The crossing number of the generalized Petersen graph $P(3k, k)$ in the projective plane

Jing Wang, Zuozheng Zhang†

Department of Mathematics and Computer Science, Changsha University, Changsha 410003, China

Abstract The crossing number of a graph $G$ in a surface $\Sigma$, denoted by $cr_\Sigma(G)$, is the minimum number of pairwise intersections of edges in a drawing of $G$ in $\Sigma$. Let $k$ be an integer satisfying $k \geq 3$, the generalized Petersen graph $P(3k, k)$ is the graph with vertex set $V(P(3k, k)) = \{u_i, v_i| i = 1, 2, \ldots, 3k\}$ and edge set $E(P(3k, k)) = \{u_iu_{i+1}, u_iv_i, v_{i+k+i}| i = 1, 2, \ldots, 3k\}$, the subscripts are read modulo $3k$. This paper investigates the crossing number of $P(3k, k)$ in the projective plane. We determine the exact value of $cr_{N_1}(P(3k, k))$ is $k - 2$ when $3 \leq k \leq 7$, moreover, for $k \geq 8$, we get that $k - 2 \leq cr_{N_1}(P(3k, k)) \leq k - 1$.

Keywords the projective plane, crossing number, the generalized Petersen graph, drawing

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1 Introduction

A surface $\Sigma$ means a compact, connected 2-manifold. It is known that there are two kinds of closed surfaces, orientable and nonorientable [1]. Every closed connected orientable surface is homeomorphic to one of the standard surfaces $S_k$ ($k \geq 0$), while each closed connected nonorientable surface is homeomorphic to one of $N_k$ ($k \geq 1$). In particular, the projective plane, $N_1$, is a 2-manifold obtained by identifying every point of the 2-sphere with its antipodal point.

Let $G = (V, E)$ be a simple graph with vertex set $V$ and edge set $E$. Let $D$ be a good drawing of the graph $G$ in a surface $\Sigma$, we denote the number of pairwise intersections of edges in $D$ by $v_D(G : \Sigma)$, or by $v(D)$ without ambiguous. The crossing number of $G$ in a surface $\Sigma$, denoted by $cr_\Sigma(G)$, is the minimum number of pairwise intersections of edges in a drawing of $G$ in $\Sigma$, i.e.,

$$cr_\Sigma(G) = \min_D v_D(G : \Sigma).$$

In particular, the crossing number of $G$ in the plane $S_0$ is denoted by $cr(G)$ for simplicity. It is well known that the crossing number of a graph in a surface $\Sigma$ is attained only in good drawings of the graph, which are the drawings where no
edge crosses itself, no adjacent edges cross each other, no two edges intersect more
than once, and no three edges have a common point.

In a drawing \( D \) of \( G = (V, E) \) in a surface \( \Sigma \), if an edge is not crossed by any
other edge, we say that it is clean in \( D \), otherwise, we say it is crossed. Moreover,
let \( F \subseteq E \), we say \( F \) is clean in \( D \) if all of the edges in \( F \) are clean, otherwise, we
say \( F \) is crossed.

Let \( A \) and \( B \) be two (not necessary disjoint) subsets of the edge set \( E \), the
number of crossings involving an edge in \( A \) and another edge in \( B \) is denoted by
\( v_D(A, B) \). In particular, \( v_D(A, A) \) is denoted by \( v_D(A) \). By counting the number
of crossings in \( D \), we have

\[ v_D(A, B \cup C) = v_D(A, B) + v_D(A, C), \]
\[ v_D(A \cup B) = v_D(A) + v_D(A, B) + v_D(B). \]

