GLOBALIZATION OF PARTIAL COHOMOLOGY OF GROUPS

MIKHAILO DOKUCHAEV, MYKOLA KHRYPCHENKO, AND JUAN JACOBO SIMÓN

ABSTRACT. We study the relations between partial and global group cohomology. We show, in particular, that given a unital partial action of a group $G$ on a ring $A$, such that $A$ is a direct product of indecomposable rings, then any partial $n$-cocycle with values in $A$ is globalizable.

INTRODUCTION

Given a partial action it is natural to ask whether there exists a global action which restricts to the partial one. This question was first considered in the PhD Thesis [1] (see also [2]) and independently in [45] and [39] for partial group actions, with subsequent developments in [3, 12, 17, 18, 21, 22, 24, 25, 30, 33, 41, 44]. More generally the problem was investigated for partial semigroup actions in [37, 38, 39, 41], for partial groupoid actions in [10, 11, 36] and around partial Hopf (co)actions in [5, 6, 7, 8, 14, 15, 16].

Globalization results help one to use known facts on global actions in the studies involving partial ones. Thus the first purely ring theoretic globalization fact [22, Theorem 4.5] stimulated intensive algebraic activity, permitting, in particular, to develop a Galois Theory of commutative rings [25]. The latter, in its turn, inspired the definition and study of the concept of a partial action of a Hopf algebra in [13], which is based on globalizable partial group actions, and which became a starting point for interesting Hopf theoretic developments. Moreover, globalizable partial actions are more manageable, so that the great majority of ring theoretic studies on the subject deal with the globalizable case. Among the recent applications of globalization facts we mention their remarkable use to paradoxical decompositions in [9] and to restriction semigroups in [41]. The reader is referred to the surveys [19, 20, 34] and to the recent book by R. Exel [31] for more information about partial actions and their applications.

In [30] R. Exel introduced the general concept of a continuous twisted partial action of a locally compact group on a $C^*$-algebra and proved that any second countable $C^*$-algebraic bundle, which is regular in a certain sense, is isomorphic to the $C^*$-algebraic bundle constructed from a twisted partial group action. The purely algebraic version of this result was obtained in [23]. The concept involves a twisting which satisfies a kind of 2-cocycle equality needed for an associativity purpose. Thus, it was natural to work out a cohomology theory, encompassing such twistings, and this was done in [26]. The partial cohomology from [26] is strongly

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related to H. Lausch’s cohomology of inverse semigroups \cite{12} and nicely fits the theory of partial projective group representations developed in \cite{27}, \cite{28} and \cite{29}.

The main globalization result from \cite{24} says that if \( A \) is a (possibly infinite) product of indecomposable rings (blocks), then any unital twisted partial action \( \alpha \) of a group \( G \) on \( A \) possesses an enveloping action, i.e. there exists a twisted global action \( \beta \) of \( G \) on a ring \( B \) such that \( A \) can be identified with a two-sided ideal in \( B \), \( \alpha \) is the restriction of \( \beta \) to \( A \) and \( B = \sum_{g \in G} \beta_g(A) \). Moreover, if \( B \) has an identity element, then any two globalizations of \( \alpha \) are equivalent in a natural sense. If \( A \) is commutative, then \( \alpha \) splits into two parts: a unital partial \( G \)-module structure on \( A \) (i.e. a unital partial action of \( G \) on \( A \)) and a twisting which is a partial 2-cocycle \( w \) of \( G \) with values in the unit group \( U \) of the multiplier algebra \( B \). The above mentioned results from \cite{24} mean in this context that given a unital \( G \)-module structure on \( A \), for any 2-cocycle \( u \) of \( G \) related to the global action on \( B \) such that \( w \) is the restriction of \( u \). In this case we say that \( u \) is a globalization of \( w \) (see Definition \ref{def:globalization}). Moreover, if \( B \) has an identity element, then any two globalizations of \( w \) are cohomologous.

The purpose of the present article is to extend the results from \cite{24} in the commutative case to arbitrary \( n \)-cocycles. The technical difficulties coming from \cite{24} are being overcome by improvements and notations. In Section \ref{sec:preliminaries} we recall some notions needed in the sequel. The main result of Section \ref{sec:main} is Theorem \ref{thm:main} in which we prove that given a unital partial \( G \)-module structure on a commutative ring \( A \), a partial \( n \)-cocycle \( w \) with values in \( A \) is globalizable if and only if \( w \) can be extended to an \( n \)-cochain \( \tilde{w} \) of \( G \) with values in the unit group \( U(A) \) which satisfies a “more global” \( n \)-cocycle identity \cite{15}. This is the \( n \)-analogue of \cite{24, Theorem 4.1} in the commutative setting. The technical part of our work is concentrated in Section \ref{sec:technical} in which we assume that \( A \) is a product of blocks, and this assumption is maintained for the rest of the paper. Our goal is to construct a more manageable partial \( n \)-cochain \( w' \) which is cohomologous to \( w \) (see Theorem \ref{thm:main'}). In Section \ref{sec:main} we prove our main existence result Theorem \ref{thm:main}. The defining formula for \( w' \) permits us to extend easily \( w' \) to an \( n \)-cochain \( w' : G^n \to U(A) \) which satisfies our “more global” \( n \)-cocycle identity (see Lemma \ref{lem:main}). Modifying \( \tilde{w} \) by a “co-boundary looking” function we define in \cite{33} a function \( \tilde{w} : G^n \to U(A) \) and show that \( \tilde{w} \) is a desired extension of \( w \) fitting Theorem \ref{thm:main} and permitting us to conclude that \( w \) is globalizable. The uniqueness of a globalization is treated in Section \ref{sec:uniqueness}. It turns out that it is possible to omit the assumption that the ring \( B \) under the global action has an identity element, imposed in \cite{24} (with \( n = 2 \)). More precisely, we prove in Theorem \ref{thm:uniqueness} that given a globalizable partial action \( \alpha \) of \( G \) on a ring \( A \), which is a product of blocks, and a partial \( n \)-cocycle \( w \) related to \( \alpha \), any two globalizations of \( w \) are cohomologous. More generally, arbitrary globalizations of cohomologous partial \( n \)-cocycles are also cohomologous. This results in Corollary \ref{cor:uniqueness} which establishes an isomorphism between the partial cohomology group \( H^n(G,A) \) and the global one \( H^n(G,U(M(B))) \), where \( U(M(B)) \) stands for the unit group of the multiplier ring \( M(B) \) of \( B \).
1. Background on globalization and cohomology of partial actions

In all what follows $G$ will stand for an arbitrary group whose identity element will be denoted by $1$, and by a ring we shall mean an associative ring, which is not unital in general. Nevertheless, our main attention will be paid to partial actions on commutative and unital rings.

In this section we recall a couple of concepts around partial actions.

**Definition 1.1.** [22] Let $\mathcal{A}$ be a ring. A partial action $\alpha$ of $G$ on $\mathcal{A}$ is a collection of two-sided ideals $\mathcal{D}_g \subseteq \mathcal{A}$ ($g \in G$) and ring isomorphisms $\alpha_g : \mathcal{D}_{g^{-1}} \to \mathcal{D}_g$ such that

(i) $\mathcal{D}_1 = \mathcal{A}$ and $\alpha_1$ is the identity automorphism of $\mathcal{A}$;

(ii) $\alpha_g(\mathcal{D}_{g^{-1}} \cap \mathcal{D}_h) = \mathcal{D}_g \cap \mathcal{D}_{gh}$;

(iii) $\alpha_g \circ \alpha_h(a) = \alpha_{gh}(a)$ for each $a \in \mathcal{D}_{h^{-1}} \cap \mathcal{D}_{h^{-1}g^{-1}}$.

An equivalent form to state (i)–(iii) is as follows:

(i) $\alpha_1 = \text{id}_{\mathcal{A}}$;

(iv) $\exists \alpha_h(a), \exists \alpha_g(a) \Rightarrow \exists \alpha_{gh}(a)$ and $\alpha_g \circ \alpha_h(a) = \alpha_{gh}(a)$, where $a \in \mathcal{A}$.

Partial actions can be obtained as restrictions of global ones as follows. Let $\beta$ be a global action of $G$ on a ring $\mathcal{B}$ and $\mathcal{A}$ a two-sided ideal in $\mathcal{B}$. Then setting $\mathcal{D}_g = \mathcal{A} \cap \beta_g(\mathcal{A})$ and denoting by $\alpha_g$ the restriction of $\beta_g$ to $\mathcal{D}_{g^{-1}}$, we readily see that $\alpha = \{ \alpha_g : \mathcal{D}_{g^{-1}} \to \mathcal{D}_g \mid g \in G \}$ is a partial action of $G$ on $\mathcal{A}$, called the restriction of $\beta$ to $\mathcal{A}$, and $\alpha$ is said to be an admissible restriction of $\beta$ if $\mathcal{B} = \sum_{g \in G} \beta_g(\mathcal{A})$.

Clearly, if $\mathcal{B} \neq \sum_{g \in G} \beta_g(\mathcal{A})$, then replacing $\mathcal{B}$ by $\sum_{g \in G} \beta_g(\mathcal{A})$, the partial action $\alpha$ can be viewed as an admissible restriction. Partial actions isomorphic to restrictions of global ones are called globalizable. The notion of an isomorphism of partial action is defined as follows.

**Definition 1.2** (see p. 17 from [2] and Definition 4 from [28]). Let $\mathcal{A}$ and $\mathcal{A}'$ be rings and $\alpha = \{ \alpha_g : \mathcal{D}_{g^{-1}} \to \mathcal{D}_g \mid g \in G \}$, $\alpha' = \{ \alpha'_g : \mathcal{D}'_{g^{-1}} \to \mathcal{D}'_g \mid g \in G \}$ be partial actions of $G$ on $\mathcal{A}$ and $\mathcal{A}'$, respectively. A morphism $(\mathcal{A}, \alpha) \to (\mathcal{A}', \alpha')$ of partial actions is a ring homomorphism $\varphi : \mathcal{A} \to \mathcal{A}'$ such that for any $g \in G$ and $a \in \mathcal{D}_{g^{-1}}$, the next two conditions are satisfied:

(i) $\varphi(\mathcal{D}_g) \subseteq \mathcal{D}'_g$;

(ii) $\varphi(\alpha_g(a)) = \alpha'_g(\varphi(a))$.

We say that a morphism $\varphi : (\mathcal{A}, \alpha) \to (\mathcal{A}', \alpha')$ of partial actions is an isomorphism if $\varphi : \mathcal{A} \to \mathcal{A}'$ is an isomorphism of rings and $\varphi(\mathcal{D}_g) = \mathcal{D}'_g$ for each $g \in G$.

By [22] Theorem 4.5] a partial action $\alpha$ on a unital ring $\mathcal{A}$ is globalizable exactly when each ideal $\mathcal{D}_g$ is a unital ring, i.e. $\mathcal{D}_g$ is generated by an idempotent which is central in $\mathcal{A}$, and which will be denoted by $1_g$. In order to guarantee the uniqueness of a globalization one considers the next.

**Definition 1.3** (Definition 4.2 from [22]). A global action $\beta$ of $G$ on a ring $\mathcal{B}$ is said to be an enveloping action for the partial action $\alpha$ of $G$ on a ring $\mathcal{A}$ if $\alpha$ is isomorphic to an admissible restriction of $\beta$.

By the above mentioned Theorem 4.5 from [22], an enveloping action $\beta$ for a globalizable partial action of $G$ on a unital ring $\mathcal{A}$ is unique up to an isomorphism.  

\[1\] This was called equivalence in [22] Definition 4.1.
Denote by \( \mathcal{F} = \mathcal{F}(G, \mathcal{A}) \) the ring of functions from \( G \) to \( \mathcal{A} \), i.e. \( \mathcal{F} \) is the Cartesian product of copies of \( \mathcal{A} \) indexed by the elements of \( G \). Note that by the proof of Theorem 4.5 from \cite{22}, the ring under the global action is a subring \( \mathcal{B} \) of \( \mathcal{F} \), and consequently \( \mathcal{B} \) is commutative if and only of so too is \( \mathcal{A} \).

Every ring is a semigroup with respect to multiplication, and if in Definition 1.4 we assume that \( \mathcal{A} \) is a (multiplicative) semigroup and the \( \alpha_g \) are isomorphisms of semigroups, then we obtain the concept of a partial action of \( G \) on a semigroup (see \cite{27}). Furthermore, the concept of a morphism of partial actions on semigroups is obtained from Definition 1.2 by assuming that \( \varphi : \mathcal{A} \to \mathcal{A}' \) is a homomorphism of semigroups.

Partial cohomology was defined in \cite{26} as follows. Let \( \alpha = \{ \alpha_g : \mathcal{D}_{g^{-1}} \to \mathcal{D}_g \mid g \in G \} \) be a partial action of \( G \) on a commutative monoid \( \mathcal{A} \). Assume that each ideal \( \mathcal{D}_g \) is unital, i.e. \( \mathcal{D}_g \) is generated by an idempotent \( 1_g = 1_g^2 \), which is central in \( \mathcal{A} \). In this case we shall say that \( \alpha \) is a unital partial action. Then \( \mathcal{D}_g \cap \mathcal{D}_h = \mathcal{D}_g \mathcal{D}_h \), for all \( g, h \in G \), so the properties (iii) and (iii) from Definition 1.1 can be replaced by

- (ii') \( \alpha_g(\mathcal{D}_{g^{-1}} \mathcal{D}_h) = \mathcal{D}_g \mathcal{D}_{gh} \);
- (iii') \( \alpha_g \circ \alpha_h = \alpha_{gh} \) on \( \mathcal{D}_{h^{-1}} \mathcal{D}_{h^{-1}g^{-1}} \).

