The Schwarzian variable associated with discrete KdV-type equations

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Abstract

We obtain a Schwarzian variable associated with discrete equations of Korteweg–de Vries-type (KdV-type). In the generic case, including the primary model Q4, the new variable satisfies the lattice Schwarzian Kadomtsev–Petviashvili equation in three dimensions. For the degenerate sub-cases of Q4 the construction reveals an invertible transformation to the lattice Schwarzian KdV equation, as well as a new auto-transformation of the Schwarzian equation itself.

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1. Introduction and synopsis

The feature of M"obius point symmetry group of an equation is synonymous with the presence of this group’s differential invariant, the Schwarzian derivative, or in the discrete setting the cross-ratio. Schwarzian forms of integrable partial differential equations appeared first in the work of Weiss through the Painlevé analysis [1] as the singularity manifold equation (see [2]). For a review of the continuous and discrete Schwarzian integrable systems see [3].

The discrete Korteweg–de Vries-type (KdV-type) equations were classified using the multidimensional consistency integrability criterion in pioneering work of Adler, Bobenko and Suris (ABS) in [4, 5] (see also [6–8]). The purpose of this paper is to introduce a new variable associated with these equations. It is defined up to the action of the M"obius group, and is therefore characterized by Schwarzian systems. In the remainder of this section we outline the main results, which contribute to the transformation theory of these equations.

It can be useful to partition the ABS list of equations into two parts. In the first part we put equations which are equivalent to the older NQC equation [9] and its parameter sub-cases, namely Q3, Q1, A2, A1, H3, and H1 in the nomenclature of [4]. Although all
ABS equations share the core characteristic of multidimensional consistency, there are several features of the NQC-type equations which do not directly transfer to the second part of the ABS list. One such feature is the connection to a discrete Kadomtsev–Petviashvili (KP)-type equation. Inherent from the author’s approach the NQC equation was connected with the discrete KP-type equation obtained in [10], specifically the natural embedding of the former equation in three dimensions satisfies the latter. This fulfills the expectation of dimensional reduction that is familiar from the continuous theory. It is an important feature; revealing the wider natural context for the integrable systems. For the second and newer part of the ABS list, namely equations Q4, Q3, Q2, H3, and H2, the connection with a discrete KP-type equation should be expected, but does not appear in the direct way as for the NQC case. One of the main aspects of the Schwarzian construction introduced here is that it provides a partial resolution of this issue, specifically, for all ABS equations the new variable satisfies the lattice Schwarzian KP equation in three dimensions.

The Schwarzian construction also inherently separates Q4 from the rest of the equations on the ABS list. Specifically it leads to new invertible transformations connecting each of the ABS equations, with the exception of Q4, to the lattice Schwarzian KdV equation (a.k.a the cross-ratio equation and Q1). This shows that allowing for non-local transformations there are at most two distinct ABS equations, which provides a discrete counterpart to the result of [11]. The obstruction to obtaining a similar transformation between Q4 and the lattice Schwarzian KdV equation is a lack of constant or oscillating singular solutions which do exist for all other ABS equations. This at first seems circumstantial, but it is consistent with the central role played by the singular solutions in the theory of this class of equations [5].

For the rest of the ABS equations, i.e. all those which appear below Q4 in the hierarchy, the Schwarzian construction serves to clarify the solution structure of the equations in relation to each other. Loosely speaking we are able to show that there is a one-to-one correspondence between solutions of all equations below Q4, modulo a (usually non-local) three-parameter symmetry group of each equation.

Further interesting features emerge when the Schwarzian construction is applied to the lattice Schwarzian KdV equation itself. The resulting auto-transformation does not coincide with the known Bäcklund transformation. In fact it connects the equation to itself, but with different values of the essential parameters of the equation (the lattice parameters). This is a feature which is new to this class of equations, but which is a fundamental part of transformations of the Painlevé equations for example, see for instance [15] and citations therein.

The paper is organized as follows. Many elements of the construction are introduced on the level of the Riccati equation in section 2. Section 3 contains the definition of the Schwarzian variable in the context of the Riccati equations associated with the Bäcklund transformations of the discrete KdV-type equations. The definition is extended to higher dimensions in section 4. The Q4 situation is considered in section 5 and the remaining systems of [4] in section 6. In the final section we describe what happens when we apply the definition to systems of consistent polynomials less symmetric than those of [4] by considering several examples, it demonstrates that situations in which the Schwarzian construction becomes degenerate are exceptional.

2. A Schwarzian variable associated with the Riccati map

Consider the scalar discrete Riccati equation

\[ R(v, \tilde{v}) := a_0 + a_1 v + a_2 \tilde{v} + a_3 v \tilde{v} = 0, \]  

(2.1)
where \( v = v(n) \) and \( \tilde{v} = v(n + 1) \) are values of the dependent variable \( v : \mathbb{Z} \to \mathbb{C} \cup \{\infty\} \) as a function of the independent variable \( n \in \mathbb{Z} \), and \( a_0 \ldots a_3 : \mathbb{Z} \to \mathbb{C} \) are non-autonomous coefficients of the polynomial \( R \). The values of the variable \( v \) are naturally associated with the vertices of a one-dimensional lattice, and the polynomial \( R \) connects consecutive vertices and so is naturally associated with the edges. A degenerate situation occurs if the polynomial on some edge is reducible, we will assume explicitly that this is not the case, specifically that
\[
(\partial_1R)(\partial_2R) - (\partial_1\partial_2R)R = a_1a_2 - a_0a_3 \neq 0,
\]
where we have introduced the convenient notation \( \partial_i \) for partial differentiation of the polynomial with respect to its \( i \)th argument.

There is a close relationship between the Riccati equation and the Möbius transformations,
\[
M := \{ x \mapsto (ax + b)/(cx + d) : a, b, c, d \in \mathbb{C}, ad \neq bc \},
\]
which form a group under composition. This is more discernible if we reformulate (2.1) as an equation for a function \( s : \mathbb{Z} \to M \),
\[
s = r \cdot s.
\]
Here \( r = x \mapsto -(a_1x + a_0)/(a_3x + a_2) \) and \( \cdot \) represents composition in the Möbius group. The values of the functions \( s \) and \( r \) are naturally associated with the vertices and edges respectively of the one-dimensional lattice. The virtue of the reformulation (2.4) is that both of these functions take values in the group \( M \). The equations (2.1) and (2.4) are equivalent in the sense that from the general solution of one we may construct the general solution of the other. This is clear because given \( s \) satisfying (2.4), \( s(c) \) satisfies (2.1) for any constant \( c \in \mathbb{C} \cup \{\infty\} \). And conversely, if the general solution of (2.1) is known, then certainly three particular solutions are known, say \( v_1, v_2 \) and \( v_3 \), and writing \( s(1) = v_1, s(\infty) = v_2 \) and \( s(0) = v_3 \) determines uniquely a function \( s \) satisfying (2.4). This is because a Möbius transformation is uniquely determined by its action on three points. Points other than 1, \( \infty \) and 0 would also be fine, provided they were distinct, however this choice results in the convenient expression
\[
s^{-1} = x \mapsto \frac{(v_1 - v_2)(v_3 - x)}{(v_1 - v_3)(v_2 - x)}.
\]
By \( s^{-1} \) we mean the unique function \( s^{-1} : \mathbb{Z} \to M \) taking the value in \( M \) which is the group inverse of the value of \( s \).

