How well can we estimate a sparse vector?

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Sparse Estimation

How well can we estimate $x$?
Applications

• Statistics
  - model selection / variable selection in high-dimensional regression

• Inverse problems

• Compressive sensing (CS)
  - matrix $\Phi$ represents a sensing system
  - typically underdetermined
  - sparsity acts as a regularizer
Core Challenges in CS

- How should we design the matrix $\Phi$ so that $M$ is as small as possible?

- How can we recover $x$ from the measurements $y$?
Answers

• Choose a *random matrix*
  - fill out the entries of $\Phi$ with i.i.d. samples from a sub-Gaussian distribution
  - project onto a “random subspace”

\[
M = O(S \log(N/S)) \ll N
\]

• Use any sparse approximation algorithm

Is this the best we can do?
Recovery from Noisy Measurements

Given $y = \Phi x + e$ or $y = \Phi(x + n)$, find $x$

- Optimization-based methods
  - basis pursuit, basis pursuit de-noising, Dantzig selector
    
    $$\hat{x} = \arg\min_{x \in \mathbb{R}^N} \|x\|_1$$
    
    s.t. $\|y - \Phi x\|_2 \leq \epsilon$

- Greedy/Iterative algorithms
  - OMP, StOMP, ROMP, CoSaMP, Thresh, SP, IHT, ...
Suppose that we observe $y = \Phi x + e$ and that $\Phi$ satisfies the RIP of order $2S$.

\[
(1 - \delta)\|x\|_2^2 \leq \|\Phi x\|_2^2 \leq (1 + \delta)\|x\|_2^2 \quad \|x\|_0 \leq 2S
\]

Typical (worst-case) guarantee

\[
\|\hat{x} - x\|_2^2 \leq C\|e\|_2^2
\]

Even if $\Lambda = \text{supp}(x)$ is provided by an oracle, the error can still be as large as $\|\hat{x} - x\|_2^2 = \|e\|_2^2/(1 - \delta)$.
Suppose now that $\Phi$ satisfies

$$A(1 - \delta)\|x\|_2^2 \leq \|\Phi x\|_2^2 \leq A(1 + \delta)\|x\|_2^2 \quad \|x\|_0 \leq 2S$$

In this case our guarantee becomes

$$\|\hat{x} - x\|_2^2 \leq \frac{C}{A} \|e\|_2^2$$

Unit-norm rows

$$\|\hat{x} - x\|_2^2 \leq C \frac{N}{M} \|e\|_2^2$$
Expected Performance

- Worst-case bounds can be pessimistic

- What about the *average* error?
  - assume $e$ is white noise with variance $\sigma^2$
    \[
    \mathbb{E} \left( \|e\|_2^2 \right) = M \sigma^2
    \]
  - for (nonadaptive) oracle
    \[
    \mathbb{E} \left( \|\hat{x} - x\|_2^2 \right) \leq \frac{S \sigma^2}{A(1 - \delta)}
    \]
  - if $e$ is Gaussian, then for $\ell_1$-minimization
    \[
    \mathbb{E} \left( \|\hat{x} - x\|_2^2 \right) \leq \frac{C'}{A} S \sigma^2 \log N
    \]
Can We Do Better?

• Better choice of $\Phi$?
• Better recovery algorithm?

Assume we have a budget for $\|\Phi\|_F^2$. If we knew the support of $x$ \textit{a priori}, then by adapting $\Phi$ to exploit this knowledge we could achieve

$$\mathbb{E} \left[ \|\hat{x} - x\|_2^2 \right] \approx \frac{S}{\|\Phi\|_F^2} S \sigma^2 \ll C' \frac{N}{\|\Phi\|_F^2} S \sigma^2 \log N$$

Is there any way to match this performance without knowing the support of $x$ in advance?

$$R_{\text{mm}}^*(\Phi) = \inf_{\hat{x}} \sup_{\|x\|_0 \leq S} \mathbb{E} \left[ \|\hat{x}(\Phi x + e) - x\|_2^2 \right]$$
No!

