The anisotropic Heisenberg model in a transverse magnetic field

D.V.Dmitriev\textsuperscript{a,b}, V.Ya.Krivnov\textsuperscript{a,b}, A.A.Ovchinnikov\textsuperscript{a,b} and A.Langari\textsuperscript{b,c}

\textsuperscript{a} Joint Institute of Chemical Physics of RAS, Kosygin str. 4, 117977, Moscow, Russia
\textsuperscript{b} Max-Planck-Institut fur Physik Komplexer Systeme, Nohantscher Str. 38, 01187 Dresden, Germany
\textsuperscript{c} Institute for Advanced Studies in Basic Sciences, Zanjan 45195-159, Iran

One dimensional spin-1/2 XXZ model in a transverse magnetic field is studied. It is shown that the field induces the gap in the spectrum of the model with easy-plane anisotropy. Using conformal invariance the field dependence of the gap at small fields is found. The ground state phase diagram is obtained. It contains four phases with different types of the long range order (LRO) and a disordered one. These phases are separated by critical lines, where the gap and the long range order vanish. Using scaling estimations and a mean-field approach as well as numerical calculations in the vicinity of all critical lines we found the critical exponents of the gap and the LRO. It is shown that transition line between the ordered and disordered phases belongs to the universality class of the transverse Ising model.

75.10.Jm - Quantized spin models

The effect of a magnetic field on an antiferromagnetic chain has been attracting much interest from theoretical and experimental points of view. In particular, the strong dependence of the properties of quasi-one-dimensional anisotropic antiferromagnets on the field orientation was experimentally observed [1]. So, it is interesting to study the dependence of properties of the one-dimensional antiferromagnet on the direction of the applied field. The simplest model of the one-dimensional anisotropic antiferromagnet is the spin-1/2 XXZ model. This model in an uniform longitudinal magnetic field (along the Z axis) was studied in a great details [2]. Since the longitudinal field commutes with the XXZ Hamiltonian the model can be exactly solved by the Bethe ansatz. This is not the case when the symmetry-breaking transverse magnetic field is applied and the exact integrability is lost. Because of a mathematical complexity of this model it has not been studied so much. From this point of view it is of a particular interest to study the ground state properties of the 1D XXZ model in the transverse magnetic field. The Hamiltonian of this model reads

$$H = \sum_{n=1}^{N} (S_n^x S_{n+1}^x + S_n^y S_{n+1}^y + \Delta S_n^z S_{n+1}^z) + h \sum_{n=1}^{N} S_n^z \quad (1)$$

with periodic boundary conditions and even N.

The spectrum of the XXZ model for $-1 < \Delta \leq 1$ is gapless. In the longitudinal field the spectrum remains gapless if the field does not exceed a saturated value $1+\Delta$. On the contrary, when the transverse magnetic field is applied a gap in the excitation spectrum seems to open up. It is supposed [3] that this effect can explain the peculiarity of low temperature specific heat in Yb\textsubscript{3}As\textsubscript{3} [1]. The magnetic properties of this compound is described by XXZ Hamiltonian with $\Delta \approx 0.98$ and it was shown [3] that the magnetic field in easy plane induces a gap in the spectrum resulting in a dramatic decrease of the linear term in the specific heat.

First of all, what do we know about the model (1)? The first part of the Hamiltonian is well-known XXZ model with the exact solution given by Bethe ansatz. In the Ising-like region $\Delta > 1$ the ground state of XXZ model has a Neel long-range order (LRO) along Z axis and there is a gap in excitation spectrum. In the region $-1 < \Delta \leq 1$ the system is in the so-called spin-liquid phase with a power-law decay of correlations and a linear spectrum. And, finally, for $\Delta < -1$ classical ferromagnetic state is a ground state of XXZ model with a gap over ferromagnetic state.

In the transverse magnetic field the total $S^z$ is not a good quantum number and the model is essentially complicated, because the transverse field breaks rotational symmetry in $X-Y$ plane and destroys the integrability of the XXZ model, except some special points. In particular, the exact diagonalization study of this model is difficult for finite systems because of non-monotonic behavior of energy levels.

The first special case of the model (1) is the limit $\Delta \to -\infty$. In this case the model (1) reduces to the 1D Ising model in a transverse field (ITF), which can be solved exactly by transformation to the system of non-interacting fermions. In both limits the system has a phase transition point $h_c = |\Delta|/2$, where the gap is closed and the LRO in Z direction vanishes. It is suggested [5] that the phase transition of the ITF type takes place for any $\Delta > 0$ at some critical value $h = h_c(\Delta)$. One can expect also that such a transition exists for any $\Delta$ and the transition line connects two limiting points $h_c = |\Delta|/2$, $\Delta \to \pm \infty$.

Similar to these limiting cases, for any $|\Delta| > 1$ and $h < h_c(\Delta)$ the system has the LRO in Z direction (Neel for $\Delta > 1$ and ferromagnetic for $\Delta < -1$). But for $|\Delta| < 1$ and $h < h_c(\Delta)$ the ground state changes and instead of the LRO in Z direction a staggered magnetization along Y axis appears at $h < h_c(\Delta)$.

This assumption is confirmed on the ‘classical’ line
The direction of the magnetic field is not important and the spectrum is gapped.

The excited states on the classical line are generally unknown, though it is assumed that it is a classical one \([4]\). The excited states on the classical line are generally unknown, though it is assumed that the spectrum is gapped.

The second case, where the model (1) remains integrable, is the isotropic AF case \(\Delta = 1\). In this case the direction of the magnetic field is not important and the ground state of the system remains spin-liquid one up to the point \(h = 2\), where the phase transition of the Pokrovsky-Talapov type takes place and the ground state becomes completely ordered ferromagnetic state.

And the last special case is \(\Delta = -1\). In this case the model (1) reduces to the isotropic ferromagnetic model in a staggered magnetic field. This model is non-integrable, but as was shown \([13]\), the system remains gapless up to some critical value \(h = h_0\), where the phase transition of the Kosterlitz-Thouless type takes place.

Summarizing all above, we expect that the phase diagram of the model (1) on \((\Delta, h)\) plane takes a form as shown in Fig.1. The phase transition contains four regions corresponding to different phases and separated by transition lines. Each phase is characterized by its own type of the LRO: the Neel order along \(X\) axis in the region (1); the ferromagnetic order along \(Z\) axis in the region (2); the Neel order along \(Y\) axis in the region (3); and in the region (4) there is no LRO except magnetization along the field direction \(X\) (which, certainly, exists in all above regions). Hereafter under LRO we mean the corresponding to given region type of LRO.

In this paper we investigate the behavior of the gap and the LRO near the transition (critical) lines. The first section is devoted to the classical line, where we review the exact ground state and construct three exact excitations. In the section 2 we study the transition line \(h_c(\Delta)\) with use of the mean-field approach and exact diagonalization of finite systems. In the section 3 we find the critical exponents in the vicinity of the line \(h = 0\). The properties of the model near the critical lines \(\Delta = \pm 1\) and in particular in the vicinity of the points \((\Delta = \pm 1, h = 0)\) are studied in sections 4 and 5.

\[ h_{c1} = \sqrt{2(1 + \Delta)} \quad (h_{c1} < h_{c}(\Delta)), \]

where the quantum fluctuations of \(XXZ\) model are compensated by the transverse field and the exact ground state of (1) at \(h = h_{c1}\) is a classical one \([4]\). The excited states on the classical line are generally unknown, though it is assumed that the spectrum is gapped.

