A linear-time algorithm for the maximum-area inscribed triangle in a convex polygon

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Abstract

Given the \( n \) vertices of a convex polygon in cyclic order, can the triangle of maximum area inscribed in \( P \) be determined by an algorithm with \( O(n) \) time complexity? A purported linear-time algorithm by Dobkin and Snyder from 1979 has recently been shown to be incorrect by Keikha, Löffler, Urhausen, and van der Hoog. These authors give an alternative algorithm with \( O(n \log n) \) time complexity. Here we give an algorithm with linear time complexity.

1 Introduction

The problem of finding a triangle inscribed in a convex polygon that maximizes the area among all inscribed triangles is a classical problem in computational geometry. Recently, Keikha, Löffler, Urhausen, and van der Hoog [KLUvdH17] showed that an algorithm of Dobkin and Snyder [DS79] that purports to find the largest triangle inscribed in a convex \( n \)-gon in \( O(n) \) computational steps is incorrect by presenting a 9-gon for which the algorithm fails to find the largest triangle. There are algorithms known that solve the problem in \( O(n \log n) \) time complexity [BDDG82, KLUvdH17]. These algorithms make use of the subproblem of rooted triangles — the maximum area triangle with some fixed vertex. Here, by focusing on a different subproblem, that of anchored triangles — the maximum area triangle with one side parallel to some fixed direction — we are able to produce an algorithm

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that finds the largest triangle inscribed in an \( n \)-gon in \( O(n) \) computational steps.

### 2 Theory

We begin by defining the notion of a triangle inscribed in \( P \) that is anchored to a direction \( u \in S^1 \), as previously introduced in [Kal14].

**Definition 1.** Let \( P \) be a convex polygon, \( u \in S^2 \) be a unit vector, and \( abc \) a triangle inscribed in \( P \). We say \( abc \) is anchored to \( u \) if \( u \) is the outer normal to the edge \( bc \) and \( abc \) achieves the maximum area out of all triangles inscribed in \( P \) with \( u \) as the outer normal to the edge \( bc \).

Implicit in the definition of a triangle anchored to \( u \) are the identities of its vertices \( a, b, \) and \( c \) which are always to be in counter-clockwise order, with the latter two delimiting the edge to which \( u \) is normal. We will also say a triangle is anchored if it is anchored to some direction.

Clearly, the maximum-area triangle inscribed in \( P \) is anchored to some \( u \in S^1 \). In fact, it is anchored to all three of its edge outer normals. Therefore, all three vertices of the maximum-area triangle inscribed in \( P \) are vertices of \( P \).
Lemma 2. 1. If \( \mathbf{abc} \) is anchored to \( \mathbf{u} \), then \(-\mathbf{u}\) is an outer normal to \(P\) at \(a\).

2. If \( \mathbf{abc} \) and \( \mathbf{a'bc'} \) are both anchored to \( \mathbf{u} \), then \( \mathbf{b} = \mathbf{b'} \) and \( \mathbf{c} = \mathbf{c'} \).

3. If \( \mathbf{abc} \) is a local maximum of the area among triangles inscribed in \(P\) with \( \mathbf{u} \) as the outer normal to the edge \( \mathbf{bc} \), then it is also a global maximum.

Proof. To prove (1), suppose that \(-\mathbf{u}\) is not an outer normal to \(P\) at \(a\), then pick some \(a'\) such that \(-\mathbf{u}\) is an outer normal to \(P\) at \(a'\). Then \(-\mathbf{u} \cdot a' > -\mathbf{u} \cdot a\), and therefore \(\mathbf{a'bc'}\) has a larger area than \(\mathbf{abc}\), so the latter cannot be anchored at \(\mathbf{u}\).

