On the Spectrum of the Resonant Quantum Kicked Rotor.

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It is proven that none of the bands in the quasi-energy spectrum of the Quantum Kicked Rotor is flat at any primitive resonance of any order. Perturbative estimates of bandwidths at small kick strength are established for the case of primitive resonances of prime order. Different bands scale with different powers of the kicking strength, due to degeneracies in the spectrum of the free rotor.

A. Introduction.

1. Background.

The Quantum Kicked Rotor (QKR) is a famous model in the study of quantum mechanical implications of classical chaotic dynamics [1, 2, 3]. Its discrete-time dynamics is generated in $L^2([0,2\pi])$ by the unitary propagator

$$\hat{U}_{\beta,\tau,\mu} = e^{-\frac{\tau}{2}(-i\frac{d}{d\theta}+\beta)^2} e^{-i\mu \cos(\theta)},$$

where $\mu$, $\beta$, and $\tau$ are real parameters, and periodic boundary conditions are understood for the momentum operator $-id/d\theta$. The QKR model owes much of its fame to dynamical localization, which is expected to occur whenever $\tau$ is sufficiently incommensurate to $2\pi$. Prior to lately established exact results [9, 10], this expectation has rested a long time on vast numerical evidence, and on formal assimilation of the QKR to a tight-binding model of motion in a disordered potential [2, 11]. The so-called QKR resonances [2, 4, 5, 6, 7, 8] occur in the opposite case, when $\tau$ is commensurate to $2\pi$, and appear to pose a simpler problem because the QKR dynamics is then assimilated to motion in a periodic potential. A band spectrum is thus expected of the propagator in (1), hence a non-empty absolutely continuous spectrum, provided one at least of the bands is not flat (i.e., reduced to a single,
infinitely degenerate eigenvalue). Precise definitions are given in Section A.2. Although band spectra at QKR resonances, and dynamical consequences thereof, were quite early discovered [4], very little is known about the band structure, which is largely determined by the arithmetics of Gaussian sums. In particular, though no instance is known of ”flat bands” besides the so-called ”anti-resonance” that is observed at order 1, their possible occurrence at higher orders has never been ruled out. Moreover, analysis has been mostly restricted to the ”traditional” KR model, which has $\beta = 0$. It was pointed out in [8] that the generalized KR model (1) presents a much richer family of resonances, and a distinction naturally arises between “primitive” and ”non-primitive” resonances, which are defined in Sect. A.2 below. Absence of flat bands is proven in this paper for primitive resonances of all orders.

Model (1) may be generalized by replacing $\cos(\theta)$ by some other ”kicking potential” $V(\theta)$. Resonances for such models have been studied in ref. [8]. In ref. [6] it was noted that flat bands at resonances are non-generic (in the sense of Baire) if $V(\theta)$ is chosen in the class of analytic potentials. Since in such models pure AC spectrum for a dense set of $\tau$ entails some continuous spectral component (presumably singular) whenever $\tau/2\pi$ is sufficiently rapidly approximated by rationals, the conclusion followed, that the latter property is generic in that class of models. The same property is now proven to hold for the straight KR model (1) itself as a byproduct of the present result.

Current knowledge of parametric dependence of bandwidths is mostly about the asymptotic regime $1 << \mu << q$, where bands are expected to be exponentially small. This behavior consistently matches the exponential localization of eigenfunctions in momentum space [2, 9], which is known to occur in the QKR model whenever $\tau/(2\pi)$ is sufficiently irrational (in the sense of rational approximation), and is heuristically understood [12] on the grounds of a similarity of the QKR to tight-binding models of Solid State physics[2]. In such models, following Thouless [13], bandwidth is related to conductance, so its exponential decay follows from Anderson localization.

Little is known about parametric dependence in other parameter ranges. The case of not too large order is in particular relevant for the investigation of the nearly-resonant dynamics, in a variant of the QKR model where peculiar transport phenomena are observed [14, 15]. For small $\mu$, the dynamics (1) is a perturbation of a free rotation. However, perturbative analysis of the band structure has nontrivial aspects, because the unperturbed quasi-energy spectrum is always degenerate at resonance, and the degree of degeneracy depends on the
arithmetics of the coprime integers $p, q$. The simplest case occurs when the divisor $q$ is prime, and is analyzed in this paper.

2. Statement of Results.

A KR resonance is said to occur whenever $\hat{U}_{\beta, \tau, \mu}$ commutes with a momentum translation $\hat{T}^Q$ ($Q$ a strictly positive integer), where $\hat{T}\psi(\theta) = e^{i\theta}\psi(\theta)$. The least such $Q$ may be termed the "length" of the resonance. Necessary and sufficient conditions for KR resonances are easily established:

**Proposition 1** (Dana, Dorofeev) $\hat{U}_{\beta, \tau, \mu}$ commutes with $\hat{T}^Q$ if, and only if,

(i) $\tau = 2\pi \frac{P}{Q}$ with $P$ integer,

(ii) $\beta = \frac{\nu}{P} + \frac{Q}{2} \mod(1)$, with $\nu$ an arbitrary integer.

