Number of cliques in random scale-free network ensembles

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Abstract

In this paper we calculate the average number of cliques in random scale-free networks. We consider first the hidden variable ensemble and subsequently the Molloy Reed ensemble. In both cases we find that cliques, i.e. fully connected subgraphs, appear also when the average degree is finite. This is in contrast to what happens in Erdős and Renyi graphs in which diverging average degree is required to observe cliques of size $c > 3$. Moreover we show that in random scale-free networks the clique number, i.e. the size of the largest clique present in the network diverges with the system size.

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When graphs are used to represent a variety of real technological, social and biological systems they are called networks. The analysis of many real networks reveals that while different networks differ one from another in their local structure, characterized by operational modules or motifs that are a signature of their function [Milo2002, Vazquez2004, Dobrin2004], many networks have some important common characteristics [Albert2002, Dorogovtsev2003, Pastor-Satorras2004]. In particular a large variety of networks have been shown to display a scale-free degree distribution $P(k) \sim k^{-\gamma}$ with non universal $\gamma$ exponents. The scale-free degree distribution strongly affects the local topology of the networks. For example, scale-free networks with an exponent $\gamma < 3$ have a very large number of small loops [Bianconi2003, Bianconi2005].

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which is a very distinctive feature with respect to Erdös and Renyi (ER) networks with finite average connectivity (Janson2000; Marinari2004). In its turn, this very peculiar local structure induce many relevant effects of the dynamics defined on these networks (Dorogovtsev2002; Leone2002; Havlin2000; Pastor-Satorras2001).

A special type of network subgraphs are cliques, i.e. fully connected subsets of nodes of the network. Cliques are relevant objects for the study of real networks, in fact cliques and overlapping sets of cliques provide relevant insights on the community structure of networks (Derenyi2005; Palla2005). In random Erdös and Renyi graphs of $N$ nodes and linking probability $p(N)$ the expected number of cliques (Janson2000) of size $c$ is given by

$$
\langle N_{c}^{ER} \rangle = \binom{N}{c} p(N)^{c(c-1)/2}.
$$

Consequently in the large $N$ limit the expected number of small cliques with $c > 3$ is different from zero only when the average degree $\langle k \rangle$ diverges as $N \to \infty$. Special attention in mathematics literature is given to the maximal size of the clique present in a graph $G$, i.e. its clique number $c_{\text{max}}$. The clique number is an important characteristic of networks and constitute also a lower bound to the coloring number, since in the coloring problem one is forced to color all the nodes of a clique with a different color. In (Bianconi2006), we show that scale-free networks with $\gamma < 3$ have many more and larger cliques than random Erdöös and Renyi networks.

In this paper we provide the complete derivation of the theoretical expectation on the number of cliques in random scale-free networks. We do this by evaluating the average number of cliques and its second moment. We found the surprising result that cliques of size $c > 3$ are present also in networks with finite average connectivity, i.e. networks with $\gamma \in (2,3)$. Moreover we can prove that the clique number $c_{\text{max}}$ of networks with $\gamma < 3$ diverge with the network size $N$ providing upper and lower bounds for the clique number. These bound arise from classical inequalities for probabilities which involve the first and the second moment of the number of cliques. These can be computed in different ensembles of random graphs (Molloy1995; Goh2001). The main section of this paper would be devoted to the calculation of the average number of cliques and its second moment in the hidden variable ensemble (Caldarelli2002; Boguna2003). Subsequently the derivation of the average number of clique is extended to the Molloy Reed ensemble (Molloy1993). The same scaling of the number of cliques is found also in this ensemble. Finally the conclusions are given.
1 Hidden variable ensemble

