The generators of 3-class group of some fields of degree 6 over $\mathbb{Q}$

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Abstract:
Let $k = \mathbb{Q}(\sqrt[3]{p}, \zeta_3)$, where $p$ is a prime number such that $p \equiv 1 \pmod{9}$, and let $C_{k,3}$ be the 3-component of the class group of $k$. In [6], Frank Gerth III proves a conjecture made by Calegari and Emerton [2] which gives necessary and sufficient conditions for $C_{k,3}$ to be of rank two. The purpose of the present work is to determine generators of $C_{k,3}$, whenever it is isomorphic to $\mathbb{Z}/9\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$.

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1 Introduction

Let $\Gamma = \mathbb{Q}(\sqrt[3]{p})$ be a pure cubic field, where $p$ is a prime number such that $p \equiv 1 \pmod{9}$. We denote by $\zeta_3 = -1/2 + i\sqrt{3}/2$ the normalized primitive third roots of unity, $k = \mathbb{Q}(\sqrt[3]{p}, \zeta_3)$ the normal closure of $\Gamma$ and $C_{k,3}$ the 3-component of the class group of $k$.

Assuming 9 divides exactly the 3-class number of $\Gamma$. Then $C_{k,3} \simeq \mathbb{Z}/9\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ if and only if $u = 1$, where $u$ is an index of units that will be defined in the notations below. In this paper, we will determine the generators of $C_{k,3}$ when $C_{k,3}$ is of type $(9, 3)$ and 3 is not a cubic residue modulo $p$. We spot that Calegari and Emerton ([2, Lemma 5.11]) proved that the rank of the 3-class group of $\mathbb{Q}(\sqrt[3]{p}, \zeta_3)$, with $p \equiv 1 \pmod{9}$, is equal to two if 9 divides the 3-class number of $\mathbb{Q}(\sqrt[3]{p})$. Moreover, in his work [6, Theorem 1, p.471], Frank Gerth III proves that the converse to Calegari-Emerton’s result is also true. The present work can be viewed as a continuation of the works [2] and [6].

After reviewing some basic properties of the norm residue symbols and prime factorization in the normal closure of a pure cubic field that will be needed later, we will establish in section 3 some preliminary results of the 3-class group $C_{k,3}$. Using this, we arrive to determine the generators of 3-class groups $C_{k,3}$ of type $(9, 3)$. All the study cases are illustrated by numerical examples and summarized in tables in section 4. The usual notations on which the work is based is as follows:
• $\Gamma = \mathbb{Q}(\sqrt[3]{d})$: a pure cubic field, where $d$ is a cube-free natural number;
• $k_0 = \mathbb{Q}(\zeta_3)$: the third cyclotomic field;
• $k = \mathbb{Q}(\sqrt[3]{d}, \zeta_3)$: the normal closure of the pure cubic field $\Gamma$;
• $u = [E_k : E_0]$: the index of the sub-group $E_0$ generated by the units of intermediate fields of the extension $k/\mathbb{Q}$ in $E_k$ the group of units of $k$;
• $\lambda = 1 - \zeta_3$ prime integer of $k_0$;
• $\langle \tau \rangle = \text{Gal}(k/\Gamma)$, $\tau^2 = id, \tau(\zeta_3) = \zeta_3^2$ and $\tau(\sqrt[3]{d}) = \sqrt[3]{d}$;
• $\langle \sigma \rangle = \text{Gal}(k/\mathbb{Q}(\zeta_3))$, $\sigma^3 = id, \sigma(\zeta_3) = \zeta_3$ and $\sigma(\sqrt[3]{d}) = \zeta_3 \sqrt[3]{d}$;
• For an algebraic number field $L$:
  - $\mathcal{O}_L$: the ring of integers of $L$;
  - $E_L$: the group of units of $L$;
  - $D_L$: the discriminant of $L$;
  - $h_L$: the class number of $L$;
  - $h_{L,3}$: the 3-class number of $L$;
  - $C_{L,3}$: the 3-class group of $L$;
  - $L_3^{(1)}$: the Hilbert 3-class field of $L$;
  - $[\mathcal{I}]$: the class of a fractional ideal $\mathcal{I}$ in the class group of $L$;
• $\left( \frac{c}{p} \right)_3 = 1 \Leftrightarrow X^3 \equiv c(\text{mod } p)$ resolved on $\mathbb{Z} \Leftrightarrow c^{(p-1)/3} \equiv 1 \pmod{p}$, where $c \in \mathbb{Z}$ and $p$ is a prime number congruent to 1($\text{mod } 3$).

2 Norm residue symbol and ideal factorization theory

2.1 The norm residue symbol

Let $L/K$ an abelian extension of number fields with conductor $f$. For each finite or infinite prime ideal $\mathcal{P}$ of $K$, we note by $f_\mathcal{P}$ the largest power of $\mathcal{P}$ that divides $f$. Let $a \in K^*$, we determine an auxiliary number $a_0$ by the two conditions $a_0 \equiv a \pmod{f_\mathcal{P}}$ and $a_0 \equiv 1 \pmod{\mathcal{P}^{b+1}}$. Let $\mathcal{Q}$ an ideal co-prime with $\mathcal{P}$ such that $(a_0) = \mathcal{P}^b \mathcal{Q}$ ($b = 0$ if $\mathcal{P}$ is infinite). We note by

$$\left( \frac{a}{\mathcal{P}} \right)_L = \left( \frac{L/K}{\mathcal{Q}} \right)$$

the Artin map in $L/K$ applied to $\mathcal{Q}$. 

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Definition 2.1. Let $K$ be a number field containing the $l^{th}$-roots of units, where $l \in \mathbb{N}$, then for each $a, b \in K^\times$ and prime ideal $\mathcal{P}$ of $K$, we define the norm residue symbol by:
\[
(a, b)_{\mathcal{P}} = \left( \frac{a, K(\sqrt[l]{b})}{\mathcal{P}} \right)^{\sqrt[l]{b}} \frac{\sqrt[l]{b}}{\sqrt[l]{b}}.
\]
Therefore, if the prime ideal $\mathcal{P}$ is unramified in the field $K(\sqrt[l]{b})$, then we write
\[
(b)_{\mathcal{P}} = \left( \frac{K(\sqrt[l]{b})}{\mathcal{P}} \right)^{\sqrt[l]{b}} \frac{\sqrt[l]{b}}{\sqrt[l]{b}}.
\]

Remark 2.1. Notice that $(a, b)_{\mathcal{P}}$ and $(b)_{\mathcal{P}}$ are two $l^{th}$-roots of units.

