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Energy of General Spherically Symmetric Solution in the Tetrad Theory of Gravitation

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We find the most general spherically symmetric solution in a special class of the tetrad theory of gravitation. The tetrad gives the Schwarzschild metric. The energy is calculated using both the superpotential method and the Euclidean continuation method. We find that unless the time-space components of the tetrad go to zero faster than $1/r$ at infinity, the two methods give different results and that these results differ from the gravitational mass of the central gravitating body. This fact implies that the time-space components of the tetrad describing an isolated spherical body must vanish faster than $1/r$ at infinity.

§ 1. Introduction

The notion of absolute parallelism was first introduced in physics by Einstein trying to unify gravitation and electromagnetism into 16 degrees of freedom of the tetrads. His attempt failed, however, because there was no Schwarzschild solution in his field equation.

Moller revived the tetrad theory of gravitation and showed that a tetrad description of a gravitational field allows a more satisfactory treatment of the energy-momentum complex than that of general relativity. The Lagrangian formulation of the theory was given by Pellegrini and Plebanski. In these attempts the admissible Lagrangians are limited by the assumption that the equations determining the metric tensor should coincide with the Einstein equation. Moller abandoned this assumption and suggested to look for a wider class of Lagrangians, allowing for possible deviation from the Einstein equation in the case of strong gravitational fields. Saez generalized Moller's theory into a scalar tetrad theory of gravitation. Meyer showed that Moller's theory is a special case of the Poincaré gauge theory.

Quite independently, Hayashi and Nakano formulated the tetrad theory of gravitation as a gauge theory of the space-time translation group. Hayashi and Shirafuji studied the geometrical and observational basis of the tetrad theory, assuming that the Lagrangian be given by a quadratic form of a torsion tensor. If invariance under the parity operations is assumed, the most general Lagrangian consists of three terms with three unknown parameters to be fixed by experiment in addition to a cosmological term. Two of the three parameters were determined by comparison with solar-system experiments, while only an upper bound has been estimated for the third.

The numerical values of the two parameters found were very small, consistent

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with a value of zero. If these two parameters are exactly equal to zero, the theory reduces to the one proposed by Hayashi and Nakano\textsuperscript{9} and Møller,\textsuperscript{4} which we shall here refer to as the HNM theory. This theory differs from general relativity only when the torsion tensor has a nonvanishing axial-vector part. It was also shown\textsuperscript{10} that the Birkhoff theorem can be extended to the HNM theory. Namely, for the spherically symmetric case in vacuum, which is not necessarily time independent, the axial-vector part of the torsion tensor should vanish due to the antisymmetric part of the field equation, and therefore, with the help of the Birkhoff theorem\textsuperscript{12} of general relativity, we see that the space-time metric is the Schwarzschild.

Mikhail et al.\textsuperscript{13} derived the superpotential of the energy-momentum complex in the HNM theory and applied it to two spherically symmetric solutions. It was found that in one of the two solutions the gravitational mass does not coincide with the calculated energy. Mikhail et al.\textsuperscript{14} also derived a spherically symmetric solution of the HNM theory starting from a tetrad which contains three unknown functions and following Mazumder and Ray.\textsuperscript{15} The solution contains one arbitrary function of the radial coordinate \( r \), and all previous solutions can be obtained from it. The physical properties of this solution have not yet been examined, however. We show that the underlying metric of this solution is just the Schwarzschild metric under certain conditions in agreement with the extended Birkhoff theorem mentioned above.

The general form of the tetrad, \( \lambda_{\mu}^{\nu} \), having spherical symmetry was given by Robertson.\textsuperscript{16} In the cartesian form it can be written as\textsuperscript{*}

\[
\begin{align*}
\lambda_{0}^{\alpha} &= iA, & \lambda_{a}^{0} &= Cx^{a}, & \lambda_{0}^{x} &= iDx^{a}, \\
\lambda_{a}^{x} &= \delta_{a}^{b}B + Fx^{a}x^{b} + \epsilon_{abc}Sx^{c},
\end{align*}
\tag{1·1}
\]

where \( A, C, D, B, F \) and \( S \) are functions of \( t \) and \( r = (x^{a}x^{a})^{1/2} \), and the zeroth vector \( \lambda_{0}^{\mu} \) has the factor \( i = \sqrt{-1} \) to preserve the Lorentz signature. We consider an asymptotically flat space-time in this paper and impose the boundary condition that for \( r \to \infty \) the tetrad (1·1) approaches the tetrad of Minkowski space-time, \((\lambda_{\mu}^{\nu}) = \text{diag} (i, \delta_{a}^{a})\).

