LTB curves with Lipschitz turn are par-regular.

É. Le Quentrec L. Mazo É. Baudrier M. Tajine
ICube-UMR 7357, 300 Bd Sébastien Brant -
CS 10413 - 67412 Illkirch Cedex FRANCE
etiennelequentrec@free.fr

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Preserving the topology during a digitization process is a requirement of first importance. To this end, it is classical in Digital Geometry to assume the shape borders to be par-regular. Par-regularity was proved to be equivalent to having positive reach or to belong to the class $C^{1,1}$ of curves with Lipschitz derivative. Recently, we proposed to use a larger class that encompasses polygons with obtuse angles, the locally turn-bounded curves. The aim of this technical report is to define the class of par-regular curves inside the class of locally turn-bounded curves using only the notion of turn, that is of integral curvature. To be more precise, in a previous article, we have already proved that par-regular curves are locally turn-bounded. Incidentally this proof lead us to show that the turn of par-regular curves is a Lipschitz function of their length. We call the class of curves verifying this latter property the curves with Lipschitz turn. In this technical report, we prove the converse assertion: locally turn-bounded curves with Lipschitz turn are par-regular. The equivalence is stated in Theorem 3.1 and the converse assertion is proved in Lemma 3.2. In section 1, we recall the definition of par-regularity and equivalently of sets with positive reach. In section 2 we present the notions of curves locally turn-bounded and of curves with Lipschitz turn. Throughout this latter section, some of intermediate steps (Lemmas 2.3 and 2.11) are proved just after the introduction of their related notions. The last section (section 3) is dedicated to the proof of the equivalence of the notions.

1 Par-regularity and equivalent notions

Let us first recall the definition of reach and of par-regularity.

Definition 1.1 (reach). The medial axis of a compact set $K$ is the set of points having at least two nearest neighbours in $K$. The reach is the minimal distance between $K$ and its medial axis.
Having positive reach is equivalent to be of class $C^{1,1}$ (the class of curves parameterized by a $C^1$ function whose derivative is Lipschitz) [Fed59].

Regarding par-regularity, we choose the same definition as in [LCG98] and [LT16].

**Definition 1.2 (par($r$)-regularity).** Let $C$ be a Jordan curve of interior $K$.

- A closed ball $\bar{B}(c_i, r)$ is an inside osculating ball of radius $r$ at point $a \in C$ if $C \cap \bar{B}(c_i, r) = \{a\}$ and $\bar{B}(c_i, r) \subset K \cup \{a\}$.

- A closed ball $\bar{B}(c_e, r)$ is an outside osculating ball of radius $r$ at point $a \in C$ if $C \cap \bar{B}(c_e, r) = \{a\}$ and $\bar{B}(c_e, r) \subset (\mathbb{R}^2 \setminus (C \cup K)) \cup \{a\}$.

- A curve $C$ is par($r$)-regular if there exist inside and outside osculating balls of radius $r$ at each $a \in C$.

The definition of par-regularity is illustrated in Figure 1.

![Figure 1: Par-regularity demands that at each point of the boundary of the blue shape, there exists an inside osculating disk and an outside osculating disk both of radius $r$.](image)

Par-regularity is equivalent to having a positive reach [LT16].

2 **Locally turn-bounded curves and curves with Lipschitz turn**

2.1 **Turn of a curve**

The definition of locally turn-bounded curves and curves with Lipschitz turn are both based on the notion of turn introduced by Milnor in [Mil50].

**Definition 2.1 (Turn of a curve [AR89]).**
• The turn $\kappa(L)$ of a polygonal line $L = [x_i]_{i=0}^{N-1}$ is defined by:

$$\kappa(L) := \sum_{i=1}^{N-2} \angle(x_i - x_{i-1}, x_{i+1} - x_i).$$

• The turn $\kappa(P)$ of a polygon $P = [x_i]_{i \in \mathbb{Z}/N\mathbb{Z}}$ is defined by (see Figure 2):

$$\kappa(P) := \sum_{i \in \mathbb{Z}/N\mathbb{Z}} \angle(x_i - x_{i-1}, x_{i+1} - x_i).$$

• A sequence $(a_j)$ of points of a simple closed curve $C$ forms a chain if for each pair $(i, j)$, the intersections of the two open arcs of $C$ from $a_i$ to $a_j$ with the set $\{a_k\}$ are exactly the subsets $\{a_k\}_{k \in [i+1,j-1]}$ and $\{a_k\}_{k \in [j+1,i-1]}$.

