QUANTUM ENTROPY OF CHARGED ROTATING BLACK HOLES

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I discuss a method for obtaining the one-loop quantum corrections to the tree-level entropy for a charged Kerr black hole. Divergences which appear can be removed by renormalization of couplings in the tree-level gravitational action in a manner similar to that for a static black hole.

1 Introduction

For more than two decades physicists have come to appreciate black holes as thermodynamic systems characterized by only a few macroscopic parameters such as mass ($m$), charge ($q$) and angular momentum ($\Omega$). The generic representative of such a hole in general relativity is the Kerr-Newman metric. The thermodynamic analogy suggests that there is an entropy associated with this hole that is proportional to the area of the event horizon.

For all other thermodynamic systems, the entropy is proportional to the logarithm of the number of hidden degrees of freedom. Are there analogous degrees of freedom for a black hole? If so, where do they come from? Statistical explanations of their origin in terms of a gas of quantum fields have been proposed. Unfortunately the resultant expressions for the entropy can be understood as one-loop corrections to the classical black hole entropy, and so do not give any explanation of the classical entropy itself.

For arbitrary static black holes, the divergences of the entropy have the same origin as the UV-divergences of the quantum effective action and can be removed by renormalization of the gravitational couplings in the tree-level gravitational action. I report here on work carried out with S. Solodukhin for the analogous problem in the stationary case. Remarkably, the UV-divergences for the one-loop entropy of a Kerr-Newman black hole are renormalized in the same way as for a static black hole.

2 The Euclidean Kerr-Newman Metric

The Euclidean Kerr-Newman metric can be in the form

$$ds^2_E = \frac{\hat{\rho}^2}{\Delta} dr^2 + \frac{\hat{\Delta}\hat{\rho}^2}{(\hat{r}^2 - \hat{a}^2)^2} \omega^2 + \hat{\rho}^2 (d\theta^2 + \sin^2 \theta \tilde{\omega}^2)$$

(1)
where the Euclidean time is $t = i\tau$ and the rotation and charge parameters have also been transformed $a = i\hat{a}$, $q = i\hat{q}$, so that the metric is purely real.

Here $\Delta(r) = (r - \hat{r}_+)(r - \hat{r}_-$), where $r_{\pm} = m \pm \sqrt{m^2 + \hat{a}^2 + \hat{q}^2}$, the quantities $\omega$ and $\tilde{\omega}$ take the form

$$\omega = \frac{(r^2 - \hat{a}^2)}{\rho^2} (d\tau - \hat{a} \sin^2 \theta d\phi) \quad \tilde{\omega} = \frac{(r^2 - \hat{a}^2)}{\rho^2} (d\phi + \frac{\hat{a}}{(r^2 - \hat{a}^2)} d\tau)$$

with $\rho^2 = r^2 - \hat{a}^2 \cos^2 \theta$. This space-time has a pair of orthogonal Killing vectors

$$K = \partial_\tau - \frac{\hat{a}}{r^2 - \hat{a}^2} \partial_\phi, \quad \tilde{K} = \hat{a} \sin^2 \theta \partial_\tau + \partial_\phi$$

which are the respective analogs of the vectors $\partial_\tau$ and $\partial_\phi$ in the (Euclidean) Schwarzschild case. The horizon surface $\Sigma$ defined by $r = \hat{r}_+$ is the stationary surface of the Killing vector $K$. Near this surface the metric is approximately

$$ds^2_E = ds^2_\Sigma + \rho^2_+ ds^2_{C_2}$$

where $\rho^2_+ = r^2_+ - \hat{a}^2 \cos^2 \theta$ and

$$ds^2_{C_2} = dx^2 + \frac{\gamma^2 x^2}{4\rho^2_+} d\chi^2$$

attached to $\Sigma$ at a point $(\theta, \psi)$, where $\chi = \tau - \hat{a} \sin^2 \theta \phi$ is an angle co-ordinate on $C_2$.

Regularity of the metric near the horizon implies the identifications $\psi \leftrightarrow \psi + 2\pi$ and $\chi \leftrightarrow \chi + 4\pi \gamma^{-1} \rho^2_+$. For this latter condition to hold independently of $\theta$ on the horizon, it is also necessary to identify $(\tau, \phi)$ with $(\tau + 2\pi \beta_H, \phi - 2\pi \Omega \beta_H)$, where $\Omega = \frac{\hat{a}}{(r^2_+ - a^2)}$ is the (complex) angular velocity and $\beta_H = (r^2_+ - \hat{a}^2) / \sqrt{m^2 + \hat{a}^2 + \hat{q}^2}$. The identified points have the same coordinate $\psi$.

Near $\Sigma$ we therefore have the following description of the Euclidean Kerr-Newman geometry: attached to every point $(\theta, \psi)$ of the horizon is a two-dimensional disk $C_2$ with coordinates $(x, \chi)$. The periodic identification of points on $C_2$ holds independently of any point on the horizon $\Sigma$, even though $\chi$ is not a global coordinate. As in static case, there is an abelian isometry generated by the Killing vector $K$, whose fixed set is $\Sigma$. Locally we have
\( K = \partial \chi \). The periodicity is in the direction of the vector \( K \) and the resulting Euclidean space \( E \) is regular manifold.

