ON DISCRETE BOUNDARY EXTENSION OF MAPPINGS IN TERMS OF PRIME ENDS

EVGENY SEVOST’YANOV

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Abstract

We study mappings that satisfy the inverse Poletsky inequality in a domain of the Euclidean space. Under certain conditions on the definition and mapped domains, it is established that they have a continuous extension to the boundary in terms of prime ends if the majorant involved in the Poletsky inequality is integrable over spheres. Under some additional conditions, the extension mentioned above is discrete.

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1 Introduction

In the publication [SSD], we studied in sufficient detail the problem of the local and global behavior of mappings satisfying the so-called inverse Poletsky inequality, in which the corresponding majorant is integrable. In particular, the possibility of continuous extension of these mappings to the boundary of the domain was shown. In this article, we will show a little more, namely that this result holds not only for integrable $Q$, but also for those that have finite integrals on spheres centered at a fixed point on a set of radii some "not very small" measure. Let us point to examples of non-integrable functions that have these finite integrals by spheres and mappings that correspond to them (see, for example, [SevSkv3, Examples 1,2]). The key point of the manuscript is also the discreteness of the extended mappings to the boundary in terms of prime ends. We would also like to point out that the manuscript is fundamentally devoted to the study of mappings in domains with bad boundaries.

Let us turn to the definitions. In what follows, $M_p(\Gamma)$ denotes the $p$-modulus of a family $\Gamma$ (see [Val, Section 6]). We write $M(\Gamma)$ instead $M_n(\Gamma)$. Let $y_0 \in \mathbb{R}^n$, $0 < r_1 < r_2 < \infty$ and

$$A = A(y_0, r_1, r_2) = \{ y \in \mathbb{R}^n : r_1 < |y - y_0| < r_2 \} .$$
Given \( x_0 \in \mathbb{R}^n \), we put
\[
B(x_0, r) = \{ x \in \mathbb{R}^n : |x - x_0| < r \}, \quad \mathbb{B}^n = B(0, 1),
\]
\[
S(x_0, r) = \{ x \in \mathbb{R}^n : |x - x_0| = r \}.
\]

Given sets \( E, F \subset \mathbb{R}^n \) and a domain \( D \subset \mathbb{R}^n \), we denote by \( \Gamma(E, F, D) \) a family of all paths \( \gamma : [a, b] \to \mathbb{R}^n \) such that \( \gamma(a) \in E, \gamma(b) \in F \) and \( \gamma(t) \in D \) for \( t \in [a, b] \). Given a mapping \( f : D \to \mathbb{R}^n \), a point \( y_0 \in f(D) \setminus \{ \infty \} \), and \( 0 < r_1 < r_2 < r_0 = \sup_{y \in f(D)} |y - y_0| \), we denote by \( \Gamma_f(y_0, r_1, r_2) \) a family of all paths \( \gamma \) in \( D \) such that \( f(\gamma) \in \Gamma(S(y_0, r_1), S(y_0, r_2), A(y_0, r_1, r_2)) \).

Let \( Q : \mathbb{R}^n \to [0, \infty] \) be a Lebesgue measurable function. We say that \( f \) satisfies the inverse Poletsky inequality at a point \( y_0 \in f(D) \setminus \{ \infty \} \) if the relation
\[
M(\Gamma_f(y_0, r_1, r_2)) \leq \int_{A(y_0, r_1, r_2) \cap f(D)} Q(y) \cdot \eta^n(|y - y_0|) \, dm(y) \tag{1.2}
\]
holds for any Lebesgue measurable function \( \eta : (r_1, r_2) \to [0, \infty] \) such that
\[
\int_{r_1}^{r_2} \eta(r) \, dr \geq 1. \tag{1.3}
\]

Using the inversion \( \psi(y) = \frac{y}{|y|^2} \), we may also define the relation (1.2) at the point \( y_0 = \infty \). A mapping \( f : D \to \mathbb{R}^n \) is called discrete if the pre-image \( \{ f^{-1}(y) \} \) of any point \( y \in \mathbb{R}^n \) consists of isolated points, and open if the image of any open set \( U \subset D \) is an open set in \( \mathbb{R}^n \). A mapping \( f \) of \( D \) onto \( D' \) is called closed if \( f(E) \) is closed in \( D' \) for any closed set \( E \subset D \) (see, e.g., [Vu, Chapter 3]). Let \( h \) be a chordal metric in \( \mathbb{R}^n \),
\[
h(x, \infty) = \frac{1}{\sqrt{1 + |x|^2}},
\]
\[
h(x, y) = \frac{|x - y|}{\sqrt{1 + |x|^2} \sqrt{1 + |y|^2}} \quad x \neq \infty \neq y. \tag{1.4}
\]
and let \( h(E) := \sup_{x,y \in E} h(x, y) \) be a chordal diameter of a set \( E \subset \mathbb{R}^n \) (see, e.g., [Va, Definition 12.1]). Everywhere further the boundary \( \partial A \) of the set \( A \) and the closure \( \overline{A} \) should be understood in the sense extended Euclidean space \( \overline{\mathbb{R}^n} \). A continuous extension of the mapping \( f : D \to \mathbb{R}^n \) also should be understood in terms of mapping with values in \( \mathbb{R}^n \) and relative to the metric \( h \) in (1.4) (if a misunderstanding is impossible). Recall that a domain \( D \subset \mathbb{R}^n \) is called locally connected at the point \( x_0 \in \partial D \), if for any neighborhood \( U \) of a point \( x_0 \) there is a neighborhood \( V \subset U \) of \( x_0 \) such that \( V \cap D \) is connected. A domain \( D \) is locally connected at \( \partial D \), if \( D \) is locally connected at any point \( x_0 \in \partial D \). The boundary of the domain \( D \) is called weakly flat at the point \( x_0 \in \partial D \), if for any \( P > 0 \) and for any neighborhood \( U \) of a point \( x_0 \) there is a neighborhood \( V \subset U \) of the same point such that
$M(\Gamma(E, F, D)) > P$ for any continua $E, F \subset D$, which intersect $\partial U$ and $\partial V$. The boundary of the domain $D$ is called weakly flat if the corresponding property is fulfilled at any point of the boundary $D$.

Recall some definitions (see, for example, [KR1] and [KR2]). Let $\omega$ be an open set in $\mathbb{R}^k$, $k = 1, \ldots, n - 1$. A continuous mapping $\sigma : \omega \to \mathbb{R}^n$ is called a $k$-dimensional surface in $\mathbb{R}^n$. A surface is an arbitrary $(n - 1)$-dimensional surface $\sigma$ in $\mathbb{R}^n$. A surface $\sigma$ is called a Jordan surface, if $\sigma(x) \neq \sigma(y)$ for $x \neq y$. In the following, we will use $\sigma$ instead of $\sigma(\omega) \subset \mathbb{R}^n$, $\sigma$ instead of $\overline{\sigma(\omega)}$ and $\partial \sigma$ instead of $\overline{\sigma(\omega)} \setminus \sigma(\omega)$. A Jordan surface $\sigma : \omega \to D$ is called a cut of $D$, if $\sigma$ separates $D$, that is $D \setminus \sigma$ has more than one component, $\partial \sigma \cap \partial D = \emptyset$ and $\partial \sigma \cap \partial D \neq \emptyset$.

