Research Article

Existence Results of Fuzzy Delay Impulsive Fractional Differential Equation by Fixed Point Theory Approach

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The main aim of this article is to study controllability and existence of solution of fuzzy delay impulsive fractional nonlocal integro-differential equation in the sense of Caputo operator. The existence and uniqueness of the solution have been carried out with the help of the Banach fixed point theorem. Moreover, for fuzzy fractional differential equations (FFDEs) driven by the Liu process, this present work introduced a concept of stability in credibility space. Finally, efficient examples are presented to demonstrate the main theoretical findings.

1. Introduction

Fractional-order dynamical equations can be used to model a huge spectrum of physical processes in modern-world observations [1]. Due to its wide range application in various areas of sciences such as physics, chemistry, biology, electronics, thermal systems, electrical engineering, mechanics, signal processing, weapon systems, electrohydraulics, population modeling, robotics, and control, the concept of fuzzy sets continues to catch the attention of researchers [2]. As a result, in recent years, scholars have been increasingly interested in it. As a concept of describing a set with uncertain boundary, the fuzzy set was developed by Zadeh et al. [3]. The concept of possibility measure was studied by Zadeh [4] in 1978. Fuzzy set theory is a very useful technique for simulating uncertain problems. In fuzzy calculus, therefore, the concept of the fractional derivative is essential. Although the possibility measure provides the theoretical basis for the measurement of fuzzy events, it does not satisfy self-duality. Liu B. and Liu Y. [5] studied the concept of credibility measure in 2002, and a sufficient and necessary condition for credibility measure was derived by Li and Liu [6] in 2006. Fractional differential equations (FDEs) are differential equations with fractional derivatives. It is known from the research on fractional derivatives that they originate uniformly from major mathematical reasons. Different types of derivatives exist, such as Caputo and RL [7]. In 1965, Zadeh used the membership function to propose the concept of fuzzy sets for the first time. The FFDE is the most fascinating field. They are useful for understanding phenomena that have an underlying effect. Kwun et al. [8] and Lee et al. [9] investigated the solution of uniqueness-existence for FDEs. Controlled processes have been explored by several researchers. In the case of the fuzzy system, Kwun et al. [10] for the impulsive semilinear FDEs, controllability in n-dimension fuzzy vector space was demonstrated. Park et al. [11] controllability of semilinear fuzzy integro-differential equations with nonlocal conditions was investigated. Park et al. [12] established controllability of impulsive semilinear fuzzy integro-differential equations. Phu and Dung [13] studied stability analysis and controllability of fuzzy control set differential equations. According to Lee et al. [14], in the n-dimensional fuzzy space $E^n_N$ of a nonlinear fuzzy control system, controllability with nonlocal initial conditions was examined.

Balasubramaniam and Dauer [15] examined the controllability of stochastic systems in Hilbert space of quasilinear stochastic evolution equations, while Feng [16] explored the controllability of stochastic with control systems associated with time-variant coefficients. Arapostathis et al. [17] analyzed the controllability of stochastic differential systems of equations with linear-controlled diffusion affected by
Lipschitz nonlinearity that is limited, smooth, and uniform. Stochastic differential equations given by Brownian motion are a well-known and well-studied area of modern mathematics. A new type of FDE was created using the Liu technique [18], which was described as follows:

\[ dX_v = f(X_v, v)dv + g(X_v, v)dC_v, \quad (1) \]

where \( C_v \) denotes Liu operation and \( f \) and \( g \) are functions that have been assigned to it. This class of equations is solved using a fuzzy technique. For homogeneous FDEs, Chen and Qin [19] studied solutions of existence-uniqueness of few special FDEs. Liu [20] investigated an approximate method for solving unknown differential equations. Abbas et al. [21, 22], Iqbal et al. [25], Shafqat et al. [26], Abuasbeh et al. [27], and Alnahdi [28] existence-uniqueness of the FFE were investigated. Arjunan et al. [29–32] worked on the fractional differential inclusions.

Using conclusions of Liu [20], Jeong et al. [33] focused on exact controllability in credibility space for FDEs. Abstract FDEs' complete controllability in credibility space is as follows:

\[ dx(v, \omega) = Ax(v, \omega)dv + f(v, x(v, \omega))d\mathcal{E}_v + Bu(v), \quad v \in [0, 3], \]

\[ x(0) = x_0. \quad (2) \]

We used the Caputo derivative to prove controllability for the fuzzy delay impulsive fractional integro-evolution equation in credibility space with nonlocal condition; as a result of the above research,

\[ t^\alpha D^\alpha_0 u(v, \zeta) = g_0(v, u(v)) + Au(v, \zeta) + \int_0^t f((v, u(v, \zeta), k(s, u(v, \zeta)))d\mathcal{E}_v + Bu(v), \quad v \in (0, \xi], \]

\[ u(0) = u_0 + h(v_1, v_2, \cdots, v_i, u_i), \quad (3) \]

where \( U(\mathbb{C}_{E_N}) \) and \( V(\mathbb{C}_{E_N}) \) are two bounded spaces. \( E_N \) is denoted for the set of numbers; all upper semicontinuously convex fuzzy on \( \mathbb{R}^m \), and \( (\Theta_1, \mathbb{P}_m, \mathcal{E}_v) \), is the credibility space.

The fuzzy coefficient is defined by the state function \( u : [0, 3] \times (\Theta_1, \mathbb{P}_m, \mathcal{E}_v) \rightarrow U, \quad f : [0, 3] \times U \rightarrow U \) is a fuzzy process. \( x : [0, 3] \times (\Theta_1, \mathbb{P}_m, \mathcal{E}_v) \rightarrow V \) is regular fuzzy function, and \( \mathcal{R} \) is linear bounded operator on \( U \) to \( V \). The initial value is \( u_0 \in E_N \), and \( \mathcal{E}_v \) denotes the Liu process.

The goal of this work is to investigate the existence and stability of results to FDEs and the exact controllability driven by the Liu process, in order to deal with a fuzzy process. Some scholars discovered FDE results in the literature, although the vast majority of them were differential equations of the first order. We discovered the results for Caputo derivatives of order \( (0, 1) \) in our research. Stability, as a part of differential equation theory, is vital in both theory and application. As a result, stability is a key subject of study for researchers, and research papers on stability for FDE have been published in the last two decades, for example, essential conditions for solution stability and asymptotic stability of FDEs. We use fuzzy delay impulsive fractional integro-evolution equations with the nonlocal condition. The theory of fuzzy sets continues to gain scholars’ attention because of its huge range of applications in different fields of sciences such as engineering, robotics, mechanics, control, thermal systems, electrical, and signal processing.

In Section 2, we go over some basic notions relating to Liu’s processes and fuzzy sets. Section 3 demonstrates the existence of solutions of FDE and shows that FDE is precisely controllable. The concept of credibility stability for FDEs driven by the Liu process was developed in Section 4. Finally, in Section 5, several theorems for FDEs driven by the Liu process that is stable in credibility space were demonstrated.

2. Preliminary

If \( M_k(\mathbb{R}^m) \) be the family of all nonempty compact convex subsets of \( \mathbb{R}^m \), then addition and scalar multiplication are commonly defined as \( M_k(\mathbb{R}^m) \). Consider two nonempty bounded subsets of \( \mathbb{R}^m, A_1 \) and \( B_1 \). The distance between \( A_1 \) and \( B_1 \) is measured using the Hausdorff metric as

\[ d(A_j, B_i) = \max \left\{ \sup_{a \in A_j} \inf_{b \in B_i} ||a - b||, \sup_{b \in B_i} \inf_{a \in A_j} ||a - b|| \right\}, \quad (4) \]

where \( ||.|| \) indicates the usual Euclidean norm in \( \mathbb{R}^m \). It follows that \( (M_k(\mathbb{R}^m), d) \) is a separable and complete metric space [20]. Satisfy the below condition:

\[ E^m = \{ j : \mathbb{R}^m \rightarrow [0, 1] | j \text{satisfies (a) – (b) below} \}, \quad (5) \]

where

(a) \( j \) is normal; there exists an \( j_0 \in \mathbb{R}^m \) such that \( j(j_0) = 1 \).

