Extraction of energy from an extremal rotating electrovacuum black hole: Particle collisions in the equatorial plane

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The collisional Penrose process received much attention when Bañados, Silk and West (BSW) pointed out the possibility of test-particle collisions with arbitrarily high center-of-mass energy in the vicinity of the horizon of an extremally rotating black hole. However, the energy that can be extracted from the black hole in this promising, if simplified, scenario, called the BSW effect, turned out to be subject to unconditional upper bounds. And although such bounds were not found for the electrostatic variant of the process, this version is also astrophysically unfeasible, since it requires a maximally charged black hole. In order to deal with these deficiencies, we revisit the unified version of the BSW effect concerning collisions of charged particles in the equatorial plane of a rotating electrovacuum black hole spacetime. Performing a general analysis of energy extraction through this process, we explain in detail how the seemingly incompatible limiting cases arise. Furthermore, we demonstrate that the unconditional upper bounds on the extracted energy are absent for arbitrarily small values of the black hole electric charge. Therefore, our setup represents an intriguing simplified model for possible highly energetic processes happening around astrophysical black holes, which may spin fast but can have only a tiny electric charge induced via interaction with an external magnetic field.

I. INTRODUCTION

Penrose [1] proposed a mechanism to extract energy from a rotating vacuum black hole through a test particle disintegration in its vicinity; one fragment can escape with more energy than the energy the original particle had, if the other fragment falls inside the black hole and reduces slightly its angular momentum [2]. However, serious doubts about the practical relevance of this original variant of the Penrose process were raised early on [3, 4]. A major obstacle is the fact that the fragments need to have relative velocity of more than half the speed of light, which is very restrictive. Nevertheless, such issues can be resolved by considering more general variants of the process. A key ingredient for one of the remedies was provided by Wald [5], who realized that a rotating black hole in an external magnetic field can become charged due to selective charge accretion. Using Wald’s weak-field solution as background, particle disintegration into oppositely charged fragments was considered, and it was shown that the requirement of high relative velocity can be circumvented [6]. This generalization, or revival, as the authors put it, of the Penrose process can be also understood as a crossover between its original variant and its electrostatic version described for nonrotating black holes [7].

Another way of fixing the shortcomings of the original Penrose process is to consider particle collisions instead of decays. The required high relative velocity of the final particles can then arise naturally as a result of a high-energy collision. Interestingly, it has been noted that the relative Lorentz factor can in fact diverge in some cases, if the
collision point is taken toward the horizon. In particular, this happens for a collision between an orbiting and an infalling particle in the case of an extremal black hole [8] and also for a collision between a radially outgoing and a radially incoming particle [9]. Neither of these options seems very realistic, as both involve particles confined to the vicinity of the horizon. However, whereas the former one generically requires a white hole horizon as explained in [10], the former one has viable variants. Notably, Bañados, Silk, and West discovered its modification with both particles coming from rest at infinity [11]. This BSW effect requires a fine-tuned particle, called critical particle, which can only asymptotically approach the horizon radius, as if approaching an orbit (see, e.g., the discussion in Section IV B in [12]). For more types of near-horizon high-energy collisional processes involving orbiting particles, see, e.g., [13, 14]. A broad overview of the collisional Penrose process and the BSW effect covering many additional aspects can be found in the work of Schnittman [15].

Since the BSW effect has been derived in the test particle approximation and it relies on fine tuning and extremality of the black hole, there has been an actual concern that it may get suppressed in more realistic circumstances [16] (see also [17] for a review). But, surprisingly, it turned out that the energy extraction is quite unsatisfactory even with all the simplifying assumptions in place. Namely, it has been established almost simultaneously both by numerical [18] and analytical means [19] (see also [20]) that there is an unconditional upper bound on the extracted energy despite the center-of-mass collision energy being unbounded. Remarkably enough, Schnittman [21] discovered a scenario that is more favorable for energy extraction than the BSW effect with its precise fine tuning. A nearly critical particle, i.e., a particle with imperfect fine tuning, can turn from incoming to outgoing motion in the radial direction before colliding with another particle in the vicinity of the black hole, which is advantageous. Nevertheless, as further clarified by additional analytical studies [22, 23], the enhancement only consists in replacing one unconditional upper bound on the extracted energy with another, higher one. Let us note that for vacuum spacetimes, such limitations can be overcome, if one considers more general objects than black holes, e.g., naked singularities, see [24, 25]. For different ways to examine the original BSW effect with improved realism, see [26, 27].

Similarly to the original Penrose process, an electrostatic variant of the BSW effect exists for maximally charged, and so nonrotating, black holes [28]; it requires fine-tuned charged particles. Surprisingly, no unconditional upper bounds on the extracted energy were found in this case [29]. Given this, it is natural to ask whether similar results can be obtained in a more realistic situation with arbitrarily small black hole charge. One such possibility is the simple case of charged particles moving along the axis of symmetry of a rotating electrovacuum black hole, which was considered in [30]. Although it was confirmed that there is no upper bound on the extracted energy regardless of how small the black hole charge might be, several caveats were found to make this setup unfeasible for microscopic particles. This motivates us to turn to the more complicated case of collisions of charged particles in the equatorial plane of a rotating electrovacuum black hole. Such a crossover between the original version and the electrostatic variant of the BSW effect has been considered in [12], yet concerning only what happens before the particle collision, i.e., the approach phase of the process. In the present paper, we shall study energy extraction in this setup; let us emphasize that the key innovation in our discussion here consists in taking into account the simultaneous influence of rotation and electric charge. Our main purpose is to show that in this case there is no unconditional upper bound on the extracted energy whenever both the black hole and the escaping particles are charged. In our analysis, we draw on some additional works [31–35].

In the context of this paper, let us mention that black holes with nonnegligible electric charge have recently seen renewed interest, as they can play a role in the mechanism behind fast radio bursts (FRBs). For example, it was suggested that a merger of black holes, at least one of which has enough electric charge, can produce FRBs due to the rapidly changing magnetic dipole moment [36]. A further study [37] investigated the possible role of magnetospheric instability in producing FRBs, both for isolated Kerr-Newman black holes and for binaries. Yet another, more conventional model describes how a FRB can result from a prompt discharge of a metastable collapsed state of a Kerr-Newman black hole [38]. Formation of Kerr-Newman black holes through collapse of rotating and magnetized neutron stars has been systematically studied in [39].

The paper is organized as follows. In Sec. II we describe the properties of motion of charged test particles around an electrovacuum black hole and classify the types of motion near an extremal horizon. We also give formulas for the collision energy in the center-of-mass frame, which diverges in the horizon limit when one of the particles is critical. In Sec. III we discuss restrictions on the parameters of critical particles that can be involved in near-horizon high-energy collisions. In particular, we determine possible bounds on these parameters. In Sec. IV we perform a full analysis for energy extraction. We consider different kinematic regimes, in which particles can be produced in near-horizon high-energy collisions, and determine which ones allow the particles to escape. Then, we study bounds on parameters of the escaping particles; we put emphasis on situations in which the energy of escaping particles is not bounded. The results are derived using a general metric form, which makes them valid also for dirty black holes, i.e., those surrounded by matter. Additionally, we explain how the previously known limiting cases can be derived from the general case. In Sec. V we apply the general results to the Kerr-Newman solution so that we can highlight the whole method using relevant figures. In Sec. VI we conclude.
II. MOTION AND COLLISIONS OF CHARGED TEST PARTICLES

A. Spacetime metric and electromagnetic potential

We shall consider a general stationary, axially symmetric spacetime representing an isolated black hole, with metric \( g \) in coordinates \((t, \varphi, r, \vartheta)\) given by

\[
g = -N^2 \, dt^2 + g_{\varphi\varphi} \, (d\varphi - \omega \, dt)^2 + g_{rr} \, dr^2 + g_{\vartheta\vartheta} \, d\vartheta^2.
\] (1)

Here, \( N^2 \) is the lapse function; \( g_{\varphi\varphi}, g_{rr}, g_{\vartheta\vartheta} \) are the respective metric potentials; and \( \omega \) is the dragging potential. We assume \( g_{\varphi\varphi} > 0 \), and also that the product \( N \sqrt{g_{rr}} \) is finite and nonvanishing even for \( N \to 0 \).

Let us further assume that our spacetime is permeated by a Maxwell field obeying the same symmetry as the metric (1). We fix the gauge for its potential \( A \) to manifest this symmetry, namely, \( A = A_t \, dt + A_\varphi \, d\varphi \), or rearranging, \( A = -\phi \, dt + A_\varphi \, (d\varphi - \omega \, dt) \). (2)

The component \( \phi = -A_t - \omega A_\varphi \) (3) is called the generalized electrostatic potential.

B. General equations of equatorial motion

Let us now consider the motion of test particles with rest mass \( m \) and electric charge \( q \) in the spacetime defined in Eq. (1). Because of the two symmetries that we assumed, there exist two quantities that are conserved during the electrogeodesic motion. They are the energy \( E \) and the axial angular momentum \( L_z \) of the test particle. We also assume that the metric (1) and the electromagnetic field are symmetric with respect to the reflections \( \vartheta \to \pi - \vartheta \). Then, we can consider motion confined to the invariant hypersurface \( \vartheta = \pi/2 \), the equatorial plane. For equatorial particles, \( L_z \) is the total angular momentum; hence, we can drop the subscript and write \( L \equiv L_z \). The energy and the axial angular momentum are given by

\[
E = -p_t - qA_t, \quad L = p_\varphi + qA_\varphi,
\] (4)

where \( p_t \) and \( p_\varphi \) are the time and azimuthal components of the particle’s 4-momentum \( p_\alpha \).

Defining two auxiliary functions \( \mathcal{X} \) and \( \mathcal{Z} \) by

\[
\mathcal{X} = E - \omega L - q\phi, \quad \mathcal{Z} = \sqrt{\mathcal{X}^2 - N^2 \left[ m^2 + \frac{(L - qA_\varphi)^2}{g_{\varphi\varphi}} \right]},
\] (5)

we can write the contravariant components of the particle’s 4-momentum \( p_\alpha \) in a compact form:

\[
p^t = \frac{\mathcal{X}}{N^2}, \quad p^\varphi = \frac{\omega \mathcal{X}}{N^2} + \frac{L - qA_\varphi}{g_{\varphi\varphi}}, \quad p^r = \frac{\sigma \mathcal{Z}}{N \sqrt{g_{rr}}},
\] (6)

The parameter \( \sigma \) has values \( \sigma = \pm 1 \) which determine the direction of the radial motion. In order for the motion to be allowed, the quantity \( \mathcal{Z} \) has to be real. Outside of the black hole, where \( N^2 > 0 \), the condition \( \mathcal{Z}^2 > 0 \) can be equivalently stated as \( |\mathcal{X}| \geq N \sqrt{m^2 + \frac{(L - qA_\varphi)^2}{g_{\varphi\varphi}}} \). It can be seen that there are two disjoint domains of allowed motion, one with \( \mathcal{X} > 0 \) and the other with \( \mathcal{X} < 0 \). These two domains touch for \( N \to 0 \), where \( \mathcal{X} \to 0 \) becomes possible. However, to preserve causality, we need to enforce \( p^t > 0 \), and thus we restrict to the \( \mathcal{X} > 0 \) domain. Then, the requirement for the motion to be allowed becomes

\[
\mathcal{X} \geq N \sqrt{m^2 + \frac{(L - qA_\varphi)^2}{g_{\varphi\varphi}}}.
\] (7)

The lower bound, i.e., the equality of Eq. (7), \( \mathcal{X} = N \sqrt{m^2 + \frac{(L - qA_\varphi)^2}{g_{\varphi\varphi}}} \), is the condition for a turning point.
The number of relevant parameters can be reduced depending on whether the particle in question is massive or massless. Kinematics of massive particles is determined by three parameters: specific energy $\varepsilon \equiv \frac{E}{m}$, specific angular momentum $L \equiv \frac{L}{m}$, and specific charge $\tilde{q} \equiv \frac{q}{m}$. Kinematics of massless particles, which are electrically neutral, e.g., photons, is characterized solely by the impact parameter defined as $b \equiv \frac{L}{\varepsilon \tilde{q}}$. Based on the distinction between massive and massless, additional features of the motion, like the existence of circular orbits, can be deduced. Effective potentials are frequently employed, both for massive particles and for massless particles (see [12] and references therein).