Note also that (iii) implies a more general equality

\[
\alpha_x(\mathcal{D}_{x^{-1}} \mathcal{D}_{y_1} \cdots \mathcal{D}_{y_n}) = \mathcal{D}_x \mathcal{D}_{xy_1} \cdots \mathcal{D}_{xy_n},
\]

for any \( x, y_1, \ldots, y_n \in G \), which easily follows by observing that \( \mathcal{D}_{x^{-1}} \mathcal{D}_{y_1} \cdots \mathcal{D}_{y_n} = \mathcal{D}_{x^{-1}} \mathcal{D}_{y_1} \cdots \mathcal{D}_{x^{-1}y_n} \).

**Definition 1.4** (see \cite{26}). A commutative monoid \( \mathcal{A} \) with a unital partial action \( \alpha \) of \( G \) on \( \mathcal{A} \) will be called a (unital) partial \( G \)-module. A morphism of (unital) partial \( G \)-modules \( \varphi : (\mathcal{A}, \alpha) \to (\mathcal{A}', \alpha') \) is a morphism of partial actions such that its restriction on each \( \mathcal{D}_g \) is a homomorphism of monoids \( \mathcal{D}_g \to \mathcal{D}_g', g \in G \).

Following \cite{26}, the category of (unital) partial \( G \)-modules and their morphisms is denoted by \( \mathcal{pMod}(G) \).

**Definition 1.5** (see \cite{26}). Let \( \mathcal{A} \in \mathcal{pMod}(G) \) and \( n \) be a positive integer. An \( n \)-cochain of \( G \) with values in \( \mathcal{A} \) is a function \( f : G^n \to \mathcal{A} \), such that \( f(x_1, \ldots, x_n) \) is an invertible element of the ideal \( \mathcal{D}(x_1, \ldots, x_n) = \mathcal{D}_{x_1} \mathcal{D}_{x_2} \cdots \mathcal{D}_{x_1, \ldots, x_n} \). By a 0-cochain we shall mean an invertible element of \( \mathcal{A} \), i.e. \( \alpha \in \mathcal{U}(\mathcal{A}) \), where \( \mathcal{U}(\mathcal{A}) \) stands for the group of invertible elements of \( \mathcal{A} \).

Denote the set of \( n \)-cochains by \( C^n(\mathcal{G}, \mathcal{A}) \). It is an abelian group under the pointwise multiplication. Indeed, its identity is \( e_n \) which is the \( n \)-cochain defined by

\[
e_n(x_1, \ldots, x_n) = 1_{(x_1, \ldots, x_n)} := 1_{x_1} 1_{x_2} \cdots 1_{x_1, \ldots, x_n},
\]

and the inverse of \( f \in C^n(\mathcal{G}, \mathcal{A}) \) is \( f^{-1}(x_1, \ldots, x_n) = f(x_1, \ldots, x_n)^{-1} \), where \( f(x_1, \ldots, x_n)^{-1} \) means the inverse of \( f(x_1, \ldots, x_n) \) in \( \mathcal{D}(x_1, \ldots, x_n) \).

The multiplicative form of the classical coboundary homomorphism now can be adapted to our context by replacing the global action by a partial one, and taking inverse elements in the corresponding ideals, as follows.

**Definition 1.6** (see \cite{26}). Let \( (\mathcal{A}, \alpha) \in \mathcal{pMod}(G) \) and \( n \) be a positive integer. For any \( f \in C^n(\mathcal{G}, \mathcal{A}) \) and \( x_1, \ldots, x_{n+1} \in G \) define

\[
(\delta^0 f)(x_1, \ldots, x_{n+1}) = \alpha_{x_1}(1_{x_1^{-1}} f(x_2, \ldots, x_{n+1}))
\]
The concept of equivalent unital twisted partial actions from [24, Def. A] multiplies ring for this purpose we remind the reader that the where for homomorphisms of right modules the usual notation is used. Thus given

Now, a unital twisted partial action (see [23, Def. 2.1]) of

If \( n = 0 \) and \( a \) is an invertible element of \( \mathcal{A} \), we set

According to Proposition 1.5 from [26] the coboundary map \( \delta^n \) is a homomorphism \( C^n(G, \mathcal{A}) \rightarrow C^{n+1}(G, \mathcal{A}) \) of abelian groups, such that

for any \( f \in C^n(G, \mathcal{A}) \). As in the classical case one defines the abelian groups of partial n-cocycles, n-coboundaries and n-cohomologies of \( G \) with values in \( \mathcal{A} \) by setting

Notice that \( H^n(G, \mathcal{A}) \) is exactly the subgroup of \( \alpha \)-invariants of \( \mathcal{U}(\mathcal{A}) \), as defined (for the case of rings) in [25, p. 79]. In order to relate partial cohomology to twisted partial actions, consider the cases \( n = 1 \) and \( n = 2 \). In the first case we have

with \( f \in C^1(G, \mathcal{A}) \), so that

Now, a unital twisted partial action (see [23, Def. 2.1]) of \( G \) on a commutative ring \( \mathcal{A} \) splits into two parts: a unital partial action of \( G \) on \( \mathcal{A} \), and a twisting which, in our terminology, is a 2-cocycle with values in the partial \( G \)-module \( \mathcal{A} \). Furthermore, the concept of equivalent unital twisted partial actions from [24, Def. 6.1] is exactly the notion of equivalence of partial 2-cocycles.

We shall use multipliers in order to define globalization of partial cocycles, and for this purpose we remind the reader that the multiplier ring of \( \mathcal{M}(\mathcal{A}) \) of an associative non-necessarily unital ring \( \mathcal{A} \) is the set

with component-wise addition and multiplication (see [4] or [22] for more details). Here we use the right-hand side notation for homomorphisms of left \( \mathcal{A} \)-modules, whereas for homomorphisms of right modules the usual notation is used. Thus given \( R : \mathcal{A} \rightarrow \mathcal{A}, L : \mathcal{A} \rightarrow \mathcal{A} \) and \( a \in \mathcal{A} \), we write \( a \leftrightarrow aR \) and \( a \leftrightarrow La \). For a
multiplier \( u = (R, L) \in \mathcal{M}(\mathcal{A}) \) and an element \( a \in \mathcal{A} \) we set \( au = aR \) and \( ua = La \), so that the associativity equality \( (au)b = a(ub) \) always holds with \( a, b \in \mathcal{A} \).

Notice that

\[
eu = ue
\]

for any \( u \in \mathcal{M}(\mathcal{A}) \) and any central idempotent \( e \in \mathcal{A} \). For

\[
eu = (e^2)u = e(\eu) = (eu)e = e(ue) = (ue)e = ue.
\]

Any \( a \in \mathcal{A} \) determines a multiplier \( u_a \) by setting \( u_a b = ab \) and \( bu_a = ba \), \( b \in \mathcal{A} \), so that \( a \mapsto u_a \) gives the canonical homomorphism \( \mathcal{A} \to \mathcal{M}(\mathcal{A}) \), which is an isomorphism if \( \mathcal{A} \) has \( 1_{\mathcal{A}} \) (in this case the inverse isomorphism is given by \( \mathcal{M}(\mathcal{A}) \ni u \mapsto u1_{\mathcal{A}} = 1_{\mathcal{A}}u \in \mathcal{A} \)). According to [22] a ring \( \mathcal{A} \) is said to be non-degenerate if the canonical map \( \mathcal{A} \to \mathcal{M}(\mathcal{A}) \) is injective. This is guaranteed if \( \mathcal{A} \) is left (or right) \( s \)-unital, i.e. for any \( a \in \mathcal{A} \) one has \( a \in \mathcal{A}a \) (respectively, \( a \in a\mathcal{A} \)).

Furthermore, given a ring isomorphism \( \phi : \mathcal{A} \to \mathcal{A}' \), the map \( \mathcal{M}(\mathcal{A}) \ni u \mapsto \phi u \phi^{-1} \in \mathcal{M}(\mathcal{A}') \), where \( \phi u \phi^{-1} = (\phi^{-1}R\phi, \phi L\phi^{-1}) \), \( u = (R, L) \), is an isomorphism of rings. In particular, an automorphism \( \phi \) of \( \mathcal{A} \) gives rise to an automorphism

\[
u \mapsto \phi u \phi^{-1}
\]

of \( \mathcal{M}(\mathcal{A}) \).

We shall also use the next.

**Remark 1.7.** If \( \mathcal{A} \) is a commutative idempotent ring, then \( \mathcal{M}(\mathcal{A}) \) is also commutative.

**Proof.** Indeed, for arbitrary \( u, v \in \mathcal{M}(\mathcal{A}) \) and \( a, b \in \mathcal{A} \) we have that \( (uv)(ab) = u(v(ab)) = u(v(a)b) = u(bv(a)) = u(b)v(a) = v(a)u(b) = v(au(b)) = v(u(ab)) = vu(ab) \). Similarly, \( (ab)(uv) = (ab)(vu) \) for all \( a, b \in \mathcal{A} \), and consequently, \( uv = vu \), showing that \( \mathcal{M}(\mathcal{A}) \) is commutative. \( \square \)

2. **The Notion of a Globalization of a Partial Cocycle and Its Relation with an Extendibility Property**

In this section we introduce the concept of a globalization of a partial \( n \)-cocycle with values in a commutative unital ring \( \mathcal{A} \) and show that a partial \( n \)-cocycle \( w \) is globalizable, provided that an extendibility property for \( w \) holds. We start with a general auxiliary result which does not involve partial actions.

Let \( \mathcal{G} \) be a group and \( \mathcal{A} \) a commutative unital ring. For \( f \in \mathcal{F} = \mathcal{F}(\mathcal{G}, \mathcal{A}) \) denote by \( f|_t \) the value \( f(t) \) and define \( \beta_x : \mathcal{F} \to \mathcal{F} \) by

\[
\beta_x(f)|_t = f(x^{-1}t),
\]

where \( x, t \in \mathcal{G} \). Then \( \beta \) is a global action of \( \mathcal{G} \) on \( \mathcal{F} \) which was used in [22] to deal with the globalization problem for partial actions on unital rings.

Let \( \tilde{\omega} : G^n \to \mathcal{U}(\mathcal{A}) \) be a function, i.e. \( \tilde{\omega} \) is an element of the group \( C^n(\mathcal{G}, \mathcal{U}(\mathcal{A})) \) of global (classical) \( n \)-cochains of \( G \) with values in \( \mathcal{U}(\mathcal{A}) \). Define \( u : G^n \to \mathcal{U}(\mathcal{F}) \) by

\[
u(x_1, \ldots, x_n)|_t = \tilde{\omega}(t^{-1}, x_1, \ldots, x_{n-1})^{-1}\tilde{\omega}(t^{-1}, x_1, x_2, \ldots, x_n)
\]

\[
\prod_{i=1}^{n-1} \tilde{\omega}(t^{-1}, x_1, \ldots, x_ix_{i+1}, \ldots, x_n)^{-1}, n > 0.
\]

We proceed with a technical fact which will be used in the main result of this section.
Lemma 2.1. The $n$-cochain $u$ is an $n$-cocycle with respect to the action $\beta$ of $G$ on $U(\mathcal{F})$, i.e. $u \in Z^n(G, U(\mathcal{F}))$.

Proof. We need to show that the function
\[ \beta_{x_i}(u(x_2, \ldots, x_{n+1})) \prod_{i=1}^{n} u(x_1, \ldots, x_i x_{i+1}, \ldots, x_{n+1})(-1)^i u(x_1, \ldots, x_n)(-1)^{n+1} \]  
(9)
is identity, i.e. it equals 1 for any $x_1, \ldots, x_{n+1} \in G$. Evaluating (9) at $t$ and using (7), we get
\[ u(x_2, \ldots, x_{n+1})|_{x_i=1} = \prod_{i=1}^{n} u(x_1, \ldots, x_i x_{i+1}, \ldots, x_{n+1})(-1)^i |_{t} u(x_1, \ldots, x_n)(-1)^{n+1} |_{t}. \]
(10)

Denote by $\tilde{\delta}^n : C^n(G, U(\mathcal{A})) \to C^{n+1}(G, U(\mathcal{A}))$ the coboundary operator which corresponds to the trivial $G$-module, i.e.
\[ (\tilde{\delta}^n w)(x_1, \ldots, x_{n+1}) = \tilde{w}(x_1, \ldots, x_{n+1}) \prod_{i=1}^{n} \tilde{w}(x_1, \ldots, x_i x_{i+1}, \ldots, x_{n+1})(-1)^i \tilde{w}(x_1, \ldots, x_n)(-1)^{n+1}. \]
(11)

We see from (8) that
\[ u(x_1, \ldots, x_n)|_{t} = \tilde{w}(x_1, \ldots, x_n)(\tilde{\delta}^n \tilde{w})(t^{-1}, x_1, \ldots, x_n)^{-1}. \]

Therefore, (10) becomes
\[ \tilde{w}(x_2, \ldots, x_{n+1})(\tilde{\delta}^n \tilde{w})(t^{-1} x_1, x_2, \ldots, x_{n+1})^{-1} \]
\[ \prod_{i=1}^{n} \tilde{w}(x_1, \ldots, x_i x_{i+1}, \ldots, x_{n+1})(-1)^i \prod_{i=1}^{n} (\tilde{\delta}^n \tilde{w})(t^{-1} x_1, \ldots, x_i x_{i+1}, \ldots, x_{n+1})(-1)^{i+1} \]
\[ \tilde{w}(x_1, \ldots, x_n)(-1)^{n+1} (\tilde{\delta}^n \tilde{w})(t^{-1}, x_1, \ldots, x_n)^{-1}. \]

