INVENTORY STRUCTURES ON LIE GROUPS

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Abstract. We approach with geometrical tools the contactization and symplectization of filiform structures and define Hamiltonian structures and momentum mappings on Lie groups.

1. Introduction. The searching of Lie groups admitting left-invariant symplectic structures started off maybe with the articles [15], [8] and was motivated by the interest of the Mathematical Physics in symplectic homogeneous spaces. Very soon symplectic concepts as reduction or geometric frameworks as fiber bundles along with the development of specific techniques as the double symplectic extension or the symplectic oxidation were contained in the characterization of symplectic structures on Lie groups, triggering a great amount of results just as much in the structure theory as in the classification; lets us cite as landmarks [5], [9], [17], [19] and [3].

Parallel with and even though the seed of contact geometry is an old one reaching back to Sophus Lie in the tools development - the contact transformations - to study systems of differential equations, its implications in the architecture of manifolds has received less attention than the symplectic geometry. Only relatively recently, the techniques acquired a geometrical body to produce global topological results. One can consult Lutz [20] for a historical display with chronological detail. The search of contact forms on odd-dimensional Lie groups was first introduced by Gromov in [14]. Even though his techniques did not produce left-invariant structures gave rise to the still open problem of finding which Lie groups admit left-invariant contact forms (e.g. see [10], [17]).

Among the nilpotent Lie algebras, there are some, distinguished by a certain algebraic property, the filiform characteristic, which makes them the least nilpotent. The term filiform was introduced by M. Vergne [22] in the study of the generation and the irreducible components of the variety of nilpotent Lie algebras. The importance of Nilpotent Lie algebras in mathematics and physics crystallized in the search of invariants to canalize the classification of this special type (e.g. see [2], [11], [6]). Particularly, the filiform structure places the Nilpotent Lie Algebras in an optimal position to be provided with contact of symplectic structures (see e.g. [17], [12]).

Roughly speaking, we can say that the symplectic structures cannot coexist with semisimple structures on Lie groups. This is an essential fact and starting point of

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the whole theory. In Sec. 2 we give a proof of this fact bounded to the intrinsic algebraic characteristic of its Lie algebra, without having to go to external constructions associated to this one, for instance its theory of representations (cf. [8]) or without going to flat structures (see Sec. 5). On the other hand, the results gathered in Sec. 3 aim to initiate the approach to the geometrization of filiform structures through fiber bundles techniques. Finally in Sec. 4, we deal with Hamiltonian vector fields, Poisson brackets and momentum mappings on symplectic Lie groups in order to consider $G$ as a self Hamiltonian $G$–space.

2. Geometric structures and connections. Let $G$ be a Lie group and let us denote with $\mathcal{G}$ the Lie algebra of left-invariant vector fields on $G$. We suppose that $G$ is connected and of even dimension $2n$. Given a left-invariant closed $2$-form $w$, its radical is the subspace $\text{rad}(w) = \{ A \in \mathcal{G} : i_A w = 0 \}$ where $i_A$ denotes the interior product with respect to $A$. We shall say that $w$ is a \emph{symplectic form} on $G$ if $\text{rad}(w) = \{ 0 \}$. In this case, we shall denote with $(G, w)$ the symplectic structure defined by $w$ on $G$. In particular we say that $G$ admits an invariant \emph{exact symplectic structure} if on $G$ there is a left-invariant $1$-form $v$ such that $dv$ is symplectic.

Let $\Omega$ be the left-invariant volume element on $(G, w)$ given by $\Omega = w \wedge \cdots \wedge w$. We claim that $\Omega$ is an exact $2n$-form if and only if it is not right-invariant. In fact, for the latter condition to happen it means that there exists $A \in \mathcal{G}$ such that $L_A \Omega \neq 0$, where $L_A$ the Lie derivative along the integral curves of $A$. But now

$$L_A \Omega = di_A \Omega \neq 0, \quad (a)$$

and as the left-hand side of (a) has the form $k \Omega$ ($k \neq 0$), we conclude that $\Omega$ is exact. Conversely if $\Omega$ is exact, it is of the form $\Omega = d\Phi$ for some left-invariant $(2n-1)$–form $\Phi$. If we denote by $A$ the left-invariant vector field on $G$ defined by $i_A \Omega = \Phi$ then it holds that $L_A \Omega = \Omega$, whereby $\Omega$ is not right-invariant.

