Benchmark for the quantum-enhanced control of reversible dynamics

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Controlling quantum systems is crucial for quantum computation and a variety of new quantum technologies. The control is typically achieved by breaking down the target dynamics into a sequence of elementary operations, whose description can be stored into the memory of a classical computer. Here we explore a different approach, initiated by Nielsen and Chuang \cite{nielsen2000quantum}, where the target dynamics is encoded in the state of a quantum system, regarded as a “quantum program”. We show that quantum strategies based on coherent interactions between the quantum program and the target system offer an advantage over all classical strategies that measure the program and conditionally operate on the system. To certify the advantage, we provide a benchmark that guarantees the successful demonstration of quantum-enhanced programming in realistic experiments.

I. INTRODUCTION

The ability to control quantum systems is at the core of quantum computing and of a new generation of quantum technologies \cite{nielsen2000quantum}. Most often, the control is achieved by decomposing the target dynamics into a sequence of elementary operations, whose execution can be controlled by a classical program, like a piece of code stored in the memory of a classical computer. This is the case, for example, in the circuit model of quantum computing and in digital quantum simulations. A different approach was put forward by Nielsen and Chuang \cite{nielsen2000quantum}, who proposed that the dynamics of a quantum system could be encoded in the state of an another quantum system. Such a state serves as a quantum program, containing the instructions needed to execute the target dynamics. What makes the program quantum is that, in general, the instructions corresponding to two distinct dynamics can be encoded into two non-orthogonal quantum states. Nielsen and Chuang’s paradigm led to the design of programmable quantum gates \cite{nielsen2000quantum, chuang1997universal} and measurements \cite{nielsen2000quantum, nielsen2006quantum}, with applications to quantum state discrimination \cite{nielsen2000quantum, yamamoto2007quantum}, quantum communication \cite{nielsen2000quantum}, and quantum learning \cite{nielsen2000quantum, chuang1996quantum}. Experimental demonstrations of programmable quantum devices have been reported in a variety of setups \cite{nielsen2000quantum, chuang1997universal}, with applications to the experimental study of commutation relations \cite{nielsen2000quantum} and to the activation of entanglement \cite{nielsen2000quantum}.

The quantum mechanical nature of the program introduces genuinely new questions. How does the size of the program affect the accuracy? How many times can one reuse the program before it loses its ability to specify the target dynamics? Is the best performance attained in a classical way, by reading out the program and conditional operating on the data, or is it attained in a quantum way, by letting program and data interact as a closed system?

In this paper we answer these three questions, focusing on the problem of rotating the spin of a quantum particle around a direction determined by the spin of another particle, as illustrated in Figure 1. We establish the ultimate quantum limit to the accuracy as a function of the size of the control spin, showing that error vanishes inverse linearly with the spin size. The limit is attained by a coherent mechanism, whereby the two spins interact with one another without leaking any information into the outside world. We show that this coherent strategy reaches a higher accuracy than every classical strategy where the control spin is measured in order to extract information about the program. For example, measurement-based strategies using a control system of spin $j = 3/2$ can achieve at most fidelity 64% in flipping a target qubit around a variable direction, while the optimal quantum strategy achieves 71% fidelity. The gap between quantum and classical strategies allows us to establish a benchmark for the demonstration of quantum-enhanced programming in realistic experiments: even if the implementation is affected by noise and imperfection, there is still a range of values of the fidelity within which the experiment can demonstrate a performance that could not be achieved classically, even with unlimited technology. While many benchmarks have been identified for quantum teleportation and cloning \cite{nielsen2000quantum, chuang1997universal}, to the best of our knowledge no benchmark for the task of
programming quantum gates has been established prior to the present work.

The paper is structured as follows. We start in Section II with the determination of the optimal tradeoff between accuracy and size of the quantum program in the case of qubit gates. The corresponding quantum benchmark is then presented in Section III. We show how to surpass the benchmark, by constructing a physical realization of the optimal programmable gate (Section IV) and showing that the advantage persists even if the program state is recycled multiple times (Section V). In Section VI we extend our result from qubits to systems of arbitrary dimension, proving that the advantage of quantum programming is generic and providing an explicit analysis for the task of programming rotation gates. The conclusions are finally drawn in Section VII.

II. ACCURACY VS SIZE

Suppose that a magnetic field is turned on for a limited period of time. During this time, the direction of the field can be encoded into the state of a magnetic material, which orients itself along the direction of the field and ideally maintains the orientation even after the field is turned off. Thanks to this property, the magnet can be used as a program to reproduce the dynamics that would have occurred if a particle were immersed in that field. This kind of reproduction can be seen as an elementary learning process, where a quantum machine learns to emulate an unknown dynamics by observing its action on a certain input. Now, the question is: how large must the magnet be in order to accurately steer the desired dynamics?

The answer depends on the size of the particle one wishes to control. Let us focus first on the case of a single qubit, embodied in a spin-$\frac{1}{2}$ particle. In this case, the target dynamics is a rotation around the direction of the field, of an angle proportional to the time of the evolution. The rotation is represented by the matrix $V_{\theta,\mathbf{n}} = \exp[-i \theta \mathbf{n} \cdot \mathbf{\sigma}/2]$, where $\theta$ is the rotation angle, depending on the strength of the field and on the evolution time, $\mathbf{n} = (n_x, n_y, n_z)$ is the rotation axis, corresponding to the direction of the field, and $\mathbf{\sigma} = \sigma_x + i \sigma_y + \sigma_z$ is a linear combination of Pauli matrices, representing the projection of the spin operator along the direction $\mathbf{n}$. The small magnet is modelled as a spin-$\frac{1}{2}$ particle, whose state $|\phi_n\rangle$ serves as an indicator of the direction $\mathbf{n}$, and as a program for the rotation gate $V_{\theta,\mathbf{n}}$. We impose that the encoding $\mathbf{n} \mapsto |\phi_n\rangle$ is consistent with the physical interpretation of $\mathbf{n}$ as a spatial direction. This means that rotating the direction $\mathbf{n}$ should be equivalent to rotating the state vector—in formula,

$$|\phi_{gn}\rangle = U^{(j)}_g |\phi_n\rangle,$$

where $g$ is an arbitrary rotation in three dimensional space and $U^{(j)}_g$ is the unitary matrix representing the action of $g$ on the Hilbert space of the spin-$j$ particle. Except for Eq. (1), we make no assumption on the program states.

