1}^n c_i^{k-1} \left( A_i'' \vec{b}_i + B_i'' \vec{a}_i \right) = 0 \quad k \in \{1, \ldots, n_2\}. \]

(104)

Given these \( n - 2 \) vector constraints, all \( n - 2 \) the derivatives \( \frac{\partial P_{n-1}}{\partial x'_1}, \ldots, \frac{\partial P_{n-1}}{\partial x'_{n_1}}, \frac{\partial P_{n-1}}{\partial x''_1}, \ldots, \frac{\partial P_{n-1}}{\partial x''_{n_2}} \) are equal to zero. Furthermore, we also have that

\[ \frac{\partial}{\partial x'_k} P_{n-2} = 0 \quad k \in \{1, \ldots, n_1 - 1\} \]

\[ \frac{\partial}{\partial x''_k} P_{n-2} = 0 \quad k \in \{1, \ldots, n_2 - 1\}, \]

so that \( P_{n-2} \) is independent of \( n - 4 \) out of the \( n - 2 \) variables \( x'_1, \ldots, x_{n_1}, x''_1, \ldots, x''_{n_2} \). Thus, we need to add another vector equation such that \( \left( w_1 \frac{\partial}{\partial x'_{n_1}} + w_2 \frac{\partial}{\partial x''_{n_2}} \right) P_{n-2} = 0 \) for some \( (w_1, w_2) \neq (0, 0) \).

Up to a variable transformation, we can set \( (w_1, w_2) = (1, 0) \) so that the additional vector equation is

\[ \sum_{i=1}^n c_i^{n_1} \left( A_i' \vec{b}_i + B_i' \vec{a}_i \right) = 0. \]

(105)

Equations (71,104,105) imply that

\[ \sum_{i=1}^n c_i^{k-1} A_i' B_i' = 0 \quad k \in \{1, \ldots, 2n_1\}, \]

(106)

\[ \sum_{i=1}^n c_i^{k-1} A_i'' B_i'' = 0 \quad k \in \{1, \ldots, 2n_2\}, \]

(107)

\[ \sum_{i=1}^n c_i^{k-1} (A_i' B_i'' + A_i'' B_i') = 0 \quad k \in \{1, \ldots, n_1 + n_2\}. \]

(108)

Since \( A_i' B_i' \) and \( A_i'' B_i' \) are not identically equal to zero (as consequence of Theorem 2), the number of Eqs. (106) and Eqs. (107) is smaller than \( n \), so that

\[ n_1 \leq \frac{n - 1}{2}, \quad n_2 \leq \frac{n - 1}{2}. \]

Without loss of generality, we can assume that \( n \) is even. Indeed, if Problem 1 can be solved for \( n \) odd, then Lemma 3 implies that it can be solve for \( n \) even, and viceversa. Since \( n_1 + n_2 = n - 2 \), we have that

\[ n_1 = n_2 = \frac{n - 2}{2}. \]

(109)

Let \( W_1, \ldots, W_n \) be \( n \) numbers defined by the equations

\[ \sum_{i=1}^n c_i^{k-1} W_i = 0 \quad k \in \{1, \ldots, n - 1\} \]

(110)
up to a constant factor. Equations \((106,107,108)\) are equivalent to the equations
\[
\begin{align*}
A'_i B'_i &= (k_0 + k_1 c_i) W_i, \\
A''_i B''_i &= (r_0 + r_1 c_i) W_i, \\
A'_i B''_i + A''_i B'_i &= (s_0 + s_1 c_i) W_i.
\end{align*}
\] (111)
(112)
(113)

These equations can be solved over the rationals for the coefficients \(c_i, B'_i\) and \(B''_i\) in terms of \(A'_i\) and \(A''_i\). The coefficients \(c_i\) take a form which is independent of \(W_i\),
\[
c_i = \frac{r_0 A'^2_i + k_0 A''^2_i - s_0 A'_i A''_i}{r_1 A'^2_i + k_1 A''^2_i - s_1 A'_i A''_i},
\] (114)
so that we first evaluate \(c_i\), then \(W_i\) by Eq. (110) and, finally, \(B'_i\) and \(B''_i\) by Eqs. (111,112). It is possible to show that condition (122) for \(l = n - 1\) implies that \((k_1, r_1) \neq (0, 0)\). Indeed, if \((k_1, r_1) = (0, 0)\), then only half of the points in \(Z_P\) satisfies the inequality in the condition. Up to a variable change, we have
\[
k_1 \neq 0, \quad s_1 = 0.
\]

Up to now we have been able to solve all the conditions of Problem 2 which refer to the last two polynomials, that is, for \(l = n - 2, n - 1\). The equations that need to be satisfied are Eqs. (104,105,111,112,113). Let us rewrite them all together.

\[
\begin{align*}
A'_i B'_i &= (k_0 + k_1 c_i) W_i, \quad A''_i B''_i = (r_0 + r_1 c_i) W_i, \quad k_1 \neq 0 \\
A'_i B''_i + A''_i B'_i &= s_0 W_i, \\
\sum_{i=1}^n c_i^{k-1} W_i &= 0 \quad k \in \{1, \ldots, n - 1\} \\
\sum_{i=1}^n c_i^{k-1} \left( A'_i \tilde{b}_i + B'_i \tilde{a}_i \right) &= 0 \quad k \in \{1, \ldots, \frac{n}{2} \} \\
\sum_{i=1}^n c_i^{k-1} \left( A''_i \tilde{b}_i + B''_i \tilde{a}_i \right) &= 0 \quad k \in \{1, \ldots, \frac{n}{2} - 1\}
\end{align*}
\] (115)

