An Axiom System for General Relativity Complete with respect to Lorentzian Manifolds *

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Abstract

We introduce several axiom systems for general relativity and show that they are complete with respect to the standard models of general relativity, i.e., to Lorentzian manifolds having the corresponding smoothness properties.

1 Introduction

In physics, the same way as in mathematics, axioms are the basic postulates of the theory. However, in physics the statements are related to the real physical world and not just to abstract mathematical constructions. Therefore, the role of the axioms (the role of statements that we assume without proofs) in physics is more fundamental than in mathematics. That is why we aim to formulate simple, logically transparent and intuitively convincing axioms. All the surprising or unusual predictions of a physical theory should be provable as theorems and not assumed as axioms. For example, the prediction “no observer can move faster than light” is a theorem in our approach and not an axiom, see e.g., [1, 3, Thm 1].

In this paper, we introduce an axiom system $\text{GenRel}$ for general relativity (GR) and show that it is complete with respect to the standard models of GR, i.e., to (continuously differentiable) Lorentzian manifolds, see Theorem 4.1. This means that any statement true in the standard models can be proved from $\text{GenRel}$, see Corollary 4.2. Then we will generalize these results for smooth (and $n$-times continuously differentiable) Lorentzian manifolds, see Theorem 7.2 and Corollary 7.3.

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In GR, Einstein’s field equations give the connection between the geometry of the spacetime and the energy-matter distribution (given by the energy-momentum tensor field). The concept of timelike geodesic and thus all the important geometric notions of spacetimes are definable in the models of our axioms, see Section 6 and [3].

Therefore, we can use Einstein’s equations as a definition of the energy-momentum tensor, see e.g., [8] or [10, §13.1, p.169], or we can extend the language of our geometric theory by the concept of energy-momentum tensor and assume Einstein’s equations as axioms. There are only methodological differences between these two approaches. In both cases, we can assume any extra condition about the energy-momentum tensor as a new axiom.

We follow in the footsteps of several great predecessors since logical axiomatization of physics, especially that of relativity theory, goes back to such leading mathematicians and philosophers as Hilbert, Gödel, Carnap, Reichenbach, Suppes and Tarski.

Logical axiomatization of relativity theory also has an extensive literature, see e.g., Ax [4], Basri [6], Benda [8], Goldblatt [13], Latzer [17], Mundy [20, 21], Pambuccian [24], Robb [26, 27], Suppes [31], Schutz [28, 29, 30], Szabó [32].

Our goals go beyond the earlier approaches in several aspects. For example, we not searching for a single monolithic axiom system, but we are building a whole flexible hierarchy of axiom systems. We also make extra effort to get a deep understanding of the connections between the elements of this hierarchy, see e.g., [3], and the relations between axiom systems formulated using different basic concepts, see e.g., [4], [18].

Another novelty in our approach is that we concentrate on the transition from special relativity (SR) to GR, we try to keep this transition logically transparent and illuminating even for the non-specialists. Starting from our streamlined axioms system SpecRel of SR, we can “derive” the axioms of GenRel in two natural steps, see [3]. The axioms of GenRel are basically the localized versions of the axioms (and some theorems) of SpecRel.

The success story of using axiomatic method and foundational thinking in the foundations of mathematics also enforces our firm belief that it worth to apply them in the foundations of spacetime theories, see also Harvey Friedman [11, 12].

For good reasons, foundations of mathematics was carried through strictly within first-order logic (FOL). For the same reasons, foundations of spacetime theories are best developed within FOL. For example, in any foundational work it is essential to avoid tacit assumptions, and one acknowledged feature of using FOL is that it helps to eliminate tacit assumptions. There are several further reasons why we work within FOL, see [11 §Why FOL?], [34, §11].

2 **Axioms for General Relativity**

First, we introduce the basic concepts of our FOL axiom system GenRel for GR. We are going to consider two sorts of objects mathematical and physical. Mathematical objects will be called quantities, they will represent physical quantities, such as speeds or coordinates. We include addition, multiplication and ordering as basic concepts on quantities. Physical
objects will be called **bodies**. We will associate a body “sitting” at the origin to every coordinate system. We will call these bodies **observers**. Light signals (**photons**) will be another special type of bodies our axioms will speak about. Coordinate systems will be represented by one relation \( W \) that we will call **worldview relation**; \( W(m, b, x_1, \ldots, x_d) \) means intuitively that “observer \( m \) coordinatizes body \( b \) by coordinates \( x_1, \ldots, x_d \) (in his coordinate system).” Here, \( d \) is a fixed natural number determining the dimension of the coordinate systems.

The above means that we will use the following formal FOL language for axiomatizing GR:

\[
\{ B, Q, +, \cdot, \leq, \Phi, \text{Ob}, \text{W} \},
\]

where \( Q \) is a sort for quantities; \( B \) is a sort for bodies; \( +, \cdot \) are binary operations of sort \( Q \) and \( \leq \) is a binary relation of sort \( Q \). \( \text{Ob} \) and \( \Phi \) are unary relations of sort \( B \) for observers and photons; finally, \( W \) is a \( 2 + d \)-place relation connecting \( B \) and \( Q \) (the first two arguments are of sort \( B \) and the rest are of sort \( Q \)). More about the intuition and the why behind our choosing of this language can be found, e.g., in [3, §2].

Now we are ready to list the axioms of GenRel. The first axiom provides some useful and widely used properties of real numbers for the quantities.

**AxEField** The structure \( \langle Q, +, \cdot, \leq \rangle \) of quantities is a Euclidean field, i.e.,

- \( \langle Q, +, \cdot \rangle \) is a field in the sense of abstract algebra;\(^2\)
- the relation \( \leq \) is a linear ordering on \( Q \) such that
  - i) \( x \leq y \rightarrow x + z \leq y + z \) and
  - ii) \( 0 \leq x \land 0 \leq y \rightarrow 0 \leq xy \) holds; and
- nonnegative elements have square roots: \( 0 \leq x \rightarrow \exists y \ y^2 = x \).

We will use \( 0, 1, -, /, \sqrt{} \) as derived (i.e., defined) operation symbols.

**AxEField** is sufficient in SR for proving the main predictions; however, in GR we will have to use more properties of real numbers, see axiom schema **CONT** on p. 7.

The next two axioms speak about the so-called worldviews of observers. The **worldline** of body \( b \) according to observer \( m \) is defined as the collection of those coordinate points where \( m \) coordinatizes \( b \), i.e.,

\[
\text{wl}_m(b) \overset{\text{def}}{=} \{ \bar{x} : W(m, b, \bar{x}) \},
\]

where \( \bar{x} \) abbreviates \( n \)-tuple \( \langle x_1, \ldots, x_n \rangle \).

In SR, the worldlines of photons are straight lines, while in GR these worldlines are more general curves. The notion of velocity for these curves is the velocity of their straight
line approximations. Our central axiom for GR will state that the velocity of a photon is 1 according to an observer when meeting it. To introduce this axiom, we need some definitions and notations.

In our formulas, we will use the usual logical connectives ¬ (not), ∧ (and), ∨ (or), → (implies), ↔ (if-and-only-if) and FOL quantifiers ∃ (exists) and ∀ (for all).

In order to define velocity for curved worldlines, let us introduce a concept of approximation. Let \( f, g : Q^m \to Q^n, m, n \geq 1 \) be partial maps and \( \bar{x} \in Q^m \). We say that \( f \) approximates \( g \) at \( \bar{x} \), in symbols \( f \sim_{\bar{x}} g \), if

\[
\forall \varepsilon > 0 \; \exists \delta > 0 \; \forall y \; \left( |\bar{y} - \bar{x}| \leq \delta \right) \rightarrow \bar{y} \in \text{Dom} f \cap \text{Dom} g \land |f(\bar{y}) - g(\bar{y})| \leq \varepsilon \cdot |\bar{y} - \bar{x}|,
\]

where \( \text{Dom} f \) is the domain of function \( f \) (see p.9) and the Euclidean length \( |\bar{z}| \) of \( \bar{z} \in Q^k \) is defined as \( \sqrt{z_1^2 + \ldots + z_k^2} \).

**Remark 2.1.** By its definition, \( f \sim_{\bar{x}} g \) implies that \( \bar{x} \) has an open neighborhood where both \( f \) and \( g \) are defined; and that \( f(\bar{x}) = g(\bar{x}) \). Approximation at a given point is an equivalence relation on functions; and if two affine maps (i.e., maps that are composition of translations and linear maps) approximate each other, then they are equal. These facts can easily be proved from AxEFIELD.

When \( f \) is a unary function, i.e., when \( m = 1 \) above, the notion of derivative\(^3\) can be defined by the above concept of approximation as:

\[
f'(x) = \bar{y} \overset{\text{def}}{=} f \sim_{x} \{(x + t, f(x) + t \cdot \bar{y}) : t \in Q\}.
\]

By this definition, the derivative of \( f \) at \( x \) is an \( n \)-dimensional vector, we call it the derivative vector of \( f \) at \( x \).

It will be convenient to use the notions of space component and time component of \( \bar{x} \in Q^d \), respectively:

\[
\bar{x}_s \overset{\text{def}}{=} \langle x_1, x_2, \ldots, x_{d-1} \rangle \quad \text{and} \quad x_t \overset{\text{def}}{=} x_d.
\]

Assume that the worldline \( \text{wl}_m(b) \) of body \( b \) is a function of time and \( \text{Dom} \text{wl}_m(b) \) is open (i.e., \( \forall \bar{x}, \bar{y} \in \text{wl}_m(b) \; [x_t = y_t \rightarrow \bar{x}_s = \bar{y}_s] \) and \( \forall \bar{x} \in \text{wl}_m(b) \exists \delta > 0 \forall t \; |t - x_t| < \delta \rightarrow \exists \bar{y} \in \text{wl}_m(b) \; y_t = t) \). Then the velocity of body \( b \) according to observer \( m \) at \( \bar{x} \in \text{wl}_m(b) \) is defined as the time-derivative of the worldline of \( b \) at \( x_t \):

\[
\text{v}_m(b, \bar{x}) \overset{\text{def}}{=} \text{wl}_m(b)'(x_t).
\]

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3Partial means that \( f \) and \( g \) are not necessarily everywhere defined on \( Q^m \).
4The derivative of \( f \) is usually defined as the limit \( \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \), this is equivalent to our definition. Its intuitive meaning is how fast and in which direction the function increases at \( x \).
5To abbreviate formulas, we use bounded quantifiers in the following way: \( \exists x \; [\varphi(x) \land \psi] \) and \( \forall x \; [\varphi(x) \rightarrow \psi] \) are abbreviated to \( \exists \bar{x} \in \varphi \land \psi \) and \( \forall \bar{x} \in \varphi \rightarrow \psi \), respectively. So \( \forall \bar{x} \bar{y} \; [W(m, b, \bar{x}) \land W(m, b, \bar{y}) \rightarrow \psi] \) is abbreviated to \( \forall \bar{x}, \bar{y} \in \text{wl}_m(b) \; \psi \).
6Both \( \bar{x} \in \text{wl}_m(b) \) and \( b \in \text{ev}_m(\bar{x}) \) below represent the same atomic formula of our FOL language, namely: \( W(m, b, \bar{x}) \).
We defined velocity \( v_m(b, \bar{x}) \) only if \( \bar{x} \in \text{wl}_m(b) \) and \( \text{wl}_m(b) \) is a function of time defined at an open interval containing \( x_t \). Let us denote these assumptions by \( \bar{x} \in \text{Dom} v_m(b) \).

