EXPONENTIAL STABILITY OF 1-D WAVE EQUATION WITH THE BOUNDARY TIME DELAY BASED ON THE INTERIOR CONTROL

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Abstract. In this paper, the stability problem of 1-d wave equation with the boundary delay and the interior control is considered. The well-posedness of the closed-loop system is investigated by the linear operator. Based on the idea of Lyapunov functional technology, we give the condition on the relationship between the control parameter $\alpha$ and the delay parameter $k$ to guarantee the exponential stability of the system.

1. Introduction. Time delays often appear in many systems such as biological systems, electrical engineering and mechanical applications [2, 22], in particular, in distributed parameter systems [3, 4, 5, 6]. In real control systems, time delays in control actions are usually caused by acquisition of response, excitation data, online data processing and computation of control forces. Efforts have been devoted to eliminating the negative influence of time delays. However, time delays cannot be eliminated due to its inherent nature. It is well known that an arbitrarily small delay can destabilize the system. For example, Datko in [3, 4, 5, 6] showed that the occurrence of delays in the control could destabilize hyperbolic control system with the boundary feedback. So it is important to study the stability problems of systems with delays. However, an appropriate time-delay effect can sometimes improve the performance of the system. Abdallah et al. [2], Kwon et al. [10] and Suh et al. [22] proposed some controllers to stabilize the systems with time delays. By now there are many methods to design control laws for systems, such as the Lyapunov function approach, Linear Matrix Inequality (LMI), Pole assignment and so on.

Concerning wave equations with the boundary or interior delays, many scholars have made great efforts to discuss the stability of these systems. For example, Li et al. [11] estimated the decay rates of the nonlinear wave equation with a time-varying delay, based on the method of the Riemannian geometry. Researchers [14, 16, 18, 21, 25, 23, 9] obtained the exponential stability of wave equations with the input delays under certain conditions by spectral analysis. Scholars [1, 8, 12, 19]
used the Lyapunov method to analyze the exponential stability of wave equations with the time-varying delays in the feedback controls. Nicaise [17] scrutinized the stability of wave equations with the time dependent delays by using the observability inequality. Pignotti [21] showed the exponential stability of locally damped wave equations with the interior time delays as the coefficients of the delay terms are sufficiently small.

In this paper, we mainly consider the interior control problem of the wave equation with the boundary time delay. The wave equation with the boundary time delay was first introduced [7]. The study of wave equations is of practical significance. For example, Lutzen [13], Miranker [15] used the wave equations to describe the vibration problem of the instruments and physical phenomena, respectively. Let us first recall the control problem of the following wave equation with the boundary delay

\[
\begin{align*}
   w_{tt}(x,t) - w_{xx}(x,t) + u(x,t) &= 0, \quad x \in (0,1), \quad t > 0, \\
   w(0,t) &= 0, \quad w_x(1,t) = kw_t(1,t - \tau), \\
   w(x,0) &= w_0(x), \quad w_t(x,0) = w_1(x), \\
   w_t(1,\theta) &= h_0(\theta), \quad \theta \in (-\tau,0),
\end{align*}
\]

(1)

where \( x \in (0,1) \) is the space variable, \( t > 0 \) is the time variable, \( w(x,t) \) is the displacement deviating from its equilibrium position, \( k \in \mathbb{R} \) is the delay parameter, \( \tau \) is the delay time and \( u(x,t) \) is the control.

It is well known that system (1) can be stabilized exponentially under the feedback control law:

\[ u(x,t) = 2\alpha w_t(x,t). \]

Note that if \( k > 0, \alpha = 0, \) and \( \tau = 0, \) system (1) is exponentially stable, and if \( \alpha > 0 \) and \( k = 0, \) system (1) is also exponentially stable. Here, if \( \alpha > 0, k \in \mathbb{R} \) and \( \tau > 0, \) we will further discuss the relationship between \( \alpha \) and \( k \) to guarantee the exponential stability of system (1).

The rest of the paper is organized as follows. In Section 2, we study the well-posedness of the closed loop system and show that the system operator generates a \( C_0 \) semigroup. In Section 3, we prove that the closed loop system is exponentially stable by establishing the relationship between the control parameter and the delay parameter. Finally, in Section 4, a brief conclusion is given.

