The Orion Star Forming Region

Molecular Gas seen in CO J=1-0 transition

Molecular Gas (contours) vs. Ionized Gas (hue map)

Star-Forming Regions in M51
Cold Molecular Cores

Optical photographic image of the Taurus molecular cloud, taken in 1907

CO(1-0) emission crosses indicate known stellar and protostellar objects
The Maximum Mass and the Stability of Bonnor-Ebert Sphere

Bonnor-Ebert Sphere Fit to Extinction Data of the Barnard 68 Core

Bonnor-Ebert Spheres

The Critical, Singular, and Ambient density profiles as functions of radius (pc). The figure illustrates the stability of Bonnor-Ebert spheres, with the critical, singular, and ambient density profiles highlighted.

The relationship between mass and the ratio of central density to ambient density ($\rho_{\text{cen}}/\rho_{\text{amb}}$) is shown, with stable and unstable regions indicated.

B68 visual extinction curves are displayed, with observations compared to a Bonnor-Ebert profile.
Star Formation Rate

\(<N_H>\) & \(T\)

| Pregalactic | \(2 \times 10^4 \text{ cm}^{-3} \left(\frac{1+z}{10}\right)^3\) | \(2 \text{ K} \left(\frac{1+z}{10}\right)^2\) |
| Halo gas | \(4 \times 10^{-2} \text{ cm}^{-3} \left(\frac{1+z}{10}\right)^3\) | \(10^4 \text{ K} \left(\frac{M_\odot}{10^8 M_\odot}\right)^{3/2} \left(\frac{1+z}{10}\right)\) |
| Solar neighborhood | \(40 \text{ cm}^{-3} \left(\frac{M_\odot/pc}{11 M_\odot/pc^3}\right)\) | N/A |

\(GMC(H_2)\) & \(10^2 \text{ cm}^{-3}\) & \(15 \text{ K} \Rightarrow M_5 = 80 M_\odot\) |
| Cores \((H_2)\) & \(10^4 \text{ cm}^{-3}\) & \(10 \text{ K} \Rightarrow M_5 = 8 M_\odot\) |
| WNM \((HI)\) & \(0.3 \text{ cm}^{-3}\) & \(10^4 \text{ K}\) \(\frac{P_K}{K} \sim 10^4 \text{ cm}^3 \text{ K}\) |
| CNM \((HI)\) & \(30 \text{ cm}^{-3}\) & \(100 \text{ K}\) |

**Jeans Mass**

\[ M_J = \left(\frac{5 kT}{\mu m_p G}\right)^{3/2} \left(\frac{3}{4 \pi \rho}\right)^{1/2} = 8 M_\odot \left(\frac{T}{10 \text{ K}}\right)^{3/2} \left(\frac{\mu}{2}\right)^{1/2} \left(\frac{N_{H_2}}{10^4 \text{ cm}^{-3}}\right)^{-1/2} \]

\(M_J \propto T^{3/2} \rho^{-1/2} \) \(\Rightarrow\) SF occurs in cold & dense gas clouds
molecular clouds are thus preferred sites of star formation

**Free Fall Timescale**

\[ t_{ff} = \sqrt{\frac{3 \pi}{32 G \rho}} = 4 \times 10^5 \text{ yr} \left(\frac{N_{H_2}}{10^4 \text{ cm}^{-3}}\right)^{1/2} \]

**SFR per core**

\[ \frac{M_5}{t_{ff}} \approx 2 \times 10^{-5} M_\odot/\text{yr} \left(\frac{T}{10 \text{ K}}\right)^{3/2} \left(\frac{\mu}{2}\right)^{-3/2} \]

**SFR of the Galaxy**

\[ \text{SFR}_{mw} = \frac{M_{H_2}}{M_5} \cdot \frac{M_5}{t_{ff}} = M_{H_2}/t_{ff} = 10^9 M_\odot/4 \times 10^5 \text{ yr} \approx 2 \times 10^3 M_\odot/\text{yr} \]

**Key assumptions:**

1. All \(H_2\) in molecular clouds are prone to gravitational collapse
2. SF proceeds on the freefall timescale
**Kelvin-Helmholtz timescale (cooling timescale)**

\[
t_{KH} = \frac{GM^2/R}{4\pi R^2 \alpha_{BB} T^4} \approx 10^7 \text{ yr} \left(\frac{M}{M_0}\right)^2 \left(\frac{R}{R_0}\right)^{-3} \left(\frac{T}{T_0}\right)^{-4} \propto M \cdot \rho^{-4}.
\]

Think about a proto star before fusion ignition.