Now we are ready for formulating the central axiom of GenRel:

\[ \text{AxPh} \quad \text{The speed of a photon an observer "meets" is 1 when they meet, and it is possible to send out a photon in each direction where the observer stands:} \]

\[ \forall m \in \text{Ob} \quad \forall p \in \text{Ph} \quad \forall \bar{x} \quad [\text{W}(m, m, \bar{x}) \land \text{W}(m, p, \bar{x}) \rightarrow \bar{x} \in \text{Dom} v_m(p) \land |v_m(p, \bar{x})| = 1], \]

\[ \forall m \in \text{Ob} \quad \forall \bar{x} \quad (\text{W}(m, m, \bar{x}) \land |v| = 1 \rightarrow \exists p \in \text{Ph} \quad [\text{W}(m, p, \bar{x}) \land \bar{x} \in \text{Dom} v_m(p) \land v_m(p, \bar{x}) = v]). \]

The next axiom talks about the worldlines of observers. Let \( \bar{o} \) denote the origin of \( Q^{d-1} \), i.e., \( \bar{o} \overset{\text{def}}{=} \langle 0, \ldots, 0 \rangle \).

\[ \text{AxSelf} \quad \text{In his own worldview, the worldline of any observer is an interval of the time-axis containing all the coordinate points of the time-axis where the observer coordinatizes something:} \]

\[ \forall m \in \text{Ob} \quad \forall \bar{x} \in \text{wl}_m(m) \quad \bar{x}_s = \bar{o}, \text{ and} \]

\[ \forall m \in \text{Ob} \quad \forall \bar{x}, \bar{y} \in \text{wl}_m(m) \quad \forall t \quad [x_t < t < y_t \rightarrow \text{W}(m, m, \bar{o}, t)], \text{ and} \]

\[ \forall m \in \text{Ob} \quad \forall t \quad [\exists b \text{ W}(m, b, \bar{o}, t) \rightarrow \text{W}(m, m, \bar{o}, t)]. \]

By the event occurring for observer \( m \) at coordinate point \( \bar{x} \), we mean the set of bodies \( m \) coordinatizes at \( \bar{x} \):

\[ \text{ev}_m(\bar{x}) \overset{\text{def}}{=} \{ b : \text{W}(m, b, \bar{x}) \}. \]

\[ \text{AxEv} \quad \text{Observers see all the events in which they participate:} \]

\[ \forall mk \bar{x} \quad (\text{Ob}(k) \land \text{W}(m, k, \bar{x}) \rightarrow \exists \bar{y} \quad \text{ev}_m(\bar{x}) = \text{ev}_k(\bar{y})[3] \]

It is convenient to introduce the worldview transformation between observers \( m \) and \( k \) as the binary relation connecting those coordinate points in which \( m \) and \( k \) see the same nonempty events:

\[ \text{w}_{mk}(\bar{x}, \bar{y}) \overset{\text{def}}{=} \text{ev}_m(\bar{x}) = \text{ev}_k(\bar{y}) \neq \emptyset. \]

We regularize worldview transformations by the following axiom.

\[ ^7 \text{ev}_m(\bar{x}) = \text{ev}_k(\bar{y}) \text{ is an abbreviation of formula } \forall b \quad [\text{W}(m, b, \bar{x}) \leftrightarrow \text{W}(k, b, \bar{y})]. \]
AxCDiff. The worldview transformations between observers are functions having linear approximations $A_{\bar{x}}$ at each coordinate point $\bar{x}$ of their domain and this linear approximation $A_{\bar{x}}$ depends continuously on point $\bar{x}$ (i.e., they are continuously differentiable maps):

$$\forall m, k \in \text{Ob} \ [w_{mk} \text{ is a function}] \wedge \forall \bar{x} \in \text{Dom} \ w_{mk} \ \exists \ \text{affine map} A_{\bar{x}} \ w_{mk} \sim_{\bar{x}} A_{\bar{x}} ,$$

and

$$\forall m, k \in \text{Ob} \ \forall \varepsilon > 0 \ \exists \delta > 0 \ \forall \bar{y}, \bar{z} \in \text{Dom} \ w_{mk} \ (|\bar{y} - \bar{z}| < \delta \rightarrow |A_{\bar{y}} - A_{\bar{z}}| < \varepsilon).$$

Remark 2.2. The physical meaning of that the worldview transformations are functions is that no observer coordinatizes an event twice, i.e., $\forall m \in \text{Ob} \ \forall \bar{x} \bar{y} \ \text{ev}_m(\bar{x}) = \text{ev}_m(\bar{y}) \rightarrow \bar{x} = \bar{y}$.

Remark 2.3. Let us note that, by the definition of $\sim_{\bar{x}}$, AxCDiff implies that the domain $\text{Dom} \ w_{mk}$ of worldview transformation $w_{mk}$ is an open set. Therefore, AxSelf$^-$ and AxCDiff imply that the worldline of observer $m$, according to him, is an open interval of the time-axis since it is the intersection of $\text{Dom} \ w_{mm}$ and the time-axis.

Our next axiom states that the derivative of worldview transformations are continuous also in the sense that the difference how they distort the Minkowski metric is small for observers in close enough events. To formulate this axiom, we have to recall some definitions.

The **Minkowski metric** $\mu$ is defined as:

$$\mu(\vec{v}, \vec{w}) \overset{\text{def}}{=} v_1 \cdot w_1 - v_2 \cdot w_2 - \ldots - v_{d-1} \cdot w_{d-1}$$

for all $\vec{v}, \vec{w} \in Q^d$. The **derivative** (or linear approximation) of map $f$ at $\bar{x} \in Q^n$, denoted by $[d_{\bar{x}}f]$, is defined as follows:

$$[d_{\bar{x}}f](\bar{y}) = A(\bar{y} + \bar{x}) - A(\bar{x}) \overset{\text{def}}{=} f \sim_{\bar{x}} A \text{ and } A \text{ is affine.}$$

In the case of unary functions, the connection between this notion of derivative and derivative vector introduced at p. 4 is the following: $f'(x) = [d_{x}f](1)$ and $[d_{x}f](t) = t \cdot f'(x)$ for all $t \in Q$.

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8That $w_{mk}$ is a function can be formalized as follows: $\forall \bar{x} \bar{y} \bar{z} [w_{mk}(\bar{x}, \bar{y}) \wedge w_{mk}(\bar{x}, \bar{z}) \rightarrow \bar{y} = \bar{z}]$.

9The quantifier “$\exists$ affine map $A$” looks like a second-order logic one, but truly it is a FOL quantifier because every affine map from $Q^d$ to $Q^d$ can be represented by a $d \times d$ matrix and a vector of $Q^d$, i.e., $d^2 + d$ elements of $Q$.

10By Remark 2.2 affine map $A_{\bar{x}}$ approximating $w_{mk}$ at $\bar{x}$ is unique. This fact justifies that we can use it as a defined concept in this formula. Also the Euclidean distance of $A_{\bar{y}}, A_{\bar{z}}$ is meaningful since $A_{\bar{y}}, A_{\bar{z}} \in Q^{d^2+d}$. 
The difference how the linear approximations of worldview transformations distort the Minkowski metric is small for observers in close enough events:

\[ \forall m \in \text{Ob} \ \forall \bar{x} \in \text{Dom } w_{mm} \ \forall \varepsilon > 0 \ \exists \delta > 0 \]

\[ \forall \bar{y} \ \forall k, h \in \text{Ob} \ \left( |\bar{x} - \bar{y}| < \delta \land W(m, k, \bar{x}) \land W(m, h, \bar{y}) \rightarrow \forall \bar{v} \bar{w} \right) \]

\[ |\mu([d_{\bar{x}} w_{mk}](\bar{v}), [d_{\bar{x}} w_{mk}](\bar{w})) - \mu([d_{\bar{y}} w_{mh}](\bar{v}), [d_{\bar{y}} w_{mh}](\bar{w}))| < \varepsilon \).

The behavior of observer \( k \)'s clock as seen by observer \( m \) is defined as follows:

\( \text{cl}_{mk} \overset{\text{def}}{=} \{ (x_t, y_t) : w_{mk}(\bar{x}, \bar{y}) \text{ and } \bar{x} \in \text{wl}_m(k) \} \).

If \( \text{cl}_{mk} \) is a function, then it is differentiable (by our previous axioms) and \( \text{cl}_{mk}(t) \) is the time \( k \)'s clock shows "when" \( m \)'s clock shows \( t \). Thus, e.g., \( \text{cl}'_{mk}(t) = 2 \) means that at \( t \) (according to \( m \)'s clock) \( k \)'s clock runs twice as fast as \( m \)'s.

AxSym\( ^- \) Meeting observers see each other’s clocks slow down with the same rate:

\[ \forall m k \in \text{Ob} \ (\text{cl}_{mk} \text{ is a function } \land \forall \bar{x} \bar{y} \left[ m, k \in \text{ev}_m(\bar{x}) = \text{ev}_k(\bar{y}) \rightarrow \text{cl}'_{mk}(x_t) = \text{cl}'_{km}(y_t) \right] ) \).

So far, we have not assumed the existence of any observer. By the next axiom, we assume the existence of some slowly moving observers in every (nonempty) event. For a more delicate assumption ensuring the existence of an observer on every definable timelike curve segment, see axiom schema COMPR on p.135.

AxThExp\( ^0_0 \) There is an observer in every nonempty event:

\[ \exists h \ \text{Ob}(h) \land \forall m \in \text{Ob} \ \exists b \left[ W(m, b, \bar{x}) \rightarrow \exists k \ \text{Ob}(k) \land W(m, k, \bar{x}) \right] \).

If the number line has some definable gaps, some key predictions of relativity, such as the twin paradox my not hold, see [19], [34, Thms. 7.1.1 and 7.1.3]. Our next assumption is an axiom excluding these gaps.

CONT\( _L \) Every subset of \( Q \) which is \( L \)-definable, bounded and nonempty has a supremum (i.e., least upper bound) with respect to \( \leq \).

See p.20 for a detailed introduction of CONT\( _L \). Let \( \mathcal{G} \) be the language of GenRel, i.e.,

\( \mathcal{G} \overset{\text{def}}{=} \{ B, Q, +, \cdot, \leq, \text{Ph}, \text{Ob}, W \} \).

Let us now introduce an axiom systems for GR as the collection of the axioms above:

\[ \text{GenRel} \overset{\text{def}}{=} \text{AxEF} + \text{AxP}^- + \text{AxSelf}^- + \text{AxEv}^- + \text{AxCDiff} \]

\[ + \text{AxC}\_0^g + \text{AxSy}^- + \text{AxThExp}^0 + \text{CONT}_{\mathcal{G}} \]
3 An axiomatic Theory of Lorentzian Manifolds

Here, we introduce a FOL axiom system \textit{LorMan} of Lorentzian manifolds, see \cite[§2]{38}, \cite[§2.2]{7} for some non FOL definition of Lorentzian manifolds. The language of $d$-dimensional Lorentzian manifolds is the following:

\[ \{ I, Q, +, \cdot, \leq, \psi, g \} \]

where $I$ (indexes) and $Q$ (quantities) are two sorts, $+$ and $\cdot$ are two-place function symbols of sort $Q$, $\leq$ is a two-place relation symbol of sort $Q$, $\psi$ (transition relation) is a $2d+2$-place relation symbol the first two arguments of which are of sort $I$ and the rest are of sort $Q$, and $g$ (metric relation) is a $3d+2$-place relation symbol the first argument of which is of sort $I$ and the rest are of sort $Q$.

Now we are ready to formulate the axioms of \textit{LorMan}.

\textbf{AxFn} The transition and the metric relations are functions in their last variables:

\[ \forall i j \bar{x} \bar{y} \bar{y}' \ \psi(i, j, \bar{x}, \bar{y}) \land \psi(i, j, \bar{x}, \bar{y}') \rightarrow \bar{y} = \bar{y}', \text{ and} \]

\[ \forall i \bar{x} \bar{v} \bar{w} \bar{a} \bar{a}' \ g(i, \bar{x}, \bar{v}, \bar{w}, \bar{a}) \land g(i, \bar{x}, \bar{v}, \bar{w}, \bar{a}') \rightarrow \bar{a} = \bar{a}' . \]
By axiom AxFn, we can speak about the transition map $\psi_{ij}$ and metric $g_i$ in the following sense:

\[
\psi_{ij}(\bar{x}) = \bar{y} \iff \psi(i, j, \bar{x}, \bar{y}) \quad \text{and} \quad g_i(\bar{x})(\bar{v}, \bar{w}) = a \iff g(i, \bar{x}, \bar{v}, \bar{w}, a).
\]

We will refer to the first and the second parts of AxFn as AxFn$\psi$ and AxFng, respectively.

