Simulation studies of three types of inverted pendulums via FOPID and PID controller

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Abstract. The inverted pendulum systems are very important problem in control literature and frequently studied. A simple and effective controller design is preferred for controlling a complex system. In this work, mathematical model of three types of inverted pendulums are derived depending upon their dynamics. Further, a fractional order PID (FOPID) controller is designed and applied for controlling and stabilizing the three types of inverted pendulums. The proposed fractional order controller is based on the fractional calculus and it is a robust controller. The responses of the pendulum system with FOPID are compared with the responses with conventional PID controllers. The overall performance of the FOPID controller is better than the conventional PID and the results are robust to the outer fast and large disturbances also.

1. Introduction

In the non-linear control system analysis, inverted pendulums (IPs) are most important problems and widely used in the control laboratories [1–6]. The inverted pendulums are studied in different forms, for example two wheeled inverted pendulums mobile robot [7,8], Joe mobile robot [9], Segway [10], Furuta pendulum [11,12], inverted 3-D pendulum etc [13]. These IPs are under-actuated, nonlinear and open-loop unstable in terms of their dynamics. Therefore, the tracking, control and stabilization of this type of system is a challenging task for control community.

The inverted pendulums may be classified in three categories depending upon how many controlling forces are present to stabilize the pendulum system [5]. When only one control force act on it for the controlling purpose, the inverted pendulum is known as x-IP (or linear IP). This type of IP is the simplest one and can move in a line. The second type of IP is planar IP (which includes x-y IP and x-z IP) in which two control forces acts on its pivot. There are two control forces present in case of the x-y IP namely $F_x$ and $F_y$ whereas $F_x$ and $F_z$ are present for the x-z IP [2,5]. The planar IP movement is restricted to the horizontal and vertical plane. Control of rocket during its take-off is based on the control of planar IP. The third type of IP is spatial inverted pendulum (SIP). There are three controlling forces $F_x$, $F_y$ and $F_z$ that are present to control this system so that the whole system can move in 3-dimensional space [4,5]. Stabilization and balancing of the SIP describe the balancing a stick or rod on one’s finger.

In the control literature, most of the researches are focused on the linear IPs which are the simplest and easy to control compared to other IPs. Thus the researches based on the linear IPs have become comprehensive and systematic. These works includes swinging up [14,15], stabilization [16] and tracking control of the IP [17]. In general, to stabilize and track the IPs, the swinging up is the first step. In the present control literature, a lot of conventional and modern control techniques have been applied in the control and stabilization of linear IP, like PID [2], sliding mode control [5,7], fuzzy control [18,19] and neural networks control [20,21]. Here, it should be noted that linear IP has one
degree of freedom while the planar IP and spatial IP have more than one degree of freedom. It is worth mentioning that only few works have been reported in literature that consider pendulum with two degree of freedom and most of them are based on inverted pendulum of one degree of freedom.

Wang has proposed PID controller for the three types of inverted pendulum but the tuning method for PID parameters are not discussed in detail [2]. In ref. [22], fractional order PID controller is designed and applied only on a linear inverted pendulum(x-IP). In ref. [23], a fractional order PID controller has been applied for controlling and stabilization of x-z inverted pendulum. Wang et. al. has applied hierarchical sliding mode control for x-z IP which parameters are optimized with the Big Bang-Big Crunch optimization technique[6]. Brisilla et. al. has proposed a non-linear control of mobile inverted pendulum [24]. Hamza et. al. has proposed linear and no-linear control algorithms for rotary inverted pendulum [25].

In this paper, we concentrate on control and stabilization of x-IP, x-y IP and x-z IP in which we deal with the dynamical modeling of each IPs. The mathematical equations of each IP are derived using the Lagrangian mechanics. MATLAB SIMULINK model of each IP is designed using the dynamical equations. Further, fractional order PID (FOPID) controller is designed and applied to each. To obtain the optimal value of the controller parameters, a nature inspired optimization technique called whale optimization algorithm [26] is used. Comparative study of the responses of the IPs with PID and FOPID controllers are analysed and discussed.

