Supersymmetric KP Hierarchy: “Ghost” Symmetry Structure, Reductions and Darboux-Bäcklund Solutions

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Abstract

This paper studies Manin-Radul supersymmetric KP hierarchy (MR-SKP) in three related aspects: (i) We find an infinite set of additional (“ghost”) symmetry flows spanning the same (anti-)commutation algebra as the ordinary MR-SKP flows; (ii) The latter are used to construct consistent reductions SKP\(_{\mathrm{r}}\) of the initial unconstrained MR-SKP hierarchy which involves a nontrivial modification for the fermionic flows; (iii) For the simplest constrained MR-SKP hierarchy SKP\(_{\mathrm{c}}\) we show that the orbit of Darboux-Bäcklund transformations lies on a supersymmetric Toda lattice being a square-root of the standard one-dimensional Toda lattice, and also we find explicit Wronskian-ratio solutions for the super-tau function.

1 Introduction

Supersymmetric integrable hierarchies of nonlinear evolution (“super-soliton”) equations were originally proposed [1] from purely mathematical motivations, but soon they attracted active interest also in theoretical physics mainly due to their close connections with superstring theory [2] (for related studies of supersymmetric integrable systems of Korteweg-de-Vries or nonlinear-Schrödinger type, see [3]).

The scope of the present paper is the supersymmetric Manin-Radul Kadomtsev-Petviashvili (MR-SKP) hierarchy [1] of integrable super-soliton nonlinear equations within the super-pseudo-differential operator formulation (see also [4]; for other formulations see [5]). We study extensions of MR-SKP hierarchy incorporating additional (anti-)commuting “ghost” symmetries, as well as reductions of MR-SKP. We use supersymmetric generalization of several basic concepts in the theory of integrable systems which up to now have been most actively pursued in the context of the ordinary (bosonic) KP hierarchy: Baker-Akhiezer wave functions and tau-functions [6, 7], eigenfunctions and squared eigenfunction potentials (see [8, 9] and references therein).

The advantage of constructing an infinite set of (anti-)commuting “ghost” symmetries in the supersymmetric context (see sect.4 below) is two-fold. On the one hand, it allows us to double the original supersymmetric hierarchy according to the “duality” concept, recently introduced in the...
context of the ordinary KP hierarchy \[ [1] \]. On the other hand, using the “ghost” symmetries we are able to define systematic reductions of the original MR-SKP model to a broad class of constrained supersymmetric KP hierarchies denoted as SKP_{2\frac{1}{2}} (see eq. (5.3) below). These hierarchies possess correct evolution under both even and odd isospectral flows. The latter turns out to be a non-trivial problem since reductions to SKP_{2\frac{1}{2}} hierarchies are incompatible with the original MR-SKP fermionic flows. We provide a solution to this problem by appropriately modifying MR-SKP fermionic flows while preserving their original (anti-)commutation algebra, i.e., preserving the integrability of the constrained SKP_{2\frac{1}{2}} systems.

The second part of the paper contains a detailed discussion of the simplest constrained MR-SKP hierarchy – SKP_{2\frac{1}{2}} (eq. (5.3) below), for which we construct Darboux-Bäcklund (DB) transformations preserving both types (even and odd) of the isospectral flows. This again is achieved thanks to the above mentioned modification of the original MR-SKP fermionic flows. Further, we study the pertinent DB-orbit and discover a new supersymmetric Toda (s-Toda) lattice structure on it. As a consequence of this result we are able to find explicit Wronskian-ratio representation for corresponding super tau-function.

Let us mention that several interesting reduced models of the supersymmetric KP hierarchy have been previously constructed in the literature in terms of super-pseudo-differential operators \[ [1, 12, 13, 14] \]. In particular, the supersymmetric version of AKNS hierarchy was found which allows a description in terms of a bosonic \[ [13] \] as well as a fermionic \[ [14] \] super-Lax operators. The various properties and superspace formulation of these models were worked out, however, their evolution equations involve only even time flows defining them effectively as reductions of the SKP hierarchy \[ [14] \], where only even time flows are present by construction.

2 Background on Manin-Radul Super-KP Hierarchy

We shall use throughout the super-pseudo-differential calculus \[ [1] \] with the following notations: \( \partial \) and \( D = \frac{\partial}{\partial \theta} + \theta \partial \) denote operators, whereas the symbols \( \partial_x \) and \( D_\theta \) will indicate application of the corresponding operators on superfield functions. As usual, \( (x, \theta) \) denote superspace coordinates. For any super-pseudo-differential operator \( A = \sum_j a_j 2^j D^j \) the subscripts (±) denote its purely differential part \( (A_+ = \sum_{j \geq 0} a_j 2^j D^j) \) or its purely pseudo-differential part \( (A_- = \sum_{j \geq 1} a_{-j/2} D^{-j}) \), respectively. For any \( A \) the super-residuum is defined as \( \text{Res}_s A = a_{-\frac{1}{2}} \). The rules of conjugation within the super-pseudo-differential formalism are as follows \[ [13] \]: \( (AB)^* = (-1)^{|A||B|} B^* A^* \) for any two elements with gradings \( |A| \) and \( |B| \); \( (\partial^k)^* = (-1)^k \partial^k \), \( (D^k)^* = (-1)^{k(k+1)/2} D^k \) and \( u^* = u \) for any coefficient superfield.

