FINITE-DIMENSIONAL REPRESENTATIONS OF HYPER LOOP ALGEBRAS

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Abstract: We study finite-dimensional representations of hyper loop algebras, i.e., the hyperalgebras over an algebraically closed field of positive characteristic associated to the loop algebra over a complex finite-dimensional simple Lie algebra. The main results are the classification of the irreducible modules, a version of Steinberg’s Tensor Product Theorem, and the construction of positive characteristic analogues of the Weyl modules as defined by Chari and Pressley in the characteristic zero setting. Furthermore, we start the study of reduction modulo $p$ and prove that every irreducible module of a hyper loop algebra can be constructed as a quotient of a module obtained by a certain reduction modulo $p$ process applied to a suitable characteristic zero module. We conjecture that the Weyl modules are also obtained by reduction modulo $p$. The conjecture implies a tensor product decomposition for the Weyl modules which we use to describe the blocks of the underlying abelian category.

Introduction

Let $G$ be a semisimple connected algebraic group over an algebraically closed field $\mathbb{F}$. One can associate to $G$ its Lie algebra $L(G)$ and its algebra of distributions $U(G)$, which we prefer to call the hyperalgebra of $G$. If $\mathbb{F}$ is of characteristic zero, the hyperalgebra coincides with the universal enveloping algebra $U(L(G))$ of $L(G)$, but this is not so in positive characteristic. $U(G)$ acts naturally on any $G$-module and it turns out that, as conjectured originally by Verma and proved by Sullivan [35], every finite-dimensional $U(G)$-module can be “lifted” to a rational finite-dimensional $G$-module. We will restrict our attention to the case when $G$ is the Chevalley group of adjoint type associated to a complex finite-dimensional simple Lie algebra $\mathfrak{g}$. In this case the algebra $U(G)$ is isomorphic to the algebra $U(\mathfrak{g})_{\mathbb{F}}$ constructed by considering Kostant’s integral form of $U(\mathfrak{g})$ and tensoring with $\mathbb{F}$ over $\mathbb{Z}$. It will suffice, for our purposes, to work over the purely algebraic setting of $U(\mathfrak{g})_{\mathbb{F}}$.

Let $\mathfrak{g}$ be as above and $\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$ the loop algebra over $\mathfrak{g}$. The finite-dimensional representation theory of $\hat{\mathfrak{g}}$ has been a very active research topic in the last decades. It is related, for instance, to integrable models and the Bethe ansatz in statistical mechanics. In [18], Garland introduced an integral form of $U(\hat{\mathfrak{g}})$ which can be used to construct what we call the hyper loop algebra $U(\hat{\mathfrak{g}})_{\mathbb{F}}$ of $\mathfrak{g}$ over $\mathbb{F}$ (see also [36, 30]). The hyperalgebra $U(\mathfrak{g})_{\mathbb{F}}$ is naturally a subalgebra of $U(\hat{\mathfrak{g}})_{\mathbb{F}}$.

The purpose of the present paper is to study some basic aspects of the category $\tilde{C}_\mathbb{F}$ of finite-dimensional $U(\hat{\mathfrak{g}})_{\mathbb{F}}$-modules such as the classification of its simple objects and its block decomposition when $\mathbb{F}$ is an algebraically closed field of positive characteristic. In the case $\mathbb{F} = \mathbb{C}$, thus $U(\hat{\mathfrak{g}})_{\mathbb{F}} = U(\hat{\mathfrak{g}})$, these questions were studied in [44, 7, 9]. It turns out that the simple finite-dimensional $\hat{\mathfrak{g}}$-modules are highest-weight modules with respect to the triangular decomposition of $\mathfrak{g}$ obtained by “looping” the usual triangular decomposition of $\mathfrak{g}$. As usual, we will call them $\ell$-highest-weight modules to distinguish from those which are highest-weight with respect to the triangular decomposition coming from the Chevalley generators of $\hat{\mathfrak{g}}$ (non-trivial highest-weight representations with respect to the later

Keywords: Loop algebras, finite-dimensional representations, hyperalgebras
decomposition are infinite-dimensional). Moreover, all the simple modules are isomorphic to suitable tensor products of the so-called evaluation representations (obtained by pulling back the simple \( \mathfrak{g} \)-modules by the evaluation map \( t \mapsto a \) for some nonzero \( a \in \mathbb{C} \)). We prove that these two results hold in positive characteristic, as well. This is done in Corollary 3.8 and Theorem 3.9, the later being a \( U(\hat{\mathfrak{g}}) \)-version of Steinberg’s Tensor Product Theorem. Using the tensor product theorem we compute the dual representation of a given irreducible one. For highest-weight representations with respect to the usual triangular decomposition in positive characteristic see [18, 19, 28, 29, 37] and references therein.

The set of \( \ell \)-highest weights can be identified with \( \text{rank}(\mathfrak{g}) \)-tuples of polynomials in \( \mathbb{F}[u] \) with constant term 1. For \( \mathbb{F} = \mathbb{C} \), it was shown in [11] that there exists a family of universal finite-dimensional \( \ell \)-highest-weight modules, called the Weyl modules. We prove that the Weyl modules for \( U(\hat{\mathfrak{g}}) \) can be defined in a similar fashion when \( \mathbb{F} \) is of positive characteristic. The reason for calling these \( \ell \)-highest-weight modules Weyl modules comes from a conjecture in [11] stating that the Weyl modules for \( U(\hat{\mathfrak{g}}) \) can be obtained as the classical limit of certain irreducible finite-dimensional modules for the corresponding quantum loop algebra, resembling the process of obtaining the Weyl modules for \( U(\hat{\mathfrak{g}}) \) by reduction modulo \( p \) of simple \( \mathfrak{g} \)-modules. This conjecture has been recently proved when \( \mathfrak{g} \) is of type \( A \) in [6] using Gelfand-Tsetlin filtrations and when \( \mathfrak{g} \) is of type \( ADE \) in [17] using Demazure modules. Moreover, H. Nakajima has pointed out that the general case can be deduced using the crystal and global basis results from [2, 23, 24, 31, 32]. Other interesting related references include [5, 16, 26, 27]. We have an analogous conjecture for the Weyl modules for \( U(\hat{\mathfrak{g}})_{\mathbb{F}} \) when \( \mathbb{F} \) is of positive characteristic (Conjecture 4.7(a)), stating that they can be obtained from the Weyl modules for \( U(\hat{\mathfrak{g}})_{\mathbb{F}^0} \) by reduction modulo \( p \), where \( \mathbb{F}^0 \) is a suitable field of characteristic zero. As \( \mathbb{Z} \)-lattices are easily seen not to be well-behaved with respect to evaluation maps, we consider more general lattices for this purpose. Namely, we consider lattices over the ring \( A \) of Witt vectors with coefficients in \( \mathbb{F} \), after changing scalars from \( \mathbb{C} \) to the fraction field \( \mathbb{F}^0 \) of \( A \). We prove that all finite-dimensional \( \ell \)-highest-weight \( U(\hat{\mathfrak{g}})_{\mathbb{F}^0} \)-modules whose coefficients of the \( \ell \)-highest weights are in \( A \) and the leading ones are units in \( A \) contain an admissible (ample) \( A \)-lattice. Thus, we obtain all of the irreducible modules as quotients of modules coming from a reduction modulo \( p \) process. This is done in Theorem 4.5 and Corollary 4.6. Combining Conjecture 4.7 with the one in [11], which is now a theorem as remarked above, we have a bridge connecting the Weyl modules for \( U(\hat{\mathfrak{g}})_{\mathbb{F}} \) with certain irreducible representations for quantum loop algebras (at generic quantization parameter).

As corollaries of Conjecture 4.7 we obtain a tensor product decomposition of the Weyl modules and the block decomposition of \( \hat{\mathfrak{c}}_{\mathbb{F}} \). Although this tensor product decomposition is the natural analogue of the one obtained in [11] for characteristic zero, the techniques used in that paper do not seem to apply to our setting. In fact, our motivation for considering the theory of reduction modulo \( p \) originated from the search for other methods which would lead to a proof of this tensor product decomposition. Indeed we expect that this decomposition holds in the context of \( A \)-lattices (Conjecture 4.7(b)), thus allowing us to transfer the problem to a characteristic zero setting. The block decomposition of \( \hat{\mathfrak{c}}_{\mathbb{F}} \) is described similarly to that of \( \hat{\mathfrak{c}}_{\mathbb{C}} \) as well, i.e., the blocks are parametrized by functions with finite support \( \chi : \mathbb{F}^\times \to P/Q \) called spectral characters. Here \( P \) and \( Q \) are the weight and root lattices of \( \hat{\mathfrak{g}} \), respectively, and \( \mathbb{F}^\times = \mathbb{F} - \{0\} \). The proof runs parallel to its characteristic zero counterpart found in [7], hence, the tensor product decomposition for Weyl modules plays a key role. However, our results on reduction modulo \( p \) are needed in order to both prove that the Weyl modules have a well-defined spectral characters and obtain a positive characteristic version of [7, Proposition 3.4] – a key ingredient in the construction of certain useful indecomposable modules.

The paper is organized as follows. In section 1 we fix the basic notation on finite-dimensional complex simple Lie algebras and their loop algebras, define the hyperalgebras, and collect some important
Multiplication establishes isomorphisms. The main part of the paper consists of sections 3 and 4. In 3.1 we define \( \ell \)-highest-weight modules and obtain the necessary relations satisfied by the finite-dimensional ones. The classification of the irreducible modules and the aforementioned tensor product and duality results are done in 3.2. The Weyl modules are constructed in 3.3. Section 4 ends the paper with the results and the conjecture on reduction modulo \( p \), as well as, their application to the description of the blocks.

Acknowledgements: D.J. is pleased to thank the Max-Planck-Institut für Mathematik in Bonn for its hospitality and support, FAPESP (processo 2006/00609-1) for supporting her visit to the University of Campinas when part of this paper was developed, and UNICAMP for its hospitality. The research of A.M. is partially supported by CNPq (processo 303349/2005-0) and FAPESP (processo 2006/00833-9). We thank V. Chari for turning our attention to this subject and for very helpful questions and suggestions. We also thank P. Russel, A. Engler, and P. Brumatti for useful discussions and pointers about discrete valuation rings and L. Scott for his interest and helpful references.

1. Hyperalgebras

1.1. Preliminaries. Let \( I \) be the set of vertices of a finite-type connected Dynkin diagram and let \( \mathfrak{g} \) be the associated simple complex Lie algebra with a fixed Cartan subalgebra \( \mathfrak{h} \) and nilpotent subalgebras \( \mathfrak{n}^\pm \). Denote by \( \mathfrak{R}^+ \) the set of positive roots so that
\[
\mathfrak{n}^\pm = \bigoplus_{\alpha \in \mathfrak{R}^+} \mathfrak{g}_{\pm\alpha}, \text{ where } \mathfrak{g}_{\pm\alpha} = \{x \in \mathfrak{g} : [h, x] = \pm \alpha(h)x, \forall h \in \mathfrak{h}\}.
\]
The simple roots will be denoted by \( \alpha_i \), the fundamental weights by \( \omega_i \), while \( Q, P, Q^+, P^+ \) will denote the root and weight lattices with corresponding positive cones, respectively. We equip \( \mathfrak{h}^* \) with the partial order \( \lambda \leq \mu \) iff \( -\mu + \lambda \in Q^+ \). The Weyl group will be denoted by \( \mathcal{W} \), its longest element by \( w_0 \), and the maximal positive root by \( \theta \). Let \( \langle , \rangle \) be the bilinear form on \( \mathfrak{h}^* \) induced by the Killing form on \( \mathfrak{g} \) and, for \( \lambda \in \mathfrak{h}^* - \{0\} \), set \( \lambda^\vee = 2\lambda/(\lambda, \lambda) \) and \( d_\lambda = \frac{1}{2}(\lambda, \lambda) \). Then \( \{\alpha_i^\vee : i \in I\} \) is a set of simple roots of the simple Lie algebra \( \mathfrak{g}^\vee \) whose Dynking diagram is obtained from that of \( \mathfrak{g} \) by reversing the arrows and \( R^\vee = \{\alpha^\vee : \alpha \in \mathfrak{R}\} \) is its root system, where \( R = \mathfrak{R}^+ \cup (-\mathfrak{R}^+) \). Moreover, if \( \alpha = \sum_i m_i \alpha_i \) and \( \alpha^\vee = \sum_i m_i^\vee \alpha_i^\vee \), then
\[
\text{m}_i^\vee = \frac{d_{\alpha_i}}{d_\alpha} \text{m}_i.
\]
S, and counit $\epsilon$. We shall denote by $U(a)^0$ the augmentation ideal, i.e., the kernel of $\epsilon$. Consider the associative $\mathbb{F}$-algebra $U(a) \otimes_{\mathbb{F}} \mathbb{F}[t, t^{-1}]$ with the obvious tensor product structure and the usual bracket. Clearly the inclusion $\tilde{a} \hookrightarrow U(a) \otimes_{\mathbb{F}} \mathbb{F}[t, t^{-1}]$ is a Lie algebra map. Therefore the next lemma is immediate from the universal property of $U(\tilde{a})$.

**Lemma 1.1.** There exists a unique algebra map $U(\tilde{a}) \to U(a) \otimes_{\mathbb{F}} \mathbb{F}[t, t^{-1}]$ which is the identity on $\tilde{a}$. □

We call the map given by this lemma the formal evaluation map and denote it by $ev$. For each $a \in \mathbb{F}^k$, consider the evaluation map $U(a) \otimes_{\mathbb{F}} \mathbb{F}[t, t^{-1}] \to U(a)$ sending $x \otimes f(t)$ to $f(a)x$ and denote by $ev_a$ the composition of this map with $ev$. Then $ev_a$ is a surjective algebra homomorphism

$$ev_a : U(\tilde{a}) \to U(a)$$

which we call the evaluation map at $a$.

**Remark.** Obviously, the existence of $ev_a$ can be proved similarly to the existence of $ev$. However, we will use the formal evaluation map in order to prove the existence of evaluation maps in the context of hyper loop algebras using $\mathbb{Z}$-lattices only (cf. Proposition 3.3 and the remark after Proposition 4.10).

### 1.2. Reduction Modulo $p$.

As usual, given any associative algebra $A$ over a field of characteristic zero, $a \in A$, and $k \in \mathbb{Z}$, we set $a^{(k)} = \frac{a^k}{k!}, (a)^{\alpha} = \frac{a(a-1)\cdots(a-k+1)}{k!} \in A$.

