Spontaneous Conformal Symmetry Breaking in Fishnet CFT

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Quantum field theories with exact but spontaneously broken conformal invariance have an intriguing feature: the vacuum energy (cosmological constant) in them is equal to zero. Up to now, the only known ultraviolet complete theories where conformal symmetry can be spontaneously broken were associated with supersymmetry (SUSY), with the most prominent example being the $\mathcal{N}=4$ SUSY Yang-Mills. In this Letter we show that the recently proposed biscalar conformal “fishnet” theory supports at the classical level a rich set of flat directions (moduli) along which conformal symmetry is spontaneously broken. We demonstrate that some of these vacua survive in the full quantum theory (in the planar limit, at the leading order of $1/N_c$ expansion) without any fine tuning. The vacuum energy is equal to zero along these flat directions, providing a first non-SUSY example of a four-dimensional quantum field theory with “natural” breaking of conformal symmetry.

INTRODUCTION

Conformal Field Theories (CFTs) represent an indispensable tool to address the behavior of many systems in the vicinity of the critical points associated with phase transitions. They also describe the limiting behavior of different quantum field theories deeply in the ultraviolet (UV) and/or infrared (IR) domains of energy. Could it be that CFTs are more important and that the ultimate theory of Nature is conformal?

At first sight, the answer to this question is negative. Indeed, conformal invariance (CI) forbids the presence of any inherent dimensionful parameters in the action of a CFT. Because of that, CFTs have neither fundamental scales nor a well defined notion of particle states. On the other hand, Nature has both.

The loophole in these arguments is that conformal symmetry can be exact, but broken spontaneously by the ground state. This breakdown introduces an energy scale determined by the vacuum expectation value of some scalar dimensionful operator. The notion of a particle is now well defined, and in addition to massive excitations, the theory contains a massless dilaton, the Goldstone mode of the broken CI.

Theories with spontaneous breaking of conformal symmetry may be relevant for the solution of the most puzzling fine-tuning issues of fundamental particle physics, namely the hierarchy and cosmological constant problems. First, the Lagrangian of the Standard Model is invariant under the full conformal group (at the classical level) if the mass of the Higgs boson is put to zero. The observed smallness of the Fermi scale in comparison with the Planck scale might be a consequence of this [1 2]. Second, if conformal symmetry is spontaneously broken, the energy of the ground state is equal to zero (see, e.g. [3 4] and below). This fact may be relevant for the explanation of the amazing smallness of the cosmological constant.

A systematic way to construct effective field theories enjoying exact but spontaneously broken CI was described in [4], following the ideas of [5 6] (for further developments see [7–9], for a review [10] and references therein). These theories are free from conformal anomalies but non-renormalizable. They remain in a weak coupling regime below the scale induced by the spontaneous conformal symmetry breaking. Their low energy limit may contain just the Standard Model fields, graviton plus the dilaton, which essentially decouples and does not lead to a long-range “fifth” force [6 11 12]. These theories are phenomenologically viable and satisfy all possible experimental constraints. Whether they can have a well-defined UV limit remains an open question.

One can try to merge the “bottom-up” approach outlined above with the “top-down” strategy, starting from a UV complete theory. All such known CFTs are always supersymmetric. The most notable and well studied example is $\mathcal{N}=4$ SUSY Yang-Mills (SYM). Although the immediate phenomenological relevance of such theories is not clear, they are widely used as “playgrounds” for studying the spontaneous breakdown of CI.

In this Letter we show that there exists a nonsupersymmetric CFT with these properties—the recently proposed strongly $\gamma$-deformed $\mathcal{N}=4$ SYM, dubbed Conformal Fishnet Theory (FCFT) [13 14]. This theory is well defined and finite at all scales and has numerous flat directions at the classical level, without fine-tuning. Moreover, a subclass of them is not lifted by quantum corrections, at least in the large-$N_c$ limit [15].

Among others, the reasons for these rather surprising properties for a non-SUSY theory are: i) FCFT has in-
herited many of the vacua of the $\mathcal{N} = 4$ SYM \[16\], which guarantees that CI may be broken even without resorting to unnatural tunings; ii) the supersymmetric stabilization mechanism of the parent theory is replaced by the absence of the dangerous loop diagrams that would normally lift the classical flat directions in the Coleman-Weinberg (CW) effective potential \[17\]; iii) the one loop CW potential in the planar limit appears to be exact, since the relevant multi-loop diagrams don’t exist.

