Dynamical mass generation of composite Dirac fermions and fractional quantum Hall effects near charge neutrality in graphene

Feng Cai¹, Yue Yu² and Ziqiang Wang¹

¹ Department of Physics, Boston College, Chestnut Hill, MA 02467, USA
² The State Key Laboratory of Theoretical Physics, Institute of Theoretical Physics, Chinese Academy of Sciences, PO Box 2735, Beijing 100190, People’s Republic of China

E-mail: caif@bc.edu

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Abstract

We develop a composite Dirac fermion theory for the fractional quantum Hall effects (QHE) near charge neutrality in graphene. We show that the interactions between the composite Dirac fermions lead to a dynamical mass generation through exciton condensation. The four-fold spin–valley degeneracy is fully lifted due to the mass generation and exchange effects such that the odd-denominator fractional QHE observed in the vicinity of charge neutrality can be understood in terms of the integer QHE of composite Dirac fermions. At a filling factor \( \nu = 1/2 \), we show that the massive composite Dirac fermion liquid is unstable against chiral p-wave pairing for weak Coulomb interactions and the ground state is a paired non-Abelian quantum Hall state described by the Moore–Read Pfaffian in the long wavelength limit.

(Some figures may appear in colour only in the online journal)

1. Introduction

Since the discovery of graphene [1], a rich set of integer [2–5] and fractional [6–9] quantum Hall effects (QHE) has been explored experimentally under the new setting of two-dimensional lattice electrons with a relativistic energy–momentum dispersion at low energy. To date, the \( \nu = 1/3 \) state has been convincingly observed [6–11]. Remarkably, initial evidence for the emergence of a \( \nu = 1/2 \) fractional quantum Hall state has been reported [8], raising the hope for realizing non-Abelian statistics in suspended graphene.

For graphene, it is known that there is a spin–valley \( SU(4) \) symmetry. This restricts the filling factors \( \nu \) of the integer QHE of the Dirac fermions to ±2, ±6, ±10, . . . , provided that the \( SU(4) \) symmetry is unbroken. The observation of the \( \nu = \pm 1 \) integer Hall plateaus [3] indicates that the \( SU(4) \) symmetry is broken. Dynamical mass generation, which lifts the spin–valley degeneracy of the zeroth Dirac Landau level (ζLL) [12], and quantum Hall ferromagnetism [13–15] have been proposed for its explanation. Despite recent theoretical efforts on the fractional QHE (FQHE) in graphene [15–20], it remains uncertain whether the observed \( \nu = 1/3 \) state implies \( SU(4) \) symmetry breaking, and is therefore a single-component Laughlin state or a multi-component Halperin state with spin–valley degeneracy [20]. Moreover, although states at even-denominator filling factors have been investigated numerically [21, 22] by exact diagonalization, the effect of Landau level mixing, which may be relevant for the stabilization of the Moore–Read Pfaffian state [23–28], was neglected when the Hilbert space is projected and restricted to that of a specific Landau level.

In this paper, we propose a mechanism of dynamical mass generation by exciton condensation and exchange-driven polarization to qualitatively describe both the Abelian and the non-Abelian FQHE in graphene. To this end, we extend the composite fermion Chern–Simons (CS) theory [29, 30]...
to the case of Dirac particles attached to an even number of flux quanta through the CS gauge field, which can be implemented by a unitary transformation. We will refer to this transformation as the CS transformation.

The CS approach is more suitable for studying the fractional quantum Hall regime where Landau level mixing is significant, since it does not restrict the quantum states to the $\zeta$ LL. The importance of Landau level mixing in graphene can be seen from the fact that, contrary to semiconductor heterostructures where the dispersion is non-relativistic, the Coulomb interaction energy and Landau level spacing for Dirac fermions both scale with $\sqrt{B}$, where $B$ is the strength of the magnetic field. Specifically, the ratio of the interaction energy and energy spacing between the zeroth and first Landau levels is given by $\alpha_g/\sqrt{2}$, where $\alpha_g = e^2/4\pi\epsilon\hbar v_F$ is the fine structure constant for graphene. For free standing graphene, the bare value of the fine structure constant is $\alpha_g \approx 2.2$ (see e.g., [33]). Therefore, the two energy scales are of the same order; the system should be in the strong coupling regime [12], where the Landau level spacing is a relevant energy scale.

A crucial step in our theory is to work with the proper particle density via a particle–hole transformation such that the vacuum state of the relativistic composite Dirac fermions (CDF) is defined by the charge neutral state with all negative energy states filled. The CDF theory introduced in this paper is rather general, involving only relations between the CDF particle density and its filling fraction $\nu$. As we will show, the latter corresponds to a unique filling fraction $\nu$ of the original electrons once the ground state is determined.

At the filling fractions $\nu = \pm 1/\phi$, with $\phi$ an even integer, we will show, by variational calculations of the ground state energy, that it is energetically favorable for the CDF to develop an exciton condensate. The latter supports single quasiparticle excitations with a CDF mass gap at low density once the ground state is determined.

