GLOBAL CLASSICAL SOLUTIONS TO TWO-DIMENSIONAL CHEMOTAXIS-SHALLOW WATER SYSTEM

YING YANG*

College of Mathematics and Statistics, Shenzhen University
Shenzhen 518060, China

(Communicated by Zhian Wang)

Abstract. We consider the Cauchy problem of two-dimensional chemotaxis-shallow water system in the present paper. For regular initial data with small energy but possibly large oscillations, we prove the global well-posedness of classical solution. Then, we show the large-time behavior of the solution using the time-independent lower-order estimates as well.

1. Introduction. Nowadays, chemotaxis are well-known to play an important role in various biological processes, including cell migration, spatial pattern formation, immune response, embryonic morphogenesis and tumor invasion. The mechanisms of chemotaxis are cells in response to a chemical signal and move toward a chemically more favorable environment. In order to theoretical understand this common phenomenon in biology, many related mathematical models appeared. In this paper, we will study the following two-dimensional chemotaxis-shallow water system

\[
\begin{align*}
    n_t + \text{div}(nu) &= D_n \Delta n - \nabla \cdot (n\chi(c)\nabla c), \\
    c_t + \text{div}(cu) &= D_c \Delta c - nf(c), \\
    h_t + \text{div}(hu) &= 0, \\
    hu_t + hu \cdot \nabla u + h^2 \nabla n + \frac{1}{2}(1+n)\nabla h^2 &= \mu \Delta u + (\mu + \lambda)\nabla (\text{div} u)
\end{align*}
\]

in space-time domain \(\mathbb{R}^2 \times (0, \infty)\). This system was derived in [4] to model the dynamics of the oxygen and aerobic bacteria in a viscous fluid with free surface. Here we denote the unknown bacterial density, the chemoattractant concentration, the fluid height and the fluid velocity field by \(n, c, h, u\), respectively. The constants \(D_n\) and \(D_c\) present the corresponding diffusion coefficients for the cells and substrate. The given functions \(\chi(c)\) and \(f(c)\) are chemotactic sensitivity and the consumption rate of the substrate by the cells. The constants \(\mu\) and \(\lambda\) are the shear viscosity and the bulk viscosity coefficients respectively with the following physical restrictions

\[ \mu > 0, \quad \mu + \lambda \geq 0. \]

2020 Mathematics Subject Classification. Primary: 35B40, 35Q35; Secondary: 76N10, 92C17.

Key words and phrases. Chemotaxis, shallow water system, global classical solution.

The author is supported by the Guangdong Basic and Applied Basic Research Foundation (No.2020A1515010446), NSFC(No.11971320, No.11671155, No.11701384) and China Scholarship Council (No.201908440614).

* Corresponding author: Ying Yang.
For the system (1) to be well-posed, it should be supplemented with the initial condition
\[ (n, c, h, u)_{|t=0} = (n_0, c_0, h_0, u_0)(x), \quad x \in \mathbb{R}^2, \] (2)
and the far-field behavior
\[ \lim_{|x| \to \infty} (n - \bar{n}, c - \bar{c}, h - \bar{h}, u)(x) = 0, \quad t \geq 0, \] (3)
where \( \bar{n} \) and \( \bar{h} \) are positive constants.

In 1970, Keller and Segel [14][15] first derived a chemotaxis model called Patlak-Keller-Segel system to describe the collective behavior of cells in response to a signal produced by the cells themselves.

\[
\begin{cases}
  u_t = \Delta u - \nabla \cdot (u \nabla v), \\
  v_t = \Delta c - v + u,
\end{cases}
\]

where \( u \) and \( v \) denote the cells density and the chemoattractant concentration respectively. From then on, mathematical analysis on chemotaxis models attracted many mathematicians to do researches in this field. Many important works on the solvability and stability of the classical Keller-Segel system and related models appeared in the past three decades. For example, we refer to [1][9][10][11][21][27]. However, the most well-studied models, containing the classical Keller-Segel system, only focus on the interplay between the cells and the chemoattractant. The liquid micro-environment of the cells here is considered to be quiescent so that its effects on the movement of the bacteria and cells are always ignored. In fact, since bacteria often live in fluids and the oxygen concentration and bacteria density are transported by the fluids and diffuse through the fluids, the biology of aerotaxis is intimately related to liquid micro-environment, for instance, cell motion and chemical diffusion.

To study the interaction mechanism between fluids, bacteria and chemicals, it is natural to consider some chemotaxis-fluids systems. By experiments, Tuval et al. [23] established a coupled system to describe the dynamics of swimming bacteria, \textit{Bacillus subtilis}, which consists of the chemotaxis model and the viscous incompressible fluid. The model is as follows
\[
\begin{align*}
  n_t + u \cdot \nabla n &= D_n \Delta n - \nabla \cdot (n \chi(c) \nabla c), \\
  c_t + u \cdot \nabla c &= D_c \Delta c - n f(c), \\
  u_t + u \cdot \nabla u + \nabla P &= D_u \Delta u + n \nabla \phi, \\
  \nabla \cdot u &= 0,
\end{align*}
\] (4)
where \( n, c, u \) are the bacteria density, the oxygen concentration and the fluid velocity, respectively. \( P = P(x, t) \) is the corresponding scalar pressure and \( \phi = \phi(x) \) is the potential function such as the gravitational force or centrifugal force. Following the development of research, the system (4) is now widely accepted and studied in biology and mathematics. The researchers usually call this coupled system chemotaxis-Navier-Stokes system or chemotaxis-Stokes system when the nonlinear convective term \( u \cdot \nabla u \) is removed. Let us recall some important works on system (1) in the field of mathematical analysis. In the two dimensional case, Winkler [26] studied the global existence of classical solution to the initial boundary value problem with the functions \( \chi(\cdot), f(\cdot) \) and \( \phi(\cdot) \) satisfied
and

\[
\begin{aligned}
\chi & \in C^2([0, \infty)), \quad \chi(\cdot) > 0 \text{ in } [0, \infty), \\
f & \in C^2([0, \infty)), \quad f(0) = 0, \quad f(\cdot) > 0 \text{ in } [0, \infty), \\
\phi & \in C^2(\Omega),
\end{aligned}
\]

(5)

and

\[
\frac{d}{dc} \frac{f(c)}{\chi(c)} > 0, \quad \frac{d^2}{dc^2} \left( \frac{f(c)}{\chi(c)} \right) \leq 0, \quad (\chi(c) \cdot f(c))' \geq 0.
\]

(6)

Under these conditions, the global solution stabilizes to the spatially uniform equilibrium \((\bar{n}_0, 0, 0)\) was established in [28]. Chae et al. [3] established global existence of classical solutions and their decay properties under the same smallness conditions of initial data. In their results, the functions \(\chi, f, \chi'\) and \(f'\) were all non-negative, \(\chi, f \in C^m(\mathbb{R}^+)\), \(m \geq 3\), \(f(0) = 0\) and \(|\nabla^l \phi|_{L^\infty(\mathbb{R}^2)} < \infty\) for \(1 \leq |l| \leq m\). Duan et al. [6] first proved the global existence of weak solutions for the chemotaxis-Stokes system under some structural conditions on \(\chi(\cdot), f(\cdot)\) and \(\phi(\cdot)\) as (5) and (6) with \(\|\phi_0\|_{L^\infty(\mathbb{R}^2)}\) is small. Then, in [16] and [7], the small assumption on \(\phi_0\) was removed. Zhang and Zheng [32] showed the global existence and uniqueness of weak solutions to the chemotaxis-Navier-Stokes system for a large class of initial data with \(\chi = 1\) and \(f(c) = c\) using microlocal analysis and a scale decomposition technique. Recently, in [5], under the basic assumptions: \(\chi(\cdot)\) and \(f(\cdot)\) are locally bounded, \(f(\cdot)\) is continuous at zero with \(f(0) = 0\), \(f(s) \geq 0\) for all \(s \in \mathbb{R}\) and \(\nabla \phi \in L^\infty(\mathbb{R}^2)\), the authors showed the global existence of weak solutions and classical solutions for both the Cauchy problem and the initial-boundary value problem. While for the three-dimensional case, Winkler [29][30] obtained the global existence of weak solutions and then showed that this weak solutions become smooth after a waiting time and uniformly converge in the large-time limit when \(\chi\) and \(f\) are sufficiently smooth given functions. We also refer to [3][7][16][26] for the similar results on the global solvability to the two-dimensional case. For more related works, including temporal decays and blow-up criteria, one can see [2][8][17][20][25][31] and references therein.