2. Mathematical Model of Three Types of IPs

2.1 x-Inverted Pendulum

The linear (or x-IP) inverted pendulum constitutes a pivot which is controlled by a horizontal force which is shown in Figure 1. In case of the linear IP, movement of cart and pendulum rod depends on the force acting on the pivot. The direction of the force may be in positive and negative x-direction.

The potential energy and kinetic energy of the IP may be written as:

\[
\begin{align*}
P &= mgz_p, \\
K &= \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m(\dot{x}_p^2 + \dot{z}_p^2)
\end{align*}
\]

(1)

Where \(x_p = x + l\sin\theta\), \(z_p = z + l\cos\theta\), \(l\) is the half length of the IP rod, \(M\) is mass of pivot and \(m\) is mass of pendulum rod, \((x, z)\) is position and \((\dot{x}, \dot{z})\) is speed of the pivot, \((x_p, z_p)\) and \((\dot{x}_p, \dot{z}_p)\) are the position and speed in x’o’z’ coordinate, \(g\) is gravity constant. Here, it is assumed that inertia of the pendulum is negligible.

From the Lagrangian equation, system Lagrangian \((L)\) is as follows:

\[
L = K - P
\]

(2)

The dynamics of the IP in the form of Lagrange’s equation is given by

\[
\begin{align*}
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} &= F_x, \\
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} &= 0
\end{align*}
\]

(3)

Where \(F_x\) is controlling force. From equation (1) and (3) we have following dynamical equation of the IP.

Where \(F_x\) is controlling force. From equation (1) and (3) we have following dynamical equation of the IP.
\[
\begin{aligned}
(M + m)\ddot{x} + ml\cos\theta \dot{\theta} - ml\sin\theta \dot{\theta}^2 &= F_x \\
cos\theta \ddot{x} + l\dot{\theta} - g\sin\theta &= 0
\end{aligned}
\]  \hspace{1cm} (4)

where \(-0.5 \leq x \geq 0.5\).

Assuming \(x_1 = x, x_2 = \dot{x}, x_3 = \theta, x_4 = \dot{\theta}\) the state space form of the equation may be written as [2]

\[
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= \frac{-mg \cos x_1 \sin x_1 + ml \sin x_1 x_2^2 + F_x}{M + ml \sin^2 x_1} + d_1 \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= \frac{-ml \cos x_1 \sin x_1 x_3 - \cos x_1 F_x + (M + m)g \sin x_1}{Ml + ml \sin^2 x_3} + d_2
\end{aligned}
\]  \hspace{1cm} (5)

where \(d_1, d_2\) are disturbances from outside, \(\theta\) is angle made by rod. Pendulum system parameters are given in Table 1.

| Mass of the cart (M kg) | Pendulum mass (m kg) | Pendulum length (l m) | Gravitational acceleration (g m/s²) |
|-------------------------|----------------------|-----------------------|-----------------------------------|
| 1                       | 0.1                  | 0.3                   | 9.8                               |

**Table 1. Parametric description of pendulum**

![Diagram of the pendulum system](image)

**Figure 1.** x-IP with control force and coordinates

### 2.2 Planar IP (x-y IP)

The planar IP consist of a pendulum rod which is connected with a cart using universal joint. There are two controlling force \(F_x\) and \(F_y\) act on the cart as shown in Figure 2. The control forces are responsible for the movement of the cart. The total energy of the IP system is given by

\[
\begin{aligned}
K &= \frac{1}{2}M(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}m(\dot{x}_p^2 + \dot{y}_p^2 + \dot{z}_p^2) \\
P &= mg\dot{z}_p
\end{aligned}
\]  \hspace{1cm} (6)
where \( x_p = x + l \sin \theta \), \( y_p = y + l \cos \theta \sin \phi \) and \( z_p = z + l \cos \theta \cos \phi \)

the mathematical model of the IP system may be written as follows [2].

\[
\begin{align*}
(M + m) \ddot{x} + ml \cos \theta \dot{\theta} - ml \sin \theta \dot{\theta}^2 &= F_x, \\
(M + m) \ddot{y} + ml \cos \theta \cos \phi \dot{\phi} - ml \sin \theta \sin \phi \dot{\phi} &= F_y, \\
-2ml \sin \theta \cos \phi \dot{\theta} \dot{\phi} - ml \cos \theta \sin \phi (\dot{\theta}^2 + \dot{\phi}^2) &= F_x, \\
(l \dot{\theta} + \cos \theta \ddot{x} - \sin \theta \sin \phi \ddot{y}) = g \sin \theta \cos \phi = 0, \\
l \cos \theta \dot{\phi} + l \cos \phi \dot{\theta} - g \sin \phi = 0
\end{align*}
\]

where \(-0.5 \leq x \geq 0.5\), and \(-0.5 \leq y \geq 0.5\).

