Fast Frequency-domain Waveforms for Spin-Precessing Binary Inspirals

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The detection and characterization of gravitational wave signals from compact binary coalescence events relies on accurate waveform templates in the frequency domain. The stationary phase approximation (SPA) can be used to compute closed-form frequency-domain waveforms for non-precessing, quasi-circular binary inspirals. However, until now, no fast frequency-domain waveforms have existed for generic, spin-precessing quasi-circular compact binary inspirals. Templates for these systems have had to be computed via a discrete Fourier transform of finely sampled time-domain waveforms, which is far more computationally expensive than those constructed directly in the frequency-domain, especially for those systems that are dominated by the inspiral part. There are two obstacles to deriving frequency-domain waveforms for precessing systems: (i) the spin-precession equations do not admit closed-form solutions for generic systems; (ii) the SPA fails catastrophically. Presently there is no general solution to the first problem, so we must resort to numerical integration of the spin precession equations. This is not a significant obstacle, as numerical integration on the slow precession timescale adds very little to the computational cost of generating the waveforms. Our main result is to solve the second problem, by providing an alternative to the SPA that we call the method of Shifted Uniform Asymptotics, or SUA, that cures the divergences in the SPA. The construction of frequency-domain templates using the SUA can be orders of magnitude more efficient than the time-domain ones obtained through a discrete Fourier transform. Moreover, this method is very faithful to the discrete Fourier transform, with mismatches on the order of $10^{-5}$.

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I. INTRODUCTION

In the two-body problem of General Relativity, the coupling of the spin and orbital angular momenta leads to Lense-Thirring precession, which causes the orbital plane as well as the spins of the bodies to change orientation. In the Solar System, such an effect is often negligible, since the magnitude of the spin angular momenta is usually much smaller than that of the orbital angular momentum. However in the late inspiral of compact objects, such as black holes and neutron stars, this is not necessarily the case, and the precession of the orbital plane can lead to very complicated motion.

Precession is both a blessing and a curse. It adds rich structure to the templates which can greatly enhance the accuracy of parameter estimation, but it also makes detecting and characterizing the signals more challenging. From a practical standpoint, part of the challenge is that it is not easy to produce accurate waveform templates for precessing systems at reasonable computational cost. These waveform templates require precise knowledge of the evolution of the orbital angular momentum, the orbital phase, and the frequency. The fact that no analytic solution to the precession equations is known in full generality has been used to justify the use of approximate solutions, which may suffice for detection purposes, but are inadequate for parameter estimation.

The importance of precession in GW parameter estimation has been recognized early on in the development of accurate templates for data analysis [1–10]. Precession not only affects the evolution of the GW phase, but it also introduces modulations to the GW amplitude that evolve on much longer timescales than the orbital period. Such modulations have been shown to break parameter degeneracies that are unavoidable in non-precessing templates [4, 5, 11, 12]. Physically, templates that can accurately reproduce the complicated structure of signals will in general be capable of extracting much more useful information than templates that do not capture this structure.

Historically, the construction and study of precessing templates dates back to the beginning of the 21st century. Vecchio in 2004 [4] showed that parameter estimation in the context of LISA could be greatly improved when using the analytical simple precession model derived by Apostolatos, et al. [3]. Later in 2006, Lang and Hughes [5] proposed a prescription for waveform templates with generic precessing spins, which was later refined to include subdominant harmonics [6] and then shown to improve parameter estimation when precession effects are moderately suppressed [9]. The inclusion of precession has recently been shown to improve parameter estimation for neutron star inspiral sources [11, 12], as well as to strengthen tests of General Relativity with...
GWs [6, 7].

Despite these efforts to construct closed-form approximate templates, current parameter estimation algorithms for advanced, ground-based interferometer data, such as advanced LIGO (aLIGO) [14] and advanced Virgo (AdV) [15], use numerical time-domain templates for precessing systems, i.e. templates constructed from the discrete Fourier transform (DFT) of time-domain, numerical solutions to the post-Newtonian (PN) expanded (and sometimes resummed) Einstein equations. In the PN approximation, the field equations are expanded in small velocities and weak fields and then possibly resummed [10]. The construction of such templates is quite computationally expensive because, when computing the DFT of a time series, one needs to discretize the signal with a uniform spacing that is capable of resolving the shortest timescales. This fixes the time resolution to one half of the inverse of the highest frequency reached by the signal, which is typically much higher than the resolution needed to resolve the signal during most of the inspiral.

Because of this computational cost issue, a significant amount of effort has recently been put into deriving precessing templates constructed directly in the frequency domain, i.e. without having to compute the DFT of a time series, but that remain as accurate as possible [8, 10, 12, 17]. Such efforts relied on approximate solutions to the precession equations. In particular, frequency-domain waveforms were constructed for binaries undergoing simple precession [17], and for generic precessing binaries but in the small misalignment approximation [10] and in the small spins approximation [13]. Templates for generic precessing binaries have also been constructed using simple precession formulae [18], but this is not mathematically well-justified because generic precessing systems exhibit more complex evolution.

Once a closed-form prescription for the temporal evolution of the angular momentum is specified, the Fourier transform of the GW response function is conventionally modeled through the stationary phase approximation (SPA). This approximation assumes the Fourier transform at a given frequency is dominated by the Fourier integral in a small neighborhood about a certain stationary point, i.e. where the first time derivative of the argument of the integrand vanishes. For generic precessing systems, however, the argument of the phase of the generalized Fourier integrand will encounter catastrophes [18], where multiple stationary points coalesce, and the second time derivatives of the phase of the integrand vanishes together with the first derivative. Such catastrophes introduce mathematical singularities in the SPA Fourier amplitude that render the SPA ill-suited for the construction of frequency-domain templates for precessing systems.

One way around these precession-induced catastrophes is through the method of Uniform Asymptotic expansion [14, 20]. When the phase oscillates, as it does here, the Fourier integrand can be expanded in a Bessel series [10, 21]. The idea is to rewrite the phase of the integrand by expanding the oscillatory terms that depend on the spin and angular momentum, and thus on trigonometric functions of the precession phases, in an infinite Bessel series. Although the full series is still problematic from a catastrophe theory standpoint, each term in the series has a well behaved SPA. One then avoids catastrophes by truncating the infinite sum at a finite order, which is justified because the contribution from the Bessel functions that multiply the amplitude at high orders in the sum are very small. The resulting frequency-domain waveform remains highly accurate and faithful [10], but the need to introduce Bessel series makes them relatively computationally expensive, which partially defeats the purpose of constructing them in the first place.

In this paper, we propose a new approach, the method of Shifted Uniform Asymptotics (SUA) that resums the Bessel function series generated by the Uniform Asymptotic expansion, resulting in a sum of time-shifted precession modulations. The SUA allows us to construct frequency-domain waveforms for generic precessing systems without the need to evaluate Bessel expansions. This method takes as input any closed-form or numerical time-domain solution to the orbital angular momentum, the orbital phase, and the orbital frequency, and outputs a remarkably simple, frequency-domain representation of the GW response. In essence, this approach resums the Bessel expansions described in Ref. [10], using the generic Taylor expansion of a shifted function; the result is a frequency-domain waveform that resembles the usual SPA result, but with a correction that consists of a sum of amplitudes shifted with respect to the SPA stationary time. The SUA frequency-domain template is simple to compute, computationally inexpensive and highly accurate relative to time-domain waveforms obtained through a DFT.

