Method Monte - Carlo for solving of non - linear integral equations.

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Abstract

We offer in this short report a simple Monte - Carlo method for solving a well - posed non - linear integral equations of second Fredholm’s and Volterra’s type and built a confidence region for solution in an uniform norm, applying the grounded Central Limit Theorem in the Banach space of continuous functions.

We prove that the rate of convergence our method coincides with the classical one.

Key words and phrases.

Non - linear integral equation of the Fredholm’s and Volterra’s second type, confidence region, uniform norm, Monte - Carlo method, metric spaces, metric entropy, compactness, Borelian distribution, weak convergence and compactness, Hölder’s condition, kernel, distance, Mittag - Leffler’s function, contraction principle, partition, random process and field (r.f.), Law of Large Numbers (LLN), Banach space, Lipschitz condition, convergence, asymptotical approach, Central Limit Theorem (CLT), Banach space of continuous functions, Gaussian distribution, iterations, estimations, well posedness, approximation, speed of convergence, Lebesgue - Riesz spaces, moments, spectral radius, modulus of the uniform continuity.
1 Definitions. Statement of problem. Notations.
Fredholm’s equations.

Let $\{T = \{t\}, B, \mu, d\}$ be compact (complete) metric probabilistic measure space: $\mu(T) = 1$, equipped with a non-trivial (bounded) distance function $d = d(t, s)$, $t, s \in T$. It will be presumed as usually that the measure $\mu$ is Borelian.

We consider in this preprint the following non-linear integral equation of the second Fredholm’s type

$$x(t) = f(t) + \int_T K(t, s, x(s)) \mu(ds). \quad (1)$$

The function $f : T \to R$ in (1) will be presumed to be continuous, $K = K(t, s, z)$, $t, s \in T$, $z \in R$ is common continuous kernel, non-linear in general case relative the last variable $z$; $x = x(t)$, $x : T \to R$ is unknown function.

The case of Volterra’s second type of integral equation will be considered in a penultimate section.

We agree to take henceforth to write

$$\int g(s) \, d\mu = \int_T g(s) \, \mu(ds)$$

for any measurable function $g : T \to R$.

We will deal with the space of all continuous numerical valued functions $C(T, d) = C(T)$ equipped with an ordinary uniform norm

$$||x|| = ||x||_{C(T)} \overset{\text{def}}{=} \sup_{t \in T} |x(t)| = \max_{t \in T} |x(t)|.$$

The case of the Lebesgue - Riesz spaces $L_p(T, \mu)$ is considered in many articles, in particular in [19].

We must introduce in this regards some additional notations and assumptions. Namely, let $f = f(t)$ be (bounded) continuous function: $f(\cdot) \in C(T)$. Further, impose the following important condition on the kernel function $K = K(t, s, z)$: it will be bounded, common continuous and satisfies relative the third argument Lipschitz condition with constant less than 1:

$$\exists \rho = \text{const} \in (0, 1) \Rightarrow |K(t, s, z_1) - K(t, s, z_2)| \leq \rho |z_1 - z_2|. \quad (2)$$

It is known, see e.g. [18], chapter 16, section 4; [22] that the source problem (1) is well-posed: the continuous solution $x = x(t)$ there exists, is unique and continuously depended on the datum $f(\cdot), K(\cdot, \cdot, \cdot)$, of course, subject to the condition (2). See also the classical monographs [7], chapters 2,3; [8], chapter 1.

Moreover, the solution $x = x(t)$ may be obtained in particular as an uniform limit as $n \to \infty$ of the following recursion: $x_0(t) := f(t)$,
\[ x_{n+1}(t) := f(t) + \int_T K(t, s, x_n(s)) \mu(ds), \ n = 0, 1, 2, \ldots \]  

(3) 

Briefly:

\[ x_{n+1}(t) := R[f, K, x_n](t); \]  

(4) 

where

\[ R[f, K, x_n](t) = \int_T K(t, s, x_n(s)) \mu(ds), \ n = 0, 1, 2, \ldots \]  

(5) 

An error estimate:

\[ ||x - x_m|| \leq ||x_1 - x_0|| \cdot \frac{\rho^m}{1 - \rho}, \ m = 1, 2, \ldots, \]  

(6) 

contraction principle.

We will agree to write for certain random field (r.f.) \( \Delta(t) \)

\[ \text{Law}(\Delta(\cdot)) = N(a(t), R(t,s)), \ t, s \in T \]  

if the r.f. \( \Delta(t) \) has a Gaussian (normal) distribution with parameters

\[ \mathbb{E}\Delta(t) = a(t), \ \text{Cov}(\Delta(t), \Delta(s)) = R(t,s). \]  

As a rule in this report \( a(t) = 0; \) — the so-called centered case.

Of course, the function \( R = R(t,s) \) must be finite, symmetrical and non-negative definite.

Further, let \( \{\Delta_N\} = \{\Delta_N(t)\}, \ t \in T, \ N = 1, 2, \ldots \) be a sequence of continuous a.e. random fields.

We will agree to write for the sequence of random fields \( \Delta_N(t) \) converges weakly (in distribution) in this space \( C(T, d) = C(T) \), as \( N \to \infty \), to some r.f. \( \Delta(t) \),

\[ \text{Law}(\Delta_N(\cdot)) \overset{\text{dist}}{\to} \text{Law}(\Delta(\cdot)). \]

As a rule, the limiting r.f. \( \Delta \) will be Gaussian and centered.

We recall further some facts about this convergence (Prokhorov - Skorokhod theory).

## 2 Description of method. Convergence.

But the question arises: how to calculate the integrals holding in (3) integrals?

Our claim in this report is to offer the consistent Monte-Carlo method, more precisely, a depending trials method, for these integral
computations, and to built an asymptotical as $N \to \infty$ confidence domain for solution in an uniform norm.

We will prove that the rate of convergence as $N \to \infty$ of this method in the uniform norm for the $x_m(t)$ calculation is equal to $1/\sqrt{N}$, where $N$ is an amount of common elapsed random numbers having the distribution $\mu$.

More precisely, we will ground that the classical normed $\sqrt{N}$, where $N$ denotes the amount of all elapsed random variables, deviation between Monte-Carlo approximation and deterministic one satisfies the Central Limit Theorem (CLT) in the space of all continuous functions.