Regrouping the factors and using (11), we obtain
\[ (\tilde{\delta}^n \tilde{w})(x_1, \ldots, x_{n+1})(\tilde{\delta}^n \tilde{w})(t^{-1} x_1, x_2, \ldots, x_{n+1})^{-1} \]
\[ \prod_{i=1}^{n} (\tilde{\delta}^n \tilde{w})(t^{-1}, x_1, \ldots, x_i x_{i+1}, \ldots, x_{n+1})(-1)^{i+1} (\tilde{\delta}^n \tilde{w})(t^{-1}, x_1, \ldots, x_n)^{-1}, \]
which is $(\tilde{\delta}^{n+1} \tilde{\delta}^n \tilde{w})(t^{-1}, x_1, \ldots, x_n) = 1$. \hfill \Box

Let now $\alpha$ be a unital partial action of $G$ on $\mathcal{A}$. Then
\[ \varphi(a)|_{t} = \alpha_{t^{-1}}(1,a), \]
(12)
a $\in \mathcal{A}$, defines an embedding of $\mathcal{A}$ into $\mathcal{F}$, and $(\beta, \mathcal{B})$ is an enveloping action for $(\alpha, \mathcal{A})$, where $\mathcal{B} = \sum_{g \in G} \beta_g(\varphi(\mathcal{A}))$ (see the proof of [22, Theorem 4.5]). Since $(\beta, \mathcal{B})$ is unique up to an isomorphism, it follows by Theorem 3.1 from [21] that $\mathcal{B}$ is left $s$-unital. Hence there is a canonical embedding of $\mathcal{B}$ into the multiplier ring $\mathcal{M}(\mathcal{B})$ and, moreover, $\mathcal{B}$ is commutative as so too is $\mathcal{A}$. In addition, $\mathcal{B}$ is idempotent because $\mathcal{B}$ is left $s$-unital, which implies that $\mathcal{M}(\mathcal{B})$ is commutative.
Remark 2.3. Observe that any partial 0-cocycle balization is the constant function \( u \). Moreover, thanks to Remark 1.7. Using the commutativity of \( \mathcal{M}(B) \), the global action \( \beta \) of \( G \) on \( B \) can be extended by (3) to a global action \( \beta^* \) of \( G \) on \( \mathcal{M}(B) \) by setting
\[
\beta^*_g(u) = \beta_g u \beta^{-1}_g = (\beta^{-1}_g R \beta_g, \beta_g L \beta^{-1}_g),
\]
where \( u = (R, L) \in \mathcal{M}(B) \) and \( g \in G \). This permits us to consider the group of units \( U(\mathcal{M}(B)) \) as a \( G \)-module via \( \beta^* \).

**Definition 2.2.** Let \( \alpha = \{\alpha_g : \mathcal{D}_{g^{-1}} \to \mathcal{D}_g \mid g \in G\} \) be a unital partial action of \( G \) on a commutative ring \( A \) and \( w \in \mathbb{Z}^n(G, A) \). Denote by \( \alpha \) the enveloping action of \( G \) on \( B \) and by \( \varphi : A \to B \) the embedding which transforms \( \alpha \) into an admissible restriction of \( \beta \). A globalization of \( w \) is a (classical) \( n \)-cocycle \( u \in \mathbb{Z}^n(G, U(\mathcal{M}(B))) \), where \( G \) acts on \( U(\mathcal{M}(B)) \) via \( \beta^* \), such that
\[
\varphi(w(1_x, \ldots, x_n)) = \varphi((1_x, \ldots, x_n))u(1_x, \ldots, x_n),
\]
for any \( 1_x, \ldots, x_n \in G \). If \( n = 0 \), then by \( 1_x, \ldots, x_n \) we mean \( 1_A \) in (14).

Observe from (3) that (14) implies
\[
\varphi(w(1_x, \ldots, x_n)) = u(1_x, \ldots, x_n)\varphi((1_x, \ldots, x_n)).
\]

With the notation in Definition 2.2 we say, following [24], that \( \alpha \) is unitally globalizable if \( B \) has \( I_B \). In this case \( \beta \) is called a unital enveloping action of \( \alpha \). Note that by the uniqueness, up to an isomorphism, of an enveloping action (see [22, Theorem 4.5]), all enveloping actions of a unitally globalizable partial action are unital. Notice, furthermore, that unitally globalizable partial actions are also called partial actions of finite type (see [35, Definition 1.1]).

It is readily seen that if \( B \) contains \( I_B \), then the isomorphism \( \mathcal{M}(B) \cong B \) transforms \( \beta^* \) into \( \beta \), and the globalization \( u \) is an \( n \)-cocycle with values in \( U(B) \).

**Remark 2.3.** Observe that any partial 0-cocycle \( w \) is globalizable, and its globalization is the constant function \( u \in U(F) \) with \( u|_F = w \). Moreover, such \( u \) is unique.

Indeed, (14) reduces to \( \varphi(w) = \varphi((1_A)u) \), which is the 0-cocycle identity for \( w \) by (12). Moreover, \( u \) is an (invertible) multiplier of \( B \), as \( \beta_g(\varphi(a))|_1 = \varphi(a)|_{g^{-1}w} = \alpha_{g^{-1}w}(1_{g^{-1}w})aw = \alpha_{g^{-1}w}(1_{g^{-1}w})1_{g^{-1}w} = \alpha_{g^{-1}w} = \beta_g(\varphi(aw))|_1 \), thanks to (4) and (12) and the 0-cocycle identity for \( w \).

Now if \( u_1, u_2 \in \mathcal{M}(U(B)) \) are globalizations of \( w \), then \( \varphi((1_A)u_1) = \varphi((1_A)u_2) \) by (14). Applying \( \beta_x \) to this equality and using the 0-cocycle identity for \( u_1 \) which means that \( \beta_x u_1 \beta^{-1}_x = u_1 \), \( i = 1, 2 \), one gets \( \beta_x(\varphi((1_A)u_1)u_2 = \beta_x(\varphi((1_A)u_1)u_2 \) for all \( x \in G \). Consequently, \( \beta_x(\varphi(a))u_1 = \beta_x(\varphi(a))u_2 \) for all \( x \in G \) and \( a \in A \). It follows that \( u_1 = u_2 \), as \( B = \sum_{g \in G} \beta_g(A) \).

Given an arbitrary \( n > 0 \), as in the case \( n = 2 \) (see [24, Theorem 4.1]), we are able to reduce the globalization problem for partial \( n \)-cocycles to an extendibility property.

**Theorem 2.4.** Let \( \alpha = \{\alpha_g : \mathcal{D}_{g^{-1}} \to \mathcal{D}_g \mid g \in G\} \) be a unital partial action of \( G \) on a commutative ring \( A \) and \( w \in \mathbb{Z}^n(G, A) \). Then \( w \) is globalizable if and only if there exists a function \( \tilde{w} : G^n \to U(A) \) which satisfies the equalities
\[
\alpha x_1 \left( 1_{x_1^{-1}} \tilde{w}(x_2, \ldots, x_{n+1}) \right) \prod_{i=1}^n \tilde{w}(x_1, \ldots, x_{i-1}, x_i x_{i+1}, \ldots, x_{n+1}) = (-1)^i.
\]
\[ w(x_1, \ldots, x_n) = 1_{(x_1, \ldots, x_n)} \bar{w}(x_1, \ldots, x_n), \quad (16) \]

for all \( x_1, \ldots, x_{n+1} \in G \).

**Proof.** We shall assume that \( n > 0 \), as \( n = 0 \) was considered in Remark 234.

Suppose that \( w \in Z^n(G, A) \) is globalizable. Denote by \((\beta, B) \) an enveloping action of \((\alpha, A) \) and let \( \beta^\ast \) be the corresponding action of \( G \) on \( M(B) \) (see 134). Let \( u \in Z^n(G, U(M(B))) \) be a globalization of \( w \) and define \( \bar{w}(x_1, \ldots, x_n) \in U(A) \) by

\[
\varphi(\bar{w}(x_1, \ldots, x_n)) = \varphi(1_A)u(x_1, \ldots, x_n) = u(x_1, \ldots, x_n)\varphi(1_A).
\]

(17)

Evidently, \( \bar{w}(x_1, \ldots, x_n) \in U(A) \), as \( u(x_1, \ldots, x_n) \) is an invertible multiplier, and \( \varphi(\bar{w}(x_1, \ldots, x_n)^{-1}) = \varphi(1_A)u^{-1}(x_1, \ldots, x_n) = u^{-1}(x_1, \ldots, x_n)\varphi(1_A) \). Then (10) clearly holds by (13), and for (16) notice first that

\[
\varphi(1_g) = \beta_g(\varphi(1_A))\varphi(1_A),
\]

and consequently (and in fact more generally),

\[
\varphi(\alpha_g(1_g^{-1}a)) = \beta_g(\varphi(a))\varphi(1_A),
\]

for all \( g \in G \) and \( a \in A \) (see 234 p. 79). The (global) \( n \)-cocycle identity for \( u \) is of the form

\[
\beta^\ast_x(u(x_2, \ldots, x_{n+1})) \prod_{i=1}^{n} u(x_1, \ldots, x_ix_{i+1}, \ldots, x_{n+1})^{-1} = 1_{M(B)}.
\]

(20)

Applying the first multiplier in (20) to \( \varphi(1_{x_1}) \) and using (13), (17) and (19), we obtain

\[
\beta^\ast_{x_1}(u(x_2, \ldots, x_{n+1}))\varphi(1_{x_1}) = (\beta_{x_1}(u(x_2, \ldots, x_{n+1})\beta^{-1}_{x_1}(\varphi(1_A))\varphi(1_A))
\]

\[
= (\beta_{x_1}(u(x_2, \ldots, x_{n+1})\varphi(1_A)))\varphi(1_A)
\]

\[
= (\beta_{x_1}[\varphi(\bar{w}(x_2, \ldots, x_{n+1}))])\varphi(1_A)
\]

\[
= \varphi(\alpha_{x_1}(1_{x_1^{-1}}\bar{w}(x_2, \ldots, x_{n+1}))).
\]

Then applying both sides of (20) to \( \varphi(1_{x_1}) \) and using axioms of a multiplier, we readily see that (15) is a consequence of (20).

Suppose now that there exists \( \bar{w} : G^n \to U(A) \) such that (15) and (16) hold. Let \( (\beta, B) \) be the globalization of \((\alpha, A) \), with \( \beta, B \in F = F(G, A) \) and \( \varphi : A \to B \) as described above. In particular, it follows from (11) that

\[
\varphi(1_{(x_1, \ldots, x_n)})|_t = 1_{(t^{-1}, x_1, \ldots, x_n)}.
\]

(21)

Taking our \( \bar{w} \), define \( u : G^n \to U(F) \) by formula (5). We are going to show that \( u \) is a globalization of \( w \). By Lemma 2.1 one has \( u \in Z^n(G, U(F)) \). We now check (14). By (12)

\[
\varphi(w(x_1, \ldots, x_n)|_t = \alpha_{t^{-1}}(1_{t}w(x_1, \ldots, x_n)),
\]
which by the partial $n$-cocycle identity for $w$ equals
\[
 w(t^{-1}x_1, x_2, \ldots, x_n) \prod_{i=1}^{n-1} w(t^{-1}, x_1, \ldots, x_i x_{i+1}, \ldots, x_n)(-1)^i
 \]
\[
 w(t^{-1}, x_1, \ldots, x_{n-1})(-1)^n.
\]

In view of (3), (16) and (21) the latter is
\[
 1_{(t^{-1}, x_1, \ldots, x_n)} u(x_1, \ldots, x_n)|_t = \varphi (1_{(x_1, \ldots, x_n)}) |_t u(x_1, \ldots, x_n)|_t,
\]
for arbitrary $t, x_1, \ldots, x_n \in G$, proving (14).

It remains to see that $u(x_1, \ldots, x_n)$ and $u(x_1, \ldots, x_n)^{-1}$ are multipliers of $B$. Notice first that using (15) for $(t^{-1}, x_1, \ldots, x_n)$ we obtain from (8) that
\[
 \alpha_{t^{-1}} (1_t \tilde{w}(x_1, \ldots, x_n)) = 1_{t^{-1}} \tilde{w}(t^{-1}x_1, x_2, \ldots, x_n)
\]
\[
 \prod_{i=1}^{n-1} \tilde{w}(t^{-1}, x_1, \ldots, x_i x_{i+1}, \ldots, x_n)(-1)^i
\]
\[
 \tilde{w}(t^{-1}, x_1, \ldots, x_{n-1})(-1)^n
\]
\[
 = 1_{t^{-1}} u(x_1, \ldots, x_n)|_t.
\]

Then by (12)
\[
 u(x_1, \ldots, x_n)|_t \varphi(a)|_t = \alpha_{t^{-1}} (1_t \tilde{w}(x_1, \ldots, x_n)) \alpha_{t^{-1}} (1_t a),
\]
so that
\[
 u(x_1, \ldots, x_n) \varphi(a) = \varphi (a \tilde{w}(x_1, \ldots, x_n)),
\]
for all $x_1, \ldots, x_n \in G$ and $a \in A$. Equalities (8) and (23) readily imply
\[
 u(x_1, \ldots, x_n)^{-1} \varphi(a) = \varphi (a \tilde{w}(x_1, \ldots, x_n)^{-1}).
\]

Furthermore, applying the $n$-cocycle identity for $u$ to $(t^{-1}, x_1, \ldots, x_n)$ we see that
\[
 \beta_{t^{-1}} (u(x_1, \ldots, x_n)) \varphi(a) = u(t^{-1}x_1, x_2, \ldots, x_n)
\]
\[
 \prod_{i=1}^{n-1} u(t^{-1}, x_1, \ldots, x_i x_{i+1}, \ldots, x_{n+1})(-1)^i
\]
\[
 u(t^{-1}, x_1, \ldots, x_{n-1})(-1)^n \varphi(a),
\]
which belongs to $\varphi(A)$ thanks to (23) and (24). Thus $\beta_{t^{-1}} (u(x_1, \ldots, x_n)) \varphi(A) \subseteq \varphi(A)$, which yields $u(x_1, \ldots, x_n) \beta_t(\varphi(A)) \subseteq \beta_t(\varphi(A))$. Since $B = \sum_{t \in G} \beta_t(\varphi(A))$, it follows that $u(x_1, \ldots, x_n) B \subseteq B$, and hence $u(x_1, \ldots, x_n) \in \mathcal{M}(B)$. Similarly, $u(x_1, \ldots, x_n)^{-1} \in \mathcal{M}(B)$, as desired. \(\square\)

Note that taking $t = 1$ in (22) we obtain $u(x_1, \ldots, x_n)|_1 = \tilde{w}(x_1, \ldots, x_n)$.

