Calculation for stability of centrally compressed rods of steered stiffness in existence of linearly deformable connections

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Abstract. The authors consider one-parameter problems of calculating the stability of centrally compressed rods of step stiffness with linearly elastically compliant support constraints. The problem of calculating the stability is set as bifurcation. Euler's exact solutions are obtained for different variants of the design scheme of a rod of two-stage rigidity. An iterative algorithm for calculating the stability beyond the limit of proportionality is proposed, based on the concept of the conditional flexibility of rod sections, using the experimentally obtained diagram $\sigma - \varepsilon$ and the table of coefficients $\varphi$. Numerical examples illustrating the proposed methods and algorithms are considered.

1. Introduction
Stepped rods are widely used in construction practice, primarily as columns of building frames. In the calculation of such rods, in addition to assessing the strength, it is necessary to assess the possible loss of stability (until the strength is exhausted). The issue of buckling of both individual rods and core systems has been the subject of a large number of works: sections in educational literature [1-4]; sections in reference and regulatory literature [5-8]; separate and monographs and articles [9-20]. However, to date, the literature has not presented general approaches and methods for calculating the stability of centrally compressed rods of variable stiffness in the presence of compliant bonds. When using the practical method of selecting sections from the stability condition [3], it is difficult to determine the reduced (calculated) lengths and flexibilities of the rods necessary to use the table of coefficients $\varphi$. Therefore the theme of present article in which approaches and techniques for overcoming of the specified difficulties are offered, is represented actual.

2. Setting research objectives
The object of the study is a centrally compressed rod of step stiffness, some of the supporting links of which are linearly elastically compliant. Examples of such rods are shown in Figure 1. It also shows possible forms of buckling of rods.
3. Solving problems of calculating the stability of the rod in the formulation of Euler

Directed by Euler problems dare at following assumptions:

- The systems in question are conservative;
- Stability loss occurs in linearly-elastic area of work of a material;
- The form of loss of the stability, connected with transition in next with initial the form of balance (branching) is realized.

In the case when, when assessing stability, the own weight of the rod is not taken into account (which is quite acceptable for racks of low and medium height), when describing the next (bending) form of equilibrium in sections, you can use differential equations of the second order:

Section 1,

\[
0 \leq x_1 \leq l_1, \quad EI_1 \frac{d^2 v_1(x_1)}{dx^2} = -P \left[v_1(x_1) - v_1(0)\right] - c \cdot v_1(0) \cdot x_1, \quad (1)
\]

where \( c \) (kn / m) is the stiffness coefficient of the linearly elastically compliant bond. We introduce the coefficient. \( k_i = \left( \frac{P}{EI_i} \right)^{\frac{1}{2}} \). Then equation (1) is converted to

\[
\frac{d^2 v_1(x_1)}{dx^2} + k_i^2 \cdot v_1(x_1) = k_i^2 \cdot v_1(0) - \frac{c \cdot v_1(0)}{EI_i} x_1, \quad (2)
\]

And the general solution (2) is written:

\[
v_1(x_1) = C_{11} \cdot \cos(k_1 x_1) + C_{12} \cdot \sin(k_1 x_1) + v_1(0) - \frac{c \cdot v_1(0)}{EI_i} x_1 \quad (3)
\]

Performing similar transformations in section 2 \((0 \leq x_2 \leq l_2)\), we get:

\[
v_2(x_2) = C_{21} \cdot \cos(k_2 x_2) + C_{22} \cdot \sin(k_2 x_2) + \frac{v_1(0) + m \cdot v_1(l_1)}{1 + m} - \frac{c \cdot v_1(0) \cdot (l_1 + x_2)}{k_i^2 EI_i} \quad (4)
\]
where \( k_2 = \left[ \frac{P \cdot (1 + m)}{EI_x} \right]^{1/2} \). To determine the integration constants and the parameter \( v_1(0) \), we use the following conditions:

1. \( x = 0, v_1(0) = C_{11} + v_1(0), C_{11} = 0 \)
2. \( x = l_1, v_1(l_1) = v_2(0) \)
3. \( \frac{dv_1(l_1)}{dx} = \frac{dv_2(0)}{dx} \)
4. \( x = l_2, v_2(l_2) = 0 \)
5. \( \frac{d^2v_2(l_2)}{dx^2} = 0 \)

From a boundary condition (d) follows:

\[
C_{21} = \tan(k_2l_2) \cdot C_{22} \quad (5)
\]

Taking into account conditions 1.-5., after transformations we will come to system of the homogeneous equations

\[
\begin{pmatrix}
  a_{11} & a_{12} & a_{13} \\
  a_{21} & a_{22} & a_{23} \\
  a_{31} & a_{32} & a_{33}
\end{pmatrix}
\begin{pmatrix}
  v_1(0) \\
  C_{12} \\
  C_{22}
\end{pmatrix} = 0,
\]

where \( a_{ij} \) – the factors depending from \( k_1, k_2 \).

