ON THE CHARACTERISTIC POLYNOMIAL OF CARTAN MATRICES AND CHEBYSHEV POLYNOMIALS

PANTELIS A. DAMIANOU

Abstract. We explore some interesting features of the characteristic polynomial of the Cartan matrix of a complex simple Lie algebra. The characteristic polynomial is closely related with the Chebyshev polynomials of first and second kind. In addition, we give explicit formulas for the characteristic polynomial of the Coxeter adjacency matrix, we compute the associated polynomials and use them to derive the Coxeter polynomial of the underlying graph. We determine the expression of the Coxeter and associated polynomials as a product of cyclotomic factors. We use this data to propose an algorithm for factoring Chebyshev polynomials over the integers. Finally, we prove an interesting sine formula which involves the exponents, the Coxeter number and the determinant of the Cartan matrix.

1. Introduction

The aim of this paper is to explore the intimate connection between Chebyshev polynomials and root systems of complex simple Lie algebras. Chebyshev polynomials are used to generate the characteristic and associated polynomials of Cartan and adjacency matrices and conversely one can use machinery from Lie theory to derive properties of Chebyshev polynomials. Some of the results in this paper are well-known but we re-derive them in the context of Chebyshev polynomials.

Cartan matrices appear in the classification of simple Lie algebras over the complex numbers. A Cartan matrix is associated to each such Lie algebra. It is an \(\ell \times \ell\) square matrix where \(\ell\) is the rank of the Lie algebra. The Cartan matrix encodes all the properties of the simple Lie algebra it represents. Let \(\mathfrak{g}\) be a simple complex Lie algebra, \(\mathfrak{h}\) a Cartan subalgebra and \(\Pi = \{\alpha_1, \ldots, \alpha_\ell\}\) a basis of simple roots for the root system \(\Delta\) of \(\mathfrak{h}\) in \(\mathfrak{g}\). The elements of the Cartan matrix \(C\) are given by

\[
c_{ij} := 2\frac{\langle \alpha_i, \alpha_j \rangle}{\langle \alpha_j, \alpha_j \rangle}
\]

where the inner product is induced by the Killing form. The \(\ell \times \ell\)-matrix \(C\) is invertible. It is called the Cartan matrix of \(\mathfrak{g}\). The Cartan matrix for a complex simple Lie algebra obeys the following properties:

\begin{enumerate}
  \item \(C\) is symmetrizable. There exists a diagonal matrix \(D\) such that \(DC\) is symmetric.
  \item \(c_{ii} = 2\).
  \item \(c_{ij} \in \{0, -1, -2, -3\}\) for \(i \neq j\).
  \item \(c_{ij} = 0 \iff c_{ji} = 0\).
\end{enumerate}

The complex simple Lie algebras are classified as:

\[A_1, B_1, C_1, D_1, E_6, E_7, E_8, F_4, G_2\].

1991 Mathematics Subject Classification. 33C45, 17B20, 20F55.

Key words and phrases. Cartan matrix, Chebyshev polynomials, simple Lie algebras, Coxeter polynomial, Cyclotomic polynomials.
Traditionally, $A_l, B_l, C_l, D_l$ are called the classical Lie algebras while $E_6, E_7, E_8, F_4, G_2$ are called the exceptional Lie algebras. Moreover, for any Cartan matrix there exists just one simple complex Lie algebra up to isomorphism giving rise to it. The classification of simple complex Lie algebras is due to Killing and Cartan around 1890. According to A. J. Coleman ([4]) the classification paper of Killing is the greatest mathematical writing of all times (after Euclid’s Elements and Newton’s Principia). Simple Lie algebras over $\mathbb{C}$ are classified by using the associated Dynkin diagram. It is a graph whose vertices correspond to the elements of $\Pi$. Each pair of vertices $\alpha_i, \alpha_j$ are connected by

$$m_{ij} = \frac{4(\alpha_i, \alpha_j)^2}{(\alpha_i, \alpha_i)(\alpha_j, \alpha_j)}$$

edges, where

$$m_{ij} \in \{0, 1, 2, 3\}.$$ 

Dynkin Diagrams for simple Lie algebras

$A_n$

\[
\begin{array}{c}
\circ \quad \circ \quad \circ \quad \cdots \quad \circ \quad \circ \\
\end{array}
\]

$B_n$

\[
\begin{array}{c}
\circ \quad \circ \quad \circ \quad \cdots \quad \circ \quad \circ \\
\end{array}
\]

$C_n$

\[
\begin{array}{c}
\circ \quad \circ \quad \circ \quad \cdots \quad \circ \quad \circ \\
\end{array}
\]

$D_n$

\[
\begin{array}{c}
\circ \quad \circ \quad \circ \quad \cdots \quad \circ \quad \circ \\
\end{array}
\]

$E_6$

\[
\begin{array}{c}
\circ \quad \circ \quad \circ \quad \cdots \\
\circ \quad \circ \\
\end{array}
\]

$E_7$

\[
\begin{array}{c}
\circ \quad \circ \quad \circ \quad \cdots \\
\circ \quad \circ \\
\end{array}
\]
If the root system is indecomposable it has a highest root $\theta$. Let $\alpha_0 = -\theta$ and let $\Pi_0 = \{\alpha_0, \alpha_1, \ldots, \alpha_l\}$. As before define
$$\tilde{C} := 2 \frac{(\alpha_i, \alpha_j)}{\langle \alpha_j, \alpha_j \rangle} \quad 0 \leq i, j \leq l.$$\
$\tilde{C}$ is called the extended Cartan matrix. It satisfies all the properties of $C$ except that $\det \tilde{C} = 0$. Extended Cartan matrices classify affine Lie algebras. We use the symbols $A^{(1)}_n, B^{(1)}_n, C^{(1)}_n, D^{(1)}_n, E^{(1)}_6, E^{(1)}_7, E^{(1)}_8, F^{(1)}_4, G^{(1)}_2$, for the corresponding affine Lie algebras. For more details on these classifications see [13], [14]. The following result due to A. Koranyi [15] is useful in computing the characteristic polynomial of the Cartan matrix.

**Theorem 1.** Let $C$ be the $n \times n$ Cartan matrix of a simple Lie algebra over $\mathbb{C}$. Let $p_n(x)$ be its characteristic polynomial. Then
$$p_n(x) = q_n \left( \frac{x}{2} - 1 \right)$$
where $q_n$ is a polynomial related to Chebyshev polynomials as follows:

- $A_n : \quad q_n = U_n$
- $B_n, C_n : \quad q_n = 2T_n$
- $D_n : \quad q_n = 4xT_{n-1}$

where $T_n$ and $U_n$ are the Chebyshev polynomials of first and second kind respectively.

We have verified similar results for affine Lie algebras:

- $A^{(1)}_{n-1} : \quad 2T_n + 2(-1)^{n-1}$
- $B^{(1)}_{n-1} : \quad 2(T_n - T_{n-4})$
- $C^{(1)}_{n-1} : \quad 2(T_n - T_{n-2})$
- $D^{(1)}_{n-1} : \quad 8x^2(T_{n-2} - T_{n-4})$.

In this paper we limit our discussion and proofs mainly for the case of simple Lie algebras. The case of affine Lie algebras will be treated in a forthcoming publication.
characteristic polynomial of $\Gamma$ is that of $A$. Similarly the norm of $\Gamma$ is defined to be the norm of $A$. One defines the spectral radius of $\Gamma$ to be
\[ \rho(\Gamma) = \max\{|\lambda| : \lambda \text{ is an eigenvalue of } A\}. \]

If the graph is a tree then the characteristic polynomial $p_A$ of the adjacency matrix is simply related to the characteristic polynomial of the Cartan matrix $p_C$.

In fact,
\[ a_n(x) = p_n(x + 2). \]

Using the fact that the spectrum of $A$ is the same as the spectrum of $-A$ it follows easily that if $\lambda$ is an eigenvalue of the adjacency matrix then $2 + \lambda$ is an eigenvalue of the corresponding Cartan matrix.

In this paper we use the following notation. Note that the subscript $n$, in all cases, is equal to the degree of the polynomial except in the case of $Q_n(x)$ which is of degree $2n$.

- $p_n(x)$ will denote the characteristic polynomial of the Cartan matrix.
- $a_n(x)$ will denote the characteristic polynomial of the adjacency matrix.
  Note that
  \[ a_n(x) = p_n(x + 2) = q_n\left(\frac{x}{2}\right). \]
- Finally we define the associated polynomial
  \[ Q_n(x) = x^n a_n(x + 1/x). \]

$Q_n(x)$ turns out to be an even, reciprocal polynomial of the form $Q_n(x) = f_n(x^2)$. The polynomial $f_n$ is the so called Coxeter polynomial of the underlying graph. For the definition and spectral properties of the Coxeter polynomials see [1], [2], [3], [6], [16], [23], [24]. The roots of $Q_n$ in the cases we consider are in the unit disk and therefore by a theorem of Kronecker, see [8], $Q_n(x)$ is a product of cyclotomic polynomials. We determine the factorization of $f_n$ as a product of cyclotomic polynomials. This factorization in turn determines the factorization of $Q_n$. The irreducible factors of $Q_n$ are in one-to-one correspondence with the irreducible factors of $a_n(x)$. As a bi-product we obtain the factorization of the Chebyshev polynomials of the first and second kind over the integers. More precisely we prove the following result:

**Theorem 2.** Let $\psi_n(x)$ be the minimal polynomial of the algebraic integer $2 \cos \frac{2\pi}{n}$. Then
\[ U_n(x) = \prod_{j|2n+2 \atop j \neq 1,2} \psi_j(2x). \]

Let $n = 2^\alpha N$ where $N$ is odd and let $r = 2^{\alpha+2}$. Then
\[ T_n(x) = \frac{1}{2} \prod_{j|N} \psi_{rj}(2x). \]

The irreducible polynomials $\psi_n$ were introduced by Lehmer in [18]. The factorization is consistent with previous results, e.g. [12], [21].

Using the factorization of the polynomials $a_n(x)$ and $p_n(x)$ we obtain the following interesting sine formula: Let $\mathfrak{g}$ be a complex simple Lie algebra of rank $\ell$, $h$ the Coxeter number, $m_1, m_2, \ldots, m_\ell$ the exponents of $\mathfrak{g}$ and $C$ the Cartan matrix. Then
The Mahler measure of a polynomial
\[ p(x) = \prod_{k=1}^{n} (x - \alpha_k) \]
is defined to be
\[ M(p) = \prod_{k=1}^{n} \max \{ 1, |\alpha_k| \} = \exp \left( \frac{1}{2\pi} \int_{0}^{2\pi} \ln(|p(e^{i\theta})| d\theta) \right). \]

According to Kronecker’s Theorem \( M(p) = 1 \) iff \( p(x) \) is a product of cyclotomic polynomials and \( x \). D. H. Lehmer posed the following question: Is there a polynomial with Mahler measure between 1 and \( 1 + \epsilon \) for each \( \epsilon > 0 \)? For more details on Lehmer problem see [11], [19].

A real algebraic integer \( \lambda > 1 \) is a Salem number if all its conjugate roots have absolute value no greater than 1, and at least one has absolute value 1. It follows that the minimal polynomial of \( \lambda \) is reciprocal. One usually associates to each Salem number a combinatorial object, i.e. a graph. The smallest known Salem number is the largest real root of Lehmer’s polynomial
\[ l(x) = x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1 \]
which is about 1.1762808. According to Lehmer’s Conjecture this is the smallest Salem number. Lehmer’s polynomial corresponds to a Dynkin graph of type \( E_{10} \).

The associated polynomials corresponding to Dynkin diagrams of simple complex Lie algebras are products of cyclotomic polynomials. In order to produce Salem numbers one has to consider more complicated types of graphs.

2. Chebyshev polynomials

To compute explicitly \( p_n(x) \) we use the following result due to A. Koranyi [15]:

**Proposition 1.** Let \( C \) be the \( n \times n \) Cartan matrix of a simple Lie algebra over \( \mathbb{C} \). Let \( p_n(x) \) be its characteristic polynomial and define \( q_n(x) = \det(2xI + A) \). Then
\[ p_n(x) = q_n \left( \frac{x}{2} - 1 \right), \quad a_n(x) = q_n \left( \frac{x}{2} \right). \]

The polynomial \( q_n \) is related to Chebyshev polynomials as follows:

\[ A_n : \quad q_n = U_n \]
\[ B_n, C_n : \quad q_n = 2T_n \]
\[ D_n : \quad q_n = 4xT_{n-1} \]

where \( T_n \) and \( U_n \) are the Chebyshev polynomials of first and second kind respectively.

**Proof.** We give an outline of the proof. Note that
\[ q_n \left( \frac{x}{2} - 1 \right) = \det \left( 2\frac{x-2}{2}I_n + A \right) \]
\[ = \det \left( xI_n - 2I_n + A \right) \]
\[ = \det \left( xI_n - 2I_n + 2I_n - C \right) \]
\[ = \det \left( xI_n - C \right) = p_n(x). \]
Furthermore, the matrix $A$ for classical Lie algebras has the form

$$A = \begin{pmatrix}
0 & 1 & & & & \\
1 & 0 & 1 & & & \\
& 1 & 0 & 1 & & \\
& & 1 & 0 & 1 & \\
& & & 1 & 0 & \\
& & & & D & \\
\end{pmatrix},$$

where $D$ is

$$(0), \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 \end{pmatrix}$$

for the cases $A_n$, $B_n$, $C_n$, $D_n$ respectively. The proof is by induction on $n$. Suppose the result is proved for $n$ and $n-1$. To get $q_{n+1}(x)$, expand the determinant of $2xI_{n+1} + A$ by the first two rows to obtain

$$q_{n+1}(x) = 2xq_n(x) - q_{n-1}.$$

This is the recurrence relation satisfied by the Chebyshev polynomials $T_n$ and $U_n$. Sections 3 and 4 cover in detail the cases $A_n$ and $B_n$ respectively. In section 5 we outline the case of $D_n$.