(b) \( j \) is fuzzy convex, such that is \( j(\lambda v + (1 - \lambda)s) \geq 1 \).

(c) \( j \) is upper semicontinuous function on \( \mathbb{R}^m \), that is, \( j(0) \geq \lim_{k \rightarrow \infty} j(v_k) \) for any \( v_k \in \mathbb{R}^m(k = 0, 1, 2, \cdots) \), \( v_k \rightarrow 0 \).

(d) \( |j| = \inf \{ u \in \mathbb{R}^m \mid j(u) > 0 \} \) is compact.

In \( \mathbb{R}^m \) [34], for \( 0 < \beta < 1 \), denote \( j[\beta] = \{ v \in \mathbb{R}^m | |u(v) \geq \beta \} \) and \( |u|^{\beta} \) are nonempty compact convex sets. Then from (a) to (b), it concludes that \( \beta \)-level set \( \{j[\beta] \in \mathbb{R}^m \} \) for all \( 0 < \beta < 1 \). Using Zadeh’s extension principle, we can have scalar multiplication and addition in fuzzy number space \( E^m \) as follows:

\[ j \oplus \varphi = \{ j[\beta] \oplus \varphi[\beta], [k]^{\beta} = k[\varphi]^{\beta}, \quad (6) \]

where \( j, \varphi \in E^m, k \in \mathbb{R}^m \) and \( 0 < \beta < 1 \). Assume \( E_N \) denotes a set of all numbers upper semicontinuously convex fuzzy on \( \mathbb{R}^m \).
Definition 1 (see [35]). Given a complete metric $D_1$ by
\[
D_1 (j, y) = \sup_{0 < c < 1} d_L \left\{ \left[ j^\beta, p^\beta \right] \right\},
\]
\[
= \sup_{0 < c < 1} \max \left\{ \left[ j^\beta - p^\beta, j^\beta - p^\beta \right] \right\},
\]
for any $u, v \in E_N$, which satisfies $D_1 (j + z, q + z) = D_1 (j, q)$ for each $z \in E_N$ and $j^\beta = [j^\beta, \mu^\beta]$, for each $b (j, q)$ where $\lambda^\beta, \mu^\beta \in R^m$ with $\beta^\beta \leq \mu^\beta$.

Definition 2 (see [36]). The fractional derivative of RL is stated as
\[
a D^\beta_a f (v) = \left( \frac{d}{d v} \right)^{n+1} \int_a^v (v - t)^{-\lambda} f (t) dt, \quad (n \leq \lambda \leq n+1). \tag{8}
\]

Definition 3 (see [37]). The fractional derivatives in the sense of Caputo $\beta^\beta D^\beta_a f (v)$ of order $\alpha \in R^+ $ are described by
\[
\beta^\beta D^\beta_a f (v) = D^\beta_a \left( f (v) - \sum_{k=0}^{n-1} \frac{f^{(k)} (a)}{k!} (v - a)^k \right), \tag{9}
\]
where $n = [\alpha] + 1$ for $\alpha \notin N_0 ; n = \sigma$ for $\sigma \in N_0$.

Definition 4 (see [37]). The Wright function $\psi_\alpha$ is defined by
\[
\psi_\alpha (\omega) = \sum_{n=0}^\infty \frac{(-\omega)^n}{n! (\pi n + 1 - \sigma)}
\]
\[
= \frac{1}{\pi} \sum_{n=1}^\infty \frac{(-\omega)^n}{n (n-1)!} \Gamma (n \sigma) \sin (n \pi \sigma),
\]
where $\omega \in C$ with $0 < \sigma < 1$.

Definition 5 (see [38]). For any $j, q \in C ([0, T], E_N)$, metric $H_1 (j, q) \in C ([0, T], E_N)$ is defined by
\[
H_1 (j, q) = \sup_{0 \leq t \leq T} D_1 (j (v), q (v)). \tag{11}
\]

Consider that $\Theta_1$ is a nonempty set and $P^m$ denotes power set on $\Theta_1$. A case is a label given to each element of $P^m$. To present an axiomatic crediblity, an idea based on the consideration of $\Theta_1$ will occur. To validate that the number $C_1 \{ A_i \}$ is applied to each $A_i$ event, representing the probability of $A_i$ happens. We accept the four main axioms to ensure that the number $C_1 \{ A_i \}$ has certain mathematical features that we predict:

(a) Normality property $C_1 \{ \Theta_1 \} = 1$,
(b) Monotonicity property $C_1 \{ A_i \} \leq C_1 \{ B_i \}$, whenever $A_i \subset B_i$.
(c) Self-duality property $C_1 \{ A_i \} + C_1 \{ A_i' \} = 1$ for any event $A_i$,
(d) Maximaliy property $C_1 \{ \cup A_i \} = \sup \{ C_1 \{ A_i \} \}$ for any events $\{ A_i \}$ with $\sup \{ C_1 \{ A_i \} \} < 0.5$.

Definition 6 (see [39]). Take $\Theta_1$ be the nonempty set, $P^m$ be the power set of $\Theta_1$, and $C_1$ be the credibility measure. After that, the triplet $(\Theta_1, P^m, C_1)$ is assigned to the set of real numbers.

Definition 7 (see [39]). A fuzzy variable is a function that is generated from a set of real numbers $(\Theta_1, P^m, C_1)$ to credibility space $(\Theta_1, P^m, C_1)$.

Definition 8 (see [39]). If $(\Theta_1, P^m, C_1)$ be credibility space $(\Theta_1, P^m, C_1)$ and $(\Theta_1, P^m, C_1)$ be an index set, a fuzzy process is a function that takes a set of real numbers and multiplies them by $T \times (\Theta_1, P^m, C_1)$.

It is a fuzzy method. $u (v, \xi)$ is a two-variable function in which $u (v, \xi^*)$ represents a fuzzy variable for each $\xi^*$. For each fixed $\xi^*$, the function $u (v, \xi)$ is termed a sample path of fuzzy process. The fuzzy process $u (v, \xi)$ is said to be sample continuous if sample ping is continuous for almost all $\xi$. Alternately of $u (v, \xi)$, we frequently use the notation $u_\xi$.

Definition 9 (see [39]). $(\Theta_1, P^m, C_1)$ is the symbol of a credibility space. The $\beta$-level set is applied for the fuzzy random variable $u_\omega$ in credibility space for each $\beta \in (0, 1)$.
\[
[u_\omega] = \left[ (u_\omega)_1, (u_\omega)_2 \right], \tag{12}
\]
is defined by
\[
(u_\omega)_1 = \inf \{ a \in R^m : u_\omega (a) \geq \beta \},
\]
\[
(u_\omega)_2 = \sup \{ a \in R^m : u_\omega (a) \geq \beta \}, \tag{13}
\]
where $\beta \in (0, 1)$.

Definition 10 (see [5]). Suppose that $\omega$ is a fuzzy variable and that $\beta$ is a real number. Then, $\omega$’s expected value is defined:
\[
E \omega = \int_0^\infty C_\beta \{ \omega \geq \beta \} dr - \int_0^\infty C_\beta \{ \omega \leq \beta \} dr, \tag{14}
\]
if at least one of the integrals is finite.

