Gauge Invariance and the Critical Properties of Quantum Hall Plateaux Transitions

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Abstract

A model consisting of a single massless scalar field with a topological coupling to a pure gauge field is defined and studied. It possesses an $SL(2, \mathbb{Z})$ symmetry as a consequence of the gauge invariance. We propose that by adding impurities the model can be used to describe transitions between Quantum Hall plateaux. This leads to a correlation length exponent of $20/9$, in excellent agreement with the most recent experimental measurements.
I. INTRODUCTION

Theoretical studies of the Quantum Hall effect can roughly be classified either as studies of the many-body physics on a plateau (Laughlin wave function, etc.) or as studies of the plateau to plateau transitions. For a review of the latter see [1]. The transitions have been viewed as metal-insulator quantum phase transitions and viewed as such as problems in Anderson localization [2] [3]. The theoretical framework developed to study these problems involves the study of electrons in a disordered potential. Averaging over disorder using replicas leads to a sigma model in 2 dimensions [4].

In applying this framework to the Quantum Hall transitions important progress was made by Pruisken [5] [6] who understood that a topological term proportional to the Hall conductivity \( \sigma_{xy} \) was essential for obtaining the correct features of the phase diagram. For the supersymmetric version see [7]. The topological term modifies significantly the renormalization group and hence the infra-red properties, in the same way that the \( O(3) \) non-linear sigma model with topological coupling at \( \theta = \pi \) has a non-trivial infra-red fixed point [8].

Despite this progress, the detailed critical properties, such as the exponents, have so far remained uncomputable. This is due to the complexity of the sigma models obtained and the usual difficulty in finding non-trivial fixed points in the infra-red. It remains unclear whether the correct critical exponents are even contained in the sigma-models.

Progress has been made by inventing a network model which is believed to be in the same universality class as the transition [9]. Here the transition resembles percolation through the impurities. The network model was recently shown [10] to be equivalent to fermions with Dirac-like hamiltonian with various random potentials of the kind studied in [11]. For these models one is again unable to compute the critical exponents analytically due to the complexities introduced by disorder averaging and one has to resort to numerical methods.

On the other hand, very general and elegant arguments, based largely on gauge invariance, were given by Laughlin and Halperin to explain the very existence of the plateaux [12] [13]. The argument does not depend on any detailed properties of the disorder; it just has to be there to localize some states. Indeed, Halperin argued that from the gauge invariance alone, one can infer the existence of both localized and extended states; the argument does not involve finding a critical strength of the disorder for a localization/delocalization phase transition. What helps the situation is the fact that in the scaling approach to Anderson localization, two dimensions is distinguished. In principle, when there is no magnetic field, states in two dimensions are always localized, no matter what the strength of the disorder [13]. The most extreme conclusion to be reached based on these observations is that perhaps even the critical exponents governing the transitions between plateaux may follow largely from gauge invariance. It would indeed be very satisfying if both the existence of the plateaux and the critical properties of the transitions between plateaux were consequences of the same fundamental principle of gauge invariance. In this paper I will construct a model based on this notion. In other words, we will simply assume disorder is irrelevant. The issue of whether disorder is relevant or irrelevant in the renormalization group will be addressed in a separate publication [14].

In the approach developed here, in a sense I start from the end (the critical theory) rather than from the beginning. Namely, we construct a conformal theory that contains the essential features of the extended states. We then add impurities as a perturbation. What is
normally considered in the literature is the reverse, i.e. one starts from the localized states and looks for a delocalization transition. Critical theories in two dimensions are strongly constrained by conformal invariance [16], and the computation of the critical exponents should amount to the proper exercise in conformal field theory. This involves identifying the appropriate class of conformal field theory. Guided by the importance of gauge invariance, the significance of topological terms (the topological term in the sigma-model approach should survive in the infra-red since it is proportional to $\sigma_{xy}$), we construct a certain model of conformal field theory which possesses the necessary ingredients. The Hamiltonian corresponds to 2-component Dirac fermions. The definition of the model includes a constraint on the zero modes that incorporates an essential feature of the $2+1$ dimensional Quantum Hall dynamics. We view this model as describing the essential features of the extended states in a pure Quantum Hall system. After bosonizing the fermions and taking the gauge field to be a pure, but singular, gauge, we obtain a model with a topological coupling similar to the multi-boson models studied originally in the context of string theory [17] [18]. However there are some important differences. Whereas the simplest (smallest) model studied in [17] [18] has Virasoro central charge $c = 2$ (two scalar fields), the gauge invariance of our model effectively reduces the degrees of freedom to a $c = 1$ theory. This is a new class of conformal field theory characterized by an electric/magnetic $SL(2,\mathbb{Z})$ duality of the kind found by Cardy [19] which acts on the modular parameter $\tau = \theta/2\pi + ig^2/2$, where $g, \theta$ are couplings in our model. A number of works have postulated a role for $SL(2,\mathbb{Z})$ in the Quantum Hall transitions, based on the phenomenology of transition selection rules [20] [21] [22] [23] [24]. Though our microscopic model does not have some of the features assumed in these works, it may perhaps still be useful towards developing these ideas. Our model is easily studied without utilizing the full machinery of conformal field theory; the elementary tools we need are bosonization [25] and the properties of exponentials of massless scalar fields, reviewed in [26].

Starting from the Kubo formula in $2+1$ dimensions we derive some formulas for the conductivities $\sigma_{xx}, \sigma_{xy}$ which involve current-current correlation functions in the 2-dimensional quantum field theory. These formulas are different from the kind of formulas found in the literature, the latter being expressed as a double product of retarded and advanced 1-particle Green functions. We argue that our formulas represent the conformal contribution to the conductivity. In our model the conductivities are functions of $g, \theta$, and when $\sigma_{xx} = 0$, $\theta = 1/\sigma_{xy}$.

We then add an impurity potential as a perturbation away from the critical point. If, as is normally done, one takes a gaussian random potential, then according to the usual approach to Anderson localization, one must average over the disorder and search for a critical point corresponding to the critical strength of the disorder for a localization/delocalization phase transition. For the reasons given above, we chose the potential so that no disorder averaging is needed. In real experiments the critical exponents are measured for a single sample, whereas disorder averaging amounts to averaging over different realizations of disorder. If we assume that disorder is irrelevant, then universality implies that any impurity potential which breaks translational invariance should suffice. Hence, viewing the sample as a very

\[ \text{For a more comprehensive review see [27].} \]
large disk, we take the impurity potential to be a circle of impurities somewhere inside the disk. This Corbino disk geometry has essentially all of the ingredients used in the gauge argument of Halperin [13]. The virtue of doing this is that the critical properties do not depend on the strength of the impurity potential. Furthermore we are already at a fixed point before perturbation. Our resulting model can be solved by mapping it onto a boundary field theory, with the impurities residing at the boundary.

In the scaling theory developed in [28], the primary critical exponents are $\mu, \nu$ defined as

$$\Delta B \propto T^\mu, \quad \left(\frac{\partial \rho_{xy}}{\partial B}\right)_{\text{max}} \propto T^{-\mu} \quad (1.1)$$

and also

$$\xi_c \propto |B - B_c|^{-\nu} \quad (1.2)$$

Here, $\rho_{xy}$ is the resistivity, $B$ the magnetic field, $\Delta B$ the width of the region between plateaux, $B_c$ the critical value at a transition, $T$ the temperature and $\xi_c$ the correlation length. The exponents are related by $\mu = p/2\nu$, where $p$ is the inelastic scattering length exponent. In the limit of zero temperature, one expects $\mu = 1/\nu$. It has been shown in experiments that the exponents are independent of which plateaux are involved, including the fractional ones. As we will see, our model has this feature. We will also argue that our model predicts the value:

$$\nu = 20/9 \quad (1.3)$$

There has been a certain prejudice for the value $\nu = 7/3 = 21/9$, though the only analytical computation supporting this value [29] is based on the percolation picture of the network model [3], and furthermore has been criticized as perhaps not corresponding to the right physics [1]. Experimental measurements of $\nu = 2.4 \pm 0.1$ and $\nu = 2.3 \pm 0.1$ were reported in [30] and [31] respectively. If we are allowed to assume $\mu = 1/\nu$, then our result is rather close the most recent measurement of $\mu = .45 \pm .05$ reported in [22]. The numerical estimate with the smallest error, $\nu = 2.35 \pm .03$ is due to Huckestein and includes data from both random Landau matrix approach to an Anderson tight-binding model in a magnetic field and from the network model [3]. A variant of the network model [3] gives $\nu = 2.43 \pm .18$. Our value is closer to the result of Ando $\nu = 2.2 \pm .1$ obtained from transmission through a disordered system [34]. Indeed if the result (1.3) is actually correct then the consistently higher values for $\nu$ obtained numerically for the network/Anderson models suggests that our model is in a different universality class. Certainly it is clear from the very definitions of the models that they are not simply equivalent.