It is the aim of the present work to find the most general, asymptotically flat solution with spherical symmetry in the HNM theory and calculate the energy of that solution. We do this using two methods and compare the results. One method is to apply the superpotential of Mikhail et al.,\textsuperscript{13} and the other is based on the Euclidean continuation method of Gibbons and Hawking.\textsuperscript{17–19}

In § 2 we briefly review the tetrad theory of gravitation. In § 3 we first study the general, spherically symmetric solution with a nonvanishing \( S \)-term (see (1·1)) and obtain a solution with one parameter. Then we study the general, spherically symmetric tetrad without the \( S \)-term. All the remaining, unknown functions are allowed to depend on \( t \) and \( r \). We find the general solution with an arbitrary function of \( t \)

\* In this paper Latin indices \((i, j, \cdots)\) represent the vector number, and Greek indices \((\mu, \nu, \cdots)\) represent the vector components. All indices run from 0 to 3. The spatial part of Latin indices are denoted by \((a, b, \cdots)\), while that of greek indices by \((\alpha, \beta, \cdots)\). In the present convention, latin indices are never raised. The tetrad \( \lambda^{a} \) is related to the parallel vector fields \( b^{a} \) of Ref. 10) by \( \lambda^{a} = ib^{a} \) and \( \lambda^{a} = b_{a}^{a} \).
and \( r \). We also study the solution of Mikhail et al.\(^{14}\) by transforming it to isotropic cartesian coordinates. It is then found that their general tetrad is just the \( t \)-independent case of our general tetrad without the \( S \)-term.

In § 4 the energy of the gravitating source is calculated using the superpotential method, assuming different types of asymptotic behavior for the unknown function involved in the tetrad. In § 5 we discuss the Euclidean continuation of the general stationary tetrad and calculate the action, the energy, and the entropy following the Gibbons-Hawking method. The final section is devoted to presenting the main results and discussion.

The computer algebra system REDUCE 3.3 is used in some calculations.

### § 2. The tetrad theory of gravitation

In this paper we follow Möller's construction\(^{41}\) of the tetrad theory of gravitation based on the Weitzenböck space-time. In this theory the field variables are the 16 tetrad components \( \gamma_{i}^{\mu} \), from which the metric is derived by

\[
g^{\mu \nu} \equiv \gamma_{i}^{\mu} \gamma_{i}^{\nu} \tag{2.1}\]

The Lagrangian \( L \) is an invariant constructed from \( \gamma_{\mu \nu \rho} \) and \( g^{\mu \nu} \), where \( \gamma_{\mu \nu \rho} \) is the contorsion tensor given by

\[
\gamma_{\mu \nu \rho} \equiv \gamma_{i}^{\mu} \gamma_{i}^{\nu} ; \rho \tag{2.2}\]

where the semicolon denotes covariant differentiation with respect to Christoffel symbols. The most general Lagrangian density invariant under the parity operation is given by the form

\[
\mathcal{L} \equiv (-g)^{1/2} \left( a_{1} \Phi^{\mu} \Phi_{\mu} + a_{2} \gamma^{\mu \nu \rho} \gamma_{\mu \nu \rho} + a_{3} \gamma^{\mu \nu \rho} \gamma_{\nu \mu \rho} \right) \tag{2.3}\]

where

\[
g \equiv \det(g_{\mu \nu}) \tag{2.4}\]

and \( \Phi_{\mu} \) is the basic vector field defined by

\[
\Phi_{\mu} \equiv \gamma_{\mu \rho} \tag{2.5}\]

Here \( a_{1} \), \( a_{2} \) and \( a_{3} \) are constants determined by Möller such that the theory coincides with general relativity in the weak fields:

\[
a_{1} = -\frac{1}{\kappa}, \quad a_{2} = \frac{\lambda}{\kappa}, \quad a_{3} = \frac{1}{\kappa} (1 - 2\lambda) \tag{2.6}\]

where \( \kappa \) is the Einstein constant and \( \lambda \) is a free dimensionless parameter.\(^{*}\) The same identification of the parameters was obtained by Hayashi and Nakano.\(^{9}\)

Möller applied the action principle to the Lagrangian density (2.3) and obtained

\(^{*}\) Throughout this paper we use the relativistic units, \( c = G = 1 \) and \( \kappa = 8\pi \).
the field equation in the form
\[ G_{\mu\nu} + H_{\mu\nu} = -\kappa T_{\mu\nu}, \] (2·7)
\[ F_{\mu\nu} = 0, \] (2·8)
where the Einstein tensor \( G_{\mu\nu} \) is defined by
\[ G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R. \] (2·9)
Here \( H_{\mu\nu} \) and \( F_{\mu\nu} \) are given by
\[ H_{\mu\nu} \overset{\text{def}}{=} \lambda \left[ \gamma_{\rho\sigma} \gamma^{\rho\sigma} + \gamma_{\rho\sigma} \gamma^{\rho\sigma} + \gamma_{\rho\sigma} \gamma^{\rho\sigma} + g_{\mu\nu} \left( \gamma_{\rho\sigma} \gamma^{\rho\sigma} - \frac{1}{2} \gamma_{\rho\sigma} \gamma^{\rho\sigma} \right) \right] \] (2·10)
and
\[ F_{\mu\nu} \overset{\text{def}}{=} \lambda \left[ \Phi_{\mu,\nu} - \Phi_{\nu,\mu} - \Phi_{\rho} \left( \gamma^{\rho}_{\mu\nu} - \gamma^{\rho}_{\nu\mu} \right) + \gamma^{\rho}_{\mu\nu;\rho} \right]. \] (2·11)
These are symmetric and skew symmetric tensors, respectively.

