Truncated expansion of $\zeta_{p^n}$ in the $p$-adic Mal’cev-Neumann field

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Abstract

Fix an odd prime $p$. In this article, we provide a mod $p$ harmonic number identity, which appears naturally in the canonical expansion of a root $\zeta_{p^n}$ of the $p^n$-th cyclotomic polynomial $\Phi_{p^n}(T)$ in the $p$-adic Mal’cev-Neumann field $L_p$. We establish a $(p-1)p^{n-2}$-truncated expansion of $\zeta_{p^n}$ via a variant of the transfinite Newton algorithm, which gives the first $\aleph_0^2$ terms of the canonical expansion of $\zeta_{p^n}$. The harmonic number identity simplifies the expression of this expansion. Moreover, as an application of the truncated expansion of $\zeta_{p^n}$, for $m \geq 3$, we construct a uniformizer $\pi_{p,1}^{m,1}$ of the false Tate curve extension $K_{p,1}^{m,1} = \mathbb{Q}_p(\zeta_{p^n}, p^{1/p})$ of $\mathbb{Q}_p$.

Keywords: Harmonic number identity, Transfinite Newton algorithm, $p$-adic Mal’cev-Neumann field, Explicit uniformizer, False-Tate curve extension of $\mathbb{Q}_p$

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1. Introduction

Fix an odd prime $p$. Let $\Phi_{p^n}(T)$ be the $p^n$-th cyclotomic polynomial. The harmonic number $H_k$ is the special value of the partial sum $\sum_{n=1}^{k} \frac{1}{n}$ of the Riemann $\zeta$ series at $s = 1$. It serves as a bridge between the theory of combinatorics and number theory. In this article, we provide a mod $p$ harmonic number identity using the $r$-restricted Stirling number of the second kind, which also encodes the congruence properites of the $r$-restricted Stirling numbers of the second kind. This harmonic number identity appears naturally in the truncated expansion of $\zeta_p^n$ in the $p$-adic Mal’cev-Neumann field $\mathbb{L}_p$, and we use it to establish the $\frac{1}{p^n-2(p-1)}$-truncated expansion of $\zeta_p^n$. Moreover, as an application of the truncated expansion of $\zeta_p^n$, for $m \geq 3$ and $p \geq 5$, we construct the uniformizers $\pi_p^{m,1}$ of the false Tate curve extension $\mathbb{K}_p^{m,1} = \mathbb{Q}_p(\zeta_{p^m}, p^{1/p})$ of $\mathbb{Q}_p$.

1.1 A mod $p$ identity for harmonic numbers

The Bell polynomials are used to study set partitions in combinatorial mathematics. Let $\alpha_l = (j_1, j_2, \ldots, j_l) \in \mathbb{N}^l$ be a multi-index. We denote its norm by $|\alpha_l| = j_1 + j_2 + \cdots + j_l$ and its factorial by $\alpha_l! = \prod_{k=1}^l j_k!$. Let $x = (x_1, \ldots, x_l)$ be a $l$-tuple of formal variables. The power of a multi-index $\alpha_l$ of $x$ is defined by

$$x^{\alpha_l} := \prod_{i=1}^{l} x_i^{j_i}.$$
Definition 1.1. For integer numbers \( n \geq k \geq 0 \), the \textit{incomplete exponential Bell polynomial} \( B_{n,k}(x_1, x_2, \ldots, x_{n-k+1}) \) with parameter \((n,k)\) is a polynomial given by the sum

\[
\sum_{\alpha_{n-k+1}=(j_1, \ldots, j_{n-k+1}) \in \mathbb{N}^{n-k+1}} \frac{n!}{\alpha_{n-k+1}!} \left( \frac{x_1}{1!} \cdot \frac{x_{n-k+1}}{(n-k+1)!} \right)^{\alpha_{n-k+1}}.
\]

With multinomial theorem, the incomplete exponential Bell polynomial can also be defined in terms of its generating function (cf. [4, P.134 Theorem A]):

\[
\frac{1}{k!} \left( \sum_{m \geq 1} x_m \frac{t^m}{m!} \right)^k = \sum_{n \geq k} B_{n,k}(x_1, \ldots, x_{n-k+1}) \frac{t^n}{n!}, \quad k = 0, 1, 2, \ldots.
\]

The special values of the incomplete exponential Bell polynomial at the points \((1, \ldots, 1)\) and \((1, \ldots, 1, 0, \ldots, 0)\), called Stirling numbers of the second kind and \(r\)-restricted Stirling numbers of the second kind (cf. [8, 11]) respectively. More precisely, we have the following definition:

Definition 1.2.

1. For integer numbers \( n \geq k \geq 0 \), the \textit{Stirling number of the second kind} is defined by

\[
\left\{ \begin{array}{c} n \\ k \end{array} \right\} = B_{n,k}(1, 1, \ldots, 1);
\]

2. For integer numbers \( n \geq k \geq 0 \) and positive integer \( r \), the \textit{\(r\)-restricted Stirling number of the second kind} is defined by

\[
\left\{ \begin{array}{c} n \\ k \end{array} \right\} \leq r = \left\{ \begin{array}{c} n \\ k \end{array} \right\}, \quad \text{if } n-k+1 \leq r;
\]

\[
B_{n,k}(1, \ldots, 1, 0, \ldots, 0), \quad \text{otherwise}.
\]

We have the following \textit{mod} \( p \) harmonic number identity:

Theorem (cf. Theorem 2.7). For an odd prime \( p \) and an integer \( 1 \leq k \leq p-1 \), we have

\[
\sum_{i=1}^{k} \frac{1}{(k-i)!} \left( \sum_{m=1}^{p} \frac{(p-1)!}{(p-m)! (i+p)!} \left\{ \begin{array}{c} i+p \\ m \end{array} \right\} \right) \left( \sum_{m=1}^{p-1} \frac{1}{(k-1)!} \right) \equiv -\frac{1}{k!} H_k \mod p,
\]

where \( H_k \) is the harmonic number.

Remark 1.3. This \textit{mod} \( p \) harmonic number identity is a translation of the congruence property of the \(r\)-restricted Stirling numbers of the second kind (cf. Proposition 2.8).
1.2. The truncated expansion of $\zeta_p^2$ in the $p$-adic Mal’cev-Neumann field

1.2.1. The $p$-adic Mal’cev-Neumann field

Let $\mathcal{O}_\mathbb{Q}_p = W(\bar{\mathbb{F}}_p)$ be the ring of Witt vectors over $\bar{\mathbb{F}}_p$ and let $L_p$ be the $p$-adic Mal’cev-Neumann field $\mathcal{O}_\mathbb{Q}_p((p^{\mathbb{Q}}))$ (cf. [12, Section 4]).

Every element $\alpha$ of $L_p$ can be uniquely written as

$$\sum_{x \in \mathbb{Q}} [\alpha_x] p^x,$$

where $[\cdot] : \bar{\mathbb{F}}_p \to W(\bar{\mathbb{F}}_p)$ is the Teichmuller character. (1.1)

For any $\alpha = \sum_{x \in \mathbb{Q}} [\alpha_x] p^x \in L_p$, we set $\text{Supp}(\alpha) = \{ x \in \mathbb{Q} : \alpha_x \neq 0 \}$, which is well-ordered by the definition of $L_p$. Thus, we can define the $p$-adic valuation $v_p$ by the formula:

$$v_p(\alpha) = \begin{cases} \inf \text{Supp}(\alpha), & \text{if } \alpha \neq 0; \\ \infty, & \text{if } \alpha = 0. \end{cases}$$

The field $L_p$ is complete for the $p$-adic topology and it is also algebraically closed. Moreover, it is the maximal complete immediate extension of $\mathbb{Q}_p$. The field $L_p$ is spherical complete and the field $\mathbb{C}_p$ of $p$-adic complex numbers is not spherical complete, which can be continuously embedded into $\mathbb{L}_p$.

**Definition 1.4.**

1. For any $\alpha \in L_p$, we call the unique expression (1.1) of $\alpha$, the **canonical expansion** of $\alpha$.

2. Let $r \in \mathbb{Q}$ and $\alpha \in L_p$, we rewrite the canonical expansion of $\alpha$ in the following way

$$\alpha = \sum_{x \in \mathbb{Q}, x < r} [\alpha_x] p^x + O(p^r),$$

and we call this expression the $r$-**truncated canonical expansion** of $\alpha$.

3. If $\sum_{x \in \mathbb{Q}, x < r} \beta_x p^x + O(p^r)$ is another element in $L_p$ with $\beta_x \in \mathcal{O}_\mathbb{Q}_p$ such that

$$\sum_{x \in \mathbb{Q}, x < r} \beta_x p^x \equiv \alpha \mod p^r,$$

then we call $\sum_{x \in \mathbb{Q}, x < r} \beta_x p^x + O(p^r)$ a $r$-**truncated expansion** of $\alpha$.

Given $\alpha \in L_p$, for $x \in \mathbb{Q}$, we denote the coefficient of $p^x$ in the expansion of $\alpha$ by $[C_x(\alpha)] \in \mathcal{O}_\mathbb{Q}_p$. Then $C_x(\alpha)$ is the coefficient of $p^x$ modulo $p$. This gives a map

$$C : \mathbb{Q} \times L_p \to \bar{\mathbb{F}}_p; (x, \alpha) \mapsto C_x(\alpha).$$

---

2 A valued field extension $(E, w)$ of $(F, v)$ is an immediate extension, if $(E, w)$ and $(F, v)$ have the same residue field. A valued field $(E, w)$ is maximally complete if it has no immediate extensions other than $(F, v)$ itself.

3 A valued field is said to be spherical complete, if the intersection of every decreasing sequence of closed balls is nonempty.
1.2.2. Transfinite Newton algorithm

Let $\Phi_p(T) = \sum_{k=0}^{p-1} T^{n^p-1+k} \in \mathbb{Q}_p[T]$ be the $p^n$-th cyclotomic polynomial. In [19], we expanded a result of Kedlaya (cf. [7] Proposition 1) into a transfinite Newton algorithm to study the canonical expansion in $\mathbb{L}_p$ of a root of a polynomial $\Phi(T) \in \mathbb{L}_p[T]$.

Let $\Lambda_{p-1} = \sum_{k=0}^{p-1} \frac{1}{[k]} \zeta_{2(p-1)}^{-k} p^{\frac{k}{p-1}}$ and $\sigma_2 = \sum_{k=2}^{\infty} p^{-1/r^k}$. We have the following explicit formula for the first $\aleph_0$ terms of a root $\zeta_{p^2}$ of the $p^2$-th cyclotomic polynomial $\Phi_{p^2}(T)$, which is a special case of the [19] Theorem 3.3.

**Theorem 1.5.** Let $\zeta_{p^2}^{(i)}$ be the $i$-th approximation of $\zeta_{p^2}$ in the transfinite Newton algorithm. Then we have

$$\zeta_{p^2}^{(i)} = \begin{cases} \sum_{k=0}^{i} \frac{1}{[k]} \zeta_{2(p-1)}^{-k} p^{\frac{k}{p-1}}, & \text{for } 0 \leq i \leq p - 1, \\ \Lambda_{p-1} + \sum_{i=p+2}^{\infty} \zeta_{2(p-1)} p^{\frac{i}{p-1} - \frac{1}{p}}, & \text{for } i \geq p. \end{cases}$$

In other words, $\zeta_{p^2} = \Lambda_{p-1} + \zeta_{2(p-1)} p^{\frac{1}{p-1}} \sigma_2 + O(p^{\frac{1}{p-1}})$.

**Remark 1.6.** Theoretically, the transfinite Newton algorithm is an “effective” method to get the canonical expansion of an algebraic number in $\mathbb{L}_p$, but the computation is relatively heavy even for finite steps. The notion of $r$-truncated canonical expansion also encodes the information about how many steps one needs to get the $r$-truncated canonical expansion using the transfinite Newton algorithm. In fact, it reduces to count the cardinality of the $r$-truncated support (i.e. the cardinality of the set $\{ x \in \mathbb{Q} : \alpha_x \neq 0, x < r \}$).

On the other hand, in practice, it is not obvious that one can get the first $\aleph_0$-terms of the canonical expansion of an algebraic number. For the first $\aleph_0$-terms of the canonical expansion of $\zeta_{p^n}$, we did try the first few terms, make a conjecture on the general formula, then prove it inductively. For a given truncated position $r \in \mathbb{Q}$, in general we don’t know if it can be achieved in finite steps or a finite $\aleph_0$ steps via the transfinite Newton algorithm, thus it is difficult to give the explicit $r$-truncated expansion of an algebraic number in $\mathbb{L}_p$.

We introduce a variant of transfinite Newton algorithm (cf. Section 3.2), which allows us to establish the $\left( \frac{1}{p-1} + \frac{1}{p(p-1)} \right)$-truncated canonical expansion of $\zeta_{p^2}$ and the $\frac{2}{p-1}$-truncated expansion of $\zeta_{p^2}$.

**Theorem 1.7.**

1. (cf. Proposition 3.11)

$$\zeta_{p^2} = \Lambda_{p-1} + \zeta_{2(p-1)} p^{\frac{1}{p-1}} \sigma_2 + \zeta_{2(p-1)}^2 p^{\frac{1}{p-1} + \frac{1}{p(p-1)}} \sigma_2 + O\left(p^{\frac{1}{p-1} + \frac{1}{p(p-1)}}\right).$$

2. (cf. Theorem 3.21)

$$\zeta_{p^2} = \left( \sum_{k=0}^{p-1} \frac{1}{[k]} \zeta_{2(p-1)}^{-k} p^{\frac{k}{p-1}} \right) \left( 1 + \zeta_{2(p-1)} p^{\frac{1}{p-1}} \sigma_2 \right) + \frac{1}{2} \zeta_{2(p-1)}^2 p^{\frac{2}{p-1}} \sigma_2^2$$
\[ + \frac{1}{2} S_{2(p-1)}p^{\frac{2}{p(p-1)}} - \frac{p-2}{p^{p-1}} \sum_{k=1}^{p-1} \frac{H_k}{k!} S_{2(p-1)}p^{\frac{1}{p(p-1)}} \left( \frac{k}{p} \right) + O\left( p^{\frac{2}{p(p-1)}} \right). \]

**Remark 1.8.** The first expansion corresponds to the first 2ℵ₀-terms of the canonical expansion of \( \zeta_{p^2} \) and the second expansion corresponds to the first ℵ₀-terms of the canonical expansion of \( \zeta_{p^2} \).

### 1.3. Uniformizer of the false Tate curve extensions \( \mathbb{K}_{p}^{m,1} \) of \( \mathbb{Q}_p \)

The interest of the construction of the explicit uniformizer of the false Tate curve extension \( \mathbb{K}_{p}^{m,n} = \mathbb{Q}_p(\zeta_{p^m}, p^{1/p^n}) \) of \( \mathbb{Q}_p \), for \( n, m \in \mathbb{N} \), is to study the field of norms (cf. \([6, 5]\)) of the \( p \)-adic Lie extension \( \mathbb{Q}^{FT}_p = \bigcup_{m,n \geq 1} \mathbb{K}_{p}^{m,n} \) of \( \mathbb{Q}_p \), which is still a mysterious (cf. \([2]\)). The theory of field of norms of Fontaine and Wintenberger tells us that

\[
X_{\mathbb{Q}_p}(\mathbb{Q}^{cycl}_p) = \lim_{\leftarrow} \mathbb{N}_{\mathbb{K}_p^{n+1,0}/\mathbb{K}_p^{n,0}}(\mathbb{K}_p^{n+1,0})^\times \cup \{0\}
\]

is a field of characteristic \( p \). We have the following result on the field of norms \( X_{\mathbb{Q}_p}(\mathbb{Q}^{FT}_p) \):

**Theorem 1.9** (Fontaine-Wintenberger). Let \( L_n = \mathbb{Q}^{cycl}_p\left( p^{1/p^n} \right) \) for all \( n \geq 1 \).

1. (cf. \([2]\) Proposition 4.2) The field of norms \( X_{\mathbb{Q}_p}(L_n) \) is a finite separable extension of \( X_{\mathbb{Q}_p}(\mathbb{Q}^{cycl}_p) \).

2. (cf. \([2]\) Proposition 4.8) Let \( X_{\mathbb{Q}_p^{cycl}/\mathbb{Q}_p}(\mathbb{Q}^{FT}_p) \) be the direct limit of the fields of norms \( X_{\mathbb{Q}_p}(L_n) \). Then, we have

\[
X_{\mathbb{Q}_p}(\mathbb{Q}^{FT}_p) = X_{\mathbb{Q}_p^{cycl}}(X_{\mathbb{Q}_p^{cycl}/\mathbb{Q}_p}(\mathbb{Q}^{FT}_p)).
\]

Moreover, if we have a norm compatible system of the uniformizers \( \pi_{p}^{m,n} \) of the tower \( \{\mathbb{K}_p^{m,n}/\mathbb{Q}_p\}_{n,m \geq 0} \), then we have

\[
X_{\mathbb{Q}_p}(\mathbb{Q}^{FT}_p) = F_p(\pi),
\]

with \( \pi = (\pi_{p}^{m,n})_{m \geq 0,n \geq 0} \).

The idea to construct the uniformizer of the false Tate curve extension \( \mathbb{K}_{p}^{2,n} \) of \( \mathbb{Q}_p \) for \( n \geq 1 \) in \([19]\) Theorem 3.23 is to use the truncated expansion of \( \zeta_{p^2} \) to construct an algebraic number with a “nice” \( p \)-adic valuation. Basically, we would like to use the same idea to construct the uniformizer of \( \mathbb{K}_{p}^{m,n} \) for \( m \geq 3 \). However, the first \( \aleph_0 \)-terms of the canonical expansion of \( \zeta_{p^n} \) established in \([19]\) Theorem 3.3] suggests that we need to expand \( \zeta_{p^m} \) further to get an element of

---

\[4\]The notation \( X_K(L) \) is used in \([6, 5]\) to represent the field of norms of the extension \( L/K \).
Lp with the desired valuation. Moreover, another difficulty appears: once we pass the limit term (for example the first $\aleph_0$-term), the truncated part of $\zeta_p^n$ may not be an algebraic number. Our strategy to construct a uniformizer of $K^{m,1}_p$ is the following:

1. Let $\sigma_n = \sum_{k=n}^{+\infty} p^{-k}$. We deduce the following $p^{-2n/2(p-1)}$-truncated expansion of $\zeta_p^n$ from that of $\zeta_p^2$ via a simple lemma (cf. Lemma 3.24):

$$\zeta_p^n = \sum_{k=0}^{p-1} \frac{(-1)^{n k}}{k!} \zeta_{2(p-1)} p^{n-1} \zeta_p^k + \sum_{k=0}^{p-1} \frac{(-1)^{n(k+1)}}{k!} \zeta_{2(p-1)} p^{n-1} \sigma_n$$

$$- \sum_{k=1}^{p-1} \frac{H_k}{k!} (-1)^{n(k+1)} \zeta_{2(p-1)} p^{n-1} \sigma_n + \frac{(-1)^n}{2} \zeta_{2(p-1)} p^{n-2} \frac{\sigma_n}{2} + O\left(p^{n-2} \frac{\sigma_n}{2}\right).$$

2. We use the uniformizer $\pi^{2,1}_p$ constructed in [19, Theorem 3.23] to represent the truncated expansion of $\zeta_p^3$ and obtain an algebraic number with the desired $p$-adic valuation. In fact, we define a serie of elements $\pi^{m,1}_p$ of $\mathbb{Q}_p^{FT}$ for $m \geq 3$ recursively: let

$$\pi^{3,1}_p = (\zeta_p^3 - 1)^{-2p+2} \cdot \left(\zeta_p^3 - \sum_{k=0}^{p-1} \frac{(-1)^k}{k!} \left(\pi^{2,1}_p\right)^k - \sum_{k=1}^{p-1} \frac{(-1)^k (k-H_k)}{k!} \left(\pi^{2,1}_p\right)^{p+k}\right).$$

and for $m \geq 4$, let

$$\pi^{m,1}_p = (\zeta_p^m - 1)^{-2p+2} \cdot \left(\zeta_p^m - \sum_{k=0}^{p-1} \frac{(-1)^k}{k!} \left(\pi^{m-1,1}_p\right)^k - \sum_{k=1}^{p-1} \frac{(-1)^k (2k-H_k)}{k!} \left(\pi^{m-1,1}_p\right)^{p+k}\right).$$

By construction, we have

$$\pi^{3,1}_p \in \mathbb{Q}(\zeta_p^3, \pi^{2,1}_p) \subset \mathbb{K}^{2,1}_p (\zeta_p^3) = \mathbb{K}^{3,1}_p.$$

Similarly, it is easy to see by induction that $\pi^{m,1}_p \in \mathbb{K}^{m,1}_p$ for $m \geq 3$.

