SEMI-PARALLEL MERIDIAN SURFACES IN $\mathbb{E}^4$

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Abstract. In the present article we study a special class of surfaces in the four-dimensional Euclidean space, which are one-parameter systems of meridians of the standard rotational hypersurface. They are called meridian surfaces. We classified semi-parallel meridian surface in 4-dimensional Euclidean space $\mathbb{E}^4$.

1. Introduction

Let $M$ be a submanifold of a $n$-dimensional Euclidean space $\mathbb{E}^n$. Denote by $\overline{R}$ the curvature tensor of the Vander Waerden-Bortoletti connection $\nabla$ of $M$ and $h$ is the second fundamental form of $M$ in $\mathbb{E}^n$. The submanifold $M$ is called semi-parallel (or semi-symmetric [15]) if $\overline{R} \cdot h = 0$ [6]. This notion is an extrinsic analogue for semi-symmetric spaces, i.e. Riemannian manifolds for which $R \cdot R = 0$ and a direct generalization of parallel submanifolds, i.e. submanifolds for which $\nabla h = 0$. In [6] J. Deprez showed the fact that the submanifold $M \subset \mathbb{E}^n$ is semi-parallel implies that $(M, g)$ is semi-symmetric. For references on semi-symmetric spaces, see [18]; for references on parallel immersions, see [8]. In [6] J. Deprez gave a local classification of semi-parallel hypersurfaces in Euclidean $n$-space $\mathbb{E}^n$.

Recently, the present authors considered the Wintgen ideal surfaces in Euclidean $n$-space $\mathbb{E}^n$. They showed that Wintgen ideal surfaces in $\mathbb{E}^n$ satisfying the semi-parallelity condition

$$R(X, Y) \cdot h = 0$$

are of flat normal connection [2]. Further, the same authors in [3] proved that the tensor product surfaces in $\mathbb{E}^4$ satisfying the semi-parallelity condition (1.1) are totally umbilical.

In [15] Ganchev and Milousheva constructed special two dimensional surfaces which are one-parameter of meridians of the rotation hypersurfaces in $\mathbb{E}^4$ and called these surfaces meridian surfaces. The geometric construction of the meridian surfaces is different from the construction of the standard rotational surfaces with two dimensional axis in $\mathbb{E}^4$ [9]. The same authors classified the meridian surfaces with constant Gauss curvature ($K \neq 0$) and constant mean curvature $H$ [13]. Recently, meridian surfaces with 1-type Gauss map is characterized by the present authors and Milousheva in [4]. Further, meridian surfaces were studied in [10] as a surface in Minkowski 4-space. For more details see also [11], [12] and [17].

In the present study we consider the meridian surfaces in 4-dimensional Euclidean space $\mathbb{E}^4$. We gave a classification of the meridian surfaces in 4-dimensional Euclidean space $\mathbb{E}^4$ satisfying the semi-parallelity condition (1.1).

Date: March 25, 2015.

2000 Mathematics Subject Classification. 53C15, 53C40.

Key words and phrases. Gaussian curvature, Meridian surface, Semi-parallel surface.
2. Basic Concepts

Let \( M \) be a smooth surface in \( n \)-dimensional Euclidean space \( \mathbb{E}^n \) given with the surface patch \( X(u, v) : (u, v) \in D \subset \mathbb{E}^2 \). The tangent space to \( M \) at an arbitrary point \( p = X(u, v) \) of \( M \) span \{ \( X_u, X_v \) \}. In the chart \((u, v)\) the coefficients of the first fundamental form of \( M \) are given by

\[
(2.1) \quad E = \langle X_u, X_u \rangle, \quad F = \langle X_u, X_v \rangle, \quad G = \langle X_v, X_v \rangle,
\]
where \( \langle , \rangle \) is the Euclidean inner product. We assume that \( W^2 = EG - F^2 \not= 0 \), i.e. the surface patch \( X(u, v) \) is regular. For each \( p \in M \), consider the decomposition \( T_p \mathbb{E}^n = T_p M \oplus T^\perp_p M \) where \( T^\perp_p M \) is the orthogonal component of the tangent plane \( T_p M \) in \( \mathbb{E}^n \), that is the normal space of \( M \) at \( p \).