We think of functions as special binary relations. Hence we compose them as relations. The composition $R \mathrel{\hat{\circ}} S$ of binary relations $R$ and $S$ is defined as:

\[
R \mathrel{\hat{\circ}} S \overset{\text{def}}{=} \{ (a, c) : \exists b \ R(a, b) \land S(b, c) \}.
\]

So $(g \mathrel{\hat{\circ}} f)(x) = f(g(x))$ if $f$ and $g$ are functions. We will also use the notation $x \mathrel{\hat{\circ}} g \mathrel{\hat{\circ}} f$ for $(g \mathrel{\hat{\circ}} f)(x)$ because it is easier to grasp. In the same spirit, we will sometimes use the notation $x \mathrel{\hat{\circ}} f$ for $f(x)$.

The domain $\text{Dom} R$ and the range $\text{Ran} R$ of a binary relation $R$ are defined respectively as:

\[
\text{Dom} R \overset{\text{def}}{=} \{ x : \exists y \ R(x, y) \} \quad \text{and} \quad \text{Ran} R \overset{\text{def}}{=} \{ x : \exists x \ R(x, y) \}.
\]

The inverse of $R$ is defined as:

\[
R^{-1} \overset{\text{def}}{=} \{ (a, b) : R(b, a) \}.
\]

Let $\text{Id}_H$ denote the identity map from $H \subseteq Q^d$ to $H$, i.e.,

\[
\text{Id}_H(\bar{x}) = \bar{x} \quad \text{for all} \quad \bar{x} \in H.
\]

AxCom$\psi$ The transition maps satisfy the following basic compatibility relations:

\[
\forall i \psi_{ii} = \text{Id}_{\text{Dom} \psi_{ii}}, \quad (3)
\]

\[
\forall ij \psi_{ij} = \psi_{ji}^{-1}, \quad (4)
\]

\[
\forall ijk \psi_{ij} \mathrel{\hat{\circ}} \psi_{jk} \subseteq \psi_{ik}. \quad (5)
\]

**Proposition 3.1.** Axiom AxFn$\psi$ and AxCom$\psi$ imply that

\[
\langle i, \bar{x} \rangle \sim \langle j, \bar{y} \rangle \iff \psi_{ij}(\bar{x}) = \bar{y}
\]

is an equivalence relation on the set $\{ \langle i, \bar{x} \rangle : \bar{x} \in \text{Dom} \psi_{ii} \text{ and } i \in I \}$, i.e., on the disjoint union of the domains of $\psi_{ii}$.

**Proof.** Let us first note that $\text{Dom} \psi_{ij} \subseteq \text{Dom} \psi_{ii}$ for all $i, j \in I$ by axiom AxCom$\psi$ since

\[
\text{Dom} \psi_{ij} = \text{Dom} \psi_{ij} \mathrel{\hat{\circ}} \psi_{ij}^{-1} \subseteq \text{Dom} \psi_{ii}.
\]
Therefore, the definition of \( \sim \) is meaningful since we can compute \( \psi_{ij}(\bar{x}) \) for all \( \bar{x} \in \text{Dom } \psi_{ii} \).

The reflectivity of \( \sim \) is equivalent to \((3)\) since \( \langle i, \bar{x} \rangle \sim \langle i, \bar{x} \rangle \) iff \( \psi_{ii}(\bar{x}) = \bar{x} \) by definition.

The symmetry of \( \sim \) is equivalent to \((4)\) since

\[
\langle i, \bar{x} \rangle \sim \langle j, \bar{y} \rangle \iff \psi_{ij}(\bar{y}) = \bar{x} \iff \psi_{ji}(\bar{x}) = \bar{y} \iff \langle j, \bar{y} \rangle \sim \langle i, \bar{x} \rangle.
\]

Finally, the transitivity of \( \sim \) is implied by \((5)\). To show this, let \( \langle i, \bar{x} \rangle \sim \langle j, \bar{y} \rangle \) and \( \langle j, \bar{y} \rangle \sim \langle k, \bar{z} \rangle \). Then \( \bar{x} \sim \psi_{ij} \bar{y} \) and \( \bar{y} \sim \psi_{jk} \bar{z} \) by definition. Hence \( \bar{x} \sim \psi_{ij} \psi_{jk} \bar{z} \). By \((5)\) of axiom \( \text{AxCom}_\psi \), \( \psi_{ik} \) extends \( \psi_{ij} \psi_{jk} \). Therefore, \( \psi_{ik}(\bar{x}) = \bar{z} \). Thus \( \langle i, \bar{x} \rangle \sim \langle k, \bar{z} \rangle \) as desired.

**Remark 3.2.** By Proposition 3.1 manifold \( M \) can be defined as a new sort in the sense of [2, p.649.], i.e., let \( M \) be the disjoint union of the domains of transition maps \( \psi_{ii} \) factorized by the equivalence relation \( \sim \). Let \( e \in M \). The maps

\[
\psi_i(e) = \bar{x} \iff \langle i, \bar{x} \rangle \in e
\]

are the so called **charts** of \( M \), see Figure 1. Chart \( \psi_i \) is well-defined since \( \bar{x} = \bar{y} \) if \( \langle i, \bar{x} \rangle \sim \langle i, \bar{y} \rangle \) because \( \psi_{ii} = \text{id}_{\text{Dom } \psi_{ii}} \).

**AxCDiff** The transition maps are continuously differentiable:

\[
\forall ij \ \forall \bar{x} \in \text{Dom } \psi_{ij} \ \exists \text{ affine map } A \ \psi_{ij} \sim_{\bar{x}} A_{\bar{x}}, \text{ and } \forall i j \ \forall \varepsilon > 0 \ \exists \delta > 0 \ \forall \bar{y} \bar{z} \in \text{Dom } \psi_{ij} \ (|\bar{y} - \bar{z}| < \delta \rightarrow |A_{\bar{y}} - A_{\bar{z}}| < \varepsilon).
\]

Axiom **AxCDiff** implies that \( \text{Dom } \psi_{ij} \) is open by the definition of \( \sim_{\bar{x}} \), see Remark 2.1.

**AxCom** The metric and the transition maps commute in the following sense:

\[
\forall i \ \forall \bar{x} \in \text{Dom } g_i \cap \text{Dom } \psi_{ij} \ \forall \bar{v} \bar{w} \ g_i(\bar{x})(\bar{v}, \bar{w}) = g_j(\psi_{ij}(\bar{x}))( [d_{\psi_i} \psi_{ij}](\bar{v}), [d_{\psi_i} \psi_{ij}](\bar{w})).
\]

We used Proposition 3.1 to define the points of manifold \( M \) as equivalence classes of coordinate points connected by the transition maps \( \psi_{ij} \). Proposition 3.3 is an analogous statement that allows us to tie the vectors of different coordinate systems into one abstract element of the tangent space at a certain point \( e \) of \( M \) by using the derivatives \([d_{\psi_i(e)} \psi_{ij}]\) of the worldview transformations \( \psi_{ij} \) at the coordinate points \( \psi_i(e) \) corresponding to \( e \).

**Proposition 3.3.** Let \( e \in M \). Axioms **AxEFiel**, **AxFn**, **AxCom_\psi** and **AxCDiff_\psi** imply that

\[
\langle i, \bar{v} \rangle \approx_e \langle j, \bar{w} \rangle \iff \exists [d_{\psi_i(e)} \psi_{ij}](\bar{v}) = \bar{w}
\]

is an equivalence relation on the set

\[
\{ \langle i, \bar{v} \rangle : \bar{v} \in Q^d, \ i \in I \text{ and } e \in \text{Dom } \psi_i \}.
\]
**Proof.** Axioms AxField, AxFn, AxComψ and AxCDiffψ ensure that the definition of $\approx_{e}$ is meaningful, i.e., $M$ is definable and $[d_{\psi_{i}(e)}\psi_{ij}]$ exists.

To prove the reflexivity of $\approx_{e}$, let $i \in I$ (such that $e \in Dom \psi_{i}$) and $\bar{v} \in Q^{d}$. We have $\langle i, \bar{v} \rangle \approx_{e} \langle i, \bar{v} \rangle$ iff $[d_{\psi_{i}(e)}\psi_{ii}](\bar{v}) = \bar{v}$. But $\psi_{ii} = Id_{\psi_{ii}}$ by AxComψ. So $[d_{\psi_{i}(e)}\psi_{ii}] = Id_{Q^{d}}$. Hence $\approx_{e}$ is reflexive.

To prove the symmetry of $\approx_{e}$, let $\langle i, \bar{v} \rangle \approx_{e} \langle j, \bar{w} \rangle$. This is equivalent to $[d_{\psi_{i}(e)}\psi_{ij}](\bar{v}) = \bar{w}$ by definition. Since $\psi_{ij}^{-1} = \psi_{ji}$, we have $[d_{\psi_{i}(e)}\psi_{ij}]^{-1} = [d_{\psi_{i}(e)}\psi_{ji}]$ by Corollary 8.3. So $[d_{\psi_{j}(e)}\psi_{ji}]\bar{w} = \bar{v}$. Hence $\langle j, \bar{w} \rangle \approx_{e} \langle i, \bar{v} \rangle$. Thus $\approx_{e}$ is symmetric.

To prove the transitivity of $\approx_{e}$, let $\langle i, \bar{v} \rangle \approx_{e} \langle j, \bar{w} \rangle$ and $\langle j, \bar{w} \rangle \approx_{e} \langle k, \bar{u} \rangle$. Then $[d_{\psi_{i}(e)}\psi_{ij}](\bar{v}) = \bar{w}$ and $[d_{\psi_{j}(e)}\psi_{jk}](\bar{w}) = \bar{u}$ by definition. By chain rule (see Theorem 8.3) $[d_{\psi_{i}(e)}(\psi_{ij} \circ \psi_{jk})] = [d_{\psi_{i}(e)}\psi_{ij}] \circ [d_{\psi_{j}(e)}\psi_{jk}]$. So $[d_{\psi_{i}(e)}\psi_{ik}](\bar{v}) = \bar{u}$. Thus $\langle i, \bar{v} \rangle \approx_{e} \langle k, \bar{u} \rangle$. Hence $\approx_{e}$ is transitive. 

**Remark 3.4.** By Proposition 3.3, the tangent space at $e \in M$ can be defined as a new sort in the sense of [2] p.649., i.e., let $T_{e}M$ be the set

$$\{ \langle i, \bar{v} \rangle : \bar{v} \in Q^{d}, i \in I \text{ and } e \in Dom \psi_{i} \} \quad (6)$$

factorized by the equivalence relation $\approx_{e}$. By AxCom, the metric $g$ can be lifted to the tangent space $T_{e}M$, i.e., $g_{e} : T_{e}M \times T_{e}M \to Q$ can be defined for all $\bar{v}, \bar{w} \in T_{e}M$ as $g_{e}(\bar{v}, \bar{w}) = g_{e}(x)(\bar{v}, \bar{w})$ if $\psi_{i}(e) = x$, $\langle i, \bar{v} \rangle \in \bar{v}$ and $\langle i, \bar{w} \rangle \in \bar{w}$.

We assume that the metric is Lorentzian by postulating that it can be transformed to the Minkowski metric $\mu$.

**AxLorg** Metric $g_{i}$ is a Lorentzian metric for all $i \in I$:

$$\forall i \forall \bar{x} \in Dom g_{i} \exists \text{ linear map } L \forall \bar{v} \bar{w} \ g_{i}(\bar{x})(\bar{v}, \bar{w}) = \mu(L\bar{v}, L\bar{w}).$$

We also assume that the metric is continuous by the next axiom.