Once the information about the rotation axis is encoded in the control spin, the problem is to devise a mechanism that emulates rotations around that axis. Mathematically, the mechanism is described by a quantum channel, describing the joint evolution of the control and target. To evaluate the accuracy of the control mechanism, we compare the output state of the target with the ideal output of the gate $V_{\theta,\mathbf{n}}$. As a figure of merit, we use the fidelity

$$F(j, \theta, \mathbf{n}, \psi) = \langle \psi | V_{\theta,\mathbf{n}} \left[ C_\theta (\phi_n \otimes \psi) \right] V_{\theta,\mathbf{n}} | \psi \rangle,$$

where $|\psi\rangle$ is the initial state of the data qubit and $C_\theta$ is the quantum channel describing the effective evolution from the control and target together to the target alone. Note that a priori the fidelity could depend on the input state $|\psi\rangle$ and on the rotation axis $\mathbf{n}$. To eliminate the dependence, one can consider the average input-output fidelity

$$F(j, \theta) = \int d\mathbf{n} \int d\psi F(j, \theta, \mathbf{n}, \psi),$$

where $d\mathbf{n}$ is the invariant probability distribution on the unit sphere and $d\psi$ is the invariant probability distribution on the pure states of the system. In actual experiments, the averages over all directions and over all states can be replaced by averages over a finite set of directions and states, using the theory of unitary designs.

Our first result is the optimal quantum scaling of the fidelity with the program size. By maximizing over the quantum channel $C_\theta$ and over the program states $|\phi_n\rangle$ we find the optimal value

$$F_{\text{opt}}(j, \theta) = \frac{1}{3} + \frac{2}{3(1 + 2j)^2} \left[ 2j^2 + \frac{2j + 1}{2} + \frac{2j + 1}{2} \cos \theta + j \sqrt{1 + 2(2j + 1) \cos \theta + (2j + 1)^2} \right],$$

valid for $j \geq 3/2$ (see Appendix A for the derivation and for the expression of the fidelity in the $j = 1/2$ and $j = 1$ cases). The dependence on the rotation angle is illustrated in Figure 2, where one can see that the fidelity is minimum for $\theta = \pi$, meaning that the rotations of 180 degrees are the hardest to program.

Eq. (4) gives the exact expression of the fidelity, but an even more insightful expression can be obtained by Taylor expansion, which yields the approximate formula

$$F_{\text{opt}}(j, \theta) = 1 - \frac{1 - \cos \theta}{3j} + O \left( \frac{1}{j^2} \right).$$

This result shows that the error (defined as 1 minus fidelity) tends to zero as the control spin becomes macroscopically large. Note that the scaling $1/j$ refers to the
average over all possible rotation axes and over all possible input states. Later, we will prove that the scaling $1/j$ is optimal even in the worst-case over all input states.

III. BENCHMARK FOR COHERENT QUANTUM CONTROL

We have established the ultimate quantum performance in programming qubit rotations. An important question is whether this performance can be achieved through a classical strategy where the program is measured and a conditional operation on the data is performed. We refer to these strategies as measure-and-operate (MO) strategies. In Appendix B we determine the maximum fidelity achievable by arbitrary MO strategies, providing a benchmark that can be used to certify the demonstration of quantum-enhanced programming in realistic experiments. The exact value of the benchmark, derived in Appendix B, is

$$F_{MO}(j, \theta) = \frac{4j + 4 + (2j + 1) \cos(\theta - \tau)}{6j + 9} + \frac{(2j + 1)(\cos \theta + \cos \tau) + \cos(\theta + \tau) + 1}{3(j + 1)(2j + 3)},$$

with $\tau = \arccot \left[ \frac{(2j)^2 + 3j + 2}{(2j)^2 + 3j} \cos \theta + 2j + 1 \right] / [(2j)^2 + 3j] \sin \theta$. Figure 3 shows the gap between the benchmark and the optimal quantum fidelity for small values of the control spin. For large spins, the benchmark takes the asymptotic value

$$F_{MO}(j, \theta) \approx 1 - \frac{2(1 - \cos \theta)}{3j} + O \left( \frac{1}{j^2} \right).$$

Note that the error (one minus fidelity) is exactly twice the error of the optimal quantum protocol, which can be read out from Eq. (5). The error goes to zero both for quantum and classical strategies, but the rate for quantum strategies is twice as fast. Intriguingly, this seems to be a recurring feature in the relation between optimal classical and quantum programming strategies, as we will see later in the paper.

IV. PHYSICAL REALIZATION OF THE OPTIMAL PROGRAMMABLE GATE

We have seen that the optimal quantum strategy offers an advantage over all classical, measurement-based strategies. But how is the advantage achieved, concretely? For the program states, we find out that the optimal choice is to use spin-coherent states [34], encoding the direction $\mathbf{n}$ in the spin coherent state with maximum projection along the direction $\mathbf{n}$. This is not unexpected, because spin coherent states are optimal for the estimation of spatial directions [35]. Still, estimation and programming are two distinct operational tasks—and indeed, estimation is not the optimal strategy for controlling rotations. It is a non-trivial open question whether any deeper connection exists that links the optimal setup for quantum estimation with the optimal setup for quantum control.

