A GENERALIZED MEAN VALUE INEQUALITY FOR SUBHARMONIC FUNCTIONS AND APPLICATIONS

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ABSTRACT. If \( u \geq 0 \) is subharmonic on a domain \( \Omega \) in \( \mathbb{R}^n \) and \( p > 0 \), then it is well-known that there is a constant \( C(n, p) \geq 1 \) such that \( u(x)^p \leq C(n, p) \mathcal{MV}(u^p, B(x, r)) \) for each ball \( B(x, r) \subset \Omega \). We recently showed that a similar result holds more generally for functions of the form \( \psi \circ u \) where \( \psi : \mathbb{R}_+ \to \mathbb{R}_+ \) may be any surjective, concave function whose inverse \( \psi^{-1} \) satisfies the \( \Delta_2 \)-condition. Now we point out that this result can be extended slightly further. We also apply this extended result to the weighted boundary behavior and nonintegrability questions of subharmonic and superharmonic functions.

1. INTRODUCTION

1.1 Previous results. If \( u \) is a nonnegative and subharmonic function on \( \Omega \) and \( p > 0 \), then there is a constant \( C = C(n, p) \geq 1 \) such that

\[
\left(1\right) \quad u(x)^p \leq C \frac{m(B(x, r))}{m(B(x, r))} \int_{B(x, r)} u(y)^p \, dm(y)
\]

for all \( B(x, r) \subset \Omega \). Here \( \Omega \) is a domain in \( \mathbb{R}^n \), \( n \geq 2 \), \( B(x, r) \) is the Euclidean ball with center \( x \) and radius \( r \), and \( m \) is the Lebesgue measure in \( \mathbb{R}^n \). See [FeSt72, Lemma 2, p. 172], [Ku74, Theorem 1, p. 529], [Ga81, Lemma 3.7, pp. 121-123], [AhRu93, (1.5), p. 210]. These authors considered only the case when \( u = |v| \) and \( v \) is harmonic function. However, the proofs in [FeSt72] and [Ga81] apply verbatim also in the general case of nonnegative subharmonic functions. This was pointed out in [Ri89, Lemma, p. 69], [Su90, p. 271], [Su91, p. 113], [Ha92, Lemma 1, p. 113], [Pa94, p. 18] and [St98, Lemma 3, p. 305]. In [AhBr88, p. 132] it was pointed out that a modification of the proof in [FeSt72] gives in fact a slightly more general result, see 2.1 below. A possibility for an essentially different proof was pointed out already in [To86, pp. 188-190]. Later other different proofs were given in [Pa94, p. 18, and Theorem 1, p. 19] (see also [Pa96, Theorem A, p. 15]), [Ri99, Lemma 2.1, p. 233] and [Ri01, Theorem, p. 188]. The results in [Pa94], [Ri99] and [Ri01] hold in fact for more general function classes than just for nonnegative subharmonic functions. See 2.1 and Theorem A below. Compare also [DBTr84] and [Do88, p. 485].

The inequality \( \left(1\right) \) has many applications. Among others, it has been applied to the (weighted) boundary behavior of nonnegative subharmonic functions [To86, p. 191], [Ha92, Theorems 1 and 2, pp. 117-118], [St98, Theorems 1, 2 and 3, pp. 301, 307], [Ri99, Theorem, p. 233] and on the nonintegrability of subharmonic and superharmonic functions [Su90, Theorem 2, p. 271], [Su91, Theorem, p. 113].

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Because of the importance of the mean value inequality (1), it is worthwhile to present a unified result which contains this mean value inequality and all its generalizations cited above. Below in Theorem 2.5 we propose such a generalization. Instead of nonnegative subharmonic functions we formulate our result slightly more generally for functions which we call quasi-nearly subharmonic functions and which will be defined in 2.1. We also give two applications.

As the first application we improve in Theorem 3.4 below our recent result [Ri99, Theorem, p. 188] on the weighted boundary behavior of nonnegative subharmonic functions. As the second application we give in Corollary 4.5 below a supplement to Suzuki’s results on the nonintegrability of superharmonic and subharmonic functions [Su91, Theorem, p. 113]. Our result is a limiting case to Suzuki’s results.