In detail. Let $\vec{\xi} = \xi[N] = \{\xi(1), \xi(2), \xi(3), \ldots, \xi(N)\}$, $\xi = \xi(1)$ be a $N$-tuple of independent random variables (r.v.-s) having the distribution $\mu$:

$$P(\xi(i) \in A) = \mu(A)$$

and $N$ is some "great" integer number $N >> 1$.

Introduce a set of integer numbers

$$S = S(N) = \{1, 2, 3, \ldots, N - 1, N\}, \quad (7)$$

and introduce also some its partition $Q = \{Q_k\}, k = 1, 2, \ldots, m$ as follows

$$Q(1) = \{1, 2, \ldots, n(1)\}; \quad Q(2) = \{n(1) + 1, n(1) + 2, \ldots, n(2)\}, \ldots, \quad (8)$$

$$Q(m - 1) = \{n(m - 1) + 1, n(m - 1) + 2, \ldots, n(m)\}. \quad (9)$$

Of course, $\{n(j)\}$ are positive integer numbers and such that $1 < n(1) < n(2) \ldots < n(m - 1) < n(m) = N$.

Introduce the following integer vector depending (in general case) on the value $N$, (and of course on the number of iterations $m$) of differences $\vec{q} = q[N] = \{q(k)\}, k = 1, 2, \ldots, m$:

$$q(1) = q[N](1), \quad q(2) = q[N](2) = n(2) - n(1),$$

$$q(3) = q[N](3) = n(3) - n(2), \ldots, q(m) = q[N](m) = n(m) - n(m - 1). \quad (10)$$

Evidently, the numbers $q[N](k)$ are positive integer and

$$\sum_{k=1}^{m} q[N](k) = n(m) = N. \quad (12)$$

We impose in the sequel throughout all the report the following important conditions on the introduced vector either
\[
\lim_{N \to \infty} q[N](k) = \infty, \quad \lim_{N \to \infty} q[N](m)/N > 0; \quad (13)
\]

or sometimes a more strong condition
\[
\lim_{N \to \infty} q[N](k) = \infty, \quad \lim_{N \to \infty} q[N](m)/N = 1. \quad (14)
\]

Let us introduce also a following vector depending on the variable \( N \) and on the value \( m \):
\[
\gamma = \vec{\gamma} = \vec{\gamma}_N = \{ \gamma_N(i) \}, \quad i = 1, 2, \ldots, m, \quad \dim \gamma = m, \quad (15)
\]
\[
\gamma(1) = \gamma_N(1) = n(1)/N, \quad \gamma(2) = \gamma_N(2) = [n(2) - n(1)]/N, \quad (16)
\]
\[
\gamma(3) = \gamma_N(m) = [n(3) - n(2)]/N, \ldots, \quad \gamma(m) = \gamma_N(m) = [n(m) - n(m-1)]/N. \quad (17)
\]

Evidently, \( \gamma_N(k) \in (0, 1) \) and
\[
\sum_{k=1}^{m} \gamma(k) = \sum_{k=1}^{m} \gamma_N(k) = 1.
\]

**We introduce also the condition that as \( N \to \infty \Rightarrow**
\[
0 < \lim_{N \to \infty} \min_k \gamma_N(k) \leq \lim_{N \to \infty} \max_k \gamma_N(k) < 1. \quad (18)
\]

In particular, the sequence \( \gamma_N(k) \) can be selected as
\[
\gamma_N(k) = 1/m, \quad k = 1, 2, \ldots, m \quad (19)
\]

Evidently, the condition (18) entails one (13).

The **optimal choice** of the variables \( q[N](k), \quad k = 1, 2, \ldots, m \) will be clarified below.

Let us offer the following **iterative** Monte Carlo procedure \( x^0_0(t) = x_0(t) := f(t), \) and for the values \( k = 1, 2, 3, 4, \ldots, m - 1, m, \)
\[
x^1_1(t) = x^{1(N)}_1(t) = f(t) + \frac{1}{n(1)} \sum_{i=1}^{n(1)} K(t, \xi(i), x^0_0(\xi(i))),
\]
\[
x^2_2(t) = x^{2(N)}_2(t) = f(t) + \frac{1}{n(2) - n(1)} \sum_{i=n(1)+1}^{n(2)} K(t, \xi(i), x^1_1(\xi(i))),
\]
\[
\]
\[ x_k(t) = x_k^{(N)}(t) = f(t) + \frac{1}{n(k) - n(k-1)} \sum_{i=n(k-1)+1}^{n(k)} K(t, \xi(i), x_{k-1}^{i}(\xi(i))), \tag{22} \]

\[ x_m(t) = x_m^{(N)}(t) = f(t) + \frac{1}{n(m) - n(m-1)} \sum_{i=n(m-1)+1}^{n(m)} K(t, \xi(i), x_{m-1}^{i}(\xi(i))), \tag{23} \]

and recall that \( n(m) = N \).

**Theorem 2.1.** Suppose in addition that the metric space \((T, d)\) is (complete) compact. Then the sequence of random fields \( x_m^{(N)}(t) \) converges as \( N \to \infty \) uniformly with probability one to the approximation \( x_m = x_m(t), t \in T \):

\[ P \left( \lim_{N \to \infty} \max_{t \in T} |x_m^N(t) - x_m(t)| \to 0 \right) = 1, \tag{24} \]

as well as this proposition holds true for all the previous values \( k \).

**Proof.** Note that the Banach space \( C(T, d) = C(T) \) of all numerical values continuous functions is complete and separable. Therefore one can apply the famous Law of Large Numbers (LLN) in this Banach space, see e.g. [10], [11], [23], chapters 4,5. As long as the kernel \( K = K(t, s, z) \) is continuous and bounded, it follows from (20) that

\[ P(||x_1^1 - x_1||C(T) \to 0) = 1. \]

By means of an induction

\[ P(||x_k^k - x_k||C(T) \to 0) = 1, \ k = 2, 3, \ldots, m. \]

The proposition of theorem 2.1 follows for the value \( k = m \).

**Example 2.1.** Let us consider the problem of computations (multiple, in general case) parametric integrals of the form

\[ I(t) = \int_T g(t, s) \mu(ds), t \in T \tag{25} \]

by means of the Monte-Carlo method

\[ I_N(t) = N^{-1} \sum_{i=1}^{N} g(t, \xi(i)), \tag{26} \]

in our notations.

This method appear at first by A.S.Frolov and N.N.Tchentzov in [12], 1962, and was named as ”depending trials method”, as long as the values \( I_n(t_1) \) and \( I_n(t_2), t_{1,2} \in T \) are in general case correlated.
The modification of this method for discontinuous functions is offered in [14], for non-linear Partial Differential Equations of Navier-Stokes type - in [27].