Before moving on, let us emphasize that there is a price to pay for these nice features: this chiral theory is not unitary. As a consequence, it is a logarithmic CFT.

**FISHNET CFT**

The FCFT model involves two interacting $N_c \times N_c$ complex matrix fields $X$ and $Z$ (and their Hermitian conjugates) in the adjoint of $SU(N_c)$; the Lagrangian at the classical level reads \[13, 18\]

$$\mathcal{L} = N_c \text{tr} \left( \partial_\mu \bar{X} \partial_\mu X + \bar{Z} \partial_\mu Z + \bar{\xi} \bar{X} \bar{Z} X Z \right).$$

Here a bar stands for Hermitian conjugation, while $\bar{\xi} = g^2 N_c e^{-i\gamma}/(4\pi)^2$; $g$ is the Yang-Mills coupling constant and $\gamma$ one of the three twists of the parent $\gamma$-deformed $\mathcal{N} = 4$ SYM \[19\]. The Lagrangian \[1\] is obtained by considering the double-scaling (DS) limit corresponding to weak coupling and at the same time large imaginary $\gamma_3$, such that $\xi$ and $\gamma_{1,2}$ remain finite.

A direct consequence of the strong imaginary deformation is the absence of the term corresponding to the Hermitian counterpart of the quartic interaction. This makes manifest the fact that the theory is not unitary. Nevertheless, it is exactly the absence of the complex conjugate interaction term that has far reaching implications. It restricts severely the number of possible planar graphs for various physical quantities, to the point that there are often none, or only one diagram contributing at each order in the perturbative expansion \[20\].

At the same time, the fixed chirality of the interaction vertex, and the absence of the vertex of opposite chirality, forces them to possess the “fishnet” structure \[21\]. This roughly means that the bulk structure of sufficiently large planar graphs is of the regular square lattice \[13\]. Importantly, the aforementioned chirality forbids the presence of certain diagrams, such as the ones that induce masses for the fields and the ones that renormalize the quartic coupling $\xi$. Consequently, the FCFT behaves as a fully-fledged logarithmic CFT, which implies the standard scaling properties for its local observables (i.e. correlators).

In addition, the theory appears to be integrable in the planar, ‘t Hooft $N_c \to \infty$ limit \[13, 22, 23\], due to the integrability of the individual “fishnet” graphs discovered long ago \[24\], see also \[25, 26\]. Hence, many of the physical quantities—such as non-trivial Operator Product Expansion (OPE) data as well as certain three- and four-point correlators—are in fact exactly calculable \[27\].

However, the model is not complete already at one-loop order: the cancellation of the divergences associated with the correlation functions of certain composite operators, such as $\text{tr}(X^2)$, $\text{tr}(\bar{X}^2)$, $\text{tr}(X\bar{Z})$, $\text{tr}(X\bar{Z})$, requires that in the classical action \[1\] new double-trace terms be included \[28\]. These read

$$L_{d,t}/(4\pi)^2 = \alpha_1^2 \left[ \text{tr}(X^2) \text{tr}(\bar{X}^2) + \text{tr}(X^2) \text{tr}(\bar{X}^2) \right] - \alpha_2^2 \left[ \text{tr}(X\bar{Z}) \text{tr}(\bar{X}\bar{Z}) + \text{tr}(X\bar{Z}) \text{tr}(\bar{X}\bar{Z}) \right],$$

with $\alpha_1$ and $\alpha_2$ couplings that, in general, depend on the renormalization scale destroying, on the quantum level, the conformal symmetry. However, the beta functions for the running double-trace couplings possess two complex conjugate fixed lines, parametrized by $\xi$, with $\alpha_1^2 = \alpha_2^2 = \pm \frac{g^2}{\pi} - \frac{\xi^4}{8} + \frac{4g^2}{x} + O(\xi^8)$ and $\alpha_2^2 = \xi^2$, for both of them \[29\].