C. Near-horizon expansions

We wish to study collisions of particles near the black hole horizon, where $N \to 0$. Let us denote the values of the various quantities on the outer black hole horizon by a subscript or a superscript $H$ depending on the convenience. As we consider solely equatorial motion, all quantities in the following are understood to be evaluated at various quantities on the outer black hole horizon by a subscript or a superscript $H$ depending on the convenience. As for the massive particle case.

In the vicinity of the horizon, we can perform expansions in variable $(r - r_H) \ll r_H$. We are interested in extremal black holes; the horizon located at $r_H$ is understood to be degenerate hereafter. Let us expand the dragging potential $\omega$ and the generalized electrostatic potential $\phi$ in first order as follows,

$$\omega = \omega_H + \hat{\omega}(r - r_H) + \ldots, \quad \phi = \phi_H + \hat{\phi}(r - r_H) + \ldots,$$

respectively, and where $\hat{\omega} = \frac{\partial \omega}{\partial r} \big|_{r=r_H}$ and $\hat{\phi} = \frac{\partial \phi}{\partial q} \big|_{r=r_H}$. For extremal black holes, we can also renormalize the lapse function as $N^2 = (r - r_H)^2 N^2$, which leads, in particular, to

$$N^2_H = \left. \frac{1}{2} \frac{\partial^2 N^2}{\partial r^2} \right|_{r=r_H}.$$

Finally, let us introduce a new set of constants $X_H$, $\chi$, and $\lambda$, which are preserved throughout the motion and useful to describe the kinematics of particles close to $r_H$. They are defined in terms of $E, L, q$ as follows:

$$X_H = E - \omega_H L - q\phi_H, \quad \chi = -\hat{\omega} L - q\hat{\phi}, \quad \lambda \equiv p_H^H = L - qA^H_\phi.$$

Note that the parameter $\lambda$ is now defined in a slightly different way than in the previous paper [12] on charged particle collisions. The formulas given here can be recast into the convention used in [12] by putting $\lambda \to -m\lambda A^H_\phi$.

Two of the new parameters, namely, $X_H$ and $\chi$, are expansion coefficients of the function $X$ given in Eq. (5), i.e.,

$$X \approx X_H + \chi(r - r_H) + \ldots.$$

The parameters $E, L, q$ can be expressed in terms of the new ones through inverse relations

$$E = X_H + \left( \frac{\omega_H \hat{\phi} - \hat{\omega} \phi_H}{\phi + \hat{\omega} A^H_\phi} \right) \lambda + \chi A^H_\phi,$$

$$L = \hat{\phi} \lambda - \chi A^H_\phi,$$

$$q = -\frac{\omega \lambda + \chi}{\phi + \hat{\omega} A^H_\phi},$$

which all contain the same expression in the denominator. Hence, when it vanishes, i.e.,

$$\hat{\phi} + \hat{\omega} A^H_\phi = 0,$$

there is clearly a problem with the definitions given in Eq. (11). Indeed, if Eq. (13) holds, $\chi$ and $\lambda$ become proportional to each other, $\chi = -\hat{\omega} \lambda$, and thus the variables $X_H, \chi, \lambda$ no longer span the whole parameter space. When this degeneracy happens, we can use $X_H, \lambda, q$ as our alternative set of parameters. Then, the inverse relations to express $E, L$ become

$$E = X_H + \omega_H \lambda - qA^H_\phi,$$

$$L = \lambda + qA^H_\phi.$$

The behavior of particles close to the horizon radius $r_H$ depends significantly on the value of $X_H$. For particles with $X_H < 0$, the condition (7) is necessarily violated near the horizon, and thus the particles cannot get arbitrarily close to $r_H$. On the other hand, particles with $X_H \geq 0$ can exist arbitrarily close to $r_H$. Let us discuss these types now.
D. Types of particles close to \( r_H \)

1. Usual (subcritical) particles

Particles with \( \chi_H > 0 \) are bound to fall into the black hole if they move inward and get near the horizon. In our discussion, we will refer to those particles as usual particles. Let us emphasize that we will not consider outgoing usual particles in the vicinity of \( r_H \), since it can be shown that such particles cannot be produced in (generic) near-horizon collisions; see [31]. In our analysis, we exclude the white hole region from which outgoing usual particles could naturally emerge. For usual particles approaching \( r_H \), the function \( Z \) of Eq. (5) can be expanded in terms of \( N^2 \), and consequently of \( \chi \), as follows:

\[
Z \approx \chi - \frac{N^2}{2\chi} \left[ m^2 + \frac{(L - qA_{\varphi})^2}{g_{\varphi\varphi}} \right] + \ldots
\]  

(15)

2. Critical particles, especially class I critical particles

We can also consider particles with \( \chi_H = 0 \), which are called critical. They are fine-tuned to be on the verge between not being able to reach the horizon and falling into the black hole. Here, we use the local notion of critical particles. For asymptotically flat spacetimes, it is also possible to define the critical particles globally, such that they are on the edge of being able to approach the horizon from infinity; see [11]. By the definition given in Eq. (10), condition \( \chi_H = 0 \) can be understood also as a constraint for parameters \( E, L, q \):

\[
E - \omega_H L - q\phi_H = 0.
\]  

(16)

The expansion around \( r_H \) of the function \( Z \) introduced in Eq. (5) looks rather different for critical particles,

\[
Z \approx \sqrt{\chi^2 - N_H^2 \left( m^2 + \frac{\lambda^2}{g_{\varphi\varphi}} \right) (r - r_H)} + \ldots
\]  

(17)

Let us emphasize that with \( \chi_H = 0 \) the causality condition \( p^t > 0 \) necessarily implies \( \chi > 0 \).

It can be shown that critical particles cannot approach the horizon unless the black hole is extremal (see, e.g., [12, 32] and references therein). Harada and Kimura [34] distinguished several subtypes of critical particles, out of which we consider chiefly the class I critical particles. The approximate trajectory of an incoming class I critical particle near \( r_H \) has the form \( r = r_H \left[ 1 + \exp \left( -\frac{\tau}{\tau_{\text{relax}}} \right) \right] \), where \( \tau \) is the proper time and \( \tau_{\text{relax}} \) is a positive constant; see [12] for details. Since critical particles of any type can never reach \( r_H \), any collisional process involving them will thus happen at some radius \( r_C > r_H \). Therefore, it makes sense to consider particles that behave approximately as critical at a given radius.

3. Nearly critical particles

A particle will behave approximately as critical at a radius \( r_C \), if the zeroth-order term in the expansion of \( \chi \) is of comparable magnitude as the first-order term. To quantify this, let us define a formal expansion,

\[
\chi_H \approx -C \left( r_C - r_H \right) - D \left( r_C - r_H \right)^2 + \ldots,
\]  

(18)

where the minus sign in front of the terms follows the usual convention. \( C, D, \) and so on, are constants that are needed for consistency of momentum conservation law at each expansion order. However, here, we are interested only in the first order, and so the constant \( C \) is enough for our purposes.

For nearly critical particles, the expansion (11) evaluated at \( r_C \) can be recast using Eq. (13) as

\[
\chi \approx (\chi - C) \left( r_C - r_H \right) + \ldots
\]  

(19)

Similarly, the expansion of function \( Z \) defined in Eq. (5) reads for them

\[
Z \approx \sqrt{(\chi - C)^2 - N_H^2 \left( m^2 + \frac{\lambda^2}{g_{\varphi\varphi}} \right) (r_C - r_H)} + \ldots
\]  

(20)
Nearly critical particles with $C > 0$ cannot fall into the black hole, and they must have a turning point at some radius smaller than $r_C$. Therefore, it makes sense to study collisional processes near the horizon involving also outgoing nearly critical particles. Furthermore, for particles with $\chi \gg C > 0$, we can neglect $C$ and treat them as precisely critical around $r_C$. Thus, we can consider outgoing critical particles, too.

4. Class II critical particles and class II nearly critical particles

There exist values of parameters of critical particles or nearly critical particles, for which the leading-order coefficient in the expansion (17) (or (20)) vanishes. The new leading order then becomes

$$ Z \sim (r_C - r_H)^{\frac{3}{2}} $$

(21)
or higher. These are the so-called class II critical particles or nearly critical particles.

Kinematics of class II critical particles represents an interesting theoretical issue, which, however, involves technical complications; cf. [12]. Moreover, since class II critical particles require fine tuning of two parameters, instead of just one, they are much less important for practical considerations. Thus, we will mostly omit details regarding class II critical particles in the following.

E. BSW effect and its Schnittman variant

We have seen that in the near-horizon region of an extremal black hole two distinct types of motion do coexist. Whereas usual particles with $X_H > 0$ cross $r_H$ and fall into the black hole, critical particles with $X_H = 0$ can only approach $r_H$ asymptotically. This leads to a divergent relative Lorentz factor due to the relative velocity between the two types of motion approaching the speed of light.

Hence, the expression for the collision energy in the center-of-mass frame,

$$ E_{CM}^2 = m_1^2 + m_2^2 - 2g_{\mu\nu}p_1^\mu p_2^\nu, $$

(22)

will be dominated by the scalar-product term, if we consider near-horizon collisions between critical and usual particles. In particular, inserting (6) for a critical particle labeled 1 and for a usual particle labeled 2 and using Eqs. (11), (15), and (17), we find that the leading-order contribution of Eq. (22) is

$$ E_{CM}^2 \approx \frac{\lambda_1^2}{r_C - r_H} \left\{ \frac{2}{\lambda_2^2} \left[ \chi_1 + \sigma_1 \sqrt{\chi_1^2 - \Lambda_H^2} \left( m_1^2 + \frac{\lambda_1^2}{g_{\phi\phi}} \right) \right] \right\}, $$

(23)

see [12] for details. A process with an incoming critical particle ($\sigma_1 = -1$) is called BSW type after Bañados, Silk, and West [11], whereas the one with reflected (nearly) critical particle ($\sigma_1 = +1$) is called Schnittman type [21]. Note that the usual particle is always incoming, i.e., $\sigma_2 = -1$. We used the aforementioned approximation $\chi_1 \gg C_1 > 0$ for the Schnittman process.

III. KINEMATICS OF PARTICLES BEFORE COLLISION

A. Admissible region in the parameter space

1. General considerations

Critical particles are the key ingredient of certain high-energy collisional processes in extremal black hole spacetimes. Nevertheless, the parameters of critical particles that can act in such processes are restricted, since the requirement of Eq. (7) must be fulfilled all the way from the point of inception to the point of collision.

Let us disregard the concern about where the critical particle originated and focus instead on the point of collision at radius $r_C$. Since we want $r_C$ very close to $r_H$, the minimum requirement is that there must be some neighborhood of $r_H$, where condition (7) is satisfied. Using a linear approximation in $r - r_H$, we get the following inequality,

$$ \chi > \Lambda_H \sqrt{m^2 + \frac{\lambda^2}{g_{\phi\phi}}}, $$

(24)
which defines the admissible region of parameters. Conversely, for parameters satisfying the inequality opposite to Eq. (24), condition (7) will be violated in some neighborhood of \( r_{H} \).