Given \(2n\) vectors \(\tilde{a}_1, \ldots, \tilde{a}_n, \tilde{b}_1, \ldots, \tilde{b}_n\) satisfying these equations, there are \(n - 1\) directions \(\tilde{u}_1, \ldots, \tilde{u}_{n-1}\) such that
\[
\tilde{u}_{2k-1} \cdot \frac{\partial}{\partial x} (\tilde{a}_i, \tilde{b}_i) = c_i^{k-1} (A'_i, B'_i) \quad k \in \{1, \ldots, \frac{n}{2} - 1\} \\
\tilde{u}_{2k} \cdot \frac{\partial}{\partial x} (\tilde{a}_i, \tilde{b}_i) = c_i^{k-1} (A''_i, B''_i) \quad k \in \{1, \ldots, \frac{n}{2}\}.
\] (116)

This can be easily verified by substitution. Let us define the coordinate system \((y_1, \ldots, y_{n+1}) \equiv \tilde{y}\) such that
\[
\tilde{u}_k \cdot \frac{\partial}{\partial x} = \frac{\partial}{\partial y_k} \quad k \in \{1, \ldots, n - 1\}.
\] (117)
Given the polynomials
\[ P_{n-1} = \sum_{i=1}^{n} \hat{a}_i \hat{b}_i \]
\[ P_{n-2} = \sum_{i=1}^{n} c_i \hat{a}_i \hat{b}_i \]
with \( \hat{a}_i = \vec{a}_i \cdot \vec{y} \) and \( \hat{b}_i = \vec{b}_i \cdot \vec{y} \), it is easy to verify that
\[ \frac{\partial P_{n-1}}{\partial y_k} = 0, \quad k \in \{1, \ldots, n-2\}, \]
\[ \frac{\partial P_{n-2}}{\partial y_k} = 0, \quad k \in \{1, \ldots, n-3\}, \]
\[ \frac{\partial^2 P_{n-2}}{\partial y_{n-2}^2} = 0. \]  
(119)

The polynomial \( P_{n-1} \) depends on 2 variables (in the affine space) and the polynomial \( P_{n-2} \) depends linearly on an additional variable \( y_{n-2} \). Thus, the algebraic set of the two polynomials admits a Gaussian parametrization, that is, the equations \( P_{n-1} = 0 \) and \( P_{n-2} = 0 \) can be solved with \( a \ la \) Gauss elimination of two variables. Note that the polynomial \( P_{n-1} \) has rational roots by construction. The next step is to satisfy the conditions of Problem 2 for the other polynomials \( P_1, \ldots, P_{n-3} \) by setting \( c_{k,i} \) and the other remaining free coefficients. It is interesting to note that it is sufficient to take \( c_{2s,i} = c_i^{n/2-s} \) with \( s \in \{1, \ldots, (n-4)/2\} \) for satisfying every condition of Problem 2 for \( l \) even. The polynomials \( P_2, P_4, \ldots, P_{n-2} \) take the form
\[ P_{2s} = \sum_{i=1}^{n} c_i^{n/2-s} \hat{a}_i \hat{b}_i, \quad s \in \{1, \ldots, (n-4)/2\}. \]  
(120)

Furthermore, we can choose \( c_{1,i} \) such that \( \partial^2 P_1 / \partial x_1^2 = 0 \). With this choice, we have that
\[ \frac{\partial P_1}{\partial y_k} = 0, \quad k \in \{1, \ldots, l-1\} \]
\[ \frac{\partial^2 P_l}{\partial y_l^2} = 0 \]
\[ l \in \{2, 4, \ldots, n-4, n-2\} \cup \{1, n-1\}. \]  
(121)

Thus, we are halfway to solve Problem 2, about half of the conditions are satisfied. The hard core of the problem is to solve the conditions for \( P_1, P_3, \ldots, P_{n-3} \). The form of Polynomials (120) is not necessarily the most general. Thus, let us take a step backward and handle Problem 2 for the polynomial \( P_{n-3} \) with the equations derived so far. We will find that this polynomial cannot satisfy the required conditions if \( n > 4 \), so that the number of parameters has to be greater than 1.

Let us denote by \( d_i \) the coefficients \( c_{n-3,i} \). Eqs. (70,71) with \( l = n-3 \) give the equations
\[ \sum_{i=1}^{n} e_i \left( A_{k,i} B_{k',i} + A_{k',i} B_{k,i} \right) = 0 \quad k, k' \in \{1, \ldots, n-3\}, \]
where

\[ \sum_{i=1}^{n} e_i c_i^{k+k'-2} A'_i B'_i = 0 \quad k, l \in \{1, \ldots, \frac{n}{2} - 1\} \]

\[ \sum_{i=1}^{n} e_i c_i^{k+k'-2} A''_i B''_i = 0 \quad k, l \in \{1, \ldots, \frac{n}{2} - 2\} \]

\[ \sum_{i=1}^{n} e_i c_i^{k+k'-2} (A'_i B'_i + A''_i B''_i) = 0 \quad \begin{cases} k \in \{1, \ldots, \frac{n}{2} - 1\} \\ k' \in \{1, \ldots, \frac{n}{2} - 2\} \end{cases} \]

that is,

\[ \sum_{i=1}^{n} e_i c_i^{k-1} A'_i B'_i = 0 \quad k \in \{1, \ldots, n - 3\} \]

\[ \sum_{i=1}^{n} e_i c_i^{k-1} A''_i B''_i = 0 \quad k \in \{1, \ldots, n - 5\} \]

\[ \sum_{i=1}^{n} e_i c_i^{k-1} (A'_i B'_i + A''_i B''_i) = 0 \quad k \in \{1, \ldots, n - 4\}. \]

