COHOMOLOGICAL CHARACTERIZATIONS OF PROJECTIVE SPACES AND HYPERQUADRICS

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1. Introduction

Projective spaces and hyperquadrics are the simplest projective algebraic varieties, and they can be characterized in many ways. The aim of this paper is to provide a new characterization of them in terms of positivity properties of the tangent bundle (Theorem 1.1).

The first result in this direction was Mori’s proof of the Hartshorne conjecture in [Mor79] (see also Siu and Yau [SY80]), that characterizes projective spaces as the only manifolds having ample tangent bundle. Then, in [Wah83], Wahl characterized projective spaces as the only manifolds whose tangent bundles contain ample invertible subsheaves. Interpolating Mori’s and Wahl’s results, Andreotti and Wiśniewski gave the following characterization:

Theorem [AW01]. Let $X$ be a smooth complex projective $n$-dimensional variety. Assume that the tangent bundle $T_X$ contains an ample locally free subsheaf $\mathcal{E}$ of rank $r$. Then $X \cong \mathbb{P}^n$ and either $\mathcal{E} \cong \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus r}$ or $r = n$ and $\mathcal{E} = T_{\mathbb{P}^n}$.

We note that earlier, in [CP98], Campana and Peternell obtained the same result for $r \geq n - 2$.

Let $\mathcal{E}$ be an ample locally free subsheaf of $T_{\mathbb{P}^n}$ of rank $p < n$. By taking its determinant, we obtain a non-zero section in $H^0(\mathbb{P}^n, \wedge^p T_{\mathbb{P}^n} \otimes \mathcal{O}_{\mathbb{P}^n}(-p))$. On the other hand, most sections in $H^0(\mathbb{P}^n, \wedge^p T_{\mathbb{P}^n} \otimes \mathcal{O}_{\mathbb{P}^n}(-p))$ do not come from ample locally free subsheaves of $T_{\mathbb{P}^n}$.

This motivates the following characterization of projective spaces and hyperquadrics, which was conjectured by Beauville in [Bea00]. Here $Q_p$ denotes a smooth quadric hypersurface in $\mathbb{P}^{p+1}$, and $\mathcal{O}_{Q_p}(1)$ denotes the restriction of $\mathcal{O}_{\mathbb{P}^{p+1}}(1)$ to $Q_p$. When $p = 1$, $(Q_1, \mathcal{O}_{Q_1}(1))$ is just $(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2))$.

Theorem 1.1. Let $X$ be a smooth complex projective $n$-dimensional variety and $\mathcal{L}$ an ample line bundle on $X$. If $H^0(X, \wedge^p T_X \otimes \mathcal{L}^{-p}) \neq 0$ for some positive integer $p$, then either $(X, \mathcal{L}) \cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$, or $p = n$ and $(X, \mathcal{L}) \cong (Q_p, \mathcal{O}_{Q_p}(1))$.

The statement of this theorem can be interpreted in the following way. Let $X$ be a smooth complex projective $n$-dimensional variety and $\mathcal{L}$ an ample line bundle on $X$. Consider the sheaf $\mathcal{L}^{-1} := T_X \otimes \mathcal{L}^{-1}$. Then Wahl’s theorem [Wah83] says that if $H^0(X, \mathcal{L}^{-1}) \neq 0$ then $X \cong \mathbb{P}^n$. Theorem 1.1 generalizes this statement to the case when one only assumes that $H^0(X, \wedge^p \mathcal{L}) \neq 0$ for some $0 < p \leq n$.

In order to prove Theorem 1.1, first notice that $X$ is uniruled by [Miy87, Corollary 8.6]. Next observe that if the Picard number of $X$ is 1, then it is necessarily a Fano variety. If the Picard number is larger than 1, then we fix a minimal covering family $H$ of rational curves on $X$, and follow the strategy in [AW01] of looking at the $H$-rationally connected quotient $\pi : X^o \to Y^o$ of $X$ (see Section 2 for definitions). We show that any non-zero section $s \in H^0(X, \wedge^p T_{X^o} \otimes \mathcal{L}^{-p})$ restricts to a non-zero section $s^o \in H^0(X^o, \wedge^p T_{X^o} \otimes \mathcal{L}^{-p})$, except in the very special case when $p = 2$ and $X \cong \mathbb{P}^2$. This is achieved in Section 5.

Afterwards we need to deal with two cases: the case when $X$ is a Fano manifold with Picard number 1, and the case in which the $H$-rationally connected quotient $\pi : X^o \to Y^o$ is either a projective space bundle or a quadric bundle, and $H^0(X^o, \wedge^p T_{X^o} \otimes \mathcal{L}^{-p}) \neq 0$.

When $X$ is a Fano manifold with Picard number $\rho(X) = 1$, the result follows from the following.

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Theorem 1.2 (=Theorem 5.3). Let $X$ be a smooth $n$-dimensional complex projective variety with $\rho(X) = 1$, $\mathcal{L}$ an ample line bundle on $X$, and $p$ a positive integer. If $H^0(X, \mathcal{L} \otimes \mathcal{O}_X(-p)) \neq 0$, then either $(X, \mathcal{L}) \simeq (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$, or $p = n \geq 3$ and $(X, \mathcal{L}) \simeq (Q_p, \mathcal{O}_{Q_p}(1))$.

The paper is organized as follows. In Section 2 we gather old and new results about minimal covering families of rational curves and their rationally connected quotients. In Section 3 we show that the relative anticanonical bundle of a generically smooth surjective morphism from a normal projective $\mathbb{Q}$-Gorenstein variety onto a smooth curve is never ample. This will be used to treat the case when the $H$-rationally connected quotient $\pi: X^0 \to Y^0$ is a quadric bundle. In Section 4 we show that $p$-derivations can be lifted to the normalization. This technical result will be used in the following section, which is the technical core of the paper. In Section 5 we study the behavior of non-zero global sections of bundles of the form $\wedge^p \mathcal{T}_X \otimes \mathcal{M}$ with respect to fibrations $X \to Y$. We also prove some general vanishing results, such as the following.

Theorem 1.3 (=Corollary 5.5). Let $X$ be a smooth complex projective variety and $\mathcal{L}$ an ample line bundle on $X$. If $H^0(X, \wedge^p \mathcal{T}_X \otimes \mathcal{L}^{-p−1−k}) \neq 0$ for integers $p \geq 1$ and $k \geq 0$, then $k = 0$ and $(X, \mathcal{L}) \simeq (\mathbb{P}^p, \mathcal{O}_{\mathbb{P}^p}(1))$.

Finally, in Section 6 we prove Theorem 1.2 and put things together to prove Theorem 1.1.

Notation and definitions. Throughout the present article we work over the field of complex numbers unless otherwise noted. By a vector bundle we mean a locally free sheaf and by a line bundle an invertible sheaf. If $X$ is a variety and $x \in X$, then $\kappa(x)$ denotes the residue field $\mathcal{O}_{X,x}/\mathfrak{m}_{X,x}$. Given a variety $X$, we denote by $\rho(X)$ the Picard number of $X$. If $\mathcal{E}$ is a vector bundle over a variety $X$, we denote by $\mathcal{E}^*$ its dual vector bundle, and by $\mathbb{P}(\mathcal{E})$ the Grothendieck projectivization $\text{Proj}_X(\text{Sym}(\mathcal{E}))$. For a morphism $f: X \to T$, the fiber of $f$ over $t \in T$ is denoted by $X_t$.

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2. Minimal rational curves on uniruled varieties

In this section we gather some properties of minimal covering families of rational curves and their corresponding rationally connected quotients. For more details see [Kol96], [Deb01], or [AK03].

Let $X$ be a smooth complex projective uniruled variety and $H$ an irreducible component of RatCurves$^n(X)$. Recall that only general points in $H$ are in $1:1$-correspondence with the associated curves in $X$.

We say that $H$ is a covering family of rational curves on $X$ if the corresponding universal family dominates $X$. A covering family $H$ of rational curves on $X$ is called unsplit if it is proper. It is called minimal if, for a general point $x \in X$, the subfamily of $H$ parametrizing curves through $x$ is proper. As $X$ is uniruled, a minimal covering family of rational curves on $X$ always exists. One can take, for instance, among all covering families of rational curves on $X$ one whose members have minimal degree with respect to a fixed ample line bundle.

Fix a minimal covering family $H$ of rational curves on $X$. Let $C$ be a rational curve corresponding to a general point in $H$, with normalization morphism $f: \mathbb{P}^1 \to C \subset X$. We denote by $[C]$ or $[f]$ the point in $H$ corresponding to $C$. We denote by $f^*T_X^+$ the subbundle of $f^*T_X$ defined by $f^*T_X^+ = \text{im}[H^0(\mathbb{P}^1, f^*T_X(-1)) \otimes \mathcal{O}_{\mathbb{P}^1}(1) \to f^*T_X] \hookrightarrow f^*T_X$.

By [Kol96 IV.2.9], if $[f]$ is a general member of $H$, then $f^*T_X \simeq \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus d} \oplus \mathcal{O}_{\mathbb{P}^1}(n-d-1)$, where $d = \deg f^*T_X - 2 \geq 0$.

Given a point $x \in X$, we denote by $H_x$ the normalization of the subscheme of $H$ parametrizing rational curves passing through $x$. By [Kol96 II.1.7, II.2.16], if $x \in X$ is a general point, then $H_x$ is a smooth
projective variety of dimension \( d = \deg(f^*T_X) - 2 \). We remark that a rational curve that is smooth at \( x \) is parametrized by at most one element of \( H_x \).

