Lattice structure and convergence of a Game of Cards *

Eric Goles¹, Michel Morvan², and Ha Duong Phan³

¹ Departamento de Ingeniería Matemática, Escuela de Ingeniería, Universidad de Chile, Casilla 170-Correo 3, Santiago, Chile,
² LIIFA Université Denis Diderot Paris 7 and Institut universitaire de France - Case 7014-2, Place Jussieu-75256 Paris Cedex 05-France,
³ LIIFA Université Denis Diderot Paris 7 - Case 7014-2, Place Jussieu-75256 Paris Cedex 05-France,

egoles@dim.uchile.cl, morvan@liafa.jussieu.fr, phan@liafa.jussieu.fr

Abstract. We study the dynamics the so-called Game of Cards by using tools developed in the context of discrete dynamical systems. We extend a result of [4] and of [10] (this last one in the context of distributed systems) who established a necessary and sufficient condition for the game to converge. We precisely describe the structure of the set of configurations (that we show to be very closed to a lattice structure) and we state bounds for the convergence time.

Keywords: Integer composition, Order, Lattice, Convergence.

1 Introduction

This paper is devoted to the study of the dynamics of a discrete system related to some self stabilizing protocol on a ring of processors. As explained in [4], this protocol can be seen in terms of a game of cards described as follows. “Assume a finite set of players sitting around a table. Initially, each player holds a finite number of non-distinguishable cards. The only move a player can make is passing a card to his/her right neighbor, provided that his neighbor has fewer cards than the player itself. The game terminates when no move is possible.” In the cited paper, the following theorem is proved.

Theorem 1. The Game of Cards terminates if the total number of cards is a multiple of the number of players.

The proof given by the authors for this theorem was simpler than the one proposed for the equivalent result in [11]. Moreover, the authors were pointing out the fact that studying some distributed protocols in terms of discrete dynamical systems could be fruitful. In this paper, we replace the game of cards

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in the broader context of the study of transition systems on compositions of a given integer, where the total number of cards which is decomposed in each configuration is the sum of the number of cards of each player. The dynamics of such transition systems on compositions has been intensively studied by various authors and provides a powerful framework to derive structural and dynamical properties.

In this paper, we are going to investigate in more details the structure of the set of all possible configurations of the game with \( n \) cards and \( p \) players. For that, let us represent a configuration by a list of \( p \) integers \( a = (a_1, \ldots, a_p) \) where \( a_i \) is the number of cards of player \( i \). At each step player \( i \) can give a card to player \( (i+1) \) (modulo \( p \)). Let \( n = kp + q \) with \( 0 \leq q < p \). Let us call \( \mathcal{G} \) the graph defined on the set of all possible configurations of the game — that is the set of \( p \)-dimensional vectors of integers such that the sum of the components is \( n \), and having for arcs the set of couples \( (a, b) \) of configurations such that \( b \) can be obtained from \( a \) in one step. Following [4], let us call dual those configuration that do not belong to a non trivial circuit. The Game of Cards can be coded by a Chip Firing Game of which many properties have been studied in [11]. Moreover, due to its special rule, the Game of Cards has other interesting properties that we will studying in this paper. We will first characterize dual configuration and will show that if \( n \) is not a multiple of \( p \), the unique non trivial strongly connected component of the graph \( \mathcal{G} \) is the set of dual configurations. We will also study the subgraph of \( \mathcal{G} \) induced by the set of configurations that can be reached from a given configuration \( O \). We will characterize the partial order naturally associated to this graph when the set of dual configurations is replaced by a unique vertex, and we will establish its lattice structure. We will finish by bounding the number of steps necessary to arrive to a dual configuration.

In the following, we are going to discuss some lattice properties of the above dynamical systems. Let us recall that a finite lattice can be described as a finite partial order such that any two elements \( a \) and \( b \) admit a least upper bound (denoted by \( sup(a, b) \)) and a greatest lower bound (denoted by \( inf(a, b) \)). \( Sup(a, b) \) is the smallest element among the elements greater than both \( a \) and \( b \). \( Inf(a, b) \) is defined similarly. A useful result about finite lattices is that a partial order is a lattice if and only if it admits a greatest element and any two elements admit a greatest lower bound. For more informations about lattice theory, see [1,3].