We remark that letting \( v \) take values in \( \mathbb{C} \cup \{\infty\} \) is now more obviously natural because elements of \( M \) permute this set.

We now introduce the object of principal concern, a function \( \varphi : \mathbb{Z} \to \mathbb{C} \cup \{\infty\} \) which is defined in terms of a fixed but as yet un-specified function \( w : \mathbb{Z} \to \mathbb{C} \cup \{\infty\} \) as
\[
\varphi := s^{-1}(w),
\]
where \( s \) is a solution of (2.4). The solution of (2.4) is unique up to the transformation \( s \to s \cdot m \) for an arbitrary constant (i.e. independent of \( n \)) \( m \in M \), and therefore the definition (2.6) determines \( \varphi \) up to the transformation \( \varphi \to m(\varphi) \). This connects the definition (2.6) with the following one-dimensional Schwarzian difference equation:
\[
\frac{(\varphi - \tilde{\varphi})(\ddot{\varphi} - \tilde{\varphi})}{(\varphi - \tilde{\varphi})(\dot{\varphi} - \tilde{\varphi})} = \frac{(\ddot{\varphi} - \tilde{\varphi})(\dot{\varphi} - \tilde{\varphi})}{(\varphi - \tilde{\varphi})(\dot{\varphi} - \tilde{\varphi})} = \frac{(\ddot{\varphi} - \tilde{\varphi})(\dot{\varphi} - \tilde{\varphi})}{(\varphi - \tilde{\varphi})(\dot{\varphi} - \tilde{\varphi})).
\]
Here we have simply evaluated the cross-ratio of four consecutive values of \( \varphi \), the right-hand side follows by substituting for \( \varphi \) from (2.6) and using (2.4) together with the Möbius invariance of the cross-ratio, therefore (2.7) is a consequence of our definition. However this equation
actually characterizes \( \phi \) because any two solutions of (2.7) are Möbius related: observe first, the equation is invariant under transformations \( \phi \rightarrow m(\phi) \) for arbitrary constant \( m \in \mathbb{M} \), and second, the equation is third order, which ensures the existence of a Möbius transformation sending any set of well-posed initial data to any other.

Notice that the right-hand side of (2.7) does not depend on \( s \), so solving this equation obtains \( \phi \) without requiring first the solution of (2.1). There is however the following subtlety: from the definition (2.6) it is easy to see that \( \phi \) is constant if and only if \( w \) is a solution of (2.1), whereas the equation (2.7) becomes ambiguous in this situation. So definition (2.6) is the more robust.

We remark that the \( \phi \) variable can play a useful role when only two particular solutions of (2.1) are known. If we label the known solutions \( v_2 \) and \( v_3 \) then the associated variable \( \phi \) may be found by integration of the homogeneous linear equation

\[
(v_3 - w)R(v_2, \tilde{w})\phi - (v_2 - w)R(v_3, \tilde{w})\phi = 0.
\]  

This equation is the result of combining (2.5) and (2.6) and demanding \( v_1 \) also satisfy (2.1). Solving (2.8) obtains \( \phi \) from \( v_2 \) and \( v_3 \). Once a particular solution of (2.8) is known (excluding \( \phi = 0 \) and \( \phi = \infty \)) we can again combine (2.5) and (2.6) to reconstruct solution \( v_1 \), and thus the general solution of (2.1). Therefore if two particular solutions are known, the transformation to the variable \( \phi \) reduces (2.1) to the homogeneous linear equation (2.8). The auxiliary function \( w \) may be chosen freely, and in particular may be chosen to make the equation for \( \phi \) (2.7) simple.

To conclude this preliminary section let us re-iterate that our intention so far has been to establish, separately from further considerations, a basic definition of \( \phi \), its Möbius invariance, as well as some explicit formulae. This is to emphasize aspects of the construction which are present already at the level of the Riccati equation. Note that the construction involves an arbitrary background function \( w \), and choosing \( w \) is really what makes the construction useful. In particular there will appear natural choices of \( w \) leading to equations characterizing \( \phi \) that provide an important alternative to solving the original equation for \( u \).

3. Application to Bäcklund transformations

Well known Bäcklund transformations for discrete KdV-type equations are of the following generic form

\[
B_1(u, \hat{u}, v, \hat{v}) = 0, \quad B_2(u, \hat{u}, v, \hat{v}) = 0.
\]  

(3.1)

Here \( B_1 \) and \( B_2 \) are autonomous polynomials of degree one in four variables, \( u, v : \mathbb{Z}^2 \rightarrow \mathbb{C} \cup \{\infty\} \) are dependent variables and \( \hat{,} \) denote shifts on these variables in the two lattice directions. Such Bäcklund transformations connect quadrilateral lattice equations in \( u \) and \( v \),

\[
P(u, \hat{u}, \hat{v}, \hat{v}) = 0, \quad P^*(v, \hat{v}, \hat{u}, \hat{u}) = 0,
\]  

(3.2)

where \( P \) and \( P^* \) are also autonomous polynomials of degree one in four variables. The key property we assume is the consistency of the polynomials when associated with the faces of a cube as described in [17, 18]—the polynomials \( P \) and \( P^* \) here lie on opposite faces. Choosing \( u \) to satisfy the equation on the left in (3.2) results in the compatibility of (3.1) as a system for \( v \), and the function \( v \) which emerges as the solution of this system then satisfies the equation on the right in (3.2).

1 This generalizes the transformation exploited in [16] which corresponds to the particular choice \( w = \infty \).
In the theory of such systems developed in [4, 5] the following polynomials play a fundamental role:

\[
\begin{align*}
\mathcal{H}_1 &:= (\partial_h B_1)(\partial_h B_1) - (\partial_h B_2)(\partial_h B_2)B_1, \\
\mathcal{H}_2 &:= (\partial_h B_2)(\partial_h B_2) - (\partial_h B_1)(\partial_h B_1)B_2,
\end{align*}
\]

(3.3)

The first two of these arise in the non-degeneracy condition for (3.1) as Riccati equations for \(v\):

\[
\mathcal{H}_1(u, \tilde{u}) \neq 0, \quad \mathcal{H}_2(u, \tilde{u}) \neq 0.
\]

(3.4)

If these conditions hold the solution \(u\) is said to be non-singular\(^2\).

If \(u\) is a non-singular solution of the equation on the left in (3.2), then the Riccati equations for \(v\) in (3.1) can be reformulated as the equivalent system

\[
\tilde{s} = r_1 \cdot s, \quad \tilde{s} = r_2 \cdot s.
\]

(3.5)

for a function \(s : \mathbb{Z}^2 \to M\). Here \(r_1, r_2 : \mathbb{Z}^2 \to \mathbb{M}\) are the Möbius transformations \(v \mapsto \tilde{v}\) and \(v \mapsto \tilde{v}^\dagger\) defined by (3.1), which are naturally associated with the edges of the quadrilateral lattice. The values of \(s\) are associated with the vertices. Compatibility of (3.5) means simply that \(r_1 \cdot r_2 = r_2 \cdot r_1\). This reformulation of the Bäcklund transformation allows the definition of a Schwarzian variable \(\psi\) associated with a non-singular solution of the equation defined by \(P\):

**Definition 1.** Let \(w : \mathbb{Z}^2 \to \mathbb{C} \cup \{\infty\}\) be a fixed function and \(u : \mathbb{Z}^2 \to \mathbb{C} \cup \{\infty\}\) be a non-singular solution of the equation on the left in (3.2). We refer to \(\psi\) as the Schwarzian variable associated with \(u\) if \(\psi = s^{-1}(w)\) for some \(s : \mathbb{Z}^2 \to M\) satisfying the Bäcklund system (3.5). We denote the set of all such pairs of functions \((u, \psi)\) by \(\mu\).

