Strengthening the Cohomological Crepant Resolution
Conjecture for Hilbert-Chow morphisms

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Abstract

Given any smooth toric surface $S$, we prove a SYM-HILB correspondence which relates the 3-point, degree 0, extended Gromov-Witten invariants of the $n$-fold symmetric product stack $[\text{Sym}^n(S)]$ of $S$ to the 3-point extremal Gromov-Witten invariants of the Hilbert scheme $\text{Hilb}^n(S)$ of $n$ points in $S$. As we do not specialize the values of the quantum parameters involved, this result proves a strengthening of Ruan’s Cohomological Crepant Resolution Conjecture for the Hilbert-Chow morphism $\text{Hilb}^n(S) \to \text{Sym}^n(S)$ and yields a method of reconstructing the cup product of $\text{Hilb}^n(S)$ from the orbifold invariants of $[\text{Sym}^n(S)]$.

0 Introduction

0.1 Overview

Let $S$ be a smooth complex surface and $n$ a positive integer. The symmetric group $\mathfrak{S}_n$ on $n$ letters acts on the $n$-fold product $S^n$ by

$$g \cdot (s_1, \ldots, s_n) = (s_{g(1)}, \ldots, s_{g(n)}).$$

The quotient scheme $S^n/\mathfrak{S}_n$, denoted by $\text{Sym}^n(S)$, is referred to as the $n$-fold symmetric product of $S$. It is singular but the quotient stack $[S^n/\mathfrak{S}_n]$ (cf. Section 1.1) is a smooth orbifold. We denote $[S^n/\mathfrak{S}_n]$ by $[\text{Sym}^n(S)]$ and call it the $n$-fold symmetric product stack of $S$. Note that $\text{Sym}^n(S)$ is the coarse moduli scheme of $[\text{Sym}^n(S)]$.

The Hilbert scheme of $n$ points in $S$, written as $\text{Hilb}^n(S)$ or $S^{[n]}$, parametrizes 0-dimensional closed subscheme $Z$ of $S$ satisfying

$$\dim_{\mathbb{C}} H^0(Z, O_Z) = n.$$

There exists a resolution of singularities $\rho: \text{Hilb}^n(S) \to \text{Sym}^n(S)$ defined by

$$\rho([Z]) = \sum_{p \in S} \ell(O_{Z,p})[p],$$

where $\ell(O_{Z,p})$, the length of $O_{Z,p}$, is simply the multiplicity of $p$ in $Z$. The resolution $\rho$ is called the Hilbert-Chow morphism and is also crepant (cf. [Bea]), i.e.,

$$K_{\text{Hilb}^n(S)} = \rho^* K_{\text{Sym}^n(S)},$$

where $K_{\text{Hilb}^n(S)}$ (resp. $K_{\text{Sym}^n(S)}$) denotes the canonical class of $\text{Hilb}^n(S)$ (resp. $\text{Sym}^n(S)$). Furthermore, Fu and Namikawa ([FuN]) show that $\rho$ provides a unique crepant resolution for $\text{Sym}^n(S)$. 

The following summarizes the relationships among these spaces:

\[ \text{Hilb}^n(S) \xrightarrow{\rho} \text{Sym}^n(S). \]

Here \( \rho \) is the canonical map to the coarse moduli space.

Theoretical physicists believe that string theory on an orbifold and string theory on any crepant resolution should belong to the same family. As for the above examples, it is expected that there is an equivalence between string theories of \([\text{Sym}^n(S)]\) and \([\text{Hilb}^n(S)]\).

The physical principle has led to various mathematical predictions; see, for example, \[R\] \[BG\] \[CoIT\] \[CoR\]. In this article, we are particularly interested in the following conjecture, which is referred to as the Cohomological Crepant Resolution Conjecture (abbreviated as CCRC).

**Conjecture 0.1** (Ruan \[R\]). Let \( \mathcal{X} \) be a smooth Gorenstein orbifold and \( \mathcal{X} \) its coarse moduli space. Assuming that \( \mathcal{X} \) admits a crepant resolution \( \mathcal{Y} \), the Chen-Ruan cohomology ring of \( \mathcal{X} \) is isomorphic to the quantum corrected cohomology ring of \( \mathcal{Y} \).

The notions of Chen-Ruan cohomology and quantum corrected cohomology will be recalled later.

There are several examples for which CCRC is known to be true. For instance, Fantechi and Göttsche \[FaG\], and independently Uribe \[U\] apply the results of Lehn and Sorger \[LS\] to establish the validity of CCRC for \([\text{Sym}^n(S)]\), where \( S \) is an arbitrary smooth complex projective surface with trivial canonical class. Moreover, J. Li and W.-P. Li \[LL\] prove CCRC for multiplication by divisor classes when \( S \) is any smooth, simply-connected, complex projective surface. However, CCRC in the case of \([\text{Sym}^n(S)]\), for an arbitrary smooth toric surface \( S \), has not yet been fully verified. This case will be the main focus of this paper.

Let \( T = (\mathbb{C}^\times)^2 \). The \( T \)-equivariant cohomology of a point is simply the polynomial algebra in two variables \( t_1, t_2 \). In what follows, we assume that all equivariant cohomology rings are with respect to the torus \( T \).

Let \( S \) be a smooth toric surface. The basic objects of this paper are 3-point, degree 0, extended Gromov-Witten invariants of \([\text{Sym}^n(S)]\) and 3-point extremal Gromov-Witten invariants of \([\text{Hilb}^n(S)]\). They are encoded in what we call the extended 3-point functions

\[ \langle \cdot, \cdot, \cdot \rangle_{[\text{Sym}^n(S)]}(u) \in \mathbb{Q}(t_1, t_2)[[u]] \quad (\text{cf. Section 1.4.1}) \]

and the extremal 3-point functions

\[ \langle \cdot, \cdot, \cdot \rangle_{[\text{Hilb}^n(S)]}(q) \in \mathbb{Q}(t_1, t_2)[[q]] \quad (\text{cf. Section 2.3.1}) \]

respectively.

Our goal is to construct a SYM-HILB correspondence, and to prove a strengthening of CCRC for symmetric products of smooth toric surfaces. The following is the main result of this article.

**Theorem 0.2.** Given any smooth toric surface \( S \) and positive integer \( n \). Let \( q = -e^{iu} \) where \( i \) is a square root of \(-1\). Then there is an isometric isomorphism \( L \) which maps the equivariant Chen-Ruan cohomology of \([\text{Sym}^n(S)]\) onto the equivariant cohomology of \([\text{Hilb}^n(S)]\), and which satisfies the identities

\[ \langle \alpha_1, \alpha_2, \alpha_3 \rangle_{[\text{Sym}^n(S)]}(u) = \langle L(\alpha_1), L(\alpha_2), L(\alpha_3) \rangle_{[\text{Hilb}^n(S)]}(q) \]

for any equivariant Chen-Ruan cohomology classes \( \alpha_1, \alpha_2, \alpha_3 \).
We will make the correspondence $L$ explicitly in Section 3. Roughly speaking, it maps the fixed-point basis for the equivariant Chen-Ruan cohomology of $[\text{Sym}^n(S)]$ to the Nakajima basis for equivariant cohomology of $\text{Hilb}^n(S)$. Our proof uses a localization technique and relies on the case of $S = \mathbb{C}^2$, which follows from the results of Bryan-Graber [BG] and Okounkov-Pandharipande [OP].

The equivariant Chen-Ruan cohomology of the symmetric product stack $[\text{Sym}^n(S)]$ is given by extended 3-point functions $\langle\cdot,\cdot,\cdot\rangle_{\text{Sym}^n(S)}(u)$ with $u$ being specialized to 0, while the equivariant quantum corrected cohomology of the Hilbert scheme $\text{Hilb}^n(S)$ is defined by extremal 3-point functions $\langle\cdot,\cdot,\cdot\rangle_{\text{Hilb}^n(S)}(q)$ with $q$ being set to $-1$. Thus, it is an immediate consequence of Theorem 0.2 that CCRC is valid for $[\text{Sym}^n(S)]$ and $\text{Hilb}^n(S)$.

**Corollary 0.3.** The equivariant Chen-Ruan cohomology ring of $[\text{Sym}^n(S)]$ is isomorphic to the equivariant quantum corrected cohomology ring of $\text{Hilb}^n(S)$.

On the other hand, by taking $q = 0$, we have the following.

**Corollary 0.4.** The cup product of $\text{Hilb}^n(S)$ can be recovered from the extended 3-point functions of $[\text{Sym}^n(S)]$.

If we have closed-form formulas for the symmetric product orbifold invariants involved, the cup product of the Hilbert scheme can be written down explicitly.

Note also that Theorem 0.2 does give a stronger correspondence than the correspondence predicted by CCRC as well as the ones obtained in [FaG] [UL] for the case of toric surfaces because we consider 3-point invariants (while [LL] focuses solely on 2-point invariants), and the extended 3-point orbifold invariants possess richer enumerative geometry than the (usual) 3-point invariants, i.e., we do not specialize the values of quantum parameters (in [FaG] [UL] [LL], $u$ and $q$ are set to be 0 and $-1$ respectively).

Furthermore, we can even use the map $L$ and the setting of this paper to compare the full Gromov-Witten theories of $[\text{Sym}^n(S)]$ and $\text{Hilb}^n(S)$ when $S = A_r$, the minimal resolution of the quotient variety $\mathbb{C}^2/\mu_{r+1}$ ($\mu_{r+1}$ is the cyclic group of order $r+1$). The reader can consult [ChG] for details.

### 0.2 Outline

We investigate the extended 3-point functions of the symmetric product stack $[\text{Sym}^n(S)]$ in Section 3 and the extremal 3-point functions of the Hilbert scheme $\text{Hilb}^n(S)$ in Section 2. In Section 3, we give a concrete description of the SYM-HILB correspondence mentioned above and use the results in Section 1 and Section 2 to show Theorem 0.2.

