Closed-form solutions for the Lévy-stable distribution

Karina Arias-Calluari[1,*] Fernando Alonso-Marroquin [1,2], and Michael S. Harré [3]

[1] School of Civil Engineering, The University of Sydney, Sydney NSW 2006
[2] Computational Physics IJB, ETH Zurich, CH-8093, Zurich, Switzerland
[3] Complex Systems Research Group, The University of Sydney, Sydney NSW 2006

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The Lévy-stable distribution is the attractor of distributions which hold power laws with infinite variance. This distribution has been used in a variety of research areas, for example in economics it is used to model financial market fluctuations and in statistical mechanics as a numerical solution of fractional kinetic equations of anomalous transport. This function does not have an explicit expression and no uniform solution has been proposed yet. This paper presents a uniform analytical approximation for the Lévy-stable distribution based on matching power series expansions. For this solution, the trans-stable function is defined as an auxiliary function which removes the numerical issues of the calculations of the Lévy-stable. Then, the uniform solution is proposed as a result of an asymptotic matching between two types of approximations called “the inner solution” and “the outer solution”. Finally, the results of analytical approximation are compared to the numerical results of the Lévy-stable distribution function, making this uniform solution valid to be applied as an analytical approximation.

I. INTRODUCTION

A wide range of natural and social phenomena exhibit a power-law in the probability distribution of large events. These tails are characterized by the asymptotic relation \( f(x) \sim 1/x^{1+\alpha} \), where \( x \) is the size of the events. For \( 0 < \alpha \leq 1 \), the distribution has an indefinite mean value. On the other hand, for \( 1 < \alpha \leq 2 \), the distribution has a defined mean value but still exhibits infinite variance [3]. These heavy-tailed distributions have been observed in economics and statistical mechanics.

In the field of economics, the statistics of price returns, trade size, and share volumes have been investigated. Heavy tailed distributions have been observed in the correlations of the absolute value of the S&P 500 returns [4, 5], the effects of networks on price returns [6], daily returns of the Dow Jones index [7], Brent crude oil returns [8] and the aggregate output growth rate distribution [9]. Even after applying five different estimation techniques, power law tails with the characteristic index \( \alpha \) were found on the cumulative distribution of trade size and share volumes of 252 US stocks over the 42-year period from 1963-2005 [10, 11]. To capture heavy tails different models have been proposed to simulate the stock price dynamics. For instance, models of anomalous diffusion of option pricing were introduced as an extension of the well-known Brownian model [12, 13]. These models are focused on different aspects such as capturing the dynamic of the price with waiting times (periods of stagnation) which are Lévy-stable distributed [12] or on the effect of “particles” representing agent’s interaction [13]. The continuous counterpart of these discrete models is the Fokker-Planck equation (FPE) that is presented in terms of fractional derivatives. The solution of FPE gives the time evolution of the probability density function (pdf) of price return [12–14].

In the field of statistical mechanics, the diffusion equation (DE) is a fundamental equation of transport dynam-
[31] and Zolotarev [32] used power series to obtain converging algorithms of the Lévy-stable distribution function in two ranges, the first for $\alpha < 1$ and the second for $\alpha > 1$ for symmetric distributions. However, some of the proposed series do not converge to the Lévy-stable function, and some of them are only applicable for extreme values $x \to 0$ or $x \to \infty$. Mantegna [33] presented a similar solution that of Elliot [31] but the algorithm is only valid when $x \to \infty$ and $0.75 < \alpha \leq 1.95$. Nolan [34] presented an algorithm for asymmetric distributions of large events $x \to \infty$ focusing only on the tail behaviour of the distribution. Thus, the Lévy-stable distribution function does not have an explicit expression [35, 36] and no uniform solution of the Lévy-stable distribution has been proposed until now [30, 32, 34].

Due to the absence of an explicit expression, numerical solutions were developed to evaluate the Lévy-stable distribution function by using numerical recursive quadrature methods [37–39]. Nolan [37, 40] develops a numerical solution for the estimation of Lévy-stable parameters through a maximum likelihood method for each data set of $x$. However, Nolan’s method converges only for $\alpha > 0.4$ and the convergence to the Lévy-stable distribution function seems to be not accurate enough. Despite this fact, Nolan’s method constitutes an important method that is still being used [38].

Apart from the numerical issues in the evaluation of the Lévy-stable distribution, some authors have pointed out its infinite variance as a drawback [41–43]. To avoid the infinite variance of the Lévy-stable distribution function, several truncations are proposed. The truncation was justified by the observed change of slope of the tails on extensive datasets [44]. For example, when evaluating the returns per minute of S&P 500 index data over the ten year period from 1985-95 a change of slope from $\alpha = 1.4$ to $\alpha = 3$ was found. The truncations make the variance finite, consequently the distribution function of the sum of independent random variables converges to the normal distribution due to the central limit theorem for large $N$. Nevertheless, a time series in some stock market indices can exhibit infinite variance, one such case is the variance of price fluctuations in Shanghai stock market index, which increases when the time frame is enlarged [45, 46].

The aim of this paper is to formulate a uniform analytical approximation for the Lévy-stable distribution function based on a series expansion. To achieve this aim we propose several regularizations of the inner and outer series expansions to ensure convergence. This will be an important tool to get the most accurate approximation reducing numerical errors (oscillations) when the Lévy-stable function is evaluated.

This paper is divided in two parts. The first part introduces the Lévy-stable distribution and the trans-stable function. They are defined by Fourier transformations in sections III and IV respectively. The trans-stable function is shown to be identical to the Lévy-stable distribution for $\alpha < 1$ and it has the same asymptotic behaviour for $\alpha > 1$ for large events. The second part refers to section V and it deals with the closed form—analytical approximations—of the Lévy-stable distribution. For this purpose, two types of approximations are developed. The first approximation refers to the inner expansion that converges asymptotically to the Lévy-stable distribution as $x \to 0$. The second approximation refers to the outer expansion that converges asymptotically as $x \to \infty$. For the outer expansion two cases are presented, one is obtained from the Lévy-stable function in subsection VB and the second one from the trans-stable function in subsection VC. Finally, the uniform solution in section VI is proposed as a result of matching the inner and the outer solution. The analytical equation of uniform solution proposed in this paper gives an approximated solution of the Lévy-stable distribution function over the range $-\infty < x < \infty$.

II. CENTRAL LIMIT THEOREM FOR LÉVY-STABLE FLIGHTS

Section 35 of the book by Gnedenko and Kolmogorov [47] shows that the normal distribution is an “attractor” of distributions with finite variances. On the other hand, the attractor of power law distributions with infinite variances corresponds to the more general “Lévy-stable law”. In other words the Lévy-stable is a specific function to which other distributions converge.

The fundamental concept of attractors is formulated as follows. If a normalized sum of a set of independent, identically distributed random variables $\{X_1, X_2, X_3, \ldots X_N\}$ satisfies:

$$\lim_{N \to \infty} \frac{1}{\sigma_N} \left( \sum_{i=1}^{N} X_i - \mu_N \right) = X, \quad (1)$$

then $X$ belongs to the stable law. The coefficients $\mu_N$ and $\sigma_N$ represent the centering and normalizing values respectively [47].