We show in [Section 4] the following theorem:
Theorem (cf. Theorem 4.1). For prime $p \geq 3$ and integer $m \geq 3$, $\pi_p^{m,1}$ is a uniformizer of $\mathbb{K}_p^{m,1}$.

Our construction of the uniformizer $\pi_p^{2,n}$ and $\pi_p^{m,1}$ relies heavily on the truncated expansion of $\zeta_p^n$ in $L_p$. It seems that the $\frac{p-2}{(p-1)}$-truncated expansion of $\zeta_p^n$ is not enough to construct the uniformizer of $\mathbb{K}_p^{m,n}$ with $m \geq 3$ and $n \geq 2$. On the other hand, Theorem 4.1 and the implementation (cf. [15]) of MacLane’s approximation of valuations (cf. [9, 10, 14]) in SageMath ([18]) inspire us to make the following conjecture:

Conjecture 1.10. (1) For each prime $p \geq 3$ and integer $n \geq 1$, there exists a series of polynomials $\{R_p^{m,n}(T)\}_{m \geq 2}$ in $\mathbb{Z}_p[T]$, such that the element

$$
\pi_p^{m+1,n} = (\zeta_{p^{m+1}} - 1)^{s_p^{m,n}} \cdot (\zeta_{p^{m+1}} - R_p^{m,n}(\pi_p^{m,n})),
$$

with $s_p^{m,n} = \frac{1}{p} - p^n(p-1) \cdot v_p(\zeta_{p^{m+1}} - R_p^{m,n}(\pi_p^{m,n})) \in \mathbb{Z}$, is a uniformizer of $\mathbb{K}_p^{m+1,n}$ for all $m \geq 2$.

(2) For given integer $n \geq 1$ and $m \geq 2$, there exists a non-negative integer $h_{m,n}$, $h_{m,n}$ power series

$$
W_0^{m,n}(T), W_1^{m,n}(T), \ldots, W_{h_{m,n}}^{m,n}(T)
$$

in $\mathbb{Q}[[T]]$ and $h_{m,n}$ polynomials

$$
u_{m,n}^{i}(T) = -1, \nu_0^{m,n}(T), \nu_1^{m,n}(T), \ldots, \nu_{h_{m,n}}^{m,n}(T)
$$

in $\mathbb{Z}[T]$, such that, for any prime $p \geq 3$, the polynomials $\{R_p^{m,n}(T)\}_{m \geq 2}$ in (1) can be defined by

$$
R_p^{m,n}(T) = \sum_{i=0}^{h_{m,n}} T^{\nu_{m,n}^{i}(T)+1} \cdot \text{Trun}_{\leq u}^{\nu_{m,n}^{i}(T)}(W_i^{m,n}(T)),
$$

where $\text{Trun}_{\leq u}(\mathcal{R})$ is the truncation of power series $\mathcal{R}$ consisting of the terms with degree $\leq u$.

(3) If one chooses the power series $W_0^{m,n}(T), W_1^{m,n}(T), \ldots, W_{h_{m,n}}^{m,n}(T)$ and the polynomials

$$
u_{m,n}^{i}(T) = -1, \nu_0^{m,n}(T), \nu_1^{m,n}(T), \ldots, \nu_{h_{m,n}}^{m,n}(T)
$$

properly in (2), the series $\{R_p^{m,n}(T)\}_{m \geq 2}$ is eventually stable, i.e.

$$
R_p^{m,n}(T) = R_p^{m+1,n}(T) = R_p^{m+2,n}(T) = \ldots
$$

for sufficiently large $m$.

---

5The series $\{R_p^{m,n}(T)\}_{m \geq 2}$ is obviously non-unique.

6We implicitly require that $-1 < \nu_0^{m,n}(p) < \nu_1^{m,n}(p) < \cdots < \nu_{h_{m,n}}^{m,n}(p)$ for any prime $p \geq 3$.  

8
2. A mod $p$ identity of harmonic numbers

2.1. Stirlings numbers of the second kind

In [Section 1.1] we define the $(r$-restricted) Stirling numbers of the second kind as the special values of the incomplete exponential Bell polynomial and the generating function formula of Bell polynomial implies the generating function formula for the $(r$-restricted) Stirling numbers of the second kind. Combining with the fact that \( \binom{n}{k} \leq r \) for \( n \geq r(k+1) \), we can write the generating function formula as

\[
\frac{1}{k!} \left( \sum_{m=1}^{r} \frac{t^m}{m!} \right)^k = \sum_{n=k}^{\infty} \binom{n}{k} \frac{t^n}{n!}, \quad k = 0, 1, 2, \ldots.
\]

We denote by \((x)_n = x(x-1)(x-2)\cdots(x-n+1)\) the falling factorials, which form a basis of the $\mathbb{Q}$-vector space $\mathbb{Q}[x]$. The Stirling numbers of the second kind may also be characterized as the coordinate of powers of the indeterminate $x$ with respect to the basis consisting of the falling factorials (cf. [4, Page 207 Theorem B]): If \( n > 0 \), one has

\[
x^n = \sum_{m=0}^{n} \binom{n}{m} (x)_m.
\]

The following proposition collects the arithmetic properties of $(r$-restricted) Stirling numbers of the second kind in our previous work.

**Proposition 2.1.**

(1) [19, Corollary 3.12]

\[
\sum_{k=1}^{n} (-1)^{k-1}(k-1)!\binom{n}{k} = \begin{cases} 
0, & n \geq 2; \\
1, & n = 1.
\end{cases}
\]

(2) [19, Lemma 3.14] Let \( p \) be an odd prime number. For an integer \( k \) that \( 1 \leq k \leq p \), one has

\[
\left\{p - 1 + k \right\}_p \equiv \begin{cases} 
1 \text{ mod } p, & \text{if } k = 1 \text{ or } p; \\
0 \text{ mod } p, & \text{otherwise}.
\end{cases}
\]

(3) [19, Lemma 3.15] Let \( p \) be an odd prime number and \( r \) be an integer number satisfying \( 1 \leq r < p - 1 \), then one has

\[
\left\{r + p \right\}_p \leq_r B_{r+p, p}(1, \cdots, 1, 0) \equiv 0 \text{ mod } p.
\]

(4) [19, Lemma 3.16] Let \( i \) be an integer that \( 1 \leq i \leq p - 1 \) and \( k \in \mathbb{Z}_{>0} \). Then for any integer \( l \geq k \), we have

\[
v_p \left( \frac{k!}{l!} \binom{i}{k} \right) \geq 0.
\]
2.2. Congruence properties of Stirling numbers of the second kind

In this paragraph, we will establish three congruence properties of the \((r\text{-restricted})\) Stirling numbers of the second kind.

**Lemma 2.2.** For a prime \(p \geq 3\) and an integer \(1 \leq i \leq p - 1\) and \(1 \leq m \leq p\), we have

\[ p \mid \binom{i + p}{m} \leq_{p-1}. \]

**Proof.** If \(m < p\), by Proposition 2.1 (4) we have

\[ v_p \left( \frac{m!}{(i + p)!} \binom{i + p}{m} \right) \geq 0. \]

As a result, we have

\[ v_p \left( \binom{i + p}{m} \leq_{p-1} \right) \geq v_p((i + p)!) - v_p(m!) = 1. \]

If \(m = p\), then by [8, Proposition 1]

\[
\binom{i + p}{p} \leq_{p-1} = p \left( \binom{i + p - 1}{p} \leq_{p-1} + \binom{i + p - 1}{p - 1} \leq_{p-1} \right)
\]

\[ - \binom{i + p - 1}{p - 1} \binom{i}{p} \leq_{p-1} \binom{i + p - 1}{p - 1} - \binom{i + p - 1}{p - 1} \binom{i}{p - 1} \leq_{p-1} \mod p. \]

Again by Proposition 2.1 (4) we have

\[ v_p \left( \frac{(p-1)!}{(i + p - 1)!} \binom{i + p - 1}{p - 1} \right) \leq_{p-1} \geq 0. \]

Therefore

\[ v_p \left( \binom{i + p - 1}{p - 1} \right) \geq v_p((i + p - 1)!) - v_p((p - 1)!) = 1. \]

Notice that

\[ v_p \left( \frac{i + p - 1}{p - 1} \right) = v_p((i + p - 1)! - v_p((p - 1)!) = 1, \]

one can conclude that \(p \mid \binom{i + p}{m} \leq_{p-1}.\)
Lemma 2.3. For a prime \( p \geq 3 \) and an integer \( 1 \leq i \leq p - 1 \), we have
\[
\sum_{m=p+1}^{i+p} (-1)^{m-1}(m-1)!\left\{\frac{i+p}{m}\right\}_{\leq p-1} = \begin{cases} p + O(p^2), & i = 1 \\ O(p^2), & i \geq 2 \end{cases}
\]

Proof. We rewrite the summation in the lemma as follows:
\[
\sum_{m=p+1}^{i+p} (-1)^{m-1}(m-1)!\left\{\frac{i+p}{m}\right\}_{\leq p-1}
= \sum_{t=1}^{i} (-1)^{t+p-1}(t+p-1)!\left\{\frac{i+p}{t+p}\right\}_{\leq p-1}
= \sum_{t=1}^{i} (-1)^{t}(t+p-1)!\left\{\frac{i+p}{t+p}\right\}_{\leq p-1}.
\]
Thus, we reduce to study the \( p \)-adic valuation of each term.

1. Since \( (p-1)! \equiv -1 \mod p \), we have
   \( (t+p-1)! = (p-1)! \cdot p \cdot (p+1) \cdot \cdots \cdot (p+t-1) \equiv -p(t-1)! \mod p^2. \)

2. As \( (i+p) - (t+p) + 1 = i - t + 1 \leq p - 1 \), we have \( \left\{\frac{i+p}{t+p}\right\}_{\leq p-1} = \left\{\frac{i+p}{t}\right\}. \)
   By \cite{17} Corollary 2.1, we have
   \[
   \left\{\frac{i+p}{t+p}\right\} = \left\{\frac{i+1}{t+p}\right\} + \left\{\frac{i}{t}\right\} \mod p.
   \]
   Since \( i + 1 < t + p \), we have \( \left\{\frac{i+1}{t+p}\right\} = 0. \)
   By combining these estimations, we have
   \[
   \sum_{m=p+1}^{i+p} (-1)^{m-1}(m-1)!\left\{\frac{i+p}{m}\right\}_{\leq p-1}
   = \sum_{t=1}^{i} (-1)^{t}(-p(t-1)! + O(p^2))\left\{\frac{i}{t}\right\} + O(p)
   = O(p^2) + p \sum_{t=1}^{i} (-1)^{t-1}(t-1)!\left\{\frac{i}{t}\right\}
   = \begin{cases} p + O(p^2), & i = 1 \\ O(p^2), & i \geq 2 \end{cases}
   \]
where the last equality follows from \cite{2} Proposition 2.1 (1). \( \square \)
Let
\[ B_n(x_1, \ldots, x_n) = \sum_{k=1}^{n} B_{n,k}(x_1, \ldots, x_{n-k+1}), n = 1, 2, \ldots \]
be the \(n\)-th complete Bell polynomial. For \(n \geq 1\), we define a map
\[ \mathbb{B}_n : \mathbb{Z}^n \to \mathbb{Z}^n, (x_1, \ldots, x_n) \mapsto (B_1(x_1), B_2(x_1, x_2), \ldots, B_n(x_1, \ldots, x_n)). \]

**Proposition 2.4.** Let \(p \geq 3\) be a prime.

(1) For \(n \in \mathbb{N}_{\geq 1}\), \(\mathbb{B}_n\) is bijective and we denote its inverse simply by \(\mathbb{B}_n^{-1}\).

(2) Let \(\delta_j = \begin{cases} 1, & j \leq p - 1, \\ 0, & j \geq p \end{cases}\), and \(1 \leq i \leq p - 1\) be an integer. We have
\[ \mathbb{B}_n^{-1}(\delta_1, \ldots, \delta_{p+i}) = (1, 0 \cdots, 0, -1, \xi_{p+1}, \ldots, \xi_{p+i}), \]
where \(\xi_{p+k} \in \mathbb{Z}\) satisfies \(\xi_{p+k} \equiv (-1)^{p+i}k \mod p^2\) for \(1 \leq k \leq i\).

**Proof.**

1. We prove the first assertion by induction on \(n\). For \(n = 1\), we have \(\mathbb{B}_1(x) = x\). Assume for \(1 \leq k \leq n\), \(\mathbb{B}_k\) are bijective. We will prove the assertion for \(k = n + 1\). For any \((y_1, \ldots, y_{n+1}) \in \mathbb{Z}^{n+1}\), there exists a unique \((x_1, \ldots, x_n) \in \mathbb{Z}^n\) such that \(\mathbb{B}_n(x_1, \ldots, x_n) = (y_1, \ldots, y_n)\). The recurrence relation
\[ B_{n+1}(x_1, \ldots, x_{n+1}) = \sum_{i=0}^{n} \binom{n}{i} B_{n-i}(x_1, \ldots, x_{n-i})x_{i+1}, \]
shows that \(\mathbb{B}_{n+1}\) is bijective. In fact, we can set
\[ x_{n+1} = y_{n+1} - \sum_{i=0}^{n-1} \binom{n}{i} B_{n-i}(x_1, \ldots, x_{n-i})x_{i+1} \]
\[ = y_{n+1} - \sum_{i=0}^{n-1} \binom{n}{i} y_{n-i}x_{i+1}. \tag{2.1} \]

2. For \(p = 3\), the result follows from a direct calculation. Therefore, in the following, we assume \(p \geq 5\). Let \((\xi_1, \ldots, \xi_{p+i}) := \mathbb{B}_p^{-1}(\delta_1, \ldots, \delta_{p+i})\). By the proof of the first assertion, we calculate the components of \((\xi_1, \ldots, \xi_{p+i})\) inductively. We have \(\xi_1 = 1\). By the formula (2.1) we have
\[ \xi_2 = \delta_2 - \sum_{i=0}^{0} \binom{1}{i} \delta_{1-i}\xi_{i+1} = 1 - B_1(\xi_1) = 0. \]

\[7\]Cited from [12.3.4]. We regard \(B_0\) as 1.
For $2 \leq l \leq p - 2$, assume that $x_2, \cdots, x_l = 0$, we calculate

$$x_{l+1} = \delta_{l+1} - \sum_{i=0}^{l-1} \binom{l}{i} \delta_{l-i} x_{i+1} = 1 - B_1(x_1) = 0.$$  

For $x_p$ and $x_{p+1}$, a direct calculation shows that

$$x_p = \delta_p - \sum_{i=0}^{p-2} \binom{p-1}{i} \delta_{p-i} x_{i+1} = 0 - B_1(x_1) = -1,$$

$$x_{p+1} = \delta_{p+1} - \sum_{i=0}^{p-1} \binom{p}{i} \delta_{p-i} x_{i+1} = 0 - \left(\frac{p}{p-1}\right) x_p = p.$$  

It rests to show the mod $p^2$ values of $x_{p+i}$ for $1 \leq i \leq p - 1$, which we will show by induction. Suppose $x_{p+t} \equiv \frac{(-1)^{t+1} p}{t} \mod p^2$ for $1 \leq t \leq k \leq p - 2$. Thus, we have the following identity

$$x_{p+k+1} = \delta_{p+k+1} - \sum_{i=0}^{p+k-1} \binom{p+k}{i} \delta_{p+k-i} x_{i+1} = - \sum_{i=p-1}^{p+k-1} \binom{p+k}{i} x_{i+1}$$

$$= \left[\frac{(p+k)}{p-1} \cdot (-1) + \sum_{i=1}^{k} \frac{(p+k)}{p+t-1} \frac{(-1)^{t+1} p}{t} + O(p^2)\right].$$

For $1 \leq t \leq k$, we have the following estimations:

$$\frac{(p+k)}{p-1} = \frac{(p+1) \cdots (p+k)}{(k+1)!} = p \cdot O(p) = \frac{p}{k+1} + O(p^2),$$

$$\frac{1}{t} \cdot \frac{(p+k)}{p+t-1} = \frac{(p+t) \cdots (p+k)}{(k-t+1)! \cdot t} = \frac{(t) \cdots (k)}{(k-t+1)! \cdot t} = O(p) + \frac{1}{k+1} \frac{(k+1)!}{(k+1-t)!} = O(p) + \frac{1}{k+1} \binom{k+1}{t}.$$  

Thus, the identity (2.2) can be written as following

$$x_{p+k+1} = \frac{p}{k+1} + \frac{p}{k+1} \sum_{t=1}^{k} \binom{k+1}{t} (-1)^t + O(p^2)$$

$$= \frac{p}{k+1} \sum_{t=0}^{k+1} \binom{k+1}{t} (-1)^t - \frac{(-1)^{k+1} p}{k+1} + O(p^2)$$

$$= \frac{(-1)^{k+2} p}{k+1} + O(p^2),$$

which is the expected formula. \(\square\)
Proposition 2.5 (Inverse relations of Bell polynomials). For an integer \( n \geq 1 \) and \( y_1, \ldots, y_n \in \mathbb{Z} \), we have \( B_{n+1}^{-1}(y_1, \ldots, y_n) = (y_1', \ldots, y_n') \), where

\[
y_i' = \sum_{k=1}^{i} (-1)^{i-k-1}(k-1)!B_{i,k}(y_1, \ldots, y_{i-k+1}), \quad i = 1, \ldots, n.
\]

Proof. See \[13\] (50) of Chapter 2. \( \square \)

By applying this proposition to \( n = p + i \), \((y_1, \ldots, y_{p+i}) = (\delta_1, \ldots, \delta_{p+i})\) and comparing with Proposition 2.4 (2), we can conclude:

Proposition 2.6. For a prime \( p \geq 3 \) and an integer \( 1 \leq i \leq p - 1 \), we have

\[\sum_{m=1}^{i+p} (-1)^{m-1}(m-1)! \binom{i+p}{m} \equiv \frac{(-1)^{i+1}p}{i} \mod p^2.\]

2.3. Harmonic numbers and the Stirling numbers of the second kind

In this paragraph, we prove the following mod \( p \) harmonic number identity:

Theorem 2.7. For a prime \( p \geq 3 \) and an integer \( 1 \leq k \leq p - 1 \), we have

\[\sum_{i=1}^{k} \frac{1}{(k-i)!} \left( \sum_{m=1}^{p} \frac{(p-1)!}{(p-m)!(i+p)!} \binom{i+p}{m} \right) \leq_{p-1} - \frac{1}{(k-1)!} \equiv - \frac{1}{k!} H_k \mod p,\]

where \( H_k \) is the harmonic number.

This theorem can be deduced from the following proposition:

Proposition 2.8. For a prime \( p \geq 3 \) and an integer \( 1 \leq i \leq p - 1 \), we have

\[\sum_{m=1}^{p} \frac{(p-1)!}{(p-m)!(i+p)!} \binom{i+p}{m} \leq_{p-1} 0, \quad i = 1 \)

\[\frac{(-1)^{i+1}p}{i} \mod p, \quad 2 \leq i \leq p - 1\]

Proof. For \( p = 3 \), the result follows from direct calculation. Therefore we assume \( p \geq 5 \).

