A HITCHIN–KOBAYASHI CORRESPONDENCE FOR KAehler Fibrations

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ABSTRACT. Let X be a compact Kaehler manifold and E → X a principal K bundle, where K is a compact connected Lie group. Let \( \mathcal{A}^{1,1} \) be the set of connections on E whose curvature lies in \( \Omega^{1,1}(E \times_{Ad} \mathfrak{k}) \). Let \( \mathfrak{k} = \text{Lie}(K) \), and fix on \( \mathfrak{k} \) a nondegenerate biinvariant bilinear pairing. This allows to identify \( \mathfrak{k} \cong \mathfrak{k}^* \). Let F be a Kaehler left K-manifold and suppose that there exists a moment map \( \mu : F \to \mathfrak{k}^* \) for the action of K on F. Let \( \mathcal{S} = \Gamma(E \times_K F) \). In this paper we study the equation

\[ \Lambda F_A + \mu(\Phi) = c \]

for \( A \in \mathcal{A}^{1,1} \) on E and a section \( \Phi \in \mathcal{S} \), where \( F_A \) is the curvature of A and \( c \in \mathfrak{k} \) is a fixed central element. We study which orbits of the action of the complex gauge group on \( \mathcal{A}^{1,1} \times \mathcal{S} \) contain solutions of the equation and we define a positive functional on \( \mathcal{A}^{1,1} \times \mathcal{S} \) which generalises the Yang-Mills-Higgs functional and whose local minima coincide with the solutions of the equation.

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1. Introduction

1.1. Let X be a compact Kaehler manifold. Let G be a connected complex reductive Lie group with maximal compact subgroup K, and let E → X be a K-principal bundle on X (with the K action on the right). Let \( \mathcal{G}_K = \Gamma(E \times_{Ad} K) \) be the real
The group \( G \) acts on \( X \) by pullback, and this action can be extended to an action of \( G \) (see subsection 2.2). Let \( \mathcal{A}^{1,1} \subset \mathcal{A} \) be the space of connections whose curvature belongs to \( \Omega^{1,1}(E \times_{\text{Ad}} \mathfrak{k}) \) (equivalently, those which define an integrable holomorphic structure on \( E_G \)). The space \( \mathcal{A}^{1,1} \) is \( G \)-invariant.

Let \( F \) be any Kaehler manifold. Suppose that there is a Hamiltonian left action of \( K \) on \( F \) which respects the complex structure, and let \( \mu : F \to \mathfrak{k}^* \) be a moment map for this action. We recall that by definition the following is satisfied: (C1) for any \( s \in \mathfrak{k} \), \( d\mu (s) = \iota_{\mathcal{X}_s} \omega_F \) (where \( \mathcal{X}_s \) is the field on \( F \) generated by \( s \in \mathfrak{k} \) and \( \omega_F \) is the symplectic form of \( F \)) and (C2) \( \mu \) is equivariant with respect to the actions of \( K \) on \( F \) and the coadjoint action on \( \mathfrak{k}^* \). The map \( \mu \) is unique up to addition of constant central elements of \( \mathfrak{k}^* \).

Since \( F \) is Kaehler, the action of \( K \) on \( F \) extends automatically to a unique holomorphic action of \( G \) (see [GS]). Let \( \mathcal{F} = E \times_K F = E_G \times_G F \to X \) be the associated bundle on \( X \) with fibre \( F \), and let \( \mathcal{J} \) be the space \( \Gamma(\mathcal{F}) \) of smooth sections of \( \mathcal{F} \). The group \( \mathcal{G}_G \) acts on \( \mathcal{J} \), and consequently also on \( \mathcal{J} \). Since \( \mu \) is \( K \)-equivariant we can extend fibrewise the moment map \( \mu \), thus obtaining for any \( \Phi \in \mathcal{J} \) a section \( \mu(\Phi) \in \Omega^0(E \times_{\text{Ad}} \mathfrak{k}^*) \).

In this paper we study the equation

\[
\Lambda F_A + \mu(\Phi) = c, \tag{1.1}
\]

where \( A \in \mathcal{A}^{1,1} \), \( F_A \in \Omega^2(E \times_{\text{Ad}} \mathfrak{k}) \) is the curvature of \( A, \Phi \in \mathcal{J} \), and \( c \in \Omega^0(E \times_{\text{Ad}} \mathfrak{k}) \) is a constant central element. Here \( \Lambda : \Omega^r(X) \to \Omega^{r-2}(X) \) is the adjoint of the map given by wedging with the symplectic form \( \omega \) of \( X \), and we identify (by means of a biinvariant metric on \( \mathfrak{k} \)) \( \Omega^0(E \times_{\text{Ad}} \mathfrak{k}^*) \) with \( \Omega^0(E \times_{\text{Ad}} \mathfrak{k}) \).

1.2. The main question which we consider is the following: for which pairs \((A, \Phi) \in \mathcal{A}^{1,1} \times \mathcal{J}\) there exist a gauge transformation \( g \in \mathcal{G}_G \) such that \((A', \Phi') = g(A, \Phi)\) satisfies equation (1.1)? We will define two conditions on pairs \((A, \Phi)\) called simplicity and \(c\)-stability, and in Theorem 2.19 we will prove that, if \((A, \Phi)\) is a simple pair, then there exist a gauge \( g \in \mathcal{G}_G \) sending \((A, \Phi)\) to a pair \( g(A, \Phi) \) which solves (1.1) if and only if \((A, \Phi)\) is \(c\)-stable. Observe that if \( g(A, \Phi) \) solves (1.1), so does \( kg(A, \Phi) \) for any \( k \in \mathcal{G}_K \). We will also prove that in each \( \mathcal{G}_K \) orbit inside \( \mathcal{A}^{1,1} \times \mathcal{J} \) there is at most one \( \mathcal{G}_K \) orbit of pairs which satisfy (1.1). This is proved in Theorem 2.19.

Such a characterization of solutions to (1.1) is typically called a Hitchin–Kobayashi correspondence, since a particular case of it \((F \text{ equal to a point})\) was independently conjectured by Hitchin and Kobayashi.

One can look at Theorem 2.19 from two different points of view. When \( X \) consists of a single point, the curvature term vanishes in equation (1.1), and so our problem reduces to a well known one in Kaehler geometry. Namely, that of studying which \( G \) orbits inside \( F \) contain zeroes of the moment map \( \mu \). More generally, one studies which \( G \) orbits have points whose image is a fixed central element in \( \mathfrak{k}^* \) or belongs to a given coadjoint orbit in \( \mathfrak{k}^* \). If \( F \) is a projective manifold, one can answer as follows: a \( G \) orbit
contains a zero of the moment map if and only if it is stable in the sense of Mumford Geometric Invariant Theory (GIT for short) [KeNe, MFK, GS]. To extend the notion of GIT stability to actions on any Kähler manifold $F$, we use the notion of analytic stability (see definition 5.1). This notion coincides with that of GIT stability in the case of projective manifolds, and characterizes the $G$-orbits in which the moment map vanishes somewhere (see Theorem 5.4). This is the content of the so-called Kempf–Ness theory. So, in this sense, our result can be viewed as a fibrewise generalisation of Kempf–Ness theory.

There is, however, another point of view which allows to look at Theorem 2.19 as a result à la Kempf–Ness in infinite dimensions. One can give a Kähler structure to the configuration space $\mathcal{A}^{1,1} \times \mathcal{F}$ (for this we use the same biinvariant metric on $\mathfrak{g}$ that was used to give a sense to equation (1.1)); then the action of the gauge group $G_k$ is symplectic and by isometries, and the left hand side in equation (1.1) is a moment map of this action (see sections 4.1, 4.2 and 4.3). This point of view was adopted for the first time in the context of gauge theories by Atiyah and Bott [AB] in their study of Yang-Mills equations over Riemann surfaces, which are a particular case of the equations that we consider. The idea of Atiyah and Bott was used by Donaldson [Do1] in his proof of the theorem of Narasimhan and Seshadri (which is a particular case of Theorem 2.19), and it has been subsequently often used in studying other particular cases of equation (1.1).

1.3. After proving Theorem 2.19 we address the problem of finding a functional on $\mathcal{A}^{1,1} \times \mathcal{F}$ which generalises the classical Yang-Mills-Higgs functional and whose (local) minima satisfy equation (1.1). We define for any connection $A$ on $E$ a covariant derivation which assigns to any section $\Phi \in \mathcal{F}$ a section $d_A \Phi \in \Omega^1(\Phi^* \mathrm{Ker} \, d\pi_F)$, where $\pi_F : F \to X$ denotes the projection. When $F$ is a vector space on which $K$ acts linearly, $\mathcal{F}$ is a vector bundle, $\mathrm{Ker} \, d\pi_F$ is canonically isomorphic to $F$, and the covariant derivation $d_A$ coincides with the usual one in differential geometry. The Yang-Mills-Higgs functional is defined as

$$\mathcal{YM}^c(A, \Phi) = \| F_A \|^2 L^2 + \| d_A \Phi \|^2 L^2 + \| c - \mu(\Phi) \|^2 L^2,$$

where $(A, \Phi) \in \mathcal{A}^{1,1} \times \mathcal{F}$. If $F$ is a representation space for $K$, the Yang-Mills-Higgs functional coincides with the usual one in gauge theories. Now, using the splitting $\Omega^1(X) \otimes \mathbb{C} = \Omega^{1,0}(X) \oplus \Omega^{0,1}(X)$ we obtain from $d_A$ an operator $\overline{\partial}_A$ which sends any $\Phi \in \mathcal{F}$ to a section $\overline{\partial}_A \Phi \in \Omega^{0,1}(\Phi^* \mathrm{Ker} \, d\pi_F)$. We then consider the two equations for a connection $A \in \mathcal{A}^{1,1}$ and a section $\Phi \in \mathcal{F}$

$$\begin{align*}
    \overline{\partial}_A \Phi &= 0, \\
    \Lambda F_A + \mu(\Phi) &= c.
\end{align*}$$

We show in section 7 that the pairs $(A, \Phi) \in \mathcal{A}^{1,1} \times \mathcal{F}$ solving these equations minimize the Yang-Mill-Higgs functional among the pairs whose section belong to a fixed homology class of sections of $\mathcal{F}$.

The way we identify the solutions of the equations with (local) minima of Yang-Mills-Higgs functional is similar to the one used in the study of holomorphic pairs (see [Br1]). The main difference is in the step where in dealing with holomorphic pairs
uses the Kaehler identities. At that point we use certain results on the coupling form on symplectic fibrations due to Guillemin, Lerman and Sternbert [GLeS].

Note that in the Hitchin–Kobayashi correspondence we ignore the first equation in (1.2), that is, $\overline{\partial} A \Phi = 0$ (this equation can be given a sense even when $F$ is not a vector space; see section 7). Indeed, this condition is not necessary in the proof: we only need $\Phi$ to be smooth. Furthermore, the equation $\overline{\partial} A \Phi = 0$ is invariant under $G_K$, while the interest of our problem stems from the fact that $\Lambda F_A + \mu(\Phi) = c$ is only $G_K$-invariant.

It is a remarkable fact that both the equations (1.2) and the results in section 4 make perfect sense even when the complex structure on $F$ is not integrable. In a forthcoming paper we will study these equations and we will show how the gauge equivalence classes of its solutions can be used to define invariants of Hamiltonian actions on compact symplectic manifold (see [Mu]).

1.4. Many particular instances of equations (1.2) have been already studied. When $F = \{pt\}$ equation (1.1) becomes the Hermite–Einstein equation, which was studied for example in [BarT1, Do1, Do2, NS8, UY]. A good reference for Hitchin–Kobayashi correspondence for Hermite–Einstein equations and its interesting history is the book by Lübke and Teleman [LT]. When $F$ is a representation space for $K$, the fibre bundle $F$ is a vector bundle. Theorem 2.19 has been proved for many particular choices of $K$ and representations $K \to U(F)$ (see for example [Br1, Br2, BrGP3, GP1, GP2, Hi, JT, Si]). In 1996 Banfield [Ba] gave a proof of the Hitchin–Kobayashi correspondence for any $K$ and any representation space $F$ of $K$.

A particular case of our construction which does not fit in Banfield’s result is that of extensions and filtrations of vector bundles. They arise when $F$ is a Grassmannian or, more generally, any flag manifold. A Hitchin–Kobayashi correspondence for extensions was studied by Bradlow and García–Prada [BrGP1], and by Daskalopoulos, Uhlenbeck and Wentworth [DaUW]; the correspondence for filtrations has been proved by Álvarez Cónsul and García–Prada [AlGP].

1.5. This paper is organised as follows. In section 2 we state the main result of this paper. Sections 3 to 6 are devoted to the proof of this result. In section 3 we explain the construction of a certain functional which will be the main tool in the proof. In section 4 we describe a Kaehler structure on the manifold $\mathcal{M}^{1,1} \times F$ and we identify our equation as a moment map for the action of $G_K$ on $\mathcal{M}^{1,1} \times F$. In section 5 we prove a particular case of our theorem, and the general proof is given in section 6. In section 7 we introduce (a generalisation of) the Yang-Mills-Higgs functional and we prove that its minima coincide with the solutions of equations (1.2). Finally, in sections 8 and 9 we work out two different examples of our correspondence.

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2. Stability and statement of the correspondence

2.1. The isomorphism $\mathfrak{k} \simeq \mathfrak{k}^*$. To give a meaning to equation (\[4\]) we need a $K$-equivariant isomorphism $\mathfrak{k} \simeq \mathfrak{k}^*$. From now on we will assume that such an isomorphism comes from a biinvariant metric on $K$ which is the pullback of the Killing metric through a faithful representation $\rho_a : K \to U(W_a)$, where $W_a$ is a Hermitian vector space. (In other words, the isomorphism $\mathfrak{k} \simeq \mathfrak{k}^*$ is a hidden parameter of the equation, and we prove the correspondence for some particular choices of it.) Our characterisation of solutions to (\[4\]) will depend on the choice of $\rho_a$ and $W_a$ (this is not strange, since the equation also depends on them).

2.2. The action of $\mathcal{G}_K$ on $\mathcal{A}$. Let $\omega$ be the symplectic form on $X$ and $I \in \text{End}(TX)$ the complex structure. In the sequel $\omega^{[k]}$ will denote $\omega^k/k!$ The volume element $\omega^{[n]}$ will be implicitly assumed in all the integrals of functions on $X$.

Let $\mathcal{C}$ be the set of $G$-invariant complex structures on $E_G = E \times_K G$ for which the map $d\pi_G : T\pi_G \to \pi_G^* TX$ is complex. We define a map $\mathcal{C} : \mathcal{C} \to \mathcal{A}$ as follows. A complex structure $I \in \mathcal{C}$ is mapped to the connection $\mathcal{C}(I)$ given by the horizontal distribution $I(TE) \cap TE \subset TE$ (this makes sense, since the inclusion $E = E \times_K K \subset E \times_K G$ given by $K \subset G$ induces an inclusion $TE \subset T\pi_G$). This defines a connection and the map $\mathcal{C}$ is a bijection (see \[1\]). We call $\mathcal{C}$ the Chern map.

The following is readily checked.

Lemma 2.1. (i) Let $G$ act holomorphically on a vector space $W$. Let us take a complex structure $I \in \mathcal{C}$. The associated bundle $V = E_G \times_G W$ is endowed by $I$ of a complex structure $I_V \in \text{End}(TV)$. Any section $\sigma \in \Omega^0(V)$ may be viewed as a map $\sigma : X \to V$. Then, the antiholomorphic part $\overline{\partial}_I(\sigma) = (d\sigma + I_V \circ d\sigma \circ I_X)/2$ can be regarded as an element in $\Omega^{0,1}(V)$.

(ii) For any $A \in \mathcal{A}$ we have $\overline{\partial}_A = \overline{\partial}_{\mathcal{C}^{-1}(A)}$, where $\overline{\partial}_A : \Omega^0(V) \to \Omega^{0,1}(V)$ is the usual $\overline{\partial}$ operator obtained from $A$.

(iii) The set $\mathcal{A}^{1,1}$ is mapped by $\mathcal{C}^{-1}$ to the set of integrable complex structures on $E_G$.

The group $\mathcal{G}_K$ acts on $\mathcal{C}$ by pullback, and using the map $\mathcal{C}$ we transfer the action of $\mathcal{G}_K$ on $\mathcal{C}$ to an action on $\mathcal{A}$. This action extends the action of $\mathcal{G}_K$ and (by (iii) in the preceding lemma) leaves invariant the subset $\mathcal{A}^{1,1} \subset \mathcal{A}$.

2.3. Maximal weights. Let $I_F \in \text{End}(TF)$ be the complex structure of $F$. We will denote by $\langle u, v \rangle = \omega_F(u, I_F v)$ the Kaehler metric on $F$.

Let $s \in \mathfrak{k}$ be any nonzero element, and let us write $\mu_s = \langle \mu, s \rangle : F \to \mathbb{R}$. (Here and in the sequel we denote by $\langle \cdot , \cdot \rangle_W : W^* \times W \to \mathbb{R}$ the canonical pairing for any vector space $W$.) Recall that $\mathcal{A}_s$ is the field generated on $F$ by $s$.

Lemma 2.2. The gradient of $\mu_s$ is $I_F \mathcal{A}_s$.

Proof. Let $x \in F$ and take any vector $v \in T_x F$. Then $\nabla_v(\mu_s) = \langle d\mu_s, v \rangle_{T_x F} = \omega_F(\mathcal{A}_s, v) = \omega_F(I_F \mathcal{A}_s, I_F v) = \langle I_F \mathcal{A}_s, v \rangle$, by the definition of moment map. □
Consider the gradient flow $\phi_s^t : F \to F$ of the function $\mu_s$. $\phi_s^t$ is defined by these properties: $\phi_s^0 = \text{Id}$ and $\frac{\partial}{\partial t} \phi_s^t = \nabla(\mu_s) = I\mathcal{X}_s$. Using the action of $G$ on $F$ we can write $\phi_s^t(x) = e^{its}x$.