In the section 2 we study the transition line \(h_c(\Delta)\) (see below).

FIG. 1. Phase diagram of the model (1). The thick solid lines denote the critical lines, thin solid line is the ‘classical’ line, and dashed line denotes the line \(h_1(\Delta)\) (see below).

\[ h_{c1} = \sqrt{2(1 + \Delta)} \]

and \(\Phi_{1,2}\) are direct product of single-site functions:

\[ |\Phi_1\rangle = |\alpha_1 \alpha_2 \alpha_3 \alpha_4 \ldots\rangle \]

\[ |\Phi_2\rangle = |\bar{\alpha}_1 \alpha_2 \bar{\alpha}_3 \alpha_4 \ldots\rangle \]

where \(|\alpha_i\rangle\) is the state of i-th spin lying in the \(XY\) plane for \(|\Delta| < 1\) (or in \(XZ\) plane for \(\Delta > 1\)) and forming an angle \(\varphi\) with \(X\) axis. These states can be written as:

\[ |\alpha_i\rangle = (e^{i\varphi S^+_i} - 1) |\downarrow\rangle, \quad |\Delta| < 1 \]

\[ |\alpha_i\rangle = (e^{i\varphi S^-_i} - 1) |\downarrow\rangle, \quad \Delta > 1 \]

with \(\cos \varphi = h_{c1}/2\) for \(|\Delta| < 1\) and \(\cosh \varphi = h_{c1}/2\) for \(\Delta > 1\).

The state \(|\bar{\alpha}_i\rangle\) is obtained by rotation of i-th spin by \(\pi\) about the axis of the magnetic field \(X\).

\[ |\bar{\alpha}_i\rangle = e^{i\varphi S^+_i} |\alpha_i\rangle \]

The ground state has a two-sublattice structure and is characterized by the presence of the LRO in the \(Y\) \((|\Delta| < 1)\) or in the \(Z\) \((\Delta > 1)\) directions. In particular, for \(|\Delta| < 1\) the staggered magnetization \(\langle S^y_n \rangle\) is

\[ \langle S^y_n \rangle = \frac{(-1)^n}{2} \sqrt{1 - \frac{h_{c1}^2}{4}} \]

However, the excited states of (1) on the classical line are nontrivial in general. Nevertheless, some of them can be found exactly. For this aim it is convenient to introduce the operator overturning the i-th spin:

\[ R_i = e^{i\pi S^+_i}, \quad |\Delta| < 1 \]

\[ R_i = e^{i\pi S^-_i}, \quad \Delta > 1 \]

so that the states of the ‘overturned’ i-th spin \(|\beta_i\rangle = R_i |\alpha_i\rangle\), \(|\bar{\beta}_i\rangle = R_i |\bar{\alpha}_i\rangle\) are orthogonal to \(|\alpha_i\rangle\), \(|\bar{\alpha}_i\rangle\):

\[ \langle \alpha_i | \beta_i \rangle = \langle \bar{\alpha}_i | \bar{\beta}_i \rangle = 0 \]
Then, the exact excited state are written as:

\[
\begin{align*}
|\psi^1_{1(2)}\rangle &= \sum_{n} R_n |\Phi_{1(2)}\rangle \\
|\psi^2_{1(2)}\rangle &= \sum_{n} (-1)^n R_n R_{n+1} |\Phi_{1(2)}\rangle \\
|\psi^3_{1(2)}\rangle &= \sum_{n,m} (-1)^n R_n R_{n+1} R_m |\Phi_{1(2)}\rangle
\end{align*}
\]

So, each of three exact excitations is also two-fold degenerated. Actually, this degeneracy is a consequence of the \( Z_2 \) symmetry describing the rotation of all spins by \( \pi \) about the axis of the magnetic field \( X \).

To show that these states are really exact ones it is convenient to rotate the coordinate system, so that one of the ground states, for example \( \Phi_1 \), will be all spins pointing down. For the case \( |\Delta| < 1 \) this transformation is the rotation of the spins on even (odd) sites by an angle \( \varphi \) (-\( \varphi \)) around the Z axis followed by the rotation by \( \pi/2 \) about the Y axis:

\[
\begin{align*}
S^y_n &= \sigma^x_n \cos \varphi + (-1)^n \sigma^y_n \sin \varphi \\
S^x_n &= (-1)^n \sigma^x_n \sin \varphi - \sigma^y_n \cos \varphi \\
S^z_n &= -\sigma^z_n
\end{align*}
\]

For \( \Delta > 1 \) the transformation of the spin operators is defined by the relation

\[
\begin{align*}
S^x_n &= \sigma^y_n \cos \varphi + (-1)^n \sigma^z_n \sin \varphi \\
S^y_n &= \sigma^z_n \\
S^z_n &= -(-1)^n \sigma^z_n \sin \varphi + \sigma^x_n \cos \varphi
\end{align*}
\]

Then the Hamiltonian (1) on the classical line becomes

\[
H_1 = \Delta \sum \sigma_n \cdot \sigma_{n+1} + (1 + \Delta) \sum \sigma^z_n
\]

\[+ h_{cl} \sqrt{1 - \frac{h_{cl}^2}{4}} \sum_{n} (-1)^n \sigma^y_n (\sigma^z_{n+1} + \sigma^z_{n-1} + 1) \tag{4}\]

for \( \Delta < 1 \) and

\[
H_2 = \sum \sigma_n \cdot \sigma_{n+1} - (1 - \Delta) \sum \sigma^z_n \sigma^z_{n+1} + 2 \sum \sigma^x_n
\]

\[+ \sqrt{h_{cl}^2 - 4} \sum_{n} (-1)^n \sigma^y_n (\sigma^z_{n+1} + \sigma^z_{n-1} + 1) \tag{5}\]

for \( \Delta > 1 \).

The ground state of both Hamiltonians as well as of (1) is two-fold degenerated. One of the ground state is obviously all spins \( \sigma_n \) pointing down \( \Phi_1 = |0\rangle \equiv |\downarrow\downarrow\ldots\rangle \). The energy of this state is:

\[E_0 = -\frac{N}{2} - \frac{N \Delta}{4}\]

The second ground state \( \Phi_2 \) in this representation has a more complicated form:

\[\Phi_2 = \prod_n \left( \cos \varphi + (-1)^n \sigma^z_n \sin \varphi \right) |0\rangle\]

Now, it is easy to see that the following three excited states are exact ones:

\[
\begin{align*}
|\psi^1_{1}\rangle &= \sum \sigma^+_n |0\rangle, \quad E_1 - E_0 = 1 + \Delta \\
|\psi^2_{1}\rangle &= \sum (-1)^n \sigma^+_n \sigma^+_m |0\rangle, \quad E_2 - E_0 = 2 + \Delta \\
|\psi^3_{1}\rangle &= \sum (-1)^n \sigma^+_n \sigma^+_m |0\rangle, \quad E_3 - E_0 = 3 + 2\Delta
\end{align*}
\]

One can check that last terms in (4) and (5) annihilate these three functions and, hence, they are exact excited states of (1) for any even \( N \). Similarly to the ground state the excited states (7) are degenerated with the states \( |\psi^k_{1}\rangle \). These states \( |\psi^k_{1}\rangle \) can be represented in the same form (7), but in the coordinate system, where the function \( \Phi_2 \) is all spins pointing down.