Møller assumed that the energy-momentum tensor of matter fields is symmetric. In the Hayashi-Nakano theory, however, the energy-momentum tensor of spin-1/2 fundamental particles has a nonvanishing antisymmetric part arising from effects due to the intrinsic spin, and the right-hand side of (2·8) does not vanish when we take into account the possible effects of intrinsic spin. Nevertheless, since in this paper we consider only solutions in the vacuum, we refer to the tetrad theory of gravitation based on the choice of the parameters, (2·6), as the Hayashi-Nakano-Møller (HNM) theory.

It can be shown\(^{10}\) that the tensors \( H_{\mu\nu} \) and \( F_{\mu\nu} \) consist of only those terms which are linear or quadratic in the axial-vector part of the torsion tensor \( a_{\rho} \), defined by
\[ a_{\rho} \overset{\text{def}}{=} \frac{1}{3} \varepsilon_{\mu\nu\rho\sigma} \gamma^{\nu\rho\sigma}, \] (2·12)
where \( \varepsilon_{\mu\nu\rho\sigma} \) is defined by
\[ \varepsilon_{\mu\nu\rho\sigma} \overset{\text{def}}{=} (-g)^{1/2} \delta_{\mu\nu\rho\sigma} \] (2·13)
with \( \delta_{\mu\nu\rho\sigma} \) being completely antisymmetric and normalized as \( \delta_{0123} = -1 \). Therefore, both \( H_{\mu\nu} \) and \( F_{\mu\nu} \) vanish if \( a_{\rho} \) vanishes. In other words, when \( a_{\rho} \) is found to vanish from the antisymmetric part of the field equation (2·8), the symmetric part (2·7) coincides with the Einstein equation.

For the spherically symmetric case which is not necessarily time-independent, it was shown\(^{10}\) that the antisymmetric part (2·8) implies that \( a_{\rho} \) should vanish. Then according to the Birkhoff theorem of general relativity, the metric of spherically symmetric space-time in the vacuum must be the Schwarzschild.

§ 3. Spherically symmetric solutions

In this section we find the most general, spherically symmetric vacuum solution
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of the form (1·1) in the HNM theory. The axial-vector part of the torsion tensor $a^a$ is vanishing, and the skew part of the field equation is satisfied identically, as explained above. We discuss two cases separately, one with $S \neq 0$ and the other with $S = 0$.

(i) The case of nonvanishing $S$-term. We start with the tetrad of (1·1) with the six unknown functions of $t$ and $r$. In order to study the condition that $a^a$ vanishes, it is convenient to start from the general expression for the covariant components of the tetrad,

$$
\begin{align*}
\lambda_0 &= i\tilde{A}, \quad \lambda_a = \tilde{C}x^a, \quad \lambda_0 = i\tilde{D}x^a, \\
\lambda_a &= \delta_{aa}\tilde{B} + \tilde{F}x^a x^a + \epsilon_{aa}Sx^a,
\end{align*}
$$

(3·1)

where the six unknown functions, $\tilde{A}$, $\tilde{C}$, $\tilde{D}$, $\tilde{F}$, $\tilde{S}$ are connected with the six functions of (1·1). We can assume without loss of generality that the two functions $\tilde{D}$ and $\tilde{F}$ are vanishing by making use of the freedom to redefine $t$ and $r$. Then the condition that $a^a$ vanishes is found to be

$$
\begin{align*}
0 &= \sqrt{(-g)} a^a = \left\{ \begin{array}{l}
2\tilde{B}\tilde{S} + \frac{2}{3}r(\tilde{B}\tilde{S}' - \tilde{B}'\tilde{S}), \quad \mu = 0, \\
\left\{ -\frac{4}{3} \tilde{C}\tilde{S} + \frac{2}{3} (\tilde{B}\tilde{S}' - \tilde{B}'\tilde{S}) \right\} x^a, \quad \mu = a
\end{array} \right. (3·2)
\end{align*}
$$

with $\tilde{S}' = d\tilde{S}/dr$ and $\tilde{S} = d\tilde{S}/dt$. This condition can be satisfied if

$$
\begin{align*}
\tilde{C} = 0, \quad \tilde{S} = -\frac{\xi}{r}\tilde{B},
\end{align*}
$$

(3·3)

where $\xi$ is a constant with the dimension of $(\text{length})^2$.

The symmetric part of the field equations now coincides with the Einstein equation and gives the Schwarzschild metric. The metric tensor formed of the tetrad (3·1) with (3·3) is not of the isotropic form, however, and the space-space components involve a term proportional to $x^a x^b$. We can eliminate such a nondiagonal term of the metric tensor by a scale change of the space coordinate from $x^a$ to $X^a = (\rho/r)x^a$. Here and henceforth we denote by $\rho$ the radial variable of the isotropic coordinate. After an elementary calculation we see that $\rho$ is given by

$$
\rho = \frac{r}{\sqrt{2}} \left(1 + \sqrt{1 + \frac{\xi^2}{r^2}} \right)^{1/2}.
$$

(3·4)

