Integrable Circular Brane Model and Coulomb Charging at Large Conduction

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Abstract

We study a model of 2D QFT with boundary interaction, in which two-component massless Bose field is constrained to a circle at the boundary. We argue that this model is integrable at two values of the topological angle, $\theta = 0$ and $\theta = \pi$. For $\theta = 0$ we propose exact partition function in terms of solutions of ordinary linear differential equation. The circular brane model is equivalent to the model of quantum Brownian dynamics commonly used in describing the Coulomb charging in quantum dots, in the limit of small dimensionless resistance $g_0$ of the tunneling contact. Our proposal translates to partition function of this model at integer charge.

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1 The model

In this note we address a model of 2D Euclidean field theory with boundary interaction which we call the circular brane model. The model contains two-component Bose field \( \mathbf{X}(z, \bar{z}) = (X^1(z, \bar{z}), X^2(z, \bar{z})) \) defined on a disk of a radius \( R \) with usual complex coordinates \((z, \bar{z}), |z| \leq R\). Its bulk dynamics is that of a free massless field, as described by the action

\[
A_{\text{disk}} = \frac{1}{\pi} \int_{|z| \leq R} d^2 z \partial_z \mathbf{X} \cdot \partial_{\bar{z}} \mathbf{X}
\]  

(d\(d^2 z = dx dy\), but it obeys nonlinear boundary condition

\[
X^2_B = \frac{1}{g_0},
\]

where \( X_B \) stands for the values of \( \mathbf{X} \) at the boundary \( |z| = R \), and \( g_0 \) is a coupling constant. Due to the nonlinear boundary condition, the theory needs renormalization. It has to be equipped with the ultraviolet (UV) cutoff, and consistent removal of the UV divergences requires that the bare coupling constant \( g_0 \) be given a dependence of the cutoff momentum \( \Lambda \), according to the Renormalization Group (RG) flow equation

\[
\Lambda \frac{dg_0}{d\Lambda} = -2 g_0^2 - 4 g_0^3 + \cdots.
\]

The leading two terms of the beta function written down in (3) were computed in [1] and [2], and indeed agree with the more general calculations in [3] and [4]. Eq. (3) shows that the theory is asymptotically free – at short distances the effective size of the circle (2) becomes large. It is believed that the theory develops a physical energy scale \( E^* \), and at the distances \( \gg 1/E^* \) from the boundary the effect of the boundary is that of the fixed boundary condition \( X_B = 0 \) (the boundary (2) “flows” to the Dirichlet boundary). According to the equation (3)

\[
E^* \sim \Lambda \ g_0^{-1} \ e^{-\frac{1}{2g_0}}.
\]

As usual, the effect of the boundary can be described in terms of the boundary state [5]. The boundary state \( |B\rangle \) is a special vector in the space of states of radial quantization (in our case the space of states of two-component free massless Bose field on a circle of the circumference \( 2\pi R \)). This state incorporates all information about the boundary conditions; any correlation function can be written in terms of the Euclidean-time Heisenberg operators sandwiched between \( |B\rangle \) and the radial-quantization vacuum.
0⟩. In particular, the overlap ⟨0|B⟩ coincides with the disk partition function, up to the factor $R^{c/6}$. In our case |0⟩ is the Fock vacuum of the free massless field $X(z,\bar{z})$ with the zero-mode momentum $P = 0$. More generally, we will be interested in the overlap amplitude ⟨P|B⟩, where |P⟩ is the Fock vacuum with the zero-mode momentum $P$; of course, this amplitude relates to the (unnormalized) one point correlation function of the exponential field inserted at the center of the disk,

$$\langle P | B \rangle = R^{-c/6 + P^2/2} \left< e^{iP \cdot X(0,0)} \right>_{\text{disk}} ,$$

(5)

Here $c = 2$ is the central charge of the bulk theory. It will be convenient to transform the spherical brane model to the geometry of semi-infinite cylinder, using the standard exponential map:

$$z/R = e^{u/R}, \quad \bar{z}/R = e^{\bar{u}/R},$$

(6)

where $u = \sigma + i\tau$, $\bar{u} = \sigma - i\tau$, with $\sigma$ ranging from $-\infty$ to 0, and $\tau$ being a periodic coordinate with the period $2\pi R$. In what follows we will write (with some abuse of notations) $X(\tau,\sigma)$ for the field $X$ in (1) expressed in terms of these cylindrical coordinates. Then, introducing the shifted field variable

$$Y(\tau,\sigma) = X(\tau,\sigma) + \frac{i\sigma}{R} P ,$$

(7)

one can relate the the amplitude ⟨P|B⟩ to the partition function:

$$Z(R, H) = \lim_{\ell \to \infty} \left\{ 2 e^{\frac{\pi}{\ell}} \int D Y e^{-A_{\text{cyl}}[Y]} \right\} ,$$

(8)

where the integration field variable $Y(\tau,\sigma)$ is still subject to the circular brane constraint

$$Y_B^2 = \frac{1}{g_0} ,$$

(9)

and

$$A_{\text{cyl}}[Y] = \frac{1}{4\pi} \int_{-\ell}^{0} d\sigma \int d\tau \left( \partial_\sigma Y \cdot \partial_\sigma Y + \partial_\tau Y \cdot \partial_\tau Y \right) - 2 \oint d\tau H \cdot Y_B .$$

(10)