As an example, we apply the SUA method to a numerical, time-domain solution for generic precessing binaries with arbitrary spin magnitude and orientation. In particular, we adopt the SpinTaylorT4 model as the time-domain solution, i.e. the numerical solution to the 2PN expanded precession and 3.5PN orbital equations\(^2\). This model yields a time series solution for the orbital angular momentum and phase, as well as a time series for the waveform itself. Using the former, we compute the SUA, frequency-domain representation to the SpinTaylorT4 waveform and compare it to the DFT of the SpinTaylorT4 time series. In the rest of this paper, we will refer to the frequency-domain SUA version of the SpinTaylorT4 waveforms as SpinTaylorT4Fourier to distinguish them from the

\(^1\) Here we are using the naming conventions of the LIGO Algorithm Library, https://www.lsc-group.phys.uwm.edu/daswg/projects/lalsuite.html.

\(^2\) An expression is said to be \(\mathcal{O}(v^{2N+1})\).
usual time-domain SpinTaylorT4 waveforms. Recall that we use the word “ waveform” to refer to frequency series as is customary in the GW data analysis community. We find excellent agreement between these two waveforms, with unfaithfulness\(^3\) of the order of 10\(^{-5}\). The SUA waveforms, however, are found to be much faster to generate than time-domain templates. All of the above is implemented and carried out in the lalsimulation open-source package of the LIGO Scientific Collaboration, including the SUA version of the SpinTaylorT4 and SpinTaylorT2 templates, which we called SpinTaylorT4Fourier and SpinTaylorT2Fourier respectively. A definition of those PN flavors can be found in e.g.\(^{22,23}\).

Figure \(\textbf{1}\) shows how faithful the SUA waveforms are. This figure shows mismatch distributions between time-domain SpinTaylorT4 waveforms and either

(i) The SpinTaylorT4Fourier SUA waveforms constructed with SpinTaylorFourier time-domain solutions to the precession equations and with the sum of shifted amplitudes in Eq. (43) truncated at either (ia) the zeroth term (dot-dashed blue) or (ib) the third term (solid black);

(ii) the single-spin, simple precession approximation of Ref. \(^{18}\) (dotted red), constructed by solving the precession equations approximately, assuming one of the two compact objects is not spinning.

The time-domain SpinTaylorT4 waveform and the frequency-domain SpinTaylorT4Fourier waveform include subdominant PN harmonic corrections. The single spin waveform contains only the leading PN order amplitude. The distributions are created through Monte-Carlo sampling over system parameters for a binary black hole inspiral, with mass range in \((5, 20)\)\(M_\odot\), spin angular momentum magnitude in \((0, m^2_A)\), where \(m_A\) is the individual BH mass, and random spin orientations. Observe that the single-spin templates have relatively poor agreement with time-domain SpinTaylorT4 waveforms, while the SUA templates show excellent agreement. Observe also that retaining only the zeroth order term in the sum of shifted amplitudes is not sufficient to ensure > 99% faithfulness.

The remainder of this paper presents the details of the SUA method and of the comparison described above. Section \[\text{II}\] derives the SUA method. Section \[\text{III}\] establishes the validity of the method by applying it and comparing it to time-domain SpinTaylorT4 templates. Section \[\text{IV}\] concludes and points to future work.

Throughout this paper, we use geometric units where \(G = c = 1\), as well as the following conventions and notation:

- Three-dimensional vectors are written in boldface and unit vectors carry a hat over them, e.g. \(\mathbf{A} = (A_x, A_y, A_z)\), with norm \(\mathbf{A} = |\mathbf{A}|\), and unit vector \(\mathbf{A} = \mathbf{A}/|\mathbf{A}|\).
- Total time derivatives are denoted with a dot, e.g. \(\dot{f} = df/dt\).
- The individual masses of the binary system components are \(m_1\) and \(m_2\), with \(m_1 > m_2\), the total mass is \(M = m_1 + m_2\), the dimensionless individual masses are \(\mu_A = m_A/M\), \(A \in \{1, 2\}\), and the symmetric mass ratio is \(\nu = m_1m_2\).
- \(L\) is the Newtonian orbital angular momentum.
- \(S_A\), \(A \in \{1, 2\}\) are the individual spin angular momentum vectors, \(a_A = S_A/(m_AM)\) are dimensionless spin vectors, and \(\chi_A = S_A/m^2_A\) are the individual spin parameters.
- \(\hat{N}\) is the sky localization vector from the detector to the source.

\[\text{II. THE SHIFTED UNIFORM ASYMPTOTICS METHOD}\]

In this section, we derive the SUA method to obtain the frequency-domain representation of a time-domain waveform for generic precessing inspirals that avoids

\[^3\] The unfaithfulness is a particular noise-weighted cross-correlation between two waveforms in the frequency domain, minimized only over unphysical parameters. An unfaithfulness of zero implies perfect agreement between the two waveforms.
catastrophes. For concreteness, we focus on GWs emitted during the quasi-circular inspiral of compact objects with generic spin magnitudes and orientations, such that the orbital plane is undergoing generic precession. We study the Fourier transform of the response of advanced, ground-based detectors to such GWs, assuming an L-shaped detector in the long wavelength approximation. Our results, of course, could easily be extended to other detector configurations, such as eLISA [24].

A. Preliminaries

The time-domain response function can be given as [3]

\[ h(t) = F_+(t)h_+(t) + F_\times(t)h_\times(t), \]  

(1)

where the time-domain GW plus- and cross-polarizations can be written as

\[ h_{+\times} = \sum_{n>0} A_{+,\times}^{(n)}(\iota) e^{-in(\phi_C+\phi_T)} + c.c., \]  

(2)

with c.c. the complex conjugate, \( n \in \mathbb{N} \) the harmonic number, \( \iota \equiv \arccos(L \cdot \hat{N}) \) the inclination angle, \( \phi_C \) the carrier GW phase, \( \phi_T \) the Thomas phase, which accounts for the precession of the greater axis of the projection of the orbital plane onto the sky inside the orbital plane as \( L \) precesses. The antenna or beam pattern functions are

\[ F_+(\theta_N, \phi_N, \psi_N) = \frac{1}{2} \left(1 + \cos^2 \theta_N \right) \cos 2\phi_N \cos 2\psi_N \]  

\[ \quad - \cos \theta_N \sin 2\phi_N \sin 2\psi_N, \]  

(3)

\[ F_\times(\theta_N, \phi_N, \psi_N) = F_+(\theta_N, \phi_N, \psi_N - \pi/4), \]  

(4)

where \( \theta_N, \phi_N, \psi_N \) are spherical angles that label the position of the binary in a frame tied to the detector, with \( \hat{x} \) and \( \hat{y} \) unit vectors along the arms of the detector, \( \hat{z} = \hat{x} \times \hat{y} \), and \( \psi_N \) the polarization angle defined through

\[ \tan \psi_N = \frac{\hat{L} \cdot \hat{z} - (\hat{L} \cdot \hat{N})(\hat{z} \cdot \hat{N})}{\hat{N} \cdot (\hat{L} \times \hat{z})}. \]  

(5)

B. Separation of Scales

A key element of the SUA method is the use of multiple-scale analysis [34]. In order to illustrate this, consider the equations of motion during the inspiral in the so-called SpinTaylorT4 form [14, 20, 34]:

\[ M\dot{\phi}_\text{orb} = \xi^3, \]  

(6)

\[ \dot{\phi}_C = \phi_\text{orb} - (6 - 3\sqrt{2})\xi^3 \log \xi, \]  

(7)

\[ M\dot{\xi} = \xi^9 \sum_{n=0}^{N} b_n \xi^n, \]  