3. From $w$ to $w'$

Our next purpose is to show that $\tilde{w}$ in Theorem 2.3 exists, provided that $A$ is a product of blocks, and we need first some technical preparation for this fact, which we do in the present section.

Suppose that $A = \prod_{\lambda \in \Lambda} A_{\lambda}$, where $A_{\lambda}$ is an indecomposable (commutative) unital ring, called a block. We identify the unity element of $A_{\mu}$, $\mu \in \Lambda$, with the indecomposable idempotent $1_{A_{\mu}}$ of $A$ which is the function $\Lambda \to \bigcup_{\lambda \in \Lambda} A_{\lambda}$ whose
value at $\mu$ is the identity of $A_{\mu}$ and the value at any $\lambda \neq \mu$ is the zero of $A_{\lambda}$. Then $A_{\mu}$ is identified with the ideal of $A$ generated by the idempotent $1_{A_{\mu}}$. Denote by $pr_{\mu}$ the projection of $A$ onto $A_{\mu}$, namely, $pr_{\mu}(a) = 1_{A_{\mu}}a$. Thus, any $a \in A$ is identified with the set of its projections $\{pr_{\lambda}(a)\}_{\lambda \in \Lambda}$, and we write $a = \prod_{\lambda \in \Lambda} pr_{\lambda}(a)$ in this situation. If there exists $\Lambda_1 \subseteq \Lambda$, such that $pr_{\lambda}(a) = 0_{A}$ for all $\lambda \in \Lambda \setminus \Lambda_1$, then we shall also write $a = \prod_{\lambda \in \Lambda_1} pr_{\lambda}(a)$, and such elements $a$ form an ideal in $A$ which we denote by $\prod_{\lambda \in \Lambda_1} A_{\lambda}$.

Since $A_{\lambda}$ is indecomposable, the only idempotents of $A_{\lambda}$ are $0_{A}$ and $1_{A_{\lambda}}$. Hence, for any idempotent $e$ of $A$ the projection $pr_{\lambda}(e)$ is either $0_{A}$, or $1_{A_{\lambda}}$. In particular,

$$eA = \prod_{\lambda \in \Lambda_1} A_{\lambda},$$

(25)

where $\Lambda_1 = \{\lambda \in \Lambda \mid pr_{\lambda}(e) = 1_{A_{\lambda}}\}$. Thus, the unital ideals of $A$ are exactly the products of blocks $A_{\lambda}$ over all $\Lambda_1 \subseteq \Lambda$.

**Lemma 3.1.** Let $I = \prod_{\lambda \in \Lambda_1} A_{\lambda}$ and $J = \prod_{\lambda \in \Lambda_2} A_{\lambda}$ be unital ideals of $A$ and $\varphi : I \rightarrow J$ an isomorphism. Then there exists a bijection $\sigma : \Lambda_1 \rightarrow \Lambda_2$, such that $\varphi(pr_{\lambda}(a)) = pr_{\sigma(\lambda)}(\varphi(a))$ for all $a \in I$ and $\lambda \in \Lambda_1$.

**Proof.** Note that $\{1_{A_{\lambda}}\}_{\lambda \in \Lambda_1}$ and $\{1_{A_{\lambda}}\}_{\lambda \in \Lambda_2}$ are the sets of indecomposable idempotents of $I$ and $J$, respectively. Since $\varphi$ is an isomorphism, $\varphi(1_{A_{\lambda}}) = 1_{A_{\sigma(\lambda)}}$ for some bijection $\sigma : \Lambda_1 \rightarrow \Lambda_2$. Then $\varphi(pr_{\lambda}(a)) = \varphi(1_{A_{\lambda}}a) = 1_{A_{\sigma(\lambda)}}\varphi(a) = pr_{\sigma(\lambda)}(\varphi(a))$. 

Let $\alpha = \{\alpha_x : D_{x^{-1}} \rightarrow D_x \mid x \in G\}$ be a unital partial action of $G$ on $A$. By the observation above each domain $D_x$ is a product of blocks, and $\alpha_x$ maps a block of $D_{x^{-1}}$ onto some block of $D_x$. As in [24] we call $\alpha$ transitive, when for any pair $\lambda', \lambda'' \in \Lambda$ there exists $x \in G$, such that $A_{\lambda'} \subseteq D_{x^{-1}}$ and $\alpha_x(A_{\lambda'}) = A_{\lambda''} \subseteq D_x$.

In all what follows, if otherwise is not stated, we assume that $\alpha$ is transitive. Then we may fix $\lambda_0 \in \Lambda$, so that each $A_{\lambda}$ is $\alpha_x(A_{\lambda_0})$ for some $x \in G$ with $A_{\lambda_0} \subseteq D_{x^{-1}}$. Observe that, whenever $A_{\lambda_0} \subseteq D_{x^{-1}}D_{x'^{-1}}$ and $\alpha_x(A_{\lambda_0}) = \alpha_x'(A_{\lambda_0})$, it follows that $A_{\lambda_0} \subseteq D_{(x')^{-1}x}$ and $\alpha_{x^{-1}x'}(A_{\lambda_0}) = A_{\lambda_0}$. Hence, introducing as in [24] the subgroup

$$H = \{x \in G \mid A_{\lambda_0} \subseteq D_{x^{-1}} \text{ and } \alpha_x(A_{\lambda_0}) = A_{\lambda_0}\}$$

and choosing a left transversal $\Lambda'$ of $H$ in $G$, one may identify $\Lambda$ with a subset of $\Lambda'$, namely, $\lambda \in \Lambda$ corresponds to (a unique) $g \in \Lambda'$, such that $A_{\lambda_0} \subseteq D_{g^{-1}}$ and $\alpha_g(A_{\lambda_0}) = A_{\lambda}$. Assume, moreover, that $\Lambda'$ contains the identity element $1$ of $G$. Then $\lambda_0$ is identified with 1 and thus

$$A_g = \alpha_g(A_1) \text{ for } g \in \Lambda \subseteq \Lambda'.$$

Given $x \in G$, we use the notation $\bar{x}$ from [24] for the element of $\Lambda'$ with $x \in \bar{x}H$. We recall the following useful fact.

**Lemma 3.2** (Lemma 5.1 from [24]). Given $x \in G$ and $g \in \Lambda'$, one has

(i) $g \in \Lambda$ $\Leftrightarrow$ $A_1 \subseteq D_{g^{-1}}$;

(ii) if $g \in \Lambda$, then $\bar{x}g \in \Lambda$ $\Leftrightarrow$ $A_g \subseteq D_{x^{-1}}$, and in this situation $\alpha_x(A_g) = A_{\bar{x}g}$.

Notice that taking $g = 1$ in (ii) one gets $\bar{x} \in \Lambda$ $\Leftrightarrow$ $A_1 \subseteq D_{x^{-1}}$. Then using (ii)

once again, we see that for any $g \in \Lambda$

$$A_g \subseteq D_x \Leftrightarrow \bar{x}g \in \Lambda \Leftrightarrow A_1 \subseteq D_{g^{-1}x}.$$

(27)
In particular, \( A_x \subseteq D_x \) for all \( x \in G \), such that \( x \in \varLambda \).

For any \( g \in \varLambda \) and \( a \in A \) define
\[
\theta_g(a) = \alpha_g(\text{pr}_1(a)).
\] (28)

Note that by (i) of Lemma 3.2 the block \( A_1 \) is a subset of \( D_{g^{-1}} \), so \( \alpha_g(\text{pr}_1(a)) \) makes sense and belongs to \( A_g \). Thus, \( \theta_g \) is a correctly defined homomorphism \( A \to A_g \).

Clearly,
\[
\theta_g(a) = \theta_g(1_x a)
\] (29)

for any \( x \in G \), such that \( A_1 \subseteq D_x \). In particular, this holds for \( x = g^{-1} \).

Observe also that
\[
\theta_g(a) = \theta_g(1_{g^{-1}} a) = \text{pr}_g(\alpha_g(1_{g^{-1}} a))
\] (30)

in view of Lemma 3.1. It follows that
\[
\alpha_g(\text{pr}_g(1_{g^{-1}} a)) = \theta_g(1_{g^{-1}} a) = \text{pr}_g(\alpha_g(1_{g^{-1}} a))
\] (31)

Lemma 3.3. Let \( n > 0 \) and \( w \in Z^n(G, A) \). Then
\[
w(x_1, \ldots, x_n) = \prod_{g \in \varLambda} \theta_g[w(g^{-1} x_1, x_2, \ldots, x_n)]
\]

As \( w \in Z^n(G, A) \), one has
\[
1_{(g^{-1} x_1, \ldots, x_n)} = (\delta^n w)(g^{-1} x_1, \ldots, x_n)
\]
\[
= \alpha_{g^{-1}}(1_g w(x_1, \ldots, x_n)) \cdot w(g^{-1} x_1, x_2, \ldots, x_n)^{-1}
\]
\[
\prod_{k=1}^{n-1} w(g^{-1} x_k x_{k+1}, \ldots, x_n)^{(1-1)^k}
\]
\[
w(g^{-1}, x_1, \ldots, x_{n-1})^{(-1)^n}.
\]

Hence,
\[
\alpha_{g^{-1}}(1_g w(x_1, \ldots, x_n)) = w(g^{-1} x_1, x_2, \ldots, x_n)
\]
\[
\prod_{k=1}^{n-1} w(g^{-1} x_k x_{k+1}, \ldots, x_n)^{(1-1)^k}
\]
\[
w(g^{-1}, x_1, \ldots, x_{n-1})^{(-1)^n}.
\]
Given \( x \in G \), denote by \( \eta(x) \) the element \( x^{-1}x \in H \). Let \( n > 0 \) and \( g \in \Lambda' \). Define \( \eta^g_n : G^n \to H \) by
\[
\eta^g_n(x_1, \ldots, x_n) = \eta(x^{-1}_n \cdots x^{-1}_1 (x_1) g)
\]
and \( \tau^g_n : G^n \to H^n \) by
\[
\tau^g_n(x_1, \ldots, x_n) = (\eta^g_n(x_1), \eta^g_n(x_1, x_2), \ldots, \eta^g_n(x_1, \ldots, x_n)).
\]
Observe that
\[
\eta^g_n(x_1) \eta^g_n(x_1, x_2) \cdots \eta^g_n(x_1, \ldots, x_n) = \eta(x^{-1}_n \cdots x^{-1}_1) = \eta^g_n(x_1 \cdots x_n).
\]
We shall also need the functions \( \sigma^g_{n,i} : G^n \to G^{n+1} \), \( 0 \leq i \leq n \), defined by
\[
\sigma^g_{n,0}(x_1, \ldots, x_n) = (g^{-1}, x_1, x_2, \ldots, x_n),
\]
\[
\sigma^g_{n,i}(x_1, \ldots, x_n) = (\tau^g_i(x_1, \ldots, x_i), (x_1^{-1} \cdots x^{-1}_1 g)^{-1}, x_{i+1}, \ldots, x_n), \quad 0 < i < n,
\]
\[
\sigma^g_{n,n}(x_1, \ldots, x_n) = (\tau^g_n(x_1, \ldots, x_n), (x_1^{-1} \cdots x^{-1}_1 g)^{-1}).
\]
In the formulas above we may allow \( n \) to be equal to zero, meaning that \( \sigma^g_{0,0} = g^{-1} \in G \).

With any \( n > 0 \) and \( w \in C^n(G, A) \) we shall associate
\[
w'(x_1, \ldots, x_n) = 1_{(x_1, \ldots, x_n)} \prod_{g \in \Lambda} \theta_g \circ w \circ \tau^g_n(x_1, \ldots, x_n),
\]
\[
\varepsilon(x_1, \ldots, x_{n-1}) = 1_{(x_1, \ldots, x_{n-1})} \prod_{g \in \Lambda} \theta_g \left( \prod_{i=0}^{n-1} w \circ \sigma^g_{n-1,i}(x_1, \ldots, x_{n-1})^{(-1)'} \right).
\]

**Lemma 3.4.** One has \( w' \in C^n(G, A) \) and \( \varepsilon \in C^{n-1}(G, A) \).

**Proof.** Notice by (34) and (35) that
\[
w \circ \tau^g_n(x_1, \ldots, x_n) \in \mathcal{U}(D_{\eta(x^{-1}_k) \cdots \eta(x^{-1}_2)}) \cdots D_{\eta(x^{-1}_n \cdots x^{-1}_1)}).
\]
Since \( \eta(x^{-1}_k \cdots x^{-1}_1) \in H \), then \( A_1 \subseteq D_{\eta(x^{-1}_k \cdots x^{-1}_1)} \), \( 1 \leq k \leq n \), so
\[
\text{pr}_1 \circ w \circ \tau^g_n(x_1, \ldots, x_n) \in \mathcal{U}(A_1)
\]
and hence by (38)
\[
\theta_g \circ w \circ \tau^g_n(x_1, \ldots, x_n) = \alpha_g \circ \text{pr}_1 \circ w \circ \tau^g_n(x_1, \ldots, x_n) \in \mathcal{U}(A_g).
\]
Therefore, the product of the values of \( \theta_g \) on the right-hand side of (39) belongs to \( \mathcal{U}(A) \) and thus \( w' \in C^n(G, A) \).