Let \(y_{\min} = \min\{x \cdot \mathbf{u} : x \in P\}\) and \(y_{\max} = \max\{x \cdot \mathbf{u} : x \in P\}\). Now, let \(\mathbf{v} = (-u_2, u_1)\), where \(\mathbf{u} = (u_1, u_2)\) and define the functions \(f(y) = \max\{z : y\mathbf{u} + z\mathbf{v} \in P\}\) and \(g(y) = -\min\{z : y\mathbf{u} + z\mathbf{v} \in P\}\) for \(y \in [y_{\min}, y_{\max}]\). If \(\mathbf{abc}\) is inscribed in \(P\) with \(\mathbf{u}\) as the outer normal to the edge \(\mathbf{bc}\), then \(\mathbf{b} = y\mathbf{u} - g(y)\mathbf{v}\) and \(\mathbf{c} = y\mathbf{u} + f(y)\mathbf{v}\) for some \(y\) or \(\mathbf{b} = y_{\max}\mathbf{u} - z'\mathbf{v}\) and \(\mathbf{c} = y_{\min}\mathbf{u} + z''\mathbf{v}\) with \(z' < f(y_{\max})\) or \(z'' < g(y_{\max})\). If the latter case holds, then \(\mathbf{abc}\) cannot be of maximal area, even locally. The functions \(f(y)\) and \(g(y)\) (and so also \(f(y) + g(y)\)) are convex and piecewise linear on \([y_{\min}, y_{\max}]\). It follows that \(s(y) = \sqrt{\frac{1}{2}(y - y_{\min})[f(y) + g(y)]}\) is also convex and has a single maximum. Therefore, if \(\mathbf{abc}\) is of maximal area among triangles with \(\mathbf{u}\) as the outer normal, then \(\mathbf{b} = y^*\mathbf{u} - g(y^*)\mathbf{v}\) and \(\mathbf{c} = y^*\mathbf{u} + f(y^*)\mathbf{v}\), where \(y^*\) is the unique maximum of \(s(y)\). Furthermore, there are no local maxima except for ones with those vertices \(\mathbf{b}\) and \(\mathbf{c}\) that achieve the global maximum. \(\square\)

When \(-\mathbf{u}\) is an outer normal of \(P\) at a unique point (necessary a vertex of \(P\)), then the triangle anchored to \(\mathbf{u}\) is unique. When \(-\mathbf{u}\) is an outer normal to \(P\) along an edge, we will choose among the anchored triangles the one such the \(a\) is the vertex at the counter-clockwise end of that edge. Therefore, we will always have a unique triangle in mind when we refer to the triangle anchored to \(\mathbf{u}\). We denote the vertices of that triangle as functions of \(\mathbf{u}\): \(a(\mathbf{u})\), \(b(\mathbf{u})\), and \(c(\mathbf{u})\).

Lemma 3. 1. \(a : S^1 \to \partial P\) is piecewise constant.

2. \(b, c : S^1 \to \partial P\) are continuous.

Proof. 1. When \(-\mathbf{u}\) is not an outer normal to \(P\) along an edge, then any vector in a neighbor of \(-\mathbf{u}\) is also normal to \(P\) at \(a(\mathbf{u})\). Therefore, \(a(\mathbf{u})\)
is locally constant. Since \( P \) has finitely many edges, \( a(u) \) is locally constant at all but finitely many points.

2. If \( b(u_k) \to b' \) and \( c(u_k) \to c' \) as \( u_k \to u \), where \( \text{area}(ab'c') < \text{area}(a(u)b(u)c(u)) \), then for sufficiently large \( k \), we could deform \( a(u)b(u)c(u) \) to create a triangle with \( u_k \) as an outer normal and area larger than \( a_k b_k c_k \). This is a contradiction, so \( \text{area}(ab'c') \geq \text{area}(ab'c') \), and by uniqueness of the maximum, the functions are continuous.

So, the anchored triangles of \( P \) form a cyclic one-parameter family, in which two vertices move continuously and the third jumps from one vertex to the next in cyclic order. If we could go through the anchored triangles in a systematic way, we could find the one that maximizes the area, and we would be done. In the next section, we will show how this can be done, but we require some more preparation.

**Definition 4.** Let \( P \) be a convex polygon, \( u \in S^1 \) be a unit vector, and \( abc \) a triangle inscribed in \( P \). We say \( abc \) is candidate-anchored to \( u \) if it satisfies the conditions

1. \( u \) is an outer normal to the edge \( bc \).
2. \(-u\) is an outer normal to \( P \) at \( a \).