These equations imply that

\[ e_i A'_i B'_i = F_{11}(c_i) W_i \]

\[ e_i A''_i B''_i = F_{22}(c_i) W_i \]

\[ e_i (A'_i B'_i + A''_i B''_i) = F_{12}(c_i) W_i, \]

where \( F_{11}(x), F_{22}(x) \) and \( F_{12}(x) \) are polynomials of degree lower than 3, 5 and 4, respectively. Thus,

\[ e_i = \frac{F_{11}(c_i)}{k_0 + k_1 c_i} = \frac{F_{22}(c_i)}{r_0 + r_1 c_i} = \frac{F_{12}(c_i)}{s_0}. \]  \hspace{1cm} (124)

The second and third equalities give polynomials of degree lower than 6 and 5, respectively. Since \( c_i \neq c_j \) for \( i \neq j \) and \( n \) is even, the coefficients of these polynomials are equal to zero for \( n > 4 \). In particular, \( k_0 + k_1 c_i \) divides \( F_{11}(c_i) \) and, thus, \( e_i \) is equal to a linear function of \( c_i \). We have that \( P_{n-3} = q_1 P_{n-2} + q_2 P_{n-1} \) for some constants \( q_1 \) and \( q_2 \), so that there is no independent polynomial \( P_{n-3} \) satisfying the required conditions for \( n > 4 \). In conclusion, we searched for a solution of Problem 2 with one parameter \((M = 1)\), but we ended up to find a solution with \( n/2 - 1 \) parameters. Let us stress that we have not proved that \( M \) cannot be less than \( n/2 - 1 \), we have only proved that \( P_{n-3} \) cannot satisfy the required conditions, so that solutions with \( M > 1 \) may exist. Furthermore, we employed the condition \( M = 1 \) in some intermediate inferences. Thus, to check the existence of better solutions, we need to consider the case \( M \neq 1 \) from scratch.

For the sake of completeness, let us write down the solution for \( n = 4 \). Eqs. (122) reduce to

\[ \sum_{i=1}^{n} e_i A'_i B'_i = 0 \]  \hspace{1cm} (125)

Up to a replacement \( P_1 \rightarrow \lambda_1 P_1 + \lambda_2 P_2 + \lambda_3 P_3 \) for some constants \( \lambda_i \) with \( \lambda_1 \neq 0 \), we have that

\[ e_i = \frac{1}{k_0 + k_1 c_i}. \]  \hspace{1cm} (126)
Thus, the 4 polynomials take the form

\[ P_0 = \hat{a}_1 \hat{b}_1, \quad P_1 = \sum_{i=1}^{4} \frac{\hat{a}_i \hat{b}_i}{k_0 + k_i c_i}, \]
\[ P_2 = \sum_{i=1}^{4} c_i \hat{a}_i \hat{b}_i, \quad P_3 = \sum_{i=1}^{4} \hat{a}_i \hat{b}_i. \]  \hspace{1cm} (127)

Let us give a numerical example with 4 polynomial, built by using the derived equations.

1. Numerical example with \( n = 4 \)

Let us set \( A'_i = i, \ A''_i = 1, \ k_0 = k_1 = r_0 = 1, \ r_1 = 2, \) and \( s_0 = 3. \) Up to a linear transformation of \( x_3 \) and \( x_4, \) this setting gives the polynomials

\[ P_3(x_3, x_4) = \\
5x_3 (8427x_4 + 9430) - 209 (3x_4 (393x_4 + 880) + 1478) \]
\[ P_2(x_2, x_3, x_4) = \\
5538425x_2^3 + 18810 (1445x_2 + 5718x_4 + 6421) x_3 - 786258 (3x_4 (267x_4 + 598) + 1004) \]
\[ P_1(x_1, x_2, x_3, x_4) = \\
2299 [205346285x_3 - 38(63526809x_4 + 35594957)] - 5[-2045057058x_2^2 + \\
1630827(1813x_3 + 1254x_4)x_2 + 2891872832x_3^2 + 49595866272x_4^2 + \\
4892481x_1 (1254x_2 - 1429x_3 - 418) - 87093628743x_3 x_4] \]
\[ P_0(x_1, x_2, x_3, x_4) = \\
(627x_1 + 627x_2 - 46x_3 + 1881(x_4 + 1)) (5016x_1 + 6270x_2 + 2555x_3 - 3762 (4x_4 + 5)) \]

Taking \( x_4 \) as the parameter \( \tau \) and solving the equations \( P_3 = P_2 = P_1 = 0 \) with respect to \( x_3, x_2 \) and \( x_1, \) we replace the result in \( P_0 \) and obtain, up to a constant factor,

\[ \mathcal{R}(\tau) = \frac{\prod_{k=1}^{16} (\tau - \tau_k)}{Q_1(\tau)Q_2(\tau)Q_3(\tau)}, \]  \hspace{1cm} (129)

where

\[ Q_1(\tau) = 8427\tau + 9430, \]
\[ Q_2(\tau) = 3\tau (393\tau + 880) + 1478, \]
\[ Q_3(\tau) = 3\tau (9\tau (7(5367293625\tau + 24273841402) + 288165964484) + 1954792734568) + 1657527934720, \]
\[ (\tau_1, \ldots, \tau_{16}) = - (\frac{86}{69}, \frac{800}{681}, \frac{122}{123}, \frac{3166}{633}, \frac{140}{2452}, \frac{318}{2163}, \frac{558}{152}, \frac{2578}{152}, \frac{158}{1070}, \frac{3932}{3932}, \frac{141}{135}, \frac{1831}{2072}, \frac{1142}{218}) \]  \hspace{1cm} (130)