It is equivalent to prove the congruence

\[\frac{i!}{(i+p)!} \sum_{m=1}^{p} \frac{(p-1)!}{(p-m)!} \binom{i+p}{m} \leq_{p-1} 0, \quad i = 1 \]

\[\frac{(-1)^{i+1}p}{i} \mod p, \quad 2 \leq i \geq 2\]

(2.3)

Notice that

\[
\frac{i!}{(i+p)!} = \frac{i!}{(p-1)! \cdot p \cdot (p+1) \cdots (p+i)} = \frac{i!}{(-1 + O(p))p(i! + O(p))}
\]
\[= - \frac{1}{p} + O(1)\]
and
\[\frac{(p-1)!}{(p-m)!} \equiv (-1)^{m-1}(m-1)! \mod p.\]

By Lemma 2.2 we can reduce the congruence (2.3) to
\[\sum_{i=1}^{p} (-1)^{m-1}(m-1)! \left\{ \begin{array}{c} i + p \\ m \end{array} \right\}_{\leq p-1} \equiv \begin{cases} 0, & i = 1 \\ (-1)^{i}, & i \geq 2 \end{cases} \mod p.\]

By Lemma 2.3 we have
\[
\sum_{m=1}^{p} (-1)^{m-1}(m-1)! \left\{ \begin{array}{c} i + p \\ m \end{array} \right\}_{\leq p-1} \\
= \sum_{m=1}^{i+p} (-1)^{m-1}(m-1)! \left\{ \begin{array}{c} i + p \\ m \end{array} \right\}_{\leq p-1} - \sum_{m=p+1}^{i+p} (-1)^{m-1}(m-1)! \left\{ \begin{array}{c} i + p \\ m \end{array} \right\}_{\leq p-1}
\equiv \begin{cases} \sum_{m=1}^{i+p} (-1)^{m-1}(m-1)! \left\{ \begin{array}{c} 1 + p \\ m \end{array} \right\}_{\leq p-1} - p, & i = 1 \\
\sum_{m=1}^{i+p} (-1)^{m-1}(m-1)! \left\{ \begin{array}{c} i + p \\ m \end{array} \right\}_{\leq p-1}, & i \geq 2 \mod p^2.
\end{cases}
\]

Then the result follows from Proposition 2.6.

Proof of Theorem 2.7 If \(k = 1\), the theorem is trivial. For \(k \geq 2\), by Proposition 2.8 we have
\[
\sum_{i=1}^{k} \frac{1}{(k-i)!} \left( \sum_{m=1}^{i+p} \frac{(p-1)!}{(p-m)!}\left\{ \begin{array}{c} i + p \\ m \end{array} \right\}_{\leq p-1} \right) \equiv \begin{cases} \sum_{i=2}^{k} \frac{1}{(k-i)!} \frac{(-1)^{i}}{i! \cdot \cdot \cdot i}, & i \geq 2 \mod p \\
1 \cdot \frac{k}{k!} \sum_{i=1}^{k} \frac{\left( \begin{array}{c} k \\ i \end{array} \right) (-1)^{i}}{i} \equiv - \frac{1}{k!} H_k,
\end{cases}
\]

where the last equality follows from [3, (9.2)].

3. Truncated expansion of \(\zeta_{p^2}\): a reflection on transfinite Newton algorithm

Let \(P(T) = a_0T^n + a_1T^{n-1} + \cdots + a_n \in \mathbb{Z}_p[T]\) be a polynomial with \(a_n \neq 0\).

Definition 3.1. For any \(\mu \in \mathbb{Z}_p\), we call \(P_\mu(T) = P(T + \mu)\) the \(\mu\)-perturbed polynomial of \(P(T)\).
Let \( P_k(T) = \sum_{k=0}^{n} b_k T^{n-k} \). Then for \( 0 \leq k \leq n \), we have
\[
b_k = \sum_{j=0}^{k} a_{k-j} \binom{n-k+j}{j} \mu^j.
\]

### 3.1. Transfinite Newton algorithm and Newton polygon

In [19], we expanded a result of Kedlaya (cf. [7, Proposition 1]) into a transfinite Newton algorithm to find the canonical expansion of a root of the polynomial \( P(T) \in \mathbb{L}_p[T] \) in the \( p \)-adic Malcev-Neumann field \( \mathbb{L}_p \).

**Definition 3.2** (Newton polygon). Suppose \((K, v)\) is a valued field with value group \( \mathbb{Q} \). Let \( J(T) = \sum_{i=0}^{n} a_n T^i \in K[T] \) be a nonzero polynomial. For \( 0 \leq i \leq n \), we have the points \((i, v(a_i)) \in \mathbb{N} \times \mathbb{R} \), where \( \mathbb{R} = \mathbb{R} \cup \{+\infty\} \). If \( a_i = 0 \), \((i, v(a_i))\) is regarded as \( Y_{+\infty} \), the point at infinity of the positive vertical axis.

1. Define the **Newton polygon** \( \widetilde{\text{Newt}}(J) \) of \( J(T) \) as the lower boundary of the convex hull of the points \((i, v(a_i)) \) for \( i = 0, \cdots, n \). As a consequence, \( \widetilde{\text{Newt}}(J) \) is a function on \( \mathbb{R}_{\geq 0} \) with values in \( \mathbb{R} \).

2. The integers \( m \) such that \((m, v(a_m)) \) are vertices of \( \widetilde{\text{Newt}}(J) \) are called the **breakpoints**, and we denote by \( m_{\max}^J \) the **largest breakpoint** less than \( n \).

3. Given two adjacent breakpoints \( m_1^J < m_2^J \), denote by \( s_{m_1}^J = \frac{v(a_{m_1}) - v(a_{m_2})}{m_2^J - m_1^J} \), the **slope** of constituent segment of \( \widetilde{\text{Newt}}(J) \) with endpoints \((m_1^J, v(a_{m_1}^J))\) and \((m_2^J, v(a_{m_2}^J))\). The **largest slope** is denoted by
\[
s_{\max}^J = s_{m_{\max}^J}^J = \frac{v(a_n) - v(a_{m_{\max}^J})}{n - m_{\max}^J}.
\]

If \((n, v(a_n)) = Y_{+\infty} \) (i.e. \( a_n = 0 \)), we regard \( s_{\max}^J = \infty \). Thus, \( s_{\max} \) is a map from \( K[T] \) to \( \mathbb{Q} \cup \{\infty\} \).

We will omit the superscript \( J \) if there is no confusion.

Apply the above notions to \( P(T) = a_0 T^n + a_1 T^{n-1} + \cdots + a_n \in \mathbb{L}_p[T] \) and we set \( s = s_{\max}^P \) and \( m = m_{\max}^P \).

**Definition 3.3.** We define the residue polynomial associated to \( P(T) \) by
\[
\text{Res}_P(T) = \sum_{k=0}^{n-m} C_0 \left( a_{n-k} p^{-v_p(a_m) - s(n-m-k)} \right) T^k \in \mathbb{F}_p[T].
\]

\(^8\text{Notice that if } m \text{ is a breakpoint, then } (m, v(a_m)) = Y_{+\infty} \Leftrightarrow m = n \text{ and } a_n = 0.\)
The transfinite Newton algorithm can be summarized in the following pseudo-code:

**Algorithm 1** transfinite Newton algorithm for \( \mathbb{L}_p \)

**INPUT:** A non-constant polynomial \( P(T) \in \mathbb{L}_p[T] \)

**OUTPUT:** A root of \( P(T) \) in \( \mathbb{L}_p \)

function Newton(\( P \))

\[
\begin{align*}
r &\leftarrow 0 \\
s_{\text{max}} &\leftarrow 0, m_{\text{max}} \leftarrow 0, c \leftarrow 0 \\
\text{Res}_{\Phi}(T) &\leftarrow 0 \\
\Phi(T) &\leftarrow f(T) \quad \triangleright \text{We denote the coefficient of } T^i \text{ in } \Phi \text{ as } b_{n-i}, \text{where } n = \deg(\Phi). \\
\text{while } \Phi(0) \neq 0 \text{ do} \\
\quad m_{\text{max}} &\leftarrow p^m_{\text{max}} \\
\quad s_{\text{max}} &\leftarrow s^p_{\text{max}} \\
\quad \text{Res}_{\Phi}(T) &\leftarrow \sum_{k=0}^{n-m_{\text{max}}} C_{v_p(b_m)+s_{\text{max}}(n-m_{\text{max}}-k)}(b_{n-k})T^k \\
\quad c &\leftarrow \text{any root of } \text{Res}_{\Phi}(T) \text{ in } \mathbb{F}_p \\
\quad r &\leftarrow r + [c] \cdot p^{s_{\text{max}}} \\
\quad \Phi(T) &\leftarrow \Phi(T + [c] \cdot p^{s_{\text{max}}}) \\
\text{end while} \\
\text{return } r \\
\end{align*}
\]

end function

**Definition 3.4.** We call the value of \( \Phi(T) \) (resp. \( \text{Res}_{\Phi}(T) \) and \( r \)) in the above pseudo-code after the loop iterates for \( l \) times, the \( l \)-th approximation polynomial (resp. the \( l \)-th residue polynomial and the \( l \)-th approximation of a root of \( \Phi(T) \)). In particular, the \( l \)-th approximation polynomial is the \( \mu \)-perturbed polynomial by taking \( \mu \) to be the \( l \)-th approximation of a root of \( \Phi(T) \).

### 3.2. A variant of transfinite Newton algorithm

As we explained in the introduction, one can repeat the transfinite Newton algorithm, and we are lucky enough to have an explicit formula for the first \( 2\aleph_0 \)-terms of the canonical expansion of a root \( \zeta_{p^2} \) of the \( p^2 \)-th cyclotomic polynomial \( \Phi_{p^2}(T) \). However, in this paragraph, we will introduce a variant of transfinite Newton algorithm and sketch the proof of the **Theorem 1.7** via this new method. The interest of this new method is that it can be used to get the first \( \aleph_0 \)-terms of the truncated expansion of \( \zeta_{p^2} \) efficiently.

#### 3.2.1. \( \left( 1 + \frac{2}{p-1} \right) \)-truncated expansion of \( \Lambda_{p-1}^p - 1 \)

Let \( \lambda = \zeta_{2(p-1)p^{\aleph_0}} \) and \( \Lambda_{p-1} = \sum_{k=0}^{p-1} \frac{\lambda^k}{|k|} \). In this paragraph, we will give the \( \left( 1 + \frac{2}{p-1} \right) \)-truncated expansion of \( \Lambda_{p-1}^p - 1 \) (cf. **Proposition 3.8**), which will be used to give the \( p\aleph_0 \)-terms of the truncated expansion of \( \zeta_{p^2} \).
Let \( \hat{\Lambda}_{p-1} = \sum_{k=0}^{p-1} \frac{\lambda^k}{k!} \). Since \( \hat{\Lambda}_{p-1} - \Lambda_{p-1} \in O\left( p^{1+\frac{2}{p-1}} \right) \), we have

\[
\Lambda_{p-1}^p - 1 = \hat{\Lambda}_{p-1}^p - 1 + O(p^2).
\]

Using binomial expansion, we can write

\[
\hat{\Lambda}_{p-1}^p - 1 = \left( \sum_{k=1}^{p-1} \frac{\lambda^k}{k!} \right)^p + \sum_{j=1}^{p-1} \binom{p}{j} \left( \sum_{k=1}^{p-1} \frac{\lambda^k}{k!} \right)^j. \tag{3.1}
\]

Thus, we reduce to study the \((1 + \frac{2}{p-1})\)-truncated expansions of \( \left( \sum_{k=1}^{p-1} \frac{\lambda^k}{k!} \right)^p \) and \( \sum_{j=1}^{p-1} \binom{p}{j} \left( \sum_{k=1}^{p-1} \frac{\lambda^k}{k!} \right)^j \), which will be established in Lemma 3.5 and Lemma 3.7 respectively.

**Lemma 3.5.** We have the following \((1 + \frac{2}{p-1})\)-truncated expansion for \( \left( \sum_{k=1}^{p-1} \frac{\lambda^k}{k!} \right)^p \):

\[
\left( \sum_{k=1}^{p-1} \frac{\lambda^k}{k!} \right)^p = \sum_{k=1}^{p-1} \frac{\lambda^{kp}}{k!} + \sum_{n=1}^{p-1} \frac{p!}{(n+p)!} \left\{ \frac{n+p}{p} \right\}_{\leq p-1} \lambda^{n+p} + O\left( p^{1+\frac{2}{p-1}} \right).
\]

**Proof.** For the term \( \left( \sum_{k=1}^{p-1} \frac{\lambda^k}{k!} \right)^p \), by the generating function formula for the restricted Stirling number of the second kind, we have

\[
\left( \sum_{k=1}^{p} \frac{\lambda^k}{k!} \right)^p = \sum_{n=p}^{p(p-1)} \frac{p!}{n!} \left\{ \frac{n}{p} \right\}_{\leq p-1} \lambda^n.
\]

Let \( \delta_j = \begin{cases} 1, & \text{if } j \leq p - 1, \\ 0, & \text{if } j \geq p, \end{cases} \) and we have

\[
\frac{p!}{n!} \left\{ \frac{n}{p} \right\}_{\leq p-1} = \sum_{\sum j_k = p, \sum k_j = n, j_1, \ldots, j_{n-p+1} \geq 0} \binom{p}{j_1, \ldots, j_{n-p+1}} \left( \frac{\delta_{j_1}}{n!} \right) \cdots \left( \frac{\delta_{j_{n-p+1}}}{(n-p+1)!} \right)^{j_{n-p+1}}.
\]

If \( p \nmid n \), then \( j_1, \ldots, j_{n-p+1} < p \) and \( p \mid \binom{p}{j_1, \ldots, j_{n-p+1}} \). Therefore, if \( p \nmid n \) and \( n \geq 2p \), together with \( v_p(\lambda) = \frac{1}{p(p-1)} \), we have \( v_p\left( \frac{p!}{n!} \left\{ \frac{n}{p} \right\}_{\leq p-1} \lambda^n \right) \geq 1 + \frac{2}{p-1} \). As
a consequence, we have:

$$
\sum_{n=p}^{p(p-1)} \frac{p!}{n!} \{ n \} \leq p^{-1} \lambda^n
$$

$$
= \sum_{k=1}^{p-1} \frac{p!}{(kp)!} \{ kp \} \leq p^{-1} \lambda^{kp} + \sum_{n=p+1}^{2p-1} \frac{p!}{n!} \{ n \} \leq p^{-1} \lambda^n + O\left(p^{1+\frac{2}{p-1}}\right)
$$

$$
= \sum_{k=1}^{p-1} \frac{p!}{(kp)!} \{ kp \} \leq p^{-1} \lambda^{kp} + \sum_{n=1}^{p} \frac{p!}{(n+p)!} \{ n+p \} \leq p^{-1} \lambda^{kp} + O\left(p^{1+\frac{2}{p-1}}\right).
$$

By [19, Lemma 3.17], one has

$$
\frac{p!}{(kp)!} \{ kp \} \leq \begin{cases}
1, & \text{if } k = 1; \\
\frac{1}{k!} \{ kp \} \leq p^{-1} + O(p) = \frac{1}{k!} + O(p), & \text{if } 2 \leq k \leq p - 1.
\end{cases}
$$

Then we have $\sum_{k=1}^{p-1} \frac{p!}{(kp)!} \{ kp \} \leq p^{-1} \lambda^{kp} = \sum_{k=1}^{p-1} \frac{p!}{(kp)!} \{ kp \} \leq p^{-1} \lambda^{kp} + O\left(p^{1+\frac{2}{p-1}}\right)$, and the result follows.

**Lemma 3.6.** Let $p \geq 3$ be a prime.

1. Let $\alpha \in O_{p}$. Then we have

$$
\sum_{s=1}^{p-1} \binom{p}{s} \sum_{t=s}^{p-1} \frac{s!}{t!} \{ t \} \leq p^{-1} \lambda^{kp} + O\left(p^{2+2v_{p}(\alpha)}\right).
$$

2. Let $U_{p} = 1 + \frac{1}{(p-1)^{2}}$. We have

$$
\frac{1}{p!} \sum_{m=1}^{p-1} \binom{p}{m} \frac{m!}{p!} \{ m \} \leq -U_{p} + O(p^{p-1}).
$$

**Proof.** 1. Since $t - s + 1 \leq p - 1 + 1 - 1 = p - 1$, one has $\binom{t}{s} \leq p^{-1} = \binom{t}{s}$.

Therefore by exchanging the order of the summations, we obtain

$$
\sum_{s=1}^{p-1} \binom{p}{s} \sum_{t=s}^{p-1} \frac{s!}{t!} \{ t \} \leq p^{-1} \sum_{t=1}^{p} \frac{\alpha^{t}}{t!} \sum_{s=1}^{t} \binom{p}{s} s! \{ t \} = \sum_{t=1}^{p} \frac{\alpha^{t}}{t!} \sum_{s=1}^{t} \binom{p}{s} \{ t \}
$$

By [19 (3.7)],

$$
\sum_{s=1}^{t} \binom{p}{s} \{ t \} = \sum_{s=0}^{t} \binom{p}{s} \{ t \} - (p) \{ 0 \} = p^{t}.
$$
Therefore
\[
\sum_{s=1}^{p-1} \left( \frac{p}{s} \right) \sum_{t=s}^{p-1} \left( \frac{s!}{t!} \left\{ t \right\} \right) \alpha^t = \sum_{t=1}^{p-1} \frac{\alpha^t}{t!} p^t = p\alpha + O\left(p^{2+2v_p(\alpha)}\right).
\]

2. Notice that \( \{p\}_{m} \) for \( m \geq 2 \), one can write
\[
\frac{1}{p!} \sum_{m=1}^{p-1} \left( \frac{p}{m} \right) m!\left\{ \frac{p}{m} \right\} \leq p = 1 + \frac{p-1}{p!} (p^p - p) = -U_p + O(p^{p-1}).
\]

\[\square\]

**Lemma 3.7.** The \((1 + \frac{2}{p-1})\)-truncated expansion of \( \sum_{j=1}^{p-1} \left( \frac{p}{j} \right) \left( \sum_{t=1}^{p-1} \frac{\lambda^t}{t!} \right)^j \) is the following:

\[
p\lambda + \zeta_{2(p-1)p^{\frac{1}{p-1}}} \cdot U_p + \frac{\lambda^{n+p}}{(n+p)!} \sum_{s=1}^{p-1} \frac{1}{s!} \left\{ \frac{n}{s} \right\} \left( \sum_{t=s}^{p-1} \left( \frac{p}{s} \right) s!\left\{ \frac{t}{s} \right\} \right) \leq p = 1 + \frac{1}{p-1}.
\]

where \( U_p = 1 + \frac{1}{p-1} \).