The Gnedenko-Kolmogorov theorem is a generalization of the classical central limit theorem which states that normalized sum of independent random variables with finite variance in Eq. (1) converges to a variable that is normally distributed [47, 48]. This is the case of distributions with power-law tails ($\alpha \geq 2$) with finite variance. The normalized coefficient is $\sigma_N = \sqrt{N}$ and the centering coefficient is $\mu_N = N E[X]$, where $N$ represents the length of the sum and $E[X]$ refers to expected value [49, 50]. On the other hand, for independent random variables power law distributions with infinite variances $^1 0 < \alpha < 2$, Uchaikin and Zorotalev [32, 50] show that

$^1$Note: Infinite variance is observed for $0 < \alpha < 2$. This characteristic occurs for $0 < \alpha \leq 1$, as a consequence of not having a well-defined expected value $E[X]$. For $1 < \alpha < 2$, the integral in the variance definition diverges [30, 32, 50].
$X$ in Eq. (1) follows a symmetric Lévy-stable law if the normalization coefficient is $\sigma_N = N^{1/\alpha}$ and the centering coefficient is $\mu_N = 0$ for $\alpha \leq 1$ or $\mu_N = NE[X]$ for $\alpha > 1$.

Lévy-stable distributions belong to a wider class of infinitely divisible distributions (ID). A random variable $X$ is ID if it can be represented as the sum of a number $N$ of independent and identically distributed random variables $Y_1, Y_2, \ldots, Y_N$.

The characteristic function is defined as the Fourier transform of the probability density function $f(x)$,

$$\varphi(t) = E(e^{itx}) = \int_{-\infty}^{\infty} f(x)e^{itx}dx. \quad (3)$$

The characteristic function of the ID distribution can be derived as follows. Consider $X$ as a sum of two independent random variables $X = Y_1 + Y_2$ with pdf’s $f_1(x)$ and $f_2(x)$ respectively. For the convolution of the two probability distributions $[43]$, the pdf of $X$ has the form,

$$f(x) = \int_{-\infty}^{\infty} f_1(k)f_2(x-k)dk. \quad (4)$$

By substituting Eq. (4) into Eq. (3) and interchanging the order of the integration the equation for the characteristic function of $X$ is obtained,

$$\varphi_X(t) = \varphi_{Y_1}(t)\varphi_{Y_2}(t). \quad (5)$$

Assuming that $Y_1$ and $Y_2$ are identically distributed, the characteristic function of $f(x)$ can be defined as $\varphi_X(t, 2) = (\varphi(t))^2$. In general, for the sum of $N$ independent and identically distributed random variables in Eq. (2), the characteristic function is given by:

$$\varphi_X(t, N) = (\varphi_N(t))^N. \quad (6)$$

Consequently, Eq. (2) and Eq. (6) are equivalent. Then, the limit is applied in Eq. (6), $\varphi_X(t) = \lim_{N \to \infty} \varphi_X(t, N)$. As a consequence, $\varphi_X(t)$ is the characteristic function of the pdf of the random variable $X$. This statement constitutes the Levy continuity theorem that guarantees pointwise convergence $[51, 52]$. The Lévy-Khintchine formula or Triple Lévy gives the general equation for ID distributions $[51]$. This formula determines the class of characteristic function where the pdf is calculated from its Fourier transform $[52-55]$. The Lévy-stable distribution constitutes a special case of the general Lévy-Khintchine in one-dimensional case that is presented in the next section $[52]$.

### III. LEVY-STABLE DISTRIBUTION FUNCTION

The Lévy-stable distribution is given by the Fourier transform of Eq. (3),

$$f(x; \alpha, \beta, \sigma, \mu) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \varphi(t; \alpha, \beta, \sigma, \mu)e^{ixt}dt. \quad (7)$$

Where $\varphi(t)$ is presented in Section 34 of the Gnedenko-Kolmogorov book $[47]$ as,

$$\varphi(t; \alpha, \beta, \sigma, \mu) = e^{(it\mu-|\sigma t|^\alpha)(1-i\beta\text{sgn}(t)\Phi)}. \quad (8)$$

The four parameters involved are: the stability parameter $\alpha \in (0, 2]$, the skewness parameter $\beta \in [-1, 1]$, the scale parameter $\sigma \in (0, \infty)$, and the location parameter $\mu \in (-\infty, \infty)$. The parameter $\alpha$ constitutes the characteristic exponent of the asymptotic power-law in the tails and it determines whether the mean value and the variance exist. The Lévy-stable distribution with $0 < \alpha \leq 1$ does not have a mean value and it has a define variance only for $\alpha = 2$ $[56]$.

The function $\text{sgn}(t)$ represents the sign of $t$ and the function $\Phi$ is defined as:

$$\Phi = \begin{cases} \tan \left( \frac{\pi \alpha}{2} \right) & \alpha \neq 1, \\ -\frac{2}{\pi} \log |t| & \alpha = 1. \end{cases} \quad (9)$$

The Lévy-stable distribution is the family of all attractors of normalized sums of independent and identically distributed random variables. The most well-known Lévy-stable distribution functions are the Cauchy distribution with $\alpha = 1$ and the normal distribution function with $\alpha = 2$. Both functions have $\beta = 0$, which means they are symmetric distributions about their mean $[32]$.

In this paper we will focus on symmetric distributions. For this case the Lévy-stable distribution can be normalized as follows:

$$f(x; \alpha, \beta, 0, \sigma, \mu) = \text{Re} \left\{ \left( S \left( \frac{x-\mu}{\sigma} \right) \right) \right\}, \quad (10)$$

where the general distribution function is given by the following equation:

$$S(x; \alpha) = \frac{1}{\pi} \int_{0}^{\infty} e^{-t^\alpha} e^{ixt}dt. \quad (11)$$

The real part of this function corresponds to the normalized Lévy-stable distribution,

$$s(x; \alpha) = \text{Re}(S(x; \alpha)).$$

Consequently, by applying Euler’s formula we arrive at $[57]$:

$$s(x; \alpha) = \frac{1}{\pi} \int_{0}^{\infty} e^{-t^\alpha} \cos(tx)dt. \quad (12)$$
IV. TRANS-STABLE FUNCTION

Zolotarev (1986) used the term “trans-stable” to refer to a power series expansion that converges to the Lévy-stable distribution for $0 < \alpha < 1$ only [32]. In this paper, trans-stable is the function which one of its solutions originates Zolotarev series when the series expansions are applied around $x \to \infty$. First we define the complex trans-stable function in the range of $0 < \alpha < 2$. For $\alpha < 1$, the Lévy-stable distribution and the trans-stable function are identical. For $\alpha > 1$, the trans-stable function and the Lévy-stable distribution present the same asymptotic behaviour for $x \to \infty$. Consequently, our trans-stable function can be used to find a numerical approximation of the Lévy-stable distribution function for $\alpha > 1$ for large events.

First, the complex trans-stable function is defined as an integral over the path $C$ in the complex plane:

$$G_C(x; \alpha) = \frac{1}{\pi} \int_C I(x, z; \alpha) dz$$

where

$$I(x, z; \alpha) = e^{-z^\alpha} e^{izx}.$$  \hspace{1cm} (14)

The relation of this function to the Lévy-stable $S(x; \alpha)$ and the trans-stable $T(x; \alpha)$ functions is obtained by choosing a particular path $C$ in the complex plane. Then, the Lévy-stable distribution and trans-stable function are given by Eq. (15) and (16) respectively:

$$S(x; \alpha) = G_{[0, \infty)}(x; \alpha) = \frac{1}{\pi} \int_0^\infty e^{-t^\alpha} e^{itx} dt$$ \hspace{1cm} (15)

$$T(x; \alpha) = G_{[0, i\infty)}(x; \alpha) = \frac{1}{\pi} \int_0^{i\infty} e^{-t^\alpha} e^{itx} dt.$$ \hspace{1cm} (16)

First it will be shown that for $0 < \alpha \leq 1$, both Lévy-stable $S(x; \alpha)$ and trans-stable $T(x; \alpha)$ functions are identical. For $1 < \alpha < 2$ it will be demonstrated that both functions exhibit the same asymptotic behaviour when $x \to \infty$.