2 Basic structure of the Game of Cards

Let us first state the following corollary of the main theorem of [4].

**Corollary 1.** If \( q = 0 \) there is no dual configuration, which means that the game terminates; if \( q > 0 \), the game does not terminates and the dual configurations are exactly the \( \binom{n}{p} \) configurations such that each player owns either \( k \) or \( k+1 \) cards.

*Proof.* The proof comes immediately from the proof of [4].

\( \square \)
See Figure 1 for two examples:

![Fig. 1. The Game of Cards in two cases: 6 cards and 3 players, and 6 cards and 4 players.](image)

Fig. 1. The Game of Cards in two cases: 6 cards and 3 players, and 6 cards and 4 players. $a \rightarrow b$ signifies that $b$ can obtained from $a$ by moving a card from a player positioned between 1 and $p−1$. $a →→ b$ signifies that $b$ can obtained from $a$ by moving a card from the $p−th$ player to the first player.

We can now state the following theorem.

**Theorem 2.** The unique non trivial strongly connected component of $G$ is the set of dual configurations.

*Proof.* The result is obvious if $q = 0$. If $q > 0$, it is clear from the definition that a non dual configuration can not belong to a strongly connected component since it does not belong to a circuit. So we just have to prove that the set of dual configurations is a strongly connected component.

Let $a = (a_1, \ldots, a_p)$ and $b = (b_1, \ldots, b_p)$ be two dual configurations. We have $\forall i \in [1, p]$, the value of $a_i$ and $b_i$ is $k$ or $k + 1$. Let $P$ be the dual configuration defined by

$$P = k + 1, \ldots, k + 1, k, \ldots, k.$$
We are going to show that there exists a path from \( a \) to \( P \) and a path from \( P \) to \( b \), which implies the existence of a path from \( a \) to \( b \). Similarly, the existence of a path from \( b \) to \( a \) could be stated.

The path from \( a \) to \( P \) is built by following an arbitrary maximal path starting from \( a \) and in which no transition is made in which player \( q \) gives a card to its right neighbor (such a path exists since in each infinite path player \( q \) plays an infinite number of times). The unique possible transition from the last configuration of this path is the one in which player \( q \) plays, which proves that this configuration is \( P \).

We have now to find a path from \( P \) to \( b \). Let \( i \) be the last index smaller than or equal to \( p \) such that \( b_i = k + 1 \). By consecutively applying the playing rule from configuration \( P \) on players \( q, q+1, \ldots, i-1 \), we obtain the following configuration:

\[
\underbrace{k+1, \ldots, k+1}_q, \underbrace{k, \ldots, k}_u, \underbrace{k+1, \ldots, k}_1.
\]

Let \( j \) be the last index smaller than \( i \) such that \( b_j = k + 1 \). We can apply the same techniques and by iterating the process, we obtain the configuration \( b \). This achieves the proof. \( \square \)

If we remark now that when \( q > 0 \) there is no fixed point, the previous result allows to state:

**Corollary 2.** There is a path from any configuration to any dual configuration.

If \( q > 0 \), any infinite sequence of transitions starts by a finite sequence of transitions applied to a non dual configuration and leading to another non dual configuration and is followed by an infinite sequence of transitions applied to dual configurations and leading to dual configurations. In some sense, these dual configurations represent a generalization of fixed points, since when the system is in such configuration it cannot reach a non dual configuration but it can reach any dual one. So it is natural to consider the reduced graph \( R(\mathcal{G}) \) obtained from \( \mathcal{G} \) by replacing the set of dual configurations by a unique vertex \( \perp \) with no outgoing arc and with one ingoing arc coming from each non dual configuration that can lead in one step to a dual configuration. \( R(\mathcal{G}) \) is then the quotient graph relatively to the equivalence relation “is in the same strongly connected component of”). We can now consider the partial order \( \prec_{ge} \) on \( R(\mathcal{G}) \) by considering the transitive closure of the graph: \( b \prec_{ge} a \) if \( b \) is reachable from \( a \) by following a path in the graph.

