A Federer-style characterization of sets of finite perimeter on metric spaces

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Abstract

In the setting of a metric space equipped with a doubling measure that supports a Poincaré inequality, we show that a set $E$ is of finite perimeter if and only if $\mathcal{H}(\partial^1 I_E) < \infty$, that is, if and only if the codimension one Hausdorff measure of the 1-fine boundary of the set’s measure theoretic interior $I_E$ is finite.

1 Introduction

Federer’s structure theorem states that a set $E \subset \mathbb{R}^k$ is of finite perimeter if and only if $\mathcal{H}(\partial^* E)$ is finite, see [11, Section 4.5.11]. Here $\mathcal{H}$ is the codimension one (in this case, $k-1$-dimensional) Hausdorff measure, and $\partial^* E$ is the measure theoretic boundary of $E$. In a complete metric space $X$ equipped with a doubling measure that supports a Poincaré inequality, the “only if” direction has been shown by Ambrosio, see [1], but the “if” direction remains open.

In this paper we define for $A \subset X$ the 1-fine boundary $\partial^1 A$, which always contains $\partial^* A$ but can be strictly larger; for example on the real line, the 1-fine boundary coincides with the topological boundary. However, using a fine continuity result for BV functions given in [21], we show that for any set of finite perimeter $E$, denoting the measure theoretic interior of $E$ by $I_E$, the

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difference $\partial^1 I_E \setminus \partial^r E$ is $\mathcal{H}$-negligible. In particular, then $\mathcal{H}(\partial^1 I_E) < \infty$. In showing this, we first prove a suitable characterization of the 1-fine boundary, in analogy with what is known in the case $p > 1$, see [6, Section 7].

Then we show that the condition $\mathcal{H}(\partial^1 I_E) < \infty$ is also sufficient for $E$ to be of finite perimeter. For this, we generalize further concepts and results of fine potential theory from the case $p > 1$ to the case $p = 1$; all such considerations appear to be new even in the Euclidean setting. In particular, we study the existence of capacitary potentials and prove weak analogs of the Cartan property for solutions of obstacle problems, and of the Choquet property for finely open sets. These have recently been studied for $p > 1$ in the metric setting in [8, 9]; see also [24] and [16] for the Euclidean theory and its history in the unweighted and weighted settings, respectively.

Our result is the following — see Section 2 for the definitions.

Theorem 1.1. For an open set $\Omega \subset X$ and a $\mu$-measurable set $E \subset X$, we have $P(E, \Omega) < \infty$ if and only if $\mathcal{H}(\partial^1 I_E \cap \Omega) < \infty$. Furthermore, then $\mathcal{H}((\partial^1 I_E \setminus \partial^r E) \cap \Omega) = 0$.

Necessity is given by Theorem 4.7 in Section 4. Sufficiency is given by Theorem 5.1 in Section 5. The results of [23] and [21] are used extensively in the proofs.

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2 Preliminaries

In this section we introduce the notation, definitions, and assumptions used in the paper.

In this paper, $(X, d, \mu)$ is a complete metric space equipped with a Borel regular outer measure $\mu$ satisfying a doubling property, that is, there is a constant $C_d \geq 1$ such that

$$0 < \mu(B(x, 2r)) \leq C_d \mu(B(x, r)) < \infty$$

for every ball $B = B(x, r)$ with center $x \in X$ and radius $r > 0$. We assume that $X$ consists of at least two points. By iterating the doubling condition,
we obtain that for any $x \in X$ and $y \in B(x, R)$ with $0 < r \leq R < \infty$, we have

$$\frac{\mu(B(y, r))}{\mu(B(x, R))} \geq \frac{1}{C_d^2} \left( \frac{r}{R} \right)^Q,$$

where $Q > 1$ only depends on the doubling constant $C_d$. When we want to specify that a constant $C$ depends on the parameters $a, b, \ldots$, we write $C = C(a, b, \ldots)$.

A complete metric space with a doubling measure is proper, that is, closed and bounded sets are compact. Since $X$ is proper, for any open set $\Omega \subset X$ we define $\text{Lip}_{\text{loc}}(\Omega)$ to be the space of functions that are Lipschitz in every open $\Omega' \Subset \Omega$. Here $\Omega' \Subset \Omega$ means that $\overline{\Omega'}$ is a compact subset of $\Omega$. Other local spaces of functions are defined similarly.

For any set $A \subset X$ and $0 < R < \infty$, the restricted spherical Hausdorff content of codimension one is defined by

$$H_R(A) := \inf \left\{ \sum_{i=1}^{\infty} \frac{\mu(B(x_i, r_i))}{r_i} : A \subset \bigcup_{i=1}^{\infty} B(x_i, r_i), \ r_i \leq R \right\}.$$

The codimension one Hausdorff measure of $A \subset X$ is given by

$$\mathcal{H}(A) := \lim_{R \to 0} \mathcal{H}_R(A).$$

The measure theoretic boundary $\partial^* E$ of a set $E \subset X$ is the set of points $x \in X$ at which both $E$ and its complement have positive upper density, i.e.

$$\limsup_{r \to 0} \frac{\mu(B(x, r) \cap E)}{\mu(B(x, r))} > 0 \quad \text{and} \quad \limsup_{r \to 0} \frac{\mu(B(x, r) \setminus E)}{\mu(B(x, r))} > 0.$$

The measure theoretic interior and exterior of $E$ are defined respectively by

$$I_E := \left\{ x \in X : \lim_{r \to 0} \frac{\mu(B(x, r) \setminus E)}{\mu(B(x, r))} = 0 \right\},$$

and

$$O_E := \left\{ x \in X : \lim_{r \to 0} \frac{\mu(B(x, r) \cap E)}{\mu(B(x, r))} = 0 \right\}.$$
A curve is a rectifiable continuous mapping from a compact interval into $X$. The length of a curve $\gamma$ is denoted by $\ell_{\gamma}$. We will assume every curve to be parametrized by arc-length, which can always be done (see e.g. [13, Theorem 3.2]). A nonnegative Borel function $g$ on $X$ is an upper gradient of an extended real-valued function $u$ on $X$ if for all curves $\gamma$, we have

$$|u(x) - u(y)| \leq \int_\gamma g \, ds,$$

(2.4)

where $x$ and $y$ are the end points of $\gamma$. We interpret $|u(x) - u(y)| = \infty$ whenever at least one of $|u(x)|$, $|u(y)|$ is infinite. Of course, by replacing $X$ with a set $A \subset X$ and considering curves $\gamma$ in $A$, we can talk about a function $g$ being an upper gradient of $u$ in $A$. Upper gradients were originally introduced in [17].

If $g$ is a nonnegative $\mu$-measurable function on $X$ and (2.4) holds for 1-almost every curve, we say that $g$ is a 1-weak upper gradient of $u$. A property holds for 1-almost every curve if it fails only for a curve family with zero 1-modulus. A family $\Gamma$ of curves is of zero 1-modulus if there is a nonnegative Borel function $\rho \in L^1(X)$ such that for all curves $\gamma \in \Gamma$, the curve integral $\int_\gamma \rho \, ds$ is infinite.

Given an open set $\Omega \subset X$, we consider the following norm

$$\|u\|_{N^{1,1}(\Omega)} := \|u\|_{L^1(\Omega)} + \inf \|g\|_{L^1(\Omega)},$$

where the infimum is taken over all 1-weak upper gradients $g$ of $u$ in $\Omega$. The substitute for the Sobolev space $W^{1,1}(\Omega)$ in the metric setting is the Newton-Sobolev space

$$N^{1,1}(\Omega) := \{u : \|u\|_{N^{1,1}(\Omega)} < \infty\}.$$ 

We understand Newton-Sobolev functions to be defined everywhere (even though $\|\cdot\|_{N^{1,1}(\Omega)}$ is, precisely speaking, then only a seminorm). For more on Newton-Sobolev spaces, we refer to [27, 5, 18].

Next we recall the definition and basic properties of functions of bounded variation on metric spaces, following [25]. See also e.g. [2, 10, 12, 28] for the classical theory in the Euclidean setting. For $u \in L^1_{\text{loc}}(X)$, we define the total variation of $u$ in $X$ by

$$\|Du\|(X) := \inf \left\{ \liminf_{i \to \infty} \int_X g_{u_i} \, d\mu : u_i \in \text{Lip}_{\text{loc}}(X), u_i \rightharpoonup u \text{ in } L^1_{\text{loc}}(X) \right\},$$

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where each \( g_{u_i} \) is an upper gradient of \( u_i \). We say that a function \( u \in L^1(X) \) is of bounded variation, and denote \( u \in BV(X) \), if \( \|Du\|(X) < \infty \). By replacing \( X \) with an open set \( \Omega \subset X \) in the definition of the total variation, we can define \( \|Du\|(\Omega) \). For an arbitrary set \( A \subset X \), we define

\[
\|Du\|(A) = \inf \{ \|Du\|(\Omega) : A \subset \Omega, \Omega \subset X \text{ is open} \}.
\]

If \( \|Du\|(X) < \infty \), \( \|Du\|(\cdot) \) is a finite Radon measure on \( X \) by [25, Theorem 3.4]. A \( \mu \)-measurable set \( E \subset X \) is said to be of finite perimeter if \( \|D\chi_E\|(X) < \infty \), where \( \chi_E \) is the characteristic function of \( E \). The perimeter of \( E \) in \( \Omega \) is also denoted by

\[
P(E, \Omega) := \|D\chi_E\|(\Omega).
\]

Similarly as above, if \( P(E, \Omega) < \infty \), then \( P(E, \cdot) \) is finite Radon measure on \( \Omega \).

