COMPACT KÄHLER MANIFOLDS ADMITTING LARGE SOLVABLE GROUPS OF AUTOMORPHISMS

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Abstract. Let $G$ be a group of automorphisms of a compact Kähler manifold $X$ of dimension $n$ and $N(G)$ the subset of null-entropy elements. Suppose $G$ admits no non-abelian free subgroup. Improving the known Tits alternative, we obtain that, up to replace $G$ by a finite-index subgroup, either $G/N(G)$ is a free abelian group of rank $\leq n - 2$, or $G/N(G)$ is a free abelian group of rank $n - 1$ and $X$ is a complex torus, or $G$ is a free abelian group of rank $n - 1$. If the last case occurs, $X$ is $G$-equivariant birational to the quotient of an abelian variety provided that $X$ is a projective manifold of dimension $n \geq 3$ and is not rationally connected. We also prove and use a generalization of a theorem by Fujiki and Lieberman on the structure of $\text{Aut}(X)$.

1. Introduction

We work over the field $\mathbb{C}$ of complex numbers. Let $X$ be a compact Kähler manifold of dimension $n$ and $\text{Aut}(X)$ the group of all (holomorphic) automorphisms of $X$. Fujiki and Lieberman proved that $\text{Aut}(X)$ is a complex Lie group of finite dimension but it may have an infinite number of connected components. The connected component $\text{Aut}_0(X)$ of $\text{Aut}(X)$ is the set of automorphisms obtained by integrating holomorphic vector fields on $X$. It is a normal subgroup of $\text{Aut}(X)$ and its action on the Hodge cohomology groups $H^{p,q}(X,\mathbb{C})$ is trivial. See [9], [13] for details.

Elements in $\text{Aut}_0(X)$ have zero (topological) entropy. Indeed, for any automorphism $g \in \text{Aut}(X)$, the entropy of $g$, defined in the theory of dynamical systems, turns out to be equal to the logarithm of the spectral radius of the pull-back operator $g^*$ acting on $\oplus_{0 \leq p \leq n} H^{p,p}(X,\mathbb{R})$. This is a consequence of theorems due to Gromov and Yomdin. Topological entropy is always a non-negative number and also equals the logarithm of the spectral radius of $g^*$ acting on the whole cohomology group $\oplus_{0 \leq p,q \leq n} H^{p,q}(X,\mathbb{C})$. This number is strictly positive if and only if the spectral radius of $g^*$ on $H^{p,p}(X,\mathbb{R})$ is strictly larger than 1 for all or for some $p$ with $1 \leq p \leq n - 1$. See [2], [11] and [16] for details.

The group $\text{Aut}(X)$ satisfies the following Tits alternative type result which was proved in [2 Theorem 1.5] (generalizing [17 Theorem 1.1]). See [2] for a survey. Recall that a
group $H$ is virtually solvable (resp. free abelian, unipotent...), if a finite-index subgroup of $H$ is solvable (resp. free abelian, unipotent...).

**Theorem 1.1.** Let $X$ be a compact Kähler manifold of dimension $n \geq 2$ and $G \subseteq \text{Aut}(X)$ a group of automorphisms. Then one of the following two alternative assertions holds:

1. $G$ contains a subgroup isomorphic to the non-abelian free group $\mathbb{Z} \ast \mathbb{Z}$, and hence $G$ contains subgroups isomorphic to non-abelian free groups of all countable ranks.
2. $G$ is virtually solvable.

In the second case, or more generally, when the representation $G|_{H^2(X, \mathbb{C})}$ is virtually solvable, $G$ contains a finite-index solvable subgroup $G_1$ such that the null-entropy subset

$$N(G_1) := \{ g \in G_1 \mid g \text{ is of null entropy} \}$$

is a normal subgroup of $G_1$ and the quotient $G_1/N(G_1)$ is a free abelian group of rank $r \leq n - 1$.

When the action of $G$ on $H^2(X, \mathbb{C})$ is virtually solvable, the integer $r$ in the theorem is called the dynamical rank of $G$ and we denote it as $r = r(G)$. It does not depend on the choice of $G_1$. When $G$ is abelian, Theorem 1.1 can be deduced from [5, Theorem 1] and the classical Tits alternative for linear algebraic groups. In [5, Remarque 4.9], the authors mentioned the interest of studying these $X$ admitting a commutative $G$ of positive entropy and maximal dynamical rank $n - 1$.

In this paper, we study virtually solvable but not necessarily commutative groups $G$ of maximal dynamical rank $r(G) = n - 1$. From Theorem 1.2 below, it turns out that $N(G)$ is then virtually contained in $\text{Aut}_0(X)$, hence $G/(G \cap \text{Aut}_0(X))$ is virtually a free abelian group of rank $n - 1$. In particular, [17, Question 2.17] is affirmatively answered by Theorem 1.2 (or Theorem 4.1) below.

**Theorem 1.2.** Let $X$ be a compact Kähler manifold of dimension $n \geq 2$ and $G \subseteq \text{Aut}(X)$ a group of automorphisms such that the null-entropy subset $N(G)$ of $G$ is a subgroup of (and hence normal in) $G$ and $G/N(G) \cong \mathbb{Z}^{\oplus n-1}$. Then either $X$ is a complex torus, or $N(G)$ is a finite group and hence $G$ is virtually a free abelian group of rank $n - 1$.

Note that $N(G)$ is normal in $G$ because if $g$ has zero entropy, by Gromov and Yomdin, $hgh^{-1}$ also has zero entropy for every $h \in \text{Aut}(X)$.

**Remark 1.3.** We can deduce from Theorem 1.1 that the representation $G|_{H^2(X, \mathbb{C})}$ is virtually solvable if and only if, for some finite-index subgroup $G_1$ of $G$, $N(G_1)$ is a normal subgroup of $G_1$ and $G_1/N(G_1) \cong \mathbb{Z}^{\oplus r}$ for some $r \leq n - 1$, (cf. [2, Theorem 2.3]). So the condition of Theorem 1.2 is, up to replace $G$ by a finite-index subgroup,
equivalent to that $G|_{H^2(X,\mathbb{C})}$ is virtually solvable and $G$ has maximal dynamical rank $n-1$. Proposition 1.4 below gives a more precise description of the present situation.

Recall that by Theorem 1.1 the hypothesis in the following result is satisfied when $G$ admits no non-abelian free subgroup. Note also that Theorem 1.2 can be applied to the following group $G_0$.

**Proposition 1.4.** Let $X$ be a compact Kähler manifold of dimension $n \geq 2$ and $G \subseteq \text{Aut}(X)$ a group of automorphisms such that $G|_{H^2(X,\mathbb{C})}$ is virtually solvable and of maximal dynamical rank $n-1$. Then there is a normal subgroup $G_0$ of $G$ such that

1. $G/G_0$ is isomorphic to a subgroup of the symmetric group $S_n$. In particular, it contains at most $n!$ elements.

2. The null-entropy subset $N(G_0)$ is a normal subgroup of $G_0$ and $G_0/N(G_0) \cong \mathbb{Z}^{\oplus n-1}$.

**Remark 1.5.** Let $X, G$ be as in Proposition 1.4 and let $H$ be the normalizer of $G$ in $\text{Aut}(X)$. Then either $|H : G| < \infty$, or $X$ is a complex torus and in the latter case $|H/(H \cap \text{Aut}_0(X)) : G/(G \cap \text{Aut}_0(X))| < \infty$.

Theorem 1.2 (or Theorem 4.1 (2)) enables us to apply [19, Theorem 1.1] and immediately get Corollary 1.6 below. When $X$ is rationally connected, we can also state some necessary and sufficient hypothesis for $X$ to be $G$-equivariant birational to a quotient of an abelian variety, see [19] for more details.