Finally, in order to avoid confusion we shall also employ the following notations: for any super-(pseudo-)differential operator \( A \) and a superfield function \( f \), the symbol \( A(f) \) will indicate application (action) of \( A \) on \( f \), whereas the symbol \( Af \) will denote just operator product of \( A \) with the zero-order (multiplication) operator \( f \).

MR-SKP hierarchy is defined through the fermionic Lax operator \( L \) :

\[
L = D + f_0 + \sum_{j=1}^{\infty} b_j \theta^{-j} D + \sum_{j=1}^{\infty} f_j \theta^{-j} \tag{2.1}
\]
expressed in terms of a bosonic “dressing” operator \( W \) :

\[
L = WDW^{-1} \quad , \quad W = 1 + \sum_{j=1}^{\infty} \alpha_j \theta^{-j} D + \sum_{j=1}^{\infty} \beta_j \theta^{-j} \tag{2.2}
\]
where \( b_j, \beta_j \) are bosonic superfield functions whereas \( f_j, \alpha_j \) are fermionic ones and where:

\[
f_0 = 2\alpha_1 \quad , \quad b_1 = -D_\theta \alpha_1 \quad , \quad f_1 = 2\alpha_2 - \alpha_1 D_\theta \alpha_1 - 2\alpha_1 \beta_1 - D_\theta \beta_1
\]

Remark. The square of MR-SKP Lax operator (2.1) is an even operator of the form:

\[
L^2 = \partial + D_\theta b_1 \partial^{-1} D + \left( 2b_2 + b_1^2 + D_\theta f_1 + b_1 D_\theta f_0 \right) \partial^{-1} + \ldots
\]

Note that the zero order term in \( L^2 \) vanishes \( D_\theta f_0 + 2b_1 = 0 \) due to (2.3).

The Lax evolution eqs. for MR-SKP read [1]:

\[
\text{Remark. Accordingly, the super-Zakharov-Shabat (super-ZS) eqs. take the following form:}
\]

\[
\frac{\partial}{\partial t} L = - \left[ L^{2l}, L \right] = \left[ L^{2l}, L \right] (2.5)
\]

\[
D_n L = - \left\{ L^{2n-1}, L \right\} = \left\{ L^{2n-1}, L \right\} - 2L^{2n} (2.6)
\]

\[
\frac{\partial}{\partial t} W = - \left( WD^{-1}W^{-1} \right) W , \quad D_n W = - \left( WD^{2n-1}W^{-1} \right) W (2.7)
\]

with the short-hand notations:

\[
D_n = \frac{\partial}{\partial \theta_n} - \sum_{k=1}^{\infty} \theta_k \frac{\partial}{\partial \theta_{n+k-1}} , \quad \{ D_k, D_l \} = -2 \frac{\partial}{\partial \theta_{k+l-1}} (t, \theta) \equiv (t_1, \theta_1) (2.8)
\]

Accordingly, the super-Zakharov-Shabat (super-ZS) eqs. take the following form:

\[
\frac{\partial}{\partial t_k} L^{2l} + \frac{\partial}{\partial t_l} L^{2k} - \left[ L^{2l+k}, L \right] = 0 , \quad \frac{\partial}{\partial t_k} L^{2l-1} - D_l L^{2k} - \left[ L^{2l-1+k}, L \right] = 0 (2.10)
\]

\[
D_k L^{2l-1} + D_l L^{2k-1} - \left\{ L^{2l-1+k}, L \right\} + 2L^{2(k+l-1)} = 0 (2.11)
\]

Remark. Let us stress that, unlike the possibility to identify \( t_1 \equiv x \) (since the zero-order term in \( L^2 \) vanishes), we cannot identify \( \theta_1 \equiv \theta \). Therefore, there is a nontrivial “evolution” already with respect to the lowest fermionic flow \( D_1 \) (which cannot in general be identified with \( D \)).

The super-Baker-Akhiezer (super-BA) and the adjoint super-BA wave functions are defined as:

\[
\psi_{BA}(t, \theta; \lambda, \eta) = W \left( \psi_{BA}^{(0)}(t, \theta; \lambda, \eta) \right) , \quad \psi_{BA}^{*}(t, \theta; \lambda, \eta) = W^{-1} \left( \psi_{BA}^{*}(0)(t, \theta; \lambda, \eta) \right) (2.12)
\]

(with \( \eta \) being a fermionic “spectral” parameter), in terms of the “free” super-BA functions:

\[
\psi_{BA}^{(0)}(t, \theta; \lambda, \eta) \equiv e^{\xi(t, \theta; \lambda, \eta)} , \quad \psi_{BA}^{*}(0)(t, \theta; \lambda, \eta) \equiv e^{-\xi(t, \theta; \lambda, \eta)} (2.13)
\]

\[
\xi(t, \theta; \lambda, \eta) = \sum_{l=1}^{\infty} \lambda^l t_l + \eta \theta + (\eta - \lambda \theta) \sum_{n=1}^{\infty} \lambda^{n-1} \theta_n (2.14)
\]

for which it holds:

\[
\frac{\partial}{\partial t_k} \psi_{BA}^{(0)} = \delta_x^k \psi_{BA}^{(0)} , \quad D_n \psi_{BA}^{(0)} = D_{\theta}^{2n-1} \psi_{BA}^{(0)} = \delta_x^{n-1} D_{\theta} \psi_{BA}^{(0)} (2.15)
\]
Accordingly, (adjoint) super-BA wave functions satisfy:

\[ \left( L^2 \right)^{(s)} \psi^{(s)}_{BA} = \pm \lambda \psi^{(s)}_{BA} , \quad \frac{\partial}{\partial t} \psi^{(s)}_{BA} = \pm \left( L^2 \right)^{(s)} \psi^{(s)}_{BA} , \quad D_n \psi^{(s)}_{BA} = \pm \left( L^{2n-1} \right)^{(s)} \psi^{(s)}_{BA} \]