(8)

\[ \dot{\phi}_T = \frac{\cos \iota}{1 - \cos^2 \iota} \left( \hat{L} \times \hat{N} \right) \cdot \hat{L}, \]  

(9)

\[ M\dot{\hat{L}} = -\xi^6 (\Omega_1 + \Omega_2), \]  

(10)

\[ M\dot{\hat{a}}_1 = \mu_2 \xi^5 \Omega_1, \]  

(11)

\[ M\dot{\hat{a}}_2 = \mu_1 \xi^5 \Omega_2, \]  

(12)

where \( \xi = \mathcal{O}(v/c) \ll 1 \) is a PN expansion parameter and \( \phi_\text{orb} \) is the orbital phase. The coefficients \( b_n = \mathcal{O}(\xi^0) \) and \( \xi^9 = \mathcal{O}(\xi^8) \) are given in appendix A. The vectors \( \Omega_1 \) satisfy \( \hat{\Omega}_1 \perp \hat{a}_1 \) and \( (\Omega_1 + \Omega_2) \perp \hat{L} \). From these equations, one can see that there are three separate timescales: the orbital timescale \( T_\text{orb} = \mathcal{O}(\xi^{-3}) \) defined through Eq. (6), the radiation-reaction timescale \( T_{\text{r.r.}} = \mathcal{O}(\xi^{-5}) \), defined through Eq. (8), and the precession timescale \( T_\text{prec} = \mathcal{O}(\xi^{-5}) \), defined through Eqs. (10-12). We thus have a natural separation of scales, \( T_{\text{r.r.}} \gg T_{\text{prec}} \gg T_{\text{orb}} \), that is tailor-made for a multiple scale analysis treatment.

With this separation of scales in mind, we can rewrite the time-domain response as

\[ h(t) = \sum_{n,k,m} A_{n,k,m} e^{-i(n\phi_T + k\iota + m\psi_N)} e^{-in\phi_C} + c.c., \]  

(13)

where we note that \( \phi_C \) varies on the orbital timescale while \( \phi_T, \iota, \) and \( \psi_N \) vary on the precession timescale. Furthermore, we can note from Eq. (10) that it is natural to expand \( (\phi_C, \iota, \psi_N) \) in a Fourier series on the precession timescale, i.e. with amplitudes \( A_j \) and phases \( \beta_j \) that satisfy \( \beta_j = \mathcal{O}(\xi^5) \) and \( A_j = \mathcal{O}(\xi) \). We can then rewrite Eq. (13) as

\[ h(t) = \sum_{n,A} A_n A e^{-i\sum_j A_j \sin \beta_j} e^{-i\phi_C} + c.c., \]  

(14)

where \( A \) is an abstract amplitude vector that represents the dependence of each harmonic on the precession phases \( \beta_j \) and varies on the radiation-reaction timescale.

Separation of scales has revealed that the calculation of the Fourier transform of \( h(t) \) is equivalent to transforming another function \( H(t) \) defined via

\[ H(t) = H_0(t) e^{-i\delta\phi(t)}, \]  

(15)

with “background” response

\[ H_0(t) = A e^{-i\phi_C}, \]  

(16)

and a precession correction

\[ \delta\phi = \sum_j A_j \sin \beta_j, \]  

(17)

that satisfy

\[ \dot{\phi}_C \sim \mathcal{O}(\xi^3), \quad \dot{\beta}_j \sim \mathcal{O}(\xi^5), \]  

(18)

\[ A \sim \mathcal{O}(\xi^2), \quad A_j \sim \mathcal{O}(\xi), \]  

(19)

where \( A, A_j, \dot{\phi}_C, \) and \( \dot{\beta}_j \) all vary on the radiation reaction timescale. We will derive the SUA method using this toy problem first, and then generalize it to the actual GW response.
C. Bessel Expansion

We begin by expanding the precession correction in a Bessel series \[10\]:

\[
H(t) = \sum_{k \in \mathbb{Z}^n} \mathcal{A} \left[ \prod_m J_{k_m}(A_m) \right] e^{-i(n\dot{\phi}_C + \sum_j k_j \dot{\beta}_j(t))}.
\]

(20)

We can now Fourier transform \( H(t) \) using the stationary phase approximation because the second derivative of the argument of the imaginary exponential is always negative, and thus, it avoids catastrophes, i.e. times at which this second derivative would vanish. Note though that, in principle, the catastrophes are actually still there when one includes arbitrarily large values of \( k_j \), since then one could have \( \sum_j k_j \dot{\beta}_j \sim n\dot{\phi}_C \). For such large values of the sum index, however, the corresponding Bessel amplitudes suppress the divergences. Using the SPA, one finds

\[
\tilde{H}(f) \approx \sum_{k \in \mathbb{Z}^n} \mathcal{A}(t_k) \left[ \prod_m J_{k_m}[A_m(t_k)] \right] e^{2\pi i \frac{t_k}{n\dot{\phi}_C(t_k)} + \sum_j k_j \dot{\beta}_j(t_k)} e^{i[2\pi ft_k - n\dot{\phi}_C(t_k) - \sum_j k_j \dot{\beta}_j(t_k) - \pi/4]},
\]

(21)

where the stationary points \( t_k \) are defined via the condition

\[
2\pi f = n\dot{\phi}_C(t_k) + \sum_j k_j \dot{\beta}_j(t_k).
\]

(22)

Let us now re-express the full Fourier transform in a product decomposition of the form \( \tilde{H}(f) = \tilde{H}_0(f) \tilde{H}_{\text{corr}}(f) \), where \( \tilde{H}_0(f) \) is the Fourier transform of the background response

\[
\tilde{H}_0(f) \approx \mathcal{A}(t_0) \sqrt{2\pi \frac{1}{n\dot{\phi}_C(t_0)}} e^{i[2\pi ft_0 - n\dot{\phi}_C(t_0) - \pi/4]},
\]

(23)

again computed in the SPA because \( \ddot{\phi}_C > 0 \), where \( t_0 \) is the stationary point defined through the condition \( 2\pi f = n\dot{\phi}_C(t_0) \). Note that \( \tilde{H}_{\text{corr}}(f) \) is not necessarily the Fourier transform of a known time-domain signal, and carries a tilde only to stress that it is a Fourier-domain quantity.

To do so, we keep only leading PN order terms in the amplitude, and neglect any factors of \( \mathcal{O}(\xi) \) or higher in the phase. First, we Taylor expand the stationary phase conditions to obtain

\[
t_k = t_0 + \Delta t_k,
\]

(24)

\[
\Delta t_k = - \frac{1}{n\dot{\phi}_C(t_0)} \sum_j k_j \dot{\beta}_j(t_0) + \mathcal{O}(\xi^{-4}).
\]

(25)

Expanding the amplitude of Eq. (21) to leading PN order, we then find

\[
\tilde{H}(f) \approx \mathcal{A}(t_0) \sqrt{2\pi \frac{1}{n\dot{\phi}_C(t_0)}} \sum_{k \in \mathbb{Z}^n} \left[ \prod_m J_{k_m}[A_m(t_0)] \right] e^{i\Psi_k},
\]

(26)

where the Fourier phases \( \Psi_k \) can be expanded using Eq. (21) to find

\[
\Psi_k = 2\pi ft_0 - n\dot{\phi}_C(t_0) - \frac{\pi}{4} - \sum_j k_j \dot{\beta}_j(t_0)
\]

\[
+ \frac{1}{2n\dot{\phi}_C(t_0)} \left[ \sum_j k_j \dot{\beta}_j(t_0) \right]^2 + \mathcal{O}(\xi).
\]

(27)

Combining these results, the precession correction to the Fourier transform is them simply

\[
\tilde{H}_{\text{corr}}(f) = \sum_{k \in \mathbb{Z}^n} \left[ \prod_m J_{k_m}[A_m(t_0)] \right] e^{i\Delta \Psi_k},
\]

(28)

with the precession phase correction

\[
\Delta \Psi_k = - \sum_j k_j \dot{\beta}_j(t_0) + \frac{1}{2} T^2 \left[ \sum_j k_j \dot{\beta}_j(t_0) \right]^2,
\]

(29)

with the new timescale \( T = [n\ddot{\phi}_C(t_0)]^{-1/2} \).