Let \( H_1, \ldots, H_k \) be minimal covering families of rational curves on \( X \). For each \( i \), let \( \overline{H}_i \) denote the closure of \( H_i \) in Chow(\( X \)). We define the following equivalence relation on \( X \), which we call \((H_1, \ldots, H_k)\)-equivalence. Two points \( x, y \in X \) are \((H_1, \ldots, H_k)\)-equivalent if they can be connected by a chain of 1-cycles from \( \overline{H}_1 \cup \cdots \cup \overline{H}_k \). By [Cam92] (see also [Kol96] IV.4.16), there exists a proper surjective morphism \( \pi^0 : X^0 \to Y^0 \) from a dense open subset of \( X \) onto a normal variety whose fibers are \((H_1, \ldots, H_k)\)-equivalence classes. We call this map the \((H_1, \ldots, H_k)\)-rationally connected quotient of \( X \). When \( Y^0 \) is a point we say that \( X \) is \((H_1, \ldots, H_k)\)-rationally connected.

**Remark 2.1.** By [Kol96] IV.4.16, there is a universal constant \( c \), depending only on the dimension of \( X \), with the following property. If \( H_1, \ldots, H_k \) are minimal covering families of rational curves on \( X \), and \( x, y \in X \) are general points on a general \((H_1, \ldots, H_k)\)-equivalence class, then \( x \) and \( y \) can be connected by a chain of at most \( c \) rational cycles from \( \overline{H}_1 \cup \cdots \cup \overline{H}_k \).

The next two results are special features of the \((H_1, \ldots, H_k)\)-rationally connected quotient of \( X \) when the families \( H_1, \ldots, H_k \) are unsplit. The first one says that \( \pi^0 \) can be extended in codimension 1 to an equidimensional proper morphism with integral fibers, but possibly allowing singular fibers. The second one describes the general fiber of the \( H \)-rationally connected quotient of \( X \) when \( H \) is unsplit and \( H_x \) is irreducible for general \( x \in X \).

**Lemma 2.2.** Let \( X \) be a smooth complex projective variety and \( H_1, \ldots, H_k \) unsplit covering families of rational curves on \( X \). Then there is an open subset \( X^0 \) of \( X \) with \( \codim_X(X \setminus X^0) \geq 2 \), a smooth variety \( Y^0 \), and a proper surjective equidimensional morphism with irreducible and reduced fibers \( \pi^0 : X^0 \to Y^0 \) which is the \((H_1, \ldots, H_k)\)-rationally connected quotient of \( X \).

**Proof.** The fact that the \((H_1, \ldots, H_k)\)-rationally connected quotient of \( X \) can be extended in codimension 1 to an equidimensional proper morphism follows from the proof of [BCD07] Proposition 1]. This holds even in the more general context of quasi-unsplit covering families on \( \mathbb{Q} \)-factorial varieties. In [BCD07] Proposition 1] this is established for a single quasi-unsplit family, but the same proof works for finitely many quasi-unsplit families. For convenience we review the construction of that extension.

Let \( \pi^0 : X^0 \to Y^0 \) be the \((H_1, \ldots, H_k)\)-rationally connected quotient of \( X \). By shrinking \( Y^0 \) if necessary, we may assume that \( \pi^0 \) is smooth. Let \( Y \to \text{Chow}(X) \) be the normalization of the closure of the image of \( Y^0 \) in \( \text{Chow}(X) \), and let \( \mathcal{U} \subset Y \times X \) be the restriction of the universal family to \( Y \). Denote by \( p : \mathcal{U} \to Y \) and \( q : \mathcal{U} \to X \) the induced natural morphisms. Notice that \( q : \mathcal{U} \to X \) is birational.

Let \( 0 \in Y \) and set \( \mathcal{U}_0 = p^{-1}(0) \). Then \( q(\mathcal{U}_0) \) is contained in an \((H_1, \ldots, H_k)\)-equivalence class. This follows from taking limits of chains of rational curves from the families \( H_1, \ldots, H_k \) (see Remark 2.1), observing the assumption that the \( H_i \)’s are unsplit, and the fact that the image of a general fiber of \( p \) in \( X \) is an \((H_1, \ldots, H_k)\)-equivalence class.

Let \( E \) be the exceptional locus of \( q \). Since \( X \) is smooth, \( E \) has pure codimension 1 in \( \mathcal{U} \). Set \( S = q(E) \subset X \). This is a set of codimension at least 2 in \( X \). We shall show that \( S \) is closed with respect to \((H_1, \ldots, H_k)\)-equivalence. From that it will follow that the morphism \( p|_{\mathcal{U} \setminus E} : \mathcal{U} \setminus E \to Y \setminus p(E) \) is proper and induces a proper equidimensional morphism \( X \setminus S \to Y \setminus p(E) \) extending \( \pi^0 \). Let \( L \) be an effective ample divisor on \( Y \). Then there exists an effective \( q \)-exceptional divisor \( F \) on \( \mathcal{U} \) and an effective divisor \( D \) on \( X \) such that \( p^*L = F = q^*D \). First we claim that \( \text{supp} \, F = E \). Indeed, let \( C \subset E \) be any curve contracted by \( q \). Then \( C \) is not contracted by \( p \) since \( \mathcal{U} \subset Y \times X \). Hence \( F \cdot C = q^*D \cdot C = p^*L \cdot C < 0 \), and so \( C \subset \text{supp} \, F \). This proves the claim. Notice that the general fiber of \( p \) does not meet \( E \). Therefore, for any curve \( C \subset \mathcal{U} \) contained in a general fiber of \( p \), we have \( q^*D \cdot C = 0 \). This shows in particular that \( D \cdot E = 0 \) for any curve \( E \) from any of the families \( H_1, \ldots, H_k \). If \( E \subset \mathcal{U} \) is mapped onto \( E \) by \( q \), then \( F \cdot E = q^*D \cdot \tilde{E} - p^*L \cdot \tilde{E} \leq 0 \). Hence either \( \tilde{E} \) is contained in \( E = \text{supp} \, F \) or it is disjoint from it. Therefore, if \( E \) is a curve from any of the families \( H_1, \ldots, H_k \), then either \( E \subset S \) or \( E \cap S = \emptyset \). In other words, \( S \) is closed with respect to \((H_1, \ldots, H_k)\)-equivalence.

Replace \( X \) with \( X \setminus S \) and \( Y \) with \( Y \setminus p(E) \), obtaining a proper equidimensional morphism \( \pi^0 : X^0 \to Y^0 \) with \( \codim(X \setminus X^0) \geq 2 \). Since \( Y \) is normal, we may also replace \( Y \) with its smooth locus and we still have the condition \( \codim(X \setminus X^0) \geq 2 \).
The locus $B$ of $Y^\circ$ over which $\pi^0$ has multiple fibers has codimension at least 2 in $Y^\circ$. To see this, compactify $Y^\circ$ to a smooth projective variety $\bar{Y}$ and take a resolution $\pi: \bar{X} \to \bar{Y}$ of the indeterminacies of $X \to \bar{Y}$ with $\bar{X}$ smooth and projective. Let $\bar{C} \subset \bar{Y}$ be a smooth projective curve obtained by intersecting $\dim \bar{Y} - 1$ general very ample divisors on $\bar{Y}$. Let $\bar{\pi}_C: \bar{X}_C \to \bar{C}$ be the corresponding morphism. Then $\bar{X}_C$ is smooth projective and the general fiber of $\bar{\pi}$ is rationally connected. Hence $\bar{\pi}_C$ has a section by [GHS03], and thus it cannot contain multiple fibers. Now, replace $\bar{Y}$ with $\pi^0 \backslash B$ to obtain an equidimensional proper morphism with no multiple fibers.

Let $F$ be a general fiber of $\pi^0$. For each $i$, denote by $H^j_i$, $1 \leq j \leq n_i$, the unsplittable covering families of rational curves on $F$ whose general members correspond to rational curves on $X$ from the family $H_i$. Let $[H^j_i]$ denote the class of a member of $H^j_i$ in $N_1(F)$ and $\mathcal{H} := \{\{H^j_i\} \mid i = 1, \ldots, k, j = 1, \ldots, n_i\}$. Then by [Ko96] IV.3.13.3, $N_1(F)$ is generated by $\mathcal{H}$.

Finally, we shall show that the locus $B'$ of $Y^\circ$ over which the fibers of $\pi^0$ are not integral has codimension at least 2 in $Y^\circ$. Let $C \subset Y^\circ$ be a smooth curve obtained by intersecting $\dim Y^\circ - 1$ general very ample divisors on $Y^\circ$. Let $\pi_C: X_C \to C$ be the corresponding morphism. Then $X_C$ is smooth. We denote the image of the classes $[H^j_i]$’s in $N_1(X_C)$ and their collection $\mathcal{H}$ by the same symbols. By taking limits of chains of rational curves from the families $H_1, \ldots, H_k$ and applying [Ko96] IV.3.13.3 (see Remark 2.1), we see that any curve contained in any fiber of $\pi_C$ is numerically proportional in $N_1(X_C)$ to a linear combination of the $[H^j_i]$’s. Hence $N_1(X_C/C)$ is generated by $\mathcal{H}$. Therefore, all fibers of $\pi_C$ are irreducible. Indeed, if $F'_0$ is an irreducible component of a reducible fiber $F_0$, then $F'_0$ is a Cartier divisor on $X_C$, and $F'_0 \cdot [H^j_i] = 0$ for every $H^j_i$. On the other hand, there is a curve $\ell \subseteq F_0$ such that $F'_0 \cdot \ell > 0$, contradiction the fact that $N_1(X_C/C)$ is generated by $\mathcal{H}$. Since there are no multiple fibers, the fibers are also reduced. Finally, we replace $Y^\circ$ with $Y^\circ \backslash B'$ and obtain a morphism with the required properties.

