Superfluidity in relativistic neutron stars

David Langlois\textsuperscript{1,2}

\textsuperscript{1} D\textsuperscript{é}partement d’Astrophysique Relativiste et de Cosmologie, Observatoire de Paris-Meudon, 92195 Meudon, France
\textsuperscript{2} Institut d’Astrophysique de Paris, 98bis Boulevard Arago, 75014 Paris, France

Abstract. The purpose of these notes is to give a brief review of superfluidity in neutron stars. After a short presentation explaining why and how superfluidity is expected in the crust and core of neutron stars, consequences on thermal evolution and rotational dynamics are discussed. The second part summarizes a formalism that has been recently developed to describe the hydrodynamics of superfluids or superconductors in the framework of general relativity. As an application, one can compute the oscillations of a two-component relativistic neutron star.

1 Introduction

The present contribution will discuss superfluidity and superconductivity in a rather extreme environment: neutron stars. Neutron stars, dense stars composed mainly of neutrons, were envisaged by Landau as soon as the neutron was discovered in 1932. In 1934, Baade and Zwicky suggested that supernovae were manifestations of a transition from an ordinary star to a very dense neutron star. In 1967, Hewish and Bell discovered the first “pulsar” (pulsating source of radio), which was soon identified as a rotating neutron star. The following year, two new pulsars, much studied until now, were discovered: the Vela pulsar with a period $P = 89$ ms and the Crab pulsar with $P = 33$ ms. Since then, more than one thousand pulsars have been detected.

Neutron stars are rather impressive objects. They contain a mass of the order of the solar mass confined in a radius $R \sim 10$ km, which implies an average mass density of the order $\rho \sim 10^{14}$ g/cm$^3$. They can rotate up to several hundred times per second. Due to these extreme conditions, neutron stars are of interest for various branches of physics. First, neutron stars are so dense and so compact that their gravitational field is very strong,

$$\frac{GM}{c^2 R} \sim 0.2,$$

and can be described correctly only with general relativity. Moreover, binary pulsars have been extraordinary “laboratories” to test the strong field predictions of general relativity, in particular to verify (indirectly) the existence of gravitational radiation.

Second, the magnetic field, typically of the order of $10^{12}$ G, plays a very important rôele in the neutron star physics, in particular in the pulsar emission mechanism (see e.g. \cite{1}). Third, neutron stars are unique places where one can
find matter in an ultra-dense state. The density is indeed higher than the atomic nuclear density. The corresponding pressure therefore depends essentially on the strong interactions, and neutron star observations are potentially a very rich source of information about the behaviour of strong interactions at high densities, which, until now, remains very poorly known from both theoretical and experimental points of view.

Finally, neutron star physics has also some connection with the low temperature physics. Studying the nucleon interactions, Migdal suggested the possibility that neutron star matter become superfluid at sufficiently low temperature. If the temperature of neutron stars, typically $T \simeq 10^6 \text{K}$, may seem huge in terrestrial standards, it is in fact smaller than the superfluid critical temperature evaluated as $T_c \sim 10^9 - 10^{10} \text{K}$.

In these notes, we will focus on two aspects of the physics of neutron stars: the property of superfluidity with its consequences on the evolution of a neutron star; the necessity to consider a neutron star as a general relativistic object. Many other interesting aspects of neutron stars will thus be left aside and the curious reader will find some useful information in e.g. [2].

The plan will be the following: we will begin with some details about the superfluid properties of the interior of neutron stars. Observable consequences of superfluidity will then be discussed: first, the impact on the thermal evolution of neutron stars, then on their rotational dynamics. In a second part, we will summarize some elements of a formalism that gives a hydrodynamical description of the interior of neutron stars in the framework of general relativity. Finally, as an illustration, oscillations of two-component superfluid neutron stars are considered.

## 2 Superfluidity and superconductivity

### 2.1 Composition of the interior of a neutron star

Let us start by describing the matter composition inside a neutron star. Because there is a strong density gradient from the exterior to the interior of the star, the composition changes dramatically with the radial distance from the center. Let us progress from the surface of the star towards its interior.

