RAMSEY-TYPE PROBLEMS IN ORIENTATIONS OF GRAPHS

BRUNO PASQUALOTTO CAVALAR

ABSTRACT. Given an acyclic oriented graph $\vec{H}$ and a graph $G$, we write $G \rightarrow \vec{H}$ if every orientation of $G$ has an oriented copy of $\vec{H}$. We define $R(\vec{H})$ as the smallest number $n$ such that there exists a graph $G$ satisfying $G \rightarrow \vec{H}$. Denoting by $R(H)$ the classical Ramsey number of a graph $H$, we show that $\vec{R}(\vec{H}) \leq 2R(H)^{c \log^2 h}$ for every acyclic oriented graph $\vec{H}$ with $h$ vertices, where $H$ is its underlying undirected graph. We also study the threshold function for the event $\{G(n, p) \rightarrow \vec{H}\}$ in the binomial random graph $G(n, p)$. Finally, we consider the isometric case, in which we require that, for every two vertices $x, y \in V(\vec{H})$ and their respective copies $x', y'$ in $\vec{G}$, the distance between $x$ and $y$ is equal to the distance between $x'$ and $y'$. We prove an upper bound for the isometric Ramsey number of an acyclic orientation of the cycle, applying the hypergraph container lemma in random graphs.

1. INTRODUCTION

Given graphs $G$ and $H$, we write $G \rightarrow H$ to denote that every two-coloring of the edges of $G$ contains a monochromatic copy of $H$. The Ramsey number $R(H)$ of a graph $H$ is defined as

$$R(H) := \inf \{ n \in \mathbb{N} : \exists G = G^n \text{ such that } G \rightarrow H \}.$$

This number was proved to be finite by Ramsey [18] and Erdős and Szekeres [11]. Finding bounds for $R(H)$ with regards to specific choices of $H$ is a classical problem in combinatorics (see e.g. the dynamic survey of Radziszowski [17]). Moreover, the threshold function for the property that a random graph $G(n, p)$ satisfies $G(n, p) \rightarrow H$ is well-studied for random graphs [19]. In this work we study these same problems for a variant of this notion in orientations of graphs. Let us begin with a few definitions.

1.1. Digraphs and oriented graphs. A directed graph or digraph $\vec{G} = (V, E)$ where $V$ is a set of vertices and $E$ is a set such that $E \subseteq (V \times V) \setminus \{(v, v) : v \in V\}$. Just as in the case of undirected graphs, an element of $E$ is called an edge. An oriented graph $\vec{G} = (V, E)$ is a digraph where $(u, v) \in E$ implies $(v, u) \notin E$ for every $u, v \in V$. Moreover, an oriented graph $\vec{G} = (V_1, E_1)$ is said to be an orientation of a graph $G = (V_2, E_2)$ if $V_1 = V_2$ and, for every $u, v \in V_1 = V_2$, we have $(u, v) \in E_2$ if and only if $(u, v) \in E_1$ or $(v, u) \in E_1$. In this case, we say that $G$ is the underlying undirected graph of $\vec{G}$. Furthermore, when $\vec{G}$ is an oriented graph, we write $G$ to denote the underlying undirected graph of $\vec{G}$. To avoid confusion, we will always denote a digraph by a capital letter with $\rightarrow$.

1.2. Oriented Ramsey number. Given a graph $G$ and an acyclic oriented graph $\vec{H}$, let us write $G \rightarrow \vec{H}$ to denote that every orientation of the edges of $G$ contains a copy of $\vec{H}$. Observe that $\vec{H}$ must be acyclic, since every undirected graph has an acyclic orientation. One may also ask for bounds on the oriented Ramsey number $\vec{R}(\vec{H})$, which is defined as

$$\vec{R}(\vec{H}) := \inf \{ n \in \mathbb{N} : \exists G = G^n \text{ such that } G \rightarrow \vec{H} \}.$$
This number was proved to be finite by Erdős and Moser [10] (see Theorem 3.1 in this paper). Unlike the classical Ramsey number, very little has been published on bounds for \( \tilde{R}(\tilde{H}) \).

In Section 3, we quickly survey some known bounds for the oriented Ramsey number of an acyclic oriented graph \( \tilde{H} \) with \( h \) vertices and apply results and concepts from Conlon, Fox, Lee, and Sudakov [9] and Balko, Cibulka, Král and Kynčl [2], so as to show that \( \tilde{R}(\tilde{H}) \leq 2R(H)^{c \log^2 h} \). We are also able to prove better bounds for specific choices of \( \tilde{H} \).

1.3. An oriented Ramsey theorem for random graphs. For a graph \( H \), we denote by \( m_2(H) \) its 2-density, defined as

\[
m_2(H) := \max_{F \subseteq H, v(F) \geq 3} \frac{e(F) - 1}{v(F) - 2}.
\]

Consider also the binomial random graph \( G(n, p) \), which is the random graph in which each edge appears independently with probability \( p \). The following is a famous result of Rödl and Ruciński [19], which determines, for an undirected graph \( H \), the threshold function for \( G(n, p) \to H \). Here we state only the 1-statement.

**Theorem 1.1** (Rödl and Ruciński [19]). Let \( H \) be a graph. There exists a constant \( C = C(H) \) such that, if \( p \geq C n^{-1/m_2(H)} \), then

\[
\lim_{n \to \infty} \mathbb{P}[G(n, p) \to H] = 1.
\]

Define \( m_2(\tilde{H}) := m_2(H) \). In Section 5, we prove the following version of Theorem 1.1 for acyclic oriented graphs.

**Theorem 5.1.** Let \( \tilde{H} \) be an acyclic oriented graph. There exists a constant \( C = C(\tilde{H}) \) such that, if \( p \geq C n^{-1/m_2(\tilde{H})} \), then

\[
\lim_{n \to \infty} \mathbb{P}[G(n, p) \to \tilde{H}] = 1.
\]

Adapting some arguments from Nenadov and Steger [16], our proof of Theorem 5.1 makes use of the hypergraph container lemma of Balogh, Morris and Samotij [3] and Saxton and Thomason [23]. In Section 4, we develop the necessary container theory for digraphs that allows us to prove Theorem 5.1 in Section 5.

The technique of using hypergraph containers in random graphs for Ramsey problems has recently been employed by Hán, Retter, Rödl and Schacht [13], Rödl, Ruciński and Schacht [20] and Conlon, Dellamonica, La Fleur, Rödl and Schacht [8]. Our approach is also inspired by theirs, and some resemblance to their arguments is to be expected.

1.4. Isometric oriented Ramsey number. Finally, we consider the isometric oriented Ramsey number \( \tilde{R}_{\text{iso}}(\tilde{H}) \) of an acyclic oriented graph \( \tilde{H} \), a concept first introduced by Banakh, Idzik, Pikhurko, Protasov and Pszczółka [4].

For an undirected graph \( G \), we denote by \( d_G(u, v) \) the distance between two vertices \( u, v \in V(G) \). Given two oriented graphs \( \tilde{H} \) and \( \tilde{F} \), we say that a copy \( f : V(\tilde{H}) \to V(\tilde{F}) \) of \( \tilde{H} \) in \( \tilde{F} \) is an isometric copy if \( d_{\tilde{H}}(x, y) = d_{\tilde{F}}(f(x), f(y)) \) for every \( x, y \in V(\tilde{H}) \). Note that the distance is taken with respect to the underlying undirected graphs.

Given an oriented graph \( \tilde{H} \) and a graph \( G \), we write \( G \xrightarrow{\text{iso}} \tilde{H} \) if every orientation of \( G \) has an isometric oriented copy of \( \tilde{H} \). The isometric oriented Ramsey number \( \tilde{R}_{\text{iso}}(\tilde{H}) \) is defined as

\[
\tilde{R}_{\text{iso}}(\tilde{H}) := \inf \left\{ n \in \mathbb{N} : \exists G = G^n \text{ such that } G \xrightarrow{\text{iso}} \tilde{H} \right\}.
\]
It was proved in [4, Theorem 2.1] that the isometric oriented Ramsey number of acyclic oriented graphs is always finite. In Section 6, we devise a bound for $\bar{R}_{iso}(\bar{H})$ when $\bar{H}$ is an acyclic orientation of a cycle, adapting a construction of Hàn, Retter, Rödl, and Schacht [13]. This proof also makes use of the results developed in Section 4.