Let \( \chi(M) \) and \( \chi^\perp(M) \) be the space of the smooth vector fields tangent and normal to \( M \) respectively. Denote by \( \nabla \) and \( \nabla^\perp \) the Levi-Civita connections on \( M \) and \( \mathbb{E}^n \), respectively. Given any vector fields \( X_i \) and \( X_j \) tangent to \( M \) consider the second fundamental form \( h : \chi(M) \times \chi(M) \to \chi^\perp(M) \);

\[
(2.2) \quad h(X_i, X_j) = \nabla_{X_i}X_j - \nabla_{X_j}X_i; \quad 1 \leq i, j \leq 2.
\]
where \( \nabla \) is the induced. This map is well-defined, symmetric and bilinear.

For any normal vector field \( N_\alpha \) \( 1 \leq \alpha \leq n - 2 \) of \( M \), recall the shape operator \( A : \chi^\perp(M) \times \chi^\perp(M) \to \chi^\perp(M) \);

\[
(2.3) \quad A_{N_\alpha}X_i = -\nabla^\perp_{N_\alpha}X_i + D_{X_i}N_\alpha; \quad 1 \leq i \leq 2.
\]
where \( D \) denotes the normal connection of \( M \) in \( \mathbb{E}^n \). This operator is bilinear, self-adjoint and satisfies the following equation:

\[
(2.4) \quad \langle A_{N_\alpha}X_i, X_j \rangle = \langle h(X_i, X_j), N_\alpha \rangle, \quad 1 \leq i, j \leq 2.
\]

The equation \( \Box \) is called Gaussian formula, and

\[
(2.5) \quad h(X_i, X_j) = \sum_{\alpha=1}^{n-2} h^\alpha_{ij}N_\alpha, \quad 1 \leq i, j \leq 2
\]
where \( h^\alpha_{ij} \) are the coefficients of the second fundamental form \( h \). If \( h = 0 \) then \( M \) is called totally geodesic. \( M \) is totally umbilical if all shape operators are proportional to the identity map. \( M \) is an isotropic surface if for each \( p \) in \( M \), \( \|h(X, X)\| \) is independent of the choice of a unit vector \( X \) in \( T_p M \).

If we define a covariant differentiation \( \nabla h \) of the second fundamental form \( h \) on the direct sum of the tangent bundle and normal bundle \( TM \oplus T^\perp M \) of \( M \) by

\[
(2.6) \quad (\nabla_X h)(X_j, X_k) = D_X h(X_j, X_k) - h(\nabla_X X_j, X_k) - h(X_j, \nabla_X X_k),
\]
for any vector fields \( X_i, X_j, X_k \) tangent to \( M \). Then we have the Codazzi equation

\[
(2.7) \quad (\nabla_{X_i}h)(X_j, X_k) = (\nabla_{X_j}h)(X_i, X_k),
\]
where \( \nabla \) is called the Vander Waerden-Bortolotti connection of \( M \).

We denote \( R \) and \( \bar{R} \) the curvature tensors associated with \( \nabla \) and \( D \) respectively;

\[
(2.8) \quad R(X_i, X_j)X_k = \nabla_{X_i}\nabla_{X_j}X_k - \nabla_{X_j}\nabla_{X_i}X_k - \nabla_{[X_i, X_j]}X_k,
\]
(2.9) \quad \bar{R}(X_i, X_j)N_\alpha = h(X_i, A_{N_\alpha}X_j) - h(X_j, A_{N_\alpha}X_i).

The equation of Gauss and Ricci are given respectively by

\[
(2.10) \quad \langle R(X_i, X_j)X_k, X_l \rangle = \langle h(X_i, X_l), h(X_j, X_k) \rangle - \langle h(X_i, X_k), h(X_j, X_l) \rangle,
\]
(2.11) \[ \langle R^i(X_i, X_j)N_\alpha, N_\beta \rangle = \langle [A_{N_\alpha}, A_{N_\beta}]X_i, X_j \rangle, \]

for the vector fields \( X_i, X_j, X_k \) tangent to \( M \) and \( N_\alpha, N_\beta \) normal to \( M \). 

