Dispersion relation of nutation surface spin waves in ferromagnets

Mikhail Cherkasskii¹ *, Michael Farle²,³, and Anna Semisalova²

¹ Department of General Physics 1, St. Petersburg State University, St. Petersburg, 199034, Russia
² Faculty of Physics and Center of Nanointegration (CENIDE), University of Duisburg-Essen, Duisburg, 47057, Germany
³ Kirensky Institute of Physics, Federal Research Center KSC SB RAS, Krasnoyarsk, 660036, Russia

* Corresponding author: macherkassii@hotmail.com

Inertia effects in magnetization dynamics are theoretically shown to result in a new type of spin waves, i.e. nutation surface spin waves, which propagate at terahertz frequencies in in-plane magnetized ferromagnetic thin films. Considering the magnetostatic limit, i.e. neglecting exchange coupling, we calculate dispersion relation and group velocity, which we find to be slower than the velocity of conventional (precession) spin waves. In addition, we find that the nutation surface spin waves are backward spin waves. Furthermore, we show that inertia causes a decrease of the frequency of the precession spin waves, namely magnetostatic surface spin waves and backward volume magnetostatic spin waves. The magnitude of the decrease depends on the magnetic properties of the film and its geometry.

I. INTRODUCTION

From the classical point of view, spin waves are collective excitations of magnetically ordered materials, that is waves of precession of the magnetization [1,2], for example in thin magnetic films [3–5], layered magnetic structures [6,7], periodic magnetic crystals [8,9], and nanometer-sized structures [10]. These waves exhibit typical linear [1,2] and nonlinear wave effects [11–13], such as excitation [14,15], propagation [16–18], reflection [19,20], and interference [21,22] in the first case, and self-focusing [23–26], formation of envelope solitons [27,28], chaotic behavior [29], as well as parametric three- and four-waves processes [30,31] in the nonlinear case.

Recently, it has been theoretically and experimentally demonstrated that the effects of inertia of magnetization should be considered in the full description of spin dynamics at pico- and femtosecond timescales [32–38]. The nutation motion of magnetization is a manifestation of inertia of the magnetic moments. A rigorous derivation including inertia in the Landau-Lifshitz-Gilbert equation was carried out by R. Mondal et al. in the Dirac-Kohn-Sham framework [33,34]. A relation between the Gilbert damping and the inertial characteristic time was investigated in Ref. [32]. In another approach M.-C. Ciornei et al. confirmed that inertia is responsible for nutation, and that this motion is superimposed on the precession of magnetization [38]. The influence of nutation on the dynamic susceptibility was analytically [39] and numerically [40] studied.

Despite these theoretical advances, the experimental study of inertial spin dynamics has only begun. Following the first indirect observation of inertial magnetization dynamics in Ni₈₀Fe₂₁ and Co films [41], the first direct experimental confirmation of nutation resonance was reported by Neeraj et al. [37].

In this paper, we predict an additional effect, that is the emergence of propagating nutation surface spin waves (NSSW) in the dipole–dipole coupling limit, and the transformation of conventional precession waves to precession-nutation spin waves. We derive dispersion relation of NSSW and calculate the spectral shift of precession-nutation spin waves with respect to precession spin waves. The emergence of nutation waves due to exchange coupling rather than dipolar interaction has been proposed by Makhfudz et al. [42] recently.

In general, the following interactions must be taken into account to describe the dynamics of spin waves: Zeeman, spin-orbit, exchange, and dipole–dipole interactions. The phase shift between precessing magnetic moments propagates as a spin wave through the ferromagnet because of dipole–dipole or exchange coupling (Fig. 1(a)). Magnetic inertia effects, which are expected to contribute to dynamics of spin waves, originate from spin-orbit coupling (coupling of the spins to the lattice via the orbital moment). In magnetization dynamics, this relativistic effect is considered with different orders of approximation. In the lowest order, one obtains the Gilbert damping of magnetization precession and the gyromagnetic ratio, i.e., the relation between angular momentum and spin. In higher order approximations, magnetic inertia appears [33,34,43], and the gyromagnetic ratio must be generalized, which leads to nutation motion of magnetic moments superimposed on their precession. Taking inertia into account one finds that the deviation of localized moments will propagate through the spin system in the form of both precession and nutation motions, i.e. in ferromagnetic materials one needs to add to all “conventional” spin wave
modes a high frequency wave-like motion with small amplitude caused by inertia. Additionally, waves having predominantly inertial nature appear in ferromagnetic thin films, which we call here \textit{nutation surface spin waves}. Since these waves have terahertz frequencies (compared to typically GHz frequencies of other spin wave modes), they can be plotted as a small deviation on top of a “frozen” precession motion (Fig. 1(b)).