The integer $Q$ will always be equal to the resonance length in this paper. The integer $q$ such that $P/Q = p/q$ with $p, q$ coprime integers will be termed the order of the resonance. A resonance will be termed "primitive" if its length is equal to its order, $Q = q$. At resonance, the Bloch index $\vartheta := \theta \mod(2\pi/Q)$ is conserved. A unitary map $b$ from $L^2([0, 2\pi])$ onto $L^2([0, 2\pi/Q]) \otimes \mathbb{C}^q$ is defined via:

$$(b\psi)_j(\vartheta) = \psi(\vartheta + 2\pi \frac{j-1}{Q}) , \quad 1 \leq j \leq Q , \quad \vartheta \in [0, 2\pi/Q].$$

Straightforward calculation shows that $b\hat{U}b^{-1}$ has a direct integral decomposition [4, 6, 16]:

$$b\hat{U}b^{-1} = \int_{[0, 2\pi/Q]} d\vartheta \; S(\vartheta),$$

that is, $(b\hat{U}\psi)(\vartheta) = S(\vartheta)(b\psi)(\vartheta)$, where $S(\vartheta)$ is the unitary $Q \times Q$ matrix, which has matrix elements:

$$S_{jk}(\vartheta) = e^{-i\mu \cos(\vartheta + 2\pi(j-1)/Q)} \; G_{jk},$$

$$G_{jk} = \frac{1}{Q} \sum_{s=0}^{Q-1} e^{2\pi is(j-k)/Q} \; a_{s+1},$$

$$a_r = e^{-i\pi p(r+\beta-1)^2/q} , \quad 1 \leq r \leq Q$$

Of all parameters $p, q, \beta, \mu, \vartheta$, only those which are strictly necessary at a given time will be specified when listing the arguments on which a quantity depends. The numbers $a_r$ in (4)
are the eigenvalues of the matrix $G = S(\vartheta, 0)$, which is defined in (3). The eigenvalues of $S(\vartheta, \mu)$ will be denoted $w_j = w_j(\vartheta, \mu)$ and numbered so that $w_j(\vartheta, 0) = a_j$, $(j = 1, 2, \ldots, Q)$. As $\vartheta$ varies in $(0, 2\pi/Q)$, each eigenangle $\text{arg}(w_j(\vartheta))$ sweeps a band in the quasi-energy spectrum of $\hat{U}$, that will be said to be "flat" in the case of a constant eigenangle.

Analysis of the band structure is easily performed in the case of primitive resonances with $Q = q = 1$, as they yield one band: $w_1(\vartheta, \mu) = e^{-i\pi p e^{\beta}} e^{-i\mu \cos(\vartheta)}$, which is not flat (if $\mu \neq 0$).

The non-primitive resonance of order $q = 1$ with $p = 1$, $Q = 2$, $\beta = 0$ is quite easily seen to yield flat bands and is known long since \[4\] under the denotation of "anti-resonance". Flatness of all bands has been proven for all non-primitive resonances of order 1 in ref.\[8\] and can be verified by direct calculation of the matrix (2).

Here it will be proven that:

**Theorem 1** No band is flat at any primitive resonance of any order, as long as $\mu \neq 0$. Hence, the spectrum of $\hat{U}_{\beta, \tau, \mu}$, $(\mu \neq 0)$, is purely absolutely continuous at primitive resonances.

Perturbative estimates of bandwidths will be proven for small $\mu$ in a special class of resonances. Perturbation theory is cumbersome, because the unperturbed ($\mu = 0$) spectrum is always degenerate, to an extent that depends on the arithmetics of $p$ and $q$. The simplest case is that of primitive resonances of prime order:

**Theorem 2** If $\tau = 2\pi \frac{p}{q}$, $q > 2$ is prime, $p$ is not a multiple of $q$, and $\beta = \frac{1}{2}$, then $a_r = a_{q-r+1}$ holds for $1 \leq r \leq q$ and all eigenvalues (4), except one, are twice degenerate. Asymptotically as $\mu \to 0$:

\[
\frac{d w_j(\vartheta, \mu)}{d \vartheta} \sim \mu^\alpha s_j(p, q) \sin(\vartheta),
\]

\[
\frac{q}{2} < \alpha_j = \max\{2j - 1, q - 2j + 1\} \leq q.
\]

There are positive constants $C, \gamma$ so that, asymptotically as $q \to \infty$:

\[
|s_j(p, q)| \lesssim C e^{-\gamma q + A_j(q)}, \quad A_j(q) = O\left(\sqrt{q \log^3(q)}\right).
\]

The present proof yields an estimate $\gamma \geq 0.0016$, which is hardly optimal. Numerical calculations, though of necessity restricted to small values of $q$, suggest an average exponent $\sim 0.6$. 
B. Proof of Thm. 1

In the following $\mu > 0$ is assumed, with no loss of generality. Lemma 1 establishes a necessary condition in order that an eigenvalue of $S(\vartheta, \mu)$ for fixed $\mu > 0$ may be constant for $\vartheta$ in a set of positive measure in $(0, 2\pi/Q)$. Lemma 2 says that this condition cannot be met at any primitive resonance.

**Lemma 1** Let $\tau$ and $\beta$ be as in Prop. 1, and let $d$ denote the integer part of $\frac{Q+1}{2}$. If $\mu > 0$ is fixed and the matrix $S(\vartheta, \mu)$ has an eigenvalue which is constant for $\vartheta$ in a subset of positive measure of $(0, 2\pi/Q)$, then the matrix $G^{(d)} := \{G_{jk}\}_{1 \leq j,k \leq d}$ is singular.

**Proof:** eqn. (2) defines a matrix-valued function on the unit circle $\rho = 1$ in the plane of the complex variable $z = \rho e^{i\vartheta}$. This function is analytically continued to $\mathbb{C} \setminus \{0\}$, in the form $S(z) = Z(z)G$, where $G$ is as in (3), and $Z(z)$ is a diagonal matrix, with diagonal elements given by:

$$Z_{jj}(z) = e^{-i\frac{\mu}{2}(z\xi^{j-1}z^{-1}\xi^{j-1})} = e^{-i\frac{\mu}{2}(\rho+\rho^{-1})\cos(2\pi(j-1)/Q+\vartheta)}e^{i\frac{\mu}{2}(\rho-\rho^{-1})\sin(2\pi(j-1)/Q+\vartheta)}.$$  \hspace{1cm} (7)

where $\xi := e^{-2\pi i/Q}$. Whenever $\vartheta$ is not a multiple of $\pi/Q$, this matrix is hyperbolic. In particular, for $z_\rho := \rho e^{i\frac{\tau}{Q}}$ and $\rho > 1$ the expanding subspace of $Z(z_\rho)$ has dimension $d$ given by the integer part of $\frac{Q+1}{2}$. Vectors $x$ in the expanding subspace have components $x_j = 0$ for $d < j \leq Q$ and projection onto such vectors will be denoted $P$.