Thus, in the present theory, the dynamical $SU(4)$ symmetry breaking mass generation offers a route to the observed FQHE states in graphene near charge neutrality [6–9]. We show that the quasi-particles above the exciton condensate have a pairing instability in the chiral p-wave channel, leading to an even-denominator paired quantum Hall state described by the non-Abelian Moore–Read Pfaffian [38, 39] in the long wavelength limit. On the other hand, at odd-denominator filling fractions, such as $\nu = 1/3$, the quasi-particles above the exciton condensate occupy fully filled Landau levels of the residual magnetic field [29], giving rise to a single-component Laughlin state for the electrons in graphene.

The key ingredient of the present theoretical framework is the dynamical mass generation of massless Dirac fermions, which enables us to approach the FQHEs near filling factors $\nu = 0, \pm 1, \pm 2$ in a unified way. This is an exciting example of the interplay between condensed matter and high energy physics; as the mechanism was introduced by Nambu and Jona-Lasinio [34], inspired by the BCS theory of superconductivity. It was used recently to explain Landau level splitting [12] in graphene. Here we extend this mechanism to the FQHE of lattice electrons in graphene, thus providing another physical realization of dynamical mass generation in a condensed matter system. We note that such an excionic mass generation intrinsically involves Landau level mixing, since it requires the formation of particle–hole pairs consisting of states from different Landau levels.

This paper is structured as follows. In section 2, we formulate the composite Dirac fermion theory by introducing a unitary transformation that implements the flux attachment. We then derive the statistical interaction mediated by the CS gauge field, focusing on the even-denominator filling factors, where the external magnetic field is canceled by the flux of the CS field. The effects of the statistical interaction are then investigated. In section 3, we show that an immediate consequence of the statistical interaction is the exciton condensation, which opens up mass gaps for the four components of the CDFs. The normal state is obtained by doping the exciton insulating state according to the filling factors of the fractional quantum Hall states. In section 4, we study the pairing instability induced by the statistical interaction. We show that the leading instability is in the complex p-wave channel, which provides a realization of the non-Abelian Moore–Read state in graphene.

2. Theory of composite Dirac fermions

We start with graphene electrons in the continuum limit under a perpendicular external magnetic field $B = -\nabla \phi z$. The effective Hamiltonian can be written down in terms of the four-component fermion operator $\psi_s(x)$ for the two-sublattice and two-valley degrees of freedom,

$$ H = -ih_F \int d^2 x \sum_s \psi_s^\dagger (\partial_0 + ie\vec{A}_s/h) \psi_s, \quad (1) $$

where $v_F$ is the Fermi velocity and $\vec{A}_s(x)$ is the vector potential: $\nabla \times \vec{A} = B$. The summations over the repeated spin
index $s$ and the spatial index $i = 1, 2$ are implied. The $4 \times 4$ matrix $\alpha_i = \gamma^0 \gamma^I_i$, with the $\gamma$ matrices given by $\gamma^0 = I \otimes \tau_1$, $\gamma^I = -\sigma_i \otimes \tau_2$, where $\sigma_i$ and $\tau_i$ are the $2 \times 2$ Pauli matrices acting in the sublattice and valley subspaces, respectively. To make the $SU(4)$ symmetry explicit, it is instructive to separate out the valley degrees of freedom and rewrite the Hamiltonian as

$$H = -i \hbar v_F \int d^2 x \psi^\dagger_{R\tau} (x) \sigma_i \left( \partial_i + \frac{e}{\hbar} A_i \right) \psi_{R\tau} (x)$$

$$+ i \hbar v_F \int d^2 x \psi^\dagger_{L\tau} (x) \sigma_i \left( \partial_i + \frac{e}{\hbar} A_i \right) \psi_{L\tau} (x).$$

(2)

Here $\psi_{s\tau}(x)$ is a two-component fermion spinor field for spin $s = \uparrow, \downarrow$ and valley $\tau = R, L$. For convenience, we denote $\psi_{s\tau} = \psi_{s\alpha}$ with $\alpha = \{1, 2, 3, 4\} = \{(\downarrow, R), (\downarrow, L), (\uparrow, R), (\uparrow, L)\}$.

The CDF field $\psi_{s\alpha}$ can be introduced by a unitary transformation

$$\begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} = \begin{pmatrix} e^{i \tau_1} & 0 & 0 & 0 \\ 0 & e^{i \tau_3} & 0 & 0 \\ 0 & 0 & e^{i \tau_3} & 0 \\ 0 & 0 & 0 & e^{i \tau_1} \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix},$$

(3)

with

$$\mathcal{I}_\alpha (x) = \int \frac{d^2 x'}{2\pi} K_{\alpha\beta} \rho_\beta(x') \arg(x - x'),$$

(4)

where $\alpha, \beta = 1, 2, 3, 4$ and $\rho_\alpha$ is the particle density operator in the spin–valley sector $\alpha$. Note that due to the presence of multiple components, a $K$-matrix must be introduced in the unitary transformation [35]. Its physical meaning is explained below. The form of the $K$-matrix can be specified by physical considerations. In order for the CDF field $\psi_{s\alpha}(x)$ to be fermionic, $K$ must be symmetric and its diagonal elements must be even [37].

