SU(1, 1) echoes for breathers in quantum gases

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Though the celebrated spin echoes have been widely used to reverse quantum dynamics, they are not applicable to systems whose constituents are beyond the control of the su(2) algebra. Here, we design echoes to reverse quantum dynamics of breathers in three-dimensional unitary fermions and two-dimensional bosons and fermions with contact interactions, which are governed by an underlying su(1, 1) algebra. Geometrically, SU(1, 1) echoes produce closed trajectories on a single or multiple Poincaré disks and thus could recover any initial states without changing the sign of the Hamiltonian. In particular, the initial shape of a breather determines the superposition of trajectories on multiple Poincaré disks and whether the revival time has period multiplication. Our work provides physicists with a recipe to tailor collective excitations of interacting many-body systems.

It is notoriously difficult to reverse quantum dynamics of many-body systems. In addition to the challenge of changing the signs of all terms in the Hamiltonian simultaneously, there exists a conceptual obstacle of reversing the dynamics of all particles in a synchronized means, reflecting the irreversible nature of the arrow of time. Nevertheless, the well-established spin echoes [1] have been widely used to overcome dephasing in spin systems, laying the foundation of many modern technologies, such as the nuclear magnetic resonance, magnetic resonance imaging, and the central spin problem in a broad range of condensed matter systems [2–4].

The study of collective excitations has been a main theme in the field of ultracold atoms and related topics [5–8]. Among such excitations, the breathing modes (or breathers) of interacting fermions and bosons have provided physicists with valuable information about superfluidity and hydrodynamics in the past two decades [9–14]. However, due to complex interaction effects in many-body systems, it is in general a grand challenge to recover the initial state once collective excitations are generated. The standard spin echoes do not apply to these breathers, whose relevant degrees of freedom do not obey the su(2) algebra, the underlying mechanism that spin echoes rely on. A crucial question then arises. Could we reverse many-body dynamics of breathers in interacting bosons and fermions?

In this work, we implement a fundamentally important group, SU(1, 1), to design echoes to reverse collective excitations of quantum gases. If the initial state is an eigenstate of a harmonic trap, SU(1, 1) echoes can be geometrized using a single Poincaré disk and guarantee that the initial state returns at 2nT, where n is an integer and T is the period of repeated drivings. When the initial state is not an eigenstate of a harmonic trap, multiple Poincaré disks are required to describe the dynamics. The interference between trajectories on these Poincaré disks determines whether the revival time is 2T or longer, the latter corresponding to period multiplication. When incommensurate frequencies exist in the dynamics, the revival time extends to infinity. These results also shed light on remarkable phenomena observed in a recent experiment by Dalibard’s group at ENS [15]. It was found that, starting from a triangular shape of the breather, the observed period is consistent with well-known results in harmonic traps when the quantum anomaly is negligible [16–21]. In sharp contrast, an initial disk shape leads to a period multiplication, quadrupling that of a triangle. Other shapes, however, do not have regular periodicities in experimentally accessible timescales. Such extraordinary behaviors of breathers can be naturally explained using the underlying algebra and the geometric representation of SU(1, 1) echoes, which allow us to infer how initial shapes of breathers lead to distinct superpositions of Poincaré disks and consequently, the revival times.

Generators of SU(1, 1) satisfy

\[ [K_1, K_2] = -iK_0, \quad [K_2, K_0] = iK_1, \quad [K_0, K_1] = iK_2. \] (1)

Geometrically, SU(1, 1)/U(1) corresponds to a Poincaré disk [22], where each point on the disk is a SU(1, 1) coherent state as analogous to a spin coherent state on a Bloch sphere, as shown in Fig. 1. Such a coherent state characterized by a complex number \(|z| < 1\) is written as

\[ |k, z⟩ = (1 - |z|^2)^k \sum_{n=0}^{∞} \frac{Γ(2k + n)}{Γ(n + 1)Γ(2k)} z^n |k, n⟩, \] (2)

where \(Γ(x)\) is the gamma function. \(k\) is determined by the Casimir operator \(C = K_0^2 - K_1^2 - K_2^2\), \(C|k, n⟩ = k(k - 1)|k, n⟩\). A single Poincaré disk is characterized by a unique \(k\). \(n\) is obtained from \(K_0|k, n⟩ = (k + n)|k, n⟩\).