**Proof.** Using the generating function formula for Stirling number of the second kind, we have
\[
\sum_{j=1}^{p-1} \left( \frac{p}{j} \right) \left( \sum_{t=1}^{p-1} \frac{\lambda^t}{t!} \right)^j = \sum_{s=1}^{p-1} \frac{1}{s!} \sum_{t=s}^{p-1} \left\{ \frac{t}{s} \right\} \frac{\lambda^t}{t!} = \sum_{s=1}^{p-1} \frac{1}{s!} \sum_{t=s}^{p-1} \left( \frac{p}{s} \right) s!\left\{ t \right\} \frac{\lambda^t}{t!} \leq p = 1 + \frac{1}{p-1}.
\]

Since \( v_p(s!\left\{ t \right\} \leq p-1) \geq 0 \) and \( v_p(\lambda) = \frac{1}{p(p-1)} \), we have:
\[
\sum_{j=1}^{p-1} \left( \frac{p}{j} \right) \left( \sum_{t=1}^{p-1} \frac{\lambda^t}{t!} \right)^j = \sum_{s=1}^{p-1} \left( \frac{p}{s} \right) \sum_{t=s}^{p-1} \left( \frac{s!}{t!} \left\{ t \right\} \leq p-1 \right) \lambda^t + O\left(p^{1+\frac{2}{p-1}}\right)
\]
\[
= \sum_{s=1}^{p-1} \left( \frac{p}{s} \right) \sum_{t=s}^{p-1} \left( \frac{s!}{t!} \left\{ t \right\} \leq p-1 \right) \lambda^t + \sum_{s=1}^{p-1} \left( \frac{p}{s} \right) \sum_{t=p}^{p-1} \left( \frac{s!}{t!} \left\{ t \right\} \leq p-1 \right) \lambda^t + O\left(p^{1+\frac{2}{p-1}}\right)
\]

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We estimate the terms

$$
\sum_{s=1}^{p-1} \binom{p}{s} \sum_{t=s}^{p-1} \frac{s! \{ t \}}{t! \{ s \} \leq p-1} \lambda^t
$$

and

$$
\sum_{s=1}^{p-1} \binom{p}{s} \sum_{t=p}^{2p-1} \frac{s! \{ t \}}{t! \{ s \} \leq p-1} \lambda^t
$$

separately:

1. By applying Lemma 3.6 (1) to $\alpha = \lambda$, we obtain

$$
\sum_{s=1}^{p-1} \binom{p}{s} \sum_{t=s}^{p-1} \frac{s! \{ t \}}{t! \{ s \} \leq p-1} \lambda^t = p\lambda + O(p^2).
$$

2. By exchanging the order of summations in $\sum_{s=1}^{p-1} \binom{p}{s} \sum_{t=p}^{2p-1} \frac{s! \{ t \}}{t! \{ s \} \leq p-1} \lambda^t$, we have

$$
\sum_{s=1}^{p-1} \binom{p}{s} \sum_{t=p}^{2p-1} \frac{s! \{ t \}}{t! \{ s \} \leq p-1} \lambda^t = \sum_{t=p}^{2p-1} \frac{\lambda^t}{t!} \left( \sum_{s=1}^{p-1} \binom{p}{s} s! \{ t \} \right) \leq p-1.
$$

Moreover, for $t = p$, by Lemma 3.6 (2) we have

$$
\frac{\lambda^p}{p!} \sum_{s=1}^{p-1} \binom{p}{s} s! \{ p \} \leq p-1 = \zeta_{2(p-1)p^{\frac{p-1}{p}}} U_p + O(p^2).
$$

Therefore, we have

$$
\sum_{j=1}^{p-1} \binom{p}{j} \left( \sum_{l=1}^{p} \lambda^l \right)^j
$$

$$
= p\lambda + \zeta_{2(p-1)p^{\frac{p-1}{p}}} U_p + \sum_{t=p+1}^{2p-1} \frac{\lambda^t}{t!} \left( \sum_{s=1}^{p-1} \binom{p}{s} s! \{ t \} \right) + O(p^{1+\frac{2}{p}})
$$

$$
= p\lambda + \zeta_{2(p-1)p^{\frac{p-1}{p}}} U_p + \sum_{t=p+1}^{2p-1} \lambda^t \left( \sum_{s=1}^{p-1} \frac{p!}{(p-s)!t!} \{ t \} \right) + O(p^{1+\frac{2}{p}})
$$

$$
= p\lambda + \zeta_{2(p-1)p^{\frac{p-1}{p}}} U_p + \sum_{n=1}^{p-1} \frac{\lambda^{n+p}}{(n+p)!} \left( \sum_{s=1}^{p-1} \binom{p}{s} s! \{ n+p \} \right) \leq p-1 + O(p^{1+\frac{2}{p}}).
$$

Let $\tilde{\lambda}_{p-1} = \sum_{k=0}^{p-1} \frac{\lambda^k}{k!}$, $\tilde{\Lambda}_{p-1} = \sum_{k=0}^{p-1} \frac{\lambda^k}{k!}$ and $\tilde{\lambda}_{p-1} = \tilde{\Lambda}_{p-1} + \zeta_{2(p-1)p^{\frac{p-1}{p}}} U_p$. 

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Proposition 3.8. We have the \((1 + \frac{2}{p^{\frac{1}{t}}})\)-truncated expansion of \(\Lambda_{p-1}^p - 1\):

\[
\Lambda_{p-1}^p - 1 = \Lambda_{p-1}^p - 1 + \sum_{n=1}^{p-1} \frac{p!}{(n+p)!} \left\{ \frac{n+p}{p} \right\}_{\leq p-1} \lambda^{n+p}
+ \zeta_2(p-1) p^{1 + \frac{1}{p^{\frac{1}{t}}}} + \zeta_2(p-1) p^{1 + \frac{1}{p^{\frac{1}{t}}}} \cdot U_p
\]

\[
+ \sum_{n=1}^{p-1} \frac{\lambda^{n+p}}{(n+p)!} \left( \sum_{s=1}^{p-1} \frac{p!}{s!} \left\{ \frac{n+p}{s} \right\}_{\leq p-1} \right)
+ O \left( p^{1 + \frac{2}{p^{\frac{1}{t}}}} \right).
\]

Proof. By combining Lemma 3.5 and Lemma 3.7 one has

\[
\Lambda_{p-1}^p - 1 = \Lambda_{p-1}^p - 1 + \sum_{n=1}^{p-1} \frac{p!}{(n+p)!} \left\{ \frac{n+p}{p} \right\}_{\leq p-1} \lambda^{n+p}
+ \zeta_2(p-1) p^{1 + \frac{1}{p^{\frac{1}{t}}}} + \zeta_2(p-1) p^{1 + \frac{1}{p^{\frac{1}{t}}}} \cdot U_p
\]

\[
+ \sum_{n=1}^{p-1} \frac{\lambda^{n+p}}{(n+p)!} \left( \sum_{s=1}^{p-1} \frac{p!}{s!} \left\{ \frac{n+p}{s} \right\}_{\leq p-1} \right)
+ O \left( p^{1 + \frac{2}{p^{\frac{1}{t}}}} \right).
\]

The result follows from Proposition 2.8. In fact, we have

\[
\sum_{n=1}^{p-1} \frac{\lambda^{n+p}}{(n+p)!} \left( \sum_{s=1}^{p-1} \frac{p!}{s!} \left\{ \frac{n+p}{s} \right\}_{\leq p-1} \right)
= \sum_{n=1}^{p-1} p^{n+p} \left( \sum_{s=1}^{p-1} \frac{(p-1)!}{(p-s)!} \left\{ \frac{n+p}{s} \right\}_{\leq p-1} \right)
= \sum_{n=1}^{p-1} \frac{(-1)^{n+1}}{n!n} \zeta_2(p-1) p^{1 + \frac{1}{p^{\frac{1}{t}}}} + \zeta_2(p-1) p^{1 + \frac{1}{p^{\frac{1}{t}}}} + \zeta_2(p-1) p^{1 + \frac{1}{p^{\frac{1}{t}}}} + O(p^2).
\]

Remark 3.9. In order to simplify the formula, we use Proposition 2.8 here to get rid of the Stirling’s numbers. As a consequence, the mod p harmonic identity will not appear directly in the computation.

Lemma 3.10. One has

\[
\frac{1}{\Lambda_{p-1}^p} = \sum_{k=0}^{p-1} \frac{k \zeta_{2(p-1)} k^{1 + \frac{1}{p^{\frac{1}{t}}}}}{k!} - \zeta_{2(p-1)} p^{1 + \frac{1}{p^{\frac{1}{t}}}} - \zeta_{2(p-1)} p^{1 + \frac{1}{p^{\frac{1}{t}}}} U_p
\]

\[
- 2\zeta_{2(p-1)} p^{1 + \frac{1}{p^{\frac{1}{t}}}} + \frac{1}{p^{\frac{1}{t}}} + \kappa + O \left( p^{1 + \frac{2}{p^{\frac{1}{t}}}} \right),
\]

where \(\kappa = - \sum_{n=2}^{p-1} \frac{(-1)^{n+1}}{n!n} \zeta_2(p-1) p^{1 + \frac{1}{p^{\frac{1}{t}}}} + \zeta_2(p-1) p^{1 + \frac{1}{p^{\frac{1}{t}}}} + \zeta_2(p-1) p^{1 + \frac{1}{p^{\frac{1}{t}}}}\).
Proof. Since $v_p\left(1 - \Lambda_{p-1}^p\right) = \frac{1}{p - \tau}$, we have

$$\frac{1}{\Lambda_{p-1}^p} = \frac{1}{1 - (1 - \Lambda_{p-1}^p)} = \sum_{k=0}^{p} (-1)^k \left(\Lambda_{p-1}^p - 1\right)^k + O\left(p^{1 + \frac{2}{p - \tau}}\right)$$

$$= \sum_{k=0}^{p} (-1)^k \left(\hat{\Lambda}_{p-1} - 1 + \zeta_{2(p-1)}p^{1 + \frac{1}{p - \tau}} - \zeta_{2(p-1)}p^{\frac{1}{p - \tau}}U_p - \kappa\right)^k$$

$$+ O\left(p^{1 + \frac{2}{p - \tau}}\right).$$

An direct computation gives

$$\left(\hat{\Lambda}_{p-1} - 1 + \zeta_{2(p-1)}p^{1 + \frac{1}{p - \tau}} + \zeta_{2(p-1)}p^{\frac{1}{p - \tau}}U_p - \kappa\right)^k$$

$$= \begin{cases} \left(\hat{\Lambda}_{p-1} - 1\right)^k + O\left(p^{1 + \frac{2}{p - \tau}}\right), & \text{if } k \geq 3; \\
\left(\hat{\Lambda}_{p-1} - 1\right)^2 - 2\zeta_{2(p-1)}p^{1 + \frac{1}{p - \tau}} + \zeta_{2(p-1)}p^{\frac{1}{p - \tau}} + O\left(p^{1 + \frac{2}{p - \tau}}\right), & \text{if } k = 2. \end{cases}$$

Therefore, we have

$$\frac{1}{\Lambda_{p-1}^p} = \sum_{k=0}^{p} (-1)^k \left(\hat{\Lambda}_{p-1} - 1\right)^k - \zeta_{2(p-1)}p^{1 + \frac{1}{p - \tau}} - \zeta_{2(p-1)}p^{\frac{1}{p - \tau}}U_p$$

$$- 2\zeta_{2(p-1)}p^{1 + \frac{1}{p - \tau}} + \zeta_{2(p-1)}p^{\frac{1}{p - \tau}} + \kappa + O\left(p^{1 + \frac{2}{p - \tau}}\right).$$

In the following, we expand the term $\sum_{k=0}^{p} (-1)^k \left(\hat{\Lambda}_{p-1} - 1\right)^k$, which will allow us to conclude the result. Let $\eta = -\zeta_{2(p-1)}p^{\frac{1}{p - \tau}}$. Using Proposition 2.1 (4) we obtain

$$\sum_{k=0}^{p} (-1)^k \left(\hat{\Lambda}_{p-1} - 1\right)^k = 1 + \sum_{k=1}^{p} (-1)^k \sum_{n=k}^{\infty} \frac{k!}{n!} \left\{ n \right\}_{n=k} \eta^n$$

$$= 1 + \sum_{k=1}^{p} (-1)^k \sum_{n=k}^{p} \frac{k!}{n!} \left\{ n \right\}_{n=k} \eta^n + O\left(p^{1 + \frac{2}{p - \tau}}\right).$$

By exchanging the order of summations, one has

$$\sum_{k=0}^{p} (-1)^k \left(\hat{\Lambda}_{p-1} - 1\right)^k = 1 + \sum_{n=1}^{p} \eta^n \sum_{k=1}^{n} (-1)^k \frac{n}{k!} \left\{ n \right\}_{n=k} \sum_{k=1}^{\infty} (-1)^k \left\{ n \right\}_{n=k} \leq p-1 + O\left(p^{1 + \frac{2}{p - \tau}}\right).$$
Together with the fact \( \{ n \atop k \} \leq p - 1 = \{ n \atop k \} \) for \( n - k + 1 \leq p - 1 \), we have

\[
\sum_{k=0}^{p} (-1)^k \left( \hat{\Lambda}_{p-1} - 1 \right)^k = 1 + \sum_{n=1}^{p} \eta^n n! \sum_{k=1}^{n} (-1)^k k! \{ n \atop k \} \\
+ \frac{\eta^p}{p!} (-1)^1 1! \{ p \atop 1 \} \leq_{p-1} \{ p \atop 1 \} + O \left( p^{1+\frac{2}{p-1}} \right) \\
= 1 + \frac{\eta^p}{p!} + \sum_{n=1}^{p} \frac{\eta^n}{n!} \sum_{k=1}^{n} (-1)^k k! \{ n \atop k \} + O \left( p^{1+\frac{2}{p-1}} \right). \\
(3.2)
\]

Notice that

\[
(p^3 - 1)_k = (p^3 - 1)(p^3 - 2) \ldots (p^3 - k) = (-1)^k k! + O(p^3),
\]

one has

\[
\sum_{k=1}^{n} (-1)^k k! \{ n \atop k \} = \sum_{k=1}^{n} ((p^3 - 1)_k + O(p^3)) \{ n \atop k \} = \sum_{k=1}^{n} (p^3 - 1)_k \{ n \atop k \} + O(p^3) \\
= (p^3 - 1)^n + O(p^3) = (-1)^n + O(p^3).
\]

where the second to last identity is [Page 207 Theorem B]. Therefore, the expansion (3.2) can be rewritten as

\[
\sum_{k=0}^{p} (-1)^k \left( \hat{\Lambda}_{p-1} - 1 \right)^k = 1 + \frac{\eta^p}{p!} + \sum_{n=1}^{p} \frac{(-\eta)^n}{n!} + O \left( p^{1+\frac{2}{p-1}} \right).
\]

As a consequence, we have

\[
\frac{1}{\hat{\Lambda}_{p-1}} = - \zeta_{2(p-1)} p^{1+\frac{2}{p-1}} - 2 \zeta_{2(p-1)}^2 p^{1+\frac{2}{p-1}} + \frac{1}{p-1} - \zeta_{2(p-1)} p^{\frac{1}{p-1}} U_p + \kappa \\
+ 1 + \frac{\eta^p}{p!} + \sum_{n=1}^{p} \frac{(-\eta)^n}{n!} + O \left( p^{1+\frac{2}{p-1}} \right) \\
= \sum_{k=0}^{p-1} \frac{\zeta_{2(p-1)}^k}{k!} p^{\frac{k}{p-1}} - \zeta_{2(p-1)} p^{1+\frac{1}{p-1}} - \zeta_{2(p-1)} p^{\frac{1}{p-1}} U_p \\
- 2 \zeta_{2(p-1)}^2 p^{1+\frac{1}{p-1}} + \frac{1}{p-1} + \kappa + O \left( p^{1+\frac{2}{p-1}} \right).
\]

3.2.2. Perturbed polynomial and its Newton polygon

The first step of our strategy is to use the transfinite Newton algorithm to get the first few terms (finite or infinite) of the canonical expansion of a root of \( P(T) \). In the case \( P(T) = \Phi_{p,2}(T) \), the first \( \aleph_0 \) terms of a root \( \zeta_{p,2} \) of \( P(T) \) is given in

\[24\]
1. Take \( \frac{1}{p-1} \)-truncated canonical expansion of \( \zeta_p^2 \) is the following:

\[
\zeta_p^2 = \Lambda_{p-1} + \zeta_{2(p-1)} p^{\frac{1}{p(p-1)}} \sigma_2 + O\left(p^{\frac{1}{p-1}}\right).
\]

The second step is to disturb \( P(T) \). Suppose \( \mu_0 \in \mathbb{L}_p \) is an approximation of a root of \( P(T) \) obtained in the first step of our strategy. We set \( r_0 = \sup \text{Supp}(\mu_0) \) and choose a rational number \( r > r_0 \). We choose randomly an element \( \alpha \in \mathbb{L}_p \) such that \( r_0 < \nu_p(\alpha) \) and \( \sup \text{Supp}(\alpha) = r \). Then we disturb \( P(T) \) by \( \mu = \mu_0 + \alpha \) and estimate the maximal slope \( s_\mu \) of the perturbed polynomial \( P_\mu(T) \). Since \( P_\mu(T) = \sum_{k=0}^n b_k T^{n-k} \) with \( b_k = \sum_{j=0}^{k} a_{k-j} (p^{k-j}) \mu^j \) for \( 0 \leq k \leq n \), we can reduce the estimation of the maximal slope to calculate the \( p \)-adic valuation of \( b_k \) for \( 0 \leq k \leq n \). There are two different cases depending on the value of \( s_\mu \):

1. if \( s_\mu \geq r \), then \( \mu \) is a \( r \)-truncated expansion of a root of \( P(T) \);
2. if \( s_\mu < r_0 \), then we add an indeterminate \( M \) to \( \mu \). We try to find the solution of \( M \) which corrects \( \mu \) such that \( \mu + M \) gives the \( r \)-truncated expansion of a root of \( P(T) \). Since \( \mu_0 \) is an approximation, we have \( \nu_p(M) \geq r_0 \) and \( \mu + M = \mu_0 + O(p^r) \). This provides a possibility to produce a truncated equation by considering \( (\mu + M) p^k = (\mu_0 + O(p^r)) p^k \) for some \( k \geq 1 \).

In the rest of this paragraph, we establish two truncated expansions of a root \( \zeta_p^2 \) of \( P(T) = \Phi_p(T) \), which exactly correspond to the two situations described as above. For any \( \mu \in \mathbb{L}_p \), we have the following formula for the coefficients of \( \mu \)-perturbed polynomial \( P_\mu(T) = \Phi_p(T + \mu) =: \sum_{k=0}^{p(p-1)} b_{p(p-1)-k} T^k \):

\[
\begin{align*}
\beta_{p(p-1)} &= \sum_{l=0}^{p-1} \mu^l \mu^p = \frac{\mu^{p^2} - 1}{\mu^p - 1}; \\
\beta_{p(p-1)-1} &= \sum_{l=1}^{p-1} \left( \frac{pl}{1} \right) \mu^{pl-1} = \frac{p}{\mu} \left( \mu^{p^2} - 1 + \mu^p - 1 \right).
\end{align*}
\]

1. Take \( \mu_0 = \Lambda_{p-1} + \zeta_{2(p-1)} p^{\frac{1}{p(p-1)}} \sigma_2 \), \( r_0 = \frac{1}{p-1} \), \( r = \frac{1}{p-1} + \frac{1}{p(p-1)} \) and \( \mu = \Lambda_{p-1} + \left( 1 + \zeta_{2(p-1)} p^{\frac{1}{p(p-1)}} \right) \zeta_{2(p-1)} p^{\frac{1}{p(p-1)}} \sigma_2 \).

With the expansion of \( \mu^p - 1 \) and \( \mu^{p^2} - 1 \) (cf. Lemma 3.16 and (3.7)), a direct calculation using the strong triangle inequality gives:

\[
v_p(\beta_{p(p-1)-1}) = v_p \left( \frac{p}{\mu} \left( \mu^{p^2} - 1 + \mu^p - 1 \right) \right) = 2 - \frac{1}{p-1};
\]

\[25\]
\[ v_p(b_{p(p-1)}) = v_p\left(\frac{\mu^p - 1}{\mu - 1}\right) = 2 + \frac{1}{p(p-1)}. \]

Notice that the line passing through the points \((p(p-1) - 1, v_p(b_{p(p-1)-1}))\) and \((p(p-1), v_p(b_{p(p-1)})\) has slope equal to or smaller than the maximal slope of \(\mathcal{N}_{\text{new}}(P_\mu)\). As a consequence, the maximal slope of \(\mathcal{N}_{\text{new}}(P_\mu)\) is greater or equal to
\[ \left(2 + \frac{1}{p(p-1)}\right) - \left(2 - \frac{1}{p-1}\right) = \frac{1}{p(p-1)} + \frac{1}{p-1}, \]
which implies that there exists a root of \(P_\mu\) with valuation \(\geq \frac{1}{p(p-1)} + \frac{1}{p-1}\).

