Quantum effects of Aharonov-Bohm type and noncommutative quantum mechanics

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Quantum mechanics in noncommutative space modifies the standard result of the Aharonov-Bohm effect for electrons and other recent quantum effects. Here we obtain the phase in noncommutative space for the Spavieri effect, a generalization of Aharonov-bohm effect which involving a coherent superposition of particles with opposite charges moving along single open interferometric path. By means of the experimental considerations a limit \( \sqrt{\theta} \approx (0,137\text{eV})^{-1} \) is achieved, improving by 10 orders of magnitude the derived by Chaichian et. al. for the Aharonov-Bohm effect. It is also shown that the noncommutative phases of the Aharonov-Casher and He-McKellar-Willkens effects are nullified in the current experimental tests.

INTRODUCTION

Recently there has been a growing interest in studying quantum mechanics in noncommutative (NC) space \([1,2,3,4,5,6]\). Because quantum nature experiments are measured with high precision, these are feasible scenarios for setting limits on the experimental manifestation of NC space. The Aharonov-Bohm (AB) effect, in which two coherent beams of charged particles encircle an infinite solenoid \([7]\), has been studied by Chaichian et. al. \([8]\), and Li and Dulat \([9]\) in the NC space. The expression of the obtained phase includes an additional term dependent on the NC space parameter, \( \theta \) (measured in units of \((\text{length})^2 \)). The limit on \( \theta \) found in the AB effect is of the order of \( \sqrt{\theta} \ll 10^6\text{GeV}^{-1} \) which corresponds to a relatively large scale of 1\(\text{A} \). This same approach was extended to the Aharonov-Casher (AC) \([10]\) effect by Li and Wang \([11]\), and Mirza and Zarei \([12]\), in this effect two coherent beams of neutral particles encircle an infinite charged wire. Considering the reported experimental error of the AC effect (\(\sim 25\% \)) \([13]\) a limit \( \sqrt{\theta} \ll 10^7\text{GeV}^{-1} \) is obtained \([12,11]\). The He-McKellar-Willkens (HWM) \([14,15]\) effect, in which neutral particles with electric dipole moment interact with an magnetic field, has been studied in the NC context by Wang and Li \([16,17]\), and by Dayi \([18]\) and in the context of the Anandan phase \([19]\) by Passos \([20]\).

There is no experimental report on the parameter limit \( \theta \) in quantum effects for electric dipoles. We consider here a new effect of AB type, proposed by Spavieri in \([21]\). In this effect, two beams of particles with charges, \(+q\) and \(-q\), move along a single side of an infinite solenoid, even though the beams do not enclose the solenoid (as in the ordinary AB effect). The advantage of this effect, called here “the S effect”, is that the size of the solenoid has no limit, so that it can be considered to be very large, such as a cyclotron. The S effect has been studied by Spavieri and Rodriguez \([22]\) in the context of massive electrodynamics (or photon mass).

In this scenario the product of two functions is replaced by the product Moyal-Weyl (or star “*”) \([25]\), so the ordinary Schrödinger equation, \( H\psi = E\psi \), is written as:

\[
H(\hat{x}_i,\hat{p}_i) * \psi = E\psi
\]  

\((1)\)

Now, the star product between two functions in an NC plane \((i,j = 1,2)\) is defined by the following expression:

\[
(f * g)(x) = \delta_{ij} \partial_i x_j f(x_i) g(x_j)
\]

\((2)\)

Under certain experimental considerations proposed and discussed in \([22]\), Spavieri and Rodriguez envisage a limit on the mass of the photon of \(m_\gamma \sim 10^{-31}\text{g} \), which is the best limit obtainable for the photon mass by means of a laboratory experiment with a quantum approach. Consequently, due to the success of the S effect in the photon mass scenario, we derive here the phase of the S effect in the context of the NC quantum mechanics as an application of the phase found by Chaichian et. al. \([8]\). Keeping the experimental proposal of \([22]\) we get a new limit on \( \theta \) in the context of the quantum effects of the AB type. In addition, recent advances in atomic interferometry have allowed obtaining measurements of the HWM phase \([23]\) which allows exploring experimentally the manifestation of the space NC in the HWM effect by means of the phases found in \([20]\) for these effects. This same analysis may be extended to the experimental configuration proposed by Sangster et. al. \([24]\) for the AC effect where the particles do not enclose the charged wire.

