Conjugate coupling-induced symmetry breaking and quenched oscillations

K. Ponrasu¹, K. Sathiyadevi¹, V. K. Chandrasekar¹ and M. Lakshmanan²

¹ Centre for Nonlinear Science & Engineering, School of Electrical & Electronics Engineering, SASTRA Deemed University - Thanjavur-613 401, Tamil Nadu, India
² Centre for Nonlinear Dynamics, School of Physics, Bharathidasan University Tiruchirappalli-620 024, Tamil Nadu, India

received 10 September 2018; accepted in final form 23 October 2018
published online 26 November 2018

PACS 05.45.Xt – Synchronization; coupled oscillators
PACS 05.65.+b – Self-organized systems
PACS 05.45.-a – Nonlinear dynamics and chaos

Abstract – Spontaneous symmetry breaking (SSB) is essential and plays a vital role in many natural phenomena, including the formation of the Turing pattern in organisms and complex patterns in brain dynamics. In this work, we investigate whether a set of coupled Stuart-Landau oscillators can exhibit spontaneous symmetry breaking when the oscillators are interacting through dissimilar variables or conjugate coupling. We find the emergence of the SSB state with coexisting distinct dynamical states in the parametric space and show how the system transits from symmetry breaking state to out-of-phase synchronized (OPS) state while admitting multistabilities among the dynamical states. Further, we also investigate the effect of the feedback factor on SSB as well as oscillation quenching states and we point out that the decreasing feedback factor completely suppresses SSB and oscillation death states. Interestingly, we also find that the feedback factor completely diminishes only symmetry breaking oscillation and oscillation death (OD) states but it does not affect the nontrivial amplitude death (NAD) state. Finally, we have deduced the analytical stability conditions for in-phase and out-of-phase oscillations, as well as amplitude and oscillation death states.

Introduction. – Coupled nonlinear oscillatory systems exhibit distinct collective dynamical behaviors including various synchronization patterns [1–4], and oscillation quenching states [5–8], which have been observed in physical, chemical and biological systems [9–11]. Among them, one of the intriguing phenomena observed recently is the symmetry breaking states which play a crucial role in many natural systems. Importantly, in biological systems the spontaneous symmetry breaking (SSB) is essential for cell movement, developmental patterning of vertebrates and also for Turing pattern formation in organisms [12,13]. In the neural network, SSB leads to the formation of diverse complex patterns [14]. Due to the above features, the mechanism underlying such SSB and the situation in which such SSB occurs are of great interest. Recently, the trade-off between the attractive and repulsive coupling-induced SSB has been reported [15].

Oscillation quenching is another emergent intriguing phenomenon found in various chemical and biological systems [16–19]. The quenching of oscillation is further subdivided into two categories i.e., i) amplitude death (AD) and ii) oscillation death (OD). The AD exists when the coupled oscillators interact in such a way as to suppress each other’s oscillations and collectively attain a homogeneous steady state [20]. In the earlier studies, it has been reported that three main factors can cause AD, namely i) coupling through dissimilar variables [21], ii) time delay [22–25] and iii) frequency mismatch of the oscillators [26]. Later, such dynamical behavior was also observed in dynamic coupling [27], direct-indirect coupling [28] and even in mean-field diffusive coupling [7]. In many practical situations the nontrivial AD is a desirable control mechanism to suppress harmful oscillations in laser systems and neuronal activity, where a constant output is needed and fluctuations should be suppressed [29–31]. Further, the coupled oscillators can exhibit another type of oscillation quenching phenomenon, namely oscillation death (OD), where the
strong interaction among the oscillators leads the system to attain an inhomogeneous steady state (IHSS) [17,32]. The OD can occur under distinct coupling schemes such as time-delayed coupling [33,34] conjugate coupling [35], mean-field diffusive [7] and repulsive interactions [36]. The OD state exhibits significant implications in many biological systems such as synthetic genetic networks [37,38], cellular differentiation [39] and cardiovascular phenomena [40]. Even though the OD state has many practical applications, in some situations the oscillation death is detrimental and should be avoided. Later, many investigations have been carried out to control oscillation death or revival of oscillation from death and several authors have reported the ubiquitous phenomenon of feedback as a mechanism to control the death state [41–43].