Let $\Phi = \{x^\pm_\alpha, h_\alpha : \alpha \in \mathbb{R}^+, i \in I\}$ be a Chevalley basis for $\mathfrak{g}$, where $x^\pm_\alpha \in \mathfrak{g}_{\pm \alpha}$, $h_\alpha = [x^+_\alpha, x^-_\alpha]$, and let $x^+_\alpha, r = x^+_\alpha \otimes t^r, h_\alpha, r = h_\alpha \otimes t^r$. When $r = 0$ we may just write $x^+_\alpha$ and $h_\alpha$. Also, we may write $x^+_\alpha, r$ and $h_\alpha, r$ in place of $x^+_\alpha, r$ and $h_\alpha, r$, respectively. Notice that the set $\tilde{\Phi} = \{x^\pm_\alpha, i \in I, r \in \mathbb{Z}\}$ is a basis for $\tilde{\mathfrak{g}}$ and define $\tilde{\mathfrak{g}}_{\mathbb{Z}}$ to be the $\mathbb{Z}$-span of $\tilde{\Phi}$. The $\mathbb{Z}$-span of $\tilde{\Phi}$ is a Lie $\mathbb{Z}$-subalgebra of $\tilde{\mathfrak{g}}_{\mathbb{Z}}$ which we denote by $\mathfrak{g}_{\mathbb{Z}}$.

If $\mathbb{F}$ is any field, set

$$\mathfrak{g}_F = \mathfrak{g}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{F} \quad \text{and} \quad \tilde{\mathfrak{g}}_F = \tilde{\mathfrak{g}}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{F},$$

so that $\mathfrak{g}_F$ and $\tilde{\mathfrak{g}}_F$ are Lie algebras over $\mathbb{F}$.

Given $\alpha \in \mathbb{R}^+, r \in \mathbb{Z}$, define elements $\Lambda_{\alpha, r} \in U(\tilde{\mathfrak{g}})$ by the following equality of formal power series in $u$:

$$\Lambda^\pm_\alpha(u) = \sum_{r=0}^{\infty} \Lambda_{\alpha, \pm r} u^r = \exp \left( - \sum_{s=1}^{\infty} \frac{h_\alpha, \pm s}{s} u^s \right).$$

We may write $\Lambda_{i, r}$ in place of $\Lambda_{\alpha, i, r}$. It follows from (1.1) that, if $\alpha = \sum_i m_i \alpha_i \in R^+$, then $h_\alpha = \sum_i m_i \gamma_i$ and

$$\Lambda^\pm_\alpha(u) = \prod_{i \in I} (\Lambda^\pm_{\alpha_i}(u))^{m_i \gamma_i}. \quad (1.3)$$

We have (cf. [15, Lemma 5.1]):

$$ev(\Lambda_{\alpha, r}) = (-1)^r \left( \frac{h_\alpha}{|r|} \right) \otimes t^r. \quad (1.4)$$

Set

$$H_\alpha(u) = ev_{-1}(\Lambda^+_\alpha(u)) = \exp \left( - \sum_{s \geq 1} \frac{h_\alpha}{s} \frac{(-u)^s}{s} \right), \quad (1.5)$$

so that $(H_\alpha(u))_k$ (the coefficient of $u^k$ in $H_\alpha(u)$) is $\binom{h_\alpha}{k}$. 

For \( k \in \mathbb{Z}, k \neq 0 \), consider also the endomorphism \( \tau_k \) of \( U(\tilde{\mathfrak{g}}) \) extending \( t \mapsto t^k \) and set \( \Lambda_{\alpha,r,k} = \tau_k(\Lambda_{\alpha,r}), \Lambda_{\alpha,r,k}(u) = \sum_{i=0}^{\infty} \Lambda_{\alpha,r,k}u^i \). Notice that \( (h_i^k) \) is a polynomial in \( h_i \) of degree \( k \). Hence, the set \\
\{\( (h_i^{k_1}) \cdots (h_i^{k_l}) : k_j \in \mathbb{Z}_+ \)\}, where \( l = |I| \), is a basis for \( U(\mathfrak{h}) \). Similarly, observe that \( \Lambda_{i,\pm r,k}, r,k \in \mathbb{N} \), is a polynomial in \( h_{i,\pm sk}, 1 \leq s \leq r \), whose leading term is \( (-h_{i,\pm k})^r \). Finally, given an order on \( \tilde{\mathfrak{g}} \) and a PBW monomial with respect to this order, we construct an ordered monomial in the elements \\
\( (x_{\alpha,r}^\pm)^{(k)}, \Lambda_{i,r,k}, (h_i^k), r,k \in \mathbb{Z}, k > 0, \alpha \in R^+, i \in I, \)
using the correspondence just discussed for the basis elements of \( U(\tilde{\mathfrak{h}}) \), as well as, the obvious correspondence \( (x_{\alpha,r}^\pm)^k \leftrightarrow (x_{\alpha,r}^\pm)^{(k)} \). The set of monomials thus obtained is then a basis for \( U(\tilde{\mathfrak{g}}) \), while the monomials involving only \( (x_{\alpha,r}^\pm)^{(k)}, (h_i^k) \) form a basis for \( U(\mathfrak{g}) \). Let \( U(\tilde{\mathfrak{g}})_\mathbb{Z} \) (resp. \( U(\mathfrak{g})_\mathbb{Z} \)) be the \( \mathbb{Z} \)-span of these monomials. The following crucial theorem was proved in [25] (\( U(\mathfrak{g}) \) case) and [18] (\( U(\tilde{\mathfrak{g}}) \) case).

**Theorem 1.2.** \( U(\tilde{\mathfrak{g}})_\mathbb{Z} \) (resp. \( U(\mathfrak{g})_\mathbb{Z} \)) is the \( \mathbb{Z} \)-subalgebra of \( U(\tilde{\mathfrak{g}}) \) generated by \( \{ (x_{\alpha,r}^\pm)^{(k)}, \alpha \in R^+, r,k \in \mathbb{Z}, k \geq 0 \} \) (resp. \( \{ (x_{\alpha}^\pm)^{(k)}, \alpha \in R^+, r, k \in \mathbb{Z}_+ \} \)).

For \( a \in \{ g, n^\pm, h, \tilde{\mathfrak{g}}, \tilde{n}^\pm \} \), let \( U(\mathfrak{a})_\mathbb{Z} \) denote the corresponding \( \mathbb{Z} \)-subalgebra of \( U(\tilde{\mathfrak{g}}) \). Given a field \( \mathbb{F} \), the \( \mathbb{F} \)-hyperalgebra of \( \mathfrak{a} \) is defined by \\
\( U(\mathfrak{a})_\mathbb{F} = U(\mathfrak{a})_\mathbb{Z} \otimes_\mathbb{Z} \mathbb{F}. \)

We will also refer to \( U(\tilde{\mathfrak{g}})_\mathbb{F} \) as the hyper loop algebra of \( \mathfrak{g} \) over \( \mathbb{F} \). Then the PBW theorem gives \\
\( U(\tilde{\mathfrak{g}})_\mathbb{F} = U(\mathfrak{n}^-)_\mathbb{F} U(\tilde{\mathfrak{h}})_\mathbb{F} U(\mathfrak{n}^+)_\mathbb{F} \) and \\
\( U(\tilde{\mathfrak{g}})_\mathbb{F} = U(\tilde{\mathfrak{h}})_\mathbb{F} U(\tilde{\mathfrak{g}})_\mathbb{F} \).

Clearly, if \( \mathbb{F} \) is of characteristic zero then \( U(\mathfrak{a})_\mathbb{F} \) is naturally isomorphic to \( U(\mathfrak{a})_\mathbb{F} \). For fields of positive characteristic we just have an algebra homomorphism \( U(\mathfrak{a})_\mathbb{F} \to U(\mathfrak{a})_\mathbb{F} \) which is neither injective nor surjective. If no confusion arises, we will write \( x \), instead of \( x \otimes 1 \), for the image of an element \( x \in U(\tilde{\mathfrak{g}})_\mathbb{Z} \) in \( U(\tilde{\mathfrak{g}})_\mathbb{F} \).

Quite clearly the Hopf algebra structure on \( U(\tilde{\mathfrak{g}}) \) preserves the \( \mathbb{Z} \)-forms \( U(\mathfrak{a})_\mathbb{Z} \) and, therefore, induces a Hopf algebra structure on \( U(\tilde{\mathfrak{g}})_\mathbb{F} \) with counit given by \( \epsilon((x_{\alpha,r}^\pm)^{(k)}) = 0, \epsilon(a) = a, a \in \mathbb{F}, \) and comultiplication given by

\begin{equation}
\Delta((x_{\alpha,r}^\pm)^{(k)}) = \sum_{l+m=k} (x_{\alpha,r}^\pm)^{(l)} \otimes (x_{\alpha,r}^\pm)^{(m)},
\end{equation}

\begin{equation}
\Delta(h_i^k) = \sum_{l+m=k} (h_i^l) \otimes (h_i^m), \text{ and } \Delta(\Lambda_{\alpha,\pm k}) = \sum_{l+m=k} \Lambda_{\alpha,\pm l} \otimes \Lambda_{\alpha,\pm m}.
\end{equation}

Moreover, the antipode on the basis of \( U(\tilde{\mathfrak{h}})_\mathbb{F} \) is determined by \\
\( S(\Lambda_{\alpha,k}^\pm(u)) = (\Lambda_{\alpha,k}^\pm(u))^{-1} \) and \\
\( S(H_\alpha(u)) = (H_\alpha(u))^{-1}, \)
where the inverses in the last two equations are the ones of formal power series.

### 1.3. Some Lemmas.

We now collect some essential identities on \( U(\tilde{\mathfrak{g}})_\mathbb{F} \), when \( \mathbb{F} \) is a field of characteristic \( p > 0 \). We begin with the following trivial observation:

\begin{equation}
(x_{\alpha,r}^\pm)^{(k)}(x_{\alpha,r}^\pm)^{(l)} = \binom{k+l}{k}(x_{\alpha,r}^\pm)^{(k+l)}.
\end{equation}

From this, one easily deduces

\begin{equation}
((x_{\alpha,r}^\pm)^{(k)})^p = 0.
\end{equation}
It is well-known (see [21]) that the elements \( \frac{h_i}{k} \) satisfy
\[
(1.11) \quad \left( \frac{h_i}{k} \right)^p = \left( \frac{h_i}{k} \right)
\]
and it is easy to see that we have
\[
(1.12) \quad \left( \frac{h_i}{l} \right)(x_{\alpha,r}^\pm)^{(k)} = (x_{\alpha,r}^\pm)^{(k)} \left( \frac{h_i \pm k\alpha(h_i)}{l} \right).
\]

Given \( \alpha \in R^+, s \in Z \), define
\[
X_{\alpha;s,\pm}(u) = \sum_{r \geq 1} x_{\alpha,\pm(r+s)}^{-} u^r.
\]

**Lemma 1.3.** We have:
\[
(1.13) \quad (x_{\alpha}^+)^{(l)}(x_{\alpha}^-)^{(k)} = \min\{k,l\} \sum_{m=0} \left( x_{\alpha}^- \right)^{(k-m)} \left( h_{\alpha} - k - l + 2m \right) \frac{k}{m} \left( x_{\alpha}^+ \right)^{(l-m)},
\]
and
\[
(1.14) \quad (x_{\alpha,\pm s}^+)^{(l)}(x_{\alpha,\pm(s+1)}^-)^{(k)} \in (-1)^l \left( (X_{\alpha;\pm,\pm}(u))^{(k-l)} \Lambda_{\alpha}^\pm(u) \right)_k + U(\mathfrak{g}) U(\mathfrak{n}^+)^0.
\]

In (1.14), \( k \geq l \geq 1 \) and the subindex \( k \) means the coefficient of \( u^k \) of the above power series.

**Proof.** It suffices to prove that the relations hold in \( U(\mathfrak{g})_Z \). For both claims the strategy is to commute the elements on the left hand side. The proof of (1.13) can be found in [20, Lemma 26.2].

Relation (1.14) was proved in [18, Lemma 7.5] for \( s = 0 \) and the choice of “±” such that we have “+” on the right hand side. Consider the subalgebra of \( U(\mathfrak{g})_Z \) generated by \( (x_{\alpha,r}^\pm)^{(k)} \) for a fixed \( \alpha \in R^+ \). It is easy to see that, for each \( s \in Z \), the assignment \( (x_{\alpha,r}^\pm)^{(k)} \mapsto (x_{\alpha,r+\pm s}^\pm)^{(k)} \) extends uniquely to an algebra automorphism of this subalgebra which is the identity when restricted to \( U(\mathfrak{h})_Z \). The general case of (1.14) (with “+” on the right hand side) follows easily from the case \( s = 0 \) using these automorphisms (see also [11, Lemma 1.3]). For the opposite choice of “±”, just apply the automorphism determined by the assignment \( (x_{\alpha,r}^+)^{(k)} \mapsto (x_{\alpha,-r}^-)^{(k)} \).

The following lemmas will be needed in the proof of Theorem 5.11. Consider monomials involving only the elements \( (x_{\alpha,r}^-)^{(k)} \). Define the degree of \( (x_{\alpha,r}^-)^{(k)} \) to be \( k \) and extend it additively.

**Lemma 1.4.** Let \( \alpha, \beta \in R^+, k, l \in Z_+, r, s \in Z \). Then \( (x_{\alpha,r}^-)^{(k)}(x_{\beta,s}^-)^{(l)} \) is in the span of \( (x_{\beta,s}^-)^{(l)}(x_{\alpha,r}^-)^{(k)} \) together with monomials of degree strictly smaller than \( k + l \).

**Proof.** Immediate from the proof of [20, Lemma 26.3.C] using that \( U(\mathfrak{n}^-)_Z \) is \( (Q^+ \times Z) \)-graded.

The next lemma is part of [18, Lemma 5.11] and shows that the elements \( \Lambda_{\alpha,r;k} \) are linear combinations of products of the elements \( \Lambda_{\alpha,s} \).

