LIOUVILLE-TYPE THEOREMS FOR FUNCTIONS OF FINITE ORDER

B.N. KHABIBULLIN

Abstract. A convex, subharmonic or plurisubharmonic function respectively on the real axis, on a finite dimensional real of complex space is called a function of a finite order if it grows not faster than some positive power of the absolute value of the variable as the latter tends to infinity. An entire function on a finite-dimensional complex space is called a function of a finite order if the logarithm of its absolute value is a (pluri-)subharmonic function of a finite order. A measurable set in an $m$-dimensional space is called a set of a zero density with respect to the Lebesgue density if the Lebesgue measure of the part of this set in the ball of a radius $r$ is of order $o(r^m)$ as $r \to +\infty$. In this paper we show that convex function of a finite order on the real axis and subharmonic functions of a finite order on a finite-dimensional real space bounded from above outside some set of a zero relative Lebesgue measure are bounded from above everywhere. This implies that subharmonic functions of a finite order on the complex plane, entire and subharmonic functions of a finite order, as well as convex and harmonic functions of a finite order bounded outside some set of a zero relative Lebesgue measure are constant.

Keywords: entire function, subharmonic function, pluri-subharmonic function, convex function, harmonic function of entire order, Liouville theorem.

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The base of this work is a classical Liouville theorem for entire functions, that is, for holomorphic on complex plane $\mathbb{C}$ or on $\mathbb{C}^n$, where $n \in \mathbb{N} := \{1, 2, \ldots\}$ functions.

Liouville theorem. A bounded entire function is constant.

The same statement holds for bounded from above subharmonic functions on $\mathbb{C}$ [1, Cor. 2.3.4] and as an obvious corollary, for plurisubharmonic functions on $\mathbb{C}^n$, convex functions on the real line $\mathbb{R}$ and as an immediate corollary, on $\mathbb{R}^m$ with $1 < m \in \mathbb{N}$, as well as for harmonic functions on $\mathbb{R}^m$ for all $m \in \mathbb{N}$ [2, Thm. 1.19].

Recently in work [3, Lm. 4.2], there was given a version of Liouville theorem for entire functions of finite order on $\mathbb{C}$ bounded not everywhere but only outside some small set $E \subset \mathbb{C}$. In [4, Lm. 4.2], its proof was corrected and before its formulation in Theorem 2.1 in [5] it was said that this theorem was established by A.A. Borichev. The proofs in [3] and [4] employ rather advanced facts and arguing from the theory of complex variable and the potential theory on the complex plane.

Theorem B. ([3, Lm. 4.2], [4, Lm. 4.2], [5, Thm. 2.1]) If an entire function of a finite order on $\mathbb{C}$ is bounded outside some set $E \subset \mathbb{C}$ measurable by the planar Lebesgue measure $\lambda$...
and this set has a zero planar density in the sense that
\[
\lim_{r \to +\infty} \frac{\lambda(\{z \in E: |z| \leq r\})}{r^2} = 0,
\]
then this function is constant.

The main result of this work develops and extends Theorem B on plurisubharmonic and entire functions on \(\mathbb{C}^n\) for all \(n \in \mathbb{N}\), as well as on convex and harmonic functions on \(\mathbb{R}^m\). At the same time, our proof is simpler in the case of entire functions of a single complex variable and it is based on an approach differing from that employed in the former proofs of Theorem B.

Let a function \(M\) with values in an extended real line \(\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm \infty\}\) is defined on a positive half-line \(\mathbb{R}^+ := \{x \in \mathbb{R}: x \geq 0\}\), in \(\mathbb{R}^m\) or in \(\mathbb{C}^n\) identified with \(\mathbb{R}^{2n}\), with the Euclidean norm \(|\cdot|\), but, generally speaking, outside some closed ball \(B(r)\) of a bounded radius \(r \in \mathbb{R}^+\) and centered at the origin. The order of the function \(M\) at infinity can be defined as [6, Sect. 2.1]
\[
\text{ord}[M] := \lim_{|x| \to \infty} \frac{\ln(1 + M^+(x))}{\ln |x|} \in \mathbb{R}^+ \cup \{+\infty\},
\]
where \(M^+: x \mapsto \max\{0, M(x)\}\) is a positive part of the function \(M\). The order of entire function \(f\) on \(\mathbb{C}^n\) is defined as order \(\text{ord}[\ln |f|]\) in the sense of (2).

**Definition.** (cf. with (1)) A relative upper Lebesgue density a subset \(E \subset \mathbb{R}^m\) measurable by the Lebesgue measure \(\lambda\) on \(\mathbb{R}^m\) is the quantity
\[
\mathbb{L}_m(E) := \lim_{r \to +\infty} \frac{\lambda(E \cap B(r))}{r^m} \in \mathbb{R}^+ \cup \{+\infty\}.
\]

If in the right hand side of the above identity the usual limit \(\lim_{r \to +\infty}\) is well-defined, we call it simply relative Lebesgue density \(\mathbb{L}_m(E) \in \mathbb{R}^+ \cup \{+\infty\}\) of the set \(E\). The definitions are obviously extended to \(\mathbb{C}^n\) identified with \(\mathbb{R}^{2n}\) and the notations are \(\mathbb{L}_{2n}\) and \(\mathbb{L}_{2n}\).