D. Bessel Resummation

The product decomposed result obtained above is similar to that of \[11\], but inefficient in practice because of the large number of terms that must be kept in the Bessel expansion. However, all of these Bessel terms can be resummed. To do so, let us first note that

\[
\partial_t (A_j \sin \beta_j) = \dot{\beta}_j A_j \left[ \cos \beta_j + \mathcal{O}(\xi^3) \right],
\]

(30)

which then implies

\[
\partial_t^q [f(\delta \phi)] = \left[ 1 + \mathcal{O}(\xi^3) \right] \left[ \sum_j \dot{\beta}_j \partial \beta_j \right]^q f(\delta \phi).
\]

(31)

This relation allows us to simplify the precession correction of the Fourier transform to
where in the first equality we have Taylor expanded the exponential produced by the second term in Eq. (24), in the second equality the $k_j$ is replaced by the $\partial_j$ derivative acting on $\exp(-i \sum_j k_j \beta_j)$, in the third equality we resummed the Bessel expansion of $\exp(-i \delta \phi)$ and in the last equality we used Eq. (24).

With this at hand, we can now use the shift relation $F(x + h) = e^{h \hat{H}} F(x)$ in the form

$$ f(t_0 + kT) = \sum_{p \geq 0} \frac{(iT^2)^p}{2^p p!} \partial_t^p f(t_0), \tag{35} $$

which is the definition of a Taylor expansion. This expression allows us to write the following double shift relation

$$ e^{-i \delta \phi(t_0 + kT)} + e^{-i \delta \phi(t_0 - kT)} = 2 \sum_{p \geq 0} \frac{(kT)^{2p}}{(2p)!} \partial_t^{2p} e^{-i \delta \phi(t_0)}, \tag{36} $$

which we can use to rewrite the precession correction of the Fourier transform as

$$ \hat{H}_{corr}(f) \approx \sum_{p=0}^{k_{\text{max}}} \frac{(-iT^2)^p}{2^p p!} \partial_t^{2p} e^{-i \delta \phi(t_0)} \tag{37} $$

$$ = \sum_{k=0}^{k_{\text{max}}} \sum_{p=0}^{k_{\text{max}}} a_k a_{k_{\text{max}}} \frac{k^{2p}}{(2p)!} \partial_t^{2p} e^{-i \delta \phi(t_0)} \tag{38} $$

$$ \approx \frac{1}{2} \sum_{k=0}^{k_{\text{max}}} a_k a_{k_{\text{max}}} \left[ e^{-i \delta \phi(t_0 + kT)} + e^{-i \delta \phi(t_0 - kT)} \right], \tag{39} $$

where the $a_k a_{k_{\text{max}}}$ are constant coefficients. The first equality is simply Eq. (34) truncated at order $k_{\text{max}}$. The second equality is true provided that the coefficients $a_k a_{k_{\text{max}}}$ satisfy

$$ \frac{(-iT^2)^p}{2^p p!} \sum_{k=0}^{k_{\text{max}}} a_k a_{k_{\text{max}}} \frac{k^{2p}}{(2p)!}, \tag{40} $$

with $p \in \{0, \ldots, k_{\text{max}}\}$. This is an easily solvable linear system of $k_{\text{max}} + 1$ equations for $k_{\text{max}} + 1$ variables. The third equality Eq. (39), is established through Eq. (30) truncated at order $k_{\text{max}}$.

Another way of deriving Eqs. (39) and (40) from Eq. (44) can shed some light on the meaning of the constants $a_k a_{k_{\text{max}}}$ and is presented in appendix B.

### E. The SUA Fourier Response

We can now go back to $h(t)$ and compute its Fourier transform $\hat{h}(f)$. We can rewrite Eq. (2) as

$$ h(t) = \sum_n A_n(t) e^{-i n \phi C} + \text{c.c.}, \tag{41} $$

with the precession-dependent amplitude

$$ A_n(t) = e^{-i n \phi C(t)} \left\{ F_+(t) A_+(n) \left| \psi(t) \right\rangle + F_\times(t) A_\times(n) \left| \psi(t) \right\rangle \right\}. \tag{42} $$

By analogy with the toy problem, the Fourier transform is then approximately

$$ \hat{h}(f) \approx \sum_n \sqrt{\frac{2 \pi}{n \phi C(t_0,n)}} \left\{ \sum_{k=0}^{k_{\text{max}}} a_k a_{k_{\text{max}}} \left[ A_n(t_{0,n} + kT_n) + A_n(t_{0,n} - kT_n) \right] e^{i(2\pi ft_{0,n} - n \phi C(t_{0,n}) - \pi/4)} \right\}, \tag{43} $$

where the stationary points are simply given by $2\pi f = n \phi C(t_0,n)$, and $T_n = (n \phi C(t_0,n))^{-1/2}$, with constants $a_k a_{k_{\text{max}}}$ that satisfy the linear system of equations defined by Eq. (10) with $p \in \{0,1,\ldots,k_{\text{max}}\}$. In the above expression, we replaced time derivatives of $\phi C$ by time derivatives of $\phi C$, because $\phi C - \dot{\phi C} = O(\xi^3) \ll \dot{\phi C} = O(\xi^3)$, as established from Eq. (7).

The Fourier transform of any time-domain GW response function is given by Eq. (13), which only requires knowledge of $\xi(t)$, $\phi C(t)$, and $\dot{L}(t)$. Notice that one does not need to specify whether the latter are derived analytically or numerically, or the waveform in a particular frame, or a particular approximant when solving the equations of motion. The SUA Fourier response only requires that timescales separate, a valid assumption that breaks down only close to plunge.
Interestingly, if we retain only the lowest order solution in Eq. (43) with $k_{\text{max}} = 0$, then $a_0 = 1$ and one recovers a waveform similar to the one proposed by Lang and Hughes in [3]. Their waveform is in fact the $k_{\text{max}} = 0$ SUA waveform, with (i) $\hat{L}(t)$ computed by numerically integrating Eqs. (10-12), (ii) the relations $t(f)$ and $\phi_{\text{orb}}(f)$ computed by analytically integrating Eq. (45) through a PN expansion, (iii) restricting the amplitudes $A^{(n)}_k$ to leading order, and (iv) replacing $\phi_C$ by $\phi_{\text{orb}}$ which is justified when one uses leading-order amplitudes. The calculation described above provides a mathematical justification for this prescription and it includes the necessary corrections to extend it. However, a crucial difference between it and the $k_{\text{max}} = 0$ solution that we used in the following section is that we used a numerical solution for the relations $t(f)$ and $\phi_{\text{orb}}(f)$ instead of an analytical one, which significantly improves the accuracy of the waveforms.