Now, the equality

\[
\chi = N_{H} \sqrt{m^{2} + \frac{\lambda^{2}}{g_{\varphi \varphi}^{H}}} \tag{25}
\]

corresponds to the breakdown of the linear approximation of Eq. (7). Comparing with Eq. (17), we see that Eq. (25) also implies the critical particles to be class II. Higher-order expansion terms are needed to decide whether motion of class II critical particles is allowed close to \( r_{H} \). (We note that such higher-order kinematic restrictions were worked out in Sections IV E and V B in [12] and that additional information on this subject can be found in Sec. II D and footnote 2 in [31] and in Sec. VII in [33].)

Now, let us consider the physical interpretation of the admissible region of parameters. In particular, we would like to distinguish different variants of the collisional processes corresponding to the previously known limiting cases. For extremal vacuum black holes, only critical particles corotating with the black hole can participate in the high-energy collisions, whereas for the nonrotating extremal black holes, the critical particles need to have the same sign of charge as the black hole. In order to identify counterparts of these limiting variants, which we will call centrifugal mechanism and electrostatic mechanism, we need to assess how to define the direction in which a charged particle orbits.

The momentum component \( p_{\varphi} \) determines the direction of motion in \( \varphi \) with respect to a locally nonrotating observer (cf. [3]). For uncharged particles, \( p_{\varphi} = L \) is constant, and thus the distinction is universal and unambiguous. Nevertheless, for charged particles, \( p_{\varphi} \) depends on \( r \) through the \( qA_{\varphi} \) term. Therefore, we essentially need to compare values of \( p_{\varphi} \) at some reference radius. A first straightforward choice would be to use \( \lambda \equiv p_{\varphi}^{H} \); see Eq. (10). However, it is clear from Eq. (25) that one can find points with any value of \( \lambda \) in the admissible region (whereas values of \( \chi \) in the admissible region are bounded from below by \( \chi \geq N_{H} m \)). Apart from the degenerate case \([13]\), no kinematic restriction on \( \lambda \) is thus possible. Hence, basing the definition of the centrifugal mechanism on \( \lambda \) would lead to a trivial result.

A second possible choice is \( L \). What is the justification to use \( L \)? We shall consider a region of our spacetime, where the influence of the dragging and of the magnetic field is insignificant, for example, a far zone of an asymptotically flat spacetime. More precisely, let us consider a region where \( \omega \) and \( A_{\varphi} \) are negligible, and thus \( p_{\varphi} \approx p^{H}_{\varphi} \approx L \). Then, it readily follows that in such a region particles with \( L = 0 \) move along trajectories of (approximately) constant \( \varphi \) and, conversely, particles with different signs of \( L \) orbit in different directions therein. Hence, we can say that \( L \) uniquely distinguishes the direction of motion in \( \varphi \) of a particle before it came under the influence of the dragging and of the magnetic field near the black hole.

We conclude that we need to view the admissible region through the parameters \( L \) and \( q \) for physical interpretation. Similarly to [12], let us focus on Eq. (25) of the border of the admissible region. Substituting relations (10) for \( \chi \) and \( \lambda \) into Eq. (25) does not generally lead to a single-valued functional dependence between \( q, L \). We circumvent this issue by plugging the condition (25) into relations (12), which yields parametric expressions for the border:

\[
q = -\frac{\omega A_{\varphi}^{H}}{\phi + \omega A_{\varphi}^{H}} \sqrt{m^{2} + \frac{\lambda^{2}}{g_{\varphi \varphi}^{H}}} + \frac{1}{\phi + \omega A_{\varphi}^{H}} \hat{\lambda} \tag{26}
\]

\[
L = \frac{\lambda}{\phi + \omega A_{\varphi}^{H}} \sqrt{m^{2} + \frac{\lambda^{2}}{g_{\varphi \varphi}^{H}}} \tag{27}
\]

\[
E = \frac{\left( \hat{\phi} - \omega \phi H \right) \lambda + N_{H} A_{\varphi}^{H}}{\phi + \omega A_{\varphi}^{H}} \sqrt{m^{2} + \frac{\lambda^{2}}{g_{\varphi \varphi}^{H}}} \tag{28}
\]

Since we are dealing with critical particles, the three expressions are not independent. In [12], different possibilities were distinguished by studying restrictions on signs of \( q \) and \( L \) in the admissible region; in particular, the centrifugal mechanism was identified as the case when only the sign of \( L \) is restricted, and the electrostatic mechanism was identified as the case when only the sign of \( q \) is restricted. Here, we employ a complementary, deeper approach and determine the precise bounds on \( q, L, \) and \( E \).

2. Bounds on parameters

Bounds on values of \( q, L, \) and \( E \) in the admissible region will appear as extrema of expressions (26)–(28) with respect to \( \lambda \).
Let us start with the electric charge \( q \). From Eq. (26), we find that a stationary point can occur at the following value of \( \lambda \):

\[
\lambda = -m \frac{g^H_{\varphi\varphi} \dot{\omega}}{\sqrt{\mathcal{N}_H^2 - g^H_{\varphi\varphi} \omega^2}}. \tag{29}
\]

Due to the square root in the denominator, we need to distinguish three possibilities.

**Case 1a:** If Eq. (29) is imaginary, Eq. (26) will take all real values, and hence there is no bound on \( q \).

**Case 1b:** If Eq. (29) is real, it will correspond to an extremum of Eq. (26) with value

\[
q_b = -\frac{m}{\dot{\phi} + \dot{\omega} A^H_{\varphi}} \sqrt{\mathcal{N}_H^2 - g^H_{\varphi\varphi} \omega^2}, \tag{30}
\]

which serves as a bound for \( q \). For Eqs. (27) and (28), the following values of \( L \) and \( E \) will be implied by Eq. (29):

\[
L = -\frac{m A^H_{\varphi}}{\dot{\phi} + \dot{\omega} A^H_{\varphi}} \sqrt{\mathcal{N}_H^2 - g^H_{\varphi\varphi} \omega^2}, \tag{31}
\]

\[
E = \frac{m A^H_{\varphi}}{\dot{\phi} + \dot{\omega} A^H_{\varphi}} \sqrt{\mathcal{N}_H^2 - g^H_{\varphi\varphi} \omega^2} \left( \omega_H \hat{\phi} - \dot{\omega} \phi_H \right). \tag{32}
\]

Looking at the \(| \lambda | \to \infty \) behavior of Eq. (26), one can deduce that

\[
\dot{\phi} + \dot{\omega} A^H_{\varphi} < 0 \tag{33}
\]

corresponds to Eq. (30) being a lower bound, whereas if the opposite inequality is satisfied, Eq. (30) will be an upper bound.

**Case 1c:** If the expression under the square root in Eq. (29) is zero (and Eq. (29) is thus an invalid expression), the values of charge in the admissible region will be bounded by \( q_b = 0 \). However, \( q = 0 \) cannot be attained for any finite value of other parameters on the border.

Let us turn to the angular momentum \( L \). From Eq. (27), we find that a value of \( \lambda \) for a candidate stationary point is

\[
\lambda = m \frac{g^H_{\varphi\varphi} \dot{\phi} \text{sgn} A^H_{\varphi}}{\sqrt{\mathcal{N}_H^2 (A^H_{\varphi})^2 - g^H_{\varphi\varphi} \dot{\phi}^2}}. \tag{34}
\]

Again, there are three possible cases.

**Case 2a:** If Eq. (34) is imaginary, there is no bound on \( L \) in the admissible region.

**Case 2b:** If Eq. (34) is real, it will correspond to an extremum of Eq. (27) with value

\[
L_b = -\frac{m \text{sgn} A^H_{\varphi}}{\dot{\phi} + \dot{\omega} A^H_{\varphi}} \sqrt{\mathcal{N}_H^2 (A^H_{\varphi})^2 - g^H_{\varphi\varphi} \dot{\phi}^2}, \tag{35}
\]

which serves as a bound for \( L \). For Eqs. (26) and (28), the following values of \( q \) and \( E \) will be implied by Eq. (34):

\[
q = -\frac{m \text{sgn} A^H_{\varphi}}{\dot{\phi} + \dot{\omega} A^H_{\varphi}} \sqrt{\mathcal{N}_H^2 (A^H_{\varphi})^2 - g^H_{\varphi\varphi} \dot{\phi}^2}, \tag{36}
\]

\[
E = \frac{m \text{sgn} A^H_{\varphi}}{\dot{\phi} + \dot{\omega} A^H_{\varphi}} \sqrt{\mathcal{N}_H^2 (A^H_{\varphi})^2 - g^H_{\varphi\varphi} \dot{\phi}^2} \left( \omega_H \hat{\phi} - \dot{\omega} \phi_H \right). \tag{37}
\]

From the \(| \lambda | \to \infty \) behavior of Eq. (27), we can infer that Eq. (35) is a lower bound, if

\[
\frac{A^H_{\varphi}}{\dot{\phi} + \dot{\omega} A^H_{\varphi}} < 0. \tag{38}
\]
When the opposite inequality is satisfied, Eq. (39) is an upper bound. Case 2c: If Eq. (44) is undefined due to the expression in the denominator being zero, the values of \( L \) in the admissible region will be bounded by \( L_h = 0 \), and this value cannot be reached for a finite value of other parameters on the border.

Combining the possibilities together, we can conclude that cases 1a2b and 1a2c correspond to the centrifugal mechanism, whereas variants 1b2a and 1c2a correspond to the electrostatic mechanism. Case 1a2a signifies the coexistence of both. Note that the combination of signs of \( q \) and \( L \) leading to \( \chi < 0 \) is excluded in any case. The other possible combinations, i.e., 1b2b and 1c2c, do not correspond to any simpler limiting cases.

Let us turn to the energy \( E \) to finish the discussion of bounds on parameters. From Eq. (28), we find that a value of \( \lambda \) for a candidate stationary point is

\[
\lambda = -m \frac{g^H_{\phi \phi} \left( \omega_H \hat{\phi} - \hat{\omega} \phi_H \right) \text{sgn} A^H_i}{\sqrt{\mathcal{N}^2_{H} \left( A^H_i \right)^2 - g^H_{\phi \phi} \left( \omega_H \hat{\phi} - \hat{\omega} \phi_H \right)^2}},
\]

Unlike in the previous two cases, this value can be adjusted using the available gauge freedom. Consequently, we can choose Eq. (39) to be real, as explained below. Furthermore, it turns out that we can also choose the corresponding stationary point of Eq. (28) to be a minimum. Its value is

\[
E_{\text{min}} = \frac{m \text{sgn} A^H_i}{\phi + \hat{\omega} A^H_\phi} \sqrt{\mathcal{N}^2_{H} \left( A^H_i \right)^2 - g^H_{\phi \phi} \left( \omega_H \hat{\phi} - \hat{\omega} \phi_H \right)^2},
\]

and for Eqs. (26) and (27), the following values of \( q \) and \( L \) are implied by Eq. (39):

\[
q = \frac{-m \text{sgn} A^H_i}{\phi + \hat{\omega} A^H_\phi} \frac{\mathcal{N}^2_{H} A^H_i - g^H_{\phi \phi} \left( \omega_H \hat{\phi} - \hat{\omega} \phi_H \right)}{\sqrt{\mathcal{N}^2_{H} \left( A^H_i \right)^2 - g^H_{\phi \phi} \left( \omega_H \hat{\phi} - \hat{\omega} \phi_H \right)^2}},
\]

\[
L = \frac{-m \text{sgn} A^H_i}{\phi + \hat{\omega} A^H_\phi} \frac{\mathcal{N}^2_{H} A^H_i A^H_\phi + g^H_{\phi \phi} \left( \omega_H \hat{\phi} - \hat{\omega} \phi_H \right)}{\sqrt{\mathcal{N}^2_{H} \left( A^H_i \right)^2 - g^H_{\phi \phi} \left( \omega_H \hat{\phi} - \hat{\omega} \phi_H \right)^2}}.
\]

What are the requirements in order to have a lower bound on \( E \) in the admissible region, and is it always possible to make these requirements satisfied simultaneously? First, we have to impose the condition

\[
\mathcal{N}^2_{H} \left( A^H_i \right)^2 - g^H_{\phi \phi} \left( \omega_H \hat{\phi} - \hat{\omega} \phi_H \right)^2 > 0
\]

(43) to make Eq. (39) real. By checking the \(|\lambda| \to \infty\) behavior of Eq. (28), we can see that we must also require

\[
\frac{A^H_i}{\phi + \hat{\omega} A^H_\phi} > 0
\]

(44), in order for Eq. (44) to be a lower bound. Next, recalling (44), one can observe that the combinations \( A^H_\phi \equiv -\phi_H - \omega_H A^H_\phi \) and \( \omega_H \hat{\phi} - \hat{\omega} \phi_H \) are linearly independent, except for the degenerate case when Eq. (13) holds, which has to be treated separately anyway. Therefore, there is always a way to choose values of \( \phi_H \) and \( \omega_H \) that make any of the conditions given in Eqs. (13) and (14) satisfied or violated.