**Theorem 1.** Let \(m \in \mathbb{N}\) and \(E \subset \mathbb{R}^m\) be a subset of zero relative Lebesgue density \(\mathbb{L}_m(E) = 0\) in \(\mathbb{R}^m\). If a subharmonic function \(v\) of a finite order on \(\mathbb{R}^m\) is bounded from above on \(\mathbb{R}_m \setminus E\), then
\[
\sup_{\mathbb{R}^m} v = \sup_{\mathbb{R}^m \setminus E} v < +\infty.
\]

Let \(n \in \mathbb{N}\). A function \(\mathbb{C}^n\) is called plurisubharmonic if its restriction on each complex straight line is a subharmonic function. In particular, as \(n = 1\), these notions coincide, while each plurisubharmonic function on \(\mathbb{C}^n\) is subharmonic on \(\mathbb{R}^{2n}\). By Theorem [1] the classical Liouville theorem for plurisubharmonic and entire functions implies the following statement.

**Theorem 2.** Let \(n \in \mathbb{N}\) and \(E \subset \mathbb{C}^n\) be a set of zero relative Lebesgue density in \(\mathbb{C}^n\) in the sense of the above definition on \(\mathbb{R}^{2n}\) identified with \(\mathbb{C}^n\), that is, \(\mathbb{L}_{2n}(E) = 0\). If a plurisubharmonic or entire function of a finite order on \(\mathbb{C}^n\) is bounded from above on \(\mathbb{C}^n \setminus E\), then it is constant.

Subharmonic functions on \(\mathbb{R}\) are exactly convex functions. For each \(m \in \mathbb{N}\), each convex of harmonic function on \(\mathbb{R}^m\) is also subharmonic. Thus, by Theorem [1] and classical Liouville theorems for convex or harmonic functions on \(\mathbb{R}^m\) we obtain immediately the following theorem.

**Theorem 3.** Let \(m \in \mathbb{N}\) and \(E \subset \mathbb{R}^m\) be a set of zero relative Lebesgue density in \(\mathbb{R}^m\). If a convex or harmonic function of entire order on \(\mathbb{R}^m\) is bounded from above on \(\mathbb{R}^m \setminus E\), then it is constant.
It remains to prove Theorem 1 and we proceed to this.

For $m \in \mathbb{N}$, $x \in \mathbb{R}^m$ and $r \in \mathbb{R}^+$ by $B(x, r) := \{x' \in \mathbb{R}^m : |x' - x| \leq r\}$ we denote a closed ball in $\mathbb{R}^m$ of radius $r$ centered at $x$, and as above, $\overline{B}(r) := B(0, r)$. Similar notation is introduced $\mathbb{C}^m$ identified with $\mathbb{R}^{2n}$. For a $\lambda$-integrable function $v : \overline{B}(x, r) \to \mathbb{R}$ we let

$$B_v(x, r) := \frac{1}{\lambda(B(x, r))} \int_{B(x, r)} v \, d\lambda = \frac{1}{\lambda((B(x, r))} \int_{B(x, r)} v \, d\lambda$$

where $b_m$ is the volume of the unit ball. These are respectively mean functions of $v$ over closed balls $B(x, r)$ and $\overline{B}(r)$. The positivity is understood as $\geq 0$, the negativity does as $\leq 0$.

**Lemma 1.** Let $0 < R \in \mathbb{R}^+$ and $v$ be a positive $\lambda$-measurable function on a closed ball $\overline{B}(R) \subset \mathbb{R}^m$, $0 < r < R$. Then

$$B_v(x, R - r) \leq \left(1 + \frac{r}{R - r}\right)^m B_v(R) \text{ for each point } x \in \overline{B}(r). \quad (6)$$

**Proof.** By definition (5) and owing to the positivity of $v$ on $\overline{B}(R)$ and the inclusions $B(x, r - R) \subset \overline{B}(R)$ for all $x \in \overline{B}(r)$ we obtain:

$$B_v(x, R - r) \overset{(5)}{=} \frac{1}{b_m(R - r)^m} \int_{B(x, R - r)} v \, d\lambda \leq \frac{1}{b_m(R - r)^m} \int_{\overline{B}(R)} v \, d\lambda = \frac{b_m R_m}{b_m(R - r)^m b_m R_m} \int_{\overline{B}(R)} v \, d\lambda \overset{(6)}{=} \left(1 + \frac{r}{R - r}\right)^m B_v(R),$$

and this completes the proof. \[\square\]