Finally, note that

$$a_n(x) = p_n(x + 2) = q_n\left(\frac{x + 2}{2} - 1\right) = q_n\left(\frac{x}{2}\right).$$

It is not clear why the Chebyshev polynomials appear. However, it seems that their properties were designed in order to fit nicely the theory of complex simple Lie algebras. In fact, the roots of these polynomials determine the spectrum of the Cartan and adjacency matrices. This information is crucial in the proof of the sine formula. The precise definition and some basic properties of $U_n$ and $T_n$ will be given in sections 3 and 4 respectively.

In the case of exceptional Lie algebras, one can directly compute (preferably using a symbolic manipulation package in the case of $E_n$) the characteristic polynomials $a_n(x)$ and $p_n(x)$ for each exceptional type. We present the calculations in section 8.

3. Cartan matrix of type $A_n$

3.1. Eigenvalues of the $A_n$ Cartan matrix. Toeplitz matrices have constant entries on each diagonal parallel to the main diagonal. Tridiagonal Toeplitz matrices are commonly the result of discretizing differential equations.

The eigenvalues of the Toeplitz matrix

$$\begin{pmatrix}
b & a & & & \\
c & b & a & & \\
& c & b & a & \\
& & c & b & a \\
& & & c & b \end{pmatrix}$$

are given by

$$\lambda_j = b + 2a \sqrt{\frac{c}{a}} \cos \frac{j\pi}{n+1}, \quad j = 1, 2 \ldots, n,$$
see e.g. [22] p. 59).

The Cartan matrix of type $A_n$ is a tri-diagonal matrix of the form.

\[
C_{A_n} = \begin{pmatrix}
2 & -1 & & & & \\
-1 & 2 & -1 & & & \\
& & & \ddots & & \\
& & & & & -1 \\
& & & & & 2 \\
& & & & & 1
\end{pmatrix}.
\]

It appears in the classification theory of simple Lie algebras over $\mathbb{C}$.

Taking $a = c = -1$, $b = 2$ in (4) we deduce that the eigenvalues of $A_n$ are given by

\[
\lambda_j = 2 - 2 \cos \frac{j \pi}{n+1} = 4 \sin^2 \frac{j \pi}{2(n+1)} \quad j = 1, 2, \ldots, n.
\]

Let $d_n$ be the determinant of $C_{A_n}$. One can compute it using expansion on the first row and induction. We obtain $d_n = 2d_{n-1} - d_{n-2}$, $d_1 = 2$, $d_2 = 3$. This is a simple linear recurrence with solution $d_n = n + 1$.

We conclude that

\[
\prod_{j=1}^{n} 4 \sin^2 \frac{j \pi}{2(n+1)} = n + 1.
\]

Equivalently

\[
2^{2n} \prod_{j=1}^{n} \sin^2 \frac{j \pi}{2(n+1)} = n + 1. \quad (A_n \ sine \ formula)
\]

We refer to this relation as the $A_n$ sine formula.

In Table 1 we list the determinants for the Cartan matrices of complex simple Lie algebras.

3.2. The characteristic polynomial. We list the formula for the characteristic polynomial of the matrix $A_n$ for small values of $n$.

\[
p_1(x) = x - 2
\]
\[
p_2(x) = x^2 - 4x + 3 = (x - 1)(x - 3)
\]
\[
p_3(x) = x^3 - 6x^2 + 10x - 4 = (x - 2)(x^2 - 4x + 2)
\]
\[
p_4(x) = x^4 - 8x^3 + 21x^2 - 20x + 5 = (x^2 - 5x + 5)(x^2 - 3x + 1)
\]
\[
p_5(x) = x^5 - 10x^4 + 36x^3 - 56x^2 + 35x - 6 = (x - 1)(x - 2)(x - 3)(x^2 - 4x + 1)
\]
\[
p_6(x) = x^6 - 12x^5 + 55x^4 - 120x^3 + 126x^2 - 56x + 7
\]
\[
p_7(x) = x^7 - 14x^6 + 78x^5 - 220x^4 + 330x^3 - 252x^2 + 84x - 8.
\]

**Proposition 2.** Let $p_n(x)$ be the characteristic polynomial of the Cartan matrix $\mathfrak{a}_n$. Then

\[
p_n(x) = \sum_{j=0}^{n} (-1)^{n+j} \binom{n+j+1}{2j+1} x^j.
\]
We will present the proof of this Proposition in the next subsection 3.3 using properties of Chebyshev polynomials.

3.3. Chebyshev polynomials of the second kind. The Chebyshev polynomials form an infinite sequence of orthogonal polynomials. The Chebyshev polynomial of the second kind of degree \( n \) is usually denoted by \( U_n \). We list some properties of Chebyshev polynomials following [20], [25].

A fancy way to define the \( n \)th Chebyshev polynomial of the second kind is

\[
U_n(x) = \det \begin{pmatrix} 2x & 1 \\ 1 & 2x & 1 \\ & & \ddots & \ddots \\ & & \ddots & \ddots \end{pmatrix},
\]

where \( n \) is the size of the matrix. By expanding the determinant with respect to the first row we get the recurrence

\[
U_0(x) = 1, \quad U_1(x) = 2x, \quad U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x).
\]

It is easy then to compute recursively the first few polynomials:

\[
\begin{align*}
U_0(x) &= 1 \\
U_1(x) &= 2x \\
U_2(x) &= 4x^2 - 1 \\
U_3(x) &= 8x^3 - 4x \\
U_4(x) &= 16x^4 - 12x^2 + 1 \\
U_5(x) &= 32x^5 - 32x^3 + 6x \\
U_6(x) &= 64x^6 - 80x^4 + 24x^2 - 1.
\end{align*}
\]

Letting \( x = \cos \theta \) we obtain

\[
U_n(x) = \sin((n+1)\theta) / \sin \theta.
\]

\( U_n(x) \) is a solution of the differential equation

\[
(1 - x^2)y''' - 3xy' + n(n+2)y = 0.
\]

There is also an explicit formula which is used in the proof of Proposition 2.

\[
U_n(x) = \sum_{j=0}^{n} (-2)^j \binom{n+j+1}{2j+1}(1-x)^j.
\]

Another formula in powers of \( x \) is

\[
U_n(x) = \sum_{j=0}^{\left\lfloor n/2 \right\rfloor} (-1)^j \binom{n-j}{j} (2x)^{n-2j}.
\]

The polynomials \( U_n \) satisfy the following properties:

\[
\begin{align*}
U_n(-x) &= (-1)^n U_n(x) \\
U_n(1) &= n + 1 \\
U_{2n}(0) &= (-1)^n \\
U_{2n-1}(0) &= 0.
\end{align*}
\]
Knowledge of the roots of $U_n$ implies

\[ U_n(x) = 2^n \prod_{j=1}^{n} \left[ x - \cos \left( \frac{j\pi}{n+1} \right) \right]. \tag{11} \]

Setting $x = 1$ in this equation we obtain again the $A_n$ sine formula $\square$.

**Lemma 1.**

\[ p_n(x) = U_n(x/2 - 1), \]

where $U_n$ is the Chebyshev polynomial of the second kind.

**Proof.** We write the eigenvalue equation in the form \( \det(xI_n - C_{A_n}) = 0 \), where $I_n$ is the $n \times n$ identity matrix. Explicitly,

\[
\det (xI_n - C_{A_n}) = \det \begin{pmatrix}
  x-2 & 1 & 1 & \cdots & 1 \\
  1 & x-2 & 1 & \cdots & 1 \\
  \vdots & \vdots & \cdots & \vdots & \vdots \\
  1 & \cdots & 1 & x-2 & 1 \\
  1 & \cdots & \cdots & 1 & x-2
\end{pmatrix} = U_n \left( \frac{x}{2} - 1 \right).
\]

\(\square\)

**Remark 1.** Note that

\[ p_n(0) = U_n(-1) = (-1)^n U_n(1) = (-1)^n (n+1), \]

which agrees (up to a sign) with the formula for the determinant of $A_n$. Also

\[ p_n(2) = U_n(0) = 0 \]

if $n$ is odd. Therefore for $n$ odd, $p_n(x)$ is divisible by $x - 2$.

We can now prove Proposition $\square$

**Proof.** We use the notation

\[ c_{nj} = \binom{n+j+1}{2j+1}. \]

\[ (-1)^n p_n(x) = (-1)^n U_n \left( \frac{x}{2} - 1 \right) = U_n \left( 1 - \frac{x}{2} \right) = \]

\[ = \sum_{j=0}^{n} (-2)^j c_{nj} \left( 1 - \frac{x}{2} \right)^j = \sum_{j=0}^{n} (-1)^j c_{nj} x^j. \]

Therefore

\[ p_n(x) = \sum_{j=0}^{n} (-1)^{n+j} c_{nj} x^j. \]

\(\square\)
4. Cartan matrix of type $B_n$ and $C_n$

4.1. Chebyshev polynomials of the first kind. The Chebyshev polynomials of the first kind are denoted by $T_n(x)$.

They are defined by the recurrence

$$T_0(x) = 1, \quad T_1(x) = x, \quad T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x).$$

Using this recursion one may compute the first few polynomials:

- $T_0(x) = 1$
- $T_1(x) = x$
- $T_2(x) = 2x^2 - 1$
- $T_3(x) = 4x^3 - 3x$
- $T_4(x) = 8x^4 - 8x^2 + 1$
- $T_5(x) = 16x^5 - 20x^3 + 5x$
- $T_6(x) = 32x^6 - 48x^4 + 18x^2 - 1$.

It is well-known that $\cos(n\theta)$ can be expressed as a polynomial in $\cos(\theta)$. For example

- $\cos(0\theta) = 1$
- $\cos(1\theta) = \cos \theta$
- $\cos(2\theta) = 2(\cos \theta)^2 - 1$
- $\cos(3\theta) = 4(\cos \theta)^3 - 3(\cos \theta)$.

More generally we have:

$$\cos(n\theta) = T_n(\cos \theta).$$

$T_n(x)$ is a solution of the differential equation

$$(1 - x^2)y'' - xy' + n^2y = 0.$$

There is also an explicit formula which is used in the proof of Propositions 3, 4.

$$T_n(x) = n \sum_{j=0}^{n} (-2)^j \frac{(n + j - 1)!}{(n - j)!(2j)!} (1 - x)^j \quad (n > 0).$$

Another formula in powers of $x$ is

$$T_n(x) = \frac{n}{2} \sum_{j=0}^{n} (-1)^j \frac{(n - j - 1)!}{j!(n - 2j)!} (2x)^{n-2j}.$$

The polynomials $T_n$ satisfy the following properties:

- $T_n(-x) = (-1)^n T_n(x)$
- $T_n(1) = 1$
- $T_{2n}(0) = (-1)^n$
- $T_{2n-1}(0) = 0$.

Also

$$T_n(x) = 2^{n-1} \prod_{j=1}^{n} \left[ x - \cos \left( \frac{(2j - 1)\pi}{2n} \right) \right].$$

Setting $x = 1$ in this equation quickly leads to the formula

$$2^{2n} \prod_{j=1}^{n} \sin^2 \left( \frac{(2j - 1)\pi}{4n} \right) = 2.$$

We refer to this relation as the $B_n$ sine formula.
We refer to this relation as the $B_n$ sine formula.

4.2. The characteristic polynomial. The Cartan matrix of type $B_n$ is a tridiagonal matrix of the form

$$
C_{B_n} = \begin{pmatrix}
2 & -1 & & \\
-1 & 2 & -1 & \\
 & \ddots & \ddots & \ddots \\
& -1 & 2 & -1 \\
& 0 & -1 & 2 \\
\end{pmatrix}.
$$

(17)

Since the Cartan matrix of type $C_n$ is the transpose of this matrix we consider only the Cartan matrix of type $B_n$. Using expansion on the first row it easy to prove that $\det(C_{B_n}) = 2$.

We list the formula for the characteristic polynomial of the matrix $B_n$ for small values of $n$.

$$
\begin{align*}
p_2(x) &= x^2 - 4x + 2 \\
p_3(x) &= x^3 - 6x^2 + 9x - 2 = (x - 2)(x^2 - 4x + 1) \\
p_4(x) &= x^4 - 8x^3 + 20x^2 - 16x + 2 \\
p_5(x) &= x^5 - 10x^4 + 35x^3 - 50x^2 + 25x - 2 = (x - 2)(x^4 - 8x^3 + 19x^2 - 12x + 1) \\
p_6(x) &= (x^2 - 4x + 1)(x^4 - 8x^3 + 20x^2 - 16x + 1) \\
p_7(x) &= x^7 - 14x^6 + 77x^5 - 210x^4 + 294x^3 - 196x^2 + 49x - 2.
\end{align*}
$$

As in the $A_n$ case we define a sequence of polynomials in the following way:

$$
q_n(x) = \det\left(\begin{array}{ccc}
2x & 1 & \\
1 & 2x & 1 & \\
\ddots & \ddots & \ddots \\
1 & 2x & 1 \\
1 & 2x & \\
\end{array}\right).
$$

By expanding the determinant with respect to the first row we get the recurrence

$$
q_1(x) = 2x, \quad q_2(x) = 4x^2 - 2, \quad q_{n+1}(x) = 2xq_n(x) - q_{n-1}(x).
$$

We may define $q_0(x) = 2$.

It is easy then to compute recursively the first few polynomials:

$$
\begin{align*}
q_0(x) &= 2 \\
q_1(x) &= 2x \\
q_2(x) &= 4x^2 - 2 = 2(2x^2 - 1) \\
q_3(x) &= 8x^3 - 6x = 2(4x^3 - 3) \\
q_4(x) &= 16x^4 - 16x^2 + 2 = 2(8x^4 - 8x + 1) \\
q_5(x) &= 32x^5 - 40x^3 + 10x = 2(16x^5 - 20x^3 + 5x) \\
q_6(x) &= 64x^6 - 96x^4 + 36x^2 - 2 = 2(32x^6 - 48x^4 + 18x^2 - 1) \\
q_7(x) &= 2(64x^7 - 112x^5 + 56x^3 - 7x).
\end{align*}
$$

It is clear that $q_n(x) = 2T_n(x)$ where $T_n$ is the $n$th Chebyshev polynomial of the first kind. Therefore

$$
p_n(x) = q_n(x/2 - 1),
$$

where $q_n = 2T_n(x)$.  