\[\nu = 7/3 = \nu_p + 1, \text{ where } \nu_p = 4/3 \text{ is the percolation exponent.}\]

\[3\text{The data from the random Landau matrices and from the network model were statistically indistinguishable. (B. Huckestein, private communication.)}\]
II. THE CRITICAL MODEL AND ITS CONFORMAL PROPERTIES

A. Definition of the Model

We consider fermions in two spatial dimensions with coordinates \( x, y \), which we often denote simply as \( x \). The constants \( e = c = \hbar = 1 \). The second quantized action one needs to study is

\[
S_{2+1} = \int dt d^2x \; \bar{\Psi} (i \partial_t - H) \Psi
\]

where \( H \) is the hamiltonian. For the purpose of studying disorder in the \( x - y \) plane arising from a disordered potential in \( H \), it is convenient to work with the time-Fourier transformed Green functions. Thus we expand

\[
\Psi(x, t) = \int \frac{d\varepsilon}{\sqrt{2\pi}} e^{-i\varepsilon t} \Psi_\varepsilon(x)
\]

and the action becomes

\[
S_{2+1} = \int d\varepsilon \int d^2x \; \bar{\Psi}_\varepsilon(x)(\varepsilon - H)\Psi_\varepsilon(x)
\]

Since the functional integral is defined by \( e^{iS} \), for a fixed energy \( \varepsilon \) one needs to study a euclidean functional integral defined by \( e^{-S} \), where

\[
S = i \int d^2x \; \bar{\Psi}(H - \varepsilon)\Psi
\]

and we have dropped the subscript \( \varepsilon \) on \( \Psi \).

Our model consists of a two component fermion

\[
\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}
\]

with the hermitian Dirac hamiltonian

\[
H = \frac{1}{\sqrt{2}} (-i \partial_x - A_x) \sigma_x + \frac{1}{\sqrt{2}} (-i \partial_y - A_y) \sigma_y + V(x, y)
\]

This kind of Dirac hamiltonian has been considered before in connection with Quantum Hall transitions [11] [10], but here the meaning is rather different. Our theory is interpreted as describing the extended states in a bulk that is free of impurities. For spinless electrons it should be a scale invariant theory with \( c = 1 \) that is rotationally invariant, and the above hamiltonian is essentially the unique one with these properties. Wen’s description of the edge states may be useful in making this connection more precise [12]. Defining the complex coordinates

\[
z = \frac{1}{\sqrt{2}}(x + iy), \quad \bar{z} = \frac{1}{\sqrt{2}}(x - iy),
\]

and the gauge fields \( A_z = (A_x - iA_y)/\sqrt{2}, A_{\bar{z}} = (A_x + iA_y)/\sqrt{2} \), upon rescaling \( \Psi \to \Psi/\sqrt{2\pi} \) one obtains
\[ S = \int \frac{d^2x}{2\pi} \left[ \psi_1^\dagger (\partial_z - iA_z) \psi_2 + \psi_2^\dagger (\partial_\bar{z} - iA_{\bar{z}}) \psi_1 - i(V + \varepsilon) \left( \psi_2^\dagger \psi_2 + \psi_1^\dagger \psi_1 \right) \right] \] (2.8)

The \( V \) and \( \varepsilon \) terms give the fermions a mass and thus break the conformal invariance. For the remainder of this section we set \( V = \varepsilon = 0 \) and focus on the conformal field theory coupled to the gauge field.

The fermions can be bosonized with a single scalar field \( \phi \) satisfying \( \partial_z \partial_z \phi = 0 \). The scalar field separates into left and right moving parts

\[ \phi(z, \bar{z}) = \phi_L(z) + \phi_R(\bar{z}) \] (2.9)

and the fermions have the bosonized expressions

\[ \psi_1 = e^{-i\phi_L}, \quad \psi_1^\dagger = e^{i\phi_L}, \quad \psi_2 = e^{i\phi_R}, \quad \psi_2^\dagger = e^{-i\phi_R} \] (2.10)

The current coupled to the gauge field then has the following bosonized expressions

\[ j_z = \frac{1}{2\pi} \psi_2^\dagger \psi_1 = \frac{i}{2\pi} \partial_z \phi \] (2.11)

\[ j_{\bar{z}} = \frac{1}{2\pi} \psi_1^\dagger \psi_2 = -\frac{i}{2\pi} \partial_{\bar{z}} \phi \]

Introducing the completely anti-symmetric tensor

\[ \epsilon_{xy} = -\epsilon_{yx}; \quad \epsilon_{z\bar{z}} = -\epsilon_{\bar{z}z} = i \] (2.12)

the current takes the form

\[ j_\mu = \frac{1}{2\pi} \epsilon_{\mu\nu} \partial_\nu \phi \] (2.13)

The current is thus a topological current which is identically conserved \( \partial_\mu j_\mu = 0 \) by virtue of the anti-symmetry of \( \epsilon_{\mu\nu} \). The bosonized action takes the form

\[ S = \int d^2x \left[ \frac{1}{8\pi} \partial_\mu \phi \partial_\mu \phi - \frac{i}{2\pi} \epsilon_{\mu\nu} \partial_\nu \phi A_\mu \right] \] (2.14)

where \( \partial_\mu \partial_\mu = \partial_z \partial_z + \partial_{\bar{z}} \partial_{\bar{z}} \).

In order to clarify some of our subsequent arguments, we introduce a coupling \( g \) by rescaling the current in the \( j_\mu A_\mu \) coupling to

\[ j_\mu = \frac{1}{2\pi g} \epsilon_{\mu\nu} \partial_\nu \phi \] (2.15)

One justification for this is to consider adding a current-current interaction

\[ \frac{\pi k}{2} \int d^2x j_\mu j_\mu = k \int d^2x \frac{1}{8\pi} (\partial_\mu \phi)^2 \] (2.16)

Such an interaction would arise for instance upon averaging over a gaussian disordered component of the gauge field \( A_\mu \), where \( k \) is proportional to the variance. This interaction merely re-scales the kinetic term. Redefining \( \phi \rightarrow \phi/\sqrt{1 + k} \) leads to the current (2.15) with
\(g^2 = 1 + k\). If one prefers, \(g\) can be viewed as a kind of bookkeeping device that will be convenient in the sequel. Our original model corresponds to \(g = 1\).

Whereas the fermionic description has manifest gauge symmetry, the bosonic description as it stands does not. However, the gauge invariance can be restored by adding an \(A^2\) term:

\[
S = \int d^2x \left[ \frac{1}{8\pi} \partial_\mu \phi \partial_\mu \phi - \frac{i}{2\pi g} \epsilon_{\mu \nu} \partial_\nu \phi A_\mu + \frac{1}{2\pi g^2} A_\mu A_\mu \right] \quad (2.17)
\]

To see this, introduce the dual field \(\tilde{\phi}\) defined as

\[
\partial_\mu \tilde{\phi} = -i\epsilon_{\mu \nu} \partial_\nu \phi
\]

In terms of left and right movers defined in Eq. (2.9), the above equation implies (up to a constant)

\[\tilde{\phi} = \phi_L - \phi_R \quad (2.18)\]

Using the relation \((\partial_\mu \phi)^2 = - (\partial_\mu \tilde{\phi})^2\) one can verify that the action has the local gauge invariance

\[\tilde{\phi}(x) \rightarrow \tilde{\phi}(x) + \frac{2}{g} \lambda(x), \quad A_\mu(x) \rightarrow A_\mu(x) + \partial_\mu \lambda(x) \quad (2.19)\]

Since \(\psi_1 = \exp(-i(\phi + \tilde{\phi})/2), \psi_2 = \exp(i(\phi - \tilde{\phi})/2)\), the gauge transformation on the fermions is

\[\psi_{1,2} \rightarrow e^{-i\lambda} \psi_{1,2} \quad (2.20)\]

for \(g = 1\), as it should be. The addition of the \(A^2\) term is analogous to the Green-Schwarz mechanism for anomaly cancelation in low energy effective string theory [35].

For application to the Quantum Hall effect, the gauge field should incorporate some magnetic flux \(\Phi\) through the \(x - y\) plane. The flux is \(\Phi = \int d^2x \nabla \times A = \oint dx_\mu A_\mu\). A non-zero flux can be obtained with \(A_\mu\) a pure gauge, \(A_\mu = \partial_\mu \chi\), as long as \(\chi\) is allowed to be discontinuous across a cut in the \(x - y\) plane extending from the origin to infinity. Namely, the flux is non-zero if \(\chi\) has winding modes

\[
\Phi = \int d\sigma \partial_\sigma \chi = \chi(\sigma = 2\pi) - \chi(0), \quad z = \frac{r}{\sqrt{2}} e^{i\sigma} \quad (2.21)
\]

For \(A_\mu = \partial_\mu \chi\) the action takes the form

\[
S = \int d^2x \left[ \frac{1}{8\pi} \partial_\mu \phi \partial_\mu \phi - \frac{i}{2\pi g} \epsilon_{\mu \nu} \partial_\nu \phi \partial_\mu A_\mu + \frac{1}{2\pi g^2} \partial_\mu \chi \partial_\mu \chi \right] \quad (2.22)
\]

The coupling between \(\chi\) and \(\phi\) is a topological term, i.e. a total derivative. A model of two independent scalar fields with the same topological coupling has been studied before as a \(c = 2\) conformal field theory [13] [13]. Our model differs in a significant way: due to the structure of the couplings the gauge invariance Eq. (2.20) allows \(\chi\) to be gauged away up
to the effects of its discontinuity across the cut. Thus we expect our model to share some of the features found in [17] [18], but in a $c = 1$ conformal field theory.