A sequence of cuts $\sigma_1, \sigma_2, \ldots, \sigma_m, \ldots$ in $D$ is called a chain, if:

(i) the set $\sigma_{m+1}$ is contained in exactly one component $d_m$ of the set $D \setminus \sigma_m$, wherein $\sigma_{m-1} \subset D \setminus (\sigma_m \cup d_m)$; (ii) $\lim_{m \to \infty} d_m = \emptyset$.

Two chains of cuts $\{\sigma_m\}$ and $\{\sigma_k\}$ are called equivalent, if for each $m = 1, 2, \ldots$ the domain $d_m$ contains all the domains $d_k$, except for a finite number, and for each $k = 1, 2, \ldots$ the domain $d_k$ also contains all domains $d_m$, except for a finite number.

The end of the domain $D$ is the class of equivalent chains of cuts in $D$. Let $K$ be the end of $D$ in $\mathbb{R}^n$, then the set $I(K) = \bigcap_{m=1}^{\infty} \overline{d_m}$ is called the impression of the end $K$. Throughout the paper, $\Gamma(E, F, D)$ denotes the family of all paths $\gamma : [a, b] \to \overline{\mathbb{R}^n}$ such that $\gamma(a) \in E$, $\gamma(b) \in F$ and $\gamma(t) \in D$ for every $t \in [a, b]$. In what follows, $M$ denotes the modulus of a family of paths, and the element $dm(x)$ corresponds to the Lebesgue measure in $\mathbb{R}^n$, $n \geq 2$, see [Na1]. Following [Na2], we say that the end $K$ is a prime end, if $K$ contains a chain of cuts $\{\sigma_m\}$ such that $\lim_{m \to \infty} M(\Gamma(C, \sigma_m, D)) = 0$ for some continuum $C$ in $D$. In the following, the following notation is used: the set of prime ends corresponding to the domain $D$, is denoted by $E_D$, and the completion of the domain $D$ by its prime ends is denoted $\overline{D}_p$.

Consider the following definition, which goes back to Näkki [Na1], see also [KR3]. We say that the boundary of the domain $D$ in $\mathbb{R}^n$ is locally quasiconformal, if each point $x_0 \in \partial D$ has a neighborhood $U$ in $\mathbb{R}^n$, which can be mapped by a quasiconformal mapping $\varphi$ onto the unit ball $\mathbb{B}^n \subset \mathbb{R}^n$ so that $\varphi(\partial D \cap U)$ is the intersection of $\mathbb{B}^n$ with the coordinate hyperplane.

For a given set $E \subset \mathbb{R}^n$, we set $d(E) := \sup_{x,y \in E} |x-y|$. The sequence of cuts $\sigma_m, m = 1, 2, \ldots$, is called regular, if $\overline{\sigma_m} \cap \overline{\sigma_{m+1}} = \emptyset$ for $m \in \mathbb{N}$ and, in addition, $d(\sigma_m) \to 0$ as $m \to \infty$. If the end $K$ contains at least one regular chain, then $K$ will be called regular. We say that a bounded domain $D$ in $\mathbb{R}^n$ is regular, if $D$ can be quasiconformally mapped to a domain with a locally quasiconformal boundary whose closure is a compact in $\mathbb{R}^n$, and, besides that, every prime end in $D$ is regular. Note that space $\overline{D}_p = D \cup E_D$ is metric, which can be demonstrated as follows. If $g : D_0 \to D$ is a quasiconformal mapping of a domain $D_0$ with
a locally quasiconformal boundary onto some domain $D$, then for $x, y \in \overline{D}_P$ we put:

$$\rho(x, y) := |g^{-1}(x) - g^{-1}(y)|,$$

(1.5)

where the element $g^{-1}(x), x \in E_D$, is to be understood as some (single) boundary point of the domain $D_0$. The specified boundary point is unique and well-defined by [IS$_2$, Theorem 2.1, Remark 2.1], cf. [Na$_2$, Theorem 4.1]. It is easy to verify that $\rho$ in (1.5) is a metric on $\overline{D}_P$, and that the topology on $\overline{D}_P$, defined by such a method, does not depend on the choice of the map $g$ with the indicated property.

We say that a sequence $x_m \in D, m = 1, 2, \ldots$, converges to a prime end of $P \in E_D$ as $m \to \infty$, if for any $k \in \mathbb{N}$ all elements $x_m$ belong to $d_k$ except for a finite number. Here $d_k$ denotes a sequence of nested domains corresponding to the definition of the prime end $P$.

Note that for a homeomorphism of a domain $D$ onto $D'$.

**Theorem 1.1.** Let $D \subset \mathbb{R}^n, n \geq 2$, be a domain with a weakly flat boundary, and let $D' \subset \mathbb{R}^n$ be a regular domain. Suppose that $f$ is open discrete and closed mapping of $D$ onto $D'$ satisfying the relation (1.2) at any point $y_0 \in \partial D'$. Suppose that, for each point $y_0 \in \partial D'$ there is $0 < r_\ast = r_\ast(y_0) < \sup_{y \in D'} |y - y_0|$ such that, for any $0 < r_1 < r_2 < r_\ast$ there is a set $E \subset [r_1, r_2]$ of a positive linear Lebesgue measure such that the function $Q$ is integrable on $S(y_0, r)$ for any $r \in E$. Then $f$ has a continuous extension $\overline{f} : \overline{D} \to \overline{D'}_P$, while $\overline{f}(D) = \overline{D'}_P$.

In [Vu], some issues related to the discreteness of a closed quasiregular map $f : \mathbb{R}^n \to \mathbb{R}^n$ in $\mathbb{R}^n$ are considered, see [Vu, Lemma 4.4, Corollary 4.5 and Theorem 4.7]. In particular, the following result holds (see [Vu, Theorem 4.7]).

**Theorem.** Let $f : \mathbb{R}^n \to G'$ be a closed non-constant quasiregular mapping and let $G'$ be locally connected on the boundary. Then $f$ can be extended to a continuous mapping $f : \overline{\mathbb{R}^n} \to \overline{\mathbb{R}^n}$ such that $N(f) = N(\overline{f})$ and hence $\overline{f}$ is discrete.

The theorem given below is devoted to a deeper study of this fact, more precisely, we extend the mentioned result not only to a wider class of mappings, but also to a wider class of domains. Here we will consider the case when this extension should be understood in terms of prime ends. Let us give some definitions.

