A NOTE ON GENERALIZED THURSTON–BENNEQUIN INEQUALITIES

NOBUO IIDA, HOKUTO KONNO, AND MASAKI TANIGUCHI

Abstract. We give a generalized Thurston–Bennequin-type inequality for links in $S^3$ using a Bauer–Furuta-type invariant for 4-manifolds with contact boundary. As a special case, we also give an adjunction inequality for smoothly embedded orientable surfaces with negative intersection in a closed oriented smooth 4-manifold whose non-equivariant Bauer–Furuta invariant is non-zero.

1. Main results

Adjunction inequalities are lower bounds on genera of smoothly embedded surfaces in a 4-manifold modeled on the adjunction formulae for algebraic curves in an algebraic surface, and they have been important tools to study 4-dimensional topology. In this paper, we give a new adjunction-type inequality. The most general result is given as a generalized Thurston–Bennequin-type inequality for links in $S^3$ (Theorem 1.4), and it deduces also an adjunction inequality for closed surfaces with negative self-intersection in a closed 4-manifold, described in Subsection 1.1.

1.1. Adjunction inequality for negative-self intersections. First, we describe our adjunction-type inequality for closed 4-manifolds.

Most known adjunction inequalities are proved for surfaces with non-negative self-intersection (see, for example, [13]). However, Ozsváth–Szabó [17, Corollary 1.7] proved the adjunction inequality for surfaces with negative self-intersection provided that the 4-manifold has non-vanishing Seiberg–Witten invariant and is of Seiberg–Witten simple type. Recently, Kato, Nakamura, and Yasui [11, Theorem 1.7] and Baraglia [1, Theorem 1.8] proved adjunction inequalities without simple type assumption provided that the non-vanishing of the (mod 2) Seiberg–Witten invariant and $b^+ \equiv 3 \mod 4$. Note that the results by Ozsváth–Szabó, Kato–Nakamura–Yasui, and Baraglia cannot be applied to connected sums of 4-manifolds with $b^+ > 0$ since the Seiberg–Witten invariants vanish for such 4-manifolds.

We give an adjunction-type inequality for surfaces with negative self-intersection also for some class of 4-manifolds obtained as connected sums of 4-manifolds with $b^+ > 0$. Also, our method does not need to require Seiberg–Witten simple type as in [1,11]. The statement is formulated in terms of the (non-equivariant) Bauer–Furuta invariant $\mathcal{F}$:

Theorem 1.1. Let $X$ be an oriented closed smooth 4-manifold with $b_1(X) = 0$ and $\mathfrak{s}$ be a Spin$^c$ structure on $X$. Suppose that the non-equivariant Bauer–Furuta
invariant $BF(X, s)$ is non-trivial. Let $\Sigma \hookrightarrow X$ be a smoothly embedded oriented closed surface with $g(\Sigma) > 0$. Then we have

$$|c_1(s) : [\Sigma]| + |\Sigma|^2 \leq 2g(\Sigma).$$

By Bauer’s calculation \cite{2} of Bauer–Furuta invariants, we may apply Theorem 1.1 to many concrete 4-manifolds. Recall that the (Seiberg–Witten) formal dimension $d(s)$ for a closed Spin$^c$ 4-manifold $(X, s)$ is defined as

$$d(s) = c_1(s)^2 - 2\chi(X) - 3\sigma(X).$$

**Corollary 1.2.** Let $1 \leq n \leq 3$. For $i \in \{1, \ldots, n\}$, let $X_i$ be an oriented closed smooth 4-manifold with $b^+(X_i) \equiv 3 \mod 4$ and $b_1(X_i) = 0$, and $s_i$ be a Spin$^c$ structure on $X_i$ with formal dimension 0. Suppose that the Seiberg–Witten invariants $\text{SW}(X_i, s_i)$ are odd numbers for all $i$. Set

$$(X, s) = \#_{i=1}^n (X_i, s_i)$$

and let $\Sigma \hookrightarrow X$ be a smoothly embedded oriented closed surface with $g(\Sigma) > 0$. Then we have

$$|c_1(s) : [\Sigma]| + |\Sigma|^2 \leq 2g(\Sigma).$$

**Remark 1.3.** We describe the nature of Theorem 1.1 and Corollary 1.2.

1. In the case that $|\Sigma|^2 \geq 0$, the results of Theorem 1.1 and Corollary 1.2 have been known (see such as \cite{1, 10}). The new parts of Theorem 1.1 and Corollary 1.2 are the statements for the case that $|\Sigma|^2 < 0$.