For any Borel sets \( E_1, E_2 \subset X \), we have by [25, Proposition 4.7]

\[
P(E_1 \cup E_2, X) \leq P(E_1, X) + P(E_2, X).
\]

The proof works equally well for \( \mu \)-measurable \( E_1, E_2 \subset X \) and with \( X \) replaced by any open set \( \Omega \). Then by approximation from the outside by open sets, we obtain for any \( A \subset X \)

\[
P(E_1 \cup E_2, A) \leq P(E_1, A) + P(E_2, A). \tag{2.5}
\]

We have the following coarea formula from [25, Proposition 4.2]: if \( U \subset X \) is an open set and \( u \in L^1_{\text{loc}}(U) \), then

\[
\|Du\|(U) = \int_{-\infty}^{\infty} P(\{u > t\}, U) \, dt. \tag{2.6}
\]

We will assume throughout that \( X \) supports a \((1, 1)\)-Poincaré inequality, meaning that there exist constants \( C_P \geq 1 \) and \( \lambda \geq 1 \) such that for every ball \( B(x, r) \), every locally integrable function \( u \) on \( X \), and every upper gradient \( g \) of \( u \), we have

\[
\int_{B(x, r)} |u - u_{B(x, r)}| \, d\mu \leq C_P r \int_{B(x, \lambda r)} g \, d\mu,
\]

where

\[
u_{B(x, r)} := \int_{B(x, r)} u \, d\mu := \frac{1}{\mu(B(x, r))} \int_{B(x, r)} u \, d\mu.
\]
The (1, 1)-Poincaré inequality implies the so-called Sobolev-Poincaré inequality, see e.g. [5, Theorem 4.21], and by applying the latter to approximating locally Lipschitz functions in the definition of the total variation, we get the following Sobolev-Poincaré inequality for BV functions. For every ball \(B(x, r)\) and every \(u \in L^1_{\text{loc}}(X)\), we have

\[
\left( \int_{B(x,r)} |u - u_{B(x,r)}|^{Q/(Q-1)} \, d\mu \right)^{(Q-1)/Q} \leq C_{SP} \frac{\|Du\|(B(x, 2\lambda r))}{\mu(B(x, 2\lambda r))},
\]

where \(Q\) is the exponent from (2.1) and \(C_{SP} = C_{SP}(C_d, C_P, \lambda) \geq 1\) is a constant. For a \(\mu\)-measurable set \(E \subset X\), this implies (see e.g. [20, Equation (3.1)])

\[
\frac{1}{2} \left( \frac{\min\{\mu(B(x,r) \cap E), \mu(B(x,r) \setminus E)\}}{\mu(B(x,r))} \right)^{(Q-1)/Q} \leq C_{SP} \frac{P(E, B(x,2\lambda r))}{\mu(B(x,2\lambda r))}.
\]

Rearranged, this implies

\[
\min\{\mu(B(x,r) \cap E), \mu(B(x,r) \setminus E)\} \leq 2C_{SP} \left( \frac{\min\{\mu(B(x,r) \cap E), \mu(B(x,r) \setminus E)\}}{\mu(B(x,r))} \right)^{1/Q} P(E, B(x,2\lambda r)).
\]

Moreover, the (1,1)-Poincaré inequality implies the following Sobolev inequality. If \(x \in X\), \(0 < r < \frac{1}{4} \text{diam}(X)\), and \(u \in N^{1,1}(X)\) with \(u = 0\) in \(X \setminus B(x,r)\), then

\[
\int_{B(x,r)} |u| \, d\mu \leq C_S r \int_{B(x,r)} g_u \, d\mu
\]

for any upper gradient \(g_u\) of \(u\) and a constant \(C_S = C_S(C_d, C_P) \geq 1\), see [5, Theorem 5.51]. By approximation, we obtain that for any \(x \in X\), any \(0 < r < \frac{1}{4} \text{diam}(X)\), and any \(\mu\)-measurable set \(E \subset B(x,r)\), we have

\[
\mu(E) \leq C_{Sr} P(E, X).
\]

The 1-capacity of a set \(A \subset X\) is given by

\[
\text{Cap}_1(A) := \inf \|u\|_{N^{1,1}(X)},
\]
where the infimum is taken over all functions $u \in N^{1,1}(X)$ such that $u \geq 1$ in $A$. We know that $\text{Cap}_1$ is an outer capacity, meaning that

$$\text{Cap}_1(A) = \inf \{ \text{Cap}_1(U) : U \supset A \text{ is open} \}$$

for any $A \subset X$, see e.g. [5, Theorem 5.31]. If a property holds outside a set $A \subset X$ with $\text{Cap}_1(A) = 0$, we say that it holds $1$-quasieverywhere. If $u \in N^{1,1}(X)$, then $\|u - v\|_{N^{1,1}(X)} = 0$ if and only if $u = v$ $1$-quasieverywhere, see [5, Proposition 1.61].

The variational $1$-capacity of a set $A \subset D$ with respect to a set $D \subset X$ is given by

$$\text{cap}_1(A, D) := \inf \int_X g_u d\mu,$$

where the infimum is taken over functions $u \in N^{1,1}(X)$ and upper gradients $g_u$ of $u$ such that $u \geq 1$ in $A$ (equivalently, $1$-quasieverywhere in $A$) and $u = 0$ in $X \setminus D$. We know that $\text{cap}_1$ is also an outer capacity, in the sense that if $\Omega \subset X$ is a bounded open set and $A \subset \Omega$, then

$$\text{cap}_1(A, \Omega) = \inf \{ \text{cap}_1(U) : U \text{ open, } A \subset U \subset \Omega \},$$

see [5, Theorem 6.19]. For basic properties satisfied by capacities, such as monotonicity and countable subadditivity, see e.g. [5].

Given a set $E \subset X$ of finite perimeter, for $\mathcal{H}$-almost every $x \in \partial^* E$ we have

$$\gamma \leq \liminf_{r \to 0} \frac{\mu(B(x,r) \cap E)}{\mu(B(x,r))} \leq \limsup_{r \to 0} \frac{\mu(B(x,r) \setminus E)}{\mu(B(x,r))} \leq 1 - \gamma,$$

(2.11)

where $\gamma \in (0, 1/2]$ only depends on the doubling constant and the constants in the Poincaré inequality, see [11, Theorem 5.4]. For an open set $\Omega \subset X$ and a $\mu$-measurable set $E \subset X$ with $P(E, \Omega) < \infty$, we know that for any Borel set $A \subset \Omega$,

$$P(E, A) = \int_{\partial E \cap A} \theta_E d\mathcal{H},$$

(2.12)

where $\theta_E : X \to [\alpha, C_d]$ with $\alpha = \alpha(C_d, C_P, \lambda) > 0$, see [11, Theorem 5.3] and [3, Theorem 4.6].

The lower and upper approximate limits of an extended real-valued function $u$ on $X$ are defined respectively by

$$u^-(x) := \sup \left\{ t \in \mathbb{R} : \lim_{r \to 0} \frac{\mu(B(x,r) \cap \{ u < t \})}{\mu(B(x,r))} = 0 \right\}$$

(2.13)
and
\[
u^>(x) := \inf \left\{ t \in \mathbb{R} : \lim_{r \to 0} \frac{\mu(B(x, r) \cap \{u > t\})}{\mu(B(x, r))} = 0 \right\}.
\] (2.14)

Note that for \( u = \chi_E \) with \( E \subset X \), we have \( x \in I_E \) if and only if \( u^\wedge(x) = u^\vee(x) = 1 \), \( x \in O_E \) if and only if \( u^\wedge(x) = u^\vee(x) = 0 \), and \( x \in \partial^* E \) if and only if \( u^\wedge(x) = 0 \) and \( u^\vee(x) = 1 \).

We understand BV functions to be \( \mu \)-equivalence classes. To consider fine properties, we need to consider the pointwise representatives \( u^\wedge \) and \( u^\vee \).

We need to generalize concepts of fine potential theory from the case \( p > 1 \) to the case \( p = 1 \). The following definition can be taken directly from e.g. [9].

**Definition 2.1.** A set \( A \subset X \) is \( 1 \)-quasiopen if for every \( \varepsilon > 0 \) there is an open set \( G \subset X \) with \( \operatorname{Cap}_1(G) < \varepsilon \) such that \( A \cup G \) is open.

Next we define the fine topology in the case \( p = 1 \).

**Definition 2.2.** We say that \( A \subset X \) is 1-thin at the point \( x \in X \) if
\[
\lim_{r \to 0} r \frac{\operatorname{cap}_1(B(x, r) \cap A, B(x, 2r))}{\mu(B(x, r))} = 0.
\]

We also say that a set \( U \subset X \) is 1-finely open if \( X \setminus U \) is 1-thin at every \( x \in U \). Then we define the 1-fine topology as the collection of 1-finely open sets on \( X \).

We denote the 1-fine interior of a set \( G \subset X \), i.e. the largest 1-finely open set contained in \( G \), by \( \text{fine-int} \ G \). We denote the 1-fine closure of a set \( G \subset X \), i.e. the smallest 1-finely closed set containing \( G \), by \( \overline{G}^1 \). We define the 1-fine boundary of a set \( G \subset X \) by \( \partial^1 G := \overline{G}^1 \setminus \text{fine-int} \ G \).

Finally, we define the 1-base \( b^1 G \) of a set \( G \subset X \) as the set of points where \( G \) is not 1-thin.

Note that always \( b^1 G \subset \overline{G}^1 \). See [21, Section 4] for motivation of the definition of 1-thinness, and for a proof of the fact that the 1-fine topology is indeed a topology.

### 3 The 1-fine boundary

In this section we give a suitable characterization of the 1-fine boundary.
Lemma 3.1. For any $A \subset X$, we have $I_A \cup \partial^* A \subset b_1 A$ and $\partial A \subset \partial^1 A$.

Proof. Suppose $x \in X \setminus b_1 A$, so that

$$
\lim_{r \to 0} r \frac{\cap_1(B(x, r) \cap A, B(x, 2r))}{\mu(B(x, r))} = 0.
$$

By the definition of the variational capacity, for every $r > 0$ we find a non-negative function $u_r \in N^{1,1}(X)$ such that $u_r \geq 1$ in $B(x, r) \cap A$, $u_r = 0$ in $X \setminus B(x, 2r)$, and $u_r$ has an upper gradient $g_r$ with

$$
\frac{r}{\mu(B(x, r))} \int_X g_r \, d\mu \to 0 \quad \text{as } r \to 0.
$$

But by the Sobolev inequality (2.8),

$$
\limsup_{r \to 0} \frac{\mu(B(x, r) \cap A)}{\mu(B(x, r))} \leq \limsup_{r \to 0} C_d \int_{B(x, 2r)} u_r \, d\mu 
\leq \limsup_{r \to 0} 2C_s C_d r \int_{B(x, 2r)} g_r \, d\mu = 0.
$$

Thus $x \notin I_A \cup \partial^* A$.