**Turbulence Support against gravitational collapse**

\[
M_J = 8 M_\odot \left(\frac{T}{10^4 \text{ cm}^3} \right)^{3/2} \left(\frac{10^4 \text{ cm}^3}{\text{cm}^3} \right)^{1/2} \quad C_s = \sqrt{\frac{8 \Phi}{\rho}} = \sqrt{\frac{8 k T}{\rho \Delta m}}
\]

\[
= 8 M_\odot \left(\frac{5}{0.3 \text{ km/s}}\right)^3 \left(\frac{10^4 \text{ cm}^3}{\text{cm}^3} \right)^{1/2} \quad = 0.54 \text{ km/s} \left(\frac{\Delta m}{\text{cm}^3} \right)^{1/2} \left(\frac{2}{\mu} \right)^{1/4} \left(\frac{T}{50 \text{ K}}\right)^{1/4}
\]

Observed velocity dispersion in clouds \(\gg\) thermal broadening, indicating supersonic turbulence in larger clouds dominates the kinetic energy.

\[
\sigma = 3 \text{ km/s} \left(\frac{R}{20 \text{ pc}}\right)^{0.5}
\]

Larson (1981)

\[
\sigma = \begin{cases} 
0.36 \text{ km/s for } R \sim 0.1 \text{ pc (core size)} \\
5.7 \text{ km/s for } R \sim 100 \text{ pc (GMC size)}
\end{cases}
\]

\(\Rightarrow M_J \sim 10^7 M_\odot\) for GMCs, which is greater than typical mass of GMCs (10^5 M_\odot).

\(M_J \sim 10^4 M_\odot\) for cores, comparable to the typical core mass (10^4 M_\odot).

Therefore, only the densest 1% molecular clouds can form stars.

**Re-evaluating MW SFR:**

\[
SFR_{MW} = N_{\text{core}} \cdot (\text{SFR per core}) = \frac{6 \ M_{\odot}}{M_{\text{core}}} \cdot \frac{M_{\text{core}}}{t_{\text{SF}}} = 6 \ M_{\odot}/t_{\text{SF}}
\]

\(\epsilon \approx 1\%\) - core mass fraction

\(t_{\text{SF}} = \max \left(t_{\text{gas replenish}}, t_{\text{cool \& ff}}\right)\)

\(t_{KH} \sim 10^7 \text{ yr} - \text{cooling timescale}\)

\(\Rightarrow SFR_{MW} = 10^{-2} \cdot \frac{10^9 M_\odot}{10^3 \text{ yr}} \approx 1 M_\odot/\text{yr}\)

\[
SFE_1 = \frac{SFR}{M_{\text{H}_2}} = \frac{6}{t_{\text{SF}}} \approx 1 \text{ Gyr}^{-1}
\]

\[
SFE_2 = \frac{SFR \cdot t_{\text{ff}}}{M_{\text{H}_2}} = \epsilon \cdot \left(t_{\text{ff}}/t_{\text{SF}}\right) \approx 10^{-5}
\]
Bonnor–Ebert Sphere

Isothermal gas, self-gravitating, hydrostatic equilibrium, spherical symmetry:

\[ \frac{dP}{dr} = -\rho \frac{d\Phi}{dr}, \quad \frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d\Phi}{dr} \right) = 4\pi G \rho, \quad P = \rho \frac{d\mu}{d\rho} = \rho c_s^2 \]

Singular isothermal sphere (SIS) is obtained when we try \( P = \rho_0 \cdot r^\alpha \), which gives

\( P = \rho_0 \cdot c_s^2 \cdot r^\alpha \)

\[ \frac{d\Phi}{dr} = -\frac{1}{\rho} \frac{dP}{dr} = -\alpha c_s^2 / r \]

\[ \nabla^2 \Phi = \frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d\Phi}{dr} \right) = -\alpha c_s^2 \cdot r^{-2} = 4\pi G \rho_0 \cdot r^\alpha \]

\[ \Rightarrow \alpha = -2 \quad & \quad \rho_0 = \frac{c_s^2}{2\pi G} \quad \Rightarrow \quad \rho = \frac{c_s^2}{2\pi G} \cdot r^{-2} \]

Other solutions exist for the same set of differential equations. In particular, we're interested in a solution where \( P(r=0) \) is finite. To do that, we first combine the three equations into one equation. First, Eqs 0 + 9 gives:

\[ \frac{dP}{dr} = c_s^2 \frac{d\Phi}{dr} = -\rho \frac{d\Phi}{dr} \Rightarrow P = p_c \exp \left( -\frac{\Phi - \Phi_c}{c_s^2} \right) \]

define the dimensionless variables:

\[ \gamma = \frac{\Phi - \Phi_c}{c_s^2}, \quad \chi = \left( \frac{4\pi G p_c}{c_s^2} \right)^{1/2} \cdot r \quad \left( \gamma = \ln \left( \frac{p_c}{p} \right) \right) \quad \text{logarithmic density} \]

