TSCHENBYSHEEFF OF THE SECOND KIND AND BERNSTEIN POLYNOMIALS CHANGE OF BASES

MOHAMMAD A. ALQUDAH

Abstract. We construct multiple representations relative to different bases of the generalized Tschebyseff polynomials of second kind. We build the change-of-basis matrices between the generalized Tschebyseff of the second kind polynomial basis and Bernstein polynomial basis. Also, we provide an explicit closed form of the generalized polynomials of degree $r \leq n$ in terms of the Bernstein basis of fixed degree $n$.

AMS Subject Classification: 42C05, 33C50, 33C45, 33C70, 05A10, 33B15

Keywords: Generalized Tschebyseff, Bernstein Basis, Basis Transformation, Bézier Coefficient, Gamma function

1. Introduction, Background and Motivation

Approximation by polynomials is the oldest and simplest way to represent complicated functions defined over finite domains, since it is possible to approximate any arbitrary continuous function by a polynomial and make the error less than a given accuracy by increasing the degree of the approximating polynomial. On the other side, polynomials can be represented in many different bases such as the monomial power, Tschebyseff polynomials of the second kind (Tschebyseff-II), Bernstein, and Hermite basis form.

It is desirable to work with more than one basis for a given polynomial; it is useful to switch bases so as to simplify problems, it is of fundamental importance in the efficiency of numerical calculations, to easily transform coordinate-wise representations of polynomials taken with respect to one basis to their equivalent representations with respect to another basis. Every type of polynomial basis has its strength and advantages, and sometimes it has disadvantages. Many problems can be solved and many difficulties can be removed by appropriate choice of the basis.

1.1. Bernstein polynomials. Bernstein polynomials are incredibly useful mathematical tools, they are simply defined by

\begin{equation}
B_n^k(x) = \begin{cases} 
\frac{n!}{k!(n-k)!} x^k (1-x)^{n-k} & k = 0, 1, \ldots, n \\
0 & \text{else}
\end{cases}
\end{equation}

The Bernstein polynomials have been studied thoroughly and there are a fair amount of literature on these polynomials, they are known for their geometric properties \cite{2, 5}, and the Bernstein basis form is known to be optimally stable.

Date: April 8, 2015.
They are all non-negative, $B^n_k(x) \geq 0$, $x \in [0,1]$, satisfy symmetry relation $B^n_k(x) = B^n_{n-k}(1-x)$, has a single unique maximum of $(\binom{n}{k})k^kn^{-n}(n-k)^{n-k}$ at $x = \frac{k}{n}$, $k = 0,\ldots,n$, their roots are $x = 0,1$ with multiplicities. The Bernstein polynomials of degree $n$ can be defined by combining two Bernstein polynomials of degree $n-1$. So, the $k$th degree Bernstein polynomial can calculated recursively be following relation $B^n_k(x) = (1-x)B^n_k-1(x) + xB^n_{k-1}(x)$, $k = 0,\ldots,n; n \geq 1$ where $B^n_0(x) = 0$ and $B^n_n(x) = 0$ for $k < 0$ or $k > n$. Moreover, the product of two Bernstein polynomials is also a Bernstein polynomial and given by

$$\binom{n+m}{i+j}B^i_n(x)B^j_m(x) = \binom{n}{i}\binom{m}{j}B^{i+m}_{n+k}(x).$$

Analytic and geometric properties of Bernstein polynomials make them important for the development of Bézier curves and surfaces. The Bernstein polynomials are actually the standard basis for the Bézier representations of curves and surfaces in Computer Aided Geometric Design. However, the Bernstein polynomials are not orthogonal and could not be used effectively in the least-squares approximation [8]. Since then the method of least squares approximation accompanied by orthogonal polynomials has been introduced, developed, and analyzed.

1.2. Least-Square Approximation. The best approximation can be defined as that which minimizes the difference between the original function and the approximation; best approximations with respect to the 2-norm are called least squares approximations.

