Fractional SIS epidemic models and their solutions

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Abstract

This work deals with the fractional SIS epidemic model in the case of constant population size. We provide a representation of the explicit solution to the fractional model under suitable assumptions and we validate the results by considering two numerical schemes. We examine the explicit representation and the numerical schemes for the limit case of the fractional order $\alpha \uparrow 1$, corresponding to the well-known ordinary SIS model, and we analyse the effects of the fractional derivatives by comparing the two models.

Keywords. Fractional epidemic model, SIS model, fractional logistic equation, series expansion, numerical solutions.

Mathematics Subject Classification. 92D30, 78A70, 26A33.

1 Introduction

The study of mathematical models for epidemiology has a long history, dating back to the early 1900s with the theory developed by Kermack and McKendrick [15]. Such theory describes compartmental models, where the population is divided into groups depending on the state of individuals with respect to disease, distinguishing for example the infected from the susceptible and the recovered. The dynamic of the disease is then described by a system of ordinary differential equations for each class of individuals. The use of mathematical models for epidemiology is particularly useful to predict the progress of an infection and take strategic decisions in emergency situations to limit the spread of the disease.

In recent years the study of epidemiological models using fractional calculus has spread widely. For example, in [18] the authors prove via numerical simulations that the proposed fractional model gives better results than the classical theory, when compared to real data. Moreover, for some diseases it is necessary to take into account the history of the system [20]. For other works on fractional epidemiological models we refer to [8, 3, 13, 10].

In this work we focus on the SIS (susceptible $\rightarrow$ infectious $\rightarrow$ susceptible) epidemiological model. This model has a long history and it describes the spread of human viruses such as influenza. SIS is a model without immunity, where the individual recovered from the infection comes back into the class of susceptibles.

Fractional calculus is nowadays used in biological models to take into account macroscopic effect. Fractional operators consider the entire history of the biological process hence their use make possible to model these non local effects often encountered in biological phenomena. Here we propose a fractional SIS model with constant population size. The use of fractional derivatives in the model means that some global effect may produce slowdown in the process. This is verified and discussed in the validation of the model.

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We give the explicit solution by series of the unknown functions in the model. We validate the
goodness of the theoretical formulas by applying two different numerical schemes. Then we compare
the fractional case results ($0 < \alpha < 1$) with the well-known ordinary case taking the limit $\alpha \uparrow 1$ and
we analyse the effects produced by the fractional derivatives.

From the technical point of view our result take advantage of the explicit representation by series of
the solution of fractional logistic equation solved in a recent paper [9]. Thanks to a fruitful formulation of the SIS model we are able to adapt the results obtained for fractional logistic equation in [9] and
to give the solution of fractional SIS model by series.

The paper is organized as follows. Section 2 is devoted to the ordinary SIS model. In Section 3 we
introduce the fractional SIS model with constant population size and we state the main results of the
paper based on a representation by series of the solution of an equivalent problem. In Section 4 we
validate the model using two numerical schemes and we provide some numerical tests also comparing
the fractional model with the classical one. In Section 5 we prove the main results of the paper.

1.1 Preliminaries on fractional calculus

Here we only introduce some basic aspects on fractional calculus and the non linear equations we deal
with from now on. A deep discussion on fractional derivative has been postponed in Section 5. Let us introduce the Caputo (sometimes termed Djrbashian-Caputo) derivative which may be defined as follows

\[ D_0^\alpha u(t) := \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{u'(s)}{(t-s)^\alpha} ds, \quad t > 0 \]  (1.1)

where $u' = du/ds$ and

\[ \Gamma(z) = \int_0^\infty s^{z-1} e^{-s} ds \]

is the Gamma function. We denote by

\[ \mathcal{I}^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s) ds \]  (1.2)

the Riemann-Liouville fractional integral, which is connected to the Caputo fractional derivative by the following relation

\[ D_0^\alpha u(t) = \mathcal{I}^{1-\alpha} u'(t). \]  (1.3)

We consider the equation

\[ D_0^\alpha u = \varphi(u) \]  (1.4)

with

\[ \varphi(u) = u(1-u) \quad \text{and} \quad \varphi(u) = -u^2. \]

In order to study the solution to (1.4) we first underline that fact that

\[ D_0^\alpha \frac{t^{\alpha k}}{\Gamma(\alpha k + 1)} = \frac{t^{\alpha k - \alpha}}{\Gamma(\alpha k - \alpha + 1)} \]  (1.5)

which is to say, for $\alpha = 1$,

\[ \frac{d}{dt} \frac{t^k}{k!} = \frac{t^{k-1}}{(k-1)!}. \]
In particular, we say that the solution to (1.4) has a representation in terms of the power series
\[ u(t) = \sum_{k \geq 0} \psi_k \frac{t^{\alpha k}}{\Gamma(\alpha k + 1)} \] (1.6)
and we assume that the following further conditions
\[ \psi_0 = u(0) \quad \text{and} \quad \psi_1 = \varphi(\psi_0) \]
must be satisfied. The last assumption on the coefficients is due to the reasonable fact that they are related by
\[ \psi_{k+1} = F(\varphi, \psi_k), \quad k \in \mathbb{N} \] (1.7)
for some relation \( F \) to be better specified. However, \( \psi_0 = u(0) \) agrees with the definition whereas \( \psi_1 = \varphi(\psi_0) \) has to be verified and it seems to be the relevant condition, covering the dependence on \( \varphi \). From the theory of power series we know that to each representation (1.6) corresponds a radius of convergence \( r_\alpha \in [0, \infty) \) such that the series converges absolutely and uniformly in \((0, r)\) for every \( r < r_\alpha \). By the root test we also have that
\[ r_\alpha = \left( \lim_{k \to \infty} \sup \left| \psi_k \frac{1}{\Gamma(\alpha k + 1)} \right|^{1/k} \right)^{-1/\alpha} \] (1.8)
and because of the relation (1.7), the radius \( r_\alpha \) depends on \( \varphi \) by means of the sequence \( \{\psi_k\}_k \).

**Definition 1.** We say that \( u \) is the solution to (1.4) if \( u \) solves pointwise (1.4) in the convergence set and satisfies the initial condition. In particular, \( u \) is the solution to (1.4) in every compact subset \( K \subseteq (0, r_\alpha) \) with radius of convergence \( r_\alpha \) depending on \( \varphi \).

We consider the following well-known result which is the simple and instructive case in which \( \varphi \) is linear.

