Quantum quasiballistic dynamics and thick point spectrum

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August 2018

Abstract

We obtain dynamical lower bounds for some types of self-adjoint operators with point spectrum in terms of the spacing properties of their eigenvalues. In particular, it is shown that systems with thick point spectrum present generically, in the set of initial conditions, a quasiballistic dynamics. Explicit applications include discrete Schrödinger operators with uniform electric fields and random operators (Anderson model).

1 Introduction

Let $T$ be a self-adjoint operator in a separable complex Hilbert space $\mathcal{H}$ and $\xi \in \mathcal{H}$. The relations between the dynamics $e^{-itT}\xi$ and spectral properties of $T$ is a classical subject of the mathematics and physics literature. Systems with pure point and purely continuous spectra present important qualitative and quantitative dynamical differences. The dense point spectrum in an interval, also called thick point spectrum, is an intermediate step from discrete to continuous spectra, and so one could expect some sort of resemblance of the continuous dynamics to thick point dynamics. However, to the best knowledge of the present authors, no such property has been yet detailed in the literature. It is the main purpose of this note to show that thick point spectrum implies Baire generically, in the set of initial conditions $\xi$, a quasiballistic dynamics (in the sense of Definition 1.1). This applies, in particular, to some arbitrarily small Hilbert-Schmidt perturbations of purely continuous operators whose spectra contain an interval (taking into account the Weyl-von Neumann Theorem [24, 25]).

It is well-known that there are explicit relations between the large time behaviour of the dynamics $e^{-itT}\xi$ and the fractal properties of the spectral measure $\mu^T_\xi$ of $T$ associated with $\xi$. In this context, we refer to [1, 13, 14, 16], among others. In order to obtain the desired
quasiballistic dynamics for systems with thick point spectra, we shall explore properties of suitable generalized fractal dimensions of their spectral measures.

Let \( \{ e_n \} \) be an orthonormal basis of \( \mathcal{H} \) and denote the (time-average) \( p \)-moments, \( p > 0 \), of the position operator at time \( t > 0 \), with initial condition \( \xi \), by

\[
\langle \langle |X|^p \rangle \rangle_{t, \xi} := \frac{1}{t} \int_0^t \sum_n |n|^p |\langle e^{-isT} \xi, e_n \rangle|^2 \, ds.
\]

These quantities describe the asymptotic behaviour of the position of the wave packet \( e^{-isT} \xi \) as \( t \) goes to infinity (see [1, 7, 13, 16] and references therein). There is a connection between these moments and some dimensional properties of \( \mu^T_\xi \). Namely, in case \( \mathcal{H} = \ell^2(\mathbb{Z}^d) \) or \( L^2(\mathbb{R}^d), d \geq 1 \), Guarneri and Schulz-Baldes [14] have proved that

\[
\alpha^+ (\xi, p) := \limsup_{t \to \infty} \frac{\ln \langle \langle |X|^p \rangle \rangle_{t, \xi}}{\ln t} \geq \dim^+_{\mu^T_\xi} p/d,
\]

where \( \dim^+_{\mu^T_\xi} \) denotes the (upper) packing dimension of \( \mu^T_\xi \) (see Definition 2.1 ahead). We note that this lower bound is, in many situations, far from being optimal; del Rio et. all. [10] have presented an example of operator with pure point spectrum (and so, with \( \dim^+_{\mu^T_\xi} = 0 \) for every \( \xi \in \mathcal{H} \)) but with quasiballistic dynamics, that is, \( \alpha^+ (\delta_0, 2) = 2 \); see Appendix 2 in [10] for details.

Barbaroux et. all. [1] obtained a refinement of the previous estimate; under some assumptions on \( \mu^T_\xi \) (see Theorem 2.1 in [1]), they proved that

\[
\alpha^+ (\xi, p) \geq D^+_{\mu^T_\xi} \left( \frac{1}{1 + p/d} \right) p/d,
\]

where \( D^+_{\mu^T_\xi} (\cdot) \) denotes an upper generalized fractal dimension of \( \mu^T_\xi \) (see Definition 2.2 ahead). This refinement is far from trivial, since although every pure point measure has packing dimension equal to zero, some of them may have non-trivial generalized fractal dimensions, as discussed in [1]. In this sense, it is clear that non-trivial dynamical lower bounds may occur even when the spectrum is pure point. For this reason, during the last decade, many authors have been exploiting the relations between dynamical lower bounds and pure point spectrum. We mention the papers [1, 5, 6, 8, 9, 10, 15, 19] for references and additional comments about important results on dynamical lower bounds and pure point spectrum.

Some words about notation: \( \mathcal{H} \) will always denote a separable complex Hilbert space and \( T \) a self-adjoint operator in \( \mathcal{H} \). For each Borel set \( \Lambda \subset \mathbb{R} \), \( P^T (\Lambda) \) represents the spectral resolution of \( T \) over \( \Lambda \). A finite Borel measure \( \mu \) on \( \mathbb{R} \) is supported on a Borel set \( \Lambda \subset \mathbb{R} \).
if \( \mu(\mathbb{R} \setminus \Lambda) = 0 \); we denote the support of \( \mu \) by \( \text{supp}(\mu) \). In this paper \( \mu \) always indicates, unless explicitly stated, a finite positive Borel measure on \( \mathbb{R} \). For each \( x \in \mathbb{R} \) and each \( \epsilon > 0 \), \( B(x, \epsilon) \) denotes the open interval \((x - \epsilon, x + \epsilon)\).