As we now describe, a further constraint on the zero modes of $\chi$ allows us to incorporate an essential feature of the $2 + 1$ dimensional Quantum Hall dynamics. Electric current conservation implies the continuity equation $\partial_t \rho + \partial_\mu J_\mu = 0$ where $\rho$ is the charge density. When the conductivity $\sigma_{xx} = 0$ we may write $J_\mu = \sigma_{xy} \epsilon_{\mu\nu} E_\nu$. Inserting this into the continuity equation and using the Maxwell equation $\vec{\nabla} \times \vec{E} = -\partial_t \vec{B}$, one obtains $\partial_t (\rho - \sigma_{xy} B) = 0$, where $B$ is the magnetic field perpendicular to the $x - y$ plane. Integrating over space:

$$\frac{d}{dt} (q - \sigma_{xy} \Phi) = 0 \quad (2.24)$$

where $q = \int d^2 x \rho$ is the electric charge and $\Phi = \int d^2 x B$ the flux.

Let $C$ denote a circular contour of arbitrary radius surrounding the origin. If the charge inside this circle is zero at $t = -\infty$, then at time $t$,

$$q(t) = -\int_{-\infty}^t dt \oint_C dx \epsilon_{\mu\nu} J_\nu \quad (2.25)$$

For $t \to \infty$, using the time dependence in Eq. (2.3),

$$\int_{-\infty}^\infty dt J_\mu(x, t) = \int d\varepsilon j_\mu(x) \quad (2.26)$$

where $j_\mu$ is the current in Eq. (2.11). Since the flux is $\Phi = \oint_C dx \partial_\mu \phi$, for fixed $\varepsilon$ the equation (2.24) leads to

$$-\oint_C dx \epsilon_{\mu\nu} j_\nu = \sigma_{xy} \oint_C dx \partial_\mu \phi \quad (2.27)$$

We take the point of view that the proportionality expressed in Eq. (2.27) captures an essential feature of the Quantum Hall dynamics. Since this is real time dynamics in $2 + 1$ dimensions, it cannot follow from our action $S$, but must be put in by hand. We will express this proportionality in terms of a fundamental parameter $\theta$:

$$\frac{\theta}{2\pi g} \oint_C dx \partial_\mu \phi = \oint_C dx \partial_\mu \chi \quad (2.28)$$

where we have used Eq. (2.13). Though Eq. (2.28) follows from Eq. (2.27) with the identification $\theta = 1/\sigma_{xy}$, we only expect this to be valid when $\sigma_{xx} = 0$, and so it is incorrect to make this identification at this stage. Rather, in the next section we will compute the conductivities $\sigma_{xx}, \sigma_{xy}$ in terms of the parameters $g, \theta$.

To summarize, our conformal model is defined by the action Eq. (2.23) with the constraint Eq. (2.28) on the zero modes.
B. The Spectrum of Conformal Fields Depends on $\theta$

The action possesses the gauge invariance
\[
\chi \to \chi + \lambda, \quad \tilde{\phi} \to \tilde{\phi} + \frac{2}{g} \lambda
\] (2.29)
which follows from Eq. (2.20). Thus, $\chi$ can be gauged away up to its effects on the zero modes. One consequence of this is that the spectrum of allowed fields is modified in a way that depends on $\theta$. Consider the conformal fields $\exp(i\alpha_L \phi_L + i\alpha_R \phi_R)$. The topological charge separates into left and right pieces:
\[
Q = - \oint_C dx_\mu \epsilon_{\mu\nu} j_\nu = Q_L - Q_R
\] (2.30)
where
\[
Q_L = \frac{1}{2\pi g} \oint dz \partial_z \phi_L, \quad Q_R = \frac{1}{2\pi g} \oint d\sigma \partial^\sigma \phi_R
\] (2.31)
The exponential fields are characterized by their charges $\alpha_{L,R}$:
\[
[Q_{L,R}, e^{i\alpha_L \phi_L + i\alpha_R \phi_R}] = \alpha_{L,R} e^{i\alpha_L \phi_L + i\alpha_R \phi_R}
\] (2.32)

To determine the spectrum of allowed $\alpha_{L,R}$ we conformally map the theory onto the cylinder by letting
\[
z = e^w, \quad w = t + i\sigma
\] (2.33)
The anti-symmetric tensor is $\epsilon_{\sigma t} = -\epsilon_{t\sigma} = 1$. The coordinate $\sigma$ is along the circumference of the cylinder and takes values $0 \leq \sigma \leq 2\pi$, whereas $t$ runs along the length of the cylinder, $-\infty \leq t \leq \infty$. In order to carry out canonical quantization, let us for the moment further rotate to Minkowski space $t \to it$. The field $\phi$ can be expanded as
\[
\phi(t, \sigma) = \phi_0(t) + gQ\sigma + i \sum_n \frac{1}{n} \left( a_n e^{-in(t+\sigma)} + \bar{a}_n e^{-in(t-\sigma)} \right)
\] (2.34)
The lagrangian for the zero modes is
\[
L = \int_0^{2\pi} d\sigma \left[ \frac{1}{8\pi} \left( \ddot{\phi}_0^2 - g^2 Q^2 \right) - \frac{1}{2\pi g} \dot{\phi}_0 \partial_\sigma \chi \right]
\] (2.35)
\[
= \frac{1}{4} \left( \ddot{\phi}_0^2 - g^2 Q^2 \right) - \frac{1}{2\pi g} \dot{\phi}_0 \Phi
\]
where $\Phi$ is again the magnetic flux.

The momentum conjugate to $\phi_0$ is
\[
p_0 = \frac{1}{2} \dot{\phi}_0 - \frac{1}{2\pi g} \Phi
\] (2.36)
Substituting \( \phi_0(t) = \phi_0 + \dot{\phi}_0 t \) into Eq. (2.34) and using Eq. (2.36) one obtains the zero mode contribution to \( \phi \). After mapping back to the \( z \) coordinate one finds

\[
\phi(z, \bar{z}) = \phi_0 - iQ_L \log z - iQ_R \log \bar{z} + \ldots
\]

(2.37)

where

\[
Q_L = \left( p_0 + \frac{\Phi}{2\pi g} \right) + \frac{gQ}{2} \tag{2.38}
\]

\[
Q_R = \left( p_0 + \frac{\Phi}{2\pi g} \right) - \frac{gQ}{2}
\]

We first impose that the charge \( Q \) is an integer \( n \). Then, since \([\phi_0, p_0] = i\), mutual locality of the exponential fields when \( \theta = 0 \) requires \( p_0 = m/g \) where \( m \) is some other integer. Finally the constraint Eq. (2.28) imposes \( \Phi = \theta Q \). Thus the spectrum of exponential fields consists of the operators

\[
\mathcal{O}_{n,m} = \exp(i\alpha_L \phi_L + i\alpha_R \phi_R) \tag{2.39}
\]

where

\[
\alpha_L = \frac{1}{g} \left( m + \frac{\theta}{2\pi} n \right) + \frac{gn}{2} \tag{2.40}
\]

\[
\alpha_R = \frac{1}{g} \left( m + \frac{\theta}{2\pi} n \right) - \frac{gn}{2}
\]

The integers \( n, m \) are electric and magnetic charges.

The structure of the fields \( \mathcal{O}_{n,m} \) can be understood as simply arising from the gauge transformation Eq. (2.20). Namely, using Eq. (2.28) to identify \( \chi = \theta \phi/2\pi g \), the gauge transformation reads

\[
\bar{\phi} \rightarrow \bar{\phi} + \frac{\theta}{\pi g^2} \phi, \quad \phi \rightarrow \phi \tag{2.41}
\]

Let us express

\[
\alpha_L \phi_L + \alpha_R \phi_R = \alpha \phi + \tilde{\alpha} \tilde{\phi} \tag{2.42}
\]

where \( \alpha = (\alpha_L + \alpha_R)/2, \tilde{\alpha} = (\alpha_L - \alpha_R)/2 \). Then, indeed one can verify that the \( \theta \) dependence in Eq. (2.40) follows from the shift Eq. (2.41):

\[
\mathcal{O}_{n,m}(\phi, \tilde{\phi}; \theta) = \mathcal{O}_{n,m} \left( \phi, \tilde{\phi} + \frac{\theta}{\pi g^2} \phi; \theta = 0 \right) \tag{2.43}
\]

In summary, our model is a free massless scalar \( \phi \) supplemented by the transformation Eq. (2.41). Any correlation function involving the fields \( \phi, \tilde{\phi} \) will be computed by first performing the transformation (2.41) and then using the identifications (2.9)(2.19) and the two-point functions:
\[ \langle \phi_L(z)\phi_L(0) \rangle = -\log z, \quad \langle \phi_R(z)\phi_R(0) \rangle = -\log z \] (2.44)

The identifications (2.9), (2.19) together with the above equation imply (when \( \theta = 0 \)):

\[ \langle \phi(z, z)\phi(0) \rangle = \langle \tilde{\phi}(z, z)\tilde{\phi}(0) \rangle = -\log(z^2) \] (2.45)

Some \( \theta \) dependent correlation functions will be computed in the sequel.