The states \( |\psi^1_{1(2)}\rangle \) are especially interesting because they define the gap of the model (1) on the classical line at small value of \( h_{cl} \). Our numerical calculations of finite systems show that at \( h_{cl} \rightarrow 0 \) (\( \Delta \rightarrow -1 \)) the lowest branch of excitations has a minimum at \( k = 0 \) and corresponding excitation energy is \( (1 + \Delta) \) (of course, due to the \( Z_2 \) symmetry there is another branch with the minimum at \( k = \pi \) and the same minimal energy, but we consider one branch only). Excitation energy at \( k = \pi \) obtained by the extrapolation of numerical calculations at \( N \rightarrow \infty \) is \( 2(1 + \Delta) \). When \( h_{cl} \) increases excitation energies at \( k = 0 \) and \( k = \pi \) are drawn together and at some \( h_{cl} \) they are equal to each other. Our numerical results give \( h_{cl} \approx 0.76 \) (\( \Delta \approx -0.79 \)). So, the gap on the classical line is \( (1 + \Delta) \) for \( -1 < \Delta < -0.79 \).

### II. THE TRANSITION LINE \( H = H_C(\Delta) \)

The existence of the transition line \( h_c(\Delta) \) passing through the whole phase diagram is quite natural, because at some value of the magnetic field all types of the LRO except the LRO along the field must vanish. The transition line connects two obvious limits \( \Delta \rightarrow \pm \infty \), when the model (1) reduces to the ITF model. The line passes through the exactly soluble point \( (\Delta = 1, h = 2) \) and the point \( (\Delta = -1, h = h_0) \) studied in [13]. We suppose that the whole line \( h_c(\Delta) \) is of ITF type with algebraically decaying of correlations with corresponding critical exponents [15].

The transition line can be also observed from the numerical calculations of finite systems. As an example, the dependencies of excitation energies of three lowest levels on \( h \) and for \( N = 10 - 18 \) are shown on Fig.2. From this figure one can see that two lowest states cross each other \( N/2 \) times and the last crossing occurs on the classical point \( h_{cl} = \sqrt{2} \). These two states form two-fold degenerated ground state in the thermodynamic limit. They have different momenta \( k = 0 \) and \( k = \pi \) and different quantum number describing the \( Z_2 \) symmetry, which remains in the system after applying the field. As for the
first excitation above the degenerated ground state, on Fig. 2 we see also numerous level crossings. These level crossings lead to the incommensurate effects which manifest itself in the oscillatory behavior of the spin correlation functions. The correlation functions at \( n \gg 1 \) have a form:

\[
\langle S^\alpha_i S^\alpha_j \rangle - \langle S^\alpha \rangle^2 = f(n)e^{-kx} \tag{8}
\]

where \( \langle S^\alpha \rangle (\alpha = x, y, z) \) is the corresponding magnetization (the LRO), and \( f(n) \) is the oscillatory function of \( n \) with the oscillating period depending on \( h \) and \( \Delta \). All crossings disappear at \( h > h_{cl}(\Delta) \) and in this region of the phase diagram the correlation functions do not contain oscillatory terms.

The energy of the first excitation near \( h_{cl} \) goes down rapidly, and after extrapolation we found that for the case \( \Delta = 0 \) at the magnetic field \( h_c \approx 1.456(6) > h_{cl} \) the gap vanishes. Inside the region \( h_{cl} < h < h_c \) the ground state remains two-fold degenerated, though there is no level crossings. At \( h > h_c \) the mass gap appears again and for a large field the gap is proportional to \( h \).

In order to determine the transition line \( h_c(\Delta) \) and to study the model in the vicinity of \( h_c(\Delta) \), we use the Fermi-representation of (1). This representation gives the exact solution in the limits \( \Delta \to \pm \infty \) and, besides, it yields the exact ground state on the classical line.

At first it is convenient to perform in (1) a rotation of the spins around the \( Y \) axis by \( \pi/2 \), so that the magnetic field is along the \( Z \) axis:

\[
H = \sum (\Delta S^\alpha_n S^\alpha_{n+1} + S^y_n S^y_{n+1} + S^z_n S^z_{n+1}) + h \sum S^z_n \tag{9}
\]

After Jordan-Wigner transformation to Fermi operators \( a^+_n, a_n \)

\[
S^\alpha_n = e^{i\pi} \sum a^+_n a^+_n a_n \frac{1}{2} \tag{10}
\]

the Hamiltonian (9) takes the form

\[
H_f = -\frac{hN}{2} + \frac{N}{4} \sum (h - 1 - \frac{1 + \Delta}{2} \cos k) a^+_k a_k + \frac{1 - \Delta}{4} \sum \sin k(a^+_k a^+_k + a_{-k} a_{-k}) + \sum a^+_n a_n a^+_n a_{n+1} + a_{n+1} a_n \tag{11}
\]

Treating the Hamiltonian \( H_f \) in the mean-field approximation we find the ground state energy \( E_0 \) and the one-particle excitation spectrum \( \varepsilon(k) \):

\[
E_0/N = (h - 1) \left( \gamma_1 - \frac{1}{2} \right) + \frac{1}{4} - \left( 1 - \frac{g}{2} \right) \gamma_2 + \frac{g}{2} \gamma_3 + \left( \gamma_1^2 - \gamma_2^2 + \gamma_3^2 \right) \tag{12}
\]

\[
\varepsilon(k) = \sqrt{a^2(k) + b^2(k)} \tag{13}
\]

where \( g = 1 - \Delta \) and

\[
a(k) = (h - 1) - \left( 1 - \frac{g}{2} \right) \cos k + 2\gamma_1 - 2\gamma_2 \cos k \tag{14}
\]

\[
b(k) = \left( \frac{g}{2} + 2\gamma_3 \right) \sin k \tag{14}
\]

Quantities \( \gamma_1, \gamma_2 \) and \( \gamma_3 \) are the ground state averages, which are determined by the self-consistent equations:

\[
\gamma_1 = \langle a^+_n a_n \rangle = \sum_{k>0} \left( 1 - \frac{a(k)}{\varepsilon(k)} \right) \tag{15}
\]

\[
\gamma_2 = \langle a^+_n a_{n+1} \rangle = -\sum_{k>0} \frac{a(k)}{\varepsilon(k)} \cos k \tag{15}
\]

\[
\gamma_3 = \langle a^+_n a_{n+1} \rangle = -\sum_{k>0} \frac{b(k)}{2\varepsilon(k)} \sin k \tag{15}
\]

Magnetization \( S = \langle S^z_n \rangle \) of the model (1) is equal to

\[
S = \frac{1}{2} - \gamma_1 \tag{16}
\]

The numerical solution of Eqs.(15) shows that the function \( \varepsilon(k) \) has a minimum at \( k_{\text{min}} \), which is changed from \( \pi/2 \) at \( h = 0 \) to zero at \( h = h_1(\Delta) \) and \( k_{\text{min}} = 0 \) for \( h > h_1(\Delta) \). The gap in the spectrum \( \varepsilon(k) \) vanishes at \( h_c(\Delta) \) \( (h_c > h_1) \) and for \( h > h_1 \) is \( m = |h - h_c| \). The dependencies of \( h_1(\Delta) \) and \( h_c(\Delta) \) are shown on Fig.1. We note that the Hamiltonian \( H_f \) differs from the domain-wall fermionic Hamiltonian which is mapped of (1) in [5]. The transition line obtained in [5] is a linear function of \( \Delta \) in contrast to \( h_c(\Delta) \) on Fig.1.

