Spherical systems in models of nonlocally corrected gravity

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The properties of static, spherically symmetric configurations are considered in the framework of two models of nonlocally corrected gravity, suggested in S. Deser and R. Woodard., Phys. Rev. Lett. 663, 111301 (2007), and S. Capozziello et al., Phys. Lett. B 671, 193–198 (2009). For the first case, where the Lagrangian of nonlocal origin represents a scalar-tensor theory with two massless scalars, an explicit condition is found under which both scalar fields are canonical (non-phantom). If this condition does not hold, one of the fields exhibits a phantom behavior. Scalar-vacuum configurations then behave in a manner known for scalar-tensor theories. In the second case, the Lagrangian of nonlocal origin exhibits a scalar field interacting with the Gauss-Bonnet (GB) invariant and contains an arbitrary scalar field potential. It is found that the GB term, in general, leads to violation of the well-known no-go theorems valid for minimally coupled scalar fields in general relativity. It is shown, however, that some configurations of interest are still forbidden — whatever be the scalar field potential and the GB-scalar coupling function, namely, “force-free” wormholes (such that $g_{tt} = \text{const}$) and black holes with higher-order horizons.

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I. INTRODUCTION

It has been recently shown (see [1–3] and references therein) that the dynamical Casimir effect manifests itself in the effective models of gravity and cosmology (owing, in particular, to the possible existence of compact extra dimensions) in the appearance of nonlocal contributions to the effective gravitational field Lagrangian. Cosmological consequences of such theories have been widely discussed (see, e.g., the same references above); less attention has been paid, however, to local configurations, such as stellar models, black holes, wormholes etc., whose existence and properties can crucially depend on nonlocal corrections to gravity and are very important in astrophysical observations. Moreover, some of these configurations can lead to interesting links between black hole physics and cosmology as, for instance,
the “black universes” described in [4, 5], which look like black holes from one of their two asymptotic regions and like an expanding de Sitter universe from the other.

In this paper, we will study the possible impact of nonlocal gravity corrections on the existence and properties of the simplest local objects, namely, static, spherically symmetric configurations, such as black holes and wormholes. To our knowledge, this is to date the first attempt to study wormholes in nonlocal gravity. Two particular models of nonlocal gravity will be considered, namely, those introduced in [3, 6].

II. EFFECTIVE MULTISCALAR-TENSOR THEORY: NORMAL AND PHANTOM BEHAVIOR

One version [2] of nonlocally corrected gravity in four dimensions is described by the Lagrangian

\[ L = \sqrt{-g} \left\{ \frac{1}{2} R \left[ 1 + f(\Box^{-1} R) \right] + L_m \right\}, \]

where \( R \) is the Ricci scalar, \( L_m \) is the Lagrangian of matter, and \( \Box \) the d’Alembertian operator. It has been shown [7] that this Lagrangian can be cast into a local form:

\[ L = \sqrt{-g} \left\{ \frac{1}{2} \left[ R (1 + f(\phi) - \xi \partial_\mu \partial^\mu \phi) \right] + L_m \right\}, \]

where \( \phi \) and \( \xi \) are scalar fields. Eq. (2) represents the Jordan frame of a scalar-tensor theory with two massless scalars. It should be noted that this kind of Lagrangians is a typical manifestation of the Casimir effect, since the nonlocal theory is a direct consequence of quantum field theory effects in the curved spacetime [8]. We are thus here considering a direct application of a semiclassical description of the Casimir effect to the current cosmological epoch, which is interesting to remark.

To be precise, the Lagrangian (2) leads to some extra solutions as compared to (1), as has been demonstrated [9] at least in the degenerate case \( f = \text{const} \) (see also [10]). A reason is that the derivation of (2) involves substitution of the second-order constraint \( \Box \phi = 0 \) into the action, possibly introducing extra degrees of freedom. Thus some extra care is required with the initial conditions for the equations of motion due to (2) for all solutions to coincide with those due to (1). Actually this will not affect our results here, in our non-degenerate case, once we restrict to the “mass shell” (the second-order constraint condition, in this case). On a more general setting of the correspondence problem between nonlocal theories and their local counterparts see, e.g., [11] and references therein.

