The Fast Wandering of Slow Birds

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I study a single “slow” bird moving with a flock of birds of a different, and faster (or slower) species. I find that if the speed $v_s$ of the “slow” bird equals a certain “magic” speed $v_2$, different from the speed of the flock, it will randomly wander transverse to the mean direction of flock motion far faster than the other birds will: its RMS transverse displacement will grow with time $t$ like $t^{5/3}$, in contrast to $t^{2/3}$ for the other birds. If $v_s \neq v_2$, the RMS displacement of the “slow” bird crosses over from $t^{5/3}$ to $t^{2/3}$ scaling at a time $t_c$ that scales according to $t_c \propto |v_s - v_c|^{-2}$.

Flocking is the collective, coherent motion of large numbers of organisms – is one of the most familiar and ubiquitous biological phenomena. This phenomenon also spans an enormous range of length scales: from kilometers (herds of wildebeest) to microns (e.g., the microorganism Dictyostelium discoideum). Following the recognition that this phenomenon is a dynamical version of ferromagnetic ordering, a phenomenological continuum theory was developed, which showed, among other things, that two-dimensional flocks did not obey the Mermin-Wagner theorem: that is, they are capable of spontaneously breaking a continuous symmetry (rotation invariance) by developing long-ranged order, even in flocks with only short ranged interactions. Such order is impossible (in spatial dimensions $d = 2$) for equilibrium systems.

This fundamental difference between flocks and equilibrium systems arises from novel “anomalous” scalings of fluctuations in flocks. Many of the quantitative predictions of the phenomenological continuum theory have been confirmed in numerical simulations. In particular, the lateral wandering of a bird in a flock of identical birds was found to grow superdiffusively in $d = 2$, scaling like $t^{5/3}$.

In this paper, I consider a single “slow” bird moving with a flock of birds of a different speed $v_s$, which attempts to follow the other birds. If $v_s \neq v_2$, the “slow” bird will wander laterally much faster than the other birds will: its RMS transverse displacement will grow with time $t$ like $t^{5/3}$, in contrast to $t^{2/3}$ for the other birds. If $v_s = v_2$, the “slow” bird will wander laterally much faster than the other birds.

Consider now the mean squared lateral displacement of the slow bird:

$$w^2(t; v_s) \equiv \langle |x^\perp_s(t) - x^\perp_s(0)|^2 \rangle$$

perpendicular to the mean direction of motion of the flock. (Here and throughout this paper, $\perp$ and $\parallel$ denote directions perpendicular to, and along, the direction of mean flock motion, respectively.)

Since the mean $x^\perp$ position $x^\perp_s$ of the slow bird obeys

$$x^\perp_s(t) = x^\perp_s(0) + \int_0^t v^\perp_s(t)dt$$

where $v^\perp_s(t)$ is the $\perp$ velocity of the slow bird at time $t$, $w^2(t; v_s)$ is given by

$$w^2(t; v_s) = \int_0^t dt' \int_0^t dt'' \langle v^\perp_s(t')v^\perp_s(t'') \rangle .$$

Now I need to relate the perpendicular velocity of the slow bird to the position and time dependent velocity...
field $v_\perp (\vec{r}, t)$ of the flock. Since the slow bird moves in the same direction as the other birds in the flock, the required perpendicular velocity is given by:

$$v_\perp^s (t) = \frac{v_\perp (\vec{r}_s (t), t) v_s}{v_0}$$

(4)

where $v_0$ is the mean speed of the flock (in the sense that, $< \vec{v} (\vec{r}, t) > = v_0 \hat{x}_\parallel$, with $\hat{x}_\parallel$ the unit vector along the direction of mean flock motion), and $\vec{r}_s (t)$ is the position of the slow bird at time $t$. This is given by

$$\vec{r}_s (t) = \vec{r}_s (0) + v_s t \hat{x}_\parallel + \delta x^s_\parallel (t) \hat{x}_\parallel + \delta x^s_\perp (t) \hat{x}_\perp$$

(5)

where $\delta x^s_\parallel (t)$ and $\delta x^s_\perp (t)$ are the deviations of the slow bird from uniform motion at speed $v_s$ along $\hat{x}_\parallel$.

