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New solvable dynamical systems

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New solvable dynamical systems are identified and the properties of their solutions are tersely discussed.

1. Introduction

We call solvable a dynamical system if its solution can be explicitly exhibited in terms of elementary functions or obtained by algebraic operations, typically by finding the \( N \) zeros \( z_n(t) \) of a time-dependent polynomial \( p_N(z;t) \) of degree \( N \) in \( z \) which is known—again, possibly via algebraic operations—in terms of the initial data of the dynamical system. Recently a somewhat novel technique to identify such models has been introduced [1], and quite a few such models have been identified and investigated [2–8].

In the following Section 2 new solvable models are presented and their solutions are tersely discussed. Proofs of these findings are provided in Section 3.

Let us end this introductory section by mentioning that all these models generally belong to the class of so-called “goldfish-type” dynamical systems and many-body problems, the solvability of which generally emerges from the relations among the \( N \) coefficients and the \( N \) zeros of a time-dependent (monic) polynomial of degree \( N \) in the (complex) variable \( z \): an approach introduced almost four decades ago [9], to which the term “goldfish” was appended about 15 years ago [10], and which has been rather extensively investigated in the following years, see for instance [11–22].

Since a Referee kindly suggested that the above outline of our approach was a bit too terse, let us add here an overview of the general context in which the results reported in the present paper should be situated. The modern interest in solvable/integrable models can be traced—of course, somewhat arbitrarily: there always are precursors of precursors—to the discovery/invention in 1967 of the Inverse Spectral Transform technique to solve the KdV equation, which is an evolution Partial Differential Equation in one-plus-one dimensions: time \( t \) and space \( x \). [23] This major achievement was soon followed by another major achievement: the discovery/invention in 1968 of the Lax pair method to identify solvable/integrable models [24]. Let us hereafter focus on dynamical systems, i.e. on nonlinear systems of Ordinary Differential Equations, which are the topic treated in the present paper. In this context the Lax pair technique allows to reformulate certain dynamical system in terms of two \( (N \times N) \)-matrices (defined in terms of the variables of the dynamical system), with one of these two matrices evolving in time in a simple manner implying that its \( N \) eigenvalues are constants of motion. In the special case when the dynamical system so identified is Hamiltonian, this generally implies that it is integrable. More generally, the Lax breakthrough opened the
way to the identification of solvable dynamical systems (as defined above): which therefore evolve nonchaotically and indeed, in some cases, remarkably simply, for instance periodically or even isochronously. Two general techniques emerged over time to identify the equations of motion of such systems. The first one takes as starting point a solvable \((N \times N)\)-matrix evolution equation and then focusses on the evolution of its \(N\) eigenvalues, which are then interpreted as the dynamical variables: its origin can be traced to a seminal 1976 paper which introduced a convenient technique to compute the corresponding equations of motion [25]. It was thereby shown that even a trivially solvable linear evolution of the matrix leads to an interesting nonlinear evolution of its eigenvalues, such as that described by the so-called Calogero-Moser many-body problem; opening thereby a line of investigation that has been extensively pursued during the last few decades (see for instance [10] and [20] and references therein; but there are also may other books covering this material, and thousands of papers...). An alternative technique (1978) focussed on the connection for instance \[10\] and \[20\] and references therein; but there are also may other books covering this material, and thousands of papers...). An alternative technique (1978) focussed on the connection among the \(N\) coefficients \(c_m(t)\) and the \(N\) zeros \(z_n(t)\) of a time-dependent (monic) polynomial of degree \(N\) in its argument \(z\) (generally, a complex number), relying on the observation that a solvable evolution of the \(N\) coefficients \(c_m(t)\) implies of course that the corresponding evolution of the \(N\) zeros \(z_n(t)\) is as well solvable [9]. As mentioned above, a recent observation [1] has significantly facilitated the identification of solvable systems via this approach, leading to the investigation of several such models, each of course with its own distinctive features (see references above). The present paper fits into this development. Let us finally mention—again, in connection with an issue raised by the Referee—that the two technologies to identify and investigate solvable dynamical systems outlined above have some common features—for instance, the computation of the eigenvalues of an \((N \times N)\)-matrix entails of course the solution of a polynomial equation of degree \(N\)—but they are quite different, so that in each specific case it is less than obvious how a model whose solvability has been identified within one approach—allowing a detailed study of its behavior—can also be investigated via the other approach. And since both approaches eventually involve the solution of a polynomial equation of degree \(N\) it is of course plain that only for \(N \leq 4\) the solution of almost all solvable dynamical systems featuring \(N\) dynamical variables can be explicitly expressed in terms of elementary functions.

