ON THE KK-THEORY OF STRONGLY SELF-ABSORBING C*-ALGEBRAS

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Abstract. Let \( D \) and \( A \) be unital and separable C*-algebras; let \( D \) be strongly self-absorbing. It is known that any two unital \(*\)-homomorphisms from \( D \) to \( A \otimes D \) are approximately unitarily equivalent. We show that, if \( D \) is also \( K_1 \)-injective, they are even asymptotically unitarily equivalent. This in particular implies that any unital endomorphism of \( D \) is asymptotically inner. Moreover, the space of automorphisms of \( D \) is compactly-contractible (in the point-norm topology) in the sense that for any compact Hausdorff space \( X \), the set of homotopy classes \([X, \text{Aut}(D)]\) reduces to a point. The respective statement holds for the space of unital endomorphisms of \( D \). As an application, we give a description of the Kasparov group \( KK(D, A \otimes D) \) in terms of \(*\)-homomorphisms and asymptotic unitary equivalence. Along the way, we show that the Kasparov group \( KK(D, A \otimes D) \) is isomorphic to \( K_0(A \otimes D) \).

0. Introduction

A unital and separable C*-algebra \( D \neq \mathbb{C} \) is strongly self-absorbing if there is an isomorphism \( D \sim \rightarrow D \otimes D \) which is approximately unitarily equivalent to the inclusion map \( D \rightarrow D \otimes D, d \mapsto d \otimes 1_D \). Strongly self-absorbing C*-algebras are known to be simple and nuclear; moreover, they are either purely infinite or stably finite. The only known examples of strongly self-absorbing C*-algebras are the UHF algebras of infinite type (i.e., every prime number that occurs in the respective supernatural number occurs with infinite multiplicity), the Cuntz algebras \( \mathcal{O}_2 \) and \( \mathcal{O}_\infty \), the Jiang–Su algebra \( \mathcal{Z} \) and tensor products of \( \mathcal{O}_\infty \) with UHF algebras of infinite type, see [14]. All these examples are \( K_1 \)-injective, i.e., the canonical map \( U(D)/U_0(D) \rightarrow K_1(D) \) is injective.

It was observed in [14] that any two unital \(*\)-homomorphisms \( \sigma, \gamma : D \rightarrow A \otimes D \) are approximately unitarily equivalent, were \( A \) is another unital and separable C*-algebra. If \( D \) is \( K_1 \)-injective, the unitaries implementing the equivalence may even be chosen to
be homotopic to the unit. When $\mathcal{D}$ is $\mathcal{O}_2$, $\mathcal{O}_\infty$, it was known that $\sigma$ and $\gamma$ are even asymptotically unitarily equivalent – i.e., they can be intertwined by a continuous path of unitaries, parametrized by a half-open interval. Up to this point, it was not clear whether the respective statement holds for the Jiang–Su algebra $\mathcal{Z}$. Theorem 2.2 below provides an affirmative answer to this problem. Even more, we show that the path intertwining $\sigma$ and $\gamma$ may be chosen in the component of the unit.

We believe this result, albeit technical, is interesting in its own right, and that it will be a useful ingredient for the systematic further use of strongly self-absorbing $C^*$-algebras in Elliott’s program to classify nuclear $C^*$-algebras by $K$-theory data. In fact, this point of view is our main motivation for the study of strongly self-absorbing $C^*$-algebras; see [8], [10], [16], [17], [18] and [15] for already existing results in this direction.

For the time being, we use Theorem 2.2 to derive some consequences for the Kasparov groups of the form $KK(\mathcal{D}, A \otimes \mathcal{D})$. More precisely, we show that all the elements of the Kasparov group $KK(\mathcal{D}, A \otimes \mathcal{D})$ are of the form $[\varphi] - n[\iota]$ where $\varphi : \mathcal{D} \to K \otimes A \otimes \mathcal{D}$ is a $*$-homomorphism and $\iota : \mathcal{D} \to A \otimes \mathcal{D}$ is the inclusion $\iota(d) = 1_A \otimes d$ and $n \in \mathbb{N}$. Moreover, two non-zero $*$-homomorphisms $\varphi, \psi : \mathcal{D} \to K \otimes A \otimes \mathcal{D}$ with $\varphi(1_D) = \psi(1_D) = e$ have the same KK-theory class if and only if there is a unitary-valued continuous map $u : [0, 1) \to e(K \otimes A \otimes \mathcal{D})e$, $t \mapsto u_t$ such that $u_0 = e$ and $\lim_{t \to 1} \|u_t \varphi(d) u_t^* - \psi(d)\| = 0$ for all $d \in \mathcal{D}$. In addition, we show that $KK_i(\mathcal{D}, \mathcal{D} \otimes A) \cong K_i(D \otimes A)$, $i = 0, 1$.