C. The Partition Function has an \( SL(2, \mathbb{Z}) \) Symmetry

The partition function on the torus possesses an \( SL(2, \mathbb{Z}) \) modular symmetry acting on the coupling constants \( g, \theta \). Demonstrating this in our \((c = 1)\) conformal field theory closely parallels the discussion in [19] [18] for 2-boson \((c = 2)\) theories with topological term.

A torus is obtained by imposing periodic boundary conditions on the cylinder described above. Let the length of the cylinder in the \( t \) direction be \( l \). Since the hamiltonian on the cylinder is \( H = L_0 + \overline{L}_0 - c/12 \), where \( L_0, \overline{L}_0 \) are the zero modes of the Virasoro algebra, one has

\[ Z = \text{Tr} e^{-l(L_0 + \overline{L}_0 - c/12)} \] (2.46)

The trace is over the Virasoro highest weight representations corresponding to the fields \( O_{n,m} \). The zero mode contribution to \( L_0 + \overline{L}_0 \) corresponds to the (anomalous) conformal scaling dimension of \( O_{n,m} \). Using

\[ \langle e^{i\alpha L\phi_L(z)} e^{-i\alpha L\phi_L(0)} \rangle = z^{-\alpha_L^2}, \quad \langle e^{i\alpha R\phi_R(z)} e^{-i\alpha R\phi_R(0)} \rangle = \overline{z}^{-\alpha_R^2} \] (2.47)

one finds

\[ d_{n,m}(g, \theta) = \frac{1}{2}(\alpha_L^2 + \alpha_R^2) = \frac{1}{g^2} \left( m + \frac{\theta n}{2\pi} \right)^2 + \frac{g^2n^2}{4} \] (2.48)

The partition function is then

\[ Z(g, \theta) = \frac{1}{|\eta|^2} \sum_{n,m} \exp \left( -ld_{n,m}(g, \theta) \right) \] (2.49)

where the Dedekind \( \eta \)-function comes from the non-zero modes \( a_n, \overline{a}_n \). (See [26] [27].)

\( Z \) is obviously invariant under \( \theta \to \theta + 2\pi \) since this just shifts the integer \( m \). It is also easy to show that

\[ d_{n,m}(g, \theta) = d_{m,-n}(g', \theta') \] (2.50)

with

\[ g'^2 = \frac{g^2}{(g^4/4 + (\theta/2\pi)^2)} \] (2.51)

\[ \theta' = -\frac{\theta}{(g^4/4 + (\theta/2\pi)^2)} \]
Thus, $Z$ is also invariant under $(g, \theta) \to (g', \theta')$.

Introduce a modular parameter $\tau$ (not to be confused with the geometrical modular parameter of the torus $\tau_{\text{torus}} = il/2\pi$)

$$\tau = \frac{\theta}{2\pi} + ig^2/2$$

Then the two above symmetries correspond to

$$T: \quad \tau \to \tau + 1, \quad S: \quad \tau \to -1/\tau$$

These two transformations generate the group $SL(2, \mathbb{Z})$, whose elements $\Gamma$ transform $\tau \to \Gamma(\tau) = (a\tau + b)/(c\tau + d)$ where $a, b, c, d$ are integers satisfying $ad - bc = 1$. The partition function has the full symmetry $Z(\Gamma(\tau)) = Z(\tau)$.

### III. CONFORMAL CONDUCTIVITY

The conductivity tensor $\sigma_{\mu\nu}$ has the following properties: $\sigma_{xx} = \sigma_{yy}$, $\sigma_{xy} = -\sigma_{yx}$. In terms of the complex coordinate $z$, this implies

$$\sigma_{zz} = \sigma_{\bar{z}z} = 0$$

(3.1)

The non-zero components are

$$\sigma_{zz} = \sigma_{xx} + i\sigma_{xy}, \quad \sigma_{\bar{z}z} = \sigma_{xx} + i\sigma_{yx} = (\sigma_{zz})^*$$

(3.2)

Thus it is natural to think of the conductivity as the single complex parameter $\sigma_{zz}$. We will also use the covariant description

$$\sigma_{xx} = \frac{1}{2}\sigma_{\mu\mu}, \quad \sigma_{xy} = \frac{1}{2}\epsilon_{\mu\nu}\sigma_{\mu\nu}$$

(3.3)

The conductivity is usually expressed in terms of retarded and advanced one-particle Green functions, which can be computed from the action (2.8) by including a small positive (negative) imaginary part to $\varepsilon$ for the retarded (advanced) Green function. Studying a localization/delocalization phase transition amounts to finding a renormalization group fixed point of the theory where the theory is conformally invariant. Our aim in this section is to understand the properties of the conductance at the critical point, thus we set the potential $V$ to zero. The $\varepsilon$-terms also break conformal invariance. In what follows we will derive some simple expressions representing conformal contributions to the conductivity when $V = \varepsilon = 0$. In a sense these "conformal conductivities" represent the conductivity at a possible fixed point of a model. Later in the paper we will restore an impurity potential as a perturbation.

We start from the Kubo formula in $2+1$ dimensions. Throughout this section $t$ is the real time in the $2+1$ dimensional world. We work at finite temperature as a computational tool, taking the zero temperature limit at the end. The AC conductivity is given by

$$\sigma_{\mu\nu}(\omega) = \frac{i}{\omega}\Pi_{\mu\nu}(\omega)$$

(3.4)
where $\Pi(\omega)$ is an analytic continuation to real time of the euclidean ($E$) Matsubara quantity:

$$\Pi_{\mu\nu}(\omega) = \Pi_{\mu\nu}^{(E)}(i\omega \to \omega + i\eta)$$

(3.5)

where $\eta$ is small and positive, and

$$\Pi_{\mu\nu}^{(E)}(i\omega) = - \int d^2x \int_0^\beta d\tau e^{i\omega\tau} \langle J_\mu(x, \tau)J_\nu(0) \rangle$$

(3.6)

Here, $\tau = it$ is euclidean time, $\beta$ is the inverse temperature, and the correlation is at finite temperature.

In the Matsubara formulation we work with the action

$$S = \int_0^\beta d\tau \int d^2x \psi^\dagger (\partial_\tau + H)\psi$$

(3.7)

and expand the fields as follows:

$$\psi(x, \tau) = \sum_\nu e^{-i\nu\tau} \psi_{i\nu}(x)$$

(3.8)

where $\nu = 2\pi(n + 1/2)/\beta$ with $n$ an integer. The action then takes the form

$$S = \sum_\nu \beta \int d^2x \psi_{i\nu}^\dagger(x)(H - i\nu)\psi_{i\nu}(x)$$

(3.9)

For notational simplicity, we carry out the computation without displaying the spacial tensorial properties, restoring them at the end. The current is a fermion bilinear:

$$J(x, \tau) = \sum_{\nu, \nu'} e^{i(\nu' - \nu)\tau} \psi_{iv'}^\dagger(x)\psi_{i\nu}(x)$$

(3.10)

Inserting this into Eq. (3.6) one obtains

$$\Pi^{(E)}(i\omega) = i(e^{i\omega\beta} - 1) \sum_{\nu, \nu'} \frac{1}{\omega + \nu' - \nu} \langle \psi_{iv'}^\dagger(x)\psi_{i\nu}(x)J(0) \rangle$$

(3.11)

There are well-known techniques for carrying out the sums over Matsubara frequencies (see e.g. [36]). The sum over $\nu'$ may be performed by considering the contour integral

$$\oint \frac{dz}{2\pi i} n_F(z) \frac{1}{z - (i\nu - i\omega)} \langle \psi_z^\dagger(x)\psi_{i\nu}(x)J(0) \rangle$$

(3.12)

where

$$n_F(z) = \frac{1}{e^{\beta z} + 1}$$

(3.13)

and the contour of integration is a circle of radius $R$ as $R \to \infty$. In this way one picks up the poles in $n_F(z)$ at $z = i\pi(2n + 1)/\beta$ with residue $-1/\beta$. Assuming the only other pole is at $z = i\nu - i\omega$, one obtains
\[ \Pi^{(E)}(i\omega) = -\beta e^{i\omega \beta} \int d^2x \sum_\nu \langle \psi_{\nu}^\dagger(x) \psi_\nu(x) J(0) \rangle \]  

(3.14)

The summation over \( \nu \) is now studied by considering the contour integral

\[ \oint \frac{dz}{2\pi i} n_F(z) \langle \psi_{z-i\omega}(x) \psi_z(x) J(0) \rangle \]  

(3.15)

The integrand is known to have branch cuts at \( z = \varepsilon + i\omega \) and \( z = \varepsilon \), where \( \varepsilon \) is real. Thus the contour must be chosen to run above and below the branch cuts along \( z = \varepsilon + i\omega \pm i\delta \) and \( z = \varepsilon \pm i\delta \). The other parts of the contour are along a circle at \( \infty \), thus the integral Eq. (3.15) is a sum over three closed contours and all poles in \( n_F \) are picked up. The sum over \( \nu \) is then expressed as the sum of four real integrations over \( \varepsilon \) coming from above and below the branch cuts.