The paper is organized as follows. In Subsection 1.1 we state the main results of this work. Subsection 1.2 is devoted to some of their applications. In Section 2, we fix some notation and present a dynamical characterization of the generalized fractal dimensions (Proposition 2.1). In Section 3, we investigate the existence of dense \( G_\delta \) sets of initial conditions (Theorem 3.1), and then present a proof of Theorem 1.1. In Appendix A, we prove Proposition A.1.

### 1.1 Main results

In order to properly present our results, we introduce the following notion.

**Definition 1.1.** Let \(-\infty \leq a < b \leq \infty \) and \((a_j) \subset [a, b]\). One says that \((a_j)\) is weakly-spaced if, for each \( \alpha > 0 \), there exists a subsequence \((a_{j_l})\) such that

i) there exists \( C_\alpha > 0 \) so that, for every \( l \geq 1 \), \( a_{j_l} - a_{j_{l+1}} \geq C_\alpha / l^{1+\alpha} \),

ii) \( c_l := a_{j_l} - a_{j_{l+1}} \) is monotone and \( \lim_{l \to \infty} (a_{j_l} - a_{j_{l+1}}) = 0 \).

**Remark 1.1.** We note that if \( \bigcup_j \{a_j\} \) is a dense subset in \([a, b]\), then \((a_j)\) is weakly-spaced (see Proposition A.1).

**Definition 1.2.** Let \( \xi \in \mathcal{H} \). The corresponding dynamics \( e^{-itT} \xi \) is called quasiballistic if \( \alpha^+(\xi, p) = p \) for all \( p > 0 \).

Breuer et. al. have obtained in [3] results on dynamical upper bounds for discrete one dimensional Schrödinger operators in terms of various spacing properties of the eigenvalues of their finite volume approximations. In contrast with such results, we show in this work that, under some assumptions on the operator, if its eigenvalues are weakly-spaced (Definition 1.1), then the dynamics of every initial condition in a robust set have quasiballistic behaviour. More specifically, we shall prove the following result.

**Theorem 1.1.** Let \( \Lambda \subset \mathbb{R} \) be a bounded Borel set and \( T \) a self-adjoint operator with pure point spectrum in \( \Lambda \). Suppose that \( \Lambda \) contains a weakly-spaced sequence of eigenvalues of \( T \). Then,

\[ G^T(\Lambda) := \{\xi \in \mathcal{H}_\Lambda \mid \alpha^+(\xi, p) = p \text{ for all } p > 0\} \]

is a dense \( G_\delta \) set in \( \mathcal{H}_\Lambda \), where \( \mathcal{H}_\Lambda := \mathcal{P}^T(\Lambda)(\mathcal{H}) \).
Corollary 1.1. Let $-\infty < a < b < \infty$, and $T$ a self-adjoint operator with thick point spectrum in $I := [a, b]$. Then,
\[ G^T(I) = \{ \xi \in \mathcal{H}_I \mid \alpha^+(\xi, p) = p \text{ for all } p > 0 \} \]
is a dense $G_\delta$ set in $\mathcal{H}_I$, where $\mathcal{H}_I = P^T(I)(\mathcal{H})$.

Proof. Let $(\lambda_j)$ be an enumeration of the eigenvalues of $T$ in $I$. Since $\bigcup_j \{ \lambda_j \}$ is a dense subset of $I$, it follows from Proposition A.1 that $(\lambda_j)$ is weakly-spaced. The result is now a direct consequence of Theorem 1.1.

We note that Theorem 1.1 gives a rather general sufficient condition for an operator with pure point spectrum to present non-trivial dynamical lower bounds. The main ingredient in the proof of this result involves a fine analysis of the generalized fractal dimensions of spectral measures of operators with pure point spectrum. Namely, in order to prove Theorem 1.1 we exploit some relations between such dimensions and the spacing properties of the eigenvalues (see Theorem 3.1).

We also remark that, under the hypotheses of Theorem 1.1, every neighborhood (in $\mathcal{H}$) of each $e_n$ contains infinitely many normalized $\xi$ such that $\alpha^+(\xi, p) = p$; this is particularly interesting when $\mathcal{H} = \ell^2(\mathbb{Z}^d)$ and, for every $n \in \mathbb{Z}^d$, $e_n(\cdot) = \delta_n(\cdot)$ (in this case, $e_n$ represents a localized state, and the result of Theorem 1.1 can be seen as a statement of the fact that arbitrarily close to any localized state, there exists a state whose dynamics is quasiballistic).

1.2 Applications

Now we illustrate our general results by presenting some explicit applications.