In this ensemble the prescription to generate a class of scale-free networks with exponent $\gamma$ is the following: \textit{i}) assign to each node $i$ of the graph a hidden continuous variable $q_i$ distributed according a $\rho(q)$ distribution. Then \textit{ii}) each pair of nodes with hidden variables $q, q'$ are linked with probability $r(q, q')$. When the hidden variable distribution is scale-free $\rho(q) = \rho_0 q^{-\gamma}$ for $q \in [m, Q]$ and the linking probability is linear in both $q$ and $q'$, i.e. $r(q, q') = qq'/(\langle q \rangle N)$ we obtain a random uncorrelated scale-free network. In this specific case a cutoff

$$Q \sim \begin{cases} N^{1/\gamma} & \text{for } \gamma \in (1, 2] \\ N^{1/2} & \text{for } \gamma \in (2, 3] \\ N^{1/(\gamma-1)} & \text{for } \gamma \in (3, \infty) \end{cases}$$

is needed to keep the linking probability smaller than one, i.e. $Q^2/(\langle q \rangle N) \leq 1$.

1.1 Average number of cliques

A clique $C$ of size $c$ is a set of $c$ distinct nodes $C = \{i_1, \ldots, i_c\}$, each one connected with all the others. For each choice of the nodes, the probability that they are connected in a clique is

$$\prod_{i \neq j \in C} r(q_i, q_j).$$

Consequently the average number of cliques of size $c$ is given by the number of ways in which we can pick $c$ nodes in the network with $n(q)$ nodes with hidden variable $q_i \in (q, q + \Delta q)$ multiplied by the probability that each couple of node of this set is linked. Since in random scale-free networks we have $r(q, q') = \frac{qq'/(\langle q \rangle N)}$ we can write

$$\langle N_c \rangle = \sum_{\{n(q)\}} \prod_q \left( \frac{N(q)}{n(q)} \right) \left( \frac{q}{\sqrt{\langle q \rangle N}} \right)^{(c-1)n(q)}$$

where $N(q) = N \rho(q)$ are the nodes of the network with hidden variable $q_i \in (q, q + \Delta q)$ and where the sum is extended to all the sequences $\{n(q)\}$ satisfying $\sum_q n(q) = c$. Introducing a integral representation of the delta function $\delta(\sum_q n(q) - c)$ and performing the summation over $n(q)$ we get
\[ \langle N_c \rangle = \int dy e^{y(c/N + N \log[1 + \theta^{-1} e^{-y}]忘记) \text{.} \quad (4) \]

where we have taken the limit \( \Delta q \to 0 \). In (4) we have introduced the variable \( \theta \) defined as

\[ \theta = \frac{q}{\sqrt{\langle q \rangle N}}, \quad (5) \]

and we have indicated with \( \langle \rangle \) the average over the distribution \( \rho(q) \). Solving the integral in (4) by saddle point method one finds

\[ \langle N_c \rangle \approx \sqrt{\frac{2\pi}{N|f''(y^*)|}} e^{Nf(y^*)} \quad (6) \]

with \( f(y) = yc/N + \langle \log[1 + \theta^{-1} e^{-y}] \rangle \) and \( y^* \) fixed by the saddle point equation

\[ \frac{c}{N} = \left( \frac{\theta^{-1} e^{-y^*}}{1 + \theta^{-1} e^{-y^*}} \right). \quad (7) \]

If we assume that the cutoff of the hidden variable distribution is equal to \( Q = \sqrt{\langle q \rangle N(1 - \epsilon)} \) with an \( \epsilon \geq 0 \), the maximal clique size depends on both the \( \gamma \) exponent and on \( \epsilon \). The dependence in \( \epsilon \) reflects the fact that when \( \epsilon = 0 \) the highest degree nodes have a probability to be linked \( r(q, q') \) which approach one. Considering the definition of \( y^* \) from Eq. (7) we can see that the asymptotic expansion

\[ e^{y^*} \approx \frac{N}{c} \langle \theta^{-1} \rangle \quad (8) \]

is valid until

\[ \frac{c}{N} \langle \theta^{2(c-1)} \rangle \langle [\theta^{(c-1)}] \rangle^2 \leq Q^{\gamma-1} \frac{c^2}{(2c-3)N} < 1, \quad (9) \]

i.e. until \( c < c^* \sim (2NQ^{1-\gamma})^{1/2} \).

