A Systematic Scan for 7-colourings of the Grid

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Abstract

We study the mixing time of a systematic scan Markov chain for sampling from the uniform distribution on proper 7-colourings of a finite rectangular sub-grid of the infinite square lattice, the grid. A systematic scan Markov chain cycles through finite-size subsets of vertices in a deterministic order and updates the colours assigned to the vertices of each subset. The systematic scan Markov chain that we present cycles through subsets consisting of $2 \times 2$ sub-grids and updates the colours assigned to the vertices using a procedure known as heat-bath. We give a computer-assisted proof that this systematic scan Markov chain mixes in $O(\log n)$ scans, where $n$ is the size of the rectangular sub-grid. This is the first time the mixing time of a systematic scan Markov chain on the grid has been shown to mix for less than 8 colours. We also give partial results that underline the challenges of proving rapid mixing of a systematic scan Markov chain for sampling 6-colourings of the grid by considering $2 \times 3$ and $3 \times 3$ sub-grids.

1 Introduction

This paper is concerned with sampling from the uniform distribution, $\pi$, on the set of proper $q$-colourings of a finite-size rectangular grid. A $q$-colouring of a graph is an assignment of a colour from a finite set of $q$ distinct colours to each vertex and we say that a colouring is a proper colouring if no two adjacent vertices are assigned the same colour. Proper $q$-colourings of the grid correspond to the zero-temperature anti-ferromagnetic $q$-state Potts model on the square lattice, a model of significant importance in statistical physics (see for example Salas and Sokal [14]).

Sampling from $\pi$ is computationally challenging, however it remains an important task and it is frequently carried out in experimental work by physicists by simulating some suitable random dynamics that converges to $\pi$. Ensuring that a dynamics converges to $\pi$ is generally straightforward, but obtaining good upper bounds on the number of steps
required for the dynamics to become sufficiently close to $\pi$ is a much more difficult problem. Physicists are at times forced to “guess” (using some heuristic methods) the number of steps required for their dynamics to be sufficiently close to the uniform distribution in order to carry out their experiments. By establishing rigorous bounds on the convergence rates (mixing time) of these dynamics computer scientists can provide underpinnings for this type of experimental work and also allow a more structured approach to be taken.

Providing bounds on the mixing time of Markov chains is a well-studied problem in theoretical computer science. However, the types of Markov chains frequently considered by computer scientists do not always correspond to the dynamics usually used in the experimental work by physicists. In computer science, the mixing time of various types of random update Markov chains have been frequently analysed; notably on the grid by Achlioptas, Molloy, Moore and van Bussel [1] and Goldberg, Martin and Paterson [9]. We say that a Markov chain on the set of colourings is a random update Markov chain when one step of the process consists of randomly selecting a set of vertices (often a single vertex) and updating the colours assigned to those vertices according to some well-defined distribution induced by $\pi$. Experimental work is, however, often carried out by cycling through and updating the vertices (or subsets of vertices) in a deterministic order. This type of dynamics has recently been studied by computer scientists in the form of systematic scan Markov chains (systematic scan for short). For results regarding systematic scan see for instance Dyer, Goldberg and Jerrum [5, 4] and Pedersen [12] although these papers are not considering the grid specifically. It is important to note that systematic scan remains a random process since the method used to update the colour assigned to the selected set of vertices is a randomised procedure drawing from some well-defined distribution induced by $\pi$.

In Section 3 we present a computer assisted proof that systematic scan mixes rapidly when considering 7-colourings of the grid. Previously eight was the least number of colours for which systematic scan on the grid was known to be rapidly mixing, due to Pedersen [12], a result which we hence improve on in this paper. We will make use of a recent result by Pedersen [12] to prove rapid mixing of systematic scan by bounding the influence on a vertex (note that the literature traditionally talks about sites rather than vertices). We will provide bounds on this influence parameter by using a heuristic to mechanically construct sufficiently good couplings of proper colourings of a $2 \times 2$ sub-grid. We will hence use a heuristic based computation in order to establish a rigorous result about the mixing time of a systematic scan Markov chain. Finally, in Section 4, we consider the possibility of proving rapid mixing of systematic scan for 6-colourings of the grid by increasing the size of the sub-grids. We give lower bounds on the appropriate influence parameter that imply that the proof technique we employ does not imply rapid mixing of systematic scan for 6-colourings of the grid when using $2 \times 2$, $2 \times 3$ and $3 \times 3$ sub-grids.

### 1.1 Preliminaries and statement of results

Let $Q = \{1, \ldots, 7\}$ be the set of colours and $V = \{1, \ldots, n\}$ the set of vertices of a finite rectangular grid $G$ with toroidal boundary conditions. Working on the torus is common practice as it avoids treating several technicalities regarding the vertices on the boundary of a finite grid as special cases and hence lets us present the proof in a more “clean” way.
We point out however that these technicalities are straightforward to deal with (more on this in Section 2). We formally say that a colouring \( \sigma \) of \( G \) is a function from \( V \) to \( Q \). Let \( \Omega^+ \) be the set of all colourings of \( G \) and \( \Omega \) be the set of all proper \( q \)-colourings. Then the distribution \( \pi \), described earlier, is the uniform distribution on \( \Omega \). If \( \sigma \in \Omega^+ \) is a colouring and \( j \in V \) is a vertex then \( \sigma_j \) denotes the colour assigned to vertex \( j \) in colouring \( \sigma \). Furthermore, for a subset of vertices \( \Lambda \subseteq V \) and a colouring \( \sigma \in \Omega^+ \) we let \( \sigma_\Lambda \) denote the colouring of the vertices in \( \Lambda \) under \( \sigma \). For each vertex \( j \in V \), let \( S_j \) denote the set of pairs \( (\sigma, \tau) \in \Omega^+ \times \Omega^+ \) of colourings that only differ on the colour assigned to vertex \( j \), that is \( \sigma_i = \tau_i \) for all \( i \neq j \).