**AxC0g** Metric $g_{i}$ is continuous for all $i$:

$$\forall i \forall \varepsilon > 0 \forall \bar{x} \in Dom g_{i} \exists \delta > 0 \forall \bar{y} \in Dom g_{i} \left( |\bar{x} - \bar{y}| < \delta \right.$$ $$\left. \rightarrow \forall \bar{v} \bar{w} \ g_{i}(\bar{x})(\bar{v}, \bar{w}) - g_{i}(\bar{y})(\bar{v}, \bar{w})| < \varepsilon \right).$$

To ensure that the metric is defined everywhere, we assume the following axiom.

**AxFullg** Metric $g_{i}$ is defined everywhere on $Dom \psi_{ii}$ for all $i \in I$:

$$\forall i \ Dom \psi_{ii} \subseteq Dom g_{i}.$$ 

Let us note that, without assuming AxFullg, it is even possible that $Dom g_{i}$ is empty for all $i \in I$.

To be able to introduce our theory LorMan, let the above language of Lorentzian manifolds be denoted by $\mathcal{M}$, i.e., $\mathcal{M} = \{ I, Q, +, \cdot, \leq, \psi, g \}$.

$$\text{LorMan} \overset{\text{def}}{=} \text{AxField} + \text{AxFn} + \text{AxComψ} + \text{AxCom} + \text{AxCDiffψ}$$ $$+ \text{AxLorg} + \text{AxC0g} + \text{AxFullg} + \text{CONT}_{\mathcal{M}}$$
4 Completeness of GenRel with respect to Lorentzian Manifolds

Here we are going to define the basic concepts of Lorentzian manifolds in terms of GenRel. This will give us a translation of all the formulas of the language of Lorentzian manifolds to that of GenRel. Then we will show that the definitional extension of the models of GenRel satisfies the axioms of Lorentzian manifolds, see Theorem 4.1. This theorem implies that the translation of any sentence of language \( \mathcal{M} \) of Lorentzian manifolds can be proved from GenRel, see Corollary 4.2.

Let \( d \geq 3 \). Let \( \mathfrak{G} \) be a model of language \( \mathcal{G} \). We are going to associate a model \( M(\mathfrak{G}) \) of language \( \mathcal{M} \) to \( \mathfrak{G} \). Let \( M(\mathfrak{G}) \) be the following structure. Let the structure \( \langle Q, +, \cdot, \leq \rangle \) in \( M(\mathfrak{G}) \) be the same as that of \( \mathfrak{G} \). Let

\[ I \overset{\text{def}}{=} \text{Ob}, \quad \psi_{mk} \overset{\text{def}}{=} w_{mk}, \]

Finally, let relation \( g \) defined as follows

\[ g(m, \bar{x}, \bar{v}, \bar{w}, a) \overset{\text{def}}{=} \exists k \text{ Ob}(k) \land W(m, k, \bar{x}) \land \mu([d_{w}w_{mk}(\bar{v})], [d_{w}w_{mk}(\bar{w})]) = a. \]  (7)

The above model construction determines a translation from language \( \mathcal{M} \) of Lorentzian manifolds to language \( \mathcal{G} \) of GenRel. We give this translation by formula induction. Quantity variables are translated to quantity variables and index variables are translated to body variables. For atomic formulas, it is defined as follows:

\[ Tr(x + y) = x + y, \quad Tr(x \cdot y) = x \cdot y \quad Tr(x \leq y) = x \leq y, \]

\[ Tr(\psi(i, j, \bar{x}, \bar{y})) = \forall b \left[ W(i, b, \bar{x}) \leftrightarrow W(j, b, \bar{y}) \right], \]

\[ Tr(g(i, \bar{x}, \bar{v}, \bar{w}, a)) = \exists j \left[ \text{Ob}(j) \land W(i, j, \bar{x}) \land \mu([d_{w}w_{ij}(\bar{v})], [d_{w}w_{ij}(\bar{w})]) = a \right]. \]

For logical connectives \( \land, \neg, \) etc.:

\[ Tr(\psi \land \varphi) = Tr(\psi) \land Tr(\varphi), \quad Tr(\neg \varphi) = \neg Tr(\varphi), \] etc.

For quantifiers \( \forall \) and \( \exists \):

\[ Tr(\forall i \phi) = \forall i \left[ \text{Ob}(i) \rightarrow Tr(\phi) \right] \quad Tr(\exists i \phi) = \exists i \left[ \text{Ob}(i) \land Tr(\phi) \right] \]

if \( i \) is an index variable and

\[ Tr(\forall x \phi) = \forall x Tr(\phi) \quad Tr(\exists x \phi) = \exists x Tr(\phi) \]

if \( x \) is a quantity variable.

As usual, let \( \mathfrak{M} \models \phi \) denote that formula \( \phi \) is valid in model \( \mathfrak{M} \) and let the class of models of theory \( Th \) is defined as the collection of structures in which all the formulas of \( Th \) are valid:

\[ \text{Mod}(Th) \overset{\text{def}}{=} \{ \mathfrak{M} : \forall \phi \in Th \mathfrak{M} \models \phi \}. \]
Theorem 4.1. Let $d \geq 3$. Then

$$M(\mathcal{G}) \models \text{LorMan} \text{ if } \mathcal{G} \models \text{GenRel}$$

or equivalently

$$M(\mathcal{G}) \in \text{Mod} (\text{LorMan}) \text{ if } \mathcal{G} \in \text{Mod} (\text{GenRel}),$$

i.e., $M$ maps models of $\text{GenRel}$ to models of $\text{LorMan}$.

Theorem 4.1 implies the following completeness of $\text{GenRel}$, where $\vdash$ denotes the usual relation of FOL deducibility. Let $Fm(\mathcal{L})$ denote the set of all formulas of language $\mathcal{L}$.

Corollary 4.2. Let $d \geq 3$. Let $\varphi \in Fm(\mathcal{M})$. Then

$$\text{GenRel} \vdash Tr(\varphi) \text{ if } \mathcal{M} \models \varphi \text{ for all } \mathcal{M} \in \text{Mod}(\text{LorMan}).$$

The meaning of Corollary 4.2 is that if a statement $\varphi$ is true in every Lorentzian manifold, then its translation $Tr(\varphi)$ is provable from our axiom system $\text{GenRel}$.

In Section 7, we generalize these results for smooth (and $n$-times continuously differentiable) Lorentzian manifolds, see Theorem 7.2 and Corollary 7.3.

5 Turning Lorentzian Manifolds into Models of GenRel

In Section 4 we have constructed a Lorentzian manifold from every model of $\text{GenRel}$. What about the converse direction? Can a model of $\text{GenRel}$ constructed from every Lorentzian manifold? In this section, we are going to show that the converse construction works for smooth Lorentzian manifolds if the structure of quantities is the field $\mathbb{R}$ of real numbers.

To outline this construction, let $\mathcal{M}$ be a smooth Lorentzian manifold over $\mathbb{R}$. Let $\langle Q, +, \cdot, \leq \rangle$ be $\langle \mathbb{R}, +, \cdot, \leq \rangle$. By this choice, $\text{AxField}$ and $\text{CONT}_G$ are satisfied.

A vector $\vec{v} \in Q^d$ is called lightlike iff the length of its time component is equal to the length of its space component, i.e., $|v_t| = |\vec{v}_s|$; or equivalently $\mu(\vec{v}, \vec{v}) = 0$. A vector $\vec{v} \in Q^d$ is called timelike iff $|\vec{v}_s| < |v_t|$; or equivalently iff $\mu(\vec{v}, \vec{v}) > 0$.

A differentiable curve is called lightlike (timelike) if all of its derivative vectors are lightlike (timelike).

Let $\mathcal{P}_h$ be the set of lightlike curves in $\mathcal{M}$. We associate an observer $m$ to a normal convex neighborhood $N_m$ and a timelike curve segment $\gamma_m$ contained by $N_m$. So let $\text{Ob}$ be the set of pairs consisting a timelike curve segment and normal convex neighborhood containing it. Let $B$ be the union of $\mathcal{P}_h$ and $\text{Ob}$, i.e., $B = \mathcal{P}_h \cup \text{Ob}$.

To define $\mathcal{W}$, we associate a coordinate system to every observer $m$. Then $\mathcal{W}(m, b, \vec{x})$ will hold true iff the curve corresponding to body $b$ crosses coordinate point $\vec{x}$ in observer $m$’s coordinate system. We define the coordinate system of $m$ as a transformed version of $N_m$.

\footnote{See, e.g., [23, pp.129-130] for a precise definition.}
To satisfy \textbf{AxSelf}⁻, we have to transform \( N_m \) (in a smooth way) such that \( \gamma_m \) is mapped to a subset of the time-axis, see Figure 2. To satisfy \textbf{AxPh}⁻, we have to transform \( N_m \) such that the light signals crossing \( \gamma_m \) have coordinate speed 1 in the moment of the crossing in \( N_m \). These can be ensured by transforming \( N_m \) such that \( \gamma_m \) goes to a subset of the time-axis and the metric restricted to the time-axis in transformed \( N_m \) is the Minkowski metric. In this case, \textbf{AxSymT}⁻ will also be satisfied because then the derivative of the worldview transformation between meeting observers will be a Lorentz transformation in the point of meeting.

All the axioms corresponding to the smoothness of coordinate transformations and that of metric are also satisfied because \( M \) was smooth and we transformed \( N_m \) smoothly.

So the only question remains whether it is possible to transform neighborhoods \( N_m \) the way described above? By Fermi–Walker transporting (see, e.g., [25, §9]) of an orthogonal basis along \( \gamma_m \), we can get a so called Fermi–Walker normal coordinates. By transforming \( N_m \) to this normal coordinates we get the required coordinate system for observer \( m \).

It is a question for further research to generalize the construction of this section for Lorentzian manifolds over Euclidean fields.\footnote{This question is not at all trivial since it requires to generalizing several classical theorems of differential geometry over Euclidean fields in the spirit of [34, §10]. This may also require extending the languages of \textit{LorMan} and \textit{GenRel}, e.g., to be able to quantify over integrals of some definable functions. This is so because the usual definition of integral (as opposed to that of derivative) is not a FOL definition in the language of Euclidean fields. In [19], we were able to prove every theorem over Euclidean fields without a general FOL definable concept of integration by quantifying over observers when we needed to ensure the existence of the integrals of some definable functions. However, this trick may not work to prove every theorem used in the construction of this section.}

\section{Geodesics}

In this section, we are going to define timelike geodesics in \textit{GenRel}.
We call the worldline of observer $m$ **timelike geodesic**, if each of its points has a neighborhood within which $m$ “maximizes measured time” between any two encountered events, i.e.,

$$\forall \bar{z} \in \text{wl}_m(m) \exists \delta > 0 \ \forall k \bar{x} \bar{y} \left( |\bar{x} - \bar{z}| < \delta \land |\bar{y} - \bar{z}| < \delta \land \text{Ob}(k) \right)$$

$$\land \bar{x}, \bar{y} \in \text{wl}_m(m) \land \text{wl}_m(k) \land \left[ \forall \bar{w} \in \text{wl}_m(k) \left| \bar{w} - \bar{z} \right| < \delta \right]$$

$$\rightarrow |x_t - y_t| \geq |w_{mk}(\bar{x})_t - w_{mk}(\bar{y})_t|, \quad (8)$$

see Fig. 3

If there are not enough observers, it may not be a big deal that the worldline of $m$ is a time-like geodesic by the above definition. Therefore, we postulate the existence of many observers by the following axiom schema of comprehension.

**COMPR** For any parametrically definable continuously differentiable timelike curve in any observer’s worldview, there is another observer whose worldline is the range of this curve.

**COMPR** can be formalized as the collection of formulas $\text{Ax}\exists \psi$ below. To introduce these formulas, let $\psi$ be a formula in the language of GenRel such that all the free variables of $\psi$ are among $t, \bar{x}$ and $\bar{y}$, where $t \in Q, \bar{x} \in Q^d$ and there is no restriction on parameter $\bar{y}$.