It is interesting to note that the mean field approximation gives the exact ground state on the classical line \( h_{cl} = \sqrt{2(1 + \Delta)} \). On this line the solution of Eqs.(15) has very simple form:

\[
\gamma_1 = \frac{1}{2} - \frac{h_{cl}}{4}, \quad \gamma_2 = -\gamma_3 = \frac{4 - h_{cl}^2}{16}, \quad |\Delta| < 1
\]

\[
\gamma_1 = \frac{1}{2} - \frac{1}{h_{cl}}, \quad \gamma_2 = \gamma_3 = \frac{h_{cl}^2 - 4}{4h_{cl}^2}, \quad \Delta > 1 \tag{17}
\]
and the energy $E_0/N = -\frac{1}{2} - \frac{\Delta}{2}$.

The gap on the classical line in the mean-field approximation is

$$
m = \frac{1}{4}(2 - h_{cl})^2, \quad |\Delta| < 1
$$

$$
m = \frac{h_{cl}^2 - 2}{2h_{cl}^2}(h_{cl} - 2)^2, \quad \Delta > 1 \quad (18)
$$

We compared (18) with the results of the extrapolation of finite systems on the classical line. The coincidence is fairly good for $\Delta > 0.5$. Eq.(18) gives rather satisfactory estimation for the gap up to $\Delta \approx -0.5$. For example, at $\Delta = 0$ ($h_{cl} = \sqrt{2}$) from Eq.(18) $m = 0.086$, while extrapolated gap is $m \approx 0.076(4)$.

The smaller the fermion density the better works the mean-field approximation. It becomes worse when the magnetization $S \to 0$. This is the reason of incorrect behavior of the gap at $h_{cl} \to 0$ ($\Delta \to -1$). According to (18) $m = 1$, while it vanishes in this limit as $m = 1 + \Delta$ (7).

The Hamiltonian $H_1$ in the mean-field approximation has a form similar to the well-known bilinear Fermi-Hamiltonian describing the anisotropic XY model or the ITF model. Using results of [15] the following facts related to the considered model can be established:

1. There is a staggered magnetization $\langle S^y_n \rangle$ along the Y axis for $|\Delta| < 1$ or $\langle S^z_n \rangle$ along the Z axis for $|\Delta| > 1$ and they vanish at $h \to h_c$, as ($h_c - h$)$^{1/8}$.

2. The magnetization $S$ has a logarithmic singularity at $h \to h_c$.

3. The spin correlation function decay exponentially (excluding the transition line) at $n \to \infty$:

$$
G^\alpha(n) = \langle S^\alpha_i S^\alpha_n \rangle - \langle S^\alpha_i \rangle^2 = f(n)e^{-\alpha n} \quad (19)
$$

The function $f(n)$ has an oscillatory behavior at $0 < h < h_{cl}$, while at $h > h_{cl}$ it is monotonic. At $h = h_{cl}$, $f(n) = 0$ and $f(n) \sim \frac{\cos \omega n}{n} \left( \omega = \frac{\sqrt{2h_{cl} - h}}{h_{cl} - h} \right)$ at $(h_{cl} - h) \ll 1$. Thus the classical line determines the boundary on the phase diagram where the spin correlation functions show the incommensurate behavior.

On the transition line $h = h_c(\Delta)$ the spin correlation functions have power-law decay:

$$
G^x(n) \sim 1/n^2, \quad G^y(n) \sim 1/n^{1/4}, \quad G^z(n) \sim 1/n^{9/4}
$$

for $|\Delta| < 1$ and

$$
G^x(n) \sim 1/n^2, \quad G^y(n) \sim 1/n^{9/4}, \quad G^z(n) \sim 1/n^{1/4},
$$

for $|\Delta| > 1$.

These results show that the transition at $h = h_c(\Delta)$ belongs to the universality class of the ITF model.

In the vicinity of the point $h = 2, \Delta = 1$ the fermion density is small ($S \approx \frac{1}{2}$) and the mean-field approximation of the four fermion term gives the accuracy, at least, up to $g^3$ or $(2 - h)^4$. For this case we give corresponding expressions (at $g \ll 1$):

$$
h_c = 2 - \frac{g^2}{2} - \frac{g^2}{32},
$$

$$
h_1 = h_c - \frac{g^2}{16},
$$

$$
m = |h - h_c|, \quad h > h_1
$$

$$
m = \frac{g}{2\sqrt{2}}\sqrt{h_c - h - \frac{g^2}{32}}, \quad h < h_1 \quad (20)
$$

The magnetization $S$ is

$$
S = \frac{1}{2} - \frac{\sqrt{2}}{\pi} \sqrt{h_c - h - \frac{g}{8\pi}}, \quad g \ll \sqrt{h_c - h}
$$

$$
S = \frac{1}{2} - \frac{g}{4\pi} - \frac{2(h_c - h)}{\pi g} \ln \left( \frac{g^2}{h_c - h} \right), \quad g \gg \sqrt{h_c - h} \quad (21)
$$

The susceptibility $\chi(h) = \frac{dS}{dh}$ is

$$
\chi(h) = \frac{2}{\pi g} \ln \left( \frac{g^2}{h_c - h} \right), \quad g \gg \sqrt{h_c - h}
$$

$$
\chi(h) = \frac{1}{\sqrt{2\pi}h_c - h}, \quad g \ll \sqrt{h_c - h} \quad (22)
$$

As follows from (22) there is a crossover from square root to logarithmic divergence of $\chi$ when the parameter $\frac{g^2}{h_c - h}$ is changed from 0 to $\infty$.

