APPROXIMATION OF HOMOGENIZED COEFFICIENTS IN DETERMINISTIC HOMOGENIZATION AND CONVERGENCE RATES IN THE ASYMPTOTIC ALMOST PERIODIC SETTING

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Dedicated to the memory of V.V. Zhikov, 1940-2017.

Abstract. For a homogenization problem associated to a linear elliptic operator, we prove the existence of a distributional corrector and we find an approximation scheme for the homogenized coefficients. We also study the convergence rates in the asymptotic almost periodic setting, and we show that the rates of convergence for the zero order approximation, are near optimal. The results obtained constitute a step towards the numerical implementation of results from the deterministic homogenization theory beyond the periodic setting. To illustrate this, numerical simulations based on finite volume method are provided to sustain our theoretical results.

1. Introduction

The purpose of this work is to establish the existence of a distributional corrector in the deterministic homogenization theory for a family of second order elliptic equations in divergence form with rapidly oscillating coefficients, and find an approximation scheme for the homogenized coefficients, without smoothness assumption on the coefficients. Under additional condition, we also study the convergence rates in the asymptotic almost periodic setting. We start with the statement of the problem (1.5).

Let \( \mathcal{A} \) be an algebra with mean value on \( \mathbb{R}^d \), that is, a closed subalgebra of the \( \mathcal{C}^* \)-algebra of bounded uniformly continuous real-valued functions on \( \mathbb{R}^d \), \( \text{BUC}(\mathbb{R}^d) \), which contains the constants, is translation invariant and is such that any of its elements possesses a mean value in the following sense: for every \( u \in \mathcal{A} \), the sequence \( (u^\varepsilon)_{\varepsilon>0} \) \( (u^\varepsilon(x) = u(x/\varepsilon)) \) weakly*-converges in \( L^\infty(\mathbb{R}^d) \) to some real number \( M(u) \) (called the mean value of \( u \)) as \( \varepsilon \to 0 \). The mean value expresses as

\[
M(u) = \lim_{R \to \infty} \int_{B_R} u(y) dy \quad \text{for } u \in \mathcal{A}
\]

where we have set \( \int_{B_R} = \frac{1}{|B_R|} \int_{B_R} \).

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For $1 \leq p < \infty$, we define the Marcinkiewicz space $\mathfrak{M}^p(\mathbb{R}^d)$ to be the set of functions $u \in L_{\text{loc}}^p(\mathbb{R}^d)$ such that

$$\limsup_{R \to \infty} \int_{B_R} |u(y)|^p \, dy < \infty.$$ 

Then $\mathfrak{M}^p(\mathbb{R}^d)$ is a complete seminormed space endowed with the seminorm

$$\|u\|_p = \left( \limsup_{R \to \infty} \int_{B_R} |u(y)|^p \, dy \right)^{1/p}.$$ 

We denote by $B_A^p(\mathbb{R}^d)$ ($1 \leq p < \infty$) the closure of $A$ in $\mathfrak{M}^p(\mathbb{R}^d)$. Then for any $u \in B_A^p(\mathbb{R}^d)$ we have that

$$\|u\|_p = \left( \lim_{R \to \infty} \int_{B_R} |u(y)|^p \, dy \right)^{1/p} = (M(|u|^p))^{1/p}.$$ 

Consider the space $B_A^{1,p}(\mathbb{R}^d) = \{ u \in B_A^p(\mathbb{R}^d) : \nabla_y u \in (B_A^p(\mathbb{R}^d))^d \}$ which is a complete seminormed space with respect to the seminorm

$$\|u\|_{1,p} = \left( \|u\|_p^p + \|\nabla_y u\|_p^p \right)^{1/2}.$$ 

The Banach counterpart of the previous spaces are defined as follows. We set $\mathcal{B}_A^p(\mathbb{R}^d) = B_A^p(\mathbb{R}^d)/\mathcal{N}$ where $\mathcal{N} = \{ u \in B_A^p(\mathbb{R}^d) : \|u\|_p = 0 \}$. We define $\mathcal{B}_A^{1,p}(\mathbb{R}^d)$ mutatis mutandis: replace $B_A^p(\mathbb{R}^d)$ by $\mathcal{B}_A^p(\mathbb{R}^d)$ and $\partial/\partial y_i$ by $\overline{\partial}/\partial y_i$, where $\overline{\partial}/\partial y_i$ is defined by

$$\frac{\overline{\partial}}{\partial y_i}(u + \mathcal{N}) := \frac{\partial u}{\partial y_i} + \mathcal{N} \text{ for } u \in B_A^{1,p}(\mathbb{R}^d).$$

It is important to note that $\overline{\partial}/\partial y_i$ is also defined as the infinitesimal generator in the $i$th direction coordinate of the strongly continuous group $T(y) : \mathcal{B}_A^p(\mathbb{R}^d) \to \mathcal{B}_A^p(\mathbb{R}^d)$; $T(y)(u + \mathcal{N}) = u(\cdot + y) + \mathcal{N}$. Let us denote by $\varphi : \mathcal{B}_A^p(\mathbb{R}^d) \to \mathcal{B}_A^{1,p}(\mathbb{R}^d) = \mathcal{B}_A^p(\mathbb{R}^d)/\mathcal{N}$; $\varphi(u) = u + \mathcal{N}$, the canonical surjection. Remark: $u \in B_A^{1,p}(\mathbb{R}^d)$ implies $\varphi(u) \in B_A^{1,p}(\mathbb{R}^d)$ and observing (1.3), \[\frac{\varphi(u)}{\varphi(y)} = \frac{\partial u}{\partial y_i} + \mathcal{N} \text{ for } u \in B_A^{1,p}(\mathbb{R}^d).\]

We assume in the sequel that the algebra $A$ is ergodic, that is, any $u \in \mathcal{B}_A^p(\mathbb{R}^d)$ that is invariant under $(T(y))_{y \in \mathbb{R}^d}$ is a constant in $\mathcal{B}_A^p(\mathbb{R}^d)$, i.e., if $\|T(y)u - u\|_p = 0$ for every $y \in \mathbb{R}^d$, then $\|u - c\|_p = 0, c$ a constant. Let us also recall the following property [23, 29]:

(1) The mean value $M$ viewed as defined on $A$, extends by continuity to a non negative continuous linear form (still denoted by $M$) on $B_A^p(\mathbb{R}^d)$. For each $u \in B_A^p(\mathbb{R}^d)$ and all $a \in \mathbb{R}^d$, we have $M(u(\cdot + a)) = M(u)$, and $\|u\|_p = (M(|u|^p))^{1/p}$.

To the space $B_A^p(\mathbb{R}^d)$ we also attach the following corrector space

$$B_{\#A}^{1,p}(\mathbb{R}^d) = \{ u \in W_{\text{loc}}^{1,p}(\mathbb{R}^d) : \nabla u \in B_A^p(\mathbb{R}^d)^d \text{ and } M(\nabla u) = 0 \}.$$ 

In $B_{\#A}^{1,p}(\mathbb{R}^d)$ we identify two elements by their gradients: $u = v$ in $B_{\#A}^{1,p}(\mathbb{R}^d)$ iff $\nabla(u - v) = 0$, i.e. $\|\nabla(u - v)\|_p = 0$. We equip $B_{\#A}^{1,p}(\mathbb{R}^d)$ with the gradient norm $\|u\|_{\#,p} = \|\nabla u\|_p$ and obtain a Banach space [13, Theorem 3.12] containing $B_A^{1,p}(\mathbb{R}^d)$. 

We recall the $\Sigma$-convergence. A sequence $(u_\varepsilon)_{\varepsilon>0} \subset L^p(\Omega)$ ($1 \leq p < \infty$) is said to:

(i) weakly $\Sigma$-converge in $L^p(\Omega)$ to $u_0 \in L^p(\Omega; \mathcal{B}^p_A(\mathbb{R}^d))$ if, as $\varepsilon \to 0$,

$$\int_\Omega u_\varepsilon(x) f \left( x, \frac{x}{\varepsilon} \right) dx \to \int_\Omega M(u_0(x, \cdot) f(x, \cdot)) dx \quad (1.4)$$

for any $f \in L^{p'}(\Omega; A)$ ($p' = p/(p-1)$);

(ii) strongly $\Sigma$-converge in $L^p(\Omega)$ to $u_0 \in L^p(\Omega; \mathcal{B}^p_A(\mathbb{R}^d))$ if (1.4) holds and further $\|u_\varepsilon\|_{L^p(\Omega)} \to \|u_0\|_{L^p(\Omega; \mathcal{B}^p_A(\mathbb{R}^d))}$.

We denote (i) by "$u_\varepsilon \rightharpoonup u_0$ in $L^p(\Omega)$-weak $\Sigma$", and (ii) by "$u_\varepsilon \to u_0$ in $L^p(\Omega)$-strong $\Sigma$".

The main properties of the above concept are:

- Every bounded sequence in $L^p(\Omega)$ ($1 < p < \infty$) possesses a subsequence that weakly $\Sigma$-converges in $L^p(\Omega)$.

- If $(u_\varepsilon)_{\varepsilon \in E}$ is a bounded sequence in $W^{1,p}(\Omega)$, then there exist a subsequence $E'$ of $E$ and a couple $(u_0, u_1) \in W^{1,p}(\Omega) \times L^p(\Omega; B^{1,p}_{\#A}(\mathbb{R}^d))$ such that

$$u_\varepsilon \rightharpoonup u_0 \text{ in } W^{1,p}(\Omega) \text{-weak } \Sigma \quad (1 \leq j \leq d)$$

- If $u_\varepsilon \to u_0$ in $L^p(\Omega)$-weak $\Sigma$ and $v_\varepsilon \to v_0$ in $L^q(\Omega)$-strong $\Sigma$, then $u_\varepsilon v_\varepsilon \to u_0 v_0$ in $L^r(\Omega)$-weak $\Sigma$, where $1 \leq p, q, r < \infty$ and $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$.

Our aim is to study the following problem

$$- \nabla \cdot \left( A \left( x, \frac{x}{\varepsilon} \right) \nabla u_\varepsilon \right) = f \text{ in } \Omega, \ u_\varepsilon \in H^1_0(\Omega) \quad (1.5)$$

where $\varepsilon > 0$ is a small parameter, $f \in L^2(\Omega)$, $\Omega$ is an open bounded set of $\mathbb{R}^d$ (integer $d \geq 1$) with smooth boundary $\partial \Omega$, and $A \in \mathcal{C}(\overline{\Omega}; L^\infty(\mathbb{R}^d)^{d\times d})$ is a symmetric matrix satisfying

$$\alpha |\lambda|^2 \leq A(x, y) \lambda \cdot \lambda \leq \beta |\lambda|^2 \text{ for all } (x, \lambda) \in \overline{\Omega} \times \mathbb{R}^d \text{ and a.e. } y \in \mathbb{R}^d; \quad (1.6)$$

$$A(x, \cdot) \in (\mathcal{B}^2_A(\mathbb{R}^d))^{d\times d} \text{ for all } x \in \overline{\Omega} \quad (1.7)$$

where $\alpha$ and $\beta$ are two positive real numbers.

It is well-known that under assumptions (1.6), problem (1.5) uniquely determines a function $u_\varepsilon \in H^1_0(\Omega)$. Under the additional assumption (1.7), the following result holds.

**Theorem 1.1.** There exists $u_0 \in H^1_0(\Omega)$ such that $u_\varepsilon \rightharpoonup u_0$ weakly in $H^1_0(\Omega)$ and strongly in $L^2(\Omega)$ (as $\varepsilon \to 0$) and $u_0$ solves uniquely the problem

$$- \nabla \cdot (A^*(x) \nabla u_0) = f \text{ in } \Omega, \quad (1.8)$$

$A^*$ being the homogenized matrix defined by

$$A^*(x) = M \left( A(x, \cdot)(I_d + \nabla \chi(x, \cdot)) \right) \quad (1.9)$$
where, \( \chi = (\chi_j)_{1 \leq j \leq d} \in C(\overline{\Omega}; B^{1,2}_{\#A}(\mathbb{R}^d)^d) \) is such that, for any \( x \in \Omega \), \( \chi_j(x, \cdot) \) is the unique solution (up to an additive constant depending on \( x \)) of the problem
\[
\nabla_y \cdot (A(x, \cdot)(e_j + \nabla_y \chi_j(x, \cdot))) = 0 \quad \text{in} \quad \mathbb{R}^d.
\]
(1.10)

If we set \( u_1(x, y) = \nabla u_0(x) \chi(x, y) = \sum_{i=1}^d \frac{\partial a_i}{\partial x_i}(x) \chi_i(x, y) \) and assume that \( u_1 \in H^1(\Omega; A^1) \) \( (A^1 = \{ v \in A : \nabla_y v \in (A^d)^d \}) \), then, as \( \varepsilon \to 0 \),
\[
u_\varepsilon - u_0 - \varepsilon u_1^\varepsilon \to 0 \quad \text{in} \quad H^1(\Omega) \quad \text{strongly}
\]
(1.11)

where \( u_1^\varepsilon(x) = u_1(x, x/\varepsilon) \) for a.e. \( x \in \Omega \).

Remark 1.1. Problem (1.10) is the corrector problem. It helps to obtain a first order approximation \( u_\varepsilon(x) \approx u_0(x) + \varepsilon u_1(x, x/\varepsilon) \) of \( u_\varepsilon \) as seen in (1.11). Its solvability is addressed in the following result, which is the first main result of this work.

Theorem 1.2. Let \( \xi \in \mathbb{R}^d \) and \( x \in \overline{\Omega} \) be fixed. There exists a unique (up to an additive function of \( x \)) function \( v_\xi \in C(\overline{\Omega}; H^1_{loc}(\mathbb{R}^d)) \) such that \( \nabla_y v_\xi \in C(\overline{\Omega}; B^2_{\#A}(\mathbb{R}^d)^d) \) and \( M(\nabla_y v_\xi(x, \cdot)) = 0 \), which solves the equation
\[
\nabla_y \cdot (A(x, \cdot)(\xi + \nabla_y v_\xi(x, \cdot))) = 0 \quad \text{in} \quad \mathbb{R}^d.
\]
(1.12)

The proof of Theorem 1.2 will be obtained as a consequence of Lemma 2.1 in Section 2 below. The progress compared to the previously known results exists in the solution of the corrector problem: it is obtained by approximation with distributional solutions of partial differential equations in sufficiently large balls. Since the approximation can be quantitatively controlled, this method also provides a basis for the numerical calculation. Theorem 1.2 is well known in the random stationary ergodic environment. However for the general deterministic setting, we believe that a detailed proof must be provided since it also covers the non ergodic algebras framework.

The next step consists in finding an approximation scheme for the homogenized matrix \( A^* \) (see (1.9)). This problem has been solved (for (1.5)) in the periodic setting, since under the periodic assumption, the corrector problem is posed on a bounded domain (namely the periodic cell \( Y = (0,1)^d \)) since in that case, the solution \( \chi_j \) is periodic. A huge contrast between the periodic setting and the general deterministic setting (as considered in this work) is that in the latter, the corrector problem is posed on the whole space \( \mathbb{R}^d \), and cannot be reduced (as in the periodic framework) to a problem on a bounded domain. As a result, the solution of the corrector problem (1.10) (and hence the homogenized matrix which depends on this solution) can not be computed directly. Therefore, as in the random setting (see e.g. (12)), truncations of (1.10) must be considered, particularly on large domains \( (-R, R)^d \) with appropriate boundary conditions, and the homogenized coefficients will therefore be captured in the asymptotic regime. This is done in Theorem 3.1 (see Section 3). We then find the rate of convergence for the approximation scheme (see Theorem 3.2). It is natural to determine the convergence rates for the approximation (1.11) setting in two cases:

1) the asymptotic periodic one represented by the algebra \( A = C_0(\mathbb{R}^d) + C_{per}(Y) \);
In case 1), the corrector function $\chi_j(x, \cdot)$ (solution of (1.10)) belongs to the Sobolev-Besicovitch space $B^{1,2}_A(\mathbb{R}^d)$ associated to the algebra $A$ and is bounded in $L^\infty(\mathbb{R}^d)$. As a result, we proceed as in the well-known periodic setting. In contrast with case 1), the corrector function in case 2) does not (in general) belong to the associated Sobolev-Besicovitch space $B^{1,2}_A(\mathbb{R}^d)$. So information is available mainly for the gradient of the corrector. To address this issue, we use the approximate corrector $\chi_{T,j}$, distributional solution to $-\nabla \cdot A(\varepsilon_j + \nabla \chi_{T,j}) + T^{-2}\chi_{T,j} = 0$ in $\mathbb{R}^d$, which belongs to $B^{1,2}_A(\mathbb{R}^d)$ as shown in Section 2. This leads to the following result, which is one of the main result of the work.

**Theorem 1.3.** Let $\Omega$ be a $C^{1,1}$ bounded domain in $\mathbb{R}^d$. Suppose that the matrix $A(x, y) \equiv A(y)$ and is asymptotic almost periodic. Assume that $A$ satisfies (1.6). For $f \in L^2(\Omega)$, let $u_\varepsilon$ and $u_0$ be the weak solutions of Dirichlet problems (1.5) and (1.8) respectively. Then there exists a function $\eta : (0, 1) \to [0, \infty)$ depending on $A$ with $\lim_{t \to 0} \eta(t) = 0$ such that

$$
\|u_\varepsilon - u_0 - \varepsilon \chi_T^T \nabla u_0\|_{L^2(\Omega)} \leq C\eta(\varepsilon) \|f\|_{L^2(\Omega)}
$$

and

$$
\|u_\varepsilon - u_0\|_{L^2(\Omega)} \leq C[\eta(\varepsilon)]^2 \|f\|_{L^2(\Omega)}
$$

where $T = \varepsilon^{-1}$ and $\chi_T$ is the approximate corrector defined by (5.5), and $C = C(\Omega, A, d)$.

The precise convergence rates in case 1) are presented in the following result.

**Theorem 1.4.** Suppose that $A$ is asymptotic periodic and satisfies ellipticity conditions (1.6) and (4.2). Assume $\Omega$, $f$, $u_\varepsilon$ and $u_0$ are as in Theorem 1.3. Denoting by $\chi$ the corrector defined by (1.10), there exists $C = C(\Omega, A, d) > 0$ such that

$$
\|u_\varepsilon - u_0 - \varepsilon \chi^T \nabla u_0\|_{L^2(\Omega)} \leq C\varepsilon^{\frac{1}{2}} \|f\|_{L^2(\Omega)}
$$

and

$$
\|u_\varepsilon - u_0\|_{L^2(\Omega)} \leq C\varepsilon \|f\|_{L^2(\Omega)} .
$$

Theorem 1.4 can be obtained as a special case of Theorem 1.3. However we provide an independent proof since we do not need the approximate corrector in this special situation. Estimate 1.16 is optimal.

The above results generalize the well known ones in the periodic and the uniformly almost periodic settings as considered in [31]. In Theorem 1.4 we assume that the matrix $A$ has the form $A = A_0 + A_{\text{per}}$ where $A_0$ has entries in $L^2(\Omega)$ and $A_{\text{per}}$ is periodic. In Theorem 1.3, we do not make any restriction on $A_0$ as above. Also, the estimate (1.14) is near optimal. The assumptions will be made precise in the latter sections.

The problem considered in Theorems 1.3 and 1.4 has been firstly addressed in the periodic framework by Avellaneda and Lin [7] (see also [22]), and in the random setting (that is, for second order linear elliptic equations with random coefficients) by Yurinskii [36], Pozhidaev and Yurinskii [28], and Bourgeat and Piatnitski [12] (see also a recent series of works by
Gloria and Otto [18, 19, 20, and the recent monograph [5]). Although it is shown in [30] that deterministic homogenization theory can be seen as a special case of random homogenization theory at least as far as the qualitative study is concerned, we can not expect to use this random formulation to address the issues of rate of convergence in the deterministic setting. Indeed, in the random framework, the rate of convergence relies systematically on the uniform mixing property (see e.g. [12, 28, 36]) of the coefficients of the equation. As proved by Bondarenko et al. [11], the almost periodic operators do not satisfy the uniform mixing property. As a result, we can not use the random framework to address the issue in the general deterministic setting. We therefore need to elaborate a new framework for solving the underlying problem. Beyond the periodic (but non-random) setting Kozlov [23] determined the rates of convergence in almost periodic homogenization by using almost periodic coefficients satisfying a frequency condition (see e.g. (6.1)). In the same vein, Bondarenko et al. [11] derived the rates of convergence by considering a perturbation of periodic coefficients (in dimension \(d = 1\)). The very first works that use the general almost periodicity assumption are a recent series of work by Shen et al. [6, 31, 32] in which they treated second order linear elliptic systems in divergence form. They used approximate correctors to derive the rates of convergence. A reason to use approximate correctors is the lack of sufficient knowledge on the corrector itself. Indeed in that case it is known that the gradient of the corrector is almost periodic. However it is not known in general whether the corrector itself is almost periodic. Under certain conditions, it is shown in [4, 32] that the corrector is almost periodic. But the approximate corrector is in general almost periodic together with its gradient.

It seems necessary to compare ours results in Theorems 1.3 and 1.4 with the existing ones in the literature. First of all, it is worth noting that the algebra of continuous asymptotic almost periodic functions is included in the Banach space of Weyl almost periodic functions; see e.g. [9]. Thus the results obtained in [32] can be seen as generalizing those in Theorems 1.3 and 1.4. However it is not exactly the case. Indeed in [32], the rates of convergence are found in terms of the modulus of Weyl-almost periodicity of the matrix \(A\), that is, in terms of the function

\[
\rho^1_A(R,L) = \sup_{y \in \mathbb{R}^d} \inf_{|z| \leq R} \left( \sup_{x \in \mathbb{R}^d} \int_{B_L(x)} |A(t + y) - A(t + z)|^2 \, dt \right)^{\frac{1}{2}} \text{ for } R, L > 0
\]

where \(B_L(x)\) stands for the open ball in \(\mathbb{R}^d\) centered at \(x\) and of radius \(L > 0\). In our work, we distinguish two cases: 1) the asymptotic periodic case in which we show that the rate of convergence is optimal, that \(\|u_\varepsilon - u_0\|_{L^2(\Omega)} = O(\varepsilon)\); 2) In the general continuous asymptotic almost periodic setting, we show as in [32], that the rate of convergence depends on the modulus of asymptotic almost periodicity defined by

\[
\rho_A(R,L) = \sup_{y \in \mathbb{R}^d} \inf_{|z| \leq R} \|A(\cdot + y) - A(\cdot + z)\|_{L^\infty(B_L(0))}.
\]

As it is easily seen, the comparison between \(\rho^1_A(R,L)\) and \(\rho_A(R,L)\) is not straightforward. So our result in Theorem 1.4 does not follows directly from its counterpart Theorem 1.4 in [32].
Our work combines the framework of [31] with the general deterministic homogenization theory introduced by Zhikov and Krivenko [39] and Nguetseng [24]. Furthermore, numerical simulations based on finite volume method are provided to sustain our main theoretical results.