2. The well-posedness. In this section, we shall study the well-posedness of system (1). To this end, we first formulate the system in an appropriate Hilbert space \( \mathcal{H} \).

Set \( z(x,t) = w_t(1,t - x\tau) \). Clearly, \( z(0,t) = w_t(1,t), \quad z(1,t) = w_t(1,t - \tau) \) and \( \tau z_t(x,t) + z_x(x,t) = 0 \).

Then system (1) can be rewritten as

\[
\begin{align*}
   w_{tt}(x,t) - w_{xx}(x,t) + 2\alpha w_t(x,t) &= 0, \quad x \in (0,1), \quad t > 0, \\
   \tau z_t(x,t) + z_x(x,t) &= 0, \\
   w(0,t) &= 0, \quad w_x(1,t) = kw_t(1,t), \\
   z(0,t) &= w_t(1,t), \quad z(x,0) = w_t(1,-x\tau), \\
   w(x,0) &= w_0(x), \quad w_t(x,0) = w_1(x), \\
   z(x,0) &= h_0(-x\tau).
\end{align*}
\]

(2)
Set \( v(x,t) = w_t(x,t) \). Then (2) becomes

\[
\begin{align*}
\begin{cases}
  w_t(x,t) = v(x,t), & x \in (0,1), \\
  v_t(x,t) = w_{xx}(x,t) - 2\alpha v(x,t), \\
  z_t(x,t) = -\frac{1}{\tau} z_x(x,t), \\
  w(0,t) = 0, & w(x,1,t) = k z(1,t), \\
  z(0,t) = v(1,t), \\
  w(x,0) = w_0(x), & v(x,0) = v_0(x) = w_1(x), \\
  z(x,0) = h_0(-x\tau).
\end{cases}
\end{align*}
\]

(3)

Let \( H^k((0,1)) \) be an usual Sobolev space, denoted by \( V^k([0,1]) = \{ H^k([0,1]) | f(0) = 0 \} \). Set

\[ \mathcal{H} = V^1([0,1]) \times L^2([0,1]) \times L^2([0,1]), \]
equipped with the inner product. For \( W_j = (w_j, v_j, z_j) \in \mathcal{H}, i = 1, 2, \)

\[ (W_1, W_2)_{\mathcal{H}} = \int_0^1 w_1'(x)w_2'(x)dx + \int_0^1 v_1(x)v_2(x)dx + \tau \int_0^1 z_1(x)z_2(x)dx. \]

A direct verification shows that (\( \mathcal{H}, \| \cdot \|_{\mathcal{H}} \)) is a Hilbert space.

Define the operator \( A \) in \( \mathcal{H} \) as follows,

\[ D(A) = \{ (w,v,z) \in V^2(0,1) \times V^1(0,1) \times H^1(0,1) | w'(1) = k z(1), z(0) = v(1) \} \]

and for \( (w,v,z) \in D(A), \)

\[ \begin{pmatrix}
  w \\
  v \\
  z
\end{pmatrix} A =
\begin{pmatrix}
  w''(x) - 2\alpha v(x) \\
  -\tau^{-1} z'(x)
\end{pmatrix}. \]

(4)

System (2) or (3) can be written as

\[
\begin{align*}
\begin{cases}
  \frac{dW(t)}{dt} = AW(t), & t > 0, \\
  W(0) = W_0,
\end{cases}
\end{align*}
\]

(5)

where \( W(t) = (w(x,t), v(x,t), z(x,t))^T \), and \( W(0) = (w_0(x), v_0(x), h_0(-x\tau))^T. \)

For the operator \( A \), we have the following result.

**Lemma 2.1.** Let \( A \) and \( \mathcal{H} \) be defined as above. Then \( 0 \in \rho(A) \), and \( A^{-1} \) is compact on \( \mathcal{H} \). Hence, \( \sigma(A) \) consists of all isolated eigenvalues of the finite multiplicity.