The transformed Hamiltonian, i.e. the Hamiltonian of the CDFs is given by

$$H = (-1)^\varphi \hbar v_F \int d^2 x \psi^\dagger_{\varphi\tau} (x) \left( \partial_i + \frac{e}{\hbar} (A_i + a^\varphi_i) \right) \psi_{\varphi\tau} (x).$$

(5)

with the CS gauge field $a^\varphi = (a^1, a^2)$:

$$a^1_i (x) = \frac{\hbar}{e} \sum_\rho K_{\alpha\rho} \int d^2 x' \frac{-(x_2 - x'_2)}{|x - x'|^2} \rho_\rho (x'),$$

$$a^2_i (x) = \frac{\hbar}{e} \sum_\rho K_{\alpha\rho} \int d^2 x' \frac{x_1 - x'_1}{|x - x'|^2} \rho_\rho (x').$$

(6)

It is straightforward to verify that $a^\varphi$ satisfies

$$\nabla \times a^\varphi (x) = \tilde{\partial} \sum_\rho K_{\alpha\rho} \rho_\rho (x) \hbar/e.$$

(7)

Physically, equation (7) describes how the CS gauge field coupled to the CDFs with the spin–valley index $s$ is generated by the flux quanta attached to other CDFs in the same or different spin–valley sectors. The number of flux quantum is specified by the matrix elements of $K$. We note that $\rho_\alpha$ can be either positive or negative, due to the particle–hole symmetry of the Dirac spectrum and our choice of vacuum associated with the charge neutral point. As a result, the matrix elements of $K$ can be positive or negative. In particular, if $K$ describes a state with filling factor $\tilde{\nu}$, $-K$ describes a state with filling factor $-\nu$. The condition for the CDFs to experience a vanishing net magnetic field on average requires

$$\sum_\rho K_{\alpha\rho} \rho_\rho (\rho) = e B / h$$

(8)

for all the values of $\alpha$. For a given filling fraction $\tilde{\nu} = \sum_\rho (\rho/\hbar) e B / h = \pm 1 / \tilde{\phi}$, there are three types of physical solution:

$$K^1 = \text{sgn}(\tilde{\nu}) \begin{pmatrix} \phi & \phi & \phi & \phi \\ \phi & -\phi & -\phi & -\phi \\ -\phi & -\phi & \phi & \phi \\ -\phi & -\phi & -\phi & -\phi \end{pmatrix},$$

(9)

$$K^2 = \text{sgn}(\tilde{\nu}) \begin{pmatrix} \phi & \phi & \phi & \phi \\ \phi & \phi & \phi & \phi \\ \phi & \phi & \phi & \phi \\ \phi & \phi & \phi & \phi \end{pmatrix},$$

$$K^3 = \text{sgn}(\tilde{\nu}) \begin{pmatrix} \phi & \phi & \phi & \phi \\ -\phi & -\phi & -\phi & -\phi \\ -\phi & -\phi & -\phi & -\phi \\ -\phi & -\phi & -\phi & -\phi \end{pmatrix}.$$
with negative eigenvalues result in wavefunctions that are not normalizable [36], as a consequence of the quadratic energy–momentum dispersion.

The expressions for $K^1$ and $K^3$ can be further simplified by noting that the two valleys for a given spin projection are degenerate in these states. As a result, the $K$-matrices in this case effectively reduces to $2 \times 2$ matrices. Letting $\Psi' = (\Psi_1, \Psi_2)^T$ and $\Psi'' = (\Psi_3, \Psi_4)^T$, the CS transformation can be expressed as

$$
\left( \begin{array}{c} \psi'_1(x) \\ \psi'_2(x) \end{array} \right) = \left( \begin{array}{cc} e^{i\tau_1(x)} & 0 \\ 0 & e^{i\tau_2(x)} \end{array} \right) \left( \begin{array}{c} \psi''_1(x) \\ \psi''_2(x) \end{array} \right),
$$

with

$$T_s(x) = \sum s' \int d^2x' K_{sx'} \rho_s(x') \arg(x - x'),$$

and

$$K^1 = \left( \begin{array}{cc} \tilde{\phi} & \text{sgn}(\tilde{\phi}) \tilde{\phi} \\ \text{sgn}(\tilde{\phi}) \tilde{\phi} & -\tilde{\phi} \end{array} \right),
$$

$$K^3 = \text{sgn}(\tilde{\phi}) \left( \begin{array}{cc} \phi & \tilde{\phi} \\ \tilde{\phi} & -\phi \end{array} \right).$$

Under the transformation,

$$H = -i\hbar \nu_F \int d^2x \Psi_s^\dagger \alpha_i(\hat{\partial}_i + i e(A_i + a_i^\dagger)/\hbar) \Psi_s,$$

To keep the presentation simple, we will focus on $K^1$ and $K^3$, and comment on $K^2$ when appropriate.

