EQUIVARIANT MODELS OF SPHERICAL VARIETIES

MIKHAIL BOROVOI
WITH AN APPENDIX BY
GIULIANO GAGLIARDI

Abstract. Let $G$ be a connected semisimple group over an algebraically closed field $k$ of characteristic 0. Let $Y = G/H$ be a spherical homogeneous space of $G$, and let $Y'$ be a spherical embedding of $Y$. Let $k_0$ be a subfield of $k$. Let $G_0$ be a $k_0$-model ($k_0$-form) of $G$. We show that if $G_0$ is an inner form of a split group and if the subgroup $H$ of $G$ is spherically closed, then $Y$ admits a $G_0$-equivariant $k_0$-model. If we replace the assumption that $H$ is spherically closed by the stronger assumption that $H$ coincides with its normalizer in $G$, then both $Y$ and $Y'$ admit $G_0$-equivariant $k_0$-models, and these models are unique.

Contents

0. Introduction 2
1. Semi-morphisms of $k$-schemes 4
2. Semi-morphisms of $G$-varieties 6
3. Quotients 11
4. Semi-morphisms of homogeneous spaces 12
5. $k$-automorphisms of homogeneous spaces 14
6. Equivariant models of $G$-varieties 15
7. Spherical homogeneous spaces and their combinatorial invariants 17
8. Action of an automorphism of the base field on the combinatorial invariants of a spherical homogeneous space 21
9. Equivariant models of automorphism-free spherical homogeneous spaces 24
10. Equivariant models of spherically closed spherical homogeneous spaces 25
11. Equivariant models of spherical embeddings of automorphism-free spherical homogeneous spaces 31

Appendix A. Algebraically closed descent for spherical homogeneous spaces 32

Appendix B. The action of the automorphism group on the colors of a spherical homogeneous space 34

References 36
0. Introduction

Let $G$ be a connected semisimple group over an algebraically closed field $k$ of characteristic 0. Let $Y$ be a $G$-variety, that is, an (irreducible) algebraic variety over $k$ together with a morphism

$$\theta: G \times_k Y \to Y$$

defining an action of $G$ on $Y$. We say that $(Y, \theta)$ is a $G$-variety or just that $Y$ is a $G$-variety.

Let $k_0 \subset k$ be a subfield. Let $G_0$ be a $k_0$-model (or $k_0$-form) of $G$, that is, an algebraic group over $k_0$ together with an isomorphism of algebraic $k$-groups

$$\simeq_G: G_0 \times_{k_0} k \xrightarrow{\sim} G.$$

By a $G_0$-equivariant $k_0$-model of the $G$-variety $(Y, \theta)$ we mean a $G_0$-$k_0$-variety $(Y_0, \theta_0)$ together with an isomorphism $\simeq_Y: Y_0 \times_{k_0} k \xrightarrow{\sim} Y$ such that the diagram (20) commutes, see Section 4 below.

From now on till the end of the Introduction we assume that $Y$ is a spherical homogeneous space of $G$. This means that $Y = G/H$ (with the natural action of $G$) for some algebraic subgroup $H \subset G$ and that a Borel subgroup $B$ of $G$ has an open orbit in $Y$.

Let $Y \hookrightarrow Y'$ be a spherical embedding of $Y = G/H$. This means that $Y'$ is a $G$-variety, that $Y'$ contains $Y$, and that $Y'$ contains $Y$ as an open dense $G$-orbit. Then $B$ has an open dense orbit in $Y'$.

Inspired by the works of Akhiezer and Cupit-Foutou [ACF14], [Akh15], [CF15], for a given $k_0$-model $G_0$ of $G$ we ask whether there exist a $G_0$-equivariant $k_0$-model $Y_0$ of $Y$ and a $G_0$-equivariant $k_0$-model $Y'_0$ of $Y'$.

Since char $k = 0$, by a result of Alexeev and Brion [AB05, Theorem 3.1], see Knop’s MathOverflow answer [Kn17b] and Appendix A below, the spherical subgroup $H$ of $G$ is conjugate to some (spherical) subgroup defined over the algebraic closure of $k_0$ in $k$. Therefore, from now on we assume that $k$ is an algebraic closure of $k_0$. We set $\Gamma = \text{Gal}(k/k_0)$ (the Galois group of $k$ over $k_0$).

Let $T$ be a maximal torus of $G$ contained in a Borel subgroup $B$. We consider the Dynkin diagram $\text{Dyn}(G) = \text{Dyn}(G,T,B)$. The $k_0$-model $G_0$ of $G$ defines the so-called $*$-action of $\Gamma = \text{Gal}(k/k_0)$ on the Dynkin diagram $\text{Dyn}(G)$, see Tits [Tits66, Section 2.3, p.39]. In other words, we obtain a homomorphism

$$\varepsilon: \Gamma \to \text{Aut} \text{Dyn}(G).$$

The $k_0$-group $G_0$ is called an inner form (of a split group) if the $*$-action is trivial, that is, if $\varepsilon_\gamma = \text{id}$ for all $\gamma \in \Gamma$. For example, if $G$ is a simple group of any of the types $A_1$, $B_n$, $C_n$, $E_7$, $E_8$, $F_4$, $G_2$, then any $k_0$-model $G_0$ of $G$ is an inner form, because in these cases $\text{Dyn}(G)$ has no nontrivial automorphisms. If $G$ is a split $k$-group, then of course $G$ is an inner form.

Let $D(Y)$ denote the set of colors of $Y = G/H$, that is, the (finite) set of the closures of $B$-orbits of codimension one in $Y$. A spherical subgroup $H \subset G$ is called spherically closed if the automorphism group $\text{Aut}^G(Y) = \mathcal{N}_G(H)/H$ acts on $D = D(Y)$ faithfully, that is, if the homomorphism

$$\text{Aut}^G(Y) \to \text{Aut}(D)$$

is injective. Here $\mathcal{N}_G(H)$ denotes the normalizer of $H$ in $G$.

Example 0.1. Let $k = \mathbb{C}$, $G = \text{PGL}_2(\mathbb{C})$, $H = T$ (a maximal torus), $Y = G/T$. Then $|\mathcal{N}_G(T)/T| = 2$, and the spherical homogeneous space $Y$ of $G$ has exactly two colors, which are swapped by the non-unit element of $\mathcal{N}_G(T)/T$. We see that the subgroup $H = T$ of $G$ is spherically closed.
Theorem 0.2. Let $G$ be a connected semisimple group over an algebraically closed field $k$ of characteristic 0. Let $Y = G/H$ be a spherical homogeneous space of $G$. Let $k_0$ be a subfield of $k$ such that $k$ is an algebraic closure of $k_0$. Let $G_0$ be a $k_0$-model of $G$. Assume that:

(i) $G_0$ is an inner form,
(ii) $H$ is spherically closed.

Then $Y$ admits a $G_0$-equivariant $k_0$-model $Y_0$.

Theorem 0.2 is a special case of the more general Theorem 0.2 below, where instead of assuming that $G_0$ is an inner form, we assume only that for all $\gamma \in \Gamma$ the automorphism $\varepsilon_\gamma$ of $\text{Dyn}(G)$ preserves the combinatorial invariants (Luna-Losev invariants) of the spherical homogeneous space $Y$. Note that we have to assume that char $k = 0$ because in the proof we use Losev’s uniqueness theorem [Lo09, Theorem 1], which has been proved only in characteristic 0. Note also that $Y = G/H$ might have no $G_0$-equivariant $k_0$-models if $H$ is not spherically closed, see Example 0.1 below.

Theorem 0.2 was inspired by Theorem 1.1 of Akhiezer [Akh15] and by Corollary 1 of Cupit-Foutou [CF15, Section 2.5].

Note that the $G_0$-equivariant $k_0$-model $Y_0$ in Theorem 0.2 is in general not unique. The following theorem is a special case of the more general theorem 0.3 below.

Theorem 0.3. In Theorem 0.2 the set of isomorphism classes of $G_0$-equivariant $k_0$-models of $Y = G/H$ is naturally a principal homogeneous space of the abelian group

$$H^1(\Gamma, \text{Aut}^G(Y)) \simeq (\text{Hom}(\Gamma, S_2))^{\Omega(2)}.$$ 

Here $S_2$ is the symmetric group on two symbols (isomorphic to $\mathbb{Z}/2\mathbb{Z}$), $\Omega(2) = \Omega(2)(Y)$ is the finite set defined in Section 7 below (before Definition 7.3), and $(\cdot)^{\Omega(2)}$ denotes the group of maps from the set $\Omega(2)$ to the group in the parentheses.

In particular, for $k_0 = \mathbb{R}$ we have $\text{Hom}(\Gamma, S_2) = S_2$, and therefore, the number of these isomorphism classes is $2^s$, where $s = |\Omega(2)|$. For $G$ and $Y$ as in Example 0.1 we have $s = 1$, hence for each of the two $\mathbb{R}$-models of $G$ there are exactly two non-isomorphic equivariant $\mathbb{R}$-models of $Y$, see Example 0.1 below.

Corollary 0.4 (Akhiezer’s theorem). In Theorem 0.2 instead of (ii) assume that

(ii') $H$ is self-normalizing, that is, $N_G(H) = H$.

Then $Y = G/H$ admits a $G_0$-equivariant $k_0$-model $Y_0$, and this model is unique up to a unique isomorphism.

Indeed, since $H$ is self-normalizing, it is spherically closed. By Theorem 0.2 $Y$ admits a $G_0$-equivariant $k_0$-model. The uniqueness assertion is obvious because $\text{Aut}^G(Y) = \{1\}$.

Corollary 0.4 generalizes Theorem 1.1 of Akhiezer [Akh15], where the case $k_0 = \mathbb{R}$ was considered.

Theorem 0.5. Under the assumptions of Corollary 0.4, any spherical embedding $Y'$ of $Y = G/H$ admits a $G_0$-equivariant $k_0$-model $Y_0'$. This $k_0$-model $Y_0'$ is compatible with the unique $G_0$-equivariant $k_0$-model $Y_0$ of $Y$ from Corollary 0.4 and hence is unique up to a unique isomorphism.

Theorem 0.5 generalizes Theorem 1.2 of Akhiezer [Akh15], who proved in the case $k_0 = \mathbb{R}$ that the wonderful embedding of $Y$ admits a unique $G_0$-equivariant $\mathbb{R}$-model. Our proof of Theorem 0.5 uses results of Huruguen [Hu11]. Note that in Theorem 0.5 we do not assume that $Y'$ is quasi-projective.
Theorems 0.2, 0.3, and 0.5 seem to be new even in the case $k_0 = \mathbb{R}$.

The plan of the rest of the paper is as follows. In Sections 1–6 we consider semilinear morphisms and models for general $G$-varieties and homogeneous spaces of $G$, not necessarily spherical. In Sections 7–8 we consider combinatorial invariants of spherical homogeneous spaces. Following ideas of Akhiezer [Akh15, Theorem 1.1] and Cupit-Foutou [CF15, Theorem 3(1), Section 2.2], for $\gamma \in \Gamma = \text{Gal}(k/k_0)$ we give a criterion of isomorphism of a spherical homogeneous space $Y = G/H$ and the "conjugate" variety $\gamma^* Y = G/\gamma(H)$ in terms of the action of $\gamma$ on the combinatorial invariants of $G/H$. In Sections 9–11 we prove Corollary 0.4, Theorem 0.2, Theorem 0.3, and Theorem 0.5. In Appendix A for a connected reductive group $G_0$ defined over an algebraically closed field $k_0$ of characteristic 0 and for an algebraically closed extension $k \supset k_0$, it is proved that any spherical subgroup $H$ of the base change $G = G_0 \times_{k_0} k$ is conjugate to a (spherical) subgroup defined over $k_0$. In Appendix B, following Friedrich Knop’s MathOverflow answer [Kn17a] to the author’s question, Giuliano Gagliardi gives a proof of an unpublished theorem of Ivan Losev that describes the image of $\text{Aut}^G(G/H) = N_G(H)/H$ in the group of permutations of $\mathcal{D}(G/H)$. Our proofs of Theorems 0.2, 0.3, and 0.5 use this result of Losev.

Acknowledgements. The author is very grateful to Friedrich Knop for answering the author’s numerous MathOverflow questions, especially for the answer [Kn17a], to Giuliano Gagliardi for writing Appendix B, and to Roman Avdeev for suggesting Example 10.9 and proving Proposition 10.10. It is a pleasure to thank Michel Brion for very helpful e-mail correspondence. The author thanks Dmitri Akhiezer, Stéphanie Cupit-Foutou, Cristian D. González-Avilés, David Harari, Boris Kunyavskiĭ, and Stephan Snigerov for helpful discussions. This preprint was written during the author’s visits to the University of La Serena (Chile) and to the Paris-Sud University, and he is grateful to the departments of mathematics of these universities for support and excellent working conditions.

Notation and assumptions. $k$ is a field. In Section 2 and everywhere starting Section 4, $k$ is algebraically closed. $k_0$ is a subfield of the algebraically closed field $k$ such that $k$ is a Galois extension of $k_0$ (except for Appendix A), hence $k_0$ is perfect. A $k$-variety is a reduced separated scheme of finite type over $k$, not necessarily irreducible. An algebraic $k$-group is a smooth $k$-group scheme of finite type over $k$, not necessarily connected. All algebraic $k$-subgroups are assumed to be smooth.

1. Semi-morphisms of $k$-schemes

Let $k$ be a field and let $\text{Spec} k$ denote the spectrum of $k$. By a $k$-scheme we mean a pair $(Y, p_Y)$, where $Y$ is a scheme and $p_Y : Y \to \text{Spec} k$ is a morphism of schemes. Let $(Y, p_Y)$ and $(Z, p_Z)$ be two $k$-schemes. By a $k$-morphism

$$\lambda : (Y, p_Y) \to (Z, p_Z)$$

we mean a morphism of schemes $\lambda : Y \to Z$ such that the following diagram commutes:

\[
\begin{array}{ccc}
Y & \xrightarrow{\lambda} & Z \\
\downarrow p_Y & & \downarrow p_Z \\
\text{Spec} k & \xrightarrow{id} & \text{Spec} k
\end{array}
\]

Let $\gamma : k \to k$ be an automorphism of $k$ (we write $\gamma \in \text{Aut}(k)$). Let

$$\gamma^* := \text{Spec} \gamma : \text{Spec} k \to \text{Spec} k$$

denote the induced automorphism of $\text{Spec} k$, then $(\gamma \gamma')^* = (\gamma')^* \circ \gamma^*$. 
Let \((Y, p_Y)\) be a \(k\)-scheme. By abuse of notation we write just that \(Y\) is a \(k\)-scheme. We define the \(\gamma\)-conjugated \(k\)-scheme \(\gamma_* (Y, p_Y) = (\gamma_* Y, \gamma_* p_Y)\) to be the base change of \((Y, p_Y)\) from \(\text{Spec} \ k\) to \(\text{Spec} \ k\) via \(\gamma^*\). By abuse of notation we write just \(\gamma_* Y\) for \(\gamma_* (Y, p_Y)\).

**Lemma 1.1.** Let \((Y, p_Y)\) be a \(k\)-scheme, and let \(\gamma \in \text{Aut}(k)\). Then the \(\gamma\)-conjugated \(k\)-scheme \(\gamma_* (Y, p_Y)\) is canonically isomorphic to \((Y, (\gamma^*)^{-1} \circ p_Y)\).

**Proof.** Write \((X, p_X) = \gamma_* (Y, p_Y)\), then \(X\) comes with a canonical morphism \(\lambda: X \to Y\) such that the following diagram commutes:

\[
\begin{array}{ccc}
X & \xrightarrow{\lambda} & Y \\
\downarrow{p_X} & & \downarrow{p_Y} \\
\text{Spec} \ k & \xrightarrow{\gamma^*} & \text{Spec} \ k
\end{array}
\]

Since \((\gamma^{-1})_* (\gamma_* (Y, p_Y))\) is canonically isomorphic to \((Y, p_Y)\), one can easily see that \(\lambda\) is an isomorphism of schemes. From the above diagram we obtain a commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\lambda} & Y \\
\downarrow{p_X} & & \downarrow{p_Y} \\
\text{Spec} \ k & \xrightarrow{\gamma^*} & \text{Spec} \ k
\end{array}
\]

which gives a canonical isomorphism of \(k\)-schemes \((X, p_X) \xrightarrow{\sim} (Y, (\gamma^*)^{-1} \circ p_Y)\). \qed

We define an action of \(\gamma: k \to k\) on \(k\)-points. Let \(y\) be a \(k\)-point of \(Y\), that is, a morphism \(y: \text{Spec} \ k \to Y\) such that \(p_Y \circ y = \text{id}_{\text{Spec} \ k}\). We denote

\[
(1) \quad \gamma_! (y) = y \circ \gamma^*: \text{Spec} \ k \to \text{Spec} \ k \to Y,
\]

then an easy calculation shows that \(\gamma_!(y)\) a \(k\)-point of \(\gamma_* Y\). Thus we obtain a bijection

\[
(2) \quad \gamma!: Y(k) \to (\gamma_* Y)(k), \quad y \mapsto \gamma!(y).
\]

Let \(G\) be a \(k\)-group scheme. Following Flicker, Scheiderer, and Sujatha [FSS98, (1.2)], we define the \(k\)-group scheme \(\gamma_* G\) to be the base change of \(G\) from \(\text{Spec} \ k\) to \(\text{Spec} \ k\) via \(\gamma^*\). Then the map \((2)\)

\[
\gamma!: G(k) \to (\gamma_* G)(k)
\]

is an isomorphism of groups (because for any field extension \(\lambda: k \leftrightarrow k'\) the corresponding map on rational points

\[
\lambda!: G(k) \to (G \times_k k')(k')
\]

is a homomorphism). If \(H \subset G\) is a \(k\)-group subscheme, then \(\gamma_* H\) is naturally a \(k\)-group subscheme of \(\gamma_* G\) (because a base change of a group subscheme is a group subscheme). From the commutative diagram

\[
\begin{array}{ccc}
H(k) & \xrightarrow{\gamma} & (\gamma_* H)(k) \\
| & & | \\
G(k) & \xrightarrow{\gamma} & (\gamma_* G)(k)
\end{array}
\]

we see that

\[
(3) \quad (\gamma_* H)(k) = \gamma!(H(k)) \subset (\gamma_* G)(k).
\]
Let \((Y, \theta)\) be a \(G\)-\(k\)-scheme (a \(G\)-scheme over \(k\)), where
\[
\theta: G \times_k Y \to Y,
\]
is an action of \(G\) on \(Y\). By abuse of notation we write just that \(Y\) is a \(G\)-\(k\)-scheme. Again we define the \(\gamma_s G\)-\(k\)-scheme \(\gamma_s(Y, \theta) = (\gamma_s Y, \gamma_s \theta)\) to be the base change of \((Y, \theta)\) from \(\text{Spec} \, k\) to \(\text{Spec} \, k\) via \(\gamma^*\).

**Definition 1.2.** Let \((Y, p_Y)\) and \((Z, p_Z)\) be two \(k\)-schemes. A semilinear morphism
\[
(\gamma, \nu): (Y, p_Y) \to (Z, p_Z)
\]
is a pair \((\gamma, \nu)\) where \(\gamma: k \to k\) is an automorphism of \(k\), and \(\nu: Y \to Z\) is a morphism of schemes such that the following diagram commutes:
\[
\begin{array}{ccc}
Y & \xrightarrow{\nu} & Z \\
p_Y & & p_Z \\
\text{Spec} \, k & \xrightarrow{\gamma^{-1}} & \text{Spec} \, k
\end{array}
\]
We shorten “semilinear morphism” to “semi-morphism”. We write \(\nu: (Y, p_Y) \to (Z, p_Z)\) is a \(\gamma\)-semi-morphism” if \((\gamma, \nu): (Y, p_Y) \to (Z, p_Z)\) is a semi-morphism. Then by abuse of notation we write just that \(\nu: Y \to Z\) is a \(\gamma\)-semi-morphism.

Note that if we take \(\gamma = \text{id}_k\), then a \(\text{id}_k\)-semi-morphism \((Y, p_Y) \to (Z, p_Z)\) is just a morphism of \(k\)-schemes.

**Lemma 1.3.** If \((\gamma, \nu): (Y, p_Y) \to (Z, p_Z)\) is a semi-morphism of nonempty \(k\)-schemes, then the morphism of schemes \(\nu: Y \to Z\) uniquely determines \(\gamma\).