Assume now that for almost all the values \( s \) the function \( g(t, s) \) is continuous relative the value \( t \) and is bounded:

\[
\sup_{i, s \in T} |g(t, s)| < \infty.
\]

Define the probability of the uniform convergence

\[
P_{uc}[g] := P(\sup_{t \in T} |I_N(t) - I(t)| \to 0).
\]

Then

\[
P_{uc}[g] = 1. \tag{28}
\]

3 Main result. Investigation of convergence. Asymptotic confidence region.

Central Limit Theorem in the space of continuous functions.

For the concrete error estimate, on the other words, for the building of the confidence domain in the uniform norm we need to evaluate the following tail probability (inside the framework of example 2.1)

\[
P_N(u) \overset{def}{=} P(\sqrt{N} \ |I_N - I| > u), \ u = \text{const} \geq 1. \tag{29}
\]

In order to evaluate the tail probability \( P_N(u) \), we need to apply the so-called Central Limit Theorem (CLT) in the Banach space of all continuous functions \( C(T) \).

Let us recall some used facts from this theory. Let \( \eta_i = \eta_i(t), \ t \in T, \ \eta(t) := \eta_1(t) \) be a sequence of centered (mean zero): \( \mathbb{E}\eta_i(t) = 0 \) independent identically distributed random fields, having uniformly bounded second moment:

\[
\sigma^2[\eta] := \sup_{t \in T} \mathbb{E}\eta^2(t) < \infty. \tag{30}
\]

Denote the normed sum

\[
S_N(t) := N^{-1/2} \sum_{i=1}^{N} \eta_i(t), \ N = 1, 2, \ldots. \tag{31}
\]

Evidently, the finite-dimensional distributions of \( S_N(t) \) converge as \( N \to \infty \) to ones for Gaussian centered r.f. \( S(t) = S_\infty(t) \) having at the same covariation function \( R_S(t, s) \) as \( \eta(t) \)
\( R_S(t, s) = R_\eta(t, s) = R_{S_N}(t, s) = ES(t)S(s) = E\eta(t)\eta(s), \text{ } t, s \in T. \)

By definition, the sequence of r.f. \( \{\eta_i(t)\} \), or simple the individual r.f. \( \eta(t) \), satisfies the CLT in the (Banach) space \( C(T) \) iff the sequence of distributions of \( S_N(\cdot) \) in the space \( C(T) \) converges weakly to one for \( S(\cdot) \): that is, for arbitrary bounded continuous functional \( G : C(T) \rightarrow R \)

\[
\lim_{N \rightarrow \infty} E G(S_N) = E G(S). \tag{32}
\]

If (32) there holds, then take place the convergence of correspondent tail functions

\[
\lim_{N \rightarrow \infty} P(||S_N|| > u) = P(||S|| > u), \text{ } u > 0. \tag{33}
\]

The asymptotical behavior as \( u \rightarrow \infty \), as well as non-asymptotical estimations for the tail probability \( P_\infty(u) = P(u) \), is investigated in many works, see e.g. [5], [9], [24], chapter 3, sections 3.1 - 3.5; [28] etc. Roughly speaking,

\[
\ln P(u) \sim -\frac{u^2}{2 \max_{t \in T} R(t, t)} = -\frac{u^2}{2 \sigma^2[\eta]}.
\]

The Central Limit Theorem (CLT) in the space of continuous functions \( C(T, d) \) and its applications are devoted many works, e.g. [3], [4], [6], [12], [14], [21], [24], chapter 4, section 4.4; [26] etc.

Let us quote as an example the following result belonging to G.Pizier [29]. Define for some value \( p \geq 2 \) the following bounded semi-distance natural function

\[
d_p[\eta](t_1, t_2) \overset{def}{=} ||\eta(t_1) - \eta(t_2)||_p, \text{ } t_1, t_2 \in T. \tag{34}
\]

Hereafter the notation ||\theta||_p denotes the usually Lebesgue - Riesz \( L(p) \) norm of the r.v. \( \theta \):

\[
||\theta||_p = \left[ E|\theta|^p \right]^{1/p}.
\]

Denote by \( N(T, d, \epsilon) \) the minimal number of \( d \) closed balls covering the whole space \( T \). Evidently, \( \forall \epsilon > 0 \Rightarrow N(T, d, \epsilon) < \infty \) iff the metric set \( (T, d) \) is pre-compact set.

The value \( H(T, d, \epsilon) = \ln N(T, d, \epsilon) \) is named as metric entropy of the set \( T \) relative the distance \( d \). Some examples of entropy evaluation may be found, e.g. in [24], chapter 3, sections 3.1 - 3.3. In particular, if the set \( T \) is closed bounded subset of the Euclidean space \( R^d \) equipped with ordinary Euclidean distance \( |t_1 - t_2| \) and for which

\[
r(t, s) \asymp |t - s|^\alpha, \text{ } \alpha = \text{const} \in (0, 1), \tag{35}
\]
then

\[ N(T, r, \epsilon) \asymp \epsilon^{-d/\alpha}, \quad \epsilon \in (0, 1). \]  

(36)

It is known, see the famous work of G. Pizier [29], that if

\[ \int_0^1 N^{1/p}(T, d_p[\eta], \epsilon) \, d\epsilon < \infty, \]  

(37)

then the sequence of random fields \{\eta_i(t)\} satisfies the CLT in the space \( C(T, d_p) \). See also a more general proposition in [24], chapter 3, section 3.17.

The last condition (37) is satisfied if for example \( T \) is closed bounded subset of whole Euclidean space \( \mathbb{R}^d \) with correspondent norm \( |t| \) and if \( \exists \beta > 0, \exists C < \infty \Rightarrow \)

\[ E|\eta(t) - \eta(s)|^p \leq C|t - s|^{d+\beta}, \]  

(38)

Kolmogorov - Slutsky condition.

Another version of the CLT in the space of continuous functions may be found in particular in [15].