We are now going to focus on the relation between non dual configurations by first showing that between two such configurations the length of all paths is the same. For that, we are going to introduce the notion of “shot vector”, following [5], which describes for each player the number of cards it gave to its neighbor.

Let \( a \) and \( b \) be two non dual configurations such that \( b \prec_{ge} a \). Let \( \mathcal{C} \) be a sequence of transitions from \( a \) to \( b \):

\[
\mathcal{C} : a \rightarrow c^1 \rightarrow c^2 \rightarrow \ldots \rightarrow c^j \rightarrow b.
\]
Let $s_i(C)$ be the number of cards given by player $i$ to its neighbor during this sequence. Let $s(C)$ be the sequence $(s_1(C), s_2(C), \ldots, s_p(C))$ and call it the shot vector of the sequence $C$. Let $|s(C)| = \sum_{i=1}^{p} s_i(C)$ be the length of the sequence $C$. It immediately comes that:

$$s_2(C) - s_1(C) = a_2 - b_2,$$
$$s_3(C) - s_2(C) = a_3 - b_3,$$
$$\vdots$$
$$s_1(C) - s_p(C) = a_1 - b_1,$$

which implies that

$$s(C) = (s_p(C), \ldots, s_p(C))+(a_1-b_1, a_1+a_2-b_1-b_2, \ldots, a_1+\ldots+a_{p-1}-b_1-\ldots-b_{p-1}, 0).$$

If we denote

$$d(a, b) = (a_1 - b_1, a_1 + a_2 - b_1 - b_2, \ldots, a_1 + \ldots + a_{p-1} - b_1 - \ldots - b_{p-1}, 0),$$

we obtain

$$s(C) = s_p(C) \ast (1, \ldots, 1) + d(a, b).$$

Let us now introduce the following partial order between shot vectors: given two shot vectors $s(C)$ and $s(D)$, $s(C) \leq s(D)$ if $\forall i \ s_i(C) \leq s_i(D)$. Let $a$ and $b$ be two elements reachable from a given configuration $O$. Let us consider two sequences of transitions, one from $O$ to $a$, the other from $O$ to $b$:

$$C : O \to c^1 \to c^2 \to \ldots \to c^n \to a,$$
$$D : O \to d^1 \to d^2 \to \ldots \to d^v \to b.$$

We can state the following lemma:

**Lemma 1.** Assume that there exists an index $j$ such that $s_j(C) \leq s_j(D)$ and $\forall j' \neq j$, $s_{j'}(C) \geq s_{j'}(D)$. If it is possible to apply the rule on position $j$ of $b$, then it is also possible to apply this rule to $a$ in the same position $j$.

**Proof.** From the shot vector definition, we obtain:

$$a_1 = O_1 - s_1(C) + s_p(C),$$
$$a_2 = O_2 - s_2(C) + s_1(C),$$
$$\vdots$$
$$a_p = O_p - s_p(C) + s_{p-1}(C).$$

Since the necessary and sufficient condition to apply the rule on position $j$ of $b$ is $b_j - b_{j+1} \geq 1$, we are going to focus on the difference $a_j - a_{j+1}$.

$$a_j - a_{j+1} = O_j - s_j(C) + s_{j-1}(C) - (O_{j+1} - s_{j+1}(C) + s_j(C))$$
We then have
\[ O_j - O_{j+1} - 2s_j(C) + s_{j-1}(C) + s_{j+1}(C) \geq O_j - O_{j+1} - 2s_j(D) + s_{j-1}(D) + s_{j+1}(D) = b_j - b_{j+1} \geq 1, \]
which proves the lemma. \( \square \)

We can now establish the following result, which states that the shot vector associated to a sequence of transitions only depends on the initial and final configurations.