Over a finite field \( \mathbb{Z}_p, \) one can check that the numerator has about 16 distinct roots for \( p \gg 16. \)

For \( p \simeq 16, \) the roots are lower because of collision or because the denominator of some rational root \( \tau_k \) is divided by \( p. \)
2. Brief excursus on retro-causality and time loops

Previously, we have built the polynomials (120). Setting them equal to zero, we have a triangular system of about $n/2$ polynomial equations that can be efficiently solved in $n/2$ variables, say $x_1$, given the value of the other variables, say $x_2$. This system is more or less symmetric, that is, the variables $x_2$ can be efficiently computed given the first block $x_1$ (up to few variables). To determine the overall set of variables, we need the missing $n/2$ polynomials in the ideal $I$. It is possible to choose the coefficients $c_{l,i}$ of these polynomials in such a way that the associated equations have again a triangular form with respect to one of the two blocks $x_1$ and $x_2$, up to few variables. Thus, we end up with two independent equations and a boundary condition,

\[
\begin{align*}
    x_2 &= R_1(x_1), \\
    x_3 &= R_2(x_2), \\
    x_3 &= x_1,
\end{align*}
\]

(131)

where $R_1$ and $R_2$ vectorial rational functions. The first two equations can be interpreted as time-forward and time-backward processes. The last equation identifies the initial state of the forward process with the final state of the backward process. The overall process can be seen also as a deterministic process in a time loop. This analogy is suggestive, since retro-causality is considered one possible explanation of quantum weirdness. Can a suitable break of causality allow for a description of quantum processes in a classical framework? To be physically interesting, this break should not lead to a computational power beyond the power of quantum computers, otherwise a fine tuning of the theory would be necessary to conceal, in a physical process, much of the power allowed by the causality break. A similar fine tuning is necessary if, for example, quantum non-locality is explained with superluminal interactions. These classical non-local theories need an artificial fine tuning to account for non-signaling of quantum theory.

B. $(n - 1)/3$ parameters at most

In the previous subsection, we have built a class of curves defined by systems of $n - 1$ polynomial equations such that about half of the variables can be efficiently solved over a finite field as functions of the remaining variables. These curves and the polynomial $P_0$ have $2^n$ rational intersection points. From a different perspective (discarding about $n/2$ polynomials), we have found a parametrizable variety with about $n/2$ parameters such that its intersection with some hypersurface has $2^n$ rational points.
In this subsection, we show that the number of parameters can be dropped to about $n/3$ so that about $2n/3$ variables can be efficiently eliminated, at least. In the following, we consider space dimensions $n$ such that $n - 1$ is a multiple of 3. Let us define the integer

$$n_1 \equiv \frac{n - 1}{3}. \quad (132)$$

Let us define the rational numbers $A_i, B_i, \tilde{A}_i, \tilde{B}_i, W_i$, and $c_i$ with $i \in \{1, \ldots, n\}$ as a solution of the equations

$$A_i B_i = W_i, \quad \tilde{A}_i \tilde{B}_i = W_i,$$

$$A_i \tilde{B}_i + \tilde{A}_i B_i = 2c_i W_i,$$

$$\sum_{i=1}^n c_i^{k-1} W_i = 0, \quad k \in \{1, \ldots, n - 1\},$$

$$i \neq j \Rightarrow c_i \neq c_j. \quad (133)$$

The procedure for finding a solution has been given previously. We define the polynomials

$$P_s = \sum_{i} c_i^{s-1} \hat{a}_i \hat{b}_i, \quad s \in \{1, \ldots, n\}. \quad (134)$$

The linear functions $\hat{a}_i$ and $\hat{b}_i$ are defined by the $n - 1$ linear equations

$$\sum_{i=1}^{n_1} c_i^{k-1} (A_i \hat{b}_i + B_i \hat{a}_i) = 0, \quad k \in \{1, \ldots, n_1\}$$

$$\sum_{i=1}^{n_1} c_i^{k-1} (\tilde{A}_i \hat{b}_i + \tilde{B}_i \hat{a}_i) = 0, \quad k \in \{1, \ldots, 2n_1\}. \quad (135)$$

These equations uniquely determine $\hat{a}_i$ and $\hat{b}_i$, up to a linear transformation of the variables $x_1, \ldots, x_{n+1}$. Up to a linear transformation, we have

$$\frac{\partial (\hat{a}_i, \hat{b}_i)}{\partial x_k} = c_i^{k-1} (\tilde{A}_i, \tilde{B}_i), \quad k \in \{1, \ldots, n_1\}$$

$$\frac{\partial (\hat{a}_i, \hat{b}_i)}{\partial x_{k+n_1}} = c_i^{k-1} (A_i, B_i), \quad k \in \{1, \ldots, n_1\}. \quad (136)$$

