A unity of QCD evolution dynamics at small $x$ range

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Abstract

The DGLAP, BFKL, modified DGLAP and modified BFKL equations are constructed in a unified partonic framework. The antishadowing effect in the recombination process is emphasized, which leads to two different small $x$ behaviors of gluon distribution. In the meantime, the BFKL equation and modified DGLAP equation are viewed as the corrections of the initial gluon correlations to the evolution dynamics at twist-2 and twist-4, respectively. A partonic explanation of Regge theory for the BFKL dynamics and the relation of the BFKL dynamics with the helicity configuration of the initial gluons are presented.

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1 Introduction

The evolution equations for parton distributions play an important role in the QCD studies of high-energy processes. A standard evolution equation is the DGLAP equation (by Dokshitzer, Gribov, Lipatov, Altarelli and Parisi in [1]), which sums large logarithmic $Q^2$ corrections ($LL(Q^2)$) to the integrated parton distributions. The DGLAP equation can be obtained by using the renormalization group or graphic techniques. The DGLAP equation successfully predicts the $Q^2$-dependence of parton distributions in a broad kinematic range. However, the DGLAP equation is unsatisfactory at small $x$, where the resummation of the leading logarithm of $x$ ($LL(1/x)$) becomes important and it leads to the BFKL equation (by Balitsky, Fadin, Kuraev and Lipatov in [2]). The BFKL equation for the unintegrated gluon distribution is originally derived by using the Reggeization of gluons. Both the BFKL and DGLAP equations predict a rapid increase of the parton densities at small $x$ due to parton splitting, and the unitarity limit is violated. Therefore, the corrections of the higher order QCD, which suppress or shadow the growth of parton densities, become a focus of intensive study in recent years. Several nonlinear evolution equations are proposed at small $x$. Some of them are the GLR-MQ equation (by Gribov, Levin and Ryskin in [3] and by Mueller and Qiu in [4]), modified DGLAP equation (by Zhu and Ruan in [5]), JIMWLK equation (by Jalilian-Marian, Iancu, McLerran, Weigert, Leonidov and Kovner in [6]), Balitsky-Kovchegov equation [7], and various versions of the evolution equations based on the color dipole picture [8]. Unfortunately, we still lack a unified picture to clearly understand the relation between the DGLAP and BFKL equations and the unitarity corrections on them.

In this paper we try to present a unified partonic framework for understanding some of QCD evolution equations at small $x$. Our idea is straightforward. We begin with an elementary amplitude Fig. 1a of deep inelastic scattering (DIS) in the parton scattering...
picture, where the dashed line implies a virtual current probing gluon. This amplitude, together with its conjugate amplitude, constructs the DGLAP equation for gluons. The correlations among initial partons are neglected in the derivation of the DGLAP equation. Obviously, this assumption is invalid in the higher density region of partons, where the parton wave functions begin to spatially overlap. Therefore, the leading corrections of the correlating initial gluons to the elementary amplitude at small $x$ should be considered.

To this end, we add the possible initial gluons to Fig. 1a step by step. The resulting three sets of amplitudes are listed in Fig. 1b-1d. It is interesting that these amplitudes produce the BFKL equation, modified DGLAP equation and modified BFKL equation, respectively.

A key problem in the restoring unitarity is the origin of the negative corrections. In the viewpoint of elementary QCD interaction, the suppression to the gluon splitting is naively from its inverse process-the gluon fusion (or the recombination process, i.e., a gluon splitting combines with the gluon fusion). DIS structure functions are the imaginary parts of the amplitudes for the forward Compton scattering of the target with a probe. According to the time-ordered perturbative expansion of the statement of unitarity of the S-matrix, the structure functions are associated with the sum of cut diagrams. These different cut graphs represent various possible sub-partonic processes due to the unitarity of the perturbative S-matrix. Therefore, the sum of cut graphs in the recombination process is necessary not only for infrared safety, but also for restoring unitarity. As Gribov, Levin and Ryskin pointed out that the $2 \rightarrow 2 \rightarrow 2$ cut-diagram of the recombination process always contributes the positive correction and it can not lead to the negative effect in parton evolution equation [3]. The negative screening effect in the recombination process can occur in the interferant or virtual cut-diagrams of the recombination amplitudes [5]. For computing the contributions from the interference processes, work [3] used the AGK
The result is the GLR-MQ equation. The GLR-MQ equation was generalized to include the multi-gluon recombination process in the Glauber-Mueller formula [8].

In the next advance, one of us (zhu) disputed in [5] the above mentioned applications of the AGK cutting rule in the GLR-MQ equation. The TOPT-cutting rule based on the time ordered perturbation theory (TOPT) replaces the AGK cutting rule to expose the relations among various cut diagrams in a recombination process [5]. Thus, we can completely compute the contributions of the gluon recombination process. Following it, the modified DGLAP equation was proposed in [5]. A remarkable property of this equation is that the positive antishadowing and negative shadowing in the nonlinear evolution equation are naturally separated. The recombination among the gluon at Bjorken variable $y \geq x$ will reduce the density of the gluon at $x$. In the meantime, a gluon in the region $[x/2, x]$ can generate a gluon at $x$ if the former combines with its companion. In consequence, the modifications in the gluon distribution at $x$ could come from either the screening (shadowing) effect of the fusion of two gluons at $y \geq x$, or the enhancement (antishadowing) effect of the fusion of two gluons at $y \geq x/2$ and $y < x$. As a result, the corrections arising from gluon recombination at small $x$ depends not only on the size of gluon density at this value of $x$, but also on the sharp of gluon density in the region $[x/2, x]$. Thus, the shadowing effect in the evolution process will be weakened by the antishadowing effect if the gluon distribution has a steeper form [5,10]. Although the gluon recombination re-distributes the parton densities in the above mentioned processes, it always keeps the momentum conservation. We note that the GLR-MQ equation also computes the positive contribution of the $2 \rightarrow 2-2 \rightarrow 2$ cut diagram, the momentum conservation of the recombination process is violated due to the mistakes in the application of the AGK cutting rule, where the contributions of all cut diagrams only differ by a
numeric weight.

We shall extend this work along the above mentioned researches. For this purpose, we give a short review about the applications of the TOPT in the factorization scheme. Following it the QCD evolution equations listed in Fig. 1 are derived in a traditional Bjorken frame. We find that the antishadowing effect, which is a direct result of the momentum conservation, is non-negligible in the corrections of the gluon recombination to the DGLAP and BFKL equations. In particular, we predict that the gluon distribution asymptotically (abruptly) approaches to the saturation limit below (above) a moderate $Q^2$-scale due to the antishadowing effect.

It is interesting that, in this framework, the BFKL equation can be derived as a correction of the correlations of initial gluons at the twist-2, and the relation of the BFKL dynamics with the helicity configuration of the initial gluons becomes transparent. Then the partonic interpretation of Regge theory for the BFKL dynamics is presented. We also show that the similarity between the modified DGLAP equation and the BFKL equation in their ladder diagrams. We compare the modified BFKL equation with the color dipole model of the Balitsky-Kovchegov equation. We find that they have a similar physical picture but they are really different evolution dynamics. As is well known, the QCD evolution dynamics for parton densities is extensively applied in high energy physics. The discussions in this work present a new way to reveal some unknown properties, which can be regarded as a supplement to our knowledge about evolution equations.

The paper is organized as follows. In Section 2 we discuss the separation of the evolution kernels from the TOPT diagrams using the semi-classical Weizsäcker-Williams ($W-W$) approximation [11]. In Sections 3 we discuss the connection between the DGLAP equation and BFKL equation. In Section 4 the modified DGLAP equation is generalized to include the BFKL equation. The discussions and concluding remarks are given in
Section 5.
2 Factorization

The partonic picture of a high energy process is frame- and gauge-dependent. The appropriate choices of the coordinate frame and gauge are helpful for extracting the evolution kernels using a simple partonic picture. A partonic scattering model takes Bjorken frame, i.e., the momentum of the nucleon is

\[ P^\mu = (P_0, \overrightarrow{P}, P_3) = (P, 0, P), \] (2.1)

while the virtual probe has almost zero-energy \( q_0 = (2M\nu + Q^2)/4P \) and zero-longitudinal momentum \( q_3 = -(2M\nu - Q^2)/4P \) as \( P \to \infty \), so that the momentum of the probe is mainly transverse to the nucleon direction. In the mean time, we work in the physical axial gauge: \( n \cdot A = 0 \), with \( n \) being the light cone vector

\[ n^\mu \equiv \frac{1}{\sqrt{2}}(1, 0, -1), \] (2.2)

and

\[ \overline{n}^\mu \equiv \frac{1}{\sqrt{2}}(1, 0, 1). \] (2.3)

The evolution kernel in a QCD evolution equation is a part of more complicated scattering diagram. In general, the correlations among the initial partons make these partons off mass-shell. One of the most important hypotheses in the derivation of the DGLAP equation is that all correlations among initial partons are negligible during the interaction. In this case, the interaction of a virtual probe with the nucleon can be factorized as the nonperturbative parton distribution and hard probe-parton matrix in the collinear factorization scheme.

On the other hand, at the higher density range of partons, these correlations among the initial partons can no longer be neglected (Fig. 2). In this case, the transverse momenta
of the initial partons are non-zero and these partons are generally off mass-shell, therefore, the $k_T$-factorization scheme [12] is necessary.