Moreover, many investigations have been accomplished to study the dynamical behavior in coupled oscillators where the coupling among the oscillations is established through similar variables. On the other hand, in many real situations coupling via dissimilar variable is also desirable and such a coupling is known as conjugate coupling. In a variety of experimental situations the coupling via a conjugate variable is essential where the subsystems are coupled by feeding the output of one into the other. For example, in laser experiments, [44] the light emitted from each laser diode is monitored by a photo-diode detector (whose AC signal is amplified or reduced) by feedback to the other. Further, in the ecological context, the dynamics of two predator-prey systems that are coupled via cross-predation is also investigated, in which each predator consumes the prey in the other system which leads to stabilizing the food web to a new equilibrium [45]. In addition, theoretical predictions reveal that the time delay is not necessary, and coupling with dissimilar variables can lead to AD even when the coupling is bidirectional and diffusive [21]. Further, the emergence of a large number of states such as homogeneous and inhomogeneous oscillation death states and homogenous oscillatory states has been found in a ring of identical Stuart-Landau oscillators [49], which are coupled through the mean field of the dissimilar variables. The governing equations can be given by

\[ \dot{z}_k = f(z_k) + \epsilon [\text{Im}(Z) - \alpha \text{Re}(z_k)], \quad k = 1, 2, \quad (1) \]

where \( f(z_k) = (\lambda + i \omega - |z_k|^2)z_k \) and \( z_k = r e^{i\phi_k} = x_k + iy_k \in \mathbb{C}, \) \((k = 1, 2)\). \( x_k \) and \( y_k \) are the state variables of the system. \( Z = z_1 + z_2 \), is the mean field of the coupled system. The intrinsic parameter of the system is denoted by \( \lambda \) which corresponds to the distance from the Hopf bifurcation point [50]. \( \omega \) is the natural frequency, \( \epsilon \) is the coupling strength and \( \alpha \) \((0 \leq \alpha \leq 1)\) is the feedback control parameter. To solve eq. (1) we use the Runge-Kutta fourth-order method with time step value of 0.01.

**Dynamical behavior and symmetry breaking dynamics.** – At first, the dynamical behavior of the collective system is inspected through time series for a fixed value of natural frequency \( \omega = 0.5 \) (for fixed values of \( \lambda = 1.0 \) and \( a = 1.0 \)) and by varying the coupling strength \( \epsilon \) in fig. 1. Emergence of in-phase synchronized (IPS) oscillations is observed at \( \epsilon = 0.2 \) as shown in fig. 1(a). In this state, both oscillators oscillate with the same amplitude and phase. On increasing the coupling strength to \( \epsilon = 0.4 \), both oscillators attain a nontrivial homogeneous steady state or amplitude death (NAD) state (see fig. 1(b)). Figure 1(c) depicts that at a critical value of the coupling strength \( \epsilon = 0.5 \), the system exhibits a spontaneous symmetry breaking (SSB) state, where both oscillators oscillate with different amplitudes. It is due to spontaneous breaking of the permutational or translational symmetry \((z_1 \pm z_2)\) of the system. Further, by increasing the coupling strength to \( \epsilon = 0.6 \), one can note that the oscillators adjust their amplitudes so as to give rise to out-of-phase synchronized (OPS) oscillations as shown in fig. 1(d). Increasing the coupling strength to a higher value \( \epsilon = 0.9 \)
leads to the oscillation death (OD) state where both oscillators attain inhomogeneous steady states (IHSS) (see fig. 1(e)).

In addition, to understand the dynamical transitions and multistability of the considered system, a bifurcation analysis has been carried out by fixing the natural frequency at $\omega = 0.5$. The corresponding bifurcation diagrams are illustrated in fig. 2 (obtained using XPPAUT software). At this frequency $\omega = 0.5$, the systems exhibit an IPS state for lower values of the coupling strength $\epsilon$ (see fig. 2) and it is evident that increasing $\epsilon$ destabilizes the IPS state and gives birth to a nontrivial amplitude death (NAD) state. The observed NAD state is stable for values of $\epsilon < 0$, and it gives birth to an IPS state for higher values of $\epsilon$. The IPS state is further destabilized while increasing $\epsilon$, which gives birth to the oscillation death (OD) state at strong values of $\epsilon$ via subcritical pitchfork bifurcation ($PB_s$). Additionally, the coexistence of the NAD state is observed in the OD region (see fig. 2). Besides, the bifurcation analysis indicates that both oscillators in the IPS state (i.e., both oscillators oscillate with the same amplitude and phase) transit to the nontrivial AD state (i.e., the oscillators attain the homogeneous steady state with nontrivial values) on increasing the strength of coupling. On the other hand, the OPS state (oscillations with the same amplitude and anti-phase) emerges from the SSB (oscillators oscillate with different amplitudes) state and transit to the oscillation death state on increasing the coupling strength. The above detailed bifurcation analysis was carried out along the line $L_1$ in fig. 3(a). The inset in fig. 2 illustrates the enlarged image of the dashed lines (grey), which delineates the dynamical transition in the symmetry breaking regions. Before the stabilization of the IPS state, the emergence of the stable SSB state is evident from the inset of fig. 2, which is further destabilized via saddle-node (SN) bifurcation. In addition, as noted before we find that the OPS state is stabilized via pitchfork bifurcation of periodic orbits ($PB_P$). The shaded region in the inset denotes the coexistence of OPS and SSB states. Comparing fig. 2 and the inset, one can note that the shaded region denotes the bistability region where SSB, OPS and NAD states are stable. The global dynamical behavior of eq. (1) is further detailed in the following section.