2. Basic facts

In this section we describe some preliminary results involving inequalities and probability theory that will be useful in what follows.

**Fact 2.1.** The following inequalities hold.

\[
\begin{align*}
1 + x &\leq e^x \quad \forall x \in \mathbb{R}, \\
1 - x &\geq \exp\left(\frac{-x}{1-x}\right) \quad \forall x \in [0,1), \\
\log x &\leq \frac{x}{2} \quad \forall x > 0.
\end{align*}
\]

One can easily check the following fact by taking derivatives.

**Fact 2.2.** Let $c > 0$ be a constant and define the function $f(x) := \left(\frac{e^x}{x}\right)^x$ for $x > 0$. The function $f(x)$ achieves its maximum value at $x = c$, and is monotonically increasing for $x \leq c$ and monotonically decreasing for $x \geq c$.

Let $V$ be a finite set. A **property of graphs** with respect to $V$ is a subset of the set of all graphs with vertex set $V$, closed under isomorphism. A property $P$ with respect to $V$ is said to be **monotone increasing** if, for every two graphs $H \in P$ and $G$ with vertex set $V$ and such that $H$ is a subgraph of $G$, we have $G \in P$. Moreover, such a property $P$ is said to be **monotone decreasing** if, for every two graphs $H$ and $G \in P$ with vertex set $V$ and such that $H$ is a subgraph of $G$, we have $H \in P$.

The following theorem is a simplified version of what is known as FKG inequality. The interested reader is pointed to Chapter 6 of [1] or Section 2.2 of [14] to learn more.

**Theorem 2.3** (FKG Inequality, Theorem 6.3.3 [1]). Let $P_1, P_2, Q_1$ and $Q_2$ be graph properties, where $P_1$ and $P_2$ are monotone increasing and $Q_1$ and $Q_2$ are monotone decreasing. We have

\[
\begin{align*}
\mathbb{P}[G(n,p) \in P_1 \cap P_2] &\geq \mathbb{P}[G(n,p) \in P_1] \cdot \mathbb{P}[G(n,p) \in P_2], \\
\mathbb{P}[G(n,p) \in Q_1 \cap Q_2] &\geq \mathbb{P}[G(n,p) \in Q_1] \cdot \mathbb{P}[G(n,p) \in Q_2].
\end{align*}
\]

By induction, one easily gets the following corollary.

**Corollary 2.4.** Let $P_1, P_2, \ldots, P_n$ and $Q_1, Q_2, \ldots, Q_n$ be graph properties, where $P_1, P_2, \ldots, P_n$ are monotone increasing and $Q_1, Q_2, \ldots, Q_n$ are monotone decreasing. We have

\[
\begin{align*}
\mathbb{P}\left[ G(n,p) \in \bigcap_{i=1}^{n} P_i \right] &\geq \prod_{i=1}^{n} \mathbb{P}[G(n,p) \in P_i], \\
\mathbb{P}\left[ G(n,p) \in \bigcap_{i=1}^{n} Q_i \right] &\geq \prod_{i=1}^{n} \mathbb{P}[G(n,p) \in Q_i].
\end{align*}
\]
3. Bounds for the oriented Ramsey number

3.1. Some known bounds. For the sake of introduction, we first survey a few known bounds for the oriented Ramsey number.

It is well-known, and not difficult to prove, that a tournament is acyclic if and only if it is transitive, and that there is exactly one transitive tournament on \( n \) vertices up to isomorphism (see e.g. Section 4.2 of [6]). Therefore, we can denote by \( \vec{K}_k \) the acyclic tournament on \( k \) vertices. To our knowledge, the following is the first bound to appear of the oriented Ramsey number of an oriented graph.

**Theorem 3.1** (Erdős and Moser [10]). Let \( \vec{K}_k \) be the acyclic orientation of \( K_k \) for some positive integer \( k \). We have

\[
2^{(k-1)/2} \leq \vec{R}(\vec{K}_k) \leq 2^{k-1}.
\]

We remark that the lower bound above can be proved by a standard application of the probabilistic method (see e.g.: Theorem 1 of [10] or Proposition 1.1.1 of [1]), and the upper bound can be proved by induction on \( k \), observing that every acyclic oriented graph has a topological ordering.

Since clearly \( \vec{R}(\vec{H}) \leq \vec{R}(\vec{K}_h) \) for every acyclic oriented graph \( \vec{H} \) on \( h \) vertices, we obtain the following corollary.

**Corollary 3.2.** Let \( \vec{H} \) be an acyclic oriented graph on \( h \) vertices. We have \( \vec{R}(\vec{H}) \leq 2^{h-1} \). In particular, the oriented Ramsey number \( \vec{R}(\vec{H}) \) is finite.

**Definition 3.3.** We denote by \( \vec{P}_k \) the directed path of length \( k \), which is the oriented graph with vertex set \( V(\vec{P}_k) := [k+1] \) and edge set \( E(\vec{P}_k) := \{(i, i+1) : i \in [k]\} \).

The following theorem is a known result of Gallai and Roy (see, for example, Theorem 14.5 of Bondy and Murty [5]).

**Theorem 3.4** (Gallai-Roy Theorem). If \( G \) is a graph such that \( \chi(G) = k + 1 \), then \( G \to \vec{P}_k \).

**Proof sketch.** Consider an arbitrary orientation \( \vec{G} \) of \( G \). Color each vertex \( v \in V(G) \) with the number of vertices contained in the largest directed path in \( \vec{G} \) which begins in \( v \). Observe that this is a proper coloring. Therefore, the largest directed path contained in \( \vec{G} \) has at least \( \chi(G) = k \) vertices. \( \square \)

Since \( \chi(K_{k+1}) = k + 1 \), we have thus completely determined the oriented Ramsey number of \( \vec{P}_k \).

**Corollary 3.5.** For every \( k \in \mathbb{N} \), we have \( \vec{R}(\vec{P}_k) = k + 1 \).

3.2. Ordered graphs. Before stating our bounds, we introduce the concept of ordered graphs and ordered Ramsey numbers, recently studied in Balko, Cibulka, Král and Kynčl [2] and Conlon, Fox, Lee, and Sudakov [9].

An **ordered graph** \( G \) is a pair \( G = (G', <_G) \) where \( G' \) is a graph and \( <_G \) is a total ordering of the vertices of \( G' \). For convenience we write \( V(G) := V(G') \) and \( E(G) := E(G') \). When a graph \( G \) is equipped with a total ordering of its vertices, we will simply refer to \( G \) as an ordered graph without further qualifications.
An ordered graph $G$ is said to contain an ordered graph $H$ if there exists a function $\phi : V(H) \to V(G)$ such that, for every $x, y \in V(H)$, we have $\phi(x) <_G \phi(y)$ if and only if $x <_H y$, and $\{i, j\}$ is an edge of $H$ only if $\{\phi(i), \phi(j)\}$ is an edge of $G$. In this case, we call $\phi$ a monotone embedding.

If the graphs $H$ and $G$ are ordered graphs, we write $G^{\text{ord}} \to H$ to denote that every two-coloring of the edges of $G$ contains an ordered monochromatic copy of $H$. When the graph $H$ is equipped with a total ordering, the ordered Ramsey number $R_<(H)$ can be defined analogously, as follows:

$$R_<(H) := \inf \left\{ n \in \mathbb{N} : \exists G = G^n : G^{\text{ord}} \to H \right\}.$$ 

The following is a general bound for the ordered Ramsey number of graph, depending on the Ramsey number of its corresponding unordered graph. In particular, this proves that the ordered Ramsey number of an ordered graph is always finite.

**Theorem 3.6** (Conlon, Fox, Lee, and Sudakov [9]). There exists a constant $c$ such that, for every ordered graph $H$ on $n$ vertices, we have

$$R_<(H) \leq R(H)^{c \log^2 n}.$$ 

More precise bounds for $R_<(H)$ for specific classes of ordered graphs can be found in Conlon, Fox, Lee, and Sudakov [9] and Balko, Cibulka, Král and Kynčl [2].

3.3. Our bounds. We now give a bound for the oriented Ramsey number of $\vec{H}$ depending on the Ramsey number of $H$. Our proof will be inspired in the proof of Theorem 2.1 of [4] (our Theorem 6.1), but, in reality, this idea already appeared in Cochand and Duchet [7] and in Rödl and Winkler [21].