Consequently for clique sizes \( c < c^* \) one has the valid asymptotic expression for \( \langle N_c \rangle \)

\[ \langle N_c \rangle \approx \sqrt{\frac{2\pi}{c}} \left( \frac{Ne}{c} \theta^{-1} \right)^c. \quad (10) \]
To find an upper bound for the clique number (the maximal clique size) is a bit more involved. We start from the classical inequality
\[ P(N_c > 0) \leq \langle N_c \rangle \] (11)

and the expression (6) together with (7) for the average number of cliques \( \langle N_c \rangle \). If \( \langle N_c \rangle \to 0 \) in the \( N \to \infty \) limit, then \( \bar{c} \) fixes an upper bound \( \bar{c} \) for the maximal clique size of the network.

In Appendix A show that the clique number \( c_{max} \leq \bar{c} \) with \( \bar{c} \) satisfy for \( \epsilon = 0 \) the condition
\[ \left\langle \frac{N_c^2}{\bar{c} \theta^{\bar{c}-1}} \right\rangle = 1. \] (12)

On the other this expression provide a upper bound also for the case \( \epsilon \neq 0 \) since in this case \( \bar{c} \) defined in Eq. (12) is still in within the validity of the asymptotic expansion (10) and correspond to an expected number of cliques \( N_c \to 0 \) as \( N \to \infty \).

The values of \( \bar{c} \) and \( c^* \) will depend both on the \( \gamma \) exponent and on the value of \( \epsilon \). In fact networks with different values of \( \gamma \) have different structural cutoffs.

- **Networks with \( \gamma > 3 \)**
  These networks have a natural cutoff \( Q = aN^{\frac{1}{\gamma-1}} \). Considering this cutoff when performing the average in equation (12), we find \( \bar{c} = 3 \) in the limit \( N \to \infty \). Therefore these networks, as well as the Erdős and Renyi networks, have maximal clique size \( c_{max} \leq 3 \).

- **Networks with \( 2 \leq \gamma < 3 \)**
  These networks have a structural cutoff defined as in the case \( 2 \leq \gamma < 3 \), i.e.
  \[ \langle N_c \rangle \approx \sqrt{\frac{2\pi}{c}} A_{\gamma,(c)} \frac{N^{(3-\gamma)/2} (1-\epsilon)^{(c-\gamma)}}{c(c-\gamma)} \] (13)

  whit \( A_{\gamma,(c)} \) been a constant depending on the power-law exponent \( \gamma \) and on the average connectivity of the graph \( \langle q \rangle \). Moreover the value of \( \bar{c} \) and \( c^* \) defined in equations (9) and (12) depend on the system size \( N \), the \( \gamma \) exponent and on \( \epsilon \) as shown in the Table 1.

  We observe that while for the case \( \epsilon > 0 \) the asymptotic expansion is valid much above the upper bound \( \bar{c} \), for \( \epsilon = 0 \) the upper bound and the limit of the validity of the asymptotic expansion \( c^* \) have the same order of magnitude, i.e. \( c^* \sim \bar{c} \sim N^{\frac{3-\gamma}{\gamma}} \) but we have \( \bar{c} > c^* \).

- **Networks with \( 1 \leq \gamma < 2 \)**
  These networks have a structural cutoff defined as in the case \( 2 \leq \gamma < 3 \), i.e.
\[ Q = (1 - \epsilon) \sqrt{\langle q \rangle N}. \] Given this expression and the divergence of the average degree with the upper cutoff \( \langle q \rangle \sim Q^{2-\gamma} \), we get that the upper cutoff \( Q \) scales with the network size \( N \) as \( Q \sim N^{1/\gamma} \). The asymptotic expansion gives for the average number of cliques of sizes \( c < c^* \)

\[ \langle N_c \rangle \approx \sqrt{\frac{2\pi c}{c^*}} \left( B_{\gamma,m} \frac{N^{1/\gamma} \epsilon(1 - \epsilon)(c+1-2/\gamma)}{c(c - \gamma)} \right)^c \]  

where \( B_{\gamma,m} \) is a function depending on the power-law exponent \( \gamma \) and on the lower cutoff \( m \) of the distribution.