We say that a function $\varphi : D \to \mathbb{R}$ has a finite mean oscillation at a point $x_0 \in D$, write $\varphi \in FMO(x_0)$, if

$$\limsup_{\varepsilon \to 0} \frac{1}{\Omega_n \varepsilon^n} \int_{B(x_0, \varepsilon)} |\varphi(x) - \overline{\varphi}_\varepsilon| \, dm(x) < \infty,$$

where $\overline{\varphi}_\varepsilon = \frac{1}{\Omega_n \varepsilon^n} \int_{B(x_0, \varepsilon)} \varphi(x) \, dm(x)$. We also say that a function $\varphi : D \to \mathbb{R}$ has a finite mean oscillation at $A \subset \overline{D}$, write $\varphi \in FMO(A)$, if $\varphi$ has a finite mean oscillation at any point
Let $D$, $x_0 \in A$. Let
\[ q_{x_0}(r) = \frac{1}{\omega_{n-1} r^{n-1}} \int_{S(x_0,r)} Q(y) \, dH^{n-1}(y), \]
and $\omega_{n-1}$ denotes the area of the unit sphere $\mathbb{S}^{n-1}$ in $\mathbb{R}^n$. The most important statement of the manuscript is the following.

**Theorem 1.2.** Let $n \geq 2$, let $D$ be a domain with a weakly flat boundary and let $D'$ be a regular domain. Let $f$ be open discrete and closed mapping of $D$ onto $D'$ for which there is a Lebesgue measurable function $Q : \mathbb{R}^n \to [0, \infty]$, equal to zero outside $D'$, such that the relations (1.2)–(1.3) hold at any point $y \in \partial D'$. Assume that, one of the following conditions hold:

1) $Q \in FMO(\partial D')$;

2) for any $y_0 \in \partial D'$ there is $\delta(y_0) > 0$ such that
\[ \int_{\varepsilon}^{\delta(y_0)} \frac{dt}{t q_{y_0}^{n-1}(t)} < \infty, \quad \int_{0}^{\delta(y_0)} \frac{dt}{t q_{y_0}^{n-1}(t)} = \infty \tag{1.7} \]
for sufficiently small $\varepsilon > 0$.

Then $f$ has a continuous extension $\overline{f} : \overline{D} \to \overline{D}'$ such that $N(f, D) = N(f, \overline{D}) < \infty$. In particular, $\overline{f}$ is discrete in $\overline{D}$, that is, $\overline{f}^{-1}(P_0)$ consists only from isolated points for any $P_0 \in E_{D'}$.

## 2 Proof of Theorem 1.1

Let $D \subset \mathbb{R}^n$, $f : D \to \mathbb{R}^n$ be a discrete open mapping, $\beta : [a, b) \to \mathbb{R}^n$ be a path, and $x \in f^{-1}(\beta(a))$. A path $\alpha : [a, c) \to D$ is called a maximal $f$-lifting of $\beta$ starting at $x$, if

1) $\alpha(a) = x$; (2) $f \circ \alpha = \beta|_{[a,c)}$; (3) for $c < c' \leq b$, there is no a path $\alpha' : [a, c') \to D$ such that $\alpha = \alpha'|_{[a,c)}$ and $f \circ \alpha' = \beta|_{[a,c')}$. The following statement holds (see [Ri, Corollary II.3.3]).

**Proposition 2.1.** Let $f : D \to \mathbb{R}^n$ be a discrete open mapping, $\beta : [a, b) \to \mathbb{R}^n$ be a path, and $x \in f^{-1}(\beta(a))$. Then $\beta$ has a maximal $f$-lifting starting at $x$. If $\beta : (a, b) \to f(D)$ is a path, and $x \in f^{-1}(\beta(b))$, then $\beta$ has a maximal $f$-lifting ending at $x$.

A path $\alpha : [a, b) \to D$ is called a total $f$-lifting of $\beta$ starting at $x$, if

1) $\alpha(a) = x$; (2) $(f \circ \alpha)(t) = \beta(t)$ for any $t \in [a, b)$. In the case when the mapping $f$ is also closed, we have a strengthened version of Proposition 2.1 (see, for example, [Yu, Lemma 3.7]).

**Proposition 2.2.** Let $f : D \to \mathbb{R}^n$ be a discrete open and closed mapping, $\beta : [a, b) \to f(D)$ be a path, and $x \in f^{-1}(\beta(a))$. Then $\beta$ has a total $f$-lifting starting at $x$.

**Proof of Theorem 1.1.** We carry out the proof according to a scheme similar to the proof of Theorem 1 in [Sev2]. Fix $x_0 \in \partial D$. It is necessary to show the possibility of
continuous extension of the mapping \( f \) to the point \( x_0 \). Using, if necessary, the transformation \( \varphi : \infty \mapsto 0 \) and taking into account the invariance of the modulus \( M \) in the left part of the relation \( \varphi : \infty \mapsto 0 \) (see \[Va\] Theorem 8.1), we may assume that \( x_0 \neq \infty \).

Assume that the conclusion about the continuous extension of the mapping \( f \) to the point \( x_0 \) is not correct. Then any prime end \( P_0 \in E_{D'} \) is not a limit of \( f \) at \( x_0 \), in other words, there is a sequence \( x_k, k = 1, 2, \ldots, x_k \to x_0 \) as \( k \to \infty \) and a number \( \varepsilon_0 > 0 \) such that \( \rho(f(x_k), P_0) \geq \varepsilon_0 \) for any \( k \in \mathbb{N} \), where \( \rho \) is one of the metrics in (1.5). Since \( D' \) is a regular domain by the assumption, it may be mapped on some bounded domain \( D_* \) with a locally quasiconformal boundary using some a mapping \( h : D' \to D_* \). Note that, there is a one-to-one correspondence between boundary points and prime ends of domains with locally quasiconformal boundaries (see, e.g., [IS, Theorem 2.1]; cf. [Na, Theorem 4.1]). Since \( D_* \) is a compactum in \( \mathbb{R}^n \), we conclude from the above that a metric space \( (\overline{D'}_p, \rho) \) is compact. Thus, we may assume that \( f(x_k) \) converges to some element \( P_1 \neq P_0, P_1 \in \overline{D'}_p \) as \( k \to \infty \). Since, by the assumption, \( f \) has no a limit at \( x_0 \), there is at least one a sequence \( y_k \to x_0 \) as \( k \to \infty \) such that \( \rho(f(y_k), P_1) \geq \varepsilon_1 \) for any \( k \in \mathbb{N} \) and some \( \varepsilon_1 > 0 \). Again, since the metric space \( (\overline{D'}_p, \rho) \) is compact, we may assume that \( f(y_k) \to P_2 \) as \( k \to \infty \), \( P_1 \neq P_2, P_2 \in \overline{D'}_p \). Since \( f \) is closed, it preserves the boundary of a domain, see \[Va\] Theorem 3.3. Thus, \( P_1, P_2 \in E_{D'} \).