(2.16)

Correspondingly, the defining equations for arbitrary (adjoint-) super-eigenfunctions (sEF’s) are:

\[ \frac{d}{dt} \Phi = L^2_+ (\Phi) , \quad D_n \Phi = L^{2n-1}_+ (\Phi) , \quad \frac{d}{dt} \Psi = - \left( L^2 \right)^* (\Psi) , \quad D_n \Psi = - \left( L^2 \right)^* (\Psi) \]

(2.17)

with supersymmetric “spectral” representations (cf. [9]):

\[ \Phi(t, \theta) = \int d\lambda \, d\eta \, \varphi(\lambda, \eta) \psi_{BA}(t; \lambda, \eta) , \quad \Psi(t, \theta) = \int d\lambda \, d\eta \, \varphi^*(\lambda, \eta) \psi^*_{BA}(t; \lambda, \eta) \]

(2.18)

For later use let us write down the explicit expression for the “free” sEF \( \Phi^{(0)} \) of the “free” \( L^{(0)} = D \). Namely, taking into account (2.13)-(2.15) and (2.18) we get (for definiteness, consider bosonic \( \Phi^{(0)} \)):

\[ \frac{d}{dt} \Phi^{(0)} = \frac{\partial}{\partial \theta} \Phi^{(0)} , \quad D_n \Phi^{(0)} = D^{2n-1}_\theta \Phi^{(0)} \]

(2.19)

\[ \Phi^{(0)}(t, \theta) = \int d\lambda \, d\eta \, \varphi^{(0)}(\lambda, \eta) e^{\xi(t, \theta ; \lambda, \eta)} \]

\[ = \int d\lambda \left[ \left( 1 - \theta \sum_{n \geq 1} \lambda^n \theta_n \right) \varphi_B(\lambda) + \left( \theta + \sum_{n \geq 1} \lambda^{n-1} \theta_n \right) \varphi_F(\lambda) \right] e^{\sum_{l \geq 1} \lambda^l \theta_l} \]

(2.20)

where \( \varphi^{(0)}(\lambda, \eta) = \varphi_F(\lambda) + \eta \varphi_B(\lambda) \) is arbitrary “spectral” density.

The super-tau-function \( \tau(t, \theta) \) is related with the super-residues of powers of the super-Lax operator [2.1] as follows:

\[ \text{Res} L^{2k} = \frac{\partial}{\partial t_k} D \ln \tau , \quad \text{Res} L^{2k-1} = D_k D \ln \tau \]

(2.21)

Eqs. (2.21) follow from the identities:

\[ \frac{\partial}{\partial t} \text{Res} L^{2k} = \frac{\partial}{\partial t_k} \text{Res} L^{2l} , \quad \frac{\partial}{\partial t} \text{Res} L^{2k-1} = D_k \text{Res} L^{2l} \]

\[ D_l \text{Res} L^{2k-1} + D_l \text{Res} L^{2l-1} + 2 \text{Res} L^{2(k+l-1)} = 0 \]

(2.22)

which in turn are easily derived from eqs. (2.3)–(2.6). In particular, for the coefficients of \( \mathcal{L} \) and \( \mathcal{W} \) we have:

\[ b_1 = \frac{\partial}{\partial t_1} \ln \tau \equiv \partial_x \ln \tau , \quad \alpha_1 = D_1 \ln \tau \]

(2.23)

In what follows we shall encounter objects of the form \( D^{-1}_\theta (\Phi \Psi) = D_\theta \partial_x^{-1} (\Phi \Psi) \) where \( \Phi, \Psi \) is a pair of sEF and adjoint-sEF. Similarly to the purely bosonic case [3] one can show that application of inverse derivative on such products is well-defined (upto an overall \( (t, \theta) \)-independent constant). Namely, there exists a unique superfield function – supersymmetric “squared eigenfunction potential” (super-SEP) \( S(\Phi, \Psi) \) such that: \( D_\theta S(\Phi, \Psi) = \Phi \Psi \). More precisely the super-SEP satisfies the relations:

\[ \frac{\partial}{\partial t_k} S(\Phi, \Psi) = \text{Res} \left( D^{-1} \mathcal{L}^{2k} \Phi \mathcal{D}^{-1} \right) , \quad D_n S(\Phi, \Psi) = \text{Res} \left( D^{-1} \mathcal{L}^{2n-1} \Phi \mathcal{D}^{-1} \right) \]

(2.24)
whose consistency follows from the super-ZS eqs. (2.10)–(2.11). In particular, eqs. (2.24) for \( k = 1 \) and \( n = 1 \) read:

\[
\partial_x S(\Phi, \Psi) = \mathcal{R} \mathcal{S}(D^{-1} \Psi \mathcal{L}^2 \Phi D^{-1}) = D_0(\Phi \Psi) = D_0(\Phi \Psi) \quad , \quad D_1 S(\Phi, \Psi) = \mathcal{R} \mathcal{S}(D^{-1} \Psi \mathcal{L} \Phi D^{-1}) = \Phi \Psi
\] (2.25)