Let us return to the source tail function (29). We need to apply mentioned before CLT in the space \( C(T) \). Denote

\[ \zeta_N(t) = \sqrt{N} (I_N(t) - I(t)). \]

Suppose that the r.f. \( \zeta_i(t) = g(t, \xi(i)) - I(t), \xi(t) = \xi_1(t) = g(t, \xi) - I(t), \xi = \xi_1 \) satisfies the Central Limit Theorem (CLT) in the space \( C(T) \). This implies by definition that the sequence of distributions of r.f. \( \zeta_N(\cdot) \) in the space \( C(T) \) converges as \( N \to \infty \) weakly, i.e. in distribution, to the centered continuous Gaussian distributed random field \( \zeta(t) = \zeta_\infty(t) \) having at the same covariation function as \( \zeta_1(t) : R[\zeta](t_1, t_2) := \)

\[ E(\zeta(t_1) - I(t_1))(\zeta(t_2) - I(t_2)) = \int_T g(t_1, s)g(t_2, s) \mu(ds) - I(t_1)I(t_2). \]

Then

\[ \lim_{N \to \infty} P_N(u) = P_\infty(u) = P(u), \quad u > 0, \]  

(39)

where

\[ P_\infty(u) = P(\sup_{t \in T} |\zeta_\infty(t)| > u). \]  

(40)

As a consequence: as \( u \to \infty \)

\[ \ln P(u) \sim -\frac{u^2}{2 \max_{t \in T} R[\zeta](t, t)}. \]
In order to make sure the CLT for the random field \( g(t, \xi) - I(t) \), it is sufficient to refer the Pizier conditions. Namely, suppose
\[
\beta = \beta^2[g] = \sup_{t \in T} \text{Var}[g(t, \xi)] < \infty, \\
\exists p \geq 2 \Rightarrow \sup_{t \in T} ||g(t, \xi)||_p < \infty,
\]
and that
\[
\int_0^1 N^{1/p} [T, d_p[g], \epsilon] d\epsilon < \infty, \tag{41}
\]
where the distance function \( d_p[g](t_1, t_2) \) is defined above
\[
d_p[g](t_1, t_2) := || [g(t_1, \xi) - g(t_2, \xi)] - [I(t_1) - I(t_2)] ||_p,
\]
As was noted above, the condition (41) is satisfied if \( T \) is bounded closed subset of whole space \( \mathbb{R}^d \) with ordinary Euclidean norm \(|t|\), and if
\[
\mathbb{E}|g(t, \xi) - g(s, \xi)|^p \leq C|t - s|^{d+\theta}, \exists \theta > 0. \tag{42}
\]
It is clear that if the metric \( d_p[g](\cdot, \cdot) \) is continuous relative the source one \( d(\cdot, \cdot) \), then the CLT in the space \( C(T, d_p[g]) \) entails one in \( C(T, d) \).

Preliminary considerations.

Let us investigate the first random approximation (20). Assume that the r.f \( K(t, \xi_i, f(\xi_i)) \) satisfies the CLT in the space \( C(T, d) \); then one can write (approximately)
\[
x_1(t) = f(t) + \frac{1}{n(1)} \sum_{i=1}^{n(1)} K(t, \xi_i, x_0^0(\xi_i)) = \\
f(t) + \frac{1}{n(1)} \sum_{i=1}^{n(1)} K(t, \xi_i, f(\xi_i)) = \\
x_1(t) + \frac{1}{\sqrt{n(1)}} \zeta_1(t) = x_1(t) + \frac{1}{\sqrt{q[N](1)}} \zeta_1(t), \tag{43}
\]
where
\[
\text{Law}(\zeta_1(\cdot)) \xrightarrow{\text{dist}} N(0, R_1[K,f](t_1, t_2)). \tag{44}
\]
and
\[
R_1[K,f](t_1, t_2) = R_1(t, s) := \int_T K(t_1, s, f(s)) K(t_2, s, f(s)) \mu(ds) -
\]
\[ \int_{T} K(t_1, s, f(s)) \mu(ds) \int_{T} K(t_2, s, f(s)) \mu(ds). \]

Let us suppose temporarily

\[ \text{Law}(\zeta_1(\cdot)) = N(0, R_1[f](t_1, t_2)). \] (45)

The case of the second iteration is more complicated. Let us impose in addition the following condition on the kernel function \( K = K(t, s, z) : \)

\[ \exists \lambda = \lambda(t, s), \theta = \theta(t, s; z, v), \delta = \text{const} \in (0, 1] \]

such that

\[ \sup_{t, s \in T} |\lambda(t, s)| < \infty, \sup_{t, s \in T, z, v \in R} |\theta(t, s; z, v)| < \infty, \Rightarrow \]

\[ K(t, s, z) - K(t, s, v) = \lambda(t, s) (z - v) + \theta(t, s; z, v)|z - v|^{1+\delta}. \] (46)

We derive under this condition (46) as \( N \to \infty \)

\[ x_2^2(t) = f(t) + \frac{1}{n(2) - n(1)} \sum_{i=n(1)+1}^{n(2)} K(t, \xi_i, x_1(x_1(\xi_i))) \sim f(t) + \]

\[ \frac{1}{q[N](1)} \sum_{i=n(1)+1}^{n(2)} K[(t, \xi_i, x_1(\xi_i)) + [n(1)]^{-1/2} \zeta_2(\xi_i)] \sim \]

\[ f(t) + \int_{T} K(t, s, x_1(s)) \mu(ds) + \frac{1}{\sqrt{n(2) - n(1)}} \zeta_2(t) = \]

\[ x_2(t) + (q[N](2)N)^{-1/2} \zeta_2(t), \]

where \( \zeta_2(t) = \zeta_2[K, f; N](t) \) is (approximately) Gaussian centered continuous random field with covariation function

\[ R_2(t_1, t_2) = R_2[f, K](t_1, t_2) = \int_{T} K(t_1, s, x_1(s)) K(t_2, s, x_1(s)) \mu(ds) - \]

\[ \int_{T} K(t_1, s, x_1(s)) \mu(ds) \cdot \int_{T} K(t_2, s, x_1(s)) \mu(ds). \]

We find quite analogously for the values \( k = 3, 4, \ldots, m, \) especially for the "final" value \( k = m \) the approximate representations

\[ x_k^k(t) = x_k(t) + (q[N](k))^{-1/2} \zeta_k(t), \] (47)

where \( \zeta_2(t) = \zeta_2[K, f; N](t) \) is (approximately) Gaussian centered continuous random field with covariation function
\[ R_k(t_1, t_2) = R_k[f, K](t_1, t_2) = \int_T K(t_1, s, x_{k-1}(s)) K(t_2, s, x_{k-1}(s)) \mu(ds) - \int_T K(t_1, s, x_{k-1}(s)) \mu(ds) \cdot \int_T K(t_2, s, x_{k-1}(s)) \mu(ds). \]

The relation (47) in the case \( k = m \) implies exactly the CLT in the space \( C(T, d) \) for the Monte Carlo approximation \( x_m^m(t) \) for the \( m^{th} \) iteration.