**Proof.** Clearly, \( D(A) \) is dense in \( \mathcal{H} \). For any \( F = (f,g,h) \in \mathcal{H} \), we consider the solvability of the equation \( AW = F \), where \( W = (w,v,z) \in D(A) \), i.e.,

\[
\begin{align*}
\begin{cases}
  v(x) = f(x), \\
  w''(x) - 2\alpha v(x) = g(x), \\
  -\tau^{-1} z'(x) = h(x),
\end{cases}
\end{align*}
\]

with the boundary conditions

\[ w(0) = 0, \quad w'(1) = k z(1), \quad z(0) = v(1). \]

Obviously

\[
\begin{align*}
\begin{cases}
  v(x) = f(x), \\
  z(x) = f(1) - \tau \int_0^x h(s)ds,
\end{cases}
\end{align*}
\]

and \( w(x) \) satisfies

\[ w''(x) = 2\alpha f(x) + g(x), \quad w(0) = 0, \quad w'(1) = k z(1). \]
Clearly
\[ xw'(1) - w(x) = \int_0^x dr \int_r^1 (2\alpha f(s) + g(s))ds. \]

Therefore
\[
\begin{align*}
\left\{ \begin{array}{l}
v(x) = f(x), \\
z(x) = f(1) - \tau \int_0^x h(s)ds, \\
w(x) = kx (f(1) - \tau \int_0^x h(s)ds) - \int_0^x dr \int_r^1 (2\alpha f(s) + g(s))ds,
\end{array} \right.
\end{align*}
\]
Note that \( W = (w,v,z) \in D(\mathcal{A}) \subset V^2(0,1) \times V^1(0,1) \times H^1(0,1) \) and \( \mathcal{A}^{-1}F = W \). So \( 0 \in \rho(\mathcal{A}) \). The Sobolev’s Embedding Theorem asserts that \( \mathcal{A}^{-1} \) is a compact operator in \( \mathcal{H} \). Consequently, \( \sigma(\mathcal{A}) \) consists of all isolated eigenvalues of the finite multiplicity.

In what follows, we discuss the generation property of the semigroup \( C_0 \). To this end, we define a new inner product in \( \mathcal{H} \), for \( W_j = (w_j, v_j, z_j) \in \mathcal{H}, i = 1, 2, \)

\[
(W_1, W_2)_{\text{new}} = \int_0^1 e^{\gamma x} [w_1'(x) - v_1(x)][w_2'(x) - v_2(x)]dx + \int_0^1 e^{\delta x} [w_1'(x) + v_1(x)]
\]

\[ w_2'(x) + v_2(x)]dx + \tau \int_0^1 e^{\beta x} z_1(x)\bar{z}_2(x)dx, \]

where \( \gamma, \delta \) and \( \beta \) are positive real numbers.

Observe that

\[
(W_1, W_2)_{\text{new}} = \int_0^1 e^{\gamma x} |w_1'(x) - v_1(x)|^2 + \int_0^1 e^{\delta x} |w_1'(x) + v_1(x)|^2 + \tau \int_0^1 e^{\beta x} |z_1(x)|^2 dx
\]

\[ \geq \int_0^1 |w_1'(x) - v_1(x)|^2 + \int_0^1 |w_1'(x) + v_1(x)|^2 + \tau \int_0^1 |z_1(x)|^2 dx \]

\[ \geq ||W_1||^2_{\mathcal{H}}, \]

so \((W_1, W_2)_{\text{new}}\) is equivalent to the inner product \((W_1, W_2)_{\mathcal{H}}\).

Obviously, there exists a positive constant \( M \), such that \( \mathcal{A} - MI \) is a dissipative operator in the sense of the new inner product.

For any real \( W = (w, v, z) \in D(\mathcal{A}), \)

\[
(\mathcal{A}W, W)_{\text{new}}
\]

\[ = \int_0^1 e^{\gamma x} [w'(x) - (w''(x) - 2\alpha v(x)) + (w''(x) - 2\alpha v(x))] dx + \int_0^1 e^{\delta x} [w'(x) + (w''(x) - 2\alpha v(x)) + (w''(x) + 2\alpha v(x))] dx - \int_0^1 e^{\beta x} z'(x)z(x) dx
\]

\[ = -\int_0^1 e^{\gamma x} (w''(x) - v'(x))(w'(x) - v(x)) dx + 2\alpha \int_0^1 e^{\gamma x} v(x)(w'(x) - v(x)) dx
\]

\[ + \int_0^1 e^{\delta x} (w''(x) + v'(x))(w'(x) + v(x)) dx - 2\alpha \int_0^1 e^{\delta x} v(x)(w'(x) + v(x)) dx
\]