To make clear under what conditions the scalars in (2) have usual kinetic terms, and whether or not they contain a phantom degree of freedom, it is helpful to pass on to the Einstein-frame metric \( \bar{g}_{\mu\nu} \) using the standard conformal mapping

\[ g_{\mu\nu} = \frac{1}{F} \bar{g}_{\mu\nu}, \]

where

\[ F = F(x^\mu) = 1 + f(\phi) - \xi \]

is the coefficient of \( R \) in (2) representing the nonminimal coupling of the scalars \( \phi \) and \( \xi \) to gravity. We assume \( F > 0 \), to provide for a positive effective gravitational constant in the theory (2). This assumption is thus necessary for a meaningful theory and manifestly holds
if $f$ and $\xi$ are small, that is, closely enough to the limit in which the theory (1) approaches general relativity. Moreover, the fact that under this assumption we afterwards arrive at the well-defined sigma-model Lagrangian (10) with two scalar fields shows that $F > 0$ does not too strongly restrict the set of solutions.

The transformation (3) results in

$$L\sqrt{-g} = \frac{1}{2}\sqrt{\bar{g}}\left\{R - \frac{1}{F^2} \left[\frac{3}{2}(\partial F)^2 + F \bar{g}^{\mu\nu}\partial_\mu \phi \partial_\nu \xi\right] + \frac{2L_m}{F^2}\right\},$$

(5)

where bars mark quantities obtained from or with the metric $\bar{g}_{\mu\nu}$, and $(\partial F)^2 = \bar{g}^{\mu\nu}\partial_\mu F \partial_\nu F$.

This Lagrangian describes a massless nonlinear sigma model with two scalars $\phi$ and $\xi$. It is a special case of nonlinear sigma models of the form

$$L = \frac{1}{2}R - h_{ab}\bar{g}^{\mu\nu}\partial_\mu \phi^a \partial_\nu \phi^b + \frac{L_m}{F^2},$$

(6)

where $h_{ab}$ are arbitrary functions of $n$ scalar fields $\phi^a$ ($a, b = 1, 2, \ldots, n$). If the matrix $h_{ab}$ is positive-definite, the set of scalar fields is normal (non-phantom) in the sense that the kinetic energy is positive. To check that one can, as usual, diagonalize $h_{ab}$ algebraically, as is conventionally done for quadratic forms. It should be noted, however, that such a procedure will not, in general, lead to a valid Lagrangian in terms of the newly introduced scalar fields, because linear combinations of derivatives as, e.g., $A(\phi, \xi)\partial_\mu \phi + B(\phi, \xi)\partial_\mu \xi$, are not always integrable, and it can be quite hard to find an integrating factor.

We therefore try to diagonalize the kinetic term in (5) by substituting

$$F = F(\phi, \eta), \quad \xi = 1 + f(\phi) - F,$$

(7)

where $\eta$ is a new field introduced instead of $\xi$; the second equality is just a rewriting of (4). Then, the kinetic term in (5) takes the form

$$\frac{3}{2F^2}\left\{(\partial \phi)^2\left[F_\phi^2 + \frac{2}{3}F(f_\phi - F_\phi)\right] + (\partial \eta)^2 F_\eta^2 + 2F_\eta \bar{g}^{\mu\nu}\partial_\mu \phi \partial_\nu \eta(F_\phi - F/3)\right\},$$

(8)

where the indices $\phi$ and $\eta$ denote $\partial/\partial \phi$ and $\partial/\partial \eta$, respectively. The expression (8) is diagonal with respect to $\phi$ and $\eta$ under the conditions $F_\eta \neq 0$, $F_\phi = F/3$, and to satisfy them we choose simply

$$F(\phi, \eta) = \eta e^{\phi/3}.$$

(9)

As a result, the Lagrangian (5) reads

$$L = \frac{1}{2}R - \frac{3}{4}\frac{(\partial \eta)^2}{\eta^2} - \frac{1}{12F}(6f_\phi - F)(\partial \phi)^2 + \frac{L_m}{F^2}.$$  

(10)

We see that the theory is free from phantom fields if $6f_\phi > F$ and that it does contain a phantom if $6f_\phi < F$. A more general version of Eq. (10) in $D$ dimensions can be found in [12], where the Newtonian limit and post-Newtonian corrections of the theory (2) were considered.