**Proof.** We may and shall assume that \(Y\) and \(Z\) are affine, \(Y = \text{Spec} \, R_Y\), \(Z = \text{Spec} \, R_Z\). Then we have a commutative diagram
\[
\begin{array}{ccc}
R_Y & \xrightarrow{\nu^*} & R_Z \\
\downarrow{\gamma^{-1}} & & \downarrow{\gamma^{-1}} \\
k & = & k
\end{array}
\]
Since \(k\) is a field, the vertical arrows are injective, and therefore, the homomorphism of rings \(\nu^*\) uniquely determines the automorphism \(\gamma^{-1}\). \(\square\)

We define an action of a semi-morphism \((\gamma, \nu): (Y, p_Y) \to (Z, p_Z)\) on \(k\)-points. If \(y: \text{Spec} \, k \to Y\) is a \(k\)-point of \((Y, p_Y)\), we set
\[
(\gamma, \nu)(y) = \nu \circ y \circ \gamma^*: \text{Spec} \, k \to Z,
\]
which is a \(k\)-point of \((Z, p_Z)\). This formula is compatible with the usual formula for the action of a \(k\)-morphism on \(k\)-points. By abuse of notation we write \(\nu(y)\) instead of \((\gamma, \nu)(y)\).

**Definition 1.4.** By a \(\gamma\)-semi-isomorphism \(\nu: (Y, p_Y) \to (Z, p_Z)\) we mean a \(\gamma\)-semi-morphism \(\nu: (Y, p_Y) \to (Z, p_Z)\) for which the morphisms of schemes \(\nu: Y \to Z\) is an isomorphism. By a \(\gamma\)-semi-automorphism of a \(k\)-scheme \((Y, p_Y)\) we mean a \(\gamma\)-semi-isomorphism \(\mu: (Y, p_Y) \to (Y, p_Y)\).

Let us fix \(\gamma \in \text{Aut}(k)\). The commutative diagram
\[
\begin{array}{ccc}
Y & \xrightarrow{\nu} & Z \\
p_Y & & p_Z \\
\text{Spec} \, k & \xrightarrow{(\gamma^{-1})} & \text{Spec} \, k
\end{array}
\]

[Raw Text Content]
shows that
\begin{equation}
(\gamma, \nu): (Y, p_Y) \to (Z, p_Z)
\end{equation}
is a semi-morphism (that is, \(\nu: (Y, p_Y) \to (Z, p_Z)\) is a \(\gamma\)-semi-morphism) if and only if
\begin{equation}
(id_k, \nu): \gamma_*(Y, p_Y) \to (Z, p_Z)
\end{equation}
is a semi-morphism (that is, \(\nu: \gamma_*(Y, p_Y) \to (Z, p_Z)\) is a \(k\)-morphism). For brevity we write
\begin{equation}
\nu_\gamma: \gamma_* Y \to Z
\end{equation}
for the \(k\)-morphism \((\ref{eq:gamma-morphism})\), then the \(k\)-morphism \(\nu_\gamma\) acts on \(k\)-points as follows:
\begin{equation}
(y': \text{Spec } k \to \gamma_* Y) \mapsto (\nu \circ y': \text{Spec } k \to Z).
\end{equation}

**Example 1.5.** Let \((Y, p_Y)\) be a \(k\)-scheme. Recall that \(\gamma_*(Y, p_Y) = (Y, (\gamma^*)^{-1} \circ p_Y)\). The commutative diagram
\[
\begin{array}{ccc}
Y & \xrightarrow{id_Y} & Y \\
\downarrow p_Y & & \downarrow p_Y \\
\text{Spec } k & (\gamma^*)^{-1} & \text{Spec } k
\end{array}
\]
shows that \((\gamma, id_Y): Y \to \gamma_* Y\) is a \(\gamma\)-semi-isomorphism. We denote this \(\gamma\)-semi-isomorphism by
\[
\gamma! : Y \to \gamma_* Y.
\]
Comparing formulas \((\ref{eq:gamma-morphism})\) and \((\ref{eq:gamma-iso})\), we see that the \(\gamma\)-semi-isomorphism \(\gamma!: Y \to \gamma_* Y\) acts on \(k\)-points as the bijective map \(\gamma!: Y(k) \to (\gamma_* Y)(k)\) defined by formula \((\ref{eq:gamma-morphism})\).

Now let \(\nu: Y \to Z\) be a \(\gamma\)-semi-morphism. The commutative diagram \((\ref{eq:gamma-morphism})\) shows that
\begin{equation}
\nu = \nu_\gamma \circ \gamma! = (id_k, \nu) \circ ((\gamma^*)^{-1}, id_Y): Y \xrightarrow{\gamma!} \gamma_* Y \xrightarrow{\nu_\gamma} Z,
\end{equation}
where \(\gamma!\) is a \(\gamma\)-semi-isomorphism and \(\nu_\gamma\) is a \(k\)-morphism (an \(id_k\)-semi-morphism). It follows that
\begin{equation}
\nu(y) = \nu_\gamma(\gamma!(y)) \quad \text{for } y \in Y(k)
\end{equation}
(this follows also from comparing formulas \((\ref{eq:gamma-morphism}), (\ref{eq:gamma-iso}),\) and \((\ref{eq:gamma-action})\)).

**Example 1.6.** Let \(Y_0\) be a \(k_0\)-scheme, where \(k_0\) is a subfield of \(k\). Let \(\gamma \in \text{Aut}(k/k_0)\), that is, \(\gamma\) is an automorphism of \(k\) that fixes all elements of \(k_0\). Consider
\[
Y := Y_0 \times_{k_0} k = Y_0 \times_{\text{Spec } k_0} \text{Spec } k
\]
and
\[
\mu_\gamma = id_{Y_0} \times ((\gamma^*)^{-1}): Y \to Y.
\]
It follows from the construction of \(\mu_\gamma\) that the following diagram commutes:
\[
\begin{array}{ccc}
Y & \xrightarrow{\mu_\gamma} & Y \\
\downarrow p_Y & & \downarrow p_Y \\
\text{Spec } k & (\gamma^*)^{-1} & \text{Spec } k
\end{array}
\]
We see that \(\mu_\gamma\) is a \(\gamma\)-semi-automorphism of \(Y\).

Let \(Y\) be an affine \(k\)-variety, \(Y = \text{Spec } R_Y\), then \(R_Y\) is the ring of regular functions on \(Y\). If \(f \in R_Y\), then for any \(y \in Y(k)\) the value \(f(y) \in k\) is defined.
Lemma 1.7. Let $\nu: (Y, p_Y) \to (Z, p_Z)$ be a $\gamma$-semi-isomorphism of affine $k$-varieties, where $\gamma: k \to k$ is an automorphism of $k$. Let $Y = \text{Spec} \, R_Y$, $Z = \text{Spec} \, R_Z$, and let $\nu^*: R_Z \to R_Y$ denote the morphism of rings corresponding to $\nu$. Let $f_Z \in R_Z$. Then
\[ f_Z(\nu(y)) = \gamma((\nu^* f_Z)(y)) \quad \text{for all } y \in Y(k). \]

Proof. The assumption that $\nu: Y \to Z$ is a $\gamma$-semi-morphism means that the diagram (14) commutes. A $k$-point $y \in Y(k)$ corresponds to a homomorphism of $k$-algebras $\varphi_y: R_Y \to k$, and the following diagram commutes:
\[
\begin{array}{ccc}
R_Y & \xrightarrow{\nu^*} & R_Z \\
\varphi_y \downarrow & & \downarrow \varphi_y \\
k & \xrightarrow{\gamma^{-1}} & k
\end{array}
\]
hence $\varphi_y(\gamma) = \gamma \circ \varphi_y \circ \nu^*$. We set $f_Y = \nu^* f_Z \in R_Y$, then $f_Y(y) = \varphi_y(f_Y)$, and (13) means that
\[ (\gamma \circ \varphi_y \circ \nu^*)(f_Z) = \gamma(\varphi_y(\nu^* f_Z)), \]
which is obvious. \hfill $\Box$

Now let $\nu: (Y, p_Y) \to (Z, p_Z)$ be a $\gamma$-semi-isomorphism of irreducible $k$-varieties, where $\gamma: k \to k$ is an automorphism of $k$. Then the isomorphism of schemes $\nu: Y \to Z$ induces an isomorphism of the fields of rational functions
\[ \nu_*: K(Y) \to K(Z), \quad f \mapsto \nu_* f. \]
For $f \in K(Y)$ and $y \in Y(k)$, the value $f(y) \in k \cup \{\infty\}$ of $f$ at $y$ is defined, where we write $f(y) = \infty$ if $f$ is not regular at $y$.

Corollary 1.8. With the above notation and assumptions we have
\[ (\nu_* f_Y)(z) = \gamma(f_Y(\nu^{-1}(z))) \quad \text{for all } f_Y \in K(Y), \quad z \in Z(k). \]

Proof. We consider the isomorphism
\[ \nu^* = \nu_*^{-1}: K(Z) \to K(Y), \quad f \mapsto \nu^* f_Z, \quad \text{where } f_Z \in K(Z). \]
Set $f_Z = \nu_* f_Y$, then $f_Y = \nu^* f_Z$. We must prove that (13) holds. We may and shall assume that $Y$ and $Z$ are affine varieties, $Y = \text{Spec} \, R_Y$, $Z = \text{Spec} \, R_Z$, $f_z \in R_Z$, and that the morphism $\nu$ corresponds to a homomorphism of rings $\nu^*: R_Z \to R_Y$. Now the corollary follows from Lemma 1.7. \hfill $\Box$

Remark 1.9. (Classical language.) In this remark we describe the variety $\gamma_* Y$ and the map $\gamma_!: Y(k) \to (\gamma_* Y)(k)$ in the language of the classical algebraic geometry. First, consider the affine space $A^n_k$, then $A^n_k(k) = k^n$. Let $k_0$ be the prime subfield of $k$, that is, the subfield generated by 1, then $A^n_k = A^n_{k_0} \times k_0$. Let $\gamma \in \text{Aut}(k) = \text{Aut}(k/k_0)$, then $\gamma$ induces a $\gamma$-semi-automorphism $\mu_{\gamma}: A^n_k \to A^n_k$, see Example 1.6. For $i = 1, \ldots, n$, let $f_i$ denote the $i$-th coordinate function on $A^n_k$, which is a regular function. Since $f_i$ comes from a regular function on $A^n_{k_0}$, we have $\mu_{\gamma} f_i = f_i$, and by Lemma 1.7 we have
\[ f_i(\mu_{\gamma}(x)) = \gamma(f_i(x)) \quad \text{for } x \in A^n_k(k) = k^n. \]
If we write $x = (x_i)_{i=1}^n \in k^n$, where $x_i = f_i(x) \in k$, then
\[ \mu_{\gamma}(x) = \gamma((x_i)_{i=1}^n). \]

Now let $Y \subset A^n_k$ be an affine variety. Let $\iota: Y \to A^n_k$ denote the inclusion morphism, then $\gamma$ induces a $k$-morphism
\[ \gamma_* \iota: \gamma_* Y \to \gamma_* A^n_k. \]
We have a $k$-isomorphism
\[ (\mu_{\gamma})_{\iota}: \gamma_* A^n_k \to A^n_k, \]
and we obtain a $k$-morphism
\[(\gamma_*t)' = (\mu_\gamma)_t \circ \gamma_*t : \gamma_*Y \to A^n_k.\]

From the commutative diagram
\[
\begin{array}{ccc}
Y(k) & \xrightarrow{\gamma} & (\gamma_*Y)(k) \\
\downarrow{\iota} & & \downarrow{\gamma_*t} \\
A^n_k(k) & \xrightarrow{\gamma} & (\gamma_*A^n_k)(k)
\end{array}
\]
we see that
\[(\gamma_*t)(\gamma!(y)) = \gamma!(\iota(y)) \quad \text{for } y \in Y(k),
\]
hence
\[(\gamma_*t)'(y) = (\mu_\gamma)_t(\gamma!(\iota(y))) = \mu_\gamma(\iota(y)).\]

Now we assume that $k$ is algebraically closed. As usual in the classical algebraic geometry, we identify $Y$ with the algebraic set
\[Y(k) \subset A^n_k(k) = k^n.\]
Furthermore, we identify $\gamma_*Y$ with the algebraic set
\[(\gamma_*t)'(Y(k)) = \mu_\gamma(Y(k)) \subset k^n.\]
We see that
\[\gamma_*Y = \mu_\gamma(Y) = \{\gamma(y_i)_{i=1}^n \mid (y_i)_{i=1}^n \in Y\},\]
and that the map
\[\gamma! : Y(k) \to (\gamma_*Y)(k)\]
sends a point $y$ with coordinates $(y_i)_{i=1}^n$ to the point with coordinates $\gamma(y_i)_{i=1}^n$. If $Y \subset k^n$ is defined by a family of polynomials $(P_\alpha)_{\alpha \in A}$, then $\gamma_*Y \subset k^n$ is defined by the family $(\gamma(P_\alpha))_{\alpha \in A}$, where $\gamma(P_\alpha)$ is the polynomial obtained from $P_\alpha$ by acting by $\gamma$ on the coefficients.

2. Semi-morphisms of $G$-varieties

2.1. In this section $k$ is an algebraically closed field, and $Y$ is a $k$-variety, that is, a reduced separated scheme of finite type over $k$.

Let $G$ be an algebraic group over $k$ (we write also “an algebraic $k$-group”), that is, a smooth group scheme of finite type over $k$. Let $(Y, \theta)$ be a $G$-variety, that is, a $k$-variety $Y$ together with an action
\[\theta : G \times_k Y \to Y\]
of $G$ on $Y$. If $g \in G(k)$ and $y \in Y(k)$, we write just $g \ast y$ or $g \cdot y$ for $\theta(g, y) \in Y(k)$.

Definition 2.2 (cf. [FSS98 (1.2)]). Let $\gamma \in \text{Aut}(k)$. A $\gamma$-semi-automorphism of an algebraic $k$-group $G$ is a $\gamma$-semi-automorphism of $k$-schemes $\tau : G \to G$ such that the corresponding isomorphism of $k$-varieties $\tau^\gamma : \gamma_*G \to G$, see [4], is an isomorphism of algebraic $k$-groups. This condition is the same as to require that certain diagrams containing $\tau$ commute, see [Brv93 1.2].

Let $H \subset G$ be an algebraic $k$-subgroup. By Definition 2.2 we have $\tau(H) = \tau^\gamma(\gamma_*H)$, hence $\tau(H)$ is a $k$-subgroup of $G$. We have $(\tau(H))(k) = \tau(H(k))$.

We shall always assume that $G = G_0 \times_{k_0} k$, where $G_0$ is an algebraic group defined over a subfield $k_0$ of $k$, and that $\gamma \in \text{Aut}(k/k_0)$, that is, $\gamma$ is an automorphism of $k$ fixing all elements of $k_0$. Then we have a $\gamma$-semi-automorphism
\[\sigma_\gamma = \text{id}_{G_0} \times (\gamma^*)^{-1} : G \to G,
\]
compare Example 1.6. If \( \alpha \) is any \( k \)-automorphism of \( G \), then
\[
\tau := \alpha \circ \sigma : G \to G
\]
is a \( \gamma \)-semi-automorphism of \( G \), and all \( \gamma \)-semi-automorphisms of \( G \) (for given \( \gamma \)) can be obtained in this way.

**Definition 2.3.** Let \( G \) be an algebraic \( k \)-group, and let \((Y, \theta_Y)\) and \((Z, \theta_Z)\) be two \( G \)-\( k \)-varieties. Let \( \gamma \in \text{Aut}(k) \), and let \( \tau : G \to G \) be a \( \gamma \)-semi-automorphism of \( G \). A \( \tau \)-equivariant \( \gamma \)-semi-morphism

\[
\nu : (Y, \theta_Y) \to (Z, \theta_Z)
\]
is a \( \gamma \)-semi-morphism \( \nu : Y \to Z \) such that the following diagram commutes:

\[
\begin{array}{ccc}
G \times_k Y & \xrightarrow{\theta_Y} & Y \\
\tau \times \nu \downarrow & & \downarrow \nu \\
G \times_k Z & \xrightarrow{\theta_Z} & Z
\end{array}
\]

where we write \( \tau \times \nu \) for the product of \( \tau \) and \( \nu \) over the automorphism \((\gamma^*)^{-1}\) of \( \text{Spec} \, k \).

Since \( k \) is algebraically closed, \( G \) is smooth (reduced), and \( Y \) and \( Z \) are reduced, we see that the diagram (14) commutes if and only if

\[
\nu(g \cdot y) = \tau(g) \cdot \nu(y) \quad \text{for all } g \in G(k), \ y \in Y(k).
\]

**Construction 2.4.** Let \( G \) be an algebraic \( k \)-group, and let \((Y, \theta_Y)\) be a \( G \)-\( k \)-variety. The group \( \gamma_* G \) naturally acts on \( \gamma_* Y \): the action \( \theta : G \times_k Y \to Y \) gives an action

\[
\gamma_* \theta : \gamma_* G \times_k \gamma_* Y \to \gamma_* Y.
\]

By definition, a \( \gamma \)-semi-automorphism \( \tau \) of \( G \) defines an isomorphism of algebraic \( k \)-groups

\[
\tau_\gamma : \gamma_* G \to G.
\]

We identify \( G \) and \( \gamma_* G \) via \( \tau_\gamma \) and obtain from (15) an action

\[
\tau^* \gamma_* \theta : G \times_k \gamma_* Y \to \gamma_* Y, \quad (g, y') \mapsto (\gamma_* \theta)(\tau_\gamma^{-1}(g), y') \quad \text{for } g \in G(k), \ y' \in (\gamma_* Y)(k).
\]

By abuse of notation we write \( \tau^* \theta \) for \( \tau^* \gamma_* \theta \) and we write \( \gamma_* Y \) for the \( G \)-\( k \)-variety \((\gamma_* Y, \tau^* \gamma_* \theta)\). We write

\[
g \circ \tau y' = \tau^* \theta(g, y'), \quad \text{where } g \in G(k), \ y' \in (\gamma_* Y)(k).
\]

By formula (12) we have

\[
\tau(g) = \tau_\gamma(\gamma_!(g)),
\]

hence

\[
g \circ \tau y' = (\gamma_* \theta)(\tau_\gamma^{-1}(g), y') = \gamma_!(\theta(\gamma^{-1}_!(\tau_\gamma^{-1}(g)), \gamma^{-1}_!(y'))) = \gamma_!(\gamma^{-1}_!(\tau^{-1}(g)), \gamma^{-1}_!(y'))) = \gamma_!(\tau^{-1}(g) \circ \gamma^{-1}_!(y')).
\]

**Lemma 2.5.** Let \( G \) be an algebraic \( k \)-group, and let \((Y, \theta)\) be a \( G \)-\( k \)-variety. Let \( \gamma \in \text{Aut}(k) \), and let \( \tau : G \to G \) be a \( \gamma \)-semi-automorphism of \( G \). Let \( y^{(0)} \in Y(k) \) be a \( k \)-point, and write \( H = \text{Stab}_G(y^{(0)}) \). Consider the action

\[
\tau^* \theta : G \times_k \gamma_* Y \to \gamma_* Y.
\]

Then the stabilizer in \( G(k) \) of the point \( \gamma_!(y^{(0)}) \in (\gamma_* Y)(k) \) under the action \( \tau^* \theta \) is \( \tau(H(k)) = (\tau(H))(k) \).
Proof. By formula (10) we have
\[ g \ast_{\tau} \gamma(y(0)) = \gamma(\tau^{-1}(g) \cdot y(0)). \]
Since the stabilizer in \( G(k) \) of \( y(0) \in Y(k) \) is \( H(k) \), the lemma follows. \( \square \)

Note that \( \nu \) in Definition 2.3 defines a \( k \)-morphism
\[ \nu_{\gamma}: \gamma_{*}Y \to Z, \]
see (11).

Lemma 2.6. Let \( \gamma \in \text{Aut}(k) \) and let \( \tau: G \to G \) be a \( \gamma \)-semi-automorphism of \( G \). Let \( (Y, \theta_{Y}) \) and \( (Z, \theta_{Z}) \) be two \( G \)-k-varieties. A morphism of schemes \( \nu: Y \to Z \) is a \( \tau \)-equivariant \( \gamma \)-semi-morphism if and only if \( \nu_{\gamma}: \gamma_{*}Y \to Z \) is a \( G \)-equivariant morphism of \( k \)-varieties.

Proof. By (11) the morphism of schemes \( \nu \) is a \( \gamma \)-semi-morphism \( Y \to Z \) if and only if it is a \( k \)-morphism \( \gamma_{*}Y \to Z \).