Since there are rational points in the curve, there is another variable, say $x_{2n_1+1}$, such that the second derivative $\partial^2 P_1/\partial x_{2n_1+1}^2$ is equal to zero. Using the above equations, we have

$$\frac{\partial P_1}{\partial x_k} = 0, \quad k \in \{1, \ldots, 2n_1 - s + 1\},$$

$$\frac{\partial^2 P_1}{\partial x_k^2} = 0, \quad k = 2n_1 - s + 2, \quad s \in \{1, \ldots, 2n_1 + 1\}. \quad (137)$$

Thus, the first $2n_1 + 1 = \frac{2n+1}{3}$ polynomials take the triangular form (32), up to a reorder of the indices. These polynomials define a parametrizable variety with $(n - 1)/3$ parameters. Stated in a different way, there is a curve and a hypersurface such that their intersection contains $2^n$ points and at least $(2n + 1)/3$ coordinates of the points in the curve can be evaluated efficiently given the value of the other coordinates. It is possible to show that all the intersection points are in the parametrizable variety, that is, they satisfy the third of Conditions (50).
VI. CONCLUSION AND PERSPECTIVES

In this paper, we have reduced prime factorization to the search of rational points of a parametrizable variety \( V \) having an arbitrarily large number \( N_P \) of rational points in the intersection with a hypersurface \( \mathcal{H} \). To reach a subexponential factoring complexity, the number of parameters \( M \) has to grow sublinearly in the space dimension \( n \). In particular, if \( N_P \) grows exponentially in \( n \) and \( M \) scales as a sublinear power of \( n \), then the factoring complexity is polynomial (subexponential) if the computation of a rational point in \( V \), given the parameters, requires a number of arithmetic operations growing polynomially (subexponentially) in the space dimension.

Here, we have considered a particular kind of rational parametrization. A set of \( M \) coordinates, say \( x_{n-M+1}, \ldots, x_n \), of the points in \( V \) are identified with the \( M \) parameters, so that the first \( n-M \) coordinates are taken equal to rational functions of the last \( M \) coordinates. In particular, the parametrization is expressed in a triangular form. The \( k \)-th variable is taken equal to a rational function \( R_k = \frac{N_k}{D_k} \) of the variables \( x_{k+1}, \ldots, x_n \), with \( k \in \{1, \ldots, n-M\} \). That is,

\[
x_k = R_k(x_{k+1}, \ldots, x_n), \quad k \in \{1, \ldots, n-M\},
\]

which parametrize a variety in the zero locus of the \( n-M \) polynomials,

\[
P_k = D_k x_k - N_k, \quad k \in \{1, \ldots, n-M\}.
\]

To reach polynomial complexity, there are two requirements on these polynomials. First, they have to contain a number of monomials scaling polynomially in \( n \), so that the computation of \( R_k \) is efficient. For example, we could require that the degree is upper-bounded by some constant.

Second, their zero locus has to share an exponentially large number of rational points with some hypersurface \( \mathcal{H} \) (a superpolynomial scaling \( N_P \sim e^{b n^\beta} \) with \( 0 < \beta < 1 \) is actually sufficient, provided that the growth of \( M \) is sufficiently slow). The hypersurface is the zero locus of some polynomial \( P_0 \). Also the computation of \( P_0 \) at a point has to be efficient.

We have proposed a procedure for building pairs \( \{V, \mathcal{H}\} \) satisfying the two requirements. First, we define the set of \( N_P \) rational points. This set can depend on some coefficients. Since \( N_P \) has to grow exponentially in the dimension, we need to define them implicitly as common zeros of a set of polynomials, say \( G_1, G_2, \ldots \). The simplest way is to take \( G_k \) as products of linear functions, like the polynomials (3). These polynomials generate an ideal \( I \). The relevant information on the generators is encoded in a satisfiability formula in conjunctive normal form without negations and a linear matroid. We have called these two objects a model. Second, we search for \( n-M \)
polynomials in $I$ with the triangular form \((139)\). These polynomials always exist. Thus, the task is to find a solution such that the polynomials contain as few monomials as possible. This procedure is illustrated with the simplest example. The generators are taken equal to reducible quadratic polynomials of the form \((3)\), whose associated algebraic set contains $2^n$ rational points. We search for polynomials $P_k$ of the form $\sum_i c_i G_i$ with $c_i$ constant. First, we prove that there is no solution for $M = 1$ and space dimension greater than 4. Then, we find a solution for $M = (n - 1)/3$. If there are solutions with $M$ scaling sublinearly in $n$, then a factoring algorithm with polynomial complexity automatically exists, since the computational complexity of the rational functions $R_k$ is polynomial by construction. The existence of such solutions is left as an open problem.