SU(1, 1) has been widely applied in algebraic methods for solving energy spectra, two modes squeezing in quantum optics, SU(1, 1) interferometers, and studies of exotic geometric phases and expansions of Fermi gases [23–34]. However, it has not been implemented to study
interactions exist between any pair of particles. As dimensions, fermions and bosons with contact interactions, there is a criterion. In the latter case, the kinetic energy. In three dimensions, both $H = e^{-i\varphi K_1}$ moves it along a diameter and is followed by a rotation $U_0 = e^{-i\varphi K_0}$. Using Eq.(4), we conclude $(U_0U_I)^2 = e^{-i2\pi K_0}$, i.e., a rotation of $2\pi$ about the origin, and thus the initial state is recovered. This echo applies to any initial states on the Poincaré disk and any $\varphi_1K_1 + \varphi_2K_2$, similar to the standard spin echoes.

We first consider the initial state as the ground state of the Hamiltonian, $H_0 = 2K_0$. To implement a $SU(1,1)$ echo, the trapping frequency is suddenly changed to $\omega_1 = \kappa\omega_0$ at $t = 0$, where $\kappa$ is an arbitrary real or imaginary number. In the latter case, it corresponds to an inverted harmonic trap. When $t = t_1$, the original harmonic trap is restored and the system evolves for another time period $t_0$. Then the above two steps are repeated. Such dynamics are governed by the Hamiltonians,

$$H_1 = (1 + \kappa^2)K_0 + (1 - \kappa^2)K_1, \quad nT < t < nT + t_1,$$

$$H_0 = 2K_0, \quad nT + t_1 < t < (n + 1)T,$$  \hspace{1cm} (5)

where $n$ is a non-negative integer, and $T = t_0 + t_1$ defines a period. The propagator, $(U_0U_I)^2 = (e^{-iH_0t}e^{-iH_1t_1})^2$, from $t = nT$ to $t = (n + 2)T$ can be rewritten as

$$e^{-i(\zeta_1 + 2t_0)K_0}e^{-i\eta_1K_1}e^{-i(2\zeta_1 + 2t_0)K_0}e^{-i\eta_1K_1}e^{-i\zeta_1K_0},$$  \hspace{1cm} (6)

where $\zeta_1 = \arctan\left(\frac{1+\kappa^2}{2\kappa}\tan\kappa t_1\right)$ and $\eta_1 = 2\arcsinh\left(\frac{1-\kappa^2}{2\kappa}\sin(\kappa t_1)\right)$. We have used the Baker-Campbell-Hausdorff decomposition,

$$e^{-i(\zeta_0K_0 + \epsilon_1K_1 + \epsilon_2K_2)} = e^{-i\epsilon_1K_0}e^{-i\eta_1K_1}e^{-i(2\zeta_1 + 2t_0)K_0}e^{-i\eta_1K_1}e^{-i\zeta_1K_0},$$  \hspace{1cm} (7)

where $\zeta = \left(\frac{\epsilon_2}{\epsilon_1}\tan\frac{\zeta_1}{2}\right)$, $\cos\phi = \left(\frac{\epsilon_2}{\epsilon_1}\right)$, $\sinh\left(\frac{\eta}{2}\right) = \sqrt{\frac{\xi^2}{\xi^2 + \zeta^2}}\sin\left(\frac{\zeta}{2}\right)$, and $\xi^2 = \xi_0^2 - \xi_1^2 - \xi_2^2$. To deliver an echo, it is required that $\pi = 2(t_0 + \zeta_1)$, or equivalently,

$$t_0 = \pi - \zeta_1 = \pi - \arctan\left(\frac{1 + \kappa^2}{2\kappa}\tan\kappa t_1\right).$$  \hspace{1cm} (8)

Under this condition, the system returns to the initial state at $t = 2nT$. It is worth pointing out that there also exist echoes allowing the initial state to return in a longer time, say $t = 3nT$ (Supplementary Materials).

The expectation value of the potential energy, $E_{pot} = \langle \frac{1}{r} \sum_i \delta_i^2 \rangle$ in the time interval $nT + t_1 < t < (n + 1)T$, or equivalently half of the mean squared distance to the center of the trap, can be written as $E_{pot} = \langle K_0 - K_1 \rangle$. Using

SU(1,1) echoes arise from the identity,

$$e^{-i(\varphi_1K_1 + \varphi_2K_2)}e^{-i\pi K_0}e^{-i(\varphi_1K_1 + \varphi_2K_2)}e^{i\pi K_0} = \mathcal{I},$$  \hspace{1cm} (4)

