Prior entanglement between senders enables perfect quantum network coding with modification

Masahito Hayashi
ERATO-SORST Quantum Computation and Information Project, Japan Science and Technology Agency, 201 Daini Hongo White Bldg. 5-28-3, Hongo, Bunkyo-ku, Tokyo 113-0033, Japan.

We find a protocol transmitting two quantum states crossly in the butterfly network only with prior entanglement between two senders. This protocol requires only one qubit transmission or two classical bits transmission in each channel in the butterfly network. It is also proved that it is impossible without prior entanglement. More precisely, an upper bound of average fidelity is given in the butterfly network when prior entanglement is not allowed.

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I. INTRODUCTION

Recently, long distance transmission of quantum state has been actively researched by many various groups. Hence, when global network of quantum communication becomes realized, the efficient use of quantum network is essential. Especially, a large scale network often has a bottleneck point that causes a transmission rate relatively small for the size of its communication resource. Hence, it is required to resolve this bottleneck problem in order to realize the high communication rate. In the classical network system, Ahlswede et al. formulated this problem as network coding, and gave its solution as a coding in typical examples like the butterfly network. That is, they showed that the two informations can be sent crossly in the butterfly network, in which, all channels are allowed to transmit only one bit, then, the bottleneck is the channel $F$. The butterfly network seems only a specific example of network coding, however, it represents the properties of networks with bottlenecks so that their solution gave a trigger for a more general solution. Concerning quantum system, Hayashi et al. initiated to study transmitting quantum state based on the quantum network as its quantum extension. In particular, they focused on the butterfly network, and proved that perfect quantum state transmission is impossible in the butterfly network, i.e., the bottleneck $F$ cannot be resolved in the quantum setting. After this research, Iwama et al. treated quantum network coding with various types of networks.

On the other hand, prior entanglement provides some miracle performances in quantum information. In dense coding, prior entanglement enables the two-bit classical information only by one qubit transmission. In quantum teleportation, prior entanglement enables the transmission of quantum state only by the sending classical information. Hence, it is an interesting problem to discuss whether prior entanglement enhances quantum network coding.

In order to resolve this bottleneck, Leung et al. proposed a network code that transmits the quantum state crossly by use of shared entanglement between any two parties. That is, they pointed out that if all pairs of the sender and the receiver share prior entanglement, combination of quantum teleportation and dense coding enables perfect quantum state transmission in this network. If two-bit transmission is allowed instead of one qubit transmission, only using quantum teleportation, perfect transmission of quantum state is available. However, their protocol requires preparing shared entanglement among four players. So, it is required to reduce the number of players sharing the prior entanglement because increase of this number yields increase of the communication cost in the preparation stage.

In this paper, we treat the butterfly network, with/without prior entanglement between only two senders. As our result, we find that prior entanglement between two senders enables perfect quantum transmission while it is impossible without prior entanglement even in the following modification of the rule of quantum network coding treated by Hayashi et al. In this paper, we allow either one qubit transmission or two-bit classical communication in Fig. although Hayashi et al. allow only one qubit transmission in all channels in Fig. This is because one qubit transmission can be exchanged with two-bit classical communication under...
the prior entanglement. We prove that perfect quantum state transmission is impossible even in such a modification. This type of use of prior entanglement seems to suggest further application of prior entanglement. Hence, future research of this direction can be expected.

Now, we give the classical protocol by Ahlswede et al. [3] for the butterfly network (Fig 1). The purpose is sending the one-bit classical information $X_i$ from the site $A_1$ to the site $B_1$, and sending the other one-bit classical information $X_2$ from the site $A_2$ to the site $B_2$. In this case, all channels can send only one-bit information. The bottleneck is the channel $F$ between the sites $C_1$ and $C_2$. The solution is given by FIG 2. That is, the receiver $B_i$ recovers the information $X_i$ by taking the sum of two received bits.

![FIG. 2: Ahlswede et al. [3]s protocol](image)

However, the same protocol is impossible in the quantum case. In this paper, all channels $D_1$, $D_2$, $E_1$, $E_2$, $F$, $G_1$, $G_2$ can transmit only one qubit or two-bit classical information. The purpose is sending the one-qubit state $\psi_1$ from the site $A_1$ to the site $B_1$, and sending the other one-qubit state $\psi_2$ from the site $A_2$ to the site $B_2$. The main result is the protocol transmitting these two quantum states crossly with two prior Bell states between the senders $A_1$ and $A_2$. Further, we prove its impossibility without prior entanglement. That is, the average $\frac{H_R + H_S}{2}$ is less than 0.9504, where $f_i$ is the average fidelity between the sent state on $A_i$ and the recovered state on $B_i$ with the uniform distribution concerning $\psi_i$. In our proof, we only use the constraint that the size of the channel $F$ is either one qubit or two classical bits. Other constraints are not essential for our proof of impossibility part. Note that Hayashi et al. [7] obtained the upper bound 0.983 of fidelity. However, they concern the worst case instead of the average case.