The properties of the theory are well illustrated by static, spherically symmetric vacuum solutions similar to Schwarzschild’s in general relativity. Let us take the Einstein-frame Lagrangian in the general form (6). Assuming $L_m = 0$ (vacuum), we can easily find the
corresponding metric, whose properties depend on whether the matrix \( h_{ab} \) is positive-definite or not.

Indeed [13], if we write the general static, spherically symmetric metric as
\[
\!ds^2 = -e^{2\gamma(u)} dt^2 + e^{2\alpha(u)} du^2 + e^{2\beta(u)} d\Omega^2,
\]
where \( d\Omega^2 = (d\theta^2 + \sin^2 \theta d\varphi^2) \) and \( u \) is an arbitrary radial coordinate, and assume \( \phi^a = \phi^a(u) \), the stress-energy tensor of the scalar fields has the form
\[
T_{\mu}^{\nu} = h_{ab}(\phi^a)'(\phi^b)' \text{diag}(1, -1, 1, 1),
\]
that is, it has the same structure as for a single massless scalar field (here and henceforth the prime denotes \( d/du \)). Therefore, the metric has the same form as in this simple case: for a normal scalar it is the Fisher solution [14], for a phantom one it was first found by Bergmann and Leipnik [15] and is sometimes called “anti-Fisher” (by analogy with anti-de Sitter). Let us reproduce it in the simplest joint form suggested in [16].

Choosing the harmonic radial coordinate \( u \), such that \( \alpha(u) = 2\beta(u) + \gamma(u) \), we easily solve two combinations of the Einstein equations for the metric (11), namely, \( R_0^0 = 0 \) (whence \( \gamma'' = 0 \)) and \( R_0^0 + R_2^2 = 0 \) (whence \( \beta'' + \gamma'' = e^{2(\beta + \gamma)} \)). As a result, the metric has the form
\[
\!ds^2_E = -e^{-2mu} dt^2 + \frac{e^{2mu}}{s^2(k, u)} \left[ \frac{du^2}{s^2(k, u)} + d\Omega^2 \right],
\]
where \( k \) and \( m \) are integration constants while the function \( s(k, u) \) is defined as
\[
s(k, u) = \begin{cases} 
  k^{-1} \sinh ku, & k > 0, \\
  u, & k = 0, \\
  k^{-1} \sin ku, & k < 0.
\end{cases}
\]

In addition, the \( \left( \right. \) Einstein equation gives
\[
k^2 \text{sign} k = m^2 + C_\phi, \quad (15)
\]
\[
C_\phi = h_{ab}(\phi^a)'(\phi^b)' = \text{const}. \quad (16)
\]

The scalar field equations read
\[
2 \left[ h_{ab}(\phi^b)' \right]' + \frac{\partial h_{bc}}{\partial \phi_a}(\phi^b)'(\phi^c)' = 0 \quad (17)
\]
and obviously cannot be solved in a general form, but (16) is their first integral. The metric (13) is defined (without loss of generality) for \( u > 0 \), it is flat at spatial infinity \( u = 0 \), and \( m \) has the meaning of a Schwarzschild mass in proper units. Its properties crucially depend on the sign of \( k \), which in turn depends on \( C_\phi \), hence on the nature of the matrix \( h_{ab} \).