5.1. The characteristic polynomial. The Cartan matrix of type $D_n$ is a matrix of the form

\[
C_{D_n} = \begin{pmatrix}
2 & -1 & & & \\
-1 & 2 & -1 & & \\
& \ddots & \ddots & \ddots & \\
& & -1 & 2 & -1 \\
& & & -1 & 2 & 0 \\
& & & & -1 & 0 & 2
\end{pmatrix}.
\]

Note that the matrix is no longer tri-diagonal. Using expansion on the first row and induction it easy to prove that $\det (C_{D_n}) = 4$.

We list some formulas for the characteristic polynomial of the matrix $C_{D_n}$ for small values of $n$.

\[
\begin{align*}
p_2(x) &= x^2 - 4x + 2 = (x - 2)^2 \\
p_3(x) &= x^3 - 6x^2 + 10x - 4 = (x - 2)(x^2 - 4x + 2) \\
p_4(x) &= x^4 - 8x^3 + 21x^2 - 20x + 4 = (x - 2)^2(x^2 - 4x + 1) \\
p_5(x) &= x^5 - 10x^4 + 36x^3 - 56x^2 + 34x - 4 = (x - 2)(x^4 - 8x^3 + 20x^2 - 16x + 2) \\
p_6(x) &= x^6 - 12x^5 + 55x^4 - 120x^3 + 125x^2 - 52x + 4 = (x - 2)^2(x^4 - 8x^3 + 19x^2 - 12x + 1)
\end{align*}
\]

5.1. The characteristic polynomial. The Cartan matrix of type $D_n$ is a matrix of the form

\[
C_{D_n} = \begin{pmatrix}
2 & -1 & & & \\
-1 & 2 & -1 & & \\
& \ddots & \ddots & \ddots & \\
& & -1 & 2 & -1 \\
& & & -1 & 2 & 0 \\
& & & & -1 & 0 & 2
\end{pmatrix}.
\]

Note that the matrix is no longer tri-diagonal. Using expansion on the first row and induction it easy to prove that $\det (C_{D_n}) = 4$.

We list some formulas for the characteristic polynomial of the matrix $C_{D_n}$ for small values of $n$.

\[
\begin{align*}
p_2(x) &= x^2 - 4x + 2 = (x - 2)^2 \\
p_3(x) &= x^3 - 6x^2 + 10x - 4 = (x - 2)(x^2 - 4x + 2) \\
p_4(x) &= x^4 - 8x^3 + 21x^2 - 20x + 4 = (x - 2)^2(x^2 - 4x + 1) \\
p_5(x) &= x^5 - 10x^4 + 36x^3 - 56x^2 + 34x - 4 = (x - 2)(x^4 - 8x^3 + 20x^2 - 16x + 2) \\
p_6(x) &= x^6 - 12x^5 + 55x^4 - 120x^3 + 125x^2 - 52x + 4 = (x - 2)^2(x^4 - 8x^3 + 19x^2 - 12x + 1)
\end{align*}
\]

5. Cartan matrix of type $D_n$

5.1. The characteristic polynomial. The Cartan matrix of type $D_n$ is a matrix of the form

\[
C_{D_n} = \begin{pmatrix}
2 & -1 & & & \\
-1 & 2 & -1 & & \\
& \ddots & \ddots & \ddots & \\
& & -1 & 2 & -1 \\
& & & -1 & 2 & 0 \\
& & & & -1 & 0 & 2
\end{pmatrix}.
\]

Note that the matrix is no longer tri-diagonal. Using expansion on the first row and induction it easy to prove that $\det (C_{D_n}) = 4$.

We list some formulas for the characteristic polynomial of the matrix $C_{D_n}$ for small values of $n$.

\[
\begin{align*}
p_2(x) &= x^2 - 4x + 2 = (x - 2)^2 \\
p_3(x) &= x^3 - 6x^2 + 10x - 4 = (x - 2)(x^2 - 4x + 2) \\
p_4(x) &= x^4 - 8x^3 + 21x^2 - 20x + 4 = (x - 2)^2(x^2 - 4x + 1) \\
p_5(x) &= x^5 - 10x^4 + 36x^3 - 56x^2 + 34x - 4 = (x - 2)(x^4 - 8x^3 + 20x^2 - 16x + 2) \\
p_6(x) &= x^6 - 12x^5 + 55x^4 - 120x^3 + 125x^2 - 52x + 4 = (x - 2)^2(x^4 - 8x^3 + 19x^2 - 12x + 1)
\end{align*}
\]

Proposition 4. Let $p_n(x)$ be the characteristic polynomial of the Cartan matrix $C_{D_n}$. Then

\[
p_n(x) = (x - 2) \sum_{j=0}^{n-1} (-1)^{n+j} \frac{2n-2(n+j-2)!}{(n-j)!(2j)!} x^j.
\]

Define a sequence of polynomials in the following way:

\[
q_n(x) = \det \begin{pmatrix}
2x & 1 & & & \\
1 & 2x & \ddots & & \\
& & \ddots & 2x & 1 \\
& & & 1 & 2x & 0 \\
& & & & 1 & 0 & 2x
\end{pmatrix}.
\]

By expanding the determinant with respect to the first row we get the recurrence

\[
q_2(x) = 4x^2, \quad q_3(x) = 8x^3 - 4x, \quad q_{n+1}(x) = 2xq_n(x) - q_{n-1}.
\]

We may define $q_1(x) = 4x$. It produces the following sequence of polynomials:
\[ q_1(x) = 4x \]
\[ q_2(x) = 4x^2 \]
\[ q_3(x) = 8x^3 - 4x = 4x(2x^2 - 1) \]
\[ q_4(x) = 16x^4 - 12x^2 = 4x^2(4x^2 - 3) \]
\[ q_5(x) = 32x^5 - 32x^3 + 4x = 4x(8x^4 - 8x^2 + 1) \]
\[ q_6(x) = 64x^6 - 80x^4 + 20x^2 = 4x^2(16x^4 - 20x^2 + 5) \]
\[ q_7(x) = 128x^7 - 192x^5 + 72x^3 - 4x = 4x(2x^2 - 1)(16x^4 - 16x^2 + 1) . \]

It is clear that \( q_n(x) = 4xT_n(x) \) where \( T_n \) is the \( n \)th Chebyshev polynomial of the first kind. Equivalently, \( q_n = 2(T_n + T_{n-2}) \). As in the previous cases, we conclude that

\[ p_n(x) = q_n(x/2 - 1) . \]

where \( q_n(x) = 4xT_{n-1}(x) \).

Note that

\[ p(0) = q_n(-1) = (-1)^n q_n(1) = (-1)^n 4T_{n-1}(1) = 4(-1)^n , \]

which agrees with the formula for the determinant of \( D_n \).

We now give the proof of Proposition 4.

**Proof.** Let \( p_n(x) \) be the characteristic polynomial of the Cartan matrix of type \( D_n \).

We use the notation

\[ c_{nj} = \frac{(n + j - 2)!}{(n - j - 1)!(2j)!} , \]

and

\[ d_{nj} = \frac{(2n - 2)(n + j - 2)!}{(n - j - 1)!(2j)!} \]

\[ (-1)^n p_n(x) = (-1)^n q_n \left( \frac{x}{2} - 1 \right) = q_n \left( 1 - \frac{x}{2} \right) = \]

\[ = 2(x - 2)T_{n-1} \left( 1 - \frac{x}{2} \right) = \]

\[ 2(x - 2)(n - 1) \sum_{j=0}^{n-1} (-2)^j c_{nj} \left( 1 - \frac{x}{2} \right)^j \]

\[ = 2(x - 2)(n - 1) \sum_{j=0}^{n-1} (-2)^j c_{nj} \left( \frac{x}{2} \right)^j \]

\[ = (x - 2) \sum_{j=0}^{n-1} (-1)^j d_{nj} x^j . \]

Therefore

\[ p_n(x) = (x - 2) \sum_{j=0}^{n-1} (-1)^{n+j} d_{nj} x^j . \]

\[ \square \]
6. The Coxeter Polynomial

A Coxeter graph is a simple graph $\Gamma$ with $n$ vertices and edge weights $m_{ij} \in \{3, 4, \ldots, \infty\}$. We define $m_{ii} = 1$ and $m_{ij} = 2$ if node $i$ is not connected with node $j$. By convention if $m_{ij} = 3$ then the edge is often not labeled. If $\Gamma$ is a Coxeter graph with $n$ vertices we define a bilinear form $B$ on $\mathbb{R}^n$ by choosing a basis $e_1, e_2, \ldots, e_n$ and setting

$$B(e_i, e_j) = -2 \cos \frac{\pi}{m_{ij}}.$$ 

If $m_{ij} = \infty$ we define $B(e_i, e_j) = -2$. We also define for $i = 1, 2, \ldots, n$ the reflection $\sigma_i(e_j) = e_j - B(e_i, e_j)e_i$.

Let $S = \{\sigma_i \mid i = 1, \ldots, n\}$. The Coxeter group $W(\Gamma)$ is the group generated by the reflections in $S$. $W$ has the presentation

$$W = \langle \sigma_1, \sigma_2, \ldots, \sigma_n \mid \sigma_i^2 = 1, (\sigma_i\sigma_j)^{m_{ij}} = 1 \rangle.$$ 

It is well-known that $W(\Gamma)$ is finite if and only if $B$ is positive definite. A Coxeter element (or transformation) is a product of the form

$$\sigma_{\alpha(1)}\sigma_{\alpha(2)}\ldots\sigma_{\alpha(n)} \quad \alpha \in S_n.$$ 

If the Coxeter graph is a tree then the Coxeter elements are in a single conjugacy class in $W$. A Coxeter polynomial for the Coxeter system $(W, S)$ is the characteristic polynomial of the matrix representation of a Coxeter element. For Coxeter systems whose graphs are trees the Coxeter polynomial is uniquely determined. This covers all the cases we investigate. We define $\rho(W)$ to be the spectral radius of the associated Coxeter adjacency matrix. A Coxeter system is called

1. Spherical if $\rho(W) < 2$
2. Affine if $\rho(W) = 2$
3. Hyperbolic or higher-rank if $\rho(W) > 2$.

In this paper we consider only spherical Coxeter systems. In this case $B$ is positive definite and the Coxeter element has finite order.

**Example 1.** Consider a Coxeter system with graph $A_3$.

The bilinear form is defined by the Cartan matrix

$$\begin{pmatrix}
2 & -1 & 0 \\
-1 & 2 & -1 \\
0 & -1 & 2
\end{pmatrix}.$$

The reflection $\sigma_1$ is determined by the action

$$\sigma_1(e_1) = -e_1, \quad \sigma_1(e_2) = e_1 + e_2, \quad \sigma_1(e_3) = e_3.$$

It has the matrix representation

$$\sigma_1 = \begin{pmatrix}
-1 & 1 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}.$$ 

Similarly

$$\sigma_2 = \begin{pmatrix}
1 & 0 & 0 \\
1 & -1 & 1 \\
0 & 0 & 1
\end{pmatrix}, \quad \sigma_3 = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 1 & -1
\end{pmatrix}.$$
### Table 2. Exponents and Coxeter number for root systems.

| Root system | Exponents | Coxeter number |
|-------------|-----------|---------------|
| $A_n$       | 1, 2, 3, ..., $n$ | $n + 1$ |
| $B_n$       | 1, 3, 5, ..., $2n - 1$ | $2n$ |
| $C_n$       | 1, 3, 5, ..., $2n - 1$ | $2n$ |
| $D_n$       | 1, 3, 5, ..., $2n - 3$, $n - 1$ | $2n - 2$ |
| $E_6$       | 1, 4, 5, 7, 8, 11 | 12 |
| $E_7$       | 1, 5, 7, 9, 11, 13, 17 | 18 |
| $E_8$       | 1, 7, 11, 13, 17, 19, 23, 29 | 30 |
| $F_4$       | 1, 5, 7, 11 | 12 |
| $G_2$       | 1, 5 | 6 |

A Coxeter element is defined by

$$R = \sigma_1 \sigma_2 \sigma_3 = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}.$$  

The Coxeter polynomial is the characteristic polynomial of $R$ which is

$$x^3 + x^2 + x + 1.$$  

Note that $R$ has order 4, $R^4 = I$. The order of the Coxeter element is called the Coxeter number. Note that the roots of the Coxeter polynomial are $-1, -i, i$. We can write $i = i^1, -1 = i^2, -i = i^3$. The integers 2, 3, 4 are the degrees of the Chevalley invariants and their product, 24, is the order of the Coxeter group.

In general the order of the Coxeter group is

$$(m_1 + 1)(m_2 + 1) \ldots (m_\ell + 1)$$

where $m_i$ are the exponents of the eigenvalues of the Coxeter polynomial. The factors $m_i + 1$ are the degrees of the Chevalley invariants. If $\zeta$ is a primitive $h$ root of unity (where $h$ is the Coxeter number) then the roots of the Coxeter polynomial are $\zeta^m$ where $m$ runs over the exponents of the corresponding root system [4], [7]. This observation allows the calculation of the Coxeter polynomial for each root system.

In table 2 we list the Coxeter number and the exponents for each root system.

#### 6.1. Exponents. Let us recall the definition of exponents for a simple complex Lie group $G$, see [3], [5], [17]. Suppose $G$ is a connected, complex, simple Lie Group $G$. We form the de Rham cohomology groups $H^i(G, \mathbb{C})$ and the corresponding Poincaré polynomial of $G$:

$$p_G(t) = \sum_{i=1}^\ell b_i t^i,$$

where $b_i = \dim H^i(G, \mathbb{C})$ are the Betti numbers of $G$. The De Rham groups encode topological information about $G$. Following work of Cartan, Ponrjagin and Brauer, Hopf proved that the cohomology algebra is isomorphic to that of a finite product of $\ell$ spheres of odd dimension where $\ell$ is the rank of $G$. This result implies that

$$p_G(t) = \prod_{i=1}^\ell (1 + t^{2m_i+1}).$$

The positive integers $\{m_1, m_2, ..., m_\ell\}$ are called the exponents of $G$. They are also the exponents of the Lie algebra $\mathfrak{g}$ of $G$. One can also extract the exponents from the root space decomposition of $\mathfrak{g}$ following methods which were developed.
by R. Bott, A. Shapiro, A.J. Coleman and B. Kostant. The exponents of a simple complex Lie algebra are given in table 2.

