ON SOME INTEGRAL REPRESENTATION OF $\zeta(n)$ INVOLVING NIelsen’S GENERALIZED POLYLOGARITHMS AND THE RELATED PARTITION PROBLEM

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ABSTRACT

In this paper, we study a family of single variable integral representations for some products of $\zeta(2n + 1)$, where $\zeta(z)$ is Riemann zeta function and $n$ is a positive integer. Such representation involves the integral $L_z(a, b) := \frac{1}{(a - 1)!b!} \int_0^1 \log^{a-1}(t) \log^b(1 - t) dt / t$ with positive integers $a, b$, which is related to Nielsen’s generalized polylogarithms. By analyzing the related partition problem, we discuss the structure of such integral representation, especially the condition of expressing products of $\zeta(2n + 1)$ by finite $\mathbb{Q}(\pi)$-linear combination of $L_z(a, b)$.

Keywords Riemann zeta function at integers · integral representation · Nielsen’s generalized polylogarithms

1 Introduction

It’s well known that many number theoretic properties of $\zeta(2n + 1)$ are nowadays still unsolved mysteries, such as the rationality (only known $\zeta(3)$ is irrational), transcendence and existence of closed-form functional equation that satisfied by $\zeta(2n + 1)$. Thanks to the basic functional relation between gamma function $\Gamma(z)$ and sine function $\sin(z)$, i.e. $\Gamma(z)\Gamma(1 - z) = \frac{\pi}{\sin(\pi z)} = \pi z \csc(\pi z)$, one can explicitly express $\zeta(2n)$ by $r_{2n}\pi^{2n}$, where $r_{2n}$ is some rational number related to Bernoulli number $B_{2n}$. Unfortunately, to find such a simple analogous formula for $\zeta(2n + 1)$ is considered to be impossible. Studying the integral and series representations for $\zeta(2n + 1)$ is somehow an important way to analyze the number theoretic properties of $\zeta(2n + 1)$, for instance, F. Beukers’ work [1] is an excellent example. In this paper we discuss a class of integral representation with

$$L_z(a, b) := \frac{1}{(a - 1)!b!} \int_0^1 \log^{a-1}(t) \log^b(1 - t) dt / t$$

Theoretically, many polynomials of $\zeta(n)$ can be represented by this integral. In fact, only $\zeta(2n + 1)$ or $\zeta(2n + 1)^d$ are interesting. Among all family of single variable integral representations that can represent polynomials of $\zeta(n)$, $L_z(a, b)$ is likely the most simple one. Via establishing some linear combination of $L_z(a, b)$ on $\mathbb{Q}(\pi)$, we are even able to express some $\zeta(2n + 1)^d$. However, this method is in somehow restricted, which we shall discuss in the last section.

In fact, $(-1)^{a+b-1}L_z(a, b)$ is exactly a special value $S_{a,b}(1)$ of Nielsen’s generalized polylogarithm $S_{a,b}(z)$, which was introduced by N. Nielsen [2].

$$S_{a,b}(z) = (-1)^{a+b-1} \frac{1}{(a - 1)!b!} \int_0^1 \log^{a}(t) \log^b(1 - zt) dt$$

In most cases, this function is known for mathematical physicists in the context of quantum electrodynamics. Only few literatures [3][4][5] concerned about the special case $S_{a,b}(1)$. However, number theoretic properties of $S_{a,b}(z)$

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have seldom been studied.

Throughout the paper, integrals $\int_0^1 f(t)\,dt$ may be regarded as improper, namely $\int_0^1 - f(t)\,dt$. $\zeta(s)$ denotes Riemann zeta function $\zeta(s) := \sum_{k=1}^{\infty} \frac{1}{k^s}$. In general, $N, m, n, k, i, j, l, a, b$ denote nonnegative integers.

2 Preliminaries

Our main result is based on the following well-known formula([3, p45]).

**Theorem 1.** For $z \in \mathbb{C}$ and $|z| < 1$, we have

$$\Gamma(1 + z) = \exp(-\gamma z + \sum_{k=2}^{\infty} (-1)^{k+1} \zeta(k) \frac{z^k}{k})$$

**Proof.** By the definition of Gamma function

$$\Gamma(z) = \frac{1}{z} e^{-\gamma z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right)^{-1} e^{z/n}$$

we can rewrite it as

$$\log \Gamma(1 + z) = -\gamma z - \sum_{n=1}^{\infty} \log(1 + \frac{z}{n}) - \frac{z}{n}$$

When $|z| < 1$, then $|z/n| < 1$ for all $n = 1, 2, \ldots$, thus we have

$$\log(1 + \frac{z}{n}) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{z^k}{k n^k} = \frac{z}{n} + \sum_{k=2}^{\infty} (-1)^{k+1} \frac{z^k}{k n^k}$$

Therefore

$$\log \Gamma(1 + z) = -\gamma z - \sum_{n=1}^{\infty} \sum_{k=2}^{\infty} (-1)^{k+1} \frac{z^k}{k n^k} + \frac{z}{n} - \frac{z}{n}$$

$$= -\gamma z + \sum_{k=2}^{\infty} \sum_{n=1}^{\infty} (-1)^k \frac{z^k}{k n^k} + \sum_{n=1}^{\infty} \frac{1}{n^k}$$

$$= -\gamma z + \sum_{k=2}^{\infty} (-1)^k \frac{\zeta(k)}{k} z^k$$

Taking $\exp$ on both sides we get what we need to prove. The validity of changing the order of double sum is based on the normal convergence of $\Gamma(1 + z)$.

About the partition problem, the related notations we adopt are following.

For any positive integer $N$, a partition of $N$ is a way written $N$ into the sum of positive integers. Two sums that differ only in the order of their summands are considered the same partition. For given $N$, each partition of $N$ can be regarded as a finite multiset $M = ([n], \mu_M)$ in which the underlying set is $[n] = \{n \in \mathbb{Z} : 1 \leq n \leq N\}$. Therefore $M$ is determined by the multiplicity function $\mu_M : [n] \to \mathbb{Z}_{\geq 0}$ that satisfies

$$\sum_{n=1}^{\infty} n \mu_M(n) = N$$
Now let $X = (x_1, ..., x_N)$, where $x_n = \mu_M(n)$, then $X$ totally determines $M = ([n], \mu_M)$. Therefore the alternative way to define the partition of $N$ is by

**Definition 2.**

$$\mathcal{P}(N) := \{X = (x_1, x_2, ..., x_N) \in \mathbb{Z}^N : \sum_{n=1}^{N} nx_n = N, 0 \leq x_n \leq N\}$$

$\mathcal{P}(N)$ is called the partition set of $N$. Its element is called a partition element of $N$, which denoted by $X$.