where $\mathcal{I}$ is the identity operator, $\varphi_1$ and $\varphi_2$ are two arbitrary real numbers. On the Poincaré disk, $e^{-i\pi K_0}$ is a rotation of $\pi$ about the origin, and $e^{-i(\varphi_1K_1 + \varphi_2K_2)}$ is a boost changing $|z|$. A simple echo is illustrated in Fig.1(b). Starting from a given initial state, $U_I = e^{-i\varphi K_1}$, such $SU(1,1)$ echo brings the system back to the initial state.
Fig. 2. (a-b) Numerical results of $F(t)$ and $\langle r^2(t) \rangle / 2$ of 2D BECs. The initial state is the ground state of $K_0$, i.e., the origin of a single Poincaré disk. $\kappa = 2.0$ correspond to a modified and vanishing harmonic trap in the time interval, $nT < t < nT + t_1$, respectively. $\kappa = i, 2i$ correspond to inverted harmonic traps. $N_g = 25600$, $\omega = 20 \times 2\pi$ Hz and $t_1 = \pi/8$. $t_0$ is determined by Eq.8. (c) Left panel: harmonic traps in different time intervals. Right panel: snapshots of densities at different times for $\kappa = 2i$. (d) Trajectories on the Poincaré disk. Dotted and solid lines are evolutions governed by $H_1$ and $H_0$, respectively. Dot-dashed lines show the trajectories if only $H_1$ is applied in a quench dynamics.

properties of $SU(1,1)$ coherent states, $\langle k, z | K_0 | k, z \rangle = k^{1+|z|^2}, \langle k, z | K_1 | k, z \rangle = 2k^{Re(z)}$, we obtain,

$$E_{\text{pot}}(t) = k^{1+|z|^2} - 2Re(z).$$

$E_{\text{pot}}$ is the time interval $nT < t < nT + t_1$ simply multiplies the above equation by $k^2$. Apparently, $E_{\text{pot}}$ is periodic with period $2T$. For the system prepared in the ground state of $H_0$ with ground state energy $E_g$, we have $k = E_g/2$. Results above are valid for any eigenstates of the initial Hamiltonian, hold for any finite temperatures at thermal equilibrium, and $k$ in Eq. (9) should be understood as $\langle K_0|\text{thermal} = Tr(K_0 e^{-\beta H_0})/Tr(e^{-\beta H_0})$, where $\beta$ is the inverse temperature.

It is useful to consider 2D bosons with a weak contact interaction as an example. Such dynamics is well captured by a Gross-Pitaevskii equation,

$$i \frac{\partial}{\partial t} \Psi(r, t) = \left( -\frac{1}{2} \nabla^2 + \frac{1}{2} \kappa(t) r^2 + gN |\Psi(r, t)|^2 \right) \Psi(r, t),$$

where $N$ is the number of bosons, $g = 4\pi a_0$ with $a_0$ being the dimensionless scattering length. To prepare the initial state, we use an imaginary time evolution to obtain the ground state of the initial Hamiltonian, $H_0$. We then let the condensate evolve based on the GP equation, in which the Hamiltonian is determined by Eq.(14). We trace both the overlap between $\Psi(r, t)$ and $\Psi(r, 0)$, $F(t) = \int dr |\Psi(r, 0)\rangle \langle \Psi(r, t)|$, and the absolute value of the potential energy, $|E_{\text{pot}}|$. Fig. 2 shows a few typical choices. (I), $\kappa$ is real, which corresponds to retaining a harmonic trap, though the frequency could be different from the initial one. (II), $\kappa = 0$, which corresponds to completely turning off the harmonic trap. (III), $\kappa$ is purely imaginary, which means an inverted harmonic trap. For a generic $H = \sum_{i=0,1,2} \xi_i K_i$, $\xi = \{\xi_0, \xi_1, \xi_2\}$ defines an external field with a strength, $\xi^2 = \xi_0^2 - \xi_1^2 - \xi_2^2$. For instance, in Eq.(14), we have $\xi = 2\kappa$. In (I), $\xi^2 > 0$, and the system follows a closed loop on the Poincaré disk. In (II), $\xi$ vanishes. Without a confining potential in the real space, the trajectory on the Poincaré disk eventually becomes tangent with the boundary circle. In (III), $\xi$ becomes purely imaginary. While a deconfining potential pushes BECs to expand in the real space, on the Poincaré disk, the trajectory becomes an open path. Though quantum dynamics governed by $H_1$ alone in (I-III) are distinct, $SU(1,1)$ echo always lead to revivals. Fig. 2 clearly shows that both $F(t)$ and $|E_{\text{pot}}|$ are periodic functions of $t$ with a period of $2T$.