Let $w$, $(|w| = 1)$, be a constant eigenvalue of $S(z)$ for all $z$ in a subset of positive measure of the circle $\{|z| = \rho = 1\}$. For any fixed value $\lambda \in \mathbb{C}$ the characteristic polynomial $\mathcal{P}(z, \lambda) := \det(S(z) - \lambda)$ is an analytic function of $z$ in $\mathbb{C} \setminus \{0\}$. This is in particular true for $\lambda = w$, and then analyticity forces $\mathcal{P}(z, w)$ to vanish at all points $z$ in $\mathbb{C} \setminus \{0\}$. Thus $\forall \rho > 1$ $w$ is a unimodular eigenvalue of $S(z_\rho)$, and there exists $x_\rho \in \mathbb{C}^Q$ with $||x_\rho|| = 1$ such that $S(z_\rho)x_\rho = wx_\rho$. Then $||PS(z_\rho)x_\rho|| = ||Px_\rho|| \leq 1$, whence, using (7):

$$1 \geq ||PS(z_\rho)x_\rho|| = ||Z(z_\rho)PGx_\rho|| \geq e^{c(\rho-\rho^{-1})}||PGx_\rho||,$$ \hspace{1cm} (8)

where $c := \mu \sin(\pi/(2Q))$. Similarly, from $||P^\perp S(z_\rho)x_\rho|| = ||P^\perp x_\rho||$ it follows that

$$||P^\perp x_\rho|| \leq e^{-c(\rho-\rho^{-1})}||P^\perp Gx_\rho|| \leq e^{-c(\rho-\rho^{-1})}.$$ \hspace{1cm} (9)
As \( \rho \) is arbitrarily large, (8) and (9) imply that there is a \( \mathbf{y} \) of unit norm, such that \( \mathbf{Py} = \mathbf{y} \) and \( \mathbf{PGy} = 0 \). As the range of \( \mathbf{P} \) is the expanding subspace of dimension \( d \), the claim is proven. \( \Box \).

**Lemma 2** Let \( \tau, \beta \) be as in Prop. 1 and let \( d \) and \( \mathbf{G}^{(d)} \) be as in Lemma 1. If \( Q = q \), i.e. \( P \) and \( Q \) are coprime, then the matrix \( \mathbf{G}^{(d)} \) is not singular.

**Proof:** For \( q = 1 \) the claim is obvious, so \( q > 1 \) will be assumed. For \( 1 \leq j \leq q \) let \( \mathbf{e}^{(j)} \) the vector in \( \mathbb{C}^q \) whose \( r \)-th component is \( e^{(j)}_r = \xi^{(j-1)(r-1)} \), where \( \xi := e^{-2\pi i/q} \neq 1 \). Let \( \mathbf{y} = (y_1, \ldots, y_d) \in \mathbb{C}^d \) satisfy \( \mathbf{G}^{(d)} \mathbf{y} = 0 \), and \( \mathbf{y} \neq 0 \). From Definition (3) it follows that the vector \( \mathbf{b} \in \mathbb{C}^q \) which is defined by:

\[
\mathbf{b}_r = a_r \sum_{k=1}^{d} y_k e^{(k)}_r , \quad 1 \leq r \leq q
\]

is orthogonal to \( \mathbf{e}^{(j)} \) for \( 1 \leq j \leq d \). As vectors \( \{\mathbf{e}^{(j)}\}_{1 \leq j \leq q} \) are a basis in \( \mathbb{C}^q \), \( \mathbf{b} \) must be a combination of vectors \( \mathbf{e}^{(j)} \) with \( d + 1 \leq j \leq q \); and so, constants \( u_k, (1 \leq k \leq q - d) \) exist, so that

\[
a_r \sum_{k=1}^{d} y_k \xi^{(k-1)(r-1)} = \xi^{d(r-1)} \sum_{k=1}^{q-d} u_k \xi^{(k-1)(r-1)} , \quad 1 \leq r \leq q .
\]

Defining polynomials \( F(z) = \sum_1^d y_k z^{k-1} \) and \( G(z) = \sum_1^{q-d} u_k z^{k-1} \), eqn. (10) may be rewritten as:

\[
G(\xi^{r-1}) = c_r F(\xi^{r-1}) , \quad c_r := a_r \xi^{d(1-r)} , \quad 1 \leq r \leq q .
\]

Replacing \( r \) with \( r + s \) in this equation one obtains:

\[
c_{r+s} G(\xi^{r-1}) F(\xi^{r+s-1}) = c_r G(\xi^{r+s-1}) F(\xi^{r-1}) .
\]

Then, using (11) and (11):

\[
\frac{c_{r+s}}{c_r} = e^{i\pi \frac{s}{q}(2d-ps-2p\beta)} e^{-2\pi is(r-1)\frac{2}{q}} ,
\]

Since \( p, q \) are mutually prime, \( s \) may be chosen such that \( ps = 1 \mod (q) \), and then:

\[
\frac{c_{r+s}}{c_r} = e^{i\gamma} \xi^{r-1} ,
\]

where \( \gamma \) is independent of \( r \). Denoting \( \lambda := \xi^s \), from (12) it follows that

\[
e^{i\gamma} z G(z) F(\lambda z) - G(\lambda z) F(z) = 0
\]
whenever \( z \) is a \( q \)-th root of unity, \( i.e. \ z = \xi^{r-1}, 1 \leq r \leq q \). By definition of \( F(z) \) and \( G(z) \), the polynomial on the lhs in (14) has degree at most \( q - 1 \) and so it must identically vanish. Hence, \( \forall z \):

\[
e^{i\gamma} z \ G(z) \ F(\lambda z) = G(\lambda z) \ F(z) .
\]