Finally, we should comment the relation between the main idea and the preceding researches. Leung et al. [5] gave the relationship between the secret sharing [8] and the capacity region of quantum network coding with the butterfly network. Our proof of impossible part is motivated by this method. However, our proof does not use any result concerning secret sharing. Only the relation $I(R : A) + I(R : B) \leq 2H(R)$ in [7] is used, where $I(R : A)$ is the mutual information $H(R) + H(A) - H(RA)$ and $H(A)$ is the von Neumann entropy of the system $A$. In fact, they conjectured that prior entanglement between neighboring parties cannot enhance the ability of quantum network code in the butterfly network. As is mentioned in discussions, we can show this conjecture as a byproduct. This is a great advantage of our method.

This paper uses several relations in quantum information, in which the corresponding relation in the book [9] is referred. Further, while their approach can treat only the asymptotic case where the fidelity goes to 1, our approach can treat the finite fidelity.

II. OUR PROTOCOL

Now, we give our protocol, which enables transmitting quantum state perfectly and crossly based on the butterfly network, whose protocol is summarized by FIG 3. In our protocol, we essentially use quantum teleportation [1]. Assume that the two senders $A_1$ and $A_2$ share two pairs of the maximally entangled state $\Phi^+$, where the first pair has two sites $A_{1,1}$ and $A_{2,1}$ and the second pair has other two sites $A_{1,2}$ and $A_{2,2}$. The two senders $A_1$ and $A_2$ prepare their states $|\psi_1\rangle$ and $|\psi_2\rangle$.

![FIG. 3: Our protocol with prior entanglement](image)

In the first step, the sender $A_i$ performs the Bell measurement $\{\Phi^+, \Phi^-, \Psi^+, \Psi^-\}$ on the joint system $A_i \otimes A_{i,i}$, and obtains his data $X_i$, where $(0,0), (1,0), (0,1), (1,1)$ correspond to $\Phi^+, \Phi^-, \Psi^+, \Psi^-$, respectively. In this case the state on the remaining site $A_{i,i+1}$ is $U(X_i \otimes X_j)^{-1}|\psi_{i+1}\rangle$, where $U(X)$ is the recovering unitary operation for teleportation with the outcome $X$.

In the second step, the sender $A_i$ performs the unitary operation $U(X_i)^{-1}$ to the remaining site $A_{i,i+1}$. Hence, the state on the system $A_{i,i+1}$ becomes $U(X_i)^{-1}U(X_{i+1})^{-1}|\psi_{i+1}\rangle = c(X_i, X_{i+1})U(X_1 \otimes X_2)^{-1}|\psi_{i+1}\rangle$, where $|c(X_i, X_{i+1})| = 1$. Then, the sender $A_i$ sends the system $A_{i,i+1}$ to $B_{i+1}$ via the channel $E_i$. 

He also sends the classical information $X_t$ via the channel $D_t$. In the third step, the site $C_t$ sends the classical information $X_t \otimes X_{2}$ via the channel $F$. Also, the site $C_{2}$ sends the same classical information $X_t \otimes X_{2}$ to $B_1$ and $B_2$ via the channels $G_1$ and $G_2$.

In the final step, the receiver $B_t$ performs the unitary operation $U(X_t \otimes X_2)$ to the received state $U(X_t \otimes X_2)^{-1} \otimes \phi_i$. Then, he recovers the original state $U(X_t \otimes X_2)U(X_1 \otimes X_2)^{-1} \otimes |\psi_i\rangle = |\psi_i\rangle$. This protocol can be extended to the qudit case.

One may think that the phase factor causes dephasing when the state to be sent is entangled. Assume that $|\phi\rangle$ is the initial state among $A_1$, $A_2$, and their reference system $R$. When the data $X_t$ and $X_2$ are obtained, the final state is $U(X_t \otimes X_2)U(X_1 \otimes X_2)^{-1} \otimes U(X_t \otimes X_2)U(X_2)^{-1} \otimes \rho_R|\phi\rangle = c(X_t, X_2)c(X_t, X_1)|\phi\rangle$. Then, the state can be recovered perfectly.

### III. PROPERTIES OF QUANTUM INFORMATION SYSTEM

For our proof of impossibility without prior entanglement, we focus on the following six properties in quantum information system: The number $(\ast, \ast)$ represents the number of equation in [9].

[P1] Monotonicity of quantum mutual information (8.38)

\[
I(A : B) := H(A) + H(B) - H(AB) 
\]

satisfies

\[
I(R_1R_2 : B_1B_2) \leq I(R_1R_2 : E_1E_2F).
\]

[P2] Sum of quantum mutual information: As was shown by Imai et al. [7], the inequality

\[
I(R_1 : E_1) + I(R_1 : B_1) \leq 2H(R_1)
\]

holds.