**Global behavior of the coupled system.** – In order to understand the global dynamical behavior of the system (1), a two-parameter phase diagram is constructed in the ($\epsilon, \omega$)-space as shown in fig. 3. From fig. 3(a), at lower values of $\omega$, for increasing coupling we have observed stable IPS states at very low ranges of $\epsilon$, and the increment of $\epsilon$ further gives rise to the NAD state for all values of the coupling strength. In this case, the IPS state loses its stability and ultimately the system transits to the stable NAD state. The increase of the coupling strength also exhibits bistability between OD and NAD states. At lower values of the coupling strength, on increasing the natural frequency the system transits from the monostable nature of the IPS state to the bistable state. In other words, the coexistence of IPS and OPS states can be observed. It is also noticed that at critical values of $\epsilon$ and $\omega$, the system.

![Bifurcation diagram](image-url)

**Fig. 2:** (Colour online) Bifurcation diagram for a pair of conjugately coupled Stuart-Landau oscillators for $\omega = 0.5$ and the enlarged image of dashed lines (grey) is shown in the inset. In the figure the dotted (black) lines and solid (red) line represent the unstable and stable oscillation death states. The filled circles and triangles denote stable in-phase synchronized and out-of-phase synchronized states, respectively. Spontaneous symmetry breaking states are denoted by diamond points (pink) and the unfilled circles represent unstable limit cycle oscillations. SN, $PB_P$ and $PB_s$ are the saddle-node, pitchfork bifurcation of periodic orbits and subcritical pitchfork bifurcation points, respectively. Figure 2 is also plotted along the line $L_1$ in fig. 3. Other parameters are the same as in fig. 1.
system exhibits SSB. In this state the transition to the OPS state takes place through symmetry breaking oscillations and the NAD and OPS states also coexist with SSB. At strong values of $\epsilon$ and $\omega$ the system admits only the NAD state. The regions $R_1$, $R_2$, $R_3$ and $R_4$ represent the bistability regions between IPS-OPS, OPS-NAD, SSB-NAD and NAD-OD, respectively. Further, to inspect the dynamical transitions in the symmetry breaking region, we enlarge the symmetry breaking region in fig. 3(b) where we find two distinct types of transitions along the lines $L_1$ and $L_2$ which are illustrated through a study of the basin of attraction in fig. 4. Interestingly, in the SSB region we find the coexistence of three stable states (i.e., tristability), namely SSB, OPS and NAD, which is denoted as $R_5$ in fig. 3(b). Additionally, we have investigated how the transition takes place from the SSB to the OPS state in the following section.