Following [9], the principal properties of the norm residue symbol are given as follows:

Properties

1. The product formula:
   \begin{align*}
   &\bullet \quad \left( \frac{a_1 a_2, b}{\mathcal{P}} \right)_l = \left( \frac{a_1, b}{\mathcal{P}} \right)_l \left( \frac{a_2, b}{\mathcal{P}} \right)_l; \\
   &\bullet \quad \left( \frac{a, b_1 b_2}{\mathcal{P}} \right)_l = \left( \frac{a, b_1}{\mathcal{P}} \right)_l \left( \frac{a, b_2}{\mathcal{P}} \right)_l;
   \end{align*}

2. The inverse formula: \( \left( \frac{a, b}{\mathcal{P}} \right)_l = \left( \frac{b, a}{\mathcal{P}} \right)_l^{-1} \);

3. \( \left( \frac{a}{\mathcal{P}} \right)_l = 1 \iff a \) is norm residue of $K(\sqrt[l]{b})$ modulo $f_b$;

4. \( \left( \frac{\sigma a, \sigma b}{\sigma \mathcal{P}} \right)_l = \sigma \left( \frac{a, b}{\mathcal{P}} \right)_l \), for each automorphism $\sigma$ of $K$;

5. If $\mathcal{P}$ is not divisible by the conductor $f_b$ of $K(\sqrt[l]{b})$ and appears in $(a)$ with the exponent $e$, then:
   \begin{align*}
   &\bullet \quad \left( \frac{a, b}{\mathcal{P}} \right)_l = \left( \frac{b}{\mathcal{P}} \right)_l^{-e}; \\
   &\bullet \quad \mathcal{P} \text{ is infinite (} e = 0 \text{)} \Rightarrow \left( \frac{a, b}{\mathcal{P}} \right)_l = 1;
   \end{align*}
6. The classical reciprocity law: let \(a, b \in K^*\), and the conductors \(f_a\) and \(f_b\) of respectively \(K(\sqrt{a})\) and \(K(\sqrt{b})\) are co-prime, then:

\[
\left( \frac{a}{(b)} \right)_l = \left( \frac{b}{(a)} \right)_l;
\]

7. \(\prod_{\mathfrak{P}} \left( \frac{a, b}{\mathfrak{P}} \right)_l = 1\), where the product is taken on the finite and infinite prime ideals;

8. Let \(L\) is a finite extension of \(K\), \(a \in L\) and \(b \in K^*\), then:

\[
\prod_{\mathfrak{P}} \left( \frac{a, b}{\mathfrak{P}} \right)_l = \left( \frac{N_{L/K}(a, b)}{\mathfrak{P}} \right)_l.
\]

**Remark 2.2.** From property (3), we have:

\(a\) is a norm in \(K(\sqrt{b})\) \(\Rightarrow \left( \frac{a, b}{\mathfrak{P}} \right)_l = 1\),

for each prime ideal \(\mathfrak{P}\) of \(K\).

For more basic properties of the norm residue symbol in the number fields, we refer the reader to the papers [3], [8] and [9]. Notice that in section 3, we will use the norm cubic residue symbols \((l = 3)\). As the ring of integer \(\mathcal{O}_{\Gamma}\) is principal, \(h_{k_0} = 1\), we will write the norm cubic residue symbol as follows:

\[
\left( \frac{a, b}{\pi} \right)_3 = \left( \frac{a, b}{\pi} \right)_3\text{ and } \left( \frac{a}{\pi} \right)_3 = \left( \frac{a}{\pi} \right)_3
\]

where \(a, b \in k_0^*\) and \(\pi\) is a prime integer of \(\mathcal{O}_{k_0}\).

### 2.2 Prime factorization in a pure cubic field and in its normal closure

Let be \(\Gamma = \mathbb{Q}(\sqrt[3]{d})\) a pure cubic field, and \(\mathcal{O}_\Gamma\) the ring of integers of \(\Gamma\). We write the natural integer \(d\) in form \(d = ab^2\), where \(a\) and \(b\) are cube-free and co-prime positive integers. In his paper [3], Dedekind has defined two different types of pure cubic fields as follows:

**Definition 2.2.** Using the same notations as above:

1. We say that \(\Gamma = \mathbb{Q}(\sqrt[3]{d})\) is of the **first kind** if \(3\mathcal{O}_\Gamma = \mathcal{P}^3\), where \(\mathcal{P}\) is a prime ideal of \(\mathcal{O}_\Gamma\), in this case, \(a^2 - b^2 \equiv 0 \pmod{9}\).

2. We say that \(\Gamma = \mathbb{Q}(\sqrt[3]{d})\) is of the **second kind** if \(3\mathcal{O}_\Gamma = \mathcal{P}^2\mathcal{P}_1\), where \(\mathcal{P} \neq \mathcal{P}_1\) are two primes of \(\mathcal{O}_\Gamma\), in this case, \(a^2 - b^2 \equiv 0 \pmod{9}\).
Now, let $p$ be a prime number. In the following Proposition, we give the decomposition of the prime $p$ in the pure cubic field $\Gamma = \mathbb{Q}(\sqrt[3]{ab^2})$. We denote by $\mathcal{P}, \mathcal{P}_i$ prime ideals of $\Gamma$, and by $\mathcal{N}$ the absolute norm $\mathcal{N}_{\Gamma/\mathbb{Q}}$.

**Proposition 2.1.**

Let $p$ a prime number such that $p \neq 3$, then:

1. If $p$ divides $ab$ and $p \neq 3$, then $p\mathcal{O}_\Gamma = \mathcal{P}^3$, $\mathcal{N}(\mathcal{P}) = p$.
2. If $p \nmid 3ab$ and $p \equiv -1 \pmod{3}$, then $p\mathcal{O}_\Gamma = \mathcal{P}\mathcal{P}_1$, with $\mathcal{N}(\mathcal{P}) = p$ and $\mathcal{N}(\mathcal{P}_1) = p^2$.
3. If $p \nmid 3ab$ and $p \equiv 1 \pmod{3}$, then:
   a. $p\mathcal{O}_\Gamma = \mathcal{P}\mathcal{P}_1\mathcal{P}_2$ with $\mathcal{N}(\mathcal{P}) = \mathcal{N}(\mathcal{P}_1) = \mathcal{N}(\mathcal{P}_2)$, if $ab^2$ is a cubic residue modulo $p$;
   b. $p\mathcal{O}_\Gamma = \mathcal{P}$ with $\mathcal{N}(\mathcal{P}) = p^3$, if $ab^2$ is not a cubic residue modulo $p$.

