ON THE CLIQUE NUMBER OF NON-COMMUTING GRAPHS OF CERTAIN GROUPS

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Abstract. Let $G$ be a non-abelian group. The non-commuting graph $\mathcal{A}_G$ of $G$ is defined as the graph whose vertex set is the non-central elements of $G$ and two vertices are joint if and only if they do not commute. In a finite simple graph $\Gamma$ the maximum size of a complete subgraph of $\Gamma$ is called the clique number of $\Gamma$ and it is denoted by $\omega(\Gamma)$. In this paper we characterize all non-solvable groups $G$ with $\omega(\mathcal{A}_G) \leq 57$, where the number 57 is the clique number of the non-commuting graph of the projective special linear group $\text{PSL}(2, 7)$. We also complete the determination of $\omega(\mathcal{A}_G)$ for all finite minimal simple groups.

1. Introduction and results

Let $G$ be a non-abelian group and $Z(G)$ be its center. Following [1] and [11], the non-commuting graph $\mathcal{A}_G$ of $G$ is defined as the graph whose vertex set is $G \setminus Z(G)$ and two distinct vertices $a$ and $b$ are joint whenever $ab \neq ba$. Let $\Gamma$ be a simple graph. The set of vertices of every complete subgraph of $\Gamma$ is called a clique of $\Gamma$. The maximum size (if it exists) of a complete subgraph of $\Gamma$ is called the clique number of $\Gamma$ and it is denoted by $\omega(\Gamma)$. Thus a clique of $\mathcal{A}_G$ is no more than a set of pairwise non-commuting elements of $G$. However, as the following results show, the clique number of the non-commuting graph of a group not only has some influence on the structure of a group but also finding it, is important in some areas such as cohomology ring of a group. By a famous result of Neumann [12] answering a question of P. Erdős, we know that the finiteness of all cliques in $\mathcal{A}_G$ implies the finiteness of the factor group $G/Z(G)$ (and so $\omega(\mathcal{A}_G)$ is finite). In [2, Theorem 1.4], non-solvable groups $G$ satisfying the condition $\omega(\mathcal{A}_G) \leq 21$ are characterized. Specifically, such a group $G$ is isomorphic to $Z(G) \times A_5$, where $A_5$ is the alternating group of degree 5. Also according to [2, Theorem 1.5], the derived length of a non-abelian solvable group $G$ is at most $2\omega(\mathcal{A}_G) - 3$. 

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For a prime number \( p \), a finite \( p \)-group \( G \) is called extra-special if the center, the Frattini subgroup and the derived subgroup of \( G \) all coincide and are cyclic of order \( p \). The clique number of extra-special \( p \)-groups is important as it provides combinatorial information which can be used to calculate their cohomology lengths (The cohomology length of a non-elementary abelian \( p \)-group is a cohomology invariant defined as a result of a Serre’s theorem \([15]\)). Chin \([7]\) has obtained upper and lower bounds for clique numbers of non-commuting graphs of extra-special \( p \)-groups, for odd prime numbers \( p \). Specifically, it is proved in \([7, \text{Theorem 2.2}]\) that if \( G \) is an extra-special group of order \( p^{2n+1} \), then 
\[ \omega(A_{G_n}) = p + 1 \text{ and } np + 1 \leq \omega(A_{G_{n+1}}) \leq \frac{p(p-1)^n - 2}{p - 2}. \]

For \( p = 2 \), it has been shown by Isaacs (see \([3, \text{p. 40}]\)) that \( \omega(A_G) = 2m + 1 \) for any extra-special group \( G \) of order \( 2^{2m+1} \).

Finding the clique number of the non-commuting graph of a group itself is of independent interest as a pure combinatorial problem. Brown in \([5] \) and \([6] \) has studied the clique and chromatic numbers of \( A_{S_n} \), where \( S_n \) is the symmetric group of degree \( n \). It is proved in \([9]\) that \( \omega(A_{S_n}) \neq \chi(A_{S_n}) \) for all \( n \geq 15 \), where \( \chi(\Gamma) \) is the chromatic number of the graph \( \Gamma \). It is easy to see that the chromatic number of \( A_G \) is equal to the minimum number (if it exists) of abelian subgroups of \( G \) whose set-theoretic union is the whole group \( G \).

