A Sub-Supersolution Method for a Class of Nonlocal Problems Involving the \( p(x) \)-Laplacian Operator and Applications

Gelson C.G. dos Santos\(^1\) · Giovany M. Figueiredo\(^2\) · Leandro S. Tavares\(^3\)

Received: 30 April 2017 / Accepted: 3 September 2017 / Published online: 20 September 2017
© Springer Science+Business Media B.V. 2017

Abstract In the present paper, we study the existence of solutions for some nonlocal problems involving the \( p(x) \)-Laplacian operator. The approach is based on a new sub-supersolution method.

Keywords Fixed point arguments · Nonlocal problems · \( p(x) \)-Laplacian · Sub-supersolution

Mathematics Subject Classification (2000) 35J60 · 35Q53

1 Introduction

In this work, we are interested in the nonlocal problem
\[
\begin{aligned}
-\mathcal{A}(x, |u|_{L^r(x)}) \Delta_{p(x)} u &= f(x, u) |u|^{\alpha(x)}_{L^q(x)} + g(x, u) |u|^{\gamma(x)}_{L^s(x)} & \text{in } \Omega, \\
u &= 0 & \text{on } \partial \Omega,
\end{aligned}
\]  

(P)

where \( \Omega \) is a bounded domain in \( \mathbb{R}^N \) (\( N > 1 \)) with \( C^2 \) boundary, \( |.|_{L^m(x)} \) is the norm of the space \( L^m(x)(\Omega) \), \( -\Delta_{p(x)} u := -\text{div}(\nabla u |^{p(x)}-2 \nabla u) \) is the \( p(x) \)-Laplacian operator, \( r, q, s, \alpha, \gamma : \Omega \to [0, \infty) \) are measurable functions and \( \mathcal{A}, f, g : \overline{\Omega} \times \mathbb{R} \to \mathbb{R} \) are continuous functions satisfying certain conditions.

\(^1\) Faculdade de Matemática, Universidade Federal do Pará, 66075-110, Belém, PA, Brazil
\(^2\) Departamento de Matemática, Universidade de Brasília, 70910-900, Brasília, DF, Brazil
\(^3\) Centro de Ciências e Tecnologia, Universidade Federal do Cariri, 63048-080, Juazeiro do Norte, CE, Brazil
In the last decades, several works related to the \( p/p(x) \)-Laplacian operator arose, see for instance [1, 2, 10, 29–31, 33–36, 38, 39] and the references therein. Partial differential equations involving the \( p(x) \)-Laplacian arise, for instance, in nonlinear elasticity, fluid mechanics, non-Newtonian fluids and image processing. See for instance [1, 8, 11, 41, 42] and the references therein for more informations.

The nonlocal term \(|.|_{L^m(x)} \) with the condition \( p(x) = r(x) \equiv 2 \) is considered in the well known Carrier’s equation

\[
\rho u_{tt} - a(x, t, |u|^2) \Delta u = 0,
\]

that models the vibrations of a elastic string when the variation of the tensions are not too small. See [9] for more details. The same nonlocal term arises also in Population Dynamics, see [12, 16] and its references; other interesting nonlocal models that are related to Population Dynamics can be found in [22, 23] and in [25].

In the literature, there are several works related to \((P)\) with \( p(x) \equiv p \) \((p \text{ constant})\), see for instance [3, 12–19, 21, 22, 24, 26–28, 37, 43]. For example in [26] the authors used a sub-supersolution argument to study the nonlocal problem

\[
\begin{aligned}
-\Delta_p u &= |u|_{L^p(x)}^{a(x)} \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]

In [3], the authors used an abstract sub-supersolution theorem whose proof is mainly based on a version of the Minty-Browder Theorem for pseudomonotone operators to study the problem

\[
\begin{aligned}
-a(\int_\Omega |u|^q) \Delta u &= h_1(x, u) f(\int_\Omega |u|^p) + h_2(x, u) g(\int_\Omega |u|^r) \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]

where \( h_1 : \Omega \times \mathbb{R}^+ \to \mathbb{R} \) are continuous functions, \( q, p, r \in [1, \infty) \) are constants and the functions \( a, f, g : [0, \infty) \to \mathbb{R}^+ \) are functions with \( f, g \in L^\infty([0, \infty)) \) and

\[
a(t), f(t), g(t) \geq a_0 > 0,
\]

for all \( t \in [0, \infty) \), where \( a_0 \) is a constant.

Recently in [43], the authors studied the existence, and multiplicity of solutions to the problem

\[
\begin{aligned}
-a(\int_\Omega |u|^q) \Delta u &= h_1(x, u) f(\int_\Omega |u|^p) \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]

where \( q \in (0, \infty) \), \( a(t) \geq a_0 > 0 \) and \( f_\lambda : \Omega \times \mathbb{R} \to \mathbb{R} \) are continuous functions, with \( f_\lambda \) depending on the parameter \( \lambda \).

In [15], the authors used the Schauder’s Fixed Point Theorem to study the boundary value problem

\[
\begin{aligned}
-\mathfrak{A}(x, u) \Delta u &= \lambda f(u) \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]

where \( f \in C^1([0, \theta], \mathbb{R}) \), \( f(0) = f(\theta) = 0 \), \( f'(0) > 0 \), \( f(t) > 0 \) in \((0, \theta)\), the function \( \mathfrak{A} : \Omega \times L^p(\Omega) \to \mathbb{R} \) is such that the mapping \( x \mapsto \mathfrak{A}(x, u) \) is measurable for all \( u \in L^p(\Omega) \) and the function \( u \mapsto \mathfrak{A}(x, u) \) is continuous from \( L^p(\Omega) \) into \( \mathbb{R} \) for almost all \( x \in \Omega \). They also considered that there are constants \( a_0, a_\infty > 0 \) such that

\[
a_0 \leq \mathfrak{A}(x, u) \leq a_\infty \quad \text{a.e. in } \Omega,
\]
for all $u \in L^p(\Omega)$. In [20] the authors studied again the problem considered in [15]. For other interesting related works see [7, 40].

Recently in [37, Theorem 1], the first two authors considered the problem $(P)$ for $p(x) \equiv 2$ (i.e., $-\Delta_{p(x)} = -\Delta$). They proved a sub-supersolution theorem for $(P)$ and applied such result in three problems. Specifically, they considered a sublinear problem, a concave-convex problem, and a logistic equation. Their arguments are mainly based on the existence of the first eigenvalue of the Laplacian operator ($-\Delta, H_0^1(\Omega)$). The $p(x)$-Laplacian operator, in general, has no first eigenvalue, that is, the infimum of the eigenvalues equals 0 (see [32]).

The lack of the existence of the first eigenvalue imply a considerable difficulty when dealing with boundary values problems involving the $p(x)$-Laplacian by using sub-supersolution methods. Papers that consider such problems by using the mentioned method are rare in the literature. Among such works we want to mention the papers [4, 5, 36, 44, 45].

The main goal of this paper is to generalize [37, Theorem 1] for the $p(x)$-Laplacian operator and the three applications in [37]. Below, we describe the main points regarding the generalization of the results of [37].

(i) In [37], the homogeneity of ($-\Delta, H_0^1(\Omega)$), and the eigenfunction associated to the first eigenvalue $\lambda_1$ are used to construct a subsolution. Differently from the operator ($-\Delta_{p(x)}, W_0^{1,p(x)}(\Omega)$), the $p(x)$-Laplacian operator is not homogeneous. Another important point is the problem of the existence of the first eigenvalue of the operator ($-\Delta_{p(x)}, W_0^{1,p(x)}(\Omega)$). To avoid these problems we used some arguments contained in [36];

(ii) We present weaker conditions on the exponents $r, q, s, \alpha$ and $\gamma$;

(iii) As an application of Theorem 1, we prove the existence of a positive solution for some nonlocal problems, which generalize the three applications in [37];

(iv) As in [37, Theorem 1] and differently from several works that consider the nonlocal term $A(x, |u|_{L^r(x)})$ satisfying $A(x, t) \geq a_0 > 0$ (where $a_0$ is a constant), our Theorem 1 allows us to study $(P)$ in the mentioned case, and in situations where $A(x, 0) = 0$.