**Theorem 3.7.** There exists a constant $c$ such that the following holds. Let $\vec{H}$ be an acyclic oriented graph with $h$ vertices and $H$ its underlying undirected graph. There exists orderings $<_0$ and $<_1$ of the vertices of $H$ such that, for $H_0 = (H, <_0)$ and $H_1 = (H, <_1)$, we have

$$R_0(\vec{H}) \leq R_<(H_0) + R_<(H_1) \leq 2R(H)^{c \log^2(h)}.$$ 

**Proof.** Let $\vec{F}$ be the oriented graph formed by two disjoint copies of $\vec{H}$, in which one has reversed edges. More formally, let $\vec{F}$ be the oriented graph with vertex set

$$V(\vec{F}) := V(\vec{H}) \times \{0, 1\}$$

and edge set

$$E(\vec{F}) := \left\{ ((u, 0), (v, 0)), ((v, 1), (u, 1)) : (u, v) \in E(\vec{H}) \right\}.$$ 

Since $\vec{H}$ is acyclic, the oriented graph $\vec{F}$ is also acyclic. Therefore, there exists an ordering $<$ of the vertices of $\vec{F}$ such that $u < v$ if $(u, v) \in E(\vec{F})$. Let $F$ be the (ordered) underlying undirected graph of $\vec{F}$ equipped with the ordering $<$. Let $<_0$ be an ordering of the vertices of $H$ such that, for $x, y \in V(H)$, we have $x <_0 y$ if and only if $(x, 0) < (y, 0)$. Define $<_1$ analogously. Let $H_0 := (H, <_0)$ and $H_1 := (H, <_1)$. Clearly, we have

$$R_<(F) \leq R_<(H_0) + R_<(H_1).$$

Let $<$ be an arbitrary ordering of the vertices of $K_N$. We thus consider $K_N$ to be an ordered complete graph. By Theorem 3.6, there exists a number $N$ such that $K_N^{\text{ord}} \to F$ and

$$N = R_<(F) \leq R_<(H_0) + R_<(H_1) \leq 2R(H)^{c \log^2(h)}.$$ 

5
Now it suffices to prove that $K_N \to \vec{H}$. Let $\vec{K}$ be an arbitrary orientation of $K_N$. Color the edges of $K_N$ in the following way: an edge $\{u, v\} \in E(K_N)$ with $u < v$ is colored blue if $(u, v) \in E(\vec{K})$ and red otherwise. By the choice of $N$, there exists an ordered monochromatic copy of $F$ in $K_N$. Let $\phi : V(F) \to V(K_N)$ be the monotone embedding of this copy. If the copy of $F$ in $K_N$ is blue, then the set of vertices $\{\phi((v, 0)) : v \in V(\vec{H})\}$ induces a directed copy of $\vec{H}$ in $\vec{K}$ with the color blue. Otherwise, if the copy is red, then the set of vertices $\{\phi((v, 1)) : v \in V(\vec{H})\}$ induces a copy with the color red. In either case we have proved $K_N \to \vec{H}$, as desired. 

**Remark 3.8.** The proof of Theorem 3.7 shows that the orderings $<_0$ and $<_1$ of $V(\vec{H})$ can be taken to be the topological ordering of $\vec{H}$ and the reverse topological ordering of $\vec{H}$, respectively.

The following theorem gives an exact formula for the classical Ramsey number of a cycle $C_k$ on $k$ vertices. One can find this result as Theorem 2 of a survey from Radziszowski [17].

**Theorem 3.9** (Rosta [22], Faudree and Schelp [12]). We have

$$R(C_k) = \begin{cases} 6, & \text{if } k = 3 \text{ or } k = 4 \\ 2k - 1, & \text{if } k \geq 5 \text{ is odd} \\ 3k/2 - 1 & \text{if } k \geq 6 \text{ is even.} \end{cases}$$

Therefore, it clearly holds that $R(C_k) \leq 2k$ for every $k \geq 3$. We now get the following corollary.

**Corollary 3.10.** There exists a constant $c$ such that the following holds. Let $k \geq 3$ and let $\vec{H}$ be an acyclic orientation of the cycle on $k$ vertices $C_k$. We have

$$\vec{R}(\vec{H}) \leq 2(2k)^{c \log^2(k)}.$$
double-counting on the pairs \((S, \vec{H}')\) where \(S \in \mathcal{S}\) and \(\vec{H}'\) is a copy of \(\vec{H}\) contained in \(S\) yields
\[
|S| \leq \varepsilon \left( \frac{n - h}{R - h} \right) \left( \frac{n}{h} \right).
\]
This implies that the set \(\overline{\mathcal{S}}\) defined as
\[
\overline{\mathcal{S}} := \left( \frac{V(\vec{K})}{R} \right) \setminus \mathcal{S}
\]
satisfies
\[
|\overline{\mathcal{S}}| = \left( \frac{n}{R} \right) - |\mathcal{S}|
\geq \left( \frac{n}{R} \right) - \varepsilon \left( \frac{n - h}{R - h} \right) \left( \frac{n}{h} \right)
= \left( \frac{n}{R} \right) \left( 1 - \varepsilon \left( \frac{n}{R} \right) \right) \left( \frac{n}{h} \right)
= \left( \frac{n}{R} \right) \left( 1 - \varepsilon \left( \frac{R}{h} \right) \right).
\]
Observe that, by definition of \(\overline{\mathcal{S}}\), every set \(S \in \overline{\mathcal{S}}\) induces at least one edge \(e \in E(\vec{K}) \setminus \vec{F}\). Moreover, every edge \(e \in E(\vec{K}) \setminus \vec{F}\) is contained in at most \(\frac{n - 2}{R - 2}\) \(R\)-element subsets. Now, double-counting on the pairs \((S, e)\) where \(S \in \overline{\mathcal{S}}\) and \(e \in E(\vec{K}[S])\) we get
\[
|E(\vec{K}) \setminus \vec{F}| \geq \frac{|\overline{\mathcal{S}}|}{\binom{n - 2}{R - 2}} \geq \left( 1 - \varepsilon \left( \frac{R}{h} \right) \right) \frac{\left( \frac{n}{R} \right)}{\binom{n - 2}{R - 2}} \geq \left( 1 - \varepsilon \left( \frac{R}{h} \right) \right) \frac{1}{R^2} n^2 = \delta n^2,
\]
by setting \(\delta := \left( 1 - \varepsilon \left( \frac{R}{h} \right) \right) \frac{1}{R^2}\).

The desired result now follows by observing \(|E(K_n) \setminus F| = |E(\vec{K}) \setminus \vec{F}|\).

\[\square\]

\textbf{Remark 4.2.} The proof of Theorem 4.1 shows that \(\delta\) can be taken as
\[
\delta := \left( 1 - \varepsilon \left( \frac{R}{h} \right) \right) \frac{1}{R^2}.
\]
Moreover, if \(\varepsilon \leq (1/2)\left( \frac{R}{h} \right)^{-1}\), then \(\delta \geq 1/(2R^2)\).

\textbf{4.2. A container lemma for digraphs.} Let \(\mathcal{H}\) be a \(\ell\)-uniform hypergraph. For a set \(J \subseteq V(\mathcal{H})\), we define the \textit{degree} of \(J\) by
\[
d(J) := |e \in E(\mathcal{H}) : J \subseteq e|.
\]
For a vertex \(v \in V(\mathcal{H})\), we let \(d(v) := d(\{v\})\). For \(j \in [\ell]\), we also define the \textit{maximum \(j\)-degree} of a vertex \(v \in V(\mathcal{H})\) by
\[
d^{(j)}(v) := \max \left\{ d(J) : v \in J \in \binom{V(\mathcal{H})}{j} \right\}.
\]
We denote the average of \(d^{(j)}(v)\) for all \(v \in V(\mathcal{H})\) by
\[
d_j := \frac{1}{v(\mathcal{H})} \sum_{v \in V(\mathcal{H})} d^{(j)}(v).
\]
Note that $d_1$ is the average degree of $\mathcal{H}$. Finally, for $\tau > 0$, we define $\delta_j$ as
\[
\delta_j := \frac{d_j}{d_1 \tau^{j-1}}
\]
and the co-degree function $\delta(\mathcal{H}, \tau)$ by
\[
\delta(\mathcal{H}, \tau) := 2^{\ell - 1} \sum_{j=2}^{\ell} 2^{-\binom{j-1}{2}} \delta_j.
\]

We now state a condensed version of the Container Lemma, as expressed in Saxton and Thomason [23]. This version can be found as Theorem 2.1 in [13].