Let \( g \in G(k) \), \( y' \in (\gamma_{*}Y)(k) \). Using formula (16) we obtain
\[ \nu_{\gamma}(g \ast_{\tau} y') = \nu(\gamma^{-1}(g \ast_{\tau} y')) = \nu(\tau^{-1}(g) \ast Y \gamma^{-1}(y')). \]
We have also
\[ g \ast Y \nu(\gamma^{-1}(y')) = g \ast Z \nu(y'). \]

If \( \nu \) is \( \tau \)-equivariant, then
\[ \nu(\tau^{-1}(g) \ast Y \gamma^{-1}(y')) = g \ast Z \nu(\gamma^{-1}(y')), \]
and we obtain that
\[ \nu_{\gamma}(g \ast_{\tau} y') = g \ast Z \nu(\gamma^{-1}(y')) \quad \text{for all } g \in G(k), \ y' \in (\gamma_{*}Y)(k), \]
hence \( \nu_{\gamma} \) is \( G \)-equivariant.

Conversely, if \( \nu_{\gamma} \) is \( G \)-equivariant, we obtain from the above calculations that
\[ \nu(\tau^{-1}(g) \ast Y \gamma^{-1}(y')) = \nu_{\gamma}(g \ast_{\tau} y') = g \ast Z \nu_{\gamma}(y') = g \ast Z \nu(\gamma^{-1}(y')). \]
Set \( y = \gamma^{-1}(y'), \ g' = \tau^{-1}(g) \), then we obtain that
\[ \nu(y' \ast_{Y} y) = \tau(g') \ast Z \nu(y) \quad \text{for all } g' \in G(k), \ y \in Y(k). \]
Thus \( \nu \) is \( \tau \)-equivariant. \( \square \)

Corollary 2.7. Let \( \gamma \) be an automorphism of \( k \), and let \( \tau: G \to G \) be a \( \gamma \)-semi-automorphism of \( G \). Let \( (Y, \theta) \) be a \( G \)-variety. There exists a \( \tau \)-equivariant \( \gamma \)-semi-automorphism \( \mu: Y \to Y \) if and only if the \( G \)-k-variety \( (\gamma_{*}Y, \tau \circ \theta) \) is isomorphic to \( (Y, \theta) \).

Proof. We take \( Z = Y \) in Lemma 2.6. \( \square \)

3. Quotients

Let \( k \) be a field (not necessarily algebraically closed). By an algebraic scheme over \( k \) we mean a scheme of finite type over \( k \). By an algebraic group scheme over \( k \) we mean a group scheme over \( k \) whose underlying scheme is of finite type over \( k \).

Let \( H \) be an algebraic group subscheme of an algebraic group \( k \)-scheme \( G \). A quotient of \( G \) by \( H \) is an algebraic scheme \( Y \) over \( k \) equipped with an action \( \theta: G \times_{k} Y \to Y \) and a point \( y^{(0)} \in Y(k) \) fixed by \( H \) satisfying certain properties (a) and (b), see Milne [M18 Definition 5.20].
By [Mi18] Theorem 5.28 there exists a quotient of \( G \) by \( H \). By [Mi18] Proposition 5.22 this quotient \((Y,\theta,y^{(0)})\) has the following universal property:

(U) Let \( Z \) be a \( k \)-scheme on which \( G \) acts, and let \( z^{(0)} \in Z(k) \) be a point fixed by \( H \). Then there exists a unique \( G \)-equivariant map \( Y \to Z \) making the following diagram commute:

\[
\begin{array}{ccc}
G & \xrightarrow{g \mapsto g \cdot y^{(0)}} & Y \\
\downarrow{g \mapsto g \cdot z^{(0)}} & & \downarrow{g \mapsto g \cdot z^{(0)}} \\
& Z \\
\end{array}
\]

Clearly the universal property (U) uniquely determines the quotient up to a unique isomorphism, so we may take (U) as a definition of the quotient.

We return to our settings: \( k \) is an algebraically closed field, \( G \) is a linear algebraic \( k \)-group (a smooth affine group scheme) and \( H \) is a smooth algebraic \( k \)-subgroup of \( G \). Since \( G \) is smooth, so is the quotient \( Y \), see [Mi18] Corollary 5.26. Since \( G \) is smooth and affine, the quotient \( Y \) is a separated algebraic scheme, see [Mi18] Theorem 7.18. Thus \( Y \) is a \( k \)-variety, and therefore, in the universal property \( U \) defining \( Y \) we may assume that \( Z \) is a \( k \)-variety. Since \( k \) is algebraically closed and \( H \) is smooth, the condition “fixed by \( H \)” is equivalent to “fixed by \( H(k) \)”.

We arrive to the following definition of Springer:

**Definition 3.1** (cf. Springer [Sp98] Section 5.5). Let \( k \) be an algebraically closed field, and let \( G \) be a linear algebraic \( k \)-group. Let \( H \subset G \) be a smooth \( k \)-subgroup. A quotient of \( G \) by \( H \) is a pointed \( G \)-\( k \)-variety \((Y,\theta: G \times_k Y \to Y, y^{(0)} \in Y(k))\) such that \( H(k) \) fixes \( y_0 \), with the following universal property:

(U') For any pointed \( G \)-\( k \)-variety \((Z,\theta_Z,z^{(0)})\) such that the \( k \)-point \( z^{(0)} \in Z(k) \) is fixed by \( H(k) \), there exists a unique morphism of pointed \( G \)-\( k \)-varieties \((Y,\theta,y^{(0)}) \to (Z,\theta_Z,z^{(0)})\).

For \( G \) and \( H \) as in Definition 3.1 let \((Y,\theta,y^{(0)})\) be a quotient of \( G \) by \( H \). The action of \( G \) on \( Y \) induces a \( G \)-\( k \)-morphism

\[
G \to Y, \quad g \mapsto g \cdot y^{(0)},
\]

where \( G \) acts on itself by left translations.

As usual, we write \( G/H \) for \( Y \) and \( g \cdot H \) or \( gH \) for \( g \cdot y^{(0)} \), where \( g \in G(k) \). In particular, we write \( 1 \cdot H \) for \( y^{(0)} \). The \( G \)-equivariant morphism \( G/H = Y \to Z \) of (U') sends \( 1 \cdot H \in (G/H)(k) \) to \( z^{(0)} \), hence for any \( g \in G(k) \) it sends the \( k \)-point \( gH \in (G/H)(k) \) to \( g \cdot z^{(0)} \in Z(k) \). Thus the quotient \( G/H \) has the following universal property:

(U'') For any pointed \( G \)-\( k \)-variety \((Z,\theta_Z,z^{(0)})\) such that the \( k \)-point \( z^{(0)} \) is fixed by \( H(k) \), there exists a unique \( G \)-\( k \)-morphism \( G/H \to Z \) sending \( gH \) to \( g \cdot z^{(0)} \) for any \( g \in G(k) \).

By [Mi18] Definition 5.20(a)] the morphism (17) induces an injective map \( G(k)/H(k) \to (G/H)(k) \) sending \( g \cdot H(k) \) to \( gH \). By [Mi18] Proposition 5.25 the morphism (17) is faithfully flat, and therefore, since \( k \) is algebraically closed, we see that the induced map \( G(k)/H(k) \to (G/H)(k) \) is surjective. We conclude that this map is bijective. Thus any \( k \)-point of \( G/H \) is of the form \( gH \), where \( g \in G(k) \).

4. Semi-morphisms of homogeneous spaces

Let \( k \) be an algebraically closed field.

**Lemma 4.1** (well-known). Let \( G \) be a linear algebraic \( k \)-group over an algebraically closed field \( k \), and let \( H_1, H_2 \) be two \( k \)-subgroups. Then \( Y_1 = G/H_1 \) and \( Y_2 = G/H_2 \) are isomorphic as \( G \)-\( k \)-varieties if and only if the subgroups \( H_1 \) and \( H_2 \) are conjugate. To be more precise, for \( a \in G(k) \) the following two assertions are equivalent:
(i) There exists an isomorphism of $G$-k-varieties $\phi_a : G/H_1 \to G/H_2$ taking $g \cdot H_1$ to $ga^{-1} \cdot H_2$ for $g \in G(k)$;
(ii) $H_1 = a^{-1}H_2a$.

Proof. (i)$\Rightarrow$(ii). Clearly $\text{Stab}_{G(k)}(1 \cdot H_1) = H_1(k)$ and $\text{Stab}_{G(k)}(a^{-1} \cdot H_2) = a^{-1} \cdot H_2(k) \cdot a$. Since $\phi_a(1 \cdot H_1) = a^{-1} \cdot H_2$, these stabilizers coincide, whence (ii).

(ii)$\Rightarrow$(i). Set $Y_2 = G/H_2$, $y_2(0) = 1 \cdot H_2 \in Y_2(k)$, $y' = a^{-1} \cdot y_2(0) \in Y_2(k)$, then $\text{Stab}_{G(k)}(y') = a^{-1}H_2(k)a = H_1(k)$, so by the property $(U''')$ of the quotient $G/H_1$ there exists a unique morphism of $G$-varieties $\phi_a : G/H_1 \to G/H_2$ such that

$$\phi_a(g \cdot H_1) = g \cdot a^{-1} \cdot H_2 \quad \text{for} \quad g \in G(k).$$

Similarly, since the stabilizer in $G(k)$ of $a \cdot H_1 \in (G/H_1)(k)$ is $H_2(k)$, there exists a unique morphism of $G$-varieties $\psi_a : G/H_2 \to G/H_1$ such that

$$\psi_a(g \cdot H_2) = g \cdot a \cdot H_1 \quad \text{for} \quad g \in G(k).$$

Clearly these two morphisms are mutually inverse, hence both $\phi_a$ and $\psi_a$ are isomorphisms. $\square$

4.2. Let $k$ be an algebraically closed field. Let $G$ be a linear algebraic group over $k$. Let $\gamma \in \text{Aut}(k)$. Let $\tau : G \to G$ be a $\gamma$-semi-automorphism of $G$.

Let $H \subset G$ be a smooth $k$-subgroup. Set $Y = G/H$, then we have a morphism $\theta : G \times_k Y \to Y$ defining the action of $G$ on $Y$. Furthermore, the variety $Y$ has a $k$-point $y(0) = 1 \cdot H$ such that $\text{Stab}_{G(k)}(y(0)) = H(k)$, and the group of $k$-points $G(k)$ acts on $Y(k)$ transitively.

Consider the variety $\gamma_\ast Y$, the action $\gamma_\ast \theta : \gamma_\ast G \times_k \gamma_\ast Y \to \gamma_\ast Y$ of $\gamma_\ast G$ on $\gamma_\ast Y$, and the $k$-point $\gamma(y(0)) \in (\gamma_\ast Y)(k)$. As in Construction 2.4 we obtain an action

$$\tau_\theta : G \times_k \gamma_\ast Y \to \gamma_\ast Y.$$

Lemma 4.3. Let $k$, $G$, $\gamma$, $\tau$, $H$ be as in Subsection 4.2. Set $Y = G/H$ and let $\theta : G \times_k Y \to Y$ denote the canonical action. Consider the map on $k$-points

$$(G/H)(k) \to (G/\tau(H))(k), \quad g \cdot H \mapsto \tau(g) \cdot \tau(H) \quad \text{for} \quad g \in G(k).$$

Then the following assertions hold:

(i) The pointed $G$-k-variety $(\gamma_\ast Y, \tau_\ast \theta, \gamma(\gamma(y(0))))$ is isomorphic to $G/\tau(H)$;
(ii) the map $(18)$ is induced by some $\gamma$-semi-isomorphism $\nu : G/H \to G/\tau(H)$.

Proof. Let $(Z, \theta_Z, z(0))$ be a pointed $G$-k-variety, and assume that $\tau(H(k))$ fixes $z(0)$. Consider the pointed $G$-k-variety

$$((\gamma^{-1})_Z, (\tau^{-1})_Z \theta_Z, \gamma^{-1}(z(0))),$$

where the action

$$(\tau^{-1})_Z \theta_Z : G \times_k (\gamma^{-1})_Z \to (\gamma^{-1})_Z$$

is defined as in Construction 2.4 but for the pair $(\gamma^{-1}, \tau^{-1})$ instead of $(\gamma, \tau)$. By Lemma 2.5 $H(k)$ fixes $\gamma^{-1}(z(0)) \in (\gamma^{-1})_Z$. For any morphism of pointed $G$-k-varieties

$$\kappa : (Y, \theta, y(0)) \to ((\gamma^{-1})_Z, (\tau^{-1})_Z \theta_Z, \gamma^{-1}(z(0)))$$

we obtain a morphism of pointed $G$-k-varieties

$$\gamma_\ast \kappa : (\gamma_\ast Y, \tau_\ast \theta, \gamma(\gamma(y(0)))) \to (Z, \theta_Z, z(0)).$$

We see that the map $\kappa \mapsto \gamma_\ast \kappa$ is a bijection between the set of morphisms as in (20) and the set of morphisms as in (21). Since $Y = G/H$ and $H(k)$ fixes $\gamma^{-1}(z(0))$ under the action (19), we conclude by the universal property $(U''')$ for the quotient $Y = G/H$ that
the former set contains exactly one element. It follows that the latter set contains exactly one element, that is, the triple \((\gamma, Y, \tau^*; \gamma(y(0)))\) has the universal property \((U'')\). This means that \((\gamma, Y, \tau^*; \gamma(y(0)))\) is a quotient of \(G\) by \(\tau(H)\), which proves (i). It follows that there exists an isomorphism of \(G\)-\(k\)-varieties

\[(22) \quad \lambda: \gamma_s Y \to G/\tau(H) \text{ such that } g * \gamma(y(0)) \mapsto g \cdot \tau(H).\]

We set

\[(23) \quad \nu = \lambda \circ \gamma: \ G/H = Y \xrightarrow{\gamma} \gamma_s Y \xrightarrow{\lambda} G/\tau(H),\]

where \(\gamma: Y \to \gamma_s Y\) is the \(\gamma\)-semi-morphism of Example 1.5. Then we have \(\nu_\gamma = \lambda\). Since \(\nu_\gamma\) is an isomorphism of \(G\)-\(k\)-varieties, by Lemma 2.6 \(\nu\) is a \(\gamma\)-equivariant \(\gamma\)-semi-isomorphism. Since

\[\nu(1 \cdot H) = \nu(y(0)) = \lambda(\gamma(y(0))) = 1 \cdot \tau(H)\]

by (22), we have

\[\nu(g \cdot H) = \nu(g \cdot (1 \cdot H)) = \tau(g) \cdot \nu(1 \cdot H) = \tau(g) \cdot \nu(H),\]

which proves (ii). \(\Box\)

**Corollary 4.4.** Let \(G\) be a linear algebraic \(k\)-group and \(H \subseteq G\) be an algebraic \(k\)-subgroup. Set \(Y = G/H\). Let \(\gamma \in \mathrm{Aut}(k)\) and let \(\tau: G \to G\) be a \(\gamma\)-semi-automorphism of \(G\). The following three conditions are equivalent:

(i) There exists a \(\tau\)-equivariant \(\gamma\)-semi-automorphism \(\mu: Y \to Y\);

(ii) The \(G\)-\(k\)-variety \(G/\tau(H)\) is isomorphic to \(G/H\);

(iii) The algebraic subgroup \(\tau(H) \subseteq G\) is conjugate to \(H\).

**Proof.** By Corollary 2.4 there exists \(\mu: Y \to Y\) as in (i) if and only if the \(G\)-\(k\)-variety \((\gamma_s Y, \tau^*; \gamma(y(0)))\) is isomorphic to \((Y, \theta)\). By construction \((Y, \theta) = G/H\), and by Lemma 4.3 \((\gamma_s Y, \tau^*; \gamma(y(0))) \cong G/\tau(H)\). Thus (i) \(\Leftrightarrow\) (ii). By Lemma 4.1 (ii) \(\Leftrightarrow\) (iii). \(\Box\)

4.5. With the assumptions of Subsection 4.2 assume also that \(G\) is connected, then the homogeneous spaces \(Y_1 = G/H\) and \(Y_2 = G/\tau(H)\) of \(G\) are irreducible \(k\)-varieties. We consider the fields of rational functions \(K(Y_1)\) and \(K(Y_2)\). The \(\gamma\)-semi-isomorphism \(\nu: Y_1 \to Y_2\) of (23) induces an isomorphism of fields

\[\nu_*: K(Y_1) \xrightarrow{\sim} K(Y_2), \quad f_1 \mapsto f_2 = \nu_* f_1.\]

and by Lemma 1.8 we have

\[(24) \quad f_2(y_2) = \gamma(f_1(\nu^{-1}(y_2))) \text{ for } y_2 \in Y_2(k).\]

5. \(k\)-AUTOMORPHISMS OF HOMOGENEOUS SPACES

Let \(G\) be a linear algebraic group over an algebraically closed field \(k\). Let \(Y\) be a \(G\)-\(k\)-variety. We denote by \(\mathrm{Aut}^G(Y)\) the group of \(G\)-equivariant \(k\)-automorphisms of \(Y\), that is, of \(k\)-automorphisms \(\psi: Y \to Y\) such that

\[\psi(g \cdot y) = g \cdot \psi(y) \quad \text{for } g \in G, \ y \in Y.\]

We assume that \(Y\) is a homogeneous space of \(G\), that is, \(Y = G/H\), where \(H\) is a \(k\)-subgroup of \(G\). Set \(N = N_G(H)\), the normalizer of \(H\) in \(G\).

**Lemma 5.1** (well-known). For \(n \in N(k)\) we define a map on \(k\)-points

\[n_*: G/H \to G/H, \ gH \mapsto gn^{-1}H \text{ for } g \in G(k).\]

Then

(i) The map \(n_*\) is induced by some automorphism \(\phi_n \in \mathrm{Aut}^G(G/H)\);
(ii) The map
\[ \phi: N(k) \rightarrow \text{Aut}^G(G/H), \quad n \mapsto \phi_n \]
is a homomorphism inducing an isomorphism
\[ N(k)/H(k) \cong \text{Aut}^G(G/H). \]

Proof. By assumption \( n^{-1} H n = H \), and by Lemma 4.1 there exists an isomorphism \( \phi_n: G/H \rightarrow G/H \) such that \( \phi_n(g \cdot H) = g n^{-1} \cdot H \), which proves (i).

Clearly the map \( \phi \) of (25) is a homomorphism with kernel \( H(k) \). To prove (ii) it remains to show that \( \phi \) is surjective.

Let \( \psi \in \text{Aut}^G(G/H) \). Write \( \psi(1 \cdot H) = a \cdot H \) with \( a \in G(k) \). Since \( \psi \) is an isomorphism of \( G \)-\( k \)-varieties, we have \( \text{Stab}_{G(k)}(a \cdot H) = \text{Stab}_{G(k)}(1 \cdot H) = H(k) \). On the other hand, \( \text{Stab}_{G(k)}(a \cdot H) = a \cdot H(k) \cdot a^{-1} \). Thus \( a \cdot H \cdot a^{-1} = H \), hence \( a \in N(k) \). We have \( \psi(g \cdot H) = g a \cdot H \). Write \( a = n^{-1} \), then \( n \in N(k) \) and \( \psi = \phi_n \). Thus the homomorphism \( \phi \) is surjective, as required. \( \square \)

Corollary 5.2. If \( N_G(H) = H \), then \( \text{Aut}^G(G/H) = \{1\} \). \( \square \)

6. EQUIVARIANT MODELS OF \( G \)-VARIETIES

Let \( k \) be an algebraically closed field, and \( k_0 \subset k \) be a subfield such that \( k \) is a Galois extension of \( k_0 \), that is, \( k_0 \) is a perfect field and \( k \) is an algebraic closure of \( k_0 \). We write \( \Gamma = \text{Gal}(k/k_0) := \text{Aut}(k/k_0) \).

Let \( Y \) be a \( k \)-variety. Let \( \gamma, \gamma' \in \Gamma \). If \( \mu \) is a \( \gamma \)-semi-automorphism of \( Y \), and \( \mu' \) is a \( \gamma' \)-semi-automorphism of \( Y \), then \( \mu \circ \mu' \) is a \( \gamma \circ \gamma' \)-semi-automorphism of \( Y \) and \( \mu^{-1} \) is a \( \gamma^{-1} \)-semi-automorphism of \( Y \).

We denote by \( \text{SAut}_{k/k_0}(Y) \) or just by \( \text{SAut}(Y) \) the group of all \( \gamma \)-semi-automorphisms \( \mu \) of \( Y \) where \( \gamma \) runs over \( \Gamma = \text{Gal}(k/k_0) \).