In this work, we will assume that the functions $r, p, q, s, \alpha$ and $\gamma$ satisfy the hypotheses below

$$(H_0) \quad p \in C^1(\overline{\Omega}), r, q, s \in L^\infty(\Omega) \quad \text{where}$$

$$L^\infty_+(\Omega) = \{ m \in L^\infty(\Omega) \text{ with } \text{ess inf } m(x) \geq 1 \},$$

and $\alpha, \gamma \in L^\infty(\Omega)$ satisfy

$$1 < p^- := \inf_{\Omega} p(x) \leq p^+ := \sup_{\Omega} p(x) < N \quad \text{and} \quad \alpha(x), \gamma(x) \geq 0 \quad \text{a.e. in } \Omega.$$ 

In order to present our main result, we need some definitions. We say that $u \in W_0^{1,p(x)}(\Omega) \cap L^\infty(\Omega)$ is a (weak) solution of $(P)$ if

$$\int_{\Omega} |\nabla u|^{p(x)-2} \nabla u \nabla \varphi = \int_{\Omega} \left( \frac{f(x, u)|u|_{L^q(x)}^{\alpha(x)}}{A(x, |u|_{L^q(x)})} + \frac{g(x, u)|u|_{L^s(x)}^{\gamma(x)}}{A(x, |u|_{L^s(x)})} \right) \varphi,$$

for all $\varphi \in W_0^{1,p(x)}(\Omega)$.

Given $u, v \in S(\Omega) := \{ u : \Omega \rightarrow \mathbb{R} : u \text{ is measurable} \}$, we write $u \leq v$ if $u(x) \leq v(x)$ a.e. in $\Omega$ and $[u, v] := \{ w \in S(\Omega) : u(x) \leq w(x) \leq v(x) \text{ a.e. in } \Omega \}$. 

© Springer
We say that \((u, \overline{u})\) is a sub-supersolution pair for \((P)\) if \(u \in W^{1,p(\cdot)}_0(\Omega) \cap L^\infty(\Omega), \overline{u} \in W^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)\) are such that \(u \leq \overline{u}, u = 0 = \overline{u}\) on \(\partial \Omega\) and if, for all \(\varphi \in W^{1,p(\cdot)}_0(\Omega)\) with \(\varphi \geq 0\), the following inequalities hold

\[
\int_\Omega |\nabla u|^{p(\cdot)-2} \nabla u \nabla \varphi \leq \int_\Omega \left( \frac{f(x,u)|u|^{p(\cdot)}_L}{A(x,|u|^{L(\cdot)}_r)} + \frac{g(x,u)|u|^{p(\cdot)}_L}{A(x,|u|^{L(\cdot)}_r)} \right) \varphi
\]

and

\[
\int_\Omega |\nabla \overline{u}|^{p(\cdot)-2} \nabla \overline{u} \nabla \varphi \geq \int_\Omega \left( \frac{f(x,\overline{u})|\overline{u}|^{p(\cdot)}_L}{A(x,|\overline{u}|^{L(\cdot)}_r)} + \frac{g(x,\overline{u})|\overline{u}|^{p(\cdot)}_L}{A(x,|\overline{u}|^{L(\cdot)}_r)} \right) \varphi,
\]

for all \(w \in [u, \overline{u}]\).

Now we present our main result.

**Theorem 1** Suppose that \(r, p, q, s, \alpha\) and \(\gamma\) satisfy \((H_0)\), \((u, \overline{u})\) is a pair of sub-supersolution for \((P)\) with \(u > 0\) a.e. in \(\Omega\), \(f(x,t), g(x,t) \geq 0\) in \(\overline{\Omega} \times [0, |\overline{u}|_{L^\infty}]\) are continuous functions and \(A : \overline{\Omega} \times (0, \infty) \rightarrow \mathbb{R}\) is continuous with \(A(x,t) > 0\) in \(\overline{\Omega} \times [0, |\overline{u}|_{L^\infty}, |\overline{u}|_{L^\infty}]\). Then \((P)\) has a weak positive solution \(u \in [u, \overline{u}]\).

## 2 Preliminaries: The Spaces \(L^{p(\cdot)}(\Omega)\), \(W^{1,p(\cdot)}(\Omega)\) and \(W_0^{1,p(\cdot)}(\Omega)\)

In this section, we point out some facts regarding to the spaces \(L^{p(\cdot)}(\Omega)\), \(W^{1,p(\cdot)}(\Omega)\) and \(W_0^{1,p(\cdot)}(\Omega)\) that will be often used in this work. For more information, see Fan-Zhang [29], and the references therein.

Let \(\Omega \subset IR^N\) \((N \geq 1)\) be a bounded domain. Given \(p \in L^\infty_+(\Omega)\), we define the generalized Lebesgue space

\[
L^{p(\cdot)}(\Omega) = \left\{ u \in S(\Omega) : \int_\Omega |u(x)|^{p(\cdot)} dx < \infty \right\}.
\]

We define in \(L^{p(\cdot)}(\Omega)\) the norm

\[
|u|_{p(\cdot)} := \inf\left\{ \lambda > 0 ; \int_\Omega \left| \frac{u(x)}{\lambda} \right|^{p(\cdot)} dx \leq 1 \right\}.
\]

The space \((L^{p(\cdot)}(\Omega), \|\cdot\|_{L^{p(\cdot)}})\) is a Banach space.

Given \(m \in L^\infty(\Omega)\), we define

\[
m^+ := \text{ess sup}_\Omega m(x) \quad \text{and} \quad m^- := \text{ess inf}_\Omega m(x).
\]

**Proposition 1** Define the quantity \(\rho(u) := \int_\Omega |u|^{p(\cdot)} dx\). For all \(u, u_n \in L^{p(\cdot)}(\Omega), n \in \mathbb{N}\), the following assertions hold.

(i) Let \(u \neq 0\) in \(L^{p(\cdot)}(\Omega)\), then \(|u|_{L^{p(\cdot)}} = \lambda \Leftrightarrow \rho(\frac{u}{\lambda}) = 1\).

(ii) If \(|u|_{L^{p(\cdot)}} < 1 (= 1; > 1)\), then \(\rho(u) < 1 (= 1; > 1)\).

(iii) If \(|u|_{L^{p(\cdot)}} > 1\), then \(|u|_{L^{p(\cdot)}} \leq \rho(u) \leq |u|_{L^{p(\cdot)}}^p\).

(iv) If \(|u|_{L^{p(\cdot)}} < 1\), then \(|u|_{L^{p(\cdot)}}^p \leq \rho(u) \leq |u|_{L^{p(\cdot)}}^p\).

(v) \(|u_n|_{L^{p(\cdot)}} \rightarrow 0 \Leftrightarrow \rho(u_n) \rightarrow 0\). and \(|u_n|_{L^{p(\cdot)}} \rightarrow \infty \Leftrightarrow \rho(u_n) \rightarrow \infty\).
We define the generalized Sobolev space as
\[ W^{1,p(x)}(\Omega) := \left\{ u \in L^{p(x)}(\Omega) : \frac{\partial u}{\partial x_j} \in L^{p(x)}(\Omega), j = 1, \ldots, N \right\}, \]
with the norm \( \|u\|_w = |u|_{L^{p(x)}} + \sum_{j=1}^N |\frac{\partial u}{\partial x_j}|_{L^{p(x)}}, u \in W^{1,p(x)}(\Omega). \) The space \( W^{1,p(x)}_0(\Omega) \) is defined as the closure of \( C_0^\infty(\Omega) \) with respect to the norm \( \|\cdot\|_w. \)

**Theorem 3** If \( p^{-} > 1 \), then \( W^{1,p(x)}(\Omega) \) is a Banach, separable and reflexive space.

**Proposition 2** Let \( \Omega \subset \mathbb{R}^N \) be a bounded domain and consider \( p, q \in C(\overline{\Omega}). \) Define the function \( p^*(x) = \frac{Np(x)}{N-p(x)} \) if \( p(x) < N \) and \( p^*(x) = \infty \) if \( N \geq p(x). \) The following statements hold.

(i) (Poincaré inequality) If \( p^{-} > 1 \), then there is a constant \( C > 0 \) such that \( |u|_{L^{p(x)}} \leq C|\nabla u|_{L^{p(x)}} \) for all \( u \in W^{1,p(x)}_0(\Omega). \)

(ii) If \( p^{-}, q^{-} > 1 \) and \( q(x) < p^*(x) \) for all \( x \in \overline{\Omega} \), the embedding \( W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega) \) is continuous and compact.

From (i) of Proposition 2, we have that \( \|u\| := |\nabla u|_{L^{p(x)}} \) defines a norm in \( W^{1,p(x)}_0(\Omega) \) which is equivalent to the norm \( \|\cdot\|_w. \)

**Definition 1** Consider \( u, v \in W^{1,p(x)}(\Omega). \) We say that
\[ -\Delta_{p(x)} u \leq -\Delta_{p(x)} v, \]
if
\[ \int_{\Omega} |\nabla u|^{p(x)-2}\nabla u \nabla \varphi \leq \int_{\Omega} |\nabla v|^{p(x)-2}\nabla v \nabla \varphi, \]
for all \( \varphi \in W^{1,p(x)}_0(\Omega) \) with \( \varphi \geq 0. \)

The following result is contained in [33, Lemma 2.2] and [34, Proposition 2.3].