To prove that \( \varepsilon \in C^{n-1}(G, A) \) for \( n > 1 \), observe first that the right-hand side of (40) depends only on \( \theta_g \) with \( g \in \Lambda \) satisfying
\[
A_g \subseteq D_{(x_1, \ldots, x_{n-1})}
\]
(if there is no such \( g \), then \( D_{(x_1, \ldots, x_{n-1})} \) is zero and thus \( \varepsilon(x_1, \ldots, x_{n-1}) \) is automatically invertible in this ideal). Now
\[
\prod_{i=0}^{n-1} w \circ \sigma^g_{n-1,i}(x_1, \ldots, x_{n-1}) \in \mathcal{U}(D_{(g^{-1}, x_1, \ldots, x_{n-1})} D_{\eta(x^{-1}_1) \cdots D_{\eta(x^{-1}_n)}}).
\]
As above, $A_1 \subseteq D_{\eta(x_1^{-1} \ldots x_n^{-1}g)}$, $1 \leq k \leq n - 1$, because $\eta(x_1^{-1} \ldots x_n^{-1}g) \in H$. Moreover, by (27) condition (41) is equivalent to $A_1 \subseteq D_{(g^{-1}, x_1, \ldots, x_{n-1})}$. The rest of the proof now follows as for $w'$. If $n = 1$, then

$$
\varepsilon = \prod_{g \in \Lambda} \theta_g(w(g^{-1})) \in \mathcal{U}(A),
$$

(42)

as $D_{g^{-1}} \supseteq A_1$ by (1) of Lemma 3.2.

The following notation will be used in the results below.

$$
\Pi(l, m) = \prod_{k=l, i=m}^{n-1} w \circ \sigma_{n-1, i}(x_1, \ldots, x_kx_{k+1}, \ldots, x_n)(-1)^{k+i}
$$

$$
\prod_{k=i}^{n-1} w \circ \sigma_{n-1, i}(x_1, \ldots, x_{n-1})(-1)^{n+i},
$$

(43)

where $1 \leq l \leq n - 1$ and $0 \leq m \leq n - 1$.

**Lemma 3.5.** For all $w \in Z^1(G, A)$ and $x \in G$ we have:

$$
(\delta^0 \varepsilon)(x)\alpha_x(1_{x^{-1}}\varepsilon)^{-1}w(x)^{-1} = \prod_{g \in \Lambda} \theta_g(w(g^{-1}x)^{-1}).
$$

(44)

Moreover, for $n > 1$, $w \in Z^n(G, A)$ and $x_1, \ldots, x_n \in G$:

$$
(\delta^n \varepsilon)(x_1, \ldots, x_n)\alpha_x(1_{x^{-1}}\varepsilon(x_2, \ldots, x_n))^{-1}w(x_1, \ldots, x_n)^{-1}
$$

$$
= \prod_{g \in \Lambda} \theta_g(w(g^{-1}x_1, x_2, \ldots, x_n)^{-1} \Pi(1, 1)).
$$

(45)

**Proof.** Indeed, by (33), (32) and (12) we see that

$$
(\delta^0 \varepsilon)(x)\alpha_x(1_{x^{-1}}\varepsilon)^{-1}w(x)^{-1} = \varepsilon^{-1}w(x)^{-1}
$$

$$
= \prod_{g \in \Lambda} \theta_g(w(g^{-1})^{-1}w(g^{-1}x)^{-1}w(g^{-1}))
$$

$$
= \prod_{g \in \Lambda} \theta_g(w(g^{-1}x)^{-1}).
$$

For (45) observe from (2), (40) and (43) that

$$
(\delta^n \varepsilon)(x_1, \ldots, x_n)\alpha_x(1_{x^{-1}}\varepsilon(x_2, \ldots, x_n))^{-1}
$$

$$
= \prod_{k=1}^{n-1} \varepsilon(x_1, \ldots, x_kx_{k+1}, \ldots, x_n)(-1)^{k}\varepsilon(x_1, \ldots, x_{n-1})(-1)^{n}
$$

$$
= \prod_{g \in \Lambda} \theta_g(\Pi(1, 0)).
$$

Now in (32) one has

$$
w(g^{-1}, x_1, \ldots, x_kx_{k+1}, \ldots, x_n)(-1)^k = w \circ \sigma_{n-1, 0}(x_1, \ldots, x_kx_{k+1}, \ldots, x_n)(-1)^{k+0},
$$

$$
w(g^{-1}, x_1, \ldots, x_{n-1})(-1)^{n} = w \circ \sigma_{n-1, 0}(x_1, \ldots, x_{n-1})(-1)^{n+0},
$$

which are the factors of $\Pi(1, 0)$ corresponding to $i = 0$ and $1 \leq k \leq n - 1$. Hence,

$$
\prod_{g \in \Lambda} \theta_g(\Pi(1, 0)) = w(x_1, \ldots, x_n) \prod_{g \in \Lambda} \theta_g(w(g^{-1}x_1, x_2, \ldots, x_n)^{-1} \Pi(1, 1)).
$$
Lemma 3.6. For all \( n \geq 1 \), \( w \in Z^n(G, \mathcal{A}) \), \( g \in \Lambda \) and \( x_1, \ldots, x_n \in G \):

\[
w(g^{-1}x_1, x_2, \ldots, x_n)^{-1}\Pi(1, 1) = \alpha_{\eta_i^g(x_1)}(1_{\eta_i^g(x_1)}^{-1}w \circ \sigma_{n-1, 0}^g(x_2, \ldots, x_n))^{-1} \\
w(\tau_i^g(x_1), (x_1^{-1}g)^{-1}x_2, x_3, \ldots, x_n)^{-1}\Pi(2, 2) \\
\prod_{i=1}^{n-1} w \circ \sigma_{n-1, 1}^g(x_1x_2, x_3, \ldots, x_n)^{(-1)^{n+1}}. \tag{46}
\]

Proof. Since \( w \) is a partial \( n \)-cocycle, one has that (see (33), (34) and (37))

\[
(\delta^n w) \circ \sigma_{n, 1}^g(x_1, \ldots, x_n) = (\delta^n w)(\tau_i^g(x_1), (x_1^{-1}g)^{-1}, x_2, \ldots, x_n) \\
= (\delta^n w)(\eta_i^g(x_1), (x_1^{-1}g)^{-1}, x_2, \ldots, x_n) \\
= (\delta^n w)(g^{-1}x_1 \cdot x_1^{-1}g, (x_1^{-1}g)^{-1}, x_2, \ldots, x_n) \tag{47}
\]

Applying (2), we expand (47) as follows:

\[
1_{g^{-1}x_1, x_1^{-1}g}(g^{-1}x_1, x_2, \ldots, x_n) = \alpha_{g^{-1}x_1, x_1^{-1}g}^{-1}w((x_1^{-1}g)^{-1}, x_2, \ldots, x_n) \\
w(g^{-1}x_1, x_2, \ldots, x_n)^{-1} \\
w(g^{-1}x_1 \cdot x_1^{-1}g, (x_1^{-1}g)^{-1}, x_2, x_3, \ldots, x_n) \\
\prod_{k=2}^{n-1} w(g^{-1}x_1 \cdot x_1^{-1}g, (x_1^{-1}g)^{-1}, x_2, \ldots, x_kx_{k+1}, \ldots, x_n)^{(-1)^{k+1}} \\
w(g^{-1}x_1 \cdot x_1^{-1}g, (x_1^{-1}g)^{-1}, x_2, \ldots, x_{n-1})^{(-1)^{n+1}}.
\]

Using our notations (33), (34), (37) and (38), we conclude that

\[
1_{\eta_i^g(x_1)}w(g^{-1}x_1, \ldots, x_n)^{-1} = \alpha_{\eta_i^g(x_1)}(1_{\eta_i^g(x_1)}^{-1}w \circ \sigma_{n-1, 0}^g(x_2, \ldots, x_n))^{-1} \tag{48}
\]

\[
w(\tau_i^g(x_1), (x_1^{-1}g)^{-1}x_2, x_3, \ldots, x_n)^{-1} \\
\prod_{k=2}^{n-1} w \circ \sigma_{n-1, 1}^g(x_1, \ldots, x_kx_{k+1}, \ldots, x_n)^{(-1)^{k}} \tag{49}
\]

\[
w \circ \sigma_{n-1, 1}^g(x_1, x_2, \ldots, x_n)^{-1} \tag{50}
\]

the lines (51) and (51) being the inverses of the factors of \( \Pi(1, 1) \), which correspond to \( i = 1 \) and \( 2 \leq k \leq n - 1 \). Thus, after the multiplication of the right-hand side of equality (48) by \( \Pi(1, 1) \), they will be reduced, and at their place we shall have the factors of \( \Pi(1, 1) \) which correspond to \( k = 1 \), and the factors of \( \Pi(2, 2) \) (i.e. those of \( \Pi(1, 1) \) with indexes \( 2 \leq i, k \leq n - 1 \)), giving the right-hand side of (46). It remains to note that \( 1_{\eta_i^g(x_1)}\Pi(1, 1) = \Pi(1, 1) \) and the idempotents which appear in the cancellations are absorbed by the element (49), except \( 1_{x_1^{-1}g} \) which is absorbed by the element in the right hand side of (38).

Lemma 3.7. For all \( 1 < j < n \), \( w \in Z^n(G, \mathcal{A}) \), \( g \in \Lambda \) and \( x_1, \ldots, x_n \in G \):

\[
w(\tau_j^g(x_1, \ldots, x_{j-1}), (x_{j-1}^{-1}x_1^{-1}g)^{-1}x_j, x_{j+1}, \ldots, x_n)^{-1}\Pi(j, j)
\]

\[
w(\tau_j^g(x_1, \ldots, x_{j-1}), (x_{j-1}^{-1}x_1^{-1}g)^{-1}x_j, x_{j+1}, \ldots, x_n)^{-1}\Pi(j, j)
\]
Expanding (53), we obtain by (2)

\[
\Pi\left(\left(\text{here by } \Pi(n,n) \text{ we mean the identity element } 1_A\right)\right)
\]

(52)

**Proof.** We use the same idea as in the proof of Lemma 3.6.

\[
(\delta^n w) \circ \sigma_{n,j}^g(x_1, \ldots, x_n)
\]

\[
= (\delta^n w)(\sigma_{j}^g(x_1, \ldots, x_j), (x_j^{-1} \cdots x_i^{-1}g)^{-1}, x_{j+1}, \ldots, x_n)
\]

\[
= (\delta^n w)(g^{-1}x_1x_1^{-1}g, (x_j^{-1}g)^{-1}x_2x_2^{-1}g, \ldots, (x_j^{-1} \cdots x_1^{-1}g)^{-1}, x_{j+1}, \ldots, x_n)
\]

\[
= 1_{g^{-1}x_1x_1^{-1}g}g^{-1}x_1x_2x_2^{-1}g^{-1} \cdots g^{-1}x_1 \cdots x_j^{-1}g^{-1}(g^{-1}x_1 \cdots x_{j+1} \cdots x_n).
\]

Expanding (53), we obtain by (2)

\[
1_{g^{-1}x_1x_1^{-1}g}g^{-1}x_1x_2x_2^{-1}g^{-1} \cdots g^{-1}x_1 \cdots x_j^{-1}g^{-1}(g^{-1}x_1 \cdots x_{j+1} \cdots x_n)
\]

\[
= \alpha^{-1}x_1g^{-1}x_1^{-1}g^{-1}x_1^{-1}g^{-1}w\left((x_1^{-1}g)^{-1}x_2x_2^{-1}g, \right.
\]

\[
\left.\cdots, (x_j^{-1} \cdots x_1^{-1}g)^{-1}x_{j+1} \cdots x_n\right)
\]

\[
= \prod_{s=2}^{j-1} w(g^{-1}x_1x_1^{-1}g, \ldots, (x_s^{-1} \cdots x_1^{-1}g)^{-1}x_sx_s+1x_{s+1} \cdots x_j^{-1}g, \ldots)
\]

\[
= \prod_{t=j+1}^{n-1} w(g^{-1}x_1x_1^{-1}g, \ldots, (x_j^{-1} \cdots x_1^{-1}g)^{-1}x_{j+1} \cdots x_n)(-1)^{j+1}
\]

Therefore,

\[
= \alpha^{-1}x_1g^{-1}x_1^{-1}g^{-1}x_1^{-1}g^{-1}(g^{-1}x_1 \cdots x_{j+1} \cdots x_n).
\]
The lines (58) and (59) are the inverses of the factors of \( \prod(\cdot) \), as 1

Note that the factors (54) and (56) may be included into the product (55), permitting thus \( s \) run from 1 to \( j - 1 \) in (55). It follows that

\[
1_{\eta_i^n}(x_1)1_{\eta_j^n}(x_1 x_2) \cdots 1_{\eta_i^n}(x_1,\ldots,x_j)1_{(g^{-1}x_1 \cdots x_{j+1},\ldots,x_n)}(1)\]

\[
= \alpha_{\eta_i^n}(x_1)(1_{\eta_j^n}(x_1)1_{\eta_j^n}(x_1)w \circ \sigma_{n-1,j-1}^g(x_2,\ldots,x_n))
\]

\[
= \prod_{s=2}^{j-2} w \circ \sigma_{n-1,j-1}^g(x_1,\ldots,x_{s}x_{s+1},\ldots,x_n)^{-1}
\]

\[
= \prod_{t=j+1}^{n-1} w \circ \sigma_{n-1,j}^g(x_1,\ldots,x_{t}x_{t+1},\ldots,x_n)^{-1}
\]