Note the duality in the set of exponents:

\[ m_i + m_{\ell+1-i} = h \]

where \( h \) is the Coxeter number.

### 6.2. Cyclotomic polynomials.

A complex number \( \omega \) is called a **primitive \( n \)th root of unity** provided \( \omega \) is an \( n \)th root of unity and has order \( n \). Such an \( \omega \) generates the group of units, i.e., \( \omega, \omega^2, \ldots, \omega^n = 1 \) coincides with the set of all roots of unity. An \( n \)th root of 1 of the form \( \omega^k \) is a primitive root of unity iff \( k \) is relatively prime to \( n \). Therefore the number of primitive roots of unity is equal to \( \phi(n) \) where \( \phi \) is Euler’s totient function.

We define the \( n \)th cyclotomic polynomial by

\[ \Phi_n(x) = (x - \omega_1)(x - \omega_2) \cdots (x - \omega_s) \]

where \( \omega_1, \omega_2, \ldots, \omega_s \) are all the distinct primitive \( n \)th roots of unity. The degree of \( \Phi_n \) is of course equal to \( s = \phi(n) \). \( \Phi_n(x) \) is a monic, irreducible polynomial with integer coefficients. The first twenty cyclotomic polynomials are given below:

\[
\begin{align*}
\Phi_1(x) &= x - 1 \\
\Phi_2(x) &= x + 1 \\
\Phi_3(x) &= x^2 + x + 1 \\
\Phi_4(x) &= x^2 + 1 = \Phi_2(x^2) \\
\Phi_5(x) &= x^4 + x^3 + x^2 + x + 1 \\
\Phi_6(x) &= x^2 - x + 1 = \Phi_3(-x) \\
\Phi_7(x) &= x^6 + x^5 + x^4 + x^3 + x^2 + x + 1 \\
\Phi_8(x) &= x^4 + 1 = \Phi_2(x^4) = \Phi_4(x^2) \\
\Phi_9(x) &= x^6 + x^3 + 1 = \Phi_3(x^3) \\
\Phi_{10}(x) &= x^4 - x^3 + x^2 - x + 1 = \Phi_5(-x) \\
\Phi_{11}(x) &= x^{10} + x^9 + \cdots + x + 1 \\
\Phi_{12}(x) &= x^4 - x^2 + 1 = \Phi_6(x^2) \\
\Phi_{13}(x) &= x^{12} + x^{11} + \cdots + x + 1 \\
\Phi_{14}(x) &= x^6 - x^5 + x^4 - x^3 + x^2 + 1 = \Phi_7(-x) \\
\Phi_{15}(x) &= x^8 - x^7 + x^5 - x^4 + x^3 - x + 1 \\
\Phi_{16}(x) &= x^8 + 1 = \Phi_2(x^8) = \Phi_8(x^2) \\
\Phi_{17}(x) &= x^{16} + x^{15} + \cdots + x + 1 \\
\Phi_{18}(x) &= x^6 - x^3 + 1 = \Phi_9(-x) \\
\Phi_{19}(x) &= x^{18} + x^{17} + \cdots + x + 1 \\
\Phi_{20}(x) &= x^8 - x^6 + x^4 - x^2 + 1 = \Phi_{10}(x^2) .
\end{align*}
\]

A basic formula for cyclotomic polynomials is

\[ x^n - 1 = \prod_{d \mid n} \Phi_d(x) , \]

where \( d \) ranges over all positive divisors of \( n \). This gives a recursive method of calculating cyclotomic polynomials. For example, if \( n = p \), where \( p \) is prime, then \( x^p - 1 = \Phi_1(x) \Phi_p(x) \) which implies that

\[ \Phi_p(x) = x^{p-1} + x^{p-2} + \cdots + x + 1 . \]

In table 3 we list the Coxeter polynomial and its factorization as a product of cyclotomic polynomials. The cases \( B_n \) and \( D_n \) are explained in remark 4.
Table 3. Coxeter polynomials for Spherical Graphs

| Dynkin Diagram | Coxeter Polynomial | Cyclotomic Factors |
|----------------|-------------------|--------------------|
| $A_n$          | $x^n + x^{n-1} + \ldots + 1$ | $\prod_{d \mid (n+1)} \Phi_d$ |
| $B_n, C_n$     | $x^n + 1$         | $\prod_{d \mid N} \Phi_{2md}(x)$ (*) |
| $D_n$          | $x^n + x^{n-1} + x + 1$ | $\Phi_2 \Phi_12$ |
| $E_6$          | $x^6 + x^5 - x^3 + x + 1$ | $\Phi_2 \Phi_18$ |
| $E_7$          | $x^7 + x^6 - x^4 - x^3 + x + 1$ | $\Phi_12$ |
| $E_8$          | $x^8 + x^7 - x^5 - x^4 - x^3 + x + 1$ | $\Phi_6$ |
| $F_4$          | $x^4 - x^3 + 1$ | $\Phi_12$ |
| $G_2$          | $x^2 - x + 1$ | $\Phi_30$ |

Remark 3. (*) $x^n + 1$ is irreducible iff $n$ is a power of 2, see [9]. If $n$ is odd it has a factor of $x + 1 = \Phi_2(x)$. To obtain the factorization of $x^n + 1$ we use the following procedure. Let $n = 2^a p_1^{k_1} p_2^{k_2} \ldots p_s^{k_s}$ where the $p_i$ are odd primes. Define $m = 2^a$ and $N = p_1^{k_1} \ldots p_s^{k_s}$. The factorization of $x^n + 1$ is given by

$$x^n + 1 = \prod_{d \mid N} \Phi_{2md}(x).$$

(**) In the case of $D_n$ we use the same procedure with $n$ replaced by $n - 1$.

Remark 4. In table 3 note the following: In the factorization of $f_n(x)$, the factor of highest degree is $\Phi_h$ where $h$ is the Coxeter number.

The roots of $U_n(x)$ are given by

$$x_k = \cos \left( \frac{k\pi}{n+1} \right) \quad k = 1, 2, \ldots, n.$$  

Therefore the roots of

$$a_n(x) = U_n \left( \frac{x}{2} \right)$$

are in the interval $[-2, 2]$. If a monic polynomial with integer coefficients has all of its roots in the interval $[-2, 2]$ then they are of a special form. We follow the proof from [10].

Proposition 5. Let $\lambda$ be a non-zero real root of a monic polynomial $p(x) \in \mathbb{Z}[x]$. If all the roots of $p(x)$ are real and in the interval $[-2, 2]$ then $\lambda = 2 \cos(2\pi q)$ where $q$ is a rational number.

Proof. Let $n = \deg(p)$ and define the associated polynomial

$$Q(x) = x^n p(x + \frac{1}{x}).$$

Denote by

$$\lambda = 2 \cos \theta_1, 2 \cos \theta_2, \ldots 2 \cos \theta_n$$

the roots of $p(x)$. Then

$$p(x) = \prod_j (x - 2 \cos \theta_j)$$

$$Q(x) = \prod_j (x^2 - 2x \cos \theta_j + 1) = \prod_j (x - e^{i\theta_j})(x - e^{-i\theta_j})$$

It follows from Kronecker’s Theorem, see [8], that $e^{i\theta_j}$ is a root of unity and therefore $\frac{\theta_j}{2\pi}$ is rational. □
Example 2. Let us consider $U_n(\frac{x}{2})$ for $n = 5$. The polynomial factors as

\[ a_5(x) = U_5(\frac{x}{2}) = x^5 - 4x^3 + 3x = x(x - 1)(x + 1)(x^2 - 3) \]

and the roots are $-\sqrt{3}, -1, 0, 1, \sqrt{3}$. As in the proof of the Proposition \ref{prop}, we can write them as

\[ 2 \cos \frac{\pi}{6}, 2 \cos \frac{\pi}{3}, 2 \cos \frac{\pi}{2}, 2 \cos \frac{2\pi}{3}, 2 \cos \frac{5\pi}{6}. \]

The associated polynomial similarly factors as

\[ Q_5(x) = x^{10} + x^8 + x^4 + x^2 + 1 = (x^2 + x + 1)(x^2 - x + 1)(x^4 - x^2 + 1). \]

The largest eigenvalue is $2 \cos \frac{\pi}{6}$. In general, and for each simple, complex, Lie algebra, the maximum eigenvalue is

\[ 2 \cos \frac{\pi}{h} \]

where $h$ is the Coxeter Number and the roots of $a_n(x)$ are $2 \cos \frac{m \pi}{n}$ where $m_i$ is an exponent of $g$.

Given a graph $\Gamma$ we define the associated polynomial $Q_n(x)$ by the formula

\[ Q_n(x) = x^n a_n(x + \frac{1}{x}), \]

where $a_n(x)$ is the characteristic polynomial of the graph. It is clear that $Q_n(x)$ is a reciprocal polynomial. In all the cases we consider $Q_n(x)$ is an even polynomial.

6.3. Associated polynomials for $A_n$. Using the same procedure as in the example, we present the factorization of $Q_n(x)$ for small values of $n$ as a product of cyclotomic polynomials. Note that $Q_n(x)$ is an even polynomial. The reason: If $n$ is even $U_n(x)$ is an even polynomial and $a_n(x) = U_n(\frac{x}{2})$ is also even. Therefore $Q_n(x)$ is even. If $n$ is odd then $U_n(x)$ and $a_n(x)$ are both odd functions. This implies that $Q_n(x)$ is even.

- $A_2 \Phi_3 \Phi_6$
- $A_3 \Phi_4 \Phi_8$
- $A_4 \Phi_5 \Phi_{10}$
- $A_5 \Phi_3 \Phi_4 \Phi_6 \Phi_{12}$
- $A_6 \Phi_7 \Phi_{14}$
- $A_7 \Phi_4 \Phi_6 \Phi_{16}$
- $A_8 \Phi_3 \Phi_6 \Phi_9 \Phi_{18}$
- $A_9 \Phi_4 \Phi_5 \Phi_{10} \Phi_{20}$
- $A_{10} \Phi_{11} \Phi_{22}$
- $A_{11} \Phi_3 \Phi_4 \Phi_6 \Phi_8 \Phi_{12} \Phi_{24}$

The characteristic polynomial of the Coxeter transformation has roots $\zeta^k$ where $\zeta$ is a primitive $h$ root of unity and $k$ runs over the exponents of a root system of type $A_n$. Therefore

\[ f_n(x) = (x - \zeta)(x - \zeta^2) \ldots (x - \zeta^n). \]

\[ \Rightarrow (x - 1)f_n(x) = x^{n+1} - 1 \]

\[ \Rightarrow f_n(x) = \frac{x^{n+1} - 1}{x - 1}. \]

Using formula \ref{formula} we obtain

\[ f_n(x) = \prod_{d|n+1, d \neq 1} \Phi_d. \]
It is not difficult to guess the factorization of $Q_n(x)$.

**Proposition 6.**

\[ Q_n(x) = \prod_{\substack{j \mid 2n+2 \atop j \neq 1, 2}} \Phi_j(x) . \]

**Proof.** Since

\[ Q_n(x) = f_n(x^2) = \prod_{\substack{j \mid n+1 \atop j \neq 1}} \Phi_j(x^2) \]

we should know what is $\Phi_j(x^2)$.

It is well-known, see [9], that

\[ \Phi_j(x^2) = \begin{cases} 
\Phi_{2j}(x), & \text{if } j \text{ is even} \\
\Phi_j(x)\Phi_{2j}(x), & \text{if } j \text{ is odd}
\end{cases} \]

To complete the proof we must show that each divisor of $2n+2$ bigger than 2 appears in the product (22). Let $d$ be a divisor of $2n+2$ bigger than 2. We consider two cases:

i) If $d$ is odd then since $d|2(n+1)$ we have that $d|n+1$. Since $\Phi_d$ is a factor of $f_n(x)$, then $f_d(x^2) = \Phi_d(x)\Phi_{2d}(x)$, and therefore $\Phi_d$ appears.

ii) If $d$ is even, then $d = 2s$ for some integer $s$ bigger than 1. Since $2s|2(n+1)$ we have that $s|n+1$. Therefore $\Phi_s$ appears in the factorization of $f_n(x)$. If $s$ is odd then $\Phi_s(x^2) = \Phi_s(x)\Phi_{2s}(x)$ and if $s$ is even $\Phi_s(x^2) = \Phi_{2s}(x)$. In either case $\Phi_{2s} = \Phi_d$ appears.

□

An alternative way to derive the formula for $f_{A_n}$ is the following: In the case of $A_n$ we have

\[ a_n(x) = U_n(x^2) . \]

Therefore,

\[ Q_n(x) = x^n U_n \left( \frac{1}{2} \left( x + \frac{1}{x} \right) \right) . \]

Set $x = e^{i\theta}$ to obtain

\[
Q_n(x) = e^{in\theta} U_n \left( \frac{1}{2} (e^{i\theta} + e^{-i\theta}) \right) \\
= e^{in\theta} U_n (\cos \theta) \\
= e^{in\theta} \frac{\sin(n+1)\theta}{\sin \theta} \\
= e^{n\theta} \left( e^{i(n+1)\theta} + e^{-i(n+1)\theta} \right) \\
= \frac{x^{2(n+1)} - 1}{x^2 - 1} .
\]

Setting $u = x^2$ we obtain

\[
\frac{u^{n+1} - 1}{u - 1} = u^n + u^{n-1} + \cdots + u + 1 .
\]

This is of course the Coxeter polynomial of the $A_n$ graph. Therefore $Q_n(x) = x^{2n} + x^{2(n-1)} + \cdots + x^2 + 1$ for all $x \in \mathbb{C}$.