**Corollary 1.6.** Assume the condition for $X, G$ in Theorem 1.2 (or 4.1). Assume further that $n \geq 3$, $X$ is a projective manifold and $X$ is not rationally connected. Then, replacing $G$ by a finite-index subgroup, we have the following:

1. There is a birational map $X \dashrightarrow Y$ such that the induced action of $G$ on $Y$ is biregular and $Y = T/F$, where $T$ is an abelian variety and $F$ is a finite group whose action on $T$ is free outside a finite subset of $T$.

2. There is a faithful action of $G$ on $T$ such that the quotient map $T \rightarrow T/F = Y$ is $G$-equivariant. Every $G$-periodic proper subvariety of $Y$ or $T$ is a point.

Generalizing Theorem 4.1 and Corollary 1.6, we propose the following. See [4, Problem 1.5] and the remarks afterwards for a related problem. See Oguiso-Truong [15, Theorem 1.4] for the first example of rational threefold or Calabi-Yau threefold admitting a primitive automorphism of positive entropy.

**Problem 1.7.** (1) Classify compact Kähler manifolds of dimension $n \geq 3$ admitting a solvable group $G$ of automorphisms of dynamical rank $r$ for some $2 \leq r \leq n-1$,
such that the pair \((X, G)\) is primitive, i.e., there is no non-trivial \(G\)-equivariant meromorphic fibration even after replacing \(G\) by a finite-index subgroup.

(2) One may strengthen the condition on \(G\) to that every non-trivial element of \(G\) is of positive entropy. So \(G\) is isomorphic to \(\mathbb{Z}^{\oplus r}\).

(3) One may also weaken the condition on \(G\) to that the representation \(G|_{H^{1,1}(X, \mathbb{C})}\) is an abelian or solvable group.

One may ask similar problems for the case of manifolds defined over fields with positive characteristic but new tools need to be developed. We refer the reader to a pioneering work in this direction by Esnault-Srinivas in [7].

The present paper is organized as follows. In Section 2, we give a generalization of a theorem by Fujiki and Lieberman that we will need in the proof of the main theorem. Theorem 2.1 and Proposition 2.6 should be of independent interest. In Section 3, we study a notion of privileged vector space which allows us to construct invariant eigenvectors with maximal eigenvalues. The main theorem, its proof, the proofs of Proposition 1.4 and Remark 1.5 are given in the last section.

2. Fujiki and Lieberman type theorem

Let \(X\) be a compact Kähler manifold of dimension \(n\). Let \(\omega\) be a Kähler form on \(X\). Its class in \(H^{1,1}(X, \mathbb{R}) := H^{1,1}(X, \mathbb{C}) \cap H^2(X, \mathbb{R})\) is denoted by \(\{\omega\}\). Let \(\text{Aut}_{\{\omega\}}(X)\) denote the group of all automorphisms \(g\) of \(X\) preserving \(\{\omega\}\), i.e., \(g^*\{\omega\} = \{\omega\}\). Fujiki and Lieberman proved in [9, Theorem 4.8] and [13, Proposition 2.2] that \(\text{Aut}_0(X)\) is a finite-index subgroup of \(\text{Aut}_{\{\omega\}}(X)\).

The purpose of this paragraph is to prove a more general version of this theorem that we will use later.

Let \(\phi\) be a differential \((p, p)\)-form on \(X\). We say that \(\phi\) is positive if in any local holomorphic coordinates \(z\) we can write \(\phi(z)\) as a finite linear combination, with non-negative functions as coefficients, of \((p, p)\)-forms of type

\[(i l_1(z) \wedge \overline{l_1(z)}) \wedge \ldots \wedge (i l_p(z) \wedge \overline{l_p(z)})],\]

where \(l_j(z)\) are \(\mathbb{C}\)-linear functions in \(z\). This notion does not depend on the choice of coordinates \(z\).

A \((p, p)\)-current \(T\) on \(X\) is said to be weakly positive if \(T \wedge \phi\) defines a positive measure for any positive \((n-p, n-p)\)-form \(\phi\). A \((p, p)\)-form is weakly positive if it is weakly positive in the sense of currents. A \((p, p)\)-current \(T\) on \(X\) is said to be positive if \(T \wedge \phi\) defines a positive measure for any weakly positive \((n-p, n-p)\)-form \(\phi\). It turns out that a form is positive if it is positive in the sense of currents and positivity implies weak positivity. These two notions coincide for \(p = 0, 1, n-1, n\) and are different otherwise. If \(\alpha\) is a
holomorphic $p$-form on $X$, then $\bar{\partial}^* \alpha \wedge \bar{\alpha}$ is an example of weakly positive $(p,p)$-form. It vanishes only when $\alpha = 0$. This can be easily checked using local coordinates on $X$.

Weakly positive and positive forms and currents are real, that is, they are invariant under the complex conjugation. If $\omega$ is a Kähler form on $X$ as above, then $\omega^p$ is a positive $(p,p)$-form for every $p$. A positive (resp. weakly positive) $(p,p)$-form or current $T$ is said to be strictly positive (resp. strictly weakly positive) if there is an $\epsilon > 0$ such that $T - \epsilon \omega^p$ is positive (resp. weakly positive). We refer to Demailly [3], or [6] for more details.

Recall from Hodge theory that the group $H^{p,q}(X, \mathbb{C})$ can be obtained as the quotient of the space of closed differential $(p,q)$-forms by the subspace of exact $(p,q)$-forms. It is not difficult to see that every closed $(p,q)$-current $T$, defines via the pairing $\langle T, \phi \rangle$ with $\phi$ an $(n - p, n - q)$-form, an element in the dual of $H^{n-p,n-q}(X, \mathbb{C})$, because $T$ vanishes on exact $(n - p, n - q)$-forms by Stokes’ theorem. So by Poincaré duality, $T$ defines a class in $H^{p,q}(X, \mathbb{C})$ that will be denoted by $\{T\}$.

Denote by $\mathcal{K}_p(X)$ (resp. $\mathcal{K}^w_p(X)$) the set of classes of strictly positive (resp. strictly weakly positive) closed $(p,p)$-forms. They are strictly (or salient) convex open cones in $H^{p,p}(X, \mathbb{R})$. Denote also by $\mathcal{E}_p(X)$ (resp. $\mathcal{E}^w_p(X)$) the set of classes of positive (resp. weakly positive) closed $(p,p)$-currents. They are strictly (or salient) convex closed cones in $H^{p,p}(X, \mathbb{R})$. Indeed, if $T$ is a (weakly) positive closed $(p,p)$-current, the quantity $\langle T, \omega^{n-p} \rangle$ is equivalent to the mass-norm of $T$ and it only depends on the cohomology class of $\{T\}$. One often call this quantity the mass of $T$. So $T$ vanishes if and only if its class $\{T\}$ vanishes. Moreover, if $T_n$ are such that $\{T_n\}$ converge to some class $c$, then since $\{T_n\}$ are bounded, the masses of $T_n$ are bounded and hence, up to extract a subsequence, we can assume $T_n$ converge to some (weakly) positive closed current $T$. It follows that $c$ is represented by $T$ and hence $\mathcal{E}_p(X)$ (resp. $\mathcal{E}^w_p(X)$) are closed.