Considering the two important facts that the cells stay at the surface of the fluid and the vertical acceleration of the fluid can be neglected, the authors in [4] derived the two-dimensional chemotaxis-shallow water system (1) from the chemotaxis-Navier-Stokes system (4). This coupled shallow water type chemotactic model presents a kind of chemotaxis-compressible fluid, which not only consists of chemotaxis and diffusion mechanism, but also includes transport and conservation laws. Up to now, there are only few analytic results on the solvability of corresponding initial(-boundary)-value problems. In [4], the authors first established the local strong solution and blow up criterion under the large initial data allowing vacuum. Tao and Yao [22] proved the global well-posedness of strong solution and investigated the asymptotic behavior of the global solution. However, the \(H^3\)-norm of the initial data needs to be small enough, which plays an important role in their proof. Later, the \(L^p\) decay estimates of the global solutions were showed in [24].

Motivated by the works for the three dimensional compressible Navier-Stokes equation [12][13], we want to look for a global classical solution to the problem (1)-(3) with small total energy but possibly large oscillations in the present paper. Compared with the previous results, there are three differences in our work. First, most of the proof on the global existence of classical solutions for compressible Navier-Stokes equations or other related systems begin with the basic energy equality. However, we know that since the diffusion mechanism, conservation laws are not satisfied for the chemotaxis part of our system (1)-(3). In order to obtain
the global existence for the coupled system, we introduce an energy inequality using the small energy condition on the initial data. Second, we remove the smallness on the high-order derivatives of the initial data, so that the initial data may contain the large oscillations. Our result improves the results in [22]. Third, we show the exponential stability of the chemoattractant concentration \( c \), based on the strong diffusion characteristic.

Before proving our main results, we first explain the notations and conventions used throughout this paper. In what follows, for simplicity, let
\[
D_n = D_c = 1, \quad \chi(c) \equiv 1, \quad f(c) = c, \quad \bar{n} = \bar{h} = 1
\]
and the results proved in this paper can be easily extended for general \( \chi \) and \( f \), as the choices in [26] and [29]. Furthermore, we need the following notations.

The initial energy is defined by
\[
E_0 = \int \left( |h_0 - 1|^2 + \frac{1}{2} |h_0| u_0|^2 + \frac{1}{2} |n_0 - 1|^2 + \frac{1}{2} |c_0|^2 \right) dx. \tag{7}
\]

Now, the main results can be stated as follows.

**Theorem 1.1.** For the given constant \( M_1 > 0 \) (not necessarily small), assume that the initial data \((n_0, c_0, h_0, u_0)\) satisfy
\[
\begin{cases}
\inf h_0 > 0, \ n_0 \geq 0, \ c_0 \geq 0, \\
(n_0 - 1, c_0, h_0 - 1, u_0) \in H^3, \\
\|\nabla n_0\|_{L^2} + \|\nabla c_0\|_{L^2} + \|\nabla h_0\|_{H^1} + \|\nabla u_0\|_{H^1} \leq M_1,
\end{cases} \tag{8}
\]
then there exists a positive constant \( \varepsilon \) depending only on \( \mu, \lambda, M_1 \) such that if
\[
E_0 \leq \varepsilon,
\]
the problem (1)–(3) has a unique global solution \((n, c, h, u)\) satisfying
\[
\begin{cases}
h > 0, \ n \geq 0, \ c \geq 0, \ x \in \mathbb{R}^2, \ t \geq 0, \\
(n - 1, c, h - 1, u) \in C([0, +\infty); H^3), \\
h_t \in C([0, +\infty); H^2), \ (n_t, c_t, u_t) \in C([0, +\infty), H^1) \cap L^2(0, +\infty; H^2),
\end{cases}
\]
which is a classical one for \( t > 0 \).

**Remark 1.** From (8) and the small initial energy, one can see that the initial data in Theorem 1.1 have small \( H^1 \)-norm for \((h_0, u_0)\), which is weaker than that in [22] and [24] where the smallness of \( H^1 \)-norm is required. Moreover, the initial bacterial density \( n_0 \) and the chemoattractant concentration \( c_0 \) allow large oscillations.

**Theorem 1.2.** Assume that the assumptions of Theorem 1.1 hold. Then, the global solution has the following large time behavior
\[
\lim_{t \to \infty} \left( \|n(\cdot, t) - 1\|^2_{L^q} + \|c(\cdot, t)\|^2_{L^q} + \|h(\cdot, t) - 1\|^2_{L^q} + \|u(\cdot, t)\|^2_{L^q} \right) = 0, \tag{9}
\]
for any \( q \in (2, \infty] \). Furthermore, there exist \( T_1 > 0 \) and \( 0 < \varepsilon^* < \varepsilon \), such that if \( E_0 \leq \varepsilon^* \), the solution \( c \) satisfies
\[
\|c(\cdot, t)\|_{L^\infty} \leq Ce^{-\frac{1}{2}t}, \quad \text{for any } t > T_1,
\]
where \( C \) is a constant depending only on \( M_1 \) and the \( H^2 \)-norm of the initial data \( n_0 \) and \( c_0 \).
The rest of this paper is organized as follows. In section 2, as preliminaries, some known inequalities and facts are shown. In section 3, we establish some a priori estimates on the solutions, which are needed for obtaining the global existence of classical solutions. First, the lower-order a priori estimates, which are independent of time, are derived. Then, we prove the time-dependent estimates on the higher-order norms of the solutions. Finally, the proof of our main results will be completed in section 4.

2. Preliminaries. In this section, we recall some elementary inequalities and known results which will be used throughout this paper. We start with the well-known Sobolev inequalities (see [19] for example).

Lemma 2.1. Let \( 0 \leq j \leq m, 1 \leq r, q \leq +\infty \) and the function \( f \in C_0^\infty(\mathbb{R}^2) \), then we have
\[
\| \nabla^j f \|_{L^p} \leq C_0 \| \nabla^m f \|_{L^r}^{\alpha} \| f \|_{L^q}^{1-\alpha},
\]
where \( \frac{1}{m} \leq \alpha \leq 1 \) satisfies
\[
\frac{1}{p} = \frac{j}{2} + \alpha \left( \frac{1}{r} - \frac{m}{2} \right) + (1-\alpha) \frac{1}{q},
\]
and \( C_0 \) is a positive constant depending on \( j, m, r, q, \alpha \).

In this paper, we shall frequently use the following inequalities which are obtained by Lemma 2.1.

Corollary 1. For the function \( f \in C_0^\infty(\mathbb{R}^2) \), it holds that
\[
\| f \|_{L^4} \leq C \| f \|_{L^2}^{\frac{1}{2}} \| \nabla f \|_{L^2}^{\frac{1}{2}} \quad \text{and} \quad \| f \|_{L^\infty} \leq C \| f \|_{L^2} \| \nabla^2 f \|_{L^2}^{\frac{1}{2}},
\]
where \( C \) is a positive constant.

We state the local existence and uniqueness theorem for the solution of the system (1)–(3) here. One can see the proof in [4] and [22].

Lemma 2.2. (Local well-posedness) Assume that the initial data \((n_0 - 1, c_0, h_0 - 1, u_0) \in H^3(\mathbb{R}^2)\), \( n_0, c_0 \geq 0 \) and \( h_0 > 0 \). Then there exist a time \( T^* > 0 \), such that the system (1)–(3) has a unique solution
\[
(n - 1, c, h - 1, u) \in C([0, T^*), H^3(\mathbb{R}^2)),
\]
\[
h_t \in C([0, T^*), H^2(\mathbb{R}^2)); \quad (n_t, c_t, u_t) \in L^\infty(0, T^*; H^1(\mathbb{R}^2)) \rightleftharpoons L^2(0, T^*; H^2(\mathbb{R}^2)).
\]

Lemma 2.3. Assume that the assumptions of Theorem 1.1 hold. Then the global solution \((n, c, h, u)\) to the Cauchy problem of the system (1)–(3) satisfies
\[
n(x, t) \geq 0, \quad c(x, t) \geq 0 \quad \text{a.e. in} \ \mathbb{R}^2 \times (0, +\infty).
\]

Proof. It follows from the maximum principle that \( n \) and \( c \) preserve the nonnegativity of the initial data, which gives (10). A similar result can be found in Lemma 2.1 in [5].