### 0.3 Setting

Throughout the paper, we let $S$ be a smooth toric surface acted upon by the torus $T = (\mathbb{C}^\times)^2$.

The surface $S$ is determined by a fan $\Sigma$ that is a finite collection of strongly convex rational polyhedral cones $\sigma$ contained in $N = \text{Hom}(M, \mathbb{Z})$, where $M \cong \mathbb{Z}^2$. That is, $S$ is obtained by gluing together affine toric varieties $S_{\sigma}$ and $S_{\tau}$ along $S_{\sigma \cap \tau}$ for $\sigma, \tau \in \Sigma$. Here, for example, $S_{\sigma}$ has coordinate ring $\mathbb{C}[\sigma^\vee \cap M]$, which is the $\mathbb{C}$-algebra with generators $\chi^m$ for $m \in \sigma^\vee \cap M$ and multiplication defined by $\chi^m \chi^{m'} = \chi^{m+m'}$. Note that $\sigma^\vee \cap M$ is, by definition, the set of elements $m \in M$ satisfying $v(m) \geq 0$ for all $v \in \sigma$. It is a finitely generated semigroup, and so the $\mathbb{C}$-algebra $\mathbb{C}[\sigma^\vee \cap M]$ is finitely generated.

In addition, $S$ has finitely many $T$-invariant subvarieties, and so it has a finite number of $T$-fixed points, denoted by $x_1, \ldots, x_s$. 

(We do not study smooth toric surfaces without $\mathbb{T}$-fixed points as they are not interesting in equivariant theory.)

For each $i$, $x_i$ is contained in

$$U_i := S_{\sigma_i}$$

for some $\sigma_i \in \Sigma$. As $S$ is smooth and $U_i$ possesses a unique $\mathbb{T}$-fixed point $x_i$, we see that $U_i$ must be isomorphic to the affine plane with $x_i$ corresponding to the origin. However, $S$ is not necessarily the union $\bigcup_{i=1}^{n} U_i$.

From here on, let us fix the above setting on the open sets $U_i$ and the fixed points $x_i$. We denote by $L_i$ and $R_i$ the tangent weights at $x_i$.

For ease of exposition, we need some other notation. Below is the one that will be used frequently.

**Notation:**

1. To avoid doubling indices, we identify $A_i(X) = H^2_i(X; \mathbb{Q})$, $A_i(X) = H^{2i}_i(X; \mathbb{Q})$ and $A_i(X; \mathbb{Z}) = H^{2i}_i(X; \mathbb{Z})$, just to name a few, for any complex variety $X$ to appear in this article (note that we drop $\mathbb{Q}$ but not $\mathbb{Z}$). They will be referred to as cohomology or homology rings rather than Chow rings.

2. (a) Denote by $t_1, t_2$ the generators of the equivariant cohomology of a point, that is, $A^*_\mathbb{T}$(point) = $\mathbb{Q}[t_1, t_2]$.

   (b) $V_m = V \otimes_{\mathbb{Q}[t_1, t_2]} \mathbb{Q}[t_1, t_2]$ for each $\mathbb{Q}[t_1, t_2]$-module $V$.

3. For any space $X$ with a $\mathbb{T}$-action, $X^\mathbb{T}$ denotes the $\mathbb{T}$-fixed locus of $X$.

4. An orbifold $X$ is a smooth Deligne-Mumford stack of finite type over $\mathbb{C}$. Denote by $c: X \rightarrow X$ the canonical map to the coarse moduli space.

5. Given any object $\mathcal{O}$, $\mathcal{O}^n$ means that $\mathcal{O}$ repeats itself $n$ times.

6. Let $\sigma$ be a partition of a nonnegative integer.

   (a) $\ell(\sigma)$ is the length of $\sigma$. Unless otherwise stated, $\sigma$ is presumed to be written as

   $$\sigma = (\sigma_1, \ldots, \sigma_{\ell(\sigma)}) \text{ with } \sigma_1 \geq \cdots \geq \sigma_{\ell(\sigma)}.$$  

   To make a emphasis, if $\sigma_k$ is another partition, it is simply $(\sigma_k, \ldots, \sigma_k)$.  

   (b) Let $\vec{\sigma} = (\alpha_1, \ldots, \alpha_{\ell(\sigma)})$ be an $\ell(\sigma)$-tuple of cohomology classes associated to $\sigma$ so that we may form a cohomology-weighted partition $\sigma(\vec{\alpha}) := \sigma_1(\alpha_1) \cdots \sigma_{\ell(\sigma)}(\alpha_{\ell(\sigma)})$. The group $\text{Aut}(\sigma(\vec{\alpha}))$ is defined to be the group of permutations on $\{1, 2, \ldots, \ell(\sigma)\}$ fixing

   $$\left(\sigma_1(\alpha_1), \ldots, (\sigma_{\ell(\sigma)}(\alpha_{\ell(\sigma)})\right).$$

   Let $\text{Aut}(\sigma)$ be the group $\text{Aut}(\sigma(\vec{\alpha}))$ when all entries of $\vec{\alpha}$ are identical.

   (c) Let $g_\sigma = |\text{Aut}(\sigma)| \prod_{i=1}^{\ell(\sigma)} \sigma_i$ be the order of the centralizer of any permutation of cycle type $\sigma$, and $|\sigma| = n$ if $\sigma_1 + \cdots + \sigma_{\ell(\sigma)} = n$.

   (d) $(2) := (2, 1^{n-2})$ and $1 := (1^n)$ are partitions of length $n-1$ and length $n$ respectively.
1 Symmetric Product Stack

1.1 Some definitions

Let $X$ be a smooth complex variety and $n$ a positive integer. Given any finite set $N$ of $\{1, \ldots, n\}$, let $\mathfrak{S}_N$ be the symmetric group on $N$ and

$$X^N = \{(s_i)_{i \in N}: s_i's \text{ are elements of } X\},$$

a set of $|N|$-tuples of elements of $X$. We denote by $\mathfrak{S}_n$ the group $\mathfrak{S}_{\{1, \ldots, n\}}$ and by $X^n$ the set $X^{(1, \ldots, n)}$.

The $n$-fold symmetric product $\text{Sym}^n(X) = X^n/\mathfrak{S}_n$ of $X$ is the coarse moduli scheme of the quotient stack $[\text{Sym}^n(X)]$ defined as follows:

- an object over $U$ is a pair $(p: P \to U, f: P \to X^n)$ where $p$ is a principal $\mathfrak{S}_n$-bundle, and $f$ is a $\mathfrak{S}_n$-equivariant morphism;
- suppose that $(p': P' \to U', f': P' \to X^n)$ is another object, a morphism from $(p', f')$ to $(p, f)$ is a Cartesian diagram

\[
\begin{array}{ccc}
P' & \xrightarrow{\alpha} & P \\
\downarrow^{p'} & & \downarrow^p \\
U' & \xrightarrow{\beta} & U
\end{array}
\]

such that $f' = \alpha \circ f$.

The symmetric product stack $[\text{Sym}^n(X)]$ is a smooth orbifold with atlas $X^n \to [X^n/\mathfrak{S}_n]$.

1.2 Chen-Ruan cohomology

Let $[\text{Sym}^n(X)]$ be the stack of cyclotomic gerbes to $[\text{Sym}^n(X)]$. It is isomorphic to a disjoint union of orbifolds

$$\coprod_{[g] \in C} [X_g^n/C(g)],$$

where $C$ is the set of the conjugacy classes of $\mathfrak{S}_n$, $C(g)$ is the centralizer of $g$, $C(g)$ is the quotient group $C(g)/\langle g \rangle$, and $X_g^n$ is the $g$-fixed locus of $X^n$. Obviously, the connected components of $\mathcal{T}[\text{Sym}^n(X)]$ can be labeled with the partitions of $n$. If $[g]$ is the conjugacy class corresponds to the partition $\lambda$, we may write

$$X(\lambda) = X_g^n/C(g), \text{ and } \overline{X(\lambda)} = X_g^n/C(g).$$

The component $[X^n/\mathfrak{S}_n]$ is called the untwisted sector while all other components of the stack $\mathcal{T}[\text{Sym}^n(X)]$ are called twisted sectors.

The Chen-Ruan cohomology $A^*_{\text{orb}}([\text{Sym}^n(X)])$ is defined to be the cohomology $A^*(\mathcal{T}[\text{Sym}^n(X)])$ of the stack of cyclotomic gerbes in $X$. Thus, it is simply $\bigoplus_{[g] \in C} A^*(X_g^n/C(g)) = \bigoplus_{[g] \in C} A^*(X_g^n/C(g))$. (For any orbifold $\mathcal{Y}$ with coarse moduli space $Y$, we identify $A^*(\mathcal{Y})$ with $A^*(Y)$ via the pushforward $c_*: A^*(\mathcal{Y}) \to A^*(Y)$ defined by $c_*([\mathcal{V}]) = \frac{1}{s}[c(\mathcal{V})]$, where $\mathcal{V}$ is a closed integral substack and $s$ is the order of the stabilizer of a generic geometric point of $\mathcal{V}$.)
Additionally, for \( \alpha \in A^i(X(\lambda)) \), the orbifold (Chow) degree of \( \alpha \) is defined to be \( i + \text{age}(\lambda) \), where \( \text{age}(\lambda) = n - \ell(\lambda) \) is the age of the sector \([X(\lambda)]\). In other words,

\[
A^*_\text{orb}(\text{Sym}^n(X)) = \bigoplus_{|\lambda|=n} A^* - \text{age}(\lambda)(X(\lambda)).
\]

When \( X \) admits a \( T \)-action, we may put the above cohomologies into an equivariant context by considering \( T \)-equivariant cohomologies as the quotient schemes and orbifolds discussed above clearly inherit \( T \)-actions from \( X \).