**Definition 2.3.** Let $x \in F$ be any point, and take an element $s \in \mathfrak{k}$. Let

$$\lambda_t(x; s) = \mu_s(e^{its}x).$$

We define the maximal weight $\lambda(x; s)$ of the action of $s$ on $x$ to be

$$\lambda(x; s) = \lim_{t \to \infty} \lambda_t(x; s) \in \mathbb{R} \cup \{\infty\}.$$

This limit always exists since by Lemma 2.2 the function $\lambda_t(x; s)$ increases with $t$. The definition of the maximal weight depends on the chosen moment map. Since this is not unique, we will sometimes write the maximal weight of $s \in \mathfrak{k}$ acting on $x \in F$ with respect to the moment map $\mu$ as $\lambda^\mu(x; s)$.

**Proposition 2.4.** The maximal weights satisfy the following properties:

1. They are $K$-equivariant, that is, for any $k \in K$, $\lambda(kx; ks^{-1}) = \lambda(x; s)$.
2. For any positive real number $t$ one has $\lambda(x; ts) = t\lambda(x; s)$.

See sections 8 and 9 for explicit computations of maximal weights in some particular situations.

2.4. **Parabolic subgroups.** A good reference for this material is [R]. Let $\mathfrak{g}$ be the Lie algebra of $G$, and split $\mathfrak{g} = \mathfrak{z} \oplus \mathfrak{g}^*$ as the sum of the centre plus the semisimple part $\mathfrak{g}^* = [\mathfrak{g}, \mathfrak{g}]$ of $\mathfrak{g}$. Take a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}^*$. Let $R \subset \mathfrak{h}^*$ be the set of roots. We can decompose

$$\mathfrak{g} = \mathfrak{z} \oplus \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha,$$

where $\mathfrak{g}_\alpha \subset \mathfrak{g}^*$ is the subspace on which $\mathfrak{h}$ acts through the character $\alpha \in \mathfrak{h}^*$.

Fixing a (irrational) linear form on $\mathfrak{h}^*$, we divide the set of roots in positive and negative roots: $R = R^+ \cup R^-$. Let us write the set of simple roots $\Delta = \{\alpha_1, \ldots, \alpha_r\} \subset R^+$. Recall that the set $\Delta$ is characterised by the following property: any root can be written as a linear combination of the elements of $\Delta$ with integer coefficients all of the same sign. Furthermore, $r$ equals $\dim_{\mathbb{C}} \mathfrak{h}$, the rank of $G$. The simple coroots are by definition $\alpha_j^\vee = 2\alpha_j/\langle \alpha_j, \alpha_j \rangle$, where $1 \leq j \leq r$.

We have taken a maximal compact subgroup $K \subset G$. From now on we will assume that the following relation holds between $K$ and the Cartan subalgebra $\mathfrak{h}$: $\mathfrak{z} \oplus \mathfrak{h}$ is the complexification of the Lie algebra $\mathfrak{t}$ of a maximal torus $T \subset K$.

**Lemma 2.5.** Chose, for any root $\alpha \in R$, a nonzero element $g_\alpha \in \mathfrak{g}_\alpha$ in such a way that $g_\alpha$ and $g_{-\alpha}$ satisfy $\langle g_\alpha, g_{-\alpha} \rangle = 1$. Let $\mathbb{R}R^* \subset \mathfrak{h}$ denote the real span of the duals (with respect to the Killing metric) of the roots. Assume that $\mathfrak{z} \oplus \mathfrak{h}$ is the complexification of the Lie algebra of a maximal torus $T$ of a maximal compact subgroup $K \subset G$. Then $\mathfrak{g}^* \cap \mathfrak{k} = i\mathbb{R}R^* \oplus \bigoplus_{\alpha \in R} \mathbb{R}(g_\alpha + g_{-\alpha}) \oplus \mathbb{R}(ig_\alpha - ig_{-\alpha})$. 
This lemma (and the following ones in this subsection) can be easily proved using basic results on reductive Lie groups (see for example [FH]).

Let $\lambda_1, \ldots, \lambda_r$ be the set of fundamental weights, which belong to $\mathfrak{h}^*$ and are the duals with respect to the Killing metric of the simple coroots. Let us denote by $\lambda'_1, \ldots, \lambda'_r$ the elements in $\mathfrak{h}$ dual to the fundamental weights through the Killing metric.

To define a parabolic subgroup of $G$, take any subset $A = \{\alpha_{i_1}, \ldots, \alpha_{i_s}\} \subset \Delta$. Let

$$D = D_A = \{\alpha \in R \mid \alpha = \sum_{j=1}^r m_j \alpha_j, \text{ where } m_i \geq 0 \text{ for } 1 \leq i \leq s\}.$$ 

**Definition 2.6.** The subalgebra $\mathfrak{p} = \mathfrak{z} \oplus \mathfrak{h} \oplus \bigoplus_{\alpha \in D} \mathfrak{g}_\alpha$ will be called the parabolic subalgebra of $\mathfrak{g}$ with respect to the set $A \subset \Delta$. The connected subgroup $P$ of $G$ whose subalgebra is $\mathfrak{p}$ will be called the parabolic subgroup of $G$ with respect to $A$. Furthermore, any positive (resp. negative) linear combination of the fundamental weights $\lambda_{i_1}, \ldots, \lambda_{i_s}$ plus an element of the dual of $i(\mathfrak{z} \cap \mathfrak{k})$ will be called a dominant (resp. antidominant) character on $\mathfrak{p}$ (or on $P$).

**Remark 2.7.** We will regard $G$ as a parabolic subgroup of itself (with respect to the empty set $\emptyset \subset \Delta$).

Observe that our definition of parabolic subgroup depends upon the choice of a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and of a linear form on $\mathfrak{h}^*$. In general, any parabolic subgroup $P \subset G$ obtained from a different choice of Cartan subalgebra and linear form will be conjugate to a parabolic subgroup obtained from our data.

### 2.5. Parabolic subgroups and filtrations.

Let $\rho : K \to U(W_\rho)$ be a representation on a Hermitian vector space $W_\rho$. We will write its (unique) lift to a holomorphic representation of the complexification $G$ of $K$ with the same letter $\rho : G \to GL(W_\rho)$. Take $P \subset G$ to be the parabolic subgroup with respect to a set $A = \{\alpha_{i_1}, \ldots, \alpha_{i_s}\} \subset \Delta$. Let $\chi$ be the dual of an antidominant character of $P$. Thanks to our conventions (Lemma 2.23), $\chi$ belongs to $i\mathfrak{k}$. So, since $\rho$ is unitary, $\rho(\chi)$ diagonalises and has real eigenvalues. Let $\lambda_1 < \cdots < \lambda_r$ be the set of different eigenvalues of $\rho(\chi)$, and let us write $W(\lambda)$ the eigenspace of eigenvalue $\lambda$. Let $W^{\lambda_k} = \bigoplus_{j \leq k} W(\lambda_j)$, and let $\mathfrak{W}_\rho(\chi)$ be the partial flag $0 \subset W^{\lambda_1} \subset \cdots \subset W^{\lambda_r} = W_\rho$.

**Lemma 2.8.** (i) The action of $P$ leaves invariant the partial flag $\mathfrak{W}_\rho(\chi)$. Suppose that the restriction of $\rho$ to the semisimple part $\mathfrak{p}^s$ of $\mathfrak{p}$ is faithful. If $\chi = z + \sum_{k=1}^s m_k \lambda'_k$, where $z \in \mathfrak{z}$, and, for any $k$, $m_k < 0$, then $P$ is precisely the antiimage by $\rho$ of the stabiliser of $\mathfrak{W}_\rho(\chi)$. (ii) Let $\chi \in i\mathfrak{k}$ be any element. There is a choice of Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ contained in $\mathfrak{p}$ such that $\chi \in \mathfrak{h}$ and $\chi$ is antidominant with respect to $P$ if and only if the stabiliser of the partial flag $\mathfrak{W}_\rho(\chi)$ contains $P$.

**Lemma 2.9.** Let $\chi$ be any element in $i\mathfrak{k}$. The antiimage by $\rho$ of the stabiliser of $\mathfrak{W}_\rho(\chi)$ is a parabolic subgroup $P_\rho(\chi)$ of $G$. Moreover, $\chi$ is the dual of an antidominant character of $P_\rho(\chi)$. 

Let us take now any subspace $W' \subset W_\rho$ belonging to the filtration $\mathfrak{M}_\rho(\chi)$. Since $P$ leaves $W'$ invariant, we may define $\overline{W'} = G \times_P W' \to G/P$ (here we view $G$ as a right $P$-principal bundle). Define also an action of $G$ on $G \times W'$ by $g' (g, w) = (g'g, g^{-1}g'gw)$. This action descends to an action on $\overline{W'}$. Repeating this for each subspace in $\mathfrak{M}_\rho(\chi)$ we obtain the following.

**Lemma 2.10.** The filtration of holomorphic vector bundles $\overline{\mathfrak{M}}_\rho(\chi) = G \times_\rho \mathfrak{M}_\rho(\chi) \to G/P$ admits a holomorphic lift of the right action of $G$ on $G/P$.

2.6. Parabolic and maximal compact subgroups. Given any parabolic subgroup $P \subset G$ with Lie algebra $\mathfrak{p}$, we will write $P_K$ (resp. $\mathfrak{p}_K$) for the subgroup $P \cap K$ (resp. the subalgebra $\mathfrak{p} \cap \mathfrak{k}$). $P_K$ is a maximal compact subgroup of $P$.

**Lemma 2.11.** Let $E_G \to X$ be a $G$-principal bundle on any topological space $X$. If $E_G$ admits reductions of its structure group from $G$ to a parabolic subgroup $P$ and to the maximal compact subgroup $K$, then it also admits a reduction of its structure group from $G$ to $P_K$.

**Lemma 2.12.** Let $P$ be a parabolic subgroup with respect to the set

$$A = \{\alpha_{i_1}, \ldots, \alpha_{i_s}\} \subset \Delta.$$

For any $j \in \{i_1, \ldots, i_s\}$, the element $\lambda'_j \in i\mathfrak{k}$ (dual with respect to the Killing metric of the fundamental weight $\lambda_j$) is left fixed by the adjoint action of $\mathfrak{p}_K$ on $\mathfrak{g}$.

2.7. Reductions of the structure group and filtrations. Following the notation in subsection 2.3, we denote $V_\rho = E \times_\rho W_\rho$. In this subsection we will see that there is a correspondence between the reductions of the structure group of $E$ to a parabolic subgroup $P$ together with an antidominant character of $P$, and certain filtrations of $V_\rho$ by subbundles. We denote $E(G/P)$ the bundle $E_G \times_G (G/P)$. The space of reductions of the structure group of $E_G$ from $G$ to $P$ is $\Gamma(E(G/P))$.

2.7.1. Fix a parabolic subgroup $P \subset G$ and take a reduction $\sigma \in \Gamma(E(G/P))$. Let $\chi$ be an antidominant character for $P$. There is a canonical reduction of the structure group $G$ of $E_G$ to $K$, since $E_G = E \times_K G$. Thanks to Lemma 2.11, this reduction, together with $\sigma$, gives a reduction $\sigma_K \in \Gamma(E(G/P_K))$, where $P_K = P \cap K$. And then, Lemma 2.12 implies that we get a section $g_{\sigma, \chi} \in \Omega^0(E \times_{Ad} i\mathfrak{k}) = i\mathfrak{g} \otimes \mathfrak{g}^*$ which is fibrewise the dual of $\chi$.

With the element $g_{\sigma, \chi}$ we can obtain a filtration of $V_\rho$ as follows. First of all, $\rho(g_{\sigma, \chi})$ has constant real eigenvalues (which are equal to those of $\rho(\chi) \in \text{End}(W_\rho)$). Let $\lambda_1 < \cdots < \lambda_r$ be the different eigenvalues, and let $V^\lambda_\rho(\chi)$ be the eigenbundle of eigenvalue $\lambda_j$. Finally, let $V^\lambda_\rho = \bigoplus_{i \leq k} V^\lambda_\rho(i)$. Denote by $\mathfrak{M}_\rho(\sigma, \chi)$ the filtration

$$0 \subset V^\lambda_\rho \subset V^{2\lambda}_\rho \subset \cdots \subset V^{r\lambda}_\rho = V_\rho.$$

Alternatively, recall that on $G/P$ there is a filtration of $G$-equivariant (holomorphic) vector bundles, $\overline{\mathfrak{M}}_\rho(\chi)$ (see Lemma 2.11). $G$-equivariance allows to define the filtration $\overline{\mathfrak{M}}_\rho(\chi) = E \times_G \overline{\mathfrak{M}}_\rho(\chi) \to E(G/P)$. Then $\mathfrak{M}_\rho(\sigma, \chi) = \sigma \mathfrak{M}_\rho(\chi)$. 

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2.7.2. Conversely, take $g \in \Omega^0(E \times_{\text{Ad}} \mathfrak{k})$. Suppose that $\rho(g)$ has constant eigenvalues, and let $\lambda_1 < \cdots < \lambda_r$ be the set of different values they take. Just as before, we consider the filtration

$$0 \subset V_{\rho}^{\lambda_1} \subset V_{\rho}^{\lambda_2} \subset \cdots \subset V_{\rho}^{\lambda_r} = V_{\rho},$$

(2.3)

Fix a point $x \in X$. After trivialising the fibre $E_x$ we can identify $g(x)$ with and element $\chi$ of $\mathfrak{k}$. Let $P = P_{\rho}(\chi)$ (see Lemma 2.3). We obtain a reduction $\sigma \in \Gamma(E(G/P))$ as follows. Let $y \in X$. Trivialise $E_y$ and identify $g(y)$ with $\chi_y \in \mathfrak{k}$. Let

$$\sigma(y) = \{g \in G | g(\mathfrak{M}_{\rho}(\chi)) = \mathfrak{M}_{\rho}(\chi_y)\}.$$

Then $\sigma(y)$ is invariant under left multiplication by elements of $P$, and in fact gives a unique point in $G/P$ (here we use Lemma 2.3). Furthermore, the definition of $\sigma(y)$ is compatible with change of trivialisation in the sense that it gives a section $\sigma \in \Gamma(E(G/P))$.

**Lemma 2.13.** The filtration (2.3) is equal to $\mathfrak{M}_{\rho}(\sigma, \chi)$.

2.7.3. Holomorphic reductions of the structure group. Suppose that there is a fixed (integrable) holomorphic structure on $E_G$. This structure induces a holomorphic structure on the total space of the associated bundle $E(G/P)$, since $G/P$ is a complex manifold and the action of $G$ on $G/P$ is holomorphic.

**Definition 2.14.** Let $\sigma \in \Gamma(E(G/P))$. A reduction $\sigma$ is holomorphic if the map $\sigma : X \to E(G/P)$ is holomorphic.

One can give an equivalent definition of holomorphicity in terms of the filtrations induced by the reduction $\sigma$ in the associated vector bundles.

**Lemma 2.15.** Let $\sigma \in \Gamma(E(G/P))$. If the reduction $\sigma$ is holomorphic then, for any antidominant character $\chi$ of $P$ and for any representation $\rho : K \to U(W)$, the filtration $\mathfrak{M}_{\rho}(\sigma, \chi)$ of $V_{\rho}$ is holomorphic. Conversely, let $g \in \Omega^0(E \times_{\text{Ad}} \mathfrak{k})$ have constant eigenvalues, and let $P \subset G$, $\sigma \in \Gamma(E(G/P))$, $\chi \in \mathfrak{k}$ and $\mathfrak{M}_{\rho}(\sigma, \chi)$ be obtained from it as in 2.7.2. Suppose that $\rho$ is faithful. If $\mathfrak{M}_{\rho}(\sigma, \chi)$ is holomorphic, then so is $\sigma$.

2.8. Total degree of a reduction of the structure group. Let $V = V_{\rho_a} = E \times_{\rho_a} W_a$ be the vector bundle associated to the representation $\rho_a$ (see subsection 2.2). We will apply the preceding results on filtrations of vector bundles to $V$. Let $P$ be a parabolic subgroup of $G$ with respect to $\{\alpha_{i_1}, \ldots, \alpha_{i_r}\} \subset \Delta$. Suppose that $\sigma \in \Gamma(E(G/P))$ is a reduction. Let $\chi$ be an antidominant character of $P$.

We begin by defining the degree of the pair $(\sigma, \chi)$. Let $0 \subset V_{\rho_a}^{\lambda_1} \subset \cdots \subset V_{\rho_a}^{\lambda_r} = V$ be the filtration $\mathfrak{M}_{\rho_a}(\sigma, \chi)$ of $V$. For any vector bundle $V'$ we denote

$$\deg(V') = 2\pi \langle c_1(V') \cup [\omega^{[n-1]}], [X] \rangle.$$

Here $[\omega^{[n-1]}]$ denotes the cohomology class represented by the form $\omega^{[n-1]}$ and $[X] \in H_{2n}(X; \mathbb{Z})$ is the fundamental class of $X$. Then we set

$$\deg(\sigma, \chi) = \lambda_r \deg(V) + \sum_{k=1}^{r-1} (\lambda_k - \lambda_{k+1}) \deg(V^{\lambda_k}).$$
2.9. Stability, simple pairs and the correspondence. Let $\sigma \in \Gamma(E(G/P))$ be a reduction. We define the maximal weight of $(\sigma, \chi)$ acting on a section $\Phi \in \mathcal{S}$ as
\[
\int_{x \in X} \lambda(\Phi(x); -ig_{\sigma,\chi}(x)),
\]
where $\lambda(\Phi(x); -g_{\sigma,\chi}(x))$ is the maximal weight of $-g_{\sigma,\chi}(x)$ acting on $\Phi(x)$ as defined in 2.3 (note that here we use the $K$-equivariance of the maximal weights, as stated in Lemma 2.4).

Finally, given any central element $c \in \mathfrak{z} \cap \mathfrak{k}$ we define the $c$-total degree of the pair $(\sigma, \chi)$ as
\[
T^c_{\sigma}(\sigma, \chi) = \deg(\sigma, \chi) + \int_{x \in X} \lambda(\Phi(x); -ig_{\sigma,\chi}(x)) + \langle i\chi, c \rangle \ Vol(X).
\]
Just as the maximal weights, the $c$-total degree is allowed to be equal to $\infty$.