Let \( \sigma_m \) and let \( \sigma'_m, m = 0, 1, 2, \ldots, \) be a sequence of cuts corresponding to prime ends \( P_1 \) and \( P_2 \), respectively. Let also cuts \( \sigma_m, m = 0, 1, 2, \ldots, \) lie on spheres \( S(z_0, r_m) \) centered at a point \( z_0 \in \partial D' \), where \( r_m \to 0 \) as \( m \to \infty \) (such a sequence \( \sigma_m \) exists by [IS, Lemma 3.1], cf. [KR, Lemma 1]). We may assume that \( r_0 < r_* = r_*(z_0) \), where \( r_* \) is the number from conditions of the theorem. Let \( d_m \) and \( g_m, m = 0, 1, 2, \ldots, \) be sequences of domains in \( D' \) corresponding to cuts \( \sigma_m \) and \( \sigma'_m \), respectively. Since \( (\overline{D'}_p, \rho) \) is a metric space, we may consider that \( d_m \) and \( g_m \) disjoint for any \( m = 0, 1, 2, \ldots, \), in particular,

\[
d_0 \cap g_0 = \emptyset. \tag{2.1}
\]

Since \( f(x_k) \) converges to \( P_1 \) as \( k \to \infty \), for any \( m \in \mathbb{N} \) there is \( k = k(m) \) such that \( f(x_k) \in d_m \) for \( k \geq k = k(m) \). By renumbering the sequence \( x_k \) if necessary, we may assume that \( f(x_k) \in d_k \) for any natural \( k \). Similarly, we may assume that \( f(y_k) \in g_k \) for any \( k \in \mathbb{N} \). Fix \( f(x_1) \) and \( f(y_1) \). Since, by the definition of a prime end, \( \bigcap_{k=1}^{\infty} d_k = \bigcap_{l=1}^{\infty} g_l = \emptyset \), there are numbers \( k_1 \) and \( k_2 \in \mathbb{N} \) such that \( f(x_1) \notin d_{k_1} \) and \( f(y_1) \notin g_{k_2} \). Since, by the definition, \( d_k \subset d_{k_0} \) for any \( k \geq k_1 \) and \( g_k \subset g_{k_2} \) for \( k \geq k_2 \), we obtain that

\[
f(x_1) \notin d_k, \quad f(y_1) \notin g_k, \quad k \geq \max\{k_1, k_2\}. \tag{2.2}
\]

Let \( \gamma_k \) be a path joining \( f(x_1) \) and \( f(x_k) \) in \( d_1 \), and let \( \gamma'_k \) be a path joining \( f(y_1) \) and \( f(y_k) \) in \( g_1 \). Let also \( \alpha_k \) and \( \beta_k \) be total \( f \)-liftings of \( \gamma_k \) and \( \gamma'_k \) in \( D \) starting at \( x_k \) and \( y_k \), respectively (such liftings exist by Proposition 2.2). Note that the points \( f(x_1) \) and \( f(y_1) \)
may have no more than a finite number of pre-images under the mapping $f$ in the domain $D$, see [Vui Lemma 3.2]. Then there exists $R_0 > 0$ such that $\alpha_k(1), \beta_k(1) \in D \setminus B(x_0, R_0)$ for any $k = 1, 2, \ldots$. Since the boundary of $D$ is weakly flat, for any $P > 0$ there is $i = i_P \geq 1$ such that

$$M(\Gamma(|\alpha_k|, |\beta_k|, D)) > P \quad \forall \ k \geq k_P. \quad (2.3)$$

Let us to show that, the condition (2.3) contradicts the definition of $f$ in (1.2). Indeed, let $\gamma : [0, 1] \to D, \gamma(0) \in |\alpha_k|$ and $\gamma(1) \in |\beta_k|$. In particular, $f(\gamma(0)) \in |\gamma_k|$ and $f(\gamma(1)) \in |\gamma'_k|$. In this case, it follows from the relations (2.1) and (2.3) that $|f(\gamma)| \cap d_1 \neq \emptyset \neq |f(\gamma)| \cap (D' \setminus d_1)$ for $k \geq \max\{k_1, k_2\}$. By [Ku] Theorem I.5.46 $|f(\gamma)| \cap d_1 \neq \emptyset$, in other words, $|f(\gamma)| \cap S(z_0, r_1) \neq \emptyset$, because $\partial d_1 \cap D' \subset \sigma_1 \subset S(z_0, r_1)$ by the definition of a cut $\sigma_1$. Let $t_1 \in (0, 1)$ be such that $f(\gamma(t_1)) \in S(z_0, r_1)$ and $f(\gamma)|_{t_1} := f(\gamma)|_{[t_1, 1]}$. Without loss of generality, we may assume that $f(\gamma)|_{1} \subset \mathbb{R}^n \setminus B(z_0, r_1)$. Arguing similarly for a path $f(\gamma)|_{1}$, we may find a point $t_2 \in (t_1, 1)$ such that $f(\gamma(t_2)) \in S(z_0, r_0)$. Put $f(\gamma)|_{2} := f(\gamma)|_{[t_1, t_2]}$. Then $f(\gamma)|_{2}$ is a subpath of $f(\gamma)$ and, in addition, $f(\gamma)|_{2} \in \Gamma(S(z_0, r_1), S(z_0, r_0), D')$. Without loss of generality, we may assume that $f(\gamma)|_{2} \subset B(z_0, r_0)$. Therefore, $\Gamma(|\alpha_k|, |\beta_k|, D) > \Gamma_f(z_0, r_1, r_0)$. From the latter relation, due to the minority of the modulus of families of paths (see e.g. [Fu Theorem 1(c)]) we obtain that

$$M(\Gamma(|\alpha_k|, |\beta_k|, D)) \leq M(\Gamma_f(z_0, r_1, r_0)). \quad (2.4)$$

Combining (2.4) with (1.2), we obtain that

$$M(\Gamma(|\alpha_k|, |\beta_k|, D)) \leq \int_{A(y_0, r_1, r_0) \cap f(D)} Q(y) \cdot \eta^n(|y - y_0|) \ dm(y), \quad (2.5)$$

where $\eta : (r_1, r_2) \to [0, \infty]$ is any Lebesgue measurable function with $\int_{r_1}^{r_0} \eta(r) \ dr \geq 1$.