**Lemma 1.5.** In \( U(\mathfrak{h})_Z \), for all \( k, s \in N \) and \( \alpha \in R^+ \), we have
\[
\Lambda_{\alpha,\pm s;k} = k \Lambda_{\alpha,\pm sk} + \sum_{(\vec{r}, \vec{n})} m_{\vec{r}, \vec{n}} \Lambda_{\alpha,\pm r_1}^{n_1} \cdots \Lambda_{\alpha,\pm r_l}^{n_l},
\]
for some \( m_{\vec{r}, \vec{n}} \in Z \). The sum is over the pairs \((\vec{r}, \vec{n})\) where \( \vec{r} = (r_1, \ldots, r_l) \) and \( \vec{n} = (n_1, \ldots, n_l) \) are such that \( l, r_j, n_j \in N, r_j \neq r_j, \sum_j n_j > 1 \), and \( \sum_j n_j r_j = sk \).
1.4. Frobenius Homomorphism.

Lemma 1.6. \( U(\mathfrak{g})_F \) (resp. \( U(\mathfrak{g})_F \)) is generated as an algebra by the elements \((x_{\alpha,r}^{\pm})^k\) (resp. \((x_{\alpha,r}^{\pm})^k\)), \(\alpha \in R^+, k, r \in \mathbb{Z}, k \geq 0\). Moreover, \(U(\mathfrak{h})_F\) is generated as an algebra by \((h_i^\pm)^k\), \(i \in I, k \in \mathbb{Z}_+\).

Proof. The first statement is immediate from Theorem [1.2] and [1.9]. The second is a statement on \(U(\mathfrak{g})_F\) and is well known. \(\square\)

It is known that there exists a Hopf algebra map \(\tilde{\phi}: U(\mathfrak{g})_F \to U(\mathfrak{g})_F\) sending \((x_{\alpha,r}^{\pm})^k\) to \((x_{\alpha,r}^{\pm})^{k-1}\) (with the convention that the later is zero when \(k = 0\)). We will denote the restriction of \(\tilde{\phi}\) to \(U(\mathfrak{g})_F\) by \(\phi\) and call both of them the (arithmetic) Frobenius homomorphisms. The first formula below is well known and the second was proved in [13].

\[
\phi \left( \left( \begin{array}{c} h_i \\ p^k \end{array} \right) \right) = \left( \begin{array}{c} h_i \\ p^{k-1} \end{array} \right) \quad \text{and} \quad \tilde{\phi}(A_{i,r}) = \begin{cases} A_{i,r/p}, & \text{if } p \text{ divides } r \\ 0, & \text{otherwise.} \end{cases}
\]

The proof of the existence of the map \(\phi\) can be found in [22], for instance. For the existence of \(\tilde{\phi}\), see [29] Lemma 1.3 and [10] Lemma 9.5.

Given a \(U(\mathfrak{g})_F\)-module (resp. \(U(\mathfrak{g})_F\)-module) \(V\), we denote by \(V^{\tilde{\phi}^m}\) (resp. \(V^{\phi^m}\)) the pull-back of \(V\) by \(\tilde{\phi}^m\) (resp. \(\phi^m\)).

2. Review of Finite-Dimensional \(U(\mathfrak{g})_F\)-Modules

In this section we review some results on finite-dimensional representations of \(U(\mathfrak{g})_F\) which will be relevant for our purposes. In the first subsection we consider the case \(F = \mathbb{C}\), where we summarize the basic results without proofs. The literature for this subsection is vast and well known (all the results we mention can be found in [20] to name but one reference). In the other subsections \(F\) will be an algebraically closed field of characteristic \(p > 0\). Essentially all of the results can be found in [22] (see also [3]), although the approach there is heavily geometric. Our approach follows that of [20] Chapter VII and [21]. Since some proofs are relevant for section 3 we consider it appropriate to sketch them.

2.1. Characteristic Zero and Lattices. Given a \(U(\mathfrak{g})\)-module \(V\), a vector \(v \in V\) is called a weight vector if \(hv = \mu(h)v\) for some \(\mu \in h^*\) and all \(h \in \mathfrak{h}\). The subspace consisting of weight vectors of weight \(\mu\) will be denoted by \(V_{\mu}\). If \(v\) is a weight vector such that \(n^+v = 0\), then \(v\) is called a highest-weight vector. If \(V\) is generated by a highest-weight vector of weight \(\lambda\), then \(V\) is said to be a highest-weight module of highest weight \(\lambda\).

The following theorem summarizes the basic facts about finite-dimensional \(U(\mathfrak{g})\)-modules.

**Theorem 2.1.** Let \(V\) be a finite-dimensional \(U(\mathfrak{g})\)-module.

(a) \(V = \bigoplus_{\mu \in h^*} V_{\mu}\) and \(\dim V_{\mu} = \dim V_{w_{\mu}}\) for all \(w \in \mathcal{W}\).

(b) \(V\) is completely reducible.

(c) For each \(\lambda \in P^+\) the \(U(\mathfrak{g})\)-module \(V^0(\lambda)\) generated by a vector \(v\) satisfying

\[
x_i^+v = 0, \quad h_iv = \lambda(h_i)v, \quad (x_i^-)^{\lambda(h_i)+1}v = 0, \quad \forall i \in I,
\]

is irreducible and finite-dimensional. If \(V\) is irreducible, then \(V\) is isomorphic to \(V^0(\lambda)\) for some \(\lambda \in P^+\).

(d) If \(\lambda \in P^+\) and \(V \cong V^0(\lambda)\), then \(V_{\mu} \neq 0\) iff \(w_{\mu} \leq \lambda\) for all \(w \in \mathcal{W}\). Furthermore, the minimal weight of \(V^0(\lambda)\) is \(w_0\lambda\). \(\square\)
An admissible lattice for a $U(g)$-module $V$ is the $\mathbb{Z}$-span of a basis for $V$ which is invariant under the action of $U(g)_{\mathbb{Z}}$. The basic results about lattices can be summarized in the following Theorem (see [20]).

**Theorem 2.2.** Let $V, W$ be finite-dimensional $U(g)$-modules.

(a) If $L$ is an additive subgroup of $V$ which is invariant under the action of $U(g)_{\mathbb{Z}}$, then $L = \bigoplus_{\mu \in P} L_\mu$, where $L_\mu = L \cap V_\mu$.

(b) There exists an admissible lattice for $V$.

(c) If $L, M$ are admissible lattices for $V, W$, respectively, then $L \otimes \mathbb{Z} M$ is an admissible lattice for $V \otimes W$.

(d) If $V$ is an irreducible module and $v$ is a highest-weight vector of weight $\lambda$, then $L = U(n^-)_{\mathbb{Z}}v$ is minimal in the set of admissible lattices for $V$ satisfying $L\lambda = \mathbb{Z}v$. □

### 2.2. Classification of Irreducible Modules in Positive Characteristic

From now on, $\mathbb{F}$ is an algebraically closed field of characteristic $p > 0$ and $\mathbb{F}_p$ denotes its prime field. In the present subsection we recall the methods used to classify the irreducible representations of $U(g)_{\mathbb{F}}$ up to isomorphism. Although the classification is the same as in the case of $U(g)$, the methods are quite different and will be used when we treat the case of $U(g)_{\mathbb{F}}$.

Let $V$ be a $U(h)_{\mathbb{F}}$-module. A nonzero vector $v \in V$ is called a weight vector if there exists $z = (z_{i,k}), z_{i,k} \in \mathbb{F}, i \in I, k \in \mathbb{Z}_+$, called the weight of $v$, such that $(h^{(i)})^{}v = z_{i,k}v$. Notice that (1.1) implies that $z_{i,k}$ must be in $\mathbb{F}_p$. We say that $z$ is integral (resp. dominant integral) if $z_{i,k} = (\mu(h_i))_{P_k}$ for some $\mu \in P$ (resp. $\mu \in P^+$). In that case we identify $z$ with $\mu$ and say that $v$ has weight $\mu$. If $V$ is a $U(g)_{\mathbb{F}}$-module and $v$ is weight vector such that $(x_\alpha^+)^{(k)}v = 0$ for all $\alpha \in R^+, k \in \mathbb{N}$, then $v$ is said to be a highest-weight vector. If $V$ is generated by a highest-weight vector, $V$ is called a highest-weight module.

Since the $(h_i^k)$ commute, we can decompose any finite-dimensional representation $V$ of $U(g)_{\mathbb{F}}$ in a direct sum of generalized eigenspaces for the action of $U(h)_{\mathbb{F}}$:

$$V = \bigoplus_z V_z.$$  

We say that $z$ is a weight of $V$ if $V_z \neq 0$ and, in that case, $V_z$ is called a weight space of $V$. In the case $z$ is integral we write $V_\mu$ instead of $V_z$.

Given $z = (z_{i,k})$ and $\mu \in P$ define $z + \mu = y$ by the equality $y_{i,k}v = (h_i^k+\mu(h_i))v$, where $v$ is some weight vector of weight $z$. It follows from (1.12) that if $v$ has weight $z$ then $(x_\alpha^+)^{(k)}v$ is either zero or has weight $z \pm k\alpha$. Hence, if $v$ is a highest-weight vector for a highest-weight representation $V$, we have dim($V_z$) = 1 and $V_y \neq 0$ only if $y \leq z$, where $y \leq z$ iff $y = z - \eta$ for some $\eta \in Q^+$. Standard arguments then show:

**Proposition 2.3.** Every highest-weight module is indecomposable and has a unique maximal proper submodule, hence, also a unique irreducible quotient. □

Recall that any nonnegative integer $m$ can be written uniquely as $m = \sum_{j \geq 0} m_j p^j$, where $0 \leq m_j < p$, so that $(m \over p^r) = m_r (\mod p)$ for all $r \geq 0$. We shall write $\overline{m}$ for the image of $m \in \mathbb{Z}$ in $\mathbb{F}_p$.

**Theorem 2.4.** If $V$ is an irreducible finite-dimensional $U(g)_{\mathbb{F}}$-module, then $V$ is a highest-weight representation with dominant integral highest weight.
Proof. Since $V$ is irreducible, the generalized eigenspaces spaces $V_z$ are in fact eigenspaces. Moreover, since $V$ is finite-dimensional, it also follows that there exists a maximal weight $z$ and, hence, $V$ is a highest-weight module. It remains to prove that $z$ is dominant integral. Let $v$ be a highest-weight vector for $V$. As we have already observed above, $(x^+_{\alpha})^{(k)} v$ is either zero or has weight $z - k \alpha$. This implies that, for every $i \in I$, there exists $N_i \in \mathbb{Z}_+$ minimal such that $(x^-_i)^{(p^k)} v = 0$ for all $k \geq N_i$. Moreover, we conclude from (1.13) with $k = l \geq p^{N_i}$ that $(e^i)^{(l)} v = 0$ for all $r \geq N_i$. Now we easily see that $z$ coincides with $\lambda \in P^+$ defined by $\lambda(h_i) = \sum_{j=0}^{N_i-1} m_{i,j} p^j$ with $0 \leq m_{i,j} < p$ such that $m_{i,r} = z_{i,r}$.

In order to complete the classification of the irreducible $U(g)_F$-modules in terms of highest weights, it remains to prove that for every $\lambda \in P^+$, there exists an irreducible $U(g)_F$-module having $\lambda$ as highest weight. We will use reduction modulo $p$ as follows.

Let $V$ be a finite-dimensional $U(g)_F$-module and $L$ an admissible lattice for $V$. Setting $L_\mathbb{Z} = L \otimes \mathbb{Z} \mathbb{F}$, we have that $L_\mathbb{Z}$ is a $U(g)_F$-module and $\dim_{\mathbb{Z}}(L_\mathbb{Z}) = \dim_{\mathbb{C}}(V)$. The $U(g)_F$-module $L_\mathbb{Z}$ is called a reduction modulo $p$ of $V$ (via $L$). If $L$ is a minimal admissible lattice for $V = V^0(\lambda)$, then $L_\mathbb{Z}$ is clearly highest-weight with highest weight $\lambda$. Hence, by Proposition 2.3 it has a finite-dimensional irreducible quotient. Let $V(\lambda)$ denote this quotient.

We end this subsection remarking that the following statement remains true in positive characteristic.

**Proposition 2.5.** Let $V$ be a finite-dimensional $U(g)_F$-module. The generalized eigenspaces $V_\mu$ are in fact eigenspaces and $\dim V_\mu = \dim V_{w_\mu}$ for all $w \in \mathcal{W}$. 

### 2.3. Weyl Modules and Duality.

**Definition 2.6.** Given $\lambda \in P^+$, let $W(\lambda)$ be the $U(g)_F$-module generated by a vector $v$ satisfying

\[(2.1) \quad (x^+_{\alpha})^{(p^k)} v = 0, \quad \left(\frac{h_i}{p^k}\right) v = \left(\lambda(h_i)\right) v, \quad (x^-_{\alpha})^{(l)} v = 0, \quad \forall \alpha \in R^+, i \in I, k, l \in \mathbb{Z}_+, l > \lambda(h_\alpha).

The modules $W(\lambda)$ are called Weyl modules. One can show that every finite-dimensional highest-weight $U(g)_F$-module is a quotient of some $W(\lambda)$. A comparison between the definition of $W(\lambda)$ and Theorem 2.1(c) hints that we have the following theorem which is a consequence of Kempf’s Vanishing Theorem and shows that $W(\lambda)$ is universal in the family of finite-dimensional highest-weight modules with highest weight $\lambda$.

**Theorem 2.7.** Let $\lambda \in P^+$ and $L$ a minimal admissible lattice for $V^0(\lambda)$. Then $W(\lambda)$ is isomorphic to $L_\mathbb{Z}$. In particular, $W(\lambda)$ is finite-dimensional. 

The notions of lowest-weight vector and lowest-weight module are defined similarly to the corresponding highest-weight notions. It is well-known that $V^0(\lambda)$ is a lowest-weight module with lowest weight $w_0 \lambda$, where $w_0$ is the longest element of $W$. Given a highest-weight vector $v$ for $V^0(\lambda)$ and a reduced expression $w_0 = s_{i_1} \cdots s_{i_k}$, set $m_{i_k} \in \mathbb{Z}_+$, $k = 1, \ldots, l$, to be $(s_{i_{k-1}} \cdots s_{i_1} \lambda)(h_{i_k})$. Then a lowest-weight vector of $V^0(\lambda)$ is given by $v' = (x^-_{i_1})^{(m_{i_1})} \cdots (x^-_{i_k})^{(m_{i_k})} v$ and, moreover, $v = (x^+_{i_1})^{(m_{i_1})} \cdots (x^+_{i_k})^{(m_{i_k})} v'$. This shows that the image of $v'$ in the irreducible quotient of $W(\lambda)$ is nonzero and moreover:

**Corollary 2.8.** For all $\lambda \in P^+$, $W(\lambda)$ and $V(\lambda)$ are lowest-weight modules with lowest weight $w_0 \lambda$. 