The value of \( \bar{c} \) and \( c^* \) defined in equations (9) and (12) depend on the system size \( N \), the \( \gamma \) exponent and on \( \epsilon \). Their scaling is shown in Table 1. Also in this range of values of \( \gamma \) for \( \epsilon > 0 \) the asymptotic expansion is valid much above the upper bound \( \bar{c} \), while for \( \epsilon = 0 \) the upper bound and the limit of the validity of the asymptotic expansion \( c^* \) have the same order of magnitude, i.e. \( c^* \sim \bar{c} \sim N^{1/\gamma} \), but we have \( \bar{c} > c^* \).

1.2 Second moment of the average number of cliques

In order to derive a lower bound on the clique number \( c_{max} \) we use a classical relation of probability theory (Janson2000), i.e.

\[ P(N_c > 0) \geq \langle N_c \rangle^2 / \langle N_c^2 \rangle \]  

where \( \langle N_c^2 \rangle \) is the second moment of the number of cliques of size \( c \) in the considered random graph ensemble. Consequently if \( \langle N_c \rangle^2 / \langle N_c^2 \rangle \geq K \) we are guaranteed that the typical graph contains cliques of size \( c \) with probability \( P(N_c > 0) \geq K > 0 \). Thus we proceed in the calculation of the second moment of the clique number \( \langle N_c^2 \rangle \). To do this calculation we count the average number of pairs of cliques of size \( c \) present in the graph with an overlap of \( o = 0, \ldots, c \) nodes. We use the notation \( \{n(q)\} \) to indicate the number of the nodes with hidden variable \( q \) belonging to the first clique, \( \{n_o(q)\} \) to indicate the number of nodes belonging to the overlap and with \( \{n'(q)\} \) to indicate the number of nodes belonging to the second clique but not to the overlap. We consider only sequences \( \{n(q)\}, \{n'(q)\}, \{n_o(q)\} \) which satisfy \( \sum_q n(q) = c, \sum_q n_o(q) = o \) and \( \sum_q n'(q) = c - o \). With these conditions, and then substituting the conditions with delta functions we get

\[ \langle N_c^2 \rangle = \sum_{o=0}^c \sum \sum \sum \prod_q \begin{pmatrix} N(q) \\ n(q) \end{pmatrix} \begin{pmatrix} N(q) - n(q) \\ n'(q) \end{pmatrix} \begin{pmatrix} n(q) \\ n_o(q) \end{pmatrix} \theta(q) \]
\[
\int dy \int dy' \int dy'' \sum_{o=0}^{c} e^{N(f(y,y',y'',q))}.
\]  
(16)

where \( g(q) = (c-1)(n(q) + n'(q)) + (c - o)n_o(q) \) and

\[
f(y, y', y'', q) = \frac{yc + y'(c - o) + y''o}{N} + \log \left[ 1 + \left( e^{-y'} + e^{-y} \right) \theta^{c-1} + e^{-(y+y'')} \theta^{2c-o-1} \right].
\]

The saddle point method, gives

\[
\begin{cases}
y = y' \\
\frac{c-o}{N} = \frac{e^{-y'} \theta^{c-1}}{1 + (e^{-y'} + e^{-y}) \theta^{c-1} + e^{-(y+y'')} \theta^{2c-o-1}} \\
\frac{o}{N} = \frac{e^{-y'} \theta^{2c-o-1}}{1 + (e^{-y'} + e^{-y}) \theta^{c-1} + e^{-(y+y'')} \theta^{2c-o-1}}.
\end{cases}
\]

Using the asymptotic expansions of these saddle point equations valid for \( c < c^* \) we found

\[
\langle N^2_c \rangle \leq \sum_{o=0}^{c} \left( \frac{N}{c-o} \langle \theta^{c-1} \rangle \right)^{2(c-o)} \left( \frac{N}{o} \langle \theta^{2c-o-1} \rangle \right)^{o} e^{2c-o} \sqrt{\frac{2\pi^3}{c(c-o)^o}}
\]
(17)