The Hamiltonian $H$ in equation (13) can be separated into two parts

$$H = H_0 + V_{st},$$

where $H_0$ is the Hamiltonian for free massless Dirac particles and $V_{st}$ describes the CDF interactions mediated by the CS gauge field. It can be obtained from the $A_i + a_i^\dagger$ terms by using the explicit expressions for $a_i^\dagger$ in equation (6) and writing the vector potential $A_i$ in terms of the average density ($\rho_s$). The final result is an interaction of the form:

$$V_{st} = \sum_s -i\hbar \nu_F \int d^2x d^2x' \left( \Psi_{R_i}^\dagger(x), M_s(x, x') \Psi_{R_i}(x) \right) - \Psi_{L_i}^\dagger(x), M_s(x, x') \Psi_{L_i}(x),$$

where

$$M_s(x, x') = \sum s' K_{sx'} \delta_{sx'}(x') \left( \begin{array}{cc} 0 & 1/(z - z') \\ -1/(\bar{z} - \bar{z}') & 0 \end{array} \right),$$

with the bilinear fermion operator $\delta_{sx} = \rho_s - \langle \rho_s \rangle$, and the holomorphic coordinates $z = x + iy$, $\bar{z} = x - iy$. We will refer to $V_{st}$, which is essentially a CDF current–density interaction, as the statistical interaction.

Since the CDF experiences zero net magnetic field, its field operator can be conveniently expanded in the helicity basis as

$$\Psi_{R_i}(x) = \frac{1}{\sqrt{2V}} \sum_k e^{i\mathbf{k} \cdot \mathbf{x}} \left[ \begin{array}{c} e^{-i\mathbf{k} \cdot \mathbf{x}} \tilde{A}_k \ + \ e^{i\mathbf{k} \cdot \mathbf{x}} B_k^\dagger \end{array} \right],$$

where $\tilde{A}_k = \text{sgn}(k_z) \sqrt{2} + \mathbf{k} \cdot \mathbf{L}$. Note that, in the operator expansion, we have performed a particle–hole transformation such that removing a particle in a negative energy state is redefined as creating a hole with positive energy. Specifically, $B_k^\dagger$ and $D_k^\dagger$ in the above expressions are the hole creation operators for the $R$- and $L$-valleys respectively. In the helicity basis, the kinetic energy part of the Hamiltonian becomes

$$H_0 = \sum_s \hbar \nu_F (A_k^\dagger A_k + B_k^\dagger B_k + C_k^\dagger C_k + D_k^\dagger D_k).$$

3. Exciton condensation and normal states

The statistical interaction derived in the previous section has important physical consequences on the nature of the ground states for the CDFs. We now show that it drives the formation of an exciton condensate of the CDFs. The normal state is obtained by populating quasiparticles on top of the exciton condensate. In other words, the normal state of the CDFs corresponds to doping an excitonic insulator. We shall use the variational approach and construct a variational wavefunction for the normal. First, at charge neutrality, the exciton vacuum can be written as follows,

$$|0\rangle = \prod \prod_s \left( \cos \varphi_{R,k} - \sin \varphi_{R,k} A_k^\dagger B_k^\dagger \right) \times \left( \cos \varphi_{L,k} - \sin \varphi_{L,k} C_k^\dagger D_k^\dagger \right) |\text{vac}\rangle,$$

where $\varphi_{R,L}$ are variational parameters and $|\text{vac}\rangle$ describes the state where the valence CDF bands are filled and the conduction bands empty. The quasiparticle operators associated with the exciton condensate $|0\rangle$ can be obtained through a Bogoliubov transformation and are given by

$$a_k^\dagger = \cos \varphi_{R,k} A_k^\dagger + \sin \varphi_{R,k} B_k^\dagger,$$

$$b_k = -\sin \varphi_{R,k} A_k^\dagger + \cos \varphi_{R,k} B_k.$$

The normal state with a nonzero particle density can be constructed by creating quasiparticles on top of the exciton vacuum $|0\rangle$. A generic normal state with a positive filling factor can be written as $|N \rangle = \prod \prod_{k \leq k_F^d} a_k^\dagger b_k^\dagger \prod_{k \leq k_F^l} c_k^\dagger d_k^\dagger |0\rangle$, where $k_F^R$ and $k_F^L$ are the Fermi wavevectors associated with the two valleys with spin $s$. At the filling factor $\tilde{\nu}$, we have

$$\sum_{s, k \leq k_F^R} |N| a_k^\dagger a_k |N\rangle + \sum_{s, k \leq k_F^L} |N| c_k^\dagger c_k |N\rangle = \tilde{\nu} eB/h.$$

Hereafter, the wavevector will be measured in units of $k_F = 1/\hbar \theta^{1/2}$, which corresponds to the spin-polarized CDF Fermi vector, and the energy in unit of $\hbar \nu_F$. We will show that the variational energy is indeed minimized at nonzero $\nu_{R,L}$ in favor of an exciton condensate.
It is useful to note that the exciton condensation energy from the statistical interaction is linear in the exciton mass. This is due to the fact that $V_{st}$ is a current–density interaction and can be expressed as

$$
V_{st} = \frac{1}{2} \sum_{\alpha, \beta} K_{\alpha \beta} \sum_{\mathbf{x}, \mathbf{x}'} \psi^\dagger_{\mathbf{x}}(\mathbf{x}) \gamma_3 \gamma_5 \psi^\dagger_{\mathbf{x}'}(\mathbf{x}')
$$