Ignoring here the surface ocean, the first layer of a neutron star is the outer crust: it is made of a lattice of nuclei with a gas of relativistic degenerate electrons. The nuclei are richer and richer in neutrons as the density increases (see e.g. [3] for a recent review). At some point ($\rho \simeq 4.3 \times 10^{11} \text{g/cm}^3$), neutrons begin to leak out of the nuclei. This is called the neutron drip transition and this marks the boundary between the outer crust and the inner crust. For the latter, one still has a lattice of nuclei immersed in an electron gas, but in addition there is a liquid of free neutrons. The density still increasing, the difference between the neutrons in nuclei and the neutrons outside becomes fainter and fainter until the nuclei simply dissolve (at a density $\rho \sim 2 \times 10^{14} \text{g/cm}^3$). One has reached the limit of the crust and one now enters the realm of the core.
In the outer core, one will find the coexistence of a gas of free neutrons, which is by far the main species, with a liquid of free protons, plus a gas of electrons (and muons) which ensures charge neutrality. Finally the deeper layers of a neutron star, the inner core, remain mysterious. Several possibilities can be envisaged, including appearance of hyperons, condensation of pions or kaons, or the transition from hadronic to quark matter.

### 2.2 Energy gaps and critical temperature

Like electrons in a superconductor, the superfluidity of neutrons in neutron stars is due to the pairing of two neutrons near the Fermi surface in momentum space, according to the Cooper mechanism (see e.g. [4]). In the inner crust, the neutron Cooper pairs are preferably in a state $^1S_0$. In the outer core, where the density is much higher, the neutron Cooper pairs are of the $^3P_2$ type (see [5] and [6]).

The evaluation of the energy gaps give typically values of the order of 1 MeV.

In the same layer, the protons are free and can undergo the same process as neutrons, i.e. form Cooper pairs. Being fewer than the neutrons, they are expected to condense into a $^1S_0$ state [7]. Note that hyperons that may exist in the inner core are also expected to be superfluid.

![Fig. 1. Schematic view of the internal structure of a neutron star](image-url)
2.3 Various equations of state

In order to determine the global structure of neutron stars, one needs the effective bulk equation of state of neutron star matter. Inserting this equation of state in Einstein’s equations, it is immediate, in the case of a non-rotating star (see subsection 6.1), to obtain the radial profile of the star, i.e. the radial profile of the energy density, of the pressure and of the metric coefficients. One can also take into account the rotation of the star.

The main problem however is that the equation of state is unknown at very high densities. The reason is the lack of experimental data as well as the theoretical and computational challenge to evaluate the interactions of high density matter. Therefore there exist many different equations of state in the literature (see e.g. [8] for a recent review) and the hope is to infer from observations of neutron stars some constraints on the high density equations of state (see [9] and [10]).

3 Cooling processes in neutron stars

Born with a temperature of the order $10^{11} - 10^{12}$ K, neutron stars cool very rapidly to temperatures of less than $10^{10}$ K within minutes. The subsequent thermal evolution is more uncertain and could be strongly affected by the presence of superfluidity. The mechanism by which neutron stars cool down is essentially neutrino emission. Neutrinos can be produced in two types of processes:

- Direct URCA processes
  
  They correspond to the simplest beta reactions
  
  \[ n \rightarrow p + e^- + \bar{\nu}_e, \quad p + e^- \rightarrow n + \nu_e. \]  

- Modified URCA processes
  
  They correspond to beta reactions with the presence of a second nucleon.

Direct URCA ractions are (not surprisingly) more efficient than the modified ones but they can occur only if conservation of energy and momentum is satisfied, which implies some constraint on the species fractions, namely that the proton fraction is sufficiently high. If this is not the case, then direct URCA processes are completely suppressed and cooling occurs only through modified URCA processes and is much slower.

However, taking into account the existence of superfluidity leads to the conclusion that the direct URCA processes could be partially suppressed: the neutrino emissivity would be reduced by a factor \( \exp(-\Delta/kT) \), where \( \Delta \) is the superfluid gap energy. The reason is that, before the beta reaction takes place, one needs to break the Cooper pair.

The existence of superfluidity therefore enables us to envisage, between the two extreme scenarios suggested above of fast cooling or slow cooling, an intermediate scenario of moderate cooling, which may be testable in the future by the temperature measurements of neutron stars (see [1] and references therein).
4 Rotational dynamics of neutron stars: glitches

Another important consequence of superfluidity concerns the rotational dynamics of neutron stars, which can be followed extremely precisely via the observation of the radio signal from pulsars. The main feature is an extremely stable periodic signal, the pulsar being analogous to a lighthouse beacon: the radio emission is indeed collimated along the magnetic axis which does not coincide with the rotation axis.

For isolated pulsars, one observes a steady, although tiny, increase of the pulse period, i.e. a steady decrease of the angular velocity. This is interpreted as a loss of angular momentum due to the emission of electromagnetic radiation. Assuming a magnetic dipole, the rotational energy loss is given by

$$\dot{E} = I \dot{\Omega} = -\frac{B^2 R^6 \Omega^4 \sin^2 \theta}{6c^3},$$

where $I$ is the moment of inertia and $\theta$ the angle between the magnetic and rotation axes.