For given $X \in \mathcal{P}^d_s(N)$, the support and the norm of $X = (x_1, x_2, ..., x_N)$ are defined by

**Definition 3.**

$$\text{Supp}(X) := \{(n, x_n) : x_n > 0\}$$

$$\|X\| := \sum_{n=1}^{N} x_n$$

The set of restricted partition of $N$ that has exactly $t$ parts and the size of each part is not less than $s(s > 1)$, is denoted by $\mathcal{P}^d_s(N)$, namely

**Definition 4.**

$$\mathcal{P}^d_s(N) := \{X = (0, ..., 0, x_s, ..., x_N) \in \mathcal{P}(N) : \|X\| = t\}$$

On could similarly define $\mathcal{P}^d_s(N), \mathcal{P}^{2t}_s(N), \mathcal{P}^{s\ell}_s(N)$ and so on. In this paper, $\mathcal{P}_2(N)$ is particularly more often used than others. That is

$$\mathcal{P}_2(N) := \{X = (0, x_2, ..., x_N) : \sum_{n=2}^{N} nx_n = N, \forall x_n \in \mathbb{Z}, 0 \leq x_n \leq N\}$$

Further, the odd partition set and even partition set are defined by following

**Definition 5.**

$$\mathcal{P}^{\ell}_s(N) := \{X = (x_1, ..., x_N) \in \mathcal{P}^d_s(N) : x_{2m} = 0 \text{ for all } m \in \mathbb{Z}\}$$

$$\mathcal{P}^{e}_s(N) := \{X = (x_1, ..., x_N) \in \mathcal{P}^d_s(N) : \forall x_{2m-1} = 0 \text{ for all } m \in \mathbb{Z}\}$$

Note that for odd $N$, $\mathcal{P}^{e}_s(N) = \emptyset$ for all $t$, $\mathcal{P}^{\ell}_s(N) = \emptyset$ for all even $t$. Similarly, For even $N$, $\mathcal{P}^{\ell}_s(N) = \emptyset$ for all odd $t$.

Before discussing the relation between $Lz(a, b)$ and $\zeta(n)$, we shall introduce $Lz(a, b)$ and $lz(a, b)$.

**Definition 6.** For nonnegative integers $a, b$, define

$$lz(a, b) := \int_{0}^{1} \log^a(t) \log^b(1-t) dt$$

For positive integers $a, b$, define

$$Lz(a, b) := \frac{1}{a!b!} (lz(a, b) + alz(a - 1, b) + blz(a, b - 1))$$

It’s obvious to see that both $lz$ and $Lz$ are symmetric, namely $lz(a, b) = lz(b, a), Lz(a, b) = Lz(b, a)$. In fact, we have

**Proposition 1.**

$$Lz(a, b) = \frac{1}{(a - 1)!b!} \int_{0}^{1} \frac{\log^{a-1}(t) \log^b(1-t)}{t} dt$$

or by the symmetry,

$$Lz(a, b) = \frac{1}{a!(b-1)!} \int_{0}^{1} \frac{\log^{b-1}(t) \log^a(1-t)}{t} dt$$
Proof. Only need to prove that

\[ lz(a, b) + alz(a - 1, b) + blz(a, b - 1) = b \int_0^1 \log^{b-1}(t) \log^a(1-t) \, dt \]

With integration by parts, one can see

\[ lz(a, b) = -\int_0^1 t(a \log a - b \log b) \, dt \]

Still by substituting \( x = 1 - t \), we obtain immediately

\[ lz(a, b) = -alz(a - 1, b) + b \int_0^1 \log^a(1-x) \, dx \]

That is what we need.

\[ \square \]

3 The relation between \( lz(a, b) \) and \( \zeta(n) \)

Recall the simple relation between gamma function and beta function with \( x, y \in \mathbb{C}, |x|, |y| < 1 \).

\[ f(x, y) := (1 + x + y)B(1 + x, 1 + y) = \frac{\Gamma(1 + x)\Gamma(1 + y)}{\Gamma(1 + x + y)} \] \hspace{1cm} (1)

Since \( Re(1 + x) > 0, Re(1 + y) > 0 \), the left hand side of the equation has the integral representation

\[ B(1 + x, 1 + y) = \int_0^1 t^x(1-t)^y \, dt \]

Our strategy is follow: Applying Taylor’s theorem for multivariate functions \( f(x, y) \). On the one hand, at \( (x, y) = (0, 0) \) any all partial derivatives of \( (1 + x + y)B(1 + x, 1 + y) \) can be evaluated explicitly. On the other hand, applying Theorem \[ \square \] for \( \Gamma(1 + x), \Gamma(1 + y) \) and \( \Gamma(1 + x + y) \), then evaluate the expansion coefficients of \( \frac{\Gamma(1 + x)\Gamma(1 + y)}{\Gamma(1 + x + y)} \), namely

\[ \frac{\Gamma(1 + x)\Gamma(1 + y)}{\Gamma(1 + x + y)} = \exp(\sum_{n=2}^{\infty} (-1)^n \frac{\zeta(n)}{n} (x^n + y^n - (x + y)^n)) \]

If we denote \( (x + y)^n - x^n - y^n \) by \( P_n(x, y) \) or \( P_n \). Let

\[ D_{n,k} := (-1)^{k+1} \frac{\zeta(n)^k}{k!n^k} \]

Thus we can rewrite the above formula as

\[ \frac{\Gamma(1 + x)\Gamma(1 + y)}{\Gamma(1 + x + y)} = \prod_{n=2}^{\infty} \exp((-1)^{n+1} \frac{\zeta(n)}{n} P_n) \]

\[ = \prod_{n=2}^{\infty} (1 + \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\zeta(n)^k}{n^k k!} P_n^k) \]

\[ = \prod_{n=2}^{\infty} (1 + \sum_{k=1}^{\infty} D_{n,k} P_n^k) \]
Due to the normal convergence we can expand and rearrange the last infinite product and rearrange terms with the order up to \( \deg(P^n_k) \), where \( \deg(\cdot) \) is the total degree of polynomial. Since \( P_n(x, y) \) is homogeneous polynomial of \( x, y \), therefore obviously \( P^n_k(x, y) \) is also homogeneous polynomial of \( x, y \) with \( \deg(P^n_k) = k \deg(P_n) = nk \).