Using (4) and (5), I see that the desired single bird autocorrelation function in [9] is

$$\langle v_\perp^s (t') v_\perp^s (t'') \rangle = \left( \frac{v_s}{v_0} \right)^2 C_\perp \left( x_\perp (t') - x_\perp (t''), v_\perp |t' - t''| + \delta x_\parallel (t') - \delta x_\parallel (t''), t' - t'' \right)$$

(6)

where $C_\perp (\vec{r}, t)$ is the real space and time perpendicular velocity field auto-correlation function.

I’ll assume (and have verified a posteriori) that both $\delta x_\parallel$ and $\delta x_\perp$ are small enough compared to the average motion $v_s t \hat{x}_\parallel$ that their effect on the velocity-velocity auto-correlation in [25] is negligible. This leaves the task of evaluating $C_\perp (R_\perp = 0, R_\parallel = v_s t, t)$.

Expressing $C_\perp$ in terms of its Fourier transform gives

$$C (\delta t) \equiv \langle v_\perp^s (t') v_\perp^s (t'') \rangle = \left( \frac{v_s}{v_0} \right)^2 C_\perp (R_\perp = 0, R_\parallel = v_s \delta t) = \left( \frac{v_s}{v_0} \right)^2 \int dq_\perp dq_\parallel d\omega e^{i (\omega - v_\perp q_\perp \delta t)} C_\perp (q_\perp, \omega)$$

(7)

where I’ve defined $\delta t \equiv t' - t''$, and the spatio-temporally perpendicular Fourier-transformed velocity-velocity correlation function $C_\perp (q_\perp, \omega)$ is predicted by the continuum theory [8] to be:

$$C_\perp (q_\perp, \omega) = \frac{\Delta (\omega - v_\perp q_\parallel)^2}{(\omega - c_+ (\theta_\parallel q_\perp))^2 (\omega - c_- (\theta_\parallel q_\perp))^2 + (\omega (\Gamma_\parallel (q_\perp) + \Gamma_\perp (q_\perp)) - q_\parallel (v_2 \Gamma_L (q_\perp) + \gamma \Gamma_\perp (q_\perp)))^2}$$

(8)

where $\Delta$ is a phenomenological parameter related to the RMS magnitude of the directional errors the birds make in trying to follow each other, and the two direction-dependent sound speeds $c_\pm (\theta_\parallel q_\perp)$ are given by

$$c_\pm (\theta_\parallel q_\perp) = \left( \frac{\gamma + v_2}{2} \right) \cos (\theta_\parallel q_\perp) \pm c_2 (\theta_\parallel q_\perp)$$

(9)

with

$$v_\pm (\theta_\parallel q_\perp) \equiv \pm \left( \frac{\gamma - v_2}{2} \right) \cos (\theta_\parallel q_\perp) + c_2 (\theta_\parallel q_\perp)$$

(10)

and

$$c_2 (\theta_\parallel q_\perp) \equiv \sqrt{\frac{(\gamma - v_2)^2 \cos^2 (\theta_\parallel q_\perp)}{4} + c_0^2 \sin^2 (\theta_\parallel q_\perp)}$$

(11)

where $\theta_\parallel q_\perp$ is the angle between $q_\perp$ and the direction of flock motion (i.e., the $x_\perp$ axis), and $\gamma$ and $v_2$ are additional phenomenological parameters with the dimensions of speed. Neither of them is generally equal to the mean speed $v_0$ of the flock. The speed $v_2$ will prove (as I’ll show below) to be the “magic” speed that leads, if $v_s = v_2$, to faster wandering of the “slow” bird; hence, the fact that
\(v_2 \neq v_0\) is crucial to making the behavior of the “slow” bird distinct from that of the other birds in the flock.