**Theorem 1.1.** Let us consider the equation (1.4) on \( K = [0, \infty) \) with \( u(0) = 1 \) and
\[ \varphi(u) = -au \]
where \( a \in \mathbb{R} \). Then,
\[ u(t) = E_\alpha(-at^\alpha) = \sum_{k \geq 0} (-a)^k \frac{t^{\alpha k}}{\Gamma(\alpha k + 1)}, \quad t \geq 0 \] (1.9)
where \( E_\alpha \) is the Mittag-Leffler function and the coefficients \( \psi_k \) are such that \( \psi_0 = 1 \) and
\[ \psi_{k+1} = \varphi(\psi_k) \quad \forall k \geq 0. \]

## 2 The SIS model

Let us consider an infective disease which does not confer immunity and which is transmitted through contact between people. We divide the population into two disjoint classes which evolve in time: the susceptibles and the infectives. The first class contains the individuals which are not yet infected but who can contract the disease; the second class contains the infected population which can transmit the disease. The SIS model [14] is a simple disease model without immunity, where the individuals recovered from the infection come back into the class of susceptibles. Such a model is used to describe the dynamic of infections which do not confer a long immunity, as the cold or influenza.
Denoting by $S(t)$ and $I(t)$ the number of susceptibles and infectives, respectively, at time $t$, the SIS model with non constant population [24, 23] is written as

$$
\begin{align*}
\dot{S}(t) &= \Lambda N(t) - \beta \frac{S(t)I(t)}{N(t)} + \gamma I(t) - \mu S(t) \\
\dot{I}(t) &= \beta \frac{S(t)I(t)}{N(t)} - \gamma I(t) - \mu I(t)
\end{align*}
$$

with $N(t) = S(t) + I(t)$, $S(0) = S_0$ and $I(0) = I_0$,

where $\Lambda$ is the birth rate, $\mu$ is the death removal rate, $\beta$ is the contact rate and $\gamma$ is the recovery removal rate. In Figure 1 we show an intuitive picture of the SIS model.

The sum of susceptibles and infectives is defined in (2.1) by the variable $N(t)$. We introduce the difference between the susceptible and infective populations

$$Z(t) = S(t) - I(t),$$

from which we recover $S$ and $I$ as

$$S(t) = \frac{N(t) + Z(t)}{2}, \quad I(t) = \frac{N(t) - Z(t)}{2}.$$

By (2.1) and some simple computations we have

$$\dot{N}(t) = (\Lambda - \mu)N(t) \quad \text{(2.3)}$$

$$\dot{Z}(t) = \left(\Lambda - \frac{\beta}{2} + \gamma\right)N(t) - (\gamma - \mu)Z \left(1 - \frac{\beta}{2N(\gamma - \mu)}Z\right). \quad \text{(2.4)}$$

The explicit solution of (2.3) with initial datum $N_0 = S_0 + I_0$ is given by

$$N(t) = N_0 e^{(\Lambda - \mu)t}. \quad \text{(2.5)}$$

In contrast, equation (2.4) is not immediately solvable. However, the identification of the explicit solution of such equation is beyond the purposes of this work.

We now slightly modify problem (2.1), by assuming that the birth rate coincides with the death rate, i.e. $\Lambda = \mu$. This hypothesis implies that the population size remains constant in time. Indeed,
if $\Lambda = \mu$ then equation (2.3) implies $\dot{N}(t) = 0$ and thus $N(t) = N$, where $N$ is the population size. We divide (2.1) by $N$ and we introduce $S(t)$ and $I(t)$ as the portion of susceptibles and infectives at time $t$, respectively. The problem becomes

$$
\begin{align*}
\dot{S}(t) &= \mu - \beta S(t)I(t) + \gamma I(t) - \mu S(t) \\
\dot{I}(t) &= \beta S(t)I(t) - \gamma I(t) - \mu I(t)
\end{align*}
$$

(2.6)

with $S(t) + I(t) = 1$, $S(0) = S_0$ and $I(0) = I_0$.

The SIS model with constant population is particularly appropriate to describe some bacterial agent diseases such as gonorrhea, meningitis and streptococcal sore throat. In Figure 2 we draw an intuitive representation of the SIS model with constant population.

![Graphical description of SIS model with constant population.](image)

Let us introduce the basic reproduction number [2] i.e. the expected number of secondary infections produced during the period of infection. For the SIS model it is defined by

$$
\sigma = \frac{\beta}{\gamma + \mu}
$$

(2.7)

where $\gamma + \mu$ is the infection period. Since $S(t) = 1 - I(t)$, we rewrite the equation for $\dot{I}(t)$ in (2.6) as

$$
\dot{I}(t) = \beta \frac{\sigma - 1}{\sigma} I \left( 1 - \frac{\sigma}{\sigma - 1} I \right).
$$

(2.8)

We observe that (2.8) is the logistic equation [22]. Hence, we introduce the carrying capacity

$$
c = \frac{\sigma - 1}{\sigma}
$$

(2.9)

and we rewrite (2.8) as

$$
\dot{I}(t) = \begin{cases} 
-\beta I^2 & \text{if } c = 0, \\
bl (1 - \frac{I}{c}) & \text{if } c \neq 0
\end{cases}
$$

(2.10)

where

$$
b = \beta c.
$$

(2.11)
Note that $c \in (-\infty, 1)$. Therefore, the solution to model (2.6) is

\[
I(t) = \begin{cases} 
\frac{I_0}{\beta I_0 t + 1} & \text{if } c = 0 \\
\frac{I_0}{I_0 + (c - I_0)e^{-bt}} & \text{if } c \neq 0.
\end{cases}
\] (2.12)

\[
S(t) = \begin{cases} 
1 - \frac{I_0}{\beta I_0 t + 1} & \text{if } c = 0 \\
1 - \frac{I_0}{I_0 + (c - I_0)e^{-bt}} & \text{if } c \neq 0.
\end{cases}
\] (2.13)

Let us focus on the infectives $I(t)$. The following proposition can be proved with standard computations.

**Proposition 2.1.** The solution $I(t)$ to (2.10), defined in (2.12), is such that

- $I(t) \xrightarrow{t \to \infty} 0$ if $c \leq 0$.
- $I(t) \xrightarrow{t \to \infty} c$ if $c > 0$.

For the logistic functions, it is useful to define the unique inflection point where the function changes its concavity. We focus on $I(t)$ in (2.12) for $c \neq 0$, for which the inflection point $t_I$ is defined as

\[
t_I = \frac{1}{b} \ln \left( \frac{c - I_0}{I_0} \right).
\] (2.14)

Moreover, we look for the point of intersection between the susceptible and the infective populations, i.e. the time $t$ such that $I(t) = 1 - I(t)$. From (2.12) we recover the time of intersection $t^*$

\[
t^* = \begin{cases} 
\frac{2I_0 - 1}{\beta I_0} & \text{if } c = 0 \\
\frac{1}{b} \ln \left( \frac{c - I_0}{(2c - 1)I_0} \right) & \text{if } c \neq 0.
\end{cases}
\] (2.15)

We note that, for $c \neq 0$,

\[
t^* = t_I + \frac{1}{b} \ln \left( \frac{1}{2c - 1} \right).
\] (2.16)

**Proposition 2.2.** Let $t^*$ be the time of intersection between $S(t)$ and $I(t)$, defined in (2.15). Then $t^* \in [0, \infty)$ if and only if one of the following conditions hold:

1. $I_0 \geq \frac{1}{2}$ and $c < \frac{1}{2}$;
2. $I_0 \leq \frac{1}{2}$ and $c > \frac{1}{2}$.

