Functional analysis/Mathematical physics

The essential spectrum of $N$-body systems with asymptotically homogeneous order-zero interactions

Le spectre essentiel des systèmes à $N$-corps avec interactions asymptotiquement homogènes d'ordre zéro

Vladimir Georgescu $^a$, Victor Nistor $^{b,c}$

$^a$ Département de mathématiques, Université de Cergy-Pontoise, 95000 Cergy-Pontoise, France
$^b$ Université de Lorraine, UFR MIM, Ile du Saulcy, CS 50128. 57045 Metz cedex 01, France
$^c$ Pennsylvania State University, Mathematics Department, University Park, PA 16802, USA

ABSTRACT

We study the essential spectrum of $N$-body Hamiltonians with potentials defined by functions that have radial limits at infinity. The results extend the HVZ theorem which describes the essential spectrum of usual $N$-body Hamiltonians. The proof is based on a careful study of algebras generated by potentials and their cross-products. We also describe the topology on the spectrum of these algebras, thus extending to our setting a result of A. Mageira. Our techniques apply to more general classes of potentials associated with translation invariant algebras of bounded uniformly continuous functions on a finite-dimensional vector space $X$.

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RÉSUMÉ

Nous étudions le spectre essentiel des hamiltoniens des systèmes à $N$ corps avec potentiels définis par des fonctions qui ont des limites radiales à l’infini. Les résultats étendent le théorème HVZ, qui décrit le spectre essentiel des hamiltoniens des systèmes à $N$ corps usuels. La preuve de notre théorème principal est basée sur une étude approfondie des algèbres générées par les potentiels avec des limites radiales à l’infini et de leurs produits croisés. Nous décrivons également la topologie sur le spectre de ces algèbres, étendant ainsi à notre cas un résultat de A. Mageira. Nos techniques s’appliquent à des classes plus générales de potentiels associées à des algèbres de fonctions uniformement continues bornées invariantes par translation.

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Soit $X$ un espace vectoriel réel de dimension finie et $S_X := (X \setminus \{0\})/\mathbb{R}_+$ la sphère à l’infini de $X$. On dit qu’une fonction $v : X \to \mathbb{C}$ a des limites radiales uniformes à l’infini si $v(\hat{a}) := \lim_{r \to \infty} v(ra)$ existe uniformément en $\hat{a} \in S_X$. Soit $V_Y : X/Y \to \mathbb{R}$ une fonction borélienne ayant des limites radiales uniformes à l’infini, pour chaque sous-espace linéaire $Y \subset X$. Nous supposons $V_Y = 0$, sauf pour un nombre fini d’espaces $Y$. On note $\pi_Y$ la surjection canonique $X \to X/Y$ et on garde la notation $V_Y$ pour la fonction $V_Y \circ \pi_Y$. Dans cet article, nous utilisons des produits croisés de $C^*$-algèbres pour étudier le spectre essentiel des opérateurs de la forme $H := h(P) + \sum_Y V_Y$, ici, $h : X^* \to [0, \infty]$ est une fonction continue et propre et $P$ est l’observable moment (formellement $P = -iV$). Soit $v : X \to \mathbb{C}$ et $\hat{a} \neq 0$ tel que $\lim_{r \to \infty} v(ra + x)$ existe pour tout $x \in X$. Cette limite est une fonction de $x \in X$, qui ne dépend que de la classe $\alpha = \hat{a}$ de $\hat{a}$ dans $S_X$, que nous noterons $t_{\alpha}(v)$. Par exemple, si $v = V_Y$ avec $V_Y$ comme plus haut, alors $t_{\alpha}(V_Y) = V_Y$ si $\alpha \subset Y$ et $\tau(V_Y) = V_Y(\pi_Y(\alpha)) \in \mathbb{R}$ si $\alpha \not\subset Y$, où $\pi(\alpha) \in S_{X/Y}$ est naturellement défini. Plus tard (voir le Théorème 3.1), nous définirons $t_{\alpha}(S)$ pour une classe générale d’opérateurs $S$, en particulier pour $S = H$, ce qui donnera une nouvelle signification à la définition de $t_{\alpha}$.