One may note the similarity to the descriptions of $KK(\mathcal{O}_\infty, \mathcal{O}_\infty \otimes A)$ ([8], [10]) and $KK(\mathbb{C}, \mathbb{C} \otimes A)$. However, we do not require that $\mathcal{D}$ satisfies the universal coefficient theorem (UCT) in KK-theory. In the same spirit, we characterize $\mathcal{O}_2$ and the universal UHF algebra $\mathcal{Q}$ using $K$-theoretic conditions, but without involving the UCT.

As another application of Theorem 2.2 (and the results of [7]), we prove in [4] an automatic trivialization result for continuous fields with strongly self-absorbing fibres over finite dimensional spaces.

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1. Strongly self-absorbing $C^*$-algebras

In this section we recall the notion of strongly self-absorbing $C^*$-algebras and some facts from [14].

1.1 Definition: Let $A$, $B$ be $C^*$-algebras and $\sigma, \gamma : A \to B$ be $*$-homomorphisms. Suppose that $B$ is unital.
(i) We say that $\sigma$ and $\gamma$ are approximately unitarily equivalent, $\sigma \approx_u \gamma$, if there is a sequence $(u_n)_{n \in \mathbb{N}}$ of unitaries in $B$ such that
\[
\|u_n \sigma(a) u_n^* - \gamma(a)\| \xrightarrow{n\to\infty} 0
\]
for every $a \in A$. If all $u_n$ can be chosen to be in $U_0(B)$, the connected component of $1_B$ of the unitary group $U(B)$, then we say that $\sigma$ and $\gamma$ are strongly approximately unitarily equivalent, written $\sigma \approx_{su} \gamma$.

(ii) We say that $\sigma$ and $\gamma$ are asymptotically unitarily equivalent, $\sigma \approx_{uh} \gamma$, if there is a norm-continuous path $(u_t)_{t \in [0,\infty)}$ of unitaries in $B$ such that
\[
\|u_t \sigma(a) u_t^* - \gamma(a)\| \xrightarrow{t\to\infty} 0
\]
for every $a \in A$. If one can arrange that $u_0 = 1_B$ and hence $(u_t \in U_0(B)$ for all $t$), then we say that $\sigma$ and $\gamma$ are strongly asymptotically unitarily equivalent, written $\sigma \approx_{suh} \gamma$.

1.2 The concept of strongly self-absorbing $C^*$-algebras was formally introduced in [14] Definition 1.3:

**Definition:** A separable unital $C^*$-algebra $D$ is strongly self-absorbing, if $D \not= \mathbb{C}$ and there is an isomorphism $\varphi : D \to D \otimes D$ such that $\varphi \approx u \text{id}_D \otimes 1_D$.

1.3 Recall [14] Corollary 1.12:

**Proposition:** Let $A$ and $D$ be unital $C^*$-algebras, with $D$ strongly self-absorbing. Then, any two unital $^*$-homomorphisms $\sigma, \gamma : D \to A \otimes D$ are approximately unitarily equivalent. In particular, any two unital endomorphisms of $D$ are approximately unitarily equivalent.

We note that the assumption that $A$ is separable which appears in the original statement of [13] Corollary 1.12 is not necessary and was not used in the proof.

1.4 Lemma: Let $D$ be a strongly self-absorbing $C^*$-algebra. Then there is a sequence of unitaries $(w_n)_{n \in \mathbb{N}}$ in the commutator subgroup of $U(D \otimes D)$ such that for all $d \in D$
\[
\|w_n(d \otimes 1_D)w_n^* - 1_D \otimes d\| \to 0 \text{ as } n \to \infty.
\]

**Proof:** Let $\mathcal{F} \subset D$ be a finite normalized set and let $\varepsilon > 0$. By [14] Prop. 1.5] there is a unitary $u \in U(D \otimes D)$ such that $\|u(d \otimes 1_D)u^* - 1_D \otimes d\| < \varepsilon$ for all $d \in \mathcal{F}$. Let $\theta : D \otimes D \to D$ be a $^*$-isomorphism. Then $\|((\theta(u^*) \otimes 1_D)u(d \otimes 1_D)u^*(\theta(u) \otimes 1_D) - 1_D \otimes d\| < \varepsilon$ for all $d \in \mathcal{F}$. By Proposition 1.3 $\theta \otimes 1_D \approx u \text{id}_D \otimes D$ and so there is a unitary $v \in U(D \otimes D)$ such that $\|\theta(v^*) \otimes 1_D - uvu^*v^*\| < \varepsilon$ and hence $\|((\theta(u^*) \otimes 1_D)u - vu^*v^*u\| < \varepsilon$. Setting $w = vu^*v^*u$ we deduce that $\|w(d \otimes 1_D)w^* - 1_D \otimes d\| < 3\varepsilon$ for all $d \in \mathcal{F}$.