In \([2]\) the authors have determined non-solvable groups \( G \) with \( \omega(A_G) \leq 21 \), where the number \( 21 \) is the clique number of the non-commuting graph of the least (with respect to the order) non-abelian simple group \( A_5 \). The clique number of the non-commuting graph of \( \text{PSL}(2, 7) \) (which is the second-least order non-abelian simple group) is \( 57 \). Here we give a characterization of non-solvable groups \( G \) with \( \omega(A_G) \leq 57 \).

**Theorem 1.1.** Let \( G \) be a finite non-solvable group such that \( \omega(A_G) \leq 57 \). Then \( G \) has one of the following structures

1. \( G \cong Z(G) \times \text{PSL}(2, p) \), where \( p \in \{5, 7\} \); 
2. \( G = Z(G)K \), where \( K \) is a subgroup of \( G \) isomorphic to \( \text{SL}(2, p) \) and \( p \in \{5, 7\} \); 
3. \( G = G''(a)S \), where \( a^2 \in Z(G) \) and \( G'' \cong A_5 \) or \( \text{SL}(2, 5) \), and \( S \) is the solvable radical of \( G \); 
4. \( G = G''(a)Z(G) \), where \( a^2 \in Z(G) \) and \( G'' \cong \text{PSL}(2, 7) \) or \( \text{SL}(2, 7) \). 

In \([1, \text{Lemma 4.4}]\), the clique numbers of \( A_G \) of all projective special linear groups \( G = \text{PSL}(2, q) \) have been obtained. A family of the minimal simple groups
CLIQUE NUMBER

(i.e. finite non-abelian simple groups all of whose proper subgroups are solvable) are projective special linear groups of degree 2 over a finite field. All minimal simple groups were completely classified by a well-known result of Thompson [17]. In Section 3, we shall find the clique number of $A_G$ for the remaining finite minimal simple groups $G$ i.e., the Suzuki groups $Sz(2^{2m+1})$ and the projective special linear group $PSL(3,3)$ over the field with 3 elements. As Thompson’s classification of the minimal simple groups is a very useful tool to obtain solvability criteria in the class of finite groups (see [18] for a recent and interesting application of Thompson’s theorem), we hope that these new information might be useful to obtain new solvability criterion.

A clique of a graph $\Gamma$ is called a maximum clique if its size is $\omega(\Gamma)$. We say that a clique $X$ of a graph can be extended to a maximum clique if there exists a maximum clique containing $X$. We will prove that every clique of $A_G$ for every minimal simple group $G$ except $PSL(3,3)$ can be extended to a maximum clique of $A_G$ (see Proposition 2.6 and Theorem 1.2).

**Theorem 1.2.** Let $G = Sz(q)$ ($q = 2^{2m+1}$ and $m > 0$) be the Suzuki group over the field with $q$ elements (see [10, p. 182]). Then

1. $\omega(Sz(q)) = (q^2 + 1)(q - 1) + \frac{q^2(q^2+1)}{2} + \frac{q^2(q^2+1)(q-1)}{4(q+2r+1)} + \frac{q^2(q^2+1)(q-1)}{4(q-2r+1)}$, where $r = 2^m$.
2. Every clique of $A_G$ can be extended to a maximum clique of $A_G$.

**Theorem 1.3.** $\omega(A_{PSL(3,3)}) = 1067$.

We use the usual notation: for example $C_G(a)$ is the centralizer of an element $a$ in a group $G$, $N_G(H)$ is the normalizer of a subgroup $H$ in $G$, $GL(n,q)$, $SL(n,q)$, $PGL(n,q)$ and $PSL(n,q)$ denote respectively, the general linear group, the special linear group, the projective general linear group, and the projective special linear group of degree $n$ over the finite field of order $q$, and $D_{2n}$ is the dihedral group of order $2n$. A family $\{G_1, \ldots, G_k\}$ of proper subgroups of a group $G$ is called a partition of $G$ if every non-identity element of $G$ belongs to exactly one of the $G_i$’s.

2. **Proofs**

To prove Theorem 1.3 we need the following lemmas.

**Lemma 2.1.** Let $G$ be a finite non-abelian group.

(i) For any non-abelian subgroup $H$ of $G$, $\omega(A_H) \leq \omega(A_G)$.

(ii) For any non-abelian factor group $G/N$ of $G$, $\omega(A_{G/N}) \leq \omega(A_G)$.

**Proof.** It is straightforward. □
Lemma 2.2. $\omega(A_{PGL(2,q)}) = \begin{cases} 4 & \text{if } q = 2 \\ 10 & \text{if } q = 3 \\ q^2 + q + 1 & \text{if } q > 3 \end{cases}$

Proof. By Lemma 2.1, we have $\omega(A_{PSL(2,q)}) \leq \omega(A_{PGL(2,q)}) \leq \omega(A_{GL(2,q)})$.