Computing the crossing number of a given graph is, in general, an elusive
problem. Garey and Johnson have proved that the problem of determining the
crossing number of an arbitrary graph in the plane is NP-complete [2]. Because of
its difficulty, there are limited results concern on this problem, see [3, 4, 5, 6, 7] and
the references therein. The generalized Petersen graph \( P(n, k) \) is a counterexample
to many conjectures and thus plays an important role in graph theory. Exoo,
Harary and Kabell began to study the crossing number of \( P(n, k) \) in
\( S_0 \) and they worked out the case when \( k = 2 \) [8]. For the case \( k = 3 \), Fiorini proved that
\( cr(P(9, 3)) = 2 \) [9], later on, Richter and Salazar [10] determined the crossing
number of \( P(3t + h, 3) \) is \( t + h \) if \( h \in \{0, 2\} \) and \( t + 3 \) if \( h = 1 \), for each \( t \geq 3 \),
with the single exception of \( P(9, 3) \). We tried to obtain \( cr(P(n, k)) \) when \( n \) can
be expressed as a function of \( k \), and proved that \( cr(P(3k, k)) = k \) for \( k \geq 4 \) [11].

As for the crossing number of graphs in a surface other than the plane, it is
not surprising that the results are even more restricted: only the crossing number
of Cartesian product graph \( C_3 \square C_n \) in \( N_1 \) [12], the crossing number of the complete
graph \( K_9 \) in \( S_2 \) [13], the crossing number of the complete bipartite graph \( K_{3, n} \) in
a surface with arbitrary genus [14], the crossing number of \( K_{4, n} \) either in \( N_1 \) or in
\( S_1 \) [15, 16], and the crossing number of the circulant graph \( C(3k; \{1, k\}) \) in \( N_1 \) [17]
have been determined. These facts motivate us to investigate the crossing number
of \( P(3k, k) \) (\( k \geq 3 \)) in the projective plane, the main theorem of this paper is

**Theorem 2** When \( 3 \leq k \leq 7 \), we have \( cr_{N_1}(P(3k, k)) = k - 2 \). Moreover, for
\( k \geq 8 \), we have \( k - 2 \leq cr_{N_1}(P(3k, k)) \leq k - 1 \).

Let us give an overview of the rest of this paper. Some basic notations are
introduced in Section 2. Section 3 is devoted to give the proof of Theorem 2 by
investigating the upper and lower bound of \( cr_{N_1}(P(3k, k)) \) independently. The
lower bound of \( cr_{N_1}(P(3k, k)) \) is based on the result of Lemma 6 which will be
proved finally in Section 4.
2 Preliminaries

For $F \subseteq E$, we denote by $G \setminus F$ the graph obtained from $G$ by deleting all edges in $F$. Furthermore, we also use $F$ to denote the subgraph induced on the edge set $F$ if there is no ambiguity.

Let $k$ be an integer greater than or equal to 3, the generalized Petersen graph $P(3k, k)$ is the graph with vertex set

$$V(P(3k, k)) = \{u_i, v_i | i = 1, 2, \ldots, 3k\}$$

and edge set

$$E(P(3k, k)) = \{u_iu_{i+1}, u_iv_i, v_iv_{k+i} | i = 1, 2, \ldots, 3k\}$$

the subscripts are read modulo $3k$.

To seek the structural of $P(3k, k)$, we find it is helpful to partite the edge set $E(P(3k, k))$ into several subsets $E_i$ and $H_i$ as follows. For $1 \leq i \leq k$, let

$$E_i = \{v_iv_{k+i}, v_{k+i}v_{2k+i}, v_{2k+i}v_i, u_iv_i, u_{k+i}v_{k+i}, u_{2k+i}v_{2k+i}\}$$

and let

$$H_i = \{u_iu_{i+1}, u_{k+i}u_{k+i+1}, u_{2k+i}u_{2k+i+1}\},$$

the subscripts are expressed modulo $3k$.

Set $E' = \bigcup_{i=1}^k E_i$. It is not difficult to see that

$$E(P(3k, k)) = E' \cup \left( \bigcup_{i=1}^k H_i \right)$$

and that $P(3k, k) \setminus E_i$ contains a subgraph homeomorphic to $P(3(k-1), (k-1))$ for each $1 \leq i \leq k$.