A contact structure in a $(2n+1)$–dimensional Lie group $G$ is given by a left-invariant $1$-form $\alpha$ such that $\text{rad}(d\alpha) \cap \ker(\alpha) = \{ 0 \}$ and $d\alpha$ defines a symplectic structure on $\ker(\alpha)$. The \emph{Reeb vector field} is the unique $A \in \mathcal{G}$ satisfying $\alpha(A) = 1$, $i_A (d\alpha) = L_A \alpha = 0$. For other equivalent definitions in general contact manifolds along with interesting details one can consult e.g. [18]. In this general case, under some conditions, the Reeb vector field can generate an action of the group $S^1$ on $M$ in such a way that the contact form on $M$ is the $1$–form of a connection in the principal bundle $\pi : M \to M/S^1$. We collect this result here for Lie groups, disseminated in the literature sometimes in confusing terms, see [4], [13], [1], [17].

**Definition 2.1.** Let $\pi_G : P \to M$ a principal fiber bundle over a manifold $M$ with group $G$. We shall say that a connection $\Gamma$ in $P$, defined by its $\mathcal{G}$–valued form $w$, determines a symplectic curvature, if its curvature form $\Omega_w = dw + \frac{1}{2}[w, w]$ verifies: if $i_X \Omega_w = 0$ for a horizontal vector field $X \in \ker(w)$ then $X = 0$.

We can now state

**Proposition 1.** There exists a bijection between left-invariant contact forms on a $(2n+1)$–dimensional Lie group $G$ and connections with symplectic curvature on a principal bundle $\pi : G \to G/H$, where $H$ is a closed subgroup of $G$ of dimension $1$.

**Proof.** In fact, if $\alpha$ is a left invariant contact form on $G$ then $H = \{X \in \mathcal{G} : L_X \alpha = 0\}$, is a Lie subalgebra of $\mathcal{G}$ whose corresponding Lie group $H = \{ h \in G : ad(h)^* \alpha = \alpha \}$ has dimension $1$ and is closed in $G$. In this way, as $\mathcal{G} = \ker(\alpha) \oplus H$, if we choose a basis $H = \{X\}$ and we write $w(Y) = \alpha(Y) \cdot X$, since $ad(H) \ker(\alpha) = \ker(\alpha)$,
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the \( \mathcal{H} \)-valued form \( w \) defines a left-invariant connection \( \Gamma \) on the principal bundle \( \pi : G \to G/H \). The curvature form \( \Omega \) of this invariant connection is given by

\[
\Omega = d\alpha \cdot X
\]

which is symplectic when restricted to \( \ker(w) \).

Conversely, let us consider a left-invariant connection \( \Gamma \) on the principal bundle \( \pi : G \to G/H \) that fibers an \( (2n+1) \)-dimensional Lie group over the \( 2n \)-homogeneous manifold \( G/H \). If \( \Gamma \) is defined by a \( \mathcal{H} \)-valued 1-form \( w \), it determines a decomposition \( \mathcal{G} = \ker(w) \oplus \mathcal{H} \). We define a left-invariant 1-form \( \alpha \) on \( G \) by expressing the \( \mathcal{H} \)-component of the connection form by \( w(Y) = \alpha(Y) \cdot X \) (\( Y \in \mathcal{G}, \mathcal{H} = \langle X \rangle \)). We have \( X \in \text{rad}(d\alpha) \). In fact, if \( Y \in \ker(w) \) then

\[
i_X d\alpha(Y) = L_X \alpha(Y) = X(\alpha(Y)) - \alpha([X,Y]) = 0,
\]