The further investigation is organized as follows. Section 2 is devoted to the proof of Theorems 1.1 and 1.2. Section 3 deals with the approximation of the homogenized coefficients. In Section 4, we prove Theorems 1.3 while in Section 5 we prove Theorem 1.4. In Section 6, we provide some examples of concrete algebras and functions for which the results, in particular those of Theorems 3.2, 1.3 and 1.4 apply. Finally, in Section 7 we present numerical results illustrating the method and supporting the proposed procedure.

2. Existence result for the corrector equation

Let the matrix $A$ satisfy (1.6) and (1.7). Our aim is to solve the corrector problem (1.10). Let $B^2_\infty(\mathbb{R}^d) = B^2_\infty(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, which is a Banach space under the $L^\infty(\mathbb{R}^d)$-norm.

Lemma 2.1. Let $h \in \mathcal{C} (\overline{\Omega}; B^2_\infty(\mathbb{R}^d))$ and $H \in \mathcal{C} (\overline{\Omega}; B^2_\infty(\mathbb{R}^d))$. For any $T > 0$, there exists a unique function $u \in \mathcal{C} (\overline{\Omega}; B^{1,2}_A(\mathbb{R}^d))$ such that

$$- \nabla_y \cdot (A(x, \cdot) \nabla_y u(x, \cdot)) + T^{-2} u(x, \cdot) = h(x, \cdot) + \nabla_y \cdot H(x, \cdot) \text{ in } \mathbb{R}^d$$

(2.1)

for any fixed $x \in \overline{\Omega}$. The solution $u$ satisfies further

$$\sup_{z \in \mathbb{R}^d} \int_{B_R(z)} (T^{-2} |u(x, y)|^2 + |\nabla u(x, y)|^2) dy \leq C \sup_{z \in \mathbb{R}^d} \int_{B_R(z)} (|H(x, y)|^2 + T^2 |h(x, y)|^2) dy$$

(2.2)

for any $R \geq T$ and all $x \in \overline{\Omega}$, where the constant $C$ depends only on $d$, $\alpha$ and $\beta$.

Proof. Since the variable $x$ in (2.1) behaves as a parameter, we drop it throughout the proof of the existence and uniqueness. Thus, in what follows, we keep using the symbol $\nabla$ instead of $\nabla_y$ to denote the gradient with respect to $y$, if there is no danger of confusion.

1. Existence. Fix $R > 0$ and define $v_{T,R} \equiv v_R \in H^1_0(B_R)$ as the unique solution of

$$- \nabla \cdot A \nabla v_R + T^{-2} v_R = h + \nabla \cdot H \text{ in } B_R.$$

Extending $v_R$ by 0 off $B_R$, we obtain a sequence $(v_{R})_R$ in $H^1_{\text{loc}}(\mathbb{R}^d)$. Let us show that the sequence $(v_{R})_R$ is bounded in $H^1_{\text{loc}}(\mathbb{R}^d)$. We proceed as in [18] (see also [28]). In the variational formulation of the above equation, we choose as test function, the function $\eta_z v_R$, where $\eta_z(y) = \exp(-c |y - z|)$ for a fixed $z \in \mathbb{R}^d$, $c > 0$ to be chosen later. We get

$$\int_{B_R} \eta_z^2 A \nabla v_R \cdot \nabla v_R + T^{-2} \int_{B_R} \eta_z^2 v_R^2 = -2 \int_{B_R} \eta_z v_R A \nabla v_R \cdot \nabla \eta_z - 2 \int_{B_R} \eta_z v_R H \cdot \nabla \eta_z$$

$$- \int_{B_R} \eta_z^2 H \cdot \nabla v_R + \int_{B_R} h \eta_z^2 v_R$$

$$= I_1 + I_2 + I_3 + I_4.$$
The left-hand side of the above equality is bounded from below by
\[ \alpha \int_{B_R} \eta_z^2 |\nabla v_R|^2 + T^{-2} \int_{B_R} \eta_z^2 v_R^2, \]
while for the right-hand side, we have the following bounds (after using the Young’s inequality and the bounds on A):
\[ |I_1| \leq \frac{\alpha \beta T^{-2}}{k} \int_{B_R} v_R^2 |\nabla \eta_z|^2 + \frac{T^2 \beta k}{\alpha} \int_{B_R} \eta_z^2 |\nabla v_R|^2, \]
\[ |I_2| \leq \frac{\alpha \beta T^{-2}}{k} \int_{B_R} v_R^2 |\nabla \eta_z|^2 + \frac{T^2 k}{\alpha \beta} \int_{B_R} \eta_z^2 |H|^2, \]
\[ |I_3| \leq \frac{T^2 \beta k}{\alpha} \int_{B_R} \eta_z^2 |\nabla v_R|^2 + \frac{T^{-2} \alpha}{4k} \int_{B_R} \eta_z^2 |H|^2, \]
\[ |I_4| \leq \frac{\alpha \beta T^{-2} c^2}{k} \int_{B_R} v_R^2 \eta_z^2 + \frac{T^2 k}{4 \alpha \beta c^2} \int_{B_R} \eta_z^2 |h|^2 \]
where \( k > 0 \) is to be chosen later. Noticing that \( |\nabla \eta_z| = c \eta_z \), we readily get after using the series of inequalities above,
\[ \int_{B_R} \eta_z^2 \left( \alpha - 2 \frac{T^2 \beta k}{\alpha} \right) |\nabla v_R|^2 + T^{-2} \int_{B_R} \eta_z^2 \left( 1 - 3 \frac{\alpha \beta c^2}{k} \right) v_R^2 \]
\[ \leq \int_{B_R} \left[ \left( \frac{T^2 k}{\alpha \beta} + \frac{T^{-2} \alpha}{4 \beta k} \right) |H|^2 + \frac{k T^2}{4 \alpha \beta c^2} |h|^2 \right] \eta_z^2. \]
Choosing therefore \( k = \frac{\alpha^2}{4 T^2} \) and \( c = \frac{1}{2 \beta T} \left( \frac{\alpha}{6} \right)^{1/2} \), we obtain the estimate
\[ \alpha \int_{B_R} \eta_z^2 |\nabla v_R|^2 + T^{-2} \int_{B_R} \eta_z^2 v_R^2 \leq \int_{B_R} \left[ \left( \frac{\alpha}{4 \beta^2} + \frac{1}{\alpha} \right) |H|^2 + \frac{3 T^2}{2} |h|^2 \right] \eta_z^2. \]
(2.3)

The inequality (2.3) above shows that the sequence \((v_R)\) is bounded in \( H^1_{\text{loc}}(\mathbb{R}^d) \); indeed, for any compact subset \( K \) in \( \mathbb{R}^d \), the left-hand side of (2.3) is bounded from below by \( c_K (\alpha \int_{B_R} |\nabla v_R|^2 + T^{-2} \int_{B_R} v_R^2) \) where \( c_K = \min_K \eta_z^2 > 0 \) while the right-hand side is bounded from above by \( C \int_{\mathbb{R}^d} \eta_z^2 \)
where
\[ C = \left( \frac{\alpha}{4 \beta^2} + \frac{1}{\alpha} \right) \|H\|^2_{C(\overline{\Omega};L^\infty(\mathbb{R}^d))} + \frac{3 T^2}{2} \|h\|^2_{C(\overline{\Omega};L^\infty(\mathbb{R}^d))}. \]

Hence there exist a subsequence of \((v_R)\) and a function \( v \in H^1_{\text{loc}}(\mathbb{R}^d) \) such that the above mentioned subsequence weakly converges in \( H^1_{\text{loc}}(\mathbb{R}^d) \) to \( v \), and it is easy to see that \( v \) is a distributional solution of (2.1) in \( \mathbb{R}^d \). Taking the \( \liminf_{R \to \infty} \) in (2.3) yields
\[ \alpha \int_{\mathbb{R}^d} \eta_z^2 |\nabla v_R|^2 + T^{-2} \int_{\mathbb{R}^d} \eta_z^2 v_R^2 \leq \int_{\mathbb{R}^d} \left[ \left( \frac{\alpha}{4 \beta^2} + \frac{1}{\alpha} \right) |H|^2 + \frac{3 T^2}{2} |h|^2 \right] \eta_z^2. \]
(2.4)

We infer from (2.4) that
\[ \sup_{z \in \mathbb{R}^d} \int_{B_R(z)} (|\nabla v|^2 + T^{-2} v^2) \leq C \]
(2.5)
where $C$ does not depend on $z$, but on $T$. Estimate (2.2) (for $R = T$) follows from \[28\] while the case $R > T$ is a consequence of Caccioppoli’s inequality; see \[32\] Lemma 3.2.

Let us show that $v \in B_{A}^{1,2}(\mathbb{R}^{d})$. It suffices to check that $v$ solves the equation

$$M(A(\xi + \nabla v) \cdot \nabla \phi + T^{-2}v \phi) = M(h \phi - H \cdot \nabla \phi), \text{ all } \phi \in B_{A}^{1,2}(\mathbb{R}^{d}).$$

(2.6)

To this end, let $\varphi \in C_{0}^{\infty}(\mathbb{R}^{d})$ and $\phi \in B_{A}^{1,2}(\mathbb{R}^{d})$. Define (for fixed $\varepsilon > 0$), $\psi(y) = \varphi(\varepsilon y)\phi(y)$. Choose $\psi$ as test function in the variational form of (2.1) and get

$$\int_{\mathbb{R}^{d}} [A \nabla u \cdot (\varepsilon \varphi(\varepsilon \cdot) + \varphi(\varepsilon \cdot) \nabla \phi) + T^{-2}u \varphi(\varepsilon \cdot) \phi] \, dy = \int_{\mathbb{R}^{d}} [h \varphi(\varepsilon \cdot) \phi - H \cdot (\varepsilon \varphi(\varepsilon \cdot) + \varphi(\varepsilon \cdot) \nabla \phi)] \, dy.$$  

The change of variables $t = \varepsilon y$ leads (after multiplication by $\varepsilon^{d}$) to

$$\int_{\mathbb{R}^{d}} [A^{\varepsilon}(\nabla_{y} u)^{\varepsilon} \cdot (\varepsilon \varphi^{\varepsilon} \nabla \phi + \varphi(\nabla_{y} \phi)^{\varepsilon}) + T^{-2}u^{\varepsilon} \varphi \phi^{\varepsilon}] \, dt = \int_{\mathbb{R}^{d}} [h^{\varepsilon} \varphi^{\varepsilon} \phi - H^{\varepsilon} \cdot (\varepsilon \varphi^{\varepsilon} \nabla \phi + \varphi(\nabla_{y} \phi)^{\varepsilon})] \, dt$$

where $w^{\varepsilon}(t) = w(t/\varepsilon)$ for a given $w$. Letting $\varepsilon \to 0$ above yields

$$\int_{\mathbb{R}^{d}} M(A \nabla u \cdot \nabla \phi + T^{-2}u \phi) \varphi dt = \int_{\mathbb{R}^{d}} M(h \phi - H \cdot \nabla \phi) \varphi dt$$

for all $\varphi \in C_{0}^{\infty}(\mathbb{R}^{d})$ and $\phi \in B_{A}^{1,2}(\mathbb{R}^{d})$.

which amounts to (2.6). So, we have just shown that, if $v \in H_{loc}^{1}(\mathbb{R}^{d})$ solves (2.1) in the sense of distributions in $\mathbb{R}^{d}$, then it satisfies (2.6). Before we proceed any further, let us first show that (2.6) possesses a unique solution in $B_{A}^{1,2}(\mathbb{R}^{d})$ up to an additive function $w \in B_{A}^{1,2}(\mathbb{R}^{d})$ satisfying $M(|w|^{2}) = 0$. First and foremost, we recall that the space $B_{A}^{1,2}(\mathbb{R}^{d}) = B_{A}^{1,2}(\mathbb{R}^{d})/\mathcal{N}$ (where $\mathcal{N} = \{u \in B_{A}^{1,2}(\mathbb{R}^{d}) : ||u||_{1,2} = 0 \}$) is a Hilbert space with inner product

$$(u + \mathcal{N}, v + \mathcal{N})_{1,2} = M(uv + \nabla u \cdot \nabla v)$$

for $u, v \in B_{A}^{1,2}(\mathbb{R}^{d})$. If $w \in \mathcal{N}$ then $M(w) = 0$, since $|M(w)| \leq M(|w|) \leq (M(|w|^{2}))^{1/2} = ||w||_{2} = 0$, so that $(,)_{1,2}$ is well defined. Now, (2.6) is equivalent to $a(v, \phi) = \ell(\phi)$ for all $\phi \in B_{A}^{1,2}(\mathbb{R}^{d})$ where

$$a(v, \phi) = M(T^{-2}v \phi + A \nabla v \cdot \nabla \phi), \quad \ell(\phi) = M(h \phi - H \cdot \nabla \phi).$$

$a(\cdot, \cdot)$ defines a continuous coercive bilinear form on $B_{A}^{1,2}(\mathbb{R}^{d})$; $\ell$ is a continuous linear form on $B_{A}^{1,2}(\mathbb{R}^{d})$. Lax-Milgram theorem implies that $v + \mathcal{N}$ is a unique solution of (2.6). This yields $v \in B_{A}^{1,2}(\mathbb{R}^{d})$.

2. Uniqueness. The uniqueness of the solution amounts to consider (2.1) with $h = 0$ and $H = 0$. We derive from (2.4)

$$\alpha \int_{\mathbb{R}^{d}} \eta_{z}^{2} |\nabla v|^{2} + T^{-2} \int_{\mathbb{R}^{d}} \eta_{z}^{2} v^{2} = 0,$$
3. Continuity. To investigate the continuity of \( v \) with respect to \( x \), we fix \( x_0 \in \Omega \) and we let \( w(x) = v(x, \cdot) - v(x_0, \cdot) \). Then \( w(x) \in B^{1,2}_A(\mathbb{R}^d) \) and

\[
-\nabla \cdot A(x, \cdot)\nabla w(x) + T^{-2}w(x) = h(x, \cdot) - h(x_0, \cdot) + \nabla \cdot (H(x, \cdot) - H(x_0, \cdot)) + \nabla \cdot (A(x, \cdot) - A(x_0, \cdot))\nabla v(x_0, \cdot),
\]

so that, using estimate \( (2.2) \), we find (for any \( R \geq T \))

\[
\sup_{z \in \mathbb{R}^d} \int_{B_R(z)} (T^{-2} |w(x)|^2 + |\nabla w(x)|^2) \, dy \leq CT^2 \sup_{z \in \mathbb{R}^d} \int_{B_R(z)} |h(x, y) - h(x_0, y)|^2 \, dy \\
+ C \sup_{z \in \mathbb{R}^d} \int_{B_R(z)} |H(x, y) - H(x_0, y)|^2 \, dy \\
+ C \sup_{z \in \mathbb{R}^d} \int_{B_R(z)} |A(x, y) - A(x_0, y)|^2 |\nabla v(x_0, y)|^2 \, dy \\
\leq CT^2 \|h(x, \cdot) - h(x_0, \cdot)\|_{L^\infty(\mathbb{R}^d)}^2 \\
+ C \|H(x, \cdot) - H(x_0, \cdot)\|_{L^\infty(\mathbb{R}^d)}^2 \\
+ C \|A(x, \cdot) - A(x_0, \cdot)\|_{L^\infty(\mathbb{R}^d)}^2.
\]

Continuity is a consequence of the following estimate

\[
T^{-2} \|v(x, \cdot) - v(x_0, \cdot)\|_2^2 + \|\nabla v(x, \cdot) - \nabla v(x_0, \cdot)\|_2^2 \\
\equiv \lim_{R \to \infty} \int_{B_R(z)} T^{-2} |w(x)|^2 + |\nabla w(x)|^2 \, dy \\
\leq CT^2 \|h(x, \cdot) - h(x_0, \cdot)\|_{L^\infty(\mathbb{R}^d)}^2 + C \|H(x, \cdot) - H(x_0, \cdot)\|_{L^\infty(\mathbb{R}^d)}^2 \\
+ C \|A(x, \cdot) - A(x_0, \cdot)\|_{L^\infty(\mathbb{R}^d)}^2.
\]

\[ \square \]

**Proof of Theorem 1.2.** 1. Existence and continuity. Let us denote by \( (\chi_{T,j}(x, \cdot))_{T \geq 1} \) (for fixed \( 1 \leq j \leq d \)) the sequence constructed in Lemma \([2.1]\) and corresponding to \( h = 0 \) and \( H = Ae_j, \ e_j \) being denoting the \( j \)th vector of the canonical basis of \( \mathbb{R}^d \). It satisfies \( (2.2) \), so that by the weak compactness, the sequence \( (\nabla \chi_{T,j}(x, \cdot))_{T \geq 1} \) weakly converges in \( L^2_{loc}(\mathbb{R}^d) \) (up to extraction of a subsequence) to some \( V_j(x, \cdot) \in L^2_{loc}(\mathbb{R}^d) \). From the equality

\[
\partial^2 \chi_{T,j}(x, \cdot)/\partial y_i \partial y_i = \partial^2 \chi_{T,j}(x, \cdot)/\partial y_i \partial y_i, \quad \text{a limit passage in the distributional sense yields}
\]

\[
\partial V_{j,i}(x, \cdot)/\partial y_i = \partial V_{j,i}(x, \cdot)/\partial y_i, \quad \text{where} \quad V_j = (V_{j,i})_{1 \leq i \leq d}.
\]

This implies \( V_j(x, \cdot) = \nabla \chi_j(x, \cdot) \) for some \( \chi_j(x, \cdot) \in H^1_{loc}(\mathbb{R}^d) \). Using the boundedness of \( (T^{-1} \chi_{T,j}(x, \cdot))_{T \geq 1} \) in \( L^2_{loc}(\mathbb{R}^d) \), we pass to the limit in the variational formulation of \( (2.1) \) (as \( T \to \infty \)) to get that \( \chi_j \) solves \( (1.12) \).

Arguing exactly as in the proof of \( (2.6) \) (in Lemma \([2.1]\)), we arrive at \( V_j(x, \cdot) \in B^{1,2}_A(\mathbb{R}^d) \). Also, since \( \chi_{T,j}(x, \cdot) \in B^{1,2}_A(\mathbb{R}^d) \), we have \( M(\nabla \chi_{T,j}(x, \cdot)) = 0 \), hence \( M(\nabla \chi_j(x, \cdot)) = 0 \). We repeat the proof of the Part 3. in the previous lemma to find that \( \nabla_y \chi_j \in C(\overline{\Omega}; B^{2}_A(\mathbb{R}^d)) \).
2. **Uniqueness** (of $\nabla_y \chi_j$). Fix $x \in \overline{\Omega}$ and assume that $\chi_j(x, \cdot) \in H^1_{\text{loc}}(\mathbb{R}^d)$ is such that $-\text{div}(A(x, \cdot)\nabla_y \chi_j(x, \cdot)) = 0$ in $\mathbb{R}^d$ and $\nabla_y \chi_j(x, \cdot) \in B^2_A(\mathbb{R}^d)^d$. Then it follows from Property (3.10) that, given $0 < \sigma < 1$, there exists $C_\sigma > 0$ independent from $r$ and $R$ such that

$$
\int_{B_r} |\nabla_y \chi_j(x, y)|^2 \, dy \leq C_\sigma \left( \frac{r}{R} \right)^\sigma \int_{B_R} |\nabla_y \chi_j(x, y)|^2 \, dy \quad \text{for all} \ 0 < r < R. \tag{2.7}
$$

Next, since $-\text{div}(A(x, \cdot)\nabla_y \chi_j(x, \cdot)) = 0$ in $\mathbb{R}^d$ and $\nabla_y \chi_j(x, \cdot) \in B^2_A(\mathbb{R}^d)^d$, we show as for (2.6) that

$$
M(A(x, \cdot)\nabla_y \chi_j(x, \cdot) \cdot \nabla_y \phi) = 0 \quad \text{for all} \ \phi \in B^1_{\#A}(\mathbb{R}^d). \tag{2.8}
$$

Choosing $\phi = \chi_j(x, \cdot)$ in (2.8), and using the ellipticity of $A$, it emerges $M(|\nabla_y \chi_j(x, \cdot)|^2) = 0$, that is, $\lim_{R \to \infty} \int_{B_R} |\nabla_y \chi_j(x, y)|^2 \, dy = 0$. Coming back to (2.7) and letting there $R \to \infty$, we are led to $\int_{B_r} |\nabla_y \chi_j(x, y)|^2 \, dy = 0$ for all $r > 0$. This gives $\nabla_y \chi_j(x, \cdot) = 0$.

We can now prove Theorem 1.1.