Regarding the control mechanism, we find that the channel $C_\theta$ has an economical realization [36–38], meaning that it can be implemented by letting the program and the data interact as a closed system, without introducing extra ancillas. Explicitly, the optimal quantum channel is realized through the unitary evolution

$$U_\theta = \exp \left[ -i f(\theta) \frac{\mathbf{J} \cdot \mathbf{\sigma}}{2j + 1} \right]$$

where $\mathbf{J} = (J_x, J_y, J_z)$ are the spin operators of the control spin, $\mathbf{J} \cdot \mathbf{\sigma} = \sum_{i=x,y,z} J_i \otimes \sigma_i$ is the Heisenberg cou-
pling, and \( f(\theta) \) is the function
\[
 f(\theta) = \arccos \left( \frac{1 + (2j + 1) \cos \theta}{\sqrt{[1 + (2j + 1) \cos \theta]^2 + [(2j + 1) \sin \theta]^2}} \right).
\] (9)

Physically, this unitary evolution can be realized by setting up an isotropic spin-spin interaction, described by the Hamiltonian \( H = \alpha \mathbf{J} \cdot \mathbf{\sigma} \), for some suitable coupling constant \( \alpha \), and by letting the two spins evolve for time
\[
 t(\theta) = \frac{f(\theta)}{(2j + 1) \alpha \hbar},
\] (10)
depending on the angle \( \theta \) of the target rotation. Remarkable, the same program states and the same interaction can be used to control the full time evolution of the target system: one has only to adjust the interaction time [determined by the angle \( f(\theta) \)] based on the evolution time in the target dynamics [determined by the angle \( \theta \)].

Eq. (8) shows that the optimal way to program the dynamics is to set up a Heisenberg interaction between the control and target spins. This result answers an open question raised by Marvian and Mann [39], who assumed the Heisenberg interaction and showed that it can be used to approximate arbitrary rotations in the limit of large \( j \) limit. Marvian and Mann asked whether the Heisenberg interaction achieves the best scaling of the error with the spin size, a question that is answered in the affirmative by our result. The optimality of the Heisenberg interaction is not limited to the average fidelity: in terms of scaling with \( j \), the unitary gate [5] is optimal also in the worst case over all input states. Indeed, one can explicitly evaluate the worst case fidelity, which takes the value
\[
 F_w(j, \theta) = 1 - \frac{1 - \cos \theta}{j} + O \left( \frac{1}{j^2} \right).
\] (11)

Note that the error scaling \( 1/j \) is the best one could have hoped for, because the average error is a lower bound to the worst case error and we know from Eq. [5] that the average error cannot vanish faster than \( 1/j \).

Knowing the value of the worst case fidelity, one can estimate the error scenarios where one or more gates are implemented with a quantum program. For example, a quantum circuit that uses \( k \) programmable single-qubit gates will have an error of size \( k/j \sqrt{j} \) with respect to the ideal functionality. This means that, in order to have a negligible error, the size of the program should be large compared to the number of programmable gates in the circuit.

V. LONGEVITY OF THE QUANTUM ADVANTAGE

The Heisenberg interaction transfers information from the program to the data. This leads to a backreaction effect, whereby the program gradually loses its ability to control operations on the target [40]. An important question is how many times the program can be reused before the accuracy drops below a certain threshold. The number of reusing times is called longevity in Ref. [40].

Another important question is how many times the control spin can be reused before the quantum advantage is lost. We will refer to this number as the longevity of the quantum advantage.

Suppose that the joint evolution of control and target is described by the same unitary gate at every step. Assuming the gate to be of the form of Eq. (6) for some fixed function \( f(\theta) \), we obtain the close-form expression
\[
 F(j, \theta, n) = 1 - \frac{1 - \cos \theta}{3j} \cdot \frac{n(1 - \cos \theta) + j}{j}
\] (12)
quantifying the average fidelity at the leading order in \( j \) (see Appendix C for the derivation). From this expression one can see that the longevity grows as \( j^2 \). However, the longevity of the quantum advantage is much shorter: comparing the above fidelity with the MO fidelity in Eq. (7), we find that the quantum advantage disappears if the number of repetitions is larger than
\[
 L(j, \theta) = \frac{j}{1 - \cos \theta} + O(1).
\] (13)

One can also consider more elaborate strategies where the interaction time between control and target is optimized at every step. However, these strategies do not increase the longevity of the quantum advantage in the large \( j \) limit.

VI. PROGRAMMING LARGER SYSTEMS

Our result establishes the existence of a quantum advantage for single-qubit gates. This finding is conceptually important, because the advantage for single qubits implies an advantage of coherent programming for quantum systems of arbitrary dimension. Indeed, one can immediately prove the advantage by using the qubit benchmark for gates that act nontrivially only in a fixed two-dimensional subspace.

Our results also give a heuristic for the control of higher dimensional spins. The idea is to encode the rotation axis in a spin coherent state and to let the control and target spin interact as closed system. Explicitly, we make two spin systems undergo the Heisenberg interaction \( U_{\theta}^{(k)} = \exp \left[ -i \theta 2J \cdot \mathbf{K} / (2j + 1) \right] \), where \( \mathbf{K} = (K_x, K_y, K_z) \) are the spin operators of the target spin. Using the unitary gate \( U_{\theta} \), in Appendix D we obtain the average fidelity
\[
 F(j, k, \theta) = 1 - \frac{k(2k + 1)(1 - \cos \theta)}{3j},
\] (14)
in the large \( j \) limit. Remarkably, the error grows quadratically—rather than linearly—with the size of the target.
spin: in order to ensure high fidelity, the size of the program must be large compared to the square of the size of data. The same conclusion holds for the worst case fidelity, which has the asymptotic expression

\[ F_w(j, k, \theta) = 1 - \frac{k(k+1) + c(k)}{j} (1 - \cos \theta), \]  

(15)

with \( c(k) = 0 \) for even \( k \) and \( c(k) = 1/4 \) for odd \( k \).