If \( h_{ab} \) is positive-definite, we have \( C_\phi > 0 \) for all nontrivial scalar field configurations and obtain the Fisher metric: in this case \( k > 0 \) and the substitution \( e^{-2ku} = 1 - 2k/r = \tilde{P}(r) \) converts (13) into
\[
\!ds^2_E = P(r)^2 dt^2 - P(r)^{-a} dr^2 - P(r)^{1-a} r^2 d\Omega^2,
\]
where \( a = m/k = (1 - C_\phi/k^2)^{1/2} \) (we assume \( m > 0 \)). The solution is defined for \( r > 2k \), and \( r = 2k \) is a central singularity. The Schwarzschild metric is restored for \( C_\phi = 0, a = 1 \).
If $h_{ab}$ is not positive-definite, $C_\phi$ can have any sign. Thus, for some nontrivial scalar field configurations we may have $C_\phi = 0$, hence the Schwarzschild metric. For others, there can be $C_\phi < 0$ which correspond to “anti-Fisher” phantom-field metrics. In addition to singular metrics, they include (in case $k > 0$) the so-called cold black holes with horizons of infinite area (see [17] and references therein). In case $k < 0$, the substitution $|k|u = \cot^{-1}(r/|k|)$ converts the metric (13) into

$$ds^2_E = e^{-2mu}dt^2 - e^{2mu}[dr^2 - (k^2 + r^2)d\Omega^2],$$

which describes a traversable wormhole with different signs of mass at its two flat asymptotics. More details about the anti-Fisher metric can be found in [13, 17].

As to the theory (2), we can assert that its vacuum static, spherically symmetric solution in the Einstein frame is characterized by the Fisher metric in case $6\frac{df}{d\phi} > F$ and contains families with both Fisher and anti-Fisher metrics in case $6\frac{df}{d\phi} < F$.

The Jordan-frame metric is obtained from (13) by the transformation (3). To perform it we must know the function $F$, which is hard to find in a general form, due to the difficulty of solving the field equations (17). But one can assert that if $F$ is everywhere positive and finite (including its limiting values at infinity and the other extreme of the $u$ range), then the mapping (3) transforms regular points to regular points, a flat infinity to a flat infinity, and a singularity to a singularity. Therefore, the main qualitative features of the (anti-)Fisher metric, e.g., the existence of wormholes, are preserved by the Jordan-frame metric as well. Also, since (anti-)Fisher metrics do not contain horizons of finite area, such horizons will also be absent in the Jordan frame. Regarding cosmological applications of this theory, we can mention that cosmologies with two scalars, one of them being phantom, and with two canonical scalars but with a nontrivial potential, have been discussed in [18, 19]; see also [20].

III. SCALAR FIELDS INTERACTING WITH THE GAUSS-BONNET INVARIANT

The initial Lagrangian can contain, in addition to (2), terms with the Gauss-Bonnet invariant [3]

$$\mathcal{G} = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma},$$

and functions of $\Box^{-1}$, so that its local form can read, e.g.,

$$L\sqrt{-g} = \sqrt{-g}\left\{\frac{1}{2}\left[R[1 + f(\phi) - \xi] - \partial_\mu\xi\partial^\mu\phi\right] - V(\phi) + h(\phi)\mathcal{G} + L_m\right\},$$

where $h(\phi)$ is a function specified by some underlying theory. Anyhow, being quadratic in the curvature, this term can play a significant role only at sufficiently large curvatures, most probably (if $h$ is not too large and not too rapidly changing) close to the Planck level. Thus, the above results are also valid for the Lagrangian (20) at reasonably small curvatures. In particular, the nonsingular metric (19) should remain a solution in the whole space in case $V \equiv 0$ at moderate parameter values; however, at strong curvatures and, in particular, near the singularities, addition of $h(\phi)\mathcal{G}$ can drastically change the geometry, and such cases deserve a further study.
Here we restrict ourselves to a somewhat more special effective Lagrangian of the above type, containing a single scalar field minimally coupled to the curvature, namely,

$$L \sqrt{-g} = \frac{\sqrt{-g}}{2\kappa^2} \left\{ R - \varepsilon g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) + h(\phi) G \right\} + \sqrt{-g} L_m, \quad (21)$$

where the coefficient $\varepsilon = \pm 1$ distinguishes normal ($\varepsilon = 1$) and phantom ($\varepsilon = -1$) scalar fields, $V(\phi)$ is a potential, and $G$ is the Gauss-Bonnet (GB) invariant. The theory (21) may be called generalized dilatonic GB gravity, its special cases have been widely discussed in the context of a low-energy limit of string theory, see [21] for a recent discussion.