We present the characteristic polynomial of the adjacency matrix and the Coxeter polynomial for small values of $n$. 
• $A_2 \ a_2 := x^2 - 1 \ f_2 = \Phi_3$
• $A_3 \ a_3 := x^3 - 2x \ f_3 = \Phi_2 \Phi_4$
• $A_4 \ a_4 := x^4 - 3x^2 + 1 \ f_4 = \Phi_5$
• $A_5 \ a_5 := x^5 - 4x^3 + 3x \ f_5 = \Phi_2 \Phi_6 \Phi_6$
• $A_6 \ a_6 := x^6 - 5x^4 + 6x^2 - 1 \ f_6 = \Phi_7$
• $A_7 \ a_7 := x^7 - 6x^5 + 10x^3 - 4x \ f_7 = \Phi_2 \Phi_8$
• $A_8 \ a_8 := x^8 - 7x^6 + 15x^4 - 10x^2 + 1 \ f_8 = \Phi_3 \Phi_9$
• $A_9 \ a_9 := x^9 - 8x^7 + 21x^5 - 20x^3 + 5x \ f_9 = \Phi_2 \Phi_5 \Phi_{10}$
• $A_{10} \ a_{10} := x^{10} - 9x^8 + 26x^6 - 35x^4 + 15x^2 - 1 \ f_{10} = \Phi_{11}$.

Note that $a_n(x)$ is explicitly given by the formula

$$a_n(x) = U_n \left( \frac{x}{2} \right)$$

due to \([10]\).

### 6.4. Associated Polynomials for $B_n$ and $C_n$

In the case of $B_n$ we have

$$a_n(x) = 2T_n(\frac{x}{2}) .$$

Therefore,

$$Q_n(x) = 2x^n T_n \left( \frac{1}{2}(x + \frac{1}{x}) \right) .$$

Set $x = e^{i\theta}$ to obtain

$$Q_n(x) = 2e^{i\theta} T_n \left( \frac{1}{2}(e^{i\theta} + e^{-i\theta}) \right)$$

$$= 2e^{i\theta} T_n(\cos \theta)$$

$$= 2e^{i\theta} \cos n\theta$$

$$= 2e^{i\theta} \frac{1}{2} (e^{i\theta} + e^{-i\theta}) = e^{2i\theta} + 1 = x^{2n} + 1 .$$

Therefore $Q_n(x) = x^{2n} + 1$ for all $x \in \mathbb{C}$. As a result the Coxeter polynomial is $f_n(x) = x^n + 1$. We present the factorization of $f_n(x)$ for small values of $n$.

• $B_2 \ a_2 = x^2 - 2 \ f_2 = \Phi_4$
• $B_3 \ a_3 = x^3 - 3x \ f_3 = \Phi_2 \Phi_6$
• $B_4 \ a_4 = x^4 - 4x^2 + 2 \ f_4 = \Phi_8$
• $B_5 \ a_5 = x^5 - 5x^3 + 5x \ f_5 = \Phi_2 \Phi_{10}$
• $B_6 \ a_6 = x^6 - 6x^4 + 9x^2 - 2 \ f_6 = \Phi_4 \Phi_{12}$
• $B_7 \ a_7 = x^7 - 7x^5 + 14x^3 - 7x \ f_7 = \Phi_2 \Phi_{14}$
• $B_8 \ a_8 = x^8 - 8x^6 + 20x^4 - 16x^2 + 2 \ f_8 = \Phi_{16}$
• $B_9 \ a_9 = x^9 - 9x^7 + 27x^5 - 30x^3 + 9x \ f_9 = \Phi_2 \Phi_6 \Phi_{18}$
• $B_{10} \ a_{10} = x^{10} - 10x^8 + 35x^6 - 50x^4 + 25x^2 - 2 \ f_{10} = \Phi_4 \Phi_{20}$.

Write $n = 2^m N$ where $N$ is odd. As we already mentioned

$$f_n(x) = x^n + 1 = \prod_{d|N} \Phi_{2md}(x) ,$$

where $m = 2^a$. Therefore

$$f_n(x) = x^n + 1 = \prod_{d|n \atop d \text{ odd}} \Phi_{2^{a+1}d}(x) = \prod_{d|N} \Phi_{2^{a+1}d}(x) .$$
Proposition 7. Let $r = 2^{\alpha + 2}$. Then

$$Q_n(x) = \prod_{d \mid n, d \text{ odd}} \Phi_{rd}(x).$$

Proof. It follows from the formula

$$\Phi_k(x^2) = \Phi_{2k}(x)$$

when $k$ is even. \qed

Example 3. Let $n = 24$. Then $24 = 2^3 \cdot 3$. Therefore $\alpha = 3$.

$$Q_{24}(x) = x^{48} + 1 = \prod_{d \mid 24, d \text{ odd}} \Phi_{32d} = \Phi_{32} \Phi_{96} = (x^{16} + 1)(x^{32} - x^{16} + 1).$$

Note that the $a_n(x)$ polynomial is explicitly given in this case by the formula

$$a_n(x) = 2T_n\left(\frac{x}{2}\right) = \sum_{j=0}^{\left\lfloor \frac{n}{2} \right\rfloor} (-1)^j \frac{n(n-1)!}{j!(n-2j)!} (x)^{n-2j}$$

due to (14).

Since $a_n(x) = 2T_n\left(\frac{x}{2}\right)$ these polynomials satisfy the recursion

$$a_{n+1} = xa_n(x) - a_{n-1}(x)$$

with $a_0(x) = 2$ and $a_1(x) = x$. We would like to mention a useful application of these polynomials. One can use them to express $x^n + x^{-n}$ as a function of $\zeta = x + \frac{1}{x}$. For $x = e^{i\theta}$ it is just the expression of $2 \cos n\theta$ as a polynomial in $2 \cos \theta$. This polynomial is clearly $a_n(x)$, the adjacency polynomial of $B_n$.

Example 4.

$$\left(x + \frac{1}{x}\right)^2 = x^2 + \frac{1}{x^2} + 2.$$  

Therefore

$$x^2 + \frac{1}{x^2} = \zeta^2 - 2 = a_2(\zeta).$$

Similarly

$$x^3 + \frac{1}{x^3} = \zeta^3 - 3\zeta = a_3(\zeta),$$

$$x^4 + \frac{1}{x^4} = \zeta^4 - 4\zeta^2 + 2 = a_4(\zeta).$$

6.5. Associated Polynomials for $D_n$. In the case of $D_n$ we have

$$q_n(x) = 4xT_{n-1}(x).$$

Therefore,

$$a_n(x) = 2xT_{n-1}\left(\frac{x}{2}\right),$$

and

$$Q_n(x) = 2x^n\left(x + \frac{1}{x}\right)T_{n-1}\left(\frac{1}{2}(x + \frac{1}{x})\right).$$

Set $x = e^{i\theta}$ to obtain
\[ Q_n(x) = 2(x^{n+1} + x^{n-1})T_{n-1}(\cos \theta) \]
\[ = 2(x^{n+1} + x^{n-1}) \cos(n-1)\theta \]
\[ = 2(x^{n+1} + x^{n-1}) \frac{1}{2} (e^{i(n-1)\theta} + e^{-i(n-1)\theta}) \]
\[ = x^{2n} + x^{2(n-1)} + x^2 + 1. \]

Therefore, \( Q_n(x) = x^{2n} + x^{2(n-1)} + x^2 + 1 \) for all \( x \in \mathbb{C} \). We conclude that \( f_n(x) = x^n + x^{n-1} + x + 1 \). We present the formula for \( a_n(x) \) and the factorization of \( f_n(x) \) for small values of \( n \).

- \( D_4 \, a_4 = x^4 - 3x^2 \quad f_4(x) = \Phi_2^2 \Phi_6 \)
- \( D_5 \, a_5 = x^5 - 4x^3 + 2x \quad f_5 = \Phi_2 \Phi_8 \)
- \( D_6 \, a_6 = x^6 - 5x^4 + 5x^2 \quad f_6 = \Phi_2^2 \Phi_{10} \)
- \( D_7 \, a_7 = x^7 - 6x^5 + 9x^3 - 2x \quad f_7 = \Phi_2 \Phi_4 \Phi_{12} \)
- \( D_8 \, a_8 = x^8 - 7x^6 + 14x^4 - 7x^2 \quad f_8 = \Phi_2^2 \Phi_{14} \)
- \( D_9 \, a_9 = x^9 - 8x^7 + 20x^5 - 16x^3 + 2x \quad f_9 = \Phi_2 \Phi_{16} \)
- \( D_{10} \, a_{10} = x^{10} - 9x^8 + 27x^6 - 30x^4 + 9x^2 \quad f_{10} = \Phi_2^2 \Phi_6 \Phi_{18} \).

Write \( n - 1 = 2^a N \) where \( N \) is odd and \( r = 2^{a+1} \).

\[ f_n(x) = (x + 1)(x^{n-1} + 1) = \Phi_2(x) \prod_{d\mid N} \Phi_{2r^d}(x). \]

**Proposition 8.**

\[ Q_n(x) = \Phi_4(x) \prod_{d\mid n-1} \Phi_{2r^d}(x). \]

7. Factorization of Chebyshev Polynomials

7.1. Chebyshev polynomials of second kind. It is well-known that the roots of \( U_n \) are given by

\[ x_k = \cos \frac{k\pi}{n+1} \quad k = 1, 2, \ldots, n, \]

as we already observed in (11).

We can write them in the form \( x_k = \cos k\theta \), where \( \theta = \frac{\pi}{n+1} \).

The roots of \( a_n(x) = U_n(x) \) are

\[ \lambda_k = 2 \cos \frac{k\pi}{n+1} = 2 \cos k\theta \quad k = 1, 2, \ldots, n, \]

i.e. the roots of \( a_n(x) \) are

\[ 2 \cos \frac{m_1 \pi}{h} \]

where \( m_1 \) are the exponents of \( A_n \) and \( h \) is the Coxeter number for \( A_n \).

The roots of \( a_n(x) = U_n(x) \) are

\[ \lambda_k = 2 \cos \frac{k\pi}{h} = 2 \cos k\theta \quad k = 1, 2, \ldots, n. \]

Denote them by

\[ \lambda_1 = 2 \cos \theta = 2 \cos \theta, \quad \lambda_2 = 2 \cos 2\theta, \ldots, \quad \lambda_n = 2 \cos n\theta. \]

Note that \( \theta_k = k\theta \) and

\[ \theta_j + \theta_{n+1-j} = \pi. \]

This implies that \( \lambda_j = -\lambda_{n-j+1} \). As a result

\[ \{ \lambda_j \mid j = 1, 2, \ldots, n \} = \{-\lambda_j \mid j = 1, 2, \ldots, n \}. \]
The roots of \( p_n(x) \) are then
\[
\xi_k = 2 + \lambda_k = 2 + 2 \cos \frac{k\pi}{n+1} = 2 + 2 \cos \theta_k = 4 \cos^2 \frac{\theta_k}{2}.
\]
It follows from (25) that the eigenvalues of \( C \) occur in pairs \( \{2 + \lambda, 2 - \lambda\} \). This is a general result which holds for each Cartan matrix corresponding to a simple Lie algebra over \( \mathbb{C} \), see [23, p. 345] for a general proof.

It follows from Theorem 5 that the roots of \( Q_n(x) \) are
\[
e^{i\theta_1}, e^{i\theta_2}, \ldots, e^{i\theta_n}, e^{-i\theta_1}, e^{-i\theta_2}, \ldots, e^{-i\theta_n},
\]
or equivalently,
\[
e^{i\theta}, e^{2i\theta}, \ldots, e^{ni\theta}, e^{(n+1)i\theta}, e^{(n+2)i\theta}, \ldots, e^{(2n+1)i\theta}.
\]
Note that
\[
e^{i\theta} = e^{i\pi} = e^{\frac{2\pi i}{n}}.
\]
As a result \( e^{i\theta} \) is a \( (2h) \)th primitive root of unity and therefore a root of the cyclotomic polynomial \( \Phi_{2h} \). The other roots of this cyclotomic polynomial are of course \( e^{ki\theta} \) where \( (k, 2h) = 1 \). This determines \( \Phi_{2h} \) as an irreducible factor of \( Q_n(x) \). To determine the other irreducible factors we proceed as follows: The roots of \( \Phi_{2h}(x) \) are of \( \Phi_h \) are \( e^{ki\theta} \) where \( (k, 2h) = 2 \). In general for each \( d \) which is a divisor of \( 2h \) (but \( d \neq 1, 2 \)) we form \( \Phi_d \) by grouping together all the \( e^{ki\theta} \) such that \( (k, 2h) = \frac{2h}{d} \). Since \( e^{(n+1)i\theta} \) and \( e^{2ni\theta} \) do not appear as roots of \( Q_n(x) \) the cyclotomic polynomials \( \Phi_1 \) and \( \Phi_2 \) do not appear in the factorization. You can consider this argument as another proof of Proposition 22.

The roots of \( a_n(x) \) are of the form \( e^{i\theta_d} + e^{-i\theta_d} \), where \( e^{i\theta_n} \) is a root of \( Q_n(x) \). It is easy to see that each irreducible factor of \( Q_n(x) \) determines an irreducible factor of \( a_n(x) \) and conversely. In fact this is the argument of Lehmer in [13]. If \( \Phi_d \) is a cyclotomic factor of \( Q_n(x) \) then \( \Phi_d \) being a reciprocal polynomial it can be written in the form
\[
\Phi_d(x) = x^m \psi_d \left( x + \frac{1}{x} \right)
\]
where \( m = \deg \psi_n = \frac{1}{2} \phi(d) \). According to Lehmer the polynomial \( \psi_d \) is irreducible. These polynomials are all the irreducible factors of \( a_n(x) \). This fact is easily established by looking at the roots of \( a_n(x) \). The way to determine the irreducible factors of \( a_n(x) \) is the following: Start with a cyclotomic factor of \( Q_n(x) \). For example, consider \( \Phi_{2h}(x) \). The roots of this polynomial are \( e^{ki\theta} \) where \( (k, 2h) = 1 \). Take only \( e^{ki\theta} \) with \( (k, 2h) = 1 \) such that \( 1 \leq k \leq n \). The corresponding roots of \( a_n(x) \) are of course \( 2 \cos k\theta \), \( 1 \leq k \leq n \) and \( (k, 2h) = 1 \). This determines the polynomial \( \psi_{2h} \). Then we repeat this procedure with the other cyclotomic factors.