Let \( \mathcal{M} \) be a Markov chain with state space \( \Omega^+ \) and stationary distribution \( \pi \). Suppose that the transition matrix of \( \mathcal{M} \) is \( P \). Then the mixing time from an initial colouring \( \sigma \in \Omega^+ \) is the number of steps, that is applications of \( P \), required for \( \mathcal{M} \) to become sufficiently close to \( \pi \). Formally the mixing time \( \text{Mix}(\mathcal{M}, \epsilon) \) of \( \mathcal{M} \) from an initial colouring \( \sigma \in \Omega^+ \) is defined, as a function of the deviation \( \epsilon \) from stationarity, by

\[
\text{Mix}_\sigma(\mathcal{M}, \epsilon) = \min\{t > 0 : d_{TV}(P^t(\cdot, \sigma), \pi) \leq \epsilon\},
\]

where

\[
d_{TV}(\theta_1, \theta_2) = \frac{1}{2} \sum_i |\theta_1(i) - \theta_2(i)| = \max_{A \subseteq \Omega^+} |\theta_1(A) - \theta_2(A)|
\]

is the total variation distance between two distributions \( \theta_1 \) and \( \theta_2 \) on \( \Omega^+ \). The mixing time \( \text{Mix}(\mathcal{M}, \epsilon) \) of \( \mathcal{M} \) is then obtained by maximising over all possible initial colourings

\[
\text{Mix}(\mathcal{M}, \epsilon) = \max_{\sigma \in \Omega^+} \text{Mix}_\sigma(\mathcal{M}, \epsilon).
\]

We say that \( \mathcal{M} \) is rapidly mixing if the mixing time of \( \mathcal{M} \) is polynomial in \( n \) and \( \log(\epsilon^{-1}) \).

We will make use of a recent result by Pedersen [12] to study the mixing time of a systematic scan Markov chain for \( 7 \)-colourings of the grid using block updates. We need the following notation in order to define our systematic scan Markov chain. Define the following set \( \Theta = \{\Theta_1, \ldots, \Theta_m\} \) of \( m \) blocks. Each block \( \Theta_k \subseteq V \) is a \( 2 \times 2 \) sub-grid and \( m \) is the smallest integer such that \( \bigcup_{k=1}^m \Theta_k = V \). For any block \( \Theta_k \) and a pair of colourings \( \sigma, \tau \in \Omega^+ \) we write “\( \sigma = \tau \) on \( \Theta_k \)” if \( \sigma_i = \tau_i \) for each \( i \in \Theta_k \) and similarly “\( \sigma = \tau \) off \( \Theta_k \)” if \( \sigma_i = \tau_i \) for each \( i \in V \setminus \Theta_k \). We also let \( \partial \Theta_k \) denote the set of vertices in \( V \setminus \Theta_k \) that are adjacent to some vertex in \( \Theta_k \), and we will refer to \( \partial \Theta_k \) as the boundary of \( \Theta_k \).

Note from our previous definitions that \( \sigma_{\partial \Theta_k} \) denotes the colouring of the boundary of \( \Theta_k \) under a colouring \( \sigma \in \Omega^+ \). We will refer to \( \sigma_{\partial \Theta_k} \) as a boundary colouring. Finally we say that a \( 7 \)-colouring of the \( 2 \times 2 \) sub-grid \( \Theta_k \) agrees with a boundary colouring \( \sigma_{\partial \Theta_k} \) if (1) no adjacent sites in \( \Theta_k \) are assigned the same colour and (2) each vertex \( j \in \Theta_k \) is assigned a colour that is different to the colours of all boundary vertices adjacent to \( j \).

For each block \( \Theta_k \) and colouring \( \sigma \in \Omega^+ \) let \( \Omega_k(\sigma) \) be the subset of \( \Omega^+ \) such that if \( \sigma' \in \Omega_k(\sigma) \) then \( \sigma' = \sigma \) off \( \Theta_k \) and \( \sigma'_{\partial \Theta_k} \) agrees with \( \sigma_{\partial \Theta_k} \). Let \( \pi_k(\sigma) \) be the uniform distribution on \( \Omega_k(\sigma) \). We then define \( P^{[k]} \) to be the transition matrix on the state space \( \Omega^+ \) for performing a so-called heat-bath move on \( \Theta_k \). A heat-bath move on a block \( \Theta_k \), given a colouring \( \sigma \in \Omega^+ \), is performed by drawing a new colouring from the distribution \( \pi_k(\sigma) \). Note in particular that applying \( P^{[k]} \) to a colouring \( \sigma \in \Omega^+ \) results in a colouring \( \sigma' \in \Omega^+ \) such that \( \sigma' = \sigma \) off \( \Theta_k \) and the colouring \( \sigma'_{\partial \Theta_k} \) of \( \Theta_k \) is proper and agrees with
the colouring $\sigma'_{\partial \Theta_k}$ of the boundary of $\Theta_k$ (which is identical to $\sigma_{\partial \Theta_k}$). We formally define the following systematic scan Markov chain for 7-colourings of $G$, which systematically performs heat-bath moves on $2 \times 2$ sub-grids, as follows. It is worth pointing out that this holds for any ordering of the set of blocks.

**Definition 1.** The systematic scan dynamics for 7-colourings of $G$ is a Markov chain $\mathcal{M}_\text{grid}$ with state space $\Omega^+$ and transition matrix $P_{\text{grid}} = \prod_{k=1}^{m} P^{[k]}$.

It can be shown that the stationary distribution of $\mathcal{M}_\text{grid}$ is $\pi$ by considering the construction of $P_{\text{grid}}$. It is customary to refer to one application of $P_{\text{grid}}$ (that is updating each block once) as one scan. One scan takes $\sum_k |\Theta_k|$ vertex updates and by construction of $\Theta$ this sum is clearly of order $O(n)$.

We will prove the following theorem and point out that this is the first proof of rapid mixing of systematic scan for 7-colourings on the grid.

**Theorem 2.** Let $\mathcal{M}_\text{grid}$ be the Markov chain from Definition 1 on 7-colourings of $G$. Then the mixing time of $\mathcal{M}_\text{grid}$ is

$$\text{Mix}(\mathcal{M}_\text{grid}, \varepsilon) \leq 63 \log(n \varepsilon^{-1}). \quad (4)$$

### 1.2 Context and related work

We now provide an overview of previous achievements for colourings of the grid. Previously it was known that systematic scan for $q$-colourings on general graphs with maximum vertex degree $\Delta$ mixes in $O(\log n)$ scans when $q \geq 2\Delta$ due to Pedersen [12]. That result is a hand-proof and uses block updates that updates the colour at each endpoint of an edge during each step. Earlier Dyer et al. [4] had shown that a single-site systematic scan Markov chain (where one vertex is updated at a time) mixes in $O(\log n)$ scans when $q > 2\Delta$ and in $O(n^2 \log n)$ scans when $q = 2\Delta$. It is hence well-established that systematic scan is rapidly mixing for $q$-colourings of the grid when $q \geq 8$ but nothing has been known about the mixing time for smaller $q$. The results of both Pedersen [12] and Dyer et al. [4] bound the mixing time by studying the influence on a vertex. We will use that technique in this paper as well, however we will construct the required couplings using a heuristic. We defer the required definitions to Section 2 which also contains the proof of Theorem 2.