In our calculation, we work in the dipole–dipole coupling limit, which allows us to use a magnetostatic approach in which Maxwell’s equations are transformed into the Walker equation [4]. To obtain the dispersion relation including inertia, we use the dynamic susceptibility derived from the Inertial Landau–Lifshitz–Gilbert (ILLG) equation [39] and substitute the result into the Walker equation.

In this paper, we consider waves propagating in thin ferromagnetic films magnetized in-plane by an external magnetic field. We focus on two particular configurations: (A) waves propagating perpendicular the external magnetic field $H_0$ (see Fig. 1(c), y-axis), and (B) waves propagating along $H_0$ (Fig. 1(c), z-axis). The latter case (B) corresponds to backward volume magnetostatic spin waves (BVMSW), when only precession is taken into account and as n-BVMSW when precession-nutation case is considered. Similarly, for precession-nutation configuration we distinguish in our nomenclature between magnetostatic surface spin waves (MSSW), i.e. in other words Damon-Eshbach modes, when inertia is neglected, and n-MSSW when inertia is included. Finally, for precession-nutation configuration our calculation predicts a new type of waves – \textit{nutation surface spin waves}.

\section{II. Dispersion Equations and Wave Characteristics}

The ferromagnetic film, magnetic field and coordinate system are shown in Fig. 1(c). The film with thickness $L$ is placed in an external magnetic field $H_0$ strong enough to saturate the magnetization of the film. We assume that the exciting magnetic field is small $|h| \ll |H_0|$, and the static magnetization vector $M_0$ and external magnetic field $H_0$ are aligned.

The Maxwell’s equations in magnetostatics are written as
\begin{align}
\nabla \times h &= 0, \\
\nabla \cdot (h + m) &= 0,
\end{align}
where $m$ is the response of the magnetization to the small driving magnetic field. Equ. (1) allows to introduce the magnetic potential using $h = \nabla \psi$, substitute this potential into equation (2) and obtain the Walker’s equation
\begin{equation}
(1 + \chi \nabla^2 + \frac{\partial^2 \psi}{\partial y^2}) + \frac{\partial^2 \psi}{\partial z^2} = 0,
\end{equation}
where $\chi$ is the diagonal component of the dynamic susceptibility tensor (see the Supplemental Material [44]). The potential obeys Laplace’s equation outside of the film
\begin{equation}
\nabla^2 \psi = 0.
\end{equation}
Following the Ansatz and interface conditions from Ref. [4], the characteristic equations from Walker’s and Laplace’s equations are obtained
\begin{equation}
\begin{align*}
(1 + \chi) \left( k_x^2 + k_y^2 \right) + k_z^2 & = 0, \\
(k_x^2 - k_y^2 - k_z^2) & = 0,
\end{align*}
\end{equation}
where superscripts denote internal and external wavenumbers. The characteristic equations (5) describe the relationship between the transverse and longitudinal wavenumbers and determine the allowed range of wavenumber of propagating spin waves. To investigate propagating waves one employs the real parts of the dynamic susceptibility in equ. (6)
\begin{equation}
\begin{align*}
(k_x^2 - 2k_yk_z(1 + \chi \cot(k/L)) \\
- (k_y^2 - 1 + \chi \cot^2(k_z/L)) - k_z^2 \chi \alpha & = 0,
\end{align*}
\end{equation}
where $\chi'$ is the real dispersive part of $\chi$, and $\chi'_s$ is the real dispersive part of the anti-diagonal component of the dynamic susceptibility tensor. The effect of inertia of the magnetization is introduced by the dynamic susceptibility deduced from the ILLG equation. The detailed derivation for a Cartesian coordinate system can be found in the Supplemental Material [44]. In the following sections, we focus on the dispersion relations for spin waves propagating in perpendicular (A) and parallel (B) direction to the magnetic field.

### A. Perpendicular configuration

For spin waves propagating in perpendicular direction to the external magnetic field the equation (6) becomes

\[
(1 + \chi') \left( 1 + \coth(k, L) \right) - \frac{\chi'^2 - \chi'^2}{2} = 0, \tag{7}
\]

since $k_z$ should be equal to zero in this configuration. The substitution of the susceptibility expressions (equations (S7)-(S11) from Ref. [44]) into (7) allows to calculate the dispersion relation between frequency and wavenumber $k$.