A \( k_0 \)-model of \( Y \) is a \( k_0 \)-variety \( Y_0 \) together with an isomorphism of \( k \)-varieties
\[ \varphi_Y: Y_0 \times_{k_0} k \rightarrow Y. \]
Note that \( \gamma \in \Gamma \) defines a \( \gamma \)-semi-automorphism of \( Y_0 \times_{k_0} k \)
\[ \text{id}_{Y_0} \times (\gamma^*)^{-1}: Y_0 \times_{\text{Spec} k_0} \text{Spec} k \rightarrow Y_0 \times_{\text{Spec} k_0} \text{Spec} k \]
and thus, via \( \varphi_Y \), a \( \gamma \)-semi-automorphism \( \mu_\gamma \) of \( Y \). We obtain a homomorphism
\[ \Gamma \rightarrow \text{SAut}(Y), \quad \gamma \mapsto \mu_\gamma. \]

Conversely:

Lemma 6.1 (Borel and Serre [BS64] Lemma 2.12). Let \( k, k_0, \Gamma, Y \) be as above. Assume that for any \( \gamma \in \Gamma \) we have a \( \gamma \)-semi-automorphism \( \mu_\gamma \) of \( Y \) such that

(i) the map \( \Gamma \rightarrow \text{SAut}_{k/k_0}(Y), \quad \gamma \mapsto \mu_\gamma \), is a homomorphism,

(ii) the restriction of this map to \( \text{Gal}(k/k_1) \) for some finite Galois extension \( k_1/k_0 \) in \( k \) comes from a \( k_1 \)-model \( Y_1 \) of \( Y \),

(iii) \( Y \) is quasi-projective.

Then there exists a \( k_0 \)-model \( Y_0 \) of \( Y \) that defines this homomorphism \( \gamma \mapsto \mu_\gamma \). \( \square \)

6.2. Let \( G \) be a linear algebraic group over \( k \). We assume that we are given a \( k_0 \)-model of \( G \), that is, a linear algebraic group \( G_0 \) over \( k_0 \) together with an isomorphism of algebraic \( k \)-groups over \( \varphi_G: G_0 \times_{k_0} k \rightarrow G \). For \( \gamma \in \Gamma \) the automorphism \( (\gamma^*)^{-1} \) of \( \text{Spec} k \) induces a \( \gamma \)-semi-automorphism \( \text{id}_{G_0} \times (\gamma^*)^{-1} \) of \( G_0 \times_{\text{Spec} k_0} \text{Spec} k \). We identify \( G \) with \( G_0 \times_{\text{Spec} k_0} \text{Spec} k \).
Spec $k$ via $x_G$; then for any $\gamma \in \Gamma$ we obtain a $\gamma$-semi-automorphism $\sigma_\gamma : G \to G$. The map

$$\Gamma \to \text{SAut}(G), \quad \gamma \mapsto \sigma_\gamma$$

is a homomorphism. We identify $\gamma_\ast G$ with $G$ using $(\sigma_\gamma)_\ast : \gamma_\ast G \to G$.

Let $(Y, \theta)$ be a $G$-$k$-variety. By a $G_0$-equivariant $k_0$-model of the $G$-$k$-variety $Y$ we mean a $G_0$-$k_0$-variety $(Y_0, \theta_0)$ together with an isomorphism $\varphi_Y : Y_0 \times_{k_0} k \sim Y$ such that the following diagram commutes:

$$\begin{array}{ccc}
G_0 \times_{k_0} k & \longrightarrow & Y_0 \times_{k_0} k \\
\varphi_Y \downarrow & & \varphi_Y \downarrow \\
G \times_k Y & \longrightarrow & Y
\end{array}$$

where $G_0 := G_0 \times_{k_0} k$ and $Y_0 := Y_0 \times_{k_0} k$. For a given $k_0$-model $G_0$ of $G$ we ask whether there exists a $G_0$-equivariant $k_0$-model $Y_0$ of $Y$.

Let $(Y, \theta)$ be a $G$-$k$-variety. We write $g \cdot y$ for $\theta(g, y)$. Recall (Definition 2.3) that a $\gamma$-semi-automorphism $\mu$ of $Y$ is $\sigma_\gamma$-equivariant if the following diagram commutes:

$$\begin{array}{ccc}
G \times_k Y & \longrightarrow & Y \\
\sigma_\gamma \times \mu \downarrow & & \mu \downarrow \\
G \times_k Y & \longrightarrow & Y
\end{array}$$

Since $k$ is algebraically closed, $G$ is smooth (reduced) and $Y$ is reduced, this is the same as to require that

$$\mu(g \cdot y) = \sigma_\gamma(g) \cdot \mu(y) \quad \text{for all } g \in G(k), \ y \in Y(k).$$

We ask whether there exists such $\mu$.

A $G_0$-equivariant $k_0$-model of $Y$ defines a homomorphism

$$\Gamma \to \text{SAut}_{k_0/Y}(Y), \quad \gamma \mapsto \mu_\gamma,$$

where for any $\gamma \in \Gamma$, the $\gamma$-semi-automorphism $\mu_\gamma$ of $Y$ is $\sigma_\gamma$-equivariant. Conversely:

**Lemma 6.3.** Let $k$, $k_0$, $\Gamma = \text{Gal}(k/k_0)$, $G$, $(Y, \theta)$ be as above and let $G_0$ be a $k_0$-model of $G$. Assume that for any $\gamma \in \Gamma$ we have a $\gamma$-semi-automorphism $\mu_\gamma$ of $Y$ such that the following conditions are satisfied:

(i) the map $\Gamma \to \text{SAut}_{k_0/Y}(Y)$, $\gamma \mapsto \mu_\gamma$ is a homomorphism,

(ii) the restriction of this map to $\text{Gal}(k/k_1)$ for some finite Galois extension $k_1/k_0$ in $k$ comes from a $G_1$-equivariant $k_1$-model $Y_1$ of $Y$, where $G_1 = G_0 \times_{k_0} k_1$,

(iii) $Y$ is quasi-projective,

(iv) for any $\gamma \in \Gamma$, the $\gamma$-semi-automorphism $\mu_\gamma$ is $\sigma_\gamma$-equivariant.

Then there exists a $G_0$-equivariant $k_0$-model $Y_0$ of $Y$ that defines this homomorphism $\gamma \mapsto \mu_\gamma$.

**Proof.** By Lemma 6.1 the homomorphism

$$\Gamma \to \text{SAut}(Y), \quad \gamma \mapsto \mu_\gamma$$

defines a $k_0$-model $Y_0$ of $Y$. Using Galois descent for morphisms (see e.g. Jahnel [Ja00 Proposition 2.8]) we obtain from condition (iv) that $\theta$ comes from some morphism $\theta_0 : G_0 \times_{k_0} Y_0 \to Y_0$, and the $k_0$-model $(Y_0, \theta_0)$ of $(Y, \theta)$ is $G_0$-equivariant. $\square$
Remark 6.4. If in Lemma 6.3 we do not assume that $Y$ is quasi-projective, then we obtain a $k_0$-model $Y_0$ in the category of algebraic $k_0$-spaces (see Wedhorn [We15, Proposition 8.1]), but not necessarily in the category of $k_0$-schemes (even when $k = \mathbb{C}$ and $k_0 = \mathbb{R}$, see Huruguen [Hu11, Theorem 2.35]).

Lemma 6.5. Let $k$, $k_0$, $\Gamma = \text{Gal}(k/k_0)$, $G$, $(Y, \theta)$ be as above and let $G_0$ be a $k_0$-model of $G$. Assume that $\text{Aut}^G(Y) = \{1\}$. Assume that for any $\gamma \in \Gamma$ there exists a $\gamma$-semi-automorphism $\mu_{\gamma}$ of $Y$ satisfying condition (iv) of Lemma 6.3. Then such $\mu_{\gamma}$ is unique, and the map $\gamma \mapsto \mu_{\gamma}$ satisfies conditions (i) and (ii) of Lemma 6.3.

Proof. If $\mu_{\gamma}'$ is another such $\gamma$-semi-automorphism, then $\mu_{\gamma}^{-1}\mu_{\gamma}' \in \text{Aut}^G(Y) = \{1\}$, hence $\mu_{\gamma} = \mu_{\gamma}'$. If $\gamma, \delta \in \Gamma$, then $\mu_{\gamma}^{-1}\mu_{\gamma}\mu_{\delta} \in \text{Aut}^G(Y) = \{1\}$, hence $\mu_{\gamma,\delta} = \mu_{\gamma}\mu_{\delta}$, hence the map $\gamma \mapsto \mu_{\gamma}$ is a homomorphism, that is, condition (i) holds.

Since $G$ and $Y$ are of finite type over $k$, there exists a finite Galois extension $k_1/k$ in $k$ and a $G_1$-equivariant $k_1$-model $(Y_1, \theta_1)$ of $(Y, \theta)$, where $G_1 := G_0 \times_{k_0} k_1$. This $k_1$-model defines a homomorphism

$$\gamma \mapsto \mu_{\gamma}' : \text{Gal}(k_1/k_1) \to \text{SAut}(Y)$$

such that $\mu_{\gamma}'$ is $\sigma_{\gamma}$-equivariant for all $\gamma \in \text{Gal}(k_1/k_1)$. Since a $\sigma_{\gamma}$-equivariant $\gamma$-semi-automorphism is unique, we see that for all $\gamma \in \text{Gal}(k_1/k_1)$ we have $\mu_{\gamma} = \mu_{\gamma}'$, and hence, the restriction of the map

$$\gamma \mapsto \mu_{\gamma} : \Gamma \to \text{SAut}(Y)$$

to $\text{Gal}(k_1/k_1)$ comes from the $k_1$-model $(Y_1, \theta_1)$ of $(Y, \theta)$, that is, condition (ii) of Lemma 6.3 is satisfied. \hfill $\square$

7. Spherical homogeneous spaces and their combinatorial invariants

Let $G$ be a connected reductive group over an algebraically closed field $k$. We describe combinatorial invariants (invariants of Luna and Losev) of a spherical homogeneous space $Y = G/H$ of $G$.

We start with combinatorial invariants of $G$. We fix $T \subset B \subset G$, where $B$ is a Borel subgroup and $T$ is a maximal torus. Let $\text{BRD}(G) = \text{BRD}(G, T, B)$ denote the based root datum of $G$. We have

$$\text{BRD}(G, T, B) = (X, X^\vee, R, R^\vee, S, S^\vee)$$

where

- $X = X^T(T) := \text{Hom}(T, \mathbb{G}_{m,k})$ is the character group of $T$;
- $X^\vee = X_*(T) := \text{Hom}(\mathbb{G}_{m,k}, T)$ is the cocharacter group of $T$;
- $R = R(G, T) \subset X$ is the root system;
- $R^\vee \subset X^\vee$ is the coroot system;
- $S = S(G, T, B) \subset R$ is the system of simple roots (the basis of $R$) defined by $B$;
- $S^\vee \subset R^\vee$ is the system of simple coroots.

There is a canonical pairing $X \times X^\vee \rightarrow \mathbb{Z}$, $(\chi, x) \mapsto \langle \chi, x \rangle$, and a canonical bijection $\alpha \mapsto \alpha^\vee : R \rightarrow R^\vee$ such that $S^\vee = \{\alpha^\vee \mid \alpha \in S\}$. See Springer [Sp79, Sections 1 and 2] for details.

We consider also the Dynkin diagram $\text{Dyn}(G) = \text{Dyn}(G, T, B)$, which is a graph with the set of vertices $S$. The edge between two simple roots $\alpha, \beta \in S$ is described in terms of the integers $\langle \alpha, \beta^\vee \rangle$ and $\langle \beta, \alpha^\vee \rangle$.

We call a pair $(T, B)$ as above a Borel pair. If $(T', B')$ is another Borel pair, then by Theorem 11.1 and Theorem 10.6(4) in Borel’s book [B91], there exists $g \in G(k)$ such that

$$g \cdot T \cdot g^{-1} = T', \quad g \cdot B \cdot g^{-1} = B'.$$
This element $g$ induces an isomorphism
\[ g^*: \text{BRD}(G, T', B') \xrightarrow{\sim} \text{BRD}(G, T, B). \]
If $g' \in G(k)$ another element as in [29], then $g = gt$ for some $t \in T(k)$, and therefore, the isomorphism
\[ (g')^*: \text{BRD}(G, T', B') \xrightarrow{\sim} \text{BRD}(G, T, B) \]
coincides with $g^*$. Thus we can canonically identify $\text{BRD}(G, T', B')$ with $\text{BRD}(G, T, B)$ and write $\text{BRD}(G)$ for $\text{BRD}(G, T, B)$. We say that $\text{BRD}(G)$ is the canonical root datum of $G$. We see that the based root datum $\text{BRD}(G)$ is an invariant of $G$. In particular, the character lattice $X = X^*(T)$ with the subset $S \subset X$ is an invariant, and the Dynkin diagram $\text{Dyn}(G)$ is an invariant.

Now we describe combinatorial invariants of a homogeneous spherical $G$-variety $Y = G/H$. Let $K(Y)$ denote the field of rational functions of $Y$. The group $G(k)$ acts on $K(Y)$ by
\[ (g \cdot f)(y) = f(g^{-1}y) \quad \text{for } f \in K(Y), \; g \in G(k), \; \text{and } y \in Y(k). \]
For $\chi \in X^*(B)$ let $K(Y)_\chi$ denote the space of $\chi$-eigenfunctions in $K(Y)$, that is, the $k$-space of rational functions $f \in K(X)$ such that
\[ b \cdot f = \chi(b) \cdot f \quad \text{for all } b \in B(k). \]
Since $B$ has an open dense orbit in $Y$, the $k$-dimension of $K(Y)_\chi$ is $\leq 1$. Let $\mathcal{X} = \mathcal{X}(Y) \subset X^*(B)$ denote the set of characters $\chi$ of $B$ such that $K(Y)_\chi \neq 0$, which is a subgroup of $X^*(B)$ called the weight lattice of $Y$. We set
\[ V = V(Y) = \text{Hom}_\mathbb{Z}(\mathcal{X}, \mathbb{Q}). \]

Let $\text{Val}(K(Y))$ denote the set of $\mathbb{Q}$-valued valuations of the field $K(Y)$ that are trivial on $k$. The group $G(k)$ naturally acts on $K(Y)$ and on $\text{Val}(K(Y))$. We will consider the set $\text{Val}^B(K(Y))$ of $B(k)$-invariant valuations, and the set $\text{Val}^G(K(Y))$ of $G(k)$-invariant valuations. We have a canonical map
\[ \rho: \text{Val}^B(K(Y)) \to V, \; v \mapsto (\chi \mapsto v(\chi)), \]
where $v \in \text{Val}^B(K(Y))$, $\chi \in \mathcal{X}$, $f_\chi \in K(Y)_\chi$, $f_\chi \neq 0$. It is known, see Knop [Kn89, Corollary 1.8], that the restriction of $\rho$ to $\text{Val}^G(K(Y))$ is injective. We denote by
\[ V = V(Y) := \rho(\text{Val}^G(K(Y))) \subset V \]
the image of $\text{Val}^G(K(Y))$ in $V$. It is a cone in $V$ called the valuation cone of $Y$.

Let $D = D(Y)$ denote the set of colors of $Y$, that is, the set of closures of $B$-orbits of codimension 1 in $Y$; it is finite. Each $D \in D$ is a $B$-invariant divisor, which defines a $B$-invariant valuation of $K(Y)$ that we denote by $\text{val}(\mathcal{D})$. Thus we obtain a map
\[ \text{val}: D \to \text{Val}^B(K(Y)). \]
By abuse of notation we denote $\rho(\text{val}(\mathcal{D})) \in V$ by $\rho(\mathcal{D})$. Thus we obtain a map
\[ \rho: D \to V, \]
which in general is not injective (for example, it is not injective for $G$ and $Y$ as in Example 0.1).

For $\mathcal{D} \in D$, let $\text{Stab}_G(\mathcal{D})$ denote the stabilizer of $D \subset Y$ in $G$. Clearly $\text{Stab}_G(\mathcal{D}) \supset B$, hence $\text{Stab}_G(\mathcal{D})$ is a parabolic subgroup of $G$. For $\alpha \in S$, let $P_\alpha \supset B$ denote the corresponding minimal parabolic subgroup of $G$ containing $B$. Let $\varsigma(\mathcal{D})$ denote the set of $\alpha \in S$ for which $P_\alpha$ is not contained in $\text{Stab}_G(\mathcal{D})$. We obtain a map
\[ \varsigma: D \to \mathcal{P}(S), \]
where $\mathcal{P}(S)$ denotes the set of all subsets of $S$. 
Lemma 7.1. Any fiber of the map \( \varsigma \) has \( \leq 2 \) elements.

Proof. Let \( D \in \mathcal{D} \). Since \( Y \) is a homogeneous \( G \)-variety, clearly \( \text{Stab}_G(D) \neq G \), hence \( \varsigma(D) \neq \emptyset \). We see that there exists \( \alpha \in \varsigma(D) \). Consider the set \( \mathcal{D}(\alpha) \) consisting of those \( D \in \mathcal{D} \) for which \( \alpha \in \varsigma(D) \). By Proposition [3.2] in Appendix [3] below we have \( |\mathcal{D}(\alpha)| \leq 2 \), and the lemma follows. \( \square \)

Consider the map
\[ \rho \times \varsigma : \mathcal{D} \rightarrow V \times \mathcal{P}(S). \]

Corollary 7.2. Any fiber of the map \( \rho \times \varsigma \) has \( \leq 2 \) elements. \( \square \)

Consider the subset \( \Omega := \text{im}(\rho \times \varsigma) \subset V \times \mathcal{P}(S) \). Let \( \Omega^{(2)} \) (resp. \( \Omega^{(1)} \)) denote the subset of \( \Omega \) consisting of the elements with two preimages (resp. with one preimage) in \( \mathcal{D} \). We obtain two subsets \( \Omega^{(1)}, \Omega^{(2)} \subset V \times \mathcal{P}(S) \), and by Corollary [22] we have \( \Omega = \Omega^{(1)} \cup \Omega^{(2)} \) (disjoint union).

Definition 7.3. Let \( G \) be a connected reductive group over an algebraically closed field \( k \). Let \( Y = G/H \) be a spherical homogeneous space of \( G \). By the combinatorial invariants of \( Y \) we mean
\[ \mathcal{X} \subset X^*(B), \quad V \subset V := \text{Hom}_Z(\mathcal{X}, Q), \quad \text{and} \quad \Omega^{(1)}, \Omega^{(2)} \subset V \times \mathcal{P}(S). \]

7.4. Let \( G \) be a connected reductive \( k \)-group. Let \( H_1 \subset G \) be a spherical subgroup, then we set \( Y_1 = G/H_1 \). We consider the set of colors \( \mathcal{D}(Y_1) \) and the canonical maps
\[ \rho_1 : \mathcal{D}(Y_1) \rightarrow V(Y_1), \quad \varsigma_1 : \mathcal{D}(Y_1) \rightarrow \mathcal{P}(S). \]
If \( H_2 \subset G \) is another spherical subgroup, then we set \( Y_2 = G/H_2 \) and consider the set of colors \( \mathcal{D}(Y_2) \) and the canonical maps
\[ \rho_2 : \mathcal{D}(Y_2) \rightarrow V(Y_2), \quad \varsigma_2 : \mathcal{D}(Y_2) \rightarrow \mathcal{P}(S). \]
Now assume that there exists \( a \in G(k) \) such that \( H_2 = aH_1a^{-1} \). Then we have an isomorphism of \( G \)-varieties of Lemma [1.1]
\[ \phi_a : Y_1 \rightarrow Y_2, \quad g \cdot H_1 \mapsto ga^{-1} \cdot H_2. \]
It follows that \( X(Y_1) = X(Y_2), V(Y_1) = V(Y_2), V(Y_1) = V(Y_2) \). Moreover, the \( G \)-equivariant map \( \phi_a : Y_1 \rightarrow Y_2 \) induces a bijection
\[ (\phi_a)_* : \mathcal{D}(Y_1) \rightarrow \mathcal{D}(Y_2) \]
satisfying
\[ \rho_2 \circ (\phi_a)_* = \rho_1, \quad \varsigma_2 \circ (\phi_a)_* = \varsigma_1. \]
It follows that
\[ \Omega^{(1)}(Y_1) = \Omega^{(1)}(Y_2) \quad \text{and} \quad \Omega^{(2)}(Y_1) = \Omega^{(2)}(Y_2). \]
Conversely:

Proposition 7.5 ( Losev’s Uniqueness Theorem [Lo09 Theorem 1]). Let \( G \) be a connected reductive group over an algebraically closed field \( k \) of characteristic 0. Let \( H_1, H_2 \subset G \) be two spherical subgroups, and let \( Y_1 = G/H_1 \) and \( Y_2 = G/H_2 \) be the corresponding spherical homogeneous spaces. If \( X(Y_1) = X(Y_2), V(Y_1) = V(Y_2), \Omega^{(1)}(Y_1) = \Omega^{(1)}(Y_2), \) and \( \Omega^{(2)}(Y_1) = \Omega^{(2)}(Y_2) \), then there exists \( a \in G(k) \) such that \( H_2 = aH_1a^{-1} \).