However, in this work we use the $W-W$ approximation rather than the $k_T$-factorization scheme. The reason is that the $W-W$ approximation allows us to extract the evolution kernels and to keep all initial and final partons of the evolution kernels on their mass-shell. For illustrating this idea, we consider a sub-process as shown in Fig. 3, where the momenta of three massless particles are parameterized as

\[ p_1 = (p, 0, p), \]

\[ p_2 = (x_2p + \frac{k^2}{2x_2p}, k, x_2p), \]

and

\[ p_3 = (p_3^0, -k, x_3p), \]

where $p \to \infty$. The particle 3 is off its mass-shell. We decompose this propagator into a forward and a backward propagator according to the TOPT. The corresponding momenta are on mass-shell

\[ \hat{p}_F = (x_3p + \frac{k^2}{2x_3p}, -k, x_3p), \]

and

\[ \hat{p}_B = (-x_3p - \frac{k^2}{2x_3p}, -k, x_3p), \]

respectively. Note that Eqs. (2.4) and (2.5) require
\[ xp \gg | k |, \]

which is consistent with the conditions of high energy- and small angle-scattering in the \( W - W \) approximation. One can find that the forward propagator contributes the power-factor \( 1/k^2 \), since

\[
\frac{1}{2E_3 E_2 + E_{p_F} - E_1} \sim \frac{1}{k^2},
\]

while the backward propagator gives the contributions \( \sim 1/p^2 \). If the dominant contributions in a process are from the terms with the power of transverse momentum, one can find that the contributions from the backward propagator are suppressed at \( p \rightarrow \infty \), and the dominant contribution is from the forward component. In this case, we can break the forward propagator since it propagates on mass-shell.

The wave function of a parton in the Bjorken frame occupies a space with transverse size \( \sim 1/k \) and longitudinal size \( \sim 1/xp \) in the nucleon. The partons are regarded as independent if the partons are dilute and the parton distributions obey the DGLAP equation (Fig. 4a). However, this evolution equation predicts a rapid rise of gluon multiplicities inside the nucleon at small \( x \). At sufficiently large gluon densities, the wave functions of the gluons overlap with each other. We call such a correlating gluon cluster with a transverse scale \( x_{ab} \) as the cold spot, which phenomenologically describes the correlation among initial partons (the dashed circles in Fig. 4b). In this work we mainly consider the cold spots consisted of two gluons, which dominates the leading small \( x \) processes.

The elementary correlation among two initial partons in a cold spot is shown in Fig. 5, where the dark circle implies all possible QCD-channels. Let us consider a configuration of two correlating partons that form a cold spot inside a nucleon. We assume that the longitudinal momentum of the initial parton is much larger than its transverse momentum
in the Bjorken frame even at small $x$. Thus we can take the $W - W$ approximation and extract the two parton correlation function from the TOPT-diagram as

$$f(p_a, p_b, x_a, x_b) \equiv \frac{E_{p_a} + E_{p_b}}{2E_P} |M_{P\rightarrow p_a p_b X}|^2 \left[ \frac{1}{E_P - E_{p_a} - E_{p_b}} \right]^2 \left[ \frac{1}{2E_{p_a}} \right]^2 \left[ \frac{1}{2E_{p_b}} \right] \prod_X \frac{d^3k_X}{(2\pi)^32E_X}. \quad (2.8)$$

Using Fourier transform with respect to the transverse momentum transfer, these partons localize in a range with the size $x_{ab}$ on the impact space. The unintegrated distribution $f(x_{ab}^2, x_a, x_b)$ can be understood as the probability of finding two partons with transverse distance $x_{ab}$ and the longitudinal momentum fractions $x_a$ and $x_b$, respectively.

The probability of forming a cold spot is proportional to the "size" of gluon. In particular, we have a straightforward bound condition

$$f(x_{ab}^2 = 0, x_a, x_b) = 0, \quad (2.9)$$

which implies that the overlap probability of two sizeless partons in the impact space vanishes.

In the impact space, the integrated gluon distribution reads

$$G(x_{Q^2}^2, x) \equiv xg(x_{Q^2}^2, x) = \int_{x_{Q^2}^2}^{R^2} \frac{dE_{ab}}{E_{ab}^2} f(x_{ab}^2, x), \quad (2.10)$$

or

$$f(x_{ab}^2, x) = -\frac{d^2}{dx_{Q^2}^2} \frac{dE_{ab}}{E_{ab}^2} \bigg|_{x_{Q^2}^2 = x_{ab}^2} \quad (2.11)$$

while in the momentum space, they are

$$G(Q^2, x) \equiv xg(Q^2, x) = \int_{Q_{min}^2}^{Q^2} \frac{d^2k^2}{k^2} f(k^2, x), \quad (2.12)$$
or

\[ f(k^2, x) = Q^2 \frac{\partial \bar{g}(Q^2, x)}{\partial Q^2} \bigg|_{Q^2=k^2}, \]  

(2.13)

where \( R_N \) is an effective radius of the nucleon. Note that the momentum \( Q \) of the probe is mainly transverse to the nucleon direction.

One can imagine that the cold spots fill the whole transverse plane of a nucleon at a higher density scale \( n_s \), which corresponds to the saturation of the gluon distributions (Fig. 4c). The saturation is a complicated state, where the long- and short-distance correlations among partons coexist in a nucleon. In this work, we focus on the region far from the saturation limit.

In the next step, we discuss the separation of the virtual probe-vertex from the DIS amplitude. In principle, in- and out-probe lines can be attached to the left- and right-hand sides of the cut line in all possible ways. However, in the Bjorken frame, one only need to pick up the diagrams with two photon lines following the cut line at the leading logarithmic \( (Q^2) \) approximation \( (LL(Q^2)A) \), since these diagrams contribute the dominant terms \( \sim \frac{dk^2}{k^2} \) [5].

Under the same (i.e., \( LL(Q^2)A \)) condition, the contributions of the backward components in the two propagators connecting with the probe are suppressed. Thus, we can take the \( W-W \) approximation to factorize the virtual probe-vertex from the cut diagram and extract the evolution kernel in TOPT. However, the BFKL dynamics works in the region beyond the \( LL(Q^2)A \). Fortunately, we could show that the dominant contributions still come from the terms with the power of the transverse momentum, i.e., we pick up the dominant terms \( \sim \frac{dk^2}{k^n} \). Therefore, the \( W-W \) approximation is still feasible in the separation of the probe-vertex for the BFKL dynamics.
3 Evolution equations at twist-2

Using the $W - W$ approximation mentioned in Sec. 2, the DIS cross section at leading twist is generally factorized into the following convolution form

$$d\sigma(probe^*p \to k'X)$$

$$= f(k_1^2, x_1) \otimes K(k_2^2, x_2, \alpha_s) \otimes d\sigma(probe^*k_2 \to k')$$

$$\equiv \Delta f(k_2^2, x_2) \otimes d\sigma(probe^*k_2 \to k'),$$

(3.1)

with the perturbative evolution kernel $K$, the $probe^*$-parton cross section $d\sigma(probe^*k_2 \to k')$ and the nonperturbative unintegrated parton distribution function(s) $f$. For simplicity, we take the fixed $\alpha_s$ in this work. We emphasize that the factorization between $K$ and $d\sigma(probe^*k_2 \to k')$ is valid because the propagators linking these two parts are on mass-shell in the $W - W$ approximation, and hence we can define $\Delta f(k_2^2; x_2)$. According to the scale-invariant parton picture of the renormalization group [13], we regard $\Delta f(k_2^2, x_2)$ as the increment of the distribution $f(k_1^2, x_1)$ when it evolves from $(k_1^2, x_1)$ to $(k_2^2, x_2)$. Thus, the connection between the unintegrated gluon distribution functions $f(k_1^2, x_1)$ and $f(k_2^2, x_2)$ due to Eq. (3.1) is

$$f(k_2^2, x_2) = f(k_1^2, x_1) + \Delta f(k_2^2, x_2)$$

$$= f(k_1^2, x_1) + \int_{k_2^2_{\text{min}}}^{k_2^2} \frac{dk_1^2}{k_1^2} \int_{x_2}^{1} \frac{dx_1}{x_1} K(k_2^2, x_2, x_1, \alpha_s) f(k_1^2, x_1),$$

(3.2)

where the factorization scales are integrated, thus the observed cross section $d\sigma(probe^*p \to k'X)$ is independent of the factorization scheme.