In addition to this slow increase of the period, a few pulsars exhibit some rare events named glitches at which the period suddenly decreases, i.e. the angular velocity suddenly increases, followed by a slow exponential relaxation with a typical time scale of the order of days to months. The two most famous ‘glitching’ pulsars are the Vela and Crab pulsars with period changes $\Delta \Omega/\Omega$ of the order of $10^{-6}$ and $10^{-8}$ respectively.

The current explanation for these glitches is the following. It is based on the idea that the neutron star interior contains a weakly coupled component that is not directly slowed down by the electromagnetic torque and therefore rotates slightly faster than the star crust. What is directly observable is the angular velocity of the crust, and if, for some reason, there is a rapid transfer of angular momentum from the faster component to the crust, then one should see a sudden increase of the pulsar period. Of course, a natural candidate for the component rotating faster is the neutron superfluid.

4.1 The two-fluid model

The relaxation after the glitch can also be explained, at least qualitatively, by the simple two-component model of Baym et al [12]. One considers two components with respective angular velocities $\Omega_c$ and $\Omega_n$, and respective moments of inertia, $I_c$ and $I_n$. The first component corresponds to the crust and to whatever is strongly coupled to it, whereas the second component corresponds to the neutron superfluid weakly coupled to the crust. Note that the pulsar angular velocity $\Omega$ which is the directly observable quantity can be identified with the crust angular velocity, so that $\Omega_c = \Omega$. The evolution of the two components is governed by the system

$$I_c \dot{\Omega} = -\frac{I_c}{\tau_c} (\Omega - \Omega_n) - \alpha,$$
\[ I_n \dot{\Omega}_n = \frac{I_c}{\tau_c} (\Omega - \Omega_n), \tag{5} \]

where the first term on the right hand side represents a coupling between the two components, coupling characterized by the timescale \( \tau_c \). \( \alpha \) corresponds to the electromagnetic torque. One can integrate the above system just after a glitch characterized by a sudden jump of the crust angular velocity \( \Delta \Omega_0 \equiv \Omega(t = 0^+) - \Omega(t = 0^-) \). The subsequent evolution is then

\[ \Omega(t) = \Omega_0(t) + \Delta \Omega_0 \left( Q e^{-t/\tau} + 1 - Q \right), \tag{6} \]

where \( Q \) is the healing parameter and \( \Omega_0(t) \equiv \Omega_0 - (\alpha/I)t \) \( \Omega_0 \) being a constant) corresponds to the evolution without glitch, and the characteristic relaxation timescale is given by

\[ \tau = \frac{I_n}{I} \tau_c, \tag{7} \]

with \( I = I_n + I_c \).

### 4.2 Rôle of the vortices

The two-fluid model gives an effective view of the rotational evolution, but it is interesting to explore in more details the underlying mechanism responsible for the coupling between the two components. This is where the vortices enter into play.

\[ \Omega(t) \]

\[ \Delta \Omega_0 \]

\[ \Omega_0(t) \]

\[ Q \Delta \Omega_0 \]

\[ t \]

**Fig. 2.** Evolution of the angular velocity during a glitch
The neutron star at its birth is a rapidly rotating object. With its cooling, the temperature reaches the critical temperature under which superfluidity appears. However, one essential property of a superfluid is that its flow is irrotational. The way the superfluid solves the contradiction between irrotationality and rotation of the star is, like superfluid Helium in laboratories, via the creation of quantized vortices that carry all the angular momentum of the superfluid. The superfluid ‘velocity’ (in fact, momentum) being the gradient of a phase,

$$ p_n = 2m_n v_n = \hbar \nabla \phi, $$

the circulation around any closed path is given by

$$ \kappa \equiv \oint_C dv_n = \frac{\hbar}{2m_n} (2\pi N), $$

where $N$ is an integer. Superfluid vortices usually correspond to one unit ($N = 1$) of quantized circulation $\kappa = \hbar/(2m_n)$. One can then easily compute the average density of vortices for a superfluid rotating at uniform angular velocity $\Omega_n$:

$$ n_V = 4 \frac{\Omega_n m_n}{\hbar}. $$

Putting numbers, one finds

$$ n_V = 6.3 \times 10^3 \left( \frac{P}{1 \text{ s}} \right)^{-1} \text{ vortices per cm}^2 $$

In the neutron star core, the existence of a proton superconductor, which is believed to be a type II superconductor, suggests the presence of magnetic vortices, or fluxoids, which carry the magnetic flux going through the neutron star. Note that one must be careful to compute the total energy of a fluxoid in a rotating superconducting background [13]. Moreover, because of the ‘entrainment effect’ by which the neutron superfluid momentum is associated with both neutron and proton currents, neutron superfluid vortices, in the neutron star core, will carry a magnetic flux as well, thus implying a strong coupling between the core superfluid and the crust [14,15].