Lemma 11 (see [5]). If $\omega$ is a fuzzy vector, then the following are properties of expected value operator $E$:
(a) If $f \leq g$, $E [f (\omega)] \leq E [g (\omega)]$
(b) $E [-f (\omega)] = -E [f (\omega)]$
(c) If $f$ and $g$ are comonotonic, we have for any nonnegative real numbers $a$ and $b$, 
\[
E [a f + b g (\omega)] = a E [f (\omega)] + b E [g (\omega)].
\]
where $f(\omega)$ and $g(\omega)$ are fuzzy variables, respectively.

**Definition 12** (see [5]). A fuzzy process $\mathcal{C}_v$ is Liu process, if

(a) $\mathcal{C}_0 = 0$,
(b) the $\mathcal{C}_v$ has independent and stationary increments,
(c) any increment $\mathcal{C}_{v_i} - \mathcal{C}_v$ is normally distributed fuzzy variable with expected value $\nu v$ and variance $\phi^2 v_i$, with membership function.

\[
\xi(u) = 2 \left(1 + \exp \left(\frac{\pi |u - \nu v|}{\sqrt{6} \phi v}\right)\right)^{-1}, u \in \mathbb{R}^m
\]  

(16)

The parameters $\phi$ and $\epsilon$ represent the diffusion and drift coefficients, respectively. If $\epsilon = 0$ and $\phi = 1$, the Liu process is standard.

**Definition 13** (see [40]). Suppose that $\mathcal{C}_v$ is a standard Liu process and $u_\lambda$ is a fuzzy process. The mesh is fixed as $\nu = \nu_0 < \cdots < \nu_n = d$ for any partition of the closed interval $[c, d]$ with $\nu = \nu_0 < \cdots < \nu_n = d$,

\[
\Delta = \max_{1 \leq i \leq n} (\nu_i - \nu_{i-1}).
\]  

(17)

After that, the fuzzy integral of $u_\lambda$ with regard to $\mathcal{C}_v$ is calculated:

\[
\int_c^d u_\lambda d\mathcal{C}_v = \lim_{\nu \to 0} \sum_{i=1}^{n} \mu(\nu_i - \nu_{i-1}) (\mathcal{C}_{\nu_i} - \mathcal{C}_{\nu_{i-1}}),
\]  

(18)

determined by the limit exists almost positively and is a fuzzy variable.

**Lemma 14** (see [40]). Consider that $\mathcal{C}_v$ represent the standard Liu process with $\mathcal{C}_v(\zeta) > 0$, and the direction $\mathcal{C}_v$ is Lipschitz continuous, employing the below inequality:

\[
|\mathcal{C}_{\nu_i} - \mathcal{C}_{\nu_j}| < \mathcal{H}(\zeta)|\nu_i - \nu_j|,
\]  

(19)

where $\mathcal{H}(\zeta)$ is Lipschitz, which is a fuzzy variable described by

\[
\mathcal{H}(\zeta) = \begin{cases} \sup_{0 \leq s \leq \nu} |\mathcal{C}_{\nu} - \mathcal{C}_v| & \nu - s, \mathcal{C}_v(\zeta) > 1, \\ \infty, \quad \text{otherwise}, \end{cases}
\]  

(20)

and $E[\mathcal{H}^p] < \infty$ for all $p > 1$.

**Lemma 15** (see [40]). Assume that $h(\nu; c)$ is a continuously differentiable function and that $\mathcal{C}_v$ is a standard Liu process. The function is defined as $u_\nu = h(\nu; \mathcal{C}_v)$. Then, there is the chain rule, which is as follows:

\[
du_\nu = \frac{\partial h(\nu; \mathcal{C}_v)}{\partial \nu} d\nu + \frac{\partial h(\nu; \mathcal{C}_v)}{\partial \mathcal{C}_v} d\mathcal{C}_v.
\]  

(21)

**Lemma 16** (see [40]). The fuzzy integral inequality exists if $f(\nu u)$ is a continuous fuzzy process:

\[
\left|\int_c^d f(\nu u) d\mathcal{C}_v\right| \leq \mathcal{H}(\zeta) |\int_c^d f(\nu) | d\nu.
\]  

(22)

In Lemma 14, the term $\mathcal{H}(\zeta)$ is defined.

### 3. Existence of Solutions

This part applies the symbol $u_\lambda$ instead of the lengthy notation $u(v, \zeta)$, as defined by Definition 8. The existence-uniqueness of solutions to FDE $1 (x \equiv 0)$ has been investigated.

\[
\begin{cases}
\int_0^d u_\lambda d\mathcal{C}_v = g_0 u_\lambda + A u_\lambda + \int_{0}^{\nu} f(v, u_\lambda) d\mathcal{C}_v, \quad s \in [a, b), \\
u(0) = u_0 + h(v_1, v_2, \cdots, v_\nu, u(\cdot)), \quad \in E_N,
\end{cases}
\]  

(23)

where $u_\lambda$ is state that includes values from the $U(\mathfrak{E}_N)$ set of values. The set of all upper semicontinuously convex fuzzy numbers on $\mathbb{R}^m$ is called $\mathfrak{E}_N$, credibility space is $(\mathcal{E}_N, \mathfrak{P}_N, \mathcal{C}_\alpha)$, fuzzy coefficient is $\alpha$, and state function $u : [0, 3] \times (\mathcal{E}_N, \mathfrak{P}_N, \mathcal{C}_\alpha) \rightarrow U$ is fuzzy process, $f : [0, 3] \times U \rightarrow \mathbb{R}$ is regular fuzzy function, $\mathcal{C}_\alpha$ is standard Liu process, and $u_0 \in \mathfrak{E}_N$ is initial value.

**Lemma 17**. If $u(v)$ is the solution of equation (3) for $u(0) = u_0 + g_0 (v_1, v_2, \cdots, v_\nu, \mathcal{C}_\alpha)$, then $u(v)$ is given by

\[
u(\alpha) = \nu^{\beta-1} P_\beta(v) [u_0 + g_0 (v_1, v_2, \cdots, v_\nu, \mathcal{C}_\alpha)]
\]  

\[
+ \int_0^\nu (v-s)^{\beta-1} P_\beta(v-s) g_0 (s, x(s)) ds
\]  

\[
+ \int_0^\nu (v-s)^{\beta-1} P_\beta(v-s) \mathcal{C}_\alpha (s, x(s)) ds
\]  

\[
+ \int_0^\nu \mathcal{F}_{\mathcal{N}}(s, u(s, \mathcal{C}_\alpha)) ds + \mathcal{B}(\mathcal{C}_\alpha)(s) ds,
\]  

(24)

holds, and then,

\[
u(\alpha) = \nu^{\beta-1} P_\beta(v) [u_0 + g_0 (v_1, v_2, \cdots, v_\nu, \mathcal{C}_\alpha)]
\]  

\[
+ \int_0^\nu (v-s)^{\beta-1} P_\beta(v-s) g_0 (s, x(s)) ds
\]  

\[
+ \int_0^\nu (v-s)^{\beta-1} P_\beta(v-s) \mathcal{C}_\alpha (s, x(s)) ds
\]  

\[
+ \int_0^\nu \mathcal{F}_{\mathcal{N}}(s, u(s, \mathcal{C}_\alpha)) ds + \mathcal{B}(\mathcal{C}_\alpha)(s) ds,
\]  

(25)
where
\[
P_q(v) = \int_0^\infty q \zeta M_q(\zeta) Q(\nu^2 \zeta) d\zeta.
\]

Suppose that the statements below are correct:

(i) For \( u_0, v_0 \in C([0, \mathcal{F}] \times (\Theta_1, P^m, \mathcal{E}_r), U), v \in [0, \mathcal{F}] \), there exists a positive number \( \beta \) such that \( \beta \) holds, \( (23) \) has an unique solution \( \Theta \).