That is, we can expand and rearrange \( \prod_{n=2}^{\infty} (1 + \sum_{k=1}^{\infty} D_{n,k} P^n_k) \) as follow

\[
\prod_{n=2}^{\infty} (1 + \sum_{k=1}^{\infty} D_{n,k} P^n_k) = 1 + (D_{2,1} P_2) + (D_{3,1} P_3) + (D_{4,1} P_3 + D_{2,2} P_2^2) + (D_{5,1} P_5 + D_{3,1} D_{2,1} P_3 P_2) + (D_{6,1} P_6 + D_{4,1} D_{2,1} P_4 P_2 + D_{4,2} P_3^2 + D_{2,3} P_2^3) + ...
\]

or

\[
f(x, y) = \prod_{n=2}^{\infty} (1 + \sum_{k=1}^{\infty} D_{n,k} P^n_k) = 1 + \sum_{N=2}^{\infty} \sum_{X \in P_2(N)} \prod_{(\alpha, \beta) \in \text{Supp}(X)} D_{\alpha, \beta} P_{\alpha}^\beta
\]

Notice that

\[
\deg\left( \prod_{(\alpha, \beta) \in \text{Supp}(X)} D_{\alpha, \beta} P_{\alpha}^\beta \right) = \sum_{(\alpha, \beta) \in \text{Supp}(X)} \alpha \beta = N
\]

For fixed positive integers \( a, b \), the term \( x^a y^b \) has degree \( a + b \). Therefore it only appears in \( \sum_{X \in P_2(a+b)} \prod_{(\alpha, \beta) \in \text{Supp}(X)} D_{\alpha, \beta} P_{\alpha}^\beta \). Now we can assume that

\[
\sum_{X \in P_2(N)} \prod_{(\alpha, \beta) \in \text{Supp}(X)} D_{\alpha, \beta} P_{\alpha}^\beta = \sum_{j=1}^{N-1} \rho_{N-j, j} x^{N-j} y^j
\]

Now we can evaluate \( \rho_{a,b} \) in two ways. The first one is integral representation.

**Lemma 1.** For positive integers \( a, b \), we have

\[
\rho_{a,b} = \frac{1}{a! b!} \left. \frac{\partial^{a+b}}{\partial x^a \partial y^b} \right|_{(0,0)} f(x, y) = Lz(a, b)
\]

**Proof.** Let \( g(x, y) = 1 + x + y, h(x, y) = B(1 + x + y) \), notice that \( \frac{\partial}{\partial x} = \frac{\partial}{\partial y} = 1 \) for all \( (x, y) \), therefore any one of second-order partial derivatives \( \frac{\partial^2 g}{\partial x^2}, \frac{\partial^2 g}{\partial y^2}, \frac{\partial^2 g}{\partial x \partial y} \) vanishes. Using Leibniz rule for \( x \)-component

\[
\frac{\partial}{\partial x^a} gh = g \frac{\partial^a}{\partial x^a} h + a \frac{\partial^{a-1}}{\partial x^{a-1}} h
\]

and then for \( y \)-component, we have

\[
\frac{\partial^{a+b}}{\partial x^a \partial y^b} gh = \frac{\partial^b}{\partial y^b} \left( g \frac{\partial^a}{\partial x^a} h + a \frac{\partial^{a-1}}{\partial x^{a-1}} h \right)
\]

\[
= \frac{\partial^{a+b}}{\partial x^a \partial y^b} h + b \frac{\partial^{a+b-1}}{\partial x^{a} \partial y^{b-1}} h + a \frac{\partial^{a+b-1}}{\partial x^{a-1} \partial y^b} h
\]

Its value at the point \( (x, y) = (0, 0) \) is

\[
\left. \frac{\partial^{a+b}}{\partial x^a \partial y^b} \right|_{(0,0)} gh = \left( \frac{\partial^{a+b}}{\partial x^a \partial y^b} h + b \frac{\partial^{a+b-1}}{\partial x^a \partial y^{b-1}} h + a \frac{\partial^{a+b-1}}{\partial x^{a-1} \partial y^b} h \right)_{(0,0)}
\]
On the other hand notice that for positive integer $a, b$

\[ \frac{\partial^{a+b}}{\partial x^a \partial y^b} |_{(0,0)} B(1 + x, 1 + y) = \int_0^1 \frac{\partial^{a+b}}{\partial x^a \partial y^b} |_{(0,0)} t^x (1 - t)^y dt \]

\[ = \int_0^1 \log^a(t) \log^b(1 - t) dt \]

\[ = l_z(a, b) \]

Therefore,

\[ \frac{\partial^{a+b}}{\partial x^a \partial y^b} |_{(0,0)} gh = l_z(a, b) + az(a - 1, b) + bz(a, b - 1) \]

Namely,

\[ \frac{1}{abl} \frac{\partial^{a+b}}{\partial x^a \partial y^b} |_{(0,0)} f(x, y) = l_z(a, b) \]

Now we discuss the second approach.

**Lemma 2.** Assume that $n_j \geq 2, k_j \geq 1$, let

\[ \prod_{j=1}^K P_{n_j}^{k_j} = \sum C_{\lambda, \mu} x^\lambda y^\mu \]

If $\lambda + \mu \neq \sum_{j=1}^K n_j k_j$ or $\lambda \mu = 0$, then $C_{\lambda, \mu} = 0$. Otherwise if $\mu = \sum_{j=1}^K n_j k_j - \lambda$ then $C_{\lambda, \mu}$ is given by

\[ C_{\lambda, \mu} = \sum_{\ell \in \mathcal{S}(\mu)} \prod_{j=1}^K \prod_{i=1}^{k_j} \binom{n_j}{\ell_{ji}} \]

with $\mathcal{S}(\mu)$ as follow

\[ \mathcal{S}(\mu) = \{ \ell : \sum_{j=1}^K \sum_{i=1}^{k_j} \ell_{ji} = \mu, \forall \ell_{ji} \in \mathbb{Z}, 1 \leq \ell_{ji} \leq n_j \} \]

**Proof.** Firstly, it is obviously that $P_{n_j}^{k_j}$ is homogeneous polynomial of degree $n_j k_j$ for each $j$, therefore $\prod_{j=1}^K P_{n_j}^{k_j}$ is also a homogeneous polynomial with $deg(\prod_{j=1}^K P_{n_j}^{k_j}) = \sum_{j=1}^K n_j k_j$. On the other hand, notice that for all $n_j \geq 2$, the coefficient of the terms $x^{n_j}$ and $y^{n_j}$ in $P_{n_j}$ are both 0. Hence only $x^\lambda y^\mu$ with the conditions $\lambda + \mu = \sum_{j=1}^K n_j k_j$ and $\lambda \mu \neq 0$ has nonzero coefficient.

