EQUITABLE PARTITIONS FOR RAMANUJAN GRAPHS

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Abstract. For \(d\)-regular graph \(G\), an edge-signing \(\sigma : E(G) \to \{-1, 1\}\) is called a good signing if the absolute eigenvalues of adjacency matrix are at most \(2\sqrt{d - 1}\). Bilu-Linial conjectured that for each regular graph there exists a good signing. In this paper, by using new concept "Equitable Partition", we solve the conjecture \textsuperscript{1} for some cases. We show how to find out a good signing for special complete graphs and lexicographic product of two graphs. In particular, if there exist two good signings for graph \(G\), then we can find a good signing for a 2-lift of \(G\).

1. Introduction

For any graph \(G\), we denote the set of all vertices and edges of \(G\) by \(V(G)\) and \(E(G)\), respectively. For two vertices \(u, v \in V(G)\), we denote \(u \sim v\) or \(uv\) for brevity, if \(u\) and \(v\) are adjacent. The degree of a vertex \(v \in V(G)\), denoted by \(d(v)\), is the number of adjacent vertices of \(v\). The maximum and minimum degree of graph \(G\) is denoted by \(\Delta(G)\) and \(\delta(G)\), respectively. An edge-signing is an edge-weighted graph obtained by signing \(-1\) and \(1\) to each edge of \(G\). In this case, the sign of the edge \(uv\) is denoted by \(e_{uv}\). For the specific sign \(\sigma : E(G) \to \{-1, 1\}\), we denote \(G^\sigma\) and \(A^\sigma\) for the edge-signed graph and associated signed adjacency matrix, respectively. In the latter case, \(A^\sigma = [a^\sigma_{ij}]\) of \(G^\sigma\) is defined as \(a^\sigma_{ij} = \sigma(ij)\), if \(ij \in E(G)\), and \(a^\sigma_{ij} = 0\), if \(ij \notin E(G)\). The spectrum of a graph \(G^\sigma\), denoted by \(\{\lambda_i\}_{1 \leq i \leq n, n = |V(G)|}\) is the multiset of the eigenvalues of \(A^\sigma\). In addition, define \(\rho(G^\sigma)\) as the maximum value of \(|\lambda_i|\), where \(1 \leq i \leq n\). A graph is called Ramanujan if all its nontrivial eigenvalues lie in \([-2\sqrt{d - 1}, 2\sqrt{d - 1}]\). Bilu and Linial proposed the following conjecture in \[2\].

**Conjecture 1.1.** (Bilu-Linial) Let \(G\) be a \(d\)-regular graph with \(d > 1\). Then \(\rho(G^\sigma) \leq 2\sqrt{d - 1}\), for some edge-signing \(\sigma\) of \(G\).

By omitting the regularity of the above conjecture, we have a stronger conjecture as follow.

\[\text{2000 Mathematics Subject Classification.} \quad 05C50, 05C21.\]

*Key words and phrases.* Edge signing, Good signing, Adjacency matrix, Lexicographic product.

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Conjecture 1.2. [7] Conjecture 2] For a graph G with $\Delta(G) > 1$, there exists an edge signing $\sigma$ for G such that

$$\rho(G^\sigma) \leq 2\sqrt{\Delta(G) - 1}. \tag{1.2}$$

For the first time Lubotzky and et. al [12] and Margulis [13] found the infinite families of Ramanujan graphs. Adam and et. al proved the Conjecture [1.1] for bipartite graphs. In addition, they showed that there are infinite families of $(c, d)$-biregular bipartite graphs such the second largest eigenvalue is less than $\sqrt{c - 1 + \sqrt{d - 1}}$ (1.4). For more information about the construction of Ramanujan graphs see [3, 4, 8, 15, 9, 16].

A 2-lift is a process on a graph which create a new graph with twice vertices as follow. For a graph $G$ define graph $G'$ with $V(G')$ and $E(G')$. For each $u \in V(G)$, we have two vertices $u_0$ and $u_1$ in $V(G')$. If $uv \in V(G)$, then $E(G')$ contains one of the following cases:

$$\{u_0v_0, u_1v_1\}$$

or

$$\{u_0v_1, u_1v_0\}.$$  

Hence, for a graph $G$, we can construct $2^{|E(G)|}$ new graphs by using 2-lift of $G$. First time, Bilu-Linial used the 2-lift method to construct infinite families of expanders. Friedmen was the first one who introduced old and new eigenvalues for 2-lift of graphs [5]. To achieve more information about 2-lift of graphs see [1, 10, 11].

