Combinatorics and topology  
of toric arrangements defined by root systems  

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Abstract

Given the toric (or toral) arrangement defined by a root system \( \Phi \), we describe the poset of its layers (connected components of intersections) and we count its elements. Indeed we show how to reduce to 0-dimensional layers, and in this case we provide an explicit formula involving the maximal subdiagrams of the affine Dynkin diagram of \( \Phi \). Then we compute the Euler characteristic and the Poincaré polynomial of the complement of the arrangement, which is the set of regular points of the torus.

1 Introduction

Let \( g \) be a semisimple Lie algebra of rank \( n \) over \( \mathbb{C} \), \( \mathfrak{h} \) a Cartan subalgebra, \( \Phi \subset \mathfrak{h}^* \) and \( \Phi^\vee \subset \mathfrak{h} \) respectively the root and coroot systems. The equations \( \{ \alpha(h) = 0, \alpha \in \Phi \} \) define in \( \mathfrak{h} \) a family \( \mathcal{H} \) of intersecting hyperplanes. Let \( \langle \Phi^\vee \rangle \) be the lattice spanned by the coroots: the quotient \( T = \mathfrak{h}/\langle \Phi^\vee \rangle \) is a complex torus of rank \( n \). Each root \( \alpha \) takes integer values on \( \langle \Phi^\vee \rangle \), hence it induces a map \( T \to \mathbb{C}/\mathbb{Z} \cong \mathbb{C}^* \) that we denote by \( e^\alpha \). This is a character of \( T \); let \( H_\alpha \) be its kernel:

\[
H_\alpha \doteq \{ t \in T \mid e^\alpha(t) = 1 \}.
\]

In this way \( \Phi \) defines in \( T \) a finite family of hypersurfaces

\[
\mathcal{T} \doteq \{ H_\alpha, \alpha \in \Phi^+ \}
\]

(since clearly \( H_\alpha = H_{-\alpha} \)). \( \mathcal{H} \) and \( \mathcal{T} \) are called respectively the hyperplane arrangement and the toric arrangement defined by \( \Phi \) (see for instance [8], [10], [23]). We call spaces of \( \mathcal{H} \) the intersections of elements of \( \mathcal{H} \), and layers of \( \mathcal{T} \) the connected components of the intersections of elements of \( \mathcal{T} \). We denote by \( \mathcal{L}(\Phi) \) the set of the spaces of \( \mathcal{H} \), by \( \mathcal{C}(\Phi) \) the set of the layers of \( \mathcal{T} \), and by \( \mathcal{L}_d(\Phi) \) and \( \mathcal{C}_d(\Phi) \) the sets of \( d \)–dimensional spaces and layers. Clearly
if $\Phi = \Phi_1 \times \Phi_2$ then $\mathcal{L}(\Phi) = \mathcal{L}(\Phi_1) \times \mathcal{L}(\Phi_2)$ and $\mathcal{C}(\Phi) = \mathcal{C}(\Phi_1) \times \mathcal{C}(\Phi_2)$, hence from now on we will suppose $\Phi$ to be irreducible. Let $W$ be the Weyl group of $\Phi$: since $W$ permutes the roots, its natural action on $T$ restricts to an action on $\mathcal{C}(\Phi)$.

$\mathcal{H}$ is a classical object, whereas $T$ has recently been shown ([8]) to provide a geometric way to compute the values of the Kostant partition function. This function counts in how many ways an element of the lattice $\langle \Phi \rangle$ can be written as sum of positive roots, and plays an important role in representation theory, since (by Kostant’s and Steinberg’s formulae [19], [25]) it yields efficient computation of weight multiplicities and Littlewood-Richardson coefficients, as shown in [3] using results from [1], [3], [7], [27]. The values of Kostant partition function can be computed as a sum of contributions given by the elements of $\mathcal{C}_0(\Phi)$ (see [6, Teor 3.2]).

Furthermore, let $R_{\Phi}$ be the complement in $T$ of the union of all elements of $T$. $R_{\Phi}$ is known as the set of the regular points of the torus $T$ and has been widely studied (see in particular [8], [20], [21]). The cohomology of $R_{\Phi}$ is direct sum of contributions given by the elements of $\mathcal{C}(\Phi)$ (see for instance [8]). Then by describing the action of $W$ on $\mathcal{C}(\Phi)$ we implicitly obtain a $W$–equivariant decomposition of the cohomology of $R_{\Phi}$, and by counting and classifying the elements of $\mathcal{C}(\Phi)$ we can compute the Poincaré polynomial of $R_{\Phi}$.

We say that a subset $\Theta$ of $\Phi$ is a subsystem if it satisfies the following conditions:

1. $\alpha \in \Theta \Rightarrow -\alpha \in \Theta$
2. $\alpha, \beta \in \Theta$ and $\alpha + \beta \in \Phi \Rightarrow \alpha + \beta \in \Theta$.

For each $t \in T$ let us define the following subsystem of $\Phi$:

$$\Phi(t) = \{\alpha \in \Phi | e^\alpha(t) = 1\}.$$ and denote by $W(t)$ the stabilizer of $t$.

The aim of Section 2 is to describe $\mathcal{C}_0(\Phi)$, which is the set of points $t \in T$ such that $\Phi(t)$ has rank $n$. We call its elements the points of the arrangement $T$. Let $\alpha_1, \ldots, \alpha_n$ be simple roots of $\Phi$, $\alpha_0$ the lowest root (i.e. the opposite of the highest root), and $\Phi_p$ the subsystem of $\Phi$ generated by $\{\alpha_i\}_{0 \leq i \leq n, i \neq p}$. Let $\Gamma$ be the affine Dynkin diagram of $\Phi$ and $V(\Gamma)$ the set of its vertices (a list of such diagrams can be found for instance in [13] or in [18]). $V(\Gamma)$ is in bijection with $\{\alpha_0, \alpha_1, \ldots, \alpha_n\}$, hence we can identify each vertex $p$ with an integer from 0 to $n$. The diagram $\Gamma_p$ obtained by removing from $\Gamma$ the vertex $p$ (and all adjacent edges) is the ordinary Dynkin diagram of $\Phi_p$. Let $W_p$ be the Weyl group of $\Phi_p$, i.e. the subgroup of $W$ generated
by all the reflections $s_{\alpha_0}, \ldots, s_{\alpha_n}$ except $s_{\alpha_p}$. Notice that $\Gamma_0$ is the Dynkin diagram of $\Phi$ and $W_0 = W$.