\[ - \int_0^1 e^{\beta x} z'(x)z(x) dx
\]
\[
\begin{align*}
&\leq -\frac{e^{\gamma x}}{2} (w'(x) - v(x))^2 |_{0}^{1} + \left(\frac{\gamma}{2} + \alpha\right) \int_{0}^{1} e^{\gamma x} (w'(x) - v(x))^2 dx + \alpha \int_{0}^{1} e^{\gamma x} v(x)^2 dx \\
&\quad + \frac{e^{\delta x}}{2} (w'(x) + v(x))^2 |_{0}^{1} + \left(\frac{\delta}{2} - \alpha\right) \int_{0}^{1} e^{\delta x} (w'(x) + v(x))^2 dx + \alpha \int_{0}^{1} e^{\delta x} v(x)^2 dx \\
&\quad - \frac{1}{2} e^{\beta x} z^2(x) |_{0}^{1} + \frac{\beta}{2} \int_{0}^{1} e^{\beta x} z^2(x) dx \\
&= \left(\frac{\gamma}{2} + \alpha\right) \int_{0}^{1} e^{\gamma x} (w'(x) - v(x))^2 dx - \left(\frac{\delta}{2} - \alpha\right) \int_{0}^{1} e^{\delta x} (w'(x) + v(x))^2 dx \\
&\quad + \frac{\beta}{2} \int_{0}^{1} e^{\beta x} z^2(x) dx + \alpha \int_{0}^{1} e^{\gamma x} v(x)^2 dx + \alpha \int_{0}^{1} e^{\delta x} v(x)^2 dx \\
&\quad - \frac{e^{\gamma x}}{2} (w'(x) - v(x))^2 |_{0}^{1} + \frac{e^{\delta x}}{2} (w'(x) + v(x))^2 |_{0}^{1} - \frac{1}{2} e^{\beta x} z^2(x) |_{0}^{1}.
\end{align*}
\]

Note that

\[
\int_{0}^{1} e^{\gamma x} v(x)^2 dx + \int_{0}^{1} e^{\delta x} v(x)^2 dx \leq 2 \max \{e^{\gamma}, e^{\delta}\} \int_{0}^{1} v(x)^2 dx
\]

\[
\leq 2 \max \{e^{\gamma}, e^{\delta}\} \int_{0}^{1} (v(x) + |w'(x)|)^2 dx
\]

\[
= \max \{e^{\gamma}, e^{\delta}\} \int_{0}^{1} [(w' - v(x))^2 + |w'(x) + v(x)|^2] dx
\]

\[
\leq \max \{e^{\gamma}, e^{\delta}\} \int_{0}^{1} e^{\gamma x} (w' - v(x))^2 + \int_{0}^{1} e^{\delta x} |w'(x) + v(x)|^2 dx.
\]

So

\[
(AW, W)_{\text{new}} \leq \left[\frac{\gamma}{2} + \alpha + \alpha \max \{e^{\gamma}, e^{\delta}\}\right] \int_{0}^{1} e^{\gamma x} (w'(x) - v(x))^2 dx
\]

\[
+ \left[\alpha \max \{e^{\gamma}, e^{\delta}\} + \alpha - \frac{\delta}{2}\right] \int_{0}^{1} e^{\delta x} (w'(x) + v(x))^2 dx
\]

\[
+ \frac{\beta}{2} \int_{0}^{1} e^{\beta x} z^2(x) dx
\]

\[
- \frac{e^{\gamma x}}{2} (w'(x) - v(x))^2 |_{0}^{1} + \frac{e^{\delta x}}{2} (w'(x) + v(x))^2 |_{0}^{1} - \frac{1}{2} e^{\beta x} z^2(x) |_{0}^{1}
\]

\[
\leq M \left[\int_{0}^{1} e^{\gamma x} (w'(x) - v(x))^2 dx + \int_{0}^{1} e^{\delta x} (w'(x) + v(x))^2 dx + \int_{0}^{1} e^{\beta x} z^2(x) dx\right]
\]

\[
- \frac{e^{\gamma x}}{2} (w'(x) - v(x))^2 |_{0}^{1} + \frac{e^{\delta x}}{2} (w'(x) + v(x))^2 |_{0}^{1} - \frac{1}{2} e^{\beta x} z^2(x) |_{0}^{1}
\]

where

\[
M = \max \left\{\frac{\gamma}{2} + \alpha + \alpha \max \{e^{\gamma}, e^{\delta}\}, \frac{\beta}{2}\right\}.
\]