**Definition 1.2.** For a function $f(x)$, continuous on $[0,1]$ the least square approximation requires finding a polynomial (Least-Squares Polynomial)

$$p_n^*(x) = \sum_{k=0}^{n} a_k \phi_k(x)$$

that minimizes the error

$$E(a_0,a_1,\ldots,a_n) = \int_0^1 [f(x) - p_n^*(x)]^2 dx.$$ 

We require the minimum of $E(a_0,a_1,\ldots,a_n)$ over all values $a_0,a_1,\ldots,a_n$. A necessary condition for $E$ to have a minimum is:

$$\frac{\partial E}{\partial a_k} = 0 = -2 \int_0^1 [f(x) - p_n^*(x)] \phi_k(x) dx, \quad k = 0,\ldots,n.$$ 

These conditions will imply the following

$$\int_0^1 f(x)\phi_i(x) dx = \int_0^1 \sum_{k=0}^{n} a_k \phi_k(x)\phi_i(x) dx.$$ 

Thus, the $a_i$ that minimize $\|f(x) - \sum_{k=0}^{n} a_k \phi_k(x)\|_2$ satisfy the system of equations given by

$$\int_0^1 f(x)\phi_i(x) dx = \sum_{k=0}^{n} a_k \int_0^1 \phi_k(x)\phi_i(x) dx, \quad i = 0,1,\ldots,n.$$ 

This gives rise to a system of $(n+1)$ equations in $(n+1)$ unknowns: $a_0,a_1,\ldots,a_n$. These equations are often called the Normal equations.
Solution of these equations will yield the unknowns: $a_0, a_1, \ldots, a_n$ of the least-squares polynomial $p_\ast^n(x)$. It is important to choose a suitable basis, for example by choosing $\phi_i(x) = x^i$, then

$$
\int_0^1 f(x)x^i \, dx = \sum_{k=0}^n a_k \int_0^1 x^{i+k} \, dx = \sum_{k=0}^n \frac{a_k}{i+k+1}.
$$

The matrix of coefficients of the normal equations is Hilbert matrix which has round-off error difficulties and notoriously ill-conditioned for even modest values of $n$. Such computations can, however, be made computationally effective by using a special type of polynomials, called orthogonal polynomials.

Choosing $\{\phi_0(x), \phi_1(x), \ldots, \phi_n(x)\}$ to be orthogonal greatly simplifies the least-squares approximation problem. The matrix of the normal equations is diagonalized, which simplifies calculations and gives a compact form for $a_i, i = 0, 1, \ldots, n$.

See [8] for more details on the least squares approximations.

1.3. Gamma functions. The gamma function $\Gamma(n)$ is an extension of the factorial function, with its argument shifted down by 1. That is, if $n$ is a positive integer:

$$
\Gamma(n) = (n-1)!. 
$$

The Eulerian integral of the first kind is useful and will be used in main result simplifications.

**Definition 1.3.** The Eulerian integral of the first kind is a function of two complex variables defined by

$$
\int_0^1 u^{x-1}(1-u)^{y-1} \, du = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \quad \Re(x), \Re(y) > 0. \tag{1.2}
$$

The double factorial of an integer $n$ is given by

$$
(2n-1)!! = (2n-1)(2n-3)(2n-5)\ldots(3)(1) \quad \text{if } n \text{ is odd} \\
n!! = (n)(n-2)(n-4)\ldots(4)(2) \quad \text{if } n \text{ is even}, \tag{1.3}
$$

where $0!! = (-1)!! = 1$.

From (1.3), we can derive the following relation for factorials

$$
n!! = \begin{cases} 
2\frac{(n-2)!}{n!} & \text{if } n \text{ is even} \\
2\frac{(n-4)!}{(n-2)!} & \text{if } n \text{ is odd}
\end{cases}. \tag{1.4}
$$

It is easy to derive the factorial of an integer plus half as

$$
\left(n + \frac{1}{2}\right)! = \frac{\sqrt{\pi}}{2^{n+1}}(2n+1)!!. \tag{1.5}
$$

From the relation (1.4) to have $(2n)!! = 2^n n!$, and $(2n)!! = (2n-1)!!2^n n!$.