**Example 5.** To determine the factorization of \( U_5(x) = 32x^5 - 32x^3 + 6x \). Since \( n = 5 \), \( h = n + 1 = 6 \) and \( \theta = \frac{\pi}{5} \). The roots of \( U_5 \) are
\[
\cos \frac{\pi}{6}, \cos \frac{2\pi}{6}, \cos \frac{3\pi}{6}, \cos \frac{4\pi}{6}, \cos \frac{5\pi}{6}.
\]
The roots of \( a_5(x) \) are
\[
\lambda_1 = 2 \cos \frac{\pi}{6}, \lambda_2 = 2 \cos \frac{2\pi}{6}, \lambda_3 = 2 \cos \frac{3\pi}{6}, \lambda_4 = 2 \cos \frac{4\pi}{6}, \lambda_5 = 2 \cos \frac{5\pi}{6}.
\]
The roots of \( Q_5(x) \) are
\[
e^{\frac{\pi}{5}}, e^{\frac{2\pi}{5}}, e^{\frac{3\pi}{5}}, e^{\frac{4\pi}{5}}, e^{\frac{5\pi}{5}}, e^{\frac{6\pi}{5}}, e^{\frac{7\pi}{5}}, e^{\frac{8\pi}{5}}, e^{\frac{9\pi}{5}}, e^{\frac{10\pi}{5}}, e^{\frac{11\pi}{5}}.
\]
We group together all $e^{ik\theta}$ such that $(k, 12) = 1$ i.e.,
$$e^{\frac{\pi}{6}}, e^{\frac{5\pi}{6}}, e^{\frac{7\pi}{6}}, e^{\frac{11\pi}{6}}$$
which are the roots of $\Phi_{12}(x)$. Note that
$$e^{\frac{\pi}{6}} + e^{\frac{11\pi}{6}} = e^{\frac{\pi}{6}} + e^{-\frac{\pi}{6}} = 2 \cos \frac{\pi}{6} = \lambda_1 = \sqrt{3},$$
and
$$e^{\frac{5\pi}{6}} + e^{\frac{7\pi}{6}} = e^{\frac{5\pi}{6}} + e^{-\frac{5\pi}{6}} = \cos \frac{5\pi}{6} = \lambda_5 = -\sqrt{3}.$$ 
These roots $\lambda_1$ and $\lambda_5$ are roots of $\psi_{12} = x^2 - 3$ which is an irreducible factor of $a_5(x)$.
Then we group together all $e^{ik\theta}$ such that $(k, 12) = 2$ i.e.,
$$e^{\frac{2\pi}{6}}, e^{\frac{10\pi}{6}}$$
which are the roots of $\Phi_6(x)$. Note that
$$e^{\frac{2\pi}{6}} + e^{\frac{10\pi}{6}} = e^{\frac{2\pi}{6}} + e^{-\frac{4\pi}{6}} = 2 \cos \frac{2\pi}{6} = \lambda_2 = 1.$$ 
Therefore $x - 1$ is the irreducible factor of $a_5(x)$ corresponding to $\Phi_6(x)$.
Then we group together all $e^{ik\theta}$ such that $(k, 12) = 3$ i.e.,
$$e^{\frac{3\pi}{6}}, e^{\frac{9\pi}{6}}$$
which are the roots of $\Phi_4(x)$. Note that
$$e^{\frac{3\pi}{6}} + e^{\frac{9\pi}{6}} = e^{\frac{3\pi}{6}} + e^{-\frac{3\pi}{6}} = 2 \cos \frac{3\pi}{6} = \lambda_3 = 0.$$ 
Therefore $\psi_4(x) = x$ is the irreducible factor of $a_5(x)$ corresponding to $\Phi_4(x)$.
Finally we group together all $e^{ik\theta}$ such that $(k, 12) = 4$ i.e.,
$$e^{\frac{4\pi}{6}}, e^{\frac{2\pi}{6}}$$
which are the roots of $\Phi_3(x)$. Note that
$$e^{\frac{4\pi}{6}} + e^{\frac{2\pi}{6}} = e^{\frac{4\pi}{6}} + e^{-\frac{4\pi}{6}} = 2 \cos \frac{4\pi}{6} = \lambda_4 = -1.$$ 
Therefore $\psi_3(x) = x + 1$ is the irreducible factor of $a_5(x)$ corresponding to $\Phi_3(x)$.
We end up with the integer factorization of $a_5(x)$ into irreducible factors:
$$a_5(x) = x(x + 1)(x - 1)(x^2 - 3).$$
Since $U_5(x) = a_5(2x)$ we obtain the factorization of $U_5(x)$:
$$U_5(x) = 2x(2x + 1)(2x - 1)(4x^2 - 3).$$
Remark 6. To compute $\psi$ to define $\psi$ since

\begin{align*}
\psi_3(x) &= x + 1 \\
\psi_4(x) &= x \\
\psi_5(x) &= x^2 + x - 1 \\
\psi_6(x) &= x - 1 \\
\psi_7(x) &= x^3 + x^2 - 2x - 1 \\
\psi_8(x) &= x^2 - 2 \\
\psi_9(x) &= x^3 - 3x + 1 \\
\psi_{10}(x) &= x^2 - x + 1 \\
\psi_{11}(x) &= x^5 + x^4 - 4x^3 - 3x^2 + 3x + 1 \\
\psi_{12}(x) &= x^2 - 3 \\
\psi_{13}(x) &= x^6 + x^5 - 5x^4 - 4x^3 + 6x^2 + 3x - 1 \\
\psi_{14}(x) &= x^3 - x^2 - 2x + 1 \\
\psi_{15}(x) &= x^4 - x^3 - 4x^2 + 4x + 1 \\
\psi_{16}(x) &= x^4 - 4x^2 + 2 \\
\psi_{17}(x) &= x^8 + x^7 - 7x^6 - 6x^5 + 15x^4 + 10x^3 - 10x^2 - 4x + 1 \\
\psi_{18}(x) &= x^3 - 3x - 1 \\
\psi_{19}(x) &= x^9 + x^8 - 8x^7 - 7x^6 + 21x^5 + 15x^4 - 20x^3 - 10x^2 + 5x + 1 \\
\psi_{20}(x) &= x^4 - 5x^2 + 5 \\
\psi_{21}(x) &= x^6 - x^5 - 6x^4 + 6x^3 + 8x^2 - 8x + 1 \\
\psi_{22}(x) &= x^5 - x^4 - 4x^3 + 3x^2 + 3x - 1 \\
\psi_{23}(x) &= x^{11} + x^{10} - 10x^9 - 9x^8 + 36x^7 + 28x^6 - 56x^5 - 35x^4 + 35x^3 + 15x^2 - 6x - 1 \\
\psi_{24}(x) &= x^4 - 4x^2 + 1 .
\end{align*}

Remark 5. Since $\psi_n(x)$ is the minimal polynomial of $2\cos\frac{2\pi}{n}$ it is reasonable to define $\psi_1(x) = x - 2$ and $\psi_2(x) = x + 2$. They correspond to the reducible $Q_1(x) = (x - 1)^2$ and $Q_2(x) = (x + 1)^2$ respectively.

Remark 6. To compute $\psi_n$ is straightforward. We give two examples.

- $n = 36$. Since

$$\Phi_{36} = x^{12} - x^6 + 1$$

the polynomial $\psi_{36}$ is of degree 6. We need

$$x^{12} - x^6 + 1 = x^6\psi_{36}(\zeta)$$

where $\zeta = x + \frac{1}{x}$. Therefore

$$\psi_{36}(\zeta) = x^6 + \frac{1}{x^6} - 1 = (\zeta^6 - 6\zeta^4 + 9\zeta^2 - 2) - 1 = \zeta^6 - 6\zeta^4 + 9\zeta^2 - 3 .$$

- $n = 60$. Since

$$\Phi_{60} = x^{16} + x^{14} - x^{10} - x^8 - x^6 + x^2 + 1$$

the polynomial $\psi_{60}$ is of degree 8. We need

$$x^{16} + x^{14} - x^{10} - x^8 - x^6 + x^2 + 1 = x^8\psi_{60}(\zeta)$$

where $\zeta = x + \frac{1}{x}$. Therefore

$$\psi_{60}(\zeta) = (x^8 + \frac{1}{x^8}) + (x^6 + \frac{1}{x^6}) - (x^2 + \frac{1}{x^2}) - 1$$

$$= a_8(\zeta) + a_6(\zeta) - a_2(\zeta) - 1$$

$$= \zeta^8 - 7\zeta^6 + 14\zeta^4 - 8\zeta^2 + 1 .$$
Example 6. To find the factorization of
\[ U_9(x) = 512x^9 - 1024x^7 + 672x^5 - 160x^3 + 10x. \]

Since
\[ Q_9(x) = \Phi_4 \Phi_5 \Phi_{10} \Phi_{20}, \]
we have that
\[ a_9(x) = \psi_4 \psi_5 \psi_{10} \psi_{20} = x(x^2 + x - 1)(x^2 - x - 1)(x^4 - 5x^2 + 5). \]

Finally
\[ U_9(x) = a_9(2x) = 2x(4x^2 + 2x - 1)(4x^2 - 2x - 1)(16x^4 - 20x^2 + 5). \]

To conclude we state the following result:

Proposition 9.
\[ U_n(x) = \prod_{j \mid 2^2n+2, j \neq 1, 2} \psi_j(2x). \]

7.2. Chebyshev polynomials of the first kind. The roots of \( T_n \) are given by
\[ x_k = \cos \left( \frac{2k-1}{2n} \pi \right) \quad k = 1, 2, \ldots, n. \]
Let \( h = 2n \) (\( h \) is the Coxeter number of the root system \( B_n \)) and \( \theta = \frac{\pi}{h} \). Then
\[ x_k = \cos(2k-1)\theta \quad k = 1, 2, \ldots, n, \]
i.e. the roots of \( T_n \) are
\[ \cos k\theta \]
where \( k \) runs over the exponents of a root system of type \( B_n \).

The roots of \( a_n(x) = 2T_n(\frac{x}{2}) \) are
\[ \lambda_k = 2 \cos \left( \frac{2k-1}{h} \pi \right) = 2 \cos (2k-1)\theta \quad k = 1, 2, \ldots, n. \]
Denote them by
\[ \lambda_1 = 2 \cos \theta_1 = 2 \cos \theta, \quad \lambda_2 = 2 \cos \theta_2, \ldots, \lambda_n = 2 \cos \theta_n. \]

Note that \( \theta_k = (2k-1)\theta \) and
\[ \theta_j + \theta_{n+1-j} = \pi. \]
This implies that \( \lambda_j = -\lambda_{n-j+1} \). As a result
\[ \{\lambda_j | j = 1, 2, \ldots, n\} = \{-\lambda_j | j = 1, 2, \ldots, n\}. \]

The roots of \( p_n(x) \) are then
\[ \xi_k = 2 + \lambda_k = 2 + 2 \cos \left( \frac{2k-1}{2n} \pi \right) = 2 + 2 \cos \theta_k = 4 \cos^2 \frac{\theta_k}{2}. \]
It follows from (26) that the eigenvalues of \( C \) occur in pairs \( \{2 + \lambda, 2 - \lambda\} \).

It follows from Theorem 5 that the roots of \( Q_n(x) \) are
\[ e^{i\theta}, e^{i2\theta}, \ldots, e^{i(n-1)\theta}, e^{-i\theta}, e^{-i2\theta}, \ldots, e^{-i(n-1)\theta}, \]
or equivalently,
\[ e^{i\theta}, e^{i3\theta}, e^{i5\theta}, \ldots, e^{i(2n-1)\theta}, e^{i(2n+1)\theta}, e^{i(2n+3)\theta}, \ldots, e^{i(4n-1)\theta}. \]

Note that
\[ e^{i\theta} = e^{\frac{2\pi i}{h}} = e^{\frac{2\pi i}{2n}} = e^{\frac{\pi i}{n}}. \]
As a result \( e^{i\theta} \) is a \((2h)\)th primitive root of unity and therefore a root of the cyclotomic polynomial \( \Phi_{2h} = \Phi_{4n} \). The other roots of this cyclotomic polynomial are of course \( e^{ki\theta} \) where \( (k, 2h) = 1 \) and \( k \) odd. This determines \( \Phi_{2h} \) as an irreducible
factor of $Q_n(x)$. To determine the other irreducible factors we proceed as follows:

Take an odd divisor $d$ of $4n$. It is of course an odd divisor of $n$ as well. If we write $n = 2^a N$ where $N$ is odd, this divisor $d$ is also a divisor of $N$. Use the notation $r = 2^a + 2$ and note that $2h = 4n = rN$. We group together all $e^{i\theta}$ where $k$ is odd and $(N, k) = \frac{N}{r}$. Note that $d = N$ corresponds to $\Phi_{4n}$. These roots define $\Psi_{r, d}$. You can consider this argument as another proof of Proposition 23.

As in the case of $A_n$ the roots of $a_n(x)$ are of the form $e^{i\theta} + e^{-i\theta}$, where $e^{i\theta}$ is a root of $Q_n(x)$. Again, the irreducible factor of $Q_n(x)$ are in one-to-one correspondence with the irreducible factors of $a_n(x)$. We denote the irreducible factors of $a_n(x)$ with $\psi_n$ as before.