The quantum strategy exhibits an advantage over the MO strategy consisting in measuring the direction \( \mathbf{n} \) from the spin coherent state pointing in direction \( \mathbf{n} \) and performing a rotation based on the outcome. Again, we find that the error of the quantum strategy vanishes in the macroscopic limit of large control systems, at a rate twice as fast as the error of the classical strategy.

VII. CONCLUSIONS

We determined the ultimate accuracy for the execution of rotations controlled by quantum spins. The ultimate accuracy limit is achieved through a Heisenberg interaction, with the interaction time depending on the rotation angle and on the spin size. Our work calls for the experimental realization of programmable setups that achieve the ultimate quantum limits to the control of rotation gates. For small values of the spin, a possible testbed is provided by NMR systems, where spin-spin interactions are naturally available [41]. Another possibility is to use quantum dots, where one can engineer a coupling between a single spin and an assembly of spins effectively behaving as a single spin \( j \) particle [42]. This scenario, named the box model, can be achieved through a uniform coupling of a central spin to the neighbouring sites. No matter what platform is used, our results provide the rigorous benchmark that can be used to validate the successful demonstration of quantum-enhanced programmable gates.

On the fundamental side, our work unveils a deep relation between the classical and quantum approaches to programming. In the classical approach, the target gate is approximated by a sequence of elementary gates, whose number grows as \( \log 1/\epsilon \) with the error parameter \( \epsilon \) [43]. At the leading order, the number of gates is equal to the number of bits used by the classical program that describes the target gate. In the quantum approach, we found that the target gate is approximated with error \( \epsilon = O(1/j) \), implying that the number of program qubits needed to achieve error \( \epsilon \) scales as \( \log(2j + 1) = O(\log 1/\epsilon) \). In other words, our result shows that the classical and quantum approaches are asymptotically equivalent in terms of tradeoff between accuracy and size. It is worth noticing, however, that our quantum strategies were constructed from a symmetry assumption, namely that the rotations in space are reflected into rotation of the program states [cf. Eq. [1]]. Although physically reasonable, this assumption may be lifted, by allowing arbitrary encodings of the target dynamics in the program states. Removing the symmetry assumption [1] might in principle lead to an improved size-accuracy tradeoff beyond the \( O(1/j) \) scaling observed in this paper. Determining whether this is the case requires the development of new techniques beyond the scope of the present investigation. While we expect the scaling \( O(1/j) \) to be remain optimal even with general encodings, we believe that the development of techniques to tackle the question has the potential to reveal new relations between quantum programming, quantum metrology, and quantum simulations.

Acknowledgements. This work is supported by the Hong Kong Research Grant Council through Grant No. 17326616 and 17300317, by National Science Foundation of China through Grant 11675136, by the HKU Seed Funding for Basic Research, and by the Canadian Institute for Advanced Research (CIFAR).

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Appendix A: Optimal quantum fidelity

The fidelity $F(j, \theta)$ in Eq. (3) of the main text is the result over two averages: the average over all pure states and the average over all rotations axes. Quite conveniently, the average over the states can be eliminated by using the well-known relation with the entanglement fidelity \[ F^{(e)}(j, \theta, n) \]
where $F^{(e)}(j, \theta, n)$ is the entanglement fidelity, given by
\[ F^{(e)}(j, \theta, n) = \langle \Phi^{+}_{\theta, n} | [ (\mathcal{C}_{\theta} \otimes \mathcal{I}) (\phi_{n} \otimes \Phi^{+}) ] | \Phi^{+}_{\theta, n} \rangle. \]  \[ (A2) \]
Here $\Phi^{+}$ denotes the projector on the canonical maximally entangled state $[\Phi^{+} \equiv ((0)|0) + (1)|1])/\sqrt{2}$ and $[| \Phi^{+}_{\theta, n} \rangle \equiv (V_{\theta} \otimes I) |\Phi^{+} \rangle]$. Using Eq. (A1), the average fidelity can be rewritten as
\[ F(j, \theta) = \frac{1}{3} + \frac{2}{3} F^{(e)}(j, \theta), \] \[ (A3) \]
where $F^{(e)}(j, \theta)$ is the average entanglement fidelity
\[ F^{(e)}(j, \theta) = \int d\psi F(j, \theta, n, \psi) = \frac{1}{3} + \frac{2}{3} F^{(e)}(j, \theta, n), \]
where $F^{(e)}(j, \theta, n)$ is the entanglement fidelity for channel $C_{\theta}$.