**Proof.** See [3].

The ramification of the prime 3 need a particular treatment, it is the purpose of the following Proposition:

**Proposition 2.2.**

The decomposition into prime factors of 3 is:

$$3\mathcal{O}_\Gamma = \begin{cases} 
\mathcal{P}^3, & \text{if } a^2 \not\equiv b^2 \pmod{9}, \\
\mathcal{P}^2\mathcal{P}_1, & \text{if } a^2 \equiv b^2 \pmod{9}.
\end{cases}$$

**Proof.** See [3].

The ideal factorization rules for the 3rd cyclotomic field $k_0$ (see [11]) is as follows:

(i) $3\mathcal{O}_{k_0} = \lambda^2 = (1 - \zeta_3)^2$;

(ii) $p\mathcal{O}_{k_0} = \pi_1\pi_2$ in $k_0$ if $p \equiv 1 \pmod{3}$;

(iii) $q\mathcal{O}_{k_0} = q$ in $k_0$ if $q \equiv -1 \pmod{3}$.

Next, let $k$ be the normal closure of $\Gamma$. We note by $\mathcal{O}_k$ the ring of integers of $k$, $\mathfrak{P}$ and $\mathfrak{P}_s$ are prime ideals of $k$, $\mathcal{N} = \mathcal{N}_{k/\mathbb{Q}}$ the norm of $k$ on $\mathbb{Q}$. Combining the ideal factorization rules for $\Gamma$ with those of the field $k_0$. The decomposition of the prime 3 in $k$ is the purpose of the following Theorem:

**Proposition 2.3.**

The prime 3 decomposes in $k$ as follows:

$$3\mathcal{O}_k = \begin{cases} 
\mathfrak{P}^6, & \text{if } a^2 \not\equiv b^2 \pmod{9}, \\
\mathfrak{P}_1^2\mathfrak{P}_2^2\mathfrak{P}_3^2, & \text{if } a^2 \equiv b^2 \pmod{9}.
\end{cases}$$
Proof.
We have 3 ramifies in the quadratic field \( k_0 = \mathbb{Q}(\zeta_3) \).

1) Suppose that \( \Gamma \) is the first kind, then by Proposition 2.2 we have \( 3\mathcal{O}_\Gamma = \mathcal{P}^3 \). Hence, \( 3\mathcal{O}_k = \mathfrak{P}^6 \).

2) Conversely, suppose that \( \Gamma \) is of second kind, then \( 3\mathcal{O}_\Gamma = \mathcal{P}^2\mathcal{P}_1 \). It follows that \( 3\mathcal{O}_k = \mathfrak{P}_1^2\mathfrak{P}_2^2 \).

However, we have the following Proposition in which we characterize the decomposition of prime ideals of \( p \not\equiv 3 \) in \( k \).

**Proposition 2.4.**
Let \( p \) a prime number such that \( p \not\equiv 3 \), then:

1. If \( p \) divides \( D_\Gamma \), then:
   
   (a) \( p\mathcal{O}_k = \mathfrak{P}_1^3\mathfrak{P}_2^3 \), with \( \mathcal{N}(\mathfrak{P}_1) = \mathcal{N}(\mathfrak{P}_2) = p \), if and only if \(-3\) is a quadratic residue modulo \( p \).
   
   (b) \( p\mathcal{O}_k = \mathfrak{P}_3^3 \), with \( \mathcal{N}(\mathfrak{P}) = p^2 \), if and only if \(-3\) is not a quadratic residue modulo \( p \).

2. If \( p \) does not divides \( D_\Gamma \) and \( p \equiv 1 \) (mod 3), then:
   
   (a) \( p\mathcal{O}_k = \mathfrak{P}_1\mathfrak{P}_2 \), with \( \mathcal{N}(\mathfrak{P}_1) = \mathcal{N}(\mathfrak{P}_2) = p^3 \), if and only if \( D_\Gamma \) is not a cubic residue modulo \( p \).

3. If \( p \) does not divides \( D_\Gamma \) and \( p \equiv -1 \) (mod 3), then: \( p\mathcal{O}_k = \mathfrak{P}_1\mathfrak{P}_2\mathfrak{P}_3 \), with \( \mathcal{N}(\mathfrak{P}_1) = \mathcal{N}(\mathfrak{P}_2) = \mathcal{N}(\mathfrak{P}_3) = p^2 \), if and only if \(-3\) is not a quadratic residue modulo \( p \).

Proof. 1. We use Proposition 2.1 and the decomposition of prime ideals in the quadratic fields \( k_0 = \mathbb{Q}(\zeta_3) \).

2. Suppose that \( p \) does not divide \( D_\Gamma \) and \( p \equiv 1 \) (mod 3), then \(-3\) is a quadratic residue modulo \( p \), then the multiplication formula gives

\[
\left( \frac{-3}{p} \right) = \left( \frac{-1}{p} \right) \left( \frac{3}{p} \right).
\]

On the one hand, by the Euler’s Theorem we have

\[
\left( \frac{-1}{p} \right) = (-1)^{(p-1)/2},
\]
On the other hand, the quadratic reciprocity law gives

$$\left(\frac{p}{3}\right) \left(\frac{3}{p}\right) = (-1)^{(p-1)/2},$$

since $p \equiv 1 \pmod{3}$, then $p$ is a square modulo $3$, which gives $\left(\frac{p}{3}\right) = 1$, so

$$\left(\frac{3}{p}\right) = (-1)^{(p-1)/2}.$$

Then

$$\left(\frac{-3}{p}\right) = ((-1)^{(p-1)/2})^2 = (-1)^{p-1} = 1.$$  

Thus, $p$ decomposes completely in $k_0$.

(a) If $\mathcal{D}_\Gamma$ is a cubic residue modulo $p$, then by Proposition 2.1 we have $p$ split completely in $\Gamma$. Hence $p$ split completely in $k$.