The solution of (3.5) is unique up to the transformation \(s \mapsto s \cdot m\) for arbitrary constant (i.e., independent of lattice position) \(m \in M\). Thus if \((u, \psi) \in \mu\), then \((u, \psi') \in \mu\) if and only if there exists \(m \in M\) such that \(\psi' = m(\psi)\). Although it is immediate from the definition, this is a key property of the relation \(\mu\), in particular it has the consequence that

\[
(u', \psi), (u, \psi), (u, \psi') \in \mu \Rightarrow (u', \psi') \in \mu.
\]

(3.6)

Property (3.6) is a generalization of transitivity to the situation where the domain of a relation is different than the co-domain (as it is for the relation \(\mu\)). It induces a partition on both domain and co-domain, and the relation lifts to a bijection between the equivalence classes. In the co-domain this is simply partition by Möbius equivalence, whereas any partition induced on the domain reveals a kind of symmetry of the equation defined by polynomial \(P\).

Based on the established Möbius invariance of the co-domain of \(\mu\), we will further investigate this co-domain using the cross-ratio. The cross-ratio of four values of \(\psi\) around an elementary quadrilateral is found to be

\[
\frac{(\psi - \tilde{\psi})(\tilde{\psi} - \hat{\psi})}{(\psi - \hat{\psi})(\tilde{\psi} - \hat{\psi})} = \frac{(|\tilde{r}_2 \cdot r_1|(w) - \tilde{r}_2(\tilde{u}))(|\tilde{r}_1(\tilde{u}) - \hat{u})}{(|r_2 \cdot r_1|(w) - r_2(\tilde{u}))(|r_1(\tilde{u}) - \hat{u})}.
\]

(3.7)

immediately from definition 1. More generally, Möbius-invariant constraints on \(\psi\) can be obtained by considering the cross-ratio of values on any set of four distinct vertices of the lattice, the vertices of a single quadrilateral considered in (3.7) are the most primitive such set. Using the definition of \(r_1\) and \(r_2\) (3.7) evaluates to

\[
\frac{(\psi - \tilde{\psi})(\tilde{\psi} - \hat{\psi})}{(\psi - \hat{\psi})(\tilde{\psi} - \hat{\psi})} = \chi \frac{B_1(u, \tilde{u}, w, \tilde{w})B_3(\tilde{u}, \hat{u}, \tilde{w}, \hat{w})}{B_2(u, \tilde{u}, w, \tilde{w})B_3(\tilde{u}, \hat{u}, \tilde{w}, \hat{w})}.
\]

(3.8)

\(^2\) The definition of singularity here, which is a singularity of the Bäcklund transformation, differs from the definition of singularity of the equation in [5], however for most examples these notions coincide.
where $\chi$ is independent of $w$. $\chi$ is easily found as a rational expression in the coefficients of $r_1$, $r_2, t_1$ and $t_2$. There are several ways to express it because the coefficients are themselves related due to the compatibility $t_1 \cdot r_2 = t_2 \cdot r_1$. Interestingly only its square seems to be expressible symmetrically, and is found to be

$$\chi^2 = \frac{H_2(u, \hat{u})H_2(\hat{u}, \hat{\hat{u}})}{H_1(u, \hat{u})H_1(\hat{u}, \hat{\hat{u}})}.$$  

(3.9)

The expression (3.8) is rather important, it will motivate the choice of the so-far arbitrary auxiliary function $w$ appearing in definition 1. In particular, the reducibility of the four terms on the right-hand side of (3.8) when they are considered as polynomials in the four variables $u, \hat{u}, \hat{\hat{u}}$ and $\hat{\hat{u}}$, is guaranteed if we choose $w$ to satisfy

$$H_1^*(w, \hat{w}) = 0, \quad H_2^*(w, \hat{w}) = 0,$$

(3.10)

where the polynomials here were defined in (3.3). This leads to the right-hand side of (3.8) being greatly simplified which is an important additional possibility for the Schwarzian construction. Before seeing how this works in practice we show how the construction has a natural interplay with the multidimensional consistency.

4. Extension to higher dimensions

In most cases the Bäcklund transformation described in the previous section embeds naturally in a three-dimensional system:

$$B_1(u, \hat{u}, v, \hat{v}) = 0, \quad B_2(u, \hat{u}, v, \hat{v}) = 0, \quad B_3(u, \hat{u}, v, \hat{v}) = 0,$$

(4.1)

where now $u, v : \mathbb{Z}^3 \to \mathbb{C} \cup \{\infty\}$ and $\cdot, \hat{\cdot}, \hat{\hat{\cdot}}$ denote shifts in the three lattice directions. Such extended Bäcklund transformations connect systems in $u$ and $v$:

$$P_{12}(u, \hat{u}, \hat{\hat{u}}, \hat{\hat{\hat{u}}}) = 0, \quad P_{12}^*(v, \hat{v}, \hat{\hat{v}}, \hat{\hat{\hat{v}}}) = 0,$$

$$P_{23}(u, \hat{u}, \hat{\hat{u}}, \hat{\hat{\hat{u}}}) = 0, \quad P_{23}^*(v, \hat{v}, \hat{\hat{v}}, \hat{\hat{\hat{v}}}) = 0,$$

$$P_{31}(u, \hat{u}, \hat{\hat{u}}, \hat{\hat{\hat{u}}}) = 0, \quad P_{31}^*(v, \hat{v}, \hat{\hat{v}}, \hat{\hat{\hat{v}}}) = 0,$$

(4.2)

where $P_{12}$ and $P_{12}^*$ are just a relabelling of $P$ and $P^*$ from before. The property required is the consistency of all these polynomials on a four-dimensional hypercube.

The generalization of the Schwarzian variable to this higher dimensional situation is fairly obvious, we give the details here in this section to demonstrate this. The generalized definition will be important in order to establish the connection between the two-dimensional systems and the three-dimensional lattice Schwarzian KP equation.

We define polynomials $H_3$ and $H_3^*$ in the natural way extending (3.3),

$$H_3 := (\partial B_3)(\partial B_3) - (\partial B_3 \partial B_3)B_3,$$

$$H_3^* := (\partial B_3)(\partial B_3) - (\partial B_3 \partial B_3)B_3.$$  

(4.3)

A solution of the system on the left in (4.2) is then said to be non-singular if

$$H_1(u, \hat{u}) \neq 0, \quad H_2(u, \hat{u}) \neq 0, \quad H_3(u, \hat{u}) \neq 0$$

(4.4)

throughout $\mathbb{Z}^3$. If $u$ is a non-singular solution of the system on the left in (4.2), then the equations (4.1) for $v$ can be reformulated as the equivalent system

$$\hat{s} = r_1 \cdot s, \quad \hat{s} = r_2 \cdot s, \quad \hat{s} = r_3 \cdot s$$

(4.5)
for a function \( s : \mathbb{Z}^3 \to M \). This reformulation of the Bäcklund equations leads naturally to the generalized definition of the Schwarzian variable:

**Definition 2.** Let \( w : \mathbb{Z}^3 \to \mathbb{C} \cup \{ \infty \} \) be fixed and \( u : \mathbb{Z}^3 \to \mathbb{C} \cup \{ \infty \} \) be a non-singular solution of the system on the left in (4.2). We refer to \( \varphi \) as the Schwarzian variable associated with \( u \) if \( \varphi = s^{-1}(w) \) for some \( s : \mathbb{Z}^3 \to M \) satisfying the Bäcklund system (4.5).