The two functions $\tilde{A}$ and $\tilde{B}$ are so determined that the metric tensor now takes the well-known isotropic form. Applying the scale change to the tetrad (3·1) with (3·3), we finally obtain the general spherically symmetric solution with nonvanishing $S$-term: the nonvanishing, covariant components of the tetrad are given by

$$
\begin{align*}
\lambda_0 = \frac{1 - \frac{m}{2\rho}}{1 + \frac{m}{2\rho}},
\end{align*}
$$

where $m$ is a constant with the dimension of mass.
\[ \lambda_a = \left(1 + \frac{m}{2\rho}\right)^2 \left[ \sqrt{1-k(\rho)^2} \delta_{aa} + (1 - \sqrt{1-k(\rho)^2}) \frac{X^a X^a}{\rho^2} + k(\rho) \epsilon_{aa} \frac{X^a}{\rho} \right], \quad (3.5) \]

where \( k(\rho) \) is given by
\[ k(\rho) = \frac{\xi}{\left(1 + \frac{x}{4\rho^2}\right)\rho^2}. \quad (3.6) \]

(ii) The case of no S-term. In this case the axial-vector part of the torsion tensor is identically vanishing. Thus, when this tetrad is applied to the field equations, the skew part is automatically satisfied in vacuum, and the solution of the symmetric part is the Schwarzschild. Therefore, the solution of the form (1.1) with \( S=0 \) can be obtained from the diagonal tetrad of the Schwarzschild metric by a local Lorentz transformation which maintains spherical symmetry,
\[ (\Lambda_{\mu}) = \begin{pmatrix} \sqrt{H^2+1} & iH \frac{X^b}{\rho} \\ -iH \frac{X^a}{\rho} & \delta_a^b + (\sqrt{H^2+1} - 1) \frac{X^a X^b}{\rho^2} \end{pmatrix}, \quad (3.7) \]

where \( H \) is an arbitrary function of \( t \) and \( \rho \). Namely, we see that
\[ \lambda^\mu = \Lambda_{\mu} \lambda_i^{(0)\mu} \quad (3.8) \]
is the most general spherically symmetric solution without the S-term. Here \( \lambda_i^{(0)\mu} \) is the diagonal tetrad in isotropic cartesian coordinates given by
\[ \lambda_0^{(0)} = i \left( \frac{1 + \frac{m}{2\rho}}{1 - \frac{m}{2\rho}} \right), \quad \lambda_a^{(0)} = i \frac{\delta_a}{(1 + \frac{m}{2\rho})^2}. \quad (3.9) \]

The explicit forms of the \( \lambda^\mu \) are then given by
\[ \lambda_0^0 = \frac{\left(1 + \frac{m}{2\rho}\right)}{\left(1 - \frac{m}{2\rho}\right)} \sqrt{H^2+1}, \]
\[ \lambda_0^a = i \frac{H}{(1 + \frac{m}{2\rho})^2} \frac{X^a}{\rho}, \]
\[ \lambda_a^0 = \frac{H \left(1 + \frac{m}{2\rho}\right)}{(1 - \frac{m}{2\rho})} \frac{X^a}{\rho}, \]
\[ \lambda_a^a = \frac{1}{(1 + \frac{m}{2\rho})^2} \left[ \delta_a^a + (\sqrt{H^2+1} - 1) \frac{X^a X^a}{\rho^2} \right]. \quad (3.10) \]
It is clear that if $\xi$ and $H(r, t)$ are equal to zero the two classes of solutions given by (3·5) and (3·10) coincide and reduce to the solution given by Hayashi and Shirafuji\textsuperscript{10} in the special case $p=q=2$.

Now let us compare the solution (3·10) with that given by Mikhail et al.\textsuperscript{14}. They started from a spherically symmetric tetrad with three unknown functions, which is given in spherical polar coordinates by

$$
\left( \mathcal{X}^\mu \right)_i = \begin{pmatrix}
  iA & iDr & 0 & 0 \\
  0 & B\sin\theta\cos\phi & \frac{B}{r}\cos\theta\cos\phi & -\frac{B\sin\phi}{r\sin\theta} \\
  0 & B\sin\theta\sin\phi & \frac{B}{r}\cos\theta\sin\phi & \frac{B\cos\phi}{r\sin\theta} \\
  0 & B\cos\theta & -\frac{B}{r}\sin\theta & 0
\end{pmatrix} \tag{3·11}
$$

and they applied it to the field equations, (2·7) and (2·8), to obtain a solution of the form

$$
A = \frac{K_1}{1 - \frac{rB'}{B}}, \quad D^2 = \frac{1}{\left(1 - \frac{rB'}{B}\right)^2} \left( \frac{B}{r} \right)^3 \left[ K_2 + \frac{rB'}{B} \left( \frac{rB'}{B} - 2 \right) \frac{r}{B} \right], \tag{3·12}
$$

where $K_1$ and $K_2$ are constants of integration, and $B$ is an arbitrary function of $r$. The line-element squared takes the form

$$
ds^2 = \frac{(B^2 - D^2 r^2)}{A^2 B^2} dt^2 - \frac{2Dr}{AB^2} dr dt + \frac{1}{B^2} (dr^2 + r^2 d\Omega^2) \tag{3·13}
$$

with $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$. We assume $B(r)$ to be nonvanishing so that the surface area of a sphere of constant $r$ is finite. We also assume that $A(r)$ and $B(r)$ satisfy the asymptotic condition $\lim_{r \to \infty} A(r) = \lim_{r \to \infty} B(r) = 1$ and $\lim_{r \to \infty} rB' = 0$. Then, we can show from (3·12) and (3·13) that (1) $K_1 = 1$, (2) $B(r) > 0$, (3) $\lim_{r \to \infty} rD(r) = 0$, and (4) if $B - rB'$ vanishes at some point, then $1 - BK_3/r < 0$ at that point.