To prove the second claim, note that by the first claim,

$$
\overline{A} \supset b_1 A \supset I_A \cup \partial^* A
$$

and

$$
\text{fine-int } A = X \setminus \overline{X \setminus A} \subset X \setminus b_1 (X \setminus A) 
\subset X \setminus (I_{X \setminus A} \cup \partial^* A) = X \setminus (O_A \cup \partial^* A) = I_A.
$$

By combining these, we obtain $\partial^1 A = \overline{A} \setminus \text{fine-int } A \supset \partial^* A$.

Next we gather some known results.

Lemma 3.2 ([21, Lemma 4.3]). Let $x \in X$, let $r > 0$, and let $G \subset X$ be a $\mu$-measurable set with

$$
\frac{\mu(B(x, 2r) \cap G)}{\mu(B(x, 2r))} \leq \frac{1}{2C_d^{\log_2(128\lambda)}}.
$$

Then for some constant $C_1 = C_1(C_d, C_P, \lambda)$,

$$
\cap_1(B(x, r) \cap I_G, B(x, 2r)) \leq C_1 P(G, B(x, 2r)).
$$
Moreover, it is straightforward to show that for any set $A \subset X$ and any ball $B(x,r)$,
\[
cap_1(B(x,r) \cap A, B(x,2r)) \leq C_d \mathcal{H}(B(x,r) \cap A).
\]

This can be deduced by using suitable cutoff functions.

In the particular case of a set of finite perimeter, the 1-fine closure is essentially just the measure theoretic closure; this is essentially contained in [21 Proposition 4.4], but we repeat the proof here.

**Lemma 3.3.** Let $E \subset X$ be a set of finite perimeter. Then there exists a $\mathcal{H}$-negligible set $N \subset X$ such that
\[
\overline{T^{-1}_E} \subset I_E \cup \partial^* E \cup N.
\]

**Proof.** By [4 Theorem 2.4.3] we know that if $\nu$ is a Radon measure on $X$, $t > 0$, and $A \subset X$ is a Borel set for which we have
\[
\limsup_{r \to 0} r \frac{\nu(B(x,r))}{\mu(B(x,r))} \geq t
\]
for all $x \in A$, then $\nu(A) \geq t \mathcal{H}(A)$. Since $E$ is of finite perimeter, we have $\mathcal{H}(\partial^* E) < \infty$ by (2.12). By using (3.2) and the above density result with $\nu = \mathcal{H}|_{\partial^* E}$, we get
\[
\limsup_{r \to 0} r \frac{\text{cap}_1(B(x,r) \cap \partial^* E, B(x,2r))}{\mu(B(x,r))} \leq C_d \limsup_{r \to 0} r \frac{\mathcal{H}(B(x,r) \cap \partial^* E)}{\mu(B(x,r))} = 0
\]
for $\mathcal{H}$-almost every $x \in X \setminus \partial^* E$, that is, for every $x \in X \setminus (\partial^* E \cup N)$ with $\mathcal{H}(N) = 0$.

By Lemma 3.2, if $x \in X$ and $r > 0$ satisfy
\[
\frac{\mu(B(x,2r) \cap E)}{\mu(B(x,2r))} \leq \frac{1}{2C_d^{\log_2(128\lambda)}},
\]
then $\text{cap}_1(B(x,r) \cap I_E, B(x,2r)) \leq C_1 P(E, B(x,2r))$. Thus we get for all
$$x \in X \setminus (I_E \cup \partial^* E \cup N)$$

$$\limsup_{r \to 0} r \frac{\text{cap}_1(B(x,r) \cap I_E, B(x,2r))}{\mu(B(x,r))} \leq C_1 \limsup_{r \to 0} r \frac{P(E, B(x,2r))}{\mu(B(x,r))}$$

$$\leq C_1 C_d \limsup_{r \to 0} r \frac{\mathcal{H}(\partial^* E \cap B(x,2r))}{\mu(B(x,r))}$$

$$\leq C_1 C_d^2 \limsup_{r \to 0} r \frac{\mathcal{H}(\partial^* E \cap B(x,2r))}{\mu(B(x,2r))}$$

$$= 0$$

by the equality in (3.3). By combining this with (3.3), and (3.2) with $A = N$, we have

$$\limsup_{r \to 0} r \frac{\text{cap}_1(B(x,r) \cap (I_E \cup \partial^* E \cup N), B(x,2r))}{\mu(B(x,r))} = 0$$

for all $x \in X \setminus (I_E \cup \partial^* E \cup N)$. Thus $I_E \cup \partial^* E \cup N \supset I_E$ is a 1-finely closed set, so that $\overline{I_E} \subseteq I_E \cup \partial^* E \cup N$. \hfill \qed

**Theorem 3.4** ([21, Corollary 5.4]). Let $u \in N^{1,1}(X)$. Then $u$ is continuous (in the sense of a real-valued function) with respect to the 1-fine topology 1-quasieverywhere.

By [14] Theorem 4.3, Theorem 5.1 we know that if $A \subset X$,

$$\text{Cap}_1(A) = 0 \quad \text{if and only if} \quad \mathcal{H}(A) = 0. \quad (3.4)$$

**Lemma 3.5.** Let $x \in X$, let $0 < r < \text{diam}(X)/8$, and let $A \subset B(x,r)$ with

$$r \frac{\text{cap}_1(A, B(x,2r))}{\mu(B(x,r))} \leq \frac{1}{8C_SC_d^{\lceil \log_2(128\lambda) \rceil}}. \quad (3.5)$$

Then we have

$$\text{cap}_1(B(x,r) \cap \overline{A}, B(x,2r)) \leq C_1 \text{cap}_1(A, B(x,2r)),$$

where $C_1 = C_1(C_d, C_P, \lambda)$ is the constant from Lemma 3.2.
Proof. Fix $\varepsilon > 0$. By the definition of the variational capacity and the fact that it is an outer capacity, we can pick $u \in N^{1,1}(X)$ with $u \geq 1$ in a neighborhood of $A$, $u = 0$ in $X \setminus B(x, 2r)$, and

$$\text{cap}_1(A, B(x, 2r)) + \varepsilon \geq \int_X g_u \, d\mu \geq \|Du\|(X),$$

where $g_u$ is an upper gradient of $u$, and where the last inequality follows from the fact that Lipschitz functions are dense in $N^{1,1}(X)$, see e.g. [5, Theorem 5.1]. By using the coarea formula (2.6), we find a number $t \in (0, 1)$ such that

$$P(\{u > t\}, X) \leq \|Du\|(X) \leq \text{cap}_1(A, B(x, 2r)) + \varepsilon. \quad (3.6)$$

By the isoperimetric inequality (2.9), we have

$$\mu(\{u > t\}) \leq 2CSr P(\{u > t\}, X) \leq 2CSr \text{cap}_1(A, B(x, 2r)) + 2CSr\varepsilon \leq \frac{\mu(B(x, r))}{8CS C_d^{\left[\log_2(128\lambda)\right]}r} + 2CSr\varepsilon$$

Thus by assuming that $\varepsilon \leq \mu(B(x, r))/(8CS C_d^{\left[\log_2(128\lambda)\right]}r)$, we have

$$\frac{\mu(\{u > t\})}{\mu(B(x, 2r))} \leq \frac{\mu(\{u > t\})}{\mu(B(x, r))} \leq \frac{1}{2C_d^{\left[\log_2(128\lambda)\right]}}$$

and now by Lemma 3.2 we have

$$\text{cap}_1(B(x, r) \cap I_{\{u > t\}}, B(x, 2r)) \leq C_1 P(\{u > t\}, B(x, 2r)) \leq C_1 P(\{u > t\}, X) \leq C_1 \text{cap}_1(A, B(x, 2r)) + C_1\varepsilon. \quad (3.7)$$

by (3.6). Again by the definition of the variational capacity, we find a function $v \in N^{1,1}(X)$ with $v \geq 1$ in $B(x, r) \cap I_{\{u > t\}}$, $v = 0$ in $X \setminus B(x, 2r)$, and

$$\int_X g_v \, d\mu \leq \text{cap}_1(B(x, r) \cap I_{\{u > t\}}, B(x, 2r)) + \varepsilon, \quad (3.8)$$

by (3.6).
where $g_v$ is an upper gradient of $v$. Note that

$$A \subset B(x, r) \cap \text{int}(\{u > t\}) \subset B(x, r) \cap I_{\{u > t\}}. \quad (3.9)$$

For a suitable $\mathcal{H}$-negligible set $N \subset X$, by (3.9) and Lemma 3.3 we have

$$B(x, r) \cap \overline{A} \subset B(x, r) \cap I_{\{u > t\}} \subset B(x, r) \cap \partial^* \{u > t\} \cup N. \quad (3.10)$$

Let $\tilde{N} \subset X$ be the set of points where $v$ is not 1-finely continuous. By Theorem 3.4, $\text{Cap}_1(\tilde{N}) = 0$, and then by (3.4), also $\mathcal{H}(\tilde{N}) = 0$. If $y \in B(x, r) \cap \partial^* \{u > t\} \setminus \tilde{N}$, we deduce $y \in B(x, r) \cap b_1 I_{\{u > t\}} \setminus \tilde{N}$ by applying Lemma 3.1 with $A = I_{\{u > t\}}$ and noting that $\partial^* \{u > t\} = \partial^* I_{\{u > t\}}$. Thus necessarily $v(y) \geq 1$. Thus $v \geq 1$ 1-quasieverywhere in the set

$$B(x, r) \cap (I_{\{u > t\}} \cup \partial^* \{u > t\} \cup N).$$

Thus by (3.10), $v \geq 1$ 1-quasieverywhere in $B(x, r) \cap \overline{A}$, and so by (3.11) and (3.8), we get

$$\text{cap}_1(B(x, r) \cap \overline{A}, B(x, 2r)) \leq \int_X g_v \, d\mu \leq C_1 \text{cap}_1(A, B(x, 2r)) + C_1 \varepsilon + \varepsilon.$$

Letting $\varepsilon \to 0$, we obtain the result. \hfill \Box

Now we can give a suitable characterization of the 1-fine interior. We take the proof almost directly from \cite[Proposition 7.8]{6}, where it is given for $p > 1$.

**Proposition 3.6.** Let $A \subset X$. Then fine-int $A = A \setminus b_1(X \setminus A)$.

**Proof.** If $x \in \text{fine-int} A$, then by definition $X \setminus \text{fine-int} A$ is 1-thin at $x$, and thus so is $X \setminus A$. Thus $x \in A \setminus b_1(X \setminus A)$.