**Proposition 2.3.** Let $X$ be a smooth complex projective variety and $H$ an unsplittable covering family of rational curves on $X$. Assume that $H_x$ is irreducible for general $x \in X$. Let $\pi^0: X^\circ \to Y^\circ$ be the $H$-rational connected quotient of $X$. Then the general fiber of $\pi^0$ is a Fano manifold with Picard number 1.

**Proof.** Let $X_t$ be a general fiber of $\pi^0$, and suppose $\rho(X_t) \neq 1$. Denote by $[H]$ the class of the members of $H$ in $N_1(X)$. By [Ko96] IV.3.13.3, every proper curve on $X_t$ is numerically proportional to $[H]$ in $N_1(X)$. There exists an irreducible component $H_t$ of $H_{X_t} = \{[C] \in H \mid C \subset X_t\}$ which is an unsplittable covering family of rational curves on $X_t$. Since $H_x$ is irreducible for general $x \in X$, such a component $H_t$ is unique. Since $\rho(X_t) \neq 1$, $X_t$ is not $H_t$-rationally connected by [Ko96] IV.3.13.3. Let $\sigma_t: X^\circ_t \to Z^\circ_t$ be the (nontrivial) $H_t$-rationally connected quotient of $X_t$. Notice that for every $z \in Z_t^\circ$ there is a curve $C_z \subset X_t$ numerically proportional to $[H]$ in $N_1(X)$, meeting the fiber of $\sigma_t$ over $z$, but not contained in it. Since $H_t$ is unique, there is a dense open subset $X'$ of $X$ and a fibration $\sigma: X' \to Z'$ whose fibers are fibers of $\sigma_t$ for some $t \in Y^\circ$. Moreover, there is a curve $C \subset X$ numerically proportional to $[H]$ in $N_1(X)$, meeting $X'$, and not contracted by $\sigma$. But this is impossible. Indeed, let $L'$ be an effective divisor on $Z'$ meeting but not containing the image of $C$ by $\sigma$. Let $L$ be the closure of $\sigma^{-1}(L')$ in $X$. Then $\sigma \cdot C > 0$ while $L \cdot \ell = 0$ for any curve $\ell$ parametrized by $H$ lying on a fiber of $\sigma$.

**Remark 2.4.** The statement of Proposition 2.3 does not hold in general if we do not assume that $H_x$ is irreducible for general $x \in X$. Indeed, one may take $\pi^0: X^\circ \to Y^\circ$ to be a suitable family of quadric surfaces in $\mathbb{P}^3$ and $H$ to be the family of lines on the fibers of $\pi^0$.

**Definition 2.5.** Let $X$ be a smooth complex projective variety, and $H$ a minimal covering family of rational curves on $X$. Let $x \in X$ be a general point. Define the tangent map $\tau_x: H_x \to \mathbb{P}(T_xX^\circ)$ by sending a curve that is smooth at $x$ to its tangent direction at $x$. Define $C_x$ to be the image of $\tau_x$ in $\mathbb{P}(T_xX^\circ)$. This is called the variety of minimal rational tangents at $x$ associated to the minimal family $H$.

The map $\tau_x: H_x \to C_x$ is in fact the normalization morphism by [Keb02] and [HM04]. If $\tau_x$ is an immersion at every point of $H_x$, then all curves parametrized by $H_x$ are smooth at $x$ by [Ko96] V.3.6] and [Ara06] Proposition 2.7], and, as a consequence, there is a one-to-one correspondence between points of $H_x$ and the associated curves on $X$. The variety $C_x$ comes with a natural projective embedding into $\mathbb{P}(T_xX^\circ)$. This embedding encodes important geometric properties of $X$. The following result was proved in [Ara06] and gives a structure theorem for varieties whose variety of minimal rational tangents is linear.
Theorem 2.6 [Ara06]. Let $X$ be a smooth complex projective variety, $H$ a minimal covering family of rational curves on $X$, and $\mathcal{C}_x \subset \mathbb{P}(T_x X^*)$ the corresponding variety of minimal rational tangents at $x \in X$. Suppose that for a general $x \in X$, $\mathcal{C}_x$ is a $d$-dimensional linear subspace of $\mathbb{P}(T_x X^*)$.

Then there exists an open subset $X^0 \subset X$ and a $\mathbb{P}^{d+1}$-bundle $\varphi^0 : X^0 \to T^0$ over a smooth base with the property that every rational curve parametrized by $H$ and meeting $X^0$ is a line on a fiber of $\varphi^0$. In particular, $\varphi^0 : X^0 \to T^0$ is the $H$-rationally connected quotient of $X$. If $H$ is unsplit, then we may take $X^0$ such that $\operatorname{codim}(X \setminus X^0) \geq 2$.

Proposition 2.7. Let $X$ be a smooth complex projective variety, $H$ a minimal covering family of rational curves on $X$, and $\pi^0 : X^0 \to Y^0$ the $H$-rationally connected quotient of $X$. Suppose that the tangent bundle $T_X$ contains a subsheaf $\mathcal{D}$ such that $f^* \mathcal{D}$ is an ample vector bundle for a general member $[f] \in H$. Then $\pi^0$ is a projective space bundle and the inclusion $\mathcal{D}|_{X^0} \hookrightarrow T_{X^0}$ factors through an inclusion $\mathcal{D}|_{X^0} \hookrightarrow T_{X^0/Y^0}$.

Proof. Let $C_x \subset \mathbb{P}(T_x X^*)$ be the variety of minimal rational tangents associated to $H$ at a general point $x \in X$. By [Ara06] Proposition 4.1, $C_x$ is a union of linear subspaces of $\mathbb{P}(T_x X^*)$ containing $\mathbb{P}(\mathcal{D} \otimes \kappa(x))$.

In [Ara06] Proposition 4.1 $\mathcal{D}$ is assumed to be ample, but the proof only uses the fact that $f^* \mathcal{D}$ is a subsheaf of $f^* T_X$ for general $[f] \in H$.

Lemma 2.8 below implies that $C_x$ is irreducible, and thus a linear subspace of $\mathbb{P}(T_x X^*)$.

Now we apply Theorem 2.6 to conclude that $\pi^0$ is a projective space bundle. Moreover, for a general point $x \in X^0$, the stalk $\mathcal{D}_x$ is contained in $(T_{X^0/Y^0})_x$. Since the cokernel of $T_{X^0/Y^0} \to T_{X^0}$ is torsion free, we conclude that there is an inclusion $\mathcal{D}|_{X^0} \hookrightarrow T_{X^0/Y^0}$ factoring $\mathcal{D}|_{X^0} \hookrightarrow T_{X^0}$. □

The following lemma is Proposition 2.2 of [Hwa07]. In [Hwa07] Proposition 2.2 $X$ is assumed to have Picard number one, but this assumption is not used in the proof.

Lemma 2.8. Let $X$ be a smooth complex projective variety, $H$ a minimal covering family of rational curves on $X$, and $\mathcal{C}_x \subset \mathbb{P}(T_x X^*)$ the corresponding variety of minimal rational tangents at $x \in X$. Suppose that for a general $x \in X$, $\mathcal{C}_x$ is a union of linear subspaces of $\mathbb{P}(T_x X^*)$.

Then the intersection of any two irreducible components of $\mathcal{C}_x$ is empty.

3. The relative anticanonical bundle of a fibration

In this section we prove that the relative anticanonical bundle of a generically smooth surjective morphism from a normal projective $\mathbb{Q}$-Gorenstein variety onto a smooth curve cannot be ample. In fact, we prove the following more general result. Note that a similar theorem was proved in [Miy93] Theorem 2.

Theorem 3.1. Let $X$ be a normal projective variety, $f : X \to C$ a surjective morphism onto a smooth curve, and $\Delta \subseteq X$ a Weil divisor such that $(X, \Delta)$ is log canonical over the generic point of $C$. Then $-(K_{X/C} + \Delta)$ is not ample.

Proof. Let $X \xrightarrow{\sigma} \tilde{C} \xrightarrow{\sigma} C$ be the Stein factorization of $f$. Then $K_{\tilde{C}} = \sigma^* K_C + R_\sigma$ where $R_\sigma$ is the ramification divisor of $\sigma$ and so $-(K_{X/C} + \Delta) = -(K_{X/C} + \Delta) + g^* R_\sigma$. Notice that $R_\sigma$ is effective and hence if $-(K_{X/C} + \Delta)$ is ample, then so is $-(K_{X/C} + \Delta)$.