Then we prove:

**Theorem 1.** There is a bijection between the $W$—orbits of $C_0(\Phi)$ and the vertices of $\Gamma$, having the property that for every point $t$ in the orbit $O_p$ corresponding to the vertex $p$, $\Phi(t)$ is $W$—conjugate to $\Phi_p$ and $W(t)$ is $W$—conjugate to $W_p$.

As a corollary we get the formula

$$|C_0(\Phi)| = \sum_{p \in V(\Gamma)} \frac{|W|}{|W_p|}. \quad (1)$$

In Section 3 we deal with layers of arbitrary dimension. For each layer $C$ of $T$ we consider the subsystem of $\Phi$

$$\Phi_C = \{ \alpha \in \Phi | e^\alpha(t) = 1 \ \forall t \in C \}$$

and its completion $\Phi_C = \langle \Phi_C \rangle R \cap \Phi$.

Let $K_d$ be the set of subsystems $\Theta$ of $\Phi$ of rank $n - d$ that are complete (i.e. such that $\Theta = \overline{\Theta}$), and let $C_{\Theta}^\Phi$ be the set of layers $C$ such that $\Phi_C = \Theta$. This gives a partition of the layers:

$$C_d(\Phi) = \bigsqcup_{\Theta \in K_d} C_{\Theta}^\Phi.$$

Notice that the subsystem of roots vanishing on a space of $H$ is always complete; then $K_d$ is in bijection with $L_d$. The elements of $L_d$ are classified and counted in \cite{22}, \cite{23}. Thus the description of the sets $C_{\Theta}^\Phi$ given in Theorem \ref{thm:classification} yields a classification of the layers of $T$. In particular we show that $|C_{\Theta}^\Phi| = n_{\Theta}^{-1} |C_0(\Theta)|$, where $n_{\Theta}$ is a natural number depending only on the conjugacy class of $\Theta$, and then

$$|C_d(\Phi)| = \sum_{\Theta \in K_d} n_{\Theta}^{-1} |C_0(\Theta)|.$$

In Section 4, using results of \cite{8} and \cite{9}, we deduce from Theorem \ref{thm:classification} that the Euler characteristic of $R_\Phi$ is equal to $(-1)^n |W|$. Moreover, Corollary \ref{cor:euler_characteristic} yields a formula for the Poincaré polynomial of $R_\Phi$:

$$P_\Phi(q) = \sum_{d=0}^n (-1)^d q^{n-d} \sum_{\Theta \in K_d} n_{\Theta}^{-1} |W(\Theta)|.$$ 

By this formula $P_\Phi(q)$ can be explicitly computed.

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2 Points of the arrangement

2.1 Statements

For all facts about Lie algebras and root systems we refer to [15]. Let
\[ g = h \oplus \bigoplus_{\alpha \in \Phi} g_\alpha \]
be the Cartan decomposition of \( g \), and let us choose nonzero elements
\[ X_0, X_1, \ldots, X_n \]
in the one-dimensional subalgebras \( g_\alpha_0, g_\alpha_1, \ldots, g_\alpha_n \); since \([g_\alpha, g_\alpha'] = g_{\alpha + \alpha'}\) whenever \( \alpha, \alpha', \alpha + \alpha' \in \Phi \), we have that \( X_0, X_1, \ldots, X_n \) generate \( g \). Let \( a_0 = 1 \) and for \( p = 1, \ldots, n \) let \( a_p \) be the coefficient of \( \alpha_p \) in \( -\alpha_0 \). For each \( p = 0, \ldots, n \) we define an automorphism \( \sigma_p \) of \( g \) by
\[ \sigma_p(X_j) = \begin{cases} X_j & \text{if } j \neq p \\ e^{2\pi i a_p} X_j & \text{if } j = p \end{cases} \]

Let \( G \) be the semisimple and simply connected linear algebraic group having root system \( \Phi \); then \( g \) is the Lie algebra of \( G \), and \( T \) is the maximal torus of \( G \) corresponding to \( h \) (see for instance [14]). \( G \) acts on itself by conjugacy, and for each \( g \in G \) the map \( k \mapsto gkg^{-1} \) is an automorphism of \( G \). Its differential \( Ad(g) \) is an automorphism of \( g \).

**Remark 2.** For every \( t \in C_0(\Phi) \), let \( g^{Ad(t)} \) be the subalgebra of the elements fixed by \( Ad(t) \). For every \( \alpha \in \Phi \) and for every \( X_\alpha \in g_\alpha \) we have that
\[ Ad(t)(X_\alpha) = e^{\alpha(t)} X_\alpha \]
and then
\[ g^{Ad(t)} = h \oplus \bigoplus_{\alpha \in \Phi(t)} g_\alpha. \]

On the other hand \( g^{\sigma_p} \) is generated by the subalgebras \( \{g_{\alpha_i}\}_{0 \leq i \leq n, i \neq p} \). Then \( g^{Ad(t)} \) and \( g^{\sigma_p} \) are semisimple algebras having root system respectively \( \Phi(t) \) and \( \Phi_p \). Our strategy will be to prove that for each \( t \in C_0(\Phi) \), \( Ad(t) \) is conjugate to some \( \sigma_p \). This implies that \( g^{Ad(t)} \) is conjugate to \( g^{\sigma_p} \) and then \( \Phi(t) \) to \( \Phi_p \), as claimed in Theorem [1].