This demonstration is based on the evaluation of the complex trans-stable integral Eq. (13) using polar representation for $\alpha \leq 1$ and rectangular representation for $\alpha > 1$ on the complex integrand. The demonstrations are presented in the following subsections.

1. For $0 < \alpha \leq 1$

Here we will show that for $0 < \alpha \leq 1$ the Lévy-stable and trans-stable functions are identical. This demonstration will be done by considering the closed contour shown in Figure 1. Since the complex function in Eq. (14) is analytical over the complex plane, the integral over the closed contour Eq. (13) is zero,

$$\oint I(x, z; \alpha) dz = 0.$$ \hspace{1cm} (17)

Let us take the contour in Figure 1 that can be divided into four straight paths so that:

$$\sum_{i=1}^4 \int_{C_i} I(x, z; \alpha) dz = 0.$$ \hspace{1cm} (18)

Now, we will take the limit when $\tau \to \infty$ in Figure 1. Using Eq. (15,16,18) the following equation is obtained:

$$S(x; \alpha) - T(x; \alpha) = -\lim_{\tau \to \infty} \sum_{i=2}^3 \int_{C_i} I(x, z; \alpha) dz.$$ \hspace{1cm} (19)

To evaluate the right hand side in Eq. (19) it is convenient to use the polar representation of the complex number $z = re^{i\theta}$ and express Eq. (14) in polar coordinates:

$$I(x, z; \alpha) = e^{g(x, r, \theta; \alpha) + ih(x, r, \theta; \alpha)},$$ \hspace{1cm} (20)

$$g(x, r, \theta; \alpha) = -r^\alpha \cos (\theta \alpha) - rx \sin \theta,$$ \hspace{1cm} (21)

$$h(x, r, \theta; \alpha) = -r^\alpha \sin (\theta \alpha) + rx \cos \theta.$$ \hspace{1cm} (22)

It will be adopted the nomenclature of signal theory, where the polar notation separates the effects of instantaneous amplitude $|I| = e^\theta$ and its instantaneous phase $h$ of a complex function [58]. Consequently, $g(x, r, \theta; \alpha)$ represents the attenuation factor and $h(x, r, \theta; \alpha)$ represents the oscillation factor.
Now let us notice that \( \lim_{r \to \infty} g(x, r, \theta; \alpha) = -\infty \) for \( 0 < \alpha \leq 1 \) at any value of \( x \). This statement is based on the fact that \( \cos(\theta \alpha) \geq 0 \) in the first quadrant for \( \alpha \leq 1 \).

Consequently, \( \lim_{r \to \infty} I(x, z; \alpha) = 0 \) so that the integral of the right side of the Eq. (19) vanishes at \( r \to \infty \), therefore:

\[
S(x; \alpha) = T(x; \alpha) \quad \text{if} \quad 0 < \alpha \leq 1.
\] (23)

So, the Eq. (23) will allow to use the trans-stable function \( T(x; \alpha) \) instead of the Lévy-stable distribution function \( S(x; \alpha) \) for \( 0 < \alpha \leq 1 \) in the numerical integration. This is with the aim to remove numerical oscillation, specifically in the tails. It is noticeable that the integration of the trans-stable function \( T(x; \alpha) \) in Eq. (16) is performed over the imaginary axis. Applying the following change of variable \( t \to -it \) (formally done by defining \( u = -it \) so that \( du = -idt \) and later replacing the dummy variable \( u \) by \( t \) inside the integral), the trans-stable function is converted into a Laplace transformation. Consequently, the integration is performed over the real axis. The Fourier and Laplace representations for \( T(x; \alpha) \) are shown in Eq. (24),

\[
T(x; \alpha) = \frac{1}{\pi} \int_0^{\infty} e^{-r^\alpha} e^{-xt} dt = \frac{1}{\pi} \int_0^{\infty} e^{-(it)^\alpha} e^{-xt} dt. \quad (24)
\]

Figure 2 compares the Fourier representation of the Lévy-stable distribution function \( S(x; \alpha) \) and the Laplace representation of trans-stable function \( T(x; \alpha) \). The integration is performed using a recursive adaptive Simpson quadrature method [59]. It is evident that the Laplace representation removes the oscillations of the Fourier representation of the Lévy-stable distribution for \( \alpha < 1 \).

It is important to add that the Lévy-stable distribution function and trans-stable function hold the same value for their Fourier and Laplace transform representations. The difference between each transform representation is the axis in which each function is integrated. The expressions are shown in Table 1.

Figure 2. Comparison of numerical integration \( 0 < \alpha < 1 \) between Fourier and Laplace transform of the Lévy-stable \( S(x; \alpha) \) and the trans-stable \( T(x; \alpha) \) functions using recursive adaptive Simpson quadrature method [59]. The absolute error tolerance of the method is \( \xi = 3.5 \times 10^{-5} \). Top plots are shown in semi-logarithmic scale and in logarithmic scale.

| \( \alpha \) | \( S(x; \alpha) \)-Fourier | \( T(x; \alpha) \)-Laplace |
|---|---|---|
| 0.40 | 0.90 | 0.40 |
| 0.50 | 0.70 | 0.50 |
| 0.60 | 0.50 | 0.60 |
| 0.70 | 0.30 | 0.70 |
| 0.80 | 0.10 | 0.80 |
| 0.90 | 0.00 | 0.90 |

Table 1. Summary of Fourier and Laplace representations for the Lévy-stable and the trans-stable functions

2. For \( 1 < \alpha < 2 \)

Here we will shown that for \( 1 < \alpha < 2 \) the Lévy-stable and trans-stable functions have the same asymptotic behaviour on large events if the integrals are appropriately truncated.

Let us recall Eq. (21) for the attenuation factor,

\[
g(x, r, \theta; \alpha) = -r^\alpha \cos(\theta \alpha) - rx \sin \theta.
\]

In the previous section, it was shown that \( \cos(\theta \alpha) \) is always positive in the first quadrant of the complex plane if \( 0 < \alpha \leq 1 \). Otherwise, if \( \alpha > 1 \), then \( \cos(\theta \alpha) < 0 \) when \( \theta = \pi/2 \). Consequently, \( \lim_{r \to \infty} I(x, r, \theta; \alpha) = \infty \) in this case.
range, so that the right hand side of Eq. (19) can not be neglected. Therefore $S(x) \neq T(x)$ if $\alpha > 1$.

We can find an approximation between these two functions if the $\tau$ value in the contour of Figure 1 is kept large but finite ($\tau < \infty$). Thus, Eq. (18) becomes:

$$S(x; \alpha, \tau) - T(x; \alpha, \tau) = -\sum_{i=2}^{3} \int_{S_i} I(x, z; \alpha) \, dz,$$

(25)

where $S(x; \alpha, \tau)$ and $T(x; \alpha, \tau)$ are the truncated integrals in Eqs. (15) and (16) respectively:

$$S(x; \alpha, \tau) = \frac{1}{\pi} \int_{0}^{\tau} e^{-\alpha t} e^{ixt} \, dt,$$

(26)

$$T(x; \alpha, \tau) = \frac{1}{\pi} \int_{0}^{i\tau} e^{-\alpha t} e^{ixt} \, dt.$$  

(27)

Now, the right hand of Eq. (25) can be evaluated in the limit where $x \to \infty$. First, notice that in the contour of integration in Figure 1 the magnitude of $r$ is bounded by the condition $0 < r < \sqrt{2} \tau$ and $\sin(\theta) > 0$ in the first quadrant, thus:

$$\lim_{x \to \infty} g(x, r, \theta; \alpha) = -\infty.$$  

Consequently, $\lim_{x \to \infty} I(x, z; \alpha) = 0$ so that the integral on the right of Eq. (25) vanishes at $x \to \infty$. Therefore, the asymptotic behaviour is obtained for $1 < \alpha < 2$,

$$S(x; \alpha, \tau) \sim T(x; \alpha, \tau) \quad \text{as} \quad x \to \infty.$$  

(28)

This demonstrates that both functions are asymptotically equivalent when the integrals are truncated.