Thus, in order to prove the statement, we may assume that $f$ has connected fibers. Let us now assume to the contrary that $-(K_{X/C} + \Delta)$ is ample. Let $\pi : \tilde{X} \to X$ be a log resolution of singularities of $(X, \Delta)$, $A$ an ample divisor on $C$, and $m \gg 0$ such that $D = -m(K_{X/C} + \Delta) - f^* A$ is very ample. Then

$$K_{\tilde{X}} + \pi_*^{-1} \Delta \sim_\mathbb{Q} \pi^*(K_X + \Delta) + E_+ - E_-,$$

where $E_+$ and $E_-$ are effective $\pi$-exceptional divisors with no common components and such that the support of $\pi_*^{-1} \Delta + E_+ + E_-$ is an snc divisor. By the log canonical assumption, $E_-$ can be decomposed as $E_- = E + [F]$ where $[E]$ is reduced and $E_-$ agrees with $E$ over the generic point of $C$. Set $\tilde{f} = f \circ \pi$ and let $\tilde{D} \in [\pi^* D]$ be a general member. Setting $\Delta = \pi_*^{-1} \Delta + \frac{1}{m} \tilde{D} + E$, we obtain that $(\tilde{X}, \Delta)$ is log canonical and that

$$(3.1.1) \quad K_{\tilde{X}} + \Delta + F \sim_\mathbb{Q} \tilde{f}^* K_C + E_+ - \frac{1}{m} \tilde{f}^* A.$$
Furthermore, since $E_+$ is $\pi$-exceptional, $\pi_*\mathcal{O}_X(lE_+)$ is an ideal sheaf in $\mathcal{O}_X$ for any $l \in \mathbb{Z}$ (see for instance [Deb01, Lemma 7.11]). Then for any $l \in \mathbb{N}$ sufficiently divisible,

$$\tilde{f}_*\mathcal{O}_X(lm(K_{X/C} + \Delta)) \xrightarrow{\sim} \tilde{f}_*\mathcal{O}_X(lm(K_{X/C} + \Delta + F)) \approx \tilde{f}_*\mathcal{O}_X(l(mE_+ - \tilde{f}^*A)) \approx \tilde{f}_*\mathcal{O}_X(lmE_+) \otimes \mathcal{O}_C(-lA) \subseteq \mathcal{O}_C(-lA).$$

Finally, observe that

- $\tilde{f}_*\mathcal{O}_X(lm(K_{X/C} + \Delta + F))$ is nonzero by (3.1.1) and because $E_+$ is effective,
- $\tilde{f}_*\mathcal{O}_X(lm(K_{X/C} + \Delta))$ is semi-positive by [Cam04, Thm. 4.13], and
- $\nu$ is an isomorphism over a nonempty open subset of $C$.

Therefore, $\tilde{f}_*\mathcal{O}_X(lm(K_{X/C} + \Delta))$ is a non-zero semi-positive sheaf contained in $\mathcal{O}_C(-lA)$, but that contradicts the fact that $A$ is ample. □

4. LIFTING $p$-DERIVATIONS TO THE NORMALIZATION

In this section we show that $p$-derivations (see definition 4.4 below) can be lifted to the normalization. This is a generalization of Seidenberg’s theorem in [Sei66]. The proofs in this section follow closely the proof of Theorem 2.1.1 in [Käl06] and we also use the following result from [Käl06].

**Lemma 4.1** [Käl06 Lemma 2.1.2]. Let $(A, m, k)$ be a local Noetherian domain and $\partial$ a derivation of $A$. Let $\nu$ be a discrete valuation on the fraction field $K(A)$ with center in $A$. Then there exists $c \in \mathbb{Z}$ such that $\nu \left( \frac{\partial(x)}{x} \right) \geq c$ for any $x \in K(A) \setminus \{0\}$.

**Definition 4.2.** Let $R$ be a ring, $A$ an $R$-algebra and $M$ an $A$-module. Denote by $\Omega_{A/R}$ the module of relative differentials of $A$ over $R$. Given a positive integer $p$, we denote by $\Omega_{A/R}^p$ the $p$-th wedge power of $\Omega_{A/R}$. A $p$-derivation of $A$ over $R$ with values in $M$ is an $A$-linear map $\partial : \Omega_{A/R}^p \to M$. Such a map $\partial$ induces a skew symmetric map $K(A)^{\otimes p} \to M \otimes_A K(A)$, where $K(A)$ denotes the fraction field of $A$. We use the same symbol $\partial$ to denote this induced map. When $M = A$ and $R$ is clear from the context, we call $\partial$ simply a $p$-derivation of $A$.

**Lemma 4.3.** Let $(A, m, k)$ be a local Noetherian domain, $p$ a positive integer, and $\partial$ a $p$-derivation of $A$. Let $\nu$ be a discrete valuation on the fraction field $K(A)$ with center in $A$. Then there exists $c \in \mathbb{Z}$ such that $\nu \left( \frac{\partial(x_1, \ldots, x_p)}{x_1 \cdots x_p} \right) \geq c$ for any $x_1, \ldots, x_p \in K(A) \setminus \{0\}$.

**Proof.** We use induction on $p$. If $p = 1$, this is Lemma 4.1. Suppose now that $p \geq 2$ and let $(A, m, k)$ be a local Noetherian domain, $\partial$ a $p$-derivation of $A$, and $\nu$ a discrete valuation on the fraction field $K(A)$ with center in $A$. Let $m_1, \ldots, m_p$ be generators of the maximal ideal $m$.

Using the formula

$$\frac{\partial(x_1, x_{1,2}, \ldots, x_{p,1, p})}{x_1, x_{1,2}, \ldots, x_{p,1, p}} = \sum \frac{\partial(x_{1,1}, \ldots, x_{p,1, p})}{x_{1,1}, \ldots, x_{p,1, p}},$$

we get

$$\nu \left( \frac{\partial(x_1, x_{1,2}, \ldots, x_{p,1, p})}{x_1, x_{1,2}, \ldots, x_{p,1, p}} \right) \geq \min \left\{ \nu \left( \frac{\partial(x_{1,1}, \ldots, x_{p,1, p})}{x_{1,1}, \ldots, x_{p,1, p}} \right) \right\}$$

for $x_{1,1}, x_{1,2}, \ldots, x_{p,1, p} \in A \setminus \{0\}$. Further observe that

$$\frac{\partial(x_1^{-1}, x_2, \ldots, x_p)}{x_1^{-1} x_2 \cdots x_p} = -\frac{\partial(x_1, \ldots, x_p)}{x_1 \cdots x_p}.$$
Suppose now that at least one of the $x_i$’s is in $m$. For simplicity we assume that $x_1, \ldots, x_l \in A \setminus m$ and $x_{l+1}, \ldots, x_p \in \{m_1, \ldots, m_r\}$, $0 \leq l < p$. We may view $\partial(x_1, \ldots, x_{l+1}, \ldots, x_p)$ as an $l$-derivation of $A$.

The result then follows by induction. □

**Definition 4.4.** Let $S$ be a scheme, $X$ a scheme over $S$, and $\mathcal{L}$ a line bundle on $X$. Denote by $\Omega_{X/S}$ the sheaf of relative differentials of $X$ over $S$, and by $\Omega^p_{X/S}$ its $p$-th wedge power for $p \in \mathbb{N}$. A $p$-derivation of $X$ over $S$ with values in $\mathcal{L}$ is a morphism of sheaves $\partial : \Omega^p_{X/S} \rightarrow \mathcal{L}$. When $S$ is the spectrum of a field and $\mathcal{L}$ is clear from the context, we drop $S$ and $\mathcal{L}$ from the notation and call $\partial$ simply a $p$-derivation on $X$.

**Proposition 4.5.** Let $X$ be a Noetherian integral scheme over a field $k$ of characteristic zero and $\eta : \widetilde{X} \rightarrow X$ its normalization. Let $\mathcal{L}$ be a line bundle on $X$, $p$ a positive integer, and $\partial$ a $p$-derivation with values in $\mathcal{L}$. Then $\partial$ extends to a unique $p$-derivation $\partial$ on $\widetilde{X}$ with values in $\eta^* \mathcal{L}$.

**Proof.** The uniqueness of $\partial$ is clear since $\mathcal{L}$ is torsion free and $\eta$ is birational. The existence of the lifting can be established locally. So we may assume that $X$ is the spectrum of an integral $k$-algebra $A$, $\mathcal{L}$ is trivial, and $\partial$ is a $p$-derivation of $A$. Let $A'$ denote the integral closure of $A$ in its fraction field $K(A)$. There exists a unique extension of $\partial$ to a $p$-derivation of $K(A)$, which we also denote by $\partial$. We must prove that $\partial(A', \ldots, A') \subset A'$.

First we reduce the problem to the case when $A$ is a 1-dimensional local ring and $A'$ is a DVR. Since $A'$ is integrally closed in $K(A)$, $A'$ is the intersection of its localizations at primes of height one [Mat80, 2. Theorem 38]. Let $p'$ be a prime of height one of $A'$, and set $p = p' \cap A$. Notice that $\partial(A_p, \ldots, A_p) \subset A_p$, and the result follows if we prove that $\partial(A'_p, \ldots, A'_p) \subset A'_p$. Hence we may assume that $A$ is a 1-dimensional local ring and $A'$ is a DVR. Denote by $m$ and $m'$ the maximal ideals of $A$ and $A'$ respectively.

Next we further reduce the problem to the case when $A$ and $A'$ are complete local rings. Let $\tilde{R}$ be the completion of $A'$ with respect to the $m'$-adic topology. Let $\tilde{A}$ be the completion of $A$ with respect to the $m$-adic topology. Since $A$ is 1-dimensional, there is an inclusion of local rings $A \subset \tilde{R}$. Let $\nu$ be a discrete valuation of $K(A')$ whose valuation ring is $A'$. By Lemma 4.3, $\partial$ is a continuous $p$-derivation of $R$ with values in $K(A')$. Hence it has a unique extension to a continuous $p$-derivation $\partial$ of $K(\tilde{R})$. Notice that the condition $\partial(A, \ldots, A) \subset A$ implies that $\partial(\tilde{A}, \ldots, \tilde{A}) \subset A'$ by the Artin-Rees Lemma. Since $K(A) \cap \tilde{R} = A'$, the result then follows if we prove that $\partial(\tilde{R}, \ldots, \tilde{R}) \subset \tilde{R}$. Therefore we may assume that $A$ and $A'$ are complete 1-dimensional local rings.