7.6. Consider the group \( \text{Aut}^G(Y) = N_G(H)/H \), this group acts on \( \mathcal{D} \). We consider the surjective map
\[ \varsigma = \rho \times \varsigma : \mathcal{D} \rightarrow \Omega. \]
By Corollary [22] each of the fibers of \( \varsigma \) has either one or two elements. We denote by \( \text{Aut}_\Omega(\mathcal{D}) \) the group of permutations \( \pi : \mathcal{D} \rightarrow \mathcal{D} \) such that \( \varsigma \circ \pi = \varsigma \). It is clear that
the group $\text{Aut}^G(Y)$, when acting on $D$, acts on the fibers of the map $\zeta$, so we obtain a homomorphism

$$\text{Aut}^G(Y) \to \text{Aut}_\Omega(D).$$

**Theorem 7.7** (Losev, unpublished). Let $G$ be a connected reductive group over an algebraically closed field $k$ of characteristic $0$. Let $Y = G/H$ be a spherical homogeneous space of $G$. Then, with the above notation, the homomorphism

$$\text{Aut}^G(Y) \to \text{Aut}_\Omega(D).$$

is surjective.

This theorem will be proved in Appendix B, see Theorem B.4.

**Corollary 7.8** (Strong version of Losev’s Uniqueness Theorem). Let $G$, $H_1$, $H_2$, $Y_1 = G/H_1$, $Y_2 = G/H_2$ be as in Proposition 7.5, in particular, $\mathcal{X}(Y_1) = \mathcal{X}(Y_2)$, $\mathcal{V}(Y_1) = \mathcal{V}(Y_2)$, $\Omega^{(1)}(Y_1) = \Omega^{(1)}(Y_2)$, and $\Omega^{(2)}(Y_1) = \Omega^{(2)}(Y_2)$. Let $\varphi : D(Y_1) \to D(Y_2)$ be any bijection satisfying

$$\rho_2 \circ \varphi = \rho_1, \quad \varsigma_2 \circ \varphi = \varsigma_1,$$

(such a bijection exists because $\Omega^{(1)}(Y_1) = \Omega^{(1)}(Y_2)$ and $\Omega^{(2)}(Y_1) = \Omega^{(2)}(Y_2)$). Then there exists $a' \in G(k)$ such that $H_2 = a'H_1(a')^{-1}$ and

$$(\phi_{a'})_* = \varphi : D(Y_1) \to D(Y_2).$$

*Proof.* By Proposition 7.5 there exists $a \in G(k)$ such that $H_2 = aH_1a^{-1}$. This element $a$ defines a map

$$(\phi_a)_* : D(Y_1) \to D(Y_2)$$

satisfying (31). Set

$$\psi = (\phi_a)^{-1}_* \circ \varphi : D(Y_1) \to D(Y_1),$$

then $\psi$ satisfies

$$\rho_1 \circ \psi = \rho_1, \quad \varsigma_1 \circ \psi = \varsigma_1,$$

hence $\psi \in \text{Aut}_\Omega D(Y_1)$. By Theorem 7.7 there exists $n \in \mathcal{N}_G(H_1)$ such that

$$(\phi_n)_* = \psi : D(Y_1) \to D(Y_1).$$

We set $a' = an$, then $a'H_1(a')^{-1} = H_2$, $\phi_{a'} = \phi_a \circ \phi_n$, and

$$(\phi_{a'})_* = (\phi_a)_* \circ (\phi_n)_* = (\phi_a)_* \circ \psi = \varphi.$$ 

*Corollary 7.9.* If in Theorem 7.7 $H$ is spherically closed, then the homomorphism (31) is an isomorphism.

*Proof.* Indeed, since $H$ is spherically closed, the homomorphism (31) is injective, and by Theorem 7.7 it is surjective, hence it is bijective, as required.

Note that

$$\text{Aut}_\Omega(D) = \prod_{\omega \in \Omega} \text{Aut}(\zeta^{-1}(\omega)),$$

where $\text{Aut}(\zeta^{-1}(\omega))$ is the group of permutations of the set $\zeta^{-1}(\omega)$. It is clear that for any $\omega \in \Omega$, the restriction homomorphism

$$(32) \quad \text{Aut}_\Omega(D) \to \text{Aut}(\zeta^{-1}(\omega))$$

is surjective.

**Corollary 7.10.** If in Theorem 7.7 $\mathcal{N}_G(H) = H$, then the surjective map $\zeta$ is bijective, hence $D$ injects into $V \times P(S)$. 
Proof. It follows from Theorem [17] and the surjectivity of the homomorphism [52], that the group \( \text{Aut}^G(Y) = \mathcal{N}_G(H)/H \) acts transitively on the fiber \( \zeta^{-1}(\omega) \) for any \( \omega \in \Omega \). Since by assumption \( \mathcal{N}_G(H)/H = \{1\} \), we conclude that each fiber of \( \zeta \) has exactly one element, hence \( \zeta \) is bijective, as required. \( \square \)

8. Action of an automorphism of the base field
on the combinatorial invariants of a spherical homogeneous space

8.1. Let \( k \) be an algebraically closed field, \( G \) be a connected reductive group over \( k \), \( H \subset G \) be a spherical subgroup, and \( Y_1 = G/H \) be the corresponding spherical homogeneous space.

Let \( k_0 \subset k \) be a subfield and let \( \gamma \in \text{Aut}(k/k_0) \). We assume that \( “G \) is defined over \( k_0” \), that is, we are given a \( k_0 \)-model \( G_0 \) of \( G \). Then we have a \( \gamma \)-semi-automorphism \( \sigma_\gamma \) of \( \sigma \), see Subsection 6.2. Set \( H_2 = \sigma_\gamma(H_1) \subset G \) and denote by \( Y_2 := G/H_2 \) the corresponding spherical homogeneous space.

We wish to know whether the spherical homogeneous spaces \( Y_1 \) and \( Y_2 \) are isomorphic, and we wish to know whether \( Y_1 \) and \( Y_2 \) are isomorphic.

We fix a Borel pair \((T, B)\), then \( T \subset B \subset G \). Consider
\[
\sigma_\gamma(T) \subset \sigma_\gamma(B) \subset G.
\]
Then \((\sigma_\gamma(T), \sigma_\gamma(B))\) is again a Borel pair, hence there exists \( g_\gamma \in G(k) \) such that
\[
g_\gamma \cdot \sigma_\gamma(T) \cdot g_\gamma^{-1} = T, \quad g_\gamma \cdot \sigma_\gamma(B) \cdot g_\gamma^{-1} = B.
\]
Set \( \tau = \text{inn}(g_\gamma) \circ \sigma_\gamma : G \rightarrow G \), then \( \tau \) is a \( \gamma \)-semi-automorphism of \( G \), and
\[
\tau(B) = B, \quad \tau(T) = T.
\]
Set \( H'_2 = \tau(H_1) \subset G \) and \( Y'_2 = G/H'_2 \). We have \( H'_2 = g_\gamma \cdot H_2 \cdot g_\gamma^{-1} \), so by Lemma 4.1 \( Y_2 \) and \( Y'_2 \) are isomorphic, and we wish to know whether \( Y_1 \) and \( Y'_2 \) are isomorphic.

By (33), \( \tau \) acts on the characters of \( T \) and \( B \); we denote the corresponding automorphism by \( \varepsilon_\gamma \). By definition
\[
(34) \quad \varepsilon_\gamma(\chi)(b) = \gamma(\chi(\tau^{-1}(b))) \text{ for } \chi \in X^*(B), \ b \in B(k),
\]
and the same for the characters of \( T \) (recall that \( X^*(B) = X^*(T) \)). Since \( \tau(B) = B \), we see that \( \varepsilon_\gamma \) when acting on \( X^*(T) \), preserves \( S = S(G, T, B) \subset X^*(T) \). It is well known (see e.g. [BKLR14] 3.2 and Proposition 3.1(a)) that the automorphism \( \varepsilon_\gamma \) does not depend on the choice of \( g_\gamma \) and that the map
\[
\varepsilon : \text{Aut}(k/k_0) \rightarrow \text{Aut}(X^*(T), S), \quad \gamma \mapsto \varepsilon_\gamma
\]
is a homomorphism. Since \( \varepsilon_\gamma \) acts on \( X^*(B) \) and on \( S \), one can define \( \varepsilon_\gamma(\Omega_1(Y_1)), \varepsilon_\gamma(\Omega_2(Y_1)), \varepsilon_\gamma(\Omega_1(Y_1)), \varepsilon_\gamma(\Omega_2(Y_1)) \).

Following Akhiezer [Akh85], we compute the combinatorial invariants of the spherical homogeneous space \( Y'_2 \). We define a map
\[
(35) \quad Y_1(k) \rightarrow Y'_2(k), \quad g \cdot H_1 \mapsto \tau(g) \cdot H'_2, \text{ where } g \in G(k).
\]
By Lemma 4.3 the map (35) is induced by some \( \gamma \)-semi-isomorphism
\[
\nu : Y_1 \rightarrow Y'_2,
\]
and so we obtain an isomorphism of the function fields
\[
\nu_* : K(Y_1) \rightarrow K(Y'_2).
\]

Lemma 8.2. Let \( \chi_1 \in X^*(B) \) and assume that \( f_1 \in K(Y_1)_{\chi_1} \). Then \( \nu_* f_1 \in K(Y'_2)_{\chi_2} \), where \( \chi_2 = \varepsilon_\gamma(\chi_1) \).
Proof. By assumption
\[ f_1(b^{-1} y_1) = \chi_1(b) \cdot f_1(y_1) \quad \text{for all } y_1 \in Y_1(k), \ b \in B(k). \]
We write \( f'_2 = \nu_1 f_1 \in K(Y'_2) \). Since \( \nu : Y_1 \to Y'_2 \) is a \( \gamma \)-semi-isomorphism, by Corollary 8.3 we have
\[ f'_2(y'_2) = \gamma(f_1(\nu^{-1}(y'_2))) \quad \text{for } y'_2 \in Y'_2(k). \]
Note that \( \tau^{-1} : G \to G \) is a \( \gamma^{-1} \)-semi-automorphism of \( G \), and \( \nu^{-1} : Y'_2 \to Y_1 \) is a \( \tau^{-1} \)-equivariant \( \gamma^{-1} \)-semi-isomorphism. Moreover, \( \tau^{-1}(T) = T \) and \( \tau^{-1}(B) = B \). We compute:
\[
\begin{align*}
f'_2(b^{-1} \cdot y'_2) &= \gamma(f_1(\nu^{-1}(b^{-1} \cdot y'_2))) = \gamma(f_1((\tau^{-1}(b))^{-1} \cdot \nu^{-1}(y'_2))) \\
&= \gamma(\chi_1(\tau^{-1}(b))) \cdot \gamma(f_1(\nu^{-1}(y'_2))) = \varepsilon_\gamma(\chi_1)(b) \cdot f'_2(y'_2).
\end{align*}
\]
Thus \( f'_2 \in K(Y'_2)^{(B)} \), where \( \chi_2 = \varepsilon_\gamma(\chi_1) \), as required. \( \square \)

Corollary 8.3. \( \mathcal{X}(Y'_2) = \varepsilon_\gamma(\mathcal{X}(Y_1)) \).

Proof. By Lemma 8.2 we have \( \varepsilon_\gamma(\mathcal{X}(Y_1)) \subset \mathcal{X}(Y'_2) \). Applying Lemma 8.2 to the triple \((\gamma^{-1}, \tau^{-1}, \nu^{-1})\), we obtain that \( \varepsilon_{\gamma^{-1}}(\mathcal{X}(Y'_2)) \subset \mathcal{X}(Y_1) \), hence \( \mathcal{X}(Y'_2) \subset \varepsilon_\gamma(\mathcal{X}(Y_1)) \). Thus \( \mathcal{X}(Y'_2) = \varepsilon_\gamma(\mathcal{X}(Y_1)) \), as required. \( \square \)

Let \( v_1 \in \text{Val}^B(K(Y_1)) \). We define \( \nu_1 v_1 \in \text{Val}^B(K(Y'_2)) \) by
\[ (\nu_1 v_1)(f'_2) = v_1(\nu^{-1}_1(f'_2)) \quad \text{for } f'_2 \in K(Y'_2). \]

We consider the maps
\[ \rho_1 : \text{Val}^B(K(Y_1)) \to V(Y_1) \quad \text{and} \quad \rho'_2 : \text{Val}^B(K(Y'_2)) \to V(Y'_2). \]

Lemma 8.4. For any \( v_1 \in \text{Val}^B(K(Y_1)) \) we have
\[ \rho'_2(\nu_1 v_1) = \varepsilon_\gamma(\rho_1(v_1)). \]

Proof. See Huruguen \cite[Proposition 2.18]{Hu11}. \( \square \)

Corollary 8.5. \( \mathcal{V}(Y'_2) = \varepsilon_\gamma(\mathcal{V}(Y_1)) \).

Remark 8.7. Propositions 2.18 and 2.19 of Huruguen \cite[Section 2.2]{Hu11} are proved in his paper under certain additional assumptions. Namely, Huruguen assumes that that \( k/k_0 \) is a Galois extension, that the triple \((G, Y, \theta)\) has a \( k_0 \)-model \((G_0, Y_0, \theta_0)\), and that \( Y_0 \) has a \( k_0 \)-point \( y^{(0)} \). Those assumptions are not used in his proof.

By abuse of notation, if \( D_1 \in \mathcal{D}(Y_1) \) and \( D'_2 \in \mathcal{D}(Y'_2) \), we write \( \rho_1(D_1) \) for \( \rho_1(\text{val}_1(D_1)) \in V(Y_1) \) and \( \rho'_2(D'_2) \) for \( \rho'_2(\text{val}_2(D'_2)) \in V(Y'_2) \).

Corollary 8.8 (from Lemma 8.3 and Lemma 8.6). For any \( D_1 \in \mathcal{D}(Y_1) \) we have
\[ \rho'_2(\nu_1 D_1) = \varepsilon_\gamma(\rho_1(D_1)). \]

Lemma 8.9. For any \( D_1 \in \mathcal{D}(Y_1) \) we have:
Proposition 8.12. □

and only if the subgroup $\sigma H$ (Uniqueness Theorem) the subgroups $0$, then by Proposition 8.11 the equalities (37) hold, and by Proposition 7.5 (Losev’s invariants of $Y$.)

Proof. (i) follows from the fact that the map $\nu: Y_1(k) \to Y_2(k)$ is $\tau$-equivariant, and (ii) follows from (i).

Corollary 8.10 (from Corollary 8.8 and Lemma 8.9).

$\Omega(1)(Y_2) = \varepsilon_\gamma(\Omega(1)(Y_1))$ and $\Omega(2)(Y_2) = \varepsilon_\gamma(\Omega(2)(Y_1))$.

Proposition 8.11.

$X(Y_2) = \varepsilon_\gamma(X(Y_1))$, $V(Y_2) = \varepsilon_\gamma(V(Y_1))$, $\Omega(1)(Y_2) = \varepsilon_\gamma(\Omega(1)(Y_1))$, $\Omega(2)(Y_2) = \varepsilon_\gamma(\Omega(2)(Y_1))$.

Proof. Since $H_1^2$ and $H_2$ are conjugate, by Lemma 4.1 the $G$-varieties $Y_2$ and $Y_1$ are isomorphic, hence they have the same combinatorial invariants, and the proposition follows from Corollaries 8.3, 8.5, and 8.10.

Note that Proposition 8.11 generalizes Propositions 5.2, 5.3, and 5.4 of Akhiezer [Akh15]. Namely, in the case when $\gamma^2 = 1$, our Proposition 8.11 is equivalent to those results of Akhiezer. Our proofs are similar to his.

Proposition 8.12. With the notation and assumptions of Subsection 8.1, if the subgroups $H_1$ and $H_2 = \sigma_\gamma(H_1)$ are conjugate, then $\varepsilon_\gamma$ preserves the combinatorial invariants of $Y_1$, that is

\[ \varepsilon_\gamma(X(Y_1)) = X(Y_1), \quad \varepsilon_\gamma(V(Y_1)) = V(Y_1), \quad \varepsilon_\gamma(\Omega(1)(Y_1)) = \Omega(1)(Y_1), \quad \varepsilon_\gamma(\Omega(2)(Y_1)) = \Omega(2)(Y_1). \]

Conversely, if equalities (36) hold and char $k = 0$, then $H_1$ and $H_2$ are conjugate.

Proposition 8.12 generalizes Theorem 3(1) of Cupit-Foutou [CF15], where the case $k_0 = \mathbb{R}$ was considered.

Proof. If $H_1$ and $H_2$ are conjugate, then by Lemma 4.1 the homogeneous spaces $Y_1 = G/H_1$ and $Y_2 = G/H_2$ are isomorphic as $G$-varieties, hence then they have the same combinatorial invariants, that is,

\[ X(Y_2) = X(Y_1), \quad V(Y_2) = V(Y_1), \quad \Omega(1)(Y_2) = \Omega(1)(Y_1), \quad \Omega(2)(Y_2) = \Omega(2)(Y_1), \]

and (36) follows from Proposition 8.11. Conversely, if equalities (36) hold and char $k = 0$, then by Proposition 8.11 the equalities (37) hold, and by Proposition 7.5 (Losev’s Uniqueness Theorem) the subgroups $H_1$ and $H_2$ are conjugate.

Corollary 8.13. With the notation and assumptions of Subsection 8.1, if there exists a $\sigma_\gamma$-equivariant $\gamma$-semi-automorphism $\mu: Y_1 \to Y_1$, then $\varepsilon_\gamma$ preserves the combinatorial invariants of $Y_1$, that is, equalities (36) hold. Conversely, if equalities (36) hold and char $k = 0$, then there exists a $\sigma_\gamma$-equivariant $\gamma$-semi-automorphism $\mu: Y_1 \to Y_1$.

Proof. By Corollary 4.4 there exists a $\sigma_\gamma$-equivariant $\gamma$-semi-automorphism $\mu: Y_1 \to Y_1$ if and only if the subgroup $\sigma_\gamma(H_1)$ of $G$ is conjugate to $H_1$. Now the corollary follows from Proposition 8.12.
9. Equivariant models of automorphism-free spherical homogeneous spaces

9.1. Let \( k_0 \) be a perfect field and let \( k \) be a fixed algebraic closure of \( k_0 \) with Galois group \( \Gamma = \text{Gal}(k/k_0) \).

Let \( G \) be a connected reductive group over \( k \). Let \( T \subset B \subset G \) be as in Section 7. We consider the based root datum \( \text{BRD}(G) = \text{BRD}(G, T, B) \).

Let \( G_0 \) be a \( k_0 \)-model of \( G \). For any \( \gamma \in \Gamma \), this model defines a \( \gamma \)-semi-automorphism \( \sigma_\gamma : G \to G \), which induces an automorphism \( \varepsilon_\gamma \in \text{AutBRD}(G) \), see Section 8. We obtain a homomorphism \( \varepsilon : \Gamma \to \text{AutBRD}(G) \), \( \gamma \mapsto \varepsilon_\gamma \).

Let \( Y = G/H \) be a spherical homogeneous space of \( G \). We consider the combinatorial invariants of \( Y \):

\[
X(Y) = X(X^*(T)), \quad V(Y) = V(Y) \subset \text{Hom}_Z(\mathcal{X}, \mathbb{Q}), \quad \Omega^{(1)}(Y) = \Omega^{(1)}(Y), \quad \Omega^{(2)}(Y) = \Omega^{(2)}(Y) \subset \text{Hom}_Z(\mathcal{X}, \mathbb{Q}) \times \mathcal{P}(S),
\]

see Section 7. Since \( \varepsilon_\gamma \) acts on \( \text{BRD}(G) \), we can define \( \varepsilon_\gamma(X), \varepsilon_\gamma(V), \varepsilon_\gamma(\Omega^{(1)}), \varepsilon_\gamma(\Omega^{(2)}) \).

Recall that \( Y = G/H \). By Lemma 4.3(i) we have \( \gamma_\ast Y \cong G/\sigma_\gamma(H) \).

**Proposition 9.2.** If \( Y = G/H \) admits a \( G_0 \)-equivariant \( k_0 \)-model \( Y_0 \), then for all \( \gamma \in \Gamma \), \( \varepsilon_\gamma \) preserves the combinatorial invariants of \( Y \), that is

\[
(38) \quad \varepsilon_\gamma(X) = X, \quad \varepsilon_\gamma(V) = V, \quad \varepsilon_\gamma(\Omega^{(1)}) = \Omega^{(1)}, \quad \varepsilon_\gamma(\Omega^{(2)}) = \Omega^{(2)}.
\]

**Proposition 9.2** follows from formulas of Huruguen [Hu11, Section 2.2]. For the reader’s convenience we prove it here.