**Ax$\exists \psi$** If formula $\psi$ defines a continuously differentiable timelike curve in observer $m$’s worldview, then there is another observer $k$ whose worldline is the range of curve $\psi$:

$$\forall \bar{y} \forall m \left( \text{Timelikecurve}(m, \psi) \rightarrow \exists k [\bar{x} \in \text{wl}_m(k) \leftrightarrow \exists t \ \psi(t, \bar{x}, \bar{y})] \right), \quad (9)$$

where $\text{Timelikecurve}(m, \psi)$ is a formula expressing that $\psi$ defines a timelike curve in observer $m$’s worldview. Formula $\text{Timelikecurve}(m, \psi)$ can be formulated as the conjunction of the
following:

“ψ defines a function,” i.e.,
\[
\forall t \exists \bar{x} \quad [\psi(t, \bar{x}, \bar{y}) \land \psi(t, \bar{z}, \bar{y}) \rightarrow \bar{x} = \bar{z}], \tag{13}
\]

“Dom ψ is an open interval,” i.e.,
\[
\forall ab \in \text{Dom} \psi \exists \delta > 0 \forall c \left[ (|a - c| < \delta \lor a < c < b) \rightarrow c \in \text{Dom} \psi \right],
\]

“ψ is differentiable,” i.e.,
\[
\forall t_0 \in \text{Dom} \psi \exists \varepsilon > 0 \forall t \left( t \in \text{Dom} \psi \land 0 < |t - t_0| < \delta \rightarrow |\psi(t) - \psi(t_0) - \bar{z}(t - t_0)| < \varepsilon |t - t_0| \right), \tag{14}
\]

“ψ’ is continuous,” i.e.,
\[
\forall t_0 \in \text{Dom} \psi' \exists \varepsilon > 0 \forall t \left( t \in \text{Dom} \psi' \land |t - t_0| < \delta \rightarrow |\psi'(t) - \psi'(t_0)| < \varepsilon \right),
\]

“ψ is timelike,” i.e.,
\[
\forall t \in \text{Dom} \psi \forall k \in \text{ev}_m (\psi(t)) \mu (d_{\psi(t)} \mathbf{w}_{mk} | \psi'(t), (d_{\psi(t)} \mathbf{w}_{mk} | \psi'(t)) > 0.
\]

The assumption of axiom schema COMPR guarantees that our definition of geodesic coincides with the usual one because of the followings.

Over the field \( \mathbb{R} \) of real numbers, a curve is timelike geodesic if it is locally the longest curve among all timelike curves, see, e.g., [14, Prop.4.5.3.]. By [8] and COMPR, the worldline of observer is timelike geodesic if it is locally the longest among all definable timelike curves. So to show that [8] gives back the usual notion of timelike geodesics, it is enough to show that every timelike curve can be approximated by a definable timelike curve. Now we are going to show this.

**Theorem 6.1.** In continuously differentiable Lorentzian manifolds over the field \( \mathbb{R} \) of real numbers, every timelike curve can be approximated (with arbitrary precision) by continuously differentiable timelike curves definable in the language of ordered fields.

**Proof.** Let \( \gamma \) be a timelike curve that we would like to approximate with precision \( \varepsilon > 0 \). Without losing generality we can assume that \( \gamma \) can be covered by one coordinate system (otherwise we cut \( \gamma \) into smaller pieces and approximate it piece by piece). So let us fix a coordinate system containing \( \gamma \).

It is well-known that all curves can be approximated by broken lines. So let \( \bar{x}_1 \bar{x}_2 \ldots \bar{x}_n \) be a broken line approximating \( \gamma \) with precision \( \varepsilon/3 \) in the fixed coordinate system containing \( \gamma \). Without losing generality, we can assume that \( \bar{x}_i \bar{x}_{i+1} \) are chords of \( \gamma \).

---

13From now on, we will denote the value of the function defined \( \psi \) at \( t \) by \( \psi(t) \).
14From now on, we will denote by \( \psi' \) the derivative of \( \psi \) defined by this FOL formula.
In Minkowski spacetime, all chords of a timelike curve are timelike (see [34, Prop.10.4.4] for a proof of this statement using AxEffield). So if the broken line approximation is fine enough, \( \bar{x}_i \bar{x}_{i+1} \) are timelike segments since the metric is continuous.

Broken line \( \bar{x}_1 \bar{x}_2 \ldots \bar{x}_n \) may not be definable. However, since the field \( \mathbb{Q} \) of rational numbers is dense in \( \mathbb{R} \) and points having rational coordinates are definable, we can replace this broken line with a definable one without changing its length more than \( \varepsilon/3 \). Let this definable broken line be \( \bar{y}_1 \bar{y}_2 \ldots \bar{y}_n \). So \( \bar{y}_1 \bar{y}_2 \ldots \bar{y}_n \) is a definable broken line which approximates \( \gamma \) with precision \( 2\varepsilon/3 \).

We have to prove that the vertexes of this definable broken line \( \bar{y}_1 \bar{y}_2 \ldots \bar{y}_n \) can be rounded by continuously differentiable definable timelike curves in small enough neighborhoods without changing its length more than \( \varepsilon/3 \). Since the vertexes of \( \bar{y}_1 \bar{y}_2 \ldots \bar{y}_n \) are points having rational coordinates, the corresponding four-velocities \( \bar{\gamma}_i = \bar{y}_i - \bar{y}_{i-1} \) are definable and the definable coordinates on \( \bar{y}_1 \bar{y}_2 \ldots \bar{y}_n \) are dense. In Minkowski spacetime, any two definable coordinate points can be connected by a continuously differentiable definable timelike curve \( \gamma^* \) such that the speeds of \( \gamma^* \) at the start and the end are arbitrary definable speeds smaller than 1 and the speed of \( \gamma^* \) between these points are smaller than \( 1 - \delta \) for some \( \delta > 0 \) (this last property guaranties that \( \gamma^* \) remains timelike if we change the metric slightly), see Lemma [8.11]. Since the metric is continuous, the metric in small enough neighborhoods around the vertexes are approximately the Minkowski metric. So we can use Lemma [8.11] to connect definable points on the edges near to the vertexes of the broken line \( \bar{y}_1 \bar{y}_2 \ldots \bar{y}_n \) in small enough neighborhoods without changing its length more than \( \varepsilon/3 \).

The resulting rounded up broken line can be parametrized such that it gives us the desired continuously differentiable definable timelike curve that approximates \( \gamma \) with precision \( \varepsilon \).

\[\blacksquare\]

7 refinements of the main theorem

In this section, we are going to refine Theorem 4.1 for smooth (and \( n \)-times continuously differentiable) Lorentzian manifolds by introducing axioms ensuring the smoothness of the worldview transformations and the metric. To do so, we need some further definitions.

Let the standard basis vectors of \( \mathbb{Q}^n \) be denoted by \( \bar{e}_i \), i.e.,

\[
\bar{e}_i \triangleq \langle 0, \ldots, 1, \ldots, 0 \rangle
\]

for all \( 1 \leq i \leq n \). Let \( f \) be a definable function from a subset of \( \mathbb{Q}^k \) to \( \mathbb{Q} \) defined by formula \( \phi_f(\bar{x}, \bar{y}) \), i.e., \( f(\bar{x}) = \bar{y} \iff \phi_f(\bar{x}, \bar{y}) \). The \( i \)-ht partial derivative of \( f \) is defined by the following FOL formula:

\[
\phi_{\partial_i f}(\bar{z}, \bar{w}) \overset{def}{\iff} \phi_f(\bar{x}, \bar{y}) \land \forall \varepsilon > 0 \exists \delta > 0 \forall h z \left( |h| \leq \delta \land \phi_f(\bar{x} + h \cdot \bar{e}_i, \bar{z}) \rightarrow |z - y - \bar{w} \cdot h| \leq \varepsilon |h| \right).
\]
Formula $\phi_{\partial_if}$ captures the usual concept of partial derivatives, i.e.,

$$\partial_if(x) \overset{\text{def}}{=} \lim_{h \to 0} \frac{f(x + h \cdot e_i) - f(x)}{h}.$$  

We say that $i$-th partial derivative of $f$ exists at $\bar{x}$ iff there is a $w$ such that $\phi_{\partial_if}(\bar{x}, w)$ holds. Since function $\partial_if$ defined by formula $\phi_{\partial_if}$ is the same type as $f$, i.e., $\text{Dom} \partial_if \subseteq \text{Dom} f$ and $\text{Ran} \partial_if \subseteq Q$, we can iterate the partial derivations and define $\partial_{i_1 \ldots i_n} f$ as $\partial_i \partial_i_2 \ldots \partial_i_n f$.

Function $f = \langle f_1, \ldots, f_m \rangle : Q^k \to Q^m$ is said to be $n$-times **continuously differentiable** if $\text{Dom} f$ is open and all $n$-th partial derivatives of all of its components (i.e., $\partial_{i_1 \ldots i_n} f_1(\bar{z}), \ldots, \partial_{i_1 \ldots i_n} f_m(\bar{z})$ for all $0 \leq i_1 \ldots i_n \leq k$) exits for all $\bar{z} \in \text{Dom} f$ and they are continuous. This concept can be defined by a FOL formula since the partial derivatives and the continuity can be defined in FOL, see [34, §10.2].

In GenRel, we assumed only the differentiability of the worldview transformations. We assume stronger differentiability properties for them by the next axioms:

**AxC**  The worldview transformations are $n$-times continuously differentiable maps:

$$\forall m, k \in \text{Ob} \ (w_{mk} \text{ is a function} \quad \wedge \text{Dom} w_{mk} \text{ is open} \quad \wedge \forall \bar{x} \in \text{Dom} w_{mk} \quad \bigwedge_{1 \leq a_1, \ldots, a_n \leq d} \partial_{a_1 \ldots a_n} w_{mk}(\bar{x}) \text{ exists and } \partial_{a_1 \ldots a_n} w_{mk} \text{ is continuous}.)$$

**Remark 7.1.** Axiom **AxCDiff** is equivalent to **AxC** because, if $f$ is a differentiable function from a subset of $Q^k$ to $Q^m$, then

$$[d_{xf}] = \begin{bmatrix} \partial_f f_1(\bar{x}) & \partial_f f_2(\bar{x}) & \ldots & \partial_f f_k(\bar{x}) \\ \vdots & \vdots & \ddots & \vdots \\ \partial_f f_m(\bar{x}) & \partial_f f_m(\bar{x}) & \ldots & \partial_f f_m(\bar{x}) \end{bmatrix}.$$  

By the following axioms, we can ensure the metric corresponding observers to be smooth enough.

**AxCg**  For all observer $m$, the metric $g_m$ is $n$-times continuously differentiable:

$$\forall m \in \text{Ob} \ \forall \bar{x} \in \text{Dom} g_m \quad \bigwedge_{1 \leq a_1, \ldots, a_n \leq d} \partial_{a_1 \ldots a_n} g_m(\bar{x}) \text{ exists and } \partial_{a_1 \ldots a_n} g_m \text{ is continuous}.$$  

---

15 The statement “definable set $H \subseteq Q^n$ is open” can be captured by the FOL formula $\forall \bar{x} \in H \exists \delta > 0 \forall \bar{y} \ (|\bar{x} - \bar{y}| < \delta \rightarrow \bar{y} \in H)$.  

16 The continuity of definable function $f$ can be captured by the following FOL formula $\forall \bar{x} \in \text{Dom} f \forall \varepsilon > 0 \exists \delta > 0 \forall \bar{y} \in \text{Dom} f \ (|\bar{x} - \bar{y}| < \delta \rightarrow |f(\bar{x}) - f(\bar{y})| < \varepsilon)$.  

18
For the smooth case, let us introduce $\text{AxC}^\infty$ as the axiom schema containing $\text{AxC}^n$ for all positive integers $n$; and let $\text{AxC}^\infty g_m$ be the axiom schema containing $\text{AxC}^n g_m$ for all positive integers $n$. Now we can introduce the promised extensions of $\text{GenRel}$:

$$\text{GenRel}^n \overset{\text{def}}{=} \text{GenRel} + \text{AxC}^n + \text{AxC}^{n-1} g_m$$

if $1 \leq n \leq \infty$. By Remark [7.1] $\text{GenRel}^1$ is equivalent to $\text{GenRel}$.