Recent results have revealed that, in a single-site setting, one is not restricted use the total influence on a vertex when analysing the mixing time of systematic scan by bounding influence parameters. In a single-site setting one can define an $n \times n$-matrix whose entries are the influences that all vertices have on each other. Hayes [10] has shown that providing a sufficiently small upper bound on the spectral gap of this matrix implies rapid mixing of both systematic scan and random update. Dyer, Goldberg and Jerrum [6] furthermore showed that an upper bound on any matrix norm also implies rapid mixing of both types of Markov chains. These techniques are however not known to apply to Markov chains using block moves. See the PhD thesis by Pedersen [13] for more comprehensive review of the above results and for the difficulties in extending them to cover block dynamics.

As random update Markov chains have received more attention than systematic scan we also summarise some mixing results of interest regarding $q$-colourings of the grid (recall
that a random update Markov chain selects randomly a subset of sites to be updated at each step). Achlioptas et al. [1] give a computer-assisted proof of mixing in $O(n \log n)$ updates when $q = 6$ by considering blocks consisting of $2 \times 3$ sub-grids. Our computations are similar in nature to the ones of Achlioptas et al. however their computations are not sufficient to imply mixing of systematic scan as we will discuss in due course. More recently Goldberg, Martin and Paterson [9] gave a hand-proof of mixing in $O(n \log n)$ updates when $q \geq 7$ using the technique of strong spatial mixing. Previously Salas and Sokal [14] gave a computer-assisted proof of the $q = 7$ case, a result which was also implied by another computer-assisted result due to Bubley, Dyer and Greenhill [3] that applies to 4-regular triangle-free graphs. Finally it is worth pointing out that, in the special case when $q = 3$, two complementary results of Luby, Randall and Sinclair [11] and Goldberg, Martin and Paterson [8] give rapid mixing of random update.

2 Bounding the mixing time of systematic scan

This section will contain a proof of Theorem 2 although the proof of a crucial lemma, which requires computer-assistance, is deferred to Section 3. We will bound the mixing time of $M_{\text{grid}}$ by bounding the influence on a vertex, a parameter which we denote by $\alpha$ and will define formally in due course. If $\alpha$ is sufficiently small then Theorem 2 from Pedersen [12] implies that any systematic scan Markov chain, whose transition matrices for updating each block satisfy two simple properties, mixes in $O(\log n)$ scans. For completeness we restate this theorem (Theorem 3 below) and in the statement we let $M_{\rightarrow}$ denote a systematic scan Markov chain whose transition matrices for each block update satisfy the required properties.

**Theorem 3.** If $\alpha < 1$ then the mixing time of $M_{\rightarrow}$ is

$$\text{Mix}(M_{\rightarrow}, \varepsilon) \leq \frac{\log(n\varepsilon^{-1})}{1 - \alpha}.$$  

(5)

For each block $\Theta_k$ the transition matrix $P^{[k]}$ needs to satisfy the following two properties in order for Theorem 3 to apply.

1. If $P^{[k]}(\sigma, \tau) > 0$ then $\sigma = \tau$ off $\Theta_k$, and

2. $\pi$ is invariant with respect to $P^{[k]}$.

It is pointed out in Pedersen [12] that if $P^{[k]}$ is a transition matrix performing a heat-bath move then both of these properties are easily satisfied. Furthermore, it is pointed out that when $\Omega$ is the set of proper colourings of a graph, then $\pi$ is the uniform distribution on $\Omega$ as we require. Since the transition matrices $P^{[k]}$ used in the definition of $M_{\text{grid}}$ perform heat-bath updates we are hence able to use Theorem 3 to bound the mixing time of $M_{\text{grid}}$.

We are now ready to formally define the parameter $\alpha$ denoting the influence on a vertex. For any pair of colourings $(\sigma, \tau) \in S_i$ let $\Psi_k(\sigma, \tau)$ be a coupling of the distributions induced by $P^{[k]}(\sigma, \cdot)$ and $P^{[k]}(\tau, \cdot)$, namely $\pi_k(\sigma)$ and $\pi_k(\tau)$ respectively. We remind the reader that a coupling of two distributions $\pi_1$ and $\pi_2$ on state space $\Omega^+$ is a joint
distribution $\Omega^+ \times \Omega^+$ such that the marginal distributions are $\pi_1$ and $\pi_2$. For ease of reference we also let $p_j(\Psi_k(\sigma, \tau))$ denote the probability that a vertex $j \in \Theta_k$ is assigned a different colour in a pair of colourings drawn from some coupling $\Psi_k(\sigma, \tau)$. We then let

$$\rho_{i,j}^k = \max_{(\sigma, \tau) \in S_i} p_j(\Psi_k(\sigma, \tau))$$

be the influence of $i$ on $j$ under $\Theta_k$. Finally the parameter $\alpha$ denoting the influence on any vertex is defined as

$$\alpha = \max_{k} \max_{j \in \Theta_k} \sum_{i} \rho_{i,j}^k.$$  \hspace{1cm} (7)

Pedersen [12] actually defines $\alpha$ with a weight associated with each vertex, however as we will not use weights in our proof we have omitted them from the above account. So, in order to upper bound $\alpha$ we are required to upper bound the probability of a discrepancy at each vertex $j \in \Theta_k$ under a coupling $\Psi_k(\sigma, \tau)$ of the distributions $\pi_k(\sigma)$ and $\pi_k(\tau)$ for any pair of colourings $(\sigma, \tau) \in S_i$ that only differ at the colour of vertex $i$. Our main task is hence to specify a coupling $\Psi_k(\sigma, \tau)$ of $\pi_k(\sigma)$ and $\pi_k(\tau)$ for each pair of colourings $(\sigma, \tau) \in S_i$ and upper bound the probability of assigning a different colour to each vertex in a pair of colourings drawn from that coupling.

Consider any block $\Theta_k$ and any pair of colourings $(\sigma, \tau) \in S_i$ that differ only on the colour assigned to some vertex $i$. Clearly the distribution on colourings of $\Theta_k$, induced by $\pi_k(\sigma)$ only depends on the boundary colouring $\sigma_{\partial \Theta_k}$. Similarly, the distribution on colourings of $\Theta_k$, induced by $\pi_k(\tau)$ depends only on $\tau_{\partial \Theta_k}$. If $i \notin \partial \Theta_k$ then the distributions on the colourings of $\Theta_k$, induced by $\pi_k(\sigma)$ and $\pi_k(\tau)$, respectively, are the same and we let $\Psi_k(\sigma, \tau)$ be the coupling in which any pair of colourings drawn from $\Psi_k(\sigma, \tau)$ agree on $\Theta_k$. That is, if the pair $(\sigma', \tau')$ of colourings are drawn from $\Psi_k(\sigma, \tau)$ then $\sigma' = \sigma$ off $\Theta_k, \tau' = \tau$ off $\Theta_k$ and $\sigma' = \tau'$ on $\Theta_k$. This gives $\rho_{i,j}^k = 0$ for any $i \notin \partial \Theta_k$ and $j \in \Theta_k$.