**Proposition 1.** Let \( a \) and \( b \) be two non dual configurations such that \( b <_{gc} a \). Then all the sequences of transitions from \( a \) to \( b \) have the same shot vector and so have the same length.

**Proof.** Let \( C \) and \( D \) be two sequences of transitions from \( a \) to \( b \):
\[
C : a \rightarrow c^1 \rightarrow c^2 \rightarrow \ldots \rightarrow c^n \rightarrow b, \\
D : a \rightarrow d^1 \rightarrow d^2 \rightarrow \ldots \rightarrow d^n \rightarrow b.
\]

Let us recall that:
\[
s(C) = s_p(C) * (1, \ldots, 1) + d(a, b), \\
s(D) = s_p(D) * (1, \ldots, 1) + d(a, b).
\]

Assume that \( s_p(D) > s_p(C) \). We have \( s(D) > s(C) \). We are going to show that there exists a path of positive length from \( b \) to \( b \), which is a contradiction. For that, we are going to build step by step a sequence of transitions from \( b \) to \( b \):
\[ b \rightarrow e_1 \rightarrow \ldots \rightarrow e_l \rightarrow b. \]

For \( i \leq v \) let us denote by \( D^i \) the following sequence:
\[
D : a \rightarrow d^1 \rightarrow d^2 \rightarrow \ldots \rightarrow d^i.
\]

There exists a first index \( i \) such that \( s(D^i) \not\leq s(C) \), which implies that there exists \( j \) such that \( s_j(D^i) > s_j(C) \) and \( \forall j' \neq j, s_j(D^i) \leq s_j(C) \). Since \( i \) is the first index having this property, we have \( s(D^{i-1}) \leq s(C) \), so \( s_j(D^{i-1}) = s_j(C) \) and \( s_j(D^i) = s_j(C) + 1 \). Since \( d^i \) and \( a \) satisfy the conditions of Lemma 1, we can apply the rule on position \( j \) to \( b \) to obtain a new configuration denoted by \( e^1 \). Let \( E^1 \) be the following sequence of transitions:
\[
a \rightarrow c^1 \rightarrow c^2 \rightarrow \ldots \rightarrow c^n \rightarrow b \rightarrow e^1.
\]

We then have \( s(D^i) \leq s(E^1) \leq s(D) \). By iterating this process, we can define \( e^2, e^3, \ldots \). Since \( |s(E^1)| - |s(C)| = i \) and \( s(C) \leq s(E^1) \leq s(D) \), after \( l = |s(D)| - |s(C)| \) steps, we will obtain \( e^l \rightarrow b \), which is the contradiction. Since the case where \( s_p(D) < s_p(C) \) is similar, \( s_p(D) = s_p(C) \) and therefore \( s(D) = s(C) \), which achieves the proof. \( \square \)

Using this result, we can define the shot vector \( s(a, b) \) for any couple of non dual configurations \( a \) and \( b \) such that \( b <_{gc} a \) as being equal to the shot vector of any sequence of transitions from \( a \) to \( b \). This shot vector will be very useful in understanding more precisely the structure and properties of the game.
3 Lattice structure of the Game of Cards

We dispose now of the tools we need for studying the structure of the set of all configurations that can be obtained from a given initial configuration $O = (O_1, \ldots, O_p)$. Let us denote by $GC(O)$ the set of all non-dual configurations reachable from $O$ to which we add $\bot$ as unique minimal element if $q > 0$ ($GC(O)$ is then the restriction of $R(G)$ to the configurations reachable from $O$). We are going to study the order $(GC(O), <_{gc})$ (in the following, for simplicity reasons, $GC(O)$ will both denote the set itself and the associated partial order). Let us first characterize this order.