Now suppose that $X_0 \subset X$ has as complement in $X$ a complex codimension 2 submanifold. Suppose also that a reduction $\sigma$ is defined only in $X_0$, that is, $\sigma \in \Gamma(X_0; E(G/P))$. In this case it also makes sense to speak about $T^c_{\sigma}(\sigma, \chi)$ for any antidominant character $\chi$. The only difficulty would be in defining the degree $\deg(\sigma, \chi)$; however, it is well known that the degree of a vector bundle can be computed by integrating the Chern-Weil form in the complement of a complex codimension 2 variety.

**Definition 2.16.** A pair $(A, \Phi) \in \mathfrak{a}^{1,1} \times \mathcal{S}$ is $c$-stable if for any $X_0 \subset X$ whose complement on $X$ is a complex codimension 2 submanifold, for any parabolic subgroup $P$ of $G$, for any holomorphic (with respect to the complex structure $\mathcal{E}^{-1}A$ on $E_G$, see Lemma 2.4) reduction $\sigma \in \Gamma(X_0; E(G/P))$ defined on $X_0$, and for any antidominant character $\chi$ of $P$ we have
\[
T^c_{\sigma}(\sigma, \chi) > 0.
\]

We will say that an element $s \in \mathfrak{g}_G$ is semisimple if, for any $x \in X$, after identifying $(E \times \text{Ad} \ g)_x \simeq \mathfrak{g}$, $s(x) \in \mathfrak{g}$ is a semisimple element. (This is independent of the chosen isomorphism $(E \times \text{Ad} \ g)_x \simeq \mathfrak{g}$, because an element of $\mathfrak{g}$ is semisimple if and only if any element in its orbit by the adjoint action of $G$ on $\mathfrak{g}$ is semisimple.)

**Definition 2.17.** A pair $(A, \Phi)$ is simple if no semisimple element in $\text{Lie}(\mathfrak{g}_G)$ leaves $(A, \Phi)$ fixed, that is, for any semisimple $s \in \text{Lie}(\mathfrak{g}_G)$, $\mathcal{F}^{(s,\Phi)}(A, \Phi) \neq 0$.

**Remark 2.18.** If $(A, \Phi)$ is simple then so is any point in the $\mathfrak{g}_G$ orbit through $(A, \Phi)$.

We are now ready to state the main theorem of this paper.

**Theorem 2.19** (Hitchin–Kobayashi correspondence). Let $(A, \Phi) \in \mathfrak{a}^{1,1} \times \mathcal{S}$ be a simple pair. There exists a gauge transformation $g \in \mathcal{G}_G$ such that
\[
\Lambda F_g(A) + \mu(g(\Phi)) = c \quad (2.4)
\]
if and only if $(A, \Phi)$ is $c$-stable. Furthermore, if two different $g, g' \in \mathcal{G}_G$ solve equation $(2.4)$, then there exists $k \in \mathcal{G}_K$ such that $g' = kg$. 

We briefly explain the idea of the proof of Theorem 2.19. We construct on $\mathcal{A}^{1,1} \times \mathcal{S} \times \mathcal{G}_K$, a functional $\Psi$ (that we will call integral of the moment map) whose critical points give the solutions of equation (2.4). We prove that the pair $(A, \Phi)$ is $c$-stable if and only if the functional $\Psi$ is, in a certain sense, proper along the slice $\{A\} \times \{\Phi\} \times \mathcal{G}_G$. Then we prove that the functional being proper along $\{A\} \times \{\Phi\} \times \mathcal{G}_G$ is equivalent to its having a critical point in $\{A\} \times \{\Phi\} \times \mathcal{G}_G$, thus achieving the proof of Theorem 2.19.

3. THE INTEGRAL OF THE MOMENT MAP

In this section we consider the following general situation. Let $H$ be a Lie group which acts on a Kaehler manifold $M$ respecting the Kaehler structure, and assume that there exists a moment map

$$\mu : M \to \mathfrak{h}^*,$$

where $\mathfrak{h} = \text{Lie}(H)$. Suppose that there exists the complexification $L = H^\mathbb{C}$ of $H$, and that the inclusion $\iota : H \to L$ induces a surjection $\iota_* : \pi_1(H) \to \pi_1(L)$. Under this assumptions, we construct a functional

$$\Psi : M \times L \to \mathbb{R}$$

which we call the integral of the moment map $\mu$, and which satisfies these two properties:

- for any $x \in M$, the critical points of the restriction $\Psi_x$ of $\Psi$ to $\{x\} \times L$ coincide with the points of the orbit $Lx$ on which the moment map vanishes and
- the restriction of $\Psi_x$ to lines of the form $\{e^{ts}t \in \mathbb{R}\}$, where $s \in \mathfrak{l} = \text{Lie}(L)$, is convex.

If $H$ is compact then $L = H^\mathbb{C}$ always exists and $\pi_1(H) \to \pi_1(L)$ is always satisfied. But note that we do not need our manifold $M$ or our groups $H, L$ to be finite dimensional. In fact, we will use this construction mainly in the infinite dimensional case $(M; H, L) = (\mathcal{A}^{1,1} \times \mathcal{S}; \mathcal{G}_K, \mathcal{G}_G)$ (in section 4 we will prove that $\mathcal{A}^{1,1} \times \mathcal{S}$ is a Kaehler manifold, that the action of $\mathcal{G}_K$ respects the Kaehler structure, and we will identify a moment map for this action). The resulting integral of the moment map will be a certain modification of Donaldson functional.

3.1. Definition of $\Psi$. Let us fix a point $x \in M$, and let $\phi : L \to M$ be the map which sends $h \in L$ to $hx \in M$. We define a 1-form on $L$, $\sigma = \sigma^x \in \Omega^1(L)$, as follows: given $h \in L$ and $v \in T_hL$,

$$\sigma_h(v) = \langle \mu(hx), -i\pi(v) \rangle_t,$$

where $\pi : T_hL = \mathfrak{h} \oplus i\mathfrak{h} \to i\mathfrak{h}$ is the projection to the second summand.

We will use the following formula, which holds for any two vector fields $X, Y$ and any 2-form $\omega$ on $M$

$$d\omega(X, Y) = L_X(\omega(Y)) - L_Y(\omega(X)) - \omega([X, Y]). \quad (3.5)$$

Equality (3.5) is a particular case of a formula which describes the exterior derivative of forms of arbitrary degree in terms of Lie derivatives (see [BeGeV] p. 18).
Lemma 3.1. The 1-form $\sigma$ is exact.

Proof. Let us first of all prove that $d\sigma = 0$. Given $g \in \mathfrak{L}$, let $\mathcal{X}_g^L \in \Gamma(TL)$ be the field generated by $g$ acting on the left on $L$ (on the other hand, $\mathcal{X}_g$ will denote the vector field generated by $g$ on $M$). We will prove that for any pair $g, g' \in \mathfrak{h} \cup i\mathfrak{h}$, $d\sigma(g, g') = 0$. This implies by linearity that $d\sigma = 0$. We will treat separately three cases, and will make use of formula (3.5), which in our case reads

$$d\sigma(\mathcal{X}_g^L, \mathcal{X}_{g'}^L) = \langle d(\sigma(\mathcal{X}_g^L)), \mathcal{X}_{g'}^L \rangle_{TL} - \langle d(\sigma(\mathcal{X}_g^L)), \mathcal{X}_{g'}^L \rangle_{TL} - \sigma([\mathcal{X}_g^L, \mathcal{X}_{g'}^L])$$

Suppose first that $g, g' \in \mathfrak{h}$. In this case, $\pi(\mathcal{X}_g^L) = \pi(\mathcal{X}_{g'}^L) = \pi([\mathcal{X}_g^L, \mathcal{X}_{g'}^L]) = 0$, hence by the formula it is clear that $d\sigma(\mathcal{X}_g^L, \mathcal{X}_{g'}^L) = 0$.

Now suppose that $g \in \mathfrak{h}$ and $g' \in i\mathfrak{h}$. Observe that $\sigma(\mathcal{X}_g^L) = 0$, so we have to prove that $\langle d(\sigma(\mathcal{X}_g^L)), \mathcal{X}_{g'}^L \rangle_{TL} - \sigma([\mathcal{X}_g^L, \mathcal{X}_{g'}^L]) = 0$. Differentiating property (C2) of the moment map (see section 1) we have $\langle d(\mu, v)_b, \mathcal{X}_g^L \rangle_{TM} + \langle \mu, [g, v]_b \rangle_M = 0$. The functoriality of the differentiation $d$ implies that $\langle d(\sigma(\mathcal{X}_g^L)), \mathcal{X}_{g'}^L \rangle_{TL} + \sigma([\mathcal{X}_g^L, \mathcal{X}_{g'}^L]) = 0$. On the other hand, since the action of $L$ on $M$ is on the left, $[\mathcal{X}_g^L, \mathcal{X}_{g'}^L] = -\mathcal{X}_g^L$ (see for example [BeGeV] p. 208), so we obtain

$$\langle d(\sigma(\mathcal{X}_g^L)), \mathcal{X}_{g'}^L \rangle_{TL} - \sigma([\mathcal{X}_g^L, \mathcal{X}_{g'}^L]) = 0,$$

which is what we wanted to prove. The case $g \in i\mathfrak{h}$ and $g' \in \mathfrak{h}$ is dealt with in a very similar way.

Finally, there remains the case $g, g' \in i\mathfrak{h}$. In this situation $[g, g'] \in \mathfrak{h}$, and so $\sigma([\mathcal{X}_g^L, \mathcal{X}_{g'}^L]) = 0$. In view of this we have to prove

$$\langle d(\sigma(\mathcal{X}_g^L)), \mathcal{X}_{g'}^L \rangle_{TL} = \langle d(\sigma(\mathcal{X}_g^L)), \mathcal{X}_{g'}^L \rangle_{TL}.$$

The left hand side is equal to $\phi^*([d(\mu, ig)_b, \mathcal{X}_g^L])$ and this, by property (C1) of the moment map, is equal to $\phi^*([\omega_M(I\mathcal{X}_g^L, \mathcal{X}_g^L)]) = \phi^*([\mathcal{X}_g^L, \mathcal{X}_g^L])$, where $\omega_M$ denotes the symplectic form on $M$. The right hand side is equal to $\phi^*([\omega_M(I\mathcal{X}_g^L, \mathcal{X}_g^L)]) = \phi^*([\mathcal{X}_g^L, \mathcal{X}_g^L])$. Both functions are the same by the symmetry of $\langle, \rangle$.

Once we know that $d\sigma = 0$, let us prove that $\sigma$ is exact. Let $\iota : H \rightarrow L$ denote the inclusion. It is clear that $\iota^*\sigma = 0$. On the other hand, by our hypothesis $\iota_* : \pi_1(H) \rightarrow \pi_1(L)$ is exhaustive. These two facts imply that $\sigma$ is exact. Indeed, if it were not exact then we could find a path $\gamma : [0, 1] \rightarrow L$, $\gamma(0) = \gamma(1) = 1 \in L$ such that

$$\int_0^1 \gamma \sigma \neq 0.$$

But then we could deform $\gamma$ to a path $\gamma' \subset H$, and, since $d\sigma = 0$, the value of the integral would not change and in particular would be nonzero. This is in contradiction with the fact that $\iota^*\sigma = 0$. So $\sigma$ is exact.

Let $\Psi_x : L \rightarrow \mathbb{R}$ be the unique function such that $\Psi_x(1) = 0$ and such that $d\Psi_x = \sigma^x$. Define also $\Psi : M \times L \ni (x, g) \mapsto \Psi_x(g)$. We will call the function $\Psi$ the integral of the moment map.
Remark 3.2. If the symplectic form in $M$ is the curvature of a line bundle $L \to M$ and there is a lift of the action of $G$ to $L$, then the integral $\Psi$ of the moment map coincides with the functional defined in section 6.5.2 of [DoKî].

3.2. Properties of $\Psi$. In this subsection we give the properties of the integral of the moment map which will be used below.

**Proposition 3.3.** Let $x \in M$ be any point, and let $s \in \mathfrak{h}$.

1. $\Psi(x,e^{is}) = \int_0^1 \langle \mu(e^{its}), s \rangle dt = \int_0^1 \lambda_t(x;s)dt$,
2. $\frac{\partial \Psi}{\partial t}(x,e^{its})|_{t=0} = \langle \mu(x), s \rangle_{\mathfrak{h}} = \lambda_0(x;s),$
3. $\forall t_0 \in \mathbb{R}$, $\frac{\partial^2 \Psi}{\partial t^2}(x,e^{its})|_{t=t_0} \geq 0$, with equality if and only if $\mathcal{L}_{x}^*(e^{it_0}\lambda x) = 0$,
4. $\forall t_0 > 0$, $\Psi(x,e^{it_0}x) \geq (l-t_0)\lambda_1(x;s) + C_s(x;t_0)$, where $C_s(x;t_0)$ is a continuous function on $x \in M$, $s \in \mathfrak{h}$ and $t_0 \in \mathbb{R}$.

**Proof.** By definition, $\Psi(x,e^{is}) = \int_0^1 \sigma^x$, where $\gamma$ is any path in $L$ joining $1 \in L$ to $e^{is}$. If we take $\gamma : [0,1] \ni t \mapsto e^{its}$, then the integral reduces to $\int_0^1 \langle \mu(e^{its}), s \rangle dt$. This proves (1). Property (2) is deduced from (1) differentiating. (3) is a consequence of (1) and the fact that $\lambda_t(x;s)$ increases with $t$. To prove (4), let $C_s(x;t_0) = \int_0^1 \lambda_t(x;s)dt$. Then:

$$\int_0^1 \lambda_t(x;ls)dt = \int_0^1 \lambda_t(x;s)dt \geq (l-t_0)\lambda_1(x;s) + C_s(x;t_0);$$

the first equality is obtained making a change of variable and using (2) in Proposition 2.4, and the inequality comes from the fact that $\lambda_t(x;s)$ increases as a function of $t$. 

**Proposition 3.4.** Let $x \in M$ be any point, and let $s \in \mathfrak{h}$.

1. If $g,h \in L$, then $\Psi(x,g) + \Psi(gx,h) = \Psi(x,hg),$
2. for any $k \in H$ and $g \in L$, $\Psi(x,kg) = \Psi(x,g)$, and $\Psi(x,1) = 0$,
3. for any $k \in H$ and $g \in L$, $\Psi(kx,h) = \Psi(x,k^{-1}gk)$.

**Proof.** To prove (1), observe that for any $g \in L$, $\sigma^{gx} = R_g^* \sigma^x$, where $R_g$ denotes right multiplication in $L$ (indeed, for any $g' \in L$ one has $\sigma^{gx}(g') = \sigma^x(g'g)$ – as usual, we identify the tangent spaces $T_g(L)$ and $T_{g'g}(L)$ making $L$ act on the right). This equivalence, together with the requirement that $\Psi_{gx}(1) = 0$ implies that, for any $h \in L$, $\Psi_{gx}(h) = \Psi_x(hg) - \Psi_x(g)$. Property (2) is a consequence of (1) together with the fact that, for any $x \in M$, $\Psi_x|_H = 0$. Finally, to prove (3) we use points (1) and (2): $\Psi(x,k^{-1}gk) = \Psi(x,gk) + \Psi(gkx,k^{-1}) = \Psi(x,k) + \Psi(kx,g) = \Psi(kx,g)$. 

**Proposition 3.5.** An element $g \in L$ is a critical point of $\Psi_x$ if and only if $\mu(gx) = 0$.

**Proof.** This is a consequence of (2) in 3.3 and (1) in 3.4. 

Just like maximal weights, the function $\Psi$ depends on the moment map, which is not unique. When it is not clear from the context which moment map we consider, we will write $\Psi^\mu$ to mean the integral of the moment map $\mu$. 
3.3. **Linear properness.** In this section we restrict to the case \((M; H, L) = (F; K, G)\). Let \(\rho_a : g \to \text{End}(W_a)\) be the complexification of the (differential of the) representation \(\rho_a : K \to U(W_a)\) chosen in subsection 2.1. We define a norm on \(g\) as follows: for any \(s \in g\),

\[
|s| = (s, s)^{1/2} = \text{Tr}(\rho_a(s)\rho_a(s)^*)^{1/2}.
\]

Let \(\log_G : G \simeq K \times \exp(\mathfrak{k}) \to \mathfrak{k}\) denote the projection to the second factor of the Cartan decomposition composed with the logarithm. For any \(g \in G\) we will call \(|g|_{\log} := |\log_G g|\) the length of \(g\).

**Definition 3.6.** We will say that \(\Psi_x\) is linearly proper if there exist positive constants \(C_1\) and \(C_2\) such that for any \(g \in G\)

\[
|g|_{\log} \leq C_1 \Psi_x(g) + C_2.
\]

**Proposition 3.7.** Let \(h \in G\) and \(x \in F\). If \(\Psi_x\) is linearly proper then \(\Psi_{hx}\) is also linearly proper.

Before giving the proof of this proposition we prove the following technical result.

**Lemma 3.8.** Let \(N = \dim W_a\) and \(h \in G\). There exists \(C \geq 1\) such that for any \(g \in G\)

\[
N^{-1/2}|gh|_{\log} - \log C \leq |g|_{\log} \leq N^{1/2}(|gh|_{\log} + \log C).
\]

Furthermore, \(C\) depends continuously on \(h \in G\).

