Corrigendum: Stationary problem related to the nonlinear Schrödinger equation on the unit ball (2012 Nonlinearity 25 2271–301)

Reika Fukuizumi¹, Fouad Hadj Selem² and Hiroaki Kikuchi³

¹ Graduate School of Information Sciences, Tohoku University, 6-3-09 Aramaki-Aza-Aoba, Aoba-ku, Sendai 980-8579, Japan
² CEA Saclay, Neurospin, I2BM, CEA, 91191 Gif-sur-Yvette, France
³ Department of Mathematics, Tsuda College, 2-1-1, Tsuda-machi, Kodaira-shi, Tokyo 187-8577, Japan

E-mail: hiroaki@tsuda.ac.jp

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The statement of proposition 9 (i) in [2] is incorrect. The space $H^{1,1}_{0,k,0,1}$ in proposition 9 should be replaced by the closed subspace $X_k$ of $H^{1,1}_{0,k,0,1}$, defined as follows. For each fixed $N \in \mathbb{N}$,

$$X_k = \{ v \in H^{1,1}_0(-1,1), \text{ there exists a function } \varphi \in H^{1,1}_0(0,1) \text{ s.t. } v(x) = G[\varphi](x) \},$$

where $G$ is defined by, for any $\varphi \in H^{1,1}_0(0,1)$,

$$G[\varphi](x) = \left( \frac{k}{2} \right)^{\frac{3}{2}} \sum_{n=1}^{2k} (-1)^{n+1} \varphi(-1)^{n+1} \frac{k}{2} \left( x + 1 - \frac{2n}{k} + \{1 + a(n)\} \frac{2}{k} \right) \chi_{[-1, \frac{2n-1}{k}, \frac{2n}{k}, \frac{2n+1}{k}, \frac{2n+2}{k}, \frac{2n+3}{k}]}(x).$$

(1)

Then the statement of proposition 9 (i) in [2] is corrected as follows.

A correct version of proposition 9 (i) in [2]. Let any $k \in \mathbb{N}$ be fixed. The $k$-th standing wave $e^{i\omega t}Q_{k,\omega}(x)$ is orbitally stable in $X_k$ for all $\omega > -\lambda_k$ if $1 < p < 5$ or $\omega$ close to $-\lambda_k$ if $p > 5$.

Remark. We remark that $Q_{k,\omega} \in X_k$, taking the corresponding $\varphi$, e.g. as the ground state $G[\varphi](x)$ on $(0,1)$ in (1). Using the fact that this ground state is even and a scaling property of the function from this work may be used under the terms of the Creative Commons Attribution 3.0 licence. Any further distribution of this work must maintain attribution to the author(s) and the title of the work, journal citation and DOI.
equation satisfied by the stationary solutions, the expression of \( Q_{k,\omega} = G^{(0,1)} \) defined by (1) is in fact coincident with (12) of [2].

The error arises in the proof of the well-posedness of Cauchy problem in section 6, page 2290 of [2]. In this corrigendum, we give a precise proof of the local well-posedness in \( X_k \). We note that even if we replace \( H^1_{0,k} \) by \( X_k \) in section 6 of [2], the statement of proposition 26 and the proof of proposition 9 (i) in [2] still work without any other modification: in fact, the argument in the above Remark applies similarly to all eigenfunctions of the linearized operator, i.e. each eigenfunction, which is unique, may be written in the form of (1), thus belongs to \( X_k \). The computation of the \( L^2 \)-norm of \( Q_{k,\omega} \) does not change thanks to the above Remark. Thus, we obtain the stability of the \( k \)-th standing wave \( e^{\pm iQ_{k,\omega}(x)} \) in \( X_k \).

Let \( I \subset \mathbb{R} \) be an open, bounded interval. Consider, for \( p > 1 \),

\[
\begin{cases}
  iu_t + u_{xx} + |u|^{p-1}u = 0, & t > 0, \quad x \in I, \\
  u(t,x) = 0, & t > 0, \quad x \in \partial I.
\end{cases}
\tag{2}
\]

Remark that the local well-posedness of (2) in \( H^1_0(I) \) is established in theorems 3.3.5 and 3.3.9 of [1]. For the later use, we introduce the result here.

**Proposition 1.** For any \( u_0 \in H^1_0(I) \), there exists a maximal time \( T_{\text{max}}(I, u_0) = T_{\text{max}}(\|u_0\|_{H^1(I)}) > 0 \) such that there exists a unique solution \( u \in C([0, T_{\text{max}}(I, u_0)], H^1(I)) \cap C^1([0, T_{\text{max}}(I, u_0)], H^{-1}(I)) \) of (2) with \( u(0) = u_0 \) and the following alternative holds:

\[
T_{\text{max}}(I, u_0) = +\infty \quad \text{or} \quad \lim_{t \to T_{\text{max}}(I, u_0)} \|\nabla u(t)\|_2 = \infty.
\]

Moreover, the solution \( u \) depends continuously on \( u_0 \) in the following sense: if \( u_{0,m} \to u_0 \) in \( H^1_0(I) \), and if \( u_m \) and \( u \) denote the corresponding solution of (2), then \( u_m \to u \) in \( C([0, T], H^1_0(I)) \) for all \( T < T_{\text{max}}(I, u_0) \).

The main purpose of this corrigendum is to prove the following proposition.

**Proposition 2.** Let \( k \in \mathbb{N} \) and \( p > 1 \) be fixed. Let \( u_0 \in X_k \) and \( I = (-1, 1) \). Then there exists a unique solution \( u \in C([-T_{\text{max}}(I, u_0)], X_k) \cap C^1([-T_{\text{max}}(I, u_0)], H^{-1}(I)) \) of (2) satisfying \( u(0) = u_0 \).

Before showing proposition 2, let us explain why the symmetry defined in \( X_k \) is conserved by the flow of (2). First, we introduce some properties of the gluing operator \( G \) whose verification can be found at the end of this corrigendum.

**Lemma 1.** Let \( k \in \mathbb{N} \) and \( p > 1 \) be fixed.

(i) For \( \varphi \in H^0_0(0, 1) \) and \( x \in (-1, 1) \),

\[
(G(\varphi)(x))_a = \left( k \right)^{\frac{p+1}{p}} \sum_{n=0}^{\left\lfloor \frac{p}{2} \right\rfloor} \varphi \left( (-1)^{n+1} k \left( x + 1 - \frac{2n}{k} + \left(1 + a(n)\right) \frac{2}{k}\right) \right) \chi_{\left[ -1 - \frac{2(n-a(n))}{k} - 1, -1 + \frac{2n}{k} \right]}(x).
\tag{3}
\]

(ii) For \( \varphi \in H^0_0(0, 1) \) and \( x \in (-1, 1) \),

\[
(G(\varphi)(x))_a = \left( k \right)^{\frac{p}{2} - 1} \sum_{n=0}^{\left\lfloor \frac{p}{2} \right\rfloor} \left( (-1)^{n+1} \varphi \left( (-1)^{n+1} k \left( x + 1 - \frac{2n}{k} + \left(1 + a(n)\right) \frac{2}{k}\right) \right) \right) \chi_{\left[ -1 - \frac{2(n-a(n))}{k} - 1, -1 + \frac{2n}{k} \right]}(x).
\tag{4}
\]

Using lemma 1, we may check the following fact.

**Lemma 2.** Let \( k \in \mathbb{N} \) and \( p > 1 \) be fixed. Let \( v \in C([0, T], H^1_0(0, 1) \cap H^1_0(0, 1)) \cap C^1([0, T], L^2(0, 1)) \) be a solution of (2) with \( I = (0, 1) \) for some \( T > 0 \). Then \( u(0, x) = G\left( \varphi \left( \frac{x}{T}, \right) \right) \) \( \chi_{\left[ -1 - \frac{2(n-a(n))}{k} - 1, -1 + \frac{2n}{k} \right]}(x) \) is of class
\[ C\left[\left[0, \frac{4}{T}\right], H^2(-1, 1) \cap H_0^1(-1, 1)\right) \cap C\left[\left[0, \frac{4}{T}\right], L^2(-1, 1)\right) \] and satisfies
\[ u_t + u_{xx} + |u|^{p-1}u = 0 \quad \text{in } H^{-1}(-1, 1) \]
for any \( t \in \left[\left[0, \frac{4}{T}\right]\right) \).

By the definition of \( X_k \), for \( u_0 \in X_k \), there exists \( \varphi \in H^1_0(0, 1) \) satisfying \( u_0 = \varphi \). Lemma 2 implies that if \( \varphi \) is smooth enough, and if the evolution starting from \( \varphi \) satisfies (2) with \( I = (-1, 1) \) and remains smooth, the symmetry of \( X_k \) is conserved through the evolution (2) with \( I = (-1, 1) \) starting from \( u_0 \). Thus, we need a regularization procedure to apply lemma 2. By density, for \( \varphi \in H^1_0(0, 1) \), there exists a sequence \( \{\varphi_m\}_{m \in \mathbb{N}} \subset C(0, T_{\max}(I, \varphi)) \cap H^1_0(0, 1) \) such that \( \varphi_m \) converges to \( \varphi \) in \( H^1_0(0, 1) \). The continuous dependence in Proposition 1 and classical regularity argument read;

**Corollary 1.** Let \( p > 1 \), \( I = (0, 1) \) and \( \varphi \in H^1_0(I) \). Let \( \{\varphi_m\}_{m \in \mathbb{N}} \subset C(0, T_{\max}(I, \varphi)) \cap H^1_0(I) \) be such that \( \varphi_m \rightarrow \varphi \) in \( H^1_0(I) \). For each \( m \in \mathbb{N} \), there exists a unique solution \( v_m \in C(0, T_{\max}(I, \varphi)) \cap H^1_0(I) \cap C^0(0, T_{\max}(I, \varphi)) \cap L^2(\mathbb{R}) \) of (2) with \( \varphi_n(0) = \varphi_m \).

Applying lemma 2 to \( G\left[v_m\left(\frac{4}{T}t, \cdot\right)\right] \) for \( t < \frac{4}{T}T_{\max}(0, 1) \), where \( v_m \) is constructed in corollary 1, we see that \( G[v_m] \) is a solution of (2) in \( H^{-1}(-1, 1) \). It thus suffices to show the convergence \( G[v_m] \rightarrow G[v] \) in \( H^1_0(-1, 1) \) where \( v \) is a solution of (2) in \( C(0, T_{\max}(0, 1), \varphi)) \cap H^1_0(0, 1) \) ensured by Proposition 1, with \( v(0) = \varphi \in H^1_0(0, 1) \).

**Proof of proposition 2.** Let \( T < T_{\max}(0, 1) \) be arbitrarily fixed. Since \( \varphi_m \rightarrow \varphi \) in \( H^1_0(I) \) as \( m \rightarrow \infty \), by the continuous dependence on initial data, for each \( t \in [0, T] \), \( v_m(t) \rightarrow v(t) \) in \( H^1_0(I) \) as \( m \rightarrow \infty \), where \( v_m \) and \( v \) are the corresponding solutions to \( \varphi_m \) and \( \varphi \) respectively. Also, by the continuity of \( G \), for each \( t \in [0, \frac{4}{T}] \), \( u_m(t) := G\left[v_m\left(\frac{4}{T}t, \cdot\right)\right] \rightarrow u(t) := G\left[v\left(\frac{4}{T}t, \cdot\right)\right] \) in \( H^1_0(I) \) as \( m \rightarrow \infty \). As a consequence, for any \( w \in H^1_0(I) \), \( \langle u_m \rangle_{p-1}u_m, w \rangle \rightarrow \langle u \rangle_{p-1}u, w \rangle \) in \( H^{-1/2} \) as \( m \rightarrow \infty \). On the other hand, lemma 2 implies that for any \( w \in H^1_0(I) \) and \( \phi \in C^0_0\left(0, \frac{4}{T}T\right) \),

\[ \int_0^{\frac{4}{T}T} \left[ \langle u_m \rangle_{H^{-1/2}} \phi'(t) + \langle u_{m,xx} + |u_m|^{p-1}u_m \rangle_{H^{-1/2}} \phi(t) \right] dt = 0. \]

Taking the limit \( m \rightarrow \infty \), by the dominated convergence theorem,

\[ \int_0^{\frac{4}{T}T} \left[ \langle u \rangle_{H^{-1/2}} \phi'(t) + \langle u_{xx} + |u|^{p-1}u \rangle_{H^{-1/2}} \phi(t) \right] dt = 0, \]

from which, we have for a.e. \( t \in \left[\left[0, \frac{4}{T}T\right]\right) \),

\[ \langle u_t + u_{xx} + |u|^{p-1}u \rangle_{H^{-1/2}} = 0. \]

But here we know already \( u \in C\left[\left[0, \frac{4}{T}T\right], H^2(I)\right) \cap C\left[\left[0, \frac{4}{T}T\right], H^{-1/2}(I)\right) \), and thus this equation holds in \( H^{-1/2}(I) \) for any \( t \in \left[\left[0, \frac{4}{T}T\right]\right) \). In conclusion, for any \( u_0 \in X_k \), there exists a solution \( u(t) = G\left[v\left(\frac{4}{T}t, \cdot\right)\right] \) of (2), which belongs to \( C\left[\left[0, \frac{4}{T}T\right], X_k\right) \cap C\left[\left[0, \frac{4}{T}T\right], H^{-1/2}(I)\right) \) with \( u(0) = u_0 \). On the other side, by proposition 1, the solutions of (2) is unique in \( C(0, T_{\max}(I, u_0), X_k) \) because \( X_k \subset H^1_0(I) \), thus this \( u(t) = G\left[v\left(\frac{4}{T}t, \cdot\right)\right] \) is the unique solution in
\(X_t\) on \(\left[0, \frac{1}{k} T \wedge T_{\max}(I, u_0)\right]\). We now define
\[
\tilde{T}_{\max} = \sup\{\tau > 0 \mid \exists u \in C([0, \tau], X_\bar{\cdot}) \text{ sol. of (2) with } u(0) = u_0\}.
\]
It is not difficult to see that this is a maximal time in the sense that \(\lim_{t \uparrow \tilde{T}_{\max}} \|u(t)\|_{L^2} = \infty\), i.e. \(\tilde{T}_{\max} = T_{\max}(I, u_0)\).

We will give a proof of lemmas 1 and 2 hereafter. We use the notation \(\langle f, g \rangle = \int f(x)g(x)dx\), for \(I \subset \mathbb{R}\).