It remains to ground the applicability of the CLT for our integrals \( x_k^k(t) \), \( k = 1, 2, \ldots, m; \ t \in T \), especially for the extremal case \( k = m \). We start as above from the case \( x_1(t) \).

Introduce the following semi - distance on the set \( T \) depending on some numerical parameter \( p(1), p(1) \geq 2 \)

\[ d_{p(1)}(t_1, t_2) := \left\{ \int_T \left[ \left| K(t_1, s, f(s)) - K(t_2, s, f(s)) \right|^{p(1)} \right] \mu(ds) \right\}^{1/p(1)}. \quad (48) \]

As we know, if

\[ \sup_{t \in T} \int_T \left| K(t, s, f(s)) \right|^{p(1)} \mu(ds) < \infty, \]

\[ \sup_{t \in T} R_1(t, t) < \infty, \]

and

\[ \int_0^1 N^{1/p(1)} \left( T, d_{p(1)}(\epsilon) \right) \mu(d\epsilon) < \infty, \]

then the CLT for \( x_1^1(t) \) in the space \( C(T, d_{p(1)}) \) holds true.

Let us consider a general case \( k = 2, 3, \ldots, m \) by means of induction. Introduce as before the following semi - distances on the set \( T \) depending on some numerical parameter \( p(k), p(k) \geq 2 \)

\[ d_{p(k)}(t_1, t_2) := \left\{ \int_T \left[ \left| K(t_1, s, x_{k-1}(s)) - K(t_2, s, x_{k-1}(s)) \right|^{p(k)} \right] \mu(ds) \right\}^{1/p(k)}. \quad (49) \]

We deduce as above that if

\[ \sup_{t \in T} \int_T \left| K(t, s, x_{k-1}(s)) \right|^{p(k)} \mu(ds) < \infty, \quad (50) \]

\[ \sup_{t \in T} R_k(t, t) < \infty, \quad (51) \]

and
then the CLT for \( x_k^t(t) \) in the space \( C(T, d_{p(k)}) \) holds true.

To summarize:

**Theorem 3.1.** Suppose that all the conditions (50), (93) and (93) are satisfied for all the values \( k = 1, 2, \ldots, m \); as well as all the restrictions formulated before. Assume also that all the introduced before distance functions \( d_{p(k)}(t_1, t_2) \) are continuous relative the source distance \( d = d(t_1, t_2) \).

Then the sequence of random fields \( x_m^N(t) \) satisfies the CLT as \( N \to \infty \) in the space \( C(T, d) \):

\[
\text{Law} \left\{ \sqrt{\log N} (x_m^N(t) - x_m(t)) \right\} \overset{\text{distr}}{\rightarrow} N(0, R_m(t, s)).
\]

(53)

4 The case of non-linear Volterra’s integral equations.

We consider in this section the non-linear Volterra’s integral equation of the ordinary form

\[
X(\tau, y) = f(\tau, y) + \int_0^\tau dv \int_T K(\tau, y, \nu, v, X(\nu, v)) \mu(dv), \quad \tau, \nu \in [0, 1],
\]

(54)

or equally

\[
X(\tau, y) = f(\tau, y) + \int_0^\tau dv \int_T K(\tau, \nu, \tau \nu, v, X(\tau \nu, v)) \mu(dv).
\]

(55)

For brevity,

\[
X(\tau, y) = f(\tau, y) + F [X](\tau, y),
\]

(56)

where again \( \tau, \nu \in [0, 1], y, v \in T \),

\[
F [X](\tau, y) = \int_0^\tau dv \int_T K(\tau, y, \nu, v, X(\nu, v)) \mu(dv)
\]

(57)

\[
\tau \int_0^\tau dv \int_T K(\tau, y, \tau \nu, v, X(\tau \nu, v)) \mu(dv).
\]

(58)

We retain all the previous notations and restrictions: \( (T, d), \mu, \{Q\} \) etc.

The particular case of the equation
\[ X(\tau) = X_0 + \int_0^\tau K(\nu, X(\nu))d\nu \]
correspondent to the well-known Cauchy problem for an ordinary differential equation
\[ dX(\tau)/d\tau = K(\tau, X(\tau)), \quad X(0) = X_0 = \text{const}. \]

Let us introduce also the sequence \( \{\eta_i\}, i = 1, 2, \ldots, N; \eta := \eta_1 \) of independent uniformly distributed on the unit interval \([0, 1]\) r.v., independent also on the source sequence \( \{\xi_j\} \):
\[
P(\eta_i < w) = P(\eta < w) = w, \quad w \in [0, 1],
\]
and of course
\[
P(\eta_i < w) = 0, \quad w < 0; \quad P(\eta_i < w) = 1, \quad w > 1.
\]

Note that
\[
F[X](\tau, y) = \tau EK(\tau, y, \eta, \xi, X(\tau, \eta, \xi)) \quad \text{(59)}
\]
with correspondent consistent as \( n \to \infty \) with probability one in the uniform norm \( C([0, 1] \otimes T) \) Monte-Carlo estimation
\[
\hat{F}_n(\tau, y) \overset{\text{def}}{=} n^{-1} \sum_{i=1}^n \tau K(\tau, y, \eta_i, \xi_i, X(\tau\eta_i, \xi_i)). \quad \text{(60)}
\]

We suppose as above that the function \( f = f(\cdot, \cdot) \) and the kernel \( K(\cdot, \cdot, \cdot, \cdot, \cdot) \) are continuous, bounded and that the kernel \( K \) satisfies the famous Lipschitz condition
\[
\exists C \in (0, \infty) \Rightarrow |K(\tau, \nu, v, z_1) - K(\tau, \nu, v, z_2)| \leq C |z_1 - z_2|, \quad \text{(61)}
\]
but we do not assume in contradiction to the previous sections that \( C < 1. \)

The applications of these equations are described in particular in the works [2], [14], [13], [22], [31].

The case of solution of super-linear growth for the kernel \( K = K(\tau, y, \nu, v, z_1) \) having in particular blow-up solution is considered in [1]; see also the reference therein.