Note that three edges $v_iv_{k+i}, v_{k+i}v_{2k+i}$ and $v_{2k+i}v_i$ form a 3-cycle in $E_i$, which is denoted by $EC_i$ throughout the following discussions. Formally,

$$EC_i = v_iv_{k+i}v_{2k+i}v_i.$$

Let $D$ be a good drawing of $P(3k, k)$ in the projective plane, we define a function $f_D(H_i)$ ($1 \leq i \leq k$) counting the number of crossings related to $H_i$ in $D$ as follows:

$$f_D(H_i) = v_D(H_i, H_i) + \frac{1}{2} \sum_{1 \leq j \leq k, j \neq i} v_D(H_i, H_j).$$

In the rest discussions, we find the following definition is useful.
Definition 3 An $E'$–clean drawing of $P(3k, k)$ is a good drawing of $P(3k, k)$ in the projective plane such that $E'$ is clean.

By Eqs. (1) and (2) and by counting the number of crossings in $D$, we get

Lemma 4 Let $D$ be an $E'$–clean drawing of $P(3k, k)$, then

$$v(D) = \sum_{i=1}^{k} f_D(H_i).$$

3 The sketch of the proof of Theorem 2

In the beginning of this section, the upper bound of $cr_{N_1}(P(3k, k))$ can be obtained easily.

Lemma 5 $cr_{N_1}(P(9, 3)) \leq 1$, and $cr_{N_1}(P(3k, k)) \leq k - 1$ for $k \geq 4$.

Proof. Wilson’s Lemma states that the crossing number of a non-planar graph in the projective plane is strictly less than its crossing number in the plane \cite{12}. Combining this fact with the result that $cr(P(9, 3)) = 2$ \cite{9} and that $cr(P(3k, k)) = k$ for $k \geq 4$ \cite{11}, the lemma follows easily.

Our main efforts are made to establish the lower bound of $cr_{N_1}(P(3k, k))$, which is based on the lemma below.

Lemma 6 For $k \geq 4$, let $D$ be an $E'$–clean drawing of $P(3k, k)$. Then $v(D) \geq k - 1$.

We postpone its proof to Section 4. By assuming Lemma 6, we can prove the lower bound of $cr_{N_1}(P(3k, k))$ immediately.

Lemma 7 $cr_{N_1}(P(3k, k)) \geq k - 2$ for $k \geq 3$.

Proof. We prove the lemma by induction on $k$. First of all, it is seen from Figure 1 and Figure 2 that $P(9, 3) \setminus \{v_3v_9, v_2v_5, v_1v_7\}$ is a subdivision of $F_{13}(12, 18)$, which is one of minimal forbidden subgraphs for the projective plane (see Appendix A in \cite{18}), therefore, the induction basis $cr_{N_1}(P(9, 3)) \geq 1$ holds. Suppose that $cr_{N_1}(P(3(k-1), (k-1))) \geq k - 3$ when $k \geq 4$, consider now the graph $P(3k, k)$. Let $D$ be a good drawing of $P(3k, k)$ in $N_1$.

Case 1. There exists an integer $i$ ($1 \leq i \leq k$) such that $E_i$ is crossed in $D$.

W.l.o.g., we may assume that $i = 1$. By deleting the edges of $E_1$ in $D$, we get a good drawing $D_0$ of $P(3(k-1), (k-1))$ in $N_1$ with at least $k - 3$ crossings by the induction hypothesis, thus

$$v(D) \geq v(D_0) + 1 \geq (k - 3) + 1 = k - 2.$$
Case 2. \( E_i \) is clean in \( D \) for every \( 1 \leq i \leq k \).
Then \( D \) is an \( E' \)-clean drawing of \( P(3k, k) \). Due to Lemma 3, \( v(D) \geq k - 1 \) holds.
Therefore, \( cr_{N_1}(P(3k, k)) \geq k - 2 \) when \( k \geq 3 \). \qed

Based on the former lemmas, we can prove Theorem 2 in the following.