2. Theorem 1.1 and Corollary 1.2 generalizes the adjunction inequality by Baraglia \cite[Theorem 1.8]{1} from 4-manifolds with non-trivial (mod 2) Seiberg–Witten invariant to 4-manifolds with non-trivial Bauer–Furuta invariant.

3. As in \cite[Theorem 1.8]{1}, the inequality (1) is slightly weaker than the usual adjunction inequality: the right-hand side of the usual adjunction inequality is $2g(\Sigma) - 2$.

4. For the case that $n = 1$ in Corollary 1.2, the statement of the corollary follows from a special case of the Ozsváth–Szabó’s adjunction inequality \cite{17} for negative-self intersection and the usual adjunction inequality for non-positive self-intersection, provided that $(X, s)$ is of simple type. Note that we do not have to impose any simple type assumption in Theorem 1.1 and Corollary 1.2 as in \cite[Theorem 1.7]{11} and \cite[Theorem 1.8]{1}.

1.2. Generalized Thurston–Bennequin-type inequality. Theorem 1.1 is generalized to a relative genus bound. Let $X$ be a compact oriented 4-manifold whose boundary is $S^3$ and let $\xi_{std}$ be the standard contact structure on $\partial X = S^3$. Let $L = \bigsqcup_{i=1}^n K_i \subset (S^3, \xi_{std})$ be an oriented Legendrian link and $\Sigma \subset X$ be a smooth connected oriented surface whose oriented boundary is $K$. We define the Thurston–Bennequin number of $L$ by $tb(L) := \sum_{i=1}^n (tb(K_i) + \sum_{j \neq i} \text{lk}(K_i, K_j))$. If we take a front projection of $L$, we can write $tb(L) = \sum_{i=1}^n \left( w(K_i) - \frac{1}{2} \# \text{cusp}(K_i) + \sum_{j \neq i} \text{lk}(K_i, K_j) \right)$, which is the same as $tb(L)$ in \cite{4}, where $w$ is the writhe and $\text{lk}$ means the linking number.
Since $H^2(S^3; \mathbb{Z}) = H^1(S^3; \mathbb{Z}) = 0$, we have canonical isomorphism between the determinant line bundle $\det s$ of the positive spinor bundle and $\xi_{std}$. The generalized rotation number $r(L, [\Sigma], s)$ is defined to be $\langle c_1(\det s, \hat{L}), [\Sigma] \rangle$, where $h$ is the isomorphism and $\hat{L}$ is a section of $\det s|_{S^3} = \xi_{std}$ on $L$ given by the tangent vector field of $L$, which is unique up to pointwise positive scaling. Note that the relative first Chern class with respect to this section $c_1(\det s, \hat{L})$ is an element of $H^2(X, L; \mathbb{Z})$.

Now we are ready to state our relative adjunction inequality:

**Theorem 1.4.** Let $X$ be an oriented closed smooth 4-manifold with $b_1(X) = 0$ and $s$ be a Spin$^c$ structure on $X$. Let $L$ be an oriented Legendrian link in $(S^3, \xi)$, where $\xi$ is the standard contact structure. Suppose that the non-equivariant Bauer–Furuta invariant $BF(X, s)$ is non-trivial. Let $\Sigma \hookrightarrow X \setminus \text{int} \ D^4$ be a smoothly and properly embedded oriented compact connected surface with $g(\Sigma) > 0$ such that $\partial \Sigma = L$. Then, we have

$$|r(L, [\Sigma], s, h)| + [\Sigma] \cdot [\Sigma] + \text{tb}(L) - n + 2 \leq 2g(\Sigma).$$

**Remark 1.5.** We describe the nature of Theorem 1.4.

(1) We first note that our inequality typically can be applied to symplectic caps $X$ whose boundary is $(S^3, \xi_{std})$ with $b^+(X) \equiv 3 \mod 4$ and $b_1(X) = 0$. The usual generalized Thurston–Bennequin inequality ([15]) holds for symplectic fillings but not for symplectic caps.