Denoting \( v = n - 1 \) and \( \rho = h - 1 \), the equations (1) can be written in the perturbation form as
The initial data are given as

\[
\begin{cases}
  v_t - \Delta v = -u \cdot \nabla v - (v + 1) \text{div} u - \nabla \cdot [(v + 1) \nabla c], \\
  c_t - \Delta c = -u \cdot \nabla c - c \text{div} u - (v + 1) c, \\
  \rho_t + \text{div} u = -\rho \text{div} u - u \cdot \nabla \rho, \\
  u_t - \mu \Delta u - (\mu + \lambda) \nabla \text{div} u + \nabla \rho = -g(\rho) [\mu \Delta u + (\mu + \lambda) \nabla (\text{div} u)] - u \cdot \nabla u \\
  - \nabla [(\rho + 1)(v + 1)],
\end{cases}
\]

where

\[ g(\rho) = \frac{\rho}{\rho + 1} \]  

(12)

The initial data are given as

\[(v, c, \rho, u)|_{t=0} = (v_0, c_0, \rho_0, u_0)(x), \ x \in \mathbb{R}^2, \]  

(13)

where \(v_0 = n_0 - 1\) and \(\rho_0 = h_0 - 1\), and the far-field behavior

\[
\lim_{|x| \to \infty} (v, c, \rho, u)(x) = 0, \ t \geq 0.
\]

(14)

3. A priori estimates. This section is devoted to establish some necessary a priori estimates of solutions to (11)-(14). Assume that \((v, c, \rho, u)\) is a smooth solution to (11)-(14) on \(\mathbb{R}^2 \times (0, T)\) for some positive time \(T > 0\). Let us define the following functions:

\[
A_1(T) = \sup_{0 \leq t \leq T} \left( \|v\|_{L^2}^2 + \|c\|_{L^2}^2 + \|\rho\|_{L^2}^2 + \|u\|_{L^2}^2 \right) \\
+ \int_0^T \left( \|\nabla v\|_{L^2}^2 + \|\nabla c\|_{L^2}^2 + \|\nabla \rho\|_{L^2}^2 \right) dt,
\]

\[
A_2(T) = \sup_{0 \leq t \leq T} \left( \|\nabla \rho\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 \right) + \int_0^T \left( \|\nabla \rho\|_{L^2}^2 + \|\nabla^2 c\|_{L^2}^2 \right) dt,
\]

\[
A_3(T) = \sup_{0 \leq t \leq T} \|\nabla^2 \rho\|_{L^2}^2.
\]

Throughout this section, we denote by \(C\) or \(C_i (i = 0, 1, 2, \ldots)\) the positive constants which may depend on \(\mu\) and \(\lambda\) but are independent of time \(T > 0\). We also sometimes use \(C(\alpha)\) to emphasize the dependence on \(\alpha\). Since we are considering the problem with small initial energy, without loss of generality, we assume \(E_0 \leq 1\) in what follows.

3.1. Time-independent lower-order estimates. In this subsection, we derive the lower-order a priori estimates on the solutions which are independent of time \(T > 0\). Now, we start with the following key a priori estimates.

**Proposition 1.** Assume that the initial data satisfy (8). Let \((v, c, \rho, u)\) be a smooth solution to (11)-(14) on \(\mathbb{R}^2 \times (0, T)\) satisfying

\[
A_1(T) \leq 2E_0^2, \ A_2(T) \leq 2M, \ A_3(T) \leq 2\{\frac{1}{\mu} + 1\}M,
\]

(15)

where \(M = \max\{2M_1, \frac{2M_1}{C_1}\}\) and the constant \(C_1\) is defined in Lemma 3.4. Then there exists a constant \(\varepsilon_1 > 0\) depending only on \(\mu, \lambda\) and \(M_1\), such that

\[
A_1(T) \leq E_0^2, \ A_2(T) \leq \frac{3}{2}M, \ A_3(T) \leq \frac{3}{2}\{\frac{1}{\mu} + 1\}M,
\]

(16)

provided \(E_0 \leq \varepsilon_1\) holds.
Lemma 3.1. Assume that all the assumptions of Proposition 1 hold. Then there exists a constant $\varepsilon_1 > 0$ depending only on $M_1$ such that
\begin{equation}
|\rho| \leq \frac{1}{2} \text{ and } 1 \leq h \leq \frac{3}{2}
\end{equation}
on $\mathbb{R}^2 \times (0, T)$, provided $E_0 \leq \varepsilon_1$.

Proof. By (15) and Sobolev inequality, we obtain
\begin{align*}
\|\rho\|_{L^\infty} &\leq C\|\rho\|_{L^2}^{\frac{1}{2}}\|\nabla^2 \rho\|_{L^2}^{\frac{1}{2}} \leq C(M)E_0^{\frac{1}{8}} \leq \frac{1}{2},
\end{align*}
provided $E_0 \leq \varepsilon_1 := \min\{1, (4C(M))^{-\frac{1}{8}}\}$.

It follows from Lemma 3.1, one can easily find that
\begin{equation}
|g(\rho)| \leq C|\rho| \text{ and } |g^{(k)}(\rho)| \leq C, \quad k \geq 1.
\end{equation}

Now, using Lemma 3.1 and the assumptions of Proposition 1, we have the following energy inequality.

Lemma 3.2. Under all the assumptions of Proposition 1, it holds that
\begin{align*}
\sup_{0 \leq t \leq T} (\|v\|_{L^2}^2 + \|c\|_{L^2}^2 + \|\rho\|_{L^2}^2 + \|u\|_{L^2}^2) + \int_0^T (\|\nabla v\|_{L^2}^2 + \|\nabla c\|_{L^2}^2 + \|\nabla u\|_{L^2}^2) dt \\
\leq CE_0 e^{CM},
\end{align*}
where $C$ depending only on $\mu$ and $\lambda$.

Proof. Multiplying (11) by $v$, $c$, $2\rho$ and $hu$ respectively, integrating over $\mathbb{R}^2$, after integration by parts, we have
\begin{align*}
\frac{1}{2} \frac{d}{dt} \int (2\rho^2 + hu^2 + v^2 + c^2) dx + \int (|\nabla v|^2 + |\nabla c|^2 + \mu|\nabla u|^2 + (\mu + \lambda)|\nabla u|^2) dx \\
= -\int (6\rho \nabla \rho \cdot u + \rho^2 \nabla u + v \nabla v \cdot u + v^2 \nabla u + c \nabla c \cdot u + c^2 \nabla u + 2\rho^2 \nabla \rho \cdot u) dx \\
+ \int ((v + 1) \nabla c \cdot \nabla v - (v + 1)c^2) dx - \int (\rho^2 \nabla v \cdot u + 2\rho \nabla v \cdot u) dx \\
- \int h^2 v \nabla \rho \cdot u dx \\
= \sum_{i=1}^4 I_i.
\end{align*}

From (14), the boundary condition which is hidden in the decay of solutions at spatial infinity can ensure that the integration by parts is valid here and in what follows. By virtue of (17), the Sobolev and Cauchy inequalities, we find that
\begin{align*}
I_1 &= -\int (6\rho \nabla \rho \cdot u + \rho^2 \nabla u + v \nabla v \cdot u + v^2 \nabla v + c \nabla c \cdot u + c^2 \nabla u + 2\rho^2 \nabla \rho \cdot u) dx \\
&\leq C\|\rho\|_{L^2}^2 \|\nabla \rho\|_{L^2} \|\nabla u\|_{L^2}^2 + C\|v\|_{L^2} \|\nabla v\|_{L^2}^2 \|\nabla u\|_{L^2}^2 + C\|c\|_{L^2} \|\nabla c\|_{L^2} \|\nabla u\|_{L^2}^2
\end{align*}
Under all the assumptions of Proposition 1, it holds that
\begin{equation}
\leq \epsilon (\|\nabla u\|_{L^2}^2 + \|\nabla v\|_{L^2}^2 + \|\nabla c\|_{L^2}^2) + C(\epsilon) \left[\|\rho\|_{L^2} \|
abla \rho\|_{L^2} + \|\nabla u\|_{L^2}^2 (\|v\|_{L^2}^2 + \|c\|_{L^2})\right],
\end{equation}
(21)

\[ I_3 = - \int (\rho^2 \nabla v \cdot u + 2\rho \nabla v \cdot u) \, dx \]
\[ \leq C\|\nabla v\|_{L^2} \|\rho\|_{L^4} \|u\|_{L^4} \]
\[ \leq C\|\nabla v\|_{L^2} \|\rho\|_{L^2} \|
abla \rho\|_{L^2} \|u\|_{L^2}^3 \|\nabla u\|_{L^2}^2 \]
\[ \leq \epsilon \|\nabla v\|_{L^2}^2 + C(\epsilon) (\|\rho\|_{L^2} \|
abla \rho\|_{L^2} + \|u\|_{L^2} \|\nabla u\|_{L^2} + \|\rho\|_{L^2} \|\nabla u\|_{L^2}^2 \]
\[ + \|u\|_{L^2}^2 \|\nabla \rho\|_{L^2}^2) \], \tag{22}

\[ I_4 = - \int h^2 v \nabla \rho \cdot u \, dx \]
\[ \leq C\|\nabla \rho\|_{L^2} \|u\|_{L^4} \|v\|_{L^4} \]
\[ \leq \epsilon (\|\nabla u\|_{L^2}^2 + \|\nabla v\|_{L^2}^2) + C(\epsilon) \|\nabla \rho\|_{L^2}^2 (\|u\|_{L^2}^2 + \|v\|_{L^2}^2). \tag{23} \]

Due to (10), we obtain
\[ I_2 = \int ((v + 1) \nabla c \cdot \nabla v - (v + 1)c^2) \, dx \]
\[ \leq \int \nabla c \cdot \nabla v \, dx + C\|v\|_{L^2} \|\nabla v\|_{L^2} \|\nabla^2 c\|_{L^2} \]
\[ \leq \frac{1}{2} \|\nabla c\|_{L^2}^2 + \frac{1}{2} \|\nabla v\|_{L^2}^2 + \epsilon \|\nabla v\|_{L^2}^2 + C(\epsilon) \|v\|_{L^2} \|\nabla^2 c\|_{L^2}^2. \tag{24} \]