Poisson Eq. then can be written in a dimensionless form:

\[ \frac{1}{\chi^2} \frac{d}{dr} \left( r^2 \frac{d\gamma}{dr} \right) = 4\pi G p_c \exp \left( -\frac{\Phi - \Phi_c}{c_s^2} \right) \]

\[ \Rightarrow \frac{1}{\chi^2} \frac{d}{dx} \left( x^2 \frac{dy}{dx} \right) = e^{-\gamma} \quad \text{isothermal Lane–Eddington Eq.} \]

Its numerical solution gives us the density profile:

\[ r = \left( \frac{c_s^2}{4\pi G p_c} \right)^{1/2} \chi, \quad \rho = \rho_c \exp (-\gamma) \]

Boundary condition: \( \gamma = 0 \) & \( \frac{dy}{dx} = 0 \) at \( r = 0 \), \( p_c \) is specified.
The profiles solved under the boundary condition is called Bonnor-Ebert spheres (1955), representing non-singular isothermal spheres. The enclosed mass within its boundary is:

\[ M = 4\pi \int_0^{x_t} \rho(r) r^2 dr = 4\pi \left( \frac{C_s^2}{4\pi \gamma \rho_c} \right)^{3/2} \int_0^{x_t} \rho(x) x^2 dx \]

\[ = \frac{1}{\sqrt{4\pi \rho_c}} \frac{C_s^2}{\gamma^{3/2}} x_t^3 \frac{d\rho}{dx} \bigg|_{x_t} \]

\[ = \frac{C_s^2}{\gamma^{3/2} \rho_c^{1/2}} M \]

where \( m \) is the dimensionless mass:

\[ m = \frac{1}{4\pi} \left( \frac{\rho_c}{\rho_t} \right)^{-2} x_t^2 \frac{d\rho}{dx} \bigg|_{x_t} \]

and \( x_t \) is the radius at which the density equals the ambient ISM density:

\[ \rho(x) = \rho_c e^{-\frac{x}{x_t}} \]

because \( x_t \) only depends on \( \rho_c/\rho_t \), the density contrast, \( M \) is only a function of \( \rho_c/\rho_t \). It first increases as the core becomes more centrally concentrated, reaches its peak at \( \rho_c/\rho_t \approx 14 \), and begins to decrease at higher contrast!

This last regime is unstable because as the core gains mass, it has to reduce central density!

The Bonnor-Ebert mass is the mass at the maximum, it is thus a lower limit for gravitational collapse, similar to Jeans Mass.

\[ M_{BE} = \frac{C_s^2}{\sqrt{\gamma} \rho_c^{1/2}} \]

\[ M_{max} = 2.2 M_\odot \left( \frac{M}{2.3} \right)^{-2} \left( \frac{N_H}{10^2 \text{ cm}^{-2}} \right)^{-1/2} \left( \frac{T}{15 \text{ K}} \right)^{3/2} \]

\[ M_J = 120 M_\odot \left( \frac{M}{2.3} \right)^{3/2} \left( \frac{N_{H_2}}{10^8 \text{ cm}^{-3}} \right)^{-1/2} \left( \frac{T}{15 \text{ K}} \right)^{3/2} \]
Size of Protostellar Disk

Jeans length: \( R_J = \left( \frac{15 kT}{4 \pi G \rho m_p} \right)^{1/2} = 0.1 \text{ pc} \left( \frac{T}{10^4 \text{ K}} \right)^{1/2} \left( \frac{\rho}{10^4 \text{ g cm}^{-3}} \right)^{-1/2} \)

Conservation of angular momentum dictates the formation of a rotating disk. The edge of the disk has a radius of \( R_f \) and circular velocity \( V_f \). The initial rotation velocity of the cloud is \( V_0 \). We have two equations:

\[
\frac{GM_J}{R_f^2} = \frac{V_f^2}{R_f}, \quad V_f \cdot R_f = V_0 \cdot R_J
\]

\[
\Rightarrow R_f = \frac{V_0^2 R_J^2}{GM_J} \approx 600 \text{ AU} \left( \frac{V_0}{0.1 \text{ km/s}} \right)^2 \left( \frac{R_J}{0.1 \text{ pc}} \right)^2 \left( \frac{M_J}{8 M_\odot} \right)^{-1}
\]

Cloud Fragmentation & the minimum Jeans Mass

\( M_J \propto T^{3/2} \rho^{-1/2} \sim \begin{cases} 
\rho^{-1/2} & \text{during isothermal collapse (T=const.)} \\
\rho^{(3\alpha-4)/2} & \text{in adiabatic collapse (T\propto \rho^{\alpha-1})}
\end{cases} \)

so the Jeans mass reaches its minimum when the collapse transitions to adiabatic.