(ii) For all \( u, \Theta \in C([0, \mathcal{F}] \times (\Theta_1, P^m, \mathcal{E}_r), U), v \in [0, \mathcal{F}] \),

\[
d_L \left( f(v, u_v), [f(v, u_v)]^\beta \right) \leq m_L \left( [u_v]^\beta, [v_v]^\beta \right)
\]

\[
f(0, X(0, 0)) = 0.
\]

Therefore, we obtain
\[
D_L(\phi \lambda_\nu, \psi \mu_\nu) = \max_{C(\theta_0, 1)} d_L \left( [\phi \lambda_\nu]^\beta, [\psi \mu_\nu]^\beta \right)
\]
\[
\leq cm \mathcal{H} \int_0^\gamma d_L \left( \phi \lambda_\nu, \psi \mu_\nu \right) d\nu.
\]

As a result, according to Lemma 11, for a.s. \( \omega \in \Theta_1 \),

\[
E(H_1(\phi \lambda_\nu, \psi \mu_\nu)) = E \left( \max_{C(\theta_0, 1)} D_L(\phi \lambda_\nu, \psi \mu_\nu) \right)
\]
\[
\leq E \left( c m \mathcal{H} \int_0^\gamma D_L(\phi \lambda_\nu, \psi \mu_\nu) d\nu \right)
\]
\[
\leq c m \mathcal{H} E(H_1(\omega, \mu)).
\]

A contraction mapping is \( \phi \) according to hypothesis \( (i) \).

3.1 Exact Controllability. In this section, we will study exact controllability for differential equation in the context of Caputo operator (3). We investigate a solution for equation \( (3) x \) in \( V(cE_k) \).

\[
\left\{ \begin{array}{l}
\phi \lambda_\nu = \nu^{\beta-1} P_\lambda(v) u_0 + h(v, v_0, \cdots, v_n, u_0) + \int_0^\gamma (v-s)^{\beta-1} P_\lambda(v-s) g(s, u_0) d\nu + \int_0^\gamma (v-s)^{\beta-1} P_\lambda(v-s) g(s, u_s) d\nu + \int_0^\gamma (v-s)^{\beta-1} P_\lambda(v-s) g(s, u_s) d\nu + \int_0^\gamma \left[ A u_s + \int_0^\gamma \mathcal{K}(s, u_s, dE_s) + B u_s \right] d\nu \\
u(0) = u_0 + h(v, v_0, \cdots, v_n, u_0).
\end{array} \right.
\]
where $S(v)$ continuous, such that $S(0) = I = S'(0)$ and $|S(v)| \leq c, c > 0, v \in [0, \mathfrak{T}]$. The term of controllability is defined for Caputo fuzzy differential equations.

**Definition 19.** Equation (3) is called a controllable on $[0, \mathfrak{T}]$, if there is control $u_\nu \in E$ for every $u_\nu \in E$, where the solution $u$ of (3) satisfies the condition $u_\nu = u_\nu \in U$, a.s. $\zeta$, that is, $[u_\nu] = [u_\nu]$. Given fuzzy $\tilde{G} : \tilde{P}(\mathbf{R}^m) \rightarrow U$ mapping such that

$$
\tilde{G}^\beta (v) = \left\{ \begin{array}{ll}
\int_0^T (v-s)^{-1}P\gamma(v-s)B(v,s) \mathcal{E}_v ds, & v \in \tilde{\Gamma}_x, \\
0, & \text{otherwise},
\end{array} \right.
$$

(34)

where $\tilde{\Gamma}_x$ is closure of support $x$ and a nonempty fuzzy subset $\tilde{P}(\mathbf{R}^m)$ of $\mathbf{R}^m$.

After that, there is a $\tilde{G}^\beta (i = m, n)$,

$$
\tilde{G}^\beta_m(p_m) = \int_0^T (v-s)^{-1}P\gamma_m(v-s)B(p_m, \mathcal{E}_v) ds, (p_m) \in \left\{ (p)m, (p)m \right\},
$$

$$
\tilde{G}^\beta_n(p_n) = \int_0^T (v-s)^{-1}P\gamma_n(v-s)B(p_n, \mathcal{E}_v) ds, (p_n) \in \left\{ (p)n, (p)n \right\}.
$$

(35)

We assume that $\tilde{G}^\beta_m, \tilde{G}^\beta_n$ are bijective functions. A $\beta$-level set of $x_\nu$ can be presented as below:

$$
[x] = \left( \tilde{G}^\beta_m, \tilde{G}^\beta_n \right) = \left( \tilde{G}^\beta_m \right)^{-1} \left( \tilde{G}^\beta_n \right)^{-1}
$$

$$
\{ (u_\nu)^m, (u_\nu)^n \}
$$

$$
\left\{ (u_\nu)^m - (u_\nu)^m \tilde{G}^\beta_1, (u_\nu)^n - (u_\nu)^n \tilde{G}^\beta_2 \right\}
$$

$$
\left\{ (u_\nu)^m - (u_\nu)^m \tilde{G}^\beta_1, (u_\nu)^n - (u_\nu)^n \tilde{G}^\beta_2 \right\}
$$

$$
\left\{ (u_\nu)^m - (u_\nu)^m \tilde{G}^\beta_1, (u_\nu)^n - (u_\nu)^n \tilde{G}^\beta_2 \right\}
$$

(36)

This expression is substituted into (33) to get the $\beta$-level of $x_\nu$. 

$$
[u_\nu]^\beta = \left[ v^0 \tilde{P}g(v)(u_0 \in \mathcal{H}(v, v_1, v_2, \ldots, v_p, u(\cdot)) 
$$

$$
+ \int_0^T (v-s)^{-1} \tilde{P}g(v-s) (u_0, u_\nu) ds
$$

$$
+ \int_0^T (v-s)^{-1} \tilde{P}g(v-s) \mathcal{H}(s, u_\nu) ds
$$

$$
+ Bu_\nu \mathcal{E}_v ds
$$

(37)
We now set \( \nu_{\beta k} \) and 
\[
\psi \left[ \frac{dL}{d\nu} \right]_n + \nu_{\beta k} \left[ \frac{dL}{d\nu} \right]_n + \left( \mathcal{H}_n(s, u_i) \mathcal{C}_i \right) \left[ \frac{dL}{d\nu} \right]_n = \left[ u_i \right]_n^a. \tag{37}
\]

Hence, this control \( x_{\psi} \) satisfies \( u_{\psi} = u_i \), a.s. \( \zeta \).