3 Issue of Darboux-Bäcklund Transformations in MR-SKP Hierarchy

Consider the “gauge” transformation of \( \mathcal{L} \) (2.1) of the form:

\[
\tilde{\mathcal{L}} = T \mathcal{L} T^{-1} \quad , \quad T = \chi \mathcal{D}^{-1}
\] (3.1)

which parallels the familiar DB-transformation in the purely bosonic case [15, 16]. Requiring the transformed Lax operator \( \tilde{\mathcal{L}} \) to obey MR-SKP evolution eqs. of the same form (2.5)–(2.6) as \( \mathcal{L} \) implies that \( T \) must satisfy:

\[
\frac{\partial}{\partial t} T T^{-1} - (T L_+^{2m} T^{-1})_+ = 0 \quad , \quad D_n T T^{-1} - (T L_+^{2n-1} T^{-1})_+ = -2 (\tilde{\mathcal{L}}^{2n-1})_+ \] (3.2)

The first eq. (3.2) is exactly analogous to the purely bosonic case and implies that \( \chi \) must be a sEF (2.17) of \( \mathcal{L} \) w.r.t. the even MR-SKP flows. However, there is a problem with the second eq. (3.2). Namely, for the general (unconstrained) MR-SKP hierarchy it does not have solutions for \( \chi \). In particular, if \( \chi \) would be a sEF also w.r.t. fermionic flows (cf. second eq. (2.17)), then the l.h.s. of second eq. (3.2) would become zero whereupon we would get the contradictory relation:

\[
(\tilde{\mathcal{L}}^{2n-1})_+ = 0.
\]

Thus, we conclude that the DB-transformations of the general MR-SKP hierarchy preserve only the bosonic flow equations. In what follows we shall look for consistent solutions of (3.2) in the framework of constrained MR-SKP systems which will be achieved thanks to a non-trivial modification of the fermionic MR-SKP flows preserving their anti-commutation algebra (2.8).

There is a further essential distinction of DB-transformations for MR-SKP hierarchy and its purely bosonic counterpart. Calculating the super-residues of the powers of the DB-transformed Lax operator we obtain:

\[
\mathcal{R} \mathcal{S} \tilde{\mathcal{L}}^s = \mathcal{D}_0 \left( \chi^{-1} L_+^s(\chi) \right) + (-1)^{s+1} \mathcal{R} \mathcal{S} \mathcal{L}^s
\] (3.3)

Note the crucial sign factor in front of the second term on the r.h.s. of eq. (3.3). Together with the first eq. (2.21) it implies for the DB-transformed super-\( \tau \) function:

\[
\tilde{\tau} = \chi \tau^{-1}
\] (3.4)

in contrast with the bosonic case (where we have \( \tilde{\tau} = \chi \tau \)).

4 Super-“Ghost” Symmetries of MR-SKP Hierarchy

Consider an infinite set \( \{ \Phi_{j/2}, \Psi_{j/2} \}_{j=0}^{\infty} \) of pairs of (adjoint-)sEF’s of \( \mathcal{L} \) where those with integer indices are bosonic, whereas those with half-integer indices are fermionic. Next, let us introduce
the following infinite set of super-pseudo-differential operators:

$$\mathcal{M}_{s/2} = \sum_{k=0}^{s-1} \Phi_{s-k/2} D^{-1} \Psi_{k/2}, \quad s = 1, 2, \ldots$$

(4.1)

which generate an infinite set of flows $\bar{\sigma}_{s/2}$ ($\bar{\sigma}_{n-1/2} = \bar{D}_n$, $\bar{\sigma}_k \equiv \frac{\partial}{\partial t_k}$):

$$\bar{\sigma}_{s/2} \mathcal{W} = \mathcal{M}_{s/2} \mathcal{W}, \quad \bar{D}_n \mathcal{L} = \{ \mathcal{M}_{n-1/2}, \mathcal{L} \}, \quad \frac{\partial}{\partial t_k} \mathcal{L} = \{ \mathcal{M}_k, \mathcal{L} \}$$

(4.2)

On (adjoint-)sEF’s entering $\mathcal{M}_{s/2}$ we allow a non-homogeneous action of the super-flows which parallels the construction of generalized “ghost” symmetry flows in the bosonic case (non-homogeneous terms are absent in the traditional approach to “ghost” symmetry flows):

$$\bar{\sigma}_s \Phi_{l/2} = \mathcal{M}_{s/2} \Phi_{l/2} - \Phi_{s+l/2}, \quad \bar{\sigma}_s \Psi_{l/2} = -\mathcal{M}_{s/2} \Psi_{l/2} + (-1)^s \Psi_{s+l/2}$$

(4.3)

$$\bar{\sigma}_s F^{(s)} = \pm \mathcal{M}_{s/2} F^{(s)}$$

(4.4)

where $F^{(s)}$ is a generic (adjoint-)sEF not belonging to the set $\{ \Phi_{j/2}, \Psi_{j/2} \}$.