Substituting (21)–(24) into (20) and choosing \( \epsilon < \frac{1}{2} \) small enough, one has
\[ \frac{1}{2} \frac{d}{dt} \int (2\rho^2 + hu^2 + v^2 + c^2) \, dx + C_0 \int (\|\nabla v\|^2 + \|\nabla c\|^2 + \|\nabla u\|^2) \, dx \]
\[ \leq C((\|v\|_{L^2}^2 + \|c\|_{L^2}^2 + \|h^2 u\|_{L^2}^2 + \|\sqrt{\rho}\|_{L^2}^2)\|\nabla u\|_{L^2}^2 \]
\[ + ((\|v\|_{L^2}^2 + \|h^2 u\|_{L^2}^2 + \|\sqrt{\rho}\|_{L^2}^2)\|\nabla \rho\|_{L^2}^2 + \|v\|_{L^2}^2 \|\nabla^2 c\|_{L^2}^2). \tag{25} \]

Integrating (25) over \((0, T)\), by virtue of (7), (15) and Gronwall's inequality, we get (19) and the lemma is proved. \( \square \)

**Lemma 3.3.** Under all the assumptions of Proposition 1, it holds that
\begin{equation}
\int_0^T \|\nabla \rho\|_{L^2}^2 \, dt \leq C(M) E_0^3 \left( \int_0^T \|\nabla^2 \rho\|_{L^2}^2 \, dt + 1 \right). \tag{26} \end{equation}

**Proof.** Multiplying (11) by \( \nabla \rho \), integrating over \( \mathbb{R}^2 \) and using (11), we have
\[ \int |\nabla \rho|^2 \, dx \]
\[ = - \frac{d}{dt} \int (u \cdot \nabla \rho) \, dx + \int (\rho + 1) |\text{div} u|^2 \, dx + \int (\text{div} u (u \cdot \nabla \rho) - u \cdot \nabla u \cdot \nabla \rho) \, dx \]
\[ + \int \nabla \rho \cdot \frac{1}{\rho + 1} [\mu \Delta u + (\mu + \lambda) \nabla (\text{div} u)] \, dx - \int (\rho + 1) \nabla v \cdot \nabla \rho \, dx \]
\[ - \int (v + 1) |\nabla \rho|^2 \, dx \]
\[ \leq \epsilon (\|\nabla u\|_{L^2}^2 + \|\nabla v\|_{L^2}^2 + \|\nabla c\|_{L^2}^2) + C(\epsilon) \left[\|\rho\|_{L^2} \|
abla \rho\|_{L^2} + \|\nabla u\|_{L^2}^2 (\|v\|_{L^2}^2 + \|c\|_{L^2})\right]. \tag{27} \]
\[ -\frac{d}{dt} \int (u \cdot \nabla \rho) dx + \sum_{i=1}^{4} J_i - \int (v + 1)|\nabla \rho|^2 dx. \]

It is easy to get from (17) that
\[ J_1 \leq C||\nabla u||^2_{L^2}. \]

Utilizing (15), (17), the Sobolev and Cauchy inequalities, after integration by parts, we get
\[ J_2 = -\int u^2 \nabla^2 \rho dx \leq C\parallel \nabla^2 \rho \parallel_{L^2} \parallel u \parallel^2_{L^2} \leq C\parallel \nabla^2 \rho \parallel_{L^2} \parallel u \parallel_{L^2} \parallel \nabla u \parallel_{L^2} \]
\[ \leq C(M)E_0 \parallel \nabla^2 \rho \parallel^2_{L^2} + C\parallel \nabla u \parallel^2_{L^2}, \]
\[ J_3 \leq C \left( \int \frac{1}{\rho + 1}|\nabla^2 \rho \cdot \nabla u| dx + \int \frac{|\nabla \rho|^2}{(\rho + 1)^2} \cdot |\nabla u| dx \right) \]
\[ \leq C\parallel \nabla^2 \rho \parallel_{L^2} \parallel \nabla u \parallel_{L^2} + \parallel \nabla^2 \rho \parallel_{L^2} \parallel \nabla \rho \parallel_{L^2} \parallel \nabla u \parallel_{L^2} \]
\[ \leq C(M)E_0^{\frac{1}{2}} \parallel \nabla^2 \rho \parallel^2_{L^2} + E_0^{-\frac{1}{2}} \parallel \nabla u \parallel^2_{L^2}, \]
\[ J_4 \leq \frac{1}{2} \parallel \nabla \rho \parallel^2_{L^2} + C\parallel \nabla v \parallel^2_{L^2}. \]

After substituting the above inequalities into (27), it yields
\[ \int |\nabla \rho|^2 dx \leq -2 \frac{d}{dt} \int (u \cdot \nabla \rho) dx + C(||\nabla u||^2_{L^2} + ||\nabla v||^2_{L^2}) + C(M)E_0^{\frac{1}{2}} \parallel \nabla^2 \rho \parallel^2_{L^2} \]
\[ + C(M)E_0^{-\frac{1}{2}} \parallel \nabla u \parallel^2_{L^2}. \]

Thus, using (15) and integrating (28) over \((0, T)\), (26) holds. The proof of this lemma is completed. \(\square\)

**Lemma 3.4.** Assume that all the assumptions of Proposition 1 hold. Then there exists a constant \(\varepsilon_2 > 0\) depending only on \(\mu, \lambda\) and \(M_1\), such that
\[ \sup_{0 \leq t \leq T} (||\nabla v||^2_{L^2} + ||\nabla c||^2_{L^2} + ||\nabla \rho||^2_{L^2} + ||\nabla u||^2_{L^2}) \]
\[ + C_1 \int_0^T (||\nabla^2 v||^2_{L^2} + ||\nabla^2 c||^2_{L^2} + ||\nabla^2 u||^2_{L^2}) dt \]
\[ \leq C(M)E_0^{\frac{1}{2}} + 2||\nabla \rho_0||^2_{L^2} + ||\nabla c_0||^2_{L^2} + ||\nabla v_0||^2_{L^2} + ||\nabla u_0||^2_{L^2}, \]
provided \(E_0 \leq \varepsilon_2\).

**Proof.** Applying \(\nabla\) to (11), then multiplying (11) by \(\nabla v, \nabla c, 2\nabla \rho\) and \(\nabla u\) respectively, integrating over \(\mathbb{R}^2\), we have
\[ \frac{1}{2} \frac{d}{dt} \int (2|\nabla \rho|^2 + |\nabla u|^2 + |\nabla v|^2 + |\nabla c|^2^2) dx \]
\[ + \int (|\nabla^2 v|^2 + |\nabla^2 c|^2 + \mu|\nabla^2 u|^2 + (\mu + \lambda)|\nabla \text{div} u|^2) dx \]
\[ = -2 \int \nabla (\rho \text{div} u + u \cdot \nabla \rho) \cdot \nabla \rho dx - \int \nabla (u \cdot \nabla v + v \text{div} u + \nabla (v + 1) \nabla c) \cdot \nabla vd x \]
\[ - \int \nabla (u \cdot \nabla c + c \text{div} u + (v + 1) c) \cdot \nabla cd x \]
\[- \int \nabla (g(\rho)[\mu \Delta u + (\mu + \lambda) \nabla (\text{div} u)] + u \cdot \nabla u + \nabla (\rho v)) \cdot \nabla u \, dx \]

\[= \sum_{i=1}^{N} K_i. \quad (30)\]

It follows from (15), (17), the Sobolev and Cauchy inequalities that

\[K_1 \leq C \|\nabla \rho\|_{L^2} \|\nabla u\|_{L^4}^2 + C \|\nabla^2 u\|_{L^2} \|\rho\|_{L^4} \|\nabla \rho\|_{L^4}\]

\[\leq C (\|\nabla u\|_{L^2} + \|\nabla^2 u\|_{L^2}) \|\rho\|_{L^4}^2 \|\nabla \rho\|_{L^4}^2 + \|\nabla \rho\|_{L^2}^2 \|\nabla^2 \rho\|_{L^2} \|\rho\|_{L^2}\]

\[\leq \epsilon \|\nabla^2 u\|_{L^2}^2 + C(\epsilon) (\|\rho\|_{L^4}^2 \|\nabla \rho\|_{L^4}^2 + \|\nabla \rho\|_{L^4}^2 \|\nabla^2 \rho\|_{L^2} \|\rho\|_{L^2})\]

\[\leq \epsilon \|\nabla^2 u\|_{L^2}^2 + C(M, \epsilon) E_0^2 \|\nabla \rho\|_{L^2}^2 + \|\nabla u\|_{L^2}^2. \quad (31)\]

