The action $\rho_{AB} \rightarrow A \otimes B \rho_{AB} A^T \otimes B^T$ (upto a normalization factor) of stochastic local operation and classical communication (SLOCC) on a two-qubit density matrix $\rho_{AB} = \frac{1}{2} \sum_{\nu} \lambda_{\nu} \rho_{\nu} \otimes \sigma_{\nu}$ manifests as Lorentz transformation on the $4 \times 4$ real matrix $\Lambda$ (defined by $\Lambda_{\mu \nu} = Tr(\rho_{AB} (\rho_{\nu} \otimes \sigma_{\nu}))$) in the form $\Lambda \rightarrow \Lambda = L_{A,B} \Lambda L_{A,B}^T$, where $L_{A,B} \in SL(2,C)$ and $L_{A,B} \in SO(3,1)$. With the help of suitable Lorentz transformations $L_{A,B},$ it is possible to arrive the Lorentz singular value decomposition $\Lambda = \Lambda_{A,B} \Lambda_{A,B} \Lambda_{A,B}^T$ of the real matrix $\Lambda$ parametrizing the two-qubit states. Verstraete et.al. [1, 2] had arrived at two different types of Lorentz singular value decomposition for the real matrix $\Lambda$ under SLOCC, employing highly technical results on matrix decompositions in $n$ dimensional spaces with indefinite metric [3]. It was pointed out that the canonical forms of Refs.[1, 2] fail to reveal the underlying geometric features in an unambiguous way [4]. In this work we have carried out a complete analysis for obtaining the algebraically different canonical forms of two-qubit states based on an entirely different approach – inspired by the techniques developed in classical polarization optics by some of us [5, 6]. Our approach leads to an elegant geometrical representation of two-qubit state on and within the Bloch ball [7].

We construct two real symmetric matrices $\Omega_A = \Lambda G A^T$, and $\Omega_B = A^T G \Lambda$ where $G = \text{diag}(1, -1, -1, -1)$ denotes the Minkowski metric. These matrices undergo Lorentz congruent transformations $\Omega_{A,B} \rightarrow \Omega_{A,B} = L_{A,B} \Omega_{A,B} L_{A,B}^T$. Following the detailed mathematical analysis carried out in Refs. [5, 6], we recognize that the matrices $G \Omega_{A,B}$ [7] are positive semidefinite and they have either time-like or light-like Minkowski eigenvectors associated with their highest eigenvalue. This results in two distinct canonical forms $\Lambda_1^{\text{lc}}$ or $\Lambda_2^{\text{lc}}$ for the real matrix $\Lambda$. The type-I canonical form, corresponding to time-like eigenvector associated with the highest eigenvalue $\lambda_0$ of $G \Omega_{A,B}$, is diagonal $\Lambda_1^{\text{lc}} = \text{diag}(1, \sqrt{r_1}, -\sqrt{r_2}, -\sqrt{r_3})$ and the associated two-qubit state is in the Bell-diagonal form [7]. On the other hand the type-II canonical form $\Lambda_2^{\text{lc}}$ (corresponding to light-like eigenvector for the highest eigenvalue of $G \Omega_{A,B}$) has a non-diagonal form given by [7] $\Lambda_2^{\text{lc}} = \left( \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & r_1 & 0 & 0 \\ 0 & 0 & -r_1 & 0 \\ 1 - r_0 & 0 & 0 & r_0 \end{array} \right)$, where $0 \leq r_1^2 \leq r_0 \leq 1$, $r_0 = \frac{\lambda_0}{\lambda_1}$, $\rho_0 = (L^T A_{O_A} L_A)_{00}$, with $\lambda_0 \geq \lambda_1$ denoting doubly degenerate eigenvalues of $G \Omega_{A,B}$. We show that under the map $\Lambda^{\text{lc}}$ the set of all four-vectors $\{(x_1, x_2, x_3) : x_1^2 + x_2^2 + x_3^2 = 1\}$ representing the Bloch sphere gets transformed to $\{\Lambda_1^{\text{lc}} (x_1, x_2, x_3) : (1, y_1, y_2, y_3) \}$ depicting ellipsoidal surface with semi-axes lengths $\sqrt{\lambda_i}/\lambda_0, i = 1, 2, 3$ centered at the origin. We also show that the type-II canonical form $\Lambda_2^{\text{lc}}$ maps the set of all four-vectors $\{(x_1, x_2, x_3) : x_1^2 + x_2^2 + x_3^2 = 1\}$ on to a spheroidal surface characterised by the four-vectors $(1, y_1, y_2, y_3) \rightarrow (1, r_0, 0, 0)$ obeying $\frac{x_1^2 + x_2^2}{r_0^2} + (\frac{y_1 - 1 - r_0}{\sqrt{\lambda_0}})^2 = 1$. The geometric intuition underlying the canonical forms is displayed in the figure.
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