A remark on the phase transition for the geodesic flow of a rank one surface of nonpositive curvature

Keith Burns\textsuperscript{a} and Dong Chen\textsuperscript{b}

\textsuperscript{a}Northwestern University, Evanston, IL, USA; \textsuperscript{b}The Ohio State University, Columbus, OH, USA

ABSTRACT
For any rank 1 nonpositively curved surface $M$, it was proved by Burns-Climenhaga-Fisher-Thompson that for any $q < 1$, there exists a unique equilibrium state $\mu_q$ for $q\phi^u$, where $\phi^u$ is the geometric potential. We show that as $q \to 1^-$, the weak-$^*$ limit of $\mu_q$ is the restriction of the Liouville measure to the regular set.

ARTICLE HISTORY
Received 21 December 2022
Accepted 22 June 2023

KEYWORDS
Equilibrium states; geodesic flow; phase transition

MATHEMATICS SUBJECT CLASSIFICATIONS:
37B40; 37C40; 37D40

This note is an appendix to the recent paper [1]. As there, $M$ is a compact Riemannian surface of nonpositive curvature and $\mathcal{F} = (f^t)_{t \in \mathbb{R}}$ is the geodesic flow on the unit tangent bundle $T^1M$. A vector in $v \in T^1M$ is called regular if the geodesic to which it is tangent passes through a point where the curvature of $M$ is negative and singular if the curvature of $M$ is 0 at all points along this geodesic. We denote the sets of regular and singular vectors by $\mathcal{R}$ and $\mathcal{S}$ respectively. It is clear that $\mathcal{R}$ is open and $\mathcal{S}$ is closed. We assume that both sets are nonempty.

As explained in [1], the unstable subbundle $E^u$ is a continuous $\mathcal{F}$-invariant one dimensional subbundle of $TT^1M$ on which the derivative of the geodesic flow is noncontracting. The unstable Jacobian potential (often called the geometric potential) $\phi^u : T^1M \to \mathbb{R}$ is defined by

$$\phi^u(v) = -\lim_{t \to 0} \frac{1}{t} \log \det d f^t|_{E^u} = -\frac{d}{dt}\bigg|_{t=0} \log \det d f^t|_{E^u}.$$

Recall that the topological pressure $P$ is defined by

$$P(\phi) = \sup_{\mu} \left( h_{\mu}(\mathcal{F}) + \int_{T^1M} \phi \, d\mu \right).$$

The graph of the function $q \mapsto P(q\phi^u)$ is shown in Figure 1, which is copied from [1]. There is a phase transition at $q = 1$. It was shown in [2] that, for each $q < 1$, there is a unique equilibrium state $\mu_q$ for $q\phi^u$. Moreover $\mu_q(\mathcal{R}) = 1$ for all $q < 1$. For $q \geq 1$, any
measure supported on $S$ is an equilibrium state for $q\varphi^u$, and these are the only equilibrium states when $q > 1$. The main result of [1] is that for $\varphi^u$, i.e. when $q = 1$, there is exactly one additional ergodic equilibrium state $\mu_L$. It is the restriction to $R$ of the Liouville measure.

It is natural to ask if $\mu_q \to \mu_L$ as $q \to 1^-$, but this question was not addressed in [1]. We answer it here.

**Theorem 0.1:**

$$
\mu_q \to \mu_L
$$

in the weak*–topology as $q \to 1^-$. 

**Proof:** Our argument is similar to the proof of part 8 of Proposition 5 in [3]. If $\nu$ is a measure on $T^1 M$, we set

$$
I(\nu) = \int_{T^1 M} \varphi^u(v) \, d\nu(v).
$$

It suffices to show that if $\mu_{q_n} \to \mu$ in weak*–topology for any sequence $q_n \to 1^-$, then $\mu = \mu_L$.

It is well known that $I(\mu_L) < 0$. (The opposite inequality is mistakenly asserted in [1] because the authors forgot about the minus sign in the definition of $\varphi^u$. Nevertheless, the claimed opposite inequality in [BBFS21] doesn’t influence the result there.) By [1, Theorem 2.2], we know that any equilibrium state for $\varphi^u$ is a convex linear combination of $\mu_L$ and a measure supported on $S$. Since $\varphi^u = 0$ on the set $S$, it follows that $I(\nu) > I(\mu_L)$ for any equilibrium state $\nu$ for $\varphi^u$ other than $\mu_L$. It will therefore suffice to show that $\mu$ is an equilibrium state for $\varphi^u$ and that $I(\mu) \leq I(\mu_L)$.

That $\mu$ is an equilibrium state for $\varphi^u$ follows from the next lemma, which is a general and well known fact. The entropy map for our geodesic flow $F$ is upper semicontinuous, because $F$ is $h$–expansive (see Proposition 3.3 of [4]), and it is obvious that the other hypotheses of the lemma hold for $F$. 