where $J_{\alpha, \beta}(\mathbf{x}, \mathbf{x}') = \psi^\dagger_{\mathbf{x}'}(\mathbf{x}') \gamma_j \psi_{\mathbf{x}}(\mathbf{x})$. The internal energy coming from the statistical interaction $\langle V_{st} \rangle$ is a linear function of the exciton order parameter $\langle \psi_{\mathbf{x}}(\mathbf{x}) \gamma_3 \gamma_5 \psi_{\mathbf{x}'}(\mathbf{x}') \rangle$. As a result, the mass has the same sign as that of the corresponding matrix element in $K$ in order to lower the energy. To gain further insight, one can isolate the interaction $V_{st}$ in the exchange channel (e.g., for the $R$-valley)

$$
F_{st} = -\frac{\pi}{\sqrt{\nu}} \sum_{\alpha, \mathbf{k}, \mathbf{p}} K_{\alpha \beta} \sin 2\nu_{\mathbf{R} \mathbf{k}} \cos 2\nu_{\mathbf{R} \mathbf{k}} a_{\mathbf{k}}^\dagger a_{\mathbf{p}}^\dagger \delta_{\mathbf{k}, \mathbf{p}},
$$

where $\mathbf{k}_>$ (\mathbf{k}_<) is the bigger (smaller) of $\mathbf{k}$ and $\mathbf{p}$. To lower the energy, i.e. to have a negative $F_{st}$, $K_{\alpha \beta}$ and $\nu_{\mathbf{R} \mathbf{k}}$ must have the same sign and a positive $\nu$ implies a positive mass.

The diagonal elements of $K^{\alpha \beta}$ have the same sign and a positive integer contribution to the filling factor $\tilde{\nu}$ the CDFs. Remarkably, this QAHE at $\tilde{\nu}$ implies that the total filling factor $\nu = 1 + \tilde{\nu}$ for the electrons, where the integer contribution to $\sigma_{xy}$ comes from the QAHE of the exciton condensate. Similarly, $K^{\alpha \beta}$ describes states at filling factor $\nu = \pm(1 - \tilde{\nu})$, since it leads to CDF bands with Chern number ±1.

![Figure 1](image_url)  
**Figure 1.** Schematic plots of the zeroth Landau levels for different $K$-matrices. The lines with four dots depict completely filled Landau levels, and lines with two dots means partially filled Landau levels. The difference between the $K$-matrices is reflected in the signs of the exciton mass gaps, and thus the relative positions of the zeroth Landau levels with respect to zero energy. $K^{\alpha \beta}$ leads to opposite masses for the two spin projections and therefore describes the filling factor $\nu = 1 + \tilde{\nu}$ (middle panel). $K^3$ results in the same sign of mass for all bands, leading to the filling factor $\nu = -2 + 1/\tilde{\nu}$ (bottom panel).

FQHE in the vicinity of charge neutrality observed in recent experiments [6–9].

We now calculate the dynamical mass for the symmetry breaking state $\langle N \rangle$ described by $K^1$ at $\nu = 1/2$ by minimizing the variational energy $E_K = \langle N \mid H \mid N \rangle$, where the Hamiltonian includes both the statistical interaction $V_{st}$ and the Coulomb interaction

$$
V_c(\mathbf{x}, \mathbf{x}') = \frac{g}{|\mathbf{x} - \mathbf{x}'|}, \quad g = \frac{e^2}{4\pi \epsilon \hbar v_F}.
$$

We focus on the spin-down bands and drop the spin indices for simplicity. The variational equation for the $R$-valley (similar for the $L$-valley) $\psi_{\mathbf{R} \mathbf{k}}$ can be expressed as a set of self-consistent equations for the quasiparticle dispersion $\epsilon_{\mathbf{R} \mathbf{k}} = \sqrt{\alpha_{\mathbf{R} \mathbf{k}}^2 + m_{\mathbf{R} \mathbf{k}}^2}$, where $m_{\mathbf{R} \mathbf{k}}$ is the mass gap, $\alpha_{\mathbf{R} \mathbf{k}}$ is the renormalized dispersion, and $\sin 2\nu_{\mathbf{R} \mathbf{k}} = m_{\mathbf{R} \mathbf{k}}/\epsilon_{\mathbf{R} \mathbf{k}}$.

$$
\alpha_{\mathbf{R} \mathbf{k}} = k + \frac{\pi}{\sqrt{\nu}} \sum_{\mathbf{p}} v_1(\mathbf{k}, \mathbf{p}) \frac{\epsilon_{\mathbf{R} \mathbf{p}}}{\epsilon_{\mathbf{R} \mathbf{k}}} a_{\mathbf{k}}^\dagger a_{\mathbf{p}}^\dagger \frac{1}{2} \epsilon_{\mathbf{R} \mathbf{p}}\delta_{\mathbf{p}, \mathbf{p}^\prime},
$$

$$
m_{\mathbf{R} \mathbf{k}} = g \frac{\pi}{\sqrt{\nu}} \sum_{\mathbf{p}} v_1(\mathbf{k}, \mathbf{p}) \frac{m_{\mathbf{R} \mathbf{p}}}{\epsilon_{\mathbf{R} \mathbf{p}}} + 2 \frac{\pi}{\sqrt{\nu}} \sum_{\mathbf{p} > \mathbf{k}} \frac{1}{2} \epsilon_{\mathbf{R} \mathbf{p}}\delta_{\mathbf{p}, \mathbf{p}^\prime}.
$$