The next step is to find the truncation value $\tau$ that leads to the best approximation of these functions. The value of $\tau$ should be chosen to minimize the truncation error and at the same time to make the domain of integration as small as possible. With this aim, the trans-stable function $T(x; \alpha, \tau)$ in Eq. (16) is expressed in its Laplace representation by using the change of variable $t \to \tau t$.

Thus,

$$T(x; \alpha, \tau) = \frac{1}{\pi} \int_{0}^{\tau} \tilde{I}(x, t; \alpha) \, dt,$$

(29)

where $\tilde{I}$ corresponds to Laplace transform integrand shown in Eq. (24) and Table I,

$$\tilde{I}(x, t; \alpha) = e^{-(x-t)} \alpha e^{-\pi t i}.$$  

(30)

Then, considering Euler’s representation for a complex exponential function $e^{\theta} = \cos(\theta) + isin(\theta)$, the following equations are obtained to express Eq. (30):

$$\tilde{I}(x, t; \alpha) = e^{\tilde{g}(x, t; \alpha) + i\tilde{h}(x, t; \alpha)},$$

(31)

The instantaneous amplitude $|\tilde{I}| = e^{\beta}$ will be determined by the attenuation factor in Eq. (32). For that reason, an analysis of $\tilde{g}(x, t; \alpha) = 0$ divides two regions, one with exponential growth ($\tilde{g} > 0$) and the other with exponential decay ($\tilde{g} < 0$).

Figure 3. Curve $\tilde{g} = 0$ separates two regions with $\tilde{g} > 0$ and $\tilde{g} < 0$. In the latter region, two zones can be distinguished: zone A with $\partial g/\partial t < 0$ and zone B with $\partial g/\partial t > 0$. The equation $x = \alpha t$ is an estimation of the boundary between zones A and B.

In Figure 3, two sub-regions can be recognized in $\tilde{g} < 0$. The first one, “Zone A” which contains negative $\tilde{g}$ values with downward trend $\partial g/\partial t < 0$ that is faster as $x \to \infty$. The second sub-region is “Zone B”, it contains smaller negative $\tilde{g}$ values that follow an upward trend and $\partial g/\partial t > 0$ displaying an increase behaviour when $x \to 0$. Considering these sub-regions, the truncation $\tau$ in Eq. (29) will depend on $x$ value as follows:

- For $x \to 0$, The integration must avoid zone B. The values of $\tilde{g}(x, t; \alpha)$ in this zone lead to an exponential growth due to an upward trend $\partial g/\partial t > 0$, consequently $|\tilde{I}| \to 0$.

- For $x \to \infty$, the integration should be restricted to zone A. The downward trend $\partial g/\partial t < 0$ leads to obtain $\tilde{g}(x, t; \alpha) \to 0$. Consequently, the convergence of $|\tilde{I}| \to 0$ occurs faster as $t \to \infty$.

For $x \to 0$, the cut off $\tau_1$ which avoids most of zone B is defined by $x = \alpha t$. This equation is an estimation of
the boundary between zones A and B for all range of \( \alpha \) values.

The cut off \( \tau_1 \) obeys a linear equation and is obtained from the following equations:

\[
e^{\bar{g}(x,\tau_1;\alpha)} = |\bar{I}| = \epsilon \quad \text{and} \quad x = \alpha \tau_1, \tag{34}
\]

where the tolerance \( \epsilon \) represents a negligible instantaneous amplitude \( |\bar{I}| \).

For \( x \to \infty \), the cut off \( \tau_1 \) will restrict the integration of \( \bar{I} \) on a closed interval \([0, t_c]\). This occurs due to a faster downward trend \( \partial \bar{g}/\partial t < 0 \). The \( t_c \) value represents the point where the instantaneous amplitude can be considered a negligible quantity \( |\bar{I}| = \epsilon \). Thus, the cut off \( \tau_1 \) obeys an equation of a vertical line \( \tau_1 = t_c \).

Notice that there are two different definitions for \( \tau_1 \). Each one corresponds to a particular sub-regions A \((x \to \infty)\) or B \((x \to 0)\). Consequently, the truncation \( \tau_1 \) for the trans-stable function is defined by two equations which depend on the \( x \) and \( \epsilon \) values. These two equations have their intersection point at \((t_c, x_c)\):

\[
\tau_1(\epsilon, x) = \begin{cases} 
t_c(\epsilon) & \text{if } x > x_c \\
\frac{x}{\alpha} & \text{if } x < x_c 
\end{cases} \quad \text{for} \quad \alpha > 1, \tag{35}
\]

where \( t_c(\epsilon) \) and \( x_c \) are given by the implicit form of the following equations:

\[
\alpha t_c^2 + t_c^\alpha \cos(\pi \alpha/2) + \ln(\epsilon) = 0, \\
x_c = \alpha t_c.
\tag{36}
\]

Figure 4 illustrates the contour plot of the instantaneous amplitude \( |\bar{I}| \) for \( \alpha = 1.4 \). The truncation \( \tau_1 \) is presented as a cut-off made when a negligible value of instantaneous amplitude is achieved \( |\bar{I}| = \epsilon = 10^{-3} \). The point \((x_c, t_c)\) is located at the intersection between the contour line of the given tolerance \( \epsilon \) and the equation \( \tau_1 = x/\alpha \). The truncation \( \tau_1 \) avoids zone B which contains negative values for \( \bar{g} \) with \( \partial \bar{g}/\partial t > 0 \). One can observe that there is an abrupt upward trend in \( |\bar{I}| \) for \( x \to 0 \). So, the truncation \( \tau_1 \) allows us to make a perfect cut off before this upward trend starts. It is noticeable that with a small tolerance \( \epsilon \) the intersection will occur in the rightmost part of the figure, consequently the interval of integration will be wider and a more accurate result can be obtained.

Figure 5 shows how the solutions of trans-stable and Lévy-stable distribution functions are quite similar after \( x_c \) value, which depends on the tolerance \( \epsilon \). For a smaller \( \epsilon \) the similarity of both asymptotic series is expected to improve due to a wider interval of integration. However, the value \( x_c \) will be higher and the similarity will start at the rightmost part of the axis.
These conditions are that the integrand must be bounded and the domain of integration is a closed interval \([0, 1]\) for \(0 < \alpha \leq 1\).

### A. Inner Expansion

The inner expansion is obtained making a substitution of \(e^{ixt}\) by its Taylor series expansion given by Eq. (37) in the integrand of the Lévy-stable distribution \(I\). After this substitution, the integrals in Eq. (15) and (26) can be analytically solved. The difference between these two equations are the truncation on the interval of integration.

For \(\alpha \leq 1\) the convergence of the series is slow, demanding a large value of order \(n\) in Eq. (37) to reach an acceptable similarity with the original integrand \(I\). For this reason, the improper integral is truncated after a small enough amplitude of \(I\) is obtained. For \(\alpha > 1\) the convergence occurs faster and truncation is not needed.