Using the coordinate transformation

$$
dT = dt + \frac{ADr}{B^2 - D^2 r^2} dr, \tag{3·14}
$$

we can eliminate the cross term of (3·13) to obtain

$$
ds^2 = -\eta_1 dT^2 + \frac{1}{\eta_1} \frac{dr^2}{A^2 B^2} + \frac{r^2}{B^2} d\Omega^2 \tag{3·15}
$$

with $\eta_1 = (B^2 - D^2 r^2)/A^2 B^2$. Taking the new radial coordinate $R = r/B$, we finally obtain

$$
ds^2 = -\eta_1 dT^2 + \frac{dR^2}{\eta_1} + R^2 d\Omega^2, \tag{3·16}
$$

where
\[ \eta(R) = \left( 1 - \frac{K_2}{R} \right). \tag{3·17} \]

Then, (3·16) coincides with the Schwarzschild metric with mass \( m = K_2/2 \), and hence the solution in the case of the spherically symmetric tetrad gives no more than the Schwarzschild solution when \( 1 - rB'/B \) has no zero and \( R \) is a monotonically increasing function of \( r \). If \( 1 - rB'/B \) has zeroes, the line-element (3·13) is singular at these zeroes, which lie inside the event horizon, as seen from property (4) mentioned above. We shall study in the future whether this singularity at zero-points of \( 1 - rB'/B \) is physically acceptable or not.

The tetrad (3·11) has been subject to two steps of coordinate transformations from \((t, r, \theta, \phi)\) to \((T, R, \theta, \phi)\). We now apply a further transformation from \((T, R, \theta, \phi)\) to the isotropic coordinate \((T, X^a)\) with \( a = 1, 2 \) and 3, where the line-element squared takes the well-known form

\[ ds^2 = -\left( \frac{1 - m}{2\rho} \right)^2 dT^2 + \left( 1 + \frac{m}{2\rho} \right)^2 (dX^a)^2 \tag{3·18} \]

with \( \rho = (X^a X^a)^{1/2} \). After a lengthy calculation, the tetrad can be shown to assume the form

\[ \lambda_0^a = \frac{(1 + \frac{m}{2\rho})^2}{(1 - \frac{m}{2\rho})} \left[ 1 - \frac{\rho B'}{B^3(1 + \frac{m}{2\rho})^4} \right], \]

\[ \lambda_0^a = \frac{2i \sqrt{\frac{m}{2\rho}}}{(1 - \frac{m}{2\rho})(1 + \frac{m}{2\rho})^3} \left[ 1 - \frac{\rho^3 B^2}{2mB^3(1 + \frac{m}{2\rho})^6} \right]^{1/2} \frac{X^a}{\rho}, \]

\[ \lambda_0^a = \frac{\sqrt{\frac{m}{2\rho}}}{(1 - \frac{m}{2\rho})} \left[ 1 - \frac{\rho^3 B^2}{mB^3(1 + \frac{m}{2\rho})^2} \right]^{1/2} \frac{X^a}{\rho}, \]

\[ \lambda_0^a = \frac{1}{(1 + \frac{m}{2\rho})^2} \left[ \delta_a + \frac{m}{(1 - \frac{m}{2\rho})} \left( 1 - \frac{\rho^3 B^2}{mB^3(1 + \frac{m}{2\rho})^2} \right) \frac{X^a X^a}{\rho} \right]. \tag{3·19} \]

It is easy to verify that this tetrad can be obtained from (3·10) by choosing the function \( H \) as

\[ H = 2 \sqrt{\frac{m}{2\rho}} \left[ 1 - \frac{\rho^3 B^2}{mB^3(1 + \frac{m}{2\rho})^2} \right]^{1/2}, \tag{3·20} \]
§ 4. Energy with Möller's method

The superpotential of the HNM theory is given by Mikhail et al.\textsuperscript{13} as

\[ U_{\mu}^{\nu} = \frac{(-g)^{1/2}}{2\kappa} P_{\rho\sigma}^{\mu\nu} \left[ \Phi^\alpha g_{\mu\nu} - \lambda g_{\rho\sigma} \gamma^{\rho\sigma} - (1 - 2\lambda) g_{\rho\nu} \gamma^{\rho\sigma} \right], \tag{4·1} \]

where \( P_{\rho\sigma}^{\mu\nu} \) is

\[ P_{\rho\sigma}^{\mu\nu} \equiv \delta_{\rho}^{\mu} g_{\rho\sigma}^{\nu} + \delta_{\rho}^{\nu} g_{\sigma\rho}^{\mu} - \delta_{\sigma}^{\mu} g_{\rho\nu}^{\rho}, \tag{4·2} \]

with \( g_{\rho\sigma}^{\mu\nu} \) being a tensor defined by

\[ g_{\rho\sigma}^{\mu\nu} = \delta_{\rho}^{\mu} \delta_{\sigma}^{\nu} - \delta_{\rho}^{\nu} \delta_{\sigma}^{\mu}. \tag{4·3} \]

The energy is expressed by the surface integral,\textsuperscript{20}

\[ E = \lim_{\rho \to \infty} \int_{\rho = \text{constant}} U_{\rho}^{\alpha} n_{\alpha} dS, \tag{4·4} \]

where \( n_{\alpha} \) is the unit 3-vector normal to the surface element \( dS \).