**Proof of Theorem 1.1.** Let $\Phi_\varepsilon = \psi_0 + \varepsilon \psi_1^\varepsilon$ with $\psi_1^\varepsilon(x) = \psi_1(x, x/\varepsilon) \ (x \in \Omega)$, where $\psi_0 \in C^\infty_0(\Omega)$ and $\psi_1 \in C^\infty_0(\Omega) \otimes A^\infty$, $A^\infty = \{u \in A : D^\alpha u \in A \text{ for all } \alpha \in \mathbb{N}^d\}$. Taking $\Phi_\varepsilon$ (which belongs to $C^\infty_0(\Omega)$) as a test function in the variational formulation of (1.5) yields

$$
\int_\Omega A^\varepsilon \nabla u_\varepsilon \cdot \nabla \Phi_\varepsilon \, dx = \int_\Omega f \Phi_\varepsilon \, dx. \tag{2.9}
$$

It is not difficult to see that the sequence $(u_\varepsilon)_{\varepsilon > 0}$ is bounded in $H^1_0(\Omega)$, so that, considering an ordinary sequence $E \subset \mathbb{R}^d_+$, there exist a couple $(u_0, u_1) \in H^1_0(\Omega) \times L^2(\Omega; B^1_{\#A}(\mathbb{R}^d))$ and a subsequence $E'$ of $E$ such that, as $E' \ni \varepsilon \to 0$,

$$
u_\varepsilon \to u_0 \text{ in } H^1_0(\Omega)-\text{weak and in } L^2(\Omega)-\text{strong}$$

$$
\nabla u_\varepsilon \to \nabla u_0 + \nabla_y u_1 \text{ in } L^2(\Omega)^d-\text{weak } \Sigma. \tag{2.10}
$$

On the other hand

$$
\nabla \Phi_\varepsilon = \nabla \psi_0 + (\nabla_y \psi_1^\varepsilon + \varepsilon (\nabla \psi_1)^\varepsilon) \to \nabla \psi_0 + \nabla_y \psi_1 \text{ in } L^2(\Omega)^d-\text{strong } \Sigma. \tag{2.11}
$$

This yields in (2.9) the following limit problem

$$
\int_\Omega M(A(\nabla u_0 + \nabla_y u_1) \cdot (\nabla \psi_0 + \nabla_y \psi_1)) \, dx = \int_\Omega f \psi_0 \, dx \quad \forall (\psi_0, \psi_1) \in C^\infty_0(\Omega) \times (C^\infty_0(\Omega) \otimes A^\infty). \tag{2.12}
$$

Problem (2.12) above is equivalent to the system

$$
\int_\Omega M(A(\nabla u_0 + \nabla_y u_1) \cdot \nabla \psi_0) \, dx = \int_\Omega f \psi_0 \, dx \quad \forall \psi_0 \in C^\infty_0(\Omega) \tag{2.13}
$$

$$
\int_\Omega M(A(\nabla u_0 + \nabla_y u_1) \cdot \nabla_y \psi_1) \, dx = 0 \quad \forall \psi_1 \in C^\infty_0(\Omega) \otimes A^\infty. \tag{2.14}
$$

Taking in (2.14) $\psi_1(x, y) = \varphi(x)v(y)$ with $\varphi \in C^\infty_0(\Omega)$ and $v \in A^\infty$, we get

$$
M(A(x, \cdot)(\nabla u_0 + \nabla_y u_1) \cdot \nabla_y v) = 0 \quad \forall v \in A^\infty, x \in \overline{\Omega}, \tag{2.15}
$$
which is, thanks to the density of $A^\infty$ in $B^{1,2}_A(\mathbb{R}^d)$, the weak form of
\[
\nabla_y \cdot (A(x,\cdot)(\nabla u_0 + \nabla_y u_1)) = 0 \quad \text{in } \mathbb{R}^d \quad \text{(for all fixed } x \in \Omega),
\]
with respect to the duality defined by (2.15). So fix $\xi \in \mathbb{R}^d$ and consider the problem
\[
\nabla_y \cdot (A(x,\cdot)(\xi + \nabla_y v_\xi(x,\cdot))) = 0 \quad \text{in } \mathbb{R}^d; \quad v_\xi(x,\cdot) \in B^{1,2}_A(\mathbb{R}^d).
\]

Thanks to Theorem 1.2, Eq. (2.17) possesses a unique solution $v_\xi$ (up to an additive constant depending on $x$) in $C(\Omega; B^{1,2}_A(\mathbb{R}^d))$. Choosing there $\xi = \nabla u_0(x)$, the uniqueness of the solution implies $u_1(x,y) = \chi(x,y) \cdot \nabla u_0(x)$ where $\chi = (\chi_j)_{1 \leq j \leq d}$ with $\chi_j = \epsilon_j$, $\epsilon_j$ the $j$th vector of the canonical basis of $\mathbb{R}^d$. Replacing in (2.13) $u_1$ by $\chi \cdot \nabla u_0$, we get
\[
\int_\Omega (M(A(I + \nabla_y \chi))\nabla u_0) \cdot \nabla \psi_0 \, dx = \int_\Omega f \psi_0 \, dx \quad \forall \psi_0 \in C_0^\infty(\Omega),
\]
that is, $-\nabla \cdot A^*(x) \nabla u_0 = f$ in $\Omega$.

It remains to verify (1.11). Define $\Phi_\epsilon(x) = u_0(x) + \epsilon u_1(x,x/\epsilon)$. Then using (1.6) we obtain
\[
\alpha \int_\Omega |\nabla u_\epsilon - \nabla \Phi_\epsilon|^2 \, dx \leq \int_\Omega A^\epsilon (u_\epsilon - \Phi_\epsilon) \cdot \nabla (u_\epsilon - \Phi_\epsilon) \, dx
\]
\[
= \int_\Omega f(u_\epsilon - \Phi_\epsilon) \, dx - \int_\Omega A^\epsilon \nabla \Phi_\epsilon \cdot \nabla (u_\epsilon - \Phi_\epsilon) \, dx.
\]

Since $u_1 \in L^2(\Omega; A^1)$, we have that $\int_\Omega f(u_\epsilon - \Phi_\epsilon) \, dx \to 0$. Indeed $\Phi_\epsilon \to u_0$ in $L^2(\Omega)$ (and hence $u_\epsilon - \Phi_\epsilon \to 0$ in $L^2(\Omega)$). Next observe that $\nabla \Phi_\epsilon \to \nabla u_0 + \nabla y u_1$ in $L^2(\Omega)$-strong $\Sigma$; in fact, $\nabla \Phi_\epsilon = \nabla u_0 + \epsilon (\nabla u_1)^\epsilon + (\nabla y u_1)^\epsilon$, and since $\nabla y u_1 \in L^2(\Omega; A)$, we obtain $(\nabla y u_1)^\epsilon \to \nabla y u_1$ in $L^2(\Omega)$-strong $\Sigma$. One gets readily $\nabla u_\epsilon \to \nabla \Phi_\epsilon \to 0$ in $L^2(\Omega)$-weak $\Sigma$. Using $A$ as a test function, $\int_\Omega A^\epsilon \nabla \Phi_\epsilon \cdot \nabla (u_\epsilon - \Phi_\epsilon) \, dx \to 0$. We have just shown that $u_\epsilon - u_0 - \epsilon u_1^\epsilon \to 0$ in $L^2(\Omega)$ and $\nabla (u_\epsilon - u_0 - \epsilon u_1^\epsilon) = \nabla u_\epsilon - \nabla \Phi_\epsilon \to 0$ in $L^2(\Omega)$. This proves (1.11) and completes the proof of Theorem 1.1.

We assume henceforth that the matrix $A$ does not depend on $x$, that is, $A(x,y) = A(y)$. Let $\chi_T = (\chi_{T,j})_{1 \leq j \leq d}$ be defined by (??).

**Lemma 2.2.** Let $T \geq 1$ and $\sigma \in (0,1)$. Assume that $A \in (A)^{d \times d}$. There exist positive numbers $C = C(A,d)$ and $C_\sigma = C_\sigma(d,\sigma,A)$ such that
\[
T^{-1} \|\chi_T\|_{L^\infty(\mathbb{R}^d)} \leq C,
\]
(2.18)
\[
\sup_{x \in \mathbb{R}^d} \left( \int_{B_r(x)} |\nabla \chi_T|^2 \, dy \right)^{1/2} \leq C_\sigma \left( \frac{T}{r} \right)^{\sigma} \quad \text{for } 0 < r \leq T,
\]
(2.19)
\[
|\chi_T(x) - \chi_T(y)| \leq C_\sigma T^{1-\sigma} |x-y|^\sigma \quad \text{for } |x-y| \leq T.
\]
(2.20)

**Proof.** Let us first check (2.18). From the inequality (2.2), we deduce that
\[
\sup_{x \in \mathbb{R}^d, R \geq T} \left( \int_{B_R(x)} |\chi_T|^2 \right)^{1/2} \leq C T
\]
(2.21)
where $C$ depends only on $d$, $\alpha$ and $\beta$. Now fix $z = (z_i)_{1 \leq i \leq d}$ in $\mathbb{R}^d$ and define
\[ u(y) = \chi_{T,j}(y) + y_j - z_j, \quad y \in \mathbb{R}^d. \tag{2.22} \]
Then $u$ solves the equation
\[ \nabla \cdot (A \nabla u) = T^{-2} \chi_{T,j} \quad \text{in} \quad \mathbb{R}^d. \tag{2.23} \]
Using the De Giorgi-Nash estimates, we obtain
\[ \sup_{B_T(z)} |u| \leq C \left( \int_{B_{2T}(z)} |u|^2 \right)^{\frac{1}{2}} + T^2 \left( \int_{B_{2T}(z)} \left| T^{-2} \chi_{T,j} \right|^2 \right)^{\frac{1}{2}} \leq CT \]
where $C = C(d, A)$. It follows that $|\chi_{T,j}(z)| \leq CT$. Whence (2.18). Now, concerning (2.20), one uses Schauder estimates: if $v \in H^1_{\text{loc}}(\mathbb{R}^d)$ is a weak solution of $-\nabla \cdot (A \nabla v) = h + \nabla \cdot H$ in $B_{2R}(x_0)$, then for each $\sigma \in (0, 1)$ and for all $x, y \in B_R(x_0)$,
\[ |v(x) - v(y)| \leq C |x - y|^\sigma \left( \int_{B_{2R}(x_0)} |v|^2 \right)^{\frac{1}{2}} + \sup_{z \in B_{2R}(x_0)} r.2^{-\sigma} \left( \int_{B_r(z)} |h|^2 \right)^{\frac{1}{2}} \tag{2.24} \]
where $C = C(\sigma, A)$ (see e.g. [15] or [31, Theorem 3.4]). Assume $x, y \in \mathbb{R}^d$ with $|x - y| \leq T$. Applying (2.24) with $2R = T$, $h = T^{-2} \chi_{T,j}$, $H = Ae_j$, $v = \chi_{T,j}$ and $x_0 = 0$,
\[ |\chi_{T,j}(x) - \chi_{T,j}(y)| \leq C |x - y|^\sigma \left( \int_{B_{2R}(x_0)} |v|^2 \right)^{\frac{1}{2}} + \sup_{z \in B_{2R}(x_0)} r.2^{-\sigma} \left( \int_{B_r(z)} |h|^2 \right)^{\frac{1}{2}} \leq CT^{1-\sigma} |x - y|^\sigma, \]
where we have used (2.18) for the last inequality above. To obtain (2.19), we use Caccioppoli’s inequality for $-\nabla \cdot (A \nabla \chi_{T,j}) + T^{-2} \chi_{T,j} = \nabla \cdot ( Ae_j)$ in $B_{2r}(x)$ and (2.20) to get
\[ \int_{B_r(x)} \left| \nabla \chi_{T,j}(y) \right|^2 dy \leq C r^{-2} \int_{B_{2r}(x)} \left| \chi_{T,j}(y) - \chi_{T,j}(x) \right|^2 dy + C \int_{B_{2r}(x)} |A|^2 dy \leq C r^{-2} (T^{1-\sigma} r) + C \left( \frac{T^{1-\sigma}}{r^{1-\sigma}} \right)^2 \quad \text{since} \quad 0 < r \leq T. \tag{2.19} \]
(2.19) follows by replacing $\sigma$ by $1 - \sigma$. This finishes the proof. 

The next result will be used in the forthcoming sections. It involves Green’s function $G : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ solution of
\[ -\nabla_x \cdot (A(x) \nabla_x G(x, y)) = \delta_y(x) \quad \text{in} \quad \mathbb{R}^d. \tag{2.25} \]
The properties of the function $G$ require the definition of the weak-$L^2$ space denoted by $L^{2, \infty}(\mathbb{R}^d)$ (see [8, Chapter 1] for its definition) together with its topological dual denoted by $L^{2,1}(\mathbb{R}^d)$ (see [34] for its definition).
Proposition 2.1. Assume the matrix $A \in L^\infty(\mathbb{R}^d)^{d \times d}$ is uniformly elliptic (see (1.6)) and symmetric. Then equation (2.25) has a unique solution in $L^\infty(\mathbb{R}^d, W_{loc}^{1,1}(\mathbb{R}^d))$ satisfying:

(i) $G(\cdot, y) \in W_{loc}^{1,2}(\mathbb{R}^d \backslash \{y\})$ for all $y \in \mathbb{R}^d$;

(ii) There exists $C = C(d) > 0$ such that

\[
\|\nabla_y G(x, \cdot)\|_{L^2(\mathbb{R}^d)} \leq C, \tag{2.26}
\]

\[
|G(x, y)| \leq \begin{cases} 
C(1 + |\log |x - y||) & \text{if } d = 2 \\
C |x - y|^{2-d} & \text{if } d \geq 3
\end{cases}
, \text{ all } x, y \in \mathbb{R}^d \text{ with } x \neq y, \tag{2.27}
\]

\[
\int_{B_{2R}(x) \backslash B_R(x)} |\nabla_y G(x, y)|^q dy \leq \frac{C}{R^{N(q-1)-q}} \text{ for all } R > 0 \text{ and } 1 \leq q \leq 2. \tag{2.28}
\]

If $A$ has Hölder continuous entries, then for $d \geq 3$ and for all $x, y \in \mathbb{R}^d$ with $x \neq y$, \n
\[
|\nabla_y G(x, y)| \leq C |x - y|^{1-d}. \tag{2.29}
\]

Properties (2.27) and (2.29) are classical; see e.g. [21] Theorems 1.1 and 3.3. (2.28) is proved in [10, Lemma 4.2].

3. APPROXIMATION OF HOMOGENIZED COEFFICIENTS: QUANTITATIVE ESTIMATES

To simplify the presentation of the results, we assume from now on that $A(x, y) = A(y)$. We henceforth denote the mean value by $\langle \cdot \rangle$.

3.1. Approximation by Dirichlet problem. In the preceding section, we saw that the corrector problem is posed on the whole of $\mathbb{R}^d$. However, if the coefficients of our problem are periodic (say the function $y \mapsto A(y)$ is $Y$-periodic ($Y = (-1/2, 1/2)^d$), then this problem reduces to another one posed on the bounded subset $Y$ of $\mathbb{R}^d$, and this yields coefficients that are computable. Contrasting with the periodic setting, the corrector problem in the general deterministic framework cannot be reduced to a problem on a bounded domain. Therefore, truncations must be considered, particularly on large domains like $Q_R$ (the closed cube centered at the origin and of side length $R$) with appropriate boundary conditions. We proceed exactly as in the random setting (see [12]). We consider the equation

\[-\nabla_y \cdot (A(e_j + \nabla_y \chi_{j,R})) = 0 \text{ in } Q_R, \quad \chi_{j,R} \in H^1_0(Q_R), \tag{3.1}\]

which possesses a unique solution satisfying

\[
\left(\int_{Q_R} |\nabla_y \chi_{j,R}|^2 dy \right)^{\frac{1}{2}} \leq C \text{ for any } R \geq 1 \tag{3.2}
\]

where $C$ is independent of $R$. Set $\chi_R = (\chi_{j,R})_{1 \leq j \leq d}$. We define the effective and approximate effective matrices $A^*$ and $A^*_R$ respectively, as follows

\[
A^* = \langle A(I + \nabla_y \chi) \rangle \quad \text{and} \quad A^*_R = \int_{Q_R} A(y)(I + \nabla_y \chi_R(y)) dy. \tag{3.3}
\]

Theorem 3.1. The generalized sequence of matrices $A^*_R$ converges, as $R \to \infty$, to the homogenized matrix $A^*$. 

**Proof.** We set, for \( x \in Q_1, \ w_j^R(x) = \frac{1}{R} \chi_{j,R}(Rx), \ A_R(x) = A(Rx) \) and consider the re-scaled version of (3.1) whose \( w_j^R \) is solution. It reads as
\[
- \nabla \cdot (A_R(e_j + \nabla w_j^R)) = 0 \quad \text{in} \ Q_1, \quad w_j^R = 0 \quad \text{on} \ \partial Q_1. \tag{3.4}
\]
Then (3.4) possesses a unique solution \( w_j^R \in H_0^1(Q_1) \) satisfying the estimate
\[
\|\nabla w_j^R\|_{L^2(Q_1)} \leq C \quad (1 \leq j \leq d) \tag{3.5}
\]
where \( C > 0 \) is independent of \( R > 0 \). Proceeding as in the proof of Theorem 1.1, we derive the existence of \( w_j \in H_0^1(Q_1) \) and \( w_{j,1} \in L^2(Q_1; B_1^{1,2} (\mathbb{R}^d)) \) such that, up to a subsequence not relabeled,
\[
w_j^R \to w_j \text{ in } H_0^1(Q_1)-\text{weak and } \nabla w_j^R \to \nabla w_j + \nabla_y w_{j,1} \text{ in } L^2(Q_1)^d-\text{weak } \Sigma \tag{3.6}
\]
and the couple \((w_j, w_{j,1})\) solves the equation
\[
\int_{Q_1} \langle A(e_j + \nabla w_j + \nabla_y w_{j,1}) \cdot (\nabla \psi_0 + \nabla_y \psi_1) \rangle \, dx = 0 \quad \forall (\psi_0, \psi_1) \in C_0^\infty(Q_1) \times (C_0^\infty(Q_1) \otimes \mathcal{A}^\infty), \tag{3.7}
\]
which can be rewritten in the following equivalent form (3.8)-(3.9)
\[
\int_{Q_1} \langle A(e_j + \nabla w_j + \nabla_y w_{j,1}) \rangle \cdot \nabla \psi_0 dx = 0 \quad \forall \psi_0 \in C_0^\infty(Q_1) \tag{3.8}
\]
and
\[
\langle A(e_j + \nabla w_j + \nabla_y w_{j,1}) \cdot \nabla_y v \rangle \, dx = 0 \quad \forall v \in \mathcal{A}^\infty. \tag{3.9}
\]
To solve (3.9), we consider its weak distributional form
\[
\nabla_y \cdot (A(e_j + \nabla w_j + \nabla_y w_{j,1})) = 0 \quad \text{in } \mathbb{R}^d. \tag{3.10}
\]
So fix \( \xi \in \mathbb{R}^d \) and consider the problem
\[
\nabla_y \cdot (A(e_j + \xi + \nabla_y \pi_j(\xi))) = 0 \quad \text{in } \mathbb{R}^d; \quad \pi_j(\xi) \in B_1^{1,2}(\mathbb{R}^d). \tag{3.11}
\]
Then \( \pi_j(\xi) \) has the form \( \pi_j(\xi) = \chi_j + \theta_j(\xi) \) where \( \chi_j \) is the solution of the corrector problem (1.10) and \( \theta_j(\xi) \) solves the equation
\[
\nabla_y \cdot (A(\xi + \nabla_y \theta_j(\xi))) = 0 \quad \text{in } \mathbb{R}^d; \quad \theta_j(\xi) \in B_1^{1,2}(\mathbb{R}^d), \tag{3.12}
\]
that is, \( \theta_j(\xi) = \xi \cdot \chi \) where \( \chi = (\chi_k)_{1 \leq k \leq d} \) with \( \chi_k \) being the solution of (1.10) corresponding to \( j = k \) therein. It follows that \( \pi_j(\xi) = \chi_j + \xi \cdot \chi \), so that the function \( w_{j,1} \), which corresponds to \( \pi_j(\nabla w_j) \), has the form \( w_{j,1} = \chi_j + \chi \cdot \nabla w_j \). Coming back to (3.8) and replacing there \( w_{j,1} \) by \( \chi_j + \chi \cdot \nabla w_j \), we obtain
\[
\int_{Q_1} \langle A(I + \nabla_y \chi) \rangle (e_j + \nabla w_j) \cdot \nabla \psi_0 dx = 0 \quad \forall \psi_0 \in C_0^\infty(Q_1). \tag{3.13}
\]
This shows that \( w_j \in H_0^1(Q_1) \) solves uniquely the equation
\[
- \nabla \cdot (A^*(e_j + \nabla w_j)) = 0 \quad \text{in } Q_1, \tag{3.14}
\]
and further we have, as \( R \to \infty \),
\[
A_R(e_j + \nabla w_j^R) \to A^*(e_j + \nabla w_j) \text{ in } L^2(Q_1)^d-\text{weak}. \tag{3.15}
\]
To see (3.15), we observe that the sequence \((A_R(e_j + \nabla w_j^R))_R\) is bounded in \(L^2(Q_1)^d\) and we choose a test function \(\Phi \in C_0^\infty(Q_1)^d\); then by the sigma-convergence (where we take \(A(y)\Phi(x)\) as a test function) we have from the second convergence result in (3.6) that
\[
\int_{Q_1} A_R(e_j + \nabla w_j^R) \cdot \Phi dx \to \int_{Q_1} \langle A(e_j + \nabla w_j + \nabla_y w_j,1) \cdot \Phi \rangle dx = \int_{Q_1} \langle A(e_j + \nabla w_j + \nabla_y w_j,1) \rangle \cdot \Phi dx.
\]
But according to (3.13), we see that
\[
\langle A(e_j + \nabla w_j + \nabla_y w_j,1) \rangle = \langle A(I + \nabla_y \chi) \rangle (e_j + \nabla w_j) = A^*(e_j + \nabla w_j).
\]
Now, since (3.14) has the form \(-\nabla \cdot (A^*\nabla w_j) = 0\) in \(Q_1\), \((A^*\) has constant entries\) we infer from the ellipticity property of \(A^*\) and the uniqueness of the solution to \(-\nabla \cdot (A^*\nabla w_j) = 0\) in \(H^1_0(Q_1)\) that \(w = (w_1, ..., w_d) = 0\). Hence the whole sequence \((w_j^R)_R\) weakly converges towards 0 in \(H^1_0(Q_1)\). Therefore, integrating (3.15) over \(Q_1\), we readily get (denoting \(w^R = (w_1^R, ..., w_d^R)\))
\[
A_R^* = \int_{Q_1} A(I + \nabla w^R) dx = \int_{Q_1} A^*(I + \nabla w) dx = A^*
\]
as \(R \to \infty\), where \(I\) is the \(d \times d\) identity matrix. This completes the proof.

### 3.2. Quantitative estimates

We study the rate of convergence for the approximation scheme of the previous subsection, under the assumption that the corrector lies in \(B^2_\lambda(\mathbb{R}^d)\). To this end, instead of considering the corrector problem (1.10) we rather consider its regularized version (2.1) which we recall here below:
\[
-\nabla \cdot A(y)(e_j + \nabla \chi_{T,j}) + T^{-2} \chi_{T,j} = 0 \text{ in } \mathbb{R}^d.
\]
We define the regularized homogenized matrix by
\[
A_T^* = \langle A(I + \nabla \chi_T) \rangle, \quad \chi_T = (\chi_{T,j})_{1 \leq j \leq d}
\]
Recalling that the homogenized matrix has the form \(A^* = \langle A(I + \nabla \chi) \rangle\), we show in (3.21) below that \(|A^* - A_T^*| \leq CT^{-1}\), so that \(A_T^* \to A^*\) as \(T \to \infty\).