Discrete Schrödinger operators with uniform electric fields. Let $\mathbb{Z}^d$, $d \geq 2$, be endowed with the norm $|k| = \sum_{j=1}^d |k_j|$. The discrete Schrödinger operator with uniform electric field of constant strength $E \in \mathbb{Z}^d$, $H^d_E$, is defined by the action
\[ (H^d_E u)_j := \sum_{|k|=1} u_{j+k} + (E, j)u_j + v_j u_j, \]
where $(\cdot, \cdot)$ denotes the ordinary scalar product in $\mathbb{R}^d$ and $\sup_{j} |v_j| < \infty$. For $d \geq 2$, under some assumptions on $(v_j)$, it is possible to show (see details in [11]) that $H^d_E$ has thick point spectrum in $\mathbb{R}$. Therefore, in this case, $H^d_E$ satisfies the hypotheses of Corollary 1.1. It was also shown that in case $v_j = 0$ for all $j$, its eigenvectors have super-exponential decay; so, we have an example with generic quasiballistic dynamics in a system with strongly localized eigenvectors (that is, with such localized states arbitrarily close to states whose dynamics are quasiballistic).
**Anderson model.** For each fixed $a > 0$, let $\Omega = [-a, a]^\mathbb{Z}$ be endowed with the product topology and with the respective Borel $\sigma$-algebra. Assume that $(\omega_j)_{j \in \mathbb{Z}} = \omega \in \Omega$ is a set of independent, identically distributed (i.i.d.) real-valued random variables with a common probability measure $\rho$ not concentrated on a single point and such that $\int |\omega_j|^\gamma d\rho(\omega_j) < \infty$ for some $\gamma > 0$. Denote by $\nu := \hat{\rho}^\mathbb{Z}$ the probability measure on $\Omega$. The Anderson model is a random Hamiltonian on $\ell^2(\mathbb{Z})$, defined for each $\omega \in \Omega$ by

$$(h_\omega u)_j := u_{j-1} + u_{j+1} + \omega_j u_j.$$  

It is not hard to check \[7, 22\] that

$$\sigma(h_\omega) = [-2, 2] + \text{supp}(\rho).$$

Furthermore, it is possible to show \[4, 23\] that $\nu$-a.s. $\omega$, $h_\omega$ has thick point spectrum. Hence, $\nu$-a.s. $\omega$, $h_\omega$ satisfies the hypotheses of Corollary 1.1.

**Remark 1.2.** We note that there are some Anderson operators \[22\] and Anderson Dirac operators \[18\] defined on $\ell^2(\mathbb{Z}^d)$, or even in $L^2(\mathbb{R}^d)$, with $d \geq 2$, satisfying the hypotheses of Corollary 1.1.

**Discrete one-dimensional Schrödinger operators.** For each fixed $a > 0$, consider the family $X$ of discrete one-dimensional Schrödinger operators $A$ on $\ell^2(\mathbb{Z})$ given by the action

$$(Au)_j := u_{j-1} + u_{j+1} + v_j u_j,$$

where $(v_j)$ is a (real-valued) sequence in $\ell^\infty(\mathbb{Z})$, such that, for each $j \in \mathbb{Z}$, $|v_j| \leq a$. Let $\nu$ be the product of infinitely (countable) many copies of the normalized Lebesgue measure on $[-a, a]$, and let

$$D := \{ A \in X \mid \sigma(A) = [-a - 2, a + 2], \sigma(A) \text{ is pure point} \}.$$ 

It is possible to show, through Anderson’s localization, that $\nu(X \setminus D) = 0$ \[20\]. Therefore, $\nu$-a.s. operator $A$ satisfies the hypotheses of Corollary 1.1.

**Discrete limit-periodic Schrödinger operators.** Let the discrete Schrödinger operator $H_\epsilon$, defined on $\mathbb{Z}^d$, $d \geq 1$, by the action

$$(H_\epsilon u)_j := \epsilon \sum_{|k|=1} u_{j+k} + v_j u_j,$$  

where $(v_j)$ is a (real-valued) sequence in $\ell^\infty(\mathbb{Z}^d)$, such that, for each $j \in \mathbb{Z}^d$, $|v_j| \leq a$. Let $\nu$ be the product of infinitely (countable) many copies of the normalized Lebesgue measure on $[-a, a]^d$, and let

$$D := \{ A \in X \mid \sigma(A) = [-a - 2, a + 2], \sigma(A) \text{ is pure point} \}.$$ 

It is possible to show, through Anderson’s localization, that $\nu(X \setminus D) = 0$ \[20\]. Therefore, $\nu$-a.s. operator $A$ satisfies the hypotheses of Corollary 1.1.
where $\epsilon$ is a small positive coupling constant. For some limit periodic potentials $v = (v_j)$, $H_v$ has thick point spectrum in $[0, 1]$ (see [17] for details). In this case, $H_v$ satisfies the hypotheses of Corollary [17].

**Continuous one-dimensional Schrödinger operators.** Consider the continuous one-dimensional Schrödinger operator

$$H_V := -\frac{d^2}{dx^2} + V$$

acting in an appropriate domain, where $V$ is a real-valued multiplication operator.