For a point \( x \in \partial P \), define its forward counter-clockwise tangent \( e_+(x) \in S^1 \) to be the direction from \( x \) to the vertex immediately counter-clockwise to \( x \). Similarly, let \( e_-(x) \), the backward counter-clockwise tangent, be the direction to \( x \) from the vertex immediately clockwise to \( x \). The two are equal except at vertices of \( P \). Let \( L_\pm(x) \) be the line \( \{ x + se_\pm(x) : s \in \mathbb{R} \} \). Let \( abc \) be candidate-anchored to \( u \), and let \( M \) be the line through \( b \) and \( c \). Let \( b'_\pm(h) \) and \( c'_\pm(h) \) be the intersections of \( M + \|c - b\|hu \) with \( L_\pm(b) \) and with \( L_\pm(c) \), respectively. Then define

\[
Q_{\pm\pm}(a, b, c) = \frac{d(\text{area}(ab'_\pm(h)c'_\pm(h)))}{dh}\bigg|_{h=0}.
\]

Note that the intersection \( b'_\pm(h) \) does not exists if \( e_\pm(b) \) is antiparallel or directly parallel to \( c - b \). Since the antiparallel case is impossible a limiting procedure suggests that we should take \( Q_{\pm\pm}(a, b, c) = -\infty \) in this case. Similarly, if \( e_\pm(c) \) is parallel to \( c - b \) (again necessarily directly parallel), we take \( Q_{\pm\pm}(a, b, c) = +\infty \). If both are true, then we leave \( Q_{\pm\pm} \) undefined. 

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Lemma 5. Let abc be candidate-anchored to u.

1. abc is anchored to u if and only if $Q_-(a, b, c) \geq 0$ and $Q_+(a, b, c) \leq 0$.

2. If neither b or c are vertices of P, then abc is anchored to u if and only if $Q_+(a, b, c) = 0$.

3. If $Q_+(a, b, c) = 0$, then abc is anchored to u.

Proof. For (1), recall the functions $f(y)$, $g(y)$, and $s(y)$ from the proof of Lemma 2. If $b = yu - g(y)v$ and $c = yu + f(y)v$, then $Q_-(a, b, c) = (1/||c - b||)d(s^2)/dy_y$ and $Q_+(a, b, c) = (1/||c - b||)d(s^2)/dy_y$ are the directional derivatives of the area $s(y)^2$. The triangle is anchored to u if and only if $y$ is a maximum of $s(y)$, which completes the proof.

(2) is a simple corollary of (1), since $Q_+(a, b, c) = Q_-(a, b, c) = 0$ when neither b or c are vertices of P.

Finally, because the tangent discontinuities at vertices of P can only be in the convex direction, $\text{area}(ab'h)c_h'(h) \leq \text{area}(ab'h)c_h'(h)$ when $h < 0$ and $\text{area}(ab'h)c_h'(h) \leq \text{area}(ab'h)c_h'(h)$ when $h > 0$. Therefore, if $Q_+(a, b, c) = 0$, then $Q_-(a, b, c) \geq 0$ and $Q_+(a, b, c) \leq 0$, and by (1), the triangle is anchored.

\[ \square \]

Theorem 6. Let abc be anchored to $u = (\cos \theta, \sin \theta)$ and suppose that if $-u$ is an outer normal to an edge of P, that a is at the counter-clockwise end of that edge. There exists $\delta > 0$ such that when $u' = (\cos \theta', \sin \theta')$ and $\theta' \in [0, \theta + \delta]$, there is a triangle $a'b'c'$ anchored to $u'$ such that $a' = a$, $b'$ is on the edge segment between b and the vertex immediately counter-clockwise to b, and $c'$ is on the edge segment between c and the vertex immediately counter-clockwise to c.

Proof. We will analyze three cases, $Q_+(a, b, c) = 0$, $Q_+(a, b, c) < 0$, and $Q_+(a, b, c) > 0$ (since abc is anchored, $e_+(c)$ is not parallel to $c - b$ so $Q_+(a, b, c)$ is not undefined).

