Analysis of the superdefomed rotational bands

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Abstract

All available experimental data for the $\Delta I = 2$ transition energies in superdeformed bands are analyzed by using a new one-point formula. The existence of deviations from the smooth behavior is confirmed in many bands. However, we stress that one cannot necessarily speak about staggering patterns as they are mostly irregular. Simulations of the experimental data suggest that the irregularities may stem from the presence of irregular kinks in the rotational spectra. This could be a clue but, at the moment, where such kinks come from is an open question.
A regular $\Delta I = 2$ staggering pattern, in which the states differing by four angular momentum units are shifted by a similar amount of energy, found for the first time in the measured $\Delta I = 2$ transition energies of the yrast superdeformed (SD) band of the nucleus $^{149}$Gd [1] attracted much interest. Since then, it is reported that similar effects have been observed for several SD bands in the nuclei of the mass regions $A \sim 130$ ($^{131-133}$Ce [2]), $A \sim 150$ ($^{147-148}$Gd, $^{148}$Eu, $^{154}$Er) and $A \sim 190$ ($^{192}$Tl, $^{191-194}$Hg, $^{195}$Pb [11]). From the theoretical point of view, a lot of effort has been devoted to understand this phenomenon [12–15]. However, no definite and decisive conclusion has been obtained till now and many questions are still open.

Usually, the experimental data is presented using multi-point formulae as for example the five-point formula [1]

$$\Delta E(\gamma)(I) \equiv \frac{1}{16}[6E(\gamma)(I) - 4E(\gamma)(I+2) - 4E(\gamma)(I-2) + E(\gamma)(I+4) + E(\gamma)(I-4)] \quad (1)$$

This kind of presentation leads to a certain regularity (staggering) in some of the observed bands. It is questionable, however, whether the produced staggering does really exist in the spectrum. It should be noted that such a multi-point formula will produce different patterns depending on its multiplicity. This makes the analysis rather formula-dependent and it could possibly lead to misinterpretation of the experimental data.

In a recent paper [16], we have called the attention to this point and proposed an alternative way of presenting the experimental data by introducing the so-called “one-point formula”. This formula is free from the ambiguities which a multi-point formula may introduce. In the present work, a new extended form of the one-point formula is proposed and applied to the systematic analysis of all available experimental data. The necessity of such an extension has been already suggested in our earlier work [16]. The new formula is capable of handling a higher order global behavior (presence of a term $I^n$ with $n > 3$) which indeed exists in the observed data.

The basic idea behind the one-point formula is to look at the deviation of the $\Delta I = 2$ transition energy

$$\Delta E(I) \equiv E(I) - E(I-2) \quad (2)$$

from its smoothly increasing part. After all, this was also the reason of using a multi-point formula to remove the smooth lower order spin dependence (i.e. lower order powers in the spin variable $I$) by evaluating a derivative using finite differences.

Instead of taking derivatives, we subtract a polynomial of order $N$ in $I$ from $\Delta E(I)$ and define what we call the $N$th order one-point formula

$$\Delta^{(1)}_N E(I) \equiv \Delta E(I) - Q_N(I), \quad Q_N(I) = \sum_{m=0}^{N} q_m I^m \quad (3)$$

where the coefficients $q_m$ are determined by minimizing the quantity

$$\chi(q_0, \cdots, q_N) \equiv \left[\Delta^{(1)}_N E(I)\right]^2 \quad (4)$$

with respect to $q_m$ ($\frac{\partial \chi}{\partial q_m} = 0$). This leads to a set of $N + 1$ equations ($m = 0, 1, \cdots, N$)

$$\sum_{n=0}^{N} S_{mn} q_n = T_m, \quad S_{mn} \equiv \sum_{I} I^m I^n, \quad T_m \equiv \sum_{I} I^m \Delta E(I). \quad (5)$$
Our previous one-point formula corresponds to $N = 3$ \[^{13}\]. We note that the smooth part $Q_N(I)$ which we subtract from $\Delta E(I)$ is nothing other than a polynomial of order $N$ determined by the $\chi$-square fit to $\Delta E(I)$. However, in practice, this formula cannot be used in the present form particularly when the order of the polynomial $N$ is larger than 3 since the equation (5) is highly ill-conditioned and this is the reason why one needs an extension. Thus, we want to transform it into another form.