**The proof of Theorem 2.** Figures 3, 4, 5 and 6 demonstrate good drawings of \( P(3k, k) \) in \( N_1 \) with \( k - 2 \) crossings when \( 4 \leq k \leq 7 \), thus \( cr_{N_1}(P(3k, k)) \leq k - 2 \). Together with Lemma 5 and Lemma 7, it is confirmed that \( cr_{N_1}(P(3k, k)) = k - 2 \) for \( 3 \leq k \leq 7 \).

For \( k \geq 8 \), we have \( k - 2 \leq cr_{N_1}(P(3k, k)) \leq k - 1 \) due to Lemma 5 and Lemma 7. The proof is completed. \qed

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Figure 1: The graph \( P(9, 3) \).

Figure 2: The graph \( F_{13}(12, 18) \).

Figure 3: A good drawing of \( P(12, 4) \) in \( N_1 \) with two crossings.

Figure 4: A good drawing of \( P(15, 5) \) in \( N_1 \) with three crossings.
4 The proof of Lemma 6

For $1 \leq i \leq k$, let

$$R_i = E_i \cup H_i \cup E_{i+1}. \quad (3)$$

The following observation is obvious, since it is appeared several times in the rest of the paper, we state it formally.

**Observation 8** Let $D$ be an $E'$—clean drawing of $P(3k,k)$, then $E_i$ is clean for all $1 \leq i \leq k$. 

![Figure 5: A good drawing of $P(18,6)$ in $N_1$ with four crossings.](image)

![Figure 6: A good drawing of $P(21,7)$ in $N_1$ with five crossings.](image)

![Figure 7: A subdrawing of $R_i$.](image)

![Figure 8: A subdrawing of $R_i$.](image)
Lemma 9 For \( k \geq 3 \), let \( D \) be an \( E' \)–clean drawing of \( P(3k,k) \) such that one of the 3-cycles \( EC_i \) and \( EC_{i+1} \) is non-contractible. Then \( f_D(H_i) \geq 1 \).

Proof. W.l.o.g., assume that \( EC_i \) is non-contractible, for the other case, the proof is analogous. Then \( EC_{i+1} \) is contractible, otherwise, two non-contractible cycles \( EC_i \) and \( EC_{i+1} \) will cross each other at least once in \( N_1 \), a contradiction with Observation 8.

Suppose to contrary that \( f_D(H_i) < 1 \), then three edges of \( H_i \) cannot cross each other by Eq. (2). By the above analyses, there are three possibilities of the subdrawing of \( R_i \), see Figure 7, Figure 8 and Figure 9. It is seen that, in each subdrawing of \( R_i \), the projective plane has been divided into four regions.

Firstly, we consider the case that the subdrawing of \( R_i \) is as drawn in Figure 7. Consider now the subgraph \( E_i+2 \). It is asserted that all of the vertices of \( E_i+2 \) lie in the same region in the subdrawing of \( R_i \) by Observation 8, furthermore, they cannot lie in the region labelled \( f_4 \), otherwise the edge \( u_{i+1}u_{i+2} \) will cross \( EC_{i+1} \) at least once, a contradiction with Observation 8. The following three cases are investigated.

Case 1. The vertices of \( E_i+2 \) lie in the region labelled \( f_1 \).

Then the edge \( u_{2k+i+1}u_{2k+i+2} \) and the path \( u_{k+i+2}u_{k+i+3} \cdots u_{2k+i} \) will cross \( H_i \) at least once, respectively, which implies that \( f_D(H_i) \geq 1 \) by Eq. (2), a contradiction.

Case 2. The vertices of \( E_i+2 \) lie in the region labelled \( f_2 \).