Now, the vortices play an essential rôle for the coupling of the superfluid to the normal component of the star, because the pure superfluid has no viscosity at all and, as such, can circulate without affecting any normal component. But the vortices interact with the normal component. Without entering the details, one can distinguish two types of interaction between the vortices and the normal part of a neutron star:

- **Pinning**
  This is the situation where a vortex, or a segment of it, is anchored to the (inner) crust. The pinning is due to the interaction between the vortex and the nucleus, which can be attractive or repulsive depending on the matter density [16,17].
Friction

This is the case where the vortex, moving through the normal component, interacts with it in some dissipative processes, which results into an effective friction force (per unit length) exerted on the vortex by the normal component of the form

\[
F_{\text{drag}} = C(v_V - v_c).
\]  

(12)

Let us assume here that we are in the friction case. The vortices, interacting with the normal component, are also sensitive to the motion of the superfluid through a Magnus force term,

\[
F_{M} = n_n \kappa \times (v_V - v_n),
\]  

(13)

which is a force per unit vortex length. The effective motion of the vortices is obtained by requiring that the two forces (12) and (13) just cancel. This implies that the vortices must have a radial motion in addition to the angular motion. One finds easily that the angular motion is given by

\[
\Omega_V = \frac{c_r^{-1} \Omega_n + c_r \Omega_c}{c_r^{-1} + c_r},
\]  

(14)

which means that the angular velocity of the vortices is simply a weighted average of the superfluid and normal angular velocities, the weight depending on the dimensionless friction coefficient

\[
c_r = \frac{C}{\kappa n_n}.
\]  

(15)

The two extreme cases are: the case where the friction is zero, the vortices being then in corotation with the superfluid; the case where the friction is huge, the vortices being then in corotation with the normal component. As for the radial velocity of the vortices, it is given by

\[
v^r_V = \frac{\Omega_n - \Omega_c}{c_r^{-1} + c_r}.
\]  

(16)

One can then compute the effective coupling timescale between the two components of the stars, by noting that the evolution of the angular velocity of the superfluid component is directly given by the radial velocity of the vortex array, according to the expression

\[
\dot{\Omega}_n = -2 \frac{v^r_V}{r} \Omega_n,
\]  

(17)

simply because the angular momentum of the superfluid is proportional to the vortex density. Let us consider now the evolution of the angular velocity lag between the two components,

\[
\omega \equiv \Omega_n - \Omega_c,
\]  

(18)
and let us define the coupling timescale as $\tau_d \equiv |\omega/\dot{\omega}|$. Combining the expressions obtained above, one finds

$$\tau_d \simeq \frac{c_r + c_r^{-1}}{2} \frac{I_c}{I_c + I_n} \Omega^{-1}.$$  \hspace{1cm} (19)

The above Newtonian analysis has been generalized to a general relativistic context in \[18\]. The coupling between the crust and the superfluid can also be evaluated when vortices are pinned, in which case vortices can move outwards by vortex creep, the temperature being a crucial parameter \[19\].

### 4.3 Origin of the glitches

There is no certainty at present on the physical origin of observed glitches. The scenario which seems to have attracted the greatest attention is the one proposed by Anderson and Itoh \[20\]. In this model, vortices are supposed to be pinned in the crust due to the nucleus-vortex interaction. Because the normal component is slowing down, a Magnus force, due to the superfluid motion relative to the vortices, progressively builds up until this force reaches a critical amplitude for which the pinning sites simply break. The vortices then can suddenly move outwards, thus transferring angular momentum from the superfluid to the crust, which explains the sudden spin-up of the crust.

Other models have been proposed to explain glitches. For example, a glitch could be induced by a sudden heat wave propagating in the star, which would increase the effective coupling between the superfluid and the crust \[21\]. Such a heating could be generated for instance by a crustquake. In this respect, one must mention that the differential rotation between the superfluid and the crust generates a centrifugal buoyancy force which might increase the crust stresses and maybe lead to a glitch \[22\].