(2) Unlike the closed case (Theorem 1.1 and Corollary 1.2), the result of Theorem 1.4 is new also for non-negative self-intersection surfaces. As a known relative genus bounds for connected sums, in [10], Mukherjee and the first and third authors gave an adjunction-type inequality for certain connected sums for surfaces with non-negative self-intersections. Let us compare (2) with the adjunction inequality given in [10, Remark 5.4]. For example, [10, Remark 5.4] implies

$$[\Sigma]^2 \leq 2g(\Sigma) - 2$$

under the same assumption of Theorem 1.4 with $TB(K) > 0$, $c_1(s) = 0$ and $[\Sigma]^2 \geq 0$. On the other hand, (2) implies

$$[\Sigma]^2 + TB(K) \leq 2g(\Sigma) - 1,$$

where $TB(K)$ is the maximal Thurston–Bennequin number. Therefore, if $TB(K) > 1$, the bound from (2) is stronger than that in [10]. Also we do not need to assume $[\Sigma]^2 \geq 0$ and $TB(K) > 0$.

(3) Define the 4-genus $g_4(L)$ as minimal genus of all connected properly and smoothly embedded surfaces in $D^4$ bounded by $L$. If $X = D^4$, the inequality (2) recovers

$$|r(L)| + \text{tb}(L) \leq 2g_4(L) + n - 2,$$

which is known as the Thurston–Bennequin inequality (see for example [16, Section 1.2, page 18]). Here $r(L)$ is the rotation number with respect to the standard contact structure on $S^3$. If $K$ is Lagrangian fillable, it is proven in [5] that the inequality (3) is attained by the equality, and therefore the inequality (3) is the best possible inequality. Moreover, as shown in [4, page 1949], this bound (3) is sharp for a positive torus link.
For other relative adjunction inequalities, see also [8, 14], for example. Corresponding to Corollary 1.2, we have:

**Corollary 1.6.** Let $1 \leq n \leq 3$. For $i \in \{1, \ldots, n\}$, let $X_i$ be an oriented closed smooth 4-manifold with $b^+(X_i) \equiv 3 \mod 4$ and $b_1(X_i) = 0$, and $s_i$ be a Spin$^c$ structure on $X_i$ with formal dimension 0. Suppose that the Seiberg–Witten invariants $SW(X_i, s_i)$ are odd numbers for all $i$. Set

$$(X, s) = \#_{i=1}^n (X_i, s_i).$$

Let $L$ be an oriented Legendrian link in $(S^3, \xi)$, where $\xi$ is the standard contact structure, and $\Sigma \hookrightarrow X \setminus \text{int} D^4$ be a smoothly and properly embedded oriented compact surface with $g(\Sigma) > 0$ such that $\partial \Sigma = L$. Then we have

$$|r(L, [\Sigma], s)| + [\Sigma] \cdot [\Sigma] + \text{tb}(L) - n + 2 \leq 2g(\Sigma).$$

(4)

Next, we note an application of Theorem 1.4 to negative-definite 4-manifolds:

**Corollary 1.7.** Let $X$ be a closed smooth oriented negative-definite 4-manifold with $b_1(X) = 0$. Let $L$ be an oriented Legendrian link in $(S^3, \xi)$, where $\xi$ is the standard contact structure, and $\Sigma \hookrightarrow X \setminus \text{int} D^4$ be a smoothly and properly embedded oriented compact surface with $g(\Sigma) > 0$ such that $\partial \Sigma = L$. Then we have

$$\max_{s \in \text{Spin}^c(X), \ c_1(s)^2 = -b_2(X)} \ |r(L, [\Sigma], s)| + [\Sigma] \cdot [\Sigma] + \text{tb}(L) - n + 2 \leq 2g(\Sigma).$$

(5)

Here the maximum is taken over spin$^c$ structures $s$ of $X$ that satisfy $c_1(s)^2 = -b_2(X)$.