The extension to higher dimensions is clear. In other words we have shown that the Schwarzian-variable construction is compatible with the multidimensional consistency.

### 5. The Q4 Schwarzian variable

In [5] it is shown that the generic non-degenerate compatible system of polynomials (3.1), (3.2) may be taken, without loss of generality, in the form

\[
B_1 = Q_{a,k}, \quad B_2 = Q_{\beta,k}, \quad P = P^* = Q_{a,\beta},
\]

where \( Q_{a,\beta} \) is the polynomial

\[
Q_{a,\beta}(u, \tilde{u}, \hat{u}, \hat{\tilde{u}}) := \text{sn}(\alpha)
\left(u\tilde{u} + \hat{u}\hat{\tilde{u}}\right) - \text{sn}(\beta)
\left(u\hat{\tilde{u}} + \hat{u}\tilde{\hat{u}}\right)
-
\text{sn}(\alpha - \beta)
\left(u\hat{\tilde{u}} + \hat{u}\hat{\tilde{u}} - k\text{sn}(\alpha)\text{sn}(\beta)\left(1 + u\hat{\tilde{u}}\right)\right).
\]

Thus the polynomials around the cube differ only in the value of parameters, these parameters enter the polynomial through the Jacobi elliptic function \( \text{sn} \) with modulus \( k \). This is the Q4 system in its Jacobi parameterization. It was originally discovered in [19] as the superposition principle for Bäcklund transformations of the Krichever–Novikov equation [20], the Jacobi parameterization was obtained in [6].

We will consider the Schwarzian variable of the Q4 system with the fixed function \( w \) taken to be in the very particular form

\[
w = \sqrt{k}\text{sn}(\zeta), \quad \zeta = \zeta_0 + n\alpha + m\beta.
\]

Here \( n, m \in \mathbb{Z} \) are the independent variables, they are incremented by the shifts \( \cdot \) and \( ^\cdot \) respectively. This function is the singular solution of (5.2), it satisfies the system (3.10), which leads to a Schwarzian variable associated with Q4 that satisfies the following.

**Proposition 1.** If (3.1), (3.2) is the Q4 system (5.1) and \( w \) is the fixed function given in (5.3), then the following equation holds between a non-singular solution \( u \) of the equation on the left in (3.2) and its associated Schwarzian variable \( \varphi \):

\[
\frac{(\varphi - \hat{\varphi})(\hat{\varphi} - \hat{\tilde{\varphi}})}{(\varphi - \hat{\varphi})(\hat{\varphi} - \hat{\tilde{\varphi}})} = \frac{pq(1 + w\tilde{w}sp)(1 + \hat{\tilde{u}}\hat{w}sp)(\hat{\tilde{u}} - \hat{\tilde{w}})(\hat{\tilde{u}} - \hat{\tilde{w}})}{qp(1 + w\tilde{w}sq)(1 + \hat{\tilde{u}}\hat{w}sq)(\hat{\tilde{u}} - \hat{\tilde{w}})(\hat{\tilde{u}} - \hat{\tilde{w}})},
\]

in which we have used the notation

\[
p = \sqrt{k}\text{sn}(\alpha), \quad q = \sqrt{k}\text{sn}(\beta), \quad s = \sqrt{k}\text{sn}(\kappa), \quad \tilde{\varphi} = \sqrt{k}\text{sn}(\zeta + \kappa).
\]

**Proof.** It is straightforward to verify that substituting the definition of \( B_1 \) and \( B_2 \) from (5.1) into the right-hand side of (3.8) leads to

\[
\frac{(\varphi - \tilde{\varphi})(\hat{\varphi} - \hat{\tilde{\varphi}})}{(\varphi - \hat{\tilde{\varphi}})(\hat{\varphi} - \hat{\tilde{\varphi}})} = \frac{qq Q_{a,k}(u, \tilde{u}, w, \tilde{w})Q_{a,k}(\hat{u}, \hat{\tilde{u}}, \hat{\tilde{w}})}{pp Q_{\beta,k}(u, \tilde{u}, w, \tilde{w})Q_{\beta,k}(\hat{u}, \hat{\tilde{u}}, \hat{\tilde{w}})}.
\]
The particular choice (5.3) leads to
\[ Q_{\alpha,\kappa}(u, \bar{u}, w, \tilde{w}) = p(1 + w\tilde{w}sp)(u - \bar{w})(\bar{u} - \tilde{w}), \]
\[ Q_{\beta,\kappa}(u, \bar{u}, w, \tilde{w}) = q(1 + w\tilde{w}sq)(u - \bar{w})(\bar{u} - \tilde{w}), \]
which together with their once-shifted versions can be substituted into (5.5) resulting in (5.4).

Thus (5.4) is how (3.7) looks in the case of the Q4 system with \( w \) chosen as (5.3). Note that due to the particular choice of \( u, \) \( u \) only appears in (5.4) through values on two opposite vertices of the quadrilateral.

The natural extension of the Q4 system to higher dimensions (4.1), (4.2) is obtained by complementing (5.1) with the associations
\[ B_3 = Q_{\gamma,\kappa}, \quad P_{23} = P_{23}^\kappa = Q_{\beta,\gamma}, \quad P_{31} = P_{31}^\kappa = Q_{\gamma,\alpha}. \]

Whilst the extension of \( w \) to three dimensions is
\[ w = \sqrt{k} \text{sn}(\zeta), \quad \zeta = \zeta_0 + n\alpha + m\beta + l\gamma, \]
where \( l \in \mathbb{Z} \) is the third independent variable, which is incremented by the \( \bar{\gamma} \) shift. This higher dimensional consideration of the Schwarzian variable associated with the Q4 equation with \( w \) taken to be the singular solution results in the following.