1.2 Notation. Our notation is more or less standard, see [Ri99]. However, for convenience of the reader we recall here the following. We use the common convention $0 \cdot \infty = 0$. We write $v_n = m(B(0,1))$. The $d$-dimensional Hausdorff (outer) measure in $\mathbb{R}^n$ is denoted by $H^d$, $0 \leq d \leq n$. In the sequel $\Omega$ is always a domain in $\mathbb{R}^n$, $\Omega \neq \mathbb{R}^n$, $n \geq 2$. The diameter of $\Omega$ is denoted by $\text{diam} \, \Omega$. The distance from $x \in \Omega$ to $\partial \Omega$, the boundary of $\Omega$, is denoted by $\delta(x)$. $L^1_{\text{loc}}(\Omega)$ is the space of locally integrable functions on $\Omega$. Our constants $C$ are always positive, mostly $\geq 1$, and they may vary from line to line.

2. QUASI-NEARLY SUBHARMONIC FUNCTIONS

2.1 The definition. We call a (Lebesgue) measurable function $u : \Omega \to [-\infty, \infty]$ quasi-nearly subharmonic, if $u \in L^1_{\text{loc}}(\Omega)$ and if there is a constant $C_0 = C_0(n,u,\Omega) \geq 1$ such that

\begin{equation}
    u(x) \leq \frac{C_0}{r^n} \int_{B(x,r)} u(y) \, dm(y)
\end{equation}

for any ball $B(x,r) \subset \Omega$. Compare [Ri99, p. 233] and [Do57, p. 430]. Nonnegative quasi-nearly subharmonic functions have previously been considered by [Pa94] (he called them "functions satisfying the $sh_K$-condition") and in [Ri99] (where they were called "pseudosubharmonic functions"). See [Do88, p. 485] for an even more general function class of (nonnegative) functions. As a matter of fact, also we will restrict ourselves to nonnegative functions.

Nearly subharmonic functions, thus also quasisubharmonic and subharmonic functions, are examples of quasi-nearly subharmonic functions. Recall that a function $u \in L^1_{\text{loc}}(\Omega)$ is nearly subharmonic, if $u$ satisfies (2) with $C_0 = \frac{1}{v_n}$. See [Her71, pp. 14, 26]. Furthermore, if $u \geq 0$ is subharmonic and $p > 0$, then by (1) above, $u^p$ is quasi-nearly subharmonic. By [Pa94, Theorem 1, p. 19] or [Ri99, Lemma 2.1, p. 233] this holds even if $u \geq 0$ is quasi-nearly subharmonic. See also [AhBr88, p. 132].

2.2 Permissible functions. In [Ri01, Theorem, p. 188] we proved the following result which contains essentially the cited results in [Pa94] and [Ri99] as special cases.

**Theorem A.** ([Ri01, Theorem, p. 188]) Let $u$ be a nonnegative subharmonic function on $\Omega$. Let $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ be a concave surjection whose inverse $\psi^{-1}$ satisfies the $\Delta_2$-condition. Then there exists a constant $C = C(n,\psi,u) \geq 1$ such that

\[ \psi(u(x_0)) \leq \frac{C}{\rho^n} \int_{B(x_0,\rho)} \psi(u(y)) \, dm(y) \]

for any ball $B(x_0,\rho) \subset \Omega$. 
Recall that a function $\psi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ satisfies the $\Delta_2$-condition, if there is a constant $C = C(\psi) \geq 1$ such that

$$\psi(2t) \leq C \psi(t)$$

for all $t \in \mathbb{R}_+$.

In order to improve our result, Theorem A above, still further, we give the following definition. A function $\psi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is permissible, if there is a nondecreasing, convex function $\psi_1 : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ and an increasing surjection $\psi_2 : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that $\psi = \psi_2 \circ \psi_1$ and such that the following conditions are satisfied:

(a) $\psi_1$ satisfies the $\Delta_2$-condition.
(b) $\psi_2^{-1}$ satisfies the $\Delta_2$-condition.
(c) The function $t \mapsto \frac{t}{\psi_2(t)}$ is quasi-increasing, i.e. there is a constant $C = C(\psi_2) \geq 1$ such that

$$\frac{s}{\psi_2(s)} \leq C \frac{t}{\psi_2(t)}$$

for all $s, t \in \mathbb{R}_+$, $0 \leq s \leq t$.

Observe that the condition (b) is equivalent with the following condition.

(b') For some constant $C = C(\psi_2) \geq 1$,  

$$\psi_2(Ct) \geq 2 \psi_2(t)$$

for all $t \in \mathbb{R}_+$.