### 1.3 Bases and fixed-point classes

In this section, we exhibit a basis for the equivariant Chen-Ruan cohomology \( A^*_\text{orb}(\text{Sym}^n(S)) \). It will be helpful for the determination of extended 3-point functions and for setting up our desired SYM-HILB correspondence later.

Given a partition \( \lambda \) of \( n \), we would like to construct a basis for the cohomology \( A^*_T(S_g^n)^C(g) \), where \( g \in \mathfrak{S}_n \) has cycle type \( \lambda \).

The permutation \( g \) has a cycle decomposition, i.e., a product of disjoint cycles (including 1-cycles),

\[ g = g_1 \cdots g_{\ell(\lambda)} \]

with \( g_i \) being a \( \lambda_i \)-cycle. For each \( i \), let \( N_i \) be the minimal subset of \( \{1, \ldots, n\} \) such that \( g_i \in \mathfrak{S}_{N_i} \). Thus \( |N_i| = \lambda_i \) and \( \prod_{i=1}^{\ell(\lambda)} N_i = \{1, \ldots, n\} \). It is clear that

\[ S^n_g = \prod_{i=1}^{\ell(\lambda)} S_{g_i}^{N_i}, \quad \text{and} \quad S_{g_i}^{N_i} \cong S. \]

To the partition \( \lambda \), we associate an \( \ell(\lambda) \)-tuple \( \vec{\eta} = (\eta_1 \cdots \eta_{\ell(\lambda)}) \) with entries in \( A^*_T(S)_m \). Let us put

\[
g(\vec{\eta}) = \frac{1}{|\text{Aut}(\lambda(\vec{\eta}))|} \prod_{i=1}^{\ell(\lambda)} \lambda_i \sum_{h \in C(g)} \bigotimes_{i=1}^{\ell(\lambda)} g^{b_i}_i(\eta_i) \in A^*_T(S_g^n)^C(g)^m. \tag{1.1}
\]

This requires some explanations:

- \( g^b_i = h^{-1} g_i h \).
- Let \( N \) be a subset of \( \{1, \ldots, n\} \). For each \( [N] \)-cycle \( \alpha \in \mathfrak{S}_N \) and \( \eta \) a class on \( S \), let \( \alpha(\eta) \) be the pullback of \( \eta \) by the obvious isomorphism \( S^N \cong S \).
- Two classes \( \bigotimes_{i=1}^{\ell(\lambda)} g^{b_1}_i(\eta_1) \) and \( \bigotimes_{i=1}^{\ell(\lambda)} g^{b_2}_i(\eta_2) \) on the space \( S^n_g \) coincide for some \( b_1, b_2 \in C(g) \), and a straightforward verification shows that each term \( \bigotimes_{i=1}^{\ell(\lambda)} g^{b_i}_i(\eta_i) \) repeats as many times. Hence, \((|\text{Aut}(\lambda(\vec{\eta}))| \prod_{i=1}^{\ell(\lambda)} \lambda_i)^{-1}\) is a normalization factor to ensure that no repetition occurs in \( \vec{\eta} \).
If \( g = k_1 \cdots k_{|\lambda|} \) is another cycle decomposition with \( \lambda \)-cycles \( k_i \)'s, then there exists \( h \in C(g) \) such that

\[
\bigotimes_{i=1}^{\ell(\lambda)} k_i(\eta_i) = \bigotimes_{i=1}^{\ell(\lambda)} g_i^h(\eta_i).
\]

Thus, the expression \( \boxed{1.1} \) is independent of the cycle decomposition.

Given a basis \( \mathcal{B} \) for \( A^*_\tau(S)_m \). The classes \( g(\overline{\eta}) \)'s, with \( \eta_i \)'s elements of \( \mathcal{B} \), form a basis for \( A^*_\tau(S)^{C(g)}_n \). (Note that if \( \hat{g} \) is another permutation of cycle type \( \lambda \), the classes \( g(\overline{\eta}) \) and \( \hat{g}(\overline{\eta}) \) are identical in \( A^*_\tau,orb([\text{Sym}^n(X)]) \).

From now on, we use the notation \( \lambda(\overline{\eta}) \) to stand for the class \( g(\overline{\eta}) \) in \( \boxed{1.1} \). The classes \( \lambda(\overline{\eta}) \)'s, running over all partitions \( \lambda \) of \( n \) and elements \( \eta_i \)'s of \( \mathcal{B} \), give a basis for the localized Chen-Ruan cohomology \( A^*_\tau,orb([\text{Sym}^n(S)])_m \).

We are going to work with the fixed-point basis \( \{[x_1], \ldots, [x_s]\} \). For partitions \( \lambda_1, \ldots, \lambda_s \), we use the multipartition \( \tilde{\lambda} := (\lambda_1, \ldots, \lambda_s) \) to indicate the class \( \lambda_1([x_1]) \cdots \lambda_s([x_s]) \) on \([\text{Sym}^n(U_i)]\) and give a basis for \( A^*_\tau,orb([\text{Sym}^n(U_i)])_m \).

### 1.4 Extended three-point functions

#### 1.4.1 Extended orbifold Gromov-Witten invariants

Given any positive integers \( m \) and \( n \), we denote by

\[
\overline{M}_{0,m}([\text{Sym}^n(S)])
\]

the moduli space parametrizing genus zero, \( m \)-pointed, twisted stable map \( f: (\mathcal{C}, \mathcal{P}_1, \ldots, \mathcal{P}_m) \rightarrow [\text{Sym}^n(S)] \) of degree 0. Note, in particular, that \( \mathcal{P}_i \cong B\mu_{r_i} \) is the classifying stack of the cyclic group \( \mu_{r_i} \) of order \( r_i \) for some positive integer \( r_i \). The map \( f \) is representable and comes equipped with the ordinary stable map \( f_c: (\mathcal{C}, c(\mathcal{P}_1), \ldots, c(\mathcal{P}_m)) \rightarrow \text{Sym}^n(S) \), where \( C \) is the coarse moduli space of \( \mathcal{C} \). For more information on twisted stable maps, consult [CR2] or [AGV].

Denote the \( i \)-th evaluation map by

\[
ev_i: \overline{M}_{0,m}([\text{Sym}^n(S)]) \rightarrow \mathcal{T}[[\text{Sym}^n(S)].
\]

At the level of \( \text{Spec}(\mathbb{C}) \)-points, \( \ev_i \) is defined by sending \( [f]: (\mathcal{C}, \mathcal{P}_1, \ldots, \mathcal{P}_m) \rightarrow [\text{Sym}^n(S)] \) to \( [f]_{\mathcal{P}_i}: \mathcal{P}_i \rightarrow [\text{Sym}^n(S)] \). For partitions \( \sigma_1, \ldots, \sigma_m \) of \( n \), let

\[
\overline{M}([\text{Sym}^n(S)], \sigma_1, \ldots, \sigma_m; a) = \bigcap_{i=1}^m \ev_i^{-1}(X(\sigma_i)) \cap \bigcap_{j=1}^a \ev^{-1}_{m+j}(X(2))
\]

be an open and closed substack of the moduli space \( \overline{M}_{m+a}([\text{Sym}^n(S)]) \). For ease of explanation, we refer to the last \( a \) markings of \([f] \in \overline{M}([\text{Sym}^n(S)], \sigma_1, \ldots, \sigma_m; a) \) as simple marked points.
Given any \( \alpha_i \in A_T^*[\text{Sym}^n(S)] \) and any partition \( \sigma_i \) of \( n \) for \( i = 1, \ldots, m \), the \( m \)-point extended Gromov-Witten invariant of degree 0 is defined by

\[
\langle \alpha_1, \ldots, \alpha_m \rangle_{\text{Sym}^n(S)} = \frac{1}{a!} \sum_{|\sigma_1|, \ldots, |\sigma_m| = n} \int_{[\text{Sym}^n(S), \sigma_1, \ldots, \sigma_m; a]} \text{ev}_1^*(\alpha_1) \cdots \text{ev}_m^*(\alpha_m)
\]

(cf. [BG]). Here \([ \ ]_{\text{vir}}^T\) represents the equivariant virtual class. Note that the underlying moduli space is not necessarily compact, but the above definition makes sense when all \( \alpha_i \)'s are \( T \)-fixed point classes because the space of twisted stable maps meeting these classes is compact. Since \( T \)-fixed point basis spans the equivariant Chen-Ruan cohomology \( A^*_T([\text{Sym}^n(S)]) \), the invariant \( \langle \alpha_1, \ldots, \alpha_m \rangle_{\text{Sym}^n(S)} \) with insertions being any Chen-Ruan classes on \([\text{Sym}^n(S)]\) can be defined by writing each \( \alpha_i \) in terms of fixed-point classes and by linearity. Another interpretation of (2.3) is to treat the integral as a sum of residue integrals over \( T \)-fixed connected components of \([\text{Sym}^n(S), \sigma_1, \ldots, \sigma_m; a]\) via virtual localization formula. The invariants in both treatments take values in \( \mathbb{Q}(t_1, t_2) \).

We are primarily interested in 3-point extended invariants, and we encode them in a generating function: For classes \( \alpha_1, \alpha_2, \alpha_3 \in A_T^*[\text{Sym}^n(S)] \), set

\[
\langle \alpha_1, \alpha_2, \alpha_3 \rangle_{\text{Sym}^n(S)}(u) = \sum_{a=0}^{\infty} \langle \alpha_1, \alpha_2, \alpha_3 \rangle_{a} \text{Sym}^n(S) u^a.
\]

We refer to these functions as extended 3-point functions.

Note that the Chen-Ruan product [CR1] is given by the (non-extended) 3-point orbifold Gromov-Witten invariants of degree 0 (i.e., setting \( u = 0 \)), and so the extended 3-point functions do provide more enumerative information than the Chen-Ruan product.

