ON GENERALIZATION OF BAILEY'S IDENTITY INVOLVING PRODUCT OF GENERALIZED HYPERGEOMETRIC SERIES

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Abstract. The aim of this research paper is to obtain explicit expressions of

(i) \( _1F_1 \left[ \frac{\alpha}{2\alpha + i}; x \right] \cdot _1F_1 \left[ \frac{\beta}{2\beta + j}; x \right] \)

(ii) \( _1F_1 \left[ \frac{\alpha}{2\alpha - i}; x \right] \cdot _1F_1 \left[ \frac{\beta}{2\beta - j}; x \right] \)

and

(iii) \( _1F_1 \left[ \frac{\alpha}{2\alpha + i}; x \right] \cdot _1F_1 \left[ \frac{\beta}{2\beta - j}; x \right] \)

in the most general form for any \( i, j = 0, 1, 2, \cdots \).

For \( i = j = 0 \), we recover well known and useful identity due to Bailey. The results are derived with the help of a well known Bailey’s formula involving products of generalized hypergeometric series and generalization of Kummer’s second transformation formulas available in the literature. A few interesting new as well as known special cases have also been given.

1. Introduction

In the theory of hypergeometric series and generalized hypergeometric series, classical summation theorems such as those of Gauss, Gauss’s second, Kummer and Bailey for the series \( _2F_1 \); Watson, Dixon, Whipple and Saalshütz for the series \( _3F_2 \) and others play a key role. Applications of these classical summation theorems are well known now.

In a very popular, useful and interesting research paper, Bailey [1] had obtained a large number of known as well as new results involving products of generalized hypergeometric series.

We recall here the following well known and useful transformation formulas due to Kummer[5]

\[ e^{-x} _1F_1 \left[ \frac{\alpha}{\beta}; x \right] = _1F_1 \left[ \frac{\beta - \alpha}{\beta}; -x \right] \] (1.1)

and

\[ e^{-\frac{x^2}{4}} _1F_1 \left[ \frac{\alpha}{2\alpha}; x \right] = _0F_1 \left[ \frac{\alpha}{\alpha + \frac{1}{2}}; \frac{x^2}{16} \right]. \] (1.2)

Bailey[1] derived (1) by employing the classical Gauss’s summation theorem[2,4,12,13]:

\[ _2F_1 \left[ \frac{a, b}{c}; 1 \right] = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)} \] (1.3)

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identities can be re-written in the following forms:

\[ 2F1 \left[ \frac{a, b}{4(a + b + 1)} ; \frac{1}{2} \right] = \frac{\Gamma \left( \frac{1}{2} \right) \Gamma \left( \frac{1}{2}a + \frac{1}{2}b + \frac{1}{2} \right)}{\Gamma \left( \frac{1}{2}a + \frac{1}{2} \right) \Gamma \left( \frac{1}{2}b + \frac{1}{2} \right)} \]  \tag{1.4}

In 1998, Rathie and Choi\[11\] derived the result (1.2) which is equivalent to

\[ 1F1 \left[ \frac{\alpha}{2}; 2x \right] = e^{x} 0F1 \left[ \frac{-x^2}{4} \right] \]  \tag{1.5}

by utilizing Gauss’s second summation theorem (1.4).

On the other hand, from the theory of differential equations, Preece \[6\] obtained the following very well known and useful identity involving product of the generalized hypergeometric series:

\[ 1F1 \left[ \frac{\alpha}{2}, x \right] \cdot 1F1 \left[ \frac{\alpha}{2}, -x \right] = 1F2 \left[ \frac{\alpha + 1}{2}, 2\alpha ; \frac{x^2}{4} \right] \]  \tag{1.6}

In 1928, Bailey generalized the Preece’s identity (1.6) in the form

\[ 1F1 \left[ \frac{\alpha}{2}, x \right] \cdot 1F1 \left[ \frac{\alpha}{2}, -x \right] = 2F3 \left[ \frac{\alpha + 1}{2}, \frac{\alpha + 1}{2}, \frac{\alpha + 1}{2}, \frac{\alpha + 1}{2}, x^2 \right] \]  \tag{1.7}

by employing the following classical Watson’s summation theorem [2]

\[ 3F2 \left[ \frac{1}{2}, \frac{1}{2}, \frac{1}{2}; \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right] = \frac{\Gamma \left( \frac{1}{2} \right) \Gamma \left( \frac{1}{2}a + \frac{1}{2}b + \frac{1}{2} \right) \Gamma \left( \frac{1}{2}a - \frac{1}{2}b + \frac{1}{2} \right)}{\Gamma \left( \frac{1}{2}a + \frac{1}{2} \right) \Gamma \left( \frac{1}{2}b + \frac{1}{2} \right) \Gamma \left( \frac{1}{2}b - \frac{1}{2} \right)} \]  \tag{1.8}

provided \( \Re(2c - a - b) > -1 \).