If $\psi$ is a permissible function, we will in the sequel use always one, fixed constant $C_1 = C_1(\psi)$, in the case of all the properties (a), (b), (c) (and (b')). If $\psi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is an increasing surjection satisfying the conditions (b) and (c), we say that it is strictly permissible. Permissible functions are necessarily continuous.

Let it be noted that the condition (c) above is indeed natural. For just one counterpart to it, see e.g. [HiPh57, Theorem 7.2.4, p. 239].

Observe that our previous definition for permissible functions in [Ri99, 1.3, p. 232] was much more restrictive: A function $\psi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ was there defined to be permissible if it is of the form $\psi(t) = \vartheta(t)^p$, $p > 0$, where $\vartheta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a nondecreasing, convex function satisfying the $\Delta_2$-condition.

2.3 Remarks The following list gives examples of permissible functions (we leave the slightly tedious verifications to the reader). Our list, especially (ii), shows that the considered class of permissible functions is wide and natural. On the other hand, in view of the simple example in (vi), one sees that functions of type (ii) are by no means the only permissible functions: There exists a huge amount of permissible functions of other types.

(i) The functions $\psi_1(t) = \vartheta(t)^p$, $p > 0$, already considered in [Ri99, 1.3, p. 232, and Lemma 2.1, p. 233].
(ii) Functions of the form $\psi_2 = \phi_2 \circ \varphi_2$, where $\phi_2 : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a concave surjective function whose inverse $\phi_2^{-1}$ satisfies the $\Delta_2$-condition, and $\varphi_2 : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a nondecreasing convex function satisfying the $\Delta_2$-condition. (Observe here that any concave function $\phi_2 : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is necessarily nondecreasing.) These (or, to be more exact, the functions $\phi_2$ defined above) were considered in [Ri91, Theorem, p. 188].
(iii) $\psi_3(t) = c t^{\alpha \gamma} (\log(\delta + t^\gamma))^{\beta}$, where $c > 0$, $0 < \alpha < 1$, $\delta \geq 1$, and $\beta, \gamma \in \mathbb{R}$ are such that $0 < \alpha + \beta \gamma < 1$, and $p \geq 1$. 

(iv) For $0 < \alpha < 1$, $\beta \geq 0$ and $p \geq 1$,

$$\psi_4(t) = \begin{cases} p^\alpha, & \text{for } 0 \leq t \leq e, \\
u_\beta(t), & \text{for } t > e. \end{cases}$$

(v) For $0 < \alpha < 1$, $\beta < 0$ and $p \geq 1$,

$$\psi_5(t) = \begin{cases} (\frac{\beta}{\alpha})^p, & \text{for } 0 \leq t \leq e^{-\beta/\alpha}, \\
u_\beta(t), & \text{for } t > e^{-\beta/\alpha}. \end{cases}$$

(vi) For $p \geq 1$,

$$\psi_6(t) = \begin{cases} 2n + \sqrt{p - 2n}, & \text{for } t^p \in [2n, 2n + 1), \quad n = 0, 1, 2, \ldots, \\
2n + 1 + [t^p - (2n + 1)]^2, & \text{for } t^p \in [2n + 1, 2n + 2), \quad n = 0, 1, 2, \ldots. \end{cases}$$

For $p = 1$ the functions in (i), (iii), (iv), (v), (vi), and also in (ii) provided $\phi_2(t) = t$, are strictly permissible.

2.4 Remark. Our previous results were restricted to the cases where $\psi$ was either of type (i) ([Ri99, (1.3), p. 232, and Lemma 2.1, p. 233]) or of type (ii) ([Ri01, Theorem, p. 188] (or Theorem A above)). Now we give a refinement to our results in a unified form. The proof is a modification of Pavlović’s argument [Pa94, proof of Theorem 1, p. 20].

2.5 Theorem. Let $u$ be a nonnegative quasi-nearly subharmonic function on $\Omega$. If $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is a permissible function, then $\psi \circ u$ is quasi-nearly subharmonic on $\Omega$.

Proof. In view of [Ri99, Lemma 2.1, p. 233] we may restrict us to the case where $\psi = \psi_2 : \mathbb{R}_+ \to \mathbb{R}_+$ is strictly permissible.