This substitution leads to the bi-quartic equation

\[
A_c \omega^4 + B_c \omega^3 + C_c \omega^2 + D_c \omega + E_c = 0, \tag{8}
\]

where

\[
A_c = 2 \alpha' \tau \left( 1 + \coth(k, L) \right), \tag{9}
\]

\[
B_c = 2 \alpha' \tau \left( -2 + 2 \alpha'^2 - \alpha \tau (4 \omega_m + \omega_S) \right) \tag{10}
\]

\[
C_c = 2 + 2 \alpha' + 2 \alpha \tau (4 \omega_m + \omega_S) \tag{11}
\]

\[
D_c = -8 \alpha \tau (4 \omega_m + \omega_S) \tag{12}
\]

\[
E_c = \omega_S \left[ 2 \omega^2 + 2 \omega_m \omega_S - \omega^2 \right] + \omega_S \coth(k, L), \tag{13}
\]

We employ Ferrari's method for finding the solutions of equation (8), and introduce the notation:

\[
a' = \frac{C_c}{A_c} - \frac{3B_c^2}{8A_c}, \tag{15}
\]

\[
b' = \frac{B_c C_c + B_c^3 + D_c}{2A_c}, \tag{15}
\]

\[
c' = \frac{B_c^2 C_c - 3B_c^2}{16A_c^2} - \frac{B_c D_c + E_c}{4A_c^2} + \frac{E_c}{A_c}. \tag{15}
\]

In Ferrari's method, one determines the root of the nested depressed cubic equation written here as:

\[
y = \frac{-5a'}{6} + U + V, \tag{16}
\]

where

\[
U = \sqrt{\frac{P_c}{27}} \frac{Q_c^2}{4} \frac{Q_c}{2}, \tag{17}
\]

\[
V = \frac{P_c}{3}, \tag{17}
\]

\[
P_c = \frac{a'^2 - c'}{12}, \tag{17}
\]

\[
Q_c = \frac{1}{3} a' c - \frac{a'^3}{108} - \frac{b'}{8}. \tag{17}
\]

The bi-quartic equation (8) has four roots describing the relationship of frequency of wavenumber, i.e. different dispersion branches. The first branch corresponds to ferromagnetic resonance (FMR). The second one is the n-MSSW branch (Fig. 2(b) and (e)). The third and fourth branches are complex conjugates and describe NSSW, the real and imaginary parts are plotted in Fig. 2(b) and (c).

The n-MSSW branch is given by the expression

\[
\omega_{n\text{-MSSW}}^2 = \left( -\frac{B_c}{4A_c} - \frac{\sqrt{a'} + 2y}{2} \right)^{1/2} \tag{18}
\]

\[
+ \frac{1}{2} \sqrt{-3a' - 2y + \frac{2b'}{\sqrt{a' + 2y}}} \right)^{1/2} \tag{18}
\]

The frequency in (18) is real, hence n-MSSW propagate as ordinary waves. There is a spectral red-shift between the n-MSSW and MSSW branches, and in the following we study in detail the shift of the spectrum limits. The upper and lower limits of the spectrum are shifted down differently. Without nutation the dispersion branch of MSSW exists in the frequency range $\omega_1 < \omega < \omega_0$, where $\omega_0 = \omega_m + \omega_S / 2$, and $\omega_1 = \sqrt{\omega_m (\omega_m + \omega_S)}$. The upper spectrum limit of the n-MSSW $\omega_{n}^0$ can be calculated with the expression (18) at $k_y \to \infty$ that yields $\coth(k, L) = 1$. For instance, we calculate the difference between upper limits for the following parameters $\mu_b M_a = 1 \text{T}$, $\mu_b H_b = 100 \text{ mT}$, $\alpha = 0.0065$, and $\tau = 10^{-11} \text{ s}$. 

3
FIG. 2. (Color online) (a) The group velocity of nutation surface spin waves. (b) The dispersion branches of nutation surface spin waves (purple curves) in terahertz range, MSSW (red curves) and n-MSSW (green curves) in microwave range. (c) The inherent losses of nutation surface spin waves. (d) The group velocity of the n-MSSW. (e) The dispersion branches of MSSW (red curves) and n-MSSW (green curves) in magnified scale. The parameters for the calculation of precession-nutation waves (n-MSSW) are $\mu_n M_0 = 1$ T, $\mu_n H_0 = 100$ mT, $\alpha = 0.0065$, and $\tau = 10^{-11}$ s and for MSSW are the same parameters except $\alpha = 0$ and $\tau = 0$.

The difference is 115 MHz or 0.7 percent, and it can be clearly seen in the Fig 2(e). The decrease of frequencies of n-MSSW is caused by nutation, which is a rotation of magnetization in the opposite direction compared to the precession [44].