2. Results

A first, rather trivial, dynamical system is characterized by the following simple system of \(N\) Ordinary Differential Equations (ODEs):

\[
\gamma'_1(\tau) = a_1 [\gamma_1(\tau)]^{1-1/r} , \quad (2.1a)
\]

\[
\gamma'_m(\tau) = a_m \gamma_{m-1}(\tau) , \quad m = 2, \ldots, N . \quad (2.1b)
\]

**Notation 2.1.** Here and hereafter \(N\) is an arbitrary positive integer \((N \geq 2)\), the \(N\) functions \(\gamma_m \equiv \gamma_m(\tau)\) with \(m = 1, 2, \ldots, N\) are the dependent variables, \(\tau\) is the independent variable, the number \(r\) in the exponent in the right-hand side of (2.1a) is hereafter assumed to be an arbitrary real rational number—\(r = p/q\) with \(p\) and \(q\) two nonvanishing coprime integers \((q > 0\) for definiteness, and \(p \neq 0\) so that \(r \neq 0\)), and the \(N\) parameters \(a_m\) are \(N\) arbitrarily assigned (generally complex, nonvanishing) numbers. The \(N\) dependent variables \(\gamma_m \equiv \gamma_m(\tau)\) should as well be generally considered complex numbers, and in the following the independent variable \(\tau\) shall also be complex, although when considering this system of ODEs as a dynamical system one might prefer...
to consider $\tau$ as a real variable ("time"). Note that in the following the explicit indication of the dependence on the independent variable $\tau$ will be occasionally omitted. Primes denote of course differentiations with respect to the argument of the function they are appended to. Indices such as $m$ and $n$ generally take integer values in the range from 1 to $N$ (unless otherwise indicated, see for instance (2.1b)). And let us finally recall—for their relevance to formulas written below—the standard convention according to which an empty sum vanishes and an empty product equals unity:

$$\sum_{S=0}^{S+} = 0 \text{ if } S > 0, \prod_{J=J_0}^{J+} = 1 \text{ if } J > J_0.$$

As shown in the following Section 3 the solution of the initial-values problem of this dynamical system—with the $N$ initial values $\gamma_m(0)$ assigned, and of course $\gamma_1(0) \neq 0$ if $1/r > 1$ (see (2.1a))—is provided by the following formula:

$$\gamma_m(\tau) = \sum_{\ell=0}^{m-2} \left\{ \frac{\tau^\ell}{\ell!} \left[ A_{m,m-\ell+1} \gamma_{m-\ell}(0) - A_{m,2} \beta_{m-\ell} \gamma_1(0) \right] \right\} + A_{m,2} \beta_m \gamma_1(0) (1 + \rho \tau)^{m-1}, \quad m = 1, 2, \ldots, N,$$

(2.2a)

with, above and hereafter (see Notation 2.1),

$$A_{m,\ell} = \prod_{s=\ell}^{m} (a_s), \quad \beta_m = \rho^{1-m} \prod_{j=1}^{m-1} \left[ (r + j)^{-1} \right],$$

(2.2b)

$$\rho = \frac{a_1}{r} \left[ \gamma_1(0) \right]^{-1/r}.$$  

(2.2c)

A, perhaps more interesting, dynamical system obtains by introducing new independent and dependent variables via the positions

$$\tau = \tau(t) = \exp(i \omega t) - 1,$$

implying

$$\dot{\tau}(t) = \exp(i \omega t),$$

(2.3a)