**Proof of lemma 1.** By density, it is enough to check (3) and (4) for \(\varphi \in C_0^\infty(0, 1)\). Let \(w \in C_0^\infty(-1, 1)\).

\[
\left(\frac{k}{2}\right)^{\frac{2}{p-1}} \langle (GL\varphi)\|, w \rangle = -\left(\frac{k}{2}\right)^{\frac{2}{p-1}} \langle (GL\varphi), w_\varphi \rangle
\]
\[
= -\sum_{n=1}^{\infty} (-1)^{n-1} \int_{-1}^{1} \varphi \left( (-1)^{n-1} \frac{k}{2} \left[ x + 1 - \frac{2n}{k} + (1 + a(n)) \frac{2}{k} \right] \right)
\times \chi_{[-1 + \frac{2(n-1)}{k}, -1 + \frac{2n}{k}]}(x)\varphi(x)dx
\]
\[
= -\sum_{n=1}^{\infty} (-1)^{n-1} \int_{-1}^{1} \varphi \left( (-1)^{n-1} \frac{k}{2} \left[ x + 1 - \frac{2n}{k} + (1 + a(n)) \frac{2}{k} \right] \right)\varphi(x)dx
\]
\[
= -\sum_{n=1}^{\infty} (-1)^{n-1} \int_{-1}^{1} \left( (-1)^{n-1} \frac{k}{2} \left[ x + 1 - \frac{2n}{k} + (1 + a(n)) \frac{2}{k} \right] \right)\varphi(x)dx
\]
\[
= -\sum_{n=1}^{\infty} (-1)^{n-1} \int_{-1}^{1} \varphi \left( (-1)^{n-1} \frac{k}{2} \left[ x + 1 - \frac{2n}{k} + (1 + a(n)) \frac{2}{k} \right] \right)\varphi(x)dx
\]
\[
\times \chi_{[-1 + \frac{2(n-1)}{k}, -1 + \frac{2n}{k}]}(x)\varphi(x)dx
\]

since \(\varphi((-1)^{n-1}(1 + a(n))) = \varphi((-1)^{n-1}a(n)) = 0\) for any \(n = 1, \ldots, k\). Here, we have used the integration by parts in the fourth equality. This implies (3). Above computation implies that for any \(w \in C_0^\infty(-1, 1)\),

\[
\left(\frac{k}{2}\right)^{\frac{2}{p-1}} \langle (GL\varphi)\|, w \rangle = \left(\frac{k}{2}\right)^{\frac{2}{p-1}} \langle (GL\varphi), w_\varphi \rangle
\]
\[
= -\frac{k}{2} \sum_{n=1}^{\infty} \int_{-1}^{1} \varphi \left( (-1)^{n-1} \frac{k}{2} \left[ x + 1 - \frac{2n}{k} + (1 + a(n)) \frac{2}{k} \right] \right)\varphi(x)dx
\]

Again, by integration by parts, the last quantity is calculated as follows.
\[-\frac{k}{2} \sum_{n=1}^{k} \int_{-1}^{1} \varphi'((-1)^{n-1}1 + a(n)) \pi(1 + \frac{2n}{k}) \left(1 + \frac{2(n-1)}{k} \right) \pi(x) dx \]

\[-\frac{k}{2} \sum_{n=1}^{k} \left\{ \varphi'((-1)^{n-1}1 + a(n)) \pi(1 + \frac{2n}{k}) - \varphi'((-1)^{n-1}a(n)) \pi(1 + \frac{2(n-1)}{k}) \right\} \]

Thus, (4) is also verified. \qed

**Proof of lemma 2.**

Proof. For any \( t \in \left[0, \frac{4}{k^2} T \right] \), we put \( u(t, x) = G \left( \sqrt{\frac{2}{k}} t, \cdot \right) \pi(x) \) for \( x \in (-1, 1) \). By the definition of \( G \), the regularity of \( u(t, x) \) follows immediately. Using Lemma 1, for each fixed \( t \in \left[0, \frac{4}{k^2} T \right] \), and for any \( w \in C^\infty_0 (-1, 1) \),

\[ \langle iu_t + u_{xx} + |u|^{p-1} u, w \rangle \]

\[ = \left( \frac{k}{2} \right)^{p+1} \sum_{n=1}^{k} \int_{-1}^{1} \left((-1)^{n-1}1 + a(n)\right) \pi \left(1 + \frac{2n}{k} \right) \left(1 + \frac{2(n-1)}{k} \right) \pi(x) dx \]

with \( \tau = \frac{k^2}{4} \) and \( X = (-1)^{n-1} \left(x + 1 - \frac{2n}{k} + 1 + a(n) \right)^\frac{2}{k} \). Since \( v \in C([0, T], H^r(0, 1) \cap H^1_0(0, 1)) \) satisfies (2) with \( I = (0, 1) \) in \( L^2(0, 1) \),

\[ \langle iu_t + u_{xx} + |u|^{p-1} u, w \rangle = 0. \]

In particular,

\[ iu_t + u_{xx} + |u|^{p-1} u = 0 \quad \text{in} \ H^1(I), \]

for any \( t \in \left[0, \frac{4}{k^2} T \right] \). \qed

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**References**

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Stationary problem related to the nonlinear Schrödinger equation on the unit ball

Reika Fukuizumi¹, Fouad Hadj Selem²,⁴ and Hiroaki Kikuchi³,⁵

¹ Graduate School of Information Sciences, Tohoku University, 6-3-09 Aramaki-Aza-Aoba, Aoba-ku, Sendai 980-8579, Japan
² Laboratoire de Mathématiques, Reims University, BP 1039, 51687 Reims Cedex 02, France
³ Department of Mathematics, Hokkaido University, Kita 10, Nishi 8, Kita-ku, Sapporo 060-0810, Japan

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Abstract

In this paper, we study the stability of standing waves for the nonlinear Schrödinger equation on the unit ball in $\mathbb{R}^N$ with Dirichlet boundary condition. We generalize the result of Fibich and Merle (2001 Physica D 155 132–58), which proves the orbital stability of the least-energy solution with the cubic power nonlinearity in two space dimension. We also obtain several results concerning the excited states in one space dimension. Specifically, we show the linear stability of the first three excited states and we give a proof of the orbital stability of the $k$th excited state, restricting ourselves to the perturbation of the same symmetry as the $k$th excited state. Finally, our numerical simulations on the stability of the $k$th excited state are presented.

Mathematics Subject Classification: 35B35, 35Q55, 35J20, 65M06

(Some figures may appear in colour only in the online journal)

1. Introduction and main result

We consider the nonlinear Schrödinger equation on the unit ball in $\mathbb{R}^N$ with Dirichlet boundary condition:

$$\begin{cases}
\partial_t u + \Delta u + |u|^{p-1}u = 0, & (t, x) \in \mathbb{R} \times B_1, \\
 u = 0, & (t, x) \in \mathbb{R} \times \partial B_1,
\end{cases}$$

(1)

⁴ Current address: Laboratoire de Mathématiques, Université Blaise Pascal, UMR 6620 – CNRS Campus des Cézeaux – B.P. 80026 63171 Aubière cedex, France
⁵ Current address: Department of Mathematics, Tsuda College, 2-1-1, Tsuda-machi, Kodaira-shi, Tokyo 187-8577, Japan
where $u = u(t, x)$ is an unknown complex-valued function for $t \in \mathbb{R}$, $x \in B_1 = \{ x \in \mathbb{R}^N \mid |x| < 1 \}$. We always assume $N \geq 1$, and $1 < p \leq 2^* - 1$. Here $2^*$ is the usual Sobolev critical exponent, that is, $2^* = 2N/(N - 2)$ if $N \geq 3$ and $2^* = \infty$ if $N = 1$ or 2.

The nonlinear Schrödinger equation models the propagation of a laser beam in an optical fibre (see, e.g., Agrawal [1]). Equation (1) on the unit ball with Dirichlet boundary condition, with $N = 2$ and $p = 3$, describes the propagation of the laser beam in the hollow-core fibre. A hollow waveguide filled with a noble gas is used as an effective technique for extending the interaction length between nonlinear optical materials and high-energy laser pulses.

In this technique, the electric field outside the core is negligible because the reflective index difference between the hollow-core and the cladding is sufficiently large, and almost all of the laser beam reflects on the interface between the core and the cladding. This implies the Dirichlet boundary condition (see [19, 20, 30, 50] for details). Note that the use of a bulk medium, whose laser beam reflects on the interface between the core and the cladding, is not allowed in this case. The authors in [39, 40] have found some novel self-focusing dynamics due to the boundary condition, which do not occur in a bulk medium.

In this paper we are interested in the influence of this boundary condition on the standing wave solutions. A standing wave solution is a solution of equation (1) with the form

$$ u(t, x) = e^{i\omega t} Q_\omega(x), $$

where $\omega \in \mathbb{R}$ and $Q_\omega$ is a solution of the stationary problem:

$$ \begin{cases} 
-\Delta Q + \omega Q - |Q|^{p-1}Q = 0, & x \in B_1, \\
Q = 0, & x \in \partial B_1. 
\end{cases} $$

(2)

Few attempts to understand the influence of the boundary have been made except for Fibich and Merle [20]. The authors [20] found that the boundary has a stabilizing effect by showing that in the case where $N = 2$ and $p = 3$, there exist stable least-energy solutions although all least-energy solutions in $\mathbb{R}^N$ are unstable. In this paper, we investigate how the boundary has an effect on the stability, and will find some stabilizing effects for more general power $p$ and general dimension $N$.

Moreover, we discuss the stability of the excited states, which have never been studied well so far. Surprisingly, we will see that there are some cases where the excited states are stable due to the boundary condition, while the excited states in $\mathbb{R}^N$ are believed to be unstable in any case (see, e.g., [51, sections 3 and 4]).

We introduce several notations. Let $\Omega$ be a domain in $\mathbb{R}^N$ and $K$ represents a field. In this paper we will take $K$ for $\mathbb{R}$ or $\mathbb{C}$. We denote by $L^2(\Omega, K)$ the space of all $K$-valued $L^2$-functions on $\Omega$ with the scalar product

$$ (u, v)_{L^2(\Omega)} = \text{Re} \int_\Omega u \bar{v} \, dx $$

for $u, v \in L^2(\Omega, K)$. We denote by $C^\infty_0(\Omega, K)$ the function space of $K$-valued infinitely many differentiable functions with compact support in $\Omega$. We define a function space $H^1_0(\Omega, K)$ as a closure of $C^\infty_0(\Omega, K)$ with the norm

$$ \| u \|_{H^1(\Omega)} = \left\{ \| \nabla u \|_{L^2(\Omega)}^2 + \| u \|_{L^2(\Omega)}^2 \right\}^{1/2}. $$

The function space $H^1_0(\Omega, K)$ is equipped with inner product

$$ (u, v)_{H^1(\Omega)} = (\nabla u, \nabla v)_{L^2(\Omega)} + (u, v)_{L^2(\Omega)}.$$

We denote by $H^{-1}(\Omega, K)$ the dual space of $H^1_0(\Omega, K)$. The duality pairing between $H^{-1}(\Omega, K)$ and $H^1_0(\Omega, K)$ is denoted by $(u, v)$ for $u \in H^{-1}(\Omega, K)$ and $v \in H^1(\Omega, K)$. If there is no fear of confusion, we write $H^1_0(\Omega)$ or $L^2(\Omega)$, for short. For the sake of definiteness, here we consider only the case of $\Omega = B_1 = \{ x \in \mathbb{R}^N \mid |x| \leq 1 \}$. We may generalize some parts of our results to more general domains (see remark 6).
Note that if we define a functional \( S_\omega \in C^2(H^1_0(B_1, \mathbb{C}), \mathbb{R}) \) by

\[
S_\omega(u) = \frac{1}{2} \| \nabla u \|_{L^2(B_1)}^2 + \frac{\omega}{2} \| u \|_{L^2(B_1)}^2 - \frac{1}{p+1} \| u \|_{L^{p+1}(B_1)}^{p+1},
\]

then we see that \( S'_\omega(Q_\omega) = 0 \) if and only if \( Q_\omega \in H^1_0(B_1, \mathbb{C}) \) is a weak solution of equation (2). We say that \( Q_\omega \in H^1_0(B_1, \mathbb{C}) \) is a least-energy solution of equation (2) if \( Q_\omega \) is a weak solution of equation (2) which minimizes the functional \( S_\omega \) for all non-trivial solution of equation (2). It is well-known that the least-energy solution exists for \( \omega > -\lambda_1 \) in the case where \( 1 < p < 2^* - 1 \) (see, e.g., Willem [54, theorem 4.2]), where \( \lambda_1 > 0 \) is the first eigenvalue of the operator \(-\Delta\) in \( B_1 \). Moreover, in the case where \( p = 2^* - 1 \), Brezis and Nirenberg [8, theorems 1.1 and 1.2] showed that the least-energy solution exists for \( \omega \in (-\lambda_1, 0) \) when \( N \geq 4 \) and for \( \omega \in (-\lambda_1, -\lambda_1/4) \) when \( N = 3 \). The least-energy solution is unique (up to phase), radially symmetric and strictly decreasing in \( r = |x| \) (see [24, theorem 1] for the symmetry and [36, theorem 2] and [56, theorems 1 and 2] for the uniqueness).

Throughout this paper, we will assume the local well-posedness of the Cauchy problem for equation (1) in the energy space \( H^1_0(B_1, \mathbb{C}) \).

**Assumption (A).** For any \( u_0 \in H^1_0(B_1, \mathbb{C}) \), there exist \( T = T(\|u_0\|_{H^1}) > 0 \) and a unique solution \( u \in C([0, T), H^1_0(B_1, \mathbb{C})) \) of equation (1) with \( u(0) = u_0 \). Moreover, the solution satisfies the following conservation laws:

\[
E(u(t)) = E(u_0), \quad \|u(t)\|_{L^2(B_1)}^2 = \|u_0\|_{L^2(B_1)}^2
\]

for \( t \in [0, T) \), where

\[
E(v) = \frac{1}{2} \| \nabla v \|_{L^2(B_1)}^2 - \frac{1}{p+1} \| v \|_{L^{p+1}(B_1)}^{p+1}.
\]

**Remark 1.**

(i) Note that in the case \( N = 1 \), we can verify assumption (A) holds (see [12, theorem 3.5.1]).

Moreover, in [41, 42], it is proved that assumption (A) holds in the case where \( N = 2 \) and \( 1 < p \leq 3 \). Recently assumption (A) was established in [7] for the case of \( N = 3 \) under a certain restriction for the power of nonlinearities (see also [3] and [12, theorem 3.6.1]).

(ii) If assumption (A) holds, then it follows from the conservation laws and the Gagliardo–Nirenberg inequality that for any \( u_0 \in H^1_0(B_1, \mathbb{C}) \), the solution \( u \) of equation (1) with \( u(0) = u_0 \) exists globally when \( 1 < p < 1 + 4/N \) (see, e.g., [26, lemma 3.2]).

The stability of standing waves is defined as follows.

**Definition 2.** We say that the standing waves \( e^{i\omega t} Q_\omega \) of equation (1) are orbitally stable in \( H^1_0(B_1) \) if for any \( \epsilon > 0 \), there exists \( \delta > 0 \) such that the following statement holds: if \( u_0 \in H^1_0(B_1, \mathbb{C}) \) and \( \| u_0 - Q_\omega \|_{H^1} < \delta \), then the solution \( u(t) \) of equation (1) with \( u(0) = u_0 \) satisfies

\[
\inf_{t \in \mathbb{R}} \| u(t) - e^{i\omega t} Q_\omega \|_{H^1} < \epsilon
\]

for all \( t \geq 0 \). Otherwise, \( e^{i\omega t} Q_\omega \) is said to be orbitally unstable. In particular, \( e^{i\omega t} Q_\omega \) is said to be strongly unstable if there exists an initial datum \( u_0 \in H^1_0(B_1, \mathbb{C}) \) and \( \| u_0 - Q_\omega \|_{H^1} < \delta \), then the solution \( u(t) \) of equation (1) with \( u(0) = u_0 \) blows up in finite time.

It is well-known that the least-energy solutions in \( \mathbb{R}^N \) are stable when \( 1 < p < 1 + 4/N \) and strongly unstable when \( 1 + 4/N \leq p < 2^* - 1 \) (see [11] for the stability and [5, 52] for the strong instability).

Concerning the stability of the least-energy solution on the unit ball in \( \mathbb{R}^N \), we obtain the following results.
Theorem 3. Let \( N \geq 1 \) and \( Q_\omega \) be the least-energy solution of equation (2). Suppose that assumption (A) holds.

(i) Let \( 1 < p \leq 1 + 4/N \). There exists \( \omega_1 > 0 \) such that standing wave \( e^{i\omega t} Q_\omega \) is orbitally stable in \( H^1_0(B_1) \) for any \( \omega \in (\omega_1, \infty) \).

(ii) Let \( 1 + 4/N < p < 2^* - 1 \). There exists \( \omega_2 > 0 \) such that standing wave \( e^{i\omega t} Q_\omega \) is orbitally unstable for any \( \omega \in (\omega_2, \infty) \).

(iii) Let \( 1 < p \leq 2^* - 1 \) if \( N \geq 3 \) and \( 1 < p < \infty \) if \( N = 1 \) or 2. There exists \( \omega_0 > 0 \) such that standing wave \( e^{i\omega t} Q_\omega \) is orbitally stable in \( H^1_0(B_1) \) for any \( \omega \in (-\lambda_1, -\lambda_1 + \epsilon_0) \).

The case where \( N = 2 \) and \( p = 3 \) has already been investigated by Fibich and Merle [20, lemma 10]. The authors proved that the least-energy solution is orbitally stable for sufficiently large \( \omega > 0 \), or \( \omega \) sufficiently close to \( -\lambda_1 \). We generalize this result to more general nonlinearity powers, adding some detailed arguments. We also remove the condition on the frequency \( \omega \) and prove the orbital stability for all \( \omega \) when \( 1 < p \leq 1 + 4/N \), in case of \( N = 1 \). More precisely, we have the following theorem.