The FCFT is completely defined by the explicitly local Lagrangian $\mathcal{L} + L_{d,t}$, with conformal symmetry persisting at the quantum level for the critical values of the $\alpha$’s.

**CLASSICAL VACUA IN TREE APPROXIMATION**

The spontaneous breaking of CI corresponds to a situation in which one or both of the fields have a non-vanishing vacuum expectation value (vev). As our CFT is non-unitary, we model this vacuum state by an extreme of the (complex) effective action.

Let us look for a (constant in spacetime) ansatz for the fields that extremizes the potential of the theory. The matrix equations of motion for such configurations are obtained by varying the action with respect to $X$, $\bar{X}$, $Z$, and $\bar{Z}$, respectively; they read

$$\kappa \text{tr}(\bar{X}^2)X + \text{tr}(\bar{X} Z) + \text{tr}(X \bar{Z}) = N_c \bar{X} \bar{X},$$
$$\kappa \text{tr}(X^2)\bar{X} + \text{tr}(XZ) + \text{tr}(XZ) = N_c XZ,$$
$$\kappa \text{tr}(Z^2)Z + \text{tr}(XZ) + \text{tr}(XZ) = N_c X\bar{X},$$
$$\kappa \text{tr}(Z^2)Z + \text{tr}(XZ) + \text{tr}(XZ) = N_c X\bar{X},$$

with $\kappa = -2\alpha_2^2/\xi^2$. The presence of the non-Hermitian single-trace interaction term, as well as the fact that $\kappa$ is complex at the conformal point, results into the equations for the fields and their Hermitian counterparts to not be related by complex conjugation.

If the vev of the fields vanish, i.e. $\langle X \rangle_{\text{tree}} = \langle Z \rangle_{\text{tree}} = 0$, the above are (trivially) satisfied and there is no spontaneous symmetry breaking. A plethora of aspects of the
theory on this conformal phase have been and are still being investigated actively; see \[13, 28, 29, 30, 31, 32\].

Turning to the existence of nontrivial vacua, we require for simplicity that \( \langle X \rangle_{\text{tree}} = 0 \), while \( Z \) acquires a nonzero diagonal vev

\[
\langle Z \rangle_{\text{tree}} = v \text{ diag } (z_1, \ldots, z_{N_c}) ,
\]

(4)

with \( v \) a (complex) parameter with dimension of mass and \( z_k \) are, in general, complex numbers subject to

\[
\sum_{k=1}^{N_c} z_k = \sum_{k=1}^{N_c} \bar{z}_k = 0 ,
\]

(5)

since the field lives in the adjoint of \( SU(N_c) \).

Plugging the above into (3), we find that the first two equations are identically satisfied, while the last two become

\[
\kappa \text{ tr } \langle \langle Z \rangle^2 \rangle_{\text{tree}} \langle Z \rangle_{\text{tree}} = 0 , \quad \text{and} \quad \kappa \text{ tr } \langle \langle Z \rangle^2 \rangle_{\text{tree}} \langle \bar{Z} \rangle_{\text{tree}} = 0 .
\]

Since \( \kappa \neq 0 \) and by construction \( \langle Z \rangle_{\text{tree}} \neq 0 \), the only option for both equations to hold is to require that

\[
\sum_{k=1}^{N_c} z_k^2 = \sum_{k=1}^{N_c} \bar{z}_k^2 = 0 .
\]

(6)

Remarkably, this theory has the nontrivial symmetry breaking solutions \([1, 5, 6]\), at any value of the coupling \( \xi \) \([15]\). This is a rather salient point that deserves some discussion. One might expect that whether or not the theory possesses ground states with nonlinearly realized conformal symmetry would crucially depend on the specific value of the coupling constant. This is precisely what happens in other nonsupersymmetric CFTs such as the massless \( \varphi^4 \) theory and its generalizations \([11]\), where finetunings are required in order for CI to be spontaneously broken down to Poincaré \([14]\), see also \([15]\). On the contrary, the FCFT has inherited from its parent \( \mathcal{N} = 4 \) SYM flat vacua with vanishing energy, without the need for finetuning. Equivalently, the dilaton—that is part of the theory’s spectrum in the Coulomb phase—has zero mass, naturally.