Below we use the following conventions: $a/\infty = 0$ for $a \neq \infty$, $a/0 = \infty$ for $a > 0$ and $0 \cdot \infty = 0$ (see, e.g., [Sa 3.I]). Put

$$I = \int_{r_2}^{r_0} \frac{dt}{\frac{1}{tq_0^n} \cdot \frac{1}{tq_0^{n-1}}(1)} \cdot (2.6)$$

By the assumption, there is a set $E \subset [r_1, r_0]$ of a positive measure such that $q_{z_0}(t)$ is finite for all $t \in E$. In this case, a function $\eta_0(t) = \frac{1}{tq_0^n} \cdot \frac{1}{tq_0^{n-1}}(1)$ satisfies the relation (1.3). Substituting this function in the right-hand part of (2.5) and using the Fubini theorem, we obtain that

$$M(\Gamma(|\alpha_k|, |\beta_k|, D)) \leq \frac{\omega_{n-1}}{t^{n-1}} < \infty. \quad (2.7)$$

The relation (2.7) contradicts with (2.3). The contradiction obtained above disproves the assumption on the absence of a continuous extension of the mapping $f$ to the boundary of the domain $D$. The proof of the equality $\overline{f(D)} = \overline{D'}$ is similar to the second part of the proof of Theorem 3.1 in [SSD]. □
Remark 2.1. The statement of Theorem 1.1 remains true, if in its formulation instead of the specified conditions on function $Q$ to require that $Q \in L^1_{\text{loc}}(\mathbb{R}^n)$, $Q(y) \equiv 0$ for $y \in \mathbb{R}^n \setminus f(D)$. Indeed, by [Sa, Theorem III.8.1], for any point $y_0 \in \mathbb{R}^n$

$$\int_{\varepsilon_1}^{\varepsilon_2} \int_{S(y_0, r)} Q(y) \, dA \, dr = \int_{\varepsilon_1 <|y-y_0|<\varepsilon_2} Q(y) \, dm(y) \quad (2.8)$$

for any $0 \leq \varepsilon_1 < \varepsilon_2$. By (2.8), it follows that $q(y_0(r)) < \infty$ for $\varepsilon_1 < r < \varepsilon_2$.

Remark 2.2. The statement of Theorem 1.1 remains true, if in its formulation instead of the specified conditions on function $Q$ to require that, for any $y_0 \in \partial D'$ there is $\delta(y_0) > 0$ such that

$$\int_{0}^{\delta(y_0)} \frac{dt}{t q(y_0)^{n-1}(t)} = \infty, \quad \int_{\varepsilon_0}^{\delta(y_0)} \frac{dt}{t q(y_0)^{n-1}(t)} < \infty \quad (2.9)$$

for sufficiently small $\varepsilon > 0$. This statement may be proved by the choosing of the admissible function $\eta$ in (2.5) and by the using the fact that the second condition in (2.9) is possible only if the inequality $q(y_0(t)) < \infty$ holds for some set $E \subset [\varepsilon, \delta(y_0)]$ of a positive linear measure.

Remark 2.3. The statement of Theorem 1.1 remains true, if in its formulation instead of the specified conditions on function $Q$ to require that, for any $y_0 \in \partial D'$ there is $\varepsilon_0 = \varepsilon_0(y_0) > 0$ and a Lebesgue measurable function $\psi : (0, \varepsilon_0) \to [0, \infty]$ such that

$$I(\varepsilon, \varepsilon_0) := \int_{\varepsilon}^{\varepsilon_0} \psi(t) \, dt < \infty \quad \forall \varepsilon \in (0, \varepsilon_0), \quad I(\varepsilon, \varepsilon_0) > 0 \quad \text{as} \quad \varepsilon \to 0, \quad (2.10)$$

and, in addition,

$$\int_{A(y_0, \varepsilon, \varepsilon_0)} Q(x) \cdot \psi^n(|y-y_0|) \, dm(y) \leq C_0 I^n(\varepsilon, \varepsilon_0), \quad (2.11)$$

as $\varepsilon \to 0$, where $C_0$ is some constant, and $A(y_0, \varepsilon, \varepsilon_0)$ is defined in (1.1).

Indeed, literally repeating the proof of the statement given in Theorem 1.1 to the ratio (2.5) inclusive, we put

$$\eta(t) = \begin{cases} \psi(t)/I(r_1, r_0), & t \in (r_1, r_0), \\ 0, & t \notin (r_1, r_0), \end{cases}$$

where $I(r_1, \varepsilon_0) = \int_{r_1}^{\varepsilon_0} \psi(t) \, dt$. Observe that $\int_{r_1}^{\varepsilon_0} \eta(t) \, dt = 1$. Now, by the definition of $f$ in (1.2) and due to the relation (2.5) we obtain that

$$M(\Gamma(|\vec{\alpha}|, |\vec{\beta}|, D)) \leq C_0 < \infty. \quad (2.12)$$

The relation (2.12) contradicts with (2.3). The resulting contradiction proves the desired statement. □
3 On the discreteness of mappings with the inverse Poletsky inequality at the boundary of a domain

In [Vu], some issues related to the discreteness of a closed quasiregular map \( f : \mathbb{B}^n \to \mathbb{R}^n \) in \( \mathbb{B}^n \) are considered, see [Vu, Lemma 4.4, Corollary 4.5 and Theorem 4.7]. In this section we talk about the discreteness of mappings that satisfy the condition (1.2). Among other things, we note that we are primarily interested here in the case when the mapped domain has a bad boundary.

We will say that \( f \) satisfies the inverse Poletsky inequality at a point \( y_0 \in f(D) \setminus \{ \infty \} \) relative to \( p \)-modulus, if the relation

\[
M_p(\Gamma_f(y_0, r_1, r_2)) \leq \int_{A(y_0, r_1, r_2) \cap f(D)} Q(y) \cdot \eta^p(|y - y_0|) \, dm(y)
\]

holds for any Lebesgue measurable function \( \eta : (r_1, r_2) \to [0, \infty] \) such that

\[
\int_{r_1}^{r_2} \eta(r) \, dr \geq 1.
\]

Using the inversion \( \psi(y) = \frac{y}{|y|^2} \), we also may defined the relation (3.1) at the point \( y_0 = \infty \).

Following [NP, Section 2.4], we say that a domain \( D \subset \mathbb{R}^n, n \geq 2 \), is uniform with respect to \( p \)-modulus, if for any \( r > 0 \) there is \( \delta > 0 \) such that the inequality

\[
M_p(\Gamma(F^*, F, D)) \geq \delta
\]

holds for any continua \( F, F^* \subset D \) with \( h(F) \geq r \) and \( h(F^*) \geq r \). When \( p = n \), the prefix "relative to \( p \)-modulus" is omitted. Note that this is the definition slightly different from the "classical" given in [NP, Chapter 2.4], where the sets \( F \) and \( F^* \subset D \) are assumed to be arbitrary connected. We prove the following statement (see its analogue for quasiregular mappings of the unit ball in [Vu, Lemma 4.4]).