(b) If $\mathcal{D}_\Gamma$ is not a cubic residue modulo $p$, we have $p$ remains prime in $\Gamma$. Hence $p\mathcal{O}_k = \mathfrak{P}_1\mathfrak{P}_2$.

3. We have $p\mathcal{O}_\Gamma = \mathcal{P}\mathcal{P}_1$, and $p$ remains inert in $k_0$, hence the result.  

\[\square\]

**Remark 2.3.** In the preceding Proposition 2.4, the situation $p\mathcal{O}_k = \mathfrak{P}_1\mathfrak{P}_2$ is never happens because if $p \equiv -1 \pmod{3}$, we have always $\left(\frac{-3}{p}\right) = -1$.

## 3 The generators of $C_{k,3}$

First, we let $C_{k,3}^{(\sigma)} = \{A \in C_{k,3} \mid A^\sigma = A\}$ be the group of ambiguous ideal classes of $k/k_0$, where $\sigma$ is a generator of Gal $(k/k_0)$, and put $q^* = 0$ or $1$ according to $\zeta_3$ is not norm or norm of an element of $k\setminus\{0\}$. Let $t$ be the number of primes ramifies in $k/k_0$. Then according to [4], we have

$$|C_{k,3}^{(\sigma)}| = 3^{t-2+q^*}.$$  

If we denote by $C_{k_0,3}$ the Sylow 3-subgroup of the ideal class group of $k_0$, $C_{k_0,3} = \{1\}$. Let be $C_{k,3}^{(1-\sigma)} = \{A^{(1-\sigma)} \mid A \in C_{k,3}\}$. By the exact sequence :

$$1 \longrightarrow C_{k,3}^{(\sigma)} \longrightarrow C_{k,3} \xrightarrow{1-\sigma} C_{k,3} \longrightarrow C_{k,3}/C_{k,3}^{(1-\sigma)} \longrightarrow 1$$

we deduce that

$$|C_{k,3}^{(\sigma)}| = |C_{k,3}/C_{k,3}^{(1-\sigma)}|.$$
Then

Lemma 3.1. The fact that \( C^{(\sigma)}_{k,3} \) and \( C_{k,3}/C^{1-\sigma}_{k,3} \) are elementary abelian 3-groups imply that:

\[
\text{rank } C^{(\sigma)}_{k,3} = \text{rank } (C_{k,3}/C^{1-\sigma}_{k,3}).
\]

Define the 3-group \( C^{(1-\sigma)^i}_{k,3} \) for each \( i \in \mathbb{N} \) by

\[
C^{(1-\sigma)^i}_{k,3} = \{ \mathcal{A}^{(i-\sigma)^i} | \mathcal{A} \in C_{k,3} \},
\]

and let \( s \) be the positive integer such that \( C^{(\sigma)}_{k,3} \subseteq C^{(1-\sigma)^{s-1}}_{k,3} \) and \( C^{(1-\sigma)^s}_{k,3} \not\subseteq C^{(1-\sigma)^{s-1}}_{k,3} \).

The following Proposition gives the structure of the 3-class group \( C_{k,3} \) when 27 divides exactly the class number of \( k \):

**Proposition 3.1.** Let be \( \Gamma \) a pure cubic field, \( k \) its normal closure and \( u \) the index of units defined as above, then:

1) \( C_{k,3} \cong \mathbb{Z}/9\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \iff [C_{\Gamma,3} \cong \mathbb{Z}/9\mathbb{Z} \text{ and } u = 1] \);

2) \( C_{k,3} \cong \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \iff [C_{\Gamma,3} \cong \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \text{ and } u = 1] \).

**Proof.** 1) Assume that \( C_{k,3} \cong \mathbb{Z}/9\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \), then \( h_{k,3} = 27 \). According to Theorem 14.1 of [1], we have \( 27 = \frac{u}{3} \cdot h_{\Gamma,3}^2 \), then \( u = 1 \) because otherwise 27 will be a square in \( \mathbb{N} \), which is a contradiction. Then \( h_{\Gamma,3}^2 = 3^4 \) and \( h_{\Gamma,3} = 9 \).

On the other hand, by Lemma 2.1 and Lemma 2.2 of [5] we have \( C_{k,3} \cong C_{\Gamma,3} \times C_{k,3}^- \), then \( |C_{k,3}^-| = 3 \). Since \( C_{k,3} \) is of type \((9,3)\), we deduce that \( C_{k,3}^- \) is a cyclic 3-group of order 3 and \( C_{k,3}^+ \) is a cyclic 3-group of order 9. Therefore:

\[
u = 1 \text{ and } C_{\Gamma,3} \cong \mathbb{Z}/9\mathbb{Z}.
\]

Reciprocally, assume that \( u = 1 \) and \( C_{\Gamma,3} \cong \mathbb{Z}/9\mathbb{Z} \). By Theorem 14.1 of [1], we deduce that \( |C_{k,3}| = \frac{1}{3} \cdot |C_{\Gamma,3}|^2 \), then \( |C_{k,3}| = 27 \) and \( |C_{k,3}^-| = 3 \). Thus:

\[
C_{k,3} \cong C_{\Gamma,3} \times C_{k,3}^- \cong \mathbb{Z}/9\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}.
\]

2) We have the same proof as above.

\[\square\]

**Lemma 3.1.** Let \( k = \mathbb{Q}(\sqrt[3]{p}, \zeta_3) \), where \( p \) is a prime number such that \( p \equiv 1 \pmod{3} \). Let \( C^{(\sigma)}_{k,3} \) be the ambiguous ideal class group of \( k/\mathbb{Q}(\zeta_3) \), where \( \sigma \) is a generator of \( \text{Gal}(k/\mathbb{Q}(\zeta_3)) \). Then \( |C^{(\sigma)}_{k,3}| = 3 \).