**Theorem 3.** Let $a$ and $b$ be two non-dual configurations of $GC(O)$, then

$$a >_{gc} b \iff s(O, a) < s(O, b).$$

**Proof.** In order to show that $s(O, a) < s(O, b) \Rightarrow a >_{gc} b$, we can consider two sequences of transitions, one from $O$ to $a$ and the other from $O$ to $b$, and then make a similar proof as the one used in Proposition 1.

On the other hand, let $a$ and $b$ be two non-dual configurations of $GC(O)$ such that $b <_{gc} a$. Let $E$ be a sequence of transitions from $a$ to $b$. The sequence $D$ built by concatenating the sequence $C$ from $O$ to $a$ and the sequence $E$ is clearly a sequence from $O$ to $b$, and so we obtain:

$$s(O, b) = s(D) = s(C) + s(E) = s(O, a) + s(a, b) > s(O, a).$$

\[\square\]

We can now establish the underlying lattice structure of the Game of Cards.

**Theorem 4.** $GC(O)$ is a lattice. If $a$ and $b$ are two elements of $GC(O)$ different from $\bot$, then the configuration $c$ such that $s(O, c) = (\max(s_i(O, a), s_i(O, b)))_{i \in [1, p]}$ is reachable from $O$; if $c$ is not dual, then $\inf_{gc}(a, b) = c$, otherwise $\inf_{gc}(a, b) = \bot$.

**Proof.** Let us assume that $s(O, a)$ and $s(O, b)$ are incomparable (otherwise $a$ and $b$ are comparable and the result is obvious). We are first going to prove that $c$ is reachable from $a$. For that, we just have to find a configuration $a'$ such that $a \rightarrow a'$ and $s(O, a') \leq s(O, c)$. Let

$$O \rightarrow d_1 \rightarrow d_2 \rightarrow \ldots \rightarrow d^i \rightarrow b$$

be a sequence of transitions from $O$ to $b$ and let $i$ be the first index such that $s(O, d^i) \leq s(O, a)$ and $s(O, d^{i+1}) \not\leq s(O, a)$. Let us call $j$ the position on which the transition is applied on $d^i$. Clearly $s_j(O, d^i) \leq s_j(O, a)$ and $s_j(O, d^{i+1}) >$
\(s_j(O,a)\). Since \(a\) and \(d^i\) verify the condition of Lemma 1, we can apply the transition on position \(j\) of \(a\) to obtain a new configuration \(a'\). We have \(\forall j' \neq j, s_{j'}(O,a') \leq s_{j'}(O,c)\) and \(s_j(O,a') = s_j(O,d^{i+1}) \leq s_j(O,b) \leq s_j(O,c)\), which proves that \(c\) is reachable from \(a\). The proof is similar for \(b\).

This implies that \(c\) is reachable from \(O\), and by definition \(c\) is the greatest configuration smaller than both \(a\) and \(b\). If \(c\) is not dual, it is the greatest lower bound of \(a\) and \(b\), and if \(c\) is dual, \(\bot\) is then this lower bound. Since \(\text{GC}(O)\) also has a greatest element, it is a lattice, which ends the proof.

\[\square\]

4 Convergence time

We are now going to focus on the time needed either to arrive to the unique stable configuration or to twice through to the same configuration. For that, we are going to use Proposition 3 which states that all the sequences between two non dual configurations have the same length. So we are going to build a particular path between a given initial configuration and either the fixed point if \(q = 0\) or a particular dual configuration if \(q > 0\).

Let us first study the case \(q = 0\) where all the sequence converge to a unique fixed point \((k, \ldots, k)\) denoted by \(P\).

**Theorem 5.** If \(q = 0\), then from any initial state \(O\), there always exists a player that never can give a card to its neighbor.