Let \( X_i \wedge X_j \) denote the endomorphism \( X_k \rightarrow \langle X_j, X_k \rangle X_i - \langle X_i, X_k \rangle X_j \). Then the curvature tensor \( R \) of \( M \) is given by the equation

\[
R(X_i, X_j)X_k = \sum_{\alpha=1}^{n-2} (A_{N_\alpha}X_i \wedge A_{N_\alpha}X_j) X_k.
\]

It is easy to show that

\[
R(X_i, X_j)X_k = K (X_i \wedge X_j) X_k,
\]

where \( K \) is the Gaussian curvature of \( M \) defined by

\[
K = \langle h(X_1, X_1), h(X_2, X_2) \rangle - \|h(X_1, X_2)\|^2
\]

(see [14]).

The normal curvature \( K_N \) of \( M \) is defined by (see [5])

\[
K_N = \left\{ \sum_{1=\alpha<\beta}^{n-2} \langle R^i(X_1, X_2)N_\alpha, N_\beta \rangle^2 \right\}^{1/2}.
\]

We observe that the normal connection \( D \) of \( M \) is flat if and only if \( K_N = 0 \), and by a result of Cartan, this equivalent to the diagonalisability of all shape operators \( A_{N_\alpha} \) of \( M \), which means that \( M \) is a totally umbilical surface in \( \mathbb{E}^n \).

3. Semi-Parallel Surfaces

Let \( M \) a smooth surface in \( n \)-dimensional Euclidean space \( \mathbb{E}^n \). Let \( \nabla \) be the connection of Vander Waerden-Bortoletti of \( M \). Denote the tensors \( \nabla \) by \( \overline{\nabla} \). Then the product tensor \( \overline{R} \cdot h \) of the curvature tensor \( \overline{R} \) with the second fundamental form \( h \) is defined by

\[
(\overline{R}(X_i, X_j) \cdot h)(X_k, X_l) = \nabla_{X_i}(\nabla_{X_j}h(X_k, X_l)) - \nabla_{X_j}(\nabla_{X_i}h(X_k, X_l))
\]

for all \( X_i, X_j, X_k, X_l \) tangent to \( M \).

The surface \( M \) is said to be semi-parallel if \( \overline{R} \cdot h = 0 \), i.e. \( \overline{R}(X_i, X_j) \cdot h = 0 \) ([15], [6], [7], [10]). It is easy to see that

\[
(\overline{R}(X_i, X_j) \cdot h)(X_k, X_l) = R^i(X_i, X_j)h(X_k, X_l)
\]

\[
-h(R(X_i, X_j)X_k, X_l) - h(R(X_i, X_j)X_l, X_k),
\]

This notion is an extrinsic analogue for semi-symmetric spaces, i.e. Riemannian manifolds for which \( R \cdot R = 0 \) and a generalization of parallel surfaces, i.e. \( \nabla h = 0 \).

Substituting (2.3) and (2.4) into (2.9) we get

\[
R^i(X_1, X_2)N_\alpha = h^\alpha_2(h(X_1, X_1) - h(X_2, X_2) + (h^\alpha_2 - h^\alpha_1)h(X_1, X_2).
\]

Further, by the use of (2.13) we get

\[
R(X_1, X_2)X_1 = -K X_2, R(X_1, X_2)X_2 = K X_1.
\]
So, substituting (3.2) and (3.3) into (3.1) we obtain

\[
\left(\mathcal{R}(X_1, X_2) \cdot h\right)(X_1, X_1) = \left(\sum_{\alpha=1}^{n-2} h_{11}^\alpha (h_{22}^\alpha - h_{11}^\alpha) + 2K\right) h(X_1, X_2)
\]

\[
+ \sum_{\alpha=1}^{n-2} h_{11}^\alpha h_{12}^\alpha (h(X_1, X_1) - h(X_2, X_2)),
\]

(3.4) \[
\left(\mathcal{R}(X_1, X_2) \cdot h\right)(X_1, X_2) = \left(\sum_{\alpha=1}^{n-2} h_{22}^\alpha (h_{22}^\alpha - h_{11}^\alpha) \right) h(X_1, X_2)
\]

\[
+ \sum_{\alpha=1}^{n-2} h_{12}^\alpha h_{12}^\alpha (h(X_1, X_1) - h(X_2, X_2)).
\]

Semi-parallel surfaces in \(E^n\) are classified by J. Deprez [6]:

**Theorem 3.1.** [6] Let \(M\) a surface in \(n\)-dimensional Euclidean space \(E^n\). Then \(M\) is semi-parallel if and only if locally:

i) \(M\) is equivalent to a 2-sphere, or

ii) \(M\) has trivial normal connection, or

iii) \(M\) is an isotropic surface in \(E^5 \subset E^n\) satisfying \(\|H\|^2 = 3K\).