A conference matrix, denoted by $C(n)$, is a weighing matrix of order $n$ and its entries on diagonal are zero and off the diagonal are 1 and -1 such that $C(n)C(n)^T = I$. It is known that a symmetric conference matrix exists if $n = q + 1$, where $q$ is a prime power with $q \equiv 1 \pmod{4}$. A symmetric conference matrix which entries on the first row are non negative is called normalized symmetric conference matrix. A matrix $H_{n-1}$ is defined by removing the first row and column of normalized symmetric conference matrix $C(n)$. For each conference matrix of order $n$, there is a signing $\sigma$ such that $\rho(C(n)) \leq \sqrt{n - 1}$.

Let $G$ and $H$ be two graphs. Then lexicographic product of G and H which is denoted by $G \circ H$ is a graph with vertex set $V(G) \times V(H)$ and two vertices $(x, y)$ and $(z, t)$ are adjacent if and only if either $x$ is adjacent to $z$ in G or $x = z$ and $y$ is adjacent to $t$ in H. A decomposition of graph $G$ is a set of edge-disjoint subgraphs of $G$ such that each edge of $G$ belongs to exactly one subgraph.

2. Preliminaries

There exists a relation between the spectrum of $G'$ and the spectrum of $G$ and $A^\sigma$ as follow.

Proposition 2.1. [2] Lemma 3.1] $\text{Spec}(G') = \text{Spec}(G) \cup \text{Spec}(A^\sigma)$

Definition 2.2. Let $G$ be a graph and $\sigma$ be an edge-signing of $G$. Then for vertex $u$ of $G$, define

$$N^+(u) = \{v | \sigma(uv) = 1\},$$

$$N^-(u) = N(u) \setminus N^+(u).$$

Moreover, for each subset $S \subseteq V(G)$, define

$$d(u, S) = |N^+(u) \cap S| - |N^-(u) \cap S|. \tag{2.2}$$
Definition 2.3. Let $G$ be a graph and $\sigma$ be a signing of it. The partitions $C_1, \ldots, C_k$ of $V(G)$ is called equitable partition, if for each two integers $1 \leq i, j \leq k$ and $u \in C_i$, the value $d(u, C_j)$ depends only on $C_i$ and $C_j$.

Definition 2.4. Given an equitable partition $\pi$ with cells $C_1, \ldots, C_k$. The characteristic matrix $P$ is a $|V(G)| \times k$ matrix with $P_{ij} = 1$ if vertex $i$ belongs to $C_j$ and $P_{ij} = 0$, otherwise. Given an equitable partition $\pi$. Then define matrix $B$ such that $B_{ij} = d(u, C_j)$, where $u \in C_i$.

Notice that the entries of $B$ can be negative. The following proposition is similar to the [6] Lemma 9.3.1. Thus, we skip the proof.

Proposition 2.5. Let $\pi$ be an equitable partition for $G^\sigma$. Then $A^\sigma P = PB$.

By Proposition 2.4, we have $(A^\sigma)^T P = PB^r$, where $r$ is a positive number. Since the entries of $B$ can be negative, it is possible the largest eigenvalue of $B$ and $A^\sigma$ be different. If $X$ is an eigenvector corresponded to eigenvalue $\lambda$ of matrix $A^{\sigma \text{sigma}}$, then $A^\sigma X = \lambda X$. By proposition 2.4, we deduce that $B^T(P^TX) = \lambda(P^TX)$. Thus, $\lambda$ is an eigenvalue of $B$ if and only if $P^TX \neq 0$. To find the spectrum $G^\sigma$, we first characterize the spectrum of $B$. After that compute $P^TX$, where $X$ is eigenvector correspond to $\lambda$. If $P^TX$ is nonzero, then $\lambda$ is eigenvalue of $A^\sigma$. If $P^TX$ is zero, we try to find a relation between entries of $X$ and the eigenvalue $\lambda$.