We now set
\[
\psi u_{\psi} = v_{\psi} P_{\beta}(\nu) \left[ u_{0} + h(v_1, v_2, \ldots, v_p, u(\cdot)) \right] + \int_0^s (v-s)^{n-1} P_{\beta}(v-s) g_{\psi}(s, u_i) ds + \int_0^s (v-s)^{n-1} P_{\beta}(v-s) B \left[ \frac{dL}{d\nu} \right]_n \left[ u_i - v_{\psi} P_{\beta}(\nu) \left[ u_{0} + h(v_1, v_2, \ldots, v_p, u(\cdot)) \right] - \int_0^s (v-s)^{n-1} P_{\beta}(v-s) g_{\psi}(s, u_i) ds + \int_0^s (v-s)^{n-1} P_{\beta}(v-s) A_{\psi} + f \left[ \mathcal{H}(s, u_i) \mathcal{C}_i \right] \right] ds.
\tag{38}
\]

Fuzzy mappings \( \hat{G}^{-1} \) holds the above equation.

\[
d_{p}\left( |\psi u_{\psi}|^p, |\nu_{\psi}|^p \right) = d_{p}\left( |v_{\psi} P_{\beta}(\nu) | u_{0} + h(v_1, v_2, \ldots, v_p, u(\cdot)) \right) + \int_0^s (v-s)^{n-1} P_{\beta}(v-s) g_{\psi}(s, u_i) ds + \int_0^s (v-s)^{n-1} P_{\beta}(v-s) B \left[ \frac{dL}{d\nu} \right]_n \left[ u_i - v_{\psi} P_{\beta}(\nu) \left[ u_{0} + h(v_1, v_2, \ldots, v_p, u(\cdot)) \right] - \int_0^s (v-s)^{n-1} P_{\beta}(v-s) g_{\psi}(s, u_i) ds + \int_0^s (v-s)^{n-1} P_{\beta}(v-s) A_{\psi} + f \left[ \mathcal{H}(s, u_i) \mathcal{C}_i \right] \right] ds.
\]

\[\text{Theorem 20. If Lemma 16 and hypotheses (I\(_{ij}\)) and (I\(_{ik}\)) are hold, then equation (3) is controllable on [0, 3].}\]

\[\text{Proof. From } C(0, 3) \times (\Theta_{\psi}, P^m, U) \text{ to } C(0, 3), \text{ we can clearly see that } \psi \text{ is continuous. We have Lemma 16 and hypotheses (I\(_{ij}\)) and (I\(_{ik}\)) for any given } \zeta \text{ with } C(0, 3) = [0, \infty), \psi_{\psi}, \rho_{\psi} \in C(0, 3) \times (\Theta_{\psi}, P^m, C_{\psi}), U.\]

Hence, by Lemma 11,
\[
E(H_{\psi}(\psi u_{\psi}, \nu_{\psi})) = E\left( \sup_{v_{\psi} \in [0, 3]} D_{\psi}(\psi u_{\psi}, \nu_{\psi}) \right) = E\left( \sup_{v_{\psi} \in [0, 3]} D_{\psi}(\psi u_{\psi}, |\psi_{\psi}|^p, |\nu_{\psi}|^p) \right) ds \leq E\left( \sup_{v_{\psi} \in [0, 3]} 2c \mathcal{F}(H_{\psi}(u, v)) \right).
\]

(40)
As a consequence, (2cm. $\mathcal{F}$ $\mathfrak{M}$) $< 1$ is a $\mathfrak{A}$, sufficient $\mathfrak{M}$. As a result, $\psi$ stands for contraction. The Banach fixed point theorem is now being applied to show that (33) has a single fixed point. $[0, \mathfrak{M}]$ can be used to control (3).

\begin{equation}
\begin{aligned}
\hat{D}_t^\beta u(v, \zeta) &= \begin{cases} 
\hat{g}(v, u(v)) + A u(v, \zeta) + f(v, u(v)) + \int_0^\zeta k(s, u(v, \zeta)) \, ds, 
\end{cases} \\
u(0) &= u_0 + h(v_1, v_2, \ldots, v_n, u(\cdot)), \quad \in \mathcal{E}_N,
\end{aligned}
\end{equation}

where states consider values from $U(\mathcal{E}_N)$ and space $V (\mathcal{E}_N)$ two bounded spaces. The set of all, upper semicontinuously convex, fuzzy numbers on $\mathbb{R}^m$ is $\mathcal{E}_N$ and $(\Theta_1, \mathbb{P}_m, \mathcal{E}_r)$ denotes credibility space.

The state function $u : [0, \mathfrak{M}] \times (\Theta_1, \mathbb{P}_m, \mathcal{E}_r) \to U$ is fuzzy coefficient. Fuzzy process $f : [0, \mathfrak{M}] \times U \to U$. $x : [0,

\begin{aligned}
\begin{cases} 
u_i = \nu_{i-1}^\beta P(\nu_i | u_0 + h(v_1, v_2, \ldots, v_n, u(\cdot)) + \int_0^\nu (v-s) P(\nu | u_0 + h(v_1, v_2, \ldots, v_n, u(\cdot))), \quad &
\hat{d}_u^\beta(u, \zeta) = (u_m, (3-\beta)(u_n)), \quad v \\
u(0) = u_0 + h(v_1, v_2, \ldots, v_n, u(\cdot)) \in \mathcal{E}_N,
\end{cases}
\end{aligned}
\end{equation}

Therefore, Lemma 17 is satisfied.

\begin{equation}
\begin{aligned}
2^\beta &= [\beta + 1, 3 - \beta] \text{ is the } \beta\text{-level, set of fuzzy, number 2, for all } \beta \in (0, 1). \beta\text{-level set of } f(v, u_i) \text{ is}
\end{aligned}
\end{equation}

\begin{equation}
\begin{aligned}
[f(v, u_i)]^\beta = v \left( (\beta + 1)(u_n)^\beta_m, (3-\beta)(u_n)^\beta_m \right).
\end{aligned}
\end{equation}

Further, we have

\begin{equation}
\begin{aligned}
d_L \left( [f(v, u_i)]^\beta, [f(v, u_i)]^\beta \right) &= d_L \left( v \left( (\beta + 1)(u_n)^\beta_m, (3-\beta)(u_n)^\beta_m \right), v \left( (\beta + 1)(u_n)^\beta_m, (3-\beta)(u_n)^\beta_m \right) \right) \\
&= v \max \left\{ (\beta + 1)(u_n)^\beta_m - (\nu_n)^\beta_m, (3-\beta)(u_n)^\beta_m - (\nu_n)^\beta_m \right\} \\
&\leq 3\mathfrak{M} \max \left\{ \left\| (u_n)^\beta_m - (\nu_n)^\beta_m \right\|, \left\| (u_n)^\beta_m - (\nu_n)^\beta_m \right\| \right\} \\
&= m d_L \left( [u_n]^\beta, [\nu_n]^\beta \right),
\end{aligned}
\end{equation}

Example 1. We investigate FFDE in credibility space:

\begin{align*}
[\mathfrak{M}] \times (\Theta_1, \mathbb{P}_m, \mathcal{E}_r) &\to V \text{ is a regular fuzzy function, } x : [0, \\
[\mathfrak{M}] \times (\Theta_1, \mathbb{P}_m, \mathcal{E}_r) &\to V \text{ is a control function, and } B \text{ is a } V \text{ to } U \text{ linear bounded operator. } u_0 \in \mathcal{E}_N \text{ is an initial value, and } \mathcal{E}_r \text{ is standard Liu process.}
\end{align*}

Assume $f(v, u_i) = 2\nu_1u_i, S^{-1}(v) = e^{-v}$, defining $u_v = S^{-1}(v)u_v$. Then, the equations of balance become

\begin{equation}
\begin{aligned}
\nu_i &= \nu_{i-1}^\beta P(\nu_i | u_0 + h(v_1, v_2, \ldots, v_n, u(\cdot)) + \int_0^\nu (v-s) P(\nu | u_0 + h(v_1, v_2, \ldots, v_n, u(\cdot)), \\
&\hat{d}_u^\beta(u, \zeta) + f(s, u(s)), \quad \in \mathcal{E}_r) + B(s) \mathcal{E}_r(s)) \quad ds,
\end{aligned}
\end{equation}

where $m = 3\mathfrak{M}$ satisfies an inequality in the (J1) and (J2) hypotheses. All conditions given in Theorem 18 are fulfilled. Assume that $\hat{1} = 0$ is the initial value for $u_0$. The plan set $u^* = \hat{2}$. $\hat{1} = \hat{1} = [\beta - 1, 1 - \beta], \beta \in (0, 1)$ is $\beta$-level set of fuzzy numbers $\hat{1}$. The $x_s$ of (41)’s $\beta$-level set is presented.