In the case of the evolution along the transverse momentum, we differentiate Eq. (3.2) with respect to $k_2^2$ and get
\[
\frac{\partial f(k_2^2, x_2)}{\partial k_2^2} = \int_{x_2}^1 \frac{dx_1}{x_1} \frac{1}{k_1^2} \mathcal{K} \left( \frac{k_2^2}{k_1^2}, \frac{x_2}{x_1}, \bar{\alpha}_s \right) f(k_1^2, x_1) \bigg|_{k_1^2=k_2^2} \\
+ \int_{k_{1_{\text{min}}}^2}^{k_2^2} \frac{dk_1^2}{k_1^2} \int_{x_2}^1 \frac{dx_1}{x_1} \frac{\partial \mathcal{K} \left( \frac{k_1^2}{k_2^2}, \frac{x_2}{x_1}, \bar{\alpha}_s \right)}{\partial k_2^2} f(k_1^2, x_1).
\]

(3.3)

Unfortunately, this equation is unsolvable in a resummation form. However, at the \(LL(k_2^2)A\), the evolution kernel \(\mathcal{K}\) in Eq. (3.3) only is the function of the longitudinal variables and the second term in the right-hand side of Eq. (3.3) vanishes. In this case, using Eq. (2.12) we have

\[
\Delta G(Q^2, x_2) \equiv \int_{k_{2_{\text{min}}}^2}^{Q^2} \frac{dk_2^2}{k_2^2} \Delta f(k_2^2, x_2) \\
= \int_{k_{2_{\text{min}}}^2}^{Q^2} \frac{dk_2^2}{k_2^2} \int_{k_{2_{\text{min}}}^2}^{k_2^2} \frac{dk_1^2}{k_1^2} \int_{x_2}^1 \frac{dx_1}{x_1} \mathcal{K} \left( \frac{x_2}{x_1}, \bar{\alpha}_s \right) f(k_1^2, x_1) \\
= \int_{k_{2_{\text{min}}}^2}^{Q^2} \frac{dk_2^2}{k_2^2} \int_{x_2}^1 \frac{dx_1}{x_1} \mathcal{K}_{DGLAP} \left( \frac{x_2}{x_1}, \bar{\alpha}_s \right) G(k_2^2, x_1),
\]

(3.4)

and

\[
G(Q^2, x_2) = G(k_2^2, x_1) + \Delta G(Q^2, x_2),
\]

(3.5)

which lead to the DGLAP equation for the gluon distribution

\[
Q^2 \frac{\partial g(Q^2, x_2)}{\partial Q^2} = \int_{x_2}^1 \frac{dx_1}{x_1} \mathcal{K}_{DGLAP} \left( \frac{x_2}{x_1}, \bar{\alpha}_s \right) g(Q^2, x_1).
\]

(3.6)

We could extract the evolution kernel \(\mathcal{K}_{DGLAP}\) in the collinear factorization scheme [14] if the transverse momenta of the initial partons are neglected.

It is interesting that the DGLAP equation Eq. (3.6) also can be derived from the renormalization group, where the gluonic structure function is factorized as
\[ F(Q^2, x_2) = \int_{x_2}^{1} \frac{dx_1}{x_1} C \left( \frac{Q}{\mu}, \frac{x_2}{x_1}, \alpha_s(\mu^2) \right) g(\mu^2, x_1). \]  

In this equation \( \mu \) is the unintegrated renormalization scale as well as the factorization scale, \( C \) is the perturbative coefficient function. Using

\[ \mu \frac{\partial F(Q^2, x_2)}{\partial \mu} = 0, \]  

one can get [15]

\[ \int_{x_2}^{1} \frac{dx_1}{x_1} \left[ \frac{dC \left( \frac{Q}{\mu}, \frac{x_2}{x_1}, \alpha_s(\mu) \right)}{d \ln \mu^2} g(\mu^2, x_1) + C \left( \frac{Q}{\mu}, \frac{x_2}{x_1}, \alpha_s(\mu) \right) \frac{dg(\mu^2, x_1)}{d \ln \mu^2} \right] = 0, \]  

at the lowest order of \( \alpha_s(\mu^2) \), it becomes

\[ \int_{x_2}^{1} \frac{dx_1}{x_1} C^{(0)} \left( \frac{Q}{\mu}, \frac{x_2}{x_1}, \alpha_s(\mu) \right) \frac{dg(\mu^2, x_1)}{d \ln \mu^2} = - \int_{x_2}^{1} \frac{dx_1}{x_1} dC^{(1)} \left( \frac{Q}{\mu}, \frac{x_2}{x_1}, \alpha_s(\mu) \right) g(\mu^2, x_1). \]  

Using

\[ C^{(0)} = \delta \left( 1 - \frac{x_2}{x_1} \right), \]  

and

\[ C^{(1)} = K_{DGLAP} \left( \frac{x_2}{x_1} \right) \ln \frac{Q^2}{\mu^2}, \]  

one can get the same results as Eq. (3.6).

One difference between the coefficient function \( C^{(1)} \) in Eq. (3.10) and the evolution kernel \( K_{DGLAP} \) in Eq. (3.6) is that two propagators connected with the probe in the coefficient function \( C^{(1)} \) are off mass-shell, but not on mass-shell like in Eq. (3.1). However,
as we have proved that the evolution kernels from two methods are really equivalent in the infrared safe theory even at twist-4. The reason is that the contributions of the backward components in these two propagators always are suppressed [5].

Considering that the scale-invariant parton picture [13] is a phenomenological interpretation of the renormalization group in the derivation of the QCD evolution equation, the above mentioned two methods are really equivalent.

We now construct the evolution equations along the longitudinal variable. As we shall show that the transverse position of the targeted gluon is uncertain in this case, we need to change the integrated phase space in Eq. (3.2), i.e.,

\[
\int \frac{dk^2_1}{k^2_1} \int \frac{dx_1}{x_1} \to \int \frac{dk^2_2}{k^2_2} \int \frac{dx_2 k^2_2 x_2}{x_2 k^2_1 x_1}.
\]

Thus, we have

\[
f(k^2_1, x_1) = f(k^2_2, x_2) - \Delta f(k^2_1, x_1)
\]

\[
= f(k^2_2, x_2) - \int \frac{dk^2_2}{k^2_2} \int_{x_{min}}^{x_1} \frac{dx_2 k^2_2 x_2}{x_2 k^2_1 x_1} \mathcal{K} \left( \frac{k^2_2}{k^2_1}, \frac{x_2}{x_1}, \alpha_s \right) f(k^2_2, x_2)
\]

\[
\equiv f(k^2_2, x_2) - \int \frac{dk^2_2}{k^2_2} \int_{x_{min}}^{x_1} \frac{dx_2}{x_2} \mathcal{K}' \left( \frac{k^2_2}{k^2_1}, \frac{x_2}{x_1}, \alpha_s \right) f(k^2_2, x_2),
\]

where we omit the integral limit for the transverse momentum since they are non-ordering. At the \(LL(1/x)A\), \(\mathcal{K}'\) is only the function of the transverse momenta. We differentiate Eq. (3.14) with respect to \(x_1\) and obtain

\[
-x \frac{\partial f(k^2_1, x)}{\partial x}
\]

\[
= \int \frac{dk^2_2}{k^2_2} \mathcal{K}' \left( \frac{k^2_2}{k^2_1}, \alpha_s \right) f(k^2_2, x)
\]

\[
\equiv \int \frac{dk^2_2}{k^2_2} \mathcal{K}_{BFKL} \left( \frac{k^2_2}{k^2_1}, \alpha_s \right) f(k^2_2, x),
\]
This is the real part of the BFKL equation. Where ‘−’ implies that the evolution along the direction of decreasing \(x\), i.e., \(x \rightarrow x - dx\).

Now let us calculate the evolution kernel. We take the square of the total amplitude in Fig. 1b and separate out the probe vertex using the \(W - W\) approximation [11], one can get the four TOPT-diagrams Fig. 6, where the evolution kernels are constructed by following amplitudes:

\[
A_{BFKL} = A_{BFKL1} + A_{BFKL2},
\]

\[
A_{BFKL1} = \sqrt{\frac{E_k}{E_{p_a}} \frac{1}{2E_k E_k + E_{l_a} - E_{p_a}}} M_{b1},
\]

and

\[
A_{BFKL2} = \sqrt{\frac{E_k}{E_{p_b}} \frac{1}{2E_k E_k + E_{l_b} - E_{p_b}}} M_{b2}.
\]

(3.16)

The momenta of the partons are parameterized as

\[
p_a = (x_1 P + \frac{(k + l_a)^2}{2x_1 P}, k + l_a, x_1 P),
\]

(3.17)

\[
k = (x_2 P + \frac{k^2}{2x_2 P}, k, x_2 P),
\]

(3.18)

\[
l_a = ((x_1 - x_2) P + \frac{l_a^2}{2(x_1 - x_2) P}, l_a, (x_1 - x_2) P),
\]

(3.19)

\[
p_b = (x_1 P + \frac{(k + l_b)^2}{2x_1 P}, k + l_b, x_1 P),
\]

(3.20)

and
\( l_b = ((x_1 - x_2)P + \frac{l_b^2}{2(x_1 - x_2)}P, l_b, (x_1 - x_2)P). \) \hspace{1cm} (3.21)

One of the matrix is

\[
M_{BFKL} = igf^{abc}[g_{\alpha\beta}(p_a + k)_\gamma + g_{\beta\gamma}(-k + l_a)_\alpha + g_{\gamma\alpha}(-l_a - p_a)_\beta]\epsilon_\alpha(p_a)\epsilon_\beta(k)\epsilon_\gamma(l_a), \hspace{1cm} (3.22)
\]

where the polarization vectors are

\[
\epsilon(p_a) = (0, \frac{\epsilon \cdot (k + l_a)}{x_1 P}), \hspace{1cm} (3.23)
\]

and

\[
\epsilon(k) = (0, \frac{\epsilon \cdot k}{x_2 P}), \hspace{1cm} (3.24)
\]

\[
\epsilon(l_a) = (0, \frac{\epsilon \cdot l_a}{(x_1 - x_2) P}). \hspace{1cm} (3.25)
\]

Taking the leading logarithmic \((1/x)\) approximation, i.e., assuming that \(x_2 \ll x_1\), one can get the total amplitude

\[
A_{BFKL}(k_{up}, k_{down}, x_1, x_2) = igf^{abc}2\sqrt{\frac{x_1}{x_2}}[\frac{\epsilon \cdot k_{up}}{k_{up}^2} + \frac{\epsilon \cdot k_{down}}{k_{down}^2}], \hspace{1cm} (3.26)
\]

where we use different indices \(up\) and \(down\) to distinguish between the different paths of the probing position along the time order on the impact space.

