Noncommutative cosmologies

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Abstract. A possible way to resolve the singularities of general relativity is to assume that the description of space-time using commuting coordinates is not valid above a certain fundamental scale; beyond that scale the space-time has noncommutative structure leading in turn to a resolution of the singularity. We present models which realize this idea within the so-called frame formalism.

1. Introduction

It is a widespread belief that singularities and divergences are only technical artifacts which indicate the limitations of a physical theory. This applies to singularities in the Einstein theory of gravity as well. One believes that these singularities are not of physical significance but only signal the existence of a new structure of space-time beyond a certain scale. This new structure might be offered by noncommutative geometry. The ultimate aim then is the construction of a generalization of the theory of general relativity which becomes essentially noncommutative in regions where the commutative limit would be singular. The physical idea we have in mind is that the description of space-time using a set of commuting coordinates is only valid at scales smaller than some fundamental one. At higher scales it is impossible to localize a point and a new geometry should be used. As the description by the commuting coordinates breaks down, they must be replaced by elements of a noncommutative algebra.

According to the general idea outlined above a singularity in the metric is due to the extrapolation of the use of commuting coordinates beyond their natural domain of definition into the region where they are physically inappropriate. From this point of view, ‘near the singularity’ space-time could be described by a noncommutative algebra $\mathcal{A}$ over the complex numbers generated by four hermitian generators $x^\lambda$. We introduce a set of $J^{\mu\nu}$ to define commutation relations

$$[x^\mu, x^\nu] = i\hbar J^{\mu\nu}(x).$$

(1.1)

The geometry of $\mathcal{A}$ is described by a noncommutative version of the Cartan moving frame $\{\theta^\alpha\}$, as it will be defined below. In spite of the rather lengthy formalism the basic idea is simple. We start with a classical geometry described by a moving frame $\theta^\alpha$ and we associate

$$\theta^\alpha \longrightarrow J^{\mu\nu}$$

(1.2)
to it a noncommutative algebra with generators $x^\mu$ and commutation relations (1.1) which we identify with position space. To this algebra we add the extra elements which are necessary in order that the derivations become inner; this is ordinary quantum mechanics. The new element is the fact that if the original algebra describes a curved space-time then the Jacobi identities force the extended algebra to be noncommutative.

Typically one would proceed in three steps. First choose a moving frame to describe a metric. Quantize it by replacing the moving frame by a noncommutative frame, as described below. Finally, look for a noncommutative algebra consistent with the resulting differential calculus; this is the image of the map (1.2).

Let $e_\alpha$ be dual to the frame 1-forms $\theta^\alpha$. The extra momenta $p_\alpha$ which must be added to the algebra in order that the derivations be inner stand in duality to the position operators $x^\mu$ by the relation

$$[p_\alpha, x^\mu] = e_\mu^\alpha. \quad (1.3)$$

The right-hand side of this identity defines the gravitational field. The left-hand side must obey Jacobi identities. These identities yield relations between quantum mechanics in the given curved space-time and the noncommutative structure of the algebra. The three aspects of reality then, the curvature of space-time, quantum mechanics and the noncommutative structure of space-time are intimately connected. Furthermore the Jacobi identities impose

$$[p_\alpha, J^{\mu\nu}] = [x^{[\mu}, [p_\alpha, x^{\nu]})]. \quad (1.4)$$

If the space is flat and the frame is the canonical flat frame then the right-hand side vanishes and it is possible to consistently choose the expression $J^{\mu\nu}$ to be equal to a constant or even to vanish. But on the other hand if the $J^{\mu\nu}$ is non-trivial, the right-hand side cannot vanish and we conclude that the space is curved.

A detailed description of the method we shall use has been given in a previous article [1] and it suffices here to outline the prescription. We suppose that a complete consistent noncommutative geometry has been given. By this we mean that the frame and the commutation relations are explicitly known. Although noncommutative ‘gravity’ in the Kaluza-Klein sense had been investigated earlier [2, 3] it would seem that the first concrete example of noncommutative ‘gravity’ was [4] an extension of the 2-sphere. In spite of the fact it is not a realistic example of gravity it clearly illustrates the relation between the commutation relations and the effective classical gravitational field. There have been several recent investigations of the same subject, at least two of which [5, 6] are not far in spirit from the present calculations.