1. **For \(0 < \alpha \leq 1\)**

The inner expansion is obtained by substituting \(e^{ixt}\) in Eq. (26) by its Taylor expansion using Eq. (37). Then:

\[
S_i(x; \alpha, \epsilon) = \frac{1}{\pi} \int_0^{\tau_2(x, \epsilon)} e^{-\epsilon t} e^{ixt} dt \sim \frac{1}{\pi} \int_0^{\tau_2(x, \epsilon)} I_n dt \text{ as } n \to \infty,
\]

where \(I_n\) is given by:

\[
I_n(x, t, \alpha) = \sum_{k=0}^{n} e^{-\epsilon t} \frac{(ixt)^k}{k!}.
\]

The upper limit \(\tau_2\) is given by the following equation:

\[
\tau_2(x, \epsilon) = -\frac{\ln(\epsilon)}{x}.
\]

This truncation results from the equation \(e^{ix\tau_2} = \epsilon\), where \(\epsilon\) represents the tolerance that needs to be small to ensure a cut-off when negligible quantities of \(|I_n|\) are obtained. Consequently, the area under the curve of both functions are similar.

The convergence of \(I_n\) to \(I\) demands a large value of order \(n\) in Eq. (37), as it can be observed in Fig (6). This occurs because of slow decay of \(e^{-\epsilon t}\) value for \(\alpha < 1\). This is the reason to evaluate the integral in the closed interval \([0, \tau_2]\), where the original integrand \(I\) and its Taylor series approximation \(I_n\) are similar.

The integrals in Eq. (38) can be solved without difficulty. Then, the inner expansion \(s_i\) is given by the real part of this solution,

\[
s_i(x; \alpha, \epsilon) = Re(S_i(x; \alpha, \epsilon)).
\]
Note: Matlab defines the incomplete gamma function as $\gamma$. Due to a computation of the incomplete gamma function, the integrand $I$ is evaluated at $x = 4.5$ for three cases of $n = 20, 30, 50$ with $\epsilon = 10^{-3}$.

Consequently,

$$s_i(x; \alpha, \epsilon) = \frac{1}{\alpha^n} \sum_{k=0}^{\infty} \frac{x^k}{k!} \gamma \left( \frac{k+1}{\alpha}, \tau_2(x; \epsilon)^\alpha \right) \cos \left( \frac{\pi k}{2} \right),$$

where $\gamma$ represents the incomplete gamma function.

Due to a computation of the incomplete gamma function $\gamma$, Eq. (41) was modified for numerical analysis in Matlab [62].

Here it is derived the inner expansion $s_i$ for $\alpha > 1$ from the non-truncated form of Lévy-stable distribution function. This derivation is made by substituting $e^{ixt}$ in Eq. (15) by its Taylor expansion in Eq. (37), then:

$$S_i(x; \alpha) = \frac{1}{\pi} \int_0^\infty e^{-\tau^\alpha} e^{ix\tau} d\tau \sim \frac{1}{\pi} \int_0^\infty I_n d\tau \quad \text{as} \quad n \to \infty.$$  \hfill (43)

For $\alpha > 1$, the convergence of integrand $I$ and the integrand after the substitution $I_n$ occurs faster than for $\alpha < 1$. This feature is observed in Figure (6), where an acceptable convergence between $I$ and $I_n$ is obtained with a small $n$ value. Consequently, the integral is evaluated without truncation or taking the limit $\epsilon \to 0$ in Eq. (38).

Then, it is only considered the real part of the solution of Eq. (43),

$$s_i(x; \alpha) = \text{Re}(S_i(x; \alpha)).$$
Consequently,

\[ s_i(x; \alpha) = \frac{1}{\pi \alpha} \sum_{k=0}^{\infty} \frac{x^k}{k!} \Gamma \left( \frac{k + 1}{\alpha} \right) \cos \left( \frac{\pi k}{2} \right), \]  

where \( \Gamma \) represents the gamma function \[62\],

\[ \Gamma(b) = \int_0^\infty x^{b-1} e^{-x} dx. \]

Examples for \( \alpha = 0.75 \) and \( \alpha = 1.80 \) are shown on Figures 7 and 8 respectively. In Figure 7, for \( \alpha \leq 1 \) the truncation \( \tau_2 \) is needed, otherwise the convergence to Lévy-stable distribution function will be ultraslow as \( n \to \infty \). This is evident when a comparison is made between subfigure (a) and (b). They represent a non-truncated and truncated Lévy-stable solution respectively. The subfigure (b) displays an acceptable convergence with a smaller order \( n \). In Figure 8, for \( \alpha > 1 \) the convergence to the Lévy-stable distribution function occurs faster and no truncation is needed. For both cases the inner expansion \( s_i \) behaves well because it converges to \( s(x; \alpha) \).

### B. Outer Expansion

The outer expansion is obtained making a substitution of the amplitude \( e^{-\tau t^\alpha} \) in the integrand of the truncated Lévy-stable distribution function \( I \) in Eq. (26) by its Taylor series expansion around \( t = 0 \). Then, the following relation is obtained:

\[ S_o(x; \alpha, \epsilon) = \frac{1}{\pi} \int_0^{\tau_3(\epsilon)} e^{-t^\alpha \epsilon ix t} dt \sim \frac{1}{\pi} \int_0^{\tau_3(\epsilon)} G_n dt \quad \text{as} \quad n \to \infty, \]  

where \( G_n \) is given by:

\[ G_n(x; \alpha) = \sum_{k=0}^{n} \frac{(-\epsilon)^k}{k!} x^k. \]  

The upper limit \( \tau_3(\epsilon) \) is given by the following equation:

\[ \tau_3(\epsilon) = \left[-\ln(\epsilon)\right]^{1/\alpha}. \]  

This truncation is calculated from \( e^{-\tau_3 n} = \epsilon \), where \( \epsilon \) is defined as tolerance and represents a negligible instantaneous amplitude when \( \epsilon \) is small. The truncation allows a faster convergence of \( G_n \) to \( I \) and reduces the error of integration due to an accurate approximation on the interval \([0, \tau_3]\).

The original integrand \( I \) and the new integrand after applying Taylor series \( G_n \) in Eq. (45) were evaluated in Figure 9. Since the convergence of \( G_n \) to \( I \) is slow, the truncation \( \tau_3 \) is considered to define the new interval of integration \([0, \tau_3]\).

To obtain the outer solution \( s_o \) a change of variable after the series expansion is applied in Eq. (45). The change of variable is \( -u = ix t \), so \( -du = ix dt \). This gives us an approximation of the form:

\[ S_o(x; \alpha, \epsilon) \sim \frac{1}{\pi} \sum_{k=0}^{n} \frac{(-1)^k}{k!} \left( \frac{-1}{ix} \right)^{k+1} e^{-ix \tau_3(\epsilon)} \int_0 u^{k-1} e^{-u} du. \]  

\[ (48) \]
Consequently, we obtain the following relation between the variables, geometric function. Then, comparing Eq. (48) and (49),

\[ t = 0 \]

1.

The plots correspond to \( x = 4.5 \) and \( n = 20 \), \( n = 30 \), \( n = 60 \) for three cases of \( \alpha = 0.75 \) (left) and \( \alpha = 1.8 \) (right). Truncation of the integral is required for both cases. The reason of that is to reach an accurate approximation between the original integrand \( I \) and the one after the series expansion \( G_n \). The error is measured by the absolute value of the difference \( |I - G_n| \). For these particular examples, the integrand \( I \) is evaluated at \( x = 4.5 \) for three cases of \( n = 20, 30, 50 \) with \( \epsilon = 10^{-9} \).