In the first case, $Q_+(a, b, c) = 0$, let $b'(tb) = b + tb e_+(b)$, let $c'(tb) = c + tb e_+(c)$, and consider the triangle $a'b'(tb)c'(tb)$. Let $u'(tb, tc)$ be perpendicular to $c'(tb) - b'(tb)$, so the triangle is candidate-anchored to $u'(tb, tc)$. For small enough nonnegative $tb$ and $tc$, $e_+(b'(tb)) = e_+(b)$ and $e_+(c'(tc)) = e_+(c)$, and the equation $Q_+(a, b'(tb), c'(tc)) = 0$ gives an implicit relation between $tb$ and $tc$ of the form $\beta tb + \gamma tc + \alpha tc = 0$, where $\beta = e_+(b) \wedge (c - \frac{1}{2}a - \frac{1}{2}b)$, $\gamma = e_+(c) \wedge (b - \frac{1}{2}a - \frac{1}{2}c)$, $\alpha = e_+(b) \wedge e_+(c)$, and $\wedge$ denotes the wedge product $(x_1, y_1) \wedge (x_2, y_2) = x_1y_2 - y_1x_2$. We now
claim that $\beta \geq 0$, $\gamma \geq 0$, and at least one of them is positive. To show this most easily, we translate and rotate $P$ so that $a = (0, 0)$ and $u = (1, 0)$. Since $Q_{++}(a, b, c) = 0$ we know neither $L_+(b)$ or $L_+(c)$ are vertical, so we can parameterize them as $\{(x, r_b + m_b x)\}$ and $\{(x, r_c + m_c x)\}$ respectively. The equations $(c - b) \cdot u = 0$ and $Q_{++}(a, b, c) = 0$ together with $b \in L_+(b)$ and $c \in L_+(c)$, determine $b$ and $c$ to have $x = (r_c - r_b) / (m_b - m_c)$. We now immediately get $\beta = r_c$ and $\gamma = -r_b$. Since $a = (0, 0)$ is contained in $P$, the support lines, $L_+(b)$ and $L_+(c)$ must intersect the $y$-axis below and above it respectively, so $r_b \leq 0$ and $r_c \geq 0$. If both lines intersect at $a$, then we have $x = 0$ for $b$ and $c$, which is impossible. Therefore, we can invert the implicit relations $\beta b - \gamma t_c + \alpha t_c = 0$ and $(c'(t_c) - b'(t_b)) \cdot u' = 0$ to obtain functions $t_b(u')$ and $t_c(u')$ defined in the neighborhood of $u$ and in particular, for $u' = (\cos \theta', \sin \theta')$, $\theta' \in [\theta, \theta + \delta]$. The triangles $\mathcal{A}b'\mathcal{B}(t_b(u'))c'(t_c(u'))$ have $Q_{++} = 0$, and are therefore anchored to $u'$.

In the second case, $Q_{++}(a, b, c) < 0$. Since $abc$ is anchored, we know that $b$ is a vertex (or else $Q_{--}(a, b, c) = Q_{++}(a, b, c) < 0$). Let $c'(t_c) = c + t_c e(c)$. Since $Q_{++}(a, b, c'(t_c))$ evolves continuously and $Q_{++}(a, b, c'(0))$ is negative, we have for small enough positive $t_c$ that $Q_{++}(a, b, c'(t_c)) = Q_{--}(a, b, c'(t_c))$ is also negative. If $Q_{--}(a, b, c'(0)) > 0$, then the same argument gives $Q_{--}(a, b, c'(t_c)) > 0$ for small enough $t_c$, and the triangle $\mathcal{A}b'\mathcal{C}(t_c(u'))$ is anchored. Therefore, the only problematic case is when $Q_{--}(a, b, c'(0)) = 0$. Again, for ease of analysis, we translate and rotate $P$ so that $a = (0, 0)$ and $u = (1, 0)$. Since $Q_{--}(a, b, c) = 0$ we know neither $L_-(b)$ or $L_+(c)$ are vertical, so we can parameterize them as $\{(x, r_b + m_b x)\}$ and $\{(x, r_c + m_c x)\}$ respectively, and we can solve for $b$ and $c$ on those lines from $Q_{--}(a, b, c) = 0$ and $(c - b) \cdot u = 0$. We find that $dQ_{--}/dt_c = 4(m_b - m_c)(-r_b) / (r_b - r_c)^2 > 0$. Therefore, $Q_{--}(a, b, c'(t_c)) \geq 0$ for small enough $t_c$ in this case too. If we let $t_c(u')$ be the solution of $(c'(t_c) - b(u')) \cdot u' = 0$, we obtain the triangle $\mathcal{A}bc'(t_c(u'))$, which we have shown to be anchored to $u' = (\cos \theta', \sin \theta')$, when $\theta' \in [\theta, \theta + \delta]$ and $\delta$ is sufficiently small.