2. General considerations

We construct the noncommutative space using the frame formalism; it is defined in close analogy to the description of gravity using the moving frames. As at each point the ordinary (Riemannian) manifold is flat, we can choose a set $\{\tilde{e}_\alpha\}$ of tangent vectors to be orthonormal. The set $\{\tilde{e}_\alpha\}$ or the set $\{\theta^\alpha\}$ of their dual 1-forms ($\tilde{\theta}^\alpha(e_\beta) = \delta^\alpha_\beta$), is called the moving frame. In terms of $\tilde{\theta}^\alpha$ the line element is given as

$$ds^2 = - (\tilde{\theta}^0)^2 + (\tilde{\theta}^1)^2 + (\tilde{\theta}^2)^2 + (\tilde{\theta}^3)^2. \quad (2.1)$$

In terms of the moving frame, the usual definition of the differential can be written as

$$df = (\partial_\mu f)dx^\mu = (\tilde{e}_\alpha f)\tilde{\theta}^\alpha. \quad (2.2)$$

The definition of the moving frame can be generalized to the case when coordinates $x^\mu$ belong to the noncommutative algebra $\mathcal{A}$. The frame is, again, a set of 1-forms, now denoted as $\theta^\alpha$. 

The noncommutative equivalent to the assumption (2.1) is that the frame commutes with the algebra, i.e. for every element \( f \) of \( A \)
\[
f \theta^\alpha = \theta^\alpha f. \tag{2.3}
\]
The differential is defined as before
\[
df = (e_\alpha f) \theta^\alpha, \tag{2.4}
\]
where \( e_\alpha \) are derivations dual to \( \theta^\alpha \), i.e. \( \theta^\alpha (e_\beta) = \delta^\alpha_\beta \). In particular
\[
dx^\mu = (e_\alpha x^\mu) \theta^\alpha = e^\mu_\alpha \theta^\alpha. \tag{2.5}
\]
For derivations one assumes further that they obey the Leibniz rule and for the differential, that \( d^2 = 0 \).

The metric \( g \) we define as bilinear extension of a map
\[
g(\theta^\alpha \otimes \theta^\beta) = g^{\alpha\beta}, \tag{2.6}
\]
where \( g^{\alpha\beta} \) is a deformation of the Minkowski metric which satisfies certain reality and symmetry conditions. One of the advantages of the formalism is that the classical limit is straightforward: it is the commutative space defined by the limit of the frame.

An important point in our construction is that the differential and the algebraic structures are related. The algebra is defined by a product restricted by the matrix of elements \( J^{\mu\nu} \) while the metric is defined by \( e^\mu_\alpha \). Consistency requirements, derived by Leibniz rules,
\[
i ke_\alpha J^{\mu\nu} = [e^\mu_\alpha, x^\nu] - [e^\nu_\alpha, x^\mu], \tag{2.7}
\]
impose relations among these two matrices which in simple situations allow us to find a one-to-one correspondence between the structure of the algebra and the metric. One can see already here a differential equation for \( J^{\mu\nu} \) in terms of \( e^\mu_\alpha \). In important special cases this equation reduces to a simple differential equation of one variable.

The relation (2.7) can be written also as Jacobi identities
\[
[p_\alpha, [x^\mu, x^\nu]] + [x^\nu, [p_\alpha, x^\mu]] + [x^\mu, [x^\nu, p_\alpha]] = 0, \tag{2.8}
\]
if one introduce the momenta \( p_\alpha \) associated to the derivation by the relation (1.3).