Then the edge \( u_{i+1}u_{i+2} \) and the path \( u_{2k+i+2}u_{2k+i+3} \cdots u_{3k}u_1 \cdots u_i \) will cross \( H_i \) at least once, respectively, which implies that \( f_D(H_i) \geq 1 \), a contradiction.

Case 3. The vertices of \( E_i+2 \) lie in the region labelled \( f_3 \).

Then \( H_i \) will be crossed by the edge \( u_{k+i+1}u_{k+i+2} \) and by the path \( u_{i+2}u_{i+3} \cdots u_{k+i} \) at least once respectively, which yields that \( f_D(H_i) \geq 1 \), a contradiction.
Similar contradictions can be made if the subdrawing of $R_i$ is as drawn in Figure 8 or Figure 9.

\[ v_{2k+i+1} v_{k+i+1} v_i + 1 \]

\[ v_{2k+i} v_{k+i} v_i \]

\[ u_{2k+i+1} u_{k+i+1} u_i + 1 \]

\[ u_{2k+i} u_{k+i} u_i \]

Figure 11: A subdrawing of $EC_i \cup EC_{i+1}$.

Figure 12: A subdrawing of $R_i$.

**Lemma 10** For $k \geq 3$, let $D$ be an $E'$-clean drawing of $P(3k, k)$. If $EC_i \cup EC_{i+1}$ is as drawn in Figure 10 or Figure 11, then $f_D(H_i) \geq 1$.

**Proof.** Suppose to contrary that $f_D(H_i) < 1$. By Eq. (2), it is claimed that

**Claim 11** The edges of $H_i$ cannot cross each other in $D$.

If $EC_i \cup EC_{i+1}$ is as drawn in Figure 10. Then $R_i$ is as shown in Figure 12 by Observation 8 and Claim 11. Consider the subgraph $E_{i+2}$, it lies in one of the regions labelled $f_1$, $f_2$ and $f_3$ by Observation 8 again. Assume that $E_{i+2}$ lies in the region labelled $f_1$, then the edge $u_{2k+i+1} u_{2k+i+2}$ and the path $u_{k+i+2} u_{k+i+3} \cdots u_{2k+i}$ will cross $H_i$ at least once respectively. Hence, by Eq. (2), we have $f_D(H_i) \geq 1$, a contradiction. Similar contradictions can be made if $E_{i+2}$ lies in the region labelled $f_2$ or $f_3$.

Using the analogous arguments, we can also show that $f_D(H_i) \geq 1$ if $EC_i \cup EC_{i+1}$ is as drawn in Figure 11.

**Lemma 12** For $k \geq 3$, let $D$ be an $E'$-clean drawing of $P(3k, k)$. If $EC_i \cup EC_{i+1}$ is as drawn in Figure 13 and $f_D(H_i) < 1$, then $R_i$ is as shown in Figure 15.

**Proof.** First of all, we conclude that the edges of $H_i$ cannot have internal crossings in $D$ by the assumption that $f_D(H_i) < 1$. Thus, there are two possibilities of the subdrawing of $R_i$ in $D$, see Figure 14 and Figure 15.

If $R_i$ is as drawn in Figure 14, Observation 8 enforces that $E_{i+2}$ lies in one of the region labelled $f_1$, $f_2$ and $f_3$. We may assume firstly that $E_{i+2}$ lies in the region labelled $f_1$, for other cases the proof is the same. Under this circumstance,
Figure 13: A subdrawing of $EC_i \cup EC_{i+1}$.

the edge $u_{2k+i+1}u_{2k+i+2}$ and the path $u_{k+i+2}u_{k+i+3}\cdots u_{2k+i}$ will cross $H_i$ at least once respectively, therefore, $f_D(H_i) \geq 1$ by Eq.(2), a contradiction.

Thus, $R_i$ is as shown in Figure 15. □

Combining Lemma 10 with Lemma 12, we have

Lemma 13 For $k \geq 3$, let $D$ be an $E'$–clean drawing of $P(3k,k)$ such that both $EC_i$ and $EC_{i+1}$ are contractible cycles. If $R_i$ is not as drawn in Figure 15, then $f_D(H_i) \geq 1$.