1.4. Univariate Tschebyscheff-II and the generalized Tschebyscheff-II polynomials. The univariate classical Tschebyscheff-II orthogonal polynomials $U_n(x)$ are special case of Jacobi polynomials $P_n^{(\alpha, \beta)}$ with $\alpha = \beta = 1/2$. The relationship between Tschebyscheff-II and Jacobi Polynomials defined as $P_n^{(\frac{1}{2}, \frac{1}{2})}(1)U_n(x) = (n+1)P_n^{(\frac{1}{2}, \frac{1}{2})}(x)$. Tschebyscheff-II polynomials are traditionally defined on $[-1, 1]$, however, it is more convenient to use $[0, 1]$.
For the convenience we recall the following explicit expressions for univariate Tschebyscheff-II polynomials of degree \( n \) in \( x \), using combinatorial notation that gives more compact and readable formulas. see Szegö [9]:

\[
U_n(x) := \frac{(n + 1)(2n)!!}{(2n + 1)!!} \sum_{k=0}^{n} \binom{n + \frac{1}{2}}{n - k} \binom{n + \frac{1}{2}}{k} \left( \frac{x + 1}{2} \right)^{n-k} \left( \frac{x - 1}{2} \right)^k ,
\]

which it can be transformed in terms of Bernstein basis on \( x \in [0, 1] \),

\[
U_n(2x - 1) := \frac{(n + 1)(2n)!!}{(2n + 1)!!} \sum_{k=0}^{n} (-1)^{n+1} \binom{n + \frac{1}{2}}{k} \binom{n + \frac{1}{2}}{n-k} \binom{n}{k} B_n^k(x).
\]

The Tschebyscheff-II polynomials \( U_n(x) \) of degree \( n \) are the orthogonal polynomials, except for a constant factor, with respect to the weight function

\[
W(x) = (1 - x^2)^{\frac{1}{2}}
\]

Also, the Tschebyscheff-II polynomials satisfy the orthogonality relation [1]

\[
\int_0^1 x^\frac{1}{2}(1 - x)^\frac{1}{2} U_n(x)U_m(x)dx = \begin{cases} 0 & \text{if } m \neq n \\ \frac{\pi}{8} & \text{if } m = n \end{cases}.
\]

The generalized Tschebyscheff-II polynomials been characterization in [1]. For \( M, N \geq 0 \), the generalized Tschebyscheff-II polynomials \( \mathcal{U}_n^{(M,N)}(x) \) are orthogonal on the interval \([-1, 1]\) with respect to the weight function [6],

\[
\frac{2}{\pi} x^\frac{1}{2}(1 - x)^\frac{1}{2} + M\delta(x + 1) + N\delta(x - 1).
\]

and defined in [1] as

\[
\mathcal{U}_n^{(M,N)}(x) = \frac{(2n + 1)!!}{2^n(n + 1)!} U_n(x) + \sum_{k=0}^{n} \lambda_k \frac{(2k + 1)!!}{2^k(k + 1)!} U_k(x),
\]

where

\[
\lambda_k = \frac{k(k + 1)(2k + 1)(M + N)}{6} + \frac{(k + 2)(k + 1)^2k^2(k - 1)MN}{9}.
\]

2. Main Results

In this section we provide a closed form for the matrix transformation of the generalized Tschebyscheff-II polynomial basis into Bernstein polynomial basis, and for Bernstein polynomial basis into generalized Tschebyscheff-II polynomial basis.

2.1. Bernstein to Generalized Tschebyscheff-II transformation and vice versa. Rababah [7] provided some results concerning the univariate Chebyshev polynomials of first kind. In the we extend the procedure in [7] to generalize the results for the generalized case. The theorem will be used to combine the superior performance of the least-squares of the generalized Tschebyscheff-II polynomials with the geometric insights of the Bernstein polynomials basis.

The next theorem, see [1] for the proof, provides a closed form for generalized Tschebyscheff-II polynomial \( \mathcal{U}_n^{(M,N)}(x) \) of degree \( r \) as a linear combination of the Bernstein polynomials \( B_i^r, i = 0, 1, \ldots, r \) of degree \( r \).
Theorem 2.1. [1] For $M, N \geq 0$, the generalized Tschebyscheff-II polynomials $\mathcal{B}_r^{(M,N)}(x)$ of degree $r$ have the following Bernstein representation:

$$\mathcal{B}_r^{(M,N)}(x) = \frac{(2r + 1)!!}{2^r(r + 1)!}\sum_{i=0}^{r}(-1)^{r-i}\vartheta_{i,r}B_i^r(x) + \sum_{k=0}^{r}\lambda_k \frac{(2k + 1)!!}{2^k(k + 1)!}\sum_{i=0}^{k}(-1)^{k-i}\vartheta_{i,k}B_i^k(x)$$

where $\lambda_k$ defined by (2.11), $\vartheta_{i,r} = \frac{(2r+1)!!}{2^r r!}(2^r r)$, and 

$$\vartheta_{i,r} = \frac{(2r + 1)^2}{2^{2r}(2r - 2i + 1)(2i + 1)}\binom{2r}{i}, \quad i = 0, 1, \ldots, r.$$

