Analysis of the Heavy Quarkonium States $h_c$ and $h_b$ with QCD Sum Rules

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Abstract

In this article, we take the tensor currents $\bar{Q}(x)\sigma_{\mu\nu}Q(x)$ to interpolate the $P$-wave spin-singlet heavy quarkonium states $h_Q$, and study the masses and decay constants with the Borel sum rules and moments sum rules. The masses and decay constants from the Borel sum rules and moments sum rules are consistent with each other, the masses are also consistent with the experimental data. We can take the decay constants as basic input parameters and study other phenomenological quantities with the three-point correlation functions via the QCD sum rules. The heavy quarkonium states $h_Q$ couple potentially to the tensor currents $\bar{Q}(x)\sigma_{\mu\nu}Q(x)$, and have the quark structure $e^{i\bar{k}_1\xi^i\sigma^k\zeta}$ besides the quark structure $ik_1^j\xi_j^i\sigma^k\zeta$. In calculations, we take into account the leading-order, next-to-leading-order perturbative contributions, and the gluon condensate, four-quark condensate contributions in the operator product expansion. The analytical expressions of the perturbative QCD spectral densities have applications in studying the two-body decays of a boson to two fermions with the vertexes $\sigma_{\mu\nu}\gamma_5$ and $\sigma_{\mu\nu}$.

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1 Introduction

In 2011, the BABAR collaboration observed evidences for the spin-singlet bottomonium state $h_b(1P)$ in the sequential decays $\Upsilon(3S) \rightarrow \pi^0 h_b(1P)$, $h_b(1P) \rightarrow \gamma h_b(1S)$ [1]. Later, the Belle collaboration reported the first observation of the spin-singlet bottomonium states $h_b(1P)$ and $h_b(2P)$ with the significances of 5.5 $\sigma$ and 11.2 $\sigma$ respectively in the collisions $e^+e^- \rightarrow h_b(nP)\pi^+\pi^-$ at energies near the $\Upsilon(5S)$ resonance, and determined the masses $M_{h_b(1P)} = (9898.3 \pm 1.1^{+1.3}_{-1.1})$ MeV and $M_{h_b(2P)} = (10259.8 \pm 0.6^{+1.4}_{-1.1})$ MeV [2]. On the other hand, the mass of the spin-singlet charmonium state $h_c(1P)$ has been updated from time to time since its first observation in the $p\bar{p}$ collisions by the R704 collaboration [3], the average value listed in the Review of Particle Physics is $M_{h_c(1P)} = (3525.41 \pm 0.16)$ MeV [4].

The heavy quarkonium states play an important role both in studying the interplays between the perturbative and nonperturbative QCD and in understanding the heavy quark dynamics due to absence of the light quark contaminations. In this article, we study the heavy quarkonium states $h_c$ and $h_b$ with the QCD sum rules, explore their quark structures, and make predictions for the masses to be confronted with experimental data. The QCD sum rules is a powerful (nonperturbative) theoretical tool in studying the heavy quarkonium states [5] [6], the existing works focus on the $S$-wave heavy quarkonium states $J/\psi$, $\eta_c$, $\Upsilon$, $\eta_b$, and the $P$-wave spin-triplet heavy quarkonium states $\chi_{cJ}$, $\chi_{bJ}$, $J = 0, 1, 2$, while the works on the $P$-wave spin-singlet heavy quarkonium states $h_c$ and $h_b$ are few [6] [7]. On the other hand, the heavy quarkonium spectrum have been studied extensively by the (potential) nonrelativistic QCD, and the existing works also focus on the $S$-wave heavy quarkonium states and $P$-wave spin-triplet heavy quarkonium states [8], the works on the $P$-wave spin-singlet heavy quarkonium states $h_Q$ are few [9]. In the (potential) nonrelativistic QCD, the fine splittings and hyperfine splittings among the heavy quarkonium states are treated perturbatively.

The tensor currents $\bar{Q}(x)\sigma_{\mu\nu}Q(x)$ and axialvector currents $\bar{Q}(x)\gamma^\mu\gamma_5Q(x)$ without derivatives

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have the following properties under the parity and charge-conjunction transforms,

\[
\begin{align*}
\bar{Q}(x)\sigma^\mu Q(x) &\xrightarrow{P} \bar{Q}(\tilde{x})\sigma_\mu Q(\tilde{x}), \\
Q(x)\gamma_5^\mu Q(x) &\xrightarrow{C} -Q(x)\sigma^\mu Q(x), \\
\bar{Q}(x)\gamma_5^\mu Q(x) &\xrightarrow{P} -\bar{Q}(\tilde{x})\gamma_\mu\gamma_5 Q(\tilde{x}), \\
\bar{Q}(x)\gamma_5^\mu Q(x) &\xrightarrow{C} \bar{Q}(x)\gamma_\mu\gamma_5 Q(x).
\end{align*}
\]

where \(x^\mu = (t, \vec{x})\) and \(\tilde{x}^\mu = (t, -\vec{x})\). The \(P\)-wave spin-singlet heavy quarkonium states \(h_Q\) have the spin-parity-charge-conjunction \(J^{PC} = 1^-\), the axialvector currents \(\bar{Q}(x)\gamma_\mu\gamma_5 Q(x)\) couple potentially to the axialvector heavy quarkonium states \(\chi_{c1}\) and \(\chi_{b1}\), which have the quantum numbers \(J^{PC} = 1^{++}\) rather than \(1^-\), the tensor currents are superior to the axialvector currents in studying the \(h_Q\). In Ref.\[6\], Reinders, Rubinstein and Yazaki study the interpolating currents \(\bar{Q}(x)\partial_\mu \gamma_5 Q(x)\) with derivatives, and obtain the prediction \(M_{h_{Q}(1P)} = (3.51 \pm 0.01)\text{ GeV}\).

In the nonrelativistic limit, the interpolating currents are reduced to the following form,

\[
\begin{align*}
Q\sigma^\mu Q &\rightarrow 2m_Q\epsilon^{ijk}\xi^i\sigma^j\zeta^k \cdots \cdots \cdots \cdots \cdots \cdots \cdot J^{PC} = 1^- , \\
\bar{Q}\gamma_\mu\gamma_5 Q &\rightarrow 2m_Q\xi^i\sigma^j\zeta^k \cdots \cdots \cdots \cdots \cdots \cdots \cdot J^{PC} = 1^{++} , \\
\bar{Q}\gamma_5^\mu Q &\rightarrow ik_1^i\xi^j\sigma \cdot (\vec{k}_1 - \vec{k}_2)\zeta \cdots \cdots \cdots \cdots \cdots \cdots \cdot J^{PC} = 1^- ,
\end{align*}
\]

where the \(\xi\) and \(\zeta\) are the two-component spinors of the heavy quark fields \(\bar{Q}\) and \(Q\) respectively; the \(k_1\) and \(k_2\) are the three-vectors of the heavy quark fields \(\bar{Q}\) and \(Q\) respectively, and the \(\sigma^i\) are the pauli matrices. From Eq.(2), we can see that the interpolating currents \(\bar{Q}\sigma^\mu Q\) and \(\bar{Q}\gamma_5^\mu Q\) both have the correct quantum numbers of the heavy quarkonium states \(h_Q\), therefor they both couple potentially to the \(h_Q\). It is interesting to study whether or not the \(h_Q\) have the quark structure \(\epsilon^{ijk}\xi^i\sigma^j\zeta^k\) besides the quark structure \(ik_1^i\xi^j\sigma \cdot (\vec{k}_1 - \vec{k}_2)\zeta\).