1The factor 2 in (8) is added partly to eliminate the spurious boundary degeneracy of the auxiliary Neumann boundary at $\sigma = -\ell$ in (10), and partly to take into account the infrared boundary entropy of the real boundary at $\sigma = 0$; with this normalization the $R \to \infty$ behavior of $Z(R, H)$ is $\sim e^{-2\pi R \ell}$, with the coefficient 1.
Here and below \( \oint d\tau \) stands for \( \int_0^{2\pi R} d\tau \). The “external field” \( \mathbf{H} \) relates to \( \mathbf{P} \) in (5) as

\[
\mathbf{H} = \frac{i \mathbf{P}}{4\pi R},
\]

and the argument \( H \) in the r.h.s. of (8) is \( \sqrt{H^2} \). Thus

\[
\langle \mathbf{P} | \mathbf{B} \rangle = \frac{1}{\sqrt{2}} Z \left( R, \sqrt{-\frac{\mathbf{P}^2}{4\pi R}} \right),
\]

where the factor \( \frac{1}{\sqrt{2}} \) is the boundary degeneracy \( \mathcal{G} \) of the infrared (Dirichlet) fixed point. We note that by the nature of the boundary condition (9) the partition function \( Z(R, H) \) is expected to be an entire function of \( H \), and its analytic continuation to pure imaginary \( H \) is unambiguous. In the form (10) the model has interpretation as 1+1 QFT on a half-line, at the temperature \( T = \frac{1}{2\pi R} \). At \( R \to \infty \) the partition function (8) develops standard linear asymptotic

\[
\log Z(R, H) \to -2\pi R \mathcal{E}(H) \quad \text{as} \quad R \to \infty,
\]

where \( \mathcal{E}(H) \) is the boundary energy. This quantity depends on \( H \) and \( g_0 \), the latter dependence being through the energy scale \( E^* \), Eq.(4). We found it convenient to fix overall normalization of \( E^* \) relating it to the zero temperature susceptibility,

\[
\frac{1}{E^*} = -\frac{1}{2} \left. \frac{\partial^2 \mathcal{E}}{\partial H^2} \right|_{H=0}.
\]

The circular brane model shows many similarities with the \( O(3) \) sigma model (or \( n \)-field), the asymptotic freedom (3) being just one of them [7]. Another similarity is apparent when one observes that the field configurations \( \mathbf{Y}(\tau, \sigma) \) in (8) can be separated into the topological classes characterized by the integer-valued winding number

\[
w = \frac{g_0}{2\pi} \oint d\tau \mathbf{Y}_B \wedge \partial_\tau \mathbf{Y}_B,
\]

just like the sigma-model field configurations \( n \) fall into their topological classes characterized by the mapping degree \( S^2 \to S^2 \). Moreover, in the sigma-model the instanton configurations minimizing the action in each topological sector are found explicitly in terms of rational functions [8]; very similar expressions exist for the instanton configurations in the circular brane model [9,10]. The calculations of the instanton determinants [11]...
also exhibit much similarity \cite{12,13,14}. As usual, existence of the integer-valued topological charge allows one to introduce another parameter, the topological angle $\theta$, 

$$Z_\theta = \sum_{w \in \mathbb{Z}} Z^{(w)} e^{i\theta w},$$

(16)

where the partition functions $Z^{(w)}$ are the functional integrals \cite{5} taken over the field configurations in the topological class $w$ only. The full partition function $Z$, as defined by Eq.\,(8), involves all topological sectors with equal weights, i.e. $Z = Z_{\theta=0}$. It is this $\theta = 0$ partition function which we had in mind in defining the boundary energy $E(H)$ and the energy scale $E^*$ through \cite{13,14}.

One of the reasons we mention this analogy here is because the $O(3)$ sigma model is known to be integrable at two special values of the topological angle, $\theta = 0$ and $\theta = \pi$ \cite{15,16,17,18}. Therefore it is natural to expect that the circular brane model is also integrable at these two special values of $\theta$ \footnote{It is worth mentioning here that the case $\theta = \pi$ is of special interest. In this case the circular brane model flows to a nontrivial conformal boundary condition which is believed to be related to the infrared fixed point of two-channel spin-$\frac{1}{2}$ Kondo model in its interacting sector \cite{19}.}. We argue in Appendix that for these values of $\theta$ the model admits a number of nontrivial higher-spin local integrals of motion. In fact, our argument there goes for the whole class of the \textquotedblleft $O(N)$ spherical brane models\textquotedblright, with $N$-component field $X$, subject to the spherical brane boundary condition \cite{2}. Moreover, in these models it is possible to describe the whole set of the local integrals of motion $I_{2k-1}^{2k}, k = 1, 2, 3 \ldots$ in some details.

The aim of this note is to present a proposal for the exact form of the partition function \cite{5} at the integrable point $\theta = 0$ in terms of solutions of certain ordinary differential equation. We will present this in the next section, but let us first make some remark and introduce suitable notations. As was mentioned above, renormalization trades the bare coupling constant $g_0$ for the RG invariant scale \cite{4}, and the partition function actually depends on the dimensionless combination $RE^*$. In fact, in the circular brane model (as well as in the $O(3)$ sigma model) the last statement is not entirely true. As was observed in \cite{12}, the functional integral \cite{5} has a specific non-perturbative divergence due to the small-size instantons, which cannot be absorbed into the renormalization of the coupling constant $g_0$. Due to this effect, the partition function \cite{5} is expected to have the form

$$Z_\theta = e^{L_0 \cos \theta} Z^*_0 (\kappa, h),$$

(17)
where
\[ L = A \log \left( \frac{B(g_0) \Lambda/E^*}{E^*} \right), \tag{18} \]
and
\[ \kappa = 2\pi R E^*, \quad h = 2\pi R H. \tag{19} \]

Here \( A \) is a constant, which is universal (actually \( A = 2 \)), but \( B(g_0) \) is not – it depends on the details of UV cutoff. The factor \( Z^*_\theta \) depends on the RG invariant parameters \( \kappa, h \) and \( \theta \) only. Similar nonperturbative small-instanton divergence is well known to be present in the \( O(3) \) sigma model \[11\], where it is blamed, for instance, for the violation of the normal scaling of the topological susceptibility \[20\]. It is important to note that the small instanton divergence does not affect the \( h \)-dependence of the free energy (interaction of small instantons with the external field \( H \) is negligible), therefore the energy scale \( E^* \) defined through (13) and (14) enjoys the normal RG behavior \(4\).