The impact parameter in the evolution equations is first introduced by A. Mueller in his color dipole approaches [8]. We imitate his method but work in the scattering model. After Fourier transforming with respect to the transverse momentum transfer, we obtain the representation of the amplitude in the impact parameter space as follows.
\[ A_{BFKL}(x_{a0}, x_{b0}, x_1, x_2) = \int \frac{d^2 k_{up} d^2 k_{down}}{(2\pi)^4} A_{BFKL}(k_{up}, k_{down}, x_1, x_2) e^{i k_{up} x_{a0} + i k_{down} x_{b0}} \]

\[ = i g f^{abc} 2 \sqrt{\frac{x_1}{x_2}} \left[ \frac{x_{a0}}{x_{a0} x_{b0}^2} - \frac{x_{b0}}{x_{b0} x_{a0}^2} \right] \cdot \xi. \]  

(3.27)

Using this amplitude we construct the evolution kernel in the BFKL equation

\[ \mathcal{K}_{BFKL} \frac{dx_{a0}^2}{x_{a0}} \frac{dx_2}{x_2} = \sum_{pol} A_{BFKL} A_{BFKL}^* \frac{d^3 l_a}{(2\pi)^2 2E_l_a} \]

\[ = < 3 >_{color} \frac{\alpha_s}{\pi^2} \frac{x_{a0}^2}{x_{a0} x_{b0}^2} \frac{d^2 x_{a0}}{x_{a0}} \frac{d^2 x_2}{x_2}, \]  

(3.28)

where the position \( x_0 \) in Eq. (3.28) is running along the transverse coordinator due to uncertainty relation although \( k_{up} \) identifies with \( k_{down} \) in the forward deep inelastic scattering. Therefore, the measurement of the unintegrated parton distributions in the BFKL dynamics is non-local and we use the dark box to indicate the above mentioned measurement. On the other hand, we can not regard the fraction \( x \) of the longitudinal momentum of the parton in the BFKL equation as the Bjorken variable since the transverse scale \( k^2 \) of this parton is irrelevant to the probe scale \( Q^2 \).

Using Eqs. (3.15) and (3.28), we get

\[ -x \frac{\partial f(x_{ab}^2, x)}{\partial x} = \frac{3\pi \alpha_s}{\pi^2} \int d^2 x_0 \frac{x_{ab}^2}{x_{a0}^2 x_{b0}^2} f(x_{a0}^2, x) \]

\[ = \frac{3\pi \alpha_s}{2\pi^2} \int d^2 x_0 \frac{x_{ab}^2}{x_{a0}^2 x_{b0}^2} [f(x_{a0}^2, x) + f(x_{ab}^2, x)]. \]  

(3.29)

Note that the distribution \( f(x_{ab}^2, x) \) of a gluon at \( x_0 \) inside a cold spot, which is scaled by a fixed \( x_{ab} \) can be represented by either \( f(x_{a0}^2, x) \) (if taking the coordinate relative to \( x_a \)), or \( f(x_{ab}^2, x) \) (if taking the coordinate relative to \( x_b \)). The right-hand side of Eq. (3.29)
is illustrated in Fig. 7b, where the un-connecting gluons in Fig. 7 means all possible connections as shown in Fig. 6 and in the last diagram the probe is included.

The equation (3.29) is the real part of the BFKL equation in the impact space. A complete BFKL equation should include the contributions from the virtual processes, which are necessary for the infrared safety of the evolution dynamics. Using the TOPT-cutting rule (see Appendix), one can prove that the virtual diagrams in Fig 8 (where we omit the conjugation diagrams) contribute the same evolution kernel as the real kernel but differ by a factor $-1/2$. In the mean time, the probed distribution $f(x_{a0}^2, x)$ is replaced by $f(x_{ab}^2, x)$ because of the change of the cut line. Thus, we can directly write the complete equation as

$$-x \frac{\partial f(x_{ab}^2, x)}{\partial x} = \frac{3\alpha_s}{2\pi^2} \int d^2 x_0 \frac{x_{ab}^2}{x_{a0}^2 x_{b0}^2} \left[ f(x_{a0}^2, x) + f(x_{b0}^2, x) - f(x_{ab}^2, x) \right].$$  \hspace{1cm} (3.30)

This form of the BFKL equation was first derived by Mueller in the color dipole model [8].

To make regularity of the evolution kernel explicit, we use (for simplicity we replace $x_{ab}^2 \rightarrow \rho, x_{a0}^2 \rightarrow \rho_1$ and $x_{b0}^2 \rightarrow \rho_2$)

$$\frac{\rho}{\rho_1 \rho_2} = \frac{1}{\rho_1} + \frac{1}{\rho_2} + \frac{\rho - \rho_1 - \rho_2}{\rho_1 \rho_2}. \hspace{1cm} (3.31)$$

The evolution equation becomes

$$x \frac{\partial f(\rho, x)}{\partial x} = \frac{3\alpha_s}{\pi} \int_0^\infty \frac{d\rho_1}{|\rho_1 - \rho|} \left[ \rho \frac{f(\rho_1, x) - f(\rho, x)}{\rho_1} \right] + \frac{3\alpha_s}{\pi} \int_0^\infty \frac{d\rho_1}{\rho_1} f(\rho, x). \hspace{1cm} (3.32)$$
The first integral of Eq. (3.32) is regular both for $\rho_1 \to 0$ and $\rho_1 \to \rho$. Defining $\rho_1 = u \rho$ when $\rho_1 < \rho$ and $\rho_1 = \rho/u$ when $\rho_1 > \rho$, one can write Eq. (3.32) as

$$\frac{-x \partial f(\rho, x)}{\partial x} = \frac{3\alpha_s}{\pi} \int_0^1 \frac{du}{1-u} [f(\rho, x) - f(\rho/u, x) - 2f(\rho, x)].$$

(3.33)

The solution at small $x$ is power behaved

$$f(x^2_{ab}, x) \sim \sqrt{x^2_{ab}}x^{-\lambda},$$

(3.34)

where

$$\lambda = 4\alpha_s \ln 2.$$  

(3.35)

Now let us discuss the relation of the BFKL dynamics with the DGLAP evolution equation. The correlations among the initial gluons disappear in the dilute parton system. In this case the contributions of the interference diagrams Figs. 6c and 6d are negligible. Thus, Eq. (3.28) reduces to sum two independent splitting functions, each of which is the splitting function in the DGLAP equation in the impact space at the small $x$ limit,

$$K_{DGLAP} \frac{dx^2_{a0}}{x_{a0}^2} \frac{dx_1}{x_1} = \sum_{pol} |A_{DGLAP}(x_{a0}, x_1, x_2)|^2 \frac{d^3l_a}{(2\pi)^3 2E_{l_a}}$$

$$= \frac{6\alpha_s}{2\pi} \frac{dx_1}{x_2} \frac{dx^2_{a0}}{L_{a0}}.$$  

(3.36)

Because the interference amplitudes disappear in the DGLAP dynamics, the transverse position of the probing parton is irrelevant to the measurement, a simple probe (dashed line in Fig. 7) can be used. On the other hand, a single transverse scale $x^2_{a0}$ in the DGLAP dynamics allows us to take $Q^2 = k^2$. Thus we can define the Bjorken variable $x_B$ and set $x_2 = x_B$ in Eq. (3.36). Returning to the momentum space and using Eq. (3.6), one can
get the real part of the DGLAP equation at the double leading logarithmic (i.e., $\ln Q^2$ and $\ln(1/x)$) approximation (DLLA)

$$\frac{\partial g(Q^2, x_B)}{\partial \ln Q^2} = \int_{x_B}^{1} \frac{dx_1}{x_1} K_{DGLAP}(z, \alpha_s)g(Q^2, x_1) - \frac{1}{2} g(Q^2, x_B) \int_0^1 dz K_{DGLAP}(z, \alpha_s)$$

$$\quad (small \ x \ limit) \longrightarrow \frac{\alpha_s}{2\pi} \int_{x_B}^{1} dx_1 \frac{6}{x_B} g(Q^2, x_1), \quad (3.37)$$

where the contributions of the virtual diagrams disappear at small $x$ since they company with the singularities at $x \to 1$ (see appendix A).