Let $\{k_j\}_j$ be an arbitrary sequence of positive real numbers. Then, by Theorem 2 in [21], there exists a potential $V$ on $[0, \infty)$ so that $\{k_j^2\}_j$ are eigenvalues of $H_V$ on $[0, \infty)$. In this case, if $\{k_j^2\}_j$ is dense in $[0, \infty)$, one has that $H_V$ satisfies the hypotheses of Corollary [17].

### 2 Preliminaries

In this section we fix some notation and present auxiliary results.

#### 2.1 Fractal dimensions

Recall that the pointwise upper scaling exponent of $\mu$ at $x \in \mathbb{R}$ is defined as

$$d_\mu^+(x) := \limsup_{\epsilon \downarrow 0} \frac{\ln \mu(B(x, \epsilon))}{\ln \epsilon},$$

if, for all $\epsilon > 0$, $\mu(B(x; \epsilon)) > 0$; if not, $d_\mu^+(x) := \infty$.

**Definition 2.1.** The upper packing dimension of $\mu$ is defined as

$$\dim \mu^+(\mu) := \mu\text{-ess sup} d_\mu^+(x).$$

**Definition 2.2.** Let $q \in \mathbb{R} \setminus \{1\}$. The lower and upper $q$-generalized fractal dimensions of $\mu$ are defined, respectively, as

$$D_\mu^-(q) := \liminf_{\epsilon \downarrow 0} \frac{\ln \int \mu(B(x, \epsilon))^{q-1} d\mu(x)}{(q - 1) \ln \epsilon} \quad \text{and} \quad D_\mu^+(q) := \limsup_{\epsilon \downarrow 0} \frac{\ln \int \mu(B(x, \epsilon))^{q-1} d\mu(x)}{(q - 1) \ln \epsilon},$$

with integrals taken on $\text{supp}(\mu)$.

**Definition 2.3.** Let $q \in \mathbb{R} \setminus \{1\}$. The lower and upper mean-$q$ dimensions of $\mu$ are defined, respectively, as

$$m_\mu^-(q) := \liminf_{\epsilon \downarrow 0} \frac{\ln \left[ \int B(x, \epsilon)^{q-1} dx \right]}{(q - 1) \ln \epsilon} \quad \text{and} \quad m_\mu^+(q) := \limsup_{\epsilon \downarrow 0} \frac{\ln \left[ \int B(x, \epsilon)^{q-1} dx \right]}{(q - 1) \ln \epsilon}.$$
Remark 2.1. If $\mu$ has a bounded support, then for all $q \in (0, 1)$, $0 \leq D^+_{\mu}(q) \leq 1$. Moreover, it is possible to show that for $q > 0$, $q \neq 1$, $D^+_{\mu}(q) = m^+_{\mu}(q)$; see [2] for a detailed discussion.

2.2 Dynamical characterization of fractal dimensions

Let $r > 0$ and let $\mu$ be a finite positive Borel measure on $\mathbb{R}$ so that supp($\mu$) $\subset [-r, r]$. Consider, for every $t > 0$ and every $q \in \mathbb{R}$,

$$C_{\mu}(q, t) := t \int_{-r}^{r} \left( \int_{-r}^{r} e^{-t|x-y|} d\mu(y) \right)^{q} \, dx.$$

Proposition 2.1. Let $\mu$ be as before and $q > 0$, $q \neq 1$. Then,

$$\lim_{t \to \infty} \inf \frac{\ln C_{\mu}(q, t)}{(q-1) \ln t} = -D^+_{\mu}(q),$$

$$\lim_{t \to \infty} \sup \frac{\ln C_{\mu}(q, t)}{(q-1) \ln t} = -D^-_{\mu}(q).$$

Although natural to specialists, we present a proof of this result for the convenience of the reader.

Proof (Proposition 2.1). We will show that

$$\lim_{t \to \infty} \inf \frac{\ln C_{\mu}(q, t)}{(q-1) \ln t} = -m^+_{\mu}(q),$$

(1)

$$\lim_{t \to \infty} \sup \frac{\ln C_{\mu}(q, t)}{(q-1) \ln t} = -m^-_{\mu}(q).$$

(2)

Since supp($\mu$) $\subset [-r, r]$, one has that, for each $t > 1$ and each $x \in [-r - 1, r + 1]^c$, $\mu(B(x, \frac{1}{t})) = 0$. Hence, it follows that, for $t > 1$,

$$C_{\mu}(q, t) = t \int_{-r}^{r} \left( \int_{-r}^{r} e^{-t|x-y|} d\mu(y) \right)^{q} \, dx \geq t \int_{-r}^{r} \left( \int_{|x-y| < \frac{1}{t}} e^{-t|x-y|} d\mu(y) \right)^{q} \, dx$$

$$\geq \frac{t}{e^q} \int_{-r}^{r} \mu(B(x, \frac{1}{t}))^q dx = \frac{t}{e^q} \int \mu(B(x, \frac{1}{t}))^q dx$$