We now need to construct $\Psi_k(\sigma, \tau)$ for the case when $i \in \partial \Theta_k$. For each $j \in \Theta_k$ we need $p_j(\Psi_k(\sigma, \tau))$ to be sufficiently small in order to avoid $\rho_{i,j}^k$ being too big. If the $\rho_{i,j}^k$-values are too big the parameter $\alpha$ will be too big (that is greater than one) and we cannot make use of Theorem 3 to show rapid mixing. Constructing $\Psi_k(\sigma, \tau)$ by hand such that $p_j(\Psi_k(\sigma, \tau))$ is sufficiently small is a difficult task. It is, however, straightforward to control the parameter $\alpha$ for a given pair of boundary colourings $\sigma_{\partial \Theta_k}$ and $\tau_{\partial \Theta_k}$. From these distributions we can then use some suitable heuristic to construct a coupling that is good enough for our purposes. We hence need to construct a specific coupling for each individual pair of colourings differing only at a single vertex. In order to do this we will make use of the following lemma, which is proved in Section 3.

**Lemma 4.** Let $v_1, \ldots, v_4$ be the four vertices in a $2 \times 2$-block and $z_1, \ldots, z_8$ be the boundary vertices of the block and let the labeling be as in Figure 7. Let $Z$ and $Z'$ be any two $7$-colourings of the boundary vertices such that $Z$ and $Z'$ agree on each vertex except on $z_1$. Let $\pi_Z$ and $\pi_{Z'}$ be the uniform distributions on proper $7$-colourings of the block that agree with $Z$ and $Z'$, respectively. For $i = 1, \ldots, 4$ let $p_{v_i}(\Psi)$ denote the probability that the colour of vertex $v_i$ differ in a pair of colourings drawn from a coupling $\Psi$ of $\pi_Z$ and $\pi_{Z'}$. Then there exists a coupling $\Psi$ such that $p_{v_1}(\Psi) < 0.283, p_{v_2}(\Psi) < 0.079, p_{v_3}(\Psi) < 0.051$ and $p_{v_4}(\Psi) < 0.079$. 

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Figure 1: General labeling of the vertices in a 2×2-block \( \Theta_k \) and the vertices \( \partial \Theta_k \) on the boundary of the block.

![Labeling of vertices](image)

Figure 2: A 2×2-block \( \Theta_k \) showing all eight positions of a vertex \( i \in \partial \Theta_k \) on the boundary of the block in relation to a vertex \( j \in \Theta_k \) in the block.

![Positions of vertices](image)

Thus if \( i \in \partial \Theta_k \) we let \( \Psi_k(\sigma, \tau) \) be the coupling of \( \pi_k(\sigma) \) and \( \pi_k(\tau) \) that draws the colouring of \( \Theta_k \) from the coupling \( \Psi \) in Lemma 4, where \( Z \) is the boundary colouring obtained from \( \sigma_{\partial \Theta_k} \) and \( Z' \) is obtained from \( \tau_{\partial \Theta_k} \), and leaves the colour of the remaining vertices, \( V \setminus \Theta_k \), unchanged. That is, if the pair \((\sigma', \tau')\) of colourings are drawn from \( \Psi_k(\sigma, \tau) \) then \( \sigma' = \sigma \) off \( \Theta_k \), \( \tau' = \tau \) off \( \Theta_k \) and the colourings of \( \Theta_k \) in \( \sigma' \) and \( \tau' \) are drawn from the coupling \( \Psi \) in Lemma 4 (see the proof for details on how to construct \( \Psi \)). It is straightforward to verify that this is indeed a coupling of \( \pi_k(\sigma) \) and \( \pi_k(\tau) \). Note that due to the symmetry of the 2×2-block, with respect to rotation and mirroring, we can always label the vertices of \( \Theta_k \) and \( \partial \Theta_k \) such that label \( z_1 \) in Figure 1 represents the discrepancy vertex \( i \) on the boundary. Hence we can make use of Lemma 4 to compute upper bounds on the parameters \( \rho_{i,j}^k \). We summarise the \( \rho_{i,j}^k \)-values in the following Corollary of Lemma 4. Note that due to the symmetry of the block we can assume that vertex \( j \in \Theta_k \) in the corollary is located in the bottom left corner, as Figure 2 shows.

**Corollary 5.** Let \( \Theta_k \) be any 2×2-block, let \( j \in \Theta_k \) be any vertex in the block and let \( i \in \partial \Theta_k \) be a vertex on the boundary of the block. Then

\[
\rho_{i,j}^k = \max_{(\sigma, \tau) \in \mathcal{S}_i} p_j(\Psi_k(\sigma, \tau)) < \begin{cases} 
0.283, & \text{if } i \text{ and } j \text{ as in Figure 2(a) or (b)}, \\
0.079, & \text{if } i \text{ and } j \text{ as in Figure 2(c) or (h)}, \\
0.051, & \text{if } i \text{ and } j \text{ as in Figure 2(e) or (f)}, \\
0.079, & \text{if } i \text{ and } j \text{ as in Figure 2(d) or (g)}. 
\end{cases}
\]

If \( i \notin \partial \Theta_k \) is not on the boundary of the block then \( \rho_{i,j}^k = 0 \).

We can then use Corollary 5 to prove Theorem 2. The proof of Theorem 2 is given here:
Proof of Theorem \[2.\] Let \( \alpha_{k,j} = \sum_i \rho_{k,i,j}^j \) be the influence on \( j \) under \( \Theta_k \). We need \( \alpha_{k,j} \) to be upper bounded by one for each block \( \Theta_k \) and vertex \( j \in \Theta_k \) in order to ensure that \( \alpha = \max_k \max_{j \in \Theta_k} \alpha_{k,j} \) is less than one. Fix any block \( \Theta_k \) and any vertex \( j \in \Theta_k \). A vertex \( i \in \partial \Theta_k \) on the boundary of the block can occupy eight different positions on the boundary in relation to \( j \) as showed in Figure 2(a)–(h). Recall that we are working on the torus, and hence every vertex on the boundary of the block will belong to \( G \). Thus, using the bounds from Corollary \[5.\] we have

\[
\alpha_{k,j} = \sum_i \rho_{k,i,j}^j < 2(0.283 + 0.079 + 0.051 + 0.079) = 0.984. \tag{9}
\]

Then \( \alpha = \max_k \max_{j \in \Theta_k} \alpha_{k,j} < \max_k 0.984 = 0.984 < 1 \) and we obtain the stated bound on the mixing time of \( M_{\text{grid}} \) by Theorem \[3.\]