3. Good signing for some complete graphs

By an equitable partition we show that if there is a conference matrix of order $n$, then there exists a good signing for complete graph $K_m$, where $m \in \{n + 1, n + 2, n + 3\}$.

Lemma 3.1. Let $C(n)$ be a normalize symmetric conference matrix. Then there is a good signing for $K_m$, where $m \in \{n + 1, n + 2, n + 3\}$.

Proof. Since we have three choices for $m$, we consider three cases.

**Case 1.** Suppose that $m = n + 1$. Define an equitable partition with three cells such that $|C_1| = 1, |C_2| = 1$ and $|C_3| = n - 1$. Moreover, $A_{C_3} = H_{n-1}$ and for each $u \in C_3$, define $d(u, C_i) = 1$, where $i = 1, 2$. Moreover, the sign of the edge between $C_1$ and $C_2$ is one. In this sense,

$$B = \begin{bmatrix} 0 & 1 & n-1 \\ 1 & 0 & n-1 \\ 1 & 1 & 0 \end{bmatrix}$$

Suppose that $\lambda$ is an eigenvalue of $A^\sigma$ and $A^\sigma X = \lambda X$. If $P^T X \neq 0$, then $\lambda$ is an eigenvalue of $B$. Since the eigenvalues of $B$ are $\frac{1}{2}(1 - \sqrt{8n - 15})$, $-1, \frac{1}{2}(\sqrt{8n - 15} + 1)$, we can see $\lambda \leq 2\sqrt{n} - 1$. If $P^T X = 0$, then $x_1 = x_2 = 0$ and $\lambda$ is an eigenvalue of $A_{C_3}$. From the fact that the largest eigenvalue of $A_{C_3}$ is $\sqrt{n} - 1$, we get $\sigma$ is a good signing for $K_{n+1}$.

**Case 2.** Assume that $m = n + 2$. Consider an equitable partition by four cells such that $|C_1| = |C_2| = |C_3| = 1$ and $|C_4| = n - 1$. Put $A_{C_4} = H_{n-1}$ and $d(u, C_i) = 1$, for each $u \in C_4$ and $1 \leq i \leq 3$. In addition, assume that the sign of the edges
between $C_1$, $C_2$ and $C_3$ is 1. Hence,

$$B = \begin{bmatrix} 0 & 1 & 1 & n-1 \\ 1 & 0 & 1 & n-1 \\ 1 & 1 & 0 & n-1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

Assume that $\lambda$ is an eigenvalue of $A^\sigma$ and $A^\sigma X = \lambda X$. If $P^T X \neq 0$, then $\lambda$ is an eigenvalue of $B$. Since the eigenvalues of $B$ are $-\sqrt{3n-2+1}, -1, -1, \sqrt{3n-2+1}$, we deduce that $\lambda$ is less than $2\sqrt{n}$. Otherwise, we have $x_1 = x_2 = x_3 = 0$ and $\lambda$ is an eigenvalue of $H_{n-1}$ which is less than $2\sqrt{n}$.

**Case 3.** Assume that $m = n + 3$. Consider an equitable partition with three cells such that $C_1 = \{u_1, v_1\}$, $C_2 = \{u_2, v_2\}$ and $|C_3| = n - 1$. Now, define a signing $\sigma$ on $E(K_m)$ such that $A_{C_i} = H_{n-1}$, $\sigma(u_1v_1) = -\sigma(u_2v_2) = 1$, $\sigma(u_1u_2) = \sigma(v_1v_2) = -\sigma(u_1v_2) = -\sigma(v_1u_2) = 1$ and $d(u, C_i) = 2$, for each $u \in C_3$ and $i = 1, 2$. In this case,

$$B = \begin{bmatrix} -1 & 0 & n-1 \\ 0 & 1 & n-1 \\ 2 & 2 & 0 \end{bmatrix}$$

Let $\lambda$ be an eigenvalue of $K_m^\sigma$ with $A^\sigma X = \lambda X$. If $P^T X \neq 0$, then $\lambda$ is an eigenvalue of $B$. Since the eigenvalues of $B$ are $-\sqrt{4n-3}, 0, \sqrt{4n-3}$, we have $|\lambda| \leq 2\sqrt{n} + 1$. If $P^T X = 0$, then $2x_2 = (\lambda - 1)x_1$ and $2x_3 = (\lambda - 1)x_2$. If $x_1 = 0$, then $\lambda$ is an eigenvalue of $A_{C_1}$. If $x_1 \neq 0$, then $\lambda = -1$ or $\lambda = 3$. Therefore, the assertion holds.