**Proposition 3** Consider \( u, v \in W^{1,p(x)}(\Omega). \) If \( -\Delta_{p(x)} u \leq -\Delta_{p(x)} v \) and \( u \leq v \) on \( \partial \Omega \), (i.e., \( (u - v)^+ \in W^{1,p(x)}_0(\Omega) \)), then \( u \leq v \) in \( \Omega. \) If \( u, v \in C(\overline{\Omega}) \) and \( S = \{x \in \Omega : u(x) = v(x)\} \) is a compact set of \( \Omega \), then \( S = \emptyset. \)

**Lemma 1** [34, Lemma 2.1] Let \( \lambda > 0 \) be the unique solution of the problem
\[ \begin{cases} -\Delta_{p(x)} z_{\lambda} = \lambda & \text{in } \Omega, \\ u = 0 & \text{on } \partial \Omega. \end{cases} \tag{3} \]
Define \( \rho_0 = \frac{p^{-}}{2|\Omega|^{\frac{1}{p^{-}}}}. \) If \( \lambda \geq \rho_0 \), then \( |z_{\lambda}|_{L^{\infty}} \leq C^* M^{\frac{1}{p^{-}-1}} \) and \( |z_{\lambda}|_{L^{\infty}} \leq C_* M^{\frac{1}{p^{-}-1}} \) if \( \lambda < \rho_0 \). Here \( C^* \) and \( C_* \) are positive constants depending only on \( p^+, p^{-}, N, |\Omega| \) and \( C_0 \), where \( C_0 \) is the best constant of the embedding \( W^{1,1}_0(\Omega) \hookrightarrow L^{\frac{N}{N-1}}(\Omega). \)

Regarding the function \( z_{\lambda} \) of the previous result, it follows from [31, Theorem 1.2] and [33, Theorem 1] that \( z_{\lambda} \in C(\overline{\Omega}) \) with \( z_{\lambda} > 0 \) in \( \Omega. \)
2.1 Proof of Theorem 1

The goal of this section is the proof of Theorem 1.

Proof of Theorem 1 Consider the operator $T : L^{p(x)}(\Omega) \to L^{\infty}(\Omega)$ defined by

$$(Tu)(x) = \begin{cases} 
    u(x) & \text{if } u(x) \leq u(\bar{x}), \\
    u(x) & \text{if } u(x) \leq u(\bar{x}) \leq \overline{u}(x), \\
    \overline{u}(x) & \text{if } u(\bar{x}) \geq \overline{u}(x). 
\end{cases}$$

The operator $T$ is well-defined, because $u, \overline{u} \in L^{\infty}(\Omega)$ and $Tu \in [u, \overline{u}]$.

Let $p'(x) = \frac{p(x)}{p(x) - 1}$ and consider the operator $H : [u, \overline{u}] \to L^{p'(x)}(\Omega)$

$$H(v)(x) = \frac{f(x, v(x)) |v|^{\alpha(x)}_{L^{p(x)}}}{A(x, |v|_{L^{p(x)}})} + \frac{g(x, v(x)) |v|^{\gamma(x)}_{L^{s(x)}}}{A(x, |v|_{L^{r(x)}})},$$

where $|.|_{L^{m(x)}}$ denotes the norm of $L^{m(x)}(\Omega)$.

Note that the operators $H$ and $u \mapsto HoT(u)$ are well-defined. In fact, since $f, g$ and $A$ are continuous functions with $A(x, t) > 0$ in the compact set $\overline{\Omega} \times [|u|_{L^{r(x)}}, |\overline{u}|_{L^{r(x)}}]$ and $|w|^{p(x)}_{L^{m(x)}} \leq |w|^{\theta_{-}}_{L^{m(x)}} + |w|^{p(x)}_{L^{m(x)}}$ for all $w \in L^{m(x)}(\Omega), \theta \in L^{\infty}(\Omega)$, then there is a constant $K_0 > 0$ such that

$$|H(v)| \leq K_0, \quad \forall v \in [u, \overline{u}].$$

Since $\Omega$ is a bounded domain, it follows that $H$ is well-defined. The operator $u \mapsto HoT(u)$ is well-defined, because the inclusion $Tu \in [u, \overline{u}]$ imply that

$$|H(Tu)| \leq K_0, \quad \forall u \in L^{p(x)}(\Omega).$$

We claim that the operator $u \mapsto HoT(u)$ is continuous. In order to show such affirmation, let $(u_n)$ be a sequence in $L^{p(x)}(\Omega)$ that converges to $u$ in $L^{p(x)}(\Omega)$. Since $Tu_n, Tu \in [u, \overline{u}]$, the Lebesgue Dominated Convergence Theorem combined with Proposition 1 imply that $Tu_n \to Tu$ in $L^{m(x)}(\Omega)$ for all $m \in L^{\infty}(\Omega)$. The continuity of $f, g$ and $A$ combined with the Lebesgue Dominated Convergence Theorem imply that $H(Tu_n) \to H(Tu)$ in $L^{p'(x)}(\Omega)$ and then we have the desired continuity.

Fix $v \in L^{p(x)}(\Omega)$. The inequality (4) imply that $(HoT)(v) \in L^{\infty}(\Omega)$, thus by [29, Theorem 4.2] the problem

$$\begin{cases}
  -\Delta_{p(x)} u = H(Tv) & \text{in } \Omega, \\
  u = 0 & \text{on } \partial \Omega,
\end{cases} \quad (P_L)$$

has a unique solution. Therefore we can define the operator $S : L^{p(x)}(\Omega) \to L^{p(x)}(\Omega)$, given by $S(v) = u$ where $u \in W^{1,p(x)}(\Omega)$ is the unique solution of $(P_L)$.

We affirm that $S$ is compact. In fact, let $(v_n)$ be a bounded sequence in $L^{p(x)}(\Omega)$ and define $u_n := S(v_n), n \in \mathbb{N}$. The definition of $S$ imply that

$$\int_{\Omega} |\nabla u_n|^{p(x)-2} \nabla u_n \nabla \varphi = \int_{\Omega} H(Tv_n) \varphi.$$
for all \( n \in \mathbb{N} \) and \( \varphi \in W^{1,p(x)}_0(\Omega) \). Using the inclusion \( T v_n \in [u, \bar{u}] \), the inequality (4) and considering the test function \( \varphi = u_n \), we have
\[
\int_{\Omega} |\nabla u_n|^{p(x)} \leq K_0 \int_{\Omega} |u_n|,
\]
for all \( n \in \mathbb{N} \).

The Poincaré inequality combined with the embedding \( L^{p(x)}(\Omega) \hookrightarrow L^1(\Omega) \) imply that
\[
\int_{\Omega} |\nabla u_n|^{p(x)} \leq C \|u_n\|,
\]
for all \( n \in \mathbb{N} \), where \( C \) is a constant that does not depend on \( n \in \mathbb{N} \).

If \( \|u_n\| > 1 \), by Proposition 1 we have
\[
\|u_n\|^{p(x)} \leq C \|u_n\|,
\]
for all \( n \in \mathbb{N} \), where the constant \( C \) does not depend on \( n \in \mathbb{N} \). Therefore the sequence \((u_n)\) is bounded in \( W^{1,p(x)}_0(\Omega) \). Thus, up to a subsequence, we have
\[
\text{up to a subsequence, we have } u_n \rightharpoonup u \text{ in } W^{1,p(x)}_0(\Omega)
\]
for some \( u \in W^{1,p(x)}_0(\Omega) \). Since the embedding \( W^{1,p(x)}_0(\Omega) \hookrightarrow L^{p(x)}(\Omega) \) is compact, we have \( u_n \rightharpoonup u \) in \( L^{p(x)}(\Omega) \). Therefore \( S \) is a compact operator.

With respect to continuity, let \((v_n)\) be a sequence in \( L^{p(x)}(\Omega) \) with \( v_n \rightharpoonup v \) in \( L^{p(x)}(\Omega) \) for \( v \in L^{p(x)}(\Omega) \). Define \( u_n := S(v_n) \) and \( u := S(v) \). Note that
\[
\int_{\Omega} |\nabla u_n|^{p(x)} - 2 \nabla u_n \nabla \varphi = \int_{\Omega} H(T v_n) \varphi
\]
and
\[
\int_{\Omega} |\nabla u|^{p(x)} - 2 \nabla u \nabla \varphi = \int_{\Omega} H(T v) \varphi,
\]
for all \( \varphi \in W^{1,p(x)}_0(\Omega) \). Such equations with \( \varphi = u_n - u \) provide
\[
\int_{\Omega} \langle |\nabla u_n|^{p(x)} - 2 \nabla u_n - |\nabla u|^{p(x)} - 2 \nabla u, \nabla (u_n - u) \rangle = \int_{\Omega} \left[ H(T v_n) - H(T v) \right] (u_n - u).
\]

The previous arguments imply that the sequence \((u_n)\) is bounded in \( L^{p(x)}(\Omega) \). Thus by Hölder inequality, we have
\[
\left| \int_{\Omega} \left[ H(T v_n) - H(T v) \right] (u_n - u) \right| \leq C \left| (HoT)(v_n) - (HoT)(v) \right|_{L^{p'(x)}},
\]
where the constant \( C \) does not depend on \( n \in \mathbb{N} \). Since \( HoT \) is continuous, we have
\[
\int_{\Omega} \langle |\nabla u_n|^{p(x)} - 2 \nabla u_n - |\nabla u|^{p(x)} - 2 \nabla u, \nabla (u_n - u) \rangle \to 0,
\]
which imply the continuity of \( S \).