The state equations of the system may be written as

\[
\begin{align*}
\dot{x}_1 &= x_2, \\
x_2 &= \frac{F_x}{M' + m \sin^2 \theta} \cos^2 x_4 \cos x_5 + M \sin^2 x_5 + d_i, \\
\dot{x}_3 &= x_4, \\
x_4 &= \frac{F_x}{M' + m \sin^2 \theta} \cos^2 x_4 \cos x_5 + M \sin^2 x_5 + d_i, \\
\dot{x}_5 &= x_6, \\
x_6 &= \frac{F_x}{(M' + m \sin^2 \theta) \cos^2 x_4 \cos x_5 + M \sin^2 x_5} l \cos \theta + d_4
\end{align*}
\]

Where \( x_1 = x, x_2 = \dot{x}, x_3 = y, x_4 = \dot{y}, x_5 = \theta, x_6 = \dot{\theta}, x_7 = \phi, x_8 = \dot{\phi} \)

\[F_i = M_m \sin x_4 x_6^2 + M_m \cos^2 x_5 \sin x_4 x_5^2 - M_m \cos^2 x_5 \sin x_4 x_5^2 + m \cos^2 x_5 \sin x_4 x_5^2, \]

\[F_6 = (M + m) F_y - m \cos^2 x_4 \sin x_5 \cos x_4 \sin x_5 F_y - M_m \cos^2 x_5 \sin x_4 x_5^2 + M_m \cos^3 x_5 \sin x_4 x_5^2 - M \cos x_4 x_5^2, \]

\[F_i = m \cos x_4 \cos^2 x_5 F_y - (M + m) \cos x_4 F_y + (M + m) \sin x_4 \sin x_5 F_y + M (m + M) g \sin x_4 \cos x_5, \]

\[F_6 = -m \sin^2 x_5 \cos x_4 F_y - M \cos x_4 F_y + m \sin x_4 \cos x_4 F_y + 2 M \sin x_5 \sin x_4 F_y + 2 M (m + M) g \sin x_4 + 2 M^2 l \sin x_5 \sin x_4, \]

\[-2 M \sin x_5 \cos x_4 \cos^2 x_5 \sin x_4 x_5 - M \cos x_4 \sin x_4 \sin x_5, \]

and \( d_1, d_2, d_3, d_4 \) are outer disturbances.

In the above equations, \( F_x \) and \( F_y \) are the controlling and \( \theta \) and \( \phi \) are the pendulum angle in the \( x \) and \( y \) directions respectively.
2.3 x-z Inverted pendulum

The x-z IP consists of a rod and a pivot. The movement of the pivot is controlled by horizontal and vertical forces (\( F_x \) and \( F_z \)) as shown Figure 2. The two forces act such a way that the pendulum rod remain stable in upright position. The mathematical model of the IP is as follows[2].

\[
\begin{align*}
(M + m)x + ml\cos \theta \ddot{\theta} - m \sin \theta \dot{\theta}^2 &= F_x \\
(M + m)\ddot{z} - m \sin \theta \ddot{\theta} - ml\cos \theta \ddot{\theta}^2 &= F_z - (M + m)g \\
\cos \theta \ddot{x} - \sin \theta \ddot{z} + i \dot{\theta} - g \sin \theta &= 0
\end{align*}
\]  