Dynamical transition through the basin of attraction in the SSB region. — In order to understand the multistabilities among the dynamical states and transition route to OPS in the SSB region, we portray the basin of attraction in fig. 4 by fixing $y_1(0) = -0.45$ and $y_2(0) = 0.5$ and by varying $x_1(0)$ and $x_2(0)$. Figure 4(A): (a)–(d) are plotted for the dashed line along $L_1$ in fig. 3(b) at $\omega = 0.5$. At coupling strength $\epsilon = 0.5$, the coexistence of SSB and NAD states in the basin of attraction, that is the SSB state for some initial conditions and the NAD state for other initial conditions (see fig. 4(A)(a)). Here, we notice that the symmetric initial conditions (i.e., both $x_1(0)$ and $x_2(0)$ are either positive or negative) lead to NAD, whereas the asymmetric initial states give rise to SSB states (i.e., in the first and third quadrants where either $x_1(0)$ or $x_2(0)$ is positive while the other one is negative). On increasing the coupling strength $\epsilon = 0.507$, the emergence of the OPS branch inside the SSB region is observed in the basin of attraction. Here, the coexistence of NAD, SSB and OPS states are noticed with respect to distinct initial conditions. On a further increment of $\epsilon$ to 0.509, the basin of attraction for the OPS state is increased and the corresponding SSB basin is decreased. By increasing the coupling strength further, it is seen that the SSB basin is completely occupied by the OPS basin. In this way the SSB state transits to the OPS state through increasing the OPS basin smoothly as a function of the coupling strength. The transition is also analyzed along the line $L_2$ in fig. 3(b) at $\omega = 0.43$ in the SSB region, and the corresponding basin of attraction is depicted in figs. 4(B)(a), (B)(b). The SSB state in the first and third quadrant is directly replaced by the OPS basin during a small increment of the coupling strength from $\epsilon = 0.576$ to $\epsilon = 0.577$. From the basin stability analysis, we find that the coexistence of distinct dynamical states in the basin of attraction causes multistability in the parametric space and we confirm that two distinct types of transition to OPS from the SSB exist as well as the coexistence of SSB-NAD, SSB-OPS-NAD and OPS-NAD states. From the basin of attraction, it is also clear that the symmetric initial states lead only to NAD and IPS states, whereas the asymmetric initial states give rise to OPS and SSB states. Further, the effects of the feedback factor on stable SSB, NAD and OD states are demonstrated in the following section.

Impact of feedback factor on spontaneous symmetry breaking (SSB) state and oscillation quenching states. — To exemplify the effect of the feedback factor on SSB we have plotted the regions of SSB for three
The feedback factor, namely $a = 1.0$, $a = 0.4$ and $a = 0.2$ in the $(\epsilon, \omega)$-space in Fig. 5(a). It is evident that there is a decrease of the SSB regions with decreasing feedback factor $a$. From Fig. 5(a), it is observed that the SSB region is relatively large for $a = 1$ compared to the SSB regions with $a = 0.4$ and $a = 0.2$. Decreasing the feedback factor decreases the SSB region and finally there will be no symmetry breaking oscillations for $a = 0$ which completely suppresses the SSB states. Additionally, the impact of the feedback factor is also analyzed in the NAD and OD regions. The shrinking of the oscillation death is evident from Fig. 5(b). The shaded region in Fig. 5(b) denotes the OD region for $a = 1$ and distinct lines represent different boundaries of NAD and OD for different values of the feedback factor. Decreasing the feedback factor only shrinks the oscillation death state but not the nontrivial AD state which remains even in the absence of the feedback factor. Figure 5(c) is plotted for $\omega$ as a function of the feedback factor $a$. With respect to the feedback factor the boundary of the OD region attains null values of $\omega$ but the nontrivial AD states take nonzero values of $\omega$, which confirms the retaining of AD regions. Furthermore, to confirm the observed results, the bifurcation diagram is plotted at $a = 0$ in Fig. 5(d) which clearly depicts the emergence of NAD even in the absence of the feedback factor. It is observed that the oscillatory region is increased compared to the presence of the feedback factor. Finally, the system shows stable NAD at large coupling strengths and it confirms that the OD state is entirely suppressed. From the above results we conclude that the feedback factor completely suppresses the SSB and OD regions only but it sustains the NAD region.

The stability of the observed in-phase and out-of-phase oscillations, NAD and OD states, is detailed in the next section.

**Stability of distinct dynamical states.** — In order to study the stability of the observed dynamical states, we first rewrite eq. (1) by separating the symmetric ($z_+$) and anti-symmetric ($z_-$) variables, where $z_+ = (z_1 + z_2)/2$ and $z_- = (z_1 - z_2)/2$. Equation (1) in terms of these new variables is given by

$$\dot{z}_+ = \frac{1}{2}(f(z_+ + z_-) + f(z_+ - z_-)) + i\epsilon \text{Im}(z_+) - \epsilon a \text{Re}(z_+),$$

$$\dot{z}_- = \frac{1}{2}(f(z_+ + z_-) - f(z_+ - z_-)) - \epsilon a \text{Re}(z_-).$$

(2a)

(2b)