Therefore we conclude

**Proposition 3.11.** There exists a \(p^2\)-th primitive root of unity \(\zeta_{p^2}\) satisfying
\[ \zeta_{p^2} = \mu + O\left(\frac{1}{p^{r-1} + \frac{1}{p}}\right). \quad (3.4) \]

2. Recall that \(\sigma_2 = \sum_{k=2}^{\infty} \frac{1}{p^k}.\) We take
\[ \mu_0 = \Lambda_{p-1} + (1+\lambda)\zeta_{2(p-1)p^{r-1}}\sigma_2, r_0 = \frac{1}{p-1} + \frac{1}{p(p-1)}, r = \frac{2}{p-1} \]
and \(\mu = \Lambda_{p-1} \left(1 + \zeta_{2(p-1)p^{r-1}}\sigma_2\right).\) Suppose \(\Lambda = \mu + \mathcal{M}\) is a \(r\)-truncated expansion of \(\zeta_{p^2}\) with an indeterminate \(\mathcal{M}\). Then we have
\[ \Lambda = \mu + \mathcal{M} = \mu_0 + O(p^{r_0}). \]

Consider the truncated expansions of \(\Lambda^p\).

(a) Using \(\Lambda = \mu_0 + O(p^{r_0})\), we get
\[ \Lambda^p = \mu_0^p + \sum_{k=1}^{p-1} \binom{p}{k} \mu_0^{p-k}O(p^{r_0})^k + O(p^{r_0})^p. \]

Note that \(r_0 \cdot p = 1 + \frac{2}{p-1}\) and \(v_p\left(\sum_{k=1}^{p-1} \binom{p}{k} \mu_0^{p-k}O(p^{r_0})^k\right) \geq 1 + \frac{1}{p-1} + \frac{1}{p^{r_0}}\).

As \(\Lambda\) is an approximation of \(\zeta_{p^2}\), the majoration of the maximal slope of the Newton polygon of \(P_\Lambda\) will improve the truncated position of \(\Lambda^p\) (cf. **Lemma 3.17**):
\[ \Lambda^p = \Lambda_{p-1}^+ + O(p^{1+r}), \quad (3.5) \]
where \(\Lambda_{p-1}^+ = \sum_{j=0}^{p-1} \frac{(-1)^j}{[p]} \zeta_{2(p-1)p^{j-1}} + \zeta_{2(p-1)p^{j-1}} U_p.\)

(b) Using \(\Lambda = \mu + \mathcal{M}\), \(v_p(\mathcal{M}) \geq r_0\) and \(v_p(\mathcal{M}^k) \geq \frac{2}{p-1}\) for \(k \geq 2\), we have
\[ \Lambda^p = \sum_{k=0}^{p} \binom{p}{k} \mu^{p-k} \mathcal{M}^k = \mu^p + p\mu^{p-1} \mathcal{M} + O(p^{1+r}). \]
Thus, together with (3.5), we have \( \mu^p + p\mu^{p-1} \mathcal{M} = \hat{\Lambda}_{p-1} + O(p^{1+r}) \). In other words, we have an equation \( p\mathcal{M} = \mu \left( \frac{\hat{\Lambda}_{p-1} + 1}{\mu^p} \right) + O(p^{1+r}) \). We calculate the \((1 + r)\)-truncated expansion of \( \mu \left( \frac{\hat{\Lambda}_{p-1} + 1}{\mu^p} \right) \) in Section 3.3.3 and obtain the value of \( \mathcal{M} \) (cf. Lemma 3.20): One has

\[
\mathcal{M} = \frac{1}{2} \zeta_2(p-1) p^{\frac{1}{p-1}} \sigma_2^2 + \frac{1}{2} \zeta_2(p-1) p^{\frac{1}{p-1}} - \frac{p-1}{2p^{1/p}} \\
- \sum_{k=1}^{p-1} \frac{H_k}{k!} \zeta_{2(p-1)} p^{\frac{k}{p-1}} + O(p^{\frac{1}{p-1}}),
\]

where \( H_k = \sum_{n=1}^{k} \frac{1}{n} \) is the harmonic number.

### 3.3. Estimations

Let \( P(T) \in \mathcal{L}_p[T] \) be a given polynomial. For any \( \Lambda \in \mathcal{L}_p \), the \( \sigma \)-perturbed polynomial \( P_\sigma(T) \), in particular the Newton polygon associated to \( P_\sigma(T) \), plays an important role in the transfinite Newton algorithm and its variant. When taking \( P(T) = \Phi_{p^2}(T) \) the \( p^2 \)-th cyclotomic polynomial, the coefficients of its \( \sigma \)-perturbed polynomial is determined by \( \sigma^p \) and \( \Lambda^{p^2} \) (cf. the formula (3.3)). In this paragraph, we will establish the truncated expansions of \( \Lambda^p \) and \( \Lambda^{p^2} \) for \( \Lambda = \Lambda_{p-1} + (1 + \lambda)\zeta_2(p-1)p^{\frac{1}{p-1}} \sigma_2 \) or \( \Lambda = \Lambda_{p-1} \left( 1 + \zeta_2(p-1)p^{\frac{1}{p-1}} \sigma_2 \right) + \mathcal{M} \).

#### 3.3.1. Two useful lemmas

The first lemma will be used in our calculation, in particular, for \( A = \sigma_n \), with \( \sigma_n = \sum_{k=n}^{\infty} p^{-1/p^k} \).

**Lemma 3.12.** Let \( A = \sum_{q \in \mathbb{Q}} [\alpha_q] p^q \in \mathcal{L}_p \) with \( \text{Supp}(A) \subset \left[ 0, \frac{1}{p} \right) \). Then

\[
A^p = \sum_{q \in \text{Supp}(A)} [\alpha_q]^p p^q + O\left( p^{1+p^\nu_\mathbb{Q}(A)} \right).
\]

**Proof.** For any \( q \in \mathbb{Q} \), let

\[
\mathcal{S}_q = \{ (g_1, \cdots, g_p) \in \text{Supp}(A)^p | g_1 + \cdots + g_p = q \}.
\]

By the definition of multiplication in \( \mathcal{L}_p \), we know that \( A^p = \sum_{q \in \mathbb{Q}} T_q p^q \), where

\[
T_q = \sum_{(g_1, \cdots, g_p) \in \mathcal{S}_q} [\alpha_{g_1} \cdots \alpha_{g_p}]
\]

is a finite sum. Fix \( q_0 \in \text{Supp}(A) \). If \( \mathcal{S}_{q_0} = \emptyset \), then \( T_{q_0} = 0 \).
Otherwise there exists a finite set $R_{q_0} = \{m_1 < \cdots < m_{n_{q_0}}\} \subseteq \text{Supp}(A)$ such that $S_{q_0} \subseteq R_{q_0}^p$, i.e.

$$T_{q_0} = \sum_{g_1, \ldots, g_p \in R_{q_0}} [\alpha_{g_1} \cdots \alpha_{g_p}],$$

In this case, $pm_{m_1} \leq q_0 \leq pm_{m_{n_{q_0}}}$. Now consider the power $(\sum_{q \in R_{q_0}} [\alpha_q]p^q)^p$, which can be expanded as following

$$\left( \sum_{q \in R_{q_0}} [\alpha_q]p^q \right)^p = \sum_{k=1}^{n_{q_0}} [\alpha_{m_k}]p^{m_k} + O \left( p^{1 + \nu_p(m_1)} \right),$$

where

$$\tilde{T}_q = \sum_{g_1, \ldots, g_p \in R_{q_0}} [\alpha_{g_1} \cdots \alpha_{g_p}].$$

By the multinomial expansion, we have

$$\left( \sum_{k=1}^{n_{q_0}} [\alpha_{m_k}]p^{m_k} \right)^p = \sum_{k=1}^{n_{q_0}} [\alpha_{m_k}]p^{m_k} + O \left( p^{1 + \nu_p(m_1)} \right),$$

which leads to the following estimation

$$\tilde{T}_q = \begin{cases} [\alpha_{q/p}] + O(p), & \text{if } q = pm_{m_1}, \ldots, pm_{m_{n_{q_0}}}; \\ O(p), & \text{otherwise}. \end{cases}$$

Therefore, we have

$$T_{q_0} = \begin{cases} [\alpha_{q_0/p}] + O(p), & \text{if } q_0 = pm_{m_1}, \ldots, pm_{m_{n_{q_0}}}; \\ O(p), & \text{otherwise}. \end{cases}$$

Since for every $q \in \mathbb{Q}$ such that $S_q \neq \emptyset$, we can modify $R_q$ by adding arbitrary finite elements of $\text{Supp}(A)$, thus we can conclude that

$$T_q = \begin{cases} [\alpha_{q/p}] + O(p), & \text{if } q \in p\text{Supp}(A); \\ O(p), & \text{if } S_q \neq \emptyset \text{ and } q \notin p\text{Supp}(A); \\ 0, & \text{if } S_q = \emptyset. \end{cases}$$

As a consequence,

$$A^p = \sum_{q \in p\text{Supp}(A)} \left( [\alpha_{q/p}] + O(p) \right) \cdot p^q + \sum_{S_q \neq \emptyset \text{ and } q \notin p\text{Supp}(A)} O(p) \cdot p^q.$$

Since $S_q = \emptyset$ for $q \leq \nu_p(A)$, one obtains

$$A^p = \sum_{q \in p\text{Supp}(A)} [\alpha_{q/p}] \cdot p^q + O \left( p^{1 + \nu_p(A)} \right) = \sum_{q \in \text{Supp}(A)} [\alpha_{q}]p^q + O \left( p^{1 + \nu_p(A)} \right).$$
Corollary 3.13. For positive integer number \( m, n \geq 1 \), we have
\[
\left( \sum_{k=n+m}^{\infty} p^{-1/p^k} \right)^p = \sum_{k=m}^{\infty} p^{-1/p^k} + O\left( p^{1-1/p^m} \right).
\]

Proof. If \( n = 1 \), by Lemma 3.12 we have
\[
p^{1/p^m} \left( \sum_{k=m+1}^{\infty} p^{-1/p^k} \right)^p = \left( \sum_{k=m+1}^{\infty} p^{1/p^{m+1}-1/p^k} \right)^p
= \sum_{k=m+1}^{\infty} p^{1/p^{m}-1/p^k} + O(p).
\]

Then by inverting \( p^{1/p^m} \) on both sides, one obtains
\[
\left( \sum_{k=m+1}^{\infty} p^{-1/p^k} \right)^p = \sum_{k=m}^{\infty} p^{-1/p^{k-1}} + O\left( p^{1-1/p^m} \right).
\]

If the statement is proved for \( 1, 2, \ldots, n \), then
\[
\left( \sum_{k=n+1+m}^{\infty} p^{-1/p^k} \right)^{p^{n+1}} = \left( \sum_{k=n+(m+1)}^{\infty} p^{-1/p^k} \right)^p
= \left( \sum_{k=m+1}^{\infty} p^{-1/p^k} + O\left( p^{1-1/p^{m+1}} \right) \right)^p
= \sum_{t=0}^{p} \binom{p}{t} \left( \sum_{k=m+1}^{\infty} p^{-1/p^k} \right)^{p-t} \cdot O\left( p^{1-1/p^{m+1}} \right)^t.
\]

By moving terms with valuation greater than 1 into the error term, we have
\[
\left( \sum_{k=n+1+m}^{\infty} p^{-1/p^k} \right)^{p^{n+1}} = \left( \sum_{k=m+1}^{\infty} p^{-1/p^k} \right)^p + O(p^1)
= \sum_{k=m}^{\infty} p^{-1/p^{k-1}} + O\left( p^{1-1/p^m} \right).
\]

The result follows from the induction. \( \square \)

The second useful lemma that we want to establish (cf. Lemma 3.15), allows us to reduce the study of \( \Lambda_{p-1}^{p^2} \) to that of \( \Lambda_{p-1}^{p} \).

Lemma 3.14.
\[
\tilde{\Lambda}_{p-1}^p - 1 = -\zeta_{2(p-1)} p^{1+\frac{1}{\nu-1}} \cdot U_p + O\left( p^{2+\frac{2}{\nu-1}} \right).
\]
Proof. Let $\eta = -\zeta_{2(p-1)}p^{\frac{1}{p-1}}$. We have

$$
\left(\sum_{l=0}^{p-1} \frac{\eta^l}{l!}\right)^p - 1 = \sum_{j=1}^{p-1} \binom{p}{j} \left(\sum_{l=0}^{p-1} \frac{\eta^l}{l!}\right)^j = \sum_{j=1}^{p-1} \binom{p}{j} \sum_{k=j}^{p-1} \frac{j!}{k!} \frac{k}{j} \leq_{p-1} \eta^k. \tag{3.6}
$$

Notice that

$$
v_p\left(\binom{p}{j} \left(\sum_{l=0}^{p-1} \frac{\eta^l}{l!}\right)^j \eta^k\right) \geq 1 - v_p(j) + \frac{k}{p-1},
$$

by assembling terms with valuation greater than or equal to $2 + \frac{2}{p-1}$, we can rewrite (3.6) as

$$
\left(\sum_{l=0}^{p-1} \frac{\eta^l}{l!}\right)^p - 1 = \sum_{j=1}^{p-1} \binom{p}{j} \sum_{k=j}^{p-1} \frac{j!}{k!} \frac{k}{j} \leq_{p-1} \eta^k + \sum_{k=p}^{2p-1} \frac{p!}{k!} \left(\frac{p}{j}\right) \eta^k + O\left(p^{2+\frac{2}{p-1}}\right)
$$

$$
= \sum_{j=1}^{p-1} \binom{p}{j} \sum_{k=j}^{p-1} \frac{j!}{k!} \frac{k}{j} \leq_{p-1} \eta^k + \sum_{j=1}^{p-1} \binom{p}{j} \frac{j!}{p!} \left(\frac{p}{j}\right) \leq_{p-1} \eta^k
$$

$$
+ \sum_{k=p}^{2p-1} \frac{p!}{k!} \left(\frac{k}{p}\right) \leq_{p-1} \eta^k + O\left(p^{2+\frac{2}{p-1}}\right).
$$

By Lemma 3.6 (1), one has

$$
\sum_{j=1}^{p-1} \binom{p}{j} \sum_{k=j}^{p-1} \frac{j!}{k!} \frac{k}{j} \leq_{p-1} \eta^k = -\zeta_{2(p-1)}p^{1+\frac{1}{p-1}} + O\left(p^{2+\frac{2}{p-1}}\right)
$$

and

$$
\eta^p \sum_{j=1}^{p-1} \binom{p}{j} \frac{j!}{p!} \left(\frac{p}{j}\right) \leq_{p-1} = -\zeta_{2(p-1)}p^{1+\frac{1}{p-1}} \cdot U_p + O\left(p^2\right).
$$

On the other hand, since $k - p + 1 \leq p - 1$ for $k \leq 2p - 2$, we have

$$
\sum_{k=p}^{2p-1} \frac{p!}{k!} \left(\frac{k}{p}\right) \leq_{p-1} \eta^k = \sum_{k=p}^{2p-2} \frac{p!}{k!} \left(\frac{k}{p}\right) \eta^k + \frac{p!}{(2p-1)!} \left(\frac{2p-1}{p}\right) \leq_{p-1}.
$$

Since $v_p(p!) = v_p(k!) = 1$ for $k = p, \ldots, 2p - 2$, by Proposition 2.1 (2) we have

$$
\frac{p!}{k!} \left(\frac{k}{p}\right) = \begin{cases} O(p), & \text{if } p < k \leq 2p - 2; \\ 1, & \text{if } k = p.
\end{cases}
$$

Besides that, by Proposition 2.1 (3) one has $v_p\left(\frac{p!}{(2p-1)!}\right) = 0$ and

$$
v_p\left(\frac{2p-1}{p}\right) \leq_{p-1} \geq 1.
$$
Therefore
\[
\sum_{k=p}^{2p-1} \frac{k!}{k^k} \leq p^p + \sum_{k=p+1}^{2p-2} O\left(p^{1+\frac{k}{p-1}}\right) + O\left(p^{1+\frac{2p-1}{p-1}}\right) = p^p + O\left(p^{2+\frac{2}{p-1}}\right).
\]

To sum up, we obtain
\[
\left(\sum_{t=0}^{p-1} \frac{\eta^t}{t!}\right)^p - 1 = -\zeta_2(p-1)p^{1+\frac{1}{p-1}} + O\left(p^{2+\frac{2}{p-1}}\right) - \zeta_2(p-1)p^{1+\frac{1}{p-1}} \cdot U_p + O\left(p^3\right)
\]
\[
+ \zeta_2(p-1)p^{1+\frac{1}{p-1}} + O\left(p^{2+\frac{2}{p-1}}\right)
\]
\[
= -\zeta_2(p-1)p^{1+\frac{1}{p-1}} \cdot U_p + O\left(p^{2+\frac{2}{p-1}}\right).
\]

\[\square\]

**Lemma 3.15.** Let \( A \in \mathbb{L}_p \). If \( A = \tilde{\Lambda}^+_{p-1} + O\left(p^{1+\frac{2}{p-1}}\right) \), then
\[
A^p - 1 = p\left(A - \tilde{\Lambda}^+_{p-1}\right) + O\left(p^{1+\frac{2}{p-1}}\right).
\]

**Proof.** Since \( \tilde{\Lambda}^+_{p-1} - \hat{\Lambda}^+_{p-1} \in O\left(p^{1+\frac{2}{p-1}}\right) \), we have
\[
A^p - 1 = \left(\tilde{\Lambda}^+_{p-1} + \zeta_2(p-1)p^{\frac{1}{p-1}} U_p + A - \tilde{\Lambda}^+_{p-1} + O\left(p^{1+\frac{2}{p-1}}\right)\right)^p - 1
\]
\[
= \tilde{\Lambda}^+_{p-1} - 1 + \sum_{t=1}^{p} \left(\begin{array}{c} p \\ t \end{array}\right) \tilde{\Lambda}^{p-t}_{p-1} \left(\zeta_2(p-1)p^{\frac{1}{p-1}} U_p + A - \tilde{\Lambda}^+_{p-1} + O\left(p^{1+\frac{2}{p-1}}\right)\right)^t.
\]

By **Lemma 3.14** we know that
\[
\tilde{\Lambda}^+_{p-1} - 1 = -\zeta_2(p-1)p^{1+\frac{1}{p-1}} \cdot U_p + O\left(p^{2+\frac{2}{p-1}}\right).
\]

Since
\[
u_p\left(\left(\zeta_2(p-1)p^{\frac{1}{p-1}} U_p + A - \tilde{\Lambda}^+_{p-1} + O\left(p^{1+\frac{2}{p-1}}\right)\right)^t\right) \geq t\left(1 + \frac{1}{p-1}\right)
\]
and \( \tilde{\Lambda}^{p-t}_{p-1} = 1 + O\left(p^{\frac{1}{p-1}}\right) \), we have
\[
A^p - 1 = -\zeta_2(p-1)p^{1+\frac{1}{p-1}} \cdot U_p + O\left(p^{2+\frac{2}{p-1}}\right)
\]
\[
+ p\left(\zeta_2(p-1)p^{\frac{1}{p-1}} U_p + A - \tilde{\Lambda}^+_{p-1} + O\left(p^{1+\frac{2}{p-1}}\right)\right)\left(1 + O\left(p^{\frac{1}{p-1}}\right)\right)
\]
\[
= p\left(A - \tilde{\Lambda}^+_{p-1}\right) + O\left(p^{2+\frac{2}{p-1}}\right).
\]

\[\square\]
3.3.2. The truncated expansions of $\Lambda^p$

1. We first study the $\left(1 + \frac{1}{p-1} + \frac{2}{p(p-1)} - \frac{1}{p^2}\right)$-truncated expansion of

$$\Lambda = \Lambda_{p-1} + (1 + \lambda)\zeta_{2(p-1)}p^{\frac{1}{p-1}}\sigma_2,$$

where $\sigma_2 = \sum_{k=2}^{+\infty} p^{-1/p^k}$.