It is no hard to derive by means of induction the following estimation
\[
|F^n[X_1](\tau) - F^n[X_2](\tau)| \leq \frac{C^n}{n!} ||X_1 - X_2|| \leq \frac{C^n}{n!} ||X_1 - X_2||, \quad n = 1, 2, \ldots, \quad \text{(62)}
\]
where \( F^n \) denotes the \( n^{th} \) iteration of the operator \( F \), therefore the continuous solution of the equation (54) there exists, is unique and may be calculated by means of iterations: 
\[
X_0 = f(\tau, y),
\]
\[
X_{n+1}(\tau, y) = f(\tau, y) + F[X_n](\tau, y), \tag{63}
\]
see e.g. [2], [13], [16], [31].

Error estimate:
\[
||X_m(\tau) - X(\tau)||C(T) \leq ||X_1(\tau) - X_0(\tau)||C(T) \cdot \sum_{n=m}^{\infty} \frac{C^n}{n!} \tau^n \to 0, \tag{64}
\]
when \( m \to \infty \).

As a consequence:
\[
\sup_{\tau \in [0,1]} ||X_m(\tau) - X(\tau)||C(T) \leq \sup_{\tau \in [0,1]} ||X_1(\tau) - X_0(\tau)||C(T) \cdot \sum_{n=m}^{\infty} \frac{C^n}{n!}, \tag{65}
\]

Each integral appears in (63) may be computed as before by means of depending trial Monte Carlo method, so that the rate of convergence in the uniform norm is equal the classical value \( 1/\sqrt{N} \).

In detail, we retain the notations and conditions of the second section, especially the vector \( \vec{y} \) and partition of the whole set \( S(N), \quad Q = \{Q_k\}, \quad k = 1, 2, \ldots, m \) (8), (9).

Define as above \( X_0^0(\tau, y) = X_0(\tau, y) := f(\tau, y) \) and recursively
\[
X_1^1(\tau, y) = f(\tau, y) + \frac{\tau}{n(1)} \sum_{i=1}^{n(1)} K(\tau, y, \tau \eta_i, \xi_i, X_0^0(\tau \eta_i, \xi_i)), \tag{66}
\]
\[
X_2^2(\tau, y) = f(\tau, y) + \frac{\tau}{n(2) - n(1)} \sum_{i=n(1)+1}^{n(2)} K(\tau, y, \tau \eta_i, \xi_i, X_1^1(\tau \eta_i, \xi_i)), \tag{67}
\]
and for the values \( k = 3, 4, \ldots, m \)
\[
X_k^k(\tau, y) = f(\tau, y) + \frac{\tau}{n(k) - n(k-1)} \sum_{i=n(k-1)+1}^{n(k)} K(\tau, y, \tau \eta_i, \xi_i, X_{k-1}^{k-1}(\tau \eta_i, \xi_i)), \tag{68}
\]
and ultimately \( X_m^m(\tau, y) =
\[
X_m^m(\tau, y) = f(\tau, y) + \frac{\tau}{n(m) - n(m-1)} \sum_{i=n(m-1)+1}^{n(m)} K(\tau, y, \tau \eta_i, \xi_i, X_{m-1}^{m-1}(\tau \eta_i, \xi_i)). \tag{69}
\]
Recall once again that $n(m) = N$.

**Again preliminary considerations.**

Let us investigate the first random approximation (66). Assume that the r.f $(\tau, v) \to \tau \, K(\tau, y, \tau \eta_i, \xi_i, f(\eta_i, \xi_i))$ satisfies the CLT in the space $C([0, 1] \otimes T)$; then one can write (approximately)

\[
X_1 \tau(y) = X_1(\tau, y) + \frac{\tau}{n(1)} \sum_{i=1}^{n(1)} K(\tau, y, \tau \eta_i, \xi_i, X_0(\tau \eta_i, \xi_i)) = \\
X_1(\tau, y) + \frac{1}{\sqrt{n(1)}} \beta_1(\tau, y) = X_1(\tau, y) + \frac{1}{\sqrt{q[N](1)}} \beta_1(\tau, y),
\]

where $\beta_1(\tau, y)$ is a random field such that

\[
\text{Law}(\beta_1(\cdot)) \xrightarrow{\text{dist}} N\left(0, R^V_1[K, f](\tau_1, \tau_2, y_1, y_2)\right),
\]

and

\[
R^V_1[K, f](\tau_1, \tau_2, y_1, y_2) = R^V_1(\tau_1, \tau_2, y_1, y_2) :=
\]

\[
\tau_1 \tau_2 \int_0^1 dv \int_T K(\tau_1, y_1, \tau_1 v, X_0(\tau_1 v, v)) \, K(\tau_2, y_2, \tau_2 v, X_0(\tau_2 v, v)) \mu(dv) -
\]

\[
\tau_1 \int_0^1 dv \int_T K(\tau_1, y_1, \tau_1 v, X_0(\tau_1 v, v)) \mu(dv) \times
\]

\[
\tau_2 \int_0^1 dv \int_T K(\tau_2, y_2, \tau_2 v, X_0(\tau_2 v, v)) \mu(dv).
\]

Let us suppose temporarily as above

\[
\text{Law}(\beta_1(\cdot)) = N\left(0, R^V_1[f](\tau_1, \tau_2, y_1, y_2)\right).
\]

The case of the second iteration is more complicated. Let us impose in addition the following condition on the kernel function $K = K(t, s, y, v, z)$:

\[
\exists \Lambda = \Lambda(\tau, \nu, y, v), \exists \Theta = \Theta(\tau, \nu; y, v, z_1, z_2), \exists \delta = \text{const} \in (0, 1]
\]

such that

\[
\sup_{\tau, \nu \in [0, 1]} \sup_{y, v \in T} |\Lambda(\tau, \nu, y, v)| < \infty, \quad \sup_{\tau, \nu \in [0, 1]} \sup_{z_1, z_2 \in R} |\Theta(\tau, \nu; y, v, z_1, z_2)| < \infty,
\]

and

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\[ K(\tau, \nu, y, v, z_1) - K(\tau, \nu, y, v, z_2) = \]
\[ \Lambda(\tau, \nu, y, v) (z_1 - z_2) + \Theta(\tau, \nu; y, v, z_1, z_2) |z_1 - z_2|^{1+\delta}. \] (78)