**Proof.** Assume that \(O \neq P\). Let \(i\) be such that \(d_i(O,P) = \min_{1 \leq j \leq p}(O,P)\).

We are going to show step by step that there exists a path from \(O\) to \(P\) in which player \(i\) never plays. Let \(m\) be such that \(O_m\) is maximal among the \(O_j\) and such that \(O_{m+1} < O_m\). Since \(O \neq P\), such \(m\) exists and \(O_m > k\). We have \(d_m(O,P) = d_{m-1}(O,P) + O_m - k \geq d_{m-1}(O,P) + 1\), so \(m \neq i\) and \(d_m(O,P) \geq d_i(O,P) + 1\). Let \(a\) be the configuration obtained from \(O\) by applying the rule on position \(m\). We are going to show that \(a\) is such that \(d_i(a,P) = \min_{1 \leq j \leq p}d_j(a,P)\).

If \(m = p\), then \(d_m(a,P) = d_m(O,P) = 0\) and for all \(j \neq p\), \(d_j(a,P) = d_j(O,P) + 1\). If \(m \neq p\), then \(d_m(a,P) = d_m(O,P) - 1\) and for all \(j \neq p\), \(d_j(a,P) = d_j(O,P)\). In the two cases \(d_m(O,P) \geq d_i(O,P) + 1\) and so \(d_m(a,P) \geq d_i(O,P)\), which \(d_i(a,P) = \min_{1 \leq j \leq p}d_j(a,P)\).

By iterating the process, we arrive to the fixed point \(P\) by a path with no transition in position \(i\). Therefore \(s_i(O,P) = 0\) and then for all configuration \(a\) between \(O\) and \(P\), \(s_i(O,a) = 0\) (for a given \(i\), \(s_i\) can only increase when following a path).

\[\square\]

We obtain now the following corollary which directly comes from the previous proof:

**Corollary 3.** If \(q = 0\) and if the initial configuration is \(O\), the game ends after \(t\) steps, with

\[t = p \ast (\max_{1 \leq i \leq p}d_i(O,P)) + \sum_{i=1}^{i=p} d_i(O,P)\]
Let us finish by considering the case $q > 0$. Here, it is more difficult to give an exact formula of the time necessary to arrive on a dual configuration, since all the path leading to such a configuration may not have the same length. However, it is possible to give an upper bound to this time. Let us consider a particular dual configuration $P = k + 1, \ldots, k + q, k, \ldots, k$. We obtain the following result (the proof is identical to the proof of Theorem 5).

**Theorem 6.** Assume $q > 0$ and let $O$ be an arbitrary configuration. Let $i$ be the first index such that $d_i(O, P)$ is a minimal component of $d(O, P)$. Then all the paths from $O$ to $P$ without circuits have the same shot vector where $s_i(O, P) = 0$, that is in which player $i$ never plays. This shot vector is given by the following formula:

$$s(O, P) = -(\min_{1 \leq i \leq p} d_i(O, P) \ast (1, \ldots, 1)) + d(O, P).$$

Moreover, the time to reach $P$ from $O$ is equal to

$$t = p \ast (\min_{1 \leq i \leq p} d_i(O, P)) + \sum_{i=1}^{p} d_i(O, P).$$

**Corollary 4.** If $q > 0$ and if the game starts from a given configuration $O$, then the configuration obtained after

$$p \ast (\min_{1 \leq i \leq p} d_i(O, P)) + \sum_{i=1}^{p} d_i(O, P) + q(p - q) + 1$$

steps has been obtained earlier.

**Proof.** Let us consider the dominance ordering on dual configurations, in which a configuration $a$ is greater or equal to a configuration $b$ if and only if $\forall j \in [1, n]$, $\sum_{i=1}^{j} a_i \geq \sum_{i=1}^{j} b_i$. The greatest element of this order is clearly $P$ and the maximal length of a chain in this order is $q(p - q)$. Let $a$ and $b$ be two dual configurations such that $a$ covers $b$ in this order. It is clearly possible to go from $a$ to $b$ by a transition, so the covering relations are a subset of the set of transitions between dual configurations. Since the dual configurations are the unique non trivial strongly connected component of $G$, it is clear that the maximal length of a path between two dual configurations in $G$ is $q(p - q) + 1$, which proves the corollary.

□

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