ON THE COPRODUCT IN AFFINE SCHUBERT CALCULUS

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Abstract. The cohomology of the affine flag variety $\hat{F}_{\mathcal{L}G}$ of a complex reductive group $G$ is a comodule over the cohomology of the affine Grassmannian $\text{Gr}_G$. We give positive formulae for the coproduct of an affine Schubert class in terms of affine Stanley classes and finite Schubert classes, in (torus-equivariant) cohomology and $K$-theory. As an application, we deduce monomial positivity for the affine Schubert polynomials of the second author.

1. Introduction

Let $G$ be a complex reductive group with maximal torus $T$ and flag variety $G/B$, and denote by $\xi^w_{G/B}$ the Schubert classes of $H^*_T(G/B)$ (all cohomology rings are taken with integer coefficients), indexed by the finite Weyl group $W$. Let $\hat{F}_{\mathcal{L}G}$ denote the affine flag variety of $G$ and $\text{Gr}_G$ denote the affine Grassmannian of $G$. There is a coaction map

$$\Delta : H^*_T(\hat{F}_{\mathcal{L}G}) \rightarrow H^*_T(\text{Gr}_G) \otimes H^*_T(\hat{F}_{\mathcal{L}G}).$$

It is induced via pullback from the product map of topological spaces $\Omega K \times LK/T_{\mathbb{R}} \rightarrow LK/T_{\mathbb{R}}$, where $K \subset G$ is a maximal compact subgroup and $T_{\mathbb{R}} = K \cap T$ is the maximal compact torus. The cohomology ring $H^*_T(\hat{F}_{\mathcal{L}G})$ has Schubert classes $\xi^w$ indexed by the affine Weyl group $\hat{W}$. The inclusion $\varphi : \Omega K \hookrightarrow LK/T_{\mathbb{R}}$ induces a “wrongway” pullback map

$$\varphi^* : H^*_T(\hat{F}_{\mathcal{L}G}) \rightarrow H^*_T(\text{Gr}_G).$$

By definition, the equivariant affine Stanley class $F^w \in H^*_T(\text{Gr}_G)$ is given by $F^w := \varphi^*(\xi^w)$. We refer the reader to \cite{LLMSSZ} for further background.

Theorem 1.1. Let $w \in \hat{W}$. Then we have

$$\Delta(\xi^w) = \sum_{uv = w} F^u \otimes \xi^v$$

and under the isomorphism $H^*_T(\hat{F}_{\mathcal{L}G}) \cong H^*_T(\text{Gr}_G) \otimes H^*_T(G/B)$,

$$\xi^w = \sum_{uv = w} F^u \otimes \xi^v_{G/B}$$

where $u \in \hat{W}$ and $v \in W$ and we write $w \doteq uv$ if $w = uv$ and $\ell(w) = \ell(u) + \ell(v)$.

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The class \( \xi_{\xi_i/\xi} \) is considered an element of \( H^*_T(\hat{F}_{g}) \) via pullback under evaluation at the identity (see (3.3)). The same formulae hold in non-equivariant cohomology.

In the majority of this article (Sections 2–6) we will work in torus-equivariant \( K \)-theory \( K^*_T(\hat{F}) \) of the affine flag variety. The coproduct formula for holds in (torus-equivariant) \( K \)-theory with Demazure product replacing length-additive products (see Theorem 4.7). Our proof relies heavily on the action of the affine nilHecke ring on \( K^*_T(\hat{F}) \). Let us note that there are a number of different geometric approaches \([KK, KS, LSSa]\) for constructing Schubert classes in \( K^*_T(\hat{F}) \), see \([LLMS, \text{Section 3}]\) for a comparison. However, our results hold at the level of Grothendieck groups and the precise geometric model (thick affine flag variety, thin affine flag variety, or based loop group) is not crucial.

In Section 4 the proofs for cohomology are indicated. In Section 6, we give examples of our formula in classical type. In particular, we prove (Theorem 6.1) that the affine Schubert polynomials \([Lee]\) are monomial positive, and we explain how the Billey–Haiman formula \([BH]\) for type \( C \) or \( D \) Schubert polynomials (see also \([LMN]\)) is a consequence of our coproduct formula.

By taking an appropriate limit, the coproduct formula for backstable (double) Schubert polynomials \([LLS]\) can be deduced from Theorem 4.7. Whereas the proofs in \([LLS]\) are essentially combinatorial, the present work relies heavily on equivariant localization and the nilHecke algebra.

2. Affine nilHecke ring and the equivariant \( K \)-theory of the affine flag variety

The proofs of our results for a complex reductive group easily reduces to that of a semisimple simply-connected group. To stay close to our main references \([KK, LSSa]\), we work with the latter. Henceforth, we fix a complex semisimple simply-connected group \( G \).

The results of this section are due to Kostant and Kumar \([KK]\). Our notation agrees with that of \([LSSa]\).

2.1. Small-torus affine \( K \)-nilHecke ring. Let \( T \subset G \) be the maximal torus with character group, or weight lattice \( P \). We have \( P = \bigoplus_{i \in \hat{I}} \mathbb{Z} \omega_i \) where \( \omega_i \) denotes a fundamental weight and \( \hat{I} \) denotes the finite Dynkin diagram of \( G \). Let \( \hat{P} = \mathbb{Z} \delta \oplus \bigoplus_{i \in \hat{I}} \mathbb{Z} \Lambda_i \) be the affine weight lattice with fundamental weights \( \Lambda_i \) for \( i \) in the affine Dynkin node set \( \hat{I} = I \cup \{0\} \), and let \( \delta \) denote the null root. The natural projection \( cl : \hat{P} \to P \) has kernel \( \mathbb{Z} \delta \oplus \mathbb{Z} \Lambda_0 \) and satisfies \( cl(\Lambda_i) = \omega_i \) for \( i \in I \). Let \( af : P \to \hat{P} \) be the section of \( cl \) given by \( af(\omega_i) = \Lambda_i - \text{level}(\Lambda_i) \Lambda_0 \) for \( i \in I \), where \( \text{level}(\Lambda) \) is the level of \( \Lambda \in \hat{P} \) \([Kac]\).

The finite Weyl group \( W \) acts naturally on \( P \) and on \( R(T) \), where \( R(T) \cong \mathbb{Z}[P] = \bigoplus_{\lambda \in P} \mathbb{Z} e^\lambda \) is the Grothendieck group of the category of finite-dimensional \( T \)-modules, and for \( \lambda \in P \), \( e^\lambda \) is the class of the one-dimensional \( T \)-module with character \( \lambda \). Let \( Q(T) = \text{Frac}(R(T)) \). The affine Weyl group \( \hat{W} \) also acts on \( P \), \( R(T) \), and \( Q(T) \) via the level-zero action, that is, via the homomorphism \( cl_{\hat{W}} : \hat{W} \cong Q^\vee \rtimes W \to W \) given by \( t_{\mu} v \mapsto v \) for \( \mu \) in the coroot lattice \( Q^\vee \) and \( v \in W \). In particular, \( s_0 = t_{\theta^\vee} s_{\theta} \) satisfies \( cl(s_0) = s_{\theta} \) where \( \theta \) is the highest root and \( \theta^\vee \) is its associated coroot. For \( w \in W \) and \( f \in R(T) \), the action of a Weyl group element \( w \) on \( f \) is sometimes denoted by \( w f \). Let \( \hat{W}^0 \) denote the minimal-length coset representatives in \( \hat{W}/W \).
We let $u \ast v \in \hat{W}$ denote the Demazure (or Hecke) product of $u, v \in \hat{W}$. It is the associative product determined by

$$s_i \ast v = \begin{cases} s_iv & \text{if } s_iv > v, \\ v & \text{otherwise.} \end{cases} \quad \text{and} \quad v \ast s_i = \begin{cases} vs_i & \text{if } vs_i > v, \\ v & \text{otherwise.} \end{cases}$$

Let $\hat{K}_{Q(T)}$ be the smash product of the group algebra $Q[\hat{W}]$ and $Q(T)$, defined by $\hat{K}_{Q(T)} = Q(T) \otimes_Q Q[\hat{W}]$ with multiplication

$$(q \otimes w)(p \otimes v) = qp \otimes vw$$

for $p, q \in Q(T)$ and $v, w \in W$. We write $qw$ instead of $q \otimes w$. Define the elements $T_i \in \hat{K}_{Q(T)}$ by

$$(2.1) \quad T_i = (1 - e^{\alpha_i})^{-1}(s_i - 1).$$

The $T_i$ satisfy

$$(2.2) \quad T_i^2 = -T_i \quad \text{and} \quad T_i T_j \cdots = T_j T_i \cdots.$$

We have the commutation relation in $\hat{K}_{Q(T)}$

$$(2.3) \quad T_i q = (T_i \cdot q) + (s_i \cdot q)T_i \quad \text{for } q \in Q(T).$$

Let $T_w = T_{i_1} T_{i_2} \cdots T_{i_N} \in \hat{K}_{Q(T)}$ where $w = s_{i_1} s_{i_2} \cdots s_{i_N}$ is a reduced decomposition; it is well-defined by (2.2). It is easily verified that

$$T_i T_w = \begin{cases} T_{s_i w} & \text{if } s_i w > w \\ -T_w & \text{if } s_i w < w \end{cases} \quad \text{and} \quad T_w T_i = \begin{cases} T_{w s_i} & \text{if } w s_i > w \\ -T_w & \text{if } w s_i < w \end{cases}$$

where $<$ denotes the Bruhat order on $\hat{W}$. The algebra $\hat{K}_{Q(T)}$ acts naturally on $Q(T)$. In particular, one has

$$(2.4) \quad T_i \cdot (qq') = (T_i \cdot q)q' + (s_i \cdot q)T_i \cdot q' \quad \text{for } q, q' \in Q(T).$$

The 0-Hecke ring $\hat{K}_0$ is the subring of $\hat{K}_{Q(T)}$ generated by the $T_i$ over $\mathbb{Z}$. It can also be defined by generators $\{ T_i \mid i \in I \}$ and relations (2.2). We have $\hat{K}_0 = \bigoplus_{w \in \hat{W}} \mathbb{Z}T_w$.