3. Additional remarks

Above, we have identified points on the border of the admissible region where a minimal or maximal value of one of the parameters \( q \), \( L \), and \( E \) is reached. Values of all the parameters at such points are proportional to the particle’s mass. This illustrates the fact that only a reduced set of parameters is needed to describe particles’ kinematics. In particular, for massive critical particles, two parameters are sufficient. These can be either \( \chi = \frac{m}{\bar{m}} \) and \( \lambda = \frac{\bar{m}}{m} \) or any two of \( \hat{q} \), \( \hat{l} \), and \( \varepsilon \). Therefore, we can understand the admissible region given in Eq. (27) as an area in a two-dimensional parameter space. Its border given in Eq. (28) can be viewed as a curve therein, namely, a branch of a hyperbola with
axes $\tilde{\chi} = 0 \text{ and } \tilde{\lambda} = 0 \text{ and with its vertex on } \tilde{\lambda} = 0$. Let us note that in variables $\tilde{q}$ and $l$ the asymptotes of this hyperbola can be expressed as

$$l = -\tilde{q} \frac{\sqrt{g^H_{\phi\phi}} \phi \pm N^H \phi}{\sqrt{g^H_{\phi\phi}} \phi} ; \quad (45)$$

see Eq. (43) in [12].

Considering the tilde parameters, i.e., the parameters normalized to unit rest mass, one excludes a priori critical photons. However, this is not a big issue, since they have trivial kinematics. Indeed, all critical photons share the same single value of impact parameter $b_{cr} = \frac{1}{\omega H}$. Therefore, the parameter space of critical photons is effectively zero dimensional, and their kinematics depends only on the properties of the spacetime itself. When are the critical photons able to approach $r_H$? The expansion given in Eq. (17) reads for those critical photons

$$Z \approx |L| \sqrt{\hat{\omega}^2 - N^2 H - g^H_{\phi\phi} (r - r_H) + \ldots} \quad (46)$$

Here, the expression under the square root is proportional to the one in Eq. (29) with a negative factor. Therefore, critical photons can be involved in the high-energy collisional processes close to $r_H$ only in the case 1a.

Finally, let us clarify the link between bounds on $q, L, E$ and restrictions on signs of those parameters. Starting with $q$, we can observe that condition (33) also determines the sign of Eq. (30). Thus, if Eq. (30) is a lower bound, its value is positive, whereas if it is an upper bound, its value is negative. Therefore, whenever values of $q$ in the admissible region are bounded by Eq. (30), they must also all have the same sign. An identical relation holds between Eq. (35) and (38). Last, the gauge condition (44), which we use to enforce a lower bound on energy, also implies that the bound given in Eq. (40) has a positive value. Hence, dividing by $m$ and rearranging the sign factors, we can express the (possible) bounds on values of $\tilde{q}, l$, and $\varepsilon$ in the admissible region as follows:

$$-\tilde{q} \text{sgn} \left( \hat{\phi} + \hat{\omega} A^H_{\phi} \right) > \frac{1}{\phi + \hat{\omega} A^H_{\phi}} \sqrt{N^2 H - g^H_{\phi\phi}} \hat{\omega}^2 , \quad (47)$$

$$-l \text{sgn} \left[ \hat{\phi} + \hat{\omega} A^H_{\phi} \right] A^H_{\phi} > \frac{1}{\phi + \hat{\omega} A^H_{\phi}} \sqrt{N^2 H \left( A^H_{\phi} \right)^2 - g^H_{\phi\phi}} \hat{\omega}^2 , \quad (48)$$

$$\varepsilon > \frac{1}{\phi + \hat{\omega} A^H_{\phi}} \sqrt{N^2 H \left( A^H_{\phi} \right)^2 - g^H_{\phi\phi} \left( \hat{\omega} - \hat{\omega}_H \phi - \hat{\omega}_H \phi \right)} \phi \quad . \quad (49)$$

### B. Degenerate case

Let us now explore the previously excluded case when the degeneracy condition (13) is satisfied. As we noted in Sec. [II C] this means that the variables $\chi$ and $\lambda$ become proportional, namely, $\chi = -\tilde{\omega} \lambda$. Because of this, Eq. (29) with Eq. (13) degenerates into an algebraic equation for one variable, which has a single solution,

$$\lambda = -m \frac{\text{sgn} \hat{\omega}}{\sqrt{N^2 H - g^H_{\phi\phi}}} . \quad (50)$$

One can see that the expressions under the square roots in the denominators of Eq. (29) and Eq. (50) are related by a negative factor. Therefore, Eq. (50) is defined in real numbers in case 1a. On the other hand, in cases 1b and 1c, there is no real solution of Eq. (29) with Eq. (13), and thus the collisional processes studied here are impossible for critical particles with any value of $\lambda$. We are unaware of a black hole spacetime where this would occur in the equatorial plane. However, a similar thing happens around the poles of the Kerr solution, as demonstrated by [34]. Note also that $\lambda \text{sgn} \hat{\omega} > 0$ certainly violates Eq. (29) with Eq. (13).

Let us now consider the physical interpretation of the degenerate case given by Eq. (50). Using Eq. (14) (with
\( \chi_H = 0 \), we find that Eq. (50) can be expressed as

\[
L = - \frac{m \text{sgn} \tilde{\omega}}{\sqrt{\frac{\tilde{\omega}^2}{N_0^H} - \frac{1}{g_{\phi \phi}}} } + q A_t^H ,
\]

(51)

\[
E = - \frac{m \omega_H \text{sgn} \tilde{\omega}}{\sqrt{\frac{\tilde{\omega}^2}{N_0^H} - \frac{1}{g_{\phi \phi}}} } - q A_t^H .
\]

(52)

The charge of the particle plays a role of a free variable; there can never be a bound on the values of \( q \) in the admissible region in the degenerate case. However, if \( A_t^H = 0 \), Eq. (51) will correspond to a single value of \( L \), which will constitute a bound on \( L \). If Eq. (13) holds together with \( A_t^H = 0 \), we have \(- \text{sgn}(\tilde{\omega} L) = \text{sgn} \chi > 0\), and therefore we can infer

\[
- l \text{sgn} \tilde{\omega} > \frac{1}{\sqrt{\frac{\tilde{\omega}^2}{N_0^H} - \frac{1}{g_{\phi \phi}}} } .
\]

(53)

Note that Eq. (13) with \( A_t^H = 0 \) implies \( \tilde{\phi} = 0 \). This would signify case \( 2c \) as defined in general (cf. [12]), yet the bound on \( l \) has a nonzero value in this case. We want a lower bound on \( E \) in the admissible region, and therefore we impose gauge conditions \( A_t^H = 0 \) and \( \omega_H \text{sgn} \tilde{\omega} < 0 \) in Eq. (52). Then, it holds that

\[
\varepsilon > \frac{|\omega_H|}{\sqrt{\frac{\tilde{\omega}^2}{N_0^H} - \frac{1}{g_{\phi \phi}}} } .
\]

(54)

### IV. ENERGY EXTRACTION

#### A. Conservation laws and kinematic regimes

1. **Conservation laws**

Now, we discuss the properties of particles than can be produced in the high-energy collisional processes described in Sec. [11] and, in particular, how much energy such particles can extract from a black hole. Let us consider a simple setup in which a critical particle or a nearly critical particle, call it particle 1, and an incoming usual particle, call it particle 2, collide close to the horizon radius \( r_H \), and their interaction leads to production of just two new particles, particle 3 and particle 4. We assume conservation of charge

\[
q_1 + q_2 = q_3 + q_4 ,
\]

(55)

and also the conservation of 4-momentum at the point of collision. From the azimuthal component of the 4-momentum, see Eq. (1), we infer the conservation of angular momentum

\[
L_1 + L_2 = L_3 + L_4 .
\]

(56)

The conservation law for the time component of the 4-momentum can be used to derive the conservation of energy

\[
E_1 + E_2 = E_3 + E_4 .
\]

(57)

There is another conserved 4-momentum component, the radial one \( p^r \). Instead of writing down the conservation of \( p^r \), we combine it together with the conservation of the time component \( p^t \) of the 4-momentum. For that, we use the combinations \( N^2 p^t \mp N \sqrt{g_{rr}} p^r \), which is advantageous because they lead to combinations of the functions \( X \) and \( Z \), both defined in Eq. [5]. Indeed,

\[
N^2 p^t \mp N \sqrt{g_{rr}} p^r = X \mp \sigma Z .
\]

(58)

Since we assumed that particle 2 is incoming, \( \sigma_2 = -1 \), the summation of the conservation laws leads to the following equation:

\[
X_1 \mp \sigma_1 Z_1 + X_2 \pm Z_2 = X_3 \mp \sigma_3 Z_3 + X_4 \mp \sigma_4 Z_4 .
\]

(59)
Let us find the leading-order terms in Eq. (60). For usual particles, $\mathcal{X}$ and $\mathcal{Z}$ differ by a term proportional to $N^2$ (see Eq. (15)), so their combinations with different signs have different leading orders in expansion around $r_H$.

$$\mathcal{X} - \mathcal{Z} \sim (r - r_H)^2, \quad \mathcal{X} + \mathcal{Z} \approx 2\mathcal{X}_H. \quad (60)$$

On the other hand, in the case of critical particles, or nearly critical particles, the leading order of expansion in $r_C - r_H$ for both combinations is

$$\mathcal{X} - \mathcal{Z} \sim (r_C - r_H). \quad (61)$$

Let us start to analyze Eq. (59) with its upper sign. We assumed particle 2 to be a usual particle, and thus the leading order is the zeroth one, so

$$2\lambda^H_2 = \lambda^H_3 - \sigma_2 \lambda^H_3 + \lambda^H_4 - \sigma_4 \lambda^H_4. \quad (62)$$

This equation can be satisfied only when one of the final particles, say 4, is usual and incoming, i.e., $\lambda^H_4 > 0$, $\sigma_4 = -1$. Let us now turn to Eq. (59) with its lower sign. We see that usual incoming particles 2 and 4 will make no contribution to zeroth order and first order. On the other hand, critical particle 1 will contribute to the first order, and this contribution will dominate the left-hand side. Therefore, the expansion of the right-hand side must also be dominated by a first-order contribution, which means that particle 3 has to be critical or nearly critical. The leading order of Eq. (59) with the lower sign thus becomes

$$\chi_1 + \sigma_1 \sqrt{\chi_1^2 - \mathcal{N}^H_2 \left( m_1^2 + \frac{\lambda^2_3}{g_{\varphi\varphi}} \right)} = \chi_3 - C_3 + \sigma_3 \sqrt{\left( \chi_3 - C_3 \right)^2 - \mathcal{N}^H_2 \left( m_3^2 + \frac{\lambda^2_3}{g_{\varphi\varphi}} \right)} . \quad (63)$$

Here, $C_3$ parametrizes deviation of particle 3 from criticality according to Eq. (18). We put $C_1 = 0$ for simplicity. One can denote the whole left-hand side of Eq. (63) as a new parameter $A_1$ such that

$$\mathcal{N}^H_1 A_1 \equiv \chi_1 + \sigma_1 \sqrt{\chi_1^2 - \mathcal{N}^H_2 \left( m_1^2 + \frac{\lambda^2_3}{g_{\varphi\varphi}} \right)}, \quad (64)$$

which will carry all the information about particle 1. Since $\chi_1 > 0$, we can make sure that $A_1 \geq 0$. The difference between BSW-type processes, $\sigma_1 = -1$, and Schnittman-type processes, $\sigma_1 = +1$, is absorbed into the definition of $A_1$, and thus the results expressed using $A_1$ hereafter will be the same for both.