Finally, we must insure that the differential is well defined. A necessary condition is that \( d[x^\mu, \theta^\alpha] = 0 \). Then it follows that
\[
d[x^\mu, \theta^\alpha] = [dx^\mu, \theta^\alpha] + [x^\mu, d\theta^\alpha] = e^\mu_\beta [\theta^\beta, \theta^\alpha] = \frac{1}{2} [x^\mu, C^{\alpha\beta\gamma}] \theta^\beta \theta^\gamma, \tag{2.9}
\]
where \( C^{\alpha\beta\gamma} \) are the Ricci rotation coefficients Then we find that multiplication of the 1-forms basis must satisfy
\[
[\theta^\alpha, \theta^\beta] = \frac{1}{2} \theta^\beta_\mu [x^\mu, C^{\alpha\beta\gamma}] \theta^\gamma \theta^\gamma. \tag{2.10}
\]
Consistency requires then that
\[
\theta^\beta_\mu [x^\mu, C^{\alpha\beta\gamma}] = 0, \tag{2.11}
\]
and
\[
\theta^\beta_\mu [x^\mu, C^{\alpha\beta\gamma}] = \text{const.} \tag{2.12}
\]

We have in general three consistency equations which must be satisfied in order to obtain a noncommutative extension. They are the Leibniz rule (2.7), the Jacobi identity and the conditions (2.11-2.12) on the differential. The first two constraints follow from the Jacobi
identities but they are not completely independent of the differential calculus since one involves the momentum operators.

To illustrate the importance of the Jacobi identities we mention that they force a modification of the canonical commutation relations and introduce a dependence

$$\hbar \delta_{\alpha} \mu \rightarrow \hbar e_{\alpha}$$

of Planck’s ‘constant’ on the gravitational field. We mentioned already that if one place the canonical commutator (1.3) in the Jacobi identity with two coordinate and one momentum entry then consistency is telling us that the coordinates in general cannot commute.

3. Inner-derivations model

In this paper we explore two noncommutative cosmological models inspired by the classical Kasner metric. The first model is based on inner derivations. The Kasner metric is the simplest anisotropic homogeneous solution to Einstein equations; it is given by

$$ds^2 = -dt^2 + t^{2p_1}(dy^1)^2 + t^{2p_2}(dy^2)^2 + t^{2p_3}(dy^3)^2.$$  (3.1)

The vacuum equations with vanishing cosmological constant impose the following constraints on the parameters $p_i$

$$p_1 + p_2 + p_3 = 1, \quad p_1^2 + p_2^2 + p_3^2 = 1,$$  (3.2)

so (3.1) is in fact a 1-parameter family of solutions. The scalar curvature of the metric (3.1) is of course zero, while the nonvanishing components of the Ricci curvature tensor are proportional to $t^{-2}$. We shall discuss noncommutative metrics of the form similar to (3.1).

The moving frame of (3.1) is given by

$$\theta^0 = dt, \quad \theta^i = t^{p_i} dy^i,$$  (3.3)

$i = 1, 2, 3$ and there is no summation in $i$. The frame can be written perhaps in a simpler way in other coordinates $x^i = t^{p_i} y^i$. We have

$$\theta^0 = dt, \quad \theta^i = dx^i - \frac{p_i}{t} x^i dt = dx^i - Q_j^i t^{-1} x^j dt,$$  (3.4)

where $Q_j^i = \text{diag}(p_1, p_2, p_3)$ is a constant $3 \times 3$ matrix. We will use $\alpha, \beta, \gamma, \alpha, \beta, \gamma$ for the frame indices while $\mu, \nu, \rho, i, j, k$ are the coordinate indices. The Ricci rotation coefficients for the classical Kasner frame are given by

$$C^\alpha_{b0} = Q^\alpha_b t^{-1},$$  (3.5)

and the nonvanishing components of the Ricci curvature tensor are

$$R^0_0 = -\text{Tr} (Q - Q^2) t^{-2}, \quad R^a_{b} = -(1 - \text{Tr} Q) Q^a_b t^{-2}.$$  (3.6)

The most predictive variant of noncommutative frame formalism is obtained assuming that the derivations $e_\alpha$ dual to the frame $\theta^\alpha$ are inner,

$$e_\alpha f = [p_\alpha, f],$$  (3.7)

given by the momenta $p_\alpha$ which generate the same algebra $\mathcal{A}$ as the coordinates. This is a strong requirement which gives additional constraints. One can show [7] that the commutation relations between the momenta in this case are of the form

$$[p_\alpha, p_\beta] = K_{\alpha \beta} + F^\gamma_{\alpha \beta} p_\gamma - 2\pi Q^\gamma_{\alpha \beta} p_\gamma p_\delta,$$  (3.8)
where $K_{\alpha \beta}$, $F_{\gamma \alpha \beta}$ and $Q_{\gamma \delta \alpha \beta}$ are constants. The measure of noncommutativity is $\bar{k}$ introduced already in (1.1), $\bar{k} \sim \hbar G_N$. The parameter $\epsilon$ is dimensionless, $\epsilon = \bar{k} \mu^2$ and we assume that in the commutative limit $\epsilon$ is small. The Ricci rotation coefficients are linear in the momenta,

$$C^\gamma_{\alpha \beta} = -4i\epsilon Q^\gamma_{\delta \alpha \beta} p_\gamma. \quad (3.9)$$