We emphasize that the initial state could be a superposition of multiple eigenstates of $C$ such that multiple Poincaré disks are required. We consider an arbitrary propagator $U$ in the $SU(1,1)$ group acting on $|\Psi\rangle = \sum c_{nk} |k, n\rangle = \sum |\psi_k\rangle$, where $|\psi_k\rangle = \sum c_{nk} |k, n\rangle$ and $\langle \psi_k | \psi_k \rangle \sim \delta_{k,k'}$. Here we have suppressed other quantum numbers for the same $k, n$. As $U|\Psi\rangle = \sum U|\psi_k\rangle$, and $U|\psi_k\rangle$ corresponds to an evolution on a single Poincaré disk, the dynamics thus correspond to superpositions of trajectories on multiple Poincaré disks. If an echo, $\langle U_0 U_1 \rangle = e^{-2\pi i \kappa K_0}$, acts on the initial state for $m$ times, where $m$ is an integer, we obtain

$$e^{-2\pi i m K_0} |\Psi\rangle = \sum c_{nk} e^{-2\pi i nk} |k, n\rangle. \quad (11)$$
Notice that $e^{-2i\pi mk}$ is independent of $n$, the trajectory on each Poincaré disk returns to the initial position with an extra $k$-dependent phase. The return probability $P(m) = |\langle \Psi(0)|\Psi(2mt)\rangle|^2$ becomes

$$P(m) = \left| \sum_k \tilde{P}_k e^{-2i\pi mk} \right|^2,$$

where $\tilde{P}_k = \sum_n |c_{nk}|^2$, $\sum_k \tilde{P}_k = 1$. From Eq.(12), it is apparent that $P(m) = 1$ only if $e^{-2i\pi mk} = e^{i\phi_0}$ for all $k$'s with a nonzero $c_{nk}$, where $\phi_0 \in [0, 2\pi)$ is independent of $k$. This is certainly satisfied if the initial state includes a single state, $|k_0, n_0\rangle$, as we have previously discussed. It is also clear that $e^{-2i\pi mk} = e^{i\phi_0}$ will never be satisfied if $|\Psi\rangle$ includes incommensurate $k$s, i.e., the difference between a pair of $k$s in Eq.(11), $k - k'$, is an irrational number. Whereas such a scenario is impossible in models with $k$s being either integers or half integers [35-37], in breathers with a continuous spectrum of $k$, dynamics controlled by incommensurate $k$s may arise.

If $k$s in Eq.(11) are commensurate, i.e., all $k$s are represented by $k = k_0 + p/Q$, where $k_0$ is a given reference with a nonzero $c_{nk_0}$, $p \in \mathbb{Z}$, $Q \in \mathbb{N}_+$, and $p$ and $Q$ are co-prime numbers, we have $P(Q) = 1$, and the system evolves back to its initial state after $2Q$ periods. Therefore, different superpositions of $|k, n\rangle$ in the initial state may lead to distinct revival times after $SU(1,1)$ echoes are applied. If $Q > 1$, period multiplication emerges in the dynamics. Fig.3 shows two examples, which correspond to $Q = 1$ and $Q = 4$, respectively. Consequently, the revival times are $2T$ and $8T$, i.e., period quadruples in the latter case.

Applying the above analysis to breathers of different initial shapes, we observe that the triangle and the disk correspond to $Q = 1$ and $Q = 4$, respectively. The initial state is chosen as the ground state of a flat-box potential with an infinite potential wall. At $t = 0$, the flat-box potential is turned off and the system evolves based on Eq.(14). $F(t)$ of a triangle recovers its initial value after $2T$. As the initial state is not an eigenstate of $K_0$, it must be a superposition of multiple $|k, n\rangle$ with differences between $k$s being integers, i.e., $Q = 1$ as shown in Fig.3(a). The exact number of Poincaré disks can be, in principle, determined by considering a particular Hamiltonian, $\tilde{H} \equiv C$, $e^{-i\tilde{H}t}|\Psi\rangle = \sum_{nk} c_{nk} e^{-ik(k-1)t}|n, k\rangle = \sum_k e^{-ik(k-1)t}|\psi_k\rangle$. A Fourier transform of $F(t) = \sum_k e^{-ik(k-1)t}\langle \psi_k|\psi_k\rangle$ to the frequency space unfolds how many $k$s are involved and their corresponding weights. Nevertheless, such calculations are not essential here, since our echoes apply to any superpositions in Eq.(11), regardless of the exact number of Poincaré disks involved.

The results of a disks shape are distinct. Fig.4 shows that the revival time of the disk is $8T$. We conclude that the superposition in the initial state must be similar to Fig.3(b). Again, the exact form of $c_{nk}$ is not important here, as any superpositions in Eq.(11) that corresponds to $Q = 4$ are reversed by $SU(1,1)$ echoes at $t = 8T$. The quench dynamics in the ENS experiment has a propagator, $e^{-iK_0t}$, corresponding to $SU(1,1)$ echoes where $t_1 = 0$. Such quench dynamics has a periodicity of $2T$ and
8T for the triangle and the disk, respectively [15]. This also confirms that the triangle and the disk correspond to a superposition of multiple Poincaré disks with $Q = 1$ and $Q = 4$, respectively. We have not found other initial shapes, such as a square, which return to the initial states within timescales of our numerical simulations, similar to results of the quench dynamics [15]. We conclude that these shapes are described by either incommensurate $ks$ or commensurate $ks$ corresponding to a very large $Q$. In the later case, the revival is not observable in relevant timescales of numerics and experiments, and breathers may behave like those with non-commensurate $ks$.