Then we want to give a bijection between vertices of \( \Gamma \) and \( W \)-orbits of \( C_0(\Phi) \) showing that, for every \( t \) in the orbit \( \mathcal{O}_p \), \( Ad(t) \) is conjugate to \( \sigma_p \). However, since some of the \( \sigma_p \) (as well as the corresponding \( \Phi_p \)) are themselves conjugate, this bijection is not canonical. To make it canonical we should merge the orbits corresponding to conjugate automorphisms: for this we consider the action of a larger group.
Let $\Lambda(\Phi) \subset \mathfrak{h}$ be the lattice of the coweights of $\Phi$, i.e.

$$\Lambda(\Phi) = \{ h \in \mathfrak{h} | \alpha(h) \in \mathbb{Z} \forall \alpha \in \Phi \}.$$ 

The lattice spanned by the coroots $\langle \Phi^\vee \rangle$ is a sublattice of $\Lambda(\Phi)$; set

$$Z(\Phi) = \frac{\Lambda(\Phi)}{\langle \Phi^\vee \rangle}.$$ 

This finite subgroup of $T$ coincides with $Z(G)$, the center of $G$. It is well known (see for instance [13, 13.4]) that

$$Ad(g) = id_g \iff g \in Z(\Phi).$$ 

Notice that

$$Z(\Phi) = \{ t \in T | \Phi(t) = \Phi \}$$

thus $Z(\Phi) \subseteq C_0(\Phi)$. Moreover, for each $z \in Z(\Phi)$, $t \in T$, $\alpha \in \Phi$,

$$e^\alpha(zt) = e^\alpha(z) e^\alpha(t) = e^\alpha(t)$$

and therefore $\Phi(zt) = \Phi(t)$. In particular $Z(\Phi)$ acts by multiplication on $C_0(\Phi)$. Notice that this action commutes with that of $W$: indeed, let

$$N = N_G(T)$$

be the normalizer of $T$ in $G$. We recall that $W \cong N/T$ and the action of $W$ on $T$ is induced by the conjugacy action of $N$. The elements of $Z(\Phi) = Z(G)$ commute with the elements of $G$, hence in particular with the elements of $N$. Thus we get an action of $W \times Z(\Phi)$ on $C_0(\Phi)$.

Let $Q$ be the set of the $\text{Aut}(\Gamma)$-orbits of $V(\Gamma)$. If $p, p' \in V(\Gamma)$ are two representatives of $q \in Q$, then $\Gamma_p \cong \Gamma_{p'}$, thus $W_p \cong W_{p'}$. Moreover we will see (Corollary [7(ii)]) that $\sigma_p$ is conjugate to $\sigma_{p'}$. Then we can restate Theorem [1] as follows.

**Theorem 3.** There is a canonical bijection between $Q$ and the set of $W \times Z(\Phi)$-orbits in $C_0(\Phi)$, having the property that if $p \in V(\Gamma)$ is a representative of $q \in Q$, then:

1. every point $t$ in the corresponding orbit $O_q$ induces an automorphism conjugate to $\sigma_p$;
2. the stabilizer of $t \in O_q$ is isomorphic to $W_p \times \text{Stab}_{\text{Aut}(\Gamma)} p$.

This theorem implies immediately the formula:

$$|C_0(\Phi)| = \sum_{q \in Q} |q| \frac{|W|}{|W_p|}$$ 

where $p$ is any representative of $q$. This is clearly equivalent to formula (1).
Remark 4. If we view the elements of $\Lambda(\Phi)$ as translations, we can define a group of isometries of $\mathfrak{h}$

$$\widetilde{W} \cong W \ltimes \Lambda(\Phi).$$

$\widetilde{W}$ is called the extended affine Weyl group of $\Phi$ and contains the affine Weyl group $\hat{W} = W \ltimes \langle \Phi^\vee \rangle$ (see for instance [16, 24]).

The action of $W \ltimes Z(\Phi)$ on $C_0(\Phi)$ is induced by that of $\widetilde{W}$. Indeed $\widetilde{W}$ preserves the lattice $\langle \Phi^\vee \rangle$ of $\mathfrak{h}$, and thus acts on $T = \mathfrak{h}/\langle \Phi^\vee \rangle$ and on $C_0(\Phi) \subset T$. Since the semidirect factor $\langle \Phi^\vee \rangle$ acts trivially, $\widetilde{W}$ acts as its quotient

$$\frac{\widetilde{W}}{\langle \Phi^\vee \rangle} \cong W \times Z(\Phi).$$

### 2.2 Examples: the classical root systems

In the following examples we denote by $S_n$, $D_n$, $C_n$ respectively the symmetric, dihedral and cyclic group on $n$ letters.

1. **Case $C_n$** The roots

$$2\alpha_i + \cdots + 2\alpha_{n-1} + \alpha_n$$

($i = 1, \ldots, n$) take integer values on the points $[\alpha_i^\vee/2], \ldots, [\alpha_n^\vee/2] \in \mathfrak{h}/\langle \Phi^\vee \rangle$, and thus on their sums, for a total of $2^n$ points of $C_0(\Phi)$. Indeed, let us introduce the following notation. Fixed a basis $h_1^*, \ldots, h_n^*$ of $\mathfrak{h}^*$, the simple roots of $C_n$ can be written as

$$\alpha_i = h_i^* - h_{i+1}^* \text{ for } i = 1, \ldots, n - 1 \text{, and } \alpha_n = 2h_n^*. \quad (4)$$

Then

$$\Phi = \{h_i^* - h_j^*\} \cup \{h_i^* + h_j^*\} \cup \{\pm 2h_i^*\} \quad (i, j = 1, \ldots, n, i \neq j)$$

and writing $t_i$ for $e^{h_i^*}$, we have that

$$e^\Phi = \{e^\alpha, \alpha \in \Phi\} = \{t_it_j^{-1}\} \cup \{t_it_j\} \cup \{t_i^{\pm 2}\}.$$

The system of $n$ independent equations

$$\begin{cases} t_1^2 = 1 \\ \cdots \\ t_n^2 = 1 \end{cases}$$
has $2^n$ solutions: $(\pm 1, \ldots, \pm 1)$, and it is easy to see that all other systems does not have other solutions. The Weyl group $W \simeq \mathfrak{S}_n \ltimes (\mathbb{C}_2)^n$ acts on $T = (\mathbb{C}^*)^n$ by permuting and inverting its coordinates; the second operation is trivial on $C_0(\Phi)$. Thus two elements of $C_0(\Phi)$ are in the same $W$–orbit if and only if they have the same number of negative coordinates. Then we can define the $p$–th $W$–orbit $O_p$ as the set of points with $p$ negative coordinates. (This choice is not canonical: we may choose the set of points with $p$ positive coordinates as well). Clearly if $t \in O_p$ then

$$W(t) \simeq (\mathfrak{S}_p \times \mathfrak{S}_{n-p}) \ltimes (\mathbb{C}_2)^n.$$ 