As \( G(z)F(\lambda z) \) and \( G(\lambda z)F(z) \) have the same degree, they must vanish in order that this equation be satisfied \( \forall z \). This implies that either \( F(z) = 0 \) or \( G(z) = 0 \) identically. The latter case implies the former because then \( F(z) \) has to vanish at the \( q \)-th roots of unity due to eqn. (11), and its degree is \( \leq d - 1 < q \). Hence \( F(z) \) vanishes identically in all cases, in contradiction to \( y \neq 0 \). \( \square \)

C. Proof of Thm.2.

1. Perturbation Theory.

For the purposes of this section it is convenient to unitarily transform the matrix \( S(\theta, \mu) \) to \( X(\theta, \mu) := F(\theta)^{-1}G(\vartheta/q, \mu)G^{-1}F(\theta) \), where the unitary matrix \( F \) is defined by the matrix elements:

\[
F_{jk}(\theta) = \frac{1}{\sqrt{q}} e^{-i(j-1)\theta/q} e^{-2\pi i (j-1)(k-1)} , \quad 1 \leq j, k \leq q .
\]

Straightforward calculation shows that :

\[
\hat{X}(\vartheta, \mu) = \hat{C} e^{-i\mu \hat{V}(\vartheta)} .
\]

where

\[
\hat{C} = \sum_{j=1}^{q} e^{-\pi i (j-1/2)^2/q} |j\rangle \langle j| ,
\]

\[
\hat{V}(\vartheta) = \frac{1}{2} \sum_{j=1}^{q-1} \left( \langle j | j+1 \rangle + \langle j+1 | j \rangle \right) + \frac{1}{2} \left( \langle 1 | q \rangle e^{i\vartheta} + \langle q | 1 \rangle e^{-i\vartheta} \right) ;
\]

Dirac notations are used, and \( \{ |j\rangle , \ j = 1, 2, \ldots, q \} \) denotes the canonical basis in \( \mathbb{C}^q \).

Let \( \gamma \) be a circular path in the complex plane, counterclockwise oriented, centered at an unperturbed eigenvalue \( a_j \), and containing no other unperturbed eigenvalue. For sufficiently small \( \mu < 2\pi/q \), this circle will contain no perturbed eigenvalue, except those which were born of \( a_j \). Then

\[
\Pi(\gamma, \theta, \mu) = -\frac{1}{2\pi i} \int_{\gamma} dz \ (X(\theta, \mu) - z)^{-1}
\]

(19)
is projection onto the eigenspace of $\mathbf{X}$ which corresponds to these eigenvalues. Consequently,
\[
\Pi(\gamma, \theta, \mu) \mathbf{X}(\theta, \mu) \Pi(\gamma, \theta, \mu) = -\frac{1}{2\pi i} \int_{\gamma} dz \left( \mathbf{X}(\theta, \mu) - z \right)^{-1}.
\]  
(20)

We denote
\[
\mathbf{R}(z) = [\mathbf{C} - z\mathbf{I}]^{-1}, \quad \mathbf{L}(z) = \mathbf{R}(z)\mathbf{C}.
\]

For sufficiently small $\mu$, the resolvent in (19) may be expanded as follows:
\[
\begin{align*}
(X(\theta, \mu) - z)^{-1} &= \left[ \mathbf{I} + L(z)(e^{-i\mu V(\theta)} - \mathbf{I}) \right]^{-1} \mathbf{R}(z) \\
&= \mathbf{R}(z) + \sum_{n=1}^{\infty} (-1)^n \left[ L(z)(e^{-i\mu V(\theta)} - \mathbf{I}) \right]^n \mathbf{R}(z)
\end{align*}
\]
(21)

Next we expand the exponential in powers of $\mu$. Let $\Omega_{n,\ell}$ denote the set of vectors $\overrightarrow{r}(\ell) \in \mathbb{N}^n$ whose components $r_1, \ldots, r_n$, are strictly positive integers that satisfy $|\overrightarrow{r}(\ell)| = r_1 + \ldots + r_n = \ell$. For $\overrightarrow{r}(\ell) \in \Omega_{n,\ell}$ let
\[
\mathbf{P}(\theta, z, \overrightarrow{r}(\ell)) = \mathbf{L}(z)V(\theta)^{r_1} \mathbf{L}(z)V(\theta)^{r_2} \ldots \mathbf{L}(z)V(\theta)^{r_n} \mathbf{L}(z)\mathbf{C}^{-1}.
\]
(22)

With such notations, one may expand the resolvent in eqn. (19) in powers of $\mu$ as follows:
\[
(X(\theta, \mu) - z)^{-1} = \mathbf{R}(z) + \sum_{\ell=1}^{\infty} (-i)^\ell \mu^\ell \mathbf{Q}_\ell(\theta, z),
\]
(23)

where:
\[
\mathbf{Q}_\ell(\theta, z) = \sum_{n=1}^{\ell} (-1)^n \sum_{\overrightarrow{r}(\ell) \in \Omega_{n,\ell}} \frac{1}{r_1!r_2!\ldots r_n!} \mathbf{P}(\theta, z, \overrightarrow{r}(\ell)).
\]
(24)