2. Kinematic regimes

Having derived Eqs. (62) and (63) from the conservation law, Eq. (59), we now turn to their physical implications in collisional Penrose processes. For a Penrose process, one of the particles must fall inside the black hole, and we can make sure that particle 4 is bound to do so according to Eq. (62). On the other hand, particle 3 can be produced in four distinct kinematic regimes, based on the combination of sign of $C_3$ and the sign variable $\sigma_3$ in Eq. (63). In accordance with (22), let us denote the regimes with $C_3 > 0$ as $\text{+}$, $C_3 < 0$ as $\text{-}$, $\sigma_3 = +1$ as $\text{out}$, and $\sigma_3 = -1$ as $\text{in}$. The four kinematic regimes are then out+, out-, in+, and in-.

There are important differences among the four kinematic regimes in several regards. First, we should determine which ones allow particle 3 to escape from the vicinity of the black hole. For simplicity, let us assume a situation when condition (2) is well approximated by linear expansion terms. In such a case, there can be at most one turning point near $r_H$. The radius $r_T$ of this turning point is defined by the condition

$$\chi_3 (r_T - r_H) - C_3 (r_C - r_H) = \mathcal{N}^H \sqrt{m_3^2 + \frac{\lambda^2_3}{g_{\varphi\varphi}}} (r_T - r_H), \quad (65)$$

which can be rearranged as follows:

$$r_C - r_T = \frac{\chi_3 - C_3 - \mathcal{N}^H \sqrt{m_3^2 + \frac{\lambda^2_3}{g_{\varphi\varphi}}}}{\chi_3 - \mathcal{N}^H \sqrt{m_3^2 + \frac{\lambda^2_3}{g_{\varphi\varphi}}}} \left( r_C - r_H \right) . \quad (66)$$
Note that Eq. (66) may imply $r_T < r_H$, and hence no turning point in the region of our interest. The motion of particle 3 must be allowed at $r_C$, where it is produced; hence,

$$\chi_3 - C_3 - N_H \sqrt{m_3^2 + \frac{\lambda^2}{g_{\varphi \varphi}}} > 0.$$  \hspace{1cm} (67)

Therefore, the numerator of the fraction in Eq. (66) is positive, and since $r_C > r_H$ by definition, we can conclude that $r_T < r_C$ for particles produced with parameters in the admissible region, whereas $r_T > r_C$ for the ones outside of it. (Note the definition of the admissible region of parameters given in Eq. (24).) However, if $r_T > r_C$, particle 3 produced at $r_C$ can never escape. Therefore, in order for particle 3 to escape, it must be produced with parameters in the admissible region.

Particles with $C_3 > 0$ cannot fall into the black hole by definition, and thus they must have a turning point at a radius $r_C < r_T < r_H$. Therefore, in regimes out+ and in+, particle 3 can be produced only with parameters in the admissible region, and it is automatically guaranteed to escape.

Particles with $C_3 < 0$, in turn, can cross the horizon; their motion is allowed both at $r_H$ and at $r_C$. Hence, there must be an even number of turning points between $r_H$ and $r_C$. However, we assumed the existence of at most one turning point, and thus there can be none. Incoming particle 3 produced with $C_3 < 0$ therefore has to fall into the black hole; i.e., escape in the in− regime is impossible. Last, in the out− regime, particle 3 can either escape or be reflected and fall into the black hole, based on whether its parameters lie in the admissible region or not.

The way in which parameters $C_3$ and $\sigma_3$ determine escape possibilities of particle 3 is actually independent of the particular system in question. This can be seen, e.g., through comparison with Sec. IV B in [30], in which particles moving along the symmetry axis are considered. However, despite being so universal and so important for escape of particle 3, parameters $C_3$ and $\sigma_3$ are quite irrelevant for all other purposes. Indeed, if particle 3 escapes, $\sigma_3$ must eventually flip to +1, whereas $C_3$ encodes only a small deviation from fine tuning of parameters of particle 3.

We shall now solve Eq. (63) for $C_3$ and $\sigma_3$, in order to view the four different kinematic regimes in terms of the other parameters, i.e., $\chi_3$, $\lambda_3$, $m_3$, and $A_1$. First, we can observe from Eq. (63) that

$$\sigma_3 = \text{sgn}(N_H A_1 - \chi_3 + C_3).$$  \hspace{1cm} (68)

Expressing $C_3$ form Eq. (63) and then substituting it back into Eq. (68), we obtain the solutions as follows:

$$C_3 = \frac{N_H}{2} \left[ A_1 + \frac{1}{A_1} \left( m_3^2 + \frac{\lambda^2}{g_{\varphi \varphi}} \right) \right],$$  \hspace{1cm} (69)

$$\sigma_3 = \text{sgn} \left[ A_1^2 - \left( m_3^2 + \frac{\lambda^2}{g_{\varphi \varphi}} \right) \right].$$  \hspace{1cm} (70)

Since we are interested only in the sign of $C_3$, and $\sigma_3$ is a sign variable per se, only ratios among the four parameters on the right-hand sides matter to us. Therefore, we have considerable freedom in choosing the relevant three variables. Nevertheless, we have seen above that we also need to consider the admissible region, for which the relevant parameters are $\tilde{\chi}$, $\tilde{\lambda}$. Thus, it is natural to understand Eqs. (69) and (70) as depending on $\tilde{\chi}_3$, $\tilde{\lambda}_3$ and on the ratio between $A_1$ and $m_3$.

A third parameter, i.e., the ratio between $A_1$ and $m_3$, clearly stands out; it tracks a comparison between properties of two particles, and it is irrelevant for the admissible region of particle 3. Therefore, we find it natural to visualize the different kinematic regimes as regions in the same two-dimensional parameter space as the admissible region, with the ratio between $A_1$ and $m_3$ serving as an external parameter. However, since we are interested in a physical interpretation, namely, in energy extraction, we will keep $A_1$ and $m_3$ separate in the equations, and we will not explicitly pass to the parameters normalized to unit rest mass.

If we treat the ratio between $A_1$ and $m_3$ as an external parameter, there are just two main possibilities, namely, a heavy regime and a light regime. In the heavy regime, defined by $m_3 > A_1$, the right-hand side of Eq. (70) is negative for any $\tilde{\lambda}_3$, and hence the in region covers the whole parameter space. In the light regime, defined by $m_3 < A_1$, the parameter space is divided into in and out regions.

**B. Structure of the parameter space**

1. **Overall picture**

Now, we should understand how the regions of parameters corresponding to different kinematic regimes are distributed across our parameter space. Let us start with the distinction between + and − regimes, which is always
present regardless of the ratio between $A_1$ and $m_3$. From the solution given in Eq. (69), we see that $C_3 > 0$ implies the inequality

$$\chi_3 > \frac{\mathcal{N}_H}{2} \left[ A_1 + \frac{1}{A_1} \left( m_3^2 + \frac{\lambda_3^2}{g_{\varphi\varphi}} \right) \right],$$

(71)

which defines the $+$ region of parameters. Conversely, the inequality opposite to Eq. (71) entails $C_3 < 0$ and defines the $-$ region. For $C_3 = 0$, one has

$$\chi_3 = \frac{\mathcal{N}_H}{2} \left[ A_1 + \frac{1}{A_1} \left( m_3^2 + \frac{\lambda_3^2}{g_{\varphi\varphi}} \right) \right],$$

(72)

which defines the border between the regions, and it corresponds to particle 3 being produced as precisely critical. In the $\chi_3, \lambda_3$ parameter space, Eq. (72) represents a parabola with axis $\lambda_3 = 0$. For a physical interpretation, let us substitute Eq. (72) into Eq. (12) to obtain parametric expressions for the border as follows:

$$q_3 = \frac{1}{\tilde{A}_\varphi} \left\{ \tilde{\omega} \lambda_3 + \frac{\mathcal{N}_H}{2} \left[ A_1 + \frac{1}{A_1} \left( m_3^2 + \frac{\lambda_3^2}{g_{\varphi\varphi}} \right) \right] \right\},$$

(73)

$$L_3 = \frac{1}{\tilde{A}_\varphi} \left\{ \tilde{\lambda}_3 - \frac{\mathcal{N}_H A_{\varphi}^H}{2} \left[ A_1 + \frac{1}{A_1} \left( m_3^2 + \frac{\lambda_3^2}{g_{\varphi\varphi}} \right) \right] \right\},$$

(74)

$$E_3 = \frac{1}{\tilde{A}_\varphi} \left\{ \left( \omega_{\varphi} \tilde{\phi} - \tilde{\omega} \phi_\varphi \right) \lambda_3 + \frac{\mathcal{N}_H A_{\varphi}^H}{2} \left[ A_1 + \frac{1}{A_1} \left( m_3^2 + \frac{\lambda_3^2}{g_{\varphi\varphi}} \right) \right] \right\}.$$

(75)

Recalling the gauge condition Eq. (44), we can make sure that Eq. (75) leads to $E_3 \to \infty$ for $|\lambda_3| \to \infty$. Therefore, we see that values of $E_3$ in neither the $+$ nor $-$ region are bounded from above. This was not possible in the previously known special cases; see [19]. Since the escape of particle 3 is guaranteed in the $+$ regime, we can also conclude that there is no upper bound on the energy extracted from the black hole. Such a possibility is often called the super-Penrose process. Furthermore, as the $+$ region exists for any value of $m_3$, we see that there is no bound on the mass of escaping particles as well.

Now, let us turn to the distinction between in and out regimes. From Eq. (70), we can see that parameters in the in region must satisfy the condition

$$|\lambda_3| > \sqrt{g_{\varphi\varphi} (A_1^2 - m_3^2)},$$

(76)

whereas the opposite inequality holds for parameters in the out region. The two values of $\lambda_3$ that separate the regions, i.e.,

$$\lambda_3 = \pm \sqrt{g_{\varphi\varphi} (A_1^2 - m_3^2)},$$

(77)

correspond to a situation when our leading-order approximation breaks down, since the square root on the right-hand side of Eq. (63) becomes zero and we cannot consistently assign a value to $\sigma_3$. This indicates that particle 3 with those values of $\lambda_3$ will be produced as class II nearly critical, and a different expansion would be needed to determine its initial direction of motion.

Let us note that particle 3 can be produced in the in+ regime, i.e., with $\chi_3$ and $\lambda_3$ satisfying both Eq. (71) and Eq. (76), for any value of the ratio between $m_3$ and $A_1$. This is another thing that was not possible in the previously studied special cases (i.e., vacuum black holes and nonrotating black holes).

2. Osculation points

Having derived borders that divide the $\chi_3, \lambda_3$ parameter space according to various criteria, we shall now consider the corners where the borders meet. We can get insight into this issue from the physical interpretation of the borders; Eq. (25) gives a set of parameters for which precisely critical particles are of class II, Eq. (72) corresponds to particle 3 being produced as precisely critical, and Eq. (15) corresponds to particle 3 being produced as class II critical or nearly critical. If any two of those eventualities happen together, the third one follows automatically. Therefore, all
three borders must meet in the same points of the parameter space. Indeed, substituting Eq. (77) into both Eq. (25) and Eq. (72) leads to \( \lambda_3 = N_H A_1 \). Conversely, in the heavy regime \( m_3 > A_1 \), in which case Eq. (74) is absent, the remaining borders given in Eqs. (25) and (72) cannot meet at any point. Note that in the \( m_3 = A_1 \) case, Eqs. (25) and (72) touch at \( \lambda_3 = 0 \).