Note that the factors (58) and (59) may be included into the product (55).}

The lines (58) and (59) are the inverses of the factors of \( \Pi(j,j) \) corresponding to \( i = j \) and \( j + 1 \leq k \leq n - 1 \). Therefore, multiplication by \( \Pi(j,j) \) replaces these two lines by the factors of \( \Pi(j,j) \) with \( j \leq i \leq n-1 \) and \( k = j \), and, whenever \( j < n-1 \), there will also appear all the factors of \( \Pi(j+1,j+1) \), giving the right-hand side of equality (58). Finally, the left-hand side of (58) coincides with (57) multiplied by \( \Pi(j,j) \), as \( 1_{\eta_i^n}(x_1,\ldots,x_j)\Pi(j,j) = \Pi(j,j) \).

\[\square\]

**Lemma 3.8.** For all \( w \in Z^1(G,A) \), \( g \in \Lambda \) and \( x \in G \):

\[
1_{\eta_i^n}(x)w(g^{-1}x)^{-1} = \alpha_{\eta_i^n}(x)(1_{\eta_j^n}(x)1_{\eta_j^n}(x)w((x^{-1}g)^{-1})^{-1})(w \circ \sigma_j^g)(x)^{-1}.
\]
Moreover, for all $n > 1$, $w \in Z^n(G,A)$, $g \in \Lambda$ and $x_1, \ldots, x_n \in G$:

\[
\begin{align*}
   w(\tau_{n-1}^g(x_1, \ldots, x_{n-1}), (x_n^{-1} \cdots x_1^{-1} g)^{-1} x_n)^{-1} \\
   = \alpha_{n_1^g(x_1)}(1_{n_1^g(x_1)-1} w \circ \sigma_{n-1,n-1}^{-1}(x_2, \ldots, x_n))^{(-1)^n} \\
   \prod_{s=1}^{n-1} w \circ \sigma_{n-1,n-1}^{-1}(x_1, \ldots, x_s x_{s+1}, \ldots, x_n)^{(-1)^{s+n}} \\
   w \circ \tau_n^g(x_1, \ldots, x_n)^{-1}.
\end{align*}
\]

(61)

Proof. For (60) write

\[
1_{n_1^g(x_1)}1_{g^{-1}x_1} = (\delta^1 w) \circ \sigma_{1,1}^g(x)
\]

\[
= (\delta^1 w) \left(1_{n_1^g(x_1)-1} w \circ \sigma_{n-1,n-1}^{-1}(x_2, \ldots, x_n)\right) \\
= \alpha_{n_1^g(x_1)}(1_{n_1^g(x_1)-1} w \circ \sigma_{n-1,n-1}^{-1}(x_2, \ldots, x_n)) \\
\prod_{s=1}^{n-1} w \circ \sigma_{n-1,n-1}^{-1}(x_1, \ldots, x_s x_{s+1}, \ldots, x_n)^{(-1)^{s+n}} \\
\]

To get (61), analyze the proof of Lemma 3.7 (we skip the details):

\[
\begin{align*}
   1_{n_1^g(x_1)}1_{n_1^g(x_1)} \cdots 1_{n_1^g(x_1)}1_{g^{-1}x_1} \ldots x_n \\
   = (\delta^1 w) \circ \sigma_{n,n}^g(x_1, \ldots, x_n) \\
   = \alpha_{n_1^g(x_1)}(1_{n_1^g(x_1)-1} w \circ \sigma_{n-1,n-1}^{-1}(x_2, \ldots, x_n)) \\
   \prod_{s=1}^{n-1} w \circ \sigma_{n-1,n-1}^{-1}(x_1, \ldots, x_s x_{s+1}, \ldots, x_n)^{(-1)^{s+n}} \\
   \]

(62)

\[
\begin{align*}
\prod_{g \in \Lambda} x_g \circ \alpha_{n_1^g(x_1)} \left(1_{n_1^g(x_1)-1} \prod_{j=0}^{n-1} w \circ \sigma_{n-1,n-1}^{-1}(x_2, \ldots, x_n)^{(-1)^{j+1}}\right) \\
\]

w'(x_1 \ldots, x_n)^{-1}.
\]

(62)

Lemma 3.9. For all $n > 0$, $w \in Z^n(G,A)$ and $x_1, \ldots, x_n \in G$:

\[
(\delta^n \varepsilon)(x_1, \ldots, x_n) \alpha_{x_1}(1_{x_1^{-1} \varepsilon}(x_2, \ldots, x_n))^{-1} w(x_1, \ldots, x_n)^{-1} \\
= \prod_{g \in \Lambda} \theta_g \circ \alpha_{n_1^g(x_1)} \left(1_{n_1^g(x_1)-1} \prod_{j=0}^{n-1} w \circ \sigma_{n-1,n-1}^{-1}(x_2, \ldots, x_n)^{(-1)^{j+1}}\right) \\
\]

w'(x_1 \ldots, x_n)^{-1}.
\]

(62)

Proof. If $n = 1$, then the result follows from (29), (30), (41) and (50) and the fact that $\eta_1^g(x) \in H$.

Let $n > 1$. Using the recursion whose base is (40), an intermediate step is (51) and the final step is (61), we have

\[
\begin{align*}
   w(g^{-1}x_1, \ldots, x_n)^{-1} \Pi(1,1) \\
   = \alpha_{n_1^g(x_1)} \left(1_{n_1^g(x_1)-1} \prod_{j=0}^{n-1} w \circ \sigma_{n-1,n-1}^{-1}(x_2, \ldots, x_n)^{(-1)^{j+1}}\right) \\
   \]

w'(x_1 \ldots, x_n)^{-1}
\]
The latter is exactly the inverse of (63). Hence,

\[ \prod_{j=1}^{n-1} \prod_{i=j}^{n-1} w \circ \sigma_{n-i,1}^g(x_1, \ldots, x_j x_{j+1}, \ldots, x_n)^{(-1)^{i+j}} \]

(63)

Now switching the order in this double product, we come to

\[ \prod_{j=2}^{n-1} \prod_{s=1}^{j-1} w \circ \sigma_{n-1,j-1}^g(x_1, \ldots, x_s x_{s+1}, \ldots, x_n)^{(-1)^{s+j}} \]

(64)

After the change of indexes \( j' = j - 1 \) the product (64) becomes

\[ \prod_{j=1}^{n-1} \prod_{s=1}^{j'} w \circ \sigma_{n-1,j'}^g(x_1, \ldots, x_s x_{s+1}, \ldots, x_n)^{(-1)^{s+j'+1}} \]

Now turning the order in this double product, we come to

\[ \prod_{s=1}^{n-1} \prod_{j'=s}^{n-1} w \circ \sigma_{n-1,j'}^g(x_1, \ldots, x_s x_{s+1}, \ldots, x_n)^{(-1)^{s+j'+1}} \]

The latter is exactly the inverse of (63). Hence,

\[ w(g^{-1} x_1, x_2, \ldots, x_n)^{-1}\prod(1,1) \]

\[ = \alpha_{\eta^g_n}(x_1) \left( \prod_{j=0}^{n-1} w \circ \sigma_{1,j}^g(x_2, \ldots, x_n)^{(-1)^{j+1}} \right) \]

\[ w \circ \tau_{(1,1)}^g(x_1, \ldots, x_n)^{-1}. \]

It remains to substitute this into (63) and to apply (39). \( \Box \)

**Lemma 3.10.** For all \( x \in G \) and \( a : \Lambda' \rightarrow \Lambda \) one has

\[ \alpha_x \left( 1_{x^{-1}} \prod_{g \in \Lambda} \theta_g(a(g)) \right) = 1_x \prod_{g \in \Lambda} \theta_x.g \circ \alpha_{\eta^g_n}(x) \left( 1_{\eta^g_n(1g^{-1})} \alpha \left( x^{-1}g \right) \right). \]

(65)

**Proof.** First of all observe using (ii) of Lemma 3.2 that

\[ 1_x \prod_{g \in \Lambda} c_g = 1_x \prod_{g \in \Lambda, g \in \Lambda \subseteq D_x} c_g = 1_x \prod_{g, g^{-1} x \in \Lambda} c_g, \]

(66)

where \( c_g \) is an arbitrary element of \( A_g \). Thus, in the right-hand side of (65) we may replace the condition \( g \in \Lambda \) by a stronger one \( g, x^{-1}g \in \Lambda \). Notice also from (27) and (29) that we may put \( 1_{g^{-1} x}^{-1} \) inside of \( \theta_x.g \) in the right-hand side of (65).

Now

\[ 1_{g^{-1} x} \alpha_{\eta^g_n}(x) \left( 1_{\eta^g_n(1g^{-1})} \alpha \left( x^{-1}g \right) \right) = \alpha_{g^{-1} x} \circ \alpha_{x^{-1}g} \left( 1_{(x^{-1}g)^{-1} x} \alpha \left( x^{-1}g \right) \right), \]

(67)

and denoting the argument of \( \alpha_{x^{-1}g} \) in (67) by \( b = b(g, x) \), we deduce from (30) that

\[ \theta_x.g \circ \alpha_{g^{-1} x} \circ \alpha_{x^{-1}g} (b) = \text{pr}_g \circ \alpha_g \left( 1_{g^{-1} x} \alpha_{g^{-1} x} \circ \alpha_{x^{-1}g} (b) \right) \]

\[ = \text{pr}_g \circ \alpha_g \circ \alpha_{g^{-1} x} \circ \alpha_x \left( 1_{x^{-1} g^{-1} x} \alpha \left( x^{-1}g \right) \right) \]

\[ = \text{pr}_g \circ \alpha_x \left( 1_{x^{-1} g^{-1} x} \alpha \left( x^{-1}g \right) \right). \]
As \( xx^{-1}g = g \in \Lambda \), by (41) of Lemma 3.2 we have \( A_{xx^{-1}g} \subseteq D_{x^{-1}} \) and \( \alpha_x \left( \frac{A_{x^{-1}g}}{x} \right) = \mathcal{A}_g \). Moreover, \( A_{xx^{-1}g} \subseteq D_{x^{-1}g} \) by (27). Hence, in view of Lemma 3.1 and (30)

\[
\text{pr}_g \circ \alpha_x \left( 1_{x^{-1}1_{x^{-1}g}} \frac{A_{x^{-1}g}}{x} (b) \right) = \alpha_x \circ \text{pr}_{x^{-1}g} \left( 1_{x^{-1}1_{x^{-1}g}} \frac{A_{x^{-1}g}}{x} (b) \right) = \alpha_x \circ \text{pr}_{x^{-1}g} (b) = \alpha_x \circ \theta_x (b),
\]

and consequently

\[
\theta_g \circ \alpha_{\eta'_l}(x) \left( 1_{\eta'_l(x)^{-1}} a \left( \frac{a}{x} g \right) \right) = \alpha_x \circ \theta_{x^{-1}g} (b) = \alpha_x \circ \theta_{x^{-1}g} \left( a \left( \frac{a}{x} g \right) \right).
\]

Here we used (29) to remove \( 1_{\eta'_l(x)^{-1}} \) and \( 1_{(x^{-1}g)^{-1}} \) from \( b \). It follows that the right-hand side of (65) is

\[
\prod_{g, x^{-1}g \in \Lambda} \alpha_x \circ \theta_{x^{-1}g} \left( a \left( \frac{a}{x} g \right) \right) = \alpha_x \left( \prod_{g, x^{-1}g \in \Lambda} \theta_{x^{-1}g} \left( a \left( \frac{a}{x} g \right) \right) \right),
\]

which is verified by checking the projection of each side of the latter equality onto an arbitrary block \( \mathcal{A}_t, t \in \Lambda \). Let \( g' = \frac{a}{x} g \in \Lambda \). Then \( g = xx^{-1}g = xy^x \in \Lambda \), so (68) becomes, in view of (60),

\[
\alpha_x \left( \prod_{g', x^{-1}g' \in \Lambda} \theta_{g'} \left( a \left( g' \right) \right) \right) = \alpha_x \left( 1_{x^{-1}} \prod_{g' \in \Lambda} \theta_{g'} \left( a \left( g' \right) \right) \right),
\]

proving (65). \( \square \)

**Lemma 3.11.** For all \( n > 0, w \in \mathbb{Z}^n(G, A) \) and \( x_1, \ldots, x_n \in G \):

\[
1_{(x_1, \ldots, x_n)} \prod_{g \in \Lambda} \theta_g \circ \alpha_{\eta'_l}(x_1) \left( 1_{\eta'_l(x_1)^{-1}} \prod_{j=0}^{n-1} w \circ \sigma_{n-1,j}^{-1}(x_2, \ldots, x_n)^{-1} \right)
\]

\[
= \alpha_{x_1} \left( 1_{x_1^{-1}} \xi(x_2, \ldots, x_n) \right).
\]

**Proof.** Let us fix \( n, w \) and \( x_2, \ldots, x_n \). For arbitrary \( g \in \Lambda' \) define

\[
a(g) = \prod_{j=0}^{n-1} w \circ \sigma_{n-1,j}^{-1}(x_2, \ldots, x_n)^{-1} j.
\]

Then the left-hand side of (69) equals

\[
1_{(x_1, \ldots, x_n)} \prod_{g \in \Lambda} \theta_g \circ \alpha_{\eta'_l}(x_1) \left( 1_{\eta'_l(x_1)^{-1}} a \left( \frac{a}{x_1} g \right) \right).
\]

Since \( 1_{(x_1, \ldots, x_n)} = 1_{(x_1, \ldots, x_n)} 1_{x_1} \), then applying Lemma 3.10 we transform this into

\[
1_{(x_1, \ldots, x_n)} \alpha_{x_1} \left( 1_{x_1^{-1}} \prod_{g \in \Lambda} \theta_g \left( \prod_{j=0}^{n-1} w \circ \sigma_{n-1,j}^{-1}(x_2, \ldots, x_n)^{-1} \right) \right).
\]

Rewriting \( 1_{(x_1, \ldots, x_n)} \) as \( \alpha_{x_1} \left( 1_{x_1^{-1}}(x_2, \ldots, x_n) \right) \) and using (40), we come to the right-hand side of (69). \( \square \)
Theorem 3.12. Let $n > 0$ and $w \in Z^n(G, \mathcal{A})$. Then $w = \delta^{n-1} \varepsilon \cdot w'$. In particular, $w' \in Z^n(G, \mathcal{A})$.