since \([X,Y]\) is a horizontal vector field, for being \( X \) an fundamental vector field and \( Y \) a horizontal one. The fact that \( \text{rad}(d\alpha) \) reduces to \( \langle X \rangle \), and therefore the 1-form \( \alpha \) defines a contact structure on \( G \), is equivalent to the fact that the 2-form defined on \( G/H \) by

\[
K(Y_1,Y_2) = d\alpha(Y_1^*,Y_2^*)
\]

where \( Y_i^* \) is the horizontal lift of \( Y_i \) with respect to \( w \) is symplectic.

The Lie group \( G \) will be called simple if \( \mathcal{G} \) is nonabelian and has no proper ideals. In the same way, the Lie group \( G \) is called semisimple if \( \mathcal{G} \) has no nonzero commutative ideals. It is obvious that in a simple Lie algebra \( [\mathcal{G},\mathcal{G}] = \mathcal{G} \) and that every semisimple Lie algebra has \( (0) \) as a center. It is not difficult to see that a Lie algebra \( \mathcal{G} \) is semisimple if and only if \( \mathcal{G} \) admits a unique decomposition in the form \( \mathcal{G} = \alpha_1 \oplus \ldots \oplus \alpha_n \) with \( \alpha_i \) ideals in \( \mathcal{G} \), being each simple Lie algebras. (And in this case, the only ideals of \( G \) are the sums of various \( \alpha_i \)). Furthermore taking into account that \([\alpha_i,\alpha_j] = \delta_{ij} \alpha_i\), it easily follows that that \([\mathcal{G},\mathcal{G}] = \mathcal{G} \) for a semisimple Lie algebra \( \mathcal{G} \).

The left-invariant \( r \)-forms on \( G \) can be naturally identified with \( \wedge^r \mathcal{G}^* \). In this way, if \( w \in \wedge^r \mathcal{G}^* \), it is easy to obtain the expression:

\[
dw(X_0,\ldots,X_r) = \sum_{0 \leq i < j \leq r} (-1)^{i+j} w([X_i,X_j],X_1,\ldots,\hat{X}_i,\ldots,\hat{X}_j,\ldots,X_r)
\]

As an easy consequence we have \( H^1(\mathcal{G}) = (\mathcal{G}/[\mathcal{G},\mathcal{G}])^* \). Hence for a semisimple Lie algebra \( H^1(\mathcal{G}) = 0 \). In addition, it is not difficult to see that the condition \([\mathcal{G},\mathcal{G}] = \mathcal{G} \) implies \( H^2(\mathcal{G}) = 0 \) for a Lie algebra \( \mathcal{G} \) (e.g. see [7]).

Next crucial result was proven by Bon-Yao Chu [8] using cohomological techniques associated with the Lie algebras representations. Here we offer an intrinsic proof.

**Proposition 2.** i) A simple Lie algebra does not admit a left-invariant symplectic form.

ii) Moreover, a semisimple Lie algebra has no left-invariant symplectic structure.

**Proof.** i) Let \( \mathcal{G} \) be a simple Lie algebra and let \( w \) be a symplectic form on \( G \). Since \( H^2(\mathcal{G}) = 0 \), we can write \( w = d\alpha \), for a left-invariant 1-form \( \alpha \) on \( G \). In this way the volume form on \( G, \Omega = w \wedge \ldots \wedge w \) is also exact, hence the set \( \alpha = \{ X \in \mathcal{G} : L_X \Omega = 0 \} \) is a proper ideal of \( \mathcal{G} \), which contradicts the fact that \( \mathcal{G} \) is simple.
ii) Now if $G$ is a semisimple Lie group endowed with a symplectic structure $\omega$, we consider the decomposition of $\mathcal{G}$ has direct sum of simple Lie algebras:

$$\mathcal{G} = \mathcal{G}_1 \oplus \ldots \oplus \mathcal{G}_n.$$ 

We contend that $\mathcal{G}_2 \oplus \ldots \oplus \mathcal{G}_n \subset \mathcal{G}_1^{\perp \omega}$. Let us see by example that $\mathcal{G}_2 \subset \mathcal{G}_1^{\perp \omega}$. Thus, let $Y \in \mathcal{G}_1$, $X_2 \in \mathcal{G}_2$. Since $[\mathcal{G}_1, \mathcal{G}_1] = \mathcal{G}_1$, there exist $X_0, X_1 \in \mathcal{G}_1$ with $[X_0, X_1] = Y$. In this way

$$0 = dw([X_0, X_1], X_2) = -w([X_0, X_1], X_2) + w([X_0, X_2], X_1) - w([X_1, X_2], X_0) = -w(Y, X_2),$$

thus obtaining our contention.

Then for the restriction $w|_{\mathcal{G}_1}$, there exists $X \in \mathcal{G}_1$ such that $i_X(w|_{\mathcal{G}_1}) = 0$, by which $X \in \mathcal{G}_1^{\perp \omega}$. Therefore as $\mathcal{G}_2 \oplus \ldots \oplus \mathcal{G}_n \subset \mathcal{G}_1^{\perp \omega}$, we have

$$X \notin \rad(w)$$

which is a contradiction. $\square$

3. Filiform structures. For the Lie algebra $\mathcal{G}$ of the Lie group $G$ we define recursively

$$\mathcal{G}_0 = \mathcal{G}, \quad \mathcal{G}_1 = [\mathcal{G}, \mathcal{G}], \ldots, \mathcal{G}_{i+1} = [\mathcal{G}, \mathcal{G}_i], \ldots$$

The decreasing sequence $\mathcal{G}_0 \supseteq \mathcal{G}_1 \supseteq \mathcal{G}_2 \supseteq \ldots$ is called the central series for $\mathcal{G}$. We say that $G$ is nilpotent if $\mathcal{G}_h = 0$ for some $h$. A nilpotent Lie algebra $\mathcal{G}$ has a nonzero center, the last nonzero $\mathcal{G}_j$ being the center. A nilpotent Lie group $G$ of dimension $\geq 3$ is called filiform if $\dim \mathcal{G}_j = \dim \mathcal{G} - j$.

Let $G$ be a filiform Lie group of dimension $2n + 1$, and let

$$\mathcal{G} = \mathcal{G}_0 \supseteq \mathcal{G}_1 \supseteq \ldots \mathcal{G}_{2n} \supseteq \mathcal{G}_{2n+1} = \{0\}$$

the central series of $\mathcal{G}$. Let $M$ be a subspace of $\mathcal{G}$ such that $\mathcal{G} = \mathcal{G}_{2n} \oplus M$. As $\ad(h)M = M$ if $h \in Z(G)$, and $\mathcal{G}_{2n}$ is the Lie algebra of the center $Z(G)$ of $G$, the $\mathcal{G}_{2n}$-component of the Maurer-Cartan form of $G$ defines a connection in the principal bundle $G \to G/Z(G)$ which is invariant by left translations of $G$. This way, we can enunciate

**Proposition 3.** Let $G$ be a filiform Lie group of dimension $2n+1$, let $\mathcal{G}_{2n}$ be the Lie algebra of the center $Z(G)$ of $G$, and let $w$ be the connection form of the connection $\Gamma$ in the principal bundle $G \to G/Z(G)$ determined by the decomposition $\mathcal{G} = \mathcal{G}_{2n} \oplus M$. If $\{g \in G: (R_g)^* w = w\} = Z(G)$, then $w$ determines a left-invariant contact form on $G$.