In what follows we discuss the universal factor \( Z^*(\kappa, h) \) in the full partition function at \( \theta = 0 \),
\[ Z(R, H) = e^{\kappa L} Z^*(\kappa, h), \tag{20} \]
which carries all interesting dependence on \( H \) and \( R \). Since any factor of the form \( e^{const \kappa} \) can be absorbed into the redefinition of \( B(g_0) \) in (18), we fix the ambiguity by assuming that \( Z^*(\kappa, 0) \to 1 \) as \( \kappa \to \infty \). With this convention, the boundary energy can be written as
\[ \mathcal{E}(H) = LE^* + \mathcal{E}^*(H), \tag{21} \]
where \( \mathcal{E}^*(0) = 0 \).

## 2 The partition function

It was discovered some years ago in Ref. \[21\] that in the case of the boundary flow in the minimal CFT with integrable boundary perturbation \[22\], the overlap analogous to (5) can be related exactly to the eigenvalue problem of certain ordinary differential operator. Despite the fact that this relation was proven \[23\], and similar relations were found in other integrable models of CFT with non-conformal boundary interactions \[24,25\], its deeper reason remains mystery to us. Nonetheless, it looks natural to assume that it might be a general phenomenon, and try to identify associated differential operator.
for the circular brane model. Below we simply present our proposals for such differential operator. The motivations came from detailed studies of somewhat more general, but still integrable “brane” model (we call it the “paperclip brane model”); it generalizes the circular model in a way similar to the “sausage” deformation of the $O(3)$ sigma model \[26\]. We will report these studies elsewhere \[27\].

Consider the following ordinary differential equation

\[
\begin{cases}
-\partial_v^2 + \kappa^2 \exp(e^v) + h^2 e^v 
\end{cases} \Psi(v) = 0 .
\]

Let $\psi_-(v)$ be the solution which decays at large negative $v$, 

\[ \psi_-(v) \to e^{\kappa v} \quad \text{as} \quad v \to -\infty . \]  

Also, let $\psi_+(v)$ be the solution decaying at large positive $v$, 

\[ \psi_+(v) \to \exp \left\{ -\frac{e^v}{4} + \kappa \text{Ei} \left( \frac{e^v}{2} \right) \right\} \quad \text{as} \quad v \to +\infty , \]

where $\text{Ei}(x)$ is the integral exponent function, $\text{Ei}(x) = \text{P.V.} \int_{-\infty}^{\infty} \frac{dt}{t} e^{-t}$. The asymptotics \[23\] and \[24\] specify the solutions $\psi_-(v)$ and $\psi_+(v)$ uniquely, including their normalizations. Then

\[
Z^* = \sqrt{\pi \kappa} \left( \frac{2 e^{\gamma E} - 2 \gamma^2}{2} \right)^{\kappa} \Gamma(1 + 2\kappa) W[\psi_+, \psi_-],
\]

where $W$ is the Wronskian $\psi_+ \partial_v \psi_- - \psi_- \partial_v \psi_+$ and $\gamma_E = 0.577216 \ldots$ is Euler’s constant. Thus determination of $Z^*$ reduces to the problem of the ordinary differential equation.

### 3 Low temperature expansion

At large $R$ (low temperatures) the equation \[22\] can be studied using WKB expansion. The leading WKB approximation yields explicit expression for the boundary energy \[24\],

\[ \mathcal{E}^*(H) = -E^* \int_0^{\infty} \frac{dt}{t} \left( \sqrt{\delta^t + \frac{t}{H/E^*}} - e^{t/2} \right) . \]

Note that the integral admits the expansion in powers of $H^2$,

\[ \mathcal{E}^*(H) = E^* \sum_{k=1}^{\infty} C_k \left( \frac{H}{E^*} \right)^{2k} , \]
with

\[ C_k = \frac{(-1)^k}{2\sqrt{\pi}} \frac{\Gamma(k - \frac{1}{2})}{k} \left( k - \frac{1}{2} \right)^{-k}. \]

This expansion should be understood in terms of the expansion (66), with the powers \( H^{2k} \) representing the highest order terms \( h^{2k} \) of the polynomials \( I_{2k-1} \), the vacuum eigenvalues of the local IM \( I_{2k-1} \). Note that the above expression for the coefficients \( C_k \) is consistent with (71). The next order of the WKB expansion provides leading correction to the linear asymptotic (13),

\[ \log Z^* = -2\pi R \mathcal{E}^* - \frac{1}{2\pi R} \mathcal{E}_2^* + O(R^{-3}), \quad (28) \]

where, by direct calculation,

\[ \mathcal{E}_2^* = \frac{1}{48H} \frac{\partial}{\partial H} \left( H \frac{\partial \mathcal{E}^*}{\partial H} \right). \quad (29) \]