We have also seen that particle 3 can be produced with \( C_3 > 0 \) only when its other parameters satisfy the condition (24). Therefore, the + region must lie inside the admissible region in the parameter space, and their borders can only osculate. One can make sure that this is indeed the case by comparing the limiting behavior of Eqs. (25) and (72) for \( |\lambda_3| \rightarrow \infty \) and their values at \( \lambda_3 = 0 \), i.e., in between the values given in Eq. (77). By putting Eq. (77) into Eqs. (73)-(75) (or into Eqs. (26)-(28)), we obtain the values of \( \tilde{\lambda}_3 \), \( \tilde{q}_3 \), and \( \tilde{E}_3 \) in the processes with \( m_3 \gg A_1 \).

Since the parametric expressions given in Eqs. (73)-(75) are mere quadratic functions of \( \lambda_3 \), they will always reach an extremum, and therefore there will always be bounds on \( \lambda_3, q_3 \), and \( E_3 \) of particles produced by our hypothetical interaction.

First, let us consider a hypothetical interaction, for which this ratio can take any value. More precisely, we shall consider an idealized scenario, in which it is possible to produce particle 3 with any value of \( \chi_3 \). Let us now search for other bounds on the parameters of particle 3 in these regions. There are multiple possibilities, depending on the ratio between \( A_1 \) and \( m_3 \).

Second, let us consider a more realistic scenario, in which only some values of the ratio between \( m_3 \) and \( A_1 \) are possible. In such a case, we can search for bounds on parameters in the + region for given values of \( m_3 \) and \( A_1 \). Since the parametric expressions given in Eqs. (73)-(75) are mere quadratic functions of \( \lambda_3 \), they will always reach an extremum, and therefore there will always be bounds on values of \( q_3 \), \( L_3 \), and \( E_3 \) in the + region. Starting with \( q_3 \), we find that for

\[
q_3 = \frac{1}{\phi + \omega A_\varphi^H} \left[ \pm \omega \sqrt{g_{\varphi\varphi}^H (A_1^2 - m_3^2) - N_H A_1} \right],
\]

(78)

Eq. (73) reaches an extremum with value

\[
q_3^b = \frac{1}{2 \left( \phi + \omega A_\varphi^H \right)} \left[ N_H \left( A_1 + \frac{m_3^2}{A_1} \right) - \frac{g_{\varphi\varphi}^H \omega^2 A_1}{N_H} \right].
\]

(82)

Turning to \( L_3 \), we can infer that for

\[
L_3 = \frac{g_{\varphi\varphi}^H \phi}{N_H A_\varphi^H A_1} ,
\]

(83)

expression (74) reaches an extremum with value

\[
L_3^b = \frac{1}{2 \left( \phi + \omega A_\varphi^H \right)} \left[ N_H A_\varphi^H \left( A_1 + \frac{m_3^2}{A_1} \right) - \frac{g_{\varphi\varphi}^H \omega^2 A_1}{N_H A_\varphi^H} \right].
\]

(84)
For $E_3$, the situation is again different due to dependence on gauge. Looking at the $|\lambda_3| \to \infty$ behavior of Eq. (75), we can make sure that the condition (44) implies that Eq. (75) will reach a minimum. It occurs for

$$\lambda_3 = -\frac{g_{\varphi\varphi}^H A_i}{N_H A_t^H} \left(\omega_H \dot{\phi} - \dot{\omega} \phi_H\right),$$

(85)

and its value is

$$E_3^{\min} = \frac{1}{2} \left(\frac{\phi + \omega A_t^H}{A_1 + m_3^2} - \frac{g_{\varphi\varphi}^H A_i}{N_H A_t^H} \left(\omega_H \dot{\phi} - \dot{\omega} \phi_H\right)^2\right).$$

(86)

4. Additional remarks on the out– region

The discussion above can be extended by analyzing bounds on parameters in further, special regions in the parameter space. The out– region is particularly interesting in this regard, since there is an upper bound on the values of energy $E_3$ in this region. As noted in Ref. (30), this can be used to illustrate the difference between the BSW-type and Schnittman-type collisional process. Let us extend this argument to our more complicated case of equatorial charged particles.

Equation (75) cannot have a maximum on account of Eq. (44), and thus the upper bound on $E_3$ in the out– region must be its value for one of the osculation points. Picking the higher of the values in Eq. (80), we can write the bound as follows:

$$E_3 < \frac{\omega_H \dot{\phi} - \dot{\omega} \phi_H}{\phi + \omega A_t^H} \sqrt{g_{\varphi\varphi}^H (A_1^2 - m_3^2)} + \frac{N_H A_t^H A_i}{\phi + \omega A_t^H}.$$

(87)

We shall maximize the bound with respect to all possible parameters in order to derive an unconditional bound in terms of $E_1$. First, we consider $m_3 \ll A_1$, which also allows us to factor out $A_1$,

$$E_3 < \left(\frac{\omega_H \dot{\phi} - \dot{\omega} \phi_H}{\phi + \omega A_t^H} \sqrt{g_{\varphi\varphi}^H} + \frac{N_H A_t^H A_i}{\phi + \omega A_t^H}\right) A_1.$$

(88)

Second, we shall express $A_1$ using $E_1$ and maximize it with respect to other parameters of particle 1. We can use (12) with $\lambda_1 = 0$ (as (30) lies on (25)) to rewrite $\chi_1$ in terms of $E_1$ and $\lambda_1$,

$$\chi_1 = E_1 \left(\frac{\dot{\phi} + \omega A_t^H}{A_t^H} - \frac{\omega_H \dot{\phi} - \dot{\omega} \phi_H}{A_t^H}\right).$$

(89)

Note that gauge condition (44) implies that the coefficient at $E_1$ is positive. In the $|\lambda_1| \to \infty$ limit, for fixed $E_1$, using Eq. (30) in Eq. (31), we find that the leading order of $A_1$ is

$$A_1 \approx -\lambda_1 \frac{\omega_H \dot{\phi} - \dot{\omega} \phi_H}{N_H A_t^H} + \sigma_1 |\lambda_1| \sqrt{\left(\frac{\omega_H \dot{\phi} - \dot{\omega} \phi_H}{N_H A_t^H}\right)^2 - \frac{1}{g_{\varphi\varphi}^H}}.$$

(90)

One can see that $A_1$ is not real due to Eq. (43). Therefore, for a given $E_1$, the parameter $A_1$ will lie in the real numbers only for a finite interval of values of $\lambda_1$, and $\partial A_1/\partial \lambda_1$ will blow up with opposite signs at the opposite ends of that interval. Thus, there will always be an extremum with respect to $\lambda_1$. (See the Appendix for details.)

For the BSW-type process, $\sigma_1 = -1$, there will be a minimum. Hence, we shall start with the following inequality (see Eq. (64)):

$$N_H A_1(E_1, \lambda_1, m_1) \leq \chi_1(E_1, \lambda_1).$$

(91)

In order to maximize $\chi_1(E_1, \lambda_1)$ of Eq. (30), we need to look at values of $\lambda_1$ that satisfy

$$N_H A_1(E_1, \lambda_1, m_1) = \chi_1(E_1, \lambda_1).$$

(92)
i.e., the ends of the interval mentioned above, and on their dependence on \( m_1 \) (cf. the Appendix). The resulting unconditional bound on \( A_1 \) with \( \sigma_1 = -1 \) is

\[
A_1 \leq E_1 \frac{|\dot{\phi} + \omega A^H_\phi|}{\mathcal{N}_H |A^H_\phi| - \sqrt{g^H_{\phi \phi} \omega_H \dot{\phi} - \omega \phi_H}}. \tag{93}
\]

In combination with Eq. (88), Eq. (93) gives us the unconditional upper bound on energy \( E_3 \) of a particle produced in the out--regime in the BSW-type process as follows:

\[
E_3 < E_1 \frac{\mathcal{N}_H |A^H_\phi| + \sqrt{g^H_{\phi \phi} \omega_H \dot{\phi} - \omega \phi_H}}{\mathcal{N}_H |A^H_\phi| - \sqrt{g^H_{\phi \phi} \omega_H \dot{\phi} - \omega \phi_H}}. \tag{94}
\]

For the Schnittman-type process, \( \sigma_1 = +1 \), we can see that we need to put \( m_1 = 0 \) to maximize \( A_1 \). Then we can find the maximum of \( A_1 \) with respect to \( \lambda_1 \) (using Eq. (89) with Eq. (90); see also the Appendix) and derive the unconditional bound on \( A_1 \),

\[
A_1 \leq 2E_1 \frac{\mathcal{N}_H \left( \dot{\phi} + \omega A^H_\phi \right) A^H_\phi}{\mathcal{N}_H^2 \left( A^H_\phi \right)^2 - g^H_{\phi \phi} \left( \omega_H \dot{\phi} - \omega \phi_H \right)^2}. \tag{95}
\]

Combining with Eq. (88), we conclude that the unconditional upper bound on the energy \( E_3 \) of a particle produced in the out--regime in the Schnittman-type process is

\[
E_3 < 2E_1 \frac{\mathcal{N}_H |A^H_\phi|}{\mathcal{N}_H |A^H_\phi| - \sqrt{g^H_{\phi \phi} \omega_H \dot{\phi} - \omega \phi_H}}. \tag{96}
\]

Let us note that for \( \omega = 0 \), the above results reduce to the ones of [29], i.e., \( E_3 < E_1 \) for the BSW-type process and \( E_3 < 2E_1 \) for the Schnittman-type process. The bound for the Schnittman-type process is higher than for the BSW-type process even in the general case, due to Eq. (43). On the other hand, also due to Eq. (43), we can see that \( E_3 > E_1 \) is generally not prevented for the BSW-type process, unlike in the \( \omega = 0 \) case. However, the biggest difference is that in the general case, the gauge-dependent factors do not cancel, and thus the bounds need to be interpreted more carefully.

We have seen above that the collisional processes analyzed here have multiple features that were absent in the previously studied special cases. Thus, now we shall discuss how the special cases follow from the general results.