Now we use induction on $p$. If $p = 1$, this is Seidenberg’s theorem [Sci66], so we may assume that $p \geq 2$. Let $k_A$ be a coefficient field in $A$, and $k_A'$ a coefficient field in $A'$ containing $k_A$ [Eis95, Theorem 7.8]. The extension $k_A/k_A'$ is finite. Let $t \in m'$ be a uniformizing parameter. It suffices to show that $\partial(x_1, \ldots, x_p) \in A'$ for $x_1, \ldots, x_p \in k_A \cup \{t\}$. Since $\partial$ is skew symmetric and $p \geq 2$, we have $\partial(t, \ldots, t) = 0$. So we may assume that $x_1 \in k_A$. Since $k_A/k_A'$ is finite and separable, there exists $P(X) = \sum a_i X^i \in k_A[X]$ such that $P(x_1) = 0$ and $P'(x_1) \neq 0$. Thus

$$0 = \partial(P(x_1), x_2, \ldots, x_p) = P'(x_1)\partial(x_1, \ldots, x_p) + \sum \partial(a_i, x_2, \ldots, x_p)x_1^i.$$

Finally, $\partial(a_1, \ldots, \ldots, \ldots)$ may be viewed as a $p - 1$ derivation of $A$ and so $\partial(x_1, \ldots, x_p) \in A'$ by the induction hypothesis. □

5. Sections of $\wedge^p T_X \otimes \mathcal{M}$

The following lemma will be used several times in this section.

**Lemma 5.1.** Let $Y$ be a smooth variety, $\pi : X \rightarrow Y$ a smooth morphism, $\mathcal{M}$ a line bundle on $X$, and $p \geq 2$ an integer. Suppose that for a general fiber, $F$, of $\pi$, $H^0(F, \wedge^i T_F \otimes \mathcal{M}|_F) = 0$ for $0 \leq i \leq p - 2$. Then there exists an exact sequence:

$$0 \rightarrow H^0(X, \wedge^p T_{X/Y} \otimes \mathcal{M}) \rightarrow H^0(X, \wedge^p T_X \otimes \mathcal{M}) \rightarrow H^0(X, \wedge^{p-1} T_{X/Y} \otimes \pi^* T_Y \otimes \mathcal{M}).$$

**Proof.** The short exact sequence

$$0 \rightarrow T_{X/Y} \rightarrow T_X \rightarrow \pi^* T_Y \rightarrow 0$$
yields a filtration $\wedge^p T_X \otimes \mathcal{M} = \mathcal{F}_0 \supseteq \mathcal{F}_1 \supseteq \mathcal{F}_2 \supseteq \cdots \supseteq \mathcal{F}_p \supseteq \mathcal{F}_{p+1} = 0$ such that

$$
\mathcal{F}_i/\mathcal{F}_{i+1} \simeq \wedge^i T_{X/Y} \otimes \pi^* \wedge^{p-i} T_Y \otimes \mathcal{M}
$$

for each $i$. In particular, one has the short exact sequence,

$$
0 \to \wedge^p T_{X/Y} \otimes \mathcal{M} \to \mathcal{F}_{p-1} \to \wedge^{p-1} T_{X/Y} \otimes \pi^* \wedge^p T_Y \otimes \mathcal{M} \to 0.
$$

The assumption that $H^0(F, \wedge^p T_F \otimes \mathcal{M}|_F) = 0$ for $0 \leq i \leq p - 2$ for a general fiber of $\pi$ implies that $H^0(X, \mathcal{F}_i/\mathcal{F}_{i+1}) = 0$ for $0 \leq i \leq p - 2$, thus $H^0(X, \wedge^p T_X \otimes \mathcal{M}) = H^0(X, \mathcal{F}_0) = \cdots = H^0(X, \mathcal{F}_{p-1})$ and the result follows from (5.1.1).

The condition that $H^0(F, \wedge^p T_F \otimes \mathcal{M}|_F) = 0$ for $0 \leq i \leq p - 2$ and $F$ a general fiber of $\pi$ is easily verified when $\pi$ is a projective space bundle and $\mathcal{M}|_F$ is sufficiently negative. In this case we get the following.

**Lemma 5.2.** Let $Y$ be a smooth projective variety of dimension $\geq 1$, $\mathcal{E}$ an ample vector bundle of rank $r + 1 \geq 2$ and $\mathcal{N}$ a nef line bundle on $Y$. Consider the projective bundle $\pi : X = \mathbb{P}(\mathcal{E}) \to Y$ with tautological line bundle $\mathcal{O}(1)$. Let $p, q \in \mathbb{N}$ and assume that $p \geq 2$. Then

$$
H^0(X, \wedge^p T_{X/Y} \otimes \mathcal{O}(\mathcal{E})(-p - q) \otimes \pi^* \mathcal{N}^{-1}) = 0.
$$

**Proof.** First observe, that if $p > r$ then the statement is trivially true, so we will assume that $p \leq r$. Let $i \in \mathbb{N}$, $i < p$. After twisting by $\mathcal{O}(\mathcal{E})(-p - q) \otimes \pi^* \mathcal{N}^{-1}$, the short exact sequence

$$
0 \to \wedge^{p-i-1} T_{X/Y} \to \wedge^{p-i} (\pi^* \mathcal{E}^*(1)) \to \wedge^{p-i} T_{X/Y} \to 0
$$

yields the exact sequence

$$
\cdots \to H^i(X, \wedge^{p-i} (\pi^* \mathcal{E}^*)(-i - q) \otimes \pi^* \mathcal{N}^{-1}) \to H^i(X, \wedge^{p-i} T_{X/Y}(-p - q) \otimes \pi^* \mathcal{N}^{-1}) \to

\to H^{i+1}(X, \wedge^{p-i-1} T_{X/Y}(-p - q) \otimes \pi^* \mathcal{N}^{-1}) \to \cdots
$$

Since $i < p \leq r$ and $R^j \pi_* \mathcal{O}(\mathcal{E})(l) = 0$ for $0 < j < r$ and for any $l \in \mathbb{Z}$, the Leray spectral sequence implies that

$$
H^i(X, \wedge^{p-i} (\pi^* \mathcal{E}^*)(-i - q) \otimes \pi^* \mathcal{N}^{-1}) = H^i(Y, \wedge^{p-i} \mathcal{E}^* \otimes \mathcal{N}^{-1} \otimes \pi_* \mathcal{O}(\mathcal{E})(-i - q)).
$$

The sheaf $\pi_* \mathcal{O}(\mathcal{E})(-i - q)$ is zero unless $i = q = 0$, in which case it is isomorphic to $\mathcal{O}_Y$. Furthermore, $H^0(Y, \wedge^p \mathcal{E}^* \otimes \mathcal{N}^{-1}) = 0$ since $\mathcal{E}$ is ample and $\mathcal{N}$ is nef, and hence

$$
H^i(X, \wedge^{p-i} (\pi^* \mathcal{E}^*)(-i - q) \otimes \pi^* \mathcal{N}^{-1}) = 0
$$

for $0 \leq i \leq p - 1$. Therefore, by (5.2.2), one has a series of injections,

$$
H^0(X, \wedge^p T_{X/Y}(-p - q) \otimes \pi^* \mathcal{N}^{-1}) \hookrightarrow H^1(X, \wedge^{p-1} T_{X/Y}(-p - q) \otimes \pi^* \mathcal{N}^{-1}) \hookrightarrow \cdots

\vdots \hookrightarrow H^i(X, \wedge^{p-i} T_{X/Y}(-p - q) \otimes \pi^* \mathcal{N}^{-1}) \hookrightarrow \cdots \hookrightarrow H^p(X, \mathcal{O}(\mathcal{E})(-p - q) \otimes \pi^* \mathcal{N}^{-1}).
$$

By the Kodaira vanishing theorem $H^p(X, \mathcal{O}(\mathcal{E})(-p - q) \otimes \pi^* \mathcal{N}^{-1}) = 0$, and the statement follows.

**Corollary 5.3.** Let $Y$ be a smooth projective variety of dimension $\geq 1$ and $\mathcal{E}$ an ample vector bundle of rank $r + 1 \geq 2$ on $Y$. Consider the projective bundle $\pi : X = \mathbb{P}(\mathcal{E}) \to Y$ with tautological line bundle $\mathcal{O}(1)$. Suppose that $H^0(X, \wedge^p T_X \otimes \mathcal{O}(\mathcal{E})(-p - q) \otimes \pi^* \mathcal{N}^{-1}) \neq 0$ for some integers $p \geq 2$, $q \geq 0$, and some nef line bundle $\mathcal{N}$ on $Y$. Then $Y \cong \mathbb{P}^1$, $\mathcal{E} \cong \mathcal{O}(p_1) \oplus \mathcal{O}(p_1)$, $p = 2$, $q = 0$, and $\mathcal{N} \cong \mathcal{O}(p_1)$.