**Proof.** Since the Cartan decomposition commutes with unitary representations, we may describe the length function as follows. Let \(x \in G\) be any element and write \(\rho_a(x) = RS\), where \(R \in U(W_a)\) and \(S = \exp(u)\), where \(u = u^*\). The matrix \(u\) diagonalises and has real eigenvalues \(\lambda_1, \ldots, \lambda_N\). So \(|x|_{\log}^2 = \sum_{j=1}^N \lambda_j^2\). Define \(\max(x) = \max\|v\|=1|\log \|\rho_a(x)v\||\). Then we have \(\max |\lambda_j| = \max(x)\) and consequently

\[
\max(x) \leq |x|_{\log} \leq N^{1/2}\max(x). \quad (3.6)
\]

Let now \(h \in G\). Then there exists \(C \geq 1\), depending continuously on \(h\), such that for any \(g \in G\) and any \(v \in V\), \(C^{-1}\|\rho_a(gh)v\| \leq \|\rho_a(g)v\| \leq C\|\rho_a(gh)v\|\), which implies

\[
|\max(gh) - \max(g)| \leq \log C. \quad (3.7)
\]

Putting \(x = gh\) in (3.6) we obtain

\[
N^{-1/2}|gh|_{\log} \leq \max(gh) \leq |gh|_{\log}, \quad (3.8)
\]

and combining (3.6) with \(x = g\) and (3.7) we get

\[
\max(gh) - \log C \leq |g|_{\log} \leq N^{1/2}(|gh|_{\log} + \log C).
\]

Finally, using (3.8) we get

\[
N^{-1/2}|gh|_{\log} - \log C \leq |g|_{\log} \leq N^{1/2}(|gh|_{\log} + \log C). \quad \square
\]

**Proof.** (Proposition 3.7) Suppose that \(\Psi_x\) is linearly proper, that is, for any \(g \in G\)

\[
|g|_{\log} \leq C_1 \Psi_x(g) + C_2,
\]
where $C_1, C_2$ are positive. Fix $h \in G$. Let $C \geq 1$ be the constant in Lemma 3.8. (1) in 3.4 tells us that $\Psi_{hx}(g) = \Psi_x(gh) - \Psi_x(h)$, so we get for any $g \in G$

\[
|g|_{\log} \leq N^{1/2}(|gh|_{\log} + \log C) \leq N^{1/2}(C_1 \Psi_x(gh) + C_2 + \log C) \\
= N^{1/2}(C_1 (\Psi_x(gh) - \Psi_x(h)) + C_1 \Psi_x(h) + C_2 + \log C) \\
= N^{1/2}(C_1 \Psi_{hx}(g) + C_1 \Psi_x(h) + C_2 + \log C),
\]

so setting $C'_1 = N^{1/2}C_1$ and $C'_2 = \max\{0, N^{1/2}(C_1 \Psi_x(h) + C_2 + \log C)\}$ then $C'_1, C'_2$ are positive and $|g|_{\log} \leq C'_1 \Psi_{hx}(g) + C'_2$. This proves that $\Psi_{hx}$ is linearly proper. \qed

4. A Kaehler structure on $\mathcal{A}^{1,1} \times \mathcal{I}$

In this section we will give, following the classical idea of Atiyah and Bott [AB], a $\mathcal{G}_K$-invariant Kaehler structure on the manifold $\mathcal{A} \times \mathcal{I}$. This structure will depend on our choice of a biinvariant metric on $\mathfrak{t}^*$, and consequently on the representation $\rho_a$ used to define it. We will identify for this structure a moment map of the action of $\mathcal{G}_K$, the maximal weights and the integral of the moment map.

4.1. Unitary connections.

4.1.1. $\mathcal{A}$ is a Kaehler manifold. Let $\mathcal{A}$ be the space of $K$-connections on $E$. It is an affine space modelled on $\Omega^1(E \times_{Ad} \mathfrak{t})$. We define a complex structure $I_{\mathcal{A}}$ on $\mathcal{A}$ as follows. Given any $A \in \mathcal{A}$, the tangent space $T_A \mathcal{A}$ can be canonically identified with $\Omega^1(E \times_{Ad} \mathfrak{t}) = \Omega^0(T^*X \otimes E \times_{Ad} \mathfrak{t})$. Then we set $I_{\mathcal{A}} = -I^* \otimes 1$. The complex structure $I_{\mathcal{A}}$ is integrable. We also define on $\mathcal{A}$ a symplectic form $\omega_{\mathcal{A}}$. Let $\Lambda : \Omega^{p,q}(X) \to \Omega^{p-1,q-1}(X)$ be the adjoint of the map given by wedging with $\omega$. Then, if $A \in \mathcal{A}$ and $\alpha, \beta \in T_A \mathcal{A} \simeq \Omega^1(E \times_{Ad} \mathfrak{t})$, we set

\[
\omega_{\mathcal{A}}(\alpha, \beta) = \int_X \Lambda(B_1(\alpha, \beta)).
\]

Here $B_1 : \Omega^1(E \times_{Ad} \mathfrak{t}) \otimes \Omega^1(E \times_{Ad} \mathfrak{t}) \to \Omega^2$ is the combination of the usual wedge product with the biinvariant nondegenerate pairing $\langle, \rangle$ on $\mathfrak{t}$ obtained from the representation $\rho_a$. It turns out that $\omega_{\mathcal{A}}$ is a symplectic form on $\mathcal{A}$, and it is compatible with the complex structure $I_{\mathcal{A}}$. Hence $\mathcal{A}$ is a Kaehler manifold. Furthermore, the action of $\mathcal{G}_K$ on $\mathcal{A}$ defined in subsection 2.2 is holomorphic and is the complexification of the action of $\mathcal{G}_K$.

4.1.2. The moment map. Recall that the Lie algebra of $\mathcal{G}_K$ is $\text{Lie}(\mathcal{G}_K) = \Omega^0(E \times_{Ad} \mathfrak{t})$. There exists a moment map for the action of $\mathcal{G}_K$ on $\mathcal{A}$, and it takes the following form (see for example [DoKl, Ko]):

\[
\mu : \mathcal{A} \longrightarrow \text{Lie}(\mathcal{G}_K)^* \\
A \mapsto \Lambda F_A.
\]

The curvature $F_A$ of $A$ lies in $\Omega^2(E \times_{Ad} \mathfrak{t})$, so $\Lambda F_A \in \Omega^0(E \times_{Ad} \mathfrak{t}) \subset \Omega^0(E \times_{Ad} \mathfrak{t})^*$, the last inclusion being given by the integral of the pairing $\langle, \rangle$.

The proof of the next lemma is an easy exercise.
Lemma 4.1. Let $A \in \mathcal{A}$ be a connection, and take $s \in \text{Lie}(\mathcal{G}_K) = \Omega^0(E \times_{\text{Ad}} \mathfrak{k})$. Then
\[
\lambda_t(A; s) = \int_X \langle \Lambda F_A, s \rangle + \int_0^t \|e^{i ts} \overline{\partial}_A(s) e^{-it} \|^2 dt. \quad (4.9)
\]

When $s \in L^2(E \times_{\text{Ad}} \mathfrak{k})$ the maximal weight is given by exactly the same formula. To prove it one needs to use a technical theorem of Uhlenbeck and Yau [UY]. This result allows to regard $s$ as a genuine smooth section of $E \times_{\text{Ad}} \mathfrak{k}$ at the complementary of a complex codimension two subvariety of $X$, and to check that the integrals appearing in Lemma 4.1 converge.

4.1.3. The integral of the moment map. The $K$-equivariance of the Cartan decomposition implies that $\mathcal{G}_G \simeq \mathcal{G}_K \times i \text{Lie}(\mathcal{G}_K)$, and from this fact, using that $\pi_1(K) \to \pi_1(G)$ is surjective, we see that $\pi_1(\mathcal{G}_K) \to \pi_1(\mathcal{G}_G)$ is a surjection (both maps are the ones induced by the inclusions). As a consequence, the results of section 3 apply to actions of $\mathcal{G}_K$ on Kaehler manifolds. So there is an integral of the moment map $\Psi^{\mathcal{G}}$ which satisfies all the properties given in section 3.2. Fix now a connection $A \in \mathcal{A}$. By (4.1) and using (1) in Proposition 3.3 we see that
\[
\Psi^{\mathcal{G}}_A(e^is) = \int_0^1 \lambda_t(A; s) = \int_X \langle \Lambda F_A, s \rangle + \int_0^1 \left( \int_0^t \|e^{i ts} \overline{\partial}_A(s) e^{-it} \|^2 dt \right) dt = \int_X \langle \Lambda F_A, s \rangle + \int_0^1 (1-l)\|e^{i ts} \overline{\partial}_A(s) e^{-its} \|^2 dt. \quad (4.10)
\]

Then, by (2) in 3.4, the function $\Psi^{\mathcal{G}}_A$ factors through
\[
\Psi^{\mathcal{G}}_A : \mathcal{G}_G/\mathcal{G}_K \to \mathbb{R}.
\]
The resulting functional may be seen as a modified Donaldson functional. In fact, when $F = \{\text{pt}\}$, it coincides (up to a multiplicative constant) with the Donaldson functional. To see this, one only has to check that the Donaldson functional satisfies property (2) in 3.3 (see [Br2, Lemma 3.3.2] for the case $F = \mathbb{C}^n$).

We will use the restriction of the integral of the moment map to $\mathcal{A}^{1,1} \times \mathcal{I} \times \mathcal{G}_G$ ($\mathcal{A}^{1,1} \subset \mathcal{A}$ is a complex subvariety, but in general it is not smooth). This functional will be the main tool in proving Theorem 2.13.

4.1.4. Maximal weights for $A \in \mathcal{A}^{1,1}$. Note that since $\mathcal{A}^{1,1} \subset \mathcal{A}$ is a $\mathcal{G}_G$ invariant subvariety, the moment map, the maximal weights and the integral of the moment map of the action of $\mathcal{G}_K$ on $\mathcal{A}^{1,1}$ are the restrictions of their counterparts in $\mathcal{A}$.

Recall that $V = E \times_{\rho_a} W_a \to X$ is the vector bundle associated to the representation $\rho_a$. For any $s \in \text{Lie}(\mathcal{G}_K)$ we can view $\rho_a(s)$ as a section of $E \times_{\text{Ad}(\rho_a)} \text{End}(W_a)$. Take a connection $A \in \mathcal{A}^{1,1}$, and consider on $V$ the holomorphic structure induced by $\overline{\partial}_A$. Using Lemma 4.1 one can prove the following (see [M]).

Lemma 4.2. Let $s \in \text{Lie}(\mathcal{G}_K)$. If $\lambda(A; s) < \infty$, then the eigenvalues of $\rho_a(s)$ are constant. Let $\lambda_1 < \cdots < \lambda_r$ be the different eigenvalues of $i \rho_a(s)$, and let $V(\lambda_j) \subset V$
be the eigenbundle of eigenvalue $\lambda_j$. Put $V^{\lambda_k} = \bigoplus_{j \leq k} V(\lambda_j)$. Then, for any $k$, $V^{\lambda_k}$ is a holomorphic subbundle of $V$. Furthermore

$$\lambda(A; s) = \lambda_r \deg(V) + \sum_{k=1}^{r-1} (\lambda_k - \lambda_{k+1}) \deg(V^{\lambda_k})$$

If we consider more generally $s \in L^2_1(E \times \text{Ad} \mathfrak{k})$, then $\lambda(A; s) < \infty$ leads to a filtration of the locally free sheaf associated to $V$ by reflexive (coherent) subsheaves, and not only holomorphic subbundles of $V$ as in the smooth case. To prove this one uses a theorem of Uhlenbeck and Yau (see [UY] and [Br2, §3.11]).

4.2. Sections of the associated bundle.

4.2.1. $\mathcal{S}$ is a Kaehler manifold. Let us define a complex structure $I_\mathcal{S}$ and a symplectic form $\omega_\mathcal{S}$ on $\mathcal{S} = \Gamma(\mathcal{F})$. Consider a section $\sigma \in \mathcal{S}$. The tangent space $T_\sigma \mathcal{S} = \Gamma(\sigma^*TF_v)$, where $TF_v \subset TF$ is the subbundle of vertical tangent vectors of $F$, that is, $TF_v = \text{Ker}(d\pi_F)$. Let $\alpha \in \Gamma(\sigma^*TF_v)$. We set by definition $I_\mathcal{S}(\alpha) = I_F\alpha$. This makes sense, since the $K$-invariance of $I_F$ implies that $TF_v$ inherits the complex structure of $F$. Now let $\alpha, \beta \in \Gamma(\sigma^*TF_v)$. We define the symplectic form $\omega_\mathcal{S}$ on $\mathcal{S}$ as

$$\omega_\mathcal{S}(\alpha, \beta) = \int_X \omega_F(\alpha, \beta).$$

The 2-form $\omega_\mathcal{S}$ is nondegenerate (this is a consequence of the nondegeneracy of $\omega_F$) and $\omega_\mathcal{S}$ and $I_\mathcal{S}$ are compatible, that is, $\langle \alpha, \beta \rangle = \omega_\mathcal{S}(\alpha, I_\mathcal{S}\beta)$ is a Riemannian pairing. The two structures are integrable, and so $\mathcal{S}$ is a Kaehler manifold.

4.2.2. The actions of $\mathcal{G}_K$ and $\mathcal{G}_G$ and the moment map. Both groups $\mathcal{G}_K$ and $\mathcal{G}_G$ act on the space of sections $\mathcal{S} = \Gamma(\mathcal{F})$, and the action of $\mathcal{G}_G$ is the complexification of the action of $\mathcal{G}_K$. On the other hand, $\mathcal{G}_K$ acts by isometries and respecting the symplectic form, and there exists a moment map $\mu_\mathcal{S}$, which is equal fibrewise to $\mu$ (the moment map of the action of $K$ on $F$). As such, it is a section of $\Omega^0(E \times \text{Ad} \mathfrak{k})^*$.

4.2.3. Maximal weights. The maximal weight of $s \in \text{Lie}(\mathcal{G}_K) = \Omega^0(E \times \text{Ad} \mathfrak{k})$ acting on a section $\Phi \in \mathcal{S}$ is given by the integral of the maximal weight in each fibre:

$$\int_{x \in X} \lambda(\Phi(x); s(x)).$$

This makes sense due to the $K$-equivariance of $\lambda$. See (1) in Lemma 2.4.

4.2.4. The integral of the moment map. The results in section 3 imply that there exists an integral $\Psi_\mathcal{S}$ of the moment map of the action of $\mathcal{G}_K$ on $\mathcal{S}$. If $\Psi : F \times G \to \mathbb{R}$ is the integral of the moment map of the action of $K$ on $F$, then, for any section $\sigma \in \mathcal{S}$ and gauge transformation $g \in \mathcal{G}_G$

$$\Psi_\mathcal{S}(\sigma, g) = \int_{x \in X} \Psi(\sigma(x), g(x)).$$

This makes sense due to the $K$-equivariance of $\Psi$: see (3) in 3.4.
4.3. **Symplectic point of view.** We saw that both $\mathcal{A}^{1,1}$ and and $\mathcal{I}$ are Kaehler manifolds, with symplectic forms $\omega_{\mathcal{A}}$ and $\omega_{\mathcal{I}}$ and with actions of $G_K$ extending to actions of the complexification $G_{\mathbb{C}}$. Hence $\mathcal{A}^{1,1} \times \mathcal{I}$ is also a Kaehler manifold, with symplectic form $\omega_{\mathcal{A}} + \omega_{\mathcal{I}}$ (we omit the pullbacks). The moment map $\mu_{\mathcal{A} \times \mathcal{I}}$ of the action of $G_K$ on $\mathcal{A} \times \mathcal{I}$ will simply be the moment map of the action on $\mathcal{A}$ plus that of the action on $\mathcal{I}$. That is,

$$\mu_{\mathcal{A} \times \mathcal{I}}(A, \Phi) = \Lambda \Lambda_{\mathcal{A}} + \mu(\Phi).$$

So equation (1.1) can be written as $\mu_{\mathcal{A} \times \mathcal{I}} = c$, where $c$ denotes the central element in $(\text{Lie}(G_K))^* = \Omega^0(E \times \text{Ad} \mathfrak{k})^*$ which is fibrewise equal to a central element $c \in \mathfrak{k}^*$. Furthermore, we have the following result.

**Lemma 4.3.** $T_{\phi}(\sigma, \chi) = \lambda^{\Lambda \Lambda_{\mathcal{A}} + \mu(\Phi) - c}((A, \Phi); -ig_{\sigma, \chi}).$

**Proof.** Combine subsections 4.1.4 and 4.2.3. □

As a final comment, note that so far we have defined the gauge group as the space of smooth sections of a certain bundle. Eventually, it will be necessary to take a metric on $G_K$ (and $G_{\mathbb{C}}$) and complete both spaces with respect to the metric, to assure the convergence of certain sequences. We will use Sobolev $L^2$ and $L^1$ norms.

5. **Analytic stability and vanishing of the moment map in finite dimension**

We will now pause to prove Theorem 2.19 in the case $X = \{\text{pt}\}$, which is much easier than the general case and is interesting per se. The results in this section (at least for the case in which $F$ is projective) have been known for many years: see [KoNe, Ki]. That they are related with Hitchin–Kobayashi correspondence was also known since the first cases of the correspondence were studied. Our intention here is to make more concrete this relation and to stress on the similarities between the finite dimensional situation $X = \{\text{pt}\}$ and the general one considered in Theorem 2.19 (which corresponds to the situation in which $F = \mathcal{A}^{1,1} \times \mathcal{I}$ with the actions of $G_K$ and $G_{\mathbb{C}}$). For example, the different versions of Donaldson functional used in the literature are in fact particular instances of a construction which works for a wide class of Kaehler actions of Lie groups on Kaehler manifolds (namely, what we have called the integral of the moment map). Moreover, the $c$-stability condition is also a particular case of a general notion of stability for group actions on Kaehler manifolds (the so-called analytic stability). And the very correspondence coincides almost word by word with Theorem 5.4 given in this section. The proof which we give here works only for Kaehler actions of compact groups, and so it can not be used in the general situation (in which the group is $G_K$). Nevertheless, the scheme of the proof will be the same in the general situation.

Let us write $\Psi : F \times G \to \mathbb{R}$ for the integral of the moment map $\mu : F \to \mathfrak{k}^*$.

**Definition 5.1.** Let $x \in F$. We will say that $x$ is analytically stable if for any $s \in \mathfrak{k}$ the maximal weight of $s$ acting on $x$ is strictly positive:

$$\lambda(x; s) > 0.$$
Lemma 5.2. A point \( x \in F \) is analytically stable if and only if \( \Psi_x \) is linearly proper.