**Example 3.2.** Consider the normalized symmetric conference matrix of order 6 given by

$$A^{\sigma_1} = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & -1 & -1 & 1 \\ 1 & 1 & 0 & 1 & -1 & -1 \\ 1 & -1 & 1 & 0 & 1 & -1 \\ 1 & -1 & -1 & 1 & 0 & 1 \\ 1 & 1 & -1 & -1 & 1 & 0 \end{bmatrix}$$

By Lemma 3.1, we have good signings $\sigma_1, \sigma_2$ and $\sigma_3$ for $K_7, K_8$ and $K_9$, respectively, such that their adjacency matrices are as below.
It is not difficult to see that

$$A^{\sigma_2} = \begin{bmatrix}
0 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 0 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 0 & 1 & -1 & -1 & 1 \\
1 & 1 & 1 & 1 & 0 & 1 & -1 & -1 \\
1 & 1 & 1 & -1 & 1 & 0 & 1 & -1 \\
1 & 1 & 1 & -1 & -1 & 1 & 0 & 1 \\
1 & 1 & 1 & 1 & -1 & -1 & 1 & 0
\end{bmatrix}$$

$$A^{\sigma_1} = \begin{bmatrix}
0 & -1 & 1 & -1 & 1 & 1 & 1 & 1 \\
-1 & 0 & -1 & 1 & 1 & 1 & 1 & 1 \\
1 & -1 & 0 & 1 & 1 & 1 & 1 & 1 \\
-1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 0 & 1 & -1 & -1 \\
1 & 1 & 1 & 1 & -1 & 1 & 0 & 1 \\
1 & 1 & 1 & 1 & -1 & -1 & 1 & 0
\end{bmatrix}$$

4. Good signing for lexicographic product of graphs

In this section, we consider graphs $G \circ K_2$ and $G \circ K_4$ to find relation between edge-signing of $G$ and its lexicographic product. In the former case, we show that for a non-bipartite regular graph $G$, if $G$ is decomposable to two regular bipartite graphs, then there exists a good signing for $G \circ K_2$. In the latter case, if $G$ is regular with a good signing, then $G \circ K_4$ has a good signing. First consider $G \circ K_2$. Assume that $V(K_2) = \{u, v\}$. For each $x, y \in V(G)$ that $x$ is adjacent to $y$, we consider three edge-signing families for the induced subgraph on vertices $\{x, y\} \times \{u, v\}$, which is isomorphic to $C_4$, as bellow:

(i) All edges have signing 1.
(ii) All edges have signing $-1$.
(iii) sign the edges alternatives by $1, -1$

Now, consider an arbitrary signing $(G \circ K_2)^\sigma$ such that $\sigma$ satisfies in above condition and define each cell as $\{x\} \times \{u, v\}$, where $x \in V(G)$. Therefore, $B$ is a matrix of order $n$ and $B_{ij} \in \{-2, 0, 2\}$. Now, define subgraphs $H_1$ and $H_2$ of $G$ such that $V(H_1) = V(G)$ and $xy \in E(H_1)$ if $xy \in E(G)$ and edge-signing of the induced subgraph $\{x, y\} \times \{u, v\}$ is not satisfy in (iii), and $V(H_2) = V(G)$ and $E(H_2) = E(G) \setminus E(H_1)$. Now, define signings $H_1^{\sigma_1}$ and $H_2^{\sigma_2}$ from $\sigma$ as bellow:

(a) $\sigma_1(xy) = \sigma(xu)$, if $xy \in E(H_1)$.
(b) $\sigma_2(xy) = \sigma(xu)$, if $xy \in E(H_2)$.