Thus $|O_p| = \binom{n}{p}$ and we get:

$$|C_0(\Phi)| = \sum_{p=0}^{n} \binom{n}{p} = 2^n.$$

Notice that if $t \in O_p$ then $-t \in O_{n-p}$, and $Ad(t) = Ad(-t)$ since $Z(\Phi) = \{\pm(1, \ldots, 1)\}$. In fact $\Gamma$ has a symmetry exchanging the vertices $p$ and $n-p$. Finally notice that $C_0(\Phi)$ is a subgroup of $T$ isomorphic to $(\mathbb{C}_2)^n$ and generated by the elements

$$\delta_i = (1, \ldots, 1, -1, 1, \ldots, 1) \ (\text{with the } -1 \text{ at the } i-th \text{ place}).$$

Then we can come back to the original coordinates observing that $\delta_i$ is the nontrivial solution of the system $t_i^2 = 1$, $t_j = 1 \forall j \neq i$, and using (6) to get:

$$\delta_i \leftrightarrow \left[ \sum_{k=i}^{n} \alpha_k^\vee / 2 \right].$$

2. Case $D_n$ We can write $\alpha_n = h_{n-1}^* + h_n^*$ and the others $\alpha_i$ as before; then

$$e^\Phi = \{t_i t_j^{-1}\} \cup \{t_i t_j\}.$$ 

Then each system of $n$ independent equations is $W$–conjugate to one of this form:

$$\begin{align*}
t_1 &= t_2 \\
\ldots \\
t_{p-1} &= t_p \\
t_{p-1} &= t_p^{-1} \\
t_{p+1} &= t_{p+2} \\
\ldots \\
t_{n-1} &= t_n \\
t_{n-1} &= t_n^{-1}
\end{align*}$$
for some $p \neq 1, n - 1$. Then we get the subset of $(C_2)^n$ composed by the following $n-$ples:
\[ \{ (\pm 1, \ldots, \pm 1) \} \setminus \{ \pm \delta_i, i = 1, \ldots, n \} \]
which are in number of $2^n - 2n$. However reasoning as before we see that each one represents two points in $\mathfrak{h}/\langle \Phi' \rangle$. Namely, the correspondence is given by:
\[ \left\{ \left[ \sum_{k=i}^{n-1} \frac{a_k^\vee}{2} \pm \frac{a_{n-1}^\vee - a_n^\vee}{4} \right] \right\} \longrightarrow \delta_i. \]
From a geometric point of view, the $t_i$s are coordinates of a maximal torus of the orthogonal group, while $T = \mathfrak{h}/\langle \Phi' \rangle$ is a maximal torus of its two-sheets universal covering. Each $W-$orbit corresponding to the four extremal vertices of $\Gamma$ is a singleton consisting of one of the four points over $\pm(1, \ldots, 1)$, all inducing the identity automorphism: indeed $Aut(\Gamma)$ acts transitively on these points. The other orbits are defined as in the case $C_n$.

3. **Case $B_n$** This case is very similar to the previous one, but now $a_n = h_n^*$, 
\[ e^\Phi = \{ t_i t_j^{-1} \} \cup \{ t_i t_j \} \cup \{ t_i^{\pm 1} \} \]
and then we get the points
\[ \{ (\pm 1, \ldots, \pm 1) \} \setminus \{ \delta_i \}_{i=1,\ldots,n}. \]
In this case the projection is
\[ \left\{ \left[ \sum_{k=i}^{n-1} \frac{a_k^\vee}{2} \pm \frac{a_{n-1}^\vee - a_n^\vee}{4} \right] \right\} \longrightarrow \delta_i. \]
then we have $2^n - n$ pairs of points in $C_0(\Phi)$.

4. **Case $A_n$** If we see $\mathfrak{h}^*$ as the subspace of $\langle h_1^*, \ldots, h_{n+1}^* \rangle$ of equation $\sum h_i^* = 0$, and $T$ as the subgroup of $(\mathbb{C}^*)^{n+1}$ of equation $\prod t_i = 1$, we can write all the simple roots as $a_i = h_i^* - h_{i+1}^*$; then $e^\Phi = \{ t_i t_j^{-1} \}$. In this case $\Phi$ has no proper subsystem of its same rank, then all the coordinates must be equal. Therefore
\[ C_0(\Phi) = Z(\Phi) = \{ (\zeta, \ldots, \zeta) | \zeta^{n+1} = 1 \} \cong \mathfrak{c}_{n+1}. \]
Then $W \cong \mathfrak{g}_{n+1}$ acts on $C_0(\Phi)$ trivially and $Z(\Phi)$ transitively, as expected since $Aut(\Gamma) \cong \mathfrak{d}_{n+1}$ acts transitively on the vertices of $\Gamma$. We can write more explicitly $C_0(\Phi) \subseteq \mathfrak{h}/\langle \Phi' \rangle$ as
\[ C_0(\Phi) = \left\{ \left[ \frac{k}{n+1} \sum_{i=1}^{n} \alpha_i^\vee \right], k = 0, \ldots, n \right\}. \]
2.3 Proofs

Motivated by Remark 2, we start to describe the automorphisms of \( \mathfrak{g} \) that are induced by the points of \( C_0(\Phi) \).

**Lemma 5.** If \( t \in C_0(\Phi) \), then \( \text{Ad}(t) \) has finite order.

**Proof.** Let \( \beta_1, \ldots, \beta_n \) linearly independent roots such that \( e^{\beta_i}(t) = 1 \): then for each root \( \alpha \in \Phi \) we have that \( m\alpha = \sum c_i\beta_i \) for some \( m \) and \( c_i \in \mathbb{Z} \), and thus
\[
e^{\alpha(t^m)} = e^{m\alpha(t)} = \prod_{i=1}^{n} (e^{\beta_i})^{c_i}(t) = 1.
\]
Then \( \text{Ad}(t^m) \) is the identity on \( \mathfrak{g} \), hence by (2) \( t^m \in Z(\Phi) \). \( Z(\Phi) \) is a finite group, thus \( t^m \) and \( t \) have finite order. \( \Box \)

The previous lemma allows us to apply the following

**Theorem 6** (Kač).

1. Each inner automorphism of \( \mathfrak{g} \) of finite order \( m \) is conjugate to an automorphism \( \sigma \) of the form
\[
\sigma(X_i) = \zeta^{s_i}X_i
\]
with \( \zeta \) fixed primitive \( m \)-th root of unity and \( (s_0, \ldots, s_n) \) nonnegative integers without common factors such that \( m = \sum s_i\alpha_i \).