The Kähler cone of $X$, denoted by $\mathcal{K}(X)$, consists of all Kähler classes. So it is equal to $\mathcal{K}_1(X)$ and also to $\mathcal{K}^w_1(X)$ because positivity and weak positivity coincide for bidegree $(1,1)$. The closure $\overline{\mathcal{K}}(X) \subset H^{1,1}(X, \mathbb{R})$ of $\mathcal{K}(X)$ is called the nef cone. The cones $\mathcal{E}_1(X)$ and $\mathcal{E}^w_1(X)$ are also equal and are denoted simply by $\mathcal{E}(X)$. This is the pseudo-effective cone. Classes in the interior of $\mathcal{E}(X)$ are called big and are the classes of Kähler currents, i.e., strictly positive closed $(1,1)$-currents. In general, we have

$$\mathcal{K}_p(X) \subset \mathcal{K}^w_p(X) \subset \overline{\mathcal{K}}^w_p(X) \subset \mathcal{E}^w_p(X) \quad \text{and} \quad \mathcal{K}_p(X) \subset \overline{\mathcal{K}}_p(X) \subset \mathcal{E}_p(X) \subset \mathcal{E}^w_p(X).$$

The classes in the interior of $\mathcal{E}_p(X)$ (resp. $\mathcal{E}^w_p(X)$) are the classes of strictly positive (resp. strictly weakly positive) closed $(p,p)$-currents. In what follows, we will write $c \leq c'$ or $c' \geq c$ for classes $c, c' \in H^{p,p}(X, \mathbb{R})$ such that $c' - c$ belongs to $\mathcal{E}^w_p(X)$.

The group $\text{Aut}(X)$ acts naturally on cohomology groups and preserves all the above cones, their closures, interiors and boundaries. We will fix a norm for each cohomology
group. Our results don’t depend on the choice of these norms. The main result in this section is the following theorem.

**Theorem 2.1.** Let $X$ be a compact Kähler manifold of dimension $n$ and $G \subseteq \text{Aut}(X)$ a group of automorphisms. Assume that for every element $g \in G$ there is a class $c$ of a strictly weakly positive closed $(p,p)$-current, $1 \leq p \leq n-1$, such that $\|(g^m)^*(c)\| = o(m)$ as $m \to +\infty$. Here, $c$ and $p$ may depend on $g$. Then $G$ is virtually contained in $\text{Aut}_0(X)$, i.e., $|G : G \cap \text{Aut}_0(X)| < \infty$.

The following immediate corollary can be applied when $c$ is a big class in $H^{1,1}(X, \mathbb{R})$. If $c$ is the class of a Kähler form, the result is exactly the theorem by Fujiki and Lieberman quoted above.

**Corollary 2.2.** Let $c$ be the class of a strictly weakly positive closed $(p,p)$-current, $1 \leq p \leq n-1$. Let $\text{Aut}_c(X)$ denote the group of all automorphisms $g$ such that $g^*(c) = c$. Then $\text{Aut}_c(X)$ is virtually contained in $\text{Aut}_0(X)$.

We will see in Lemma 2.3 below that the condition that $g^*(c) = c$ for $c$ as in the corollary is equivalent to the condition that $g^*(c)$ is parallel to $c$.

We prove now Theorem 2.1. We will identify $H^{n,n}(X, \mathbb{R})$ with $\mathbb{R}$ by taking the integrals of $(n,n)$-forms on $X$. The cup-product of two cohomology classes $c, c'$ is simply denoted by $cc'$ or $c \cdot c'$. We need the following lemmas.

**Lemma 2.3.** Let $g \in \text{Aut}(X)$. Assume there is a class $c$ of a strictly weakly positive closed $(p,p)$-current for some $1 \leq p \leq n-1$, such that $\|(g^m)^*(c)\| = o(m)$ as $m \to +\infty$. Then, for every $0 \leq q \leq n$, the norm of $(g^m)^*$ acting on $H^{q,q}(X, \mathbb{R})$ is bounded by a constant independent of $m \in \mathbb{N}$. In particular, the conclusion holds if $g^*(c)$ is parallel to $c$ and in this case we have $g^*(c) = c$.

**Proof.** Using the Jordan canonical form of $g^*$ acting on $H^{p,p}(X, \mathbb{C})$, we see that the hypothesis $\|(g^m)^*(c)\| = o(m)$ implies that $\|(g^m)^*(c)\|$ is bounded by a constant independent of $m \in \mathbb{N}$. Let $c'$ be any class in $\mathcal{E}^{w}_p(X)$. Since $c$ is in the interior of $\mathcal{E}^w_p(X)$, there is a constant $\lambda > 0$ such that $c' \leq \lambda c$. Since $g^*$ preserves $\mathcal{E}^w_p(X)$ and this cone is salient and closed, we deduce that the sequence $(g^m)^*(c')$ is also bounded. We will use this property for the class $\{\omega^p\}$ of $\omega^p$.

For $0 \leq q \leq n$, we define

$$I_{q,m} := \int_X (g^m)^*(\omega^q) \wedge \omega^{n-q} = (g^m)^*\{\omega^q\} \wedge \{\omega^{n-q}\} \quad \text{and} \quad J_q := \lim_{m \to +\infty} \frac{\log I_{q,m}}{\log m}.$$  

By the Jordan canonical form of $g^*$ acting on $H^{q,q}(X, \mathbb{C})$, we see that the limit in the definition of $J_q$ exists in $\mathbb{Z}_{\geq 0} \cup \{\pm \infty\}$. The above discussion implies that $J_p \leq 0$. We also have $J_0 = J_n = 0$ since $g^*$ is the identity on $H^{0,0}(X, \mathbb{R}) \cong \mathbb{R}$ and $H^{n,n}(X, \mathbb{R}) \cong \mathbb{R}$.
We apply Theorem 2.3 below to \(\omega_1 := \omega, \omega_2 := (g^m)^*(\omega)\) and \(\omega_i := (g^m)^*(\omega)\) for \(q - 1\) indices \(3 \leq i \leq n\) and \(\omega_i = \omega\) for \(n - q - 1\) other indices \(3 \leq i \leq n\). We obtain that \(I_{q,m}^2 \geq I_{q-1,m}I_{q+1,m}\). It follows that the function \(q \mapsto J_q\) is concave, i.e., \(2J_q \geq J_{q-1} + J_{q+1}\). We then deduce that \(J_q = 0\) for every \(0 \leq q \leq n\). In particular, using again the Jordan canonical form of \(g^*\) acting on \(H^{q,q}(X, \mathbb{C})\), we see that the sequence \(I_{q,m}\) is bounded.

Since the integral \(I_{q,m}\) is the mass of the positive closed \((q,q)\)-current \((g^m)^*(\omega^q)\), we deduce that the sequence of cohomology classes \((g^m)^*\{\omega^q\}\) is also bounded. In the same way as it was just done for \(\mathcal{E}^w_p(X)\) at the beginning of the proof, we obtain that the sequence \((g^m)^*(\alpha')\) is bounded for any \(\alpha' \in \mathcal{E}^w_q(X)\). Thus, the property holds for any \(\alpha' \in H^{q,q}(X, \mathbb{R})\) because \(\mathcal{E}^w_q(X)\) generates \(H^{q,q}(X, \mathbb{R})\). So the first assertion follows.

Suppose now that \(g^*c = \lambda c\) for some \(\lambda \in \mathbb{R}\). Since \(g^*\) preserves the weak positivity, we have \(\lambda > 0\). Replacing \(g\) by \(g^{-1}\) if necessary, we may assume that \(\lambda \leq 1\). Then the condition of the first assertion is satisfied, so the norm of \((g^m)^*\) acting on all \(H^{q,q}(X, \mathbb{R})\) is bounded. Thus the spectral radius of \(g^*\) on all \(H^{q,q}(X, \mathbb{R})\) are all less than or equal to 1. The same is true for \((g^{-m})^*\), because by Poincaré duality, the action of \((g^{-m})^*\) on \(H^{q,q}(X, \mathbb{R})\) is the adjoint action of \((g^m)^*\) on \(H^{n-q,n-q}(X, \mathbb{R})\). Thus, all eigenvalues of \(g^*\) on all \(H^{q,q}(X, \mathbb{R})\) are of modulus 1. In particular, if \(g^*c = \lambda c\), then \(\lambda = 1\).

Recall the following result by Gromov, that we used above, see [4, Corollary 2.2] and [10]. It is a consequence of the mixed version of the classical Hodge-Riemann theorem.