Figure 9. Comparison of \( I = e^{tv} e^{ixt} \) and \( G_n = \frac{(-v)^k}{k!} e^{ixt} \) of Eq. (45), where \( G_n \) is the Taylor expansion of \( I \) around \( t = 0 \). The plots correspond to \( \alpha = 0.75 \) (left) and \( \alpha = 1.8 \) (right). Truncation of the integral is required for both cases. The reason of that is to reach an accurate approximation between the original integrand \( I \) and the one after the series expansion \( G_n \). The error is measured by the absolute value of the difference \( |I - G_n| \). For these particular examples, the integrand \( I \) is evaluated at \( x = 4.5 \) for three cases of \( n = 20, 30, 50 \) with \( \epsilon = 10^{-9} \).

To solve the integral, the incomplete gamma function of imaginary argument \( \gamma(v, iz) \) is used [62, 64]. The following solution is presented by Barak as a special case of confluent hypergeometric function [64],

\[
\gamma(v, iz) = \int_0^{iz} t^{v-1} e^{-t} dt \tag{49}
\]

\[ = (iz)^v v^{-1} {}_1F_1(v, 1 + v, -iz), \]

where \( {}_1F_1(v, 1 + v, -iz) \) represents the Confluent Hypergeometric function. Then, comparing Eq. (48) and (49), we obtain the following relation between the variables, \( v = k\alpha + 1, z = -\epsilon \tau_3 \) and \( t = u \).

Finally, the real part of the solution is:

\[ s_o(x; \alpha, \epsilon) = Re(S_o(x; \alpha, \epsilon)). \]

Consequently,

\[
s_o(x; \alpha, \epsilon) = -\frac{1}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \left( \frac{\cos(\pi k \alpha)}{k\alpha + 1} \right) \ldots \]

\[ \ldots (-\tau_3(\epsilon))^{k\alpha + 1} {}_1F_1(k\alpha + 1, k\alpha + 2, ix\tau_3(\epsilon)). \tag{50} \]
Due to a slow convergence of $K_n$ to $\bar{I}$ the cut-off $\tau_1$ is applied. The truncation $\tau_1$ has two different expressions. For $\alpha \leq 1$, the truncation $\tau_1$ depends on the tolerance $\epsilon$ and for $\alpha > 1$ it depends on the tolerance $\epsilon$ and $x$ values. These expressions will be explained in the following subsections.

To solve the integral in Eq. (51), the following change of variable is applied: $xt = u$ and $xdt = du$. This leads to the following series expansion:

$$T_o(x; \alpha, \epsilon) \sim \frac{1}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left(\frac{-1}{ix}\right)^{k\alpha+1} \frac{x^{\tau_1(x,\epsilon)}}{\int_0 u^{(k\alpha+1)-1} e^{-u} du.}$$

(53)

The upper limit of the integral changes from $\tau_1$ to $x\tau_1$, but still remains on the real axis. The integral above can be solved using the incomplete gamma function defined in Eq. (42). Then, the real part of the result is obtained,

$$t_o(x; \alpha, \epsilon) = Re(T_o(x; \alpha, \epsilon)).$$

Consequently,

$$t_o(x; \alpha, \epsilon) = -\frac{1}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \left(\frac{1}{x}\right)^{k\alpha+1} \sin\left(\frac{\pi\alpha k}{2}\right) \gamma(k\alpha + 1, x\tau_1(x, \epsilon)).$$

(54)

The determination of $\tau_1$ for $\alpha \leq 1$ and $\alpha > 1$ is presented in the following subsections.

1. For $0 < \alpha \leq 1$

For $\alpha \leq 1$, the cut-off $\tau_1$ in Eq. (54) is given by the following equation:

$$\tau_1(\epsilon) = \left[-\ln(\epsilon)\right]^{1/\alpha} \text{ for } \alpha \leq 1.$$

(55)

This truncation is obtained from $e^{-\tau_1^\alpha} = \epsilon$, where the tolerance $\epsilon$ represents a negligible instantaneous amplitude for the integrands in Eq. (51).

2. For $1 < \alpha < 2$

The truncation $\tau_1$ in Eq. (54) for $1 < \alpha < 2$ was already obtained in subsection IV-2 and defined by Eq. (35) as:

$$\tau_1(x, \epsilon) = \begin{cases} t_c(\epsilon) & \text{if } x > x_c, \\ x/\alpha & \text{if } x < x_c, \end{cases} \text{ for } \alpha > 1,$$

where $t_c$ and $x_c$ were defined by Eq. (36). As indicated in subsection IV-2, the value of $\tau_1$ is used to minimize the truncation error and at the same time to make the domain of integration as small as possible.

C. Outer expansion by Trans-Stable distribution

Because of the slow convergence of the outer expansion $s_o$ and its wave-like behaviour, an alternative approximation is obtained using the trans-stable function $T(x; \alpha)$. As it was previously explained in section IV, the solutions of trans-stable $T(x; \alpha)$ and Lévy-stable $S(x; \alpha)$ functions are identical for $0 < \alpha \leq 1$ and similar for $1 < \alpha < 2$ after the $x_c$ value. Consequently, the improper integral in Eq. (24) is used to calculate the series expansions for $0 < \alpha \leq 1$ and the truncated trans-stable integral in its Laplace representation in Eq. (29) for $1 < \alpha < 2$.

This outer expansion $t_o$ is given by the analytical solution of the trans-stable function after applying the Taylor series of $e^{-it\tau_1}$ around $t = 0$ in the trans-stable integrand $\bar{I}$ using Eq. (37) and (24). Then, the following equation is shown:

$$T_o(x; \alpha, \epsilon) = \frac{1}{\pi} \int_0^{\tau_1(x,\epsilon)} e^{-(it)^\alpha} e^{-xt} dt \sim \frac{1}{\pi} \int_0^{\tau_1(x,\epsilon)} K_n dt,$$

(51)

where $K_n$ is given by:

$$K_n(x) = \sum_{k=0}^{n} \frac{(-it)^\alpha}{k!} e^{-xt} \frac{1}{k!}.$$

(52)
The outer expansion by the trans-stable function converges to the original trans-stable function. Examples are shown in Figure 11 for $\alpha \leq 1$ and Figure 12 for $\alpha > 1$. Note that in both cases the truncation $r_1$ allows a faster and more accurate convergence to the real part of the trans-stable distribution $t(x; \alpha)$. Consequently the outer solution $t_o$ shows an identical solution as $s(x; \alpha)$ for $\alpha \leq 1$ and the same asymptotic behaviour for $\alpha > 1$. For a smaller $\epsilon$ the convergence of these outer expansions to the trans-stable function will occur faster. Also in Figure 11 the non-truncated trans-stable expansion is shown as an expansion that converges extremely slowly requiring a higher order $n$ than truncated trans-stable expansion to obtain an acceptable convergence. In Figure 12 the non-truncated trans-stable expansion does not converge to trans-stable function at all.

Figure 11. Outer expansion of the trans-stable function for $\alpha = 0.75$. This result is obtained from the Taylor expansion of the integrand around $t = 0$ in Eq. (54) and (55). The subfigure (a) is the non-truncated integral that shows slow convergence. The subfigure (b) corresponds to truncated integral with tolerance $\epsilon = 10^{-6}$. The subfigure (b) displays a faster convergence to the trans-stable function as a result of the truncation of the integral.