**Theorem 4.3** ([23], Corollary 3.6). Let $0 < \varepsilon, \tau < 1/2$. Let $\mathcal{H} = (V, E)$ be a $\ell$-uniform hypergraph. Suppose that $\tau$ satisfies $\delta(\mathcal{H}, \tau) \leq \varepsilon/12!$. Then for integers $K = 800(\ell!)^3$ and $s = \lceil K \log(1/\varepsilon) \rceil$ the following holds.

For every independent set $I \subseteq V$ in $\mathcal{H}$ there exists a $s$-tuple $T = (T_1, \ldots, T_s)$ of subsets of $V$ and a subset $C = C(T) \subseteq V$ depending only on $S$ such that

(a) $\bigcup_{i \in [s]} T_i \subseteq I \subseteq C$,

(b) $e(C) \leq \varepsilon \cdot e(\mathcal{H})$, and

(c) for every $i \in [s]$ we have $|T_i| \leq K \tau |V|$.

Here we prove a version of the container lemma for $\vec{H}$-free orientations of graphs. First, we need the following definitions.

**Definition 4.4.** Let $\vec{H}$ be an oriented graph and let $n \in \mathbb{N}$. Denote by $\vec{D}_n$ the digraph with vertex set $[n]$ and edge set
\[
E(\vec{D}_n) := ([n] \times [n]) \setminus \{(v, v) : v \in [n]\}.
\]

**Definition 4.5** ([15], Definition 3.5). Let $\vec{H}$ be an oriented graph with $\ell$ edges and let $n \in \mathbb{N}$. The hypergraph $\mathcal{D}(n, \vec{H}) = (V, \mathcal{E})$ is a $\ell$-uniform hypergraph with vertex set $V := E(\vec{D}_n)$ and edge set
\[
\mathcal{E} := \left\{B \in \binom{V}{\ell} : \text{the edges of } B \text{ form a digraph isomorphic to } \vec{H}\right\}.
\]

**Definition 4.6.** Let $\vec{H}$ be an oriented graph with $h$ vertices. In what follows, we denote by $\text{emb}_{\vec{H}}(h)$ the number of copies of $\vec{H}$ in $\vec{D}_h$.

Our container lemma for $\vec{H}$-free orientations of graphs is as follows. We give a more general statement than needed for this section only because we are going to need this result also in Section 6.

**Theorem 4.7** (Container lemma for $\vec{H}$-free orientations). Let $0 < \varepsilon < 1/2$ and let $\vec{H}$ be an acyclic oriented graph with $\ell$ edges. There exists positive integers $s$ and $K$ and a real number $\delta > 0$ such that, for every $n \geq R(\vec{H})$, the following holds.

Suppose $0 < \tau < 1/2$ satisfies $\delta(\mathcal{D}(n, \vec{H}), \tau) \leq \varepsilon/(12!)$. For every graph $G$ on $n$ vertices such that $G \not\rightarrow \vec{H}$ there exists a $s$-tuple $T = (T_1, \ldots, T_s) \subseteq E(G)$ and a set $C = C(T) \subseteq E(K_n)$ depending only on $T$ such that

(a) $\bigcup_{i \in [s]} T_i \subseteq E(G) \subseteq C$,

(b) $|E(K_n) \setminus C| \geq \delta n^2$, and
(c) $\left| \bigcup_{i \in [s]} T_i \right| \leq sK\tau n^2$.

Proof. Let $\vec{H}$ be an acyclic oriented graph with $\ell$ edges. Let $n_0$ and $\delta$ be as given by Theorem 4.1 for $\varepsilon_0 := \varepsilon \cdot \text{emb}_H$ and $\vec{H}$. By Remark 4.2, we can take $n_0 = \overline{R}(\vec{H})$. Fix $n \geq n_0$ and set $\mathcal{H} := \mathcal{D}(n, \vec{H})$. Since $\delta(\mathcal{H}, \tau) \leq \varepsilon_0 / 12\ell$, Theorem 4.3 gives us numbers $s$ and $K$ for $\mathcal{H}$, $\varepsilon$ and $\tau$. Let $G$ be a graph on $n$ vertices such that $G \not\cong \vec{H}$. There exists an orientation $\vec{G}$ of $G$ such that $\vec{G}$ contains no copy of $\vec{H}$. Therefore, there exists a $s$-tuple $\vec{C}$ of oriented edges and $\vec{C} = \vec{C}(\vec{T})$ such as Theorem 4.3 gives for $E(\vec{G})$. For $i \in [s]$, let $T_i$ be the underlying set of undirected edges of $\vec{T}_i$. Define $C$ analogously for $C_i$. By item (a) of Theorem 4.3, we have

$$\bigcup_{i \in [s]} T_i \subseteq E(G) \subseteq C.$$ 

Observe now that $\text{emb}_H$ counts the number of copies of $\vec{H}$ in any subset of $h$ vertices of $\vec{D}_n$, whence it follows that

$$e(\mathcal{H}) = \binom{n}{h} \text{emb}_H. \quad (4)$$

Therefore, by item (b) of Theorem 4.3 we conclude that $\vec{C}$ has at most $\varepsilon e(\mathcal{H}) = \varepsilon_0 \binom{n}{h}$ copies of $\vec{H}$. By the choice of $\delta = \delta(\vec{H}, \varepsilon_0)$, Theorem 4.1 now gives

$$|E(K_n) \setminus C| \geq \delta n^2.$$ 

Finally, we get by item (c)

$$\left| \bigcup_{i \in [s]} T_i \right| \leq sK\tau \nu(\mathcal{H}) \leq sK\tau n^2.$$ 

Therefore, there exists a $s$-tuple $T$ and a set $C$ as promised. This finishes the proof. $\square$

Remark 4.8. In light of Remark 4.2, we see that in Theorem 4.7 the value of $\delta$ can be taken as

$$\delta := \left(1 - \varepsilon \cdot \text{emb}_H \left(\frac{R}{h}\right)\right) \frac{1}{R^2},$$

where $R := \overline{R}(\vec{H})$. Moreover, if $\varepsilon \leq \left(2 \text{emb}_H \left(\frac{R}{h}\right)\right)^{-1}$, then $\delta \geq 1/(2R^2)$. Finally, the values of $s$ and $K$ are just as in Theorem 4.3.

4.3. Checking degree conditions. To apply Theorem 4.7, it is necessary to prove a bound on $\delta(\mathcal{D}(n, \vec{H}), \tau)$ for a suitable value of $\tau$. This is done by the following lemma.

Lemma 4.9. Let $\vec{H}$ be an oriented graph with $h$ vertices and $\ell \geq 2$ edges. Let also $D_r \geq 1$ and write $\tau := D_r \alpha^{-1}$. We have

$$\delta(\mathcal{D}(n, \vec{H}), \tau) \leq 2^\ell h^{-2} D_r^{-1}.$$ 

Proof. For convenience, set $\mathcal{H} := \mathcal{D}(n, \vec{H})$. Let $J \subseteq V(\mathcal{H})$. Define

$$V_J := \bigcup_{(a, b) \in J} \{a, b\} \subseteq [n].$$

Note that $(V_J, J)$ is the subdigraph of $\vec{D}_n$ induced by the set of edges $J$. For a set $S \subseteq [n]\setminus V_J$ such that $|S| = h - |V_J|$, let $\text{emb}_H(J, S)$ denote the number of copies $\vec{F}$ of $\vec{H}$ such that $V(\vec{F}) = V_J \cup S$.
and \( J \subseteq E(\tilde{F}) \). Since \( \text{emb}_{\tilde{H}}(J, S) \) is the same number for any choice of \( S \) as above, we can write only \( \text{emb}_{\tilde{H}}(J) \) to refer to this number.