\begin{equation}
[x_s] = \left( (x_s)^\beta_m, (x_s)^\beta_n \right)
\end{equation}

\begin{equation}
\left( G_s^\beta \right)^{-1} \begin{Bmatrix} (\beta + 1) - S_m^\beta (\mathfrak{M} - s)(\beta - 1) \\
- \int_0^s S_m^\beta (\mathfrak{M} - s)s(\beta + 1)(u_n)^\beta_m \mathcal{E}_r \end{Bmatrix}, \left( G_n^\beta \right)^{-1}
\end{equation}

This expression is then substituted into (42) to get the $\beta$-level of $u_i$.


\[ [u_\nu]^{\beta} = \left[ s_0^\beta (\mathcal{F} (\beta - 1) + \int_0^\beta s_0^\beta (\mathcal{F} - s) s (\beta + 1) (u_\nu)_s^\beta \, ds \right] + \int_0^\beta s_0^\beta (\mathcal{F} - s) B \left( C_n^\beta \right)^{-1} \left\{ (\beta + 1) - s_0^\beta (\mathcal{F} (\beta - 1) - \beta \right\} \\
- \int_0^\beta s_0^\beta (\mathcal{F} - s) s (\beta + 1) (u_\nu)_s^\beta \, ds, s_0^\beta (\mathcal{F} (1 - \beta) \\
+ \int_0^\beta s_0^\beta (\mathcal{F} - s) s (1 - \beta) (u_\nu)_s^\beta \, ds \right] \\
= [(\beta + 1), (3-\beta)] = [2]^{\beta}. \tag{46} \]

Following that, conditions in Theorem 20 have been fulfilled. As a result, (41) on \([0, T]\) can be controlled.

4. Definition of Stability in Credibility

We shall provide a concept of credibility stability for FFDEs driven by the Liu process in this part.

**Definition 21.** The FDE 1 is said to be stability in credibility if for any two, solutions \(u_\nu\) and \(v_\nu\), corresponding to different initial values \(u_0 + h(v_1, v_2, \ldots, v_p, u(.))\) and \(v_0 + h(v_1, v_2, \ldots, v_p, v(.))\), we have

\[
\lim_{|u_\nu - v_\nu| \to 0} \mathcal{C}_\nu \{|u_\nu - v_\nu| < \varepsilon\} = 1, \text{ for all } \nu \geq 0, \tag{47} \]

where \(\varepsilon\) is any given number and \(\varepsilon > 0\).

\[
\lim_{|u_\nu - v_\nu| \to 0} \mathcal{C}_\nu \{|u_\nu - v_\nu| < \varepsilon\} = \lim_{|u_\nu - v_\nu| \to 0} \mathcal{C}_\nu \{|u_\nu + h(v_1, v_2, \ldots, v_p, u(.)) - (v_0 + h(v_1, v_2, \ldots, v_p, v(.))| < \varepsilon\} = 1, \forall \nu \geq 0. \tag{50} \]

As a result, the credibility of FFDE is stable.

**Definition 22.** The \(n\)-dimensional FDE 1 is called stable in credibility, if for any two solutions \(u_\nu\) and \(v_\nu\), corresponding to different initial values \(v_0 + h(v_1, v_2, \ldots, v_p, u(.))\) and \(v_0 + h(v_1, v_2, \ldots, v_p, v(.))\), we have

\[
\lim_{|u_\nu - v_\nu| \to 0} \mathcal{C}_\nu \{|u_\nu - v_\nu| < \varepsilon\} = 1, \forall \nu \geq 0. \tag{51} \]

**Example 2.** Take the FFDE to better understand the concept of credibility stability.

\[ u_\nu = v^{\beta - 1} P_\beta (v) (u_\nu + h(v_1, v_2, \ldots, v_p, u(.)) \\
+ \int_0^\nu (v-s)^{\beta - 1} P_\beta (v-s) g(s, x(s)) \, ds \\
+ \int_0^\nu (v-s)^{\beta - 1} P_\beta (v-s) [A u(s, \zeta) \\
+ f(s, v(s, \zeta), \int_0^s h(s, u(s, \zeta)) \, ds] + B(s) \, ds, \tag{48} \]

respectively. Then, we have

\[
|u_\nu - v_\nu| = \left| (u_\nu + h(v_1, v_2, \ldots, v_p, u(.)) - (v_0 + h(v_1, v_2, \ldots, v_p, v(.)) \right|, \tag{49} \]

Deduce to, for any given \(\varepsilon > 0\), we always have

\[
|u_\nu - v_\nu| = \left| (u_\nu + h(v_1, v_2, \ldots, v_p, u(.)) - (v_0 + h(v_1, v_2, \ldots, v_p, v(.)) \right| < \varepsilon = 1, \forall \nu \geq 0. \tag{50} \]

**Example 3.** Take an \(m\)-dimensional FFDE:

\[ \begin{align*}
\frac{\partial}{\partial v} D_\nu^\beta u(v, \zeta) &= g_\nu(v, u(v)) + A u(v, \zeta) \\
+ &\int_0^\nu \left( (v, u(v, \zeta), \int_0^v k(s, u(v, \zeta)) \, ds \right) \, dv, \tag{52} \end{align*} \]

The two solutions corresponding to different initial values are
Thus, $m$-dimensional FFDE is stability in credibility. Note that some fuzzy differential equations driven by the Liu process are not stable in credibility. It will be demonstrated in the following example.

5. Theorems of Stability in Credibility

In this part, we will discuss the necessary criteria for a FFDE driven by the Liu process to achieve credibility stability.

**Theorem 23.** Assume the FFDE 1 for each initial value has a unique solution. Then, it is stable in credibility space, if coefficients $f(v, u)$ and $g(v, u)$ satisfy strongly Lipschitz condition

$$D(f(v, u) - f(v, v)) + (g(v, u) + g(v, v)) \leq L(v)D(u - v), \forall u, v \in R^m, v \geq 0,$$

for some integrable function $L(v)$ on $[0, +\infty)$.

**Proof.** Let $u_0$ and $v_0$ be two solutions corresponding to differential initial values $(u_0 + h(v_1, v_2, \ldots, v_p, u(\cdot)))$ and $(v_0 + h(v_1, v_2, \ldots, v_p, v(\cdot)))$, respectively. Then, for each $\delta \in \Theta_1$,

$$D(u_0 - v_0) = \int_0^\nu (v - s)^{\beta-1}P_\beta(v - s)\,ds,$$

where $\mathcal{K}(\delta)$ is the Lipschitz constant of the Liu process. When we take integral on both sides of equation (57),

$$\int_0^\nu (v - s)^{\beta-1}P_\beta(v - s)\,ds,$$

respectively. Then, we have

$$\| u_0 - v_0 \| = \| (u_0 + h(v_1, v_2, \ldots, v_p, u(\cdot))) - (v_0 + h(v_1, v_2, \ldots, v_p, v(\cdot))) \|.$$

As a result, we always have

$$\lim_{\| u_0 - v_0 \| \to 0} \mathcal{C}_r \{ |u_0 - v_0| < \epsilon \} = 1, \forall \nu \geq 0.$$

For any given $\epsilon > 0$, we always have

$$\mathcal{C}_r \{ |u_0 - v_0| < \epsilon \} \geq \{ (\nu_0 + h(v_1, v_2, \ldots, v_p, u(\cdot))) - (v_0 + h(v_1, v_2, \ldots, v_p, v(\cdot))) \| \exp \left( 1 + |\mathcal{K}(\delta)| \int_0^\nu L(s)\,ds \right) < \epsilon.\]
Lipschitz condition, the following corollary can be used to determine whether the FFDE is stable in credibility space.