With this in mind, we define the approximate regularized coefficients
\[
A_{R,T}^* = \int_{Q_R} A(I + \nabla \chi_T^R), \quad \chi_T^R = (\chi_{T,j}^R)_{1 \leq j \leq d}
\]
where \(\chi_{T,j}^R\) (the regularized approximate corrector) solves the problem
\[
-\nabla \cdot A(e_j + \nabla \chi_{T,j}^R) + T^{-2} \chi_{T,j}^R = 0 \text{ in } Q_R, \quad \chi_{T,j}^R \in H^1_0(Q_R).
\]
Then
\[
A_{R,T}^* \xrightarrow{\overrightarrow{R \to \infty}} A_T^* \xrightarrow{\overrightarrow{T \to \infty}} A^*.
\]
Convergence (***) will result from (3.21) below, while for convergence (**), we proceed exactly as in the proof of Theorem 3.1.
The aim here is to estimate the expression $|A^* - A_{R,T}^*|$ in terms of $R$ and $T$, and next take $R = T$ to get the suitable rate of convergence. The following theorem is the main result of this section.

**Theorem 3.2.** Suppose $\chi \in B_A^2(\mathbb{R}^d)$. Let $\delta \in (0,1)$. There exist $C = C(d, \delta, A)$ and a continuous function $\eta_\delta : [1, \infty) \to [0, \infty)$, which depends only on $A$ and $\delta$, such that

$$|A^* - A_{T,T}^*| \leq C \eta_\delta(T) \text{ for all } T \geq 1. \quad (3.19)$$

The proof breaks down into several steps which are of independent interest.

**Lemma 3.1.** Let $u \in B_A^2(\mathbb{R}^d)$. For any $0 < R < \infty$,

$$\left| \int_{Q_R} u - \langle u \rangle \right| \leq \sup_{y \in \mathbb{R}^d, |y| < R} \int_{Q_R} |u(t + y) - u(t)| \, dt. \quad (3.20)$$

**Proof.** Let $u \in B_A^2(\mathbb{R}^d)$. We know that, for any $y \in \mathbb{R}^d$,

$$\int_{Q_R(y)} u - \int_{Q_R} u = \int_{Q_R} (u(t + y) - u(t)) \, dt.$$

Now, let $k > 1$ be an integer; we have $Q_{kR} = \bigcup_{i=1}^{k^d} Q_R(x_i)$ for some $x_i \in \mathbb{R}^d$, so that

$$\left| \int_{Q_{kR}} u - \int_{Q_R} u \right| \leq \frac{1}{k^d} \sum_{i=1}^{k^d} \left| \int_{Q_{kR}(x_i)} u - \int_{Q_R} u \right| \leq \sup_{y \in \mathbb{R}^d} \left| \int_{Q_{kR}(y)} u - \int_{Q_R} u \right|.$$

Letting $k \to \infty$ we are led to (3.20). $\blacksquare$

The next result evaluates the difference between $A^*$ and $A_T^*$.

**Lemma 3.2.** Assume that $\chi_j$ (defined by (1.10)) belongs to $B_A^2(\mathbb{R}^d)$. There exists $C = C(d, A)$ such that

$$|A^* - A_T^*| \leq CT^{-1}. \quad (3.21)$$

**Proof.** First, let us set $v = \chi_{T,j} - \chi_j$. Then $v$ solves the equation $-\nabla \cdot (A \nabla v) + T^{-2}v = -T^{-2}\chi_j$ in $\mathbb{R}^d$. It follows from Lemma 2.1 that

$$\sup_{x \in \mathbb{R}^d} \int_{Q_T(x)} (|\nabla v|^2 + T^{-2} |v|^2) \leq CT^{-2} \sup_{x \in \mathbb{R}^d} \int_{Q_T(x)} |\chi_j|^2 \leq CT^{-2}.$$ 

In the last inequality above, we have used the fact that $\chi_j \in B_A^2(\mathbb{R}^d)$, so that

$$\sup_{x \in \mathbb{R}^d, T > 0} \int_{Q_T(x)} |\chi_j|^2 \leq C.$$
Lemma 3.2 yields

\[ \sup_{x \in \mathbb{R}^d} \left( \int_{Q_T(x)} |A \nabla (\chi_{T,j} - \chi_j)|^2 \right)^{\frac{1}{2}} \leq \|A\|_{\infty} \sup_{x \in \mathbb{R}^d} \left( \int_{Q_T(x)} |\nabla (\chi_{T,j} - \chi_j)|^2 \right)^{\frac{1}{2}} \leq CT^{-1}. \tag{3.22} \]

Now, using Lemma 3.1 with \( u = A \nabla (\chi_T - \chi) \), we obtain

\[ \left| \int_{Q_T} A \nabla (\chi - \chi_T) - (A^* - A^*_T) \right| \leq \sup_{y \in \mathbb{R}^d} \left| \int_{Q_T} A \nabla (\chi - \chi_T)(t + y) - A \nabla (\chi - \chi_T)(t) \right| dt. \tag{3.23} \]

However, from the equality

\[ \int_{Q_T} A \nabla (\chi - \chi_T)(t + y) dt = \int_{Q_T} A \nabla (\chi - \chi_T) dt \]

associated to the inequality

\[ \int_{Q_T(y)} |A \nabla (\chi - \chi_T)(t)| dt \leq \left( \int_{Q_T(y)} |A \nabla (\chi - \chi_T)|^2 \right)^{\frac{1}{2}}, \]

we deduce that the right-hand side of (3.23) is bounded by \( 2 \sup_{y \in \mathbb{R}^d} \left( \int_{Q_T(y)} |A \nabla (\chi - \chi_T)|^2 \right)^{\frac{1}{2}} \).

Taking into account (3.22), we get immediately

\[ \left| \int_{Q_T} A \nabla (\chi - \chi_T) - (A^* - A^*_T) \right| \leq CT^{-1}. \]

It follows that

\[ |A^* - A^*_T| \leq \left| \int_{Q_T} A \nabla (\chi - \chi_T) - (A^* - A^*_T) \right| + \int_{Q_T} |A \nabla (\chi - \chi_T)| \leq CT^{-1}. \]

We are now in a position to prove the theorem.

**Proof of Theorem 3.2.** We decompose \( A^* - A^*_R, T \) as follows:

\[ A^* - A^*_R, T = (A^* - A^*_T) + (A^*_T - A^*_R, T). \]

We consider each term separately.

Lemma 3.2 yields \( |A^* - A^*_T| \leq CT^{-1} \). As regard the term \( A^*_T - A^*_R, T \), we observe that

\[ v = \chi_{T,j} - \chi_{R,j} \]

solves the equation

\[ -\nabla \cdot A \nabla v + T^{-2} v = 0 \quad \text{in} \ Q_R \quad \text{and} \quad v = \chi_{T,j} \quad \text{on} \ \partial Q_R, \]

so that, proceeding exactly as in [12] Proof of Lemma 1] we obtain

\[ |A^*_T - A^*_R, T|^2 \leq C \left( T^2 \exp(-c_1 TR^\delta) + R^{\delta-1} \right) \tag{3.24} \]

where \( 0 < \delta < 1 \), and \( C \) and \( c_1 > 0 \) are independent of \( R \) and \( T \). We emphasize that in [12], the above inequality has been obtained without any help stemming from the random
character of the problem. It relies only on the bounds of the Green function of the operator
\(-\nabla \cdot A\nabla + T^{-2}\) and on the bounds of the regularized corrector \(\chi_T\).

Choosing \(R = T\) in (3.24), we define the function
\[
\eta_\delta(t) = \frac{1}{t} + t \exp \left(-\frac{c_1}{2} t^1 + \delta \right) + t^{\frac{1}{2}(\delta - 1)} \quad \text{for } t \geq 1.
\]

Then \(\eta_\delta\) is continuous with \(\lim_{t \to \infty} \eta_\delta(t) = 0\). We see that
\[
|A^* - A_{T,T}^*| \leq C\eta_\delta(T) \quad \text{for any } T \geq 1.
\]

This concludes the proof of the theorem. \(\blacksquare\)

4. Convergence rates: the asymptotic periodic setting

4.1. Preliminary results. Let us consider the corrector problem (1.10) in which \(A\) satisfies
in addition the assumptions (4.1) and (4.2) below: \(A = A_0 + A_{\text{per}}\) where
\[
A_{\text{per}} \in L^2_{\text{per}}(Y)^{d \times d} \quad \text{and} \quad \begin{cases} 
A_0 \in L^2(\mathbb{R}^d)^{d \times d} & \text{for } d \geq 3, \\
A_0 \in (L^2(\mathbb{R}^2) \cap L^{2,1}(\mathbb{R}^2))^2 \times 2 & \text{for } d = 2. 
\end{cases}
\] (4.1)

The matrix \(A_{\text{per}}\) is symmetric and further
\[
\alpha |\lambda|^2 \leq A_{\text{per}}(y) \lambda \cdot \lambda \leq \beta |\lambda|^2 \quad \text{for all } \lambda \in \mathbb{R}^d \text{ and a.e. } y \in \mathbb{R}^d.
\] (4.2)

Let \(H_{\text{per}}^1(\mathbb{R}^d) = \{u \in L^2_{\text{per}}(\mathbb{R}^d) : \nabla u \in L^2_{\text{per}}(\mathbb{R}^d)^d\}\) where \(L^2_{\text{per}}(\mathbb{R}^d) = L^2(\mathbb{R}^d) + L^2_{\text{per}}(Y)\) and \(L^2_0(\mathbb{R}^d)\) is the completion of \(C_0(\mathbb{R}^d)\) with respect to the seminorm (1.2).

**Proposition 4.1.** Let \(H\) be a function such that \(H \in L^2(\mathbb{R}^d)^d\) for \(d \geq 3\) and \(H \in (L^2(\mathbb{R}^d) \cap L^{2,1}(\mathbb{R}^d))^d\) for \(d = 2\). Assume \(A\) satisfies (1.6). Then there exists \(u_0 \in L^p(\mathbb{R}^d)\) with \(\nabla u_0 \in L^2(\mathbb{R}^d)^d\) such that \(u_0\) solves the equation
\[
- \nabla \cdot A\nabla u_0 = \nabla \cdot H \text{ in } \mathbb{R}^d
\] (4.3)
where \(p = 2^* \equiv 2d/(d - 2)\) for \(d \geq 3\) and \(p = \infty\) for \(d = 2\).

**Proof.** 1) We first assume that \(d \geq 3\). Let \(Y^{1,2} = \{u \in L^{2^*}(\mathbb{R}^d) : \nabla u \in L^2(\mathbb{R}^d)^d\}\) (where \(2^* = 2d/(d - 2)\)), and equip \(Y^{1,2}\) with the norm \(\|u\|_{Y^{1,2}} = \|u\|_{L^{2^*}(\mathbb{R}^d)} + \|\nabla u\|_{L^2(\mathbb{R}^d)}\), which makes it a Banach space. By the Sobolev’s inequality (see [3, Theorem 4.31, page 102]), there exists a positive constant \(C = C(d)\) such that
\[
\|u\|_{L^{2^*}(\mathbb{R}^d)} \leq C \|\nabla u\|_{L^2(\mathbb{R}^d)} \quad \forall u \in Y^{1,2}.
\] (4.4)

We deduce from (4.4) that (4.3) possesses a unique solution in \(Y^{1,2}\) satisfying the inequality
\[
\|u_0\|_{Y^{1,2}} \leq C \|H\|_{L^2(\mathbb{R}^d)}.
\] (4.5)

2) Now assume that \(d = 2\). We use \(G(x, y)\) defined by (2.25) to express \(u_0\) as
\[
u_0(x) = - \int_{\mathbb{R}^d} \nabla_y G(x, y) \cdot H(y) dy.
\] (4.6)
The expression (4.6) makes sense since we may proceed by approximation by assuming first that \( H \in C_0^\infty(\mathbb{R}^2)^2 \) and next using the density of \( C_0^\infty(\mathbb{R}^2) \) in \( L^{2,1}(\mathbb{R}^2) \) together with property (2.27) to conclude. So, using the generalized Hölder inequality, we get

\[
\|u_0\|_{L^\infty(\mathbb{R}^2)} \leq \sup_{x \in \mathbb{R}^2} \|\nabla_y G(x, \cdot)\|_{L^2,\infty(\mathbb{R}^2)} \|H\|_{L^{2,1}(\mathbb{R}^2)}.
\]

(4.7)

This completes the proof. \( \blacksquare \)

**Lemma 4.1.** Assume that \( A = A_0 + A_{per} \) where \( A \) and \( A_{per} \) are uniformly elliptic (see (1.6) and (4.2)) with \( A_0 \) and \( A_{per} \) being as in (4.1). Assume further that \( A_{per} \) and \( A \) are Hölder continuous. Let the number \( p \) be as in Proposition 4.1. Let \( \chi_{j,per} \in H^1_{per}(Y) \) be the unique solution of

\[
- \nabla_y \cdot (A_{per}(\epsilon_j + \nabla_y \chi_{j,per})) = 0 \text{ in } Y, \quad \int_Y \chi_{j,per} dy = 0.
\]

(4.8)

Then (1.10) possesses a unique solution \( \chi_j \in H^1_{\infty,per}(Y) \) (in the sense of Theorem 1.2) satisfying \( \chi_j = \chi_{j,0} + \chi_{j,per} \) where \( \chi_{j,0} \in L^p(\mathbb{R}^d) \) with \( \nabla_y \chi_{j,0} \in L^2(\mathbb{R}^d)^d \), and

\[
\|\chi_j\|_{L^\infty(\mathbb{R}^d)} \leq C
\]

(4.9)

where \( C = C(d, A) \).

**Proof.** First, we notice that if \( \chi_{j,per} \) solves (4.8) then \( \chi_{j,0} \) solves

\[
- \nabla_y \cdot (A \nabla_y \chi_{j,0}) = \nabla_y \cdot (A_0(\epsilon_j + \nabla_y \chi_{j,per})) \text{ in } \mathbb{R}^d.
\]

Assuming that \( A_{per} \) is Hölder continuous, we get \( \nabla_y \chi_{j,per} \in L^\infty(Y) \). Because of the property of \( A_0 \) given by (1.1), it follows that \( g = A_0(\epsilon_j + \nabla_y \chi_{j,per}) \) belongs to \( L^2(\mathbb{R}^d)^d \) (resp. \( L^2(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)^2 \)) for \( d \geq 3 \) (resp. \( d = 2 \)). Proposition 4.1 implies that \( \chi_{j,0} \in L^p(\mathbb{R}^d) \) with \( \nabla_y \chi_{j,0} \in L^2(\mathbb{R}^d)^d \) for \( d \geq 3 \). Hence in that case one has \( \langle \chi_{j,0} \rangle = 0 \) and \( \langle \nabla_y \chi_{j,0} \rangle = 0 \). This proves that \( \chi_j = \chi_{j,per} + \chi_{j,0} \in H^1_{\infty,per}(Y) \) for \( d \geq 3 \). Now, for \( d = 2 \) we have \( \chi_{j,0} \in L_0^2(\mathbb{R}^2) \) since \( \chi_{j,0} \) vanishes at infinity. Indeed, we use (4.7) to get

\[
\|\chi_{j,0}\|_{L^\infty(\mathbb{R}^2)} \leq \sup_{x \in \mathbb{R}^d} \|\nabla_y G(x, \cdot)\|_{L^2,\infty(\mathbb{R}^2)} \|g\|_{L^{2,1}(\mathbb{R}^2)}
\]

and proceed as in [10, Section 3, page 14] (first approximate \( g \) by smooth functions in \( C_0^\infty(\mathbb{R}^2)^2 \) to show that \( \chi_{j,0} \in L_0^2(\mathbb{R}^2) \)).

Let us now verify (4.9). We drop for a while the index \( j \) and just write \( \chi = \chi_0 + \chi_{per} \), where the couple \( (\chi_{per}, \chi_0) \) solves the system

\[
- \nabla_y \cdot (A_{per}(\epsilon_j + \nabla_y \chi_{per})) = 0 \text{ in } Y,
\]

(4.10)

\[
- \nabla_y \cdot (A \nabla_y \chi_0) = \nabla_y \cdot (A_0(\epsilon_j + \nabla_y \chi_{per})) \text{ in } \mathbb{R}^d.
\]

(4.11)

It is well known that \( \chi_{per} \) is bounded in \( L^\infty(\mathbb{R}^d) \). Let us first deal with \( \chi_0 \). Let \( g = A_0(\epsilon_j + \nabla_y \chi_{per}) \) and use the Green function defined in Proposition 2.1 to express \( \chi_0 \) as

\[
\chi_0(y) = - \int_{\mathbb{R}^d} \nabla_x G(y, x) g(x) dx.
\]

(4.12)
We recall that $G$ satisfies the inequality (2.29) for $d \geq 3$ and (2.26) for $d = 2$, respectively. We first assume that $d \geq 3$. Let $y \in \mathbb{R}^d$ and choose $\gamma \in C_0^\infty(B_2(y))$ such that $\gamma = 1$ on $B_1(y)$ and $0 \leq \gamma \leq 1$. We write $\chi_0$ as

$$\chi_0(y) = -\int_{\mathbb{R}^d} \nabla_x G(y, x) \cdot g(x) \gamma(x) dx - \int_{\mathbb{R}^d} \nabla_x G(y, x) \cdot g(x)(1 - \gamma(x)) dx = v_1(y) + v_2(y).$$

As for $v_1$, owing to (2.29), we have

$$|v_1(y)| \leq C \|g\|_{L^\infty(\mathbb{R}^d)} \int_{B_2(y)} |x - y|^{1-d} dx \leq C \|g\|_{L^\infty(\mathbb{R}^d)}$$

where $C = C(d)$. As for $v_2$, (2.29) and Hölder’s inequality imply,

$$|v_2(y)| \leq C \|g\|_{L^2(\mathbb{R}^d)} \left( \int_{\mathbb{R}^d \setminus B_2(y)} |x - y|^{2-2d} dx \right) \leq C \|g\|_{L^2(\mathbb{R}^d)}$$

since $2d - 2 > d$ for $d \geq 3$.

When $d = 2$, we use (2.26) to get

$$\|\chi_0\|_{L^\infty(\mathbb{R}^2)} \leq \sup_{x \in \mathbb{R}^2} \|\nabla_y G(x, \cdot)\|_{L^2, \infty(\mathbb{R}^2)} \|g\|_{L^{2,1}(\mathbb{R}^2)} \leq C \|g\|_{L^{2,1}(\mathbb{R}^2)}.$$

Lemma 4.2. (i) Let $g \in L^2(\mathbb{R}^d) + L^2_{per}(Y)$ be such that $\langle g \rangle = 0$. Then there exists at least one function $u \in H^1_{\infty, per}(Y)$ such that

$$\Delta u = g \text{ in } \mathbb{R}^d, \quad \langle u \rangle = 0. \tag{4.13}$$

(ii) Assume further that $g \in L^\infty(\mathbb{R}^d)$ and $u$ is bounded; then $u, \nabla u \in B_{\infty, per}(\mathbb{R}^d)$ and

$$\|\nabla u\|_{L^\infty(\mathbb{R}^d)} \leq C \|g\|_{L^\infty(\mathbb{R}^d)}, \tag{4.14}$$

where $C > 0$ depends only on $d$.

Proof. (i) We write $g = g_0 + g_{per}$ with $g_0 \in L^2(\mathbb{R}^d)$ and $g_{per} \in L^2_{per}(Y)$. Since $\langle g \rangle = 0$, we have $\langle g_{per} \rangle = 0$. So let $v_{per} \in H^1_{per}(Y)$ be the unique solution of

$$\Delta v_{per} = g_{per} \text{ in } Y, \quad \langle v_{per} \rangle = 0.$$

We observe that if $u$ solves (4.13), then $u$ has the form $u = v_0 + v_{per}$ where $v_0 \in H^1(\mathbb{R}^d)$ solves the problem

$$\Delta v_0 = g_0 \text{ in } \mathbb{R}^d, \quad v_0(x) \to 0 \text{ as } |x| \to \infty.$$

Since $g_0 \in L^2(\mathbb{R}^d)$, $v_0$ easily expresses as

$$v_0(x) = -\int_{\mathbb{R}^d} \Gamma_0(x - y)g_0(y) dy$$

where $\Gamma_0$ denotes the fundamental solution of the Laplacian in $\mathbb{R}^d$ (with pole at the origin). This shows the existence of $u$ in $H^1(\mathbb{R}^d) + H^1_{per}(Y) \subset H^1_{\infty, per}(\mathbb{R}^d)$. 


Let us check (ii). First, since (4.13) is satisfied, $u$ is thus the Newtonian potential of $g$ in $\mathbb{R}^d$, and by [17], page 71, Problem 4.8 (a), $\nabla u \in C^{1/2}_{\text{loc}}(\mathbb{R}^d)$. Using therefore the continuity of $\nabla u$ together with the fact that $\nabla u$ also lies in $L^2_{\infty, \text{per}}(\mathbb{R}^d)$, we infer that $\nabla u \in \mathcal{B}_{\infty, \text{per}}(\mathbb{R}^d) = C_0(\mathbb{R}^d) \oplus \mathcal{C}_{\text{per}}(Y)$. We then proceed as in the proof of Lemma 4.1 to obtain $u \in \mathcal{B}_{\infty, \text{per}}(\mathbb{R}^d)$. This completes the proof. 

The following result is a mere consequence of the preceding lemma. Its proof is therefore left to the reader.