Secondly, Assume that for each $j$ we have

\[ P_{n_j}^{k_j} = \left( \sum_{\ell_{ji} \in \mathcal{S}(\mu)} \binom{n_j}{\ell_{ji}} x^{n_j - \ell_{ji} y^{\ell_{ji}}} \right)^{k_j} \]

In this way, the coefficient of $x^\lambda y^\mu$ in the expansion of $\prod_{j=1}^K P_{n_j}^{k_j}$ should be the sum of all product of $\binom{n_j}{\ell_{ji}}$ that by choose $k_j$ coefficients from $P_{n_j}^{k_j}$ respectively and satisfying that

\[ \sum_{j=1}^K \sum_{i=1}^{k_j} \ell_{ji} = \mu, \forall \ell_{ji} \in \mathbb{Z}, 1 \leq \ell_{ji} \leq n_j \]

We denote such constraint by $\mathcal{S}(\mu)$. In fact it is coincide with

\[ \sum_{j=1}^K \sum_{i=1}^{k_j} n_j - \ell_{ji} = \lambda \]
since $\mu = \sum_{j=1}^{K} n_j k_j - \lambda$. Now for fixed $n_j, k_j, (1 \leq j \leq K)$, $C$ only determined by $\lambda$ or $\mu$, we can form now on simplify this notation by $C_\mu$.

Above Lemma is for general integers $a_j, b_j$, now we reformulate it and only aim to $\text{Supp}(X)$. Assume that $X \in \mathcal{P}_2(N)$, let

$$\prod_{(n,k)\in \text{Supp}(X)} P_n^k = \sum C_b(X) x^a y^b$$

Then if $b > N - \|X\|$ or $b < \|X\|$, then $\sum C_b(X) = 0$. Otherwise, it is given by

$$\sum_{\ell \in \mathcal{S}(b)} \prod_{(n,k)\in \text{Supp}(X)} \prod_{(n) \in \text{Supp}(X)} P_n^k$$

Now we are able to represent $\zeta(N)$ and some $\prod \sum_{n_j=N} \zeta(n_j)$ by $Lz(a, b)$ with $a + b = N$. Following we provide the representation structure of $Lz(a, b)$.

**Theorem 7.** (The Partition represented Relation) Assume that $a \leq b$, then

$$Lz(a, b) = \sum_{X \in \mathcal{P}_2(N), \|X\| \leq b} (c_b(X) \prod_{(n,k)\in \text{supp}(X)} \zeta(n)^k)$$

(4)

where $c_X$ is rational number related to $X$, and it can be evaluated by Lemma 2

**Proof.** By the expansion (3) and Lemma 1, we have

$$\sum_{j=1}^{N-1} Lz(N-j, j) x^{N-j} y^j = \sum_{j=1}^{N-1} \rho_{N-j,j} x^{N-j} y^j$$

$$= \sum_{X \in \mathcal{P}_2(N)} \prod_{(n,k)\in \text{supp}(X)} D_{n,k} P_n^k$$

$$= \sum_{X \in \mathcal{P}_2(N)} \prod_{(n,k)\in \text{supp}(X)} D_{n,k} \prod_{(n) \in \text{supp}(X)} P_n^k$$

It remains to show that the coefficient of $x^{N-b} y^b$ on the right hand side has the form (4). On the other hand, by Lemma 2, we notice that for any term $Q$ of $\prod_{(n,k)\in \text{supp}(X)} P_n^k$

$$\deg_y(Q) = \sum_{(n,k)\in \text{supp}(X)} k = \|X\|$$

and

$$\deg_y(Q) \leq \sum_{(n,k)\in \text{supp}(X)} (n-1)k = N - \|X\|$$

Therefore

$$\prod_{(n,k)\in \text{supp}(X)} P_n^k = \sum_{j=\|X\|}^{N-\|X\|} \sum_{X \in \mathcal{P}_2(N)} C_j(X) x^{N-j} y^j$$

That is to say,

$$\sum_{j=1}^{N-1} Lz(N-j, j) x^{N-j} y^j = \sum_{X \in \mathcal{P}_2(N)} \sum_{j=\|X\|}^{N-\|X\|} q_X C_j(X) x^{N-j} y^j$$
where \( q_X = \prod_{(n,k) \in \text{supp}(X)} D_{n,k} \). Now compare the coefficients on both sides, we have

\[
L_z(N - b, b) = \sum_{X \in \mathcal{P}_2(N), \|X\| \leq b} C_b(X) q_X
\]

Recalling that \( D_{n,k} := (-1)^{k(n+1)} \frac{\zeta(n)^k}{kn^k} \), so there is rational number \( \tilde{C}(X) \) such that

\[
q_X = \prod_{(n,k) \in \text{supp}(X)} D_{n,k} = \prod_{(n,k) \in \text{supp}(X)} \zeta(n)^k
\]

Therefore

\[
L_z(N - b, b) = \sum_{X \in \mathcal{P}_2(N), \|X\| \leq b} C_b(X) \tilde{C}(X) \prod_{(n,k) \in \text{supp}(X)} \zeta(n)^k
\]

Let \( C_b(X) \tilde{C}(X) \) denoted by \( c_b(X) \). It’s obviously rational. We finally have

\[
L_z(a, b) = \sum_{X \in \mathcal{P}_2(N), \|X\| \leq b} (c_b(X) \prod_{(n,k) \in \text{supp}(X)} \zeta(n)^k)
\]

The expression of \( c_b(X) \) is given by following

**Theorem 8.** Assume that \( \text{Supp}(X) = \{(n_j, k_j) : 1 \leq j \leq J\} \), then

\[
c_b(X) = (-1)^{N+\|X\|} \prod_{j=1}^{J} \frac{1}{k_j! n_j^{k_j}} \sum_{\ell \in \mathcal{S}(b)} \prod_{i=1}^{k_j} \binom{n_j}{\ell_{j_i}}
\]

or rewrite as

\[
c_b(X) = (-1)^{N+\|X\|} \prod_{(n,k) \in \text{supp}(X)} \frac{1}{k!} \sum_{\ell \in \mathcal{S}(b)} \prod_{i=1}^{k} \frac{(n-1)!}{\ell!(n-\ell)!}
\]

where \( \mathcal{S}(b) \) is given by

\[
\mathcal{S}(b) = \{ \ell \in \mathbb{Z} : \sum_{j=1}^{K} \sum_{i=1}^{k_j} \ell_{j_i}, 1 \leq \ell \leq b - 1 \}
\]