**Proof.** Let $F \simeq \mathbb{P}^r$ denote a general fiber of $\pi$ and set $\mathcal{M} = \mathcal{O}(\mathcal{E})(-p - q) \otimes \pi^* \mathcal{N}^{-1}$. Then by Bott’s formula $H^0(F, \wedge^p T_F \otimes \mathcal{M}|_F) = 0$ for every $0 \leq i \leq p - 2$. Then Lemma 5.1 and Lemma 5.2 imply that $H^0(X, \wedge^p T_{X/Y} \otimes \pi^*(\mathcal{N} \otimes \mathcal{N}^{-1}) \otimes \mathcal{O}(\mathcal{E})(-p - q)) \neq 0$. By Bott’s formula again $H^0(F, \wedge^p T_F(-p - q)) \neq 0$ implies that $q = 0$ and $r = p - 1$. Therefore we have

$$
0 \neq H^0(X, \wedge^r T_{X/Y} \otimes \pi^*(\mathcal{N} \otimes \mathcal{N}^{-1}) \otimes \mathcal{O}(\mathcal{E})(-r - 1)) =

H^0(X, \pi^*(\mathcal{N} \otimes \det \mathcal{E}^* \otimes \mathcal{N}^{-1})) \simeq H^0(Y, \pi_* \pi^*(\mathcal{N} \otimes \det \mathcal{E}^* \otimes \mathcal{N}^{-1})) \simeq

H^0(Y, T_Y \otimes (\det \mathcal{E} \otimes \mathcal{N})^{-1}).
Now Wahl’s theorem \cite{Wah83} yields that $Y \simeq \mathbb{P}^m$ for some $m > 0$. Then we immediately obtain that $\deg(\det \mathcal{E} \otimes N) \leq 2$. Since $\mathcal{E}$ is ample on a projective space,

$$2 \leq r + 1 = \mathrm{rk} \mathcal{E} \leq \deg(\det \mathcal{E} \otimes N) - \deg N \leq 2 - \deg N \leq 2.$$

Thereore all of these inequalities must be equalities and we have that $r + 1 = p = 2$, $q = 0$ and $N \simeq \mathcal{O}_Y$. Furthermore, this implies that then $\mathcal{O}_{\mathbb{P}^m}(2) \simeq \det \mathcal{E} \to T_{\mathbb{P}^m}$ and hence $m = 1$. \hfill \Box

**Proposition 5.4.** Let $X$ be a smooth projective variety, $H \subset \text{RatCurves}^n(X)$ a minimal covering family of rational curves on $X$, $\mathcal{L}$ an ample line bundle on $X$, and $\mathcal{M}$ a nef line bundle on $X$ such that $c_1(\mathcal{M}) \cdot C > 0$ for every $[C] \in H$. Suppose that $H^0(X, \Lambda^pT_X \otimes \mathcal{L}^{-p} \otimes \mathcal{M}^{-1}) \neq 0$ for some integer $p \geq 1$. Then $(X, \mathcal{L}, \mathcal{M}) \simeq (\mathbb{P}^p, \mathcal{O}_{\mathbb{P}^p}(1), \mathcal{O}_{\mathbb{P}^p}(1))$.

**Proof.** Let $[f] \in H$ be a general member and write $f^*T_X \simeq \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus d} \oplus \mathcal{O}_{\mathbb{P}^1}^{d-n-1}$. The condition that both $f^*\mathcal{L}$ and $f^*\mathcal{M}$ are ample and that $H^0(X, \Lambda^pT_X \otimes \mathcal{L}^{-p} \otimes \mathcal{M}^{-1}) \neq 0$ implies that $f^*\mathcal{L} \simeq \mathcal{O}_{\mathbb{P}^1}(1) \simeq f^*\mathcal{M}$, and thus $H$ is unsplit. A non-zero section $s \in H^0(X, \Lambda^pT_X \otimes \mathcal{L}^{-p} \otimes \mathcal{M}^{-1})$ and the contraction $\mathcal{E}_\theta : \Lambda^pT_X \otimes \mathcal{L}^{-p} \otimes \mathcal{M}^{-1} \to \Lambda^pT_X \otimes \mathcal{L}^{-p} \otimes \mathcal{M}^{-1}$ induced by a differential form $\theta \in \Omega_X$, gives rise to a non-zero map

$$\Omega_X \to \Lambda^pT_X \otimes \mathcal{L}^{-p} \otimes \mathcal{M}^{-1}, \quad \theta \mapsto \mathcal{E}_\theta(s),$$

the dual of which is the non-zero map

$$(\mathcal{L}^{-1} \otimes \mathcal{M}) \to T_X.$$  

The sheaf $f^*(\Omega_{\mathbb{P}^1}^{-1} \otimes \mathcal{L} \otimes \mathcal{M})$ is ample. Thus, by Proposition \cite{5.7} and Theorem \cite{2.9} there is an open subset $X^o \subset X$, with $\text{codim}_X(X \setminus X^o) \geq 2$, a smooth variety $Y^o$, and a $\mathbb{P}^{d+1}$-bundle $\pi^o : X^o \to Y^o$ such that any rational curve from $H$ meeting $X^o$ is a line on a fiber of $\pi^o$. Moreover, the restriction of $s$ to $X^o$ lies in $H^0(X^o, \Lambda^pT_{X^o} \otimes \mathcal{L}^{-p} \otimes \mathcal{M}^{-1})$, and its restriction to a general fiber $F$ yields a non-zero section in $H^0(F, \Lambda^pT_F \otimes \mathcal{L}^{-p}_{F} \otimes \mathcal{M}_{F})$. On the other hand, by Bott’s formula, $H^0(\mathbb{P}^{d+1}, \Lambda^pT_{\mathbb{P}^{d+1}}(-p - 1)) = 0$ unless $p = d + 1$.

Suppose $\dim(Y^o) > 0$. Since $\text{codim}_X(X \setminus X^o) \geq 2$, $Y^o$ contains a complete curve through a general point. Let $g : B \to Y^o$ be the normalization of a complete curve passing through a general point of $Y^o$. Set $X_B := X^o \times_{Y^o} B$, and denote by $\mathcal{L}_{X_B}$ and $\mathcal{M}_{X_B}$ the pullbacks of $\mathcal{L}$ and $\mathcal{M}$ to $X_B$ respectively. Then $X_B \to B$ is a $\mathbb{P}^p$-bundle, and the section $s$ induces a non-zero section in $H^0(X_B, \Lambda^pT_{X_B/B} \otimes \mathcal{L}_{X_B}^{-p} \otimes \mathcal{M}_{X_B}^{-1})$. But this is impossible by Corollary \cite{5.3} Thus $\dim(Y^o) = 0$ and $X \simeq \mathbb{P}^p$.

**Corollary 5.5.** Let $X$ be a smooth projective variety and $\mathcal{L}$ an ample line bundle on $X$. If $H^0(X, \Lambda^pT_X \otimes \mathcal{L}^{-p-1-k}) \neq 0$ for integers $p \geq 1$ and $k \geq 0$, then $k = 0$ and $(X, \mathcal{L}) \simeq (\mathbb{P}^p, \mathcal{O}_{\mathbb{P}^p}(1))$.

**Proof.** Note that $X$ is uniruled by \cite{Miy87}. The result follows easily from Proposition \cite{5.4} \hfill \Box

Here is how we are going to apply these results under the assumptions of Theorem \cite{1.1}. Suppose that $H^0(X, \Lambda^pT_X \otimes \mathcal{L}^{-p}) \neq 0$ for some ample line bundle $\mathcal{L}$ on $X$ and integer $p \geq 2$. Then $X$ is uniruled by \cite{Miy87} and we fix a minimal covering family $H$ of rational curves on $X$. Let $\pi : X^o \to Y^o$ be the $H$-rational quotient of $X$. By shrinking $Y^o$ if necessary, we may assume that $Y^o$ and $\pi$ are smooth. Corollary \cite{5.5} provides the vanishing required to apply Lemma \cite{5.1} to $\pi : X^o \to Y^o$, yielding the following.

**Lemma 5.6.** Let $Y$ be a smooth variety, $\pi : X \to Y$ a smooth morphism with connected fibers, and $\mathcal{L}$ a line bundle on $X$. Let $F$ be a general fiber of $\pi$. Suppose that $F$ is projective and that the restriction $\mathcal{L}|_F$ is ample. If $H^0(X, \Lambda^pT_X \otimes \mathcal{L}^{-p}) \neq 0$ for some integer $p \geq 2$, then either $(F, \mathcal{L}|_F) \simeq (\mathbb{P}^{p-1}, \mathcal{O}_{\mathbb{P}^{p-1}}(1))$ and $H^0(X, \Lambda^{p-1}T_{X/Y} \otimes \mathcal{L}^{-p}) \neq 0$, or $\dim(F) \geq p$ and $H^0(X, \Lambda^{p-1}T_{X/Y} \otimes \mathcal{L}^{-p}) \neq 0$.

**Proof.** Corollary \cite{5.5} implies that $H^0(F, \Lambda^iT_F \otimes \mathcal{L}^{-p}_F) = 0$ for $0 \leq i \leq p - 2$. So we may apply Lemma \cite{5.1} with $\mathcal{M} = \mathcal{L}^{-p}$ to conclude that either $H^0(X, \Lambda^{p-1}T_{X/Y} \otimes \pi^*T_Y \otimes \mathcal{L}^{-p}) \neq 0$, or $\dim F \geq p$ and $H^0(X, \Lambda^{p-1}T_{X/Y} \otimes \mathcal{L}^{-p}) \neq 0$. In the first case we have $H^0(F, \Lambda^{p-1}T_F \otimes \mathcal{L}^{-p}_F) \neq 0$, and Corollary \cite{5.5} implies that $(F, \mathcal{L}|_F) \simeq (\mathbb{P}^{p-1}, \mathcal{O}_{\mathbb{P}^{p-1}}(1))$ and so the desired statement follows. \hfill \Box
Let $X, H,$ and $\pi : X^o \to Y^o$ be as in the above discussion. If we are under the first case of Lemma 5.6 then Theorem 2.6 implies that the $\mathbb{P}^{p-1}$-bundle $\pi : X^o \to Y^o$ can be extended in codimension 1. Next we show that in this case we must have $X \simeq \mathbb{P}^{1}$.

**Lemma 5.7.** Let $X$ be a smooth projective variety and $\mathcal{L}$ an ample line bundle on $X$. Let $X^o \subset X$ be an open subset whose complement has codimension at least 2 in $X$. Let $\pi : X^o \to Y^o$ be a smooth projective morphism with connected fibers onto a smooth quasi-projective variety. If $H^0(X^o, \mathcal{O}_{\mathbb{P}^{p-1}T_{X^o}^o} \otimes \pi^*T_{Y^o} \otimes \mathcal{L}|_{X^o}^p) \neq 0$ for some integer $p \geq 2$, then $p = 2$, $X^o = X \simeq \mathbb{P}^{2}$, and $Y^o \simeq \mathbb{P}^1$.