Theorem 4. Let \( N = 1, 1 < p \leq 5 \) and \( Q_\omega \) be the least-energy solution of equation (2). Then, the standing wave \( e^{i\omega t} Q_\omega \) is orbitally stable for all \( \omega \in (-\lambda_1, \infty) \).

To prove theorems 3 and 4, we invoke the general theory in [28, theorem 3], which gives a sufficient condition for the orbital stability of solitary waves to the abstract Hamiltonian system. If we apply it to our equation, linearizing around the least-energy solution \( Q_\omega \), that sufficient condition may be interpreted as follows.

Proposition 5. Assume that \( Q_\omega \in H^1_0(B_1, \mathbb{C}) \) is the least-energy solution of equation (2). Suppose that assumption (A) holds. Assume that

(i) the positive spectrum of \( S_\omega''(Q_\omega) \) is bounded away from zero;
(ii) the kernel is spanned by \( iQ_\omega \), i.e. \( \text{Ker} S_\omega''(Q_\omega) = \text{Span}[iQ_\omega] \);
(iii) the operator \( S_\omega''(Q_\omega) \) has only one negative simple eigenvalue.

Then if \( \partial_\omega \|Q_\omega\|^2_{L^2} > 0 \) (respectively \( < 0 \)), the standing wave \( e^{i\omega t} Q_\omega \) is orbitally stable in \( H^1_0 \) (respectively unstable).}

All spectra of the operator \( S_\omega''(Q_\omega) \) are discrete, and a modification of the proof in [31, theorem 0.2] shows that the linearized operator \( S_\omega''(Q_\omega) \) satisfies the properties (ii) and (iii).

Accordingly, in order to show the orbital stability, it is enough to investigate the slope condition, i.e. the sign of \( \partial_\omega \|Q_\omega\|^2_{L^2} = 2 \int Q_\omega v_\omega dx \), where \( v_\omega = \partial_\omega Q_\omega \). Note that the differentiability of \( Q_\omega \) with respect to \( \omega \) is ensured since \( \omega \in (-\lambda_1, \infty) \mapsto Q_\omega \in H^1_0(B_1) \) is \( C^1 \) for any \( p > 1 \). (See [45, theorem 18] for the proof).

We see that \( v_\omega \in H^1_0(B_1) \) satisfies the following equation:

\[
\begin{cases}
-\Delta v + \omega v - p Q_\omega^{p-1} v = -Q_\omega, & x \in B_1, \\
v = 0 & x \in \partial B_1.
\end{cases}
\]

(3)

In the case of equation (1) in \( \mathbb{R}^N \), the slope condition is explicitly written as a function of \( \omega \), making use of the scale invariance of equation (1). However, we cannot expect such a scaling property in the present case, and thus we try to obtain the information by a scaling limit in \( \omega \) of solutions. In fact, this idea was already used in the works by Esteban and Strauss [18], the first author and Ohta [22, 23], Fibich and Merle [20] to analyse the orbital stability for some equations that do not possess the scaling invariance.
We rescale the solutions as follows.

\[
Q_{\omega}(x) = \omega^{\frac{4}{p-2}} \tilde{Q}_{\omega}(\sqrt{\omega}x), \quad v_{\omega}(x) = \omega^{\frac{2}{p-1}} \tilde{v}_{\omega}(\sqrt{\omega}x), \quad \omega > 0. \tag{4}
\]

Then \(\tilde{Q}_{\omega}\) and \(\tilde{v}_{\omega}\) satisfy the following equations respectively:

\[
- \Delta \tilde{Q} + \tilde{Q} - |\tilde{Q}|^{p-1} \tilde{Q} = 0, \quad x \in B_{\sqrt{\omega}}. \tag{5}
\]

\[
- \Delta \tilde{v} + \tilde{v} - p|\tilde{Q}|^{p-1} \tilde{v} = - \tilde{Q}, \quad x \in B_{\sqrt{\omega}}. \tag{6}
\]

We extend both functions \(\tilde{Q}_{\omega}\) and \(\tilde{v}_{\omega}\) to the entire space by defining \(\tilde{Q}_{\omega}(x) = \tilde{v}_{\omega}(x) = 0\) for \(x \in \mathbb{R}^N \setminus B_{\sqrt{\omega}}\). We will show that \(\tilde{Q}_{\omega}\) and \(\tilde{v}_{\omega}\) converge to \(R_1\) and \(R_1/(p-1) + x \cdot \nabla R_1/2\) in \(H^1(\mathbb{R}^N)\) as \(\omega\) goes to infinity, respectively. Here \(R_1 \in H^1(\mathbb{R}^N)\) is the unique positive radial solution of

\[
- \Delta R + R - |R|^{p-1} R = 0, \quad x \in \mathbb{R}^N. \tag{7}
\]

See [6, theorem 1], and [47, theorem 1] for the existence and [35, section 5] for the uniqueness. Moreover, we see that both functions \(Q_{\omega}/\|Q_{\omega}\|_{L^2}\) and \(v_{\omega}/\|v_{\omega}\|_{L^2}\) converge to \(\chi_1 \in H^1_0(B_1)\) as \(\omega\) goes to \(-\lambda_1\) in \(H^1(B_1)\). Here \(\chi_1 \in H^1_0(B_1)\) is the first eigenfunction of the operator \(-\Delta\) in \(B_1\) with \(\|\chi_1\|_{L^2} = 1\).

In fact, as in [22, 23], the asymptotics of \(Q_{\omega}\) (or \(\tilde{Q}_{\omega}\)) with respect to \(\omega\) is enough for the study of the case \(p \neq 1 + 4/N\). However, in the case where \(p = 1 + 4/N\), we need more details about the asymptotics. With this aim, we use the second approximation \(v_{\omega}\). As a side effect, we can give a simpler proof than those of [22, 23] also for the case of \(p \neq 1 + 4/N\).

We briefly mention how to check the slope condition if \(p = 1 + 4/N\). Note that by the Pohozaev identity, we have \(E(\partial_{\omega}Q_{\omega}, Q_{\omega}(1)) = 1/4\) when \(p = 1 + 4/N\), as was seen in [20]. Moreover, in this paper, we use the relation \(\omega \partial_{\omega} \|Q_{\omega}\|_{L^2(B_1)}^2 = -2 \partial_{\omega} E(\partial_{\omega}Q_{\omega})\), and we obtain

\[
\omega \partial_{\omega} \|Q_{\omega}\|_{L^2(B_1)}^2 = - \partial_{\omega} Q_{\omega}(1) \partial_{\omega} v_{\omega}(1). \tag{8}
\]

Since \(\partial_{\omega} Q_{\omega}(1) < 0\), our interest is attracted towards the sign of \(\partial_{\omega} v_{\omega}(1)\). If \(N = 1\), we can analyse the behaviour of \(v_{\omega}\) by an ODE approach, more precisely than the case of \(N \geq 2\), which yields theorem 4.

**Remark 6.** Theorem 3 would be applicable for more general bounded domains \(\Omega\), so long as it is smooth and star-shaped.

Next, we turn our attention to the excited states in one space dimension. For \(k \in \mathbb{N} \setminus \{0\}\) let \(\lambda_k \in \mathbb{R}\) and \(\chi_k \in H^1_0((-1, 1), \mathbb{R})\) be the \(k\)th eigenvalue and \(k\)th eigenfunction of the operator 

\[-d^2/dx^2 \text{ in } (-1, 1).\]

Then we can write \(\chi_k \in \mathbb{R}\) and \(\chi_k \in H^1_0((-1, 1), \mathbb{R})\) explicitly as follows

\[\lambda_k = (k\pi)^2/4 \text{ and } \chi_k = \sin(k\pi(x - 1)/2).\]

Moreover, it is known that the eigenvalue \(\lambda_k\) is simple for each \(k \in \mathbb{N} \setminus \{0\}\). Therefore, we can apply the local bifurcation theorem by Crandall and Rabinowitz [14, theorem 1.7] and we see that each \(k \in \mathbb{N} \setminus \{0\}\) there exists \(s_0 > 0\) and the unique solution \(Q_{k,\omega}(s) \in H^1((-1, 1), \mathbb{R})\) of equation (2) such that

\[Q_{k,\omega}(s) = s\chi_k + s\nu_k(s), \quad \omega(s) = -\lambda_k + \lambda_k(s) \tag{9}\]

for \(s \in (-s_0, s_0)\), where \(\lambda_k(s) \in C((-s_0, s_0))\) and \(v_k \in C((-s_0, s_0), H^1_0(-1, 1))\) with \(\lambda_k(0) = \nu_k(0) = 0\). Moreover, we infer from (12) below that \(Q_{k,\omega}\) is an even function if \(k\) is odd and an odd function if \(k\) is even for all \(\omega > -\lambda_k\). Note that the least-energy solution \(Q_{k,\omega}\) has been denoted by \(Q_{\omega}\) in the above theorems.

The linearized equation of (1) around the \(k\)th standing wave \(e^{i\omega t}Q_{k,\omega}(s)\) is written as follows.

\[
\frac{dy}{dt} = JL_{k,\omega}(s)y \quad \text{in } H^{-1},
\]
where

\[
J = -i \begin{pmatrix} \text{Re } u & \text{Im } u \\ \text{Im } u & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \text{Re } u & \text{Im } u \\ \text{Im } u & -1 \end{pmatrix}
\]

\[
\mathcal{L}_{k,\omega(s)} = S_{\omega(s)}(Q_{k,\omega(s)}) = \begin{pmatrix} L^+_{k,\omega(s)} & 0 \\ 0 & L^-_{k,\omega(s)} \end{pmatrix}.
\]

Here the operators \( L^\pm_{k,\omega(s)} : L^2((-1, 1), \mathbb{R}) \rightarrow L^2((-1, 1), \mathbb{R}) \) are defined by

\[
L^+_{k,\omega(s)} = -\frac{d^2}{dx^2} + \omega(s) - |Q_{k,\omega(s)}|^{p-1},
\]

\[
L^-_{k,\omega(s)} = -\frac{d^2}{dx^2} + \omega(s) - |Q_{k,\omega(s)}|^{p-1}
\]

with the domain \( D(L^\pm_{k,\omega(s)}) = H^2((-1, 1), \mathbb{R}) \cap H^1_0((-1, 1), \mathbb{R}) \). We show the following:

**Theorem 7.** Let \( N = 1, p > 1 \) and \( Q_{k,\omega(s)} \in H^1_0((-1, 1)) \) be the solution of equation (2) given by (9) with \( k = 2, 3, 4 \). Then there exists \( s_1 > 0 \) such that if \( |s| < s_1 \), then the standing wave \( e^{is\omega(s)}Q_{k,\omega(s)} \) is linearly stable, that is, \( \sigma(J \mathcal{L}_{k,\omega(s)}) \subset i \mathbb{R} \).

**Remark 8.** If there exists \( \lambda \in \sigma(J \mathcal{L}_{k,\omega(s)}) \) such that \( \text{Re } \lambda \neq 0 \), then we see that the standing wave \( e^{is\omega(s)}Q_{k,\omega(s)} \) is orbitally unstable by the argument in [27, appendix]. When \( k \geq 5 \), it would be possible that some linearly unstable bound states exist. This number \( k \geq 5 \) is derived from the appearance of some eigenvalues with high geometric multiplicity (see remark 24).

On the other hand, in the case of the cubic nonlinear Schrödinger equation with a harmonic potential in \( \mathbb{R} \) where we have a ‘similar’ spectrum distribution, the excited states for \( k = 3 \) and 4 are already observed to be linearly unstable in [55]. This difference arises in the fact that the eigenvalue distribution is equidistant in the case of harmonic potential, and as a result an eigenvalue with high geometric multiplicity takes place for \( k \geq 2 \). The stability issue for \( k = 2 \), benefiting from an additional symmetry of harmonic potential, is also discussed in [33].

Note that the linear stability (theorem 7 in the case of \( k = 2, 3, 4 \)) does not imply the nonlinear (or) stability. However, we can see rigorously that the \( k \)th excited state \( e^{is\omega(s)}Q_{k,\omega} \) is orbitally stable if we restrict ourselves to the perturbation of the same symmetry as \( Q_{k,\omega} \) for any \( k \in \mathbb{N} \), under certain conditions on \( \omega \) and \( p \). A similar result but only for the first excited state is obtained by Kurth [34] for the nonlinear Schrödinger equation with the harmonic potential.

**Proposition 9.** Let \( N = 1 \). Then the \( k \)th \( (k \geq 2) \) bound states \( Q_{k,\omega}(x) \) of equation (2) can be written as follows. For \( \omega > -\lambda_k \),

\[
Q_{k,\omega}(x) = \sum_{n=1}^{k} (-1)^{n-1} k^{\frac{2}{p}} Q_{\frac{k}{p}} \left( k \left( x + 1 + \frac{2n}{k} \right) \right).
\]

Here, we extend the least-energy solution \( Q_{\omega}(=Q_{1,\omega}) \) to the line \((-\infty, \infty)\) by defining \( Q_{\omega}(x) = 0 \) for \( x \in (-\infty, -1) \cup (1, \infty) \).

Moreover,

(i) the \( k \)th standing wave \( e^{is\omega(s)}Q_{k,\omega}(x) \) is orbitally stable in \( H^1_{0,k}((-1, 1)) \) for all \( \omega > -\lambda_k \),

\[
(1 \leq p \leq 5) \text{ or } \omega \text{ close to } -\lambda_k \text{ if } p > 5,
\]

where \( H^1_{0,k}((-1, 1)) \) is a closed subspace of \( H^1_0((-1, 1)) \), defined by

\[
H^1_{0,k}((-1, 1)) = \{ v \in H^1_0((-1, 1)) | v \left( -1 + \frac{2n}{k} \right) = 0 \text{ for } n = 1, 2, \ldots, k-1 \}.
\]
(ii) Let $N=1$ and $p>5$. Then the $k$th ($k \geq 1$) standing wave $e^{i\omega t}Q_{k,\omega}$ is orbitally unstable for sufficiently large $\omega>0$.

As a result of proposition 9 we can say that the stability of the $k$th excited state is due to the Dirichlet boundary condition compared with $\mathbb{R}^N$-case even if it is limited to the $H^{1}_{0,\omega}$-symmetric perturbation. However, the stability of the excited states under non-$H^{1}_{0,\omega}$-symmetric perturbation is not completely covered yet. Thus, using a numerical simulation, we examine the orbital stability of the excited states $Q_{2,\omega}$ and $Q_{3,\omega}$ more closely.

Our numerical approach is similar to those in [21, 37]. We numerically observe the profile of the solution of equation (1) with the following initial conditions: for each $k=2,3$,

$$u_{0,p}(x) = (1 + \delta_p)Q_{k,\omega}(x),$$

$$u_{0,c}(x) = Q_{k,\omega}(x - \delta_c \theta(x)),$$

where $\delta_p, \delta_c > 0$ and $\theta \in C^\infty(-1,1)$ is defined by

$$\theta(x) = \begin{cases} 
\exp\left(-\frac{1}{1 - |x|^2}\right) & \text{for } |x| < 1, \\
0 & \text{for } |x| = 1.
\end{cases}$$

The positive constant $\delta_p$ is used to compute the uniform multiplicative perturbation (14) of the excited states $Q_{k,\omega}$, perturbs the power, that is, $L^2$-norm of the excited state. It preserves its symmetry with respect to $x=0$. On the other hand, the positive constant $\delta_c$ perturbs the centre of the standing waves while it preserves the $L^2$-norm. This analysis provides not only the stability property but also the information on the character of the dynamics in the stable and unstable cases.

Here, we state briefly what our simulations suggest.

**Observation.** Let $N=1$ and $k=2$ or $3$.

(i) Let $p>5$. There exists a unique $\tilde{\omega}_{\lambda,k} > -\lambda_k$ such that the standing wave $e^{i\omega t}Q_{k,\omega}$ is stable for any $\omega \in (-\lambda_k, \tilde{\omega}_{\lambda,k})$ and strongly unstable for any $\omega \in (\tilde{\omega}_{\lambda,k}, \infty)$ under the perturbation (14).