Replacing (23) in (19) yields the expansion of the spectral projector (19) in powers of $\mu$. Matrix elements of the operator $\mathbf{P}$ between arbitrary states $|j\rangle$ and $|k\rangle$ may be written as sums over paths, specified by strings $\mathbf{m} \equiv (m_0, \ldots, m_\ell)$ of integers taken from $\{1, 2, \ldots, q\}$, such that $m_0 = j$ and $m_\ell = k$. Moreover, from the explicit form of $\mathbf{V}$ it is apparent that only those paths contribute, which jump by $\pm 1$(mod $q$) at each step. The set of such paths will be denoted by $\Lambda(\ell, j, k)$. For $\mathbf{m} \in \Lambda(\ell, j, k)$, let the integer $\nu(\mathbf{m})$ count the number of jumps $q \rightarrow 1$, minus the number of jumps $1 \rightarrow q$. Then, using (18):
\[
\begin{align*}
\langle j | \mathbf{P}(\theta, z, \overrightarrow{r}(\ell)) | k \rangle &= \sum_{\mathbf{m} \in \Lambda(\ell, j, k)} G(\mathbf{m}, \overrightarrow{r}(\ell), z) \times \\
	imes \langle m_0 | \mathbf{V}(\theta) | m_1 \rangle \langle m_1 | \mathbf{V}(\theta) | m_2 \rangle \ldots \langle m_{\ell-1} | \mathbf{V}(\theta) | m_\ell \rangle \\
&= 2^{-\ell} \sum_{\mathbf{m} \in \Lambda(\ell, j, k)} G(\mathbf{m}, \overrightarrow{r}(\ell), z) e^{i\nu(\mathbf{m})\theta},
\end{align*}
\]
(25)
where
\[
G(m, \overline{r}^{(n)}, z) = g(m_0, z) g(m_{r_1}, z) g(m_{r_1+r_2}, z) \ldots g(m_\ell, z) h(m_\ell),
\]
\[
g(m, z) = a_m/(a_m - z), \quad h(m) = a_m^{-1}.
\]
From (25) the definitions of \(L, V, \) and \(C\) it follows that:
\[
\langle j | P(\theta, z, \overline{r}^{(n)}) | k \rangle = \langle k | P(2\pi - \theta, z, \overline{r}^{(n)}) | j \rangle
\]
where \(\overline{r}^{(n)}\) is the reverse of \(\overline{r}^{(n)}\), that is, \(r'_j = r_{n+1-j}, (1 \leq j \leq n)\).

2. Degenerate case.

Let \(j < (q + 1)/2\), so that \(a_j = a_j'\) \((j' = q - j + 1)\). Let
\[
|j_+\rangle = \frac{1}{\sqrt{2}} \Pi(\gamma, \theta, \mu) (|j\rangle + |j'\rangle), \quad |j_-\rangle = \frac{1}{\sqrt{2}} \Pi(\gamma, \theta, \mu) (|j\rangle - |j'\rangle);
\]
Gram-Schmidt orthonormalization on \(\{|j_\pm\rangle\}\) yields orthonormal vectors \(\{\tilde{j}_\pm\rangle\}\) and we shall calculate the corresponding \(2 \times 2\) matrix of \(X(\theta, \mu)\). To this end we shall compute matrix element of the operators \(\Pi(\gamma, \theta, \mu)\) and \(\Pi(\gamma, \theta, \mu)X\Pi(\gamma, \theta, \mu)\) using eqs.(19) and (20), along with expansion (23), (24), (25), (26). Such elements may be written as the sum of a \(\theta\)-independent part (given by the average over \(\theta\)), plus a \(\theta\)-dependent part. The former part is determined by paths \(m\) with index \(\nu(m) = 0\). The shortest such path from \(j\) to \(j'\) is:
\[
\tilde{m}_j := (j, j + 1, \ldots, j' - 1, j'), \quad \text{of length} \ q - 2j + 1.
\]
The latter part is determined by paths \(m\) with index \(\nu(m) \neq 0\), and the shortest such path from \(j\) to \(j'\) is
\[
\tilde{m}_j := (j, j - 1, \ldots, 1, q - 1, \ldots, j'), \quad \text{of length} \ 2j - 1.
\]
with \(\nu(\tilde{m}_j) = 1\). On the other hand, the shortest loops with \(\nu \neq 0\) from \(j\) to \(j\) (or from \(j'\) to \(j'\)) have length \(q\). There are 2 such loops, reverse of each other. Hence we may write:
\[
\langle j | \Pi(\gamma, \theta, \mu) | j \rangle \sim 1 + D_j(\mu) + \mu^q c_j \cos(\theta) + o(\mu^q),
\]
\[
\langle j | \Pi(\gamma, \theta, \mu) | j' \rangle \sim F_j(\mu) + \mu^{2j-1} u_j e^{i\theta} + o(\mu^{2j-1}),
\]
\[
\langle j | \Pi(\gamma, \theta, \mu)X \Pi(\gamma, \theta, \mu) | j \rangle \sim a_j + G_j(\mu) + \mu^q b_j \cos(\theta) + o(\mu^q),
\]
\[
\langle j | \Pi(\gamma, \theta, \mu)X \Pi(\gamma, \theta, \mu) | j' \rangle \sim H_j(\mu) + \mu^{2j-1} v_j e^{i\theta} + o(\mu^{2j-1}),
\]
where $D_j(\mu), F_j(\mu), G_j(\mu)$ and $H_j(\mu)$ are independent of $\theta$, and what follows on their right is the $\theta$-dependent parts. To leading orders,

$$D_j(\mu) \sim d_j \mu^2, \quad F_j \sim f_j \mu^{q-2j+1}, \quad G_j \sim g_j \mu^2, \quad H_j \sim h_j \mu^{q-2j+1}. \quad (34)$$

Coefficients $f_j$ and $h_j$ are computed by picking the contribution of path (28) from eqn. (25), by inserting it in eq. (24) and then in eq. (23), and finally integrating along $\gamma$. Coefficients $u_j$ and $v_j$ are computed in a similar way, using path (29); and coefficients $d_j$ and $g_j$ are determined by loops of length 2. Using (30) and (34) we find:

$$\langle j_\pm | \Pi(\gamma, \theta, \mu) | j_\pm \rangle \sim 1 + D_j(\mu) \pm F_j(\mu) \pm u_j \mu^{2j-1} \cos(\theta), \quad (35)$$

and, similarly,

$$\langle j_\pm | \Pi(\gamma, \theta, \mu) X \Pi(\gamma, \theta, \mu) | j_\pm \rangle \sim a_j + G_j(\mu) \pm H_j(\mu) \pm \mu^{2j-1} v_j \cos(\theta), \quad (36)$$

$$\langle j_\pm | \Pi(\gamma, \theta, \mu) X \Pi(\gamma, \theta, \mu) | j_\mp \rangle \sim \mp i \mu^{2j-1} v_j \sin(\theta). \quad (37)$$