**Lemma 2.1.** The ring $\hat{K}_0$ acts on $R(T)$.

**Proof.** $\hat{K}_0$ acts on $Q(T)$, and $T_i$ preserves $R(T)$ by (2.4) and the following formulae for $\lambda \in P$:

$$T_i \cdot e^\lambda = \begin{cases} e^\lambda(1 + e^{\alpha_i} + \cdots + e^{((\alpha_i^\vee, \lambda) - 1)\alpha_i}) & \text{if } \langle \alpha_i^\vee, \lambda \rangle > 0 \\ 0 & \text{if } \langle \alpha_i^\vee, \lambda \rangle = 0 \quad \Box \\ -e^\lambda(1 + e^{\alpha_i} + \cdots + e^{(-\langle \alpha_i^\vee, \lambda \rangle - 1)\alpha_i}) & \text{if } \langle \alpha_i^\vee, \lambda \rangle < 0. \end{cases}$$

Define the $K$-NilHecke ring $\hat{K}$ to be the subring of $\hat{K}_{Q(T)}$ generated by $\hat{K}_0$ and $R(T)$. We have $\hat{K}_{Q(T)} \cong Q(T) \otimes_{R(T)} \hat{K}$. By (2.3), we have

$$\hat{K} = \bigoplus_{w \in \hat{W}} R(T)T_w.$$
2.2. $\hat{K}$-$\hat{K}$-bimodule structure on equivariant $K$-theory of affine flag variety. We have an isomorphism $K_T^*(\text{pt}) \cong R(T)$. We have an $R(T)$-algebra injection $\prod_{w \in W} \iota_w^* : K_T^*(\hat{F}) \hookrightarrow \prod_{w \in W} K_T^*(\text{pt}) \cong \text{Fun}(\hat{W}, R(T))$, where $\text{Fun}(\hat{W}, R(T))$ denotes the space of $R(T)$ valued functions on $R(T)$. For $\psi \in K_T^*(\hat{F})$, write $\psi(w) := \iota_w^*(\psi)$, so that $\psi$ is identified with an element of $\text{Fun}(\hat{W}, R(T))$. The subalgebra of $R(T)$-valued functions on $\hat{W}$ that can be obtained this way is characterized by the small torus affine GKM condition $[\text{LSSa}]$. We denote the product on $K_T^*(\hat{F})$ by $\cup$. Under the injection $K_T^*(\hat{F}) \hookrightarrow \text{Fun}(\hat{W}, R(T))$, it becomes the pointwise product on $\text{Fun}(\hat{W}(R(T)))$.

There is a $R(T)$-bilinear perfect pairing $\langle \cdot, \cdot \rangle : \hat{K} \times K_T^*(\hat{F})$ characterized by

$$
\langle w, \psi \rangle = \psi(w)
$$

for $w \in \hat{W}$ and $\psi \in K_T^*(\hat{F})$. Abusing notation, for $a = \sum_{w \in W} a_w w \in \hat{K}$ with $a_w \in Q(T)$, let

$$
\psi(a) = \langle a, \psi \rangle = \sum_w a_w \psi(w).
$$

Thinking of $K_T^*(\hat{F})$ as a subalgebra of $\text{Hom}_S(\hat{K}_{Q(T)}, R(T))$, a function $\psi$ lies in $K_T^*(\hat{F})$ if and only if $\psi(\hat{K}) \subseteq R(T)$.

There is a left action $\psi \mapsto a \cdot \psi$ of $\hat{K}$ on $K_T^*(\hat{F})$ given by the formulae (see $[\text{LLMSSZ}]$, Chapter 4, Proposition 3.16] for the very similar cohomology case)

$$
(q \cdot \psi)(b) = q \psi(b)
$$

(2.8)

$$
(T_i \cdot \psi)(b) = T_i \psi(s_i b) + \psi(T_i b)
$$

(2.9)

$$
(w \cdot \psi)(b) = w \psi(w^{-1} b)
$$

(2.10)

for $b \in \hat{K}$, $\psi \in K_T^*(\hat{F})$, $q \in R(T)$, $i \in \hat{I}$, and $w \in \hat{W}$. Here, $T_i$ acts on $R(T)$ as in (2.4) and (2.5).

There is another left action $\psi \mapsto a \bullet \psi$ of $\hat{K}$ on $K_T^*(\hat{F})$ given by $[\text{LSSa}]$ §2.4

$$
(a \bullet \psi)(b) = \psi(b a)
$$

(2.11)

for $a, b \in \hat{K}$ and $\psi \in K_T^*(\hat{F})$.

Remark 2.2. For those familiar with the double Schubert polynomial $\mathfrak{S}_w(x; a)$ (or also the double Grothendieck polynomial), the $\cdot$ action is on the equivariant variables $a_i$ and the $\bullet$ action is on the $x$ variables.

Let $p : \hat{F} \to \text{Gr}$ be the natural projection and $p^* : K_T^*(\text{Gr}) \to K_T^*(\hat{F})$ the pullback map, which is an injection. A class $\psi \in K_T^*(\hat{F})$ lies in the image of $p^*$ if and only if $\psi(wv) = \psi(w)$ for all $w \in \hat{W}$ and $v \in W$. We abuse notation by frequently identifying a class $\psi_{\text{Gr}} \in K_T^*(\text{Gr})$ with its image under $p^*$.

Let $[\mathcal{L}_\lambda] \in K_T^*(\hat{F})$ denote the class of the $T$-equivariant line bundle with weight $\lambda \in P$ on $\hat{F}$. Explicitly using the level zero action of $\hat{W}$ on $R(T)$ we have $[\text{KS}]$ (2.5)]

$$
\langle t_\mu v, [\mathcal{L}_\lambda] \rangle = v \cdot e^\lambda = e^{\mu \lambda}
$$

(2.12) \quad \mu \in Q^+, v \in W.$$
Lemma 2.3. For any $\lambda \in P$ and $\psi \in K^*_T(\hat{F}l)$,

\begin{equation}
(2.13) \quad e^\lambda \bullet \psi = [\mathcal{L}_\lambda] \cup \psi.
\end{equation}

Proof. Localizing at $t_\mu v$ for $\mu \in Q^\vee$ and $v \in W$, we compute that $\langle t_\mu v, e^\lambda \bullet \psi \rangle$ is equal to $\langle t_\mu v e^\lambda, \psi \rangle = \langle e^\lambda t_\mu v, \psi \rangle = \langle t_\mu v, [\mathcal{L}_\lambda] \rangle = \langle t_\mu v, [\mathcal{L}_\lambda] \cup \psi \rangle$. $\square$

3. Two endomorphisms of $K^*_T(\hat{F}l)$

3.1. Wrong-way map and Peterson subalgebra. Recall that $K \subset G$ is the maximal compact subgroup and $T^R := K \cap T$ is the maximal compact torus. We have $T^R$-equivariant homotopy equivalences between $Gr$ and the based loop group $\Omega K$, and between $\hat{F}l$ and the space $LK/T^R$. There is a $T^R$-equivariant map $\varphi : \Omega K \to LK \to LK/T^R$ given by inclusion followed by projection. It induces an $R(T)$-algebra homomorphism $\varphi^* : K^*_T(\hat{F}l) \to K^*_T(Gr)$ which is called the wrong-way map, and characterized by (see Lemma 3.3)

$$\varphi^*(\psi)(v) = \psi(t_\mu) \quad \text{for } v \in \dot{W}^0 \text{ and } t_\mu \in vW.$$ 

Let $K^*_T(Gr)$ and $K^*_T(\hat{F}l)$ denote the equivariant $K$-homology groups (see [LSSa]). There is a map of $R(T)$-modules $\varphi_* : K^*_T(Gr) \to K^*_T(\hat{F}l)$, dual to $\varphi^*$. The pairing (2.7) induces an isomorphism of $R(T)$-algebras $\tau : K^*_T(\hat{F}l) \to \hat{K}$ given by $\tau(\psi_w) = T^w$, where $\psi_w$ is the Schubert basis element (see Section 4.1).

Let $L = Z_{\hat{K}}(R(T))$ be the centralizer of $R(T)$ in $\hat{K}$, called the $K$-Peterson subalgebra. We have the following basic result [LSSa, Lemma 5.2].

Lemma 3.1. We have $L = \bigoplus_{\mu \in Q^\vee} Q(T) t_\mu \cap \hat{K}$.

Theorem 3.2 ([LSSa Theorem 5.3]). There is an isomorphism $k : K^*_T(Gr) \to L$ making the following commutative diagram of ring and left $R(T)$-module homomorphisms:

$$
\begin{array}{ccc}
K^*_T(Gr) & \xrightarrow{k} & L \\
\varphi_* \downarrow & & \downarrow \\
K^*_T(\hat{F}l) & \xrightarrow{\tau} & \hat{K}
\end{array}
$$

3.2. Pullback from affine Grassmannian. Recall that $p : \hat{F}l \to Gr$ denotes the natural projection. Define $\theta := p^* \circ \varphi^*$, so that $\theta : K^*_T(\hat{F}l) \to K^*_T(\hat{F}l)$ is the pullback map in equivariant $K$-theory of the following composition

$$
(3.1) \quad LK/T^R \xrightarrow{p} \Omega K \xrightarrow{\varphi} LK/T^R
$$

where abusing notation, we are denoting also by $p$ the natural quotient map $LK/T^R \to LK/K \simeq \Omega K$.