Recall that \( d(J) \) is the number of copies of \( \tilde{H} \) in \( \tilde{D}_n \) which contain the set \( J \). Observe now that

\[
d(J) = \binom{n - |V_J|}{h - |V_J|} \text{emb}_{\tilde{H}}(J).
\]

(5)

For every \( j \in [\ell] \), let

\[
f(j) := \min_{H' \subseteq \tilde{H}, v(H') = j} v(H').
\]

(6)

It follows from (5) that

\[
d(J) = \binom{n - |V_J|}{h - |V_J|} \text{emb}_{\tilde{H}}(J) \leq \binom{n - f(j)}{h - f(j)} \text{emb}_{\tilde{H}}(J).
\]

Note now that, for every \( e \in V(\tilde{H}) \), we have \( d^{(1)}(e) = d(e) = \binom{n - 2}{h - 2} \text{emb}_{\tilde{H}}(\{e\}) \). Therefore, the average \( d_1 \) of all \( d^{(1)}(e) \) satisfies \( d_1 = \binom{n - 2}{h - 2} \text{emb}_{\tilde{H}}(\{e\}) \), for some fixed \( e \in V(\tilde{H}) \). It follows that

\[
d(J) \leq \binom{n - f(j)}{h - f(j)} \text{emb}_{\tilde{H}}(J) \leq \binom{n - f(j)}{h - f(j)} \text{emb}_{\tilde{H}}(\{e\}) = \frac{(h - 2)(h - 3) \ldots (h - f(j) + 1)}{(n - 2)(n - 3) \ldots (n - f(j) + 1)} \leq \left( \frac{h}{n} \right)^{f(j) - 2}.
\]

Therefore, we have \( d^{(j)}(v)/d_1 \leq h^{f(j) - 2}n^{2 - f(j)} \). Since \( f(j) \leq h \), this gives us

\[
\frac{d_j}{d_1} = \frac{1}{v(\tilde{H})} \sum_{v \in V(\tilde{H})} \frac{d^{(j)}(v)}{d_1} \leq \frac{1}{v(\tilde{H})} \sum_{v \in V(\tilde{H})} h^{f(j) - 2}n^{2 - f(j)} = h^{f(j) - 2}n^{2 - f(j)} \leq h^{h - 2}n^{2 - f(j)}.
\]

We furthermore obtain

\[
d_j = \frac{d_j}{d_1} \leq h^{h - 2}n^{2 - f(j)} \tau^{j - 1} \leq h^{h - 2}n^{2 - f(j)} \tau^{j - 1} \leq h^{h - 2}n^{2 - f(j)} \tau^{j - 1}.
\]

(7)

Observe now that, by definition of \( m_2(\tilde{H}) \), we have \( m_2(\tilde{H}) \geq (j - 1)/(f(j) - 2) \). From this we may derive \( 2 - f(j) + (j - 1)/m_2(\tilde{H}) \leq 0 \). Therefore, we can conclude from (7) that

\[
d_j \leq h^{h - 2}D^{-1} \tau^{j - 1}.
\]

(8)

Now we can finally bound the co-degree function \( \delta(\mathcal{H}, \tau) \) by observing that

\[
\delta(\mathcal{H}, \tau) = 2^\ell \binom{\ell - 1}{2} \sum_{j=2}^{\ell} 2^{-(\ell-2)} \delta_j \leq 2^\ell \binom{\ell - 1}{2} h^{h - 2} D^{-1} \sum_{j=2}^{\ell} 2^{-(\ell-2)} \leq 2^\ell h^{h - 2} D^{-1}.
\]

This finishes the proof. \( \square \)

4.4. A probabilistic lemma. We will now prove a lemma that applies the results of this section to yield an upper bound on the probability that \( G(n, p) \not\rightarrow \tilde{H} \). This result will be useful both in Section 5 and Section 6.

For convenience, given numbers \( n, s \) and \( t \), define

\[
T(n, s, t) := \left\{ (T_1, \ldots, T_s) : \bigcup_{i \in [s]} T_i \subseteq E(K_n) \text{ and } \left| \bigcup_{i \in [s]} T_i \right| \leq t \right\}.
\]

Lemma 4.10. Let \( 0 < \varepsilon < 1/2 \) and let \( \tilde{H} \) be an acyclic oriented graph with \( \ell \) edges. There exists positive integers \( s \) and \( K \) and a real number \( \delta > 0 \) such that, for every \( n \geq \overline{R}(\tilde{H}) \), the following holds. For every \( 0 < \tau < 1/2 \) satisfying \( \delta(\mathcal{D}(n, \tilde{H}), \tau) \leq \varepsilon/(12!) \) and for any choice
of \( p \in (0,1) \), we have
\[
\mathbb{P}[G(n,p) \nleftrightarrow \vec{H}] \leq \exp(-\delta n^2 p) \left( 1 + \sum_{k=1}^{t} \left( \frac{2^{k-1}n^2 p}{k} \right)^k \right),
\]
where \( t := sK\tau n^2 \).

Proof. Let \( s, K \) and \( \delta \) be as given by Theorem 4.7 for \( \varepsilon, \tau \) and \( \vec{H} \). If a graph \( G \) satisfies \( G \nleftrightarrow \vec{H} \), by Theorem 4.7 there exists a \( s \)-tuple \( T = (T_1, \ldots, T_s) \in \mathcal{T}(n,s,t) \) and a set \( C(T) \subseteq E(K_n) \) such that
\[
\bigcup_{i \in [s]} T_i \subseteq E(G) \subseteq C(T)
\]
and
\[
|E(K_n) \setminus C(T)| \geq \delta n^2.
\]
Let us set for convenience \( D(T) := E(K_n) \setminus C(T) \). Since \( E(G) \subseteq C(T) \), we have
\[
E(G) \cap D(T) = \emptyset.
\]
Let \( \mathcal{G} \) be the family of all graphs \( G \) on \( n \) vertices such that \( G \nleftrightarrow \vec{H} \). For a \( s \)-tuple \( T = (T_1, \ldots, T_s) \in \mathcal{T}(n,s,t) \), let
\[
\mathcal{G}'_T := \{ G = G^n : T_i \subseteq E(G) \forall i \in [s] \},
\]
and let
\[
\mathcal{G}''_T := \{ G = G^n : E(G) \cap D(T) = \emptyset \}.
\]
Observations (9) and (10) show that
\[
\mathcal{G} \subseteq \bigcup_{T \in \mathcal{T}(n,s,t)} \mathcal{G}'_T \cap \mathcal{G}''_T.
\]
As the sets \( T_i \) and \( D(T) \) have empty intersection for every \( i \in [s] \), it follows that the events \( [G(n,p) \in \mathcal{G}'] \) and \( [G(n,p) \in \mathcal{G}''] \) are independent. We conclude
\[
\mathbb{P}[G(n,p) \in \mathcal{G}] \leq \sum_{T \in \mathcal{T}(n,s,t)} \mathbb{P}\left[ G(n,p) \in \mathcal{G}'_T \right] \cdot \mathbb{P}\left[ G(n,p) \in \mathcal{G}''_T \right].
\]
Since \( |D(T)| \geq \delta n^2 \) for every \( T \in \mathcal{T}(n,s,t) \), we have
\[
\mathbb{P}\left[ G(n,p) \in \mathcal{G}''_T \right] \leq (1-p)^{\delta n^2} \leq \exp(-\delta n^2 p).
\]
Moreover, we also have
\[
\sum_{T \in \mathcal{T}(n,s,t)} \mathbb{P}\left[ G(n,p) \in \mathcal{G}'_T \right] \leq \sum_{T \in \mathcal{T}(n,s,t)} p^{|\bigcup_{i \in [s]} T_i|}.
\]
It follows that
\[
\mathbb{P}[G(n,p) \in \mathcal{G}] \leq \exp(-\delta n^2 p) \cdot \sum_{T \in \mathcal{T}(n,s,t)} p^{|\bigcup_{i \in [s]} T_i|}.
\]
We now proceed to bound the sum in (11). For every integer \( k \) such that \( 0 \leq k \leq t \), define
\[
S(k) := \left\{ T \in \mathcal{T}(n,s,t) : \left| \bigcup_{i \in [s]} T_i \right| = k \right\}.
\]
Observe that $|S(k)| = \binom{n}{k} (2^s)^k$. Indeed, there are $\binom{n}{k}$ ways of choosing $k$ edges from $E(K_n)$, and $(2^s)^k$ ways of assigning these edges to the sets of the $s$-tuples, which gives the desired equation. Therefore,
\[
\sum_{T \in \mathcal{T}(n,s,t)} p^{\bigcup_{i \in [s]} T_i} = \sum_{k=0}^{t} |S(k)| p^k \leq \sum_{k=0}^{t} \binom{n}{k} (2^s)^k p^k \leq 1 + \sum_{k=1}^{t} \left( \frac{e^{2s-1} n^2 p}{k} \right)^k .
\]
Because of (11), this finishes the proof. \qed

**Remark 4.11.** The same quantitative remarks of Remark 4.8 hold for Lemma 4.10.

5. **An Oriented Ramsey Theorem for Random Graphs**

As promised in the introduction, we will prove in this section the following theorem, applying the results developed in Section 4.