2. Two such automorphisms are conjugate if and only if there is an automorphism of \( \Gamma \) sending the parameters \( (s_0, \ldots, s_n) \) of the first in the parameters \( (s'_0, \ldots, s'_n) \) of the second.

3. Let \( (i_1, \ldots, i_r) \) be all the indices for which \( s_{i_1} = \cdots = s_{i_r} = 0 \). Then \( \mathfrak{g}^\sigma \) is the direct sum of an \( (n-r) \)-dimensional center and of a semisimple Lie algebra whose Dynkin diagram is the subdiagram of \( \Gamma \) of vertices \( i_1, \ldots, i_r \).

This is a special case of a theorem proved in [17] and more extensively in [13] X.5.15 and 16]. We only need the following

**Corollary 7.**

1. Let \( \sigma \) be an inner automorphism of \( \mathfrak{g} \) of finite order \( m \) such that \( \mathfrak{g}^\sigma \) is semisimple. Then there is \( p \in V(\Gamma) \) such that \( \sigma \) is conjugate to \( \sigma_p \). In particular \( m = a_p \) and the Dynkin diagram of \( \mathfrak{g}^\sigma \) is \( \Gamma_p \).

2. Two automorphisms \( \sigma_p, \sigma_{p'} \) are conjugate if and only if \( p, p' \) are in the same \( \text{Aut}(\Gamma) \)-orbit.
Proof. If $g^\sigma$ is semisimple, then in the third part of Theorem\n$n = r$, hence all parameters of $\sigma$ but one are equal to 0, and the nonzero parameter $s_p$ must be equal to 1, otherwise there would be a common factor, contradicting the first part of the Theorem. Thus we get the first statement. Then the second statement follows from Theorem\n(ii).

Let be $t \in C_0(\Phi)$: by Remark\n$Ad(t)$ is semisimple, hence by Corollary\nAd(t)$ is conjugate to some $\sigma_p$. Then there is a canonical map\n$$\psi : C_0(\Phi) \to Q$$\nt $\psi(t) = \{p \in V(\Gamma)$ such that $\sigma_p$ is conjugate to $Ad(t)\}$.

Notice that $\psi(t)$ is a well-defined element of $Q$ by Corollary (ii).

We now prove the fundamental

Lemma 8. Two points in $C_0(\Phi)$ induce conjugate automorphisms if and only if they are in the same $W \times Z(\Phi)-$orbit.

Proof. We recall that $W \simeq N/T$ and the action of $W$ on $T$ is induced by the conjugation action of $N$; it is also well known that two points of $T$ are $G-$conjugate if and only if they are $W-$conjugate. Then $W-$conjugate points induce conjugate automorphisms. Moreover by (2)\n$$Ad(t) = Ad(s) \Leftrightarrow Ad(ts^{-1}) = id \Leftrightarrow ts^{-1} \in Z(\Phi).$$

Finally suppose that $t, t' \in C_0(\Phi)$ induce conjugate automorphisms, i.e.\n$$\exists g \in G | Ad(t') = Ad(g)Ad(t)Ad(g^{-1}) = Ad(gtg^{-1}).$$

Then $zt' = gtg^{-1}$ for some $z \in Z(\Phi)$. Thus $zt'$ and $t$ are $G-$conjugate elements of $T$, and hence they are $W-$conjugate, proving the claim.

We can now prove the first part of Theorem\nIndeed by the previous lemma there is a canonical injective map defined on the set of the orbits of $C_0(\Phi)$:\n$$\psi : C_0(\Phi) \to Q.$$\n
We must show that this map is surjective. The system\n$$\alpha_i(h) = 1 (\forall i \neq 0, p), \ \alpha_p(h) = a_p^{-1}$$

is composed of $n$ linearly independent equations, then it has a solution $h \in \mathfrak{h}$. Notice that $\alpha_0(h) \in \mathbb{Z}$. Let $t$ be the class of $h$ in $T$; then\n$$e^\alpha(t) = 1 \Leftrightarrow \alpha \in \Phi_p.$$
Then by Remark 2 $Ad(t)$ is conjugate to $\sigma_p$ and $\Phi(t)$ to $\Phi_p$.

In order to relate the action of $Z(\Phi)$ with that of $Aut(\Gamma)$, we introduce the following subset of $W$. For each $p \neq 0$ such that $a_p = 1$, set $z_p = w_p^0w_0$, where $w_0$ is the longest element of $W$ and $w_p^0$ is the longest element of the parabolic subgroup of $W$ generated by all the simple reflections $s_{\alpha_1}, \ldots, s_{\alpha_n}$ except $s_{\alpha_p}$. Then we define

$$W_Z = \{1\} \cup \{z_p\}_{p=1,\ldots,n|a_p=1}$$

$W_Z$ has the following properties (see [16 1.7 and 1.8]):

**Theorem 9** (Iwahori-Matsumoto).

1. $W_Z$ is a subgroup of $W$ isomorphic to $Z(\Phi)$.

2. For each $z_p \in W_Z$, we have that $z_p.\alpha_0 = \alpha_p$, and $z_p$ induces an automorphism of $\Gamma$ that sends the $0$–th vertex to the $p$–th one; this defines an injective morphism $W_Z \hookrightarrow Aut(\Gamma)$.

3. The $W_Z$–orbits of $V(\Gamma)$ coincide with the $Aut(\Gamma)$–orbits.

Therefore $Q$ is the set of $W_Z$–orbits of $V(\Gamma)$, and the bijection $\overline{\psi}$ between $Q$ and the set of $Z(\Phi)$–orbits of $C_0(\Phi)/W$ can be lifted to a noncanonical bijection between $V(\Gamma)$ and $C_0(\Phi)/W$. Then we just have to consider the action of $W$ on $C_0(\Phi)$ and prove the

**Lemma 10.** If $t \in C_p$, then $W(t)$ is conjugate to $W_p$.

*Proof.* Notice that the centralizer $C_N(t)$ of $t$ in $N$ is the normalizer of $T = C_T(t)$ in $C_G(t)$. Then $W(t) = C_N(t)/T$ is the Weyl group of $C_G(t)$. $C_G(t)$ is the subgroup of $G$ of points fixed by the conjugacy by $t$, then its Lie algebra is $g^{Ad(t)}$, which is conjugate to $g^{\sigma_p}$ by the first part of Theorem 3. Therefore $W(t)$ is conjugate to $W_p$.