We claim that there exists \( R > 0 \) such that if \( u = \theta S(u) \) with \( \theta \in [0, 1] \), then \( |u|_{L^{p(x)}} < R \). In fact, if \( \theta = 0 \), then \( u = 0 \). Suppose that \( \theta \neq 0 \). In this case, we have \( S(u) = \frac{u}{\theta} \) and such equality imply the identity
\[
\int_{\Omega} \left| \nabla \left( \frac{u}{\theta} \right) \right|^{p(x)} - 2 \nabla \left( \frac{u}{\theta} \right) \nabla \varphi = \int_{\Omega} H(T u) \varphi,
\]
for all $\varphi \in W^{1,p(x)}_0(\Omega)$. Using the test function $\varphi = \frac{u}{\theta}$, the inequality (4) and the embedding $L^{p(x)}(\Omega) \hookrightarrow L^1(\Omega)$, we get

$$\int_\Omega \left| \nabla \left( \frac{u}{\theta} \right) \right|^{p(x)} \leq K_0 \int_\Omega \frac{|u|}{\theta} \leq C |u|_{L^{p(x)}},$$

where $C > 0$ is a constant that does not depend on $u$ and $\theta$. If $|\nabla u|_{L^{p(x)}} > 1$, we have by the Poincaré inequality, and Proposition 1 that $|u|_{L^{p(x)}}^{p-1} \leq \theta^{p-1}C$, where $C$ is a constant that does not depend on $u$ and $\theta$.

Since $\theta \in (0, 1]$, by Schaefer’s fixed Point Theorem, there exists $u \in L^{p(x)}(\Omega)$ such that $u = S(u)$. Thus

$$\int_\Omega |\nabla u|^{p(x)-2} \nabla u \nabla \varphi = \int_\Omega \left( \frac{f(x, Tu)|Tu|^{q(x)}}{A(x, |Tu|_{L^{r(x)}})} - \frac{g(x, Tu)|Tu|^{r(x)}}{A(x, |Tu|_{L^{r(x)}})} \right) \varphi,$$

for all $\varphi \in W^{1,p(x)}_0(\Omega)$.

We claim that $u \in [u, \bar{u}]$. Considering $w = Tu$ in (1) and subtracting from (5), we get

$$\int_\Omega (|\nabla u|^{p(x)-2} \nabla u - |\nabla u|^{p(x)-2} \nabla u, \nabla \varphi) \leq \int_\Omega \left( \frac{f(x, u)|u|^{q(x)}}{A(x, |u|_{L^{r(x)}})} - \frac{g(x, u)|u|^{r(x)}}{A(x, |u|_{L^{r(x)}})} \right) \varphi,$$

for all $\varphi \in W^{1,p(x)}_0(\Omega)$ with $\varphi \geq 0$. Using the test function $\varphi := (u - u)_+ = \max\{u - u, 0\}$, and using that $f, g \geq 0$ in $[0, \|u\|_{L^\infty}]$, $Tu = u$ in $\{u \geq u\} := \{x \in \Omega : u(x) \geq u(x)\}$, we get

$$\int_{\{u \geq u\}} (|\nabla u|^{p(x)-2} \nabla u - |\nabla u|^{p(x)-2} \nabla u, \nabla (u - u)) \leq \int_{\{u \geq u\}} \frac{f(x, u)(|u|^{q(x)} - |Tu|^{r(x)})}{A(x, |Tu|_{L^{r(x)}})} \varphi + \int_{\{u \geq u\}} \frac{g(x, u)(|u|^{r(x)} - |Tu|^{r(x)})}{A(x, |Tu|_{L^{r(x)}})} \varphi \leq 0,$$

which imply that $u \leq u$. A similar reasoning provides the inequality $u \leq \bar{u}$. □

### 3 Applications

The main goal of this section is to apply Theorem 1 to some classes of nonlocal problems.

#### 3.1 A Sublinear Problem

In this section, we use Theorem 1 to study the nonlocal problem

$$\begin{cases}
- \mathcal{A}(x, \|u\|_{L^{r(x)}}) \Delta_{p(x)} u = u^{\beta(x)} |u|^{q(x)}_{L^{q(x)}} & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega.
\end{cases} \quad (Ps)$$

© Springer
The above problem in the case $p(x) \equiv 2$, was considered recently in [37]. The result of this section generalizes [37, Theorem 3].

**Theorem 4** Suppose that $r$, $p$, $q$, $\alpha$ satisfy $(H_0)$ and let $\beta \in L^\infty(\Omega)$ be a nonnegative function. Consider also that $\alpha^+ + \beta^+ < p^− − 1$. Let $a_0 > 0$ be a positive constant. Suppose that one of the conditions holds

$(A_1)$ $\mathcal{A}(x,t) \geq a_0$ in $\Omega \times [0, \infty)$,

$(A_2)$ $0 < \mathcal{A}(x,t) \leq a_0$ in $\Omega \times (0, \infty)$, and $\lim_{t \to +\infty} \mathcal{A}(x,t) = a_\infty > 0$ uniformly in $\Omega$.

Then $(Ps)$ has a positive solution.

**Proof** Suppose that $(A_1)$ holds, that is, $\mathcal{A}(x,t) \geq a_0$ in $\Omega \times [0, +\infty)$. We will start by constructing $\tilde{u}$. Let $\lambda > 0$ and let $z_\lambda \in W^{1,p(x)}_0(\Omega) \cap L^\infty(\Omega)$ be the unique solution of (3), where $\lambda$ will be chosen later.

For $\lambda > 0$ sufficiently large, by Lemma 1 there is a constant $K > 1$ that does not depend on $\lambda$ such that

$$0 < z_\lambda(x) \leq K \lambda^{\frac{1}{p^− − 1}} \text{ in } \Omega. \quad (6)$$

Since $\alpha^+ + \beta^+ < p^− − 1$, we can choose $\lambda > 1$ such that (6) occurs and

$$\frac{1}{a_0} K^{\beta^+} \lambda^{\frac{a^+ + \beta^+}{p^− − 1}} \max\{ |K \frac{a^−}{L^q(x)}, |K \frac{\beta^+}{L^q(x)} | \} \leq \lambda. \quad (7)$$

By (6) and (7), we get

$$\frac{1}{a_0} z_\lambda^{\beta(x)} |z_\lambda|_{L^q(x)}^{\alpha(x)} \leq \lambda.$$

Therefore

$$\begin{cases} -\Delta_{p(x)} z_\lambda \geq \frac{1}{\mathcal{A}(x, |w|_{L^{r(x)}})} z_\lambda^{\beta(x)} |z_\lambda|_{L^{q(x)}}^{\alpha(x)} \text{ in } \Omega, \\
               z_\lambda = 0 \text{ on } \partial \Omega, \end{cases}$$

for all $w \in L^\infty(\Omega)$.

Define $A_\lambda := \max\{ \mathcal{A}(x,t) : (x,t) \in \Omega \times [0, |z_\lambda|_{L^{r(x)}}] \}$. We have

$$a_0 \leq \mathcal{A}(x, |w|_{L^{r(x)}}) \leq A_\lambda \text{ in } \Omega,$$

for all $w \in [0, z_\lambda]$.

