‘TWISTED DUALITY’ IN THE C* CLIFFORD ALGEBRA

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Abstract. Let $V$ be a real inner product space and $C[V]$ its $C^*$ Clifford algebra. We prove that if $Z$ is a subspace of $V$ then $C[Z^\perp]$ coincides with the supercommutant of $C[Z]$ in $C[V].$

0. Introduction

Let $V$ be a real vector space on which $(\cdot|\cdot)$ is an inner product. Denote by $C(V)$ the associated complex Clifford algebra: thus, $C(V)$ is a unital associative complex algebra containing and generated by its real subspace $V$ subject to the Clifford relations

$$(\forall v \in V) \ v^2 = (v|v)1$$

and it admits a unique involution $^*$ for which each element of $V$ is selfadjoint. This involutive algebra carries a natural norm $||\cdot||$ that satisfies the $C^*$ property; the $C^*$-algebra obtained upon its completion is the $C^*$ Clifford algebra $C[V].$

For our purposes, it is important to note that these Clifford algebras are naturally $\mathbb{Z}_2$-graded: they are superalgebras. Explicitly, the complex algebra $C(V)$ carries a unique automorphism $\gamma$ that restricts to $V \subseteq C(V)$ as $-\text{Id}$; accordingly, $C(V)$ has an eigendecomposition

$$C(V) = C(V)_+ \oplus C(V)_-$$

in which

$$C(V)_+ = \ker(\gamma - \text{Id})$$

is the even Clifford algebra and

$$C(V)_- = \ker(\gamma + \text{Id})$$

is the odd subspace. The grading automorphism $\gamma$ extends continuously to the $C^*$ Clifford algebra $C[V]$ and thereby yields a corresponding eigendecomposition

$$C[V] = C[V]_+ \oplus C[V]_-.$$ We remark that the closure of $V$ in $C[V]$ is a copy of its Hilbert space completion, for which also $C[V]$ serves as the $C^*$ Clifford algebra; accordingly, we may and shall assume that $V$ is a real Hilbert space. Likewise, we may and shall take subspaces of $V$ that appear below to be closed.

Thus, let $Z$ be a (closed) subspace of $V$ and $Z^\perp$ its orthocomplement, so that

$$V = Z \oplus Z^\perp$$

is an orthogonal direct sum decomposition of the real Hilbert space $V$. Regarding $C[V]$ as a superalgebra, the supercommutant $C[Z]'$ of $C[Z]$ within $C[V]$ is

$$C[Z]' = C[Z]'_+ \oplus C[Z]'_-$$

with even part

$$C[Z]'_+ = \{ a \in C[V]_+ | (\forall b \in C[Z]) \ ba = ab \}$$

and odd part

$$C[Z]'_- = \{ a \in C[V]_- | (\forall b \in C[Z]) \ ba = a\gamma(b) \}.$$
Thus: even elements of $C[Z]'$ commute with each element of $C[Z]$; odd elements of $C[Z]'$ commute with even elements of $C[Z]$ and anticommute with odd elements of $C[Z]$. As elements of $V$ itself are odd, it follows that 

$$C[Z]' = \{ a \in C[V]|(\forall z \in Z) za = \gamma(a)z \}.$$ 

Our purpose in this note is to prove the following version of ‘twisted duality’.

**Theorem 1.** $C[Z]' = C[Z^\perp]$.

For the origins and development of ‘twisted duality’ see [2] [6] [3] [1]; several proofs of ‘twisted duality’ for the plain complex Clifford algebra in line with the present account are presented in [5]; and [4] is a convenient reference for the theory of Clifford algebras.

1. **Conditional expectations and twisted duality**

Our approach to twisted duality will be via conditional expectations: we shall construct a conditional expectation

$$E_Z : C[V] \rightarrow C[Z^\perp]$$

that fixes the subalgebra $C[Z]'$ pointwise; this establishes the inclusion $C[Z]' \subseteq C[Z^\perp]$ while the reverse inclusion $C[Z^\perp] \subseteq C[Z]'$ is immediate from the (linearized) Clifford relations.