Whereas it is difficult to evaluate $c_{nk}$ exactly for a many-body system, such coefficients can be straightforwardly obtained in few-body systems. For instance, in a two-body problem, $c_{nk}$ are exactly solvable for any initial states. In particular, the eigenvalues of the Casimir operator are directly related to the angular momentum. In this case, the relation between the initial shape of the breather and the revival time can be easily established (Supplementary Materials).

Since our central results are obtained by an algebraic method, independent on the representation, they apply to any systems that have the $SU(1,1)$ symmetry. We hope that our work will stimulate more interests from different disciplines to use geometric approaches to control quantum dynamics in few-body and many-body systems.

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[1] E. L. Hahn, Spin echoes, Phys. Rev. 80, 580 (1950).
[2] E. L. Hahn and D. E. Maxwell, Spin echo measurements of nuclear spin coupling in molecules, Phys. Rev. 88, 1070 (1952).
[3] E. O. Stejskal and J. E. Tanner, Spin diffusion measurements: Spin echoes in the presence of a timedependent field gradient, The Journal of Chemical Physics 42, 288 (1965).
[4] W. Yao, R.-B. Liu, and L. J. Sham, Theory of electron spin decoherence by interacting nuclear spins in a quantum dot, Phys. Rev. B 74, 195301 (2006).
[5] N. Bogoliubov, On the theory of superfluidity, J. Phys 11, 23 (1947).
[6] S. Stringari, Collective excitations of a trapped bosenoncondensed gas, Phys. Rev. Lett. 77, 2360 (1996).
[7] M. R. Matthews, B. P. Anderson, P. C. Haljan, D. S. Hall, C. E. Wieman, and E. A. Cornell, Vortices in a bosenoncondensed gas, Phys. Rev. Lett. 83, 2498 (1999).
[8] C. J. Pethick and H. Smith, Bose-Einstein condensation in dilute gases (Cambridge university press, 2008).
[9] A. Griffin, W.-C. Wu, and S. Stringari, Hydrodynamic modes in a trapped bosenoncondensed gas above the bosenoncondensed transition, Phys. Rev. Lett. 78, 1838 (1997).
[10] H. Heiselberg, Collective modes of trapped gases at the bcs-bcs crossover, Phys. Rev. Lett. 93, 040402 (2004).
[11] J. Kinast, S. L. Hemmer, M. E. Gehm, A. Turlapov, and J. E. Thomas, Evidence for superfluidity in a resonantly interacting fermi gas, Phys. Rev. Lett. 92, 150402 (2004).
[12] A. Turlapov, J. Kinast, B. Clancy, L. Luo, J. Joseph, and J. E. Thomas, Is a gas of strongly interacting atomic fermions a nearly perfect fluid? Journal of Low Temperature Physics 150, 567 (2007).
[13] J. Kinast, A. Turlapov, and J. E. Thomas, Breakdown of hydrodynamics in the radial breathing mode of a strongly interacting fermi gas, Phys. Rev. A 70, 051401 (2004).
[14] A. Altmeyer, S. Riedl, M. J. Wright, C. Kohstall, J. H. Denschlag, and R. Grimm, Dynamics of a strongly interacting fermi gas: The radial quadrupole mode, Phys. Rev. A 76, 033610 (2007).
[15] R. Saint-Jalm, P. C. M. Castillo, É. Le Cerf, B. Baklal-Hassani, J.-L. Ville, S. Nascimbene, J. Beugnon, and J. Dalibard, Dynamical symmetry and breathers in a two-dimensional bosenoncondensed gas, Phys. Rev. X 9, 021035 (2019).
[16] L. P. Pitaevskii and A. Rosch, Breathing modes and hidden symmetry of trapped atoms in two dimensions, Phys. Rev. A 55, R853 (1997).
[17] M. Holten, L. Bayha, A. C. Klein, P. A. Murthy, P. M. Preiss, and S. Jochim, Anomalous breaking of scale invariance in a two-dimensional fermi gas, Phys. Rev. Lett. 121, 120401 (2018).
[18] T. Pepper, P. Dyke, M. Zamorano, I. Herrera, S. Hoinka, and C. J. Vale, Quantum anomaly and 2d-3d crossover in strongly interacting fermi gases, Phys. Rev. Lett. 121, 120402 (2018).
[19] Y. Kagan, E. L. Surkov, and G. V. Shlyapnikov, Evolution of a bosenoncondensed gas under variations of the confining potential, Phys. Rev. A 54, R1753 (1996).
[20] F. Chevy, V. Bretin, P. Rosenbusch, K. W. Madison, and J. Dalibard, Transverse breathing mode of an elongated bosenoncondensed condensate, Phys. Rev. Lett. 88, 250402 (2002).
[21] E. Vogt, M. Feld, B. Fröhlich, D. Pertot, M. Koschorreck, and M. Köhl, Scale invariance and viscosity of a two-dimensional fermi gas, Phys. Rev. Lett. 108, 070404 (2012).
[22] M. Novaes, Some basics of su(1, 1), Revista Brasileira de Ensino de Física 26, 351 (2004).
[23] C. Quesne and M. Moshinsky, Canonical transformations and matrix elements, Journal of Mathematical Physics 12, 1780 (1971).
[24] M. Bander and C. Itzykson, Group theory and the hydrogen atom (i), Rev. Mod. Phys. 38, 330 (1966).
[25] R. M. y Romero, H. Núñez-Yépez, and A. Salas-Brito, An algebraic solution for the relativistic hydrogen atom (i), Rev. Mod. Phys. 38, 330 (1966).
[26] P. K. Aravind, Pseudospin approach to the dynamics and squeezing of su(2) and su(1, 1) coherent states, J. Opt. Soc. Am. B 5, 1545 (1988).
[27] K. Wodkiewicz and J. H. Eberly, Coherent states, squeezed fluctuations, and the su(2) am su(1,1) groups in
quantum-optics applications, J. Opt. Soc. Am. B 2, 458 (1985).