For the general form (13) of the static, spherically symmetric metric, with an arbitrary radial coordinate $u$, the GB invariant is calculated to be (the prime denotes $d/du$)

$$G = \frac{8F'(u)}{r^2 e^{\alpha + \gamma}}, \quad F(u) := e^{-\alpha + \gamma}(e^{-2\alpha}r'^2 - 1), \quad (22)$$

in agreement with the fact that a pure GB term in the Lagrangian makes a full divergence and does not contribute to the field equations. In the Lagrangian (21) the invariant $G$ appears with the factor $h(\phi)$ and therefore does contribute to the scalar and gravitational field equations which can be written in the following form:

$$\varepsilon \left( r^2 e^{\gamma - \alpha} \phi' \right)' - r^2 e^{\alpha + \gamma} V_\phi + 4h_\phi F' = 0, \quad (23)$$

$$- G'_i = \frac{\varepsilon}{2} e^{-2\alpha} \phi'^2 + V - \frac{4}{r^2} \left[ e^{-\alpha} h'(e^{-2\alpha} r'^2 - 1) \right]' , \quad (24)$$

$$- G'_r = -\frac{\varepsilon}{2} e^{-2\alpha} \phi'^2 + V + \frac{4}{r^2} e^{-2\alpha} h' \gamma'(1 - 3 e^{-2\alpha} r'^2), \quad (25)$$

$$- G'_\theta = \frac{\varepsilon}{2} e^{-2\alpha} \phi'^2 + V - \frac{4}{r} e^{-\alpha - \gamma} \left( e^{\gamma - 3\alpha} h' r' \gamma' \right)', \quad (26)$$

where the subscript $\phi$ denotes derivatives $d/d\phi$ and $G'_{\mu}$ are components of the Einstein tensor, $G'_{\mu} = R'_{\mu} - \frac{1}{3} \delta'_{\mu} R$.

It is straightforward to check that the last terms in (24)–(26) form a conservative tensor $T_{\mu}(G)$, so that $\nabla_{\nu} T_{\mu}(G) = 0$. Therefore, as is usually the case with static, spherically symmetric systems with scalar fields, the scalar equation (23) is a consequence of the Einstein equations (24)–(26). With the functions $h(\phi)$ and $V(\phi)$ specified and a coordinate (gauge) condition chosen, (23)–(26) is a well determined set of equations for the field $\phi(u)$ and two metric functions.

**Weak fields**

Since $G$ is quadratic in the curvature, it can play a significant role only at sufficiently large curvatures, most probably (if $h(\phi)$ is not too large and not too rapidly changing) close to the Planck level. In particular, if we consider static, spherically symmetric configurations governed by the Lagrangian (21) and assume that the curvatures are reasonably small (which in most cases is true, except in an immediate neighborhood of singularities), then the general properties of such configurations, known for (21) without $G$ (see, e.g., [4, 22, 23]) or, equivalently, with $h(\phi) = \text{const}$, remain valid in the presence of $G$. 
Let us recall some of these properties, by writing the metric (13) with the aid of the so-called quasi-global radial coordinate \( u \) (that is, taking the gauge condition \( \alpha + \gamma = 0 \) and denoting \( e^{2\gamma} = A(u) \)) and assuming \( \phi = \phi(u) \):

\[
ds^2 = -A(u)dt^2 + \frac{du^2}{A(u)} + r^2(u)(d\theta^2 + \sin^2 \theta d\varphi^2).
\] 

(27)

Properties of interest are then given by the theorems:

1. If \( \varepsilon = +1 \), the function \( r(u) \) cannot have a regular minimum, whatever be the potential \( V(\phi) \). In other words, wormhole throats are impossible, to say nothing of wormholes as global entities [22].