**Lemma 3.1.** Let \( n \geq 2, n - 1 < p \leq n \), let \( D \) be a domain which is uniform with respect to \( p \)-modulus, and let \( D' \) be a regular domain. Let \( f : D \to \mathbb{R}^n \) be an open discrete and closed mapping in \( D \), for which there is a Lebesgue measurable function \( Q : \mathbb{R}^n \to [0, \infty] \), equals to zero outside of \( D' \), such that the relations (3.1)–(3.2) hold for any \( y_0 \in \partial D' \). Assume that, for any \( y_0 \in \partial D' \) there is \( \varepsilon_0 = \varepsilon_0(y_0) > 0 \) and a Lebesgue measurable function \( \psi : (0, \varepsilon_0) \to [0, \infty] \) such that

\[
I(\varepsilon, \varepsilon_0) := \int_{\varepsilon}^{\varepsilon_0} \psi(t) \, dt < \infty \quad \forall \, \varepsilon \in (0, \varepsilon_0), \quad I(\varepsilon, \varepsilon_0) \to \infty \quad \text{as} \quad \varepsilon \to 0,
\]

and, in addition,

\[
\int_{A(y_0, \varepsilon, \varepsilon_0)} Q(y) \cdot \psi^p(|y - y_0|) \, dm(y) = o(I^p(\varepsilon, \varepsilon_0)),
\]
as $\varepsilon \to 0$, where $A(y_0, \varepsilon, \varepsilon_0)$ is defined in (1.1). Let $C_j$, $j = 1, 2, \ldots$, be a sequence of continua such that $h(C_j) \geq \delta > 0$ for some $\delta > 0$ and any $j \in \mathbb{N}$ and, in addition, $\rho(f(C_j)) \to 0$ as $j \to \infty$. Then there is $\delta_1 > 0$ such that $\rho(f(C_j), P_0) \geq \delta_1 > 0$ for any $j \in \mathbb{N}$ and for any $P_0 \in E_{D'}$, where the metrics $\rho$ is defined in (1.2).

Here, as usually,

$$\rho(A) = \sup_{x,y \in A} \rho(x,y),$$
$$\rho(A, B) = \inf_{x \in A, y \in B} \rho(x,y).$$

Proof. Suppose the opposite, namely, let $\rho(f(C_{j_k}), P_0) \to 0$ as $k \to \infty$ for some $P_0 \in E_{D'}$ and for some increasing sequence of numbers $j_k$, $k = 1, 2, \ldots$. Let $F \subset D$ be any continuum in $D$, and let $\Gamma_k := \Gamma(F, C_{j_k}, D)$. Due to the definition of the uniformity of the domain with respect to $p$-modulus, we obtain that

$$M_p(\Gamma_k) \geq \delta_2 > 0$$

(3.6)

for any $k \in \mathbb{N}$ and some $\delta_2 > 0$. On the other hand, let us to consider the family of paths $f(\Gamma_k)$. Let $d_l$, $l = 1, 2, \ldots$, be a sequence of domains which corresponds to the prime end $P_0$, and let $\sigma_l$ be a cut corresponding to $d_l$. We may assume that $\sigma_l$, $l = 1, 2, \ldots$, lie on spheres $S(y_0, r_l)$ centered at some point $y_0 \in \partial D'$, where $r_l \to 0$ as $l \to \infty$ (see [IS2, Lemma 3.1], cf. [KR2, Lemma 1]).

Let us to prove that, for any $l \in \mathbb{N}$ there is a number $k = k_l$ such that

$$f(C_{j_k}) \subset d_l,$$

$$k \geq k_l.$$  

(3.7)

Suppose the opposite. Then there is $l_0 \in \mathbb{N}$ such that

$$f(C_{j_{m_l}}) \cap (\mathbb{R}^n \setminus d_{l_0}) \neq \emptyset$$

(3.8)

for some increasing sequence of numbers $m_l$, $l = 1, 2, \ldots$. In this case, there is a sequence $x_{m_l} \in f(C_{j_{m_l}}) \cap (\mathbb{R}^n \setminus d_{l_0})$, $l \in \mathbb{N}$. Since by the assumption $\rho(f(C_{j_k}), P_0) \to 0$ for some sequence of numbers $j_k$, $k = 1, 2, \ldots$, we obtain that

$$\rho(f(C_{j_{m_l}}), P_0) \to 0 \quad \text{as} \quad l \to \infty.$$  

(3.9)

Since $\rho(f(C_{j_{m_l}}), P_0) = \inf_{y \in f(C_{j_{m_l}})} h(y, P_0)$ and $f(C_{j_{m_l}})$ is a compact set in $\overline{D'}_p$ as a continuous image of the compactum $C_{j_{m_l}}$ under the mapping $f$, it follows that $\rho(f(C_{j_{m_l}}), P_0) = \rho(y_l, P_0)$, where $y_l \in f(C_{j_{m_l}})$. Due to the relation (3.9) we obtain that $y_l \to y_0$ as $l \to \infty$ in the metric $\rho$. Since by the assumption $\rho(f(C_j)) = \sup_{y, z \in f(C_j)} \rho(y, z) \to 0$ as $j \to \infty$, we have that $\rho(y_l, x_{m_l}) \leq \rho(f(C_{j_{m_l}})) \to 0$ as $l \to \infty$. Now, by the triangle inequality, we obtain that

$$\rho(x_{m_l}, P_0) \leq \rho(x_{m_l}, y_l) + \rho(y_l, P_0) \to 0 \quad \text{as} \quad l \to \infty.$$
The latter contradicts with (3.8). The contradiction obtained above proves (3.7).

The following considerations are similar to the second part of the proof of Lemma 2.1 in\textsuperscript{[Sev1]}. Without loss of generality we may consider that the number \( l_0 \in \mathbb{N} \) is such that \( r_1 < \varepsilon_0 \) for any \( l \geq l_0 \), and

\[
f(F) \subset \mathbb{R}^n \setminus d_1.
\]  

(3.10)

In this case, we observe that, for \( l \geq 2 \)

\[
f(\Gamma_{ki}) > \Gamma(S(y_0, r_i), S(y_0, r_1), A(y_0, r_1, r_i)) \). 
\]  

(3.11)

Indeed, let \( \tilde{\gamma} \in f(\Gamma_{ki}) \). Then \( \tilde{\gamma}(t) = f(\gamma(t)) \), where \( \gamma \in \Gamma_{ki}, \gamma : [0, 1] \to D, \gamma(0) \in F \), \( \gamma(1) \in C_{j_{k1}} \). Due to the relation (3.10), we obtain that \( f(\gamma(0)) \in f(F) \subset \mathbb{R}^n \setminus B(y_0, \varepsilon_0) \). On the other hand, by (3.7), \( \gamma(1) \in C_{j_{k1}} \subset d_t \subset d_1 \). Thus, \( |f(\gamma(t))| \cap d_t \neq \emptyset \neq |f(\gamma(t))| \cap (\mathbb{R}^n \setminus d_1) \). Now, by [Ku, Theorem 1.5.16] we obtain that, there is \( 0 < t_1 < 1 \) such that \( f(\gamma(t_1)) \in \partial d_t \cap D \subset S(y_0, r_1) \). Set \( \gamma_1 := \gamma|_{[t_1, 1]} \). We may consider that \( f(\gamma(t)) \in d_t \) for any \( t \geq t_1 \). Arguing similarly, we obtain \( t_2 \in [t_1, 1] \) such that \( f(\gamma(t_2)) \in S(y_0, r_1) \). Put \( \gamma_2 := \gamma|_{[t_1, 1]} \). We may consider that \( f(\gamma(t)) \in d_t \) for any \( t \in [t_1, t_2] \). Now, a path \( f(\gamma_2) \) is a subpath of \( f(\gamma) = \tilde{\gamma} \), which belongs to \( \Gamma(S(y_0, r_1), S(y_0, r_1), A(y_0, r_1, r_1)) \). The relation (3.11) is established.