III. IMPLEMENTATION AND VALIDATION

In this section, we will use the SUA method to construct a particular example that we can then validate through certain data analysis measures. For this particular example, we will use the SpinTaylorT4 model to construct the time-domain evolution of the orbital angular momentum and phase, which will serve as input into the SUA method. We will then validate these templates against time-domain SpinTaylorT4 waveforms. The comparisons will be done through the faithfulness, one minus the unfaithfulness measure shown in the introduction. We stress however that the SUA method is generic; one can apply this method to any analytic or numeric evolution of the orbital angular momentum and phase.

A. Preliminaries

The data analysis measure we will use in this section to compare templates will be the faithfulness $F$ between waveforms $h_1$ and $h_2$, defined as

$$F(h_1, h_2) = \max_{\lambda_{\text{sp}}} \int_{f_{\text{min}}}^{f_{\text{max}}} \frac{\hat{h}_1(f)\hat{h}_2^*(f)}{S_n(f)} df,$$

where $S_n(f)$ is the noise spectral density of the detector we are considering, $f_{\text{min}}$ and $f_{\text{max}}$ are detector dependent frequency cutoffs, and the normalized waveform $\hat{h}_A(f)$ is

$$\hat{h}_A(f) = \left[\int_{f_{\text{min}}}^{f_{\text{max}}} \frac{|\hat{h}_A(f)|^2}{S_n(f)} df\right]^{-1/2} \hat{h}_A(f),$$

for $A = 1$ or 2 and where $\hat{h}_A(f)$ is the Fourier transform of $h_A(t)$. For concreteness, we here focus on aLIGO, with $S_n$ given by an analytical fit found in [32], and $(f_{\text{min}}, f_{\text{max}}) = (10, 10^4)$ Hz.

The faithfulness is maximized over all unphysical parameters $\lambda_{\text{sp}}$. For SPA frequency-domain waveforms, these parameters correspond to a global time shift and a global orbital phase shift, which appear as constants of integration when computing the Fourier phase. Maximization over these parameters is necessary when comparing models for which the frequency and phase evolutions $\omega_{\text{orb}}(t)$ and $\phi_{\text{orb}}(t)$ differ. For the SUA templates and the seed time-domain waveform, these two parameters correspond to the initial conditions of the evolution equations; therefore, by construction, the same parameters lead to the same evolution in both and one does not need to maximize the faithfulness with respect to them.

We chose to use the faithfulness instead of the fitting factor, defined similarly but maximized over all parameters, because our waveforms are designed for fast parameter estimation. The fitting factor is a better suited measure for waveforms aimed at detection, since in that case one does not worry about parameter biases. Notice, of course, that the faithfulness between two identical templates is simply unity, i.e. $F(h_1, h_1) = 1 = F(h_2, h_2)$, while the unfaithfulness used in the introduction is simply defined as $U = 1 - F$.

An often cited bound for the fitting factor or similar measures like the faithfulness is 0.97. The origin of it is that the average fitting factor for a waveform needs to be $0.91/3 ≈ 0.965$ for it to recover 90% of the total number of signals in an experiment [33, 34]. However, the requirements on the faithfulness for parameter estimation (PE) studies are different, and depend on the signal-to-noise ratio (SNR). A sensible requirement for a PE study is to ask that the systematic (or mismodelling) error coming from using an approximate waveform is lower than the statistical error coming from the noise in the data. While the former is SNR-independent, the latter does depend on the SNR, and therefore the faithfulness requirement on a waveform for a PE study is SNR-dependent: the faithfulness requirement scales like the inverse SNR $1 - F_{\text{req}} \sim 1/$SNR.

The calculation of the faithfulness requires the choice of at least two templates. One of them will always be the time-domain SpinTaylorT4 waveform. The other will be one of the following

(i) the SpinTaylorT4-Fourier SUA, frequency-domain templates computed with the SpinTaylorT4 numerical evolution for the orbital angular momentum and phase for values of $k_{\text{max}}$ equal to every integer between 0 and 10;

(ii) the small-spins, double precessing waveforms (DP) [13], computed by solving the precession equations analytically assuming the individual dimensionless spin magnitudes are much smaller than unity, i.e. $\chi_A \ll 1$;

(iii) the single-spin, simple precession waveforms (SP), also known as the PhenomP model [18], constructed by assuming only one object is spinning and the other is not.
The PN order to which each of these templates is valid is a tricky issue. The time-domain SpinTaylorT4 waveform, in principle, contains all valid terms only up to 2PN order; point-mass terms and spin-orbit terms are known to higher PN order (see e.g. [31, 32, 33]), but not all of them are used in the SpinTaylorT4 model. In the DP and SP templates, we choose to limit the accuracy of the PN Fourier phase to 3.5PN order. Although in principle some higher PN order terms could be kept, they would not be consistent with full General Relativity beyond 3.5PN.

The waveform templates can differ also in the number of terms kept in the wave amplitude. The restricted PN approximation, or restricted waveforms (RWF) for short, consists of keeping only the leading, PN order term in the wave amplitude, i.e. the $n = 2$ harmonic in a multipolar decomposition for a quasi-circular inspiral, without PN corrections to it. Full waveforms (FWF) consists of keeping as many PN corrections in the wave amplitude as possible. In what follows, the time-domain SpinTaylorT4 waveforms will include these higher PN order harmonics, while the SP model will keep only the leading PN order term in the amplitude. The SUA and DP models will be studied both with RWFs and FWFs.

When comparing frequency-domain waveforms to time-domain ones that end abruptly, one needs to pay attention to spectral leakage. Time-domain waveforms ending abruptly are effectively multiplied by a Heaviside function, and thus their Fourier transform will be convoluted with the Fourier transform of it. This results in unwanted oscillations near the beginning and the end of the signal. To reduce that effect, we use a Tukey window on the signal, defined by

$$W(t) = \begin{cases} 0, & t < t_1 \\ \sin^2 \left( \frac{t - t_1}{t_2 - t_1} \right), & t_1 \leq t < t_2 \\ 1, & t_2 \leq t < t_3 \\ \sin^2 \left( \frac{t - t_3}{t_4 - t_3} \right), & t_3 \leq t < t_4 \\ 0, & t_4 \leq t \end{cases} \quad (46)$$

This will reduce leakage but will also reduce the available power. A good compromise for $t_1, \ldots, t_4$ is for $\sim 10$ cycles to occur between $t_1$ and $t_2$, and as many between $t_3$ and $t_4$. If that condition is satisfied, then the window is a slowly varying function from the point of view of the SPA, and can therefore easily be taken into account in the frequency-domain waveforms. We chose $t_1$ and $t_2$ to be too small to have an influence on the frequency-domain waveforms, i.e. $n_{\text{max}} \phi_{\text{orb}}(t_1, t_2) < 10 \text{ Hz}$, where $n_{\text{max}}$ is the highest harmonic number. We chose $t_3$ and $t_4$ so that $\xi(t_3) = 15^{-1/2}$ and $\xi(t_4) = 6^{-1/2}$. The frequency-domain waveforms also need a $t(f)$ relation in order to use the window. We chose to use the post-Newtonian relations $t_n(f)$ to an order consistent with the phase where $t_n(f)$ is the time at which harmonic $n$ emits radiation at frequency $f$, so that each harmonic is multiplied by $W[t_n(f)]$.

The calculation of the faithfulness also requires a choice of systems to study. We here focus on the following:

(a) Highly spinning neutron star-neutron star systems (HSNSNS);
(b) Realistically spinning neutron star-neutron star systems (RSNSNS);
(c) Highly spinning black hole-neutron star systems (HSBHNS),
(d) Realistically spinning black hole-neutron star systems (RSBHNS),
(e) Spinning black hole-black hole systems (BHBH).

