The average cardinality of the minimal teaching set of a threshold function on a two-dimensional rectangular grid

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Abstract

It is known that the minimal teaching set of any threshold function defined on a two-dimensional rectangular grid consists of 3 or 4 points. Exact formulae for the numbers of functions corresponding to these values are proposed. Also, we prove that the average cardinality of the minimal teaching set of a threshold function is asymptotically \( \frac{7}{2} \). Corollaries of these results concerning special arrangements of lines in the plane are discussed.
Introduction

A function, mapping the domain \( \{0, 1, \ldots, m - 1\} \times \{0, 1, \ldots, n - 1\} \) into \( \{0, 1\} \), is called to be threshold iff there exists a line separating the sets of its ones and zeros (points in which the function is 1 and 0, respectively).

We study cardinality properties of a teaching set of a threshold function. The teaching set of a threshold function is a subset of the domain such that the values of the function in the points of the set allow to identify the values of the function in all other points of the domain. A teaching set of \( f \) is called to be minimal (or irreducible) iff no its proper subset is teaching for \( f \). It is known (see, for example, [3, 10, 14]) that the minimal teaching set \( T(f) \) for any threshold function \( f \) is unique (it is also true for threshold functions for more than 2 dimensions). In [10, 12] it is proved that \( T(f) \) consists of 3 or 4 points if \( m \geq 2, n \geq 2 \). Here we propose exact formulae for the number of threshold functions corresponding to these values. We prove that the average cardinality of the minimal teaching set is asymptotically \( \frac{7}{2} \) (if \( m, n \to \infty \)). Also, we discuss some corollaries of this result concerning a special arrangements of lines in the plane.

We note that the cardinality of the minimal teaching set of threshold functions defined on many-dimensional grid is studied in a number of papers; see, for example bibliography in [13]. In particular, the bounds of the average cardinality of the minimal teaching set of threshold functions is studied in [3, 11].
1 The cardinality of the minimal teaching set

Let \( m \geq 2, n \geq 2, E_m = \{0, 1, \ldots, m - 1\}, f : E_m \times E_n \to \{0, 1\} \). We denote by \( M_0(f), M_1(f) \) the set of zeros and the set of ones of \( f \), respectively, i.e.

\[
M_\nu(f) = \{ x = (x_1, x_2) \in E_m \times E_n : f(x) = \nu \} \quad (\nu = 0, 1).
\]

The function \( f \) is called to be \textit{threshold} iff there exist real numbers \( a_0, a_1, a_2 \) such that

\[
M_0(f) = \{ x \in E_m \times E_n : a_1 x_1 + a_2 x_2 \leq a_0 \}. \tag{1}
\]

Without loss of generality we can assume that the numbers \( a_0, a_1, a_2 \) are integer and \( a_1, a_2 \) are not zero simultaneously. We call the line \( a_1 x_1 + a_2 x_2 = a_0 \) the \textit{separation line} for the threshold function \( f \). Note that, with any line \( a_1 x_1 + a_2 x_2 = a_0 \), two threshold functions associate: a function \( f \) such that (1) holds and a function \( f' \) such that \( M_0(f) = \{ x \in E_m \times E_n : a_1 x_1 + a_2 x_2 \geq a_0 \} \). We denote \( \mathcal{T}(m, n) \) the set of all threshold functions defined on \( E_m \times E_n \). Let \( t(m, n) = |\mathcal{T}(m, n)| \).

A set \( T \subseteq E_m \times E_n \) is called to be \textit{teaching} for \( f \in \mathcal{T}(m, n) \) iff for any other function \( f' \in \mathcal{T}(m, n) \) there exists \( x \in T \) such that \( f(x) \neq f'(x) \). A teaching set of \( f \) is called to be \textit{minimal} or \textit{irreducible} iff no its proper subset is teaching for \( f \). A point \( x \in E_m \times E_n \) is called to be \textit{essential} for \( f \in \mathcal{T}(m, n) \) iff there exists \( f' \in \mathcal{T}(m, n) \) such that \( f(x) \neq f'(x) \) and \( f(y) = f'(y) \) for all \( y \neq x \).