Using also for \( \langle N_L \rangle \) the asymptotic expression (10) then we can express the ratio \( \frac{\langle N^2 \rangle}{\langle N_c \rangle} \) as

\[
\frac{\langle N^2 \rangle}{\langle N_c \rangle} \leq \sum_{o=0}^{c} \frac{e^{2c}}{(c-o)^{2(c-o)}o^o} \left( \langle N \theta^{c-1} \rangle \right)^{-2o} \left( N \langle \theta^{2c-o-1} \rangle \right)^{o} e^{2c-o} \sqrt{\frac{2\pi^3}{o(c-o)}}
\]
\[
\leq \sum_{o=0}^{c} \frac{e^{2c}}{(c-o)^{2(c-o)}o^o} \left( \frac{(c-o)^2}{2c-o-\gamma N \langle q \rangle N^{1-\gamma}} \right)^{o} \frac{1}{o(c-o)^o} e^{2c-o} \sqrt{\frac{2\pi^3}{o(c-o)}}.
\]
(18)

We notice that in the limit \( c \to \infty \) we have

\[
\frac{e^c}{e^{o(c-o)^{c-o}}} \leq c^o \frac{1}{e^{o(1 - \frac{o}{c})^c}} \to c^o
\]
(19)

Using this limit behavior and Stirling approximations for factorials, we get
\[
\frac{\langle N_c^2 \rangle}{\langle N_c \rangle^2} \leq \left[ 1 + \frac{c(c - \gamma)(1 - \epsilon)(\bar{c} - e)}{\bar{c}(\bar{c} - \gamma)} \right]^c
\]  
\text{(20)}

- If \( \epsilon = 0 \) it is useful to define the clique size \( \hat{c} \) satisfying

\[
\frac{\bar{c}(\hat{c} - \gamma)e}{\hat{c}(\hat{c} - \gamma)} = \frac{1}{\hat{c}}
\]  
\text{(21)}

i.e. \( \hat{c} \sim \bar{c}^{2/3} \). Then if \( c = \alpha \hat{c}^{1 - \eta} \) we have in the limit \( N \to \infty, \bar{c} \to \infty, \)

\[
\frac{\langle N_c^2 \rangle}{\langle N_c \rangle^2} \leq \begin{cases} 
1 & \text{if } \eta > 0 \\
\epsilon & \text{if } \eta = 0 \\
\infty & \text{if } \eta < 0.
\end{cases}
\]

From Eqs. (10) and (22) for \( c = \hat{c} \) defined in (21) one finds that with \( c = \alpha \hat{c}^{1 - \eta} \)

\[
P(N_c > 0) \geq \begin{cases} 
1 & \text{if } \eta > 0 \\
\frac{1}{\epsilon} & \text{if } \eta = 0 \\
0 & \text{if } \eta < 0.
\end{cases}
\]

Consequently the network contains almost surely cliques of sizes \( c \leq \xi \) with

\[
\xi = \alpha \hat{c} = \alpha' \bar{c}^{2/3}
\]  
\text{(22)}

and \( \alpha > 0 \)

- If \( \epsilon > 0 \), and \( c = \bar{c} - \alpha \bar{c}^\eta \) with \( \eta > 0 \), we have in the limit \( N \to \infty, \bar{c} \to \infty, \)

\[
\frac{\langle N_c^2 \rangle}{\langle N_c \rangle^2} < \begin{cases} 
1 & \text{if } \eta > 0 \\
\infty & \text{if } \eta \leq 0.
\end{cases}
\]

From Eqs. (10) and (23) it follows that as long as \( c = \bar{c} - \alpha \bar{c}^\eta \)

\[
P(N_c > 0) \geq \begin{cases} 
1 & \text{if } \eta > 0 \\
0 & \text{if } \eta = 0, \alpha = 0
\end{cases}
\]

Consequently we have that the network contains almost surely cliques of sizes \( c \leq \zeta \) with

\[
\zeta = \bar{c} - \alpha \bar{c}^\eta
\]  
\text{(23)}

and \( \alpha, \eta > 0 \)
Since if a graph contains a clique of size $\mathcal{c}$ it contains clearly also cliques of smaller size we proved that typical networks have a finite probability to get any cliques of size $c \leq c_\gamma$.