Let us now introduce the corresponding axioms for $n$-times continuously differentiable Lorentzian manifolds.

**AxC$^n\psi$** The transition maps are $n$-times continuously differentiable:

$$\forall ij \text{ Dom } \psi_{ij} \text{ is open } \land \forall \bar{x} \in \text{ Dom } \psi_{ij} \bigwedge_{1 \leq a_1, \ldots, a_n \leq d} \partial_{a_1 \ldots a_n} \psi_{ij}(\bar{x}) \text{ exists and } \partial_{a_1 \ldots a_n} \psi_{ij} \text{ is continuous.}$$

**AxC$^n g$** Metric $g_i$ is $n$-times continuously differentiable for all $i$:

$$\forall i \forall \bar{x} \in \text{ Dom } g_i \bigwedge_{1 \leq a_1, \ldots, a_n \leq d} \partial_{a_1 \ldots a_n} g_i(\bar{x}) \text{ exists and } \partial_{a_1 \ldots a_n} g_i \text{ is continuous.}$$

Let $\text{AxC}^\infty \psi$ be the axiom schema containing $\text{AxC}^n \psi$ for all positive integers $n$; and let $\text{AxC}^\infty g$ be the axiom schema containing $\text{AxC}^n g$ for all positive integers $n$.

Now we can introduce the FOL theories for Lorentzian manifolds corresponding to $\text{GenRel}^n$:

$$\text{LorMan}^n \overset{\text{def}}{=} \text{LorMan} + \text{AxC}^n \psi + \text{AxC}^{n-1} g$$

if $1 \leq n \leq \infty$.

**Theorem 7.2.** Let $d \geq 3$ and $1 \leq n \leq \infty$. Then

$$M(\mathcal{G}) \models \text{LorMan}^n \text{ if } \mathcal{G} \models \text{GenRel}^n$$

or equivalently

$$M(\mathcal{G}) \in \text{Mod}(\text{LorMan}^n) \text{ if } \mathcal{G} \in \text{Mod}(\text{GenRel}^n),$$

i.e., $M$ maps models of $\text{GenRel}^n$ to models of $\text{LorMan}^n$.

Theorem 7.2 implies the following completeness of $\text{GenRel}$, where $\vdash$ denotes the usual relation of FOL deducibility.

**Corollary 7.3.** Let $d \geq 3$ and $1 \leq n \leq \infty$. Let $\varphi \in Fm(\mathcal{M})$. Then

$$\text{GenRel}^n \vdash \text{Tr}(\varphi) \text{ if } \mathcal{M} \models \varphi \text{ for all } \mathcal{M} \in \text{Mod}(\text{LorMan}^n).$$
8 Proof of Theorem 7.2

In this section, we are going to prove our main result Theorem 7.2 and some earlier used statements. To do so, let us first give a detailed introduction of axiom schema \(\text{CON} \ L\).

Let \(\mathcal{L}\) be a many sorted language containing sort \(Q\) and a binary relation \(\leq\) on \(Q\). Let \(Fm(\mathcal{L})\) be the set of FOL formulas of \(\mathcal{L}\).

To introduce a \(\text{CON} \ L\) precisely, we have to introduce some notations. Let \(M\) be a model of language \(\mathcal{L}\) and \(\varphi \in Fm(\mathcal{L})\). Let \(U\) be the union of the sorts of \(M\). We use \(M \models \varphi\) in the usual sense of mathematical logic to denote that formula \(\varphi\) is valid in the structure \(M\) and \(M \models \varphi[a_1, \ldots, a_n]\) to denote that \(a_1, \ldots, a_n \in U\) satisfies \(\varphi\) in \(M\). We say that a subset \(H\) of \(Q\) is (parametrically) \(\mathcal{L}\)-definable by \(\varphi\) iff there are \(a_1, \ldots, a_n \in U\) such that

\[
H = \{ d \in Q : M \models \varphi[d, a_1, \ldots, a_n] \}.
\]

We say that a subset of \(Q\) is \(\mathcal{L}\)-definable iff it is definable by a formula of \(\mathcal{L}\). More generally, an \(n\)-ary relation \(R \subseteq Q^n\) is said to be \(\mathcal{L}\)-definable in \(M\) by parameters iff there is a formula \(\varphi \in Fm(\mathcal{L})\) with only free variables \(x_1, \ldots, x_n, y_1, \ldots, y_k\) and there are \(a_1, \ldots, a_k \in U\) such that

\[
R = \{ (p_1, \ldots, p_n) \in Q^n : M \models \varphi[p_1, \ldots, p_n, a_1, \ldots, a_k] \}.
\]

By the next axiom, for all formulas \(\varphi \in Fm(\mathcal{L})\) defining a subset of the quantities, we introduce an axiom postulating the existence of the supremum of the defined set if it is not empty and bounded.

\(\text{AxSup}_\varphi\) Every subset of \(Q\) definable by \(\varphi\) (when using \(y_1, \ldots, y_n\) as fixed parameters) has a supremum if it is nonempty and bounded:

\[
\forall y_1, \ldots, y_n \ (\exists x \ \varphi) \land [ (\exists b \ \forall x \ \varphi \to x \leq b) \to (\exists s \ \forall b \ [\forall x \ \varphi \to x \leq b] \leftrightarrow s \leq b)],
\]

where \(x\) is a variable of sort \(Q\). Now we can introduce \(\text{CON} \ L\) at the following axiom schema:

\[
\text{CON}_\mathcal{L} \overset{\text{def}}{=} \{ \text{AxSup}_\varphi : \varphi \text{ is a FOL formula of language } \mathcal{L} \}.
\]

Let us note that \(\text{CON} \ L\) is true in any model whose structure of quantities is the field of real numbers.

Let us also recall here the definition of Lorentz transformation. A linear transformation \(L\) is called \textit{Lorentz transformation} iff it preserves the Minkowski metric \(\mu\), i.e., \(\mu(\bar{v}, \bar{w}) = \mu(L(\bar{v}), L(\bar{w}))\) for all \(\bar{v}, \bar{w} \in Q^d\).

Theorem 8.1 states that \(\text{GenRel}\) implies that the derivatives of the worldview transformations between observers at the events of meeting are Lorentz transformations.

**Theorem 8.1.** Let \(d \geq 3\). Assume \(\text{GenRel}\). Let \(\forall m, k \in \text{Ob}\) and \(\bar{x} \in w_{m}(k) \cap w_{n}(m)\). Then \(w_{mk}\) is differentiable at \(\bar{x}\) and \([d_\bar{x}w_{mk}]\) is a Lorentz transformation. Here we are going to prove Theorem 8.1. To do so, first we introduce some definitions and lemmas we will use in the proof.
**Lemma 8.2.** Assume AxEField, AxEv, and AxCDiff. Let \( m, k \in \text{Ob} \) and \( \bar{x} \in \text{wl}_m(k) \). Then \( w_{mk} \) is a function differentiable at \( \bar{x} \).

**Proof.** Since \( \bar{x} \in \text{wl}_m(k) \), there is a \( \bar{y} \) such that \( \text{ev}_m(\bar{x}) = \text{ev}_k(\bar{y}) \) by AxEv. We have that \( \text{ev}_m(\bar{x}) \neq \emptyset \) since \( k \in \text{ev}_m(\bar{x}) \). Hence \( \bar{x} \in \text{Dom} w_{mk} \). So by AxCDiff, \( w_{mk} \) is a function differentiable at \( \bar{x} \).

Let us recall here that the chain rule of real analysis can be proved using axiom AxEField only, see [34, §10.3].

**Theorem 8.3** (chain rule). Assume AxEField. Let \( g : Q^n \to Q^m \) and \( f : Q^m \to Q^k \). If \( g \) is differentiable at \( \bar{x} \in Q^n \) and \( f \) is differentiable at \( g(\bar{x}) \), then \( g \circ f \) is differentiable at \( \bar{x} \) and its derivative is \( [d_\bar{x}(g \circ f)] = [d_\bar{x}g] \circ [d_{g(\bar{x})}f] \), i.e.,

\[
[d_\bar{x}(g \circ f)] = [d_\bar{x}g] \circ [d_{g(\bar{x})}f].
\]

In particular, if \( g : Q \to Q^m \), and \( g \) is differentiable at \( \bar{x} \in Q \), and \( f \) is differentiable at \( g(\bar{x}) \), then

\[
(g \circ f)'(\bar{x}) = [d_{g(\bar{x})}f](g'(\bar{x})).
\]

**Corollary 8.4.** Assume AxEField. Let \( f : Q^n \to Q^n \) be an injective function such that \( f^{-1} \) is differentiable at \( \bar{x} \). Then

\[
[d_\bar{x}f^{-1}] = [d_{f^{-1}(\bar{x})}f]^{-1}.
\]

In particular, if \( n = 1 \),

\[
(f^{-1})'(\bar{x}) = \frac{1}{f'(f^{-1}(\bar{x}))}.
\]

**Lemma 8.5.** Assume AxEField, AxEv, and AxCDiff. Let \( m, k \in \text{Ob} \) and \( \bar{x} \in \text{wl}_m(k) \cap \text{wl}_m(m) \). Then \( [d_\bar{x}w_{mk}] \) is invertible and \( [d_\bar{x}w_{mk}]^{-1} = [d_{\bar{y}}w_{km}] \), where \( \bar{y} = w_{mk}(\bar{x}) \).

**Proof.** By AxCDiff, \( w_{mk} \) and \( w_{km} \) are differentiable functions. Since \( \bar{x} \in \text{wl}_m(k) \), there is a \( \bar{y} \) such that \( \text{ev}_m(\bar{x}) = \text{ev}_k(\bar{y}) \) by AxEv. We have that \( \text{ev}_m(\bar{x}) \neq \emptyset \) since \( m, k \in \text{ev}_m(\bar{x}) \). Hence \( \bar{x} \in \text{Dom} \text{w}_{mk} \), \( \bar{y} = w_{mk}(\bar{x}) \) and \( \bar{y} \in \text{Dom} w_{km} \). Thus \( w_{mk} \) is differentiable at \( \bar{x} \) and \( w_{km} \) is differentiable at \( \bar{y} \). Since \( w_{km} \) is the inverse of \( w_{mk} \) by definition, \( [d_\bar{y}w_{km}] \) is invertible and its inverse is \( [d_{\bar{y}}w_{km}] \) by Corollary 8.4.

The **restriction** of function \( f : A \to B \) to set \( H \), denoted by \( f \upharpoonright_H \), is defined as follows:

\[
f \upharpoonright_H \overset{\text{def}}{=} \{ (a, b) : a \in H \cap \text{Dom} f \land f(a) = b \}.
\]

The **f-image of set** \( H \), is defined as follows:

\[
f[H] = \{ b : \exists a \in H \cap \text{Dom} f \land f(a) = b \}.
\]

21
Lemma 8.6. Assume $\text{AxEField}$, $\text{AxEv}^-$, $\text{AxCDiff}$, and $\text{AxPh}^-$. Let $m, k \in \text{Ob}$ and $\bar{x} \in \text{wl}_m(k) \cap \text{wl}_m(m)$. Then $[d_{\bar{x}} w_{mk}]$ is a linear bijection taking lightlike vectors to lightlike vectors.

Proof. By Lemma 8.5, $[d_{\bar{x}} w_{mk}]$ is a linear bijection.