**Corollary 4.1.** Let $g$ be a solenoidal vector in $(L^2(\mathbb{R}^d) + L^2_{\text{per}}(Y))^d$ (i.e. $\nabla \cdot g = 0$) with $\langle g \rangle = 0$. Then there exists a skew symmetric matrix $G$ with entries in $L^2_{\infty, \text{per}}(Y)$ such that $g = \nabla \cdot G$. If further $g$ belongs to $L^\infty(\mathbb{R}^d)^d$, then $G$ has entries in $\mathcal{B}_{\infty, \text{per}}(\mathbb{R}^d)$ and

$$
\|G\|_{L^\infty(\mathbb{R}^d)} \leq C \|g\|_{L^\infty(\mathbb{R}^d)}.
$$

(4.15)

4.2. **Convergence rates: proof of Theorem 1.4.** Let $u_\varepsilon$, $u_0 \in H^1_0(\Omega)$ be the weak solutions of (1.3) and (1.8) respectively. Assume further that $u_0 \in H^2(\Omega)$. We suppose in addition that $\Omega$ is sufficiently smooth. For any function $h \in L^2_{\text{loc}}(\mathbb{R}^d)$ and $\varepsilon > 0$ we define $h^\varepsilon$ by $h^\varepsilon(x) = h(x/\varepsilon)$ for $x \in \mathbb{R}^d$. We define the first order approximation of $u_\varepsilon$ by $v_\varepsilon = u_\varepsilon + \varepsilon \chi^\varepsilon \nabla u_0$. Let $w_\varepsilon = u_\varepsilon - v_\varepsilon + z_\varepsilon$ where $z_\varepsilon \in H^1(\Omega)$ is the weak solution of the following problem

$$
- \nabla \cdot A^\varepsilon \nabla z_\varepsilon = 0 \text{ in } \Omega, \quad z_\varepsilon = \varepsilon \chi^\varepsilon \nabla u_0 \text{ on } \partial \Omega.
$$

(4.16)

$z_\varepsilon$ will be used to approximate the difference of $u_\varepsilon$ and its first order approximation $v_\varepsilon$.

**Lemma 4.3.** The function $w_\varepsilon$ solves the problem

$$
\begin{cases}
- \nabla \cdot (A^\varepsilon \nabla w_\varepsilon) = \nabla \cdot (A^\varepsilon (\nabla u_0 + (\nabla y \chi)^\varepsilon \nabla u_0 - \langle A(\nabla u_0 + \nabla y \chi \nabla u_0)\rangle) \\
\quad + \varepsilon \nabla \cdot (A^\varepsilon \nabla^2 u_0 \chi^\varepsilon) \text{ in } \Omega \\
w_\varepsilon = 0 \text{ on } \partial \Omega.
\end{cases}
$$

(4.17)

**Proof.** Let $y = x/\varepsilon$. Then

$$
A(y) \nabla w_\varepsilon = A(y)(\nabla u_\varepsilon - \nabla u_0 - \nabla y \chi(y) \nabla u_0 - \varepsilon (\nabla^2 u_0) \chi(y) + \nabla z_\varepsilon),
$$

hence

$$
\nabla \cdot A(y) \nabla w_\varepsilon = \nabla \cdot A(y) \nabla u_\varepsilon - \nabla \cdot A(y) \nabla u_0 - \nabla \cdot A(y)(\nabla y \chi(y) \nabla u_0) \\
\quad - \varepsilon \nabla \cdot (A(y)(\nabla^2 u_0) \chi(y)) \\
= \nabla \cdot A^\varepsilon \nabla u_0 - \nabla \cdot A(y) \nabla u_0 - \nabla \cdot A(y)(\nabla y \chi(y) \nabla u_0) \\
\quad - \varepsilon \nabla \cdot (A(y)(\nabla^2 u_0) \chi(y)).
$$

But

$$
A^\varepsilon \nabla u_0 = \langle A(\nabla u_0 + \nabla y \chi \nabla u_0)\rangle = \langle A(I + \nabla y \chi) \nabla u_0 \rangle.
$$
Thus

\[-\nabla \cdot A^\varepsilon \nabla w_\varepsilon = \nabla \cdot [A(y) (\nabla u_0 + \nabla y \chi \nabla u_0) - \langle A(\nabla u_0 + \nabla y \chi \nabla u_0) \rangle] + \varepsilon \nabla \cdot (A(y)(\nabla^2 u_0) \chi(y)),\]

which is the statement of the lemma. ■

Set

\[a_{ij}(y) = b_{ij}(y) + \sum_{k=1}^{d} b_{ik}(y) \frac{\partial \chi_j}{\partial y_k}(y) - b^*_i \]

where \(A^* = (b^*_ij)_{1 \leq i,j \leq d}\) is the homogenized matrix, and let \(a_j = (a_{ij})_{1 \leq i \leq d}\). Then \(a_j \in [L^\infty(\mathbb{R}^d) \cap L^2_{\infty,\text{per}}(Y)]^d\) with \(\nabla \cdot a_j = 0\) and \(\langle a_j \rangle = 0\). Hence by Corollary 4.1, there is a skew-symmetric matrix \(G_j\) with entries in \(A = B_{\infty,\text{per}}(Y)\) such that \(a_j = \nabla_y \cdot G_j\). Moreover in view of (4.15) in Corollary 4.1, we have

\[\|G_j\|_\infty \leq C \|a_j\|_\infty.\]

With this in mind and recalling that \(G_j\) is skew-symmetric, Eq. (4.17) becomes

\[\begin{aligned}
- \nabla \cdot A \left( \frac{x}{\varepsilon} \right) \nabla w_\varepsilon &= \varepsilon \nabla \cdot (r_1^\varepsilon + r_2^\varepsilon) \tag{4.18}
\end{aligned}\]

where

\[r_1^\varepsilon(x) = \sum_{j=1}^{d} G_j(y) \nabla \frac{\partial u_0}{\partial x_j}(x)\text{ and } r_2^\varepsilon(x) = A(y) \nabla^2 u_0(x) \chi(y)\text{ with } y = \frac{x}{\varepsilon}.\]

Now, since \(w_\varepsilon \in H^1_0(\Omega)\), it follows from the ellipticity of \(A\) (see (1.6)) that

\[\alpha \|\nabla w_\varepsilon\|_{L^2(\Omega)} \leq \varepsilon \left( \|r_1^\varepsilon\|_{L^2(\Omega)} + \|r_2^\varepsilon\|_{L^2(\Omega)} \right) \leq C \varepsilon \|u_0\|_{H^2(\Omega)}\]

where \(C = C(d, A, \Omega)\).

We have just proved the following result.

**Proposition 4.2.** Let \(\Omega\) be a smooth bounded domain in \(\mathbb{R}^d\). Suppose that \(A = A_0 + A_{\text{per}}\) and \(A\) and \(A_{\text{per}}\) are uniformly elliptic (see (1.6) and (4.2)). For \(f \in L^2(\Omega)\), let \(u_\varepsilon\), \(u_0\) and \(v_\varepsilon\) be weak solutions of Dirichlet problems (1.5), (1.8) and (4.16), respectively. Assume \(u_0 \in H^2(\Omega)\). There \(C = C(d, A, \Omega)\) such that

\[\|u_\varepsilon - u_0 - \varepsilon \chi \varepsilon \nabla u_0 + z_\varepsilon\|_{H^1_0(\Omega)} \leq C \varepsilon \|u_0\|_{H^2(\Omega)}. \tag{4.19}\]

The estimate of the deviation of \(u_\varepsilon\) and \(v_\varepsilon\) is a consequence of the following lemma whose proof is postponed to the next section and is obtained as a special case of the proof of a general result formulated as Lemma 5.3. Observe that in Lemma 5.3 we replace \(T^{-1} \|\chi T\|_{L^\infty(\mathbb{R}^d)}\) by \(\varepsilon\) (see Remark 5.3).
Lemma 4.4. Assume \( u_0 \in H^2(\Omega) \). Let \( z_\varepsilon \) be the solution of problem (4.16). There exists \( C = C(d, A, \Omega) \) such that

\[
\|z_\varepsilon\|_{H^1(\Omega)} \leq C\varepsilon^{\frac{1}{2}} \|u_0\|_{H^2(\Omega)}.
\]

**Proof of Theorem 1.4.** Since \( \Omega \) is a \( C^{1,1} \)-bounded domain in \( \mathbb{R}^d \) and the matrix \( A^\varepsilon \) has constant entries, it is known that \( u_0 \) satisfies the inequality

\[
\|u_0\|_{H^2(\Omega)} \leq C\|f\|_{L^2(\Omega)}, \quad C = C(d, \alpha, \Omega) > 0.
\]

Using (4.19) together with (4.20) and (4.21), we arrive at

\[
\|u_\varepsilon - u_0 - \varepsilon \chi^\varepsilon \nabla u_0\|_{H^1(\Omega)} = \left\|u_\varepsilon - u_0 - \varepsilon \chi^\varepsilon \nabla u_0 + z_\varepsilon\right\|_{H^1(\Omega)} + \|z_\varepsilon\|_{H^1(\Omega)}
\leq C\varepsilon^{\frac{1}{2}} \|u_0\|_{H^2(\Omega)} + C\varepsilon^{\frac{1}{2}} \|f\|_{L^2(\Omega)},
\]

and derive the statement of (1.15) in Theorem 1.4. As for (1.16) we proceed exactly as in the proof of (1.14) in the proof of Theorem 1.3; see in particular Remark 5.4 in the next section. This concludes the proof of Theorem 1.4. \( \square \)

5. Convergence rates: the asymptotic almost periodic setting

5.1. Preliminaries. We treat the asymptotic almost periodic case in a general way, dropping restrictions (4.1) and (4.2). The results in this section extend those of the preceding section as well as those in the almost periodic setting obtained in [31].

We recall that a bounded continuous function \( u \) defined on \( \mathbb{R}^d \) is asymptotically almost periodic if there exists a couple \((v, w) \in AP(\mathbb{R}^d) \times C_0(\mathbb{R}^d) \) such that \( u = v + w \). We denote by \( B_{\infty, AP}(\mathbb{R}^d) = AP(\mathbb{R}^d) + C_0(\mathbb{R}^d) \) the Banach algebra of such functions. We denote by \( H_{\infty, AP}^1(\mathbb{R}^d) \) the Sobolev-type space associated to the Besicovitch space \( B_{\infty, AP}^2(\mathbb{R}^d) = L^2_{\infty, AP}(\mathbb{R}^d) \) the completion of \( C_0(\mathbb{R}^d) \) with respect to the seminorm (1.2) while \( B_{\infty, AP}^2(\mathbb{R}^d) \) is the Besicovitch space associated to the algebra \( AP(\mathbb{R}^d) \). We also denote by \( B_{\infty, AP}^1(\mathbb{R}^d) \) the algebra of real-valued bounded continuous functions defined on \( \mathbb{R}^d \).

The following characterization of \( B_{\infty, AP}(\mathbb{R}^d) \) is a useful tool for the considerations below.

**Proposition 5.1.** Let \( u \in C_0(\mathbb{R}^d) \). Then \( u \in B_{\infty, AP}(\mathbb{R}^d) \) if and only if

\[
\sup_{y \in \mathbb{R}^d} \inf_{z \in \mathbb{R}^d, |z| \leq L} \|u(\cdot + y) - u(\cdot + z)\|_{L^\infty(\mathbb{R}^d \setminus B_R)} \rightarrow 0 \text{ as } L \rightarrow \infty \text{ and } R \rightarrow 0.
\]

**Proof.** A set \( E \) in \( \mathbb{R}^d \) is relatively dense if there exists \( L > 0 \) such that \( \mathbb{R}^d = E + B_L \) (where we recall that \( B_L = B(0, L) \)), that is, any \( x \in \mathbb{R}^d \) expresses as a sum \( y + z \) with \( y \in E \) and \( z \in B_L \). This being so, it is known that \( u \in C_0(\mathbb{R}^d) \) lies in \( B_{\infty, AP}(\mathbb{R}^d) \) if and only if for any \( \varepsilon > 0 \), there is \( R = R(\varepsilon) > 0 \) such that the set

\[
\{ \tau \in \mathbb{R}^d : |u(t + \tau) - u(t)| < \varepsilon \quad \forall |t| \geq R \}
\]
Remark 5.1. We notice that, for any $u \in C_b(\mathbb{R}^d)$,
\[
\lim_{R \to \infty} \left( \sup_{|y| \leq R} |u(y)| \right) = \lim_{R \to 0} \left( \sup_{|y| \geq R} |u(y)| \right).
\]
In view of the above equality we may replace (5.1) by
\[
\sup_{y \in \mathbb{R}^d} \inf_{|z| \leq L} \|u(\cdot + y) - u(\cdot + z)\|_{L^\infty(B_R)} \to 0 \text{ as } L, R \to \infty \quad (5.2)
\]
since the limits in (5.1) and (5.2) are the same. In practice we will rather use (5.2).

Definition 5.1. For a function $u \in B_{\infty, AP}(\mathbb{R}^d)$ we define the modulus of asymptotic almost periodicity of $u$ by
\[
\rho_u(L, R) = \sup_{y \in \mathbb{R}^d} \inf_{|z| \leq L} \|u(\cdot + y) - u(\cdot + z)\|_{L^\infty(B_R)} \text{ for } L, R > 0. \quad (5.3)
\]
In particular we set
\[
\rho(L, R) = \sup_{y \in \mathbb{R}^d} \inf_{|z| \leq L} \|A(\cdot + y) - A(\cdot + z)\|_{L^\infty(B_R)}, \text{ } L, R > 0. \quad (5.4)
\]

Remark 5.2. Observe that if $R = \infty$ (that is, $B_R = \mathbb{R}^d$) in (5.3), then $u \in B_{\infty, AP}(\mathbb{R}^d)$ is almost periodic if and only if $\rho_u(L, \infty) \to 0$ as $L \to \infty$.

5.2. Estimates of approximate correctors. First we recall that the approximate corrector $\chi_T = (\chi_{T,j})_{1 \leq j \leq d}$ is defined as the distributional solution of
\[
- \nabla \cdot (A(e_j + \nabla \chi_{T,j})) + T^{-2} \chi_{T,j} = 0 \text{ in } \mathbb{R}^d, \quad \chi_{T,j} \in H^1_{\infty, AP}(\mathbb{R}^d) \quad (5.5)
\]
where $A \in (L^2_{\infty, AP}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d))^{d \times d}$ is symmetric and uniformly elliptic.

In all that follows in this section we assume that $A \in (B_{\infty, AP}(\mathbb{R}^d))^{d \times d}$.

Theorem 5.1. Let $T \geq 1$. Then $\chi_T \in B_{\infty, AP}(\mathbb{R}^d)$ and for any $x_0, y, z \in \mathbb{R}^d$,
\[
\|\chi_T(\cdot + y) - \chi_T(\cdot + z)\|_{L^\infty(B_R(x_0))} \leq CT \|A(\cdot + y) - A(\cdot + z)\|_{L^\infty(B_R(x_0))} \quad (5.6)
\]
for any $R > 2T$, where $C = C(d, A)$.

Proof. Fix $R > 2T$. We need to show that, for any $x_0, y, z \in \mathbb{R}^d$ and $t \in B_R(x_0)$,
\[
|\chi_T(t + y) - \chi_T(t + z)| \leq CT \|B(\cdot + y) - B(\cdot + z)\|_{L^\infty(B_R(x_0))}.
\]
We follow the same approach as in the proof of [31, Theorem 6.3]. Without restriction, assume $x_0 = 0$. We choose $\varphi \in \mathcal{C}_0^\infty(B_{\frac{3}{2}T})$ such that $\varphi = 1$ in $B_{\frac{3}{2}T}$, $0 \leq \varphi \leq 1$ and $|\nabla \varphi| \leq CT^{-1}$. We also assume that $d \geq 3$ (the case $d = 2$ follows from the case $d = 3$ by
adding a dummy variable). Define \( u(x) = \chi_{T,j}(x + y) - \chi_{T,j}(x + z) \ (x \in \mathbb{R}^d) \) and note that \( u \) solves the equation

\[
-\nabla \cdot (A(\cdot + y) \nabla u) + T^{-2}u = \nabla \cdot (A(\cdot + y) - A(\cdot + z))e_j + \nabla \cdot [(A(\cdot + y) - A(\cdot + z))\nabla v] \text{ in } \mathbb{R}^d
\]

where \( v(x) = \chi_{T,j}(x + z) \). We have

\[
-\nabla \cdot (A(\cdot + y) \nabla u) = -T^{-2}u + \nabla \cdot (\varphi(A(\cdot + y) - A(\cdot + z))e_j) + \nabla \cdot [\varphi(A(\cdot + y) - A(\cdot + z))\nabla v] - (A(\cdot + y) - A(\cdot + z))e_j \nabla \varphi - A(\cdot + y) \nabla u \cdot \nabla \varphi - \nabla \cdot (uA(\cdot + y) \nabla \varphi).
\]

Denoting by \( G^y \) the fundamental solution of the operator \(-\nabla \cdot (A(\cdot + y) \nabla)\) in \( \mathbb{R}^d \), we use \( \) the representation formula in (5.7) \( \) to get, for \( x \in B_T \),

\[
u(x) = -T^{-2} \int_{\mathbb{R}^d} G^y(x, t)u(t)\varphi(t)dt - \int_{\mathbb{R}^d} \nabla_t G^y(x, t)\varphi(t)(A(t + y) - A(t + z))e_j dt
\]

\[- \int_{\mathbb{R}^d} \nabla_t G^y(x, t)\varphi(t)(A(t + y) - A(t + z))\nabla v(t)dt
\]

\[- \int_{\mathbb{R}^d} G^y(x, t)(A(t + y) - A(t + z))e_j \nabla \varphi(t)dt
\]

\[- \int_{\mathbb{R}^d} G^y(x, t)A(t + y) \nabla u(t) \cdot \nabla \varphi(t)dt
\]

\[+ \int_{\mathbb{R}^d} \nabla_t G^y(x, t)A(t + y)u(t)\nabla \varphi(t)dt.
\]

It follows that

\[
|u(x)| \leq C T^{-2} \int_{B_T} |G^y(x, t)||u(t)||dt +
\]

\[+ C \|A(\cdot + y) - A(\cdot + z)\|_{L^\infty(B_T)} \int_{B_T} |\nabla_t G^y(x, t)||dt
\]

\[+ C \|A(\cdot + y) - A(\cdot + z)\|_{L^\infty(B_T)} \int_{B_T} |\nabla_t G^y(x, t)||\nabla \varphi(t)||dt
\]

\[+ C \|A(\cdot + y) - A(\cdot + z)\|_{L^\infty(B_T)} \int_{B_T} |G^y(x, t)||\nabla \varphi(t)||dt
\]

\[+ C \left( \int_{B_T} |G^y(x, t)||\nabla \varphi(t)||^2 dt \right)^{\frac{1}{2}} \left( \int_{B_T} |u||^2 dt \right)^{\frac{1}{2}}
\]

\[+ C \left( \int_{B_T} |\nabla_t G^y(x, t)||\nabla \varphi(t)||^2 dt \right)^{\frac{1}{2}} \left( \int_{B_T} |u||^2 dt \right)^{\frac{1}{2}}.
\]

Let us first deal with the last two terms in (5.8). Let \( 0 < \tau < 1 \) be such that \( B_{\tau T}(x) \subset B_T \) (recall that \( x \in B_T \)). Then \( B_{2T} \setminus B_{\tau T}(x) \subset B_{3T}(x) \setminus B_{\tau T}(x) \) and since \( \nabla \varphi = 0 \) in \( B_T \) (and
hence in $B_{\tau T}(x)$, it holds that

$$
\left( \int_{B_{2T}} |G^y(x, t)|^2 |\nabla \varphi(t)|^2 \, dt \right)^{\frac{1}{2}} \leq C T^{-1} \left( \int_{B_{\tau T}(x) \setminus B_{\tau T}(x)} \frac{dt}{|x - t|^{2(d-2)}} \right)^{\frac{1}{2}} \leq C T^{1 - \frac{d}{2}};
$$

$$
\left( \int_{B_{2T}} |\nabla_t G^y(x, t)|^2 |\nabla \varphi(t)|^2 \, dt \right)^{\frac{1}{2}} \leq C T^{-1} \left( \int_{B_{\tau T}(x) \setminus B_{\tau T}(x)} |\nabla_t G^y(x, t)|^2 \, dt \right)^{\frac{1}{2}} \leq C T^{-1} \left( \sum_{i=[\frac{\ln \tau}{\ln 2}]}^{2} \int_{B_{2iT}(x) \setminus B_{2iT}(x)} |\nabla_t G^y(x, t)|^2 \, dt \right)^{\frac{1}{2}} \leq C T^{-\frac{d}{2}},
$$

where $[\frac{\ln \tau}{\ln 2}]$ stands for the integer part of $\frac{\ln \tau}{\ln 2}$. We infer that the last two terms in (5.8) are bounded from above by $T \left( \int_{B_{2T}} |\nabla u|^2 \right)^{\frac{1}{2}} + \left( \int_{B_{2T}} |u|^2 \right)^{\frac{1}{2}}$. Next, for any $R > 2T$, we appeal to (2.2) in Lemma 2.1 to get in (5.7),

$$
\int_{B_{2T}} (|\nabla u|^2 + T^{-2} |u|^2) \leq \sup_{x \in \mathbb{R}^d} \int_{B_{\tau T}(x)} (|\nabla u|^2 + T^{-2} |u|^2) \leq C \sup_{x \in \mathbb{R}^d} \int_{B_{\tau T}(x)} |A(t + y) - A(t + z)|^2 \, dt + C \sup_{x \in \mathbb{R}^d} \int_{B_{\tau T}(x)} |A(t + y) - A(t + z)|^2 |\nabla v|^2 \, dt \leq C \|A(\cdot + y) - A(\cdot + z)\|_{L^\infty(B_R)}^2,
$$

where we have used the facts that $R > 2T$ and

$$
\sup_{x \in \mathbb{R}^d} \int_{B_{\tau T}(x)} |\nabla u|^2 \, dt \leq C \text{ (see (2.2) in Lemma 2.1).}
$$