**Theorem 2.4** (Gromov). Let \(\omega_i\) be Kähler forms on \(X, 1 \leq i \leq n\). Define for \(1 \leq i, j \leq 2\)

\[
I_{ij} := \int_X \omega_i \wedge \omega_j \wedge \omega_3 \wedge \ldots \wedge \omega_n = \{\omega_i\}\{\omega_j\}\{\omega_3\}\ldots\{\omega_n\}.
\]

Then we have \(I_{12}^2 \geq I_{11}I_{22}\).

**Lemma 2.5.** Let \(g \in \text{Aut}(X)\). Assume that the norm of \((g^m)^*\) acting on \(H^{1,1}(X, \mathbb{R})\) is bounded by a constant independent of \(m \in \mathbb{N}\). Then the action \(g^*|_{H^2(X, \mathbb{R})}\) has finite order.

**Proof.** We first show that the action \((g^m)^*\) on \(H^2(X, \mathbb{R})\) is bounded independently of \(m \in \mathbb{N}\). Recall that \((g^m)^*\) preserves the Hodge decomposition

\[
H^2(X, \mathbb{C}) = H^{2,0}(X, \mathbb{C}) \oplus H^{1,1}(X, \mathbb{C}) \oplus H^{0,2}(X, \mathbb{C}).
\]

So it is enough to prove that the action of \((g^m)^*\) on \(H^{2,0}(X, \mathbb{C})\) is bounded independently of \(m \in \mathbb{N}\). The similar property for \(H^{0,2}(X, \mathbb{C})\) is then obtained by complex conjugation.

Observe that if \(v\) is a class in \(H^{2,0}(X, \mathbb{C})\), it can be represented by a unique closed holomorphic 2-form \(\alpha\). The form \(\alpha \wedge \bar{\alpha}\) is weakly positive and closed, which vanishes if and only if \(\alpha = 0\). We deduce that \(v\bar{v} = 0\) if and only if \(v = 0\). Hence, \(\|v\|\) is bounded if and only if \(\|v\bar{v}\|\) is bounded.
For any \((2, 0)\)-class \(v\) in \(H^{2,0}(X, \mathbb{C})\), there is a Kähler class \(u\) satisfying \(v\bar{v} \leq u^2\). Indeed, since the class \(\{\omega^2\}\) is in the interior of the cone \(E^2_2(X)\), it is enough to choose \(u\) as a large multiple of \(\{\omega\}\). We deduce that

\[
(g^m)^*v \cdot (g^m)^*\bar{v} = (g^m)^*(v\bar{v}) \leq (g^m)^*(u^2) = (g^m)^*(u)^2.
\]

Therefore, by hypothesis, \(\|(g^m)^*v \cdot (g^m)^*\bar{v}\|\) is bounded independently of \(m \in \mathbb{N}\), thus \(\|(g^m)^*v\|\) satisfies the same property.

We conclude that the action \((g^m)^*\) on \(H^2(X, \mathbb{R})\) is bounded independently of \(m \in \mathbb{N}\). In particular, all eigenvalues of \(g^*|_{H^2(X, \mathbb{C})}\) are of modulus \(\leq 1\), hence they are of modulus 1, and indeed are roots of unity by Kronecker’s theorem, since \(g^*\) preserves (and is an isomorphism of) \(H^2(X, \mathbb{Z})\) and \(H^2(X, \mathbb{C}) = H^2(X, \mathbb{Z}) \otimes \mathbb{C}\). This and the above boundedness result imply that the action \(g^*|_{H^2(X, \mathbb{R})}\) is periodic, by looking at the Jordan canonical form of the action.

End of the proof of Theorem \(2.1\). By Lemmas \(2.3\) and \(2.5\), for any \(g \in G\), the endomorphism \(g^*|_{H^2(X, \mathbb{Z})}\) is periodic. Thus the minimal polynomials of \(g^*|_{H^2(X, \mathbb{C})}\) over \(\mathbb{Q}\) for all \(g \in G\) are all cyclotomic, hence have bounded coefficients (and degrees). So the group \(G|_{H^2(X, \mathbb{Z})}\) has bounded exponent. Thus it is a finite group by the classical Burnside’s theorem. Set \(v := \sum h^*\{\omega\}\), where \(h\) runs through the finite group \(G|_{H^2(X, \mathbb{R})}\) and \(\omega\) is a fixed Kähler form of \(X\). Then \(G\) is contained in

\[
\text{Aut}_v(X) := \{g \in \text{Aut}(X) \mid g^*(v) = v\}.
\]

Since \(v\) is a Kähler class, the theorem by Fujiki and Lieberman quoted above says that \(|\text{Aut}_v(X) : \text{Aut}_0(X)| < \infty\). Now

\[
G/(G \cap \text{Aut}_0(X)) \cong (G \cdot \text{Aut}_0(X))/\text{Aut}_0(X) \leq \text{Aut}_v(X)/\text{Aut}_0(X)
\]

and the latter is a finite group. The theorem follows.

Since \(\text{Aut}_0(X)\) acts trivially on \(H^{p,q}(X, \mathbb{C})\), the above discussion immediately implies the following:

**Proposition 2.6.** Let \(X\) be a compact Kähler manifold of dimension \(n\) and \(G \subseteq \text{Aut}(X)\) a group of automorphisms. Then \(G\) is virtually contained in \(\text{Aut}_0(X)\) if and only if the representation of \(G\) on \(H^{p,p}(X, \mathbb{R})\) for some \(1 \leq p \leq n - 1\) (or on \(\oplus_{0 \leq p,q \leq n} H^{p,q}(X, \mathbb{C})\)) is a finite group.
3. Invariant privileged vector spaces

Let $X$ be a compact Kähler manifold of dimension $n$. By Hodge theory, every class $v$ in $H^2(X, \mathbb{R})$ admits a unique decomposition

$$v = v_{20} + v_{11} + v_{02},$$

where $v_{11} \in H^{1,1}(X, \mathbb{R})$, $v_{20} \in H^{2,0}(X, \mathbb{C})$, $v_{02} \in H^{0,2}(X, \mathbb{C})$ with $v_{02} = \overline{v}_{20}$. For any vector subspace $V$ of $H^2(X, \mathbb{R})$ define

$$V^0 := V \cap (H^{2,0}(X, \mathbb{C}) \oplus H^{0,2}(X, \mathbb{C})), \quad V^{11} := V \cap H^{1,1}(X, \mathbb{R}).$$

In general, we have $V^0 \oplus V^{11} \subseteq V$, but the inclusion may be strict.

**Definition 3.1.** We say that $V$ is privileged if it satisfies the following properties:

(a) $V$ is defined over $\mathbb{Q}$, $V \neq 0$ and $V = V^0 \oplus V^{11}$.

(b) The intersection of $V^{11}$ with the nef cone $\overline{K}(X)$ has non-empty interior in $V^{11}$, or equivalently this intersection spans $V^{11}$ (because $\overline{K}(X)$ is convex).

(c) For every $v \in V^0$ there is a nef class $u$ in $V^{11}$ such that $v_{20} \overline{v}_{20} \leq u^2$.

We have the following result.

**Proposition 3.2.** Let $G$ and $Z$ be subgroups of $\text{Aut}(X)$. Assume that $Z$ is normalized by $G$ and that $Z|_{H^2(X, \mathbb{C})}$ (or equivalently $Z|_{H^2(X, \mathbb{R})}$) is a unipotent commutative matrix group. Then, there is a privileged subspace $F \subseteq H^2(X, \mathbb{R})$ which is $G$-stable such that the action of $Z$ on $F$ is the identity.

Recall here that the representation $Z|_{H^2(X, \mathbb{R})}$ of $Z$ is defined over $\mathbb{Q}$ because $\text{Aut}(X)$ acts on $H^2(X, \mathbb{Z})$ and $H^2(X, \mathbb{R}) = H^2(X, \mathbb{Z}) \otimes \mathbb{R}$.