NC QUANTUM MECHANICS

In NC quantum mechanics, the commutation relationships of the position operators satisfy the relation, \([\hat{x}_i,\hat{x}_j] = i\theta_{ij}\), where \(\{\theta_{ij}\}\) is a fully antisymmetric real matrix representing the property noncommutativity of space and \(\hat{x}_i\) represents the coordinate operator (\(\hat{p}_i\) is the corresponding moment operator) in the space NC. In this scenario the product of two functions is replaced by the product Moyal-Weyl (or star “*”) \([25]\), so the ordinary Schrödinger equation, \( H\psi = E\psi \), is written as:

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Now, the star product between two functions in an NC plane \((i,j = 1,2)\) is defined by the following expression:

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(f * g)(x) = \delta_{ij} \partial_i x_j f(x_i) g(x_j)
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\((2)\)

where \(f(x)\) and \(g(x)\) are two arbitrary functions. Usually, the NC operators are expressed by means of the formulation of the Bopp shift \([20]\) (equivalent to \([22]\)). This
formalism maps the NC problem in the usual commutative space using new NC variables defined in terms of the commutative variables. That is to say,

$$\hat{x}_i = x_i - \frac{1}{2\hbar}\theta_{ij}p_j, \quad i, j = 1, 2,$$

(3)

where the variables \(x_i\) and \(p_i\) satisfy the usual canonical commutation relations, \([x_i, x_j] = 0, [x_i, p_j] = i\hbar\delta_{ij}\) and \([p_i, p_j] = 0\). With these considerations the Hamiltonian undergoes a coordinate transformation, \(H(\hat{x}_i, \hat{p}_i) = H(x_i - \frac{1}{2\hbar}\theta_{ij}p_j, p_i)\). Note that \(\theta_{ij} \ll 1\), so that the effects of the NC space can always be treated as a disturbance. If we consider a particle of mass \(m\) and charge \(q\) in the presence of a magnetic field (or potential vector \(A_i\)), then the Hamiltonian in the space NC, \(H(\hat{x}_i, \hat{p}_i, \hat{A}_i)\) undergoes a Bopp shift in both \(\hat{x}_i\) and \(\hat{A}_i\). Therefore, in the NC space and with a magnetic field the equation of Schrödinger takes the following form:

$$\frac{\hbar^2}{2m} \left( p_i - qA_i - \frac{1}{2}\theta_{ij}p_jA_i \right)^2 \psi = E\psi,$$

(4)

whose solution is:

$$\psi = \psi_0 \exp \left[ \frac{i}{\hbar} \int_{x_0}^{x} \left( A_i + \frac{1}{2}\theta_{ij}p_jA_i \right) dx_i \right].$$

(5)

where \(\psi_0\) is the solution of (4) when \(A_i = 0\).

**PHASE OF EFFECT S IN NC QUANTUM MECHANICS AND LIMIT OVER \(\theta\)**

In [21] Spavieri has pointed out that the amount observable in the AB effect is actually the phase difference

$$\Delta \varphi = \frac{e}{\hbar} \int \mathbf{A} \cdot d\mathbf{l} - \int \mathbf{A}_0 \cdot d\mathbf{l},$$

(6)

where the integral can be taken open over an open path integral. For the usual closed path \(C\) encircling the solenoid and limiting the surface \(S\), the observable quantity is the phase-shift variation, \(\Delta \phi \propto \oint_C \mathbf{A} \cdot d\mathbf{l} - \oint_C \mathbf{A}_0 \cdot d\mathbf{l} = \oint_C \mathbf{B} \cdot d\mathbf{S} - \oint_C \mathbf{B}_0 \cdot d\mathbf{S}\). In fact, in interferometric experiments involving the AB and AC effects [21] [24] the direct measurement of the phase \(\varphi \propto \int \mathbf{A} \cdot d\mathbf{l}\) or phase shift \(\varphi \propto \oint \mathbf{A} \cdot d\mathbf{l}\) is impossible in principle without the comparison of the actual interference pattern, due to \(\mathbf{A}\), with an interference reference pattern, due to \(\mathbf{A}_0\). Thus, \(\varphi \propto \Delta \phi\) are not observable, but the variations \(\Delta \varphi\) and \(\Delta \phi\) are both gauge-invariant observable quantities [21]. Therefore, with these considerations introduced by Spavieri [21], it is possible consider a new effect of AB type without particles encircling of solenoid. In this case the particles must has opposite charges, \(\pm e\), moving along one side of solenoid, i. e., along path \(b\). Thus, the phase of this new effect called Spavieri effect, \(S\), is:

$$\Delta \varphi_S = \frac{e}{\hbar} \int_b \mathbf{A} \cdot d\mathbf{l} - \int_b \mathbf{A}_0 \cdot d\mathbf{l} = \frac{2e}{\hbar} \int_b \mathbf{A} \cdot d\mathbf{l}.$$  