[P3] Chain rule of quantum mutual information (5.75):
The conditional quantum mutual information $I(A : B|C) := H(AC) + H(BC) - H(ABC) - H(C)$ satisfies

\[
I(A : BC) = I(A : B|C) + I(A : C).
\]

[P4] Convexity of quantum transmission information (8.42):

For a channel $\kappa$ from $A$ to $B$, the quantum transmission information $I_\kappa$ is defined by $I(\kappa) := I(B : R)$ for the state $\kappa \otimes I_R(|\Phi\rangle \langle \Phi|)$, where $|\Phi\rangle$ is the maximally entangled state between the input system and the reference system $R$. Then, the convexity:

\[
\lambda I(\kappa_1) + (1 - \lambda) I(\kappa_2) \geq I(\lambda \kappa_1 + (1 - \lambda) \kappa_2)
\]

holds.

[P5] Quantum Fano inequality (5.51):
The entanglement fidelity $f_e$ concerning the channel $\kappa$ from $A$ to $B$, is given by $\langle \Phi^+ | \kappa \otimes I(B) (|\Phi^+\rangle \langle \Phi^+|) | \Phi^+\rangle$. This quantity satisfies

\[
H_{TW}(RB) \leq \eta(f_e),
\]

where $R$ is the reference system, and $\eta(x) := -x \log_2 x - (1 - x) \log_2 \frac{1}{2}$.  

[P6] Twirling of channel: For any channel $\kappa$, its twirling $\bar{\kappa}$ is defined by $\bar{\kappa}(\rho) := \int U \kappa(U \rho U^\dagger) U^\dagger d(U)$, where $p(dU)$ is the invariant distribution on $SU(2)$. From the convexity, the quantum transmission information of the original channel $\kappa$ is greater than that of its twirling $\bar{\kappa}$. Further, the entanglement fidelity of $\kappa$ is equal to that of $\bar{\kappa}$.

### IV. IMPOSSIBILITY WITHOUT PRIOR ENTANGLEMENT

In this section, we prove that the perfect quantum state transmission is impossible in the butterfly network without prior entanglement. For this purpose, we will prove the entanglement fidelity $f_{e,i}$ concerning the channel $\kappa_i$ from $A_i$ to $B_i$ satisfies the following inequality:

\[
\frac{1}{2} \leq \eta(f_{e,1} + f_{e,2}).
\]

Solving this inequality yields $f_{e,1} + f_{e,2} \leq 0.9256$. Since the average $f_i$ of the fidelity $E_{\psi_i}(\langle \psi_i | (\psi_i(\langle \psi_i | (\psi_i|) \psi_i)$ is equal to $\frac{f_{e,1} + f_{e,2}}{2}$, we obtain $f_{e,1} + f_{e,2} \leq 0.9504$.

Now, we prove the inequality (1). Let $R_t$ be the reference system of $A_t$. We focus on the channel with the input $E_1$ and the output $E_2 \otimes B_1$. Since $H(R_t) = 2$, $P_2$ implies that

\[
I(R_1 : E_1) \leq 2 - I(R_1 : B_1). \tag{2}
\]

Similarly,

\[
I(R_2 : E_2) \leq 2 - I(R_2 : B_2). \tag{3}
\]

As there is no prior entanglement, the system $R_1E_1D_1$ is independent of the system $R_2E_2D_2$. Hence,

\[
I(R_1 : R_2), I(E_1 : E_2) \leq I(R_1E_1 : R_2E_2) \leq I(R_1E_1 : R_2D_1) = 0. \tag{4}
\]

Thus,

\[
I(R_1R_2 : E_1E_2) = I(R_1 : E_1) - I(R_2 : E_2)
\]

\[
= H(R_1R_2) + H(E_1E_2) - H(R_1E_1) - H(R_2E_2)
\]

\[
- H(R_1) - H(E_1) + H(R_1E_1)
\]

\[
- H(R_2) - H(E_2) + H(R_2E_2)
\]

\[
= - I(R_1 : R_2) - I(E_1 : E_2) + I(R_1E_1 : R_2E_2) = 0. \tag{5}
\]