**Proposition 2.** If \( \phi \) is the Schwarzian variable associated with the three-dimensional Q4 system (5.1), (5.7) and fixed function \( w \) given in (5.8), then it satisfies the equation
\[ \frac{(\phi - \hat{\phi})(\phi - \tilde{\phi})(\hat{\phi} - \tilde{\phi})}{(\phi - \hat{\phi})(\phi - \tilde{\phi})(\hat{\phi} - \tilde{\phi})} = 1. \]

**Proof.** Complement (5.4) with similar relations from the other pairs of lattice directions:
\[ \frac{(\phi - \hat{\phi})(\phi - \tilde{\phi})}{(\phi - \hat{\phi})(\phi - \tilde{\phi})} = \frac{qr(1 + w\tilde{w}s\bar{r})(1 + w\tilde{w}s\bar{r})(u - \bar{w})(\bar{u} - \tilde{w})}{\bar{r}q(1 + w\tilde{w}s\bar{r})(1 + w\tilde{w}s\bar{r})(u - \bar{w})(\bar{u} - \tilde{w})}. \]
\[ \frac{(\phi - \hat{\phi})(\phi - \tilde{\phi})}{(\phi - \hat{\phi})(\phi - \tilde{\phi})} = \frac{rp(1 + w\tilde{w}s\bar{r})(1 + w\tilde{w}s\bar{r})(u - \bar{w})(\bar{u} - \tilde{w})}{\bar{r}p(1 + w\tilde{w}s\bar{r})(1 + w\tilde{w}s\bar{r})(u - \bar{w})(\bar{u} - \tilde{w})}. \]

They can be found by cyclic permutation, we have used the further notation
\[ r = \sqrt{k} \text{sn}(\gamma), \quad r = \sqrt{k} \text{sn}(\gamma - \kappa). \]
The left-hand side of the product of (5.4), (5.10) and (5.11) is exactly the left-hand side of (5.9). The right-hand side of this product is
\[ \frac{(1 + \tilde{w},\tilde{w},s\tilde{p})(1 + \tilde{w},\tilde{w},s\tilde{q})(1 + \tilde{w},\tilde{w},s\bar{r})}{(1 + \tilde{w},\tilde{w},s\tilde{p})(1 + \tilde{w},\tilde{w},s\tilde{q})(1 + \tilde{w},\tilde{w},s\bar{r})} \]
which is equal to 1. This is an elliptic function identity which is a consequence of the addition formula for the Jacobi sn function.

The right-hand side of (5.9) is equal to 1, in particular it is independent of \( u, \) which is due to the special choice of \( w. \) The three-dimensional equation (5.9) is very well known, it is the lattice Schwarzian KP equation. It was first identified as the discrete analogue of the Schwarzian KP equation in [21], and is gauge-equivalent to the earlier equation given in [10] for generic values of the parameters. A geometric-incidence interpretation and connection to the Hirota-Miwa equation was established in [22].

A natural characterization of lattice Schwarzian KP solutions which are also a Q4 Schwarzian variable is at this time an open question.

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3 This gauge-equivalence was independently known to G. W. R. Quispel.
It is clear from (6.2) that the Schwarzian variable of the Q3 non-singular solution of the equation on the left in (3.2), \( \phi \) variable.

The remaining systems classified in [4] are all degenerate cases of the Q4 system. They share the feature that polynomials around the cube differ only in values of parameters, but with the difference that the parameters appear rationally, we denote them by \( \delta \). The scenario for systems defined by the remaining polynomials in table 1 is slightly different. This is due to the nature of system (3.10), which in these cases admits solutions of a quite different character. However amongst the solutions of system (3.10) there do not appear any that are consistent with the Q4 system. For this reason we list the polynomials in table 2.

### Table 1. The polynomials listed in [4] which define consistent systems on a cube, where it appears \( \delta \in \mathbb{C} \) is a constant parameter.

| \( Q_{p,q} (\dot{u}, \ddot{u}, \hat{u}) \) | \( Q^3 \) | \( (p - 1/p)(u\ddot{u} + \dddot{u}u) - (q - 1/q)(u\dddot{u} + \dddot{u}u) \) 
| --- | --- | --- |
| | \( Q^2 \) | \( p(u - \ddot{u}) \dot{u} - q(u - \dddot{u}) \dot{u} \) 
| | \( Q^1 \) | \( p(u - \ddot{u}) \dot{u} - q(u - \dddot{u}) \dot{u} + \delta^2 p/q \) 
| | \( H^3 \) | \( p(u\ddot{u} + \dddot{u}u) - q(u\dddot{u} + \dddot{u}u) + \delta(p^2 - q^2) \) 

### Table 2. Data relevant to proposition 3.

| \( w \) | \( \psi \) | \( \psi \) | \( A(p) \) |
| --- | --- | --- | --- |
| \( Q^3 \) | \( \xi + \delta^3/\xi^2 \), \( \xi = \xi_0 p^n q^m \) | \( w(\xi) \) | \( w(\xi/s) \) | \( (p - 1/p)/(p/s - s/p) \) |
| \( Q^2 \) | \( \xi^2 \), \( \xi = \xi_0 + np + mq \) | \( w(\xi/s) \) | \( w(\xi - s) \) | \( p/(p - s) \) |
| \( Q^1 \) | \( u_0 + n\delta p + m\delta q \) | \( w + \delta s \) | \( w - \delta s \) | \( p/(p - s) \) |

### 6. The degenerate sub-cases of Q4

The remaining systems classified in [4] are all degenerate cases of the Q4 system. They share the feature that polynomials around the cube differ only in values of parameters, but with the difference that the parameters appear rationally, we denote them by \( p, q \) and \( s \).

\[
B_1 = Q_{p,s}, \quad B_2 = Q_{q,s}, \quad P = P^* = Q_{p,q}.
\]

The polynomials and the names they were given in [4] are reproduced here in table 1. The ‘type-Q’ (Q3*, Q2 and Q1*) in table 1) polynomials are the most similar to Q4. For these systems the analogue of proposition 1, which is proven by a similar calculation, is as follows:

**Proposition 3.** Suppose the polynomials in (3.1) and (3.2) are given by (6.1) where \( Q_{p,q} \) is the polynomial Q3*, Q2 or Q1* listed in table 1. Then the following equation holds between a non-singular solution of the equation on the left in (3.2), \( u \), and its associated Schwarzian variable, \( \varphi \):

\[
\frac{(\varphi - \dot{\varphi})(\ddot{\varphi} - \ddot{\varphi})}{(\varphi - \dot{\varphi})(\ddot{\varphi} - \ddot{\varphi})} = \frac{A(p)}{A(q)} \frac{(\dddot{u} - \ddot{u})(\dot{u} - \ddot{u})}{(\dddot{u} - \ddot{u})(\dot{u} - \ddot{u})}
\]

where the corresponding choice of function \( w \), the meaning of notation \( \dot{w} \) and \( \ddot{w} \), and the function \( A \) are listed in table 2.

It is clear from (6.2) that the Schwarzian variable of the Q3*, Q2 and Q1* systems extended to three dimensions also satisfies (5.9).

The scenario for systems defined by the remaining polynomials in table 1 is slightly different. This is due to the nature of system (3.10), which in these cases admits solutions of a quite different character. However amongst the solutions of system (3.10) there do...
appear constant or oscillating\(^4\) functions, and these result in the right-hand side of (3.7) being constant:

**Proposition 4.** Suppose the polynomials in (3.1) and (3.2) are given by (6.1) where \(Q_{p,q}\) is one of the polynomials listed in table 1. Also suppose the functions \(w\) and \(A\) are those corresponding to that polynomial in table 3. Then the associated Schwarzian variable satisfies

\[
\frac{(\psi - \tilde{\psi})(\hat{\psi} - \tilde{\hat{\psi}})}{(\psi - \hat{\psi})(\hat{\psi} - \tilde{\hat{\psi}})} = A(p)A(q). \tag{6.3}
\]

Equation (6.3) may be written as \(Q_{A(p),A(q)}(\psi, \hat{\psi}, \tilde{\psi}, \tilde{\hat{\psi}}) = 0\) where \(Q_{p,q}\) is the polynomial \(Q_{10}\) appearing in table 1, this is the lattice Schwarzian KdV equation. It was first identified as the discrete analogue of the Schwarzian KdV equation in [12] and is a parameter sub-case of the earlier NQC equation [9]. A geometric interpretation connected with discrete conformal maps was given in [18], and the symmetries of this equation were analysed in [23]. The reduction from the lattice Schwarzian KP equation to equation (6.3) was characterized in terms of the geometric-incidence picture in [22], this reduction can also be characterized in terms of the lattice Schwarzian KP Bäcklund transformation as a two-cycle solution, which was shown in [24].