Using (4.3) we arrive at the following:

**Proposition 1** The infinite set of super-flows $\bar{\sigma}_{s/2}$ (4.4) (anti-)commute both with the ordinary super-flows of MR-SKP (2.3)–(2.4) as well as among themselves:

$$\left[ \frac{\partial}{\partial t_s}, \frac{\partial}{\partial t_l} \right] = \left[ \frac{\partial}{\partial t_s}, D_n \right] = 0, \quad \left[ \bar{D}_s, \frac{\partial}{\partial t_l} \right] = \{ \bar{D}_s, D_n \} = 0$$

(4.5)

$$\left[ \frac{\partial}{\partial t_s}, \frac{\partial}{\partial t_l} \right] = \left[ \frac{\partial}{\partial t_s}, \bar{D}_n \right] = 0, \quad \{ \bar{D}_l, \bar{D}_j \} = -2 \frac{\partial}{\partial t_{l+j-1}}$$

(4.6)

meaning that $\mathcal{M}_{s/2}$ obey the following eqs.:

$$\frac{\partial}{\partial t_k} \mathcal{M}_{s/2} = \left[ \mathcal{L}_{s/2}^k, \mathcal{M}_{s/2} \right], \quad D_n \mathcal{M}_k = \left[ \mathcal{L}_{s/2}^{n-1}, \mathcal{M}_k \right], \quad D_n \mathcal{M}_{k-1/2} = \left[ \mathcal{L}_{s/2}^{n-1}, \mathcal{M}_{k-1/2} \right]$$

(4.7)

$$\frac{\partial}{\partial t_k} \mathcal{M}_l - \frac{\partial}{\partial t_l} \mathcal{M}_k - \left[ \mathcal{M}_k, \mathcal{M}_l \right] = 0, \quad \frac{\partial}{\partial t_k} \mathcal{M}_{l-1/2} - \bar{D}_l \mathcal{M}_k - \left[ \mathcal{M}_k, \mathcal{M}_{l-1/2} \right] = 0$$

(4.8)

$$\bar{D}_k \mathcal{M}_{l-1/2} + \bar{D}_l \mathcal{M}_{k-1/2} - \left[ \mathcal{M}_{k-1/2}, \mathcal{M}_{l-1/2} \right] = -2 \mathcal{M}_{k+l-1}$$

(4.9)

In checking eqs.(4.7)–(4.9) we make use of several useful identities for super-pseudo-differential operators:

$$\left[ B_b, \Phi_{s/2} D^{-1} \Psi_{l/2} \right] = B_b (\Phi_{s/2} D^{-1} \Psi_{l/2} - \Phi_{s+l/2} D^{-1} B_b (\Psi_{l/2}))$$

(4.10)

$$\left[ B_f, \Phi_{s/2} D^{-1} \Psi_{l/2} \right]^{(s)} = B_f (\Phi_{s/2} D^{-1} \Psi_{l/2} + (-1)^s \Phi_{s+l/2} D^{-1} B_f (\Psi_{l/2}))$$

(4.11)

$$\left( \Phi_{s/2} D^{-1} \Psi_{l/2} \right) \left( \Phi_{s/2} D^{-1} \Psi_{l/2} \right)^* = \mathcal{X}_{(s,k)} (\Phi_{s/2} D^{-1} \Psi_{l/2} + (-1)^{k(l+j+1)} \Phi_{s/2} D^{-1} \mathcal{X}_{(s,l)} (\Psi_{l/2}))$$

(4.12)

$$\left( \Phi_{s/2} D^{-1} \Psi_{l/2} \right)^* = (-1)^{lj+j+l} \Phi_{s/2} D^{-1} \Phi_{s/2}$$

(4.13)

where $B_b, B_f$ indicate arbitrary bosonic/fermionic purely differential super-operators, and $[\cdot, \cdot]^{(s)}$ denotes commutator or anticommutator whenever the second element is bosonic/fermionic.
5 Constrained MR-SKP Hierarchies

The super-"ghost"-symmetry flows and the corresponding generating operators $M_{\mathcal{S}}^{(4.1)-(1.3)}$ can be used to construct reductions of the full (unconstrained) MR-SKP hierarchy. Namely, since according to Prop.1 the super-"ghost" flows obey the same algebra (4.6) as the original MR-SKP flows, we can identify an infinite subset of the latter with a corresponding infinite subset of the former:

$$\partial_n = -\bar{\partial}_n \quad , \quad n = 1, 2, \ldots; \quad \partial_k \equiv \frac{\partial}{\partial t_k} \quad , \quad \partial_{k-\frac{1}{2}} \equiv D_k \quad , \quad \bar{\partial}_k \equiv \frac{\partial}{\partial \bar{t}_k} \quad , \quad \bar{\partial}_{k-\frac{1}{2}} \equiv \bar{D}_k$$

where $(r, m)$ are some fixed positive integers of equal parity, and retain only these flows as Lax evolution flows (this is a supersymmetric extension of the usual reduction procedure in the purely bosonic case [18]). Eqs.(5.1) imply the identification $(L_{r}^{\ell} \Phi_{m}^{2} - \Psi_{m}^{2})$ for any $\ell$ and, therefore, the corresponding reduced MR-SKP hierarchy denoted as $SKP^{(r, m)}_{1/2}$ is described by the following constrained super-Lax operator:

$$L_{(r, m)}^{(1/2)} = L_{r, 1/2}^{r} + \sum_{j=0}^{m-1} \Phi_{m-1-j}^{2} \bar{D}^{1} \Psi_{j}^{1/2}$$

The two simplest constrained MR-SKP Lax operators read:

$$L_{(1/2)}^{(1/2)} = \mathcal{L} = \mathcal{D} + f_{0} + \Phi_{0} \bar{D}^{-1} \Psi_{0}$$

$$L_{(1, 1)} = \partial + \mathcal{D}_{0} f_{0} + 2b_{1} + \Phi_{0} \bar{D}^{-1} \Psi_{1}^{1/2} + \Phi_{1}^{2} \bar{D}^{-1} \Psi_{0}$$

where $\Phi_{0}, \Psi_{0}$ and $\Phi_{1/2}, \Psi_{1/2}$ are pairs of bosonic and fermionic (adjoint-)sEF's w.r.t. the bosonic flows (about the fermionic flows, see below).