Not trivial decision (6) corresponds to a condition:

\[
\begin{vmatrix}
  a_{11} & a_{12} & a_{13} \\
  a_{21} & a_{22} & a_{23} \\
  a_{31} & a_{32} & a_{33}
\end{vmatrix} = 0
\quad (7)
\]

Opening the determinant in (7), we obtain an equation, from the solution of which the critical value of the load parameter is found. Solution of the stability problem in the Euler formulation for case 2 (see figure 1) is carried out in the same way.

Results of the decision of test problems for cases 1a), 1б) and a case 2 (Figure 1) at \( c = 1000 \text{kN/m} \); \( l_1 = 4 \text{ m}; l_2 = 8 \text{ m}; EI_1 = 10000 \text{ kN·m}^2; EI_2 = 20000 \text{ kN·m}^2; m = 2 \) have appeared the following: the case 1a), \( P_{cr} = 257.03 \text{kN} \) the case 2, \( P_{cr} = 926.3 \text{kN} \); the case 1b) at the set initial data is not realized. These results will quite be coordinated with calculations of cores of constant rigidity.

4. Verifying the applicability of Euler’s statement

The tests of the applicability of the Euler formulation can be carried out by preliminary linear-elastic calculation of the critical parameter of the load and the subsequent calculation of the critical stresses.

Consider case 1a) and case 2 (Figure 1). The following cross-sectional shapes are adopted in the sections (Figure 2).
Figure 2. Forms of cross-sections of a stepped rod in areas.

Geometrical characteristics and rigidity of sections:

\[ A_1 = 76 \cdot \delta^2; \quad I_1 = 4585.3 \cdot \delta^4; \quad A_2 = 138.24 \cdot \delta^2; \quad I_2 = 19143.47 \cdot \delta^4 \]

Let value of parameter is accepted \( \delta = 1 \text{ cm} \). Values of rigidity of sections are equal:

\[ E = 2 \cdot 10^4 \text{kN/m}^2, \quad (\text{steel}), \quad EI_1 = 9170 \text{ kN} \cdot \text{m}^2; \quad EI_2 = 38286 \text{ kN} \cdot \text{m}^2; \quad A_1 = 0.0076 \text{ m}^2; \]
\[ A_2 = 0.013824 \text{ m}^2. \]

The calculations for the above algorithms led to the following values: case 1a): \( \sigma_{cr} = 432.16 \text{ kN} \); case 2: \( \sigma_{cr} = 1649 \text{ kN} \); pressure in a case 1a): \( \sigma_{cr} < \sigma_{p} \); pressure in a case 2: \( \sigma_{cr} > \sigma_{p} \);

Thus, to apply the formulation of Euler in case 2 is impossible.

5. Calculation of the stepped rod for stability beyond the proportionality limit. Selection of sections from the conditions of stability

Below we propose an algorithm for calculating the stability of complex rods, whose reduced lengths and flexibility are difficult to determine directly. The algorithm is based on the use of an experimental diagram \( \sigma + \varepsilon \) and a table of longitudinal bending coefficients \( \varphi \). The algorithm is described from the point of view of selection of sections from the conditions of stability.

Let us consider, for example, case 2, when the calculation according to the Euler method leads to the values \( \sigma_{cr} > \sigma_{p} \). The calculation algorithm is iterative and contains the following steps at iteration \( j \).

1. Choose values of buckling coefficients \( \varphi_i \), \( i = 1,2 \) at iteration.
2. Calculate the required cross-sectional areas in the areas: The demanded areas of cross-section 1,2 are calculated:

\[ A_{1j} = \frac{P}{\varphi_j \cdot R}; \quad A_{2j} = \frac{3 \cdot P}{\varphi_j^2 \cdot R} \quad (8) \]

where \( P \) – The set parameter of loading, \( R \) – Resistance of a material is settlement.

Taking into account the given form of sections on the plots, we find the required value of the parameter \( \delta_i \) at iteration:
3. Calculation of a given rod for stability in a linear-elastic formulation using the algorithm described earlier and the tangents of modules of materials $E_{ji}, i=1,2$, is carried out at iteration. It is defined $P_{cr}$ at iteration.

4. Calculate the values of critical stresses in the areas:

$$\sigma_{cr} = \frac{P_{cr}}{A_{ji}}, \sigma_{cr}^2 = \frac{3 \cdot P_{cr}}{A_{ji}}$$

(10)

5. Using the found values of the critical stresses $\sigma_{cr}$, we determine the values of the tangent modules $E_{ji}, i=1,2$, in the sections according to the diagram $\sigma + \varepsilon$.

6. Taking into account the found values of the tangent module $E_{ji}, i=1,2$, in the sections, the conditional flexibility of the sections is calculated using

$$\lambda_j = \left( \frac{\pi^2 \cdot E_{ji}}{\sigma_{cr}} \right)^{\frac{1}{2}}$$

(11)

7. According to the table of coefficients $\phi$ [1], the values $\phi_j$ corresponding to the found conditional flexibilities of sections (11) are found.

8. Analysis of the end of search

$$|\phi_j - \phi_j'| \leq \varepsilon, i=1,2$$

(12)

where $\varepsilon$ - the set small size.