Proof. Since both $EC_i$ and $EC_{i+1}$ are contractible cycles, there are three possibilities of the subdrawing of $EC_i \cup EC_{i+1}$, see Figure 10, Figure 11, and Figure 13. Lemma 10 implies that $f_D(H_i) \geq 1$ if $EC_i \cup EC_{i+1}$ is as drawn in Figure 10 or Figure 11. Furthermore, Lemma 12 implies that $f_D(H_i) \geq 1$ if $R_i$ is not as drawn in Figure 15. □

Figure 15: A subdrawing of $R_i$.

Figure 16: A subdrawing of $R_1 \cup H_2 \cup E_3$. 
Now, we are ready to prove Lemma \[6\].

**The proof of Lemma \[6\].** Suppose to contrary that

\[ v(D) < k - 1. \] (4)

By Lemma \[4\], there exists an integer \( i \) such that \( f_D(H_i) < 1 \), otherwise,

\[ v(D) = \sum_{i=1}^{k} f_D(H_i) \geq k, \]

a contradiction with Eq.(4). The following two cases are discussed: Case 1. there exists an integer \( i \) such that \( f_D(H_i) = 0 \) and Case 2. \( f_D(H_i) > 0 \) for every \( 1 \leq i \leq k \) and there exists an integer \( i \) such that \( f_D(H_i) = \frac{1}{2} \).

**Case 1.** There exists an integer \( i \) such that \( f_D(H_i) = 0 \).

W.l.o.g., let \( f_D(H_1) = 0 \). Thus, we can get that

**Claim 14** \( R_1 \) is clean in \( D \).

Furthermore, there exists another integer \( j \) \( (j \neq 1) \) satisfying \( f_D(H_j) < 1 \), otherwise,

\[ v(D) = \sum_{i=2}^{k} f_D(H_i) \geq k - 1, \]

a contradiction with Eq.(4).

**Subcase 1.1.** \( j = 2 \) or \( j = k \).

By symmetry, we only need to consider the case that \( j = 2 \). It follows from Lemma \[9\] that both \( EC_1 \) and \( EC_2 \) are contractible cycles since \( f_D(H_1) = 0 \). Moreover, Lemma \[13\] enforces that the subdrawing of \( R_1 \) is as shown in Figure 15 by replacing all the indices \( i \) by 1.

Now, we consider the subgraph \( E_3 \). All of the vertices of \( E_3 \) lie in the region labeled \( f_1 \), on whose boundary lies the vertices \( u_2, u_{k+2} \) and \( u_{2k+2} \), otherwise, we have \( v_D(H_2, R_1) \geq 1 \), contradicting Claim 14. Moreover, the edges of \( H_2 \) cannot have internal crossings by Eq.(2). All these arguments confirm that the only possibility of the subdrawing of \( R_1 \cup H_2 \cup E_3 \) is as shown in Figure 16. Note that the region labeled by \( f_1 \) in Figure 15 has been separated into three regions, which are labelled by \( f_{11}, f_{12} \) and \( f_{13} \).

Next, we consider the subgraph \( E_k \) (note that \( k \geq 4 \) is crucial here for \( E_k \) being not equal to \( E_3 \)). By Observation 8, \( E_k \) lies in one of \( f_{11}, f_{12} \) and \( f_{13} \). We may assume that \( E_k \) lies in \( f_{11} \), for other cases the proof is the same. Under this circumstance, \( H_2 \) will be crossed by the edge \( u_{2k}u_{2k+1} \) and by the path \( u_{k+3}u_{k+4} \cdots u_{2k} \) at least once respectively, therefore, \( f_D(H_2) \geq 1 \) by Eq.(2), a contradiction.
Subcase 1.2. \( j \notin \{2, k\} \).