The above derivations also reveal a relation between two evolution equations from a new view point: the evolution kernel of the DGLAP equation is the un-interferant part of the BFKL kernel. Or reversely, the BFKL equation can be regarded as the corrections from the initial parton correlations to the DGLAP equation on the twist-2 level.
4 Evolution equations at twist-4

We consider the evolution kernel based on Fig. 1d and call it as the modified BFKL equation. Note that two pairs of initial gluons, which are hidden in the correlation function, for example in Fig. 9a, should be indicated as Fig. 9b. A set of cut diagrams based on the Fig. 1d are listed in Fig. 10, where the probe vertices have been separated out using the $W - W$ approximation. Similar to the derivation of Eq. (3.28), we write the evolution kernel in this equation as

$$
\mathcal{K}_{MD-BFKL} = \sum_{pol} A_{MD-BFKL} A_{MD-BFKL}^* \frac{d^3 l_a}{(2\pi^3)2 E_{l_a}}.
$$

(4.1)

The amplitudes

$$
A_{MD-BFKL} = A_{MD-BFKL1} + A_{MD-BFKL2},
$$

(4.2)

where

$$
A_{MD-BFKL1} = \sqrt{\frac{E_k}{E_p + E_p}} \frac{1}{2 E_k} \frac{1}{E_k + E_{l_a} - E_p - E_p} M_{MD-BFKL1},
$$

(4.3)

and

$$
A_{MD-BFKL2} = \sqrt{\frac{E_k}{E_p + E_p}} \frac{1}{2 E_k} \frac{1}{E_k + E_{l_d} - E_p - E_p} M_{MD-BFKL2}.
$$

(4.4)

For simplicity, in Fig. 10 we assume that all cold spots have a same size

$$
\mathcal{L}_{ab} = \mathcal{L}_{a'b'} = \mathcal{L}_{cd} = \mathcal{L}_{c'd'},
$$

and

$$
\mathcal{L}_{bc} = \mathcal{L}_{b'c'}.
$$

(4.5)
The momenta of the partons, for example, are parameterized as

\[ p_a = (x_1 P + \frac{(l_a - m)^2}{2x_1 P}, l_a - m, x_1 P), \quad (4.6) \]

\[ p_b = (x_1 P + \frac{(k + m)^2}{2x_1 P}, k + m, x_1 P), \quad (4.7) \]

\[ k_b = (x_2 P + \frac{k^2}{2x_2 P}, k, x_2 P), \quad (4.8) \]

\[ l_a = ((2x_1 - x_2) P + \frac{k^2}{2(2x_1 - x_2) P}, l_a (2x_1 - x_2) P), \quad (4.9) \]

and in the t-channel

\[ m = p_b - k_b = ((x_1 - x_2) P + \frac{(k + m)^2}{2x_1 P} - \frac{k^2}{2x_2 P}, m, (x_1 - x_2) P). \quad (4.10) \]

The matrix in Eq. (4.3) is

\[ M_{MD-BFKL1} = ig f^{abc} C^{\alpha\beta\gamma} - \frac{d\gamma}{m^2} ig f^{def} C^{\rho\sigma\eta} \epsilon_\alpha(p_a) \epsilon_\rho(p_b) \epsilon_\beta(l_a) \epsilon_\sigma^*(k), \quad (4.11) \]

where \( C^{\alpha\beta\gamma} C^{\rho\sigma\eta} \) are the triple gluon vertices and polarization vectors are

\[ \epsilon(p_a) = (0, \epsilon, -\frac{\epsilon \cdot (l_a - m)}{x_1 P}), \quad (4.12) \]

\[ \epsilon(p_b) = (0, \epsilon, -\frac{\epsilon \cdot (k + m)}{x_1 P}), \quad (4.13) \]

\[ \epsilon(k_b) = (0, \epsilon, -\frac{\epsilon \cdot k}{x_2 P}), \quad (4.14) \]

and
\[ \epsilon(l_a) = (0, \mathbf{\epsilon} - \frac{\epsilon \cdot L_a}{2(x_1 - x_2)}), \quad (4.15) \]

Thus, we have

\[ A_{MD-BFKL}(k_{up}, k_{down}, m, m', x_1, x_2) \]

\[ = g^2 f_{abc} f_{dce} \frac{x_1}{2x_2} \left[ \frac{6 \epsilon \cdot k_{up} \epsilon \cdot m}{k_{up}^2 m^2} + \frac{6 \epsilon \cdot k_{down} \epsilon \cdot m'}{k_{down}^2 m'^2} \right], \quad (4.16) \]

Taking the Fourier transformation

\[ A_{MD-BFKL}(x_0, x_0, x_{ab}, x_{cd}, x_1, x_2) \]

\[ = \int \frac{d^2 k_{up} d^2 k_{down} d^2 m d^2 m'}{(2\pi)^8} A_{MD-BFKL}(k_{up}, k_{down}, m, m', x_1, x_2) e^{i k_{up} x_{ab} + i k_{down} x_{cd} + i m x_{ab} + i m' x_{cd}}, \quad (4.17) \]

we get

\[ A_{MD-BFKL}(x_0, x_0, x_{ab}, x_1, x_2) = 6g^2 f_{abc} f_{dce} \frac{x_1}{2x_2} \left[ \frac{x_{ab} x_{cd}}{x_0 x_{ab}} - \frac{x_0 x_{ab}}{x_{cd} x_{ab}} \right] \epsilon \cdot \mathcal{L}. \quad (4.18) \]

Summing all channels, we derived the evolution kernel corresponding to Fig. 10 as

\[ K_{MD-BFKL} \frac{d^2 x_0}{x_2} \frac{d^2 x_{ab}}{x_2} \frac{d x_2}{x_2} \]

\[ = \sum_{pol} A_{MD-BFKL} A_{MD-BFKL}^* \left[ \frac{1}{16\pi^3} \frac{dx_2}{x_2} \frac{d^2 x_{cd}}{x_2} \frac{d^2 x_{cd}}{x_2} \right] \]

\[ = 2(\frac{9}{32})^{color} \pi \frac{36\pi^2}{x_0 x_{ab}} \frac{d^2 x_{cd}}{x_2} \frac{d^2 x_{cd}}{x_2} \frac{d x_2}{x_2}. \quad (4.19) \]

In the case of \(|x_{bc}| \gg |x_{ab}|\), one can neglect the correlations between two cold spots with the sizes \(x_{ab}\) and \(x_{cd}\). In consequence, the contributions of the interferant terms (Figs. 10c and 10d) disappear and Fig. 1d backs to Fig. 1c. Thus, Eq. (4.19) reduces to the real part of the modified DGLAP-kernel.
$$K_{MD-DGLAP} \frac{dx_2^2}{x_2^2} \frac{dx_1}{x_1} = 2 \left( \frac{9}{32} \right) \frac{36\pi^2}{\alpha_s} \frac{d^2 x_2}{x_2^2} \frac{d^2 x_1}{x_1^2}.$$  

(4.20)

Similar to Eq. (3.5) we derive the modified DGLAP equation using

$$G(Q_2^2, x_2) = G(Q_1^2, x_1) + \Delta G(Q_2^2, x_2)$$

$$= G(Q_1^2, x_1) + \int_{Q_1^2}^{Q_2^2} \frac{dQ_1^2}{Q_1^2} \int_{x_2}^{1/2} \frac{dx_1}{x_1} \frac{x_2}{x_1} K_{MD-DGLAP} \left( \frac{x_2}{x_1}, \alpha_s \right) G(2)(Q_1^2, x_1),$$  

(4.21)

where a power suppressed factor $1/Q_1^2$ has been extracted from the evolution kernel. The 4-gluon correlation function $G^{(2)}$ is a generalization of the gluon distribution beyond the leading twist. It is usually modelled as the square of the gluon distribution [3,5]. For example, $G^{(2)} = C_1 G^2$ and $C_1 = 1/(\pi R_N^2)$. The evolution constructed by $K_{MD-DGLAP}$ is illustrated by Fig. 7c.

The complete modified DGLAP equation combining DGLAP dynamics at small $x$ and in the momentum space was written as [5]

$$\frac{\partial G(Q^2, x_B)}{\partial \ln Q^2} = \frac{6\pi_s}{2\pi} \int_{x_B}^{1} \frac{dx_1}{x_1} G(Q^2, x_1) + \frac{81}{4} \frac{\alpha_s^2}{\pi Q^2 R_N^2} \int_{x_B/2}^{1/2} \frac{dx_1}{x_1} G^{(2)}(Q^2, x_1)$$

$$- \frac{81}{2} \frac{\alpha_s^2}{\pi Q^2 R_N^2} \int_{x_B}^{1/2} \frac{dx_1}{x_1} G^2(Q^2, x_1),$$  

(4.22)

where the right-hand second term of (4.22) is positive antishadowing and comes from the contributions from Fig. 11a, while the third term is negative shadowing and arises from the contributions of the $1 \to 2-3 \to 2$ cut diagrams (see Fig. 11b, which is the same order as Fig. 11a but we haven’t indicated it in Fig. 1c). Note that the shadowing and antishadowing terms are defined on different kinematics domains $[x, 1/2]$ and $[x/2, 1/2]$, respectively. Comparing with the GLR-MQ equation, there are several features in the modified DGLAP equation: (i) The momentum conservation of partons is restored in a
complete modified DGLAP equation; (ii) Because of the shadowing and antishadowing effects in the modified DGLAP equation have different kinematic regions, the net effect depends not only on the local value of the gluon distribution at the observed point, but also on the shape of the gluon distribution when the Bjorken variable goes from $x$ to $x/2$. In consequence, the shadowing effect in the evolution process will be obviously weakened by the antishadowing effect if the distribution is steeper [5]; (iii) Both the GLR-MQ and modified DGLAP equations assume that the fusion only occurs among gluons with the same value of $x$. However, there is a priori no reason to forbid the recombination of two gluons with different values $x$. If the recombination of gluons with different $x$ is included, we find that this modification unreasonably enhances the shadowing effect in the GLR-MQ equation, while it does not change the predictions of the modified DGLAP equation. Therefore, a correct antishadowing correction is important in gluon recombination [10]; (iv) Finally, we emphasize that, as we have shown in [10], the existence in the recombination process is a general result of the momentum conservation and it is irrelevant to the concrete form of the evolution dynamics.