In the in-phase subspace, $Z_+ = \{(z_+, z_-)|z_+ \equiv 0\}$, and in the anti-symmetric subspace, $Z_- = \{(z_+, z_-)|z_+ \equiv 0\}$. Thus, in the symmetric and anti-symmetric subspaces, the dynamical equations can be reduced, respectively, to

$$\dot{z}_+ = f(z_+) + i\epsilon \text{Im}(z_+) - \epsilon a \text{Re}(z_+), \quad \dot{z}_- = 0,$$

(3)

and

$$\dot{z}_- = f(z_-) - \epsilon a \text{Re}(z_-), \quad \dot{z}_+ = 0.$$  

(4)

We find the explicit expressions for the different oscillatory states, steady states and their stabilities in the following. 

a) *Dynamical states in symmetric subspace*: The solution of the corresponding dynamical equation in the symmetric subspace, eq. (3), is found to be expressed as

$$x(t) = \frac{e^{\psi t} \cos[\delta - \frac{1}{2} \psi t]}{C - \frac{1}{\psi^2} e^{\psi t} [U_0 + U_1 \cos \psi t + U_2 \sin \psi t]},$$

$$y(t) = \frac{1}{2\omega} \left(-ea + \psi \tan \left(\frac{1}{2} \psi t - \delta\right)\right) x(t),$$

(5)

where $\psi = \sqrt{4\omega \omega - \epsilon^2 a^2}$, $\lambda = 2\lambda - \epsilon a$, $\beta_1 = -\lambda((2+\epsilon^2)\cos 2\omega + a(1+a^2)\sin 2\omega - 2\omega^2) + a(1+a^2)\epsilon^2 +\epsilon \omega - 2\omega^2 \epsilon$, $\beta_2 = (-a\lambda + \epsilon(1+a^2) - \omega)\psi$ and $\omega = \omega - \epsilon$. $C$ and $\delta$ are integration constants. The other constants $U_0$, $U_1$ and $U_2$ are

$$U_0 = \frac{2(\cos - 2\omega)\omega}{\lambda}, \quad U_1 = \frac{\epsilon(\beta_1 \cos 2\delta + \beta_2 \sin 2\delta)}{2(\lambda^2 - \epsilon a + \omega^2)},$$

$$U_2 = \frac{\epsilon(\beta_1 \sin 2\delta - \beta_2 \cos 2\delta)}{2(\lambda^2 - \epsilon a + \omega^2)}.$$  

(6)

Further, we can find that the solution of eq. (5) will be periodic when $4\omega \omega > \epsilon^2 a^2$. In this case, we can write the state variables of $x_k(t)$ and $y_k(t)$, $k = 1, 2$, in the asymptotic limit ($t \to \infty$) as

$$x_1(t) = \frac{\cos(\delta - \frac{1}{2} \psi t)}{(U_0 + U_1 \cos \psi t + U_2 \sin \psi t)^\frac{1}{2}},$$

$$y_1(t) = \frac{1}{2\omega} \left(-ea \cos(\delta - \frac{1}{2} \psi t) + \psi \sin(\delta - \frac{1}{2} \psi t)\right) \frac{1}{(U_0 + U_1 \cos \psi t + U_2 \sin \psi t)^\frac{1}{2}},$$

(7)

with $x_2 = x_1$ and $y_2 = y_1$. Using the Floquet theory and by integrating the above equations from $t = 0$ to $t = T = \frac{2\pi}{\sqrt{4\omega \omega - \epsilon^2 a^2}}$, we determine the Floquet multipliers.
ρ_i (i = 1, ..., 4) and we conclude that when all the four eigenvalues lie within the unit circle, then the corresponding periodic orbit is stable. Using the Floquet multipliers we have obtained the boundary for in-phase synchronized oscillations.

Whenever 4ωω < ε^2a^2, the solution in eq. (5) tends to a homogeneous steady state which is given by

\[ x_1^* = \pm \frac{1}{\sqrt{2}} \left( \frac{1}{a^2} (\epsilon a^2 \beta_3 + \Omega(2\lambda + \psi_1) - a(\beta_4 + \lambda \psi_1)) \right)^{\frac{1}{2}}, \]
\[ y_1^* = \pm \frac{1}{\sqrt{2}} (\epsilon a + \psi_1)x_1^*, \] (8)

with \( x_2^* = x_1^* \) and \( y_2^* = y_1^* \). Other constants are \( \psi_1 = \sqrt{\epsilon^2 a^2 - 4\omega^2} \), \( \beta_3 = (\epsilon a + \lambda + \psi) \), \( \beta_4 = (\epsilon^2 + \epsilon \omega - 2\omega^2) \). The eigenvalues corresponding to eq. (8) can be expressed as