Conversely, assume that $X \setminus A$ is 1-thin at $x \in A$, i.e.

$$\lim_{r \to 0} r \frac{\text{cap}_1(B(x, r) \setminus A, B(x, 2r))}{\mu(B(x, 2r))} = 0. \quad (3.11)$$

For every $r > 0$, let $F_r := \overline{B(x, r) \setminus A}$. Fix $s > 0$. We show that $F_s$ is 1-thin at $x$. By (3.11) it suffices to show that for sufficiently small $0 < r \leq s$,

$$\text{cap}_1(B(x, r) \cap F_s, B(x, 2r)) \leq C \text{cap}_1(B(x, r) \setminus A, B(x, 2r)) \quad (3.12)$$
for some constant $C > 0$. Note that for $0 < r \leq s$, $F_r \cup (X \setminus B(x, r))$ is 1-finely closed and contains $X \setminus A$, and hence also contains $F_s$. Thus
\[
B(x, r) \cap F_s \subset B(x, r) \cap (F_r \cup (X \setminus B(x, r))) = B(x, r) \cap F_r.
\]
Thus
\[
\text{cap}_1(B(x, r) \cap F_s, B(x, 2r)) \leq \text{cap}_1(B(x, r) \cap F_r, B(x, 2r)) \leq C \text{cap}_1(B(x) \setminus A, B(x, 2r))
\]
for sufficiently small $0 < r \leq s$ by Lemma 3.5. This establishes (3.12), and thus $F_s$ is 1-thin at $x$. The set $B(x, s) \setminus F_s$ is 1-finely open and contained in $A$, and since $F_s$ is 1-thin at $x$, the set $(B(x, s) \setminus F_s) \cup \{x\}$ is also 1-finely open, and contained in $A$. Thus $(B(x, s) \setminus F_s) \cup \{x\} \subset \text{fine-int } A$, and so $x \in \text{fine-int } A$. 

Now we can characterize the 1-fine closure and the 1-fine boundary in the following way.

**Corollary 3.7.** Let $A \subset X$. Then $\overline{A}^1 = A \cup b_1A$ and
\[
\partial^1 A = (A \cap b_1(X \setminus A)) \cup ((X \setminus A) \cap b_1A).
\]

**Proof.** Note that $\overline{A}^1 = X \setminus \text{fine-int}(X \setminus A)$. Thus by Proposition 3.6
\[
\overline{A}^1 = X \setminus ((X \setminus A) \setminus b_1A) = A \cup b_1A.
\]
Then
\[
\partial^1 A = \overline{A}^1 \setminus \text{fine-int } A = (A \cup b_1A) \setminus (A \setminus b_1(X \setminus A))
\]
\[
= (A \cup b_1A) \cap ((X \setminus A) \cup b_1(X \setminus A))
\]
\[
= (A \cap b_1(X \setminus A)) \cup (b_1A \cap (X \setminus A)).
\]

The following proposition can be taken directly from [9, Lemma 4.8], where it is given in the case $p > 1$. The proof is also verbatim the same, except that instead of referring to [9, Theorem 4.3] we refer to Theorem 3.4.

**Proposition 3.8.** Let $A \subset X$. Then $\text{Cap}_1(\overline{A}^1) = \text{Cap}_1(A)$. If $A \subset D \subset X$, then
\[
\text{cap}_1(A, D) = \text{cap}_1(\overline{A}^1 \cap D, D).
\]
If furthermore $\text{cap}_1(A, D) < \infty$, then $\text{Cap}_1(\overline{A}^1 \setminus \text{fine-int } D) = 0$ and
\[
\text{cap}_1(A, D) = \text{cap}_1(\overline{A}^1 \cap \text{fine-int } D, \text{fine-int } D).
\]
4 Necessity of $\mathcal{H}(\partial^1 I_E) < \infty$

In this section we consider the quasicontinuity and fine continuity properties of BV functions and in particular sets of finite perimeter, and show that every set of finite perimeter $E$ satisfies the condition $\mathcal{H}(\partial^1 I_E) < \infty$.

The following quasicontinuity-type result is essentially given by [23, Theorem 1.1], and later proved in precisely the given form in the below reference. Recall the definitions of the lower and upper approximate limits $u^\wedge$ and $u^\vee$ from (2.13) and (2.14).

**Theorem 4.1** ([22, Corollary 4.3]). Let $\Omega \subset X$ be an open set, let $E \subset X$ be a $\mu$-measurable set with $P(E, \Omega) < \infty$, and let $\varepsilon > 0$. Then there exists an open set $G \subset \Omega$ with $\text{Cap}_1(G) < \varepsilon$ such that if $y_k \to x$ with $y_k, x \in \Omega \setminus G$, then

\[
\min\{|\chi_E^\wedge(y_k) - \chi_E^\wedge(x)|, |\chi_E^\vee(y_k) - \chi_E^\vee(x)|\} \to 0
\]

and

\[
\min\{|\chi_E^\vee(y_k) - \chi_E^\wedge(x)|, |\chi_E^\wedge(y_k) - \chi_E^\vee(x)|\} \to 0.
\]

Recall that a set $A \subset X$ is 1-quasiopen if for every $\varepsilon > 0$ there is an open set $G \subset X$ with $\text{Cap}_1(G) < \varepsilon$ such that $A \cup G$ is open.

**Proposition 4.2.** Let $\Omega \subset X$ be an open set and let $E \subset X$ be a $\mu$-measurable set with $P(E, \Omega) < \infty$. Then the sets $I_E \cap \Omega$ and $O_E \cap \Omega$ are 1-quasiopen.

**Proof.** Let $\varepsilon > 0$. Let $G \subset \Omega$ be an open set obtained by applying Theorem 4.1 so that $\text{Cap}_1(G) < \varepsilon$. Let $V := (I_E \cap \Omega) \cup G$. To check that $V$ is open, note first that if $x \in G$, then of course $B(x, r) \subset G \subset V$ for some $r > 0$. If $x \in (I_E \cap \Omega) \setminus G$, then by Theorem 4.1 there exists $r > 0$ with $B(x, r) \setminus G \subset (I_E \cap \Omega) \setminus G$, and thus

\[
B(x, r) \subset (I_E \cap \Omega) \cup G = V.
\]

Thus $V$ is open.

Since $P(X \setminus E, \Omega) = P(E, \Omega)$ and $O_E = I_{X \setminus E}$, also $O_E \cap \Omega$ is 1-quasiopen.

The following fact and its proof are essentially the same as in the case $p > 1$, see [8, Theorem 1.4].
Proposition 4.3. Every 1-quasiopen set $A \subset X$ is the union of a 1-finely open set and a $\mathcal{H}$-negligible set.

Proof. Take open sets $G_j \subset X$ with $\text{Cap}_1(G_j) < 2^{-j}$, $j \in \mathbb{N}$, such that each $A \cup G_j$ is an open set. By Proposition 3.8 we have $\text{Cap}_1(G_j) = \text{Cap}_1(G_j) < 2^{-j}$. Let $D := A \cap \bigcap_{j \in \mathbb{N}} \overline{G_j}^1$, so that $\text{Cap}_1(D) = 0$, and then also $\mathcal{H}(D) = 0$ by (3.3). Then for each $j \in \mathbb{N}$, $A \setminus \overline{G_j}^1 = A \cup G_j \setminus \overline{G_j}^1$ is a 1-finely open set, since it is the intersection of an open set and a 1-finely open set. The set

$$V := \bigcup_{j \in \mathbb{N}} (A \setminus \overline{G_j}^1) = A \setminus D$$

is a 1-finely open set, since it is the union of 1-finely open sets, and $A = V \cup D$. \hfill \Box

Proposition 4.4. Let $\Omega \subset X$ be an open set and let $E \subset X$ be a $\mu$-measurable set with $P(E, \Omega) < \infty$. Then each of the sets $I_E \cap \Omega$ and $O_E \cap \Omega$ is the union of a 1-finely open set and a $\mathcal{H}$-negligible set.

Proof. Combine Propositions 4.2 and 4.3. \hfill \Box

Recall the characterizations of the 1-fine closure and the 1-fine boundary given in Corollary 3.7. When we consider measure theoretic interiors, things are simplified somewhat further.

Lemma 4.5. For any $\mu$-measurable set $E \subset X$, we have $\overline{T_E}^1 = b_1 I_E$ and $X \setminus \overline{T_E}^1 = b_1 (X \setminus I_E)$.

Proof. Applying Corollary 3.7 and the first claim of Lemma 3.1 with $A = I_E$, and noting that $I_A = I_E$, we obtain

$$\overline{T_E}^1 = I_E \cup b_1 I_E = b_1 I_E.$$ 

Similarly, by applying Corollary 3.7,

$$X \setminus \overline{T_E}^1 = (X \setminus I_E) \cup b_1 (X \setminus I_E) = O_E \cup \partial^* E \cup b_1 (X \setminus I_E) = b_1 (X \setminus I_E),$$

where the last equality follows from the first claim of Lemma 3.1 with $A = X \setminus I_E$, by noting that $I_A = O_E$ and $\partial^* A = \partial^* E$. \hfill \Box
Lemma 4.6. For any $\mu$-measurable set $E \subset X$, we have

$$\partial^1 I_E = b_1 I_E \cap b_1 (X \setminus I_E).$$

Proof. By the definition of the 1-fine boundary and Lemma 4.5, we have

$$\partial^1 I_E = \overline{I_E} \cap X \setminus \overline{I_E} = b_1 I_E \cap b_1 (X \setminus I_E).$$

Note that by Lemma 3.1, for any $\mu$-measurable set $E \subset X$ we have $\partial^* E \subset \partial^1 I_E$. On the other hand, when $E$ is of finite perimeter, these sets almost coincide. This is the content of the following theorem that is the main result of this section.

Theorem 4.7. Let $\Omega \subset X$ be an open set and let $E \subset X$ be a $\mu$-measurable set with $P(E, \Omega) < \infty$. Then $\mathcal{H}(\partial^1 I_E \setminus \partial^* E) \cap \Omega) = 0$, and so in particular $\mathcal{H}(\partial^1 I_E \cap \Omega) < \infty$.

