Topological Properties of Spatial Coherence Function

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Abstract

Topology of the spatial coherence function is considered in details. The phase singularity (coherence vortices) structures of coherence function are classified by Hopf index and Brouwer degree in topology. The coherence flux quantization and the linking of the closed coherence vortices are also studied from the topological properties of the spatial coherence function.

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I. INTRODUCTION

Coherence function, which was first introduced by Wolf in 1954, is a very important quantity in describing the cross correlation between the fluctuating fields at different space-time points in coherence optics[1]. Due to its theoretical importance and practical interest, the concept of optical coherence has been studied extensively, and considerable progress has been made during the past five decades. More recently, the existence of coherence vortices or the phase singularities of a complex coherence function has been theoretically predicted, and examined by experiments[2, 3, 4, 5]. The intriguing characteristics of coherence vortex have drawn great interest. A great deal of works on the coherence vortex have been done by many physicists[2, 3, 4, 5].

On the other hand, the existence of the coherence vortices is a topological phenomena for the complex coherence function, but their topological properties are not clear. The coherence vortex is a new type of topological objects. In three dimensional space, these topological objects form string-like vortices, i.e., the coherence vortex lines. In a general case, these vortex lines could be closed, linked together, and even knotted. The coherence vortices carry rich topological information. It is what inspires us to use topological method to study the coherence vortices.

In this paper, by making use of the $\phi$-mapping method[6, 7, 8, 9], we study the topological current of coherence vortex and topological knot invariant of the knotted coherence vortex lines in detail. Meanwhile, we also investigate the coherence flux quantization of the coherence vortex. This paper is arranged as follows. In Sec.II, we construct a coherence vorticity field. The topological coherence current of the coherence vortex lines can be naturally introduced from this coherence vorticity field. The topological coherence current don’t vanish only when the coherence vortex lines exist. The topological charges of these vortex lines are expressed by the topological quantum number, the Hopf index and Brouwer degree of the $\phi$-mapping. In Sec. III, we introduce the coherence flux by analogy to the magnetic flux, and find that this coherence flux is quantized in topological level, which is similar to magnetic flux quantization in superconductor and superfluid systems[8]. Then we research the closed and knotted coherence vortex lines. An exact expression of the topological knot invariant is given. Sec. IV is our concluding remarks.
II. TOPOLOGICAL COHERENCE CURRENT AND COHERENCE VORTEX

In the theory of coherence vortex, the coherence current is a new concept which was introduced in Ref. [5]. For the spatial coherence function, the coherence current density can be expressed as

$$\mathbf{T}(\mathbf{x}_1, \mathbf{x}_2) = \text{Im}[\Gamma^*(\mathbf{x}_1, \mathbf{x}_2)\nabla_1 \Gamma(\mathbf{x}_1, \mathbf{x}_2)],$$

(1)

where $\Gamma(\mathbf{x}_1, \mathbf{x}_2)$ is the mutual spatial coherence function, and $\nabla_1$ is the differential operations to be performed with respect to the point $\mathbf{x}_1$. In Eq. (1), we see that there are two spatial variables $\mathbf{x}_1$ and $\mathbf{x}_2$. In fact the coherence domain is just the one of variable spaces for $\mathbf{x}_1$ or $\mathbf{x}_2$, in our discussions, we will restrict to the case for the variation of location $\mathbf{x}_1$ by keeping $\mathbf{x}_2$ fixed.

The complex mutual coherence function $\Gamma(\mathbf{x}_1, \mathbf{x}_2)$ can be written as

$$\Gamma(\mathbf{x}_1, \mathbf{x}_2) = \phi_1 + i\phi_2,$$

(2)

in which $\phi^a$ ($a = 1, 2$) are real functions. Using this $\phi$-field, we can define an unit vector field: $n^a = \frac{\phi^a}{||\phi||}$ ($||\phi||^2 = \phi^a\phi^a = ||\Gamma||^2, a = 1, 2$). By using of this $n$-field, the coherence current density in Eq.(1) can be expressed as

$$\mathbf{T}(\mathbf{x}_1, \mathbf{x}_2) = ||\Gamma||^2 \epsilon_{ab} n^a \nabla_1 n^b.$$  

(3)

For convenience, we will instead $\mathbf{x}_1$ by variation $\mathbf{x}$ in our following discussions.