\[ \mu_{1,2} = \frac{1}{2} (-\lambda - 2\psi_1 \mp \sqrt{\lambda^2}), \quad \mu_{3,4} = \frac{1}{2} (-\lambda - 2\psi_1 \pm \sqrt{\lambda^2}) \] (9)

where \( \beta_5 = a^2(\epsilon - 4a\lambda) + a^2(\epsilon^2 + 4\lambda^2) \) and \( \beta_6 = \lambda^2 + (\epsilon - 2\omega)\omega \). The stable region of NAD emerges when \( \omega < \frac{1}{2}(\epsilon + \sqrt{\epsilon^2 + a^2\epsilon^2}) \). It is also clear that when varying the coupling strength \( \epsilon \), there exists a direct transition from stable IPS to the NAD state. The analytical boundaries for IPS and NAD states get well fitted with the numerically obtained boundaries (see fig. 3).

b) Dynamical states in the anti-symmetric subspace: In this section, we have also found the solution for the dynamical equation in the anti-symmetric subspace, eq. (4), which can be written as

\[ x(t) = \frac{e^{\frac{1}{2}(\lambda - \psi) t}}{(C + e^{\lambda t}(V_0 + V_1 \cos(2\delta - \theta t) + V_2 \sin(2\delta - \theta t)))^{\frac{1}{2}}}, \]
\[ y(t) = \frac{1}{2\omega} \left( -\epsilon a + \theta \tan \left( \frac{1}{\sqrt{2}} (\Delta - \delta) \right) \right) x(t), \] (10)

where \( \theta = \sqrt{4\omega^2 - \epsilon^2 a^2} \). \( C \) and \( \delta \) are integration constants. The other constants \( V_0, V_1 \) and \( V_2 \) are

\[ V_0 = 2\lambda - \epsilon a, \quad V_1 = \frac{\epsilon a(\epsilon a - \epsilon^2 a^2 + 2\omega^2)}{4\omega^2(\lambda^2 - \epsilon a + \omega^2)}, \]
\[ V_2 = \frac{\epsilon a(\epsilon a - \epsilon^2 a^2)}{4\omega^2(\lambda^2 - \epsilon a + \omega^2)}. \] (11)

The solution given by eq. (10) is periodic when \( 4\omega^2 > \epsilon^2 a^2 \). In the asymptotic limit \( (t \rightarrow \infty) \), the state variables \( x_k \) and \( y_k \), \( k = 1, 2 \), can be written as

\[ x_1(t) = \frac{\cos(\delta - \frac{1}{2}\theta t)}{(V_0 + V_1 \cos(2\delta - \theta t) - V_2 \sin(2\delta - \theta t))^\frac{1}{2}}, \]
\[ y_1(t) = \frac{-1}{\omega} \left( \frac{V_0 + V_1 \cos(2\delta - \theta t) + V_2 \sin(2\delta - \theta t)}{(V_0 + V_1 \cos(2\delta - \theta t) - V_2 \sin(2\delta - \theta t))^\frac{1}{2}} \right). \] (12)

with \( x_2 = -x_1 \) and \( y_2 = -y_1 \). Note that the solution given in eq. (12) is periodic with respect to the period \( t = T = \frac{2\pi}{\sqrt{4\omega^2 - \epsilon^2 a^2}} \). Then, using the Floquet multipliers, we have figured out the boundary of the stable OPS state.

Further, the solution in eq. (10) tends toward a steady state when \( 4\omega^2 < \epsilon^2 a^2 \), which can be expressed as

\[ x_1^* = \pm \frac{1}{\sqrt{2}} \left( \frac{1}{a^2} (\epsilon a^2 \beta_3 + \Omega(2\lambda + \psi_1) - a(\beta_4 + \lambda \psi_1)) \right)^{\frac{1}{2}}, \]
\[ y_1^* = \pm \frac{1}{\sqrt{2}} (\epsilon a + \psi_1)x_1^*, \] (13)

with \( x_2^* = -x_1^* \) and \( y_2^* = -y_1^* \). \( \theta_1 = \sqrt{\epsilon^2 a^2 - 4\omega^2} \); we have studied the stability of such anti-symmetric steady states, eq. (13), too and found that the eigenvalues corresponding to the states are given by

\[ \mu_{1,2} = \frac{1}{2} (-\lambda - 2\theta_1 \mp \sqrt{\lambda^2}), \quad \mu_{3,4} = \frac{1}{2} (-\lambda - 2\theta_1 \pm \sqrt{\lambda^2}) \] (14)

where \( \lambda = (\lambda - \epsilon a) \). From the above eigenvalues we have found the stability condition for the oscillation death state. When \( \omega < \frac{4\epsilon a}{\pi} \), the stable OD region emerges. The obtained analytical stability curves of the OPS and OD regions exactly match with the numerically obtained boundaries (see fig. 3). From this, we have verified the stability curves of the observed dynamical states of IPS, OPS, NAD and OD regions analytically, too.