It is not difficult to see that $A^{\sigma_1} = \frac{1}{2}B$. Suppose that $A^{\sigma}X = \lambda X$, Depending on $P^TX$ is zero or not, consider the following two cases:

Case 1. If $P^TX \neq 0$, then $\lambda$ is an eigenvalue of $B$. Hence, $2\lambda$ is an eigenvalue of $H_1^{\sigma_1}$.
Case 2. If $P^TX = 0$, then we get $x_i = -x_{i+1}$, for each odd number $i, 1 \leq i \leq 2n$. By above signing, for the adjacent vertices $x$ and $y$ in $V(G)$, the submatrix
correspond to the vertices \( \{x, y\} \times \{u, v\} \) is one of the following matrices:

\[
\begin{bmatrix}
1 & 1 \\
1 & 1 \\
1 & 1 \\
1 & 1
\end{bmatrix}
\begin{bmatrix}
-1 & -1 \\
-1 & -1 \\
-1 & -1 \\
-1 & -1
\end{bmatrix}
\begin{bmatrix}
1 & 1 \\
1 & 1 \\
1 & 1 \\
1 & 1
\end{bmatrix}
\]

By multiplying the first or second matrices by the vector \([x_i, x_{i+1}]^T\), we get the vector \(0\). If we multiply the third or fourth matrices by \([x_i, x_{i+1}]^T\), then we have \(2[x_i, x_{i+1}]^T\) or \(-2[x_i, x_{i+1}]^T\), respectively. Now, by multiplying \(A^\sigma\) by \(X\), we deduce that \(\lambda\) is twice of an eigenvalue of matrix corresponded to graph \(H_{2}^\sigma\). Thus, we can deduce that \(\rho(G^\sigma) \leq 2 \max\{\rho(H_{1}^\sigma), \rho(H_{2}^\sigma)\}\).

Assume that \(G\) is a non-bipartite graph and it decomposes to two bipartite graphs \(H_1\) and \(H_2\).

**Lemma 4.1.** Let non-bipartite graph \(G\) be decomposable to \(H_1\) and \(H_2\) such that \(\Delta(G) = 2\Delta(H_1) = 2\Delta(H_2)\). If there are good signings for \(H_1\) and \(H_2\), then there is a good signing for \(G \circ K_2\).

**Corollary 4.2.** If \(G\) is a non-bipartite \(d\)-regular graph and decomposable to \(d/2\)-regular bipartite graphs \(H_1\) and \(H_2\), then \(G \circ K_2\) has a good signing.

**Example 4.3.** Consider two cycles of length six \(H_1\) and \(H_2\) such that they isomorphic to \(u_1v_1u_2v_2u_3v_3u_1\) and \(u_1v_2v_1v_3u_2u_3u_1\), respectively (Fig. 1). Clearly, \(H_1\) and \(H_2\) are 2-regular bipartite graphs of a non-bipartite 4-regular graph Fig. 1.

![Figure 1. A non-bipartite graph of order four](image)

**Theorem 4.4.** If there is a good signing for a \(d\)-regular graph \(G\), then there is a good signing for \(G \circ K_4\).

**Proof.** Suppose that \(V(K_4) = \{u_1, u_2, u_3, u_4\}\) and \(\sigma\) is a signing of \(G\) such that \(\rho(G^\sigma) \leq 2\sqrt{d-1}\). If \(xy \in E(G)\), then the subgraph induced by \(\{x, y\} \times \{u_1, u_2, u_3, u_4\}\) is isomorphic to \(K_{4,4}\). Now, let \(xy \in E(G)\) and define a signing \(\sigma'\) of \(G \circ K_4\) as follow. If \(\sigma(xy) = 1\) or \(\sigma(xy) = -1\), then sign of the subgraph induced by \(\{x, y\} \times \{u_1, u_2, u_3, u_4\}\) is defined as (i) and (ii), respectively.

(i) All edges have signing 1 except the edges between \((x, u_i)\) and \((y, u_i)\), for \(1 \leq i \leq 4\).

(ii) All edges have signing \(-1\) unless the edges between \((x, u_i)\) and \((y, u_i)\), for \(1 \leq i \leq 4\).

