Research Article

Solutions of Conformable Fractional-Order SIR Epidemic Model

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In this paper, the conformable fractional-order SIR epidemic model are solved by means of an analytic technique for nonlinear problems, namely, the conformable fractional differential transformation method (CFDTM) and variational iteration method (VIM). These models are nonlinear system of conformable fractional differential equation (CFDE) that has no analytic solution. The VIM is based on conformable fractional derivative and proved. The result revealed that both methods are in agreement and are accurate and efficient for solving systems of OFDE.

1. Introduction

The problem of spreading of a (non) fatal disease in a population which is assumed to have a constant size over the period of the epidemic is considered in COVID-19. The goal of the epidemic model is to understand and if possible control the spread of disease [1]. At time $t$, suppose the population consists of

(i) $s(t)$: the number of susceptible, who do not have the disease but could get it.

(ii) $i(t)$: the number of infectives, who have the disease and can transmit it to others.

(iii) $r(t)$: the number of removed, who cannot get the disease or transmit it; either they have a natural immunity or they have recovered from the disease and are immune from getting it again or they have been placed in isolation or they have died. The mathematical model does not distinguish between these possibilities.

Assume there is a steady constant rate between susceptible and infectives and that a constant proportion of this constant results in transmission. Then, in time $\delta t$, the susceptible becomes infective:

$$\delta s = -\beta si \delta t,$$

where $\beta$ is a positive constant. If $\gamma > 0$ is the rate at which current infectives become isolated, then

$$\delta i = \beta si \delta t - \gamma i \delta t,$$

and the number of new isolated $\delta r$ is given by

$$\delta r = \gamma i \delta t.$$

If we let $\delta t \rightarrow 0$, then the following nonlinear system of ODEs determines the progress of the disease:

$$\frac{d}{dt}s(t) = -\beta s(t)i(t),$$

$$\frac{d}{dt}i(t) = \beta s(t)i(t) - \gamma i(t) = 0,$$

$$\frac{d}{dt}r(t) = \gamma i(t),$$

with initial conditions

$$s(0) = N_s,$$

$$i(0) = N_i,$$

$$r(0) = N_r.$$

The following simple SIR model [2–5] is transformed to conformable fractional differential equation and is tested to
show the efficiency of the variational iteration method [6] and differential transformation method [7–11] to solve such models.

Fractional differentiation and integration operators have different kinds of definitions which we can mention, the Riemann–Liouville definition [12, 13], the Caputo definition [14], and so on. Lately, Khalil et al. [15] introduced a new simple definition of the fractional derivative named the conformable fractional derivative (CFD), which can redress shortcomings of the other definitions. The main advantages of the CFD can be summarized as follows [16–19]:

1. It satisfies all concepts and rules of an ordinary derivative such as: quotient, product, and chain rules, while the other fractional definitions fail to meet these rules.

2. It can be extended to solve exact and numerical fractional differential equations and systems easily and efficiently.

The reason for considering a fractional-order system instead of its integer order counterpart is that fractional-order differential equations are generalizations of integer order differential equations. Also, using fractional-order differential equations can help us to reduce the errors arising from the neglected parameters in modelling real-life phenomena.

We like to argue that fractional-order equations are more suitable than integer order ones in modelling biological, economical, and social systems (generally complex adaptive systems) where memory effects are important.

The main objective of our work is to introduce the conformable fractional-order approach for the study of a particular SIR model in a constant population. In this case, the conformable fractional-order system of the SIR model will be transformed to one conformable fractional equation and are solved using the variational iteration method and the conformable differential transformation method for numerical comparison.