Now we estimate the boundary term

\[
B = -\frac{e^{\gamma x}}{2} (w'(x) - v(x))^2 |_{0}^{1} + \frac{e^{\delta x}}{2} (w'(x) + v(x))^2 |_{0}^{1} - \frac{1}{2} e^{\beta x} z^2(x) |_{0}^{1}.
\]

Using the boundary conditions

\[
v(0) = 0, \quad w'(1) = k z(1), \quad z(0) = v(1),
\]
then

\[
B = -\frac{e\gamma}{2}(w'(1) - v(1))^2 + \frac{e\delta}{2}(w'(1) + v(1))^2 - \frac{1}{2}e\delta x^2(1) + \frac{1}{2}(w'(0) - v(0))^2 - \frac{1}{2}(w'(0) + v(0))^2 + \frac{1}{2}z^2(0)
\]

\[
= -\frac{e\gamma}{2}(kz(1) - z(0))^2 + \frac{e\delta}{2}(kz(1) + z(0))^2 - \frac{1}{2}e\delta z^2(1) + \frac{1}{2}z^2(0)
\]

\[
= -\frac{1}{2}[(k^2e\gamma + e\delta - k^2e\delta)z^2(1) - 2k(e\delta + e\gamma)z(1)z(0) + (e\gamma - 1 - e\delta)z^2(0)].
\]

We get the following result.

**Lemma 2.2.** Let

\[
F(x) = (k^2e\gamma + e\delta - k^2e\delta) - 2k(e\delta + e\gamma)x + (e\gamma - 1 - e\delta)x^2.
\]

If \(e\gamma > 1 + e\delta\) and \(e\delta \geq k^2\frac{e\gamma - e\delta}{e\gamma - 1} - 1\), then for any \(x \in \mathbb{R}\), \(F(x) \geq 0\).

**Proof.** If \(e\gamma > 1 + e\delta\), we get \(k^2e\gamma + e\delta - k^2e\delta > 0\) and

\[
\frac{F(x)}{e\gamma - 1 - e\delta} = \left(x - k\frac{e\gamma + e\delta}{e\gamma - 1 - e\delta}\right)^2
\]

\[
+ \frac{(k^2e\gamma - k^2e\delta + e\delta)(e\gamma - 1 - e\delta) - k^2(e\gamma + e\delta)^2}{(e\gamma - 1 - e\delta)^2}
\]

\[
= \left(x - k\frac{e\gamma + e\delta}{e\gamma - 1 - e\delta}\right)^2 + \frac{e\delta(e\gamma - e\delta - 1) - k^2(e\gamma - e\delta)^2}{(e\gamma - 1 - e\delta)^2},
\]

where

\[
(k^2e\gamma - k^2e\delta + e\delta)(e\gamma - 1 - e\delta) - k^2(e\gamma + e\delta)^2
\]

\[
= e\delta(e\gamma - e\delta - 1) - k^2(e\gamma - e\delta)^2 - 4k^2e\gamma + e\delta.
\]

So, \(F(x) \geq 0\) provided \(e\delta \geq k^2\frac{e\gamma - e\delta}{e\gamma - 1} + 4e\gamma + e\delta\).

Summarizing the discussion above we get the following result.

**Theorem 2.3.** Let \(A\) and \(H\) be defined as above. Let \(\gamma, \delta\) and \(\beta\) be positive constants and satisfy

\[
e\gamma - e\delta - 1 > 0, \quad e\delta \geq k^2\frac{e\gamma - e\delta}{e\gamma - 1} + 4e\gamma + e\delta.
\]

Set

\[
M = \max\left\{\frac{\gamma}{2} + \alpha + \alpha e\gamma, \frac{\beta}{2\tau}\right\}.
\]

Then \(A - MI\) is a dissipative operator in the space \((H, (\cdot, \cdot)_{new})\). Hence \(A\) generates a \(C_0\) semigroup of the bounded linear operators in \(H\).

**Theorem 2.3** is a direct consequence of Lemmas 2.1, 2.2 and Lummer-Philips [20].
3. **Exponential stability of the system.** In this section, the Lyapunov function approach is used to discuss the stability result of system (1). Here the key point is to construct an appropriate Lyapunov function of (1).