**Lemma 3.16.** The $\left(1 + \frac{1}{p-1} + \frac{2}{p(p-1)} - \frac{1}{p^2}\right)$-truncated expansion of $\Lambda^p - 1$ is the following:

$$\hat{\Lambda}_{p-1} - 1 + \zeta_{2(p-1)}p^{\frac{1}{p-1}} \cdot U_p + \zeta_{2(p-1)}p^{1+\frac{1}{p-1}+\frac{1}{p(p-1)}} + O(p^{1+\frac{1}{p-1}+\frac{2}{p(p-1)}-\frac{1}{p^2}}).$$

**Proof.** Since $\Lambda = \Lambda_{p-1} + \left(1 + \zeta_{2(p-1)}p^{\frac{1}{p-1}}\right)\zeta_{2(p-1)}p^{\frac{1}{p-1}}\sigma_2$, we have

$$\Lambda^p - 1 = \Lambda_{p-1}^p - 1 + \sum_{j=1}^{p} \binom{p}{j} \Lambda_{p-1}^{p-j} \left(1 + \zeta_{2(p-1)}p^{\frac{1}{p-1}}\right)\zeta_{2(p-1)}p^{\frac{1}{p-1}}\sigma_2^j. $$

Notice that, the Corollary 3.13 shows that $\sigma_2^p = \sigma_1 + O(p^{1-1/p})$. Thus,

(a) for $2 \leq j \leq p - 1$, we have

$$v_p\left(\binom{p}{j} \Lambda_{p-1}^{p-j} \left(1 + \zeta_{2(p-1)}p^{\frac{1}{p-1}}\right)\zeta_{2(p-1)}p^{\frac{1}{p-1}}\sigma_2^j\right) \geq 1 + \frac{2}{p-1} - \frac{2}{p^2} \geq 1 + \frac{1}{p-1} + \frac{2}{p(p-1)} - \frac{1}{p^2}.$$

(b) for $j = p$, we have

$$\left(1 + \zeta_{2(p-1)}p^{\frac{1}{p-1}}\right)\zeta_{2(p-1)}p^{\frac{1}{p-1}}\sigma_2^p = \left(1 + \zeta_{2(p-1)}p^{\frac{1}{p-1}}\right)^p \left(-\zeta_{2(p-1)}p^{1+\frac{1}{p-1}}\sigma_1 + O\left(p^{1-1/p}\right)\right) = \left(1 - \zeta_{2(p-1)}p^{\frac{1}{p-1}} + O(p)\right) \cdot \left(-\zeta_{2(p-1)}p^{1+\frac{1}{p-1}} - \zeta_{2(p-1)}p^{1+\frac{1}{p-1}}\sigma_2 + O\left(p^{2+\frac{1}{p-1}}\right)\right) = \left(1 - \zeta_{2(p-1)}p^{\frac{1}{p-1}}\right) \cdot \left(-\zeta_{2(p-1)}p^{1+\frac{1}{p-1}} - \zeta_{2(p-1)}p^{1+\frac{1}{p-1}}\sigma_2 + O\left(p^{2+\frac{1}{p-1}}\right)\right).$$

(c) for $j = 1$, we have

$$p\Lambda_{p-1}^{p-1} \left(1 + \zeta_{2(p-1)}p^{\frac{1}{p-1}}\right)\zeta_{2(p-1)}p^{\frac{1}{p-1}}\sigma_2 = p\left(1 + (p-1)\zeta_{2(p-1)}p^{\frac{1}{p-1}} + O\left(p^{\frac{2}{p-1}}\right)\right)$$
2. Now, we turn to study the second case. Recall that, we set \( \mu = \Lambda_{p-1} + (1 + \lambda) \zeta_{2(p-1)p^{\frac{1}{p-1}}} \sigma_2 \), \( r_0 = \frac{1}{p-1} + \frac{1}{p(p-1)} \), \( r = \frac{2}{p-1} \) and \( \mu = \Lambda_{p-1} \left( 1 + \zeta_{2(p-1)p^{\frac{1}{p-1}}} \sigma_2 \right) \). Suppose \( \Lambda = \mu + \mathcal{M} \) is a \( r \)-truncated expansion of \( \zeta_{p^2} \) with an indeterminate \( \mathcal{M} \). By applying the \( r_0 \)-truncation \( \Lambda = \mu_0 + O(p^{r_0}) \) of \( \Lambda \), we have

\[
\Lambda^p = \hat{\Lambda}^p_{p-1} + O(p^{\frac{1}{p-1}}).
\]

\[
\Lambda^p - 1 = \hat{\Lambda}^p_{p-1} - 1 - \zeta_{2(p-1)p^{\frac{1}{p-1}}} + \zeta_{2(p-1)p^{\frac{1}{p-1}}} \cdot U_p
\]

Thus, we have

\[
\Lambda^p - 1 = \hat{\Lambda}^p_{p-1} - 1 + \zeta_{2(p-1)p^{\frac{1}{p-1}}} + \zeta_{2(p-1)p^{\frac{1}{p-1}}} \cdot U_p
\]

By applying Lemma 3.15 to \( \Lambda^p \), we know that

\[
\Lambda^p - 1 = \hat{\Lambda}^p_{p-1} - 1 + \zeta_{2(p-1)p^{\frac{1}{p-1}}} + \zeta_{2(p-1)p^{\frac{1}{p-1}}} \cdot U_p
\]

Therefore

\[
\Lambda^p - 1 = \hat{\Lambda}^p_{p-1} - 1 + \zeta_{2(p-1)p^{\frac{1}{p-1}}} \cdot U_p + \zeta_{2(p-1)p^{\frac{1}{p-1}}}^2 + \zeta_{2(p-1)p^{\frac{1}{p-1}}} \cdot U_p
\]

By Proposition 3.8

\[
\Lambda^p - 1 = \hat{\Lambda}^p_{p-1} - 1 + \zeta_{2(p-1)p^{\frac{1}{p-1}}} + \zeta_{2(p-1)p^{\frac{1}{p-1}}} \cdot U_p
\]

\[
\Lambda^p - 1 = \hat{\Lambda}^p_{p-1} - 1 + \zeta_{2(p-1)p^{\frac{1}{p-1}}} + \zeta_{2(p-1)p^{\frac{1}{p-1}}}^2 + \zeta_{2(p-1)p^{\frac{1}{p-1}}} \cdot U_p
\]

By applying Lemma 3.15 to \( \Lambda^p \), we know that

\[
\Lambda^p - 1 = \hat{\Lambda}^p_{p-1} - 1 + \zeta_{2(p-1)p^{\frac{1}{p-1}}} \cdot U_p + \zeta_{2(p-1)p^{\frac{1}{p-1}}}^2 + \zeta_{2(p-1)p^{\frac{1}{p-1}}} \cdot U_p
\]

Therefore

\[
\Lambda^p - 1 = \hat{\Lambda}^p_{p-1} - 1 + \zeta_{2(p-1)p^{\frac{1}{p-1}}} + \zeta_{2(p-1)p^{\frac{1}{p-1}}} \cdot U_p
\]

By applying Lemma 3.15 to \( \Lambda^p \), we know that

\[
\Lambda^p - 1 = \hat{\Lambda}^p_{p-1} - 1 + \zeta_{2(p-1)p^{\frac{1}{p-1}}} + \zeta_{2(p-1)p^{\frac{1}{p-1}}} \cdot U_p
\]
where \( \hat{\Lambda}^+_{p-1} = \sum_{i=0}^{p-1} \frac{(-1)^i}{i!} \zeta_{2(p-1)} p \frac{1}{\sigma^2} U_p \). The following lemma confirms the assertion \((3.5)\).

**Lemma 3.17.** We have \( \Lambda^p = \hat{\Lambda}^+_{p-1} + O(p^{1+r}) \).

**Proof.** Since \( \Lambda^p = \hat{\Lambda}^+_{p-1} + o\left(p^{1+r}\right) \), we obtain the following formula by Lemma 3.15

\[ \Lambda^p - 1 = p \left( \Lambda^p - \hat{\Lambda}^+_{p-1} \right) + O\left(p^{2+r}\right). \] (3.8)

On the other hand, since \( \Lambda \) is an approximation of \( \zeta_{p^2} \), the maximal slope of the Newton polygon of \( \Phi_{p^2}(T + \Lambda) \) is equal to or greater than \( r = \frac{2}{p-1} \).

Denote by \( P = (p(p-1) - 1, y_p) \) the intersection of the line \( x = p(p-1) - 1 \) and the segment \( \mathcal{L} \) in \( \mathcal{Neut}(\Phi(T + \Lambda)) \) with the maximal slope. By [19, Lemma 2.4], the endpoints of \( \mathcal{L} \) are on or above the Newton polygon \( \mathcal{Neut}\left(\Phi(k,2)\right) \), for any integer number \( k \geq p \).

Denote by \( \mathcal{P} = (p(p-1) - 1, y_p) \) the intersection of the line \( x = p(p-1) - 1 \) and the segment \( \mathcal{L} \) in \( \mathcal{Neut}(\Phi(T + \Lambda)) \) with the maximal slope. By [19, Fig. 3], for every \( k \in \mathbb{Z}_{\geq p} \), we can calculate that the line \( x = p(p-1) - 1 \) intersects with the segment with maximal slope in \( \mathcal{Neut}\left(\Phi(k,2)\right) \) at \( \left(p(p-1) - 1, 2 - \frac{1}{p-1} - \frac{p-1}{p-1} \right) \). As a consequence, \( y_p \geq 2 - \frac{1}{p-1} - \frac{p-1}{p-1} \) for every \( k \in \mathbb{Z}_{\geq p} \), which implies \( y_p \geq \frac{2}{p-1} \).

Therefore the maximal slope of \( \mathcal{Neut}(\Phi(T + \Lambda)) \) is \( s_{\Lambda} = v_p\left(\frac{\Lambda^p - 1}{\Lambda^p - 1}\right) \geq \frac{2}{p-1} \), and consequently

\[ v_p\left(\Lambda^p - 1\right) \geq v_p\left(\Lambda^p - 1\right) + y_p + \frac{2}{p-1} \]

\[ \geq \frac{1}{p-1} + 2 - \frac{1}{p-1} + \frac{2}{p-1} \]

\[ = 2 + \frac{2}{p-1}. \]

Compare this inequality with \((3.8)\) we know that

\[ v_p\left(p\left(\Lambda^p - \hat{\Lambda}^+_{p-1}\right)\right) \geq 2 + \frac{2}{p-1}, \]

in other words, we have

\[ \Lambda^p = \hat{\Lambda}^+_{p-1} + O(p^{1+r}). \]

\( \square \)

3.3.3. Solve the congruence equation

Recall that, we set

\[ \mu_0 = \Lambda_{p-1} + (1 + \lambda)\zeta_{2(p-1)} p \frac{1}{\sigma^2}, r_0 = \frac{1}{p-1} + \frac{1}{p(p-1)}, r = \frac{2}{p-1} \]

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\[ \mu = \Lambda_{p-1} \left( 1 + \zeta_{2(p-1)p^r} \right). \]

Suppose \( \Lambda = \mu + \mathcal{M} \) is a \( r \)-truncated expansion of \( \zeta_{p^2} \) with an indeterminate \( \mathcal{M} \). In the following, we compute the \((1+r)\)-truncated expansion of \( \mu \left( \frac{\Lambda_{p-1}^+}{\mu^p} - 1 \right) \) and solve the equation

\[ p\mathcal{M} = \mu \left( \frac{\Lambda_{p-1}^+}{\mu^p} - 1 \right) + O(p^{1+r}). \]

**Lemma 3.18.** We have

\[ \frac{\Lambda_{p-1}^+}{\mu^p} = \mathcal{W} + O\left(p^{1+\frac{2}{p-1}}\right), \]

where

\[ \mathcal{W} = \sum_{l=0}^{p-1} \frac{(-1)^l}{l!} \zeta_{2(p-1)p^{\frac{r}{p-1}}} + \zeta_{2(p-1)p^{\frac{r}{p-1}}} U_p + \zeta_{2(p-1)p^{1+\frac{1}{p-1}}} \]

\[ - \zeta_{2(p-1)p^{1+\frac{1}{p-1}}} + \zeta_{2(p-1)p^{1+\frac{1}{p-1}}} \sigma_2 + O\left(p^{1+\frac{2}{p-1}}\right). \]

**Proof.** By the definition of \( \mu \), we only need to prove the truncated expansion of

\[ \frac{\Lambda_{p-1}^+}{\left( 1 + \zeta_{2(p-1)p^{\frac{1}{p-1}}} \sigma_2 \right)^p + O\left(p^{1+\frac{2}{p-1}}\right)} \]

is \( \mathcal{W} \). By binomial expansion,

\[ \left( 1 + \zeta_{2(p-1)p^{\frac{1}{p-1}}} \sigma_2 \right)^p = 1 + \zeta_{2(p-1)p^{\frac{1}{p-1}}} \sigma_2^p + \sum_{k=1}^{p-1} \binom{p}{k} \zeta_{2(p-1)p^{\frac{1}{p-1}}}^k \sigma_2^k. \]

Since

\[ v_p\left( \zeta_{2(p-1)p^{\frac{1}{p-1}}} \sigma_2^k \right) \geq \frac{2}{p-1} \text{ for } k \geq 3, \]

we can write

\[ \left( 1 + \zeta_{2(p-1)p^{\frac{1}{p-1}}} \sigma_2 \right)^p = 1 + \zeta_{2(p-1)p^{\frac{1}{p-1}}} \sigma_2^p + \zeta_{2(p-1)p^{1+\frac{1}{p-1}}} \sigma_2 \]

\[ + \frac{p}{2} \zeta_{2(p-1)p^{1+\frac{1}{p-1}}} \sigma_2 + O\left(p^{1+\frac{2}{p-1}}\right). \]

\[ = 1 - \zeta_{2(p-1)p^{1+\frac{1}{p-1}}} \left( \sigma_2^p - \sigma_2 \right) \]

\[ - \frac{1}{2} \zeta_{2(p-1)p^{1+\frac{1}{p-1}}} \sigma_2^2 + O\left(p^{1+\frac{2}{p-1}}\right). \]

(3.9)

By **Corollary 3.13**

\[ \sigma_2^p - \sigma_2 = p^{-1/p} + \sigma_2 + O\left(p^{1-1/p}\right) - \sigma_2 = p^{-1/p} + O\left(p^{1-1/p}\right). \]

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Therefore, we have

\[ 1 - \zeta_2(p-1) p^{1 + \frac{1}{p-1}} \left( p^{-1/p} + O\left(p^{1-1/p}\right) \right) \]
\[ - \frac{1}{2} \zeta_2(p-1) p^{1 + \frac{2}{p-1}} \sigma^2 + O\left(p^{1 + \frac{2}{p-1}}\right) \]
\[ = 1 - \zeta_2(p-1) p^{1 + \frac{1}{p(p-1)}} - \frac{1}{2} \zeta_2(p-1) p^{1 + \frac{2}{p-1}} \sigma^2 + O\left(p^{1 + \frac{2}{p-1}}\right). \]

By taking multiplicative inverse and expanding into power series, we have

\[
\frac{1}{1 - \zeta_2(p-1) p^{1 + \frac{1}{p(p-1)}}} = 1 - \zeta_2(p-1) p^{1 + \frac{1}{p(p-1)}} + \frac{1}{2} \zeta_2(p-1) p^{1 + \frac{2}{p-1}} \sigma^2 + O\left(p^{1 + \frac{2}{p-1}}\right).
\]

As a result, the truncated expansion of \( \frac{\tilde{\Lambda}_{p-1}}{1 + \zeta_2(p-1) p^{1 + \frac{1}{p(p-1)}} \sigma^2} \) is the following:

\[
\tilde{\Lambda}_{p-1} \left( 1 + \zeta_2(p-1) p^{1 + \frac{1}{p(p-1)}} + \frac{1}{2} \zeta_2(p-1) p^{1 + \frac{2}{p-1}} \sigma^2 + O\left(p^{1 + \frac{2}{p-1}}\right) \right) = \sum_{l=0}^{p-1} \frac{(-1)^l}{l!} \zeta_2(p-1) p^{1 + \frac{l}{p-1}} U_p + \zeta_2(p-1) p^{1 + \frac{1}{p(p-1)} + \frac{2}{p-1}} \sigma^2 + O\left(p^{1 + \frac{2}{p-1}}\right)
\]
\[
= \sum_{l=0}^{p-1} \frac{(-1)^l}{l!} \zeta_2(p-1) p^{1 + \frac{l}{p-1}} U_p + \zeta_2(p-1) p^{1 + \frac{1}{p(p-1)}} + \frac{1}{2} \zeta_2(p-1) p^{1 + \frac{2}{p-1}} \sigma^2 + O\left(p^{1 + \frac{2}{p-1}}\right)
\]

which is exactly \( W + O\left(p^{1 + \frac{2}{p-1}}\right) \).

Recall we set \( \kappa = - \sum_{n=2}^{p-1} \frac{(-1)^{n+1}}{n!m} \zeta_2(p-1) p^{1 + \frac{1}{p(p-1)} + \frac{m}{p-1}} \).