Proof. This is an immediate consequence of Lemmas 3.9 and 3.11.

4. Existence of a globalization

In this section we construct the cocycle $\bar{w}$ whose existence was announced above. Keeping the notations of Section 2, we begin with some auxiliary formulas whose proof will be left to the reader.

Lemma 4.1. Let $g \in \Lambda'$. Then

$$\eta^n_0(x_1, \ldots, x_n) = \eta^{n-1}_1(x_2, \ldots, x_n), \quad n \geq 2,$$  
(70)

$$\eta^n_i(x_1, \ldots, x_i, x_{i+1}, \ldots, x_n) = \eta^{n-2}_i(x_1, \ldots, x_i, x_{i+1}, x_{i+2}, \ldots, x_n), \quad 1 \leq i \leq n-2,$$  
(71)

$$\eta^n_n(x_1, \ldots, x_{n-1}, x_n, x_{n+1}) = \eta^n_1(x_1, \ldots, x_n) \eta^n_{n+1}(x_{n-1}, x_n, x_{n+1}), \quad n \geq 1.$$  
(72)

We now define a function $\tilde{w} : G^n \to \mathcal{A}$ by removing the idempotent $1(x_1, \ldots, x_n)$ from the right-hand side of (39), that is

$$\tilde{w}(x_1, \ldots, x_n) = \prod_{g \in \Lambda} \theta_g \circ w \circ \tau^g_n(x_1, \ldots, x_n).$$  
(73)

As it was observed in the proof of Lemma 3.4, $\tilde{w}(x_1, \ldots, x_n) \in \mathcal{U}(\mathcal{A})$, so $\tilde{w}$ is a classical $n$-cochain from $C^n(G, \mathcal{A})$. It turns out that $\tilde{w}$ satisfies the “quasi” $n$-cocycle identity (33).

Lemma 4.2. Let $n > 0$, $w \in Z^n(G, A)$ and $x_1, \ldots, x_n \in G$. Then

$$\alpha_{x_1} \left(1_{x_1} \tilde{w}(x_2, \ldots, x_{n+1})\right) \prod_{i=1}^{n} \tilde{w}(x_1, \ldots, x_i, x_{i+1}, \ldots, x_{n+1})^{(-1)^i} \tilde{w}(x_1, \ldots, x_n)^{(-1)^{n+1}} = 1_{x_1}.$$  
(74)

Proof. According to (73), the left-hand side of (74) is

$$\alpha_{x_1} \left(1_{x_1} \prod_{g \in \Lambda} \theta_g \circ w \circ \tau^g_n(x_2, \ldots, x_{n+1})\right)$$  
(75)

$$\prod_{i=1}^{n} \prod_{g \in \Lambda} \theta_g \circ w \circ \tau^g_n(x_1, \ldots, x_i, x_{i+1}, \ldots, x_{n+1})^{(-1)^i} \prod_{g \in \Lambda} \theta_g \circ w \circ \tau^g_n(x_1, \ldots, x_n)^{(-1)^{n+1}}.$$  
(76)

Using Lemma 3.10 we rewrite (75) as

$$1_{x_1} \prod_{g \in \Lambda} \theta_g \circ \alpha_{x_1} \eta^n_1(x_1) \left(1_{x_1} \eta^n_1(x_1) \tilde{w}(x_2, \ldots, x_{n+1}) \right).$$  
(77)

Moreover, since $\theta_g$ is a homomorphism, (76) coincides with

$$\prod_{g \in \Lambda} \theta_g \left(\prod_{i=1}^{n} w \circ \tau^g_n(x_1, \ldots, x_i, x_{i+1}, \ldots, x_{n+1})^{(-1)^i}\right).$$
Therefore, in order to prove \((74)\), one suffices to check the equality
\[
1_{\eta(x_1^{-1}g)} \cdots 1_{\eta(x_n^{-1} \cdots x_1^{-1}g) = \alpha_{\eta}(x_1) \left( 1_{\eta(x_1^{-1}w \circ \tau_n^{x_1^{-1}g}(x_2, \ldots, x_{n+1})} \right)}
\]
(78)
\[
\prod_{i=1}^{n} w \circ \tau_n^g(x_1, \ldots, x_i x_{i+1}, \ldots, x_{n+1})^{(-1)^i} + 1_{\tau_n^g(x_1, \ldots, x_n)^{(-1)^n+1}}
\]
(79)
Indeed, each \(\eta(x_i^{-1} \cdots x_1^{-1}g)\) belongs to \(H\), so by \((28)\),
\[
\theta_g \left( 1_{\eta(x_1^{-1} \cdots x_i^{-1}g)} \right) = \alpha_g \circ \operatorname{pr}_1 \left( 1_{\eta(x_i^{-1} \cdots x_1^{-1}g)} \right) = \alpha_g(1_{\Delta_1}) = 1_{\Delta_0},
\]
and consequently,
\[
\prod_{g \in \Lambda} \theta_g \left( 1_{\eta(x_1^{-1}g)} \cdots 1_{\eta(x_n^{-1} \cdots x_1^{-1}g)} \right) = \prod_{g \in \Lambda} 1_{\Delta_0} = 1_{\Delta}.
\]
We show that \((78)\) is exactly the partial n-cocycle identity
\[
(\delta^n w) \circ \tau_{n+1}^g(x_1, \ldots, x_{n+1}) = 1_{\eta(x_1^{-1}g)} \cdots 1_{\eta(x_n^{-1} \cdots x_1^{-1}g)}.
\]
(81)

By \((31)\) and \((70)\) one has
\[
\tau_n^g(x_2, \ldots, x_{n+1}) = (\eta_n^g(x_1, x_2), \ldots, \eta_n^g(x_1, \ldots, x_{n+1})),
\]
so the right-hand side of \((78)\) is the first factor of the left-hand side of \((81)\) expanded in accordance with \((2)\). Now, the \(i\)-th factor of the product \((79)\) is of the form
\[
w(\tau_{i-1}^g(x_1, \ldots, x_{i-1}), \eta_i^g(x_1, \ldots, x_{i-1}, x_i x_{i+1}), \ldots, \eta_n^g(x_1, \ldots, x_i x_{i+1}, \ldots, x_{n+1}))^{(-1)^i},
\]
which coincides with the \(i\)-th factor of the analogous product of the expansion of the left-hand side of \((31)\) thanks to \((71)\) and \((72)\). Finally, \((80)\) is literally the last factor of the above mentioned expansion. \(\square\)

We proceed now with the construction of \(\bar{w}\) needed in Theorem \(2.4\). Given \(n > 0\) and \(x_1, \ldots, x_n \in G\), we define
\[
\tilde{\varepsilon}(x_1, \ldots, x_n-1) = \varepsilon(x_1, \ldots, x_{n-1}) + 1_{\Delta} - 1_{(x_1, \ldots, x_{n-1})} \in \mathcal{U}(\Delta),
\]
(82)
understanding that \(\tilde{\varepsilon} = \varepsilon \in \mathcal{U}(\Delta)\) if \(n = 1\). Define also
\[
\bar{w}(x_1, \ldots, x_n) = (\tilde{\delta}^{n-1} \tilde{\varepsilon})(x_1, \ldots, x_n) w(x_1, \ldots, x_n) \in \mathcal{U}(\Delta),
\]
(83)
where
\[
(\tilde{\delta}^{n-1} \tilde{\varepsilon})(x_1, \ldots, x_n) = \tilde{\alpha}_{x_1}(\tilde{\varepsilon}(x_2, \ldots, x_n))
\]
\[
\prod_{i=1}^{n-1} \tilde{\varepsilon}(x_1, \ldots, x_i x_{i+1}, \ldots, x_n)^{(-1)^i} = \tilde{\varepsilon}(x_1, \ldots, x_{n-1})^{(-1)^n},
\]
(84)
and
\[
\tilde{\alpha}_x(a) = \alpha_x(1_{x^{-1}a}) + 1_{\Delta} - 1_x,
\]
(85)
with \(x \in G\) and \(a \in \Delta\).

Our main result is as follows.
Theorem 4.3. Let $A$ be a commutative unital ring which is a (possibly infinite) direct product of indecomposable rings, and let $\alpha = \{ \alpha_g : D_{g^{-1}} \to D_g \mid g \in G \}$ be a (non-necessarily transitive) unital partial action of $G$ on $A$. Then for any $n \geq 0$ each partial cocycle $w \in Z^n(G, A)$ is globalizable.

Proof. Since the case $n = 0$ has been explained in Remark 2.3, we assume $n > 0$. Consider first the transitive case. We will show that our $\tilde{w}$ defined in (83) satisfies (13) and (16). It directly follows from (89), (83), (82), (81) and (85) that

$$1_{(x_1, \ldots, x_n)} \tilde{w}(x_1, \ldots, x_n) = (\delta^{n-1} \epsilon)(x_1, \ldots, x_n) \cdot w'(x_1, \ldots, x_n)$$

for all $x_1, \ldots, x_n \in G$. By Theorem 3.12 this yields that $w$ satisfies (13). As to (15), we see that

$$\alpha_{x_i} \left( 1_{x_i^{-1}} \tilde{w}(x_2, \ldots, x_{n+1}) \right) \prod_{i=1}^{n} \tilde{w}(x_1, \ldots, x_i x_{i+1}, \ldots, x_{n+1})^{(-1)^i} \tilde{w}(x_1, \ldots, x_n)^{(-1)^{n+1}}$$

(86)

can be written as product of the following two factors

$$\alpha_{x_i} \left( 1_{x_i^{-1}} \tilde{w}'(x_2, \ldots, x_{n+1}) \right) \prod_{i=1}^{n} \tilde{w}'(x_1, \ldots, x_i x_{i+1}, \ldots, x_{n+1})^{(-1)^i} \tilde{w}'(x_1, \ldots, x_n)^{(-1)^{n+1}}$$

(87)

and

$$\alpha_{x_i} \left( 1_{x_i^{-1}} (\delta^{n-1} \epsilon)(x_2, \ldots, x_{n+1}) \right) \prod_{i=1}^{n} (\delta^{n-1} \epsilon)(x_1, \ldots, x_i x_{i+1}, \ldots, x_{n+1})^{(-1)^i} (\delta^{n-1} \epsilon)(x_1, \ldots, x_n)^{(-1)^{n+1}}.$$ 

(88)

Thanks to Lemma 4.2, the factor (87) is $1_{x_i}$, whereas the expansion of (88) has the same form as the usual $\delta^n \circ \delta^{n-1}$ in homological algebra, with the difference that instead of a global action we have a mixture of $\alpha$ with $\tilde{\alpha}$. Consequently, all factors in (88) to which neither $\alpha$, nor $\tilde{\alpha}$ is applied, cancel amongst themselves resulting in $1_A$. The remaining factors of the expansion are those of

$$\alpha_{x_i} \left( 1_{x_i^{-1}} (\delta^{n-1} \epsilon)(x_2, \ldots, x_{n+1}) \right)$$

(89)

and the first factors in each

$$(\delta^{n-1} \epsilon)(x_1, \ldots, x_i x_{i+1}, \ldots, x_{n+1})^{(-1)^i}, \ 1 \leq i \leq n,$$

(90)

and in

$$(\delta^{n-1} \epsilon)(x_1, \ldots, x_n)^{(-1)^{n+1}}.$$ 

(91)

The factors in the expansion of (89) are exactly

$$\alpha_{x_i} \left( 1_{x_i^{-1}} \tilde{\alpha}_{x_2}(\tilde{\epsilon}(x_3, \ldots, x_{n+1})) \right),$$

(92)

$$\alpha_{x_i} \left( 1_{x_i^{-1}} \tilde{\epsilon}(x_2, \ldots, x_i x_{i+1}, \ldots, x_{n+1})^{(-1)^i-1} \right), \ 2 \leq i \leq n,$$

(93)

and

$$\alpha_{x_i} \left( 1_{x_i^{-1}} \tilde{\epsilon}(x_2, \ldots, x_n)^{(-1)^n} \right),$$

(94)

whereas the first factors in (90) and (91) are

$$\tilde{\alpha}_{x_1 x_2}(\tilde{\epsilon}(x_3, \ldots, x_{n+1}))^{-1},$$

(95)
which comes from the case $i = 1$ in (90).

$$\tilde{\alpha}_{x_1}(\tilde{\varepsilon}(x_2, \ldots, x_i x_{i+1}, \ldots, x_{n+1}))(-1)^{j'}, \ 2 \leq i \leq n,$$

and

$$\tilde{\alpha}_{x_1}(\tilde{\varepsilon}(x_2, \ldots, x_n)).$$

(97)

Multiplying the elements in (96) and (97) by $1_{x_1}$, we see that they are canceled with those in (93) and (94), respectively. Now (92) equals $1_{x_1} \tilde{\alpha}_{x_1} x_2 \tilde{\varepsilon}(x_3, \ldots, x_{n+1})$ due to the commutative version of (19) from [24], so that it cancels with (95). It follows that (88) also equals $1_{x_1}$, and we conclude that $\tilde{w}$ satisfies (15). It remains to apply Theorem 2.4.