1.5 Remark: In the situation of Proposition 1.3, suppose that the commutator subgroup of $U(D)$ is contained in $U_0(D)$. This will happen for instance if $D$ is assumed to be $K_1$-injective. Then one may choose the unitaries $(u_n)_{n \in \mathbb{N}}$ which implement the approximate
unitary equivalence between $\sigma$ and $\gamma$ to lie in $U_0(A \otimes D)$. This follows from [14] (the proof of) Corollary 1.12, since the unitaries $(u_n)_{n \in \mathbb{N}}$ are essentially images of the unitaries $(w_n)_{n \in \mathbb{N}}$ of Lemma 1.4 under suitable unital *-homomorphisms.

2. ASYMPTOTIC VS. APPROXIMATE UNITARY EQUIVALENCE

It is the aim of this section to establish a continuous version of Proposition 1.3.  

2.1 Lemma: Let $\mathcal{D}$ be separable unital strongly self-absorbing C*-algebra. For any finite subset $\mathcal{F} \subset \mathcal{D}$ and $\varepsilon > 0$, there are a finite subset $\mathcal{G} \subset \mathcal{D}$ and $\delta > 0$ such that the following holds:

If $A$ is another unital C*-algebra and $\sigma : \mathcal{D} \to A \otimes \mathcal{D}$ is a unital *-homomorphism, and if $w \in U_0(A \otimes \mathcal{D})$ is a unitary satisfying

\[ \|[w, \sigma(d)]\| < \delta \]

for all $d \in \mathcal{G}$, then there is a continuous path $(w_t)_{t \in [0,1]}$ of unitaries in $U_0(A \otimes \mathcal{D})$ such that $w_0 = w$, $w_1 = 1_{A \otimes \mathcal{D}}$ and

\[ \|[w_t, \sigma(d)]\| < \varepsilon \]

for all $d \in \mathcal{F}$, $t \in [0,1]$.

Proof: We may clearly assume that the elements of $\mathcal{F}$ are normalized and that $\varepsilon < 1$. Let $u \in \mathcal{D} \otimes \mathcal{D}$ be a unitary satisfying

\[ \|u(d \otimes 1_{\mathcal{D}})u^* - 1_{\mathcal{D}} \otimes d\| < \frac{\varepsilon}{20} \]

for all $d \in \mathcal{F}$. There exist $k \in \mathbb{N}$ and elements $s_1, \ldots, s_k, t_1, \ldots, t_k \in \mathcal{D}$ of norm at most one such that

\[ \|u - \sum_{i=1}^{k} s_i \otimes t_i\| < \frac{\varepsilon}{20}. \]

Set

\[ \delta := \frac{\varepsilon}{k \cdot 10} \]

and

\[ \mathcal{G} := \{s_1, \ldots, s_k\} \subset \mathcal{D}. \]

Now let $w \in U_0(A \otimes \mathcal{D})$ be a unitary as in the assertion of the lemma, i.e., $w$ satisfies

\[ \|[w, \sigma(s_i)]\| < \delta \]

for all $i = 1, \ldots, k$. We proceed to construct the path $(w_t)_{t \in [0,1]}$.

By [14] Remark 2.7 there is a unital *-homomorphism

$\varphi : A \otimes \mathcal{D} \otimes \mathcal{D} \to A \otimes \mathcal{D}$
such that

\[(6) \quad \| \varphi(a \otimes 1_D) - a \| < \frac{\varepsilon}{20} \]

for all \( a \in \sigma(F) \cup \{w\} \).