It follows at once that

$$
T \left( \int_{B_{2T}} |\nabla u|^2 \right)^{\frac{1}{2}} + \left( \int_{B_{2T}} |u|^2 \right)^{\frac{1}{2}} \leq CT \|A(\cdot + y) - A(\cdot + z)\|_{L^\infty(B_R)}, \quad (5.9)
$$
Concerning the second term in the right-hand side of (5.8), we have
\[
\int_{B_{2T}} |\nabla_t G^y(x, t)| \, dt \leq C \int_{B_{3T}(x)} |\nabla_t G^y(x, t)| \, dt \tag{5.10}
\]
\[
\leq C \sum_{i=-\infty}^{1} \int_{B_{2i+1T}(x) \setminus B_{2iT}(x)} |\nabla_t G^y(x, t)| \, dt \leq C \sum_{i=-\infty}^{1} 2^iT \leq CT,
\]
where we have used for the first inequality in (5.10), the fact that $B_{2T} \subset B_{3T}(x)$ (recall that $x \in B_T$), and for the last inequality, \[2.28\] (for $q = 1$). It follows that
\[
C \|A(\cdot + y) - A(\cdot + z)\|_{L^\infty(B_R)} \int_{B_{2T}} |\nabla_t G^y(x, t)| \, dt \leq CT \|A(\cdot + y) - A(\cdot + z)\|_{L^\infty(B_R)}.
\]
As for the third term in the right-hand side of (5.8) is concerned, we concentrate on the control of the integral
\[
I = \int_{B_{2T}} |\nabla_t G^y(x, t)| |\nabla v(t)| \, dt.
\]
First, we note that the function $v$ solves the equation
\[
-\nabla \cdot (A(\cdot + z) \nabla v) + T^{-2}v = \nabla \cdot (A(\cdot + z)e_j) \quad \text{in } \mathbb{R}^d
\]
so that appealing to (2.2),
\[
\left( \int_{B_{2T}} |\nabla v|^2 \right)^{\frac{1}{2}} \leq C. \tag{5.11}
\]
Next, Hölder inequality and (5.11) lead to
\[
I \leq CT^{\frac{d}{2}} \left( \int_{B_{2T}} \left| \nabla_t G^y(x, t) \right|^2 \, dt \right)^{\frac{1}{2}} \leq CT^{\frac{d}{2}} \left( \int_{B_{3T}(x) \setminus B_{T}(x)} \left| \nabla_t G^y(x, t) \right|^2 \, dt \right)^{\frac{1}{2}} \tag{5.12}
\]
\[
\leq CT^{\frac{d}{2}} \left( \sum_{i=\frac{\ln T}{\ln 2}}^{2} \int_{B_{2i+1T}(x) \setminus B_{2iT}(x)} \left| \nabla_t G^y(x, t) \right|^2 \, dt \right)^{\frac{1}{2}} \leq CT^{\frac{d}{2}} \left( \sum_{i=\frac{\ln T}{\ln 2}}^{2} (2^iT)^{2-d} \right)^{\frac{1}{2}} \leq CT^{\frac{d}{2}}T^{1-\frac{d}{2}} = CT.
\]
For the fourth term in the right-hand side of (5.8), we have
\[
\int_{B_{2T}} |G^y(x, t)| |\nabla \varphi(t)| \, dt \leq CT^{-1} \int_{B_{3T}(x)} \frac{dt}{|x-t|^{d-2}} \leq CT.
\]
We have therefore shown that
\[
|u(x)| \leq CT^{-2} \int_{B_{2T}} \frac{|u(t)|}{|x-t|^{d-2}} \, dt + CT \|A(\cdot + y) - A(\cdot + z)\|_{L^\infty(B_R)} \tag{5.12}
\]
Using the well known fractional integral estimates, (5.12) yields
\[
\left( \int_{B_T} \left| u \right|^p \right)^{\frac{1}{p}} \leq C \left( \int_{B_{2T}} \left| u \right|^p \right)^{\frac{1}{p}} + CT \|A(\cdot + y) - A(\cdot + z)\|_{L^\infty(B_R)}
\]
where \(1 < p < q \leq \infty\) with \(\frac{1}{p} - \frac{1}{q} < \frac{2}{d}\). However from (5.9) we derive the estimate
\[
\left( \int_{B_T} |u|^2 \right)^{\frac{1}{2}} \leq CT \|A(\cdot + y) - A(\cdot + z)\|_{L^\infty(B_R)},
\]
so that by an iteration argument, we are led to
\[
\|u\|_{L^\infty(B_T)} \leq CT \|A(\cdot + y) - A(\cdot + z)\|_{L^\infty(B_R)}.
\]
This yields (recalling that \(x_0 = 0\))
\[
|u(0)| \leq CT \|A(\cdot + y) - A(\cdot + z)\|_{L^\infty(B_R)}.
\]
Recalling that 0 may be replaced by any \(t \in B_R\), this completes the proof.

**Theorem 5.2.** Let \(T \geq 1\) and \(R > 2T\). For any \(0 < L \leq T\) and \(\sigma \in (0, 1)\), there is \(C_\sigma = C_\sigma(\sigma, A)\) such that
\[
T^{-1} \|\chi_T\|_{L^\infty(\mathbb{R}^d)} \leq C_\sigma \left( \rho(L, R) + \left( \frac{L}{T} \right)^\sigma \right). \tag{5.13}
\]

**Proof.** Let \(y, z \in \mathbb{R}^d\) with \(|z| \leq L \leq T\). Then
\[
|\chi_T(y)| \leq |\chi_T(y) - \chi_T(0)| + |\chi_T(0)|
\]
and
\[
|\chi_T(y) - \chi_T(0)| \leq |\chi_T(y) - \chi_T(z)| + |\chi_T(z) - \chi_T(0)|
\]
\[
= |\chi_T(0 + y) - \chi_T(0 + z)| + |\chi_T(z) - \chi_T(0)|
\]
\[
\leq \sup_{x \in B_R} |\chi_T(x + y) - \chi_T(x + z)| + |\chi_T(z) - \chi_T(0)|
\]
\[
\leq CT \|A(\cdot + y) - A(\cdot + z)\|_{L^\infty(B_R)} + C_\sigma T^{-1-\sigma} L^{\sigma},
\]
where for the last inequality above we have used (2.20) (in Lemma 2.2) and (5.6) (in Theorem 5.1). It follows readily that
\[
\sup_{y \in \mathbb{R}^d} |\chi_T(y) - \chi_T(0)| \leq T \left( C\rho(L, R) + C_\sigma \left( \frac{L}{T} \right)^\sigma \right). \tag{5.14}
\]
On the other hand, observing that
\[
|\chi_T(0)| \leq \left| \int_{B_r} (\chi_T(t) - \chi_T(0)) dt \right| + \left| \int_{B_r} \chi_T(t) dt \right|
\]
\[
\leq \sup_{y \in \mathbb{R}^d} |\chi_T(y) - \chi_T(0)| + \left| \int_{B_r} \chi_T(t) dt \right|
\]
and letting \(r \to \infty\), we use the fact that \(\langle \chi_T \rangle = 0\) to get
\[
|\chi_T(0)| \leq \sup_{y \in \mathbb{R}^d} |\chi_T(y) - \chi_T(0)|.
\]
The above inequality associated to (5.14) yield (5.13).
Now, we set (for $T \geq 1$ and $\sigma \in (0, 1]$)

$$
\Theta_\sigma(T) = \inf_{0 < L < T} \left( \rho(L, 3T) + \left( \frac{L}{T} \right)^\sigma \right)
$$

(5.15)

where $\rho(L, R)$ is given by (5.4). Then $T \mapsto \Theta_\sigma(T)$ is a continuous decreasing function satisfying $\Theta_\sigma(T) \to 0$ when $T \to \infty$ (this stems from the asymptotic almost periodicity of $A$, so that $\rho(L, 3T) \to 0$ as $T \to \infty$). We infer from (5.13) that

$$
T^{-1} \| \chi_T \|_{L^\infty(\mathbb{R}^d)} \leq C_\sigma \Theta_\sigma(T)
$$

(5.16)

and hence

$$
T^{-1} \| \chi_T \|_{L^\infty(\mathbb{R}^d)} \to 0 \text{ as } T \to \infty.
$$

As in [31] we state the following result.

**Lemma 5.1.** Let $g \in L^2_{\infty, AP}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ with $\langle g \rangle = 0$ and

$$
\sup_{x \in \mathbb{R}^d} \left( \int_{B_r(x)} |g|^2 \right)^{\frac{1}{2}} \leq C_0 \left( \frac{T}{r} \right)^{1-\sigma} \text{ for } 0 < r \leq T
$$

(5.17)

where $\sigma \in (0, 1]$. Then there is a unique $u \in H^1_{\infty, AP}(\mathbb{R}^d)$ such that

$$
- \Delta u + T^{-2}u = g \text{ in } \mathbb{R}^d, \quad \langle u \rangle = 0
$$

(5.18)

and

$$
T^{-2} \| u \|_{L^\infty(\mathbb{R}^d)} + T^{-1} \| \nabla u \|_{L^\infty(\mathbb{R}^d)} \leq C,
$$

(5.19)

where $C = C(d)$ and $C_\sigma = C_\sigma(d, \sigma)$. Moreover $u$ and $\nabla u$ belong to $B^\infty_{\infty, AP}(\mathbb{R}^d)$ with

$$
T^{-2} \| u \|_{L^\infty(\mathbb{R}^d)} \leq \Theta_1(T)
$$

(5.20)

and

$$
T^{-1} \| \nabla u \|_{L^\infty(\mathbb{R}^d)} \leq C \Theta_\sigma(T)
$$

(5.21)

where $\Theta_\sigma(T)$ is defined by (5.15) and $C = C(d, \sigma, g)$.

**Proof.** If we proceed as in the proof of Lemma 2.1, we derive the existence of a unique $u \in H^1_{\infty, AP}(\mathbb{R}^d)$ solving (5.18); we may also refer to [28] for another proof. Next using the fundamental solution of $-\Delta + T^{-2}$, we easily get (5.19). We infer from (5.19) that $u, \nabla u \in B^\infty_{\infty, AP}(\mathbb{R}^d)$. In order to obtain (5.20) we use (5.17) and proceed as in [31, Lemma 7.1]. It remains to check (5.21) and (5.22). To that end, we apply (5.19) to the function

$$
\frac{u(\cdot + z) - u(\cdot + z)}{\| A(\cdot + y) - A(\cdot + z) \|_{L^\infty(B_R)}}
$$

with $u$ solution of (5.18). Then

$$
T^{-2} \| u(\cdot + y) - u(\cdot + z) \|_{L^\infty(B_R)} \leq C \| A(\cdot + y) - A(\cdot + z) \|_{L^\infty(B_R)}
$$

(5.23)
and
\[ T^{-1} \| \nabla u(\cdot + y) - \nabla u(\cdot + z) \|_{L^\infty(B_R)} \leq C \| A(\cdot + y) - A(\cdot + z) \|_{L^\infty(B_R)}. \]  \hfill (5.24)

Using the boundedness of the gradient (see (5.19)), we obtain
\[ |u(x) - u(t)| \leq CT |x - t| \quad \forall x, t \in \mathbb{R}^d. \]  \hfill (5.25)

Next assuming that \(|z| \leq L \leq T\), we have
\[ T^{-2} |u(y) - u(0)| \leq T^{-2} |u(y) - u(z)| + T^{-2} |u(z) - u(0)| \]
\[ \leq C \| A(\cdot + y) - A(\cdot + z) \|_{L^\infty(B_R)} + CT^{-1}L \]
where we used (5.23) and (5.25). Hence
\[ \sup_{y \in \mathbb{R}^d} T^{-2} |u(y) - u(0)| \leq C(\rho(L, R) + T^{-1}L) \]  \hfill (5.26)

for any \( R > 2T \) and \( L > 0 \). Also, using the inequality
\[ T^{-2} |u(0)| \leq T^{-2} \left( \int_{B_r} (u(t) - u(0))dt \right) + T^{-2} \left( \int_{B_r} u(t)dt \right) \]
\[ \leq T^{-2} \sup_{y \in \mathbb{R}^d} |u(y) - u(0)| + T^{-2} \left( \int_{B_r} u(t)dt \right) \]
together with the fact that \( \langle u \rangle = 0 \), we get (after letting \( r \to \infty \))
\[ T^{-2} |u(0)| \leq C(\rho(L, R) + T^{-1}L) \quad \forall 0 < L \leq T \]  \hfill (5.27)
where we have also used (5.26). Putting together (5.26) and (5.27), and choosing in the resulting inequality \( R = 3T \), and finally taking the inf \( 0 < L < T \), we are led to (5.21).

Proceeding as above using this time (5.20) and (5.24) we arrive at (5.22).

**Lemma 5.2.** Let \( \chi_{T,j} \) be defined by (5.5), and let \( \Omega \) be an open bounded set of class \( C^{1,1} \) in \( \mathbb{R}^d \). Then
\[ \int_\Omega \left| \left( \nabla y \chi_{T,j} \right) \left( \frac{x}{\varepsilon} \right) w(x) \right|^2 dx \leq C \int_\Omega (|w|^2 + \delta^2 |\nabla w|^2) dx, \quad \text{all } w \in H^1(\Omega) \]  \hfill (5.28)

where \( \delta = T^{-1} \| \chi_T \|_{L^\infty(\mathbb{R}^d)} \) with \( T = \varepsilon^{-1} \), and \( C = C(A, \Omega, d) > 0 \).

**Proof.** By a density argument, it is sufficient to prove (5.28) for \( w \in C_0^\infty(\Omega) \). We recall that \( \chi_{T,j} \) solves the equation
\[ -\nabla \cdot (A(e_j + \nabla \chi_{T,j})) + T^{-2} \chi_{T,j} = 0 \quad \text{in } \mathbb{R}^d. \]  \hfill (5.29)

Testing (5.29) with \( \psi(y) = \varphi(\varepsilon y) \) where \( \varphi \in H^1_{loc}(\mathbb{R}^d) \) with compact support, and next making the change of variable \( x = \varepsilon y \), we get
\[ \int_{\mathbb{R}^d} \left[ (A^\varepsilon(e_j + (\nabla y \chi_{T,j})^\varepsilon) \cdot \nabla \varphi + T^{-2} \chi_{T,j}^\varphi \right] dx = 0 \]
where \( u^\varepsilon(x) = u(x/\varepsilon) \) for \( u \in H^1_{\text{loc}}(\mathbb{R}^d) \). Choosing \( \varphi(x) = \chi_{T,j}(x/\varepsilon) |w(x)|^2 \) with \( w \in C_0^\infty(\Omega) \), we obtain
\[
\int_\Omega \left[ (A^\varepsilon(e_j + (\nabla_y \chi_{T,j})^\varepsilon) \cdot \left( \frac{1}{\varepsilon} |(\nabla_y \chi_{T,j})^\varepsilon| w|^2 + 2w \chi_{T,j}^\varepsilon \nabla w \right) + T^{-2} |\chi_{T,j}^\varepsilon|^2 |w|^2 \right] dx = 0,
\]
or
\[
\int_\Omega A^\varepsilon(\nabla_y \chi_{T,j})^\varepsilon \cdot (\nabla_y \chi_{T,j})^\varepsilon w dx = -2\varepsilon \int_\Omega A^\varepsilon(\nabla_y \chi_{T,j})^\varepsilon w \cdot \chi_{T,j}^\varepsilon \nabla w dx - \int_\Omega w(A^\varepsilon e_j) \cdot (\nabla_y \chi_{T,j})^\varepsilon w dx - 2\varepsilon \int_\Omega w(A^\varepsilon e_j) \cdot \chi_{T,j}^\varepsilon \nabla w dx - \varepsilon T^{-2} \int_\Omega |\chi_{T,j}^\varepsilon|^2 |w|^2 dx = I_1 + I_2 + I_3 + I_4.
\]
The left hand-side of (5.30) is estimated from below by \( \alpha \int_\Omega \left| (\nabla_y \chi_{T,j})^\varepsilon w \right|^2 dx \) while, for the respective terms of the right hand-side of (5.30) we have, after the use of Hölder and Young inequalities,
\[
|I_1| \leq \frac{\alpha}{3} \int_\Omega \left| (\nabla_y \chi_{T,j})^\varepsilon w \right|^2 dx + C\varepsilon^2 \int_\Omega |\chi_{T,j}^\varepsilon|^2 |\nabla w|^2 dx;
\]
\[
|I_2| \leq \frac{\alpha}{3} \int_\Omega \left| (\nabla_y \chi_{T,j})^\varepsilon w \right|^2 dx + C \int_\Omega |w|^2 dx;
\]
\[
|I_3| \leq C \int_\Omega |w|^2 dx + C\varepsilon^2 \int_\Omega |\chi_{T,j}^\varepsilon|^2 |\nabla w|^2 dx \quad \text{and} \quad |I_4| \leq C \int_\Omega |w|^2 dx.
\]
It follows that
\[
\int_\Omega \left| (\nabla_y \chi_{T,j})^\varepsilon w \right|^2 dx \leq C\varepsilon^2 \int_\Omega |\chi_{T,j}^\varepsilon|^2 |\nabla w|^2 dx + C \int_\Omega |w|^2 dx \leq C \varepsilon^2 \left\| \chi_{T,j} \right\|_{L^\infty(\mathbb{R}^d)}^2 \int_\Omega |\nabla w|^2 dx + C \int_\Omega |w|^2 dx.
\]
Since \( T = \varepsilon^{-1} \), we get (5.28), taking into account that \( T^{-1} \left\| \chi_{T,j} \right\|_{L^\infty(\mathbb{R}^d)} \leq T^{-1} \left\| \chi_T \right\|_{L^\infty(\mathbb{R}^d)} \).

**Remark 5.3.** In the case of asymptotic periodic functions, we replace \( \chi_{T,j} \) by \( \chi_j \in H^1_{\text{loc,per}}(Y) \) solution of the corrector problem (1.10) and we have (in view of Lemma 4.1) \( \left\| \chi_j \right\|_{L^\infty(\mathbb{R}^d)} \leq C \).

It follows that
\[
\int_\Omega \left| (\nabla_y \chi_j) \left( \frac{x}{\varepsilon} \right) w(x) \right|^2 dx \leq C \int_\Omega \left( |w|^2 + \varepsilon^2 |\nabla w|^2 \right) dx, \quad \text{for all} \ w \in H^1(\Omega)
\]
where \( C = C(A, \Omega, d) \).
Let $u_0 \in H^1_0(\Omega)$ be the weak solution of (1.8). Let $z_\varepsilon \in H^1(\Omega)$ be the unique weak solution of
\[- \nabla \cdot (A^\varepsilon \nabla z_\varepsilon) = 0 \text{ in } \Omega, \quad z_\varepsilon = \varepsilon \chi_T^\varepsilon \nabla u_0 \text{ on } \partial \Omega\] (5.31)
where $\Omega$ is as in Lemma 5.2. Then we have

**Lemma 5.3.** Let $z_\varepsilon$ be as in (5.31) with $T = \varepsilon^{-1}$. Then there exists $\varepsilon_0 \in (0, 1)$ such that
\[
\|z_\varepsilon\|_{H^1(\Omega)} \leq C \left( T^{-1} \|\chi_T\|_{L^\infty(\mathbb{R}^d)} \right)^{\frac{1}{2}} \|u_0\|_{H^2(\Omega)}, \quad 0 < \varepsilon \leq \varepsilon_0,
\] (5.32)
where $C = C(A, \Omega) > 0$.

It follows from (5.32) that for any $\sigma \in (0, 1)$, there exists $C_\sigma = C_\sigma(\sigma, A, \Omega) > 0$ such that
\[
\|z_\varepsilon\|_{H^1(\Omega)} \leq C_\sigma(\Theta_\varepsilon) \left( \|\chi_T\|_{L^\infty(\mathbb{R}^d)} \right)^{\frac{1}{2}} \|u_0\|_{H^2(\Omega)}, \quad 0 < \varepsilon \leq \varepsilon_0
\] (5.33)
where $\Theta_\varepsilon$ is defined by (5.15).

For the proof of Lemma 5.3, we need the following result whose proof can be found in [26].

**Lemma 5.4** ([26, Lemma 5.1]). Let $\Omega$ be as in Lemma 5.2. Then there exists $\delta_0 \in (0, 1]$ depending on $\Omega$ such that, for any $u \in H^1(\Omega)$,
\[
\int_{\Gamma_\delta} |u|^2 \, dx \leq C \delta \|u\|_{L^2(\Omega)} \|u\|_{H^1(\Omega)}, \quad 0 < \delta \leq \delta_0
\] (5.34)
where $C = C(\Omega)$ and $\Gamma_\delta = \Omega_\delta \cap \Omega$ with $\Omega_\delta = \{ x \in \mathbb{R}^d : \text{dist}(x, \partial \Omega) < \delta \}$.