**Proof.** A \( G_0 \)-equivariant \( k_0 \)-model \( Y_0 \) of \( Y \) defines, for any \( \gamma \in \Gamma \), a \( \sigma_\gamma \)-equivariant \( \gamma \)-semi-automorphism \( \mu_\gamma \) of \( Y \), hence an isomorphism of \( G \)-\( k \)-varieties \( (\mu_\gamma)^\natural : G/\sigma_\gamma(H) = \gamma_\ast Y \to Y \). We see that the \( G \)-varieties \( G/H \) and \( G/\sigma_\gamma(H) \) are isomorphic, hence they have the same combinatorial invariants. By Proposition 8.11 the combinatorial invariants of the spherical homogeneous space \( G/\sigma_\gamma(H) \) are

\[
\varepsilon_\gamma(X), \quad \varepsilon_\gamma(V), \quad \varepsilon_\gamma(\Omega^{(1)}), \quad \varepsilon_\gamma(\Omega^{(2)}),
\]

and (38) follows. \( \square \)

The next theorem is a partial converse of Proposition 9.2.

**Theorem 9.3.** Let \( k, \ k_0, \ \Gamma, \ G, \ H, \ G_0 \) be as in 9.1. Assume that:

(i) For all \( \gamma \in \Gamma \), \( \varepsilon_\gamma \) preserves the combinatorial invariants of \( Y \), that is, equalities (38) hold;
(ii) \( N_G(H) = H \);
(iii) \( \text{char} \ k = 0 \).

Then \( Y = G/H \) admits a \( G_0 \)-equivariant \( k_0 \)-model \( Y_0 \). This \( k_0 \)-model is unique up to a unique isomorphism.

**Proof.** Let \( \gamma \in \Gamma \). Since \( \text{char} \ k = 0 \) and \( \varepsilon_\gamma \) preserves the combinatorial invariants of \( Y \), by Corollary 8.13 there exists a \( \sigma_\gamma \)-equivariant \( \gamma \)-semi-automorphism

\[
\mu_\gamma : Y \to Y.
\]
Thus condition (i) of Lemma 9.3 is satisfied.

Since $N_G(H) = H$, by Corollary 5.2 $\text{Aut}^G(Y) = \{1\}$, and by Lemma 6.3 conditions (ii) and (iii) of Lemma 9.3 are satisfied.

The variety $Y = G/H$ is quasi-projective, hence condition (iv) of Lemma 6.3 is satisfied.

By Lemma 6.3 there exists a $G_0$-equivariant $k_0$-model $Y_0$ of $Y$. Since $\text{Aut}^G(Y) = \{1\}$, for any given $\gamma \in \Gamma$ the $\gamma$-semi-automorphism $\mu_{\gamma}$ is unique, hence the model $Y_0$ is unique up to a unique isomorphism.

Recall that a $k_0$-model $G_0$ of a connected reductive $k$-group $G$ is called an inner form (of a split group) if $\varepsilon_{\gamma} = 1$ for all $\gamma \in \Gamma = \text{Gal}(k/k_0)$.

**Lemma 9.4.** Let $k$, $k_0$, $\Gamma$, $G$, $H$, $G_0$ be as in [4.1]. Then each of the conditions below imply condition (i) of Theorem 9.3.

(i) $G_0$ is an inner form;
(ii) $G_0$ is absolutely simple (that is, $G$ is simple) of one of the types $A_1$, $B_n$, $C_n$, $E_7$, $E_8$, $F_4$, $G_2$;
(iii) $G_0$ is split.

**Proof.** (i) If $G_0$ is an inner form, then $\varepsilon_{\gamma} = 1$ for any $\gamma \in \Gamma$, hence condition (i) of Theorem 9.3 is clearly satisfied.

(ii) In these cases $\text{Dyn}(G)$ has no nontrivial automorphisms, hence $\Gamma$ acts trivially on $\text{Dyn}(G)$. We see that (ii) implies (i).

(iii) In this case clearly $\varepsilon_{\gamma} = 1$ for all $\gamma \in \Gamma$.

**Corollary 9.5.** If $\text{char} k = 0$, $N_G(H) = H$, and at least one of the conditions (i–iii) of Lemma 9.4 is satisfied, then $Y$ admits a $G_0$-equivariant $k_0$-model, and this $k_0$-model is unique.

**Remark 9.6.** Assume that $k = \mathbb{R}$ and $N_G(H) = H$. The assertion that if condition (iii) of Lemma 9.4 is satisfied, then $Y$ has a unique $G_0$-equivariant $\mathbb{R}$-model $Y_0$, is Theorem 4.12 of Akhiezer and Cupit-Foutou [ACF14]. The similar assertion when only condition (i) of Lemma 9.4 is satisfied, is Theorem 1.1 of Akhiezer [Akh15]. Our paper is inspired by this result of Akhiezer, and our proof of Theorem 9.3 is similar to his proof.

10. Equivariant Models of Spherically Closed Spherical Homogeneous Spaces

In this section we do not assume that $N_G(H) = H$.

Let $k$, $G$, $H$, $Y = G/H$, $T \subset B \subset G$ be as in Section 4 in particular, $k$ is algebraically closed, and we assume that $\text{char} k = 0$. The group $\text{Aut}^G(Y) = N_G(H)/H$ acts on $Y$ and on the set $D$ of colors of $Y$.

**Definition 10.1.** A spherical homogeneous space $Y = G/H$ is called spherically closed if $N_G(H)/H$ acts on $D$ faithfully, that is, if the homomorphism

$$\text{Aut}^G(Y) \to \text{Aut}(D)$$

is injective. (Here $\text{Aut}(D)$ denotes the group of permutations of the finite set $D$.)

Let $k_0 \subset k$ be a subfield such that $k$ is an algebraic closure of $k_0$. Let $G_0$ be a $k_0$-model of $G$, and for $\gamma \in \Gamma := \text{Gal}(k/k_0)$ let $\sigma_\gamma : G \to G$ be the $\gamma$-semi-automorphism defined by $G_0$. Let $\varepsilon_{\gamma} : X^*(T) \to X^*(T)$ be as in [4.1].

**Theorem 10.2.** Let $k$ be an algebraically closed field. Let $G$, $H$, $Y = G/H$ be as in Section 4. Let $G_0$ be a $k_0$-model of $G$, where $k_0 \subset k$ is a subfield such that $k$ is an algebraic closure of $k_0$. Assume that
Lemma 10.5. The following lemma is obvious:

An automorphism $\gamma$ is a homomorphism. Assume that for all $\mu\nu\sigma$

Then $Y$ admits a $G_0$-equivariant $k_0$-model.

Theorem 10.2 generalizes the existence assertion of Theorem 9.3. It was inspired by Corollary 1 of Cupit-Foutou [CF15, Section 2.5], where the case $k_0 = \mathbb{R}$ was considered.

In order to prove the theorem we need a few lemmas.

**Lemma 10.3.** Let $\zeta: D \to \Omega$ be a mapping of nonempty finite sets. Let $\Gamma$ be a group acting on $\Omega$ by a homomorphism

$$s: \Gamma \to \text{Aut}(\Omega), \quad \gamma \mapsto s_\gamma.$$

Assume that for any $\gamma \in \Gamma$ there exists a permutation $m_\gamma: D \to D$ covering $s_\gamma$, that is, such that the following diagram commutes:

$$\begin{array}{ccc}
D & \xrightarrow{m_\gamma} & D \\
\zeta & \downarrow & \zeta \\
\Omega & \xrightarrow{s_\gamma} & \Omega
\end{array}$$

Then there exists a homomorphism $m': \Gamma \to \text{Aut}(D)$ such that:

(i) for any $\gamma \in \Gamma$ the permutation $m'_\gamma$ covers $s_\gamma$;
(ii) for any $\gamma \in \Gamma$ we have $m'_\gamma = a_\gamma \circ m_\gamma$, where $a_\gamma \in \text{Aut}_\Omega(D)$;
(iii) $m'_\gamma = \text{id}_D$ for all $\gamma \in \ker s$.

**Proof.** We may and shall assume that $\Gamma$ acts transitively on $\Omega$. Let $\omega, \omega' \in \Omega$, then there exists $\gamma \in \Gamma$ such that $s_\gamma(\omega) = \omega'$. By hypotheses there exists $m_\gamma \in \text{Aut}(D)$ covering $s_\gamma$, then $m_\gamma$ induces a bijection $\zeta^{-1}(\omega) \to \zeta^{-1}(\omega')$, hence the cardinalities of $\zeta^{-1}(\omega)$ and $\zeta^{-1}(\omega')$ are equal. We see that $\omega \mapsto |\zeta^{-1}(\omega)|$ is a constant function on $\Omega$; we denote its value by $n$. For each $\omega \in \Omega$ we fix some bijection between $\zeta^{-1}(\omega)$ and the set $\{1, \ldots, n\}$; we denote the element of $\zeta^{-1}(\omega) \subset D$ corresponding to $i \in \{1, \ldots, n\}$ by $d^{(i)}(\omega)$. Then we define $m'_\gamma \in \text{Aut}(D)$ by

$$m'_\gamma(d^{(i)}(\omega)) = d^{(i)}(s_\gamma(\omega)).$$

Since $s: \gamma \mapsto s_\gamma$ is a homomorphism, we see that $m': \gamma \mapsto m'_\gamma$ is a homomorphism, and clearly $m'_\gamma$ covers $s_\gamma$, which proves (i). Set $a_\gamma = m'_\gamma \circ m_\gamma^{-1}$, then clearly (ii) holds, and the assertion (iii) holds by construction.

**10.4.** Write

$$\zeta = \rho \times \zeta: D \to V \times \mathcal{P}(S), \quad \Omega = \text{im} \zeta, \quad s_\gamma = \varepsilon_\gamma|\Omega: \Omega \to \Omega,$$

then the map

$$\Gamma \to \text{Aut}(\Omega), \quad \gamma \mapsto s_\gamma$$

is a homomorphism. Assume that for all $\gamma \in \Gamma$ there exists a $\sigma_\gamma$-equivariant $\gamma$-semi-automorphism $\mu: Y \to Y$, that is, a $\gamma$-semi-automorphism satisfying

$$\mu(g \cdot y) = \sigma_\gamma(g) \cdot \mu(y) \quad \text{for all } g \in G(k), \; y \in Y(k).$$

The following lemma is obvious:

**Lemma 10.5.** If $\gamma, \delta \in \text{Aut}(k/k_0)$, $\mu$ is a $\sigma_\gamma$-equivariant $\gamma$-semi-automorphism, and $\nu$ is a $\sigma_\delta$-equivariant $\delta$-semi-automorphism, then $\mu \nu$ is a $\sigma_{\gamma \delta}$-equivariant $\gamma \delta$-semi-automorphism and $\mu^{-1}$ is a $\sigma_{\gamma^{-1}}$-equivariant $\gamma^{-1}$-semi-automorphism. 


10.6. Consider $\sigma_\gamma(T) \subset \sigma_\gamma(B) \subset G$. There exists $g_\gamma \in G(k)$ such that if we set $\sigma'_\gamma = \inn(g_\sigma) \circ \sigma_\gamma$, then

$$\sigma'_\gamma(T) = T \quad \text{and} \quad \sigma'_\gamma(B) = B. \quad \tag{40}$$

For $\mu: Y \to Y$ as in (39), we define a $\gamma$-semi-automorphism

$$\mu' = g_\gamma \circ \mu: Y \to Y, \quad y \mapsto g_\gamma \cdot \mu(y) \quad \text{for} \ y \in Y(k).$$

Then for $g \in G(k), \ y \in Y(k)$ we have

$$\mu'(g \cdot y) = g_\gamma \cdot \mu(g \cdot y) = g_\gamma \cdot \sigma_\gamma(g) \cdot \mu(y) = (g_\gamma \cdot \sigma_\gamma(g) \cdot g_\gamma^{-1}) \cdot (g_\gamma \cdot \mu(y)) = \sigma'_\gamma(g) \cdot \mu'(y). \quad \tag{41}$$

Let $D \in \mathcal{D} = \mathcal{D}(Y)$ be a color, this means that $D$ is the closure of a codimension one $B$-orbit in $Y$. Since $\sigma'_\gamma(B) = B$, it follows from (41) that the divisor $\mu'(D)$ in $Y$ is the closure of a codimension one $B$-orbit, that is, a color. We obtain a permutation

$$m_\mu: \mathcal{D} \to \mathcal{D}, \quad D \mapsto \mu'(D), \quad \tag{42}$$

covering $s_\gamma$. Since $g_\gamma$ for which (10) holds is defined uniquely up to multiplication on the left by an element $t \in T(k) \subset B(k)$, we see that $m_\mu$ depends only on $\mu$ and does not depend on the choice of $g_\gamma$.

**Lemma 10.7.** The map $\mu \mapsto m_\mu$ is a homomorphism: for $\gamma, \mu, \delta, \nu$ as in Lemma 10.5 we have $m_{\mu \nu} = m_\mu \circ m_\nu$.

**Proof.** Let $(\gamma, \mu)$ and $(\delta, \nu)$ be as in Lemma 10.5. Choose $g_\gamma, g_\delta \in G(k)$ such that

$$g_\gamma \cdot \sigma_\gamma(T, B) \cdot g_\gamma^{-1} = (T, B), \quad g_\delta \cdot \sigma_\delta(T, B) \cdot g_\delta^{-1} = (T, B). \quad \tag{43}$$

Set

$$\mu' = g_\gamma \circ \mu: Y \to Y, \quad \nu' = g_\delta \circ \nu: Y \to Y.$$

Then for $y \in Y(k)$ we have

$$\mu'(\nu'(y)) = \mu'(\nu(y)) = g_\gamma \cdot \mu(g_\delta \cdot \nu(y)) = g_\gamma \cdot \sigma_\gamma(g_\delta) \cdot (\mu \nu)(y). \quad \tag{44}$$

On the other hand, from (43) we obtain

$$g_\gamma \cdot \sigma_\gamma(g_\delta \cdot \sigma_\delta(T, B) \cdot g_\delta^{-1}) \cdot g_\gamma^{-1} = (T, B),$$

hence

$$g_\gamma \sigma_\gamma(g_\delta \cdot \sigma_\delta(T, B) \cdot (g_\gamma \sigma_\gamma(g_\delta))^{-1} = (T, B).$$

Thus we may set

$$(\mu \nu)' = (g_\gamma \sigma_\gamma(g_\delta)) \circ \mu \nu,$$

that is,

$$(\mu \nu)'(y) = g_\gamma \sigma_\gamma(g_\delta) \cdot (\mu \nu)(y) \quad \text{for} \ y \in Y(k).$$

Comparing with (44), we see that with this $(\mu \nu)'$ we have

$$(\mu \nu)' = \mu' \circ \nu'.$$

hence

$$m_{\mu \nu} = m_\mu \circ m_\nu,$$

as required. \qed
Proof of Theorem 10.2. Let \( \gamma \in \Gamma \). Since \( \text{char } k = 0 \) and \( \varepsilon_\gamma \) preserves the combinatorial invariants of \( Y = G/H \), by Corollary 8.13 there exists a \( \sigma_\gamma \)-equivariant \( \gamma \)-semi-automorphism \( \mu_\gamma : Y \to Y \). Set
\[
m_\gamma = m_{\mu_\gamma} \in \text{Aut}(D),
\]
see Subsection 10.8. Then \( m_\gamma \) covers \( s_\gamma \), where \( s_\gamma \in \text{Aut}(\Omega) \) is the restriction of \( \varepsilon_\gamma \) to \( \Omega \). By Lemma 10.3 there exists a \textit{homomorphism}
\[
m' : \Gamma \to \text{Aut}(D), \quad \gamma \mapsto m'_\gamma
\]
such that for any \( \gamma \in \Gamma \) the permutation \( m'_\gamma \in \text{Aut}(D) \) covers \( s_\gamma \) (property (i)) and we have \( m'_\gamma = a_\gamma \circ m_\gamma \), where \( a_\gamma \in \text{Aut}_k(D) \) (property (ii)). By Theorem 7.7 there exists an automorphism \( \tilde{a}_\gamma \in \text{Aut}^G(Y) \) inducing \( a_\gamma \) on \( D \). We set \( \mu'_\gamma = \tilde{a}_\gamma \circ \mu_\gamma \), then \( \mu'_\gamma \) is a \( \sigma_\gamma \)-equivariant \( \gamma \)-semi-automorphism of \( Y \), and by Lemma 10.7 \( \mu'_\gamma \) acts on \( D \) by \( a_\gamma \circ m_\gamma = m'_\gamma \).

Let \( \gamma, \delta \in \Gamma \), then \( \mu'_\gamma \mu'_\delta (\mu'_\delta)^{-1} \in \text{Aut}^G(Y) \) and by Lemma 10.7 it acts on \( D \) by \( m'_\gamma m'_\delta (m'_\delta)^{-1} = \text{id}_D \). Since \( Y \) is spherically closed, we conclude that \( \mu'_\gamma \mu'_\delta (\mu'_\delta)^{-1} = \text{id}_Y \), hence \( \mu_{\gamma \delta} = \mu_\gamma \circ \mu_\delta \). Thus the map \( \gamma \mapsto \mu'_\gamma \) is a homomorphism.

It is easy to see that \( Y \) admits a \( G_{k_2} \)-equivariant \( k_2 \)-model \( Y_2 \) over some finite Galois extension \( k_2/k_0 \) in \( k \). Let \( \Gamma_2 = \text{Gal}(k/k_2) \), and for \( \gamma \in \Gamma_2 \) let \( \mu''_\gamma \) denote the \( \gamma \)-semi-automorphism of \( Y \) defined by the \( k_2 \)-model \( Y_2 \). After passing to a finite extension, we may assume that for \( \gamma \in \Gamma_2 \) we have \( s_\gamma = \text{id}_\Omega \), and by property (iii) of Lemma 10.3 we have \( m'_\gamma = \text{id}_D \). Moreover, we may assume that for \( \gamma \in \Gamma_2 \) the semi-automorphism \( \mu''_\gamma \) acts trivially on \( D \). It follows that \( (\mu''_\gamma)^{-1} \mu'_\gamma \) acts trivially on \( D \), and clearly \( (\mu''_\gamma)^{-1} \mu'_\gamma \in \text{Aut}^G(Y) \). Since \( Y \) is spherically closed, we conclude that \( (\mu''_\gamma)^{-1} \mu'_\gamma = \text{id}_Y \), hence \( \mu''_\gamma = \mu'_\gamma \) for \( \gamma \in \Gamma_2 \). We see that the homomorphism \( \gamma \mapsto \mu'_\gamma \) satisfies condition (iii) of Lemma 6.3. Note that \( Y = G/H \) is quasi-projective, that is, condition (iv) of Lemma 6.3 is satisfied as well. By Lemma 6.3 the homomorphism \( \gamma \mapsto \mu'_\gamma \) defines a \( G_0 \)-equivariant \( k_0 \)-model \( Y_0 \) of \( Y \), which completes the proof of the theorem. \( \square \)

10.8. In Example 10.1 we considered a spherically closed spherical variety \( Y = G/H \), where \( G = \text{SL}_2(k) \) and \( H = T \), a maximal torus in \( G \). In this case it is obvious that for any \( k_0 \)-model \( G_0 \) of \( G \) there exists a \( G_0 \)-equivariant \( k_0 \)-model \( Y_0 \) of \( Y = G/H \). Indeed, there exists a maximal torus \( T_0 \subset G_0 \) defined over \( k_0 \), and it is clear that \( Y_0 := G_0/T_0 \) is a \( G_0 \)-equivariant \( k_0 \)-model of \( Y = G/T \). In the following example we consider a spherically closed subgroup that is not conjugate to a subgroup defined over \( k_0 \).

Example 10.9. Let \( k = \mathbb{C} \), \( k_0 = \mathbb{R} \). Following a suggestion of Roman Avdeev, we take \( G = \text{SO}_{2n+1, \mathbb{C}} \), where \( n \geq 2 \), and we take for \( H \) a Borel subgroup of \( \text{SO}_{2n, \mathbb{C}} \), where \( \text{SO}_{2n, \mathbb{C}} \subset \text{SO}_{2n+1, \mathbb{C}} = G \). By Proposition 10.10 below, \( H \) is a spherically closed spherical subgroup of \( G \) and \( N_G(H) \neq H \). Take \( G_0 = \text{SO}_{2n+1, \mathbb{R}} \), then \( G_0 \) is an anisotropic (compact) \( \mathbb{R} \)-model of \( G \). Since the Dynkin diagram of \( G \) has no nontrivial automorphisms, \( G_0 \) is an inner form. We wish to show that \( Y = G/H \) admits a \( G_0 \)-equivariant \( \mathbb{R} \)-model. Clearly \( H \) is not conjugate to any subgroup of \( G_0 \) defined over \( \mathbb{R} \) because \( H \) is not reductive, and therefore, we cannot argue as in Subsection 10.8. Since \( N_G(H) \neq H \), we cannot apply Theorem 10.2 the homogeneous variety \( Y = G/H \) does admit a \( G_0 \)-equivariant \( \mathbb{R} \)-model.