We make the following remark. In the proof of Theorem \[2.\] above, we assume that \( G \) is a finite rectangular grid with toroidal boundary conditions. Hence, every block is a \( 2 \times 2 \)-sub-grid and each vertex on the block boundary belongs to \( V \). We note that if \( G \) is a finite rectangular grid without toroidal boundary conditions then some vertices on the boundary \( \partial \Theta_k \) of a block \( \Theta_k \) might fall outside \( G \). The sum in Equation \[9.\] is over boundary vertices \( i \) that do belong to \( V \), and hence the number of terms in this sum is reduced if some boundary vertices do not belong to \( V \), making \( \alpha \) smaller. Furthermore, if \( G \) is a non-rectangular region of the grid then a block next to the boundary might be smaller than \( 2 \times 2 \) vertices. Suppose \( \Theta_k \) is a block that is smaller than \( 2 \times 2 \) vertices. Then the vertices that are missing in order to make \( \Theta_k \) a full \( 2 \times 2 \)-block are boundary vertices. Suppose \( i \in \partial \Theta_k \) belongs to \( V \) and \( i' \in \partial \Theta_k \) does not belong to \( V \). When constructing couplings \( \Psi_k(\sigma, \tau) \), where \( (\sigma, \tau) \in S_i \), we must consider the vertex \( i' \) as “colourless”, which would decrease the value of \( \rho_{k,i,j}^j \). A more rigorous analysis yields that our mixing result with seven colours and \( 2 \times 2 \)-blocks holds for arbitrary finite regions \( G \) of the grid.

Of course we have yet to establish a proof of Lemma \[4.\] and the rest of this paper will be concerned with this. Our method of proof uses some ideas of Goldberg, Jalsenius, Martin and Paterson \[7.\] in so far as it is computer assisted and we will be focusing on minimising the probability of assigning different colours to vertex \( v_1 \) in the constructed couplings. We will however be required to construct a coupling on the \( 2 \times 2 \) sub-grid, rather than establishing bounds on the disagreement probability of a vertex adjacent to the initial discrepancy and then extending this to a coupling on the whole block recursively. Our approach is similar to the one Achlioptas et al. \[1.\] take, however we do not have the option of constructing an “optimal” coupling using a suitable linear program (even when feasible) since our probabilities will be maximised over all boundary colourings. The crucial difference between the approaches is that Achlioptas et al. \[1.\] are using path coupling (see Bubley and Dyer \[2.\]) as a proof technique which requires them to bound the expected Hamming distance between a pair of colourings drawn from a coupling. This in turn enables them to, for a given boundary colouring, specify an “optimal” coupling which minimises Hamming distance. We are, however, required to bound the influence of \( i \) on \( j \) for each boundary colouring and sum over the maximum of these influences. The reason for this is the inherit maximisation over boundary colourings in the definition of \( \rho_{k,i,j}^j \) as described above.

Finally it is worth mentioning that providing bounds on the expected Hamming distance is similar to showing that the influence of a vertex is small and it is known that
this condition implies rapid mixing of a random update Markov chain, see for example Weitz [15]. In a single-site setting the condition “the influence of a vertex is small” also implies rapid mixing of systematic scan (Dyer et al. [4]), however, in a block setting this condition is not sufficient to give rapid mixing of systematic scan (Pedersen [13]), which is why we need to bound the influence on a vertex.

3 Constructing the coupling by machine

In order to prove Lemma 4 we will construct a coupling $\Psi$ of $\pi_Z$ and $\pi_Z'$ for all pairs of boundary colourings $Z$ and $Z'$ that are identical on all boundary vertices but vertex $z_1$, on which $Z$ and $Z'$ differ. For each coupling constructed we verify that the probabilities $p_{v_i}(\Psi), i = 1, \ldots, 4$, are within the bounds of the lemma. The method is well suited to be carried out with the help of a computer and we have implemented a program in C to do so. Before stating the proof of Lemma 4 we will discuss how a coupling can be represented by an edge-weighted complete bipartite graph. We make use of this representation of $\Psi$ in the proof of the lemma.

3.1 Representing a coupling as a bipartite graph

Let $S$ be a set of objects and let $W$ be a set of pairs $(s, w_s)$ such that $s \in S$ and $w_s \geq 0$ is a non-negative value representing the weight of $s$. Each element $s \in S$ is contained in exactly one of the pairs in $W$. If the value $w_s$ is an integer (which it is in our case) it can be regarded as the multiplicity of $s$ in a multiset. The set $W$ is referred to as a weighted set of $S$. Let $\pi_{S,W}$ be the distribution on $S$ such that the probability of $s$ is proportional to $w_s$, where $(s, w_s)$ is a pair in $W$. More precisely, the probability of $s$ in $\pi_{S,W}$ is $P_{\pi_{S,W}}(s) = w_s / \sum_{(t, w_t) \in W} w_t$. For example, let $W$ be a weighted set of $S$ and let $S' \subseteq S$ be a subset of $S$. Assume the weight $w_s = 0$ if $s \in S \setminus S'$ and $w_s = k$ if $s \in S'$, where $k > 0$ is a positive constant. Then $\pi_{S,W}$ is the uniform distribution on $S'$.

The reason for introducing the notion of a weighted set is that it can be used when specifying a coupling of two distributions. Let $S$ be a set and let $W$ and $W'$ be two weighted sets of $S$ such that the sum of the weights in $W$ equals the sum of the weights in $W'$. Let $w_{\text{tot}}$ denote this sum. That is, $w_{\text{tot}} = \sum_{(s, w_s) \in W} w_s = \sum_{(s', w_{s'}) \in W'} w_{s'}$. The two weighted sets $W$ and $W'$ define two distributions $\pi_{S,W}$ and $\pi_{S,W'}$. Let $K_{[S],[S]}$ be an edge-weighted complete bipartite graph with vertex sets $W$ and $W'$. That is, for each pair $(s, w_s) \in W$ there is an edge to every pair in $W'$. Every edge $e$ of $K_{[S],[S]}$ has a weight $w_e \geq 0$ such that the following condition holds. Let $(s, w_s)$ be any pair in $W \cup W'$ and let $E$ be the set of all $|S|$ edges incident to $(s, w_s)$. Then $\sum_{e \in E} w_e = w_s$. It follows that the sum of the edge weights of all $|S|^2$ edges in $K_{[S],[S]}$ equals $w_{\text{tot}}$, the sum of the weights in $W$ (and $W'$). The idea is that $K_{[S],[S]}$ represents a coupling $\Psi$ of $\pi_{S,W}$ and $\pi_{S,W'}$. In order to draw a pair of elements from $\Psi$ we randomly select an edge $e$ in $K_{[S],[S]}$ proportional to its weight. The endpoints of $e$ represent the elements in $S$ drawn from $\pi_{S,W}$ and $\pi_{S,W'}$. More precisely, the probability of choosing edge $e$ in $K_{[S],[S]}$ with weight $w_e$ is $w_e / w_{\text{tot}}$. If edge $e = ((s, w_s), (s', w_{s'}))$ is chosen it means that we have drawn $s$ from $\pi_{S,W}$ and $s'$ from $\pi_{S,W'}$, the marginal distributions of $\Psi$.  