and, therefore,

$$\lim_{t \to \infty} \inf \frac{\ln C_{\mu}(q, t)}{(q-1) \ln t} \leq -m^+_{\mu}(q), \quad \lim_{t \to \infty} \sup \frac{\ln C_{\mu}(q, t)}{(q-1) \ln t} \leq -m^-_{\mu}(q).$$
Let $0 < \delta < 1$. Then, for each $x \in \mathbb{R}$ and $t > 0$,
\[
\int_{\mathbb{R}} e^{-t|x-y|} d\mu(y) = \int_{|x-y| < \frac{1}{t^{1-\delta}}} e^{-t|x-y|} d\mu(y) + \int_{|x-y| \geq \frac{1}{t^{1-\delta}}} e^{-t|x-y|} d\mu(y) \leq \mu(B(x, \frac{1}{t^{1-\delta}})) + e^{-t^\delta} \mu(\mathbb{R}).
\]
Therefore,
\[
\left( \int_{\mathbb{R}} e^{-t|x-y|} d\mu(y) \right)^q \leq 2^q \max \left\{ \mu(B(x, \frac{1}{t^{1-\delta}})), \mu(\mathbb{R}) e^{-t^\delta} \right\}^q \leq 2^q \mu(B(x, \frac{1}{t^{1-\delta}}))^q + 2^q \mu(\mathbb{R})^q e^{-qt^\delta}. \tag{3}
\]
Since $m_\mu^-(q) \geq 0$ (see Remark 2.1), by (3), one gets, for sufficiently large $t$,
\[
C_\mu(q, t) \leq 2^q t \int_{\mathbb{R}} \mu(B(x, \frac{1}{t^{1-\delta}}))^q dx + (2r + 2)2^q \mu(\mathbb{R})^q e^{-qt^\delta} \leq 2^{q+1} t \int_{\mathbb{R}} \mu(B(x, \frac{1}{t^{1-\delta}}))^q dx.
\]
Thus,
\[
(1 - \delta) \liminf_{t \to \infty} \frac{\ln C_\mu(q, t)}{(q-1) \ln t} \geq -m_\mu^+(q),
\]
\[
(1 - \delta) \limsup_{t \to \infty} \frac{\ln C_\mu(q, t)}{(q-1) \ln t} \geq -m_\mu^-(q).
\]
Since $0 < \delta < 1$ is arbitrary, the complementary inequalities in (1) and (2) follow. The results are now a consequence of Remark 2.1. \hfill \Box

## 3 Lower bounds and fractal dimensions

In this section, our main goal is to prove Theorem 1.1. We begin investigating the existence of $G_\delta$ sets.

### 3.1 $G_\delta$ sets

**Proposition 3.1.** Let $T$ be a bounded self-adjoint operator on $\mathcal{H}$ and $q \in (0, 1)$. Then, for each $\gamma \geq 0$,

i) $G_{\gamma^-}^T := \{ \xi \in \mathcal{H} \mid D_{\mu_\xi}^- (q) \leq \gamma \}$ is a $G_\delta$ set in $\mathcal{H},$

ii) $G_{\gamma^+}^T := \{ \xi \in \mathcal{H} \mid D_{\mu_\xi}^+ (q) \geq \gamma \}$ is a $G_\delta$ set in $\mathcal{H}.
Proof. We just present the proof of item i). For each \( j \geq 1 \), let \( g_j : (0, \infty) \to (0, \infty), \)
\( g_j(t) := t^\frac{1}{j} + \gamma \). Since, for each \( j \geq 1 \) and each \( t > 0 \), the mapping
\[
\mathcal{H} \ni \xi \mapsto g_j(t)C_{\mu_\xi}^\mu T_\xi(q, t)^{1/q-1}
\]
is continuous (by dominated convergence), it follows that, for each \( j \geq 1 \) and each \( t \in (0, \infty) \),
the set
\[
\bigcup_{t \geq k} \{ \xi \in \mathcal{H} \mid g_j(t)C_{\mu_\xi}^\mu T_\xi(q, t)^{1/q-1} > n \}
\]
is open; thus, by Proposition 2.1,
\[
G_{T-}^\mu_\xi = \bigcap_{j \geq 1} \{ \xi \in \mathcal{H} \mid \limsup_{t \to \infty} g_j(t)C_{\mu_\xi}^\mu T_\xi(q, t)^{1/q-1} = \infty \}
\]
is a \( G_\delta \) set in \( \mathcal{H} \).

### 3.2 Generic minimal \( D_{\mu_\xi}^- (q) \) and maximal \( D_{\mu_\xi}^+ (q) \)

Next, we relate some spacing properties of the eigenvalues of self-adjoint operators with pure point spectrum to the generalized fractal dimensions of their spectral measures. The typical value of such dimensions (in Baire’s sense) is obtained if the eigenvalues of these operators are weakly-spaced (in the sense of Definition 1.1).