Since $U(g)_F$ is a Hopf algebra, given a $U(g)_F$-module $V$, one can equip the dual vector space $V^*$ with a structure of $U(g)_F$-module where the action of $x \in U(g)_F$ on $f \in V^*$ is given by

\[(xf)(v) = f(S(x)v)\]
for all \( v \in V \).

**Proposition 2.9.** Let \( V \) be a finite-dimensional \( U(\mathfrak{g})_\mathbb{F} \)-module. Then

(a) The natural isomorphism of vector spaces \( V \to V^{**} \) is a \( U(\mathfrak{g})_\mathbb{F} \)-module isomorphism.
(b) If \( V = V(\lambda), \lambda \in P^+ \), then \( V^* \cong V(-w_0\lambda) \).

**Proof.** Part (a) is an immediate consequence of the fact that \( S^2 \) is the identity. Now if \( V \) is irreducible, since duality preserves exact sequences, it follows from (a) that \( V^* \) is also irreducible. From (1.8), we conclude that \( V_\mu \neq 0 \) iff \( V^*_{-\mu} \neq 0 \). The final claim is now immediate from Corollary 2.8.

\[ \square \]

2.4. Tensor Products. We now recall Steinberg’s Tensor Product Theorem [34]. We sketch only the part of the proof which will be relevant for section 3. Our argument essentially follows the one given in [12]. Let \( P^+_p = \{ \lambda \in P^+ : \lambda(h_i) < p, \forall i \in I \} \). We shall use the following lemma and refer to the aforementioned references for its proof.

**Lemma 2.10.** Let \( \lambda, \mu \in P^+_p - \{ 0 \} \). Then \( V(\lambda) \) is irreducible as \( \mathfrak{g}_\mathbb{F} \)-module and \( V(\lambda) \otimes V(\mu) \) is reducible as \( U(\mathfrak{g})_\mathbb{F} \)-module.

**Theorem 2.11.** For \( \lambda \in P^+ \), let \( \lambda_k \) be the unique elements of \( P^+_p \) such that \( \lambda = \sum_{k=0}^m p^k \lambda_k \). Then \( V(\lambda) \cong \otimes_k V(p^k \lambda_k) \). Moreover, if \( \mu_j \in P^+_p - \{ 0 \} \) and \( l_j \in \mathbb{Z}_+ \), \( j = 0, \ldots, n \), are such that \( \otimes_{j=0}^n V(p^j \mu_j) \cong V(\lambda) \), then \( m = n \) and (up to reordering) \( \mu_k = \lambda_k \) and \( l_k = k \) for all \( k \).

**Proof.** First observe that for any \( \mu \in P^+_p \) and \( k \in \mathbb{Z}_+ \) we have \( V(p^k \mu) \cong V(\mu)^{\otimes k} \) (see section 1.4). Therefore, \( x_\alpha^{(p)}(\mu) V(p^k \mu) = 0 \) if \( l < k \). Now let \( v_\lambda \) be highest-weight vectors for \( V(p^k \lambda_k) \), \( V' = \otimes_{k=0}^m V(p^k \lambda_k) \), and \( v = \sum_i w_i \otimes w_i' \in V(\lambda_0) \otimes V' \), where \( w_i' \) are linearly independent. Then \( x_\alpha^{(p)} v = \sum_i (x_\alpha^{(p)} w_i) \otimes w_i' \). Since \( V(\lambda_0) \) is irreducible as \( \mathfrak{g}_\mathbb{F} \)-module, it follows that \( x_\alpha^{(p)} v = 0 \) only if \( v = v_0 \otimes v' \) for some \( v' \in V' \). Now let \( V'' = \otimes_{k=2}^m V(p^k \lambda_k) \), and \( v = v_0 \otimes (\sum_i w_i' \otimes w_i'') \in V(\lambda_0) \otimes V(\mu_1) \otimes V'' \), where \( w_i'' \) are linearly independent. Then \( (x_\alpha^{(p)})^{(p)} v = \sum_i ((x_\alpha^{(p)})^{(p)} w_i') \otimes w_i'' \). Since \( V(\lambda_1) \) is irreducible as \( \mathfrak{g}_\mathbb{F} \)-module, it follows that \( (x_\alpha^{(p)})^{(p)} v = 0 \) only if \( v = v_0 \otimes v_1 \otimes v'' \) for some \( v'' \in V'' \). Continuing like this we see that \( \otimes_{k=0}^m V(p^k \lambda_k) \) is irreducible. Since it is clearly a highest-weight module with highest-weight \( \lambda \), the first statement is proved. On the other hand, we must have \( \lambda_k = \sum_{j \in I_k} \mu_j \) where \( I_k = \{ j : l_j = k \} \). Therefore, if \( \{ \mu_j \} \) were not as stated, there would clearly exist \( j \neq j' \) such that \( l_j = l_{j'} \). The lemma above would then imply \( V(p^j \mu_j) \otimes V(p^{j'} \mu_{j'}) \) is reducible and, hence, also \( \otimes_{j=0}^n V(p^j \mu_j) \).

**Remark.** One of the reasons Theorem 2.11 is important comes from the fact that the (finitely many) modules \( V(\lambda), \lambda \in P^+_p \), are irreducible as modules for the subalgebra of \( U(\mathfrak{g})_\mathbb{F} \) generated by \( x_\alpha^{(p)} \) (since they are irreducible as \( \mathfrak{g}_\mathbb{F} \)-modules). This is a finite-dimensional algebra, called the restricted universal enveloping algebra of \( \mathfrak{g}_\mathbb{F} \). Hence, the study of finite-dimensional irreducible \( U(\mathfrak{g})_\mathbb{F} \)-modules is reduced to the study of finitely many modules for a finite-dimensional algebra.

3. Finite-Dimensional \( U(\mathfrak{g})_\mathbb{F} \)-Modules

In this section we establish some basic results about the category of finite-dimensional \( U(\mathfrak{g})_\mathbb{F} \)-modules such as the classification of the irreducible ones and the characterization of the universal highest-weight modules.
3.1. \textbf{ℓ-Highest-Weight Modules}. Let $V$ be a $U(\mathfrak{g})_F$-module. We say $v \in V$ is an \ℓ-weight vector if it is an eigenvector for the action of $U(\mathfrak{h})_F$, i.e., if there exist $z_{i,k}, \varpi_{i,r} \in F$ such that

\begin{equation}
\left(\begin{array}{c} h_i \\ p^k \end{array}\right)v = z_{i,k}v, \quad \Lambda_{i,r}v = \varpi_{i,r}v,
\end{equation}

for all $i \in I$ and all $r, k \in \mathbb{Z}, k \geq 0$. In that case the corresponding functional $\varpi \in (U(\mathfrak{h}))^*$ is called the \ℓ-weight of $v$. If $v$ is an \ℓ-weight vector and $(x_{\alpha,r}^+)^{(k)}v = 0$ for all $\alpha \in R^+$ and all $r, k \in \mathbb{Z}, k > 0$, we say $v$ is an \ℓ-highest-weight vector. If $V$ is generated by an \ℓ-highest-weight vector, we say $V$ is an \ℓ-highest-weight module.

Given a finite-dimensional $U(\mathfrak{g})_F$-module $V$ we know from section 2 that $V$ can be written as the direct sum of its weight spaces when regarded as $U(\mathfrak{g})_F$-module:

\[ V = \bigoplus_{\mu \in \mathcal{P}} V_\mu. \]

Moreover, since $U(\mathfrak{h})_F$ is a commutative algebra, we can also write the following decomposition of $V$ into direct sum of generalized eigenspaces for the action of $U(\mathfrak{h})_F$:

\[ V = \bigoplus_{\varpi \in (U(\mathfrak{h}))^*} V_\varpi. \]

The next proposition establishes a set of relations satisfied by all finite-dimensional \ℓ-highest-weight modules.

\begin{proposition}
Let $V$ be a finite-dimensional $U(\mathfrak{g})_F$-module, $\lambda \in P^+$, and $v \in V_\lambda$ be such that

\[ (x_{\alpha,s}^+)^{(k)}v = 0 \quad \text{and} \quad \Lambda_{i,r}v = \omega_{i,s}v, \]

for all $\alpha \in R^+, i \in I, k, s \in \mathbb{Z}, k > 0$, and some $\omega_{i,s} \in F$. Then

\[ (x_{\alpha,s}^-)^{(k)}v = \Lambda_{i,\pm r}v = 0 \quad \text{for all } k > \lambda(h_\alpha), r > \lambda(h_\alpha), s \in \mathbb{Z}. \]

Moreover, $\omega_{i,\pm \lambda(h_\alpha)} \neq 0$ and there exist polynomials $f_i \in F[t_0, t_1, \cdots, t_{\lambda(h_i)}]$, depending only on $\lambda(h_i)$, such that

\[ \omega_{i,-r} = f_i(\omega_{i,\lambda(h_i)}, \omega_{i,1}, \cdots, \omega_{i,\lambda(h_i)}) \]

for all $r = 1, \cdots, \lambda(h_i)$.

\end{proposition}

\begin{proof}
For each $r \in \mathbb{Z}, \alpha \in R^+$, the elements $(x_{\alpha,s}^+)^{(k)}, k \in \mathbb{Z}_+$, generate a subalgebra $U(\tilde{\mathfrak{g}}_{\alpha,r})_F$ of $U(\tilde{\mathfrak{g}})_F$ isomorphic to $U(\mathfrak{sl}_2)_F$. Hence, the equality $(x_{\alpha,s}^-)^{(k)}v = 0$ for $k > \lambda(h_\alpha)$ follows from the fact that $v$ generates a (finite-dimensional) highest-weight module for this subalgebra, which is then isomorphic to a quotient of the Weyl module $W(\lambda(h_i))$.

Setting $\alpha = \alpha_i, s = 0, l = k = r$ in (1.14) we get $\Lambda_{i,\pm r}v = 0$ for $r > |\lambda(h_i)|$. Now, choosing $r = \lambda(h_i)$, we see that $\omega_{i,\pm \lambda(h_i)} \neq 0$. In fact, since $W = U(\tilde{\mathfrak{g}})_Fv$ is a finite-dimensional representation for $U(\tilde{\mathfrak{g}})_F$ having $W_\lambda = Fv$ as its highest-weight space by equation (1.12), it follows that $\lambda - (r + m)\alpha_i$ is not a weight of $W$ for any $m > 0$. Therefore $(x_{\alpha,s}^-)^{(m)}(x_{i,\pm 1}^+)^{(r)}v = 0$ for all $m \in \mathbb{N}$. On the other hand, by considering the subalgebra $U(\tilde{\mathfrak{g}}_{\alpha_i,\mp 1})_F$, we see that $(x_{\alpha,s}^-)^{(r)}v \neq 0$. It follows that $(x_{i,\pm 1}^+)^{(r)}v$ generates a lowest weight finite-dimensional representation of $U(\tilde{\mathfrak{g}}_{\alpha_i,0})_F$ and, in particular, $0 \neq (x_{i,\pm 1}^+)^{(r)}(x_{i,\pm 1}^-)^{(r)}v = \Lambda_{i,\pm r}v$. 

\end{proof}
For the last statement we proceed by induction on \( r = 1, \ldots, \lambda(h_i) = m_i \). Setting \( \alpha = \alpha_i, s = 0, l = m_i \) and \( k = l + r \) in (1.14) we get
\[
\omega_i,m_i(x_{i,1}^{-1})^{(r)}v + \sum_{j=1}^{m_i/m} \omega_i,m_i-jY_jv = 0,
\]
where \( \omega_{i,0} = 1 \) and \( Y_j \) is the sum of the monomials \( (x_{i,1}^{-1})^{(k_1)} \cdots (x_{i,m_i+1}^{-1})^{(k_{m_i+1})} \) such that \( \sum_n n k_n = r \) and \( \sum_n nk_n = r + j \). Now, since \( -r < r + j < -2r \) for all \( m_i \), it is not difficult to see that \( (x_{i,-2}^{-1})^{(r)}Y_j \in U(\hat{\mathfrak{g}})U(\hat{\mathfrak{n}}^+)U_\mathfrak{g}^+ + H_j \), where \( H_j \) is a linear combination of monomials of the form \( \Lambda_{i,r} \cdots \Lambda_{i,r,m} \) such that \( -r < r_j < m_i \). Moreover, \( (x_{i,-2}^{-1})^{(r)}(x_{i,1}^{-1})^{(r)} \in (-1)^r \Lambda_{i,-r} + U(\hat{\mathfrak{g}})U(\hat{\mathfrak{n}}^+)U_\mathfrak{g}^+ \) by (1.14). Hence,
\[
0 = (x_{i,-2}^{-1})^{(r)} \left( \omega_i,m_i(x_{i,1}^{-1})^{(r)}v + \sum_{j=1}^{m_i/m} \omega_i,m_i-jY_jv \right) = (-1)^r \omega_i,-r\omega_i,m_i v + \sum_{j=1}^{m_i/m} \omega_i,m_i-jH_jv.
\]
From here it is easy to deduce the last statement.

We would like to be more precise about the last statement of the previous proposition (cf. [11] Proposition 1.1(\( v \))). Namely, we want to prove that
\[
\Lambda_{i,\lambda(h_i)}\Lambda_{i,-r}v = \Lambda_{i,\lambda(h_i)-r}v \quad \text{for all } i, 0 \leq r \leq \lambda(h_i).
\]
In other words, given \( v, \lambda, \) and \( \omega_{i,r} \) as in the proposition and setting
\[
\omega_i(u) = 1 + \sum_{r=1}^{\lambda(h_i)} \omega_{i,r}u^r,
\]
we want to show that
\[
\Lambda_{i}^{-}(u)v = \omega_{i}^{-}(u)v,
\]
where for a polynomial \( f(u) = \prod_j (1 - a_j u) \in \mathbb{F}[u] \), we set \( f^{-}(u) = \prod_j (1 - a_j^{-1} u) \) (when convenient we shall also write \( f = f^{+} \)). The element \( \omega_{i,\lambda} \in I \) is called the Drinfeld polynomial of the \( \ell \)-highest-weight module generated by \( v \). We denote by \( P_\mathfrak{g}^+ \) the multiplicative monoid consisting of all \( |I| \)-tuples of the form \( \omega = (\omega_i)_{i \in I} \) where each \( \omega_i \) is a polynomial in \( \mathbb{F}[u] \) with constant term 1. The differential equations techniques used in [11] for proving (3.2) do not work in positive characteristic. However, in light of Proposition 3.3 it suffices to exhibit, for each \( \omega \in P_\mathfrak{g}^+ \), one finite dimensional \( \ell \)-highest-weight module with \( \ell \)-highest weight \( \omega \) on which (3.2) is satisfied. This will be done in the next subsection.