Since \( w \in U_0(A \otimes D) \), there is a path \((\tilde{w}_t)_{t \in [\frac{1}{2}, 1]}\) of unitaries in \( A \otimes D \) such that

\[(7) \quad \tilde{w}_{\frac{1}{2}} = w \text{ and } \tilde{w}_1 = 1_{A \otimes D}. \]

For \( t \in [\frac{1}{2}, 1] \) define

\[(8) \quad w_t := \varphi((\sigma \otimes \text{id}_D)(u)^*(\tilde{w}_t \otimes 1_D)\sigma \otimes \text{id}_D)(u) \in U(A \otimes D); \]

then \((w_t)_{t \in [\frac{1}{2}, 1]}\) is a continuous path of unitaries in \( A \otimes D \). For \( t \in [\frac{1}{2}, 1] \) and \( d \in F \) we have

\[
\|[(w_t, \sigma(d))]\| \\
= \|w_t \sigma(d)w_t^* - \sigma(d)\| \\
\leq \|w_t \varphi(\sigma(d) \otimes 1_D)w_t^* - \varphi(\sigma(d) \otimes 1_D)\| + 2 \cdot \frac{\varepsilon}{20} \\
\leq \|((\sigma \otimes \text{id}_D)(u)^*(\tilde{w}_t \otimes 1_D)\sigma \otimes \text{id}_D)(u(1_D \otimes d)u^*)((\tilde{w}_t^* \otimes 1_D) \\
- (\sigma \otimes \text{id}_D)(d \otimes 1_D))\| + \frac{\varepsilon}{10} \\
\leq \|((\sigma \otimes \text{id}_D)(u)(\tilde{w}_t \otimes 1_D)(\sigma \otimes \text{id}_D)(1_D \otimes d)(\tilde{w}_t^* \otimes 1_D) \\
- (\sigma \otimes \text{id}_D)(d \otimes 1_D))\| + \frac{\varepsilon}{10} + \frac{\varepsilon}{20} \\
= \|((\sigma \otimes \text{id}_D)(u)^*(1_D \otimes d)(1_D \otimes d)u - d \otimes 1_D)\| + \frac{\varepsilon}{10} + \frac{\varepsilon}{20} \\
\leq \frac{\varepsilon}{20} + \frac{\varepsilon}{10} + \frac{\varepsilon}{20} \\
\leq \frac{\varepsilon}{3}. \]
where for the last equality we have used that the \( \bar{w}_t \) are unitaries and that \( \sigma \) is a unital \(*\)-homomorphism. Furthermore, we have

\[
\|w - w\| \leq \|\varphi((\sigma \otimes \text{id}_D)\bar{w}_t)(\sum_{i=1}^k \sigma(s_i) \otimes t_i) - w\| + \frac{\varepsilon}{20}
\]

The above estimate allows us to extend the path \((w_t)_{t \in [\frac{1}{2}, 1]}\) to the whole interval \([0, 1]\) in the desired way: We have \(\|w_{\frac{1}{2}}w^* - 1_D\| < \frac{\varepsilon}{3} < 2\), whence \(-1\) is not in the spectrum of \(w_{\frac{1}{2}}w^*\). By functional calculus, there is \(a = a^* \in A \otimes D\) with \(\|a\| < 1\) such that \(w_{\frac{1}{2}}w^* = \exp(\pi i a)\). For \(t \in [0, \frac{1}{2})\) we may therefore define a continuous path of unitaries

\[
w_t := (\exp(2\pi i t a))w \in \mathcal{U}(A \otimes D).
\]

It is clear that \(w_0 = w\) and \(w_t \to w_{\frac{1}{2}}\) as \(t \to (\frac{1}{2})^-,\) whence \((w_t)_{t \in [0, \frac{1}{2}]}\) is a continuous path of unitaries in \(A\) satisfying \(w_0 = w\) and \(w_1 = 1_A \otimes D\). Moreover, it is easy to see that

\[
\|w_t - w\| \leq \|w_{\frac{1}{2}} - w\| < \frac{\varepsilon}{3}
\]

for all \(t \in [0, \frac{1}{2})\), whence

\[
\|[w_t, \sigma(d)]\| < \|[w_{\frac{1}{2}}, \sigma(d)]\| + \frac{2}{3} \varepsilon < \varepsilon
\]

for \(t \in [0, \frac{1}{2}), d \in \mathcal{F}\).

We have now constructed a path \((w_t)_{t \in [0, 1]} \subset \mathcal{U}(A)\) with the desired properties.

2.2 Theorem: Let \(A\) and \(D\) be unital \(C^*\)-algebras, with \(D\) separable, strongly self-absorbing and \(K_1\)-injective. Then, any two unital \(*\)-homomorphisms \(\sigma, \gamma : D \to A \otimes D\) are strongly asymptotically unitarily equivalent. In particular, any two unital endomorphisms of \(D\) are strongly asymptotically unitarily equivalent.
Proof: Note that the second statement follows from the first one with $A = D$, since $D \cong D \otimes D$ by assumption.