Importantly proposition 4 applies to every polynomial in table 1, the type-Q polynomials recur here but we have made a restricted choice for the function \(w\). The reason for including them again is because of a stronger result which applies: loosely speaking, (6.3) actually characterizes the co-domain of \(\mu\). This result is most simply formulatied by complementing the transformation to the lattice Schwarzian KdV equation described in proposition 4 with explicit equations for the inverse transformation. As indicated in section 2 it is fairly convenient to proceed by writing

\[
s(1) = v_1, \quad s(\infty) = v_2, \quad s(0) = v_3, \tag{6.4}
\]

\(^4\) By the term oscillating here and subsequently we mean a function which takes only two distinct values depending on whether \(n + m\) is even or odd.
so that rather than \( s \) itself we deal with the three scalar functions \( v_1, v_2, v_3 : \mathbb{Z}^2 \to \mathbb{C} \cup \{\infty\} \) in terms of which the inverse of \( s \) is given by (2.5). The forward transformation to the Schwarzian variable \( \varphi \) can then be written

\[
B_1(u, \hat{u}, v_1, \hat{v}_1) = 0, \quad B_2(u, \hat{u}, v_1, \hat{v}_1) = 0, \quad i \in \{1, 2, 3\}.
\]

(6.5)

That is, obtaining first three particular solutions \( v_1, v_2 \) and \( v_3 \) of the Bäcklund system (3.1) which corresponds to the first part of system (6.5), and then constructing \( \varphi \) rationally from them as in the second part. Elimination of \( v_1 \) from (6.5) results in the following explicit system for the inverse transformation:

\[
(v_3 - w)B_1(u, \hat{u}, v_2, \hat{w})\varphi - (v_2 - w)B_1(u, \hat{u}, v_3, \hat{w})\varphi = 0,
\]

\[
B_1(u, \hat{u}, v_2, \hat{v}_2) = 0, \quad B_1(u, \hat{u}, v_3, \hat{v}_3) = 0,
\]

(6.6)

\[
(v_3 - w)B_2(u, \hat{u}, v_2, \hat{w})\varphi - (v_2 - w)B_2(u, \hat{u}, v_3, \hat{w})\varphi = 0,
\]

\[
B_2(u, \hat{u}, v_2, \hat{v}_2) = 0, \quad B_2(u, \hat{u}, v_3, \hat{v}_3) = 0.
\]

(6.7)

**Proposition 5.** Make again the suppositions of proposition 4. If \( \varphi \) is a non-singular solution of (6.3) then the systems (6.6) and (6.7) define compatible bi-rational mappings \( (u, v_2, v_3) \mapsto (\hat{u}, \hat{v}_2, \hat{v}_3) \) and \( (u, v_2, v_3) \mapsto (\hat{u}, \hat{v}_2, \hat{v}_3) \). If \( u, v_2 \) and \( v_3 \) are functions determined by these mappings and we define a further function \( v \) by the equation

\[
\varphi = \frac{(v - v_2)(v_3 - w)}{(v - v_3)(v_2 - w)}, \quad c \in \mathbb{C} \cup \{\infty\},
\]

(6.8)

then \( (u, v) \) satisfy (3.1), (3.2), and the Schwarzian variable associated with \( u \) is \( \varphi \).

Thus system (6.6), (6.7) re-constructs \( u \) from its Schwarzian variable \( \varphi \). The proof of proposition 5 is a straightforward case-by-case calculation. The non-singularity condition is required for the mappings to be bi-rational. By examination of (3.4) for the lattice Schwarzian KdV equation (6.3), the non-singularity just means that \( \varphi \) takes distinct values on any pair of adjacent vertices.

The first equations of (6.6) and (6.7) are just (2.8) when \( R \) is the polynomial defining the Riccati-type Bäcklund equations (3.1). In writing the inverse transformation explicitly we have made several choices. First, there is some preference in lattice orientation, the mappings described are in the direction of forward lattice shifts, however by inspection one sees that the same system yields mappings in the other directions by a simple re-arrangement (they are bi-rational). Second, we have chosen to write the transformation as coupled rank-3 mappings (coupled first-order systems), elimination of \( v_2 \) or \( v_2 \) and \( v_3 \) would lead to rank-2 or scalar equations respectively, but they would be on a larger lattice stencil. Clearly applying the inverse transformation described yields intermediary functions \( v_2 \) and \( v_3 \). The re-construction of \( v_1 \) from \( v_2, v_3 \) and \( \varphi \) is immediate from the last equality in (6.5), the equation for \( v \) (6.8) is obtained by writing \( v = s(c) \).

Consider an example. In the particular case of polynomial \( H1 \) in table 1 the transformation described in propositions 4 and 5 connects the pair of equations

\[
\frac{(\varphi - \tilde{\varphi})(\tilde{\varphi} - \hat{\varphi})}{(\varphi - \hat{\varphi})(\hat{\varphi} - \tilde{\varphi})} = \frac{q - s}{p - s}, \quad (u - \hat{u})(\tilde{u} - \hat{u}) = p - q.
\]

(6.9)

First of all suppose that \( u \) satisfies the equation on the right in (6.9). If \( v_1, v_2 \) and \( v_3 \) are three particular solutions obtained from \( u \) by the natural auto-Bäcklund transformation of this equation, with Bäcklund parameter \( s \), then \( \varphi \) defined by

\[
\varphi = \frac{(v_1 - v_2)(v_3 - \infty)}{(v_1 - v_3)(v_2 - \infty)} = \frac{v_1 - v_2}{v_1 - v_3}.
\]

(6.10)
satisfies the equation on the left of (6.9). Conversely, suppose that \( \phi \) satisfies the equation on the left in (6.9). The mappings defined by (6.6) and (6.7) for this example are as follows:

\[
\begin{align*}
\tilde{u} &= \frac{\hat{\phi}v_2 - \hat{\phi}v_3}{\hat{\phi} - \phi}, & \tilde{v}_2 &= u + \frac{(p-s)(\phi - \hat{\phi})}{\phi(v_2 - v_3)}, & \tilde{v}_3 &= u + \frac{(p-s)(\phi - \hat{\phi})}{\phi(v_2 - v_3)}, \\
\tilde{u} &= \frac{\hat{\phi}v_2 - \hat{\phi}v_3}{\hat{\phi} - \phi}, & \tilde{v}_2 &= u + \frac{(q-s)(\phi - \hat{\phi})}{\phi(v_2 - v_3)}, & \tilde{v}_3 &= u + \frac{(q-s)(\phi - \hat{\phi})}{\phi(v_2 - v_3)}.
\end{align*}
\]

(6.11)

These three-component mappings commute due to the supposition that \( \phi \) satisfies the equation on the left in (6.9), and their solution yields three functions \( u, v_2 \) and \( v_3 \) each of which satisfies the equation on the right in (6.9). Furthermore the solutions \( v_2 \) and \( v_3 \) are related to \( u \) by the natural auto-Bäcklund transformation of this equation with Bäcklund parameter \( s \). And using (6.8) functions \( v_2 \) and \( v_3 \) can also be combined rationally with \( \phi \) to obtain a function \( v \),

\[
v = \frac{cv_2 - \hat{\phi}v_3}{c - \phi},
\]

(6.12)

which is the most general solution of the equation on the right in (6.9) that can be obtained by the auto-Bäcklund transformation (with parameter \( s \)) applied to the constructed solution \( u \).