Lemma 3.3. For all $\mu \in Q^\vee$, $v \in W$, and $\psi \in K^*_T(\hat{F}l)$ we have

$$
(3.2) \quad (\theta \psi)(t_\mu v) = \psi(t_\mu).
$$
Proof. It is enough to observe that the $T_\mathbb{R}$-fixed point $t_\mu v$ maps to $t_\mu$ under the composition (3.1). □

3.3. Coaction. The inclusion $\Omega K \hookrightarrow LK$ induces an action $\Omega K \times LK/T_\mathbb{R} \to LK/T_\mathbb{R}$ of $\Omega K$ on $LK/T_\mathbb{R}$. This action is $T_\mathbb{R}$-equivariant where $T_\mathbb{R}$ acts diagonally on the direct product, acting on $\Omega K$ by conjugation and on $LK/T_\mathbb{R}$ by left translation. This in turn induces a map $K^*_T(\text{Gr}) \otimes_{R(T)} K^*_T(\hat{\mathbb{F}}L) \to K^*_T(\hat{\mathbb{F}}L)$. This map corresponds to the action of $L$ on $\hat{K}$ by left multiplication, giving a commutative diagram

$$
K^*_T(\text{Gr}) \otimes_{R(T)} K^*_T(\hat{\mathbb{F}}L) \longrightarrow K^*_T(\hat{\mathbb{F}}L)
$$

By duality, we also have a coproduct map

$$
\Delta : K^*_T(\hat{\mathbb{F}}L) \longrightarrow K^*_T(\text{Gr}) \otimes_{R(T)} K^*_T(\hat{\mathbb{F}}L).
$$

Note that $\Delta|_{K^*_T(\text{Gr})}$ is the usual coproduct of $K^*_T(\text{Gr})$, part of the $R(T)$-Hopf algebra structure of $K^*_T(\text{Gr})$. Often, we will think of the image of $\Delta$ inside $K^*_T(\hat{\mathbb{F}}L)$ via the inclusion $p^* : K^*_T(\text{Gr}) \to K^*_T(\hat{\mathbb{F}}L)$.

Proposition 3.4. For all $a \in L$, $b \in \hat{\mathbb{K}}$, and $\psi \in K^*_T(\hat{\mathbb{F}}L)$ we have

$$
\langle ab, \psi \rangle = \sum_{(\psi)} \langle a, \psi_1 \rangle \langle b, \psi_2 \rangle
$$

where $\Delta(\psi) = \sum_{(\psi)} \psi_1 \otimes \psi_2$.

Proof. By definition, $\langle ab, \psi \rangle = \sum_{(\psi)} \langle a, \psi_1^\text{Gr} \rangle_{\text{Gr}} \langle b, \psi_2 \rangle \in K^*_T(\text{Gr}) \otimes_{R(T)} K^*_T(\hat{\mathbb{F}}L)$, where $\Delta(\psi) = \sum_{(\psi)} \psi_1 \otimes \psi_2$ and $\langle a, \psi \rangle_{\text{Gr}}$ is the pairing between $L$ and $K^*_T(\text{Gr})$ induced by Theorem 3.2 and the duality between $K^*_T(\text{Gr})$ and $K^*_T(\text{Gr})$. But then since $\varphi^* \circ p^*$ is the identity, we have that $\langle a, \psi^\text{Gr} \rangle_{\text{Gr}} = \langle a, (\varphi^* \circ p^*)(\psi^\text{Gr}) \rangle_{\text{Gr}} = \langle a, p^*(\psi^\text{Gr}) \rangle$. In the second equality, we have used the projection formula

$$
\langle a, \varphi^*(b) \rangle_{\text{Gr}} = \langle \varphi_*(a), b \rangle_{\hat{\mathbb{F}}L} \quad \text{for } a \in K^*_T(\text{Gr}), b \in K^*_T(\hat{\mathbb{F}}L).
$$

This gives the desired formula. □

Lemma 3.5. Let $\psi \in K^*_T(\hat{\mathbb{F}}L)$. If $\Delta(\psi) = \sum_{(\psi)} \psi_1 \otimes \psi_2$, then $\theta(\psi_1) = \psi_1$.

3.4. Loop evaluation at identity. Let $\text{ev}_1 : LK/T_\mathbb{R} \to K/T_\mathbb{R}$ be induced by evaluation of a loop at the identity. Since this is a $T_\mathbb{R}$-equivariant map (via left translation) it induces an $R(T)$-algebra homomorphism

$$
\text{ev}_1^* : K^*_T(G/B) \to K^*_T(\hat{\mathbb{F}}L).
$$

Let $q : K/T_\mathbb{R} \to LK/T_\mathbb{R}$ be the natural inclusion; it is $T_\mathbb{R}$-equivariant for left translation. The algebraic analogue of $q$ identifies $G/B$ with the finite-dimensional Schubert variety $X_{w_0} \subset \mathbb{F}L$. 

Define $\eta := ev_1^* \circ q^*$ so that $\eta : K_T^*(\hat{\text{Fl}}) \to K_T^*(\hat{\text{Fl}})$ is the pullback map in equivariant $K$-theory of the following composition

\begin{equation}
LK/T_R \xrightarrow{ev_1} K/T_R \xrightarrow{q} LK/T_R.
\end{equation}

**Lemma 3.6.** For all $\mu \in Q^\vee$, $v \in W$, and $\psi \in K_T^*(\hat{\text{Fl}})$ we have

\begin{equation}
(\eta \psi)(t_\mu v) = \psi(v).
\end{equation}

**Proof.** It suffices to note that the $T_R$-fixed point $t_\mu v$ is sent to $v$ under the composition (3.6). \qed

**Lemma 3.7.** For all $\lambda \in X$,

\begin{equation}
ev_1^*([L^G/B]_\lambda) = [L^\hat{\text{Fl}}_\lambda].
\end{equation}

**Proof.** For all $\mu \in Q^\vee$ and $u \in W$ we have

\[i_{t_\mu u}^*(ev_1^*([L^G/B]_\lambda)) = i_{t_\mu}^*(L^G/B) = v \cdot e^\lambda = (t_\mu v) \cdot e^\lambda = i_{t_\mu u}^*([L^\hat{\text{Fl}}_\lambda]).\] \qed

### 3.5. Coproduct identity.

The following identity is the main result of this section.

**Proposition 3.8.** For $\psi \in K_T^*(\hat{\text{Fl}})$ and $a \in \hat{K}$, we have

\[a \bullet \psi = \sum_{(\psi)} \psi(1) \cup \eta(a \bullet \psi(2))\]

where $\Delta(\psi) = \sum_{(\psi)} \psi(1) \otimes \psi(2)$. In particular, taking $a = 1$, we have the identity

\[\cup \circ (1 \otimes \eta) \circ \Delta = 1\]

in $\text{End}_{R(T)}(K_T^*(\hat{\text{Fl}}))$.

**Proof.** For $\mu \in Q^\vee$ and $v \in W$, we compute

\[\langle t_\mu v, a \bullet \psi \rangle = \langle t_\mu v a, \psi \rangle\]

\[= \sum_{(\psi)} \langle t_\mu, \psi(1) \rangle \langle va, \psi(2) \rangle\]

by Proposition 3.4

\[= \sum_{(\psi)} \langle t_\mu, \psi(1) \rangle \langle v, a \bullet \psi(2) \rangle\]

\[= \sum_{(\psi)} \langle t_\mu v, \psi(1) \rangle \langle t_\mu v, \eta(a \bullet \psi(2)) \rangle\]

by Lemmas 3.3 and 3.6

\[= \langle t_\mu v, \sum_{(\psi)} \psi(1) \cup \eta(a \bullet \psi(2)) \rangle.\] \qed

### 3.6. Commutation relations.

We record additional commutation relations involving the nilHecke algebra actions, and the endomorphisms $\theta$ and $\eta$.

Let $\kappa : K_T^*(\hat{\text{Fl}}) \to K_T^*(\hat{\text{Fl}})$ be the pullback map in equivariant $K$-theory induced by the composition

\begin{equation}
\hat{\text{Fl}} \to \text{id} \to \hat{\text{Fl}}
\end{equation}

where id denotes the basepoint of $\hat{\text{Fl}}$. It is an $R(T)$-algebra homomorphism.
Lemma 3.9. For all $\mu \in \mathbb{Q}^\vee$, $v \in W$, and $\psi \in K^*_T(\text{Gr})$ we have

\begin{equation}
\kappa(\psi)(t_\mu v) = \psi(\text{id}).
\end{equation}

Lemma 3.10. As $R(T)$-module endomorphisms of $K^*_T(\hat{\text{Fl}})$, we have the relations

$$\theta^2 = \theta, \quad \eta^2 = \eta, \quad \kappa^2 = \kappa;$$

$$\theta \eta = \eta \theta = \kappa \theta = \eta \kappa = \kappa \eta = \kappa.$$

Proof. Straightforward from Lemmas 3.3, 3.6, and 3.9. \qed

For $w \in \hat{\text{W}}$, define the endomorphism

$$w \circ : = (w\cdot) \circ (w\bullet) = (w\bullet) \circ (w\cdot)$$

of $K^*_T(\hat{\text{Fl}})$.

Proposition 3.11. The map $\theta$ interacts with the two actions $\cdot$ and $\bullet$ of $\hat{K}$ on $K^*_T(\hat{\text{Fl}})$ in the following way:

1. $(q\cdot) \circ \theta = \theta \circ (q\cdot)$
2. $(t_\mu \cdot) \circ \theta = \theta \circ (t_\mu \cdot)$
3. $(w\cdot) \circ \theta = \theta \circ (w \circ)$
4. $(w \bullet) \circ \theta = \theta$

where $q \in R(T)$, $w \in W$, and $\mu \in \mathbb{Q}^\vee$. By (1), (2), (3), we see that $\theta(K^*_T(\hat{\text{Fl}})) = p^*(K^*_T(\text{Gr}))$ is a $\hat{K}$-submodule of $K^*_T(\hat{\text{Fl}})$ under the $\cdot$ action.