Again, expanding (28) in powers of \( H \) we have

\[ \log Z^* \simeq -\sum_{k=1}^{\infty} C_k \left( 2\pi \mathcal{E}^* \right)^{1-2k} I_{2k-1}, \quad (30) \]

with

\[ I_{2k-1} = \frac{1}{R^{2k-1}} \left( h^{2k} + \frac{k^2}{12} h^{2k-2} + \ldots \right), \quad (31) \]

in perfect agreement with Eqs. (66), (69). Further terms in (31) can be obtained by computing yet higher orders of the WKB expansion. This way one finds

\[ I_1 = \frac{1}{R} \left( h^2 + \frac{1}{12} \right), \]
\[ I_3 = \frac{1}{R^3} \left( h^4 + \frac{1}{3} h^2 + \frac{1}{40} \right), \quad (32) \]
\[ I_5 = \frac{1}{R^5} \left( h^6 + \frac{3}{4} h^4 + \frac{19}{100} h^2 + \frac{71}{5040} \right), \]

again in agreement with (68).
4 High temperature expansion

It is also possible to develop short distance (high temperature) expansion of the partition function (25). For small $\kappa$ the solutions $\psi_-(v)$ and $\psi_+(v)$ can be evaluated using appropriate versions of perturbation theory. To make the resulting formulae compact we shall describe here these expansions in the case $h = 0$ only. The solution $\psi_-(v)$ can be written as

$$\psi_-(v) = e^{\kappa v} \left\{ 1 + \sum_{n=1}^{\infty} a_n(\kappa) e^{nv} \right\}, \quad (33)$$

where the coefficients $a_n(\kappa) = O(\kappa^2)$ and admit systematic expansions in powers of $\kappa$. The $\kappa \to 0$ expansion of the solution $\psi_+(v)$ is more subtle. Instead of powers of $\kappa$, it expands in powers of the “running coupling constant” $g = g(\kappa)$ defined by the equation

$$\kappa = g^{-1} e^{-\frac{1}{2g}}. \quad (34)$$

After change of variables

$$v = g x - \log(g) \quad (35)$$

the equation (22) becomes

$$\left\{ -\partial_x^2 + e^x + \delta U(x) \right\} \Psi = 0, \quad (36)$$

where

$$\delta U(x) = \exp \left( \frac{e^{gx} - 1}{g} \right) - e^x. \quad (37)$$

For $|x| \sim 1$ the term $\delta U(x)$ is $O(g)$; in this domain solutions of (36) admit systematic expansion in powers of $g$. Expansion of $\psi_+$ is obtained by iterations of (36), starting with $\psi_+^{(0)} = C(g) K_0(2e^{x/2})$, where $K_0$ is the Macdonald function and the normalization constant $C(g)$ must be adjusted to match the asymptotic form (24). It turns out that both expansions, of $\psi_-$ in the powers of $\kappa$, and of $\psi_+$ in the powers of $g$, have common domain of validity at $v \sim -\log(g)$, and can be used there for evaluation of the Wronskian in (25). As the result, the following form of the small $\kappa$ expansion of the partition function (20) emerges

$$Z(R,0) \simeq e^{2\kappa \log(2\pi B(g_0) R \Lambda)} \frac{g^\kappa}{\sqrt{g}} \sum_{n=0}^{\infty} \kappa^n z_n(g), \quad (38)$$
where the coefficients $z_n(g)$ are power series in $g$. The term $n = 0$ in (38) should be interpreted as the perturbative contribution to the partition function, the term $n = 1$ corresponds to the one-instanton contribution, and the higher nonperturbative terms $n = 2, 3 \ldots$ presumably describe the contributions of multiple instanton and anti-instanton configurations. Explicit computation along the lines described above yields

\[
\begin{align*}
  z_0(g) &= 1 - (1 + \gamma_E) g + O(g^2), \\
  z_1(g) &= \log(2) - 2 + 3\gamma_E - 2\gamma_E g - 2(2 + \gamma_E) g^2 + O(g^3).
\end{align*}
\] (39)

5 Dissipative quantum rotator

The circular brane model has useful interpretation in terms of Brownian dynamics of quantum rotator. It was noticed a while ago in Ref. [28] that the free massless bulk dynamics [10] is equivalent to the Caldeira-Leggett model of quantum thermostat [29]. Upon fixing the boundary values $Y_B(\tau)$ and integrating out the bulk part of the field $Y(\tau, \sigma)$, Eq. (8) reduces to

\[
Z(R, H) = \int \mathcal{D}\eta \, e^{-A_{\text{diss}}[\eta]},
\] (40)

with

\[
A_{\text{diss}}[\eta] = -\frac{2H}{\sqrt{g_0}} \int d\tau \cos(\eta) + \frac{1}{8\pi^2 R^2 g_0} \int d\tau \int d\tau' \frac{\sin^2 \left( \frac{\eta(\tau) - \eta(\tau')}{2} \right)}{\sin^2 \left( \frac{\tau - \tau'}{2R} \right)},
\] (41)

where $\eta(\tau)$ is the angular field defined through $H \cdot Y_B(\tau) = H/\sqrt{g_0} \cos \eta(\tau)$. The model similar\(^3\) to (41) was introduced in [30] (see e.g. [31] for a review) as an effective field theory describing tunneling of quasiparticles between superconductors. More recently it was subject of much interest in studying of the phenomenon of Coulomb blockade in quantum dots [32], in the regime where the dimensionless resistance of the tunneling contact $g_0$ is small [33]. [12][2][13][14]. In this context, the effective action in fact contains another term,

\[
A_{\text{CB}} = A_{\text{diss}} + \frac{1}{4E_C} \int d\tau \eta^2, \quad (42)
\]

\(^3\)The model of [30] has the potential term $\cos(2\eta)$ instead of $\cos(\eta)$ in [11].
where \( E_C \) has the meaning of the charging energy of the dot in the absence of the tunneling, \( E_C = e^2/2C \). Of course, \( \tau \) is the Matsubara time, and \( 2\pi R = 1/T \), the inverse temperature. When \( T \ll E_C \) this term just provides explicit UV cutoff, with the cutoff energy

\[
\Lambda = E_C/g_0.
\]  
(43)