What is normally done is to factorize the correlation function in Eq. (3.14) into a product of two 1-particle Green functions; the result is an expression involving a product of the difference between retarded and advanced Green functions, i.e. the spectral density. The retarded (advanced) Green functions correspond to taking \( \eta = 0^{+} \) (0−) in Eq. (3.5). This would lead one to believe that when \( \eta = 0 \) the conductivities are zero. However as explained in \[38\] a non-zero density of states requires a vacuum expectation value which survives when \( \eta = 0 \), and an analogy has been made with spontaneous symmetry breaking. Our aim is to extract the conformal (critical) contribution to the conductivity, i.e. the part that survives as \( \varepsilon, \eta \) go to zero, so we follow a different procedure than the usual one. We are interested in the DC conductivity at \( \omega = 0 \). As \( \omega \) tends to zero, in particular when \( \omega = 2\delta \), the integrations along \( z = \varepsilon + i\omega - i\delta \) and \( z = \varepsilon + i\delta \) coalesce, and there is effectively only one branch cut at \( z = \varepsilon \). Integrations above and below this cut give:

\[ \Pi^{(E)}(i\omega) = \beta^2 e^{i\omega \beta} \int d^2x \int \frac{d\varepsilon}{2\pi i} (n_F(\varepsilon + i\omega) - n_F(\varepsilon)) \langle J_\varepsilon(x) J_\varepsilon(0) \rangle \]  

(3.16)

with \( J_\varepsilon = \psi_{\varepsilon}^\dagger \psi_\varepsilon \) where \( \varepsilon \) has the same meaning as in Eq. (2.3). We can now make the analytic continuation (3.5) and take the \( \omega \to 0 \) limit. Using \( \partial_\varepsilon n_F(\varepsilon) = -\delta(\varepsilon) \) at zero temperature, the integration over \( \varepsilon \) sets \( \varepsilon = 0 \). Recall \( \varepsilon = 0 \) corresponds to the conformal limit. Finally, rescaling \( J_{\varepsilon=0}(x) \to j(x)/\beta \), the zero temperature limit may be taken. Our final result is the simple formula

\[ \sigma'_{\mu\nu} = -\int d^2x \langle j_\mu(x) j_\nu(0) \rangle \]  

(3.17)

We denote the above quantity as \( \sigma'_{\mu\nu} \) to emphasize that it is the critical contribution to \( \sigma_{\mu\nu} \). (The disappearance of the 1/2\pi in Eq. (3.16) comes from the extra 2\pi in Eq. (2.8).) The same rescaling of the currents by 1/\beta removes the \( \beta \) in Eq. (3.9). Thus in the formula Eq. (3.17), the currents \( j_\mu \) are those in Eq. (2.11) and the correlation function is computed with

---

4For a discussion in the context of disordered electrons, see [37].

5See also section IVC.
respect to the action (2.23) with $\varepsilon = 0$. The conformal currents $j_\mu$ have dimension 1, so that $\sigma'_{\mu\nu}$ is dimensionless as it should be.

We now evaluate $\sigma'_{\mu\nu}$ in our model. The current is $j_\mu = i\partial_\mu \tilde{\phi}/2\pi g$. We evaluate the current-current correlator in the presence of the coupling to $\chi$ by simply performing the gauge transformation Eq. (2.41):

$$
\sigma'_{xx} = \frac{1}{8\pi^2 g^2} \int d^2x \left( \partial_\mu \tilde{\phi}(x) + \frac{\theta}{\pi g^2} \phi(x) \right) \partial_\mu \left( \tilde{\phi}(0) + \frac{\theta}{\pi g^2} \phi(0) \right)
$$

(3.18)

$$
\sigma'_{xy} = \frac{1}{8\pi^2 g^2} \int d^2x \epsilon_{\mu\nu} \left( \partial_\mu \tilde{\phi}(x) + \frac{\theta}{\pi g^2} \phi(x) \right) \partial_\nu \left( \tilde{\phi}(0) + \frac{\theta}{\pi g^2} \phi(0) \right)
$$

We take the geometry to be a disk of radius $r$ with $r$ going to $\infty$, and we don’t impose any specific boundary condition at infinity. The integrals in Eq. (3.18) can be moved to the boundary of the disk using the divergence and Stokes theorems

$$
\int d^2x \partial_\mu f_\mu = - \oint_C dx \epsilon_{\mu\nu} f_\nu, \quad \int d^2x \epsilon_{\mu\nu} \partial_\mu f_\nu = \oint_C dx f_\mu
$$

(3.19)

Consider for instance the contribution:

$$
\int d^2x \langle \partial_\mu \phi(x) \partial_\nu \phi(0) \rangle = - \oint_C dx \epsilon_{\mu\nu} \langle \phi(x) \partial_\nu \phi(0) \rangle
$$

(3.20)

where the contour $C$ is the boundary of the disk. Using Eq. (2.45) one finds

$$
\int d^2x \langle \partial_\mu \tilde{\phi}(x) \partial_\nu \tilde{\phi}(0) \rangle = 2\pi \left( \oint_C dz \frac{1}{2\pi i} \frac{1}{z} + \oint_C dz \frac{1}{2\pi i} \frac{1}{\overline{z}} \right) = 4\pi
$$

(3.21)

Similar reasoning gives

$$
\int d^2x \langle \partial_\mu \tilde{\phi}(x) \partial_\nu \phi(0) \rangle = 4\pi, \quad \int d^2x \langle \partial_\mu \phi(x) \partial_\nu \tilde{\phi}(0) \rangle = 0
$$

(3.22)

We also need

$$
\int d^2x \epsilon_{\mu\nu} \langle \partial_\mu \phi(x) \partial_\nu \tilde{\phi}(0) \rangle = \oint_C dx_\mu \langle \phi(x) \partial_\mu \tilde{\phi}(0) \rangle = 4\pi i
$$

(3.23)

The same result holds with $\phi$ and $\tilde{\phi}$ interchanged. Putting this all together we obtain

$$
\sigma'_{xx} = \frac{1}{2\pi g^2} \left( 1 + \left( \frac{\theta}{\pi g^2} \right)^2 \right)
$$

(3.24)

$$
\sigma'_{xy} = \frac{i}{2\pi \pi g^4}
$$

---

6The integral over $d\overline{z}$ originally has opposite sign due to $\epsilon_{zz} = -\epsilon_{z\overline{z}}$, but for the contour $C$, the sense of integration over $\overline{z}$ is reversed in comparison to $z$; the integrals in Eq. (3.21) are both usual Cauchy integrals.
The overall $1/2\pi$ is expected from the fact that $\sigma_{\mu\nu}$ has units of $e^2/h = 1/2\pi$.

That $\sigma'_{xy}$ is imaginary for real $\theta$ can be traced back to the "i" in the coupling to the gauge field in Eq. (2.17). We now give arguments supporting the idea that for the computation done in this section, we should analytically continue from euclidean to Minkowski space. The reason has to do with the difference between the hermiticity properties in first quantization versus second. In first quantization, the hamiltonian $H$ is a hermitian operator $H^\dagger = H$. For our particular hamiltonian (2.6) this follows from $(-i\partial_\mu)^\dagger = -i\partial_\mu$ and $A_{x,y}^\dagger = A_{x,y}$. However, when working with the action $S$ as a conformal field theory, the natural reality properties of the functional integral are different from this. Note for instance that the bosonization (2.10) is at odds with the hermiticity properties of $\psi_1, \psi_2$. Let us focus on the current $j_\mu$ in Eq. (2.11). In the first quantized viewpoint, $(\psi_1, \psi_2)^\dagger = \psi_1^\dagger, \psi_2^\dagger$ implies $(j_z)^\dagger = j_z$, which requires $(\partial_z \phi)^\dagger = \partial_z \phi$. The latter is not the usual hermiticity assumed in conformal field theory which is tied to the functional integral. In conformal field theory one considers rather $(\partial_z \phi)^\dagger = \partial_z \phi$. This suggests that to perform the computations in this section meaningfully, one should analytically continue to Minkowski space. The complex analytic structure of the conductivity tensor as displayed in Eq. (3.2) also allows the interpretation of the analytic continuation of $\sigma_{xy}$ as a continuation from euclidean to Minkowski space for the coordinates $x,y$. From the point of view of first quantization, this seems to correspond having an imaginary vector potential. We point out that the importance of imaginary vector potentials for delocalization transitions has recently been recognized in both electronic and biological systems [39] [40] [41]. For more discussion on this rather delicate point, see [14].