Conclusion. – We have investigated the dynamical behavior in a pair of conjugately coupled Stuart-Landau oscillators. Interestingly, we found the emergence of spontaneous symmetry breaking due to breaking of permutational symmetries. We also noticed the coexistence of distinct dynamical states such as SSB-NAD, NAD-OD, and IPS-OPS states in the parametric space. Through a basin of attraction study, the dynamical transition to the OPS state was illustrated in the symmetry breaking region. In particular, we report that the transition to the OPS was found as a smooth and sudden transition as a function of the coupling strength. Further, we have analyzed the effect of the feedback factor on the SSB, NAD and OD states. We found that the decreasing feedback factor completely shrinks the symmetry breaking oscillation state and oscillation death state but it does not completely suppress the nontrivial AD state. Finally, we have also deduced the analytical expressions for the observed in-phase and out-of-phase oscillations as well as nontrivial amplitude death and oscillation death states. We also find that the feedback factor entirely shrinks the oscillation death states and nontrivial AD states sustained even in the absence of the feedback factor. The obtained results are in good agreement with laser applications where we need to suppress unwanted fluctuations and attain constant output. We believe that the present study will shed light on
the dynamics of symmetry breaking states and control of such dynamical behavior, which will be helpful in many physical and biological systems.

***

KP and VKC are supported by SERB-DST Fast Track scheme for young scientists under Grant No. YSS/2014/000175. KS sincerely thanks the CSIR for the fellowship under SRF Scheme (09/1095(0037)/18-EMR-I). The work of ML is supported by a DST-SERB Distinguished Fellowship program.