**Proof.** If $c = 0$, then the claim follows by (2.15). If $c \neq 0$, then we have $t^* \in [0, \infty)$ if

\[
\frac{1}{\beta c} \geq 0 \quad \text{and} \quad \frac{c - I_0}{(2c - 1)I_0} \geq 1
\]
or

\[
\frac{1}{\beta c} < 0 \quad \text{and} \quad \frac{c - I_0}{(2c - 1)I_0} < 1
\]

The thesis follows by simple computations.
3 Main results

In this section we reformulate the SIS problem (2.1) using the Caputo fractional derivative introduced in (1.1). We obtain
\[
\begin{aligned}
\mathcal{D}_t^\alpha S(t) &= \Lambda N(t) - \beta \frac{S(t)I(t)}{N(t)} + \gamma I(t) - \mu S(t) \\
\mathcal{D}_t^\alpha I(t) &= \beta \frac{S(t)I(t)}{N(t)} - \gamma I(t) - \mu I(t)
\end{aligned}
\] (3.1)
with \( N(t) = S(t) + I(t), \) \( S(0) = S_0 \) and \( I(0) = I_0, \)

where \( \Lambda, \mu, \beta \) and \( \gamma \) represent the birth, death, contact and recovery removal rate, respectively, as in the ordinary SIS model (2.6).

Again we introduce the difference between susceptibles and infectives \( Z(t) \). By the linearity (P3) of the Caputo derivative (see Section 5), we have
\[
\mathcal{D}_t^\alpha N(t) = (\Lambda - \mu)N(t)
\] (3.2)
\[
\mathcal{D}_t^\alpha Z(t) = \left( \Lambda - \frac{\beta}{2} + \gamma \right) N(t) - (\gamma - \mu)Z \left( 1 - \frac{\beta}{2N(\gamma - \mu)} \right).
\] (3.3)

Proposition 3.1. The solution of (3.2) with initial datum \( N_0 = S_0 + I_0 \) is
\[
N(t) = N_0 E_\alpha ( (\mu - \Lambda)t^\alpha ),
\] (3.4)
where \( E_\alpha \) is the Mittag-Leffler function, defined in (1.9).

Also in the fractional case the explicit solution of (3.3) is not immediate to be found, thus we proceed as for the classical theory.

We repeat the same procedure of the classical case and we assume \( \Lambda = \mu \) in (3.1). Dividing the two equations by the constant population size \( N \) we obtain
\[
\begin{aligned}
\mathcal{D}_t^\alpha S(t) &= \mu - \beta S(t)I(t) + \gamma I(t) - \mu S(t) \\
\mathcal{D}_t^\alpha I(t) &= \beta S(t)I(t) - \gamma I(t) - \mu I(t)
\end{aligned}
\] (3.5)
with \( S(t) + I(t) = 1, \) \( S(0) = S_0 \) and \( I(0) = I_0, \)

We now provide an explicit representation of the solution of (3.5) under particular assumptions. We recall that \( \sigma \) is the basic reproduction number defined in (2.7), \( c \) is the carrying capacity defined in (2.9), and \( b = \beta c. \) In the following theorems, \( B(x, y) \) denotes the Beta function.

Theorem 3.1. Let \( \alpha \in (0, 1), \) \( c \neq 0 \) and \( b^{1/\alpha} < 1. \) An explicit representation of the solution of the fractional SIS model (3.5) with initial condition \( I_0 = c/2 \) and \( S_0 = 1 - I_0 \) is given by
\[
I(t) = c \sum_{k \geq 0} E_0^\alpha y_{\alpha k} \frac{t^{\alpha k}}{\Gamma(\alpha k + 1)}
\] (3.6)
\[
S(t) = 1 - I(t),
\] (3.7)
with \( E_0^\alpha = \frac{1}{2}, \) \( E_1^\alpha = E_0^\alpha - (E_0^\alpha)^2, \) \( E_2^\alpha = 0 \) and
\[
E_{2k+1}^\alpha = -\frac{1}{\alpha k + 1} \sum_{i+j=k} \frac{E_i^\alpha E_j^\alpha}{B(\alpha i + 1, \alpha j + 1)} \quad \forall k \geq 1.
\]

The series is absolutely and uniformly convergent on any compact subset \( K \subseteq (0, r_\alpha), \) where
\[
r_\alpha = \frac{1}{b^{1/\alpha}} \left( \frac{\Gamma(\alpha + 1)\Gamma(3\alpha + 1)}{\Gamma(2\alpha + 1)} \right)^{1/\alpha}.
\] (3.8)
Theorem 3.2. Let $\alpha \in (0, 1)$, $c = 0$. An explicit representation of the solution of the fractional SIS model (3.5) with initial condition $I_0 = 1/(2\beta)$ and $S_0 = 1 - I_0$ is given by

$$I(t) = \frac{1}{\beta} \sum_{k \geq 0} A^\alpha_k \frac{t^\alpha}{\Gamma(\alpha k + 1)},$$

(3.9)

$$S(t) = 1 - I(t),$$

(3.10)

with $A^\alpha_0 = \frac{1}{2}$, $A^\alpha_1 = -(A^\alpha_0)^2$ and

$$A^\alpha_{k+1} = -\frac{1}{\alpha k + 1} \sum_{i,j} \frac{A^\alpha_i A^\alpha_j}{B(\alpha i + 1, \alpha j + 1)} \forall k \geq 1.$$

The series converges absolutely and uniformly in $K \subset (0, r_\alpha)$ with $r_\alpha \leq (1/2)^{1/\alpha}$.

4 Comparison between models and numerical validation

In this section we proceed with the validation of the previous results on the fractional SIS model by means of numerical approximations, and we analyse the effects of fractional derivatives by comparing the ordinary and fractional SIS model.

4.1 Numerical approximation

The explicit solution (3.6)-(3.7) to the fractional SIS model (3.5) for $c \neq 0$ is defined for $b^{1/\alpha} < 1$ and initial datum $I_0 = c/2$. The explicit solution (3.9)-(3.10) to the fractional SIS model (3.5) for $c = 0$ is defined for the initial datum $I_0 = 1/(2\beta)$. In order to compute the solution to the fractional SIS model for any set of parameters and any initial datum we propose and compare two numerical schemes to approximate (3.5). To this end, let us consider the following problem

$$\mathcal{D}^\alpha u(t) = f(u(t))$$

(4.1)

on a time interval $[0, T]$ uniformly divided into $N + 1$ time steps of length $\Delta t$. Our aim is to define the discrete solution $u_n = u(t_n)$ for $n = 1, \ldots, N$, where $t_n = n\Delta t$ and $u_0$ is known.

We refer to the following method as the Method 1. Following [1], we observe that equations (1.3) and (4.1) imply

$$\mathcal{I}^{1-\alpha} u' = f(u)$$

$$\mathcal{I}^\alpha \mathcal{I}^{1-\alpha} u' = \mathcal{I}^\alpha f(u)$$

$$\mathcal{I}^1 u' = \mathcal{I}^\alpha f(u),$$

and thus we rewrite (4.1) as

$$u(t) = u(0) + \mathcal{I}^\alpha f(u).$$

(4.2)

We introduce a Predictor-Evaluate-Corrector-Predictor (PECE) method [7]. Specifically, we use the implicit one-step Adams-Moulton method [19, Chapter 11], i.e.

$$u_{n+1} = u_0 + \frac{1}{\Gamma(\alpha)} \left( \sum_{j=0}^n a_{j,n+1} f(u_j) + a_{n+1,n+1} f(\tilde{u}_{n+1}) \right),$$

(4.3)

where the coefficients $a_{j,n+1}$ and $\tilde{u}_{n+1}$ are defined below.