First, let us consider the case with vanishing \( S \)-term, for which the tetrad (3·10) asymptotically takes the following form:

\[ \lambda_{\alpha}^{0} = i\left[ 1 + \left( \frac{m}{\rho} + \frac{H^{2}}{2} \right) \right], \]

\[ \lambda_{\alpha}^{a} = iH \frac{X^{a}}{\rho}, \]

\[ \lambda_{\alpha}^{0} = H \frac{X^{a}}{\rho}, \]

\[ \lambda_{\alpha}^{a} = \left( 1 - \frac{m}{\rho} \right) \delta_{\alpha}^{a} + \frac{H^{2}}{2} \frac{X^{a}X^{b}}{\rho^{2}}, \tag{4·5} \]

where we understand that \( H \) denotes the leading term of the function \( H(\rho, t) \) for \( \rho \to \infty \). We discuss three different cases separately according to the asymptotic form of \( H \).

(i) The case \( H \sim f(t)/\sqrt{\rho^{1-\epsilon}} \) for a constant \( \epsilon \) satisfying \( 1 > \epsilon > 0 \). The superpotential of (4·1) behaves for large \( \rho \) as

\[ U_{\rho}^{\alpha} = -\frac{f^{2}}{\kappa \rho^{2}} \frac{X^{a}}{\rho} \rho^{\epsilon}. \tag{4·6} \]

Substituting (4·6) into (4·4), we see that the integral (4·4) is divergent. Thus, this case should be rejected from our consideration.

(ii) The case \( H \sim f(t)/\sqrt{\rho} \). The superpotential of (4·1) behaves as

\[ U_{\rho}^{\alpha} = \frac{2m + f^{2}}{\kappa \rho^{2}} \frac{X^{a}}{\rho}. \tag{4·7} \]
in this case. Substituting (4·7) into (4·4), we obtain

\[ E = m + \frac{f^2}{2}. \]  

(4·8)

Therefore, if \( f \neq 0 \) the value of the energy differs from the gravitational mass \( m \).

(iii) The case \( H \sim f(t)\sqrt{\rho^{1+\epsilon}} \) for a positive constant \( \epsilon \). The superpotential of (4·1) behaves as

\[ q_0 = \frac{2m}{k\rho^3} \frac{x^a}{\rho} \]  

(4·9)

for this case. Calculating the energy from (4·4), we find

\[ E = m \]  

(4·10)

in agreement with the gravitational mass.

Now let us consider the solution with non-vanishing \( S \)-term. The asymptotic behavior of the tetrad (3·5) is given by (4·5) with vanishing \( H \). Therefore, the energy is given by (4·10) also in this case.

§ 5. Thermal properties of the spherically symmetric solution

Gibbons and Hawking\(^{17-19}\) discussed the thermal properties of the Schwarzschild solution, for which the line-element squared takes the positive-definite standard form

\[ ds^2 = +\left(1 - \frac{2m}{r}\right)dr^2 + \left(1 - \frac{2m}{r}\right)^{-1}d\Omega^2, \]  

(5·1)

after the Euclidean continuation of the time variable, \( t = -i\tau \). By the transformation \( x = 4m(1 - 2m/r)^{1/2} \), the line-element squared becomes

\[ ds^2 = +\left(\frac{x}{4m}\right)^2d\tau^2 + \left(\frac{r^2}{4m^2}\right)^2dx^2 + r^2d\Omega^2, \]  

(5·2)

which shows that \( \tau \) can be regarded as an angular variable with period \( 8\pi m \). Now the Euclidean section of the Schwarzschild solution is the region defined by \( 8\pi m \geq \tau \geq 0 \) and \( x > 0 \), where the metric is positive definite, asymptotically flat and non-singular. They calculated the Euclidean action \( \tilde{I} \) of general relativity from the surface term as follows:

\[ \tilde{I} = 4\pi m^2 = \frac{\beta^2}{16\pi}, \]  

(5·3)

where \( \beta = 8\pi m = T^{-1} \), with \( T \) being interpreted as the absolute temperature of the Schwarzschild black hole.