C. Special cases and the degenerate case

1. Quasiradial limit

First, let us investigate how to recover the results for radially moving particles [29]. Similarly as in Sec. III A, we can choose to consider either particles that move radially with respect to a locally nonrotating observer very close to the horizon, i.e., \( \lambda_3 = 0 \), or particles that would move radially in a region devoid of the influence of dragging and of magnetic field, i.e., \( L_3 = 0 \). However, both choices lead to a trivial transition, unlike in Sec. III A. Considering particles with a fixed value of \( \lambda_3 \), the condition \( C_3 > 0 \), see Eq. (44), can be restated (using Eq. (89) for particle 3) as follows:

\[
E_3 > \frac{1}{\dot{\phi} + \omega A^H_\phi} \left\{ \left( \omega_H \dot{\phi} - \omega \phi_H \right) \lambda_3 + \frac{\mathcal{N}_H A^H_\phi}{2} \left[ A_1 + \frac{1}{A_1} \left( m_3^2 + \frac{\lambda_3^2}{g^H_{\phi \phi}} \right) \right] \right\}. \tag{97}
\]

The key feature we want to reproduce is the existence of a threshold value \( \mu \), such that \( E_3 > \mu \) corresponds to the + regime and \( E_3 < \mu \) corresponds to the -- regime. An indeed, by setting \( \lambda_3 = 0 \), we get a threshold value \( \mu \), given by

\[
\mu \equiv \frac{\mathcal{N}_H A^H_\phi}{2 \left( \dot{\phi} + \omega A^H_\phi \right)} \left( A_1 + \frac{m_3^2}{A_1} \right). \tag{98}
\]

Moreover, \( \lambda_3 = 0 \) lies in the out region whenever it exists. Thus, we can also see that for \( \lambda_3 = 0 \), the heavy regime \( m_3 > A_1 \) coincides with the in regime and the light regime \( m_3 < A_1 \) with the out regime. This also replicates the results of [29].
2. Geodesic limit

Second, we discuss the transition to geodesic particles, i.e., \( q_3 = 0 \). We shall rewrite \( C_3 \) of Eq. (99) in terms of \( E_3 \) and \( q_3 \) (using Eq. (10) and dropping the contribution proportional to \( A_H \)),

\[
C_3 = -\frac{1}{\omega_H} \left[ \hat{\omega} E_3 + q_3 \left( \omega_H \hat{\phi} - \hat{\omega} \phi_H \right) \right] - \frac{N_H}{2} \left[ A_1 + \frac{1}{A_1} \left( m_3^2 + \frac{(E_3 + q_3 A^H_1)^2}{g_{\phi \phi}^H \omega_H^2} \right) \right]. \tag{99}
\]

(More precisely, by using Eq. (10) with (13) in the derivation of (99), one can make sure that the \( A_H \) term influences only higher orders of expansion.) The resulting expression (99) admits a factorization,

\[
C_3 = -\frac{N_H}{2g_{\phi \phi}^H \omega_H^2 A_1} (E_3 - R_) (E_3 - R_+), \tag{100}
\]

where \( R_\pm \) stand for

\[
R_\pm = -q_3 A^H_1 + \frac{g_{\phi \phi}^H A_1}{N_H} \left[ -\omega_H \hat{\omega} \pm |\omega_H| \sqrt{\hat{\omega}^2 - \frac{2q_3 N_H}{g_{\phi \phi}^H A_1} \left( \hat{\phi} + \omega A^H_1 \right) - \frac{N_H^2}{g_{\phi \phi}^H} \left( 1 + \frac{m_3^2}{A_1^2} \right)} \right]. \tag{101}
\]

Since \( R_+ > R_- \), the + regime corresponds to \( R_- < E_3 < R_+ \) for a fixed value of \( q_3 \). Let us now rewrite \( \sigma_3 \) of Eq. (70) in terms of \( E_3 \) and \( q_3 \),

\[
\sigma_3 = \text{sgn} \left[ A_1^2 - \left( m_3^2 + \frac{(E_3 + q_3 A^H_1)^2}{g_{\phi \phi}^H \omega_H^2} \right) \right]. \tag{102}
\]

The result again admits a factorization,

\[
\sigma_3 = -\text{sgn} [(E_3 - S_+) (E_3 - S_-)] \tag{103}
\]

where \( S_\pm \) are

\[
S_\pm = -q_3 A^H_1 \pm |\omega_H| \sqrt{g_{\phi \phi}^H (A_1^2 - m_3^2)} \tag{104}
\]

As \( S_+ > S_- \), the out regime corresponds to \( S_- < E_3 < S_+ \) for a fixed value of \( q_3 \).

Now, let us put \( q_3 = 0 \) in the equations above to find the geodesic limit. For geodesic particles, i.e., for \( q_3 = 0 \), it should be possible to produce particles with high values of \( E_3 \) or \( m_3 \) only in the in– regime, which prevents their escape. In putting \( q_3 = 0 \), we denote the resulting values of \( R_\pm \) for the geodesic limit as \( R_\pm^g \), so that

\[
R_\pm^g = \frac{g_{\phi \phi}^H A_1}{N_H} \left[ -\omega_H \hat{\omega} \pm |\omega_H| \sqrt{\hat{\omega}^2 - \frac{N_H^2}{g_{\phi \phi}^H} \left( 1 + \frac{m_3^2}{A_1^2} \right)} \right]. \tag{105}
\]

Since \( S_- \) becomes negative for \( q_3 = 0 \) and \( E_3 > 0 \), we need to consider only \( S_+ \) in the geodesic limit, i.e., \( S_+^g \), which reads

\[
S_+^g = |\omega_H| \sqrt{g_{\phi \phi}^H (A_1^2 - m_3^2)}. \tag{106}
\]

If a geodesic particle 3 has sufficiently high energy, such that it satisfies both \( E_3 > R_+^g \) and \( E_3 > S_+^g \), it will be produced in the in– regime and fall into the black hole. Conversely, we can see that \( R_+^g \) and \( S_+^g \) all become imaginary for \( m_3 \gg A_1 \), and thus the in– regime is the only possible regime in that case. Hence, we reproduced the results of \( [19, 20] \) that the mass and energy of escaping geodesic particles is bounded. Note that in \( [19, 20] \), the symbols \( \lambda_\pm \) were used for \( R_\pm^g \) and \( \lambda_0 \) for \( S_+^g \).
3. Degenerate case

Third, let us return to the degenerate case given in Eq. (13), which was so far excluded from our discussion of energy extraction. We shall use the same parametrization as in the geodesic case. If we apply Eq. (13) to $R^{\pm}_{\pm}$ given in Eq. (101), they go over to degenerate $R^{\pm}_{\pm}$, i.e., $R^{d}_{\pm}$, which read

$$R^{d}_{\pm} = -q_{3}A^{H}_{t} - \frac{g^{H}_{\phi\phi}A^{H}_{l}}{N^{H}_{l}} \left[ \omega_{H} \mp |\omega_{H}| \sqrt{\omega^{2} - \frac{N^{2}_{H}}{g^{H}_{\phi\phi}}} \left( 1 + \frac{m^{2}}{A^{2}_{l}} \right) \right].$$

(107)

We have determined in Sec. III B that we need to impose the gauge condition $A^{H}_{l} = 0$ in the degenerate case. However, with this condition, it holds that $R^{d}_{\pm} = R^{g}_{\pm}$. Furthermore, putting $A^{H}_{l} = 0$ has the same effect on $S^{\pm}$ given in Eq. (104) as putting $q_{3} = 0$. Thus, we see that upon the gauge condition $A^{H}_{l} = 0$, the degenerate case completely coincides with the geodesic case. Therefore, the degenerate case corresponds to a situation when the spacetime behaves locally as vacuum close to $r_{H}$. However, it can be shown that the spacetime does not need to be globally vacuum in order for Eq. (13) to be satisfied (see, e.g., [35]).

V. RESULTS FOR KERR-NEWMAN SOLUTION

A. List of relevant quantities

Let us now apply the framework developed above to the case of extremal Kerr-Newman solution with mass $M$, angular momentum $aM$, and charge $Q$. The extremal case is defined by the condition

$$M^{2} = Q^{2} + a^{2},$$

(108)

which implies

$$r_{H} = M.$$

(109)

We also list the quantities relevant for our discussion as follows:

$$N^{H}_{l} = \sqrt{\frac{Q^{2} + a^{2}}{Q^{2} + 2a^{2}}},$$

$$g^{H}_{\phi\phi} = \frac{(Q^{2} + 2a^{2})^{2}}{Q^{2} + a^{2}},$$

$$\omega_{H} = \frac{a}{Q^{2} + 2a^{2}},$$

$$\phi_{H} = \frac{Q\sqrt{Q^{2} + a^{2}}}{Q^{2} + 2a^{2}},$$

$$A^{H}_{l} = -\frac{Q}{\sqrt{Q^{2} + a^{2}}},$$

$$A^{H}_{\phi} = \frac{Qa}{\sqrt{Q^{2} + a^{2}}},$$

$$\dot{\omega} = -\frac{2a\sqrt{Q^{2} + a^{2}}}{(Q^{2} + 2a^{2})^{2}}$$

$$\dot{\phi} = -\frac{Q^{3}}{(Q^{2} + 2a^{2})^{2}}.$$

(113)

B. Admissible region in the parameter space

Critical particles can approach $r = M$, whenever their parameters lie inside the admissible region in the parameter space. Equations (26)-(28) for the border of the admissible region go over to

$$q = \frac{\sqrt{Q^{2} + a^{2}}}{Q(Q^{2} + 2a^{2})} \left[ -2a\lambda + \sqrt{(Q^{2} + 2a^{2})^{2}m^{2} + (Q^{2} + a^{2})^{2}} \lambda^{2} \right],$$

(114)

$$L = \frac{Q^{2}\lambda + a\sqrt{(Q^{2} + 2a^{2})^{2}m^{2} + (Q^{2} + a^{2})^{2}} \lambda^{2}}{Q^{2} + 2a^{2}},$$

(115)

$$E = \frac{-a\lambda + \sqrt{(Q^{2} + 2a^{2})^{2}m^{2} + (Q^{2} + a^{2})^{2}} \lambda^{2}}{Q^{2} + 2a^{2}}.$$

(116)
As we discussed in Sec. II A 2, bounds on values of \( q \), \( L \), and \( E \) in the admissible region are given by extrema of Eqs. (14)-(16) as functions of \( \lambda \). If \( |a|/\lambda < \frac{1}{2} \), then Eq. (14) reaches an extremum, which has a value given in Eq. (30), i.e.,

\[
q_b = m\frac{\sqrt{Q^2 - 3a^2}}{Q},
\]

and the corresponding values given in Eqs. (31) and (32) of \( L \) and \( E \) become

\[
L = \frac{ma}{\sqrt{Q^2 + a^2}} \frac{3Q^2 + a^2}{\sqrt{Q^2 - 3a^2}} , \quad \quad \quad E = \frac{m}{\sqrt{Q^2 + a^2}} \frac{Q^2 - a^2}{\sqrt{Q^2 - 3a^2}} .
\]

For \( |a|/\lambda > \sqrt{3} - 1 \), there exists an extremum of Eq. (116), which has a value given in Eq. (35), i.e.,

\[
L_b = m\text{sgn} a \frac{\sqrt{a^4 + Q^2 a^2 - Q^4}}{\sqrt{Q^2 + a^2}},
\]

and the corresponding values given in Eqs. (36) and (37) of \( q \) and \( E \) go over to

\[
q = m\frac{|a|}{|Q|} \frac{3Q^2 + a^2}{\sqrt{a^4 + Q^2 a^2 - Q^4}}, \quad \quad \quad E = \frac{m |a|}{\sqrt{Q^2 + a^2}} \frac{2Q^2 + a^2}{\sqrt{a^4 + Q^2 a^2 - Q^4}} .
\]

In the standard gauge vanishing at spatial infinity, the dragging potential and the electromagnetic potential of an extremal Kerr-Newman solution satisfy the conditions given in Eqs. (36) and (44). Therefore, Eq. (116) always has a minimum, see Eq. (10),

\[
E_{\text{min}} = \frac{m |q|}{\sqrt{Q^2 + a^2}} ,
\]

and the corresponding values given in Eqs. (11) and (42) of \( q \) and \( L \) turn into

\[
q = m\frac{Q^2 - a^2}{|Q|}, \quad \quad \quad L = \frac{ma}{|Q|} \frac{2Q^2 + a^2}{\sqrt{Q^2 + a^2}} .
\]

The degenerate case of Eq. (13) corresponds to the extremal Kerr solution, i.e., \( a = 0 \). Let us note that for the extremal Reissner-Nordström solution, the border of the admissible region has a symmetry with respect to change \( l \rightarrow -l \). Such a possibility was labeled as case 3 in [12]. A summary of the general results on bounds on parameters in the admissible region is given in Table I.