Proof. By Proposition 4.4 we know that there exist 1-finely open sets $A_1, A_2 \subset X$ and $\mathcal{H}$-negligible sets $N_1, N_2 \subset X$ such that $I_E \cap \Omega = A_1 \cup N_1$ and $O_E \cap \Omega = A_2 \cup N_2$. Take $x \in \Omega \setminus (\partial^* E \cup N_1 \cup N_2)$. Thus $x \in A_1 \cup A_2$. Suppose $x \in A_1$. Then

$$r \frac{\operatorname{cap}_1 (B(x, r) \setminus I_E, B(x, 2r))}{\mu(B(x, r))} \leq r \frac{\operatorname{cap}_1 (B(x, r) \setminus A_1, B(x, 2r))}{\mu(B(x, r))} \to 0$$

as $r \to 0$, since $A_1$ is a 1-finely open set. Thus $x \notin b_1 (X \setminus I_E)$, and by Lemma 4.6 $x \notin \partial^1 I_E$. If $x \in A_2$, similarly

$$r \frac{\operatorname{cap}_1 (B(x, r) \cap I_E, B(x, 2r))}{\mu(B(x, r))} \leq r \frac{\operatorname{cap}_1 (B(x, r) \setminus O_E, B(x, 2r))}{\mu(B(x, r))} \leq r \frac{\operatorname{cap}_1 (B(x, r) \setminus A_2, B(x, 2r))}{\mu(B(x, r))} \to 0$$

as $r \to 0$, since $A_2$ is a 1-finely open set. Thus $x \notin b_1 I_E$, and so $x \notin \partial^1 I_E$. In conclusion, $(\partial^1 I_E \setminus \partial^* E) \cap \Omega \subset N_1 \cup N_2$, so that $\mathcal{H}(\partial^1 I_E \setminus \partial^* E) \cap \Omega) = 0$. By (2.12) we know that $\mathcal{H}(\partial^* E \cap \Omega) < \infty$, so now also $\mathcal{H}(\partial^1 I_E \cap \Omega) < \infty$.

Note that we have the following.
Lemma 4.8. For any $\mu$-measurable set $E \subset X$, we have $\partial^1 I_E = \partial^1 O_E$.

**Proof.** Fix $x \in \partial^1 O_E$. Then $x \in b_1 O_E$ by Lemma 4.6 and thus $x \in b_1(X \setminus I_E)$. We need to show that also $x \in b_1 I_E$. Note that for any $r > 0$, $B(x, r) \cap b_1 I_E \subset b_1(B(x, r) \cap I_E)$. By using Proposition 3.8 and the first claim of Lemma 3.1 with $A = I_E$ (note that $I_A = I_E$), we obtain that

$$
\limsup_{r \to 0} r \frac{\text{cap}_1(B(x, r) \cap I_E, B(x, 2r))}{\mu(B(x, r))} \\
\geq \limsup_{r \to 0} r \frac{\text{cap}_1(b_1(B(x, r) \cap I_E), B(x, 2r))}{\mu(B(x, r))} \\
\geq \limsup_{r \to 0} r \frac{\text{cap}_1(B(x, r) \cap b_1 I_E, B(x, 2r))}{\mu(B(x, r))} \\
\geq \limsup_{r \to 0} r \frac{\text{cap}_1(B(x, r) \cap (I_E \cup \partial^* I_E), B(x, 2r))}{\mu(B(x, r))} \\
= \limsup_{r \to 0} r \frac{\text{cap}_1(B(x, r) \setminus O_E, B(x, 2r))}{\mu(B(x, r))} > 0,
$$

since $x \in b_1(X \setminus O_E)$ by Lemma 4.6. Thus $x \in \partial^1 I_E$. \hfill \qed

**Remark 4.9.** Despite the above lemma, $\partial^1 I_E$ seems in some way a strange set to consider, since we seem to obtain it by first taking an open set in the measure topology, and by then taking the boundary in a different topology, namely the 1-fine topology. To clarify the issue somewhat, let us see what happens if we define the measure topology in a more axiomatic way than is used in defining $I_E$, $O_E$, and $\partial^* E$. We say that a set $U \subset X$ is 0-finely open if

$$
\lim_{r \to 0} \frac{\mu(B(x, r) \setminus U)}{\mu(B(x, r))} = 0
$$

for every $x \in U$, that is, $U \subset I_U$. Then as in Definition 2.2 for any $G \subset X$ we can define the 0-fine interior 0-fine-int $G$, the 0-fine closure $\overline{G}^0$, and the 0-fine boundary $\partial^0 G := \overline{G}^0 \setminus 0\text{-fine-int } G$. Moreover, let $b_0 G$ be the set of points where the density of $G$ is not zero (somewhat confusingly), i.e. $b_0 G = X \setminus O_G$. Analogously to Proposition 3.6 and Corollary 3.7 we can then show that

$$
0\text{-fine-int } G = G \setminus b_0(X \setminus G), \quad \overline{G}^0 = G \cup b_0 G,
$$

and

$$
\partial^0 G = (G \cap b_0(X \setminus G)) \cup ((X \setminus G) \cap b_0 G).
$$
Then as in Lemma 4.6, we have at least for any $\mu$-measurable set $E \subset X$
\[ \partial^0 I_E = b_0 I_E \cap b_0 (X \setminus I_E). \]
But this is exactly $\partial^* I_E = \partial^* E$. Thus the measure theoretic boundary $\partial^* E$ is the boundary of $I_E$ in the 0-fine topology (and not of $E$). Moreover, $I_E$ is not the same as 0-fine-int $E = E \cap I_E$, so perhaps $I_E$ should be viewed only as a measure theoretic concept, and not a topological one. Now the conclusion of Theorem 4.7 can be reformulated in the more symmetric fashion
\[ \mathcal{H}(\partial^1 I_E \setminus \partial^0 I_E) = 0. \]

**Example 4.10.** Let $X = \mathbb{R}^2$ (unweighted), and consider the slit disk
\[ E = B(0, 1) \setminus \{x = (x_1, x_2) : x_1 > 0, x_2 = 0\}. \]
Note that the 1-dimensional Hausdorff measure $\mathcal{H}^1$ is comparable to the codimension one Hausdorff measure $\mathcal{H}$. Now $\partial^* E = \partial B(0, 1)$ and $P(E, \mathbb{R}^2) = \mathcal{H}^1(\partial^* E) = 2\pi$. From Corollary 3.7 it follows that $\partial^1 E = \partial E$, so that $\partial^1 E \setminus \partial^* E$ consists of the slit, and so $\mathcal{H}(\partial^1 E \setminus \partial^* E) > 0$. Similarly, fine-int $E = E$, and thus also $\mathcal{H}(\partial^1 \text{fine-int } E \setminus \partial^* E) > 0$.

Thus in Theorem 4.7, we cannot replace $I_E$ by either $E$ or fine-int $E$.

**Example 4.11.** Let $X = \mathbb{R}$ (unweighted). For any $x \in \mathbb{R}$, $r > 0$, and $A \subset \mathbb{R}$ with $B(x, r) \cap A \neq \emptyset$ we have
\[ \text{cap}_1(B(x, r) \cap A, B(x, 2r)) \geq 2. \]
Thus we find that the metric topology and the 1-fine topology coincide, and so for any $A \subset \mathbb{R}$ we have $\partial^1 A = \partial A$. For a set of finite perimeter $E \subset \mathbb{R}$, the result $\mathcal{H}(\partial^1 I_E \setminus \partial^* E) = 0$ thus implies that $\partial I_E = \partial^* E$, since $\mathcal{H}$ is now (comparable to) the counting measure. Thus $I_E$ is a *good representative* of $E$, a well-known result on the real line, see e.g. [2, Sections 3.2 & 3.5].

On the other hand, if we define
\[ E := \mathbb{R} \setminus \bigcup_{j \in \mathbb{N}} [2^{-j}, 2^{-j} + 2^{-2j}], \]
then clearly $P(E, \mathbb{R}) = \infty$ and $0 \in I_E$. If $U \supset I_E$ is an open set, then $U$ contains a neighborhood of the origin and thus $U \setminus I_E \neq \emptyset$. Thus $\text{Cap}_1(U \setminus I_E) \geq 2$, since every point has 1-capacity 2. Thus $I_E$ is not a 1-quasiopen set. Moreover, $\mathcal{H}(\partial^1 I_E \setminus \partial^* E) = \mathcal{H}([0]) > 0$, so the conclusions of Proposition 4.2 and Theorem 4.7 do not necessarily hold unless $E$ is of finite perimeter.
Given a set of finite perimeter $E \subset X$, if we denote by $\Sigma \gamma E$ the subset of $\partial^* E$ where (2.11) holds, we know that $\mathcal{H}(\partial^* E \setminus \Sigma \gamma E) = 0$. Theorem 4.7 then shows that the difference between $\partial^* E$ and the a priori larger set $\partial^1 I_E$ is also $\mathcal{H}$-negligible. In conclusion, the boundary of a set of finite perimeter is quite regular in the sense that all of these sets almost coincide.

5 Sufficiency of $\mathcal{H}(\partial^1 I_E) < \infty$

In this section we prove that the condition $\mathcal{H}(\partial^1 I_E) < \infty$ is also sufficient for $E$ to be of finite perimeter. 

**Theorem 5.1.** Let $\Omega \subset X$ be an open set, let $E \subset X$ be a $\mu$-measurable set, and assume that $\mathcal{H}(\partial^1 I_E \cap \Omega) < \infty$. Then $P(E, \Omega) < \infty$.

In proving this result, we will need to study the concept of capacitary potential in the case $p = 1$. Given an open set $U \subset X$ and an arbitrary set $H \subset U$, we define the variational BV-capacity by

$$\text{cap}_{BV}(H, U) := \inf \|Du\|(X),$$

where the infimum is taken over functions $u \in L^1_{\text{loc}}(X)$ with $u \geq 1$ in a neighborhood of $H$ and $u = 0$ in $X \setminus U$.

Note that by the coarea formula (2.6), we know that in fact

$$\text{cap}_{BV}(H, U) = \inf P(D, X),$$

where the infimum is taken over $\mu$-measurable sets $D \subset U$ containing a neighborhood of $H$. Now we give a new characterization of the variational BV-capacity. First we take note of the following lemmas.