**Proof.** The proof is straightforward. Recall that \( c_b(X) = C_b(X) \tilde{C}(X) \), now on the one hand we have,

\[
\tilde{C}(X) = \prod_{(n,k) \in \text{supp}(X)} (-1)^{k(n+1)} \frac{1}{k! n^{k}}
\]

\[
= (-1)^{\sum_{j=1}^{J} k_j(n_j+1)} \prod_{j=1}^{J} \frac{1}{k_j! n_j^{k_j}}
\]

\[
= (-1)^{N+\|X\|} \prod_{j=1}^{J} \frac{1}{k_j! n_j^{k_j}}
\]

On the other hand, by Lemma 2

\[
C_b(X) = \sum_{\ell=b}^{J} \prod_{j=1}^{J} \prod_{i=1}^{k_j} \binom{n_j}{\ell_{j_i}}
\]

Multiply \( C_b(X) \) and \( \tilde{C}(X) \) together, then we have what we need.

\[
\square
\]
4 Properties of $Lz(a, b)$ and $l_z(a, b)$

Proposition 2.

$$Lz(a, 1) = \frac{(-1)^a}{a!} \int_0^\infty \frac{e^z}{e^z - 1}dz = (-1)^a \zeta(a + 1).$$

The proof is also easy, consider the substitution of $t = 1 - e^{-z}$. This formula connects $Lz(a, 1)$ to the well-known formula about $\zeta(n)$.

Proposition 3.

$$Lz(2n - 1, 2) = n\zeta(2n + 1) - \sum_{j=2}^{n} \zeta(j)\zeta(2n - j + 1)$$

$$Lz(2n - 2, 2) = (n - \frac{1}{2})\zeta(2n) - \sum_{j=2}^{n} \zeta(j)\zeta(2n - j)$$

or mixing them, as

$$Lz(a, 2) = \frac{a + 1}{2} \zeta(a + 2) - \sum_{j=2}^{[\frac{a}{2}]+1} \zeta(j)\zeta(a + 2 - j)$$

Proof. By Theorem 7, only need to consider the restricted partition that has merely one or two parts. For the case I. $N = 2n + 1$

For exactly one-part partition, there is only one element $X_1 \in \mathcal{P}_2(N)$, and $Supp(X_1) = \{(N, 1)\}$. Similarly, for two-part partition, there are $n - 1$ elements: $X_{2,j} \in \mathcal{P}_2(N)$ with $Supp(X_{2,j}) = \{(j, 1), (N - j, 1)\},$ where $j = 2, ..., n$. Therefore

$$Lz(2n - 1, 2) = c_2(X_1)\zeta(2n + 1) + \sum_{j=2}^{n} c_2(X_{2,j})\zeta(j)\zeta(2n - j + 1)$$

By Theorem 8, it easy to get the coefficients

$$c_2(X_1) = \binom{N}{2} \frac{N}{2} = n$$

$$c_2(X_{2,j}) = -\binom{j}{1} \binom{N - j}{1} \frac{1}{j \cdot N - j} = -1$$

Therefore

$$Lz(2n - 1, 2) = n\zeta(2n + 1) - \sum_{j=2}^{n} \zeta(j)\zeta(2n - j + 1)$$

case II. $N = 2n$.

For exactly one-part partition, there is only one element $X_1 \in \mathcal{P}_2(N)$, and $Supp(X_1) = \{(N, 1)\}$. Similarly, for two-part partition, there are $n - 1$ elements: $X_{2,j} \in \mathcal{P}_2(N)$ with $Supp(X_{2,j}) = \{(j, 1), (N - j, 1)\},$ where $j = 2, ..., n$. Therefore

$$Lz(2n - 2, 2) = c_2(X_1)\zeta(2n) + \sum_{j=2}^{n} c_2(X_{2,j})\zeta(j)\zeta(2n - j)$$

By Theorem 8, it easy to get the coefficients

$$c_2(X_1) = \binom{N}{2} \frac{N}{2} = \frac{2n - 1}{2}$$

$$c_2(X_{2,j}) = -\binom{j}{1} \binom{N - j}{1} \frac{1}{j \cdot N - j} = -1$$
Therefore

\[ L_z(2n - 2, 2) = (n - \frac{1}{2})\zeta(2n) - \sum_{j=2}^{n} \zeta(j)\zeta(2n - j) \]

Let \( a = 2n - 1 \) or \( a = 2n - 2 \), rewrite those two formulas we have

\[ L_z(a, 2) = a + 1 \cdot \zeta(a + 2) - \sum_{j=2}^{n} \zeta(j)\zeta(a + 2 - j) \]

there is a relation between \( L_z(a, b) \) and multiple zeta function. Following example shows that the symmetric property of \( L_z(a, b) \) implies a nontrivial relation between \( L_z(a, b) \) and multiple zeta function for argument of integers \( \zeta(n_1, n_2) \). For larger \( a, b \), we need more techniques.

**Example 9.** Consider \( L_z(2, 1) = L_z(1, 2) \). One the one hand,

\[
L_z(2, 1) = \int_{0}^{1} \frac{\log(t) \log(1-t)}{t} dt = - \int_{0}^{1} \log(t)(1 + \sum_{n=1}^{\infty} \frac{t^n}{n+1}) dt = \cdots = \zeta(3)
\]

However, on the other hand, with the similar trick, for \(|t| < 1\), we have the expansion

\[
\frac{\log^2(1-t)}{t} = \sum_{n=1}^{\infty} S_{n+1}^{(2)} t^n
\]

where \( S_{n}^{(2)} \) denotes \( \sum_{i+j=n, i,j \geq 1} \frac{1}{ij} \). then

\[
L_z(1, 2) = \frac{1}{2!} \int_{0}^{1} \frac{\log^2(1-t)}{t} dt = \frac{1}{2} \sum_{n=2}^{\infty} \frac{S_{n}^{(2)}}{n}
\]

Notice that for \( S_{n}^{(2)} \)

\[
S_{n}^{(2)} = \frac{1}{n} \sum_{k=1}^{n-1} \frac{n}{k(n-k)} = \frac{1}{n} \sum_{k=1}^{n-1} \frac{1}{k} + \frac{1}{n-k} = 2 \sum_{k=1}^{n-1} \frac{1}{k}
\]

Therefore, above \( L_z(1, 2) \) can be reformulated as

\[
L_z(1, 2) = \frac{1}{2} \sum_{n=2}^{\infty} \frac{S_{n}^{(2)}}{n} = \sum_{n>m \geq 1} \frac{1}{n^2m} = \zeta(2, 1)
\]

Hence we prove Euler identity \( \zeta(3) = \zeta(2, 1) \) by using \( L_z(a, b) = L_z(b, a) \).