(9)

where \(-0.5 \leq x \geq 0.5\). and \(-0.5 \leq z \geq 0.5\). State space form of (9) is as follows:

\[
\begin{align*}
\dot{x}_1 &= x_1 \\
\dot{x}_2 &= \frac{-m \sin x_1 \cos x_2 F_z + MF_z - m \sin^2 x_1 F_z + Mnl \sin x_1 \dot{x}_1^2 + d_1}{M^2 + Mm} \\
\dot{x}_3 &= x_3 \\
\dot{x}_4 &= \frac{mF_z - m \cos^2 x_1 F_z - m \sin x_1 \cos x_2 F_z + MF_z - M^2 g + Mnl \cos x_1 \dot{x}_1^2}{M^2 + Mm} + d_2 \\
\dot{x}_5 &= x_4 \\
\dot{x}_6 &= \frac{-\cos x_1 F_z + \sin x_1 F_z + d_1}{Ml} + d_1
\end{align*}
\]  

(10)

3. FOPID calculus and controller design
3.1 Fractional calculus

It is the branch of mathematics which deals with the integral and differential calculus having order (powers) in fractional numbers [27]. Although these operators are complex in nature but gives more generalized form of the calculus. There are three commonly known definitions which are given as follows [28]. The first equation is called Reimann Liouville

\[ aD_t^\alpha f(t) = \frac{1}{\Gamma(n - \alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(\tau)d\tau}{(t - \tau)^{\alpha-n+1}} \]  

(11)

The second equation (Grünwald-Letnikov equation) is

\[ aD_t^\alpha f(t) = \lim_{h \to 0} \frac{1}{h^\alpha} \sum_{j=0}^{(t-a)/h} (-1)^j \binom{\alpha}{j} f(t - jh) \]  

(12)

Where \( \binom{\alpha}{j} = \frac{\Gamma(\alpha + 1)}{\Gamma(j + 1)\Gamma(\alpha - j + 1)} \)

And the third is Caputo expression, which is given by

\[ aD_t^\alpha f(t) = \frac{1}{\Gamma(n - \alpha)} \int_a^t \frac{f^n(\tau)d\tau}{(t - \tau)^{\alpha-n+1}} \]  

(13)

where \( n - 1 < \alpha < n \), with \( a \) as initial condition and \( n \) is an interger

3.2 Fractional order PID controller

Model of FOPID controller \( PI^\mu D^\alpha \) is as shown in Figure 4, where \( \lambda > 0 \) and \( \mu > 0 \). The FOPID equation is as follows:

\[ u(t) = k_p e(t) + k_i D^{-\lambda} e(t) \]  
\[ + k_d D^\mu e(t) \]  

(14)

Above equation may be written in s-domain as follows:

\[ C(s) = k_p + k_i / s^\lambda \]  
\[ + k_d s^\mu \]  

(15)

There are five parameters \( k_p, k_i, k_d, \lambda \) and \( \mu \) which are optimized by PSO technique.

![Figure 4. FOPID controller design](image-url)
4. Particle Swarm Optimization (PSO) Technique

To solve complex engineering problems with large search space, swarm based modern optimization is widely used to find the optimal solutions. Particle swarm optimization (PSO) is a population-based swarm intelligence method to solve these problems. The PSO was first given by Kennedy and Eberhart[29]. This technique mimics the swarm behavior of birds and school of fish with their collision free movement. Each bird is called a ‘particle’ in the algorithm of PSO which have specific position and velocity. These particles move in multi-dimensional space with their own experience or neighbouring particle. The velocity and position of particles get updated regularly with their movement. The fitness function parameters of the PSO algorithm is given in the Table 2. The values of other parameters of the PSO technique is given in Table 3. For optimization of the parameters of the proposed controller, we need to set a range of values which is given in Table 4.

\begin{table}[h]
\centering
\caption{Parameters of fitness function}
\begin{tabular}{cccc}
\hline
$T$ & $w_1$ & $w_2$ & $w_3$ \\
\hline
30 & 1 & 1 & 0.05 \\
\hline
\end{tabular}
\end{table}

\begin{table}[h]
\centering
\caption{Values of PSO parameters}
\begin{tabular}{cccccccc}
\hline
PSO Values & Number of particles & Number of iterations & $c_1$ & $c_2$ & $W_{\text{max}}$ & $W_{\text{min}}$ & $R_1$ & $R_2$ \\
\hline
parameters & 20 & 30 & 2 & 2 & 0.9 & 0.1 & 0.1 & 0.1 \\
\hline
\end{tabular}
\end{table}

\begin{table}[h]
\centering
\caption{Controller parameters}
\begin{tabular}{lc}
\hline
Controller parameters & Range \\
\hline
$K_p, K_i, K_d$ & $\{-20, 20\}$ \\
$\lambda, \mu$ & $\{0, 2\}$ \\
\hline
\end{tabular}
\end{table}