1.3 Average number of cliques passing through a node

To find the expected number of cliques of size $c$ passing through a given node, with hidden variable $q$, we can repeat the arguments proposed for the calculation of the first moment with the difference that we integrate over all the hidden variables of the nodes in the cliques except for the hidden variable $q_i = q$ of the chosen node. Following these arguments one finds for cliques $c < c^*$

$$N_c(q) \approx \left( \frac{q}{\sqrt{\langle q \rangle N}} \right)^{c-1} N_{c-1}. \quad (24)$$

Consequently nodes with higher hidden variable $q$ are expected to be part of more cliques.
2 Molloy Reed ensembles

The counting of the number of cliques in the Molloy-Reed (Molloy 1995) follows a procedure much similar to the one considered for the hidden variable ensemble giving similar results. To construct a Molloy-Reed network one proceed as follows: i) a degree is assigned to each node of the network following the desired degree distribution with cutoff $K$

$$K \sim \begin{cases} N^{1/\gamma} & \text{for } \gamma \in (1, 2] \\ N^{1/2} & \text{for } \gamma \in (2, 3] \\ N^{1/3} & \text{for } \gamma \in (3, \infty) \end{cases}$$

Degree distributions which do not satisfy the parity of $\langle k \rangle N = \sum_i k_i$ are disregarded; ii) the edges coming out of the nodes are randomly matched until all edges are connected. The structural cutoff for $\gamma \leq 3$ ensures that the probability of double links and tadpoles is small (Bianconi 2005).

To calculate $\langle N_c \rangle$ in this ensemble first one has to count in how many ways it is possible to have a clique of size $c$ in the network and weight the results with the fraction of possible networks in the ensemble which contains the clique. Let us first state that the total number of graphs in the Molloy-Reed ensemble is given by $((\langle k \rangle N - 1))!!$. Indeed when constructing the network by linking $\langle k \rangle N$ unconnected edges one start by taking one edge at random and connecting it to one of the $((\langle k \rangle N - 1))$ possible connections. Then one proceed taking another edge and linking it to one of the remaining $((\langle k \rangle N - 3))$ possible connections thus giving rise of one of the $((\langle k \rangle N - 1))!!$ possible networks. By similar arguments one shows that the total number of networks containing a given clique of size $c$ are $[(\langle k \rangle N - c(c - 1) - 1))!!$. On the other side the total number of cliques of size $c$ in the Molloy-Reed ensemble is given by the number of ways one can choose $c$ nodes $\{i_1, \ldots, i_c\}$ of connectivity $\{k_1, k_2, \ldots, k_c\}$ and connect each pair of them. The number of ways one can choose the edges coming out of the nodes to form the clique is given by

$$\Pi_{i=1}^c \frac{k_i!}{(k_i - c + 1)!}.$$ 

Consequently the average number of cliques in the Molloy-Reed ensemble will be given by

$$N_c = \sum_{\{n_k\}} \prod_{k=c}^K \left( \frac{N(k)}{n(k)} \right) \left( \frac{k!}{(k - c + 1)!} \right)^{n_k} W_{N,c}$$ (25)
where \( N(k) = NP(k) (n(k)) \) is the number of nodes with connectivity \( k \) present in the network (loop), \( K \) is the cutoff of the degree distribution and the sum over \( \{n(k)\} \) is restricted to \( \{n(k)\} \) such that \( \sum_k n(k) = c \). Moreover we use the definition \( W_{N,c} = (\langle k \rangle N - c(c-1) - 1)!/(\langle k \rangle N)! \). If we use the Stirling approximation for \( W_{N,c} \) we get the expression

\[
W_{N,c} \sim (\langle k \rangle N)^{-c(c-1)/2} e^{N g(\omega)}
\]

with \( \omega = c(c-1)/N \) and

\[
g(\omega) = \frac{1}{2}(\langle k \rangle - 2\omega) \log \left( \frac{\langle k \rangle - 2\omega}{\langle k \rangle} \right) + \omega \sim \frac{3\omega^2}{\langle k \rangle}
\]