Now, if $q > 5$ or $q = 4$, then by [1, Lemma 4.4] $q^2 + q + 1 = \omega(A_{PSL(2,q)}) \leq \omega(A_{PGL(2,q)}) \leq \omega(A_{GL(2,q)}) = q^2 + q + 1$.

Thus $\omega(A_{PGL(2,q)}) = q^2 + q + 1$, for $q = 4$ and $q > 5$. Also since $PGL(2,5) \cong S_5$ and $PGL(2,3) \cong S_4$ it follows from [5, p. 2] that $\omega(A_{PGL(2,5)}) = 31$, $\omega(A_{PGL(2,3)}) = 10$ and as $PGL(2,2) \cong PSL(2,2)$, by [1] Lemma 4.4 we have that $\omega(A_{PGL(2,2)}) = 4$. This completes the proof. □

Theorem 2.3. Let $G$ be a non-abelian simple group such that $\omega(A_G) \leq 57$. Then $G \cong A_5$ or $G \cong PSL(2,7)$.

Proof. By Neumann’s result [12], $G/Z(G)$ is finite and since $G$ is a non-abelian simple group, we have that $Z(G) = 1$. Thus $G$ is finite. Suppose that the result is false, and let $M$ be a minimal counter example. Thus every proper non-abelian simple section of $M$ is isomorphic to $A_5$ or $PSL(2,7)$. By [4] Proposition 4] $M$ is isomorphic to one of the following:

- PSL(2,2$^m$), $m = 4$ or a prime;
- PSL(2,3$^p$), PSL(2,5$^p$), PSL(2,7$^p$), $p$ a prime;
- PSL(2,$p$), $p > 7$;
- PSL(3,3), PSL(3,5), PSL(3,7);
- PSU(3,3), PSU(3,4), PSU(3,7) (the projective special unitary group of degree 3 over the finite field of order 3,4 and 7 respectively) or $Sz(2^p)$, $p$ an odd prime.

Now, for every prime number $p$ and every integer $n \geq 0$, by [1] Lemma 4.4], $\omega(A_{PSL(2,p^n)}) = p^{2n} + p^n + 1$. Thus since $PSL(2,2^2) \cong A_5$, among the projective special linear groups, we only need to investigate $PSL(3,3)$, $PSL(3,5)$ and $PSL(3,7)$.

For each prime divisor $p$ of $|G|$, let $\nu_p(G)$ be the number of Sylow $p$-subgroups of $G$.

If $p$ is a prime number dividing $|G|$ such that the intersection of any two distinct Sylow $p$-subgroups is trivial, then by [8] Lemma 3, we must have $\nu_p(G) \leq 57 (*)$.

Now $PSL(3,3)$ has order $2^4 \times 3^3 \times 13$, so $\nu_{13}(PSL(3,3)) > 57$.

$PSL(3,5)$ has order $2^5 \times 3 \times 5^3 \times 31$, so $\nu_{31}(PSL(3,5)) > 57$.

$PSL(3,7)$ has order $2^5 \times 3 \times 7^3 \times 19$, so $\nu_{19}(PSL(3,7)) > 57$. 
Proposition 2.4. Let $G = \text{PGL}(2, q)$, where $q$ is a power of a prime number $p$ and let $k = \gcd(q - 1, 2)$. Then

1. A Sylow $p$-subgroup $P$ of $G$ is an elementary abelian group of order $q$ and the number of Sylow $p$-subgroups of $G$ is $q + 1$.
2. $G$ contains a cyclic subgroup $D$ of order $q - 1$ such that the number of conjugates of $D$ is $\frac{q(q+1)}{2}$.
3. $G$ contains a cyclic subgroup $I$ of order $q + 1$ such that the number of conjugates of $I$ is $\frac{q(q-1)}{2}$.
4. The set $\{P^x, D^x, I^x \mid x \in G\}$ is a partition for $G$. If $q$ is odd, then the following hold for non-trivial elements $a \in D$ and $b \in P$.
   (a) If $a$ is not of order 2, then $C_G(a) = D$.
   (b) If $a$ is of order 2, then $C_G(a) \cong D_{2(q-1)}$.
   (c) $C_G(b) = P$.
5. If $q \equiv 0 \pmod{4}$, then $G = \text{PGL}(2, q) \cong \text{PSL}(2, q)$ and by Proposition 3.21 of [1], if $a$ is a non-trivial element of $G$, then

$$C_G(a) = \begin{cases} P^x & \text{if } a \in P^x \\ D^x & \text{if } a \in D^x \\ I & \text{if } a \in I^x \end{cases}$$

Proof. The proof follows from the results in Chapter II of [1] concerning projective linear groups. \hfill \square

A group $G$ is called an AC-group if the centralizer of every non-central element is abelian.