Now let us discuss the modified BFKL equation using Eq. (4.19), which is the corrections of gluon recombination to the BFKL equation and it is illustrated in Fig. 7d. Unfortunately, it is difficult to write a simple form of the modified BFKL equation due to the complicated evolution kernel (4.19). For the sake of simplification, we take the following approximations (see Fig. 12): At the first step, we take the $W-W$ approximation to factorize Fig. 12a to a four gluon correlation function (Fig. 12b) and the general recombination function (Fig. 12c). Comparing Eq. (3.36) with Eq. (3.28), one can find that the kernel $K_{DGLAP}$ is a part of the kernel $K_{BFKL}$ and they differ only by a phase space. Similarly, the modified DGLAP kernel is evolving along the transverse momentum in the modified DGLAP equation, but, as a part of the modified BFKL-kernel (Fig. 10a and
10b), they also participate the evolution along the longitudinal momentum. Therefore, at the next step, we only keep the contributions of Fig. 10a in Fig. 12c, i.e., using the modified DGLAP kernel to replace the kernel (4.19). According to Fig. 12, we rewrite $\Delta G(Q^2_2, x_2)$ in Eq. (4.21) in the impact space as

$$
\Delta G(x^2_{ab}, x_2) = \int_{x^2_{ab}} dx^2 \int_{x^2_{2} \rightarrow x_2} \frac{dx_1 x_2}{x_1 x_1} K_{MD-DGLAP} \left( \frac{x_2}{x_1}, \alpha_s \right) f^{(2)}(x^2_{ab}, x_1). \tag{4.23}
$$

Thus, we have

$$
\Delta f(x^2_{ab}, x_1) = \frac{x^2_{ab}}{x^2_{ab}} \left. \frac{\partial \Delta G(x^2_{ab}, x_2)}{\partial x^2_{ab}} \right|_{x^2_{ab} = x^2_{ab}}
$$

$$
= \frac{1}{x^2_{ab}} \int_{x^2_{2} \rightarrow x_2} \frac{dx_1 x_2}{x_1 x_1} K_{MD-DGLAP} \left( \frac{x_2}{x_1}, \alpha_s \right) f^{(2)}(x^2_{ab}, x_1), \tag{4.24}
$$

which gives a correction to the evolution of the unintegrated distribution along small $x$ direction

$$
-x \frac{\partial f(x^2_{ab}, x)}{\partial x^2_{ab}} = \frac{81 \pi^2}{4} f^{(2)}(x^2_{ab}, \frac{x}{2}) - \frac{81 \pi^2}{2} f^{(2)}(x^2_{ab}, x). \tag{4.25}
$$

Combining with the BFKL equation, we obtain an approximate form of the modified BFKL equation

$$
-x \frac{\partial f(x^2_{ab}, x)}{\partial x} = \frac{\alpha_s}{2 \pi^2} \int d^2 x_0 6 \frac{x^2_{ab}}{x^2_{ab} x^2_{ab}} f(x^2_{ab}, x) - \frac{1}{2 \pi^2} f(x^2_{ab}, x) \int d^2 x_0 6 \frac{x^2_{ab}}{x^2_{ab} x^2_{ab}}
$$

$$
+ \frac{81 \pi^2}{4} f^{(2)}(x^2_{ab}, \frac{x}{2}) - \frac{81 \pi^2}{2} f^{(2)}(x^2_{ab}, x), \tag{4.26}
$$

where the nonlinear evolution kernels are the derivative of the modified DGLAP kernels with respect to $x$; in the meantime, we assume that $f^{(2)} = C_2 f^2$ and $C_2$ is a unknown correlation coefficient.
Comparing with the modified DGLAP equation (4.22), the antishadowing effect in the modified BFKL equation (4.26) is more sensitive to the shape of the gluon distribution at small $x$. For example, in the case of a typical BFKL solution Eq. (3.34), the antishadowing effect may cancel, or even outweigh than the shadowing effect in the modified BFKL equation (4.26). Such faster increasing gluon distribution impels the multi-gluon recombination (or correlation of multi-cold spots) to participate the evolution and reach the saturation limit earlier.

There is a difficulty in the numerical calculation of Eq. (4.26): the value of $f(x_2^{ab}, x/2)$ is unknown when the equation evolves to small $x$. To avoid this problem, we take following approximation at the beginning modified BFKL-renge, for example, in the evolution from $x_1$ to $x_2 = x_1 - \Delta x$, the increment of the unintegrated gluon distribution due to Eq. (4.26) is taken as

$$
\Delta f(x_2^{ab}, x_1) = BFKL[f(x_2^{ab}, x_0)] + AS[f(x_2^{ab}, x_0/2)] - S[f(x_2^{ab}, x_0)]
$$

$$
\sim BFKL[f(x_2^{ab}, x_0)] + AS[f_{BFKL}(x_2^{ab}, x_0/2)] - S[f(x_2^{ab}, x_0)],
$$

where $BFKL$, $AS$ and $S$ indicate the contributions from the BFKL kernel, antishadowing term and shadowing term in Eq. (4.26) and $f_{BFKL}$ implies the solution of the linear BFKL equation. A schematic solution of Eq. (4.26) at this approximation is illustrated in Fig. 13. Where we take a typical BFKL input distribution as

$$
f(x_2^{ab}, x_0) = \sqrt{x_2^{ab}k_s^2} \exp \left[-\frac{\log(x_2^{ab}/k_s^2)^2}{5}\right],
$$

and $k_s^2 = 1 GeV^2$, $x_2^{ab} = 5 GeV^{-2}$. Comparing the coefficients of the nonlinear terms of Eq. (4.26) with that of Eq. (4.22), we take $C_2 = 1/(25\pi) GeV^{-2}$.

The dashed-point curve in Fig. 13 is the imaginary saturation limit. One can find that the antishadowing effect is un-negligible in such abrupt distribution. For comparison, we
draw a corresponding BFKL-solution (dashed curve) and a solution of Eq. (4.26) without the antishadowing corrections (point curve) in Fig. 13.

The schematic kinematic ranges of four relating evolution equations are shown in Fig. 14. Although both the BFKL equation and modified DGLAP equation are regarded as the corrections of the correlations among initial partons to the DGLAP dynamics, the BFKL equation dominates the higher $Q^2$-range since the modified DGLAP equation is order of $\alpha_s^2$ and has a power suppression factor. However, the modified DGLAP equation will replace the BFKL equation below a moderate $Q^2$-scale because the $1/x$ factor in $K_{MD-DGLAP}$ (see Eq. (4.20)). It is interesting that because the gluon distribution becomes flatter in the modified DGLAP-region in Fig. 14, the antishadowing contributions in Eq. (4.26) can be neglected when the evolution enter into the modified BFKL-region. Thus, we predict that the gluon distribution asymptotically or abruptly approach to the saturation limit below or above a moderate $Q^2$-scale.
5 Discussions

At first, we point out that our nonlinear equation (4.26) has a similar picture of the Balitsky-Kovchegov equation [7], but they have different interpretations. The Balitsky-Kovchegov equation is one of nonlinear evolution equation incorporating with the screening corrections to the BFKL equation. Besides, this equation is regarded as an approximation form of the JIMWLK equation at the lowest order. According to Ref. [8], assuming that the scattering matrix $S(x, x)$, which is the part of dashed box in Fig. 7b but use quark-antiquark dipole to replace the gluonic dipole, obeys the BFKL equation

$$-x \frac{\partial S(x^2, x)}{\partial x} = \frac{3\alpha_s}{2\pi^2} \int d^2x_0 \frac{x_0^2}{x^2_0 - x^2} [S(x^2_0, x) + S(x^2_0, x) - S(x^2_0, x)].$$

(5.1)

Then assuming that

1. The gluon decay inside the dipole leads to the dipole splitting at large $N_c$, i.e.,

$$S(x^2_0, x) + S(x^2_0, x) \rightarrow S(x^2_0, x)S(x^2_0, x),$$

(5.2)

in Eq. (5.1);

2. Defining the scattering amplitude

$$S = 1 - T,$$

(5.3)

one can simply obtain the Balitsky-Kovchegov equation

$$-x \frac{\partial T(x^2, x)}{\partial x} = \frac{3\alpha_s}{2\pi^2} \int d^2x_0 \frac{x_0^2}{x^2_0 - x^2} [T(x^2_0, x) + T(x^2_0, x) - T(x^2_0, x) - T(x^2_0, x)T(x^2_0, x)],$$

(5.4)
where $T$ is relative to the gluon distribution [7]. Figure 15 is the one-step evolution containing the $4 \to 2$ recombination amplitude.

Comparing Fig. 15 with Fig. 7d, one can find that the Balitsky-Kovchegov equation and the modified BFKL equation have a similar physical picture. However, they are really different nonlinear evolution dynamics. In fact, the replacements Eq. (5.2) and (5.3) result the negative term $TT$ in Eq. (5.4). This term is graphically regarded as the simultaneous scattering of two dipole. Such consideration of the shadowing effect is relevant neither to the AGK- nor to TOPT-cutting rules, but it is similar to a shadowing mechanism in the multi-scattering process (for example, the Glauber-type scattering).

Different from the recombination process, the antishadowing effect is absent in the Glauber-type scattering since the shadowed (or absorbed) momenta by target are unmeasured [17]. Besides, the linear parts and nonlinear part in the Balitsky-Kovchegov equation share the same BFKL-kernel due to Eqs. (5.2) and (5.3). On the other hand, the nonlinear evolution kernel in the modified BFKL equation (4.26) is different from the linear BFKL-kernel since the former including the recombination process. Thus, we present two different shadowing mechanisms: the QCD gluon recombination process and the absorptive effect in the multi-scattering process.