The last equation can be considered as the one providing the commutative limit in terms of properties of the tangent space.

One noncommutative generalization of the Kasner space, that is, the frame and the momentum algebra, was introduced in [8]. Here we just briefly recall these results and derive the corresponding position algebra. It can be shown that the noncommutative frame of the form (3.3-3.4) is inconsistent if $J^{\mu \nu} \neq 0$ and thus one would like to modify it for example by introducing a function $\tau(t)$ instead of $t^{-1}$ in (3.5). The Ansatz we analyze is the following: for $\theta^0$ we take $\theta^0 = dt$, which means

$$e_0 t = [p_0, t] = 1, \quad e_a t = [p_a, t] = 0, \quad (3.10)$$

while for the rotation coefficients we assume

$$C^a_{b0} = Q^a_b \tau(t). \quad (3.11)$$

At the same time from (4.7) it follows that

$$C^a_{b0} = k^a p_a. \quad (3.12)$$

These two assumptions, combined with the quadratic form of the algebra (3.8) and with the Jacobi identities, determine the algebra almost uniquely. It can be written as [8]

$$[p_0, p_1] = K_{01} - qp_1 \tau, \quad [p_0, p_2] = K_{02} - qp_2 \tau, \quad [p_0, p_3] = \frac{1}{\bar{k}} (1 + \frac{q^2 \tau^2}{\mu^2}),$$

$$[p_1, p_2] = \frac{1}{\bar{k}} (1 + \frac{q^2 \tau^2}{\mu^2}), \quad [p_1, p_3] = 0, \quad [p_2, p_3] = 0. \quad (3.13)$$

This corresponds to the choice $k^a = (0, 0, -i\epsilon q \mu^2)$. Constraints impose that the vector $k^a$ is an eigenvector of the matrix $Q$

$$Q^a_b k^b = q k^a.$$

One also obtains

$$\text{Tr} Q = 3q.$$

If for simplicity we take that $Q$ is proportional to identity, $Q = qI$, for $\tau(t)$ we obtain the equation

$$\dot{\tau} + q \tau^2 + q \mu \tau = 0. \quad (3.14)$$

We will consider its periodic solution

$$\tau = \mu \cot q \mu t. \quad (3.15)$$

There is also a hyperbolic solution $\tau = \mu \coth q \mu t$ which can be obtained from (3.15) by $\mu \to i\mu$. For $\mu = 0$ the solution is $\tau = (qt)^{-1}$. In all cases the rotation coefficients have the correct or 'classical' behavior at the origin $t \to 0$

$$\tau \to t^{-1}. \quad (3.16)$$
Assuming that coordinates can be expressed as functions of momenta, the duality relations (3.10) and the Jacobi identities can be considered as differential equations for $x^\mu = x^\mu(p_\alpha)$ and solved. One of the ‘Fourier transformations’ was obtained already:

$$p_3 = -\frac{1}{ik q\mu} \cot q\mu t.$$  

(3.17)

The general solution to the remaining equations is

$$\begin{align*}
p_1 &= \frac{K_{10}}{q\mu} \cot q\mu t - \frac{1}{ik \sin q\mu t} y, \\
p_2 &= \frac{K_{20}}{q\mu} \cot q\mu t + \frac{1}{ik \sin q\mu t} x, \\
z &= \frac{1}{2} i k (F(t)p_0 + p_0 F(t)) + G(t)(x - y),
\end{align*}$$  

(3.18)

provided that

$$\begin{align*}
[p_0, x] &= 0, \quad [p_0, y] = 0, \\
[t, x] &= 0, \quad [t, y] = 0.
\end{align*}$$  

(3.19)