Corollary 3.19. We have

\[
\frac{\tilde{\Lambda}_{p-1}}{\mu^p} - 1 = - \zeta_2(p-1) p^{1 + \frac{1}{p(p-1)} + \frac{1}{p-1}} + \frac{1}{2} \zeta_2(p-1) p^{1 + \frac{2}{p-1}} \sigma^2 + \kappa + O\left(p^{1 + \frac{2}{p-1}}\right).
\]
The product can be expanded as

By expanding the product, one has the expansion of \( \frac{1}{\Lambda_{p-1}} \) (cf. Lemma 3.10), we have

\[
\frac{\Lambda_{p-1}^+}{\mu p} - 1 = \mathcal{W} \cdot \frac{1}{\Lambda_{p-1}^+} - 1
\]

\[
= \left( \sum_{l=0}^{p-1} \frac{\lambda^l p}{l!} + \zeta_2(p-1) p^{1 + \frac{1}{p+\mu p}} + \zeta_2(p-1) p^{1 - \frac{1}{p+\mu p}} U_p \right)
\]

\[
- \frac{2}{\zeta_2(p-1)} p^{1 + \frac{1}{p+\mu p} + \frac{1}{p+\mu p}} - 2 \zeta_2(p-1) p^{1 + \frac{1}{p+\mu p} + \frac{1}{p+\mu p}} U_p
\]

\[
- 2 \zeta_2(p-1) p^{1 + \frac{1}{p+\mu p} + \frac{1}{p+\mu p}} - 2 \zeta_2(p-1) p^{1 + \frac{1}{p+\mu p} + \frac{1}{p+\mu p}} U_p
\]

\[
- 1 + O\left(p^{1 + \frac{2}{p+\mu p}}\right).
\]

By expanding the product, one has

\[
\frac{\Lambda_{p-1}^+}{\mu p} - 1 = -1 - \zeta_2(p-1)^2 p^{1 + \frac{1}{p+\mu p} + \frac{1}{p+\mu p}} + \frac{1}{2} \zeta_2(p-1)^2 p^{1 + \frac{2}{p+\mu p} + \frac{2}{p+\mu p}} + \zeta_2(p-1)^2 p^{1 + \frac{2}{p+\mu p} + \frac{2}{p+\mu p}} U_p
\]

\[
+ \left( \sum_{i=0}^{p-1} \frac{(-1)^i}{i!} \zeta_2(p-1) p^{1 + \frac{1}{p+\mu p}} \right) \left( \sum_{j=0}^{p-1} \frac{\zeta_2(p-1)^2 j!}{p^{1 + \frac{2}{p+\mu p}}} \right) + O\left(p^{1 + \frac{2}{p+\mu p}}\right).
\]

(3.10)

The product can be expanded as

\[
\left( \sum_{i=0}^{p-1} \frac{(-1)^i}{i!} \zeta_2(p-1) p^{1 + \frac{1}{p+\mu p}} \right) \left( \sum_{j=0}^{p-1} \frac{\zeta_2(p-1)^2 j!}{p^{1 + \frac{2}{p+\mu p}}} \right)
\]

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\[
= \sum_{m=0}^{2p-2} c_2^{(p-1)} B_m \sum_{i+j=m, i,j \leq p-1} \frac{(-1)^i}{i!} \frac{(-1)^j}{j!}
= \sum_{m=0}^{p-1} c_2^{(p-1)} B_m \sum_{i=0}^{m} \frac{(-1)^i}{i!(m-i)!} + \frac{1}{p!} c_2^{(p)} B_{p-1} \sum_{j=0}^{p-1} \frac{(-1)^{p-j}}{j!} + O\left(p^{1+\frac{2}{r+1}}\right)
= \sum_{m=0}^{p-1} \frac{1}{m!} c_2^{(p-1)} B_m \sum_{i=0}^{m} \binom{m}{i} (-1)^i + O\left(p^{1+\frac{2}{r+1}}\right).
\]

Since \[
\sum_{i=0}^{m} \binom{m}{i} (-1)^i = \begin{cases} 
1, & m = 0 \\
0, & m > 0
\end{cases}
\]
we know that
\[
\left(\sum_{i=0}^{p-1} \frac{(-1)^i}{i!} c_2^{(p-1)} B_i \right) \left(\sum_{j=0}^{p-1} \frac{c_2^{(p-1)} B_{p-1}}{j!} \right) = 1 + O\left(p^{1+\frac{2}{r+1}}\right).
\]
As a result, we have

\[
(3.10) = -c_2^{(p-1)} B^{1+\frac{1}{r+1}} + \frac{1}{2} c_2^{(p-1)} B^{1+\frac{2}{r+1}} + \kappa + O\left(p^{1+\frac{2}{r+1}}\right).
\]

**Lemma 3.20.** One has
\[
\mathcal{M} = \frac{1}{2} c_2^{(p-1)} B^{1+\frac{1}{r+1}} + \frac{1}{2} c_2^{(p-1)} B^{1+\frac{2}{r+1}} \sigma_2^2 + \kappa + O\left(p^{1+\frac{2}{r+1}}\right).
\]

**Proof.** Recall \[ M = \frac{1}{p} \left(1 + \frac{1}{r+1}\right) A_{p-1} \left(\frac{\lambda_{p-1}}{\mu p} - 1\right). \] By Corollary 3.19, we have
\[
A_{p-1} \left(\frac{\lambda_{p-1}}{\mu p} - 1\right) = A_{p-1} \cdot \left( -c_2^{(p-1)} B^{1+\frac{1}{r+1}} + \frac{1}{2} c_2^{(p-1)} B^{1+\frac{2}{r+1}} \sigma_2^2 + \kappa + O\left(p^{1+\frac{2}{r+1}}\right) \right),
\]
where
\[
\kappa = -\sum_{n=2}^{p-1} \frac{(-1)^{n+1}}{n! n} c_2^{(p-1)} B^{1+\frac{1}{r+1}} + \frac{\mu p}{r+1}.
\]

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By expanding the product, we obtain

\[
\Lambda_p \left( \frac{\hat{p}^{+\mu^{-1}}}{\mu^p} - 1 \right) = O(1^{+\frac{2}{p-1}}) - \zeta^2_{2(p-1)} \sum_{n=2}^{p-1} \frac{1}{n!} \sum_{k=0}^{p-2} \zeta^k_{2(p-1)} \frac{1}{k!} + \frac{1}{\mu^{-\frac{1}{p-1}}} \sum_{k=0}^{p-2} \zeta^k_{2(p-1)} \frac{1}{k!} \sum_{k=0}^{p-2} \zeta^k_{2(p-1)} \frac{1}{k!} \\
+ \frac{1}{\mu^{-\frac{1}{p-1}}} \sum_{k=0}^{p-2} \zeta^k_{2(p-1)} \frac{1}{k!} \sum_{k=0}^{p-2} \zeta^k_{2(p-1)} \frac{1}{k!} \\
- \sum_{n=2}^{p-1} \frac{1}{n!} \sum_{k=0}^{p-2} \zeta^k_{2(p-1)} \frac{1}{k!} \sum_{k=0}^{p-2} \zeta^k_{2(p-1)} \frac{1}{k!} \\
= O(1^{+\frac{2}{p-1}}) - \sum_{k=1}^{p-1} \frac{1}{(k-1)!} \sum_{k=0}^{p-2} \zeta^k_{2(p-1)} \frac{1}{k!} \sum_{k=0}^{p-2} \zeta^k_{2(p-1)} \frac{1}{k!} \\
+ \frac{1}{2} \zeta^2_{2(p-1)} \sum_{n=2}^{p-1} \frac{1}{n!} \sum_{k=0}^{p-2} \zeta^k_{2(p-1)} \frac{1}{k!} \sum_{k=0}^{p-2} \zeta^k_{2(p-1)} \frac{1}{k!} \\
- \sum_{n=2}^{p-1} \zeta^2_{2(p-1)} \sum_{n=2}^{p-1} \frac{1}{n!} \sum_{k=0}^{p-2} \zeta^k_{2(p-1)} \frac{1}{k!} \sum_{k=0}^{p-2} \zeta^k_{2(p-1)} \frac{1}{k!}. 
\]

By \( \text{[3] (9.2)} \), we know that \( \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k} = H_n \). Therefore

\[
\Lambda_p \left( \frac{\hat{p}^{+\mu^{-1}}}{\mu^p} - 1 \right) = \frac{1}{2} \zeta^2_{2(p-1)} + \frac{1}{2} \zeta^3_{2(p-1)} + \frac{1}{2} \zeta^3_{2(p-1)} + O(1^{+\frac{2}{p-1}}) \\
= \frac{1}{2} \zeta^2_{2(p-1)} + \frac{1}{2} \zeta^3_{2(p-1)} + \frac{1}{2} \zeta^3_{2(p-1)} + O(1^{+\frac{2}{p-1}}) \\
= \frac{1}{2} \zeta^2_{2(p-1)} + \frac{1}{2} \zeta^3_{2(p-1)} + \frac{1}{2} \zeta^3_{2(p-1)} + O(1^{+\frac{2}{p-1}}). 
\]

As a consequence,

\[
M = \left( 1 + \zeta_{2(p-1)} \frac{2}{p-1} \frac{1}{\mu^{-\frac{1}{p-1}}} \right) \\
\times \left( \frac{1}{2} \zeta^2_{2(p-1)} + \frac{1}{2} \zeta^3_{2(p-1)} + \frac{1}{2} \zeta^3_{2(p-1)} + O(1^{+\frac{2}{p-1}}) \right).
\]
Then there exists a
expanding the product in Theorem 3.21 and replace the coefficient of each term
by its Teichmuller lifting.
that we can get a simple and useful expression.
given by
Newton algorithm to get furthur truncated expansion of
Theorem 3.21. We have
Remark 3.22. One obtains the
coefficientsof
the Newton polygon of
of a
Proof. Let
Finally, we obtain the \( \frac{2}{p-1} \)-truncated expansion of \( \zeta_{p^2} \):
\[
\zeta_{p^2} = \left( \sum_{k=0}^{p-1} \frac{1}{|k|!} \zeta_{2(p-1)p^{\frac{1}{p-1}}\sigma_2^k} \right) \left( 1 + \zeta_{2(p-1)p^{\frac{1}{p-1}}\sigma_2} \right) + \frac{1}{2} \zeta_{2(p-1)p^{\frac{2}{p-1}}\sigma_2^2}
\]
\[
+ \frac{1}{2} \zeta_{2(p-1)p^{\frac{2}{p-1}}\sigma_2^2} - \frac{p-2}{p-1} - \sum_{k=0}^{p-1} H_n \zeta_{2(p-1)p^{\frac{1}{p-1}}\sigma_2^k} + O\left( p^{\frac{2}{p-1}} \right).
\]
Remark 3.23. Theoretically, we can iteratively repeat the variant of transfinite Newton algorithm to get further truncated expansion of \( \zeta_{p^2} \). But it is not clear that we can get a simple and useful expression.

3.4. The truncated expansion of \( \zeta_{p^n} \)
Lemma 3.24. Suppose \( A \in \mathcal{O}_p \) has 1-truncated canonical expansion
\[
A = \sum_{g \in \text{Supp}(A) \cap [0,1)} [a_g]p^g + O(p^1).
\]
Then there exists a \( p \)-th root of \( A \) in \( \mathbb{L}_p \) with \( \frac{1}{p} \)-truncated canonical expansion given by
\[
B = \sum_{g \in \text{Supp}(A) \cap [0,1)} [a_g^\frac{1}{p}]p^\frac{g}{p} + O(p^\frac{1}{p}).
\]
Proof. Let \( B_0 = \sum_{g \in \text{Supp}(A) \cap [0,1)} [a_g^\frac{1}{p}]p^\frac{g}{p} \). We claim that \( B_0 \) is an approximation of a \( p \)-th root \( A^\frac{1}{p} \) of \( A \) in \( \mathbb{L}_p \).
Consider the polynomial \( f(T) = (T + B_0)^p - A \in \mathbb{L}_p[T] \). To describe the Newton polygon of \( f(T) \), we need to estimate the \( p \)-adic valuation of the coefficients of \( f(T) \). For \( 1 \leq i \leq p-1 \), we have \( v_p\left( \binom{p}{i} B_0^{p-i} \right) \geq 1 \). By Lemma 3.12 one has
\[
B_0^p = \sum_{g \in \text{Supp}(A) \cap [0,1)} [a_g]p^g + O(p^{1+v_p(B_0)}),
\]
and consequently \( v_p(B_0^p - A) \geq 1 \). Thus, except that the coefficient of leading term of \( f(T) \) has valuation 0, the \( p \)-adic valuation of every coefficients of other terms in \( f(T) \) is at least 1. This implies that the maximal slope of \( N_{\text{weil}}(f) \) is at least \( \frac{1}{p} \). Therefore there exists a root \( U \in \mathbb{F}_p \) of \( f(T) \) with \( v_p(U) \geq \frac{1}{p} \). By setting \( B = B_0 + U \), we know that \( B_0^p = A \) and

\[
V = \sum_{g \in \text{Supp}(A) \cap [0,1)} \lfloor a_g \rfloor p^g + O\left(p^{\frac{1}{2}}\right),
\]

as promised.

As an application of the preview lemma, we can deduce the \( p^{\frac{2}{p-1}} \)-truncated expansion of \( \zeta_{p^n} \) from the \( p^{\frac{1}{p-1}} \)-truncated expansion of \( \zeta_{p^n} \) (cf. Theorem 3.21):

**Proposition 3.25.** For \( n \geq 2 \), we have the following \( p^{\frac{2}{p-1}} \)-truncated expansion of \( \zeta_{p^n} \):

\[
\zeta_{p^n} = \sum_{k=0}^{p-1} \frac{(-1)^n k!}{k!} \zeta_{2(p-1)}^{2(p-1)} p^{\frac{2}{p-1}(p-1)} + \sum_{k=0}^{p-1} \frac{(-1)^{n(k+1)}}{k!} \zeta_{2(p-1)}^{2(p-1)} p^{\frac{2}{p-1}(p-1)} \sigma_n
\]

\[
- \sum_{k=1}^{p-1} H_k \frac{(-1)^{n(k+1)}}{k!} \zeta_{2(p-1)}^{2(p-1)} p^{\frac{2}{p-1}(p-1)} + \frac{1}{2} \zeta_{2(p-1)}^{2(p-1)} \sigma_n^2 + \frac{(-1)^n}{2} \zeta_{2(p-1)}^{2(p-1)} p^{\frac{2}{p-1}(p-1)} - \zeta_{2(p-1)}^{2(p-1)} + O\left(p^{\frac{2}{p-1}(p-1)}\right).
\]

**Remark 3.26.** We obtain the \( p^{\frac{2}{p-1}} \)-truncated canonical expansion of \( \zeta_{p^n} \) from Proposition 3.25 immediately by expanding the product and replacing the coefficient of each term by its Teichmuller lifting. This is based on that fundamental fact about the Teichmuller lifting:

\[
[k] = \begin{cases} 
  k, & \text{if } k = 0, 1; \\
  k + O(p), & \text{if } k \geq 2.
\end{cases}
\]

Taking account of the support of \( \zeta_{p^n} \), we know that in general for an odd prime \( p \), taking off or adding on the Teichmuller lifting does not affect the correctness of the truncated expansion if and only if the truncated position is not greater than 1 + \( p^{\frac{2}{p-1}(p-1)} \). Thus, when one tries to improve the results to a better truncated position, the Teichmuller lifting in the calculation may not be neglectable anymore. For example, \( \Lambda_{p-1} \) is no longer a truncated expansion of \( \Lambda_{p-1} \) when the truncated position is greater than 1 + \( \frac{2}{p(p-1)} \). On the other hand, since combinatorial techniques are our primary tools, the truncated (noncanonical) expansion without the Teichmuller liftings is more convenient for us.

\[\text{There exists some exceptional primes such that the assertion is not true.}\]
4. Uniformizer of the false Tate curve extension of $\mathbb{Q}_p$

Let

$$\pi_{2,1}^{p,1} = p^{-1/p} \left( \zeta_{p^2} - \sum_{k=0}^{p-1} \frac{1}{k!} \zeta_{p^{2(p-1)}}^{k} \right)$$

be the uniformizer of $\mathbb{K}_p^{2,1}$ constructed in [19, Theorem 3.23]. We set

$$\pi_{3,1}^{p,1} = (\zeta_{p^3} - 1)^{-2p+2} \left( \zeta_{p^3} - \sum_{k=0}^{p-1} \frac{(-1)^k}{k!} (\pi_{p,1}^{2,1})^k \right) - \sum_{k=1}^{p-1} \frac{(-1)^k(k-H_k)}{k!} (\pi_{p,1}^{2,1})^{p+k}$$

which is an element of $\mathbb{K}_p^{3,1}$, and for $m \geq 4$, we set

$$\pi_{m,1}^{p,1} = (\zeta_{p^m} - 1)^{-2p+2} \left( \zeta_{p^m} - \sum_{k=0}^{p-1} \frac{(-1)^k}{k!} (\pi_{p,1}^{m-1,1})^k \right) - \sum_{k=1}^{p-1} \frac{(-1)^k(2k-H_k)}{k!} (\pi_{p,1}^{m-1,1})^{p+k}$$

which is an element of $\mathbb{K}_p^{m,1}$.

**Theorem 4.1.** For prime $p \geq 3$ and integer $m \geq 3$, $\pi_{m,1}^{p,1}$ is a uniformizer of $\mathbb{K}_p^{m,1}$.

4.1. Preliminary

Let $\sigma_n = \sum_{k=n}^{\infty} p^{-k}$ and $H_k = \sum_{i=1}^{k} \frac{1}{i}$ be the harmonic number. In this section, we establish a key truncated expansion of certain type element in $\mathbb{L}_p$, which is a slight generalization of the element appearing in the construction of $\pi_{p,1}^{2,1}$.

Suppose $A_{p,n}^{(\beta)} \in \mathbb{L}_p$ has $\left(\frac{2}{p^{n-1}(p-1)}\right)$-truncated expansion

$$A_{p,n}^{(\beta)} = (\beta^{2(p-1)} p^{n-1(p-1)}) \sigma_n \left( 1 + (-1)^n \beta \sum_{i=1}^{n} \frac{1}{i} \right) + O\left( p^{n-1(p-1)} \right),$$

where $\beta \in \mathbb{Z}_p$. The following lemma gives the $\left(\frac{2}{p^{n-1}(p-1)}\right)$-truncated expansion of $(A_{p,n}^{(\beta)})^k$, for $k = 1, \cdots, 2p - 1$. To simplify the statement, for every integer $s \geq 1$, we represent the indicator function of the set $\{1, 2, \cdots, s\}$ by $1_{\leq s}(x)$, i.e.

$$1_{\leq s}(x) = \begin{cases} 1, & \text{if } x \in \{1, 2, \cdots, s\}; \\ 0, & \text{otherwise}. \end{cases}$$

Besides that, the indicator functor of the set $\{s\}$ is denoted by $1_s(x) = 1_{\leq 1}(x - s + 1)$.\[42\]
Lemma 4.2. Let prime $p \geq 3$ and integer $n \geq 2$. For integer $k = 1, \cdots, 2p - 1$, we have

$$
\left( A_{p,n}^{(β)} \right)^k = (-1)^{nk} \zeta_{2(p-1)}^{k} p^{\frac{k}{p^n - t(p-1)}} \cdot \sum_{\omega = 1}^{k} \frac{\beta^{\omega}}{\omega!} k^{\omega} \left( 1 + \beta(-1)^n \zeta_{2(p-1)}^{k} p^{\frac{k-1}{p^n - t(p-1)}} \right)^k
$$

Proof. We expand

$$
\left( A_{p,n}^{(β)} \right)^k = (-1)^{nk} \zeta_{2(p-1)}^{k} p^{\frac{k}{p^n - t(p-1)}} \cdot \sum_{\omega = 1}^{k} \frac{\beta^{\omega}}{\omega!} k^{\omega} \left( 1 + \beta(-1)^n \zeta_{2(p-1)}^{k} p^{\frac{k-1}{p^n - t(p-1)}} \right)^k
$$

Since the condition $\frac{k+1}{p^n - t(p-1)} - \frac{k}{p^n} < \frac{2}{p^n - t(p-1)}$ implies $k < p$, we can rewrite the expansion of $\left( A_{p,n}^{(β)} \right)^k$ as

$$
(-1)^{nk} \zeta_{2(p-1)}^{k} p^{\frac{k}{p^n - t(p-1)}} \cdot \sum_{\omega = 1}^{k} \frac{\beta^{\omega}}{\omega!} k^{\omega} \left( 1 + \beta(-1)^n \zeta_{2(p-1)}^{k} p^{\frac{k-1}{p^n - t(p-1)}} \right)^k
$$

The lemma follows from the following estimation of the terms:

- For the term $(-1)^{nk} \zeta_{2(p-1)}^{k} p^{\frac{k}{p^n - t(p-1)}} \sigma_n^k$, one has

$$
(-1)^{nk} \zeta_{2(p-1)}^{k} p^{\frac{k}{p^n - t(p-1)}} \sigma_n^k = (-1)^{nk} \zeta_{2(p-1)}^{k} p^{\frac{k}{p^n - t(p-1)}} \left( p^{-1/p^n} + \sigma_{n+1} \right)^k
$$

The condition $\frac{k}{p^n - t(p-1)} - \frac{k-t}{p^n} < \frac{2}{p^n - t(p-1)}$ implies

$$
t = \begin{cases} 
0,1, & \text{if } k \leq p + 1, k \neq 2,3; \\
0,1,2, & \text{if } k = 2,3; \\
0, & \text{if } p + 2 \leq k \leq 2p - 1.
\end{cases}
$$

For $k = 3$ and $t = 2$, we have

$$
(-1)^{n} \zeta_{2(p-1)}^{3} \left( \frac{3}{2} \right) p^{\frac{3}{p^n - t(p-1)}} \sigma_{n+1}^2
$$
\[
= 3(-1)^n \zeta_{2(p-1)}^k p^{\frac{2k+1}{p-1}} \left( p^{-\frac{2k-1}{p-1}} + O \left( p^{-\frac{k+1}{p-1}} \right) \right) \\
= 3(-1)^n \zeta_{2(p-1)}^k p^{\frac{2k+2}{p-1}} + O \left( p^{\frac{k+1}{p-1}} \right).
\]

As a consequence, we have
\[
(-1)^n \zeta_{2(p-1)}^k p^{\frac{k}{p-1}} a_n^k \\
= (-1)^n \zeta_{2(p-1)}^k p^{\frac{k}{p-1}} \cdot \frac{1}{p^k} + \sum_{t=0}^{k+1} \binom{k}{t} p^{-\frac{k-t}{p-1}} \sigma_n^t.
\]