In the QCD sum rules, additional partial derivative \(\partial^\mu\) in the interpolating currents lead to additional power of \(s\) in the spectral densities \(\rho(s)\) of the two-point correlation functions, which enhances the continuum contributions even if the Borel depression is taken into account, see Fig.1. In the limit \(m_Q \rightarrow 0\), the spectral densities \(\rho(s)\) are of the orders \(O(1)\) and \(O(s^2)\) for the currents \(\bar{Q}\sigma^\mu Q\) and \(\bar{Q}\gamma_5^\mu Q\), respectively, and we prefer constructing quark currents without partial derivatives. In this article, we interpolate the singlet heavy quarkonium states \(h_Q\) with the tensor currents \(\bar{Q}\sigma^\mu Q\), calculate the masses and decay constants (or pole residues). The decay constants are basic input parameters in studying the \(h_c D D^*\), \(h_c D_s D_s^*\), \(h_c D_s^* D_s^*\), \(h_c D_s D_s^* D_s^*\) vertexes and \(h_c \rightarrow D, D_s, D^*, D_s^*\) form-factors with three-point correlation functions using the QCD sum rules.

The article is arranged as follows: we derive the QCD sum rules for the masses and decay constants of the heavy quarkonium states \(h_Q\) in Sect.2; in Sect.3, we present the numerical results and discussions; and Sect.4 is reserved for our conclusions.

2 QCD sum rules for the heavy quarkonium states \(h_Q\)

In the following, we write down the two-point correlation functions \(\Pi_{\mu\nu\alpha\beta}(p)\) in the QCD sum rules,

\[
\begin{align*}
\Pi_{\mu\nu\alpha\beta}(p) &= i \int d^4x e^{ipx} \langle 0 | T \left\{ J_{\mu\nu}(x) J_{\alpha\beta}^\dagger(0) \right\} | 0 \rangle, \\
J_{\mu\nu}^1(x) &= \bar{Q}(x)\sigma_{\mu\nu}\gamma_5 Q(x), \\
J_{\mu\nu}^2(x) &= \bar{Q}(x)\sigma_{\mu\nu}Q(x).
\end{align*}
\]
states representation [5, 6]. After isolating the ground state contribution from the heavy quarkonium
Figure 1: The Borel parameter depressed density
where the $T^2$ is the Borel parameter and the $\rho(x)$ is the spectral density.

where $J_{\mu\nu}(x) = J^{1}_{\mu\nu}(x), J^{2}_{\mu\nu}(x)$, the two interpolating currents are related with each other through
the relation $\sigma^{\mu\nu}\gamma_5 = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}\sigma_{\alpha\beta}$. We decompose the correlation functions $\Pi_{\mu\nu\alpha\beta}(p)$ as

$$
\Pi_{\mu\nu\alpha\beta}(p) = \Pi(p) (\tilde{g}_{\mu\alpha}p_{\nu}\beta + \tilde{g}_{\nu\beta}p_{\mu}\alpha - \tilde{g}_{\mu\beta}p_{\nu}\alpha - \tilde{g}_{\nu\alpha}p_{\mu}\beta) + \tilde{\Pi}(p) (g_{\mu\alpha}g_{\nu\beta} - g_{\mu\beta}g_{\nu\alpha}) \tag{5}
$$

according to Lorentz covariance, where

$$
\tilde{g}_{\mu\nu} = -g_{\mu\nu} + \frac{p_{\mu} p_{\nu}}{p^2}. \tag{6}
$$

Then we project the components $\Pi(p)$ and $\tilde{\Pi}(p)$,

$$
\Pi(p) = \frac{1}{2(1-D)p^2} \left( g_{\mu\alpha}g_{\nu\beta} - \frac{D}{D-2} \tilde{g}_{\mu\alpha}\tilde{g}_{\nu\beta} \right) \Pi_{\mu\nu\alpha\beta}(p),
$$

$$
\tilde{\Pi}(p) = \frac{1}{(D-1)(D-2)} g_{\mu\alpha}\tilde{g}_{\nu\beta} \Pi_{\mu\nu\alpha\beta}(p) \tag{7}
$$

where $D$ is the spacetime dimension.

We can insert a complete set of intermediate hadronic states with the same quantum numbers as the current operators $J_{\mu\nu}(x)$ into the correlation functions $\Pi_{\mu\nu\alpha\beta}(p)$ to obtain the hadronic representation [5, 6]. After isolating the ground state contribution from the heavy quarkonium states $h_Q$, we get the following result,

$$
\Pi_{\mu\nu\alpha\beta}(p) = \frac{f_{Q}^2}{M^2_{h_Q} - p^2} (\tilde{g}_{\mu\alpha}p_{\nu}\beta + \tilde{g}_{\nu\beta}p_{\mu}\alpha - \tilde{g}_{\mu\beta}p_{\nu}\alpha - \tilde{g}_{\nu\alpha}p_{\mu}\beta) + \cdots,
$$

$$
= \Pi(p) (\tilde{g}_{\mu\alpha}p_{\nu}\beta + \tilde{g}_{\nu\beta}p_{\mu}\alpha - \tilde{g}_{\mu\beta}p_{\nu}\alpha - \tilde{g}_{\nu\alpha}p_{\mu}\beta) + \cdots, \tag{8}
$$

where the decay constants $f_{h_Q}$ are defined by

$$
\langle 0 | J^{1}_{\mu\nu}(0) | h_Q(p) \rangle = f_{h_Q} (\varepsilon_{\mu} p_{\nu} - \varepsilon_{\nu} p_{\mu}),
$$

$$
\langle 0 | J^{2}_{\mu\nu}(0) | h_Q(p) \rangle = if_{h_Q} \varepsilon_{\mu} \varepsilon_{\nu} \varepsilon^{\lambda} p^{\lambda}, \tag{9}
$$

and the $\varepsilon_{\mu}$ are the polarization vectors of the heavy quarkonium states $h_Q$. We choose the tensor structure $\tilde{g}_{\mu\alpha}p_{\nu}\beta + \tilde{g}_{\nu\beta}p_{\mu}\alpha - \tilde{g}_{\mu\beta}p_{\nu}\alpha - \tilde{g}_{\nu\alpha}p_{\mu}\beta$ to study the heavy quarkonium states $h_Q$. In this article, we take a simple ground state plus continuum ansatz to approximate the phenomenological spectral densities. Experimentally, the first few radial excited quarkonium (or bottomonium)
states are narrow and appear as resonance-like states rather than as continuum-like states. As the dominant contributions come from the perturbative terms and the gluon condensates play a minor important role, the higher resonance-like states can also be described by the perturbative terms and attributed to the continuum states, such a simple approximation (or ansatz) works well.

One may concerns the possible contaminations come from the $J = 2$ tensor mesons $\chi_{Q2}$ couple potentially to the interpolating currents $\eta_{\mu\nu}(x)$,

$$\langle 0 | \eta_{\mu\nu}(0) | \chi_{Q2}(p) \rangle = \lambda \varepsilon_{\mu\nu},$$

where

$$\eta_{\mu\nu}(x) = \frac{i}{2} \{ \overline{Q}(x) \gamma_\mu \left[ \overline{D}_\nu(x) - \overline{D}_\nu(x) \right] Q(x) + \overline{Q}(x) \gamma_\nu \left[ \overline{D}_\mu(x) - \overline{D}_\mu(x) \right] Q(x) \},$$

$$\overline{D}_\mu(x) = \overline{D}_\mu(x) - ig_s G_\mu(x), \quad \overline{D}_\mu(x) = \overline{D}_\mu(x) + ig_s G_\mu(x), \quad G_\mu = \frac{\lambda^a}{2} G^a_\mu,$$

the $\lambda^a$ are the Gell-Mann matrices, the $\varepsilon_{\mu\nu}$ are the polarization tensors of the $\chi_{Q2}$ mesons with the property,

$$\sum_\lambda \varepsilon_{\alpha\beta}^\mu(\lambda, p) \varepsilon_{\mu\nu}(\lambda, p) = -\frac{2}{3} \frac{g_{\alpha\beta}}{\varepsilon}\frac{\varepsilon_{\mu\nu}}{2}.$$

The $J = 2$ tensor mesons $\chi_{Q2}$ have no contaminations \[10\].