It follows from (3.11) that

\[
\Gamma_{ki} > \Gamma_f(S(y_0, r_1), S(y_0, r_1), A(y_0, r_1, r_1)) \). 
\]  

(3.12)

Set

\[
\eta(t) = \begin{cases} 
\psi(t)/I(r_1, r_1), & t \in (r_1, r_1), \\
0, & t \notin (r_1, r_1),
\end{cases}
\]

where \( I(r_1, r_1) = \int_{r_1}^{r_1} \psi(t) \, dt \). Observe that \( \int_{r_1}^{r_1} \eta(t) \, dt = 1 \). Now, by the relations (3.5) and (3.12), and due to the definition of \( f \) in (3.1), we obtain that

\[
M_p(\Gamma_{ki}) \leq M_p(\Gamma_f(S(y_0, r_1), S(y_0, r_1), A(y_0, r_1, r_1))) \leq \frac{1}{I_p(r_1, r_1)} \int_{A(y_0, r_1, r_1)} Q(y) \cdot \psi_p(\|y - y_0\|) \, dm(y) \to 0 \quad \text{as} \quad l \to \infty.
\]  

(3.13)

The relation (3.13) contradicts with (3.6). The contradiction obtained above proves the lemma. \( \square \)

**Corollary 3.1.** The statement of Lemma 3.7 is fulfilled if we put \( D = \mathbb{B}^n \).

**Proof.** Obviously, the domain \( D = \mathbb{B}^n \) is locally connected at its boundary. We prove that this domain is uniform with respect to the \( p \)-modulus for \( p \in (n - 1, n) \). Indeed, since \( \mathbb{B}^n \) is a Loewner space (see [He, Example 8.24(a)]), the set \( \mathbb{B}^n \) is Ahlfors regular with respect to the Euclidean metric \( d \) and Lebesgue measure in \( \mathbb{R}^n \) (see [He, Proposition 8.19]). In addition,
in \( \mathbb{B}^n \), \((1;p)\)-Poincaré inequality holds for any \( p \geq 1 \) (see e.g. [HaK, Theorem 10.5]). Now, by [AS, Proposition 4.7] we obtain that the relation

\begin{equation}
M_p(\Gamma(E, F, \mathbb{B}^n)) \geq \frac{1}{C} \min\{\text{diam } E, \text{diam } F\},
\end{equation}

holds for any \( n - 1 < p \leq n \) and for any continua \( E, F \subset \mathbb{B}^n \), where \( C > 0 \) is some constant, and diam denotes the Euclidean diameter. Since the Euclidean distance is equivalent to the chordal distance on bounded sets, the uniformity of the domain \( D = \mathbb{B}^n \) with respect to the \( p \)-modulus follows directly from (3.14). □

We need the following statement (see [Na1, Theorem 4.2]).

**Proposition 3.1.** Let \( \mathcal{F} \) be a family of connected sets in \( D \) such that \( \inf_{F \in \mathcal{F}} h(F) > 0 \), and let \( \inf_{F \in \mathcal{F}} M(\Gamma(F, A, D)) > 0 \) for some continuum \( A \subset D \). Then

\[ \inf_{F, F' \in \mathcal{F}} M(\Gamma(F, F', D)) > 0. \]

Let \( p \geq 1 \). Due to [MRSY, Section 3] we say that a boundary \( D \) is called strongly accessible with respect to \( p \)-modulus at \( x_0 \in \partial D \), if for any neighborhood \( U \) of the point \( x_0 \in \partial D \) there is a neighborhood \( V \subset U \) of this point, a compactum \( F \subset D \) and a number \( \delta > 0 \) such that \( M_p(\Gamma(F, D)) \geq \delta \) for any continua \( E \subset D \) such that \( E \cap \partial U \neq \emptyset \neq E \cap \partial V \). The boundary of a domain \( D \) is called strongly accessible with respect to \( p \)-modulus, if this is true for any \( x_0 \in \partial D \). When \( p = n \), prefix “relative to \( p \)-modulus” is omitted. The following lemma is valid (see the statement similar in content to [Na1, Theorem 6.2]).

**Lemma 3.2.** A domain \( D \subset \mathbb{R}^n \) has a strongly accessible boundary if and only if \( D \) is uniform.

**Proof.** The fact that uniform domains have strongly accessible boundaries has been proved in [SevSkv1, Remark 1]. It remains to prove that domains with strongly accessible boundaries are uniform.

We will prove this statement from the opposite. Let \( D \) be a domain which has a strongly accessible boundary, but it is not uniform. Then there is \( r > 0 \) such that, for any \( k \in \mathbb{N} \) there are continua \( F_k \) and \( F_k^* \subset D \) such that \( h(F_k) \geq r \), \( h(F_k^*) \geq r \), however,

\[ M(\Gamma(F_k, F_k^*, D)) < 1/k. \]  

(3.15)

Let \( x_k \in F_k \). Since \( \overline{D} \) is compact in \( \overline{\mathbb{R}^n} \), we may assume that \( x_k \to x_0 \in \overline{D} \). Note that the strongly accessibility of the domain \( D \) at the boundary points is assumed to be, and at the inner points it is even weakly flat, which is the result of Väisälä’s lemma (see e.g. [Va, Sect. 10.12], cf. [SevSkv2, Lemma 2.2]). Let \( U \) be a neighborhood of the point \( x_0 \) such that \( h(x_0, \partial U) \leq r/2 \). Then there is a neighborhood \( V \subset U \), a compactum \( F \subset D \) and a number \( \delta > 0 \) such that the relation \( M(\Gamma(F, D)) \geq \delta \) holds for any continuum \( E \subset D \) such that \( E \cap \partial U \neq \emptyset \neq E \cap \partial V \). By the choice of the neighborhood \( U \), we obtain that
Given a mapping $f : D \to \mathbb{R}^n$, a set $E \subset D$ and $y \in \mathbb{R}^n$, we define the multiplicity function $N(y, f, E)$ as a number of preimages of the point $y$ in a set $E$, i.e.

$$N(y, f, E) = \text{card } \{x \in E : f(x) = y\},$$

$$N(f, E) = \sup_{y \in \mathbb{R}^n} N(y, f, E). \quad (3.18)$$

Note that, the concept of a multiplicity function may also be extended to sets belonging to the closure of a given domain. Finally, we formulate and prove a key statement about the discreteness of mapping (see [Vu, Theorem 4.7]).

**Lemma 3.3.** Suppose that, $p = n$, the domain $D$ is weakly flat and the domain $D'$ is regular. Assume that, there is a Lebesgue measurable function $Q : \mathbb{R}^n \to [0, \infty]$, equals to zero outside of $D'$, such that the relations (3.1)–(3.2) hold for any $y_0 \in \partial D'$. Assume that, for any $y_0 \in \partial D'$ there is $\varepsilon_0 = \varepsilon_0(y_0) > 0$ and a Lebesgue measurable function $\psi : (0, \varepsilon_0) \to [0, \infty]$ such that (3.4)–(3.5) as $\varepsilon \to 0$, where $A(y_0, \varepsilon, \varepsilon_0)$ is defined in (1.1).