In addition, the wavevector dependent longitudinal, and \(\rho\) dampings \(\Gamma_{L,\rho}\) are given by:

\[
\Gamma_L(\vec{q}) = D_L(\vec{q})q_\perp^2 + D_0 q_\parallel^2 , \tag{12}
\]

\[
\Gamma_\rho(\vec{q}) = D_0^0 q_\parallel^2 , \tag{13}
\]

where the renormalized, wavevector \((\vec{q})\)-dependent diffusion coefficient \(D_L(\vec{q})\) obeys the scaling law:

\[
D_L(\vec{q}) = q_\perp^2 f_L \left( \frac{(q_\perp)}{(q_\Lambda)^2} \right)
\]

where the summation index \(\sigma = \pm\) labels the two sound modes with speeds \(c_\pm(\theta_{\vec{q}})\) (Eqn. 2), and I’ve defined

\[
\Gamma_\sigma(\vec{q}) = \Gamma_L \left( \frac{v_\sigma(\theta_{\vec{q}})}{2c_2(\theta_{\vec{q}})} \right) + \Gamma_\rho \left( \frac{v_{-\sigma}(\theta_{\vec{q}})}{2c_2(\theta_{\vec{q}})} \right) = (D_\parallel q_\parallel^2 + D_L(\vec{q}) q_\perp^2) \left( \frac{v_\sigma(\theta_{\vec{q}})}{2c_2(\theta_{\vec{q}})} \right) + \frac{D_\rho q_\parallel^2}{(v_{-\sigma}(\theta_{\vec{q}}))} \left( \frac{v_{-\sigma}(\theta_{\vec{q}})}{2c_2(\theta_{\vec{q}})} \right) . \tag{16}
\]

Because of the anisotropy of the damping \(\Gamma_\sigma\), this integral is dominated, as \(t \to \infty\), by \(q_\parallel \approx (q_\perp/\Lambda)^\frac{5}{3} \Lambda \gg q_\perp\); hence, \(\vec{q}_\parallel \to 0\), and I can replace \(v_\sigma(\theta_{\vec{q}})\) and \(c_\sigma(\theta_{\vec{q}})\) everywhere they appear with \(v_\sigma(\theta_{\vec{q}_\parallel = 0})\) and \(c_\sigma(\theta_{\vec{q}_\parallel = 0})\). I can likewise replace \(|\vec{q}| = q\) with \(|q_\parallel|\).

Doing this, we easily see that the scaling form Eqn. 14 for \(D_L(\vec{q})\) implies a similar scaling form for \(\Gamma_\sigma(\vec{q})\):

\[
\Gamma_\sigma(\vec{q}) = q_\perp^2 f_\sigma \left( \frac{(q_\perp)}{(q_\Lambda)^2} \right) \propto \left\{ \begin{array}{l}
\frac{q_\perp}{q_\parallel} \ll \left( \frac{q_\perp}{q_\Lambda} \right)^\frac{5}{3} \text{ } q_\parallel = \frac{\Lambda}{t} > \left( \frac{q_\Lambda}{t} \right)^\frac{5}{3} \frac{q_\perp}{q_\parallel} \leq \left( \frac{q_\perp}{q_\Lambda} \right)^\frac{5}{3} \text{ } q_\parallel = \frac{\Lambda}{t} \leq \left( \frac{q_\Lambda}{t} \right)^\frac{5}{3}
\end{array} \right. \tag{17}
\]

Putting \(\theta_{\vec{q}} = 0\) in Eqn. 19, I see immediately that one of the two \(c_\sigma(\theta_{\vec{q}} = 0)\)’s is \(v_2\), while the other = \(\gamma\). Clearly, only the latter value of \(\sigma\) survives, due to the \(c_\sigma q - v_2 q_\parallel\) factor in equation 15. Keeping only the appropriate term then gives:

\[
C(t) = 2 \left( \frac{v_\gamma}{v_0} \right)^2 \Delta \int d^2 q \frac{q_\perp^2 e^{i((v_\gamma - v_2)q_\parallel - \Gamma_\sigma(\vec{q}))t}}{(2\pi)^2} \Gamma_\sigma(\vec{q}) , \tag{18}
\]

where the value of \(\sigma\) is that for which \(c_\sigma(\theta_{\vec{q}}) = \gamma\).