Now, we construct $u$. Since $\partial \Omega$ is $C^2$, there is a constant $\delta > 0$ such that $d \in C^2(\Omega_{3\delta})$ and $|\nabla d(x)| \equiv 1$, where $d(x) := \text{dist}(x, \partial \Omega)$ and $\Omega_{3\delta} := \{ x \in \Omega; d(x) \leq 3\delta \}$. From [36, p. 12], we have that, for $\sigma \in (0, \delta)$ sufficiently small, the function $\phi = \phi(k, \sigma)$ defined by

$$\phi(x) = \begin{cases} e^{kd(x)} - 1 & \text{if } d(x) < \sigma, \\
e^{k\sigma} - 1 + \int_\sigma^{d(x)} k e^{k\sigma} \left( \frac{2 \delta - t}{2 \delta - \sigma} \right) \frac{t^2}{2^2 - 1} dt & \text{if } \sigma \leq d(x) < 2\delta, \\
e^{k\sigma} - 1 + \int_\sigma^{2\delta} k e^{k\sigma} \left( \frac{2 \delta - t}{2 \delta - \sigma} \right) \frac{t^2}{2^2 - 1} dt & \text{if } 2\delta \leq d(x), \end{cases}$$

is a super solution of $(Ps)$ for all $\sigma \in (0, \delta)$ sufficiently small.
belongs to $C^1_0(\Omega)$, where $k > 0$ is an arbitrary number. They also proved that

$$-\Delta_{p(x)}(\mu \phi) = \begin{cases} -k(\mu e^{k(x)p(x)-1})[p(x)(p(x) - 1) + (d(x) + \frac{1}{k}\mu \Delta_{d(x)}(x))] & \text{if } d(x) < \sigma, \\ \frac{1}{2\Delta_{\sigma}^k} - \frac{2(p(x)-1)}{\sigma^2} \Delta_{d(x)}(x) & \text{if } d(x) < \sigma, \\ \frac{1}{2\Delta_{\sigma}^k} - \frac{2(p(x)-1)}{\sigma^2} \Delta_{d(x)}(x) & \text{if } d(x) < \sigma, \\ 0 & \text{if } 2\delta < d(x), \end{cases}$$

for all $\mu > 0$.

Let $\sigma = \frac{1}{k} \ln \frac{1}{\sigma}$ and $\mu = e^{-ak}$, where $a = \frac{p^{-1}}{\max_{\Omega} \vert x \vert}$. Then $e^{k\sigma} = 2^{\frac{1}{\sigma}}$ and $k\mu \leq 1$ if $k > 0$ is sufficiently large. From [36, p. 12], we have

$$-\Delta_{p(x)}(\mu \phi) \leq 0 \leq \frac{1}{\Lambda_k} (\mu \phi)^{\beta(x)} \mu \phi_{L_\eta(x)}^{\alpha(x)}$$

and

$$-\Delta_{p(x)}(\mu \phi) \leq \tilde{C}(k\mu)^{p^{-1}} \ln k \mu \mid \sigma < d(x) < 2\delta. \quad (8)$$

Since $\alpha^+ + \beta^+ < p^+ - 1$, an application of the L'Hospital’s rule imply that

$$\lim_{k \to +\infty} \frac{\tilde{C}k^{p^{-1}}}{e^{ak(p^+ - (\alpha^+ + \beta^+))}} \ln k \frac{k}{e^{ak}} = 0. \quad (9)$$

If $\sigma \leq d(x) < 2\delta$, we have $\phi(x) \geq 2^{\frac{1}{\sigma}} - 1$ for all $k > 0$, because $e^{k\sigma} = 2^{\frac{1}{\sigma}}$. Thus, there is a constant $C_0 > 0$ that does not depend on $k$ such that $|\phi|_{L_\eta(x)}^{p(x)} \geq C_0$ if $\sigma \leq d(x) < 2\delta$. By (9), we can choose $k > 0$ large enough such that

$$\frac{C_1k^{p^{-1}}}{e^{ak(p^+ - (\alpha^+ + \beta^+))}} \ln k \frac{k}{e^{ak}} \leq \frac{C_0}{\Lambda_k} (2^{\frac{1}{\sigma}} - 1)^{\beta^+}. \quad (10)$$

It is possible to choose $k > 0$ large such that $\mu \phi(x) \leq 1$ for all $x \in \Omega$ satisfying $\sigma < d(x) < \delta$. Therefore from (8) and (10), we have

$$-\Delta_{p(x)}(\mu \phi) \leq \frac{1}{\Lambda_k} (\mu \phi)^{\beta(x)} \mu \phi_{L_\eta(x)}^{\alpha(x)}$$

for $k > 0$ large enough. Fix $k > 0$ satisfying the above property, and the inequality $-\Delta_{p(x)}(\mu \phi) \leq 1$. For $\lambda > 1$, we have $-\Delta_{p(x)}(\mu \phi) \leq -\Delta_{p(x)}(\mu \phi)_{z,\lambda}$. Therefore $\mu \phi \leq \varepsilon \lambda$. The first part of the result is proved.

Now suppose that $0 < A(x, t) \leq a_0$ in $\Omega \times (0, \infty)$. Let $\delta, \sigma, \mu, \alpha, \lambda, \varepsilon$ and $\phi$ as before. From the previous arguments, there exist $k > 0$ large enough and $\mu > 0$ sufficiently small such that

$$-\Delta_{p(x)}(\mu \phi) \leq 1 \quad \text{and} \quad -\Delta_{p(x)}(\mu \phi) \leq \frac{1}{a_0} (\mu \phi)^{\beta(x)} \mu \phi_{L_\eta(x)}^{\alpha(x)}$$

in $\Omega$.

In particular, for $w \in L_\infty(\Omega)$ with $\mu \phi \leq w$, we have

$$-\Delta_{p(x)}(\mu \phi) \leq \frac{1}{A(x, \mu \phi_{L_\eta(x)}^{\alpha(x)})} (\mu \phi)^{\beta(x)} \mu \phi_{L_\eta(x)}^{\alpha(x)}$$

in $\Omega$. \quad (11)
Since \( \lim_{t \to \infty} A(x, t) = a_\infty > 0 \) uniformly in \( \Omega \), there is a constant \( a_1 > 0 \) such that \( A(x, t) \geq \frac{a_\infty}{t} \) in \( \Omega \times (a_1, \infty) \). Let \( m_k := \min\{A(x, t) : \Omega \times \{\mu \phi |_{L^1(\Omega)}, a_1\}\} > 0 \) and \( A_k := \min\{m_k, \frac{a_\infty}{2}\} \). Then \( A(x, t) \geq A_k \) in \( \Omega \times \{\mu \phi |_{L^1(\Omega)}, \infty\} \).

Fix \( k > 0 \) satisfying (11). Let \( \lambda > 1 \) such that (6) occurs and

\[
\frac{1}{A_k} K^{\beta^+} \lambda^{-\frac{p-1}{p}} \max\{|K|_{L^q(\Omega)}^\alpha, |K|_{L^q(\Omega)}^\gamma\} \leq \lambda,
\]

where \( K > 1 \) is a constant that does not depend on \( k \) and \( \lambda \) (see Lemma 1). Thus for all \( w \in [\mu \phi, z_\lambda] \), we have

\[
-\Delta_p(x) z_\lambda \leq \frac{1}{A(x, w |_{L^q(\Omega)})} z_\lambda^{\beta(x)} |z_\lambda|_{L^q(\Omega)}^{\gamma(x)} \text{ in } \Omega.
\]

From the weak comparison principle, we have \( \mu \phi \leq z_\lambda \). Therefore \((\mu \phi, z_\lambda)\) is a sub-supersolution pair for \((P)\).

\[\square\]

3.2 A Concave-Convex Problem

In this section, we consider the following nonlocal problem with concave-convex nonlinearities

\[
\begin{cases}
-\mathcal{A}(x, u |_{L^{r}(\Omega)}) \Delta_{p(x)} u = \lambda |u|^{\beta(x)-1}u |u|^{\alpha(x)} + \theta |u|^{\gamma(x)-1}u |u|^{\eta(x)} & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega.
\end{cases}
\]

((P)\(\lambda, \theta\))

The local version of ((P)\(\lambda, \theta\)) with \( p(x) \equiv 2 \) and constant exponents was considered in the famous paper by Ambrosetti-Brezis-Cerami [6] in which a sub-supersolution argument is used. In [37], the problem (P)\(\lambda, \theta\) was studied with \( p(x) \equiv 2 \). The following result generalizes [37, Theorem 4].

**Theorem 5** Suppose that \( r, p, q, s, \alpha \) and \( \gamma \) satisfy \((H_0)\) and \( \beta, \eta \in L^\infty(\Omega) \) are nonnegative functions with \( 0 < \alpha^- + \beta^- \leq \alpha^+ + \beta^+ < p^- - 1 \). Let \( a_0, b_0 > 0 \) positive numbers. The following assertions hold.

\((A_1)\) If \( p^- - 1 < \eta^- + \gamma^- \) and \( \mathcal{A}(x, t) \geq a_0 \) in \( \Omega \times [0, b_0] \), then given \( \theta > 0 \), there exists \( \lambda_0 > 0 \) such that for each \( \lambda \in (0, \lambda_0) \), the problem ((P)\(\lambda, \theta\)) has a positive solution \( u_{\lambda, \theta} \).