2. No-hair theorem [24]. Suppose \( \varepsilon = +1 \) and \( V \geq 0 \). Then the only asymptotically flat black hole solution to the field equations in the range \( (u_h, \infty) \), where \( u = u_h \) is the event horizon, comprises the Schwarzschild metric, \( \phi = \text{const} \) and \( V \equiv 0 \).

3. An asymptotically flat solution with a regular center (i.e., a particle-like, or star-like solution) is impossible if \( \varepsilon = +1 \) and \( V(\phi) \geq 0 \), or if \( \varepsilon = -1 \) and \( V(\phi) \leq 0 \) [23].

4. Global structure theorem [22]. The function \( B(u) = A(u)/r^2 \) cannot have a regular minimum. It follows that the system can contain no more than two Killing horizons, which are described as regular zeros of \( A(u) \), and then the horizons are simple and bound a static region \( (A > 0) \). A double horizon is possible, but it then separates two nonstatic (T) regions, and a static region is absent.

Theorem 4 is the most universal: it holds for any \( \varepsilon \) and \( V(\phi) \) and does not depend on assumptions about the asymptotic behavior; however, for an asymptotically flat configuration it implies that there can be no more than one horizon. Thus the whole set of possible kinds of global causal structure is the same as in the Schwarzschild-de Sitter solution, despite the existence of a scalar field.

As mentioned above, nonsingular solutions for (21) without \( G \), such as those describing wormholes and regular black holes which can emerge without violating the no-go theorems, are likely to satisfy, approximately, the full equations due to (21), if the corresponding curvatures are very small as compared to the Planck one.

In the full theory (21) the above theorems are no longer valid. This can be checked directly, using the explicit expression of \( G \) for static, spherically symmetric metrics. In particular, the so-called dilatonic GB black hole solutions with nontrivial scalar fields and \( V \equiv 0 \) are well known (see, e.g., [25–28] and references therein), so that the no-hair theorem does not hold. We will show, however, that some general restrictions can be obtained for any choice of \( h(\phi) \) and \( V(\phi) \).

### Possible wormholes

Let us begin with possible wormhole solutions and use the form (27) of the metric. The expression (22) then simplifies to give

\[
G = \frac{F'}{r^2}, \quad F(u) := A'(Ar^2 - 1),
\] 

(28)
where, as before, the prime stands for \( d/du \). The difference between Eqs. (24) and (25) reads

\[
2rr'' = -\varepsilon r^2 \phi'^2 + 4h''(Ar'^2 - 1) + 8Ar' r''h',
\]

(29)

while the difference between (24) and (26) can be written in the form

\[
A''r^2 - (r^2)'A' \equiv (r^4B')' = -2 - 8\sqrt{A'}(\sqrt{A}(Ar'^2 - 1)h')' + 4r[A'A' r'h']',
\]

(30)

where \( B(u) = A/r^2 \).

With a phantom field \( \phi (\varepsilon = -1) \), wormholes manifestly do exist, and the simplest example of such an exact solution is given by the Ellis wormhole \([29]\) for which

\[
A(u) \equiv 1, \quad r^2(u) = u^2 + k^2, \quad k = \text{const} > 0,
\]

\[
\phi(u) = \sqrt{2} \arctan(u/k), \quad V(\phi) \equiv 0,
\]

\[
h(\phi) = h_0 + h_1(u^3 + 3k^2u), \quad u = k \tan(\phi/\sqrt{2}),
\]

(31)

where \( h_0 \) and \( h_1 \) are integration constants. The metric is a special case \((m = 0)\) of \((19)\) (with a changed notation, \( r \mapsto u \)). In case \( h_1 = 0 \) we have \( h = \text{const} \), thus returning to the usual Einstein-scalar equations leading to the anti-Fisher solution discussed above, but in the general case \( h_1 \neq 0 \) \((31)\) is an exact solution of the full theory \((21)\).