We have shown a new approach to derive a set of consistent evolution equations in a partonic framework. We present the connections among the DGLAP equation (3.37), BFKL equation (3.30), modified DGLAP equation (4.22) and modified BFKL equation (4.26) in a clear physical picture (Fig. 1). We also show that these evolution equations are uniformly derived from the factorizability of DIS cross sections. The four evolution equations mentioned above have similar structures: The positive part in an evolution equation is completely separated from the negative part. The positive part is the contributions from the real diagrams, while the negative part comes from the virtual diagrams.
in Eqs. (3.30) and (A-9), or from the interference diagrams in Eqs. (4.22) and (4.26), respectively.

The BFKL equation is traditionally understood as an improvement of the DGLAP equation at small $x$, where the summation of the leading logarithm of $x (LL(1/x))$ terms is important. The derivation of the BFKL equation in this work shows a new relation with the modified DGLAP equation. We re-draw the cut diagrams Figs. 16a-16d in our derivation of the BFKL equation as the ladder diagrams in Figs. 16e-16h. On the other hand, a typical diagram in Regge approach of the BFKL equation is shown in Fig. 17, where the vertical (dashed) lines are reggeized gluons, which can be explained as the gluon ladders in perturbative QCD. Thus, Figs. 16e-16h give a possible partonic explanation of the Regge theory for the BFKL equation.

As is well known, the BFKL equation could also be derived in the color dipole model [8]. Now let us compare the evolution equations in our partonic model with that in the color dipole model. We point out that these two approaches have different partonic configurations. The dipole is constructed by two heavy quarks, which represent a whole nucleon in the color dipole model, while the cold spot in our approach is generally the multi-parton cluster (as one of the constituents of the nucleon). In the concrete, we compare Fig. 18a in our partonic scattering model with the color dipole picture Figs. 18b or 18c, where the transverse coordinators of the dipole are assumed to be freezed during interaction. Note that the dashed lines in Fig. 18a are integrated out as the final state in the inclusive DIS processes, therefore, some of them are omitted in Fig. 1. On the other hand, the splitting kernel in the color dipole model is constructed by the gluon-quark vertex, while it is the three gluon vertex in our partonic scattering approach. The latter dominates the small $x$ behavior since it ($\sim 1/z_2$).

It is interesting that a cut diagram Fig. 19a of the modified DGLAP equation is
equivalent to the ladder diagram Fig. 19b. Comparing Fig. 19b with Figs. 17e-17h, we find that the BFKL and modified DGLAP equations have the same (two ladders) structure in the initial gluon configuration, but the probe partially couples with two ladders in the BFKL equation at twist-2, while it fully couples with two ladders though the gluon fusion mechanics in the modified DGLAP equation at twist-4.

We now discuss the relation of the BFKL dynamics with the helicity configuration of the initial gluons. As we have pointed out in [5], the polarized form of two correlating initial partons is relevant to the structure of the spin-independent evolution kernel. For ease of representation, we draw all possible helicity amplitudes in Fig. 20. Obviously, the processes illustrated Figs. 20(a1)-20(a4) should be inhibited due to all initial and final states in Fig. 20 having fixed helicities in the $W - W$ approximation. Thus, the BFKL kernel (3.22) is a result, where the contributions from Figs. 20(b1)-20(b4) being excluded, i.e., we should assume that two gluons in a cold spot possess the same helicity. On the other hand, the BFKL kernel reduces to the DGLAP kernel if these two correlating gluons have the opposite helicities, or the BFKL kernel becomes

\[
\mathcal{K}_{BFKL} \frac{dx_1^2}{x_1^2} \frac{dx_2^2}{x_2^2} = \langle 3 \rangle_{color} \frac{\alpha_s}{\pi^2} \left[ \frac{x_{1b}^2}{x_{10}^2 x_{20}^2} + \frac{2}{x_{20}^2} \right] d^2x_0 \frac{dx_2}{x_2},
\]

if two gluons in a cold spot can take any helicities.

In summary, we discussed the corrections of the initial gluon correlations to the evolution dynamics in a unified framework. We presented the following results: (1) a possible connection among a set of evolution equations at small $x$; (2) the antishadowing effect, which is a direct result of the momentum conservation in elementary QCD process, is un-negligible in the corrections of the gluon recombination to either the DGLAP or BFKL equation; (3) the gluon distribution asymptotically (abruptly) approaches to the saturation...
tion limit in lower (higher) $Q^2$ regions is predicted; (4) the BFKL equation and modified DGLAP equation are viewed as the corrections of the initial gluon correlations to the evolution dynamics at twist-2 and twist-4, respectively; (5) a partonic interpretation of Regge theory for the BFKL dynamics and the relation of the BFKL dynamics with the helicity configuration of the initial gluons are presented.

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Appendix:

For illustrating the applications of the TOPT-cutting rule, which are proposed in [5], we write a complete derivation of the DGLAP equation for gluon. The contributions of the real diagrams (Fig. 21a) to the hard part of Eq. (3.1) are

\[
H(\text{probe}^* l \rightarrow \text{probe}^* l) = \frac{1}{2} \frac{M_{l \rightarrow k'}}{2E_l} \frac{1}{E_L - E_k - E_{l'}} \frac{1}{2E_k} \frac{1}{E_l - E_k - E_{l'}} M_{l \rightarrow k'}^{*} \frac{d^3 l'}{(2\pi)^3 2E_{l'}} \\
\times \frac{1}{4E_{\text{probe}}} |M_{\text{probe}^* k \rightarrow k'}|^2 (2\pi)^4 \delta(p_{\text{probe}^*} + k - k') \frac{d^3 k'}{(2\pi)^3 2E_{k'}} \\
= \mathcal{K}_{\text{DGLAP}}(x_1, x_2, x_3, \alpha_s) d\mathbf{x}_3 \frac{d l^2_\perp}{l^2_\perp} \delta(x_2 - x_B) d x_2 C(\text{probe}^* k \rightarrow k'),
\]

where the factor \(1/(2E_l)\) is extracted from the definition of the gluon distribution \(g(Q^2, x_B)\), and

\[
\mathcal{K}_{\text{DGLAP}}(x_1, x_2, x_3, \alpha_s) d\mathbf{x}_3 \frac{d l^2_\perp}{l^2_\perp} = \frac{E_k}{E_l} |M_{l \rightarrow k'}|^2 \left[ \frac{1}{E_l - E_k - E_{l'}} \right]^2 \left[ \frac{1}{2E_k} \right]^2 \frac{d^3 k'}{(2\pi)^3 2E_{l'}}, \quad (A - 2)
\]

\[
C(\text{probe}^* k \rightarrow k') \delta(x_2 - x_B) d x_2 = \frac{1}{8E_k E_{\text{probe}}^*} |M_{\text{probe}^* k \rightarrow k'}|^2 (2\pi)^4 \delta(p_{\text{probe}^*} + k - k') \frac{d^3 k'}{(2\pi)^3 2E_{k'}}, \quad (A - 3)
\]
is the contributions from \(d\sigma(\text{probe}^* k \rightarrow k')\). Thus,

\[
\Delta g(Q^2, x_B) = \int \frac{dl^2_\perp}{l^2_\perp} d\mathbf{x}_1 d\mathbf{x}_2 d\mathbf{x}_3 \mathcal{K}_{\text{DGLAP}}(x_1, x_2, x_3, \alpha_s) g(l^2_\perp, x_1) \delta(x_2 - x_B) \delta(x_1 - x_2 - x_3),
\]

where we inset \(\delta(x_1 - x_2 - x_3) d\mathbf{x}_1\). We obtain
\[ \frac{\partial g(Q^2, x_B)}{\partial \ln Q^2} = \int dx_1 dx_2 dx_3 K_{DGLAP}(x_1, x_2, x_3, \alpha_s) g(Q^2, x_1) \delta(x_2 - x_B) \delta(x_1 - x_2 - x_3) \]
\[ = \int dx_1 dz K_{DGLAP}(z, \alpha_s) g(Q^2, x_1) \delta(x_1 z - x_B) \]
\[ = \int \frac{dx_1}{x_1} K_{DGLAP}(\frac{x_B}{x_1}, \alpha_s) g(Q^2, x_1) . \quad (A - 5) \]

On the other hand, the contributions from one of the virtual diagrams, for say, Fig. (21b) are

\[ H(probe^*l \rightarrow probe^*l) \]
\[ = \frac{1}{2} \frac{1}{2 E_l} \frac{1}{2 E_k} \frac{1}{2 E_{k'}} \frac{1}{E_L - E_k - E_{k'}} \frac{1}{E_l + E_{l'} - E_l} d^3l' \]
\[ \times \frac{1}{4 E_{probe}^*} |M_{probe^*l \rightarrow k'}|^2 (2\pi)^4 \delta(p_{probe^*} + l - k') \frac{d^3k'}{(2\pi)^32E_{k'}} \]
\[ = -\frac{1}{2} K_{DGLAP}(x_1, x_2, x_3, \alpha_s) dx_3 \frac{dl_1^2}{l_1^2} \delta(x_1 - x_B) dx_1 C(probe^*l \rightarrow k'), \quad (A - 6) \]

where

\[ C(probe^*l \rightarrow k') \delta(x_1 - x_B) dx_1 \]
\[ = \frac{1}{8 E_l E_{probe}^*} |M_{probe^*l \rightarrow k'}|^2 (2\pi)^4 \delta(p_{probe^*} + l - k') \frac{d^3k'}{(2\pi)^32E_{k'}} , \quad (A - 7) \]

is the contributions from \( d\sigma(probe^*l \rightarrow k') \). Therefore, we have

\[ \frac{\partial g(Q^2, x_B)}{\partial \ln Q^2} = -\frac{1}{2} \int dx_1 dx_2 dx_3 K_{DGLAP}(x_1, x_2, x_3, \alpha_s) g(Q^2, x_1) \delta(x_1 - x_B) \delta(x_1 - x_2 - x_3) \]
\[ = -\frac{1}{2} g(Q^2, x_B) \int dz K_{DGLAP}(z, \alpha_s) . \quad (A - 8) \]