(2.3b)

and

$$c_m = c_m(t) = \exp\left( -i \lambda_m \omega t \right) \gamma_m(\tau), \quad m = 1, 2, \ldots, N,$$

(2.4a)

with, above and hereafter,

$$\lambda_m = r + m - 1.$$  

(2.4b)

**Notation 2.2.** Here and hereafter $t$ ("time") is a real variable, $\omega$ is an arbitrary real (nonvanishing) parameter to which the basic period

$$T_0 = 2\pi/|\omega|$$

(2.5)

is associated, and note that in (2.3b) we used—and we will do so hereafter—the standard notation according to which superimposed dots denote differentiations with respect to the real variable $t$ ("time").
It is then plain that the time-dependent variables $c_m(t)$ evolve according to the autonomous dynamical system

$$\dot{c}_1(\tau) = -i \lambda_1 \omega c_1(t) + a_1 [c_1(t)]^{1-1/r},$$

(2.6a)

$$\dot{c}_m(\tau) = -i \lambda_m \omega c_m(t) + a_m c_{m-1}(t), \quad m = 2, \ldots, N,$$

(2.6b)

and that the solution of this dynamical system is provided by the formula

$$c_m(t) = \exp(-i \lambda_m \omega t) \left\{ \sum_{\ell=0}^{m-2} \left\{ \frac{[\exp(i \omega t) - 1]^\ell}{(i \omega)^\ell \ell!} \right\} \right.$$

$$\cdot [A_{m,m-\ell+1} c_{m-\ell}(0) - A_{m,2} \beta_{m-\ell} c_1(0)]$$

$$+ A_{m,2} \beta_m c_1(0) \left[ 1 + \rho \frac{\exp(i \omega t) - 1}{i \omega} \right]^{\lambda_m} \right\}, \quad m = 1, 2, \ldots, N,$$

(2.7a)

with $A_{m,\ell}$, $\beta_m$ and $\lambda_m$ defined again by (2.2b) and (2.4b), but now of course with

$$\rho = \frac{a_1}{r} [c_1(0)]^{-1/r}.$$ 

(2.7b)

**Remark 2.1.** It is plain that this solution—for arbitrary initial data—is nonsingular and periodic with period $T_0$, see (2.5), if $r$ is a nonnegative integer, i.e. if $q = 1$ and $p \geq 0$ (see Notation 2.1), implying that all the numbers $\lambda_m$ are as well nonnegative integers. Otherwise the solution might become singular, but only for the nongeneric initial datum $c_1(0)$ such that there hold the equality $\left| (i \omega / \rho) - 1 \right| = 1$ (see (2.7b) and (2.3a)) namely

$$\left| \frac{(i \omega r)}{a_1} [c_1(0)]^{1/r} - 1 \right| - 1 \neq 0.$$ 

(2.8a)

It is moreover plain from (2.7) that if there holds instead the inequality

$$\left| \frac{(i \omega r)}{a_1} [c_1(0)]^{1/r} - 1 \right| - 1 = 0$$

(2.8b)

all the functions $c_m(t)$ are (nonsingular and) periodic with period $T = qT_0$,

$$c_m(t + T) = c_m(t), \quad T = \frac{2q \pi}{|\omega|}, \quad m = 1, 2, \ldots, N,$$

(2.9)

of course with $q$ the denominator of the rational number $r$, see Notation 2.1 and the definition (2.4b) of $\lambda_m$. Note that this outcome holds in spite of the fact that the time evolution of $(1 + \rho \tau)^{\lambda_m}$ with (2.3a) is somewhat different—due to the rational branch point at $\tau = -1/\rho$—depending on the sign of the left-hand side of (2.8b).
Proposition 2.1. The following set of $N$ nonlinearly coupled ODEs characterizing the time-evolution of the $N$ time-dependent variables $z_n(t)$,  
\[ \dot{z}_n = - \left[ \prod_{\ell=1, \ell \neq n}^N (z_n - z_{\ell})^{-1} \right] \left[ \left\{ -i \lambda_m \omega c_m(t) + a_m \left[ c_1(t) \right]^{1 - 1/n} \right\} (z_n)^{N-1} + \sum_{m=2}^{N} \left\{ [-i \lambda_m \omega c_m(t) + a_m \left[ c_{m-1}(t) \right] (z_n)^{N-m}] \right\} \right], \tag{2.10} \]