**Lemma 5.2 ([21, Lemma 3.1]).** For any $G \subset X$, we can find an open set $V \supset G$ with $\text{Cap}_1(V) \leq C_2 \text{Cap}_1(G)$ and $P(V, X) \leq C_2 \text{Cap}_1(G)$, where $C_2 = C_2(C_d, C_P, \lambda)$.

**Lemma 5.3 ([22, Lemma 3.9]).** Let $\Omega \subset X$ be an open set and let $u \in L^1_{\text{loc}}(\Omega)$ with $\|Du\|(\Omega) < \infty$. Then for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $A \subset \Omega$ with $\text{Cap}_1(A) < \delta$, then $\|Du\|(A) < \varepsilon$.

**Proposition 5.4.** Given an open set $U \subset X$ and $H \subset U$, we have

$$\text{cap}_{BV}(H, U) = \inf P(D, X),$$

where the infimum is taken over $\mu$-measurable sets $D \subset U$ with $H \subset I_D$. 

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Proof. One inequality is clear. To prove the opposite inequality, we can assume that there exists a \( \mu \)-measurable set \( D \subset U \) with \( H \subset I_D \) and \( P(D, X) < \infty \). By applying the coarea formula (2.6), we find open sets \( U_i \Subset U \) with \( U = \bigcup_{i \in \mathbb{N}} U_i \), and \( P(U_i, X) < \infty \) for each \( i \in \mathbb{N} \). Fix \( \varepsilon > 0 \), and then fix \( i \in \mathbb{N} \). By Lemma 5.3 there exists \( \delta \in (0, \varepsilon) \) such that if \( A \subset X \) with \( \text{Cap}_1(A) < \delta \), then

\[
P(U_i, A) < \frac{2^{-i-1}\alpha\varepsilon}{C_d};
\]

(5.1)

recall the constant \( \alpha \) from (2.12). By Proposition 4.2 we find a set \( G_i \subset X \) with \( \text{Cap}_1(G_i) < \frac{2^{-i-1}\alpha\delta}{C_dC_2} \) such that \( (I_D \cap U_i) \cup G_i \) is an open set. By Lemma 5.2 we find an open set \( V_i \supset G_i \) with \( \text{Cap}_1(V_i) < \frac{2^{-i-1}\alpha\delta}{C_dC_2} \), and \( P(U_i, X) < \infty \), and then by (2.12),

\[
P(U_i \cap V_i, X) \leq C_d \mathcal{H}(\partial^*(U_i \cap V_i))
\]

\[
\leq C_d(\mathcal{H}(\partial^*U_i \cap I_{V_i}) + \mathcal{H}(\partial^*V_i))
\]

\[
\leq C_d\alpha^{-1}(P(U_i, I_{V_i}) + P(V_i, X))
\]

(5.2)

\[
\leq C_d\alpha^{-1}(2^{-i-1}\alpha\varepsilon/C_d + 2^{-i-1}\alpha\varepsilon/C_d)
\]

\[
= 2^{-i}\varepsilon.
\]

This can be done for each \( i \in \mathbb{N} \), and then the set

\[
(I_D \cap U) \cup \bigcup_{i \in \mathbb{N}} (U_i \cap V_i) = \bigcup_{i \in \mathbb{N}} (I_D \cup V_i) \cap U_i = \bigcup_{i \in \mathbb{N}} ((I_D \cap U_i) \cup V_i) \cap U_i
\]

contains \( H \), is contained in \( U \), and is an open set since each \( (I_D \cap U_i) \cup V_i \) and each \( U_i \) is an open set. Moreover, by the fact that \( \mu((I_D \cap U)\Delta D) = 0 \), and by the lower semicontinuity and subadditivity (2.5) of perimeter,

\[
P \left( (I_D \cap U) \cup \bigcup_{i \in \mathbb{N}} (U_i \cap V_i), X \right) = P \left( D \cup \bigcup_{i \in \mathbb{N}} (U_i \cap V_i), X \right)
\]

\[
\leq P(D, X) + \sum_{i \in \mathbb{N}} P(U_i \cap V_i, X)
\]

\[
\leq P(D, X) + \varepsilon
\]

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by \((5.2)\). Since \(\varepsilon > 0\) was arbitrary, we obtain the result. \qed

**Remark 5.5.** The above proposition essentially states that the BV-capacity turns out to be an outer capacity even if we do not define it as such. This is similar to \([7, \text{Theorem 4.1}]\), where it is shown that the variational capacity \(\text{cap}_p\) is an outer capacity under very weak assumptions.

Moreover, Proposition \([5.4]\) and another application of the coarea formula give

\[
\text{cap}_{BV}(H, U) = \inf \|Du\|(X),
\]

where the infimum is taken over functions \(u \in L^1_{\text{loc}}(X)\) with \(u^\wedge \geq 1\) in \(H\) and \(u = 0\) in \(X \setminus U\) (more precisely, \(u = 0\) \(\mu\)-almost everywhere in \(X \setminus U\); recall that we understand BV functions to be \(\mu\)-equivalence classes).

Following the definitions and terminology used in the case \(p > 1\), we give the following definition.

**Definition 5.6.** Given an open set \(U \subset X\) and \(H \subset U\), we say that a \(\mu\)-measurable set \(D \subset U\) is a 1-capacitary potential for \(H\) in \(U\) if \(H \subset I_D\) and

\[
P(D, X) = \text{cap}_{BV}(H, U) < \infty.
\]

Of course, Proposition \([5.4]\) guarantees that this definition makes sense. Almost any example on the real line shows that a 1-capacitary potential is not unique, contrary to the case \(p > 1\). However, we have the following existence result.

**Proposition 5.7.** Let \(U \subset X\) be an open set and let \(H \subset X\) with \(I_H \subset U\). If \(\text{cap}_{BV}(I_H, U) < \infty\), then a 1-capacitary potential for \(I_H\) in \(U\) exists.

Note that here we do not even need \(H\) to be \(\mu\)-measurable. However, it can be shown that \(I_H, O_H,\) and \(\partial^* H\) are still Borel sets.

**Proof.** Take a sequence of \(\mu\)-measurable sets \(D_i \subset U\) with \(I_H \subset I_{D_i}\) and

\[
P(D_i, X) \to \text{cap}_{BV}(I_H, U).
\]

By the weak compactness of BV functions, see \([25, \text{Theorem 3.7}]\), there exists \(D \subset U\) such that by passing to a subsequence (not relabeled), \(\chi_{D_i} \to \chi_D\) in \(L^1_{\text{loc}}(X)\). By the lower semicontinuity of perimeter with respect to convergence in \(L^1_{\text{loc}}(X)\), we have

\[
P(D, X) \leq \liminf_{i \to \infty} P(D_i, X) = \text{cap}_{BV}(I_H, U).
\]
Fix $x \in X$ and $R > 0$. Since also $\chi_{X \setminus D_i} \to \chi_{X \setminus D}$ in $L^1_{\text{loc}}(X)$, we have

$$\mu(B(x,R) \cap I_H \setminus D) = \int_{B(x,R) \cap I_H} \chi_{X \setminus D} \, d\mu = \lim_{i \to \infty} \int_{B(x,R) \cap I_H} \chi_{X \setminus D_i} \, d\mu = 0$$

since $\mu(I_H \setminus D_i) = 0$ for all $i \in \mathbb{N}$ (by the Lebesgue differentiation theorem).

By letting $R \to \infty$, we get $\mu(I_H \setminus D) = 0$, and so $I_H \subset I_D$. Then $D$ is a 1-capacitary potential for $I_H$ in $U$.

The crux of the proof of Theorem 5.1 is obtaining the following Choquet-type property in the case $p = 1$. For this, we combine ideas from the proof of the case $p > 1$ given in [8, Theorem 7.1] with methods of metric space BV theory, see especially [1, Theorem 5.3].

**Proposition 5.8.** Let $A \subset X$ be a $\mu$-measurable set and let $\varepsilon > 0$. Then there exists an open set $U \subset X$ with $U \cup b_1A = X$ and

$$\text{Cap}_1(U \cap \overline{A}) < \varepsilon.$$

**Proof.** Let $\{B(x_j, r_j)\}_{j \in \mathbb{N}}$ be a covering of $X$ by balls such that every point is covered by arbitrarily small balls; this is possible since the space is separable.

We can assume that $r_j < \text{diam}(X)/8$ for all $j \in \mathbb{N}$. For each $j \in \mathbb{N}$, by [19, Lemma 6.2] there exists $s \in [r_j, 2r_j]$ such that

$$P(B(x_j, s)) \leq 2C_d \frac{\mu(B(x_j, s))}{s} < \infty.$$

In particular, $\text{cap}_{\text{BV}}(I_{B(x_j, r_j) \cap A}, B(x_j, 2r_j)) < \infty$, and so by Proposition 5.7, for each $j \in \mathbb{N}$ we can let $D_j \subset X$ be a 1-capacitary potential for $I_{B(x_j, r_j) \cap A}$ in $B(x_j, 2r_j)$. Then $B(x_j, r_j) \cap I_A \subset I_{D_j}$, and thus

$$B(x_j, r_j) \cap O_{D_j} \cap I_A = \emptyset \quad \text{for each } j \in \mathbb{N}.$$  \(5.4\)

**Claim:** We have $\mathcal{H} \left( X \setminus (b_1A \cup \bigcup_{j \in \mathbb{N}} (B(x_j, r_j) \cap O_{D_j})) \right) = 0$.