It is interesting to mention here that the Bailey identity (1.7) reduces to the Preece’s identity (1.6) by taking \( \beta = \alpha \).

Further, if we use Kummer’s first transformation (1.1), the Preece’s and Bailey’s identities can be re-written in the following forms:

\[ \left\{ 1F1 \left[ \frac{\alpha}{2}, x \right] \right\}^2 = e^x 1F2 \left[ \frac{\alpha}{2}, 2\alpha ; \frac{x^2}{4} \right] \]  \tag{1.9}

and

\[ 1F1 \left[ \frac{\alpha}{2}, x \right] \cdot 1F1 \left[ \frac{\beta}{2}, -x \right] = e^x 2F3 \left[ \frac{\alpha + 1}{2}, \frac{\alpha + 1}{2}, \frac{\alpha + 1}{2}, x^2 \right] \]  \tag{1.10}

In 1997, Rathie\[9\] gave a very short proof of Preece’s identity (1.9) by using Kummer’s second transformation (1.2) and the following Bailey’s product formula [1]

\[ 0F1 \left[ \frac{-}{\rho}, x \right] \cdot 0F1 \left[ \frac{-}{\sigma}, x \right] = 2F3 \left[ \frac{\rho + \sigma}{2}, \frac{\rho + \sigma - 1}{2}, \rho, \rho + \sigma - 1 ; 4x \right] \]  \tag{1.11}
and obtained the following two results closely related to (1.9)

\[ 1F_1 \left[ \frac{\alpha}{2\alpha} ; x \right] \cdot 1F_1 \left[ \frac{\alpha}{2\alpha + 1} ; x \right] = e^x \left\{ 1F_2 \left[ \frac{\alpha}{\alpha + \frac{1}{2} 2\alpha} ; \frac{x^2}{4} \right] - \frac{x}{2(2\alpha + 1)} 1F_2 \left[ \frac{\alpha + 1}{\alpha + \frac{1}{2} 2\alpha + 1} ; \frac{x^2}{4} \right] \right\} \] (1.12)

and

\[ 1F_1 \left[ \frac{\alpha}{2\alpha} ; x \right] \cdot 1F_1 \left[ \frac{\alpha}{2\alpha - 1} ; x \right] = e^x \left\{ 1F_2 \left[ \frac{\alpha}{\alpha + \frac{1}{2} 2\alpha - 1} ; \frac{x^2}{4} \right] + \frac{x}{2(2\alpha - 1)} 1F_2 \left[ \frac{\alpha}{\alpha + \frac{1}{2} 2\alpha} ; \frac{x^2}{4} \right] \right\} \] (1.13)

In 1998, Choi and Rathie established a very short proof of Bailey’s result (1.10) by the same technique developed by Rathie and obtained a few results closely related to (1.10).

Very recently, Choi and Rathie [10] established the generalization of Preece’s identity (1.9) and obtained explicit expressions of

(i) \( \left\{ 1F_1 \left[ \frac{\alpha}{2\alpha + i} ; x \right] \right\}^2 \) .

(ii) \( \left\{ 1F_1 \left[ \frac{\alpha}{2\alpha - i} ; x \right] \right\}^2 \) .

and

(iii) \( 1F_1 \left[ \frac{\alpha}{2\alpha + i} ; x \right] \cdot 1F_1 \left[ \frac{\alpha}{2\alpha - i} ; x \right] \) (1.16)

in the most general case for any \( i = 0, 1, 2 \ldots \)

These results have been obtained with the help of the following results recorded in [7].

\[ e^{-\frac{1}{2}x} 1F_1 \left[ \frac{\alpha}{2\alpha + i} ; x \right] = \sum_{m=0}^{i} \frac{(-i)_m (2\alpha - 1)_m}{(2\alpha + i)_m (\alpha - \frac{1}{2})_m m! 2^m} x^m 0F_1 \left[ \frac{\alpha + m + \frac{i}{2}}{2} ; \frac{x^2}{16} \right] \] (1.17)

and

\[ e^{-\frac{1}{2}x} 1F_1 \left[ \frac{\alpha}{2\alpha - i} ; x \right] = \sum_{m=0}^{i} \frac{(-i)_m (2\alpha - 2i - 1)_m}{(2\alpha - i)_m (\alpha - i - \frac{1}{2})_m m! 2^m} x^m 0F_1 \left[ \frac{\alpha + m - i + \frac{1}{2}}{2} ; \frac{x^2}{16} \right] \] (1.18)

for each \( i \in N_0 \).