These physical systems are defined by the minimum and maximum value of the masses and spins, which we collect in Table I.

| Type       | $m_1_{\text{min}}$ | $m_1_{\text{max}}$ | $m_2_{\text{min}}$ | $m_2_{\text{max}}$ | $\chi_1_{\text{min}}$ | $\chi_1_{\text{max}}$ | $\chi_2_{\text{min}}$ | $\chi_2_{\text{max}}$ |
|------------|---------------------|---------------------|--------------------|--------------------|------------------------|------------------------|------------------------|------------------------|
| HSNSNS     | 1                   | 2.5                 | 1                  | 2.5                | 0                      | 1                      | 0                      | 1                      |
| RSNSNS     | 1                   | 2.5                 | 1                  | 2.5                | 0.1                    | 0                      | 0.1                    | 0                      |
| HSBHNS     | 5                   | 20                  | 1                  | 2.5                | 0                      | 1                      | 0                      | 1                      |
| RSBHNS     | 5                   | 20                  | 1                  | 2.5                | 0                      | 1                      | 0.1                    | 0                      |
| BHBH       | 5                   | 20                  | 5                  | 20                 | 0                      | 1                      | 0                      | 1                      |

TABLE I: Minimum and maximum values of the masses and spin magnitudes for each system type in our simulations.

For each system type, we randomized over $10^4$ systems with the following distributions:

- the masses $m_i$ are uniformly distributed in log space between a maximum and a minimum value;
- the spin magnitudes $\chi_i$ are uniformly distributed between a maximum and a minimum value;
- all angles are uniformly distributed on the sphere.

We did not randomize over distance, as it is factored out in the faithfulness.

### B. Results

Let us first discuss the computational efficiency of our new model. This is difficult because it is highly dependent on technical factors, such as the duration of the time signal, the Nyquist frequency of the time-domain waveform, the discretization of the frequency series, the choice of $k_{\text{max}}$, etc. The efficiency is also significantly dependent on the physical parameters of the system considered, such as the spin magnitudes, the spin orientation, and the mass ratio. To give a feeling of the computational efficiency of our waveform, we have collected
in Table [I] the average computation time for our waveforms and for the corresponding time-domain waveform, computed from a simulation of 1000 systems of each type described above on a modern computer. We have chosen to start our waveform generation at 10 Hz and stop it at the Schwarzschild ISCO, i.e. when the PN parameter $\xi$ defined in Eq. (3) reaches $6^{-1/2}$. We used a frequency resolution of 0.1 Hz for the SUA waveforms and a Nyquist frequency of 3 times the orbital frequency at the Schwarzschild ISCO for the time-domain waveforms. We did not use any filter on the time series for these efficiency comparisons, and used for each waveform presented the code available in lalsimulation. Note that time-domain waveforms are faster to compute for black hole-black hole binaries than for neutron star-neutron star ones, so much so that they become prohibitively expensive for the latter, which is precisely the system type for which the inspiral part is the most important. On the other hand, the SUA waveforms remain relatively cheap for all systems, with an up to 2 orders of magnitude improvement in computational expense relative to time-domain waveforms for neutron star binaries.

As a reference, we also ran the same efficiency comparisons for non-precessing systems, to be able to compare our model with existing spin-aligned waveforms. As an example we compared our code to the SpinTaylorF2 model, fixing the spin magnitudes to zero. As long as the spins are aligned the effect of the spin magnitudes on the computation time is negligible for all waveforms we compared here. We collect the results in Table [III]. Note that we included the computation times for $k_{\text{max}} > 0$ to give a more complete description of the computational efficiency of our model, even though for non-precessing systems the SUA waveform is independent on $k_{\text{max}}$. Notice that the computation times are much smaller than in Table [I]. The reason for that is that when the spins are aligned or zero, the precession timescale disappears from the problem, and the relevant timescale for the numerical integration of the equations of motion becomes the radiation reaction timescale, which is much longer. This renders not only the solving of the equation of motion faster, but also the construction and the evaluation of the necessary interpolation functions constructed from the solution. Note that for neutron star-neutron star systems, the $k_{\text{max}} = 0$ SpinTaylorT4Fourier waveforms are slower than the SpinTaylorF2 waveforms by only a factor 3-9, depending on whether we include subdominant harmonics or not. For higher-mass systems which are not inspiral dominated, this ratio can go up to a factor of about 20.

Let us now discuss the faithfulness between the different models listed in the previous subsection and the time-domain SpinTaylorT4 waveforms. Table [IV] shows the median, lower 1-$\sigma$ and upper 1-$\sigma$ quantiles of the distributions of the faithfulness between the latter waveforms and the SUA model, the DP [13] and the SP models [18]. In all calculations, the physical parameters are kept fixed, as described in the previous subsection. Overall, we find that the SUA method offers unprecedented levels of accuracy, and can reproduce the results of much more costly time-domain models with mismatches (or unfaithfulness $U = 1 - F$) smaller than $10^{-4}$, and typically of the order of $10^{-5}$.

Let us discuss Table [IV] in more detail. First, focus on the first three rows only and observe that the faithfulness distributions for restricted waveform models do not change much between $k_{\text{max}} > 0$ and $k_{\text{max}} = 0$. This implies that the inaccuracies coming from restricting the sum over $k$ in Eq. (43) to only $k = 0$ are comparable or smaller than the inaccuracies coming from neglecting PN amplitude corrections.

Second, now concentrating on the first six rows, observe that the faithfulnesses for the restricted waveform models are worse for black hole-neutron star systems than for other system types, while this is not true for the full waveform models. This is because the amplitude of the subdominant harmonics appearing at next-to-leading PN order, $n = 1$ and $n = 3$, have an overall $(m_1 - m_2)/M$ factor. These corrections are suppressed for systems with similar masses, but are important for black hole-neutron star systems, which typically have higher mass ratios.

Third, comparing the fourth to sixth rows, observe that the faithfulnesses for the full waveform models with $k_{\text{max}} = 0$ are significantly smaller than those for the same models with $k_{\text{max}} = 3$. This is much less so when one compares the $k_{\text{max}} = 3$ results to the $k_{\text{max}} = 10$ ones. Analyzing the data from our simulations for all $0 \leq k_{\text{max}} \leq 10$, we find that the faithfulness increase with increasing $k_{\text{max}}$ starts slowing significantly at about $k_{\text{max}} = 2$ or 3. This means that the inaccuracies coming from restricting the sum over $k$ in Eq. (43) to $k = k_{\text{max}}$ dominate over other sources of error for $k_{\text{max}} \leq 2$ or 3.

Fourth, let us finally compare the first six rows to the last three and observe that the faithfulness distributions are comparable for the DP and SP models. This implies that the main source of inaccuracy is here in the discrepancy between the TaylorT4 phase and the TaylorF2 phase, which is common to the DP and SP models, and has been shown to be substantial [22, 28]. Unlike for the SUA model, including subdominant harmonics does not here appear to yield a significant increase in faithfulness. Furthermore, the only system type for which these models offer sufficiently high faithfulnesses is the realistically spinning neutron star-neutron star type, for which both spin magnitudes are smaller than 0.1 and precession effects are thus suppressed.