III. THE LINE $H = 0, |\Delta| < 1$

A. Scaling estimations

The XXZ model is integrable and its low-energy properties are described by a free massless boson field theory with the Hamiltonian

$$
H_0 = \frac{v}{2} \int dx \left[ \Pi^2 + (\partial_x \Phi)^2 \right] \quad (23)
$$

where $\Pi(x)$ is the momentum conjugate to the boson field $\Phi(x)$, which can be separated into left and right moving terms $\Phi = \Phi_L + \Phi_R$. The dual field $\tilde{\Phi}$ is defined as a difference $\tilde{\Phi} = \Phi_L - \Phi_R$. The spin-density operators are represented as

$$
S_n^z \approx \frac{1}{2\pi R} \partial_x \Phi + \text{const.}(1)^n \cos \frac{\Phi}{R}
$$

$$
|\Delta S_n^z| \approx \cos \left( 2\pi R \frac{\Phi}{R} \right) \left[ C(1)^n + \text{const.} \cos \frac{\Phi}{R} \right] \quad (24)
$$

with constant $C$ found in [7]. The compactification radius $R$ is known from the exact solution

$$
2\pi R^2 = \theta = 1 - \frac{\arccos(\Delta)}{\pi}
$$

The non-oscillating part of the operator $S_n^z$ in Eq.(24) has scaling dimension $d = \theta/2 + 1/2\theta$ and conformal spin
$S = 1$. The non-zero conformal spin of the perturbation operator $S^x$ can cause the incommensurability in the system [8], which is in accord with Eq.(19). As was shown in [9], the common formula for the mass gap

$$m \sim h^\nu, \quad \nu = \frac{1}{2 - d} = \frac{2}{4 - \theta - 1/\theta}$$  \hspace{1cm} (25)$$

is not applicable in the whole region $|\Delta| < 1$. Due to non-zero conformal spin of the non-oscillating part of the operator $S^x$ it is necessary to consider higher-order effects in $h$. The analysis shows [9] that the original perturbation with nonzero conformal spin generates another perturbation with zero conformal spin

$$V = h^2 \cos \left(4\pi R \hat{\phi} \right)$$  \hspace{1cm} (26)$$

This perturbation gives the critical exponent for the mass gap

$$m \sim h^\gamma, \quad \gamma = \frac{1}{1 - \theta}$$  \hspace{1cm} (27)$$

Comparing Eq.(25) and (27) we see that the perturbation (26) becomes more relevant in the region $\Delta < \cos[\pi \sqrt{2}] \approx -0.266$.

It turns out that the oscillating part of the operator $S^x$ gives another, more relevant index for the gap at $\Delta < 0$. Let us reproduce usual conformal line of arguments for this oscillating part. The perturbed action for the model is

$$S = S_0 + h \int dt dx S^x(x,t)$$  \hspace{1cm} (28)$$

where $S_0$ is the Gaussian action of XXZ model. The time-dependent correlation functions of the XXZ chain show the power-law decay at $|\Delta| < 1$ and have the asymptotic form [6]

$$\langle S^x(x,\tau) S^x(0,0) \rangle \sim \frac{(-1)^x A_1}{(x^2 + v^2 \tau^2)^{\frac{2}{2}}} - \frac{A_2}{(x^2 + v^2 \tau^2)^{\frac{2}{2} + \frac{\theta}{2}}}$$  \hspace{1cm} (29)$$

with $A_1, A_2$ are known constants [7] and $\tau = it$ is imaginary time. Therefore, we can estimate the large-distance contribution to the action of the oscillating part of the operator $S^x(x,\tau)$ as

$$h \int dt dx S^x(x,\tau) \sim h \int dx \sum n \frac{(-1)^n}{(n^2 + v^2 \tau^2)^{\theta/4}} \sim h \int dx \sum n \frac{\theta}{(n^2 + v^2 \tau^2)^{\frac{2}{2} + \frac{\theta}{2}}}$$

The relevant field $S^x(x,\tau)$ gives rise to a finite correlation length $\xi$. This correlation length is such that the contribution of the field $S^x(x,\tau)$ to the action is of order of 1. That is

$$h \int \frac{d\tau}{\xi} \sum n \frac{\theta}{(n^2 + v^2 \tau^2)^{\frac{2}{2} + \frac{\theta}{2}}} \sim h \int \frac{\theta}{(x^2 + v^2 \tau^2)^{\frac{2}{2} + \frac{\theta}{2}}}$$

FIG. 3. The dependence of the critical exponents $\nu, \mu$ and $\gamma$ on $\Delta$. The smallest exponent gives the most relevant type of perturbation and defines the index for the mass gap.

$$h \int_0^{\xi/\nu} \int_0^\xi \frac{\theta x}{(x^2 + v^2 \tau^2)^{\theta/4 + 1}} \sim \theta h \xi^{1-\theta/2} \sim 1$$

which gives for the mass gap

$$m \sim v/\xi \sim h^\mu, \quad \mu = \frac{1}{1 - \theta/2}$$  \hspace{1cm} (30)$$

In fact, the oscillating factor $(-1)^n$ in the correlator in some sense eliminates one singular integration over $x$, and into common conformal formula $m \sim h^{\frac{\theta - 1}{\theta}}$, where $D$ is the dimension of space and $d$ is the scaling dimension of perturbation operator, one should put $D = 1$ instead of conventional $D = 2$.

The comparison of the expressions Eqs.(25), (27) and (30) shows that for $0 < \Delta < 1$ the leading term is given by Eq.(25), while for $-1 < \Delta < 0$ by Eq.(30). Thus, one has:

$$m \sim h^\nu, \quad 0 < \Delta < 1$$
$$m \sim h^\mu, \quad -1 < \Delta < 0$$  \hspace{1cm} (31)$$

The functions $\nu(\Delta), \mu(\Delta), \gamma(\Delta)$ are shown on Fig.3. In this respect the model (1) is different from the XXZ model in the staggered transverse field for which $m \sim h^{2/(4-\theta)}$ for all $|\Delta| < 1$ [10].

The staggered magnetization (LRO) along $Y$ axis behaves as:

$$\langle S^y_\nu \rangle \sim (-1)^n / \xi^{\theta/2} \sim (-1)^n m^{\theta/2}$$  \hspace{1cm} (32)$$

Hence, the LRO has also two different critical exponents:

$$\langle |S^y_\nu| \rangle \sim h^{\frac{\theta}{2-\theta}} \quad 0 < \Delta < 1$$
$$\langle |S^y_\nu| \rangle \sim h^{\frac{\theta}{2+\theta}} \quad -1 < \Delta < 0$$  \hspace{1cm} (33)$$
B. Perturbation series

The critical exponents $\nu$ and $\mu$ can be also derived from the analysis of infrared divergences of the perturbation theory in $h$. Obviously only even orders in $h$ give contribution. Let us estimate the large-distance behavior of the following operator, defining the order of the perturbation series:

$$ U = \frac{1}{E_0 - H_0} V \frac{1}{E_0 - H_0} V $$

with $V = h \sum S^x_i$ and $H_0$ is the Hamiltonian of XXZ model. The perturbation series for the ground state energy has a form:

$$ \delta E \sim V \frac{1}{E_0 - H_0} V (1 + U + U^2 + \ldots ) $$

(35)

Hereafter we consider large but finite system of the length $N$. We shall care about indices at $N$ and $h$ only, omitting all factors.