**Proof.** Suppose that for some $p \geq 2$ there is a non-zero section

$$s \in H^0(X^o, \wedge^{p-1}T_{X^o}^o \otimes \pi^*T_{Y^o} \otimes \mathcal{L}|_{X^o}^p) \neq 0.$$ 

By Corollary 5.5 the fibers of $\pi$ are isomorphic to $\mathbb{P}^{p-1}$, and the restriction of $\mathcal{L}$ to each fiber is isomorphic to $\mathcal{O}_{\mathbb{P}^{p-1}}(1)$. Since $\pi$ has relative dimension $p-1$, there exists an inclusion $\wedge^{p-1}T_{X^o}^o \otimes \pi^*T_{Y^o} \subseteq \wedge^pT_{X^o}$, and thus $s$, as in (5.4.1), yields a map $\varphi : \Omega_{X^o}^{p-1} \otimes \mathcal{L}|_{X^o}^p \to T_{X^o}$ of rank $p$ at the generic point. Since $\text{codim}_X(X \setminus X^o) \geq 2$, $s$ extends to a section $\tilde{s} \in H^0(X, \wedge^pT_X \otimes \mathcal{L}^p)$. Denote by $\tilde{\varphi} : \Omega_{X}^{p-1} \otimes \mathcal{L}^p \to T_X$ the associated map, which has rank $p$ at the generic point.

Let $E = \pi_*\mathcal{L}$. By [Fuj75, Corollary 5.4], $X^o \simeq \mathbb{P}(E)$ over $Y^o$ and then $\wedge^{p-1}T_{X^o}^o \otimes \mathcal{L}^p \simeq \pi^*(\det \mathcal{E})^*$, and $s$ is the pullback of a global section $s_{Y^o} \in H^0(Y^o, T_{Y^o} \otimes \det \mathcal{E})$. This implies that the distribution $\mathcal{G}$ defined by $s$ is integrable. Moreover, its leaves are the pullbacks of the leaves of the foliation $\mathcal{F}$ defined by the map $\det \mathcal{E} \hookrightarrow T_{Y^o}$ associated to $s_{Y^o}$.

Since $\text{codim}_X(X \setminus X^o) \geq 2$, we can find complete curves sweeping out a dense open subset of $Y^o$. Let $C$ be a general complete curve on $Y^o$. Compactify $Y^o$ to a smooth variety $Y$, and let $\mathcal{F}$ be an invertible subsheaf of $T_Y$ extending $\mathcal{F}$. Then $\mathcal{F}|_C = \det \mathcal{E}|_C$ is ample. By [BM01, Theorem 0.1] (see also [KSCT07, Theorem 1]), the leaf of the foliation $\mathcal{F}$ through any point of $C$ is rational. We conclude that the leaves of $\mathcal{F}^o$ are (possibly noncomplete) rational curves. Thus the closures of the leaves of the distribution $\mathcal{G}$ defined by $\tilde{\varphi}$ are algebraic.

Let $F \subset X$ be the closure of a leaf of $\mathcal{G}$ that meets $X^o$ and let $\eta : \tilde{F} \to F$ be its normalization. Then there exists a morphism $\tilde{F} \to B$ onto a smooth rational curve. The general fiber of this morphism is isomorphic to $\mathbb{P}^{p-1}$ and the restriction of $\mathcal{L}$ to the general fiber is isomorphic to $\mathcal{O}_{\mathbb{P}^{p-1}}(1)$. The fibers are thus generically reduced and finally reduced since fibers satisfy Serre’s condition $S_1$. By [Fuj75, Corollary 5.4], $\tilde{F} \to B$ is a $\mathbb{P}^{p-1}$-bundle and, in particular, $\tilde{F}$ is smooth.

The section $\tilde{s} \in H^0(X, \wedge^pT_X \otimes \mathcal{L}^p)$ defines a non-zero map $\Omega_{X}^{p-1} \to \mathcal{L}^p$. Since $F$ is the closure of a leaf of $\mathcal{G}$ and $\mathcal{L}|_F$ is torsion free, the restriction of this map to $F$ factors through a map $\Omega_{F}^{p-1} \to \mathcal{L}^p|_F$. By Lemma 4.5 this map extends to a map $\Omega_{\tilde{F}}^{p-1} \to \eta^*\mathcal{L}|_{F}^p$. Corollary 5.5 then implies that $p = 2$ and $\tilde{F} \simeq \mathbb{P}^{2}$. Moreover $\eta^*\mathcal{L}|_F \simeq \mathcal{O}_{Q_2}(1)$. In particular, $\pi : X^o \to Y^o$ is a $\mathbb{P}^{1}$-bundle. Denote by $H$ the unsplit covering family of rational on $X$ whose general member corresponds to a fiber of $\pi$.

We claim that the general leaf of $\mathcal{F}^o$ is a complete rational curve. From this it follows that the general leaf of $\mathcal{G}$ is compact, and contained in $X^o$. Let $\tilde{F}$ denote the normalization of the closure of a general leaf of $\mathcal{G}$. Since $\tilde{F} \simeq Q_2$ and $\eta^*\mathcal{L}|_F \simeq \mathcal{O}_{Q_2}(1)$, $X$ admits an unsplit covering family $H'$ of rational curves whose general member corresponds to a ruling of $\tilde{F} \simeq Q_2$ that is not contracted by $\pi$. Since $\text{codim}(X \setminus X^o) \geq 2$, the general member of $H'$ corresponds to a complete rational curve contained in $X^o$. Its image in $Y^o$ is a complete leaf of $\mathcal{F}^o$. As we noted above, this implies that $F = \tilde{F} \simeq Q_2$. Notice that the section $\tilde{s}$ does not vanish anywhere on a general leaf $F \simeq Q_2$ of $\mathcal{F}^o$.

Let $\varphi : X^o \to Z'$ be the $(H, H')$-rationally connected quotient of $X$. Then the general fiber of $\varphi$ is a leaf $\tilde{F} \simeq Q_2$ of $\mathcal{F}^o$. By Lemma 2.2 we may assume that $\text{codim}(X \setminus X^o) \geq 2$, $Z'$ is smooth, and $\varphi$ is a proper surjective equidimensional morphism with irreducible and reduced fibers. Therefore $\varphi : X' \to Z'$ is a quadric bundle by [Fuj75, Corollary 5.5]. Since the families $H$ and $H'$ are distinct, $\varphi$ is in fact a smooth quadric bundle.
We claim that in fact \( X = F \) and \( Z' \) is a point. Suppose otherwise, and let \( g : C \to Z' \) be the normalization of a complete curve passing through a general point of \( Z' \). Set \( X_C = X' \times_{Z'} C \), denote by \( \varphi_C : X_C \to C \) the corresponding (smooth) quadric bundle, and write \( L_{X_C} \) for the pullback of \( L \) to \( X_C \). The section \( \tilde{s} \) induces a non-zero section in \( H^0(X_C, \omega_{X_C/C}^{-1} \otimes L_{X_C}^{-2}) \) that does not vanish anywhere on a general fiber of \( \pi_C \). Thus \( \omega_{X_C/C}^{-1} \) is ample, contradicting Proposition 3.1. \( \square \)

6. Proof of Theorem 1.1

In order to prove the main theorem, we shall reduce it to the case when \( X \) has Picard number \( \rho(X) = 1 \). To treat that case, we will recall some facts about slopes of vector bundles that will be used later.

**Definition 6.1.** Let \( X \) be an \( n \)-dimensional projective variety and \( \mathcal{H} \) an ample line bundle on \( X \). Let \( \mathcal{E} \) be a torsion-free sheaf on \( X \). We define the slope of \( \mathcal{E} \) with respect to \( \mathcal{H} \) to be \( \mu_{\mathcal{H}}(\mathcal{E}) = \frac{c_1(\mathcal{E}) \cdot c_1(\mathcal{H})^{n-1}}{\text{rk}(\mathcal{E})} \). We say that a torsion-free sheaf \( \mathcal{F} \) on \( X \) is \( \mu_{\mathcal{H}} \)-semistable if for any subsheaf \( \mathcal{E} \) of \( \mathcal{F} \) we have \( \mu_{\mathcal{H}}(\mathcal{E}) \leq \mu_{\mathcal{H}}(\mathcal{F}) \).

**Lemma 6.2.** Let \( X \) be a smooth \( n \)-dimensional projective variety and \( \mathcal{H} \) an ample line bundle on \( X \). Let \( \mathcal{F} \) be a vector bundle on \( X \), \( p \) a positive integer, and \( \mathcal{N} \) an invertible subsheaf of \( \mathcal{F} \otimes^p \). Then \( \mathcal{F} \) contains a (torsion-free) subsheaf \( \mathcal{E} \) such that \( \mu_{\mathcal{H}}(\mathcal{E}) \geq \frac{\mu_{\mathcal{H}}(\mathcal{N})}{p} \).