Since $\psi$ is continuous, $\psi \circ u$ is measurable and $\psi \circ u \in L^1_{\text{loc}}(\Omega)$. It remains to show that $\psi \circ u$ satisfies the generalized mean value inequality (2). But this can be seen exactly as in [Ri01, proof of Theorem, pp. 188-189], the only difference being that instead of the property 2.4 in [Ri01, p. 188] of concave functions, one now uses the above property (c) in 2.2 of permissible functions. □

3. Weighted boundary behavior

3.1 Stoll’s result. Improving previous results of Gehring [Ge57, Theorem 1, p. 77] and Hal- lenbeck [Ha92, Theorems 1 and 2, pp. 117-118], Stoll [St98, Theorem 2, p. 307] gave the following result.

Theorem B. Let $f$ be a nonnegative subharmonic function on a domain $G$ in $\mathbb{R}^n$, $G \neq \mathbb{R}^n$, $n \geq 2$, with $C^1$ boundary. Let

$$\int_G f(x)^p \delta(x)^\gamma dm(x) < \infty$$

for some $p > 0$ and $\gamma > -1 - \beta(p)$. Let $0 < d \leq n - 1$. Then for each $\tau \geq 1$ and $\alpha > 0$ ($\alpha > 1$ when $\tau = 1$), there exists a subset $E_\tau$ of $\partial G$ with $H^d(E_\tau) = 0$ such that

$$\lim_{\rho \to 0} \sup_{x \in \Gamma_{\tau,\alpha,\rho}(\zeta)} [\delta(x)^{n + \gamma - 1 - \beta(p)} f(x)^p] = 0$$

for all $\zeta \in \partial G \setminus E_\tau$.

Above, for $\zeta \in \partial G$ and $\rho > 0$,

$$\Gamma_{\tau,\alpha,\rho}(\zeta) = \Gamma_{\tau,\alpha}(\zeta) \cap G_{\rho},$$
where
\[ \Gamma_{t,\alpha}(\zeta) = \{ x \in G : |x - \zeta|^t < \alpha \delta(x) \}, \quad G_\rho = \{ x \in G : \delta(x) < \rho \}. \]
Moreover, \( \beta(p) = \max\{ (n-1)(1-p), 0 \} \). Stoll makes the assumption \( \gamma > -1 - \beta(p) \) in order to exclude the trivial case \( f \equiv 0 \). As a matter of fact, it follows from a result of Suzuki [Su90, Theorem 2, p. 271] that (3) together with the condition \( \gamma \leq -1 - \beta(p) \) implies indeed that \( f \equiv 0 \), provided \( G \) is a bounded domain with \( C^2 \) boundary.

Our previous improvement [Ri99, Theorem, p. 233] to Stoll's result, Theorem B above, can now be refined slightly further, just using our above result Theorem 2.5. This refinement will be given in Theorem 3.4 below. For this purpose we first recall, and just for the convenience of the reader, some terminology from [Ri99, pp. 231–232].

### 3.2 Admissible functions

A function \( \varphi : \mathbb{R}_+ \to \mathbb{R}_+ \) is admissible, if it is increasing (strictly), surjective, and there are constants \( C_2 > 1 \) and \( r_2 > 0 \) such that
\[ \varphi(2t) \leq C_2 \varphi(t) \quad \text{and} \quad \varphi^{-1}(2s) \leq C_2 \varphi^{-1}(s) \quad \text{for all} \quad s, t \in \mathbb{R}_+, \quad 0 \leq s, t \leq r_2. \]

Nonnegative, nondecreasing functions \( \varphi_1(t) \) which satisfy the \( \Delta_2 \)-condition and for which the functions \( t \mapsto \frac{\varphi_1(t)}{t} \) are nondecreasing, are examples of admissible functions. Further examples are \( \varphi_2(t) = c t^\alpha [\log(\delta + t^\gamma)]^\beta \), where \( c > 0, \alpha > 0, \delta \geq 1, \) and \( \beta, \gamma \in \mathbb{R} \) are such that \( \alpha + \beta \gamma > 0 \).