Here \(\Lambda\) is an ultraviolet cutoff of order an inverse interbird distance.

Performing the integral over \(\omega\) by complex contour techniques, and using Eqn. 8 for \(C_\perp(\vec{q},\omega)\), gives

\[
\propto \left\{ \begin{array}{l}
q_\perp \frac{q_\parallel}{q_\perp} \ll \left( \frac{q_\perp}{q_\Lambda} \right)^\frac{5}{3} \text{ } q_\parallel \ll \left( \frac{q_\perp}{q_\Lambda} \right)^\frac{5}{3} \text{ } q_\parallel \gg \left( \frac{q_\perp}{q_\Lambda} \right)^\frac{5}{3}
\end{array} \right. \tag{14}
\]

Now clearly, if \(v_\gamma \neq v_2\), the first term in the exponential is much larger, at small \(q\), than the \(\Gamma_L(\vec{q})\) term. Since the former scales as \(\vec{q}_\parallel\), while the latter scales as \(q_\parallel^2\), in the dominant regime \(q_\parallel \sim q_\perp^\frac{2}{5}\). Hence, I can drop the latter term in the exponential, since it is small \(q\)’s which dominate the integral at large times (as we’ll see in a moment). Dropping this term, and using the scaling law[17] for \(\Gamma_\sigma(\vec{q})\), it is easy to see that rescaling the components \(q_\parallel\) and \(q_\perp\) of \(\vec{q}\) according to

\[
q_\parallel \equiv \frac{Q_\parallel}{t} , \text{ } q_\perp \equiv \frac{Q_\perp}{t^\frac{1}{5}} \tag{19}
\]

pulls all of the time dependence out of the integral \(d^2 q\), giving the scaling law

\[
C(t) \propto t^{-\frac{2}{5} - 1 + 2} \propto t^{-\frac{1}{5}} \tag{20}
\]

for \(C(t)\). Inserting this into 7 and 7 into 3 gives

\[
\propto t^{-\frac{2}{5}} \tag{21}
\]

a known result for birds in flocks of identical birds.

However, for slow birds, we now see that the results change dramatically if \(v_\gamma = v_2\), since the \((v_\gamma - v_2)q_\parallel t\) term in the exponential in Eqn. 18 vanishes. As a result, I must keep the \(\Gamma_\sigma\) term in the exponential in that equation. Doing so, and again using Eqn. 17, I see that the rescaling to a new vector variable \(\vec{Q}\) via

\[
q_\parallel \equiv \frac{Q_\parallel}{\sqrt{t}} \tag{22}
\]

\[
q_\perp \equiv \frac{Q_\perp}{t^{\frac{1}{5}}/6} \tag{23}
\]
pulls all of the time dependence out of the integral when \( v_s = v_2 \), and gives the scaling law
\[
C(t) = At^{-1/3},
\]
where \( A \) is an unimportant constant. Note that this decay is much slower than that (eqn. (20)) for a bird moving at a speed other than the magic speed \( v_s = v_2 \). Note further that since, in general, the mean speed \( v_0 \) of the flock differs from \( v_2 \) (i.e., \( v_0 \neq v_2 \)) that the identical birds that make up the bulk of the flock will exhibit the more rapidly decaying correlations of eqn. (20). It is only a bird that differs from the generic members of the flock, specifically by flying at a speed \( v_s \neq v_0 \), that has a chance to achieve \( v_s = v_2 \), and the much slower decay eqn. (24). This slower decay of velocity correlations leads immediately to faster lateral wandering. This can be seen by inserting (24) into (3), which gives
\[
w^2(t; v_s = v_2) = A \int_0^t dt' \int_0^t dt'' |t' - t''|^{-4} = \frac{9A}{5} t^\frac{2}{3} (25)
\]
Since \( t^\frac{2}{3} \gg t^\frac{1}{3} \) as \( t \to \infty \), this result eqn. (25) confirms the fundamental conclusion of this paper: birds moving at the magic speed \( v_2 \) wander laterally much faster than birds moving at any other speed. In particular, they wander much faster than the “normal” birds in the flock.