Integrating by parts and using (15), it can be obtained that

\[K_2 = \int \left( \nabla u \cdot |\nabla v|^2 + v \nabla^2 u \cdot \nabla v \cdot dx - \frac{1}{2} \|\nabla v\|^2 + \nabla^2 u \cdot \nabla v \cdot dx \right) \]

\[\leq C (\|\nabla u\|_{L^2} \|\nabla v\|_{L^2}^2 + C \|\nabla^2 u\|_{L^2} \|\nabla v\|_{L^4} + C \|\nabla^2 c\|_{L^2} \|\nabla v\|_{L^4})\]

\[\leq C (\|u\|_{L^4} \|\nabla v\|_{L^2} \|\nabla^2 c\|_{L^2} + \frac{1}{2} \|\nabla^2 v\|_{L^2}^2 + \frac{1}{2} \|\nabla^2 u\|_{L^2}^2)\]

\[\leq C (\|\nabla u\|_{L^2} \|\nabla v\|_{L^2}^2 + C \|\nabla^2 u\|_{L^2} \|\nabla v\|_{L^2} \|\nabla^2 v\|_{L^2})\]

\[\leq \epsilon \|\nabla^2 v\|_{L^2}^2 + C(M, \epsilon) E_0 \|\nabla u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 \|\nabla^2 c\|_{L^2} + \frac{1}{2} \|\nabla^2 v\|_{L^2}^2. \quad (32)\]

In virtue of Lemma 2.3, we deduce

\[K_3 = - \int \frac{3}{2} \nabla u \cdot |\nabla c|^2 + c \nabla c \cdot \nabla^2 u - \frac{1}{2} \|\nabla v\|^2 + (v + 1) \|\nabla v\|\,

\[\leq C \|\nabla u\|_{L^2} \|\nabla v\|_{L^2} \|\nabla^2 c\|_{L^2} + C \|\nabla^2 u\|_{L^2} \|\nabla v\|_{L^4} \|\nabla c\|_{L^4}\]

\[\leq \epsilon (\|\nabla^2 c\|_{L^2}^2 + \|\nabla v\|_{L^2}^2) + C(M, \epsilon) E_0 \|\nabla^2 c\|_{L^2} + C(M, \epsilon) E_0 \|\nabla u\|_{L^2} \|\nabla^2 c\|_{L^2} \]

\[\quad \leq C (\|\nabla u\|_{L^2} \|\nabla v\|_{L^2}^2 + C(\epsilon) E_0 \|\nabla^2 c\|_{L^2} + \|\nabla u\|_{L^2} \|\nabla^2 c\|_{L^2} \}

\[\quad \leq \epsilon (\|\nabla^2 u\|_{L^2}^2 + \|\nabla^2 v\|_{L^2}^2) + C(M, \epsilon) E_0 \|\nabla u\|_{L^2} \|\nabla^2 c\|_{L^2} \]

Then, the last terms of (30) can be estimated as follows.

\[K_4 = \int (g(\rho)[\mu \Delta u + (\mu + \lambda) \nabla (\text{div} u)] + u \cdot \nabla u + \nabla (\rho v)) \cdot \nabla^2 u \, dx \]

\[\leq C \|\rho\|_{L^\infty} \|\nabla^2 u\|_{L^2} + C \|\nabla^2 u\|_{L^2} \|\rho\|_{L^4} \|\nabla u\|_{L^4} + C \|\nabla^2 u\|_{L^2} \|\rho\|_{L^4} \|\nabla u\|_{L^4}\]

\[\leq C \|\rho\|_{L^4} \|\nabla^2 u\|_{L^2} \|\nabla^2 u\|_{L^2} + C \|\nabla^2 u\|_{L^2} \|\nabla u\|_{L^2} \|\rho\|_{L^4}. \quad (34)\]
exists a constant 

Assume that all the assumptions of Proposition 1 hold. Then there

Lemma 3.5. Then we immediately get that

thus, if

which implies the desired estimate (29), after integrating (36) over \((0, T)\). The proof of Lemma 3.4 is completed. \qed

**Lemma 3.5.** Assume that all the assumptions of Proposition 1 hold. Then there exists a constant \(\varepsilon_0 > 0\), depending only on \(\mu, \lambda, M_1\), such that

\[
\sup_{0 \leq t \leq T} \left( \|\nabla^2 \rho\|_{L^2}^2 + \|\nabla^2 u\|_{L^2}^2 \right) + C_2 \int_0^T \|\nabla^3 u\|_{L^2}^2 \, dt \\
\leq C(M) E_0^{\frac{1}{4}} + \frac{2}{3 \mu} \int_0^T \|\nabla^2 v\|_{L^2}^2 \, dt + 2 \|\nabla^2 \rho_0\|_{L^2}^2 + \|\nabla^2 u_0\|_{L^2}^2,
\]

provided \(E_0 \leq \varepsilon_3\). In addition,

\[
\int_0^T \|\nabla^2 \rho\|_{L^2}^2 \, dt \leq C(M)
\]

and

\[
\int_0^T \|\nabla \rho\|_{L^2}^2 \, dt \leq C(M) E_0^\frac{1}{2}
\]

hold.
Proof. Multiplying (11)4 by $\nabla^4 \rho$, integrating over $\mathbb{R}^2$ and using (11)3, we have
\[
\int |\nabla^2 \rho|^2 dx = -\frac{d}{dt} \int \nabla u \cdot \nabla^2 \rho dx
\]
\[
+ \int (\nabla^2 u \cdot \nabla (\rho + 1) \text{div} u - \nabla^2 u \cdot \nabla (u \cdot \nabla \rho) - \nabla (u \cdot \nabla \rho) \cdot \nabla^2 \rho) dx
\]
\[
+ \int \nabla^2 \rho \cdot \nabla (\frac{1}{\rho + 1} [\mu \Delta u + (\mu + \lambda) \nabla (\text{div} u)]) dx
\]
\[
- \int [\nabla ((\rho + 1) v) \cdot \nabla^2 \rho + \nabla ((v + 1) \nabla \rho) \cdot \nabla^2 \rho] dx
\]
\[
= -\frac{d}{dt} \int \nabla u \cdot \nabla^2 \rho dx + \sum_{i=1}^{3} F_i. \tag{40}
\]

By (15), (17), the Sobolev and Cauchy inequalities, we have
\[
F_1 \leq C \|\nabla^2 u\|_{L^2}^2 + C \|\nabla^2 u\|_{L^2} \|\nabla u\|_{L^2} \|\nabla \rho\|_{L^4} + C \|u\|_{L^\infty} \|\nabla^2 u\|_{L^2} \|\nabla^2 \rho\|_{L^2}
\]
\[
+ C \|\nabla^2 \rho\|_{L^2} \|\nabla^2 u\|_{L^2} \|\nabla u\|_{L^2}
\]
\[
\leq C \|\nabla^2 u\|_{L^2}^2 + C \|\nabla^2 u\|_{L^2} \|\nabla \rho\|_{L^2} \|\nabla u\|_{L^2} \|\nabla \rho\|_{L^2}
\]
\[
+ C \|\nabla^2 \rho\|_{L^2} \|\nabla^2 u\|_{L^2} \|\nabla u\|_{L^2} \tag{41}
\]

Using integration by parts, it infers from (15) and (17) that
\[
F_2 = \frac{1}{2} \int \frac{1}{(\rho + 1)^2} (\nabla \rho)^3 \cdot \nabla^2 u dx + \frac{1}{2} \int \frac{1}{(\rho + 1)^2} (\nabla \rho)^2 \cdot \nabla^3 u dx
\]
\[
+ \int \frac{1}{(\rho + 1)^2} \nabla^2 \rho \cdot \nabla^3 u dx
\]
\[
\leq C \|\nabla \rho\|_{L^2} \|\nabla u\|_{L^2} \|\nabla^2 \rho\|_{L^2} \|\nabla^3 u\|_{L^2}
\]
\[
+ C \|\nabla \rho\|_{L^2} \|\nabla^3 u\|_{L^2}
\]
\[
\leq C \|\nabla^2 \rho\|_{L^2} \|\nabla u\|_{L^2} \|\nabla \rho\|_{L^2} \|\nabla^3 u\|_{L^2}
\]
\[
+ C \|\nabla^2 \rho\|_{L^2} \|\nabla^3 u\|_{L^2}
\]
\[
\leq \epsilon \|\nabla^2 \rho\|_{L^2}^2 + C(M, \epsilon) (\|\nabla^2 u\|_{L^2}^2 + \|\nabla^3 u\|_{L^2}^2). \tag{42}
\]

Taking the advantage of Lemma 2.3, the term $F_3$ can be estimated as following
\[
F_3 \leq -2 \int \nabla \rho \cdot \nabla v \cdot \nabla^2 \rho dx - \int (\rho + 1) \nabla^2 v \cdot \nabla^2 \rho dx
\]
\[
\leq C \|\nabla^2 \rho\|_{L^2} \|\nabla \rho\|_{L^4} \|\nabla v\|_{L^4} + C \|\nabla^2 v\|_{L^2} \|\nabla \rho\|_{L^2}
\]
\[
\leq \epsilon \|\nabla^2 \rho\|_{L^2}^2 + C(M, \epsilon) \|\nabla^2 v\|_{L^2}^2. \tag{43}
\]