4. **Meridian Surfaces in \(E^4\)**

In the following sections, we will consider the meridian surfaces in \(E^4\) which is first defined by Ganchev and Milousheva [9]. The meridian surfaces are one-parameter systems of meridians of the standard rotational hypersurface in \(E^4\).

Let \(\{e_1, e_2, e_3, e_4\}\) be the standard orthonormal frame in \(E^4\), and \(S^2(1)\) be a 2-dimensional sphere in \(E^3 = \text{span}\{e_1, e_2, e_3\}\), centered at the origin \(O\). We consider a smooth curve \(C : r = r(v), v \in J, J \subset \mathbb{R}\) on \(S^2(1)\), parameterized by the arc-length \(\left\|\left(\left(\frac{dr}{dv}\right)^2\right)\right\| = 1\). We denote \(t = r'\) and consider the moving frame field \(\{t(v), n(v), r(v)\}\) of the curve \(C\) on \(S^2(1)\). With respect to this orthonormal frame field the following Frenet formulas hold good:

\[
r' = t;
\]

\[
t' = \kappa n - r;
\]

\[
n' = -\kappa t,
\]

where \(\kappa\) is the spherical curvature of \(C\).

Let \(f = f(u), g = g(u)\) be smooth functions, defined in an interval \(I \subset \mathbb{R}\), such that

\[
(f')^2(u) + (g')^2(u) = 1, \ u \in I.
\]
In [9] Ganchev and Milousheva constructed a surface $M^2$ in $\mathbb{E}^4$ in the following way:

\begin{equation}
M^2 : X(u, v) = f(u) r(v) + g(u) e_4, \quad u \in I, \ v \in J.
\end{equation}

The surface $M^2$ lies on the rotational hypersurface $M^3$ in $\mathbb{E}^4$ obtained by the rotation of the meridian curve $\alpha : u \to (f(u), g(u))$ around the $Oe_4$-axis in $\mathbb{E}^4$. Since $M^2$ consists of meridians of $M^3$, we call $M^2$ a meridian surface [9]. If we denote by $\kappa_\alpha$ the curvature of meridian curve $\alpha$, i.e.,

\begin{equation}
\kappa_\alpha = f'(u)g''(u) - f''(u)g(u) = \frac{-f''(u)}{\sqrt{1 - f'^2(u)}},
\end{equation}

We consider the following orthonormal moving frame fields, $X_1, X_2, N_1, N_2$ on the meridian surface $M^2$ such that $X_1, X_2$ are tangent to $M^2$ and $N_1, N_2$ are normal to $M^2$. The tangent space of $M^2$ is spanned by the vector fields:

\begin{equation}
X_1 = \frac{\partial X}{\partial u}, \quad X_2 = \frac{1}{r} \frac{\partial X}{\partial v},
\end{equation}

\begin{equation}
N_1 = n(v), \quad N_2 = -g'(u) r(v) + f'(u) e_4.
\end{equation}

By a direct computation we have the components of the second fundamental forms as;

\begin{equation}
\begin{aligned}
h_{11}^1 &= h_{12}^1 = h_{21}^1 = 0, & h_{22}^1 &= \frac{2}{r}, \\
h_{11}^2 &= \kappa_\alpha, & h_{12}^2 = h_{21}^2 = 0, & h_{22}^2 &= \frac{\kappa'}{r}.
\end{aligned}
\end{equation}

Therefore the shape operator matrices of $M^2$ are of the form

\begin{equation}
A_{N_1} = \begin{bmatrix} 0 & 0 \\ 0 & \frac{\kappa'}{r} \end{bmatrix}, \quad A_{N_2} = \begin{bmatrix} \kappa_\alpha & 0 \\ 0 & \frac{\kappa'}{r} \end{bmatrix}
\end{equation}

and hence we have

\begin{equation}
K = \frac{\kappa_\alpha \kappa'}{r}, \quad K_N = 0,
\end{equation}

which implies that the meridian surface $M^2$ is totally umbilical surface in $\mathbb{E}^4$.