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The bipartite graph representation of a coupling will be used when we construct couplings of colourings of $2 \times 2$-blocks in the proof of Lemma 4.

### 3.2 The proof of Lemma 4

Here is the proof of Lemma 4.

**Proof of Lemma 4.** Fix two colourings $Z$ and $Z'$ of the boundary that differ on vertex $z_1$. Let $c$ be the colour of vertex $z_1$ in $Z$ and let $c' \neq c$ be the colour of $z_1$ in $Z'$. Let $C_1$ and $C_2$ be the two sets of proper 7-colourings of the block that agree with $Z$ and $Z'$, respectively. Let $C^+$ be the set of all 7-colourings of the block. Let $W_1$ and $W_2$ be two weighted sets of $C^+$. The weights are assigned as follows.

- For the pair $(\sigma, w_\sigma) \in W_1$ let the weight $w_\sigma = |C_1|$ if $\sigma \in C_1$, otherwise let $w_\sigma = 0$.
- For the pair $(\sigma, w_\sigma) \in W_2$ let the weight $w_\sigma = |C_2|$ if $\sigma \in C_2$, otherwise let $w_\sigma = 0$.

It follows from the assignment of the weights that the distribution $\pi_{C^+,W_1}$ is the uniform distribution on $C_1$. That is, $\pi_{C^+,W_1} = \pi_Z$. Similarly, $\pi_{C^+,W_2}$ is the uniform distribution on $C_2$. Note that the sum of the weights is $|C_1||C_2|$ in both $W_1$ and $W_2$. Then a coupling $\Psi$ of $\pi_{C^+,W_1}$ and $\pi_{C^+,W_2}$ can be specified with an edge-weighted complete bipartite graph $K = K_{|C_1|,|C_2|}$. For a given valid assignment of the weights of the edges of $K$, making $K$ represent a coupling $\Psi$, we can compute the probabilities of having a mismatch on a vertex $v_i$ of the block when two colourings are drawn from $\Psi$. Let $E$ be the set of all edges $e = ((\sigma, w_\sigma), (\sigma', w_{\sigma'}))$ in $K$ such that $\sigma$ and $\sigma'$ differ on vertex $v_i$. Then $p_{v_i}(\Psi) = \sum_{e \in E} w_e / |C_1||C_2|$.

In order to obtain sufficiently small upper bounds on $p_{v_i}(\Psi)$ for the four vertices $v_1, \ldots, v_4$ in the block we would like to assign weights to the edges of $K$ such that much weight is assigned to edges between colourings that agree on many vertices in the block. In general it is not clear exactly how to assign weights to the edges. For instance, if we assign too much weight to edges between colourings that are identical on vertex $v_2$ we might not be able to assign as much weight as we would like to on edges between colourings that are identical on vertex $v_1$. Thus, the probability of having a mismatch on $v_1$ would increase. Intuitively a good strategy would be to assign as much weight as possible to edges between colourings that are identical on the whole block. This implies that we try to assign as much weight as possible to edges between colourings that are identical on vertex $v_1$, the vertex adjacent to the discrepancy vertex $z_1$ on the boundary. If there is a mismatch on vertex $v_1$ it should be a good idea to assign as much weight as possible to edges between colourings that are identical on the whole block apart from vertex $v_1$. This idea leads to a heuristic in which the assignment of the edge weights is divided into three phases. The exact procedure is described as follows.

In phase one we match identical colourings. For all colourings $\sigma \in C^+$ of the block the edge $e = ((\sigma, w_\sigma), (\sigma, w'_\sigma))$ in $K$ will be given weight $w_e = \min(w_\sigma, w'_\sigma)$. That is, we maximise the probability of drawing the same colouring $\sigma$ from both $\pi_{C^+,W_1}$ and $\pi_{C^+,W_2}$.

For the following two phases we define an ordering of the colourings in $C^+$. We order the colourings lexicographically with respect to the vertex order $v_3, v_2, v_4, v_1$. That is, if the seven colours are $1, \ldots, 7$ the colouring of $v_3, v_2, v_4, v_1$ will start with $1, 1, 1, 1,$
respectively. The next colouring will be $1, 1, 1, 2$, and so on. This ordering of colourings in $C^+$ carries over to an ordering of the pairs in $W_Z$ and $W_{Z'}$. That is, we order the pairs $(\sigma, w_\sigma)$ in $W_Z$ with respect to the lexicographical ordering of $\sigma$. Similarly we order the pairs in $W_{Z'}$. This ordering of the pairs will be important in the next two phases. It provides some control of how colourings are being paired up in terms of the assignment of the weights on edges between pairs. Edges will be considered with respect to this ordering because choosing an arbitrary ordering of the edges would not necessarily result in probabilities $p_{v_i}(\Psi)$ that would be within the bounds of the lemma.

In the second phase we ignore the colour of vertex $v_1$ and match colourings that are identical on all of the remaining three vertices $v_2$, $v_3$ and $v_4$. More precisely, for each pair $(\sigma, w_\sigma) \in W_Z$, considered in the ordering explained above, we consider the edges $e = (\sigma, w_\sigma', (\sigma', w_\sigma'))$ where $\sigma$ and $\sigma'$ are identical on all vertices but $v_1$. The edges are considered in the ordering of the second component $(\sigma, w_\sigma') \in W_{Z'}$. We assign as much weight as possible to $e$ such that the total weight on edges incident to $(\sigma, w_\sigma) \in W_Z$ does not exceed $w_\sigma$ and such that the total weight on edges incident to $(\sigma', w_\sigma') \in W_{Z'}$ does not exceed $w_\sigma'$. Note that in the lexicographical ordering of the colourings, vertex $v_1$ is the least significant vertex and therefore the ordering provides some level of control of pairing up colourings that are similar on the remaining three vertices. It turns out that the resulting coupling is sufficiently good for proving the lemma.

In the third and last phase we assign the remaining weights on the edges. As in phase two, for each pair $(\sigma, w_\sigma) \in W_Z$ we consider the edges $e = (\sigma, w_\sigma, (\sigma', w_\sigma'))$. The pairs and edges are considered in accordance with the ordering explained above. The difference between the second and third phase is that now we do not have any restrictions on the colourings $\sigma$ and $\sigma'$. We assign as much weight as possible to $e$ such that the total weight on edges incident to $(\sigma, w_\sigma) \in W_Z$ does not exceed $w_\sigma$ and such that the total weight on edges incident to $(\sigma', w_\sigma') \in W_{Z'}$ does not exceed $w_\sigma'$. After phase three we have assigned all weights to the edges of $K$ and hence $K$ represents a coupling $\Psi$ of $\pi_Z$ and $\pi_{Z'}$.