### 1.4.2 The product formula

First of all, we use the notation

\( \vec{g} = (g_1, \ldots, g_s) \)

to represent an \( s \)-tuple of partitions or an \( s \)-tuple of nonnegative integers. For the case of integers, if all entries of \( \vec{g} \) sum to \( \ell \), we say that

\[ |\vec{g}| = \ell. \]

**Fixed loci.** Let \( f : C \to [\text{Sym}^n(S)] \) be any \( T \)-fixed twisted stable map. It naturally comes with the following diagram

\[ P \xrightarrow{f'} S^n \]

\[ \pi \downarrow \quad \downarrow \]

\[ C \xrightarrow{f} [\text{Sym}^n(S)] \]

\[ c \downarrow \quad \downarrow c \]

\[ C \xrightarrow{f_c} \text{Sym}^n(S). \]

Here \( P = C \times_{[\text{Sym}^n(S)]} S^n \), and by the representability of \( f \), the fiber product \( P \) is a scheme. Moreover, taking \( f' \) modulo \( S_{n-1} \) and composing with the \( n \)-th projection, we have a map

\[ f : \tilde{C} \to S^\ell \subseteq S \]
where $\tilde{C} = P/\mathfrak{S}_{n-1}$ is in fact a degree $n$ admissible cover of $C$.

Given $s$-tuples $\sigma_1, \ldots, \sigma_m$ of partitions and an $s$-tuple $\vec{a}$ of nonnegative integers with $|\vec{a}| = d$. Let

$$\mathcal{M}(\vec{\sigma}_1, \ldots, \vec{\sigma}_m; \vec{a})$$

be the locus in $\overline{\mathcal{M}}([\text{Sym}^n(S)], \sigma'_1, \ldots, \sigma'_m; d)$ (here $\sigma'_i$ admits a decomposition $(\sigma_{i1}, \ldots, \sigma_{is})$) which parametrizes $T$-fixed, $(m + d)$-pointed, twisted stable maps $f: C \to [\text{Sym}^n(S)]$ of degree zero such that the associated admissible cover $\tilde{C}$ is ramified with monodromy $\sigma_1, \ldots, \sigma_m, (2)^d$ over $C$ and with the following configuration:

- The cover $\tilde{C}$ may be written as a disjoint union

$$\prod_{k=1}^s \tilde{C}_k$$

of curves, but $\tilde{C}_k$ is possibly empty or disconnected. Each $\tilde{C}_k$, if nonempty, is contracted by $\tilde{f}$ to $x_k$.

- The cover $\tilde{C}_k \to C$ is ramified with monodromy $\sigma_{1k}, \ldots, \sigma_{mk}, (2)^{a_k}$ for every $k = 1, \ldots, s$.

Suppose that $|\sigma_{ik}| = n_k$ for some $n_k$, $\forall i = 1, \ldots, m$. We have a natural morphism

$$\phi: \mathcal{M}(\vec{\sigma}_1, \ldots, \vec{\sigma}_m; \vec{a}) \to \prod_{k=1}^s \overline{\mathcal{M}}([\text{Sym}^{n_k}U_k], \sigma_{1k}, \ldots, \sigma_{mk}; a_k)^T$$

defined as follows: Let $[f: C \to [\text{Sym}^n(S)]]$ be an element of $\mathcal{M}(\vec{\sigma}_1, \ldots, \vec{\sigma}_m; \vec{a})$. For each $k = 1, \ldots, s$, we stabilize the target of the covering $\tilde{C}_k \to C$ and the domain accordingly (by forgetting those simple markings of $C$ over which the points of $\tilde{C}_k$ are unramified). The output is the following setting

$$\begin{array}{c}
\tilde{C}_k^{\text{st}} \\
\downarrow
\end{array} \to \{x_i\}$$

with the vertical map being an admissible covering of degree $n_k$. It gives rise to a $T$-fixed twisted stable map, which we denote by

$$f_k: C_k \to [\text{Sym}^{n_k}(U_k)].$$

The map $f_k$ represents a $T$-fixed point of $\overline{\mathcal{M}}([\text{Sym}^{n_k}U_k], \sigma_{1k}, \ldots, \sigma_{mk}; a_k)$. We then take $\phi([f]) = ([f_1], \ldots, [f_s])$.

Now we focus on $m = 3$, in which case the morphism $\phi$ is surjective, and both its source and target have dimension $d$. Given any $T$-fixed connected component $F(\vec{a}) \subseteq \mathcal{M}(\vec{\sigma}_1, \vec{\sigma}_2, \vec{\sigma}_3; \vec{a})$, and denote by $\pi_k$ the $k$th projection, we let

$$F_k(\vec{a}) = \pi_k \circ \phi(F(\vec{a})).$$

The collection $\prod_{k=1}^s F_k(\vec{a})'$s form a complete set of $T$-fixed connected components of the product space $\prod_{k=1}^s \overline{\mathcal{M}}([\text{Sym}^{n_k}U_k], \sigma_{1k}, \sigma_{2k}, \sigma_{3k}; a_k)$.
Determination. We want to investigate the invariant $\langle \tilde{\lambda}, \tilde{\mu}, \tilde{\nu} \rangle_{d}[\text{Sym}^n(S)]$. It is clearly zero if the condition

$$|\lambda_k| = |\mu_k| = |\nu_k| = n_k \text{ for each } k = 1, \ldots, s$$

fails. In general, we have the following product formula.

**Proposition 1.1.** Given any $T$-fixed point classes $\tilde{\lambda}, \tilde{\mu}, \tilde{\nu}$,

$$\langle \tilde{\lambda}, \tilde{\mu}, \tilde{\nu} \rangle_{d}[\text{Sym}^n(S)] = \sum_{a_1 + \cdots + a_s = d} \prod_{k=1}^s \langle \lambda_k, \mu_k, \nu_k \rangle_{a_k}[\text{Sym}^n(U_k)].$$

**Proof.** The statement is obvious when (1.3) does not hold. So let us assume (1.3). The only fixed loci that can make contribution to the 3-point extended invariant are $M(x, x, x)$'s with $|\vec{a}| = d$. Precisely, (1.4) is given by

$$\frac{1}{d!} \sum_{|\vec{a}| = d} \sum_{F(\vec{a})} \int_{F(\vec{a})} \frac{\iota_F^*(\text{ev}_1^*(\tilde{\lambda}) \cdot \text{ev}_2^*(\tilde{\mu}) \cdot \text{ev}_3^*(\tilde{\nu}))}{e_T(N_{F(\vec{a})}^\text{vir})},$$

where $F(\vec{a}) \subset M(\tilde{x}, \tilde{\mu}, \tilde{\nu}; \vec{a})$ runs over all $T$-fixed connected components. For an arbitrary $T$-fixed component $F(\vec{a}) \subset M(\tilde{x}, \tilde{\mu}, \tilde{\nu}; \vec{a})$ and $[f] \in F(\vec{a})$, we have

$$e_T(H^i(C, f^*T[\text{Sym}^n(S)])) = \phi^* \otimes \prod_{k=1}^s e_T(H^i(C_k, f_k^*T[\text{Sym}^n(U_k)]))$$

for $i = 0, 1$, following the notation of (1.2). As a result,

$$e_T(N_{F(\vec{a})}^\text{vir}) = \phi^* \otimes \prod_{k=1}^s e_T(N_{F_k(\vec{a})}^\text{vir}).$$

Moreover, for each fixed-point class $\tilde{\sigma}$,

$$e_T(T[\tilde{x}]I[\text{Sym}^n(S)]) = \prod_{k=1}^s e_T(T_{\tilde{x}_k}U_k)^{\ell(\sigma_k)} = \prod_{k=1}^s e_T(T[\tilde{x}_k]I[\text{Sym}^n(U_k)]).$$

This forces

$$\iota_F^*(\text{ev}_1^*(\tilde{\lambda}) \cdot \text{ev}_2^*(\tilde{\mu}) \cdot \text{ev}_3^*(\tilde{\nu})) = \prod_{k=1}^s \iota_{F_k(\vec{a})}^*(\text{ev}_1^*(\lambda_k) \cdot \text{ev}_2^*(\mu_k) \cdot \text{ev}_3^*(\nu_k)).$$

Hence the contribution of $M(\tilde{\lambda}, \tilde{\mu}, \tilde{\nu}; \vec{a})$ to (1.4) equals

$$\frac{1}{a_1! \cdots a_s!} \sum_{F(\vec{a})} \prod_{k=1}^s \iota_{F_k(\vec{a})}^*(\text{ev}_1^*(\lambda_k) \cdot \text{ev}_2^*(\mu_k) \cdot \text{ev}_3^*(\nu_k))}{e_T(N_{F(\vec{a})}^\text{vir})}.$$
where the prefactor accounts for the distribution of simple marked points. The sum is nothing but

$$\prod_{k=1}^{s} \frac{1}{a_k!} \sum_{F_k(\vec{a})} \int_{F_k(\vec{a})} t_{F_k(\vec{a})}^i \left( \frac{e_T^{\nu_k} (\tilde{\lambda}_k \cdot \tilde{\mu}_k \cdot \tilde{\nu}_k)}{e_T (\Lambda_{F_k(\vec{a})}^{\nu_k})} \right).$$

Since $F_k(\vec{a})$’s run through all connected components of $\overline{M}([\text{Sym}^n U_k], \tilde{\lambda}_k, \tilde{\mu}_k, \tilde{\nu}_k; a_k)^T$ (we treat $\tilde{\lambda}_k, \tilde{\mu}_k, \tilde{\nu}_k$ as partitions of $n_k$), \cite{A3} and $\sum_{|\vec{a}|=d} \prod_{k=1}^{s} (\tilde{\lambda}_k, \tilde{\mu}_k, \tilde{\nu}_k)^{\text{[Sym}^n (U_k)]}$ coincide.

Put another way, we have the following.