Proof. Suppose first that \( x \) is analytically stable. We have to prove that there exists two positive constants \( C_1, C_2 \in \mathbb{R} \) such that, for any \( s \in \mathfrak{t} \), \( |s| \leq C_1 \Psi_x(e^{is}) + C_2 \). Assume that there are not such constants. Then, we can find sequences \( \{s_j\} \subset \mathfrak{t} \) and \( \{c_j\} \subset \mathbb{R} \) such that \( |s_j| \to \infty \), \( C_j \to \infty \) and, for any \( j \), \( |s_j| \geq C_j \Psi_x(e^{is_j}) \). Let \( u_j = s_j/|s_j| \). After passing to a subsequence, we can assume that \( u_j \to s \). Take now any \( t > 0 \). By our hypothesis, and making use of (4) in Proposition 3.3,

\[
\frac{1}{C_j} \geq \frac{\Psi_x(e^{is_j})}{|s_j|} \geq \left( \frac{|s_j| - t}{|s_j|} \right) \lambda_t(x; u_j) + \frac{C_{u_j}(x; t)}{|s_j|}.
\]

Now, making \( j \to \infty \), we obtain \( 0 \geq \lambda_t(x; s) \), since, by the compactness of \( B_t(1) = \{ s \in \mathfrak{t} | |s| = 1 \} \), \( C_{u_j}(x; t) \) is uniformly bounded. This is true for any \( t > 0 \), so passing to the limit \( t \to \infty \) we get \( 0 \geq \lambda(x; s) \), which contradicts analytic stability.

Now suppose that there exists positive \( C_1, C_2 \) such that for any \( s \in \mathfrak{t} \)

\[
|s| \leq C_1 \Psi_x(e^{is}) + C_2. \tag{5.11}
\]

We have to prove that \( x \) is analytically stable. So take \( s \in \mathfrak{t} \) and assume that \( \lambda(x; s) \leq 0 \). In this case, for any \( t \geq 0 \), \( \Psi_x(e^{its}) = \int_0^t \lambda_t(x; s)dl \leq 0 \), which, for \( t \) big enough, contradicts (5.11). This proves that \( x \) is analytically stable. \( \square \)

Corollary 5.3. Let \( x \in F \). Then \( x \) is analytically stable if and only if \( hx \) is analytically stable for any \( h \in G \).

Proof. This is a consequence of the preceeding lemma together with Lemma 3.7. \( \square \)

Theorem 5.4. Let \( x \in F \) be any point. There is at most one \( K \) orbit inside the orbit \( Gx \subset F \) on which the moment map vanishes. Furthermore, \( x \) is analytically stable if and only if: (1) the stabiliser \( G_x \) of \( x \) in \( G \) is finite and (2) there exists a \( K \) orbit inside \( Gx \) on which the moment map vanishes.

Proof. We first prove uniqueness. Assume that there are two different \( K \) orbits inside a \( G \) orbit on which the moment map vanishes, say \( Kx \) and \( Kgx \), where \( g \in G \). By the polar decomposition we can assume that \( g = e^{is} \), where \( s \in \mathfrak{t} \). Consider the function \( \Psi_x : G \to \mathbb{R} \). By Proposition 3.3, since \( \mu(x) = 0 \), both \( 1, g \in G \) are critical points of \( \Psi_x \). Consider now the path \( \gamma(t) = e^{its} \) connecting \( 1 \) and \( g \). (3) of the proposition tells us that the restriction \( \psi \) of \( \Psi_x \) to this path has second derivative \( \geq 0 \). Since \( 0 \) and \( 1 \) are critical points of \( \psi \), the second derivative must vanish at any point between \( 0 \) and \( 1 \). In particular, \( \frac{\partial^2 \psi}{\partial t^2}(x, e^{its})|_{t=0} = 0 \); but this implies (again, (3) of the proposition), that the vector field \( \mathcal{X}_s(x) = 0 \), which gives \( \mathcal{X}_s(x) = I \mathcal{X}_s(x) = 0 \). So \( e^{is}x = e^{is}x = x \), and the two orbits \( Kgx \) and \( Kx \) coincide.

Suppose now that the point \( x \) is analytically stable. Let us see that there is a \( K \) orbit inside \( Gx \) on which \( \mu \) vanishes. By Lemma 3.2, the function \( \Psi_x \) is linearly proper. Using (2) in 3.4, we conclude that there must exist a critical point in the \( G \) orbit of \( x \). Indeed, if \( \{s_j\} \subset \mathfrak{t} \) are such that \( e^{is_j} \) is a minimising sequence for \( \Psi_x \), then by the
preceeding lemma the set \(\{s_j\}\) is bounded; so it has a subsequence converging to a certain \(s \in \mathfrak{k}\), and \(e^{is}\) is a minimum of \(\Psi_x\) (of course, here we use that \(\mathfrak{k}\) has finite dimension). At this point (even more, at the \(K\) orbit through this point) the moment map must vanish. Let now \(y = e^{is}x\). By Lemma 5.3 \(y\) is analytically stable. If the stabiliser \(K_y\) of \(y\) in \(K\) were not finite, then, since \(K\) is compact, its closure would be a Lie subgroup of \(K\) of dimension greater than zero. In particular, there would exist an \(s \in \mathfrak{k}\) such that \(\mathcal{H}_s(y) = 0\). But then \(e^{ts}y = y\) for any \(t\), so that the gradient flow \(\phi^t_s\) leaves \(y\) fixed. This means that \(\lambda(y; s) = -\lambda(y; -s)\), so that either \(\lambda(y; s)\) or \(\lambda(y; -s)\) (or both) is \(\leq 0\). This contradicts analytic stability. So \(K_y\) is finite.

Finally, since \(\mu(y)\) is invariant under the coadjoint action of \(K\) in \(\mathfrak{k}^*\), it turns out that \(G_y\) is the complexification of \(K_y\). Let us see why (we copy the proof of \([Sj, \text{Proposition 1.6}]\)). One inclusion is easy: \(G_y\) contains the complexification of \(K_y\). For the other inclusion, let \(ge^{is}\) be an arbitrary element of \(G_x\), where \(g \in K\) and \(s \in \mathfrak{k}\). We want to show that \(g \in K_x\) and \(s \in \mathfrak{k}_x\) (where \(\mathfrak{k}_x\) is the infinitesimal stabiliser of \(x\)). Using the fact that \(\mu\) is \(K\)-equivariant we have

\[
\mu(e^{is}x) = g^{-1}\mu(ge^{is}x) = g^{-1}\mu(x) = \mu(x).
\]

Now, Lemma 2.2 implies that \(s \in \mathfrak{k}_x\), from which we deduce that \(g \in K_x\). This finishes the proof. So \(G_y\) is finite and in consequence \(G_x\) is also finite.

To prove the converse, let \(x \in F\). Assume that \(G_x\) is finite and that there exists \(g \in G\) such that \(\mu(gx) = 0\). Then \(G_{gx}\) is finite and consequently so is \(K_{gx}\). This implies that, for any \(s \in \mathfrak{k}\), \(\mathcal{H}_{is}(gx) \neq 0\), so (Lemma 2.2), \(\lambda(gx; s) > \mu_s(gx) = 0\). This means that \(gx\) is analytically stable, hence so is \(x\).

It is an exercise to verify that the property on analytically stable points of \(F\) of being simple (see subsection 2.9) is equivalent to that of having finite stabiliser in \(G\).

Using the results in this section one can also study the equation \(\mu = c\), where \(c \in \mathfrak{k}^*\) is any central element. Indeed, \(\mu - c\) is a moment map, and so one only has to consider the maximal weights \(\lambda^{t-c}\) and the integral \(\Psi^{t-c}\).

**Remark 5.5.** Suppose that \(F \subset \mathbb{P}^n\) is a projective manifold and that the Kaehler structure on \(F\) is that induced by the Kaehler structure on \(\mathbb{P}^n\). Using the Hilbert-Mumford numerical criterium, one can easily prove that in this context the property of being analytically stable and having finite stabiliser is the same as being stable in the sense of Mumford Geometric Invariant Theory (see \([\text{MFK}]\) and lemma 8.8 and remark 8.9 in \([\text{Kr}]\)).

6. Proof of the correspondence

6.1. The length of elements of the gauge group. There are several ways to extend the notion of length to elements of the gauge group. We will use these two definitions: if \(g \in \mathcal{G}_G\), then \(|g|_{\log,C^0} = |||g|||_{\log,C^0}\) and \(|g|_{\log,L^1} = |||g|||_{\log,L^1}\) (to give this a sense we use the \(K\) invariance of the length function, which is a consequence of the fact that the Cartan decomposition \(G \simeq K \times \exp(i\mathfrak{k})\) is \(K\)-equivariant). Define a norm \(||\cdot||_{L^p}\).
in \( \text{Lie}(\mathcal{G}) = \Omega^0(K \times \text{Ad} \mathfrak{g}) \) as the \( L^p \) norm of \(| \cdot |\): if \( s \in \Omega^0(K \times \text{Ad} \mathfrak{g}) \) then

\[
\|s\|_{L^p} = \left( \int_{x \in X} |s(x)|^p \right)^{1/p}.
\]

We will usually write \( \| \cdot \| \) instead of \( \| \cdot \|_{L^2} \).

### 6.2. Stability implies existence of solution.

Here we will follow the scheme in section 3. Fix a pair \((A, \Phi) \in \mathcal{A}^{1,1} \times \mathcal{I}\). We will make use of the integral of the moment map \( \mu^c(A, \Phi) = \Lambda F_A + \mu(\Phi) - c \); \( \Psi = (\Psi^\mathcal{A} \times \mathcal{I})^{\mu^c}_{\Lambda}(\Phi) = (\Psi^\mathcal{A})^{\mu^c}_{\Lambda}(\Phi) + (\Psi^\mathcal{I})^{\mu^c}_{\Phi} \); and will see that if the pair \((A, \Phi)\) is simple and \(c\)-stable, then there exists a \( \mathcal{G}_K \) orbit inside the \( \mathcal{G}_G \) orbit of \((A, \Phi)\) on which \( \Psi^c \) attains its minimum. The main step will be to prove that if the condition of \(c\)-stability is satisfied, then the map \( \Psi^c \) satisfies an inequality like that in Lemma 5.2. This method of proof is exactly the same that appears in [Si, Br2, BrGP1, DaUW] (and in many other places where similar results are proved), though here we have tried to remark the similarities with the finite dimensional case, so our notation changes a little bit. However, in some steps of the proof we will only give a sketch, refering to [Br2] for details.

Recall that on \( \mathfrak{g} \) we have a Hermitian pairing \( \langle , \rangle : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C} \) and a norm \(| \cdot |\), both obtained by means of the representation \( \rho^\mathfrak{g} \). We will use the following \( L^p \) norm on \( \Omega^0(E \times \text{Ad} \mathfrak{g}) \):

\[
\|s\|_{L^p} = \left( \int_X |s(x)|^p \right)^{1/p},
\]

and Sobolev norm

\[
\|s\|_{L^p_S} = \|s\|_{L^p} + \|d_A s\|_{L^p} + \|\nabla d_A s\|_{L^p},
\]

where \( \nabla : \Omega^0(T^*X \otimes E \times \text{Ad} \mathfrak{g}) \to \Omega^1(T^*X \otimes E \times \text{Ad} \mathfrak{g}) \) is \( \nabla_{LC} \otimes d_A, \nabla_{LC} \) being the Levi-Civita connection. As usual, \( L^p_S(E \times \text{Ad} \mathfrak{g}) \) will denote the completion of \( \Omega^0(E \times \text{Ad} \mathfrak{g}) \) with respect to the norm \( \| \cdot \|_{L^p_S} \).

#### 6.2.1. Suppose from now on that \((A, \Phi)\) is simple and \(c\)-stable. Our aim is to minimise \( \Psi^c \) in \( \mathcal{G}_G / \mathcal{G}_K \). Through the exponential map we can identify \( \mathcal{G}_G / \mathcal{G}_K \) with \( \Omega^0(E \times \text{Ad} i\mathfrak{k}) \). Fix from now on \( p > 2n \) and define

\[
\mathcal{M}_{2,B}^p = L^p_S(E \times \text{Ad} i\mathfrak{k}).
\]

The first thing to do is to restrict ourselves to the subset of \( \mathcal{M}_{2,B}^p \) defined as follows:

\[
\mathcal{M}_{2,B}^p = \{ s \in \mathcal{M}_{2,B}^p \mid \|\mu^c(e^s(A, \Phi))\|_{L^p} \leq B \}.
\]

Here \( B \) is any strictly positive real constant. We prove that if a metric minimizes the functional in \( \mathcal{M}_{2,B}^p \), then it also minimizes it in \( \mathcal{M}_{2,B}^p \). For that it is enough to see that any minimum in \( \mathcal{M}_{2,B}^p \) lies away from the boundary of \( \mathcal{M}_{2,B}^p \); to verify this claim one needs the hypothesis that the pair \((A, \Phi)\) is simple. Let us briefly explain how this goes (see also [Br2], Lemma 3.4.2).

Suppose that \( s \) minimizes the functional inside \( \mathcal{M}_{2,B}^p \). Let \( B = e^s(A), \Theta = e^s(\Phi) \). Define the differential operator \( L : L^p_S(E \times \text{Ad} i\mathfrak{k}) \to L^p(E \times \text{Ad} i\mathfrak{k}) \) as

\[
L(u) = i \frac{\partial}{\partial t} \mu^c(e^{tu}(B, \Theta)) \bigg|_{t=0} = i \langle d \mu_c, u \rangle_{T(\mathcal{A} \times \mathcal{I})}(B, \Theta).
\]
Now, if we can see that there exists an \( u \) such that
\[
L(u) = -i\mu^c(B, \Theta),
\]
then we can deduce that \( \mu^c(B, \Theta) = 0 \) and, hence, that \( s \) minimizes the functional in the whole space of metrics \( \mathcal{M}_{2}^{p} \) (see [Br2, Lemma 3.4.2] for a proof of this fact). The operator \( L \) is Fredholm and has index zero. Indeed, modulo a compact operator it is
\[
i\Lambda \partial_B \partial_B.
\]
Using the Kaehler identities this is equal to \( \partial_B^* \partial_B \), which is an elliptic self adjoint operator. This implies that if \( \text{Ker}(L) = 0 \) then \( L \) is surjective and so, in particular, equation (6.12) has a solution. Assume that \( L(u) = 0 \), where \( u \in \mathcal{M}_{2}^{p} \).

Then, by Lemma 2.2,
\[
0 = \langle -iL(u), -iu \rangle = \langle (d\mu^c, u)_{T(\mathfrak{a} \times \mathfrak{r}),} -iu \rangle_{\text{Lie}(G_K)}(B, \Theta)
= \| \mathcal{A}_{-iu}^{s, 1 \times \mathfrak{r}}(B, \Theta) \|^2.
\]
And this implies that \( -iu \) leaves \((B, \Theta)\) invariant. Hence if \( u \neq 0 \) then, since \( u \) is semisimple, \((B, \Theta)\) is not simple, so neither is \((A, \Phi)\); and this is a contradiction.

6.2.2. The next step is to prove that the functional \( \Psi^c \) is linearly proper with respect to the \( C^0 \) norm in \( G_G \).

**Lemma 6.1.** There exist positive constants \( C_1, C_2 \) such that for any \( s \in \mathcal{M}_{2,B}^{p} \) one has
\[
\sup |s| \leq C_1 \Psi^c(e^s) + C_2.
\]

**Remark 6.2.** It makes sense to speak about \( \sup |s| \) because, since we took \( p > 2n \), the Sobolev embedding theorem implies that \( L_2^p \hookrightarrow C^0 \) continuously (in fact, this is a compact embedding).

Just as in Lemma 5.2, it is here that one uses the stability of the pair \((A, \Phi)\). First of all one sees that such a bound is equivalent to an \( L^1 \) bound:
\[
\|s\|_{L^1} \leq C_1 \Psi^c(e^s) + C_2
\]
(the constants in both inequalities need not be the same!). One uses that pointwise
\[
|s| \Delta |s| \leq \langle \Lambda F_{e^s(A)} - \Lambda F_A, -is \rangle.
\]
This is proved in full detail in ([Br2], Prop. 3.7.1) for \( G = GL(n; \mathbb{C}) \) and the metric induced by the fundamental representation. In our case, we use the representation \( \rho_a \) to apply this result to our \( G \).

**Lemma 6.3.** For any point \( x \in X \)
\[
0 \leq \langle \mu(e^s \Phi(x)) - \mu(\Phi(x)), -is(x) \rangle_t.
\]

**Proof.** The gradient flow of \( \mu_{-is} \) is precisely \( e^s \) (see Lemma 2.2). \( \square \)

Summing the inequalities (6.14) and (6.15), using Cauchy-Schwartz, and dividing by \( |s| \) we obtain the pointwise bound \( \Delta |s| \leq |\mu^c(e^s, \Phi)) - \mu^c(A, \Phi)| \). Now, by a result of Donaldson (see [Br2], Lemma 3.7.2), this bound allows to relate the \( C^0 \) and \( L^1 \) norms of \( s \) provided \( s \in \mathcal{M}_{2,B}^{p} \). More precisely, we conclude that there exists a constant \( C_B \) such that for any \( s \in \mathcal{M}_{2,B}^{p} \) one has
\[
\|s\|_{C^0} \leq C_B \|s\|_{L^1}.
\]
6.2.3. In order to prove the existence of constants $C_1$ and $C_2$ such that $\|s\|_{L^1} \leq C_1 \Psi^c(e^s) + C_2$, we suppose the contrary and try to deduce that in this case the pair $(A, \Phi)$ cannot be $c$-stable. If there exist not such constants, then we can find a sequence of real numbers $C_j \to \infty$ and elements $s_j \in \mathcal{M}et^p_{A,B}$ with $\|s_j\|_{L^1} \to \infty$ such that $\|s_j\|_{L^1} \geq C_j \Psi^c(e^s)$ (see [Br2, Proposition 3.2.2]). Finally, making $s_j \to \infty$ we get strong convergence $\lim_{s_j \to \infty} \lambda((A, \Phi); -iu_{\infty}) = 0$.