Lemma 2.5. Let $G$ be a non-abelian AC-group such that $\omega(\mathcal{A}_G)$ is finite. Then every non-empty clique of $\mathcal{A}_G$ can be extended to a maximum clique set of $G$.

Proof. Let $\omega(\mathcal{A}_G) = n$. Then there exist elements $a_1, \ldots, a_n$ in $G$ such that $[a_i, a_j] \neq 1$, for all $i \neq j$. Thus $G = C_G(a_1) \cup \cdots \cup C_G(a_n)$ and also $C_G(a_i) \cap C_G(a_j) = Z(G)$, since $G$ is an AC-group. Therefore $\{C_G(a_i) \setminus Z(G) \mid i = 1, \ldots, n\}$ is a partition for $G \setminus Z(G)$. Let $X$ be a clique of $\mathcal{A}_G$. Then, for each $i$, $1 \leq i \leq m$, $\omega(X) = a_i$.
proof of Proposition 3.21 of [1]). For \( q = 5 \), it is not hard to see that \( q \) of order \( a \in \mathbb{Z} \) is a cyclic subgroup of order \( 2^k \), where \( k = \gcd(q - 1, 2) \). Let \( a \) be a non-trivial element of \( G \). Then \( a \in M \), for some \( M \in \mathcal{P} \). Now take an arbitrary non-trivial element \( b_N \) in each member \( N \in \mathcal{P} \) which is different from \( M \). Let \( X \) be such a set of elements. For \( q > 5 \), it is not hard to see that \( \{a\} \cup X \) is a maximum clique set for \( \mathcal{A}_G \) (see the proof of Proposition 3.21 of [1]). For \( q \leq 5 \), see the proof of Proposition 3.21 of [1].

**Proposition 2.6.** Let \( G = \text{PSL}(2, q) \) or \( \text{PGL}(2, q) \), where \( q \) is a power of a prime \( p \). Then any singleton containing a non-central element of \( G \) can be extended to a maximum clique set of \( \mathcal{A}_G \).

**Proof.** We give only the proof for \( G = \text{PSL}(2, q) \), for the other group the proof is similar and Lemma 2.4 may be used in the proof.

By [1] Proposition 3.21, \( \mathcal{P} = \{P^x, A^x, B^x | x \in G\} \) is a partition for \( G \), where \( P \) is a Sylow \( p \)-subgroup, \( A \) is a cyclic subgroup of order \( \frac{2^k - 1}{k} \) and \( B \) is a cyclic subgroup of order \( \frac{2^k - 1}{k} \), where \( k = \gcd(q - 1, 2) \). Let \( a \) be a non-trivial element of \( G \). Then \( a \in M \), for some \( M \in \mathcal{P} \). Now take an arbitrary non-trivial element \( b_N \) in each member \( N \in \mathcal{P} \) which is different from \( M \). Let \( X \) be such a set of elements. For \( q > 5 \), it is not hard to see that \( \{a\} \cup X \) is a maximum clique set for \( \mathcal{A}_G \) (see the proof of Proposition 3.21 of [1]). For \( q \leq 5 \), see the proof of Proposition 3.21 of [1].

**Theorem 2.7.** Let \( G \) be a semi-simple group, such that \( \omega(\mathcal{A}_G) \leq 57 \). Then \( G \cong \text{A}_5, \text{S}_5, \text{PSL}(2, 7) \) or \( \text{PGL}(2, 7) \).