**Corollary 1.2.** For any $\mathbb{T}$-fixed point classes $\tilde{\lambda}, \tilde{\mu}, \tilde{\nu},$

$$\langle \tilde{\lambda}, \tilde{\mu}, \tilde{\nu} \rangle^{\text{[Sym}^n (S)]}_{\text{Sym}^n U_k} = \prod_{k=1}^{s} \langle \tilde{\lambda}_k, \tilde{\mu}_k, \tilde{\nu}_k \rangle^{\text{[Sym}^n (U_k)]}_{\text{Sym}^n U_k}.$$

Moreover, any extended 3-point function is a rational function in $t_1, t_2, v^n$, where $i^2 = -1$.

**Proof.** The first statement is immediate from Proposition 1.1. The second statement is due to the fact that each $\langle \tilde{\lambda}_k, \tilde{\mu}_k, \tilde{\nu}_k \rangle^{\text{[Sym}^n (U_k)]}_{\text{Sym}^n U_k}$ is an element of $\mathbb{Q}(t_1, t_2, v^n)$ \cite{OP, BG}.

The orbifold Poincaré pairing $\langle \bullet | \bullet \rangle$ on $A^*_{\mathbb{T}, \text{orb}}([[\text{Sym}^n U_k]])_m$ is determined by

$$\langle \tilde{\lambda}_k | \tilde{\mu}_k \rangle = \delta_{\lambda_k, \mu_k} (L_k R_k)^{f(\lambda_k)} \frac{1}{\delta_{\lambda_k}}, |\lambda_k| = |\mu_k| = n_k$$

where $\delta_{\lambda_k, \mu_k}$ stands for the Kronecker delta. The argument of Proposition 1.1 may be applied to show that the orbifold pairing on $\text{Sym}^n (S)$ is expressible in terms of those on $\text{Sym}^n U_k$’s.

**Proposition 1.3.** The equivariant orbifold Poincaré pairing on $\text{Sym}^n (S)$ is determined by the formula:

$$\langle \tilde{\lambda} | \tilde{\mu} \rangle = \prod_{k=1}^{s} \delta_{\lambda_k, \mu_k} (L_k R_k)^{f(\lambda_k)} \frac{1}{\delta_{\lambda_k}}.$$

Thus, $\tilde{\lambda}$’s provide an orthogonal basis for $A^*_{\mathbb{T}, \text{orb}}([[\text{Sym}^n (S)])_m$.

## 2 Hilbert scheme of points

### 2.1 Fixed-point basis and Nakajima basis

**Fixed-point basis.** As seen in \cite{ES}, there is a one-to-one correspondence between partitions of $n$ and $\mathbb{T}$-fixed points of $\text{Hilb}^n (\mathbb{C}^2)$. Since $U_i \cong \mathbb{C}^2$, for every partition $\lambda$, the corresponding $\mathbb{T}$-fixed point $\lambda(x_i) \in \text{Hilb}^{|\lambda|} (U_i)$ can be described as follows: Suppose $\mathbb{C}[u, v]$ is the coordinate ring of $U_i$, then $\lambda(x_i)$ is the subscheme of $U_i$ with ideal $I_{\lambda(x_i)}$ being

$$(u^{\lambda_1}, u^{\lambda_2}, \ldots, v^{f(\lambda)} - 1 u^{\lambda_{e(\lambda)}}, v^{f(\lambda)}).$$

The point $\lambda(x_i)$ is supported at $x_i$ and is mapped to $|\lambda| \cdot [x_i] \in \text{Sym}^{|\lambda|} (U_i)$ by the Hilbert-Chow morphism.
The action of \( \mathbb{T} \) on \( S \) lifts to \( \text{Hilb}^n(S) \). Each element of the fixed locus \( \text{Hilb}^n(S)^\mathbb{T} \) has support in \( S^2 = \{ x_1, \ldots, x_s \} \), and \( \text{Hilb}^n(S)^\mathbb{T} \) is isolated. Each \( \mathbb{T} \)-fixed point of \( \text{Hilb}^n(S) \) is the sum
\[
\lambda_1(x_1) + \cdots + \lambda_s(x_s)
\]
for some partitions \( \lambda_i \)’s satisfying \( \sum_{i=1}^s |\lambda_i| = n \) (by “the sum” we mean the disjoint union of \( \lambda_i(x_i) \)'s). For \( \tilde{\lambda} = (\lambda_1, \ldots, \lambda_s) \), we write
\[
I_{\tilde{\lambda}} = [\lambda_1(x_1) + \cdots + \lambda_s(x_s)],
\]
which are \( \mathbb{T} \)-fixed point classes of \( \text{Hilb}^n(S) \) and form a basis for the localized cohomology \( A^*_T(\text{Hilb}^n(S))_m \).

**Nakajima basis.** Another important basis for \( A^*_T(\text{Hilb}^n(S)) \) is the Nakajima basis, which we now describe. For further details, consult [Gro, N1, N2, V, LQW].

Given a partition \( \lambda \) of \( n \) and an \( \ell(\lambda) \)-tuple \( \vec{\eta} = (\eta_1, \ldots, \eta_{\ell(\lambda)}) \) with entries in \( \mathbb{A}^*_T(\text{Hilb}^n(S)) \). We define
\[
a_\lambda(\vec{\eta}) = \frac{1}{|\text{Aut}(\lambda(\vec{\eta}))|} \prod_{i=1}^{\ell(\lambda)} \frac{1}{\lambda_i} p_{-\lambda_i}(\eta_i)|0\rangle,
\]
where \( |0\rangle = 1 \in A^0_T(\mathbb{S}^{[0]}) \), and \( p_{-\lambda_i}(\eta_i) : A^*_T(S^{[k]}) \to A^{*+\lambda_i-1+\text{deg}(\eta_i)/2}_T(S^{[k+\lambda_i]}) \) are Heisenberg creation operators. (We also denote the class \( a_\lambda(\vec{\eta}) \) by \( a_{\lambda_1}(\eta_1) \cdots a_{\lambda_{\ell(\lambda)}}(\eta_{\ell(\lambda)}) \)).

Choose a basis \( \mathfrak{B} \) for \( A^*_T(\mathbb{S}) \). The classes \( a_\lambda(\vec{\eta}) \)'s, running through all partitions \( \lambda \) of \( n \) and all \( \eta_i \in \mathfrak{B} \), give a basis for \( A^*_T(\text{Hilb}^n(S)) \). They are referred to as the Nakajima basis with respect to \( \mathfrak{B} \).

We may also work with the Nakajima basis with respect to the \( \mathbb{T} \)-fixed point classes
\[
[x_1], \ldots, [x_s].
\]
For partitions \( \lambda_1, \ldots, \lambda_s \) of \( n_1, \ldots, n_s \) respectively, we define \( \ell(\tilde{\lambda}) = \sum_{i=1}^s \ell(\lambda_i) \) and
\[
a_{\tilde{\lambda}} = a_{\lambda_1}(\{x_1\}) \cdots a_{\lambda_{\ell(\lambda_1)}}(\{x_1\}) \cdots a_{\lambda_{\ell(\lambda_s)}}(\{x_s\}) \cdot a_{\lambda_{\ell(\lambda_s)}}(\{x_s\}).
\]
The Chow degree of \( a_{\tilde{\lambda}} \) is
\[
\sum_{i=1}^s (|\lambda_i| - \ell(\lambda_i)) + 2\ell(\tilde{\lambda}) = (n - \ell(\tilde{\lambda})) + 2\ell(\tilde{\lambda}) = n + \ell(\tilde{\lambda}).
\]
In the case of \( U_i \), \( x_i \) is the unique \( \mathbb{T} \)-fixed point. Hence,
\[
a_{\lambda_i},
\]
denoting the classes \( a_{\lambda_{\ell(\lambda_1)}}(\{x_1\}) \cdots a_{\lambda_{\ell(\lambda_s)}}(\{x_s\}) |0\rangle \) (with \( |\lambda_i| = n_i \) on \( \text{Hilb}^{n_i}(U_i) \)), form a basis for \( A^*_T(\text{Hilb}^{n_i}(U_i))_m \). The equivariant Poincaré pairing \( \langle \bullet | \bullet \rangle \) of \( A^*_T(\text{Hilb}^{n_i}(U_i))_m \) is determined by the formula
\[
\langle a_{\tilde{\lambda}} | a_{\tilde{\mu}} \rangle = \delta_{\lambda_i, \mu_i} (-1)^{\ell(\lambda_i) - \ell(\mu_i)} (L_i R_i)^{\ell(\lambda_i)} \frac{1}{\delta_{\lambda_i}} \big| \lambda_i = |\mu_i| = n_i, \quad (2.1)
\]
2.2 Comparison to symmetric functions

Let \( p_i(z) = \sum_{k=1}^\infty z_k^i \) be the \( i \)th power sum. Given a partition \( \mu \), write

\[
p_\mu(z) = \frac{1}{|\text{Aut}(\mu)|} \prod_{i=1}^{\ell(\mu)} \mu_i p_{\mu_i}(z).
\]

This family of symmetric functions forms a basis for the ring \( \mathcal{R}_{\text{Sym}} \) of symmetric functions over \( \mathbb{Q}(t_1, t_2) \). Let \( \alpha_i = R_i/L_i \). We denote the integral Jack symmetric functions corresponding to \( \alpha_i \) and the partitions \( \mu \) by \( J_{\alpha_i \mu}(z) \), which actually provide an orthogonal basis for \( \mathcal{R}_{\text{Sym}} \); see [S].

The relationship between \( T \)-fixed point basis and Nakajima basis is indeed the relationship between Jack symmetric functions and the power sums. More precisely, the Nakajima basis element \( a_{(\lambda_1, \ldots, \lambda_s)}(z) \) is identified with

\[
\bigotimes_{i=1}^s L^{f(\lambda_i)}_{\lambda_i}(z(i))
\]

while the \( T \)-fixed point class \( I_{(\mu_1, \ldots, \mu_s)} \) is identified with

\[
\bigotimes_{i=1}^s L^{J_{\mu_i}(\mu_i)}_{\mu_i}(z(i)).
\]

For more details, see [NL], [V] or [LQW].