This work can proceed in different directions. First, it is necessary to investigate whether the studied model admits solutions with a much smaller set of parameters. The search has been performed in a subset of the ideal. Thus, if these solutions do not exist, we can possibly expand this subset (if it is sufficiently large, there is for sure a solution, but the polynomial complexity of $R_k$ is not guaranteed anymore). We could also relax other hypotheses such as the distinguishability of each of the $2^n$ rational points and their membership of the parametrizable variety. More general ideals are another option. In this context, we have shown that classes of models can be reduced to smaller classes by preserving the computational complexity of the associated factoring algorithms. This reduction makes the search space smaller. It is interesting to determine what is the minimal class of models obtained by this reduction. This is another problem left open. Apart from the search of better inputs of the procedure, there is a generalization of the procedure itself. The variety $V$ has the parametrization \((138)\). However, there are more general parametrizable varieties which can be taken in consideration. It is also interesting to investigate if there is some deeper relation with retro-causality, time loops and, possibly, a connection with Shor algorithm. Indeed, in the attempt of lowering the geometric genus of one of the non-parametrizable curves derived in the previous section, we found a set of solutions for the coefficients over the cyclotomic number field, so that the resulting polynomials have terms taking the form of a Fourier transform. Quantum Fourier transform is a key tool in Shor’s algorithm. This solution ends up to break the curve into the union of an exponential large number of parametrizable curves, thus it is not useful for our purpose. Nonetheless, the Fourier-like forms in the polynomials remains suggestive. Finally, the overall framework has some interesting relation with the satisfiability problem. Using a particular matroid, we have seen that there is a one-to-one correspondence between the points of an algebraic set and the solutions of a satisfiability formula (including also negations). To prove that a formula is satisfiable is equivalent to prove that a certain algebraic set is not empty. This mapping of
SAT problems to an algebraic-geometry problem turns out to be a generalization of previous works using the finite field $\mathbb{Z}_2$, see for example Ref. [12]. It can be interesting to investigate whether part of the machinery introduced here can be used for solving efficiently some classes of SAT formulae.

VII. ACKNOWLEDGMENTS

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Appendix A: Basics of algebraic geometry

1. Ideals in polynomial rings and algebraic sets

Given a field $K$, a polynomial ring $K[X_1, \ldots, X_n]$ is a ring whose elements are polynomials in $n$ variables. An ideal $I$ in the polynomial ring is a subset of the ring closed with respect to the addition of elements in $I$ and the multiplication of elements in $I$ by elements of the ring. Every ideal $I$ in the polynomial ring is generated by a finite set of polynomials in $I$ (Hilbert’s basis theorem). Given a set of generators $G_1, \ldots, G_m$ of $I$, every element of the ideal can be written in the form $\sum_{k=1}^{m} F_k G_k$, where $F_k$ are polynomials in the ring. An ideal has infinite possible sets of generators. A set of generators is a particular representation of the ideal and a change of generators is similar to a change of basis in a vector space. There is a correspondence between ideals and algebraic sets. Algebraic sets are subsets of an affine space whose elements are all the common zeros of some set of polynomials in the algebraically closed extension $\bar{K}$ of $K$. This set of polynomials generates an ideal. By definition of ideals, the common zeros of a set of polynomials are common zeros of every polynomial in the generated ideal. Thus, we can associate ideals with algebraic sets. This takes to the following equivalent definition of algebraic set.

Definition 6. A subset $V$ of an affine space is an algebraic set if there is an ideal $I$ such that the common zeros of the polynomials in $I$ are all the elements in $S$. An algebraic set associated with an ideal $I$ is denoted by $V(I)$.

This definition of algebraic set is more appealing as it does not refer to a particular representation of the ideal. The correspondence between ideals and algebraic sets is not one-to-one and an algebraic set is associated with many different ideals. For example, let $V(I)$ be the algebraic set of the ideal $I$ with generators $G_1, \ldots, G_m$. Let $J$ be the ideal generated by $G_1^k, G_2^k, \ldots, G_m^k$, where
$k$ is some integer greater than 1. In general, the ideal $J$ is a strict subset of $I$. It is easy to find examples for which $J \subset I$, such as $G_i = x_i$ with $i \in \{1, \ldots, n\}$ and $m = n$. The common zeros of the polynomials in $J$ are all the elements in $V(I)$. Thus $V(I) = V(J)$, but $J \subset I$. A stricter correspondence between ideals and algebraic sets is obtained by associating an algebraic set $V$ with the largest ideal $I$ such that $V = V(I)$, called the vanishing ideal.

**Definition 7.** Given an algebraic set $V$, we define the vanishing ideal of $V$ as

$$I(V) \equiv \bigcup_{V(I)=V} I.$$  \hspace{1cm} (A1)

In other words, $I(V)$ is the set of all the polynomials which are zero in $V$.

The composition $I(V(I)) \equiv I^*$ maps an ideal $I$ to the largest ideal with same associated algebraic set. This map is idempotent, that is, $(I^*)^* = I^*$. We have

$$I(V(I)) = I(V(I^*)) = I^*.$$  

Thus, there is bijective correspondence between algebraic sets and ideals of the form $I^*$.

Hilbert’s Nullstellensatz identifies the operation $I^*$ with the radicalization of $I$.

**Definition 8.** The radical of an ideal $I$, denoted by $\sqrt{I}$, is an ideal satisfying the double implication

$$a \in \sqrt{I} \iff \exists k \in Z^+ \text{ s.t. } a^k \in I.$$  

The radical of an ideal $I$ contains $I$. If $\sqrt{I} = I$, $I$ is called radical ideal or semiprime ideal.

Also the radicalization of an ideal is an idempotent operation, that is, $\sqrt{\sqrt{I}} = \sqrt{I}$. Since the polynomials $a$ and $a^k$ share the same zeros, it comes that

$$V(I) = V(\sqrt{I}).$$

**Theorem 3.** (Hilbert’s Nullstellensatz) Given the algebraic set $V \equiv V(I)$ of an ideal $I$, a polynomial is zero in $V$ if and only if it is an element of $\sqrt{I}$, that is, $\sqrt{I} = I^*$.

Thus,

$$I(V(I)) = I(V(\sqrt{I})) = \sqrt{I}.$$  

**Definition 9.** The intersection of $I \cap J$ of two ideals $I$ and $J$ is an ideal whose elements are both in $I$ and $J$.  