**Proof.** By Neumann’s result [12], \( G \) is finite, since in a semi-simple group the center is trivial. Let \( R \) be the centerless CR-Radical of \( G \). Then \( R \) is a direct product of a finite number of finite non-abelian simple groups, say \( R \cong S_1 \times \ldots \times S_m \). By Lemma 2.1 for each \( i \in \{1, \ldots, m\} \), \( \omega(S_i) \leq 57 \). Now by Theorem 2.3, for each \( i \in \{1, \ldots, m\} \), \( S_i \cong \text{A}_5 \) or \( S_i \cong \text{PSL}(2, 7) \). Since \( \omega(\mathcal{A}_{S_i}) \leq 21 \), it follows from [2] Lemma 2.2 that \( m = 1 \). Therefore \( R \cong \text{A}_5 \) or \( R \cong \text{PSL}(2, 7) \). We know that \( C_G(R) = 1 \) and so \( G \) is embedded into \( \text{Aut}(R) \). If \( R \cong \text{A}_5 \), \( \text{Aut}(R) \cong \text{S}_5 \) and so \( G \cong \text{A}_5 \) or \( G \cong \text{S}_5 \); if \( R \cong \text{PSL}(2, 7) \), then \( \text{Aut}(R) \cong \text{PGL}(2, 7) \) and \( G \cong \text{PSL}(2, 7) \) or \( G \cong \text{PGL}(2, 7) \). This completes the proof.

For a finite group \( G \), \( \text{Sol}(G) \) denotes the solvable radical of \( G \), i.e., the largest normal solvable subgroup of \( G \).

**Corollary 2.8.** Let \( G \) be a finite group such that \( \omega(\mathcal{A}_{\text{Sol}(G)}) = 57 \). Then \( \frac{G}{\text{Sol}(G)} \cong \text{PSL}(2, 7) \) or \( \frac{G}{\text{Sol}(G)} \cong \text{PGL}(2, 7) \).

**Proof.** Since for any finite group \( M \), \( M/\text{Sol}(M) \) has no non-trivial and proper normal abelian subgroup, the proof follows from Theorem 2.7.
Lemma 2.9. Let $G$ be a finite non-solvable group such that $\omega(A_G) \leq 57$ and $\frac{G}{Z(G)} \cong A_5$. Then $G \cong Z(G) \times A_5$ or $G = Z(G)\text{SL}(2, 5)$.

Proof. Let $S = \text{Sol}(G)$. Suppose that $C_G(S) = G$. Thus $S \leq Z(G)$ and so $S = Z(G)$. Now, consider the central extension $Z(G) \rightarrow G \rightarrow \frac{G}{Z(G)}$. By a similar argument as in [2 Lemma 4.2], we have that $K = G' \cap Z(G)$ is of order no more than 2, $G = G'Z(G)$ and $\frac{G}{K} \cong A_5$. Thus [2 Lemma 4.2] implies that there is a subgroup $L$ of $G'$ such that $G' = K \times L$ and $L \cong A_5$ or $G' \cong \text{SL}(2, 5)$. Therefore $G = G'Z(G) = LKZ(G) = LZ(G)$ and it is clear that $L \cap Z(G) = 1$ or $G = G'Z(G) \cong \text{SL}(2, 5)Z(G)$. Thus $G \cong A_5 \times Z(G)$ or $G = Z(G)\text{SL}(2, 5)$.

Now suppose that $C_G(S)$ is a proper (normal) subgroup of $G$. If $C_G(S)$ is solvable, $C_G(S) \leq S$. Now by [2 Remark 2.9], $\frac{G}{S} = \bigcup_{i=1}^{21} P_i$, where $P_1, \ldots, P_{21}$ are all the Sylow subgroups of $\frac{G}{S}$. Assume that $P_1, \ldots, P_{10}$ are Sylow 3-subgroups, $P_{11}, \ldots, P_{17}$ are Sylow 5-subgroups, and $P_{18}, \ldots, P_{21}$ are Sylow 2-subgroups of $G$. Now if we choose any element $a_iS \in P_i \setminus \{1\}$ ($i = 1, \ldots, 21$), then the set $\{a_1S, \ldots, a_{21}S\}$ is a maximum clique set for $\frac{G}{S}$ and $P_i = C_{\frac{G}{S}}(a_iS)$. For all $i \in \{1, \ldots, 10\}$, $|a_iS| = 3$ and for $i \in \{11, \ldots, 17\}$, $|a_iS| = 5$. Thus $C_{\frac{G}{S}}(a_iS) = \frac{(a_iS)^2}{S} = \langle a_iS \rangle$. Since $a_i \not\in S$ and for $i \in \{1, \ldots, 17\}$, $|a_iS|$ is prime, $a_i \not\in C_G(S)$ for each $i \in \{1, \ldots, 17\}$. Thus there exists $s_i \in S$ such that $a_is_i \not= s_ia_i$ for each $i \in \{1, \ldots, 17\}$. It is now easy to see that the set $\{a_is_i, a_i^2s_i \mid i = 1, \ldots, 10\} \cup \{a_ja_js_j, a_j^2s_j, a_j^3s_j, a_j^4s_j \mid j = 11, \ldots, 17\}$ is a clique set of $A_G$. It follows that $\omega(A_G) \geq 65$ which is a contradiction.