### 3.3 Accessible boundary points and approach regions

Let \( \varphi : \mathbb{R}_+ \to \mathbb{R}_+ \) be an admissible function and let \( \alpha > 0 \). We say that \( \zeta \in \partial \Omega \) is \((\varphi, \alpha)\)-accessible, if
\[ \Gamma_{\varphi}(\zeta, \alpha) \cap B(\zeta, \rho) \neq \emptyset \]
for all \( \rho > 0 \). Here
\[ \Gamma_{\varphi}(\zeta, \alpha) = \{ x \in \Omega : \varphi(|x - \zeta|) < \alpha \delta(x) \}, \]
and we call it a \((\varphi, \alpha)\)-approach region in \( \Omega \) at \( \zeta \).

Mizuta [Mi91] has considered boundary limits of harmonic functions in Sobolev-Orlicz classes on bounded Lipschitz domains \( U \) of \( \mathbb{R}^n, n \geq 2 \). His approach regions are of the form
\[ \Gamma_{\phi}(\zeta, \alpha) = \{ x \in U : \phi(|x - \zeta|) < \alpha \delta(x) \}, \]
where now \( \phi : \mathbb{R}_+ \to \mathbb{R}_+ \) is a nondecreasing function which satisfies the \( \Delta_2 \)-condition and is such that \( t \mapsto \frac{\phi(t)}{t} \) is nondecreasing. As pointed out above, such functions are admissible in our sense. In fact, they form a proper subclass of our admissible functions.

### 3.4 Theorem

Let \( H^d(\partial \Omega) < \infty \) where \( 0 \leq d \leq n \). Suppose that \( u \) is a nonnegative quasinearly subharmonic function in \( \Omega \). Let \( \varphi : \mathbb{R}_+ \to \mathbb{R}_+ \) be an admissible function and \( \alpha > 0 \). Let \( \psi : \mathbb{R}_+ \to \mathbb{R}_+ \) be a permissible function. Suppose that
\[ \int_{\Omega} \psi(u(x)) \delta(x)^\gamma dm(x) < \infty \]
for some \( \gamma \in \mathbb{R} \). Then
\[ \lim_{\rho \to 0} \left( \sup_{x \in \Gamma_{\varphi,\rho}(\zeta, \alpha)} \{ \delta(x)^{n+\gamma}[\varphi^{-1}(\delta(x))]^{-d} \psi(u(x)) \} \right) = 0 \]
for \( H^d \)-almost every \((\varphi, \alpha)\)-accessible point \( \zeta \in \partial \Omega \). Here
\[ \Gamma_{\varphi,\rho}(\zeta, \alpha) = \{ x \in \Gamma_{\varphi}(\zeta, \alpha) : \delta(x) < \rho \}. \]
The proof is verbatim the same as [Ri99, proof of Theorem, pp. 235–238], except that now we just replace [Ri99, Lemma 2.1, p. 233] by the more general Theorem 2.5 above.

3.5 Remark. Unlike Stoll, we have imposed no restrictions on the exponent \( \gamma \) in order to exclude the trivial case \( u \equiv 0 \). We refer, however, to such possibilities in Remark 4.7 below, after having given in Corollary 4.5 a limiting case result for a result of Suzuki.

4. A Limiting Case Result to Nonintegrability Results of Suzuki

4.1 Suzuki’s result. Suzuki [Su91, Theorem and its proof, pp. 113–115] gave the following result.

**Theorem C.** Let \( 0 < p \leq 1 \). If a superharmonic (resp. nonnegative subharmonic) function \( v \) on \( \Omega \) satisfies

\[
\int_{\Omega} |v(x)|^p \delta(x)^{np-n-2p} dm(x) < \infty,
\]

then \( v \) vanishes identically.

Suzuki pointed also out that his result is sharp in the following sense: If \( p, 0 < p \leq 1 \), is fixed, then the exponent \( \gamma = np - n - 2p \) cannot be increased. On the other hand, clearly \( -n < \gamma \leq -2 \), when \( 0 < p \leq 1 \). Since the class of permissible functions include, in addition the functions \( t^p \), \( 0 < p \leq 1 \), also a large amount of essentially different functions (see 2.3 above), one is tempted to ask whether there exists any limiting case result for Suzuki’s results, corresponding to the case \( p = 0 \). To be more precise, we pose the following question:

Let \( \Omega \) and \( v \) be as above. Let \( \gamma \leq -n \) and let \( \psi : \mathbb{R}_+ \to \mathbb{R}_+ \) be permissible. Does the condition

\[
\int_{\Omega} \psi(|v(x)|) \delta(x)^\gamma dm(x) < \infty,
\]

imply \( v \equiv 0 ? \)

Observe that the least severe form of above integrability condition occurs when \( \gamma = -n \).