First of all, we compute the term $\tilde{u}_{n+1}$ with the one-step Adams-Bashforth method. We introduce
Now we compute the coefficients $a_j$. In particular, for our uniform discretization of the time interval $[0, T]$, we have

$$b_{j,n} = \int_{t_{j-1}}^{t_j} (t_{n+1} - s)^{\alpha-1} \, ds = \frac{1}{\alpha} \left((t_{n+1} - t_j)^\alpha - (t_{n+1} - t_{j-1})^\alpha\right).$$

In particular, for our uniform grid, the coefficients are

$$b_{j,n} = \frac{\Delta t^\alpha}{\alpha} \left((n + 1 - j)^\alpha - (n - j)^\alpha\right).$$

Therefore,

$$\tilde{a}_{n+1} = a_0 + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n} b_{j,n} f(u_j). \quad (4.4)$$

Now we compute the coefficients $a_{j,n+1}$, thus we approximate $I^\alpha g$ as

$$\int_{t_0}^{t_{n+1}} (t_{n+1} - s)^{\alpha-1} g(s) \, ds \approx \int_{t_0}^{t_{n+1}} (t_{n+1} - s)^{\alpha-1} g_{n+1}(s) \, ds.$$

By using the product trapezoidal quadrature formula on the nodes $t_j$, equation (4.1) becomes

$$\int_{t_0}^{t_{n+1}} (t_{n+1} - s)^{\alpha-1} g_{n+1}(s) \, ds = \sum_{j=0}^{n+1} a_{j,n+1} g(t_j),$$

where $a_{j,n+1}$ are defined as

$$a_{j,n+1} = \int_{t_{j-1}}^{t_j} \frac{s - t_{j-1}}{t_j - t_{j-1}} (t_{n+1} - s)^{\alpha-1} \, ds + \int_{t_j}^{t_{j+1}} \frac{t_{j+1} - s}{t_{j+1} - t_j} (t_{n+1} - s)^{\alpha-1} \, ds.$$

We observe that, from integration by parts, we have

$$\int_{t_{j-1}}^{t_j} \frac{s - t_{j-1}}{t_j - t_{j-1}} (t_{n+1} - s)^{\alpha-1} \, ds = -\frac{(t_{n+1} - t_j)^\alpha}{\alpha} + \int_{t_{j-1}}^{t_j} \frac{(t_{n+1} - s)^\alpha}{\alpha(t_j - t_{j-1})} \, ds,$$

$$\int_{t_{j-1}}^{t_j} \frac{t_{j+1} - s}{t_{j+1} - t_j} (t_{n+1} - s)^{\alpha-1} \, ds = \frac{(t_{n+1} - t_j)^\alpha}{\alpha(t_{j+1} - t_j)} - \int_{t_j}^{t_{j+1}} \frac{(t_{n+1} - s)^\alpha}{\alpha(t_{j+1} - t_j)} \, ds,$$

and therefore

$$a_{0,n+1} = \frac{(t_{n+1} - t_0)^\alpha}{\alpha} - \int_{t_0}^{t_1} \frac{(t_{n+1} - s)^\alpha}{\alpha(t_1 - t_0)} \, ds,$$

$$a_{n+1,n+1} = \int_{t_n}^{t_{n+1}} \frac{(t_{n+1} - s)^\alpha}{\alpha(t_{n+1} - t_n)} \, ds,$$

$$a_{j,n+1} = \int_{t_{j-1}}^{t_j} \frac{(t_{n+1} - s)^\alpha}{\alpha(t_{j+1} - t_j)} \, ds - \int_{t_j}^{t_{j+1}} \frac{(t_{n+1} - s)^\alpha}{\alpha(t_{j+1} - t_j)} \, ds \quad \text{for } j = 1, \ldots, n.$$

Finally, in our uniform grid, the coefficients are

$$a_{0,n+1} = \frac{\Delta t^\alpha}{\alpha(\alpha + 1)} \left((n+1)^\alpha - (n - \alpha)(n + 1)^\alpha\right). \quad (4.5)$$
\[ a_{n+1} = \frac{\Delta t^\alpha}{\alpha(\alpha + 1)} \]
\[ a_{j,n+1} = \frac{\Delta t^\alpha}{\alpha(\alpha + 1)} ((n - j + 2)^{\alpha+1} - 2(n - j + 1)^{\alpha+1} + (n - j)^{\alpha+1}) \quad \text{for } j = 1, \ldots, n. \quad (4.7) \]

**Remark 4.1.** The numerical scheme described above works for any \( \alpha \in [0, 1] \).

We now introduce a method to which we refer as Method 2. Let \( \alpha \in (0, 1) \). In [12] the authors give the following approximation of the Caputo derivative
\[ D_\alpha^\Delta t u_n = \frac{1}{\Gamma(2 - \alpha)\Delta t^\alpha} \left( u_n - \sum_{j=0}^{n-1} C_{n,j} u_j \right), \quad (4.8) \]
with
\[ C_{n,0} = g(n), \quad C_{n,j} = g(n - j) - g(n - (j - 1)) \quad \text{for } j = 1, \ldots, n - 1 \]
and \( g(r) = r^{1-\alpha} - (r - 1)^{1-\alpha} \) for \( r \geq 1 \). The numerical scheme to solve (4.1) is then given by
\[ u_{n+1} = \sum_{j=0}^{n-1} C_{n,j} u_j + \Gamma(2 - \alpha)\Delta t^\alpha f(u_n). \quad (4.9) \]

We refer to [12] for further details on the properties of the scheme.

**Remark 4.2.** The numerical scheme above described works for \( \alpha \in (0, 1) \), with the extreme values excluded.

To summarize, in this section we have introduced two numerical schemes which we denote here by \( M_1 \) and \( M_2 \) for notational convenience. The solution to the fractional SIS model (3.5) with the first numerical scheme (that is Method 1) is
\[ I(t_{n+1}) = M_1(I(t_n)) \quad (4.10) \]
\[ S(t_{n+1}) = 1 - I(t_{n+1}), \quad (4.11) \]
where \( M_1 \) is defined in (4.3), and the solution with the second numerical scheme (that is Method 2) is
\[ I(t_{n+1}) = M_2(I(t_n)) \quad (4.12) \]
\[ S(t_{n+1}) = 1 - I(t_{n+1}), \quad (4.13) \]
where \( M_2 \) is defined in (4.9) and \( n = 1, \ldots, N \). Note that the function \( f(u) \) in (4.1), used for both the numerical schemes, is defined in (2.10), while \( u_0 = I_0 \).