(7)

Now, we interesting in the phase (7) in the context of NC quantum mechanics. Substituting the phase (5) in (7), and retaining only term related with the parameter \(\theta\), we obtained the correction due to NC space for the effect \(S\), thus:

$$\Delta \varphi_{SNC} = \frac{e}{\hbar} \oint_b \theta_{ij}p_i\partial_jA_idx_i.$$  

(8)

Writing (5) en Cartesian coordinates, we obtained, the phase-shift of effect \(S\) in NC space:

$$\Delta \varphi_{SNC} = -\frac{en}{4\pi^2} \theta \cdot \int \left[ (\mathbf{v} \times \hat{\nabla}A_i) - \frac{e}{m} (\mathbf{A} \times \hat{\nabla}A_i) \right] dx_i,$$

(9)

where \(i = 1, 2\) are Cartesian components \(x\) and \(y\), \(m\) is the mass of electron and \(\mathbf{v}\) is the velocity of particles. Although the effect for \(\pm q\) charged particles is viable [21], the technology and interferometry for the test of this effect needs improvements. It is worth recalling that not long ago the technology and interferometry for beams of particles with opposite magnetic \(\pm m\) or electric \(\pm d\) dipole moments was likewise unavailable, but is today a reality [24] [28]. Discussions on this subject may act as a stimulating catalyst for further studies and technological advances that will lead to the experimental test of this quantum effect. An important step in this direction has already been made [21] by showing that, at least in principle and as far as gauge invariance requirements are concerned, this effect is physically feasible.

In the experimental setups detecting the traditional AB effect there are limitations imposed by the suitable type of interferometer related to the electron wavelength, the corresponding convenient size of the solenoid or toroid, and the maximum achievable size \(\rho\) of the coherent electron beam encircling the magnetic flux [21]. In the analysis made by Boulware and Deser [29] in the context of the limit of mass photon, the radius of the solenoid is \(a = 0.1cm\), and \(\rho\) is taken to be about 10cm, implying that the electron beam keeps its state of coherence up to a size \(\rho = 10^2a\), i.e., fifty times the solenoid diameter. The advantage of the new approach for the \(\pm q\) beam of particles is that the dimension of the solenoid has no upper limits and is conditioned only by practical limits of the experimental setup, while the size of the coherent beam of particles plays no important role. Due to these advantages of the new approach introduced by Spavieri and its success in the exploration of the limit of the mass of the photon, it is asked which limit could be reached for the parameter \(\theta\) of the NC quantum mechanics. To answer this question, we consider that the electrons move
along the straight line \( y = y_0 \) from \( x = -x_0 \) to \( x = x_0 \) (open path \( b \) in [53]), with a velocity \( \mathbf{v} = v \mathbf{i} \), thus \( i = x \) in [9]. In addition, as in [8], here it is considered that 

\[ \Phi = \Phi_0 = 2.06 \times 10^{-15} \text{ T m}^2. \]

are the following: 

\[ \lambda_c = \frac{\Phi_0}{\Phi} c \]

that the kinetic term is up to 5 times smaller than the geometric term. This contrasts with the analysis made by Chiachian et. al. [8] for the AB effect where the kinetic term is five orders greater than the geometric term. Consequently, for the estimation of the limit on \( \theta \) we consider only the first term in brackets of expression

\[ \Delta \phi_S^{NC} \approx \frac{1}{8} \left( \frac{\Phi}{\Phi_0} \right)^2 \arctan \left( \frac{x}{y} \right) \]

which is 10 orders of magnitude smaller than the value of Chiachian et. al. [3] for the effect AB.