Let $A$ be a unital $C^*$-algebra such that $A \cong A \otimes D$ and let $\sigma, \gamma : D \to A$ be unital $^*$-homomorphisms. We shall prove that $\sigma$ and $\gamma$ are strongly asymptotically unitarily equivalent. Choose an increasing sequence

$$F_0 \subset F_1 \subset \ldots$$

of finite subsets of $D$ such that $\bigcup F_n$ is a dense subset of $D$. Let $1 > \varepsilon_0 > \varepsilon_1 > \ldots$ be a decreasing sequence of strictly positive numbers converging to 0.

For each $n \in \mathbb{N}$, employ Lemma 2.1 (with $F_n$ and $\varepsilon_n$ in place of $F$ and $\varepsilon$) to obtain a finite subset $G_n \subset D$ and $\delta_n > 0$. We may clearly assume that

$$F_n \subset G_n \subset G_{n+1}$$

and that $\delta_{n+1} < \delta_n < \varepsilon_n$ for all $n \in \mathbb{N}$.

Since $\sigma$ and $\gamma$ are strongly approximately unitarily equivalent by Proposition 1.3 and Remark 1.5, there is a sequence of unitaries $(u_n)_{n \in \mathbb{N}} \subset \mathcal{U}_0(A)$ such that

$$\|u_n \sigma(d) u_n^* - \gamma(d)\| < \frac{\delta_n}{2}$$

for all $d \in G_n$, $n \in \mathbb{N}$. Let us set

$$w_n := u_{n+1}^* u_n, \quad n \in \mathbb{N}.$$  

Then $w_n \in \mathcal{U}_0(A)$ and

$$\|[[w_n, \sigma(d)]]\| = \|w_n \sigma(d) w_n^* - \sigma(d)\| \leq \|u_{n+1}^* u_n \sigma(d) u_n^* u_{n+1} - u_{n+1}^* \gamma(d) u_{n+1}\| + \|u_{n+1}^* \gamma(d) u_{n+1} - \sigma(d)\| < \frac{\delta_n}{2} + \frac{\delta_{n+1}}{2} < \frac{\delta_n}{2}$$

for $d \in G_n$, $n \in \mathbb{N}$. Now by Lemma 2.1 (and the choice of the $G_n$ and $\delta_n$), for each $n$ there is a continuous path $(w_{n,t})_{t \in [0,1]}$ of unitaries in $\mathcal{U}_0(A)$ such that $w_{n,0} = w_n$, $w_{n,1} = 1_A$ and

$$\|[[w_{n,t}, \sigma(d)]]\| < \varepsilon_n$$

for all $d \in F_n$, $t \in [0,1]$.

Next, define a path $(\tilde{w}_t)_{t \in [0,\infty)}$ of unitaries in $\mathcal{U}_0(A)$ by

$$\tilde{w}_t := u_{n+1} w_{n,t-n} \text{ if } t \in [n, n+1).$$
We have that
(13) \( \bar{u}_n = u_{n+1}w_n = u_n \)
and that
\( \bar{u}_t \to u_{n+1} \)
as \( t \to n + 1 \) from below, which implies that the path \((\bar{u}_t)_{t \in [0, \infty)}\) is continuous in \( U_0(A) \).
Furthermore, for \( t \in [n, n+1) \) and \( d \in \mathcal{F}_n \) we obtain
\[
\| \bar{u}_t \sigma(d) \bar{u}_t^* - \gamma(d) \|
\leq \| u_{n+1}w_{n,t-n}\sigma(d)w_{n,t-n}^*u_{n+1}^* - \gamma(d) \|
\leq \| u_{n+1}\sigma(d)u_{n+1}^* - \gamma(d) \| + \varepsilon_n
\leq \frac{\delta_{n+1}}{2} + \varepsilon_n
\leq 2\varepsilon_n.
\]
Since the \( \mathcal{F}_n \) are nested and the \( \varepsilon_n \) converge to 0, we have
(14) \( \| \bar{u}_t \sigma(d) \bar{u}_t^* - \gamma(d) \| \xrightarrow{t \to \infty} 0 \)
for all \( d \in \bigcup_{n=0}^{\infty} \mathcal{F}_n \); by continuity and since \( \bigcup_{n=0}^{\infty} \mathcal{F}_n \) is dense in \( D \), we have (14) for all \( d \in D \). Since \( \bar{u}_0 \in U_0(A) \) we may arrange that \( \bar{u}_0 = 1_A \).

3. The group \( KK(D, A \otimes D) \) and some applications

3.1 For a separable \( C^* \)-algebra \( D \) we endow the group of automorphisms \( \text{Aut} (D) \) with the point-norm topology.

Corollary: Let \( D \) be a separable, unital, strongly self-absorbing and \( K_1 \)-injective \( C^* \)-algebra. Then \([X, \text{Aut}(D)]\) reduces to a point for any compact Hausdorff space \( X \).