The final case, $Q_{++}(a, b, c) > 0$ is symmetric to the previous case: necessarily $c$ is a vertex, and we consider the triangle $\mathcal{A}b'(t_b(u'))\mathcal{C}$, where $b'(t_b) = b + t_b e_+(b)$. By a similar analysis as in the previous case, we conclude that this triangle is anchored for small enough positive $t_b$. 

The observation that the vertices of the triangle anchored to $u$ move monotonically in a counter-clockwise way as $u$ moves in a counter-clockwise way is crucial in demonstrating that the algorithm presented in the next section takes time linear in the number of vertices.
3 Algorithmic implementation

We first show that given a vector $u$ and a polygon $P$ with $n$ vertices, it is possible to obtain a triangle inscribed in $P$ anchored to $u$ in linear time. Start by setting $a$ to the vertex $p$ of $P$ minimizing $u \cdot p$, and set both $b$ and $c$ to the vertex maximizing $u \cdot p$. We evolve $b$ to $b' = b - t_b e_-(b)$ and $c$ to $c' = c + t_c e_+(c)$, maintaining $(c' - b') \cdot u = 0$ and stopping when the first of the following conditions arises:

1. $b'$ reaches the vertex immediately clockwise to $b$,
2. $c'$ reaches the vertex immediately counter-clockwise to $c$, or
3. $Q_{+-}(a, b', c')$ reaches 0.

We repeat this iterative evolution until we have $Q_{+-}(a, b, c) \geq 0$ and $Q_{++}(a, b, c) \geq 0$.

To find the maximal inscribed triangle in $P$, we start with a triangle anchored to some arbitrary $u$. We then determine which of the cases of the proof of Theorem 6 the triangle belongs to (shift $a$ to the next vertex in the counterclockwise direction if needed to satisfy the hypothesis of the theorem), and obtain the formula for evolving $b' = b + t_b e_+(b)$ and $c' = c + t_c e_+(c)$ such that the triangle $ab'c'$ is anchored; we evolve forward to the point where the first of the following stopping conditions arises:

1. $b'$ reaches the vertex immediately counter-clockwise to $b$,
2. $c'$ reaches the vertex immediately counter-clockwise to $c$,
3. $c' - b'$ becomes parallel to $e_+(a)$, or
4. $Q_{++}(a, b', c')$ reaches 0 when it was previously not 0.

We continue this iterative evolution until we have evolved through all the anchored triangles of $P$. Whenever the vertices of the triangle all coincide with vertices of $P$, we record the area of the triangle, and store for output the triangle with the largest area.

We give a pseudocode representation of the algorithm for finding an anchored triangle in Listing 1 and for finding the largest area inscribe triangle in Listing 2. Some cumbersome formulas were moved to Listing 3 for neatness. A C++ implementation of the algorithm is available alongside the source code of this eprint, and also available at https://github.com/ykallus/max-triangle/releases/tag/v1.0.
Theorem 7. On the input \( P = (p_1, \ldots, p_n) \) representing a convex \( n \)-gon, the algorithm represented in Listing 2 outputs the vertex indices of a triangle inscribed in \( P \) that has the maximal area among all inscribed triangles. The algorithm halts after \( O(n) \) iterations.

Proof. The triangle represented by the algorithm evolves continuously through all the anchored triangles of \( P \), and any new vertex reached triggers a new iteration of the main loop. Therefore, the maximum area inscribed triangle is encountered by the algorithm at the top of the loop on some iteration, and its vertex indices will be recorded and returned.

The initial task of finding an anchored triangle takes \( O(n) \) iterations because at each iteration of \texttt{ANCHORED_TRIANGLE}, either \texttt{b} or \texttt{c} visits a new vertex of \( P \), or the algorithm halts. Since \texttt{b} and \texttt{c} never visit the same vertex more than once, this task is completed in \( O(n) \) iterations.

To see that \texttt{LARGEST_INSCRIBED_TRIANGLE} goes through \( O(n) \) iterations, note that on each iteration either one of the three triangle vertices reaches some new vertex of \( P \) or \( Q \) switches from being zero to being nonzero. Since \texttt{a} visits all \( n \) vertices in order, returning to the one it started on, and since \texttt{b} and \texttt{c} only move counter-clockwise and never cross each other or \texttt{a}, it follows that each of them can reach a new vertex at most \( 2n \) times. Since \( Q \) cannot switch from being zero to being nonzero on two consecutive iterations, the algorithm halts after \( O(n) \) iterations.