From $K$ we compute the probabilities $p_{v_1}(\Psi), p_{v_2}(\Psi), p_{v_3}(\Psi)$ and $p_{v_4}(\Psi)$ as described above. We have written a C-program which loops through all colourings $Z$ and $Z'$ of the boundary of the block and constructs the bipartite graph $K$ as described above. For each boundary the probabilities $p_{v_1}(\Psi), p_{v_2}(\Psi), p_{v_3}(\Psi)$ and $p_{v_4}(\Psi)$ are successfully verified to be within the bounds of the lemma. For details on the C-program, see http://www.csc.liv.ac.uk/~markus/systematicscan/.

\begin{proof}

4 Partial results for 6-colourings of the grid

In previous sections we have seen that systematic scan on the grid using $2 \times 2$-blocks and seven colours mixes rapidly. An immediate question is whether we can do better and show rapid mixing with six colours. This matter will be discussed in this section and we will show that, even with bigger block sizes (up to $3 \times 3$), it is not possible to show rapid mixing using the technique of this paper. More precisely, we will establish lower bounds on the parameter $\alpha$ for $2 \times 2$-blocks, $2 \times 3$-blocks and $3 \times 3$-blocks. All three lower bounds are greater than one and hence we cannot make use of Theorem 3 to show rapid mixing.
4.1 Establishing lower bounds for $2 \times 2$ blocks

We start by examining the $2 \times 2$-block again but this time with six colours. Lemma 4 provides upper bounds (under any colourings of the boundary) on the probabilities of having discrepancies at each of the four vertices of the block when two 7-colourings are drawn from the specified coupling. For six colours we will show lower bounds on these probabilities under any coupling and a specified pair of boundary colourings. Once again, let $v_1, \ldots, v_4$ be the four vertices in a $2 \times 2$-block and let $z_1, \ldots, z_8$ be the boundary vertices of the block and let the labeling be as in Figure 4. Let $Z$ and $Z'$ be any two 6-colourings of the boundary vertices that assign the same colour to each vertex except for $z_1$. Let $\pi_Z$ and $\pi_{Z'}$ be the uniform distributions on the sets of proper 6-colourings of the block that agree with $Z$ and $Z'$, respectively. Let $\Psi_{v_k}^\text{min}(Z,Z')$ be a coupling of $\pi_Z$ and $\pi_{Z'}$ that minimises $p_{v_k}(\Psi)$. That is, $p_{v_k}(\Psi) \geq p_{v_k}(\Psi_{v_k}^\text{min}(Z,Z'))$ for all couplings $\Psi$ of $\pi_Z$ and $\pi_{Z'}$. Also let $p_{v_k}^\text{low} = \max_{Z,Z'} p_{v_k}(\Psi_{v_k}^\text{min}(Z,Z'))$. We can hence say that there exist two 6-colourings $Z$ and $Z'$ of the boundary of a $2 \times 2$ block, that assign the same colour to each vertex except for $z_1$, such that $p_{v_k}(\Psi) \geq p_{v_k}^\text{low}$ for any coupling $\Psi$ of $\pi_Z$ and $\pi_{Z'}$. We have the following lemma, which is proved by computation.

**Lemma 6.** Consider 6-colourings of the $2 \times 2$-block in Figure 4. Then $p_{v_1}^\text{low} \geq 0.379$, $p_{v_2}^\text{low} \geq 0.107$, $p_{v_3}^\text{low} \geq 0.050$ and $p_{v_4}^\text{low} \geq 0.107$.

**Proof.** Fix one vertex $v_k$ in the block and fix two colourings $Z$ and $Z'$ of the boundary of the block that differ only on the colour of vertex $z_1$. Let $C_Z$ and $C_{Z'}$ be the two sets of proper 6-colourings of the block that agree with $Z$ and $Z'$, respectively. For $c = 1, \ldots, 6$ let $n_c$ be the number of colourings in $C_Z$ in which vertex $v_k$ is assigned colour $c$. Similarly let $n'_c$ be the number of colourings in $C_{Z'}$ in which vertex $v_k$ is assigned colour $c$. It is clear that the probability that $v_k$ is assigned colour $c$ in a colouring $\sigma$ drawn from $\pi_Z$ is $\Pr_{\pi_Z}(\sigma_{v_k} = c) = n_c/|C_Z|$. For $c = 1, \ldots, 6$ define $m_c = n_c|C_{Z'}|$, $m'_c = n'_c|C_Z|$ and $M = |C_Z||C_{Z'}|$. It follows that $\Pr_{\pi_Z}(\sigma_{v_k} = c) = m_c/M$ and $\Pr_{\pi_{Z'}}(\sigma'_{v_k} = c) = m'_c/M$, where $\sigma'$ and $\sigma'$ are colourings drawn from $\pi_Z$ and $\pi_{Z'}$, respectively. Observe that the quantities $m_c$, $m'_c$ and $M$ can be easily computed for a given pair of boundary colourings.

Now let $\Psi$ be any coupling of $\pi_Z$ and $\pi_{Z'}$. It is easy to see that the probability that vertex $v_k$ is coloured $c$ in both colourings drawn from $\Psi$ can be at most $\min(m_c, m'_c)/M$. Therefore, the probability of drawing two colourings from $\Psi$ such that the colour of vertex $v_k$ is the same in both colourings is at most $\sum_{c=1,\ldots,6} \min(m_c, m'_c)/M$, and the probability of assigning different colours to vertex $v_k$ is at least $p_{v_k}(\Psi) \geq 1 - \sum_{c=1,\ldots,6} \min(m_c, m'_c)/M$. We have successfully verified the bounds in the statement of the lemma by maximising the lower bound on $p_{v_k}(\Psi)$ over all boundary colourings $Z$ and $Z'$ for each vertex $v_k$ in the block. The computations are carried out with the help of a computer program written in C. For details on the program, see http://www.csc.liv.ac.uk/~markus/systematics.c.

For seven colours, Corollary 5 makes use of Lemma 4 to establish upper bounds on the influence parameters $\rho_{i,j}^k$. These parameters are used in the proof of Theorem 2 to obtain an upper bound on the parameter $\alpha$. The upper bound on $\alpha$ is shown to be less than one which implies rapid mixing for seven colours when applying Theorem 3. We can use Lemma 6 to obtain lower bounds on the influence parameters $\rho_{i,j}^k$ by completing the coupling in a way analogous to the coupling in Corollary 5. This in turn will result in a
lower bound on the parameter $\alpha$ that is greater than one. That is, following the proof of Theorem 2 and making use of Lemma 6, a lower bound on $\alpha$ will be

$$\alpha \geq 2(0.379 + 0.107 + 0.050 + 0.107) = 1.286 > 1.$$  

(10)

Hence we fail to show rapid mixing of systematic scan with six colours using $2 \times 2$-blocks.