2. Conformable Fractional Derivative and Some Properties

In this part, we review some definitions and some results of conformable fractional derivative. For more details, the reader can refer to [15, 18, 20–25]. The conformable fractional derivative of order \( \alpha \) is defined as

\[
T_\alpha f(t) = \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon t^{1-\alpha}) - f(t)}{\varepsilon},
\]

for all \( t > 0, \alpha \in (0, 1) \), and the fractional derivative at 0 is defined as

\[
(T_\alpha f)(0) = \lim_{t \to 0^+} (T_\alpha f)(t).
\]

Let \( \alpha \in 0, 1 \) and \( f \) be \( \alpha \)-differentiable at a point \( t > 0 \), and if \( f \) is differentiable, then

\[
T_\alpha f(t) = t^{1-\alpha} \frac{d}{dt} f(t), \quad (8)
\]

The fractional integral of order \( \alpha \) is defined by

\[
I_\alpha (f)(t) = \int_0^t s^{\alpha-1} f(s) \, ds, \quad \text{for all } \alpha \in (0, 1),
\]

where the integral is the usual Riemann improper integral.

3. Mathematical Modeling of the Conformable Fractional SIR Model

The conformable fractional model of actual evolution of this epidemic in a population of large size \( N \) is given by the following conformable fractional differential system:

\[
\begin{align*}
T_\alpha s(t) &= -\beta s(t) i(t), \\
T_\alpha i(t) &= \beta s(t) i(t) - \gamma i(t), \\
T_\alpha r(t) &= \gamma i(t),
\end{align*}
\]

with initial conditions

\[
\begin{align*}
s(0) &= N, \\
i(0) &= N, \\
r(0) &= N.
\end{align*}
\]

3.1. The Conformable Fractional Differential Transformation Method. In [9, 11], assume \( f \) is infinitely \( \alpha \)-differentiable function, for some \( 0 < \alpha \leq 1 \). \( f(t) \) can be expanded in fractional power series expansion about a point \( t = 0 \) as

\[
f(t) = \sum_{k=0}^{\infty} \frac{t^\alpha}{\alpha^k k!} \left[ (T_\alpha f)^{(k)} \right]_{t=0^+}. \quad (12)
\]

Here, \( \left[ (T_\alpha f)^{(k)} \right]_{t=0^+} \) denotes the application of the fractional derivative for \( k \) times. Conformable fractional differential transform of \( f(t) \) is defined as

\[
F_\alpha(k) = \frac{1}{\alpha^k k!} \left[ (T_\alpha f)^{(k)} \right]_{t=0^+}. \quad (13)
\]

Then, the inverse conformable fractional differential transform of \( F(k) \) is defined as

\[
f(t) = \sum_{k=0}^{\infty} \frac{t^\alpha}{\alpha^k k!} F_\alpha(k) = \sum_{k=0}^{\infty} \frac{t^\alpha}{\alpha^k k!} \left[ (T_\alpha f)^{(k)} \right]_{t=0^+}. \quad (14)
\]

The fundamental mathematical operations performed by conformable fractional differential transform method are listed.

Proposition 1

(i) If \( f(t) = u(t) \pm v(t) \), then \( F_\alpha(k) = U_\alpha(k) \pm V_\alpha(k) \).

(ii) If \( f(t) = cu(t), c \in \mathbb{R} \), then \( F_\alpha(k) = cU_\alpha(k) \).

(iii) If \( f(t) = u(t)v(t) \), then \( F_\alpha(k) = \sum_{l=0}^{k} U_\alpha(l)V_\alpha(k-l) \).
3.1.1. Application of Conformable Fractional Differential Transform Method. Equation (10) can be rewritten as follows:

\[
\alpha (k + 1)S_n (t) = - \beta \sum_{l=0}^{k} S_n (l) I_a (k - l),
\]
\[
\alpha (k + 1)I_n (t) = \beta \sum_{l=0}^{k} S_n (l) I_a (k - l) - \gamma I_a (k),
\]
\[
\alpha (k + 1)R_n (t) = \gamma I_a (k).
\]