In what follows we shall consider in some detail the simplest constrained $SKP^{(1/2)}_{1/2}$ hierarchy (5.3), and henceforth we shall skip the subscript $(1/2)$ of (5.3) for brevity.

Using identities (4.10)-(4.12) we find the identity for any integer power $N$ (for an analogous formula in the purely bosonic case, see [19]) :

$$\left(\mathcal{L}^{N}\right)_{-} = \sum_{j=0}^{N-1} \mathcal{L}^{N-j-1}(\Phi_{0}) \mathcal{D}^{-1} \mathcal{L}^{j*}(\Psi_{0})$$

In particular, for the square of (5.3) we get:

$$\mathcal{L}^{2} = \partial + \mathcal{L}(\Phi_{0}) \mathcal{D}^{-1} \Psi_{0} + \Phi_{0} \bar{D}^{-1} \mathcal{L}^{*}(\Psi_{0})$$

where again the zero-order term $\mathcal{D}_{0} f_{0} + 2 \Phi_{0} \Psi_{0} = 0$ as a particular case of (2.3).

The constrained MR-SKP Lax operator (5.3) satisfies consistently the bosonic flow eqs.(2.5). However, we need to make a non-trivial modification of the original fermionic flows (2.6) in order to keep them compatible with the reduction from the general to the constrained MR-SKP hierarchy. Indeed, taking the ($-$) part of eqs.(2.6) for the constrained $\mathcal{L}$ (5.3) and using identity (4.11) together with (5.5), we obtain:

$$\left( D_{n} \Phi_{0} - \mathcal{L}^{2n-1}(\Phi_{0}) \right) \mathcal{D}^{-1} \Psi_{0} - \Phi_{0} \mathcal{D}^{-1} \left( D_{n} \Psi_{0} + \left( \mathcal{L}^{2n-1} \right)^{*}_{-}(\Psi_{0}) \right) = -2 \left( \mathcal{L}^{2n} \right)_{-}$$

$$= -2 \sum_{j=0}^{2n-1} \mathcal{L}^{2n-1-j}(\Phi_{0}) \mathcal{D}^{-1} \mathcal{L}^{j*}(\Psi_{0})$$

7
which leads to apparent contradiction.

Motivated by our previous work \footnote{In \cite{8} we solved the problem of incompatibility of the standard Orlov-Schulman additional non-isospectral symmetry flows \cite{20} with the reductions of the full bosonic KP hierarchy by appropriately modifying the original Orlov-Schulman flows.} we arrive at the following important:

**Proposition 2** There exists the following consistent modification of MR-SKP flows $D_n$ \cite{2,4} for constrained SKP $\varphi_{1,2}$ hierarchy:

\[
D_n L = - \left\{ \mathcal{L}^{2n-1} - X^{(2n-1)}, \mathcal{L} \right\} = \left\{ \mathcal{L}_+^{2n-1}, \mathcal{L} \right\} + \left\{ X^{(2n-1)}, \mathcal{L} \right\} - 2\mathcal{L}^{2n} \quad (5.8)
\]

\[
X^{(2n-1)} = 2 \sum_{l=0}^{n-2} \mathcal{L}^{2(n-l)-3}(\Phi_0) \mathcal{D}^{-1} \left( \mathcal{L}^{2l+1} \right)^* (\Psi_0) \quad (5.9)
\]

\[
D_n \Phi_0 = \mathcal{L}_+^{2n-1}(\Phi_0) - 2\mathcal{L}^{2n-1}(\Phi_0) + X^{(2n-1)}(\Phi_0) \quad (5.10)
\]

\[
D_n \Psi_0 = - \left( \mathcal{L}^{2n-1} \right)^* (\Psi_0) + 2 \left( \mathcal{L}^{2n-1} \right)^* (\Psi_0) - \left( X^{(2n-1)} \right)^* (\Psi_0) \quad (5.11)
\]

The modified $D_n$ flows obey the same anti-commutation algebra \cite{2,4} as in the original unconstrained case.

In checking the correct anti-commutation algebra for $D_n$ \cite{5.8} one has to verify the identities:

\[
D_k X^{(2l-1)} + D_l X^{(2k-1)} - \left\{ X^{(2k-1)}, X^{(2l-1)} \right\} - \left\{ X^{(2k-1)}, \mathcal{L}^{2l-1} \right\} - \left\{ X^{(2l-1)}, \mathcal{L}^{2k-1} \right\} = 0 \quad (5.12)
\]

which in turn follow from the definition of $X^{(2n-1)}$ \cite{5.9} together with identities \cite{4.10,4.13}.

**Remark.** It is straightforward to generalize Prop.2 for arbitrary constrained SKP $\varphi_{1,2}$ hierarchy \cite{5.2}. Namely, the modified fermionic flows have the same form as in \cite{5.8} where in the expression for $X^{(2n-1)}$ (cf. \cite{5.9}) one has to sum over all pairs of (adjoint-)sEF’s entering the purely pseudo-differential part of $\mathcal{L}(\varphi_{1,2})$ in \cite{5.2}.