The fidelity can be conveniently rewritten using the “double ket notation”
\[ |\Psi\rangle := \sum_{n=1}^{\infty} \sum_{n} \langle m | \Psi | n \rangle | m \rangle | n \rangle. \] \[ (A5) \]
Denoting by $C_{\theta}$ the Choi operator for channel $C_{\theta}$, we obtain
\[ F^{(e)}(j, \theta) = \frac{1}{2} \int d\mathbf{\psi} \text{Tr} \left[ C_{\theta} | \tilde{\phi}_{\theta} \rangle \langle \tilde{\phi}_{\theta} | \otimes | \Phi^{+}_{\theta, n} \rangle \langle \Phi^{+}_{\theta, n} | \right] \]
\[ = \frac{1}{4} \langle \tilde{\phi} | \langle V_{\theta} | C_{\theta} \otimes \mathcal{I} | V_{\theta} \rangle \rangle, \] \[ (A6) \]
having defined
\[ C_{\theta}^{*} := \int d\mathbf{\psi} \left( U^{(j)}_{n}^{T} \otimes U^{T}_{n} \otimes U^{T}_{n} \right) C_{\theta} \left( U^{(j)}_{n} \otimes U_{n} \otimes \mathcal{I} \right). \] \[ (A7) \]
Furthermore, it is convenient to define the operator
\[ \tilde{C}_{\theta} = \langle \sigma^{(j)}_{y} \otimes \mathcal{I} \otimes \sigma_{y} \rangle C_{\theta}^{*} \left( \sigma^{(j)}_{y} \otimes \mathcal{I} \otimes \sigma_{y} \right). \] \[ (A8) \]
With this definition, it is easy to prove the relation
\[ \left[ \tilde{C}_{\theta}, U^{(j)}_{g} \otimes U_{g} \otimes U_{g} \right] = 0, \quad \forall g \in SU(2). \] \[ (A9) \]
Now, using Schur’s lemma we obtain the expression
\[
\tilde{C}_\theta = \alpha P_{j+1} + \beta P_{j-1} + P_j \otimes M_j ,
\] (A10)
where \( P_j \) is the projection on the subspace with total angular momentum \( l \), \( \alpha, \beta, \gamma \) are complex coefficients, and \( M_j = \left( \begin{array}{cc} \gamma_A & \gamma_B \\ \gamma_C & \gamma_D \end{array} \right) \) is a non-negative matrix. The trace-preserving condition on the channel is equivalent to the constraint
\[
\text{Tr}_{\text{target}}[C_\theta] = \text{I}_{\text{control}} \otimes \text{I}_{\text{target}} ,
\] (A11)
on the Choi operator. In terms of the coefficient, this implies the condition
\[
\begin{cases}
2j + 3 & \alpha + \frac{2j + 1}{2j + 2} \gamma_A = 1 \\
2j - 1 & \beta + \frac{2j + 1}{2j} \gamma_D = 1 .
\end{cases}
\] (A12)

Now, we insert the Eqs. (A10) and (A12) into the expression of the fidelity [Eq. (A6)]. After a long calculation using Clebsch-Gordan coefficients we can get the expression
\[
F^{(\theta)}(j, \theta) = A \sin^2 \frac{\theta}{2} + B \cos^2 \frac{\theta}{2} + C \bar{J}_z \sin \frac{\theta}{2} \cos \frac{\theta}{2} + D \bar{J}_z^2 \sin^2 \frac{\theta}{2} ,
\] (A13)
where \( \bar{J}_z \) (\( J_z^2 \)) is the average (of the square of the) \( z \)-component of the angular momentum, while the constants \( A, B, C, D \) are as follows:
\[
\begin{align*}
A &= \frac{1}{2(1 + 2j)} \left[ (j + 1) \alpha + j \beta \right] \\
B &= \frac{1}{2(1 + 2j)} \left[ (j + 1) \gamma_A + j \gamma_D \\
&\quad - \sqrt{j(j + 1)}(\gamma_B + \gamma_C) \right] \\
C &= \frac{1}{2 \sqrt{j(j + 1)}} \left( \gamma_B - \gamma_C \right) \\
D &= \frac{1}{2(1 + 2j)} \left[ -\frac{\alpha}{j + 1} - \frac{\beta}{j} \\
&\quad + \frac{\gamma_A}{j + 1} + \frac{\gamma_B + \gamma_C}{\sqrt{j(j + 1)}} \right] ,
\end{align*}
\]
For \( j \geq 3/2 \), taking into account that the maximum expectation value \( \bar{J}_z \) is equal to \( j \) and optimizing over the coefficients \( \alpha \) and \( \beta \) by \( \gamma_A \) and \( \gamma_D \) we obtain the optimal fidelity
\[
F^{(\theta)}_{\text{opt}}(j, \theta) = \frac{1}{(1 + 2j)^2} \left[ 2j^2 + \frac{2j + 1}{2} + \frac{2j + 1}{2} \cos \theta \\
+ j \sqrt{1 + 2(2j + 1) \cos \theta + (2j + 1)^2} \right] ,
\] (A14)
Note that the maximization of the expectation value \( \bar{J}_z \) requires the program state to be the spin-coherent state \( |j, \theta \rangle \).

Eq. (A14) gives the optimal value of the entanglement fidelity. The optimal value of the average fidelity can then be obtained from Eq. (A3), which yields
\[
F(j, \theta)_{\text{opt}} = \frac{1}{3} + \frac{2}{3(1 + 2j)^2} \left[ 2j^2 + \frac{2j + 1}{2} + \frac{2j + 1}{2} \cos \theta + j \sqrt{1 + 2(2j + 1) \cos \theta + (2j + 1)^2} \right] .
\] (A15)

The cases of \( j = 1/2 \) and \( j = 1 \) must be treated separately. In these two cases the optimal fidelity exhibits critical points, as illustrated in Figure 4.

FIG. 4. Optimal quantum fidelity for \( j = 1/2 \) and \( j = 1 \). For \( j = 1/2 \), a transition occurs when \( |\theta - \pi| = 2 \arctan \sqrt{4 + \sqrt{7}} \). For \( j = 1 \), the transition occurs when \( |\theta - \pi| = 23/100 \pi \).