For each \(x \in V(G)\) assume that \(\{x\} \times \{u_1, u_2, u_3, u_4\}\) is one cell. By the signing \(\sigma'\), this partition is an equitable partition. In this case, \(B\) is a matrix of order \(n\) and \(B_{ij} \in \{-2, 0, 2\}\). Now, let \(\lambda\) be an eigenvalue of \((G \circ K_4)^{\sigma'}\). Then there exists non-zero vector \(X\) such that \(A^\sigma X = \lambda X\). Now, consider the following two cases:
Case 1. If $P^T X \neq 0$, then $\lambda$ is an eigenvalue of $B$. Since $B = 2A^{\sigma}$, we have $\lambda = 2\lambda'$, where $\lambda'$ is an eigenvalue of $G^{\sigma}$.

Case 2. If $P^T X = 0$, then by using corresponding entries of $X$ for each cell we have $x_i + x_{i+1} + x_{i+2} + x_{i+3} = 0$, where $i = 4q + 1$ and $0 \leq q \leq n - 1$. Clearly, the submatrix correspond to the vertices $\{x, y\} \times \{u_1, u_2, u_3, u_4\}$ is one of the following matrices:

$$
\begin{bmatrix}
-1 & 1 & 1 & 1 \\
1 & -1 & 1 & 1 \\
1 & 1 & -1 & 1 \\
1 & 1 & 1 & -1 \\
\end{bmatrix}
\begin{bmatrix}
1 & -1 & -1 & -1 \\
-1 & 1 & -1 & -1 \\
-1 & -1 & 1 & -1 \\
-1 & -1 & -1 & 1 \\
\end{bmatrix}
$$

In this case, $\lambda = -2\lambda'$, where $\lambda'$ is an eigenvalue of $G^{\sigma}$. Thus, we can state $\rho(A^{\sigma'}) = 2\rho(A^{\sigma})$. From the fact $\rho(A^{\sigma}) \leq 2\sqrt{d-1}$, we deduce that

$$
\rho(A^{\sigma'}) = 2\rho(A^{\sigma}) \leq 2\sqrt{4d-1}.
$$

Definition 4.5. Suppose that $\sigma$ and $\sigma'$ be two good signing for graph $G$. If there is a diagonal matrix $D$ with entries of 1 and $-1$ such that $DA^{\sigma}D = A^{\sigma'}$, then $\sigma$ and $\sigma'$ is equivalent.

Definition 4.6. Let $A$ and $B$ be two matrices of order $n$. Then define a new product $A * B$ as follow:

$$
[A * B]_{ij} = [A]_{ij} [B]_{ij}
$$

Let $\sigma$ and $\sigma'$ be two good signings for $G$ which are not equivalent. Now, we define a new signing $\tau$ for $G$ such that adjacency matrix of $G^{\tau}$ is equal to $A^{\sigma} * A^{\sigma'}$. In other words, the sign of $uv \in E(G)$ equals $\sigma(uy)\sigma'(uv)$. Consider a 2-lift graph of $G$ by $\tau$, say $G'$. Assume that

$$
V(G') = \bigcup_{v \in V(G)} \{v_0, v_1\}.
$$

Now, define a new signing $\phi$ for graph $G'$ such that the sign of the subgraph induced by $\{v_0, v_1, u_1, v_1\}$ is the same as $\sigma'(uv)$.

Theorem 4.7. If $\sigma$ and $\sigma'$ are two good signings for $G$, then $\phi$ is a good signing for $G'$.

Proof. For each $v \in V(G)$, suppose that $\{v_0, v_1\}$ is a cell of $G'$. In this case, $G'$ has $n$ cells as $C_1, \ldots, C_n$. This partition is equitable and for each vertex $v_i \in V(G')$, we have $d(v_i, C_j) \in \{1, -1\}$. Clearly, $B$ is a matrix of order $n$ and is equal to adjacency matrix of $G^{\sigma'}$. Suppose that $\lambda$ is an eigenvalue of $G^{\sigma'}$ and $X$ is a non-zero vector of $\lambda$. Depends on $P^T X$ is zero or not, we have the following cases:

Case 1. Assume that $P^T X$ is non-zero. Hence, $\lambda$ is an eigenvalue of matrix $B$. Since $B = A^{\sigma'}$, we deduce that $\lambda$ is an eigenvalue of $A^{\sigma}$. By the fact that, $\sigma'$ is a good signing for $G$, we get $|\lambda| < 2\sqrt{d-1}$.