Using all the above formulae, one computes

$$\langle \tilde{j}_\pm | X | \tilde{j}_\pm \rangle \sim K_{j\pm}(\mu) \pm \mu^{2j-1} t_j \cos(\theta),$$

$$\langle \tilde{j}_\pm | X | \tilde{j}_\mp \rangle \sim \mp i \mu^{2j-1} t_j \sin(\theta), \quad (38)$$

where:

$$K_{j\pm}(\mu) = (a_j + G_j(\mu) \mp H_j(\mu))(1 - D_j(\mu) \mp F_j(\mu)),$$

$$t_j = v_j - a_j u_j. \quad (39)$$

Let $\xi_j^\pm = \xi_j^\pm(\theta, \mu)$ denote the eigenvalues of the matrix in (38), labeled so that $w_j(\theta, \mu) \sim \xi_j^+(\theta, \mu)$ and $w_j'(\theta, \mu) \sim \xi_j^-(\theta, \mu)$ as $\mu \to 0$.

$$\frac{d \xi_j^\pm}{d \theta} \sim \mp \mu^{2j-1} t_j \Delta_j(\mu) \frac{\mu^{q-2j+1}}{\xi_j^+ - \xi_j^-} \sin(\theta), \quad (40)$$

where, in the leading order,

$$\Delta_j(\mu) = K_{j+}(\mu) - K_{j-}(\mu) \sim 2(h_j - a_j f_j) \mu^{q-2j+1}. \quad (41)$$

as follows from (31) and (39). Moreover,

$$\xi_j^+ - \xi_j^- = \sqrt{\Delta_j(\mu)^2 + 4 \mu^{2j-1} t_j \Delta_j(\mu) \cos(\theta) + 4 \mu^{4j-2} t_j^2}. \quad (42)$$
On account of (41), the leading term in (42) is $\Delta_j(\mu)$ whenever $2j - 1 > q - 2j + 1$, and is instead $2t_j\mu^{2j-1}$, in the opposite case. Replacing in eqn.(40) we find that

$$\frac{d\xi_j^\pm}{d\theta} \sim \mp \mu^{\alpha_j} s_j \sin(\theta),$$

where $\alpha_j$ is as in (5), and

$$s_j = \begin{cases} v_j - a_j u_j, & \text{if } 2j - 1 > q - 2j + 1; \\ h_j - a_j f_j, & \text{if } 2j - 1 < q - 2j + 1. \end{cases}$$

### 3. Nondegenerate case.

In the case when $j = (q + 1)/2$, $a_j$ is a nondegenerate eigenvalue for $\mu = 0$, and then Eqs. (30) and (32) lead to:

$$w_j(\theta, \mu) = \langle j|\Pi(\gamma, \theta, \mu)X\Pi(\gamma, \theta, \mu)|j\rangle \sim a_j + G_j(\mu) - a_j D_j(\mu) + \mu^q s_j \cos(\theta),$$

where $s_j = b_j - a_j c_j$.

### 3. Proof of Estimate (6)

From eqs.(25),(26), (30), and (44) it follows that:

$$s_j = \frac{1}{2\pi} \frac{1}{2^{\alpha_j}} \sum_{n=1}^{\alpha_j} (-1)^{n+j} \sum_{r^{(n)}} \frac{1}{r^{(n)}!} \int_\gamma dz (z - a_j) G(m_j, r^{(n)}, z),$$

where $r^{(n)}!$ is shorthand for $r_1! \ldots r_n!$, $\alpha_j$ is given in eqn.(5), and $m_j$ is either the path $\overrightarrow{m}_j$ (when $j < (q + 2)/4$), or the path $\overleftarrow{m}_j$ (when $j \geq (q + 2)/4$). The following proof is for the former path and is split in Lemmata 3 and 4 below. The proof for the latter path is essentially identical, apart from notations.

**Lemma 3**: There are constants $C > 0$, $\gamma > 0$ so that

$$|s_j| \lesssim C q^{1/2} e^{-\gamma q} \prod_{l=1}^{\alpha_j-1} \frac{1}{|a_j + l a_j^* - 1|}.$$  \hfill (47)

**Proof**: using definition (26), and Cauchy’s Integral formula,

$$\frac{1}{2\pi i} \int_\gamma dz (z - a_j) G(\overrightarrow{m}_j, r^{(n)}, z) = -\frac{a_j a_{j+1} \ldots a_{j+l_n-1}}{(a_j + l_1 - a_j) \ldots (a_j + l_{n-1} - a_j)},$$

\hfill (48)
where $l_s = r_1 + \ldots + r_s$. Replacing this in (46),

$$|s_j| \leq \frac{1}{2^{\alpha_j}} \sum_{n=1}^{\alpha_j} \Xi_n,$$

where:

$$\Xi_n := \sum_{\bar{r}^{(n)} \in \Omega(n, \alpha_j)} \bar{g}_n(j, \bar{r}^{(n)}) \text{ and } G_n(j, \bar{r}^{(n)}) := \frac{1}{\bar{r}^{(n)}} \prod_{s=1}^{n-1} \left| \bar{a}_{n+s} a_j^s - 1 \right|,$$  

If $\bar{r}^{(n)} \in \Omega(n, \alpha_j)$ and $n < \alpha_j$, then there is $s' \in \{1, \ldots, n\}$ such that $r_{s'}$ is the sum of two positive integers $t', t''$. Define $\bar{r}^{(n+1)} := (r_1, \ldots, r_{s'-1}, t', t'', r_{s'+1}, \ldots, r_n) \in \Omega(n+1, \alpha_j)$, and $\omega^{(n+1)} = a_j + l_{s'-1} + r a_j^s$. Then

$$G_{n+1}(j, \bar{r}^{(n+1)}) = G_n(j, \bar{r}^{(n)}) \left( \frac{r_{s'}}{t'} \right) \left| \frac{1}{\omega^{(n+1)} - 1} \right| \geq \frac{2}{\left| \omega^{(n+1)} - 1 \right|} G_n(j, \bar{r}^{(n)}),$$