REFERENCES

[1] Pikovsky A. S., Rosenblum M. G. and Kurths J., Synchronization. A Universal Concept in Nonlinear Sciences (Cambridge University Press, Cambridge, UK) 2001.
[2] Tinsley M. R., Nkomo S. and Showalter K., Nat. Phys., 8 (2012) 662.
[3] Williams C. R. S., Murphy T. E., Roy R., Sorrentino F., Dahms T. and Schöll E., Phys. Rev. Lett., 110 (2013) 064104.
[4] Pecora L. M., Sorrentino F., Hagerstrom A. M., Murphy T. E. and Roy R., Nat. Commun., 5 (2014) 4079.
[5] Aronson D. G., Ermentrout G. B. and Kopell N., Physica D, 41 (1990) 403.
[6] Mirollo R. E. and Strogatz S. H., J. Stat. Phys., 60 (1990) 245.
[7] Banerjee T. and Ghosh D., Phys. Rev. E, 89 (2014) 062902; 052912.
[8] Zou W., Senthilkumar D. V., Duan J. and Kurths J., Phys. Rev. E, 90 (2014) 032906.
[9] Matthews P. C. and Strogatz S. H., Phys. Rev. Lett., 65 (1990) 1701.
[10] Dolnik M. and Epstein I. R., Phys. Rev. E, 54 (1996) 3361.
[11] Hoh A., Gavrielides A., Erneux T. and Kovanis V., Phys. Rev. Lett., 78 (1997) 4745.
[12] Sawi S., Maeda Y. and Sawada Y., Phys. Rev. Lett., 85 (2000) 2212.
[13] Pismen L. M., J. Chem. Phys., 101 (1994) 3135.
[14] Singh R., Menon S. N. and Sinha S., Sci. Rep., 6 (2016) 22074.
[15] Satihyadevi K., Kartigha S., Chandrasekar V. K., Senthilkumar D. V. and Lakshmanan M., Phys. Rev. E, 95 (2017) 042301.
[16] Hynne F. and Graae Soeren P., J. Phys. Chem., 91 (1987) 6573.
[17] Koskela A., Volkov E. and Kurths J., Phys. Rep., 531 (2013) 173.
[18] Zou W., Senthilkumar D. V., Nagao R., Kiss I. Z., Tang Y., Koskela A., Duan J. and Kurths J., Nat. Commun., 6 (2015) 7709.
[19] Chen J., Liu W., Zhu Y. and Xiao J., EPL, 115 (2016) 20011.
[20] Saxena G., Prasad A. and Ramaswamy R., Phys. Rep., 521 (2012) 205.
[21] Karnatak R., Ramaswamy R. and Prasad A., Phys. Rev. E, 76 (2007) 035201.
[22] Ramana Reddy D. V., Sen A. and Johnston G. L., Phys. Rev. Lett., 80 (1998) 5109; 85 (2000) 3381.
[23] Strogatz S. H., Nature (London), 394 (1998) 316.
[24] Prasad A., Phys. Rev. E, 72 (2005) 056204; Vicente R., Tang S., Mulet J., Mirasso C. R. and Liu J. M., Phys. Rev. E, 73 (2006) 047201.
[25] Bera B. K., Hens C. R. and Ghosh D., Phys. Lett. A, 380 (2016) 2366.
[26] Mirolo R. E. and Strogatz S. H., J. Stat. Phys., 60 (1990) 245.
[27] Konishi K., Phys. Rev. E, 68 (2003) 067202.
[28] Majhi S., Bera B. K., Bhowmick S. K. and Ghosh D., Phys. Lett. A, 380 (2016) 3617.
[29] Kim M. Y., Roy R., Aron J. L., Carr T. W. and Schwartz I. B., Phys. Rev. Lett., 94 (2005) 088101.
[30] Prasad A., Lai Y. C., Gavrielides A. and Kovanis V., Phys. Lett. A, 318 (2003) 71.
[31] Zhai Y., Kiss I. Z. and Hudson J. L., Phys. Rev. E, 69 (2004) 026208.
[32] Koseska A., Volkov E. and Kurths J., Phys. Rev. Lett., 111 (2013) 024103.
[33] Zou W., Senthilkumar D. V., Tang Y. and Kurths J., Phys. Rev. E, 86 (2012) 036210.
[34] Zou W., Senthilkumar D. V., Duan J. and Kurths J., Phys. Rev. E, 90 (2014) 032906.
[35] Han W., Cheng H., Dai Q., Li H., Ju P. and Yang J., Commun. Nonlinear Sci. Numer. Simul., 39 (2016) 73.
[36] Hens C. R., Olusola O. I., Pal P. and Dana S. K., Phys. Rev. E, 88 (2013) 034902.
[37] Koseska A., Volkov E., Zaikin A. and Kurths J., Phys. Rev. E, 75 (2007) 031916.
[38] Ullner E., Zaikin A., Volkov E. I. and Garca-Ojalvo J., Phys. Rev. Lett., 99 (2007) 148103; Ullner E., Koseska A., Kurths J., Volkov E., Kantz H. and Garca-Ojalvo J., Phys. Rev. E, 78 (2008) 031904.
[39] Goto Y. and Kaneko K., Phys. Rev. E, 88 (2013) 032718.
[40] Sured-Vargas J. J., Gonzalez J. A., Stefanovska A. and McClintock P. V. E., EPL, 85 (2009) 38008.
[41] Deng T., Liu W., Xiao J. and Kurth J., Chaos, 26 (2016) 094813.
[42] Chandrasekar V. K., Kartigha S. and Lakshmanan M., Phys. Rev. E, 92 (2015) 012903.
[43] Majhi S. and Ghosh D., EPL, 118 (2017) 40002.
[44] Kim M. Y., Roy R., Aron J. L., Carr T. W. and Schwartz I. B., Phys. Rev. Lett., 94 (2005) 088101.
[45] Karnatak R., Ramaswamy R. and Feudel U., Chaos, Solitons Fractals, 68 (2014) 48.
[46] Zhao N., Sun Z., Yang X. and Xu W., EPL, 118 (2017) 30005.
[47] Zhao N., Sun Z., Yang X. and Xu W., Phys. Rev. E, 97 (2018) 062203.
[48] Raksit S., Bera B. K., Majhi S., Hens C. R. and Ghosh D., Sci. Rep., 7 (2017) 45909.
[49] Schmidt I., Schönherr K., Krischer K. and García-Morales V., Chaos, 24 (2014) 013102.
[50] Kundu S., Majhi S., Karmakar P., Ghosh D. and Raksit B., EPL, 123 (2018) 30001.