Now we are going to show that $[d_{\bar{x}} w_{mk}]$ takes lightlike vectors to lightlike ones. To do so, let $v \in Q^{d-1}$ for which $|v| = 1$. Since $\bar{x} \in \text{wl}_m(m)$, there is a photon $p$ in event $\text{ev}_m(\bar{x})$ such that $v = \text{wl}_m(p)'(x_t)$ by $\text{AxPh}^-$, see Figure 4. Let $\bar{y}$ be the $w_{mk}$ image of $\bar{x}$. By $\text{AxPh}^-$, $\text{wl}_k(p)$ is a function defined in an open neighborhood of $y_t$. Since $\text{Dom} w_{mk}$ and $\text{Ran} w_{mk}$ are open, there is an open set $U \subseteq Q^d$ such that $\bar{x} \in U$ and

$$\text{Ran} w_{mk}[\text{wl}_m(p) \cap U] \subseteq \text{wl}_k(p).$$

Therefore, the tangent line of $\text{wl}_m(p)$ is mapped into the tangent line of $\text{wl}_k(p)$ by $[d_{\bar{x}} w_{mk}]$. Thus $[d_{\bar{x}} w_{mk}](\langle v, 1 \rangle)$ is parallel to $\langle \text{wl}_k(p)'(y_t), 1 \rangle$, which is a lightlike vector since $|\text{wl}_k(p)'(y_t)| = 1$ by $\text{AxPh}^-$. Therefore, $[d_{\bar{x}} w_{mk}](\langle v, 1 \rangle)$ is a lightlike vector. Since for any lightlike vector $\bar{v} \in Q^d$ there is a $v \in Q^{d-1}$ and $c \in Q$ such that $\bar{v} = c \cdot \langle v, 1 \rangle$, we have that $[d_{\bar{x}} w_{mk}]$ is a linear transformation taking lightlike vectors to lightlike vectors. \hfill ■

We say that a linear bijection $A$ has the **sym-time property** if

$$A(\bar{e}_d)_t = A^{-1}(\bar{e}_d)_t.$$  

Lemma 8.7. Assume axioms $\text{AxEField}$, $\text{AxSelf}^-$, $\text{AxEv}^-$, and $\text{AxCDiff}$. Then $\text{AxSymt}^-$ implies that $[d_{\bar{x}} w_{mk}]$ has the sym-time property for all observers $m$ and $k$ and coordinate point $\bar{x}$ for which $m, k \in \text{ev}_m(\bar{x})$.

Let $m$ and $k$ be observers and let $\bar{x}$ be a coordinate point such that $m, k \in \text{ev}_m(\bar{x})$. By $\text{AxCDiff}$, $w_{mk}$ is a differentiable function. By Lemma 8.2, $w_{mk}$ is differentiable at $\bar{x}$, i.e., $\bar{x} \in \text{Dom} w_{mk}$. Let $\bar{y}$ be $w_{mk}(\bar{x})$. Let $t : Q \to Q^d$ be the linear map $t(t) \overset{\text{def}}{=} \langle 0, \ldots, 0, t \rangle$ for
all $t \in Q$ and let the projection $\pi_t : Q^d \to Q$ be defined as $\pi_t(\bar{x}) \overset{\text{def}}{=} x_t$ for all $\bar{x} \in Q^d$. By axiom $\text{AxSelf}^-$, $\bar{y} = \iota(y_t)$ since $W(k,k,\bar{y})$. Let us note that

$$\text{cl}_{mk}(t) = (\iota \circ w_{km} \circ \pi_t)^{-1}(t)$$

for all $t \in \text{Dom cl}_{mk}$ by definitions and axiom $\text{AxSelf}^-$, see Figure 5. So $\text{cl}_{mk}(x_t) = y_t$ since $\iota(y_t) = \bar{y}$, $w_{km}(\bar{y}) = \bar{x}$ and $\pi_t(\bar{x}) = x_t$. Thus, by Corollary 8.4,

$$\text{cl}'_{mk}(x_t) = \frac{1}{(\iota \circ w_{km} \circ \pi_t)'(y_t)}$$

since $\text{cl}_{mk}$ is differentiable at $x_t$ by $\text{AxSym}^-$ and its inverse $\iota \circ w_{km} \circ \pi_t$ is a differentiable map by $\text{AxCDiff}$ and the fact that $\pi_t$ and $\iota$ are linear maps. Since $\iota$ and $\pi_t$ are linear maps, $[d_\zeta \pi_t] = \pi_t$ for all $\zeta \in Q^d$ and $\iota'(t) = [d_\zeta \iota](1) = \iota(1) = \bar{e}_d$ for all $t \in Q$. Thus, by chain rule (Theorem 8.3), we have

$$((d_\zeta \iota \circ w_{km}) \circ \pi_t)'(y_t) = ((d_\zeta y \circ w_{km}) \circ \pi_t)'(y_t) = ((d_\zeta y \circ w_{km}) \circ (\bar{e}_d))'_t.$$  
Therefore,  

$$\text{cl}'_{mk}(x_t) = \frac{1}{([d_\zeta y \circ w_{km}] \circ (\bar{e}_d))'_t}.$$  

Similarly,  

$$\text{cl}'_{km}(y_t) = \frac{1}{([d_\zeta x \circ w_{mk}] \circ (\bar{e}_d))'_t}.$$  

By $\text{AxSym}^-$, we have $\text{cl}'_{mk}(x_t) = \text{cl}'_{km}(y_t)$. Consequently,

$$([d_\zeta y \circ w_{km}] \circ (\bar{e}_d))'_t = ([d_\zeta x \circ w_{mk}] \circ (\bar{e}_d))'_t.$$  

By Lemma 8.5, $[d_\zeta x \circ w_{mk}]^{-1} = [d_\zeta y \circ w_{km}]$. Thus

$$([d_\zeta x \circ w_{mk}] \circ (\bar{e}_d))'_t = ([d_\zeta x \circ w_{mk}]^{-1} \circ (\bar{e}_d))'_t.$$  

Therefore, $[d_\zeta x \circ w_{mk}]$ has the sym-time property; and this is what we wanted to prove. □
We call a linear bijection of \( Q^d \) **space isometry** iff it is an isometry on the space part of \( Q^d \) fixing \( \bar{e}_d \), i.e., \( M(\bar{e}_d) = \bar{e}_d \), \( |M(\bar{x})| = |\bar{x}| \) and \( M(\bar{x})_t = 0 \) for all \( \bar{x} \in Q^d \) for which \( x_t = 0 \).

**Lemma 8.8.** Assume \( \text{AxEField} \). Any linear bijection \( M \) taking lightlike vectors to lightlike ones fixing \( \bar{e}_d \) is a space isometry.

**Proof.** To prove this, let us consider the \( M \)-images of the other standard basis vectors \( \bar{e}_i \), \( 1 \leq i \leq d - 1 \). First \( M(\bar{e}_i) \) has to be orthogonal (in the Euclidean sense) to \( \bar{e}_d \), this is so since both \( \bar{e}_d + M(\bar{e}_i) \) and \( \bar{e}_d - M(\bar{e}_i) \) has to be lightlike, see Figure 6.

The Euclidean length of \( M(\bar{e}_i) \) also has to be 1 since \( \bar{e}_d + M(\bar{e}_i) \) is lightlike, see Figure 6.

Finally, \( M(\bar{e}_i) \) is orthogonal to \( M(\bar{e}_j) \) if \( 1 \leq i < j < d \). If \( d = 2 \), there is nothing to be proved. If \( d \geq 3 \), the \( M \)-image of the lightlike vector \( \bar{e}_d + \frac{2}{5} \bar{e}_i + \frac{4}{5} \bar{e}_j \) has to be lightlike. By the linearity of \( M \), this \( M \)-image is \( \bar{e}_d + \frac{2}{5} M(\bar{e}_i) + \frac{4}{5} M(\bar{e}_j) \), which is lightlike iff \( M(\bar{e}_i) \) is orthogonal to \( M(\bar{e}_j) \).

These facts imply that \( M \) is an isometry on the space part of \( Q^d \) fixing \( \bar{e}_d \). Hence \( M \) is a space isometry.

**Lemma 8.9.** Let \( d \geq 3 \) and assume \( \text{AxEField} \). Any linear bijection \( A \) from \( Q^d \) to \( Q^d \) taking lightlike vectors to lightlike vectors is a Lorentz transformation composed by a dilation.\(^{17}\)

**Proof.** Let us first note that \( A \) takes timelike vectors to timelike ones. This is so since timelike vectors can be defined by the following property: \( \bar{t} \) is a timelike vector iff \( \bar{t} \neq \bar{0} \) and for any lightlike vector \( \bar{p} \) there is another lightlike vector \( \bar{q} \) such that \( \bar{p} + \bar{q} = \lambda \cdot \bar{t} \) for some \( 0 \neq \lambda \in Q \). This fact can be proved from \( \text{AxEField} \) since \( d \geq 3 \).

---

\(^{17}\)Lemma 8.9 can also be proved by using the Alexandrov-Zeeman theorem generalized over ordered fields, see [37] or [24].
Hence $A(\bar{e}_d)$ is timelike. Timelike vector $\bar{e}_d$ can be transformed to timelike vector $A(\bar{e}_d)$ by a Lorentz boost (hyperbolic rotation) $B$, space isometry $S$, and dilation $D$. Let $M$ be $A \circ (B \circ S \circ D)^{-1}$. $M(\bar{e}_d) = \bar{e}_d$ and $M$ takes lightlike vectors to lightlike ones (by the properties of its decomposition). By Lemma 8.8 we have that $M$ is a space isometry.

Thus $A = M \circ B \circ S \circ D$. This completes the proof since $M \circ B \circ S$ is a Lorentz transformation and $D$ is a dilation.

**Proof of Theorem 8.1.** Let $m$ and $k$ be two observers and let $\bar{x} \in \text{wl}_m(k) \cap \text{wl}_m(m)$. By Lemma 8.6 we have that $w_{mk}$ is differentiable at $\bar{x}$ and $[d_{\bar{x}} w_{mk}]$ is a linear bijection taking lightlike vectors to lightlike vectors. Hence, by Lemma 8.9 $[d_{\bar{x}} w_{mk}]$ has to be a Lorentz transformation $L$ composed by a dilation $D$, i.e., $[d_{\bar{x}} w_{mk}] = L \circ D$.

By Lemma 8.7 $\text{AxSymt}^-$ implies that $[d_{\bar{x}} w_{mk}]$ has the sym-time property. So dilation $D$ has to be the identity map because of the followings.

The sym-time property is true for Lorentz transformation $L$, i.e., $L(\bar{e}_d)_t = L^{-1}(\bar{e}_d)_t$. Therefore, if $D$ is a nontrivial dilation, $L \circ D$ does not have the sym-time property. For example, if $D$ is an enlargement in the decomposition $[d_{\bar{x}} w_{km}] = L \circ D$,

$$(L \circ D)(\bar{e}_d)_t > L(\bar{e}_d)_t = L^{-1}(\bar{e}_d)_t > (D^{-1} \circ L^{-1})(\bar{e}_d)_t.$$  

An analogous calculation works in the case when $D$ is a shrinking. Therefore, $D$ has to be the identity map. So $[d_{\bar{x}} w_{mk}]$ is a Lorentz transformation as stated.

**Proposition 8.10.** Let $d \geq 3$. GenRel implies that $g_m$ defined by (7) is a function for all $m \in \text{Ob}$, i.e., $a$ does not depend on the choice of observer $k \in \text{ev}_m(\bar{x})$.

**Proof.** Let $k$ and $h$ be observers such that $k, h \in \text{ev}_m(\bar{x})$. Then, by Lemma 8.2 $w_{mk}$ and $w_{mh}$ are functions differentiable at $\bar{x}$. Let $\bar{y}$ be $w_{mk}(\bar{x})$. Since $k, h \in \text{ev}_m(\bar{x})$, we have that $k, h \in \text{ev}_k(\bar{y})$, i.e., $\bar{y} \in \text{wl}_k(h) \cap \text{wl}_k(k)$. Therefore, by Theorem 8.1 $[d_{\bar{y}} w_{kh}]$ is a Lorentz transformation, i.e., it preserves the Minkowski metric. Hence

$$\mu([d_{\bar{x}} w_{mk}](\bar{v}), [d_{\bar{x}} w_{mk}](\bar{w})) = \mu([d_{\bar{x}} w_{mk}](\bar{v}), [d_{\bar{x}} w_{mk}](\bar{w}))$$

for all $\bar{v}, \bar{w} \in Q^d$ since $[d_{\bar{x}} w_{mk}] = [d_{\bar{x}} w_{mk}] \circ [d_{\bar{y}} w_{kh}]$.