Now let $G$ be a $2n$ dimensional filiform Lie group and

$$\mathcal{G}_0 \supseteq \mathcal{G}_1 \supseteq \ldots \supseteq \mathcal{G}_{2n-1} \supseteq \mathcal{G}_{2n} = \{0\}$$

the descending central series for $\mathcal{G}$. Let $G_1$ be the corresponding Lie subgroup of $G$ to the Lie subalgebra $\mathcal{G}_1$ of $\mathcal{G}$. Then $G_{2n-1}$ is a closed subgroup in $G_1$ and the projection

$$G_1 \to G_1/G_{2n-1}$$

defines a differentiable principal bundle. Let $w$ be a connection form in $G_1$ associated with a decomposition $\mathcal{G}_1 = \mathcal{G}_{2n-1} \oplus M$. As we know, if the isotropy subgroup of $w$ under the right-action of $G_1$ (which always contains $G_{2n-1}$) coincides with $G_{2n-1}$, then $w$ defines a contact structure $\eta$ on $\mathcal{G}$. Let $X \in [\mathcal{G}, \mathcal{G}]$ be the Reeb
vector field and let $e_0 \in \mathcal{G}$ such that $\mathcal{G} = [\mathcal{G}, \mathcal{G}] \oplus \mathbb{R}e_0$. We shall say that the endomorphism $\Psi \in \text{End}([\mathcal{G}, \mathcal{G}])$ and the left-invariant 1-form $\theta \in [\mathcal{G}, \mathcal{G}]^*$ define the Lie algebra structure of $\mathcal{G}$ from its derived ideal $[\mathcal{G}, \mathcal{G}]$ if we have

$$[Y, e_0] = \Psi(Y) + \theta(Y)e_0, \quad \forall Y \in [\mathcal{G}, \mathcal{G}].$$

Then the Jacobi identity provides

$$\Psi([A, B], [C, D]) = [\Psi(A, B), [C, D]] + [[A, B], \Psi(C, D)]$$

and the fact that $\theta$ is a closed 1-form.

In this way closely following Diatta in [10], if we write $\Omega = d(\eta + se_0^\ast)$, $(s \in \mathbb{R})$, where $e_0^\ast \in \mathcal{G}^*$ verifies $e_0^\ast(e_0) = 1, \; e_0([\mathcal{G}, \mathcal{G}]^* = 0$, the necessary and sufficient condition so that $\Omega^n = \Omega \wedge \Omega \wedge \Omega$ is a volume form, is that

$$\eta(\Psi(X)) \neq -s\theta(X). \quad (*)$$

Consequently if $\theta(X) \neq 0$, for any $s \in \mathbb{R}$ that verifies $(*)$ we have that $\Omega = d(\eta + se_0^\ast)$ is an exact symplectic form on $G$.

On the other hand, if $\theta(X) = 0$, the condition $(*)$ is equivalent to the fact that $\Psi(X) \notin \ker(\eta)$, and in this case for any real number $s$, the form $\Omega = d(\eta + se_0^\ast)$ is symplectic on $G$.

In this way, we have

**Proposition 4.** Let $G$ be a $2n$-dimensional filiform Lie group with Lie algebra $\mathcal{G}$. If the principal bundle $G_1 \to G_1/G_{2n-1}$ is endowed with a connection with symplectic curvature, then there exists a left-invariant contact form $\eta$ on $[\mathcal{G}, \mathcal{G}]$. Let $X$ be its Reeb vector field. Let $e_0 \in \mathcal{G}$ such that $\mathcal{G} = [\mathcal{G}, \mathcal{G}] \oplus \mathbb{R}e_0$ and $\Psi \in \text{End}([\mathcal{G}, \mathcal{G}])$, $\theta \in [\mathcal{G}, \mathcal{G}]^*$ which define the Lie algebra structure of $\mathcal{G}$ from that of $[\mathcal{G}, \mathcal{G}]$. If the constant $\theta(X)$ is different from zero, then $G$ admits a left-invariant symplectic structure. On the contrary, if $\theta(X) = 0$ a necessary and sufficient condition for the existence of a left-invariant symplectic structure on $G$ is that $\Psi(\text{rad}(d\eta)) \not\subseteq \ker(\eta)$.