Substituting (41)-(43) into (40) and choosing $\epsilon = \frac{1}{6}$, we arrive at
\[
\|\nabla^2 \rho\|_{L^2}^2 \leq -2 \frac{d}{dt} \int \nabla u \cdot \nabla^2 \rho dx + C(M) (\|\nabla^2 u\|_{L^2}^2 + \|\nabla^3 u\|_{L^2}^2 + \|\nabla^2 v\|_{L^2}^2). \tag{44}
\]

On the other hand, applying $\nabla^2$ to (11)4 and (11)3, multiplying by $2\nabla^2 \rho$ and $\nabla^2 u$, respectively, integrating over $\mathbb{R}^2$ and then adding them together, it yields that
\[
\frac{1}{2} \frac{d}{dt} \int (2|\nabla^2 \rho|^2 + |\nabla^2 u|^2) dx + \int \nabla^3 u \cdot \nabla (\mu \Delta u + (\mu + \lambda) \nabla \div \mathbf{u}) dx
\]
\[
= -6 \int \nabla \rho \cdot \nabla^2 \mathbf{u} \cdot \nabla \mathbf{y} dx - \int 2\rho \nabla^3 \mathbf{u} \cdot \nabla \rho \]
\[
- \int (6|\nabla^2 \rho|^2 \cdot \nabla \mathbf{u} + 2\rho \cdot \nabla^3 \rho \cdot \nabla^2 \rho) dx
\]
\[
+ \int (g(u) \cdot \nabla^3 u dx + \int \nabla (g(\rho) [(\mu \Delta u + (\mu + \lambda) \nabla \div \mathbf{u})]) \cdot \nabla^3 \mathbf{u} dx
\]
\[
+ \int (\nabla v \cdot \nabla \rho + v \nabla^2 \rho) \cdot \nabla^3 \mathbf{u} dx
\]
\[
= \sum_{i=1}^{7} G_i.
\]

We obtain from (15), (17), the Sobolev and Cauchy inequalities that
\[
G_1 = 3 \int |\nabla \rho|^2 \cdot \nabla^3 \mathbf{u} dx \leq C \|\nabla^3 u\|_{L^2} \|\nabla \rho\|_{L^2} \|\nabla^2 \rho\|_{L^2}
\]
\[
\leq C \|\nabla^3 u\|_{L^2} \|\nabla \rho\|_{L^2} \|\nabla^2 \rho\|_{L^2} \leq \epsilon \|\nabla^3 u\|_{L^2}^2 + C(M, \epsilon) E_0 \frac{1}{\lambda} \|\nabla^2 \rho\|_{L^2}^2,
\]
\[
G_2 \leq C \|\rho\|_{L^\infty} \|\nabla^3 u\|_{L^2} \|\nabla^2 \rho\|_{L^2} \leq C \|\nabla^3 u\|_{L^2} \|\rho\|_{L^2} \|\nabla^2 \rho\|_{L^2} \leq \epsilon \|\nabla^3 u\|_{L^2}^2 + C(M, \epsilon) E_0 \frac{1}{\lambda} \|\nabla^2 \rho\|_{L^2}^2,
\]
\[
G_3 = -\frac{5}{2} \int |\nabla \rho|^2 \cdot \nabla \mathbf{u} dx \leq C \|\nabla \mathbf{u}\|_{L^\infty} \|\nabla^2 \rho\|_{L^2}^2
\]
\[
\leq \epsilon \|\nabla^3 u\|_{L^2}^2 + C(M, \epsilon) \|\nabla \mathbf{u}\|_{L^2} \|\nabla^2 \rho\|_{L^2} \leq \epsilon \|\nabla^3 u\|_{L^2}^2 + E_0 \frac{1}{\lambda} \|\nabla^2 \rho\|_{L^2}^2 + C(M, \epsilon) \|\nabla \mathbf{u}\|_{L^2}^2,
\]

and
\[
G_4 \leq C \|\nabla^3 u\|_{L^2} (\|\nabla^2 u\|_{L^2} \|\nabla u\|_{L^1} + \|\nabla \mathbf{u}\|_{L^2} \|\nabla^2 \mathbf{u}\|_{L^2})
\]
\[
\leq C \|\nabla^3 u\|_{L^2} \|\nabla^2 u\|_{L^2} \|\nabla u\|_{L^2} + C \|\nabla \mathbf{u}\|_{L^2} \|\nabla^3 \mathbf{u}\|_{L^2} \leq \epsilon \|\nabla^3 u\|_{L^2}^2 + C(M, \epsilon) E_0 \|\nabla^2 \mathbf{u}\|_{L^2}^2 + C(M, \epsilon) \|\nabla \mathbf{u}\|_{L^2}^2.
\]

Moreover,
\[
G_5 \leq C \|\nabla^3 u\|_{L^2} \|\nabla^2 \mathbf{u}\|_{L^1} \|\nabla \rho\|_{L^1} \leq C \|\nabla^3 u\|_{L^2} \|\nabla^2 \mathbf{u}\|_{L^2} \|\nabla \rho\|_{L^2} \|\nabla \mathbf{u}\|_{L^2} \|\nabla^2 \mathbf{u}\|_{L^2} \leq \epsilon \|\nabla^3 u\|_{L^2}^2 + C(M, \epsilon) E_0 \|\nabla^2 \mathbf{u}\|_{L^2}^2,
\]

and
\[
G_6 \leq C \|\nabla^3 u\|_{L^2} \|\nabla \mathbf{v}\|_{L^1} \|\nabla \mathbf{u}\|_{L^1} \|\nabla \mathbf{v}\|_{L^2} + C \|\nabla^3 u\|_{L^2} \|\rho\|_{L^\infty} \|\nabla \mathbf{v}\|_{L^2} + \frac{3\mu}{4} \|\nabla^3 \mathbf{u}\|_{L^2}^2
\]
\[
+ \frac{1}{3\mu} \|\nabla^2 \mathbf{v}\|_{L^2}^2
\]
\[
\leq C \|\nabla^3 u\|_{L^2} (\|\nabla^2 \rho\|_{L^2} + \|\nabla^2 \mathbf{v}\|_{L^2}) \|\mathbf{v}\|_{L^2} \|\rho\|_{L^2} \|\nabla \mathbf{v}\|_{L^2} + \frac{3\mu}{4} \|\nabla^3 \mathbf{u}\|_{L^2}^2 + \frac{1}{3\mu} \|\nabla^2 \mathbf{v}\|_{L^2}^2.
\]
Proof of Proposition 1

It follows from Lemma 3.2 that

\[ A_1(T) \leq CE_0e^{CM} \leq E_0^T, \]

provided \( E_0 \leq \varepsilon_4 := \min\{\varepsilon_3, C^{-2}e^{-2CM}\} \).
If $E_0$ is chosen to be sufficiently small such that
\[ E_0 \leq \varepsilon := \min\{\varepsilon_4, (\frac{C_1M}{4}C(M)^{-1})^8, (\frac{M}{4}C(M)^{-1})^{20}\}, \]
with the help of Lemma 3.4 and Lemma 3.5, we have
\[ A_2(T) \leq \frac{C(M)}{C_1}E_0^\frac{1}{8} + \frac{2M_1}{C_1} + C(M)E_0 \leq \frac{3M}{2}. \]
For $A_3(T)$, noticing Lemma 3.4 and Lemma 3.5, we obtain
\[ A_3(T) \leq C(M)E_0^{\frac{1}{4}} + \frac{2}{3\mu} \int_0^T \|\nabla^2 v\|_{L^2}^2 dt + 2M_1 \leq \frac{3}{2}(\frac{1}{\mu} + 1)M, \]
provided that $E_0 \leq \varepsilon$. The proof of Proposition 1 is completed. □

In addition, one can obtain the following time-independent estimates immediately.

**Lemma 3.6.** Assume that all the assumptions of Theorem 1.1 hold. Then, for any given $T > 0$, it holds that
\[ \sup_{0 \leq t \leq T} (\|\nabla^2 v\|_{L^2}^2 + \|\nabla^2 c\|_{L^2}^2) + \int_0^T (\|\nabla^3 v\|_{L^2}^2 + \|\nabla^3 c\|_{L^2}^2) dt \leq C(M) + \|\nabla^2 v_0\|_{L^2}^2 + \|\nabla^2 c_0\|_{L^2}^2, \]
and
\[ \|\nabla \rho_t\|_{L^2}^2 + \int_0^T \|\nabla \rho_t\|_{L^2}^2 dt \leq C(M). \]

**Proof.** Following the proof of (37) for velocity field $u$, by Lemma 2.3, it is easy to get (53). We omit the details.