Repeating the argument, a sequence $\omega^{(n+1)}, \ldots, \omega^{(\alpha_j-1)}, \omega^{(\alpha_j)}$ can be constructed, so that

$$G_n(j, \bar{r}^{(n)}) \leq 2^{n-\alpha_j} G_{\alpha_j}(j, \bar{r}^{(\alpha_j)} \prod_{l=n+1}^{\alpha_j} \left| \omega^{(l)} - 1 \right|,$$  

where $\bar{r}^{(\alpha_j)} = (1, 1, \ldots, 1)$ is the one vector in $\Omega(\alpha_j, \alpha_j)$. Denote $P(j, n, \bar{r}^{(n)})$ the product on the rhs in (52). All numbers $\omega^{(l)}$ are $q$-th roots of unity, so $P(j, n, \bar{r}^{(n)})$ is a product of diagonals in the regular $q$-gon inscribed in the unit circle in the complex plane, drawn from the vertex at 1. The largest possible value of such a product is attained, if the vertices $\neq 1$ of the diagonals are taken as close as possible to $-1$; keeping in mind, that each diagonal may occur twice, and at most twice, in the product, due to the symmetry $a_j = a_{q-j+1}$; and that two different diagonals may have the same length, because for odd $q$ the regular $q$-gon is symmetric with respect to any diameter drawn through one of the vertices. It follows that

$$P(j, n, \bar{r}^{(n)}) < 2^{\alpha_j-n} \prod_{m=1}^{L(j,n)} \cos^4((m-1/2)\pi/q),$$

where $L(j, n) := \text{Int}((\alpha_j - n)/4)$; therefore,

$$P(j, n, \bar{r}^{(n)}) \lesssim 2^{\alpha_j-n} C_1 e^{q F(4L(j,n)/q)},$$

where $C_1 > 0$ is a numerical constant, and

$$F(x) = \int_0^x dt \log(\cos(\pi t/4)).$$
From (49), (50), (52), and (54):

\[ |s_j| < C_3 G_{\alpha_j}(j, \bar{r}^{(\alpha_j)}) \sum_{n=1}^{\alpha_j} \frac{1}{2 \alpha_j} \left( \alpha_j - 1 \right) e^{qF(4L(j,n)/q)} , \]

because there are \( \binom{\alpha_j - 1}{n - 1} \) elements in \( \Omega(n, \alpha_j) \). The sum on the rhs in (56) is an average over a binomial distribution. Since \( \alpha_j > q/2 \) (see eqn.(5)), as \( q \to \infty \) this distribution approaches a Gaussian distribution with mean \( \alpha_j/2 \) and variance \( \alpha_j/4 \), so, replacing \( (\alpha_j - n)/q \) by a continuous variable \( x \), and using again that \( q > \alpha_j > q/2 \),

\[ |s_j| \approx C_2 q^{1/2} G_{\alpha_j}(j, \bar{r}^{(\alpha_j)}) \int_{0}^{1} dx e^{q[-2(x - \lambda_j)^2 + F(x)]} , \]

where \( \lambda_j = \alpha_j/(2q) \), and so \( 1/4 < \lambda_j < 1/2 \). Analysis of the function \(-2(x - \lambda)^2 + F(x)\) in \((x, \lambda) \in [0, 1] \times (1/4, 1/2)\) yields the upper bound \(-\gamma \approx -0.0016\). This proves the Lemma, because \( G_{\alpha_j}(j, \bar{r}^{(\alpha_j)}) \) in (57) is just the product which appears in (47). □

We are left with estimating the product on the rhs of (47):

**Lemma 4** For \( q \to \infty \),

\[ \left| \sum_{l=1}^{\alpha_j - 1} \log \left( |a_j + a_j^*| - 1 \right) \right| = O \left( \sqrt{q \log^3(q)} \right) . \]

**Proof:** define:

\[ S(\phi, \rho) := \log(|1 - \rho e^{i\phi}|) = \sum_{N \in \mathbb{Z}} \sigma_N(\rho) e^{iN\phi} , \quad (0 < \rho \leq 1 , 0 \leq \phi \leq 2\pi) , \]

where [17]:

\[ \sigma_N(\rho) = -\frac{\rho^{|N|}}{|N|} \quad (N \neq 0) , \quad \sigma_0(\rho) = 0 \]

From eq.(58):

\[ \log \left( \prod_{l=1}^{\alpha_j - 1} |\rho a_j + a_j^*| - 1 \right) = \sum_{N \in \mathbb{Z}} \sigma_N(\rho) \sum_{l=1}^{\alpha_j} a_j^N a_j^{N*} . \]

We write the sum over \( N \) as \( \Sigma' + \Sigma'' \), where \( \Sigma' \) is the sum restricted to \( N \in q\mathbb{Z} \). In order to estimate \( \Sigma' \) and \( \Sigma'' \) standard facts about sums of the Gauss type are used. These are reviewed in Lemma 5 below. Using that \( \alpha_j \leq q \), and \( \sigma_N < 0 \) for \( N \neq 0 \),

\[ |

\begin{align*}
\Sigma'' &= \left| \sum_{l=1}^{\alpha_j - 1} \sigma_N(\rho) \sum_{l=1}^{\alpha_j} a_j^N a_j^{N*} \right| \\
&\leq -\sqrt{2q \log(q)} \sum_{l=1}^{\alpha_j} \sigma_N(\rho) \\
&\leq -\sqrt{2q \log(q)} \cdot S(0, \rho) = -\log(1 - \rho) \cdot \sqrt{2q \log(q)} .
\end{align*}
\]
Using Lemma 2, and (59), the remaining sum is estimated as:

\[ |\Sigma'| \leq -q \sum_{n \in \mathbb{Z}} \sigma_{nq}(\rho) = -2 \log(1 - \rho^q). \]  