This completes the proof of Theorem 3 and also of Theorem 1, since by Remark 2 the map $\psi$ defined in (7) can also be seen as the map

$$t \mapsto \psi(t) = \{p \in V(\Gamma) \text{ such that } \Phi_p \text{ is conjugate to } \Phi(t)\}.$$
3 Layers of the arrangement

3.1 From hyperplane arrangements to toric arrangements

Let $S$ be a $d$–dimensional space of $\mathcal{H}$. The set $\Phi_S$ of the elements of $\Phi$ vanishing on $S$ is a complete subsystem of $\Phi$ of rank $n - d$. Then the map $S \to \Phi_S$ gives a bijection between $L_d$ and $K_d$, whose inverse is

$$\Theta \to S(\Theta) = \{ h \in H_1 | \alpha(h) = 0 \ \forall \alpha \in \Theta \}.$$

In [23, 6.4 and C] (following [22] and [5]) the spaces of $\mathcal{H}$ are classified and counted, and the $W$–orbits of $L_d$ are completely described. This is done case-by-case according to the type of $\Phi$. We now show a case-free way to extend this analysis to the layers of $T$.

Given a layer $C$ of $T$ let us consider

$$\Phi_C = \{ \alpha \in \Phi | e^\alpha(t) = 1 \ \forall t \in C \}.$$ 

In contrast with the case of linear arrangements, $\Phi_C$ in general is not complete. For each $\Theta \in K_d$, define $C^\Theta_\Phi$ as the set of layers $C$ such that $\Phi_C = \Theta$. This is clearly a partition of the set of $d$–dimensional layers of $T$, i.e.

$$C_d(\Phi) = \bigsqcup_{\Theta \in K_d} C^\Theta_\Phi \tag{5}$$

Given any $C \in C^\Theta_\Phi$, we call $S(\Theta)$ the tangent space at the layer $U$. Then by [23] the problem of classifying the layers of $T$ reduces to classify the layers of $T$ having a given tangent space, i.e. the elements of $C^\Theta_\Phi$. In the next section we show that this amounts to classify the points of a smaller toric arrangement, namely that defined by $\Theta$.

3.2 Theorems

Let $\Theta$ be a complete subsystem of $\Phi$ and $W^\Theta$ its Weyl group. Let $\frak{k}$ and $K$ be respectively the semisimple Lie algebra and the semisimple and simply connected algebraic group of root system $\Theta$, $\frak{d}$ a Cartan subalgebra of $\frak{k}$, $\langle \Theta^\vee \rangle$ and $\Lambda(\Theta)$ the coroot and coweight lattices, $Z(\Theta) \doteq \frac{\Lambda(\Theta)}{\Theta^\vee}$ the center of $K$, $D$ the maximal torus of $K$ defined by $\frak{d}/\langle \Theta^\vee \rangle$, $D$ the toric arrangement defined by $\Theta$ on $D$ and $C_0(\Theta)$ the set of its points.

We also consider the adjoint group $K_a \doteq K/Z(\Theta)$ and its maximal torus $D_a \doteq D/Z(\Theta) \simeq \frak{d}/\Lambda(\Theta)$. We recall from [14] that $K$ is the universal covering of $K_a$, and if $D'$ is an algebraic torus having Lie algebra $\frak{d}$, then $D' \simeq \frak{d}/L$ for some lattice $\Lambda(\Theta) \supset L \supset \langle \Theta^\vee \rangle$; then there are natural covering projections $D \to D' \to D_a$ with kernels respectively $L/\langle \Theta^\vee \rangle$ and $\Lambda(\Theta)/L$. Notice that $\Theta$ naturally defines an arrangement on each torus $D'$, and that
for $D' = D_a$ the set of its 0-dimensional layers is $C_0(\Theta)/Z(\Theta)$. Given a point $t$ of some $D'$ we set
$$\Theta(t) = \{ \alpha \in \Theta | e^\alpha(t) = 1 \}.$$  

**Theorem 11.** There is a $W^\Theta$-equivariant surjective map

$$\varphi : C^\Phi_0 \to C_0(\Theta)/Z(\Theta)$$

such that $\ker \varphi \simeq Z(\Phi) \cap Z(\Theta)$ and $\Phi_C = \Theta(\varphi(C))$.

**Proof.** Let $S(\Theta)$ be the subspace of $\mathfrak{h}$ defined in the previous section, and $H$ the corresponding subtorus of $T$. $T/H$ is a torus with Lie algebra $\mathfrak{h}/S(\Theta) \simeq \mathfrak{a}$, then $\Theta$ defines an arrangement $D'$ on $D' \cong T/H$. The projection $\pi : T \to T/H$ induces a bijection between $C^\Phi_0$ and the set of 0-dimensional layers of $D'$, because $H \in C^\Phi_0$ and for each $C \in C^\Phi_0$, $\Phi_C = \Theta(\pi(C))$.

Moreover the restriction of the projection $d\pi : \mathfrak{h} \to \mathfrak{h}/S(\Theta)$ to $\langle \Phi^\vee \rangle$ is simply the map that restricts the coroots of $\Phi$ to $\Theta$. Set $R^\Phi(\Theta) = d\pi(\langle \Phi^\vee \rangle)$; then $\Lambda(\Theta) \supseteq R^\Phi(\Theta) \supseteq \langle \Theta^\vee \rangle$ and $D' \cong \partial/R^\Phi(\Theta)$. Denote by $p$ the projection $\Lambda(\Phi) \to \Lambda(\Theta)$ and embed $\Lambda(\Theta)$ in $\Lambda(\Phi)$ in the natural way. Then the kernel of the covering projection of $D' \to D_a$ is isomorphic to

$$\frac{\Lambda(\Theta)}{R^\Phi(\Theta)} \simeq p(\Lambda(\Theta)) \simeq Z(\Phi) \cap Z(\Theta).$$

We set

$$n_\Theta = \frac{|Z(\Theta)|}{|Z(\Phi) \cap Z(\Theta)|}.$$  

The following corollary is straightforward from Theorem 11.