Nous énonçons maintenant un cas particulier de notre résultat principal : si les fonctions $V_Y : X/Y \to \mathbb{R}$ sont bornées et ont des limites radiales uniformes à l’infini et si, pour chaque $\alpha \in S_X$, on pose $t_{\alpha}(H) = h(P) + \sum_{Y \supset \alpha} V_Y + \sum_{Y \not\supset \alpha} V_Y(\pi_Y(\alpha))$, alors le spectre essentiel de $H$ est $\sigma_{ess}(H) = \bigcup_{\alpha \in S_X} (t_{\alpha}(H))$.

1. Introduction

Let $X$ be a finite dimensional real vector space and, for each linear subspace $Y$ of $X$, let $V_Y : X/Y \to \mathbb{R}$ be a Borel function. We assume $V_Y = 0$, except for a finite number of $Y$. We keep the notation $V_Y$ for the function on $X$ given by $V_Y \circ \pi_Y$, where $\pi_Y : X \to X/Y$ is the natural map. In this paper, we use crossed-products of $C^*$-algebras to study the essential spectrum of Hamiltonians of the form $H := h(P) + \sum_Y V_Y$, under certain conditions on the potentials $V_Y$. Here $h : X^* \to [0, \infty]$ is a continuous, proper function and $P$ is the momentum observable (recall that proper means that $\lim_{|x| \to \infty} h(k) = +\infty$). More precisely, $h(P) = \mathcal{F}^{-1} M_h \mathcal{F}$, where $\mathcal{F} : L^2(X) \to L^2(X^*)$ is the Fourier transform and $M_h$ is the operator of multiplication by $h$ (formally $P = -i\mathcal{F}$). Operators of this form cover the Hamiltonians that are currently the most interesting (from a physical point of view) Hamiltonians of $N$-body systems. Here are two main examples. In a generalized version of the non-relativistic case, a scalar product is given on $X$, so, by taking $h(\xi) = |\xi|^2$, we get $h(P) = \Delta$, the positive Laplacian. In the simplest relativistic case, $X = (\mathbb{R}^3)^N$ and, writing the momentum $P = (P_1, \ldots, P_N)$, we have $h(P) = \sum_{k=1}^{N} (p_k^2 + m_k^2)^{1/2}$ for some real numbers $m_k$. We refer to [3] for a thorough introduction to the subject and study of these systems.

Let $S_X := (X \setminus \{0\})\!/\mathbb{R}_+$ be the sphere at infinity of $X$, i.e. the set of all half-lines $\hat{a} := \mathbb{R}_+ a$. A function $v : X \to \mathbb{C}$ is said to have uniform radial limits at infinity if $v(\hat{a}) := \lim_{r \to \infty} v(ra)$ exists uniformly in $\hat{a} \in S_X$. From the definition of the topology on $S_X$, we get $v(\hat{a}) = \lim_{r \to \infty} v(ra + x)$, $\forall x \in X$. From the definition of the topology on $S_X$, we get $v(\hat{a}) = \lim_{r \to \infty} v(ra + x)$, $\forall x \in X$. The limit may depend on $x$ and defines a function $t_{\alpha}(v) : X \to \mathbb{C}$, where $\alpha := \hat{a}$. For example, let us consider $v = V_Y$. Then $t_{\alpha}(V_Y)(x) = \lim_{r \to \infty} V_Y(r \pi_Y(\alpha) + \pi_Y(x))$. In particular, $t_{\alpha}(V_Y) = V_Y$ whenever $\alpha := \hat{a} \subset Y$ (i.e. $a \in Y$). On the other hand, if $V_Y : X/Y \to \mathbb{C}$ has uniform radial limits at infinity $\hat{a} = a \not\subset Y$, then $\pi_Y(\alpha) := \mathbb{R}_+ \pi_Y(\alpha) \in S_{X/Y}$ is well defined and $\tau_{\alpha}(V_Y)(x) = \pi_Y(\pi_Y(\alpha))$ turns out to be a constant.

Theorem 1.1. Let $V_Y : X/Y \to \mathbb{R}$ be bounded with uniform radial limits at infinity. If $\alpha \in S_X$ set

$$\tau_{\alpha}(H) = h(P) + \sum_Y t_{\alpha}(V_Y) = h(P) + \sum_{Y \supset \alpha} V_Y + \sum_{Y \not\supset \alpha} V_Y(\pi_Y(\alpha)).$$

Then $\sigma(\tau_{\alpha}(H)) = [c_{\alpha}, \infty)$ for some real $c_{\alpha}$ and $\sigma_{ess}(H) = \bigcup_{\alpha \in S_X} \sigma(\tau_{\alpha}(H)) = [\inf_{\alpha} c_{\alpha}, \infty)$.