### 4.2 Numerical tests

In this section we compare the solutions to the fractional SIS model (3.5) computed with the explicit representation and the two numerical schemes, testing both the case \( c \neq 0 \) and \( c = 0 \). In what follows, we denote by
- \( I^C, S^C \) the solutions (2.12)-(2.13) to the SIS model (2.6),
- \( I^F, S^F \) the solutions (3.6)-(3.7) or (3.9)-(3.10) to the fractional SIS model (3.5) defined by Theorems 3.1 or 3.2 respectively (depending on the carrying capacity \( c \)),
- \( I^N_1, S^N_1 \) the numerical solutions (4.10)-(4.11) computed with the methodology proposed as Method 1,
- \( I^N_2, S^N_2 \) the numerical solutions (4.12)-(4.13) computed with the methodology proposed as Method 2.
4.2.1 Test with $c \neq 0$

We start our numerical analysis with the case of carrying capacity $c \neq 0$. We fix this set of parameters: $\beta = 0.7$, $\gamma = 0.05$, $\mu = 0.12$, $\sigma = 4$ and $c = 0.75$. The initial data are $I(0) = c/2$ and $S(0) = 1 - I(0)$, the final time is $T = 5$ and the time step $\Delta t = 0.05$.

First of all we compare the exact fractional solutions (3.6)-(3.7) and the two numerical solutions (4.10)-(4.11) and (4.12)-(4.13) for $\alpha = 0.99$, which approximately corresponds to the classical derivative. Note that we do not use $\alpha \equiv 1$ since the second numerical scheme works for $\alpha \in (0, 1)$, as already observed in Remark 4.2. In Figure 3 we show the results. As expected, the exact fractional solution and the two numerical solutions to (3.5) overlap the solution to (2.6).

![Figure 3](image3)

(a) Comparison between the solutions (2.12)-(2.13) and the fractional solutions (3.6)-(3.7).
(b) Comparison between the solutions (2.12)-(2.13) and the numerical solutions (4.10)-(4.11).
(c) Comparison between the solutions (2.12)-(2.13) and the numerical solutions (4.12)-(4.13).

Figure 3. Comparison between the solutions to (2.6) and the explicit and numerical fractional solutions to (3.5) with $\alpha = 0.99$.

In Figures 4 and 5 we show the results obtained with $\alpha = 0.7$ and $\alpha = 0.3$. In the first case the two density curves are closer each other and the intersection point between them slightly moves to the right with respect to the solution shown in Figure 3. Such behavior is further emphasized by lower values of $\alpha$, as shown for example in Figure 5. Note that, in both cases the three methodologies produces almost identical results.

![Figure 4](image4)

(a) Fractional solutions (3.6)-(3.7).
(b) Numerical solutions (4.10)-(4.11).
(c) Numerical solutions (4.12)-(4.13).

Figure 4. Comparison between the explicit and numerical fractional solutions to (3.5) with $\alpha = 0.7$. 
To further investigate on the three methodologies, we compute the $L^\infty$-norm of the difference between the exact fractional solutions (3.6)-(3.7) and the two numerical solutions (4.10)-(4.11) and (4.12)-(4.13) and between the two numerical solutions each others, as shown in Table 1. We observe that the errors range from orders of $10^{-5}$ to $10^{-3}$, increasing with respect to the decrease of $\alpha$. This fact further certifies the similarity between the three proposed methodologies.

| $\alpha$ | $\| I^F - I^N_1 \|_\infty$ | $\| I^F - I^N_2 \|_\infty$ | $\| I^N_1 - I^N_2 \|_\infty$ |
|----------|-----------------|-----------------|-----------------|
| 0.99     | 1e−05           | 9e−04           | 9e−04           |
| 0.7      | 1e−05           | 2e−03           | 2e−04           |
| 0.3      | 3e−05           | 8e−03           | 8e−03           |

Table 1. Comparison of the $L^\infty$-norm between the solutions computed with the three methodologies for different values of $\alpha$.

### 4.2.2 Test with $c = 0$

We focus now on the case of carrying capacity $c = 0$. We fix this set of parameters: $\beta = 0.7$, $\gamma = 0.07$, $\mu = 0.63$, $\sigma = 1$ and $c = 0$. Moreover, the initial data are $I(0) = 1/(2\beta)$ and $S(0) = 1 - I(0)$, the final time is $T = 1$ and the time step $\Delta t = 0.01$.

In Figure 6 we compare the exact fractional solutions (3.9)-(3.10) and the two numerical solutions (4.10)-(4.11) and (4.12)-(4.13) for $\alpha = 0.99$. Again, we observe that the fractional solutions, both explicit and numerical, perfectly overlap the solution to the SIS model (2.6). In Figure 7 we show the results obtained with $\alpha = 0.7$. Analogously to the example with $c \neq 0$, the point of intersection between the two densities of population slightly moves to the right with respect to the solution shown in Figure 6. Moreover, the three different methodologies produce again almost identical results. Finally, in Figure 8 we show the results obtained with $\alpha = 0.5$. In this case, the explicit fractional solutions (3.9)-(3.10) blow up in finite time, since the final time $T$ is greater than the radius of convergence, while the two numerical solutions show that the intersection point between the two curves further moves to the right with respect to Figure 7.
Figure 6. Comparison between the solutions to (2.6) and the fractional solutions to (3.5) with $\alpha = 0.99$.

Figure 7. Comparison between the fractional solutions to (3.5) with $\alpha = 0.7$.

Figure 8. Comparison between the fractional solutions to (3.5) with $\alpha = 0.5$. 

(a) Fractional solution to (3.5) (3.9)-(3.10).
(b) Numerical solution to (3.5) described as Method 1.
(c) Numerical solution to (3.5) described as Method 2.
5 Proof of the main results

In this section we collect the proof of the results presented in the work. Moreover, we recall some basic definitions and propositions on fractional calculus which will be used in the proof.

Let us focus on the following equation

\[ K_{\Phi} \ast u' = \varphi(u) \]

where

\[ (K_{\Phi} \ast u')(t) := \int_0^t K_{\Phi}(t-s) u'(s) \, ds \]

for some well-defined function \( \varphi : \mathbb{R} \to \mathbb{R} \). The symbol \( \Phi \) defines the convolution kernel \( K_{\Phi} \) and the operator \( K_{\Phi} \ast u' \) turns out to be a convolution-type derivative for a suitable choice of \( \Phi \).

**Definition 2.** A function \( \Phi : (0, \infty) \to \mathbb{R} \) is a Bernstein function if \( \Phi \) is of class \( C^\infty \), \( \phi(\lambda) \geq 0 \) for all \( \lambda > 0 \) and

\[-(-1)^n \Phi^{(n)}(\lambda) \geq 0 \quad \text{for all } n \in \mathbb{N} \text{ and } \lambda > 0.\]

The convolution kernel we are interested in are therefore written in term of a Bernstein function \( \Phi \), in particular we have the following representation

\[ \Phi(\lambda) = \int_0^\infty \left( 1 - e^{-\lambda z} \right) \Pi(dz) \]  \hspace{1cm} (5.1)

with

\[ \frac{\Phi(\lambda)}{\lambda} = \int_0^\infty e^{-\lambda z} \Pi((z, \infty))dz \]  \hspace{1cm} (5.2)

where \( \Pi \) on \((0, \infty)\) with \( \int (1 \wedge z) \Pi(dz) < \infty \) is the associated Lévy measure (and \( \Pi((z, \infty)) \) is the tail of the Lévy measure \( \Pi \)).