At first we consider non-oscillating part of the correlator (29). Taking into account only low-lying excitation of linear spectrum of the XXZ model, giving the most divergent part, and estimating large-distance behavior of non-oscillating part of the correlator (29), we arrive at

$$ U \sim h^2 \sum_{i,j} \frac{\langle S^x_i S^x_j \rangle}{(1/N)^2} \sim h^2 N^2 \frac{N^2}{N^{\theta+1/\theta}} = h^2 N^{4-\theta-1/\theta} $$

(36)

Now we see, that if $4 - \theta - 1/\theta > 0$, then each next order in perturbation theory (35) diverges more and more strongly. To absorb these infrared divergences one has to introduce the scaling parameter $y = Nh^\nu$ and guess that the series $(1 + U + U^2 + \ldots)$ in (35) forms some function of the scaling parameter $y$. In our case $\nu = 2/(4 - \theta - 1/\theta)$ (see Eq.(25)) and $U \sim y^{2/\nu}$.

The leading divergence of the second order of the ground state energy can be found in a similar way as in (36):

$$ \delta E^{(2)} = V \frac{1}{E_0 - H_0} V \sim h^2 N^{3-\theta-1/\theta} $$

(37)

Combining the formulae (36) and (37), we can rewrite

$$ \delta E \sim Nh^{2\nu} f(y) $$

with some unknown function $f(y)$ having the expansion at small $y$:

$$ f(y) = \frac{1}{y^2} \sum_{n=1}^{\infty} c_n y^{2n/\nu} $$

In the thermodynamic limit $N \rightarrow \infty$ the scaling parameter $y = Nh^\nu$ also tends to infinity $y \rightarrow \infty$. Since the energy is proportional to $N$, the function $f(y)$ has a finite limit $f(\infty) = a$. Thus, in the thermodynamic limit for the correction to the ground state energy one has:

$$ \delta E \sim aNh^{2\nu} $$

(38)

For the first excitation state the divergences of the orders of the perturbation theory have the same form as in (36) and (37). So, for the gap one finds the same scaling parameter $y = Nh^\nu$ and

$$ m \sim Nh^{2\nu} g(y) $$

In the thermodynamic limit the mass gap is of order of unity (in terms of $N$), and, therefore, the function $g(y) \sim 1/y$ at $y \rightarrow \infty$. Thus, finally we arrive at the Eq.(25).

Now we consider more subtle, oscillating part of the correlator (29). For the oscillating part at large distances one can write:

$$ \sum_{i,j} \langle S^x_i S^x_j \rangle \sim N \sum_r \frac{(-1)^r}{r^\theta} \sim N \sum_r \frac{1}{r^{\theta+1}} \sim N \frac{1}{N^{\theta}} $$

It turns out that the oscillating part of the perturbation operator $V$ connects the low-lying gapless states with the finite energy states. That is, each second level in all orders of the perturbation series has a finite gap from the ground state. Therefore, for the operator $U$ one has

$$ U \sim h^2 \sum_{i,j} \frac{\langle S^x_i S^x_j \rangle}{(1/N)} \sim h^2 N^{2-\theta} $$

Since $\theta$ is always less than 2 the divergences grow with the order of the perturbation theory. To eliminate these divergences we introduce the scaling parameter $y = Nh^\mu$, with $\mu$ defined in Eq.(30), so that $U \sim y^{2/\mu}$.

The second order for the ground state energy in this case is

$$ \delta E^{(2)} \sim h^2 \sum_{i,j} \frac{\langle S^x_i S^x_j \rangle}{1} \sim h^2 N^{1-\theta} $$

and the total correction to the ground state energy has a form

$$ \delta E \sim Nh^{2\mu} f(y) $$

with some unknown function $f(y)$, having the finite limit $f(\infty) = b$.

Thus, in the thermodynamic limit the ground state energy behaves as

$$ \delta E \sim bNh^{2\mu} $$

The mass gap is found similarly

$$ m \sim Nh^{2\nu} g(y) $$

with the function $g(y) \sim 1/y$ at $y \rightarrow \infty$. Thus, in the thermodynamic limit we reproduce Eq.(30).
We note that we have estimated only long wave-length divergent part of the perturbation theory. Besides, there is a regular part of the perturbation theory, which gives the leading term \( \sim h^2 \). So, combining all the above facts we arrive at

\[
\frac{\delta E}{N} = -\frac{\chi}{2} h^2 + ah^{2\nu} + bh^{2\mu}
\]  

As it can be seen from Eq. (39), \( \delta E \) consists of a regular term \( h^2 \) and two singular terms. Since \( \nu > 1 \) and \( \mu > 1 \), the susceptibility \( \chi \) is finite for any \( \Delta \) in contrast to the model with staggered transverse field [10], where the singular term is \( h^n \) with \( n = 4/(4 - \theta) < 2 \).

From Eq. (25) and (30) it follows that \( \nu \to 1 \) at \( \Delta \to 1 \) and \( \mu \to 1 \) at \( \Delta \to -1 \). Hence, in both limits one of the singular terms becomes proportional to \( h^2 \), and, therefore, gives contribution to the susceptibility. It implies that in the symmetric points \( \Delta = \pm 1 \) the susceptibility has a jump.

IV. THE LINE \( \Delta = 1 \)

In the vicinity of the line \( \Delta = 1 \) it is convenient to rewrite the Hamiltonian (1) in the form

\[
H = H_0 + V
\]

\[
H_0 = \sum_n S_n \cdot S_{n+1} + h \sum_n S_n^y
\]

\[
V = -g \sum_n S_n^z S_{n+1}^z
\]

where the parameter \( g = 1 - \Delta \ll 1 \) is small. On the isotropic line \( \Delta = 1 \) the model (1) is exactly solvable by Bethe ansatz. The properties of the system remain critical up to the transition point \( h_c = 2 \), where the ground state becomes ferromagnetic. Therefore, for \( h < 2 \) and small perturbation \( V \) we can use conformal estimations.

The asymptotic of the correlation function on this line is

\[
\langle S_i^z S_{i+n}^z \rangle \sim \frac{(-1)^n}{n\alpha(h)},
\]

where \( \alpha(h) \) is known function obtained from Bethe ansatz [11] and having the following limits:

\[
\alpha(h) \sim 1 - \frac{1}{2\ln (1/h)}, \quad h \to 0
\]

\[
\alpha(h) \sim \frac{1}{2}, \quad h \to 2
\]

So, the scaling dimension of operator \( S^z \) is \( d_z = \alpha(h)/2 \), and the scaling dimension of operator \( S_i^z S_{i+1}^z \) is four times greater \( d_{zz} = 4d_z = 2\alpha(h) \). Since \( \alpha(h) < 1 \), then the perturbation \( V \) is relevant and leads to the mass gap and the staggered magnetization

\[
m \sim |g|^{1/(2 - d_{zz})} = |g|^{1/(2 - 2\alpha(h))}
\]

\[
\langle |S^y| \rangle \sim |g|^{\alpha/(4 - 4\alpha)}, \quad \Delta < 1
\]

\[
\langle |S^z| \rangle \sim |g|^{\alpha/(4 - 4\alpha)}, \quad \Delta > 1
\]

From the general expressions for the mass gap (43), in the limit \( h \to 2 \) we obtain that \( m \sim g \), which is in accord with the result of the mean field approximation (20).