By $\text{sbh}(S)$ we denote the class of all subharmonic (locally convex as $m = 1$) functions on some open neighbourhoods of the set $S \subset \mathbb{R}^m$. The role of means over balls in (5) for subharmonic functions is due to the inequality on mean over ball [1], [2]; this inequality characterizes them completely under the upper semi-continuity and local integrability in the sense of Lebesgue measure $\lambda$. In particular,

$$v(x) \leq B_v(x, r) \text{ as } v \in \text{sbh}(\overline{B}(x, r)). \quad (7)$$

**Lemma 2.** Let $0 < R \in \mathbb{R}^+$ and $v$ be a subharmonic function on a closed ball $\overline{B}(R) \subset \mathbb{R}^m$, $0 < r < R$, and $E \subset \overline{B}(r)$ be a $\lambda$-measurable set. Then

$$\int_E v \, d\lambda \leq \left(1 + \frac{r}{R - r}\right)^m \lambda(E) B_v+(R). \quad (8)$$

**Proof.** By inequality (7) on mean over ball we obtain

$$v(x) \leq B_v(x, R - r) \leq B_v+(x, R - r) \text{ for each point } x \in \overline{B}(r).$$

Integrating this inequality over the set $E$ by the Lebesgue measure $\lambda$ gives the inequality

$$\int_E v \, d\lambda \leq \int_E B_v+(x, R - r) \, d\lambda(x).$$

Hence, by inequality (6) in Lemma 1 applied to the integrand with a positive function $v^+$ in the latter integral, we obtain

$$\int_E v \, d\lambda \leq \int_E \left(1 + \frac{r}{R - r}\right)^m B_v+(R) \, d\lambda(x) = \left(1 + \frac{r}{R - r}\right)^m B_v+(R) \lambda(E),$$

and this proves (8). The proof is complete. \[\square\]
**Lemma 3.** Let $0 < R \in \mathbb{R}^+$, and $v$ be a subharmonic function on a closed ball $\overline{B}(R) \subset \mathbb{R}^m$. Then for each number $r \in (0, R)$ and for each $\lambda$-measurable subset $E \subset \overline{B}(r)$ we have the inequality

$$B_v(r) \leq \frac{1}{b_m r^m} \int_{\overline{B}(r) \setminus E} v \, d\lambda + \frac{1}{b_m r^m} \int_{E} v \, d\lambda,$$

and by inequality (8) in Lemma 2 applied to the latter integral, we arrive at (9). The proof is complete. □

**Proof of Theorem 2**. We let

$$M := \sup_{\mathbb{R}^m \setminus E} v \in \mathbb{R}.$$  

(10)

Thanks to the boundedness from above of the function $v$ on $\mathbb{R}^m \setminus E$, we can consider a subharmonic function $v - M$ negative on $\mathbb{R}^m \setminus E$. We apply Lemma 3 for arbitrary $0 < r \in \mathbb{R}^+$ with $R = 2r$ and with the set obtained by the intersection $E \cap \overline{B}(r) \subset \overline{B}(r)$ as the set $E$ to a subharmonic function $(v - M)^+ \geq 0$, where the first integral in the right hand side in (9) vanishes. As a result we obtain:

$$B_{(v-M)^+}(r) \leq \frac{1}{b_m} \left(1 + \frac{r}{2r - r}\right)^m \lambda(E \cap \overline{B}(r)) \frac{B_{(v-M)^+}(2r)}{\gamma^m}$$

$$= \frac{2^m \lambda(E \cap \overline{B}(r))}{b_m \gamma^m} B_{(v-M)^+}(2r) \quad \text{for all } 0 < r \in \mathbb{R}^+.$$  

By condition $L_m(E) = 0$ for the function

$$r \quad \mapsto \quad B_{(v-M)^+}(r) \in \mathbb{R}^+$$

(11)

this yields

$$B_{(v-M)^+}(r) = o(B_{(v-M)^+}(2r)) \quad \text{as } r \to +\infty.$$  

(12)

Function (12) is of a finite order $\text{ord}[B_{(v-M)^+}] \in \mathbb{R}^+$ since $\text{ord}[(v - M)^+] \in \mathbb{R}^+$ thanks to the finiteness of the order $\text{ord}[v]$. Hence, (12) is possible only in the case $B_{(v-M)^+} \equiv 0$ and, as an implication, $(v - M)^+ \equiv 0$. Together with (10) this implies (4). The proof is complete. □

**Remark.** The condition of zero Lebesgue density $L_m(E) = 0$ in Theorems 2 and 3, as well as the same condition with $m := 2n$ in Theorem 4, can be replaced by a formally weaker condition: there exists an unbounded sequence of positive numbers $(r_k)_{k \in \mathbb{N}}$, for which

$$\limsup_{k \to \infty} \frac{r_{k+1}}{r_k} < +\infty \quad \text{and} \quad \lim_{k \to \infty} \frac{\lambda(E \cap B(r_k))}{\gamma^m r_k} = 0,$$

since the latter implies $L_m(E) = 0$.

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Bulat Nurmievich Khabibullin
Bashkir State University,
Zaki Validi str. 32,
450000, Ufa, Russia
E-mail: khabib-bulat@mail.ru