(ii) Let $1<p<5$. There exist a unique $\omega_{\lambda,k} \in (-\lambda_k, \infty)$ and a $\omega_{\nu,k} \in (\omega_{\lambda,k}, \infty]$ such that the standing wave $e^{i\omega t}Q_{k,\omega}$ is orbitally stable for any $\omega \in (-\lambda_k, \omega_{\lambda,k})$ and orbitally unstable for any $\omega \in (\omega_{\lambda,k}, \omega_{\nu,k})$ under the perturbation (15).

(iii) Let $p \geq 5$. There exists a unique $\hat{\omega}_{\lambda,k} > -\lambda_k$ such that the standing wave $e^{i\omega t}Q_{k,\omega}$ is orbitally stable for any $\omega \in (-\lambda_k, \hat{\omega}_{\lambda,k})$ and strongly unstable for any $\omega \in (\hat{\omega}_{\lambda,k}, \infty)$ under the perturbation (15).

It follows from observations (i) and (iii) that the nature of the instability in proposition 9 (ii) would be in the sense of blow-up. Observation (ii) does not give any information about the case of large frequencies $\omega>0$. This is because our simulation does not indicate clearly whether $Q_{k,\omega}$ remains orbitally unstable or not for very large $\omega$ (for the details see section 7.2.2). According to observation (iii), even if $p \geq 5$ and under the non-symmetric perturbation (15), the excited states are stable for $\omega$ sufficiently close to $-\lambda_k$.

**Remark 10.** The fact that there are some cases where the excited states are stable was weakly recognized in section 7.1 of Fibich and Merle [20]. Precisely, letting $N_c = \|R_1\|_{L^2}^2$, the authors gave a remark that the excited states whose power is sufficiently below $N_c$ appear to be numerically stable for quite a long time, and those whose power is above $N_c$ are strongly unstable. We give more precise observations and in the case where $p=5$ and $N=1$, we find that there exists the unique threshold $\hat{\omega}_{\lambda,k}$ which separate the stable excited states from the
unstable excited states. Our numerical calculation indicates that $\hat{\omega}_{1,2} = -8.2$ is different from the frequency ($=-4.8$) of the excited state having the squared $L^2$ norm whose value equals to $N_c$. However, we have to be careful with the situation such that we are in one-dimensional case, and the threshold $\hat{\omega}_{k,k}$ arises only by the non-symmetric perturbation (15) unlikely to the case of Fibich and Merle [20].

This paper is organized as follows. Rigorous discussions will be found from section 2 to section 6. Section 7 is dedicated to the observations by numerical simulations: in section 2, we study the asymptotic behaviour of the least-energy solution $Q_\omega$ and its derivative $v_\omega$ with respect to $\omega$. Using the convergence properties of $Q_\omega$ and $v_\omega$ proved in section 2, we investigate the slope condition and show the stability of the least-energy solution for all $\omega$ which are very large or sufficiently close to $-\lambda_1$ in general dimension $N \geq 1$ in section 3. This stability result will be extended for any $\omega$ for the case $N = 1$ in section 4. Section 5 is devoted to the theoretical study on the linear stability of some excited states. In section 6, we show rigorously that the $k$-excited state $e^{\omega t}Q_{k,\omega}$ is orbitally stable if we restrict ourselves to $H^1_0$-perturbation. The (nonlinear) stability of the excited states is observed by numerical simulations in section 7. Numerically, we study the first and second excited states in one space dimension in sections 7.2 and 7.3. Following [21, 37], the profiles of the solutions are investigated in each case classifying the instability type and the dynamics of the stability.

2. Convergence properties

In this section, we show several convergence properties of $\tilde{Q}_\omega$ and $\tilde{v}_\omega$. Recall that we consider only the least-energy solution $Q_\omega$ in this section, and $\tilde{Q}_\omega$ and $\tilde{v}_\omega$ are defined in (4).

**Proposition 11.** Let $N \geq 1$ and $1 < p < 2^* - 1$. Then, we obtain the following convergence properties.

(i) $\tilde{Q}_\omega$ converges to $R_1$ strongly in $H^1(\mathbb{R}^N)$ as $\omega$ goes to $\infty$,
(ii) $\tilde{v}_\omega$ converges to $R_1/\lambda_1 + x \cdot \nabla R_1/2$ strongly in $H^1(\mathbb{R}^N)$ as $\omega$ goes to $\infty$.

**Proposition 12.** Let $N \geq 1$, $1 < p \leq 2^* - 1$ if $N \geq 3$ and $1 < p < \infty$ if $N = 1$ or 2. Then, we obtain the following convergence properties for $\omega$ close to $-\lambda_1$.

(i) $Q_\omega/\|Q_\omega\|_{L^2}$ converges to $\chi_1$ strongly in $H^1_0(B_1)$ as $\omega$ goes to $-\lambda_1$,
(ii) $v_\omega/\|v_\omega\|_{L^2}$ converges to $\chi_1$ strongly in $H^1_0(B_1)$ as $\omega$ goes to $-\lambda_1$,

where $\chi_1 \in H^1_0(B_1)$ is the first eigenfunction of the operator $-\Delta$ in $B_1$.

The idea of the proof of proposition 11 (i) follows the arguments in [17, theorem 4] and [22, lemma 2.1]. However, we repeat the argument for the sake of completeness, adding a slight modification adapted to the case of bounded domains.

**Proof of proposition 11 (i).** Using a standard variational method (see [54, theorem 4.2] for $1 < p < 2^* - 1$ and [8, theorems 1.1 and 1.2] for $p = 2^* - 1$), we see that $Q_\omega \in H^1_0(B_1)$ satisfies the following minimization problem:

$$
\tilde{d}(\omega) = \inf \left\{ \tilde{S}_\omega(u) \mid \tilde{K}_\omega(u) = 0, \ u \in H^1_0(B_1) \right\}
= \inf \left\{ \frac{p-1}{2(p+1)} \|u\|_{L^{p+1}(B_1)}^{p+1} \mid \tilde{K}_\omega(u) \leq 0, \ u \in H^1_0(B_1) \right\},
$$
where
\[ \tilde{S}_\omega(u) = \frac{1}{2} \| \nabla u \|_{L^2(B, \omega)}^2 + \frac{1}{2} \| u \|_{L^2(B, \omega)}^2 - \frac{1}{p + 1} \| u \|_{L^{p+1}(B, \omega)}^{p+1}, \]
\[ \tilde{K}_\omega(u) = \| \nabla u \|_{L^2(B, \omega)}^2 + \| u \|_{L^2(B, \omega)}^2 - \| u \|_{L^{p+1}(B, \omega)}^{p+1}. \]

Moreover, it is well-known that \( R_1 \in H^1(\mathbb{R}^N) \) is the unique minimizer for the following minimization problem:
\[ d_* = \inf \left\{ S_\infty(u) \mid K_\infty(u) \leq 0, \ u \in H^1(\mathbb{R}^N) \right\} \]
\[ = \inf \left\{ \frac{p - 1}{2(p + 1)} \| u \|_{L^{p+1}(\mathbb{R}^N)}^{p+1} \mid K_\infty(u) \leq 0, \ u \in H^1(\mathbb{R}^N) \right\}. \]

where
\[ S_\infty(u) = \frac{1}{2} \| \nabla u \|_{L^2(\mathbb{R}^N)}^2 + \frac{1}{2} \| u \|_{L^2(\mathbb{R}^N)}^2 - \frac{1}{p + 1} \| u \|_{L^{p+1}(\mathbb{R}^N)}^{p+1}, \]
\[ K_\infty(u) = \| \nabla u \|_{L^2(\mathbb{R}^N)}^2 + \| u \|_{L^2(\mathbb{R}^N)}^2 - \| u \|_{L^{p+1}(\mathbb{R}^N)}^{p+1}. \]

We show that \( \lim_{\omega \to \infty} \| \tilde{Q}_\omega \|_{L^{p+1}(\mathbb{R}^N)} = \| R_1 \|_{L^{p+1}(\mathbb{R}^N)}. \) It is easily seen that
\[ K_\infty(\tilde{Q}_\omega) = 0, \] for all \( \omega > 0. \) (16)

Thus, by the variational characterization of \( R_1, \)
\[ \| R_1 \|_{L^{p+1}(\mathbb{R}^N)} \leq \| \tilde{Q}_\omega \|_{L^{p+1}(\mathbb{R}^N)}, \] (17)

Now, we use a cut-off function \( \eta \in C_0^\infty(\mathbb{R}^N) \) such that
\[ \eta(x) = \begin{cases} 1 & |x| \leq 1/2, \\ 0 & |x| \geq 1, \end{cases} \]
and satisfies \(|\nabla \eta(x)| \leq 4\) for all \( x \in \mathbb{R}^N. \) Then we set \( \eta_\omega(x) = \eta(x/\sqrt{\omega}). \) It follows that for any \( \eta > 1, \)
\[ \tilde{K}_\omega(\mu \eta_\omega R_1) = \mu^2 \int_{B, \omega} |\nabla (\eta_\omega R_1)|^2 \, dx + \mu^2 \int_{B, \omega} |\eta_\omega R_1|^2 \, dx - \mu^{p+1} \int_{B, \omega} |\eta_\omega R_1|^{p+1} \, dx \]
\[ \leq \frac{16 \mu^2}{\omega} \| R_1 \|_{L^2(\mathbb{R}^N)}^2 + \frac{8 \mu^2}{\sqrt{\omega}} \| \nabla R_1 \|_{L^2(\mathbb{R}^N)} \| R_1 \|_{L^2(\mathbb{R}^N)} \]
\[ + (\mu^2 - \mu^{p+1}) \| R_1 \|_{L^{p+1}(\mathbb{R}^N)}^{p+1} + \mu^2 \int_{\mathbb{R}^N} (|\eta_\omega|^2 - 1) |\nabla R_1|^2 \, dx \]
\[ + \mu^2 \int_{\mathbb{R}^N} (|\eta_\omega|^2 - 1) |R_1|^2 \, dx + \mu^{p+1} \int_{\mathbb{R}^N} (1 - |\eta_\omega|^{p+1}) |R_1|^{p+1} \, dx. \]

We see that \((\mu^2 - \mu^{p+1}) \| R_1 \|_{L^{p+1}(\mathbb{R}^N)} < 0\) and the other terms vanish if \( \omega \) goes to infinity.
This yields that \( \tilde{K}_\omega(\mu \eta_\omega R_1) < 0\) for sufficiently large \( \omega > 0. \) It follows from the variational characterization of \( \tilde{Q}_\omega, \) that, for any \( \eta > 1, \)
\[ \limsup_{\omega \to \infty} \| \tilde{Q}_\omega \|_{L^{p+1}(\mathbb{R}^N)} \leq \mu^{p+1} \limsup_{\omega \to \infty} \| \eta_\omega R_1 \|_{L^{p+1}(\mathbb{R}^N)} \leq \mu^{p+1} \| R_1 \|_{L^{p+1}(\mathbb{R}^N)}. \]

Using this, together with (17), we have \( \lim_{\omega \to \infty} \| \tilde{Q}_\omega \|_{L^{p+1}(\mathbb{R}^N)} = \| R_1 \|_{L^{p+1}(\mathbb{R}^N)}. \)

On the other hand, we know that \( \lim_{\omega \to \infty} K_\infty(\tilde{Q}_\omega) = 0 \) by (16). Let \( \{ \omega_j \} \subset \mathbb{R}^+ \) be any sequence such that \( |\omega_j| \) goes to \( \infty \) as \( j \) goes to \( \infty. \) Thus, we have
\[ K_\infty(\tilde{Q}_{\omega_j}) = 0, \quad \lim_{j \to \infty} \| \tilde{Q}_{\omega_j} \|_{L^{p+1}(\mathbb{R}^N)} = \| R_1 \|_{L^{p+1}(\mathbb{R}^N)}. \]
This yields that \( \{ \tilde{Q}_{\omega_j} \} \subset H^1(\mathbb{R}^N) \) is a minimizing sequence for \( d_\infty \). Then by a standard variational argument, there exists a subsequence of \( \{ \tilde{Q}_{\omega_j} \} \) (we still denote it by the same letters) such that \( \tilde{Q}_{\omega_j} \) converges to \( R_1 \) strongly in \( H^1(\mathbb{R}^N) \) as \( \omega_j \) goes to \( \infty \). This completes the proof. \( \square \)

Next, we give a proof of proposition 11 (ii). We prepare the following lemma, which will be needed later.

**Lemma 13.** Let \( N \geq 1 \) and \( 1 < p < 2^* - 1 \). There exist constants \( \omega_0 > 0 \) and \( C_0 > 0 \) such that for any \( \omega \in (\omega_0, \infty) \) we have

\[
\| L_{\omega} g \|_{L^2(B_{\sqrt{\omega}})} \geq C_0 \| g \|_{H^1(B_{\sqrt{\omega}})}
\]  

(18) for all \( g \in H^2_{\text{rad}}(B_{\sqrt{\omega}}) \), where the operator \( L_{\omega} : L^2(B_{\sqrt{\omega}}) \to L^2(B_{\sqrt{\omega}}) \) is defined by \( L_{\omega} := -\Delta + 1 - p\tilde{Q}_\omega^{p-1} \) with the domain \( D(L_{\omega}) = H^2(\Omega) \). Here the constant \( C_0 \) is independent of \( \omega \).

**Proof.** We show this lemma by contradiction. Suppose that (18) would not hold. Then, there exist sequences \( \{ \omega_j \} \subset \mathbb{R}^n \) and \( \{ g_j \} \subset H^2_{\text{rad}}(B_{\sqrt{\omega_j}}) \) such that \( \omega_j \to \infty \), and

\[
\| g_j \|_{H^1(B_{\sqrt{\omega_j}})} = 1, \quad \text{and} \quad \| L_{\omega_j} g_j \|_{L^2(B_{\sqrt{\omega_j}})} \to 0
\]  

(19) as \( j \) goes to infinity. We extend \( g_j \in H^2_{\text{rad}}(B_{\sqrt{\omega_j}}) \) to the entire space by defining \( g_j(x) = 0 \) for \( x \in \mathbb{R}^N \setminus B_{\sqrt{\omega_j}} \). Since \( \{ g_j \} \subset H^2_{\text{rad}}(\mathbb{R}^N) \) is bounded, there exist a subsequence of \( \{ g_j \} \subset H^1_{\text{rad}}(\mathbb{R}^N) \) and \( g_\infty \in H^1_{\text{rad}}(\mathbb{R}^N) \) such that \( g_j \) converges to \( g_\infty \) weakly in \( H^1(\mathbb{R}^N) \) as \( j \to \infty \). Here, we denote the operator \(-\Delta + 1 - pR_1^{p-1} \) by \( L_\infty \). This operator \( L_\infty \) is a self-adjoint operator from \( L^2(\mathbb{R}^N) \) to \( L^2(\mathbb{R}^N) \) with the domain \( D(L_\infty) = H^2(\mathbb{R}^N) \). For any \( \varphi \in C_0^\infty(\mathbb{R}^N) \), we have

\[
\langle L_\infty g_\infty, \varphi \rangle + p \int_{\mathbb{R}^N} (\tilde{Q}^{p-1}_\omega - R_1^{p-1}) g_j \varphi \, dx \quad = \quad \lim_{j \to \infty} \langle L_{\omega_j} g_j, \varphi \rangle = 0
\]  

(20)

from proposition 11 (i). This yields, using the elliptic bootstrap argument, that \( g_\infty \in \text{Ker} \ L_\infty \vert_{H^1_{\text{rad}}(\mathbb{R}^N)} \). Hence it is seen that \( g_\infty = 0 \) since \( \text{Ker} \ L_\infty \vert_{H^1\cap C_0^\infty(\mathbb{R}^N)} = [0] \) (see [38, lemma 4.2 (i)] and [53, proposition 2.8]). On the other hand, we have

\[
1 = \| g_j \|_{H^1(\mathbb{R}^N)}^2 = \langle (-\Delta + 1) g_j, g_j \rangle \leq \langle L_{\omega_j} g_j, g_j \rangle + p \int_{\mathbb{R}^N} (\tilde{Q}^{p-1}_\omega - R_1^{p-1}) g_j \, dx
\]  

(21)

Since \( \tilde{Q}_{\omega_j} \) converges to \( R_1 \) strongly in \( H^1(\mathbb{R}^N) \) and \( \tilde{g}_j \) converges to 0 weakly in \( H^1(\mathbb{R}^N) \) as \( j \) goes to \( \infty \), we have \( \lim_{j \to \infty} \int_{\mathbb{R}^N} (\tilde{Q}^{p-1}_\omega - R_1^{p-1}) g_j \, dx = 0 \). Therefore, the right-hand side of (21) tends to zero when \( j \) goes to \( \infty \) because of (19), which is absurd. Thus, (18) holds. \( \square \)

We now give a proof for proposition 11 (ii).