Now suppose that $C_G(S)$ is not solvable. Thus $\frac{C_G(S)S}{S}$ is not solvable and so $C_G(S)S = G$. Let $N$ be a non-solvable subgroup of $C_G(S)$ of the least order. It follows that $NS = G$,

\[
\frac{N}{N \cap S} \cong \frac{NS}{S} = \frac{G}{S} \cong A_5,
\]

$\text{Sol}(N) = N \cap S$ and every proper subgroup of $N$ is solvable.

If $\text{C}_N(\text{Sol}(N)) = N$, then $Z(N) = \text{Sol}(N)$. By the first part of the proof, $N = Z(N) \times A_5$ or $N = Z(N)\text{SL}(2, 5)$ which imply that $G = SN = S \times A_5$ or $G = \text{SSL}(2, 5)$, respectively. If $G = S \times A_5$, then by [2 Lemma 2.2], $S$ is abelian. It follows that $G = \text{SC}_{C_G}(S) = C_G(S)$, a contradiction, as we are assuming $G \not= C_G(S)$.

Therefore $G = \text{SSL}(2, 5)$. Let $\{s_1, s_2, s_3\}$ be a clique of $A_S$ and $\{b_1Z, b_2Z, \ldots, b_{21}Z\}$ be a (maximum) clique set of $\frac{\text{SL}(2, 5)}{Z} \cong A_5$, where $Z = Z(\text{SL}(2, 5))$. Then $[b_i, b_j] \not\in Z$, whenever $i \not= j$ and $i, j \in \{1, 2, \ldots, 21\}$. Now $(b_is_i)(b_js_k) = (b_js_k)(b_is_r)$ if and only if $[b_i, b_j] = [s_r^{-1}, s_i^{-1}] \in S' \cap \text{SL}(2, 5) \subseteq Z$, where $i, j \in \{1, 2, \ldots, 21\}$ and $r, k \in \{1, 2, 3\}$. It follows that $\{b_1s_i, b_2s_i, b_3s_i \mid i = 1, 2, 3\}$ is a clique set for
homomorphism $\delta : M(G/B) \to B$ so that $\text{Im} \, \delta = G' \cap B$, where $M(G/B)$ is the Schur multiplier of $G/B$ (see [14] page 354, Exercise 10). On the other hand, we know that the Schur multiplier of $\text{PSL}(2,7)$ is $\mathbb{Z}_2$. Hence $G' \cap B = B$ and so $B \leq G'$. It follows that $G$ is a perfect group of order 336. It is well-known that the only perfect group of order 336 is $\text{SL}(2,7)$. This completes the proof. \qed
Lemma 2.13. Let \( G \) be a finite non-solvable group such that \( \omega(A_G) \leq 57 \) and \( \frac{G}{\mathbb{Z}(G)} \cong \text{PSL}(2,7) \). Then \( G \cong \text{Z}(G) \times \text{PSL}(2,7) \) or \( G \cong \text{Z}(G)\text{SL}(2,7) \).

Proof. Since \( 57 = \omega(A_G) \leq \omega(A_G) \leq 57 \), we have \( \omega(A_G) = \omega(A_G) = 57 \). By Lemma 2.11 \( S = \text{Z}(G) \) and since \( \frac{G}{\mathbb{Z}(G)} \cong \text{PGL}(2,7) \) and it follows from Lemma 2.13 that \( \text{PSL}(2,7) \). Thus Lemma 2.12 implies that there is a subgroup \( L \) of \( G' \) such that \( G' = K \times L \) or \( G' \cong \text{SL}(2,7) \) and \( L \cong \text{PSL}(2,7) \). Now if \( G' = K \times L \), then \( G = G'\text{Z}(G) = KL\text{Z}(G) = L\text{Z}(G) \) and it is clear that \( L \cap \text{Z}(G) = 1 \). So \( G = L \times \text{Z}(G) \cong \text{PSL}(2,7) \times \text{Z}(G) \). Otherwise \( G' \cong \text{SL}(2,7) \), and so \( G \cong \text{Z}(G)\text{SL}(2,7) \). \( \square \)

Lemma 2.14. Let \( G \) be a finite non-solvable group such that \( \omega(A_G) \leq 57 \) and \( \frac{G}{\mathbb{Z}(G)} \cong \text{PGL}(2,7) \). Then \( G = G'^{(a)}\text{Z}(G) \), where \( a^2 \in \text{Z}(G) \) and \( G'^{(a)} \cong \text{PSL}(2,7) \) or \( G'^{(a)} \cong \text{SL}(2,7) \).