Let us now consider DB-transformations on $\mathcal{L} \equiv \mathcal{L}(\varphi_{1,2})$ \cite{5.3} preserving its constrained form:

\[
\mathcal{L} = \mathcal{T} \mathcal{L} \mathcal{T}^{-1} = \mathcal{D} + \tilde{f}_0 + \tilde{\Phi}_0 \mathcal{D}^{-1} \tilde{\Psi}_0 , \quad \mathcal{T} = \Phi_0 \mathcal{D} \Phi_0^{-1} \quad (5.13)
\]

\[
\tilde{f}_0 = - f_0 - 2\mathcal{D}_0 \ln \Phi_0 , \quad \tilde{\Phi}_0 = \mathcal{T} \mathcal{L}(\Phi_0) = \Phi_0 \partial_x \ln \Phi_0 + \Phi_0 \mathcal{D}_0 f_0 + \Phi_0 \tilde{\Psi}_0 , \quad \tilde{\Psi}_0 = \Phi_0^{-1} \quad (5.14)
\]

We have the following useful identities for DB-transformed quantities:

\[
\tilde{\mathcal{L}}^s(\tilde{\Phi}_0) = \mathcal{T} \mathcal{L}^{s+1}(\Phi_0) , \quad \left( \tilde{\mathcal{L}}^{s+1} \right)^* (\tilde{\Psi}_0) = (\Phi_0)^{s+1} \mathcal{T}^{-1} \mathcal{L}^s \mathcal{L}^* (\Psi_0) = (-1)^s \Phi_0^{-1} \mathcal{D}_0^{-1} (\Phi_0 \mathcal{L}^s \Psi_0) \quad (5.15)
\]

There is a further crucial property of the modified $D_n$ flows \cite{5.8,5.9}:

**Proposition 3** The conditions for preserving the fermionic flow eqs.\cite{5.8,5.3} by the Darboux-Bäcklund -transformations on $\mathcal{L} \equiv \mathcal{L}(\varphi_{1,2})$ \cite{5.3} (cf. second eq.\cite{2.2}):

\[
D_n \mathcal{T} \mathcal{T}^{-1} - \left( \mathcal{T} \mathcal{L}^{2n-1} \mathcal{T}^{-1} \right)_- = - 2 \left( \mathcal{L}^{2n-1} \right)_- + X^{(2n-1)} + \mathcal{T} X^{(2n-1)} \mathcal{T}^{-1} \quad (5.16)
\]

where $\mathcal{T} = \Phi_0 \mathcal{D} \Phi_0^{-1}$ and the “tilde” refers to DB-transformed objects, are now satisfied.

The proof of \cite{5.16} proceeds by using the modified $D_n$ flow definitions \cite{5.9,5.11} together with identities \cite{4.10,4.13} and \cite{5.13}.\footnote{In \cite{8} we solved the problem of incompatibility of the standard Orlov-Schulman additional non-isospectral symmetry flows \cite{20} with the reductions of the full bosonic KP hierarchy by appropriately modifying the original Orlov-Schulman flows.}
6 The Darboux-Bäcklund Orbit of the Constrained MR-SKP Hierarchy

The recursive expression for the chain of the DB-transformations (6.13)–(6.14) of the constrained SKP hierarchy, starting from the “free” initial $L_0 = D$, reads (the subscript $k$ indicating the step of DB-iteration):

$$L_{k+1} = T_k L_k T_k^{-1} = D + f_{k+1} + \Phi_{k+1} D^{-1} \Psi_{k+1}, \quad T_k = \Phi_k D \Phi_k^{-1}$$

(6.1)

$$L_1 = T_0 D T_0^{-1} = D - 2D \ln \Phi_0 + \Phi_0 (\partial_x \ln \Phi_0) D^{-1} \Phi_0^{-1}$$

(6.2)

where:

$$f_{k+1} = -2D \ln \Phi_k - f_k, \quad \Psi_{k+1} = \Phi_k^{-1}$$

(6.3)

$$\Phi_{k+1} = \Phi_k \partial_x \ln \Phi_k + \Phi_k D \theta f_k + \Phi_k^2 \Psi_k$$

(6.4)

and where $\Phi_0$ is a sEF of the initial “free” $L_0 = D$ satisfying the “free” version of eq.(5.10) (no $X^{(2n-1)}$ term). Therefore, its explicit expression is given by eq.(2.20) with substituting $\theta_n \rightarrow -\theta_n$.

Further we have:

$$\Phi_1 = \partial_x \Phi_0, \quad \Psi_1 = \Phi_0^{-1}, \quad f_1 = -2D \ln \Phi_0$$

(6.5)

Note, that from (6.3)–(6.4) we find:

$$2\Phi_{k+1} \Psi_{k+1} + D \theta f_{k+1} = 2\Phi_k \Psi_k + D \theta f_k = \ldots = 0$$

(6.6)

which is consistent with the absence of a zero-order term in the square of $L_k$ in (6.1).