By analogy to fluid mechanics system, the coherence velocity field associated with the $n$-field can be defined as $\mathbf{u} = \epsilon_{ab} n^a \nabla n^b$, then the coherence vorticity is given by $\mathbf{\omega} = \frac{1}{2} \nabla \times \mathbf{u}$. This coherence vorticity is a very essential topological quantity in our discussion, the topological interpretation of this vorticity is due to the existence of coherence vortices, and only in this case the vorticity does not vanish [8]. So the exact expression for $\mathbf{\omega}$ plays an essential role in topology, in our following discussions, we will show what the exact expression for $\mathbf{\omega}$ is. By using of $n$-field, it can be expressed as

$$\omega^i = \frac{1}{2} \epsilon^{ijk} \epsilon_{ab} \partial_j n^a \partial_k n^b.$$  

(4)

Obviously, this is a topological current (which we term the topological coherence current). This current $\omega^i$ play an important role in the topological properties of the coherence optical
systems. Here we will use the $\phi$-mapping method to research the properties of this current. Using $\partial_\mu(\phi^a/\|\phi\|) = (\partial_\mu]\phi^a)/\|\phi\| + \phi^a\partial_\mu(1/\|\phi\|)$ and the Green function relation in $\phi$ space: $\partial_a\partial_a\ln\|\phi\| = 2\pi\delta^2(\vec{\phi})$ $\left(\partial_a = \partial/\partial\phi^a\right)$, we can find a non-zero expression for $\omega^i$:

$$\omega^i = \delta^2(\vec{\phi})D^i\left(\frac{\phi}{x}\right),$$

where the Jacobian vector is defined as

$$\epsilon^{ab}D^i\left(\frac{\phi}{x}\right) = \epsilon^{ijk}\partial_j\phi^a\partial_k\phi^b.$$

From the expression in Eq. (5), we can come to an important conclusion that the topological coherence current $\omega^i$ does not vanish only when $\vec{\phi} = 0$. The zeros of $\vec{\phi}$ determine the topological properties of the topological coherence current $\omega^i$, i.e., the coherence vorticity. So it is necessary to study the zero points of $\vec{\phi}$ to determine the nonzero solutions of $\omega^i$.

The implicit function theory \[10\] shows that under the regular condition $D^i\left(\frac{\phi}{x}\right) \neq 0,$

the general solutions of

$$\phi^1(x^1, x^2, x^3) = 0, \quad \phi^2(x^1, x^2, x^3) = 0$$

can be expressed as

$$\vec{x} = \vec{x}_k(s),$$

which represent the $N$ isolated singular strings $L_j \ (j = 1, 2, \ldots, N)$ with parameter $s$. These singular string solutions are just the coherence vortex lines, i.e., the phase singularities of spatial coherence function. These coherence vortex lines have the coherence function equal to zero, and the phase of coherence function undefined. In the core of the coherence vortices, the intensities of the wavefield do not vanish, this is quite different from the traditional optical vortices in the coherence field.

In $\delta$-function theory \[11\], one can prove that in three dimensional space

$$\delta^2(\vec{\phi}) = \sum_{k=1}^{N} \beta_k \int_{L_k} \frac{\delta^3(\vec{x} - \vec{x}_k(s))}{|D(\frac{\phi}{u})|_{\Sigma_k}} ds,$$

where $D(\phi/u)_{\Sigma_k} = \frac{1}{2}\epsilon^{ik}\epsilon_{mn}(\partial\phi^m/\partial u^i)(\partial\phi^n/\partial u^k)$, and $\Sigma_k$ is the $k$th planar element transverse to $L_k$ with local coordinates $(u^1, u^2)$. The positive integer $\beta_k$ is the Hopf index of $\phi$-mapping,
which means that when $\vec{x}$ covers the neighborhood of the zero point $\vec{x}_k(s)$ once, the vector field $\vec{\phi}$ covers the corresponding region in $\phi$ space $\beta_k$ times. Meanwhile taking notice of the definition of the Jacobian, one can obtain the direction vector of $L_k$