Proof. Since \( 57 = \omega(A_G) \leq \omega(A_G) \leq 57 \), we have \( \omega(A_G) = \omega(A_G) = 57 \). By Lemma 2.11 \( S = \text{Z}(G) \) and so \( \frac{G}{\mathbb{Z}(G)} \cong \text{PGL}(2,7) \) and it follows from Lemma 2.13 that \( \frac{G'}{\mathbb{Z}(G)} \cong \text{PSL}(2,7) \) or \( G'\mathbb{Z}(G) = \text{Z}(G)\text{SL}(2,7) \) and \( |\frac{G}{\mathbb{Z}(G)} : \frac{G'\mathbb{Z}(G)}{\mathbb{Z}(G)}| = 2 \). Thus \( G'^{(a)} \cong \text{PSL}(2,7) \) or \( G'^{(a)} \cong \text{SL}(2,7) \). Suppose that \( a\mathbb{Z}(G) \) is an element of \( \frac{G}{\mathbb{Z}(G)} \) \( \frac{G'\mathbb{Z}(G)}{\mathbb{Z}(G)} \) of order 2. Then \( G = G'^{(a)}\text{Z}(G) \), where \( a^2 \in \text{Z}(G) \) and \( G'^{(a)} \cong \text{PSL}(2,7) \) or \( G'^{(a)} \cong \text{SL}(2,7) \). \( \square \)

Proof of Theorem 1.1. This follows from Lemmas 2.8, 2.11, 2.13 and 2.14.

3. Clique Numbers of the Non-Commuting Graphs of the Minimal Simple Groups

For a non-trivial abelian group \( A \), we define \( \omega(A_A) = 1 \).

Lemma 3.1. Let \( G \) be a group such that there exist non-trivial subgroups \( A_1, \ldots, A_n \) of \( G \) with \( G = \bigcup_{i=1}^{n} A_i \) and \( A_i \cap A_j = \text{Z}(G) \) for \( i \neq j \).

1. If \( C_G(g) \leq A_i \) for all \( g \in A_i \text{Z}(G) \), then \( \omega(A_G) = \sum_{i=1}^{n} \omega(A_{A_i}) \).

2. If every clique of \( A_{A_i} \) can be extended to a maximum clique of \( A_{A_i} \) for each \( i \in \{1, \ldots, n\} \), then the same property for \( A_G \) is true. In particular, if all \( A_i \)'s are either abelian or AC-groups, the mentioned property holds for \( A_G \).

Proof. (1) If \( X \) is any clique of \( A_G \), then \( X = \bigcup_{i=1}^{n} X_i \), where \( X_i \subset A_i \text{Z}(G) \) for each \( i \in \{1, \ldots, n\} \). By hypothesis, \( |X| = \sum_{i=1}^{n} |X_i| \) and since \( |X_i| \leq \omega(A_{A_i}) \), it follows that \( |X| \leq \sum_{i=1}^{n} \omega(A_{A_i}) \). Now let \( W_i \) be a maximum clique of \( A_{A_i} \) for each \( i \in \{1, \ldots, n\} \). We claim that \( W = \bigcup_{i=1}^{n} W_i \) is a maximum clique for \( A_G \). Suppose, for
a contradiction, that there exist two distinct commuting elements \( a \) and \( b \) in \( \bigcup_{i=1}^{n} W_i \). Thus there exist \( i \neq j \) such that \( a \in A_i \) and \( b \in A_j \). Therefore \( a, b \in C_G(b) \), and so \( a \in A_i \cap A_j = Z(G) \), which is impossible. Since \( |W| = \sum_{i=1}^{n} \omega(A_{A_i}) \), the proof of (1) is complete.

(2) It is straightforward.

\[ \square \]

**Proof of Theorem 1.2** (i) The Suzuki group \( G \) contains subgroups \( F, A, B \) and \( C \) such that \( |F| = q^2, |A| = q - 1, |B| = q - 2r + 1 \) and \( |C| = q + 2r + 1 \) (see [10], Chapter XI, Theorems 3.10 and 3.11). Also by [10] pp. 192-193, Theorems 3.10 and 3.11, the conjugates of \( A, B, C \) and \( F \) in \( G \) form a partition for \( G \), and \( A, B, C \) are cyclic. These subgroups are all centralizers of some elements in \( G \) and \( F \) is a Sylow \( 2 \)-subgroup of \( G \).