For \( j = 1/2 \), the optimal fidelity is
\[
F_{\text{opt}}(j = 1/2, \theta) = \frac{1}{3} + \frac{1}{6} \left[ \frac{2 + 7 \cos \theta}{6 + 12 \cos \theta} - \frac{\cos \theta}{2} \right] \] (A16)
for \( |\theta - \pi| \leq 2 \arctan \sqrt{4 + \sqrt{7}} \), and
\[
F_{\text{opt}}(j = 1/2, \theta) = \frac{1}{3} + \frac{3 + 2 \cos \theta + \sqrt{5 + 4 \cos \theta}}{12}
\] (A17)
otherwise. For \( j = 1 \), the transition happens for \( |\theta - \pi| \leq \delta \), with \( \delta \approx 0.23 \pi \). The optimal fidelity is
\[
F_{\text{opt}}(j = 1, \theta) = \frac{1}{3} + \frac{2 \sin^2 \theta}{5}
\] (A18)
for \( |\theta - \pi| \leq \delta \), and
\[
F_{\text{opt}}(j = 1, \theta) = \frac{1}{3} + \frac{7 + 3 \cos \theta + \sqrt{10 + 6 \cos \theta}}{27}
\] (A19)
otherwise. Quite surprisingly, optimal program state for \( |\theta - \pi| \leq \delta \) is not the spin coherent state, but rather a \( p \)-orbital \( |1, 0 \rangle_n \) pointing in the direction of the rotation axis.
Appendix B: Optimal measure-and-operate fidelity

In a generic MO strategy, the measurement is described by a Positive Operator-Valued Measure (POVM) \( \{P_x \}_{x \in X} \), where \( X \) is the set of all possible outcomes and \( P_x \) is the positive operator associated to the outcome \( x \). The conditional operation is described by quantum channel \( C_{\theta,x} \) acting on the data qubit. On average over all possible outcomes, the action of the classical protocol is represented by the MO channel

\[
C_{\theta,\text{MO}}(\rho) = \sum_{x \in X} \text{Tr}[P_x \phi_n] C_{\theta,x}(\rho). \tag{B1}
\]

When the data qubit is in the state \( |\psi\rangle \), the fidelity between the output of the MO channel and the desired output is

\[
F_{\text{MO}}(j, \theta, n, \psi) = \langle \psi | V^\dagger_{\theta,n} [C_{\theta,\text{MO}}(\phi_n \otimes |\psi\rangle)] V_{\theta,n} |\psi\rangle. \tag{B2}
\]

Now, let \( F_{\text{MO}}(j, \theta) \) be the average of the fidelity over all input states and over all possible rotation axes. Using the relation with the entanglement fidelity, we obtain

\[
F_{\text{MO}}(j, \theta) = \frac{1}{3} + \frac{2}{3} F_{\text{MO}}^{(e)}(j, \theta), \tag{B3}
\]

with

\[
F_{\text{MO}}^{(e)}(j, \theta) = \sum_x \int d\theta \langle \psi | U_{\theta,n} C_{\theta,x} [V_{\theta,n}] |\phi_n \rangle P_x |\phi_n\rangle, \tag{B4}
\]

where \( C_{\theta,x} \) is the Choi operator of the channel \( C_{\theta,x} \). The entanglement fidelity can be rewritten as

\[
F_{\text{MO}}^{(e)}(j, \theta) = \sum_x \int d\theta \langle \psi | U_{\theta,n} C_{\theta,x} [U_{\theta,n}] \rangle \frac{\langle \phi_n | P_x |\phi_n\rangle}{4}, \tag{B5}
\]

At this point, we define the operators

\[
P_g^{(x)} = (2j + 1) U_g^{(j)}(P_x) / \text{Tr}[P_x] \tag{B6}
\]

and we note that they satisfy the normalization condition

\[
\int d\theta P_g^{(x)} = I \quad \forall x \in X. \tag{B7}
\]

In other words, the operators \( \{P_g^{(x)}\} \) define a POVM with outcome \( g \). Similarly, we define the operators \( C_g^{(x)} = (U_g \otimes \bar{U}_g)(C_{\theta,x}) \) and note that each of them is the Choi operator of a quantum channel \( C_g \). Hence, the POVM with operators \( \{P_g^{(x)}\} \) and the conditional channels \( C_g \) form an MO strategy. The entanglement fidelity can then be rewritten as

\[
F_{\text{MO}}^{(e)}(j, \theta) = \sum_x p_x F_{\text{MO}}^{(e)}(j, \theta, x), \tag{B8}
\]

where \( p_x \) is the probability \( p_x := \text{Tr}[P_x]/(2j + 1) \) and \( F_{\text{MO}}^{(e)}(j, \theta, x) \) is the fidelity of the \( x \)-th MO strategy. Hence, we obtain the bound

\[
F_{\text{MO}}^{(e)}(j, \theta) \leq \max_x F_{\text{MO}}^{(e)}(j, \theta, x)
= (2j + 1) \max_{C, \rho, \phi} \int d\theta \frac{\langle \psi | C_g [U_{\theta,n}] \rangle \langle \phi | \rho_g \rangle \phi}{4},
\]

where the maximization runs over all Choi operators \( C \) representing qubit channels and all density matrices \( \rho \) representing spin-\( j \) states, and we defined \( C_g = (U_g \otimes \bar{U}_g)(C_{\theta,x}) \) and \( \rho_g = U_{g,3}(P_x) \).

Using the relation \( U_{g} = \sigma_y U_g \sigma_y \), the bound on the fidelity becomes

\[
F_{\text{MO}}^{(e)}(j, \theta) \leq (2j + 1) \max_{C, \rho, \phi} \text{Tr}[(C \otimes \rho) \Omega], \tag{B9}
\]

with

\[
\Omega = \int d\theta \left( U_{\theta} \otimes \bar{U}_{\theta} \otimes U_{g,3}^{(j)} \right) \frac{[|W_{\theta}\rangle \langle W_{\theta}| \otimes \phi]}{4} \tag{B10}
\]

and \( W_{\theta} = \cos \frac{\theta}{2} \sigma_y + \sin \frac{\theta}{2} \sigma_x \). Note that the maximization can be restricted to pure states, of the form \( \rho = |\psi\rangle \langle \psi| \).