$$\frac{dx^i}{ds}|_{x_k} = \frac{D^i(\phi/x)}{D(\phi/u)}|_{x_k}. \quad (9)$$

Then from Eq. (8) and (9) we obtain the topological inner structure of the topological coherence current $\omega^i$:

$$\omega^i = \delta^2(\vec{\phi})D^i \left( \frac{\phi}{x} \right) = \sum_{k=1}^{N} W_k \int_{L_k} \frac{dx^i}{ds} \delta^3(\vec{x} - \vec{x}_k(s))ds, \quad (10)$$

in which $W_k = \beta_k \eta_k$ is the winding number of $\vec{\phi}$ around $L_k$, with $\eta_k =$sgn$D(\phi/u)_{x_k} = \pm 1$ being the Brouwer degree of $\phi$-mapping. The signs of Brouwer degree are very important, the $\eta_k = +1$ corresponds to the vortex, and $\eta_k = -1$ corresponds to the anti-vortex. The integer number $W_k$ measures windings of the phase around the phase singularities of the spatial coherence function, and is called the topological charge of the coherence vortices. Hence the topological charge of the coherence vortex line $L_k$ is

$$Q_k = \int_{\Sigma_k} \omega^i d\sigma_i = W_k. \quad (11)$$

By analogy to the optical vortices whose topological charge play the role of an angular momentum, here the topological charge of coherence vortex also associated with angular momentum, this is the unique characteristics for the generic vortices.

In the above discussion, we have known that the coherence vortices are zeros of the coherence function. The coherence current $\vec{T}$ and the topological coherence current $\omega^i$ are both expressed by use of the coherence function. In the theory of coherence function, there is another important quantity which called spectral degree of coherence and denoted $\mu(x_1, x_2)\quad [2,3]$. This spectral degree of coherence is a measure of the spatial coherence of the optical wavefield and takes on values between 0 and 1, zero representing complete incoherence, unity representing complete coherence. As theoretically predicted, the coherence vortices have the spectral degree of coherence equal to zero, so the coherence vortices can also derive from the topological coherence current defined by $\mu(x_1, x_2)$. The discussions by use of $\mu(x_1, x_2)$ are very similar with the discussions by coherence function.
III. COHERENCE FLUX AND GEOMETRY OF COHERENCE VORTEX LINES

In this section we begin to study the coherence flux and geometry structure of the coherence vortex lines in three dimensional coherence domain.

From the definition of the coherence vorticity and by analogy to the definition of magnetic flux, the coherence flux of $\vec{\omega}$ through a surface $S$ can be defined as

$$\Phi = \int_S \vec{\omega} \cdot d\vec{S}. \quad (12)$$

It is also called coherence circulation in Ref.[5]. It is known from above section that $\vec{\omega}$ does not vanish only when the coherence vortices exist, so the coherence flux associated with the coherence vortex lines through the surface $S$. According to Eq.(11) and Eq.(12), each coherence vortex line $L_k$ carries a coherence flux, i.e.,

$$\Phi_k = \int_{\Sigma_k} \omega^i d\sigma_i = W_k = Q_k, \quad (13)$$

which will lead to the phenomenon of coherence flux quantization. So the total flux through a surface $S$ can be expressed as a quantized form

$$\Phi = \sum_{k=1}^{N} W_k = \sum_{k=1}^{N} \beta_k \eta_k = \sum_{k=1}^{N} \Phi_k. \quad (14)$$

This is the topological essence of flux quantization. Here, the topological interpretation is clear, that is the flux can be quantized due to the topological properties of the coherence function and expressed by the topological quantum numbers: the Hopf index and Brouwer degree, which are important topological information carried by the coherence vortices. This is similar to the topological quantization of the magnetic flux in superconductor and super-fluid systems.