**Theorem:**

If \( y = \Phi x + e \) with \( e \sim \mathcal{N}(0, \sigma^2 I) \), then

\[
R_{mm}^*(\Phi) \geq C \frac{N}{\|\Phi\|_F^2} S \sigma^2 \log(N/S).
\]

If \( y = \Phi(x + n) \) with \( n \sim \mathcal{N}(0, \sigma^2 I) \), then

\[
R_{mm}^*(\Phi) \geq C \frac{N}{M} S \sigma^2 \log(N/S).
\]

\( \Phi = U \Sigma V^* \quad y' = \Sigma^{-1} U^* y = V^* x + V^* n \quad \|V^*\|_F^2 = M \)

See also: Raskutti, Wainwright, and Yu (2009)
Ye and Zhang (2010)
Suppose that $y = x + e$ with $e \sim \mathcal{N}(0, I)$ and that $S = 1$.

$$R^*_mm(I) \geq C \log(N).$$

$$\mu = \sqrt{\log N}$$

$$\|e\|_\infty = O \left( \sqrt{\log N} \right)$$
Proof Recipe

Ingredients  [Makes $\sigma^2 = 1$ servings]

- Lemma 1: Suppose $\mathcal{X}$ is a set of $S$-sparse points such that
  \[ \|x_i - x_j\|_2^2 \geq 8NR_{mm}^*(\Phi) \text{ for all } x_i, x_j \in \mathcal{X}. \]
  Then \( \frac{1}{2} \log |\mathcal{X}| - 1 \leq \frac{1}{2|\mathcal{X}|^2} \sum_{i,j} \|\Phi x_i - \Phi x_j\|_2^2. \)

- Lemma 2: There exists a set $\mathcal{X}$ of $S$-sparse points such that
  \begin{itemize}
  \item $|\mathcal{X}| = (N/S)^{S/4}$
  \item $\|x_i - x_j\|_2 \geq \frac{1}{2}$ for all $x_i, x_j \in \mathcal{X}$
  \item $\left\| \frac{1}{|\mathcal{X}|} \sum_i x_i x_i^* - \frac{1}{N} I \right\| \leq \frac{\beta}{N}$ for some $\beta > 0$
  \end{itemize}

Instructions

Combine ingredients and add a dash of linear algebra.
Proof Outline

\[ \mu = \frac{1}{|\mathcal{X}|} \sum_i x_i \quad Q = \frac{1}{|\mathcal{X}|} \sum_i x_i x_i^* \]

\[ \frac{S}{4} \log(N/S) - 2 \leq \frac{1}{|\mathcal{X}|^2} \sum_{i,j} \| \Phi x_i - \Phi x_j \|^2 \]

\[ = \text{Tr} \left( \Phi^* \Phi \left( \frac{1}{|\mathcal{X}|^2} \sum_{i,j} (x_i - x_j)(x_i - x_j)^* \right) \right) \]

\[ = \text{Tr} \left( \Phi^* \Phi \left( 2(Q - \mu \mu^*) \right) \right) \]

\[ \leq 2 \text{Tr} (\Phi^* \Phi Q) \]

\[ \leq 2 \text{Tr} (\Phi^* \Phi) \| Q \| \]

\[ \leq 2 \| \Phi \|^2_F \cdot 16 R_{mm}^*(\Phi)(1 + \beta) \]

\[ R_{mm}^*(\Phi) \geq \frac{S \log(N/S)}{128(1 + \beta) \| \Phi \|^2_F} \]
Recall: Lemma 2

Lemma 2: There exists a set $\mathcal{X}$ of $S$-sparse points such that

- $|\mathcal{X}| = (N/S)^{S/4}$
- $\|x_i - x_j\|_2 \geq \frac{1}{2}$ for all $x_i, x_j \in \mathcal{X}$
- $\|1/|\mathcal{X}| \sum_i x_i x_i^* - 1/N I\| \leq \frac{\beta}{N}$ for some $\beta > 0$

Strategy

Construct $\mathcal{X}$ by sampling (with replacement) from

$$\mathcal{U} = \left\{ x \in \{0, \sqrt{1/S}, -\sqrt{1/S}\}^N : \|x\|_0 \leq S \right\}$$

Repeat for $|\mathcal{X}| = (N/S)^{S/4}$ iterations.

With probability $> 0$, the remaining properties are satisfied.