**Proof of proposition 11 (ii).** It follows from equation (6), proposition 11 (i) and lemma 13 that there exists a constant \( C_0 > 0 \) such that

\[
C_0 \| \tilde{v}_\omega \|_{H^1(\mathbb{R}^N)} \leq \| \tilde{L}_{\omega} \tilde{v}_\omega \|_{L^2(\mathbb{R}^N)} \leq \| \tilde{Q}_{\omega} \|_{L^2(\mathbb{R}^N)} \leq 2 \| R_1 \|_{L^2(\mathbb{R}^N)}
\]

for sufficiently large \( \omega > 0 \). Let \( \{ \omega_j \} \subset \mathbb{R}^n \) such that \( \omega_j \to \infty \) as \( j \) goes to \( \infty \). Since \( \{ \tilde{v}_{\omega_j} \} \) is bounded, there exist a subsequence of \( \{ \tilde{v}_{\omega_j} \} \) (we still denote it by the same letters) and \( V \in H^1(\mathbb{R}^N) \) such that \( \tilde{v}_{\omega_j} \) converges to \( V \) weakly in \( H^1(\mathbb{R}^N) \) as \( j \) goes to \( \infty \). Suppose that
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$V = 0$. Let $\varphi \in C_0^\infty(\mathbb{R}^N)$ be a real-valued, positive function on $\mathbb{R}^N$. Multiplying equation (3) by $\varphi$ and integrating the resulting equation, we have

$$\int_{\mathbb{R}^N} \nabla \tilde{v}_{w_j} \cdot \nabla \varphi \, dx + \int_{\mathbb{R}^N} \tilde{v}_{w_j} \varphi \, dx - p \int_{\mathbb{R}^N} \tilde{Q}_{w_j}^{p-1} \tilde{v}_{w_j} \varphi \, dx = - \int_{\mathbb{R}^N} \tilde{Q}_{w_j} \varphi \, dx.$$  

By proposition 11 (i), we see that $- \int_{\mathbb{R}^N} \tilde{Q}_{w_j} \varphi \, dx \rightarrow - \int_{\mathbb{R}^N} R_1 \varphi \, dx < 0$ as $j \rightarrow \infty$. On the other hand, since $V = 0$, we have

$$\int_{\mathbb{R}^N} \nabla \tilde{v}_{w_j} \cdot \nabla \varphi \, dx + \int_{\mathbb{R}^N} \tilde{v}_{w_j} \varphi \, dx - p \int_{\mathbb{R}^N} \tilde{Q}_{w_j}^{p-1} \tilde{v}_{w_j} \varphi \, dx \rightarrow 0$$

as $j \rightarrow \infty$, which is a contradiction. Hence, $V \neq 0$. Then $V \in H^1(\mathbb{R}^N)$ is a non-trivial solution of

$$L_\infty V = -R_1,$$  

where $L_\infty = -\Delta + 1 - p R_1^{p-1}$. Since Ker $L_\infty|_{H^2_{rad}(\mathbb{R}^N)} = \{0\}$, the solution of equation (22) is unique in $H^2_{rad}(\mathbb{R}^N)$, and it is not difficult to check that $R_1/(p-1) + x \cdot \nabla R_1/2$ satisfies equation (22). Thus, we find that $V = R_1/(p-1) + x \cdot \nabla R_1/2$. Moreover, we have, multiplying equation (6) by $\tilde{v}_{w_j}$, and letting $j$ go to $\infty$,

$$\|\tilde{v}_{w_j}\|_{H^1(\mathbb{R}^N)} = p \int_{\mathbb{R}^N} \tilde{Q}_{w_j}^{p-1} |\tilde{v}_{w_j}|^2 \, dx - \int_{\mathbb{R}^N} \tilde{Q}_{w_j} \tilde{v}_{w_j} \, dx \rightarrow p \int_{\mathbb{R}^N} R_1^{p-1} |V|^2 \, dx$$

by (22). This yields that $\tilde{v}_{w_j}$ converges to $V = R_1/(p-1) + x \cdot \nabla R_1/2$ strongly in $H^1(\mathbb{R}^N)$ as $j$ goes to $\infty$. This completes the proof.

Next, we consider the case where the frequency $\omega$ is sufficiently close to $-\lambda_1$. Using a similar argument in [23, lemma 4.3], we obtain the following lemma.

**Lemma 14.** Let $N \geq 1$, $1 < p \leq 2^* - 1$ if $N \geq 3$ and $1 < p < \infty$ if $N = 1$ or $2$. Let $Q_\omega \in H^1_0(B_1)$ be the least-energy solution of equation (2). Then the following assertion holds:

(i) $\|Q_\omega\|_{L^{p+1}(B_1)} \leq (\lambda_1 + \omega)^{\frac{1}{p+1}} \|\chi_1\|_{L^{p+1}(B_1)}$ for all $\omega \in (-\lambda_1, -\lambda_1/4)$,

(ii) $\|\nabla Q_\omega\|_{L^2(B_1)}^2 \leq (\lambda_1 + \omega)^{\frac{p+1}{2}} \|\chi_1\|_{L^{p+1}(B_1)}$ for all $\omega \in (-\lambda_1, -\lambda_1/4)$,

(iii) $\lim_{\omega \rightarrow -\lambda_1} \|Q_\omega\|_{L^{p+1}(B_1)}^{p+1} / \|Q_\omega\|_{L^2(B_1)}^2 = 0$.

**Remark 15.** In [23], the authors gave a proof only for the Sobolev subcritical case $1 < p < 2^* - 1$, but we can extend it to the Sobolev critical case $p = 2^* - 1$ without any modification.

In order to prove proposition 12, we need the following lemma, which is straightforward from the standard elliptic regularity argument (see, e.g., [48, p 269]).

**Lemma 16.** Let $N \geq 1$, $1 < p \leq 2^* - 1$ if $N \geq 3$ and $1 < p < \infty$ if $N = 1$ or $2$. Let $Q_\omega \in H^1_0(B_1)$ be the least-energy solution of equation (2). Then there exists $\omega_0 \in \mathbb{R}$ and $C > 0$ (which is independent of $\omega$) such that

$$\|Q_\omega\|_{L^\infty(B_1)} \leq C$$  

for $\omega \in (-\lambda_1, \omega_0)$.

Using lemmata 14 and 16, we prove proposition 12.
Proof.

(i) We set \( \hat{Q}_0 = Q_ω/\|Q_ω\|_{L^2(Ω)} \). Multiplying equation (2) by \( Q_ω/\|Q_ω\|_{L^2(Ω)}^2 \) and integrating the resulting equation on \( B_1 \), we have

\[
\|\nabla \hat{Q}_0\|_{L^2(Ω)}^2 + \omega = \frac{\|Q_ω\|_{L^2(Ω)}^2}{\|Q_ω\|_{L^2(Ω)}^2}. \tag{24}
\]

Let \( \{ω_j\} \subset \mathbb{R}^+ \) such that \( ω_j \) goes to \(-λ_1\) as \( j \) goes to \( ∞ \). Since \( \{\hat{Q}_0\}_j \subset H^1_0(Ω) \) is bounded by (24) and lemma 14 (iii), there exists a subsequence of \( \{\hat{Q}_0\}_j \) (we still denote it by the same letters) and \( \hat{Q}_0 \in H^1_0(Ω) \) such that \( \hat{Q}_0 \) converges to \( \hat{Q}_0 \) weakly in \( H^1_0(Ω) \) as \( j \) goes to \( ∞ \). Then by the Rellich–Kondrachov theorem, we have

\[
\|\hat{Q}_0\|_{L^2(Ω)} = \lim_{j→∞} \|\hat{Q}_0\|_{L^2(Ω)} = 1. \]

It follows from (24) and lemma 14 (iii) that

\[
\lambda_1 = \liminf_{j→∞} \left\{ \frac{\|Q_ω\|_{H^1_0}(Ω)}{\|Q_ω\|_{L^2(Ω)}^2} - ω_j \right\} = \liminf_{j→∞} \|\nabla Q_ω\|_{L^2(Ω)}^2 \geq \|\nabla \hat{Q}_0\|_{L^2(Ω)}^2 \geq \lambda_1.
\]

Thus, we see that \( \|\nabla \hat{Q}_0\|_{L^2(Ω)}^2 = \lambda_1 \) and \( \|\hat{Q}_0\|_{L^2(Ω)}^2 = 1 \), which yields that \( \hat{Q}_0 \) is the first eigenfunction of the operator \(-Δ\) in \( B_1 \) with \( \|\hat{Q}_0\|_{L^2(Ω)} = 1 \). By the uniqueness, we have \( \hat{Q}_0 = χ_1 \). Moreover, (25) implies that \( \|\nabla Q_ω\|_{L^2(Ω)} \) converges to \( \|\nabla χ_1\|_{L^2(Ω)} \) as \( ω \) goes to \(-λ_1\), which concludes that \( \hat{Q}_ω \) converges to \( χ_1 \) strongly in \( H^1_0(Ω) \) as \( j \) goes to \( ∞ \).

(ii) Multiplying equation (2) by \( v_ω \) and equation (3) by \( Q_ω \), integrating the resulting equations and taking the difference, we have

\[
(p - 1) \int_{Ω} Q_ω v_ω \, dx = \int_{Ω} v_ω^2 - \|\nabla v_ω\|_{L^2(Ω)}^2
\]

\[
\times \frac{Q_ω}{\|Q_ω\|_{L^2(Ω)}^2} \|Q_ω\|_{L^2(Ω)}^2 \frac{Q_ω}{\|Q_ω\|_{L^2(Ω)}^2} \frac{Q_ω}{\|Q_ω\|_{L^2(Ω)}^2} \frac{Q_ω}{\|Q_ω\|_{L^2(Ω)}^2}.
\]

This yields that

\[
\frac{\|Q_ω\|_{L^2(Ω)}^2}{\|v_ω\|_{L^2(Ω)}^2} \leq \frac{(p - 1)}{\|Q_ω\|_{L^2(Ω)}^2} \|Q_ω\|_{L^2(Ω)}^2 \|Q_ω\|_{L^2(Ω)}^2 \frac{Q_ω}{\|Q_ω\|_{L^2(Ω)}^2} \frac{Q_ω}{\|Q_ω\|_{L^2(Ω)}^2} \frac{Q_ω}{\|Q_ω\|_{L^2(Ω)}^2} \frac{Q_ω}{\|Q_ω\|_{L^2(Ω)}^2}.
\]

Thus, we see that \( \|\nabla Q_ω\|_{L^2(Ω)}^2 \) and \( \|\hat{Q}_0\|_{L^2(Ω)}^2 \) go to \( 1 \) as \( j \) goes to \( ∞ \). Therefore, we have

\[
\lambda_1 = \liminf_{j→∞} \left\{ \frac{\|Q_ω\|_{H^1_0}(Ω)}{\|Q_ω\|_{L^2(Ω)}^2} - ω_j \right\} = \liminf_{j→∞} \|\nabla Q_ω\|_{L^2(Ω)}^2 \geq \|\nabla \hat{Q}_0\|_{L^2(Ω)}^2 \geq \lambda_1.
\]

Thus, we see that \( \|\nabla \hat{Q}_0\|_{L^2(Ω)}^2 = \lambda_1 \) and \( \|\hat{Q}_0\|_{L^2(Ω)}^2 = 1 \), which yields that \( \hat{Q}_0 \) is the first eigenfunction of the operator \(-Δ\) in \( B_1 \) with \( \|\hat{Q}_0\|_{L^2(Ω)} = 1 \). By the uniqueness, we have \( \hat{Q}_0 = χ_1 \). Moreover, (25) implies that \( \|\nabla Q_ω\|_{L^2(Ω)} \) converges to \( \|\nabla χ_1\|_{L^2(Ω)} \) as \( ω \) goes to \(-λ_1\), which concludes that \( \hat{Q}_ω \) converges to \( χ_1 \) strongly in \( H^1_0(Ω) \) as \( j \) goes to \( ∞ \).
In this section, we calculate \( \partial \omega \) |

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we have the following.

Theorem 17. Let \( N \geq 1 \), \( p > 1 \) and \( Q_\omega \) be the least-energy solution of equation (2). Then we have the following.

(i) There exists \( \omega_0 > 0 \) such that \( \partial \omega \) \( \| Q_\omega \|_{L^2(B_1)}^2 \) is positive if \( 1 < p < 1 + 4/N \), and negative if \( 1 + 4/N < p < 2^* - 1 \) for any \( \omega \in (\omega_0, \infty) \).

(ii) Let \( 1 < p \leq 2^* - 1 \) if \( N \geq 3 \) and \( 1 < p < \infty \) if \( N = 1 \) or 2. There exists \( \epsilon_0 > 0 \) such that \( \partial \omega \) \( \| Q_\omega \|_{L^2(B_1)}^2 \) is positive for any \( \omega \in (-\lambda_1, -\lambda_1 + \epsilon_0) \).

Proof.

(i) Let \( \alpha = 2/(p - 1) - (N + 2)/2 \). Then it follows from proposition 11 that

\[
\lim_{\omega \to \infty} \frac{\partial \omega \| Q_\omega \|_{L^2(B_1)}^2}{2\omega^\alpha} = \lim_{\omega \to \infty} \frac{1}{\omega^\alpha} \int_{B_1} Q_\omega v_\omega \, dx = \lim_{\omega \to \infty} \int_{B_1} \hat{Q}_\omega \hat{v}_\omega \, dx
\]

\[
= \int_{\mathbb{R}^N} R_1 \left( \frac{R_1}{p - 1} + \frac{x \cdot \nabla R_1}{2} \right) \, dx = \left( \frac{1}{p - 1} - \frac{N}{4} \right) \| R_1 \|_{L^1(B_1)}^2.
\]

Therefore, there exists \( \omega_0 > 0 \) such that \( \partial \omega \) \( \| Q_\omega \|_{L^2(B_1)}^2 \) > 0 if \( 1 < p < 1 + 4/N \) and \( \leq 0 \) if \( 1 + 4/N < p < 2^* - 1 \) for \( \omega \in (\omega_0, \infty) \).

(ii) From proposition 12, we obtain

\[
\lim_{\omega \to -\lambda_1} \frac{\partial \omega \| Q_\omega \|_{L^2(B_1)}^2}{2\omega \| Q_\omega \|_{L^2(B_1)} \| v_\omega \|_{L^2(B_1)}} = \lim_{\omega \to -\lambda_1} \int_{B_1} \hat{Q}_\omega \hat{v}_\omega \, dx = \int_{B_1} \chi_{B_1}^2 \, dx.
\]

Thus, there exists \( \epsilon_0 > 0 \) such that \( \partial \omega \) \( \| Q_\omega \|_{L^2(B_1)}^2 \) > 0 for \( \omega \in (-\lambda_1, -\lambda_1 + \epsilon_0) \). \( \square \)

Next, we consider the case where \( p = 1 + 4/N \) and frequency \( \omega > 0 \) is sufficiently large. We obtain the following theorem.

Theorem 18. Let \( N \geq 1 \) and \( p = 1 + 4/N \). Then there exists \( \omega_0' > 0 \) such that \( \partial \omega \) \( \| Q_\omega \|_{L^2(B_1)}^2 \) > 0 for any \( \omega \in (\omega_0', \infty) \).