It comes from the definition of intersection that
\[ \mathbf{V}(I \cap J) = \mathbf{V}(I) \cup \mathbf{V}(J). \tag{A2} \]

**Definition 10.** A prime ideal \( p \) of a polynomial ring \( K[X_1, \ldots, X_n] \) is a strict subset of the ring such that, for every \( a \) and \( b \) in the ring, \( ab \in p \) implies that \( a \) or \( b \) are in \( p \).

A prime ideal generalizes the concept of prime integers. The ideal generated by a prime integer in the ring \( \mathbb{Z} \) is a prime ideal. Excluding the ring among the prime ideals is like excluding 1 among prime numbers, whose ideal is the whole ring \( \mathbb{Z} \).

**Lemma 8.** The radical of an ideal \( I \) is equal to the intersection of every prime ideal containing \( I \), that is,
\[ \sqrt{I} = \bigcap_{p \text{ prime}, p \supseteq I} p \tag{A3} \]

Let us illustrate this lemma with an example in one dimension. Let \( I \) be the ideal generated by the polynomial \( P = (x_1 - k_1 x_0)^{a_1} \ldots (x_1 - k_m x_0)^{a_m} \), where the exponents are positive integers and \( i \neq j \Rightarrow k_i \neq k_j \). Its radical is generated by \( P_r = (x_1 - k_1 x_0) \ldots (x_1 - k_m x_0) \). There are \( m \) prime ideals containing \( I \) and they are generated by the linear polynomials \((x_1 - k_1 x_0),(x_1 - k_2 x_0), \ldots, (x_1 - k_m x_0)\). Their intersection is the ideal generated by \( P_r \), in accordance with the lemma.

A direct consequence of Lemma 8 is the following.

**Corollary 2.** If \( I \) and \( J \) are radicals, then also \( I \cap J \) is radical.

This lemma and Eq. (A2) imply that the order of the operations of intersection and radicalization can be exchanged, that is,
\[ \sqrt{I} \cap \sqrt{J} = \sqrt{I \cap J}. \tag{A4} \]

2. **Product of ideals**

The product of two ideals \( I \) and \( J \) in the polynomial ring \( K[X_1, \ldots, X_n] \), denoted by \( I \cdot J \), is an ideal generated by the product of the generators of \( I \) and \( J \). Namely, if \( G_1, \ldots, G_r \) and \( H_1, \ldots, H_s \) generates \( I \) and \( J \), respectively, then the polynomials \( G_i H_j \) with \( i \in \{1, \ldots, r\} \) and \( j \in \{1, \ldots, s\} \) generate \( I \cdot J \). That is,
\[ I \cdot J = \left\{ \sum_{i,j} P_{i,j} G_i H_j | P_{i,j} \in K[X_1, \ldots, X_n] \right\}. \tag{A5} \]
It is easy to show that

\[ V(I \cdot J) = V(I) \cup V(J) = V(I \cap J), \quad (A6) \]

that is, the algebraic set associated with \( I \cdot J \) is equal to the union of the algebraic sets associated with \( I \) and \( J \). Thus, the multiplication of ideals acts on the associated algebraic sets like the intersection of the ideals. The relation between product of radical ideals and their intersection is given by the equation

\[ I, J \text{ radical ideals} \Rightarrow I \cap J = \sqrt{I \cdot J}. \quad (A7) \]

In general, we have

\[ \sqrt{I \cap J} = \sqrt{I} \cap \sqrt{J} = \sqrt{I} \cdot J. \quad (A8) \]

On one side, multiplication of ideals has the advantage of being much simpler than intersection. On the other side, the product of radicals is not generally a radical ideal. As we will see, the intersection of ideals can have fewer generators with lower degree than the product. Furthermore, the intersection contains all the polynomials that are zero in the associated algebraic set, which can be a desired property.

### Appendix B: Linear-algebra tools

In this appendix, we discuss some direct consequences of Jordan’s theorem that are used in Sec. V. Every \( n \times n \) square matrix \( \mathbf{M} \) can be transformed to a Jordan normal form over an algebraically closed field through a basis change. This means that there is a basis of vectors grouped in \( m \) sets \( \{\vec{v}_{1,k}|k \in \{1, \ldots, n_1\}\}, \{\vec{v}_{2,k}|k \in \{1, \ldots, n_2\}\}, \ldots, \{\vec{v}_{m,k}|k \in \{1, \ldots, n_m\}\} \) with \( \sum_{k=1}^{m} n_m = n \) such that the application of \( \mathbf{M} \) acts as follows

\[
\begin{align*}
\hat{M}\vec{v}_{s,k} &= \lambda_s \vec{v}_{s,k} + \vec{v}_{s,k+1} & k \in \{1, \ldots, n_s - 1\} \\
\hat{M}\vec{v}_{s,k} &= \lambda_s \vec{v}_{s,k} & k = n_s
\end{align*}
\quad (B1)
\]

In particular, if \( n_s = 1 \) for every \( s \), then the matrix is diagonalizable. The existence of this basis is stated by Jordan’s theorem.

Let us apply Jordan’s theorem to the following equation

\[ A\vec{w}_k = \mathcal{B} \sum_{l=1}^{n} \lambda_{k,l} \vec{w}_l \quad k \in \{1, \ldots, n - \delta\}, \quad (B2) \]
where \( \delta \in \{0, \ldots, n - 1\} \). The vectors \( \vec{w}_1, \ldots, \vec{w}_n \) are linearly independent, on which the matrices \( A \) and \( B \) act. The task is to find a set \( \{\vec{w}_1, \ldots, \vec{w}_n\} \) satisfying Eq. (B2), given \( A \) and \( B \). In the following, we denote by boldface characters matrices acting on the labels of the vectors \( \vec{w}_k \).