Since \( f_D(H_1) = 0 \) and \( f_D(H_j) < 1 \), all of the cycles \( EC_1, EC_2, EC_j \) and \( EC_{j+1} \) are contractible by Lemma 9. By Lemma 13, \( R_1 \) (resp. \( R_j \)) is as drawn in Figure 15 by replacing all the indices \( i \) by 1 (resp. \( j \)).

Notice that both \( u_{k+1}v_{k+1}u_1u_2v_2v_{k+2}u_{k+1} \) and \( u_{k+j}v_{k+j}v_ju_ju_{j+1}v_{j+1}v_{k+j+1}u_{k+j+1} \) are non-contractible curves, therefore, they must cross each other in \( N_1 \), which yields that \( v_D(H_1, H_j) \geq 1 \) since \( E' \) is clean in \( D \), a contradiction with \( f_D(H_1) = 0 \).

Case 2. \( f_D(H_i) > 0 \) for every \( 1 \leq i \leq k \) and there exists an integer \( i \) such that \( f_D(H_i) = \frac{1}{2} \).

W.l.o.g., let \( f_D(H_1) = \frac{1}{2} \). There exists an integer \( j \) (\( j \notin \{2, k\} \)) such that \( f_D(H_j) = \frac{1}{2} \), otherwise

\[
v(D) = \sum_{i=1}^{k} f_D(H_i) \geq 3 \times \frac{1}{2} + \sum_{i=4}^{k} f_D(H_i) \geq \frac{3}{2} + (k - 3),
\]

and thus \( v(D) \geq k - 1 \) since \( v(D) \) is an integer, a contradiction with Eq.1.

![Figure 17: A subdrawing of \( R_1 \cup R_j \).](image)

Lemma 9 tells that, all of the cycles \( EC_1, EC_2, EC_j \) and \( EC_{j+1} \) are contractible. By Lemma 13, \( R_1 \) (resp. \( R_j \)) is as drawn in Figure 15 by replacing all the indices \( i \) by 1 (resp. \( j \)). Note that both \( u_{k+1}v_{k+1}v_1u_1u_2v_2u_{k+2}v_{k+2}u_{k+1} \) and \( u_{k+j}v_{k+j}v_ju_ju_{j+1}v_{j+1}v_{k+j+1}u_{k+j+1} \) are non-contractible curves, then

\[
v_D(H_1, H_j) = 1
\]

since \( f_D(H_1) = f_D(H_j) = \frac{1}{2} \), furthermore, the crossed edge of \( H_1 \) (resp. \( H_j \)) is \( u_{k+1}u_{k+2} \) (resp. \( u_{k+j}u_{k+j+1} \)), see Figure 17. It is seen that two vertices \( u_2 \) and \( u_j \) don’t lie on the boundary of a same region, thus the path \( u_2u_3 \cdots u_j \) will cross \( H_1 \) or \( H_j \) by Observation 8 a contradiction with the assumption that \( f_D(H_1) = f_D(H_j) = \frac{1}{2} \).

All of the above contradictions enforce that \( v(D) \geq k - 1 \) if \( D \) is an \( E' \)-clean drawing of \( P(3k, k) \).
5 Conclusions

This paper studies the crossing number of the generalized Petersen graph $P(3k, k)$ in the projective plane. However, when $k \geq 8$, the exact value of $cr_{N_1}(P(3k, k))$ remains open. Does $cr_{N_1}(P(3k, k)) = k - 2$ or $cr_{N_1}(P(3k, k)) = k - 1$? From the former proof, we know the problem of "Deciding the exact value of $cr_{N_1}(P(3k, k))$" is highly related to the problem of "Whether there exists a good drawing of $P(3k, k)$ in the projective plane with $k - 2$ crossings or not?" If the answer to the latter problem is "Yes", then we conclude that $cr_{N_1}(P(3k, k)) = k - 2$, otherwise, we have $cr_{N_1}(P(3k, k)) = k - 1$.

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