Eq. (6.3) can easily be rewritten as follows:

$$f_{k+1} = -2D \sum_{i=0}^{k} (-1)^{k-i} \ln \Phi_i$$

(6.7)

Recalling identity (6.4) we can alternatively rewrite eq. (6.4) as:

$$\Phi_{k+1} = -\frac{1}{2} \Phi_k D \theta f_{k+1} + \Phi_k \partial_x \ln \Phi_k - \Phi_k^2 \Psi_k$$

(6.8)

from which we obtain:

$$\Phi_{k+1} = \Phi_k \sum_{i=0}^{k} (-1)^{k-i} \partial_x \ln \Phi_i$$

(6.9)

After making the standard substitution $\Phi_k = e^{\varphi_k}$, we find from the second eq. in (6.8) a new super-Toda (s-Toda) lattice equation:

$$\partial_x \varphi_k = e^{\varphi_{k+1} - \varphi_k} + e^{\varphi_k - \varphi_{k-1}}$$

(6.10)

Note, that by acting on (6.10) with $\partial_x$ we get:

$$\partial_x^2 \varphi_k = e^{\varphi_{k+2} - \varphi_k} - e^{\varphi_k - \varphi_{k-2}}$$

(6.11)

which has the form of the ordinary one-dimensional Toda lattice equation but with a doubled lattice spacing and, of course, the Toda variables $\varphi_k = \varphi_k(x,t_2,\ldots;\theta_1,\ldots)$ are now superfields.

Eq. (6.10) can also be rewritten as:

$$e^{\varphi_{k+1} - \varphi_k} = \sum_{i=0}^{k} (-1)^{k-i} \partial_x \varphi_i$$

(6.12)
\[ \varphi_{k+1} = \varphi_k + \ln \left( \sum_{i=0}^{k} (-1)^{k-i} \partial_x \varphi_i \right) \]  

(6.13)

We now discuss the Wronskian representation for the sEF’s \( \Phi_k \). The s-Toda lattice (6.10) can apparently be thought of as the square-root of the standard Toda lattice. We can use this idea to proceed without any technical calculations. According to the construction given in [2] the EF’s \( \Phi_{2n} \) associated with even lattice points can be given the usual Wronskian expressions with the starting “point” \( \Phi_0 \). For the same reason, EF’s \( \Phi_{2n+1} \) associated with odd lattice points of the s-Toda lattice will have the usual Wronskian expressions with the starting “point” \( \Phi_1 = \partial_x \Phi_0 \equiv \Phi_0^{(1)} \) (6.3).

Generally, for \( n = 0, 1, \ldots \) we find by the above arguments:

\[
\Phi_{2n} = \frac{W_{n+1}[\Phi_0, \Phi_0^{(1)}, \ldots, \Phi_0^{(n)}]}{W_n[\Phi_0, \Phi_0^{(1)}, \ldots, \Phi_0^{(n-1)}]}, \quad \Phi_{2n+1} = \frac{W_{n+1}[\Phi_0^{(1)}, \Phi_0^{(2)}, \ldots, \Phi_0^{(n+1)}]}{W_n[\Phi_0^{(1)}, \Phi_0^{(2)}, \ldots, \Phi_0^{(n)}]} \tag{6.14}
\]

where \( W_k [f_1, \ldots, f_k] \equiv \det \| \partial_x^{i-1} f_j \|, \, i, j = 1, \ldots, k \), denotes standard Wronskian determinant (however, with superfield entries in (6.14)) and where \( \Phi^{(k)}_0 \equiv \partial_x^k \Phi_0 \) with \( \Phi_0 \) as in (2.24) (with \( \theta \to -\theta \)).

Using (6.4) and the above Wronskians expressions (6.14) we find by iteration the super-tau functions obtained by 2n recursive steps of the DB-transformations:

\[
\tau^{(2n)} = \frac{W_n[\Phi_0^{(1)}, \ldots, \Phi_0^{(n)}]}{W_{n+1}[\Phi_0, \Phi_0^{(1)}, \ldots, \Phi_0^{(n)}]}, \quad \tau^{(2n+1)} = \frac{W_{n+1}[\Phi_0^{(1)}, \ldots, \Phi_0^{(n)}]}{W_n[\Phi_0^{(1)}, \ldots, \Phi_0^{(n)}]} \tag{6.15}
\]

Moreover, since for (6.3) \( \partial_x \ln \tau = \Phi_0 \Psi_0 \), for the \( k \)-step DB iteration we have \( \partial_x \ln \tau^{(k)} = \frac{\Phi_k}{\Phi_{k-1}} \) by taking into account (6.4). The latter equation together with the relation \( \tau^{(k+1)} = \Phi_k \tau^{(k-1)} \) true for any DB-step \( k \) (cf. (3.4)) yields an alternative super-tau-function form of s-Toda lattice:

\[
\partial_x \ln \tau^{(k)}(t, \theta) = \frac{\tau^{(k+1)}(t, \theta)}{\tau^{(k-1)}(t, \theta)} \tag{6.16}
\]

with the short-hand notation (2.9).

In a subsequent paper we plan to discuss several interesting issues connected with extending the present results: (a) construction of a “doubled” MR-SKP hierarchy by providing a super-Lax formulation for the super-“ghost” symmetry flows (cf. (6.3)–(6.4)) – a supersymmetric extension of the double-KP construction of [4]; (b) general treatment of arbitrary constrained \( SKP \) hierarchies, including derivation of more general Wronskian-type solutions for the super-tau function and elucidating their Berezinian origin; (c) obtaining consistent formulation of supersymmetric two-dimensional Toda lattice as Darboux-Bäcklund orbit on the “doubled” MR-SKP hierarchy (similar to the purely bosonic case [10]) and of supersymmetric analogs of random (multi-)matrix models; (d) study of possible connections of super-tau functions, on one hand, and partition functions and joint distribution functions in random matrix models in condensed matter physics (cf. ref. [22]), on the other hand.

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