Below in Corollary 4.5 we answer the question in the affirmative, in the case of any strictly permissible function \( \psi \). In order to achieve this, we first formulate below in Theorem 4.3 a general result for arbitrary \( \gamma \leq -2 \) which is, for \( -n < \gamma \leq -2 \), however, essentially more or less just Suzuki’s above result (see Remarks 4.4 (b) below). Our formulation has the advantage that, unlike Suzuki’s result, it contains a certain limiting case, Corollary 4.5, too.

The proof which we below write down (and in quite detail, just for the convenience of the reader) is merely a slight modification of Suzuki’s argument, combined with our version for the generalized mean value inequality (Theorem 2.5 above), and also some additional estimates.

4.2 Lemma. Let \( u \) be a nonnegative subharmonic function on \( \Omega \). Suppose \( \psi : \mathbb{R}_+ \to \mathbb{R}_+ \) is a permissible function such that

\[
\int_{\Omega} \psi(u(x)) \delta(x)^\gamma dm(x) < \infty
\]

for some \( \gamma \in \mathbb{R} \). Then \( \psi(u(x)) = o(\delta(x)^{-n-\gamma}) \) as \( \delta(x) \to 0 \).

**Proof.** By Theorem 2.5, \( \psi \circ u \) is quasi-nearly subharmonic on \( \Omega \). Write for \( x \in \Omega \), \( B = B(x, \delta(x)) \) and \( B_0 = B(x, \frac{\delta(x)}{2}) \). Since

\[
\frac{1}{2} \delta(x) < \delta(y) < \frac{3}{2} \delta(x)
\]

(7)
for all $y \in B_0$, we get
\[
\psi(u(x)) \leq 2^n + |\gamma|C_0\delta(x)^{-n-\gamma} \int_{B_0} \psi(u(y))\delta(y)^\gamma dm(y) \\
\leq C\delta(x)^{-n-2} \int_{\Omega_\delta(x)} \psi(u(y))\delta(y)^\gamma dm(y),
\]
where $C = C(\gamma, n, \psi, u) > 0$ and $\Omega_{\delta(x)} = \{y \in \Omega : \delta(y) < \delta(x)\}$. The claim follows. □

4.3 Theorem. Let $\Omega$ be bounded. Let $v$ be a superharmonic (resp. nonnegative subharmonic) function on $\Omega$. Let $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ be a strictly permissible function. Suppose
\[
(8) \quad \int_{\Omega} \psi(|v(x)|)\delta(x)^\gamma dm(x) < \infty,
\]
where $\gamma \leq -2$ is such that there is a constant $C = C(\gamma, n, \psi, \Omega) > 0$ such that
\[
(9) \quad s^{n+\gamma} \leq \psi(Cs^{n-2}) \quad \text{for all } s > \frac{1}{\text{diam}\Omega}.
\]
Then $v$ vanishes identically.

4.4 Remarks. Next we consider the assumptions in Theorem 4.3.

(a) Our assumption $\gamma \leq -2$ is unnecessary, and it could be dropped: If $\gamma \in \mathbb{R}$, then it follows easily from (9) and from the property (c) in 2.2 of strictly permissible functions that indeed $\gamma \leq -2$.

(b) Suppose that $-n < \gamma \leq -2$. If, instead of (9), one supposes that
\[
s^{n+\gamma} \leq \psi(Cs^{n-2}) \quad \text{for all } s > 0,
\]
then clearly
\[
\psi(|v(x)|) \geq C^{-\frac{n+\gamma}{n-2}}|v(x)|^{\frac{n+\gamma}{n-2}}
\]
for all $x \in \Omega$. Thus (8) implies that
\[
\int_{\Omega} |v(x)|^{\frac{n+\gamma}{n-2}}\delta(x)^\gamma dm(x) < \infty,
\]
and hence $v \equiv 0$ by Suzuki’s result, Theorem C above. Recall that here $0 < p = \frac{n+\gamma}{n-2} \leq 1$ and $\gamma = np - n - 2p$. Thus Theorem 4.3, but now the assumption (9) replaced with the aforesaid assumption, is just a restatement of Suzuki’s result for bounded domains.

(c) If $\gamma \leq -n$, then the condition (9) clearly holds, since $\psi$ is strictly permissible. This case gives indeed the already referred limiting case for Suzuki’s result. See Corollary 4.5 below.