Now we will show that this phenomenon persists at the quantum level in all orders of planar perturbation theory, at least for some of these vacua.

**EXACT COLEMAN-WEINBERG EFFECTIVE POTENTIAL**

Whether or not quantum corrections jeopardize the CI by uplifting the flat directions is apparent already at the first loop order, by investigating the CW effective potential \([17]\). To proceed with the computation, we follow the standard approach when dealing with the large-\( N_c \) limit \([16]\). To this end, we rewrite the potential of the theory as

\[
N_c V/\xi^2 = \text{tr } (XZ XZ) - \overline{A} \text{tr } (X^2) - A \text{tr } (X^2) - B \text{tr } (Z^2) - B \text{tr } (\bar{Z}^2) - C \text{tr } (XZ) - C \text{tr } (\bar{X} \bar{Z}) - D \text{tr } (X \bar{Z}) - D \text{tr } (\bar{X} Z) + \frac{2N_c}{\kappa} \overline{A} A + \frac{2N_c}{\kappa} B B + N_c \overline{C} C + N_c D D ,
\]

(7)

with \( A, B, \ldots \) Lagrange multipliers. Upon integrating out the (nondynamical) auxiliary fields, one recovers the initial expression for the Lagrangian.

Let us study excitations on top of a symmetry-breaking vacuum \( X = \langle X \rangle + \delta X, Z = \langle Z \rangle + \delta Z \), where \( \langle X \rangle \) and \( \langle Z \rangle \) are, in general, arbitrary constant \( SU(N_c) \) matrices.

The one-loop correction to the effective potential is given by

\[
N_c^2 V_{1-\text{loop}} = \frac{1}{2} \int \frac{d^4p}{(2\pi)^4} \log \det \mathcal{D} ,
\]

(8)

where we moved to momentum space and denoted with \( \mathcal{D} \) the matrix of quadratic fluctuations \([17]\).

The above is to be evaluated on a saddle point. Minimizing the full potential with respect to the fields, one can show that this corresponds to an “asymmetric” configuration, in which all the Lagrange multipliers vanish, and at the same time \( \langle X \rangle = 0 \). It is easy to see that only the quadratic in \( \delta X \) terms coming from the single-trace interaction contribute. Therefore, \( \mathcal{D} \) becomes a diagonal matrix with determinant

\[
\det \mathcal{D} = \prod_{k,l=1}^{N_c} p^4 (p^2 + \xi^2 |v|^2 z_k \bar{z}_l)^2 .
\]

(9)

Plugging the above into (8), we obtain

\[
V_{1-\text{loop}} = -\frac{\dot{\xi}^4}{32\pi^2 N_c^2} \left[ \frac{1}{2} \log \xi^2 \text{tr } ((Z)^2) \text{tr } ((\bar{Z})^2) + \text{tr } ((Z)^2) \text{tr } ((\bar{Z})^2) \right] + \text{tr } ((Z)^2) \text{tr } \left( \frac{(Z)^2}{\mu} \log \left( \frac{Z}{\mu} \right) \right) + \left( \langle Z \rangle \leftrightarrow \langle \bar{Z} \rangle \right) ,
\]

(10)

with \( \mu \) the ’t Hooft-Veltman renormalization scale \([33]\).

It is clear that for \( \langle Z \rangle \neq 0 \) of the form \([4]\) and subject to \([5, 6]\), the above vanishes. Nevertheless, we have to make sure that the diagonal ansatz corresponds to an extremum of the potential including the one-loop correction. Evaluating the first derivatives of \( V_{\text{eff}} = V + V_{1-\text{loop}} \) and requiring that they be zero, it is easy to see that the configurations under consideration should also be subject to

\[
\sum_{k=1}^{N_c} z_k^2 \log z_k = \sum_{k=1}^{N_c} \bar{z}_k^2 \log \bar{z}_k = 0 .
\]

(11)
This means that a subclass of the vacua discussed in the previous section is singled out, since these are not lifted by quantum effects. As a result, the vacuum energy of the loop corrected theory on top of these flat directions is zero, or in other words, the masslessness of the dilaton persists at one-loop level. It should be stressed that this is a unique situation for a non-SUSY four-dimensional theory.