Proof: Let \( \varphi, \psi : X \to \text{Aut} (D) \) be continuous maps. We identify \( \varphi \) and \( \psi \) with unital \( * \)-homomorphisms \( \varphi, \psi : D \to C(X) \otimes D \). By Theorem 2.2 \( \varphi \) is strongly asymptotically unitarily equivalent to \( \psi \). This gives a homotopy between the two maps \( \varphi, \psi : X \to \text{Aut} (D) \).

3.2 Remark: The conclusion of Corollary 3.1 was known before for \( D \) a UHF algebra of infinite type and \( X \) a CW complex by [13], for \( D = \mathcal{O}_2 \) by [8] and [10], and for \( D = \mathcal{O}_\infty \) by [2]. It is new for the Jiang–Su algebra.

3.3 For unital \( C^* \)-algebras \( D \) and \( B \) we denote by \([D, B]\) the set of homotopy classes of unital \( * \)-homomorphisms from \( D \) to \( B \). By a similar argument as above we also have the following corollary.
Corollary: Let $D$ and $A$ be unital $C^*$-algebras. If $D$ is separable, strongly self-absorbing and $K_1$-injective, then $[D, A \otimes D]$ reduces to a singleton.

3.4 For separable unital $C^*$-algebras $D$ and $B$, let $\chi_i : KK_i(D, B) \to KK_i(\mathbb{C}, B) \cong K_i(B)$, $i = 0, 1$ be the morphism of groups induced by the unital inclusion $\nu : \mathbb{C} \to D$.

Theorem: Let $D$ be a unital, separable and strongly self-absorbing $C^*$-algebra. Then for any separable $C^*$-algebra $A$, the map $\chi_i : KK_i(D, A \otimes D) \to K_i(A \otimes D)$ is bijective, for $i = 0, 1$. In particular both groups $KK_i(D, A \otimes D)$ are countable and discrete with respect to their natural topology.

Proof: Since $D$ is KK-equivalent to $D \otimes O_\infty$, we may assume that $D$ is purely infinite and in particular $K_1$-injective by [11] Prop. 4.1.4. Let $C_\nu D$ denote the mapping cone $C^*$-algebra of $\nu$. By [3] Cor. 3.10, there is a bijection $[D, A \otimes D] \to KK(C_\nu D, SA \otimes D)$ and hence $KK(C_\nu D, SA \otimes D) = 0$ for all separable and unital $C^*$-algebras $A$ as a consequence of Corollary 3.3. Since $KK(C_\nu D, A \otimes D)$ is isomorphic to $KK(C_\nu D, S^2A \otimes D)$ by Bott periodicity and the latter group injects in $KK(C_\nu D, SC(T) \otimes A \otimes D) = 0$, we have that $KK_i(C_\nu D, D \otimes A) = 0$ for all unital and separable $C^*$-algebras $A$ and $i = 0, 1$. Since $KK_i(C_\nu D, D \otimes A)$ is a subgroup of $KK_i(C_\nu D, D \otimes \tilde{A}) = 0$ (where $\tilde{A}$ is the unitization of $A$) we see that $KK_i(C_\nu D, D \otimes A) = 0$ for all separable $C^*$-algebras $A$. Using the Puppe exact sequence, where $\chi_i = \nu^*$,

$$KK_{i+1}(C_\nu D, A \otimes D) \to KK_i(D, A \otimes D) \xrightarrow{\chi_i} KK_i(\mathbb{C}, A \otimes D) \to KK_i(C_\nu D, A \otimes D)$$

we conclude that $\chi_i$ is an isomorphism, $i = 0, 1$. The map $\chi_i = \nu^*$ is continuous since it is given by the Kasparov product with a fixed element (we refer the reader to [12], [9] or [1] for a background on the topology of the Kasparov groups). Since the topology of $K_i$ is discrete and $\chi_i$ is injective, it follows that the topology of $KK_i(D, A \otimes D)$ is also discrete. The countability of $KK_i(D, A \otimes D)$ follows from that of $K_i(A \otimes D)$, as $A \otimes D$ is separable.

3.5 Remark: In contrast to Theorem 3.4, if $D$ is the universal UHF algebra, then $KK(D, \mathbb{C}) \cong \text{Ext}(\mathbb{Q}, \mathbb{Z}) \cong \mathbb{Q}^N$ has the power of the continuum [6] p. 221.

3.6 Let $D$ and $A$ be as in Theorem 3.4 and assume in addition that $D$ is $K_1$-injective and $A$ is unital. Let $\iota : D \to A \otimes D$ be defined by $\iota(d) = 1_A \otimes d$.