QUANTUM EFFECT FOR ELECTRIC AND MAGNETIC DIPOLES IN NC QUANTUM MECHANICS

The phase to magnetic dipoles, \( \mathbf{m} \) (effect AC) and for electric dipoles, \( \mathbf{d} \), (effect HMW) in noncommutative space also has been calculated by Passos et al. [20]. The expressions are as follows:

\[ \phi_{AC} = \frac{i}{2} \int (\mathbf{m} \times \mathbf{E}) \cdot d\mathbf{r} + \frac{i}{2} m \int \mathbf{\mathbf{\nabla}} \cdot \left( \mathbf{m} \times \mathbf{E} \right) \cdot d\mathbf{r} \]

\[ \phi_{HMW} = -\frac{i}{2} \int (\mathbf{d} \times \mathbf{B}) \cdot d\mathbf{r} - \frac{i}{2} m \int \mathbf{\mathbf{\nabla}} \cdot \left( \mathbf{d} \times \mathbf{B} \right) \cdot d\mathbf{r} \]

where \( m \) is the mass of electric or magnetic dipoles. In the experimental setup proposed by Sangster et. al. [24], magnetic dipoles with opposing dipole moments are moving on the same interferometric path. With this configuration of magnetic moments the beams do not need to enclose a charged wire (as in the ordinary AC effect), but are moving in the presence of a homogeneous electric field produced by a capacitor of parallel plates. In the configuration of Sangster [24] the magnetic dipole moments, \( \mathbf{m} \), are perpendicular to the electric field, \( \mathbf{E} \),
thus the terms $\nabla \cdot (m \times E)$ in (13) vanish and the effects of the NC space can not be observed in this configuration. In the same sense, in a recent experiment carried out to observe the HMW phase \cite{32}, the electric dipole moments, $d$, of the beams are perpendicular to the magnetic field, $B$, involved in the effect. Thus, the terms $\nabla \cdot (d \times B)$ in the expression (13) vanish and the NC effect, as a function of the expression (13) derived by Passos et. al. (14), is not evidiential in this configuration. Another proposed configuration for observing the AB effect for electric dipoles is known as Takchuk effect \cite{30}. In this configuration, it is consider two infinite wires with an opposite magnetic polarization dependent on the length of the wire, that is, $M(z) = -q z$, where $q$ can be treated as a linear magnetic charge density. If the wires are sufficiently long, the magnetic vector potential can be written as $A_T = z A_{AB}$, $A_{AB}$ being the ordinary vector potential of the AB effect. Thus, the magnetic field is $B = \nabla \times A_T = z (\nabla \times A_{AB}) - A_{AB} \times z$. In this effect the beams of magnetic dipoles move in the middle plane of the wires, that is, $z = 0$, with their polarization, $d$, parallel to the wire axis. The term of interest in the NC context according to (12) is $d \times B$. Therefore, $d \times B = d z \times (z (\nabla \times A_{AB}) - A_{AB} \times z)$, evaluated at $z = 0$, we obtain that $d \times B = d A_{AB}$, but $\nabla \cdot (d \times B) = \nabla \cdot A_{AB} = 0$ for the ordinary AB effect (Coulomb’s gauge). This implies that in the scenario presented by Takchuk \cite{30}, the NC effect can not be observed.

CONCLUSIONS

We have derived the AB phase for open path, or S effect, in the context of NC quantum mechanics (eq. 12). Considering the experimental parameters of the S effect to obtained a limit on mass photon, here we obtained an upper limit on the NC parameter, $\sqrt{\theta} \leq (0.13 \text{TeV})^{-1}$, 10 orders of magnitude smaller than in previous scenarios of the AB type, and three order of magnitude smaller that the limit derived by Moummi et. al. $\leq (0.16 \text{GeV})^{-1}$, in the context of the energy lines of the hydrogen atom, which is also a quantum scenario. It can also be observed that the kinetic term in our result, which includes the speed and mass of the particle, is not relevant for $\theta$ calculation since it is several orders of magnitude smaller than the geometric terms, which is opposite to the result found Chaichian et. al. \cite{8}. Also we have shown that the NC effects are not manifested in the experimental configurations (for to interferometric open path) proposed by Sangster et. al. \cite{24} for the AC effect and the proposal of Lepoutre et. al. \cite{23,35} and Takchuk \cite{30} for the HMW effect from the point of view of the phases derived by Passos et. al. \cite{21} for these purposes. Finally, it is important to mention that these results can be improved in the future due to the development of new atomic interferometers, with more precision and longer interferometric paths \cite{39}.

[1] K. Ma and J.-H. Wang, Ann. Phys. 383, 120 (2017).