Now, the vector \( |W_{\theta}\rangle \) can be rewritten as

\[
|W_{\theta}\rangle = \sqrt{2} \left[ \cos \frac{\theta}{2} |0,0\rangle + \sin \frac{\theta}{2} |1,0\rangle \right], \tag{B11}
\]

having used the notation \( |l,m\rangle \) for the eigenstates of the \( z \)-component of the total spin. Using this fact, we obtain that every vector \( (U_{\theta} \otimes U_{\theta}) |W_{\theta}\rangle \) can be expanded in the basis \( B = \{ |0,0\rangle, |1,0\rangle, |1,0\rangle, |1,0\rangle \} \) with

\[
|1,0\rangle_z = |1,0\rangle,
|1,0\rangle_x = \frac{|1,1\rangle + i|1,1\rangle}{\sqrt{2}},
|1,0\rangle_y = \frac{|1,1\rangle - i|1,1\rangle}{\sqrt{2}}, \tag{B12}
\]

and all the expansion coefficients are real. Hence, the Choi operator \( C \) in Eq. \( \text{(B9)} \) can be chosen to have real matrix elements in the same basis. Moreover, we can restrict the maximization to the Choi operators \( C \) that are extreme points of the convex set of Choi operators with real matrix elements in this basis. Using Choi’s characterization of the extreme points \( \{(B5)-(B13)\} \), we find out that the extreme real Choi operators are rank-one—that is, they represent unitary gates.

Explicitly, we can write the Choi operator as \( C = |V_x\rangle \langle V_x| \), with

\[
|V_x\rangle = \sqrt{2} \left[ \cos \frac{\tau}{2} |0,0\rangle + \sin \frac{\tau}{2} \left( r_x |1,0\rangle + r_y |1,0\rangle + r_z |1,0\rangle_z \right) \right], \tag{B13}
\]
where $\tau$ is an angle and $r = (r_x, r_y, r_z)^T$ is a unit vector in $\mathbb{R}^3$. Now, there must exist a rotation $h$ that transforms the vector $r$ into the $z$ axis. For this particular rotation, we have

$$
(U_h \otimes U_h) |V_r\rangle = \sqrt{2} \left[ i \cos \frac{\tau}{2} |0,0\rangle + \sin \frac{\tau}{2} |1,0\rangle \right] = |W_r\rangle.
$$

(B14)

Since the operator $\Omega$ in Eq. (B10) is invariant under rotations, the bound on the fidelity becomes

$$
F_{\text{MO}}^{(c)}(j, \theta) \leq (2j + 1) \max_{\tau} \langle \Omega |W_r\rangle |\psi\rangle.
$$

(B15)

Now, note that we have

$$
\langle W_r | U_g \otimes U_g |W_\theta\rangle = 2 \cos \frac{\tau}{2} \cos \frac{\theta}{2} + 2 \sin \frac{\tau}{2} \sin \frac{\theta}{2} (1,0)|U_g^{(1)}|1,0\rangle.
$$

(B16)

Using the above relation, we obtain

$$
\langle \Omega |W_r\rangle |\psi\rangle = \int d\theta \left| \langle W_r | U_g \otimes U_g |W_\theta\rangle \right|^2 |\psi\rangle |U_g^{(j)}\rangle = 2 \cos \frac{\tau}{2} \cos \frac{\theta}{2} + 2 \sin \frac{\tau}{2} \sin \frac{\theta}{2} (1,0)|U_g^{(1)}|1,0\rangle.
$$

(B17)

By direct inspection, we find that the above expression reaches its maximum for $m = m' = j$. Moreover, we find that the maximum over the angle $\tau$ is attained for

$$
cot \tau = \frac{(2j^2 + 3j + 2) \cos \theta + 2j + 1}{(2j^2 + 3j) \sin \theta}, \quad \theta \in [0, \pi].
$$

(B22)

For this value of $\tau$, we obtain the maximum fidelity

$$
F_{\text{MO}}^{(c)}(j, \theta) = \frac{(2j + 1)(1 + \cos(\theta - \tau))}{2(j + 3)} + \frac{(2j + 1)(\cos \theta + \cos \tau) + \cos(\theta + \tau) + 1}{3(j + 1)(2j + 3)}.
$$

(B23)

In terms of average input-output fidelity, we obtain the value

$$
F_{\text{MO}}(j, \theta) = \frac{(2j + 1)(1 + \cos(\theta - \tau))}{2(j + 3)} + \frac{(2j + 1)(\cos \theta + \cos \tau) + \cos(\theta + \tau) + 1}{3(j + 1)(2j + 3)}.
$$

(B24)

The maximum fidelity is achieved by using the program state $U_g^{(m)|j, j\rangle}$, measuring the coherent state POVM $\{P_{n'} = U_{g^{(n')}}|j, j\rangle\langle j, j|U_{g^{(n')}}\}$, and rotating around the axis $n'$ of the angle $\tau$ determined by Eq. (B22). Note that the angle $\tau$ converges to $\theta$ in the large $j$ limit.