**Theorem 5.1.** Let $\vec{H}$ be an acyclic oriented graph. There exists a constant $C = C(\vec{H})$ such that, if $p \geq C n^{-1/m_2(\vec{H})}$, then
\[
\lim_{n \to \infty} P[G(n,p) \to \vec{H}] = 1 .
\]

**Proof.** Let $\varepsilon$ be sufficiently small. Suppose $n \geq R(\vec{H})$. In Lemma 4.9, set
\[
D_\varepsilon := \frac{12 ! e^{(\ell)} \theta^{h-2}}{\varepsilon} ,
\]
and let $\tau := D_\varepsilon n^{-1/m_2(\vec{H})}$. By Lemma 4.9, this yields $\delta(D(n,\vec{H}),\tau) \leq \varepsilon/(12 !)$, where $\ell := e(\vec{H})$. We are, therefore, in the conditions of Lemma 4.10. Let $s, K$ and $\delta$ be as in Lemma 4.10 for $\varepsilon, \tau$ and $\vec{H}$. Set $c := s K D_\tau$ and $p := C n^{-1/m_2(\vec{H})}$, for some constant $C$ sufficiently large with respect to $c$. By Lemma 4.10, we have
\[
P[G(n,p) \not\rightarrow \vec{H}] \leq \exp(-\delta n^2 p) \left( 1 + \sum_{k=1}^{t} \left( \frac{e^{2s-1} n^2 p}{k} \right)^k \right) ,
\]
where $t := s K \tau n^2 = cn^{2-1/m_2(\vec{H})} = cn^2 p/C$.

Let $f(k)$ be the function which maps $k$ to $(eb/k)^k$, where $b = 2^{s-1} n^2 p$. Note that this is the function in the final sum above. Since $2^{s-1} n^2 p \geq cn^2 p/C$ for $C$ sufficiently large with respect to $s$ and $c$, Fact 2.2 yields
\[
1 + \sum_{k=1}^{\frac{cn^2 p}{C}} \left( \frac{e^{2s-1} n^2 p}{k} \right)^k \leq 1 + \frac{cn^2 p}{C} \left( \frac{C e^{2s-1} n^2 p}{cn^2 p} \right)^{cn^2 p/C} \leq n^2 \left( \frac{C e^{2s-1}}{c} \right)^{cn^2 p/C} \leq n^2 \exp \left( \frac{cn^2 p}{C} \log C + 1 + (s-1) \log 2 - \log c \right) = n^2 \exp \left( \frac{cn^2 p}{C} \log C + 1 + (s-1) \log 2 - \log c \right) \leq n^2 \exp \left( \frac{c n^2 p}{C} \log C + 1 + (s-1) \log 2 - \log c \right) \leq n^2 \exp \left( \frac{\delta n^2 p}{3} \right) \leq \exp \left( \frac{\delta n^2 p}{2} \right) .
\]

n sufficiently large.

12
We may now conclude
\[ \mathbb{P}[G(n,p) \in \mathcal{G}] \leq \exp(-\delta n^2 p) \exp\left(\frac{\delta n^2 p}{2}\right) = \exp\left(-\frac{\delta n^2 p}{2}\right) = o(1), \]
as desired. \qed

6. The Isometric Oriented Ramsey Number

6.1. Introduction. Recently, Banakh, Idzik, Pikhurko, Protasov and Pszczola [4] introduced the concept of isometric oriented Ramsey number, and proved the following.

Theorem 6.1 ([4], Theorem 2.1). For every acyclic oriented graph \( \vec{H} \), the isometric oriented Ramsey number \( \vec{R}_{iso}(\vec{H}) \) is finite.

Moreover, they posed the problem of estimating \( \vec{R}_{iso}(\vec{H}) \) for acyclic oriented graphs \( \vec{H} \). As promised in the introduction, in this section we give an upper bound on \( \vec{R}_{iso}(\vec{H}) \) when \( \vec{H} \) is an acyclic orientation of the cycle on \( k \) vertices \( C_k \). In particular, we prove the following theorem.

Theorem 6.2. There exists a positive constant \( c \) such that the following holds. Let \( \vec{H} \) be an acyclic orientation of \( C_k \) and set \( R := \vec{R}(\vec{H}) \). Then
\[ \vec{R}_{iso}(\vec{H}) \leq c k^{12k^3} R^{8k^2}. \] (12)

Remark 6.3. In light of Corollary 3.10 and Theorem 6.2, one readily sees that there exists constants \( c_1 \) and \( c_2 \) such that, for any acyclic orientation \( \vec{H} \) of the cycle \( C_k \), we have
\[ \vec{R}_{iso}(\vec{H}) \leq c_1 k^{c_2 k^3}. \]

The approach employed in this section to prove Theorem 6.2 is very similar to the proof of Theorem 1.1 in Hàn, Retter, Rödl, and Schacht [13]. In what follows, we will use the notation already developed in Section 5.

6.2. Proof of Theorem 6.2. We begin by observing that, for every orientation \( \vec{H} \) of the cycle \( C_k \), we have
\[ m_2(\vec{H}) = m_2(C_k) = \frac{k-1}{k-2}. \] (13)
This will justify the choice of constants we will make in the rest of this section.

We now prove the following Lemma, which is a slightly improved version of Lemma 4.9 adjusted for orientations of cycles. Our proof makes uses of some arguments and results of the proof of Lemma 4.9. The reader is recommended to read first that proof if some steps in the following proof are unclear.

Lemma 6.4. Let \( \vec{H} \) be an orientation of the cycle \( C_k \). Let also \( D_\tau \geq 1 \) and define \( \tau \) as \( \tau := D_\tau n^{-(k-2)/(k-1)} \). For every \( n \geq D_\tau^{(k-1)^2} \), we have
\[ \delta(D(n,\vec{H}),\tau) \leq 2^\binom{k}{2} k^{k-2} D_\tau^{-(k-1)}. \]

Proof. Fix \( j \in [k] \). Let \( f(j) \) be as defined in (6). Since \( \vec{H} \) is an orientation of the cycle on \( k \) vertices, we have \( f(j) = j + 1 \) for every \( j \in [k-1] \) and \( f(k) = k \). Furthermore, by (7) we obtain
\[ \delta_j \leq k^{k-2} n^{2-f(j)+(j-1)(k-2)/(k-1)} D_\tau^{1-j}. \]
Therefore, for \( j \in [k-1] \) we have
\[
\delta_j \leq k^{k-2} n^{1-j+(j-1)(k-2)/(k-1)} D_\tau^{1-j} \\
= k^{k-2} n^{-(j-1)/(k-1)} D_\tau^{1-j} \\
\leq k^{k-2} k^{-1/(k-1)} D_\tau^{-1} \\
\leq k^{k-2} n^{-1/(k-1)}. \tag{14}
\]
Moreover, we obtain from (8) that
\[
\delta_k \leq k^{k-2} D_\tau^{-(k-1)}. \tag{15}
\]
Since, by assumption, we have \( n \geq D_\tau^{(k-1)^2} \), inequalities (14) and (15) now give us
\[
\max_{j \in [k]} \delta_j = \delta_k.
\]
We therefore conclude
\[
\delta(H, \tau) = 2^{\left(\frac{j}{j-1}\right)} \sum_{j=2}^{\ell} 2^{-(j-1)} \delta_j \leq 2^{\left(\frac{j}{j-1}\right)} k^{k-2} D_\tau^{-(k-1)} \sum_{j=2}^{\ell} 2^{-(j-1)} \leq 2^{\left(\frac{j}{j-1}\right)} k^{k-2} D_\tau^{-(k-1)},
\]
as promised. \( \square \)

We may now proceed to the proof of Theorem 6.2. The proof will be as follows. We will consider the random graph \( G(n, p) \) and, imitating the proof of Theorem 5.1, we will prove that, with positive probability, we have \( G(n, p) \xrightarrow{\text{iso}} \vec{H} \), for a number \( n \) that satisfies (12) and a suitable choice of \( p \). Our strategy will be to prove that the graph \( G(n, p) \) has girth at least \( k \) and satisfies \( G(n, p) \xrightarrow{\text{acyclic}} \vec{H} \) for an acyclic orientation \( \vec{H} \) of \( C_k \), which implies \( G(n, p) \xrightarrow{\text{iso}} \vec{H} \).