Here $\bigcup_{\alpha}$ is the closure of the union. Sometimes the union is already closed [11]. Unbounded potentials are considered in Theorem 3.2. If all the radial limits are zero, which is the case of the usual $N$-body potentials, then the terms corresponding to $\alpha \not\subset Y$ are dropped in Eq. (1). Consequently, if $h(P) = \Delta$ is the non-relativistic kinetic energy, we recover the Hunziker, van Winter, Zhishlin (HVZ) theorem. Descriptions of the essential spectrum of various classes of self-adjoint operators in terms of limits at infinity of translates of the operators have already been obtained before, see for example [7,12,8] (in historical order). Our approach is based on the “localization at infinity” technique developed in [5,6] in the context of crossed-product $C^*$-algebras.

Let us sketch the main idea of this approach. Let $C_0^b(X)$ be the algebra of bounded uniformly continuous functions, $C_0(X)$ the ideal of functions vanishing at infinity, and $C^b(X^*) = C + C_0(X)$. Consider a translation invariant $C^*$-subalgebra $\mathcal{A} \subset C_0^b(X)$ containing $C(X^*)$ and let $\hat{A}$ be its character space. Note that $\hat{A}$ is a compact topological space that naturally contains $X$ as an open dense subset and $\delta(A) = \hat{A} \setminus X$ can be thought of as a boundary of $X$ at infinity. Recall that a self-adjoint operator $H$ on a Hilbert space $\mathcal{H}$ is said to be affiliated to a $C^*$-algebra $\mathcal{A}$ of operators on $\mathcal{H}$ if one has $(H + i)^{-1} \in \mathcal{A}$. Then with each self-adjoint operator $H$ affiliated to the crossed product $\mathcal{A} \times X$ by the action of $X$, one may associate a family of self-adjoint operators $H_x$ affiliated to $\mathcal{A} \times X$ indexed by the characters $x \in \delta(A)$. This family
completely describes the image of $H$ (in the sense of affiliated operators) in the quotient of $A \times X$ with respect to the ideal of compact operators. In particular, the essential spectrum of $H$ is the closure of the union of the spectra of the operators $H_x$. These operators are the localizations at infinity of $H$, more precisely, $H_x$ is the localization of $H$ at point $x$.

Once chosen the algebra $A$, in order to use these techniques of this paper, we also need: (1) to have a good description of the character space of the Abelian algebra $A$, and (2) to have an efficient criterion for affiliation to the crossed product $A \times X$. We also indicate how to achieve (1) and (2).

2. Crossed products and localizations at infinity

For $p \in X^*$ and $q \in X$ let $(S_p f)(x) = e^{ix(p)} f(x)$ and $(T_q f)(x) = f(x + q)$. We say that $A \in B(L^2(X))$ has the position-momentum limit property if $\lim_{q \to 0} \| (S_p, A) \| = 0$ and $\lim_{q \to 0} \| (T_q - 1) A (q) \| = 0$ (where $A (q)$ means that the relation holds for $A$ and $A^q$). The set of such operators is a $C^*$-algebra equal to the crossed product $C^b(X) \rtimes X$ [5]. Note that if $A$ is a translation invariant $C^*$-subalgebra of $C^b(X)$, then there is a natural realization of the abstract crossed product $A \times X$ as a $C^*$-algebra of operators on $L^2(X)$ and we do not distinguish the two algebras. We describe this concrete version of $A \times X$ below.

If $\varphi : X \to C$ and $\psi : X^* \to C$ are measurable functions, then $\varphi (Q)$ and $\psi (P)$ are the operators on $L^2(X)$ defined as follows: $\varphi (Q) := M_\varphi$, acts as multiplication by $\varphi$ and $\psi (P) = F^{-1} M_\psi F$, where $F$ is the Fourier transform $L^2(X) \to L^2(X^*)$ and $M_\varphi$ is the operator of multiplication by $\varphi$. Then $\varphi \mapsto \psi (P)$ is an isomorphism between $C^b(X)$ and the group $C^*$-algebra $C^*(X)$ and $A \times X$ is the norm closed linear space of bounded operators on $L^2(X)$ generated by the products $\varphi (Q) \psi (P)$ with $\varphi \in A$ and $\psi \in C^b(X)$. In particular, $A \times X$ consists of operators that have the position-momentum limit property.