If $\alpha$ is not transitive, then we represent $A$ as a product of ideals, on each of which $\alpha$ acts transitively, so that the construction of $\tilde{w}$ reduces to the transitive case by means of the projection on such an ideal (see [24, Proposition 8.4]). □

5. Uniqueness of a globalization

Our aim is to show that the globalization of $w$ constructed in Section 4 is unique up to cohomological equivalence.

We would like to use item (iii) of [24, Lemma 8.3], whose proof was not sufficiently well explained. To clarify it, we need some new terminology. Let $\mathcal{R}$ be a ring and $\mathcal{R}_\mu \subseteq \mathcal{R}$, $\mu \in M$, a collection of its unital ideals. Observe from the definition of a direct product that there is a unique homomorphism $\phi : \mathcal{R} \to \prod_{\mu \in M} \mathcal{R}_\mu$, such that $\phi$ followed by the natural projection $\prod_{\mu \in M} \mathcal{R}_\mu \to \mathcal{R}_\mu'$ coincides with the multiplication by $1_{\mathcal{R}_\mu'}$ in $\mathcal{R}$ for any $\mu' \in M$. In this situation we say that the homomorphism $\phi$ respects projections.

Lemma 5.1. Let $\mathcal{C}$ be a non-necessarily unital ring and $\{C_\mu \mid \mu \in M\}$ a family of pairwise distinct unital ideals in $\mathcal{C}$. Suppose that $I$ and $J$ are unital ideals in $\mathcal{C}$ such that

$$I \cong \prod_{\mu \in M_1} C_\mu \text{ and } J \cong \prod_{\mu \in M_2} C_\mu,$$

(98)

where $M_1, M_2 \subseteq M$, $C_\mu \subseteq I$ for all $\mu \in M_1$ and $C_\mu' \subseteq J$ for all $\mu' \in M_2$. If the isomorphisms (98) respect projections, then there is a (unique) isomorphism

$$I + J \cong \prod_{\mu \in M_1 \cup M_2} C_\mu,$$

(99)

which also respects projections.

Proof. It is readily seen that $I + J$ is a unital ring with unity element $1_I + 1_J - 1_I 1_J$ and $I + J = I \oplus J'$, where $J' = J(1_J - 1_I 1_J)$. Therefore, the isomorphism $J \cong \prod_{\mu \in M_2} C_\mu$ restricts to

$$J' \cong \prod_{\mu \in M_2 \setminus M_1} C_\mu \subseteq \prod_{\mu \in M_2} C_\mu$$

(see [25]). Then

$$I + J = I \oplus J' \cong \left( \prod_{\mu \in M_1} C_\mu \right) \oplus \left( \prod_{\mu \in M_2 \setminus M_1} C_\mu \right) \cong \left( \prod_{\mu \in M_1} C_\mu \right) \times \left( \prod_{\mu \in M_2 \setminus M_1} C_\mu \right),$$
the latter being isomorphic to $\prod_{u \in M_u \cup (M_u \setminus M_u)} C_{\mu}$, which proves (99). Moreover, the isomorphism can be chosen in such a way that it respects projections, provided that the isomorphisms (98) have this property. □

**Proposition 5.2.** Let $A$ be a product $\prod_{g \in A'} A_g$ of indecomposable unital rings, $\alpha$ a transitive unital partial action of $G$ on $\mathcal{A}$ and $(\beta, B)$ an enveloping action of $(\alpha, A)$ with $A \subseteq B$. Then $B$ embeds as an ideal into $\prod_{g \in A'} A_g$, where $A'$ was defined before formula (20) and $A_g$ denotes the ideal $\beta_g(A_1)$ in $B$. Moreover, $M(B) \cong \prod_{g \in A'} A_g$, and $\beta^*$ is transitive, when seen as a partial action of $G$ on $\prod_{g \in A'} A_g$.

**Proof.** As it was explained before Lemma 5.1 there is a unique homomorphism $\phi : B \to \prod_{g \in A'} A_g$, which respects projections. We shall prove that $\phi$ is injective. Since $B = \sum_{g \in G} \beta_g(A_1)$, each element of $B$ belongs to an ideal $I$ of $B$ of the form $\sum_{i=1}^k \beta_{x_i}(A_1), x_1, \ldots, x_k \in G$. Therefore, it suffices to show that the restriction of $\phi$ to any such $I$ is injective. Using (ii) of [24, Lemma 8.3], we may construct an isomorphism

$$\beta_{x_i}(A) = \beta_{x_i} \left( \prod_{g \in A} A_g \right) \cong \prod_{g \in A} \beta_{x_i}(A_g) = \prod_{g \in A} \overline{\beta_{x_i}(A_g)}$$

which respects projections. Notice that it follows from the definition of $A'$ that the ideals $A_g, g \in A'$, are pairwise distinct. Hence by Lemma 5.1 there is an isomorphism

$$\psi : I \to \prod_{g \in A' \setminus A'} A_g,$$  \hspace{1cm} (100)

where $A' = \{ x_i A_g | g \in A, i = 1, \ldots, k \} \subseteq A'$, and it also respects projections. We claim that the restriction of $\phi$ to $I$ coincides with $\psi$, if one understands the product in the right-hand side of (100) as an ideal in $\prod_{g \in A'} A_g$ (see [25]). Indeed, for all $g \in A'$ and $b \in I$ one has

$$\text{pr}_g \circ \psi(b) = 1_{A_g} b = \text{pr}_g \circ \phi(b),$$

because $\phi$ and $\psi$ respect projections. Now if $g \in A' \setminus A''$, then $x_i^{-1} g \notin A$ for all $i = 1, \ldots, k$, since otherwise $g = \overline{x_i^{-1} g} \in A''$. Hence, for all $b = \sum_{i=1}^k \beta_{x_i}(a_i) \in I (a_i \in A)$ in view of (ii) of [24, Lemma 8.3]

$$\text{pr}_g \circ \phi(b) = 1_{A_g} b = \sum_{i=1}^k \beta_{x_i} \left( 1_{A_g} \left( x_i^{-1} g \right) a_i \right) = \sum_{i=1}^k \beta_{x_i}(0) = 0.$$

This proves the claim, and thus injectivity of $\phi$. Moreover, since $\phi(I) = \prod_{g \in A'} A_g$ is an ideal in $\prod_{g \in A'} A_g$, it follows that $\phi(I)$ is also an ideal in $\prod_{g \in A'} A_g$.

Regarding the second statement of the proposition, notice that each element of $\prod_{g \in A'} A_g$ acts as a multiplier of $B$, as $\phi(B)$ is an ideal in $\prod_{g \in A'} A_g$. Conversely, let $w \in M(B)$. Then $w 1_{A_g} = w 1_{A_g} \cdot 1_{A_g} \in A_g$ for all $g \in A'$. Define $a \in \prod_{g \in A'} A_g$ by $\text{pr}_g(a) = w 1_{A_g}$. We show that $\phi(wa) = wa$. Indeed, using the fact that $\phi$ respects projections, we get

$$\text{pr}_g(\phi(wa)) = 1_{A_g} \cdot wb = w 1_{A_g} \cdot 1_{A_g} b = w 1_{A_g} \cdot \text{pr}_g(\phi(b)) = \text{pr}_g(a \phi(b))$$

This does not confuse with (20), because $\alpha_g(A_1) = A_g \subseteq A$ for $g \in A$, so $\beta_g(A_1) = \alpha_g(A_1)$.
for all $g \in \Lambda'$. The transitivity of $\beta^*$ easily follows from the definition of $A_g$ for $g \in \Lambda'$.

**Theorem 5.3.** Let $A$ be a product $\prod_{g \in \Lambda} A_g$ of indecomposable unital rings, $\alpha$ a partial action of $G$ on $A$ and $w_i \in Z^n(G,A)$, $i = 1, 2$ ($n > 0$). Suppose that $(\beta, B)$ is an enveloping action of $\alpha, A$ and $u_i \in Z^n(G, \mathcal{U}(\mathcal{M}(B)))$ is a globalization of $w_i$, $i = 1, 2$. If $w_1$ is cohomologous to $w_2$, then $u_1$ is cohomologous to $u_2$. In particular any two globalizations of the same partial $n$-cocycle are cohomologous.

**Proof.** Let $\alpha$ be transitive. Thanks to Proposition 5.2 we may assume, up to an isomorphism, that $\mathcal{M}(B) = \prod_{g \in \Lambda'} A_g \cong A$. Define

$$u'_i(x_1, \ldots, x_n) = \prod_{g \in \Lambda'} \vartheta_g \circ u_i \circ \tau^n_g(x_1, \ldots, x_n), \quad i = 1, 2,$$

(101)

where $\vartheta_g$ is a homomorphism $\mathcal{M}(B) \to \mathcal{M}(B)$ given by

$$\vartheta_g = \beta_g \circ \text{pr}_1.$$

(102)

Since $u'_i$ has the same construction as $u'$ from Section 3 (see (39)), one has by Theorem 5.1 that $u'_1 \in Z^n(G, \mathcal{U}(\mathcal{M}(B)))$ and $u_i$ is cohomologous to $u'_i$, $i = 1, 2$.

It suffices to prove that $u'_1$ is cohomologous to $u'_2$, provided that $w_1$ is cohomologous to $w_2$. Observe, in view of (14), that for arbitrary $h_1, \ldots, h_n \in H$

$$\text{pr}_1 \circ u_i(h_1, \ldots, h_n) = \text{pr}_1 \left(u_i(h_1, \ldots, h_n)(1_{h_1}, \ldots, 1_{h_n})\right) = \text{pr}_1 \circ w(h_1, \ldots, h_n).$$

Together with (101) and (102) this implies that

$$u'_i(x_1, \ldots, x_n) = \prod_{g \in \Lambda'} \vartheta_g \circ w_i \circ \tau^n_g(x_1, \ldots, x_n), \quad i = 1, 2,$$

(103)

Let $w_2 = w_1 \cdot \delta^{n-1} \xi$ for some $\xi \in C^{n-1}(G, A)$. Since $\vartheta_g$ is a homomorphism, one immediately sees from (103) that $u_2' = u_1' \cdot (\delta^{n-1} \xi)'$, where

$$(\delta^{n-1} \xi)'(x_1, \ldots, x_n) = \prod_{g \in \Lambda'} \vartheta_g \circ (\delta^{n-1} \xi) \circ \tau^n_g(x_1, \ldots, x_n).$$

We shall show that

$$(\delta^{n-1} \xi)' = \delta^{n-1} \xi'$$

(104)

with

$$\xi'(x_1, \ldots, x_{n-1}) = \prod_{g \in \Lambda'} \vartheta_g \circ \xi \circ \tau^n_{g-1}(x_1, \ldots, x_{n-1}).$$

Taking into account the fact that $\vartheta_g$ is a homomorphism once again and interchanging the left-hand side and the right-hand side of (103), we may reduce (104) to

$$\beta_{x_1} \left(\prod_{g \in \Lambda'} \vartheta_g \circ \xi \circ \tau^n_{g-1}(x_2, \ldots, x_n)\right)$$

$$= \prod_{g \in \Lambda'} \vartheta_g \circ \beta_{\eta^g(x_1)} \circ \xi (\eta^g_2(x_1, x_2), \ldots, \eta^n_2(x_1, \ldots, x_n)), \quad (105)$$

whose right-hand side is

$$\prod_{g \in \Lambda'} \vartheta_g \circ \beta_{\eta^g(x_1)} \circ \xi \circ \tau^{x^{-1}_1 g}(x_2, \ldots, x_n)$$
by (70). Now it is readily seen that (105) follows from the global case of Lemma 3.10 (with $\alpha$ and $\theta$ replaced by $\beta^*$ and $\vartheta$, respectively).

The non-transitive case reduces to the transitive one, using the same argument as in Theorem 4.3. □

**Corollary 5.4.** Let $A$ be a product $\prod_{g \in \Lambda} A_g$ of indecomposable unital rings, $\alpha$ a partial action of $G$ on $A$ and $(\beta, B)$ an enveloping action of $(\alpha, A)$. Then the partial cohomology group $H^n(G, A)$ is isomorphic to the classical (global) cohomology group $H^n(G, U(M(B)))$.

Indeed, when $n > 0$, it follows from Theorems 4.3 and 5.3 that there is a well-defined injective map $H^n(G, A) \to H^n(G, U(M(B)))$. The constructions of $\tilde{w}$ and $u$ clearly respect products (see (8) and (73)), so the map is a monomorphism of groups. It is evidently surjective, as any $u \in Z^n(G, U(M(B)))$ restricts to $w \in Z^n(G, A)$ by means of (14), and a globalization of $w$ is cohomologous to $u$ thanks to Theorem 5.3. For the case $n = 0$ (which holds in a more general situation) see Remark 2.3.

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Instituto de Matemática e Estatística, Universidade de São Paulo, Rua do Matão, 1010, São Paulo, SP, CEP: 05508–090, Brazil

E-mail address: dokucha@gmail.com

Departamento de Matemática, Universidade Federal de Santa Catarina, Campus Reitor João David Ferreira Lima, Florianópolis, SC, CEP: 88040–900, Brazil

E-mail address: nskhripchenko@gmail.com

Departamento de Matemáticas, Universidad de Murcia, 30071 Murcia, España

E-mail address: jsimon@um.es