\((A_2)\) If \( p^- - 1 < \eta^+ + \gamma^+ \) and \( 0 < \mathcal{A}(x, t) \leq a_0 \) in \( \Omega \times (0, \infty) \) and \( \lim_{t \to \infty} \mathcal{A}(x, t) = b_0 \) uniformly in \( \Omega \), then given \( \lambda > 0 \), there exists \( \theta_0 > 0 \) such that for each \( \theta \in (0, \theta_0) \), the problem ((P)\(\lambda, \theta\)) has a positive solution \( u_{\lambda, \theta} \).

**Proof** Suppose that \((A_1)\) occurs. Let \( z_\lambda \in W^{1, p(x)}_0(\Omega) \cap L^\infty(\Omega) \) be the unique solution of (3), where \( \lambda \in (0, 1) \) will be chosen before.

Lemma 1 imply that for \( \lambda > 0 \) small enough, there exists a constant \( K > 1 \) that does not depend on \( \lambda \) such that

\[
0 < z_\lambda(x) \leq K \lambda^{\frac{1}{p-1}} \text{ in } \Omega.
\]

Let \( \overline{K} := \max\{|K|_{L^{q(x)}(\Omega)}^\alpha, |K|_{L^{q(x)}(\Omega)}^\gamma, |K|_{L^{q(x)}(\Omega)}^{\beta(x)}, |K|_{L^{q(x)}(\Omega)}^{\gamma(x)}\} \). For each \( \theta > 0 \), we can choose \( 0 < \lambda_0 < 1 \) small enough, depending on \( \theta \), such that the inequalities
\[
\lambda \geq \frac{1}{d_0} \left( \lambda \frac{p^+ - 1 + \beta^- + \alpha^-}{p^+ - 1} K^{\beta^+} \theta \lambda \frac{\eta^- + \gamma^-}{p^+ - 1} K^{\eta^-} \lambda^{-1} K \right), \quad \lambda \in (0, \lambda_0),
\]
and (12) hold, because \( \alpha^- + \beta^- > 0 \) and \( p^+ - 1 < \eta^- + \gamma^- \).

There is \( \lambda_0 > 0 \) small such that
\[
\frac{1}{d_0} \left( \lambda z_\lambda^{\beta(x)} |z_\lambda|^{\alpha(x)} L_{q(x)} + \theta z_\lambda^{\eta(x)} |z_\lambda|^{\gamma(x)} L_{s(x)} \right) 
\leq \lambda \left( K \lambda \frac{1}{p^+ - 1} \right)^{\beta(x)} |z_\lambda|^{\alpha(x)} L_{q(x)} + \theta \left( K \lambda \frac{1}{p^+ - 1} \right)^{\eta(x)} |z_\lambda|^{\gamma(x)} L_{s(x)} 
\leq \lambda,
\]
for all \( \lambda \in (0, \lambda_0) \). Thus for all \( \lambda \in (0, \lambda_0) \), we get
\[
\frac{1}{d_0} \left( \lambda z_\lambda^{\beta(x)} |z_\lambda|^{\alpha(x)} L_{q(x)} + \theta z_\lambda^{\eta(x)} |z_\lambda|^{\gamma(x)} L_{s(x)} \right) \leq \lambda.
\]

If necessary, one can consider a smaller value for \( \lambda_0 \) such that \( |z_\lambda|_{L^r(x)} \leq |K| L_{r(x)} \lambda^{-1} b_0 \). Thus for all \( w \in [0, |z_\lambda|_{L^r(x)}] \), we have \( A(x, |w| L_{r(x)}) \geq a_0 \). Therefore
\[
-\Delta_{p(x)} z_\lambda \geq \frac{1}{A(x, |w| L_{r(x)})} \left( \lambda z_\lambda^{\beta(x)} |z_\lambda|^{\alpha(x)} L_{q(x)} + \theta z_\lambda^{\eta(x)} |z_\lambda|^{\gamma(x)} L_{s(x)} \right) \quad \text{in } \Omega,
\]
for all \( \lambda \in (0, \lambda_0) \).

Now consider \( \phi, \delta, \sigma, \mu \) and \( a \) as in the proof of Theorem 4. Fix \( \lambda \in (0, \lambda_0) \) such that (13) holds. Let \( \lambda_0 := \max \{ A(x, t) : (x, t) \in \Omega \times \{0, b_0\} \} \).

Since \( \alpha^+ + \beta^+ < p^- - 1 \), the arguments of the proof of Theorem 4 imply that, if \( \mu = \mu(\lambda) > 0 \) is small enough, then
\[
-\Delta_{p(x)} (\mu \phi) \leq \lambda \quad \text{in } \Omega
\]
and
\[
-\Delta_{p(x)} (\mu \phi) \leq \frac{1}{A_0} \lambda (\mu \phi)^{\beta(x)} |\mu \phi|^{\alpha(x)} L_{q(x)} 
\leq \frac{1}{A_0} \lambda (\mu \phi)^{\beta(x)} |\mu \phi|^{\alpha(x)} L_{q(x)}
\]
in \( \Omega \), for all \( w \in [0, |z_\lambda|_{L^r(x)}] \). The weak comparison principle imply that \( \mu \phi \leq z_\lambda \) for \( \mu > 0 \) small enough. Therefore \( (\mu \phi, z_\lambda) \) is a sub-supersolution pair for \( (P)_{\lambda, \theta} \).

Now we will prove the theorem in the second case. Consider again \( \phi, \delta, \sigma, \mu \) and \( a \) as in the proof of Theorem 4. Let \( \lambda \in (0, \infty) \). Since \( \alpha^+ + \beta^+ < p^- - 1 \), we can repeat the arguments of Theorem 4 to obtain \( \mu = \mu(\lambda) > 0 \) small enough, depending only on \( \lambda \), such that
\[
-\Delta_{p(x)} (\mu \phi) \leq 1 \quad \text{and} \quad -\Delta_{p(x)} (\mu \phi) \leq \frac{\lambda}{d_0} (\mu \phi)^{\beta(x)} |\mu \phi|^{\alpha(x)} L_{q(x)} \quad \text{in } \Omega.
\]

Let \( z_M \in W_0^{1, p(x)}(\Omega) \cap L^\infty(\Omega) \) be the unique solution of (3), where \( M > 0 \) will be chosen later.
For $M \geq 1$ large enough, there is a constant $K > 1$ that does not depend on $M$ such that

$$0 < z_M(x) \leq KM^{\frac{1}{p'-1}} \quad \text{in } \Omega. \quad (14)$$

We want to obtain $M > 1$ such that, for each $w \in L^\infty(\Omega)$ with $\mu \phi \leq w$, the inequality

$$M \geq \frac{1}{A(x, |w|_{L^p(\Omega)})} \left( \lambda z_M^{\alpha(x)} z_M^{\beta(x)} + \theta z_M^{\eta(x)} \right) \quad \text{in } \Omega, \quad (15)$$

occurs.

Since $A$ is continuous and $\lim_{t \to +\infty} A(x, t) = b_0 > 0$ uniformly in $\Omega$, there is a constant $a_1 > 0$ such that $A(x, t) \geq \frac{b_0}{2}$ in $\Omega \times (a_1, +\infty)$. Consider

$$m_\lambda = \min \{A(x, t) : (x, t) \in \Omega \times [[\mu \phi]_{L^p(\Omega)}, a_1]\} > 0$$

and $A_\lambda = \min \{m_\lambda, \frac{b_0}{2}\}$. Then $A(x, t) \geq A_\lambda$ in $\Omega \times [[\mu \phi]_{L^p(\Omega)}, +\infty)$. Thus, there exists a constant $A_\lambda > 0$ with $A_\lambda \leq A(x, |w|_{L^p(\Omega)}) \leq a_0$, for all $w \in L^\infty(\Omega)$ with $\mu \phi \leq w$.

By (14), we have

$$\left( \lambda z_M^{\beta(x)} z_M^{\alpha(x)} + \theta z_M^{\eta(x)} \right) \leq \frac{(\lambda z_M^{\beta(x)} + \theta z_M^{\eta(x)}) z_M^{\gamma(x)}}{A(x, |w|_{L^p(\Omega)})}, \quad (16)$$

with $C = \max\{K^{\beta^+} K, K^{\eta^+} K\}$, where $K = \max\{|K|_{L^p(\Omega)}, |K|_{L^q(\Omega)}, |K|_{L^s(\Omega)}, |K|_{L^r(\Omega)}\}$.