We define now the critical condition

$$g^2/2 = \pm \theta/2\pi$$  \hspace{1cm} (3.25)

There are two interpretations of this condition depending on the prescription taken to make $\sigma_{xy}$ real, as we now describe.

We can perform the analytic continuation to Minkowski space by letting $\theta \to -i\theta$. The critical condition then leads to $\sigma' = \sigma^c$ with

$$\sigma^c_{xx} = 0, \quad \sigma^c_{xy} = 1/\theta$$  \hspace{1cm} (3.26)

Note that this is precisely the identification we made in arriving at the zero mode constraint Eq. (2.28). In terms of $\tau$, the critical condition leads to

$$\tau^c = \frac{\theta}{2\pi} (1 \pm i)$$  \hspace{1cm} (3.27)

For our original fermion model with $g = 1$, the critical condition gives $\theta = \pm \pi$, so that $\tau^c = (1+i)/2, (-1+i)/2$. These are known to be the non-trivial fixed points of the $SL(2, \mathbb{Z})$, one being related to the other by $T$. Our interpretation of the conformal field theory with the critical condition is that it represents a pure, ideal system consisting only of the extended states, which are the extension of the edge states into the bulk. This pure system is known to have the same conductance properties as a physical plateau in the presence of impurities [4], so Eq. (3.26) leads us to this interpretation. We emphasize that we have not added any impurities yet, so that even though Eq. (3.26) has the same properties as a plateau, there are no real plateaux yet in our model since there are no localized states. We cannot speak of the critical values of $\sigma_{xx}, \sigma_{xy}$ at a transition until impurities are added.
The other possibility for making $\sigma_{xy}$ real is to analytically continue $y \rightarrow iy$, keeping $\theta$ real as before, under which $\sigma_{xy} \rightarrow \theta/\pi^2 g^4$, and $\sigma_{xx}$ remains unchanged. Here the critical condition still leads to $\tau^c$, and

$$\sigma^c_{xy} = \pm \sigma^c_{xx} = 1/\theta$$  \hspace{1cm} (3.28)

For this prescription of analytic continuation, since $\sigma^c_{xx} \neq 0$, the interpretation would be that we are building into the model some features of the transition with impurities, i.e. the property that $\sigma^c_{xx} \neq 0$. Since we will next add impurities as a perturbation, the previous interpretation is preferable.

The $SL(2, \mathbb{Z})$ symmetry \((2.52)\) does not have a simple action on the conductivities since the latter are not a modular transformation of $\tau$. We can however point out the following. We are mainly interested in couplings satisfying the critical condition (3.28). Let us choose the positive sign in (3.27), and construct a modular parameter out of $\sigma_{xx}, \sigma_{xy}$:

$$\varsigma = 2\pi i \sigma_{\pi \pi}$$  \hspace{1cm} (3.29)

Then one has:

$$\varsigma = ST^n(\tau^c), \quad \text{for } \theta = -2\pi n$$  \hspace{1cm} (3.30)

So, for this restriction of the parameters, modular transformations of $\tau$ induce modular transformations of $\varsigma$, and this should lead to certain features of the phase diagram.

**IV. ADDING A CIRCULAR DEFECT OF IMPURITIES**

**A. The Impurity Potential**

We now introduce a potential $V$ representing some impurities in the system. For the reasons explained in the Introduction we do not chose a random potential. We will take the potential to be of the form

$$V(x, y) = V_0 \delta(r - r_0)$$  \hspace{1cm} (4.1)

where $r^2 = x^2 + y^2$ and $r_0$ is some arbitrary radius. This corresponds to a circular defect line of impurities at $r = r_0$. As before, the sample is a disk of radius $r \rightarrow \infty$. This potential certainly incorporates a fixed realization of impurities. As we will see, the important advantage of our choice of potential is that the critical exponents do not depend on $V_0$.

The potential (4.1) leads to a term in the action Eq. (2.8)

$$S_V = -iV_0 \int \frac{d^2 x}{2\pi} \delta(r - r_0) \mathcal{O}_V(x, y)$$  \hspace{1cm} (4.2)

$$\mathcal{O}_V = \psi_1^\dagger \psi_1 + \psi_2^\dagger \psi_2$$

After bosonization, the model we will study has the action Eq. (2.23) with the additional perturbation $S_V$, where $\mathcal{O}_V = \cos \phi$. 

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B. The Defect Theory can be Mapped into a Boundary Field Theory

Let us map the theory to the euclidean cylinder as in section II, where the coordinates on the cylinder \( t, \sigma \) are defined in Eq. (2.33). Since \( r^2 = x^2 + y^2 = 2z^2 = 2e^{2t} \), the defect is now a circle along the circumference of the cylinder (the \( \sigma \)-direction) at \( t_0 \) satisfying \( r_0 = \sqrt{2}e^{t_0} \). Without loss of generality we let \( t_0 = 0 \).

The theory can be solved by folding it onto a boundary field theory. This folding procedure has been previously applied to other physical problems [43] [44] [45] [46]. We first set \( V_0 = 0 \) and fold the conformal field theory. In the defect version of the problem any field can be separated into its pieces on either side of the defect:

\[
\phi(t, \sigma) = \theta(t)\phi^+(t, \sigma) + \theta(-t)\phi^-(t, \sigma)
\]

(4.3)

where \( \theta(t) \) is the step function, \( \theta(t) = 1 \) for \( t > 0 \), zero otherwise. Let \( \phi_{L,R}^{(\pm)} \) denote the \( L-R \) components of the scalar field on either side of the defect:

\[
\phi^{(\pm)} = \phi_L^{(\pm)}(t + i\sigma) + \phi_R^{(\pm)}(t - i\sigma)
\]

(4.4)

From these we define the following fields for \( t > 0 \) only:

\[
\varphi = \varphi_L + \varphi_R, \quad \varphi_L(t, \sigma) = \phi_L^{(+)}(t, \sigma), \quad \varphi_R(t, \sigma) = \phi_L^{(-)}(-t, \sigma)
\]

\[
\varphi' = \varphi_L' + \varphi_R', \quad \varphi'_L(t, \sigma) = \phi_R^{(-)}(-t, \sigma), \quad \varphi'_R(t, \sigma) = \phi_R^{(+)}(t, \sigma)
\]

(4.5)

As defined, the fields \( \varphi_L, \varphi'_L \) (\( \varphi_R, \varphi'_R \)) are functions of \( t + i\sigma \) (\( t - i\sigma \)), hence their L/R designations. Since they are defined only for \( t > 0 \), they are fields in a theory with a boundary at \( t = 0 \). Define now the even/odd combinations:

\[
\varphi^{(e)} = \varphi + \varphi', \quad \varphi^{(o)} = \varphi - \varphi'
\]

(4.6)

The action for the boundary field theory corresponding to the free hamiltonian can be written as

\[
S_{\text{free}} = \frac{1}{2} \int_0^{\infty} dt \int d\sigma \frac{1}{8\pi} \left( (\partial_\mu \varphi^{(e)})^2 + (\partial_\mu \varphi^{(o)})^2 \right)
\]

(4.7)

The theory is not fully defined until the boundary conditions in the conformal field theory are specified. General properties of boundary conformal field theory were studied by Cardy [47]. We first set the coupling to the gauge field to zero. In the defect description, when \( V = 0 \) the appropriate boundary condition is \( \partial_t \phi = 0 \) since this corresponds to \( \phi \) being continuous across the defect. Since \( \phi_L, \phi_R \) are analytic functions of \( w = t + i\sigma, \overline{w} = t - i\sigma \) this implies that on the defect \( \phi_L + \phi_R \) is a constant, which we take to be zero. Letting \( \phi = (\phi^{(+)} + \phi^{(-)})/2 \) on the defect, in terms of the boundary fields \( \phi_L + \phi_R = 0 \) reads

\[
\phi_L^{(+)} + \phi_L^{(-)} + \phi_R^{(+)} + \phi_R^{(-)} = \varphi^{(e)} = 0, \quad \text{at } t = 0
\]

(4.8)

Consider next the impurity operator \( O_V \). In the defect formulation it is appropriate to take \( \cos \phi \) at the defect to be the average of its values on either side of the defect:
\[
\cos \phi = \frac{1}{2} \left( \cos \phi^{(+)} + \cos \phi^{(-)} \right) = \cos \left[ \frac{(\phi^{(+)})}{2} \right] \cos \left[ \frac{(\phi^{(-)})}{2} \right], \quad (t = 0)
\]

(4.9)

Translating this to the boundary description, on the boundary one has
\[
\phi^{(+)} - \phi^{(-)} = \phi^{(+)}_{L} + \phi^{(+)}_{R} - \phi^{(-)}_{L} - \phi^{(-)}_{R} = \varphi_{L} - \varphi_{R} - \varphi'_{L} + \varphi'_{R} = \bar{\varphi}^{(o)}
\]