Proposition 10.10 (Roman Avdeev, private communication). Let \( G = \text{SO}_{2n+1, \mathbb{C}} \), where \( n \geq 2 \). Let \( H \) be a Borel subgroup of \( \text{SO}_{2n, \mathbb{C}} \), where \( \text{SO}_{2n, \mathbb{C}} \subset \text{SO}_{2n+1, \mathbb{C}} = G \). Then \( H \) is a spherically closed spherical subgroup of \( G \), and \( N_G(H) \neq H \).

Proof. Set \( g = \text{Lie}(G) \). Choose a Borel subgroup \( B \subset G \) and a maximal torus \( T \subset B \). Let \( X = X^*(T) \) denote the character lattice of \( T \) and let \( R = R(G, T) \subset X \) be the root system. The Borel subgroup \( B \) defines a set of positive roots \( R^+ \subset R \) and the corresponding set of
simple roots $S \subset R^+ \subset R$. Let $U$ denote the unipotent radical of $B$ and put $u = \text{Lie}(U)$. We have
\[ g = \text{Lie}(T) \oplus \bigoplus_{\beta \in R} g_{\beta}, \quad u = \bigoplus_{\beta \in R^+} g_{\beta}, \]
where $g_{\beta}$ is the root subspace corresponding to a root $\beta$.

Let $R_l \subset R$ denote the root subsystem consisting of the long roots. Observe that $R$ is a root system of type $B_n$, and $R_l$ is a root system of type $D_n$. Set $R_l^+ = R^+ \cap R_l$. We set
\[ g_l = \text{Lie}(T) \oplus \bigoplus_{\beta \in R_l} g_{\beta}, \quad u_l = \bigoplus_{\beta \in R_l^+} g_{\beta}. \]

Let $G_l$ (resp., $U_l$) be the connected algebraic subgroup of $G$ with Lie algebra $g_l$ (resp., $u_l$). Set $H = TU_l$. Then $G_l \simeq \text{SO}_{2n,C}$ and $H$ is a Borel subgroup of $G_l$.

It is well known that $H$ is a spherical subgroup of $G$. For example, this fact follows from Theorem 1 of Avdeev [Av11] (to apply this theorem one needs to check that all the short positive roots in $R$ are linearly independent). By [Av13, Proposition 5.25] $H$ is spherically closed.

We consider the Weyl group $W = W(G,T) = W(R)$. Let $r \in W = N_G(T)/T$ denote the reflection with respect to the short simple root, and let $\rho$ be a representative of $r$ in $N_G(T)$. Since $r$ preserves $R^+$, we see that $r \in N_G(H)$. Since $r \in N_G(T) \setminus T$ and $N_G(T) \setminus B = T$, we see that $\rho \notin B$. By construction $H \subset B$, and we conclude that $\rho \notin H$, hence $N_G(H) \neq H$. In fact, $N_G(H) = H \cup \rho H$ by [Av13, Theorem 3].

The following example shows that $G/H$ might have no $G_0$-equivariant $k_0$-model when $H$ is not spherically closed.

**Example 10.11.** Let $k = C$, $k_0 = R$. Let $n \geq 1$, $G = \text{Sp}_{2n,C} \times_C \text{Sp}_{2n,C}$, $Y = \text{Sp}_{2n,C}$, the group $G$ acts on $Y$ by
\[(g_1, g_2) \ast y = g_1 y g_2^{-1}, \quad g_1, g_2, y \in \text{Sp}_{2n}(C).\]

Let $H$ denote the stabilizer in $G$ of $1 \in \text{Sp}_{2n}(C) = Y(C)$, then $H = \text{Sp}_{2n,C}$ embedded diagonally in $G$. We have $Y = G/H$, and $Y$ is a spherical homogeneous space of $G$. We have $N_G(H) = Z(G) \cdot H$, where $Z(G)$ denotes the center of $G$. It follows that $N_G(H)/H \simeq \{ \pm 1 \} \neq \{1\}$. Clearly $N_G(H)/H$ acts trivially on $D(G/H)$, so $H$ is not spherically closed.

Consider the following real model of $G$:
\[ G_0 = \text{Sp}_{2n,R} \times_R \text{Sp}(n), \]
where $\text{Sp}(n)$ is the compact real form of $\text{Sp}_{2n}$. We show that $Y$ cannot have a $G_0$-equivariant real model, although $G_0$ is an inner form of a split group.

Indeed, assume for the sake of contradiction that such a real model $Y_0$ of $Y$ exists. We have $Y = \text{Sp}_{2n,C}$, and $Y_0$ is simultaneously a principal homogeneous space of $\text{Sp}_{2n,R}$ and of $\text{Sp}(n)$. Since $H^1(R, \text{Sp}_{2n,R}) = 1$, we see that $Y_0(R)$ is not empty. It follows that the topological space $Y_0(R)$ is simultaneously a principal homogeneous space of $\text{Sp}(2n,R)$ and of $\text{Sp}(n)$. Thus $Y_0(R)$ is simultaneously homeomorphic to the noncompact Lie group $\text{Sp}(2n,R)$ and to the compact Lie group $\text{Sp}(n)$, which is clearly impossible. Thus, there is no $G_0$-equivariant real model $Y_0$ of $Y$.

**10.12.** Let $k$, $k_0$, $\Gamma$, $G$, $H$, $G_0$ be as in Subsection 9.11 in particular, $\Gamma = \text{Gal}(k/k_0)$. We do not assume that char $k = 0$. We assume that $H$ is spherically closed and that $Y = G/H$ admits a $G_0$-equivariant $k_0$-model $Y_0$. Then by Corollary 8.12, $\varepsilon_\gamma$ preserves the combinatorial invariants of $Y$ for all $\gamma \in \Gamma$, in particular, $\Gamma$ acts on the finite set $\Omega(2) = \Omega(2)(Y)$. Let $U_1, U_2, \ldots, U_r$ be the orbits of $\Gamma$ in $\Omega(2)$. For each $i = 1, 2, \ldots, r$, let us choose a point $u_i \in U_i$. Set $\Gamma_i = \text{Stab}_\Gamma(u_i)$. 
Theorem 10.13. With the notation and assumptions of [10.12] we have:

(i) The set of isomorphism classes of $G_0$-equivariant $k_0$-models of $Y$ is canonically a principal homogeneous space of the abelian group $H^1(\Gamma, \text{Aut}_0(D))$;

(ii) $H^1(\Gamma, \text{Aut}_0(D)) \simeq \prod_{i=1}^r \text{Hom}(\Gamma_i, S_2)$, where $S_2$ is the symmetric group on two symbols.

Note that $S_2 \cong \{\pm 1\} \cong \mathbb{Z}/2\mathbb{Z}$.

Corollary 10.14. In Theorem [10.13] assume that $|\Gamma| = 2$. Then the number of isomorphism classes of $G_0$-equivariant $k_0$-models of $Y = G/H$ is $2^s$, where $s$ is the number of fixed points of $\Gamma$ in $\Omega^{(2)}$.

Proof. Let $U_i$ be an orbit of $\Gamma$ in $\Omega^{(2)}$. If $|U_i| = 2$, then $\Gamma_i = \{1\}$, hence $|\text{Hom}(\Gamma_i, S_2)| = 1$. If $|U_i| = 1$, then $\Gamma_i = \Gamma$, and hence, $|\text{Hom}(\Gamma_i, S_2)| = 2$. Now the corollary follows from the theorem. □

Proof of Theorem [10.13] Since $Y$ is quasi-projective, the set of the isomorphism classes in the theorem is in a canonical bijection with the pointed set $H^1(\Gamma, \text{Aut}^G(Y))$; see Serre [Se97], Proposition 5 in Section III.1.3. By Theorem 2 of Losev [Lo09] (see also Theorem B.1 below), the group $\text{Aut}^G(Y)$ is abelian, hence $H^1(\Gamma, \text{Aut}^G(Y))$ is an abelian group, and the set of isomorphism classes in the theorem is canonically a principal homogeneous space of this abelian group. By Corollary [7.9] there is a canonical isomorphism of abelian groups $\text{Aut}^G(Y) \xrightarrow{\sim} \text{Aut}_0(D)$, and (i) follows.

We compute $H^1(\Gamma, \text{Aut}_0(D))$. Recall that we have a surjective map $\zeta : D \rightarrow \Omega$. Set $D^{(2)} = \zeta^{-1}(\Omega^{(2)})$, then clearly

$$\text{Aut}_0(D) = \text{Aut}_0(D^{(2)}) = \prod_{\omega \in \Omega^{(2)}} S_2 = \prod_{i=1}^r \left( \prod_{\omega \in U_i} S_2 \right),$$

hence

$$H^1(\Gamma, \text{Aut}_0(D)) = \prod_{i=1}^r H^1(\Gamma, \prod_{\omega \in U_i} S_2).$$

Since $\Gamma$ acts on $U_i$ transitively, by the lemma of Faddeev and Shapiro, see Serre [Se97, I.2.5, Proposition 10], we have

$$H^1(\Gamma, \prod_{\omega \in U_i} S_2) \simeq H^1(\Gamma_i, S_2) = \text{Hom}(\Gamma_i, S_2).$$

Thus

$$H^1(\Gamma, \text{Aut}_0(D)) \simeq \prod_{i=1}^r \text{Hom}(\Gamma_i, S_2),$$

which proves (ii). □

Example 10.15. Let $G = SO_3, \mathbb{C} \simeq \text{PGL}_2, \mathbb{C}$. Let $T \subset G$ be a maximal torus. We take $H = T$ and consider $Y = G/H = G/T$, which is a spherical homogeneous space of $G$. We have

$$\text{Aut}^G(G/T) = \mathcal{N}_G(T)/T \simeq \{\pm 1\}.$$
Y has exactly two $G_0$-equivariant $\mathbb{R}$-models. We describe such models for each $\mathbb{R}$-model of $G = \text{SO}_3(C)$.

Consider the indefinite real quadratic form in three variables

$$F_{2.1}(x_1, x_2, x_3) = x_1^2 + x_2^2 - x_3^2, \quad x_i \in \mathbb{R}.$$ Set $G_0 = \text{SO}(F_{2.1}) = \text{SO}_{2,1}$, which is a noncompact (split) $\mathbb{R}$-model of $G$. We consider the affine quadric $Y_{2,1}^+ \subset \mathbb{A}^3_{\mathbb{R}}$ given by the equation $F_{2.1}(x) = +1$, and the affine quadric $Y_{2,1}^- \subset \mathbb{A}^3_{\mathbb{R}}$ given by the equation $F_{2.1}(x) = -1$. Then $Y_{2,1}^+$ and $Y_{2,1}^-$ are $\text{SO}_{2,1}$-equivariant $\mathbb{R}$-models of $Y = G/T$. It is well known that $Y_{2,1}^+(\mathbb{R})$ is a hyperboloid of one sheet, hence it is connected, while $Y_{2,1}^-(\mathbb{R})$ is a hyperboloid of two sheets, hence it is not connected. It follows that the $\mathbb{R}$-varieties $Y_{2,1}^+$ and $Y_{2,1}^-$ are two non-isomorphic $\text{SO}_{2,1}$-equivariant $\mathbb{R}$-models of $Y = G/T$.

Now consider the positive definite real quadratic form in three variables

$$F_3(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2, \quad x_i \in \mathbb{R}.$$ Set $G_0 = \text{SO}(F_3) = \text{SO}_{3,\mathbb{R}}$, which is a compact (anisotropic) $\mathbb{R}$-model of $G$. We consider the affine quadric $Y_3^+ \subset \mathbb{A}^3_{\mathbb{R}}$ given by the equation $F_3(x) = +1$, and the affine quadric $Y_3^- \subset \mathbb{A}^3_{\mathbb{R}}$ given by the equation $F_3(x) = -1$. Then $Y_3^+$ and $Y_3^-$ are $\text{SO}_{3,\mathbb{R}}$-equivariant $\mathbb{R}$-models of $Y = G/T$. Clearly, $Y_3^+(\mathbb{R})$ is the unit sphere in $\mathbb{R}^3$, hence it is nonempty, while $Y_3^-(\mathbb{R})$ is empty. It follows that the $\mathbb{R}$-varieties $Y_3^+$ and $Y_3^-$ are two non-isomorphic $\text{SO}_{3,\mathbb{R}}$-equivariant $\mathbb{R}$-models of $Y = G/T$.

11. Equivariant models of spherical embeddings of automorphism-free spherical homogeneous spaces

In this section we assume that $\mathcal{N}_G(H) = H$.

**Theorem 11.1.** Let $k$, $G$, $H$, $Y = G/H$, $k_0$, $\Gamma$, $G_0$ be as in Section 2.4. We assume that

(i) $G_0$ is an inner form of a split group,
(ii) $\mathcal{N}_G(H) = H$,
(iii) char $k = 0$.

Let $Y \hookrightarrow Y'$ be an arbitrary spherical embedding of $Y$. Then $Y'$ admits a $G_0$-equivariant $k_0$-model $Y_0'$. This model is compatible with the $k_0$-model $Y_0$ of $Y$ from Theorem 9.3 and is unique up to a canonical isomorphism.

This theorem generalizes Theorem 1.2 of Akhiezer [Akh15], who considered the case $k_0 = \mathbb{R}$. Note that Akhiezer considered only the wonderful embedding of $Y$, while we consider an arbitrary spherical embedding, so our result is new even in the case $k_0 = \mathbb{R}$.

**Proof.** We show that a $k_0$-model of $Y'$, if exists, is unique. Indeed, let $Y_0'$ be such a $k_0$-model. For $\gamma \in \Gamma := \text{Gal}(k/k_0)$, let $\mu'_\gamma : Y' \to Y'$ be the corresponding $\gamma$-semi-automorphism of $Y'$. Since $Y$ is the only open $G$-orbit in $Y'$, it is stable under $\mu'_\gamma$ for all $\gamma \in \Gamma$. Since $k_0$ is a perfect field, this defines a $G_0$-equivariant $k_0$-model $Y_0$ of $Y$, which is unique because $\mathcal{N}_G(H) = H$ and hence $\text{Aut}^G(Y) = \{1\}$. Since $Y$ is Zariski-dense in $Y'$, we conclude that the model $Y_0'$ of $Y'$ is unique. (This argument does not use the assumption that char $k = 0$.)

We prove the existence. By Theorem 9.3, $Y$ admits a unique $G_0$-equivariant $k_0$-model $Y_0$. The model $Y_0$ defines an action of $\Gamma$ on the finite set $\mathcal{D}$, see e.g. Huruguen [Hu11, 2.2.5]. Namely, for any $\gamma \in \Gamma$ we have a $\sigma_\gamma$-equivariant $\gamma$-semi-automorphism $\mu_\gamma$, which induces an automorphism $m_\gamma : D \to D$ covering $s_\gamma : \Omega \to \Omega$, see [42]. We show that this
action of $\Gamma$ on $D$ is trivial. Indeed, since $N_G(H) = H$, by Corollary 4.10 the surjective map
$$\zeta : D \to \Omega$$
is bijective. Since by assumption $G_0$ is an inner form, for all $\gamma \in \Gamma$ we have $\varepsilon_\gamma = 1$, hence $s_\gamma = 1$. Thus $\Gamma$ acts trivially on $\Omega$ and on $D$.

Let $CF(Y')$ denote the colored fan of $Y'$ (see Knop [Kn89] or Perrin [Pe14, Definition 3.1.9]) which is a set of colored cones $(C, F) \in CF(Y')$, where $C \subset V$ and $F \subset D$. We know that $\Gamma$ acts trivially on $V = Hom_S(X, Q)$ and on $D$. It follows that for any $\gamma \in \Gamma$ and for any colored cone $(C, F) \in CF(Y')$, we have
$$\gamma_*(C) = C, \quad \gamma_*(F) = F.$$ (45)
It follows that the colored fan $CF(Y')$ is $\Gamma$-stable. Moreover, it follows from (45) that the hypothesis of Theorem 2.26 of Huruguen [Hu11] is satisfied, that is, $Y'$ has a covering by $G$-stable and $\Gamma$-stable open quasi-projective subvarieties. By this theorem $Y'$ admits a $G_0$-equivariant $k_0$-model compatible with $Y_0$. $\square$

**Remark 11.2.** Huruguen [Hu11] assumes that $Y_0$ has a $k_0$-point, but he does not use that assumption.

**Remark 11.3.** In Theorem [Hu11] we do not assume that $Y'$ is quasi-projective.

**APPENDIX A. ALGEBRAICALLY CLOSED DESCENT FOR SPHERICAL HOMOGENEOUS SPACES**

The proofs in this appendix were communicated to the author by experts.

**Theorem A.1.** Let $G_0$ be a connected reductive group defined over an algebraically closed field $k_0$ of characteristic 0. Let $k \supset k_0$ be a larger algebraically closed field. We set $G = G_0 \times_{k_0} k$, the base change of $G_0$ from $k_0$ to $k$. Let $H \subset G$ be a spherical subgroup of $G$ (defined over $k$). Then $H$ is conjugate to a (spherical) subgroup defined over $k_0$.

**Proof.** The theorem will be proved in five steps.

1) Let $X_0$ be a variety equipped with an action of $G_0$. Then $X_0$ is the disjoint union of locally closed $G_0$-stable subvarieties $X_0^m$ consisting of all orbits of a fixed dimension $m$. The orbits of maximal (resp. minimal) dimension form an open (resp. closed) subvariety.

To see this, consider the product $G_0 \times X_0$ and the subvariety $Y_0$ consisting of the pairs $(g, x)$ such that $gx = x$, equipped with the projection to $X_0$. By a theorem of Chevalley, see EGA [Gr66] 13.1.3, the dimension of fibers of this projection is an upper semicontinuous function on $Y_0$. Restricting this function to $X_0$ (viewed as the closed subvariety of $Y_0$ on which $g = e$), it follows that the dimension of the $G_0$-orbit is an upper semicontinuous function on $X_0$.

2) Take for $X_0$ the variety of Lie subalgebras of $g_0 = Lie G_0$ of a fixed codimension, say $r$, and let $X$ be the $k$-variety obtained from $X_0$ by scalar extension. Then $X$ is the variety of Lie subalgebras of codimension $r$ in $g = Lie G$. Moreover, the stabilizer of a $k$-point $\mathfrak{h}$ in $X$ is the normalizer of $\mathfrak{h}$ (viewed as a Lie algebra) in $G$. So the dimension of the orbit of $\mathfrak{h}$ is
$$\dim(G) - \dim N_G(\mathfrak{h}) = \dim(G) - \dim(\mathfrak{h}) - (\dim N_G(\mathfrak{h}) - \dim(\mathfrak{h})) = r - \dim n_0(\mathfrak{h})/\mathfrak{h},$$
where $n_0(\mathfrak{h})$ denotes the normalizer of $\mathfrak{h}$ in $g$. Thus, if there exists $\mathfrak{h}$ such that $\mathfrak{h} = n_0(\mathfrak{h})$, then the Lie subalgebras $\mathfrak{h}$ satisfying this property are the $k$-points of the open subset of orbits of maximal dimension. Note that $n_0(\mathfrak{h})$ is the Lie algebra of $N_G(\mathfrak{h})$. So if $\mathfrak{h} = n_0(\mathfrak{h})$, then $\mathfrak{h}$ is an algebraic Lie algebra.
3) Let $H$ be a spherical subgroup of $G$ with Lie algebra $\mathfrak{h}$ such that $\mathfrak{n}_G(\mathfrak{h}) = \mathfrak{h}$. We claim that the orbit $G \cdot \mathfrak{h}$ in $X$ is open.

To prove this, recall that $G \cdot \mathfrak{h}$ has maximal dimension among $G$-orbits in $X$. Since $\mathfrak{h}$ is a spherical Lie subalgebra, all Lie subalgebras $\mathfrak{h}'$ in an open neighborhood $U$ of $\mathfrak{h}$ in $X$ are spherical and their orbits are of the same (maximal) dimension; thus, $N_G(\mathfrak{h}')$ is a spherical subgroup of $G$, with Lie algebra $\mathfrak{h}'$ by Step 2. By Theorem 3.1 of Alexeev and Brion [AB05], only finitely many conjugacy classes of spherical subgroups of the form $N_G(\mathfrak{h}')$ for $\mathfrak{h}' \in U$ are obtained in this way; let $H_1, \ldots, H_r$ be representatives of the conjugacy classes. We write $\mathfrak{h}_i = \text{Lie } H_i$, then we see that any the spherical Lie algebra $\mathfrak{h}' \in U$ is conjugate to one of $\mathfrak{h}_1, \ldots, \mathfrak{h}_r$; in particular, $U$ intersects nontrivially with finitely many $G$-orbits in $X$, and all these orbits are of the same dimension. It follows that all these orbits are open, in particular, the orbit $G \cdot \mathfrak{h}$ in $X$ is open.