We end this subsection introducing additional notation. The multiplicative group corresponding to \( P_\mathfrak{g}^+ \) will be denoted by \( P_\mathfrak{g} \). We let \( w : P_\mathfrak{g} \to P \) be the unique group homomorphism such that \( \text{wt}(\omega) = \sum_{i \in I} \deg(\omega_i)\omega_i \) for all \( \omega \in P_\mathfrak{g}^+ \). We also have an injective group homomorphism \( P_\mathfrak{g} \to (U(\hat{\mathfrak{n}})_{\mathfrak{g}})^* \) given as follows. Any element \( \omega \in P_\mathfrak{g} \) can be written uniquely as \( \omega\pi^{-1} \), where \( \omega, \pi \in P_\mathfrak{g}^+ \) are such that \( \omega_i, \pi_i \) are coprime for all \( i \in I \). Its image \( \overline{\omega} \in (U(\hat{\mathfrak{n}})_{\mathfrak{g}})^* \) is defined by
\[
\overline{\omega}(\left( \begin{array}{c} h_i \\ \ell \end{array} \right)) = \left( \frac{\text{wt}(\omega)(h_i)}{\ell} \right), \quad \overline{\omega}(\Lambda_i^+(u)) = \omega_i^+(u),
\]
for all \( k \in \mathbb{Z}_+ \), and where \( \omega_i^+ = \omega_i, \omega_i^- = \omega_i^-(\pi_i)^{-1} \). The second equality is that of power series in \( u \), obtained by expanding \( (\pi_i^+)^{-1} \) as a product of geometric power series. We shall identify \( P_\mathfrak{g} \) with its image in \( (U(\hat{\mathfrak{n}})_{\mathfrak{g}})^* \) and refer to its elements as the integral \( \ell \)-weights. Similarly, the elements in \( P_\mathfrak{g}^+ \) will be referred to as the dominant integral \( \ell \)-weights.
3.2. Classification of Irreducible Modules. If \( V \) is a finite-dimensional irreducible \( U(\mathfrak{g})_F \)-module, proceeding as in the proof of Theorem 2.4, we see that \( V \) is generated by a vector \( v \) satisfying

\[
(x_{\alpha,r}^+)^{(p^k)} v = 0, \quad (h_i)^{(p^k)} v = (\lambda(h_i))^{p^k} v, \quad \Lambda_{i,r} v = \omega_{i,r} v,
\]

for all \( \alpha \in R^+, i \in I, r, k \in \mathbb{Z}, k \geq 0 \) and some \( \lambda \in P^+, \omega_{i,r} \in F \). In particular, we have the following immediate corollary of Proposition 3.1:

**Corollary 3.2.** Every finite-dimensional irreducible \( U(\mathfrak{g})_F \)-module is an \( \ell \)-highest-weight module whose \( \ell \)-highest weight lies in \( P^+_F \).

We now introduce an important class of \( U(\mathfrak{g})_F \)-modules called evaluation representations.

**Proposition 3.3.** For \( a \in \mathbb{F}^\times \), there exists a surjective algebra homomorphism \( \text{hev}_a : U(\mathfrak{g})_F \to U(\mathfrak{g})_F \) mapping \( (x_{\alpha,r}^+)^{(k)} \) to \( a^{rk}(x_{\alpha,r}^+)^{(k)} \). In particular, \( \text{hev}_a(\Lambda_{\alpha,r}) = (-a)^r \lambda_{|\alpha|} \).

We call \( \text{hev}_a \) the hyper evaluation map at \( a \).

**Proof.** First observe that the formal evaluation map \( \text{ev} \) on \( U(\mathfrak{g}) \) (see Lemma 1.1) sends \( U(\mathfrak{g})_F \) to \( U(\mathfrak{g})_F \otimes \mathbb{Z}[t, t^{-1}] \). Hence, by reducing \( \text{ev} \) modulo \( p \), we obtain the formal hyper evaluation map \( \text{hev} : U(\mathfrak{g})_F \to U(\mathfrak{g})_F \otimes \mathbb{F}[t, t^{-1}] \). The statements of the proposition are now obvious (cf. definition of \( \text{ev}_a \) and \( 1.4 \)).

Given any \( U(\mathfrak{g})_F \)-module \( V \), let \( V(a) \) be the pull-back of \( V \) by \( \text{hev}_a \). \( V(a) \) is called the evaluation representation with spectral parameter \( a \) corresponding to \( V \). For \( a \in \mathbb{F}^\times \) and \( \mu \in P \), let \( \omega_{\mu,a} \) be the element in \( P^+_F \) whose \( i \)-th entry is \( (1 - au)^{\mu(h_i)} \), \( i \in I \). If \( V \) is a \( U(\mathfrak{g})_F \)-highest-weight module of highest weight \( \lambda \in P^+ \), it is easy to see that \( V(a) \) is an \( \ell \)-highest-weight module with Drinfeld polynomial \( \omega_{\mu,a} \in P^+_F \) and that the action of \( \Lambda_i^{-1}(u) \) on the \( \ell \)-highest vector is given by \( 3.2 \). We shall denote the evaluation representation by \( V(\lambda, a) \) when \( V = V(\lambda) \) and by \( W(\lambda, a) \) when \( V = W(\lambda) \).

If \( \lambda \in P^+_F \), it is easy to see that \( V(p^k\lambda, a) \) is isomorphic to \( V(\lambda, a^{pk})^{\tilde{\phi}} \), where \( \tilde{\phi} \) is the Frobenius homomorphism defined in section 1.4. Moreover, for any \( \lambda \in P^+ \), Theorem 2.11 implies

\[
V(\lambda, a) \cong \otimes_k V(p^k\lambda_k, a), \quad \text{where } \lambda_k \in P^+_p \text{ are such that } \lambda = \sum k p^k \lambda_k.
\]

We now prove the following version of Steinberg’s Tensor Product Theorem for hyper loop algebras.

**Theorem 3.4.** If \( \mu_j \in P^+_p - \{0\}, a_j \in \mathbb{F}^\times, \) and \( l_j \in \mathbb{Z}_+, j = 0, \cdots, n \), then \( V = \otimes_j V(p^{l_j} \mu_j, a_j) \) is irreducible if and only if \( a_j \neq a_j' \) whenever \( l_j = l_j' \).

**Proof.** The proof is a combination of the arguments used in Theorem 2.11 and [9, Theorem 1.7]. First consider the case \( V = V(p^l \lambda, a) \otimes V(p^l \mu, b) \), where \( \lambda, \mu \in P^+_p \), and let \( v = \sum_j v_j \otimes w_j \in V \) be such that \( w_j \) are linearly independent. Using \( 1.6 \) we get

\[
(x_{\alpha,r}^+)^{(k)} v = \sum_j \sum_l \sum m \phi(rm) \left( x_{\alpha}^+ \right)^{(l)} v_j \otimes \left( x_{\alpha}^+ \right)^{(m)} w_j.
\]

Hence, if \( a = b \), this implies \( (x_{\alpha,r}^+)^{(k)} v = a^{rk}(x_{\alpha,r}^+)^{(k)} v \), and it follows that, if \( v \) generates a \( U(\mathfrak{g})_F \)-submodule of \( V \), it also generates a \( U(\mathfrak{g})_F \)-submodule of \( V \). This proves the “only if” part.

Conversely, for each \( l \in \mathbb{Z}_+ \), let \( J_l = \{ j : l_j = l \} \) and \( V_l = \otimes_{j \in J_l} V(p^{l} \mu_j, a_j) \), so that \( V \cong \otimes_{l \in J} V_l \). Now observe that \( V_l \cong \left( \otimes_{j \in J_l} V(p^{l} \mu_j, a_j) \right)^{\tilde{\phi}} \). The same arguments used in [9, Theorem 1.7] show that
Theorem 2.11, we conclude that $\otimes$ vectors.

Proof. It is immediate from Theorem 3.4 that Corollary 3.5.

The proof of (3.2) is completed in a similar way by computing the action of $\Lambda^+_{\omega}$ using (1.7) and observing that each tensor factor is an evaluation representation. In this case we have $V$ is irreducible and, therefore, has an $\ell$-highest-weight vector using (1.7) and the last corollary, it follows from (3.3) and the corollary above that $V(\omega) \cong \otimes_j V(\lambda_j, a_j)$. 

Let us also record the following corollary.

Corollary 3.6. If $V$ is a finite-dimensional $U(\mathfrak{g})_F$-module, then $V\omega \neq 0$ only if $\omega \in \mathcal{P}_F$ and $V_\mu = \bigoplus_{\omega : \text{wt}(\omega) = \mu} V\omega$.

Proof. It suffices to prove the claim for irreducible representations. Using (1.7) and the last corollary, it is sufficient to consider the irreducible evaluation representations $V = V(\lambda, a)$ with $\lambda \in P^+$. But in this case we have $V_\mu = V_{\omega_{\mu, a}}$ (see also Proposition 3.7 below and its corollary).

We end the subsection computing the dual representation of a given irreducible one. Let $V$ be a finite-dimensional $U(\mathfrak{g})_F$-module. Exactly as in the case of $U(\mathfrak{g})_F$, we see that the dual vector space $V^*$ can be equipped with a $U(\mathfrak{g})_F$-module structure and $V^{**}$ is naturally isomorphic to $V$. Moreover, if $W$ is another finite-dimensional $U(\mathfrak{g})_F$-module, usual Hopf algebra arguments prove that we have a natural isomorphism of $U(\mathfrak{g})_F$-modules

$$(3.4) \quad (V \otimes W)^* \cong W^* \otimes V^*.$$ 

Given $\omega = \prod_j \omega_{\mu_j, a_j} \in \mathcal{P}_F$, set $\omega^* = \prod_j \omega_{-w_0\mu_j, a_j}$. We have:

Proposition 3.7. Let $\omega \in \mathcal{P}_F^+$ and $V = V(\omega)$. Then $V^* \cong V(\omega^*)$. 

As a corollary, we obtain the classification of the irreducible representations for $U(\mathfrak{g})_F$ (cf. [4, 9, 11]). It is easy to see that every element $\omega \in \mathcal{P}_F$ can be uniquely decomposed as $\omega = \prod_j \omega_{\mu_j, a_j}$ for some $\mu_j \in P$ and $a_i \neq a_j$.

Corollary 3.5.

(a) If $\omega = \prod \omega_{\lambda_j, a_j} \in \mathcal{P}_F^+$ with $a_i \neq a_j, i \neq j$, and $\lambda_j = \sum_k p^j \lambda_{j,k}$ with $\lambda_{j,k} \in P_F^+$, then $V = \otimes_j V(p^j \lambda_{j,k})$ is an irreducible $U(\mathfrak{g})_F$-module with $\ell$-highest weight $\omega$. In particular, (3.2) holds for $V$.

(b) The isomorphism classes of irreducible finite-dimensional $U(\mathfrak{g})_F$-modules are in one-to-one correspondence with the elements of $\mathcal{P}_F^+$.
Proof. Due to Theorem 3.3 and (3.3), it suffices to consider the case \( \omega = \omega_{\lambda, a} \) for some \( a \in \mathbb{F}^\times \) and \( \lambda \in P^+ \). Since in this case \( V \) is an evaluation representation, every weight vector of \( V \) is also an \( \ell \)-weight vector and \( V \ell = V \omega_{\lambda, a} \). Choose a basis for \( V \) consisting of weight vectors. Then it is easy to see using (1.8) and (1.4) that if \( v \) is a basis element of weight \( \mu \), then its dual vector \( v^* \) is an \( \ell \)-weight vector of \( \ell \)-weight \( \omega_{w_0 \mu, a} \). In particular, since \( V^* \cong V(-w_0 \lambda) \) as \( U(\mathfrak{g})_F \), we conclude that \( V^* \) is the evaluation representation \( V(-w_0 \lambda, a) \).

\( \square \)

3.3. The Weyl Modules. We now study the universal finite-dimensional \( \ell \)-highest-weight \( U(\mathfrak{g})_F \)-modules motivated by [11].

Definition 3.8. Given \( \omega = (\omega_i)_{i \in I} \in \mathcal{P}_F^+ \), let \( W(\omega) \) be the \( U(\mathfrak{g})_F \)-module generated by a vector \( v \) satisfying

\[
(x^+_{\alpha, r})^{(p_k)} v = 0, \quad (h_i)^{-1} v = \left( \frac{\text{wt}(\omega)(h_i)}{p_k} \right) v, \quad \Lambda_{i, \pm s} v = (\omega_\pm^+(u))_s v, \tag{3.5}
\]

and

\[
(x^-_{\alpha, r})^{(0)} v = 0, \tag{3.6}
\]

for all \( \alpha \in R^+, i \in I, k, l, r, s \in \mathbb{Z}, s, k \geq 0, l > \text{wt}(\omega)(h_\alpha) \). Here, as before, \( (\omega_\pm^+(u))_s \) means the coefficient of \( u^s \). We call \( W(\omega) \) the Weyl module with \( \ell \)-highest weight \( \omega \).

It follows from (1.12) that

\[
W(\omega) = \bigoplus_{\mu \leq \text{wt}(\omega)} W(\omega)_\mu.
\]

Standard arguments show:

Proposition 3.9. \( W(\omega) \) has a unique maximal submodule and, hence, a unique irreducible quotient.

\( \square \)

In particular, \( V(\omega) \) is the irreducible quotient of \( W(\omega) \), \( \omega \in \mathcal{P}_F^+ \). Moreover, it follows from Proposition 3.1 that every finite-dimensional \( \ell \)-highest weight module of \( \ell \)-highest weight \( \omega \) is isomorphic to a quotient of \( W(\omega) \). Hence, in order to complete the proof of the universality of \( W(\omega) \), it remains to show that it is finite dimensional. We begin with:

Proposition 3.10. If \( W(\omega)_\mu \neq 0 \), then \( W(\omega)_w \neq 0 \) for all \( w \in W \). In particular, \( W(\omega)_\mu \neq 0 \) only if \( w_0 \text{wt}(\omega) \leq \mu \leq \text{wt}(\omega) \).