For \( i = 1, \ldots, s \), let \( \lambda_i \) be a partition of \( n_i \). As the fixed-point classes \( [\mu_i(x_i)] \)’s (with \( |\mu_i| = n_i \)) span \( A^*_T(\text{Hilb}^n(S)) \), we can write \( a_{\lambda_i} = \sum_{|\mu_i| = n_i} c_{\lambda_i, \mu_i} [\mu_i(x_i)] \) for some \( c_{\lambda_i, \mu_i} \in \mathbb{Q}(t_1, t_2) \). By the above identifications,

\[
a_{(\lambda_1, \ldots, \lambda_s)} = \sum c_{\lambda_1, \mu_1} \cdots c_{\lambda_s, \mu_s} I_{(\mu_1, \ldots, \mu_s)}
\]  \hspace{1cm} (2.2)

where the sum is over partitions \( \mu_i \) of \( n_i \), \( i = 1, \ldots, s \).

2.3 Extremal three-point functions

2.3.1 Extremal invariants and quantum corrected cohomology

The kernel of the morphism

\[
\rho_* : A_1(\text{Hilb}^n(S); \mathbb{Z}) \to A_1(\text{Sym}^n(S); \mathbb{Z})
\]

is one-dimensional and is generated by an effective rational curve class \( \beta_n \) that is dual to \( -a_2(1)a_1(1)^{n-2} \). For every positive integer \( k \), we let

\[
\overline{\mathcal{M}}_{0,k}(\text{Hilb}^n(S), d)
\]

be the moduli space parametrizing stable maps from genus zero, \( k \)-pointed, nodal curves to \( \text{Hilb}^n(S) \) of degree \( di\beta_n \).

Let \( e_i : \overline{\mathcal{M}}_{0,k}(\text{Hilb}^n(S), d) \to \text{Hilb}^n(S) \) be the evaluation map at the \( i \)th marked point. Although \( \text{Hilb}^n(S) \) is not necessarily compact, the \( k \)-point, \( T \)-equivariant, extremal Gromov-Witten invariant

\[
\langle \alpha_1, \ldots, \alpha_k \rangle_{\text{Hilb}^n(S)} = \int_{\overline{\mathcal{M}}_{0,k}(\text{Hilb}^n(S), d)} e_1^*([\alpha_1]) \cdots e_k^*([\alpha_k])
\]  \hspace{1cm} (2.3)
is well-defined by a similar treatment in Section \textsuperscript{[22]}. We will explore the following extremal 3-point functions of Hilb\textsuperscript{n}(S):  
\[ \langle \alpha_1, \alpha_2, \alpha_3 \rangle_{\text{Hilb}\textsuperscript{n}(S)} = \sum_{d=0}^{\infty} \langle \alpha_1, \alpha_2, \alpha_3 \rangle_{d}^{\text{Hilb}\textsuperscript{n}(S)} q^d. \]

The quantum corrected cohomology \textsuperscript{[R]} is defined by the above generating function with some specialization. Indeed, the quantum corrected product \( \cup_{\text{qc}} \) for \( A^*_c(\text{Hilb}\textsuperscript{n}(S)) \) is defined as follows: Given any classes \( a, b \in A^*_c(\text{Hilb}\textsuperscript{n}(S)) \), \( a \cup_{\text{qc}} b \) is defined to be the unique element satisfying 
\[ \langle a \cup_{\text{qc}} b | c \rangle = \langle a, b, c \rangle_{\text{Hilb}\textsuperscript{n}(S)} q^{d_{\text{qc}}}, \quad \forall c \in A^*_c(\text{Hilb}\textsuperscript{n}(S)). \]

The vector space \( A^*_c(\text{Hilb}\textsuperscript{n}(S)) \otimes_{\mathbb{Q}[t_1,t_2]} \mathbb{Q}(t_1,t_2) \) endowed with the multiplication \( \cup_{\text{qc}} \) is referred to as the quantum corrected cohomology ring of Hilb\textsuperscript{n}(S). Note also that the cup product \( \cup \) on the Hilb\textsuperscript{n}(S) is given by
\[ \langle a \cup b | c \rangle = \langle a, b, c \rangle_{\text{Hilb}\textsuperscript{n}(S)} q^{d}. \]

### 2.3.2 The product formula

For each \( \mathbb{T} \)-fixed connected component \( \Gamma \), we denote by \( \gamma : \Gamma \to \overline{\mathcal{M}}_{0,3}(\text{Hilb}\textsuperscript{n}(S), \delta) \) the natural inclusion and by \( N_\nu^\mathbb{T} \) the virtual normal bundle to \( \Gamma \).

In this section, we will see that the 3-point invariant in \( a_\lambda, a_\mu, a_\nu \) may be expressed in terms of Gromov-Witten invariants of Hilbert schemes of points in \( \mathbb{C}^2 \). First of all, let us see a vanishing statement.

**Proposition 2.1.** \( \langle a_\lambda, a_\mu, a_\nu \rangle_{d}^{\text{Hilb}\textsuperscript{n}(S)} \) does not vanish only if
\[ |\lambda_i| = |\mu_i| = |\nu_i| \quad \text{for each} \quad i = 1, \ldots, s. \quad (2.4) \]

**Proof.** Suppose (2.4) fails, we would like to show \( \gamma^*(e^1(I_\delta) \cdot e^2(I_\delta) \cdot e^3(I_\delta)) = 0 \) for every connected component \( \Gamma \). By (2.2), we merely have to verify
\[ \gamma^*(e^1(I_\delta) \cdot e^2(I_\delta) \cdot e^3(I_\delta)) = 0 \quad (2.5) \]
for \( |\sigma_i| = |\lambda_i|, |\tau_i| = |\mu_i|, |\theta_i| = |\nu_i|, \forall i = 1, 2, 3. \) We note that the images of all \( \mathbb{T} \)-fixed stable maps in \( \Gamma \) go to the same point after composition with the Hilbert-Chow morphism. That is, \( \rho \circ e_i(\delta) \)'s are the same for each \( i \), which means that at least one of \( e^1(I_\delta) \cdot e^2(I_\delta) \cdot e^3(I_\delta) \) does not meet \( \Gamma \). Thus, (2.5) follows. \( \square \)

It remains to study the 3-point invariants
\[ \langle a_\lambda, a_\mu, a_\nu \rangle^{\text{Hilb}\textsuperscript{n}(S)} \]
under condition (2.4): \( n_i := |\lambda_i| = |\mu_i| = |\nu_i| \) for each \( i = 1, \ldots, s \) and \( \sum_{i=1}^{s} n_i = n \). We fix such \( n_i \) and partitions \( \lambda_i, \mu_i, \nu_i \) of \( n_i \) throughout the remainder of this section. Let
\[ U = \text{Hilb}\textsuperscript{n_1}(U_1) \times \cdots \times \text{Hilb}\textsuperscript{n_s}(U_s), \quad P = \rho_{s}^{-1}(n_1[x_1] + \cdots + n_{s}[x_{s}]). \]

In fact, \( P \cong \rho_{s}^{-1}(n_1[x_1]) \times \cdots \times \rho_{s}^{-1}(n_{s}[x_{s}]) \subseteq U \), where \( \rho_s : \text{Hilb}\textsuperscript{n_i}(S) \to \text{Sym}\textsuperscript{n_i}(S) \) is the Hilbert-Chow morphism, \( \forall i \). (In case \( n_i = 0 \), \( \rho_{s}^{-1}(n_i[x_i]) \) will be missing from the product.)

Let \( N = \{ i = 1, \ldots, s | n_i \geq 1 \} \). As each \( \rho_{s}^{-1}(n_i[x_i]) \) is irreducible and has complex dimension \( n_i - 1 \) for \( i \in N \), \( P \) is then irreducible and has dimension \( n - |N| \).
Let $\xi = \mu_1(x_1) + \cdots + \mu_s(x_s) \in S[3]$ and $\xi_{\bar{m}} = \mu_1(x_1) \times \cdots \times \mu_s(x_s) \in U$. We have $T_{\xi_{\bar{m}}}U = T_{\xi}S[3]$. Indeed,

$$T_{\xi_{\bar{m}}}U = \bigoplus_{i \in N} \text{Hom}_{O_T}(I_{\mu_i(x_i)}, O_{\mu_i(x_i)})$$

$$= \bigoplus_{i \in N} \text{Hom}_{O_T}(I_{\xi, x_i}, O_{\xi, x_i})$$

$$= \text{Hom}_{O_T}(I_{\xi}, O_{\xi})$$

$$= T_{\xi}S[3].$$

Denote by $\iota_P, j_P$ the inclusion of $P$ into $\text{Hilb}^n(S)$ and $U$ respectively. We have a simple lemma.