We carry out the Borel transforms (and the derivatives) with respect to the variable $P^2 = -p^2$ to obtain the Borel sum rules (and the moments sum rules), and write down the following results at the phenomenological side,

$$\Pi(T^2) = \frac{1}{\Gamma(n)} P^{2n} \left( -\frac{d}{dP^2} \right)^n \Pi(P^2)|_{P^2 \rightarrow \infty, n \rightarrow \infty; P^2/n = T^2},$$

$$= \frac{1}{T^2} \int_{4m_Q^2}^{s_0} \frac{d \Pi(s)}{\pi} e^{-s/T^2} = \frac{f_{h_Q}^2}{T^2} e^{-M_{h_Q}^2/T^2},$$

$$\Pi(n, \xi) = \frac{1}{\Gamma(n + 1)} \left( -\frac{d}{dP^2} \right)^n \Pi(P^2)|_{P^2 = 4m_Q^2 \xi},$$

$$= \frac{1}{\pi} \int_{4m_Q^2}^{s_0} ds \frac{d \Pi(s)}{(s + 4m_Q^2 \xi)^{n+1}} = \frac{f_{h_Q}^2}{(M_{h_Q}^2 + 4m_Q^2 \xi)^{n+1}},$$

where the $s_0$ are the continuum threshold parameters.

In the following, we briefly outline the operator product expansion for the correlation functions $\Pi_{\mu\nu\alpha\beta}(p)$ in perturbative QCD. The Feynman diagram for the leading-order perturbative contribution is shown in Fig.2. We calculate the diagram using the Cutkosky’s rule to obtain the leading-order spectral densities $\rho_0(s)$,

$$\rho_0(s) = \frac{\text{Im} \Pi_0(s)}{\pi} = \frac{\sqrt{\lambda(s, m_Q^2, m_Q^2)(s + 4m_Q^2)}}{4\pi^2 s^2},$$

where $\lambda(a, b, c) = a^2 + b^2 + c^2 - 2ab - 2bc - 2ca$.

The Feynman diagrams for the next-to-leading-order perturbative contributions are shown in Fig.3. Again we calculate the diagrams using the Cutkosky’s rule to obtain the spectral densities. There are two routines in application of the Cutkosky’s rule (or optical theorem), we resort to the routine used in Ref.[6], not the one used in Ref.[11].

There are ten possible cuts, see Fig.4 and Fig.5. The six cuts shown in Fig.4 attribute to virtual gluon emissions and correspond to the self-energy corrections and vertex corrections respectively. We calculate the one-loop quark self-energy corrections directly using the dimensional regularization and choose the on-shell renormalization scheme to subtract the divergences so as to implement
the wave-function renormalization and mass renormalization. Then we take into account all contributions come from the six cuts shown in Fig.4 by the following simple replacement for each vertex $\sigma_{\mu\nu}\gamma_5$ in the interpolating currents,

$$\bar{u}(p_1)\sigma_{\mu\nu}\gamma_5 u(p_2) \rightarrow \bar{u}(p_1)\sigma_{\mu\nu}\gamma_5 u(p_2) + \bar{u}(p_1)\Gamma_{\mu\nu}\gamma_5 u(p_2) - \delta Z_{\sigma}\bar{u}(p_1)\sigma_{\mu\nu}\gamma_5 u(p_2)$$

where

$$Z_i = 1 + \delta Z_i = 1 + \frac{4}{3} \frac{\alpha_s}{\pi} \left(-\frac{1}{4 \varepsilon_{\text{UV}}} + \frac{1}{2 \varepsilon_{\text{IR}}} + 3 \log \frac{m_Q^2}{4\pi\mu^2} + \frac{3}{4} \gamma - 1\right),$$

are the wave-function renormalization constants come from the self-energy corrections, see Fig.6;

$$\Gamma_{\mu\nu} = \frac{4}{3} g_s^2 \int_0^1 dx \int_0^{1-x} dy \int \frac{d^D k_E}{(2\pi)^D} \frac{2\Gamma(3)}{[k_E^2 + (xp_1 + yp_2)^2]^3} [s - 2m_Q^2 - (x+y)(s - 4m_Q^2)] \sigma_{\mu\nu} - (x+y)m_Q i [(p_1 + p_2)_{\mu}\gamma_{\nu} - (p_1 + p_2)_{\nu}\gamma_{\mu}],$$

comes from the vertex corrections, see Fig.7; the counterterm $\delta Z_{\sigma}$ comes from renormalization of the operator $\bar{Q}\sigma_{\mu\nu}\gamma_5 Q$,

$$Z_1^{-\frac{1}{2}} Z_2^{-\frac{1}{2}} (\bar{Q}\sigma_{\mu\nu}\gamma_5 Q)_0 = Z_{\sigma} (\bar{Q}\sigma_{\mu\nu}\gamma_5 Q)_{\text{r}} = (1 + \delta Z_{\sigma}) (\bar{Q}\sigma_{\mu\nu}\gamma_5 Q)_{\text{r}},$$

where the subindex 0 denotes the bare quantity and the r denotes the renormalized quantity. Here $\gamma$ is the Euler constant, $\mu^2$ is the energy scale, and the Euclidean momentum $k_E = (k_1, k_2, k_3, k_4)$. In this article, we take the dimension $D = 4 - 2\varepsilon_{\text{UV}} = 4 + 2\varepsilon_{\text{IR}}$ to regularize the ultraviolet and infrared divergences respectively, and add the energy scale factors $\mu^{2\varepsilon_{\text{UV}}}$ or $\mu^{-2\varepsilon_{\text{IR}}}$ when necessary.
Figure 4: Six possible cuts correspond to virtual gluon emissions.

Figure 5: Four possible cuts correspond to real gluon emissions.

We carry out the integral over the variables $x$, $y$ and $k_E$ to obtain

$$\tilde{\Gamma}_{\mu\nu}\gamma_5 = \frac{1}{3} \alpha_s \sigma_{\mu\nu} \gamma_5 f(s) + \frac{1}{3} \alpha_s i \left[(p_1 + p_2)_\mu \gamma_\nu - (p_1 + p_2)_\nu \gamma_\mu\right] \gamma_5 \frac{4m_Q}{\sqrt{\lambda(s, m_Q^2, m_Q^2)}} \log \left(\frac{1 + \omega}{1 - \omega}\right)$$

$$- \delta Z_\sigma \sigma_{\mu\nu} \gamma_5, \quad (20)$$

where

$$f(s) = \mathcal{F}(s) - \frac{1}{\varepsilon_{UV}} + \frac{2}{\varepsilon_{IR}} + 3 \log \frac{m_Q^2}{4\pi\mu^2} + 3\gamma - 4 - \frac{2(s - 2m_Q^2)}{\sqrt{\lambda(s, m_Q^2, m_Q^2)}} \log \left(\frac{1 + \omega}{1 - \omega}\right)$$

$$\mathcal{F}(s) = \frac{2(s - 2m_Q^2)}{\sqrt{\lambda(s, m_Q^2, m_Q^2)}} \left\{ \frac{1}{2} \log^2(1 - \omega^2) - 2 \log^2(1 + \omega) + 2 \log 2 \log \left(\frac{1 + \omega}{1 - \omega}\right) \right.$$

$$- 2 \text{Li}_2 \left(\frac{2\omega}{1 + \omega}\right) + \pi^2 \right\} + \frac{4(s - 4m_Q^2)}{\sqrt{\lambda(s, m_Q^2, m_Q^2)}} \log \left(\frac{1 + \omega}{1 - \omega}\right), \quad (21)$$

$s = p^2$, $\omega = \sqrt{1 - \frac{4m_Q^2}{s}}$, and $\text{Li}_2(x) = - \int_0^x dt \frac{\log(1-t)}{t}$.