Here, $\alpha_{\mathbf{R} \mathbf{k}}^2 = 1 - n_{\mathbf{R} \mathbf{k}}$ is the occupation number of the $R$-valley, and $v_1(\mathbf{k}, \mathbf{p})$ is the coefficient of the angular expansion of the Coulomb interaction $V_c$ in the $\ell$th angular momentum channel, with the expansion given by

$$
F_{\mathbf{k} - \mathbf{p}} = \sum_{\ell = -\infty}^{\infty} v_1(\mathbf{k}, \mathbf{p}) e^{i(\mathbf{k} - \mathbf{p}) \cdot \phi}.
$$

Since $a_{\mathbf{R} \mathbf{k}}^\dagger$ projects out the filled states, only the states above the Fermi level contribute to the dynamical mass. Note
that a natural ultraviolet energy cutoff for equations (23) is the energy spacing between the ξ LL and the first LL, which is the largest energy scale in the problem. Restoring the unit, the mass can be expressed as $M_{\text{Rk}} = m_{\text{Rk}} \hbar v_F / \sqrt{B}$. The magnitude of $M_{\text{Rk}}$ depends on the coupling constants. In particular, it depends on $k_F \phi \propto \phi^{1/2}$. With increasing $\phi$, $M_{\text{Rk}}$ will also increase. Therefore, for smaller filling fractions, the dynamical mass will be larger. This agrees with the fact that when the filling fraction is small, there will be more states available for exciton pairs, leading to larger masses. In the magnetic catalysis theory [12], it is found that in the strong coupling regime where Landau level mixing is relevant, the mass gap is proportional to the Landau level spacing $\sqrt{E_{2L}/\hbar}$, the same scaling behavior as the mass gap obtained here. Since the CDFs are not confined to a single Landau level, the current approach agrees well with the magnetic catalysis theory in the strong coupling regime. It is worth emphasizing that, in contrast to the non-relativistic composite fermion theory, where an appropriate composite fermion mass remained elusive [30], the mass of the relativistic CDF theory naturally arises from the interactions through exciton condensation.

We found that, due to the exchange interaction, the valley polarized state ($k_{yF} = \sqrt[3]{2} k_{yF}, k_{xF} = 0$) has lower energy than the unpolarized state ($k_{yF} = k_{xF} = k_{yF}$). For example, for a Coulomb strength $g = 0.3$, $\phi = 2$ and a momentum cutoff $\Lambda = 2k_F$, the energy density of the polarized state is approximately $-0.88$, while that of the unpolarized state is $-0.85$. The solution for the complete dispersion of the exciton mass $m_{\text{Rk}}$ in the spin–valley polarized state is shown in figure 3(a). Numerically, when $k$ is small, $M_{\text{Rk}} \approx 0.55 E_{\text{LL}}$, where $E_{\text{LL}} = \sqrt{2 \hbar v_F / \hbar}$ is the Landau level spacing between the $n = 0$ and $n = 1$ Landau levels. However, due to its momentum cutoff dependence, the exact magnitude of $M_{\text{Rk}}$ must be determined experimentally. The resulting CDF band structure is shown schematically in figure 2. The unoccupied spin-up bands have a mass with the same magnitude but opposite sign as the unoccupied spin-down band dictated by the diagonal elements of $K^3$, as discussed before. It should be stressed that the mass gap of the empty CDF bands generated by the statistical interaction is large enough so that the chemical potential lies inside the gap, making the emergence of the spin–valley polarized state fully self-consistent with $K^3$.

In the case of $\nu = \pm 1/3$, these $K$ matrices with $\bar{\phi} = 2$ do not lead to the complete cancellation of the external magnetic field. Rather, the CDFs fill the Landau levels of the residual magnetic field $B^* = B - 2|\rho|/e$ at an integer filling. These CDF Landau levels will develop mass gaps through the statistical interaction and Coulomb interaction via a magnetic catalyst at integer filling factors [12]. Under exchange splitting, only one CDF Landau level will be completely filled and exhibit an integer QHE. This corresponds to a FQHE state at $\nu = \pm 1/3$ of the electrons [29] described by the single-component Laughlin state. In principle, other states at $\nu = q/(2pq \pm 1)$ can be constructed by following the composite fermion approach [42]. Within the current framework, however, we can only study states within $\zeta$LLs due to the restriction that a fictitious magnetic field generated through flux attachment must partially or completely cancel the external magnetic field.

Before turning to the pairing instability, we comment on the relation between magnetic catalysis and quantum Hall ferromagnetism. Recent experiments [9, 41] showed that $SU(4)$ symmetry of the $n = \pm 1$ Landau levels is fully lifted. This seems to favor the quantum Hall ferromagnetism theories, since magnetic catalysis cannot explain the extra plateaus at $\nu = \pm 3, \pm 5$. However, it is possible that the two mechanisms (magnetic catalysis and quantum Hall ferromagnetism) play leading roles in different regions and may even work in a collaborative way. For example, it is plausible that magnetic catalysis may be the driving force in the zeroth Landau levels while quantum Hall ferromagnetism is responsible for Landau level splitting beyond the zeroth Landau level. Furthermore, it was shown in [45, 46] that the order parameters for these two mechanisms can both be nonzero.

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**Figure 2.** Schematic plot of the spin-down CDF bands before and after exciton condensation (EC) and exchange induced polarization.