**Example 7.** To determine the factorization of

$$ T_5(x) = T_5(x) = 16x^5 - 20x^3 + 5x . $n

Since $n = 5$, $h = 2n = 10$ and $\theta = \frac{\pi}{10}$. The roots of $T_5$ are

$$ \cos \frac{\pi}{10}, \cos \frac{3\pi}{10}, \cos \frac{5\pi}{10}, \cos \frac{7\pi}{10}, \cos \frac{9\pi}{10} . $$

The roots of $a_5(x)$ are

$$ \lambda_1 = 2 \cos \frac{\pi}{10}, \lambda_2 = 2 \cos \frac{3\pi}{10}, \lambda_3 = 2 \cos \frac{5\pi}{10}, \lambda_4 = 2 \cos \frac{7\pi}{10}, \lambda_5 = 2 \cos \frac{9\pi}{10} . $$

Therefore, the roots of $Q_5(x)$ are

$$ e^{\frac{\pi}{10}}, e^{\frac{3\pi}{10}}, e^{\frac{5\pi}{10}}, e^{\frac{7\pi}{10}}, e^{\frac{9\pi}{10}} . $$

We group together

$$ e^{\frac{\pi}{10}}, e^{\frac{3\pi}{10}}, e^{\frac{5\pi}{10}}, e^{\frac{7\pi}{10}}, e^{\frac{9\pi}{10}} . $$

which are the roots of $\Phi_{20}(x)$. These are all the exponentials $e^{ik\theta}$ with $k$ odd and $(k, 20) = 1$.

Note that

$$ e^{\frac{\pi}{10}} + e^{\frac{9\pi}{10}} = e^{\frac{\pi}{10}} + e^{\frac{\pi}{10}} = 2 \cos \frac{\pi}{10} = \lambda_1 , $$

$$ e^{\frac{3\pi}{10}} + e^{\frac{7\pi}{10}} = e^{\frac{3\pi}{10}} + e^{-\frac{\pi}{10}} = \cos \frac{3\pi}{10} = \lambda_2 , $$

$$ e^{\frac{5\pi}{10}} + e^{\frac{5\pi}{10}} = e^{\frac{5\pi}{10}} + e^{\frac{5\pi}{10}} = \cos \frac{7\pi}{10} = \lambda_4 , $$

$$ e^{\frac{7\pi}{10}} + e^{\frac{9\pi}{10}} = e^{\frac{7\pi}{10}} + e^{-\frac{5\pi}{10}} = \cos \frac{9\pi}{10} = \lambda_5 . $$

These roots $\lambda_1, \lambda_2, \lambda_4$ and $\lambda_5$ are roots of $\psi_{20} = x^4 - 5x^2 + 5$ which is an irreducible factor of $a_5(x)$.

The only other odd divisor of 20 is 5. Therefore we group together

$$ e^{\frac{\pi}{10}}, e^{\frac{3\pi}{10}} . $$

which are the roots of $\Phi_5(x)$. These are all the exponentials $e^{ik\theta}$ with $k$ odd and $(k, 20) = 5$. Noting that $20 = 2^2 \cdot 5$ we see that 5 and 3 are just the positive divisors of 5.

Note that

$$ e^{\frac{\pi}{10}} + e^{\frac{3\pi}{10}} = e^{\frac{\pi}{10}} + e^{-\frac{\pi}{10}} = 2 \cos \frac{5\pi}{10} = \lambda_3 = 0 . $$

Therefore $x$ is the irreducible factor of $a_5(x)$ corresponding to $\Phi_5(x)$.

We end up with the integer factorization of $a_5(x)$ into irreducible factors:

$$ a_5(x) = x(x^4 - 5x^2 + 5) . $$
Table 4. Roots of polynomials in terms of $\theta_j = \frac{m_1 \pi}{h}$.

| Polynomial | Roots |
|------------|-------|
| $q_n$      | $\cos \theta_j$ |
| $a_n$      | $2 \cos \theta_j$ |
| $p_n$      | $4 \cos^2 \frac{\theta_j}{2}$ |
| $Q_n$      | $e^{i\theta_j}$ |
| $f_n$      | $e^{2i\theta_j}$ |

Since $T_5(x) = \frac{1}{2} a_5(2x)$ we obtain the factorization of $T_5(x)$:

$$T_5(x) = x(16x^4 - 20x^2 + 5).$$

Example 8. To find the factorization of

$$T_9(x) = 256x^9 - 576x^7 + 432x^5 + 20x^3 + 9x.$$

Since $Q_9(x) = \Phi_4 \Phi_{12} \Phi_{36}$, we have that

$$a_9(x) = \psi_4 \psi_{12} \psi_{36}.$$

Since $\psi_4(x) = x$ and $\psi_{12}(x) = x^2 - 3$ and $\psi_{36} = x^6 - 6x^4 + 9x^2 - 2$ (Remark 2) we have

$$a_9(x) = x(x^2 - 3)(x^6 - 6x^4 + 9x^2 - 3).$$

Finally

$$T_9(x) = \frac{1}{2} a_9(2x) = x(4x^2 - 3)(64x^6 - 96x^4 + 36x^2 - 3).$$

To conclude we state the following result:

Proposition 10. Let $n = 2^\alpha N$ where $N$ is odd and let $r = 2^\alpha + 2$. Then

$$T_n(x) = \frac{1}{2} \prod_{j|N} \psi_{rj}(2x).$$

Remark 7. In the case of $D_n$

$$q_n(x) = 4xT_{n-1}(x).$$

Therefore

$$a_n(x) = 2xT_{n-1}(x).$$

$$a_n(x_0) = 0 \quad \Rightarrow \quad 2x_0T_{n-1}(x_0) = 0$$

$$\Rightarrow \quad x_0 = 0, \quad \text{or} \quad x_0 = 2\cos\left(\frac{2k - 1}{2(n - 1)} \pi\right) \quad k = 1, 2, \ldots, n - 1.$$ 

In summary: The roots of $a_n(x)$ are

$$2\cos \frac{m_1 \pi}{h}$$

where $h$ is the Coxeter number for $D_n$ and $m_1$ are the exponents. It follows that 0 is always a root and $a_n(x) = xg_n(x)$ where $g_n(x)$ is the $a_{n-1}$ characteristic polynomial for $B_{n-1}$.

In table 4 we list the roots of the various polynomials that we considered.
8. Exceptional Lie Algebras

8.1. $G_2$ graphs. The Cartan matrix for $G_2$ is

$$
\begin{pmatrix}
2 & -1 \\
-3 & 2
\end{pmatrix}.
$$

We can define a generalization of the Cartan matrix of $G_2$ by defining the matrix $G_n$ to be the same as the matrix of $A_n$ except at the $(n,n-1)$ position we replace $-1$ with $-3$. Then det $G_n = 3 - n$ and we find that

$$q_n(x) = 2T_n(x) - U_{n-2}(x).$$

The characteristic polynomial for a Lie algebra of type $G_2$ is

$$p_2(x) = x^2 - 4x + 1,$$

since

$$q_2(x) = 2T_2(x) - U_0(x) = 4x^2 - 3$$

and

$$p_2(x) = q_2(\frac{x}{2} - 1) = x^2 - 4x + 1.$$  

The roots of $a_2(x) = x^2 - 3$ are

$$2 \cos \frac{m_1 \pi}{h},$$

where $m_1 = 1$ and $m_2 = 5$ are the exponents of root system of type $G_2$. The Coxeter number $h$ is 6.

Finally,

$$Q_2(x) = x^4 - x^2 + 1 = \Phi_{12}(x),$$

and

$$f_2(x) = x^2 - x + 1 = \Phi_6(x).$$

Note that 1, 5 are the positive integers less than six and relatively prime to 6. This explains the appearance of $\Phi_6$.

For $n = 3$ we have the affine Lie algebra $g_2^{(1)}$. We record the following formulas:

$$q_3(x) = 8x^3 - 8x$$

$$p_3(x) = x^3 - 6x^2 + 8x$$

$$a_3(x) = x^3 - 4x = x(x^2 - 4) = \psi_1(x)\psi_2(x)\psi_4(x)$$

$$Q_3(x) = x^6 - x^4 - x^2 + 1 = \Phi_1^2\Phi_2^2\Phi_4$$

$$f_3(x) = x^3 - x^2 - x + 1 = \Phi_1^2\Phi_2.$$  

The spectrum of the graph is $-2, 0, 2$.

For $n = 4$ we have the following formulas:

$$q_4(x) = 16x^4 - 20x^2 + 3$$

$$p_4(x) = x^4 - 8x^3 + 19x^2 - 12x - 1$$

$$a_4(x) = x^4 - 5x^2 + 3$$

$$Q_4(x) = x^8 - x^6 - x^4 - x^2 + 1$$

$$f_4(x) = x^4 - x^3 - x^2 - x + 1.$$  

Note that the Coxeter polynomial of the graph $G_4$ is no longer cyclotomic. It has Malher measure approximately 1.72208.
8.2. Graph of type $F_4$. The Cartan matrix for $F_4$ is

$$
\begin{pmatrix}
2 & -1 & 0 & 0 \\
-1 & 2 & -2 & 0 \\
0 & -1 & 2 & -1 \\
0 & 0 & -1 & 2
\end{pmatrix}.
$$

$p_4(x) = x^4 - 8x^3 + 20x^2 - 16x + 1$,

and

$$a_4(x) = x^4 - 4x^2 + 1 = \psi_{24}(x).$$

The roots of $a_4(x)$ are

$$\frac{1}{2}(\pm \sqrt{6} \pm \sqrt{2})$$

i.e.

$$2 \cos \frac{m_i \pi}{12}$$

where $m_i \in \{1, 5, 7, 11\}$. These are the exponents for $F_4$ and being the numbers less than 12 and prime to 12 imply

$$f_4(x) = x^4 - x^2 + 1 = \Phi_{12}(x).$$

8.3. $E_n$ graphs. One can defined a generalized $E_n$ diagram of the same form as $A_n$ except that $a_{21} = a_{12} = a_{23} = a_{32} = 0$ and $a_{13} = a_{31} = a_{24} = a_{42} = -1$. It turns out that $q_n(x)$ is equal to $2x$ times the $q_n$ of $D_{n-1}$ minus the $q_n$ of $A_{n-2}$. We therefore obtain

$$q_n(x) = (2x) \cdot 4xT_{n-2}(x) - U_{n-2}(x) = 8x^2T_{n-2}(x) - U_{n-2}(x).$$

- $n = 6$

  The Cartan matrix for $E_6$ is

  $$
  \begin{pmatrix}
  2 & 0 & -1 & 0 & 0 & 0 \\
  0 & 2 & 0 & -1 & 0 & 0 \\
  -1 & 0 & 2 & -1 & 0 & 0 \\
  0 & -1 & -1 & 2 & -1 & 0 \\
  0 & 0 & 0 & -1 & 2 & -1 \\
  0 & 0 & 0 & 0 & -1 & 2
  \end{pmatrix}.
  $$

  $q_6(x) = 64x^6 - 80x^4 + 20x^2 - 1 = (2x + 1)(2x - 1)(16x^4 - 16x^2 + 1)$

  $p_6(x) = (x - 1)(x - 3)(x^4 - 8x^3 + 20x^2 - 16x + 1)$

  $a_6(x) = x^6 - 5x^4 + 5x^2 - 1 = (x + 1)(x - 1)(x^4 - 4x^2 + 1) = \psi_3(x)\psi_6(x)\psi_{24}(x)$

  $Q_6(x) = (x^2 + x + 1)(x^4 - x + 1)(x^6 - x^4 + 1) = \Phi_3(x)\Phi_6(x)\Phi_{24}(x)$.

  The exponents of $E_6$ are $\{1, 4, 5, 7, 8, 11\}$ and the Coxeter number is 12.

  The subset $\{1, 5, 7, 11\}$ produces $\Phi_{12}$ and $\{4, 8\}$ produces $\Phi_3$. Therefore

  $$f_6(x) = \Phi_3(x)\Phi_{12}(x).$$

  The roots of $a_6(x)$ are

  $$\pm 1, \quad \frac{1}{2}(\pm \sqrt{6} \pm \sqrt{2})$$

  i.e.

  $$2 \cos \frac{m_i \pi}{12}$$

  where $m_i \in \{1, 4, 5, 7, 8, 11\}$. These are the exponents for $E_6$. The Coxeter number is 12.
• \( n = 7 \)

The Cartan matrix for \( E_7 \) is

\[
\begin{pmatrix}
2 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & -1 & 0 & 0 & 0 \\
-1 & 0 & 2 & -1 & 0 & 0 & 0 \\
0 & -1 & -1 & 2 & -1 & 0 & 0 \\
0 & 0 & 0 & -1 & 2 & -1 & 0 \\
0 & 0 & 0 & 0 & -1 & 2 & -1 \\
0 & 0 & 0 & 0 & 0 & -1 & 2
\end{pmatrix}
\]

\( q_7(x) = 128x^7 - 192x^5 + 72x^3 - 6x = 2x(64x^6 - 96x^4 + 36x^2 - 3) \)

\( p_7(x) = (x - 2)(x^6 - 12x^5 + 54x^4 - 112x^3 + 105x^2 + 1) \)

\( a_7(x) = x^7 - 6x^5 + 9x^3 - 3x = x(x^6 - 6x^4 + 9x^2 - 3) = \psi_4(x)\psi_{30}(x) \)

\( Q_7(x) = (x^2 + 1)(x^{12} - x^6 + 1) = \Phi_4(x)\Phi_{30}(x) \).

The exponents of \( E_7 \) are \{1, 5, 7, 9, 11, 13, 17\} and the Coxeter number is 18. The subset \{1, 5, 7, 11, 13, 17\} produces \( \Phi_{18} \) and \{9\} produces \( \Phi_2 \). Therefore

\( f_7(x) = \Phi_2(x)\Phi_{18}(x) \).