The LRO in the vicinity of the point \( \Delta = 1 \) vanishes on both lines: at \( \Delta = 1 \) from (43) as \( g^{1/4} \); and at \( h = h_c \) as \( |h_c - h|^{1/8} \). Besides, one has the exact expression for LRO on the classical line

\[
\langle |S^y| \rangle_c = \frac{\sqrt{g}}{2\sqrt{2}}
\]

Combining all these facts we arrive at the following formula:

\[
\langle |S^y| \rangle = 2^{-7/8} g^{1/4} |h_c - h|^{1/8}
\]

The behavior of the system near the point \( \Delta = 1 \), \( h = 0 \) is more subtle. As it follows from Eq. (31), for very small \( h \) the mass gap is \( m \sim h \), while on the other hand from Eq. (43) one obtains another scaling \( m \sim g^{\ln(1/h)} \).

So, there are two regions near this point with different behavior of mass gap. The boundary between these two regions can be found from the following consideration. Let us rewrite the perturbation in the Hamiltonian (40) in the form:

\[
V = V_1 + V_2
\]

\[
V_1 = -\frac{g}{2} \sum_n (S_n^y S_{n+1}^y + S_n^z S_{n+1}^z)
\]

\[
V_2 = \frac{g}{2} \sum_n (S_n^y S_{n+1}^y - S_n^z S_{n+1}^z)
\]

The part of the Hamiltonian \( H_0 + V_1 \) corresponds to the XXZ model in the longitudinal magnetic field, which is gapless when the magnetic field \( h > \exp(-\pi^2/2\sqrt{g}) \). Therefore, in the region of very small magnetic field \( h < \exp(-\pi^2/2\sqrt{g}) \) the perturbation \( V_1 \) is relevant, leading to the mass gap \( m \sim h \). The two-cutoff scaling procedure [8] [9] gives rise to the mass gap \( m \sim h \exp(-\pi^2/2\sqrt{g}) \) for \( h > \exp(-\pi^2/2\sqrt{g}) \). And, finally, when \( g \) is much less than \( h \), the scaling dimension of operator \( V_2 \) defines the exponent for the gap (43). Summarizing all above, the mass gap in the vicinity of the isotropic point \( \Delta = 1 \), \( h = 0 \) is:

\[
m \sim h, \quad \ln h \ll -\frac{1}{\sqrt{g}}
\]

\[
m \sim h e^{-\pi^2/2\sqrt{g}}, \quad \frac{1}{\sqrt{g} \ln g} \gg \ln h \gg -\frac{1}{\sqrt{g}}
\]

\[
m \sim g^{-\ln h}, \quad \ln h \gg \frac{1}{\sqrt{g} \ln g}
\]

V. THE LINE \( \Delta = -1 \)

In this section we consider the model (1) in the vicinity of the line \( \Delta = -1 \), where \( (1 + \Delta) = \delta \ll 1 \) is a small
The spectrum of low-lying excitations of the magnetic state (degenerated with respect to total $S^z$ with zero momentum. The states which can be reached from the ground state by means of the transition operator $\sum (-1)^n S^z_n$ are the states with $q = \pi$ and finite gap over the ground state. When $\delta \ll 1$ the transition operator connects the low-energy states and the states with energies $\varepsilon_s \approx 2$. The second order correction to low-energy states is

$$ \delta E^{(2)}_l = \frac{\hbar^2}{2} \sum_{s,n,m} \frac{\langle l | (-1)^n S^z_n | s \rangle \langle s | (-1)^m S^z_m | l \rangle}{E_l - E_s} $$

(48)

where $\langle l |$ is the low-energy state and $\langle s |$ is the state with high energy $\approx 2$. So, for $\delta \ll 1$ Eq.(48) can be rewritten as

$$ \delta E^{(2)}_l = -\frac{\hbar^2 N}{8} - \frac{\hbar^2}{2} \sum_{n \leq m} \langle l | (-1)^{n-m} S^z_n S^z_m | l \rangle $$

(49)

The spin correlation function $\langle l | S^z_n S^z_m | l \rangle$ is slow varying function of $|m - n|$ at $\delta \ll 1$. Therefore,

$$ \sum_{n \leq m} \langle l | (-1)^{n-m} S^z_n S^z_m | l \rangle \approx -\frac{1}{2} \sum_n \langle l | S^z_n S^z_{n+1} | l \rangle $$

(50)

Thus, as follows from Eqs.(48-50), the low-lying states of (47) at $\delta \ll 1$ and $\hbar \ll 1$ are described by the XY model at $\Delta = 1$ where the mapping of (47) to the XY model (51) as well as initial model (1) is described by the XY Hamiltonian

$$ H = -\frac{\hbar^2 N}{8} - \sum [(1 - \frac{\hbar^2}{2}) S^x_n S^x_{n+1} + S^y_n S^y_{n+1} - \Delta S^z_n S^z_{n+1}] $$

(51)

The coincidence of the low-energy spectra of (47) and (51) in the vicinity of ferromagnetic point $\Delta = -1$, $\hbar = 0$ has been checked numerically for finite systems. The spectrum of low-lying excitations of the $s = \frac{1}{2}$ XY model (51) as well as initial model (1) near the ferromagnetic point $\Delta = -1$, $\hbar = 0$ can be described asymptotically exactly by the spin-wave theory, which gives

$$ m = \hbar \sqrt{(1 + \Delta)/2}, \quad \Delta > -1 $$

(52)

$$ m = \sqrt{(1 + \Delta)(1 + \Delta + h^2/2)}, \quad \Delta < -1 $$

It can be checked that Eq.(52) yields the exact gap of the XY model [14] at $|\delta| \ll 1$. The validity of the spin-wave approximation is quite natural because in the vicinity of the ferromagnetic point $\Delta = -1$, $\hbar = 0$ the number of magnons forming the ground state is small.

We note also that the gap (52) for $\Delta \geq -1$ is in accord with the conformal theory result (31) and gives us the preexponential factor for the gap. On the classical line $h_{cl} = \sqrt{2(1 + \Delta)}$ Eq.(52) yields the gap $m = 1 + \Delta$ which confirms that the function $\psi_1^{(0)}$ in Eq.(7) gives the exact gap.

The similar mapping of the model (1) with arbitrary spin $s$ to XYZ model can be done for $\Delta \approx -1$, $\hbar \ll 1$. Taking into account that $\varepsilon_s = 4s$ corresponding XYZ Hamiltonian is

$$ H = -\sum [(1 - \frac{\hbar^2}{2}) S^x_n S^x_{n+1} + S^y_n S^y_{n+1} - \Delta S^z_n S^z_{n+1}] $$

$$ -\frac{\hbar^2}{4s} \sum (S^z_n)^2 $$

(53)

where $S^z_n$ are spin-$s$ operators.

The leading term of the gap of the model (1) with arbitrary spin $s$ in the vicinity of the point $\Delta = -1$, $\hbar = 0$ is given exactly by the spin-wave theory:

$$ m = h \sqrt{(1 + \Delta)/2}, \quad \Delta > -1 $$

(54)

$$ m = 2s \sqrt{(1 + \Delta)(1 + \Delta + h^2/8s^2)}, \quad \Delta < -1 $$

On the classical line $h_{cl} = 2s \sqrt{2(1 + \Delta)}$ Eq.(54) gives the correct result $m = 2s\delta$.