**Theorem 3.1.** Let \( T \) be a bounded self-adjoint operator with pure point spectrum. Suppose that there exists an enumeration of the eigenvalues of \( T \) which is weakly-spaced. Then, for each \( q \in (0, 1) \),
\[
\{ \xi \in \mathcal{H} \mid D_{\mu_\xi}^- (q) = 0 \text{ and } D_{\mu_\xi}^+ (q) = 1 \}
\]
is a dense \( G_\delta \) set in \( \mathcal{H} \).

**Proof.** Let \( (b_j) \subset \mathbb{C} \) be a sequence such that \( \sum_{j=1}^{\infty} |b_j|^{2q} < \infty \). Let \( (e_j) \) be an orthonormal family of eigenvectors of \( T \), that is, \( Te_j = \lambda_j e_j \) for every \( j \geq 1 \). Given \( \xi \in \mathcal{H} \), write \( \xi = \sum_{j=1}^{\infty} a_j e_j \), and then define, for each \( k \geq 1 \),
\[
\xi_k := \sum_{j=1}^{k} a_j e_j + \sum_{j=k+1}^{\infty} b_j e_j.
\]
It is clear that $\xi_k \to \xi$. Moreover, for each $k \geq 1$ and each $\epsilon > 0$,

$$
\int_{\text{supp}(\mu_{\xi_k}^T)} \mu_{\xi_k}^T(B(x, \epsilon))^{q-1} \, d\mu_{\xi_k}^T(x) = \sum_{j=1}^{\infty} \mu_{\xi_k}^T(B(\lambda_j, \epsilon))^{q-1} \mu_{\xi_k}^T(\{\lambda_j\}) \\
\leq \sum_{j=1}^{\infty} \mu_{\xi_k}^T(\{\lambda_j\}) = \sum_{j=1}^{k} |a_j|^{2q} + \sum_{j=k+1}^{\infty} |b_j|^{2q}, \quad \text{(4)}
$$

from which it follows that $D_{\nu_{\xi_k}}^+(q) = 0$. Hence, $G_{\nu_{\xi_k}}^+ = \{\xi \in \mathcal{H} \mid D_{\nu_{\xi_k}}^+(q) = 0\}$ is a dense set and, therefore, by Proposition 3.1, a dense $G_\delta$ set in $\mathcal{H}$.

Now, fix an $n \in \mathbb{N}$ so that $n > \frac{q}{1-q}$ and let $(\lambda_{j_1})$ be a subsequence of $(\lambda_j)$ so that: i) there exists a $C_n > 0$ such that, for every $l \geq 1$, $\lambda_{j_l} - \lambda_{j_{l+1}} \geq C_n/(1+\frac{1}{q})$; ii) $\lim_{l \to \infty}(\lambda_{j_l} - \lambda_{j_{l+1}}) = 0$ monotonically. Define, for each $k \geq 1$,

$$
\xi_{k} := \sum_{l=1}^{k} a_l e_l + \sum_{l=r(k)}^{\infty} \frac{1}{\sqrt{l^{1+\frac{1}{q}}}} e_{j_l},
$$

where we set $r(k)$ large enough so that $\{e_1, \ldots, e_k, e_{j_{r(k)}}, e_{j_{r(k)+1}}, \ldots\}$ is an orthonormal set. Again, $\xi_k \to \xi$ in $\mathcal{H}$.

For each $m \geq 1$, put $\epsilon_m := |\lambda_{j_m} - \lambda_{j_{m+1}}|/2$. Then, for each $m > M(k)$ and each $1 \leq l \leq m$,

$$
\mu_{\xi_k}^T(B(\lambda_{j_l}, \epsilon_m)) = \mu_{\xi_k}^T(\{\lambda_{j_l}\}),
$$

where $M(k)$ is large enough so that for each $m > M(k)$, each $l \geq 1$ and each $1 \leq i \leq k$, $\lambda_i \not\in B(\lambda_{j_l}, \epsilon_m)$. Hence, for $m > \max\{M(k), r(k)\} := s(k)$,

$$
\int_{\text{supp}(\mu_{\xi_k}^T)} \mu_{\xi_k}^T(B(x, \epsilon_m))^{q-1} \, d\mu_{\xi_k}^T(x) = \sum_{l=1}^{\infty} \mu_{\xi_k}^T(B(\lambda_l, \epsilon_m))^{q-1} \mu_{\xi_k}^T(\{\lambda_l\}) \\
\geq \sum_{l=s(k)}^{m} \mu_{\xi_k}^T(B(\lambda_{j_l}, \epsilon_m))^{q-1} \mu_{\xi_k}^T(\{\lambda_{j_l}\}) \\
= \sum_{l=s(k)}^{m} \mu_{\xi_k}^T(\{\lambda_{j_l}\}) q = \sum_{l=s(k)}^{m} \frac{1}{l^{1+\frac{1}{q}} q} \\
\geq B_k m^{1-(1+\frac{1}{q}) q} \geq B_k \left( \frac{C_n}{2 \epsilon_m} \right)^{1-(1+\frac{1}{q}) q/(1+\frac{1}{q})},
$$

where $B_k$ is a constant depending only of $k$, which results in

$$
D_{\nu_{\xi_k}}^+(q) \geq \frac{1-(1+\frac{1}{q}) q}{(1-q)(1+\frac{1}{m})} := t_{n,q}.
$$
Thus, $G_{(t_n,q)}^{T}$ is a dense set and, therefore, by Proposition 3.1, a dense $G_δ$ set in $H$. Since

$$G_{1}^{T} = \bigcap_{n>\frac{1}{q}} G_{(t_n,q)}^{T}$$

and $G_{1}^{T} = \{\xi \in H \mid D_{\mu_{1}^{T}}^{+}(q) = 1\}$ (see Remark 2.1), the result is proven.