**Remark 1.8.** We note that Theorem 1.4 recovers a special case of the relative adjunction inequality for quasi-positive knots given by Baraglia [11, Theorem 1.2], and generalize the results to connected sums as follows. If $K$ is Lagrangian fillable, the equality $\text{tb}(K) = 2g_4(K) - 1$ is proven in [5]. Thus, for a closed oriented 4-manifold $X$ with non-zero mod 2 Seiberg–Witten invariant for a spin$^c$ structure $s$, $b^+(X) \equiv 3 \mod 4$ and $b_1(X) = 0$, and for a properly embedded oriented surface $\Sigma$ in $X \setminus \text{int} D^4$ bounded by $K$, our inequality (11) can be written as

$$|c_1(s) \cdot [\Sigma]| + [\Sigma]^2 + 2g_4(K) \leq 2g(\Sigma),$$

which is the same inequality in [11, Theorem 1.2]. Note that a Lagrangian fillable knot is known to be a quasi-positive knot, which is proven in [7]. Thus Corollary 1.6 recovers [11, Theorem 1.2] for Lagrangian fillable knots. Moreover, we can treat connected sums of such 4-manifolds. For example, we can consider $X' := K3\#K3\#K3$. Then, for a Lagrangian fillable knot $K$ and a properly embedded oriented surface $\Sigma$ in $X' \setminus \text{int} D^4$ bounded by $K$, we have

$$[\Sigma] \cdot [\Sigma] + 2g_4(K) \leq 2g(\Sigma),$$

which is a new bound even for $[\Sigma] \cdot [\Sigma] \geq 0$.

We also note [11, Theorem 1.9] can be recovered from Corollary 1.7 for Lagrangian fillable knots.
2. Proof of the theorems

In this section, we will prove Theorem 1.1 and Theorem 1.4. Noting (2), it is straightforward to see that Theorem 1.1 follows from Theorem 1.4 by considering connected sum with the standard Stein 4-ball $D^4$ and setting $K = U$, the unknot with the standard Legendrian representation, considered as the boundary of the surface obtained as the connected sum of an embedded 2-disc whose boundary is $U$ and the given surface. Thus we give a proof of Theorem 1.4 below. The invariant introduced in [9] is used in the proof, so let us explain it. This invariant $\Psi(X, \xi, s)$ is the Bauer–Furuta type homotopical version of Kronheimer-Mrowka’s invariant for a 4-manifold with contact boundary. While Kronheimer-Mrowka’s invariant is defined for an arbitrary compact oriented Spin$^c$ 4-manifold $(X, s)$ equipped with a contact structure $\xi$ and identification $s|_{\partial X} = s_\xi$, the invariant $\Psi(X, \xi, s)$ is now defined under additional constraint $b_3(X) = 0$, in particular the boundary $\partial X$ must be connected. The invariant is an element

$$\Psi(X, \xi, s) \in \pi^H_{\ast}(X)/(\{1\})$$

of the non-equivariant $d(s, \xi) := \langle e(S^+, \Phi_0), [X, \partial X]\rangle$-th stable homotopy group of the sphere $S^0$, defined up to sign, where $\Phi_0$ is a section on the boundary of the positive spinor bundle $S^+$ constructed from the contact structure as in [12].

Proof of Theorem 1.4. We mainly follow an argument in [15] Subsubsection 4.2.3]. Denote the components of $L$ by $L = \bigcup_{i=1}^n K_i$. Set

$$N = \begin{cases} 0, & [\Sigma]^2 \geq 0, \\ -[\Sigma]^2, & [\Sigma]^2 < 0 \end{cases}.$$

Form the connected sum of the first component $K_1$ of $L$ with $\#_N T_{2,3}$, where $T_{2,3}$ is a positive torus knot of type $(2, 3)$. Precisely, we consider Legendrian representations of $K_1$ and $\#_N T_{2,3}$, and form the connected sum along cusps as in [15] Subsubsection 4.2.3]. Let us set $L' := K_1 \#(\#_N T_{2,3}) \bigcup_{i=2}^n K_i$.