VI. UNIFORM SOLUTION

The uniform solution is presented as the combination of the inner solution and the outer solution to construct an approximation valid for all $x \in [-\infty, \infty]$. To construct the uniform solution an asymptotic matching method based on boundary-layer theory is applied [66, 67]. This method is based on superposing the inner and outer solution and subtracting the overlap between them,

$$s_u(x) = y_{out}(x) + y_{in}(x) - y_{overlap}(x).$$  \hspace{1cm} (56)

The overlap is defined as the limit of the rightmost edge of $y_{in}$ and the leftmost edge of $y_{out}$,

$$y_{overlap} = \lim_{x \to 0} y_{out} = \lim_{x \to \infty} y_{in}. \hspace{1cm} (57)$$

For this case, our proposed uniform solution $s_u$ is constructed based on our inner expansion $s_i$ and our outer expansion $t_o$. These previous solutions were already defined in section V.

For a better understanding of our uniform solution $s_u$, two sub-sections A and B are presented. Sub-section A...
Table II. Summary of Inner and Outer Solutions

| Range of $\alpha$ | $0 < \alpha \leq 1$ | $1 < \alpha < 2$ |
|-------------------|---------------------|------------------|
| **Normalized Lévy-stable distribution ($s$)** | $s(x; \alpha) = \frac{1}{\pi} \int_0^\infty e^{-t^\alpha} \cos(tx) dt$ | |
| **Normalized trans-stable distribution ($t$)** | $t(x; \alpha) = \frac{1}{\pi} \int_0^\infty e^{-t^\alpha \cos(\frac{\pi}{2} z)} - t^\alpha \sin(t^\alpha \sin(\frac{\pi}{2} z)) dt$ | |
| **Inner expansion ($s_i^n$)** | $s_i^n(x; \alpha, \epsilon) = \frac{1}{\pi \alpha} \sum_{k=0}^n \frac{x^k}{k!} \gamma \left( \frac{k + 1}{\alpha}, \tau_2 \right) \cos \left( \frac{\pi k}{2} \right)$ $\tau_2 = -\ln(\epsilon) / x$ | $s_i^n(x; \alpha) = \frac{1}{\pi \alpha} \sum_{k=0}^n \frac{x^k}{k!} \Gamma \left( \frac{k + 1}{\alpha} \right) \cos \left( \frac{\pi k}{2} \right)$ |
| **Outer expansion ($s_o^n$)** | $s_o^n(x; \alpha, \epsilon) = -\frac{1}{\pi} \sum_{k=1}^n \frac{(-1)^k}{k!} \left( \frac{\cos(\pi \alpha k)}{\alpha + 1} \right) (-\tau_3)^{k\alpha + 1} F_1(k\alpha + 1, k\alpha + 2, ix\tau_3)$ $\tau_3 = \left[-\ln(\epsilon)\right]^{1/\alpha}$ | |
| **Outer solution ($t_o^n$)** | $t_o^n(x; \alpha, \epsilon) = -\frac{1}{\pi} \sum_{k=1}^n \frac{(-1)^k}{k!} \left( \frac{1}{x} \right)^{k\alpha + 1} \gamma(k\alpha + 1, x\tau_1) \sin \left( \frac{\pi \alpha k}{2} \right)$ $\tau_1 = \left[-\ln(\epsilon)\right]^{1/\alpha}$ | $\tau_1 = \begin{cases} t_c & \text{if } x > x_c \\ x/\alpha & \text{if } x < x_c \end{cases}$ |
| **Complete and incomplete gamma functions ($\Gamma$) & ($\gamma$)** | $\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} dx$ $\gamma(z, b) = \int_0^b x^{z-1} e^{-x} dx$ | |

contains a summary of inner and outer expansions previously obtained. In sub-section B the steps taken to obtain $s_u$ are explained.

### A. Summary of inner and outer expansions

Table II contains the normalized Lévy-stable and trans-stable distribution and the summary of previous results obtained from Lévy-stable and trans-stable functions by applying Taylor expansions. The series refers to one inner expansion $s_i$ and two outer expansions $s_o$ and $t_o$.

For the inner expansion $s_i$, the solution for $\alpha \leq 1$ corresponds to a truncated Lévy-stable solution which allows a faster convergence. For $\alpha > 1$ the series is obtained from the non-truncated Lévy-stable solution. The only difference between them is the use of the incomplete gamma function $\gamma$ in the solution for $\alpha \leq 1$, where $\Gamma(z) = \lim_{b \to \infty} \gamma(z, b)$. Consequently, for both cases the truncated series can provide a good approximation. However, in the case of $\alpha \leq 1$ we must take the limit as:

$$s(x; \alpha) = \lim_{\epsilon \to 0} \left( \lim_{n \to \infty} s_i^n(x; \alpha, \epsilon) \right) \text{ for } x < \infty.$$  

In general the order how we apply the limits cannot be exchanged. However, in the case of $\alpha > 1$ the order of the limits does not affect the convergence. Taking a small value of $\epsilon$ ensures a faster convergence.

For the outer expansion two expressions were derived.

\[\text{Note: Refer to equation Eq. (35) to obtain } t_c \text{ and } x_c \text{ value for } 1 < \alpha < 2\]
The first outer expansion \( s_o \) is obtained by performing the Taylor expansion around \( t = 0 \) on the truncated Lévy-stable distribution. This solution displays a slow convergence for \( n \to \infty \). The second outer expansion \( t_o \) is obtained by applying the Taylor expansion on the truncated trans-stable function for \( x \to \infty \). The truncation of \( t_o \) depends on \( \alpha \) and there are two different cases. For \( \alpha \leq 1 \) it converges to the exact solution of \( s(x; \alpha) \) and for \( \alpha > 1 \) it converges to the same solution at the tails of \( s(x; \alpha) \). To guarantee convergence, we need to take the limit as,

\[
 t(x; \alpha) = \lim_{\epsilon \to 0} \left( \lim_{n \to \infty} t_o^n(x; \alpha, \epsilon) \right) \quad \text{for} \quad x > 0.
\]

Exchanging the order of the limits will affect the convergence. The outer expansion that will be used is \( t_o \), because it displays a faster convergence and it does not exhibit wavelike behaviour.

### B. Steps to obtain the uniform solution

To obtain the uniform solution \( s_u \), the condition in Eq. (57) needs to be satisfied. The inner expansion \( s_i \) and the outer expansion \( t_o \) have to be multiplied with an appropriate coefficient \( A(x) \) to obtain the asymptotic solutions with a common matching value \( y_m \). These operations will allow us to obtain \( y_{out} \) and \( y_{in} \). Consequently, Eq. (56) will be applied to obtain the closed-form solution of the Lévy-stable distribution function.

Below the steps are explained to obtain the location of the matching between the inner and the outer solutions \((x_m, y_m)\), the coefficient \( A(x) \), and the uniform solution \( s_u \).

#### 1. Finding inner and outer limit \((x_m, y_m)\)

Considering \( s_i \) and \( t_o \) as good approximations to the Lévy-stable distribution function, we must require that the inner and the outer expansions will be close enough before matching them [68]. Consequently, the point where the matching between \( s_i \) and \( t_o \) takes place is \((x_m, y_m)\) and it represents the location where the minimal vertical distance between the inner \( s_i \) and the outer solution \( t_o \) occurs.