In [13] Ganchev and Milousheva constructed three main classes of meridian surfaces:

I. $\kappa = 0$; i.e. the curve $C$ is a great circle on $S^2(1)$. In this case $N_1 = \text{const}$. and $M^2$ is a planar surface lying in the constant 3-dimensional space spanned by $\{X_1, X_2, N_2\}$. Particularly, if in addition $\kappa_\alpha = 0$, i.e. the meridian curve is a part of a straight line, then $M^2$ is a developable surface in the 3-dimensional space spanned by $\{X_1, X_2, N_2\}$.

II. $\kappa_\alpha = 0$, i.e. the meridian curve is a part of a straight line. In such case $M^2$ is a developable ruled surface. If in addition $\kappa = \text{const.}$, i.e. $C$ is a circle on $S^2(1)$, then $M^2$ is a developable ruled surface in a 3-dimensional space. If $\kappa \neq \text{const.}$,i.e. $C$ is not a circle on $S^2(1)$, then $M^2$ is a developable ruled surface in $\mathbb{E}^4$.

III. $\kappa_\alpha \kappa \neq 0$, i.e. $C$ is not a circle on $S^2(1)$ and $\alpha$ is not a straight line. In this general case the parametric lines of $M^2$ given by [13] are orthogonal and asymptotic.

We proved the following Theorem.
Theorem 4.1. Let $M^2$ be a meridian surface in $E^4$ given with the parametrization (4.3). Then $M^2$ is semi-parallel if and only if one of the following holds:

i) $M^2$ is a developable ruled surface in $E^3$ or $E^4$ which considered in Case II of the classification above,

ii) the curve $C$ is a circle on $S^2(1)$ with non-zero constant spherical curvature and the meridian curve is determined by

$$f(u) = \pm \sqrt{u^2 - 2au + 2b}, \quad g(u) = -\sqrt{2b - a^2} \ln\left(u - a - \sqrt{u^2 - 2au + 2b}\right),$$

where $a = \text{const}, b = \text{const}$. In this case $M^2$ is a planar surface lying in 3-dimensional space spanned by $\{X_1, X_2, N_2\}$.

Proof. Let $M^2$ be a meridian surface in $E^4$ given with the parametrization (4.3). Then by the use of (2.5) with (4.6) we see that

$$h(X_1, X_2) = 0,$$

$$(X_1, X_2) = -\frac{\kappa}{f}N_1 + \left(\kappa_\alpha - \frac{g'}{f}\right) N_2.$$ 

Further, substituting (4.9) and (4.10) into (3.4) and after some computation one can get

$$(\overline{R}(X_1, X_2) \cdot h)(X_1, X_1) = 0,$$

$$-K \left( -\frac{\kappa}{f}N_1 + \left(\kappa_\alpha - \frac{g'}{f}\right) N_2 \right),$$

$$\overline{R}(X_1, X_2) \cdot h)(X_2, X_2) = 0.$$

Suppose that, $M^2$ is semi-parallel then by definition

$$(\overline{R}(X_1, X_2) \cdot h)(X_i, X_j) = 0, \quad 1 \leq i, j \leq 2,$$

is satisfied. So, we get

$$K \left( -\frac{\kappa}{f}N_1 + \left(\kappa_\alpha - \frac{g'}{f}\right) N_2 \right) = 0.$$

Hence, two possible cases occur; $K = 0$ or $\kappa = 0$ and $\kappa_\alpha - \frac{g'}{f} = 0$. For the first case $\kappa_\alpha = 0$, i.e. the meridian curve is a part of a straight line. In such case $M^2$ is a developable ruled surface given in the Case II. For the second case $\kappa = 0$ means that the curve $c$ is a great circle on $S^2(1)$. In this case $M^2$ lies in the 3-dimensional space spanned by $\{X_1, X_2, N_2\}$. Further, using (4.4) the equation $\kappa_\alpha - \frac{g'}{f} = 0$ can be rewritten in the form

$$f(u)f''(u) - (f'(u))^2 + 1 = 0,$$

which has the solution

$$f(u) = \pm \sqrt{u^2 - 2au + 2b}. \quad (4.10)$$

Consequently, by substituting (4.10) into (4.2) one can get

$$g(u) = -\sqrt{2b - a^2} \ln\left(u - a - \sqrt{u^2 - 2au + 2b}\right).$$

This completes the proof of the theorem. \qed
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