where $\lambda_m$ is defined by (2.4b) and  
\[ c_1(t) = - \sum_{n=1}^{N} [z_n(t)], \tag{2.11a} \]
\[ c_m(t) = (-1)^m \sum_{n_1, n_2, \ldots, n_m = 1; \ \ n_1 < n_2 < \cdots < n_m}^{N} [z_{n_1}(t) z_{n_2}(t) \cdots z_{n_m}(t)], \tag{2.11b} \]

is a new example of solvable dynamical system: indeed its solution is provided by the $N$ zeros $z_n(t)$ of the polynomial (see (3.2))  
\[ p_N(z; \bar{c}(t); \bar{z}(t)) = z^N + \sum_{m=1}^{N} [c_m(t) z^{N-m}] , \tag{2.12} \]
the coefficients $c_m(t)$ of which are provided by the formulas (2.7), with the $N$ initial data $c_m(0)$ expressed of course in terms of the $N$ initial data $z_n(0)$ by the formulas (2.11) at $t = 0$.

Remark 2.2. This dynamical system, (2.10), is isochronous, all its generic (hence nonsingular) solutions being periodic with period  
\[ T = N! T = N! \left( \frac{2 q \pi}{|\omega|} \right) , \tag{2.13a} \]
\[ z_n(t + T) = z_n(t), \quad n = 1, 2, \ldots, N. \tag{2.13b} \]

The (nongeneric) singular solutions are those characterized by the nongeneric set of initial data $z_n(0)$ such that the equality (2.8a) hold with $c_1(0) = - \sum_{n=1}^{N} [z_n(0)]$, or such that the polynomial (2.12) feature at least one multiple zero (in which case the time-derivative $\dot{z}_n(t)$ eventually blows up, see (2.10)).

We trust the reader to consider this Remark 2.2 obvious on the basis of the developments reported above: see in particular the right-hand side of (2.10)), and the well-known fact that, if a time-dependent polynomial $p_N(z;t)$ of degree $N$ in $z$ is periodic in $t$, $p_N(z;t+T) = p_N(z;t)$, the unordered set $\bar{z}(t)$ of its $N$ zeros is of course as well periodic with the same period, $\bar{z}(t+T) = \bar{z}(t)$, while each of the $N$ zeros $z_n(t)$, considered as a continuous function of $t$, is also periodic but possibly with a larger period $\nu T$ (with $\nu$ a positive integer, $\nu \leq N!$) due to the possibility that the zeros, as it were, exchange their roles over their time evolution. Of course the genericity of the initial
data \( z_n(0) \) is in the context of evolutions of the dynamical system \((2.10)\) taking place in the complex \(z\)-plane, corresponding to the fact that a generic polynomial of degree \(N\) in the complex variable \(z\) features \(N\) distinct zeros \(z_n\). And it is also clear that the set of initial data \(z(0)\) can be partitioned in the complex \(z\)-plane into \(N!\) different ensembles of initial data yielding solutions periodic with periods \(\tilde{T}/\nu\) with \(\nu\) a positive integer, \(\nu \leq N!\), these ensembles being separated from each other by special (i.e., nongeneric) sets of initial data \(z(t)\) yielding solutions of the dynamical system which are singular due to "particle collisions", i.e. such that, at some time \(t = t_c \mod(T)\), two, or more, different coordinates \(z_n(t)\) coincide, say \(z_n(t_c) = z_{\ell}(t_c)\) with \(\ell \neq n\).