A nice feature of this example is that the equivalence relation induced by \( \mu \) (see the earlier discussion of property (3.6)) on the set of non-singular solutions of the equation on the right in (6.9) is also a local symmetry:

\[
u \rightarrow (a + 1/a)(u + b)/2 + (-1)^m(a - 1/a)(u + c)/2, \quad a \neq 0, b, c \in \mathbb{C}.
\]

(6.13)

In particular, the solution \( u \) obtained from \( \phi \) is unique up to the transformation (6.13), whilst the solution \( \psi \) obtained from \( u \) is unique up to transformations \( \phi \rightarrow m(\phi) \) for constant \( m \in \mathbb{M} \). Thus the transformation to the Schwarzian variable is a bijective correspondence between non-singular solutions of the two equations in (6.9) modulo the Möbius group and the symmetry group (6.13) respectively.

Glancing at table 3 it can be seen that propositions 4 and 5 connect all of the integrable equations classified in [4], with the exception of Q4, to the lattice Schwarzian KdV equation (6.3). By analogy with the situation for continuous KdV-type equations discovered in [11, 25], a transformation between Q4 and the lattice Schwarzian KdV equation should not be expected. Here this is connected to the non-existence of constant or oscillating singular solutions of Q4. To see this consider that independence from \( u \) of the right-hand side of (5.4) requires \( \dot{w} = \psi \). For other type-Q equations this constraint can be solved simultaneously with the definition of \( w \) and independently from the choice of Bäcklund parameter resulting in the choice of \( w \) listed in table 3 which brings (6.2) to (6.3). In the case of Q4 the condition \( \dot{w} = \psi \) reads \( \text{sn}(\zeta + \kappa) = \text{sn}(\zeta - \kappa) \) which is satisfied only if \( \kappa \in \{0, 2K, iK', 2K + iK'\} \). This is not a valid choice of the Bäcklund parameter because it results in reducibility of polynomials \( B_1 \) and \( B_2 \) that define the Bäcklund transformation (3.1).

As we have described in section 4, the transformation to the Schwarzian variable is compatible with the multidimensional consistency. In the cases where the transformation connects to the lattice Schwarzian KdV equation, as in propositions 4, 5 (and proposition 6 later on) this implies commutativity with the natural auto-Bäcklund transformation as illustrated in figure 1.

In the case of the fifth entry in table 3, the propositions 4 and 5 define an auto-transformation of the lattice Schwarzian KdV equation. Observe however that \( A(p) \neq p \), therefore the transformation connects instances of the same equation, but with different values of its

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5 We are grateful to S Butler for pointing out this symmetry, which can be deduced from the soliton solution given in [14].
The Schwarzian variable associated with discrete KdV-type equations

Figure 1. Arrows to the right correspond to the transformation to the Schwarzian variable, the arrows downwards correspond to the natural auto-Bäcklund transformation of the equation for variable $u$ (left) and the lattice Schwarzian KdV equation (right). The labels on arrows are the Bäcklund parameters.

parameters. Also note that this transformation does not commute with the Möbius point symmetry of the equation. In fact the parameter $\theta$ of this transformation can be taken as $\infty$ without loss of generality by composition with such a point symmetry.

7. Less symmetric systems

We now consider the situation for systems of consistent polynomials of the kind described in section 3 with different symmetry than those in [4].

Several systems where $P \neq P^*$ in (3.2) were listed in [13], although the polynomials $P$ and $P^*$ are distinct, each can be taken as one of those listed in table 1. The Bäcklund equations themselves are of the form

$$B_1 = f_p, \quad B_2 = f_q,$$

(7.1)

and can be extended to higher dimensions by associating $B_3 = f_r$ and so on for some set of parameters $p, q, r \ldots$. Some but not all of these systems admit a choice of function $w$ such that the transformation to the Schwarzian variable is invertible in the same way as described in propositions 4 and 5.

**Proposition 6.** Suppose the polynomials in (3.1) are given by (7.1) where $f_p$ is one of the polynomials listed in table 4. Also suppose the polynomials $P, P^*$ and functions $w$ and $A$ are those corresponding to $f_p$ in the same table. Then the Schwarzian variable associated with a solution of the equation on the left in (3.2) satisfies (6.3). Conversely, if $\varphi$ is a non-singular solution of (6.3) then the systems (6.6) and (6.7) define compatible bi-rational mappings $(u, v_2, v_3) \mapsto (\tilde{u}, \tilde{v}_2, \tilde{v}_3)$ and $(u, v_2, v_3) \mapsto (\hat{u}, \hat{v}_2, \hat{v}_3)$. If $u, v_2$ and $v_3$ are functions determined by these mappings and we define a further function $v$ by (6.8) then $(u, v)$ satisfy (3.1), (3.2), and the Schwarzian variable associated with $u$ is $\varphi$.

Let us exhibit a simple example: the fifth entry in table 4. The transformation to the Schwarzian variable in this asymmetric case is most fully described as connecting the Schwarzian equation to the full system (3.1), (3.2):

$$\frac{(\varphi - \tilde{\varphi})(\hat{\varphi} - \tilde{\varphi})}{(\varphi - \hat{\varphi})(\tilde{\varphi} - \tilde{\varphi})} = \frac{p^2}{q^2}, \quad pe\hat{v} - uv - \hat{u}\hat{v} = \delta, \quad q\hat{v}\hat{u} - uv - \hat{u}\hat{v} = \delta,$$

$$p(u\hat{u} + \hat{u}\hat{u}) - q(u\hat{u} + \hat{u}\hat{u}) = \delta(q^2 - p^2),$$

$$p(v\hat{v} + \hat{v}\hat{v}) - q(v\hat{v} + \hat{v}\hat{v}) = 0.$$  

(7.2)
In this instance the mappings described in proposition 6 are as follows:

$$\tilde{u} = p \frac{v_2 \tilde{\varphi} - v_3 \varphi}{\tilde{\varphi} - \varphi}, \quad \tilde{v}_2 = \frac{\tilde{\varphi} - \varphi}{p\tilde{\varphi}(v_3 - v_2)}, \quad \tilde{v}_3 = \frac{\tilde{\varphi} - \varphi}{p\tilde{\varphi}(v_3 - v_2)},$$

$$\tilde{u} = q \frac{v_2 \tilde{\varphi} - v_3 \varphi}{\tilde{\varphi} - \varphi}, \quad \tilde{v}_2 = \frac{\tilde{\varphi} - \varphi}{q\tilde{\varphi}(v_3 - v_2)}, \quad \tilde{v}_3 = \frac{\tilde{\varphi} - \varphi}{q\tilde{\varphi}(v_3 - v_2)}.$$  \hfill (7.3)

The equation on the left of (7.2) arises as the condition for compatibility of the three-component mappings defined in (7.3), which themselves yield three functions $u$, $v_2$, and $v_3$ defined on $\mathbb{Z}^2$. The resulting solution $(u, v, \varphi)$ of the system on the right in (7.2) is obtained by determining $v$ rationally from $v_2$, $v_3$, and $\varphi$ using (6.8). Here the partition induced by $\mu$ on the set of non-singular solutions of the system on the right in (7.2) is not equivalent to a local symmetry.