**Proof.** Since \( p \equiv 1 \pmod{3} \), then according to section 2.2, we have \( p = \pi_1 \pi_2 \), where \( \pi_1 \) and \( \pi_2 \) are two primes of \( k_0 \) such that \( \pi_2 = \pi_1^2 \) and \( \pi_1 \equiv \pi_2 \equiv 1 \pmod{3\mathcal{O}_{k_0}} \). We study all cases depending on the congruence class of \( p \) modulo 9, then:
• If \( p \equiv 4 \) or 7 (mod 9), then according to section 2.2, the prime 3 is ramified in the field \( L \), so the prime ideal \((1 - \zeta_3)\) is ramified in \( k/k_0 \). Also \( \pi_1 \) and \( \pi_2 \) are totally ramified in \( k \). So \( t = 3 \). In addition, the fact that \( p \equiv 4 \) or 7 (mod 9) imply that \( \pi_i \neq 1 \) (mod \( 1 - \zeta_3^3 \)) for \( i = \{1, 2\} \), then according to section 5 of [4] we obtain

\[
\left( \frac{\zeta_3, p}{\pi} \right)_3 \neq 1
\]

where the symbol \((\doteqdot)\) is the cubic Hilbert symbol. We deduce that \( \zeta_3 \) is not a norm in the extension \( k/k_0 \), so \( q^* = 0 \). Hence rank \( C_{k,3}^{(\sigma)} \) = 1 and then \( |C_{k,3}^{(\sigma)}| = 3 \).

• If \( p \equiv 1 \) (mod 9), the prime ideals which ramified in \( k/k_0 \) are \( \pi_1 \) and \( \pi_2 \), so \( t = 2 \). Moreover, \( \pi_1 \equiv \pi_2 \equiv 1 \) (mod \( 1 - \zeta_3^3 \)), then according to [4], the cubic Hilbert symbol:

\[
\left( \frac{\zeta_3, p}{\pi_1} \right)_3 = \left( \frac{\zeta_3, p}{\pi_2} \right)_3 = 1,
\]

We conclude that \( \zeta_3 \) is a norm in the extension \( k/k_0 \), then \( q^* = 1 \), so rank \( C_{k,3}^{(\sigma)} \) = 1 and \( |C_{k,3}^{(\sigma)}| = 3 \).

\[\square\]

The basic result for determining the generators of the 3-class group of \( k = \mathbb{Q}(\sqrt[3]{p}, \zeta_3) \) when the 3-class number of \( k \) is divisible by 27 exactly, where \( p \) is a prime number such that \( p \equiv 1 \) (mod 9), is summarized in the following Theorem:

**Theorem 3.1.** Let \( \Gamma = \mathbb{Q}(\sqrt[3]{p}) \), where \( p \) is a prime number such that \( p \equiv 1 \) (mod 9), \( k = \mathbb{Q}(\sqrt[3]{p}, \zeta_3) \) its normal closure and \( C_{k,3} \) the 3-class group of \( k \). Assuming 9 divides the 3-class number of \( \Gamma \) exactly, then:

The 3-class group \( C_{k,3} \) is isomorphic to \( \mathbb{Z}/9\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \) if and only if \( u = 1 \).

**Proof.** \( \Rightarrow \) By Proposition 3.1, it is clear that if \( C_{k,3} \) is isomorphic to \( \mathbb{Z}/9\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \) then \( u = 1 \).

\( \Leftarrow \) Assume that \( u = 1 \), then according to Theorem 14.1 of [1], \( h_{k,3} = 27 \). Since 9 divides the 3-class number of \( \Gamma \), then by Lemma 5.11 of [2] we have rank \( C_{k,3} = 2 \). Hence \( C_{k,3} \cong \mathbb{Z}/9\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \).

\[\square\]

**Proposition 3.2.** Let \( \Gamma = \mathbb{Q}(\sqrt[3]{p}) \), where \( p \) is a prime number such that \( p \equiv 1 \) (mod 9), \( k = \mathbb{Q}(\sqrt[3]{p}, \zeta_3) \) its normal closure, and \( C_{k,3} \) be the 3-class group of \( k \). Assume that 9 divides the 3-class number of \( \Gamma \) exactly and \( u = 1 \). Put \( \langle \mathcal{A} \rangle = C_{k,3}^+ \), where \( \mathcal{A} \in C_{k,3} \) such that \( \mathcal{A}^0 = 1 \) and \( \mathcal{A}^3 \neq 1 \). Let \( C_{k,3}^{(\sigma)} \) be the 3-group of ambiguous ideal classes of \( k/k_0 \) and \( C_{k,3}^{1-\sigma} = \{ \mathcal{A}^{1-\sigma} | \mathcal{A} \in C_{k,3} \} \) be the principal genus of \( C_{k,3} \). Then:
1. \(C^{(\sigma)}\) is a subgroup of \(C_{k,3}^+\), \(A \notin C^{(\sigma)}_{k,3}\) and \(C^{(\sigma)}_{k,3} = \langle A^3 \rangle = \langle B^{1-\sigma} \rangle\), where \(B \in C_{k,3}\) such that \(C_{k,3}^- = \langle B \rangle\).

2. \(C_{k,3}^- = \langle (A^2)^{\sigma-1} \rangle\), and we have \(C_{k,3}^{1-\sigma} = C_{k,3}^- \times C^{(\sigma)}_{k,3}\) is a 3-group of type \((3,3)\), where \(C_{k,3}^+\) and \(C_{k,3}^-\) are defined in Lemma 2.1 of [5].

**Proof.** 1. Since 9 divides the 3-class number of \(\Gamma\) exactly and \(u = 1\), then according to Theorem 3.1, \(C_{k,3}\) is of type \((9,3)\), this implies by [6] that the integer \(s\) defined above is equal 3, and according to Case 4 of [6], we conclude that \(|C^{(\sigma)}_{k,3}| = 3\) and \(|C^{(\sigma)}_{k,3}^-| = 1\), this implies that \(C^{(\sigma)}_{k,3}\) is a subgroup of \(C_{k,3}^+\). Therefore, \(\langle A^3 \rangle\) is the unique subgroup of order 3 of \(C_{k,3}^+\) and \(C^{(\sigma)}_{k,3}\) is cyclic of order 3, then \(C^{(\sigma)}_{k,3} = \langle A^3 \rangle\).