**Notation 3.3.** From here till Proposition 3.5, for simplicity, the pull-back action of $g \in \text{Aut}(X)$ on $H^*(X, \mathbb{C})$ will be denoted by the same $g$, instead of $g^*$. So for $g, h \in \text{Aut}(X)$ and $v \in H^*(X, \mathbb{C})$ the identity $(gh)^*v = h^*(g^*(v))$ will be written as $(gh)v = h(gv)$.

**Lemma 3.4.** There is a privileged $Z$-stable subspace $V \subseteq H^2(X, \mathbb{R})$ such that the action of $Z$ on $V$ is the identity.

**Proof.** Set $V_0 := H^2(X, \mathbb{R})$. Set also $K_0 := \overline{K}(X)$ which has non-empty interior in $V_0^{11} = H^{1,1}(X, \mathbb{R})$. Take any $z \in Z$. Let $l \geq 0$ be the minimal integer such that the norm of $z^m$ acting on $V_0$ satisfies $\|z^m\| = O(m^l)$ as $m \to +\infty$, noting that the action of $z$ on $H^2(X, \mathbb{R})$ is unipotent. If $l = 0$ then $z$ is the identity on $V_0$. This can be seen by using...
the Jordan canonical form of $z$. So we can take $V = V_0$ when the property $l = 0$ holds for all $z$.

Otherwise, fix a $z \in Z$ such that $l \geq 1$. Define $\pi : V_0 \rightarrow V_0$ to be the limit of $m^{-l}z^m$ and $V_1$ the image of $V_0$ by $\pi$. Using the Jordan canonical form of $z$ with a basis of $V_0$ defined over $\mathbb{Q}$, we see that $V_1 \neq 0$, $V_1 \neq V_0$ and $V_1$ is defined over $\mathbb{Q}$. Since $z^m$ preserves the decomposition $V_0^0 \oplus V_0^{11}$, our $V_1$ satisfies Property 3.1(a).

Moreover, $\pi(\mathcal{K}_0)$ has non-empty interior in $V_1^{11}$ by open mapping theorem applied to $\pi : V_0^{11} \rightarrow V_1^{11}$. By the definition of $\pi$, we have $\pi(\mathcal{K}_0) \subseteq \mathcal{K}_0$. Denote by $\mathcal{K}_1$ the intersection $V_1^{11} \cap \mathcal{K}_0$. It has non-empty interior in $V_1^{11}$. So $V_1$ satisfies Property 3.1(b).

For any $v \in V_0^0$, there is a $v^* \in V_0^0$ such that $v = \lim m^{-l}z^m(v^*)$. Let $u^* \in \mathcal{K}_0$ such that $v_{20}^*v_{20}^* \leq u^2$. Define $u := \lim m^{-l}z^m(u^*)$. We have $v_{20}v_{20} \leq u^2$ because $u^2 - v_{20}v_{20} = \lim m^{-2l}z^m(u^{*2} - v_{20}^*v_{20}^*)$.

Indeed, the class $m^{-2l}z^m(u^{*2} - v_{20}^*v_{20}^*)$ is represented by some weakly positive closed current and the cone $\mathcal{E}_2^w(X)$ of weakly positive classes in $H^{2,2}(X, \mathbb{R})$ is closed. So Property 3.1(c) holds for $V_1$. And eventually our $V_1$ is privileged.

For any arbitrary $w \in Z$, as actions on $H^2(X, \mathbb{R})$, we have $wz^m = z^mw$. It follows that $w\pi = \pi w$. So $V_1$ is $w$-stable and hence $Z$-stable. We can now apply the same arguments in order to construct inductively $V_{i+1} \subset V_i$ and $\mathcal{K}_{i+1} \subset \mathcal{K}_i$. For dimension reason, there is an $i$ such that $Z$ is the identity on $V_i$. Take $V = V_i$. We are done. \hfill \Box

**Proof of Proposition 3.2.** Let $V$ be as in Lemma 3.3. We show that for any $g \in G$, $g(V)$ is also privileged. Property 3.1(a) is clear because $g$ is defined over $\mathbb{Q}$ and preserves the Hodge decomposition $H^{2,0}(X, \mathbb{C}) \oplus H^{1,1}(X, \mathbb{C}) \oplus H^{0,2}(X, \mathbb{C})$. Property 3.1(b) is a consequence of the fact that $g$ is an isomorphism of $H^{1,1}(X, \mathbb{C})$ and preserves the nef cone $\overline{\mathcal{K}}(X)$. Property 3.1(c) is also clear because $g$ preserves the weak positivity. For any $z \in Z$ and $g \in G$, we have $z' := zg^{-1} \in Z$ and hence for any $v \in V$ (see Notation 3.3)

$$z(g(v)) = g(z'(v)) = g(v).$$

Thus, the action of $Z$ on $g(V)$ is the identity.

Consider $F = \sum V$, where $V$ runs over all subspaces satisfying Lemma 3.4. We have $g(F) \subseteq F$ and hence $g(F) = F$ for dimension reason. Also by dimension reason, the last
sum is in fact a finite sum. Therefore, $F$ is defined over $\mathbb{Q}$ because it has a system of generators defined over $\mathbb{Q}$. Like on $V$, the group $Z$ acts as the identity on $F$, and $F^{11}$ is generated by nef classes. Hence $F$ satisfies Properties 3.1(a) and 3.1(b).

To finish the proof of Proposition 3.2, we still have to show that $F$ satisfies Property 3.1(c). For this purpose, it is enough to show that if $v_{20}, v_{20}' \in H^2(X, \mathbb{C})$ and $u, u' \in K(X)$ satisfy $v_{20} \bar{v}_{20} \leq u^2$ and $v_{20}' \bar{v}_{20}' \leq u'^2$, then $(v_{20} + v_{20}')(\bar{v}_{20} + \bar{v}_{20}') \leq 2(u + u')^2$. We can represent $v_{20} - v_{20}'$ by a closed holomorphic 2-form $\alpha$. Since the $(2, 2)$-form $\alpha \wedge \bar{\alpha}$ is weakly positive, we deduce the Cauchy-Schwarz inequality $(v_{20} - v_{20}') (\bar{v}_{20} - \bar{v}_{20}') \geq 0$.

It follows that $(v_{20} + v_{20}')(\bar{v}_{20} + \bar{v}_{20}') \leq 2v_{20}\bar{v}_{20} + 2v_{20}'\bar{v}_{20}' \leq 2u^2 + 2u'^2$.

On the other hand, we can approximate $u$ and $u'$ by classes of Kähler forms and hence $uu'$ by classes of positive $(2, 2)$-forms. Since the cone $E\omega^2(X)$ is closed in $H^{2,2}(X, \mathbb{R})$, we obtain $uu' \geq 0$. Therefore, the right-hand side class in the latter chain of inequalities is bounded by $2(u + u')^2$. This ends the proof of Proposition 3.2.\qed

**Proposition 3.5.** Let $G, Z$ and $F$ be as in Proposition 3.2. Then, for every $g \in G$ there exists a nef class $v \neq 0$ in $F^{11} \subseteq F$ such that $gv = \lambda v$, where $\lambda$ is the spectral radius of $g$ acting on $F$.

**Proof.** If $\gamma$ is a complex eigenvalue of $g$ acting on $F^0$, then up to replace $\gamma$ with $\bar{\gamma}$, there are vectors $v, v' \in F^0$ such that $v''_{20} := v_{20} + iv_{20}' \neq 0$ and $g(v''_{20}) = \gamma(v''_{20})$. Therefore, we have $g^m(v''_{20}\bar{v}''_{20}) = |\gamma|^{2m}v''_{20}\bar{v}''_{20}$.

Let $u, u' \in F^{11}$ be as in Property 3.1(c) for $v, v'$ respectively. As in the proof of Proposition 3.2 for $u'' = \sqrt{2}(u + u')$, we have $|\gamma|^{2m}v''_{20}\bar{v}''_{20} \leq g^m(u''^2)$.