Let \( M > 0 \) and \( w \geq 0 \). Let \( \mathcal{M}_\omega \) be the set of (piecewise) continuous function on \([0, \infty)\) of exponential order \( \omega \) such that \( |u(t)| \leq Me^{\omega t} \). Denote by \( \tilde{u} \) the Laplace transform of \( u \). Then, we define the operator \( D_t^{\Phi} : \mathcal{M}_\omega \to \mathcal{M}_\omega \) such that

\[ \int_0^\infty e^{-\lambda t} D_t^{\Phi} u(t) \, dt = \Phi(\lambda) \tilde{u}(\lambda) - \frac{\Phi(\lambda)}{\lambda} u(0), \quad \lambda > \omega \]  \hspace{1cm} (5.3)

where \( \Phi \) is given in (5.1). Since \( u \) is exponentially bounded, the integral \( \tilde{u} \) is absolutely convergent for \( \lambda > \omega \). The inverse Laplace transforms \( u \) and \( D_t^{\Phi} u \) are uniquely defined (Lerch’s theorem). Since we have that

\[ \Phi(\lambda) \tilde{u}(\lambda) - \frac{\Phi(\lambda)}{\lambda} u(0) = (\lambda \tilde{u}(\lambda) - u(0)) \frac{\Phi(\lambda)}{\lambda} \]  \hspace{1cm} (5.4)

the operator \( D_t^{\Phi} \) can be written as a convolution involving the ordinary derivative and the inverse transform of (5.2) iff \( u \in \mathcal{M}_\omega \cap C([0, \infty), \mathbb{R}_+) \) and \( u' \in \mathcal{M}_\omega \). We mainly focus on the following two special cases:

1. \( \Phi(\lambda) = \lambda \), that is we deal with the ordinary derivative

\[ D_t^\lambda u(t) = \frac{du}{dt}(t); \]
2. \( \Phi(\lambda) = \lambda^\alpha \) with \( \alpha \in (0, 1) \) and the operator \( \mathcal{D}_t^\Phi \) becomes the Caputo fractional derivative

\[
\mathcal{D}_t^\Phi u(t) = \mathcal{D}_t^\alpha u(t) := \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{u'(s)}{(t-s)^\alpha} ds, \quad t > 0
\]  

(5.5)

with \( u'(s) = du/ds \). In this case we have that

\[
\Pi(dz) = \frac{\beta}{\Gamma(1-\beta)} z^{-\beta-1} dz \quad \text{and} \quad \Pi((z, \infty)) = \frac{z^{-\beta}}{\Gamma(1-\beta)}
\]

(5.6)

with (see formula (5.3))

\[
\int_0^\infty e^{-\lambda t} \mathcal{D}_t^\alpha u(t) dt = \lambda^\alpha \tilde{u}(\lambda) - \lambda^{\alpha-1} u(0).
\]

(5.7)

For different definitions and representations of the operator \( \mathcal{D}_t^\Phi \) the interested reader can also see the recent works [4, 5, 21] and [16, 17] for fractional calculus and derivatives.

For the reader’s convenience we recall that

\[
B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt
\]

is the Beta function. The Beta function and the Gamma function are such that

\[
B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.
\]

Furthermore we define the Riemann-Liouville fractional derivative of \( u(t) \) as

\[
\mathcal{D}_t^\alpha u(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{u(s)}{(t-s)^\alpha} ds, \quad t > 0
\]

(5.8)

and we observe that Caputo and the Riemann-Liouville fractional derivatives are linked by

\[
\mathcal{D}_t^\alpha u(t) = \mathcal{D}_t^\alpha u(t) - \frac{t^{-\alpha}}{\Gamma(1-\alpha)} u(0) = \mathcal{D}_t^\alpha (u(t) - u(0))
\]

(5.9)

Formula (5.9) will be useful further on. Thus, we recall the following result ([6, Theorem 3.1] and [6, Definition 3.2]) concerning the definition of the Caputo derivative in (5.9). Let \( T_{m-1}[u] \) be the Taylor polynomial of degree \( m - 1 \) for the function \( u \), centred at zero. That is,

\[
T_{m-1}[u](t) = \sum_{n=0}^{m-1} \frac{u^{(n)}(0)}{n!} t^n
\]

and \( u = T_{m-1}[u] + R_{m-1}[u] \) where \( R_{m-1} \) denotes the remainder term.

**Theorem 5.1.** Let \( \alpha \in (m - 1, m) \) with \( m \in \mathbb{N} \). Assume that \( u \in C^{m-1}([a, b]) \) with \( u' \in L^1([a, b]) \). Then,

\[
\mathcal{D}_t^\alpha u(t) = \mathcal{D}_t^\alpha (u(t) - T_{m-1}[u](t))
\]

almost everywhere.

Here we list some useful properties of the Caputo derivative for the reader’s convenience.

**Proposition 5.1.** The following properties hold true.
Let \( u \) be a constant function. Then \( \mathcal{D}_t^\alpha u(t) = 0 \).

Let \( u : [a, b] \to \mathbb{R} \) such that \( u(a) = 0 \) and \( \mathcal{D}_t^\alpha u \), \( \mathcal{D}_t^\beta u \) exist almost everywhere. Then, \( \mathcal{D}_t^\gamma u = \mathcal{D}_t^\delta u \).

Let \( u, v : [a, b] \to \mathbb{R} \) such that \( \mathcal{D}_t^\alpha u(t) \) and \( \mathcal{D}_t^\beta v(t) \) exist almost everywhere in \( [a, b] \). Let \( c, d \in \mathbb{R} \). Then, \( \mathcal{D}_t^\gamma (cu(t) + dv(t)) \) exists almost everywhere in \( [a, b] \). In particular,

\[
\mathcal{D}_t^\gamma (cu(t) + dv(t)) = c\mathcal{D}_t^\gamma u(t) + d\mathcal{D}_t^\gamma v(t).
\]

Let \( u \in C^1([a, b]) \). Then,

\[
\mathcal{D}_t^\gamma u(t) \to u'(t), \quad \text{as } \alpha \to 1^-
\]
pointwise in \( (a, b) \).

Proof. (P1) and (P3) are immediate consequences of the definition of the Caputo derivative. (P2) can be obtained from (5.9). (P4) in general follows from (5.7) which is given for \( \alpha \in (0, 1) \). Our discussion here is based on the result in [6, Theorem 2.20] for the Riemann-Liouville derivative and the definition (5.9) above of the Caputo derivative. The interested reader can also consult [11, page 20] in which the connection with the Marchaud derivative is considered.

Theorem 1.1 can be regarded as a by-product of a well-known result concerning the Mittag-Leffler function. However, we present the following proof in order to obtain a self-contained organization of the work.