We recall the definition of localizations at infinity for such operators. Assume $C(X^*) \subset A$, so $A$ is a compactification of $X$ and $\delta (A) = \hat{A} \setminus X$ is a compact. If $q \in X$ and $\varphi$ is a function on $X$ then $T_q \varphi$ is its translation by $q$. We extend this definition of $T_q$ by replacing $q \in X$ with $x \in \hat{A}$: $(T_q \varphi)(x) = \varphi (T_q x)$, for any $\varphi \in A$, $x \in \hat{A}$, and, $x \in X$. It is clear that $T_q \varphi \in C^b(X)$ and that its definition coincides with the previous one if $x = q \in X$. Moreover, we also get “translations at infinity” of $\varphi \in A$ by elements $x \in \delta(A)$; note however that such a translation does not belong to $A$ in general. Also, the function $x \mapsto T_q \varphi \in C^b(X)$ defined on $\hat{A}$ is continuous if $C^b(X)$ is equipped with the topology of local uniform convergence, hence $T_q \varphi \mapsto \lim_{q \to x} T_q \varphi$ in this topology for any $x \in \delta(A)$. If $A$ is an operator on $L^2(X)$, let $T_q A = T_q AT_q^*$ be its translation by $q \in X$. Clearly $T_q \varphi (P) = (T_q \varphi (Q))$. If $A \in A \times X$, then we may also consider ‘translations at infinity’ by elements of the boundary $\delta(A)$ of $X$ in $\hat{A}$ and we get a useful characterization of the compact operators. The following are mainly consequences of [6, Theorem 1.15]:

(i) For each $x \in \hat{A}$, there is a unique morphism $\tau_x : A \times X \to C^b(X) \times X$ such that $\tau_x (\varphi (Q) \psi (P)) = (T_x \varphi (Q)) \psi (P)$, $\varphi \in C^b(X)$, $\psi \in C^b(X)$. (ii) $\bigcap_{x \in \delta(A)} \ker \tau_x = C^0(X) \times X = \mathcal{X} (X) = \text{ideal of compact operators on } L^2(X)$. (iii) If $H$ is a self-adjoint operator on $L^2(X)$ affiliated to $A$ then for each $x \in \delta(A)$ the limit $\tau_x (H) := s \lim_{q \to x} T_q HT_q^*$ exists and $\tau_x (H) = \bigcup_{x \in \delta(A)} \tau_x (H)$.

To be precise, the last strong limit means: $\tau_x (H)$ is a self-adjoint operator (not necessarily densely defined) on $L^2(X)$ and $s \lim_{q \to x} T_q HT_q^* = \theta (\tau_x (H))$ for all $\theta \in C^b(X)$, it is clear that in the last three statements above one may replace $\delta(A)$ by a subset $\pi$ if for each $A \in A \times X$ we have: $\tau_x (A) = 0 \forall x \in \pi \Rightarrow \tau_x (A) = 0 \forall x \in \delta(A)$. In the case of groupoid (pseudo)differential algebras (that is, when $A$ is a manifold with corners), the morphisms $\tau_x$ can be defined using restrictions to fibers, as in [9], and the last three statements above (i)–(iii) remain valid.

3. Main results

As a warm-up and in order to introduce some general notation, we treat first the two-body case, where complete results may be obtained by direct arguments. The algebra of interactions in the standard two-body case is $C(X^*)$, and hence the Hamiltonian algebra is

$$C(X^*) \rtimes X = C \times X \times C^0(X) \times X = C^*(X) \times \mathcal{X}(X)$$

where the sums are direct. Thus $C(X^*) \times X / \mathcal{X}(X) = C^*(X)$, which finishes the theory. Another elementary case, which has been considered as an example in [5], is $X = \mathbb{R}$ with $C(\mathbb{R})$ replaced by the algebra $C(\mathbb{R})$ of continuous functions that have limits (distinct in general) at $\pm \infty$. Then there is no natural direct sum decomposition of $C(\mathbb{R}) \otimes \mathbb{R}$ as in (2), but one has, by standard arguments, $C(\mathbb{R}) \otimes \mathcal{X}(\mathbb{R}) \simeq C^*(\mathbb{R}) \oplus C^*(\mathbb{R})$. Our purpose in this section is to extend this equation to arbitrary $X$.