Due to the fact that the lower spectrum limit of the MSSW corresponds to upper spectrum limit of the BVMSW, we discuss the spectral red-shift of the lower limit of n-MSSW below.

The dispersion branch of NSSW is complex, and the real part of the branch describing propagation of the waves is determined by

$$\omega^{\text{SS}} = \frac{w_1 + w_2}{2},$$  \hspace{1cm} (19)

The imaginary part corresponding to the inherent losses of NSSW is written as

$$\omega^{\text{iSS}} = \frac{w_1 - w_2}{2i},$$  \hspace{1cm} (20)

where

$$w_1 = \left( \frac{B}{4A} + \frac{\sqrt{a_z + 2y}}{2} \right),$$

$$w_2 = \left( \frac{B}{4A} + \frac{\sqrt{a_z + 2y}}{2} \right) + \frac{1}{2} \sqrt{-3a_z - 2y - \frac{2b}{\sqrt{a_z + 2y}}} \quad (22)$$

The NSSW have inherent losses, since these waves exist only if $\alpha \neq 0$. This fact is the direct consequence of their inertial nature – these waves do not exist, if one neglects inertia and damping.

The magnetostatic potential of spin waves is concentrated close to the surface of the ferromagnetic film if the magnetic field is applied perpendicular to the wavevector, hence in this configuration conventional spin waves and nutation spin waves are surface waves. However, NSSW have lower group velocity than the precession waves that can be seen from the Fig. 2(a) and (d). The group velocity is calculated using the standard expression $v_g = \frac{\partial \omega}{\partial k_x}$. The negative value of the group velocity means that NSSW are backward waves.

The highest frequency of the NSSW is the nutation resonance frequency, the exact expression of the frequency was derived in Ref. [39] which can be approximated to $\Omega_N = (2\pi\alpha)^{-1}$. 

\[
\frac{\partial \omega}{\partial k_x} = \frac{B}{4A} + \frac{\sqrt{a_z + 2y}}{2} + \frac{1}{2} \sqrt{-3a_z - 2y - \frac{2b}{\sqrt{a_z + 2y}}} \quad (22)
\]
B. Parallel configuration

If one considers waves propagating parallel to the external magnetic field direction, \( k_y \approx 0 \), and the equation (6) can be simplified as

\[
1 + \frac{k_y L}{2} \cosh \left( \frac{k_y L}{\sqrt{1 + \chi^2}} \right) = 0. \tag{23}
\]

We substitute the susceptibility expressions (S7)-(S11) from Ref. [44] and employ numerical damped Newton's method for finding the roots of the algebraic equation (23) to calculate the relations between frequency and wavenumber in both precession and precession-nutation cases. These relations demonstrate a set of dispersion branches, and the first three branches are plotted in Fig. 3.

It is clearly seen from the Fig. 3, that the dispersion of n-BVMSW is shifted relatively to the BVMSW. To investigate this, we compare the spectral limits of the precession and precession-nutation waves. Note that the volume waves exist only in the frequency range where \( 1 + \chi' > 0 \). This condition is a consequence of equation (5), since propagating waves have real wavenumber \( k'_y \in \mathbb{R} \). Therefore, the spectrum limits are defined by

\[
1 + \chi' = 0, \tag{24}
\]

and in the non-damping and non-inertia case this equation determines the frequency range \( \omega_h < \omega < \omega_s \), where \( \omega_s = \sqrt{\omega_h (\omega_s + \omega_d)} \). If one takes nutation into account, the numerator of the equation (24) allows to find the frequency range. This numerator can be written as

\[
A_1 \omega^4 + B_1 \omega^2 + C_1 \omega^4 + D_1 \omega^2 + E_1 = 0, \tag{25}
\]

where

\[
A_1 = \alpha^2 \tau^4, \\
B_1 = \alpha^2 \tau^2 \left( -2 + 2 \alpha^2 - \alpha \tau \left( 4 \omega_h + \omega_d \right) \right), \\
C_1 = 1 + \alpha \tau \left( 2 \omega_h + \omega_d \right) + \alpha^2 \left( 2 + 3 \tau \omega_h \left( 2 \omega_h + \omega_d \right) \right) - \alpha \tau \left( 4 \omega_h + \omega_d \right) + \alpha^2, \\
D_1 = -\omega_h \left[ 4 \alpha \tau \omega_d + \alpha \tau \omega_d - \alpha^2 \omega_d \\
+ \omega_h \left( 2 - 2 \alpha^2 + 3 \alpha \tau \omega_d \right) \right], \\
E_1 = \omega_h \left( \omega_h + \omega_d \right).
\]