![Figure 1. Graph of the pressure.](image-url)
Lemma 0.2: Let $\mathcal{F}$ be a continuous flow on a compact metric space $X$ for which the entropy map $v \mapsto h_v(\mathcal{F})$ is upper semicontinuous. Consider a sequence $\{\varphi_n : X \to \mathbb{R}\}_{n \in \mathbb{N}}$ of continuous functions that converge uniformly to a continuous function $\varphi : X \to \mathbb{R}$. For each $n$, let $\nu_n$ be an equilibrium state for $\varphi_n$, and assume that $\nu_n \to \nu$ in the weak-* topology. Then $\nu$ is an equilibrium state for $\varphi$.

Proof: For each $n$ we have
$$h_{\nu_n}(\mathcal{F}) + \int_X \varphi_n(x) \, d\nu_n(x) = P(\varphi_n)$$
as $\nu_n$ —is the equilibrium state. The hypotheses of the lemma give us
$$\liminf_{n \to \infty} \left[ h_{\nu_n}(\mathcal{F}) + \int_X \varphi_n(x) \, d\nu_n(x) \right] \leq h_\nu(\mathcal{F}) + \int_X \varphi(x) \, d\nu(x).$$
On the other hand, since $P$ depends continuously on $\varphi$ with respect to the $C^0$ topology (see e.g. [5, Proposition 10.3.6]), and $\varphi_n \to \varphi$ uniformly, we have $P(\varphi_n) \to P(\varphi)$.

We now prove that $I(\mu) \leq I(\mu_L)$. The following lemma summarizes well-known consequences of the variational principle.

Lemma 0.3: Let $\mathcal{F}$ be a continuous flow on a compact metric space $X$ and $\varphi : X \to \mathbb{R}$ a continuous function. Then

1. [3, Proposition 5] The function $P : \mathbb{R} \to \mathbb{R}, q \mapsto P(q\varphi')$ is convex.
2. [3, Proposition 5(4)] If $\nu$ is an equilibrium state for $Q\varphi$, then the graph of the function $P_\nu$ defined by $P_\nu(q) = h_\nu(\mathcal{F}) + q \int_X \varphi(x) \, d\nu(x)$ is a supporting line for the graph of $P$ at $(Q, P(Q))$. Namely, $P_\nu(Q) = P(Q)$ and $P_\nu \geq P$.
3. If $Q_1 \leq Q_2$, and $\nu_i$ is an equilibrium state for $Q_i\varphi$, $i = 1, 2$, then
$$\int_X \varphi(x) \, d\nu_1(x) \leq \int_X \varphi(x) \, d\nu_2(x).$$

Proof: (1)(2) are proved in [3, Proposition 5]. Denote by $D_L P$ and $D_R P$ the left and right derivatives of $P$. By (2) and [3, Proposition 5(5)], for $i = 1, 2$, we have
$$D_L P(Q_i) \leq \int_X \varphi(x) \, d\nu_i(x) \leq D_R P(Q_i).$$
By (1), we have
$$\int_X \varphi(x) \, d\nu_1(x) \leq D_R P(Q_1) \leq D_L P(Q_2) \leq \int_X \varphi(x) \, d\nu_2(x).$$

To finish the proof of Theorem 0.1, we take $Q_1 = q_n$ and $Q_2 = 1$ in Lemma 0.3(3) and get $I(\mu_{q_n}) \leq I(\mu_L)$ for all $n$. Since $I(\mu_{q_n}) \to I(\mu)$ as $n \to \infty$, we obtain $I(\mu) \leq I(\mu_L)$ as desired.

While we were writing this paper, we learned with sorrow of the recent passing of Todd Fisher. This paper builds on his work in [1, 2], and we would like to dedicate it to his memory.
Acknowledgments
The authors would like to thank Federico Rodriguez Hertz for bringing up this question and for suggesting improvements in the exposition. We are also grateful to the referee for many helpful comments and useful remarks.

Disclosure statement
No potential conflict of interest was reported by the author(s).

References
[1] K. Burns, J. Buzzi, T. Fisher, and N. Sawyer, Phase transitions for the geodesic flow of a rank one surface with nonpositive curvature, Dyn. Syst. 36(3) (2021), pp. 527–535. MR4304640
[2] K. Burns, V. Climenhaga, T. Fisher, and D. J. Thompson, Unique equilibrium states for geodesic flows in nonpositive curvature, Geom. Funct. Anal. 28(5) (2018), pp. 1209–1259. MR3856792
[3] K. Burns and K. Gelfert, Lyapunov spectrum for geodesic flows of rank 1 surfaces, Discrete Contin. Dyn. Syst. 34(5) (2014), pp. 1841–1872. MR3124716
[4] G. Knieper, The uniqueness of the measure of maximal entropy for geodesic flows on rank 1 manifolds, Ann. Math. 148(1) (1998), pp. 291–314. MR1652924
[5] M. Viana and K. Oliveira Foundations of Ergodic Theory, Cambridge Studies in Advanced Mathematics Vol. 151. Cambridge University Press, Cambridge, 2016. p. xvi+530. MR3558990