• A polygonal line (or a polygon) is said to be inscribed in $C$ if its ordered sequence of vertices forms a chain of $C$.

• The turn $\kappa(C)$ of a simple curve $C$ (respectively of a Jordan curve) is the supremum of the turn of its inscribed polygonal lines (respectively of its inscribed polygons).

![Figure 2: The turn of the inscribed polygon is the sum of the green angles. The turn of the blue Jordan curve is the supremum of the turn of the inscribed polygons](image)

The properties of the turn used in rest of the report are recalled in Property 2.2.

### Property 2.2 ([AR89]).

- The turn coincides with the integral of the usual curvature on $C^2$ curves.

- (Fenchel’s Theorem) The turn of a Jordan curve is greater than or equal to $2\pi$. The equality case occurs if and only if the interior of $C$ is convex.

- Every curve of finite turn has left-hand and right-hand tangent vectors $e_l(c)$ and $e_r(c)$ at each of its points.
For any arc $C_{a,b}$ of finite turn containing a point $c$,
\[ \kappa(C_{a,b}) = \kappa(C_{a,c}) + \kappa(C_{c,b}) + \angle(e_l(c), e_r(c)). \]

For any Jordan curve $C$ of finite turn containing a point $c$,
\[ \kappa(C) = \kappa(C \setminus \{c\}) + \angle(e_l(c), e_r(c)). \]

The aim of the report is to characterize par-regularity in terms of turn. Par-regular curves are defined by the property of bypassing disks of controlled radius. The following lemma will permit us to transfer the knowledge of curvature of such circles to the curvature of the par-regular curve. The lemma is a slight improvement of Lemma 2 [LQMBT20]. This version is nevertheless necessary to get the result. The proof remains essentially the same.

**Lemma 2.3.** Let $C$ be a curve with endpoints $a$, $b$ such that the straight segment $(a, b)$ does not intersect the curve $C$. Let $C'$ be a simple curve from $a$ to $b$ such that $C'$ lies in the closure of the interior of the Jordan curve $C \cup [a, b]$ and $C' \cup [a, b]$ is convex. Then $\kappa(C) \geq \kappa(C')$.