Let $C(X)$ be the closure in $C(X)$ of the subalgebra of functions homogeneous of degree zero outside a compact set. Then $C(X) = \{ u \in C(X) \mid \lim_{\| u \| \to \infty} \| (\alpha u) \| = 0 \}$, where, we recall, $\alpha \in \mathbb{R}$, $\alpha$ and $S_X := (X \setminus \{ 0 \})/\mathbb{R}_+$, so $\alpha \in S_X$. As a set, the character space of $C(X)$ can be identified with the disjoint union $X = \bigcup_{\alpha} S_X$. The topology induced by the character space on $X$ is the usual one and the intersections with $X$ of the neighborhoods of some $\alpha \in S_X$ are the sets that contain a truncated cone $C$ such that there is $\alpha \in \alpha$ such $\alpha \in \alpha$ if $\alpha \geq 1$. The set of such subsets is a filter $\mathcal{F}$ on $X$ and, if $Y$ is a Hausdorff space and $u : X \to Y$, then $\lim_{\alpha \to \infty} u = y$ means that $u^{-1}(V) \in \mathcal{F}$ for any neighborhood $V$ of $u$. We shall
write $\lim_{r \to a} u(x)$ instead of $\lim u$. We have that $C(X)$ is a translation invariant $C^*$-subalgebra of $C_0(X)$ and so the crossed product $C(X) \rtimes X$ is well defined. We have the following explicit description of this algebra.

**Proposition 3.1.** The algebra $C(X) \rtimes X$ acting on $L^2(X)$ consists of bounded operators $A$ that have the position-momentum limit property and are such that the limit $\tau_a(A) = \lim_{r \to a} T_a A T_a^*$ exists for each $\alpha = \hat{a} \in S_X$. If $A \in C(X) \rtimes X$ and $\alpha \in S_X$, then $\tau_a(A) \in C^*(X)$ and the map $\tau : C(X) \rtimes X \to C(S_X) \otimes C^*(X)$ is a surjective morphism whose kernel is the set of compact operators on $L^2(X)$, which gives $C(X) \rtimes X / \mathcal{K}(X) \cong C(S_X) \otimes C^*(X)$. If $H$ is a self-adjoint operator affiliated to $C(X) \rtimes X$ then $\tau_a(H) = \lim_{r \to a} T_a T_a^*$ exists for all $\alpha \in S_X$ and $\sigma_{ess}(H) = \bigcup_{\alpha} \sigma(\tau_a(H))$.

In the next two examples $H = h(P) + V$ with $h : X^* \to [0, \infty)$ continuous and proper. We denote by $| \cdot |$ a fixed norm on $X^*$ and by $H^\dagger$ we denote the usual Sobolev spaces on $X$ ($s \in \mathbb{R}$).

**Example 1.** Let $V$ be a bounded symmetric operator satisfying: (1) $\lim_{p \to 0} ||[S_p, V]|| = 0$ and (2) the limit $\tau_a(V) = s$-lim$q \to a$ $T_a V T_a^*$ exists for each $\alpha \in S_X$. Then $H$ is affiliated to $C(X) \rtimes X$ and $\tau_a(H) = h(P) + \tau_a(V)$. Moreover, if $V$ is a function, then $\tau_a(V)$ is a number, but in general we have $\tau_a(V) = v_a(P)$ for some function $v_a \in C_0^0(X^*)$.

**Example 2.** Assume that $h$ is locally Lipschitz and that there exist $c, s > 0$ such that, for all $p$ with $|p| > 1$, $|\nabla h(p)| \leq c(1 + h(p))$ and $c^{-1} |p|^s \leq (1 + h(p))^{s/2} \leq |p|^s$. Let $V : H^\dagger \to H^{-\dagger}$ such that $\pm V \leq \mu h(P) + v$ for some numbers $\mu, v$ with $\mu < 1$ and satisfying the next two conditions: (1) $\lim_{p \to 0} ||[S_p, V]||_{H^\dagger} \to 0$, (2) $\forall \alpha \in S_X$ the limit $\tau_a(V) = s$-$\lim_{q \to a}$ $T_a V T_a^*$ exists strongly in $B(H^\dagger, H^{-\dagger})$. Then $h(P) + V$ and $h(P) + \tau_a(V)$ are symmetric operators $H^\dagger \to H^{-\dagger}$ that induce self-adjoint operators $H$ and $\tau_a(H)$ in $L^2(X)$ affiliated to $C(X) \rtimes X$ and $\sigma_{ess}(H) = \bigcup_{\alpha} \sigma(\tau_a(H))$.