Here, we only need to establish the estimate for $\nabla \rho_t$. By (11) and using Proposition 1 and Sobolev inequality, we obtain
\[ \|\nabla \rho_t\|_{L^2}^2 \leq C \int (|\nabla^2 \rho|^2 u^2 + |\nabla \rho|^2 |\nabla u|^2 + |\rho|^2 |\nabla^2 u|^2) dx \leq C (\|u\|_{L^\infty}^2 \|\nabla^2 \rho\|_{L^2}^2 + \|\nabla \rho\|_{L^4}^2 \|\nabla u\|_{L^4}^2 + \|\rho\|_{L^\infty}^2 \|\nabla^2 u\|_{L^2}^2) \leq C(M) \|\nabla^2 \rho\|_{L^2}^2. \]
Together with Lemma 3.4 and Lemma 3.5, it yields (54). □

**3.2. Time-dependent higher-order estimates.** In this subsection, we will establish the time-dependent higher estimates of the smooth solution $(v, c, \rho, u)$ satisfying Proposition 1. In what follows, we always assume the initial energy $E_0 \leq \varepsilon$ and denote the positive constant by $C_T$ which will depend on $\mu, \lambda, M, T$ and the initial data.

First of all, it follows from Lemma 3.1-Lemma 3.6 that

**Lemma 3.7.** Assume that all the assumptions of Theorem 1.1 hold. Then for any given $T > 0$, there exists a constant $C_T > 0$ such that
\[ \sup_{0 \leq t \leq T} (\|\nabla^3 v\|_{L^2}^2 + \|\nabla^3 u\|_{L^2}^2) + \int_0^T \|\nabla^4 u\|_{L^2}^2 dt \leq C_T. \]
Applying $\nabla^3$ to (11)$_3$ and then multiplying the equation by $\nabla^3 \rho$, after integrating over $\mathbb{R}^2$, we have
\[
\frac{1}{2} \frac{d}{dt} \int |\nabla^3 \rho|^2 \, dx
= - \int (\nabla^4 u \cdot \nabla^3 \rho + |\nabla^3 \rho|^2 \, \text{div}u + 6\nabla^2 \rho \cdot \nabla^2 u \cdot \nabla^3 \rho + 4\nabla \rho \cdot \nabla^3 u \cdot \nabla^3 \rho)
+ \rho \nabla^4 u \cdot \nabla^3 \rho + 6|\nabla^3 \rho|^2 \cdot \nabla u + u \cdot \nabla^4 \rho \cdot \nabla^3 \rho) \, dx
\leq \varepsilon \|\nabla^4 u\|^2_{L^2} + C(\varepsilon)\|\nabla^3 \rho\|^3_{L^2} + C \|\nabla^3 u\|^2_{L^2} \|\nabla^3 \rho\|^2_{L^2}. \tag{56}
\]
Next, applying $\nabla^2$ to (11)$_4$, squaring both sides of resulting equation, then integrating over $\mathbb{R}^2$, we get
\[
\int |\nabla^2 u| - \mu \nabla^2 \Delta u - (\mu + \lambda) \nabla^3 \text{div}u|^2 \, dx
\leq |\nabla^3 \rho|^2 |dx + \int |\nabla^2 (g(p)|\mu \Delta u + (\mu + \lambda) \nabla (\text{div}u)])|^2 \, dx + \int |\nabla^2 (u \cdot \nabla u)|^2 \, dx
\leq \|\nabla^3 \rho\|^2_{L^2} + \sum_{i=1}^3 N_i. \tag{57}
\]
Using (17), (19), the Sobolev and Cauchy inequalities, we give the bounds of $N_i, i = 1, 2, 3$,
\[
N_1 \leq C \int (p^2 |\nabla^4 u|^2 + |\nabla^3 u|^2 + |\Delta u|^2) \, dx
\leq |\rho|_{L^2} \|\nabla \rho\|_{L^2} \|\nabla^4 u\|_{L^2} + C \|\nabla^3 u\|^2_{L^2} + C \|\nabla^2 u\|^2_{L^2} \tag{58}
\leq \varepsilon \|\nabla^4 u\|^2_{L^2} + C(\varepsilon) \|\nabla \rho\|^2_{L^2} + C \|\nabla^3 u\|^2_{L^2} + C \|\nabla^2 u\|^2_{L^2},
\]
and
\[
N_2 \leq C \|u\|^2_{L^\infty} \|\nabla^3 u\|^2_{L^2} + C \|\nabla u\|^2_{L^\infty} \|\nabla^2 u\|^2_{L^2} \leq C \|\nabla^3 u\|^2_{L^2} + C \|\nabla^2 u\|^2_{L^2} \tag{59}
\]
and
\[
N_3 \leq |(\rho + 1)|_{L^\infty} \|\nabla^3 v\|^2_{L^2} + 2 |\nabla \rho|_{L^\infty} \|\nabla^2 v\|^2_{L^2} + 2 |\nabla v|_{L^\infty} \|\nabla^2 \rho\|^2_{L^2}
+ (|v|^2_{L^\infty} + 1) |\nabla^3 \rho|_{L^2}^2
\leq C \|\nabla^3 v\|^2_{L^2} + C \|\nabla \rho\|^2_{L^2} + C \|\nabla^2 u\|^2_{L^2} + C \|\nabla^2 \rho\|^2_{L^2}. \tag{60}
\]
The combination of (56)-(22) gives
\[
\frac{d}{dt} (\|\nabla^3 \rho\|^2_{L^2} + \|\nabla^3 u\|^2_{L^2}) \, dx + C \|\nabla^4 u\|^2_{L^2}
\leq C(1 + \|\nabla^3 v\|^2_{L^2} + \|\nabla^3 u\|^2_{L^2} + \|\nabla^3 u\|^2_{L^2} + \|\nabla^3 \rho\|^2_{L^2}),
\]
where we have chosen $\varepsilon$ small enough. This, together with Lemma 3.4, Lemma 3.5 and Gronwall’s inequality, completes the proof of (55). \qed

Then, similar to the proof of (55), using (11)$_3$ and (11)$_4$, we can obtain the following lemma immediately. Therefore, we omit the proof for brevity.
Lemma 3.8. Assume that all the assumptions of Theorem 1.1 hold. Then, for any given $T > 0$, it holds that
\[
\sup_{0 \leq t \leq T} (\| \nabla^3 v \|_{L^2}^2 + \| \nabla^3 c \|_{L^2}^2) + \int_0^T (\| \nabla^4 v \|_{L^2}^2 + \| \nabla^4 c \|_{L^2}^2) dt \leq C_T. \tag{61}
\]

4. Proof of Theorem 1.1 and Theorem 1.2. Proof of Theorem 1.1. By Lemma 2.2, there exists a small time $T > 0$ such that the Cauchy problem (1)–(3) has a unique solution $(n - 1, c, h - 1, u)$ on $\mathbb{R}^2 \times (0, T_0)$. Then, by all the estimates in section 3 and using the standard continuity argument, we can extend the local solution to be global one. Moreover, the unique global solution is a classical solution. Here, we refer to [18] for details.

\[\square\]

Now, we only need to show the large time behavior (9) of the global solution. 

Proof of Theorem 1.2. Since the solution $(n - 1, c, h - 1, u)$ is global and from Lemma 3.2 and Lemma 3.5, it is easy to find that
\[
\int_0^\infty (\| \nabla v \|_{L^2}^2 + \| \nabla c \|_{L^2}^2 + \| \nabla \rho \|_{L^2}^2 + \| \nabla u \|_{L^2}^2) dt \leq C(M). \tag{62}
\]

In virtue of the equations (11), (62), Lemma 3.5, Lemma 3.6 and Young’s inequality, we have
\[
\int_0^\infty \left| \frac{d}{dt} (\| \nabla v \|_{L^2}^2 + \| \nabla c \|_{L^2}^2 + \| \nabla \rho \|_{L^2}^2 + \| \nabla u \|_{L^2}^2) \right| dt \\
\leq \int_0^\infty (\| \nabla v_t \|_{L^2}^2 + \| \nabla c_t \|_{L^2}^2 + \| \nabla \rho_t \|_{L^2}^2 + \| \nabla u_t \|_{L^2}^2) dt \\
+ \int_0^\infty (\| \nabla v_t \|_{L^2}^2 + \| \nabla c_t \|_{L^2}^2 + \| \nabla \rho_t \|_{L^2}^2 + \| \nabla u_t \|_{L^2}^2) dt \\
\leq C(M). \tag{63}
\]

Thus, (62) and (63) give us
\[
\lim_{t \to \infty} (\| \nabla v \|_{L^2}^2 + \| \nabla c \|_{L^2}^2 + \| \nabla \rho \|_{L^2}^2 + \| \nabla u \|_{L^2}^2) = 0.
\]

Combining the above limit, the following time-independent lower-order estimates
\[
\| v \|_{H^2}^2 + \| c \|_{H^2}^2 + \| \rho \|_{H^2}^2 + \| u \|_{H^2}^2 \leq C(M)
\]
and Sobolev inequality, we arrive at (9).