**Proof:** For any set of finite perimeter $E \subset X$, denote by $\Sigma_\gamma E$ the subset of $\partial^* E$ where (2.11) holds. Then let

$$N := \bigcup_{j \in \mathbb{N}} \partial^* D_j \setminus \Sigma_\gamma D_j,$$
so that $\mathcal{H}(N) = 0$. Fix $x \in X \setminus (b_1 A \cup N)$. If for some $j \in \mathbb{N}$ we have $x \in B(x_j, r_j)$ and

$$\liminf_{r \to 0} \frac{\mu(B(x, r) \cap D_j)}{\mu(B(x, r))} < \gamma,$$

then $x \in B(x_j, r_j) \cap O_{D_j}$. We assume that

$$\liminf_{r \to 0} \frac{\mu(B(x, r) \cap D_j)}{\mu(B(x, r))} \geq \gamma \quad (5.5)$$

for all $j \in \mathbb{N}$ with $x \in B(x_j, r_j)$, and derive a contradiction, thus proving the claim. Define $\delta > 0$ by

$$\delta := \min \left\{ \frac{\gamma}{2}, \frac{1}{(3C_{SP})^Q C_d (1 + \lfloor \log_2 (4\lambda) \rfloor) (Q+1)} \right\}.$$

Here $[a]$ is the smallest integer at least $a \in \mathbb{R}$, and $C_{SP}$ is the constant appearing in the relative isoperimetric inequality (2.7). Fix $j \in \mathbb{N}$ such that $B(x_j, r_j) \ni x$, and $r_j > 0$ is so small that

$$r \frac{\text{cap}_1(B(x, r) \cap A, B(x, 2r))}{\mu(B(x, r))} < \frac{\delta}{5C_d^3 C_S^2} \quad (5.6)$$

for all $r \in (0, 2r_j]$, where $C_S$ is the constant from the Sobolev inequality (2.8). For any such $r$, using the definition of the variational capacity, we find $u_r \in N^{1,1}(X)$ with $u_r \geq 1$ in $B(x, r) \cap A$, $u = 0$ in $X \setminus B(x, 2r)$, and

$$\frac{\delta}{5C_d^3 C_S^2} \geq \frac{r}{\mu(B(x, r))} \int_X g_{u_r} \frac{1}{d\mu} \geq \frac{r}{\mu(B(x, r))} \|Du_r\|(X),$$

where $g_{u_r}$ is an upper gradient of $u_r$, and where the last inequality follows from the fact that Lipschitz functions are dense in $N^{1,1}(X)$, see e.g. [5, Theorem 5.1]. By using the coarea formula (2.6), we find a number $t \in (0, 1)$ such that

$$\frac{r}{\mu(B(x, r))} P\{u_r > t\}, X \leq \frac{\delta}{5C_d^3 C_S^2}.$$

In conclusion, for any $r \in (0, 2r_j]$ there exists a set $D \subset B(x, 2r)$ covering $B(x, r) \cap A$ such that

$$\frac{r}{\mu(B(x, r))} P(D, X) \leq \frac{\delta}{5C_d^3 C_S^2}.$$
On the other hand, by using [5, Lemma 11.22], the doubling property of the measure, and (5.6) with $r = 2r_j$, we obtain

\[
\frac{r_j \text{cap}_1(B(x_j, r_j) \cap A, B(x_j, 2r_j))}{\mu(B(x_j, r_j))} \leq 5C_S r_j \frac{\text{cap}_1(B(x_j, r_j) \cap A, B(x_j, 4r_j))}{\mu(B(x_j, r_j))} \\
\leq 5C_S C_d^2 r_j \frac{\text{cap}_1(B(x, 2r_j) \cap A, B(x, 4r_j))}{\mu(B(x, 2r_j))} \\
< \frac{\delta}{2C_d C_S}.
\]

(5.8)

Then as above, we conclude that there exists a set $\tilde{D} \subset B(x_j, 2r_j)$ with $\tilde{D} \supset B(x_j, r_j) \cap A$ and

\[
\frac{r_j}{\mu(B(x_j, r_j))} P(D_j, X) < \frac{\delta}{2C_d C_S}.
\]

Note that $I_{B(x_j, r_j) \cap A} \subset I_{\tilde{D}}$, and so $\tilde{D}$ is admissible for $\text{cap}_{\text{BV}}(I_{B(x_j, r_j) \cap A}, B(x_j, 2r_j))$. Then since $D_j \subset X$ is a 1-capacitary potential for $I_{B(x_j, r_j) \cap A}$ in $B(x_j, 2r_j)$, we necessarily have also

\[
\frac{r_j}{\mu(B(x_j, r_j))} P(D_j, X) < \frac{\delta}{2C_d C_S}.
\]

(5.9)

By (5.8) we have

\[
\liminf_{r \to 0} \frac{\mu(B(x, r) \cap D_j)}{\mu(B(x, r))} > \delta.
\]

However,

\[
\frac{\mu(B(x, r) \cap D_j)}{\mu(B(x, r))} < \delta \quad \text{for all } r \geq r_j;
\]

(5.10)

this can be seen as follows. Let $r \geq r_j$, and note that $D_j \subset B(x_j, 2r_j)$. By the isoperimetric inequality (2.9) and (5.9),

\[
\mu(B(x, r) \cap D_j) \leq \mu(D_j) \leq 2C_S r_j P(D_j, X) \\
< 2C_S r_j \left( \frac{\delta \mu(B(x_j, r_j))}{2C_d C_S r_j} \right) = \frac{\delta \mu(B(x_j, r_j))}{C_d}.
\]

But now $B(x_j, r_j) \subset B(x, 2r)$, and so by the doubling property of the measure,

\[
\mu(B(x_j, r_j)) \leq C_d \mu(B(x, r)),
\]

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establishing (5.10). Thus we can define \( R \in (0, r_j] \) by
\[
R := \inf \left\{ r > 0 : \frac{\mu(B(x, r) \cap D_j)}{\mu(B(x, r))} < \delta \right\}.
\]
Clearly
\[
\frac{\mu(B(x, R) \cap D_j)}{\mu(B(x, R))} < C_d \delta.
\] (5.11)
It follows that
\[
\frac{\mu(B(x, R/(4\lambda)) \cap D_j)}{\mu(B(x, R/(4\lambda)))} \leq C_{d}^{1+[\log_2(4\lambda)]} \delta \leq \frac{1}{2}.
\] (5.12)
Thus by the definition of \( R \), the relative isoperimetric inequality (2.7), and (5.12), we obtain
\[
\delta \frac{\mu(B(x, R))}{C_{d}^{[\log_2(4\lambda)]}} \leq \delta \mu(B(x, R/(4\lambda)))
\]
\[
\leq \mu(B(x, R/(4\lambda)) \cap D_j)
\]
\[
\leq 2C_{SP} \frac{R}{4\lambda} \left( \frac{\mu(B(x, R/(4\lambda)) \cap D_j)}{\mu(B(x, R/(4\lambda)))} \right)^{1/Q} P(D_j, B(x, R/2))
\]
\[
\leq C_{SP} \frac{R}{2\lambda} \left( C_{d}^{1+[\log_2(4\lambda)]} \delta \right)^{1/Q} P(D_j, B(x, R/2)).
\]
By the choice of \( \delta \), this implies that
\[
P(D_j, B(x, R/2)) \geq 6C_d \delta \frac{\mu(B(x, R))}{R}.
\] (5.13)
But by (5.7), we find a \( \mu \)-measurable set \( D \subset B(x, 2R) \) covering \( B(x, R) \cap A \) such that
\[
P(D, X) \leq \frac{\delta}{5C_d^3 C_S^2} \frac{\mu(B(x, R))}{R}.
\] (5.14)
Then by the isoperimetric inequality (2.9), we have
\[
\mu(D) \leq 2C_S R P(D, X) \leq \frac{\delta \mu(B(x, R))}{C_d C_S}.
\] (5.15)
Take \( \eta \in \text{Lip}_c(X) \) with \( 0 \leq \eta \leq 1 \), \( \eta = 1 \) in \( B(x, R/2) \), \( \eta = 0 \) in \( X \setminus B(x, R) \), and with an upper gradient \( g_\eta \leq 2/R \). Let
\[
w := \eta \chi_D + (1 - \eta) \chi_{D_j}.
\] 26
Then by a Leibniz rule, see [15, Lemma 3.2],
\[ \|Dw\|_\mathcal{E}(X) \leq P(D, B(x, R)) + \int \mathbb{1}_{D \cup D_j} g_n \, d\mu \]
by (5.11) and (5.15). Thus by (5.13)
\[ \|Dw\|_\mathcal{E}(X) - P(D_j, B^c(x, R)) \leq \frac{\delta \mu(B(x, R))}{5C_3^2C_5^2R} + \int \mathbb{1}_{D \cup D_j} g_n \, d\mu \]
by (5.11) and (5.15). Thus by (5.13)
\[ \|Dw\|_\mathcal{E}(X) - P(D_j, B^c(x, R)) \leq \frac{\delta \mu(B(x, R))}{5C_3^2C_5^2R} + \int \mathbb{1}_{D \cup D_j} g_n \, d\mu \]
by (5.11) and (5.15). Thus by (5.13)
\[ \|Dw\|_\mathcal{E}(X) - P(D_j, B^c(x, R)) \leq \frac{\delta \mu(B(x, R))}{5C_3^2C_5^2R} + \int \mathbb{1}_{D \cup D_j} g_n \, d\mu \]
Recall that \( I_{B(x, r_j) \cap A} \subset I_{D_j} \). Since \( B(x, R) \cap A \subset D \), also \( I_{B(x, r_j) \cap A} \subset B(x, R) \subset I_D \). Thus we have \( I_{B(x, r_j) \cap A} \subset \{ w = 1 \} \), and so \( I_{B(x, r_j) \cap A} \subset \{ (w^\varepsilon = 1) \). Since also \( B(x, R) \subset B(x_j, 2r_j) \), we have \( w = 0 \) in \( X \setminus B(x_j, 2r_j) \). But now (5.16) is a contradiction by (5.3), since \( D_j \) is a 1-capacitary potential for \( I_{B(x, r_j) \cap A} \) in \( B(x_j, 2r_j) \). Thus the claim is proved.