It is of interest whether or not wormhole solutions can exist for \( \varepsilon = +1 \). If \( h = \text{const} \) (i.e., \( G \) does not contribute to the field equations), then in this case \( r'' \leq 0 \), and \( r(u) \) cannot have a minimum, i.e., Theorem 1 holds. But in the general case minima of \( r \) are not excluded. At an extremum of \( r \), where \( r' = 0 \), we have from \((29)\)

\[
2rr'' = -\varepsilon r^2 \phi'^2 - 4h'',
\]

(32)

and, with a properly chosen function \( h(\phi) \), the second term can lead to \( r'' > 0 \) even for \( \varepsilon = +1 \).

This does not mean, however, that such objects necessarily exist. To illustrate it, consider a simplified system with \( A(u) \equiv 1 \), i.e., spaces without a gravitational force acting on bodies at rest. The system as a whole remains nontrivial, since Eq. \((24)\) still contains a contribution from \( h(\phi) \). However, Eqs. \((25)\) and \((26)\) do not contain \( h \) (because now \( \gamma' = A'/2A \equiv 0 \) and, thus, the terms with \( h \) are eliminated), and their difference reads

\[
\varepsilon \phi'^2 = \frac{1}{r^2}(-1 + r'^2 - rr'').
\]

(33)

It follows that a minimum of \( r \), i.e., a wormhole throat, where \( r' = 0 \) and \( r'' > 0 \), can only occur with \( \varepsilon = -1 \), just as was the case without a GB term. It should be stressed that this result does not depend on the choice of the coupling function \( h(\phi) \) and the potential \( V(\phi) \).

In the general case \( A(u) \neq \text{const} \), wormhole solutions are not excluded but it seems that, if any, they cannot be of great physical interest for the following reasons. Considering for certainty symmetric wormholes for which simultaneously \( r'(u) \) and \( A'(u) \) vanish on the throat, from Eq. \((30)\) we obtain

\[
r^2A'' = -2 + 2Arr'' + 8Ah'' \quad \text{on the throat}.
\]

Combined with \((32)\), this leads for \( \varepsilon = +1 \) to the following inequality:

\[
r^2A'' < -2 - 2Arr''
\]

(34)
on the throat where, by definition, \( r'' > 0 \). It means that \( A(u) \) has quite a strong maximum there, i.e., the throat gravitationally repels test bodies. And, in any case, it is clear that in such wormholes the curvature components should be of sub-Planckian order near the throat, hence either its size or the magnitude of tidal forces (or maybe both) make these wormholes actually non-traversable for any macroscopic bodies.

### Possible horizons

Let us return to Eq. (30). In the absence of the Gauss-Bonnet invariant \( G \), and consequently terms containing \( h \), we have there only \(-2\) on the r.h.s., which prevents a minimum of \( B(u) \) and thus leads to Theorem 4. Indeed, a horizon is a zero of \( A(u) \) at finite \( r \), hence a zero of \( B(u) \), and the absence of its minima leaves only a restricted list of possible allocations of such zeros.

The terms with \( h(\phi) \) substantially change the situation, and one may expect configurations with more complex global structures. However, it can be shown that double horizons, like that of the extremal Reissner-Nordström black hole as well as horizons of orders higher than 2, cannot exist in our system, whatever be the choice of \( h(\phi) \) and \( V(\phi) \).

Indeed, let there be a horizon at \( u = 0 \) and consider near-horizon Taylor expansions of all functions involved in Eq. (30), namely,

\[
\begin{align*}
A(u) &= A_1 u + \frac{1}{2} A_2 u^2 + \ldots, \\
r(u) &= r_0 + r_1 u + \frac{1}{2} r_2 u^2 + \ldots, \\
h(\phi) &= h(\phi(u)) = h_0 + h_1 u + \frac{1}{2} h_2 u^2 + \ldots, \\
\end{align*}
\]  

(35)

and substitute them into (30). Note that the expansion (35) in terms of the quasiglobal coordinate \( u \) is a necessary feature of horizons: attempts to find other kinds of near-horizon behavior of the metric lead to singularities [30].