The condition \(\frac{k+1}{p-1} - \frac{k-t}{p^t} < \frac{2}{p^{t-1}(p-1)}\) implies
\[
t = \begin{cases} 
0, & \text{if } k = 1; \\
0, & \text{if } 2 \leq k \leq p-1.
\end{cases}
\]

Therefore
\[
\beta k(-1)^n \zeta_{2(p-1)}^k p^{\frac{k+1}{p-1}} a_n^k \\
= \sum_{t=0}^{k+1} \binom{k}{t} p^{-\frac{k-t}{p-1}} \sigma_n^t.
\]

Recall that by Proposition 3.25 for \(n \geq 2\), we have the \(\left( p^{\frac{2}{p-1}} \right)\)-truncated expansion of a \(p^n\)-th root of unity \(\zeta_{p^n}\) in \(L_p\): For \(n \geq 2\), we have the following
where

\[
\begin{align*}
\left(\frac{p^{n-2}}{p-1}\right)^{-1} \text{-truncated expansion of } \zeta_p^n: \\
\zeta_p^n &= \sum_{k=0}^{p-1} \frac{(-1)^n}{k!} \zeta_{2(p-1)}^{k} P^{\frac{k}{p-1}} + \sum_{k=0}^{p-1} \frac{(-1)^n(k+1)}{k!} \zeta_{2(p-1)}^{k+1} P^{\frac{k+1}{p-1}} \\
&\quad - \sum_{k=1}^{p-1} H_k \frac{(-1)^n}{k!} \zeta_{2(p-1)}^{k+1} P^{\frac{k+1}{p-1}} \\
&\quad + \frac{1}{2} \left(2_p(p-1) P^{\frac{1}{p-1}}\right) \sigma_n + \left(\frac{-1}{2}\right)^{n+1} \zeta_{2(p-1)}^{3} P^{\frac{3}{p-1}} + O\left(p^{n-2(p-1)}\right).
\end{align*}
\]

**Corollary 4.3.** For prime \( p \geq 3 \) and integer \( n \geq 2 \), we have

\[
\zeta_{p^n+1} = \sum_{k=0}^{p-1} \left(\frac{-1}{k!}\right)^{k} \left(A_{p,n}\right)^{k}
\]

\[
\begin{align*}
&= (-1)^{(n+1)} \zeta_{2(p-1)} P^{\frac{2}{p-1}} \sigma_n + \sum_{k=1}^{p-1} \frac{(-1)^n}{k!} \zeta_{2(p-1)}^{k+1} P^{\frac{k+1}{p-1}} \\
&\quad + \beta \zeta_{2(p-1)}^{2} P^{\frac{2}{p-1}} \sigma_n - \mathbb{I}_3(p) \cdot \frac{(-1)^n}{2} \zeta_{2(p-1)}^{3} P^{\frac{3}{p-1}} + O\left(p^{n-2(p-1)}\right).
\end{align*}
\]

**Proof.** By Lemma 4.2 the summation \( \sum_{k=0}^{p-1} \left(\frac{-1}{k!}\right)^{k} \left(A_{p,n}\right)^{k} \) can be expanded as:

\[
\begin{align*}
&\sum_{k=0}^{p-1} \frac{(-1)^k}{k!} \left(A_{p,n}\right)^{k} \\
&= \sum_{k=0}^{p-1} \frac{(-1)^n}{k!} \zeta_{2(p-1)}^{k} P^{\frac{k}{p-1}} + \sum_{k=1}^{p-1} \frac{k(-1)^{n+1}}{k!} \zeta_{2(p-1)}^{k+1} P^{\frac{k+1}{p-1}} \\
&\quad + \beta \sum_{k=1}^{p-1} \frac{k(-1)^{k+n(k+1)}}{k!} \zeta_{2(p-1)}^{k+1} P^{\frac{k+1}{p-1}} + \frac{1}{2} \zeta_{2(p-1)}^{2} P^{\frac{2}{p-1}} \sigma_n^2 - \frac{1}{6} \mathbb{I}_3(p) \cdot 3(-1)^n \zeta_{2(p-1)}^{3} P^{\frac{3}{p-1}} + O\left(p^{n-2(p-1)}\right),
\end{align*}
\]

where \( 1 - \mathbb{I}_3(p) \) is the indicator function of the set of primes \( p \geq 5 \).

Using the identity \( \frac{k(-1)^{k+n(k+1)}}{k!} = - \frac{k(-1)^{k+n(k+1)}}{k!} \), we have

\[
\beta \sum_{k=1}^{p-1} \frac{k(-1)^{k+n(k+1)}}{k!} \zeta_{2(p-1)}^{k+1} P^{\frac{k+1}{p-1}} = - \sum_{k=1}^{p-1} \frac{k(-1)^{k+n(k+1)}}{k!} \zeta_{2(p-1)}^{k+1} P^{\frac{k+1}{p-1}}.
\]

\[
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\]
Together with the shifting of the index in the summation
\[
\sum_{k=1}^{p-1} \frac{k(-1)^{(n+1)k}}{k!} \zeta_{2(p-1)}^{k} p^{\frac{k+p-1}{2}} \sigma_{n+1}^{k+p-1},
\]
we can rewrite the summation (4.1) as following
\[
\sum_{k=0}^{p-1} \frac{(-1)^k}{k!} \left( A_{(\beta)}^{(\gamma)} \right)^k
\]
\[
= \sum_{k=0}^{p-1} \frac{(-1)^{(n+1)k}}{k!} \zeta_{2(p-1)}^{k} p^{\frac{k+p-1}{2}} + \sum_{k=0}^{p-2} \frac{(-1)^{(n+1)(k+1)}}{k!} \zeta_{k+1}^{k+1} p^{\frac{k+p}{2}} \sigma_{n+1}^{k+p+1}
\]

\[
- \sum_{k=1}^{p-1} k \beta(-1)^{(k+1)(n+1)} k^{k+1} \zeta_{2(p-1)}^{k} p^{\frac{k+p}{2}}
\]

\[
+ \frac{1}{2} \zeta_{2(p-1)}^{k+1} p^{\frac{k+2}{2}} - \frac{1}{2} \zeta_{2(p-1)}^{k+2} p^{\frac{k+1}{2}} - \beta \zeta_{2(p-1)}^{k+1} p^{\frac{k+2}{2}} \sigma_{n+1}^{k+p+2} + O(p^{p+1-\frac{2}{2}}).
\]

Therefore, together with the truncated expansion of \( \zeta_{p+1}^{k+p} \), we have
\[
\zeta_{p+1}^{k+p} = \sum_{k=0}^{p-1} \frac{(-1)^k}{k!} \left( A_{(\beta)}^{(\gamma)} \right)^k
\]
\[
= \sum_{k=0}^{p-1} \frac{(-1)^{(n+1)k}}{k!} \zeta_{2(p-1)}^{k} p^{\frac{k+p-1}{2}} + \sum_{k=0}^{p-2} \frac{(-1)^{(n+1)(k+1)}}{k!} \zeta_{k+1}^{k+1} p^{\frac{k+p}{2}} \sigma_{n+1}^{k+p+1}
\]

\[
- \sum_{k=1}^{p-1} \frac{H_k}{k!} (-1)^{(n+1)(k+1)} \zeta_{2(p-1)}^{k+1} p^{\frac{k+p}{2}} + \sum_{k=0}^{p-1} \frac{(-1)^{(n+1)k}}{k!} \zeta_{2(p-1)}^{k+1} p^{\frac{k+p}{2}} \sigma_{n+1}^{k+p+1}
\]

\[
- \sum_{k=1}^{p-1} \beta k(-1)^{(k+1)(n+1)} k^{k+1} \zeta_{2(p-1)}^{k+1} p^{\frac{k+p}{2}}
\]

\[
+ \frac{1}{2} \zeta_{2(p-1)}^{k+2} p^{\frac{k+2}{2}} - \frac{1}{2} \zeta_{2(p-1)}^{k+2} p^{\frac{k+1}{2}} - \beta \zeta_{2(p-1)}^{k+1} p^{\frac{k+2}{2}} \sigma_{n+1}^{k+p+2} + O(p^{p+1-\frac{2}{2}}).
\]

By combining terms, this identity can be simplified as
\[
\zeta_{p+1}^{k+p} = \sum_{k=0}^{p-1} \frac{(-1)^k}{k!} \left( A_{(\beta)}^{(\gamma)} \right)^k
\]
\[ (-1)^{(n+1)} \zeta_2(p-1) p^{2n-3} \sigma_{n+1} + \sum_{k=1}^{n-1} \left( \frac{(-1)^k (k \beta - H_k)}{k!} \right) (-1)^{n+k+1} \zeta_2(p-1) p^{k+p} \]

\[ + \beta \zeta_2(p-1) p^{2n-1} \sigma_{n+1} - 1 \sum_{l=0}^{n} \left( \frac{(-1)^n}{2} \zeta_2(p-1) p^{2l} + O(p^{2n-1}) \right) \]

We also need the \( \left( \frac{2}{p^{n-1} - 2} \right) \)-truncated expansion of \( \zeta_{p+1} - 1 \)^{-2p+2}:

**Proposition 4.4.** For prime \( p \geq 3 \) and integer \( n \geq 2 \), we have

\[ \left( \zeta_{p+1} - 1 \right)^{-2p+2} = p^{-\frac{2}{p}} \left( 1 - (-1)^n \zeta_2(p-1) p^{n-1} \sum_{l=0}^{n} \left( \frac{(-1)^n}{2} \zeta_2(p-1) p^{2l} + O(p^{2n-1}) \right) \right) \]

**Proof.** For \( p \geq 5 \), one has

\[ \left( \zeta_{p+1} - 1 \right)^{-2p+2} = \left( (-1)^n \zeta_2(p-1) p^{n-1} + \frac{1}{2} \zeta_2(p-1) p^{2n-2} + O(p^{3n-3}) \right)^{-2p+2} \]

\[ = \left( (-1)^n \zeta_2(p-1) p^{n-1} \right)^{-2p+2} \cdot \left( 1 - \frac{(-1)^n}{2} \zeta_2(p-1) p^{2n-2} + O(p^{2n-1}) \right)^{-2p+2} \]

\[ = p^{-\frac{2}{p}} \left( \sum_{l=0}^{n} \left( \frac{(-1)^n}{2} \zeta_2(p-1) p^{2l} + O(p^{2n-1}) \right) \right)^{2p-2} \]

Since we have the estimation

\[ \sum_{l=0}^{\infty} \left( \frac{(-1)^n}{2} \zeta_2(p-1) p^{2l} + O(p^{2n-1}) \right)^{l} = \left( 1 + \frac{(-1)^n}{2} \zeta_2(p-1) p^{2n-2} + O(p^{2n-1}) \right) \]

thus, we can rewrite (4.2) as following:

\[ (4.2) = p^{-\frac{2}{p}} \left( 1 + (2p-2) \frac{(-1)^n}{2} \zeta_2(p-1) p^{2n-2} + O(p^{2n-1}) \right) \]

\[ = p^{-\frac{2}{p}} \left( 1 - (-1)^n \zeta_2(p-1) p^{2n-2} + O(p^{2n-1}) \right). \]

For \( p = 3 \), we have an extra term in the truncated expansion:

\[ \zeta_{3n+1} - 1 \]
\[ \begin{align*}
&= (-1)^{n+1} \zeta_4 3^{\frac{n+1}{2}} + \frac{(-1)^{2n+2}}{2} \zeta_4^2 3^{\frac{n}{2}} + (-1)^{n+1} \zeta_4 3^{\frac{n+1}{2}} \sigma_{n+1} + O\left(3^{\frac{n+1}{2}}\right) \\
&= (-1)^{n+1} \zeta_4 3^{\frac{n+1}{2}} \left(1 - \frac{(-1)^n}{2} \zeta_4 3^{\frac{n}{2}} + 3 \pi^2 \sigma_{n+1} + O\left(3^{\frac{n}{2}}\right)\right).
\end{align*} \]

Therefore
\[
(\zeta_{3n+1} - 1)^{-4} = 3^{-\frac{2}{3}} \left(1 - \frac{(-1)^n}{2} \zeta_4 3^{\frac{n}{2}} - 3 \pi^2 \sigma_{n+1} + O\left(3^{\frac{n}{2}}\right)\right)^4
\]
\[
= 3^{-\frac{2}{3}} \left(1 + (-1)^n \zeta_4 3^{\frac{n}{2}} - 4 \cdot 3 \pi^2 \sigma_{n+1} + O\left(3^{\frac{n}{2}}\right)\right)
\]
\[
= 3^{-\frac{2}{3}} \left(1 - (-1)^n \zeta_4 3^{\frac{n}{2}} - 3 \pi^2 \sigma_{n+1} + O\left(3^{\frac{n}{2}}\right)\right).
\]

\[
\Box
\]

**Proposition 4.5.** For prime \( p \geq 3 \) and integer \( n \geq 2 \), the \( \frac{2}{p^{n(p-1)}} \)-truncated expansion of
\[
(\zeta_{p^{n+1}} - 1)^{-2p+2} \left(\zeta_{p^{n+1}} - \sum_{k=0}^{p-1} \frac{(-1)^k}{k!} \left(A_{p,n}\right)^k - \sum_{k=1}^{p-1} \frac{(-1)^k(k\beta - H_k)}{k!} \left(A_{p,n}\right)^{p+k}\right)
\]
is given by
\[
(1-1)^{(n+1)} \zeta_{2(p-1)} p^{\frac{1}{p^{-(p-1)}}} \sigma_{n+1} + 2 \zeta_{2(p-1)} p^{\frac{2}{p^{-(p-1)}}} \sigma_{n+1} + O\left(p^{\frac{2}{p^{-(p-1)}}}\right).
\]

**Proof.** For \( k = 1, \ldots, p - 1 \), we have
\[
\left(A_{p,n}\right)^{p+k} = (-1)^{n(k+p)} \zeta_{2(p-1)} p^{\frac{k+p}{p^{-(p-1)}}}
\]
\[
+ \frac{1}{1} (k+1)(1-1)^{(n+1)} \zeta_{2(p-1)} p^{\frac{k+p}{p^{-(p-1)}}} \sigma_{n+1}
\]
\[
= (-1)^{n(k+p)+1} \zeta_{2(p-1)} p^{\frac{k+p}{p^{-(p-1)}}} - \frac{1}{1} (k+1)(1-1)^{(n+1)} \zeta_{2(p-1)} p^{\frac{k+p}{p^{-(p-1)}}} \sigma_{n+1}
\]

Therefore, together with the truncated expansion of \( \zeta_{p^{n+1}} \), we have
\[
\begin{align*}
\zeta_{p^{n+1}} - \sum_{k=0}^{p-1} \frac{(-1)^k}{k!} \left(A_{p,n}\right)^k - \sum_{k=1}^{p-1} \frac{(-1)^k(k\beta - H_k)}{k!} \left(A_{p,n}\right)^{p+k}
&= (-1)^{(n+1)} \zeta_{2(p-1)} p^{\frac{1}{p^{-(p-1)}}} \sigma_{n+1} + \zeta_{2(p-1)} p^{\frac{2}{p^{-(p-1)}}} \sigma_{n+1} + O\left(p^{\frac{2}{p^{-(p-1)}}}\right)
\end{align*}
\]
\[
+ (1-1)^{2(p-1)} p^{\frac{2}{p^{-(p-1)}}} \sigma_{n+1}
\]
\[
+ O\left(p^{\frac{2}{p^{-(p-1)}}}\right).
\]

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we have the following truncated expansion:

\[- \frac{1}{2} \zeta_3(p) \cdot 2^2 \xi(p^{2(p-1)} \sigma_{n+1}) + O\left( p^{2p-1(p-1)} \right). \]

Together with the identity (cf. Proposition 4.4)

\[(\zeta_{p^n+1}^n - 1)^{-2p+2} = p^{-\frac{2}{n+1}} \left( 1 - (-1)^n \zeta(p^{2(p-1)} \sigma_{n+1}) - \frac{1}{2} \zeta_3(p) \cdot 3 \pi^2 \sigma_{n+1} + O\left( p^{2p-1(p-1)} \right) \right), \]

we have the following truncated expansion:

\[\left( \zeta_{p^n+1}^n - 1 \right)^{-2p+2} \cdot \left( 1 - \frac{1}{k!} \left( A_{(\beta)}^{(\beta)} \right) - \frac{1}{k!} (-1)^k (k \beta - H_k) \left( A_{(p,n)}^{(\beta)} \right) p^{k+k} \right) = p^{-\frac{2}{n+1}} \left( 1 - (-1)^n 2^2 \xi(p^{2(p-1)} \sigma_{n+1}) + \frac{1}{2} \zeta_3(p) \cdot 2^2 \xi(p^{2(p-1)} \sigma_{n+1}) + O\left( p^{2p-1(p-1)} \right) \right). \]

By expanding the product and combining terms, we obtain

\[\left( \zeta_{p^n+1}^n - 1 \right)^{-2p+2} = (-1)^n \zeta_3(p) \cdot 3 \pi^2 \sigma_{n+1} + (-1)^n \zeta_4^{3/4} - 3^{4/3}. \]

For \( p \geq 5, \zeta_3(p) = 0 \) and we get the desired result in this case. For \( p = 3 \), the result follows from the following calculation of extra terms:

\[\left( -1 \right)^n \zeta_4^{3/4} = \left( -1 \right)^n \zeta_4^{3/4} - \left( -1 \right)^n \zeta_4^{5/6} + 3^{2/3} + \left( -1 \right)^n \zeta_4^{3/4} - 3^{4/3} \]

\[= (-1)^n \zeta_4^{3/4} + \left( 3^{2/3} + O\left( 3^{2/3} \right) \right) - (-1)^n \zeta_4^{3/4} + \left( -1 \right)^n \zeta_4^{5/6} + 3^{2/3} + O\left( 3^{2/3} \right) + O\left( 3^{2/3} \right) \]

\[= O\left( 3^{2/3} \right). \]
4.2. Proof of **Theorem 4.1**

Recall that one has

\[
\pi_p^{2,1} = p^{-1/p} \left( \zeta_{p^2} + \sum_{k=0}^{p-1} \frac{1}{|\mathcal{O}|^k} \zeta_{p^2(1-p)^k} \right).
\]

We deduce the \( \left( \frac{2}{p(p-1)} \right) \)-truncated expansions of powers of \( \pi_p^{2,1} \) from Proposition 3.25

\[
\pi_p^{2,1} = \zeta_{2(p-1)p^{m-1}} \sigma_2 + \zeta_{2(p-1)p^{m-2}} \sigma_2 + O\left(p^{m-2}\right).
\]

Take \( A^{(1)}_{p,2} = \pi_p^{2,1} \), by Proposition 4.5 we know that

\[
\pi_p^{3,1} = -\zeta_{2(p-1)p^{m-1}} \sigma_3 + 2\zeta_{2(p-1)p^{m-2}} \sigma_3 + O\left(p^{m-2}\right).
\]

Assume we have proved that for \( m = 3, \ldots, n \),

\[
\pi_p^{m,1} = (-1)^m \zeta_{2(p-1)p^{m-1}} \sigma_m + 2\zeta_{2(p-1)p^{m-2}} \sigma_m + O\left(p^{m-2}\right),
\]

then by taking \( A^{(2)}_{p,n} = \pi_p^{n,1} \), one can deduce from Proposition 4.5 that

\[
\pi_p^{n+1,1} = (-1)^{(n+1)} \zeta_{2(p-1)p^{m-1}} \sigma_{n+1} + 2\zeta_{2(p-1)p^{m-2}} \sigma_{n+1} + O\left(p^{m-2}\right).
\]

Therefore for every \( m \geq 3 \), we obtain

\[
\nu_p\left(\pi_p^{m,1}\right) = \frac{1}{p^{m-1}(p-1)} - \frac{1}{p^m} = \frac{1}{p^m(p-1)} = e_{K_p^{m,1}}/Q_p.
\]

Thus, we can conclude that \( \pi_p^{m,1} \) is a uniformizer of \( K_p^{m,1} \) for every \( m \geq 3 \).

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