We now treat the $N$-body case. We first indicate a general way of constructing $N$-body Hamitlonians. For each linear subspace $Y \subset X$, let $\mathcal{A}(X/Y) \subset C_0^0(X/Y)$ be a translation invariant $C^*$-subalgebra containing $C_0(X/Y)$ with $A(X/Y) = A(0) = \mathbb{C}$. We embed $A(X/Y) \subset C_0^0(X)$ as usual by identifying $v$ with $v \circ \pi_Y$. Then the $C^*$-algebra $A$ generated by these algebras is a translation invariant $C^*$-subalgebra of $C_0^0(X)$ containing $C(X^*)$ and thus we may consider the crossed product $A \rtimes X$ which is equal to the $C^*$-algebra generated by the crossed products $A(X/Y) \rtimes X$. The operators affiliated to $A \rtimes X$ are $N$-body Hamiltonians. The standard $N$-body algebra corresponds to the minimal choice $A(X/Y) = C_0(X/Y)$ and has remarkable properties, which makes its study relatively easy (it is graded by the lattice of subspaces of $X$). Our purpose in this paper is to study what could arguably be considered to be the simplest extension of the classical $N$-body obtained by choosing $A(X/Y) = C(X/Y)$ for all $Y$. The next more general case would correspond to the choice $A(X/Y) = V(X/Y)$ (slowly oscillating functions, i.e. the closure in sup norm of the set of bounded functions of class $C^1$ with derivatives tending to zero at infinity).

**Definition 3.2.** Let $\mathcal{E}(X)$ be the $C^*$-subalgebra of $C_0^0(X)$ generated by $\bigcup_{\alpha} C^0(X/Y)$.

Clearly $\mathcal{E}(X)$ is a translation invariant $C^*$-subalgebra of $C_0^0(X)$ containing $C(X^*) := C_0(X) + \mathbb{C}$. If $Y$ is a linear subspace of $X$ then the crossed product $X \rtimes Y$ is well defined and naturally embedded in $\mathcal{E}(X)$: it is the $C^*$-algebra generated by $\bigcup_{Y \subset X} C(X/Y)$. We have $C = C(0) = \mathcal{E}(X/Y) \subset \mathcal{E}(X/Y) \subset \mathcal{E}(X/Z) \subset \mathcal{E}(X)$. If $\alpha \in S_X$, we shall denote by abuse of notation $X/\alpha$ be the quotient $X/|\alpha|$ by the subspace $|\alpha| := \mathbb{R} \alpha$ generated by $\alpha$ and let us set $\pi_\alpha = \pi_{|\alpha|}$. It is clear that $\tau_a(u)(x) = \lim_{y \to a} u(xa + x)$ exists $u \in \mathcal{E}(X)$ and that the resulting function $\tau_a(u)$ belongs to $\mathcal{E}(X)$. The map $\tau_a$ is an endomorphism of $\mathcal{E}(X)$ and a linear projection of $\mathcal{E}(X)$ onto the $C^*$-subalgebra $\mathcal{E}(X/\alpha)$.