Moreover, if \(C_{k,3}^- = \langle B \rangle\) where \(B \in C_{k,3}\), then \(B \notin C^{(\sigma)}_{k,3}\), so \(B^\sigma \neq B\). Furthermore, \(B^\sigma \neq B^2\) because otherwise we will have \(B^{2\sigma} = (B^2)^\sigma = (B^\sigma)^2 = B^4\), as \(B \in C_{k,3}^-\), then \(B^3 = 1\). Therefore, \(B^{\sigma^2} = B\), so \(B^{\sigma^3} = B^\sigma\), since \(\sigma^3 = 1\), then \(B^\sigma = B\). This is impossible because \(B^\sigma \neq B\). As \(B^2 = 1\) and \(B^{1+\sigma+\sigma^2} = 1\), then \(B^{\sigma^2} = B^{2+2\sigma}\). This equality makes it possible to show that \(B^{1-\sigma}\) is an ambiguous class. We conclude that \(C_{k,3} = \langle B^{1-\sigma} \rangle\).

2. We reason as in the assertion 1. Since \(A^2 \notin C_{k,3}^-\), we deduce that \(C_{k,3}^- = \langle (A^2)^{1-\sigma} \rangle\). then \(C_{k,3}^-\) and \(C^{(\sigma)}_{k,3}\) are contained in \(C_{k,3}^{1-\sigma}\) which is of order 9, because \(|C_{k,3}| = 27\) and \(|C^{(\sigma)}_{k,3}| = 3\). Consequently,

\[C_{k,3}^{1-\sigma} = C_{k,3}^- \times C^{(\sigma)}_{k,3} = \langle A^3, B \rangle.\]

\[\Box\]

Our principal result can be stated as follows:

**Theorem 3.2.** Let \(k = \mathbb{Q}(\sqrt[3]{p}, \zeta_3)\), where \(p\) is a prime number such that \(p \equiv 1 \mod 9\). The prime 3 decomposes in \(k\) as \(3\mathcal{O}_k = \mathcal{P}^2 \mathcal{Q}^2 \mathcal{R}^2\), where \(\mathcal{P}\), \(\mathcal{Q}\) and \(\mathcal{R}\) are prime ideals of \(k\). Put \(h = \frac{h_k}{27}\), where \(h_k\) is the class number of \(k\). Assume that 9 divides exactly the 3-class number of \(\mathbb{Q}(\sqrt[3]{p})\) and \(u = 1\). If 3 is not a cubic residue modulo \(p\), then:

1. The class \([\mathcal{R}^h]\) generate \(C_{k,3}^+\);

2. The 3-class group \(C_{k,3}\) is generated by classes \([\mathcal{R}^h]\) and \([\mathcal{R}^h][\mathcal{P}^h]^2\), and we have:

\[C_{k,3} = \langle [\mathcal{R}^h] \rangle \times \langle [\mathcal{R}^h][\mathcal{P}^h]^2 \rangle = \langle [\mathcal{R}^h], [\mathcal{P}^h]^2 \rangle.\]

In Appendix of this paper, we illustrated this results by the numerical examples with the aid of Pari programming [12] and summarized in some tables in section 4.
Proof.
We start our proof by showing that $[R^h]$ is of order 9:
Since the field $\Gamma = \mathbb{Q}(\sqrt[3]{p})$ with $p \equiv 1 \pmod{9}$ is of second kind, then by Proposition 2.2 we have $3\mathcal{O}_\Gamma = \mathcal{H}^2\mathcal{S}$, where $\mathcal{H}$ and $\mathcal{S}$ are prime of $\Gamma$, since $\mathcal{H}\mathcal{O}_k = \mathcal{P}\mathcal{Q}$ and $\mathcal{S}\mathcal{O}_k = \mathcal{R}^2$, then $3\mathcal{O}_k = \mathcal{P}^2\mathcal{Q}^3\mathcal{R}^2$, where $\mathcal{P}$, $\mathcal{Q}$ and $\mathcal{R}$ are prime ideals of $k$. Moreover, the prime ideal $\mathcal{R}$ is invariant by $\tau$, then $[\mathcal{R}] \in \{\chi \in \mathbb{C}_{k,3} | \chi^7 = \chi\}$.
If 9 divides the 3-class number of $\mathbb{Q}(\sqrt[3]{p})$ exactly and $u = 1$, then by Theorem 3.1 we have $C_{k,3}$ is of type $(9,3)$. According to Proposition 3.1, we have $C_{k,3}^+\mathcal{O}_k$ is cyclic of order 9, thus $[R^h]^9 = 1$. Hence the class $[R^h]$ is of order 9 if and only if $R^h$ and $R^{3h}$ are not principal.

We argue by the absurd: assume that $R^h$ is principal, we have

$$[R^h] = 1 \Rightarrow \exists \alpha \in k \mid R^h = \alpha \mathcal{O}_k,$$

$$\Rightarrow \mathcal{N}_{k|k_0}(R^h) = \mathcal{N}_{k|k_0}(\alpha \mathcal{O}_k),$$

$$\Rightarrow \lambda^h \mathcal{O}_{k_0} = \mathcal{N}_{k|k_0}(\alpha) \mathcal{O}_{k_0}, \text{ where } \lambda = 1 - \zeta_3,$$

$$\Rightarrow \exists \epsilon \in \mathcal{E}_{k_0} \mid \lambda^h = \epsilon \cdot \mathcal{N}_{k|k_0}(\alpha),$$

$$\Rightarrow \exists \beta \in \mathcal{O}_k \mid \lambda^h = \mathcal{N}_{k|k_0}(\beta), \text{ because } \mathcal{E}_{k_0} \subseteq \mathcal{N}_{k|k_0}(k^*),$$

that is to say $\lambda^h$ is a norm in $k = k_0(\sqrt[3]{p}) = k_0(\sqrt[3]{\pi_1\pi_2})$, where $\pi_1$ and $\pi_2$ are two primes of $k_0$ such that $p = \pi_1\pi_2$. Hence, by property (5) we have:

$$\left(\frac{\lambda^h, \pi_1 \pi_2}{\mathcal{P}}\right)_3 = 1,$$

for all ideal $\mathcal{P}$ of $k_0$.

In particular, we calculate this symbol for $\mathcal{P} = \pi_1 \mathcal{O}_{k_0}$ or $\mathcal{P} = \pi_2 \mathcal{O}_{k_0}$.