Proof of Theorem 4.3. We write the proof down only for the case $n \geq 3$. Write $v^+ = \max\{v, 0\}$ and $s = v^- = -\min\{v, 0\}$. Then $|v| = v^+ + v^-$ and $s \geq 0$ is subharmonic. (Resp. if $v$ is nonnegative and subharmonic, let $s = v$.) Proceeding as Suzuki, but using also some additional estimates, we will show that $s \equiv 0$.

By (8),
\[
\int_{\Omega} \psi(s(x))\delta(x)^\gamma dm(x) < \infty,
\]
thus
\[
v_{\psi, \gamma}(x) = \int_{\Omega} G_{\Omega}(x, y)\psi(s(y))\delta(y)^\gamma dm(y) < \infty
\]
is a potential by [Hel69, Theorem 6.3, p. 99]. Here $G_{\Omega}$ is the Green function of $\Omega$. 

By Lemma 4.2, \( \psi(s(y)) \leq C \delta(y)^{-n-\gamma} \). Thus also

(10) \[
s(y) \leq \psi^{-1}(C \delta(y)^{-n-\gamma}) \leq C \psi^{-1}(\delta(y)^{-n-\gamma})
\]
for all \( y \in \Omega \), where \( C = C(\gamma, n, \psi, s, \Omega) \geq 1 \). Let \( x \in \Omega \) be fixed for a while. Let \( B = B(x, \delta(x)) \) and \( B_0 = B(x, \frac{\delta(x)}{2}) \). Using (10) and (7) one gets

\[
s(y) \leq C \psi^{-1}(\delta(y)^{-n-\gamma}) \leq C \psi^{-1}(2^{n+\gamma} \delta(x)^{-n-\gamma}) \leq C \psi^{-1}(\delta(x)^{-n-\gamma})
\]
for all \( y \in B_0 \). Therefore

\[
\frac{\psi(s(y))}{s(y)} \geq C \frac{\psi(C \psi^{-1}(\delta(x)^{-n-\gamma}))}{\psi^{-1}(\psi^{-1}(\delta(x)^{-n-\gamma}))} \geq C \frac{\delta(x)^{-n-\gamma}}{\psi^{-1}(\delta(x)^{-n-\gamma})}
\]
for all \( y \in B_0 \). With the aid of this and of a standard estimate for the Green function \( G_B(x, \cdot) \) in \( B_0 \) and of (10), one gets

\[
v_{\psi, \gamma}(x) = \int_{\Omega} G_\Omega(x, y) \psi(s(y)) \delta(y)^\gamma dm(y) \geq \int_{B_0} G_B(x, y) \psi(s(y)) \delta(y)^\gamma dm(y) \\
\geq C \frac{\delta(x)^{-n-\gamma}}{\psi^{-1}(\delta(x)^{-n-\gamma})} \int_{B_0} |x - y|^{2-n} \delta(y)^\gamma s(y) dm(y) \\
\geq C \frac{\delta(x)^{-n-2} \psi^{-1}(\delta(x)^{-n-\gamma})}{\delta(x)^{-n-\gamma}} s(x)
\]

By (9) we see that there is a constant \( C_3 \geq 1 \) such that

\[
\delta(y)^{n-2} \psi^{-1}(\delta(y)^{-n-\gamma}) \leq C_3
\]
for all \( y \in \Omega \). Combining this with the above estimate for \( v_{\psi, \gamma} \), one gets

\[
v_{\psi, \gamma}(x) \geq C s(x),
\]
where \( C = C(\gamma, n, \psi, s, \Omega) > 0 \). Remembering that \( x \in \Omega \) was arbitrary, that \( v_{\psi, \gamma} \) is a potential and \( s \) subharmonic, it follows from [Hel69, Corollary 6.19, p. 117] that \( s \equiv 0 \). Thus \( v = v^+ \geq 0 \). It remains to show that \( v \equiv 0 \).