[28] B. Yurke, S. L. McCall, and J. R. Klauder, Su(2) and su(1,1) interferometers, Phys. Rev. A 33, 4033 (1986).

[29] C. C. Gerry, Berry’s phase in the degenerate parametric amplifier, Phys. Rev. A 39, 3204 (1989).

[30] R. Y. Chiao and T. F. Jordan, Lorentz-group berry phases in squeezed light, Physics Letters A 132, 77 (1988).

[31] F. Werner and Y. Castin, Unitary gas in an isotropic harmonic trap: Symmetry properties and applications, Phys. Rev. A 74, 053604 (2006).

[32] D. T. Son, Vanishing bulk viscosities and conformal invariance of the unitary fermi gas, Phys. Rev. Lett. 98, 020604 (2007).

[33] S. Deng, Z.-Y. Shi, P. Diao, Q. Yu, H. Zhai, R. Qi, and H. Wu, Observation of the efimovian expansion in scale-invariant fermi gases, Science 353, 371 (2016).

[34] E. Elliott, J. A. Joseph, and J. E. Thomas, Observation of conformal symmetry breaking and scale invariance in expanding fermi gases, Phys. Rev. Lett. 112, 040405 (2014).

[35] C. Lyu, C. Lv, and Q. Zhou, Geometrizing quantum dynamics of a bose-einstein condensate, (2020), arXiv:2005.04815.

[36] Y.-Y. Chen, P. Zhang, W. Zheng, Z. Wu, and H. Zhai, Many-body echo, Phys. Rev. A 102, 011301 (2020).

[37] Y. Cheng and Z.-Y. Shi, Many-body dynamics with time-dependent interaction, (2020), arXiv:2004.12754.
Supplementary Materials for “SU(1,1) echoes for breathers in quantum gases”

Echoes with periodicity $3T$

In the main text, we have discussed the $su(1,1)$ echoes with a periodicity of $2T$, which arise from the identity in Eq. (4) of the main text. Here, we consider echoes with a periodicity of $3T$ and prove the following identity.

\[
(U_0U_1)^3 = e^{-i2\pi K_0}.
\]

(13)

We consider the same Floquet sequence as the main text, where the Hamiltonians controlling breathers in a harmonic trap are given by

\[
H_1 = (1 + \kappa^2)K_0 + (1 - \kappa^2)K_1, \quad nT < t < nT + t_1,
\]

\[
H_0 = 2K_0, \quad nT + t_1 < t < (n + 1)T,
\]

(14)

where $n$ is an integer, and $T = t_0 + t_1$ defines a period. Using the BCH decomposition, we rewrite \((U_0U_1)^3\) as

\[
(U_0U_1)^3 = e^{i\zeta_1K_0} (e^{-i\gamma K_0} e^{-i\eta K_1})^3 e^{-i\zeta_1K_0},
\]