For $\mathcal{P} = \pi_1 \mathcal{O}_{k_0}$, using the property (1) of the norm residue symbol, we have:

$$\left(\frac{\lambda^h, \pi_1 \pi_2}{\pi_1}\right)_3 = \left(\frac{\lambda^h, \pi_1 \pi_2}{\pi_1}\right)_3 = \left(\frac{\lambda^h, \pi_1}{\pi_1}\right)_3 \cdot \left(\frac{\lambda^h, \pi_2}{\pi_1}\right)_3,$$

the properties (2) and (5) imply that:

$$\left(\frac{\lambda^h, \pi_2}{\pi_1}\right)_3 = \left(\frac{\lambda, \pi_2}{\pi_1}\right)_1^h = \left(\frac{\lambda}{\pi_1}\right)_3^{0 \times h} = 1,$$

and from the properties (1) and (6) we have

$$\left(\frac{\lambda^h, \pi_1}{\pi_1}\right)_3 = \left(\frac{\lambda, \pi_1}{\pi_1}\right)_3^h = \left(\frac{\lambda}{\pi_1}\right)_3^h,$$

consequently

$$\left(\frac{\lambda^h, \pi_1 \pi_2}{\pi_1}\right)_3 = \left(\frac{\lambda}{\pi_1}\right)_3^h.$$
Since the two primes $\pi_1$ and $\pi_2$ play symmetric roles, then we obtain a similar relation when $P = \pi_2$:

$$\left(\frac{\lambda^h, \pi_1\pi_2}{\pi_2}\right)_3 = \left(\frac{\lambda}{\pi_2}\right)_3^h.$$ 

The equation (*) imply that

$$\left(\frac{\lambda}{\pi_1}\right)_3 = \left(\frac{\lambda}{\pi_2}\right)_3^h = 1.$$ 

The fact that $3$ is not a cubic residue modulo $p$ imply that

$$\left(\frac{\lambda}{\pi_1}\right)_3 \neq 1$$

then

$$\left(\frac{\lambda}{\pi_1}\right)_3 \neq 1 \text{ or } \left(\frac{\lambda}{\pi_2}\right)_3 \neq 1.$$ 

Since $3$ does not divide $h$, then

$$\left(\frac{\lambda}{\pi_1}\right)_3 \neq 1 \text{ or } \left(\frac{\lambda}{\pi_2}\right)_3 \neq 1.$$ 

which is a contradiction. Consequently, the ideal $R^h$ is not principal.

Since the class $[R^h]$ is invariant by $\tau$, we deduce that the ideal $R^{3h}$ is principal if and only if $([R^h]) = C_{k,3}^{(\sigma)}$. Since $9$ divides exactly the $3$-class number of $\mathbb{Q}(\sqrt[p]{\beta})$ and $u = 1$, then by we get $|C_{k,3}| = 27$, so the positive integer $s$ defined above is equal $3$, then $C_{k,3}^{(1-\sigma)^3} = 1$, this implies that $C_{k,3}^{(\sigma)} = C_{k,3}^{(1-\sigma)^2}$. Suppose that $[R^h] \in C_{k,3}^{(\sigma)}$, then $[R^h] = [L^{(1-\sigma)^2}]$ with $L$ is prime ideal of $k$, then there exist $\alpha \in k^*$ such that $R^h = (\alpha) \cdot L^{(1-\sigma)^2}$, so $N_{k|k_0}(R^h) = N_{k|k_0}(\alpha) L^{(1-\sigma)^2}$, since $N_{k|k_0}(L^{(1-\sigma)^2}) = L^{(1-\sigma)(1-\sigma^3)} = 1$, then $\lambda^h \mathcal{O}_k = N_{k|k_0}(\alpha) \mathcal{O}_k$, where $\lambda = 1 - \zeta_3$, so there exist $\varepsilon \in E_k$ such that $\lambda^h = \varepsilon \cdot N_{k|k_0}(\alpha)$, as $\lambda^h$ and $N_{k|k_0}(\alpha)$ are in $k_0$ then $\varepsilon \in E_{k_0}$, since $E_{k_0} \subseteq N_{k|k_0}(k^*)$ then $\lambda^h = N_{k|k_0}(\alpha_1)$ where $\alpha_1 \in \mathcal{O}_k$, that means $\lambda^h$ is a norm in $k = k_0(\sqrt[p]{\beta})$ which is impossible. Finally, $[R^h]$ is of order $9$. This completes the proof of the first statement.

The second step in the proof is showing that the class $[R^h][P^h]^2$ is of order $3$. We know that $(R^h)^\tau = R^h$ and $(P^h)^\tau = Q^h$, then:

$$\left(\frac{R^h \cdot (P^h)^2}{3}\right)^{1+\tau} = \left(\frac{R^h}{3}\right)^{1+\tau} \cdot \left(\frac{(P^h)^2}{3}\right)^{1+\tau}$$

$$= \frac{(P^h)^2}{3} \cdot \left(\frac{R^h}{3}\right)^2 \cdot \left(\frac{Q^h}{3}\right)^2$$

$$= 3^h \mathcal{O}_k,$$
which imply that $[R^h \cdot (P^h)^2]^{1+\tau} = 1$. Hence $[R^h \cdot (P^h)^2] \in C_{k,3}^-$. On the other hand, $R^h \cdot (P^h)^2$ is not principal, because otherwise we have $[R^h] = [P^h]^7$, the fact that $[(R^h)^2 \cdot (P^h)^2 \cdot (Q^h)^2] = 1$ imply that $[(Q^h)^2] = 1$, which is a contradiction because the class $[Q^h]$ is of order 9 (reasoning as $R^h$). Hence $[R^h][P^h]^2$ is of order 3 and generate the group $C_{k,3}^-$. Since $[R^h]$ is a generator of $C_{k,3}^+$, we deduce that

$$C_{k,3} = \langle [R^h], [R^h][P^h]^2 \rangle.$$

\[\square\]

**Corollary 3.2.1.** Using the same notation as above, we have the following properties:

1. $P^\sigma = Q$, $Q^\sigma = R$;
2. $R^\tau = R$ and $\langle [R] \rangle = \{\chi \in C_{k,3} | \chi^\tau = \chi \}$;
3. $P^{\tau \sigma} = P$ and $\langle [P] \rangle = \{\chi \in C_{k,3} | \chi^{\tau \sigma} = \chi \}$;
4. $Q^{\tau \sigma^2} = Q$ and $\langle [Q] \rangle = \{\chi \in C_{k,3} | \chi^{\tau \sigma^2} = \chi \}$;
5. The 3-class group can be generated also by:

$$C_{k,3} = \langle [P^h], [P^h][Q^h]^2 \rangle = \langle [Q^h], [Q^h][R^h]^2 \rangle.$$

6. The 3-group $C_{k,3}^{(\sigma)}$ of ambiguous ideal classes is given by:

$$C_{k,3}^{(\sigma)} = \langle [R^3h] \rangle = \langle [P^3h] \rangle = \langle [Q^3h] \rangle.$$

7. The principal genus $C_{k,3}^{1-\sigma} = \{A^{1-\sigma} | A \in C_{k,3}\}$ is of type $(3, 3)$ and generated by:

$$C_{k,3}^{1-\sigma} = \langle [R^3h], [R^h][P^h]^2 \rangle.$$

**Proof.**

The fact that the ideals $P^h$, $Q^h$ and $R^h$ are not principals, we prove the assertions (1), (2), (3) and (4) by applying the decomposition of 3 in the normal closure $k$.