We shall find it convenient to denote by $F(V)$ the set of all finite-dimensional subspaces of $V$ directed by inclusion and to observe that

$$C(V) = \bigcup_{M \in F(V)} C(M).$$

To begin, let $u \in V$ be a unit vector and $u^\perp \subseteq V$ its orthogonal space. The Clifford algebra of $V$ decomposes naturally as

$$C(V) = C(u^\perp) \oplus u C(u^\perp).$$

To see this, let $c \in C(V)$ and choose $M \in F(V)$ containing $u$ so that $c \in C(M)$; augment $u$ to an orthonormal basis for $M$ and expand $c$ relative to this basis. Suppose that in this decomposition,

$$c = a + ub$$

with $a, b \in C(u^\perp)$. Then $\gamma(a)u = ua$ and $\gamma(b)u = ub$ on account of the (linearized) Clifford relations, so that

$$u\gamma(c)u = u\gamma(a)u + u\gamma(ub)u = a - ub$$

because the unit vector $u$ has square $1$ and is odd. It follows that the projector

$$E_u : C(V) \rightarrow C(u^\perp) \subseteq C(V)$$

of $C(V)$ on $C(u^\perp)$ along $uC(u^\perp)$ is given by

$$(\forall c \in C(V)) E_u(c) = \frac{1}{2}(c + u\gamma(c)u).$$

Let $M \in F(V)$ be a finite-dimensional subspace of $V$. If $\{u_1, \ldots, u_m\}$ is an orthonormal basis for $M$ then the projectors $E_{u_1}, \ldots, E_{u_m}$ commute, as is shown by direct calculation using the Clifford relations; the product

$$E_M = E_{u_1} \circ \cdots \circ E_{u_m}$$

projects $C(V)$ on $C(u^\perp_{u_1}) \cap \cdots \cap C(u^\perp_{u_m}) = C(M^\perp)$. 

Let $Z$ be an arbitrary (closed) subspace of $V$. Let $c \in C(V)$ and choose $M \in \mathcal{F}(V)$ so that $c \in C(M)$. Let $X \in \mathcal{F}(Z)$ be the orthogonal projection of $M$ on $Z$ and $Y \in \mathcal{F}(Z^\perp)$ its orthogonal projection on $Z^\perp$; thus $M \subseteq X \oplus Y$. Apply $E_X$ to $c \in C(X \oplus Y)$ to obtain
\[ E_X(c) \in C((X \oplus Y) \cap X^\perp) = C(Y) \subseteq C(Z^\perp); \]
if $u \in Z$ is any unit vector then $u E_X(c) u = E_X(c)$ by the Clifford relations, so
\[ E_u \circ E_X(c) = E_X(c). \]
Now, if $N \in \mathcal{F}(Z)$ is any finite-dimensional subspace of $Z$ containing $X$ then taking the product as $u$ runs over an orthonormal basis for $N \cap X^\perp$ reveals that
\[ E_N(c) = E_X(c). \]
This proves that the net $(E_N(c)|N \in \mathcal{F}(Z))$ is eventually constant and hence converges.

Thus, we construct a linear map \( E_Z : C(V) \to C(Z^\perp) \subseteq C(V) \)
by the rule
\[ (\forall c \in C(V)) \ E_Z(c) = \lim_{N \uparrow \mathcal{F}(Z)} E_N(c). \]

**Theorem 2.** If $Z$ is a (closed) subspace of $V$ then $E_Z$ is contractive:
\[ (\forall c \in C(V)) \ |E_Z(c)| \leq |c| \]
and has the (conditional expectation) property:
\[ (\forall c \in C(V))(\forall \ell, r \in C(Z)') \ E_Z(\ell cr) = \ell E_Z(c) r. \]