(4.10)

where as before the dual field \( \bar{\varphi}^{(o)} \) is defined as \( \partial_{\mu} \bar{\varphi}^{(o)} = -i\epsilon_{\mu\nu} \partial_{\nu} \varphi^{(o)} \). Thus on the boundary one has
\[
O_{V} = \cos \left( \frac{\bar{\varphi}^{(o)}}{2} \right), \quad (t = 0)
\]

(4.11)

Since the field \( \varphi^{(e)} \) decouples from the boundary, we henceforth drop it. In order to match the normalization of the previous sections we scale out the factor of 1/2 in Eq. (4.7) by now defining
\[
\varphi \equiv \varphi^{(o)}/\sqrt{2}
\]

(4.12)

The gauge field can now be restored; for \( A_{\mu} = \partial_{\mu} \chi \), the \( \chi \) terms can also be folded as for the \( \phi \) field. We finally obtain the action for the boundary theory:
\[
S = \int_{0}^{\infty} dt \int d\sigma \left( \frac{1}{8\pi} (\partial_{\mu} \varphi)^{2} - \frac{i}{2\pi \hat{g}} \epsilon_{\mu\nu} \partial_{\nu} \varphi \partial_{\mu} \chi + \frac{1}{2\pi \hat{g}^{2}} (\partial_{\mu} \chi)^{2} \right) - iV_{0} \int d\sigma \cos \left( \frac{\bar{\varphi}(0, \sigma)}{\sqrt{2}} \right)
\]

(4.13)

where in the above equation \( \chi \equiv \chi^{(o)} \). The coupling \( \hat{g} \) is related to \( g \) as follows
\[
\hat{g} = \sqrt{2} g
\]

(4.14)

Finally we need to impose a zero mode constraint as in Eq. (2.28). The original charge \( Q \) in the defect theory can be expressed as \( Q = \int d\sigma \partial_{\sigma} \phi^{(+)}/2\pi \hat{g} \). At \( t = 0 \) this is \( Q = \int d\sigma \partial_{\sigma} (\varphi^{(e)} + \varphi^{(o)})/4\pi \hat{g} \). Dropping \( \varphi^{(e)} \) and using Eq. (4.12) one has \( Q = \int d\sigma \partial_{\sigma} \varphi^{(o)}/2\pi \hat{g} \). Thus we impose
\[
\frac{\theta}{2\pi \hat{g}} \oint dx_{\mu} \partial_{\mu} \varphi = \oint dx_{\mu} \partial_{\mu} \chi
\]

(4.15)

where \( \theta \) is the same as in the unfolded theory. (Here \( x_{\mu} \) denotes \( t, \sigma \).) The couplings \( g, \theta \) are real, which leads to real anomalous dimensions of operators, and the critical condition is \( 3/2 \).

The original fermion model is characterized by the values \( \hat{g} = \sqrt{2}, \theta = \pi \). When \( \theta = 0, \hat{g} = \sqrt{2} \) is the self-dual point of the \( SL(2, \mathbb{Z}) \) symmetry described in section II, and \( V_{0} \) corresponds to a (boundary) marginal perturbation of scaling dimension 1.

Our model still possesses the gauge invariance (2.29) with \( g \) replaced by \( \hat{g} \). A 2-boson model with topological coupling and a similar boundary interaction was considered in [18]; again the important difference is the gauge invariance of our model.
The new feature of the boundary version in comparison to what we had in section II is the boundary condition at \( t = 0 \), which now depends on \( \theta \). Letting \( \varphi \to \varphi + \delta \varphi \) and requiring \( \delta S = 0 \) on the boundary leads to the equation of motion (when \( V = 0 \)):

\[
- \frac{1}{4\pi} \partial_t \varphi + \frac{i}{2\pi \hat{g}} \partial_\sigma \chi = 0, \quad (t = 0)
\] (4.16)

Imposing the zero mode constraint Eq. (4.15) one finds

\[
(\hat{g}^2/2 + \theta/2\pi) \partial_w \varphi_L = -(\hat{g}^2/2 - \theta/2\pi) \partial_w \varphi_R \quad (t = 0)
\] (4.17)

This is an interesting relation since it links the modular properties of the couplings \( g, \theta \) with the spacial coordinates. Namely, going to Minkowski space and defining the light-cone coordinates \( v = t + \sigma, \ \bar{v} = t - \sigma \), the boundary condition reads

\[
(\tau \partial_v - \tau \partial_{\bar{v}}) \varphi = 0
\] (4.18)

where \( \tau \) is the modular parameter \([2.52] \) with \( g \to \hat{g} \) and \( \tau = \tau^* \).

We will need the anomalous scaling dimension of the boundary operator \( O_V \). We first gauge away \( \chi \); as before the zero mode constraint leads to the transformation \([2.41] \) with \( g \to \hat{g} \) so that

\[
O_V \to \cos \left( \left[ (1 + \theta/\pi \hat{g}^2) \varphi_L - (1 - \theta/\pi \hat{g}^2) \varphi_R \right] / \sqrt{2} \right)
\] (4.19)

Viewing \( \sigma \) as the “time” the hamiltonian can be written as

\[
H = \left( \frac{1}{8\pi} \int_0^\infty dt (\partial_w \varphi_L)^2 + (\partial_w \varphi_L)^2 \right) + \frac{iV_0}{2\pi} O_V(t = 0)
\] (4.20)

The theory can now be rewritten using only a left-moving field. Using arguments found in \([47] \), the boundary condition Eq. (4.17) allows us to view \( \varphi_R \) as an analytic continuation of \( \varphi_L \). Using Eq. (4.17) we make the substitution

\[
\varphi_R = \frac{-(\hat{g}^2/2 + \theta/2\pi)}{(\hat{g}^2/2 - \theta/2\pi)} \varphi_L
\] (4.21)

into Eq. (4.20). Defining a rescaled field

\[
\hat{\varphi}_L = \frac{\sqrt{1 + (\theta/\pi \hat{g}^2)^2}}{(1 - \theta/\pi \hat{g}^2)} \varphi_L
\] (4.22)

one finds

\[
H = \left( \frac{1}{4\pi} \int d\sigma (\partial_w \hat{\varphi}_L)^2 \right) + \frac{iV_0}{2\pi} \cos \left( \sqrt{2}a \hat{\varphi}_L \right)
\] (4.23)

where

\[
a = \frac{1 - (\theta/\pi \hat{g}^2)^2}{\sqrt{1 + (\theta/\pi \hat{g}^2)^2}}
\] (4.24)
Since the rescaled field has the two-point function \( \langle \hat{\phi}_L(z)\hat{\phi}_L(0) \rangle = -\log z \), and
\[
\langle e^{i\sqrt{2a}z\hat{\phi}_L(z)}e^{-i\sqrt{2a}\hat{\phi}_L(0)} \rangle = 1/z^{2a^2} ,
\]
the anomalous scaling dimension is
\[
[O_V] \equiv \text{dim}(O_V) = a^2
\]

\textbf{C. Critical Exponents}

The model of the last section has a single energy scale \( M \) with units of mass set by \( V_0 \). Since the action is dimensionless, the dimension of \( V_0 \) is \( 1 - [O_V] \). The energy scale \( M \) defines a correlation length \( M = 1/\xi_c \), thus
\[
\xi_c \propto V_0^{-1/(1-[O_V])}
\]

Let \( \rho(E) \) denote the density of states, so that \( \rho(E)dE \) represents a number of states per unit volume. It can be expressed as
\[
\rho(E) = \frac{1}{V} \text{Tr} \delta(H - E) = \frac{1}{\pi V} \lim_{\eta \to 0^+} \text{Im} \text{Tr} \left( \frac{1}{H - E - i\eta} \right)
\]
where \( V \) is the 2d volume. This in turn can be expressed as a retarded Green function
\[
\rho(E) = \frac{1}{\pi} \lim_{\eta \to 0^+} \text{Im} \langle \psi^\dagger_\varepsilon(x)\psi_\varepsilon(x) \rangle \quad \varepsilon = E + i\eta
\]
In accordance with the discussion in section III, we define a critical density of states \( \rho^c \) from the above formula with \( \varepsilon = 0 \). As we argued in section III for \( \sigma^\prime_{\mu\nu} \), this quantity should represent the density of states near the critical point. Since both fields in Eq. (4.29) are at the same point \( x \), \( \rho^c \) is a one-point function of the operator \( \psi^\dagger\psi = \psi^\dagger_1\psi_1 + \psi^\dagger_2\psi_2 \). From Eq. (4.2) this operator is the operator \( O_V \):
\[
\rho^c \propto \langle O_V \rangle
\]
From the scaling dimension of \( \langle O_V \rangle \) we know that
\[
\langle O_V \rangle \propto (\xi_c)^{-[O_V]}
\]

Next, recall that the number of states per unit volume in a Landau level is \( B/2\pi \). Thus, \( \rho^c \) should scale with \( B \), so that \( \rho^c \propto |B - B_c| \). Using Eqs. (4.30)-(4.31) one obtains the relation (1.2) with
\[
\nu = 1/[O_V]
\]
Alternatively we can argue that since since \( \rho(E)dE \) has units of inverse volume, in two dimensions \( \rho \) has dimensions of energy. Letting \( \rho^c \propto E \), one then finds
\[
\xi_c \propto |E - E_c|^{-1/[O_V]}
\]

where here $E_c = 0$. The connection between the two equations (1.33) and (1.2) is due to the fact that the Fermi energy is proportional $B$ in a system with Landau levels.

