Counterexamples to a conjecture of Merker on 3-connected cubic planar graphs with a large cycle spectrum gap

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Abstract. Merker conjectured that if \( k \geq 2 \) is an integer and \( G \) a 3-connected cubic planar graph of circumference at least \( k \), then the set of cycle lengths of \( G \) must contain at least one element of the interval \([k, 2k+2]\). We here prove that for every even integer \( k \geq 6 \) there is an infinite family of counterexamples.

Key words. Cycles; Cycle spectrum; 3-connected; Cubic; Planar graphs

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1 Introduction

For a graph \( G \), we denote by \( C(G) \) the set of lengths of cycles in \( G \), i.e. its cycle spectrum. The circumference of \( G \) is the length of a longest cycle in \( G \). Merker [2] recently proved that for any non-negative integer \( k \) every 3-connected cubic planar graph \( G \) of circumference at least \( k \) satisfies \( C(G) \cap [k, 2k+9] \neq \emptyset \). He conjectured that for any integer \( k \geq 2 \) and any 3-connected cubic planar graph \( G \) of circumference at least \( k \), we have \( C(G) \cap [k, 2k+2] \neq \emptyset \). We shall abbreviate this conjecture of Merker with (†).

By Euler’s formula, every plane graph of minimum degree at least 3 contains a face of length 3, 4, or 5, so (†) holds for \( k \in \{2, 3\} \). Suppose (†) is untrue for \( k = 5 \). Then there exists a 3-connected cubic plane graph \( G \) of circumference at least 5 with \( C(G) \cap [5, 12] = \emptyset \). Any 3- or 4-cycle in \( G \) must be the boundary of a face of \( G \), and any two faces in \( G \) of size 3 or 4 are disjoint since 5 \( \notin C(G) \) and 6 \( \notin C(G) \). We contract every triangle and every quadrilateral of \( G \) to a vertex and obtain the graph \( G' \). If we exclude 3- and 4-cycles, cycles in \( G \) have length at least 13, so \( G' \) is a planar 2-connected graph with no cycle of length less than 7 (as on any \( \ell \)-cycle \( C \) of \( G \), \( C \) shares at most \( \lfloor \ell/2 \rfloor \) edges with a 3- or 4-cycle), a contradiction. The argument for \( k = 4 \) is very similar (and simpler). This yields that (†) holds for \( k \in \{2, 3, 4, 5\} \). However, we now show that for any even integer \( k \geq 6 \) there is an infinite family of counterexamples to (†).

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2 Result

Theorem. For any even integer \( k \geq 6 \) there exists an infinite family of 3-connected cubic planar graphs of circumference at least \( k \) whose cycle spectrum contains no element of \([k, 2k + 2]\).

Proof. Consider the graph \( H \) depicted in Fig. 1. Its left-most and right-most parts should be identified in the obvious way, where the boundary cycles of the two faces incident only with pentagons (top and bottom of Fig. 1) may have any length of at least \( 2r + 8 \), where \( r \) is a non-negative integer denoting the number of “rungs”, defined in the next paragraph (this yields the advertised infinite family). The vertices of \( H \) are either black or white, as illustrated in Fig. 1.

![Figure 1: The graph H.](image)

Consider the operations \( A \) and \( B \) defined in Fig. 2. We shall call a rung any edge depicted as a horizontal line-segment in Fig. 2. In each operation, we replace a cubic vertex with the plane graph \( A_r \) and \( B_r \) (in which we ignore the three dangling edges), respectively, where \( r \) denotes the number of rungs.

![Figure 2: Operations A (left-hand side) and B (right-hand side).](image)

Using operations \( A \) and \( B \), replace in \( H \) each black vertex with a copy of \( A_{r+2} \) and each white vertex with a copy of \( B_r \), respecting the orientations given in Fig. 1 by the numbers 1, 2, 3. We obtain a planar graph \( G \) that is clearly 3-connected and cubic. The circumference of \( A_{r+2} \) and \( B_r \) is \( 2r + 5 \). By construction, any cycle in \( G \) of length greater than \( 2r + 5 \) has length at least \( 4r + 15 \), which is the length of the cycle bounding the face \( F \) and also of the cycle bounding the face \( F' \). Thus, for every \( \ell \in \{2r + 6, \ldots, 4r + 14\} \),
the graph $G$ contains no cycle of length $\ell$. Setting $k := 2r + 6$, the proof is complete, since $G$ clearly has circumference at least $k$.

\[ \square \]

**Remark.** After this note was submitted, the article [1] by Cui and Lo appeared. It extends the results given in this note and fully settles the problem by determining tight gaps for cycle spectra in the class of 3-connected cubic planar graphs. It also establishes tight gaps in the cycle spectra in the larger family of all 3-connected planar graphs, thereby confirming a conjecture of Merker [2].

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**References**

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