**Corollary 24.** Assume $f(v, u)$ and $g(v, u)$ be bounded real valued functions on $[0, +\infty)$. If $f(v, u)$ and $g(v, u)$ have derivatives with respect to $u$ and satisfy

$$|f'_u(v, u)| + |g'_u(v, u)| \leq L(v), \forall v \geq 0,$$

(62)

for some integrable function $L(v)$ on $[0, +\infty)$, then FFDE 1 is stability in credibility.

**Proof.** For the bounded real valued functions $f(v, u)$ and $g(v, u)$,

$$|f(v, u)| + |g(v, u)| < \mathcal{H}(1 + |u|),$$

(63)

where $\mathcal{H}$ is constant which satisfy $|f(v, u)| + |g(v, u)| < \mathcal{H}$. We can derive from the mean value theorem that

$$|f(v, u') - f(v, u'')| + |g(v, u') - g(v, u'')|$$

$$= f'_u(v, \xi)|u' - u''| + g'_u(v, \eta)|u' - u''|$$

$$\leq L(v)|u' - u''| + L(v)|u' - u''| = 2L(v)|u' - u''|,$$

(64)

where $\xi, \eta \in (u' - u'')$ existence-uniqueness theorem demonstrates that FFDE has a unique solution. We can deduce from Theorem 23 that FFDE is stable in credibility. Different from Theorem 23 and Corollary 24, we have below corollary when FFDE is general linear FFDE driven by the Liu process.

**Corollary 25.** Suppose that $u_{1v}, u_{2v}, v_{1v},$ and $v_{2v}$, are bounded functions, with respect to $v$ on $[0, +\infty)$. If $u_{1v}$ and $v_{1v}$ are integrable, on $[0, +\infty)$, then linear FFDE driven by Liu process

$$du_v = (u_{1v}u_v + u_{2v})dv + (v_{1v}u_v + v_{2v})d\xi_v,$$

(65)

is stability in credibility.

**Proof.** For the linear FFDE 7, we have $f(v, x) = u_{1v}x + u_{2v}$ and $g(v, x) = v_{1v} + v_{2v}$, since

$$|u_{1v}u_v + u_{2v}| + |v_{1v}u_v + v_{2v}|$$

$$\leq |u_{1v}||u_v| + |u_{2v}| + |v_{1v}||u_v| + |v_{2v}|$$

$$< \mathcal{H}|u_v| + \mathcal{H} + \mathcal{H}|u_v| + \mathcal{H} = 2\mathcal{H}(|u_v| + 1),$$

$$|(u_{1v}u_v + u_{2v}) - (u_{1v}v_v + u_{2v})| + |(v_{1v}u_v + v_{2v}) - (v_{1v}u_v + v_{2v})|$$

$$= |u_{1v}(u_v - v_v)| + |v_{1v}(u_v + v_v)|$$

$$\leq |u_{1v}||u_v - v_v| + |v_{1v}||u_v + v_v|$$

$$= (|u_{1v}| + |v_{1v}|)|u_v - v_v| \leq 2\mathcal{H}(u_v - v_v),$$

(66)

where $\mathcal{H}$ is a constant which make $u_{1v} < \mathcal{H}, u_{2v} < \mathcal{H}, v_{1v} < \mathcal{H}, v_{2v} < \mathcal{H}$ hold. The existence-uniqueness theorem shows that FDE 7 has a unique solution. Since $L(v) = |u_{1v}| + |v_{1v}|$ is integrable function on $[0, +\infty)$, from Theorem 23, the credibility of FFDE can be determined.

According to Definition 22, Theorem 23 can be used to $n$-dimensional FFDEs driven by the Liu process.

**Theorem 26.** Assume that each initial value of the $n$-dimensional FFDE 1 has a unique solution. If coefficients $f(v, u)$ and $g(v, u)$ satisfy Lipschitz’s strong condition, then it is stable in credibility space:

$$||f(v, u) - f(v, v)|| + ||g(v, u) - g(v, v)||$$

$$\leq L(v)||u - v||, \forall u, v \in \mathbb{R}^n, \forall v \geq 0,$$

(67)

for some integrable function $L(v)$ on $[0, +\infty)$.

6. Conclusion

Accurate controllability for FFDEs can be used as a standard when analyzing controllability for semilinear integro-differential equations in the credibility space and fuzzy delay integro-differential equations. Therefore, the research’s theoretical conclusions can be applied to construct stochastic extensions on credibility space. The FFDEs driven by the Liu process have an important role in both theory and practice as a technique for dealing with dynamic systems in a fuzzy environment. There have been some proposed stability approaches for FFDEs driven by the Liu process up until now. This is a rewarding field with numerous research projects that can lead to a variety of applications and theories. We hope to learn more about fuzzy fractional evolution problems in future projects. We can discover uniqueness and existence with uncertainty using the Caputo derivative. Future work could include expanding on the mission concept, including observability, and generalizing other activities. This is an interesting area with a lot of study going on that could lead to a lot of different applications and theories. This is a path in which we intend to invest significant resources.

Data Availability

Data is original and references are given where required.

Conflicts of Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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References

[1] K. S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, Wiley, Hoboken, 1993.

[2] S. S. Mansouri, M. Gachpazan, and O. S. Fard, "Existence, uniqueness and stability of fuzzy fractional differential equations with local Lipschitz and linear growth conditions," *Advances in Difference Equations*, vol. 2017, no. 1, p. 13, 2017.

[3] L. A. Zadeh, G. J. Klin, and B. Yuan, *Fuzzy Sets, Fuzzy Logic, and Fuzzy Systems: Selected Papers*, vol. 6, World Scientific, 1996.

[4] L. A. Zadeh, "Fuzzy sets as a basis for a theory of possibility," *Fuzzy Sets and Systems*, vol. 1, no. 1, pp. 3–28, 1978.

[5] B. Liu and Y. K. Liu, "Expected value of fuzzy variable and fuzzy expected value models," *IEEE Transactions on Fuzzy Systems*, vol. 10, no. 4, pp. 445–450, 2002.

[6] X. Li and B. Liu, "A sufficient and necessary condition for credibility measures," *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, vol. 14, no. 5, pp. 527–535, 2006.

[7] V. Lakshmikantham and A. S. Vatsala, "Basic theory of fractional differential equations," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 69, no. 8, pp. 2677–2682, 2008.

[8] Y. C. Kwun, J. S. Kim, J. S. Hwang, and J. H. Park, "Existence of Solutions for the impulsive semilinear fuzzy integro-differential equations with nonlocal conditions and forcing term in memory in n-dimensional fuzzy vector space (EEn, de)," *International Journal of Fuzzy Logic and Intelligent Systems*, vol. 11, no. 1, pp. 25–32, 2011.

[9] B. Y. Lee, Y. C. Kwun, Y. C. Ahn, and J. H. Park, "The existence and uniqueness of fuzzy solutions for semilinear fuzzy integrodifferential equations using integral contractor," *International Journal of Fuzzy Logic and Intelligent Systems*, vol. 9, no. 4, pp. 339–342, 2009.