3.3 Proof of Theorem 1.1

We need the following

Theorem 3.2 (Theorem 2.1 in [1]). Let $T$ be a bounded self-adjoint operator on $H$. Then, for each $\xi \in H$,

$$\alpha^{+}(\xi, p) \geq D_{\mu_{1}^{T}}^{+}\left(\frac{1}{1+p}\right)p.$$

Theorem 3.3. Let $T$ be a bounded self-adjoint operator with pure point spectrum. Suppose that there exists an enumeration of the sequence of the eigenvalues of $T$ which is weakly-spaced. Then,

$$G^{T} := \{\xi \in H \mid \alpha^{+}(\xi, p) = p \text{ for all } p > 0\}$$

is a dense $G_δ$ set in $H$.

Proof. Denote by $Q^{+} = \{x \in Q \mid x > 0\}$. Since, for each $\xi \in H$ and $p > 0$, $\alpha^{+}(\xi, p) \leq p$ (see [12]), and $D_{\mu_{1}^{T}}^{+}\left(\frac{1}{1+p}\right)$ increases monotonically with respect to $p$ (see [2]), it follows from Theorems 3.1 and 3.2 that

$$G^{T} = \bigcap_{r \in Q^{+}} \{\xi \in H \mid \alpha^{+}(\xi, r) = r\}$$

contains a dense $G_δ$ set in $H$.

Finally, the proof that $G^{T}$ is a $G_δ$ set follows closely the proof of Proposition 3.2 in [6]; we, therefore, omit it.

Proof (Theorem 1.1). It is a direct consequence of Theorem 3.3 applied to $P^{T}(\Lambda)H$.

A Appendix

Proposition A.1. Let $-\infty \leq a < b \leq \infty$. If $\cup_{j}\{a_{j}\}$ is a dense subset in $[a, b]$, then $(a_{j})$ is weakly-spaced.

Proof. Let $\alpha > 0$. Firstly, we note that, for each $x > 1$,

$$\left(\frac{x}{x-1}\right)^{\alpha} + \left(\frac{x}{x+1}\right)^{\alpha} > 2.$$  \hspace{1cm} (5)
Namely, set
\[ f(\alpha) := \left( \frac{x}{x-1} \right)^\alpha + \left( \frac{x}{x+1} \right)^\alpha. \]

So,
\[
\left( \frac{x-1}{x} \right)^\alpha f'(\alpha) = \ln \left( \frac{x}{x-1} \right) - \left( \frac{x-1}{x+1} \right)^\alpha \ln \left( \frac{x}{x-1} \right)
> \ln \left( \frac{x}{x-1} \right) \left( 1 - \left( \frac{x-1}{x+1} \right)^\alpha \right) > 0.
\]

Since \( f(0) = 2 \), the inequality in (5) follows.

Given \( a < c < b \), set, for each \( l \geq 1 \),
\[ b_l := \frac{1}{l^\alpha} + c. \]

So,
\[
\lim_{l \to \infty} l^{1+\alpha} (b_l - b_{l+1}) = \alpha. \tag{6}
\]

Moreover, by (5), for \( l \geq 2 \), \( b_{l-1} - 2b_l + b_{l+1} > 0 \). Now, for \( l \) sufficiently large such that \( b_l \in [a, b] \), pick \( a_{jl} \) satisfying
\[
0 \leq a_{jl} - b_l \leq \min \left\{ \frac{(b_{l-1} - 2b_l + b_{l+1})}{2}, \frac{\alpha}{4l^{1+\alpha}} \right\}. \tag{7}
\]

Then, by (6) and (7), for \( l \) sufficiently large, one has
\[
a_{jl} - a_{j+1} = (a_{jl} - b_l) - (a_{j+1} - b_{l+1}) + (b_l - b_{l+1})
\geq -\frac{\alpha}{4(l+1)^{1+\alpha}} + \frac{3\alpha}{4l^{1+\alpha}} \geq \frac{\alpha}{2l^{1+\alpha}}.
\]
\[
a_{jl} - a_{j+1} = (a_{jl} - b_l) - (a_{j+1} - b_{l+1}) + (b_l - b_{l+1})
\leq \frac{\alpha}{4l^{1+\alpha}} + \frac{7\alpha}{4l^{1+\alpha}} = \frac{2\alpha}{l^{1+\alpha}}.
\]

Hence,
\[
\frac{\alpha}{2l^{1+\alpha}} \leq a_{jl} - a_{j+1} \leq \frac{2\alpha}{l^{1+\alpha}}.
\]