**Remark 1.** There is another proof using a series involving Striling numbers, see [7].

In fact, generally we have
Theorem 10. *(Symmetry of Series Representation)* For any positive integer \(a, b\), we have

\[
\frac{1}{a!} \sum_{n=a}^{\infty} S_n^{(a)} n^b = \frac{1}{b!} \sum_{n=b}^{\infty} S_n^{(b)} n^a
\]

where

\[
S_n^{(k)} = \sum_{m_1, \ldots, m_k} \prod_{j=1}^{k} m_j^{-1}, \quad m_j \in \mathbb{Z}^+, \quad j = 1, \ldots, k,
\]

In particular, \(S_n^{(1)} = \frac{1}{n}\)

Proof. Given positive integer \(a, b\),

\[
L_z(a, b) = \frac{1}{(a-1)! b!} \int_0^1 \frac{\log^{a-1}(t) \log^b(1-t)}{t} dt
\]

\[
= \frac{1}{(a-1)! b!} \int_0^1 \log^{a-1}(t) (-\sum_{n=1}^{\infty} t^n)^b \frac{1}{t} dt
\]

\[
= \frac{(-1)^b}{(a-1)! b!} \int_0^1 \log^{a-1}(t) \sum_{n=b}^{\infty} S_n^{(b)} t^{n-1} dt
\]

\[
= \frac{(-1)^b}{(a-1)! b!} \sum_{n=b}^{\infty} S_n^{(b)} \int_0^1 \log^{a-1}(t) t^{n-1} dt
\]

By the substitution \(t = e^{-z}\), it turns out to be

\[
L_z(a, b) = \frac{(-1)^b}{(a-1)! b!} \sum_{n=b}^{\infty} S_n^{(b)} \int_0^{\infty} (-z)^{a-1} e^{-nz} dz
\]

\[
= \frac{(-1)^b}{b!} \sum_{n=b}^{\infty} S_n^{(b)} n^{a}
\]

On the other hand, by the similar method, we have

\[
L_z(b, a) = \frac{(-1)^{a+b-1}}{a!} \sum_{n=a}^{\infty} S_n^{(a)} n^b
\]

Since \(L_z(a, b) = L_z(b, a)\) holds for all positive integers \(a, b\), therefore

\[
\frac{(-1)^{a+b-1}}{b!} \sum_{n=b}^{\infty} S_n^{(b)} n^a = \frac{(-1)^{a+b-1}}{a!} \sum_{n=a}^{\infty} S_n^{(a)} n^b
\]

namely

\[
\frac{1}{a!} \sum_{n=a}^{\infty} S_n^{(a)} n^b = \frac{1}{b!} \sum_{n=b}^{\infty} S_n^{(b)} n^a
\]

A straightforward corollary is that, if \(b = 1\), we have

\[
\zeta(a + 1) = \frac{1}{a!} \sum_{n=a}^{\infty} S_n^{(a)} n
\]

On the other hand, by the substitution of \(t = \sin^2(\theta)\) we have
Proposition 4.

\[ l_z(a, b) = 2^{a+b+1} \int_0^{\frac{\pi}{2}} \sin(\theta) \cos(\theta) \log^a \sin(\theta) \log^b \cos(\theta) d\theta \]

\[ L_z(a, b) = \frac{2^{a+b}}{a!(b-1)!} \int_0^{\frac{\pi}{2}} \cot(\theta) \log^{b-1} \sin(\theta) \log^a \cos(\theta) d\theta \]

5 Examples of Establishing the Integral Representations

- \( N = 3 \)

As the first example, for \( N = 3 \), the integral representation is trivial. But as it already demonstrated in the last section, those two equivalent representations imply some nontrivial relation \( \zeta(3) = \zeta(2, 1) \).

\[ \zeta(3) = L_z(1, 2) = \frac{1}{2} \int_0^1 \log^2(1-t) dt \]

\[ \zeta(3) = L_z(2, 1) = \int_0^1 \log(t) \log(1-t) \frac{dt}{t} \]

- \( N = 4 \)

\[ \zeta(4) = -L_z(3, 1) = -\frac{1}{2} \int_0^1 \log^2(t) \log(1-t) \frac{dt}{t} \]

\[ \zeta(4) = -4L_z(2, 2) = -2 \int_0^1 \log(t) \log^2(1-t) \frac{dt}{t} \]

\[ \zeta(4) = -L_z(1, 3) = -\frac{1}{6} \int_0^1 \log^3(1-t) \frac{dt}{t} \]

They correspond to following series representations respectively. Let \( S_n^{(3)} \) denotes \( \sum_{n_1+n_2+n_3=1} \frac{1}{n_1 n_2 n_3} \), then

\[ \zeta(4) = \sum_{n=1}^{\infty} \frac{1}{n^4} \]

\[ \zeta(4) = 2 \sum_{n=2}^{\infty} \frac{S_n^{(2)}}{n^2} = 4\zeta(3, 1) \]

\[ \zeta(4) = \frac{1}{6} \sum_{n=3}^{\infty} \frac{S_n^{(3)}}{n} \]

Following we only concern about the integral representations.

- \( N = 5 \)

\[ L_z(4, 1) = \zeta(5) \]

\[ L_z(3, 2) = 2\zeta(5) - \zeta(2)\zeta(3) \]

By mixing them, we obtain

\[ \zeta(3) = \frac{1}{\zeta(2)} (2L_z(4, 1) - L_z(3, 2)) = \frac{1}{2\pi^2} \int_0^1 \log^2(t) \log(1-t) \log \frac{t^4}{(1-t)^3} dt \]

This is a new nontrivial integral representation of Apéry’s constant. On the other hand, we have another integral representation for \( \zeta(5) \).

\[ \zeta(5) = \frac{1}{8} \int_0^1 \frac{\log^2(1-t)}{t} \left( \log^2(t) + \frac{\pi^2}{3} \right) dt \]
Once notice that

\[
\frac{1}{8} \int_0^1 \frac{\log^2(1-t)}{t} \left( \log(t) \frac{2\pi}{\sqrt{3}} \right) dt = -\frac{\pi \zeta(4)}{8\sqrt{3}}
\]