We derive under this condition (78) as \( N \to \infty \) as before

\[ X_2^2(\tau, y) = f(\tau, y) + \frac{1}{n(2) - n(1)} \sum_{i=n(1)+1}^{n(2)} K(\tau, \tau \eta_i, y, \xi_i, X_1(\tau \eta_i, \xi_i)) \sim f(\tau, y) + \]
\[ \frac{1}{\gamma(2) N} \sum_{i=n(1)+1}^{n(2)} K \left[ (\tau, \tau \eta_i, y, \xi_i, X_1(\tau \eta_i, \xi_i) + [n(1)]^{-1/2} \beta_1(\tau \eta_i, \xi_i)) \right] \sim \]
\[ f(\tau, y) + \tau \int_0^1 d\nu \int_T K(\tau, \nu, y, v, X_1(\tau \nu, v)) \mu(d\nu) + \frac{1}{\sqrt{n(2) - n(1)}} \beta_2(\tau, y) = \]
\[ X_2(\tau, y) + (q[N](2))^{-1/2} \beta_2(\tau, y), \]
where \( \beta_2(\tau, y) = \beta_2[K, f; N](\tau, y) \) is (approximately) Gaussian centered continuous random field

\[ \text{Law}(\beta_2(\cdot, \cdot)) \overset{\text{dist}}{\to} N \left( 0, R^V_2[K, f](\tau_1, \tau_2, y_1, y_2) \right), \] (79)

with covariation function

\[ R^V_2[K, f](\tau_1, \tau_2, y_1, y_2) = R^V_2(\tau_1, \tau_2, y_1, y_2) := \] (80)

\[ \tau_1 \tau_2 \int_0^1 d\nu \int_T K(\tau_1, y_1, \tau_1 \nu, v, X_1(\tau_1 \nu, v)) K(\tau_2, y_2, \tau_2 \nu, v, X_1(\tau_2 \nu, v)) \mu(dv) - \] (81)

\[ \tau_1 \int_0^1 d\nu \int_T K(\tau_1, y_1, \tau_1 \nu, v, X_1(\tau_1 \nu, v)) K(\tau_2, y_2, \tau_2 \nu, v, X_1(\tau_2 \nu, v)) \mu(dv) \times \] (82)

\[ \tau_2 \int_0^1 d\nu \int_T K(\tau_2, y_2, \tau_2 \nu, v, X_1(\tau_2 \nu, v)) \mu(dv). \] (83)

We find quite analogously for the values \( k = 3, 4, \ldots, m \), especially for the "ultimate" value \( k = m \) the approximate representations \( X_k^m(\tau, y) = \)

\[ X_k(\tau, y) + (q[N](k))^{-1/2} \beta_k(\tau, y), \] (84)
where \( \beta_k(\tau, y) = \beta_k[K, f; N](\tau, y) \) is as \( N \to \infty \) approximately Gaussian centered continuous random field with covariation function
\[ R_k^V[K, f](\tau_1, \tau_2, y_1, y_2) = R_k^V(\tau_1, \tau_2, y_1, y_2) := \tau_1 \tau_2 \times (85) \]

\[ \int_0^1 d\nu \int_T K(\tau_1, y_1, \tau_1 \nu, v, X_{k-1}(\tau_1 \nu, v)) K(\tau_2, y_2, \tau_2 \nu, v, X_{k-1}(\tau_2 \nu, v)) \mu(d\nu) = (86) \]

\[ \tau_1 \int_0^1 d\nu \int_T K(\tau_1, y_1, \tau_1 \nu, v, X_{k-1}(\tau_1 \nu, v)) \mu(d\nu) \times (87) \]

\[ \tau_2 \int_0^1 d\nu \int_T K(\tau_2, y_2, \tau_2 \nu, v, X_{k-1}(\tau_2 \nu, v)) \mu(d\nu). (88) \]

The relation (84) in the case \( k = m \) implies exactly the CLT in the space \( C([0,1] \otimes T) \) for the Monte Carlo approximation \( X_m^m(\tau, y) \) for the \( m^{th} \) iteration.

It remains to ground the applicability of the CLT in the space \( C([0,1] \otimes T) \) for our integrals \( X_k^k(t, y) \), \( k = 1, 2, \ldots, m; \ t \in [0,1], \ y \in T \) especially for the extremal case \( k = m \). We start as above from the case \( X_1^1(\tau, y) \).

Introduce the following semi-distance on the set \( U := [0,1] \otimes T \) depending on some numerical parameter \( p(1), \ p(1) \geq 2 \)

\[ r_{p(1)}(\tau_1, y_1; \tau_2, y_2) \overset{\text{def}}{=} \left\{ \int_0^1 d\nu \int_T \left| K(\tau_1, \nu, y_1, f(\nu, v)) - K(\tau_2, \nu, y_2, f(\nu, v)) \right|^{p(1)} \mu(d\nu) \right\}^{1/p(1)}. (89) \]

As we know, if

\[ \sup_{\tau \in [0,1]} \sup_{y \in T} \int_0^1 d\nu \int_T |K(\tau, \nu, y, f(\nu, v))|^{p(1)} \mu(d\nu) < \infty, \]

\[ \sup_{\tau \in [0,1]} \sup_{y \in T} R_1^V(\tau, y; \tau, y) < \infty, \]

and

\[ \int_0^1 N^{1/p(1)} \left( U, r_{p(1)}(\epsilon) \right) d\epsilon < \infty, \]

then the CLT for \( X_1^1(\tau, y) \) in the space \( C(U, r_{p(1)}) \) holds true.

Let us consider a general case \( k = 2, 3, \ldots, m \) by means of induction. Introduce as before the following semi-distances on the set \( U \) depending on some numerical parameter \( p(k) \) greatest or equal than 2: \( p(k) \geq 2 \)

\[ r_{p(k)}(\tau_1, y_1; \tau_2, y_2) \overset{\text{def}}{=} \]
\[ \left\{ \int_0^1 d\nu \int_T \left[ |K(\tau_1, \nu, y_1, v, X_{k-1}(\nu, v)) - K(\tau_2, \nu, y_2, v, X_{k-1}(\nu, v))|^{p(k)} \right] \mu(d\nu) \right\}^{1/p(k)}. \]

We deduce as above that if
\[ \sup_{\tau \in [0,1]} \sup_{y \in T} \int_0^1 d\nu \int_T |K(\tau, \nu, X_{k-1}(\nu, y))|^{p(k)} \mu(d\nu) < \infty, \] (91)
\[ \text{sup}_{\tau \in [0,1]} \text{sup}_{y \in T} R^V_k(\tau, \tau, y, y) < \infty, \] (92)
and
\[ \int_0^1 N^{1/p(k)} \left( U, d_{p(k)}, \epsilon \right) d\epsilon < \infty, \] (93)
then the CLT for \( X_k^m(t, y) \) in the space \( C(U, d_{p(k)}) \) holds true.