**Proof of Theorem 6.2.** We begin by setting the following numbers we are going to use in the proof:
\[
\varepsilon = \frac{1}{2R^2}, \tag{16}
\]
\[
D_\tau = 4 \cdot \frac{2^{k/2} \cdot k^2}{\varepsilon^{1/(k-1)}} \leq 8R^2 k^{k+2}, \tag{17}
\]
\[
K = 800k(k!)^3 \leq 800k^{3k+1}, \tag{18}
\]
\[
s = \lfloor K \log(1/\varepsilon) \rfloor \leq 1600k^{3k+2} R, \tag{19}
\]
\[
D_p = KD_\tau s^2 10R^2 \log(5R^2), \tag{20}
\]
\[
n = D_p^2, \tag{21}
\]
\[
\tau = D_\tau n^{-\frac{k+2}{k-1}}, \tag{22}
\]
\[
p = D_\tau n^{-\frac{k+2}{k-1}}. \tag{23}
\]
Observe that, for some positive constant \( c > 0 \), we have
\[
D_p \leq c \cdot k^{10k+7} R^8 \leq k^{12k} R^8,
\]
which implies
\[
n \leq c k^{12k} R^{8k^2}.
\]
Let us first prove the following claim. The proof goes just as in the proof of Claim 3.1 of [13].

**Claim 6.5.** We have \( \mathbb{P}[\text{girth}(G(n,p)) \geq k] \geq \exp(-kD_p^{k-1}n) \).

**Proof of Claim 6.5.** Let \( C(n,k) \) be the set of all cycles \( C \subseteq E(K_n) \) of length at most \( k - 1 \). Let 
\[
X := |\{C \in C(n,k) : C \subseteq E(G(n,p))\}|
\]
be the random variable counting the number of cycles of length at most \( k - 1 \) in \( G(n,p) \). For each cycle \( C \subseteq E(K_n) \) of length at most \( k - 1 \), let \( X_C \) be the indicator function of the event \( E_C := \{C \subseteq E(G(n,p))\} \). Clearly, \( X \) is the sum of all such \( C \). Therefore, 
\[
\mathbb{E}[X] = \sum_{C \in C(n,k)} p^{\lvert C \rvert} = \sum_{j=3}^{k-1} \frac{(j-1)!}{2} \binom{n}{j} p^j \leq \sum_{j=3}^{k-1} \frac{(pn)^j}{2j} \leq \frac{k}{6} (pn)^{k-1} = \frac{k}{6} D_p^{k-1}n.
\]
Moreover, the set of all graphs \( G \) on \( n \) vertices such that \( C \nsubseteq E(G) \) is a monotone decreasing property. Therefore, using the FKG inequality (Corollary 2.4), and applying inequality (2), we get 
\[
\mathbb{P}[\text{girth}(G(n,p)) \geq k] = \prod_{C \in C(n,k)} (1 - p^{\lvert C \rvert}) \geq \prod_{C \in C(n,k)} \exp \left( -\frac{p^{\lvert C \rvert}}{1 - p^{\lvert C \rvert}} \right) \geq \exp \left( -\frac{\mathbb{E}[X]}{1 - p} \right).
\]
One may now easily check that 
\[1 - p^3 = 1 - n^{-2(k-2)/(k-1)} + 1/k^2 > 1/6 ,\]
since \( n > 11 \), and the claim follows. \( \square \)

We now prove the following claim. Our proof will be similar to that of Theorem 5.1, with the difference that the calculations will be more involved.

**Claim 6.6.** We have 
\[
\mathbb{P}[G(n,p) \nrightarrow \overline{H}] \geq 1 - \exp \left( -\frac{n^2p}{4R^2} \right).
\]

**Proof of Claim 6.6.** We want to apply Theorem 4.7. We begin by observing that our choice of \( D_\tau \) implies 
\[
D_\tau^{k-1} = \left( \frac{4 \cdot 2^{k/2} \cdot k^2}{\varepsilon^{1/(k-1)}} \right)^{k-1} = \frac{4^{k-1} \cdot 2^{k(k+1)} \cdot k^{2(k+1)}}{\varepsilon^k} \geq \frac{12 \cdot 2^{k} \cdot k^{k-1} \cdot k!}{\varepsilon}.
\]
Hence, since clearly \( n \geq D_\tau^{(k-1)^2} \), Lemma 6.4 now yields \( \delta(D(n,\overline{H}),\tau) \leq \varepsilon/(12k!) \). Observe, moreover, that 
\[
\varepsilon = \frac{1}{2R \varepsilon^k} \leq \frac{1}{2k! \binom{k}{n}^2} \leq \frac{1}{2 \text{emb}_{\overline{H}} \binom{k}{n}^2}.
\]
Now Lemma 4.10, together with Remark 4.8 and Remark 4.11, gives us 
\[
\mathbb{P}[G(n,p) \nrightarrow \overline{H}] \leq \exp \left( -\frac{n^2p}{2R^2} \right) \left( 1 + \sum_{j=1}^{sK\tau n^2} \left( \frac{e2^{s-1}n^2p}{j} \right)^j \right).
\]
We now proceed to bound the sum in (24). Let \( f(k) \) be the function which maps \( j \) to \( (eb/j)^j \), where \( b = 2^{s-1}n^2p \). Note that this is the function in the final sum above. Observe moreover
that
\[ 2^{s-1}n^2p = 2^{s-1}D_p n^{-\frac{k-2}{s-1}} n^2 \geq sK D_\tau n^{-\frac{k-2}{s-1}} n^2 = sK \tau n^2, \]
whence it follows by Fact 2.2 that
\[ 1 + \sum_{k=1}^{sK \tau n^2} \left( \frac{e^{2s-1}n^p}{k} \right)^k \leq 1 + sK \tau n^2 \left( \frac{e^{2s-1}n^p}{sK D_\tau} \right)^{sK \tau n^2} = 1 + sK \tau n^2 \left( \frac{e^{2s-1}D_p}{sK D_\tau} \right)^{sK \tau n^2}. \]
Moreover, since
\[ sK \tau = sKD_\tau n^{-(k-2)/(k-1)} = sKD_\tau D_p^{-k^2(k-2)/(k-1)} \leq sKD_\tau D_p^{-1} < 1, \]
we obtain
\[ 1 + sK \tau n^2 \left( \frac{e^{2s-1}D_p}{sK D_\tau} \right)^{sK \tau n^2} \leq n^2 \left( \frac{e^{2s-1}D_p}{sK D_\tau} \right)^{sK \tau n^2} \leq n^2 \exp \left( sK \tau n^2 \log \frac{e^{2s-1}D_p}{sK D_\tau} \right) \leq n^2 \exp \left( n^2 p \cdot \frac{sK D_\tau}{D_p} \log \frac{e^{2s-1}D_p}{sK D_\tau} \right). \]
Observe now that
\[ \frac{sKD_\tau}{D_p} \log(e^{2s-1}) = \frac{\log(e^{2s-1})}{s10R^2 \log(5R^2)} \leq \frac{1}{10R^2 \log(5R^2)} \leq \frac{1}{10R^2}. \quad (25) \]
Let now \( x := D_p/(sKD_\tau) \) and set \( y := x/s \). Since the function \( \log(x)/x \) is decreasing for \( x > e \), we have \( \log(x)/x \leq \log(y)/y \). Note also that \( y = 10R^2 \log(5R^2) \leq (5R^2)^2 \) by inequality (3). Therefore, applying (3) once again, we obtain \( \log y \leq \log(5R^2) \). These observations allow us to conclude that
\[ \frac{\log(D_p/(sKD_\tau))}{D_p/(sKD_\tau)} = \frac{\log(x)}{x} \leq \frac{\log(y)}{y} \leq \frac{\log(5R^2)}{10R^2 \log(5R^2)} = \frac{1}{10R^2}. \quad (26) \]
Hence, by inequalities (25) and (26) we obtain
\[ n^2 \exp \left( n^2 p \cdot \frac{sK D_\tau}{D_p} \left( \log(e^{2s-1}) + \log \frac{D_p}{sK D_\tau} \right) \right) \leq n^2 \exp \left( \frac{n^2 p}{5R^2} \right). \]
Observe now that
\[ \frac{n^2 p}{\log n} = \frac{D_p^{2k^2} D_p}{k^2 \log(D_p)} n^{-k^2(k-2)/(k-1)} = \frac{D_p^{k^2} D_p}{k^2} n^{-k^2} \geq \frac{D_p}{k^2} \geq \frac{10R^2 D_\tau}{k^2} \geq 40R^2. \]
From this we obtain
\[ 2 \log n \leq \frac{n^2 p}{20R^2} = \frac{n^2 p}{4R^2} - \frac{n^2 p}{5R^2}, \]
which implies
\[ n^2 \exp \left( \frac{n^2 p}{5R^2} \right) \leq \exp \left( \frac{n^2 p}{4R^2} \right). \]
All our work so far therefore implies
\[ 1 + \sum_{j=1}^{sK\tau n^2} \left( \frac{\epsilon^{2s-1}n^2p}{j} \right)^j \leq \exp \left( \frac{n^2p}{4R^2} \right), \]
which, in view of (24), yields
\[ \mathbb{P}[G(n, p) \not\rightarrow \vec{H}] \leq \exp \left( -\frac{n^2p}{4R^2} \right). \]
This finishes the proof of the claim. \( \square \)