4. Hamiltonian structures. In this section, in order to be able to define momentum mappings associated to a symplectic structure $w$ on $G$ and to assume the infinitesimal symmetry with respect to the elements of $\mathcal{G}$ that this fact supposes, we make the assumption that the Lie group $G$ is endowed with a right-invariant closed nondegenerate smooth 2-form $w$ which provides it of a structure of symplectic manifold $(G, w)$. For every $A \in \mathcal{G}$ its 1-parameter group of transformations of $G$ is given by $R_a$, where $a_t = \exp(tA)$. Due to $(R_a)^*w = w$, we have $L_Aw = 0$. This condition is equivalent to the fact that $i_Aw$ is a closed form on $G$.

To define the momentum mapping associated to $(G, w)$ we need to suppose that for every $A \in \mathcal{G}$ the closed 1-form $i_Aw$ is exact, that is, we can write

$$i_Aw = df_A$$

for a unique smooth function $f_A$ on $G$ verifying $f_A(e) = 1$, $e$ being the identity element in $G$. We say that $w$ determines a Hamiltonian structure on $G$.

Now we define

$$\mu : G \to \mathcal{G}^*$$

by

$$\mu(g)(A) = f_A(g), \quad A \in \mathcal{G}.$$
Since for a fixed element \( g \in G \) the symplectic form \( w \) is \( R_g \)-invariant and the left-invariant vector field \( (R_g)_* A \ (A \in G) \) is \( \text{ad}(g^{-1})A \), it follows
\[
d (R_g^* f_A - f_{\text{ad}(g)A}) = 0.
\]
This implies, in particular, that being \( R_g^* f_A - f_{\text{ad}(g)A} \) constant on \( G \) we can define a map
\[
\Omega : G \to G^* : g \mapsto \Omega_g,
\]
\[
\Omega_g(A) = R_g^* f_A - f_{\text{ad}(g)A}, \ (A \in G)
\]
We shall call Poisson bracket of two functions \( f, g \in C^\infty(G) \) to the function defined by
\[
[f, g] = w(X_f, X_g)
\]
where \( X_f \) is the vector field such that \( i_{X_f} w = df \). It is a simple computation that given \( f, g, h \in C^\infty(G) \) it holds
\[
[f, [g, h]] + [h, [f, g]] + [g, [h, f]] = 0.
\]
In this way, the Poisson bracket provides \( C^\infty(G) \) of a Lie algebra structure.

Given \( A, B \in G \), an easy computation proves that \( [f_A, f_B] - f_{[A,B]} \) is constant on \( G \):
\[
\begin{align*}
d \left( [f_A, f_B] - f_{[A,B]} \right) &= d(Af_B) - i_{[A,B]} w \\
&= d(L_Af_B) - [L_A, i_B] w \\
&= L_A(df_B) - (L_Ai_B - i_B L_A) w = 0.
\end{align*}
\]

In this way it is a natural question to ask if there are situations \((G, w)\) for which (in duality with the Poisson manifolds \( M \) for which \( C^\infty(M) \to \mathfrak{X}(M) : f \mapsto X_f \) is a morphism of Lie algebras), the map
\[
\mathcal{G} \to C^\infty(G) \quad A \mapsto f_A
\]
is a morphism of Lie algebras. In this case we say that \( w \) endows \( G \) with a faithfully Hamiltonian structure.

We can now formulate a classic fact that we prove with our tools.

**Proposition 5.** Let \((G, w)\) be a connected faithfully Hamiltonian Lie group, the map \( \mu : G \to \mathcal{G}^* \) verifies
\[
\mu \circ R_g = \text{ad}(g)^* \circ \mu.
\]