We repeat the procedure for finding the solutions using Ferrari's method. The spectrum limits of n-BVMSW must be found in the same way as provided in (15)-(17) with the corresponding replacement of variables, i.e. subscript \( s \) is replaced by \( v \), which denotes volume waves. Thus, the upper limit of the spectrum is determined by the expression

\[
\omega^v_h = \left( -\frac{B_1}{4A_1} - \frac{\sqrt{a_v + 2y_v}}{2} \right) + \frac{1}{2} \sqrt{\frac{-3a_v - 2y_v + \frac{2b_v}{\sqrt{a_v + 2y_v}}}{\sqrt{a_v + 2y_v}}} \right)^{1/2}. \tag{27}
\]

This upper spectrum limit of n-BVMSW equals to the lower spectrum limit of n-MSSW. The lower limit of n-BVMSW is written as

\[
\omega^v_s = \left( -\frac{B_1}{4A_1} - \frac{\sqrt{a_s + 2y_s}}{2} \right) - \frac{1}{2} \sqrt{\frac{-3a_s - 2y_s + \frac{2b_s}{\sqrt{a_s + 2y_s}}}{\sqrt{a_s + 2y_s}}} \right)^{1/2}. \tag{28}
\]

The expression (28) corresponds to the FMR frequency taking damping and nutation into account. The lower limit is shifted down by 3.1 MHz compared to the case without damping and nutation effects \( \omega_h = |\rho| |H_0| \), and the upper limit decreases by 64 MHz. These values are calculated for the same parameters which are used in Figs. 2, 3. Thus, similarly to the parallel configuration of the wave vector and magnetization, the main effect of magnetization nutation for BVMSW is the shift of their dispersion branches.

III. CONCLUSION

We theoretically predict the emergence of nutation surface spin waves due to magnetization inertia in the dipolar coupling limit for in-plane magnetized ferromagnetic thin films, propagating perpendicular to the direction of the external magnetic field. These waves are backward waves and propagate at terahertz frequencies with a group velocity lower
than the velocity of conventional spin waves. Inertia leads to a red-shift of precession-nutation spin waves compared to precession spin waves. The upper spectral limit of the dispersion branches of the precession-nutation waves undergo a greater shift then the lower spectral limit.

ACKNOWLEGEMENTS

In part funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) – Project No. 392402498 (SE 2853/1-1) and Project No. 405553726 CRC/TRR 270, by the government of the Russian Federation (agreement No. 075-15-2019-1886). Helpful discussions with U. Nowak and R. Mondal are acknowledged.