With the cutoff procedure thus specified, the “small-instanton” factor \( e^{\kappa L} \) in \( (20) \) becomes unambiguous. By comparing \( (38), (39) \) with direct one-instanton calculation in \( (42) \) \cite{12, 13, 14} one finds for small \( g_0 \)

\[
L = \frac{1}{g_0} + 5 \log(g_0) + O(1).
\]  
(44)

Eqs.\( (20), (25) \) then provide the partition function of the model \( (42) \) at \( \theta = 0 \), in the limit of small \( g_0 \). One only has to relate the energy scale \( E^* \) in \( (19) \) to the parameters \( g_0 \) and \( E_C \) in the above action \( (42) \). When \( E_C/g_0 \gg H \gg E^* \) and \( g_0 \) is small, the boundary energy \( \mathcal{E}(H) \) \cite{21} can be computed directly from \( (42) \), using the standard perturbative expansion in \( g_0 \). By developing two-loop order in this perturbative expansion and comparing it with the \( H/E^* \gg 1 \) behavior of the integral in \( (26) \), we found

\[
E^* = \frac{E_C}{2\pi^2} \frac{g_0}{g_0} e^{-\frac{\pi}{g_0}} \left\{ 1 - \frac{3\pi^2}{4} g_0 + O(g_0^2) \right\}.
\]  
(45)

Finally, let us make a remark on the topological susceptibility in the model \( (42) \). As was already mentioned, the topological susceptibility

\[
E_C^* = -\frac{\pi}{R} \frac{\partial^2}{\partial \theta^2} \log(Z_\theta)\bigg|_{\theta=H=0}
\]  
(46)
in the circular brane model suffers from the small-instanton divergence, which leads to the non-universal factor \( e^{\kappa L} \cos \theta \) in \( (17) \). On the other hand, in applications to the Coulomb blockade, the topological susceptibility in \( (42) \) has direct physical meaning – it coincides with measurable capacitive energy of the quantum dot. Explicit cutoff in the action \( (42) \) makes the small-instanton factor unambiguous. According to \( (17) \), at zero temperature and zero \( H \), \( E_C^* \) has the form \( 2\pi^2 (L + \text{const}) E^* \), where the constant comes from the \( \theta \)-dependence of the second factor in \( (17) \). Since our proposal is strictly limited to the case \( \theta = 0 \), it does not allow for determination of this constant. However, at small \( g_0 \) the dominating contribution comes from the term \( LE^* \), and therefore in view of \( (14) \) and \( (15) \), the charging energy at zero temperature is estimated as

\[
E_C^*|_{T=0} = \frac{E_C}{g_0} e^{-\frac{1}{g_0}} \left\{ \frac{1}{g_0} + 5 \log g_0 + O(1) \right\}.
\]  
(47)
Of course, the leading behavior \( (1/g_0 \text{ in the brackets}) \) here is consistent with the one-instanton analysis in the Gaussian approximation \([12,13,14]\). Nontrivial prediction which follows from our proposal is the logarithmic term in (47).

6 On other spherical brane models

The circular brane model is a particular case \( N = 2 \) of the “\( O(N) \) spherical brane model”, the latter containing \( N \)-component field \( X \), subject to the same boundary condition \([2]\), so that \( X_B \in S^{N-1} \). As we argue in Appendix, the spherical brane model is integrable for any \( N \geq 1 \). Although the bulk part of this paper is devoted to the circular case \( N = 2 \), here we would like to say few words about other cases. First, for generic \( N \neq 2 \) there are no topological sectors, and the small-instanton divergence is not present. For \( N \neq 2 \) the overlap (5) \((c = N)\) can be written as

\[
\langle P \mid B \rangle = 2^{-\frac{N}{2}} e^{-\kappa C_0} Z^*(\kappa, h),
\]

where \( \kappa \) and \( h \) are still the dimensionless parameters \([19]\), and \( E^* \) is defined in Eq. (14). The factor \( Z^*(\kappa, h) \) here is normalized as in (20), i.e. \( Z^*(\kappa, 0) \to 1 \) as \( \kappa \to \infty \), and the \( N \)-dependent constant \( C_0 \) is the same as in Eqs. (66),(70).

For generic \( N \) we were not able to find differential equations which could possibly generate the overlap (48), as (22) does for \( N = 2 \). However, for two other cases, \( N = 1 \) and \( N = 3 \), representations of the function \( Z^* \) in (48) in terms of solutions of certain differential equations exist. Here we would like to describe what these differential equations are.

For \( N = 1 \), it is useful to note that the model is equivalent to the interacting sector of the one-channel spin-\( \frac{1}{2} \) Kondo model. At \( N = 1 \) the sphere degenerates to two points, which represent two states of the impurity spin. Exact equivalence is established by bosonizing the Kondo fermions (see e.g. Ref. [34]). As is well known, the Kondo model is solvable by means of Bethe ansatz technique \([35,36]\). Alternative (and equivalent) representation of the partition function can be given in terms of the solutions of the differential equation \([25]\)

\[
\left\{ -\partial_v^2 + 2\pi\kappa^2 v e^v - h^2 \right\} \Psi(v) = 0.
\]

The function \( Z^* \) coincides with associated Stokes multiplier (see [25] for the details).
For $N = 3$, consider the differential equation
\[
\left\{ -\partial^2_v + \pi \kappa^2 e^{v^2/2} + h^2 \right\} \Psi(v) = 0.
\] (50)