Following equation gives the updated velocity and position of the particle in the PSO algorithm

\begin{equation}
V_{i}^{k+1} = W \times V_{i}^{k} + c_1 \times rand( ) \times (p_{\text{best},i} - X_{i}^{k}) + c_2 \times rand( ) \times (g_{\text{best},i} - X_{i}^{k}) \\
X_{i}^{k+1} = X_{i}^{k} + V_{i}^{k+1}
\end{equation}

(16)

Where:

- $V_i$ = velocity of $i$th particle,
- $W = \text{inertia weight factor}$,
- $c_1 = \text{cognitive acceleration factor}$,
- $c_2 = \text{social acceleration factor}$,
- $rand( )$ = random numbers in range (0,1) and
- $X_i$ = position of $i$th particle. The inertia factor($w$) expression is given bellow:

\begin{equation}
w = W_{\text{max}} - \left[\frac{W_{\text{max}} - W_{\text{min}}}{k_{\text{max}}}\right] \times k
\end{equation}

(18)

where $W_{\text{max}}, W_{\text{min}} = \text{maximum and minimum values of } W \text{ respectively,}$

$k_{\text{max}} = \text{maximum number of iterations.}$
For obtaining the optimal values of proposed control scheme, an objective function is used in the PSO technique. The objective function needed a performance index for the minimization or maximization of the function. There are various types of performance index in the control literature. In this work, integral time absolute error (ITAE) in the MATLAB codes of the PSO technique.

\[ ITAE = \int_0^\infty t|e(t)|dt \] (19)

5. Simulation Results

All the mathematical equations of the inverted pendulums are modeled in MATLAB environment and simulation is carried out for each of the inverted pendulums. For the optimization purpose of the controller parameters, a PSO based MATLAB code is used. Simulation results are given in the following subsections.

5.1 Simulation results for x-IP

(a) First step with single FOPID controller design

In this case, we have designed one FOPID controller for the control of angle of x-IP. In this case we have not considered the control of position. The control structure of the x IP is given in Figure 5. Simulation results of the pendulum with FOPID controller are shown in Figure 6 and 7. The initial angle is assumed to be 0.5. The FOPID parameters obtained using PSO are given by:

\[ \lambda = 1.2, \quad \mu = 0.99, \quad k_p = 26, \quad k_i = 16, \quad \text{and} \quad k_d = 4 \]

PID parameters taken from [2] are given by:

\[ k_p = 25, \quad k_i = 15, \quad k_d = 3 \]

From the figures 6 and 7 we observe that FOPID design gives better response than the PID in terms of settling time and undershoot.

(b) Second step with two FOPID controller design

On the basis of the FOPID1 considered in the first case, we have added FOPID2 controller to control the pivot position. The new control structure is shown in Figure 8. Since there are two output which need to control, we need two FOPID control structure as shown in figure. In this case FOPID1 need not require any change, however parameters of the FOPID2 may be adjusted with optimized value. The parameters of the FOPID are obtained using optimization technique which is as follows:
For FOPID2, \[ \lambda = 1.05, \mu = 1.2, k_p = -1.3, k_i = -0.8, k_d = -1 \]
For PID1 \[ k_p = 25, k_i = 15, k_d = 3 \]
For PID2 \[ k_p = -2.4, k_i = -1, k_d = -0.75 \]

Figure 9 and 10 shows the angle and position of the pendulum respectively. From the figure, one can observe that the FOPID system give better response. Control force is shown in the Figure 11 which is satisfactory for both the controllers. Further, the figures 12, 13 and 14 are the responses under the influence of outer disturbances which is equal to \[ d_1 = d_2 = 20\sin(20\pi t) \]. From the figures it is observed that the FOPID has satisfactory response.