The total contributions come from the virtual gluon emissions (see Fig.4) to imaginary parts.
Figure 6: The quark self-energy correction.

Figure 7: The vertex correction.

Figure 8: The amplitudes for the real gluon emissions.
of the correlation functions can be expressed in the following form,

\[ \frac{\text{Im} \Pi^R(s)}{\pi} = \frac{4 \alpha_s}{3 \pi} \rho_0(s) \left\{ \frac{1}{2 \varepsilon_{\text{UV}}} + \frac{1}{\varepsilon_{\text{IR}}} + \frac{3}{2} \log \frac{m_Q^2}{4 \pi \mu^2} + \log \frac{\lambda(s, m_Q^2, m_Q^2)}{4 \pi s \mu^2} + \frac{5}{2} \gamma - 4 + \frac{1}{2} \delta Z(s) \right\} \]

\[-\frac{s - 2m_Q^2}{\sqrt{\lambda(s, m_Q^2, m_Q^2)}} \log \left( \frac{1 + \omega}{1 - \omega} \right) \left\{ \frac{1}{\varepsilon_{\text{IR}}} + 2 \gamma - 2 + \log \frac{\lambda(s, m_Q^2, m_Q^2)}{16 \pi^2 \mu^4} \right\} \]

\[+ \frac{4 \alpha_s}{3 \pi} \frac{s - 4m_Q^2}{12 \pi^2 s^2} \left\{ \sqrt{\lambda(s, m_Q^2, m_Q^2)} - (s + 4m_Q^2) \log \left( \frac{1 + \omega}{1 - \omega} \right) \right\} - 2 \rho_0(s) \delta Z_\sigma, \quad (22) \]

where

\[ \delta Z_\sigma = \frac{4 \alpha_s}{3 \pi} \left( \frac{1}{4 \varepsilon_{\text{UV}}} + \frac{1}{4} \log \frac{m_Q^2}{4 \pi \mu^2} + \frac{\gamma}{4} \right). \quad (23) \]

The four cuts shown in Fig.5 correspond to real gluon emissions. The scattering amplitudes for the real gluon emissions are shown explicitly in Fig.8. From Fig.8, we can write down the scattering amplitudes \( T_{\mu\nu\alpha}^a(p) \),

\[ T_{\mu\nu\alpha}^a(p) = \tilde{u}(p_1) \left\{ i g_s \frac{\gamma^\mu}{2} \frac{1}{p_1 + \vec{K} - m_Q} \sigma_{\mu\nu} \gamma_5 + i g_s \frac{\gamma^\mu}{2} \frac{1}{p_2 - k - m_Q} \right\} v(p_2), \quad (24) \]

then we obtain the corresponding contributions \( \text{Im} \Pi^R(s) \) to the imaginary parts of the correlation functions with optical theorem.

\[ \frac{\text{Im} \Pi^R(s)}{\pi} = \frac{1}{2 \pi} \int \frac{d^{D-1}k}{(2\pi)^{D-1}E_k} \frac{d^{D-1}\tilde{p}_1}{(2\pi)^{D-1}E_{\tilde{p}_1}} \frac{d^{D-1}\tilde{p}_2}{(2\pi)^{D-1}E_{\tilde{p}_2}} \left( 2\pi \right)^D \delta^D(p - k - p_1 - p_2) \]

\[ \text{Tr} \left\{ T_{\mu\nu\lambda}(p) T_{\alpha\beta\tau}^{\dagger}(p) \right\} g^{\lambda \tau} \frac{1}{2(1 - D)p^2} \left( g^{\mu \alpha} g^{\nu \beta} - \frac{D}{2} g^{\mu \alpha} g^{\nu \beta} \right) \]

\[ = \frac{8g_s^2}{3s} \int \frac{d^{D-1}k}{(2\pi)^{D-1}E_k} \frac{d^{D-1}\tilde{p}_1}{(2\pi)^{D-1}E_{\tilde{p}_1}} \frac{d^{D-1}\tilde{p}_2}{(2\pi)^{D-1}E_{\tilde{p}_2}} \left( 2\pi \right)^D \delta^D(p - k - p_1 - p_2) \]

\[ \left\{ (s + 2m_Q^2) \left[ \frac{s - 2m_Q^2}{k \cdot p_1 k \cdot p_2} - \frac{m_Q^2}{(k \cdot p_1)^2} - \frac{m_Q^2}{(k \cdot p_2)^2} - \frac{s - K^2}{k \cdot p_1 k \cdot p_2} \right] + \frac{(s - K^2)^2}{2k \cdot p_1 k \cdot p_2} \right\} \]

\[ - \frac{8(s - K^2)}{s} + \varepsilon_{\text{IR}} s - 4m_Q^2 \kappa \frac{1}{3} \left\{ \frac{s - 2m_Q^2}{k \cdot p_1 k \cdot p_2} - \frac{m_Q^2}{(k \cdot p_1)^2} - \frac{m_Q^2}{(k \cdot p_2)^2} \right\}, \quad (25) \]

where we have used the identities \( \sum u(p_1) \tilde{u}(p_1) = p_1 + m_Q \) and \( \sum v(p_2) \tilde{v}(p_2) = p_2 - m_Q \) for the particle and antiparticle respectively, and take the notation \( K^2 = (p_1 + p_2)^2 \). We carry out the integrals in Eq.(25) in \( D = 4 + 2\varepsilon_{\text{IR}} \) dimension to obtain the spectral densities,

\[ \frac{\text{Im} \Pi^R(s)}{\pi} = \frac{4 \alpha_s}{3 \pi} \rho_0(s) \left\{ \frac{1}{\varepsilon_{\text{IR}}} - 2 \gamma + 2 - \log \frac{\lambda^3(s, m_Q^2, m_Q^2)}{16 \pi^2 m_Q^2 s^2 \mu^4} + \frac{3s}{\sqrt{\lambda(s, m_Q^2, m_Q^2)}} \log \left( \frac{1 + \omega}{1 - \omega} \right) \right\} \]

\[+ \frac{s - 2m_Q^2}{\sqrt{\lambda(s, m_Q^2, m_Q^2)}} \log \left( \frac{1 + \omega}{1 - \omega} \right) \left\{ \frac{1}{\varepsilon_{\text{IR}}} + 2 \gamma - 2 + \log \frac{\lambda^3(s, m_Q^2, m_Q^2)}{16 \pi^2 m_Q^2 s^2 \mu^4} \right\} \]

\[+ (s - 2m_Q^2) R_{12} - R_{11} \] + \frac{4 \alpha_s}{3 \pi} \frac{1}{s^2} \left\{ \frac{1}{8} \sqrt{\lambda(s, m_Q^2, m_Q^2)} R_{12}^2 - 2R_0 \right\}

\[ - \frac{s - 4m_Q^2}{12} \left[ \sqrt{\lambda(s, m_Q^2, m_Q^2)} - (s - 2m_Q^2) \right], \quad (26) \]
the expressions of the $\overline{R}_{12}(s)$, $R_{12}^1(s)$, $R_{12}^2(s)$ and $R_{3}^1$ are given explicitly in the appendix. 