We remark that for the fourth entry in table 4, proposition 6 defines an auto-transformation for the lattice Schwarzian KdV equation. However in this case $A(p) = p$ unlike the transformation of the previous section (of which the transformation here can be viewed as a degeneration).

Loosely speaking, the fact that the constraint on functions in the co-domain of $\mu$ is two-dimensional is what leads to the existence of the inverse transformation described in propositions 5 and 6. What tends to happen in more degenerate systems of compatible polynomials is that the transformation to the Schwarzian variable reduces dimensionality, specifically $\varphi$ is constant along some direction on the lattice, which leads to non-invertibility of the transformation.

As an example of this we take a system which looks very much like those listed in table 4. In fact we will consider again the fifth entry of table 4, but with the roles of $u$ and $v$ interchanged so that

$$f_p = puu - uv - u\tilde{v} - \delta, \quad P = H^3, \quad P^* = H^3.$$  \hfill (7.4)

We take the singular solution $w = \infty$ and again the right-hand side of (3.7) is constant and thus independent of the solution $u$, so (3.7) can be used to characterize the co-domain of $\mu$. But this constant is equal to 1. Examination of (3.7) reveals that this implies either $\tilde{\varphi} = \varphi$ or $\tilde{\varphi} = \tilde{\varphi}$, and in fact it turns out that both are true, thus $\varphi$ is an oscillating function and all information about the original solution $u$ is lost. Therefore the Bäcklund transformation defined by the fifth entry in table 4 can in this way be distinguished from its inverse (7.4), and in fact the same feature occurs in all other entries of this table except the fourth.

---

Table 4. Data relevant to proposition 6. Arbitrary constants $\varphi$ and $\varphi'$ are chosen from $\mathbb{C} \cup \{\infty\} \setminus \{0\}$ and $\mathbb{C} \cup \{\infty\}$, respectively. † indicates application of the point transformation $p \rightarrow p^2, q \rightarrow q^2$ to the parameters. The Bäcklund transformations here are a subset of those given originally in table 3 of [13].

| $f_p$ | $P$ | $P^*$ | $w$ | $A(p)$ |
|-------|-----|------|-----|--------|
| $(u - \tilde{v})(u - \tilde{u}) - p(2v\tilde{v} - u - \tilde{u}) - p^2(v + \tilde{v} + p)$ | $Q^0_3$ | $Q^0_1$ | $0^{+1}$ | $1 - p^{+2}$ |
| $(v - \tilde{v})(u - \tilde{u}) - p(2v\tilde{v} - u - \tilde{u}) - p^2(v + \tilde{v} + p)$ | $Q^1_0$ | $Q^1_1$ | $\infty$ | $p$ |
| $(v - \tilde{v})(u - \tilde{u}) - p(2v\tilde{v} - u - \tilde{u}) - p^2(v + \tilde{v} + p)$ | $Q^1_1$ | $Q^1_0$ | $\varphi'$ | $\varphi$ |
| $p\tilde{v} - uv - u\tilde{v} - \delta$ | $H^3_5$ | $H^3_0$ | $\infty$ | $p^2$ |
| $(v + \tilde{v})(u + \tilde{u}) - p(v\tilde{v} + \tilde{v}^2)$ | $A^1_0$ | $A^0_1$ | $(-1)^{+4n} \varphi' - p$ |
| $(v + \tilde{v})(u + \tilde{u}) - p(1 - v/2)(1 - \tilde{v}/2)$ | $H^3_1$ | $A^0_1$ | $(-1)^{+4n} \varphi' - p^2$ |
| $(v + \tilde{v})(u + \tilde{u}) - p(v - \tilde{v})$ | $Q^1_1$ | $A^0_1$ | $(-1)^{+4n} \varphi' - p$ |
A more obvious degenerate system is the linear multidimensionally consistent equation defined by the polynomial
\[ Q_{p,q}(u, \tilde{u}, \hat{u}, \hat{\tilde{u}}) = (p_1 - q_1)u - (p_2 - q_1)\tilde{u} - (p_1 - q_2)\hat{u} + (p_2 - q_2)\hat{\tilde{u}}, \tag{7.5} \]
whence the compatible system (3.1), (3.2) is defined by (6.1) (see [7]). Here the singular solution is \( w = \infty \) which leads to a Schwarzian variable that is constant.

Suffice it to say that for systems listed in [13] but which are not included in table 4 or gauge-related to systems in table 3, either \( \varphi \) is a constant or oscillating function (like in the examples above), or else (3.7) cannot be used to characterize the co-domain of \( \mu \) because the right-hand side depends on \( u \) (like in the case of Q4 described in section 5). It may be beneficial to apply the Schwarzian construction to more examples by considering the new systems of consistent polynomials (that also appear to be less symmetric around the cube) which were found recently in [8]. In particular the Schwarzian construction provides a method to transform such new systems into the better studied lattice Schwarzian KdV equation.

8. Conclusions

In this paper we have introduced a technique that contributes to the transformation theory of equations from the ABS list, and more generally to systems of multi-affine quad equations that are consistent around a cube.

For the primary such model, Q4, our construction provides a natural connection with the lattice Schwarzian KP equation (see propositions 1 and 2), specifically, from a generic solution of Q4 when embedded in three dimensions (see section 4) we obtain a solution of the lattice Schwarzian KP equation. Although such a connection should be expected, on the fully discrete level this is actually the first connection between Q4 and a lattice KP-type system. Our construction is however one-directional, it is still an open problem to find a natural characterization the lattice Schwarzian KP solutions which are also a Schwarzian variable associated with Q4.

For all other equations from the ABS list the associated Schwarzian variable satisfies the lattice Schwarzian KdV equation (see proposition 4), and in those cases a stronger result has been proved, specifically we provide also the inverse transformation (proposition 5). In fact, due to an intrinsic transitivity property of the transformation, property (3.6), the invertible transformations actually provide a one-to-one correspondence. It is not directly between solutions of the equations, rather it is between equivalence-classes of solutions defined by a (usually non-local) three-parameter symmetry group. This is a very strong characterization of the relationship between the equations and is one of the primary results of our paper.

The same construction for asymmetric consistent systems on the cube, which illustrates the first bounds on the applicability of this construction, is described in proposition 6. In fact the consistent systems for which this construction fails to provide a transformation to the discrete Schwarzian KdV model are exceptional, and therefore understanding this situation through more examples can also be of interest. For instance the construction has the potential of forming the basis of a classifying device to distinguish between asymmetric models from genuinely different classes.

At the other end of the spectrum, for the fully symmetric and non-degenerate case, no such transformation connecting Q4 to the lattice Schwarzian KdV equation should really be expected. The obstruction to obtaining such transformation through our method seems to be connected to the nature of the singular solutions of the models. The primary model Q4 is exceptional in that for all other models in the ABS list there exist constant or oscillating singular solutions, and those special solutions play a key role in construction of the transformation.
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