**Proof.** Throughout the proof, the half-line with initial point $a$ and passing through $b$ will be noted $\overrightarrow{ab}$. Firstly, assume that $C'$ is a polygonal line. We set $C' := [a, p_1, \ldots, p_m, b]$. Let $c$ be any point in $(a, b)$. For any $i \in [1, m]$, let $q_i$ be the first intersection between the half-line $cp_i$ and $C$. Let $Q$ be the polygonal line $[a, q_1, \ldots, q_m, b]$. Let us show that $Q$ is inscribed in $C$, i.e., by definition that the sequence of its vertices is a chain. Assume by contradiction that $(a, q_1, \ldots, q_m, b)$ is not a chain of $C$. Then there exists $(i, j, k)$ such that $i < j < k$ and $(q_i, q_j, q_k)$ is not a chain of $C$ or equivalently $(q_i, q_k, q_j)$ is a chain of $C$ (up to consider $q_0 := a$ and $q_{m+1} := b$). Observe that this assumption in particular implies that $C'$ has more than two vertices: $C' \neq [a, b]$. Therefore, the interior of $C' \cup [a, b]$ is not empty. Let $C_{q_i, q_k}$ be the closed arc of $C$ delimited by $q_i$ and $q_k$. Let $T$ be the closed angular sector delimited by the half-lines $cp_i$ and $cp_k$ and containing the segment $[p_i, p_k]$. Since $T$ contains points inside and other points outside the Jordan curve $C \cup [a, b]$, the set $T \setminus C$ has at least two connected components. Let $S$ be the topological closure of the connected component of $T \setminus C$ containing $c$. Notice that $\partial S \subset [c, q_i] \cup C_{q_i, q_k} \cup [q_k, c]$ since $C$ does not intersect $[c, q_i]$ nor $[c, q_k]$. Let us show that $p_j \in S$. Since $(c, p_i, p_j, p_k)$ is a chain of the convex curve $C' \cup [a, b]$, it defines a convex polygon. Then the point $p_j$ belongs to the angular sector $T$. Therefore, $p_j$ belongs to $S$. Let $q'_j$ be a point in the intersection between the half-line $cp_j$ and the curve arc $C_{q_i, q_k}$. By its definition, the point $q'_j$ belongs to $[c, q'_j]$ and since $[c, q_j]$ does not intersect $C$, $q_j$ belongs to $S$. Let $C_{q_j, b}$ be the arc of $C$ between $q_j$ and $b$. Since the curve $C$ is simple, the arc $C_{q_j, b}$ does not intersect $C_{q_i, q_k}$. Moreover, the arc $C_{q_j, b}$ does not intersect the half-open segments $[c, q_i]$ and $[c, q_k]$ by definition of $q_i$ and $q_k$. Then, the arc $C_{q_j, b}$ has its end $q_j$ in $S$, its other end $b$ outside $S$ but does not intersect $\partial S$. Contradiction! Then $(a, q_1, \ldots, q_m, b)$ is a chain of $C$. 

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Then, $\kappa(C) \geq \kappa(Q)$ by definition of $\kappa(C)$, $\kappa(Q \cup [b, a]) \geq \kappa(C' \cup [b, a])$ by Fenchel’s Theorem (Property 2.2) and
\[
\angle(a - b, p_1 - a) \geq \angle(a - b, q_1 - a) \\
\angle(b - a, p_m - b) \geq \angle(b - a, q_m - b)
\]
for $C'$ is inside $Q \cup [a, b]$. Since
\[
\kappa(C' \cup [b, a]) = \kappa(C') + \angle(a - b, p_1 - a) + \angle(a - b, b - p_m)
\]
and
\[
\kappa(Q \cup [b, a]) = \kappa(Q) + \angle(a - b, q_1 - a) + \angle(a - b, b - q_m)
\]
by definition of the turn of a polygon, the result holds if $C'$ is a polygonal line. If $C'$ is not a polygonal line, then, from the first part of the proof, $\kappa(P) \leq \kappa(C)$ for any $P$ inscribed in $C'$. By definition of the turn and since the supremum is the smallest upper bound,
\[
\kappa(C') \leq \kappa(C).
\]

![Figure 3: Blue: the curve $C$ and the line segment $[a, b]$. Black: the polygonal line $C' = [a, p_1, p_2, b]$. Black, dashed: the projection of $p_1$ and $p_2$ on $C$ yields the points $q_1$ and $q_2$. Red: the polygonal line $Q = [a, q_1, q_2, b]$.](image)

### 2.2 Locally turn-bounded curves

We introduced in [LQMBT19] a local geometric feature based on the turn. It consists in locally bounding the turn of the curve in order to forbid some artifacts. This feature allows us to consider a wider class than the par-regular curves usually used for estimation in digital geometry.
Definition 2.4 (LTB curves [LQMBT19]). A Jordan curve \( C \) is \((\theta, \delta)\)-locally turn-bounded \(((\theta, \delta)\text{-LTB})\) if, for any two points \( a \) and \( b \) in \( C \) such that the Euclidean distance \( d(a, b) < \delta \), the turn of one of the arcs of the curve \( C \) delimited by \( a \) and \( b \) is less than or equal to \( \theta \). A Jordan curve \( C \) is \( \delta \)-locally turn-bounded \((\delta\text{-LTB})\) if it is \((\pi/2, \delta)\)-LTB.