4) By Step 3, the Lie algebras $\mathfrak{h}$ of spherical subgroups $H$ of $G$ such that $\mathfrak{n}_G(\mathfrak{h}) = \mathfrak{h}$ form finitely many $G$-orbits, and the closures of these orbits are irreducible components of the variety $X$, which is defined over $k_0$. It follows that every such orbit is defined over $k_0$. Since $k_0$ is algebraically closed, every such $G$-orbit has a $k_0$-point, which proves the theorem for spherical subgroups such that $\mathfrak{n}_G(\mathfrak{h}) = \mathfrak{h}$. Also

$$N_G(\mathfrak{h}) = N_G(H^0) = N_G(H),$$

where the latter equality follows from Corollary 5.2 of Brion and Pauer [BP87]. Thus the condition that $\mathfrak{n}_G(\mathfrak{h}) = \mathfrak{h}$ is equivalent to the condition that $N_G(H)/H$ is finite.

5) To handle the case of an arbitrary spherical $k$-subgroup $H$ of $G$, consider the spherical closure of $H$, that is, the algebraic subgroup $\overline{H}$ of $N_G(H)$ containing $H$ such that

$$\overline{H}/H = \ker [N_G(H)/H \to \text{Aut } \mathcal{D}(G/H)].$$

By Corollary A.3 below, the spherical closure $\overline{H}$ is spherical closed, that is, $N_G(\overline{H})/\overline{H}$ acts faithfully on the finite set of colors of $G/\overline{H}$, hence the group $N_G(\overline{H})/\overline{H}$ is finite, and therefore, $\mathfrak{n}_G(\text{Lie } \overline{H}) = \text{Lie } \overline{H}$. By Step 4 we may assume that $\overline{H}$ is defined over $k_0$. Now $H$ is an intersection of kernels of characters of $\overline{H}$ (since the quotient $\overline{H}/H$ is diagonalizable) and every such character is defined over $k_0$, so $H$ is defined over $k_0$, as required.

An alternative proof, also based on Theorem 3.1 of Alexeev and Brion [AB05], is sketched in Knop’s MathOverflow answer [Kn17b].

From now on till the end of this appendix we follow Avdeev [Av15]. Let $G$ be a connected reductive group over an algebraically closed field $k$ of characteristic $0$. Fix a finite covering group $\widetilde{G} \to G$ such that $\widetilde{G}$ is a direct product of a torus with a simply connected semisimple group. For every simple $\widetilde{G}$-module $V$, the corresponding projective space $\mathbb{P}(V)$ has the natural structure of a $G$-variety. Every $G$-variety arising in this way is said to be a simple projective $G$-space.

**Proposition A.2** (Bravi and Luna [BL11] Lemma 2.4.2, see also Avdeev [Av15] Corollary 3.24). For any spherical subgroup $H$ of a connected reductive group $G$ over an algebraically closed field $k$ of characteristic $0$, the spherical closure $\overline{H}$ of $H$ is the common stabilizer in $G$ of all $H$-fixed points in all simple projective $G$-spaces.

**Corollary A.3** (well known). Let $H$ be a spherical subgroup of a connected reductive group $G$ defined over an algebraically closed field $k$ of characteristic $0$. Let $H'$ denote the spherical closure of $H$, and let $H''$ denote the spherical closure of $H'$. Then $H'' = H'$, that is, $H'$ is spherical closed.

This result was stated without proof in Section 6.1 of Luna [Lu01] (see also Avdeev [Av15] Corollary 3.25).
Deduction of Corollary 4.3 from Proposition A.2. Let \( P(V) \) be a simple projective \( G \)-space. Let \( P(V)^H \) denote the set of fixed points of \( H \) in \( P(V) \). Since \( H \subset H' \), we have \( P(V)^{H'} \subset P(V)^H \). By Proposition A.2 applied to \( H \), we have \( P(V)^{H'} = P(V)^H \). Thus \( P(V)^{H'} = P(V)^H \).

By Proposition A.2 applied to \( H' \), the group \( H''(k) \) is the set of \( g \in G(k) \) that fix \( P(V)^{H''} \) for all simple projective \( G \)-spaces \( P(V) \). By Proposition A.2 applied to \( H \), the group \( H'(k) \) is the set of \( g \in G(k) \) that fix \( P(V)^H \) for all simple projective \( G \)-spaces \( P(V) \). Since \( P(V)^{H'} = P(V)^H \), we have \( H'' = H' \), as required.

\[ \square \]

APPENDIX B. THE ACTION OF THE AUTOMORPHISM GROUP ON THE COLORS OF A SPHERICAL HOMOGENEOUS SPACE

By Giuliano Gagliardi

In this appendix we prove Theorem [7.4] which we restate below as Theorem [B.4]. Our proof is based on Friedrich Knop’s MathOverflow answer [Kn17a] to Borovoi’s question. Knop writes that Theorem [7.4] was communicated to him by Ivan Losev.

Let \( G \) be a connected reductive group over an algebraically closed field \( k \) of characteristic 0. Let \( Y = G/H \) be a spherical homogeneous space.

We present results of Knop [Kn96] and Losev [Lo09] describing \( \text{Aut}^G(Y) \), the notation of Section 7. Consider the uniquely determined set \( \Sigma \subset X \) of linearly independent primitive elements in the lattice \( X \) such that

\[ \mathcal{V} = \bigcap_{\gamma \in \Sigma} \{ v \in V : \langle \gamma, v \rangle \leq 0 \}. \]

The elements of \( \Sigma \) are called the spherical roots of \( Y \). For the description of \( \text{Aut}^G(Y) \) a related subset \( \Sigma' \subset X \) is used, see Losev [Lo09]. Losev has shown how \( \Sigma \) can be computed from \( \Sigma' \) and \( D \).

**Theorem B.1** (Knop [Kn96, Theorem 5.5]). For every \( \phi \in \text{Aut}^G(Y) \) and every \( \chi \in X \) there exists \( a_{\phi, \chi} \in k^\times \) such that

\[ \phi|_{\mathcal{V}(y)} = a_{\phi, \chi} \text{id}. \]

The resulting homomorphism

\[ \text{Aut}^G(Y) \to \text{Hom}(X, k^\times) \]

is injective and its image is \( \{ \psi \in \text{Hom}(X, k^\times) : \psi(\Sigma) = \{1\} \} \).

For \( \alpha \in S \), let \( D(\alpha) \) denote the set of colors \( D \in D \) such that the parabolic subgroup \( P_\alpha \) moves \( D \), that is, \( \alpha \in \varsigma(D) \). We need the following results of Luna [Lu97], [Lu11]:

**Proposition B.2.** Let \( \alpha \in S \).

1. We have \( |D(\alpha)| \leq 2 \). Moreover, \( |D(\alpha)| = 2 \) if and only if \( \alpha \in \Sigma \cap S \).
2. Assume \( |D(\alpha)| = 2 \) and write \( D(\alpha) = \{ D_\alpha^+, D_\alpha^- \} \). If \( \rho(D_\alpha^+) = \rho(D_\alpha^-) \), then:
   (i) \( \langle \rho(D_\alpha^+), \chi \rangle = \langle \rho(D_\alpha^-), \chi \rangle = \frac{1}{2} \langle \alpha^\vee, \chi \rangle \) for all \( \chi \in X \), where \( \alpha^\vee \in X_\ast(T) \) is the corresponding simple coroot;
   (ii) we have \( \varsigma(D_\alpha^+) = \varsigma(D_\alpha^-) = \{ \alpha \} \).

**Proof.** For (1), see Luna [Lu97, Sections 2.6 and 2.7] or Timashev [Tim11, 30.10]. For (2), we use that [Lu01, Theorem 2] or [Tim11, Theorem 30.22] implies that the invariants of a spherical homogeneous space satisfy the axioms of a homogeneous spherical datum. These axioms are stated in [Lu01, Sections 2.1 and 2.2] and [Tim11, Definition 30.21]. In particular, we have \( \rho(D_\alpha^+) + \rho(D_\alpha^-) = \alpha^\vee \big|_Y \) and for every \( \beta \in \varsigma(D_\alpha^\pm) \) we have \( \beta \in X \) and \( \langle \rho(D_\alpha^\pm), \beta \rangle = 1 \). With the assumption \( \rho(D_\alpha^+) = \rho(D_\alpha^-) \), we obtain (i) and then (ii). \[ \square \]
We need the following result of Losev:

**Proposition B.3.** The set $\Sigma$ has the following properties:

1. The set $\Sigma$ can be obtained from $\Sigma$ by replacing some elements $\gamma \in \Sigma$ by $2\gamma$ and leaving the other elements $\gamma \in \Sigma$ unchanged.
2. If $\alpha \in \Sigma \cap S$ and $\langle \rho(D_\alpha^+), \chi \rangle = \langle \rho(D_\alpha^-), \chi \rangle = \frac{1}{2} (\alpha^\vee, \chi)$ for all $\chi \in \mathcal{X}$, then $2\alpha \in \Sigma$ (hence $\alpha \notin \Sigma$).

**Proof.** See Losev [Lo09, Theorem 2 and Definition 4.1.1(1)]. □

The following theorem is the main result of this appendix:

**Theorem B.4** (Losev, unpublished). The homomorphism

\[ \text{Aut}^G(Y) \rightarrow \text{Aut}_G(D) \]

is surjective.

**Proof of Theorem B.4.** Let $\mathcal{A}$ denote the set of simple roots $\alpha \in S$ such that $|D(\alpha)| = 2$ and $\rho(D_\alpha^+) = \rho(D_\alpha^-)$. By Proposition B.2 for every $\alpha \in \mathcal{A}$ we have $\varsigma(D_\alpha^+) = \varsigma(D_\alpha^-) = \{\alpha\}$, hence the map $\alpha \mapsto \{D_\alpha^+, D_\alpha^-\}$ is a bijection between $\mathcal{A}$ and the set of unordered pairs $\{D_\alpha^+, D_\alpha^-\}$ such that $\langle \rho \times \varsigma(D_\alpha^+) = \langle \rho \times \varsigma(D_\alpha^-)$. Note that there is a canonical bijection

\[ A \rightarrow \Omega(2), \quad \alpha \mapsto \langle \rho \times \varsigma(D_\alpha^+). \]

By Proposition B.2(1), for every $\alpha \in \mathcal{A}$ we have $\alpha \in S \cap \Sigma \subset \mathcal{X}$, hence there exists $f_\alpha \in K(Y)^{(B)}$, $f_\alpha \neq 0$. Moreover, from Propositions B.2 and B.3 we obtain $2\alpha \in \Sigma$ (and $\alpha \notin \Sigma$).

We want to show that for any $\alpha \in \mathcal{A}$ there exists $\phi_\alpha \in \text{Aut}^G(Y)$ such that $\phi_\alpha$ swaps $D_\alpha^+$ and $D_\alpha^-$, but fixes all $D_\beta^+$ and $D_\beta^-$ for $\beta \in \mathcal{A}$, $\beta \neq \alpha$.

Let $\alpha \in \mathcal{A}$. Since the field $k$ is algebraically closed and the set $\Sigma \subset \mathcal{X}$ is linearly independent, we can construct a homomorphism $\psi_\alpha : \mathcal{X} \rightarrow k^\times$ with $\psi_\alpha(\alpha) = -1$ and $\psi_\alpha(\gamma) = 1$ for every $\gamma \in \Sigma \setminus \{\alpha\}$, and such that $\psi_\alpha$ is of finite order in the group $\text{Hom}(\mathcal{X}, k^\times)$. Then we have $\psi_\alpha(\Sigma) = \{1\}$. By Theorem B.3 there exists an automorphism of finite order $\phi_\alpha \in \text{Aut}^G(Y)$ with

\[ \phi_\alpha(f_\beta) = \begin{cases} -f_\beta & \text{for } \beta = \alpha, \\ f_\beta & \text{for } \beta \in \mathcal{A} \setminus \{\alpha\}. \end{cases} \]

Let $\bar{H} \subset N_G(H)$ denote the subgroup containing $H$ such that

\[ \bar{H}/H = \langle \phi_\alpha \rangle \subset N_G(H)/H = \text{Aut}^G(Y), \]

where $\langle \phi_\alpha \rangle$ denotes the finite subgroup generated by $\phi_\alpha$. We set $\bar{Y} = G/\bar{H}$. We use the same notation for the combinatorial objects associated to the spherical homogeneous space $\bar{Y}$ as for $Y$, but with a tilde above the respective symbol. The morphism of $G$-varieties $Y \rightarrow \bar{Y}$ induces an embedding $K(\bar{Y}) \hookrightarrow K(Y)$, and $K(\bar{Y})$ is the fixed subfield of $\phi_\alpha$. Since $K(\bar{Y})$ is a $G$-invariant subfield of $K(Y)$, we have $K(\bar{Y})^{(B)} \subset K(Y)^{(B)}$ and $\bar{X} \subset \mathcal{X}$.

By (17) we have $\phi_\alpha(f_\alpha) = -f_\alpha \neq f_\alpha$. We see that $f_\alpha \notin K(\bar{Y})$, hence $\alpha \in \mathcal{X} \setminus \bar{X}$, in particular $\alpha \notin \Sigma$. By Proposition B.2(1) we have $|D(\alpha)| \leq 1$, hence the two colors in $D(\alpha)$ are mapped to one color by the map $Y \rightarrow \bar{Y}$, that is, $\phi_\alpha$ swaps $D_\alpha^+$ and $D_\alpha^-$. On the other hand, for any $\beta \in \mathcal{A} \setminus \{\alpha\}$, by (17) we have $\phi_\alpha(f_\beta) = f_\beta$, hence $f_\beta \in K(\bar{Y})$ and $\beta \in \bar{X}$. Since $\beta$ is a primitive element of $\mathcal{X}$, it is a primitive element of $\bar{X} \subset \mathcal{X}$. The natural map $V \rightarrow \bar{V}$ induced by $Y \mapsto \bar{Y}$ is bijective and identifies $\mathcal{Y}$ and $\bar{V}$ (see Knop [Kn98, Section 4]). Since $\beta \in \Sigma$ is dual to a wall of $-\mathcal{Y}$, it is dual to a wall of $-\mathcal{V} = -\mathcal{V}$.  

EQUIVARIANT MODELS OF SPHERICAL VARIETIES 35
It follows that $\beta \in S \cap \tilde{\Sigma}$, hence $|\tilde{D}(\beta)| = 2$, and the two colors in $D(\beta)$ are mapped to distinct colors under $Y \to \tilde{Y}$, that is, $\phi_\alpha$ fixes $D^+_{\beta}$ and $D^-_{\beta}$. 

References

[Akh15] Dmitri Akhiezer, Satake diagrams and real structures on spherical varieties, Internat. J. Math. 26 (2015), no. 12, 1550103, 13 pp.

[ACF14] Dmitri Akhiezer and Stéphanie Cupit-Foutou, On the canonical real structure on wonderful varieties, J. reine Math., 693 (2014), 231–244.

[AB05] Valery Alexeev and Michel Brion, Moduli of affine schemes with reductive group action, J. Algebraic Geom. 14 (2005), 83–117.

[Av11] Roman Avdeev, On solvable spherical subgroups of semisimple algebraic groups, Trans. Moscow Math. Soc. 2011, 1–44.

[Av13] Roman Avdeev, Normalizers of solvable spherical subgroups, Math. Notes 94 (2013), 20–31.

[Av15] Roman Avdeev, Strongly solvable spherical subgroups and their combinatorial invariants, Selecta Math. (N.S.) 21 (2015), 931–993.

[B91] Armand Borel, Linear Algebraic Groups, Second edition, Graduate Texts in Mathematics 126, Springer-Verlag, New York, 1991.

[BS64] Armand Borel et Jean-Pierre Serre, Théorèmes de finitude en cohomologie galoisienne, Comm. Math. Helv., 39 (1964), 111–164.

[Brv93] Mikhail Borovoi, Abelianization of the second nonabelian Galois cohomology, Duke Math. J. 72 (1993), 217–239.

[BKLR14] M. Borovoi, B. Kunyavskii, N. Lemire, and Z. Reichstein, Stably Cayley groups in characteristic zero, Internat. Math. Res. Notices, 2014, 5340–5397.

[BL11] Paolo Bravi and Domingo Luna, An introduction to wonderful varieties with many examples of type $F_4$, J. Algebra 329 (2011), 4–51.

[BP87] Michel Brion et Franz Pauer, Valuations des espaces homogènes sphériques, Comment. Math. Helv. 62 (1987), 265–285.

[CF15] Stéphanie Cupit-Foutou, Anti-holomorphic involutions and spherical subgroups of reductive groups, Transform. Groups 20 (2015), 909–984.

[FSS98] Yuval Z. Flicker, Claus Scheiderer, and R. Sujatha, Grothendieck’s theorem on non-abelian $H^2$ and local-global principles, J. Amer. Math. Soc. 11 (1998), 731–750.

[Gr66] Alexandre Grothendieck, Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas, Troisième partie, Inst. Hautes Études Sci. Publ. Math. No. 28, 1966.

[Hu11] Mathieu Huruguen, Toric varieties and spherical embeddings over an arbitrary field, J. Algebra 342 (2011), 212–234.

[Ja00] Jörg Jahnel, The Brauer-Severi variety associated with a central simple algebra: a survey, https://www.math.uni-bielefeld.de/LAG/man/052.pdf.

[Kn89] Friedrich Knop, The Luna-Vust theory of spherical embeddings, Proceedings of the Hyderabad Conference on Algebraic Groups, December 1989. Madras: Manoj Prakashan (1991), 225–249.

[Kn96] Friedrich Knop, Automorphisms, root systems, and compactifications of homogeneous varieties, J. Amer. Math. Soc. 9 (1996), 153–174.

[Kn17a] Friedrich Knop [https://mathoverflow.net/users/89948/friedrich-knop], Action of $N(H)/H$ on the colors of a spherical homogeneous space $G/H$, URL (version: 2017-06-09): https://mathoverflow.net/q/271795.

[Kn17b] Friedrich Knop [https://mathoverflow.net/users/89948/friedrich-knop], Is any spherical subgroup conjugate to a subgroup defined over a smaller algebraically closed field?, URL (version: 2017-08-01): https://mathoverflow.net/q/277708.

[Lo09] Ivan Losev, Uniqueness property for spherical homogeneous spaces, Duke Math. J. 147 (2009), 315–343.

[Lu97] Domingo Luna, Grosses cellules pour les variétés sphériques, Algebraic groups and Lie groups, Austral. Math. Soc. Lect. Ser., vol. 9, Cambridge Univ. Press, Cambridge, 1997, pp. 267–280.

[Lu01] Domingo Luna, Variétés sphériques de type A, Publ. Math. Inst. Hautes Etudes Sci. 94 (2001), 161–226.

[Mi18] James S. Milne, Algebraic Groups. The Theory of Group Schemes of Finite Type over a Field, Cambridge Studies in Advanced Mathematics 170, Cambridge University Press, Cambridge, 2017.

[Pe14] Nicolas Perrin, On the geometry of spherical varieties, Transform. Groups 19 (2014), 171–223.

[Se97] Jean-Pierre Serre, Galois cohomology, Springer-Verlag, Berlin, 1997.
[Sp79] Tonny A. Springer, *Reductive groups*, in: Automorphic forms, representations and \( L \)-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977) Part 1, pp. 3–27, Proc. Sympos. Pure Math. 33, Amer. Math. Soc., Providence, RI, 1979.

[Sp98] Tonny A. Springer, *Linear Algebraic Groups*, Second ed., Progress in Math., 9, Birkhäuser, Boston, MA, 1998.

[Tim11] Dmitry A. Timashev, *Homogeneous spaces and equivariant embeddings*, Encyclopaedia of Mathematical Sciences, 138, Berlin, Springer, 2011.

[Tits66] Jacques Tits, *Classification of algebraic semisimple groups*, in: Algebraic Groups and Discontinuous Subgroups (Proc. Sympos. Pure Math. 9, Boulder, Colo., 1965) pp. 33–62 Amer. Math. Soc., Providence, R.I., 1966.

[We15] Thorsten Wedhorn, *Spherical spaces*, to appear in Annales de l’Institut Fourier, arXiv:1512.01972[math.AG]

RAYMOND AND BEVERLY SACKLER SCHOOL OF MATHEMATICAL SCIENCES, TEL AVIV UNIVERSITY, 6997801 TEL AVIV, ISRAEL

E-mail address: borovoi@post.tau.ac.il