Denoting by $I$ the right-hand side of (16), we have $I \leq M$ if and only if

$$1 \geq \frac{1}{A_\lambda} \left( \lambda C M^{\beta^+ + \eta^+} + \theta C M^{\eta^+ + \nu^+} \right). \quad (17)$$

Since $\alpha^+ + \beta^+ < p^- - 1 < \eta^+ + \gamma^+$, the function

$$\Psi(t) = \frac{\lambda C t^{\beta^+ + \nu^+}}{A_\lambda t^{\frac{p^- - 1}{p^-}} + \theta C t^{\frac{\eta^+ + \gamma^+}{p^- - 1}}}, \quad t > 0,$$

belongs to $C^1((0, \infty), \mathbb{R})$ and attains a global minimum at

$$M_{\lambda, \theta} := M(\lambda, \theta) = L \left( \frac{\lambda}{\theta} \right)^{\frac{p^- - 1}{(\eta^+ + \gamma^+ - (\beta^+ + \alpha^+))}}, \quad (18)$$

where $L = \left( \frac{p^- - 1}{(\eta^+ + \gamma^+ - (\beta^+ + \alpha^+))} \right)^{\frac{p^- - 1}{(\eta^+ + \gamma^+ - (\beta^+ + \alpha^+)}}}$. The inequality (17) is equivalent to find $M_{\lambda, \theta} > 0$ such that $\Psi(M_{\lambda, \theta}) \leq 1$. By (18), we have $\Psi(M_{\lambda, \theta}) \leq 1$ if and only if

$$\frac{\lambda C P}{A_\lambda} \left( \frac{\lambda}{\theta} \right)^{\frac{\alpha^+ + \gamma^+ - (p^- - 1)}{\eta^+ + \gamma^+ - (\beta^+ + \alpha^+)}} + \theta C Q \left( \frac{\lambda}{\theta} \right)^{\frac{\eta^+ + \gamma^+ - (p^- - 1)}{\eta^+ + \gamma^+ - (\beta^+ + \alpha^+)}} \leq 1,$$

where $P = L^{\frac{\alpha^+ + \beta^+ - (p^- - 1)}{p^-}}$ and $Q = L^{\frac{\eta^+ + \gamma^+}{p^-}}$. Notice that the above inequality holds if $\theta > 0$ is small enough, because $\alpha^+ + \beta^+ < p^- - 1 < \eta^+ + \gamma^+$. Thus for $\lambda > 0$ fixed, there exists
\[ \theta_0 = \theta_0(\lambda) \text{ such that for each } \theta \in (0, \theta_0), \text{ there is a number } M = M_{\lambda, \theta} > 0 \text{ such that (17) occurs. Consequently, we have (15). Therefore} \]

\[ -\Delta_{p(x)} z_M \geq \frac{1}{A(x, |u|_{p(x)})} \left( \lambda z_M^\beta(x) |z_M|^\alpha_{L^q(x)} + \theta z_M^\eta(x) |z_M|^\eta_{L^r(x)} \right) \text{ in } \Omega. \]

Considering, if necessary, a smaller \( \theta_0 > 0 \), we get \( M \geq 1. \) Therefore \( -\Delta_{p(x)} (\mu \phi) \leq \mu \phi \) in \( \Omega \). The weak comparison principle imply that \( \mu \phi \leq z_M. \) Then \( (\mu \phi, z_M) \) is a sub-supersolution pair for \((P)_{\lambda, \theta}\). The proof is finished. \( \square \)

### 3.3 A Generalization of the Logistic Equation

In the previous sections, we considered at least one of the conditions \( A(x, t) \geq a_0 > 0 \) or \( 0 < A(x, t) \leq a_\infty, t > 0. \) In this last section, we study a generalization of the classic logistic equation, where the function \( A(x, t) \) can satisfy

\[ A(x, 0) \geq 0, \lim_{t \to 0^+} A(x, t) = \infty \text{ and } \lim_{t \to +\infty} A(x, t) = \pm \infty. \]

We will attack the problem

\[
\begin{aligned}
\begin{cases}
-\Delta_{p(x)} u & = \lambda f(u) |u|^{\alpha(x)}_{L^q(x)} \\
 u & = 0
\end{cases} \text{ in } \Omega, \\
& \text{ on } \partial \Omega.
\end{aligned}
\]

Problem \((P)_\lambda\) is a generalization of the problems studied in [15, 20, 37]. The next result generalizes [37, Theorem 5].

**Theorem 6** Suppose that \( r, p, q, \alpha \) satisfy \((H_0)\). Consider also that \( f \) satisfies \((f_1), (f_2)\) and that \( A(x, t) > 0 \) in \( \Omega \times (0, |t|_{L^r(x)}) \). Then there exists \( \lambda_0 > 0 \) such that \( \lambda \geq \lambda_0, ((P)_\lambda) \) has a positive solution \( u_\lambda \in [0, \theta] \).

**Proof** Consider the function \( \tilde{f}(t) = f(t) \) for \( t \in [0, \theta] \), and \( \tilde{f}(t) = 0, \) for \( t \in \mathbb{R} \setminus [0, \theta] \). The functional

\[ J_\lambda(u) = \int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} dx - \lambda \int_{\Omega} \tilde{F}(u) dx, \ u \in W_{1,p(x)}^1(\Omega), \]

where \( \tilde{F}(t) = \int_0^t \tilde{f}(s) ds \) is of class \( C^1(W_{1,p(x)}^1(\Omega), \mathbb{R}) \). Since \( |\tilde{f}(t)| \leq C \) for \( t \in \mathbb{R} \), we have that \( J \) is coercive. Thus \( J \) has a minimum \( z_\lambda \), which is a weak solution of the problem

\[ \begin{cases}
-\Delta_{p(x)} z = \lambda \tilde{f}(z) \text{ in } \Omega, \\
 z = 0 \text{ on } \partial \Omega.
\end{cases} \]

Consider a function \( \varphi_0 \in W_{1,p(x)}^1(\Omega) \) such that \( \tilde{F}(\varphi_0) > 0. \) Define \( z_0 := z_{\lambda_0}, \) where \( \lambda_0 > 0 \) satisfies

\[ \int_{\Omega} \frac{1}{p(x)} |\nabla \varphi_0|^{p(x)} < z_0 \int_{\Omega} \tilde{F}(\varphi_0). \]
Thus $J_{\tilde{\lambda}_0}(z_0) \leq J_{\tilde{\lambda}_0}(\varphi_0) < 0$. Since $J_{\lambda_0}(0) = 0$, we have $z_0 \neq 0$. By [30, Theorem 4.1], we have $z_0 \in W^{1,p(x)}_0(\Omega) \cap L^{\infty}(\Omega)$, and using [31, Theorem 1.2], we obtain that $z_0 \in C^{1,\alpha}(\Omega)$. Considering the test function $\varphi = z_0^- := \min\{z_0, 0\}$, we get $z_0 = z_0^+ \geq 0$. By Proposition 3, we have $z_0 > 0$.

Considering the test function $\varphi = (z_0 - \theta)^+ \in W^{1,p(x)}_0(\Omega)$, we have

$$\int_{\Omega} |\nabla z_0|^{p(x)-2}\nabla z_0 \nabla (z_0 - \theta)^+ = \tilde{\lambda}_0 \int_{\{z_0 > \theta\}} \tilde{f}(z_0)(z_0 - \theta) = 0.$$ 

Therefore

$$\int_{\{z_0 > \theta\}} (|\nabla z_0|^{p(x)-2}\nabla z_0 - |\nabla \theta|^{p(x)-2}\nabla \theta, \nabla (z_0 - \theta)) = 0,$$

which imply $(z_0 - \theta)^+ = 0 \in \Omega$. Thus $0 < z_0 \leq \theta$.

Note that there is a constant $C > 0$ such that $|z_0|^{\alpha(x)} \geq C$. Define $A_0 := \max\{A(x, t) : (x, t) \in \Omega \times [0, 1]\}$ and $\mu_0 = \frac{A_0}{C}$. Then, we have

$$-\Delta_{p(x)} z_0 = \tilde{\lambda}_0 f(z_0) = \frac{1}{A_0} \tilde{\lambda}_0 \mu_0 f(z_0) |z_0|^{\alpha(x)} \leq \frac{1}{A_0} \tilde{\lambda}_0 \mu_0 |z_0|^{\alpha(x)}.$$ 

Thus for each $\lambda \geq \tilde{\lambda}_0 \mu_0$ and $w \in [\varphi, \theta]$, we get

$$-\Delta_{p(x)} z_0 \leq \frac{1}{A(x, |w|_{L^{\theta(x)}})} \lambda f(z_0) |z_0|^{\alpha(x)}.$$

Since $f(\theta) = 0$, it follows that $(z_0, \theta)$ is sub-solution pair for $(P)_\lambda$ and the result is proved.

Remark We would like to point out that it is possible to use the function $\phi$ from the proof of Theorem 4 to consider problem $(P)_\lambda$. However, more restrictions on the functions $p$ and $f$ are needed.

Acknowledgements The authors would like to thank the anonymous referees and professor Eduardo Hitomi (UNICAMP & Princeton Univ.) for their valuable comments which helped to improve this work.