If $\alpha \in S_X$ and $\beta \in S_X/\alpha$, then $\beta$ generates a one-dimensional linear subspace $[\beta] := \mathbb{R} \beta \subset X/\alpha$, as above, and hence $\pi^{-1}_\alpha([\beta])$ is a two-dimensional subspace of $X$ that we shall denote $[\alpha, \beta]$. We shall identify $(X/\alpha)/\beta$ with $X/\alpha/\beta$. Then we have two idempotent morphisms $\tau_\alpha : \mathcal{E}(X) \to \mathcal{E}(X/\alpha)$ and $\tau_\beta : \mathcal{E}(X/\alpha) \to \mathcal{E}(X/[\alpha, \beta])$. Thus $\tau_\beta \circ \tau_\alpha : \mathcal{E}(X) \to \mathcal{E}(X/[\alpha, \beta])$ is an idempotent morphism. This construction extends in an obvious way to families $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $n \leq \dim X$ and $\alpha_1 \in S_X, \alpha_2 \in S_X/\alpha_1, \alpha_3 \in S_X/\alpha_1, \alpha_2, \ldots$ (we allow $n = 0$ and denote $A$ the set of all such families). The endomorphism $\tau_\alpha$ of $\mathcal{E}(X)$ is defined by induction: $\tau_{\alpha_2} \circ \tau_{\alpha_1} : \mathcal{E}(X) \to \mathcal{E}(X/[\alpha_1, \alpha_2])$.

**Proposition 3.3.** If $\alpha \in A$ and $a \in X/\alpha$, then $\chi(u) = (\tau_\alpha u)(\alpha)$ defines a character of $\mathcal{E}(X)$. Conversely, each character of $\mathcal{E}(X)$ is of this form.

**Remark 1.** A natural Abelian $C^*$-algebra in the present context is the set $\mathcal{R}(X)$ of all bounded uniformly continuous functions $v : X \to C$ such that $\lim_{r \to x} v(ra + x)$ exists locally uniformly in $x \in X$ for each $a \in X$. It would be interesting to find an explicit description of its spectrum.

This description of the spectrum of $\mathcal{E}(X)$ extends [10]. We now state our main results.
Theorem 3.1. Let $H$ be a self-adjoint operator on $L^2(X)$ affiliated to $\mathcal{E}(X) \times X$. Then for any $a \in X \setminus \{0\}$ the limit $s \lim_{r \to \infty} T_{ra}H T_{ra}^* =: \tau_a(H)$ exists and $\sigma_{\text{ess}}(H) = \bigcup_{a \in S_X} \sigma(\tau_a(H))$.

Theorem 3.2. Let $h$ be as in Example 2 and $V = \sum V_Y$ with $V_Y : \mathcal{H}^\ell \to \mathcal{H}^{-\ell}$ symmetric operators such that $V_Y = 0$ but for a finite number of $Y$ and satisfying: (i) $\exists \mu_Y, \nu_Y \geq 0$ with $\sum \mu_Y < 1$ such that $\pm V_Y \leq \mu_Y h(p) + \nu_Y$, (ii) $\lim_{p \to 0} ||S_p \cdot V_Y||_{H^\ell \to \mathcal{H}^{-\ell}} = 0$, (iii) $[T_y, V_Y] = 0$ for all $y \in Y$, (iv) $\tau_a(V_Y) := s \lim_{a \to \infty} \tau_a V_Y T_a^*$ exists in $B(\mathcal{H}^\ell, \mathcal{H}^{-\ell})$ for all $a \in S_{X \setminus Y}$. Then the maps $\mathcal{H}^\ell \to \mathcal{H}^{-\ell}$ given by $h(p) + V$ and $h(p) + \sum Y \tau_a(V_Y)$ induce self-adjoint operators $H$ and $\tau_a(H)$ in $L^2(X)$ affiliated to $\mathcal{E}(X)$ and $\sigma_{\text{ess}}(H) = \bigcup_{a \in S_X} \sigma(\tau_a(H))$.

Example 3. Using [2], we also obtain that Theorem 3.2 covers uniformly elliptic operators of the form $H = \sum_{|k|, |\ell| \leq s} p^k q_{kl} p^\ell$, where $q_{kl}$ are finite sums of functions of the form $V_Y \circ \pi_Y$ with $V_Y : X/Y \to \mathbb{R}$ bounded measurable such that $\lim_{s \to 0} V_Y(2)$ exists uniformly in $a \in S_{X \setminus Y}$. The fact that we allow $q_{kl}$ to be only bounded measurable for $|k| = |\ell| = s$ is not trivial.

In addition to the above-mentioned results, we also use general results on cross-product $\mathbb{C}^*$-algebras, their ideals, and their representations [4, 13]. The maximal ideal spectrum of the algebra $\mathcal{E}(X)$ is of independent interest and can be used to study the regularity properties of the eigenvalues of the $N$-body Hamiltonian [1]. Its relation to the constructions of Vasy in [14] will be studied elsewhere.

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