Now, in view of Claim 6.5 and Claim 6.6, we can deduce
\[ \mathbb{P}[\text{girth } G(n, p) \geq k \land G(n, p) \to \vec{H}] \geq \mathbb{P}[\text{girth } G(n, p) \geq k] + \mathbb{P}[G(n, p) \to \vec{H}] - 1 \]
\[ \geq \exp(-kD_p^{k-1}n) - \exp \left( -\frac{n^2p}{4R^2} \right). \] (27)
Since we also have
\[ \frac{n^2p}{4R^2} = n \cdot \frac{D_p^{k+1} - k^2(1/(1-k))}{4R^2} > n \cdot \frac{D_p^{k+1}}{4R^2} > kD_p^{k-1}n, \]
we may now conclude from (27) that
\[ \mathbb{P}[\text{girth } G(n, p) \geq k \land G(n, p) \to \vec{H}] > 0, \]
which finishes the proof. \( \square \)

7. Further directions

In this work, we have seen some bounds for the oriented Ramsey number of acyclic oriented graphs (Section 3), and have shown how to apply the hypergraph container method to study the oriented Ramsey problem in random graphs (Section 5). Moreover, we explained the concept of isometric oriented Ramsey number, and we showed how the container method applied to random graphs can be used to prove actual bounds on the isometric oriented Ramsey number of concrete graphs (Section 6).

We think our work leaves some interesting problems open for further research. Firstly, it would be interesting if better bounds were found for the oriented Ramsey number of concrete graphs. It is not clear how far from optimal are the bounds given by using ordered Ramsey numbers.

Secondly, one could also consider not only orientations of graphs, but also orientations and colorings of edges, and require the oriented copy to be monochromatic. We believe our techniques can easily handle this case, and we are already working on this.

Finally, one could also try to apply the techniques of Section 6 to derive bounds for the isometric Ramsey number of other graphs, like paths and Moore graphs.

References

[1] Noga Alon and Joel H. Spencer. The probabilistic method. Wiley-Interscience Series in Discrete Mathematics and Optimization. John Wiley & Sons, Inc., Hoboken, NJ, third edition, 2008. 2.3, 3.1

[2] Martin Balko, Josef Cibulka, Karel Král, and Jan Kynčl. Ramsey numbers of ordered graphs. Electronic Notes in Discrete Mathematics, 49:419 – 424, 2015. The Eight European Conference on Combinatorics, Graph Theory and Applications, EuroComb 2015. 1.2, 3.2, 3.2
[3] József Balogh, Robert Morris, and Wojciech Samotij. Independent sets in hypergraphs. *J. Amer. Math. Soc.*, 28(3):669–709, 2015. 1.3

[4] Taras Banakh, Adam Idzik, Oleg Pikhurko, Igor Protasov, and Krzysztof Pszczola. Isometric copies of directed trees in orientations of graphs. June 2017. 1.4, 3.3, 6.1, 6.1

[5] J. A. Bondy and U. S. R. Murty. *Graph theory*, volume 244 of *Graduate Texts in Mathematics*. Springer, New York, 2008. 3.1

[6] Gary Chartrand, Linda Lesniak, and Ping Zhang. *Graphs & digraphs*. CRC Press, Boca Raton, FL, fifth edition, 2011. 3.1

[7] M. Cochand and P. Duchet. A few remarks on orientations of graphs and Ramsey theory. In *Irregularities of partitions (Fertőd, 1986)*, volume 8 of *Algorithms Combin. Study Res. Texts*, pages 39–46. Springer, Berlin, 1989. 3.3

[8] D. Conlon, D. Dellamonica, Jr., S. La Fleur, V. Rödl, and M. Schacht. A note on induced Ramsey numbers. *arXiv e-prints*, June 2016. 1.3

[9] David Conlon, Jacob Fox, Choongbum Lee, and Benny Sudakov. Ordered Ramsey numbers. *J. Combin. Theory Ser. B*, 122:353–383, 2017. 1.2, 3.2, 3.6, 3.2

[10] P. Erdős and L. Moser. On the representation of directed graphs as unions of orderings. *Magyar Tud. Akad. Mat. Kutató Int. Közl.*, 9:125–132, 1964. 1.2, 3.1

[11] P. Erdős and G. Szekeres. A combinatorial problem in geometry. *Compositio Math.*, 2:463–470, 1935. 1

[12] R. J. Faudree and R. H. Schelp. All Ramsey numbers for cycles in graphs. *Discrete Math.*, 8:313–329, 1974. 3.9

[13] H. Hàn, T. Retter, V. Rödl, and M. Schacht. Ramsey-type numbers involving graphs and hypergraphs with large girth. April 2016. 1.3, 1.4, 4.2, 6.1, 6.2

[14] Svante Janson, Tomasz Łuczak, and Andrzej Ruciński. *Random graphs*. Wiley-Interscience Series in Discrete Mathematics and Optimization. Wiley-Interscience, New York, 2000. 2

[15] Daniela Kühn, Deryk Osthus, Timothy Townsend, and Yi Zhao. On the structure of oriented graphs and digraphs with forbidden tournaments or cycles. *J. Combin. Theory Ser. B*, 124:88–127, 2017. 4.5

[16] Rajko Nenadov and Angelika Steger. A short proof of the random Ramsey theorem. *Combin. Probab. Comput.*, 25(1):130–144, 2016. 1.3

[17] Stanislav P. Radziszowski. Small Ramsey numbers. *Electron. J. Combin.*, 1:Dynamic Survey 1, 30, 1994. 1, 3.3

[18] F. P. Ramsey. On a Problem of Formal Logic. *Proc. London Math. Soc.*, S2-30:264–286, 1930. 1

[19] Vojtěch Rödl and Andrzej Ruciński. Threshold functions for Ramsey properties. *J. Amer. Math. Soc.*, 8(4):917–942, 1995. 1, 1.3, 1.1

[20] Vojtěch Rödl, Andrzej Ruciński, and Mathias Schacht. An exponential-type upper bound for folkman numbers. *Combinatorica*, May 2016. 1.3

[21] Vojtěch Rödl and Peter Winkler. A Ramsey-type theorem for orderings of a graph. *SIAM J. Discrete Math.*, 2(3):402–406, 1989. 3.3

[22] Vera Rosta. On a Ramsey-type problem of J. A. Bondy and P. Erdős. I, II. *J. Combinatorial Theory Ser. B*, 15:94–104; ibid. 15 (1973), 105–120, 1973. 3.9

[23] David Saxton and Andrew Thomason. Hypergraph containers. *Invent. Math.*, 201(3):925–992, 2015. 1.3, 4.2, 4.3

**Instituto de Matemática e Estatística, Universidade de São Paulo, Rua do Matão 1010, 05508–090 São Paulo, SP**

*Email addresses: brunopc@ime.usp.br*