**Remark:** For \( i = 0 \), (1.17) or (1.18) reduces to the well-known Kummer’s second transformation (1.2).

The aim of this research paper is to obtain generalization of Bailey’s identity (1.10) in the form of three general results.

(i) \( 1F_1 \left[ \frac{\alpha}{2\alpha + i} ; x \right] \cdot 1F_1 \left[ \frac{\beta}{2\beta + j} ; x \right] \)
Theorem 1. Each of the following formulas hold true for $i, j$ for each $N_0$.

(i) $\binom{\alpha}{2\alpha + i} \cdot \binom{\beta}{2\beta + j}$ and

(ii) $\binom{\alpha}{2\alpha - i} \cdot \binom{\beta}{2\beta - j}$

for each $i, j \in N_0$.

The results are established with the help of the results (1.17), (1.18) and the Bailey’s product formula (1.11) by the same technique developed by Rathie[9]. A few known as well as new results have also been given.

2. Main Results

In this section, we present three general formulas asserted in the following theorem.

Theorem 1. Each of the following formulas hold true for $i, j \in N_0$.

\[
\binom{\alpha}{2\alpha + i} \cdot \binom{\beta}{2\beta + j} = e^x \sum_{m=0}^{i} \binom{-i}{m} \binom{-j}{n} \binom{2\alpha - 1}{2\beta + 1} \binom{2\alpha - 2}{2\beta - 1} \binom{2\alpha - 3}{2\beta - 2} \ldots \binom{2\alpha - 1 - i}{2\beta - 1 - j} \frac{x^{m+n}}{m!n!} \times 2F3 \left[ \frac{1}{2} \alpha + \beta + m + n - i - j + 1, \frac{1}{2} \alpha + \beta + m + n - i - j \mid \frac{x^2}{4} \right] \quad (2.1)
\]

\[
\binom{\alpha}{2\alpha - i} \cdot \binom{\beta}{2\beta - j} = e^x \sum_{m=0}^{i} \binom{-1}{m} \binom{-1}{n} \binom{2\alpha - 1}{2\beta - 1} \binom{2\alpha - 2}{2\beta - 1} \binom{2\alpha - 3}{2\beta - 2} \ldots \binom{2\alpha - i}{2\beta - i} \frac{x^{m+n}}{m!n!} \times 2F3 \left[ \frac{1}{2} \alpha + \beta + m + n - i - j + 1, \frac{1}{2} \alpha + \beta + m + n - i - j \mid \frac{x^2}{4} \right] \quad (2.2)
\]

and

\[
\binom{\alpha}{2\alpha + i} \cdot \binom{\beta}{2\beta - j} = e^x \sum_{m=0}^{i} \binom{-1}{m} \binom{-1}{n} \binom{2\alpha - 1}{2\beta - 1} \binom{2\alpha - 2}{2\beta - 1} \binom{2\alpha - 3}{2\beta - 2} \ldots \binom{2\alpha - i}{2\beta - i} \frac{x^{m+n}}{m!n!} \times 2F3 \left[ \frac{1}{2} \alpha + \beta + m + n - j + 1, \frac{1}{2} \alpha + \beta + m + n - j \mid \frac{x^2}{4} \right] \quad (2.3)
\]
Proof: The proof of the results asserted in theorem are quite simple. In order to prove result (2.1), it is sufficient to prove

\[ e^{-x} {}_1F_1 \left[ \frac{\alpha}{2\alpha + i}; x \right] \cdot {}_1F_1 \left[ \frac{\beta}{2\beta + j}; x \right] \]

\[ = \sum_{m=0}^{i} \sum_{n=0}^{j} \frac{(-i)_m (-j)_n (2\alpha - 1)_m (2\beta - 1)_n x^{m+n}}{(2\alpha + i)_m (2\beta + j)_n (\alpha - \frac{i}{2})_m (\beta - \frac{j}{2})_n 2^{2m+2n} m! n!} \]

\[ \times {}_1F_3 \left[ \frac{1}{2} (\alpha + \beta + m + n + 1), \frac{1}{2} (\alpha + \beta + m + n); \frac{x^2}{4} \right] \]  \hspace{1cm} (2.4)  