**Proof of Theorem 7.2.** Since the ordered field reduct $\langle Q, +, \cdot, \leq \rangle$ of $\mathcal{G}$ and $M(\mathcal{G})$ is the same, AxEField is valid in $M(\mathcal{G})$.

Axiom $\text{AxCDiff}$ or $\text{AxC}^n$ for any $n \geq 1$ contains that $w_{mk}$ is a function. So axiom $\text{AxFn}_{\psi}$ is valid in $M(\mathcal{G})$. By Proposition 8.10 $g_m$ is a function. Therefore, $\text{AxFn}_{\psi}$ is also valid in $M(\mathcal{G})$.

Let $m, k$ and $h$ be observers. We have that $w_{mm} = \text{ld}_{\text{Dom} w_{mm}}$, $w_{mk} = w_{km}^{-1}$ and $w_{mk} \circ w_{kh} \subseteq w_{mh}$ by the definition of worldview transformation and the fact that they are functions (by axiom $\text{AxCDiff}$ or $\text{AxC}^n$). Therefore, $\text{AxCom}_{\psi}$ is valid in $M(\mathcal{G})$.

$\text{AxCDiff}$ is $Tr(\text{AxCDiff} \wedge \text{AxFn}_{\psi})$; axiom $\text{AxC}^n$ is $Tr(\text{AxC}^n \wedge \text{AxFn}_{\psi})$. Hence $\text{AxCDiff}_{\psi}$ ($\text{AxC}^n_{\psi}$) is valid in $M(\mathcal{G})$ iff $\text{AxCDiff}$ ($\text{AxC}^n$) is valid in $\mathcal{G}$.  

25
Axiom **AxFullg** is valid in \( M(\mathfrak{G}) \) because of the followings: by axiom **AxThExp** there is an observer \( k \) such that \( W(m, k, x) \) for all \( m \in \text{Ob} \) and \( x \in \text{Dom} \, w_{mm} \); therefore, \( g_m \) is defined on \( \text{Dom} \, w_{mm} \) for all observer \( m \).

Axiom **AxC^0g** follows from **AxC^0g_m** by Proposition 8.10. Axiom **AxC^n_g_m** is \( Tr(AxC^n_g) \) for all \( n \geq 1 \). Therefore, **AxC^g** is valid in \( M(\mathfrak{G}) \) iff **AxC^n_g_m** is valid in \( \mathfrak{G} \).

To prove that axiom **AxLorg** is valid in \( M(\mathfrak{G}) \), let \( m \) be an observer and let \( x \in \text{Dom} \, g_m \). By the definition of \( g_m \), \( g_m(x)(\bar{v}, \bar{w}) = \mu\left( [d_x w_m k]((\bar{v}), [d_x w_m k]((\bar{w})) \right) \) for some observer \( k \) for which \( x \in w_m(k) \). Since \( x \in w_m(k) \), linear map \( [d_x w_m k] \) exists by Lemma 8.2. So we can choose \( [d_x w_m k] \) to be \( L \) in **AxLorg**. Hence **AxLorg** is valid in \( M(\mathfrak{G}) \).

To prove that axiom **AxComg** is valid in \( M(\mathfrak{G}) \), let \( m \) and \( h \) be observers and let \( x \in \text{Dom} \, g_m \cap \text{Dom} \, w_{mh} \). We have to show that \( g_m(x)(\bar{v}, \bar{w}) = g_h(w_{mh}(x)) \left( [d_x w_{mh}](\bar{v}), [d_x w_{mh}](\bar{w}) \right) \) for all \( \bar{v}, \bar{w} \in Q^d \). Since \( x \in \text{Dom} \, g_m \), there is an observer \( k \) in the event \( ev_m(x) \). By the definition of \( g_m \),

\[
g_m(x)(\bar{v}, \bar{w}) = \mu\left( [d_x w_{mk}](\bar{v}), [d_x w_{mk}](\bar{w}) \right)
\]

and

\[
g_h(w_{mh}(x)) \left( [d_x w_{mh}](\bar{v}), [d_x w_{mh}](\bar{w}) \right) = \mu\left( [d_{x mh}(\bar{x})](\bar{v}), [d_{x mh}(\bar{x})](\bar{w}) \right) \) \left( [d_{x mh}(\bar{x})](\bar{v}), [d_{x mh}(\bar{x})](\bar{w}) \right) \).  

So it is enough to show that \( [d_x w_{mk}] = [d_x w_{mh}] \) and \( [d_{x mh}(\bar{x})] \), which is true by chain rule, see Theorem 8.3. 

Finally, we show that **Cont_M** is valid in \( M(\mathfrak{G}) \). By formula induction, it is easy to prove that \( M(\mathfrak{G}) \models \varphi \) iff \( \mathfrak{G} \models Tr(\varphi) \). For example, \( M(\mathfrak{G}) \models g(i, x, \bar{v}, \bar{w}, a) \) holds iff

\[
\mathfrak{G} \models \exists j \, \text{Ob}(j) \land W(i, j, x) \land \mu\left( [d_x w_{ij}](\bar{v}), [d_x w_{ij}](\bar{w}) \right) = a
\]

which holds iff \( \mathfrak{G} \models Tr\left( g(i, x, \bar{v}, \bar{w}, a) \right) \) by the definitions of \( M(\mathfrak{G}) \) and \( Tr \); and \( M(\mathfrak{G}) \models \exists i \, \varphi \) iff there is an \( a \in Q \cup I \) such that \( M(\mathfrak{G}) \models \varphi[a] \) iff there is an \( a \in Q \cup B \) such that \( \mathfrak{G} \models Tr(\varphi[a]) \) iff \( \mathfrak{G} \models \exists i \, Tr(\varphi) \).

Let \( \varphi(x, y) \) be a formula in the language of **LorMan** such that \( x \) is a free variable of \( \varphi \) of sort \( Q \) and all the other free variables of \( \varphi \) are amongst \( y \). Quantity \( a \) is in the set defined by \( \varphi \) and parameter \( \tilde{p} \) iff \( M(\mathfrak{G}) \models \varphi[a, \tilde{p}] \). By the above, this is equivalent to that \( \mathfrak{G} \models Tr(\varphi[a]) \). This means that \( a \) is in the set defined by \( Tr(\varphi) \) using \( \tilde{p} \) as parameters.

By the construction of model \( M(\mathfrak{G}) \) we have that the structure \( \langle Q, +, \cdot, \leq \rangle \) of quantities in \( \mathfrak{G} \) and \( M(\mathfrak{G}) \) is the same. Consequently, the supremum of the set defined by \( \varphi \) by parameters \( \tilde{p} \) and the supremum of the set defined by \( Tr(\varphi) \) by parameters \( p \) are the same. This means that \( AxSupTr(\varphi) \in \text{CONT}_G \) implies \( AxSup(\varphi) \in \text{CONT}_M \). Hence, \( \text{CONT}_M \) is true in \( M(\mathfrak{G}) \) since \( \text{CONT}_G \) is true in \( \mathfrak{G} \).

**Lemma 8.11.** Assume **AxEFielld**. Let \( x, y, \bar{v}, \bar{w} \in Q^d \) such that \( y - x, \bar{v} \) and \( \bar{w} \) are definable timelike vectors for which \( y_t > x_t, v_t > 0 \) and \( w_t > 0 \). Then there is a continuously differentiable definable timelike curve \( \gamma \) such that \( \gamma(0) = x, \gamma(1) = y, \) and \( \gamma'(0) = \alpha \bar{v} \) and \( \gamma'(1) = \beta \bar{w} \) for some positive \( \alpha \) and \( \beta \). Moreover, there is a positive \( \delta \) such that \( |\gamma'(t)| \leq (1 - \delta) |\gamma'_0(t)| \) for all \( t \in [0, 1] \).
Proof. We can assume without losing generality that \( \bar{x} = \bar{o} \) and \( \bar{y} = \langle 1, 0, \ldots, 0 \rangle \) because by a composition of a definable translation, a definable Lorentz transformation, and a definable scaling we can map \( \bar{x} \) to \( \bar{o} \) and \( \bar{y} \) to \( \langle 1, 0, \ldots, 0 \rangle \) without changing the required properties of \( \gamma \).

Let
\[
\gamma(t) \overset{\text{def}}{=} \left< t, \frac{\bar{v}_s}{v_t} (t^3 - 2t^2 + t) + \frac{\bar{w}_s}{w_t} (t^3 - t^2) \right>,
\]
for all \( t \in \mathbb{Q} \). Then
\[
\gamma'(t) = \left< t, \frac{\bar{v}_s}{v_t} (3t^2 - 4t + 1) + \frac{\bar{w}_s}{w_t} (3t^2 - 2t) \right>.
\]
It is straightforward to verify that \( \gamma(0) = \bar{o}, \gamma(1) = \langle 1, 0, \ldots, 0 \rangle, \gamma'_s(0) = \bar{v}_s/v_t, \gamma'_s(1) = \bar{w}_s/w_t, \gamma'_t(t) = 1 \) for all \( t \in [0, 1] \). Hence \( \gamma'(0) = \alpha \bar{v} \) and \( \gamma'(1) = \beta \bar{w} \) for \( \alpha = v_t \) and \( \beta = w_t \), which are positive quantities.

It is also clear that \( \gamma \) is continuously differentiable. Let us now show that \( \gamma' \) is a timelike vector for all \( t \in [0, 1] \).

\[
|\gamma'_s(t)| = \left| \frac{\bar{v}_s}{v_t} (3t^2 - 4t + 1) + \frac{\bar{w}_s}{w_t} (3t^2 - 2t) \right|
\leq \max \left( |\bar{v}_s/v_t|, |\bar{w}_s/w_t| \right) \left( |3t^2 - 4t + 1| + |3t^2 - 2t| \right)
\leq \max \left( |\bar{v}_s/v_t|, |\bar{w}_s/w_t| \right)
\]
since \( |3t^2 - 4t + 1| + |3t^2 - 4t + 1| < 1 \) if \( t \in [0, 1] \). Consequently, there is a \( \delta > 0 \) such that \( |\gamma_s(t)| < 1 - \delta \) because \( |\bar{v}_s| < |v_t|, |\bar{w}_s| < |w_t| \). Therefore,
\[
|\gamma'_s(t)| < (1 - \delta)|\gamma'_s(t)| \text{ for all } t \in [0, 1]
\]
since \( |\gamma'_s(t)| = 1 \) for all \( t \in [0, 1] \).

9 Concluding Remarks

We have introduced several FOL axiom systems GenRel\(^n\) for general relativity and showed that they are complete with respect to Lorentzian manifolds having the corresponding smoothness properties, see Theorem 7.2. From [3], we recalled our FOL definition of timelike geodesic formulated in the language of GenRel, see (3), and justified this definition by showing that our FOL definition coincides with the usual notion of geodesic over the field \( \mathbb{R} \) of real numbers, see Theorem 6.1. Since all the other key notions of GR, such as curvature or Riemannian tensor field, are definable from timelike geodesics, we can also define all these notions in GenRel.

A future task is building our axiomatic hierarchy of relativity theories further, i.e., finding natural axiom systems similar to GenRel which are complete with respect to certain
spacetime classes, such as black holes, cosmological spacetimes, etc. For example, see [22] for an axiom capturing Malament–Hogarth spacetimes in the language of GenRel.

Another task is taking alternative axiom systems for general relativity (possibly in a completely different language, such as the language of causality, see e.g., [16]) and logically compare these axiom systems to GenRel, e.g., interpreting one in the another or proving their definitional equivalence using the techniques of [4] and [18]. This task is a part of the so called conceptual analysis of the relativity theory and it helps to understand the roles and connections of the possible basic concepts of the theory.

A third task is taking some (preferably surprising) predictions of GR and finding a minimal set of (natural) axioms implying this prediction. This task is a kind of answering why-type questions of relativity theory, see e.g., [36]. For this kind of reverse analysis in SR, see [1, §3.4], [3] on impossibility of faster than light motion, [19], [35], [34] on the twin paradox.

Doing research in any of the three tasks above will lead us to a deeper (more structured, axiomatic) understanding of the theory of GR.

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