The total spectral densities $\rho_1(s)$ come from the virtual and real gluon emissions are $\rho_1(s) = \frac{\Im\Pi_{1}^{\Gamma}(s)}{\pi} + \frac{\Im\Pi_{2}^{\Gamma}(s)}{\pi}$, 

$$\rho_1(s) = \frac{4\alpha_s}{3\pi}\rho_0(s) \left\{ \frac{1}{2}f(s) + (s - 2m_Q^2)\overline{R}_{12}(s) - R_{12}^1 + \frac{3s}{\sqrt{\lambda(s,m_Q^2,m_Q^2)}}\log\left(\frac{1 + \omega}{1 - \omega}\right) \right.$$ 

$$+ \frac{s - 2m_Q^2}{\sqrt{\lambda(s,m_Q^2,m_Q^2)}}\log\left(\frac{1 + \omega}{1 - \omega}\right) \log\left(\frac{\lambda^2(s,m_Q^2,m_Q^2)}{m_Q^4s^2}\right) - \log\left(\frac{\lambda^2(s,m_Q^2,m_Q^2)}{m_Q^4s}\right) - 2 \right\}$$ 

$$+ \frac{4\alpha_s}{3\pi}\frac{1}{\pi^2s^2} \left\{ \frac{1}{8}\sqrt{\lambda(s,m_Q^2,m_Q^2)}R_{12}^2 - 2R_{12}^3 \right\}$$ 

$$+ \frac{4\alpha_s}{3\pi}\frac{s - 4m_Q^2}{12\pi^2s^2} \left\{ s - 2m_Q^2 - (s + 4m_Q^2)\log\left(\frac{1 + \omega}{1 - \omega}\right) \right\},$$ 

which are free of divergence. The spectral densities $\rho_1(s)$ have direct applications in studying the $O(\alpha_s)$ corrections for the decays of a boson into massive fermion-antifermion pairs with the vertices $\sigma_{\mu\nu}\gamma_5$ and $\sigma_{\mu\nu}$, see Figs.6-8.

In Figs.9-10, we present all the Feynman diagrams contribute to the gluon condensates and a typical Feynman diagram contributes to the four-quark condensates. We calculate those diagrams straightforwardly with help of the full quark propagator $S_{ij}(x)$, 

$$S_{ij}(x) = \frac{i}{(2\pi)^4} \int d^4k e^{-ikx} \left\{ \frac{\delta_{ij}}{k - m_Q} - \frac{g_s G_{ij}^a}{4} \frac{\sigma^{\alpha\beta}(k + m_Q) + (k + m_Q)\sigma^{\alpha\beta}}{(k^2 - m_Q^2)^2} \right.$$ 

$$\left. \frac{m_Q^2k^2 + m_Q^2k}{(k^2 - m_Q^2)^4} + \frac{g_s D_{\alpha\beta}G_{ij}^a}{3} \frac{(f^{\lambda\alpha\beta} + f^{\lambda\alpha\beta})(k + m_Q)}{(k^2 - m_Q^2)^4} + \ldots \right\},$$ 

$$f^{\lambda\alpha\beta} = \gamma^\lambda (k + m_Q)\gamma^\alpha (k + m_Q)\gamma^\beta,$$ 

$t^n = \frac{\Lambda^n}{\sqrt{2}}$, the $i, j$ are color indexes, the $g_s^2 GG$ is the gluon condensate [6], and obtain the spectral densities $\rho_{con}(s)$, 

$$\rho_{con}(s) = -\frac{s}{12T^4}\left(\frac{\alpha_s GG}{T}\right) \int_0^1 dx \frac{x^3 + (1 - x)^3}{x(1 - x)}\delta(s - \bar{m}_Q^2)$$ 

$$-\frac{1}{12T^2}\left(\frac{\alpha_s GG}{T}\right) \int_0^1 dx\delta(s - \bar{m}_Q^2) - \frac{16\alpha_s^2(\bar{q}\bar{q})^2}{81T^4} \int_0^1 dx \left[ \frac{1}{3} \left( \frac{1 + \frac{s}{T^2}}{T^2} \right) \right] \delta(s - \bar{m}_Q^2),$$ 

where $\bar{m}_Q^2 = \frac{m_Q^2}{x(1 - x)}$. We have used the equation of motion, $D^{\mu}G^{\mu\nu}_{\mu\nu} = \sum_{q=u,d,s} g_s q\gamma_{\mu}t^aq$, and taken the approximation $\langle \bar{s}s \rangle = \langle \bar{q}q \rangle$ to obtain the contributions of the four-quark condensates. In calculations, we observe that the contributions of the four-quark condensates are depressed by inverse powers of the large Euclidean momentum $P^2$ (thereafter the Borel parameter $T^2$) and play minor important roles, so we can neglect other diagrams contribute to the four-quark condensates of the order $O(\alpha_s^2)$. Furthermore, we also neglect the contributions come from the three gluon condensates, as they are also depressed by inverse powers of the large Euclidean momentum $P^2$ and numerical coefficients. The old value (or the experiential value) estimated by the instanton model is $\langle g_s^2 f^{abc}G_{a}G_{b}G_{c} \rangle = 0.045 GeV^6$ [12], while recent studies based on the moments sum rules indicate $\langle g_s^3 f^{abc}G_{a}G_{b}G_{c} \rangle \approx (8.8 \pm 5.5)\langle \alpha_s GG \rangle \approx 0.62 \pm 0.39 GeV^6$ [13]. If we set the Borel parameters
as $T^2 = 6 \text{GeV}^2$, the three-gluon condensate can be counted as $\langle g_s^3 f^{aBc} G_a G_b G_c \rangle / T^6 = 0.0002$ or $0.003 \pm 0.002$, the contributions are very small.

Once analytical expressions of the QCD spectral densities are obtained, then we can take the quark-hadron duality and perform the Borel transforms (and the derivatives) with respect to the variable $P^2 = -p^2$ to obtain the Borel sum rules (and the moments sum rules):

$$f_{h_Q}^2 \exp \left( - \frac{M_{h_Q}^2}{T^2} \right) = \int_{4m_Q^2}^{s_0} ds \left[ \rho_0(s) + \rho_1(s) + \rho_{\text{con}}(s) \right] \exp \left( - \frac{s}{T^2} \right),$$

\[ (30) \]

$$\frac{f_{h_Q}^2}{(M_{h_Q}^2 + 4m_{Q}^2 \xi)^{n+1}} = \int_{4m_Q^2}^{s_0} ds \frac{\rho_0(s) + \rho_1(s)}{(s + 4m_{Q}^2 \xi)^{n+1}} - \frac{1}{12} \langle \alpha_s G G \rangle \left( \frac{1}{\pi} \int_0^1 dx \frac{x^3 + (1 - x)^3}{x(1 - x)} \right)$$

\[ \frac{(n + 1)(n + 2)m_Q^2}{(m_Q^2 + 4m_{Q}^2 \xi)^{n+3}} - \frac{1}{12} \langle \alpha_s G G \rangle \left( \frac{1}{\pi} \int_0^1 dx \frac{n + 1}{(m_Q^2 + 4m_{Q}^2 \xi)^{n+2}} \right) \]

\[ \frac{16 \alpha_s^2 \langle \bar{q}q \rangle^2}{81} \]

\[ \int_0^1 dx \left[ \frac{1}{3} - \frac{1}{2x(1 - x)} \right] \left( \frac{(n + 1)(n + 2)}{(m_Q^2 + 4m_{Q}^2 \xi)^{n+3}} + \left( \frac{1}{3} - \frac{x^2 + (1 - x)^2}{2x(1 - x)} \right) \right) \]

\[ \frac{(n + 1)(n + 2)(n + 3)m_Q^2}{(m_Q^2 + 4m_{Q}^2 \xi)^{n+4}} \].