Thus we get

\[
N_c = \sum_{\{n_k\}} \prod_{k=c-1}^{K} \left( \frac{N(k)}{n(k)} \right) (\kappa^{c-1})^{n(k)} e^{N g(\omega)}
\]

where

\[
\kappa^{c-1} = \frac{k!}{(k - c + 1)! (\langle k \rangle N)^{c-1/2}}
\]

Expression (28) for the average number of cliques in a Molloy Reed ensemble differs from the equivalent expression in the hidden variable ensemble 4 i) for the substitution \( \theta^{c-1} \rightarrow \kappa^{c-1} \); ii) for the factor \( \exp(N g(\omega)) \) and iii) for the fact that the average is performed only on the nodes with connectivity \( k \geq c - 1 \). Following the same steps as in the hidden variable ensemble, we get

\[
N_c = \int_{-\infty}^{\infty} \frac{dy}{2\pi} e^{\langle k \rangle N g(\omega)} e^{\langle k \rangle N [\log[1 + (\kappa^{c-1}) e^{-y}] - y]} c_{c-1} + N g(\omega)
\]

with \( g(\omega) \) given by Eq. (27) and the average performed of the \( N(k) \) distribution with a lower cutoff at \( k = c - 1 \).

Evaluating (30) by the saddle point method and following the steps described in the preceding section, we get the following approximate expression for the average number of cliques \( N_c \)

\[
N_c = \left( e^{(\kappa^{c-1}) c_{c-1}} \right)^c
\]
where this approximation is valid asymptotically for cliques of sizes \( c < c^* \). We note that \( c^* \) is fixed by the condition

\[
\frac{c \left\langle (\kappa(c-1))^2 \right\rangle}{N \left\langle (\kappa(c-1)) \right\rangle^2} \leq K^{\gamma-1} \frac{c^2}{(2c-3)N} < 1.
\] (32)

Similar results to the one found for the hidden-variable ensemble also apply for the second-moment of the number of cliques in the Molloy-Reed ensemble.

3 Conclusions

In conclusion we have have calculated the first and the second moment of the number of cliques in random scale-free network ensembles. This calculation shows these networks, provided that the power-law exponent \( \gamma < 3 \) have many small cliques and a large clique number. In particular the clique number diverges with the network size as long as \( \gamma < 3 \) which is a surprising results since in Erdős and Renyi random networks with finite average degree the maximal clique size is \( c_{\text{max}} = 3 \). Moreover we have shown that in the case in which the cutoff is the maximal allowed cutoff (i.e. following the terminology of the paper when \( \epsilon = 0 \)) there can be large fluctuations of the clique number wherever for \( \epsilon \neq 0 \) the fluctuations are small.

A Calculation of the upper bound for the clique number in the case \( \epsilon = 0 \) in the hidden variable ensemble

The evaluation of the upper bound for the clique number in the subtle case \( \epsilon = 0 \) deserve a particular attention. To address this problem we start by rewriting in the following the main results for the average number of cliques in the hidden variable ensemble. The expression (6) for the average number of cliques is given by

\[
\left\langle N_c \right\rangle = e^{Nf(y^*)} \sqrt{\frac{2\pi}{Nf''(y^*)}}
\] (A.1)

where \( y^* \) is provided by the saddle point equation (7)

\[
\frac{c}{N} = \left\langle \frac{\theta^{c-1}e^{-y^*}}{1 + \theta^{c-1}e^{-y^*}} \right\rangle.
\] (A.2)
and \( f(y^*) = y^*c + N\langle \log[1 + \theta c^{-1}e^{-y^*}] \rangle \) while \( \theta \) is given by

\[
\theta = \left( \frac{q}{\sqrt{\langle q \rangle N}} \right). \tag{A.3}
\]