**Proof of Lemma 5.3.** We set $w = \nabla u_0$ and $u = z_\varepsilon$. Assuming $u_0 \in H^2(\Omega)$, we have that $w \in H^1(\Omega)^d$. Since $\delta := T^{-1} \|\chi_T\|_{L^\infty(\mathbb{R}^d)} \to 0$ as $T \to \infty$, we may assume that $0 < \delta \leq \delta_0$ where $\delta_0$ is as in Lemma 5.4. Let $\theta_\delta$ be a cut-off function in a neighborhood of $\partial \Omega$ with support in $\Omega_{2\delta}$ (a $2\delta$-neighborhood of $\partial \Omega$), $\Omega_\rho$ being defined as in Lemma 5.4.

\[
\theta_\delta \in C^\infty_0(\mathbb{R}^d), \quad \text{supp}\theta_\delta \subset \Omega_{2\delta}, \quad 0 \leq \theta_\delta \leq 1, \quad \theta_\delta = 1 \text{ on } \Omega_\delta, \quad \theta_\delta = 0 \text{ on } \mathbb{R}^d \setminus \Omega_{2\delta} \text{ and } \delta \|\nabla \theta_\delta\| \leq C.
\] (5.35)

We set $\Phi_\varepsilon(x) = \varepsilon \theta_\delta(x) \chi_T(x/\varepsilon)w(x)$. Then
\[
\|u\|_{H^1(\Omega)} \leq C\varepsilon \|\chi_T w\|_{H^{1/2}(\partial \Omega)} \leq C \|\Phi_\varepsilon\|_{H^1(\Omega)}.
\]
So we need to estimate $\|\nabla \Phi_\varepsilon\|_{L^2(\Omega)}$. But
\[
\nabla \Phi_\varepsilon = \varepsilon \chi_T w \nabla \theta_\delta + (\nabla y \chi_T)^\varepsilon w \theta_\delta + \varepsilon \chi_T^\varepsilon \theta_\delta \nabla w = J_1 + J_2 + J_3.
\]
We have
\[
\|J_1\|_{L^2(\Omega)}^2 \leq C\varepsilon^2 \|\chi_T\|_{L^\infty(\mathbb{R}^d)}^2 \delta^{-2} \int_{\Gamma_{2\delta}} |w|^2 \, dx \\
\leq C \int_{\Gamma_{2\delta}} |w|^2 \, dx \leq C\delta \|w\|_{L^2(\Omega)} \|w\|_{H^1(\Omega)}
\]
where we have used (5.34) for the last inequality above. For $J_2$, we have (using (5.28) and (5.34))

$$
\|J_2\|^2_{L^2(\Omega)} \leq \int_{\Omega} |(\nabla y\chi_T)^\varepsilon w_{\delta}|^2 \, dx \leq C \int_{\Omega} \left( |w_{\delta}|^2 + \delta^2 |\nabla (w_{\delta})|^2 \right) \, dx
$$

$$
\leq C \int_{\Gamma_{2\delta}} |w|^2 \, dx + C\delta^2 \int_{\Omega} |\nabla (w_{\delta})|^2 \, dx
$$

$$
\leq C\delta \|w\|_{L^2(\Omega)} \|w\|_{H^1(\Omega)} + C\delta^2 \int_{\Omega} |\nabla w|^2 \, dx.
$$

But $\nabla (w_{\delta}) = w\nabla \phi + \phi \nabla w$, and

$$
\int_{\Omega} |\nabla (w_{\delta})|^2 \, dx \leq C \int_{\Gamma_{2\delta}} |\nabla \phi|^2 |w|^2 \, dx + C \int_{\Omega} |\phi \nabla w|^2 \, dx
$$

$$
\leq C\delta^{-1} \|w\|_{L^2(\Omega)} \|w\|_{H^1(\Omega)} + C \int_{\Omega} |\nabla w|^2 \, dx.
$$

Hence

$$
\|J_2\|^2_{L^2(\Omega)} \leq C\delta \|w\|_{L^2(\Omega)} \|w\|_{H^1(\Omega)} + C\delta^2 \|w\|_{H^1(\Omega)}^2.
$$

As for $J_3$,

$$
\|J_3\|^2_{L^2(\Omega)} \leq C\varepsilon^2 \int_{\Omega} |\chi_T^\varepsilon|^2 |\nabla w|^2 \, dx \leq C\delta^2 \|w\|_{H^1(\Omega)}^2.
$$

Finally, using Young’s inequality together with the fact that $\delta^2 \leq \delta$ we are led to

$$
\|\nabla \Phi^\varepsilon\|^2_{L^2(\Omega)} \leq C\delta \|w\|_{L^2(\Omega)} \|w\|_{H^1(\Omega)} + C\delta^2 \|w\|_{H^1(\Omega)}^2 \tag{5.36}
$$

$$
\leq C\delta \|w\|_{H^1(\Omega)}^2 + C\delta^2 \|w\|_{H^1(\Omega)}^2
$$

$$
\leq C\delta \|w\|_{H^1(\Omega)}^2.
$$

So we choose $\varepsilon_0$ such that $0 < \delta \leq \delta_0$ for $0 < \varepsilon \leq \varepsilon_0$ (recall that $0 \leq \delta \to 0$ as $0 < \varepsilon \to 0$). We thus derive (5.32) since $\|\Phi^\varepsilon\|^2_{L^2(\Omega)} \leq \delta \|w\|_{H^1(\Omega)}^2$. 

5.3. Convergence rates: proof of Theorem 1.3. Assume that $\Omega$ is of class $C^{1,1}$. Let $u_\varepsilon$, $u_0 \in H^1_0(\Omega)$ be the weak solutions of (1.5) and (1.8) respectively. Let $\chi_T^\varepsilon(x) = \chi_T(x/\varepsilon)$ for $x \in \Omega$ and define

$$
w_\varepsilon = u_\varepsilon - u_0 - \varepsilon \chi_T^\varepsilon \nabla u_0 + z_\varepsilon \tag{5.37}
$$

where $T = \varepsilon^{-1}$ and $z_\varepsilon \in H^1(\Omega)$ is the weak solution of (5.31).

Theorem 5.3. Suppose that $A$ is as in the preceding subsection. Assume that $u_0 \in H^2(\Omega)$. Then for any $\sigma \in (0, 1)$ there exists $C_\sigma = C_\sigma(\sigma, A, \Omega)$ such that

$$
\|w_\varepsilon\|_{H^1(\Omega)} \leq C_\sigma \left( \|\nabla \chi_T - \nabla \chi_{\varepsilon^{-1}}\|_2 + \Theta_\sigma(\varepsilon^{-1}) \right) \|u_0\|_{H^2(\Omega)}. \tag{5.38}
$$

Proof. Set

$$
A_T = A + A \nabla y \chi_T - A^*.
$$

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where $A^*$ is the homogenized matrix and where we have taken $T = \varepsilon^{-1}$. Then by simple computations as in Lemma 4.3 we get

$$- \nabla \cdot (A^\varepsilon \nabla w_\varepsilon) = \nabla \cdot (A_T^\varepsilon \nabla u_0) + \varepsilon \nabla \cdot (A^\varepsilon \nabla^2 u_0 \chi_T^\varepsilon).$$

This implies that

$$\|\nabla w_\varepsilon\|_{L^2(\Omega)} \leq C \|A_T^\varepsilon \nabla u_0\|_{L^2(\Omega)} + C\varepsilon \|A^\varepsilon \nabla^2 u_0 \chi_T^\varepsilon\|_{L^2(\Omega)}.$$

We use (5.16) to get

$$\varepsilon \|A^\varepsilon \nabla^2 u_0 \chi_T^\varepsilon\|_{L^2(\Omega)} \leq C\varepsilon \|\chi_T\|_{L^\infty(\mathbb{R}^d)} \|\nabla^2 u_0\|_{L^2(\Omega)} \leq C\Theta_\sigma(T) \|\nabla^2 u_0\|_{L^2(\Omega)}.$$ (5.40)

Concerning the term $\|A_T^\varepsilon \nabla u_0\|_{L^2(\Omega)}$, we need to replace $A_T$ by a matrix $A_T$ whose mean value is zero. So, we let $A_T = A_T - \langle A_T \rangle$ so that $\langle A_T \rangle = 0$ and $A_T^\varepsilon \nabla u_0 = A_T^\varepsilon \nabla u_0 + \langle A_T \rangle \nabla u_0$. The inequality $|\langle A_T \rangle| \leq C \|\nabla \chi - \nabla \chi_T\|_2$ yields readily

$$\|\langle A_T \rangle \nabla u_0\|_{L^2(\Omega)} \leq C \|\nabla \chi - \nabla \chi_T\|_2 \|\nabla u_0\|_{L^2(\Omega)}.$$ (5.41)

It remains to estimate $\|A_T^\varepsilon \nabla u_0\|_{L^2(\Omega)}$. We denote by $a_{T,ij}$ the entries of $A_T$: $a_{T,ij} = b_{T,ij} - \langle b_{T,ij} \rangle \equiv a_{ij}$ where

$$b_{T,ij}(y) = b_{ij}(y) + \sum_{k=1}^d b_{ik}(y) \frac{\partial \chi_T}{\partial y_k}(y) - b_{ij}^*.$$ 

In view of Lemma 5.1, let $f_{T,ij} \equiv f_{ij} \in H^{1,AP}(\mathbb{R}^d)$ be the unique solution of

$$-\Delta f_{ij} + T^{-2} f_{ij} = a_{ij} \text{ in } \mathbb{R}^d, \quad \langle f_{ij} \rangle = 0.$$

Owing to (2.19), we see that $a_{ij}$ verifies (5.17), so that (5.21) and (5.22) are satisfied, that is:

$$T^{-2} \|f_{ij}\|_{L^\infty(\mathbb{R}^d)} \leq C\Theta_1(T) \quad \text{and } T^{-1} \|\nabla f_{ij}\|_{L^\infty(\mathbb{R}^d)} \leq C\Theta_\sigma(T).$$ (5.42)

We set $f = (f_{ij})_{1 \leq i, j \leq d}$. Then writing (formally)

$$a_{ij} = - \sum_{k=1}^d \left( \frac{\partial}{\partial y_k} \left( \frac{\partial f_{ij}}{\partial y_k} - \frac{\partial f_{kj}}{\partial y_i} \right) + \frac{\partial}{\partial y_i} \left( \frac{\partial f_{kj}}{\partial y_k} \right) \right) + T^{-2} f_{ij}$$

and using the fact that

$$\sum_{i,k=1}^d \frac{\partial^2}{\partial y_i \partial y_k} \left( \frac{\partial f_{ij}}{\partial y_k} - \frac{\partial f_{kj}}{\partial y_i} \right) = 0,$$
we readily get
\[- \nabla \cdot (A^T \nabla u_0) = \nabla \cdot ((\Delta f)^\varepsilon \nabla u_0) - T^{-2} \nabla \cdot (f^\varepsilon \nabla u_0) \tag{5.43}\]
\[
= \sum_{i,j,k=1}^d \frac{\partial}{\partial x_i} \left( \frac{\partial f_{ij}}{\partial x_k} \left( \frac{x_j}{\varepsilon} \right) \frac{\partial u_0}{\partial x_j} \right) \nabla x_k - \frac{\partial f_{kj}}{\partial x_i} \left( \frac{x_j}{\varepsilon} \right) \frac{\partial u_0}{\partial x_j} - T^{-2} \nabla \cdot (f^\varepsilon \nabla u_0)
\]
\[
+ \sum_{i,j,k=1}^d \frac{\partial}{\partial x_i} \left( \frac{\partial^2 f_{kj}}{\partial x_k \partial x_j} \left( \frac{x_j}{\varepsilon} \right) \frac{\partial u_0}{\partial x_j} \right) - T^{-2} \nabla \cdot (f^\varepsilon \nabla u_0)
\]
\[
= - \sum_{i,j,k=1}^d \frac{\partial}{\partial x_i} \left( \varepsilon \left( \frac{\partial f_{ij}}{\partial x_k} \frac{\partial u_0}{\partial x_j} \right) \left( \frac{x_j}{\varepsilon} \right) \frac{\partial^2 u_0}{\partial x_k \partial x_j} \right)
\]
\[
+ \sum_{i,j,k=1}^d \frac{\partial}{\partial x_i} \left( \frac{\partial^2 f_{kj}}{\partial x_k \partial x_j} \left( \frac{x_j}{\varepsilon} \right) \frac{\partial u_0}{\partial x_j} \right) - T^{-2} \nabla \cdot (f^\varepsilon \nabla u_0).
\]

Testing (5.43) with \( \varphi \in H_0^1(\Omega) \), we obtain
\[
\| A^T \nabla u_0 \|_{L^2(\Omega)} \leq C \varepsilon \left( \int_\Omega \left| \nabla f \left( \frac{x}{\varepsilon} \right) \right|^2 \left| \nabla u_0 \right|^2 dx \right)^{\frac{1}{2}} \tag{5.44}
\]
\[
+ C \sum_{j=1}^d \left( \int_\Omega \left| \nabla h_{T,j} \left( \frac{x}{\varepsilon} \right) \right|^2 \left| \nabla u_0 \right|^2 dx \right)^{\frac{1}{2}} + |\langle A_T \rangle | \left\| \nabla u_0 \right\|_{L^2(\Omega)}
\]
\[
+ C T^{-2} \left( \int_\Omega |f^\varepsilon|^2 \left| \nabla u_0 \right|^2 dx \right)^{\frac{1}{2}}
\]
\[
= I_1 + I_2 + I_3 + I_4
\]

where \( h_{T,j} = \sum_{k=1}^d \frac{\partial f_{kj}}{\partial x_k} \in L^2_{\infty,AP}(\mathbb{R}^d) \). We estimate each term above separately. Let us first deal with \( I_2 \). Observe that \( h_{T,j} = \text{div} f_j \) where \( f_j = (f_{kj})_{1 \leq k \leq d} \). It follows from the definition of \( f_{ij} \) that
\[- \Delta f_j + T^{-2} f_j = A(e_j + \nabla \chi_{T,j}) - \langle A(e_j + \nabla \chi_{T,j}) \rangle, \]
so that, owing to the definition of \( \chi_{T,j} \),
\[- \Delta h_{T,j} + T^{-2} h_{T,j} = T^{-2} \chi_{T,j}. \tag{5.45}\]

Next, since the function \( g = T^{-1} \chi_{T,j} \) satisfies assumption [5.17] of Lemma 5.1 with \( \sigma = 1 \), it follows that \( h_{T,j} \) satisfies estimate [5.22], that is,
\[
T^{-1} \| \nabla h_{T,j} \|_{L^\infty(\mathbb{R}^d)} \leq C \Theta_{\tau}(T) \quad \forall \tau \in (0,1).
\]

Therefore
\[
|I_2| \leq C \varepsilon \| \nabla h_{T,j} \|_{L^\infty(\mathbb{R}^d)} \| \nabla u_0 \|_{L^2(\Omega)} \leq C \Theta_{\sigma}(T) \| \nabla u_0 \|_{L^2(\Omega)}.
\]

As regard \( I_1 \), we infer from (5.42) that
\[
|I_1| \leq C \varepsilon \| \nabla f \|_{L^\infty(\mathbb{R}^d)} \| \nabla^2 u_0 \|_{L^2(\Omega)} \leq C \Theta_{\sigma}(T) \| \nabla^2 u_0 \|_{L^2(\Omega)}.
\]
Concerning $I_4$, we use the first inequality in (5.42) to get
\[ |I_4| \leq C \Theta_1(T) \| \nabla u_0 \|_{L^2(\Omega)} \]
where we have put $T = \varepsilon^{-1}$. Finally, using the inequality $|\langle A_T \rangle| \leq C \| \nabla \chi - \nabla \chi_T \|_2$ we get
\[ |I_3| \leq C \| \nabla \chi - \nabla \chi_T \|_2 \| \nabla u_0 \|_{L^2(\Omega)}. \]
The result follows thereby. $lacksquare$

We are now in a position to prove Theorem 1.3.

**Proof of Theorem 1.3.** Using (5.38) together with (5.33) we get, for any $\sigma \in (0, 1)$,
\[
\begin{align*}
\| u_\varepsilon - u_0 - \varepsilon \chi_{T=\varepsilon^{-1}} \nabla u_0 \|_{H^1(\Omega)} & \leq \| u_\varepsilon - u_0 - \varepsilon \chi_{T=\varepsilon^{-1}} \nabla u_0 + z_\varepsilon \|_{H^1(\Omega)} + \| z_\varepsilon \|_{H^1(\Omega)} \\
& \leq C (\| \nabla \chi - \nabla \chi_{\varepsilon^{-1}} \|_2 + \Theta_\sigma(\varepsilon^{-1})) \| u_0 \|_{H^2(\Omega)} + C_\sigma(\Theta_\sigma(\varepsilon^{-1})) \| u_0 \|_{H^2(\Omega)} \\
& \leq C (\| \nabla \chi - \nabla \chi_{\varepsilon^{-1}} \|_2 + (\Theta_1(\varepsilon^{-1}))^\sigma + (\Theta_1(\varepsilon^{-1}))^{\frac{1}{2}}) \| u_0 \|_{H^2(\Omega)} \\
& \leq C (\| \nabla \chi - \nabla \chi_{\varepsilon^{-1}} \|_2 + (\Theta_1(\varepsilon^{-1}))^\sigma \frac{1}{2} \| u_0 \|_{H^2(\Omega)}),
\end{align*}
\]
the last inequality above stemming from the fact that $\| \nabla \chi - \nabla \chi_{\varepsilon^{-1}} \|_2 + (\Theta_1(\varepsilon^{-1}))^\sigma \rightarrow 0$
when $\varepsilon \rightarrow 0$, so that we may assume
\[ \| \nabla \chi - \nabla \chi_{\varepsilon^{-1}} \|_2 + (\Theta_1(\varepsilon^{-1}))^\sigma < 1 \]
for sufficiently small $\varepsilon$.
Choosing $\sigma = \frac{1}{2}$, we obtain
\[ \| u_\varepsilon - u_0 - \varepsilon \chi_{T=\varepsilon^{-1}} \nabla u_0 \|_{H^1(\Omega)} \leq C \left( \| \nabla \chi - \nabla \chi_{\varepsilon^{-1}} \|_2 + (\Theta_1(\varepsilon^{-1}))^{\frac{1}{2}} \right)^{\frac{1}{2}} \| u_0 \|_{H^2(\Omega)}. \tag{5.46} \]
We recall that, since $\Omega$ is a $C^{1,1}$-bounded domain in $\mathbb{R}^d$ and the matrix $A^*$ has constant entries, it holds that
\[ \| u_0 \|_{H^2(\Omega)} \leq C \| f \|_{L^2(\Omega)}, \quad C = C(d, \alpha, \Omega) > 0. \tag{5.47} \]
Next, set for $\varepsilon \in (0, 1],$
\[ \eta(\varepsilon) = \left( \| \nabla \chi - \nabla \chi_{\varepsilon^{-1}} \|_2 + (\Theta_1(\varepsilon^{-1}))^{\frac{1}{2}} \right)^{\frac{1}{2}}. \]
Since $\eta(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$, we obtain from (5.46) and (5.47), the statement of (1.13) in Theorem 1.3.

It remains to check the near optimal convergence rates result (1.14). We proceed in two parts.

**Part I.** We first check that
\[ \| u_\varepsilon \|_{H^1(\Gamma_{2\delta})} \leq C \eta(\varepsilon) \| f \|_{L^2(\Omega)} \] where $\delta = (\eta(\varepsilon))^2$. \tag{5.48}
Indeed, we have $u_\varepsilon = (u_\varepsilon - u_0 - \varepsilon \chi_{T} \nabla u_0) + u_0 + \varepsilon \chi_{T} \nabla u_0$, so that
\[ \| u_\varepsilon \|_{H^1(\Gamma_{2\delta})} \leq \| u_\varepsilon - u_0 - \varepsilon \chi_{T} \nabla u_0 \|_{H^1(\Gamma_{2\delta})} + \| u_0 \|_{H^1(\Gamma_{2\delta})} + \| \varepsilon \chi_{T} \nabla u_0 \|_{H^1(\Gamma_{2\delta})}. \]
It follows from (1.13) and (5.47) that
\[ \| u_\varepsilon - u_0 - \varepsilon \chi_\varepsilon^T \nabla u_0 \|_{H^1(\Omega)} \leq C \eta(\varepsilon) \| u_0 \|_{H^2(\Omega)} \leq C \eta(\varepsilon) \| f \|_{L^2(\Omega)}. \tag{5.49} \]
Using (5.34) we obtain
\[ \| u_0 \|_{H^1(\Omega)} \leq C \delta^{\frac{1}{2}} \| u_0 \|_{H^2(\Omega)} \leq C \delta^{\frac{1}{2}} \| f \|_{L^2(\Omega)}. \tag{5.50} \]
To estimate \( \| \varepsilon \chi_\varepsilon^T \nabla u_0 \|_{H^1(\Omega)} \), we consider a cut-off function \( \theta_{2\delta} \) of the same form as in (5.35), but with \( \delta \) replaced there by \( 2\delta \). Letting \( w = \nabla u_0 \), we observe that \( \varepsilon \chi_\varepsilon^T w = \varepsilon \theta_{2\delta} \chi_\varepsilon^T w \) on \( \Omega_{2\delta} \), so that
\[ \nabla (\varepsilon \chi_\varepsilon^T w) = \varepsilon \chi_\varepsilon^T w \nabla \theta_{2\delta} + (\nabla \chi_\varepsilon^T)^T w \theta_{2\delta} + \varepsilon \chi_\varepsilon^T \theta_{2\delta} \nabla w \) on \( \Omega_{2\delta} \).
Following the same procedure as in the proof of Lemma 5.3, we get
\[ \| \varepsilon \chi_\varepsilon^T \nabla u_0 \|_{H^1(\Omega_{2\delta})} \leq C \delta^{\frac{1}{2}} \| u_0 \|_{H^2(\Omega)} \leq C \delta^{\frac{1}{2}} \| f \|_{L^2(\Omega)}. \tag{5.51} \]
Choosing \( \delta = (\eta(\varepsilon))^2 \) in (5.50) and (5.51), and taking into account (5.49), we readily get (5.48).

**Part II.** Note that (5.38) implies
\[ \| u_\varepsilon - u_0 - \varepsilon \chi_\varepsilon^T \nabla u_0 + z_\varepsilon \|_{L^2(\Omega)} \leq C (\eta(\varepsilon))^2 \| f \|_{L^2(\Omega)}. \tag{5.52} \]
Thus, using the inequality
\[ \| \varepsilon \chi_\varepsilon^T \nabla u_0 \|_{L^2(\Omega)} \leq C (\Theta_1(\varepsilon^{-1}))^{\frac{1}{2}} \| u_0 \|_{H^2(\Omega)} \leq C (\eta(\varepsilon))^2 \| f \|_{L^2(\Omega)}, \tag{5.53} \]
we see that proving (1.14) amounts to prove that
\[ \| z_\varepsilon \|_{L^2(\Omega)} \leq C (\eta(\varepsilon))^2 \| f \|_{L^2(\Omega)} \tag{5.54} \]
where \( C = C(d, A, \Omega) \). To that end, we consider the function
\[ v_\varepsilon = z_\varepsilon - \Phi_\varepsilon, \quad \text{where} \quad \Phi_\varepsilon = \varepsilon \theta_{\delta} \chi_\varepsilon^T \nabla u_0 \text{ with } \delta = (\eta(\varepsilon))^2. \tag{5.55} \]
Then \( v_\varepsilon \in H^1_0(\Omega) \) and \( -\nabla \cdot (A^\varepsilon \nabla v_\varepsilon) = F_\varepsilon \equiv -\nabla \cdot (A^\varepsilon \nabla \Phi_\varepsilon) \) in \( \Omega \). As shown in (5.36) (where we use the inequality (4.21)), we have
\[ \| \nabla \Phi_\varepsilon \|_{L^2(\Omega)} \leq C \delta^{\frac{1}{2}} \| f \|_{L^2(\Omega)} \text{ and } \| \Phi_\varepsilon \|_{L^2(\Omega)} \leq C \delta \| f \|_{L^2(\Omega)}. \tag{5.56} \]
Now, let \( F \in L^2(\Omega) \) be arbitrarily fixed, and let \( t_\varepsilon \in H^1_0(\Omega) \) be the solution of
\[ -\nabla \cdot (A^\varepsilon \nabla t_\varepsilon) = F \text{ in } \Omega. \tag{5.57} \]
Following the homogenization process of (1.5) (see the proof of Theorem 1.1 in Section 2), we deduce the existence of a function \( t_0 \in H^1_0(\Omega) \) such that \( t_\varepsilon \to t_0 \) in \( H^1_0(\Omega) \)-weak and \( t_0 \) solves uniquely the equation \( -\nabla \cdot (A^\varepsilon \nabla t_0) = F \) in \( \Omega \). It follows from (5.48) that
\[ \| \nabla t_\varepsilon \|_{L^2(\Omega)} \leq C \eta(\varepsilon) \| F \|_{L^2(\Omega)}. \tag{5.58} \]
Taking in the variational form of (5.57) test function, we obtain

\[ \int_{\Omega} F v_{\varepsilon} dx = \int_{\Omega} A^\varepsilon \nabla t_{\varepsilon} \cdot \nabla v_{\varepsilon} dx = \int_{\Omega} \nabla t_{\varepsilon} \cdot A^\varepsilon \nabla v_{\varepsilon} dx = (F_{\varepsilon}, t_{\varepsilon}) \]  
\[ = -\int_{\Omega} A^\varepsilon \nabla \Phi_{\varepsilon} \cdot \nabla t_{\varepsilon} dx = -\int_{\Gamma_{2s}} A^\varepsilon \nabla \Phi_{\varepsilon} \cdot \nabla t_{\varepsilon} dx \]

where in (5.59), the second equality stems from the fact that the matrix \( A \) is symmetric, and in the last equality we have used the definition and properties of \( \Phi_{\varepsilon} \). Hence, using together (the first inequality in) (5.56) and (5.58), we are led to

\[ \left| \int_{\Omega} F v_{\varepsilon} dx \right| \leq C \| \nabla \Phi_{\varepsilon} \|_{L^2(\Omega)} \| \nabla t_{\varepsilon} \|_{L^2(\Gamma_{2s})} \leq C \delta^{\frac{1}{2}} \| f \|_{L^2(\Omega)} \delta^{\frac{1}{2}} \| F \|_{L^2(\Omega)} \]

Since \( F \) is arbitrary, it emerges

\[ \| v_{\varepsilon} \|_{L^2(\Omega)} \leq C \delta \| f \|_{L^2(\Omega)} \] with \( \delta = (\eta(\varepsilon))^2 \). \hspace{1cm} (5.60)

Combining (5.60) with the second estimate in (5.56) yields (5.54). This concludes the proof of Theorem 1.3.