Case 2. Suppose that $P^T X = 0$. Since

$$
P^T = \begin{bmatrix}
1 & 1 & 0 & \cdots & 0 \\
0 & 1 & 1 & \cdots & 0 \\
& \ddots & & & \\
0 & \cdots & 0 & 1 & 1
\end{bmatrix}_{n \times 2n}
$$
and 

\[ X^T = [x_1, x_2, \cdots, x_{2n-1}, x_{2n}] \]

we have \( x_i = -x_{i+1} \), for each odd number \( i, 1 \leq i \leq 2n \). So, we have 

\[ X^T = [x_1, -x_1, x_3, -x_3, \cdots, x_{2n-1}, -x_{2n-1}] \]

By removing even rows of \( A^\phi \), we have a matrix of order \( n \times 2n \), say \( D \). It is not difficult to see that 

\[ DX = (A^\tau A^\sigma')^T [x_1, x_3, \cdots, x_{2n-1}]^T \]

On the other hand, 

\[ (A^\tau A^\sigma') = (A^\sigma A^\sigma') A^\sigma = A^\sigma (A^\sigma' A^\sigma') \]

Since \( A^\sigma' A^\sigma = A \), where \( A \) is adjacency matrix of non-signing \( G \). The entries of matrix \( A \) is one or zero. Hence, 

\[ (A^\tau A^\sigma') = A^\sigma \]

Since eigenvalues of \( A^\phi \) is the same as \( D \), and 

\[ DX = (A^\sigma)[x_1, x_3, \cdots, x_{2n-1}]^T, \]

we can infer that \( \lambda \) is an eigenvalue of matrix \( A^\sigma \). From the assumptions, \( \rho(A^\sigma), \rho(A^\sigma') \leq 2\sqrt{d-1} \) and we deduce that \( \rho(A^\tau) \leq 2\sqrt{d-1} \). \( \square \)

Let \( G'' \) be a 2-lift graph that is obtained from \( G' \) by signing \( \phi \). Now, we can state the following theorem.

**Example 4.8.** Consider graph \( G \) with four vertices as Fig. ......

![Figure 2. Graph G.](image)

Suppose that \( \sigma \) and \( \sigma' \) are two good edge-signing such that 

\[
A^\sigma = \begin{bmatrix}
0 & 1 & 0 & 1 \\
1 & 0 & 1 & -1 \\
0 & 1 & 0 & 1 \\
1 & -1 & 1 & 0
\end{bmatrix}, \quad A^{\sigma'} = \begin{bmatrix}
0 & -1 & 0 & 1 \\
-1 & 0 & 1 & -1 \\
0 & 1 & 0 & 1 \\
1 & -1 & 1 & 0
\end{bmatrix}
\]

By Theorem 4.7 we have 

\[
A^\tau = A^\sigma A^{\sigma'} = \begin{bmatrix}
0 & -1 & 0 & 1 \\
-1 & 0 & 1 & 1 \\
0 & 1 & 0 & 1 \\
1 & 1 & 1 & 0
\end{bmatrix}
\]

In this case, graph \( G' \) as Fig....
In this sense, we have

$$A^\phi = \begin{bmatrix}
0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 \\
0 & -1 & 0 & 0 & 1 & 0 & -1 & 0 \\
-1 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
1 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & -1 & 0 & 1 & 0 & 0 \\
\end{bmatrix}$$

Note that the eigenvalues of $A^\phi$ are \( \{ \sqrt{\frac{-17-1}{2}}, -2, -1, -1, 0, 1, 2, \sqrt{\frac{-17-1}{2}} \} \). Therefore, \( \rho(G'^\phi) = \frac{\sqrt{-17-1}}{2} < 2\sqrt{3-1} \).

We can see $\phi$ is a good signing for $G'$.

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