Hence, recurrence relation is obtained as

\[
S_n (t) = \frac{1}{\alpha (k + 1)} \left[ - \beta \sum_{l=0}^{k} S_n (l) I_a (k - l) \right],
\]
\[
I_n (t) = \frac{1}{\alpha (k + 1)} \left[ \beta \sum_{l=0}^{k} S_n (l) I_a (k - l) - \gamma I_a (k) \right],
\]
\[
R_n (t) = \frac{1}{\alpha (k + 1)} \left[ \gamma I_a (k) \right].
\]

With initial conditions \( N_0 = 2000, N_1 = 300, \) and \( N_2 = 200 \) and parameters \( N = 2500, \beta = 0.00012, \) and \( \gamma = 0.1 \), apply the condition in (17); then, the closed form of the solution where \( k = 4 \) can be written as

\[
s(t) = \sum_{l=0}^{k} S(l) t^\alpha = 2000 - 72 \frac{t^\alpha}{\alpha} - 3.744 \frac{t^{2\alpha}}{\alpha^2} + 0.03436 \frac{t^{3\alpha}}{\alpha^3} + 0.013309 \frac{t^{4\alpha}}{\alpha^4},
\]
\[
i(t) = \sum_{l=0}^{k} I(l) t^\alpha = 300 + 42 \frac{t^\alpha}{\alpha} + 1.644 \frac{t^{2\alpha}}{\alpha^2} - 0.089168 \frac{t^{3\alpha}}{\alpha^3} - 0.011080 \frac{t^{4\alpha}}{\alpha^4},
\]
\[
r(t) = \sum_{l=0}^{k} R(l) t^\alpha = 200 + 30 \frac{t^\alpha}{\alpha} + 2.1 \frac{t^{2\alpha}}{\alpha^2} + 0.0548 \frac{t^{3\alpha}}{\alpha^3} - 0.002229 \frac{t^{4\alpha}}{\alpha^4}.
\]

3.2. Variational Iteration Method. To illustrate the basic idea of the variational iteration method, we consider the following nonlinear differential equation in the operator form:

\[
L(u(t)) + N(u(t)) = g(t),
\]

where \( L \) is a linear operator, \( N \) is a nonlinear operator, and \( g \) is an arbitrary real function which is called the inhomogeneous term. Then, the corresponding correction functional for equation (19) is given by

\[
u_{m+1}(t) = u_n(t) + \int_0^t \lambda [L u_n (s) + N u_n (s) - g(s)] \, ds,
\]

where \( \lambda \) is the general Lagrange multiplier [26], which can be identified optimally via the variational theory [27] and \( \delta N u_n = 0 \). Consider the stationary condition of the above correction functional.

**Theorem 1.** Consider the conformable fractional differential equation (10). Then, the variational iteration formula is given by

\[
s_{m+1}(t) = s_n(t) - I_a [T_a s_n(t) + \beta s_n(t)i_n(t)],
\]
\[
i_{m+1}(t) = i_n(t) - I_a [T_a i_n(t) - \beta s_n(t)i_n(t) + \gamma i_n(t)],
\]
\[
r_{m+1}(t) = r_n(t) - I_a [T_a r_n(t) - \gamma i_n(t)],
\]

where \( s_n, i_n, \) and \( r_n \) are the \( n \)th approximation, \( T_a \) is the conformal fractional derivative of order \( \alpha \), and \( I_a \) is the fractional integral of order \( \alpha \in 0, 1 \).

**Proof.** Equation (10) may be rewritten equivalently as

\[
T_a s_n(t) + \beta s_n(t)i_n(t) = 0,
\]
\[
T_a i_n(t) - \beta s_n(t)i_n(t) + \gamma i_n(t) = 0,
\]
\[
T_a r_n(t) - \gamma i_n(t) = 0.
\]

Multiplying equation (22) by a general Lagrange multiplier \( \lambda_1(t), \lambda_2(t), \) and \( \lambda_3(t) \) yields

\[
\lambda_1(t) [T_a s_n(t) + \beta s_n(t)i_n(t)] = 0,
\]
\[
\lambda_2(t) [T_a i_n(t) - \beta s_n(t)i_n(t) + \gamma i_n(t)] = 0,
\]
\[
\lambda_3(t) [T_a r_n(t) - \gamma i_n(t)] = 0.
\]