**Lemma 2.2.** $\iota_P^*(a_{\overline{\xi}}) = j_P^*(a_{\overline{\xi}} \otimes \cdots \otimes a_{\overline{\xi}})$.

**Proof.** By (2.2), it suffices to show that

$$\iota_P^* I_{(\mu_1, \ldots, \mu_s)} = j_P^* [\iota_1^*(\mu_1(x_1)) \otimes \cdots \otimes \iota_s^*(\mu_s(x_s))].$$

(2.6)

Let $\xi$ and $\xi_{\bar{m}}$ be the points as in the discussion preceding the lemma. We can see that the left side of (2.6) is given by

$$\sum_{\eta \in P^T} i_{\eta} i_{\iota_P}^* (\iota_P \circ i_\eta)^* I_{(\mu_1, \ldots, \mu_s)} e_T(N_{\eta/P}) = i_{\xi} e_T(T_{\xi}S[3]),$$

where $i_\eta : \{\eta\} \to P$ is the natural inclusion. Similarly, the right side of (2.6) coincides with $i_{\xi_{\bar{m}}} e_T(T_{\xi_{\bar{m}}}U) / e_T(N_{\xi_{\bar{m}}/(P)})$. Thus, the equality (2.6) follows from $T_{\xi_{\bar{m}}}U = T_{\xi}S[3]$. \(\blacksquare\)

**Proposition 2.3.** The 3-point extremal Gromov-Witten invariants of $\text{Hilb}(S)$ can be expressed in terms of the invariants of $\text{Hilb}(U_i)$’s. Precisely,

$$\langle a_{\overline{\lambda}}, a_{\overline{\mu}}, a_{\overline{\nu}} \rangle_{\text{Hilb}^n(S)} = \sum_{d_1 + \cdots + d_s = d} \prod_{i=1}^s \langle a_{\overline{\lambda}}, a_{\overline{\mu}}, a_{\overline{\nu}} \rangle_{\text{Hilb}^n(U_i)}.$$

**Proof.** If condition (2.4) fails, the equation clearly holds by Proposition 2.1.

Now we assume condition (2.4), i.e., $n_i := |\lambda_i| = |\mu_i| = |\nu_i|$ for each $i$. To determine the 3-point invariant $\langle a_{\overline{\lambda}}, a_{\overline{\mu}}, a_{\overline{\nu}} \rangle_{\text{Hilb}^n(S)}$, we only need to consider those connected components of $\overline{M}_{0,3}(\text{Hilb}^n(S), d)^T$ whose images under the map $\rho \circ e_i$ are the point

$$n_1[x_1] + \cdots + n_s[x_s], \ \forall \ i = 1, 2, 3.$$

Observe that any $T$-fixed stable map $f$, representing an element in these components, factors through $P$. Hence, we may calculate $\langle a_{\overline{\lambda}}, a_{\overline{\mu}}, a_{\overline{\nu}} \rangle_{d}$ over connected components lying in

$$\prod_{d_1 + \cdots + d_s = d} \overline{M}_{0,3}(P, (d_1, \ldots, d_s))^T.$$

Write $\Gamma_{d_1, \ldots, d_s}$ for $\Gamma$ whenever $\Gamma$ is contained in the moduli space $\overline{M}_{0,3}(P, (d_1, \ldots, d_s))$. On the other hand, we also note that $\Gamma_{d_1, \ldots, d_s}$’s form a complete set of $T$-fixed connected components of
\( \overline{M}_{0,3}(U, (d_1, \ldots, d_s)) \). The following diagram summarizes their relationships:

\[
\begin{array}{ccc}
\overline{M}_{0,3}(U, (d_1, \ldots, d_s)) & \xrightarrow{\gamma_U} & \overline{M}_{0,3}(P, (d_1, \ldots, d_s)) \xleftarrow{\gamma_P} \overline{M}_{0,3}(\text{Hilb}^n(S), d_1 + \cdots + d_s) \\
\end{array}
\]

Here \( \gamma_U, \gamma_P \) are the natural inclusion \( \gamma \) with the target replaced with \( \overline{M}_{0,3}(U, (d_1, \ldots, d_s)) \) and \( \overline{M}_{0,3}(P, (d_1, \ldots, d_s)) \) respectively.

We would like to show that

\[
\langle \gamma \rangle \gamma_U^s \gamma_P^s \gamma = \sum_{d_1 + \cdots + d_s = d} \langle \otimes_{i=1}^{s} \partial_{\gamma}, \otimes_{i=1}^{s} \partial_{\gamma_i} \rangle_{(d_1, \ldots, d_s)}.
\]  

(2.7)

Denote by \( e_i : \overline{M}_{0,3}(P, (d_1, \ldots, d_s)) \to P \) the evaluation map at the \( i \)-th marked point. The invariant \( \langle \partial_{\gamma}, \partial_{\gamma_i} \rangle_{d_1 + \cdots + d_s = d} \) is

\[
\sum_{d_1 + \cdots + d_s = d} \int_{\Gamma_{d_1, \ldots, d_s}} \frac{\gamma_P^* (e_1^* \nu_P^*(\partial_{\gamma_i}) \cdot e_2^* \nu_P^*(\partial_{\gamma}) \cdot e_3^* \nu_P^*(\partial_{\gamma}))}{e_T(N_{d_1, \ldots, d_s}^{vir})}.
\]

On the other hand, we find that

\[
\sum_{\Gamma_{d_1, \ldots, d_s}} \int_{\Gamma_{d_1, \ldots, d_s}} \frac{\gamma_P^* (e_1^* \nu_P^*(\otimes_{i=1}^{s} \partial_{\gamma_i}) \cdot e_2^* \nu_P^*(\otimes_{i=1}^{s} \partial_{\gamma_i}) \cdot e_3^* \nu_P^*(\otimes_{i=1}^{s} \partial_{\gamma_i}))}{e_T(N_{d_1, \ldots, d_s}^{vir})}.
\]

where \( N_{d_1, \ldots, d_s}^{vir} \) is the virtual normal bundle to \( \Gamma_{d_1, \ldots, d_s} \) in \( \overline{M}_{0,3}(U, (d_1, \ldots, d_s)) \). By Lemma [2.2] it is given by

\[
\sum_{\Gamma_{d_1, \ldots, d_s}} \int_{\Gamma_{d_1, \ldots, d_s}} \frac{\gamma_P^* (e_1^* \nu_P^*(\partial_{\gamma_i}) \cdot e_2^* \nu_P^*(\partial_{\gamma_i}) \cdot e_3^* \nu_P^*(\partial_{\gamma_i}))}{e_T(N_{d_1, \ldots, d_s}^{vir})}.
\]

Hence (2.7) is a consequence of the equality

\[
\frac{1}{e_T(N_{d_1, \ldots, d_s}^{vir})} = \frac{1}{e_T(N_{d_1, \ldots, d_s}^{vir})} \text{ for each } \Gamma_{d_1, \ldots, d_s}. \tag{2.8}
\]

Now we prove (2.8). Given any \( T \)-fixed stable map \( f : (\Sigma, p_1, p_2, p_3) \to P \) in the \( T \)-fixed connected component \( \Gamma_{d_1, \ldots, d_s} \). For

\[
X = S^{[n]} \text{ or } X = U,
\]

in order to verify (2.8), we need only examine the infinitesimal deformations of \( f \) with the source curves fixed. In fact, it suffices to check two things:

\[
\frac{e_T(H^0(\Sigma_{\nu}, f^*TX)^{\text{mov}})}{e_T(H^1(\Sigma_{\nu}, f^*TX)^{\text{mov}})} \text{ and } \frac{e_T(H^0(\Sigma_{\nu}, f^*TX)^{\text{mov}})}{e_T(H^1(\Sigma_{\nu}, f^*TX)^{\text{mov}})}
\]

are independent of \( X \) for every connected contracted component \( \Sigma_{\nu} \) and noncontracted irreducible component \( \Sigma_{\nu} \). Here \( (\cdot)^{\text{mov}} \) denotes the moving part. The first independence holds due to the fact that \( \Sigma_{\nu} \) is of genus \( 0 \) and \( T_{f(\Sigma_{\nu})} S^{[n]} = T_{f(\Sigma_{\nu})} U \). Thus, it remains to justify
the second independence. Since \( \Sigma_e \cong \mathbb{P}^1 \), \( f^*TX \) is a direct sum of line bundles over \( \Sigma_e \), i.e.,

\[
f^*TX = \bigoplus_{i=1}^{2n} \mathcal{O}_{\Sigma_i}(\ell_i^X)
\]

for some integers \( \ell_i^X \)'s, and we also have

\[
e_{\mathcal{T}}(H^0(\Sigma_e, f^*TX)^{\text{mov}}) = \prod_{i=1}^{2n} e_{\mathcal{T}}(H^0(\Sigma_i, \mathcal{O}_{\Sigma_i}(\ell_i^X))^{\text{mov}}).
\]

(2.9)

Note that the \( \mathcal{T} \)-action on \( f(\Sigma_e) \) (independent of \( X \)) induces a natural action on \( \mathcal{O}_{\Sigma_i}(\ell_i^X) \)'s. Suppose that \( \mathcal{T} \) acts on \( \mathcal{O}_{\Sigma_i}(1) \) with weight \( w \), then the \( \mathcal{T} \)-weight of \( \mathcal{O}_{\Sigma_i}(\ell)^{|p_0} \) is \( \ell w \). This means that \( T_f(p_0)X \) comes with weights \( \ell_1^X w, \ldots, \ell_{2n}^X w \). But \( T_f(p_0)S^{[n]} = T_f(p_0)U \), so the numbers \( \ell_1^X, \ldots, \ell_{2n}^X \) are exactly \( \ell_1^U, \ldots, \ell_{2n}^U \) after a suitable reordering. By (2.9), this shows the independence and ends the proof of (2.5).

On the other hand, the equality

\[
\left\langle \otimes_{i=1}^{s} a_{\lambda_i}, \otimes_{i=1}^{s} a_{\mu_i}, \otimes_{i=1}^{s} a_{\nu_i} \right\rangle_{\text{Hilb}^{n_i}(U_i)^{\text{mov}}} = \prod_{i=1}^{s} \left\langle a_{\lambda_i}, a_{\mu_i}, a_{\nu_i} \right\rangle_{\text{Hilb}^{n_i}(U_i)}
\]

holds. This is clear for \( (d_1, \ldots, d_s) = (0, \ldots, 0) \); in general, the result follows from the Product formula [Beh] in equivariant context. Combining it with (2.7), we obtain the proposition. \( \Box \)

**Proposition 2.4.** The 3-point function \( \left\langle a_{\lambda}, a_{\mu}, a_{\nu} \right\rangle_{\text{Hilb}^n(S)} \) is an element of \( \mathbb{Q}(t_1, t_2, q) \) and is given by

\[
\prod_{i=1}^{s} \left\langle a_{\lambda_i}, a_{\mu_i}, a_{\nu_i} \right\rangle_{\text{Hilb}^{n_i}(U_i)}(q).
\]

**Proof.** Each term \( \left\langle a_{\lambda}, a_{\mu}, a_{\nu} \right\rangle_{\text{Hilb}^n(S)}(q) \) is a rational function in \( t_1, t_2, q \) (cf. [OP]). By Proposition 2.4 and 2.3

\[
\left\langle a_{\lambda}, a_{\mu}, a_{\nu} \right\rangle_{\text{Hilb}^n(S)}(q) = \sum_{d=0}^{\infty} \sum_{d_1+\ldots+d_s=d} \prod_{i=1}^{s} \left\langle a_{\lambda_i}, a_{\mu_i}, a_{\nu_i} \right\rangle_{\text{Hilb}^{n_i}(U_i)}(q)^{d_i} = \prod_{i=1}^{s} \sum_{d_i=0}^{\infty} \left\langle a_{\lambda_i}, a_{\mu_i}, a_{\nu_i} \right\rangle_{\text{Hilb}^{n_i}(U_i)}(q)^{d_i} = \prod_{i=1}^{s} \left\langle a_{\lambda_i}, a_{\mu_i}, a_{\nu_i} \right\rangle_{\text{Hilb}^{n_i}(U_i)}(q),
\]

and is a rational function in \( t_1, t_2, q \) as well. \( \Box \)

Together with (2.7), we have the following result on Poincaré pairings.