In particular, a \((\theta, \delta)\)-LTB curve cannot have angular points of turn greater than \( \theta \), i.e. points \( c \) for which \( \angle(e_l(c), e_r(c)) > \theta \) (see [LQMBT19, Proposition 3]).

Property 2.5 ([LQMBT20, Lemma 2]). Let \( C \) be a \( \delta \)-LTB curve. Let \( a, b \) points of \( C \) such that \( d(a, b) < \delta \). Then there exists a unique arc of \( C \) delimited by the points \( a \) and \( b \) and whose turn is less than or equal to \( \pi/2 \).

Definition 2.6 (Straightest arc, [LQMBT20, Definition 6]). Let \( C \) be a \( \delta \)-LTB curve. Let \( a, b \) points of \( C \) such that \( d(a, b) < \delta \). The unique arc of \( C \) delimited by the points \( a \) and \( b \) and whose turn is less than or equal to \( \pi/2 \) is called the straightest arc between \( a \) and \( b \) and noted \( C_{a,b} \).

Property 2.7 ([LQMBT20], Proposition 5). Let \( C \) be a \( \delta \)-LTB Jordan curve and \( a \in C \). Then, for any \( \epsilon \leq \delta \), the intersection of \( C \) with the open disk \( B(a, \epsilon) \) is path-connected and is therefore an arc of \( C \).

From Property 2.7 we derive that LTB curves have no local U-turns.

Property 2.8 ([LQMBT20, Proposition 12]). Let \( C \) be a \( \delta \)-LTB curve. Let \( \gamma: [0, t_M) \to C \) be an injective parametrization of the curve \( C \) and \( t_m \in (0, t_M) \) be such that the arc \( \gamma([0, t_m]) \) is included in \( B(\gamma(0), \frac{\delta}{2}) \). Then, the restriction of the function \( t \mapsto ||\gamma(t) - \gamma(0)|| \) to \([0, t_m]\) is increasing.

2.3 Curves with Lipschitz turn

In the framework of LTB curves, smoothness can be expressed by a Lipschitz behavior of the turn.

Definition 2.9. A curve \( C \) has a \( k \)-Lipschitz turn if for every subarc \( A \) of \( C \),

\[ \kappa(A) \leq kL(A). \]

where \( L(A) \) stands for the length of arc \( A \).

Observe that a curve \( C \) has a \( k \)-Lipschitz turn if and only if the turn of any subarc of \( C \) is upper bounded by the turn of an arc of circle of radius \( \frac{1}{k} \) and same length. That is why the constant \( k \) will often be noted by \( \frac{1}{r} \). In particular, there is no spike in a curve with a \( k \)-Lipschitz turn.

One of the main idea of our characterization of par-regular curves in terms of integral curvature is to compare the ratio between the arc length and the Euclidean distance between the arc end points in the case of a curve with Lipschitz turn and in the reference case of an arc of circle. Lemma 2.11 gives us an upper bound on the length of a curve with Lipschitz turn. Its proof relies on the Schur’s Comparison Theorem —that we recall below— and is similar to the one of Proposition 11 in [LQMBT20].
Property 2.10 (Schur’s Comparison Theorem: [BSSM08], p. 150). Let $\gamma$ and $\bar{\gamma}$ be two simple curves parameterized by arc length on $[0, L]$ such that:

- $[\bar{\gamma}(0), \bar{\gamma}(L)] \cup \bar{\gamma}([0, L])$ is a convex Jordan curve,
- for each subinterval $I \subset [0, L]$,
  \[ \kappa(\gamma(I)) \leq \kappa(\bar{\gamma}(I)). \]

Then,
\[ \|\bar{\gamma}(L) - \bar{\gamma}(0)\| \leq \|\gamma(L) - \gamma(0)\|. \]

Lemma 2.11. Let $C$ be a $\delta$-LTB curve having a $\frac{1}{\delta}$-Lipschitz turn with $\delta \geq 2r$. Given two points $a, b$ in $C$ such that $\|b - a\| < 2r$, the straightest arc $C_{a,b}$ from $a$ to $b$ has its length smaller than $2r \arcsin \left(\frac{\|b - a\|}{2r}\right)$.