This transition occurs when radiative cooling becomes inefficient:

\( t_{\text{ff}} < t_{\text{cool}} \) where \( t_{\text{cool}} = \frac{GM^2}{4\pi R^2 c_s T} \) and \( t_{\text{ff}} = \frac{3\pi}{32G^2 \rho} \)

We also know \( \rho = \frac{M}{4\pi R^3} \) and \( R = \frac{GM}{5c_s^2} \) from virial theorem. Plug these in to replace \( \rho \) & \( R \), we have:

\( M > 0.1M_\odot \left( \frac{T}{10^4 \text{ K}} \right)^{3/4} \left( \frac{\mu}{1} \right)^{-9/4} \equiv M_{J,\text{min}} \)
Measuring the IMF

Observables: \( \Psi(M_v) \) the present-day luminosity function

Models: \( M - M_v \) relation & its derivative \( dm/dM_v \)

Equations:

\[ \Psi(M_v) \, dM_v = dN = \Xi(m) \, dm, \quad \Xi(m): \text{present mass-fun}. \]

\[ \Xi(m) = \xi(m) \cdot \frac{1}{T_G} \int_{T_{G-m}}^{T_G} b(t) \, dt \approx \xi(m) \cdot \frac{\tau(m)}{T_G} \text{ or } \xi(m) \]

where \( T_G \text{ is the age of the system (e.g. the MW)}, \text{ and } \tau(m) \text{ is the}

main-sequence lifetime of the star w/ mass of } m.

\( b(t) \text{ is the normalized star formation history:} \)

\[ \frac{1}{T_G} \int_{0}^{T_G} b(t) \, dt = 1.0 \]

For MW disk stars, it's assumed to be \( b(t) = 1.0 \text{ (const. SF)} \)

For star clusters, it's assumed to be \( b(t) = T_G \cdot \delta(t-t_0) \text{ (burst)} \)

Combining the two equations:

\[ \xi(m) = \Psi(M_v) \cdot \left( \frac{dm}{dM_v} \right)^{-1} \times \begin{cases} 1.0 & \tau(m) > T_G, \text{ low-mass stars} \\ \frac{T_G}{\tau(m)} & \tau(m) < T_G, \text{ high-mass stars} \end{cases} \]

Kroupa (2002) IMF:

\[ \xi(m) \propto \begin{cases} m^{-0.3} & m < 0.08 \, M_\odot \\ m^{-1.3} & 0.08 < m < 0.5 \, M_\odot \\ m^{-2.3} & m > 0.5 \, M_\odot \end{cases} \]

Brown Dwarfs: \( m < 0.072 \, M_\odot \)

Very low mass: \( 0.072 - 0.5 \, M_\odot \)

Low mass stars: \( 0.5 - 1 \, M_\odot \)

Intermediate mass: \( 1 - 8 \, M_\odot \)

Massive stars: \( 8 - 120 \, M_\odot \)
Initial Mass Function

$$\Phi(m) \equiv \frac{dN}{dm} = \phi_0 \cdot m^{-2.35} \quad \text{Salpeter (1955)}$$

Mass per dex = \( \int_{0.1M_\odot}^{100M_\odot} \phi(m) \cdot m \, dm \) \quad \text{unit: } M_\odot / \text{pc}^3

$$\frac{dN}{d\log m} = \ln 10 \cdot m^2 \cdot \phi = \ln 10 \cdot \phi_0 \cdot m^{-0.35} \sim \text{const.}$$

A PDF with a powerlaw slope of -2 is the same as predicting the mass of any star using its rank:

$$M(i) = \frac{M_\odot}{\text{rank}(i)} \quad \text{or rank} \propto m^{-1}$$

because

$$\frac{dN}{dm} = \frac{d\text{rank}}{dm} = m_\odot \cdot m^{-2}$$

A similar PDF is seen in the size of cities:

| City   | Pop | Ratio | 1/rank |
|--------|-----|-------|--------|
| NY     | 8.5 | 1.0   | 1.0    |
| LA     | 3.9 | 0.46  | 0.5    |
| CHI    | 2.7 | 0.32  | 0.33   |
| Houston| 2.2 | 0.26  | 0.25   |
| Phil   | 1.6 | 0.18  | 0.20   |
| Phoenix| 1.5 | 0.18  | 0.17   |

A natural result of exponential growth \( (\frac{dM}{dt} \propto m^2) \) in environs w/ limited resources?

Bondi - Hoyle Accretion: \( \dot{M} \approx \pi R_b^2 \rho v \), \( R_b \approx 2GM/c_s^2 \), \( v \sim c_s \)

\[
\Rightarrow \dot{M} \approx \frac{\pi \rho G M^2}{c_s^3} \propto M^2
\]