4.2 Bigger blocks

We failed to show rapid mixing of systematic scan with six colours and $2 \times 2$-blocks and we will now show that increasing the block size to both $2 \times 3$ and $3 \times 3$ will not be sufficient either. Lemma 7 below considers $2 \times 3$-blocks and is analogous to Lemma 6. We make use of the same notation as for Lemma 6, only the block is bigger and the labeling of the vertices is different (see Figure 3(a)). Lemma 7 is proved by computation in the same way as Lemma 6. For details on the C-program used in the proof, see http://www.csc.liv.ac.uk/~markus/systematicscan/.

**Lemma 7.** Consider 6-colourings of the $2 \times 3$-block in Figure 3(a). Then $p_{v_1}^{low} \geq 0.3671$, $p_{v_3}^{low} \geq 0.0298$, $p_{v_4}^{low} \geq 0.0997$ and $p_{v_6}^{low} \geq 0.0174$.

We will now use Lemma 7 to show that $\alpha > 1$ for $2 \times 3$ blocks. Let $\Theta_k$ be any $2 \times 3$-block and let $j \in \Theta_k$ be a vertex in a corner of the block. A vertex $i \in \partial \Theta_k$ on the boundary of the block can occupy ten different positions on the boundary in relation to $j$. See Figure 3(b) and (c). We can again determine lower bounds on the influences $\rho_{i,j}^k$ of $i$ on $j$ under $\Theta_k$ from Lemma 7. However, Lemma 7 provides lower bounds on $\rho_{i,j}^k$ only when $i \in \partial \Theta_k$ is adjacent to a corner vertex of the block, as in Figure 3(b). If $i$ is located as in Figure 3(c) we do not know more than that $\rho_{i,j}^k$ is bounded from below by zero. Nevertheless, the lower bound on $\alpha$ exceeds one. Let $\alpha_{k,j} = \sum_i \rho_{i,j}^k$ be the influence on $j$ under $\Theta_k$. Following the proof of Theorem 2 and using the lower bounds in Lemma 7 we have

$$\alpha_{k,j} = \sum_{i \text{ in Fig. 3(b)}} \rho_{i,j}^k + \sum_{i \text{ in Fig. 3(c)}} \rho_{i,j}^k \geq 2(0.3671 + 0.0298 + 0.0997 + 0.0174) = 1.028,$$

(11)

where we set the lower bound on the second sum to zero. Now,

$$\alpha = \max_k \max_{j \in \Theta_k} \alpha_{k,j} \geq 1.028 > 1.$$  

(12)
Figure 4: (a)–(b) General labeling of the vertices in a 3×3-block $\Theta_k$ and two different labellings of the vertices $\partial \Theta_k$ on the boundary of the block. The discrepancy vertex on the boundary has label $z_1$. (b)–(c) All twelve positions of a vertex $i \in \partial \Theta_k$ on the boundary of the block in relation to a vertex $j \in \Theta_k$ in the corner of the block.

Hence we cannot use Theorem 3 to show rapid mixing of systematic scan with six colours and 2×3-blocks. It is interesting to note that considering 2×3-blocks was sufficient for Achlioptas et al. [1] to prove mixing of a random update Markov chain for sampling 6-colourings of the grid.

Lastly, we increase the block size to 3×3 and show that a lower bound on $\alpha$ is still greater than one. We have the following lemma which is proved by computation in the same way as Lemmas 6 and 7. For details on the C-program used in the proof see http://www.csc.liv.ac.uk/~markus/systematicscan/.

**Lemma 8.** For 6-colourings of the 3×3-block with vertices labeled as in Figure 4(a) we have $p_{v_2}^{\text{low}} \geq 0.3537$, $p_{v_3}^{\text{low}} \geq 0.0245$, $p_{v_5}^{\text{low}} \geq 0.0245$ and $p_{v_7}^{\text{low}} \geq 0.0071$. Furthermore, for 6-colourings of the 3×3-block in Figure 4(b) we have $p_{v_1}^{\text{low}} \geq 0.0838$, $p_{v_3}^{\text{low}} \geq 0.0838$, $p_{v_5}^{\text{low}} \geq 0.0138$ and $p_{v_9}^{\text{low}} \geq 0.0138$.

Note that Lemma 8 provides lower bounds on the probabilities of having a mismatch on a corner vertex of the block when the discrepancy vertex on the boundary (labeled $z_1$) is adjacent to a corner vertex (Figure 4(a)) and adjacent to a middle vertex (Figure 4(b)). Let $\Theta_k$ be any 3×3-block and let $j \in \Theta_k$ be a vertex in a corner of the block. A vertex $i \in \partial \Theta_k$ on the boundary of the block can occupy twelve different positions on the boundary in relation to $j$. See Figure 4(c) and (d). Analogous to Corollary 5 lower bounds on the influences $\rho_{i,j}^k$ of $i$ on $j$ under $\Theta_k$ can be determined from Lemma 8. Let $\alpha_{k,j} = \sum_i \rho_{i,j}^k$ be the influence on $j$ under $\Theta_k$. Following the proof of Theorem 2 and using the lower bounds in Lemma 8 we have

$$\alpha_{k,j} = \sum_{i \text{ in Fig. 4(c)}} \rho_{i,j}^k + \sum_{i \text{ in Fig. 4(d)}} \rho_{i,j}^k \geq 2(0.3537 + 0.0245 + 0.0245 + 0.0071) + (0.0838 + 0.0838 + 0.0138 + 0.0138) = 1.0148. \quad (13)$$

Thus, $\alpha = \max_k \max_{j \in \Theta_k} \alpha_{k,j} \geq 1.0148 > 1$. Hence, we cannot use Theorem 3 to show rapid mixing of systematic scan with six colours and 3×3-blocks.

A natural question is whether we can show rapid mixing using even bigger blocks. It seems possible to do this although the computations rapidly become intractable as the block size increases. Already with a 3×3-block the number of boundary colourings we need to consider (after removing isomorphisms) is in excess of $10^8$ and for each boundary colouring there are more than $10^7$ colourings of the block to consider. In addition to
simply generating the distributions on colourings of the block, the time it would take
to actually construct the required couplings, as we did in the proof of Lemma 4, would
also increase. Finally when using a larger block size, different positions of vertex \( j \) in
the block need to be considered whereas we could make use of to the symmetry of the
\( 2 \times 2 \)-block to only consider one position of vertex \( j \) in the block. If different positions of
\( j \) have to be considered this has to be captured in the construction of the coupling and
would likely require more computations. The conclusion is that in order to show rapid
mixing for six colours of systematic scan on the grid we would most likely have to rely
on a different approach than the one presented in this paper.

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