The coefficients $\vartheta_{i,r}$ satisfy the recurrence relation

$$\vartheta_{i,r} = \frac{(2r - 2i + 3)}{(2i + 1)}\vartheta_{i-1,r}, \quad i = 1, \ldots, r.$$

The following theorem used to combine the superior performance of the least-squares of the generalized Tschebyscheff-II polynomials with the geometric insights of the Bernstein polynomials basis:

Theorem 2.2. The entries $M^n_{i,r}, i, r = 0, 1, \ldots, n$ of the matrix transformation of the generalized Tschebyscheff-II polynomial basis into Bernstein polynomial basis of degree $n$ are given by

$$M^n_{i,r} = \Phi_{i,n}^r + \sum_{k=0}^{r}\lambda_k \Phi_{i,n}^k,$$

where $\lambda_k$ defined in (1.11) and 

$$\Phi_{i,n}^r = \frac{(2r + 1)!!}{2^r(r + 1)!}\sum_{k=\max(0,i+r-n)}^{\min(i,r)}(-1)^{r-k}\binom{n-r}{i-k} \binom{r+k}{i} \binom{r+k}{k}.$$

Proof. A polynomial $p_n(x), x \in [0,1]$ of degree $n$, can be written as a linear combination of the Bernstein polynomial basis $p_n(x) = \sum_{r=0}^{n}c_r B_r^n(x)$ and the generalized Tschebyscheff-II polynomials $p_n(x) = \sum_{r=0}^{n}d_r \mathcal{B}_r^{(M,N)}(x)$.

We are interested in the matrix $M$ where $M$ where $c = M.d$ that maps the generalized Tschebyscheff-II coefficients $\{d_r\}_{r=0}^{n}$ into the Bernstein coefficients $\{c_r\}_{r=0}^{n}$.

$$c_i = \sum_{r=0}^{n}M^n_{i,r}d_r,$$

which can be written in matrix format as

$$\begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} M^n_{0,0} & M^n_{0,1} & M^n_{0,2} & \cdots & M^n_{0,n} \\ M^n_{1,0} & M^n_{1,1} & M^n_{1,2} & \cdots & M^n_{1,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ M^n_{n,0} & M^n_{n,1} & M^n_{n,2} & \cdots & M^n_{n,n} \end{bmatrix} \begin{bmatrix} d_0 \\ d_1 \\ \vdots \\ d_n \end{bmatrix}.$$

Moreover, we can write the generalized Tschebyscheff-II polynomials (1.10) as a linear combination of the Bernstein polynomial basis as

$$\mathcal{B}_r^{(M,N)}(x) = \sum_{i=0}^{n}N^n_{r,i}B_i^r(x), \quad r = 0, 1, \ldots, n,$$
where the matrix $N$ formed by $N_{r,i}^n$ is the $(n + 1) \times (n + 1)$ basis conversion matrix. This enables writing the elements of $c$ in the form

$$c_i = \sum_{r=0}^{n} d_r N_{r,i}^n. \tag{2.7}$$

Using equations (2.4) and (2.7) with easy comparison shows that $M = N^T$.