Appendix C: Longevity of the quantum advantage

The state of the control spin after the interaction can be obtained by application of the complementary channel $\tilde{C}_\theta$, defined by

$$
\tilde{C}_\theta(\rho^{(j)}) = \text{Tr}_{\text{target}} \left[ U_\theta \left( \rho^{(j)} \otimes \frac{I}{2} \right) U_\theta^\dagger \right].
$$

(C1)

To evaluate this state, it is convenient to look at the evolution of the basis states $|j, m\rangle_n$. By explicit calculation, we obtain the relation

$$
\tilde{C}_\theta(|j, m\rangle_n|j, m\rangle_n) = \sum_{i=-1}^1 c_{m+i,m} |j, m+i\rangle \langle j, m+i|_n,
$$

(C2)

where the coefficients $c_{m+i,m}$ are given by

$$
c_{m-1,m} = \frac{(j + m)(1 + j - m)}{(1 + 2j)^2} (1 - \cos \theta - \frac{\sin^2 \theta}{2j}),
$$

$$
c_{m,m} = 1 - c_{m-1,m} - c_{m+1,m},
$$

$$
c_{m+1,m} = \frac{(j - m)(1 + j + m)}{(1 + 2j)^2} (1 - \cos \theta - \frac{\sin^2 \theta}{2j}).
$$

At the first step, the program starts in the state $|j, j\rangle_n$. By repeatedly applying Eq. (C1), we then obtain the program state at every step. Explicitly, the state at the $n$-th
The average fidelity $F(j, \theta, n)$ can be computed in terms of the entanglement fidelity, using the relation

$$F(j, \theta, n) = \frac{1}{3} + \frac{2}{3} F_{\text{e}}(j, \theta, m),$$  \hspace{1cm} (C6)

where $F_{\text{e}}(j, \theta, m)$ is the fidelity when using $|j, m\rangle_n$ as program state.

One can see directly that in asymptotics, $F(j, \theta, m)$ is a geometric progression and $p(n, m, \theta)$ is a geometric progression. Inserting the above expressions into Eq. (C5), we obtain

$$F(j, \theta, n) = 1 - \frac{1 - \cos \theta}{3j} \cdot \frac{n(1 - \cos \theta) + j}{j} + O \left( \frac{1}{j^2} \right).$$  \hspace{1cm} (C11)

Comparing with the MO fidelity, we obtain that the longevity of the quantum advantage tends to $L(j, \theta) = \frac{j}{1 - \cos \theta}$.

We showed the explicit calculation of $F(j, \theta, m)$ and $p(n, m, \theta)$ when the interaction time is fixed at every step. More general strategies where the interaction time is optimized at every step can be studied in the same way. In the large $j$ limit, we find that such step-by-step optimization is not needed: the fidelity tends to the same value, no matter whether the interaction time is optimized at every step or once for all. As a result, the longevity is the same in both scenarios.

Appendix D: Controlling spin-$k$ particles

Following the structure of the optimal control mechanism for spin $1/2$, we choose the program state to be $|j, j\rangle_n$ and we let the two spins undergo the unitary gate

$$U_{\theta}^{(k)} = \exp \left[ -i \theta \frac{2J \cdot K}{2j+1} \right].$$  \hspace{1cm} (D1)
Using the above strategy, we can explicitly compute the entanglement fidelity, given by

\[
F^{(e)}(j, k, \theta) = \int dn \ F_{\nu}(j, k, \theta, n)
= \int dn \langle \Phi_{\theta, n}^{(k)+} | \text{Tr}_{\text{control}} \left( (U_{\theta} \otimes I)(\phi_{n}^{(j)} \otimes \Phi_{\theta, n}^{(k)+}) \right) | \Phi_{\theta, n}^{(k)+} \rangle.
\]

(D2)

Here \(\Phi_{\theta, n}^{(k)+}\) denotes the projector on the canonical maximally entangled state and \(|\Phi_{\theta, n}^{(k)+}\rangle\) is the rotated maximally entangled state defined by

\[
|\Phi_{\theta, n}^{(k)+}\rangle := (V_{\theta, n}^{(k)} \otimes I)|\Phi_{\theta, n}^{(k)+}\rangle.
\]

(D3)

Putting the formula of \(U_{\theta}^{(k)}\) in Eq. (D2), using the expressions of the Clebsch-Gordan coefficients, we arrive to the asymptotic expression

\[
F^{(e)}(j, k, \theta) = 1 - \frac{2(k+1)}{3j}(1 - \cos \theta) + O \left( \frac{1}{j^2} \right).
\]

(D4)

The average fidelity is then given by

\[
F(j, k, \theta) = 1 - \frac{k(2k+1)}{3j}(1 - \cos \theta) + O \left( \frac{1}{j^2} \right).
\]

(D5)

A similar calculation can be done for the MO strategy consisting in measuring the direction with the coherent state POVM and then performing the conditional operation \(V_{\theta, n}^{(k)}\) on the target. The entanglement fidelity of this strategy is given by

\[
F_{\text{MO}}^{(e)}(j, k, \theta) = \int dn \int dn' \text{Tr} [P_{n'}\phi_{n'}] F^{(e)}(j, k, \theta, n, n')
\]

with

\[
F^{(e)}(j, k, \theta, n, n') = \frac{1}{(2k+1)^2} \text{Tr} \left[ V_{\theta, n}^{(k)} V_{\theta, n'}^{(k)} \right]^2.
\]

(D6)

By denoting \(\varphi\) as the angle between axis \(n\) and \(n'\), and by \(\tau\) the rotation angle for the rotation \(V_{\theta, n}^{(k)}\), the entanglement fidelity can be rewritten as

\[
F_{\text{MO}}^{(e)}(j, k, \theta) = \frac{\int_{0}^{\pi} d\varphi \sin \varphi (\cos \varphi)^{4j} \sin^2 \frac{(2k+1)}{2} \varphi}{(2k+1)^2 \int_{0}^{\pi} d\varphi \sin \varphi (\cos \varphi)^{4j} \sin^2 \frac{\varphi}{2}}.
\]

(D7)

Performing the average, we obtain the asymptotic expression

\[
F_{\text{MO}}^{(e)}(j, k, \theta) = 1 - \frac{4k(2k+1)}{3j}(1 - \cos \theta) + O \left( \frac{1}{j^2} \right),
\]

(D8)

which can then be used to evaluate the average fidelity as

\[
F_{\text{MO}}(j, k, \theta) = 1 - \frac{2k(2k+1)}{3j}(1 - \cos \theta) + O \left( \frac{1}{j^2} \right).
\]

(D9)