**Corollary 12.**

$$|C^\Phi_0| = n_\Theta^{-1} |C_0(\Theta)|$$

and then by 12,

$$|C_d(\Phi)| = \sum_{\Theta \in \mathcal{K}_d} n_\Theta^{-1} |C_0(\Theta)|.$$  

Notice that two layers $C, C'$ of $T$ are $W-$conjugate if and only if the two conditions below are satisfied:

1. their tangent spaces are $W-$conjugate, i.e. $\exists w \in W$ such that $\overline{\Phi_C} = w.\overline{\Phi_{C'}}$;
2. $C$ and $w.C'$ are $W_{\overline{\Phi_C}}-$conjugate.

Then the action of $W$ on $\mathcal{C}(\Phi)$ is described by the following remark.
Remark 13.

1. By Theorem 11, \( \varphi \) induces a surjective map \( \overline{\varphi} \) from the set of the \( W^\Theta \)-orbits of \( C^\Phi_\Theta \) to the set of the \( W^\Theta \times \mathcal{Z}(\Theta) \)-orbits of \( C_0(\Theta) \), that are described by Theorem 3.

2. In particular if \( \Theta \) is irreducible, set \( \Gamma^\Theta \) its affine Dynkin diagram, \( Q^\Theta \) the set of the \( \text{Aut}(\Gamma) \)-orbits of its vertices, \( \Gamma_p^\Theta \) the diagram that we obtain from \( \Gamma^\Theta \) removing the vertex \( p \), and \( \Theta_p \) the associated root system. Then there is a surjective map

\[
\hat{\varphi} : C^\Phi_\Theta \rightarrow Q^\Theta
\]

such that, if \( \hat{\varphi}(C) = q \) and \( p \) is a representative of \( q \), then \( \Phi_C \simeq \Theta_p \).

3.3 Examples

Case \( F_4 \). \( Z(\Phi) = \{1\} \), thus \( n_\Theta = |Z(\Theta)| \). Therefore in this case \( n_\Theta \) does not depend on the conjugacy class, but only on the isomorphism class of \( \Theta \).

We say that a space \( S \) of \( \mathcal{H} \) (respectively a layer \( C \) of \( \mathcal{T} \)) is of a given type if the corresponding subsystem \( \Phi_S \) (respectively \( \Phi_C \)) is of that type. Then by [23, Tab. C.9] and Corollary 12 there are:

1. one space of type "A_0", tangent to one layer of the same type (the whole spaces);
2. 24 spaces of type A_1, each tangent to one layer of the same type;
3. 72 spaces of type A_1 \times A_1, each tangent to one layer of the same type;
4. 32 spaces of type A_2, each tangent to one layer of the same type;
5. 18 spaces of type B_2, each tangent to one layer of the same type and one layer of type A_1 \times A_1;
6. 12 spaces of type C_3, each tangent to one layer of the same type and 3 of type A_2 \times A_1;
7. 12 spaces of type B_3, each tangent to one layer of the same type, one of type A_3 and 3 of type A_1 \times A_1 \times A_1;
8. 96 spaces of type A_1 \times A_2, each tangent to one layer of the same type;
9. one space of type F_4 (the origin), tangent to: one layer of the same type, 12 of type A_1 \times C_3, 32 of type A_2 \times A_2, 24 of type A_3 \times A_1, and 3 of type C_4.

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Case $A_{n-1}$. It is easily seen that each subsystem $\Theta$ of $\Phi$ is complete and is a product of irreducible factors $\Theta_1, \ldots, \Theta_k$, with $\Theta_i$ of type $A_{\lambda_i-1}$ for some positive integers $\lambda_i$ such that $\lambda_1 + \cdots + \lambda_k = n$ and $n - k$ is the rank of $\Theta$. In other words, as is well known, the $W$-conjugacy classes of spaces of $H$ are in bijection with the partitions $\lambda$ of $n$, and if a space has dimension $d$ then corresponding partition has length $|\lambda| \equiv k$ equal to $d + 1$. The number of spaces of partition $\lambda$ is easily seen to be equal to $n! / b_\lambda$, where $b_\lambda = \prod i^{\lambda_i} b_i$ (see [23, 6.72]). Now let $g_\lambda$ be the greatest common divisor of $\lambda_1, \ldots, \lambda_k$. By Example 4 in Section 2.2 we have that

$$|Z(\Theta)| = \lambda_1 \ldots \lambda_k = |C_0(\Theta)|$$

and $|Z(\Phi) \cap Z(\Theta)| = g_\lambda$. Then by Corollary [12] $|C_\Phi^\Theta| = g_\lambda$ and

$$|C_d(\Phi)| = \sum_{|\lambda| = d+1} \frac{n! g_\lambda}{b_\lambda}.$$

This could also be seen directly as follows. We can view $T$ as the subgroup of $(C^*)^n$ given by the equation $t_1 \ldots t_n - 1 = 0$. Then $\Theta$ imposes the equations

$$\begin{cases}
t_1 = \cdots = t_{\lambda_1} \\
\cdots \\
t_{\lambda_1 + \cdots + \lambda_k - 1 + 1} = \cdots = t_n.
\end{cases}$$

Thus we have the relation

$$x_1^{\lambda_1} \cdots x_k^{\lambda_k} - 1 = 0.$$

If $g_\lambda = 1$ this polynomial is irreducible, because the vector $(\lambda_1, \ldots, \lambda_k)$ can be completed to a basis of the lattice $\mathbb{Z}^k$. If $g_\lambda > 1$ this polynomial has exactly $g_\lambda$ irreducible factors over $\mathbb{C}$. Then in every case it defines an affine variety having $g_\lambda$ irreducible components, which are precisely the elements of $C_\Phi^\Theta$.

4 Topology of the complement

4.1 Theorems

Let $R_\Phi$ be the complement of the toric arrangement:

$$R_\Phi \equiv T \setminus \bigcup_{\alpha \in \Phi^+} H_\alpha.$$

In this section we prove that the Euler characteristic of $R_\Phi$, denoted by $E_\Phi$, is equal to $(-1)^n|W|$. This may also be seen as a consequence of [11]
Prop. 5.3]. Furthermore, we give a formula for the Poincaré polynomial of $\mathcal{R}_\Phi$, denoted by $P_\Phi(q)$.