The complete evolution equation for the gluons is
\[
\frac{\partial g(Q^2, x_B)}{\partial \ln Q^2} = \int_{x_B}^{x_1} \frac{dx_1}{x_1} K_{DGLAP}(z, \alpha_s) g(Q^2, x_1) - \frac{1}{2} g(Q^2, x_B) \int_0^1 dz K_{DGLAP}(z, \alpha_s),
\]

where

\[
K_{DGLAP}(z) = \frac{\alpha_s}{\pi} C_A \left[ \frac{z}{1-z} + \frac{1-z}{z} + z(1-z) \right],
\]

Note that the factor 1/2 in Eq. (A-8) is the considerations of the symmetry under exchange of two internal gluons in the virtual diagrams (see Ref. [5]). The expressions of the evolution equation in the TOPT form, without the calculations of the matrixes, show that the real and virtual diagrams contribute the same evolution kernel but with the different factors. This simply form of the equation consists with a more complicate derivation of the DGLAP equation using the covariant perturbation theory in [19]. In fact, from Eqs. (A-9) and (A-10) we have

\[
\frac{\partial x_B g(Q^2, x_B)}{\partial \ln Q^2} = \frac{3\alpha_s}{\pi} \int_{x_B}^{1} dz \left[ \frac{z}{1-z} + \frac{1-z}{z} + z(1-z) \right] x_1 g(Q^2, x_1)
\]

\[
- \frac{3\alpha_s}{2} x_B g(Q^2, x_B) \int_0^1 dz \left[ \frac{z}{1-z} + \frac{1-z}{z} + z(1-z) \right],
\]

Note that \( \int_0^1 dz (1-z) = \int_0^1 dz (1-z)/z \), we obtain

\[
\frac{\partial x_B g(Q^2, x_B)}{\partial \ln Q^2} = \frac{3\alpha_s}{\pi} \int_{x_B}^{1} dz \left[ \frac{z x_1 g(Q^2, x_1) - x_B g(Q^2, x_B) + (1-z)(1+z)^2}{1-z} x_1 g(Q^2, x_1) \right]
\]

\[
- \frac{3\alpha_s}{\pi} x_B g(Q^2, x_B) \int_0^1 dz \left[ \frac{z}{1-z} + \frac{1}{2} z(1-z) \right] + \frac{3\alpha_s}{\pi} x_B g(Q^2, x_B) \int_{x_B}^{1} dz \frac{1}{1-z}
\]

\[
= \frac{3\alpha_s}{\pi} \int_{x_B}^{1} dz \left[ \frac{x_1 g(Q^2, x_1) z - x_B g(Q^2, x_B) + (1-z)(1+z)^2}{1-z} x_1 g(Q^2, x_1) \right]
\]

\[
+ \frac{\alpha_s}{\pi} \left[ \frac{11}{4} + 3 \ln(1-x) \right] x_B g(Q^2, x_B).
\]

This equation is equivalent to the following traditional form of the DGLAP equation...
\[
\frac{\partial g(Q^2, x_B)}{\partial \ln Q^2} = \int_{x_B}^1 dx_1 \frac{1}{x_1} K_{^{\text{tradoctional}}}^{DGLAP}(z, \alpha_s) g(Q^2, x_1), \quad (A - 13)
\]

where

\[
K_{^{\text{tradoctional}}}^{DGLAP}(z) = \frac{\alpha_s}{\pi} C_A \left[ \left( \frac{z}{1-z} \right)_+ + \frac{1-z}{z} + z(1-z) + \delta(1-z) \frac{11}{12} \right]. \quad (A - 14)
\]

In fact, using

\[
\int_0^1 dz \frac{f(z)}{(1-z)_+} \equiv \int_0^1 dz \frac{f(z) - f(1)}{1-z}, \quad (A - 15)
\]

Equation (A-14) becomes

\[
\frac{\partial x_B g(Q^2, x_B)}{\partial \ln Q^2} = \frac{3\alpha_s}{\pi} \int_{x_B}^1 dz \left[ \frac{z}{1-z} + \frac{1-z}{z} + z(1-z) \right] + \frac{11}{4\pi} \alpha_s x_B g(Q^2, x_B) - \frac{3\alpha_s}{\pi} \delta(1-z) \int_0^1 dz \frac{1}{1-z} \\
= \frac{3\alpha_s}{\pi} \int_{x_B}^1 dz \left[ \frac{x_1 g(Q^2, x_1)}{1-z} - x_B g(Q^2, x_B) \right] + \frac{(1-z)(1+z^2)}{z} x_1 g(Q^2, x_1) \\
+ \frac{\alpha_s}{\pi} \left[ \frac{11}{4} + 3 \ln(1-x) \right] x_B g(Q^2, x_B). \quad (A - 16)
\]

Both Eqs. (A-12) and (A-16) at small \( x \) predict Eq. (3.37). One can find that the infrared divergence of the real diagram at the end point \( (x_B \to 1) \) is exactly cancelled by the contributions of the virtual diagram. Therefore, the leading terms of the virtual diagram in Eq. (3.37) disappear at the small \( x \) region. It is different from Eq. (3.37), the contributions of the virtual diagram exist in Eq. (3.30), which regularize the singularities of the real diagram on the transverse space. The above results also can be applied in the BFKL dynamics. A difference in this case is that the absence of the conditions \( \delta(x_1 - x_B) \) and \( \delta(x_2 - x_B) \) in Eqs. (A-1) and (A-6), respectively.
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Figure Captions

Fig. 1 The corrections of the initial gluons to the basic amplitude of the DGLAP equation and they lead to (b) BFKL-, (c) modified DGLAP- and (d) modified BFKL equations, respectively.

Fig. 2 Application of the $W - W$ approximation. The dark circles denote perturbative QCD interaction with the correlation of the initial partons.

Fig. 3 A sub-vertex in a complex Feynman diagram.

Fig. 4 The parton distributions on the impact space of a nucleon: (a) the dilute parton system, (b) some of partons form the cold spots (dashed circles) and (c) the saturation, where the long- and short-distance correlations among partons coexist in a nucleon.

Fig. 5 The QCD correlations between two initial gluons.

Fig. 6 The TOPT-diagrams consisted by the elemental amplitudes in Fig. 1b on the impact space. These diagrams lead to the BFKL equation. The dashed lines are the time ordered lines in the TOPT and the dark box represent a non-local measurement of the unintegrated distribution.

Fig. 7 The illustration of the right-hand side (real part) of (a) DGLAP equation, (b) BFKL equation, (c) modified DGLAP equation and (d) modified BFKL equation, where un-connecting gluons imply all possible connections inside the cold spot like in Fig. 6. The last sub-diagrams are added probe, which contribute $\delta(x - 2 - x_B)$ or Eq. (2.10) in (a) and (c), or in (b) and (d), respectively.

Fig. 8 The virtual diagrams corresponding to Fig. 6.

Fig. 9 A cutting diagram originating from Fig. 1d.

Fig. 10 The TOPT-diagrams consisted by the elemental amplitudes in Fig. 1d in the impact space. These diagrams lead to the modified BFKL equation.

Fig. 11 Two cutting diagrams contributing to the modified DGLAP equation.
Fig. 12 The factorization of Fig. 10, which leads to an approximate form of the modified BFKL equation.

Fig. 13 The unintegrated gluon distribution $f(x^2_{ab}, x)$ at $x^2_{ab} = 5 GeV^{-2}$ as a solution of the modified BFKL equation (4.26) taking the approximation (4.27) (solid curve), a corresponding solution of the BFKL equation (dashed curve) and a solution of Eq. (4.26) but without antishadowing (point curve). The broken-point line is an imaginal saturation limit.

Fig. 14 A schematic kinematic region of four evolution equations in this work.

Fig. 15 One-step evolution including the $4 \rightarrow 2$ recombination amplitude, which is regarded as a graphic description of the nonlinear terms in the Balitsky-Kovchegov equation.

Fig. 16 Schematic cut diagrams for the BFKL equation (a-d) and corresponding ladder diagrams (e-h).

Fig. 17 A typical ladder diagram in the traditional BFKL theory, where the vertical (dashed) lines are reggeized gluons.

Fig. 18 A comparison of our partonic picture with the color dipole approaches.

Fig. 19 (a) A cut diagrams of the modified DGLAP equation and (b) corresponding ladder diagram.

Fig. 20 The helicity amplitudes contributing to the spin-independent BFKL kernel, where the processes in (a1)-(a4) should be inhibited at the $W-W$ approximation.

Fig. 21 Complete TOPT diagrams containing the probe for the DGLAP equation: (a) real diagram and (b) virtual diagrams.