Now, denoting left hand side of (2.4) by \( S \) and writing in the following form,

\[ S = \left\{ e^{-\frac{1}{2}x} {}_1F_1 \left[ \frac{\alpha}{2\alpha + i}; x \right] \right\} \cdot \left\{ e^{-\frac{1}{2}x} {}_1F_1 \left[ \frac{\beta}{2\beta + j}; x \right] \right\} \] \hspace{1cm} (2.5)  

Then using the known result (1.17) to each term of (2.5), we get after some simplification

\[ S = \sum_{m=0}^{i} \sum_{n=0}^{j} \frac{(-i)_m (-j)_n (2\alpha - 1)_m (2\beta - 1)_n x^{m+n}}{(2\alpha + i)_m (2\beta + j)_n (\alpha - \frac{i}{2})_m (\beta - \frac{j}{2})_n 2^{2m+2n} m! n!} \]

\[ \times {}_0F_1 \left[ \frac{1}{2}; \frac{x^2}{16} \right] \cdot {}_0F_1 \left[ \frac{1}{2}; \frac{x^2}{16} \right] \] \hspace{1cm} (2.6)  

Finally, using Bailey’s product formula (1.11), it is very easy to see that the \( S \) is equal to the right-hand side of (2.1).

This completes the proof of (2.1).

In exactly the same manner, the result (2.2) and (2.3) can be established. So the details are omitted.

3. Special Cases

In (2.1), (2.2) and (2.3), if we set \( \beta = \alpha \), we get the following interesting results which are also of general nature. These are

\[ {}_1F_1 \left[ \frac{\alpha}{2\alpha + i}; x \right] \cdot {}_1F_1 \left[ \frac{\alpha}{2\alpha + j}; x \right] \]

\[ = e^{x} \sum_{m=0}^{i} \sum_{n=0}^{j} \frac{(-i)_m (-j)_n (2\alpha - 1)_m (2\alpha - 1)_n x^{m+n}}{(2\alpha + i)_m (2\alpha + j)_n (\alpha - \frac{i}{2})_m (\alpha - \frac{j}{2})_n 2^{2m+2n} m! n!} \]

\[ \times {}_2F_3 \left[ \frac{1}{2} (2\alpha + m + n + 1), \frac{1}{2} (2\alpha + m + n); \frac{x^2}{4} \right] \] \hspace{1cm} (3.1)
Further in (3.1), (3.2) and (3.3), if we set $j$ recently obtained by Choi and Rathie[3].

and

$$1F_1 \left[ \frac{\alpha}{2\alpha - i}; x \right] \cdot 1F_1 \left[ \frac{\alpha}{2\alpha - j}; x \right]$$

$$= e^x \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^{m+n} (-i)_m (-j)_n (2\alpha - 2i - 1)_m (2\alpha - 2j - 1)_n x^{m+n}}{(2\alpha - i)_m (2\alpha - j)_n (\alpha - i - \frac{1}{2})_m (\alpha - j - \frac{1}{2})_n 2^{2m+2n} m! n!}$$

$$\times 2F_3 \left[ \frac{1}{2}(2\alpha + m + n - i - j + 1), \frac{1}{2}(2\alpha + m + n - i - j - \frac{1}{2}); \frac{2\alpha + m + n - i - j}{4} \right]$$

(3.2)

Further in (3.1), (3.2) and (3.3), if we set $j = i$, we immediately recover the results recently obtained by Choi and Rathie[3].

**Concluding Remark:** In this paper, we have obtained explicit expressions of

(i) $1F_1 \left[ \frac{\alpha}{2\alpha + i}; x \right] \cdot 1F_1 \left[ \frac{\beta}{2\beta + j}; x \right]$

(ii) $1F_1 \left[ \frac{\alpha}{2\alpha - i}; x \right] \cdot 1F_1 \left[ \frac{\beta}{2\beta - j}; x \right]$

and

(iii) $1F_1 \left[ \frac{\alpha}{2\alpha + i}; x \right] \cdot 1F_1 \left[ \frac{\beta}{2\beta - j}; x \right]$

in the most general form for any $i, j \in N$

Applications of these results in obtaining a large number of results (reduction and transformation formulas) involving Exton's and Srivastava's series and also for multiple series have also been obtained. The results will be published soon.

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