If \( y^* > 0 \) then we have that

\[
y^*c + N\langle \log[1 + \theta c^{-1}e^{-y^*}] \rangle \leq y^*c + N\langle \log[1 + \theta c^{-1}] \rangle. \tag{A.4}
\]

On one side, from the saddle point equation Eq. (A.2) we have

\[
e^{y^*} \leq \left( \frac{N\theta c^{-1}}{c} \right)^c. \tag{A.5}
\]

on the other side we have that,

\[
N\langle \log[1 + \theta c^{-1}] \rangle \leq 2N\left( \frac{\theta c^{-1}e^{-y^*}}{1 + \theta c^{-1}e^{-y^*}} \right) = 2c. \tag{A.6}
\]

Moreover the second derivative \( f''(y^*) \) satisfy

\[
N f''(y^*) = N\left( \frac{\theta c^{-1}e^{-y^*}}{(1 + e^{-y^*}\theta c^{-1})^2} \right) \\
\geq N\left( \frac{e^{-y^*}\theta c^{-1}}{(1 + e^{-y^*}\theta c^{-1})} \right) \frac{1}{1 + e^{-y^*}} \\
\geq \frac{c}{2}. \tag{A.7}
\]

Consequently, putting together Eqs. (A.4), (A.5) (A.6) and finally Eq. (A.7) the average number of cliques \( \langle N_c \rangle \) (A.1) satisfy

\[
\langle N_c \rangle \leq \sqrt{\frac{2}{c}} \left( \frac{N\theta c^{-1} e^2}{c} \right)^c, \tag{A.8}
\]

which together with the inequality (11) provides the upper bound (12) for the clique number scales with the system size as shown in Table 1.

At this point we must check self-consistently that indeed is \( c < \bar{c} \) then \( y^* > 0 \). To prove this we suppose on the contrary that \( y^* < 0 \). In this eventuality, the saddle point equation can be rewritten as
\[
\frac{c}{N} = \sum_{q < \bar{q}} \rho(q) \frac{\theta^{c-1} e^{-y^*}}{1 + \theta^{c-1} e^{-y^*}} + \sum_{q > \bar{q}} \rho(q) \frac{\theta^{c-1} e^{-y^*}}{1 + \theta^{c-1} e^{-y^*}} \tag{A.9}
\]

where \(\bar{q} = Q e^{\frac{e - y^*}{c}} < Q\). Expanding the two terms in series we get
\[
\frac{c}{N} = e^{-y^*} \frac{1}{c} N \frac{Q^{1-\gamma}}{c} (F_{\gamma,c} + G_{\gamma,c}) \tag{A.10}
\]

with
\[
F_{\gamma,c} = m^{\gamma-1}(\gamma - 1) \sum_{n=1}^{\infty} (-1)^n \frac{c}{[(c-1)n + 1 - \gamma]} \rightarrow F_{\gamma}^*
\]
\[
G_{\gamma,c} = m^{\gamma-1}(\gamma - 1) \sum_{n=1}^{\infty} (-1)^n \frac{c}{[(c-1)n + \gamma - 1]} \rightarrow G_{\gamma}^* \tag{A.11}
\]

Therefore for \(c \gg 1\) we have
\[
e^{-y^*} = \left(\frac{c^2}{N(F_{\gamma}^* + G_{\gamma}^*)^2}\right)^{\frac{1}{\gamma-1}} \tag{A.12}
\]

Let’s observe that \(\bar{c}\) in given by the value in the table 1 always satisfy \(\bar{c} \ll N^{1/2}\) for \(\gamma > 1\). Moreover as long as \(c \rightarrow \infty\) with \(c \ll N^{1/2}\), we get form expression (??) that \(y^* \rightarrow 0^+\). Consequently we assuming \(y^* > 0\) for \(c > \bar{c}\) we have reached a contradiction. This proves that in the hypothesis \(c > \bar{c}\) the saddle point solution \(y^*\) is always positive, i.e. \(y^* > 0\) as we assume at the beginning of the paragraph.

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