When $F$ is a one-qubit quantum channel, the relation $H(F) < H(E_1E_2) + H(E_2)$ holds. Now, let $H$ be the auxiliary system of $FR_1R_2E_1E_2$, then $H(R_1R_2E_1E_2) - H(FR_1R_2E_1E_2) = H(F) - H(H) \leq H(F)$. Hence,

\[
I(R_1R_2 : F|E_1E_2)
\]

\[
= H(R_1R_2E_1E_2) - H(FR_1R_2E_1E_2)
\]

\[
+ H(F) - H(E_1E_2) \leq 2H(F) \leq 2. \tag{6}
\]
When $F$ is a two-bit classical channel, the relation $H(R_1R_2E_1E_2) \leq H(FR_1R_2E_1E_2)$ holds. Thus,

\begin{align*}
I(R_1R_2 : F|E_1E_2) &= H(R_1R_2E_1E_2) - H(FR_1R_2E_1E_2) \\
&+ H(FE_1E_2) - H(E_1E_2) \\
&\leq H(F) \leq 2.
\end{align*}

(7)

Combining the above relations with P3 and P1, we obtain

\begin{align*}
I(R_1R_2 : B_1B_2) &\leq I(R_1R_2 : E_1E_2F) \\
&= I(R_1R_2 : F|E_1E_2) + I(R_1R_2 : E_1E_2) \\
&\leq 2 + I(R_1 : E_1) + I(R_2 : E_2) \\
&\leq 2 + 2 - I(R_1 : B_1) + 2 - I(R_2 : B_2).
\end{align*}

(8)

where the second inequality follows from (5) - (7), and the final inequality follows from \ref{eq:p6} and \ref{eq:p5}.

Now, we focus on the twirling of $\kappa_i$ by $\pi_i$, and denote the transmission informations in the case of $\pi_1$, $\pi_2$, and $\pi_1 \otimes \pi_2$ by $I_{TW}(R_1 : B_1)$, $I_{TW}(R_2 : B_2)$, and $I_{TW}(R_1R_2 : B_1B_2)$, respectively. In the case of the twirling channel, the entropy of system $X$ is described by $H_{TW}(X)$. Note that the entanglement fidelity of $\kappa_i$ is equal to that of $\pi_i$. Using \ref{eq:p6}, we have

\begin{align*}
6 &\geq I(R_1R_2 : B_1B_2) + I(R_1 : B_1) + I(R_2 : B_2) \\
&\geq I_{TW}(R_1R_2 : B_1B_2) + I_{TW}(R_1 : B_1) + I_{TW}(R_2 : B_2) \\
&= 4 - H_{TW}(R_1R_2B_1B_2) \\
&+ 2 - H_{TW}(R_1B_1) + 2 - H_{TW}(R_2B_2) \\
&= 8 - 2H_{TW}(R_1B_1) + H_{TW}(R_2B_2).
\end{align*}

Thus,

\begin{align*}
1 &\leq H_{TW}(R_1B_1) + H_{TW}(R_2B_2).
\end{align*}

(9)

\ref{eq:p6} and \ref{eq:p5} imply

\begin{equation*}
H_{TW}(R_iB_i) \leq \eta(f_{e,i}).
\end{equation*}

Therefore,

\begin{equation*}
1 \leq \eta(f_{e,1}) + \eta(f_{e,2}).
\end{equation*}

Using the convexity of $\eta$, we obtain \ref{eq:ineq1}.

In fact, our discussion on impossible part can be applied to the asymptotic case. We assume that $N$ times use of the butterfly network, $n_i$ qubit is sent from $A_i$ to $B_i$ perfectly. As a generalization of \ref{eq:ineq1}, the relation

\begin{align*}
I(R_1R_2 : B_1B_2) &\leq 2N + 2n_1 - I(R_1 : B_1) + 2n_2 - I(R_2 : B_2)
\end{align*}

holds. Since $I(R_1 : B_1) \rightarrow 2n_1$, and $I(R_1R_2 : B_1B_2) \rightarrow 2n_1 + 2n_2$, we obtain

\begin{equation*}
N \geq n_1 + n_2,
\end{equation*}

which has been obtained by Leung et al. \cite{ref5}.

\section*{V. DISCUSSIONS}

In this paper, we focused on quantum network coding in the butterfly network, and considered the effect of the existence of the prior entanglement between the senders $A_1$ and $A_2$. As the first result, we found a protocol transmitting two quantum states crossly using two prior Bell states. In the second result, we proved the impossibility when no prior entangled state is allowed. Our proof is based on information theoretical method, while Hayashi et al. \cite{ref4}’s evaluation is based on computational method. In our proof, we use the non-existence of prior entanglement only in \ref{eq:p6}. Other parts do not require this property. Hence, even if the prior entanglement between neighboring parties in the network, the discussion in Sec. IV holds. This argument was conjectured by Leung et al. \cite{ref5}.

In our protocol with prior entanglement, $I(R_1E_1 : R_2E_2) = 2$, while $I(R_1 : R_2) = I(E_1 : E_2) = 0$. Hence, the property $I(R_1E_1 : R_2E_2) = 0$ is essential for our proof. This fact indicates that good protocols should have non-zero mutual information $I(R_1E_1 : R_2E_2)$. That is, this fact may become a good indicator for seeking good network code in the quantum case.

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