Now, upon applying \( I_a \) to the both sides of equation (23) will give

\[
I_a [\lambda_1(t) [T_a s_n(t) + \beta s_n(t)i_n(t)]] = 0,
\]
\[
I_a [\lambda_2(t) [T_a i_n(t) - \beta s_n(t)i_n(t) + \gamma i_n(t)]] = 0,
\]
\[
I_a [\lambda_3(t) [T_a r_n(t) - \gamma i_n(t)]] = 0.
\]

Then, the correction functional of equation (10) will be read as follows:

\[
s_{m+1}(t) = s_n(t) + I_a [\lambda_1(t) [T_a s_n(t) + \beta s_n(t)i_n(t)]]
\]
\[
i_{m+1}(t) = i_n(t) + I_a [\lambda_2(t) [T_a i_n(t) - \beta s_n(t)i_n(t) + \gamma i_n(t)]]
\]
\[
r_{m+1}(t) = r_n(t) + I_a [\lambda_3(t) [T_a r_n(t) - \gamma i_n(t)]]
\]
In this case, the value of \( \lambda_j(t) \), for \( j = \{1, 2, 3\} \), cannot be evaluated easily from equation (25), which will be a functional with fractional integral. Equation (25) can be expressed as follows:

\[
\begin{align*}
\delta s_{n+1}(t) &= \delta s_n(t) + \delta \left( \lambda(t) s_n(t) - \int_0^t \lambda'_j(r) s_n(r) \, dr \right) \\
&= (1 + \lambda_1(t)) \delta s_n(t) - \int_0^t \lambda'_j(r) s_n(r) \, dr, \\
\delta i_{n+1}(t) &= (1 + \lambda_2(t)) \delta i_n(t) - \int_0^t \lambda'_j(r) i_n(r) \, dr, \\
\delta r_{n+1}(t) &= (1 + \lambda_3(t)) \delta r_n(t) - \int_0^t \lambda'_j(r) r_n(r) \, dr.
\end{align*}
\]

(30)

The Lagrange multipliers \( \lambda_1(t), \lambda_2(t), \) and \( \lambda_3(t) \) can be obtained by \( \lambda'_j(t) = 0 \) for all \( j = \{1, 2, 3\} \) with boundary condition: \( 1 + \lambda'_j(t) = 0 \), for all \( j = \{1, 2, 3\} \). Solving the last initial value problem for \( \lambda_j \) for all \( j = \{1, 2, 3\} \), the general Lagrange multiplier \( \lambda_j \) is found to be

\[
\lambda_j = -1, \quad \text{for all } j = \{1, 2, 3\}.
\]

Hence, substituting the value of \( \lambda_j \) into the corresponding correction functional (25) will give the following iteration formula:

\[
\begin{align*}
s_{n+1}(t) &= s_n(t) - I_a \left[ T_a s_n(t) + \beta \bar{s}_n(t) \bar{i}_n(t) \right], \\
i_{n+1}(t) &= i_n(t) - I_a \left[ T_a i_n(t) - \beta \bar{s}_n(t) \bar{i}_n(t) + \gamma \bar{i}_n(t) \right], \\
r_{n+1}(t) &= r_n(t) - I_a \left[ T_a r_n(t) - \gamma \bar{i}_n(t) \right].
\end{align*}
\]

(32)

3.2.1. Application of the Variational Iteration Method.

Applying the variational iteration method using Theorem 1, we obtain

\[
\begin{align*}
s_{n+1}(t) &= s_n(t) - I_a \left[ T_a s_n(t) + \beta \bar{s}_n(t) \bar{i}_n(t) \right], \\
i_{n+1}(t) &= i_n(t) - I_a \left[ T_a i_n(t) - \beta \bar{s}_n(t) \bar{i}_n(t) + \gamma \bar{i}_n(t) \right], \\
r_{n+1}(t) &= r_n(t) - I_a \left[ T_a r_n(t) - \gamma \bar{i}_n(t) \right].
\end{align*}
\]