Proposition 3.12. The map $\eta$ interacts with the two actions $\cdot$ and $\bullet$ of $\hat{K}$ on $K^*_T(\hat{\text{Fl}})$ in the following way:

1. $(q\cdot) \circ \eta = \eta \circ (q\cdot)$
2. $(t_\mu \cdot) \circ \eta = \eta$
3. $(w\cdot) \circ \eta = \eta \circ (w\cdot)$
4. $(q\bullet) \circ \eta = \eta \circ (q\bullet)$
5. $(t_\mu \bullet) \circ \eta = \eta$
6. $(w\bullet) \circ \eta = \eta \circ (w\bullet)$

where $q \in R(T)$, $w \in W$, and $\mu \in \mathbb{Q}^\vee$. By (1)-(6) we see that $\eta(K^*_T(\hat{\text{Fl}})) = \text{ev}^*_1(K^*_T(G/B))$ is a $\hat{K}$-submodule of $K^*_T(\hat{\text{Fl}})$ under either the $\cdot$ or the $\bullet$ action.

Proposition 3.13. The map $\kappa$ interacts with the two actions $\cdot$ and $\bullet$ of $\hat{K}$ on $K^*_T(\hat{\text{Fl}})$ in the following way:

1. $(q\cdot) \circ \kappa = \kappa \circ (q\cdot)$
2. $(t_\mu \cdot) \circ \kappa = \kappa$
3. $(w\cdot) \circ \kappa = \kappa \circ (w \circ)$
4. $(t_\mu \bullet) \circ \kappa = \kappa$
5. $(w\bullet) \circ \kappa = \kappa$

where $q \in R(T)$, $w \in W$, and $\mu \in \mathbb{Q}^\vee$. 
3.7. Action of $\hat{K}$ on tensor products. Define $\hat{K}_{Q(T)} \otimes_{Q(T)} \hat{K}_{Q(T)}$ to be the left $Q(T)$-bilinear tensor product such that

$$q(a \otimes b) = qa \otimes b = a \otimes qb$$

for all $a, b \in \hat{K}_{Q(T)}$ and $q \in Q(T)$. Define $\Delta : \hat{K}_{Q(T)} \to \hat{K}_{Q(T)} \otimes_{Q(T)} \hat{K}_{Q(T)}$ by

$$\Delta \left( \sum_{w \in W} a_w w \right) = \sum_{w} a_w w \otimes w$$

for $a_w \in Q(T)$. Then for all $i \in \hat{I}$ we have

$$\Delta(T_i) = T_i \otimes 1 + 1 \otimes T_i + (1 - e^{\alpha_i})T_i \otimes T_i.$$  

This restricts to a left $R(T)$-bilinear tensor product $\Delta : \hat{K} \to \hat{K} \otimes_{R(T)} \hat{K}$. If $M$ and $N$ are left $\hat{K}$-modules then $M \otimes_{R(T)} N$ is a left $\hat{K}$-module via

$$a(m \otimes n) = \sum_{(a)} a_{(1)}(m) \otimes a_{(2)}(n)$$

for all $a \in \hat{K}$, $m \in M$ and $n \in N$.

**Lemma 3.14.** For $\psi_1, \psi_2 \in K^*_T(\hat{F}l)$ and $a \in \hat{K}$, we have

$$a \cdot (\psi_1 \cup \psi_2) = \sum_{(a)} (a_{(1)} \cdot \psi_1) \cup (a_{(2)} \cdot \psi_2)$$

$$a \bullet (\psi_1 \cup \psi_2) = \sum_{(a)} (a_{(1)} \bullet \psi_1) \cup (a_{(2)} \bullet \psi_2).$$

**Proof.** We have

$$w \cdot (\psi_1 \cup \psi_2)(x) = w(\psi_1(w^{-1}x)\psi_2(w^{-1}x)) = ((w \cdot \psi_1) \cup (w \cdot \psi_2))(x)$$

$$w \bullet (\psi_1 \cup \psi_2)(x) = \psi_1(xw)\psi_2(xw) = ((w \bullet \psi_1) \cup (w \bullet \psi_2))(x),$$

consistent with $\Delta(w) = w \otimes w$. Next, we check that the formulae are compatible with $R(T)$-linearity. It is enough to work with the algebra generators $e^\lambda$ of $R(T)$. We have $\Delta(e^\lambda w) = e^\lambda w \otimes w$ and

$$(e^\lambda w) \cdot (\psi_1 \cup \psi_2) = e^\lambda \cdot (w \cdot (\psi_1 \cup \psi_2)) = e^\lambda \cdot ((w \cdot \psi_1) \cup (w \cdot \psi_2)) = ((e^\lambda w) \cdot \psi_1) \cup (w \cdot \psi_2).$$

Using Lemma 2.3 we have

$$(e^\lambda w) \bullet (\psi_1 \cup \psi_2) = e^\lambda \bullet (w \bullet (\psi_1 \cup \psi_2)) = [\mathcal{L}_\lambda] \cup (w \bullet \psi_1) \cup (w \bullet \psi_2) = ((e^\lambda w) \bullet \psi_1) \cup (w \bullet \psi_2).$$

3.8. Finite nilHecke algebra. The finite nilHecke ring $K$ is the subring of $\hat{K}$ generated by $R(T)$ and $T_i$ for $i \in I$. There are left actions $\cdot$ and $\bullet$ of $K$ on $K^*_T(G/B)$ that are similarly to the actions of $\hat{K}$ on $K^*_T(\hat{F}l)$.

There is a $K$-$K$-bimodule and ring homomorphism $cl_K : \hat{K} \to K$ defined (for convenience from $\hat{K}_{Q(T)} \to K_{Q(T)}$) by

$$cl_K(t_\mu a) = a$$

for $\mu \in Q^\vee$ and $a \in K$. 

\[\text{(3.15)}\]
In particular,
\[ \text{cl}_\hat{K}(T_0) = \text{cl}_\hat{K}((1 - e^{-\theta})^{-1}(s_0 - 1)) = \text{cl}_\hat{K}((1 - e^{-\theta})^{-1}(t_0 s_\theta - 1)) = (1 - e^{-\theta})^{-1}(s_\theta - 1) =: T_{-\theta}. \]

Thus we have \( \ast \) and \( \cdot \) actions of \( \hat{K} \) on \( K^*_T(G/B) \) that factor through \( \text{cl}_\hat{K} : \hat{K} \to K \).

### 3.9. Tensor product decomposition of \( K^*_T(\hat{Fl}) \)

The equivariant \( K \)-theory ring \( K^*_T(\hat{Fl}) \) is a left \( \hat{K} \)-submodule of \( K^*_T(\hat{Fl}) \) under the \( \cdot \)-action. Thinking of \( \psi_{Gr} \in K^*_T(\hat{Fl}) \) as a function from cosets \( \hat{W}/W \) to \( R(T) \), we have \( (w \cdot \psi_{Gr})(xW) = w(\psi_{Gr}(w^{-1}xW)) \).

The left \( \hat{K} \)-module structures via \( \cdot \) on \( K^*_T(Gr) \) and \( K^*_T(G/B) \) give a left \( \hat{K} \)-module structure on \( K^*_T(Gr) \otimes R(T) K^*_T(G/B) \) via \( \text{ev}_1 \).

**Theorem 3.15.** There is an \( R(T) \)-algebra isomorphism
\[
(3.16) \quad K^*_T(Gr) \otimes R(T) K^*_T(G/B) \cong K^*_T(\hat{Fl}) \]
\[
(3.17) \quad a \otimes b \mapsto p^*(a) \cup \text{ev}_1^*(b)
\]
with componentwise multiplication on the tensor product. This map is also an isomorphism of left \( \hat{K} \)-modules under the \( \cdot \) action.

The proof is delayed to after Theorem 4.7.

### 4. Affine Schubert classes

#### 4.1. Schubert bases

The \( R(T) \)-algebras \( K^*_T(\hat{Fl}) \), \( K^*_T(Gr) \), and \( K^*_T(G/B) \) have equivariant Schubert bases \( \{ \psi^x \mid x \in \hat{W} \} \), \( \{ \psi^u_{Gr} \mid u \in \hat{W}^0 \} \), and \( \{ \psi^w_{G/B} \mid w \in W \} \) respectively. The basis \( \{ \psi^x \mid x \in \hat{W} \} \subset K^*_T(\hat{Fl}) \) is uniquely characterized by
\[
(4.1) \quad \psi^x(T_w) = \delta_{v,w}.
\]

We have
\[
(4.2) \quad p^*(\psi^x_{Gr}) = \psi^x \quad \text{for all} \ z \in \hat{W}^0,
\]
\[
(4.3) \quad q^*(\psi^x) = \begin{cases} 
\psi^x_{G/B} & \text{for} \ x \in W, \\
0 & \text{for} \ x \in \hat{W} \setminus W.
\end{cases}
\]

In particular \( \eta(\psi^x) = \text{ev}_1^*(\psi^x_{G/B}) \) for \( x \in W \).

Similarly, let \( \{ \psi^x_x \mid x \in \hat{W} \} \), \( \{ \psi^u_{Gr} \mid u \in \hat{W}^0 \} \), and \( \{ \psi^w_{G/B} \mid w \in W \} \) denote homology Schubert bases of \( K^*_T(\hat{Fl}) \), \( K^*_T(Gr) \), and \( K^*_T(G/B) \). We write \( \langle \cdot , \cdot \rangle_{\hat{Fl}}, \langle \cdot , \cdot \rangle_{Gr} \), and \( \langle \cdot , \cdot \rangle_{G/B} \) for the \( R(T) \)-bilinear pairings between \( T \)-equivariant \( K \)-homology and \( K \)-cohomology, so that for example \( \langle \psi^x_x , \psi^y_y \rangle_{\hat{Fl}} = \delta_{xy} \).