**Figure 3.** (a) The dynamical masses $m_{\text{Rk}}$ and $m_{\text{Rk}}$ in the chiral $p$-wave paired state. The corresponding exciton masses in the normal state are very close and drawn in dashed lines. (b) Solution of the BdG equation for the pairing gap function $\Delta_k$, the condensate amplitude $\bar{\phi}_k = \sqrt{2} \nu_a$, and the momentum distribution function $\tilde{n}_k$. (c) The CDF dispersion in the normal state and that of quasiparticle excitations in the paired state; (d) pairing wavefunctions (unnormalized) in real space. All results are for $\tilde{\phi} = 2$ and $g = 0.3$ with a momentum cutoff $\Lambda = 2k_F$. Due to valley splitting, the Fermi vector is $k_F^* = \sqrt{2} k_F$. 
4. Paired quantum Hall states

We now show that, at even-denominator filling fractions, the spin–valley polarized composite massive Dirac fermion liquid has a pairing instability where the quasiparticles on top of the exciton condensate form spin-triplet pairs in the chiral p-wave channel. In terms of the R-valley, the statistical pairing interaction is dominated by the $\ell = 1$ angular momentum channel and has the form,

$$
P_{\ell=1} = \frac{\pi}{V} \sum_{i,k,p} K_{ikp} \sin 2\varphi_{i,p,k} \cos 2\varphi_{i,p,k} \frac{1}{k_> - k_<} \times e^{i(\xi_k - \mu_k)} a_{ik}^\dagger a_{ik} a_{ik}^\dagger a_{ik}.
$$

(25)

It is remarkable that this pairing interaction is present only if there is an exciton condensate, i.e., when $\varphi_{i,p,k} \neq 0$. Since the latter requires Landau level mixing, this implies that Landau level mixing is crucial for the pairing to occur. The variational wavefunction for the paired state has the BCS form $|\Omega\rangle = \prod_{i,k} (u_{ik} + v_{ik} a_{ik}^\dagger a_{ik})|0\rangle$, where $\varphi_{i,p,k} = 1$, and $|0\rangle$ is the exciton vacuum defined in (17). Note that the variational wavefunction contains both the exciton and pairing order parameters, which must be determined self-consistently by minimizing the ground state energy. The self-consistent equations for the dynamical mass have the same form as in (23) with $\xi_k = |v_k|^2$, but unlike in the normal state, exciton pairs exist even for $k < k_F$ due to pairing. The variation of the energy with respect to $u_{ik}$ and $v_{ik}$ leads to the familiar BdG equations (dropping the valley index),

$$
E_k u_k = \xi_k u_k + \Delta_k v_k,
$$

$$
E_k v_k = -\xi_k v_k + \Delta_k u_k,
$$

(26)

where $\xi_k = \epsilon_k - \beta_k - \mu$, $E_k = \sqrt{\xi_k^2 + |\Delta_k|^2}$, and $\beta_k = g \frac{\pi}{V} \sum_p v_0(k,p)(1 + \eta_p^2)$ comes from the Coulomb exchange. The gap function $\Delta_k$ is determined by the gap equation

$$
\Delta_k = \frac{2\pi}{V} \sum_p e^{i(\phi - \eta_p)} u_p v_p \left[ \phi \frac{1}{k_>} m_{k_>} \phi_{k_>} \phi_{k_>} - \frac{g}{2} \frac{1}{k_>} m_{k_>} \phi_{k_>} \frac{1}{k_>} v_1(k,p) - \frac{g}{2} \frac{1}{k_>} \phi_{k_>} \frac{1}{k_>} v_1(k,p) \right].
$$

Figure 3 displays the numerical solution of $E_k$ and $\Delta_k$ at $\phi = 2$ and $g = 0.3$, where the ground state is indeed a chiral $p + ip$ paired state. Because the Coulomb interaction is pair breaking, the pairing gap $\Delta_{sc} = 2 \min(E_k)$ reduces with increasing $g$ and vanishes at a critical value $g_c \approx 0.53$ and 1.28 for $\phi = 2$ and 4 respectively. For $g > g_c$, the massive Dirac fermions form a stable Fermi liquid state.

To gain further insights into the paired state, we study the two-particle pairing wavefunction by projecting the BCS state to real space: $\Psi(x_1, x_2, \ldots, x_N) = \langle 0 | \Psi(x_1) \Psi(x_2) | \Omega \rangle$. Because of the spinor structure, the upper and lower components are obtained separately,

$$
g_1(x) = \frac{1}{V} \sum_k (\cos \varphi_{k_F} + \sin \varphi_{k_F}) e^{-2i\varphi_{k_F} x} g_k,
$$

$$
g_2(x) = \frac{1}{V} \sum_k (\cos \varphi_{k_F} - \sin \varphi_{k_F}) e^{i\varphi_{k_F} x} g_k.
$$

(27)

where $g_k = v_k/|u_k|$. In the long wavelength limit, $g_k \approx e^{i\varphi_{k_F}} k$ and $m_{k_F} \approx m_0$ is essentially a constant (cf figure 3).