Corollary: If $e \in K \otimes A \otimes D$ is a projection, and $\varphi, \psi : D \to e(K \otimes A \otimes D)e$ are two unital $^*$-homomorphisms, then $\varphi \approx_{\text{suh}} \psi$ and hence $[\varphi] = [\psi] \in KK(D, A \otimes D)$. Moreover:

$$KK(D, A \otimes D) = \{[\varphi] - n[\iota] \mid \varphi : D \to K \otimes A \otimes D \text{ is a } \ast \text{-homomorphism, } n \in \mathbb{N}\}.$$
PROOF: Let \( \varphi, \psi \) and \( e \) be as in the first part of the statement. By [14, Cor. 3.1], the unital \( C^* \)-algebra \( e(\mathcal{K} \otimes A \otimes D)e \) is \( D \)-stable, being a hereditary subalgebra of a \( D \)-stable \( C^* \)-algebra. Therefore \( \varphi \approx_{\text{sa}} \psi \) by Theorem 2.2.

Now for the second part of the statement, let \( x \in KK(D, A \otimes D) \) be an arbitrary element. Then \( \chi_0(x) = [e] - n[1_{A \otimes D}] \) for some projection \( e \in \mathcal{K} \otimes A \otimes D \) and \( n \in \mathbb{N} \). Since \( e(\mathcal{K} \otimes A \otimes D)e \) is \( D \)-stable, there is a unital \( * \)-homomorphism \( \varphi : D \to e(\mathcal{K} \otimes A \otimes D)e \). Then

\[
\chi_0([\varphi] - n[i]) = [\varphi(1_D)] - n[1_D] = [e] - n[1_{A \otimes D}] = \chi_0(x),
\]

and hence \( [\varphi] - n[i] = x \) since \( \chi_0 \) is injective by Theorem 3.4.

In the remainder of the paper we give characterizations for the Cuntz algebra \( O_2 \) and for the universal UHF-algebra which do not require the UCT. The latter result is a variation of a theorem of Effros and Rosenberg [5].

3.7 Proposition: Let \( D \) be a separable unital strongly self-absorbing \( C^* \)-algebra. If \( [1_D] = 0 \) in \( K_0(D) \), then \( D \cong O_2 \).

PROOF: Since \( D \) must be nuclear (see [14]), \( D \) embeds unitally in \( O_2 \) by Kirchberg’s theorem. \( D \) is not stably finite since \( [1_D] = 0 \). By the dichotomy of [14, Thm. 1.7] \( D \) must be purely infinite. Since \( [1_D] = 0 \) in \( K_0(D) \), there is a unital embedding \( O_2 \to D \), see [11, Prop. 4.2.3]. We conclude that \( D \) is isomorphic to \( O_2 \) by [14, Prop. 5.12] .

3.8 Proposition: Let \( D, A \) be separable, unital, strongly self-absorbing \( C^* \)-algebras. Suppose that for any finite subset \( F \) of \( D \) and any \( \varepsilon > 0 \) there is a u.c.p. map \( \varphi : D \to A \) such that \( ||\varphi(cd) - \varphi(c)\varphi(d)|| < \varepsilon \) for all \( c, d \in F \). Then \( A \cong A \otimes D \).

PROOF: By [14, Thm. 2.2] it suffices to show that for any given finite subsets \( F, G \) of \( A \) and any \( \varepsilon > 0 \) there is a u.c.p. map \( \Phi : D \to A \) such that (i) \( ||\Phi(cd) - \Phi(c)\Phi(d)|| < \varepsilon \) for all \( c, d \in F \) and (ii) \( ||[\Phi(d), a]|| < \varepsilon \) for all \( d \in F \) and \( a \in G \). We may assume that \( ||d|| \leq 1 \) for all \( d \in F \). Since \( A \) is strongly self-absorbing, by [14, Prop. 1.10] there is a unital \( * \)-homomorphism \( \gamma : A \otimes A \to A \) such that \( ||\gamma(a \otimes 1_A) - a|| < \varepsilon/2 \) for all \( a \in G \). On the other hand, by assumption there is a u.c.p. map \( \varphi : D \to A \) such that \( ||\varphi(cd) - \varphi(c)\varphi(d)|| < \varepsilon \) for all \( c, d \in F \). Let us define a u.c.p. map \( \Phi : D \to A \) by \( \Phi(d) = \gamma(1_A \otimes \varphi(d)) \). It is clear that \( \Phi \) satisfies (i) since \( \gamma \) is a \( * \)-homomorphism. To conclude the proof we check now that \( \Phi \) also satisfies (ii). Let \( d \in F \) and \( a \in G \). Then