What happens to a bird moving at a speed \( v_s \) close to, but not equal to, \( v_2 \)? For sufficiently short times, it should be impossible to tell that this bird was not moving at exactly the magic speed \( v_2 \), and would therefore expect to recover the \( w^2 \propto t^\frac{2}{3} \) scaling I just found for that case. For longer times, though, one would expect the difference between the speed \( v_s \) of the slow bird and the magic speed \( v_2 \) to become apparent, leading to a crossover to the \( w^2 \propto t^\frac{2}{3} \) scaling found in that case.

This proves to be precisely the case. To see this, note that the rescaling eqns. (22), (23) imply that the integral eqn (18) for \( C(t; v_s) \) is dominated by parallel wavevectors
\[
q_{||}^{dom} \propto t^{-\frac{2}{3}}. (26)
\]
Comparing the \( \delta v q_{||} \) term in the exponent of eqn (18) with the \( \Gamma_L \) term, where \( \delta v \equiv v_s - v_2 \), and using the scaling law (17) for \( \Gamma_L \) and (20) for the dominant \( q_{||} \), I see that \( \delta v q_{||}^{dom} \sim \Gamma_L(q_{dom}) \) at a time \( t_c \) which scales according to \( \delta v t_c^\frac{2}{3} \propto (t_c^\frac{2}{3})^2 \), which implies
\[
t_c(\delta v) \propto \delta v^{-2}. (27)
\]
For smaller times \( t \ll t_c \), the \( \Gamma_L \) term in eqn. (18) dominates the \( \delta v q_{||} \) term, and the \( w^2(t) \propto t^\frac{2}{3} \) law found in eqn (25) holds. In the opposite limit \( t \gg t_c \), the scaling \( w^2(t) \propto t^\frac{2}{3} \) holds.

To summarize, for flocks in \( d = 2 \) spatial dimensions,
\[
w^2(t; v_s) \propto \begin{cases} t^\frac{2}{3}, & t \ll t_c(\delta v) \\ t^\frac{2}{3}, & t \gg t_c(\delta v) \end{cases} (28)
\]
with \( t_c(\delta v) \) given by (27).

The above results all hold in spatial dimension \( d = 2 \). In \( d = 3 \), unfortunately, the exponents characterizing scaling laws like eqns. (14), and (17) are unknown. Furthermore, other quantities like \( \Delta, D_{||}, \) and \( D_\rho \) could potentially also become strongly wavevector dependent in \( d = 3 \), as \( \Delta \) is in \( d = 2 \). However, if it proves that only a particular one of the four potentially relevant non-linearities in the phenomenological equations of motion (8, 9, 10) is actually relevant in \( d = 3 \) (namely, the “convective” one), which is certainly consistent with all calculations that have been done up to now, though by no means definitively established by them, then these complications disappear, and scaling laws analogous to eqns. (14), and (17) apply, but with different, but known, exponents (these are called the “canonical” exponents in 8, 9, 10).

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[13] The speed $v_2$ may, in some flocks, actually be greater than the mean speed $v_0$ of the flock; hence, in some cases, the “slow” bird may have to actually move faster than the other birds. I will describe the different bird in either case as the “slow” bird.

[14] This ignores weak, logarithmic corrections in precisely four dimensions; see [6, 7, 8, 9]; in practice, such corrections should be hard to see.

[15] Of course, this is not exactly true: the slow bird, like the other birds, could make mistakes in following the local direction of flock motion. However, such mistakes would simply add a conventional, “drunkard’s walk” lateral wandering (the RMS amplitude of which would scale like time $t$) to the much larger wandering due to following that flock that I am computing here.