For a canonical ensemble, the energy is given by

\[ E = \frac{\sum_ne^{-\beta E_n}}{\sum_ne^{-\beta E_n}} = -\frac{\partial}{\partial \beta}\log Z, \]  

(5·4)

where \( E_n \) is the energy in the \( n \)th state, and \( Z \) is the partition function, which is in the
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Tree approximation related to the Euclidean action of the classical solution as

\[ \tilde{I} = -\log Z. \]  \hspace{1cm} (5·5)

Use of (5·3) and (5·5) in (5·4) gives

\[ E = \frac{\beta}{8\pi} = m. \]  \hspace{1cm} (5·6)

Gibbons and Hawking also calculated the entropy of the Schwarzschild black hole to obtain

\[ S = -\sum_n P_n \log P_n = \beta E + \log Z = 4\pi m^2 = \frac{1}{4} A, \]  \hspace{1cm} (5·7)

where \( P_n = Z^{-1} e^{-\beta E} \), and \( A \) is the area of the event horizon of the Schwarzschild black hole.

Let us apply the above procedure of Gibbons and Hawking to our spherically symmetric stationary solutions, namely, to the tetrad (3·5) with the S-term and to the tetrad (3·8) without the S-term, where the arbitrary function \( H \) is assumed to be independent of \( t \). Since these solutions give the Schwarzschild metric, the variable \( r \) can be regarded as an angular variable with period \( 8\pi m \) after the Euclidean continuation, \( t = -i\tau \). The Euclidean continuation of the diagonal tetrad in isotropic cartesian coordinates is given by

\[ \lambda_i^{(u)} = \frac{1 - \frac{m}{2\rho}}{1 + \frac{m}{2\rho}}, \quad \lambda_a^{(u)} = \left(1 - \frac{m}{2\rho}\right) \delta_{aa}, \]  \hspace{1cm} (5·8)

where we use the index 4 instead of 0 in the Euclidean section. The Euclidean continuation of the general tetrad with vanishing S-term is then obtained from this diagonal one by local \( SO(4) \) rotations preserving spherical symmetry,

\[ (A_{kl}) = \begin{pmatrix} \sqrt{1-H^2} & \frac{\bar{H}X^b}{\rho} \\ -\frac{\bar{H}X^a}{\rho} & \delta_{aa} + \left(\sqrt{1-H^2} - 1\right) \frac{X^aX^b}{\rho^2} \end{pmatrix}, \]  \hspace{1cm} (5·9)

where the indices \( k \) and \( l \) run over 4, 1, 2 and 3 in the Euclidean section, and \( \bar{H} \) is an arbitrary function of \( \rho \). Here the matrix (5·9) is related to the local Lorentz transformation matrix (3·7) by the continuation \( H = -i\bar{H} \).

As for the solution with nonvanishing S-term, its Euclidean continuation is obtained by simply removing the factor \( i \) from the first equation of (3·5).

For these solutions in the Euclidean section, the axial-vector part of the torsion tensor vanishes, and the Euclidean action is given by

\[ \tilde{I} = -\frac{1}{2\kappa} \int \sqrt{g} (R - 2\Phi^\nu;_\mu) d^4x = \frac{1}{\kappa} \int \sqrt{g} \Phi^\nu;_\mu d^4x, \]  \hspace{1cm} (5·10)

where \( R \) is the Riemann-Christoffel scalar curvature, and \( \Phi^\nu \) is the basic vector field.
defined by (2·5). As in the previous section, we divide the solutions in the Euclidean section into three cases, according to the asymptotic behavior of the arbitrary function $\tilde{H}(\rho)$ in (5·9).

(i) The case $\tilde{H}(\rho) \sim f'/\sqrt{\rho^{1-\epsilon}}$ for $1 > \epsilon > 0$.* The action of (5·10) is divergent. This justifies our conclusion of the previous section that this case should be rejected.

(ii) The case $\tilde{H}(\rho) \sim f'/\sqrt{\rho}$. The surface integral (5·10) is calculated to give

$$\tilde{I} = 4\pi m(m - f^2),$$

and using this in (5·4) gives the energy

$$E = m - \frac{\partial}{\partial m}\left(\frac{m f^2}{2}\right).$$

We note that this value of the energy is different from that given by using the superpotential in § 4 and also from that of general relativity. The entropy is obtained from the second equation of (5·7) as

$$S = 4\pi m^2\left(1 - \frac{\partial f^2}{\partial m}\right).$$

(iii) The case $\tilde{H}(\rho) \sim f'/\sqrt{\rho^{1+\epsilon}}$ for $\epsilon > 0$. The Euclidean action is given by

$$\tilde{I} = 4\pi m^2$$

in this case, and the energy is

$$E = m.$$  

This value is the same as that given in § 4, and the value of the entropy is

$$S = 4\pi m^2 = \frac{1}{4} A,$$

showing that both the energy and the entropy are in agreement with those of general relativity.

Finally, we note that the result obtained in case (iii) is obtained also for the solution with nonvanishing $S$-term.

§ 6. Main results and discussion

In this paper we have studied the most general spherically symmetric solutions in the HNM tetrad theory of gravity. According to the Birkhoff theorem of this theory,¹⁰ the axial-vector part of the torsion tensor, $a^a$, should vanish for any spherically symmetric solution, and accordingly the underlying space-time metric must be the Schwarzschild.