To prove this theorem, we make use of the symmetry property of the solutions of (2) and (3). \( Q_\omega, v_\omega \) are radial, and \( C^2([0, 1]) \), thus we rewrite equation (2) and equation (3) under the following form.

\[
\begin{cases}
-\partial_r^2 Q_\omega - \frac{N - 1}{r} \partial_r Q_\omega + \omega Q_\omega - Q_\omega^p = 0, & r \in (0, 1) \\
Q_\omega(1) = 0, & \partial_r Q_\omega(0) = 0.
\end{cases}
\]  

\[
\begin{cases}
-\partial_r^2 v_\omega - \frac{N - 1}{r} \partial_r v_\omega + \omega v_\omega - p Q_\omega^{p-1} v_\omega = -Q_\omega, & r \in (0, 1) \\
v_\omega(1) = 0, & \partial_r v_\omega(0) = 0.
\end{cases}
\]  

We begin by showing the following lemma.
Lemma 19. Let $N \geq 1, 1 < p \leq 2^* - 1$ if $N \geq 3$ and $1 < p < \infty$ if $N = 1$ or 2. Suppose that there exists $\omega_0 \in (\lambda_1, \infty)$ such that $\partial_r v_{\omega_0}(1) = 0$. Then we see that $v_{\omega_0}(r) > 0$ if $r > 0$ is sufficiently close to 1.

Proof. Since $v_{\omega_0}(1) = \partial_r v_{\omega_0}(1) = 0$, we have $\partial^2_r v_{\omega_0}(1) = 0$ from equation (28). Therefore, it is enough to show that $\partial^2_r v_{\omega_0}(1) < 0$. Differentiating equation (28) with respect to $r$, we obtain

$$
- \partial_r^2 v_{\omega_0} = -\frac{N - 1}{r} \partial_r v_{\omega_0} + Q^p_{\omega_0} \omega_0 - (p - 1) \partial_{\omega_0}^{} Q^p_{\omega_0} v_{\omega_0} - p Q^p_{\omega_0}^{-1} \partial_r Q_{\omega_0} v_{\omega_0}.
$$

If $p \geq 2$, we can easily find that $\partial^2_r v_{\omega_0}(1) < 0$ from equation (29). Thus, we consider the case where $1 < p < 2$. By l’Hôpital’s rule, we can compute that

$$
\lim_{r \to 1} \frac{\partial_r^2 Q_{\omega_0} v_{\omega_0}}{\partial r Q_{\omega_0}} = \lim_{r \to 1} \frac{\partial_r^2 Q_{\omega_0} v_{\omega_0} + \partial_r Q_{\omega_0} \partial_r v_{\omega_0}}{(2 - p) \partial r Q_{\omega_0}} = \lim_{r \to 1} \frac{\partial_r^2 Q_{\omega_0} v_{\omega_0} + \partial_r Q_{\omega_0} \partial_r v_{\omega_0}}{(2 - p) \partial r Q_{\omega_0}} = 0.
$$

Therefore, we see that $\partial^2_r v_{\omega_0}(1) < 0$ from equation (29).

Proof of theorem 18. Thanks to the identity (8), it is enough to show that $\partial_r v_{\omega}(1) > 0$ for sufficiently large $\omega > 0$. By [24, proposition 4.1], there exists a constant $C > 0$ such that

$$
\lim_{r \to \infty} \partial_r R_1 / R_1 = -C.
$$

This yields that

$$
V(r) = R_1(r) \left( \frac{4}{N} + \frac{\partial_r R_1}{2 R_1} \right) < 0
$$

for sufficiently large $r > 0$. Moreover, since $V(0) = (4/N)R_1(0) > 0$ and $V(r) \to 0$ as $r \to \infty$, there exists $r_0 \in (0, \infty)$ satisfying $\partial_r V(r_0) = 0$ and $V(r) < 0$ for $r \in [r_0, \infty)$.

Suppose that $\partial_r v_{\omega}(1) \leq 0$. Recalling the fact that $v_{\omega}(x) = \omega^{(3 - p)/(p - 1)} \tilde{v}_\omega(\sqrt{\omega}x)$, lemma 19, implies that $\tilde{v}_\omega(r) > 0$ if $r$ is sufficiently close to $\sqrt{\omega}$. On the other hand, we have seen in proposition 11 (ii) that $\tilde{v}_\omega$ converges to $V$ in $H^2(\mathbb{R}^N)$ as $\omega$ tends to infinity, we thus have $\tilde{v}_\omega(r_0) < 0$ for sufficiently large $\omega > 0$. Thus, there exists $r_1(\omega) \in (r_0, \sqrt{\omega})$ such that $\partial_r \tilde{v}_\omega(r_1(\omega)) = 0$ and $\tilde{v}_\omega(r_0) > 0$ for $r \in (r_1(\omega), \sqrt{\omega})$.

We claim that $r_1(\omega) \to \infty$ as $\omega$ goes to $\infty$. Suppose that there exists a constant $M > r_0$ such that $r_1(\omega) \leq M$ for all $\omega \in (-\lambda_1, \infty)$. Using the fact that $\tilde{v}_\omega(r_0) > 0$ and $V(r) < 0$ for $r \geq M$, we have

$$
0 \leq \| \tilde{v}_\omega - V \|_{L^2(\mathbb{R}^N)}^2 \geq \int_{|x| \geq M} |\tilde{v}_\omega - V|^2 \, dx \geq \int_{|x| \geq M} |V|^2 \, dx
$$

as $\omega \to \infty$, which is a contradiction. Thus, we see that $r_1(\omega) \to \infty$ as $\omega \to \infty$. 

Multiplying equation (5) by $r^{N-1}$ and integrating the resulting equation from $r_1(\omega)$ to $\sqrt{\omega}$, we obtain
\[
\left[r^{N-1}\partial_r \tilde{v}_\omega(r)\right]_{r_1(\omega)}^{\sqrt{\omega}} = \int_{r_1(\omega)}^{\sqrt{\omega}} \left\{ \tilde{Q}_\omega + \tilde{v}_\omega - \left(1 + \frac{4}{N}\right) \tilde{Q}_{\omega}^{\frac{2}{N}} \tilde{v}_\omega \right\} r^{N-1} \, dr.
\]
We have, from $\partial_r v_\omega \leq 0$,
\[
\left[r^{N-1}\partial_r \tilde{v}_\omega(r)\right]_{r_1(\omega)}^{\sqrt{\omega}} = \omega^{\frac{N-1}{2}} \partial_r \tilde{v}_\omega(\sqrt{\omega}) \leq 0.
\]
On the other hand, since $r_1(\omega) \to \infty$ as $\omega \to \infty$ and $\tilde{Q}_\omega(r) \to 0$ as $r \to \infty$, choosing $\omega$ sufficiently large so that $1 - (1 + \frac{4}{N}) \tilde{Q}_\omega^{\frac{2}{N}} > 1/2$,
\[
\int_{r_1(\omega)}^{\sqrt{\omega}} \left\{ \tilde{Q}_\omega + \tilde{v}_\omega - \left(1 + \frac{4}{N}\right) \tilde{Q}_{\omega}^{\frac{2}{N}} \tilde{v}_\omega \right\} r^{N-1} \, dr \geq \int_{r_1(\omega)}^{\sqrt{\omega}} \left\{ \tilde{Q}_\omega + \frac{1}{2} \tilde{v}_\omega \right\} r^{N-1} \, dr > 0.
\]
Comparison between these two inequalities gives a contradiction. In conclusion, we infer that $\partial_r \tilde{v}(\sqrt{\omega}) > 0$ for sufficiently large $\omega > 0$, which completes the proof. \(\square\)

4. Calculation of $\partial_\omega \|Q_\omega\|^2_{L^2(B_1)}$ for $N = 1$

This section is devoted to show the following:

**Proposition 20.** Let $N = 1$ and $1 < p \leq 5$. Then we have $\partial_\omega \|Q_\omega\|^2_{L^2(B_1)} > 0$ for all $\omega \in (-\lambda_1, \infty)$.

We prepare several lemmata, which will be needed later.

**Lemma 21.** Let $N = 1$ and $1 < p < \infty$. Then we obtain the following identity:
\[
\left[ \partial_r^2 Q_\omega v_\omega - \partial_r v_\omega \partial_r Q_\omega \right]_{r_1}^2 = -\frac{1}{2} \left[ Q_\omega^2 \right]_{r_1}^2
\]
for $0 \leq r_1 < r_2 \leq 1$.

**Proof.** Differentiating equation (2) with respect to $r$, we obtain
\[-\partial_r^3 Q_\omega + \omega \partial_r Q_\omega - p Q_\omega^{p-1} \partial_r Q_\omega = 0.\]
Multiplying the above equation by $v_\omega$ and equation (3) by $\partial_r Q_\omega$, respectively, and taking the difference, we obtain
\[
\partial_r \left[ \partial_r^2 Q_\omega v_\omega - \partial_r v_\omega \partial_r Q_\omega \right] = -\frac{1}{2} \partial_r \left[ Q_\omega^2 \right].
\]
Then integrating from $r_1$ to $r_2$, we obtain (30). \(\square\)

**Lemma 22.** Let $N = 1$ and $1 < p < \infty$. Then, for any $\omega \in (-\lambda_1, \infty)$, the following properties are satisfied.

(i) $v_\omega(0) > 0$.
(ii) If $v_\omega(\cdot)$ has a zero in $(0, 1)$, then $\partial_r v_\omega(1) > 0$. 

Proof.

(i) Since \( v_\omega / \| v_\omega \|_2 \to \chi_1 \) in \( H_0^1 \) as \( \omega \to -\lambda_1 \), we see that \( v_\omega(0) > 0 \) if the frequency \( \omega \) is sufficiently close to \(-\lambda_1\). Suppose that there exists \( \omega_1 > 0 \) such that \( v_\omega(0) = 0 \). Then it follows from equation (28) that \( \partial_r^2 v_\omega(0) = Q_\omega(0) > 0 \). Thus, we see that \( v_\omega(r) > 0 \) if \( r > 0 \) is sufficiently close to 0. On the other hand, from (30), we obtain

\[
-\partial_r Q_\omega(1) \partial_r v_\omega(1) = \left[ v_\omega \partial_r^2 Q_\omega - \partial_r Q_\omega \partial_r v_\omega \right]_0 = -\frac{1}{2} [Q_\omega^2]_0
\]

which implies that \( \partial_r v_\omega(1) > 0 \). Thus, we see that \( v_\omega(r) < 0 \) if \( r > 0 \) is sufficiently close to 1. It follows that there exists \( r_1 \in (0, 1) \) such that \( \partial_r v_\omega(r_1) \leq 0 \) and \( v_\omega(r_1) = 0 \).

On the other hand, using (30) again, we have

\[
0 \geq -\partial_r Q_\omega(r_1) \partial_r v_\omega(r_1) = [v_\omega \partial_r^2 Q_\omega - \partial_r Q_\omega \partial_r v_\omega]_0 = -\frac{1}{2} [Q_\omega^2]_0
\]

where we used the fact that \( Q_\omega(0) \) is a decreasing function in \((0, 1)\). This leads a contradiction. Thus, \( v_\omega(0) > 0 \) for all \( \omega \in (-\lambda_1, \infty) \).

(ii) Suppose that the continuous function \( v_\omega \) has a zero in \((0, 1)\). Then one of the following can occur:

(Case 1) \( v_\omega \) changes the sign more than two times.

(Case 2) \( v_\omega \) changes the sign exactly one times.

(Case 3) \( v_\omega \) does not change sign.

In case 2, it suffices to show that \( \partial_r v_\omega(1) \neq 0 \) since \( v_\omega(0) > 0 \). The fact \( \partial_r v_\omega(1) \neq 0 \) immediately follows from the proof of lemma 19.

Next, we show that case 3 does not occur. Suppose that case 3 occurs and let \( r_0 \in (0, 1) \) be the zero of \( v_\omega \). Then since \( v_\omega \) is non-negative, we see that \( \partial_r v_\omega(r_0) = 0 \). It follows from (30) that

\[
0 \geq -\partial_r Q_\omega(r_0) \partial_r v_\omega(r_0) = [v_\omega \partial_r^2 Q_\omega - \partial_r Q_\omega \partial_r v_\omega]_0 = -\frac{1}{2} [Q_\omega^2]_0 = \frac{Q_\omega^2(r_0)}{2} > 0.
\]

This is a contradiction.

Finally, we consider case 1. Since \( v_\omega \) changes sign more than two times, there exists \( r_2 \in (0, 1) \) such that \( v_\omega(r_2) = 0 \) and \( \partial_r v_\omega(r_2) > 0 \). It follows from (30) that

\[
-\partial_r Q_\omega(r_2) \partial_r v_\omega(r_2) - v_\omega(0) \partial_r^2 Q_\omega(0) = [v_\omega \partial_r^2 Q_\omega - \partial_r Q_\omega \partial_r v_\omega]_0 = -\frac{1}{2} [Q_\omega^2]_0 < \frac{Q_\omega^2(0)}{2}.
\]

Therefore, we obtain

\[
0 < -\partial_r Q_\omega(r_2) \partial_r v_\omega(r_2) < v_\omega(0) \partial_r^2 Q_\omega(0) + \frac{Q_\omega^2(0)}{2}.
\]

On the other hand, putting \( r_2 = 1 \) and \( r_1 = 0 \) in (30), we have

\[
-\partial_r^2 Q_\omega(0) v_\omega(0) - \partial_r v_\omega(1) \partial_r Q_\omega(1) = [\partial_r^2 Q_\omega v_\omega - \partial_r Q_\omega \partial_r v_\omega]_0 = -\frac{1}{2} [Q_\omega^2]_0 = \frac{Q_\omega^2(0)}{2}.
\]

Then from (31), we see that \( \partial_r v_\omega(1) \partial_r Q_\omega(1) = -Q_\omega^2(0)/2 - \partial_r^2 Q_\omega(0) v_\omega(0) < 0 \). Therefore, we conclude that \( \partial_r v_\omega(1) > 0 \). \( \square \)
We continue to investigate some properties of \( v_{\omega} \), but from now on, as a function of \( \omega \). Let us recall the results of proposition 12 (ii), and we define \( \omega_\ast \) as follows.

\[
\omega_\ast = \sup \{ \omega \in (-\lambda_1, \infty) \mid \text{if } \omega' \in (-\lambda_1, \omega), \text{then } v_{\omega'}(r) > 0 \text{ for all } r \in (0, 1) \}.
\]

By the definition, we find that \( \partial_\omega \| Q_\omega \|_{L^2(B_1)}^2 = 2 \int_{B_1} Q_\omega v_\omega \, dx > 0 \) for \( \omega \in (-\lambda_1, \omega_\ast) \). From proposition 12 and the proof of theorem 18, we see that \( -\lambda_1 < \omega_\ast < \infty \). Moreover, we obtain the following:

**Lemma 23.** Let \( \omega_\ast \in (-\lambda_1, \infty) \) be defined as above. Then the following properties hold:

(i) \( v_{\omega_\ast}(r) > 0 \) for all \( r \in (0, 1) \).

(ii) \( \partial_r v_{\omega_\ast}(1) = 0 \).

**Proof.**

(i) The fact that the mapping \( \omega : (-\lambda_1, \infty) \mapsto v_\omega \in H^1(B_1) \) is \( C^0 \) implies that for each \( r \in [0, 1] \), \( v_\omega(r) \) is continuous in \( \omega \), using a similar argument of the elliptic regularity in the proof of proposition 12 (ii). Since \( v_{\omega}(r) \) is positive for \( \omega \in (-\lambda_1, \omega_\ast) \), we obtain \( v_{\omega_\ast}(r) = \lim_{\omega \to \omega_\ast} v_\omega(r) > 0 \) for each \( r \in [0, 1] \). It follows from the proof of lemma 22 (ii) (see case 3) that \( v_{\omega_\ast} \) does not have zero, that is, \( v_{\omega_\ast}(r) > 0 \) for all \( r \in (0, 1) \).