(33)

With initial conditions \( N_3 = 2000, N_4 = 300, \) and \( N_5 = 200 \) and parameter \( N = 2500, \beta = 0.00012, \) and \( \gamma = 0.1, \) we obtain
\begin{align*}
s_1(t) &= 2000 - 72\frac{t^\alpha}{\alpha} \\
i_1(t) &= 300 + 42\frac{t^\alpha}{\alpha}, \\
r_1(t) &= 200 + 30\frac{t^\alpha}{\alpha}, \\
s_2(t) &= 2000 - 72\frac{t^\alpha}{\alpha} - 3.744\frac{t^{2\alpha}}{\alpha^2} + 0.12096\frac{t^{3\alpha}}{\alpha^3}, \\
i_2(t) &= 300 + 42\frac{t^\alpha}{\alpha} + 1.644\frac{t^{2\alpha}}{\alpha^2} - 0.12096\frac{t^{3\alpha}}{\alpha^3}, \\
r_2(t) &= 200 + 30\frac{t^\alpha}{\alpha} + 2.1\frac{t^{2\alpha}}{\alpha^2}, \\
s_3(t) &= 2000 - 72\frac{t^\alpha}{\alpha} - 3.744\frac{t^{2\alpha}}{\alpha^2} + 0.034368\frac{t^{3\alpha}}{\alpha^3} + 0.014437\frac{t^{4\alpha}}{\alpha^4} - 0.000183\frac{t^{5\alpha}}{\alpha^5} - 0.000013\frac{t^{6\alpha}}{\alpha^6} + 2.51 \times 10^{-7}\frac{t^{7\alpha}}{\alpha^7}, \\
i_3(t) &= 300 + 42\frac{t^\alpha}{\alpha} + 1.644\frac{t^{2\alpha}}{\alpha^2} - 0.089168\frac{t^{3\alpha}}{\alpha^3} - 0.014413\frac{t^{4\alpha}}{\alpha^4} + 0.000183\frac{t^{5\alpha}}{\alpha^5} + 0.000013\frac{t^{6\alpha}}{\alpha^6} + 2.51 \times 10^{-7}\frac{t^{7\alpha}}{\alpha^7}, \\
r_3(t) &= 200 + 30\frac{t^\alpha}{\alpha} + 2.1\frac{t^{2\alpha}}{\alpha^2} + 0.0548\frac{t^{3\alpha}}{\alpha^3} - 0.003024\frac{t^{4\alpha}}{\alpha^4} - 0.000013\frac{t^{5\alpha}}{\alpha^5} - 2.51 \times 10^{-7}\frac{t^{6\alpha}}{\alpha^6} + 0.000013\frac{t^{7\alpha}}{\alpha^7}.
\end{align*}

Figure 1: Comparison of VIM solutions with the CDTM solutions of \( S(t) \) for system (10) at different \( \alpha \).
It can be observed that the result of the epidemic system of equation (10) is in complete agreement with the result obtained by the conformable fractional differential transformation method. Figures 1, 2 and 3.

4. Conclusion

In this study, we have found out approximate solutions with two numerical methods for the SIR epidemic model. These methods are based on conformable derivative which is extremely popular in the last years. Firstly, by using the $\alpha$-derivative, we have redefined the conformable differential transformation method (CDTM) and variational iteration method (VIM). Then, we have demonstrated the efficiencies and accuracies of the proposed methods by applying them to the SIR epidemic model. It is found that the approximate solution generated the VIM by our method which is in complete agreement with the corresponding approximate solution CDTM. Besides, in view of their usability, our methods are applicable to many epidemic models SEIR/SEIRS and SIS of fractional order.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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