Note that $T_{2,3}$ bounds a smoothly embedded genus-1 surface in $D^4$. Form a boundary connected sum of $\Sigma$ with a null-homologous punctured genus $n$-surface bounded by $\#_N T_{2,3}$ in $X \setminus \text{int} D^4$ and denote the resulting surface by $\Sigma'$, which is bounded by $L'$. Next, we attach $(tb(K'_1) - 1)$-framed 2-handles along $K'_1$ to $X \setminus \text{int} D^4$, where $K'_1 = K_1 \#(\#_N T_{2,3})$ and $K'_i = K_i$ for $i \geq 2$. We denote the resulting 4-manifold by $W$. Note that $W$ is decomposed as

$$W = X \# W_{(tb(K'_1) - 1, \ldots, tb(K'_n) - 1)}(L'),$$

where $W_{(tb(K'_1) - 1, \ldots, tb(K'_n) - 1)}(L')$ is the Stein filling [6] obtained from $D^4$ by attaching $n$ Weinstein 2-handles along $L'$. Denote by $\omega$ the Stein structure. Note that $b_3(W) = 0$. Define

$$\Sigma'' := \Sigma' \cup (\text{the cores of the 2-handles of } W_{(tb(K'_1) - 1, \ldots, tb(K'_n) - 1)}(L')).$$

Let $s_\omega$ be the canonical $Spin^c$ structure, and form a $Spin^c$-structure $s_W$ on $W$ as the connected sum of $s$ with $s_\omega$. We also write by $\xi_\omega$ the induced contact structure on $\partial(W_{(tb(K'_1) - 1, \ldots, tb(K'_n) - 1)}(L'))$ from $\omega$.

Now we claim that the Bauer–Furuta type invariant $\Psi(W, s_W, \xi_\omega)$ is non-trivial. Indeed, we can use a connected sum formula as follows:

$$\Psi(W, s_W, \xi_\omega) = BF(X, s) \wedge \Psi(W_{(tb(K'_1) - 1, \ldots, tb(K'_n) - 1)}(L'), s_\omega, \xi_\omega),$$

where

$$BF(X, s) = \langle e(S^+, \Phi_0), [X, \partial X]\rangle.$$
where $BF(X, s)$ is the non-equivariant Bauer–Furuta invariant. Since the 4-manifold with boundary $(W_{tb(K'_1)-1,\ldots,tb(K'_n)-1}(L'), \omega)$ is a Stein filling, from [9], we have
$$
\Psi(W_{tb(K'_1)-1,\ldots,tb(K'_n)-1}(L'), \omega, \xi_\omega) = \text{Id}.
$$
Thus we conclude that $\Psi(W, s, \omega, \xi_\omega)$ is the same as $BF(X, s)$, which is non-trivial.

By the construction $\Sigma''$, we have
$$
g(\Sigma'') = g(\Sigma) + N,
$$
$$
|\Sigma''|^2 = |\Sigma|^2 + \sum_{i=1}^n \left(tb(K'_i) - 1\right) + 2 \sum_{i<j} \text{lk}(K'_i, K'_j)
= |\Sigma|^2 + tb(L') - n
= |\Sigma|^2 + tb(L) + 2N - n,
$$
$$
\langle c_1(\omega W), [\Sigma''] \rangle = r(K, [\Sigma], \omega).
$$
Furthermore, from the adjunction inequality proven in [10] for surfaces with non-negative self-intersection applied to $\Sigma''$, we obtain
$$
|\langle c_1(\omega W), [\Sigma''] \rangle| + |\Sigma''|^2 \leq 2g(\Sigma'') - 2.
$$
This completes the proof of Theorem 1.4. □

Acknowledgement. We would like to thank David Baraglia for giving comments for an earlier version of this paper. In particular, Corollary 1.7 was pointed out by Baraglia. We wish also thank Kouichi Yasui for informing us about his papers related to adjunction inequalities.