We have, to quadratic order, $(\cos \varphi_{k_F} + \sin \varphi_{k_F}) \approx 2 - \nu_e^2 k^2/2m_0^2$ and $(\cos \varphi_{k_F} - \sin \varphi_{k_F}) \approx \nu_e^2 k^2/2m_0^2$, where $\nu_e$ is the renormalized Fermi velocity. Thus, the large distance behaviors of the pairing wavefunction are $g_1(r) \propto 1/r + c\sqrt{2/\pi} \cos(\kappa r - \pi/4)\sqrt{r}$ and $g_2(r) \propto -c\sqrt{2/\pi} \cos(\kappa r - \pi/4)\sqrt{r}$, where $c$ and $\kappa$ are numerical constants. The oscillatory terms in the above equations are due to the $k^2$ terms in the expansions, originating from the $k$ dependence of the mass, which should be distinguished from the oscillatory wavefunction in [40]. The many-body real space wavefunction for CDFs can be obtained by [39]

$$
\Psi_{CDF}(x_1, x_2, \ldots, x_N) = \langle 0 | \Psi(x_1) \Psi(x_2) \ldots \Psi(x_N) | \Omega \rangle
$$

$$
= \frac{1}{2^{N/2}} \sum_P \mathrm{sgn}(P) \prod_{i=1}^{N/2} \left( g_1(x_{P(2i-1)} - x_{P(2i)}) \right)
$$

$$
= \frac{1}{2^{N/2}} \sum_P \mathrm{sgn}(P) \prod_{i=1}^{N/2} \left( g_2(x_{P(2i-1)} - x_{P(2i)}) \right)
$$

$$
\approx \frac{1}{2^N} \sum_P \mathrm{sgn}(P) \prod_{i=1}^{N} \left( 1/(z_{P(2i-1)} - z_{P(2i)}) \right)
$$

(27)

where $N$ is an even number representing the number of CDFs. In the last line of the above equation, quadratic and higher order terms have been dropped. Note that, except for the subleading oscillatory contributions that decay faster at large distances, the pairing wavefunction resides predominantly on one component of the spinor (i.e. on one of the sublattices) and has the form of the Moore–Read Pfaffian state, in agreement with the numerical results shown in figure 3. It is remarkable that, although the large holomorphic part on the upper component is indeed dominated by contributions from the $\zeta$LL subspace, the wavefunction of the non-Abelian ground state does not entirely lie in the $\zeta$LL since the nonholomorphic, oscillatory contributions, although small, enter both the upper and the lower components of the wavefunction and can be attributed to the effects of Landau level mixing.

Similar to [30, 31], from the transformation in equation (3), the wavefunction for Dirac fermions is related to that of CDF by

$$
\Psi_{MF}(x_1, \ldots, x_N) = \prod_{i<j} (z_i - z_j)^{\frac{1}{\phi}} \Psi_{CDF}(x_1, x_2, \ldots, x_N).
$$

The full Jastrow factor $\prod_{i<j} (z_i - z_j)^{\frac{1}{\phi}}$ may be recovered by taking into account fluctuations beyond the mean field level [32].

5. Conclusion

In this paper, we presented a composite relativistic fermion theory for the FQHE in graphene. We showed that the ground state has spontaneous spin–valley polarization, leading to
a single-component Laughlin-like state at $\nu = 1/3$ and a chiral $p$-wave pairing state of the Moore–Read Pfaffian at $\nu = 1/2$. A crucial prediction of the present theory is the dynamical mass generation for the Dirac fermions through the formation of an exciton condensate in the zeroth Landau level. It originates from Landau level mixing and facilitates the spin–valley polarization and the pairing interaction for the paired states at even dominant filling fractions. This CDF mass, scaling with $\sqrt{B}$, should be detectable by scanning tunneling microscopy in a high magnetic field. It would also be desirable to see if and how the inclusion of Landau level mixing affects the results of the numerical diagonalization studies [21, 22]. In a recent experiment [8] on suspended graphene, a plateau-like feature near $\nu = 1/2$ is observed in some but not all samples. Future experiments are desirable to explore whether a true single-component quantum Hall state emerges at $\nu = 1/2$ that would realize non-Abelian statistics in graphene.

We end this paper with a few comments on some experimental results. For definiteness, we assumed a near charge neutral state, the spin is polarized. A recent experiment [41] shows that the $\nu = 0$ state is spin unpolarized. This suggests that the valley anisotropy is the dominant force behind the symmetry breaking near charge neutrality. This effect can be easily incorporated in the current framework by including a valley anisotropy term. In [9], it is found that the $\nu = 5/3$ state is missing. This may be attributed to the different environment this state lies in. For the $\nu = 1/3$ state, the associated $K$-matrix breaks the $SU(4)$ symmetry. In contrast, the $K$-matrix for $\nu = 5/3$ preserves the symmetry. Due to the $SU(4)$ symmetry, the associated Goldstone modes for $\nu = 5/3$ may destabilize the state. A discussion of this issue can be found in [42, 43]. A more surprising result in [10] is that the energy gaps are linear in $B$. Given that the energy scales in the problem are the Coulomb interaction energy and the Landau level spacing, both scaling as $B^{1/2}$, any theory based on these energy scales should produce an energy gap that scales as $B^{1/2}$. Thus, this result presents a great puzzle. Further experimental and theoretical studies are needed to resolve it.

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