**Lemma 6.4.** After passing to a subsequence, there exists $u_{\infty} \in L^2(E \times \text{Ad } \mathfrak{a})$ such that $u_j \to u_{\infty}$ weakly in $L^2(E \times \text{Ad } \mathfrak{a})$ and such that $\lambda((A, \Phi); -iu_{\infty}) < 0$.

**Proof.** Just as in Lemma 5.2, take $t > 0$. Then (4) in proposition 3.3 gives

$$\frac{1}{C_j} \geq \frac{\Psi^c(e^{s_j})}{\|s_j\|} \geq \frac{l_j - t}{l_j} \int_0^t \lambda_t((A, \Phi); -iu_j) + \frac{1}{l_j} \int_0^t \lambda_t((A, \Phi); -iu_j)dl$$

$$= \frac{l_j - t}{l_j} (\lambda_t(A; -iu_j) + \lambda_t(\Phi; -iu_j))$$

$$+ \frac{1}{l_j} \int_0^t (\lambda_t(A; -iu_j) + \lambda_t(\Phi; -iu_j))dl.$$  \hspace{1cm} (6.16)

Now, since $\|u_j\|_{C^0} \leq C_B$, and $X$ is compact, $\lambda_t(\Phi; -iu_j)$ and $\int_0^t \lambda_t((A, \Phi); -iu_j)dl$ are both bounded. Hence, there exists $C$ such that for any $j$

$$\frac{l_j - t}{l_j} \lambda_t(A; -iu_j) + \frac{1}{l_j} \int_0^t \lambda_t(A; -iu_j)dl < C.$$

Using again the boundedness of $\|u_j\|_{C^0}$ and taking into account Lemma 4.1, we obtain

$$\|\theta_A(u_j)\|_{L^2} < C_1.$$

Now, $u_j = u_j$ (because the Cartan involution leaves $\mathfrak{a}$ fixed), and this implies that $\|u_j\|_{L^2}^2$ is also bounded. So we can take a subsequence (which we again call $\{u_j\}$) that converges weakly to $u_{\infty} \in L^2$. We can also assume that there exists the limit $\lim_{j \to \infty} \lambda_t((A, \Phi); -iu_j)$. On the other hand, since the embedding $L^2 \to \mathcal{L}^2$ is compact, we get strong convergence $u_j \to u_{\infty}$ in $L^2$. $\|u_j\|_{L^2} = 1$ and the uniform bound $\|u_j\|_{C^0} \leq C_B$ imply that $\|u_j\|_{L^2} > C_B^{-1} > 0$, so $u_{\infty} \neq 0$. To see that $\lambda_t((A, \Phi); -iu_{\infty}) \leq \lim_{j \to \infty} \lambda_t((A, \Phi); -iu_j)$, we observe that

$$u_j \in L^2_{0, C_B}(E \times \text{Ad } \mathfrak{a}) = \{s \in L^2(E \times \text{Ad } \mathfrak{a}) \mid |s(x)| \leq C_B \text{ a.e.} \}.$$  

This implies that $u_{\infty} \in L^2_{0, C_B}(E \times \text{Ad } \mathfrak{a})$, and this is enough to get the inequality (see [Br2, Proposition 3.2.2]). Finally, making $j \to \infty$ in formula (6.16) we obtain

$$\lim_{j \to \infty} \lambda_t((A, \Phi); -iu_j) \leq 0,$$

so in particular $\lambda_t((A, \Phi); -iu_{\infty}) \leq 0$. Since this is true for any $t > 0$, we get $\lambda((A, \Phi); -iu_{\infty}) \leq 0$. \hfill \square

The next steps are rather standard. One can prove that $\rho_d(u_{\infty})$ has almost everywhere constant eigenvalues and that it defines a filtration of $V$ by holomorphic subbundles in the complement of a complex codimension 2 subvariety of $X$. This follows exactly the same lines as [Br2, §3.9, 3.10], the main technical point being the
use of a theorem of Uhlenbeck and Yau \cite{UY} on weak subbundles of vector bundles (see \cite[\S 3.11]{Br2}). The filtration of $V$ on $X_0$ and the gauge transformation $u_\infty$ lead to a reduction of the structure group $\sigma \in \Gamma(X_0; E(G/P))$ defined on $X_0$ by \ref{2.7.2} which will be holomorphic thanks to the results in subsection \ref{2.7.3}, and an antidominant character $\chi$ of $P$. The degree of the pair $(\sigma, \chi)$ equals $\lambda(\langle A, \Phi \rangle; -i\mu_\infty) \leq 0$. And this contradicts the stability condition, thus finishing the proof of Lemma \ref{6.4}.

6.2.4. With the inequality of Lemma \ref{6.1} in our hands, we finish the proof of existence of solution to the equations exactly as is done in \cite[\S 3.14]{Br2}. This consists of two steps: the first one is to verify that there exists an element $s \in \mathcal{M}_{t^2, B}$ minimising $\Psi^c$ and the second one is to prove the smoothness of this solution $s$.

6.3. **Existence of solutions implies stability.** The method we will follow in this section will be exactly the same as in the finite dimensional case in section \ref{5.3}. Let us take a simple pair $(A, \Phi) \in \mathfrak{a'}^{1,1} \times \mathcal{S}$. Suppose that there exists a gauge transformation $h \in \mathcal{G}_G$ such that $h(A, \Phi)$ satisfies equation \ref{2.4}. We want to prove that $(A, \Phi)$ is analytically stable.

Take $X_0 \subset X$ with complement of complex codimension 2, $P \subset G$ parabolic, $\chi$ an antidominant character of $P$ and fix a reduction $\sigma \in \Gamma(X_0; E(G/P))$. Thanks to \ref{2.7.1} we get a section $g_{\sigma, \chi} \in \Omega^0(X_0; E \times \text{Ad} i\mathfrak{t})$, and we have to check that $\lambda((A, \Phi); -ig_{\sigma, \chi}) > 0$. In the following two lemmata it will be necessary to take into account that $X_0$ has finite volume and that it has no nonconstant holomorphic functions (the last claim follows from Hartog theorem).

**Lemma 6.5.** For any semisimple $s \in L^p(X_0; E \times \text{Ad} i\mathfrak{t})$ we have $\lambda(h(A, \Phi); s) > 0$.

**Proof.** Suppose the contrary: $\lambda(h(A, \Phi); s) \leq 0$. Arguing as in \cite[\S 3.11]{Br2} (see also Lemma \ref{1.2}) we deduce that the eigenvalues of $s$ are constant. Suppose that $s$ fixes $h(A, \Phi)$. Let $A' = h^*A$. Then $d_{A'} s = 0$, so $\overline{\mathfrak{s}}_{A'} s = 0$. Now, Hartog theorem implies that $s$ extends to a global section $\overline{s} \in L^p_t(X; E \times \text{Ad} i\mathfrak{t})$. By continuity $\overline{s}$ leaves $h(A, \Phi)$ fixed, and it is semisimple (for this we need to use that the eigenvalues of $s$ are constant). This contradicts the fact that $(A, \Phi)$ (and so $h(A, \Phi)$) is semisimple. So $s$ does not fix $h(A, \Phi)$. Finally, to prove that $\lambda(h(A, \Phi); s) > 0$ we argue as in the proof of Theorem \ref{5.3}, using Lemma \ref{2.2}. \hfill $\Box$

**Lemma 6.6.** Fix a positive constant $C_B$. There exist positive constants $C_1, C_2$ such that the following holds. Let $g \in \mathcal{G}_G(X_0) = \Omega^0(X_0; E \times \text{Ad} G)$ be such that $|g|_{\log, C^0} \leq C_B |g|_{\log, L^1} < \infty$. Then $|g|_{\log, C^0} \leq C_1 \Psi_{h(A, \Phi)}^c(g) + C_2$.

**Proof.** Since $h(A, \Phi)$ is analytically stable, given any $B > 0$ there exist constants $C_1$ and $C_2$ such that for any $s \in \mathcal{M}_{t^2, B}$ there is an inequality

$$\sup |s| \leq C_1 \Psi_{h(A, \Phi)}^c(e^s) + C_2. \quad (6.17)$$

Thanks to the preceding lemma, this inequality is valid not only for $s \in \mathcal{M}_{t^2, B}$, but also for any

$$s \in \mathcal{M}_{t^2, B}(C_B) = \{ s \in L^p_t(X_0; E \times \text{Ad} i\mathfrak{t}) | \|s\|_{C^0} \leq C_B \|s\|_{L^1} \},$$

as one can see tracing the proof of Lemma \ref{6.4}. \hfill $\Box$
Lemma 6.7. There is a positive constant $C'$ such that for any $g_{\sigma,\chi}$ and $h$ and for big enough (depending on $g_{\sigma,\chi}$ and $h$) $t > 0$,

\[
|e^{tg_{\sigma,\chi}h^{-1}}|_{\log, C^0} \leq C' \left( |e^{tg_{\sigma,\chi}h^{-1}}|_{\log, L^1} + 1 \right).
\]

(6.18)

Proof. This is a consequence of Lemma 3.8 and the fact that $X$ is compact (so $|h|$ and $|h^{-1}|$ are bounded functions on $X$).

By the properties of the integral of the moment map we have that

\[
\Psi_{h(A,\Phi)}(e^{tg_{\sigma,\chi}h^{-1}}) = \Psi_{h(A,\Phi)}(e^{tg_{\sigma,\chi}}) - \Psi_{(A,\Phi)}(h).
\]

(6.19)

Now, putting $C_B = C'$ in Lemma 6.6, we conclude that

\[
t \sup |g_{\sigma,\chi}| = |e^{tg_{\sigma,\chi}h^{-1}}|_{\log, C^0} \leq C'(C_1 \Psi_{h(A,\Phi)}(e^{tg_{\sigma,\chi}h^{-1}}) + C_2 + 1) \quad \text{by Lemma 6.6}
\]

\[
\leq C'(C_1 \Psi_{h(A,\Phi)}(e^{tg_{\sigma,\chi}}) + C_2') \quad \text{by (6.19)}.
\]

This implies, reasoning like in Theorem 5.4, that $\lambda((A,\Phi); -ig_{\sigma,\chi}) > 0$. By Lemma 4.3, this is equivalent to $T_0^F(\sigma,\chi) > 0$. Hence $(A,\Phi)$ is $c$-stable.

Remark 6.8. When $F$ is a vector space the proof that existence of solution implies stability is much easier if one uses the principle that curvature increases in subbundles (see for example [Br2]). This is a consequence of the fact that the maximal weights of a linear action of $K$ on a vector space are very simple (see Lemma 8.1) and, specially, that the maximal weight of any element $s \in \text{Lie}(G_K)$ is constant along $G_K$ orbits in $\mathcal{A}^{1,1} \times \mathcal{S}$ (see section 8).

6.4. Uniqueness of solutions. The proof is exactly as in the finite dimensional case: it follows from the convexity of the integral of the moment map.

6.5. Nonsimple pairs. The Hitchin–Kobayashi correspondence which we have proved applies only to simple pairs $(A,\Phi)$. This restriction is not always satisfied. As an example, suppose that there are elements in the centre $Z = Z(g)$ of $G$ which leave $F$ fixed (trivial example: $F$ equal to a point). Any element $z \in Z$ gives an element of the Lie algebra of the gauge group, which we still denote by $z$. This element is semisimple and for any $t$ the exponential $\exp(tz)$ fixes all connections in $\mathcal{A}$, and by our assumption fixes also $\Phi$. In this situation, the pair $(A,\Phi)$ is not simple.

When our group $G$ is $\text{GL}(V)$, there is a standard way to solve this problem. We assume that the whole center $Z$ leaves $\Phi$ fixed. We have to split the equation in the $Z$ part and in the $G/Z$ part. This is done as follows. Define $\mathcal{G}_G^0$ to be the set of gauge transformation with determinant pointwise equal to 1, and suppose that there are no semisimple elements in the Lie algebra of $\mathcal{G}_G^0$ which leave $(A,\Phi)$ fixed; under this assumption we can find an element $g \in \mathcal{G}_G^0$ so that $g(A,\Phi)$ solves the trace-free part of the equation (observe that our proof applies to this situation); then Hodge theory gives a central element in $\mathcal{G}_G$ which, composed with $g$, solves the complete equation.
This idea applies for any reductive Lie group $G$. We just need to give a generalisation of the condition of having determinant pointwise equal to 1 which we imposed to the elements in $G_0$. This is given by the following

**Lemma 6.9.** Let $G$ be a reductive Lie group. There exists $k \geq 1$ and a morphism $\phi : G \to (\mathbb{C}^*)^k$ such that $\ker \phi \cap Z$ is a discrete subgroup of $G$.

**Proof.** Take a faithful representation $\rho : G \to GL(W)$. Split $W$ in eigenspaces of the roots of $Z$ acting on $W$: $W = W_1 \oplus \cdots \oplus W_k$, so that any central element $z \in Z$ acts on any piece $W_j$ by homotecies. Then $\rho(G) \subset GL(W_1) \times \cdots \times GL(W_k)$, so that for any $g \in G$ we have $\rho(g) = (g_1, \ldots, g_k)$. Let $\phi : G \to (\mathbb{C}^*)^k$ be defined as $\phi(g) = (\det g_1, \ldots, \det g_k)$. Now suppose that there exists $s \in Z(\mathfrak{g})$ such that, for any $t$, $\phi(e^{ts}) = (1, \ldots, 1)$. Since $e^{ts}$ acts by homotecies on each piece, we must have $\rho(e^{ts}) \in Z(SL(W_1)) \times \cdots \times Z(SL(W_k)) \simeq \mathbb{Z}/w_1\mathbb{Z} \times \cdots \times \mathbb{Z}/w_k\mathbb{Z}$ for any $t$, where $w_j = \dim W_j$. This implies that $\rho(e^{ts}) = (1, \ldots, 1)$ and, since $\rho$ is faithful, $z = 0$. This proves that $\ker \phi \cap Z$ is discrete. $\square$

Suppose now for simplicity that the whole center $Z(G)$ leaves $\Phi$ fixed. We then define $G_0$ to be the set of gauge transformations which fibrewise belong to $\ker \phi$, and proceed as in the case $G = GL(V)$: we find $g \in G_0$ such that the center free part of the equation is solved and then use Hodge theory to solve the complete equation.

## 7. Yang-Mills-Higgs Functional

In order to define the Yang-Mills-Higgs functional for pairs in $\mathcal{A}^{1,1} \times \mathcal{S}$ it will be necessary to extend the definition of covariant derivations on vector bundles to general fibre bundles. Recall that the subbundle $T\mathcal{F}_\nu$ of vertical tangent vectors to $\mathcal{F}$ is by definition $\ker d\pi_F$, where $\pi_F : \mathcal{F} \to X$ is the projection. Using the Kaehler metric on $TF$ we get an induced metric on $T\mathcal{F}_\nu$ (recall that the action of $K$ respects the Kaehler structure and so in particular the Kaehler metric is kept fixed by $K$). In this section we will not use the fact that the complex structure on $F$ is integrable, so that all the results remain valid when $F$ is an almost-Kaehler manifold (in fact we could also consider connections in $\mathcal{A}$).

**Definition 7.1.** Let $A \in \mathcal{A}^{1,1}$ be a connection on $E$. This connection induces a projection $\alpha : T\mathcal{F} \to T\mathcal{F}_\nu$, since $\mathcal{F}$ is a fibre bundle associated to $E$. Take a section $\Phi \in \mathcal{S} = \Gamma(\mathcal{F})$. We define the covariant derivation of $A$ on $\Phi$ as

$$d_A \Phi = \alpha(d\Phi) \in \Omega^1(\Phi^*T\mathcal{F}_\nu).$$

On the other hand, since the complex structure $I_F$ on $F$ is left fixed by the action of $K$, the bundle $\Phi^*T\mathcal{F}_\nu$ has an induced complex structure. This justifies the following definition.

**Definition 7.2.** Let $(A, \Phi) \in \mathcal{A}^{1,1} \times \mathcal{S}$. We define the $\overline{\partial}$-operator of $A$ on $\Phi$ (resp. the $\partial$-operator of $A$ on $\Phi$) to be $\overline{\partial}_A \Phi = \pi^{0,1}d_A \Phi$, (resp. $\partial_A \Phi = \pi^{1,0}d_A \Phi$), where $\pi^{0,1}$ (resp. $\pi^{1,0}$) denotes the projection of $\Omega^1(\Phi^*T\mathcal{F}_\nu)$ to the second (resp. first) summand in the decomposition $\Omega^1(\Phi^*T\mathcal{F}_\nu) = \Omega^{1,0}(\Phi^*T\mathcal{F}_\nu) \oplus \Omega^{0,1}(\Phi^*T\mathcal{F}_\nu)$. 
When $F$ is a vector space, the operators $d_A$, $\partial A$, and $\overline{\partial} A$ coincide with the usual ones for vector bundles (in this case there is a canonical identification $TF_v \simeq F$).

Recall that we have on $\mathfrak{k}$ a nondegenerate biinvariant positive definite pairing $\langle \cdot, \cdot \rangle$. This pairing gives a $K$-equivariant isomorphism $\mathfrak{k} \simeq \mathfrak{k}^*$ and an Euclidean metric on $\mathfrak{k}$ and $\mathfrak{k}^*$.

**Definition 7.3.** Fix a central element $c \in \mathfrak{k}$. The Yang-Mills-Higgs functional $\mathcal{YMH}_c : \mathcal{A}^{1,1} \times \mathcal{S} \to \mathbb{R}$ is defined as

$$\mathcal{YMH}_c(A, \Phi) = \|F_A\|_{L^2}^2 + \|d_A \Phi\|_{L^2}^2 + \|c - \mu(\Phi)\|_{L^2}^2,$$

where $\Phi \in \mathcal{S}$ is a section and $A \in \mathcal{A}^{1,1}$ a connection on $E$.

We will say that two sections $\Phi_0, \Phi_1 \in \mathcal{S}$ are homologous if they induce the same map in cohomology, i.e., $\Phi_0^* = \Phi_1^* : H^*(F) \to H^*(X)$.