(15)

where \(\zeta_1 = \arctan\left(\frac{1+\kappa^2}{2\kappa} \tan \kappa t_1\right), \eta_1 = \arcsin\left(\frac{1-\kappa^2}{2\kappa} \sin(\kappa t_1)\right),\) and \(\gamma = 2\zeta_1 + 2t_0\). To obtain

\[
(e^{-i\gamma K_0} e^{-i\eta K_1})^3 = e^{-i\alpha K_0},
\]

(16)

we adopt the $2 \times 2$ representation of $su(1,1)$ algebra. Specifically, using Pauli matrices, we have

\[
K_0 \rightarrow \sigma_z/2, \quad K_1 \rightarrow i\sigma_x/2, \quad K_2 \rightarrow i\sigma_y/2,
\]

(17)

and Eq. (16) becomes

\[
\left(\cos \frac{\gamma}{2} \cosh \frac{\eta_1}{2} \left(\cos \frac{\gamma}{2} - 2\right)\right) + \left[\cos \frac{\gamma}{2} \sinh \frac{\eta_1}{2} \sigma_x + \sin \left(\frac{\eta_1}{2}\right) \sinh \left(\frac{\gamma}{2}\right) \sigma_y - i \cosh \left(\frac{\eta_1}{2}\right) \sin \left(\frac{\gamma}{2}\right) \sigma_z\right] f(\gamma, \eta_1) = \cos \frac{\alpha}{2} - i \sin \frac{\alpha}{2} \sigma_z,
\]

(18)

where

\[
f(\gamma, \eta_1) = \cos \gamma + (1 + \cos \gamma) \cosh \eta_1.
\]

(19)

Since \(\cos(\gamma/2)\) and \(\sin(\gamma/2)\) cannot vanish simultaneously, we require \(f(\gamma, \eta_1) = 0\), which yields

\[
\gamma_0 = \arccos\left(-\frac{\cosh \eta_1}{1 + \cosh \eta_1}\right).
\]

(20)

Such \(\gamma_0\) also satisfies \(\cos \frac{\gamma_0}{2} \cosh \frac{\eta_1}{2} = \pm 1/2\). Thus, we obtain

\[
\left(\cos \frac{\gamma_0}{2} - i \sin \frac{\gamma_0}{2} \sigma_z\right) \left(\cosh \frac{\eta_1}{2} - \sinh \frac{\eta_1}{2} \sigma_z\right)^3 = -1 = \cos \pi - i \sin \pi \sigma_z,
\]

(21)

and \(\alpha = 2\pi\). Therefore, we conclude that

\[
(U_0U_1)^3 = e^{i\theta K_0} e^{-i2\pi K_0} e^{-i\theta K_0} = e^{-i2\pi K_0},
\]

(22)

if we choose

\[
t_0 = \frac{1}{2} \arccos\left(-\frac{\cosh \eta_1}{1 + \cosh \eta_1}\right) - \zeta_1.
\]

(23)

We have verified these results by numerically solving the GP equation. As shown in Fig. 5 and Fig. 6, the triangle and the disk need $3T$ and $12T$ to return to the initial state, respectively. Whereas echoes with even longer periods also exist, echoes of periods of $2T$ discussed in the main text are the simplest ones to implement in practice.
FIG. 5. A trajectory on the Poincaré Disk, where dotted and solid lines are evolutions governed by $H_1$ and $H_0$, respectively. We choose $\kappa = 2$, $t_1 = \pi/8$. $t_0$ is determined by Eq. 23.

Two unitary fermions in a harmonic trap

We consider one spin-up and one spin-down fermion in a three-dimensional harmonic trap, whose relative motion and the center of mass are decoupled. The $su(1,1)$ algebra applies to both degrees of freedom. The Hamiltonian is written as $H = H_{CM} + H_{rel}$,

$$H_{CM} = -\frac{\hbar^2}{2M} \nabla_R^2 + \frac{1}{2} M \omega_0^2 R^2,$$

$$H_{rel} = -\frac{\hbar^2}{2\mu} \nabla_r^2 + \frac{1}{2} \mu \omega_0^2 r^2.$$

(24)

FIG. 6. (a) $F(t)$ for a triangle. $Ng = 25600$, $\omega_0 = 40 \times 2\pi$Hz and $t_1 = \pi/8$. (b) $F(t)$ for a disk. $Ng = 12800$, $\omega_0 = 40 \times 2\pi$Hz and $t_1 = \pi/8$. For both cases $t_0$ is determined by Eq. 23.
where $M = 2m$, $\mu = m/2$, $\mathbf{R} = (\mathbf{r}_1 + \mathbf{r}_2)/2$, $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$, $m$ is the mass of fermions. The interaction in the unitary limit is replaced by the boundary condition of the relative motion, namely

$$\Psi_{rel}(\mathbf{r}) \propto \frac{1}{r},$$

(25)