**Remark 5.4.** In the asymptotic periodic setting of the preceding section, we replace \( \chi_T \) by \( \chi \) so that \( \| \nabla \chi - \nabla \chi_{\varepsilon-1} \|_2 = 0 \). Moreover, if we look carefully at the proof of (1.14), we notice that, in view of Remark 5.3, we may replace \( \eta(\varepsilon) \) by \( \varepsilon^{1/2} \), so that (1.14) becomes

\[ \| u_{\varepsilon} - u_0 \|_{L^2(\Omega)} \leq C \varepsilon \| f \|_{L^2(\Omega)} \]

where \( C = C(d, \alpha, \Omega) \). This shows the optimal \( L^2 \)-rates of convergence in Theorem 1.4.

6. **Some examples**

6.1. **Applications of Theorem 3.2.** Theorem 3.2 has been proved under the assumption that the corrector \( \chi_j \) lies in \( B^2_A(\mathbb{R}^d) \) for each \( 1 \leq j \leq d \). We provide some examples in which this hypothesis is fulfilled.

6.1.1. **The almost periodic setting.** We assume here that the entries of the matrix \( A \) are almost periodic in the sense of Besicovitch [9]. Then this falls into the scope of Theorem 1.1 by taking there \( \mathcal{A} = AP(\mathbb{R}^d) \).

Now, we distinguish two special cases.

**Case 1.** The entries of \( A \) are continuous quasi-periodic functions and satisfy the frequency condition (see [27]). We recall that a function \( b \) defined on \( \mathbb{R}^d \) is quasi-periodic if \( b(y) = B(\omega_1 \cdot y, ..., \omega_m \cdot y) \) where \( B \equiv B(z_1, ..., z_m) \) is a 1-periodic function with respect to every argument \( z_1, ..., z_m \). The \( \omega^1, ..., \omega^m \) are the frequency vectors, and \( \omega_j \cdot y = \sum_{i=1}^d \omega^j_i y_i \) is the inner product of vectors in \( \mathbb{R}^d \). The frequency condition on the vectors \( \omega^1, ..., \omega^m \in \mathbb{R}^d \) amounts to the following assumption:
(FC) There is \( c_0, \tau > 0 \) such that
\[
\left| \sum_{j=1}^{m} k_j \omega_i^j \right| \geq c_0 |k|^{-\tau} \quad \text{for all } k \in \mathbb{Z}^m \setminus \{0\} \text{ and } 1 \leq i \leq d.
\] (6.1)

It is clear that if (FC) is satisfied, then the vectors \( \omega^1, ..., \omega^m \) are rationally independent, that is,
\[
\sum_{j=1}^{m} k_j \omega_i^j \neq 0 \quad \text{for every } 1 \leq i \leq d \text{ and all } k \in \mathbb{Z}^m \setminus \{0\}.
\]

Then as shown in [27, Lemma 2.1], the corrector problem (1.10) possesses a solution which is quasi-periodic. So, it belongs to \( B^2_{AP}(\mathbb{R}^d) \) (the space \( B^2_A(\mathbb{R}^d) \) with \( A = AP(\mathbb{R}^d) \)) since any quasi-periodic function is almost periodic. We may hence apply Theorem 3.2.

Case 2. The entries of \( A \) are continuous almost periodic functions. In [4, Theorem 1.1] are formulated the assumptions implying the existence of bounded almost periodic solution to the problem (1.10). Hence the conclusion of Theorem 3.2 holds. Notice that this class of solutions contains continuous quasi-periodic ones (provided that the assumptions of [4, Theorem 1.1] are satisfied) but also some other almost periodic functions that are not quasi-periodic as shown in [4, Section 4].

6.1.2. The asymptotic periodic setting. We assume that \( A = A_0 + A_{per} \) where \( A_0 \in L^2(\mathbb{R}^d)^{d\times d} \) and \( A_{per} \in L^2_{per}(Y)^{d\times d} \). We are here in the framework of asymptotic periodic homogenization corresponding to \( A = B^2_{\infty,per}(\mathbb{R}^d) = C_0(\mathbb{R}^d) \oplus C_{per}(Y) \). In the proof of Lemma 4.1 we showed that the corrector lies in \( L^2_{\infty,per}(Y) = L^2_0(\mathbb{R}^d) + L^2_{per}(Y) \), which is nothing else but the space \( B^2_{A}(\mathbb{R}^d) \) with \( A = B^2_{\infty,per}(\mathbb{R}^d) \). So Theorem 3.2 applies to this setting.

Remark 6.1. Assume (i) \( A = A_0 + A_{ap} \) with \( A_0 \in C_0(\mathbb{R}^d)^{d\times d} \) and \( A_{ap} \in AP(\mathbb{R}^d)^{d\times d} \), (ii) the entries of \( A_{ap} \) either are quasi-periodic and satisfy the frequency condition, or fulfill the hypotheses of [4, Theorem 1.1]. we may use the same trick as in Lemma 4.1 to show that the corrector lies, in each of these cases, in \( B^2_{\infty,AP}(\mathbb{R}^d) = L^2_0(\mathbb{R}^d) + B^2_{A_{ap}}(\mathbb{R}^d) \). Therefore the conclusion of Theorem 3.2 holds true.

6.2. Applications of Theorems 1.3 and 1.4. Here we give some concrete examples of functions for which Theorems 1.3 and 1.4 hold. Let \( I_d \) denote the identity matrix in \( \mathbb{R}^{d\times d} \).

6.2.1. The asymptotic periodic setting. We assume that \( A = A_0 + A_{per} \) where \( A_0 = b c I_d \) with \( b_c(y) = \exp(-c |y|^2) \) for any fixed \( c > 0 \). \( A_{per} \) is any continuous periodic symmetric matrix function satisfying the ellipticity condition (1.2). In the special 2-dimension setting, we may take \( A_0 = b_1 I_2 \) and
\[
A_{per} = \begin{pmatrix}
  a_1 & 0 \\
  0 & a_2
\end{pmatrix}
\]
with \( a_1(y) = 4 + \cos(2\pi y_1) + \sin(2\pi y_2), \quad a_2(y) = 3 + \cos(2\pi y_1) + \cos(2\pi y_2) \). This special example is used for numerical tests in the next section.
6.2.2. **The asymptotic almost periodic setting.** As in the preceding subsection, we take $A_0 = b_0 I_d$ with $b_c(y) = \exp(-c |y|^2)$. We assume that $A = A_0 + A_{\text{ap}}$ with $A_{\text{ap}}$ being any matrix with continuous almost periodic entries such that $A$ satisfies hypothesis (1.6). In the special 2-dimension setting used for numerical tests below, we take $A_0 = b_1 I_2$ and

$$A_{\text{ap}} = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}$$

with $a_1(y) = 4 + \sin(2\pi y_1) + \cos(\sqrt{2}\pi y_2)$, $a_2(y) = 3 + \sin(\sqrt{3}\pi y_1) + \cos(\pi y_2)$.

7. **Numerical simulations**

Our goal in this section is to check numerically the theoretical results derived in the previous sections. We will consider the finite volume method with two-point flux approximation. Of course multi-point flux approximation can be considered when the matrix $A$ is non-diagonal. Even we will not provide similar results for the discrete problem from numerical approximation, similar results should normally be observed when the space discretization step is small enough (fine grid) as the convergence of the finite volume method for such elliptic problems is well known [14].

7.1. **Finite volume methods.** The finite volume methods are widely applied when the differential equations are in divergence form. To obtain a finite volume discretization, the domain $\Omega$ is subdivided into subdomains $(K_i)_{i \in \mathcal{I}}$, $\mathcal{I}$ being the corresponding set of indices, called control volumes or control domains such that the collection of all those subdomains forms a partition of $\Omega$. The common feature of all finite volume methods is to integrate the equation over each control volume $K_i$, $i \in \mathcal{I}$ and apply Gauss’s divergence theorem to convert the volume integral to a surface integral. An advantage of the two-point approximation is that it provides monotonicity properties, under the form of a local maximum principle. It is efficient and mostly used in industrial simulations. The main drawback is that finite volume method with two-point approximation is applicable in the so called admissible mesh [14, 33] and not in a general mesh. This drawback has been filled by finite volume methods with multi-point flux approximations [1, 2] which allow to handle anisotropy in more general geometries.

For illustration, we consider the problem find $u \in H^1_0(\Omega)$

$$- \nabla \cdot (A(x)\nabla u) = f \quad \text{in} \quad \Omega. \quad (7.1)$$

We assume that $f \in L^2(\Omega)$ and that $A$ is diagonal, so a rectangular grid should be an admissible mesh [14, 33]. Consider an admissible mesh $\mathcal{T}$ with the corresponding control volume $(K_i)_{i \in \mathcal{I}}$, we denote by $\mathcal{E}$ the set of edges of control volumes of $\mathcal{T}$, $\mathcal{E}_{\text{int}}$ the set of interior edges of control volume of $\mathcal{T}$, $u_i$ the approximation of $u$ at the center (or at any point) of the control volume $K_i \in \mathcal{T}$ and $u_\sigma$ the approximation of $U$ at the center (or at any point) of the edge $\sigma \in \mathcal{E}$. For a control volume $K_i \in \mathcal{T}$, we denote by $\mathcal{E}_i$ the set of edges of $K_i$, so that $\partial K_i = \bigcup_{\sigma \in \mathcal{E}_i} \sigma$. 
We integrate (7.1) over any control volume $K_i \in \mathcal{T}$, and use the divergence theorem to convert the integral over $K_i$ to a surface integral,

$$ -\int_{\partial K_i} A(x) \nabla u \cdot \mathbf{n}_{i,\sigma} ds = \int_{K_i} f(x) dx. $$

To obtain the finite volume scheme with two-point approximation, the following finite difference approximations are needed

$$ \sum_{\sigma \in E_i} F_{i,\sigma} \approx \int_{\partial K_i} A(x) \nabla u \cdot \mathbf{n}_{i,\sigma} ds $$

(7.2)

$$ F_{i,\sigma} = -\text{meas}(\sigma) C_{i,\sigma} \frac{u_\sigma - u_i}{d_{i,\sigma}} $$

(7.3)

$$ C_{i,\sigma} = |C_{K_i \mathbf{n}_{i,\sigma}|}, \quad A_{K_i} = \frac{1}{\text{meas}(K_i)} \int_{K_i} A(x) dx $$

(7.4)

Here $\mathbf{n}_{i,\sigma}$ is the normal unit vector to $\sigma$ outward to $K_i$, $\text{meas}(\sigma)$ is the Lebesgue measure of the edge $\sigma \in E_i$ and $d_{i,\sigma}$ the distance between the center of $K_i$ and the edge $\sigma$. Since the flux is continuous at the interface of two control volumes $K_i$ and $K_j$ (denoted by $i \mid j$) we therefore have $F_{i,\sigma} = -F_{j,\sigma}$ for $\sigma = i \mid j$, which yields

$$ \begin{cases} 
F_{i,\sigma} = -\tau_\sigma (u_j - u_i) = -\mu_\sigma \frac{\text{meas}(\sigma)}{d_{i,j}} (u_j - u_i), \sigma = i \mid j \\
\tau_\sigma = \text{meas}(\sigma) \frac{C_{i,\sigma} C_{j,\sigma}}{C_{i,\sigma} d_{i,\sigma} + C_{j,\sigma} d_{j,\sigma}} \quad \text{(transmissibility through } \sigma) 
\end{cases} $$

with

$$ \mu_\sigma = d_{i,j} \frac{C_{i,\sigma} C_{j,\sigma}}{C_{i,\sigma} d_{i,\sigma} + C_{j,\sigma} d_{j,\sigma}}, $$

where $d_{i,j}$ is the distance between the center of $K_i$ and center of $K_j$. We will set $d_{i,j} = d_{i,\sigma}$ for $\sigma = E_i \cap \partial \Omega$. For $\sigma \subset \partial \Omega$ ($\sigma \in \mathcal{E}_{\text{int}}$), we also write

$$ F_{i,\sigma} = -\tau_\sigma (u_\sigma - u_i) $$

$$ = -\frac{\text{meas}(\sigma) \mu_\sigma}{d_{i,\sigma}} (u_\sigma - u_i). $$

The finite volume discretization is therefore given by

$$ \sum_{\sigma \in E_i} F_{i,\sigma} = f_{K_i} $$

(7.5)

$$ f_{K_i} = \int_{K_i} f(x) dx $$

(7.6)

Let $h = \text{size}(\mathcal{T}) = \sup_{i \in \mathcal{I}} \sup_{(x,y) \in K_i^2} |x - y|$ be the maximum size of $\mathcal{T}$. We set $u_h = (u_i)_{i \in \mathcal{I}}$, $N_h = |\mathcal{I}|$ and $F = (f_{K_i})_{i \in \mathcal{I}} + bc$, $bc$ being the contribution of the boundary condition.$^{2}$

$^{1}$interface of the control volumes $K_i$ and $K_j$

$^{2}$Here $bc$ is null as we are looking for solution in $H^1_0(\Omega)$
Applying (7.5) through all control volumes, the corresponding finite volume scheme is given by

\[ A_h u_h = F, \]

where \( A_h \) is an \( N_h \times N_h \) matrix. The structure of \( A_h \) depends on the dimension \( d \) and the geometrical shape of the control volume. For diagonal \( A \), if \( \Omega \) is a rectangular or parallelepiped domain, any rectangular grid \((d = 2)\) or parallelepiped grid \((d = 3)\) is an admissible mesh and yields a 5-point scheme \((d = 2)\) or 7-point scheme \((d = 3)\) for the problem (7.1). To solve efficiently the linear system (7.7), we have used the Matlab linear solver bicgstab with ILU(0) preconditioners.

### 7.2. Simulations in dimension 2.

#### 7.2.1. The Asymptotic periodic setting.

For the numerical tests, we consider problems (1.5) and (1.8) in dimension \( d = 2 \) with the finite volume method scheme (7.7). We denote by \( I_d \) the square identity matrix in \( \mathbb{R}^{d \times d} \). We take \( A_0 = A_0^0 \) with \( A_0^0 = b_0 I_2 \) and

\[ A_{per} = \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix} \]

with \( b_1 = 4 + \cos(2\pi x_1) + \sin(2\pi x_2) \), \( b_2 = 3 + \cos(2\pi x_1) + \cos(2\pi x_2) \).

The right-hand side function \( f \) is given by \( f = 1 \). The computational domain is \( \Omega = (-1, 1)^2 \).

The aim in this section is to compute numerically the ”exact solution” \( u_\varepsilon \) (for a fixed \( \varepsilon > 0 \)) coming from the finite volume scheme with small \( h \), and compare it with its first order asymptotic periodic approximation \( u_\varepsilon(x) = u_0(x_1, x_2) + \varepsilon \chi(x) \cdot \nabla u_0(x_1, x_2) \).

For this purpose, the strategy is carried out as follows:

1. We compute the exact solution of (1.5) with our finite volume scheme on a rectangular fine mesh of size \( h > 0 \), with \( h \) sufficiently small to ensure that the discretization error is much smaller than \( \varepsilon \), which is the order of the error associated to the homogenization approximation (see either Proposition 4.2 or Theorem 1.4).

2. We compute the corrector functions \( \chi_1 \) and \( \chi_2 \) associated to the respective directions \( e_1 = (1, 0) \) and \( e_2 = (0, 1) \). To this end, we rather consider their approximations by the finite volume scheme (7.7), which are solutions to Eq. (3.1), and we perform this computation on the domain \( Q_6 = (-6, 6)^2 \) with Dirichlet boundary conditions (as in (3.1)). We also compute their gradients \( \nabla \chi_1 \) and \( \nabla \chi_2 \). Here we take the mesh size \( h = 8 \times 10^{-3} \) independent of \( \varepsilon \).

3. With \( \nabla \chi_1 \) and \( \nabla \chi_2 \) computed as above, we compute the homogenized matrix \( A_6^* \) as in (3.3), namely

\[ A_6^* = \left( \frac{1}{12} \right)^2 \int_{Q_6} A(x)(I_2 + \nabla \chi(x))dx \]

where here, \( \chi = (\chi_1, \chi_2) \) so that \( \nabla \chi \) is the square matrix with entries \( c_{ij} = \frac{\partial \chi_j}{\partial x_i} \).
(4) With $A^*_6$ now being denoted by $A^*$, we compute the exact solution $u_0$ of (1.8).

(5) Finally we compute the first order approximation $v_\varepsilon(x) = u_0(x) + \varepsilon\chi(x/\varepsilon) \cdot \nabla u_0(x)$ and we compare it to the exact solution $u_\varepsilon$, which has been computed at step 1.

The goal is to check the convergence result in Theorem 1.4 given by (1.15), but with the numerical solution using finite volume method. Indeed we want to evaluate the following error

$$\text{Err}(\varepsilon) = \frac{\|u_\varepsilon - u_0 - \varepsilon\chi\varepsilon\nabla u_0\|_{H^1(\Omega)}}{\|u_0\|_{H^2(\Omega)}} = \frac{\|u_\varepsilon - v_\varepsilon\|_{H^1(\Omega)}}{\|u_0\|_{H^2(\Omega)}}. \quad (7.8)$$

As we already mentioned, $u_0$, $u_\varepsilon$ and $v_\varepsilon$ are computed numerical using the finite volume scheme for a fixed $h = 8 \times 10^{-3}$ independent of a fixed $\varepsilon$. All the norms involved in (7.8) are computed using their discrete forms \[14, 33\]. The coefficients of $A$ and $f$ are $C^\infty(\Omega)$, so the corresponding solutions $u_0$, $u_\varepsilon$ and $v_\varepsilon$ should be regular enough. Their graphs are given in Figure 1. As we can observe in Table 1, the error decreases when $\varepsilon$ decreases, and therefore the convergence of $u_\varepsilon$ and $v_\varepsilon$ towards $u_0$ when $\varepsilon \to 0$ is ensured. We can also observe that the corrector plays a key role as graph of $u_\varepsilon$ is close to the one of $v_\varepsilon$. The numerical value of $A^*_6 \equiv A^*$ obtained and used for $u_0$ and $v_\varepsilon$ is given by

$$A^*_6 = \begin{pmatrix} 3.895923 & 0.00001 \\ 0 & 2.849959 \end{pmatrix}.$$ 

| $1/\varepsilon$ | 2       | 3       | 4       | 5       | 6       |
|----------------|---------|---------|---------|---------|---------|
| Err($\varepsilon$) | 0.5298  | 0.1382  | 0.0620  | 0.0577  | 0.0573  |

Table 1. $\text{Err}(\varepsilon)$ with the corresponding $1/\varepsilon$ for a fixed $h = 2 \times 10^{-3}$ independent of a fixed $\varepsilon$.

7.2.2. The asymptotic almost periodic setting. Here we take $A = A_0 + A_{ap}$ with

$$A_0 = b_0 I_2 \text{ with } b_0(x_1, x_2) = \exp(-x_1^2 + x_2^2)$$

and

$$A_{ap} = \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix} \text{ with } b_1 = 4 + \sin(2\pi x_1) + \cos(\sqrt{2}\pi x_2), \quad b_2 = 3 + \sin(\sqrt{3}\pi x_1) + \cos(\pi x_2).$$

The right-hand side function $f$ is given by $f(x_1, x_2) = \cos(\pi x_1) \cos(\sqrt{5}\pi x_2)$. The computational domain is as above, that is, $\Omega = (-1, 1)^2$. We follow the same steps as above. The corresponding value of $A_6^*$ is

$$A_6^* = \begin{pmatrix} 4.0118 & 0.0002 \\ 0.0032 & 3.0206 \end{pmatrix}.$$

We solve (1.8) using finite volume method with multi-point flux approximation \[1, 2\]. From Table 2 and Figure 2 we can draw the same conclusion as in Section 7.2.1.

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Figure 1. The graphs of $u_\varepsilon$, $u_0$, $v_\varepsilon$, $|u_\varepsilon - u_0|$ and $|u_\varepsilon - v_\varepsilon|$ in the asymptotic periodic setting, are shown in (a), (b), (c), (d) and (e) respectively for $\varepsilon = 1/6$ and $h = 2 \times 10^{-3}$.
Figure 2. The graphs of $u_\varepsilon$, $u_0$, $v_\varepsilon |u_\varepsilon - u_0|$ and $|u_\varepsilon - v_\varepsilon|$ in the asymptotic almost periodic setting, are shown in (a), (b), (c), (d) and (e) respectively for $\varepsilon = 1/6$ and $h = 2 \times 10^{-3}$. 