Nobuo Iida was supported by JSPS KAKENHI Grant Numbers 19J23048, 22J00407 and the Program for Leading Graduate Schools, MEXT, Japan. Hokuto Konno was supported by JSPS KAKENHI Grant Numbers 17H06461, 19K23412, and 21K13785. Masaki Taniguchi was supported by JSPS KAKENHI Grant Numbers 20K22319, 22K13921 and RIKEN iTHEMS Program.

References
[1] David Baraglia, On the slice genus of quasipositive knots in indefinite 4-manifolds (2022), available at arXiv:2204.09886.
[2] Stefan Bauer, A stable cohomotopy refinement of Seiberg-Witten invariants. II, Invent. Math. 155 (2004), no. 1, 21–40. MR2025299.
[3] Stefan Bauer and Mikio Furuta, A stable cohomotopy refinement of Seiberg-Witten invariants. I, Invent. Math. 155 (2004), no. 1, 1–19. MR2025298.
[4] Alberto Cavallo, The concordance invariant tau in link grid homology, Algebr. Geom. Topol. 18 (2018), no. 4, 1917–1951. MR3797061.
[5] Baptiste Chantraine, Lagrangian concordance of Legendrian knots, Algebr. Geom. Topol. 10 (2010), no. 1, 63–85. MR2580429.
[6] Yakov Eliashberg, Topological characterization of Stein manifolds of dimension $>2$, Internat. J. Math. 1 (1990), no. 1, 29–46. MR1044658.
[7] Kyle Hayden and Joshua M. Sabloff, Positive knots and Lagrangian fillability, Proc. Amer. Math. Soc. 143 (2015), no. 4, 1813–1821. MR3314092.
[8] Matthew Hedden and Katherine Ruozzi, Knot floer homology and relative adjunction inequalities (2020), available at arXiv:2009.05462.
[9] Nobuo Iida, A Bauer–Furuta-type refinement of Kronheimer and Mroscia’s invariant for 4-manifolds with contact boundary, Algebr. Geom. Topol. 21 (2021), no. 7, 3303–3333. MR4357606.
[10] Nobuo Iida, Anubhav Mukherjee, and Masaki Taniguchi, *An adjunction inequality for the Bauer-Furuta type invariants, with applications to sliceness and 4-manifold topology* (2021), available at arXiv:2102.02076.

[11] Tsuyoshi Kato, Nobuhiro Nakamura, and Kouichi Yasui, *The simple type conjecture for mod 2 seiberg-witten invariants* (2020), available at arXiv:2009.06791 to appear in J. Eur. Math. Soc.

[12] P. B. Kronheimer and T. S. Mrowka, *Monopoles and contact structures*, Invent. Math. **130** (1997), no. 2, 209–255. MR1474156

[13] Terry Lawson, *The minimal genus problem*, Exposition. Math. **15** (1997), no. 5, 385–431. MR1486407

[14] Ciprian Manolescu, Marco Marengon, and Lisa Piccirillo, *Relative genus bounds in indefinite four-manifolds* (2020), available at arXiv:2012.12270.

[15] Tomasz Mrowka and Yann Rollin, *Legendrian knots and monopoles*, Algebr. Geom. Topol. **6** (2006), 1–69. MR2199446

[16] Burak Ozbagci and András I. Stipsicz, *Surgery on contact 3-manifolds and Stein surfaces*, Bolyai Society Mathematical Studies, vol. 13, Springer-Verlag, Berlin; János Bolyai Mathematical Society, Budapest, 2004. MR2114165

[17] Peter Ozsváth and Zoltán Szabó, *The symplectic Thom conjecture*, Ann. of Math. (2) **151** (2000), no. 1, 93–124. MR1745017

**Department of Mathematics Tokyo Institute of Technology 2-12-1, Ookayama, Meguro, Tokyo 152-8551 Japan**

*Email address: iida.n.ad@m.titech.ac.jp*

**Graduate School of Mathematical Sciences, the University of Tokyo, 3-8-1 Komaba, Meguro, Tokyo 153-8914, Japan**

*Email address: konno@ms.u-tokyo.ac.jp*

**2-1 Hirosawa, Wako, Saitama 351-0198, Japan**

*Email address: masaki.taniguchi@riken.jp*