Moreover,
\[
(a_{jl} - 2a_{j+1} + a_{j+2}) = a_{jl} - b_l - 2(a_{j+1} - b_{l+1}) + a_{j+2} - b_{l+2} + (b_l - 2b_{l+1} + b_{l+2})
\geq -2(a_{j+1} - b_{l+1}) + (b_l - 2b_{l+1} + b_{l+2}) \geq 0,
\]

which implies that \( a_{jl} - a_{j+1} \) goes to zero monotonically. Therefore, it follows that \( (a_j) \) is weakly-spaced. \( \square \)
Acknowledgments

M.A. was supported by CAPES (a Brazilian government agency). S.L.C. thanks the to partial support by FAPEMIG (a Brazilian government agency; Universal Project 001/17/CEX-APQ-00352-17).

References

[1] J.-M. Barbaroux, F. Germinet, and S. Tcheremchantsev, Fractal dimensions and the phenomenon of intermittency in quantum dynamics. Duke Math. J. 110 (2001), 161–194.

[2] J.-M. Barbaroux, F. Germinet, and S. Tcheremchantsev, Generalized fractal dimensions: equivalence and basic properties. J. Math. Pure et Appl. 80 (1997), 977–1012.

[3] J. Breuer, Y. Last, and Y. Strauss, Eigenvalue spacings and dynamical upper bounds for discrete one-dimensional Schrödinger operators. Duke Math. J. 157 (2011), 425–460.

[4] R. Carmona, A. Klein and F. Martinelli, Anderson Localization for Bernoulli and other Singular Potentials. Commun. Math. Phys. 108 (1987), 41–66.

[5] S. L. Carvalho and C. R. de Oliveira, Correlation Wonderland Theorems. J. of Math. Phys. 57 (2016), 063501.

[6] S. L. Carvalho and C. R. de Oliveira, Generic quasilocalized and quasiballistic discrete Schrödinger operators. Proc. Amer. Math. Soc. 144 (2016), 129–141.

[7] D. Damanik, Schrödinger operators with dynamically defined potentials. Ergod. Th. & Dynam. Sys. 37 (2017), 1681–1764.

[8] C. R. de Oliveira and R. A. Prado, Dynamical delocalization for the 1D Bernoulli discrete Dirac operator. J. Phys. A: Math. Gen. 38 (2005), L115–L119.

[9] C. R. de Oliveira and R. A. Prado, Quantum Hamiltonians with quasi-ballistic dynamics and point spectrum. Journal of Differential Equations 235 (2007), 85–100.

[10] R. del Rio, S. Jitomirskaya, Y. Last, and B. Simon B, Operators with singular continuous spectrum, IV: Hausdorff dimensions, rank one perturbations and localization. J. Anal. Math. 69 (1996), 153–200.

[11] E. I. Dinaburg, Stark effect for a difference Schrödinger operator. Theoret. and Math. Phys. 78 (1989), 50–57.
[12] F. Germinet and A. Klein, A characterization of the Anderson metal-insulator transport transition. Duke Math. J. 124 (2004), 309–350.

[13] I. Guarneri, Spectral properties of quantum diffusion on discrete lattices. Europhys. Lett. 10 (1989), 95–100.

[14] I. Guarneri and H. Schulz-Baldes, Lower bounds on wave-packet propagation by packing dimensions of spectral measures. Math. Phys. Elect. J. 5 (1999), 1–16.

[15] J. S. Howland, Perturbation Theory of Dense Point Spectra. J. Funct. Anal. 74 (1987), 52–80.

[16] Y. Last, Quantum dynamics and decomposition of singular continuous spectra. J. Funct. Anal. 142 (2001), 406–445.

[17] J. Pöschel, Examples of Discrete Schrödinger Operators with Pure Point Spectrum. Commun. Math. Phys. 88 (1983), 447–463.

[18] R. A. Prado, C. R. de Oliveira, and S. L. Carvalho, Dynamical localization for discrete Anderson Dirac operators. J. Stat. Phys. 167 (2017), 260–296.

[19] B. Simon, Absence of ballistic motion. Commun. Math. Phys. 134 (1990) 209–212.

[20] B. Simon, Operators with singular continuous spectrum: I. General operators. Ann. of Math. 141 (1995), 131–145.

[21] B. Simon, Some Schrödinger operators with dense point spectrum. Proc. Amer. Math. Soc. 125 (1997), 203–208.

[22] G. Stolz, An introduction to the mathematics of Anderson Localization, Entropy and the Quantum II. Contemp. Math. 552 (2011), 71–108.

[23] H. von Dreifus and A. Klein, A New Proof of Localization in the Anderson Tight Binding Model. Commun. Math. Phys. 124 (1989), 285–299.

[24] J. von Neumann, Charakterisierung des Spektrums eines Integraloperators. Actualité Sci. Indust. Paris, (1935).

[25] H. Weyl, Über beschränkte quadratische Formen, deren Differenz vollstetig ist, Rend. Circ. Mat. Palermo 27 (1909), 373–392.

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