Proof. Using an argument identical to the one used in characteristic zero, it follows from (3.6) that every vector \( w \in W(\omega) \) lies inside a finite-dimensional \( U(\mathfrak{g})_F \)-module of \( U(\mathfrak{g})_F \)-modules of \( W(\omega) \). Now all the claims follow from the corresponding results for finite-dimensional \( U(\mathfrak{g})_F \)-modules.

\( \square \)

We are ready to prove:

Theorem 3.11. \( W(\omega) \) is finite-dimensional for all \( \omega \in \mathcal{P}_F^+ \).

This was proved in [11] for characteristic zero and for quantum groups, the later under the assumption that \( \mathfrak{g} \) is simply laced (for non simply laced it follows from [2]).

Proof. Set \( \lambda = \text{wt}(\omega) \) and let \( v \) be an \( \ell \)-highest-weight vector of \( W(\omega) \). It suffices to prove that \( W(\omega) \) is spanned by the elements

\[
(x^-_{\beta_1, s_1})^{(k_1)} \cdots (x^-_{\beta_m, s_m})^{(k_m)} v,
\]
with \( m, s_j, k_j \in \mathbb{Z}_+ \) and \( \beta_j \in R^+ \) such that \( s_j < \lambda(h_{\beta_j}) \) and \( \sum_j k_j \beta_j \leq \lambda - w_0 \lambda \). The last condition is immediate from the previous proposition. Moreover, the elements \((x_{\beta_1, s_1}^-)^{(k_1)} \cdots (x_{\beta_m, s_m}^-)^{(k_m)} v\) with no restriction on \( s_j \) clearly span \( W(\omega) \).

Let \( \mathcal{R} = R^+ \times \mathbb{Z} \times \mathbb{Z}_+ \) and \( \Xi \) be the set of functions \( \xi : \mathbb{N} \to \mathcal{R} \) given by \( j \mapsto \xi_j = (\beta_j, s_j, k_j) \), such that \( k_j = 0 \) for all \( j \) sufficiently large. Let also \( \Xi' \) be the subset of \( \Xi \) consisting of the elements \( \xi \) such that \( 0 \leq s_j < \lambda(h_{\beta_j}) \). Given \( \xi \in \Xi \) we associate an element \( v_\xi \in W(\omega) \) as above in the obvious way, i.e., if \( k_j = 0 \) for \( j > m \), then \( v_\xi = (x_{\beta_1, s_1}^-)^{(k_1)} \cdots (x_{\beta_m, s_m}^-)^{(k_m)} v \). Define the degree of \( \xi \) to be \( d(\xi) = \sum_j k_j \) and the maximal exponent of \( \xi \) to be \( e(\xi) = \max \{k_j\} \). Clearly \( e(\xi) \leq d(\xi) \) and \( d(\xi) \neq 0 \) implies \( e(\xi) \neq 0 \). Since there is nothing to be proved when \( d(\xi) = 0 \) we assume from now on that \( d(\xi) > 0 \). Thus, let \( \Xi_{d,e} \) be the subset of \( \Xi \) consisting of those \( \xi \) satisfying \( d(\xi) = d \) and \( e(\xi) = e \), and set \( \Xi_d = \bigcup_{1 \leq e \leq d} \Xi_{d,e} \).

We prove by induction on \( d \) and sub-induction on \( e \) that if \( \xi \in \Xi_{d,e} \) is such that there exists \( j \) with either \( s_j < 0 \) or \( s_j \geq \lambda(h_{\beta_j}) \), then \( v_\xi \) is in the span of vectors associated to elements in \( \Xi' \). More precisely, given \( 0 < e \leq d \in \mathbb{N} \), we assume, by induction hypothesis, that this statement is true for every \( \xi \) which belongs either to \( \Xi_{d,e'} \) with \( e' < e \) or to \( \Xi_{d'} \) with \( d' < d \).

Observe that (1.14) implies

\[
(3.7) \quad \left((X_{\beta,r}^-)^{(k-l)}(u)\right)_k v = 0 \quad \forall \beta \in R^+, k, l, r \in \mathbb{Z}, k > \lambda(h_{\beta}) = 1 \leq l \leq k.
\]

We split the proof in 2 cases according to whether \( e = d \) or \( e < d \).

When \( e = d \), it follows that \( v_\xi = (x_{\beta,s}^-)^{(e)} v \) for some \( \beta \in R^+ \) and \( s \in \mathbb{Z} \). Suppose first that \( e = 1 \) and let \( l = \lambda(h_{\beta}) \) and \( k = l + 1 \) in (3.7) to get

\[
(3.8) \quad (x_{\beta,r+1}^- \lambda_{\beta,0} + x_{\beta,r+2}^- \lambda_{\beta,1} + \cdots + x_{\beta,r+l+1}^-) v = 0.
\]

We consider the cases \( s \geq l \) and \( s < l \) separately and prove the statement by a further induction on \( s \) and \( |s| \), respectively. If \( s \geq l \) this is easily done by setting \( r = s - l - 1 \) in (3.8). Similarly, after observing that \( \lambda_{\beta,l} v \neq 0 \), the case \( s < l \) is dealt with by setting \( r = s - 1 \) in (3.8). If \( e > 1 \) let \( l = e\lambda(h_{\beta}) \) and \( k = l + e \) in (3.7) to obtain

\[
(3.9) \quad \sum_{n=0}^{\lambda(h_{\beta})} (x_{\beta,r+1}^-)^{(e)} \lambda_{\beta,l-\lambda(h_{\beta})} v + \text{ other terms} = 0,
\]

where the other terms belong to the span of elements \( v_{\xi'} \) with \( \xi' \in \Xi_{e',e'} \) for \( e' < e \). As before we argue by induction on \( s \) and \( |s| \) by setting \( r = s - 1 - \lambda(h_{\beta}) \) and \( r = s - 1 \) in (3.9), respectively.

For the case \( e < d \) we can assume, by inductions hypothesis, that \( 0 \leq s_j < \lambda(h_{\beta_j}) \) for \( j > 1 \). An easy application of Lemma 1.4 completes the argument in this case.

4. Reduction Modulo \( p \)

4.1. Introductory Remarks and Notation. In this section we start the theory of reduction modulo \( p \) for \( U(\mathfrak{g})_K \)-modules, where \( K \) is a field of characteristic zero. In the case of \( U(\mathfrak{g})_K \) it sufficed to prove the existence of admissible lattices for the irreducible modules because the underlying abelian category was semisimple. The category \( \mathcal{C}_K \) is not semisimple, so, even if it is possible to obtain a nice lattice theory for all irreducible modules, one could not guarantee that all of the objects in \( \mathcal{C}_K \) would contain such a lattice. In fact, even for irreducible modules the story is more subtle than the one in the \( U(\mathfrak{g})_K \)-case since the evaluation maps hev\(a\) do not preserve \( \mathbb{Z} \)-lattices unless \( a = \pm 1 \). Still, we will prove that all the \( \ell \)-highest-weight modules whose coefficients of their Drinfeld polynomials are “good”
with respect to \( p \) can indeed be reduced modulo \( p \). In particular, it will follow that every irreducible \( U(\hat{g})\mathbb{F} \)-module can be constructed as a quotient of a module obtained by a reduction modulo \( p \) process.

We consider two kinds of lattice theories. The first one is a natural generalization of the one reviewed in Theorem 2.2 for \( U(\hat{g}) \). Namely, in subsection 4.2, we consider modules which contain finitely generated free \( \mathbb{Z} \)-submodules which are invariant under the action of \( U(\hat{g})_\mathbb{Z} \). However, the modules \( V^0(\lambda, a) \) with \( a \in \mathbb{Z} \), \( a \neq \pm 1 \), are easily seen not to contain such a lattice. Then, in subsection 4.3, we consider lattices over rings other than \( \mathbb{Z} \), namely, over torsion free discrete valuation rings. We think these lattices are more suitable for studying reduction modulo \( p \) in the present context.

Let us fix some general notation to be used below. If \( \mathbb{A} \) is any commutative ring with identity, let \( \mathcal{P}_\mathbb{A}, \mathcal{P}_{\mathbb{A}}^+ \) be defined in the obvious way (cf. definition of \( \mathcal{P}_F \)). Define also \( \mathcal{P}_{\mathbb{A}}^+ \) as the subset of \( \mathcal{P}_\mathbb{A}^+ \) consisting of the elements \( \omega \) such that the coefficient of the leading term of \( \omega_i \) belongs to \( \mathbb{A}^\times \) for all \( i \in I \). Recall that \( \mathbb{A} \) is a discrete valuation ring if it is a local principal ideal domain which is not a field and that its residue field is the quotient of \( \mathbb{A} \) by its unique maximal ideal. If \( \mathbb{A} \) is a discrete valuation ring with residue field \( F \), \( a \in \mathbb{A} \), and \( \omega \in \mathcal{P}_{\mathbb{A}}^+ \), we let \( \bar{a} \) and \( \bar{\omega} \) be the images of \( a \) in \( F \) and of \( \omega \) in \( \mathcal{P}_F^+ \), respectively. As before, \( F \) denotes an algebraically closed field of characteristic \( p > 0 \). We shall also denote by \( \bar{\omega} \) the image of \( \omega \in \mathcal{P}_F^+ \) in \( \mathcal{P}_F^+ \). We fix a torsion free discrete valuation ring \( \mathbb{A} \) with residue field \( F \) (for instance, the ring of Witt vectors with coefficients in \( F \) [34, Section II.5]) and denote by \( F^0 \) the algebraic closure of the fraction field of \( \mathbb{A} \). Given \( \omega \in \mathcal{P}_{F^0}^+ \), we denote by \( W^0(\omega) \) the corresponding \( U(\hat{g})_F^0 \)-Weyl module [11] and by \( V^0(\omega) \) its irreducible quotient.

4.2. \( \mathbb{Z} \)-Lattices.

**Definition 4.1.** If \( V \) is a finite-dimensional \( F^0 \)-vector space we say that a finitely generated free \( \mathbb{Z} \)-submodule \( L \) of \( V \) is an ample lattice for \( V \) if \( L \) spans \( V \) over \( F^0 \). If the rank of \( L \) is equal to the dimension of \( V \), then we say \( L \) is a lattice for \( V \). If \( V \) is a \( U(\hat{g})_F^0 \)-module, we say that an (ample) lattice for \( V \) is admissible if \( L \) is invariant under the action of \( U(\hat{g})_\mathbb{Z} \).

If \( L \) is an ample admissible lattice for a \( U(\hat{g})_F^0 \)-module \( V \), we set \( L_F = L \otimes_\mathbb{Z} F \). Thus, \( L_F \) is a \( U(\hat{g})_F^0 \)-module and rank(\( L \)) = dim(\( L_F \)) \geq dim(\( V \)). It is trivial to see that the modules \( V^0(\lambda, a) \) with \( a \neq \pm 1 \) do not contain a finitely generated \( \mathbb{Z} \)-submodule invariant under the action of \( U(\hat{g})_\mathbb{Z} \). In fact, if \( v \) is the \( \ell \)-highest-weight vector, then \( (A_{i_1, \pm \lambda_{h_{i_1}}})^kv = (-a)^{\pm k\lambda(h_i)}v \) is not a finitely generated \( \mathbb{Z} \)-module in that case.

**Proposition 4.2.** Let \( V \) be a finite-dimensional \( \ell \)-highest weight \( U(\hat{g})_F^0 \)-module with \( \ell \)-highest-weight \( \omega \in \mathcal{P}_{\mathbb{Z}}^+ \) and \( \ell \)-highest-weight vector \( v \). Then \( L = U(\hat{g})_\mathbb{Z}v \) is an ample admissible lattice for \( V \) and \( L_F \) is isomorphic to a quotient of \( W(\bar{\omega}) \). Moreover, if \( V = W^0(\omega) \), then \( L \) is a lattice.

**Proof.** It is easy to see from [1.3], Lemma [1.5] and [3.2] that \( U(\hat{h})_\mathbb{Z}v = \mathbb{Z}v \) and, therefore, \( L = U(\hat{h})_\mathbb{Z}v \). Also, \( L \) is quite clearly a torsion free \( \mathbb{Z} \)-submodule of \( V \) which is invariant under the action of \( U(\hat{g})_\mathbb{Z} \). The proof of Theorem \( \text{3.11} \) together with the hypothesis \( \omega \in \mathcal{P}_{\mathbb{Z}}^+ \) shows that \( L \) is a finitely generated \( \mathbb{Z} \)-module which spans \( V \) over \( F^0 \) (the hypothesis \( \omega \in \mathcal{P}_{\mathbb{Z}}^{++} \) is used to replace the remark \( A_{\beta, l}v \neq 0 \) by \( A_{\beta, l}v = av \) with \( a \in \mathbb{Z}^\times \)). This completes the proof that \( L \) is an ample admissible lattice. Since the image of \( v \) in \( L_F \) is clearly an \( \ell \)-highest-weight vector with \( \ell \)-highest weight \( \omega \), the second statement follows immediately. The last statement is clear since \( L \otimes_\mathbb{Z} F^0 \cong W^0(\omega) \) is an \( \ell \)-highest-weight \( U(\hat{g}) \)-module of \( \ell \)-highest weight \( \omega \) and of dimension at least that of \( W^0(\omega) \), thus \( L \otimes_\mathbb{Z} F^0 \cong W^0(\omega) \). □

Clearly the only irreducible \( U(\hat{g})_\mathbb{F} \)-modules which can be obtained as a quotient of some \( L_F \) where \( L \) is as in the proposition are precisely those whose Drinfeld polynomials \( \omega \) lie in \( \mathcal{P}_F^+ \) and the coefficient
of the leading term of $\omega_i$ is $\pm 1$ for all $i \in I$. However, all of the $\ell$-highest-weight $U(\tilde{\mathfrak{g}})_{\mathbb{F}}$-modules $V$ whose Drinfeld polynomial is of the form $\omega_{\lambda, \ell}$ can be obtained in this way. In the next section we will see that, for each $a \in F^x$, there exists an automorphism $\psi_a$ of $U(\tilde{\mathfrak{g}})_{\mathbb{F}}$ determined by the assignment $(x_{a,r}^\pm)^{(k)} \mapsto a^{rk}(x_{a,r}^\pm)^{(k)}$ for all $a \in R^+, k, r \in \mathbb{Z}, k > 0$. One can then show that the pull-back of such $V$ by $\psi_a$ is an $\ell$-highest-weight module with Drinfeld polynomial $\omega_{\lambda, \ell}$. Hence, up to twisting by $\psi_a$, we obtain all of the evaluation modules $V(\lambda, a)$. The other irreducible modules are then obtained using tensor products.