It follows that $|\gamma|^m \lesssim \|g^m(u'')\|$.

Therefore, $|\gamma| \leq \rho(g|_{F^{11}})$, where $\rho(g|_{F^{11}})$ is the spectral radius of $g$ acting on $F^{11}$. So $g$ acts on $F^{11}$ with spectral radius $\lambda$ and preserves the cone $C := F^{11} \cap \overline{K(X)}$. Since this cone has non-empty interior, the proposition follows from Theorem 3.6 below.\qed

We quote Birkhoff’s generalization of the Perron-Frobenius theorem [1].
Theorem 3.6 (Birkhoff). Let $C$ be a strictly (or salient) convex closed cone of a finite-dimensional $\mathbb{R}$-vector space $V$ such that $C$ spans $V$ as a vector space. Let $g : V \to V$ be an $\mathbb{R}$-linear endomorphism such that $g(C) \subseteq C$. Then the spectral radius $\rho(g)$ is an eigenvalue of $g$ and there is an eigenvector $L_g \in C$ corresponding to the eigenvalue $\rho(g)$.

Before the end of this section, we recall the cone theorem of Lie-Kolchin type in [12], which will be used in the proof of our main Theorem 4.1.

Theorem 3.7 (cf. [12, Theorem 1.1]). Let $V$ be a finite-dimensional real vector space and $\{0\} \neq C \subset V$ a strictly (or salient) convex closed cone. Suppose that $G \leq \text{GL}(V)$ is a solvable group, and it has connected Zariski closure in $\text{GL}(V_C)$ (always possible by replacing $G$ with a finite-index subgroup) and $G(C) \subseteq C$. Then $G$ has a common eigenvector in the cone $C$.

Remark 3.8. Note that in this theorem the connectedness of the Zariski closure of $G$ in $\text{GL}(V_C)$ is necessary. Indeed, we have the following example. Let $G$ be the subgroup of $\text{GL}(2, \mathbb{R})$ consisting of all invertible $2 \times 2$ matrices $(a_{ij})_{1 \leq i,j \leq 2}$ with non-negative entries which are diagonal (i.e., $a_{12} = a_{21} = 0$) or anti-diagonal (i.e., $a_{11} = a_{22} = 0$). Then $G$ is a solvable Lie group and its Zariski closure in $\text{GL}(2, \mathbb{C})$ has two connected components. It preserves the cone of vectors with non-negative coordinates. However, there is no common eigenvector of $G$ in the above cone.

4. Generalization of Theorem 1.2 and proof of Proposition 1.4

Theorem 1.2 will follow from the more general form below.

Theorem 4.1. Let $X$ be a compact Kähler manifold of dimension $n \geq 2$ and $G \subseteq \text{Aut}(X)$ a group of automorphisms such that the null-entropy subset $N(G)$ of $G$ is a subgroup of (and hence normal in) $G$ and $G/N(G) \cong \mathbb{Z}^\oplus n-1$. Then

1. The representation of $N(G)$ on $\oplus_{0 \leq p,q \leq n} H^{p,q}(X, \mathbb{C})$ is a finite group. Hence $N(G)$ is virtually contained in $\text{Aut}_0(X)$.

2. Either $N(G)$ is a finite group and hence $G$ is virtually a free abelian group of rank $n-1$, or $X$ is a complex torus and $G \cap \text{Aut}_0(X) = N(G) \cap \text{Aut}_0(X)$ is Zariski-dense in $\text{Aut}_0(X) \cong X$.

3. $G/(G \cap \text{Aut}_0(X))$ and $G|_E$ are virtually free abelian groups of rank $n-1$ for any $G$-stable (real or complex) vector subspace $E$ of $\oplus_{0 \leq p,q \leq n} H^{p,q}(X, \mathbb{C})$ containing $H^{p,p}(X, \mathbb{R})$ for some $1 \leq p \leq n-1$, e.g., $H^2(X, \mathbb{R})$ or $H^{1,1}(X, \mathbb{R})$. 
Remark 4.2. The condition of Theorem 4.1 (except the requirement for the dynamical rank of $G$) is, up to replace $G$ by a finite-index subgroup, equivalent to that the representation $G|_{H^{1,1}(X,\mathbb{C})}$ (or $G|_{NS(X)}$ when $X$ is projective) is virtually solvable (see Remark [13] [17] Theorem 1.2, or [2] Theorem 2.3).

Proof of Theorem 4.1 (1). By Proposition [2.6] it is enough to prove that $N(G)|_{H^{2}(X,\mathbb{C})}$ is finite. Suppose the contrary that $N(G)|_{H^{2}(X,\mathbb{C})}$ is infinite. Since $N(G)$ is of null entropy, $N(G)|_{H^{2}(X,\mathbb{C})}$ is virtually unipotent (cf. [14] Proof of Proposition 2.2, or [2] Theorem 2.2). Now the unipotent subgroup of $N(G)|_{H^{2}(X,\mathbb{C})}$ is nilpotent and hence has a non-trivial centre, furthermore, this centre is infinite, since in characteristic zero every unipotent element has infinite order. We denote the inverse image in $G$ of this centre as $Z$. Then $Z \triangleleft G$ and $Z|_{H^{2}(X,\mathbb{C})}$ is an infinite unipotent commutative matrix group.

We apply Proposition [32] So there is a non-trivial $G$-stable privileged vector subspace $F$ of $H^{2}(X,\mathbb{R})$. Set $F_{Z} := F \cap (H^{2}(X,\mathbb{Z})/(\text{torsion}))$. Since $F$ is defined over $\mathbb{Q}$, this $F_{Z}$ is a free abelian group of finite rank with $F_{Z} = F_{Z} \otimes_{\mathbb{Z}} \mathbb{R}$. Since $F$ is privileged, $F_{Z}$ (like $F$) is also $G$-stable and the action of $Z$ on $F_{Z}$ is the identity. Write $F = F^{0} \oplus F^{11}$, where

$$F^{0} = F \cap (H^{2,0}(X,\mathbb{C}) \oplus H^{0,2}(X,\mathbb{C})), \quad F^{11} = F \cap H^{1,1}(X,\mathbb{R}).$$

Since $G$ preserves the Hodge decomposition, $F^{11}$ is $G$-stable, non-trivial and is spanned by nef classes, because $F$ is privileged.

By the proof of [2] Lemma 2.7] with $G$ there replaced by the subgroup $G|_{H^{1,1}(X,\mathbb{C})}$ of the linear algebraic group $GL(H^{1,1}(X,\mathbb{C}))$ or [14] Lemma 5.5], our $G|_{H^{1,1}(X,\mathbb{C})}$ is virtually solvable, since $N(G)|_{H^{2}(X,\mathbb{C})}$ is virtually unipotent (and hence virtually solvable) and by assumption $G/N(G)$ is abelian (and hence virtually solvable). Replacing $G$ by a finite-index subgroup, we may assume that $G|_{H^{1,1}(X,\mathbb{C})}$ is solvable and having connected Zariski closure in $GL(H^{1,1}(X,\mathbb{C}))$, so is its restriction on $F^{11} \otimes_{\mathbb{R}} \mathbb{C}$ as the image of a solvable group with connected Zariski closure. By Theorem 3.7 we can choose $0 \neq L_{1} \in F^{11}$ to be a common nef eigenvector of $G$.