Since each Bernstein polynomial of degree $r$ where $r \leq n$ can be written in terms of Bernstein polynomials of degree $n$ using the following degree elevation defined by [3]:

$$B_r^k(x) = \frac{(r+k)!}{(r+1)!} B_i^n(x), \quad k = 0, 1, \ldots, r. \tag{2.8}$$

Substituting (2.8) into

$$U_{r}^{(M,N)}(x) = \frac{(2r+1)!!}{2^r (r+1)!} \sum_{i=0}^{r} (-1)^{r-i} \eta_{r,i} B_i^n(x) + \sum_{k=0}^{r} \lambda_k \frac{(2k+1)!!}{2^k (k+1)!} \sum_{i=0}^{k} (-1)^{k-i} \eta_{k,i} B_i^k(x)$$

and rearrange the order of summations, we find that the entries of the matrix $N$ for $r = 0, 1, \ldots, n$ are given by

$$N_{r,i} = \binom{n}{i}^{-1} \frac{(2r+1)!!}{2^r (r+1)!} \sum_{k=\max(0,i+r-n)}^{\min(i,r)} (-1)^{r-k} \binom{n-r}{i-k} \binom{r+\frac{1}{2}}{k} \binom{r+k}{r-k}$$

$$+ \sum_{k=0}^{r} \lambda_k \binom{n}{i}^{-1} \frac{(2k+1)!!}{2^k (k+1)!} \sum_{j=\max(0,i+k-n)}^{\min(i,k)} (-1)^{k-j} \binom{n-k}{i-j} \binom{k+\frac{1}{2}}{j} \binom{k+k-j}{k-j}. \tag{2.9}$$

Define

$$\Phi_{i,n}^k = \frac{(2k+1)!!}{2^k (k+1)!} \sum_{j=\max(0,i+k-n)}^{\min(i,k)} (-1)^{k-j} \binom{n-k}{i-j} \binom{k+\frac{1}{2}}{j} \binom{k+k-j}{k-j}. \tag{2.10}$$

By transposing the entries of the matrix $N$, we get the entries of matrix $M$.

$$M_{i,r} = \Phi_{i,n}^r + \sum_{k=0}^{r} \lambda_k \Phi_{i,n}^k. \tag{2.11}$$

Now, we have the following corollary which enables us to write Tschebyscheff-II polynomials of degree $r$ where $r \leq n$ in terms of Bernstein polynomials of degree $n$.

**Corollary 2.1.** The generalized Tschebyscheff-II polynomials $U_{0}^{(M,N)}(x), \ldots, U_{n}^{(M,N)}(x)$ of degree less than or equal to $n$ can be expressed in the Bernstein basis of fixed degree $n$ by the following formula

$$U_{r}^{(M,N)}(x) = \sum_{i=0}^{n} N_{r,i}^n B_i^n(x), \quad r = 0, 1, \ldots, n. \tag{2.12}$$
we have

\[
\Lambda(i) = \frac{2r+1}{2^r(r+1)!} \sum_{k=\max(0,i+r-n)}^{\min(i,r)} (-1)^{r-k}(2r+1)^k \frac{(2^r)_{r-k}}{\binom{r}{i}} \cdot \frac{2^{2r}(2r-2k+1)(2k+1)}{2^{2k}(2k-2j+1)(2j+1)} \binom{n-k}{i-j} \binom{2k}{2j}.
\]

Proof. From (2.6) it is clear that each Tschebyscheff-II polynomial of degree \( r \) where \( r \leq n \) can be written in terms of Bernstein polynomials of degree \( n \).

Applying (1.5) with some simplifications, we have

\[
\left( r + \frac{1}{2} \right) \left( r + \frac{1}{2} \right) = \frac{(2r+1)!}{2^r(2r-2k+1)(r-k)!} \cdot \frac{(2r-1)!}{(2k-1)!} \cdot \frac{(2r-1)!}{(2r-k-1)!} \cdot \frac{(2k+1)!}{(2r-k)!}.
\]

Using the fact \((2n)! = (2n-1)!2^n n!\) we get

\[
\left( r + \frac{1}{2} \right) \left( r + \frac{1}{2} \right) = \frac{(2r+1)^2}{2^{2r}(2r-2k+1)(2k+1)} \frac{2^r}{r} \frac{2r}{2k}.
\]

Substituting the last identity into (2.9) we get the desired result. \(\square\)

The following theorem introduced in [1] will be used to simplify a main result.