First, let us note that the replacement $I \rightarrow I - I_0$ does not change the fitting procedure since the shape of the polynomial $Q_N(I)$ is unchanged by a parallel translation. Thus, the origin of the spin values can be shifted freely. Secondly, the spin variable can be scaled too ($I \rightarrow aI$) since the order of the polynomial remains the same. These properties can be used to rewrite the polynomial in a different form.

Shifting and scaling the spin values can be achieved most generally by a linear mapping $I = ax + b$. The increment of $x$ is thus $\Delta x = \frac{\Delta I}{a}$ ($\Delta I = 2$). We will choose $a = \frac{I_{\text{max}} - I_{\text{min}}}{2}$ and $b = \frac{I_{\text{max}} + I_{\text{min}}}{2}$ so that the range of $x$ becomes $[-1,+1]$, where $x = -1$ (+1) corresponds to $I = I_{\text{min}}$ ($I_{\text{max}}$). The polynomial in question may thus be written in the form

$$Q_N(I) = \sum_{n=0}^{N} p_n P_n(x). \quad (6)$$

Here, we use the Legendre polynomial $P_n(x)$ instead of $x^n$. The reason will be explained below. The resulting set of equations is similar to (5) but $q_n$ is replaced by $p_n$ and $I^n (I^m)$ by $P_n(x) (P_m(x))$. This representation has an advantage that there holds the relation

$$S_{mn} = \sum_{x=-1}^{+1} P_m(x)P_n(x) = 0 \quad \text{if} \ m + n = \text{odd}. \quad (7)$$

It means that the whole set of equations splits into two independent sets of equations of smaller dimensions, one with $m, n = \text{even}$ and the other with $m, n = \text{odd}$:

$$\sum_{n=\text{even or odd}}^{N} S_{mn}p_n = T_m, \quad S_{mn} \equiv \sum_{x=-1}^{+1} P_m(x)P_n(x), \quad T_m \equiv \sum_{x=-1}^{+1} P_m(x)\Delta E(I). \quad (8)$$

This set of equations determines the coefficients $p_n$ and accordingly the polynomial (6) which represents the smooth part of $\Delta E(I)$.

As mentioned before, the original set of equations (5) is highly ill-conditioned. It is indeed so ill-conditioned that even the double precision algorithm is not free from the numerical instability caused by large losses of accuracy if $N$ is greater than 3. This problem can be avoided by shifting and scaling the spin values as presented above. In fact, the situation improves slightly if one uses the power series $x^n$ thanks to a property analogous to (7). Nevertheless, this does not fully resolve the numerical instability. The reason lies basically in the fact that the zeros of $x^n$ are multiple and are all concentrated at $x = 0$. In contrast, all zeros of $P_n(x)$ are simple and never coincide with one another for different $n$'s. The use of the Legendre polynomial $P_n(x)$ instead of the power series $x^n$ is thus essential for the numerical reliability (both accuracy and stability).

In the present work, all available experimental data are analyzed using a 7th order one-point formula ($N = 7$). Before doing this, let us first introduce the corresponding “filtered” one-point formula by setting the quantity $\Delta_N^{(1)} E(I)$ to zero if its absolute value is smaller than or equal to the corresponding errorbar. This formula is quite useful in
practice. By construction, it shows whether the deviation of the transition energy $\Delta E(I)$ from its smooth part is physically significant or not.