If \( A_1 \neq 0 \), i.e., there is a simple horizon, at order \( O(1) \) we obtain a relation between the constants,  

\[
A_2 r_0^2 - 4 A_1 h_1 - 4 r_0 r_1 A_2^2 h_1 + 2 = 0,
\]

and the next orders involve coefficients from other terms of the expansion. Simple horizons are thus possible, in accordance with the known examples [25, 26, 28].

If \( A_1 = 0 \) and \( A_2 \neq 0 \), i.e., there is a double horizon, Eq. (35) at order \( O(1) \) reads

\[
r_0^2 A_2 + 2 = 0,
\]

whence \( A_2 < 0 \), i.e., a double horizon is possible but \( A < 0 \) in its neighborhood. Thus such a horizon can separate two \( T \) regions but not two \( R \) regions, a horizon like the extreme Reissner-Nordström one is impossible. The situation is, in this respect, the same as the one described by Theorem 4.

If the first nonzero coefficient \( A_n \) is with \( n > 2 \) (which means a higher-order or ultra-extremal horizon), the only term of order \( O(1) \) in (35) is \(-2\), making the equation inconsistent. Such horizons are thus impossible.

### IV. CONCLUDING REMARKS

For the Lagrangian (2) of nonlocal origin, representing the Jordan frame of a scalar-tensor theory with two massless scalars, we have found an explicit condition under which
both scalar fields are canonical (non-phantom). If this condition does not hold, one of the fields exhibits a phantom behavior. The properties of the corresponding scalar-vacuum static, spherically symmetric configurations are well known, and we have briefly described them here for clarity. They include geometries with naked singularities and, in the phantom case, traversable wormholes.

For the Lagrangian (21) of nonlocal origin, containing a scalar field interacting with the Gauss-Bonnet invariant and a nonzero scalar field potential, we have found that the Gauss-Bonnet term, in general, leads to violation of the well-known no-go theorems valid for minimally coupled scalar fields in general relativity. This means that many configurations of interest, forbidden by these theorems, can appear, but only if curvatures are sufficiently large, approaching the Planck value.

We have shown, however, that some configurations of interest are still forbidden even in the full theory, whatever be the scalar field potential $V(\phi)$ and the GB-scalar coupling function $h(\phi)$. Among these configurations there are “force-free” wormholes with a normal ($\varepsilon = +1$) scalar field (i.e., wormholes with $g_{tt} = \text{const}$) and black holes with higher-order (extremal or ultraextremal) horizons. The fact that these considerations are based on the local form of the Lagrangian—which may lead in principle to extra solutions as compared to the nonlocal form (2)—poses no problem, in the end. To wit, in the more standard theory the class of solutions is wider but each of them is given by a single-valued (ordinary) function, while in the nonlocal theory the class of solutions (which is more restricted) contains multi-valued functions. Eventually, this does not affect our “on shell” results (once the second-order constraint condition relating the two formulations is imposed).

According to [27], double (extremal) horizons are possible in the theory (21) in configurations with an electric charge. They are found and discussed in the special case $h(\phi) \sim e^{a\phi}$, $a = \text{const}$, $V(\phi) \equiv 0$. We have shown that such horizons cannot exist in vacuum (without an electromagnetic field) in the general case of the theory (21), for any $h(\phi)$ and any potential $V(\phi)$. Such extremal black hole solutions, in general, separate configurations without horizons (e.g., with naked singularities) from black holes with two simple horizons. This happens, for instance, in the Reissner-Nordström metric and in the family of solutions of Einstein gravity coupled to nonlinear electrodynamics (see, e.g., [31]). In the present case, the non-existence of double horizons probably means that black holes with two simple horizons and a Reissner-Nordström-like causal structure are also absent in the general case of the theory (21). But this is a conjecture yet to be proved.

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