Now consider closing the trajectory of \( K \) with an arbitrary period \( \beta \neq \beta_H \). This implies the identification \((\tau + 2\pi \beta, \phi - 2\pi \Omega \beta)\), and the metric on \( C_2 \) becomes

\[
\begin{align*}
    ds^2_{C_2, \alpha} = dx^2 + \alpha^2 x^2 d\chi^2
\end{align*}
\]

where \( \chi = \beta \hat{\rho}^2 (\hat{r}^2_+ - \hat{a}^2)^{-1} \) is a new angle coordinate with period \( 2\pi \). This is the metric of a two dimensional cone with angular deficit \( \delta = 2\pi (1 - \alpha) \), \( \alpha \equiv \frac{\beta}{\beta_H} \). With this new identification the metric \( C_2 \) now describes the Euclidean conical space \( E_\alpha \) with singular surface \( \Sigma \).

For static metrics it is known that curvature tensors behave as \( \alpha \)-dependent distribution functions. This can also be shown to be true for stationary metrics by regulating (6) so that

\[
\begin{align*}
    ds^2_{C_2, \alpha, b} = f(x, b) dx^2 + \alpha^2 x^2 d\chi^2
\end{align*}
\]

where \( f(x, b) \) is some smooth regulating function such that \( \lim_{b \to 0} f(x, b) = 1 \). An evaluation of the curvature tensors then yields

\[
\begin{align*}
    R^{\mu\nu}_{\alpha\beta} &= \bar{R}^{\mu\nu}_{\alpha\beta} + 2\pi (1 - \alpha) ((n^\mu n_\alpha)(n^\nu n_\beta) - (n^\mu n_\beta)(n^\nu n_\alpha)) \delta_\Sigma \\
    R^\mu_{\nu} &= \bar{R}^\mu_{\nu} + 2\pi (1 - \alpha)(n^\mu n_\nu) \delta_\Sigma \\
    R &= \bar{R} + 4\pi (1 - \alpha) \delta_\Sigma
\end{align*}
\]

in the \( b \to 0 \) limit, where \( \delta_\Sigma \) is the delta-function \( \int_M f \delta_\Sigma = \int_\Sigma f \) and \( (n^\mu n_\nu) = \sum_{a=1}^2 n^\mu_a n^n_a \), where \( n^\mu_a \) are both normal to \( \Sigma \). Barred quantities denoted tensors evaluated with the unregulated metric (6). Quadratic curvature invariants may also be shown to have a structure that is formally identical to the static case (1).

3 Heat Kernel Expansion

In the Euclidean path integral approach to a statistical field system at temperature \( T = (2\pi \beta)^{-1} \) one considers the fields which are periodic with respect to imaginary time \( \tau \) with period \( 2\pi \beta \). For a (regulated) rotating black hole metric this entails closing the integral curves of \( K \) for arbitrary \( \beta \). The partition function \( Z(\beta) \) then becomes the functional integral of the matter Euclidean action on \( E_\alpha \), with periodicity conditions imposed on the matter field(s).

For the matter action \( I_E = \frac{1}{2} \int_{E_\alpha} (\nabla \phi)^2 \) standard techniques yield

\[
\begin{align*}
    \ln Z(\beta) = -\frac{1}{2} \ln det(-\square_{E_\alpha}) \frac{1}{(4\pi s)^2} \sum_{n=0}^\infty a_n s^n
\end{align*}
\]

where the heat-kernel coefficients \( a_n \) are a sum of standard and conical coefficients. An evaluation of these indicates that the UV-divergent part of the entropy is renormalized in a manner identical to that for static black
holes. Once this is taken into account, the contribution to the entropy \( S = -(\beta \partial_\beta - 1) \ln Z(\beta) |_{\beta = \beta H} \) for a Kerr-Newman black hole becomes

\[
S_{\text{div}} = \frac{1}{48\pi^2} A_\Sigma + \frac{1}{45} \left( 1 - \frac{3q^2}{4r_+^2} \left( 1 + \frac{(r_+^2 + a^2)}{ar_+} \tan^{-1} \left( \frac{a}{r_+} \right) \right) \right) \ln \frac{L}{\hat{c}} \tag{9}
\]

in the Lorentzian section, where \( A_\Sigma = 4\pi (r_+^2 + a^2) \) is area of the horizon \( \Sigma \). When \( q = 0 \) the quantum-corrected part of the entropy in (9) is the same as that for a Schwarzschild black hole. At present there is no explanation for this.

4 Concluding Remarks

It is still an open question as to what degrees of freedom are counted by the entropy of black hole. The conical-deficit methods employed here clearly indicate that the entropy of a Kerr-Newman black hole is associated with the horizon. A proper treatment of the statistical-mechanical calculation of the quantum entropy should provide us with a better understanding of the relationship between the different methods of assigning entropy to a black hole.

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