To summarize:

**Theorem 4.1.** Suppose that all the formulated above conditions are satisfied for all the values \( k = 1, 2, \ldots, m \); as well as all the restrictions formulated before. Assume also that all the introduced before distance functions \( r_{p(k)}(t_1, t_2) \) are continuous relative the source distance \( d = d(\tau_1, y_1, \tau_2, y_2) \).

Then the sequence of random fields \( X_m^m[N](\tau, y) \) satisfies the CLT as \( N \to \infty \) in the space \( C(U, d_{p(k)}) \):

\[
\text{Law}\left\{ \sqrt{q[N](m)} \left( X_m^m[N](\tau, y) - X_m(\tau, y) \right) \right\} \xrightarrow{\text{distr}} N(0, R^V_m(\tau_1, \tau_2, y_1, y_2)).
\] (94)

### 5 Optimal choice of partition

Let us discuss in this section stated above the problem of optimal choice of the partition \( Q \), or equally the sequence of numbers \( \{q(k)\} = \{q[N](k)\} \). Recall the restriction (12):

\[ \sum_{k=1}^m q[N](k) = n(m) = N \] (95)

and a (strong) condition

\[ \lim_{N \to \infty} q[N](k) = \infty, \quad \lim_{N \to \infty} q[N](m)/N = 1. \] (96)
It follows from the grounding of Theorem 3.3 that under formulated in this theorem conditions \( Z = Z_N(q) \) defined \( \operatorname{Var}(x_m^N(t)) \propto \)

\[
[q(m)]^{-1} + [q(m)q(m-1)]^{-1} + [q(m)q(m-1)q(m-2)]^{-1} + \ldots +
\]

\[
\left[\prod_{k=1}^{m-1} q(m-k)\right]^{-1}.
\]

One can solve the optimization problem \( Z_N(q) \to \min \) subject to the limitation (95), as well as taking into account the positivity of \( \{q\} \) and restriction (96), as ordinary by means of Lagrange’s factors method. We will prefer to offer right away the asymptotically as \( N \to \infty \) optimal choice of an optimal value \( \vec{q} = \vec{q}_0 \) omitting some cumbersome computations:

\[
q_0(m) = q_0[N](m) = \operatorname{Ent}\left[N^{1/2} - C_mN^{1/4}\right],
\]

\[
q_0(m-1) = q_0[N](m-1) = \operatorname{Ent}\left[N^{1/4} - C_{m-1}N^{1/8}\right], \ldots ,
\]

\[
q_0(m-k) = q_0[N](m-k) = \operatorname{Ent}\left[N^{2^{-k-1}} - C_{m-k}N^{2^{-k-2}}\right],
\]

for the values \( k = 2, 3, \ldots , m-1 \); end finally \( q_0(1) = q_0[N](1) = \operatorname{Ent}\left[C_1N^{2^{-m}}\right] \), where \( \operatorname{Ent}[z] \) denotes the integer part of the real number \( z \) and \( \{C_k\} \) are appropriate positive constants.

Note that

\[
\forall k = m, m-1, \ldots , 1 \Rightarrow \lim_{N \to \infty} q_0[N](k) = \infty
\]

and especially

\[
\lim_{N \to \infty} \frac{q_0(m)}{\sqrt{N}} = 1.
\]

Therefore, all the conditions of Theorem 3.1 as well as of Theorem 4.1 are satisfied and following the rate of convergence of the offered methods by using of elaborated in this section partition \( q_0[N](k) \) is equal to the value \( 1/\sqrt{N} \), alike in the classical Monte Carlo method, and one can apply the Central Limit Theorem in the space of all continuous functions for the error estimation.

6 Concluding remarks.

**Remark A.** The needed for building of confidence region for \( x_m(t) \) covariation function \( R_m(t_1, t_2) \) may be consistently estimated alike the function \( x_m^N(t) \) itself: \( R_m(t_1, t_2) \approx \tilde{R}_m(t_1, t_2) \), where
\[ \hat{R}_m(t_1, t_2) \overset{def}{=} N^{-1} \sum_{i=1}^{N} K(t_1, \xi_i, x_{m-1}^m(\xi_i)) \sum_{i=1}^{N} K(t_2, \xi_i, x_{m-1}^m(\xi_i)) - \]

\[ N^{-1} \sum_{i=1}^{N} K(t_1, \xi_i, x_{m-1}^m(\xi_i)) \times N^{-1} \sum_{i=1}^{N} K(t_2, \xi_i, x_{m-1}^m(\xi_i)). \]

Analogous approach may be offered for estimation of covariation function for solving by means of the Monte Carlo method of the Volterra’s equation.

**Remark B.** The case of systems of non-linear integral equations may be investigated quite analogously.

**Remark C.** It is interest in our opinion to derive the non-asymptotical confidence domain for \(x_m(\cdot)\) in the uniform norm having at the same range \(N^{-1/2}\) as in the asymptotical case.

**Remark D.** The important for us the distance function \(d_{p(k)}(\cdot, \cdot)\) in (49) may be estimated as follows. Note first of all that

\[ \|x_k\| \leq S(f, \rho) \overset{def}{=} \frac{\|f\|}{1 - \rho}. \]

Denote by \(B(f, \rho)\) the ball in the space \(C(T)\) with the center at origin having radii \(S(f, \rho): m\)

\[ B(f, \rho) = \overset{def}{=} \{ g, g \in B(f, \rho) \cap C(T, d) \}. \]

Then

\[ d_{p(k)}(t_1, t_2) \leq \sup_{g \in B(f, \rho)} \left\{ \int_T \left[ |K(t_1, s, g(s)) - K(t_2, s, g(s))|^{p(k)} \right] \mu(ds) \right\}^{1/p(k)}. \]

**Remark E.** The remainder term appearing in (64) may be estimated by means of equality

\[ E_{1,m}(z) = z^{1-m} \left( e^z - \sum_{k=0}^{m-2} \frac{z^k}{k!} \right), \ m \geq 3, \]

where \(E_{\alpha,\beta}(z)\) is so-called generalized Mittag-Leffler’s function of the form

\[ E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \ \alpha > 0, \beta > 0, \]

see [30]. This function is entire having an order \(1/\alpha\) and type 1.

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