The distance function between \( s_i \) and \( s \) is defined as \( \delta_i \) and the distance function between \( t_o \) and \( s \) is \( \delta_o \). Consequently, \((x_m, y_m)\) is the point where the Pythagorean addition of these distances is minimal Eq. (58):

\[
\begin{align*}
\delta_i^2(x; \alpha, \epsilon) &= (s(x; \alpha) - s_i(x; \alpha, \epsilon))^2, \\
\delta_o^2(x; \alpha, \epsilon) &= (s(x; \alpha) - t_o(x; \alpha, \epsilon))^2, \\
\delta^2(x; \alpha, \epsilon) &= \delta_o^2(x; \alpha, \epsilon) + \delta_i^2(x; \alpha, \epsilon),
\end{align*}
\]

\[
\frac{d}{dx} \left( \frac{\delta^2(x; \alpha, \epsilon)}{dx} \right) \bigg|_{x = x_m} = 0. \tag{58}
\]

The \( x_m \) value is obtained from the previous equation. Then, \( y_m \) is defined by the equidistant point between both functions,

\[
y_m = \frac{s_i(x_m) + t_o(x_m)}{2}. \tag{59}
\]

#### 2. Defining the inner and the outer solutions \( y_{in} \) and \( y_{out} \)

To obtain the uniform solution \( s_u \), the asymptotic matching method based on boundary layer theorem [66] is applied. Consequently, the inner solution \( y_{in} \) and the outer solution \( y_{out} \) must have a matching asymptotic behaviour. More precisely, the limit of the outer solution \( y_{out} \) when \( x \to 0 \) should correspond to the limit of the inner solution \( y_{in} \) when \( x \to \infty \). To obtain \( y_{in} \) and \( y_{out} \) solutions, the series expansions \( s_i \) and \( t_o \) are multiplied by an appropriate coefficient to meet the requirements of matching asymptotic expansions, so the \( y_{in} \) and \( y_{out} \) are defined as follows:

\[
y_{in}(x; \alpha, \epsilon, \mu) = (s_i(x) - y_m) (1 - A(x; \mu)) + y_m, \tag{60}
\]

\[
y_{out}(x; \alpha, \epsilon, \mu) = (t_o(x) - y_m) A(x; \mu) + y_m, \tag{61}
\]

where the overlapping factor \( A(x) \) is defined as:

\[
A(x; \mu) = \frac{1}{2} \left( 1 + \tanh \left( \frac{x - x_m}{\mu} \right) \right). \tag{62}
\]

The \( A(x; \mu) \) is used to smooth \( s_i \) and \( t_o \) and provides them with a symmetric overlap section around \( x_m \) and gives \( y_{in} \) and \( y_{out} \) an asymptotic behaviour. The variable \( \mu \) determines the width of the overlap between \( y_{in} \) and \( y_{out} \).

It is easy to see that Eq. (60) and Eq. (61) satisfy Eq. (57), where the limits of \( y_{out} \) and \( y_{in} \) converge to a constant value \( y_m \).

#### 3. Defining the Uniform Solution \( s_u \)

The inner solution \( y_{in} \) Eq. (60) and the outer solution \( y_{out} \) Eq. (61) were defined to fulfill the requirements for matching asymptotic expansions. Then, Eq. (56) is applied to obtain the uniform solution \( s_u \).

\[
s_u^{n_i, n_o}(x; \alpha, \epsilon, \mu) = \frac{t_o^{n_o}(x; \alpha, \epsilon)}{2} + \frac{s_i^{n_i}(x; \alpha, \epsilon)}{2} + \ldots + \tanh \left( \frac{x - x_m}{\mu} \right) \left( \frac{t_o^{n_o}(x; \alpha, \epsilon)}{2} - \frac{s_i^{n_i}(x; \alpha, \epsilon)}{2} \right). \tag{63}
\]
4. Find the best $s_u$ by choosing the most appropriate $\mu$

The width of the overlap between $y_{in}$ and $y_{out}$ can be optimized to obtain the closest solution $s_u$ of the Lévy-stable distribution function. The most appropriate value of $\mu$ needs to be obtained for each particular value of $\alpha$. For that, the least square method will be applied between the original $s(x; \alpha)$ and the new closest solution $s_u(x; \alpha, \epsilon, \mu)$. Applying Eq. (12) and (63) the following equation is obtained:

$$L(\mu) = \sum_{i=1}^{N} (s_u(x_i; \alpha, \epsilon, \mu) - s(x_i; \alpha))^2,$$

where the $N$ value represents the length of the sample used to minimize $L$.

The similarity between the exact solution of $s(x; \alpha)$ and the uniform solution $s_u(x; \alpha, \epsilon, \mu)$ is observed in Figure 13 and 14 for $\alpha = 0.75$ and $\alpha = 1.80$ respectively. For $\alpha < 1$, a good approximation between $s(x; \alpha)$ and $s_u(x; \alpha, \epsilon, \mu)$ is obtained in the tails after mixing two different orders. The order for the inner solution is $n_i = 6$, which makes the solution concave upward. The order for the outer solution is $n_o = 17$, which makes the solution concave downward. This combination of orders will ensure a good matching asymptotic behaviour. For $\alpha > 1$, the uniform solution works well, and a good uniform solution is obtained quickly with a lower order $n = 6$.

Lower orders can be used for both cases, where the most important aspect to consider is the different concavity between $y_{in}$ and $y_{out}$ for the matching asymptotic behaviour. The concavity of the inner and outer solution is defined by the trigonometric element in each solution.

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**VII. CONCLUSIONS**

In this paper we presented a uniform solution of the Lévy-stable distribution. This solution converges to the Lévy-stable distribution function in the full range of $x$ values $-\infty < x < \infty$. This condition makes our uniform solution more robust than previous analytical expressions that were only applicable for extreme values $x \to 0$ or $x \to \infty$. Also, our uniform solution removes the negative values obtained in previous numerical solutions of the Lévy-stable distribution function for all $\alpha$ values, which makes this solution more reliable because a probability density function must be always positive.

The uniform solution is the result of an asymptotic matching between the inner and outer expansions. The inner expansion results from the Taylor series expansion of the characteristic function of the Lévy-stable distribution around $x = 0$. The outer expansion is obtained from the Taylor expansion of the integrand of the trans-stable function around $t = 0$. The convergence of these expansions is guaranteed if the integrands are truncated, and the speed of convergence depends on how is the truncation implemented.

For $\alpha \leq 1$, the uniform solution provides a good approximation for all the range of $x$ values. Also, the numerical integration of the trans-stable function constitutes a second option which allows us to obtain a robust numerical solution of the Lévy-stable distribution function and removes the oscillations. For $\alpha > 1$, the uniform solution provides an analytical solution of the Lévy-stable distribution function based on fast converging series. Consequently, the closed-form solution presented in this paper will provide an analytical solution of the fractional kinetic equations (FDE,FDAE,FFPE).

Additionally, having an analytical solution for the Lévy-stable distribution will contribute on modelling stock markets. To achieve this, Lévy-stable noise will be generated numerically. The following procedure is described to generate Lévy-stable noise. First random points between 0 and 1 are generated. Then, the inverse of the cumulative distribution function (CDF) of
the Lévy-stable distribution is applied to these points. Consequently, the corresponding image of the uniformly generated points will be Lévy-stable distributed. Different compromises between accuracy and efficiency in the random number generation can be attained by changing the order $n$ in Eq. (63). Hence, a computational efficiency and high precision are achieved during the generation of large sets of points.

For modelling stock markets, Lévy-stable noise represents the net trading volume—difference between buy and sell stocks’ volume—and will feed macroscopic models of the stock markets. On the other hand, to develop microscopic model of stock markets, the Lévy-stable noise can be used to represent the order book (OB) —list of request for buy and sell orders with prices and volumes. The use of Levy-Stable noise is justified by the fact that volumes, lifetime of orders, and the placement of limit orders in an OB present power-law decays with coefficients on Lévy-stable range. In consequence, a more realistic microscopic model can be developed by the use of our closed-form solution of the Lévy-stable distribution function.

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