By plus to above formula, we obtain

\[
\zeta(5) - \frac{\pi \zeta(4)}{8\sqrt{3}} = \frac{1}{8} \int_0^1 \frac{\log^2(1-t) \log^2(e^{\pi/\sqrt{3}} t)}{t} dt
\]

• \( N = 6 \)

\( \mathcal{P}_2(6) \) has 4 elements:

\[
X_1 = (0, 0, 0, 0, 0, 1), \quad X_2 = (0, 1, 0, 1, 0, 0), \quad X_3 = (0, 0, 2, 0, 0, 0), \quad X_4 = (0, 3, 0, 0, 0, 0)
\]

\[
\text{Supp}(X_1) = \{(6, 1)\}
\]

\[
\text{Supp}(X_2) = \{(2, 1), (4, 1)\}
\]

\[
\text{Supp}(X_3) = \{(3, 2)\}
\]

\[
\text{Supp}(X_4) = \{(2, 3)\}
\]

By (3)

\[
Lz(5, 1) x^5 y + Lz(4, 2) x^4 y^2 + Lz(3, 3) x^3 y^3 = D_{6,1} P_6 + D_{2,1} D_{4,1} P_2 P_4 + D_{3,2} P_3^2 + D_{2,3} P_2^3
\]

Comparing the coefficients, we have

\[
Lz(5, 1) = -\zeta(6)
\]

\[
Lz(4, 2) = \frac{1}{2} \zeta(3)^2 + \zeta(2) \zeta(4) - \frac{5}{2} \zeta(6) = \frac{1}{2} \zeta(3)^2 - \frac{\pi^6}{1260}
\]

\[
Lz(3, 3) = \zeta(3)^2 + \frac{3}{2} \zeta(2) \zeta(4) - \frac{10}{3} \zeta(6) - \frac{1}{6} \zeta(2)^3 = \zeta(3)^2 - \frac{23 \pi^6}{15120}
\]

By mixing above two equations, we can rewrite a more interesting but sophisticated formula.

\[
\int_0^1 \frac{\log^2(t) \log^2(1-t) \log(1-t)^{12}}{t^{12}} dt = 6 \zeta(3)^2
\]

• \( N = 7 \)

\[
Lz(6, 1) = \zeta(7)
\]

\[
Lz(5, 2) = 3 \zeta(7) - \zeta(2) \zeta(5) - \zeta(4) \zeta(3)
\]

\[
Lz(4, 3) = 5 \zeta(7) - 2 \zeta(2) \zeta(5) - \frac{5}{4} \zeta(4) \zeta(3)
\]

reformulate them, we have interesting similar representation of \( \zeta(3) \) and \( \zeta(5) \).

\[
\frac{3}{5} \zeta(2) \zeta(5) = Lz(6, 1) + Lz(5, 2) - \frac{4}{5} Lz(4, 3)
\]

\[
\frac{3}{4} \zeta(4) \zeta(3) = Lz(6, 1) - 2 Lz(5, 2) + Lz(4, 3)
\]

• \( N = 8 \)
When $N$ become larger, $a, b$ become closer, the expression of $Lz(a, b)$ would be more complicated. In fact, one can prove following statement

**Theorem 11.** If integer $N > 20$, then there always exist partition $X \in \mathcal{P}_2(N)$, such that $\prod_{(n,k) \in \text{Supp}(X)} \zeta(n)^k$ cannot be represented by finite $\mathbb{Q}(\pi)$-linear combination of $Lz(a, b)$ for all $a, b \in \mathbb{Z}^+$ with $a + b \leq N$ by using the partition represented relation (Theorem 7).

**Remark 2.** It’s still unknown whether there is any functional equation in closed form that satisfied by $\zeta(n)$ for any different $n$, therefore such $\mathbb{Q}(\pi)$-linear combination of $Lz(a, b)$ may be constructed in other ways that differ from the partition represented relation. Hence all the representations in the following proof are referred to the representations by only using the partition represented relation.

**Proof.** Firstly, for fixed $N$, let

$$
\Pi : \mathcal{P}(N) \to \mathbb{R}; X \mapsto \prod_{(n,k) \in \text{Supp}(X)} \zeta(n)^k
$$

Assume that $X = X_1 + X_2 \in \mathcal{P}_2(N)$ with $X_1 \in \mathcal{P}\mathcal{E}_2(N)$, $X_2 \in \mathcal{P}\mathcal{O}_3(N)$. Since for all positive even number $2n$, $\zeta(2n)$ can be represented as $g_n\pi^{2n}$ with $g_n \in \mathbb{Q}$, then in other words $\Pi(X_1) \in \mathbb{Q}(\pi)$. Conversely, if $\Pi(X_2)$ cannot be represented as finite $\mathbb{Q}(\pi)$-linear combination of $Lz(a, b)$ for $a + b \leq N$, then $\Pi(X)$ neither.

Therefore, it remains to prove that there always exist $X \in \mathcal{P}\mathcal{O}_3(N)$ for sufficiently large $N$, such that $\Pi(X)$ cannot be represented by finite $\mathbb{Q}(\pi)$-linear combination of $Lz(a, b)$. For even or odd $N$, such $X$ is constructed by different way.

**Case I.** Suppose that $N = 2M + 1$.

Let

$$
T(N) = \bigcup_{\ell=1}^{\infty} \mathcal{P}\mathcal{O}_3^{2\ell+1}(N)
$$
Notice that for \( Y \in \mathcal{P}_2(N) \setminus T(N) \), we can always assume that \( \Pi(\vec{X}) \) can be represented by finite \( \mathbb{Q}(\pi) \)-linear combination of \( L_z(a, b) \) for \( a + b < N \). Because otherwise, we come to a smaller \( N_1 < N \), and therefore we could repeat the process by starting from \( N_1 \) instead of \( N \). That is, it reasonable to assume that if \( Y \in \mathcal{P}_2(N) \setminus T(N) \), then

\[
\sum Y = \sum p_j L_z(a_j, b_j)
\]

with \( a_j + b_j < N \). Now Suppose that

\[
\mathcal{P}_j^{2t+1}(N) = \{ X_1^{(2t+1)}, X_2^{(2t+1)}, ..., X_{2t+1}^{(2t+1)} \}
\]

with \( |\mathcal{P}_j^{2t+1}(N)| = r_{2t+1} \). According to theorem 7, \( \Pi(X_i^{(2t+1)}) \) only appears in the representation formulas \( L_z(N - 2t - 1, 2t + 1), L_z(N - 2t - 2, 2t + 2), ..., L_z(2t + 1, N - 2t - 1) \). In fact, due to the symmetric property of \( L_z(a, b) \), only \( L_z(N - 2t - 1, 2t + 1), L_z(N - 2t - 2, 2t + 2), ..., L_z(M + 1, M) \) provide the representations that differ from each other. Therefore by Theorem 7 following linear equations system are derived, if we regard \( \Pi(X_i^{(2t+1)}) \) as unknowns, \( L_z(a, b) \) as coefficients