**Proof.** It suffices to see
\[
\mu(g) = \text{ad}(g)^* \circ \mu(1).
\]

By expressing the map \( \Omega \) in terms of the momentum map, we have
\[
\Omega : G \to G^* : g \mapsto \Omega_g = \mu(g) - \text{ad}(g)^* \circ \mu(1).
\]
Then, if \( A, B \in \mathcal{G} \)
\[
\Omega_*(A)(B) = \left. \frac{\partial}{\partial t} \right|_{t=0} \left( \mu(\exp(tA)) - \text{ad}(\exp(tA))^* \circ \mu(1) \right) (B)
\]
\[
= [f_A, f_B](1) - f_{[A,B]}(1) = 0.
\]
In this way, as \( \Omega_*(A) = 0 \), \( \Omega \) must be constant, and as \( \Omega(1) = 0 \), it follows that \( \Omega \equiv 0. \)

\[\square\]
Coadjoint Orbits. Let us denote by \( O_\alpha = G \cdot \alpha \) the orbit of an element \( \alpha \in G^* \) by the coadjoint representation of \( G \). Then \( O_\alpha \) is diffeomorphic to the homogeneous space \( G/G_\alpha \) where \( G_\alpha = \{ g \in G : ad(g)^* \alpha = \alpha \} \). Since the fundamental vector field of an element \( A \in G \), is \( ad(A)^* \), we define the symplectic structure on the tangent space at \( \alpha \) in the orbit \( O_\alpha \) by

\[
w(ad(A)^*, ad(B)^*) = \alpha([A, B]).
\]

In this way, \( w \) is nondegenerate since if \( \alpha([A, \cdot]) = L_A \alpha = 0 \), then \( A \) belongs to the Lie algebra of \( G_\alpha \); also \( w \) is a closed 2-form, due to the fact that \( w(ad(A)^*, ad(B)^*) = -d\alpha(A, B) \).

**Lemma 4.1.** For a connected faithfully Hamiltonian Lie group \((G, w)\) the image of the momentum mapping \( \mu : G \to G^* \) is a coadjoint orbit \( O_\alpha = G \cdot \alpha \), and \( \alpha = \mu(1) \) cannot be a closed form in \( G^* \).

**Proof.** The first part of the statement is clear. On the other hand, if \( d\alpha = 0 \), as \( \mu_* = \frac{\partial}{\partial t}_{|_{t=0}} \mu(\exp tA) = \frac{\partial}{\partial t}_{|_{t=0}} ad(\exp tA)^* \circ \mu(1) = ad(A)^* \circ \alpha = -i_A d\alpha \)

we would get that \( \mu \) is constant, which is a contradiction. \( \square \)

Now, for \( A, B \in G \), we have

\[
\mu_*(A)(B) = \frac{\partial}{\partial t}_{|_{t=0}} \mu(\exp tA)(B) = \frac{\partial}{\partial t}_{|_{t=0}} f_B(\exp tA) = A(df_B) = A(i_B w) = w(B, A).
\]

This way, we have \( w = d\alpha \) and consequently \( w \) is an exact and bi-invariant 2-form.

But a bi-invariant closed differential 2-form on Lie group is necessarily degenerate. This is a contradiction. We have thus proved:

**Theorem 4.2.** No right-invariant symplectic form \( w \) defined on a Lie group \( G \) can endow this with a faithfully Hamiltonian structure.

5. Additional remarks. The result of Bon-Yao Chu on the non-existence of left-invariant symplectic structures on semisimple Lie groups was generalized by J. Helmstetter [16] to perfect groups, that is, Lie Groups \( G \) whose Lie algebras are equal to their derived ideal \([G, G]\). The relevance of his proof, lays down in the existence of a bilinear product \((A, B) \mapsto AB \) on \( G \) defined by

\[
w(AB, C) = -w(B, [A, C])
\]

in such a way that \( AB - BA = [A, B] \). This take us to the concept of flat Lie group as a Lie group endowed with a left-invariant torsion-free flat connection \( \nabla \). In this way, any attempt to mirror general constructions for arbitrary symplectic manifolds as Lagrangian or isotropic subgroups must me channeled through this flat structure, see [21], [3]. In this sense, any intent of translating the Hamilton-Jacobi problem to symplectic Lie groups must go through the problem of the existence of Lagrangian subgroups, which is only partially solved, see e.g. [9], [3].
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