**Theorem 7.4.** Fix a section $\Phi_0 \in \mathcal{S}$. The pairs $(A, \Phi) \in \mathcal{A}^{1,1} \times \mathcal{S}$ which minimize the functional $\mathcal{YMH}_c$ among the pairs whose section is homologous to $\Phi_0$ are those which satisfy the following pair of equations

$$\begin{cases} \overline{\partial} A \Phi = 0 \\ \Lambda F_A + \mu(\Phi) = c. \end{cases}$$

(7.20)

The proof of this theorem will be given at the end of this section.

**7.1.** The symplectic form $\omega_F$ gives an element of $\Omega^0(\Lambda^2(TF_v)^*)$, since the action of $K$ keeps $\omega_F$ fixed. On the other hand, the connection $A$ on $E$ induces a projection

$$\alpha : TF \to TF_v$$

onto the subbundle of vertical tangent vectors. From this we obtain a map $\alpha^* : \Lambda^2(TF_v)^* \to \Lambda^2 T^* F$, and we set $\tilde{\omega}_F^A = \alpha^*(\omega_F) \in \Omega^0(\Lambda^2 T^* F) = \Omega^2(F)$. This 2-form is not in general closed. Consider the 2-form $\omega_F^A = \tilde{\omega}_F^A - \langle \pi_F^* F_A, \mu \rangle$.

**Proposition 7.5.** The 2-form $\omega_F^A \in \Omega^2(F)$ is closed, and the cohomology class it represents is independent of the connection $A$.

**Proof.** The form $\omega_F^A$ coincides with the coupling form $\omega_{A,F}$ of the symplectic fibration $F \to X$ and the connection $A$ as defined in [GLeS, Theorem 1.4.1]. This is proved in [GLeS, Example 2.3]. In [GLeS, Theorem 1.4.1] it is proved that $\omega_F^A$ is closed and in [GLeS, Theorem 1.6.1] it is shown that the cohomology of $\omega_F^A$ is independent of the connection $A$. $\square$

**Remark 7.6.** One can prove that $\omega_F^A$ is the image by the generalised Chern-Weil homomorphism (see [BeGeV, Chapter 7]) of the equivariant de Rham form $\mathcal{F}_F = \omega_F - \mu$. This gives another proof of Proposition 7.5 (see [Mu]).

In the sequel we will denote by $[\omega_F]$ the cohomology class represented by $\omega_F^A$. By a slight abuse of notation we will also denote by $[\omega_F]$ any de Rham form representing it.
Proposition 7.7. For any section $\Phi \in \mathcal{F}$ and for any connection $A \in \mathcal{A}^{1,1}$, the following equality holds:

$$\int_X \langle \Lambda F_A, \mu(\Phi) \rangle = \frac{1}{2}(\|\partial_A \Phi\|_{L^2}^2 - \|\bar{\partial}_A \Phi\|_{L^2}^2) - \int_X \Phi^* [\omega_F] \wedge \omega^{[n]}.$$

To prove Proposition 7.7 we will use the following elementary lemma.

Lemma 7.8. Let $V$ and $W$ be two Euclidean vector spaces with scalar products $\langle \cdot, \cdot \rangle_V$ and $\langle \cdot, \cdot \rangle_W$. Suppose that there are complex structures $I_V \in \text{End}(V)$, $I_W \in \text{End}(W)$ and symplectic forms $\omega_V \in \Lambda^2 V^*$, $\omega_W \in \Lambda^2 W^*$ which satisfy the following: $\langle \cdot, \cdot \rangle_V = \omega_V(\cdot, I_V \cdot)$ and $\langle \cdot, \cdot \rangle_W = \omega_W(\cdot, I_W \cdot)$. Take a linear map $f : V \to W$ and let $f^{1,0}$ (resp. $f^{0,1}$) be $(f + I_W \circ f \circ I_V)/2$ (resp. $(f - I_W \circ f \circ I_V)/2$). Let $2n = \dim_{\mathbb{R}} V$. Then $f^* \omega_W \wedge \omega_V^{[n-1]} = \frac{1}{2}(|f^{1,0}|^2 - |f^{0,1}|^2) \omega_W^{[n]}$, where, for any $g \in \text{Hom}(V, W)$, $|g|^2 = \text{Tr} g^* g$.

**Proof.** (Proposition 7.7) Using Lemma 7.8 we have

$$\int_X \Phi^* \omega_F^A \wedge \omega^{[n-1]} = \frac{1}{2}(\|\partial_A \Phi\|_{L^2}^2 - \|\bar{\partial}_A \Phi\|_{L^2}^2)$$

for any section $\Phi : X \to \mathcal{F}$. To apply the lemma we set, for any $x \in X$, $V = T_x X$ and $W = T_{\Phi(x)} \mathcal{F}$, with the induced Kaehler structures, and $f = d_A \Phi(x)$. With these identifications $f^{1,0} = \partial_A \Phi(x)$ and $f^{0,1} = \bar{\partial}_A \Phi(x)$. As a consequence,

$$\frac{1}{2}(\|\partial_A \Phi\|_{L^2}^2 - \|\bar{\partial}_A \Phi\|_{L^2}^2) - \int_X \langle \Lambda F_A, \mu(\Phi) \rangle = \int_X \Phi^* \omega_F^A - \Phi^* (\pi_F \Lambda F_A, \mu(\Phi)) \wedge \omega^{[n-1]} = \int_X \Phi^* [\omega_F] \wedge \omega^{[n-1]}.$$

This proves Proposition 7.7.

7.2. **Proof of Theorem 7.4.** The following computation has its origins in an idea of Bogomolov in studying vortex equations on $\mathbb{R}^2$. Here we mimic [Br1], except that where he uses the Kaehler identities we use Proposition 7.7.

Lemma 7.9. For any section $\Phi \in \mathcal{F}$ and any connection $A \in \mathcal{A}^{1,1}$

$$\mathcal{U}\mathcal{M}\mathcal{H}_c(A, \Phi) = \|\Lambda F_A + \mu(\Phi) - c\|_{L^2}^2 + \|\bar{\partial}_A \Phi\|_{L^2}^2 + 2 \int_X \langle \Lambda F_A, c \rangle + 2 \int_X \Phi^* [\omega_F] \wedge \omega^{[n-1]} - \int_X B_2(F_A, F_A) \wedge \omega^{[n-2]},$$

where $B_2 : \Omega^2(E \times_{\text{Ad}} \mathfrak{k}) \otimes \Omega^2(E \times_{\text{Ad}} \mathfrak{k}) \to \Omega^4(X)$ denotes the combination of the wedge product with the biinvariant pairing on $\mathfrak{k}$.

**Proof.** Throughout the proof $\| \cdot \|$ will denote $L^2$ norm. For any connection $A \in \mathcal{A}$ we have

$$|F_A|^2 \omega^{[n]} = |\Lambda F_A|^2 \omega^{[n]} - B_2(F_A, F_A) \wedge \omega^{[n-2]} + 4 |F_A^{0,2}|^2 \omega^{[n]}$$

$$+ 2 \int_X \Phi^* \omega_F^A \wedge \omega^{[n-1]} - \int_X B_2(F_A, F_A) \wedge \omega^{[n-2]}.$$

$$\mathcal{U}\mathcal{M}\mathcal{H}_c(A, \Phi) = \|\Lambda F_A + \mu(\Phi) - c\|_{L^2}^2 + \|\bar{\partial}_A \Phi\|_{L^2}^2 + 2 \int_X \langle \Lambda F_A, c \rangle + 2 \int_X \Phi^* [\omega_F] \wedge \omega^{[n-1]} - \int_X B_2(F_A, F_A) \wedge \omega^{[n-2]},$$

where $B_2 : \Omega^2(E \times_{\text{Ad}} \mathfrak{k}) \otimes \Omega^2(E \times_{\text{Ad}} \mathfrak{k}) \to \Omega^4(X)$ denotes the combination of the wedge product with the biinvariant pairing on $\mathfrak{k}$. 

**Proof.** Throughout the proof $\| \cdot \|$ will denote $L^2$ norm. For any connection $A \in \mathcal{A}$ we have

$$|F_A|^2 \omega^{[n]} = |\Lambda F_A|^2 \omega^{[n]} - B_2(F_A, F_A) \wedge \omega^{[n-2]} + 4 |F_A^{0,2}|^2 \omega^{[n]}$$

$$+ 2 \int_X \Phi^* \omega_F^A \wedge \omega^{[n-1]} - \int_X B_2(F_A, F_A) \wedge \omega^{[n-2]}.$$
We now develop using Proposition 7.7 and taking into account that $F_{\mathcal{A}}^0 = 0$
\[
\|\Lambda F + \mu(\Phi) - c\|^2 + 2\|\mathcal{J}_A\Phi\|^2 \\
= \|\Lambda F\|^2 + \|\mu(\Phi) - c\|^2 + 2\sum_X \langle \Lambda F, \mu(\Phi) \rangle - 2\sum_X \langle \Lambda F, c \rangle + 2\|\mathcal{J}_A\Phi\|^2 \\
= \|F\|^2 + \|\mu(\Phi) - c\|^2 + \|\partial A\Phi\|^2 + \|\mathcal{J}_A\Phi\|^2 - 2\sum_X \Phi^* [\omega_F] \wedge \omega^{[n-1]} - 2\sum_X \langle \Lambda F, c \rangle \\
+ \sum_X B_2(F_A, F_A) \wedge \omega^{[n-2]} \\
= \|F\|^2 + \|\mu(\Phi) - c\|^2 + \|d_A\Phi\|^2 - 2\sum_X \Phi^* [\omega_F] \wedge \omega^{[n-1]} - 2\sum_X \langle \Lambda F, c \rangle \\
+ \sum_X B_2(F_A, F_A) \wedge \omega^{[n-2]}.
\]

Theorem 7.4 follows easily from the preceding lemma. Indeed,
\[
2\sum_X \langle \Lambda F, c \rangle + 2\sum_X \Phi^* [\omega_F] \wedge \omega^{[n-1]} - \sum_X B_2(F_A, F_A) \wedge \omega^{[n-2]}
\]
is a topological quantity, that is, it only depends on the homology class of $\Phi$. That
this is true for the second summand is clear; as for the first summand, by Chern-
Weil theory one sees that it is equal to a linear combination whose coefficients depend
on $c$ of first Chern classes of line bundles obtained from $E$ through representations
$K \rightarrow \text{U}(1)$. Finally, the form $B(F_A, F_A)/8\pi^2$ represents the second Chern character
$\text{ch}_2 \in H^4(X; \mathbb{R})$ of $V = E \times_{\rho_a} W_a$ (see [Br2, p. 209]); hence the third summand is also
topological.

Finally, we obtain from 7.4 the following corollary à la Bogomolov

**Corollary 7.10.** Suppose that a pair $(A, \Phi)$ is gauge equivalent to a pair satisfying
equations (7.20). Then the following inequality holds
\[
\sum_X \langle \Lambda F, c \rangle + \sum_X \Phi^* [\omega_F] \wedge \omega^{[n-1]} - \frac{1}{2} \sum_X B_2(F_A, F_A) \wedge \omega^{[n-2]} \geq 0.
\]

8. **Example: the theorem of Banfield**

Suppose that $F$ is a Hermitian vector space and that $K$ acts on $F$ through a unitary
representation $\rho : K \rightarrow U(F)$. D. Banfield [Br] has recently proved a general Hitchin–
Kobayashi correspondence for this situation. The work of Banfield generalises existing
results on vortex equations, Hitchin equations, and on other equations arising from
particular choices of $K$ and $\rho$. In this section we will see how the result of Banfield
can be deduced from Theorem 2.19.
8.1. The stability condition. Let $h$ be the Hermitian metric on $F$. The imaginary part of $h$ with reversed sign defines a symplectic form $\omega_F$ compatible with the complex structure and hence a Kaehler structure. The action of $K$ on $F$ respects the Kaehler structure and admits a moment map $\mu : F \to P^*$

$$\mu(x) = -\frac{i}{2} \rho^*(x \otimes x^*).$$

In other words, for any $s \in P$, $(\mu(x), s)_P = -\frac{i}{2} h(x, \rho(s)x)$. Let $x \in F$ and take an element $s \in P$. Since $\rho(s) \in \mathfrak{u}(F)$, the endomorphism $\rho(s)$ diagonalizes in a basis $e_1, \ldots, e_n$: $i\rho(s)e_k = \lambda_k e_k$, where $\lambda_k$ is a real number for any $k$. Write $x = x_1e_1 + \cdots + x_ne_n$.

**Lemma 8.1.** If $\lambda_k \leq 0$ for every $k$ such that $x_k \neq 0$, then the maximal weight $\lambda(x; s)$ is equal to zero. Otherwise it is $\infty$.

Let us assume that the representation $\rho$ is contained in the representation $\rho_\alpha$. Let $E \to X$ be a $G$-principal bundle on a compact Kaehler manifold $X$. Let $\mathcal{F} = E \times_\rho F$ be the vector bundle associated to $E$ through the representation $\rho$. Take a pair $(A, \Phi) \in \mathfrak{a}^{1,1} \times \mathcal{F}$, and fix a central element $c \in P$. Consider on $E$ the holomorphic structure given by $\mathcal{J}_A$. According to definition 2.16, $(A, \Phi)$ is $c$-stable if and only if for any parabolic subgroup $P \subset G$, for any holomorphic reduction $\sigma \in \Gamma(X_0; E(G/P))$ defined on the complement of a complex codimension 2 submanifold $X_0$ of $X$ and for any antidominant character $\chi$ of $\bar{P}$, the total degree is positive:

$$T^c_\chi(\sigma, \chi) > 0.$$  

The total degree is the sum of $\deg(\sigma, \chi)$ plus the maximal weight of the action of $g_{\sigma, \chi}$ on $\Phi$ plus $\langle i\chi, c \rangle \text{Vol}(X)$. The maximal weight is

$$\int_{x \in X} \lambda(\Phi(x); -ig_{\sigma, \chi}(x)). \quad (8.21)$$

Define now $\mathcal{F}^- = \mathcal{F}^- (\sigma, \chi) \subset \mathcal{F}$ to be the subset given by the vectors in $\mathcal{F}$ on which $g_{\sigma, \chi}(x)$ acts negatively, that is, $v \in \mathcal{F}_x$ belongs to $\mathcal{F}^-$ if and only if you can write $v = \sum v_n$ such that $g_{\sigma, \chi}(x)(v_n) = \lambda_n v_n$ and $\lambda_n \leq 0$. Since the eigenvalues of $g_{\sigma, \chi}$ are constant, $\mathcal{F}^-$ is a subbundle. And since the parabolic reduction is holomorphic, so is $\mathcal{F}^-$. If $\Phi \subset \mathcal{F}^-$, then the maximal weight at each fibre is equal to zero by Lemma 8.1, so the stability condition reduces to $\deg(\sigma, \chi) > 0$. On the other hand, if $\Phi(x) \notin \mathcal{F}^-$, then there is an open neighbourhood $U$ of $x$ such that $\Phi(y) \notin \mathcal{F}^-_y$ for any $y \in U$. In this situation Lemma 8.1 tells us that, for any $y \in U$, $\lambda(\Phi(y); -ig_{\sigma, \chi}(y)) = \infty$. Since this happens in an open set, the integral $(8.21)$ is infinite (since $X$ is compact, $\Phi$ is bounded and so $\lambda(\Phi(x); -ig_{\sigma, \chi}(x))$ is bounded below). But the degree $\deg(\sigma, \chi)$ is always a finite number, so the total degree will be positive (infinite, in fact) in this case. To sum up,

**Proposition 8.2.** The pair $(A, \Phi)$ is stable if and only if for any $P, \sigma, \chi$ as above, if $\Phi$ is contained in $\mathcal{F}^- (\sigma, \chi)$, then $\deg(\sigma, \chi) + \langle i\chi, c \rangle \text{Vol}(X) > 0$.

This is precisely Banfield stability condition.
8.2. Simple pairs. To give a characterisation of simple pairs we use the following definition due to Banfield [Ba]:

**Definition 8.3.** Suppose that the vector bundle $F$ decomposes into a nontrivial direct sum $\bigoplus_k F_k$ of holomorphic vector bundles and that there is a reduction of the structure group of $E$ to $G' \subset G$, compatible with the splitting. Suppose further that a central element of the Lie algebra $g'$ of $G'$ annihilates the section $\Phi$ but acts nontrivially on $F$. Then we say $(A, \Phi)$ is a decomposable pair. If no such splitting exists, we say that $(A, \Phi)$ is an indecomposable pair.

**Lemma 8.4.** The pair $(A, \Phi)$ is simple if and only if it is indecomposable.

*Proof.* Suppose that $0 \neq s \in \Omega^0(E \times \text{Ad } g)$ is semisimple and stabilises $(A, \Phi)$. In particular $\mathcal{R}_s^\alpha(A) = 0$, and this implies that $\overline{\partial}_A(s) = 0$. So the eigenvalues of $\rho(s)$ are constant, and since $s$ is semisimple $\rho(s)$ diagonalises. Let the different eigenvalues of $\rho(s)$ be $\lambda_1 < \cdots < \lambda_r$, and consider the decomposition $F = F(\lambda_1) \oplus \cdots \oplus F(\lambda_r)$ in eigenbundles, which are holomorphic, and every $F_k = F(\lambda_k)$ having as structure group a subgroup $G_k \subset G$. Since $s$ leaves $\Phi$ fixed $\Phi$ must belong to $F(0)$. On the other hand, $0$ in obviously not the unique eigenvalue of $\rho(s)$, so the decomposition

$$F = F_1 \oplus \cdots \oplus F_r$$

is not trivial. Finally, the section $s$ provides the central element killing $\Phi$.

The proof of the converse is similar. \qed

8.3. The equations. Our equation (2.4) in the case of linear representations is the same one given by Banfield (note that Banfield also considers the holomorphicity condition $\overline{\partial}_A \Phi = 0$).