Proof of Theorem 1.1. From (5.7), the equation takes the form

\[
\lambda^\alpha \tilde{u}(\lambda) - \lambda^{\alpha - 1}u(0) = (\Lambda - \mu)\tilde{u}(\lambda)
\]

that is

\[
\tilde{u}(\lambda) = u(0) \frac{\lambda^{\alpha - 1}}{(\Lambda - \mu) + \lambda^\alpha} = \int_0^\infty e^{-\lambda t} E_\alpha((\Lambda - \mu) t^\alpha) dt, \quad \lambda > 0,
\]
since \( u(0) = 1 \). From the Stirling’s formula for Gamma function we have

\[
\left( \frac{a^k}{\Gamma(\alpha k + 1)} \right)^{1/k} \sim a \left( \frac{e}{\alpha k + 1} \right)^{\alpha + 1} (2\pi(\alpha k + 1))^{-1/(2k)} (1 + o(1))
\]

we get that

\[
\left( \frac{a^k}{\Gamma(\alpha k + 1)} \right)^{1/k} \to 0 \quad \text{as } k \to \infty.
\]

Thus, by the root criterion, we get an infinite radius of convergence. The interesting relation between coefficients is easily verified by considering that \( \varphi(v_k) = (-a)^k \).

Proof of Proposition 3.1. We compute the Laplace transform of (3.2) and we exploit (5.7) to obtain

\[
\lambda^\alpha \tilde{N}(\lambda) - \lambda^{\alpha - 1}N_0 = (\Lambda - \mu)\tilde{N}(\lambda),
\]

from which we have

\[
\tilde{N}(\lambda) = \frac{\lambda^{\alpha - 1}}{\lambda^\alpha - (\Lambda - \mu)} N_0.
\]

Since

\[
\frac{\lambda^{\alpha - 1}}{\lambda^\alpha + \mu} = \int_0^\infty e^{-\lambda t} E_\alpha(\mu^\alpha t) dt, \quad \lambda > 0,
\]
the thesis follows.
Proof of Theorem 3.1. Similarly to the classical case, by the linearity (P3) of the Caputo derivative defined in Proposition 5.1, we exploit
\[ S(t) = 1 - I(t) \]
to reduce problem (3.5) to
\[ D^\alpha_t I(t) = \beta c I \left( 1 - \frac{I}{c} \right). \]  
(5.10)

We rewrite (5.10) as
\[ D^\alpha_t u(t) = \frac{1}{M^\alpha} u(t)(1 - u(t)), \]  
(5.11)

where \( u = I/c \) and \( M = (\beta c)^{-1/\alpha} = b^{-1/\alpha} \). Equation (5.11) is the fractional logistic equation. In [9], the authors give the explicit solution of (5.11) for \( M > 1 \) and \( u(0) = 1/2 \) as
\[ u(t) = \sum_{k \geq 0} \frac{E^\alpha_k}{M^{\alpha k} \Gamma(\alpha k + 1)} t^{\alpha k}. \]  
(5.12)

In particular, they proved that the series in (5.12) converges in any compact subset of \((0, r_\alpha)\). From \( u(t) \) in (5.12) solution of (5.11) we recover \( I(t) = \frac{c u(t)}{\beta} \) solution of the fractional SIS model. \( \square \)

Proof of Theorem 3.2. By the linearity (P3) of the Caputo derivative defined in Proposition 5.1, we exploit \( S(t) = 1 - I(t) \) to reduce the problem (3.5) to
\[ D^\alpha_t I(t) = -\beta I^2. \]  
(5.13)

Let \( u = \beta I \) such that
\[ D^\alpha_t u(t) = -u^2(t). \]  
(5.14)

We prove that
\[ u(t) = \sum_{k \geq 0} A^\alpha_k \frac{t^{\alpha k}}{\Gamma(\alpha k + 1)} \]  
(5.15)
solves (5.14), so that we can recover the solution of (5.13) as \( I = u/\beta \). Indeed,
\[ D^\alpha_t u = \beta D^\alpha_t I = -\beta^2 I^2. \]  
(5.16)

The Riemann-Liouville fractional derivative of \( u(t) \) in (5.15) is
\[
D^\alpha_t u(t) = \sum_{k=0}^{\infty} A^\alpha_k \frac{t^{\alpha k}}{\Gamma(\alpha k + 1)} \\
= A^\alpha_0 \frac{t^{-\alpha}}{\Gamma(1 - \alpha)} + \sum_{k=0}^{\infty} A^\alpha_{k+1} \frac{t^{\alpha k}}{\Gamma(\alpha k + 1)} \\
= A^\alpha_0 \frac{t^{-\alpha}}{\Gamma(1 - \alpha)} + A^\alpha_1 \frac{t^0}{\Gamma(\alpha + 1)} + A^\alpha_2 \frac{t^{2\alpha}}{\Gamma(2\alpha + 1)} + A^\alpha_3 \frac{t^{3\alpha}}{\Gamma(3\alpha + 1)} + A^\alpha_4 \frac{t^{4\alpha}}{\Gamma(4\alpha + 1)} + \ldots
\]

By (5.9), we have
\[ D^\alpha_t u(t) = A^\alpha_1 + A^\alpha_2 \frac{t^\alpha}{\Gamma(\alpha + 1)} + A^\alpha_3 \frac{t^{2\alpha}}{\Gamma(2\alpha + 1)} + A^\alpha_4 \frac{t^{3\alpha}}{\Gamma(3\alpha + 1)} + A^\alpha_5 \frac{t^{4\alpha}}{\Gamma(4\alpha + 1)} + \ldots \]  
(5.17)
Now we compute \( u^2(t) \)

\[
u^2(t) = \sum_{k=0}^{\infty} \sum_{s=0}^{\infty} A_k^\alpha A_s^\alpha \frac{\mu^{(k+s)}}{\Gamma(\alpha k + 1) \Gamma(\alpha s + 1)}
\]

\[
= A_0^\alpha A_0^\alpha \\
+ \frac{2A_1^\alpha A_0^\alpha}{\Gamma(\alpha + 1)} t^\alpha \\
+ \frac{A_1^\alpha A_1^\alpha}{\Gamma(\alpha + 1)^2} + \frac{2A_0^\alpha A_2^\alpha}{\Gamma(2\alpha + 1)} t^{2\alpha} \\
+ \frac{A_2^\alpha A_2^\alpha}{\Gamma(2\alpha + 1)^2} + \frac{2A_0^\alpha A_3^\alpha}{\Gamma(3\alpha + 1)} t^{3\alpha} \\
+ \frac{A_3^\alpha A_3^\alpha}{\Gamma(3\alpha + 1)^2} + \frac{2A_0^\alpha A_4^\alpha}{\Gamma(4\alpha + 1)} t^{4\alpha} + \ldots
\]