Now we use the quasi-nef sequence as in [17] §2.2. So for all $1 \leq k \leq n - 1$, there are nonzero products $L_{1} \ldots L_{k}$ in the closure of $L_{1} \ldots L_{k-1} \cdot \overline{K}(X) \subseteq H^{k,k}(X,\mathbb{R})$, with $L_{k} \in H^{1,1}(X,\mathbb{R})$, such that

$$g^{*}(L_{1} \ldots L_{k}) = \chi_{1}(g) \ldots \chi_{k}(g)(L_{1} \ldots L_{k})$$

for some characters $\chi_{i} : G \to (\mathbb{R}_{>0}, \times)$. By [17] Proof of Theorem 1.2, the homomorphism

$$\varphi : G \to (\mathbb{R}^{\oplus n-1}, +), \quad g \mapsto (\log \chi_{1}(g), \ldots, \log \chi_{n-1}(g))$$

satisfies the following (with $r = n - 1$ now):

$$\text{Ker}(\varphi) = N(G), \quad G/N(G) \cong \text{Im}(\varphi) = \mathbb{Z}^{\oplus r}.$$
Let \( g_i \in G \) such that \( \varphi(g_i) = (0, \ldots, 0, 1, 0, \ldots, 0) \), where the \( i \)-th entry is 1.

For each automorphism \( g \), denote by \( d_i(g) \) the first dynamical degree of \( g \), i.e., the spectral radius of \( g^* \) acting on \( H^{1,1}(X, \mathbb{R}) \). Applying Theorem 3.6, there are nef classes \( L_{g_i^{\pm 1}} \) such that \( (g_i^{\pm 1})^* L_{g_i^{\pm 1}} = d_i(g_i^{\pm 1}) L_{g_i^{\pm 1}} \). By [18] Lemma 2.9 (5), we may assume that \( L_i = L_{g_i}, 1 \leq i < n \). Since the lattice \( F_Z \) is \( G \)-stable, we have \( \det(g_i|_F) = \pm 1 \). Since \( L_{g_1} = L_1 \in F \) with \( g_1^* L_{g_1} = d_1(g_1) L_{g_1} \) and \( d_1(g_1) > 1 \), this \( g_1|_F \) has spectral radius > 1, so does \( g_1^{-1}|_F \) because \( \det(g_1|_F) = \pm 1 \). Thus the privileged \( F \) contains a nef class \( N_1 \) such that \( (g_1^{-1})^* N_1 = \lambda N_1 \) with \( \lambda > 1 \) the spectral radius of \( g_1^{-1}|_F \), see Proposition 3.5.

By [18] Lemma 2.10, the following property holds for any \( 1 \leq i < n \).

**Uniqueness property.** Let \( N \) be a nef class such that \( g_i^* N = \lambda N \) for some \( \lambda > 0 \). If \( \lambda > 1 \), then \( N \) is parallel to \( L_{g_i} \), and if \( \lambda < 1 \), then \( N \) is parallel to \( L_{g_i^{-1}} \).

So the above class \( N_1 \) is parallel to \( L_{g_i^{-1}} \). As in [18] Lemma 2.13, we may assume that \( L_{g_i^{-1}} \) is parallel to \( L_{g_i^{-1}} \) (and hence to \( N_1 \)) for all \( 1 \leq i < n \). Thus \( L_{g_i^{-1}} \) is in \( F \). Hence \( g_i^{-1}|_F \) has spectral radius > 1, so does \( g_i|_F \) because \( \det(g_i|_F) = \pm 1 \). Proposition 3.5 again implies the existence of a nef class \( M_i \) in \( F \) such that \( g_i^* M_i = \mu_i M_i \) with \( \mu_i > 1 \) the spectral radius of \( g_i|_F \). Hence \( M_i \) is parallel to \( L_{g_i} \) by the uniqueness property above.

Now all \( L_{g_i^{\pm 1}} \) are in \( F \) and \( Z \) acts as the identity on \( F \). So \( Z \) fixes the nef class

\[
P := \left( \sum_{i=1}^{n-1} L_{g_i} \right) + L_{g_1^{-1}} = \sum_{i=1}^{n} L_i \quad \text{with} \quad L_n := L_{g_1^{-1}}
\]

which is big because \( P^n \geq L_1 \ldots L_{n-1} \cdot L_{g_1^{-1}} \) > 0, see [18] Lemma 2.9 (8). By Corollary 2.2, \( Z \) is virtually contained in \( \text{Aut}_0(X) \). Hence its representation \( Z|_{H^2(X,k)} \) is finite for \( k = \mathbb{Z} \) and also \( k = \mathbb{C} \). This contradicts the early assumption on \( Z \) and completes the proof of Theorem 4.1 (1).

**Proof of Theorem 4.1 (2).** By the assertion (1), \( N(G) \) is virtually contained in \( \text{Aut}_0(X) \), i.e., \( |N(G) : N(G) \cap \text{Aut}_0(X)| < \infty \). Since \( G/N(G) \cong \mathbb{Z}^m \) and by [2] Lemma 2.4, we only have to consider the case that the group \( N(G) \cap \text{Aut}_0(X) \) is infinite. We will prove that \( X \) is a complex torus in this case.

To do so, let \( H \) be the identity connected component of the Zariski-closure of the latter group in \( \text{Aut}_0(X) \). Then \( H \) is normalized by \( G \) because \( N(G) \cap \text{Aut}_0(X) \) is normal in \( G \). Consider the quotient map \( X \to X/H \) (and its graph) as in [9] Lemma 4.2, where \( G \) acts on \( X/H \) bi-regularly. The maximality of \( r(G) \) implies that \( H \) has a Zariski-dense open orbit in \( X \), see [17] Lemma 2.14.

Consider the case that the irregularity \( q(X) = 0 \). Then \( \text{Aut}_0(X) \) is a linear algebraic group, see [9] Theorem 5.5 or [13] Theorem 3.12. Hence \( X \) is an almost homogeneous variety under the action of a linear algebraic group \( H \) (\( \subseteq \text{Aut}_0(X) \)). Then \( |\text{Aut}(X) : \)]
\[ \text{Aut}_0(X) < \infty \text{ by [8] Theorem 1.2.} \] Hence \( G \leq \text{Aut}(X) \) is of null entropy, contradicting the assumption \( r(G) = n - 1 \geq 1. \)

Therefore, we may assume that \( q(X) > 0. \) Then the Albanese map \( \text{alb}_X : X \to A := \text{Alb}(X) \) is bimeromorphic, see [17, Lemma 2.13]. Since the action \( H|_A \) of \( H \) on \( A \) also has a Zariski-dense open orbit in \( A \), we have \( H|_A = \text{Aut}_0(X) \cong A. \) Let \( B \subset A \) be the locus over which \( \text{alb}_X \) is not an isomorphism. Note that \( B \) and its inverse in \( X \) are both \( H \)-stable. Since \( H|_A = \text{Aut}_0(X) \cong A \), we have \( B = \emptyset. \) Hence \( \text{alb}_X \) is an isomorphism. This proves Theorem 4.1 (2). \( \square \)

**Proof of Theorem 4.1 (3).** Let \( \psi \) be one of the following natural homomorphisms:
\[ G \to G/(G \cap \text{Aut}_0(X)) \quad \text{or} \quad G \to G|_E. \]

Then \( K := \ker(\psi) \subseteq \text{Aut}_c(X) \) for any class \( c \) in \( H^{p,p}(X,\mathbb{R}). \) Choose a class \( c \) in the interior of \( E^n_p(X). \) Since \( |\text{Aut}_c(X):\text{Aut}_0(X)| < \infty \) by Corollary 2.2, this \( K \) is of null entropy, see Proposition 2.6. Thus \( K \leq N(G). \) Now
\[ \mathbb{Z}^{\oplus n-1} \cong G/N(G) \cong (G/K)/(N(G)/K) \cong \psi(G)/\psi(N(G)). \]

By Theorem 4.1 (1), \( \psi(N(G)) \) is a finite group. So there is a finite-index subgroup \( G_1 \) of \( G \) such that \( \psi(G_1) \cong \mathbb{Z}^{\oplus n-1}, \) see [2, Lemma 2.4]. This proves Theorem 4.1 (3), hence the whole of Theorem 4.1. \( \square \)