It is known (see, for example, [3, 10, 14]), that the minimal teaching set is unique and contains all essential points and only them. Note that this is also true for threshold functions defined on many-dimensional grid.

Denote by \( T(f) \) the minimal teaching set of \( f \). In [10, 12] it is shown that \( |T(f)| \in \{3, 4\} \) for any function \( f \in \mathcal{T}(m, n) \) (see examples on Figures 1, 2).
Figure 1: Example of a function $f$ with $|T(f)| = 3$

Figure 2: Example of a function $f$ with $|T(f)| = 4$
We define the average cardinality of the minimal teaching set as

\[ \sigma(m, n) = \frac{1}{t(m, n)} \sum_{f \in T(m, n)} |T(f)|. \]

Denote

\[ f_q(m, n) = \sum_{-m<i<m} \sum_{n<j<n} \sum_{\gcd(i,j)=q} (m-|i|)(n-|j|). \]

Two different points \( p, p' \) in \( E_m \times E_n \) is called to be adjacent iff the segment \([p, p']\) contains no other point in \( E_m \times E_n \). It is easy to see that \( f_1(m, n) \) is equal to the number of ordered pairs \((p, p')\) of adjacent points in \( E_m \times E_n \).

**Lemma 1** [6] If \( m \leq n \) then

\[ f_q(m, n) = \frac{6}{\pi^2 q^2} (mn)^2 + O(mn^2) \]

Denote by \( l(m, n) \) the number of lines through at least 2 points in \( E_m \times E_n \).

**Lemma 2** [8, 5]

\[ l(m, n) = \frac{1}{2} (f_1(m, n) - f_2(m, n)) \] \hspace{1cm} (2)

**Theorem 1** [7]

\[ t(m, n) = f_1(m, n) + 2. \] \hspace{1cm} (3)

**Proof.** [7] (Sketch) We associate a threshold function to each ordered pair \((p, p')\) of adjacent points in \( E_m \times E_n \) as follows. Draw a line through the points \( p, p' \) and turn it around \( p \) through a small angle in a clockwise direction; see Fig.3. The new line is related to two threshold functions. Let us consider the function such that the set of its zeros is for definiteness “to the right” of the vector \( \overrightarrow{pp'} \). One can show [7, 15] that this
Figure 3: Illustration to the proof of Theorem 1

association is the bijection of the set of all ordered pairs in $E_m \times E_n$ into the set of all functions in $\mathcal{T}(m, n)$ not identically equal to 0 or 1. As already mentioned the number of ordered pairs in $E_m \times E_n$ is $f_1(m, n)$. Taking into account identical 0 and identical 1 we get (3).

Corollary 1

$$t(m, n) = 2 + \sum_{-m<i<m\atop-n<j<n\atop(i,j)=1} (m-|i|)(n-|j|) = \frac{6}{\pi^2}(mn)^2 + O(mn^2) \quad (4)$$

(the asymptotics is true for $m \leq n$).

We note that the asymptotic behavior of $t(m, n)$ was studied by many authors (sometimes with using another terminology), for example, [7, 11, 8, 2, 15]. In particular,
it is stated in [7] that
\[ t(n, n) = \frac{6}{\pi^2} n^4 + O(n^3 \log n). \]

In [1] the asymptotics
\[ t(m, n) = \frac{6}{\pi^2} m^2 n^2 + O(m^2 n \log n) + O(mn^2 \log \log n) \]

is obtained if \( m \leq n \). The asymptotics (4) is found in [6].

**Theorem 2**

\[ \sigma(m, n) = \frac{4f_1(m, n) - 2f_2(m, n)}{f_1(m, n) + 2}. \]  (5)

**Proof.** Denote by \( h(m, n, i, j) \) the number of threshold functions defined on the domain \( E_m \times E_n \) such that the point \((i, j)\) is essential. Then
\[ \sigma(m, n) = \frac{1}{t(m, n)} \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} h(m, n, i, j). \]  (6)

Let us consider all lines containing \((i, j)\) and at least one more point in \( E_m \times E_n \). Denote by \( l(m, n, i, j) \) the number of such lines.