\[ (31) \]

We can eliminate the decay constants $f_{h_Q}$ and obtain the QCD sum rules for the masses of the heavy quarkonium states $h_Q$,

$$M_{h_Q}^2 = \frac{\int_{4m_Q^2}^{s_0} ds \left[ \rho_0(s) + \rho_1(s) + \rho_{\text{con}}(s) \right] \exp \left( - \frac{s}{T^2} \right)}{\int_{4m_Q^2}^{s_0} ds \left[ \rho_0(s) + \rho_1(s) + \rho_{\text{con}}(s) \right] \exp \left( - \frac{s}{T^2} \right)},$$

\[ (32) \]

$$M_{h_Q}^2 = \frac{\Pi(n - 1, \xi)}{\Pi(n, \xi)} - 4m_Q^2 \xi.$$

\[ (33) \]
choose the values \( m \) pole masses and the \( \langle n \rangle \) updated value for the flavors \( \hbar \) sum rules and Lattice QCD \([4, 7, 12, 14]\). The values listed in the Review of Particle Physics \( \Lambda = \bar{2}13 \) MeV, \( \Lambda = 296 \) MeV \([4]\), we obtain the continuum threshold parameters \( s^0 \) (16 ± 1) GeV\(^2\) and \( s^0 \) (105 ± 2) GeV\(^2\) approximately. The quark condensate is determined by the Gell-Mann-Oakes-Renner relation, we take the standard value \(\langle \bar{q}q \rangle = - (0.24 \pm 0.01 \) GeV\(^3\)\) at the energy scale \( \mu = 1 \) GeV \([5, 6, 12]\). The value of the gluon condensate \(\langle \alpha_s \bar{G}G \rangle \) has been updated from time to time, and changes greatly \([7]\), we take the recently updated value \(\langle \alpha_s \bar{G}G \rangle = (0.022 \pm 0.004) \) GeV\(^4\), and neglect the uncertainty.

In this article, we calculate the perturbative \( O(\alpha_s) \) corrections \( \rho_t(s) \) in the on-shell renormalization scheme, and take the pole masses. The pole masses and the \( \overline{\text{MS}} \) masses have the relation \( m_Q = \overline{m}(m_Q^2) \left[ 1 + \frac{b_1}{b_0} \right] \) + \( \cdots \). The \( \overline{\text{MS}} \) masses have been studied extensively by the QCD sum rules and Lattice QCD \([4, 7, 12, 14]\). The values listed in the Review of Particle Physics are \( \overline{m}^2(m_0^2) = 1.275 \pm 0.025 \) GeV and \( \overline{m}^2(m_0^2) = 4.18 \pm 0.03 \) GeV \([4]\), which correspond to the pole masses \( m_c = (1.67 \pm 0.07) \) GeV and \( m_b = (4.78 \pm 0.06) \) GeV. The recent studies based on the QCD sum rules \([13, 15]\), the nonrelativistic large-n \( T \) sum rules with renormalization group improvement \([10]\) and the lattice QCD \([17]\) indicate (slightly) different values. In this article, we choose the values \( m_b = 4.80 \) GeV and \( m_c = 1.55 \) GeV, the uncertainties will be discussed later. Furthermore, we set the energy scale to be \( \mu = m_c \) and \( m_b \) for the heavy quarkonium states \( \hbar \) and \( \hbar \), respectively, and take the \( \alpha_s(\mu) \) from the Particle Data Group,

\[
\alpha_s(\mu) = \frac{1}{b_0 t} \left[ 1 - \frac{b_1}{b_0} \log \frac{t}{b_0} + \frac{b_2^2}{b_0^2} (\log^2 t - \log t - 1) + b_0 b_2 \right],
\]

where \( t = \log \frac{\mu^2}{\Lambda^2}, b_0 = \frac{33 - 2 n_f}{12 \pi}, b_1 = \frac{153 - 19 n_f}{24 \pi}, b_2 = \frac{2857 - 1033 n_f + 225 n_f^2}{128 \pi^2}, \Lambda = 213 \) MeV, \( 296 \) MeV and \( 339 \) MeV for the flavors \( n_f = 5, 4 \) and \( 3 \), respectively \([4]\).

If we take the Borel parameters as \( T^2 = (5.5 - 6.5) \) GeV\(^2\) and \( (11 - 13) \) GeV\(^2\) in the channels \( h_c \) and \( h_b \), respectively, the pole contributions are about \((51 - 69)\% \) and \((50 - 69)\% \), respectively, see Fig.11, it is reliable to extract the ground state masses. In Fig.11, we plot the pole contributions with variations of the Borel parameters \( T^2 \) and threshold parameters \( s_0 \). On the other hand, the dominant contributions come from the perturbative terms, the operator product expansion is well convergent.

In Fig.12, we plot the \( h_Q \) masses \( M_{h_Q} \) and decay constants \( f_{h_Q} \) with variations of the Borel parameters \( T^2 \) and threshold parameters \( s_0 \). From the figure, we can see that the values are stable with variations of the Borel parameters \( T^2 \). In Fig.13, we plot the masses \( M_{h_Q} \) and decay constants \( f_{h_Q} \) with variations of the moment parameters \( n \) and threshold parameters \( s_0 \) in the moments sum

![Figure 11: The pole contributions with variations of the Borel parameters \( T^2 \).](image)

3 Numerical results and discussions

From the experimental data \( M_{h_c(1P)} = (9898.3 \pm 1.1^{+1.0}_{-1.1}) \) MeV, \( M_{h_c(2P)} = (10259.8 \pm 0.6^{+1.4}_{-1.0}) \) MeV \([2]\), \( M_{h_c(1P)} = (3525.41 \pm 0.16) \) MeV, \( M_{X_c(2P)} = (3927.2 \pm 2.6) \) MeV \([4]\), we obtain the continuum threshold parameters \( s^c_h = (16 \pm 1) \) GeV\(^2\) and \( s^c_{h_b} = (105 \pm 2) \) GeV\(^2\) approximately.
rules. In the moments sum rules for the $P$-wave heavy quarkonium states, $\xi > 1$ 10, in this article, we take $\xi = 2$, and choose $n = 3 - 7$ and $n = 17 - 23$ for the $h_c$ and $h_b$, respectively. From Figs.12-13, we can see that the values from the moments sum rules are consistent with that from the Borel sum rules.