$\Psi_{rel}(\mathbf{r})$ is the relative wave function and $\Psi(\mathbf{R}, \mathbf{r}) = \Psi_{CM}(\mathbf{R}) \Psi_{rel}(\mathbf{r})$. Since the center of mass has the same simple dynamics of a single particle in a harmonic trap, we focus on the relative motion, which has the $SU(1,1)$ generators [1],

$$K_0^r = \frac{1}{2} \left( -\frac{1}{2} \nabla^2 + \frac{1}{2} r^2 \right), \quad K_1^r = \frac{1}{2} \left( -\frac{1}{2} \nabla^2 - \frac{1}{2} r^2 \right), \quad K_2^r = \frac{1}{4i} (\mathbf{r} \cdot \nabla + \nabla \cdot \mathbf{r}).$$

(26)

We have chosen the harmonic length $l_{ho} = \sqrt{\hbar/(\mu \omega_0)}$ as the unit length and $\hbar \omega_0$ as the unit energy. We first evaluate the Casimir operator of the $su(1,1)$ algebra,

$$C = (K_0^r)^2 - (K_1^r)^2 - (K_2^r)^2 = \frac{L^2}{4\hbar^2} - \frac{3}{16},$$

(27)

where $L$ is the angular momentum operator for the relative motion. $C$ has the eigenvalue $l(l + 1)/4 - 3/16$ for the angular momentum eigenstate $|l, m\rangle$. Therefore we denote the eigenstate of $C$ by $|k, n; l, m\rangle$. Here $n$ is the principle quantum number, $k$ is the Casimir quantum number, $l$ and $m$ are the angular momentum and magnetic quantum number, respectively. Since $C |k, n; l, m\rangle = (k(k - 1)|k, n; l, m\rangle = (l(l + 1)/4 - 3/16)|k, n; l, m\rangle$, we obtain

$$k = \frac{l}{2} + \frac{3}{4}, \quad l \geq 0; \quad k = \frac{1}{4} \text{ or } \frac{3}{4}, \quad l = 0,$$

(28)

where we only consider the positive discrete series for the representation of $SU(1,1)$. Using $K_- |k, 0; 0, 0\rangle = 0$, we find

$$\langle \mathbf{r} | \frac{1}{4}; 0, 0 \rangle \propto r \rightarrow 0 r, \quad \langle \mathbf{r} | \frac{3}{4}; 0, 0 \rangle \propto r \rightarrow 0 1.$$

(29)

We conclude that the ground state of the relative motion of two unitary fermions in the s-wave channel corresponds to $|1/4, n; 0, 0\rangle$. As for the center of mass, the same argument leads to $|3/4, n; 0, 0\rangle$. The spectrum of the relative motion becomes

$$E_r = l + \frac{3}{2} + 2n, \quad l \geq 1; \quad E_r = \frac{1}{2} + 2n, \quad l = 0.$$

(30)

The dynamics on multiple Poincaré disks can be generated by choosing an initial state as a superposition of different angular momentums. For example, we consider the initial state as a mixture of $s$ and $d$-wave for their relative motion. The s-wave subspace has $k_s = 1/4$ and the d-wave subspace has $k_d = 7/4$ for any magnetic quantum number $m'$. The initial state and the state at the end of $2m$ Floquet periods are

$$|\Psi(0)\rangle = |\Psi_{CM}\rangle \otimes \sum_n \left( s_n |1/4, n; 0, 0\rangle + \sum_{m'} d_{nm'} |7/4, n; 2, m'\rangle \right),$$

$$|\Psi(2mT)\rangle = |\Psi_{CM}\rangle \otimes \sum_n \left( e^{-i(1/4 + n)2\pi m} s_n |1/4, n; 0, 0\rangle + e^{-i(7/4 + n)2\pi m} \sum_{m'} d_{nm'} |7/4, n; 2, m'\rangle \right),$$

(31)

respectively. Since $k_s = k_d - 3/2$, $Q = 2$, using Eq. (31), we conclude that it takes 4 Floquet periods for the systems to recover its initial state.

If the s-wave scattering length vanishes, the s-wave subspace has $k'_s = 3/4$, while d-wave subspace still has $k'_d = 7/4$. Therefore, we obtain $k'_s = k'_d - 1$, $Q = 1$. It takes the same initial state 2 Floquet periods to return to the initial state.

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[1] F. Werner, and Y. Castin, Unitary gas in an isotropic harmonic trap: Symmetry properties and applications, Phys. Rev. A 74, 053604 (2006).