9. Example: filtrations of vector bundles

In this section we study Theorem 2.19 in the particular case in which $F$ is a Grassmannian or, more generally, a flag manifold. We assume, for simplicity, that $X$ is a Riemann surface. For the higher dimensional case everything that follows remains valid if we consider reflexive subsheaves and not only subbundles in the definition of stability (this reflects the need of considering reductions of the structure group defined on the complement of a complex codimension 2 submanifold of $X$ in the general definition of stability).

9.1. Projective manifolds with actions of Lie groups. Let $F \subset \mathbb{P}(\mathbb{C}^n)$ be any smooth complex subvariety. Let us take on $\mathbb{C}^n$ the canonical Hermitian metric. This allows to define on $\mathbb{P}(\mathbb{C}^n)$ the Fubini-Study Kaehler structure. We consider on $F$ the induced structure. Suppose that a compact Lie group $K$ acts on $\mathbb{P}(\mathbb{C}^n)$ through a representation $\rho : K \to \text{U}(n; \mathbb{C})$ leaving $F$ fixed. Since $\rho(K) \subset \text{U}(n; \mathbb{C})$, the action of $K$ on $\mathbb{P}(\mathbb{C}^n)$ (and hence on $F$) respects the Kaehler structure. A moment map $\mu_F : F \to \mathfrak{k}^*$ for this action is

$$\mu_F(x) = -\frac{i}{2} \rho^* \left( \frac{x \otimes \bar{x}^*}{\|x\|^2} \right), \quad (9.22)$$
where \( \hat{x} \in \mathbb{C} \setminus \{0\} \) denotes any lift of \( x \in F \). Take a point \( x \in F \) and consider an element \( s \in \mathfrak{t} \). We can take a basis \( e_1, \ldots, e_n \) of \( \mathbb{C}^n \) in which the action of \( s \) diagonalizes: for any \( k \), \( \iota \rho(s)e_k = \lambda_k e_k \), where \( \lambda_k \) is a real number. Fix a lifting \( \hat{x} \in \mathbb{C} \setminus \{0\} \) of \( x \in F \) and write \( \hat{x} = x_1 e_1 + \cdots + x_n e_n \).

**Lemma 9.1.** The maximal weight of \( s \) acting on \( x \) is \( \lambda(x; s) = \max \{ \lambda_k | x_k \neq 0 \} \).

The manifold \( F \) will be in this section either a Grassmannian or a flag manifold. The Lie group \( K \) will be \( U(R; \mathbb{C}) \), where \( R \geq 1 \) is an arbitrary integer, and we will take the standard representation in \( \mathbb{C}^R \) as our representation. We will assume for simplicity that \( \text{Vol}(X) = 1 \).

### 9.2. Subbundles

Let \( E \to X \) be a principal \( U(R; \mathbb{C}) \) bundle on \( X \). Consider the standard representation on \( \mathbb{C}^R \). The associated bundle is a vector bundle \( V \to X \) of rank \( R \). Using Theorem 2.19, we will find a Hitchin–Kobayashi correspondence for subbundles \( V_0 \) of \( V \) of fixed rank \( 0 < k < R \). This correspondence has already been proved in \[\text{BrGP}\] and in \[\text{DaUW}\].

Using an idea of \[\text{DaUW}\] we identify the inclusion \( V_0 \hookrightarrow V \) with a section \( \Phi \) of the bundle with fibres the Grassmannian of \( k \)-subvectorspaces \( \text{Gr}_k(\mathbb{C}^R) \) associated to \( E \) by the usual action of \( \text{GL}(R; \mathbb{C}) \) on \( \text{Gr}_k(\mathbb{C}^R) \):

\[
\mathcal{F} = E \times_{\text{GL}(R; \mathbb{C})} \text{Gr}_k(\mathbb{C}^R).
\]

The Plücker embedding maps \( \text{Gr}_k(\mathbb{C}^R) \) in a \( \text{GL}(R; \mathbb{C}) \)-equivariant way into \( \mathbb{P}(\Lambda^k \mathbb{C}^R) \), and the action of \( \text{GL}(R; \mathbb{C}) \) in \( \mathbb{P}(\Lambda^k \mathbb{C}^R) \) lifts to the obvious action in \( \Lambda^k \mathbb{C}^R \). So we are in the situation described at the beginning of this section. Observe that the centre of \( \text{GL}(R; \mathbb{C}) \) acts trivially on the Grassmannian. In consequence, the comments in subsection 6.5 are relevant in this situation.

If \( \omega \) is the symplectic form in \( \text{Gr}_k(\mathbb{C}^R) \) inherited by the Fubini-Study symplectic form on \( \mathbb{P}(\Lambda^k \mathbb{C}^R) \), then \( \tau \omega \) also gives \( \text{Gr}_k(\mathbb{C}^R) \) a Kaehler structure when \( \tau > 0 \) and everything gets multiplied by \( \tau \): the moment map, the maximal weights and the integral of the moment map. We fix from now on a constant \( \tau > 0 \) and we work with the symplectic form \( \tau \omega \). The constant \( \tau \) can be identified with the parameter appearing in the notion of stability and in the equations in \[\text{BrGP}\], \[\text{DaUW}\].

### 9.3. The moment map for the Grassmannian with the action of \( U(n) \)

The action of \( U(n; \mathbb{C}) \) on \( \text{Gr}_k(\mathbb{C}^R) \) is symplectic. Making use of formula (9.22) one easily verifies that if \( \pi \in \text{Gr}_k(\mathbb{C}^R) \), then the moment map of the action of \( U(n; \mathbb{C}) \) at the point \( \pi \) is the element in \( u(n; \mathbb{C})^* \) which sends \( \xi \in u(n; \mathbb{C}) \) to \( \mu(\pi)(\xi) = -i\tau \text{Tr}(\pi \circ \xi) \), where \( \pi \) denotes the orthogonal projection onto \( \pi \) (see \[\text{DaUW}\], p. 485).

### 9.4. Maximal weights of \( U(n) \) acting on the Grassmannian.

Consider the standard action of \( U(n) \) on \( \mathbb{P}(\Lambda^k \mathbb{C}^R) \). Take an element \( s \in u(n) \). We now give the maximal weight \( \lambda(v; s) \) in the case when \( v = v_1 \wedge \cdots \wedge v_k \neq 0 \), for \( v_j \in \mathbb{C}^R \). This case is enough for our purposes, since the image of the Grassmannian \( \text{Gr}_k(\mathbb{C}^R) \) given by the Plücker embedding into \( \Lambda^k \mathbb{C}^R \) is precisely the set of points of that form.
Let $\pi$ be the $k$-subspace of $\mathbb{C}^R$ spanned by $\{v_j\}$. Let $\lambda_1 < \cdots < \lambda_r$ be the eigenvalues of $\iota s$ acting on $\Lambda^k \mathbb{C}^R$, and for any $1 \leq j \leq r$ write $E_j = \bigoplus_{i \leq j} \text{Ker}(\iota s - \lambda_i \text{Id})$. Set $\alpha_j = \lambda_j - \lambda_{j+1}$. Then
\begin{equation}
\lambda(v; s) = \tau \left( \dim(\pi) \lambda_r + \sum_{j=1}^{r-1} \dim(\pi \cap E_j) \alpha_j \right). \tag{9.23}
\end{equation}

The proof of this formula is an easy exercise which follows from Lemma 9.1.

9.5. Simple extensions. Reasoning similarly as in Lemma 8.4 one can prove this

**Lemma 9.2.** The pair $(A, \Phi)$ is not simple if and only if one can find a holomorphic (with respect to $\overline{\partial}_A$) splitting $V = V' \oplus V''$ such that the subbundle $V_0$ given by the section $\Phi$ is contained in $V'$.

9.6. The stability condition. Let $c \in \mathbb{R}$ be a real number. Fix a pair $(A, \Phi)$, which gives a holomorphic structure on $\mathcal{V}$ and an inclusion of bundles $V_0 \subset \mathcal{V}$. In this section we will study the $-ic \text{Id}$-stability condition for the pair in terms of $V_0 \subset \mathcal{V}$.

A (holomorphic) parabolic reduction $\sigma$ of the structure group of $E$ is the same as giving a (holomorphic) filtration $0 \subset V^1 \subset \cdots \subset V^{r-1} \subset V^r = V$, and an antidominant character $\chi$ for this reduction is of the form $\chi = z \text{Id} + \sum_{j=1}^{r-1} m_j \lambda_j \alpha_j$, where $R^j = \text{rk}(V^j)$, $\lambda_{R^j} = \pi_j - \frac{R^j}{R} \text{Id}$, $\pi_j$ is the projection onto $\mathbb{C}^{R^j}$, $z$ is any real number, and the $m_j$ are real negative numbers. Taking into account that the representation is just the standard representation of $\text{GL}(n; \mathbb{C})$ in $\mathbb{C}^R$ we deduce that the degree of the pair $(\sigma, \chi)$ is
\begin{equation}
\text{deg}(\sigma, \chi) = z \text{deg}(V) + \sum_{j=1}^{r-1} m_j \left( \text{deg}(V^j) - \frac{R^j}{R} \text{deg}(V) \right).
\end{equation}

To calculate the maximal weight of the action of $\chi$ on the section $\Phi$ we use formula (9.23). The parameters that appear there are related to ours as follows: $\alpha_j = m_j$ for any $1 \leq j \leq r-1$ and $\lambda_r = z - \sum_{j=1}^{r-1} m_j \frac{R^j}{R}$. We get, after integration (recall that the volume of $X$ has been normalized to $1$):
\begin{equation}
\int_{x \in X} \mu(\Phi(x); -g_{\sigma, \chi}(x)) = \text{rk}(V_0) \left( z - \sum_{j=1}^{r-1} m_j \frac{R^j}{R} \right) + \sum_{j=1}^{r-1} m_j \text{rk}(V_0 \cap V^j). \tag{9.24}
\end{equation}

Hence, the stability notion is as follows: for any filtration $0 \subset V^1 \subset \cdots \subset V^{r-1} \subset V^r = V$ and any set of negative weights $\alpha_1, \ldots, \alpha_{r-1}$ we must have
\begin{align}
0 < z \text{deg}(V) + & \sum_{j=1}^{r-1} m_j \left( \text{deg}(V^j) - \frac{R^j}{R} \text{deg}(V) \right) \\
& + \tau \left( \text{rk}(V_0) \left( z - \sum_{j=1}^{r-1} m_j \frac{R^j}{R} \right) + \sum_{j=1}^{r-1} m_j \text{rk}(V_0 \cap V^j) \right) - z c R. \tag{9.25}
\end{align}
(Observe that thanks to our assumption that $\text{Vol}(X) = 1$, $\langle i\chi, c \rangle \text{Vol}(X) = -zcR$.) If this is to be satisfied by all possible choices of $z$, then

$$c = \frac{\deg(V) + \tau \text{rk}(V_0)}{R}.$$ 

So, given the symplectic form $\tau\omega$, there is a unique central element $c \in \mathfrak{u}(n; \mathbb{C})$ such that the pair can be $c$-stable. Putting the value of the central element inside (9.25) we get

$$0 < \sum_{j=1}^{r-1} m_j \left( \deg(V^j) - \frac{R^j}{R} \deg V - \tau \text{rk}(V_0) \frac{R^j}{R} + \tau \text{rk}(V_0 \cap V^j) \right)$$

$$= \sum_{j=1}^{r-1} m_j R^j \left( \frac{\deg(V^j) + \tau \text{rk}(V_0 \cap V^j)}{R^j} - \frac{\deg(V) + \tau \text{rk}(V_0)}{R} \right),$$

and using the fact that the numbers $m_j$ are arbitrary negative numbers, we see that a necessary and sufficient condition for $(E, \Phi)$ to be stable is that for any nonzero proper subbundle (in fact, reflexive subsheaf) $V^1 \subset V$

$$\frac{\deg(V^1) + \tau \text{rk}(V_0 \cap V^1)}{\text{rk}(V^1)} < \frac{\deg(V) + \tau \text{rk}(V_0)}{R},$$

and this is the same condition that appears in [DaUW, BrGP1].

In what concerns the equations, they are exactly those in [DaUW]. Instead of writing them in terms of a gauge transformation, we will put as the variable a metric $h$ in the bundle $V$. This is equivalent to our setting, since the relevant space in our case is the gauge group of complex transformations modulo unitary gauge transformations, and this coset space can be identified with the space of metrics. Taking into account the precise form of the moment map for the action of $\text{GL}(n; \mathbb{C})$ in $\text{Gr}_k(\mathbb{C}^R)$ we can write the equations as $\Lambda F_A - i\tau \pi_{V_0}^h = -ic \text{Id}$, where $\pi_{V_0}^h$ is the $h$-orthogonal projection onto $V_0$. The equations considered in [BrGP1] are written in a different way, but in [DaUW] it is proved that they are equivalent to ours.

9.7. Filtrations. Here we generalise the preceeding results to the case of filtrations (see [AGP]). Our trick is to identify a filtration $0 \subset V_1 \subset \cdots \subset V_s \subset V$ with a section $\Phi$ of the associated bundle with fibre the flag manifold $F_{i_1, \ldots, i_s}$, where $i_k = \text{rk}(V_k)$. This manifold is embedded in a product of Grassmannians. The Kaehler structure in the flag manifold is not unique. We can in fact take as symplectic form any weighted sum of the pullbacks of the symplectic forms in the Grassmannians, provided the weights are positive. So the Kaehler structure depends on a $s$-uple of positive parameters $\tau = (\tau_1, \ldots, \tau_s)$. We can now work out the stability notion analogously to the case of extensions, and obtain that (here we write $0 \subset V_1 \subset \cdots \subset V_s \subset V$ for the filtration represented by the section $\Phi$)

- the equation is $\Lambda F_A - i \sum \tau_k \pi_{V_k}^h = -ic \text{Id}$, where $\pi_{V_k}^h$ is the $h$-orthogonal projection onto $V_k$ and where $c$ is a real constant;

- the pair $(A, \Phi)$ is simple unless there exists a holomorphic (with respect to $\bar{\partial}_A$) splitting $V = V' \oplus V''$ such that $V_k \subset V'$ for any $k \leq s$;
the only value of \( c \) for which we can expect our filtration to be \( c \)-stable is

\[
  c = \frac{\deg(V) + \sum \tau_k \text{rk}(V_k)}{R}.
\]

- the stability notion is as follows: for any nonzero proper reflexive subsheaf \( V^1 \subset V \),

\[
  \frac{\deg(V^1) + \sum \tau_k \text{rk}(V_k \cap V^1)}{\text{rk}(V^1)} < \frac{\deg(V) + \sum \tau_k \text{rk}(V_k)}{R}.
\]

9.8. **Bogomolov inequality.** In this subsection we state the Bogomolov inequality given in Corollary 7.10 for the case of filtrations. For that we need to compute the cohomology class \( \Phi^* \phi_A(\omega_F) \).

We begin with some general observations. When the cohomology class represented by the symplectic form \( \omega_F \) of \( F \) belongs to \( H^2(F; i2\pi\mathbb{Z}) \), there exists a line bundle \( L \to F \) with a connection \( \nabla \) whose curvature coincides with \(-i\omega_F\). Assume that the action of \( K \) on \( F \) lifts to a linear action on \( L \). Then \( \nabla \) can be assumed to be \( K \)-equivariant (by just averaging if it is not). Using the action of \( K \) on \( L \) we can define a line bundle \( \mathcal{L} \to F \) as \( \mathcal{L} = E \times_K L \). Denote \( \pi_X : \mathcal{L} \to X \) and \( \pi_F : \mathcal{L} \to F \) the projections. Let \( A \) be a connection on \( E \). The connection \( A \) induces a connection on the associated bundle \( \mathcal{L} \), which may be seen as a projection \( \alpha : T\mathcal{L} \to \text{Ker} d\pi_X^2 \). Since \( \nabla \) is \( K \)-equivariant, we may extend it fiberwise to obtain a projection \( \beta : \text{Ker} d\pi_X^2 \to \text{Ker} d\pi_F^2 \). The composition \( \gamma = \beta \circ \alpha : T\mathcal{L} \to \text{Ker} d\pi_F^2 \) defines a connection \( \nabla^A \) on \( \mathcal{L} \to F \). It is an exercise to verify that \( \omega_F^A = iF_{\nabla^A} \), where \( F_{\nabla^A} \) is the curvature of \( \nabla^A \).

If \( F = \text{Gr}_k(\mathbb{C}^R) \) is a Grassmannian everything in the preceding paragraph works. In particular, the line bundle \( L \to F \) can be identified with the dual of the determinant bundle, that is, with the line bundle whose fiber on \( V \in \text{Gr}_k(\mathbb{C}^R) \) is \( \Lambda^k V^* \). More generally, if \( F = F_{i_1,\ldots,i_s} \), and \( F \) has the Kaehler structure induced by the parameters \( \tau = (\tau_1,\ldots,\tau_s) \), then for any \( (A,\Phi) \in \mathcal{G}^{1,1} \times \mathcal{G} \) we have

\[
\int_X \Phi^*[\omega_F] \wedge \omega^{n-1} = -\sum_{k=1}^s \tau_k \deg(V_k),
\]

where \( V_1 \subset \cdots \subset V_s \subset V \) is the filtration represented by the section \( \Phi \).

So Corollary 7.10 takes the following form in this case:

**Corollary 9.3.** Let \( A \) be a connection on \( E \), and consider a filtration \( 0 \subset V_1 \subset \cdots \subset V_s \subset V \) which is holomorphic with respect to \( \mathcal{O}_X \). Let us write \( \Phi \) for the section of \( \mathcal{F} \) which represents this filtration. If the pair \( (A,\Phi) \) is \( \mathcal{G}_G \) equivalent to a solution of

\[
\Lambda F_A - i \sum \tau_k \pi^h_{\Gamma_k} = -i \text{Id},
\]

then the following holds

\[
\deg(V) \left( \frac{\deg(V) + \sum \tau_k \text{rk}(V_k)}{R} \right) - \sum_{k=1}^s \tau_k \deg(V_k) - 4\pi^2 \langle ch_2(V) \cup \omega^{n-2} \rangle \geq 0.
\]
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