If the criterion (12) is carried out, calculation comes to the end. Otherwise $\phi_{j+1}$ are calculated, for example, as $\phi_{j+1} = 0.5 \cdot (\phi_j + \phi_j')$, $i=1,2$ and transition to iteration $j+1$ is carried out.

6. The technique of definition of tangents of modules

Let, for example, low-alloy steel is selected as the material of the section (see Figure 3), the strength limits of which are specified in reference materials [2]. We believe that the strength limits of steel under tension-compression are the same, and the conditional yield strength (points B, B') corresponds to a residual strain of 0.002. The function of the tangent module $E^*(\sigma)$ on the site is approximated with a cubic parabola:

$$E^*(\sigma) = a \sigma + b \sigma^2 + c \sigma^3$$

(13)
We will also assume that the values of the tangent module $E^*(\sigma_{0.2})$ at the points B, B’ (Figure 3) are the same, and for convenience we give a fragment of the diagram (Figure 4) in the stretch region.

![Image of a stretching-compression of the alloyed steel.](image1)

![Image of definition of the tangential module $E^*(\sigma_{0.2})$.](image2)

The calculations showed that the approximation (13) in section A, B, C provides high accuracy, and in the section BD leads to physically unrealistic results. By approximating a straight line diagram $\sigma \div \varepsilon$ in the section BD (due to the large flatness of this part of the diagram), we obtain the

$$\sigma_{0.4} \approx \sigma_{0.2} + \frac{\sigma_{u} - \sigma_{0.2}}{\delta + \frac{\sigma_{u}}{E} - 0.002 - \frac{\sigma_{0.2}}{E}}$$

(14)

where $\delta$ is the residual plastic deformation after the sample breaks. The value of the tangent modulus $E^*(\sigma_{0.2})$ is taken equal $\tan \beta$ and is found from the expression

$$E^*(\sigma_{0.2}) = \tan \beta = \frac{\sigma_{0.4} - \sigma_{pr}}{0.004 + \frac{\sigma_{0.4} - \sigma_{pr}}{E}}$$

(15)

The approximation coefficients (13) are found from the conditions:
Let us calculate the values $a$, $b$, $c$ for steel 09G2S whose physical and mechanical properties are given in [4]:

$\sigma_{pr} = 200$ MPa; $\sigma_{0.2} = 305$ MPa; $\sigma_{0.4} = 306.48$ MPa; $E = 200000$ MPa; $\delta = 21\% (0.21)$.

Using (14), (15), we will receive: $E'(\sigma_{0.2}) = 0.235 \cdot 10^3$ MPa. Solving the system (16), we get:

$a = 4703.33; b = -24.8888; c = 0.0318787$.

The approximation of the function by the relation (13) allows determining with a high accuracy the tangent module in the interval $\sigma_{pr} \leq \sigma \leq \sigma_{0.2}$. In the interval $\sigma_{0.2} < \sigma < \sigma_u$ it is proposed to use linear approximation:

$$E'(\sigma) = E'(\sigma_{0.2}) \cdot \left(1 - \frac{\sigma - \sigma_{0.2}}{\sigma_u - \sigma_{0.2}}\right)$$

Consider an example of calculation from the point of view of selection of sections from the stability condition by the proposed method. Let the given load parameter for the rod, considered in case 2, be $P = 0.5P_{\text{lim}} = 0.5 \cdot 1649.4 = 824.7$ kN. Estimated resistance of steel 09G2S is assumed to be $254.17$ MPa.

In this example, the process of selection of sections according to the conditions of stability co-went at iteration 7: In a considered example process of selection of sections (Figure 2) on sites 1, 2 on stability conditions has converged on iteration 7. Results of calculation have appeared the following:

$\varphi_1 = 0.4587; \varphi_2 = 0.4542; \varphi_3 = 0.7703; \varphi_4 = 0.7635; \delta_7 = 0.0096$ m. As follows from the obtained values of the coefficients, the conditions $|\varphi_1 - \varphi_4| < \varepsilon = 0.015; |\varphi_3 - \varphi_2| < \varepsilon = 0.015$ for completing the search are fulfilled. The tangent modules in the sections are equal $E_{\gamma}^1 = 2 \cdot 10^3$ MPa; $E_{\gamma}^2 = 1.1176 \cdot 10^3$ MPa;

The obtained values of the tangent models indicate that when selecting sections from the condition of stability, the material in section 1 operates within the limits of proportionality, and in section 2 beyond the limit of proportionality.

7. Conclusions

1. Conducted studies of the stability of rods of step-variable stiffness with linearly elastically compliant bonds showed that the traditional method of the practical method for calculating the stability is not applicable due to the difficulty of determining the calculated lengths of the flexibilities of the rods.

2. A method is proposed for calculating the stability of rods with step-variable-same-stiffness with linearly elastic-responsive bonds. The technique is based on a combination of an analytical method for calculating the stability, analyzing the experimental diagram $\sigma \div \varepsilon$ for the material used in compression, and also on using a table of coefficients $\varphi$ in the bending of the rods.

3. The proposed method of calculation and the algorithm implementing it can be used (with appropriate changes and extensions) and in other (except for considered) cases of calculation for stability of rods of variable stiffness with compliant connections.

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