5.2 Simulation results of x-y IP

On the basis of the FOPID controller design designed for the x inverted pendulum, we can design a FOPID controller for the x-y inverted pendulum. It is observed from the model of x-y IP that there are four output variables that need to be controlled and two input variables $F_x$ and $F_y$. Therefore, Planar IP is a multi-input multi-output (MIMO) system for which four VSC based controllers are needed for the control of four output variables. At equilibrium (that is $\theta = \phi = 0$), we have following equations:

$$\ddot{x} = \frac{F_x}{M} \quad \ddot{y} = \frac{F_y}{M} \quad \ddot{\theta} = -\frac{F_x}{Ml} \quad \text{and} \quad \ddot{\phi} = -\frac{F_y}{Ml}$$

Figure 14. Control force of the x − IP

Above kinematic equations state that variables $x$ and $\theta$ depend on $F_x$ whereas $y$ and $\phi$ are functions of $F_y$ only near the equilibrium point (origin). From the above observations, the control structure employing four VSC based system may designed for the PIP as shown in Figure 15.

Figures 16 to 21 are the various responses for the x-y inverted pendulum in which we observed that the FOPID controller design give good results. Further to check the robustness of the controller we set the disturbances equal to $d_1 = d_2 = d_3 = d_4 = 20\sin(20\pi t)$ and corresponding results are shown in Figures 22 to 27 which show good robustness.

The parameters of the FOPID is found with PSO technique which is as follows:

For FOPID1, $\lambda = 1.1, \mu = 0.99, k_p = 25.10, k_i = 14.95, k_d = 2.98$

For FOPID2, $\lambda = 1.05, \mu = 0.95, k_p = -2.41, k_i = -1, k_d = -0.75$

PID (Wang) parameters are as follows

For PID1 $k_p = 25, k_i = 15, k_d = 3$

For PID2 $k_p = -2.4, k_i = -1, k_d = -0.75$

Figure 15. Two FOPID controller design for x-y-IP
Figure 16. Angle of the x − y IP

Figure 17. Angle of the x − y IP

Figure 18. x − position of the x − y IP

Figure 19. y position of x − y IP

Figure 20. Control force in x direction of the x − y IP

Figure 21. Control force in y direction of the x − y IP

Figure 22. Angle of the x − y IP with disturbance

Figure 23. Angle of the x − y IP with disturbance
5.3 Simulation results of x-z IP
Depending upon the controller design in the previous sections, we design a FOPID3 for the control of the x-z inverted pendulum as shown in Figure 28. The parameters of the FOPID1 and FOPID2 controllers are unchanged as previous. FOPID3 controller parameters are obtained using the optimization technique as follows.
For FOPID3, \( \lambda = 0.97, \mu = 0.98, k_p = 15.01, k_i = 30.02 \) and \( k_d = 12.01 \)
For PID \( k_p = 15, k_i = 30 \) and \( k_d = 12 \)
Figures 29 to 33 show the various responses of the inverted pendulum system with proposed controller design. From the figures we observed that the VSC system gives better response compared to the PID.

Figure 24. x position of the x − y IP with disturbances
Figure 25. y position of the x − y IP with disturbances
Figure 26. Control force in x direction of the x − y IP with disturbances
Figure 27. Control force in y direction of the x − y IP with disturbances

Figure 28. Three FOPID controller design for x-z IP
To check the robustness of the pendulum system, the outer disturbances in Equation 10 are considered is \( d_1 = d_2 = d_3 = 20\sin(20t) \). Corresponding results in presence of the disturbances are shown in Figures 34 to 38. From these responses, it is observed that the proposed VSC system has satisfactory response.
6. Conclusion
In this work, FOPID controller has been applied to three types of inverted pendulum (IP) systems for its stabilization. From the simulation results, we observed that the FOPID controller performs better compared to the conventional PID controller in terms of stabilization and tracking control of the inverted pendulums.

The major contribution of this work may be summarized as follows:

- Mathematical model and state equation of the IPs are derived using Lagrange’s equation.
- FOPID controller has been proposed for the stabilization of the three types of IPs. Further, for the controller parameters adjustment, PSO technique is used with its MATLAB program.
- The stabilization of the IPs are performed with zero reference values and found that the proposed FOPID controller give better response compared to the PID controller.
- The proposed FOPID control scheme gives good dynamic performance as well as had very good robustness towards the outer disturbances.

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