Proof. By Property 2.7, the intersection of the open disk $B(a, 2r)$ and $C$ is path connected. Let $\gamma$ be the parametrization by arc length of the arc of $C$ from $a$ to $b$ in $B(a, 2r)$. Then, $\gamma(0) = a$ and $\gamma(s_1) = b$ for some $s_1 > 0$. By contradiction, assume that $s_1 > s_0$ where $s_0 = 2r \arcsin \left(\frac{\|b - a\|}{2r}\right)$ and put $c = \gamma(s_0)$. Let $\bar{\gamma}$ be the parametrization by arc length of some circle of radius $r$.

By hypothesis, for any subinterval $I$ of $[0, s_0]$,
\[ \kappa(\gamma(I)) \leq \frac{1}{r}|I|. \]

In other words, for any subinterval $I$ of $[0, s_0]$,
\[ \kappa(\gamma(I)) \leq \kappa(\bar{\gamma}(I)). \]

Hence, Schur’s Comparison Theorem [BSSM08] applies:
\[ \|c - a\| \geq \|\bar{\gamma}(s_0) - \bar{\gamma}(0)\| \]
\[ \geq \|b - a\| \quad \text{by definition of } s_0 \text{ and } \bar{\gamma}. \]

The last inequality contradicts the quasi-convexity of $s \mapsto \|\gamma(s) - \gamma(0)\|$ (Property 2.8).

Observe that the bound of the inequality in Lemma 2.11 is sharp: the equality case holds for a circle arc.

3 Equivalence between Lipschitz turn and par-regularity

The goal of this section is to characterize par-regular curves as a subset of LTB-curves thanks to the notion of Lipschitz turn. The example depicted in Figure 4 proves that having a Lipschitz turn with known parameter $k$ is not
Figure 4: The blue curve —made of half-circles of radius $r$ and straight segments— is $\delta$-LTB and has $\frac{1}{r}$-Lipschitz turn. Its reach is got at the point represented by a bullet and is equal to $\frac{b}{2}$ (then its radius of par-regularity is smaller than $\frac{b}{2}$). By choosing smaller and smaller values of $\delta$ and huge values of $r$, we get a family of curves not par($r$)-regular but whose turn is $\frac{1}{r}$-Lipschitz with arbitrarily big value of $r$ whose radius of par-regularity cannot be inferred from the Lipschitz parameter $\frac{1}{r}$.

sufficient to determine a radius of par-regularity. The value $\delta$ provided by the LTB-hypothesis is then useful to quantified the equivalence between the two notions.

Let us now state our equivalence theorem between Lipschitz turn and par-regularity.

**Theorem 3.1.**

- Let $r > 0$, any par($r$)-regular curve is $(\theta, 2r \sin(\frac{\theta}{2}))$-LTB and has a $\frac{1}{r}$ Lipschitz turn for $\theta \in (0, \pi]$.

- Conversely, any $(\frac{\pi}{2}, \delta)$-LTB curve with $\frac{1}{r}$-Lipschitz turn is par($r_1$)-regular for any $r_1 < \min(\frac{\delta}{2}, r)$.

The first implication has already been proved in [LQMBT20, Theorem 2 and Lemma 6]. From Lemmas 2.11 and 2.3, we derive in Lemma 3.2 the converse implication of Theorem 3.1.

**Lemma 3.2.** Let $C$ be a $\delta$-LTB curve having a $\frac{1}{r}$-Lipschitz turn with $\delta > 0$. Then the reach of $C$ is greater than or equal to $\min(\frac{\delta}{2}, r)$.