Let us end this section by suggesting that assignments for interested researchers might be: (i) to write out more explicitly than we did above the equations of motion \((2.10)\) for \(N = 2, 3\) and perhaps also \(N = 4\), and their explicit solutions as detailed in Proposition 2.1; (ii) to make graphs—for various assignments of the parameters and of the initial data \(z_n(0)\)—of the real and imaginary parts of the coordinates \(z_n(t)\) as functions of time and of the trajectories of these coordinates in the complex \(z\)-plane; (iii) to compare these data with analogous data obtained via numerical solutions of the equations of motion \((2.10)\). Note that for \(N = 2, 3, 4\) the zeros of a polynomial of degree \(N\) can be explicitly expressed in terms of the coefficients of the polynomial, although the resulting expressions are somewhat cumbersome for \(N = 3\) and especially for \(N = 4\). And it is of course always possible to manufacture additional solvable dynamical systems by iterating the technique that has allowed the transition from the solvable model \((2.6)\) to the solvable model \((2.10)\)—as generally described in [2].

### 3. Proofs

In this Section 3 we prove the results reported without their proofs in the preceding Section 2.

Our first task is to prove that the formula \((2.2)\) provides the solution of the dynamical system \((2.1)\).

The first step to this end is to integrate from 0 to \(\tau\) the ODE \((2.1a)\), or equivalently the ODE

\[
\gamma_1'(\tau) \left[ \gamma_1(\tau) \right]^{-1+1/r} = a_1 .
\]

This clearly yields (see \((2.2c)\))

\[
\gamma_1(\tau) = \gamma_1(0) \left( 1 + \rho \tau \right)^r ,
\]

which coincides with the \(m = 1\) case of \((2.2)\) (see Notation 2.1).

A proof by recursion of the formula \((2.2)\) is then achieved by showing—as the diligent reader will easily verify—that the assumed validity of this formula (with \(m\) replaced by \(m - 1\)) to replace \(\gamma_{m-1}(\tau)\) in the right-hand side of the ODE \((2.1b)\) implies, by a standard integration of this ODE from 0 to \(\tau\), the validity of the solution \((2.2)\) for \(m\).

Next, let us derive the equations of motion of the dynamical system \((2.10)\) in order to prove Proposition 2.1.

To this end it is convenient to introduce the following time-dependent (monic) polynomial of degree \(N\) in \(z\),

\[
p_N(z; \vec{c}(t); \vec{z}(t)) = z^N + \sum_{m=1}^N [c_m(t) \, z^{N-m}] = \prod_{n=1}^{N} [z - z_n(t)] ,
\]
where clearly the \( N \) coefficients \( c_m(t) \) of this polynomial are the \( N \) components of the time-dependent \( N \)-vector \( \vec{c}(t) \) and the \( N \) zeros \( z_n(t) \) of this polynomial are the \( N \) components of the unordered set \( \vec{z}(t) \): see the left-hand side of (3.2), and note that the notation \( p_N(z; \vec{c}(t); \vec{z}(t)) \) is somewhat redundant, since this monic polynomial is equally well defined by assigning either its \( N \) coefficients \( c_m(t) \) or its \( N \) zeros \( z_n(t) \) (indeed the \( N \) coefficients \( c_m(t) \) are themselves defined in terms of the \( N \) zeros \( z_n(t) \) by the well-known formulas (2.11)).

Let us now assume that the \( N \) coefficients \( c_m(t) \) of the polynomial \( p_N(z; \vec{c}(t); \vec{z}(t)) \) evolve according to the dynamical system (2.6); it is then plain that—as implied by the formula

\[
\dot{z}_n = -\left[ \prod_{\ell=1, \ell\neq n}^{N} (z_n - z_\ell) \right]^{-1} \sum_{m=1}^{N} \left[ \dot{c}_m z_n^{N-m} \right]
\]

(3.3)

which relates the time evolution of the \( N \) zeros \( z_n(t) \) of any time-dependent polynomial such as (3.2) to the time evolution of its \( N \) coefficients \( c_m(t) \) (if need be see [1] for a proof)—the corresponding evolution of the \( N \) zeros \( z_n(t) \) is characterized just by the set of \( N \) coupled nonlinear ODEs (2.10). The rest of Proposition 2.1 clearly follows. \( \square \)

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