For the assertion (5), since the ideals $P^h$, $Q^h$ and $R^h$ are not principal, we obtain the result by the same reasoning above.

The assertions (6) and (7) follows by using Proposition 3.2. \[\square\]
4 Appendix

Using the Pari programming [12], we illustrate the results of our main Theorem 3.2 by numerical examples. We have

\[ C_{k,3} = \langle [\mathcal{R}^h], [\mathcal{R}^h][\mathcal{P}^h]^2 \rangle \]

The following table verifies, for each prime number \( p \equiv 1 \pmod{9} \) such that \( \left( \frac{3}{p} \right) \neq 1 \) and 9 divides the 3-class number of \( \mathbb{Q}(\sqrt[3]{p}) \) exactly and \( u = 1 \), that the ideals \( \mathcal{R}^h \) and \( \mathcal{R}^{3h} \) are not principal. Therefore, the ideal \( \mathcal{R}^{3h} \) is always principal.

Table 1

| \( p \) | Type of \( C_{k,3} \) | Is principal \( \mathcal{R}^h \) | Is principal \( \mathcal{R}^{3h} \) | Is principal \( \mathcal{R}^{3h} \) |
|-------|------------------|-----------------|-----------------|-----------------|
| 199   | [9, 3]           | [8, 0]          | [6, 0]          | [0, 0]          |
| 487   | [9, 3]           | [10, 0]         | [12, 0]         | [0, 0]          |
| 1297  | [9, 3]           | [16, 0]         | [12, 0]         | [0, 0]          |
| 1693  | [9, 3]           | [2, 2]          | [6, 0]          | [0, 0]          |
| 1747  | [9, 3]           | [8, 0]          | [6, 0]          | [0, 0]          |
| 1999  | [9, 3]           | [8, 0]          | [6, 0]          | [0, 0]          |
| 2017  | [9, 3]           | [8, 0]          | [6, 0]          | [0, 0]          |
| 2143  | [9, 3]           | [14, 0]         | [6, 0]          | [0, 0]          |
| 2377  | [9, 3]           | [7, 0]          | [3, 0]          | [0, 0]          |
| 2467  | [9, 3]           | [20, 0]         | [15, 0]         | [0, 0]          |
| 2593  | [9, 3]           | [4, 2]          | [3, 0]          | [0, 0]          |
| 2917  | [9, 3]           | [8, 0]          | [6, 0]          | [0, 0]          |
| 3511  | [9, 3]           | [10, 0]         | [12, 0]         | [0, 0]          |
| 3673  | [9, 3]           | [8, 0]          | [6, 0]          | [0, 0]          |
| 3727  | [9, 3]           | [5, 0]          | [6, 0]          | [0, 0]          |
| 4159  | [9, 3]           | [4, 2]          | [12, 0]         | [0, 0]          |
| 4519  | [9, 3]           | [4, 4]          | [12, 0]         | [0, 0]          |
| 4591  | [9, 3]           | [1, 2]          | [3, 0]          | [0, 0]          |
| 4789  | [9, 3]           | [25, 5]         | [30, 0]         | [0, 0]          |
| 5347  | [9, 3]           | [8, 0]          | [6, 0]          | [0, 0]          |
| 5437  | [9, 3]           | [77, 0]         | [33, 0]         | [0, 0]          |
| 6799  | [9, 3]           | [7, 2]          | [3, 0]          | [0, 0]          |
| 8209  | [9, 3]           | [2, 2]          | [6, 0]          | [0, 0]          |
| 8821  | [9, 3]           | [4, 0]          | [3, 0]          | [0, 0]          |

However, we verify in the following table that the ideal \( \mathcal{R}^h \mathcal{P}^{2h} \) is not principal and \( \mathcal{R}^h \mathcal{P}^{2h} \) is of order 3.
Table 2

| $p$  | Type of $C_{k,3}$ | Is principal $\mathcal{K}^{h \mathcal{P}^{2h}}$ | Is principal $(\mathcal{K}^{h \mathcal{P}^{2h}})^3$ |
|------|-------------------|---------------------------------|---------------------------------|
| 199  | [9, 3]            | [0, 1]                          | [0, 0]                          |
| 487  | [9, 3]            | [0, 2]                          | [0, 0]                          |
| 1297 | [9, 3]            | [6, 4]                          | [0, 0]                          |
| 1693 | [9, 3]            | [6, 2]                          | [0, 0]                          |
| 1747 | [9, 3]            | [0, 1]                          | [0, 0]                          |
| 1999 | [9, 3]            | [0, 2]                          | [0, 0]                          |
| 2017 | [9, 3]            | [0, 2]                          | [0, 0]                          |
| 2143 | [9, 3]            | [0, 4]                          | [0, 0]                          |
| 2377 | [9, 3]            | [3, 2]                          | [0, 0]                          |
| 2467 | [9, 3]            | [0, 10]                         | [0, 0]                          |
| 2593 | [9, 3]            | [0, 2]                          | [0, 0]                          |
| 2917 | [9, 3]            | [0, 1]                          | [0, 0]                          |
| 3511 | [9, 3]            | [0, 2]                          | [0, 0]                          |
| 3673 | [9, 3]            | [0, 1]                          | [0, 0]                          |
| 3727 | [9, 3]            | [3, 1]                          | [0, 0]                          |
| 4159 | [9, 3]            | [6, 2]                          | [0, 0]                          |
| 4519 | [9, 3]            | [24, 4]                         | [0, 0]                          |

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