(ii) From (i), we know that \( \partial_r v_{\omega_\ast}(1) \leq 0 \). By the definition of \( \omega_\ast \), there exists a sequence \( \{\omega_n\} \subset (\omega_\ast, \infty) \) such that \( \lim_{\omega \to \infty} \omega_n = \omega_\ast \) and \( v_{\omega_n} \) has a zero in \((0, 1)\). From lemma 22 (ii), we have \( \partial_r v_{\omega_n}(1) > 0 \) for all \( n \in \mathbb{N} \). Using a similar argument in the proof of (i), we see that \( \partial_r v_{\omega_n}(r) \) is continuous in \( \omega \) for each \( r \in [0, 1] \). Therefore, we obtain \( \partial_r v_{\omega_\ast}(1) = \lim_{n \to \infty} \partial_r v_{\omega_n}(1) \geq 0 \). This completes the proof.

Now, we are in position to prove proposition 20.

**Proof of proposition 20.** Using the Pohozaev identity, we have

\[
E(Q_\omega) = \frac{p-5}{2(p+3)} \omega \| Q_\omega \|_{L^2(B_1)}^2 + \frac{p-1}{2(p+3)} |\partial_r Q_\omega(1)|^2.
\]

Moreover, since \( -\omega \partial_\omega \| Q_\omega \|_{L^2(B_1)}^2 = 2\partial_\omega E(Q_\omega) \), it follows that

\[
-\omega \frac{2(p-1)}{p+3} \int Q_\omega v_\omega \, dx = \frac{p-5}{2(p+3)} \| Q_\omega \|_{L^2(B_1)}^2 + \frac{p-1}{p+3} \partial_r Q_\omega(1) \partial_r v_\omega(1). \tag{32}
\]

Putting \( \omega = \omega_\ast \) in (32), we see that, using (ii) of lemma 23,

\[
-\omega_\ast \frac{2(p-1)}{p+3} \int Q_{\omega_\ast} v_{\omega_\ast} \, dx = \frac{p-5}{2(p+3)} \| Q_{\omega_\ast} \|_{L^2(B_1)}^2 \leq 0.
\]

It then follows \( \omega_\ast \geq 0 \) from the fact \( \int Q_{\omega_\ast} v_{\omega_\ast} \, dx > 0 \).

Let \( \omega \geq \omega_\ast \). Suppose that \( \partial_\omega \| Q_\omega \|_{L^2(B_1)}^2 = 2 \int Q_\omega v_\omega \, dx \leq 0 \). Then, the identity (32) leads to \( \partial_r v_{\omega_\ast}(1) \leq 0 \). On the other hand, the contraposition of lemma 22 (ii) implies that \( v_{\omega}(r) > 0 \) for all \( r \in (0, 1) \). Since \( Q_{\omega}(r) > 0 \) for all \( r \in (0, 1) \), we have \( \int Q_{\omega} v_{\omega} \, dx > 0 \), which is a contradiction. Therefore, \( \partial_\omega \| Q_\omega \|_{L^2(B_1)}^2 > 0 \) for \( \omega \in [\omega_\ast, \infty) \).
From the definition of $\omega_*$, we see that $v_\omega(r) > 0$ for all $r \in (0, 1)$ if $\omega < \omega_*$. Then we clearly see that $\partial_\omega \|Q_\omega\|_{L^2(B_1)}^2 = 2 \int Q_\omega v_\omega \, dx > 0$. This completes the proof. □

5. Linear stability of the excited states

Following section 6.2 of [15], we give a proof of theorem 7. For this purpose, we consider the space $L^2(B_1)$ of $\mathbb{R}^2$-valued functions of $L^2(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$ and $\tilde{L}^2(B_1) = L^2(B_1) + iL^2(B_1)$, which is identified with the space of $C^2$-valued, square integrable functions, endowed with the inner product

$$(\mathcal{U}, \mathcal{V}) = \langle u_1, v_1 \rangle + \langle u_2, v_2 \rangle$$

for $\mathcal{U}, \mathcal{V} \in \tilde{L}^2(B_1)$ with

$$\mathcal{U} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad \mathcal{V} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}. $$

Here $\langle u, v \rangle$ is defined by

$$\langle u, v \rangle = \int_{B_1} u(x) \overline{v(x)} \, dx. $$

We write the operator $J\mathcal{L}_{k,\omega(s)}$ as follows

$$J\mathcal{L}_{k,\omega(s)} = \begin{pmatrix} 0 & L_{k,\omega(s)}^- \\ -L_{k,\omega(s)}^+ & 0 \end{pmatrix} = \begin{pmatrix} 0 & H_k \\ -H_k & 0 \end{pmatrix} + \begin{pmatrix} \lambda_k - \omega(s) + p|Q_{k,\omega(s)}|^p & \lambda_k + \omega(s) - |Q_{k,\omega(s)}|^{p-1} \\ 0 & 0 \end{pmatrix}. $$

We set, for each $k \in \mathbb{N}$,

$$\mathcal{H}_k = \begin{pmatrix} H_k & 0 \\ 0 & H_k \end{pmatrix}$$

and $\xi_{n,k}^\pm = \pm(\lambda_n - \lambda_k)i = \pm(n^2 - k^2)\pi^2/4, n = 1, 2, \ldots$. It may be seen that the spectrum of $J\mathcal{H}_k$ consists only of the discrete eigenvalues, and the set of spectrum is expressed as $\sigma(J\mathcal{H}_k) = \{\xi_{n,k}^\pm \mid n \in \mathbb{N}\}$. Moreover, if $k \leq 4$, each eigenvalue $\xi_{n,k}^\pm$ for $n \neq k$ is simple and the corresponding eigenfunction is given by

$$\tilde{\phi}_{n,k}^\pm = \frac{1}{\sqrt{2}} \begin{pmatrix} \pm \chi_n \\ \mp i\chi_n \end{pmatrix}, \quad \text{for } n \neq k. $$

and $\ker^g J\mathcal{H}_k = \text{Span}\{\tilde{\phi}_k^+, \tilde{\phi}_k^-\}$, where

$$\tilde{\phi}_k^+ = \begin{pmatrix} 0 \\ \chi_k \end{pmatrix}, \quad \tilde{\phi}_k^- = \begin{pmatrix} \chi_k \\ 0 \end{pmatrix}. $$

Remark 24. If $k \geq 5$, we have at least, except $n = k$, two eigenvalues whose geometric multiplicities are two. Indeed, if we look for a pair $(n_1, n_2), n_1, n_2 \in \mathbb{N}$ satisfying the condition for high geometric multiplicity $n_1^2 + n_2^2 = 2k^2, n_1 \neq k, n_2 \neq k$ for fixed $k \in \mathbb{N}$, we have only one pair for $k \leq 4$, on the other hand we have more than three pairs for $k \geq 5$, which cause the eigenvalues with geometric multiplicities.
Assume \( k \leq 4 \). In order to consider the spectrum of the linearized operator \( J L_k,\omega(s) \) for \( \omega \) sufficiently close to \(-\lambda_k\), we choose the following contour:

\[
\Gamma_{n,k}^\pm = \left\{ z \in \mathbb{C} \mid |z - \xi_{n,k}^\pm| = \frac{\pi^2}{8} \right\}
\]

and the projection

\[
\Pi_{n,k}^\pm (J L_k,\omega(s)) = \frac{1}{2\pi i} \int_{\Gamma_{n,k}^\pm} (J L_k,\omega(s) - z)^{-1} \, dz.
\]

Then we have the following lemma.

**Lemma 25.** For \( \omega(s) \) sufficiently close to \(-\lambda_k\), we have \( \dim \mathcal{R}(\Pi_{n,k}^\pm (J L_k,\omega(s))) = 1 \) for \( n \neq k \) and \( \dim \mathcal{R}(\Pi_{k,k}^\pm (J L_k,\omega(s))) = 2 \).

**Proof.** It follows from [43, p 14 lemma] that if

\[
\| \Pi_{n,k}^\pm (J L_k,\omega(s)) - \Pi_{n,k}^\pm (J H_k) \|_{L^2(\mathbb{C})} < 1,
\]

we have \( \dim \mathcal{R}(\Pi_{n,k}^\pm (J L_k,\omega(s))) = \dim \mathcal{R}(\Pi_{n,k}^\pm (J H_k)) \). We shall show that (35) holds for \( \omega \) sufficiently close to \(-\lambda_k\).

For any \( z \in \rho(J L_k,\omega(s)) \cap \rho(J H_k) \), using the Neumann series, we have

\[
\| (J L_k,\omega(s) - z)^{-1} - (J H_k - z)^{-1} \|_{L^2(\mathbb{C})} \leq \sum_{j=1}^{\infty} \| J L_k,\omega(s) - J H_k \|_{L^2(\mathbb{C})} \| (J H_k - z)^{-1} \|_{L^2(\mathbb{C})}^j.
\]

By the spectrum theorem, we have

\[
\| (J H_k - z)^{-1} \|_{L^2(\mathbb{C})} < \frac{1}{\operatorname{dist}(z, \sigma(J H_k))} < \frac{8}{\pi^2}.
\]

Moreover, as in [15], for \( \omega(s) \) sufficiently close to \(-\lambda_k\),

\[
\| J L_k,\omega(s) - J H_k \|_{L^2(\mathbb{C})} \leq C(p)(|\omega(s) + \lambda_k| + \| Q_k,\omega(s) \|_{L^\infty}^{p-1}) \leq \frac{1}{8},
\]

since \( \| Q_k,\omega(s) \|_{L^\infty} \) vanishes when \( \omega \) tends to \(-\lambda_k\) by (9) and Sobolev embedding.

Hence, we obtain

\[
\| \Pi_{n,k}^\pm (J L_k,\omega(s)) - \Pi_{n,k}^\pm (J H_k) \|_{L^2(\mathbb{C})} \leq \frac{1}{2\pi} \int_{\Gamma_{n,k}^\pm} \| (J L_k,\omega(s) - z)^{-1} - (J H_k - z)^{-1} \|_{L^2(\mathbb{C})} |dz| \leq \frac{1}{2\pi} \sum_{j=1}^{\infty} \left( \frac{1}{8} \right)^j \left( \frac{8}{\pi^2} \right)^j \| \Gamma_{n,k}^\pm \| = \frac{1}{2\pi} \frac{1}{\pi^2 - 1} < 1.
\]

From lemma 25, we see that

\[
\ker \partial_\omega Q_k,\omega(s) = \left\{ \begin{pmatrix} \partial_\omega Q_k,\omega(s) \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ Q_k,\omega(s) \end{pmatrix} \right\}.
\]

**Proof of theorem 7.** All eigenvalues of \( J H_k \) are discrete, and on the imaginary axis. They are simple except the zero eigenvalue. It is known (see [27, theorem 1.1 (ii)]) that if \( J L_k,\omega(s) \)
has an eigenvalue with positive real part, then it has another eigenvalue being symmetric with respect to the real and imaginary axes. Hence, the perturbed eigenvalues for \( n \neq k \) should be on the imaginary axis, otherwise a contradiction to lemma 25 occurs. We may treat the case where \( n = k \) similarly. This completes the proof.  

\[ \square \]

6. Orbital stability of the excited states

In this section, we prove proposition 9. First, we note that we can show the Cauchy problem for equation (1) is locally well-posed in \( H^1_{0,k}((-1, 1)) \). Indeed, for each \( u_0 \in H^1_{0,k}((-1, 1)) \), we put

\[
u_{0,n}(x) = \begin{cases} u_0(x) & \text{for } x \in I_n, \\ 0 & \text{for } x \in (-1, 1) \setminus I_n, \end{cases}
\]

for \( n = 1, 2, \ldots, k \), where \( I_n = \left( -\frac{1}{2} + \frac{2}{k}(n-1), -\frac{1}{2} + \frac{2n}{k} \right) \). Since \( u_{0,n} \in H^1_0(I_n) \), it follows from [12, theorem 3.5.1] that there exist \( T_n > 0 \) and a unique solution \( u_n \in C([0, T_n), H^1_0(I_n)) \) of equation (1) with \( u_n(0) = u_{0,n} \). We set \( T = \min \{ T_n \mid n = 1, 2, \ldots, k \} \) and

\[ u = \sum_{n=1}^k u_n, \]

where we extend \( u_n \) to the interval \((-1, 1)\) by defining \( u_n(\cdot, x) = 0 \) for \( x \in (-1, 1) \setminus I_n \). Then we see that \( u \in C([0, T), H^1_{0,k}((-1, 1))) \) and satisfies equation (1).

To obtain the stability in proposition 9, we use the general theory of [28, theorem 3] again.

**Proposition 26.** Let \( Q_{k,\omega} \in H^1_{0,k} \) be the \( k \)th bound state of equation (2). Let \( T = \mathcal{S}_{\omega}(Q_{k,\omega})|_{H^1_{0,k}} \). Assume that

(i) the positive spectrum of \( T \) is bounded away from zero;

(ii) the kernel is spanned by \( iQ_{k,\omega} \), that is, \( \text{Ker} T = \text{Span}\{iQ_{k,\omega}\} \);

(iii) the operator \( T \) has exactly one negative simple eigenvalue.

Then if \( \partial_{\omega}\|Q_{k,\omega}\|_2^2 > 0 \) (respectively \( < 0 \)), the standing wave \( e^{i\omega t} Q_{k,\omega} \) is orbitally stable (respectively unstable) in \( H^1_{0,k} \).

Using proposition 26, we give a proof of proposition 9.

**Proof of (i) of proposition 9.** We first show the explicit form of (12). It was shown in [4, theorem 7] and [49, remark 3.1] that the \( k \)th bound state \( Q_{k,\omega} \) of equation (2) has \((k-1)\)-zeros in \((-1, 1)\) and is unique up to phase. In addition, we can easily see that the function

\[
\sum_{n=1}^k (-1)^{n-1}k^{\frac{n}{k}} Q_{\omega} \left( k \left( x + 1 + \frac{1-2n}{k} \right) \right)
\]

satisfies equation (2) and has \((k-1)\)-zeros in \((-1, 1)\) (see [13, section 1.2]). Therefore, (12) holds.

Next, we shall show the stability in \( H^1_{0,k}((-1, 1)) \). Recall that all spectra of the operator \( T \) are discrete. Moreover, it follows from [4, lemma 8] that \( \text{Ker} L_{\omega,\omega} = \{0\} \) for all \( \omega > -\lambda_k \). Moreover, we can easily see that the all eigenvalues of the operator \( L_{\omega,\omega} \) are simple and \( L_{\omega,\omega} Q_{k,\omega} = 0 \). Thus, we infer that the operator \( T \) satisfies the assumptions (i) and (ii) in proposition 26.
Then we shall prove that the operator $T$ has only one negative eigenvalue in $H^{1}_{0,k}(-1, 1)$. Let

$$
\Psi_{k,\omega} = \sum_{n=1}^{k} (-1)^{n-1} \psi_{n}\left(k\left(x + 1 + \frac{1-2n}{k}\right)\right)
$$

for any $\omega > -\lambda_k$, where $\Psi_{\omega/k^2} \in H^{1}_{0,k}(-1, 1)$ is the first eigenfunction of the operator $L^{+}_{1,\omega/k^2}$ with $\|\Psi_{\omega/k^2}\|_{L^2} = 1$, and we extend the eigenfunction $\Psi_{\omega/k^2}$ to the line $(-1, 1)$ by setting $\Psi_{\omega/k^2}(x) = 0$ for $x \in (-\infty, -1) \cup (1, \infty)$. We can easily see that $\Psi_{k,\omega} \in H^{1}_{0,k}(-1, 1)$ and $L^{+}_{k,\omega}\Psi_{k,\omega} = -k^2 \nu_{\omega/k^2} \Psi_{k,\omega}$, where $-\nu_{\omega/k^2} \in \mathbb{R}$ is the first eigenvalue of the operator $L^{+}_{1,\omega/k^2}$.

Recall that [4, lemma 8] showed that the number of negative eigenvalue of the operator $L^{+}_{k,\omega}$ equals to $k$. Moreover, it follows from [43, theorem XIII. 8 (c)] that for $1 \leq n \leq k$, $n$th eigenfunction has $(n-1)$-zeros. Therefore, the $n$th eigenfunction with $1 \leq n \leq k-1$ does not belong to $H^{1}_{0,k}(-1, 1)$ since by definition (13) a function in $H^{1}_{0,k}(-1, 1)$ should have $(k-1)$ zeros. These imply that the operator $L^{+}_{k,\omega}$ has exactly one negative eigenvalue in $H^{1}_{0,k}(-1, 1)$.