\[
\begin{align*}
L_z(N - 3, 3) - c_3 L_z(N - 1, 1) &= \sum p_{3j} \pi^{2j} L_z(a_j, b_j) + \sum_{i=1}^{r_3} q_3^{(3)} \Pi(X_i^{(3)}) \\
L_z(N - 4, 4) - c_4 L_z(N - 1, 1) &= \sum p_{4j} \pi^{2j} L_z(a_j, b_j) + \sum_{i=1}^{r_3} q_4^{(3)} \Pi(X_i^{(3)}) \\
L_z(N - 5, 5) - c_5 L_z(N - 1, 1) &= \sum p_{5j} \pi^{2j} L_z(a_j, b_j) + \sum_{i=1}^{r_3} q_5^{(3)} \Pi(X_i^{(3)}) + \sum_{i=1}^{r_5} q_5^{(5)} \Pi(X_i^{(5)}) \\
&... \\
L_z(M + 1, M) - c_M L_z(N - 1, 1) &= \sum p_{Mj} \pi^{2j} L_z(a_j, b_j) + \sum_{i=1}^{r_3} q_3^{(3)} \Pi(X_i^{(3)}) + ... + \sum_{i=1}^{r_M} q_3^{(3)} \Pi(X_i^{(3)})
\end{align*}
\]

where \( c, p, q \in \mathbb{Q}, a_j + b_j < N, \pi \) is the largest \( 2t + 1 \) such that \( \mathcal{P}_j^{2t+1}(N) \neq \emptyset \). There are \( M - 3 + 1 \) equations. It obvious, the number of unknowns \( |T(N)| = r_3 + ... + r_\pi \geq r_3 \). Therefore if \( M - 3 + 1 < r_3 \), due to the unknowns are more than the number of equations, then \( \Pi(X) = \prod_{(n, k) \in \text{supp}(X)} \zeta(n)^k \) cannot be solved by above equations system.

It’s well-known that \( r_3 = |\mathcal{P}_7^{3}(2M + 1)| \) increases faster than \( M \). Hence there exist \( M_1 \), such that if \( M > M_1 \), then \( M_1 - 3 + 1 < |\mathcal{P}_7^{3}(2M_1 + 1)| \). In fact, it’s not hard to find out, if \( M > 9 \), then \( M - 3 + 1 < |T(2M + 1)| \).

- Case II. Suppose that \( N = 2M \)

Let

\[
T(N) = \bigcup_{t=1}^{\infty} \mathcal{P}_j^{2t}(N)
\]

Similar to the case of odd \( N \), it reasonable to assume that if \( Y \in \mathcal{P}_2(N) \setminus T(N) \), then

\[
\sum Y = \sum p_j L_z(a_j, b_j)
\]

with \( a_j + b_j < N \). Now Suppose that

\[
\mathcal{P}_j^{2t}(N) = \{ X_1^{(2t)}, X_2^{(2t)}, ..., X_{2t}^{(2t)} \}
\]
with \(|PO^2_3(N)| = r_2t\). According to theorem 7, analogous linear equations system are derived

\[
Lz(N - 2, 2) - c_2Lz(N - 1, 1) = \sum_{i=1}^{r_2} p_{2j} \pi^{2j} Lz(a_j, b_j) + \sum_{i=1}^{r_2} q_{2i}^{(2)} \Pi(X_i^{(2)})
\]

\[
Lz(N - 3, 3) - c_3Lz(N - 1, 1) = \sum_{i=1}^{r_2} p_{3j} \pi^{2j} Lz(a_j, b_j) + \sum_{i=1}^{r_2} q_{3i}^{(2)} \Pi(X_i^{(2)})
\]

\[
Lz(N - 4, 4) - c_4Lz(N - 1, 1) = \sum_{i=1}^{r_2} p_{4j} \pi^{2j} Lz(a_j, b_j) + \sum_{i=1}^{r_2} q_{4i}^{(2)} \Pi(X_i^{(2)}) + \sum_{i=1}^{r_3} q_{4i}^{(4)} \Pi(X_i^{(4)})
\]

\[
Lz(N - 5, 5) - c_5Lz(N - 1, 1) = \sum_{i=1}^{r_2} p_{5j} \pi^{2j} Lz(a_j, b_j) + \sum_{i=1}^{r_2} q_{5i}^{(2)} \Pi(X_i^{(2)}) + \sum_{i=1}^{r_3} q_{5i}^{(4)} \Pi(X_i^{(4)})
\]

\[
Lz(M, M) - c_MLz(N - 1, 1) = \sum_{i=1}^{r_2} p_{Mj} \pi^{2j} Lz(a_j, b_j) + \sum_{i=1}^{r_2} q_{Mj}^{(2)} \Pi(X_i^{(2)}) + \sum_{i=1}^{r_3} q_{Mj}^{(4)} \Pi(X_i^{(4)})
\]

where \(c, p, q \in \mathbb{Q}, a_j + b_j < N\), \(\pi\) is the largest \(2t\) such that \(|PO^2_3(N)| \neq \emptyset\). There are \(M - 2 + 1\) equations. It obvious, the number of unknowns \(|T(N)| = r_2 + \cdots + r_\tau \geq r_2 + r_4\) if \(N\) is large enough. Therefore if \(M - 2 + 1 < r_2 + r_4\), due to the unknowns are more than the number of equations, then \(\Pi(X) = \prod_{(n,k) \in supp(X_j)} \zeta(n)^k\) cannot be solved by above equations system. It’s well-known that \(r_4 = |PO^4_3(2M)|\) increases faster than \(M\). Hence there exist \(M_2\), such that if \(M > M_2\), then \(M_2 - 2 + 1 < |PO^2_3(2M_2)|\). In fact, it’s not hard to find out, if \(M > 10\), then \(M - 2 + 1 < |T(2M)|\).

Finally, by above discussion we obtain \(N_0 = 20\). If \(N > N_0\), whenever \(N\) is odd or even, there always exist \(X \in \mathcal{F}_2(N)\) such that \(X\) cannot be represented by finite \(\mathbb{Q}(\pi)\)-linear combination of \(Lz(a, b)\) with \(a + b \leq N\).

\[\square\]

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