Then the mapping $f$ has a continuous extension $\overline{f} : \overline{D} \to \overline{D'}_p$ such that $N(f, D) = N(f, D) < \infty$. In particular, $\overline{f}$ is discrete in $\overline{D}$, that is, $\overline{f}^{-1}(P_0)$ consists only from isolated points for any $P_0 \subset E_{D'}$.

**Proof.** First of all, the possibility of continuous extension of $f$ to a mapping $\overline{f} : \overline{D} \to \overline{D'}_p$ follows by Remark [2.3]. Note also that $N(f, D) < \infty$, see [MS, Theorem 2.8]. Let us to prove that $N(f, D) = N(f, D)$. Next we will reason using the scheme proof of Theorem 4.7 in [Vu]. Assume the contrary. Then there are points $P_0 \in E_{D'}$ and $x_1, x_2, \ldots, x_k, x_{k+1} \in \partial D$ such
that \( f(x_i) = P_0, i = 1, 2, \ldots, k+1 \) and \( k := N(f, D) \). Since by the assumption \( D' \) is regular, there is a mapping \( g \) of some domain with a locally quasiconformal boundary \( D_0 \) onto \( D' \). Let us consider the mapping \( F := f \circ g^{-1} \). Note that, by the definition of a domain with a locally quasiconformal boundary, \( D_0 \) is locally connected on \( \partial D_0 \). Note that, the mapping \( F \) has a continuous extension \( \ov{F} : \ov{D} \to \ov{D}_0 \) to \( \ov{D} \), and \( \ov{F}(D) = \ov{D}_0 \) (this follows from the fact that each of the mappings \( f \) and \( g^{-1} \) has a continuous extension to \( \ov{D} \) and \( \ov{D'}_p \), respectively).

Set \( y_0 := \ov{F}(P_0) \in D_0 \). Now, for any \( p \in \mathbb{N} \) there is a neighborhood \( \ov{U}^i_p \subset B(y_0, 1/p) \) of \( y_0 \) such that the set \( \ov{U}^i_p \cap D_0 = U^i_p \) is connected.

Let us to prove that, for any \( i = 1, 2, \ldots, k+1 \) there is a component \( V^i_p \) of the set \( F^{-1}(U^i_p) \) such that \( x_i \in \ov{V}^i_p \). Fix \( i = 1, 2, \ldots, k+1 \). By the continuity of \( F \) in \( \ov{D} \), there is \( r_i = r_i(x_i) > 0 \) such that \( f(B(x_i, r_i) \cap D) \subset U^i_p \). By [MRSY] Lemma 3.15, a domain with a weakly flat boundary is locally connected on its boundary. Thus, we may find a neighborhood \( W_i \subset B(x_i, r_i) \) of the point \( x_i \) such that \( W_i \cap D \) is connected. Then \( W_i \cap D \) belongs to one and only one component \( V^i_p \) of the set \( F^{-1}(U^i_p) \), while \( x_i \in W_i \cap D \subset \ov{V}^i_p \), as required.

Next we show that the sets \( \ov{V}^i_p \) are disjoint for any \( i = 1, 2, \ldots, k+1 \) and large enough \( p \in \mathbb{N} \). In turn, we prove for this that \( h(\ov{V}^i_p) \to 0 \) as \( p \to \infty \) for each fixed \( i = 1, 2, \ldots, k+1 \). Let us prove the opposite. Then there is \( 1 \leq i_0 \leq k+1 \), a number \( r_0 > 0 \), \( r_0 < \frac{1}{2} \min_{1 \leq i,j \leq k+1, i \neq j} h(x_i, x_j) \)

and an increasing sequence of numbers \( p_m, m = 1, 2, \ldots, \) such that \( S_h(x_{i_0}, r_0) \cap \ov{V}^{i_0}_{p_m} \neq \emptyset \), where \( S_h(x_0, r) = \{ x \in \mathbb{R}^n : h(x, x_0) = r \} \), and \( h \) denotes the chordal metric in \( \mathbb{R}^n \). In this case, there are \( a_m, b_m \in \ov{V}^{i_0}_{p_m} \) such that \( a_m \to x_{i_0} \) as \( m \to \infty \) and \( h(a_m, b_m) \geq r_0/2 \). Join the points \( a_m \) and \( b_m \) by a path \( C_m \), which entirely belongs to \( \ov{V}^{i_0}_{p_m} \). Then \( h(|C_m|) \geq r_0/2 \) for \( m = 1, 2, \ldots \). On the other hand, since \( |C_m| \subset f(\ov{V}^{i_0}_{p_m}) \subset B(y_0, 1/p_m) \), then simultaneously \( h(F(|C_m|)) \to 0 \) as \( m \to \infty \) and \( h(F(|C_m|), y_0) \to 0 \) as \( m \to \infty \). Now, by the definition of the metric \( \rho \) in (1.5) and of the mapping \( g \), we obtain that \( \rho(f(|C_m|)) \to 0 \) as \( m \to \infty \) and \( \rho(f(|C_m|)), y_0) \to 0 \) as \( m \to \infty \), that contradicts with Lemma 3.3. The resulting contradiction indicates the incorrectness of the above assumption.

By [Vn] Lemma 3.6] \( F \) is a mapping of \( \ov{V}^i_p \) onto \( U^i_p \) for any \( i = 1, 2, \ldots, k+1 \). Thus, \( N(f, D) = N(F, D) \geq k+1 \), which contradicts the definition of the number \( k \). The obtained contradiction refutes the assumption that \( N(f, \ov{D}) > N(f, D) \). The lemma is proved. \( \square \)

Let us now turn to the main results of this section.

\textbf{Proof of Theorem 1.2} \ In the case 1), we choose \( \psi(t) = \frac{1}{t \log \frac{1}{t}} \), and in the case 2), we set

\[
\psi(t) = \begin{cases} 
\frac{1}{[t q_{y_0^+}] (t)}, & t \in (\varepsilon, \varepsilon_0), \\
0, & t \notin (\varepsilon, \varepsilon_0), 
\end{cases}
\]

Observe that, the relations (3.4)–(3.5) hold for these functions \( \psi \), where \( p = n \) (the proof of this facts may be found in [Sevi] Proof of Theorem 1.1]). The desired conclusion follows from Lemma 3.3. \( \square \)
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Evgeny Sevost’yanov
1. Zhytomyr Ivan Franko State University,
40 Bol’shaya Berdichevskaya Str., 10 008 Zhytomyr, UKRAINE
2. Institute of Applied Mathematics and Mechanics
of NAS of Ukraine,
1 Dobrovol’skogo Str., 84 100 Slavyansk, UKRAINE
esevostyanov2009@gmail.com