Let $\psi_+$ and $\psi_-$ be its solutions uniquely determined by the asymptotics
\[
\psi_+ \rightarrow \exp\left\{ -\frac{v^2}{8} - \pi \kappa \text{Erfi}(v/2) \right\} \quad \text{as} \quad v \rightarrow +\infty, \quad (51)
\]
\[
\psi_- \rightarrow \exp\left\{ -\frac{v^2}{8} - \pi \kappa \text{Erfi}(-v/2) \right\} \quad \text{as} \quad v \rightarrow -\infty,
\]
where $\text{Erfi}(x)$ is the imaginary error function, $\text{Erfi}(x) = \frac{2}{\sqrt{\pi}} \int_0^x dt \, e^{t^2}$. The suggested relation is
\[
Z^*(\kappa, h) = (2\kappa \sqrt{\pi})^{-1} W[\psi_+, \psi_-].
\] (52)

Like in the case $N = 2$, the $\kappa \rightarrow \infty$ expansion of (52) can be obtained by WKB analysis of the solutions of (51). Thus, the leading WKB approximation yields the boundary energy (70), with the coefficients $C_k$ given exactly by (71) with $N = 3$. Moreover, higher orders of the WKB expansion reproduce the eigenvalues (68), (69) perfectly.

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7 Appendix

7.1 The $O(N)$ spherical brane model

In this appendix we present arguments in favor of integrability of the circular brane model at $\theta = 0$ and $\theta = \pi$. In fact, the arguments apply without much
modification to more general model, which contains $N$-component Bose field $X = (X^1, \ldots, X^N)$, with $N \geq 1$, but otherwise is described by the same action (11) and the “spherical brane” boundary condition (2). Therefore throughout this appendix we mostly address this $N$-component spherical brane model. Presence of arbitrary number of components allows for a number of checks within the $1/N$ expansion.

7.2 Integrals of motion

In the bulk, the spherical brane model is a free Bose field, and as such it certainly has infinite number of integrals of motion. The derivative $\partial X$, where $\partial \equiv \partial_u$ stands for the derivative over $u = \sigma + i\tau$, is a holomorphic field in the bulk. Hence any local polynomial $P(u) = P(\partial X, \partial^2 X, \ldots)$ of this and higher derivatives with respect to $u$ gives rise to a local integral of motion

$$\mathbb{I}[P] = \oint \frac{d\tau}{2\pi} P(\partial X, \partial^2 X, \ldots) ,$$

(53)

where we have assumed the coordinate $\sigma$ along the cylinder to be the Euclidean time. Here and below the bulk composite fields entering these polynomials are always understood in terms of the normal ordering with respect to the standard Wick pairing $\langle X^a(u, \bar{u}) X^b(u', \bar{u}') \rangle = -\delta^{ab} \log |u - u'|$ corresponding to a bulk free field in an infinite space. Certainly, there is a “left-moving” counterpart to each of the integrals (53),

$$\bar{\mathbb{I}}[P] = \oint \frac{d\tau}{2\pi} P(\bar{\partial} X, \bar{\partial}^2 X, \ldots) ,$$

(54)

where $\bar{\partial} \equiv \partial_{\bar{u}}$. The integrals (53),(54) are operators acting in the space of states of free $N$-component massless Bose field quantized on a spatial circle of the circumference $2\pi R$, i.e.

$$\mathcal{H} = \int_{\mathcal{P}} \mathcal{F}_{\mathcal{P}} \otimes \bar{\mathcal{F}}_{\mathcal{P}} ,$$

(55)

where $\mathcal{F}_{\mathcal{P}}$ is the Fock space of the right-moving bosons with the zero-mode momentum $\mathcal{P}$. Large set of integrals (53) contains many infinite subsets $\mathbb{I}[P_s]$ (corresponding to special sequences of the polynomials $P_s$) of mutually commutative operators,

$$[\mathbb{I}[P_s], \mathbb{I}[P_{s'}]] = 0 .$$

(56)
The polynomial fields $P_s$ can always be chosen to have definite spin, and it is convenient to label them accordingly; below we always assume that $s$ indicates the spin of $P_s(u)$. We will also use conventional notation $I_{s-1}$ for the integral $\langle P_s \rangle$.

As is discussed in [37], integrability of the field theory (1) in the presence of the boundary requires that the boundary state satisfies the equations

$$ (I_s - \bar{I}_s) |B\rangle = 0 ,$$

for all $I_s$ in one such subset. These equations must follow from special properties of the boundary conditions, which should be such that the corresponding fields $P_s(u)$ and $P_s(\bar{u})$, being brought to the boundary $\sigma = 0$, satisfy the conditions

$$ (P_s(u) - P_s(\bar{u}))|_{\sigma=0} = i \frac{d}{d\tau} \Theta_s(\tau) ,$$

where $\Theta_s(\tau)$ are some (renormalized) local boundary fields. In the case of the spherical brane boundary condition (2), existence of the first few polynomials satisfying (58) can be demonstrated by an argument similar to that previously used to support integrability of the $O(N)$ sigma model [15]. The $O(N)$ symmetry of the boundary condition (2) suggests that the polynomials $P_s$ must have this symmetry as well. We thus can look for the corresponding integrals $I_{s-1}$ in the form:

$$ I_1 = \oint \frac{d\tau}{2\pi} \partial X \cdot \partial X ,$$

$$ I_3 = \oint \frac{d\tau}{2\pi} \left[ (\partial X \cdot \partial X)^2 + b_3 (\partial^2 X \cdot \partial^2 X) \right] ,$$

$$ I_5 = \oint \frac{d\tau}{2\pi} \left[ (\partial X \cdot \partial X)^3 + b_5 (\partial^2 X \cdot \partial^2 X)(\partial X \cdot \partial X) + c_5 (\partial^2 X \cdot \partial X)^2 + d_5 (\partial^3 X \cdot \partial^3 X) \right] ,$$

and for generic $k$

$$ I_{2k-1} = \oint \frac{d\tau}{2\pi} \left[ (\partial X \cdot \partial X)^k + b_{2k-1} (\partial^2 X \cdot \partial^2 X)(\partial X \cdot \partial X)^{k-2} + c_{2k-1} (\partial^2 X \cdot \partial X)^2 (\partial X \cdot \partial X)^{k-3} + \ldots \right] .$$