**Remark 4.1.** \( p^* \) is an isomorphism of \( K^*_T(Gr) \) with its image \( \bigoplus_{u \in \hat{W}^0} R(T) \psi^u \), whose elements are \( \hat{W} \)-\( \bullet \)-invariant by Proposition 3.11.

The localization values of Schubert classes are determined by the following triangular relation. For all \( w \in \hat{W} \), in \( \hat{K} \) we have \[ \text{[KK]} \]
\[
(4.4) \quad w = \sum_{v \leq w} \langle w , \psi^v \rangle T_v.
\]
The Schubert basis \( \{ \psi^w \mid w \in \hat{W} \} \) interacts with the \( \cdot \) and \( \bullet \) actions of \( \hat{K} \) as follows. For \( i \in \hat{I} \), define

\[
y_i := 1 + T_i = \frac{1}{1 - e^{-\alpha_i}}(1 - e^{-\alpha_i}s_i) \quad \tilde{y}_i := 1 - e^{\alpha_i}T_i = \frac{1}{1 - e^{\alpha_i}}(1 - e^{\alpha_i}s_i).
\]

**Proposition 4.2.** For \( \lambda \in P \) and \( T_i \) for \( i \in \hat{I} \), on the Schubert basis element \( \psi^w \in K^*_T(\hat{F}l) \) for \( w \in \hat{W} \), we have:

\[
\tilde{y}_i \cdot \psi^w = \begin{cases} 
\psi^{s_iw} & \text{if } s_iw < w \\
\psi^w & \text{otherwise.}
\end{cases}
\]

\[
y_i \cdot \psi^w = \begin{cases} 
\psi^{ws_iw} & \text{if } ws_i < w \\
\psi^w & \text{otherwise.}
\end{cases}
\]

\[
e^\lambda \cdot \psi^w = e^\lambda \psi^w
\]

\[
e^\lambda \cdot \psi^w = [L^\lambda] \cup \psi^w.
\]

**Proof.** (4.7) is [LSSa, Lemma 2.2] and (4.6) has a similar proof. (4.8) follows from the definition and (4.9) follows from (2.13). \( \square \)

### 4.2. Equivariant affine \( K \)-Stanley classes.

Theorem 3.2 interacts with Schubert classes as follows.

**Theorem 4.3.** [LSSa] For every \( w \in \hat{W}^0 \), \( k_w := k(\psi^w_{Gr}) \) is the unique element of \( L \) of the form

\[
k_w = T_w + \sum_{z \in \hat{W} \setminus \hat{W}^0} k^z_w T_z
\]

for some \( k^z_w \in R(T) \).

**Remark 4.4.** For all \( x \in \hat{W}^0 \),

\[
\cl_{\hat{K}}(k_x) = \delta_{id,x}T_{id}.
\]

This can be proved using Theorem 3.2 and (4.4).

For \( w \in \hat{W} \) the equivariant affine \( K \)-Stanley class \( G^w \in K^*_T(Gr) \) is defined by

\[
G^w := \varphi^*(\psi^w).
\]

Thus \( p^*(G^w) = \theta(\psi^w) \).

**Lemma 4.5.** For \( w \in \hat{W} \), we have

\[
G^w = \sum_{u \in \hat{W}^0} k^w_u \psi^u_{Gr}
\]

where the \( k^w_u \) are defined in (4.10).

**Proof.** For \( u \in \hat{W}^0 \), by (3.4) and Theorems 3.2 and 4.3 we have

\[
\langle \psi^w_{Gr}, G^w \rangle_{Gr} = \langle \psi^w_{Gr}, \varphi^*(\psi^w) \rangle_{Gr} = \langle \varphi^*(\psi^w_{Gr}), \psi^w \rangle_{\hat{F}l} = \langle \sum_{z \in \hat{W}} k^z_u T_z, \psi^w \rangle = k^w_u. \quad \square
\]
Recall that \( u \ast v \) denotes the Demazure product of \( u \) and \( v \).

**Proposition 4.6.** For \( w \in \hat{W} \), we have
\[
(4.14) \quad \Delta(\psi^w) = \sum_{w_1 \ast w_2 = w} (-1)^{\ell(w_1)+\ell(w_2)-\ell(w)} G^{w_1} \otimes \psi^{w_2}.
\]

**Proof.** For \( u \in \hat{W}^0 \) and \( v \in \hat{W} \), we have
\[
(4.15) \quad k_u T_v = \sum_{x \in \hat{W}} k_x^* T_x T_v = \sum_{w \in \hat{W}} \sum_{x \in \hat{W}} (-1)^{\ell(x)+\ell(v)-\ell(w)} k_x^* T_w.
\]

This gives a formula for the matrix of the multiplication map \( L \otimes \hat{K} \to \hat{K} \) with respect to the bases \( k_u \otimes T_v \) and \( T_w \). The dual map \( K^*_T(\hat{F}_L) \to K^*_T(\hat{F}_L) \) has the transposed matrix of Schubert matrix coefficients, giving the stated formula. \(\square\)

### 4.3. Coproduct formula for affine Schubert classes

The following formula decomposes \( \psi^w \) according to the tensor product isomorphism of Theorem 3.15.

**Theorem 4.7.** For \( w \in \hat{W} \), we have
\[
(4.16) \quad \psi^w = \sum_{(w_1,w_2) \in \hat{W} \times W} (-1)^{\ell(w_1)+\ell(w_2)-\ell(w)} p^*(G^{w_1}) \cup \ev^*_1(\psi^{w_2}_{G/B})
\]
\[
(4.17) \quad = \sum_{(w_1,w_2) \in \hat{W} \times W} (-1)^{\ell(w_1)+\ell(w_2)-\ell(w)} \theta(\psi^{w_1}_{G/B}) \cup \eta(\psi^{w_2}).
\]

**Proof.** Apply Proposition 3.8 with \( a = 1 \) and \( \psi = \psi^w \), and use Proposition 4.6. \(\square\)

**Proof of Theorem 3.15.** As \( p^* \) and \( \ev^*_1 \) are \( R(T) \)-algebra homomorphisms, so is \( 3.17 \). Note that for \( u \in \hat{W}^0 \), \( G^u = \psi^u_{Gr} \). To show that \( 3.17 \) is an isomorphism, it suffices to show that the image of the basis \( \{ \psi^u_{Gr} \otimes \psi^v_{G/B} \mid (u,v) \in \hat{W}^0 \times W \} \) of \( K^*_T(Gr) \otimes_{R(T)} K^*_T(G/B) \), namely, \( \{ p^*(G^u) \cup \ev^*_1(\psi^v_{G/B}) \mid (u,v) \in \hat{W}^0 \times W \} \), is an \( R(T) \)-basis of \( K^*_T(\hat{F}_L) \). But the latter collection of elements is unipotent with the Schubert basis of \( K^*_T(\hat{F}_L) \), by Theorem 4.7. Thus \( 3.17 \) is a \( R(T) \)-algebra isomorphism.

Finally, \( 3.16 \) is left \( (\hat{K}, \cdot) \)-module homomorphism, due to Lemma 3.14 and the fact that \( \ev^*_1 \) and \( p^* \) are left \( (\hat{K}, \cdot) \)-module homomorphisms. \(\square\)

**Corollary 4.8.** For \( i \in \hat{I} \), we have
\[
(4.18) \quad \psi^{s_i} = \begin{cases} p^*(G^{s_0}) & \text{if } i = 0 \\ p^*(G^{s_i}) - \ev^*_1(\psi^{s_i}_{G/B}) + \ev^*_1(\psi^{s_1}_{G/B}) & \text{otherwise.} \end{cases}
\]

**Proposition 4.9.** For all \( i \in \hat{I} \) we have
\[
(4.19) \quad 1 - p^*(G^{s_i}) = (1 - p^*(G^{s_0}))^\ell
\]
where \( \ell = \text{level}(\Lambda_i) \).
Proof. Let \( \{ \psi^x_T \mid x \in \hat{W} \} \) denote the equivariant Schubert basis of \( K^*_T(\hat{\text{Fl}}) \), where \( \hat{T} \cong T \times \mathbb{C}^\times \) denotes the affine maximal torus. For all \( i \in I \), in \( K^*_T(\hat{\text{Fl}}) \) we have \( \ref{KS} \)

\[
\psi^{s_i}_T(w) = 1 - e^{A_i - w \cdot A_i} \quad \text{for all } w \in \hat{W}.
\]

For all \( \mu \in Q^\vee \) and \( v \in W \) we have

\[
p^s(G^s)(t_\mu v) = \psi^{s_i}_T(t_\mu) = \text{cl}(\psi^{s_i}_T(t_\mu)) = \text{cl}(1 - e^{A_i - t_\mu \cdot A_i}).
\]

Applying this equation twice, we have

\[
\frac{1 - p^s(G^s)(t_\mu v)}{(1 - p^s(G^s)(t_\mu v))^\ell} = \text{cl}(e^{A_i - t_\mu \cdot A_i - \ell A_0 + \ell t_\mu \cdot A_0}) = \text{cl}(e^{\text{aff}(w_i) - t_\mu \cdot \text{aff}(w_i)}) = 1
\]

since for any level zero element \( \lambda \) we have \( t_\mu(\lambda) = \lambda - \langle \mu, \lambda \rangle \delta \).

\[
\square
\]

4.4. Ideal sheaf classes. For a reduced word \( w = s_{i_1} \cdots s_{i_\ell} \), define \( y_w := y_{i_1} \cdots y_{i_\ell} \in \hat{K} \), which does not depend on the choice of reduced word. By \( \ref{LSSaLemmaA3} \), we have \( \psi^w \in K^*_T(\hat{\text{Fl}}) \mid w \in \hat{W} \} \) denote the dual basis to \( \{ y_w \mid w \in \hat{W} \} \). Thus \( \langle y_w, \psi^v \rangle = \delta_{w,v} \). The element \( \psi^w \) is denoted \( \psi_{K\bar{K}} \) in \( \ref{LSSaAppendixA} \).