Thus, we are in a position to show the exponential decay characteristic of the chemoattractant concentration $c$. Multiplying (11)$_2$ by $c$ and integrating it over $\mathbb{R}^2$, and using Proposition 1, Sobolev and Cauchy inequalities, we have
\[
\frac{1}{2} \frac{d}{dt} \int c^2 dx + \int | \nabla c |^2 dx + \int c^2 dx \\
= -\int vc^2 dx - \int u \cdot \nabla c \cdot cdx - \int c^2 \text{div}u dx \\
\leq C \| v \|_{L^4} \| c \|_{L^4} \| c \|_{L^2} + C \| \nabla c \|_{L^2} \| c \|_{L^4} \| u \|_{L^4} \\
\leq C_4 (\| \nabla c \|_{L^2}^2 + \| c \|_{L^2}^2) \| v \|_{L^4} + C_4 E_0^T (\| \nabla c \|_{L^2}^2 + \| \nabla u \|_{L^2}^2 \| c \|_{L^2}^2).
\]

By virtue of (9), there exists $T_1 > 0$ such that
\[
C_5 \| v(\cdot, t) \|_{L^4} < \frac{1}{4}, \quad \text{for any } t > T_1. \tag{64}
\]
At the same time, by Proposition 1, one can get
\[
C_4 E_0 \left( \|\nabla c\|^2_{L^2} + \|\nabla u\|^2_{L^2} \right) \leq \frac{1}{4} \left( \|\nabla c\|^2_{L^2} + \|c\|^2_{L^2} \right),
\]
(provided \( E_0 \leq \varepsilon^* := \min\{\varepsilon, (4C_4)^{-8}, (8C_4M)^{-8}\} \)). It follows from (64) and (65) that
\[
\frac{d}{dt} \int c^2 dx + \int |\nabla c|^2 dx + \int c^2 dx \leq 0, \quad \text{for any } t > T_1,
\]
which implies that
\[
\|c(\cdot,t)\|^2_{L^2} \leq \|c(\cdot,T_1)\|^2_{L^2} e^{-t-T_1}, \quad \text{for any } t > T_1.
\]
Using Sobolev inequality and (53), we obtain
\[
\|c\|_{L^\infty} \leq \|c\|_{L^2} \|\nabla^2 c\|_{L^2} \leq C e^{-\frac{1}{4}t}, \quad \text{for any } t > T_1.
\]
Therefore, the proof of Theorem 1.2 is completed. \(\square\)

Acknowledgments. The author would like to thank the anonymous referees sincerely for their valuable suggestions and comments in improving the paper.

REFERENCES

[1] N. Bellomo, A. Bellouquid, Y. Tao and M. Winkler,, Toward a mathematical theory of Keller-Segel models of pattern formation in biological tissues, Math. Models Methods Appl. Sci., 25 (2015), 1663–1763.
[2] M. Chae, K. Kang and J. Lee, On existence of the smooth solutions to the coupled chemotaxis-fluid equations, Discrete Contin. Dyn. Syst. A, 33 (2013), 2271–2297.
[3] M. Chae, K. Kang and J. Lee, Global existence and temporal decay in Keller-Segel models coupled to fluid equations, Comm. Partial Differential Equations, 39 (2014), 1205–1235.
[4] J. H. Che, L. Chen, B. Duan and Z. Luo, On the existence of local strong solutions to chemotaxis-shallow water system with large data and vacuum, J. Differential Equations, 261 (2016), 6758–6789.
[5] R. J. Duan, X. Li and Z. Y. Xiang, Global existence and large time behavior for a two-dimensional chemotaxis-Navier-Stokes system, J. Differential Equations, 263 (2017), 6284–6316.
[6] R. Duan, A. Lorz and P. Markowich, Global solutions to the coupled chemotaxis-fluid equations, Comm. Partial Differential Equations, 35 (2010), 1635–1673.
[7] R. J. Duan and Z. Y. Xiang, A note on global existence for the chemotaxis-Stokes model with nonlinear diffusion, Int. Math. Res. Not. IMRN, 2014 (2014), 1833–1852.
[8] E. Espejo and M. Winkler, Global classical solvability and stabilization in a two-dimensional chemotaxis-Navier-Stokes system modeling coral fertilization, Nonlinearity, 31 (2018), 1227–1259.
[9] T. Hillen and K. J. Painter, A user’s guide to PDE models for chemotaxis, J. Math. Biol., 58 (2009), 183–217.
[10] D. Horstmann, From 1970 until present: The Keller-Segel model in chemotaxis and its consequences. I, Jahresber. Deutsch. Math.-Verein., 105 (2003), 103–165.
[11] D. Horstmann and M. Winkler, Boundedness vs. blow-up in a chemotaxis system, J. Differential Equations, 215 (2005), 52–107.
[12] X. D. Huang and J. Li, Global classical and weak solutions to the three-dimensional full compressible Navier-Stokes system with vacuum and large oscillations, Arch. Ration. Mech. Anal., 227 (2018), 995–1059.
[13] X. D. Huang, J. Li and Z. P. Xin, Global well-posedness of classical solutions with large oscillations and vacuum to the three-dimensional isentropic compressible Navier-Stokes equations, Comm. Pure Appl. Math., 65 (2012), 549–585.
[14] E. F. Keller and L. A. Segel, Initiation of slide mold aggregation viewed as an instability, J. Theor. Biol., 26 (1970), 399–415.
[15] E. F. Keller and L. A. Segel, Model for chemotaxis, J. Theor. Biol., 30 (1971), 225–234.
[16] J.-G. Liu and A. Lorz, A coupled chemotaxis-fluid model: Global existence, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 28 (2011), 643–652.

[17] A. Lorz, Coupled chemotaxis fluid model, *Math. Models Methods Appl. Sci.*, 20 (2010), 987–1004.

[18] A. Matsumura and T. Nishida, The initial value problems for the equations of motion of viscous and heat-conductive gases, *J. Math. Kyoto Univ.*, 20 (1980), 67–104.

[19] L. Nirenberg, On elliptic partial differential equations, *Ann. Sc. Norm. Super. Pisa Cl. Sci.*, 13 (1959), 115–162.

[20] Z. Tan and X. Zhang, Decay estimates of the coupled chemotaxis-fluid equations in $\mathbb{R}^3$, *J. Math. Anal. Appl.*, 410 (2014), 27–38.

[21] Y. S. Tao and M. Winkler, Boundedness in a quasilinear parabolic-parabolic Keller-Segel system with subcritical sensitivity, *J. Differential Equations*, 252 (2012), 692–715.

[22] Q. Tao and Z.-A. Yao, Global existence and large time behavior for a two-dimensional chemotaxis-shallow water system, *J. Differential Equations*, 265 (2018), 3092–3129.

[23] I. Tuval, L. Cisneros, C. Dombrowski, C. Wolgemuth, J. Kessler and R. Goldstein, Bacterial swimming and oxygen transport near contact lines, *Proc. Natl. Acad. Sci. USA*, 102 (2005), 2277–2282.

[24] W. K. Wang and Y. C. Wang, The $L^p$ decay estimates for the chemotaxis-shallow water system, *J. Math. Anal. Appl.*, 474 (2019), 640–665.

[25] Y. L. Wang, M. Winkler and Z. Y. Xiang, Global classical solutions in a two-dimensional chemotaxis-Navier-Stokes system with subcritical sensitivity, *Ann. Sc. Norm. Super. Pisa Cl. Sci.*, 18 (2018), 421–466.

[26] M. Winkler, Global large-data solutions in a chemotaxis-(Navier-)Stokes system modeling cellular swimming in fluid drops, *Comm. Partial Differential Equations*, 37 (2012), 319–351.

[27] M. Winkler, Finite-time blow-up in the higher-dimensional parabolic-parabolic Keller-Segel system, *J. Math. Pure. Appl.*, 100 (2013), 748–767.

[28] M. Winkler, Stabilization in a two-dimensional chemotaxis-Navier-Stokes system, *Arch. Ration. Mech. Anal.*, 211 (2014), 455–487.

[29] M. Winkler, Global weak solutions in a three-dimensional chemotaxis-Navier-Stokes system, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 33 (2016), 1329–1352.

[30] M. Winkler, How far do chemotaxis-driven forces influence regularity in the Navier-Stokes system?, *Trans. Amer. Math. Soc.*, 369 (2017), 3067–3125.

[31] M. Winkler, A three-dimensional Keller-Segel-Navier-Stokes system with logistic source: Global weak solutions and asymptotic stabilization, *J. Funct. Anal.*, 276 (2019), 1339–1401.

[32] Q. Zhang and X. X. Zheng, Global well-posedness for the two-dimensional incompressible chemotaxis-Navier-Stokes equations, *SIAM J. Math. Anal.*, 46 (2014), 3078–3105.

Received December 2019; revised April 2020.

E-mail address: yiyiying729@163.com