Note that in writing these expressions we have fixed the normalization of the currents $P_{2k}$: the term $(\partial X \cdot \partial X)^k$, having the highest power of $X$, comes
with the coefficient 1. In what follows we always assume this normalization of the integrals $I_{2k-1}$. The terms omitted in (60) have powers of $X$ equal $2k - 4$ or lower. In general case, the difference $\left( P_s(u) - P_s(\bar{u}) \right)_{\sigma=0}$ is a combination of local boundary fields having appropriate symmetries. These include the $O(N)$ symmetry and the anti-symmetry with respect to the reflection $\tau \to -\tau$. Note that for $N = 2$ the last reflection symmetry is violated unless $\theta = 0$ and $\theta = \pi$. Also, only the fields of right scale dimensions (i.e. equal or below the scale dimension of $P_s$) are admitted.

Elementary counting shows that for $s = 2$ the only admissible field is a total derivative over $\tau$, of the form of the r.h.s. of (58). For $s = 4$ there is only one admissible field which is not a total derivative, and for $s = 6$ there are three admissible non-derivative fields. It follows that $P_2$ always satisfy (58), and the coefficients in (59) can be adjusted to ensure that $P_4$ and $P_6$ satisfy (58) as well. The role of the boundary condition (2) in this counting is in reducing the number of independent boundary fields: the boundary conditions determine the number of redundant boundary fields. As usual, this kind of argumentation fails for higher spins, since the number of admissible fields grows too fast. Nonetheless, in many other models similar “low-spin test” was successful in detecting full integrability, with no known (to us) exceptions. Therefore we regard the above argument as a strong indication of integrability of the spherical brane model.

In fact, the coefficients in (59) for $P_4$ and $P_6$ can be determined explicitly from the commutativity condition (56). While $I_1$ automatically commutes with any $[P]$ of the form (53), the condition $[I_3, I_5] = 0$ turns out to be rather rigid. As it turns, it has only one solution (for the coefficients $b, c, d$ in (59)) with suitable properties\(^4\), namely
\begin{align*}
    b_3 &= \frac{(N + 2)}{3}, \\
    b_5 &= 3 \frac{(N + 4)}{5}, \\
    c_5 &= 7 \frac{(N + 4)}{5}, \\
    d_5 &= \frac{(N + 4)(36N + 59)}{600}.
\end{align*}

At large $N$ the integrals $I_3$ and $I_5$ reduce to their quadratic in $X$ terms, in

---

\(^4\) In fact, there are exactly three solutions. The other two correspond to previously known systems of commuting integrals. One represents the trivial free-field case, where all integrals $I_{2k-1}$ are quadratic in $X$ (obviously, normalization assumed in (59) has to be changed to accommodate this case); such integrals can be compatible only with the free field boundary conditions (i.e. no nonlinear constraints on $X_B(\tau)$, and possibly a term $\sim \int d\tau X_B(\tau)$ added to the action (10)). Another solution corresponds to the “KdV series”, in which the currents $P_s(u)$ are composite fields built from the energy-momentum tensor $T(u) = -\partial X \cdot \partial X$, as was described in [22].
agreement with easily established fact that in the $N \to \infty$ limit the spherical
brane model becomes a free theory \[38\]. Moreover, we have checked that
with this choice the integrals (60) indeed satisfy (57) in the first nontrivial
order of the $1/N$ expansion.

It looks likely that ones $I_3$ is fixed through Eqs.(59),(61), the higher-spin
integrals $I_{2k-1}$ with $k = 3, 4, 5 \ldots$ are determined uniquely by the commu-
tativity condition $[I_s, I_3] = 0$. Although for higher spins the brute-force
computation of the commutators becomes difficult, it is possible to deter-
mine this way the coefficients explicitly written in (60):

$$b_{2k-1} = \frac{k(k-1)(2k+N-2)}{2(2k-1)},$$
$$c_{2k-1} = \frac{k(k-1)(k-2)(2k+1)(2k+N-2)}{6(2k-1)}.$$  

7.3 Boundary state

In view of the structure (55) of the space $\mathcal{H}$, it is convenient to think of
the boundary state in terms of associated boundary operator. Natural iso-
morphism between $\mathcal{F}_P$ and $\bar{\mathcal{F}}_P$ (the right movers are replaced by the left
movers) makes it possible to to establish one to one correspondence be-
tween the states in $\mathcal{F}_P \otimes \bar{\mathcal{F}}_P$ and operators in $\mathcal{F}_P$. Thus the boundary state $|B\rangle$ can be re-interpreted as an operator $B : \mathcal{F}_P \to \mathcal{F}_P$. This idea was
extensively used in the study of conformal boundary conditions since the
original works \[39, 40\]. In this interpretation the equation (58) reduces to
the statement of commutativity,

$$[B, I_s] = 0.$$  

The boundary state therefore can be written as

$$|B\rangle = \int \mathcal{P} \sum_n B_n(P) \ |n, P\rangle \otimes |n, P\rangle,$$