Remark 4.10. The Schubert basis element \( \psi^w \in K^*_T(\hat{\text{Fl}}) \) represents the class of the structure sheaf \( \mathcal{O}_w \) of a Schubert variety in the thick affine flag variety. The element \( \psi^w \) represents the ideal sheaf \( \mathcal{I}_w \) of the boundary \( \partial X_w \) in a Schubert variety \( X_w \). See \( \ref{LSSaAppendixA} \).

Define

\[
\hat{G}_w := \varphi^*(\psi^w) \in K^*_T(\text{Gr}).
\]

Following \( \ref{LLMS} \), define \( l_u := \sum_{v \in \hat{W}^0 : v \leq u} k_v \in \mathbb{L} \) and define \( l^w_u \in R(T) \) by

\[
l_u := \sum_{w \in \hat{W}} l^w_u y_w.
\]

(4.20)

The coefficients \( l^w_u \) are related to \( k_v \) by the formula

\[
l^w_u = \sum_{x \leq w} (-1)^{\ell(w) - \ell(x)} \sum_{v \in \hat{W}^0 \atop v \leq u} k^x_v.
\]

We have the following variants of Lemma \( \ref{4.5} \), Proposition \( \ref{4.6} \) and Theorem \( \ref{4.7} \) with identical proofs.

Lemma 4.11. For \( w \in \hat{W} \), we have

\[
\hat{G}^w = \sum_{u \in \hat{W}^0} l^w_u \psi^u_{\text{Gr}}
\]

(4.21)

where the \( l^w_u \) are defined in \( \ref{4.20} \), and \( \psi^u_{\text{Gr}} \) is determined by \( \langle l_u, \psi^u_{\text{Gr}} \rangle = \delta_{v,u} \).

Proposition 4.12. For \( w \in \hat{W} \), we have

\[
\Delta(\psi^w) = \sum_{u = w_1 \ast w_2} \hat{G}^{w_1} \otimes \psi^{w_2}.
\]

\[\footnote{The conventions here differ by a sign to those in \( \ref{KS} \). For example, for us \( \psi^{s_i}(s_i) = 1 - e^{s_i} \).} \]}
Proof. \[\theta\]enough to separately describe how \(\theta(\psi^x)\) for \(x \in \hat{W}\), and \(\eta(\psi^w)\) for \(w \in W\), behave under the •-action. For \(\eta(\psi^w)\), Proposition 3.12 gives the following.

Proposition 4.14. For \(a \in \hat{K}\) and \(w \in W\), we have

\[
(a \cdot \eta(\psi^w)) = \eta(c_{\hat{K}}(a) \cdot \psi^w)
\]

and in particular,

\[
s_0 \cdot \eta(\psi^w) = \eta(s_0 \cdot \psi^w)
\]

\[
y_0 \cdot \eta(\psi^w) = \eta(y_{-\theta} \cdot \psi^w)
\]

where \(y_{-\theta} := 1 + T_{-\theta}\).

Since \(\theta(\psi^x)\) can be expanded in the basis \(\psi^u\) for \(u \in \hat{W}^0\), it is enough to consider the •-action on \(\psi^u\).

Theorem 4.15. For \(\lambda \in P, i \in \hat{I}\), and \(u \in \hat{W}^0\), we have

1. \(e^\lambda \cdot \psi^u = [\mathcal{L}_{\lambda}] \cup \psi^u\),
2. \(y_i \cdot \psi^u = \psi^u\) and \(s_i \cdot \psi^u = \psi^u\) if \(i \in I\),
3. For \(u \in \hat{W}^0 \setminus \{id\}\), we have

\[
y_0 \cdot \psi^u = \psi^{u_{s_0}} = \sum_{(x_1, x_2) \in \hat{W} \times W \atop x_1 \neq x_2 \atop x_1, x_2 \in \hat{W}^0} (-1)^{\ell(x_1) - \ell(x_2) - \ell(u) - 1} \theta(\psi^{x_1}) \cup \eta(\psi^{x_2}).
\]

4. For \(u \in \hat{W}^0 \setminus \{id\}\), we have

\[
s_0 \cdot \psi^u = e^{-\theta} \psi^u + \sum_{(x_1, x_2) \in \hat{W} \times W \atop x_1 \neq x_2 \atop x_1, x_2 \in \hat{W}^0} (-1)^{\ell(x_1) + \ell(x_2) - \ell(u) - 1} \theta(\psi^{x_1}) \cup \eta((1 - e^{-\theta}) \cdot \psi^{x_2}).
\]

Proof. These formulae may be deduced from Proposition 4.12 using \(s_0 = e^{-\theta} + (1 - e^{-\theta})y_0\). \(\square\)

Remark 4.16. The • and • actions of \(\hat{K}\) make \(K_T^*(\text{Gr}) \otimes_{R(T)} K_T^*(G/B)\) into a left \((\hat{K} \times \hat{K})\)-module such that the map \([3.16]\) is a left \((\hat{K} \times \hat{K})\)-module isomorphism.

4.6. Recursion. The affine Schubert classes in the tensor product \(K_T^*(\text{Gr}) \otimes_{R(T)} K_T^*(G/B)\) are determined by the following recursion.

1. \(\psi^u = \psi^u_{\text{Gr}} \otimes 1\) for \(u \in \hat{W}^0\), and
2. For all \(i \in \hat{I}\),

\[
y_i \cdot \psi^w = \begin{cases} \psi^{ws_i} & \text{if } ws_i < w \\ \psi^w & \text{otherwise.} \end{cases}
\]
The operator $y_i$ acts on $K^*_T(\text{Gr}) \otimes_{R(T)} K^*_T(G/B)$ by
$$\Delta(y_i) = (1 - e^{\alpha_i}) y_i \otimes y_i + e^{\alpha_i} (y_i \otimes 1 + 1 \otimes y_i - 1 \otimes 1)$$
which follows from (3.13).

5. Cohomology

In this section, we indicate the modifications necessary for the preceding results to hold in cohomology.

5.1. Small-torus affine nilHecke ring. Instead of $R(T)$, we work over $S = \text{Sym}_Z(P) \cong H^*_T(\text{pt})$. The algebra $K$ is replaced by the small-torus affine nilHecke ring $\hat{A}$, as defined in [LLMS] Chapter 4. The $S$-algebra $\hat{A}$ is generated by symbols $S$ and symbols $A_i$ (satisfying the nilCoxeter relations), and we have the analogue of (2.6) $\hat{A} = \bigoplus_{w \in W} S A_w$.

Instead of the Demazure product, we will make use of length-additive products. Write $uv = \bigoplus_{j} A_j$. The algebra $\hat{A}$ acts on $H^*_T(\hat{Fl})$ with a function $\hat{A}$ taking values $\xi(v) = \iota_w^*(\xi)$. For the small torus affine GKM condition see [LLMS].

There is a $S$-bilinear perfect pairing $\langle \cdot, \cdot \rangle : \hat{A} \times H^*_T(\hat{Fl})$ characterized by $\langle w, \xi \rangle = \xi(w)$.

There is a left action $\xi \mapsto a \cdot \xi$ of $\hat{A}$ on $H^*_T(\hat{Fl})$ given by the formulae [LLMSSZ] Chapter 4, Proposition 3.16]

(5.1) $\quad (q \cdot \xi)(a) = q \xi(a)$
(5.2) $\quad (A_i \cdot \xi)(a) = A_i \cdot \xi(s_i a) + \xi(A_i a)$
(5.3) $\quad (w \cdot \xi)(a) = w \xi(w^{-1} a)$

for $a \in \hat{A}$, $\xi \in H^*_T(\hat{Fl})$, $q \in S$, $i \in \hat{I}$, and $w \in \hat{W}$. Here, $A_i$ acts on $S$ via

(5.4) $\quad A_i(\lambda) = (\alpha_i^\vee, \lambda) \text{id}$
(5.5) $\quad A_i(q q') = A_i(q) q' + (s_i \cdot q) A_i(q')$.

There is another left action $\xi \mapsto a \bullet \xi$ of $\hat{A}$ on $H^*_T(\hat{Fl})$ given by [LLMSSZ] Chapter 4, Section 3.3]

(5.6) $\quad (a \bullet \xi)(b) = \xi(ba)$

for $a, b \in \hat{A}$ and $\xi \in H^*_T(\hat{Fl})$.

Let $c_1(\mathcal{L}_\lambda) \in H^*_T(\hat{Fl})$ denote the first Chern class of the $T$-equivariant line bundle with weight $\lambda \in X$ on $\hat{Fl}$. Explicitly [LLMSSZ] Chapter 4, Section 3]

(5.7) $\quad \langle t_\mu v, c_1(\mathcal{L}_\lambda) \rangle = v \cdot \lambda \quad \mu \in Q^\vee, v \in W$

Lemma 5.1. For any $\lambda \in X$ and $\xi \in H^*_T(\hat{Fl})$, we have $\lambda \bullet \xi = c_1(\mathcal{L}_\lambda) \cup \xi$. 
5.3. Endomorphisms. Let $\mathbb{P} = \mathbb{Z}_{\hat{A}}(S)$ be the centralizer of $S$ in $\hat{A}$, called the Peterson subalgebra. We have the cohomological wrong way map $\varphi^*: H^*_T(\hat{F}l) \to H^*_T(Gr)$.