(62)

Finally, for \( z \) complex with \( |z| = 1, \) and \( 0 < \rho < 1, \)

\[ |\log(|1 - \rho z|) - \log(|1 - z|)| \leq \frac{1 - \rho}{|1 - z|}, \]

whence it follows that:

\[
\left| \log \left( \prod_{l=1}^{\alpha_j} |a_{j+l}^* - 1| \right) - \log \left( \prod_{l=1}^{\alpha_j} |\rho a_{j+l}^* - 1| \right) \right| \leq (1 - \rho) \sum_{l=1}^{\alpha_j} \frac{1}{|a_{j+l}^* - 1|} < 16q(1 - \rho) \sum_{r=1}^{q} r^{-1} < 16q(1 - \rho) \log(2q). \]  

(63)

and so

\[
\left| \log \left( \prod_{l=1}^{\alpha_j} |a_{j+l}^* - 1| \right) \right| \leq \Sigma' + \Sigma'' + 16q(1 - \rho) \log(2q); \]

The proof is concluded on substituting (61) and (62) in the last estimate, and choosing \( 1 - \rho = 1/(q \log(q)) \). □

**Lemma 5** Let \( q \) prime, \( p \) not a multiple of \( q, \) \( N \in \mathbb{Z}, \) and \( T \) an integer \( \leq q. \) Then:

\[
\left| \sum_{n=j}^{j+T-1} a_n^N \right|^2 \begin{cases} 
= T^2, & \text{if } N \text{ is a multiple of } q; \\
= q, & \text{if } N \text{ is not a multiple of } q, \text{ and } T = q; \\
\leq 2q(1 + \log(q)), & \text{otherwise.} 
\end{cases}
\]

The 1st claim is obvious and the 2nd is well known. If \( N \) is not a multiple of \( q, \) and \( T < q, \) then:

\[
\left| \sum_{n=j}^{j+T-1} a_n^N \right|^2 = \left| \sum_{r=0}^{T-1} \sum_{s=0}^{T-1} e^{i\pi p N(r-s)/q} e^{i\pi p N(r-s)(r+s)/q} \right| 
= T + 2\Re \sum_{h=1}^{T-1} e^{i\pi p N[h^2-h]/q} \sum_{s=0}^{T-1-h} e^{2i\pi ps Nh/q} 
\leq T + 4 \sum_{h=1}^{T-1} \frac{1}{\sin((\pi p Nh)/q)} \leq T + 2 \sum_{h=1}^{T-1} \frac{q}{(p Nh) \mod (q)}. \]

(64)  

(65)  

(66)
As \( pN \) is prime to \( q \), \( pN^h \mod(q) \) takes all values in \( \{1, \ldots, q-1\} \) as \( h \) varies from 1 to \( q-1 \), and the claim follows from \( T < q \). \( \square \)

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[1] G. Casati, B.V. Chirikov, F. M. Izrailev, J. Ford, *Stochastic behavior of a quantum pendulum under a periodic perturbation*, Lecture Notes in Physics, Springer, Berlin, 93: 334 (1979).

[2] For reviews see, e.g., S. Fishman, *Quantum Localization*, in *Quantum Chaos*, Proc. Int. School of Physics E. Fermi, Vol. Course CXIX, ed. Casati et al. (Amsterdam: North Holland 1993), p. 187; F. M. Izrailev, *Statistics of Quasi-Energy Spectrum*, ibidem, p. 265.

[3] F. M. Izrailev, *Simple models of quantum chaos: Spectrum and eigenfunctions*, Phys. Rep. 196 (1990) 299.

[4] F. M. Izrailev and D. L. Shepelyansky, *Quantum Resonance for a Rotator in a Nonlinear Periodic Field*, Theor. Mat. Phys. 43 (1980) 353.

[5] S. Wimberger, I. Guarneri and S. Fishman, *Quantum Resonances and Decoherence for \( \delta \)-Kicked Atoms*, Nonlinearity 16 (2003) 1381, and references therein.

[6] G. Casati and I. Guarneri, *Non-Recurrent Behaviour in Quantum Dynamics*, Comm. Math. Phys. 95 (1984) 121.

[7] S. J. Chang and K. J. Shi, *Evoultion and Exact Eigenstates of a Resonant Quantum System*, Phys. Rev. A 34 (1986) 7.

[8] I. Dana and D. L. Dorofeev, *General Quantum Resonances of the Kicked Particle*, Phys. Rev. E 73 (2006) 026206.

[9] J. Bourgain, *Estimates on Greens functions, localization and the quantum kicked rotor model*, Ann. of Math. (2) 156 (2002), no. 1, 249-294.

[10] S. Jitomirskaya, *Non-Perturbative Localization*, in *Proceedings of the ICM, Beijing 2002*, vol. 3, 445-456; arXiv/math-ph/0304044.

[11] S. Fishman, D. R. Grempel and R. E. Prange, *Quantum Recurrences and Anderson Localization*, Phys. Rev. Lett. 49, (1982) 509.

[12] F. M. Izrailev, in ref. 1.

[13] D. Thouless, *Electrons in Disordered Systems and the Theory of Localization*, Phys. Rep. 13 (1974) 93.
[14] S.Fishman, I.Guarneri, and L.Rebuzzini, *A Theory for Quantum Accelerator Modes in Atom Optics*, J. Stat. Phys. **110**, (2003) 911.

[15] I.Guarneri and L.Rebuzzini, *Quantum Accelerator Modes near Higher-Order Resonances*, Phys. Rev. Lett. **100** (2008) 234103.

[16] M.Reed and B. Simon, *Methods of Modern Mathematical Physics IV: Analysis of Operators*, Academic Press (1975).

[17] I.S. Gradshteyn and I.M.Ryzhik, *Tables of Integrals, Series, and Products*, 4th edition, Academic Press 1965: 4.397.6, 4.397.16.