### 5 Relativistic description

Until very recently, only Newtonian theory was used in the studies of superfluidity in neutron stars, whereas, in parallel, general relativistic studies of neutron stars, even numerical, were based on perfect fluid matter. The purpose of the work summarized here is to establish a bridge between these two approaches, taking into account both the highly relativistic nature of neutron stars and their superfluid interior. We shall present a macroscopic formalism \[23\] allowing for the average effect of vorticity quantisation in a rotating superfluid, which can then be extended \[24\] to describe as well superconducting fluids, such as protons in the core of neutron stars. The Newtonian version of this formalism can be found in \[25\]. In the next section, a simplified relativistic two-component model of neutron star, which allows for differential rotation of the superfluid component, will be used to study oscillations of neutron stars.
5.1 Perfect fluid in general relativity

Let us start by recalling the simplest case, that of the perfect fluid. In general relativity, a perfect fluid is characterized by

- a four-velocity vector $u^\mu$ (where $\mu = 0, 1, 2, 3$ is a spacetime index denoting the time coordinate and the three space coordinates), which satisfies the normalization $g_{\rho\sigma}u^\rho u^\sigma = -1$ where $g_{\rho\sigma}$ is the metric tensor describing the geometry of spacetime (and which is used to raise or lower the indices, e.g. $u_\rho \equiv g_{\rho\sigma}u^\sigma$).
- a particle number density scalar field $n(x^\mu)$ giving in each spacetime point the density of particles as seen by an observer comoving with the fluid.
- an equation of state of the form $\rho = \rho(n)$ giving the energy density $\rho$ as a function of the particle number density $n$.

Note that we have chosen here the simpler case of a barotropic perfect fluid where the equation of state depends on only one parameter. Note also that one could characterize the fluid by its energy density instead of its particle number density, and then derive the number density by inverting the equation of state.

The equations of motion for the relativistic fluid will be generalizations of the well-known Newtonian equations of fluid mechanics, namely,

- the matter conservation equation, which can be written in the present context in the very simple form,\[ \nabla_\mu n^\mu = 0, \tag{20} \]
  where $n^\mu \equiv nu^\mu$ is the particle current and $\nabla_\mu$ stands for the covariant derivative.
- the relativistic Euler equation, which can be written in the very condensed form\[ n^\rho \nabla_{\lbrack \rho \mu \sigma \rbrack} = 0, \tag{21} \]
  where the brackets represent antisymmetrization on the indices and where one has conveniently introduced the momentum covector $\mu_\rho \equiv \mu u_\rho$, $\mu$ being the relativistic chemical potential defined by\[ \mu = \frac{d\rho}{dn}. \tag{22} \]

It is possible to derive directly the above equations of motion from a variational principle, the Lagrangian density being simply the energy density considered as a function of the particle number current $n^\mu$. Any variation of $n^\mu$ is not allowed, but only the convective variations, which correspond to variations of the particle flow lines (see [26] for more details).
5.2 Relativistic superfluid

What will characterize a “pure superfluid” (i.e. a superfluid at zero temperature, without gas of excitations), with respect to the more general class of perfect fluids, is the fact that its motion is locally irrotational. In the standard language, this is a consequence of the fact that the ‘superfluid velocity’ can be expressed as the gradient of a quantum scalar phase. Strictly speaking, the standard ‘superfluid velocity’ is in fact a momentum divided by a somewhat arbitrary mass. In the relativistic context, this will be generalized by the expression

$$\mu_\sigma = \hbar \nabla_\sigma \varphi, \quad (23)$$

so that the property of irrotational flow can be written as the vanishing of the vorticity tensor

$$w_{\rho\sigma} \equiv 2\nabla_{[\rho}\mu_{\sigma]}. \quad (24)$$

The previous condition is valid locally. However, as explained above, a superfluid can exhibit a non-irrotational flow at the price of being threaded by quantized vortices. This means that macroscopically, on distances bigger than the typical intervortex separation, one would like to describe the superfluid as well as the vortices in an average way. This can be done by considering the vorticity tensor $w_{\rho\sigma}$, now non-zero since there are vortices, as a fundamental quantity. In fact, this tensor contains information about the density of vortices as well as about the local average direction of the vortex array.

The idea then is to construct a generalized Lagrangian density that depends not only on the particle current $n^\sigma$ but also on the vorticity tensor $w_{\rho\sigma}$. The variations with respect to these fundamental variables will define the canonical momenta,

$$\delta \Lambda = \mu_\sigma \delta n^\sigma - \frac{1}{2} \lambda^{\rho\sigma} \delta w_{\rho\sigma}. \quad (25)$$

This can be seen as a generalization of the perfect fluid where $\Lambda = -\rho$ and only the first term is present on the right hand side.

The equations of motion are the matter conservation equation (20), as in the perfect fluid case, and a generalized Euler-type equation of the form,

$$w_{\mu\nu}(n^\nu - \nabla_\rho \lambda^{\rho\nu}) = 0. \quad (26)$$

Of course, the expression for $\lambda^{\rho\sigma}$, which essentially represents the energy density and momentum of an individual vortex, must be specified by resorting to a ‘microphysical’ model of the vortices (see [27] and [28] for relativistic descriptions of vortices).