**Proof.** By contradiction assume that $\text{reach}(C) < r_1 = \min(\frac{\delta}{2}, r)$. Then there exist a point $o$ on the medial axis of $C$ and two points $a$ and $b$ on $C$ such that $d(o, a) = d(o, b) = d(o, C) < r_1$. Thus, $\|a - b\| < 2r_1$. Let $C_{a,b}$ be the straightest arc of $C$ between $a$ and $b$. On the one hand, by Lemma 2.11 (noting that $C$ is $(1/r_1)$-Lipschitz for $r_1 \leq r$),

$$\mathcal{L}(C_{a,b}) \leq 2r_1 \arcsin \left( \frac{\|b - a\|}{2r_1} \right).$$

In other words, since the sine function is increasing on $[0, \pi/2]$,

$$\|b - a\| \geq 2r_1 \sin \left( \frac{\mathcal{L}(C_{a,b})}{2r_1} \right). \quad (1)$$
On the other hand, let \( r' = d(o, a) \) (\( r' < r \)). By definition of \( o, a \) and \( b \), the curve \( C \) does not intersect the interior of the circle of center \( o \) and radius \( r' \). So, let \( \bar{C} \) be the arc of this last circle bounded by \( C_{a,b} \cup [oa] \cup [ob] \). Since \( C \cup [a, b] \) is convex, by Lemma 2.3 we have,

\[
\kappa(\bar{C}) \leq \kappa(C_{a,b}).
\]

Hence,

\[
\|b - a\| = 2r' \sin\left(\frac{\kappa(\bar{C})}{2}\right) \\
\leq 2r' \sin\left(\frac{\kappa(C_{a,b})}{2}\right).
\]

The last inequality holds for the sine function is increasing on \([0, \pi/2]\) and \( \kappa(C_{a,b}) \leq \pi \) because \( C_{a,b} \) is the straightest arc between \( a \) and \( b \).

By Inequality 1 and Inequality 2

\[
2r_1 \sin\left(\frac{\mathcal{L}(C_{a,b})}{2r_1}\right) \leq 2r' \sin\left(\frac{\kappa(C_{a,b})}{2}\right) < 2r_1 \sin\left(\frac{\kappa(C_{a,b})}{2}\right).
\]

Thus,

\[
\frac{1}{r} \mathcal{L}(C_{a,b}) \leq \frac{1}{r_1} \mathcal{L}(C_{a,b}) < \kappa(C_{a,b}),
\]

which contradicts the Lipschitz hypothesis.

The bound of Lemma 3.2 is sharp. Indeed, if \( C \) is a circle of radius \( r \) then the reach is exactly \( r \) and the reach of the \( \delta \)-LTB curve depicted in Figure 4 is exactly \( \delta/2 \).

Notice that we have proved a “qualitative equivalence” between the notions of positive reach and LTB curve with Lipschitz turn but we failed to obtain a “quantitative equivalence”. Indeed, starting from a par(\( r \))-regular curve \( C \) and applying [LQMBT20, Theorem 2 and Lemma 6], we derive that, for any \( 0 \leq \theta \leq \pi/2 \), \( C \) is a \( (\theta, 2r\sin(\theta/2)) \)-LTB curve having a \( (1/r) \)-Lipschitz turn. Then, from Lemma 3.2 we get that \( C \) is a par(\( r\sin(\theta/2) \))-regular curve with \( \theta \leq \frac{\pi}{2} \). Hence, at best \( C \) is proved to be par(\( \sqrt{2}/2r \))-regular. We do not retrieve the starting parameter.

### 4 Conclusion

Theorem 3.1 permits to split the definition of par-regularity into two parts: a control of the curvature of the boundary (a par-regular curve has a Lipschitz turn) and a thickness of shape bounded by the curve (a par-regular curve is locally turn-bounded). Other generalizations of par-regularity exist in the literature. If, in [LQMBT21], we have already established the link with the quasi-regularity introduced in [NKDP17], we also consider to prove that LTB curves have a positive \( \mu \)-reach (a generalization of reach for polygonal and regular curves) [CCLT09, CCL09].
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