[1] A. Prabhakar and D. D. Stancil, Spin Waves: Theory and Applications (Springer US, 2009).
[2] A. G. Gurevich and G. A. Melkov, Magnetization Oscillations and Waves (CRC press, 1996).
[3] B. A. Kalinikos and A. N. Slavin, J. Phys. C Solid State Phys. 19, 7013 (1986).
[4] R. Damon and J. R. Eshbach, J. Phys. Chem. Solids 19, 308 (1961).
[5] R. Damon and H. Van De Vaart, J. Appl. Phys. 36, 3453 (1965).
[6] B. Hillebrands, Phys. Rev. B 41, 530 (1990).
[7] M. A. Cherkasskii and B. A. Kalinikos, JETP Lett. 97, 611 (2013).
[8] S. L. Vysotskii, S. A. Nikitov, and Y. A. Filimonov, J. Exp. Theor. Phys. 101, 547 (2005).
[9] A. V. Chumak, A. A. Serga, B. Hillebrands, and M. P. Kostylev, Appl. Phys. Lett. 93, 022508 (2008).
[10] S. Neusser and D. Grundler, Adv. Mater. 21, 2927 (2009).
[11] M. G. Cottam, Linear and Nonlinear Spin Waves in Magnetic Films and Superlattices (1994).
[12] P. E. Wigen, in Nonlinear Phenom. Chaos Magn. Mater. (World scientific, 1994), pp. 1–12.
[13] V. S. L’vov, Wave Turbulence under Parametric Excitation: Applications to Magnets (Springer Science & Business Media, 2012).
[14] S. Wintz, V. Tiberkevich, M. Weigand, J. Raabe, J. Lindner, A. Erbe, A. Slavin, and J. Fassbender, Nat. Nanotechnol. 11, 948 (2016).
[15] G. Dieterle, J. Förster, H. Stoll, A. S. Semisalova, S. Finizio, A. Gangwar, M. Weigand, M. Noske, M. Fähnle, I. Bykova, J. Gräfe, D. A. Bozhko, H. Y. Musienko- Shmarova, V. Tiberkevich, A. N. Slavin, C. H. Back, J. Raabe, G. Schütz, and S. Wintz, Phys. Rev. Lett. 122, 117202 (2019).
[16] J. R. Eshbach, Phys. Rev. Lett. 8, 357 (1962).
[17] M. Baillieu, D. Olligs, C. Fermont, and S. O. Demokritov, Europhys. Lett. 56, 741 (2001).
[18] V. E. Demidov, J. Jersch, S. O. Demokritov, K. Rott, P. Krzysteczko, and G. Reiss, Phys. Rev. B 79, 54417 (2009).
[19] J. Gouzerh, A. A. Stashkevich, N. G. Kovshikov, V. V Matyushov, and J. M. Desvignes, J. Magn. Magn. Mater. 101, 189 (1991).
[20] S.-K. Kim, S. Choi, K.-S. Lee, D.-S. Han, D.-E. Jung, and Y.-S. Choi, Appl. Phys. Lett. 92, 212501 (2008).
[21] K. Perdza, G. Woltersdorf, and C. H. Back, Phys. Rev. B 77, 54425 (2008).
[22] S. Choi, K.-S. Lee, and S.-K. Kim, Appl. Phys. Lett. 89, 62501 (2006).
[23] M. Bauer, C. Mathieu, S. O. Demokritov, B. Hillebrands, P. A. Kolodin, S. Sure, H. Dötsch, V. Grimalsky, Y. Rapoport, and A. N. Slavin, Phys. Rev. B 56, R8483 (1997).
[24] O. Büttner, M. Bauer, S. O. Demokritov, B. Hillebrands, Y. S. Kivshar, V. Grimalsky, Y. Rapoport, M. P. Kostylev, B. A. Kalinikos, and A. N. Slavin, J. Appl. Phys. 87, 5088 (2000).
[25] R. Khomeriki, Eur. Phys. J. B - Condens. Matter Complex Syst. 41, 219 (2004).
[26] V. E. Demidov, S. O. Demokritov, K. Rott, P. Krzysteczko, and G. Reiss, Appl. Phys. Lett. 91, 252504 (2007).
[27] Z. Wang, M. Cherkasskii, B. A. Kalinikos, L. D. Carr, and M. Wu, New J. Phys. 16, (2014).
[28] M. A. Cherkasskii, A. A. Nikitin, and B. A. Kalinikos, J. Exp. Theor. Phys. 122, 727 (2016).
[29] M. Wu, A. M. Hagerstrom, R. Eykholt, A. Kondrashov, and B. A. Kalinikos, Phys. Rev. Lett. 102, 237203 (2009).
[30] M. A. Cherkasskii, N. G. Kovshikov, and B. A. Kalinikos, Phys. Solid State 52, 2123 (2010).
[31] R. N. Costa Filho, M. G. Cottam, and G. A. Farias, Phys. Rev. B 62, 6545 (2000).
[32] M. Fähnle, D. Steiauf, and C. Illg, Phys. Rev. B 84, 172403 (2011).
[33] R. Mondal, M. Berritta, and P. M. Oppeneer, J. Phys. Condens. Matter 30, 265801 (2018).
[34] R. Mondal, M. Berritta, A. K. Nandy, and P. M. Oppeneer, Phys. Rev. B 96, (2017).
[35] S. Bhattacharjee, L. Nordström, and J. Fransson, Phys. Rev. Lett. 108, 057204 (2012).
[36] D. Thonig, O. Eriksson, and M. Pereiro, Sci. Rep. 7, 931 (2017).
[37] K. Neeraj, N. Awari, S. Kovalev, D. Polley, N. Zhou Hagström, S. S. P. K. Arekapudi, A. Semisalova, K. Lenz, B. Green, J. C. Deinert, I. Ilyakov, M. Chen, M. BauwTra, V. Scarla, M. D’Aquino, C. Serpico, O. Hellwig, J. E. Wegrowe, M. Gensch, and S. Bonetti, Nat. Phys. 17, 245 (2020).
[38] M.-C. Ciornei, J. M. Rubi, and J.-E. Wegrowe, Phys. Rev. B 83, 020410 (2011).
[39] M. Cherkasskii, M. Farle, and A. Semisalova, Phys. Rev. B 102, 184432 (2020).
[40] E. Olive, Y. Lansac, and J.-E. Wegrowe, Appl. Phys. Lett. 100, 192407 (2012).
[41] Y. Li, A.-L. Barra, S. Auffret, U. Ebels, and W. E. Bailey, Phys. Rev. B 92, 140413 (2015).
[42] I. Makhfudz, E. Olive, and S. Nicolis, Appl. Phys. Lett. 117, 132403 (2020).
[43] J.-E. Wegrowe and M.-C. Ciornei, Am. J. Phys. 80, 607 (2012).
[44] See Supplemental Material at [URL will be inserted by publisher] for the detailed derivation of the dynamic susceptibility for a Cartesian coordinate system.
Supplemental Material to “Dispersion relation of nutation surface spin waves in ferromagnets”