Now [10] Chapter XI, Theorems 3.10 and 3.11 implies that the number of conjugates of \( C, B, A \) and \( F \) in \( G \) are respectively, \( \alpha = \frac{q^2(q^2(q^2+1)}{4(q+2r+1)}, \beta = \frac{q^2(q^2(q^2+1)}{4(q-2r+1)}, \gamma = \frac{q^2(q^2+1)}{2} \) and \( \delta = q^2 + 1 \) and also

\[ G = \bigcup_{i=1}^{\beta} C_G(f_i) \bigcup_{i=1}^{\gamma} C_G(a_i) \bigcup_{i=1}^{\beta} C_G(b_i) \bigcup_{i=1}^{\gamma} C_G(c_i). \]

Now by [10] Chapter XI, proof of Lemma 5.9, \( |C_F(g) : Z(F)| = 2 \), for all \( g \in F \setminus Z(F) \). If \( C_F(g) = H \), then \( |Z(F)F| = 2 \) which implies that \( H \) is abelian. It follows that \( F \) is an AC-group. Let \( \{a_1,a_2,\ldots,a_n\} \) be a clique of \( A_F \). Then \( F = C_F(a_1) \cup \cdots \cup C_F(a_n) \) and the set \( \{C_F(a_i) \mid i = 1,2,\ldots,n\} \) forms a partition for \( F \). Thus \( \omega(A_F) = q - 1 \). Now it follows from Lemma 3.11 that

\[ \omega(A_{S_2(q)}) = (q^2 + 1)(q - 1) + \frac{q^2(q^2+1)}{2} + \frac{q^2(q^2+1)(q - 1)}{4(q+2r+1)} + \frac{q^2(q^2+1)(q - 1)}{4(q-2r+1)}. \]

(ii) It follows from Lemma 3.1 and the proof of part (i).

\[ \square \]

**Proof of Theorem 1.3** Let \( G = \text{PSL}(3, 3) \). It is easy to see (e.g., by GAP [16]) that the set of order elements of \( G \) is \( \{1,2,3,4,6,8,13\} \) and if \( A = \{C_G(g) \mid g \in G, |C_G(g)| = 6\}, B = \{C_G(g) \mid g \in G, |C_G(g)| = 8\}, C = \{C_G(g) \mid g \in G, |C_G(g)| = 9\} \) and \( D = \{C_G(g) \mid g \in G, |C_G(g)| = 13\} \), then \( |A| = 468, |B| = 351, |C| = 104 \) and \( |D| = 144 \). Also we know that if \( |C_G(g)| \in \{6,8\}, then \( C_G(g) \) is a cyclic subgroup of \( G \) and so there exists \( a \in G \) such that \( C_G(g) = \langle a \rangle \).

It follows that \( \langle a \rangle = C_G(a) = C_G(g) \). Thus there exist elements \( a_i, b_j, c_k, d_l \in G \) such that \( |C_G(a_i)| = 6 \) for \( 1 \leq i \leq 468, |C_G(b_j)| = 8 \) for \( 1 \leq j \leq 351, |C_G(c_k)| = 9 \) for \( 1 \leq k \leq 104 \) and \( |C_G(d_i)| = 13 \) for \( 1 \leq i \leq 144 \). Now it is easy to see (e.g., by
GAP [10] that
\[ G = \bigcup_{x \in X} C_G(x), \]
where \( X = \{a_1, \ldots, a_{468}, b_1, \ldots, b_{351}, c_1, \ldots, c_{104}, d_1, \ldots, d_{144}\} \). Since the set of order elements of \( G \) is \( \{1, 2, 3, 4, 6, 8, 13\} \), it follows that \( X \) is a clique for \( A_G \). Also since for all \( x \in X, C_G(x) \) is abelian, we have \( \omega(A_G) = |X| = 468+351+104+144 = 1067 \). This completes the proof. □

**Remark 3.2.** It is not true that every clique of the non-commuting graph of \( G = PSL(3, 3) \) can be extended to a maximum clique. It can be seen that there are two distinct elements \( x_1, x_2 \in X \) such that \( C_G(x_1) \cap C_G(x_2) \) contains a non-trivial element \( a \). Now \( \{a\} \) cannot be extended to a maximum clique. On the other hand it is easy to see that every clique containing only elements of orders in \( \{6, 8, 13\} \) can be extended to a maximum clique. We leave the easy proof to the reader.

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