Moreover, the one-loop CW effective potential \( \mathcal{V}_{\text{eff}} \) appears to be large-\( N_c \) exact. Indeed, in the planar limit, the graphs that could in principle contribute, such as the ones of Fig. 1(a), are simply not present in the fishnet CFT due to the chiral structure of single-trace interaction. This property can be easily seen to also persist in the next orders of perturbation theory, but we leave the general proof of this statement for future work. In the \( 1/N_c^2 \) order of the ’t Hooft expansion the graphs with cylindrical topologies, of the type given in Fig. 1(b), will modify the CW potential, nevertheless, we can drop them in the planar approximation.

Let us stress that at large \( N_c \), none of the physical quantities—such as correlators of local fields—can depend on the normalization scale \( \mu \) for the chosen background fields \( \langle Z \rangle, \langle Z^2 \rangle \), since in the UV regime the theory behaves like in the unbroken phase, which is UV finite. The CW potential is yet another example of such a quantity.

\[ \langle Z \rangle = v \text{ diag}(z_1, z_2, z_3, z_4, z_5, z_6, \ldots) \]  

\[ (12) \]

Plugging [12] into the system of transcendental eqs. [9], [10] and [11], we numerically find a complex (as a consequence of the non-unitarity) solution

\[ z_1 = -0.587849 - 0.808971 i, \quad z_2 = 0.260305 + 1.45187 i, \quad z_3 = 1.32754 - 0.642903 i, \quad z_4 = -1, \]  

\[ (13) \]

where the overall rescaling \( z_j \to \text{const} \times z_j \) was absorbed into the complex modulus \( v \) labeling the one-parameter family of flat vacua.

The masses generated on top of this vacuum can be calculated from the quadratic variation of the full effective potential \( \mathcal{V}_{\text{eff}} \) w.r.t. matrix fields \( X, \bar{X}, X, \bar{X} \). One can easily see that, again, the part depending on the Lagrange multipliers \( A, B, C, D \) in \( \mathcal{V}_{\text{eff}} \), does not contribute at \( N_c \to \infty \). The spectrum of the theory in the leading order at this limit comprises: i) \( N_c^2 - 1 \) complex massive excitations of the matrix scalar \( X \) whose masses are proportional to \( \xi^2 |v|^2 z_i z_j \) with \( z_i \)s from [12], [13]; ii) \( N_c^2 - 1 \) gapless modes—including the dilaton which is proportional to \( tr(\langle Z \rangle \delta Z + \langle Z \rangle \delta \bar{Z}) \)—associated with \( Z \). Note that beyond the planar approximation, the excitations of \( Z \) will acquire masses, as follows from the variation of the CW action.

There is no difficulty in finding more examples for larger block matrices of the form [12], and thus with more of the parameters labeling the flat vacua. For instance if we solve the system of eqs. [5], [6] and [11] for \( \langle Z \rangle \) made of \( N_c/5 \) sub-blocks of dimensions \( 5 \times 5 \), we will have an extra parameter, in addition to \( v \), parametrizing the flat directions. We can also mix sub-blocks of different sizes.

**CONCLUSIONS AND OPEN PROBLEMS**

In this work we initiated the study of spontaneous conformal symmetry breaking in the recently proposed fishnet CFT. We showed that the theory admits a plethora of classical flat directions along which conformal symmetry is nonlinearly realized without fine-tunings. We also studied the quantum corrections and found that the classical conformal invariance is not violated, at least in some subclass of the classical solutions. Furthermore, we argued that the CW effective potential is one-loop exact at large \( N_c \).