The functions $F$ and $G$ have an arbitrary dependence on time. In the simplest case, for $K_{10} = 0$, $K_{20} = 0$, the position algebra reads

$$\begin{align*}
[t, x] &= 0, \quad [t, y] = 0, \quad [x, y] = ik, \\
[t, z] &= -ik F(t), \quad [x, z] = -ik G(t), \quad [y, z] = ik G(t).
\end{align*}$$  

(3.20)

One can further set $G(t) = 0$ given that in the corresponding classical metric this function has no importance as it can be eliminated by the appropriate change of coordinates. $F(t)$ on the other hand must be nonzero. The position algebra (3.20) which we obtained is a direct product of two subalgebras, one generated by $(t, z)$ and the other by $(x, y)$. To obtain the frame we use the formula $dx^\mu = [p_\alpha, x^\mu] \theta^\alpha$. We obtain

$$\begin{align*}
\theta^0 &= dt, \\
\theta^1 &= \sin q\mu t \, dx, \\
\theta^2 &= \sin q\mu t \, dy, \\
\theta^3 &= \sin^2 q\mu t \left( \frac{1}{2} z \, dt - q\mu \cot q\mu t \, y \, dx + q\mu \cot q\mu t \, x \, dy - \frac{1}{2} q\mu \, dz \right).
\end{align*}$$  

(3.21)

The scalar curvature for this frame is given by

$$R = (q\mu)^4 \frac{x^2 + y^2}{2 \sin^2 q\mu t} + (q\mu)^2 \left( \frac{12}{\sin^2 q\mu t} - 20 \right).$$  

(3.22)

Note that $R$ does not depend on $F(t)$; it is singular for $t = 0$, but the space is inhomogeneous. The position algebra (3.20) turned out to be regular apart for $F(t)$. However, the simplest choice $F(t) = 1$ is consistent, and in this case the singular point $t = 0$ is not regularized any better than any other point on the manifold.

### 4. Outer-derivations model

Noncommutative spaces with the differential structure based on inner derivations have many nice uniqueness properties to which we do not enter. However, the momentum algebra (3.20) in some cases seems to be too restrictive as demonstrated in the previous model. We saw that it fixes geometry almost completely, and although the Ansatz (3.10-3.11) was rather general the geometry (3.21-3.22) we obtained at the end was rather specific and could not accommodate...
any of the usual cosmological solutions. Therefore we will construct yet another cosmological model, removing the assumption that derivations be inner. This means that we will preserve the defining relations (2.3-2.4), but we will not assume that $e_\alpha f = [p_\alpha, f]$, that is the momenta $p_\alpha$ might not belong to the original algebra $\mathcal{A}$ any more. It is easy to see however that also in this case we have considerable restrictions. The differential defined by $df = (e_\alpha f)\theta^\alpha$ has to be consistent with the relation $[x^\mu, x^\nu] = i\hbar J^{\mu\nu}$. If we denote

$$dx^\mu = \Lambda^\mu_\alpha(x)\theta^\alpha,$$

(4.1)

by differentiating (1.1) we obtain a set of linear differential equations for $J^{\mu\nu}$

$$dJ^{\mu\nu} = J^{\mu\alpha}(e_\alpha\Lambda^\nu_\beta)\theta^\beta.$$

(4.2)

Additionally, $J^{\mu\nu}$ are constrained by the Jacobi identities:

$$[x^\lambda, J^{\mu\nu}] + [x^\nu, J^{\lambda\mu}] + [x^\mu, J^{\nu\lambda}] = 0,$$

(4.3)

while the equation (2.12) relates the frame and the commutators further:

$$\theta^\mu_\beta[x^\nu, C^{\alpha}_{\beta\gamma}] = \text{const.}$$

(4.4)

Any solution to the equations (4.2-4.4) gives a consistently defined noncommutative space. Of course, it is not possible to solve these equations in general. However in simple cases, for example when there are additional symmetries, the equations can be solved. In order to get a cosmological solution we propose an Ansatz

$$dt = \theta^0, \quad dx^j = \Lambda^j_0\theta^0 + \Lambda^j_\alpha\theta^\alpha,$$