As above,

\[
\int_{\Omega} G_\Omega(x, y) \delta(y)^{-2} dm(y) \geq \int_{B_0} G_B(x, y) \delta(y)^{-2} dm(y) \\
\geq C \int_{B_0} |x - y|^{2-n} \delta(y)^{-2} dm(y) \\
\geq C \delta(x)^{2-n} \delta(x)^{-2} v_n \left( \frac{\delta(x)}{2} \right)^n = C
\]
where \( C = C(n) > 0 \). Thus by [Hel69, Lemma 6.1, p. 98, and Corollary 6.19, p. 117],

(11) \[
\int_{\Omega} G_\Omega(x, y) \delta(x)^{-2} dm(y) = \infty
\]
for all \( x \in \Omega \). Consider next an arbitrary potential \( w \) on \( \Omega \),

(12) \[
w(x) = \int_{\Omega} G_\Omega(x, y) d\lambda(y)
\]
where \( \lambda \neq 0 \) is a measure on \( \Omega \). From (11) it follows that

\[
\int_{\Omega} w(x) \delta(x)^{-2} dm(x) = \int_{\Omega} \left[ \int_{\Omega} G_\Omega(x, y) d\lambda(y) \right] \delta(x)^{-2} dm(x) \\
= \int_{\Omega} \left[ \int_{\Omega} G_\Omega(x, y) \delta(x)^{-2} dm(x) \right] d\lambda(y) = \infty.
\]
Using this and the facts that $\Omega$ is bounded and $w$, as a superharmonic function, is locally integrable, one sees that

\begin{equation}
\int_{\Omega_1} w(x) \delta(x)^{-2} \, dm(x) = \infty
\end{equation}

where $\Omega_1 = \{ x \in \Omega : \delta(x) < 1 \}$.

Suppose in particular that $w$ in (12) is the potential of the superharmonic function $v_M = \inf\{ v, M \}$, where $M > 0$. Then $v_M \geq w$, and one has by (13) and by the fact that $\gamma \leq -2$,

\[
\infty > \int_{\Omega} \psi(v(x)) \delta(x)^{\gamma} \, dm(x) \geq \int_{\Omega} \psi(v_M(x)) \delta(x)^{\gamma} \, dm(x) \\
\geq \int_{\Omega} v_M(x) ^{\frac{\psi(v_M(x))}{v_M(x)}} \delta(x)^{\gamma} \, dm(x) \\
\geq C \frac{\psi(M)}{M} \int_{\Omega} v_M(x) \delta(x)^{\gamma} \, dm(x) \\
\geq C \frac{\psi(M)}{M} \int_{\Omega_1} w(x) \delta(x)^{-2} \, dm(x) = \infty,
\]

a contradiction unless $w \equiv 0$. Since $w \equiv 0$, the nonnegative superharmonic functions $v_M$, $M > 0$, are in fact harmonic, e.g. by the Riesz Decomposition Theorem [Hel69, Theorem 6.18, p. 116]. Since this is impossible, one has $v \equiv 0$, concluding the proof.

**4.5 Corollary.** Let $\Omega$ be bounded. Let $v$ be a superharmonic (resp. nonnegative subharmonic) function on $\Omega$. Let $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ be any strictly permissible function and let $\gamma \leq -n$. If

\[
\int_{\Omega} \psi(|v(x)|) \delta(x)^{\gamma} \, dm(x) < \infty,
\]

then $v$ vanishes identically.

For the proof observe that the condition (9) is indeed satisfied for $\gamma \leq -n$, since $\Omega$ is bounded and $\psi$ is increasing.

**4.6 Remark.** The result of Theorem 4.3 does not, of course, hold any more, if one replaces strictly permissible functions by permissible functions. For a counterexample, set, say, $v(x) = |x|^{2-n}$, $\psi(t) = t^p$, where $\frac{n-1}{n-2} < p < \frac{n}{n-2}$, $\gamma = np - n - 2p$ or just $\gamma > 1$. Then clearly

\[
\int_{B} v(x)^{p} \delta(x)^{\gamma} \, dm(x) < \infty
\]

but $v \not\equiv 0$.

**4.7 Remark.** Provided $\Omega$ is bounded and $\psi$ is strictly permissible, one can, with the aid of Theorem 4.3 and Corollary 4.5, exclude some trivial cases $v \equiv 0$ from the result of Theorem 3.4 by imposing certain restrictions on the exponent $\gamma$. We point out only two cases:

(i) By Corollary 4.5, $\gamma > -n$, regardless of $\psi$.

(ii) By Suzuki’s result, Theorem C above, $\gamma > np - n - 2p$, in the case when $\psi(t) = t^p$, $0 < p \leq 1$.

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