Key: Matrix Bernstein Inequality  [Ahlswede and Winter, 2002]
Recap

Noise added to the *measurements*

\[
\mathbb{E} \left[ \| \hat{x} - x \|_2^2 \right] \leq C' \frac{N}{\| \Phi \|_F^2} S \sigma^2 \log N
\]

\[
\mathbb{E} \left[ \| \hat{x} - x \|_2^2 \right] \geq C \frac{N}{\| \Phi \|_F^2} S \sigma^2 \log(N/S)
\]

Noise added to the *signal*

\[
\mathbb{E} \left[ \| \hat{x} - x \|_2^2 \right] \leq C' \frac{N}{M} S \sigma^2 \log N
\]

\[
\mathbb{E} \left[ \| \hat{x} - x \|_2^2 \right] \geq C \frac{N}{M} S \sigma^2 \log(N/S)
\]
Noise Folding

![Graph showing SNR (dB) vs $\log_2 (N/M)$]

- 3dB per octave
- Oracle CS
- CoSaMP CS

[Note: Graph indicates performance metrics for different signal processing methods.]

[Davenport, Laska, Treichler, and Baraniuk - 2011]
Adaptivity to the Rescue?

What if we adapt the measurements to the particular signal?

If we are too greedy, our support estimate might be wrong...

Does adaptivity really help?
Sometimes...

- Information-based complexity: “Adaptivity doesn’t help!”
  - assumes signal $x$ lies in a set $K$ satisfying certain conditions
  - noise-free measurements
  - adaptivity reduces minimax error over $K$ by at most 2

- Nevertheless, adaptivity can still help [Indyk et al. - 2011]
  - reduced number of measurements in a probabilistic setting
  - still requires noise-free measurements

- What about noise?
  - distilled sensing (Haupt, Castro, Nowak, and others)
  - message seems to be that adaptivity really helps in noise
Adaptive Compressive Sensing

Suppose we have a budget of $M$ measurements of the form

$$y_i = \langle \phi_i, x \rangle + e_i$$

where $\|\phi_i\|_2 = 1$ and $e_i \sim \mathcal{N}(0, \sigma^2)$.

The vector $\phi_i$ can have an arbitrary (but deterministic) dependence on the measurements $y_1, y_2, \ldots, y_{i-1}$.

Consider the minimax MSE

$$R_{mm}^* = \inf_{\hat{x}} \sup_{\|x\|_0 \leq S} \mathbb{E} \left[ \|\hat{x}(\Phi x + e) - x\|_2^2 \right]$$
Main Result

Possibilities include

- Adaptive oracle rate: $R^*_{mm} \approx \frac{S}{M} S \sigma^2$

- Nonadaptive rate: $R^*_{mm} \approx \frac{N}{M} S \sigma^2 \log(N/S)$

- Somewhere in-between?

\[
R^*_{mm} \geq \tilde{C} \frac{N}{M} S \sigma^2
\]

In general, adaptivity does not significantly help!!

[Arias-Castro, Candès, and Davenport - 2011]
Underlying Ideas

Step 1: Consider sparse signals with nonzeros of amplitude
\[ \mu = \sqrt{N/M}. \]

Step 2: Show that if you have fewer than \( M \) measurements, then with high probability you will fail to recover a significant fraction of the support.

Step 3: Immediately translate this into a lower bound on the MSE.

[Arias-Castro, Candès, and Davenport - 2011]
Suppose that $\mathcal{S} = 1$ and that $x_{j^*} = \mu$.

Algorithm 1 [Castro et al. - 2008]
- start with random (Rademacher) measurements
- after each measurement, compute posterior distribution $p$
- re-weight subsequent measurements using $p$

Algorithm 2 [Iwen and Tewfik - 2011]
- split measurements into $\log N$ stages
- in each stage, use measurements to decide if the nonzero is in the left or right half of the “active set”
- after subdividing $\log N$ times, return support
Phase Transition in the Posterior

\[ \lambda = \frac{p_{j^*}}{\max_{j \neq j^*} p_j} \]

\[ n = 512 \quad \sigma^2 = 1 \]

[Arias-Castro, Candès, and Davenport - 2011]
Phase Transition in the MSE

$n = 512$

$\sigma^2 = 1$

$m_d = 128$

$m_e = 128$

[Phase Transition in the MSE: Arias-Castro, Candès, and Davenport - 2011]
Conclusions

• In some scenarios, CS can be sensitive to noise
  - inherent lower bound that applies to any possible sensing scheme
  - if you can average out noise, that will always help
  - sparsity is still helping a lot

• Surprisingly, adaptive algorithms cannot overcome this obstacle!

• Adaptivity might still be very useful in practice
  - practical adaptive algorithms that achieve the minimax rate for all values of $\mu$?