where $|n, P\rangle$ are the (orthonormalized) simultaneous eigenvectors of the
operators $I_s$ in the space $\mathcal{F}_P$, and $B_n(P)$ are corresponding eigenvalues of
$B$,

$$B |n, P\rangle = B_n(P) |n, P\rangle.$$  

The eigenvalue $B_0(P)$ corresponding to the Fock vacuum $|0, P\rangle$ in $\mathcal{F}_P$ coinci-
des with the overlap $\langle 0 | P \rangle$. This structure of $|B\rangle$ emphasizes importance
of the problem of simultaneous diagonalization of the integrals of motion \( I_s \). Similar problem was addressed in [22] for somewhat simpler model of integrable boundary interaction – the minimal CFT perturbed by the boundary field \( \Phi_{1,3} \). In that case associated system of integrals \( I_s \) was the KdV series (see footnote \#4). It was observed in [22] that the eigenvalues of \( I_s \) are related to corresponding eigenvalues of \( B \) in the following way. When the length of the boundary \( 2\pi R \to \infty \), the operator \( \log B \) admits asymptotic expansion in terms of the local integrals \( I_5 \). We expect similar relation to hold in the case of the spherical brane model, namely

\[
\mathbb{B} \simeq \mathbb{B}_{IR} \exp \left\{ - \sum_{k=0}^{\infty} C_k (2\pi E^*)^{1-2k} I_{2k-1} \right\}, \tag{66}
\]

where \( \mathbb{B}_{IR} \) is the boundary operator associated with the infrared fixed point, \( E^* \) is the energy scale (14), \( I_{2k-1} \) with \( k \geq 1 \) are the integrals (59), (60), \( I_{-1} \equiv \oint \frac{d\tau}{2\pi} R \), and the coefficients \( C_k \) are just numbers, independent of the parameters \( g_0 \), \( R \), and \( P \). The expansion (66) is expected to hold in the sense of asymptotic \( R \to \infty \) series (note that \( I_{2k-1} \sim R^{1-2k} \) by dimensional counting). The dependence of the matrix elements (66) on \( R \) and \( P \) comes through the integrals \( I_{2k-1} \). For instance, for the vacuum-vacuum matrix element

\[
I_{2k-1} \equiv \langle 0, P \mid I_{2k-1} \mid 0, P \rangle \tag{67}
\]

we have, from Eqs. (59, 62)

\[
\begin{align*}
I_1 &= \frac{1}{R} \left( h^2 + \frac{N}{24} \right), \\
I_3 &= \frac{1}{R^3} \left( h^4 + \frac{N+2}{12} h^2 + \frac{N(N+2)}{320} \right), \\
I_5 &= \frac{1}{R^5} \left( h^6 + \frac{N+4}{8} h^4 + \frac{(N+4)(37N + 78)}{4800} h^2 \\
&\quad + \frac{N(N+4)(143N + 282)}{483840} \right). \tag{68}
\end{align*}
\]

\footnote{This structure appears in integrable boundary flows down to the “basic” conformal boundary, the one which admits no primary boundary fields but the identity. In those cases the expansion in \( I_s \) corresponds to expansion of the infrared effective action in terms of descendents of the identity. Clearly, the Dirichlet boundary \( X_B = 0 \) is of that kind. For integrable boundary flows down to “elevated” fixed points, which admit nontrivial boundary primaries, the asymptotic expansions of \( B \) at large \( R \) should contain “dual” nonlocal integrals of motion \( \Phi \) as well. This is what we would expect to have in the case of \( \theta = \pi \) of the circular brane.}

and for general $k$,

$$I_{2k-1} = \frac{1}{R^{2k-1}} \left( h^{2k} + \frac{k(2k + N - 2)}{24} h^{2k-2} + \ldots \right), \quad (69)$$

where $h^2 = \frac{-P^2}{4}$. In the limit when $R \to \infty$ with $H = \frac{h}{2\pi R}$ kept fixed, these eigenvalues become simply $I_{2k-1} \to R (2\pi H)^{2k}$, and therefore $C_k$ in (66) are just the coefficients of the power series expansion of the boundary energy $\mathcal{E}(H)$ in (13),

$$\mathcal{E}(H) = E^* \sum_{k=0}^{\infty} C_k \left( \frac{H}{E^*} \right)^{2k}. \quad (70)$$

At the moment we have only a conjecture about exact form of these coefficients in the spherical brane model,

$$C_k = (-1)^k \left( 2\Delta \right)^{1-k} \frac{\Gamma \left( (2k - 1)\Delta \right)}{k! \Gamma^{2k-1}(\Delta)} \left( 2k - 1 \right)^{(1-2\Delta)+\Delta-1}, \quad (71)$$

with $\Delta = \frac{1}{N-1}$ and $k \geq 1$. It is possible to check that this expression agrees with the leading order of $1/N$ expansion in the $O(N)$ spherical brane model. The conjecture (71) concerns with the coefficients $C_k$ with $k \geq 1$, but it does not directly apply to the coefficient $C_0$. According to (70), the latter relates to the boundary energy at zero external field $H$, $\mathcal{E}(0) = C_0 E^*$. Presently, we do not have definite idea what exact value of $C_0$ is. Simply setting $k = 0$ in (71) is not quite acceptable, since doing so results in complex numbers for generic $N$. On the other hand, it is likely that exact $C_0$ is closely related to the expression (71) with $k = 0$ (its real part = $2\pi \cot(\pi \Delta)$?). One indication is that (71) with $k = 0$ develops a pole at $N = 2$, presumably signifying the small-instanton divergence specific to this value of $N$. Similar pole at $N = 3$ is present in the bulk vacuum energy of $O(N)$ sigma model [42].

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