**Theorem 2.3.** Let \( B_r^n(x) \) be the Bernstein polynomial of degree \( n \) and \( \mathcal{U}_i^{(M,N)}(x) \) be the generalized Tschebyscheff-II polynomial of degree \( i \), then for \( i, r = 0, 1, \ldots, n \) we have

\[
(2.10) \quad \int_0^1 x^\frac{1}{2}(1-x)^\frac{1}{2} B_r^n(x) \mathcal{U}_i^{(M,N)}(x) dx = \Lambda_{i,n}^r + \sum_{d=0}^{i} \lambda_d A_{r,n}^d,
\]

where

\[
(2.11) \quad A_{r,n}^d = \frac{(n)!}{r!} \frac{(2d+1)!}{2^{d}(d+1)!} \sum_{j=0}^{d} (-1)^d \cdot \frac{\left(\frac{1}{2}\right)^{d-j}}{j} \cdot \frac{\left(\frac{1}{2}\right)^{j}}{d-j} \cdot \frac{\Gamma(r+j+\frac{1}{2}) \Gamma(n+d-r-j+\frac{1}{2})}{\Gamma(n+d+3)}.
\]

\(\lambda_d\) defined in (1.11), and \(\Gamma(x)\) is the Gamma function.

**Theorem 2.4.** The entries \( M_{i,r}^{-1} \), \( i, r = 0, 1, \ldots, n \) of the matrix of transformation of the Bernstein polynomial basis into the generalized Tschebyscheff-II polynomial basis of degree \( n \) are given by

\[
(2.12) \quad M_{i,r}^{-1} = \frac{8}{\pi(1+\lambda_i)^2} \left( \frac{2^{i+1}}{(2i+1)!} \right)^2 \left( \Lambda_{i,n}^r + \sum_{d=0}^{i} \lambda_d A_{r,n}^d \right),
\]

where \(\Lambda_{i,n}^d\) defined in (2.11).

Proof. To write the Bernstein polynomial basis into generalized Tschebyscheff-II polynomial basis of degree \( n \), we invert (2.5) to get

\[
(2.13) \quad d = M^{-1} c.
\]
Let $M_{i,r}^{n-1}$, $N_{i,r}^{n-1}$, $i, r = 0, \ldots, n$ be the entries of $M^{-1}$ and $N^{-1}$ respectively, then the transformation of Bernstein polynomial into generalized Tschebyscheff-II polynomial basis of degree $n$ can then be written as

\begin{equation}
B^n_r(x) = \sum_{i=0}^{n} N_{i,r}^{n-1} \mathcal{U}_i^{(M,N)}(x). \tag{2.14}
\end{equation}

To find $N_{i,r}^{n-1}$, $i, r = 0, 1, \ldots, n$, multiply (2.14) by $x^{\frac{r}{2}} (1 - x)^{\frac{r}{2}} \mathcal{U}_i^{(M,N)}(x)$ and integrate over $[0,1]$ to have

\begin{equation}
\int_0^1 x^{\frac{r}{2}} (1 - x)^{\frac{r}{2}} B^n_r(x) \mathcal{U}_i^{(M,N)}(x) dx = \sum_{i=0}^{n} N_{i,r}^{n-1} \int_0^1 x^{\frac{r}{2}} (1 - x)^{\frac{r}{2}} \mathcal{U}_i^{(M,N)}(x) \mathcal{U}_i^{(M,N)}(x) dx. \tag{2.15}
\end{equation}

By using the orthogonality relation (1.8) we get

\begin{equation}
\int_0^1 B^n_r(x) (1 - x)^{\frac{r}{2}} x^{\frac{r}{2}} \mathcal{U}_i^{(M,N)}(x) dx = \frac{\pi}{8} \left( \frac{(2i + 1)!!}{2^i (i + 1)!} \right)^2 N_{i,r}^{n-1} (1 + \lambda_i)^2. \tag{2.16}
\end{equation}

Taking into account equation (1.2), Theorem 2.3, $M = N^T$, and $A_{r,n}^d$ defined in (2.11) we have

\begin{equation}
M_{i,r}^{n-1} = \frac{8}{\pi (1 + \lambda_i)^2} \left( \frac{2^i (i + 1)!!}{(2i + 1)!!} \right)^2 \left( A_{i,n}^1 + \sum_{d=0}^{i} \lambda_d A_{r,n}^d \right). \tag{2.17}
\end{equation}

\[\square\]

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Department of Mathematics, Northwood University, Midland, MI 48640 USA
E-mail address: alqudahm@northwood.edu