By (5.17) and (5.18) and by \( A_0^\alpha = 1/2 \) we have

\[
A_0^\alpha = -A_0^\alpha A_0^\alpha = -1/4 \\
A_1^\alpha = -2A_1^\alpha A_0^\alpha = \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha + 1)} \\
A_2^\alpha = A_0^\alpha A_1^\alpha \frac{\Gamma(2\alpha + 1)}{\Gamma(2\alpha + 1)} + 2A_0^\alpha A_2^\alpha \frac{\Gamma(2\alpha + 1)}{\Gamma(2\alpha + 1)} \\
A_3^\alpha = A_0^\alpha A_2^\alpha \frac{\Gamma(3\alpha + 1)}{\Gamma(3\alpha + 1)} + 2A_0^\alpha A_3^\alpha \frac{\Gamma(3\alpha + 1)}{\Gamma(3\alpha + 1)} \\
A_4^\alpha = A_0^\alpha A_3^\alpha \frac{\Gamma(4\alpha + 1)}{\Gamma(4\alpha + 1)} + 2A_0^\alpha A_4^\alpha \frac{\Gamma(4\alpha + 1)}{\Gamma(4\alpha + 1)},
\]

and thus

\[
A_{k+1}^\alpha = - \sum_{j=0}^{k} \frac{\Gamma(k\alpha + 1)}{\Gamma((k-j)\alpha + 1) \Gamma(j\alpha + 1)} A_j^\alpha A_{k-j}^\alpha.
\]

We use the fact that \( \forall k \in \{0,1,\ldots,\} \),

\[
\frac{\Gamma(k\alpha + 1)}{\Gamma((k-j)\alpha + 1) \Gamma(j\alpha + 1)} =: R_k \leq \Gamma(k\alpha + 1).
\]

From the definition above of the coefficients \( \{A_k^\alpha\}_k \) we get

\[
\left| \frac{A_{k+1}^\alpha}{\Gamma((k+1)\alpha + 1)} \right| \leq \frac{1}{\Gamma((k+1)\alpha + 1)} \sum_{j=0}^{k} R_j |A_j^\alpha A_{k-j}^\alpha| \\
\leq \frac{\Gamma(k\alpha + 1)}{\Gamma((k+1)\alpha + 1)} \sum_{j=0}^{k} |A_j^\alpha A_{k-j}^\alpha|,
\]

By iteration we obtain that \( A_k^\alpha \sim |A_0^\alpha|^k \). Since \( (0,1) \supset A_0^\alpha \leq 1/A_0^\alpha \) we write

\[
\left| \frac{A_{k+1}^\alpha}{\Gamma((k+1)\alpha + 1)} \right| \leq \frac{\Gamma(k\alpha + 1)}{\Gamma((k+1)\alpha + 1)} (k+1) \left( \frac{1}{A_0^\alpha} \right)^k := \vartheta_k, \quad k \in \mathbb{N}_0.
\]

We now consider the fact that

\[
x^{x^\gamma} < \frac{e^{x^{-1}}}{e^{x}} < x^{x^{-1/2}}, \quad x > 1
\]
(where $\gamma \approx 0.5$ is the Mascheroni constant) and we get

$$\sqrt[k]{|y_k|} \sim \frac{1}{|A^0_0|} \left( (k+1) \frac{(k\alpha + 1)^{k\alpha+1/2}}{((k+1)\alpha + 1)^{(k+1)\alpha+1-\gamma}} \right)^{1/k}.$$ 

Since

$$(k\alpha + 1)^{1/2(k\alpha+1/2)} \sim \exp \left( (\alpha + \frac{1}{2k}) \ln(k\alpha + 1) \right)$$

and

$$(k+1)^{(k+1)\alpha+1-\gamma} \sim \exp \left( (\alpha + \frac{1}{k} - \gamma) \ln((k+1)\alpha + 1) \right)$$

we get that

$$\sqrt[k]{|y_k|} \sim \frac{1}{|A^0_0|}.$$

Thus, we get the radius of convergence

$$r^q_\alpha = \left( \lim_{k \to \infty} |y_k|^{1/k} \right)^{-1/\alpha} = (|A^0_0|)^{1/\alpha}$$

for the series

$$\sum_{k \geq 0} y_k.$$ 

The convergence of the majorant series determines the uniform convergence in $(0, r_\alpha) \subseteq (0, r^q_\alpha)$ of the series we are interested in. This concludes the proof. 

**Remark 5.1.** The solution in Theorem 3.1 has been given only for the initial datum $c/2$. This is because of the representation given in [9] in terms of Euler polynomials. We notice that from the proof of the previous theorem we are able to deal with $A^0_0 \in (0, 1)$. For the series

$$v(t) = u(t/2^q) = \sum_{n \geq 0} A^q_n \frac{(t/2^q)^{n\alpha}}{\Gamma(n\alpha + 1)}, \quad t \in K^q \subseteq (0, r^q_\alpha)$$

where

$$q = \begin{cases} q = \frac{1}{|A^0_0|}, & |A^0_0| < \frac{1}{2} \\ q = 4 + \frac{1}{2} \left( \frac{1}{A^0_0} - 4 \right), & |A^0_0| \geq \frac{1}{2}. \end{cases}$$

we obtain

$$r^q_\alpha = 2^q \left( |A^0_0| \right)^{1/\alpha}.$$ 

This is the solution in $(0, r^q_\alpha)$ to

$$\mathcal{D}_t^q v = -\frac{1}{2q} v^2, \quad v(0) = A^0_0 \in (0, 1)$$
(see the proof of Theorem 3.1 in [9]). In the special case $\alpha = 1$ we know that

$$w(t) = \left(\frac{1}{A_0} - t\right)^{-1} = A_0 \sum_{k \geq 0} (-A_0)^k t^k \quad t \in (0, 1/A_0)$$

solves $w' = -w^2$ with $w(0) = A_0 \in (0, 1)$. In particular, for $A_0^\alpha = A_0 = 1/2$ we obtain convergence in any compact sets $K \subset (0, 2)$ for both solutions $v$ and $w$. This underlines the fact that introducing non-locality we may deal with solutions quite far from their non-linear analogues.

**Remark 5.2.** We observe that equations (5.11) and (5.14) fit in the framework introduced in Section 1.1 by setting

$$\varphi(u) = au(1 - u)$$

and

$$\varphi(u) = -au^2,$$

respectively.

### 6 Conclusions

In this work we have studied the fractional SIS model with constant population size. We have proposed an explicit representation of the solution to the fractional model under particular assumptions on parameters and initial data. By considering the basic reproduction number we rearrange the SIS model and obtain a logistic equation. In the new formulation of the problem the carrying capacity has a new meaning based on the parameters of the SIS model. We exploit such a formulation in order to study the fractional SIS model and obtain a fruitful characterization of the problem, despite of many difficulties introduced by non-locality. In our formulation the carrying capacity can equal zero and this brings our attention to a different non-linear problem which in turns, is related to the underlined SIS model. We have introduced two different numerical schemes to approximate the model and perform numerical simulations, with which we have tested the proposed explicit solution.

Future investigations will concern the study of the SIS model by considering the following aspects:

- the operator $\mathcal{D}_t^\alpha$ with a Bernstein symbol $\Phi$ which well agrees with some underlying properties of the model;

- non constant population size. In the literature, some models have been investigated in this direction and the population size is well-described in terms of the Mittag-Leffler function in the fractional framework. Based on the formulation above, our extension of the SIS model will be concerned with the study of the non-homogeneous (with a forcing term) fractional logistic equation. As far as we know, this is an open problem.

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