Similarly, using the fact that $L^{-}_{k,\omega}Q_{k,\omega} = 0$ and $Q_{k,\omega}$ has $k-1$ zeros, we see that the operator $L^{-}_{k,\omega}$ is non-negative in $H^{1}_{0,k}(-1, 1)$. Thus, the assumption (iii) in proposition 26 holds.

Finally, let us explain how we check the slope condition $\partial_{\omega}\|Q_{k,\omega}\|_{L^2}^2 > 0$. It follows from the form (12) that

$$
\|Q_{k,\omega}\|_{L^2}^2 = k^{p/2}\|Q_{\omega/k^2}\|_{L^2}^2
$$

for all $k \geq 2$. Then by the result of theorem 17 (ii) and proposition 20, we see that

$$
\partial_{\omega}\|Q_{k,\omega}\|_{L^2}^2 = k^{p/2}\partial_{\omega}\|Q_{\omega/k^2}\|_{L^2}^2 > 0
$$

for all $\omega > -\lambda_k$ if $1 < p \leq 5$ or $\omega$ close to $-\lambda_k$ if $p > 5$.

Thus, (i) of proposition 9 holds. $\square$

**Proof of (ii) of proposition 9.** It follows from theorem 17 (i) that

$$
\partial_{\omega}\|Q_{k,\omega}\|_{L^2}^2 = k^{p/2}\|Q_{\omega/k^2}\|_{L^2}^2 < 0
$$

for sufficiently large $\omega > 0$ if $p > 5$.

Then by a similar way in [27, theorem 3.2], we see that $k$th standing wave $e^{i\omega t}Q_{k,\omega}$ is linearly unstable, that is, there exists $\lambda \in \sigma(JL_{k,\omega})$ such that $Re\lambda \neq 0$. This implies that the standing wave $e^{i\omega t}Q_{k,\omega}$ is also orbitally unstable from the argument in [27, appendix]. $\square$

7. Numerical results

7.1. Numerical methods

In this section, numerically, we investigate the case of higher-order excited states for which few results are known not only from a theoretical point of view but also a numerical point of view (see [2, 20, 21, 32, 34, 55]). Precisely, we treat the case of the first and second excited states $Q_{2,\omega}$ and $Q_{3,\omega}$ in one space dimension for the purpose of analysing the stability in the cases that are not proved in proposition 9.

Our approach here is similar to those in [21, 37]. We solve equation (1) in one space dimension numerically using the symmetric Crank–Nicolson scheme with the initial datum (14) and (15).
First, we briefly explain the scheme for equation (1). We recall $N=1$. Let $\Delta t$ and $\Delta x$, respectively, stand for the time and space steps. We look for the approximate value, say $u^n_j$, of the solution at $t_n = n\Delta t$ and the $x_j = j\Delta x$. Then, equation (1) can be discretized as follows

$$i\Delta t (u^{n+1}_j - u^n_j) + \frac{1}{2\Delta x^2} (u^{n+1}_{j+1} - 2u^n_j + u^{n+1}_{j-1} + u^{n+1}_{j+1} - 2u^n_j + u^{n+1}_{j-1})$$

$$+ \frac{1}{p+1} \frac{|u^{n+1}_j|^{p+1} - |u^n_j|^{p+1}}{|u^n_j|^2} (u^n_j + u^{n+1}_j) = 0.$$ 

Moreover, we add the condition $u^n_j = 0$ for all $n \geq 0$ and $j$ such that $|j\Delta x| \geq 1$. This assumption ensures that the computed solution satisfies the Dirichlet boundary condition.

We employ this symmetric nonlinear Crank–Nicolson scheme when studying the stability of the standing waves even if it requires to perform a nonlinear algebraic inversion at each step, which we use a fixed-point algorithm (see [9, p. 674]). The above form of the discretization of the nonlinear term ensures that the discrete $L^2$-norm

$$N_n = \Delta x \sum_{j \in \mathbb{Z}} |u^n_j|^2$$

and the discrete energy

$$E_n = \Delta x \left( \frac{1}{2} \sum_{j \in \mathbb{Z}} \left| \frac{u^{n+1}_j - u^n_j}{\Delta x} \right|^2 - \frac{1}{p+1} \sum_{j \in \mathbb{Z}} |u^n_j|^{p+1} \right)$$

are conserved as in the continuous case. This is an important property because the stability analysis is related to these conservation laws as seen in the previous sections.

When we consider the solution with the initial datum (14), we make an additional procedure in the scheme in order to obtain a better approximation. Note that the initial datum $u_0, p = (1+\delta_p)Q_k, \omega$ belongs to the function space $H^1_{0, k}(-1, 1)$ for each $k \in \mathbb{N}$ and we can prove theoretically that the Cauchy problem for equation (1) is locally well-posed in $H^1_{0, k}(-1, 1)$. This yields that the position of zeros of the solution $u^p$ preserves. Therefore, we may force the approximate solution to satisfy

$$u^p \left( -1 + \frac{2}{k} n \right) = 0 \quad (36)$$

for $n = 1, 2, \ldots, k$ at each time step.

Next, let us explain how we seek an approximate solution of equation (2) in one space dimension. We first consider the least-energy solution. Since the positive solution of equation (2) is radially symmetric, we can reduce equation (2) to the following ordinary differential equation:

$$Q'' - \omega Q + |Q|^{p-1} Q = 0, \quad 0 < r < 1, \quad (37)$$

$$Q(0) = \beta > 0, \quad Q'(0) = 0, \quad (38)$$

$$Q(1) = 0. \quad (39)$$

We note that equation (37) can be written as a general first-order system of the form

$$X'(r) = F(r, X(r))$$

with $X(r) = (Q(r), Q'(r)) \in \mathbb{R}^2$. Here,

$$F(r, X(r)) = (Q'(r), \omega Q(r) - |Q(r)|^{p-1} Q(r)).$$

Therefore, for each $\beta > 0$, we can solve equation (37) with the initial datum (38) using the fourth-order Runge-Kutta method. Then it is enough to choose $\beta > 0$ so that the obtained
solution satisfies the boundary condition (39). This can be solved by employing the well-known shooting method (see, e.g., [16]). Consequently, we can find the profile of the least-energy solution, which enables us to check the slope condition. Once we obtain the profile of the least-energy solution, we can find that of the kth excited state from (12).

In order to check the agreement of the numerics with the rigorous stability theory, we need to observe that \( \| u(t, .) - Q_{k,\omega} \|_{H^1} \) remains ‘small’ or not. However, an increase in the norm \( \| u(t, .) - Q_{k,\omega} \|_{H^1} \) does not ensure us to know the dynamics of solution \( u \). Therefore, instead of investigating the \( H^1 \)-norm, we plot the time evolution of the maximal amplitude and the \( L^2 \)-norm of the solution. In addition to the stability of standing waves, these two quantities provide us information about the dynamics of the solution \( u \).

7.2. Stability of the first excited state

We now study the stability of the first excited state \( Q_{2,\omega} \) in one space dimension. We recall that it is observed in [2, section 3.1.2] and [32, section 3.1] that the first excited state \( e^{i\omega t} Q_{2,\omega} \) is linearly stable for \( \omega \in (-3, 5) \) in the case of the cubic nonlinear Schrödinger equation with a harmonic potential, but with a different numerical approach. Here, we study the problem of the orbital (nonlinear) stability to investigate the profile of the solution of equation (1) with the initial data (14) and (15) in case of \( k = 2 \), using the same approach as in [21]. We examine the general case \( p > 1 \) and \( \omega > -\lambda_2 \).

7.2.1. Stability under the perturbation (14). By proposition 9, we know that \( e^{i\omega t} Q_{2,\omega} \) is orbitally stable in \( H_{0,2}^1(-1, 1) \) if the frequency \( \omega \) is sufficiently close to \( -\lambda_2 = -\pi^2 \approx -9.86 \) in the case of \( p > 5 \) and for all \( \omega > -\lambda_2 \) in the case of \( 1 < p \leq 5 \). Also we know that from (ii) of proposition 9, \( e^{i\omega t} Q_{2,\omega} \) is orbitally unstable for sufficiently large \( \omega \) in the case of \( p > 5 \). Thus, we observe only the nature of the instability in the supercritical case \( p > 5 \) for large \( \omega \) with the perturbation (14). Figure 1 shows that the profiles of solution \( u_p \) with the initial datum (14) for \( p = 8, \omega = 7 \) and \( \delta_p = 0.001 \). The obtained solution numerically blows up even for small \( \delta_p > 0 \). We also verified that the collapse occurs faster when we increase frequency \( \omega \). Thus, we expect that the standing wave \( e^{i\omega t} Q_{2,\omega} \) is orbitally unstable in the sense of blow-up for all large \( \omega \) in as stated in observation (i).
7.2.2. Stability under the perturbations (15). This section is concerned with the stability of the standing wave $e^{i\omega t}Q_{2,\omega}$ under the perturbation (15). First, we study the standing wave $e^{i\omega t}Q_{2,\omega}$ for $\omega$ close to $-\lambda_2$ in the subcritical case $1 < p < 5$. Figure 2 shows the profile of the observed solution $u_c$ with the initial datum (15) for $p = 3$, $\delta_c = 0.01$ and $\omega = -9$. The amplitude of the observed solution $u_c$ remains close to $|Q_{2,\omega}|$. Furthermore, performing the same calculation with $\delta_c = 0.001$ implies that both curves $|u_c|$ and $|Q_{2,\omega}|$ are almost the same.

Let us examine the profile of the solution $u_c$ from another point of view. Figure 3 shows that the values of two quantities $\int_0^1 |u_c|^2 dx / \int_{-1}^1 |u_c|^2 dx$ and $\int_{-1}^0 |u_c|^2 dx / \int_{-1}^1 |u_c|^2 dx$. Since
the centre of the initial data shifts to right from the origin, only a small amount of the power ($\approx 0.6\%$) flows from the left side ($x < 0$) to the right side ($x > 0$). Then, the two quantities $\int_{0}^{1} |u_c|^2 \, dx / \int_{-1}^{1} |u_c|^2 \, dx$ and $\int_{0}^{1} |u_c|^2 \, dx / \int_{-1}^{1} |u_c|^2 \, dx$ increase only 2% of the initial values, without changing the profile of the solution.

Next, we consider the standing wave $e^{i\omega t} Q_{2,\omega}$ for somehow large frequency $\omega$ in the subcritical case $1 < p < 5$. Figure 4 shows the profile of obtained solution $u_c$ with the initial datum (15) for $p = 3$, $\omega = -3.8$ and $\delta_c = 0.001$. The profile of the solution $u_c$ in figure 4 is significantly different from that of the solution $u_c$ in figure 2 although the size of $\delta_c$ in
Figure 5. Normalized powers $\int_0^1 |u_c|^2 \, dx / \int_{-1}^1 |u_c|^2 \, dx$ (solid line) and $\int_0^0 |u_c|^2 \, dx / \int_{-1}^1 |u_c|^2 \, dx$ (dashed line). Here, $p = 3$, $\omega = -3.8$ and $\delta_c = 0.001$ as in figure 4.

Figure 6. Normalized powers $\int_0^1 |u_c|^2 \, dx / \int_{-1}^1 |u_c|^2 \, dx$ (solid line) and $\int_0^0 |u_c|^2 \, dx / \int_{-1}^1 |u_c|^2 \, dx$ (dashed line) for $p = 7$, $\omega = -9.3$ and $\delta_c = 0.01$.

Figure 4 is only one-tenth as large as that in figure 2. The effect of the perturbation (15) causes a dramatic change in the profile of the solution $u_c$. Indeed, the value of the two quantities $\int_0^1 |u_c|^2 \, dx / \int_{-1}^1 |u_c|^2 \, dx$ and $\int_0^0 |u_c|^2 \, dx / \int_{-1}^1 |u_c|^2 \, dx$ changes quickly when the solution $u_c$ reflects the boundaries (see figure 5). About 95% of the $L^2$-norm is moving between the left and the right side by $t = 4$. As a result, we suggest the standing wave $e^{i\omega t} Q_{2,\omega}$ is orbitally unstable if $\omega$ is large. The profile of solution $u_c$ shown in figure 4 suggests that the orbital instability holds for some frequencies around $-3.8$.

Finally, we focus on the stability of standing wave $e^{i\omega t} Q_{2,\omega}$ in critical and supercritical cases $p \geq 5$. Our simulations show that the same phenomenon as in the subcritical case ($1 < p < 5$) holds if the frequency $\omega$ is close to $-\lambda_2$. For example, figure 6 shows the values of two quantities $\int_0^1 |u_c|^2 \, dx / \int_{-1}^1 |u_c|^2 \, dx$ and $\int_0^0 |u_c|^2 \, dx / \int_{-1}^1 |u_c|^2 \, dx$, where $u_c$ is the
solution with the initial datum \((15)\) for \(p = 7, \omega = -9.3\) and \(\delta_c = 0.01\). Figure 6 seems similar to figure 3. Moreover, we note that less than 3\% of the \(L^2\)-norm moves between the left and right side. Similar profiles of the solution with initial datum \((15)\) are observed for any other \(\omega\) which is close to \(-\lambda_2\).

Repeating the same simulation when varying the frequency \(\omega\), we see that the behaviour of the solution \(u_c\) significantly changes at \(\omega = -7\) and we see that the strong instability of the standing wave \(e^{i\omega t} Q_{2, \omega}\) occurs. Moreover, the collapse occurs faster when we increase \(\omega\).
Figure 9. $|u_c(t, x)|$ (solid line) and $|Q_{2,\omega}|$ (dashed line) as a function of $x$, at various values of $t$ in the subcritical case. Here, $p = 1.9$, $\omega = 200 \gg -\lambda_2$ and $\delta_c = 0.002$.

Figure 10. $|u_c(t, x)|$ (solid line) and $|Q_{2,\omega}|$ (dashed line) as a function of $x$, at various values of $t$ in the subcritical case. Here, $p = 1.9$, $\omega = 10$ and $\delta_c = 0.0001$.

Indeed, as shown in figures 7 and 8, there exists a concentration of the $L^2$-norm of the solution $u_c$.

Therefore, we expect the statements of (ii) with the case $k = 2$ of observation in section 1.

7.2.3. Remark for observation (iii) with $k = 2$. It is natural to address the question of stability of the standing wave $e^{i\omega t}Q_{2,\omega}$ for sufficiently large $\omega$ in the subcritical case $1 < p < 5$. Figure 9 shows the profile of the solution $u_c$ with the initial datum (15) for $p = 1.9$, $\omega = 200$ and $\delta_c = 0.002$. The obtained solution $u_c$ keeps almost the same profile except some oscillations until $t = 63$. On the other hand, figure 10 shows that the profile of the solution with the initial datum (15) for $p = 1.9$, $\omega = 10$ and $\delta_c = 0.0001$ changes before $t = 7$. We may interpret this difference as follows. Since the $L^2$-norm of the solution $u_c$ becomes very large in figure 9, the oscillations do not occur except for very large values of time which makes difficult the numerical observation of these phenomena and requires an unreachable time of calculation. That is why we are not sure whether $Q_{2,\omega}$ remains orbitally unstable under the perturbation (15) for sufficiently large $\omega$.

Remark 27. In [37, 46], the authors observed the drift instability, that is, the centre of mass defined by $\int x |u(t, x)|^2/\|u(0, . )\|_2^2 \, dx$ drifts away from its initial location even for arbitrarily small perturbation. In their case, the principal amount of the $L^2$-norm moves away to the right side if it is initially perturbed to the right direction (see figures 11, 13, and 16 in [37]). In our
case, the centre of mass oscillates between $-1$ and $1$. The principal amount of the $L^2$-norm keeps moving between the left and right side.

7.3. Second excited state

Concerning the stability of the second excited state $e^{i\omega t} Q_{3,0}$, our numerical computation shows that the same results as the case $k = 2$ hold for all $p > 1$ and both perturbations (14) and (15). Thus, we refrain from explaining the details.

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