Let us now try to determine in more detail the rea-
TABLE II: Average computation times for the generation of a (left) restricted and (right) full waveform of each system type. TD corresponds to the time-domain SpinTaylorT4 waveform, and the other three are SUA SpinTaylorT4Fourier waveforms. The Nyquist frequency of the time-domain waveforms is three times the ISCO orbital frequency, and the resolution of the frequency-domain waveforms is 0.1 Hz.

| System  | TD  | $k_{\text{max}} = 0$ | $k_{\text{max}} = 3$ | $k_{\text{max}} = 10$ |
|---------|-----|---------------------|---------------------|---------------------|
| HSNNS  | 2.08 s | 38.4 ms | 68.1 ms | 104 ms |
| RSNSNS | 1.96 s | 29.2 ms | 38.1 ms | 52.7 ms |
| HSBHNS | 146 ms | 18.4 ms | 23.3 ms | 33.4 ms |
| RSBHNS | 149 ms | 15.8 ms | 21 ms | 30.2 ms |
| BHBH   | 25 ms | 8.58 ms | 11.6 ms | 17 ms |

TABLE III: Average computation times for the generation of a (left) restricted and (right) full waveform of each system type with zero spins. TD corresponds to the time-domain SpinTaylorT4 waveform, F2 corresponds to the spin-aligned SpinTaylorF2 waveform, and the other three are SUA SpinTaylorT4Fourier waveforms. The Nyquist frequency of the time-domain waveforms is three times the ISCO orbital frequency, and the resolution of the frequency-domain waveforms is 0.1 Hz.

| System  | TD  | $F_2$  | $k_{\text{max}} = 0$ | $k_{\text{max}} = 3$ | $k_{\text{max}} = 10$ |
|---------|-----|-------|---------------------|---------------------|---------------------|
| NSNS    | 2.18 s | 2.11 ms | 7.57 ms | 11.3 ms | 19.1 ms |
| BHNS    | 118 ms | 0.646 ms | 4.65 ms | 5.12 ms | 6.44 ms |
| BHBH    | 18.9 ms | 0.326 ms | 4.25 ms | 4.69 ms | 5.65 ms |

| System  | TD  | $F_2$  | $k_{\text{max}} = 0$ | $k_{\text{max}} = 3$ | $k_{\text{max}} = 10$ |
|---------|-----|-------|---------------------|---------------------|---------------------|
| NSNS    | 2.58 s | 2.01 ms | 16.5 ms | 15.6 ms | 71.7 ms |
| BHNS    | 196 ms | 0.591 ms | 6.81 ms | 10.3 ms | 18.3 ms |
| BHBH    | 25.5 ms | 0.328 ms | 6.78 ms | 10.2 ms | 17.1 ms |

IV. CONCLUSION

We have presented here a new Shifted Uniform Asymptotic method for constructing the Fourier transform of the GW response function of an inspiraling binary system. This method requires as input only the time evolution of the orbital angular momentum and of the orbital phase and frequency, which can be provided either analytically or numerically. The output of the method, the Fourier transform of the GW response, is highly accurate when compared to waveforms obtained through discrete Fourier transforms of time-domain signals, even for gravitational waves generated in generic spin precessing inspirals. This method avoids the catastrophes that lead to singularities in the Fourier amplitude obtained with a simple stationary phase approximation. Moreover, the method is computationally efficient, allowing for the construction of frequency-domain templates 50 times faster than time-domain waveforms.

We then provided an example of this method by applying the SUA method to a particular model for the time evolution of the orbital angular momentum, the orbital phase and the frequency, the SpinTaylorT4 model. We then compared the resulting Fourier transform to the corresponding time-domain waveforms produced by the discrete Fourier transform of a time signal through the faithfulness measure. We found unfaithfulnesses of order $10^{-5}$, a 4 orders of magnitude improvement over other frequency-domain templates currently used in GW data analysis.

We expect the SUA method to be very useful in GW astronomy, as it is a highly computationally efficient and generic method that can be applied to any model
|                | HSNSNS | RSNSNS | HSBHNS | RSBHNS | BHBH |
|----------------|--------|--------|--------|--------|------|
| \(k_{\text{max}} = 0\), RWF | 16%    | 50%    | 84%    | 16%    | 50%  |
| \(k_{\text{max}} = 3\), RWF | 2.47   | 2.98   | 3.53   | 2.77   | 3.36 |
| \(k_{\text{max}} = 10\), RWF | 2.79   | 3.37   | 3.86   | 2.78   | 3.39 |
| \(k_{\text{max}} = 0\), FWF | 2.72   | 3.48   | 4.22   | 4.5    | 4.92 |
| \(k_{\text{max}} = 3\), FWF | 5.16   | 5.8    | 6.27   | 5.96   | 6.29 |
| \(k_{\text{max}} = 10\), FWF | 5.29   | 5.97   | 6.43   | 6.18   | 6.47 |
| small spins, RWF | 0.916  | 1.63   | 2.36   | 2.5    | 2.75 |
| small spins, FWF | 0.913  | 1.62   | 2.34   | 2.28   | 2.83 |
| single spin, RWF | 0.765  | 1.44   | 2.28   | 2.48   | 2.88 |

TABLE IV: Lower 1-\(\sigma\) (16%), median (50%), and upper 1-\(\sigma\) (84%) quantiles of the faithfulness distributions between the time-domain SpinTaylorT4 waveforms and (i) the SUA waveforms with a SpinTaylorT4 model for the evolution of the orbital angular momentum and phase with different values of \(k_{\text{max}}\) and either with the RWF or the FWF model (first 6 rows); (ii) the small-spin DP model with either the RWF or the FWF model (seventh and eighth rows); and (iii) the single-spin SP model with the RWF model (ninth row). All numbers quoted are \(-\log_{10}(1 - F)\), where \(F\) is the faithfulness, e.g., a value of 4 corresponds to \(F = 1 - 10^{-4} = 0.9999\). We put the 16% quantiles of the FWF SUA results in boldface, as they provide a worst-case scenario estimate for the faithfulness of our model.

FIG. 2: Dephasing in radians as a function of GW frequency between the time-domain SpinTaylorT4 waveforms and the SUA waveforms (RWF on the left and FWF on the right) with different values of \(k_{\text{max}}\) (\(k_{\text{max}} = 0\) in dashed green, \(k_{\text{max}} = 3\) in dotted red, and \(k_{\text{max}} = 10\) in solid blue). This figure uses the HSNSNS system with \((m_1, m_2) = (2, 1.4)\,M_\odot\) and \(\chi_1 = 0.9 = \chi_2\) with random orientations. The unfaithfulness \(U = 1 - F\) for these systems is \(10^{-2.49}\), \(10^{-2.93}\), and \(10^{-2.93}\) for \(k_{\text{max}} = 0, 3,\) and 10 respectively all with the RWF model, while it is equal to \(10^{-2.67}, 10^{-4.92},\) and \(10^{-5.19}\) for the FWF model.

in the time domain. SUA waveforms are faithful to full time-domain waveforms obtained through discrete Fourier transforms, and thus, they allow for small parameter biases induced by template mismodeling. Moreover, their computational efficiency should allow for large-scale parameter estimation studies with advanced, ground detectors.

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Appendix A: Equations of motion

For completeness, we here give the full expression for the equations of motion in the TaylorT4 form, in terms of the dimensionless individual masses \(\mu_1 = m_1/M\), and \(\mu_2 = m_2/M\) and the dimensionless spins \(a_1 = S_1/m_1 M\) and \(a_2 = S_2/m_2 M\).