Theorem 5.2 ([Pet] [Lam08] [LLMSSZ, Chapter 4, Theorem 4.9]). There is an isomorphism $j: H^*_T(Gr) \to \mathbb{P}$ making the following commutative diagram of ring and left $\mathbb{R}(T)$-module homomorphisms:

$$
\begin{array}{ccc}
H^*_T(Gr) & \xrightarrow{j} & \mathbb{P} \\
\downarrow & & \downarrow \\
H^*_T(\hat{F}l) & \xrightarrow{\varphi^*} & \mathbb{K}
\end{array}
$$

The maps

- $p^*: H^*_T(Gr) \to H^*_T(\hat{F}l)$
- $\theta: H^*_T(\hat{F}l) \to H^*_T(\hat{F}l)$
- $\Delta: H^*_T(\hat{F}l) \to H^*_T(Gr) \otimes_S H^*_T(\hat{F}l)$
- $ev^*_1: H^*_T(G/B) \to H^*_T(\hat{F}l)$
- $\eta: H^*_T(\hat{F}l) \to H^*_T(\hat{F}l)$
- $\kappa: H^*_T(\hat{F}l) \to H^*_T(\hat{F}l)$

are defined as for $K$-theory. Lemma 3.3, Proposition 3.4, Lemma 3.5, Lemma 3.6 hold in cohomology with the obvious modifications. Lemma 3.7 holds with $c_1(\mathcal{L}_\lambda)$ replacing $[\mathcal{L}_\lambda]$.

Proposition 5.3. For $\xi \in H^*_T(\hat{F}l)$ and $a \in \hat{A}$, we have

$$
a \cdot \xi = \sum_{(\xi)} \xi_{(1)} \cup \eta(a \cdot \xi_{(2)})
$$

where $\Delta(\xi) = \sum_{(\xi)} \xi_{(1)} \otimes \xi_{(2)}$. In particular, taking $a = 1$, we have the identity

$$
\cup \circ (1 \otimes \eta) \circ \Delta = 1
$$

in $\text{End}_S(H^*_T(\hat{F}l))$.

Lemmas 3.9, 3.10, and Propositions 3.11, 3.12, 3.13 hold in cohomology.

5.4. Action of $\hat{A}$ on tensor products. Equation (3.13) is replaced by

$$
\Delta(A_i) = A_i \otimes 1 + s_i \otimes A_i = 1 \otimes A_i + A_i \otimes s_i
$$

(5.8)

Lemma 3.14 holds with no change in cohomology.
5.5. **Tensor product decomposition of** $H^*_T(\hat{F}l)$. The left $\hat{A}$-module structures via $\cdot$ on $H^*_T(Gr)$ and $H^*_T(G/B)$ give a left $\hat{A}$-module structure on $H^*_T(Gr) \otimes_S H^*_T(G/B)$.

**Theorem 5.4.** There is an $S$-algebra isomorphism
\begin{align}
H^*_T(Gr) \otimes_S H^*_T(G/B) & \cong H^*_T(\hat{F}l) \\
a \otimes b & \mapsto p^*(a) \cup \ev_1^*(b)
\end{align}
with componentwise multiplication on the tensor product. This map is also an isomorphism of left $\hat{A}$-modules under the $\cdot$ action.

5.6. **Schubert bases.** The $S$-algebras $H^*_T(\hat{F}l)$, $H^*_T(Gr)$, and $H^*_T(G/B)$ have equivariant Schubert bases $\{\xi^x \mid x \in \hat{W}\}$, $\{\xi^u_{Gr} \mid u \in \hat{W}^0\}$, and $\{\xi^w_{G/B} \mid w \in W\}$ respectively.

Equations (4.2) and (4.3) hold for cohomology Schubert classes. The analogue of Proposition 4.2 is as follows.

**Proposition 5.5.** [LLMSSZ, Chapter 4, Section 3.3] For $\lambda \in \mathcal{X} \subset S$ and $A_i$ for $i \in \hat{I}$, on the Schubert basis element $\xi^w \in H^*_T(\hat{F}l)$ for $w \in \hat{W}$, we have:
\begin{align}
A_i \cdot \xi^w &= \begin{cases} 
\xi^{s_i w} & \text{if } s_i w < s \\
0 & \text{otherwise.}
\end{cases} \\
A_i \bullet \xi^w &= \begin{cases} 
\xi^{ws_i} & \text{if } ws_i < w \\
0 & \text{otherwise.}
\end{cases} \\
\lambda \cdot \xi^w &= \lambda \xi^w \\
\lambda \bullet \xi^w &= c_1(L_\lambda) \cup \xi^w.
\end{align}

5.7. **Equivariant affine Stanley classes.**

**Theorem 5.6** (Pet, Lam08, LLMSSZ). For every $w \in \hat{W}^0$, $j_w = k(\xi^w_{Gr})$ is the unique element of $\mathbb{P}$ of the form
\begin{align}
j_w &= A_w + \sum_{z \in W \setminus \hat{W}^0} j_z^w A_z
\end{align}
for some $j_z^w \in S$.

**Remark 5.7.** By Pet, LS the coefficients $j_z^w$ are equivariant Gromov-Witten invariants for $G/B$.

For $w \in \hat{W}$ the equivariant affine Stanley class $F^w \in H^*_T(Gr)$ is defined by
\begin{align}
F^w := \varphi^*(\xi^w).
\end{align}

**Lemma 5.8.** For $w \in \hat{W}$, we have
\begin{align}
F^w = \sum_{u \in \hat{W}^0} j_u^w \xi^u_{Gr}
\end{align}
where the $j_u^w$ are defined in (5.15).

**Proposition 5.9.** For $w \in \hat{W}$, we have $\Delta(\xi^w) = \sum_{w = w_1 w_2} F^{w_1} \otimes \xi^{w_2}$. 
5.8. Coproduct formula for affine Schubert classes.

**Theorem 5.10.** For \( w \in \hat{W} \), we have
\[
\xi^w = \sum_{(w_1, w_2) \in \hat{W} \times \hat{W}} p^*(F^{w_1}) \cup \ev_1^*(\xi^{w_2}_{G/B}) = \sum_{(w_1, w_2) \in \hat{W} \times \hat{W}} \theta(\xi^{w_1}) \cup \eta(\xi^{w_2}).
\]

5.9. Formulae for Schubert divisors.

**Corollary 5.11.** For \( i \in \hat{I} \) we have
\[
(5.17) \quad \xi^{si} = \begin{cases} p^*(F^{s_0}) & \text{if } i = 0 \\ p^*(F^{si}) + \ev_1^*(\xi^{s_i}_{G/B}) & \text{otherwise.} \end{cases}
\]

**Proposition 5.12.** For all \( i \in \hat{I} \) and \( \lambda \in Q^\vee \) we have
\[
(5.18) \quad p^*(F^{si}) = \text{level}(\Lambda_i) \cdot p^*(F^{s_0}).
\]

5.10. \( \bullet \)-action on affine Schubert classes. Proposition 4.14 holds with \( A_0 \) replacing \( y_0 \) and \( A_{-\theta} := -\theta^{-1}(1 - s_\theta) \) replacing \( y_\theta \).

**Theorem 5.13.** For \( \lambda \in X \subset S \), \( i \in \hat{I} \), and \( u \in \hat{W}^0 \), we have
\begin{enumerate}
  \item \( \lambda \bullet \xi^u = c_1(\mathcal{L}_\lambda) \cup \xi^u \),
  \item \( A_i \bullet \xi^u = 0 \) and \( s_i \bullet \xi^u = \xi^u \) if \( i \in I \),
  \item For \( u \in \hat{W}^0 \setminus \{\text{id}\} \)
    \[
    A_0 \bullet \xi^u = \xi^{us_0} = \sum_{(x_1, x_2) \in \hat{W} \times \hat{W}} \theta(\xi^{x_1}) \cup \eta(\xi^{x_2}).
    \]
  \item For \( u \in \hat{W}^0 \setminus \{\text{id}\} \)
    \[
    s_0 \bullet \xi^u = \xi^u + \sum_{(x_1, x_2) \in \hat{W} \times \hat{W}} \theta(\xi^{x_1}) \cup \eta(-\text{cl}(\alpha_0) \bullet \xi^{x_2}).
    \]
\end{enumerate}

**Remark 5.14.** The \( \cdot \) and \( \bullet \) actions of \( \hat{A} \) make \( H^*_T(\text{Gr}) \otimes_S H^*_T(G/B) \) into a left \( (\hat{A} \times \hat{A}) \)-module such that the map \( [5.9] \) is a left \( (\hat{A} \times \hat{A}) \)-module isomorphism.