4.3. Lattices Over Discrete Valuation Rings. We begin by giving a motivation for considering lattices over discrete valuation rings. Let $\mathbb{P} = \mathbb{Z}(p)$ be the localization of $\mathbb{Z}$ at $\mathbb{Z} - p\mathbb{Z}$ and $U(\tilde{\mathfrak{g}})_{\mathbb{P}} = U(\tilde{\mathfrak{g}})_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{P}$. Then $\mathbb{P}$ is a torsion free discrete valuation ring with residue field $F_p$ and $U(\tilde{\mathfrak{g}})_{\mathbb{P}} \otimes_{\mathbb{P}} F \cong U(\tilde{\mathfrak{g}})_{\mathbb{F}}$. Let $a \in \mathbb{P}^x$, $v$ an $\ell$-highest-weight vector of $V = V^0(\lambda, a)$, and $L = U(\tilde{\mathfrak{g}})_{\mathbb{F}}v$. It is easy to see from (1.3), (1.4), and Lemma 4.3 that $U(\tilde{\mathfrak{g}})_{\mathbb{P}}v = \mathbb{P}v$ and, therefore, $L = U(\tilde{\mathfrak{g}})_{\mathbb{F}}v = \mathbb{P}(U(\tilde{\mathfrak{g}})_{\mathbb{F}}v) = PL'$ where $L' = U(n^-_{\mathbb{F}})_{\mathbb{P}}$ and $PL'$ is its $\mathbb{P}$-span. Since $L'$ is the $\mathbb{P}$-span of a basis for $V$ by Theorem 2.2, it follows that $L$ is the $\mathbb{P}$-span of the same basis. Thus, setting $L_{\mathbb{F}} = L \otimes_{\mathbb{P}} F$, we obtain a $U(\tilde{\mathfrak{g}})_{\mathbb{F}}$-module isomorphic to $W(\lambda, \tilde{a})$, where $\tilde{a}$ is the image of $a$ in $F_p$. This way we are able to obtain all evaluation representations of the form $V(\lambda, b), b \in F_p$, as quotients of the reduction modulo $p$ of the irreducible $U(\tilde{\mathfrak{g}})$-modules $V^0(\lambda, a)$, where $a$ is such that $\tilde{a} = b$. In order to obtain $V(\lambda, b)$ for all $b \in F$, we will have to use in place of $\mathbb{P}$ the bigger discrete valuation ring $A$ in place of $\mathbb{Z}$.

Definition 4.3. If $V$ is a finite-dimensional $F^0$-vector space, we say that a finitely generated free $A$-submodule $L$ of $V$ is an ample $A$-lattice for $V$ if $L$ spans $V$ over $F^0$. If the rank of $L$ is equal to the dimension of $V$, then we say $L$ is an $A$-lattice for $V$. If $V$ is a $U(\tilde{\mathfrak{g}})_{\mathbb{F}}$-module, we say that an (ample) lattice for $V$ is admissible if $L$ is invariant under the action of $U(\tilde{\mathfrak{g}})_A = U(\tilde{\mathfrak{g}})_{\mathbb{Z}} \otimes_{\mathbb{Z}} A$.

If $L$ is an ample $A$-lattice for a $U(\tilde{\mathfrak{g}})_{\mathbb{F}}$-module $V$, we set $L_{\mathbb{F}} = L \otimes_{A} F$. Then $U(\tilde{\mathfrak{g}})_{\mathbb{F}} \cong U(\tilde{\mathfrak{g}})_A \otimes_{A} F$ and $L_{\mathbb{F}}$ is a $U(\tilde{\mathfrak{g}})_{\mathbb{F}}$-module. The next lemma is immediate.

Lemma 4.4. Let $V$ and $W$ be finite-dimensional $U(\tilde{\mathfrak{g}})_{\mathbb{F}}$-modules, $L$ and $M$ (ample) admissible lattices for $V$ and $W$, respectively. Then $L \otimes_{A} M$ is an (ample) admissible lattice for $V \otimes W$ and $(L \otimes_{A} M)_{\mathbb{F}} \cong L_{\mathbb{F}} \otimes M_{\mathbb{F}}$ as $U(\tilde{\mathfrak{g}})_{\mathbb{F}}$-modules. \hfill $\square$

Theorem 4.5. Let $V$ be a finite-dimensional $U(\tilde{\mathfrak{g}})_{\mathbb{F}}$-$\ell$-highest-weight module with Drinfeld polynomial $\omega \in P^+_A$ and $\ell$-highest-weight vector $v$. If $L = U(\tilde{\mathfrak{g}})_A v$ we have:

(a) $L$ is an ample admissible $A$-lattice for $V$ and $L_{\mathbb{F}}$ is isomorphic to a quotient of $W(\omega)$.
(b) If $V = W^0(\omega)$, then $L$ is a lattice.
(c) If $V = V^0(\omega)$ and $\omega = \prod_{j=1}^{m} \omega_{\lambda_j, a_j}$ with $\lambda_j \in P^+, a_j \in A^x$, $a_i \neq a_j$ when $i \neq j$, then $L$ is a lattice.

Proof. The proof of parts (a) and (b) are analogous to that of Proposition 4.2 with $A$ in place of $\mathbb{Z}$.

We now prove (c). When $V$ is an evaluation representation, i.e. when $m = 1$, we proceed similarly to the motivational discussion at the beginning of this subsection replacing $P$ with $A$, $U(\tilde{\mathfrak{g}})$ with $U(\tilde{\mathfrak{g}})_{\mathbb{F}}$, and regarding $U(\tilde{\mathfrak{g}})_{\mathbb{Z}}$ as embedded in $U(\tilde{\mathfrak{g}})_{\mathbb{F}}$. In the general case we have $V = V^0(\lambda_1, a_1) \otimes \cdots \otimes V^0(\lambda_m, a_m)$. Let $v_j$ be $\ell$-highest-weight vectors of $V^0(\lambda_j, a_j)$ so that $v = v_1 \otimes \cdots \otimes v_m$ and set $L' = L_1 \otimes_A \cdots \otimes_A L_m$, where $L_j = U(\tilde{\mathfrak{g}})_A v_j$. Then $L_j$ are admissible lattices for $V^0(\lambda_j, a_j)$ by the $m = 1$ case and, by Lemma 4.3 $L'$ is an admissible lattice for $V$. It is clear from (1.3) that $L$ is an $A$-submodule of $L'$. Moreover, by part (a), $L$ is a finitely generated free $A$-module which spans $V$ and, hence, $L = L'$ since $A$ is a principal ideal domain. \hfill $\square$
The next corollary states that we have accomplished the task of constructing all of the irreducible $U(\hat{\mathfrak{g}})_F$-modules directly as quotients of some $U(\hat{\mathfrak{g}})_{p^0}$-modules by a reduction modulo $p$ process.

**Corollary 4.6.** For every $\varpi \in \mathcal{P}^+_F$ there exists $\omega \in \mathcal{P}^+_F$ such that $\varpi = \omega$ and $V(\varpi)$ is isomorphic to a quotient of $L_F$, where $L = U(\hat{\mathfrak{g}})_A v$ and $v$ is an $\ell$-highest-weight vector for $W(\omega)$.

**Proof.** Write $\varpi = \prod_j \omega_{\lambda,b_j}, b_j \in \mathbb{F}^\times, b_i \neq b_j$ for $i \neq j$, and let $a_j \in \mathbb{A}^\times$ be lifts of $b_j$ to $\mathbb{A}$. Then the corollary follows from Theorem 4.5 with $\omega = \prod_j \omega_{\lambda_j,a_j}$. \hfill $\Box$

Let $\omega, v$, and $L$ be as in Theorem 4.5 suppose $V = W(\omega)$, and write $\omega = \prod_{j=1}^m \omega_{\lambda_j,a_j}$ with $a_i \neq a_j, i \neq j$, so that $W(\omega) \cong \otimes_j W(\omega_{\lambda_j,a_j})$ (see [11]). Choose $\ell$-highest-weight vectors $v_j$ of $W(\omega_{\lambda_j,a_j})$ such that $v = v_1 \otimes \cdots \otimes v_m$ and set $L_j = U(\hat{\mathfrak{g}})_{\lambda_j} v_j, L' = L_1 \otimes \cdots \otimes L_m$. As before, it follows from (1.6) that $L \subseteq L'$.

**Conjecture 4.7.** In the notation above we have:

(a) $W(\varpi) \cong L_F$.

(b) If $\bar{a}_i \neq \bar{a}_j$ for $i \neq j$, then $L = L'$.

Part (a) is the analogous statement of the conjecture in [11] mentioned in the introduction of the paper. Notice that Theorem 4.5 implies that $\dim_F(L_F) = \dim_{\mathcal{P}}(W(\omega))$. Hence, for proving (a), it suffices to prove that $\dim_F(W(\varpi)) \leq \dim_{\mathcal{P}}(W(\omega))$.

Now part (b) is rather unusual since for $\mathcal{Z}$-lattices the appropriate analogous statement is false (as a counter-example one can take $\mathfrak{g} = sl_2, p \neq 2$, and $\omega = (1 - u)(1 + u)$). Below we give an example showing that equality can indeed happen when working with $\mathcal{A}$-lattices. This is actually the main point behind the choice of working with discrete valuation rings: they have plenty of units. We have the following corollary of the Conjecture:

**Corollary 4.8.** Let $\lambda_j \in P^+, b_j \in \mathbb{F}^\times, j = 1, \ldots, k$, be such that $b_i \neq b_j$ for $i \neq j$, and $\varpi = \prod_j \omega_{\lambda_j,b_j}$. Then:

(a) $W(\varpi) \cong \otimes W(\omega_{\lambda_j,b_j})$.

(b) If $M_j$ is a quotient of $W(\omega_{\lambda_j,b_j}), M = \otimes_j M_j$ is a quotient of $W(\varpi)$.

**Proof.** Let $a_j \in \mathbb{F}^0$ be such that $\bar{a}_j = b_j$. From part (a) of the conjecture we have $W(\varpi) \cong L_F$ and from part (b) $L_F = L'_F$. Now Lemma 4.4 implies $L'_F \cong \otimes_j (L_j)_F$. Thus, applying part (a) of the conjecture to $(L_j)_F$ we conclude part (a) of the corollary.

Once we have part (a), the proof of part (b) is standard. Namely, let $V_j$ be the kernel of the projection $W(\omega_{\lambda_j,b_j}) \to M_j$. Proceeding recursively on $j = 1, \cdots, k$, we obtain short exact sequences

$$0 \to \left( \bigotimes_{i=1}^{j-1} M_i \right) \otimes V_j \otimes \left( \bigotimes_{i=j+1}^k W(\omega_{\lambda_i,b_i}) \right) \to \left( \bigotimes_{i=1}^{j-1} M_i \right) \otimes \left( \bigotimes_{i=j}^k W(\omega_{\lambda_i,b_i}) \right) \to$$

$$\to \left( \bigotimes_{i=1}^j M_i \right) \otimes \left( \bigotimes_{i=j+1}^k W(\omega_{\lambda_i,b_i}) \right) \to 0.$$

$\Box$

The characteristic zero counterpart of part (a) of the corollary was proved in [11, Section 3]. So far we did not manage to adapt or compliment those techniques. By transferring the problem to the
setting of \( \mathbb{A} \)-lattices, we expect that other characteristic zero arguments, e.g., as in [15], will lead to a proof of part (b) of the Conjecture.

Let us record the following proposition which follows immediately from Theorem 2.2 (a).

**Proposition 4.9.** Let \( V \) be a finite-dimensional \( U(\mathfrak{g})_\mathcal{P} \)-module. Every additive subgroup of \( V \) which is invariant under the action of \( U(\mathfrak{g})_\mathcal{A} \) is the direct sum of its intersection with the weight-spaces of \( V \).

We now give the example showing that part (b) of Conjecture 4.7 in the setting of discrete valuation rings may hold. Let \( \mathfrak{g} = \mathfrak{sl}_2 \). Since \( \tilde{I} \) is a singleton, we shall drop the index referring to the roots and write \( x^\pm_i, h_r, \) and \( \Lambda_\nu \) instead of \( x^\pm_i, \), etc., and shall also identify \( P \) with \( \mathbb{Z} \). We will verify part (b) of Conjecture 4.7 for the Weyl module \( V = W^0((1 - au)^2(1 - bu)) \) where \( a, b \in \mathbb{A}^\times \) for some discrete valuation ring \( \mathbb{A} \) such that \( \tilde{a} \neq \tilde{b} \). In particular \( W^0((1 - au)^2(1 - bu)) \cong W^0((1 - au)^2) \otimes W^0(1 - bu) \). Let \( v_0, w_0 \) be \( \ell \)-highest weight vectors of \( W^0((1 - au)^2) \) and \( W^0(1 - bu) \), respectively.

\( W^0(1 - bu) \) is isomorphic to the evaluation representation \( V^0(1, b) \). It is then easy to see that \( x_1^+ w_0 = b\alpha x_0^- w_0 \) for all \( s \in \mathbb{Z} \). Thus, letting \( w_1 = x_0^- w_0 \), the set \( \{w_0, w_1\} \) is an \( \mathbb{A} \)-basis for \( L_2 = U(\mathfrak{g})_\mathcal{A} w_0 \).

Now consider \( W^0((1 - au)^2) \) and let \( L_1 = U(\mathfrak{g})_\mathcal{A} v_0 \). Since \( \text{wt}((1 - au)^2) = 2 \), letting \( k > 2 \) in (1.14) we get

\[
(\text{X}_{-s+1}^-(u))^{(k-2)} \Lambda^+_\alpha(u) \right]_{l} v_0 = 0 \quad \forall \ l, s \in \mathbb{Z}, 1 \leq l \leq k.
\]

Setting \( k = 3, l = 2 \) above, we get \( (x_{s+1}^- L_2 + x_{s+2}^- L_1 + x_{s+3}^-) v_0 = 0 \). Since \( L_2 v_0 = a^2 v_0 \) and \( \Lambda_1 v_0 = -2av_0 \), one easily proves inductively that

\[
x_{s}^- v_0 = sa^{s-1}x_1^- v_0 - (s - 1)a x_0^- v_0, \quad \text{for all} \ s \in \mathbb{Z}.
\]

Let \( v_1 = x_0^- v_0 \) and \( v_3 = x_1^- v_0 \). Thus we see that \( \{v_1, v_3\} \) is an \( \mathbb{A} \)-basis for the zero-weight space of \( W^0((1 - au)^2) \cap L_1 \). Now setting \( k = 3, l = 1 \) in (1.11), we get \( (x_{s+1}^-)^{(2)} \Lambda_1 v_0 + x_{s+1}s_{s+2} v_0 = 0 \). Setting \( s = -1 \) we get

\[
x_{1}^- x_0^- v_0 = 2a(x_0^-)^{(2)} v_0
\]

and setting \( s = 0 \) we get

\[
2a(x_1^-)^{(2)} v_0 = x_1^- x_0^- v_0.
\]

Now using (1.2) and then (1.3) on the right hand side of the last equation gives

\[
(x_1^-)^{(2)} v_0 = a^2(x_0^-)^{(2)} v_0.
\]

Finally, using (1.2), (1.3), and (1.5) we get

\[
x_{r}^- x_{s}^- v_0 = 2 a^{r+s}(x_0^-)^{(2)} v_0 \quad \forall \ r, s \in \mathbb{Z}.
\]

Hence, \( v_2 = (x_0^-)^{(2)} v_0 \) completes an \( \mathbb{A} \)-basis for \( L_1 \), i.e., \( L_1 \) is the \( \mathbb{A} \)-span of \( \{v_0, v_1, v_2, v_3\} \).