| Kerr-Newman black hole parameters | General case | Bounds on \( \tilde{q} \) | Bounds on \( l \) | Bounds on \( \varepsilon \) |
|-----------------------------------|-------------|-----------------|---------------|----------------|
| | | Vacuum | No bound | \( l\text{sgn} a \geq \frac{2|a|}{\sqrt{Q^2 + a^2}} \) | \( \varepsilon > \frac{|Q|}{\sqrt{Q^2 + a^2}} \) |
| \( |a|/\lambda > \frac{1}{2} \) | | \( 0 < |Q|/\lambda < \sqrt{3} - 1 \) | \( 1a2b \) | \( l\text{sgn} a \geq \sqrt{a^2 + Q^2 a^2 - Q^4} / (Q^2 + a^2) \) | \( l\text{sgn} a > 0 \) |
| \( \sqrt{3} - 1 \) | | \( \sqrt{3} - 1 < |Q|/\lambda < \sqrt{3} \) | \( 1a2c \) | | |
| \( \frac{1}{2} \) | | \( \sqrt{3} - 1 < |Q|/\lambda < \sqrt{3} \) | \( 1a2a \) | | |
| \( 0 < |a|/\lambda < \frac{1}{2} \) | \( 0 < |Q|/\lambda < \frac{\sqrt{3} - 1}{2} \) | \( 1c2a \) | \( \tilde{q}\text{sgn} Q > 0 \) | No bound |
| \( \frac{1}{2} \) | \( \frac{\sqrt{3}}{2} < |Q|/\lambda < 1 \) | \( 1b2a \) | \( \tilde{q}\text{sgn} Q \geq \sqrt{Q^2 - 3a^2} / (Q) \) | \( \varepsilon > \frac{|Q|}{\sqrt{Q^2 + a^2}} \) |
| 0 | 1 | \( 1b2a3 \) | \( \tilde{q}\text{sgn} Q > 1 \) | | |
TABLE II. Bounds on the parameters $\tilde{q}$ and $l$ of critical particles with $\varepsilon \leq 1$ that can approach $r = M$ in an extremal Kerr-Newman spacetime. (The placement of nonstrict inequalities is based on results about class II critical particles; see [12].)

| $|a|/M$ | $|Q|/M$ | Bounds on $\tilde{q}$ | Bounds on $l$ |
|-------|--------|----------------------|---------------|
| 1     | 0      | No bound             | $2a/\sqrt{\tilde{q}} < l \text{ sgn } a \leq 2|a|$ |
| $\sqrt{2}/3 < |a|/M < 1$ | $0 < |Q|/M < \sqrt{\tilde{q}/3}$ | $-\sqrt{Q^2 + a^2} \leq \tilde{q} \text{ sgn } Q < \sqrt{Q^2 + a^2}$ | $|a| < l \text{ sgn } a \leq |a| \frac{3Q^2 + 2a^2}{Q^2}$ |
| $0 < |a|/M < \sqrt{2}/3$ | $\sqrt{\tilde{q}/3} < |Q|/M < 1$ | $0 \leq \tilde{q} \text{ sgn } Q < \sqrt{\tilde{q}/3}$ | |

One can observe that $E \geq m$ for the the values of energy given in Eqs. (118) and (120), whereas $E_{\text{min}} \leq m$. This implies that the expressions, given in Eqs. (114) and (115), for the values of $q$ and $L$ on the border, defined by Eq. (25), of the admissible region, are monotonic along the part of the border corresponding to $E_{\text{crit}} \leq m$. Therefore, the values of $q$ and $L$ in the part of the admissible region with $E_{\text{crit}} \leq m$ are bounded by the values of $q$ and $L$ for points on Eq. (25) with $E_{\text{crit}} = m$. Using the expressions for these points obtained in [12], we present the resulting bounds in Table II.

C. Structure of the parameter space with regard to energy extraction

As we examined in Sec. IV B, the parameter space of nearly critical particles can be divided into various regions corresponding to different kinematic regimes, in which particle 3 can be produced in our collisional process. In the extremal Kerr-Newman spacetime, Eqs. (73)-(75) for the border separating the + and − regions become

$$q_3 = \sqrt{Q^2 + a^2} \frac{Q}{Q^2 + 2a^2} \left\{ -2\frac{a\lambda_3}{Q^2 + 2a^2} + \frac{1}{2} \left[ A_1 + \frac{1}{A_1} \left( m_3 + \frac{(Q^2 + a^2)\lambda_3^2}{(Q^2 + 2a^2)^2} \right) \right] \right\},$$

$$L_3 = \frac{Q^2\lambda_3}{Q^2 + 2a^2} + \frac{a}{2} \left[ A_1 + \frac{1}{A_1} \left( m_3^2 + \frac{(Q^2 + a^2)\lambda_3^2}{(Q^2 + 2a^2)^2} \right) \right],$$

$$E_3 = -\frac{a\lambda_3}{Q^2 + 2a^2} + \frac{1}{2} \left[ A_1 + \frac{1}{A_1} \left( m_3^2 + \frac{(Q^2 + a^2)\lambda_3^2}{(Q^2 + 2a^2)^2} \right) \right].$$

The two values of $\lambda_3$ that separate in and out regions given in Eq. (77) are

$$\lambda_3 = \pm \frac{Q^2 + 2a^2}{\sqrt{Q^2 + a^2}} \sqrt{A_1^2 - m_3^2}.$$  

Equations (78)-(80) for the osculation points, where the curves given by Eqs. (114)-(116) and Eqs. (123)-(125) touch, turn out to be

$$q_3 = \frac{1}{Q} \left[ \sqrt{Q^2 + a^2}A_1 \pm 2a\sqrt{A_1^2 - m_3^2} \right],$$

$$L_3 = aA_1 \pm Q^2 \frac{\sqrt{A_1^2 - m_3^2}}{\sqrt{Q^2 + a^2}},$$

$$E_3 = A_1 \pm a \frac{\sqrt{A_1^2 - m_3^2}}{\sqrt{Q^2 + a^2}}.$$  

As we noted in Sec. IV B, the values of $q_3$, $L_3$, and $E_3$ in the + region are always bounded. Equations (81), (84),
and \( E_3^{\min} \) for bounds on these parameters go over to

\[
q_3^b = \frac{1}{2 Q} \left[ \frac{Q^2 - 3 a^2}{\sqrt{Q^2 + a^2}} A_1 + \sqrt{Q^2 + a^2} m_3^2 A_1^{-1} \right],
\]

\[
L_3^b = \frac{1}{2} \left[ \frac{a^4 + Q^2 a^2 - Q^4}{a (Q^2 + a^2)} A_1 + a m_3^2 A_1^{-1} \right],
\]

\[
E_3^{\min} = \frac{1}{2} \left[ \frac{Q^2}{Q^2 + a^2} A_1 + m_3^2 A_1^{-1} \right].
\]

The structure of the parameter space is visualized for a black hole with \( \frac{a}{M} = \frac{1}{2} \) and a process with \( m_3 < A_1 \) in the left panel of Figure 1 and for a black hole with \( \frac{a}{M} = \frac{\sqrt{15}}{6} \) and a process with \( m_3 > A_1 \) on the right panel. There, it is also shown how special limiting cases discussed in Sec. IV C correspond to different sections of the parameter space.

**VI. CONCLUSIONS**

We have studied high-energy collisions of equatorial charged test particles near the horizon of an extremal rotating electrovacuum black hole, i.e., the generalized BSW effect. Such collisions are only possible when critical particles can reach the vicinity of the horizon. Consequently, we distinguished different variants of the process based on whether there exist bounds on the charge and the angular momentum of the critical particles that can approach the horizon. (Since the values of angular momentum measured at the horizon are bounded solely in the vacuum case, we used the angular momentum measured at infinity as our reference.) Geometrically, these bounds can be seen as extrema of a hyperbola curve in the two-dimensional parameter space of the critical particles.

We proceeded to examine the possibilities of energy extraction from the black hole by particles produced in the BSW collisions, using a 2 → 2 model process. We first discussed which kinematic regimes imply escape of one of the produced particles and noted that this picture is model independent. Then, we investigated to what regions in the parameter space do these kinematic regimes correspond, and we found several situations that were not possible in the previously studied cases with fewer parameters. We also explained how these limiting cases can be obtained...
as different sections of the full parameter space. This is visualized in Fig. 1 for the case of extremal Kerr-Newman solution. Leaving the technical intricacies aside, one main result stands out: there are no unconditional bounds on the extracted energy as long as both the black hole and the escaping particles are charged. (And the same actually holds for the absence of unconditional upper bounds on the mass of the escaping particle.) Thus, the influence of the electromagnetic field on the energy extraction is more important in the present setup, as it prevails for arbitrarily small black hole charge.

Although these results are very promising, we have to acknowledge that a lot of effects have not been taken into account in our considerations, most notably the electromagnetic backreaction. Additionally, as shown in [30], despite the lack of unconditional bounds, there can be caveats which make the energy extraction unfeasible even within the test particle approximation. Although we leave the details for future work, we can nevertheless show by inspection that several of the concerns mentioned in [30] for the axial case do not apply to the equatorial case. In particular, some of the difficulties in the axial case arose from the fact that critical particles needed to be highly relativistic, whereas in the present setup, this is not an issue whenever the black hole is rotating; in that case, the energy of critical or nearly critical particles does not need to be large compared to the mass of the black hole. However, in the present case, we have shown that nonrelativistic critical particles can always approach the horizon, due to $|q_3| > |q_1| > 0$, in order to have $E_3 > E_1$. In the present setup, this is not an issue whenever the black hole is rotating; in that case, the energy of critical or nearly critical particles does not need to be proportional or approximately proportional to their charge. This suggests that the presence of the frame dragging is important for the energy extraction as well, as we shall examine in follow-up work.

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Appendix A: Auxiliary formulas

Let us introduce an expression $P$ as follows:

$$P = \frac{-\hat{g}_{\varphi\varphi}^H \left( \hat{\phi} + \hat{\omega} A^H_{\varphi} \right) \left( \omega_H \hat{\phi} - \hat{\omega}\phi_H \right) E_1 + \delta \sqrt{W}}{\mathcal{N}^2_H \left( A^H_1 \right)^2 - \hat{g}_{\varphi\varphi}^H \left( \omega_H \hat{\phi} - \hat{\omega}\phi_H \right)^2}. \quad (A1)$$

We can define $P_{\pm}$ by putting $\delta = \pm 1$ and $W = W_{\text{end}}$, where $W_{\text{end}}$ is given by

$$W_{\text{end}} = \mathcal{N}^2_H \hat{g}_{\varphi\varphi}^H \left( A^H_1 \right)^2 \left\{ \left( \hat{\phi} + \hat{\omega} A^H_{\varphi} \right)^2 E_1^2 - m_1^2 \left[ \mathcal{N}^2_H \left( A^H_1 \right)^2 - \hat{g}_{\varphi\varphi}^H \left( \omega_H \hat{\phi} - \hat{\omega}\phi_H \right)^2 \right] \right\}. \quad (A2)$$

Then, it holds that $A_1(E_1, \lambda_1, m_1)$ is real whenever $\lambda_1 \in [P_-, P_+]$. Note that $A_1(E_1, \lambda_1, m_1)$ is defined by (64) with $\mathcal{N}^2_H$ and also that $\lambda_1 = P_{\pm}$ satisfies Eq. (62), i.e., the square root in Eq. (64) being equal to zero. It is possible to further define $P_{\text{ex}}$ by putting $\delta = -\sigma_1 \text{sgn} \left[ \left( A^H_1 \right) \left( \omega_H \hat{\phi} - \hat{\omega}\phi_H \right) \right]$ and $W = W_{\text{ex}}$, where $W_{\text{ex}}$ has the following relation to $W_{\text{end}}$:

$$W_{\text{ex}} = \frac{\hat{g}_{\varphi\varphi}^H \left( \omega_H \hat{\phi} - \hat{\omega}\phi_H \right)^2}{\mathcal{N}^2_H \left( A^H_1 \right)^2} W_{\text{end}}. \quad (A3)$$

One can make sure that $A_1(E_1, \lambda_1, m_1)$ reaches an extremum with respect to $\lambda_1$ for $\lambda_1 = P_{\text{ex}}$. 

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