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function $\text{ANCHORED\_TRIANGLE}((p_1, \ldots, p_n), u)$

\[
\begin{align*}
i_a & \leftarrow \arg\min_{i=1, \ldots, n} u \cdot p_i \\
i_c & \leftarrow \arg\max_{i=1, \ldots, n} u \cdot p_i \\
s_c & \leftarrow 0, i_b \leftarrow i_b - 1, s_b \leftarrow 1 \\
a & \leftarrow p_{i_a} \\
\text{loop} & \\
& \text{if } s_b = 0 \text{ then } i_b \leftarrow i_b - 1, s_b \leftarrow 1 \\
& \text{if } s_c = 1 \text{ then } i_c \leftarrow i_c + 1, s_c \leftarrow 0 \\
b & \leftarrow p_{i_b}(1 - s_b) + p_{i_b+1}s_b, c \leftarrow p_{i_c}(1 - s_c) + p_{i_c+1}s_c \\
e_b & \leftarrow p_{i_b+1} - p_{i_b}, e_c & \leftarrow p_{i_c+1} - p_{i_c} \\
& \text{if } u \cdot e_b = 0 \text{ then} \\
& \quad i_b \leftarrow i_b - 1, s_b \leftarrow 1 \\
& \quad \text{go to top of loop} \\
& \text{end if} \\
& \text{if } u \cdot e_c = 0 \text{ then} \\
& \quad i_c \leftarrow i_c + 1, s_c \leftarrow 0 \\
& \quad \text{go to top of loop} \\
& \text{end if} \\
& \text{if } e_b \wedge e_c \leq 0 \text{ then break out of loop} \\
& \quad t_q \leftarrow \text{CALCULATE\_TQ}(a, b, e_b, c, e_c, u) \\
& \quad t_b \leftarrow -t_q/(u \cdot e_b), t_c \leftarrow t_q/(u \cdot e_c) \\
& \quad \text{if } t_b \leq 0 \text{ or } t_c \leq 0 \text{ then break out of loop} \\
& \quad \text{if } t_b < s_b \text{ and } t_c < 1 - s_c \text{ then} \\
& \quad \quad s_b \leftarrow s_b - t_b, s_c \leftarrow s_c + t_c \\
& \quad \quad \text{break out of loop} \\
& \quad \text{else if } (1 - t_c)t_b < s_b t_b \text{ then} \\
& \quad \quad s_b \leftarrow t_b - (1 - s_c)t_b/t_c \\
& \quad \quad i_c \leftarrow i_c + 1, s_c \leftarrow 0 \\
& \quad \text{else} \\
& \quad \quad s_c \leftarrow t_c + s_b t_c/t_b \\
& \quad \quad i_b \leftarrow i_b - 1, s_b \leftarrow 1 \\
& \quad \text{end if} \\
& \text{end loop} \\
& \text{return } i_a, i_b, s_b, i_c, s_c \\
\end{align*}
\]

Listing 1: An $O(n)$-time algorithm to find a triangle in an $n$-gon $P$ anchored to $u$. 