Strictly on the line $\Delta = -1$ the model (1) reduces to the isotropic ferromagnet in the staggered magnetic field. This model is non-integrable, but it was suggested [13], that the system is governed by a $c = 1$ conformal field theory up to some critical value $h = h_0$, where the phase transition of the Kosterlitz-Thouless type takes place.

At $\hbar \ll 1$ where the mapping of (47) to the XYZ model (51) is valid, the line $\Delta = -1$ is described by the XYZ model and the correlation functions have a power-law decay:

$$ \langle S^z_n S^z_{n+1} \rangle \sim \frac{(-1)^n}{n^\beta(h)} $$

$$ \langle S^x_n S^x_{n+1} \rangle \sim \frac{(-1)^n}{n^{1/\beta(h)}} $$

(55)

We believe that the relation between indices of $X$ and $Y$, $Z$ correlators on the line $\Delta = -1$ has the form (55) for $0 < h < h_0$. So, the scaling dimensions of operators $S^x_n$ and $S^y_n$, $S^z_n$ on this line are $d_x = \beta/2$ and $d_y = d_x = 1/2\beta$.

On the line $\Delta = -1$ the model (1) is gapless for $h < h_0$. It means that the magnetic field term is irrelevant at $h < h_0$ ($\beta(h) > 4$) and becomes marginal at $h = h_0$, where $d_x = 2$ and $\beta(h_0) = 4$. So, at the point $h = h_0$ the transition is of the Kosterlitz-Thouless type, and for $h > h_0$ the mass gap is exponentially small.

In the vicinity of the line $\Delta = -1$ the term $\delta \sum S^x_n S^x_{n+1}$ in (47) can be considered as a perturbation and the scaling dimension of the perturbation operator $S^x_n S^x_{n+1}$ is $d_{zz} = 4d_x = 2/\beta(h)$. Since $\beta(h) \geq 4$ at $h < h_0$, the
perturbation is relevant and leads to the mass gap and the LRO:

\[ m \sim |\delta|^{2/3} \]

\[ \langle |S^y| \rangle \sim \delta^{\pi/3} \], \quad \delta > 0 \]

\[ \langle |S^z| \rangle \sim |\delta|^{\pi/3} \], \quad \delta < 0 \]

(56)

In particular, \( m \sim |\delta|^{2/3} \) and \( \langle |S^y| \rangle \sim |\delta|^{1/12} \) at \( h \to h_0 \).

The function \( \beta(h) \) is generally unknown, except the case \( h \ll 1 \), where the mapping to XXZ model is valid, and \( \beta(h) = \left[ 1 - \frac{1}{\pi} \arccos \left( \frac{h^2}{2} - 1 \right) \right]^{-1} \approx \pi/h \). However, since the model (1) at \( \Delta = -1 \) and \( h < h_0 \) is conformal invariant, we can use a finite-size scaling analysis to determine the exponent \( \beta(h) \) and the value of \( h_0 \). According to the standard scaling approach [16] \( \beta(h) = \frac{2\pi}{\Delta} \), where \( \nu \) is the sound velocity and \( \Delta \) is the difference between the two lowest energies of the system. We carried out calculations of \( \beta(h) \) for finite systems. The extrapolated function \( \beta(h) \) agrees well with the dependence \( \pi/h \) at \( h \ll 1 \) and \( \beta = 4 \) at \( h_0 \approx 0.52 \). This estimation is close to our direct numerical estimations \( h_0 \approx 0.549 \). On the other hand, the mean-field approach gives rather crude value \( h_0 = h_c(-1) = 0.69 \).

VI. CONCLUSION

In summary, we have studied the effect of the symmetry-breaking transverse magnetic field on the \( s = 1/2 \) XXZ chain. On the contrary to the longitudinal field the transverse field generates the staggered magnetization in \( Y \) direction and the gap in the spectrum of the model with the easy-plane anisotropy. Using the conformal invariance we have found the critical exponents of the field dependence of the gap and the LRO. It was shown that the spectrum of the model is gapped on the whole \( (h, \Delta) \) plane except several critical lines, where the gap and the LRO vanish. The behavior of the gap and the LRO in the vicinity of the critical lines \( \Delta = \pm 1 \) is considered on the base of the conformal field theory. We note that in the vicinity of the points \( (\Delta = 1, h = 0) \) and \( (\Delta = 1, h = 2) \) there is crossover between different regimes of the system behavior. It is shown that near the point \( (\Delta = -1, h = 0) \) the initial model can be mapped to the effective exactly solvable 1D XYZ model and has the spin-wave spectrum. The transition line \( h_c(\Delta) \) between the ordered phases and the disordered one is studied in the mean-field approximation. This study shows that this transition is similar to that in the Ising model in the transverse field. However, the behavior of the model on the transition line near the Kosterlitz-Thouless point \( (\Delta = -1, h = h_0) \) is not so clear. The mean-field approximation becomes worse at \( \Delta \to -1 \) and more sophisticated theory is needed.

We thank Prof. P.Fulde for many useful discussions. We are grateful to Max-Planck-Institut fur Physik Komplexer Systeme for kind hospitality. This work is supported under RFFR Grants No 00-03-32981 and No 00-15-97334.

[1] R.Helfrich, M.Koppen, M.Lang, F.Steglich and A.Ochiku, J.Magn.Magn.Mat. 177, 309 (1998).
[2] C.N.Yang and C.P.Yang, Phys.Rev.B 150, 321, 327 (1966).
[3] S.Lukyanov and A.Zamolodchikov, Nucl.Phys.B 493, 571 (1997); T.Hikihara and A.Furusaki, Phys.Rev.B 58, R583 (1998).
[4] A.O.Gogolin, A.A.Nersesyan, A.M.Tsvelik, Bosonization and Strongly Correlated Systems (Cambridge University Press, Cambridge, 1998).
[5] A.A.Nersesyan, A.Luther and F.V.Kusmartsev, Phys.Lett.A 176, 363 (1993).
[6] I.Affleck and M.Oshikawa, Phys.Rev.B 60, 1038 (1999).
[7] N.M.Bogoliubov, A.G.Izergin and V.E.Korepin, Nucl.Phys.B 275, 687 (1986).
[8] T.Giamarchi, J.H.Schulz, J.de Phys. 49, 819 (1988).
[9] F.C.Alcaraz and A.L.Malvezzi, J.Phys.A 28, 1521 (1995); M.Tsukano, K.Nomura, J.Phys.Soc.Jpn.67, 302 (1998).
[10] J.D.Johnson, S.Krinsky and B.M.McCoy, Phys.Rev.B 58, R583 (1998).
[11] E.Barouch and B.M.McCoy, Phys.Rev.A 3, 876 (1971).
[12] J.L.Cardy in Phase transition and critical phenomena, eds. Domb and Lebowitz, Vol.XI, (Academic, New York, 1986).