References

1. Acerbi, E., Mingione, G.: Regularity results for stationary electro-rheological fluids. Arch. Ration. Mech. Anal. 164, 213–259 (2002)
2. Afrouzi, G.A., Mirzapour, M., Radulescu, V.D.: Qualitative analysis of solutions for a class of anisotropic elliptic equations with variable exponent. Proc. Edinb. Math. Soc. (2) 59(3), 541–557 (2016)
3. Alves, C.O., Covei, D.P.: Existence of solutions for a class of nonlocal elliptic problem via sub-supersolution. Nonlinear Anal., Real World Appl. 23, 1–8 (2015)
4. Alves, C.O., Moussaoui, A., Tavares, L.S.: An elliptic system with logarithmic nonlinearity. arXiv:1702.06244 (2017)
5. Alves, C.O., Moussaoui, A.: Existence and regularity of solutions for a class of singular $(p(x), q(x))$-Laplacian systems. arXiv:1608.00217 (2016)
6. Ambrosi, A., Brezis, H., Cerami, G.: Combined effects of concave and convex nonlinearities in some elliptic problems. J. Funct. Anal. 122(2), 519–543 (1994)
7. Baraket, S., Bisci, G.M.: Multiplicity results for elliptic Kirchhoff-type problems. Adv. Nonlinear Anal. 6(1), 85–93 (2017)
8. Bisci, G.M., Radulescu, V., Servadei, R.: Variational Methods for Nonlocal Fractional Problems. Encyclopedia of Mathematics and Its Applications, vol. 162. Cambridge University Press, Cambridge (2016)
9. Carrier, G.F.: On the nonlinear vibration problem of the elastic string. Q. Appl. Math. 3, 157–165 (1945)
10. Cencelj, M., Repovs, D., Virk, Z.: Multiple perturbations of a singular eigenvalue problem. Nonlinear Anal. 119, 37–45 (2015)
11. Chen, Y., Levine, S., Rao, M.: Variable exponent, linear growth functionals in image restoration. SIAM J. Appl. Math. 66(4), 1383–1406 (2006)
12. Chipot, M., Chang, N.H.: On some model diffusion problems with a nonlocal lower order term. Chin. Ann. Math., Ser. B 24(2), 147–166 (2003)
13. Chipot, M., Chang, N.H.: On some mixed boundary value problems with nonlocal diffusion. Adv. Math. Sci. Appl. 14(1), 1–24 (2004)
14. Chipot, M., Chang, N.H.: Nonlinear nonlocal evolution problems. Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat. 97(3), 423–445 (2003)
15. Chipot, M., Corrêa, F.J.S.A.: Boundary layer solutions to functional elliptic equations. Bull. Braz. Math. Soc. 40(3), 381–393 (2009)
16. Chipot, M., Lovat, B.: Some remarks on nonlocal elliptic and parabolic problems. Nonlinear Anal. 119, 37–45 (2015)
17. Chipot, M., Levine, S., Rao, M.: Variable exponent, linear growth functionals in image restoration. SIAM J. Appl. Math. 66(4), 1383–1406 (2006)
18. Chipot, M., Molinet, L.: Asymptotic behaviour of some nonlocal diffusion problems. Appl. Anal. 80(3–4), 279–315 (2001)
19. Chipot, M., Rodrigues, J.F.: On a class of nonlinear elliptic problems. Modél. Math. Anal. Numér. 26(3), 447–467 (1992)
20. Chipot, M., Roy, P.: Existence results for some functional elliptic equations. Differ. Integral Equ. 27(3–4), 289–300 (2014)
21. Chipot, M., Savitska, T.: Nonlocal \( p \)-Laplace equations depending on the \( L^p \) norm of the gradient. Adv. Differ. Equ. 19(11–12), 997–1020 (2014)
22. Chipot, M., Savitska, T.: Asymptotic behaviour of the solutions of nonlocal \( p \)-Laplace equations depending on the \( L^p \) norm of the gradient. J. Elliptic Parabolic Equ. 1, 63–74 (2015)
23. Chipot, M., Siegwart, M.: On the asymptotic behaviour of some nonlocal mixed boundary value problems. In: Nonlinear Analysis and Its Applications, to V. Lakshmikantam on His 80th Birthday, Vols. 1–2, pp. 431–449. Kluwer Academic, Dordrecht (2003)
24. Chipot, M., Valente, V., Caffarelli, G.V.: Remarks on a nonlocal problem involving the Dirichlet energy. Rend. Semin. Mat. Univ. Padova 110, 199–220 (2003)
25. Chipot, M., Zheng, S.: Asymptotic behavior of solutions to nonlinear parabolic equations with nonlocal terms. Asymptot. Anal. 45(3–4), 301–312 (2005)
26. Corrêa, F.J.S.A., Figueiredo, G.M., Lopes, P.F.M.: On the existence of positive solutions for a nonlocal elliptic problem involving the p-Laplacian and the generalized Lebesgue space \( L^{p(\cdot)}(\Omega) \). Differ. Integral Equ. 21(3–4), 305–324 (2008)
27. Corrêa, F.J.S.A., Menezes, S.D.B.: Positive solutions for a class of nonlocal problems. In: Contributions to Nonlinear Analysis. Progr. Nonlinear Differential Equations Appl., vol. 66, pp. 195–206. Birkhäuser, Basel (2006)
28. Corrêa, F.J.S.A., Suarez, A.: Combining local and nonlocal terms in a nonlinear elliptic problem. Math. Methods Appl. Sci. 35(5), 547–563 (2012)
29. Fan, X.L., Zhang, Q.H.: Existence of solution for \( p(x) \)-Laplacian Dirichlet problem. Nonlinear Anal. 52(8), 843–1852 (2003)
30. Fan, X.L., Zhao, D.: A class of De Giorgi type and Hölder continuity. Nonlinear Anal. 36(3), 295–318 (1999)
31. Fan, X.L.: Global \( C^{1,\alpha} \) regularity for variable exponent elliptic equations in divergence form. J. Differ. Equ. 235(2), 397–417 (2007)
32. Fan, X.L., Zhang, Q., Zhao, D.: Eigenvalues of \( p(x) \)-Laplacian Dirichlet problem. J. Math. Anal. Appl. 302(2), 306–317 (2005)
33. Fan, X.L., Zhao, Y.Z., Zhang, Q.H.: A strong maximum principle for \( p(x) \)-Laplace equations. Chin. J. Contemp. Math. 24(3), 277–282 (2003)
34. Fan, X.L.: On the sub-super solution method for \( p(x) \)-Laplacian equations. J. Math. Anal. Appl. 330(1), 665–682 (2007)
35. Fu, Y., Shan, Y.: On the removability of isolated singular points for elliptic equations involving variable exponent. Adv. Nonlinear Anal. 5(2), 121–132 (2016)
36. Liu, J., Zhang, Q., Zhao, C.: Existence of positive solutions for \( p(x) \)-Laplacian equations with a singular nonlinear term. Electron. J. Differ. Equ. 2014, 155 (2014), 21 pp.
37. dos Santos, G.C.G., Figueiredo, G.M.: Positive solutions for a class of nonlocal problems involving Lebesgue generalized spaces: scalar and system cases. J. Elliptic Parabolic Equ. 2(1–2), 235–266 (2016)
38. Mihailescu, M., Repovs, D.: On a non-homogeneous eigenvalue problem involving a potential: an Orlicz-Sobolev space setting. J. Math. Pures Appl. (9) 93(2), 132–148 (2010)
39. Pucci, P., Zhang, Q.: Existence of entire solutions for a class of variable exponent elliptic equations. J. Differ. Equ. 257(5), 1529–1566 (2014)
40. Pucci, P., Xiang, M., Zhang, B.: Existence and multiplicity of entire solutions for fractional $p$-Kirchhoff equations. Adv. Nonlinear Anal. 5(1), 27–55 (2016)
41. Radulescu, V., Repovs, D.: Partial Differential Equations with Variable Exponents. Variational Methods and Qualitative Analysis. Monographs and Research Notes in Mathematics. CRC Press, Boca Raton (2015)
42. Růžička, M.: Electrorheological Fluids: Modelling and Mathematical Theory. Springer, Berlin (2000)
43. Yan, B., Wang, D.: The multiplicity of positive solutions for a class of nonlocal elliptic problem. J. Math. Anal. Appl. 442(1), 72–102 (2016)
44. Yin, H., Yang, Z.: Existence and asymptotic behavior of positive solutions for a class of $(p(x), q(x))$-Laplacian systems. Differ. Equ. Appl. 6(3), 403–415 (2014)
45. Zhang, Q.: Existence and asymptotic behavior of positive solutions to $p(x)$-Laplacian equations with singular nonlinearities. J. Inequal. Appl. 20, 19349 (2007), 9 pp.