From Eq.(4) we can easily obtain that $\nabla \cdot \vec{\omega} = 0$. In the coherence domain $V$ with a closed surface $S$, by using the Gauss theorem, we can obtain

$$\Phi = \oint_S \vec{\omega} \cdot d\vec{S} = \int_V \nabla \cdot \vec{\omega} dV = 0, \quad (15)$$

which suggests that the coherence vortex lines are constrained to form closed loops within the finite spatial volume of the coherence function, or to terminate at infinity [5]. This closed geometry also appears in fluid mechanics and classical Electromagnetism, in which the fluid vortices and magnetic lines are closed.
In fluid mechanics or classical Electromagnetism, the closed vortices or magnetic lines usually form a knotlike structure [12, 13]. Knotted configurations exist ubiquitously in nature, it also appears as knotted optical vortices in optical system [14, 15, 16, 17]. For the closed coherence vortex lines in three dimensional coherence domain, we expect that they can also form knot structure. Notice that they can be understood as the linking of two quantized coherence fluxes of the knotted vortex lines. It is known that for a knot family there are important characteristic numbers to describe its topology, such as the self-linking and the linking numbers. In fluid mechanics and classical Electromagnetism, there is an important topological knot invariant, Helicity, which measures the linking of the closed lines [12, 13]. In the case of coherence vortex lines, the helicity for the coherence (we term coherence helicity) defined as

$$H = \frac{1}{2\pi} \int \vec{u} \cdot \vec{\omega} dV.$$  \hspace{1cm} (16)

Substituting Eq.(10) into Eq.(16), one can obtain

$$H = \frac{1}{2\pi} \sum_{k=1}^{N} \int_{L_k} \vec{u} \cdot d\vec{x}.$$  \hspace{1cm} (17)

It can be seen that when these $N$ coherence vortex lines are $N$ closed curves, i.e., a family of $N$ knots $\gamma_k (k = 1, \cdots, N)$, Eq.(17) leads to

$$H = \frac{1}{2\pi} \sum_{k=1}^{N} \oint_{\gamma_k} \vec{u} \cdot d\vec{x}.$$  \hspace{1cm} (18)

This is a very important expression. Consider a transformation of coherence function $\Gamma'$: $\Gamma' = e^{i\theta} \Gamma$, this gives the U(1) gauge transformation of $\vec{u}$: $\vec{u}' = \vec{u} + \nabla \theta$, where $\theta \in \mathbb{R}$ is a phase factor denoting the U(1) gauge transformation. It is seen that the $\nabla \theta$ term in Eq.(18) contributes nothing to the integral $H$ when the coherence vortex lines are closed, hence the expression (18) is invariant under the U(1) gauge transformation. In the above discussions, we have proved that the coherence vortices are closed loops in the finite spatial coherence domain, therefore the integral $H$ is a spontaneous topological invariant for the coherence vortex lines in the coherence vortex theory.

It was proved that the integral $H$ is related to the linking and self-linking number [7]

$$H = \sum_{k=1}^{N} W_k^2 SL(\gamma_k) + \sum_{k,l=1 \ (k \neq l)}^{N} W_k W_l Lk(\gamma_k, \gamma_l),$$  \hspace{1cm} (19)
where $SL(\gamma_k)$ is self-linking number of $\gamma_k$ and $Lk(\gamma_k, \gamma_l)$ is the Gauss linking number between different knotted vortex lines $\gamma_k$ and $\gamma_l$. Obviously two coherence fluxes linked together can not be separated by any continuous deformation of the field configuration. This provides the topological stability of the knots. Since the self-linking number and the Gauss linking number are both the invariant characteristic numbers of the knotted closed curves in topology, $H$ is an important topological invariant required to describe the knotted coherence vortex lines in coherence optical systems.

IV. CONCLUSION

First, we introduce the topological coherence current and obtain the inner structure of the coherence vortices. The coherence vortices have found at the every zero point of the complex spatial coherence function under the condition that the Jacobian determinate $D^i(\phi) \neq 0$, and the topological coherence current does not vanish only when the coherence vortices exist. One also shows that the vortex structures are classified by Hopf index and Brouwer degree in topology. Second, by analogy to the magnetic flux quantization in superconductor an superfluid systems, we conclude that the coherence flux is topologically quantized. The coherence flux is quantized due to the topological properties of the coherence function and can be expressed by the topological quantum numbers of the coherence vortices. We also conclude that the coherence vortex lines are constrained to form closed loops within the finite spatial volume of the coherence function, or to terminate at infinity. This means that the coherence vortex lines may be knotted and the coherence helicity is a spontaneous topological invariant for the coherence vortex lines. The exact expression of the coherence helicity is also given.

Acknowledgments

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