5.11. Recursion. The affine Schubert classes in the tensor product \( H^*_T(\text{Gr}) \otimes_S H^*_T(G/B) \) are determined by the following recursion.
\begin{enumerate}
  \item \( \xi^u = \xi^{u}_{\text{Gr}} \otimes 1 \) for \( u \in \hat{W}^0 \), and
  \item For all \( i \in \hat{I} \)
    \[
    A_i \bullet \xi^w = \begin{cases} \xi^{ws_i} & \text{if } ws_i < w \\ 0 & \text{otherwise.} \end{cases}
    \]
\end{enumerate}

Here, the operator \( A_i \) acts on \( H^*_T(\text{Gr}) \otimes_S H^*_T(G/B) \) by \( A_i \bullet (\zeta \otimes \psi) = \zeta \otimes (A_i \bullet \psi) \) if \( i \neq 0 \) and \( A_0 \bullet (\zeta \otimes \psi) \) is computed via \([5.8]\) and Theorem 5.13.
6. Examples

6.1. Type A in cohomology. Letting $G = \text{SL}(n)$, we now consider the affine Schubert polynomials [Lee]. Recall the isomorphism $H^\bullet(\tilde{F}_L) \cong H^\bullet(\text{Gr}, G) \otimes_{H^\bullet(\text{pt})} H^\bullet(G/B)$. By [Lam08], the cohomology $H^\bullet(\text{Gr}, G)$ is isomorphic to $\Lambda/I_n$ where $\Lambda$ is the ring of symmetric functions and $I_n$ is the ideal $\langle m_\lambda \mid \lambda_1 \geq n \rangle$ in $\Lambda$. Also, $H^\bullet(G/B) = \mathbb{Z}[x_1, \ldots, x_n]/\langle e_j(x_1, \ldots, x_n) \mid j \geq 1 \rangle$ where $e_j(x_1, \ldots, x_n)$'s are elementary symmetric functions. Hence we have

$$H^\bullet(\tilde{F}_L) \cong \Lambda/I_n \otimes_{\mathbb{Z}} \mathbb{Z}[x_1, \ldots, x_n]/\langle e_j(x_1, \ldots, x_n) \mid j \geq 1 \rangle.$$

We list some affine Schubert polynomials for $n = 3$.

\[
\begin{array}{|c|c|c|c|c|c|c|}
\hline
w & s_0 & s_1 & s_2 & s_1s_0 & s_2s_1 & s_2s_1s_0 \\
\hline
\tilde{\mathcal{G}}_w & 1 & h_1 & h_1 + x_1 & h_1 + x_1 + x_2 & h_2 & h_2 + h_1x_1 + x_1^2 \\
\hline
& & & & & m_{2,1} + m_{1,1,1} & \\
\hline
\end{array}
\]

The polynomial $\tilde{\mathcal{G}}_{s_2s_1}$ can be computed in a number of different ways. First, we can start from $\tilde{\mathcal{G}}_{s_2s_1s_0}$ which is the same as the affine Schur function indexed by $s_2s_1s_0$, and use the monomial expansion of the affine Schur functions [Lam06]. Then one can act with the divided difference operator $A_i \bullet$ to obtain $\tilde{\mathcal{G}}_{s_2s_1}$. The action of $A_i \bullet$ is explicitly given in [Lee].

On the other hand, using the coproduct formula (Theorem 5.10) directly give $\tilde{\mathcal{G}}_{s_2s_1}$:

$$\tilde{\mathcal{G}}_{s_2s_1} = F_{s_2s_1} + F_{s_2} \mathcal{S}_1 + \mathcal{S}_{s_2s_1} = h_2 + h_1x_1 + x_1^2$$

where $F_w$ is the affine Stanley symmetric function, the non-equivariant version of $F^w$ in Section 5, and $\mathcal{S}_v(x)$ is the Schubert polynomial. Using the coproduct formula together with monomial expansions of $F_w$ [Lam06] and $\mathcal{S}_v(x)$ [BJS] provides the following theorem:

**Theorem 6.1.** Affine Schubert polynomials are monomial-positive.

The same coproduct formulae hold in equivariant cohomology, with the affine double Stanley symmetric function $F^w$ [LS2] replacing $F_w$, and the double Schubert polynomial $\mathcal{S}_v(x, y)$ [LaSc] replacing $\mathcal{S}_v(x)$. However, there is no combinatorially explicit formula for the equivariant affine Stanley classes $F^w$, see [LS2] Remark 23.

6.2. Type A in K-theory. Let $G = \text{SL}(n)$. We now consider affine versions of the Grothendieck polynomials. We have the isomorphism $K^\bullet(\tilde{F}_L) \cong K^\bullet(\text{Gr}, G) \otimes_{K^\bullet(\text{pt})} K^\bullet(G/B)$ and identifications $K^\bullet(G/B) = \mathbb{Z}[x_1, \ldots, x_n]/\langle e_j(x_1, \ldots, x_n) \mid j \geq 1 \rangle$ [LSSa] and $K^\bullet(\text{Gr}, G) \cong \Lambda/I_n$, where $\Lambda/I_n$ denotes the graded completion. By Theorem 4.7, we have the formula

$$\tilde{\mathcal{G}}_w = \sum_{(w_1, w_2) \in W \times W} (-1)^{\ell(w_1) + \ell(w_2) - \ell(w)} G^{w_1} \mathcal{G}_{w_2}$$

in $\Lambda/I_n \otimes_{\mathbb{Z}} \mathbb{Z}[x_1, \ldots, x_n]/\langle e_j(x_1, \ldots, x_n) \mid j \geq 1 \rangle$, where $\tilde{\mathcal{G}}_w$ is the affine Grothendieck polynomial, $G^{w_1} \in \Lambda/I_n$ denotes the affine stable Grothendieck polynomial [LSSa], and $\mathcal{G}_{w_2}$ is the Grothendieck polynomial of Lascoux and Schützenberger. For example, let
From \[\text{(LSSa, A.3.6)}\], we have expansions in terms of Schur functions:
\[
C_n^{(1)}(s) \Rightarrow s - \cdots - s_n \Leftrightarrow s_n
\]
\[
D_n^{(1)}(s) \Rightarrow s - \frac{1}{2} - \cdots - \frac{1}{n-2} - \frac{1}{n-1} \Rightarrow s_n
\]

**Figure 1.** Affine Dynkin diagrams

For classical type, this recovers the Schubert class formulas of Billey and Haiman \[\text{[BH]}\] for nonequivariant cohomology and those of Ikeda, Mihalcea, and Naruse \[\text{[IMN]}\] for equivariant cohomology. Consider the affine Dynkin diagrams of types \(C_n^{(1)}\) and \(D_n^{(1)}\) in Figure 1. Let \(W' = W'_n\) be the subgroup of \(\hat{W}\) generated by \(s_j\) for \(j \in \hat{I} \setminus \{n\}\). Since the node \(n\) is conjugate to the node 0 by a length-zero element of the extended affine Weyl group, \(W'\) is isomorphic to the finite Weyl group \(W\). Let \(G\) and \(G'\) denote the subgroups of the corresponding loop group (or affine Kac-Moody group) with Weyl groups \(W\) and \(W'\) respectively, and let \(G/B\) and \(G'/B'\) be the two finite-dimensional flag varieties (either the symplectic flag variety or the orthogonal flag variety). Finally, note that the subgroup of \(\hat{W}\) generated by \(s_j\) for \(j \in \hat{I} \setminus \{0, n\}\) is isomorphic to the type \(A_{n-1}\) Weyl group \(W_{A_{n-1}}\).

For \(w \in W'\), if we have \(w = w_1w_2\) for \((w_1, w_2) \in \hat{W} \times W\), then \((w_1, w_2) \in W' \times W_{A_{n-1}}\). Applying the affine coproduct formula (Theorem 5.10) and pulling back to \(H_T^*(G'/B')\), we have in \(H_T^*(G'/B')\) (with \(G' = \text{Sp}(2n)\) or \(G' = \text{SO}(2n)\)) the equality, for \(w \in W'\),

\[
\xi^{\tau w} = \sum_{(w_1, w_2) \in W' \times W_{A_{n-1}}} F^{w_1 \cup \tau * w_2}.
\]

Here, \(F^{w_1}\) is the pullback to \(H_T^*(G'/B')\) (under the natural projection from the flag variety to a Grassmannian) of an element of the torus-equivariant cohomology \(H_T^*(\text{LG}(n, 2n))\) of the Lagrangian Grassmannian in the \(C_n^{(1)}\) case, or an element of the torus-equivariant cohomology \(H_T^*(\text{OG}(n, 2n))\) of the orthogonal Grassmannian in the \(D_n^{(1)}\) case. Also,
\( \xi^w \) denotes a Schubert class in \( H^*_T(G/B) \) and \( \tau^* \) is the composition of pullback maps \( H^*_T(G/B) \to H^*_T(\Fl_G) \to H^*_T(G'/B') \) (the first one being \( \ev^*_1 \)).

Now, there is a natural inclusion \( \GL(n)/B \to G/B \), inducing a quotient map \( H^*_T(G/B) \to H^*_T(\GL(n)/B) \). (We write \( GL(n) \) instead of \( SL(n) \) as we are using the same \( n \)-dimensional maximal torus \( T \).) Under this surjection, \( \xi^w \) for \( w \in W_{\lambda_{n-1}} \) is mapped to the Schubert class \( \xi^w_{W_{\lambda_{n-1}}} \in H^*_T(\GL(n)/B) \) while \( \xi^w \) for \( w \notin W_{\lambda_{n-1}} \) is sent to 0. Any identity in the projective limit \( \lim_{\rightarrow m} H^*_T(\GL(m)/B) \) can thus be lifted to \( H^*_T(G/B) \) for sufficiently large \( n \).

Taking the limit as \( n \) goes to infinity, (6.1) becomes the classical Schubert class formula of [BH] in nonequivariant cohomology and that of [IMN] in equivariant cohomology (see also [AF, Tam]). Nonequivariantly, in the limit \( F^{w_1} \) becomes a type \( C \) or type \( D \) Stanley symmetric function. These were studied in the classical setting in [BH, La95, La96] and in the affine setting in [LSSb, Pon]. Equivariantly, in the limit \( F^{w_1} \) is a double analogue of type \( C \) or \( D \) Stanley symmetric function. Our definition of (equivariant) affine Stanley class gives a precise geometric description of the Grassmannian components of the formulas in [IMN]. Finally, in the \( n \to \infty \) limit, \( \tau^* \xi^{w_2} \) can be identified with the usual Lascoux and Schützenberger double Schubert polynomial [LaSc].

6.4. **Classical type in \( K \)-theory.** Our coproduct formula in equivariant \( K \)-theory should be compared with the classical type double Grothendieck polynomials of A. N. Kirillov and H. Naruse [KN, HIMN] just as our cohomological formula relates to the work of Billey and Haiman. There is a Pieri formula [Tak] in the \( K \)-homology of the type \( A \) affine Grassmannian, which gives some coproduct structure constants for \( K \)-cohomology Schubert classes.

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