[10] Y. C. Kwun, M. J. Park, J. S. Kim, J. S. Park, and J. H. Park, "Controllability for the impulsive semilinear fuzzy differential equation in n-dimension fuzzy vector space," *International Journal on Fuzzy Systems and Knowledge Discovery*, vol. 19, no. 1, pp. 45–48, 2009.

[11] J. H. Park, J. S. Park, and Y. C. Kwun, "Controllability for the semilinear fuzzy integrodifferential equations with nonlocal conditions," in *International Conference on Fuzzy Systems and Knowledge Discovery*, pp. 221–230, Berlin, Heidelberg, 2006.

[12] J. H. Park, J. S. Park, Y. C. Ahn, and Y. C. Kwun, "Controllability for the impulsive semilinear fuzzy integrodifferential equations," in *Fuzzy Information and Engineering*, pp. 704–713, Springer, Berlin, Heidelberg, 2007.

[13] N. D. Phu and L. Q. Dung, "On the stability and controllability of fuzzy control set differential equations," *International Journal of Reliability and Safety*, vol. 5, no. 3/4, pp. 320–335, 2011.

[14] B. Y. Lee, D. G. Park, G. T. Choi, and Y. C. Kwun, "Controllability for the nonlinear fuzzy control system with nonlocal initial condition in EnN," *International Journal of Fuzzy Logic and Intelligent Systems*, vol. 6, no. 1, pp. 15–20, 2006.

[15] P. Balasubramaniam and J. P. Dauer, "Controllability of semilinear stochastic evolution equations in Hilbert space," *Journal of Applied Mathematics and Stochastic Analysis*, vol. 14, no. 4, pp. 329–339, 2001.

[16] Y. Feng, "Convergence theorems for fuzzy random variables and fuzzy martingales," *Fuzzy Sets and Systems*, vol. 103, no. 3, pp. 435–441, 1999.

[17] A. Arapostathis, R. K. George, and M. K. Ghosh, "On the controllability of a class of nonlinear stochastic systems," *Systems & Control Letters*, vol. 44, no. 1, pp. 25–34, 2001.

[18] B. Liu, "Fuzzy process, hybrid process and uncertain process," *Journal of Uncertain Systems*, vol. 2, no. 1, pp. 3–16, 2008.

[19] X. Chen and Z. Qin, "A new existence and uniqueness theorem for fuzzy differential equations," *International Journal of Fuzzy Systems*, vol. 13, no. 2, 2011.

[20] Y. Liu, "An analytic method for solving uncertain differential equations," *Journal of Uncertain Systems*, vol. 6, no. 4, pp. 244–249, 2012.

[21] A. Abbas, R. Shafqat, M. B. Jeelani, and N. H. Alharthi, "Significance of chemical reaction and Lorentz force on third-grade fluid flow and heat transfer with Darcy–Forchheimer law over an inclined exponentially stretching sheet embedded in a porous medium," *Symmetry*, vol. 14, no. 4, p. 779, 2022.

[22] A. Abbas, R. Shafqat, M. B. Jeelani, and N. H. Alharthi, "Convective heat and mass transfer in third-grade fluid with Darcy–Forchheimer relation in the presence of thermal-diffusion and diffusion-thermo effects over an exponentially inclined stretching sheet surrounded by a porous medium: a CFD study," *Processes*, vol. 10, no. 4, p. 776, 2022.

[23] A. U. K. Niazi, J. He, R. Shafqat, and B. Ahmed, "Existence, uniqueness, and Eq–Ulam-type stability of fuzzy fractional differential equation," *Fractal and Fractional*, vol. 5, no. 3, p. 66, 2021.

[24] A. U. K. Niazi, N. Iqbal, R. Shah, F. Wannaloookhhe, and K. Nonlaopon, "Controllability for fuzzy fractional evolution equations in credibility space," *Fractal and fractional*, vol. 5, no. 3, p. 112, 2021.

[25] N. Iqbal, A. U. K. Niazi, R. Shafqat, and S. Zaland, "Existence and uniqueness of mild solution for fractional-order controlled fuzzy evolution equation," *Journal of Function Spaces*, vol. 2021, Article ID 5795065, 8 pages, 2021.

[26] R. Shafqat, A. U. K. Niazi, M. B. Jeelani, and N. H. Alharthi, "Existence and uniqueness of mild solution where a ∈ (1, 2) for fuzzy fractional evolution equations with uncertainty," *Fractal and Fractional*, vol. 6, no. 2, p. 65, 2022.

[27] K. Abusabheb, R. Shafqat, A. U. K. Niazi, and M. Awadalla, "Local and global existence and uniqueness of solution for time-fractional fuzzy Navier–Stokes equations," *Fractal and Fractional*, vol. 6, no. 6, p. 330, 2022.

[28] A. S. Alnahdi, R. Shafqat, A. U. K. Niazi, and M. B. Jeelani, "Pattern formation induced by fuzzy fractional-order model of COVID-19," *Axioms*, vol. 11, no. 7, p. 313, 2022.

[29] M. M. Arjunan, T. Abdeljawad, V. Kavitha, and A. Yusuf, "On a new class of Atangana-Baleanu fractional Volterra-Fredholm integro-differential inclusions with non-instantaneous impulses," *Chaos, Solitons & Fractals*, vol. 148, article 111075, 2021.

[30] M. M. Arjunan, A. Hamiaz, and V. Kavitha, "Existence results for Atangana-Baleanu fractional neutral integro-differential systems with infinite delay through sectorial operators," *Chaos, Solitons & Fractals*, vol. 149, article 111042, 2021.

[31] M. M. Arjunan, V. Kavitha, and D. Baleanu, "A new existence results on fractional differential inclusions with state-dependent delay and Mittag-Leffler kernel in Banach space," *Analele Științifice ale Universității "Ovidius" Constanța. Seria Matematică*, vol. 30, no. 2, pp. 5–24, 2022.
[32] M. M. Arjunan, P. Anbalagan, and Q. Al-Mdallal, “Robust uniform stability criteria for fractional-order gene regulatory networks with leakage delays,” Mathematical Methods in the Applied Sciences, pp. 1–18, 2022.

[33] J. H. Jeong, J. S. Kim, H. E. Youm, and J. H. Park, “Exact controllability for fuzzy differential equations using extremal solutions,” Journal of Computational Analysis and Applications, vol. 23, pp. 1056–1069, 2017.

[34] P. Diamand and P. E. Kloeden, Metric Space of Fuzzy Sets, World Scientific, 1994.

[35] G. Wang, Y. Li, and C. Wen, “On fuzzy n-cell numbers and n-dimension fuzzy vectors,” Fuzzy Sets and Systems, vol. 158, no. 1, pp. 71–84, 2007.

[36] I. Podlubny, “Fractional differential equations,” Mathematics in Science and Engineering, vol. 198, pp. 41–119, 1999.

[37] F. Mainardi, P. Paradisi, and R. Gorenflo, "Probability distributions generated by fractional diffusion equations,” 2007, http://arxiv.org/abs/0704.0320.

[38] W. Fei, "Uniqueness of solutions to fuzzy differential equations driven by Liu’s process with non-Lipschitz coefficients,” in 2009 Sixth International Conference on Fuzzy Systems and Knowledge Discovery, vol. 6, pp. 565–569, 2009.

[39] C. C. Travis and G. F. Webb, "Cosine families and abstract nonlinear second order differential equations,” Acta Mathematica Hungarica, vol. 32, no. 1-2, pp. 75–96, 1978.

[40] Y. Zhou and J. W. He, "New results on controllability of fractional evolution systems with order $\alpha \in (1, 2)$,” Evolution Equations & Control Theory, vol. 10, no. 3, p. 491, 2021.