Now the finiteness of \( N(G) \) in [5, Theorem 1] when \( r(G) = n - 1, \) has a shorter proof:

**Corollary 4.3.** Let \( X \) be a compact Kähler manifold of dimension \( n \geq 2 \) and \( G \) a group of automorphisms of \( X. \) Assume that \( G \) is commutative of maximal dynamical rank \( n - 1. \) Then \( N(G) \) is a finite group and hence \( G \) is virtually isomorphic to \( \mathbb{Z}^{\oplus n-1}. \)

**Proof.** Suppose the contrary that \( N(G) \) is infinite. By Theorem 4.1 (2), \( X \) is a complex torus and \( N(G) \cap \text{Aut}_0(X) \) is Zariski-dense in \( \text{Aut}_0(X). \) Take \( g_0 \in G \) of positive entropy. Since \( g_0 \) commutes with all elements in \( N(G), \) it commutes with any translation \( T_b \in \text{Aut}_0(X). \) Write \( g_0 = h \circ T_a \) for some translation \( T_a \) and group automorphism \( h \) for \( X, \) considered as an additive group. Then for every \( x \in X, \) we have
\[ h(a + b + x) = (g_0 \circ T_b)(x) = (T_b \circ g_0)(x) = b + h(a + x). \]

Hence \( h(b) = b \) for all \( b \in X. \) Thus \( h = \text{id}_X. \) So \( g_0 = T_a, \) and \( g_0 \) is of null entropy, a contradiction. This proves Corollary 4.3. \( \square \)

**Proof of Proposition 4.4.** By Theorem 4.1, \( G \) admits a finite-index subgroup \( G_1 \) such that \( N(G_1) \) is a normal subgroup of \( G_1 \) and \( G_1/N(G_1) \cong \mathbb{Z}^{\oplus n-1}. \) Observe that this property does not change if we replace \( G_1 \) by a finite-index subgroup of \( G_1. \) Write
$G = \bigcup_{i=1}^{m} a_iG_1$ with $a_i \in G$. Replacing $G_1$ by $\cap_{i=1}^{m} (a_iG_1a_i^{-1})$ allows to assume that $G_1$ is normal in $G$.

Theorem 1.1 can be applied to $G_1$ instead of $G$. For simplicity, we still use the same notation introduced in the proof of that theorem. In particular, the classes $L_i$, $1 \leq i \leq n$, are now common eigenvectors for the action of $G_1$ and they form a basis of an $n$-dimensional vector space that we denote by $W \subseteq H^{1,1}(X, \mathbb{R})$.

**Claim 4.4.** For every $g \in G$, $g^*$ permutes the half-lines $\mathbb{R}_+ L_i$ for $i = 1, \ldots, n$. In particular, $W$ is invariant by $G$.

**Proof.** Since $G_1$ is normal in $G$, we have $h_i := g^{-1}g_i g \in G_1$. It follows that $L_j$ are eigenvectors of $h_i^*$. In particular, $W$ is $h_i^*$-stable, and if $\lambda$ is the spectral radius of $h_i^*|_W$, then $h_i^*(L_j) = \lambda L_j$ for some $j$. Note that the eigenvalue associated with $L_j$ is always a positive number because $L_j$ is nef. Since $W$ contains big classes, we deduce that $\lambda$ is the first dynamical degree of $h_i$. Moreover, since $h_i$ and $g_i$ are conjugated, they have the same first dynamical degree, i.e., $\lambda = d_1(g_i)$.

We deduce from the definition of $h_i$ that $(g^{-1})^*(L_j)$ is an eigenvector of $g_i^*$ associated with the maximal eigenvalue $\lambda = d_1(g_i)$. By the uniqueness property quoted in the proof of Theorem 1.1 $(g^{-1})^*(L_j)$ is parallel to $L_i$. It follows that $g^*(L_i)$ is parallel to $L_j$. This proves the claim. \qed

By Claim 4.4 we can associate $g$ with an element $\sigma_g$ in the symmetric group $S_n$ that we identify with the group of all permutations of the half-lines $\mathbb{R}_+ L_i$. So we have a natural group homomorphism from $G$ to $S_n$. Let $G_0$ denote its kernel. Recall that we use the basis $L_1, \ldots, L_n$ for the vector space $W$. We see that $G_0$ is the set of $g \in G$ such that the action of $g^*$ on $W$ is given by an $n \times n$ diagonal matrix with non-negative entries and determinant 1. Indeed, the entries of this matrix are non-negative because the classes $L_j$ are nef; the determinant is 1 because $g^*$ is identity on $H^{n,n}(X, \mathbb{R})$ and hence preserves the class $L_1 \ldots L_n$. We also see that $N(G_0)$ is the set of all elements $g \in G_0$ which act as the identity on $W$, see Theorem 2.1. Hence $N(G_0)$ is a normal subgroup of $G_0$.

Now, $G_0/N(G_0)$ can be identified with a subgroup of the group of all the diagonal $n \times n$ matrices with non-negative entries and determinant 1. The later one is isomorphic to the torsion-free additive group $\left(\mathbb{R}^{\oplus n-1}, +\right)$. Moreover, since $G_0$ contains the finite-index subgroup $G_1$, the group $G_0/N(G_0)$ contains a finite-index subgroup isomorphic to $G_1/N(G_1) \cong \mathbb{Z}^{\oplus n-1}$. We necessarily have $G_0/N(G_0) \cong \mathbb{Z}^{\oplus n-1}$. This completes the proof of Proposition 1.4. \qed

**Proof of Remark 1.5.** In Theorem 1.1 we can let $G_1$ be the preimage in $G$ of the identity connected component of the Zariski closure of $G|_{H^{1,1}(X, \mathbb{C})}$ in $\text{GL}(H^{1,1}(X, \mathbb{C}))$, via
the natural homomorphism $G \to G|_{H^{-1}(X,C)}$, see [17] Proof of Theorem 1.2. We have $G_1/N(G_1) \cong \mathbb{Z}^\oplus n-1$ and $|G : G_1| < \infty$. Further, by definition, $G_1$ is normalized by $H$. Now we follow the proof of Proposition 1.4, with $G$ there, replaced by $H$ here. In particular, Claim 4.4 holds for all $g \in H$. In the same way as for $G_0$, we construct a group $H_0$ such that $G_1 \triangleleft H_0 \triangleleft H$ and $H/H_0$ is identified with a subgroup of $S_n$. By Theorem 4.1, $N(H_0)$ and $N(G_1)$ are virtually contained in $\text{Aut}_0(X)$. Thus we have a commutative diagram:

$$
\begin{array}{ccc}
H/(H \cap \text{Aut}_0(X)) & \xrightarrow{\text{finite-index}} & H_0/(H_0 \cap \text{Aut}_0(X)) \\
\downarrow & & \downarrow \\
G/(G \cap \text{Aut}_0(X)) & \xrightarrow{\text{finite-index}} & G_1/(G_1 \cap \text{Aut}_0(X))
\end{array}
$$

$$
\begin{array}{ccc}
 & & H_0/N(H_0) \cong \mathbb{Z}^\oplus n-1 \\
\text{finite-cokernel} & & \text{finite-cokernel} \\
\downarrow & & \downarrow \\
& & G_1/N(G_1) \cong \mathbb{Z}^\oplus n-1.
\end{array}
$$

It follows that $H/(H \cap \text{Aut}_0(X))$ is a finite extension of $G/(G \cap \text{Aut}_0(X))$ and both are virtually free abelian groups of rank $n - 1$, see [2, Lemma 2.4]. If $G_1 \cap \text{Aut}_0(X)$ (or equivalently $N(G_1)$) is infinite, then it is Zariski-dense in $\text{Aut}_0(X)$ and $X$ is a complex torus, see Theorem 4.1. The same property holds if we replace $G_1$ by $H_0$. Otherwise, $|H_0 : G_1| < \infty$. The remark follows.

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