(4.5)

with

$$\Lambda^j_0 = \frac{1}{2}(H^j_k(t)x^k + x^kH^j_k(t)), \quad \Lambda^j_k = U^j_k(t),$$

(4.6)

where $H(t), U(t)$ are matrix-valued functions of time which are to be determined or constrained. For the analogous commutative frame the rotation coefficients are

$$C^{\alpha}_{\beta\gamma} = (U^{-1}(H - E)U)^\alpha_\beta, \quad E^j = \dot{U}^j_kU^{-1}k.$$

(4.7)

We see that the choice $U(t) = 1, H(t) = Q(t) - 1$ corresponds to the classical Kasner. The Ansatz (4.5-4.6) means that the derivatives $e_\alpha$ act on coordinates in the following way:

$$e_0t = 1, \quad e_0x^i = \frac{1}{2}(H^i_kx^k + x^kH^i_k),$$

$$e_kt = 0, \quad e_kx^i = U^i_k.$$

(4.8)

Inserting (4.5) in the equations for $J^{\mu\nu}$ we get

$$e_0J^{0j} = [t, \Lambda^j_0] = H^j_kJ^{0k},$$

(4.9)

$$e_kJ^{0j} = [t, \Lambda^j_k] = 0,$$

(4.10)

$$e_0J^{ij} = [x^i, \Lambda^j_0] = \frac{1}{2}(H^i_kJ^{0j}x^k + H^i_kJ^{0k}x^j) + \text{h. c.},$$

(4.11)

$$e_kJ^{ij} = [x^i, \Lambda^j_k] = \dot{U}^i_kJ^{0j}.$$  (4.12)
The first two equations (4.9-4.10) have as solution $J^0_j(t)$ which is a function of time only and has to obey
\[ J^0_j = H_k^j J^0_k. \] (4.13)

The equation (4.12) has a solution of the form
\[ J^{ij} = \frac{1}{2} (J^{[i0} E_{jk]} x^k + x^k J^{[i0} E^{j]}_k^|) + S^{ij}(t). \] (4.14)

Inserting (4.14) into (4.11) we obtain two equations
\[ J^{[i0} (\dot{E}^{j]}_k + [E, H]^{j]}_k^|) x^k + h. c. = 0, \] (4.15)
\[ \dot{S}^{ij} - H_k^{[j} S^{i]k} = 0. \] (4.16)

The equation (4.15) is valid if the matrices $E$ and $H$ obey
\[ \dot{E} - \dot{H} + [E, H] = Y \pi, \] (4.17)
where $\pi$ is the projection to $J^{k0}$:
\[ \pi^j_k = J^{0j} J_{0k}, \] (4.18)
and $Y$ can be an arbitrary element of $A$.

In addition, we have the Jacobi constraints (4.3). They give the following equations:
\[ J^{0[i} (H - E)^{j]}_k^| J^{0k} = 0, \] (4.19)
\[ J^{0[i} (H - E)^{j]}_m^| S^{mk} + \text{(cyclic } ijk) = 0, \] (4.20)
\[ J^{0[i} (H - E)^{j]}_m^| J^{0m} E^k \pi + \text{(cyclic } ijk) = 0. \] (4.21)

All three are satisfied for
\[ H - E = X \pi, \] (4.22)
where $X$ is an arbitrary element of the algebra.

To recapitulate, the equations (4.2-4.3) give a set of coupled equations. They are:
\[ \dot{J}^{0j} = H_k^j J^{0k}, \] (4.23)
\[ J^{ij} = \frac{1}{2} (J^{[i0} E_{jk]} x^k + x^k J^{[i0} E^{j]}_k^|) + S^{ij}, \] (4.24)
\[ E - H = X \pi, \] (4.25)
\[ \dot{E} - \dot{H} + [E, H] = Y \pi, \] (4.26)
\[ \dot{S}^{ij} = H_k^{[j} S^{i]k}. \] (4.27)