The 3.5PN radiation-reaction equation with 2PN spin-
spin coupling is given by \[26, 29\]

\[
M_i = \xi^0 \sum_{n=0}^{7} b_n \xi^n,
\]

where \(\xi = \sqrt{\frac{11}{2}}\).

\[
b_0 = \frac{32\nu}{5}, \quad b_1 = 4\pi - \beta_3, \quad b_2 = -\frac{743}{336} - \frac{11}{4} \nu, \quad b_3 = \frac{34103}{18144} + \frac{31319}{2016} \nu + \frac{59}{18} \nu^2 - \sigma_4,
\]

\[
b_4 = \frac{-\pi}{b_0} \left( \frac{4159}{672} + \frac{189}{8} \nu \right) - \beta_5,
\]

\[
b_5 = \frac{1644732263}{139708800} - \frac{56198689}{217728} \nu + \frac{541}{896} \nu^2 - \frac{5605}{2592} \nu^3 + \frac{16}{3} \nu - \frac{451}{48} \nu^2 \right) - \beta_7,
\]

where \(\gamma_E\) is the Euler constant, and the spin-orbit couplings \(\beta_1\) and the spin-spin coupling \(\sigma_4\) are given by

\[
\beta_3 = \sum_{A \neq B} \left( \frac{113}{12} \mu_A + \frac{25}{4} \mu_B \right) \hat{L} \cdot a_A,
\]

\[
\beta_5 = \sum_{A \neq B} \left( \frac{31319}{1008} - \frac{1159}{24} \nu \right) \mu_A + \left( \frac{809}{84} - \frac{281}{8} \right) \mu_B \right) \hat{L} \cdot a_A,
\]

\[
\beta_6 = \pi \sum_{A \neq B} \left( \frac{75}{2} \mu_A + \frac{151}{6} \mu_B \right) \hat{L} \cdot a_A,
\]

\[
\beta_7 = \sum_{A \neq B} \left[ \left( \frac{130325}{756} - \frac{796069}{2016} \nu + \frac{100019}{864} \nu^2 \right) \mu_A + \left( \frac{1195759}{18144} - \frac{257023}{1008} \nu + \frac{2903}{32} \nu^2 \right) \mu_B \right] \hat{L} \cdot a_A,
\]

\[
\sigma_4 = \frac{247}{96} (a_1 + a_2)^2 - \frac{721}{96} \left( \hat{L} \cdot (a_1 + a_2) \right)^2 - \sum_A \left( \frac{7}{48} a_A^2 - \frac{1}{48} \left( \hat{L} \cdot a_A \right)^2 \right),
\]

The full 2PN spin-spin and 3.5 PN spin-orbit precession equations are given by \[30, 31\]

\[
M_i \dot{\Omega} = -\xi^6 (\Omega_1 + \Omega_2),
\]

\[
M_i \dot{\Omega}_1 = \mu_2 \xi^5 \Omega_1,
\]

\[
M_i \dot{\Omega}_2 = \mu_1 \xi^5 \Omega_2,
\]

\[
\Omega_A = (C_{A,0} + C_{A,2} \xi^2 + C_{A,4} \xi^4 + D_A \xi) \hat{L} \times a_A,
\]

\[
C_{A,0} = 2 \mu_A + \frac{3}{2} \mu_B,
\]

\[
C_{A,2} = 3 \mu_A^2 + \frac{35}{6} \mu_A \mu_B + 4 \mu_A \mu_B^2 + \frac{9}{8} \mu_B^3,
\]

\[
C_{A,4} = \frac{27}{4} \mu_A^2 + \frac{3}{2} \mu_A \mu_B + \frac{137}{12} \mu_A^2 \mu_B + \frac{19}{4} \mu_A \mu_B^3 + \frac{27}{16} \mu_B^5,
\]

\[
D_A = \frac{3}{2} \hat{L} \cdot a_B,
\]

where it is understood that \(A, B \in \{1, 2\}\) and \(A \neq B\).

### Appendix B: Alternative derivation of the SUA constants

A \((2k_{\text{max}} + 1)\)-point stencil of the point \(t_0\) is the collection of points

\[S_{k_{\text{max}}} = \{t_0 - k_{\text{max}} T, t_0 - (k_{\text{max}} - 1) T, \ldots, t_0 + k_{\text{max}} T\}.\]

\[(B1)\]

\(S_{k_{\text{max}}}\) can be used to approximate the first \(2k_{\text{max}} + 1\) derivatives of a function \(f(t)\) at \(t = t_0\) (including the zeroth order derivative):

\[T^m f(t_0) \approx \sum_{k=-k_{\text{max}}}^{k_{\text{max}}} b_{k,m,k_{\text{max}}} f(t_0 + k T).\]

\[(B2)\]

To compute the coefficients \(b_{k,m,k_{\text{max}}}\), one needs to expand the functions \(f(t_0 + k T)\) on the right hand side of the equation above as a power series in \(T\) up to order \(2k_{\text{max}}\). Solving the equation order by order in \(T\), we get a unique solution for the \(b_{k,m,k_{\text{max}}}\). We can also extend this procedure to arbitrary high order derivatives, and we get \(b_{k,m,k_{\text{max}}} = 0\) when \(m > 2k_{\text{max}}\). Symmetry ensures that \(b_{k,m,k_{\text{max}}} = (-1)^m b_{-k,m,k_{\text{max}}}\). The system of equations defining the constants \(b_{k,m,k_{\text{max}}}\) is

\[\sum_{k=-k_{\text{max}}}^{k_{\text{max}}} \frac{k^n}{n!} b_{k,m,k_{\text{max}}} = \delta_{n,m}.\]

\[(B3)\]

With this in hand, we start from Eq. \[34\], and approximate each time derivative using the stencil \(S_{k_{\text{max}}}\):

\[\tilde{H}_{\text{corr}}(f) \approx \sum_{p=0}^{k_{\text{max}}} \frac{(-i)^p}{2p!} \sum_{k=-k_{\text{max}}}^{k_{\text{max}}} b_{k,2p,k_{\text{max}}} e^{-i \delta \phi(t_0 + k T)}
\]

\[= \sum_{k=0}^{k_{\text{max}}} \left( \sum_{p=0}^{k_{\text{max}}} \frac{(-i)^p}{2p!} b_{k,2p,k_{\text{max}}} \right) \left( e^{-i \delta \phi(t_0 + k T)} + e^{-i \delta \phi(t_0 - k T)} \right),\]

\[(B4)\]
where in the last equality we used the facts that all $b_{k,2p,k_{\text{max}}}$ vanish when $p > k_{\text{max}}$ and that all derivatives are of even order, and we reordered the sums. We can rewrite Eq. (B5) in a form similar to Eq. (B9), with the coefficients $c_{k,k_{\text{max}}}$ taking the place of the $a_{k,k_{\text{max}}}$, and defined by

$$c_{k,k_{\text{max}}} = 2\sum_{p=0}^{k_{\text{max}}} \frac{(-i)^p}{2^p p!} b_{k,2p,k_{\text{max}}}.$$  \hspace{1cm} (B6)

We can multiply by $k^{2n}/(2n)!$, and sum over $k$. We get

$$\sum_{k=0}^{k_{\text{max}}} c_{k,k_{\text{max}}} \frac{k^{2n}}{(2n)!} = \sum_{p=0}^{k_{\text{max}}} \frac{(-i)^p}{2^p p!} \times 2 \sum_{k=0}^{k_{\text{max}}} \frac{k^{2n}}{(2n)!} b_{k,2p,k_{\text{max}}} = \sum_{p=0}^{k_{\text{max}}} \frac{(-i)^p}{2^p p!} \delta_{n,p} = \frac{(-i)^n}{2^n n!}.$$  \hspace{1cm} (B7)

This system of equations accepting only one solution, we have $c_{k,k_{\text{max}}} = a_{k,k_{\text{max}}}$ for any $\{k, k_{\text{max}}\}$.

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