Clearly the set \( A = \{v_i \otimes w_j : i = 0, 1, 2, 3 \text{ and } j = 0, 1\} \) is an \( \mathbb{A} \)-basis for \( L' = L_1 \otimes_\mathcal{A} L_2 \). Since \( L = U(\mathfrak{g})_\mathcal{A} (v_0 \otimes w_0) \subseteq L' \), we are left to show that \( A \subseteq L \). Using (1.2), (1.4) and \( x_{s}^- w_0 = b^s x_0^- w_0 \) we compute:

\[
x_0^- (v_0 \otimes w_0) = v_1 \otimes w_0 + v_0 \otimes w_1,
\]

\[
x_1^- (v_0 \otimes w_0) = v_3 \otimes w_0 + bv_0 \otimes w_1,
\]

\[
x_2^- (v_0 \otimes w_0) = 2av_3 \otimes w_0 - a^2v_1 \otimes w_0 + b^2v_0 \otimes w_1.
\]
Proposition 4.10. end of section 4.2.

We call $Q$ the restriction of $W$ to $\mathbb{A}^\times$. Let $x$ be a lift of $\mathbb{A}^\times$ to $\mathbb{A}^\times$. Notice that the same kind of argument can be used to give an alternate proof of Proposition 3.3 without using the formal evaluation map, but using the $W$-simple roots). Let $\mathbb{A}$ be the algebra automorphism extending $u$ onto itself. Now let $\psi_a$ be the restriction of $\psi_a$ to $U(\tilde{\mathfrak{g}})_\mathbb{A}$. □

Remark. Notice that the same kind of argument can be used to give an alternate proof of Proposition 3.3 without using the formal evaluation map, but using the $W$-form $U(\tilde{\mathfrak{g}})_\mathbb{A}$.

4.4. Block Decomposition. We now assume Conjecture 4.7 in order to obtain the block decomposition of the category of finite-dimensional $U(\tilde{\mathfrak{g}})_\mathbb{F}$-modules. We begin with the following proposition on the Jordan-Hölder series of Weyl modules.

Proposition 4.11. The $\ell$-weights of $W(\omega_{\lambda,a})$ are of the form $\omega_{\mu,a}$ with $\mu \in P$ such that $\mu \leq \lambda$.

Proof. Let $\mathbb{A}$ be a lift of $a$ to $\mathbb{A}$, consider $W^0(\omega_{\lambda,b})$, and let $L = U(\tilde{\mathfrak{g}})_\mathbb{F}v$ for some choice of $\ell$-highest-weight vector $v$ of $W^0(\omega_{\lambda,b})$. It is well-known that the $\ell$-weights of $W^0(\omega_{\lambda,b})$ are of the form $\omega_{\mu,b}$ with $\mu \in P$ such that $\mu \leq \lambda$ (cf. [71 Proposition 3.3] and [8] for instance). In particular, the weight spaces of $W^0(\omega_{\lambda,b})$ coincide with its $\ell$-weight spaces and, therefore, using Proposition 4.9 we conclude that $L$ is equal to its intersection with the $\ell$-weight spaces of $W^0(\omega_{\lambda,b})$. Since, by Conjecture 4.7(a), $W(\omega_{\lambda,a})$ is isomorphic to $L_\mathbb{F}$, the claim of the proposition is now easily deduced. □

For each $a \in \mathbb{F}^\times$ and $i \in I$, set $\omega_{i,a} = \omega_{\omega_i,a}$ (the $\ell$-fundamental weights) and $\alpha_{i,a}(u) = \omega_{a_i,a}$ (the $\ell$-simple roots). Let $Q_\mathbb{F}$ (resp. $Q^+_\mathbb{F}$) be the subgroup (resp. submonoid) of $\mathcal{P}_\mathbb{F}$ generated by all $\alpha_{i,a}(u)$. We call $Q_\mathbb{F}$ the $\ell$-root lattice. We have the following Corollary of the preceding proposition together with Corollary 4.8.
Corollary 4.12. If $V$ is a finite-dimensional $\ell$-highest-weight $U(\tilde{g})_P$-module with $\ell$-highest weight $\omega$, $V_{\omega} \neq 0$ only if $\omega \in \omega(Q\tilde{P}^{-1})$.

Proof. Proposition 4.11 implies the result holds for $W(\omega_{\lambda,a})$. Then we are done using Corollary 4.8. In fact (1.7) implies that the $\ell$-weights of the tensor product are products of the $\ell$-weights of each tensor factor (cf. [8, Lemma 4.4]).

Definition 4.13. A spectral character is a function $\chi: \mathbb{F}^\times \to P/Q$ with finite support. Equipping the space of all spectral characters $\Xi_F \subset P$ with the usual abelian group structure, one sees that the assignment $\omega_{i,a} \mapsto \chi_{i,a}$, where $\chi_{i,a}(b) = \delta_{a,b}\omega_i$, determines a group homomorphism $P_F \to \Xi_F, \omega \mapsto \chi_{\omega}$, with kernel $Q_F$. We say that a $U(\tilde{g})_F$-module $V$ has spectral character $\chi$ if $\chi_{\omega} = \chi$ whenever $V_{\omega} \neq 0$. Let $\tilde{C}_\chi$ be the category of all finite-dimensional $U(\tilde{g})_F$-modules with spectral character $\chi$.

We will denote by $\chi_{\mu,a}$ the spectral character corresponding to $\omega_{\mu,a}, \mu \in P, a \in \mathbb{F}^\times$. We use additive notation for the group operation of $\Xi_F$.

Proposition 4.14.

(a) For all $\omega \in P_F^+$, $W(\omega) \in \tilde{C}_\omega \omega$.
(b) $\tilde{C}_{\chi_1} \otimes \tilde{C}_{\chi_2} \subseteq \tilde{C}_{\chi_1 + \chi_2}$ for all $\chi_1, \chi_2 \in \Xi_F$.
(c) If $V \in \tilde{C}_\chi$ then $V^* \in \tilde{C}_{-\chi}$.

Proof. Parts (a) and (b) are immediate from Corollary 4.12 and its proof. Part (c) follows from Proposition 3.7.

Let $\tilde{C}_F$ be the category of all finite-dimensional $U(\tilde{g})_F$-modules. In the rest of the section we prove that the block decomposition of $\tilde{C}_F$ is described just as in the characteristic zero case [7] and quantum group case [8] [14]. Namely:

Theorem 4.15. The categories $\tilde{C}_\chi, \chi \in \Xi_F$, are the blocks of $\tilde{C}_F$.

Once we have the statements of Propositions 4.14 and 2.9 available, exactly the same arguments used in [7, section 5] show that every indecomposable object from $\tilde{C}_F$ belongs to some $\tilde{C}_\chi$, proving that we have the decomposition

$$\tilde{C}_F = \bigoplus_{\chi \in \Xi_F} \tilde{C}_\chi.$$ 

It remains to see that $\tilde{C}_\chi$ are indecomposable abelian subcategories. To do this it suffices to show that for any two given irreducible $U(\tilde{g})_F$-modules $V$ and $W$ having the same spectral character, there exists a finite sequence of indecomposable objects $M_1, \ldots, M_k$ such that $V$ is a simple constituent of $M_1, W$ is a simple constituent of $M_k$ and, for every $j$, $M_j$ has a common simple constituent with $M_{j+1}$ (cf. [14 Section 1]). Let us begin with the case when $V = V(\lambda, a)$ and $W = V(\mu, b)$ for some $\lambda, \mu \in P^+$ and $a, b \in \mathbb{F}^\times$. Quite clearly $\chi_{\lambda,a} = \chi_{\mu,b}$ if $\lambda - \mu \in Q$ and, if $\lambda \notin Q$, also $a = b$.

Proposition 4.16. Let $a \in \mathbb{F}^\times$ and suppose $\lambda, \mu \in P^+$ are such that $\text{Hom}_{\tilde{g}_F}(\tilde{g}_F \otimes V^0(\lambda), V^0(\mu)) \neq 0$ and $\lambda > \mu$. Then there exists a quotient $M$ of $W(\omega_{\lambda,a})$ having $V(\mu, a)$ as simple constituent.

Proof. Let $b$ be a lift of $a$ to $A$. By [7 Proposition 3.4], there exists a non-split short exact sequence of $\tilde{g}_F$-modules:

$$0 \to V^0(\mu, b) \to M \to V^0(\lambda, b) \to 0$$
for some $\ell$-highest-weight module $M^0$. From Theorem 4.15, there exists an ample admissible lattice $L$ for $M^0$ such that $M = L_\varnothing$ is a quotient of $W(\omega_{\lambda,0})$. It remains to show that there exists an $\ell$-highest-weight vector $v'$ for $V^0(\mu, b)$ in $M^0$ such $v' \in L$ and its image in $L_\varnothing$ is non-zero. Thus, let $v$ be an $\ell$-highest-weight vector for $M^0$. From the proof of Theorem 4.11, using that $b \in A^\times$ as in the proof of Proposition 4.2, we see that there exists an $A$-basis for $L$ formed of vectors which are $A$-linear combinations of elements of the form $(x_{\alpha_1, r_1})^{(k_1)} \cdots (x_{\alpha_m, r_m})^{(k_m)} v$. Let $v_1, \ldots, v_n$ be an $A$-basis for $L_\mu$. Any $\ell$-highest-weight vector for $V^0(\mu, b)$ is a solution $\sum_{j=1}^n c_j v_j$, for some $c_j \in \mathbb{F}^0$, of the linear system

\[(x_+^{(k)}) \left( \sum_{j=1}^n c_j v_j \right) = 0\]

for all $\alpha \in R^+, r \in \mathbb{Z}, k \in \mathbb{Z}_+$. Since $L$ is admissible and the $\ell$-weights of $M^0$ are in $\mathcal{P}_A$ (Proposition 4.11), it follows that there exists a solution with the $c_j$ lying in the field of fractions of $A$. Since $A$ is a unique factorization domain, it follows that we can choose $c_j$ in $A$ such that the non-zero $c_j$ are coprime. This completes the proof. \hfill \Box

**This proposition and Corollary 4.8(b) imply:**

**Corollary 4.17.** Let $a = a_0, \lambda, \mu, M$ be as in Proposition 4.16 and, for $j = 1, \ldots, k$, let $\nu_j \in P^+$ and $a_j \in \mathbb{F}^\times$ be such that $a_i \neq a_l$ for all $i, l = 0, \ldots, k, i \neq l$. Then the $U(\mathfrak{g})_{\mathbb{F}}$-module $M \otimes (\otimes_j V(\nu_j, a_j))$ is $\ell$-highest-weight and has $V(\mu, a) \otimes (\otimes_j V(\nu_j, a_j))$ and $V(\lambda, a) \otimes (\otimes_j V(\nu_j, a_j))$ as simple constituents. \hfill \Box

Now let $a \in \mathbb{F}^\times$ and $\lambda, \mu \in P^+$ be such that $\lambda - \mu \in \mathcal{Q} - \{0\}$. Then by [7 Proposition 1.2], there exists a finite sequence $\lambda = \nu_1, \nu_2, \ldots, \nu_k = \mu$ such that $\nu_j \neq \nu_{j+1}$ and $\text{Hom}_{\mathfrak{g}_{\mathbb{F}}} (\mathfrak{g}_{\mathbb{F}} \otimes V^0(\nu_j), V^0(\nu_{j+1})) \neq 0$. Since $\text{Hom}_{\mathfrak{g}_{\mathbb{F}}} (\mathfrak{g}_{\mathbb{F}} \otimes V^0(\nu_j), V^0(\nu_{j+1})) = \text{Hom}_{\mathfrak{g}_{\mathbb{F}}} (\mathfrak{g}_{\mathbb{F}} \otimes V^0(\nu_j), V^0(\nu_j))$, we conclude that there exists a sequence of $U(\mathfrak{g})_{\mathbb{F}}$-$\ell$-highest-weight modules $M_j, j = 1, \ldots, k-1$, having both $V(\nu_j, a)$ and $V(\nu_{j+1}, a)$ as simple constituents. From here it is quite clear how to complete the proof of Theorem 4.15 using the last corollary (cf. [7 Section 4]).

**Remark.** We give an informal reasoning to justify why it should be expected that the block decomposition of $\mathcal{C}_{\mathbb{F}}$ is described similarly to that of $\mathcal{C}_{\mathbb{F}_0}$, contrary to what happens with the block decompositions of $\mathcal{C}_\mathbb{F}$ and $\mathcal{C}_{\mathbb{F}_0}$ (the categories of finite-dimensional representations for $U(\mathfrak{g})_{\mathbb{F}}$ and $U(\mathfrak{g})_{\mathbb{F}_0}$, respectively). While the blocks of $\mathcal{C}_{\mathbb{F}_0}$ are as small as possible ($\mathcal{C}_{\mathbb{F}_0}$ is a semisimple category), the blocks of $\mathcal{C}_{\mathbb{F}_0}$ are as large as one can expect (for instance, when $P/Q$ is trivial, $\mathcal{C}_{\mathbb{F}_0}$ is itself an indecomposable abelian category). Hence, while the blocks of $\mathcal{C}_{\mathbb{F}}$ have space to become “larger” (and they indeed become so, but still not as large as possible [22 Chapter II.7]), that is not the case for $\mathcal{C}_{\mathbb{F}}$.

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