By the claim, we have
\[ b_1 A \cup \bigcup_{j \in \mathbb{N}} (B(x_j, r_j) \cap O_{D_j}) \cup N = X, \quad (5.17) \]
where \( \mathcal{H}(N) = 0 \). Since each \( B(x_j, r_j) \cap O_{D_j} \) is 1-quasiopen by Proposition 4.2 there exist open sets \( G_j \subset X \) with \( \text{Cap}_1(G_j) < 2^{-j-1}\varepsilon \) such that each
\[ U_j := (B(x_j, r_j) \cap O_{D_j}) \cup G_j \]
is open. For the exceptional set \( N \) with \( \mathcal{H}(N) = 0 \), we have also \( \text{Cap}_1(N) = 0 \) by (3.4), and then since \( \text{Cap}_1 \) is an outer capacity, we find an open set \( V \supset N \)
with $\text{Cap}_1(V) < \varepsilon/2$. Let $U := \bigcup_{j \in \mathbb{N}} U_j \cup V$. Then by (5.14), $b_1 A \cup U = X$. By (5.4),

$$\text{Cap}_1(U \cap I_A) = \text{Cap}_1 \left( \left( \bigcup_{j \in \mathbb{N}} G_j \cup V \right) \cap I_A \right) \leq \sum_{j \in \mathbb{N}} \text{Cap}_1(G_j) + \text{Cap}_1(V) < \varepsilon.$$ 

Note that since $U$ is open,

$$U \cap b_1 I_A \subset b_1(U \cap I_A).$$

Moreover, $\overline{T_A} = b_1 I_A$ by Lemma 4.5. Combining these,

$$U \cap \overline{T_A} = U \cap b_1 I_A \subset b_1(U \cap I_A) \subset \overline{U \cap I_A}.$$

Now by Proposition 3.8

$$\text{Cap}_1(U \cap \overline{T_A}) \leq \text{Cap}_1(U \cap \overline{T_A}) = \text{Cap}_1(U \cap I_A) < \varepsilon.$$ 

\[ \square \]

**Remark 5.9.** The Claim whose proof took up the bulk of the proof of Proposition 5.8 can be seen as a weak analog of the *Cartan property* that holds in the case $p > 1$, see [8, Theorem 1.1].

A more exact analog of the Choquet property that is known to hold for $p > 1$, see [8, Theorem 7.1], is given in the following open problem.

**Open Problem.** Let $A \subset X$ and $\varepsilon > 0$. Can we find an open set $U \subset X$ such that $U \cup b_1 A = X$ and $\text{Cap}_1(U \cap A) < \varepsilon$?

**Proposition 5.10.** For any $\mu$-measurable set $E \subset X$, $X \setminus \overline{T_E}$ is a 1-quasiopen set.

**Proof.** Fix $\varepsilon > 0$. By applying Proposition 5.8 with $A = I_E$, and noting that $I_A = I_E$, we find an open set $U \supset X \setminus b_1 I_E \supset X \setminus \overline{T_E}$ such that

$$\text{Cap}_1(U \cap \overline{T_E}) < \varepsilon.$$ 

Since $\text{Cap}_1$ is an outer capacity, we find an open set $G \supset U \cap \overline{T_E}$ with $\text{Cap}_1(G) < \varepsilon$. Thus the set

$$(X \setminus \overline{T_E}) \cup G = U \cup G$$

is open, and thus $X \setminus \overline{T_E}$ is a 1-quasiopen set. \[ \square \]
Open Problem. Is every 1-finely open set 1-quasiopen?

Note that Proposition 5.10 gives a partial result in this direction, since \( X \setminus I_E^1 \) is of course a 1-finely open set. Moreover, a positive answer to this open problem would follow from a positive answer to the previous open problem, just as Proposition 5.10 follows from Proposition 5.8.

Corollary 5.11. For any \( \mu \)-measurable set \( E \subset X \), \( I_E \setminus \partial^1 I_E \) and \( O_E \setminus \partial^1 I_E \) are 1-quasiopen sets.

Proof. By the definition of the 1-fine boundary and the second claim of Lemma 3.1, and noting that \( \partial^* E = \partial^* I_E \),

\[
X \setminus I_E^1 = X \setminus (I_E \cup \partial^1 I_E) = O_E \setminus \partial^1 I_E.
\]

Thus by Proposition 5.10 \( O_E \setminus \partial^1 I_E \) is a 1-quasiopen set. Since \( \partial^1 I_E = \partial^1 O_E \) by Lemma 4.8 \( I_E \setminus \partial^1 I_E \) is also a 1-quasiopen set.

Proposition 5.12. Let \( E \subset X \) be a \( \mu \)-measurable set and let \( \Gamma \) be the family of curves for which \( \gamma(0) \in I_E \) and \( \gamma(\ell) \in O_E \), but \( \gamma \) does not intersect \( \partial^1 I_E \).

Then \( \text{Mod}_1(\Gamma) = 0 \).

Proof. By Corollary 5.11 \( I_E \setminus \partial^1 I_E \) and \( O_E \setminus \partial^1 I_E \) are 1-quasiopen sets. Thus by [26] Remark 3.5 they are also 1-path open sets, meaning that for 1-almost every curve \( \gamma \), the sets \( \gamma^{-1}(I_E \setminus \partial^1 I_E) \) and \( \gamma^{-1}(O_E \setminus \partial^1 I_E) \) are relatively open subsets of \([0, \ell] \). Pick such \( \gamma \), and suppose that \( \gamma(0) \in I_E \) and \( \gamma(\ell) \in O_E \).

Since \([0, \ell] \) is connected, there exists \( t \in (0, \ell) \) for which

\[
\gamma(t) \in (X \setminus (I_E \setminus \partial^1 I_E)) \cap (X \setminus (O_E \setminus \partial^1 I_E)) = (O_E \cup \partial^* E \cup \partial^1 I_E) \cap (I_E \cup \partial^* E \cup \partial^1 I_E) = (O_E \setminus \partial^1 I_E) \cap (I_E \setminus \partial^1 I_E) = \partial^1 I_E.
\]

Using Proposition 5.12 we can now prove Theorem 5.1 by an argument very similar to one used previously in [23] Theorem 6.5].
Proof of Theorem 5.1. Fix $\varepsilon > 0$. Since $\mathcal{H}(\partial^1 I_E \cap \Omega) < \infty$, we can find a covering of $\partial^1 I_E \cap \Omega$ by balls $B_j = B(x_j, r_j)$ with radii $r_j \leq \varepsilon$ such that
\[
\sum_{j \in \mathbb{N}} \frac{\mu(B_j)}{r_j} \leq \mathcal{H}(\partial^1 I_E \cap \Omega) + \varepsilon.
\]
Denote $2B_j := B(x_j, 2r_j)$. For each ball $B_j$ in the cover, we fix a $1/r_j$-Lipschitz function $v_j$ such that $0 \leq v_j \leq 1$ on $X$, $v_j = 1$ in $B_j$, and $v_j = 0$ in $X \setminus 2B_j$. Now let
\[
u(x) := \begin{cases} 1 & \text{if } x \in I_E, \\ \min \left\{ 1, \sum_{j \in \mathbb{N}} v_j(x) \right\} & \text{otherwise.} \end{cases}
\]
Furthermore, let $v(x) := \min \left\{ 1, \sum_{j \in \mathbb{N}} v_j(x) \right\}$, and
\[g := \sum_{j \in \mathbb{N}} \frac{1}{r_j} \chi_{2B_j}.
\]
Clearly $g$ is an upper gradient of $v$. We will show that $g$ is also a 1-weak upper gradient of $u$ in $\Omega$. Take a curve $\gamma \notin \Gamma$ in $\Omega$ with end points $x, y \in \Omega$, where $\Gamma$ was defined in Proposition 5.12. If $x, y \in \Omega \setminus I_E$, then
\[|u(x) - u(y)| = |v(x) - v(y)| \leq \int_{\gamma} g ds.
\]
If $x, y \in I_E \cap \Omega$, then $u(x) = u(y)$, and hence the upper gradient inequality
\[|u(x) - u(y)| \leq \int_{\gamma} g ds
\]
is satisfied.

Finally, if $x \in I_E \cap \Omega$ and $y \in \Omega \setminus I_E$, then since $\gamma \notin \Gamma$, there is some $t \in [0, \ell_\gamma]$ such that $\gamma(t) \in \partial^1 I_E$, and thus $\gamma(t) \in B_k$ for some $k \in \mathbb{N}$. Note that $u(\gamma(0)) = u(x) = 1$, $u(\gamma(t)) = v(\gamma(t)) = 1$, and $u(y) = v(y)$. It follows that
\[
|u(x) - u(y)| \leq |u(\gamma(0)) - u(\gamma(t))| + |u(\gamma(t)) - u(\gamma(\ell_\gamma))| \leq |v(\gamma(t)) - v(\gamma(\ell_\gamma))| \leq \int_{\gamma} g ds.
\]
Thus in all cases the pair $u, g$ satisfies the upper gradient inequality for $\gamma \notin \Gamma$ in $\Omega$, and so $g$ is a 1-weak upper gradient of $u$ in $\Omega$. Furthermore,

$$\int_{\Omega} g \, d\mu \leq \sum_{j \in \mathbb{N}} \frac{\mu(2B_j)}{r_j} \leq C \sum_{j \in \mathbb{N}} \frac{\mu(B_j)}{r_j} \leq C(H(\partial^{1}I_E \cap \Omega) + \varepsilon) < \infty.$$ 

Now for each $i \in \mathbb{N}$, use the above construction to obtain functions $u_i \in N_{\text{loc}}^{1,1}(\Omega)$, $g_i \in L^1(\Omega)$ corresponding to $\varepsilon = 1/i$. Note that in order to show that $P(E, \Omega) < \infty$, it is enough to show that $u_i \to \chi_E$ in $L^1(\Omega)$ and that $\|g_i\|_{L^1(\Omega)}$ is a bounded sequence, since for every $u_i \in N_{\text{loc}}^{1,1}(\Omega)$ we can find a function $w_i \in \text{Lip}_{\text{loc}}(\Omega)$ with $\|w_i - u_i\|_{N^{1,1}(\Omega)} < 1/i$, see [5, Theorem 5.47]. The sequence $\|g_i\|_{L^1(\Omega)}$ is clearly bounded, and moreover (note that below, the balls $B_j$ also depend on $i$)

$$\int_{\Omega} |u_i - \chi_E| \, d\mu \leq \int_{\Omega} \chi_{\bigcup_{j \in \mathbb{N}} 2B_j} \, d\mu \leq \sum_{j \in \mathbb{N}} \frac{\mu(2B_j)}{r_j} \leq \frac{1}{i} \sum_{j \in \mathbb{N}} \frac{\mu(2B_j)}{r_j} \leq (H(\partial^{1}I_E \cap \Omega) + 1)/i \to 0$$

as $i \to \infty$. \hfill \Box

**Remark 5.13.** We still do not know whether $H(\partial^*E \cap \Omega) < \infty$ implies $P(E, \Omega) < \infty$. It is also not clear whether Proposition 5.12 would hold with $\partial^1I_E$ replaced by $\partial^*E$. If the answer to the latter question is yes, however, the proof would need to be something different, since Corollary 5.11 is not true with $\partial^1I_E$ replaced by $\partial^*E$, as demonstrated by the latter part of Example 4.11.

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