In the following, we write down the masses and decay constants of the heavy quarkonium states $h_c$ and $h_b$,

$$
\begin{align*}
M_{h_c} &= 3.530 \pm 0.006 \pm 0.050 \pm 0.090 \text{ GeV}, \\
M_{h_b} &= 9.894 \pm 0.005 \pm 0.035 \pm 0.064 \text{ GeV}, \\
f_{h_c} &= 0.490 \pm 0.002 \pm 0.040 \pm 0.045 \text{ GeV}, \\
f_{h_b} &= 0.549 \pm 0.002 \pm 0.050 \pm 0.045 \text{ GeV},
\end{align*}
$$

(35)

from the Borel sum rules, and

$$
\begin{align*}
M_{h_c} &= 3.521 \pm 0.025 \pm 0.050 \pm 0.098 \text{ GeV}, \\
M_{h_b} &= 9.899 \pm 0.006 \pm 0.040 \pm 0.063 \text{ GeV}, \\
f_{h_c} &= 0.490 \pm 0.008 \pm 0.040 \pm 0.044 \text{ GeV}, \\
f_{h_b} &= 0.552 \pm 0.003 \pm 0.047 \pm 0.046 \text{ GeV},
\end{align*}
$$

(36)

from the moments sum rules. The uncertainties come from the Borel parameters (or moment parameters), threshold parameters, heavy quark masses, sequentially. The integral ranges $4m_Q^2 \sim s_0$ and the QCD spectral densities change quickly with variations of the heavy quark masses, small variations $\delta m_Q$ can lead to relatively large uncertainties $\delta M_{h_Q}$ and $\delta f_{h_Q}$. In this article, we take $\delta m_c^2 = 8m_c\delta m_c = \pm 1 \text{ GeV}^2$ and $\delta m_b^2 = 8m_b\delta m_b = \pm 2 \text{ GeV}^2$, just like the uncertainties of the

Figure 12: The masses $M_{h_Q}$ and decay constants $f_{h_Q}$ with variations of the Borel parameters $T^2$. 
continuum threshold parameters $\delta s_{hc}^0 = \pm 1$ GeV\(^2\) and $\delta s_{hb}^0 = \pm 2$ GeV\(^2\). The masses from both QCD sum rules are consistent with the experimental data, $M_{h_c(1P)} = (9898.3 \pm 1.1^{+1.0}_{-1.1})$ MeV [2] and $M_{h_b(1P)} = (3525.41 \pm 0.16)$ MeV [2]. The heavy quarkonium states $h_Q$ couple potentially to the tensor currents $\bar{Q}\sigma_{\mu\nu}Q$, the $h_Q$ have the quark structure $\varepsilon^{ijk}\xi^i\sigma^{jk}\zeta$ besides the quark structure $ik\xi\sigma \cdot (\vec{k}_1 - \vec{k}_2)\zeta$.

For the heavy quarkonium states, especially for the bottomonium states, the relative velocities $\omega$ of the quarks are small, we should account for the Coulomb-like corrections. After taking into account all the Coulomb-like contributions, we obtain the coefficient $f(\omega)$ to dress the leading-order spectral densities $\rho_0(s)$ [15],

$$f(\omega) = \frac{4\pi\alpha_s^c}{3\omega} \frac{1}{1 - \exp \left(-\frac{4\pi\alpha_s^c}{3\omega}\right)} = 1 + \frac{2\pi\alpha_s^c}{3\omega} + \cdots \quad (37)$$

In Fig.14, we plot the coefficients $f(\omega) = 1 + \frac{\rho_1(s)}{\rho_0(s)}$ and $1 + \frac{2\pi\alpha_s^c}{3\omega}$ for the heavy quarkonium states $h_c$ and $h_b$, respectively, and take the approximation $\alpha_s^c = \alpha_s$. From the figure, we can see that $1 + \frac{\rho_1(s)}{\rho_0(s)} \approx 1 + \frac{2\pi\alpha_s^c}{3\omega}$, the perturbative $\alpha_s$ corrections $\rho_1(s)$ can be approximated by $\rho_0(s) \frac{2\pi\alpha_s^c}{3\omega}$. We can account for all the Coulomb-like contributions by multiplying the leading-order spectral densities $\rho_0(s)$ by the coefficient $f(\omega)$ tentatively. If we take the Borel parameters as $T^2 = (6.8 - 7.8)$ GeV\(^2\) and $(12.9 - 14.9)$ GeV\(^2\) in the channels $h_c$ and $h_b$, respectively, again we
Figure 14: The coefficients $f(\omega) = 1 + \frac{\rho(\omega)}{\rho_0(\omega)}$ and $1 + \frac{2\pi\alpha_s}{3\omega}$ for I and II, respectively.

obtain the pole contributions $(51 - 67)\%$ and $(50 - 67)\%$, respectively. The central values

$$
M_{h_c} = 3.516 \text{ GeV},
$$

$$
M_{h_b} = 9.884 \text{ GeV},
$$

$$
f_{h_c} = 0.576 \text{ GeV},
$$

$$
f_{h_b} = 0.657 \text{ GeV},
$$

(38)

come from the Borel sum rules indicate the shifts $\delta M_{h_c} = -0.014 \text{ GeV}$, $\delta M_{h_b} = -0.010 \text{ GeV}$, $\delta f_{h_c} = 0.086 \text{ GeV}$, $\delta f_{h_b} = 0.108 \text{ GeV}$ compared to the predictions in Eq.(35). The mass-shifts are mild, while the decay constant shifts are large.

In the $q\bar{q}$ quark model, the party $P = (-1)^L+1$, the charge conjunction $C = (-1)^L+1$, where $L$ and $S$ are the orbital and spin angular momenta, respectively. The heavy quarkonium states $h_Q$ have $J^{PC} = 1^{+-}$, so they have the quantum numbers $S = 0$, $L = 1$ and $J = L$, the spins of the quark $Q$ and antiquark $\bar{Q}$ should be antiparallel. The quark structures $\epsilon^{ijk}\xi^k\zeta$ and $ik_2\xi^i\sigma \cdot (\vec{k}_1 - \vec{k}_2)\zeta$ both satisfy the requirement, the heavy quarkonium states $h_Q$ have two possible quark structures. We can study the mixing of the two structures with the two-point correlation functions $\Pi_{\mu\alpha}(p)$,

$$
\Pi_{\mu\alpha}(p) = i \int d^4x e^{ip\cdot x} \langle 0 \{ \eta_\mu(x)\eta^\dagger_\alpha(0) \} | 0 \rangle,
$$

$$
\eta_\mu(x) = \cos \theta \bar{Q}(x)\sigma_{\mu\nu}\gamma_5 Q(x) p^\nu + \sin \theta \bar{Q}(x) \partial_\mu \gamma_5 Q(x),
$$

(39)

and search for the optimal value of the mixing angular $\theta$.

4 Conclusion

In this article, we take the tensor currents $\bar{Q}(x)\sigma_{\mu\nu}Q(x)$ to interpolate the $P$-wave spin-singlet heavy quarkonium states $h_Q$, study the masses and decay constants with the Borel sum rules and the moments sum rules, and explore whether or not the $h_Q$ have the quark structure $\epsilon^{ijk}\xi^k\sigma\zeta$ besides the quark structure $ik_2\xi^i\sigma \cdot (\vec{k}_1 - \vec{k}_2)\zeta$. The masses and decay constants come from the Borel sum rules and moments sum rules are consistent with each other, the masses are also consistent with the experimental data. The heavy quarkonium states $h_Q$ couple potentially to the tensor currents $\bar{Q}(x)\sigma_{\mu\nu}Q(x)$, and have the quark structure $\epsilon^{ijk}\xi^k\sigma\zeta$ besides the quark structure $ik_2\xi^i\sigma \cdot (\vec{k}_1 - \vec{k}_2)\zeta$. We can take the decay constants as basic input parameters and study
the relevent hadronic processes with the QCD sum rules, for example, we can study the $h_c D D^*$, $h_c D_s D_s^*$, $h_c D_s^* D_s^*$, $h_c \to D_s, D_s^*, D_s^*$ form-factors with three-point correlation functions. In calculations, we take into account the leading-order, next-to-leading-order perturbative contributions, and the gluon condensate, four-quark condensate contributions in the operator product expansion. The analytical expressions of the perturbative spectral densities have applications in studying the two-body decays of a boson to two fermions with the vertexes $\sigma_{\mu\nu}\gamma_5$ and $\sigma_{\mu\nu}$.