The FCFT is integrable in ’t Hooft limit, which offers a unique opportunity to elucidate various aspects of the dynamics behind spontaneous symmetry breaking in this particular theory. At the same time, it can serve as an inspiring example for CFTs with such behavior in general. A first step towards this direction is to check the validity of the constraints that were derived in [39]. For instance, the deep infrared limit of the two-point functions of scalar primary operators \( \mathcal{O}_I \) were shown to obey the identity

\[ \langle \mathcal{O}_I \rangle / \langle \mathcal{O}_J \rangle \sim \lim_{x \to \infty} \sum_{K} \frac{c_{IJK}}{x^{2N + \Delta_J - \Delta_K}} \langle \mathcal{O}_K \rangle \]  

\[ (14) \]
with $c_{IJK}$ the OPE coefficients and $\Delta$’s the corresponding scaling dimensions. As a test of this relation in the context of the FCFT, we can consider the dimension-two operators $\text{tr}(XZ)$, $\text{tr}(XZ)$, $\text{tr}(XZ)$, whose two-point correlators in the planar limit are protected against quantum corrections and decay as $\sim |x|^{-4}$ [27].

The fact that the vev of these operators vanish for our vacua, immediately implies the validity of [14]. The OPE data for these operators in the unbroken vacuum have been computed in [27]. A more detailed study of various consistency conditions is left for the future work [50]. In particular, the scalar one-point functions of the operators entering the r.h.s. of these operators might be computable, using the methods developed in [27] [50].

Let us also point out that some of the (classical) vacua we discussed in this work (see also footnote [14]) are present in the full $\gamma$-deformed $\mathcal{N} = 4$ SYM and propagate all the way to the fishnet CFT. One can, for instance, assume that $(X)_\text{tree} = c(Z)_\text{tree}$, with $c$ a constant. Requiring that the above satisfies the equations of motion of the $\gamma$-deformed theory even before the fishnet double scaling limit is taken, translates into the coefficient $\alpha_2$ of the double-trace terms involving $Z$ and $X$ in [2] being completely fixed $\alpha_2 = -4g^2\sin^2\left(\frac{\gamma_3}{2}\right)$. As a sanity check, note that $\lim_{\gamma_3 \to 0} \alpha_2 \sim \xi^2$, while $\lim_{\gamma_3 \to 0} \alpha_0 = 0$ as it should. To put it differently, the mere requirement that the full $\gamma$-deformed $\mathcal{N} = 4$ SYM theory possesses flat directions smoothly connected to the ones of its fishnet “descendant,” completely determines one of the coefficients appearing in the action.

Finally, it would be interesting to study to what extent the discussed properties of the FCFT survive in the next $1/N_c$ orders, or even for finite $N_c$.

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[15] The name of the theory stems from the characteristic regular square lattice form of its planar Feynman graphs.

[16] To our best knowledge, this is a unique behavior for a four-dimensional theory, though a three-dimensional CFT with flat directions that persist at the quantum level was presented in [3].

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[20] In this theory, the $SO(6)$ R-symmetry is broken down to $U(1)^3$, with $\gamma_1, \gamma_2, \gamma_3$ being the parameters (twists) of the deformation.

[21] Strictly speaking, depending on the physical quantity under consideration there may be a few diagrams that
need to be taken into account, but in very limited numbers.

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[43] The fishnet CFT has a plethora of possible flat vacua different from the simple ones that we have discussed so far. Their complete classification, however, lies well beyond the scope of the present paper. A few examples that we have been able to find are the following. As long as $\langle X \rangle_{\text{tree}} = 0$, there also exist “exotic” symmetry breaking solutions to (3), again for all values of $\xi$. These comprise nilpotent matrices, i.e., $\langle Z \rangle_{\text{tree}} \neq 0$, while $\langle Z \rangle_{\text{tree}} = 0$, which quite interestingly are not present in the full $\mathcal{N} = 4$ SYM nor in its $\gamma$-deformed descendant, nevertheless, they emerge when the strong imaginary $\gamma$-deformation limit—leading to the fishnet CFT—is considered. We can also relax the requirement that $\langle X \rangle_{\text{tree}} = 0$ and require that both fields have nonzero vev. Field configurations such that $\langle X \rangle_{\text{tree}} \propto \langle Z \rangle_{\text{tree}}$, with $\langle Z \rangle_{\text{tree}}$ given by (4), provide yet another set of acceptable vacua along which CI is nonlinearly realized, as it can be immediately checked. Finally, additional flat directions open up at isolated values of $\xi$. In what follows we will only focus on the simplest possible symmetry breaking ansatz and leave the search and study for more complicated flat vacua for the future.

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