\[
||[\Phi(d), a]|| \\
\leq ||[\Phi(d), a - \gamma(a \otimes 1_A)]|| + ||[\Phi(d), \gamma(a \otimes 1_A)]|| \\
\leq 2||\Phi(d)||||a - \gamma(a \otimes 1_A)|| + ||[\gamma(1_A \otimes \varphi(d)), \gamma(a \otimes 1_A)]|| \\
< 2\varepsilon/2 + 0 = \varepsilon.
\]
3.9 Proposition: Let \( \mathcal{D} \) be a separable, unital, strongly self-absorbing \( C^* \)-algebra. Suppose that \( \mathcal{D} \) is quasidiagonal, it has cancellation of projections and that \([1_{\mathcal{D}}] \in nK_0(\mathcal{D})^+\) for all \( n \geq 1 \). Then \( \mathcal{D} \) is isomorphic to the universal UHF algebra \( \mathcal{Q} \) with \( K_0(\mathcal{Q}) \cong \mathbb{Q} \).

Proof: Since \( \mathcal{D} \) is separable unital and quasidiagonal, there is a unital \(*\)-representation \( \pi: \mathcal{D} \to B(H) \) on a separable Hilbert space \( H \) and a sequence of nonzero projections \( p_n \in B(H) \) of finite rank \( k(n) \) such that \( \lim_{n \to \infty} \|[p_n, \pi(d)]\| = 0 \) for all \( d \in \mathcal{D} \). Then the sequence of u.c.p. maps \( \varphi_n: \mathcal{D} \to p_nB(H)p_n \cong M_{k(n)}(\mathbb{C}) \subset \mathcal{Q} \) is asymptotically multiplicative, i.e \( \lim_{n \to \infty} \|[\varphi_n(cd) - \varphi_n(c)\varphi_n(d)]\| = 0 \) for all \( c,d \in \mathcal{D} \). Therefore \( \mathcal{Q} \cong \mathcal{Q} \otimes \mathcal{D} \) by Proposition 3.8.

In the second part of the proof we show that \( \mathcal{D} \cong \mathcal{D} \otimes \mathcal{Q} \). Let \( E_n: \mathcal{Q} \to M_{n!}(\mathbb{C}) \subset \mathcal{Q} \) be a conditional expectation onto \( M_{n!}(\mathbb{C}) \). Then \( \lim_{n \to \infty} \|E_n(a) - a\| = 0 \) for all \( a \in \mathcal{Q} \).

By assumption, for each \( n \) there is a projection \( e \) in \( \mathcal{D} \otimes M_m(\mathbb{C}) \) (for some \( m \)) such that \( n![e] = [1_{\mathcal{D}}] \) in \( K_0(\mathcal{D}) \). Let \( \varphi: M_{n!}(\mathbb{C}) \to M_{n!}(\mathbb{C}) \otimes \mathcal{D} \otimes M_m(\mathbb{C}) \) be defined by \( \varphi(b) = b \otimes e \). Since \( \mathcal{D} \) has cancellation of projections and since \( n![e] = [1_{\mathcal{D}}] \), there is a partial isometry \( v \in M_{n!}(\mathbb{C}) \otimes \mathcal{D} \otimes M_m(\mathbb{C}) \) such that \( v^*v = 1_{M_{n!}(\mathbb{C})} \otimes e \) and \( vv^* = e_{11} \otimes 1_{\mathcal{D}} \otimes e_{11} \). Therefore \( b \mapsto v\varphi(b)\) gives a unital embedding of \( M_{n!}(\mathbb{C}) \) into \( \mathcal{D} \). Finally, \( \psi_n(a) = v(\varphi \circ E_n(a))v^* \) defines a sequence of asymptotically multiplicative u.c.p. maps \( \mathcal{Q} \to \mathcal{D} \). Therefore \( \mathcal{D} \cong \mathcal{D} \otimes \mathcal{Q} \) by Proposition 3.8.

3.10 Remark: Let \( \mathcal{D} \) be a separable, unital, strongly self-absorbing and quasidiagonal \( C^* \)-algebra. Then \( \mathcal{D} \otimes \mathcal{Q} \cong \mathcal{Q} \) by the first part of the proof of Proposition 3.8. In particular \( K_1(\mathcal{D}) \otimes \mathcal{Q} = 0 \) and \( K_0(\mathcal{D}) \otimes \mathcal{Q} \cong \mathcal{Q} \) by the Künneth formula (or by writing \( \mathcal{Q} \) as an inductive limit of matrices).

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