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**Appendix**

We take the notation

$$\int \frac{d\mathbf{s}}{k} = \int \frac{d^D \mathbf{k}}{2E_k} \frac{d^D \mathbf{p}_1}{2E_{p_1}} \frac{d^D \mathbf{p}_2}{2E_{p_2}} \delta^D(p - k - p_1 - p_2),$$

for simplicity, and write down the analytical expressions of the three-body phase-space integrals,

$$R_{12}(s) = \frac{s}{\pi^2 \sqrt{\lambda(s, m_Q^2, m_Q^2)}} \int d\mathbf{s} \frac{1}{k \cdot p_1 \cdot p_2} \frac{(2\pi)^{-4\epsilon_{IR}}}{\mu^{-2\epsilon_{IR}}} \int d\mathbf{p}_1 \frac{1}{2E_{p_1}} d\mathbf{p}_2 \frac{1}{2E_{p_2}} \delta^D(p - k - p_1 - p_2),$$

$$= \sqrt{\lambda(s, m_Q^2, m_Q^2)} \left\{ \log \left( \frac{1 + \omega}{1 - \omega} \right) \left[ \frac{1}{\epsilon_{IR}} - 2 \log 4\pi + 2\gamma - 2 + 2 \log \sqrt{\frac{\lambda(s, m_Q^2, m_Q^2)}{m_Q^2 s p^2}} \right] 
- \log^2 \left( \frac{1 + \omega}{1 - \omega} \right) - 4 \text{Li}_2 \left( \frac{2 \omega}{1 + \omega} \right) - \text{Li}_2 \left( \frac{1 + \omega}{2} \right) - 2 \text{Li}_2 \left( \frac{1 + \omega}{2} \right) + \log 2 \log(1 + \omega) - \frac{\log^2 2}{2} + \frac{\pi^2}{12} \right\},$$

$$= \bar{R}_{12}(s) + \frac{1}{\sqrt{\lambda(s, m_Q^2, m_Q^2)}} \log \left( \frac{1 + \omega}{1 - \omega} \right) \left[ \frac{1}{\epsilon_{IR}} - 2 \log 4\pi + 2\gamma - 2 + 2 \log \sqrt{\frac{\lambda(s, m_Q^2, m_Q^2)}{m_Q^2 s p^2}} \right],$$

$$R_{12}^1(s) = \frac{s}{\pi^2 \sqrt{\lambda(s, m_Q^2, m_Q^2)}} \int d\mathbf{s} \frac{s - K^2}{k \cdot p_1 \cdot p_2},$$

$$= \frac{s}{\sqrt{\lambda(s, m_Q^2, m_Q^2)}} \left\{ \log^2 (1 - \omega) - \log^2 (1 + \omega) + 2 \log 2 \log \left( \frac{1 + \omega}{1 - \omega} \right) + 2 \text{Li}_2 \left( \frac{1 - \omega}{2} \right) 
- 2 \text{Li}_2 \left( \frac{1 + \omega}{2} \right) \right\},$$

$$R_{12}^2(s) = \frac{s^2}{\pi^2 \sqrt{\lambda(s, m_Q^2, m_Q^2)}} \int d\mathbf{s} \frac{(s - K^2)^2}{k \cdot p_1 \cdot p_2},$$

$$= \frac{s^2}{\sqrt{\lambda(s, m_Q^2, m_Q^2)}} \left\{ \log^2 (1 - \omega) - \log^2 (1 + \omega) + 2 \log 4 \log \left( \frac{1 + \omega}{1 - \omega} \right) + 2 \text{Li}_2 \left( \frac{1 - \omega}{2} \right) 
- 2 \text{Li}_2 \left( \frac{1 + \omega}{2} \right) + 2 \omega - (1 + \omega^2) \log \left( \frac{1 + \omega}{1 - \omega} \right) \right\}.$$
\[ R_0^1(s) = \frac{1}{\pi^2} \int dps (s - K^2) \]
\[ = \frac{s}{\sqrt{\lambda(s, m_Q^2, m_Q^2)}} \left\{ \omega(15 - 4\omega^2 - 3\omega^4) + \frac{3}{2}(\omega^6 + \omega^4 + 3\omega^2 - 5) \log \left( \frac{1 + \omega}{1 - \omega} \right) \right\}. \]

\[ (40) \]

References

[1] J. P. Lees et al, Phys. Rev. D84 (2011) 091101.
[2] I. Adachi et al, Phys. Rev. Lett. 108 (2012) 032001.
[3] C. Baglin et al, Phys. Lett. B171 (1986) 135.
[4] J. Beringer et al, Phys. Rev. D86 (2012) 010001.
[5] M. A. Shifman, A. I. Vainshtein and V. I. Zakharov, Nucl. Phys. B147 (1979) 385.
[6] L. J. Reinders, H. Rubinstein and S. Yazaki, Phys. Rept. 127 (1985) 1.
[7] S. Narison, Camb. Monogr. Part. Phys. Nucl. Phys. Cosmol. 17 (2002) 1.
[8] N. Brambilla, A. Pineda, J. Soto and A. Vairo, Phys. Lett. B470 (1999) 215; N. Brambilla and A. Vairo, Phys. Rev. D62 (2000) 094019; N. Brambilla, Y. Sumino and A. Vairo, Phys. Rev. D65 (2002) 034001; A. A. Penin and M. Steinhauser, Phys. Lett. B538 (2002) 335; N. Brambilla and A. Vairo, Phys. Rev. D71 (2005) 034020; A. A. Penin, V. A. Smirnov and M. Steinhauser, Nucl. Phys. B716 (2005) 303.
[9] S. Recksiegel and Y. Sumino, Phys. Lett. B578 (2004) 369.
[10] T. M. Aliev, K. Azizi and M. Savci, Phys. Lett. B690 (2010) 164; Z. G. Wang, Mod. Phys. Lett. A26 (2011) 2761.
[11] S. Bauberger, M. Bohm, G. Weiglein, F. A. Berends and M. Buza, Nucl. Phys. Proc. Suppl. 37B (1994) 95; S. Bauberger, F. A. Berends, M. Bohm and M. Buza, Nucl. Phys. B434 (1995) 383.
[12] P. Colangelo and A. Khodjamirian, hep-ph/0010175.
[13] S. Narison, Phys. Lett. B706 (2012) 412.
[14] B. L. Ioffe, Prog. Part. Nucl. Phys. 56 (2006) 232.
[15] K. G. Chetyrkin, J. H. Kuhn, A. Maier, P. Maierhofer, P. Marquard, M. Steinhauser and C. Sturm, Phys. Rev. D80 (2009) 074010; S. Bodenstein, J. Bordes, C. A. Dominguez, J. Penarrocha and K. Schilcher, Phys. Rev. D83 (2011) 074014; S. Bodenstein, J. Bordes, C. A. Dominguez, J. Penarrocha and K. Schilcher, Phys. Rev. D85 (2012) 034003; B. Dehnadi, A. H. Hoang, V. Mateu and S. M. Zebarjad, [arXiv:1102.2264].
[16] A. Hoang, P. Ruiz-Femenia and Ma. Stahlhofen, JHEP 1210 (2012) 188.
[17] C. McNeile, C. T. H. Davies, E. Follana, K. Hornbostel and G. P. Lepage, Phys. Rev. D82 (2010) 034512.
[18] V. V. Kiselev, Int. J. Mod. Phys. A11 (1996) 3689; V. V. Kiselev, A. K. Likhoded and A. I. Onishchenko, Nucl. Phys. B569 (2000) 473; V. V. Kiselev, A. E. Kovalsky and A. K. Likhoded, Nucl. Phys. B585 (2000) 353.