Tetrads with spherical symmetry are classified into two groups according to whether or not the space-space components $a^a$, possess the $S$-term, namely the term $S(t, \rho)\epsilon_{aab}x^b$. When the $S$-term is non-vanishing, the tetrad is severely restricted by

* Here $\tilde{f}$ is a constant in contrast with the $f(t)$ introduced in the previous section.
Table I. Summary of main results. The general solution with spherical symmetry is classified into two groups according to whether or not the space-space components $\delta^a_b$ have the term $S\delta^{00} (\text{referred to as the S-term for short})$. The general solution without the S-term has an arbitrary function, so it is further classified into three classes according to the asymptotic behavior of $f(t)$ or $\rho(t)$ in the Euclidean section: (i) $\frac{\rho}{t} \sim \rho^{-(1+\epsilon)/2}$ for $1 > \epsilon > 0$, (ii) $\frac{\rho}{t} \sim f(t)^{-1/2}$ (or $\frac{\rho}{t} \sim f(t)^{-1/2}$) and (iii) $\frac{\rho}{t} \sim \rho^{-1+\epsilon/2}$ for $\epsilon > 0$. The general solution with the S-term has a constant parameter, and its components $\delta^a_b$ (or $\frac{\rho}{t}$ in the Euclidean section), are vanishing.

| Field equation | Symmetric part | Superpotential method | Gibbons-Hawking method |
|----------------|---------------|----------------------|------------------------|
| Tetrad without S-term | Satisfied identically | Schwarzschild metric | (i) | (ii) | (iii) |
| | | | Divergent $m + \frac{f^2}{2}$ | $m$ | $m$ |
| Tetrad with S-term | Gives $a^\alpha = 0$ | Schwarzschild metric | $m$ | $m$ |

the condition that $a^\alpha$ be vanishing, and we obtain a family of solutions with a constant parameter. On the other hand, when the S-term is vanishing, $a^\alpha$ identically vanishes, and accordingly we obtain a family of solutions with an arbitrary function of $t$ and $\rho$, and establish its relation with the solution of Mikhail et al.\textsuperscript{14}

We have applied the superpotential given by Mikhail et al.\textsuperscript{13} to calculate the energy of the central gravitating body. As for the tetrad without the S-term, we discussed three cases separately according to the asymptotic behavior of $\frac{\rho}{t} (t, \rho)$. (i) When $\frac{\rho}{t} \sim \rho^{-(1+\epsilon)/2}$ for $1 > \epsilon > 0$, we found that the energy is divergent. We reject this case from consideration. (ii) When $\frac{\rho}{t} \sim f(t)^{-1/2}$, the energy is given by $E = m + f^2/2$. This case has many problems, however. First the energy differs from the gravitational mass $m$, and second the energy now depends on time, because $f(t)$ is a function of $t$ in general. Is this physically acceptable? We will leave this problem to another paper, but our preliminary investigation suggests that the answer will be negative, because the equivalence between the gravitational mass and the inertial mass is violated. (iii) When $\frac{\rho}{t} \sim \rho^{-1+\epsilon/2}$ for $\epsilon > 0$, the energy agrees with $m$, and this case is very satisfactory. We also find that the tetrad with the S-term gives the same result as in case (iii).

We then used the method of Gibbons and Hawking to calculate the energy for the stationary solutions. We classified the tetrad without the S-term in the Euclidean section into three cases according to the asymptotic behavior of $\frac{\rho}{t} (\rho)$. (i) When $\frac{\rho}{t} \sim \rho^{-(1-\epsilon)/2}$ for $1 > \epsilon > 0$, the action in the Euclidean section diverges, and hence this case must be rejected. (ii) When $\frac{\rho}{t} \sim \rho^{-1/2}$ the calculated energy differs both from the gravitational mass and from the value obtained by the superpotential method. (iii) Finally when $\frac{\rho}{t} \sim \rho^{-1+\epsilon/2}$ for $\epsilon > 0$, the energy is found to coincide with the gravitational mass. As for the tetrad with the S-term, the energy agrees with that of case (iii).

Is there any inconsistency between the two methods to calculate the energy? If so, which is the correct formula for calculating energy, that given by Möller or that by Gibbons and Hawking? If the two methods are to be consistent, our result implies that we must reject case (ii), requiring that $\frac{\rho}{t}$ (or $\rho$ in the Euclidean section) vanishes faster than $1/\sqrt{\rho}$ at infinity.
Finally, we make a brief comment concerning the geometrical meaning of the above result. The spherically symmetric vacuum solutions discussed in this paper have the following property in common: They give the Schwarzschild metric and define the axial-vector part of the torsion tensor $\alpha^\nu$ which is identically vanishing. This implies that these solutions are indistinguishable observationally, as far as one uses photons or spin-1/2 fundamental particles as test particles to explore the underlying structure of space-time. In this sense, these solutions are all physically equivalent. Geometrically speaking, any one of these solutions can be chosen as parallel vector fields to define extended absolute parallelism, and then the underlying space-time becomes an extended Weizenböck space-time,\(^{10,\ast}\), in which local Lorentz transformations preserving the condition $\alpha^\nu = 0$ are allowed. According to the above result, allowed local Lorentz transformations must also preserve the required asymptotic behavior, $\rho^2 \sim \rho^{-(1+\epsilon)/2}$ for $\epsilon > 0$.

We have summarized these results in Table I.

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\(*\) For more details of extended Weizenböck space-time see § 8 of Ref. 10.)