• \( n = 8 \)

The Cartan matrix for \( E_8 \) is

\[
\begin{pmatrix}
2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & -1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 2 & -1 & 0 & 0 & 0 & 0 \\
0 & -1 & -1 & 2 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 2
\end{pmatrix}
\]

\( q_8(x) = 256x^8 - 448x^6 + 224x^4 - 32x^2 + 1 \)

\( p_8(x) = x^8 - 16x^7 + 105x^6 + 364x^5 + 714x^4 - 784x^3 + 440x^2 - 96x + 1 \)

\( a_8(x) = x^8 - 7x^6 + 14x^4 - 8x^2 + 1 = \psi_{60}(x) \)

\( Q_8(x) = x^{16} + x^{14} - x^{10} - x^8 - x^6 + x^2 + 1 = \Phi_{60}(x) \).

The exponents of \( E_8 \) are \{1, 7, 11, 13, 17, 19, 23, 29\} which are the positive integers less than 30 and prime to 30. Therefore

\( f_8(x) = \Phi_{30}(x) \).

Since

\( p_n(x) = q_n\left(\frac{x}{2} - 1\right) \)

we have that

\( p_n(0) = q_n(-1) = (-1)^{n-2}(8T_{n-2}(1) - U_{n-2}(1)) \).

Therefore the determinant of a diagram of type \( E_n \) is \( 8T_{n-2}(1) - U_{n-2}(1) = 9 - n \). This gives determinants 3, 2, 1 respectively for \( E_6, E_7, E_8 \).
Example 9. Let us consider \( n = 9 \). Of course \( E_9 \) is the same as \( E_8^{(1)} \) the affine \( E_8 \) diagram. In this case
\[
q_9(x) = 512x^9 - 1024x^7 + 640x^5 - 136x^3 + 8x
\]
and
\[
a_9(x) = x(x-1)(x+1)(x+2)(x-2)(x^2-x-1)(x^2+x-1) = \psi_1(x)\psi_2(x)\psi_3(x)\psi_4(x)\psi_5(x)\psi_9(x)
\]
\( a_9(x) \) is the characteristic polynomial of the adjacency matrix, so the spectrum of the \( E_8^{(1)} \) graph is
\[
0, 1, -1, 2, -2, \tau, \frac{1}{\tau}, -\tau, -\frac{1}{\tau}
\]
where \( \tau = \frac{1 + \sqrt{5}}{2} = 2 \cos \frac{\pi}{5} \).

For the reader who is familiar with the Mackay correspondence, these are the values in the character table of the binary icosahedral group \( SL(2,5) \).

Note that
\[
Q_9 = \Phi_2^4\Phi_3^2\Phi_4^4\Phi_9 \Phi_{10}
\]
and
\[
f_9(x) = x^9 + x^8 - x^6 - x^5 - x^4 - x^3 + x - 1 = \Phi_2^2(x)\Phi_3^2(x)\Phi_5(x)
\]

Example 10. Let us consider \( E_{10} \). In this case we obtain
\[
q_{10}(x) = 1024x^{10} - 2304x^8 + 1728x^6 - 496x^4 + 48x^2 - 1
\]
\[
a_{10}(x) = x^{10} - 9x^8 + 27x^6 - 31x^4 + 12x^2 - 1
\]
\[
Q_{10}(x) = x^{20} + x^{18} - x^{14} - x^{12} - x^{10} - x^8 - x^6 + x^2 + 1
\]
\[
f_{10}(x) = x^{10} + x^9 - x^7 - x^5 - x^4 - x^3 + x + 1
\]

We recognize \( f_{10} \) as the famous Lehmer polynomial \([3]\). Its largest real root is about 1.17628 which is the smallest known Salem number.

9. The sine formula

To prove the sine formula we only will need the following Lemma which was already proved by case to case verification:

**Lemma 2.** The roots of \( a_n(x) \) are
\[
2 \cos \frac{m_i \pi}{h}
\]
where \( m_i \) are the exponents of \( g \) and \( h \) is the Coxeter number of \( g \).

We give some references and history on how to prove this Lemma without a case by case verification: There is an empirical procedure due to H.M. Coxeter which can be used to find the roots of the Coxeter polynomial. If \( \zeta \) is a primitive \( h \) root of unity (where \( h \) is the Coxeter number) then the roots of the Coxeter polynomial are \( \zeta^m \) where \( m \) runs over the exponents of the corresponding root system \([3, 4]\). This observation allows the calculation of the Coxeter polynomial for each root system. It also explains the duality \([20]\) since non-real eigenvalues of \( R \) appear in conjugate pairs. Coxeter also observed that
\[
h\ell = 2r
\]
where \( r \) is the number of positive roots. Using \([27]\) as the only empirical fact Coleman proved in \([3]\) the procedure of Coxeter. The proof of \([27]\) is in the classic paper of Kostant of 1958 \([17]\). Knowing the roots of the Coxeter polynomial, it is straightforward to determine the spectrum of the Cartan matrix \( C \), see e.g. \([2]\) Theorem 2]. This in turns determines the roots of \( a_n(x) \) via the relation \([1]\).
We have seen in Lemma (2) that the roots of \( a_n(x) \) are
\[
2 \cos \frac{m_i \pi}{h}
\]
where \( m_i \) are the exponents of \( g \) and \( h \) is the Coxeter number of \( g \). Let \( \theta_i = \frac{m_i \pi}{h} \).
Then the roots of \( a_n(x) \) are \( \lambda_i = 2 \cos \theta_i, \quad i = 1, 2, \ldots, \ell \).
Recall the duality property of the exponents (20).
\[
m_i + m_{\ell+1-i} = h.
\]
It follows that
\[
m_i \pi/h + m_{\ell+1-i} \pi/h = \pi.
\]
As a result:
\[
\theta_i + \theta_{\ell+1-i} = \pi.
\]
Using this formula, we can infer a relationship satisfied by the roots of \( p_n(x) \) which are:
\[
\xi_i = 4 \cos^2 \frac{\theta_i}{2}.
\]
Namely:
\[
\xi_i + \xi_{\ell+1-i} = 4 \cos^2 \frac{\theta_i}{2} + 4 \cos^2 \frac{\theta_{\ell+1-i}}{2} = 4 \left( \cos^2 \frac{\theta_i}{2} + \cos^2 \frac{\theta_{\ell+1-i}}{2} \right) = 4 \left( \cos^2 \frac{\theta_i}{2} + \sin^2 \frac{\theta_i}{2} \right) = 4.
\]
It follows that the eigenvalues of the Cartan matrix occur in pairs \( \xi, 4 - \xi \).

**Remark 8.** The fact that the eigenvalues of \( C \) occur in pairs \( \{ \xi, 4 - \xi \} \), and a different line of proof can be found in [2, p. 345].

**Theorem 3.** Let \( g \) be a complex simple Lie algebra of rank \( \ell \), \( h \) the Coxeter number, \( m_1, m_2, \ldots, m_\ell \) the exponents of \( g \) and \( C \) the Cartan matrix. Then
\[
2^{2\ell} \prod_{i=1}^{\ell} \sin^2 \frac{m_i \pi}{2h} = \det C.
\]

**Proof.** It follows from Lemma (2) that
\[
a_n(x) = \prod_{i=1}^{\ell} \left( x - 2 \cos \frac{m_i \pi}{h} \right).
\]
Set \( x = 2 \).
\[
a_n(2) = \prod_{i=1}^{\ell} \left( 2 - 2 \cos \frac{m_i \pi}{h} \right)
\]
\[
= 2^{\ell} \prod_{i=1}^{\ell} \left( 1 - \cos \frac{m_i \pi}{h} \right)
\]
\[
= 2^{\ell} \prod_{i=1}^{\ell} 2 \sin^2 \frac{m_i \pi}{2h}
\]
\[
= 2^{2\ell} \prod_{i=1}^{\ell} \sin^2 \frac{m_i \pi}{2h}.
\]
To prove the formula we calculate \( a_n(2) \). We have
\[
p_n(x) = (x - \xi_1)(x - \xi_2) \ldots (x - \xi_\ell) = (x - (4 - \xi_1))(x - (4 - \xi_2)) \ldots (x - (4 - \xi_\ell)).
\]
This implies that $p_n(4) = \xi_1 \xi_2 \ldots \xi_\ell = \det C$. Since $a_n(x) = p_n(x + 2)$ we have

$$a_n(2) = p_n(4) = \det C.$$ 

\[ \Box \]

Remark 9. The formula can also be written in the form:

$$2^\ell \prod_{i=1}^\ell \sin \frac{m_i \pi}{2h} = \sqrt{\det C}.$$ 

It is not clear what is the significance of the factor $2^\ell$. We note that the sum of the Betti numbers is $2^\ell$.

Remark 10. One can compute $a_n(2)$ case by case using properties of Chebyshev polynomials:

\begin{enumerate}
  \item[(1)] $A_n$, $a_n(x) = U_n \left( \frac{x}{2} \right) \Rightarrow a_n(2) = U_n(1) = n + 1 = \det C$.
  \item[(2)] $B_n$, $a_n(x) = 2T_n \left( \frac{x}{2} \right) \Rightarrow a_n(2) = 2T_n(1) = 2 = \det C$.
  \item[(3)] $C_n$, $a_n(x) = 2T_n \left( \frac{x}{2} \right) \Rightarrow a_n(2) = 2T_n(1) = 2 = \det C$.
  \item[(4)] $D_n$, $a_n(x) = 2xT_n \left( \frac{x}{2} \right) \Rightarrow a_n(2) = 4T_n(1) = 4 = \det C$.
  \item[(5)] $G_2$, $a_2(x) = x^2 - 3 \Rightarrow a_2(2) = 1 = \det C$.
  \item[(6)] $F_4$, $a_4(x) = x^4 - 4x^2 + 1 \Rightarrow a_4(2) = 1 = \det C$.
  \item[(7)] $E_6$, $a_6(x) = x^6 - 5x^4 + 5x^3 - 1 \Rightarrow a_6(2) = 3 = \det C$.
  \item[(8)] $E_7$, $a_7(x) = x^7 - 6x^5 + 9x^3 - 3x \Rightarrow a_7(2) = 2 = \det C$.
  \item[(9)] $E_8$, $a_8(x) = x^8 - 7x^6 + 14x^4 - 8x^2 + 1 \Rightarrow a_8(2) = 1 = \det C$.
\end{enumerate}

References

1. N. A.Campo, Sur les valeurs propres de la transformation de Coxeter, *Invent. Math.* 33, (1976), 61-67.
2. S. Berman, Y. S. Lee, R. V. Moody, The spectrum of a Coxeter transformation, affine Coxeter transformations and the defect map, *J. Algebra* 121, no. 2, (1989), 339-357.
3. A. J. Coleman, Killing and the Coxeter transformation of Kac-Moody algebras, *Invent. Math.* 95, no. 3, (1989), 447-477.
4. Coleman, A. J. The Betti numbers of the simple Lie groups. Canad. J. Math. 10 (1958) 349–356
5. Collingwood, David H.; McGovern, William M. Nilpotent orbits in semisimple Lie algebras. Van Nostrand Reinhold Mathematics Series. Van Nostrand Reinhold Co., New York, 1993.
6. H. S. M. Coxeter, Discrete groups generated by reflections, *Ann. of Math. (2)* 35 (1934), 588-621.
7. Coxeter, H. S. M. Regular Polytopes. Methuen and Co., Ltd., London, 1948
8. P. A. Damianou, Monic polynomials in $\mathbb{Z}[x]$ with roots in the unit disc, *Amer. Math. Monthly* 108 (2001), 253–257.
9. P. A. Damianou, On prime values of cyclotomic polynomials, *Int. Math. Forum* 6, no 29, (2011), 1445–1456.
10. F. M. Goodman, P. de la Harpe, and V. F. R. Jones, *Coxeter Graphs and Towers of Algebras*, Mathematical Sciences Research Institute Publications, 14, Springer-Verlag, New York, 1989.
11. E. Hironaka, Lehmer’s problem, McKay’s correspondence, and 2,3,7. Topics in algebraic and noncommutative geometry (Luminy/Annapolis, MD, 2001), *Contemp. Math.*, 324, Amer. Math. Soc., Providence, RI, (2003), 123-138.
12. H. J. Hsiao, On factorization of Chebyshev's polynomials of the first kind, *Bulletin of the Institute of Mathematics, Academia Sinica* 12 (1), (1984), 89–94.
13. James E. Humphreys, *Introduction to Lie algebras and Representation Theory*, Springer, New York, 1972.
14. Victor G. Kac, *Infinite dimensional Lie algebras*, Cambridge University Press, 1994.
15. Koranyi, Adam Spectral properties of the Cartan matrices. Acta Sci. Math. (Szeged) 57 (1993), no. 1-4, 587-592.
16. B. Kostant, The McKay correspondence, the Coxeter element and representation theory, The mathematical heritage of Elie Cartan (Lyon, 1984), Asterisque 1985, Numero Hors Serie, 209-255.
17. Kostant, Bertram. The principal three-dimensional subgroup and the Betti numbers of a complex simple Lie group. Amer. J. Math. 81 1959 973–1032.

18. D. H. Lehmer A Note on Trigonometric Algebraic Numbers. Amer. Math. Monthly 40 (1933) 165–166.

19. M. Mossinghoff, Polynomials with small mahler measure, Mathematics of Computation 67 no. 224, (1998), 1697-1705.

20. J.C. Mason and D.C. Handscomb, Chebyshev Polynomials, Chapman and Hall, 2002.

21. M. O. Ruyes, V. Trevisan, P. S. Wang, Factorization properties of Chebyshev polynomials, Comput. Math. Appl. 50 (2005), 1231-1240.

22. Smith, G. D. Numerical solution of partial differential equations. Finite difference methods. Third edition. The Clarendon Press, Oxford University Press, New York, 1985.

23. R. Steinberg, Finite subgroups of SU2, Dynkin diagrams and affine Coxeter elements, Pacific J.Math. 118 no. 2, (1985), 587-598.

24. R. Stekolshchik, Notes on Coxeter transformations and the McKay correspondence, Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2008.

25. Wikipedia, Chebyshev Polynomials, Wikipedia, the free encyclopedia.

E-mail address: damianou@ucy.ac.cy

Department of Mathematics and Statistics, University of Cyprus, P.O. Box 20537, 1678 Nicosia, Cyprus