For example, in Fig. 4 all such lines through \((1, 1)\) in \( E_4^2 \) are drawn and we have \( l(4, 4, 1, 1) = 8 \).

These lines gives the partition of the plane into \( 2l(m, n, i, j) \) sectors.

It is easy to see that any new line through \((i, j)\) that belongs to a pair of “vertical” sector and does not coincides with no initial line is associated with 2 threshold functions for which the point \((i, j)\) is essential and the value of the function in \((i, j)\) is 0. It is obvious that lines belonging to different pairs of “vertical” sectors define different threshold functions. Moreover, there is no other function for which the point \((i, j)\) is essential and the value of the function in \((i, j)\) is 0.
In Fig. 4 one of such lines is drawn and one of two different ways to define a threshold function is chosen.

Thus, we have $2l(m, n, i, j)$ threshold functions for which the point $(i, j)$ is essential and the value of the function in $(i, j)$ is 0. We have the same number of functions for which the point $(i, j)$ is essential and the value of the function in $(i, j)$ is 1. Hence $h(m, n, i, j) = 4l(m, n, i, j)$. Now from (6) we get

$$\sigma(m, n) = \frac{4}{t(m, n)} \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} l(m, n, i, j).$$

(7)

Denote by $L(m, n)$ the set of line through at least two points in $E_m \times E_n$. Recall that $l(m, n) = |L(m, n)|$. Consider $\ell \in L(m, n)$. Denote by $z(m, n, \ell)$ the number of
points in \( E_m \times E_n \) belonging to \( \ell \). Now from (7) we get

\[
\bar{\sigma}(m, n) = \frac{4}{t(m, n)} \sum_{\ell \in L(m, n)} z(m, n, \ell) = 4 \cdot \frac{l(m, n) + f_1(m, n)}{t(m, n)}.
\]

Substituting here the expressions (2), (3) we obtain (6). \( \blacksquare \)

**Corollary 2** For \( m \leq n \)

\[
\bar{\sigma}(m, n) = \frac{7}{2} + O \left( \frac{1}{m} \right).
\]

We remark that the asymptotics \( \bar{\sigma}(n, n) \sim \frac{7}{2} \) was also obtained in [4], where another approach [2] was used.

Denote by \( t_\kappa(m, n) \) the number of functions \( f \in T(m, n) \) such that \( |T(f)| = \kappa \) (\( \kappa = 3, 4 \)). From Corollary 2 we get that each of the quantity \( t_3(m, n) \) and \( t_4(m, n) \) is asymptotically \( \frac{1}{2} t(m, n) \). Let us give exact formulae for \( t_3(m, n) \) and \( t_4(m, n) \).

**Corollary 3**

\[
t_3(m, n) = 2f_2(m, n) + 8, \quad t_4(m, n) = f_1(m, n) - 2f_2(m, n) - 6. \quad (8)
\]

**Proof.** From the conditions

\[
t_3(m, n) + t_4(m, n) = t(m, n), \quad \bar{\sigma}(m, n) = \frac{3t_3(m, n) + t_4(m, n)}{t(m, n)}
\]

we find

\[
t_3(m, n) = (4 - \bar{\sigma}(m, n))t(m, n), \quad t_4(m, n) = (\bar{\sigma}(m, n) - 3)t(m, n).
\]

Substituting here the expressions (3), (5) we get (8). \( \blacksquare \)
2 Special arrangements of lines

For a threshold function $f$ the set of vectors $(a_0, a_1, a_2)$, where $a_0, a_1, a_2$ are coefficients of its separating line, is a polyhedral cone defined by the following system of linear inequalities:

$$
\begin{align*}
   a_1 x_1 + a_2 x_2 &\leq a_0 & \text{for each } x \in M_0(f), \\
   a_1 x_1 + a_2 x_2 &> a_0 & \text{for each } x \in M_1(f).
\end{align*}
$$