**Proposition 2.5.** The equivariant Poincaré pairing on \( \text{Hilb}^n(S) \) can be written in terms of the pairings on the Hilbert schemes of points in the affine plane. More precisely,

\[
\left\langle a_{\mu} | a_{\nu} \right\rangle = \prod_{i=1}^{s} \delta_{\mu_i, \nu_i} (-1)^{\left| \mu_k \right| - \ell(\mu_k)} \left( L_i R_i \right)^{\ell(\mu_k)} \frac{1}{\delta_{\mu_i}}.
\]

In other words, \( a_{\mu} \)'s give an orthogonal basis for \( A^*_T(\text{Hilb}^n(S))_m \). \( \Box \)
3 SYM-HILB correspondence

3.1 Proof of the main theorem

In this section, we give a SYM-HILB correspondence that relates the theories of the symmetric product and the Hilbert scheme.

Let \( q = -e^{iu} \), where \( i^2 = -1 \), and \( F = \mathbb{Q}(i, t_1, t_2, q) \). Define

\[
L: A^*_T,\text{orb}([\text{Sym}^n(S)]) \to A^*_T(\text{Hilb}^n(S))
\]

by

\[
L(\tilde{\lambda}) = (-i)^{\text{age}(\tilde{\lambda})}a_{\tilde{\lambda}}.
\]

As we have a bijection between the bases on both sides, \( L \) extends to a \( \mathbb{Q}(i, t_1, t_2)((u)) \)-linear isomorphism. However, extended 3-point functions of \([\text{Sym}^n(S)]\) and extremal 3-point functions of \( \text{Hilb}^n(S) \) are elements of \( F \). So we may view \( L \) as an \( F \)-linear isomorphism. Note also that \( \tilde{\lambda} \) has orbifold Chow degree

\[
2\ell(\tilde{\lambda}) + \text{age}(\tilde{\lambda}) = n + \ell(\tilde{\lambda}),
\]

which matches the Chow degree of \( a_{\tilde{\lambda}} \).

We now verify Theorem 0.2, which is restated in the following.

**Theorem 3.1.** Under the substitution \( q = -e^{iu} \), \( L \) equates the extended 3-point function of \([\text{Sym}^n(S)]\) to extremal 3-point functions of \( \text{Hilb}^n(S) \). More precisely, for any Chen-Ruan cohomology classes \( \alpha_1, \alpha_2, \alpha_3 \),

\[
(\alpha_1, \alpha_2, \alpha_3)^{[\text{Sym}^n(S)]}(u) = (L(\alpha_1), L(\alpha_2), L(\alpha_3))^{\text{Hilb}^n(S)}(q).
\]

Moreover, \( L \) is an isometric isomorphism.

**Proof.** For every fixed-point class \( \tilde{\lambda} \), Proposition 1.3 and Proposition 2.5 say that \( (\tilde{\lambda} | \tilde{\lambda}) = (-1)^{\text{age}(\tilde{\lambda})}a_{\tilde{\lambda}}|a_{\tilde{\lambda}} \). Hence, \( (\tilde{\lambda} | \tilde{\lambda}) = (L(\tilde{\lambda}) | L(\tilde{\lambda})) \), and \( L \) is an isometric isomorphism.

The proof of the first assertion relies on the \( \mathbb{C}^4 \)-case. Indeed, Okounkov-Pandharipande and Bryan-Graber determine the structures of equivariant quantum cohomology rings of \( \text{Hilb}^n(\mathbb{C}^2) \) and \([\text{Sym}^n(\mathbb{C}^2)]\) respectively. Also, these rings are related by the correspondence

\[
L_{C^2}: A^*_T,\text{orb}([\text{Sym}^n(\mathbb{C}^2)]) \otimes_{\mathbb{Q}[t_1, t_2]} F \to A^*_T(\text{Hilb}^n(\mathbb{C}^2)) \otimes_{\mathbb{Q}[t_1, t_2]} F
\]

which sends \( \mu_1([0]) \cdots \mu_\ell([0]) \) to \((-i)^{\text{age}(\mu)}a_{\mu_1}([0]) \cdots a_{\mu_\ell([0])}([0]) \). Bryan and Graber show that \( L_{C^2} \) preserves 3-point functions. That is, for any Chen-Ruan cohomology classes \( \delta_1, \delta_2, \delta_3 \) on the orbifold \([\text{Sym}^n(\mathbb{C}^2)]\), we have

\[
\langle \delta_1, \delta_2, \delta_3 \rangle^{[\text{Sym}^n(\mathbb{C}^2)]}(u) = \langle L_{C^2}(\delta_1), L_{C^2}(\delta_2), L_{C^2}(\delta_3) \rangle^{\text{Hilb}^n(\mathbb{C}^2)}(q). \tag{3.1}
\]

As \( U_k \cong \mathbb{C}^2 \), we denote the corresponding map by

\[
L_{U_k}: A^*_T,\text{orb}([\text{Sym}^n(U_k)]) \otimes_{\mathbb{Q}[t_1, t_2]} F \to A^*_T(\text{Hilb}^n(U_k)) \otimes_{\mathbb{Q}[t_1, t_2]} F.
\]

To show the first assertion, it is enough to establish that for all \( \tilde{\lambda}, \tilde{\mu}, \tilde{\nu} \) satisfying Condition 1.3,

\[
(\tilde{\lambda}, \tilde{\mu}, \tilde{\nu})^{[\text{Sym}^n(S)]}(u) = (L(\tilde{\lambda}), L(\tilde{\mu}), L(\tilde{\nu}))^{\text{Hilb}^n(S)}(q). \tag{3.2}
\]
In fact, by (3.1) and Corollary 1.2, the invariant
\[\langle \tilde{\lambda}, \tilde{\mu}, \tilde{\nu} \rangle_{\text{Sym}^n(S)}(u)\]
is given by
\[\prod_{k=1}^{s} (\tilde{\lambda}_k, \tilde{\mu}_k, \tilde{\nu}_k)_{\text{Sym}^n(U_k)}(u) = \prod_{k=1}^{s} (L_{U_k}(\tilde{\lambda}_k), L_{U_k}(\tilde{\mu}_k), L_{U_k}(\tilde{\nu}_k))_{\text{Hilb}^n(U_k)}(\mu).\]
Clearly, \(\text{age}(\tilde{\sigma}) = \sum_{k=1}^{s} \text{age}(\sigma_k)\) for any fixed-point class \(\tilde{\sigma}\). By applying Proposition 2.4 we obtain (3.2).

As explained earlier, this theorem also indicates that \(L\) provides a ring isomorphism between the equivariant Chen-Ruan cohomology of \([\text{Sym}^n(S)]\) and the equivariant quantum corrected cohomology of \(\text{Hilb}^n(S)\) after setting \(u = 0\) and \(q = -1\).

3.2 The cup product structure on the Hilbert scheme

An upshot of Theorem 3.1 is that 3-point degree zero invariants of Hilbert schemes is expressible in terms of degree zero invariants of symmetric product stacks.

**Corollary 3.2.** Given cohomology-weighted partitions \(\lambda_1(\tilde{\eta}_1), \lambda_2(\tilde{\eta}_2), \lambda_3(\tilde{\eta}_3)\), the degree 0 invariant
\[\langle \lambda_1(\tilde{\eta}_1), \lambda_2(\tilde{\eta}_2), \lambda_3(\tilde{\eta}_3) \rangle_{\text{Hilb}^n(S)}\]is given by
\[i^{\sum_{k=1}^{s} \text{age}(\lambda_k)} \lim_{u \to i\infty} \langle \lambda_1(\tilde{\eta}_1), \lambda_2(\tilde{\eta}_2), \lambda_3(\tilde{\eta}_3) \rangle_{\text{Sym}^n(S)}(u).\]  

**Proof.** It follows immediately from Theorem 3.1 by taking \(q \to 0\). □

This is the precise statement of Corollary 0.4. Since the ordinary cup product of \(\text{Hilb}^n(S)\) is defined by 3-point degree zero invariants, the corollary says that the 3-point extended \([\text{Sym}^n(S)]\)-invariants in degree zero completely determine the cup product of \(\text{Hilb}^n(S)\). This is not covered by CCRC and may shed new light on the explicit calculation of the ordinary cohomology ring of the Hilbert scheme of points.

Finally, we remark that the relative Gromov-Witten theory of the threefold \(S \times \mathbb{P}^1\) is very close to the orbifold theory. Indeed, studying the relative 3-point invariants of \(S \times \mathbb{P}^1\) in degree \((0, n)\) is tantamount to studying extended 3-point functions of \([\text{Sym}^n(S)]\). In other words, the relative theory may yield an alternative way to compute the cup product on \(\text{Hilb}^n(S)\).

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