Let $d_1, \ldots, d_n$ be the degrees of $W$, i.e., the degrees of the generators of the ring of $W$-invariant regular functions on $\mathfrak{h}$; it is well known that $d_1 \ldots d_n = |W|$. The numbers $d_1 - 1, \ldots, d_n - 1$ are known as the exponents of $W$; we denote by $\mathcal{P}(\Phi)$ their product:

$$\mathcal{P}(\Phi) = (d_1 - 1) \ldots (d_n - 1).$$

Then we have:

**Theorem 14.**

$$P_\Phi(q) = \sum_{C \in \mathcal{C}(\Phi)} P(\Phi_C)(q + 1)^{d(C)} q^{n - d(C)}$$

where $d(C)$ is the dimension of the layer $C$.

**Proof.** Let $nbc(\Phi)$ be the number of no-broken circuit bases of $\Phi$: by [?], $nbc(\Phi)$ equals the leading coefficient of the Poincaré polynomial of the complement of $\mathcal{H}$ in $\mathfrak{h}$; moreover by [2] this coefficient is equal to $\mathcal{P}(\Phi)$ (these facts can be found also in [10, 10.1]).

Then the claim is a restatement of a known result. Indeed the cohomology of $\mathcal{R}_\Phi$ can be expressed as a direct sum of contributions given by the layers of $\mathcal{T}$ (see for example [8, Theor. 4.2] or [10, 14.1.5]). In terms of Poincaré polynomial this expression is:

$$P_\Phi(q) = \sum_{C \in \mathcal{C}(\Phi)} nbc(\Phi_C)(q + 1)^{d(C)} q^{n - d(C)}.\quad \Box$$

Now we use the theorem above to compute the Euler characteristic of $\mathcal{R}_\Phi$.

**Lemma 15.**

$$E_\Phi = (-1)^n \sum_{p=0}^{n} \frac{|W_p|}{|W_p|} \mathcal{P}(\Phi_p)$$

**Proof.** We have

$$E_\Phi = P_\Phi(-1) = (-1)^n \sum_{t \in \mathcal{C}_0(\Phi)} \mathcal{P}(\Phi(t))$$

(6)

because the contributions of all positive-dimensional layers vanish at $-1$. Obviously isomorphic subsystems have the same degrees, thus Theorem [1] yields the statement. \quad \Box
Theorem 16. 

\[ E_\Phi = (-1)^n |W| \]

Proof. By the previous lemma we must prove that 

\[ \sum_{p=0}^{n} \frac{P(\Phi_p)}{|W_p|} = 1 \]

If we write \( d_1^p, \ldots, d_n^p \) for the degrees of \( W_p \), the previous identity becomes 

\[ \sum_{p=0}^{n} \frac{(d_1^p - 1) \ldots (d_n^p - 1)}{d_1^p \ldots d_n^p} = 1. \]

This identity has been proved in [9], and later with different methods in [12].

Notice that \( W \) acts on \( R_\Phi \) and then on its cohomology. Then we can consider the equivariant Euler characteristic of \( R_\Phi \), that is, for each \( w \in W \),

\[ \tilde{E}_\Phi(w) = \sum_{i=0}^{n} (-1)^i \text{Tr}(w, H^i(R_\Phi, \mathbb{C})). \]

Let \( g_W \) be the character of the regular representation of \( W \). From Theorem 16 we get the following

**Corollary 17.**

\[ \tilde{E}_\Phi = (-1)^n g_W \]

Proof. Since \( W \) is finite and acts freely on \( R_\Phi \), it is well known that \( \tilde{E}_\Phi = k g_W \) for some \( k \in \mathbb{Z} \). Then to compute \( k \) we just have to look at \( \tilde{E}_\Phi(1_W) = E_\Phi \).

Finally we give a formula for \( P_\Phi(q) \) which, together with the mentioned results in [23], allows its explicit computation.

**Theorem 18.**

\[ P_\Phi(q) = \sum_{d=0}^{n} (q + 1)^d q^{n-d} \sum_{\Theta \in \mathcal{K}_d} n_{\Theta}^{-1} |W^{\Theta}| \]

Proof. By formula (5) we can restate Theorem 14 as 

\[ P_\Phi(q) = \sum_{d=0}^{n} (q + 1)^d q^{n-d} \sum_{\Theta \in \mathcal{K}_d} \sum_{C \in \mathcal{C}_{\Theta}} P(\Phi_C) \]
Moreover by Theorem 11 and Corollary 12 we get
\[ \sum_{C \in \mathcal{C}} \mathcal{P}(\Phi_C) = n^{-1} \sum_{t \in \mathcal{C}_0(\Theta)} \mathcal{P}(\Theta(t)). \]

Finally the claim follows by formula (9) and Theorem 16 applied to \( \Theta \):
\[ \sum_{t \in \mathcal{C}_0(\Theta)} \mathcal{P}(\Theta(t)) = (-1)^d \chi_\Theta = |W^\Theta|. \]

\[ \square \]

4.2 Examples

Case \( F_4 \). In Section 3.3 we have given a list of all possible types of complete subsystems, together with their multiplicities. Then we just have to compute the coefficient \( n^{-1} |W^\Theta| \) for each type. This is equal to:

- 1 for types 1., 2. and 3.
- 2 for types 4. and 8.
- 4 for type 5.
- 24 for types 6. and 7.
- 1152 for type 9.

Thus
\[ P_\Phi(q) = 2153q^4 + 1260q^3 + 286q^2 + 28q + 1. \]

Case \( A_{n-1} \). By Section 1.3.3, \( n^{-1} = \frac{\Theta}{\lambda_1 \cdots \lambda_k} \) and \( |W^\Theta| = \lambda_1! \cdots \lambda_k! \).

Hence by Theorem 17
\[ P_\Phi(q) = \sum_{d=0}^{n} (q+1)^d q^{n-d} \sum_{|\lambda|=d+1} n! b_\lambda^{-1} g_\lambda(\lambda_1 - 1)! \cdots (\lambda_k - 1)!. \]

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