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function LARGEST_INScribed_TRIANGLE((p1, ..., pn))
    ia, ib, sc, gc ← ANCHORED_TRIANGLE((p1, ..., pn),(1, 0))
    ia(init) ← ia, A_max ← 0
    while ia ≤ ia(init) + n do
        if sb = 1 then ib ← ib + 1, sb ← 0
        if sc = 1 then ic ← ic + 1, sc ← 0
            a ← pi, b ← pi(b - sb) + pi(b+1) - sb, c ← pi(c) + pi(c+1) - sb
            ea ← pi(a+1) - pi(a), eb ← pi(b+1) - pi(b), ec ← pi(c+1) - pi(c)
        if area(abc > A_max) and sb = 0 and sc = 0 then
            A_max ← area(abc), i_a(max) ← ia, i_b(max) ← ib, i_c(max) ← ic
        end if
        if ea ∧ (c - b) = 0 then ia ← ia + 1, a ← pi
        Q ← CALcuLATE_Q(a, b, e, c, e)
        if Q > 0 then
            t_b1 ← 1 - sb
            t_b3 ← (ea ∧ (c - b))/(ea ∧ eb)
            t_b4 ← CALcuLATE_TB4(a, b, e, c, e)
            t_b ← smallest positive value among t_b1, t_b3, and t_b4.
            sb ← sb + tb
        else if Q < 0 then
            t_c2 ← 1 - sc
            t_c3 ← (ea ∧ (b - c))/(ea ∧ ec)
            t_c4 ← CALcuLATE_TC4(a, b, e, c, e)
            tc ← smallest positive value among t_c2, t_c3, and t_c4.
            sc ← sc + tc
        else if Q = 0 then
            α, β, γ ← CALcuLATE_ALPHA_BETA_GAMMA(a, b, e, c, c)
            t_b1 ← 1 - sb, t_c1 ← (βtb1)/(γ - αtb1)
            t_c2 ← 1 - sc, t_b2 ← (γtc2)/(β + αtc2)
            t_b3, t_c3 ← CALcuLATE_TB3_TC3(a, e, b, e, b, c, c)
            t_b, t_c ← pair with smallest t_b among the pairs (t_b1, t_c1),
            (t_b2, t_c2), and (t_b3, t_c3) with tb ≥ 0 and tc ≥ 0.
            sb ← sb + tb, sc ← sc + tc
        end if
    end while
    return i_a(max), i_b(max), i_c(max)
end function

Listing 2: An O(n)-time algorithm to find a triangle inscribed in an n-gon P of maximal area.
function CALCULATE_Q(a, b, v, c, w)
    return (c_2 - b_2) [(b_2 - c_2) v_1 w_1 - (a_1 - c_1) v_2 w_1 + (a_1 - b_1) v_1 w_2] +
            (c_1 - b_1) [(b_1 - c_1) v_2 w_2 - (a_2 - c_2) v_1 w_2 + (a_2 - b_2) v_2 w_1]
end function

function CALCULATE_TB4(a, b, v, c, w)
    return \{v_1 w_1 (b_2 - c_2) (b_2 - c_2) + v_2 w_2 (b_1 - c_1) (b_1 - c_1) + v_1 w_2 (a_1 -
          b_1) b_2 + (c_1 - b_1) a_2 + (2b_1 - a_1 - c_1) c_2 + v_2 w_1 (b_1 - c_1) a_2 - (c_1 - a_1) c_2 +
          (2c_1 - a_1 - b_1) b_2] / [(v_1 w_2 - v_2 w_1) (v_1 (a_2 + b_2 - 2c_2) - v_2 (a_1 + b_1 - 2c_1))]
end function

function CALCULATE_TC4(a, b, v, c, w)
    return -(v_1 w_1 (b_2 - c_2) (b_2 - c_2) + v_2 w_2 (b_1 - c_1) (b_1 - c_1) + v_1 w_2 (a_1 -
          b_1) b_2 + (c_1 - b_1) a_2 + (2b_1 - a_1 - c_1) c_2 + v_2 w_1 (b_1 - c_1) a_2 - (c_1 - a_1) c_2 +
          (2c_1 - a_1 - b_1) b_2] / [(v_1 w_2 - v_2 w_1) (w_1 (a_2 + c_2 - 2b_2) - w_2 (a_1 + c_1 - 2b_1))]
end function

function CALCULATE_ALPHA_BETA_GAMMA(a, b, v, c, w)
    return 2 (v_1 w_2 - w_1 v_2), v_1 (2c_2 - a_2 - b_2) - v_2 (2c_1 - b_1 - a_1),
           w_1 (2b_2 - a_2 - c_2) - w_2 (2b_1 - c_1 - a_1)
end function

function CALCULATE_TB3_TC3(a, e, b, v, c, w)
    t_{b,3} \leftarrow \{v_1 w_1 e_2 (b_2 - c_2) + v_2 w_2 e_1 (b_1 - c_1) + v_1 w_2 (e_1 (b_2 - a_2) -
          e_2 (2b_1 - a_1 - c_1)) + v_2 w_1 (e_2 (b_1 - a_1) - e_1 (2b_2 - a_2 - c_2)) \}/2(v_1 w_2 -
          w_1 v_2) (e_2 v_1 - e_1 v_2)
end function

Listing 3: Cumbersome formulas removed from Listings 1 and 2 for neatness.