Mikhail Cherkasskii1, *, Michael Farle2,3, and Anna Semisalova2

1 Department of General Physics 1, St. Petersburg State University, St. Petersburg, 199034, Russia
2 Faculty of Physics and Center of Nanointegration (CENIDE), University of Duisburg-Essen, Duisburg, 47057, Germany
3 Kirensky Institute of Physics, Federal Research Center KSC SB RAS, Krasnoyarsk, 660036, Russia

* Corresponding author: macherkasskii@hotmail.com

The dynamic susceptibility in Cartesian coordinates is derived based on the Inertial Landau–Lifshitz–Gilbert (ILLG) equation given by

\[
\frac{d\mathbf{M}}{dt} = -\gamma \mathbf{M} \times \left[ -\mathbf{H}_{\text{eff}} - \alpha \frac{d\mathbf{M}}{dt} + \tau \frac{d^2\mathbf{M}}{dt^2} \right],
\]

(S1)

where \( \gamma \) is the gyromagnetic ratio, \( \mathbf{M} \) is the magnetization vector, \( M_0 \) is the magnetization at saturation, \( \alpha \) is the Gilbert damping, \( \tau \) is the inertial relaxation time, and \( \mathbf{H}_{\text{eff}} \) is the effective magnetic field. The spin-orbit interaction is the reason for Gilbert damping and magnetic inertia, which are described by the terms \( \mathbf{M} \times \frac{d\mathbf{M}}{dt} \), and \( \frac{d}{dt} \left( \mathbf{M} \times \mathbf{H}_{\text{eff}} \right) \) and corresponding coefficients. Note that the susceptibility was derived for circular variables of magnetization and magnetic field in Ref. [1].

We assume that the ferromagnet is magnetically saturated by a uniform magnetic field \( \mathbf{H}_0 \) acting along the z-axis. The time-varying driving field \( \mathbf{h} \) is superimposed on \( \mathbf{H}_0 \) and the linearization of ILLG can be performed since \( |\mathbf{h}| << |\mathbf{H}_0| \).

With the magnetization parallel to \( \mathbf{H}_0 \) we write the magnetization and magnetic field in the generalized form using the Fourier transformation

\[
\mathbf{M}(t) = M_0 \hat{z} + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega' \, \mathbf{m}(\omega') e^{i\omega' t},
\]

(S2)

\[
\mathbf{H}_{\text{eff}}(t) = H_0 \hat{z} + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega' \, \mathbf{h}(\omega') e^{i\omega' t},
\]

(S3)

where \( \hat{z} \) is the unit vector along the z-axis. After several transformations the ILLG equation can be simplified to

\[
\begin{align*}
&i\omega \mathbf{m}(\omega) = -\gamma M_0 \hat{z} \times \mathbf{h}(\omega) + \\
&+ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega' \left[ H_0 \hat{z} \times \mathbf{m}(\omega') + i\alpha \omega \hat{z} \times \mathbf{m}(\omega) \right]
\end{align*}
\]

(S4)

By projecting to Cartesian coordinates one obtains

\[
\mathbf{m} = \hat{z} \mathbf{h},
\]

\[
\hat{z} = \begin{bmatrix} \chi & i\chi_s & 0 \\ -i\chi_s & \chi & 0 \\ 0 & 0 & 0 \end{bmatrix}
\]

(S5)

We introduced the following notations: \( \omega_H = |\mathbf{h}| / (2\pi) \), \( \mu_0 M_0 = 1 \) T, \( \mu_0 H_0 = 100 \) mT, \( \alpha = 0.0065 \), and \( \tau = 10^{-11} \) s. (e) The orientation of vectors in the ILLG equation.

\[
\chi = \frac{-\alpha \omega_H \omega^2 + i\omega \omega_H \omega + \omega \omega_H^2}{D},
\]

\[
\chi_s = \frac{-\alpha \omega_H \omega s}{D},
\]

\[
D = \alpha^2 \tau^2 \omega^4 - 2i\alpha^2 \tau^3 \omega^2 - (1 + \alpha^2 + 2\alpha \omega_H) \omega^2 + 2i\alpha \omega_H \omega + \omega_H^2.
\]

(S6)