5.3 Superfluid-Superconducting mixtures

It is useful to extend the previous formalism to the case where the superfluid particles (or Cooper pairs) are electrically charged, like electrons in laboratory superconductors, in order to be able to describe the protons in the cores of
neutron stars. Let us consider several species labelled by the index $X$, with respective particle currents $n_X^\rho$ and respective electric charge per particle $e^X$. The total electric current is then given by

$$ j^\rho = \sum_X e^X n_X^\rho. \quad (27) $$

Proceeding as before, in order to obtain the equations of motion for the global system of fluids, one uses a variational principle based on a Lagrangian density $\mathcal{L}$ that will be the sum of three contributions:

- a “matter” contribution $A_M$, which depends only on the ‘hydrodynamical’ part of the system, and which is a function only of the particle currents $n_X^\rho$, or more exactly of all scalar combinations obtained by their mutual contractions,
- an electromagnetic interaction term depending on the electromagnetic gauge form $A_\rho$,
  $$ A_I = j^\rho A_\rho, \quad (28) $$
- an electromagnetic field contribution
  $$ A_F = \frac{1}{16\pi} F_{\rho\sigma} F^{\sigma\rho}. \quad (29) $$

where the electromagnetic field tensor $F_{\rho\sigma}$ is defined in terms of the gauge form by the usual expression $F_{\rho\sigma} = 2\nabla_{[\rho} A_{\sigma]}$.

Considering the variations of $\mathcal{L} = A_M + A_F + A_I$ with respect to the particle currents $n_X^\rho$ naturally suggests to define the canonical momentum covectors

$$ \pi_X^\rho = \mu_X^\rho + e^X A_\rho \quad (30) $$

which are the sum of a pure ‘hydrodynamical’ (or ‘chemical’) part and of the electromagnetic gauge form weighted by the electric charge of the species. From this, it is convenient to define generalized vorticity tensors defined by

$$ w_X^\rho = 2\nabla_{[\rho} \pi_X^{\sigma]} = 2\nabla_{[\rho} \mu_X^{\sigma]} + e^X F_{\rho\sigma}. \quad (31) $$

The equations of motion for the system consist of the separate matter conservation equation for each species (one can generalize to allow for chemical reactions between various species),

$$ \nabla_{\rho} n_X^\rho = 0, \quad (32) $$

and of Euler-type equations, which can be written in the very compact form

$$ n_X^\sigma w_X^{\alpha\rho} = 0. \quad (33) $$

Note that in the case of a charged component, this equation contains the electromagnetic field tensor and thus automatically includes for instance the Lorentz force exerted on the fluid.
It is also useful to write for the system under investigation the associated total stress-energy-momentum tensor, which appears on the right hand side of the Einstein’s equations,

\[ R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{34} \]

if one needs to determine the spacetime metric. Once again, the variational principle is very useful because variation of the Lagrangian density with respect to the metric directly yields

\[ T^{\rho\sigma} = T^{\rho\sigma}_M + T^{\rho\sigma}_F, \tag{35} \]

where the part derived from the material Lagrangian density contribution \( \Lambda_M \) is given by

\[ T^{\rho\sigma}_M = \sum_X n^\rho_X \mu^\sigma_X + s\Theta u^\rho u^\sigma + \Psi_M g^\rho_{\sigma}, \quad \Psi_M = \Lambda_M - \sum_X n^\rho_X \mu^\sigma_X + s\Theta, \tag{36} \]

while the electromagnetic contribution has the usual Maxwellian form

\[ T^{\rho\sigma}_F = \frac{1}{4\pi} \left( F^{\nu\mu} F_{\nu\sigma} - \frac{4}{3} F^{\mu\nu} F_{\mu\nu} g^\rho_{\sigma} \right). \tag{37} \]

The above relations are valid for ordinary perfectly conducting fluids. The condition characterizing superconductors, generalizing the superfluid condition (23), is that the generalized momentum should be the gradient of a quantum phase,

\[ \pi^X_\rho \equiv \mu^X_\rho + c^X_\rho = \hbar \nabla_\rho \phi^X, \tag{38} \]

so that superconductors (including the case of superfluids) are characterized \textit{locally} by a vanishing generalized vorticity tensor. Of course, it is then possible, in analogy with the treatment of the previous subsection, to extend the formalism in order to describe on average a superfluid-superconducting mixture threaded by superfluid vortices as well as magnetic flux tubes.

6 Relativistic neutron stars

Let us now see how one can construct a concrete model of neutron star in a general relativistic framework. To simplify, we will assume that the neutron star is made of simply two components, each being treated as a perfect fluid. First, we will consider the case of a static, i.e. nonrotating, neutron star. Then we will consider small, i.e. linear, oscillations of the two-component neutron star.