For the sake of comparison, we first show both the one-point formula and the corresponding filtered one in Fig. 1 for some selected cases. The filtered formula indeed shows clearly where deviations occur. For this reason, we will use only this formula for the presentation of data in what follows. In Figs. 2 – 6, we present the results for all the observed SD bands. To avoid possible confusion, we have used the same notation to label the SD bands as in the original experimental papers. Since the spin assignment is not known in most of cases, our spin values refer to $I - I_{ref} = 2, 4, \cdots$ where $I_{ref}$ is an unknown reference spin. For the nucleus $^{192}$Tl, however, the reference spin assumes the value $I_{ref} = 0$ because spins are suggested by the experiment [7].

It is seen that, in addition to the well known case of $^{149}$Gd, there are also several other cases where the deviation from the smooth behavior is beyond the experimental errorbars. However, we should like to emphasize that one cannot necessarily speak about the “regular staggering” patterns despite the presence of deviations as they are mostly irregular. For $^{191}$Hg, we have used the data from a new experiment [14], but we observed no significant difference between the new and old data [4]. It is also interesting to note that there are two different experiments for the same band. Namely, for the SD band 6 of $^{149}$Gd, we have the measurements from Berkeley with the gammasphere [4] and from Legnaro using the GASP spectrometer [4]. Our results are shown in Fig. 3, see the middle diagram of the upper- and lowermost panels, respectively. However, one finds no similarity between these two diagrams. In fact, a close examination of these two data shows that they do not quite agree with each other and that their differences are beyond errorbars. This is a serious problem since one does not know which data one should adopt. This example shows that it is worth remeasuring not only this nucleus but also all others (as they have nothing to compare) at different laboratories for the sake of confirmation of data.

Finally, we show that one can simulate the experimental data. Let us take the SD band C of $^{192}$Tl. We device a simple minded ‘model’ by assuming a smooth rotational spectrum $E(I) = AI(I + 1)$, which has two kinks created by shifting the energy ‘down’ at $I=16$ and ‘up’ at $I=32$ by certain amount. In Fig. 7, we compare the experimental data (left side) and the one obtained by such a “theory” (right side). One sees that the basic feature is quite well reproduced. This suggests that the presence of irregular kinks in the spectra may be responsible for the appearance of the irregularity. For more complicated cases, we believe that the patterns can also be simulated by similar models with more kinks (not necessarily regular ones). As a matter of fact, such a simulation has been recently done for diatomic molecular rotational spectra [17].

To summarize, we have analyzed all available experimental data in terms of an extended one-point formula and found many irregularities in the measured $\Delta I = 2$ transition energies. While the physics behind this phenomenon is still unclear, one thing is certain: We cannot necessarily speak about the presence of regular staggering. We have also pointed out that it may be worth reconfirming all data at different laboratories. Finally, we have simulated the SD band C of the nucleus $^{192}$Tl in terms of a simple minded model, which seems to suggest that the observed irregularities may stem from the presence of irregular kinks in the spectrum. This may imply that a more careful estimate of the experimental errorbars would be necessary.
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FIGURE CAPTIONS

Fig. 1 Seventh order one-point formula (left side) and the corresponding filtered formula (right side) applied to some SD bands

Fig. 2 Seventh order filtered one-point formula applied to SD bands of $^{131-133}$Ce and $^{147}$Gd

Fig. 3 Seventh order filtered one-point formula applied to SD bands of $^{147-148}$Gd and $^{148}$Eu

Fig. 4 Seventh order filtered one-point formula applied to SD bands of $^{149}$Gd, $^{154}$Er and $^{191}$Hg

Fig. 5 Seventh order filtered one-point formula applied to SD bands of $^{191-194}$Hg and $^{192}$Tl

Fig. 6 Seventh order filtered one-point formula applied to SD bands of $^{195}$Pb

Fig. 7 Comparison between the experiment and the “kink theory” for the SD band C of $^{192}$Tl, see the text
Experiment

"Kink Theory"