**Proof.** Consider the Harder-Narasimhan filtration of \( \mathcal{F} \):

\[
0 = \mathcal{E}_0 \subsetneq \mathcal{E}_1 \subsetneq \ldots \subsetneq \mathcal{E}_r = \mathcal{F},
\]

with \( \mathcal{Q}_i = \mathcal{E}_i/\mathcal{E}_{i-1}, \mu_{\mathcal{H}} \)-semistable for \( 1 \leq i \leq r \), and \( \mu_{\mathcal{H}}(\mathcal{Q}_1) > \mu_{\mathcal{H}}(\mathcal{Q}_2) > \ldots > \mu_{\mathcal{H}}(\mathcal{Q}_k) \). We claim that \( \mathcal{E} = \mathcal{E}_1 = \mathcal{Q}_1 \) satisfies the desired condition. In order to prove this, first let \( m \in \mathbb{N} \) be such that \( \mathcal{H} \otimes^m \)

is very ample and let \( C \subset X \) be a curve that is the intersection of the zero sets of \( n-1 \) general sections of \( \mathcal{H} \otimes^m \). Observe that for this curve \( C \), and for any torsion-free sheaf \( \mathcal{E} \) on \( X \),

\[
\mu_{\mathcal{H}}(\mathcal{E}|_{C}) = m^{n-1} \cdot \mu_{\mathcal{H}}(\mathcal{E}).
\]

Notice that by abuse of notation we denote the restriction of \( \mathcal{H} \) to \( C \) by the same symbol. Let \( \mathcal{G}_i = \mathcal{E}_1|_C \) and \( \mathcal{P}_i = \mathcal{Q}_1|_C \). By the Mehta-Ramanathan Theorem ([MR82, 6.1], [HL97, 7.2.1]) the Harder-Narasimhan filtration of \( \mathcal{F}|_C \) is exactly the restriction to \( C \) of the Harder-Narasimhan filtration of \( \mathcal{F} \) (we may assume that \( m \) was already chosen large enough for this theorem to apply as well):

\[
0 = \mathcal{G}_0 \subsetneq \mathcal{G}_1 \subsetneq \ldots \subsetneq \mathcal{G}_r = \mathcal{F}|_C.
\]

As \( X \) is smooth, so is \( C \) and hence all torsion-free sheaves on \( C \), in particular the \( \mathcal{G}_i \) and the \( \mathcal{P}_i \), are locally free. Then for each \( 1 \leq i \leq r \) there exists a filtration

\[
\mathcal{G}_i \otimes^p = \mathcal{G}_i, 0 = \mathcal{G}_i, 1 \geq \ldots \geq \mathcal{G}_i, p \geq \mathcal{G}_i, p+1 = 0,
\]

with quotients \( \mathcal{G}_{i,j}/\mathcal{G}_{i,j+1} \simeq \mathcal{G}_{i,j} \otimes^p \mathcal{P}_i \otimes^{(p-j)} \). From these filtrations, we see that the inclusion \( \mathcal{N} \hookrightarrow \mathcal{F} \otimes^p \)

induces an inclusion \( \mathcal{N}|_{C} \hookrightarrow \mathcal{P}_1 \otimes^p \mathcal{P}_k \), for suitable non-negative integers \( i_j \)'s such that \( \sum i_j = p \). Since each \( \mathcal{P}_i \) is \( \mu_{\mathcal{H}} \)-semistable (on \( C \)), so is the tensor product \( \mathcal{P}_1 \otimes^p \mathcal{P}_k \) [HL97, Theorem 3.1.4]. Hence

\[
\mu_{\mathcal{H}}(\mathcal{N}|_{C}) = \frac{\mu_{\mathcal{H}}(\mathcal{N}|_{C})}{m^{n-1}} \leq \frac{\mu_{\mathcal{H}}(\mathcal{P}_1 \otimes^p \mathcal{P}_k)}{m^{n-1}} = \frac{\sum i_j \mu_{\mathcal{H}}(\mathcal{P}_j)}{m^{n-1}} \leq \frac{p \mu_{\mathcal{H}}(\mathcal{Q}_1)|_{C}}{m^{n-1}} = p \mu_{\mathcal{H}}(\mathcal{Q}_1),
\]

and so \( \mathcal{E} = \mathcal{E}_1 = \mathcal{Q}_1 \) does indeed satisfy the required property. \( \square \)

Now we can prove our main theorems.
**Theorem 6.3.** Let $X$ be a smooth $n$-dimensional projective variety with $\rho(X) = 1$, $\mathcal{L}$ an ample line bundle on $X$, and $p$ a positive integer. Suppose that $H^0(X, T_X^{\otimes p} \otimes \mathcal{L}^{-p}) \neq 0$. Then either $(X, \mathcal{L}) \simeq (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$, or $p = n \geq 3$ and $(X, \mathcal{L}) \simeq (Q_p, \mathcal{O}_{Q_p}(1))$.

**Proof.** First notice that $X$ is uniruled by [Miy87], and hence a Fano manifold with $\rho(X) = 1$. The result is clear if $\dim X = 1$, so we assume that $n \geq 2$. Fix a minimal covering family $H$ of rational curves on $X$. By Lemma 6.2, $T_X$ contains a torsion-free subsheaf $\mathcal{E}$ such that $\mu_0(\mathcal{E}) > \frac{\rho(X) - 1}{n} = \mu_0(\mathcal{L})$. This implies that $\deg f^* \mathcal{L} \geq \deg f^* \mathcal{E}$ for a general member $[f] \in H$. If $r = \text{rk} \mathcal{E} = 1$, then $\mathcal{E}$ is ample and we are done by Wahl’s theorem. Otherwise, as $f^* \mathcal{E}$ is a subsheaf of $f^* T_X \simeq \mathcal{O}_{P^1}(2) \oplus \mathcal{O}_{P^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(-n - d - 1)$, we must have $\deg f^* \mathcal{E} = 1$ and either $f^* \mathcal{E}$ is ample, or $f^* \mathcal{E} \simeq \mathcal{O}_{P^1}(2) \oplus \mathcal{O}_{P^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(-n - d - 1)$ for a general $[f] \in H$.

In the first case $X$ is smooth and $\rho(X) = 1$, and thus $\mathcal{L}$ is unsplit. Let $\pi : X \rightarrow Y$ be the normalization of a complete curve passing through a general point of $X$. By shrinking $Y$ if necessary, we may assume that $\pi^0$ is smooth. Since $\rho(X) \geq 2$, we must have $\dim Y^0 \geq 1$ by [Kol96, IV,3.13.3].

Let $F$ be a general fiber of $\pi^0$ and set $k = \dim F$. By Lemma 6.6, either

- $k = p - 1$, $(F, \mathcal{L}|_F) \simeq (\mathbb{P}^{p-1}, \mathcal{O}_{\mathbb{P}^{p-1}}(1))$, and $H^0(X^0, \wedge^{p-1} T_{X^0/Y^0} \otimes \pi^* \mathcal{L}|_F) \neq 0$,
- or $k \geq p$ and $H^0(X^0, \wedge^{p} T_{X^0/Y^0} \otimes \mathcal{L}^{-p}) \neq 0$.

In the first case $\pi : X^0 \rightarrow Y^0$ is a $\mathbb{P}^{p-1}$-bundle and we may assume that $\text{codim} X(X \setminus X^0) \geq 2$ by Theorem 2.6. Then we apply Lemma 5.4 and conclude that $X \simeq Q_2$.

In the second case, the induction hypothesis implies that either $(F, \mathcal{L}|_F) \simeq (\mathbb{P}^k, \mathcal{O}_{\mathbb{P}^k}(1))$, or $k = p$ and $(F, \mathcal{L}|_F) \simeq (Q_p, \mathcal{O}_{Q_p}(1))$. If $F \simeq \mathbb{P}^k$, again by Theorem 2.6, $\pi : X^0 \rightarrow Y^0$ is a $\mathbb{P}^k$-bundle, and we may assume that $\text{codim} X(X^0) \geq 2$. As in the end of the proof of Proposition 5.4, we reach a contradiction by applying Corollary 5.3 to $X^0 \times_{Y^0} B \rightarrow B$, where $B \rightarrow Y^0$ is the normalization of a complete curve passing through a general point of $Y^0$.

Suppose now that $F \simeq Q_p$. Then, by Lemma 2.2 and [Fuj75, Corollary 5.5], $\pi^0$ can be extended to a quadric bundle $\pi : X' \rightarrow Y'$ with irreducible and reduced fibers, where $X'$ is an open subset of $X$ with $\text{codim} X(X \setminus X') \geq 2$, and $Y'$ is smooth. Denote by $X''$ the open subset of $X'$ where $\pi$ is smooth. Notice that $\text{codim}_{X'}(X' \setminus X'') \geq 2$. A non-zero global section of $\wedge^p T_{X'}/Y' \otimes \mathcal{L}^{-p}$ restricts to a non-zero global section of $\wedge^p T_{X''/Y'} \otimes \mathcal{L}^{-p}$, which, in turn, extends to a non-zero global section $s \in H^0(X', \omega_{X'/Y'}^{-1} \otimes \mathcal{L}^{-p})$ since $X'$ is smooth. The section $s$ does not vanish anywhere on a general fiber of $\pi$.

Let $g : C \rightarrow Y'$ be the normalization of a complete curve passing through a general point of $Y'$. Set $X_C = X' \times_{Y'} C$, denote by $\pi_C : X_C \rightarrow C$ the corresponding quadric bundle, and write $\mathcal{L}|_{X_C}$ for the pullback of $\mathcal{L}$ to $X_C$. The general fiber of $\pi_C$ is smooth. Now notice that $X_C$ is a local complete intersection variety, and nonsingular in codimension one, since the fibers of $\pi$ are reduced. In particular, $X_C$ is a normal Gorenstein variety, and the morphism $\pi_C$ is generically smooth. The section $s$ induces a non-zero section in $H^0(X_C, \omega_{X_C/C}^{-1} \otimes \mathcal{L}_{X_C}^{-p})$ that does not vanish anywhere on the general fiber of $\pi_C$. Thus $\omega_{X_C/C}^{-1}$ is ample, contradicting Proposition 3.1. \qed
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