Note that $J^{ij}$ has a form as a total ‘angular momentum’ as it is expressed as the sum of an ‘orbital’ part and a ‘spin’. Equations (4.23-4.27) have many solutions: in order to keep as close as possible to the original Kasner space we choose
\[ U = I, \quad \text{i.e.} \quad E = 0, \quad H = Q \tau. \] (4.28)
This reduces the number of degrees of freedom to one, \( \tau(t) \). If we assume further that the \( J_{0i} \) stay along one axis, for example \( J_{0i} = (0, 0, j(t)) \), then \( Q \) has to be a diagonal matrix with two eigenvalues vanishing, \( q_1 = q_2 = 0 \), and

\[
H = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & q\tau(t)
\end{pmatrix}.
\]  
(4.29)

We have then

\[
-q\dot{\tau} = j^2Y, \quad \frac{dj}{dt} = q\tau j.
\]  
(4.30)

As \( Y \) is an arbitrary element of the algebra it can be an arbitrary function of \( t \) and the first equation in (4.30) is void. However, the condition (4.4) gives an additional equation. From (4.7) we calculate the nonvanishing Ricci rotation coefficients:

\[
C_{30}^3 = -C_{03}^3 = q\tau.
\]  
(4.31)

Therefore the equation (4.4) reads

\[
j \frac{d\tau}{dt} = \text{const}.
\]  
(4.32)

Combining (4.30) and (4.32) we get

\[
\dot{\tau}e^{j q\tau dt} = \text{const},
\]  
(4.33)

or

\[
\dot{\tau} + \frac{q}{2}\tau^2 + \frac{q}{2}\mu^2 = 0,
\]  
(4.34)

an equation the same as (3.14) with \( q \rightarrow \frac{q}{2} \). Its periodic solution is

\[
\tau = \mu \cot \frac{q\mu t}{2}.
\]  
(4.35)

We also obtain \( J^{0i} \):

\[
J^{01} = 0, \quad J^{02} = 0, \quad J^{03} = ika \sin^2 \frac{q\mu t}{2}.
\]  
(4.36)

The remaining equations for \( J^{ij} \) can also be solved easily. As \( E = 0 \) we have \( J^{ij} = S^{ij} \) and therefore

\[
J^{12} = ikb, \quad J^{13} = ikc \sin^2 \frac{q\mu t}{2}, \quad J^{23} = ikd \sin^2 \frac{q\mu t}{2}.
\]  
(4.37)

The \( a, b, c, d \) are arbitrary constants. Note that the symplectic form \( J^{\mu\nu} \) is nondegenerate: \( \det J = k^4 a^2 b^2 \sin^4 \frac{q\mu t}{2} \).

The frame for this algebra is

\[
\theta^0 = dt, \quad \theta^1 = dx, \quad \theta^2 = dy,
\]
\[
\theta^3 = dz + \frac{qw}{2} z \cot \frac{q\mu t}{2} dt,
\]  
(4.38)

with the line element

\[
ds^2 = -(1 - \frac{q^2 \mu^2 z^2}{4} \cot^2 \frac{q\mu t}{2}) dt^2 + dx^2 + dy^2 + dz^2 + q\mu z \cot \frac{q\mu t}{2} dt dz.
\]  
(4.39)
The scalar curvature of this space is isotropic

\[ R = \frac{(q\mu)^2}{4\sin^2 \frac{q\mu}{2}} \left( 3 + \cos \frac{q\mu}{2} \right). \]  

(4.40)

The other curvature invariants also have just the time dependence and are given by

\[ I_1 = \frac{1}{48} R^2, \quad I_2 = \frac{1}{96} R^3. \]  

(4.41)

As before, there is a cosmological singularity at \( t = 0 \): the rotation coefficients behave near the singularity as \( t^{-1} \), the curvature as \( t^{-2} \). However the commutators (4.36-4.37) are regular. Furthermore, we see that in this case, too, the consistency conditions determine the frame completely.

5. Conclusions and discussion

In conclusion we propose to resolve the singularities of general relativity by assuming that space-time becomes fuzzy beyond a certain scale. In the specific examples we have given here the Kasner manifold has been replaced by a noncommutative algebra, whose Jacobi identities force a modification of the time dependence of the metric. In the second model which realizes the idea more successfully, the scalar curvature and other curvature invariants depend only on the time. Therefore one might consider this metric to be a cosmological model.

Finally, it should be stressed that no use is made of field equations. The restrictions on the solutions find their origin in the requirement that the noncommutative algebra be an associative one and appear as Jacobi identities on the commutation relations.

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