That is, \( A \) denotes an \( i \times j \) matrix transforming a set of vectors \( \vec{w}_1, \ldots, \vec{w}_j \) to the set of vectors \( \sum_l A_{1,l} \vec{w}_l, \ldots, \sum_l A_{i,l} \vec{w}_l \). The \( n \times n \) matrices acting on the vectors \( \vec{w}_k \) are denoted by calligraphic characters. The left-hand side of Eq. (B2) has only vectors \( \vec{w}_k \) with \( k \in \{1, \ldots, n - \delta\} \). Thus, a transformation \( T \) on \( \vec{w}_k \) preserves the form of the equation if the matrix has the form

\[
T = \begin{pmatrix} R & 0 \\ S & D \end{pmatrix}, \tag{B3}
\]

where \( R, S, \) and \( D \) are \((n - \delta) \times (n - \delta)\), \( \delta \times (n - \delta)\), and \( \delta \times \delta \) matrices, respectively. Let us define the basis of vectors

\[
\vec{v}_k \equiv \sum_{l=1}^{n} T_{k,l} \vec{w}_l \quad k \in \{1, \ldots, n\}.
\tag{B4}
\]

Denoting by \( \lambda \) the \((n - \delta) \times n\) matrix with elements \( \lambda_{k,l} \), the change of basis from \( \vec{w}_k \) to \( \vec{v}_k \) takes to the transformation

\[
\lambda \to R \lambda T^{-1}. \tag{B5}
\]

Jordan’s theorem implies that there are \( m \) sets of vectors \( \{\vec{v}_{1,k} | k \in \{1, \ldots, n_1\}\}, \{\vec{v}_{2,k} | k \in \{1, \ldots, n_2\}\}, \ldots, \{\vec{v}_{m,k} | k \in \{1, \ldots, n_m\}\} \) such that

\[
\begin{align*}
s \in \{1, \ldots, \delta\} : & \quad A \vec{v}_{s,k} = B \vec{v}_{s,k+1} \quad k \in \{1, \ldots, n_s - 1\} \\
\text{s } \in \{\delta + 1, \ldots, m\} : & \quad \begin{cases} A \vec{v}_{s,k} = B(\lambda_s \vec{v}_{s,k} + \vec{v}_{s,k+1}) & k \in \{1, \ldots, n_s - 1\} \\
A \vec{v}_{s,n_s} = B \lambda_s \vec{v}_{s,n_s} \end{cases}
\end{align*} \tag{B6}
\]

with

\[
\begin{align*}
s \in \{1, \ldots, \delta\} : & \quad \vec{v}_{s,k} \in \text{span}\{\vec{w}_1, \ldots, \vec{w}_{n-\delta}\} \quad k \in \{1, \ldots, n_s - 1\} \\
\text{s } \in \{\delta + 1, \ldots, m\} : & \quad \vec{v}_{s,n_s} \notin \text{span}\{\vec{w}_1, \ldots, \vec{w}_{n-\delta}\}.
\end{align*} \tag{B7}
\]

If the matrix \( B \) is invertible, these equations provide a simple way for building the vectors \( \vec{v}_{s,k} \). We have that the vectors take the form

\[
\begin{align*}
s \in \{1, \ldots, \delta\} : & \quad \vec{v}_{s,k} = (B^{-1}A)^{k-1} \vec{v}_s \\
s \in \{\delta + 1, \ldots, m\} : & \quad \begin{cases} \vec{v}_{s,k} = (B^{-1}A - \lambda_s)^{k-1} \vec{v}_s \\
(B^{-1}A - \lambda_s)^{n_s} \vec{v}_s = 0 \end{cases} \quad k \in \{1, \ldots, n_s\}.
\end{align*} \tag{B8}
\]
where \( \vec{v}_1, \ldots, \vec{v}_m \) are free vectors. Let us take \( \mathbf{B}^{-1} \mathbf{A} \) is diagonalizable. Thus, the last line is equivalent to the equation

\[
(B^{-1}A - \lambda_s)\vec{v}_s = 0 \quad s \in \{\delta + 1, \ldots, m\}
\]

so that \( \vec{v}_{s,k} = 0 \) for \( k \in \{2, \ldots, n_s\} \). Since the vectors \( \vec{v}_{s,k} \) are linearly independent and, thus, different from zero, we must have

\[
n_s = 1 \quad s \in \{\delta + 1, \ldots, m\}.
\]

Thus, the general solution is given by the set of vectors \( \{\vec{v}_{s,k} \mid s \in \{1, \ldots, \delta\}, k \in \{1, \ldots, n_s\}\} \) and \( \vec{v}_{\delta+1}, \ldots, \vec{v}_m \) defined by the equations

\[
\begin{cases}
  s \in \{1, \ldots, \delta\} : & \vec{v}_{s,k} = (B^{-1}A)^{k-1}\vec{v}_s \quad k \in \{1, \ldots, n_s\}, \\
  s \in \{\delta + 1, \ldots, m\} : & (B^{-1}A - \lambda_s)\vec{v}_s = 0
\end{cases}
\]

with

\[
m - \delta + \sum_{s=1}^{\delta} n_s = n.
\]

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