Furthermore, minimal teaching set $T(f)$ consists of the points that correspond to ir-redundant inequalities in (9). If we are interested in the lines that strictly separate the sets $M_0(f)$ and $M_1(f)$, then all inequalities in (9) have to be substituted by strict ones. Thus, there is a bijection the set $T(m, n)$ into the set of open cones in the partition of the space of parameters $a_0, a_1, a_2$ by all the planes $a_1 x_1 + a_2 x_2 = a_0$, where $(x_1, x_2) \in E_m \times E_n$. Moreover, the planes that form the boundary of each of these cones correspond to the points in $T(f)$. This construction is well-known in threshold logic; see, for example, [10, 14].

Figure 5: The partition of the plane $(a_1, a_2)$ by the set of lines $a_1 x_1 + a_2 x_2 = 1$, where $(x_1, x_2) \in E_m \times E_n$
Denote $T_{\nu}(m, n) = \{ f \in T(m, n) : f(0) = \nu \} \ (\nu = 0, 1)$. It is obvious that the formula $f \leftrightarrow f' = 1 - f$ defines a bijection between the sets $T_{0}(m, n), T_{1}(m, n)$, with any teaching set of $f$ corresponding to a teaching set of $f'$.

If $f \in T_{\nu}(m, n)$ then without loss of generality we can assume that $a_0 = 1$. In this case the set of $(a_1, a_2)$ such that the line $a_1 x_1 + a_2 x_2 = 1$ is separating for $f$ is the set of all solutions to the following system:

$$\begin{align*}
a_1 x_1 + a_2 x_2 &\leq a_0 \quad \text{for all } x \in M_0(f), \\
a_1 x_1 + a_2 x_2 &> a_0 \quad \text{for all } x \in M_1(f).
\end{align*}$$

So we have the bijection of the set $T_{0}(m, n)$ into the set of all open polygonal regions in the partition of the plane $(a_1, a_2)$ by the set of lines $a_1 x_1 + a_2 x_2 = 1$, where $(x_1, x_2) \in E_m \times E_n$ (see. Fig. [5]). Note that the partition of the plane by the lines (10) can be obtained also as a result of intersecting the plane $a_0 = 1$ with the partition of the space by the planes (9).

We call a convex polygon (possibly unbounded) by a generalized $n$-polygon if it can be obtained by intersecting $n$-faced polyhedral cone with a plane that does not contain the vertex of the cone.

From the remarks above and from results in the previous section we obtain the following.

**Corollary 4** Among all regions in the partition of the plane $(a_1, a_2)$ by lines $a_1 x_1 + a_2 x_2 = 1$, where $(x_1, x_2) \in E_m \times E_n$, there are only generalized 3- and 4-polygons. Moreover, the numbers of them are, respectively,

$$\frac{1}{2} t_3(m, n) = f_2(m, n) + 4 = \frac{3}{2\pi^2} m^2 n^2 + O(mn^2),$$

$$\frac{1}{2} t_4(m, n) = \frac{1}{2} f_1(m, n) - f_2(m, n) - 6 = \frac{3}{2\pi^2} m^2 n^2 + O(mn^2).$$
Figure 6: The partition of the plane \((a_1, a_2)\) by the set of lines \(a_1 x_1 + a_2 x_2 = 1\), where 
\((x_1, x_2) \in \{1, 2, \ldots, m\} \times \{1, 2, \ldots, n\}\)  

(\textit{the asymptotics is for } m \leq n). 

Analogous results can be obtained for separations of the plane \(a_1, a_2\) by the lines 
\(a_1 x_1 + a_2 x_2 = 1\), where \((x_1, x_2) \in \{1, 2, \ldots, m\} \times \{1, 2, \ldots, n\}\) etc. In particular, Fig. 4 shows such a separation of the first quarter of the plane.

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