**Proof.** The norm here is the natural one, completion of \( C(V) \) relative to which yields \( C(V) \).
If $u \in V$ is a unit vector then $|c| = 1$ so that if $c \in C(V)$ then $|E_u(c)| \leq |c|$ because $\gamma$ is isometric. Taking the product over an orthonormal basis shows that $E_N$ is contractive when $N \in \mathcal{F}(V)$. Thus, $E_Z$ is contractive follows immediately. If $u \in Z$ is a unit vector then $u \gamma(\ell) = \ell u$ and $\gamma(r)u = ur$ so that $u \gamma(\ell cr) u = \ell ur(c) ur$ and therefore $E_u(\ell cr) = \ell E_u(c) r$. Taking the product over an orthonormal basis proves the conditional expectation property for $E_N$ when $N$ is a finite-dimensional subspace of $Z$. The conditional expectation property for $E_Z$ itself follows upon taking $M \in \mathcal{F}(V)$ so large that $c, \ell, r$ lie in $C(M)$ and taking $N \in \mathcal{F}(Z)$ to contain the orthogonal projection of $M$ on $Z$. \( \square \)

We remark that the purely algebraic conditional expectation $E_Z : C(V) \to C(Z^\perp)$ fixes the supercommutant $C(Z)' \subseteq C(V)$ pointwise: if $a \in C(Z)'$ then $E_Z(a) = a$ as follows at once from Theorem 2 by taking (say) $\ell = a$ and $c = r = 1$. In particular, this implies that $C(Z)' \subseteq C(Z^\perp)$; the Clifford relations yield the reverse $C(Z^\perp) \subseteq C(Z)'$. In this way, we establish Theorem 1 for the plain complex Clifford algebra $C(V)$.

As the linear map $E_Z : C(V) \to C(Z^\perp) \subseteq C(V) \subseteq C[V]$ is a contraction, it extends continuously to a contraction
\[ E_Z : C[V] \to C[Z^\perp]. \]

Note that if $N \in \mathcal{F}(V)$ has orthonormal basis $\{u_1, \ldots, u_n\}$ then the obvious factorization
\[ E_N = E_{u_1} \circ \cdots \circ E_{u_n} \]
holds by continuous extension of its counterpart on the plain complex Clifford algebra, where if $u \in V$ is a unit vector then
\[ E_u : C[V] \to C[V] : c \mapsto \frac{1}{2}(c + u \gamma(c) u). \]
**Theorem 3.** If $Z$ is a (closed) subspace of $V$ and $c \in C[V]$ then the net
\[(E_N(c) | N \in \mathcal{F}(Z))\]
converges to $E_Z(c)$:
\[E_Z(c) = \lim_{N \uparrow \mathcal{F}(Z)} E_N(c).\]

**Proof.** Let $\varepsilon > 0$. Choose $c_\varepsilon \in C(V)$ so that $\|c - c_\varepsilon\| \leq \varepsilon$. Choose $M_\varepsilon \in \mathcal{F}(V)$ so that $c_\varepsilon \in C(M_\varepsilon)$ and let $X_\varepsilon \in \mathcal{F}(Z)$ be the orthogonal projection of $M_\varepsilon$ on $Z$. Now, let $N \in \mathcal{F}(Z)$ contain $X_\varepsilon$: as $E_Z$ and $E_N$ are contractions, $\|E_Z(c) - E_Z(c_\varepsilon)\| \leq \varepsilon$ and $\|E_N(c_\varepsilon) - E_N(c)\| \leq \varepsilon$; also, $E_Z(c_\varepsilon) = E_Z(\varepsilon c_\varepsilon) = E_N(c_\varepsilon) = E_N(\varepsilon c_\varepsilon)$. According to the triangle inequality, it follows that if $\mathcal{F}(Z) \ni N \supset X_\varepsilon$ then $\|E_Z(c) - E_N(c)\| \leq 2\varepsilon$ and the proof is complete. \(\square\)

We are now able to establish Theorem 1 in full.