In order to satisfy the critical condition $\sigma'_{xx} = 0$, from Eq. (3.25) we impose $\theta/\pi g^2 = \pm 1$. Using Eq. (1.19) one has $\theta/\pi g^2 = \pm 1/2$. Inserting this into Eq. (1.20) we obtain the value $\nu = 20/9$ quoted in the Introduction. Note that this exponent only depends on the ratio $\theta/g^2$. This value of $\nu$ holds for any $g, \theta$ with $\sigma'_{xx} = 0$. In particular $\sigma'_{xy} = 1/\theta$ is left unconstrained, and this indicates that the exponent $\nu$ is universal.

V. S-MATRICES DESIGNED FROM THE MODULAR PARAMETER $\tau$

In this section we propose an intriguing connection with the so-called staircase model, whose interesting properties were first witnessed by Al. Zamolodchikov [49], and were considered mysterious at the time.

The conformal model of section II is essentially that of a single massless scalar field with coupling constants defining a modular parameter $\tau$ and possessing an $SL(2, \mathbb{Z})$ symmetry. The original staircase model was characterized by a bulk S-matrix for a single massive particle. Though it is the boundary version that is more appropriate to our problem, let us begin by showing how the structure of the S-matrix follows quite naturally from what we have done. The energy and momentum of a relativistic particle can be parameterized by a rapidity $\beta$:

$$E = m \cosh \beta, \quad P = m \sinh \beta$$

(5.1)

where $m$ is the mass of the particle. The bulk S-matrix describing the 2-particle to 2-particle scattering must satisfy crossing symmetry and unitarity [50]:

$$S(\beta) = S(i\pi - \beta), \quad S(\beta)S(-\beta) = 1$$

(5.2)

S-matrices satisfying the above functional equations are usually built out of products of the minimal factors:

$$S(\beta) = \frac{\sinh \beta - i \sin \pi \gamma}{\sinh \beta + i \sin \pi \gamma}$$

(5.3)

where $\gamma$ is a parameter related to the coupling constants of the theory. Let us attempt to relate the S-matrix to the physics of our problem by using the $SL(2, \mathbb{Z})$ symmetry to relate $\gamma$ to $\tau$. The S-matrix has the following symmetries:

$$\gamma \rightarrow \gamma + 2, \quad \gamma \rightarrow 1 - \gamma$$

(5.4)

It does not seem possible for the S-matrix to possess the full $SL(2, \mathbb{Z})$ symmetry. The first symmetry in the above formula suggests the identification $\gamma = \tau$, since it would correspond to the transformation $T_2$, where $T$ is defined in Eq. (2.53). Next consider the transformation $\gamma \rightarrow 1 - \gamma$. The modular parameters $\tau$ satisfying the critical condition Eq. (3.25) come in complex conjugate pairs Eq. (1.27). It is natural then that the S-matrix not distinguish between these two critical $\tau$’s. Requiring then that $\gamma \rightarrow 1 - \gamma$ is equivalent to $\tau_+^\pm \rightarrow \tau_-^\mp$ requires $g = 1$, or equivalently $\theta = \pi$, in which case $\gamma = \tau_+^\pm = (1 \pm i)/2$. Finally we perturb
away from the critical condition by deforming $g$ away from 1 in $\tau$, but keeping $\theta = \pi$. Because of the form of the critical condition, we can just as well view this deformation as a deformation of $\theta$ away from $\pi$. Thus we write this as $g^2/2 \to \theta_0/2\pi$, and identify $\gamma$ as follows:

$$\gamma = \tau_{\pm} = \tau(\theta = \pi, g^2/2 = \pm\theta_0/2\pi) = \frac{1}{2} \pm i \frac{\theta_0}{2\pi} \quad (5.5)$$

The S-matrix is then

$$S(\beta) = \frac{\sinh \beta - i \cosh \theta_0/2}{\sinh \beta + i \cosh \theta_0/2} \quad (5.6)$$

It was discovered by Al. Zamolodchikov that the above S-matrix leads to a free energy with some remarkable properties. The model can be studied on a cylinder of length $l$ and radius $R$. Imposing periodic boundary conditions in the $l$-direction, the free energy $\mathcal{E}(R)$ was computed starting from the S-matrix by means of the thermodynamic Bethe ansatz [31]. It was found in [49] that as one varies $R$ the free energy $\mathcal{E}(R)$ goes through a series of plateaux, i.e. the free-energy as a function of $R$ resembles a staircase.

Let us try and be more specific now about the relation with our model. The boundary theory we obtained is defined by the action (4.13). Let us first decouple the gauge field, $\chi = 0$. Noting that $(\partial_\mu \varphi)^2 = - (\partial_\mu \tilde{\varphi})^2$, letting $\tilde{\varphi} \to i\phi$ we obtain from (4.13):

$$S = \int_0^\infty dt \int d\sigma \left[ \frac{1}{8\pi} (\partial_\mu \phi)^2 + \Lambda \cosh(\tilde{\beta}\phi) \right] + \frac{V_0}{2\pi} \int d\sigma \cosh(\tilde{\beta}\phi/2) \quad (5.7)$$

with $\tilde{\beta} = \sqrt{2}$. Our model has zero bulk coupling $\Lambda = 0$. We have included a $\Lambda$ term since as written the above action defines the boundary sinh-Gordon model. It is an integrable model that can be studied using the framework developed in [52]. The bulk S-matrix is known to be of the form Eq. (5.3) with

$$\gamma = \frac{\tilde{\beta}^2}{2 + \tilde{\beta}^2} \quad (5.8)$$

First note that our model leads to $\tilde{\beta} = \sqrt{2}$ and $\gamma = 1/2$. This is precisely the same as in Eq. (5.3) when $\theta_0 = 0$. It is therefore clear that our model (4.13) is very closely related to the boundary sinh-Gordon model with $\tilde{\beta} = \sqrt{2}$ with $\Lambda = 0$ and $\gamma$ analytically continued as in (5.3). This analytic continuation of $\gamma$ incorporates the topological coupling to the gauge field $\partial \chi$, and amounts to continuing the coupling $g^2/2$ to the modular parameter $\tau$. In terms of $\tilde{\beta}$, the above analytic continuation of $\gamma$ corresponds to a simple phase:

$$\tilde{\beta} = \sqrt{2} e^{i\alpha}, \quad \cos 2\alpha = \frac{1 - (\theta_0/\pi)^2}{1 + (\theta_0/\pi)^2} \quad (5.9)$$

A boundary version of the staircase model was studied in [53]. Indeed it was found that the boundary entropy reveals a series of plateaux as a function of renormalization group scale.

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7This paper also includes a useful reproduction of the bulk results in [49].
Keeping within the scope of this paper we refrain from going any more deeply into the integrability structures that allow a detailed study of this proposal, but will return to this in a future publication.
VI. DISCUSSION

In summary, we have constructed a reasonably simple model which we argued describes the critical properties of Quantum Hall plateau to plateau transitions. It is a $c = 1$ conformal field theory mainly characterized by a gauge symmetry and the $SL(2,\mathbb{Z})$ symmetry which follows from it. Adding impurities singles out a particular operator $\mathcal{O}_V$, whose scaling dimension, as computed in the conformal field theory, follows largely from the gauge invariance, and leads to the exponent $\nu = 20/9$.

Experimental errors at this time are perhaps too large to distinguish between $\nu = 2.3$ and $\nu = 2.2$. However if the measurements continue to indicate values closer to $\nu = 20/9$, as in $[32]$, this suggests that the transition is in a different universality class than the percolative class of the network/Anderson models, assuming the numerical work on the latter is correct. Our result appears to be more consistent with the model used in Ando’s simulation $[34]$, which gave $\nu = 2.2 \pm .1$.

Since the gauge symmetry is an important feature of our model it can perhaps be viewed as a simplified version of the Yang-Mills theories in higher dimensions which exhibit the $SL(2,\mathbb{Z})$ electric/magnetic duality $[54]$.

The boundary staircase model described in section V is a promising candidate for a model that exhibits a series of plateaux transitions.

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VIII. ADDENDUM

After the first version of this paper appeared, we reported a study of the relevance of disorder in $[14]$. There we gave evidence for two possible universality classes in the presence of disorder, in one of which all disorder is driven irrelevant by the presence of disorder in the gauge field.
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