Let us call, loosely, the component corresponding to the free neutron superfluid the “neutron” component and the component corresponding to the normal fluid the “proton” component. Their respective particle currents will be denoted \( n^\rho = n u^\rho \) and \( p^\rho = p v^\rho \), where \( u^\rho \) and \( v^\rho \) are unit four-velocities, which are not aligned if the two components are not comoving. Their equations of motion consist of two separate conservation equations of the type (20) and of two Euler equations of the type (21). Here, the two components are treated as ordinary perfect fluids and one ignores the refinements due to the presence of vortices in the superfluid component.
6.1 Static neutron star

We first consider the configuration corresponding to a non-rotating neutron star, which is thus static and spherically symmetric and for which the metric can be written in the form

\[ ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -e^\nu(r) dt^2 + e^{\lambda(r)} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2). \] (39)

The particle currents, since the two fluids are motionless, are necessarily of the form

\[ n^\mu = n^\mu, \quad p^\mu = p^\mu, \quad \bar{u}^\mu = \{ u^0, 0, 0, 0 \}. \] (40)

Writing Einstein’s equations with the above ansatz, one obtains the well-known TOV (Tolman-Oppenheimer-Volkov) equations:

\[ \frac{dP}{dr} = -G \left( \frac{\rho(r) + P(r)}{r^2(1 - 2Gm(r)/r)} \right) = -\frac{1}{2} \frac{d}{dr} \left( \rho(r) + P(r) \right) \frac{d\nu}{dr}, \] (41)

and

\[ e^{\lambda(r)} = \left( 1 - 2 \frac{m(r)}{r} \right)^{-1}, \] (42)

with \( m(r) \equiv 4\pi G \int_0^r dr' \rho(r')r'^2 \). The mass of the star is given by \( M = m(r) \), where \( R \) is the star radius defined by the condition that the pressure vanishes.

6.2 Oscillations of superfluid neutron stars

Oscillations of relativistic stars have been studied by many authors in the case of a single perfect fluid. Here, we summarize the first investigation for a two-component relativistic star \[29\], thus generalizing two previous studies of two-component Newtonian stars \[30\].

In a relativistic approach, one must consider not only the perturbations of the quantities describing the matter, for example the particle density and the velocity, but also the perturbations of the spacetime metric \( g_{\mu\nu} \). After decomposition of the perturbations into spherical harmonics, labelled by \( l \) and \( m \), it is convenient to distinguish even-parity (or polar) perturbations and odd-parity (or axial) modes depending on their transformation under parity. We will restrict ourselves here to the case of even-parity perturbations, which is the most interesting since scalar quantities, such as the matter densities, give only this type of perturbations. Also, one needs to study only \( m = 0 \) perturbations, since there is a degeneracy in \( m \) for each given \( l \) because of the spherical symmetry of the background.

Choosing a specific gauge, one will consider metric perturbations of the form

\[ \delta g_{00} = -e^{\nu} r^l H_0 e^{i\omega t} P_l(\theta), \quad \delta g_{0r} = \delta g_{r0} = -i\omega r^{l+1} H_1 e^{i\omega t} P_l(\theta), \]
\[ \delta g_{rr} = -e^{\lambda} r^l H_2 e^{i\omega t} P_l(\theta), \quad \delta g_{\theta\theta} = \delta g_{\phi\phi}/\sin^2 \theta = -r^{l+2} K e^{i\omega t} P_l(\theta), \] (43)
where $P_l$ stands for the Legendre polynomial of order $l$. The matter perturbations will be described, in a Lagrangian way, by the matter displacements,

$$
\xi_r^n = r^{l-1}e^{-\lambda/2}W_n e^{i\omega t} P_l(\theta), \quad \xi_\theta^n = -r^{l-2}V_n e^{i\omega t} \partial_\theta P_l(\theta),
$$

$$
\xi_r^p = r^{l-1}e^{-\lambda/2}W_p e^{i\omega t} P_l(\theta), \quad \xi_\theta^p = -r^{l-2}V_p e^{i\omega t} \partial_\theta P_l(\theta),
$$

from which one can compute the perturbed velocities, by taking the time derivative or the variations $\delta n$ and $\delta p$ by using the perturbed conservation equations. Inserting the above expressions for the perturbations into the perturbed Einstein’s equations as well as the perturbed Euler equations, one ends up with a system of linear equations, consisting of two constraints (one is $H_2 = H_0$, the other expresses $H_0$ in terms of the other perturbations) and of first order differential equations of the form

$$
\frac{dY}{dr} = Q_{l,\omega} Y,
$$

where $Y$ is 6-dimensional column vector containing $H_1$, $K$, $W_n$, $V_n$, $W_p$ and $V_p$, and $Q_{l,\omega}$ is a $6 \times 6$ matrix with $r$-dependent coefficients that depend only on the background configuration, as well as on $l$ and $\omega$. Considering the boundary conditions at the center and at the surface of the star, this system can be solved, up to a global amplitude, for any value of $\omega$. The physically relevant modes, however, also called quasi-normal modes, correspond the specific values of $\omega$ for which the metric outside the star represents only outgoing gravitational waves.