**Proof of Theorem 1.** As noted previously, the Clifford relations yield $C[Z^\perp] \subseteq C[Z]'$. We need only prove the reverse inclusion, so let $c \in C[Z]'$. Recall the formulae displayed prior to Theorem 3: if $u \in Z$ is a unit vector, then $E_u(c) = c$; taking the product as $u$ runs over an orthonormal basis, it follows that if $N \in \mathcal{F}(Z)$ then $E_N(c) = c$. Finally, Theorem 3 allows us to pass to the limit as $N \uparrow \mathcal{F}(Z)$ to conclude that $c = E_Z(c) \in C[Z']$.

We announced that $E_Z : C[V] \to C[Z^\perp]$ would be a conditional expectation; indeed it is. Let $c \in C[V]$ and let $\ell, r \in C[Z^\perp] = C[Z]'$; choose sequences $C(V) \ni c_n \to c$, $C(Z^\perp) \ni r_n \to \ell$ and $C(Z^\perp) \ni r_n \to r$. The conditional expectation property of $E_Z$ in Theorem 2 justifies the middle step in
\[E_Z(\ell c_n r_n) = E_Z(\ell r_n c_n) = \ell r_n E_Z(c_n) = \ell r_n E_Z(c_n)\]
whence continuity of $E_Z$ and passage to the $n \to \infty$ limit yield the required identity
\[E_Z(\ell c r) = \ell E_Z(c) r.\]

Further, $E_Z$ preserves the involution $\ast$ and in fact preserves positivity. Let $c \in C(V)$: if $u \in V$ is a unit vector then
\[E_u(c^\ast c) = \frac{1}{2} (c^\ast c + (\gamma(c)u)^\ast (\gamma(c)u))\]
is a convex combination of terms $d^\ast d$ for $d \in C(V)$ so the same is true of $E_N(c^\ast c)$ whenever $N \in \mathcal{F}(Z)$ and therefore true of $E_Z(c^\ast c)$; by continuity, it follows that $E_Z(c^\ast c) \geq 0$ whenever $c \in C[V]$. Of course, $E_Z$ is idempotent.

We close by offering what is perhaps an eccentric application of ‘twisted duality’.

**Theorem 4.** If $\{Z_\lambda | \lambda \in \Lambda\}$ is any family of (closed) subspaces of $V$ then
\[\bigcap_{\lambda \in \Lambda} C[Z_\lambda] = C\left[\bigcap_{\lambda \in \Lambda} Z_\lambda\right].\]

**Proof.** Only $\subseteq$ is in question, so let $c \in \bigcap_{\lambda \in \Lambda} C[Z_\lambda]$. If $\lambda \in \Lambda$ then $c \in C[Z_\lambda] \subseteq C[Z_\lambda^\perp]'$ by (easier direction) ‘twisted duality’ so that if also $v \in Z_\lambda^\perp$ then $vc = \gamma(c)v$. By linearity, it follows that
\[v \in \sum_{\lambda \in \Lambda} Z_\lambda^\perp \implies vc = \gamma(c)v\]
whence continuity yields
\[v \in \sum_{\lambda \in \Lambda} Z_\lambda^\perp \implies vc = \gamma(c)v.\]

Recalling the Hilbert space identity
\[\sum_{\lambda \in \Lambda} Z_\lambda^\perp = \left(\bigcap_{\lambda \in \Lambda} Z_\lambda\right)^\perp\]
we conclude from (harder direction) ‘twisted duality’ that
\[ c \in C[\left( \sum_{\lambda \in \Lambda} Z_{\lambda}^{\perp} \right)^{\perp}] \subseteq C[\left( \sum_{\lambda \in \Lambda} Z_{\lambda}^{\perp} \right) \perp] = C[\bigcap_{\lambda \in \Lambda} Z_{\lambda}] \]