We calculated the susceptibility tensor \( \chi \) and its dissipative part \( \chi_s \) using the following equations:

\[
\chi = \frac{-\alpha \omega_H \omega^2 + i\omega \omega_H \omega + \omega \omega_H^2}{D},
\]

\[
\chi_s = \frac{-\alpha \omega_H \omega s}{D},
\]

\[
D = \alpha^2 \tau^2 \omega^4 - 2i\alpha^2 \tau^3 \omega^2 - (1 + \alpha^2 + 2\alpha \omega_H) \omega^2 + 2i\alpha \omega_H \omega + \omega_H^2.
\]

(S6)

The calculation was performed for \( |\mathbf{h}| / (2\pi) = 28 \) GHz T\(^{-1}\), \( \mu_0 M_0 = 1 \) T, \( \mu_0 H_0 = 100 \) mT, \( \alpha = 0.0065 \), and \( \tau = 10^{-11} \) s.
to susceptibility in circular variables as $\chi_s = \chi \pm \chi_a$. Note that the precession resonance occurs for $\chi$, corresponding right-hand rotation, i.e. positive polarization, while nutation resonance has negative polarization. This can be also verified considering the vector orientations of ILLG equation (Fig. S1(e)). In the low damping limit, the vector orientation of $\frac{d\mathbf{M}}{dt}$ is identical to $\mathbf{M} \times \mathbf{H}_p$, and it is easy to find the orientation of $\frac{d^2\mathbf{M}}{dt^2}$ using this rule. It is clearly seen that the directions of precession torque $\mathbf{M} \times \mathbf{H}_p$, and nutation $(\alpha \tau / M_{ax}) \mathbf{M} \times d^2\mathbf{M}/dt^2$ are opposite, and this corresponds to reversed polarizations of ferromagnetic and nutation resonances.

The dispersive and dissipative parts of the susceptibility can be separated with $\chi = \chi' - i\chi''$, $\chi_a = \chi' + i\chi''$, then

$$\chi' = \frac{1}{N} \left[ -\alpha^3 \tau \omega_p \omega^6 + \left( \alpha \tau \omega_p - \alpha^3 \tau + 3 \alpha^2 \tau \omega_p \omega_p \right) \omega^4 + \left( -\omega_p \omega_p + \alpha \omega_p \omega_p - 3 \alpha \tau \omega_p \omega_p \right) \omega^2 + \omega_p \omega_p \right],$$  

(S7)

$$\chi'' = \frac{\omega}{N} \left[ \alpha \tau \omega_p \omega^4 + \left( \alpha \omega_p + \alpha^3 \omega_p - 2 \alpha \tau \omega_p \omega_p \right) \omega^2 + \alpha \omega_p \omega_p \right],$$

(S8)

$$\chi_a' = \frac{1}{N} \left[ -\alpha^3 \tau \omega_p \omega^6 - \left( \omega_p + \alpha^3 \omega_p + 2 \alpha \tau \omega_p \omega_p \right) \omega^4 + \omega_p \omega_p \right],$$

(S9)

$$\chi_a'' = \frac{2 \alpha \omega_p \omega^6 \left( \omega_p - \alpha \omega_p \right)}{N},$$

(S10)

$$N = \alpha^4 \tau \omega^8 + 2 \alpha^2 \tau^2 \left( 1 + \alpha^2 - 2 \alpha \tau \omega_p \right) \omega^6 + \left( 1 + 4 \alpha \tau \omega_p + 2 \alpha^2 + 6 \alpha^2 \tau \omega_p \right) \omega^4 - 4 \alpha \tau \omega_p \alpha + \alpha^4 \omega^4 + 2 \omega_p \left( 1 + \alpha^2 - 2 \alpha \tau \omega_p \right) \omega^2 + \omega_p^4,$$

(S11)

The relation between $\chi$ and $\chi_a$, $\chi_s$ is elucidated in Fig. S1(a-d). Note that the ferromagnetic resonance (FMR) is observed for $\chi = \chi + \chi_a$, whereas nutation resonance appears for $\chi = \chi - \chi_a$. Hence one can consider the sum and the difference of the $\chi$ and $\chi_a$ curves. The frequency dependence of $\chi_a$ in the terahertz range (Fig. S1(b)) is inverse to $\chi$ (Fig. S1(d)), and if one takes sum of the curves $\chi$ and $\chi_a$, then FMR peak increases in the microwave range, but nutation peak vanishes in terahertz range. One can perform a similar consideration for $\chi_s$.

[1] M. Cherkasskii, M. Farle, and A. Semisalova, Phys. Rev. B 102, 184432 (2020).