A numerical investigation for a very crude, and unrealistic, model of two independent polytropes (with different adiabatic indices) has shown that the two-component star will exhibit new modes, superfluid modes, which are specific of the existence of two components since the two fluids are counter-moving for these modes.

References

1. F.C. Michel, *Theory of neutron star magnetospheres*, University of Chicago Press, Chicago (1991).
2. S.L. Shapiro, S.A. Teukolsky, *Black holes, white dwarfs and neutron stars: the physics of compact objects*, John Wiley & Sons, New York (1983).
3. C.J. Pethick, D.G. Ravenhall, *Annu. Rev. Nucl. Part. Sci. 45, 429 (1995*)
4. J. A. Sauls, p. 457, in *Timing neutron stars*, Eds H. Ögelman, E.P.J. van den Heuvel; Dordrecht, Kluwer (1989).
5. M. Hoffberg, A.E. Glassgold, R.W. Richardson, M. Ruderman, *Phys. Rev. Lett.* 24, 775 (1970)
6. T. Takatsuka, *Prog. Theor. Phys. 48, 1517 (1972*)
7. N.C. Chao, J.W. Clark, C.H. Yang, *Nucl. Phys. PA 179, 320 (1972*)
8. H. Heiselberg, V. Pandharipande, “Recent progress in neutron star theory”, astro-ph/0003276
9. N.K. Glendenning, *Compact stars*, Springer-Verlag (1997)
10. J. M. Lattimer, M. Prakash, “Neutron Star Structure and the Equation of State”, astro-ph/0002232
11. D. Page, M. Prakash, J.M. Lattimer, A. Steiner, “Prospects of Detecting Baryon and Quark Superfluidity from Cooling Neutron Stars”, hep-ph/0005099.
12. G. Baym, C.J. Pethick, D. Pines, M. Ruderman, Nature 224, 872 (1969).
13. B. Carter, R. Prix, D. Langlois, “Energy of flux tubes in rotating superconductor”, to appear in Phys. Rev. B (2000) cond-mat/991024.
14. G.A. Vardanian, D.M. Sedrakian, Zh. Eksp. Teor. Fiz. 81, 1731 (1981) [English translation: Soviet Phys. JETP 54, 919 (1981)]
15. M.A. Alpar, S.A. Langer, J.A. Sauls, Astrophys. J. 282, 533 (1984)
16. R.I. Epstein, G. Baym, ApJ 328, 680 (1988)
17. P.M. Pizzochero, L. Viverit, R.A. Broglia, Phys. Rev. Lett. 79, 3347 (1997)
18. D. Langlois, D. M. Sedrakian, B. Carter, Month. Not. R.A.S. 297, 1189 (1998)
19. M.A. Alpar, P.W. Anderson, D. Pines, J. Shaham, ApJ 276, 325 (1984)
20. P.W. Anderson, N. Itoh, Nature 256, 25 (1975)
21. B. Link, R.I. Epstein, ApJ 457, 844 (1996) “Thermally driven neutron star glitches”
22. B. Carter, D. Langlois, D. Sedrakian, “Centrifugal buoyancy as a mechanism for neutron star glitches”, to appear in Astro. Astrophys. (2000) astro-ph/0004121
23. B. Carter, D. Langlois, Nuclear Physics B 454, 402 (1995).
24. B. Carter, D. Langlois, Nuclear Physics B 531, 478 (1998)
25. G. Mendell, L. Lindblom, Ann. Phys. 205, 110 (1991).
26. B. Carter, in Relativistic Fluid Dynamics, ed. A. Anile, M. Choquet Bruhat, Lecture Notes in Mathematics 1385, pp 1-64 (Springer -Verlag, Heidelberg, 1989).
27. B. Carter, D. Langlois, Phys. Rev. D 52, 4640 (1995).
28. R. Prix, to appear in Phys. Rev. D (2000) gr-qc/0004076
29. G.L. Comer, D. Langlois, Lap Ming Lin, Phys. Rev. D 60, 104025 (1999)
30. L. Lindblom, G. Mendell, ApJ 421, 689 (1994); U. Lee, AA 303, 515 (1995)