At first, we consider some basic functions. Let $w(x,t)$ be a solution of system (1). We define

$$E_0(t) = \frac{1}{2} \int_0^1 \left[ w_x^2(x,t) + (w_t(x,t) + \alpha w(x,t))^2 \right] dx + \frac{\alpha^2}{2} \int_0^1 w^2(x,t) dx.$$  \hspace{1cm} (6)

**Lemma 3.1.** Let $E_0(t)$ be defined as (6). Then

$$\frac{\sqrt{1 + \alpha^2}}{2(\alpha + \sqrt{1 + \alpha^2})} \int_0^1 (w_x^2(x,t) + w_t^2(x,t)) dx \leq E_0(t) \leq \max \left\{ \frac{2+\alpha^2}{2}, (1 + \alpha) \right\} \int_0^1 (w_x^2(x,t) + w_t^2(x,t)) dx$$

and

$$\dot{E}_0(t) = w_x(1,t)w_1(1,t) + \alpha w_t(1,t)w(1,t) - \alpha \int_0^1 (w_x^2(x,t) + w_t^2(x,t)) dx.$$ \hspace{1cm} (7)

**Proof.** Since

$$\int_0^1 w^2(x,t) dx \leq \frac{1}{2} \int_0^1 w_x^2(x,t) dx,$$

it is easy to estimate

$$E_0(t) \leq \max \left\{ \frac{2+\alpha^2}{2}, (1 + \alpha) \right\} \int_0^1 (w_x^2(x,t) + w_t^2(x,t)) dx.$$  \hspace{1cm} (8)

Furthermore

$$\int_0^1 [(w_t(x,t) + \alpha w(x,t))^2 + \alpha^2 w(x,t)] dx$$

$$= \int_0^1 w_t^2(x,t) dx + 2\alpha \int_0^1 w_t(x,t)w(x,t) dx + 2\alpha^2 \int_0^1 w^2(x,t) dx$$

$$\geq \int_0^1 w_t^2(x,t) dx + 2\alpha^2 \int_0^1 w^2(x,t) dx - \alpha \left[ \delta \int_0^1 w_t^2(x,t) dx + \frac{1}{\delta} \int_0^1 w^2(x,t) dx \right]$$

$$= (1 - \alpha \delta) \int_0^1 w_t^2(x,t) dx + \left( 2\alpha^2 - \frac{\alpha}{\delta} \right) \int_0^1 w^2(x,t) dx,$$
Let \( \delta \) be defined as (9). Then
\[
E_0(t) = \frac{1}{2} \int_0^1 \left[w_x^2(x,t) + (w_t(x,t) + \alpha w(x,t))^2\right] dx + \frac{\alpha^2}{2} \int_0^1 w^2(x,t) dx \\
\geq \frac{1}{2} \int_0^1 w_x^2(x,t) dx + \frac{(1-\alpha \delta)}{2} \int_0^1 w_t^2(x,t) dx \\
+ \frac{1}{2} \left(2\alpha^2 - \frac{\alpha}{\delta}\right) \int_0^1 w^2(x,t) dx \\
= \frac{(1-\alpha \delta)}{2} \int_0^1 w_x^2(x,t) dx + \frac{(1-\alpha \delta)}{2} \int_0^1 w_t^2(x,t) dx \\
+ \frac{\alpha \delta}{2} \int_0^1 w_x^2(x,t) dx + \frac{1}{2} \left(2\alpha^2 - \frac{\alpha}{\delta}\right) \int_0^1 w^2(x,t) dx \\
\geq \frac{(1-\alpha \delta)}{2} \int_0^1 (w_x^2(x,t) + w_t^2(x,t)) dx + \frac{\alpha}{2} \left(2\alpha + \delta - \frac{1}{\delta}\right) \int_0^1 w^2(x,t) dx.
\]

Taking \( \delta = \sqrt{1 + \alpha^2} - \alpha \), we have \( 2\alpha + \delta - \frac{1}{\delta} = 0 \) and
\[
1 - \alpha \delta = \frac{\sqrt{1 + \alpha^2}}{\alpha + \sqrt{1 + \alpha^2}} > \frac{1}{2},
\]
hence
\[
E_0(t) \geq \frac{\sqrt{1 + \alpha^2}}{2 (\alpha + \sqrt{1 + \alpha^2})} \int_0^1 (w_x^2(x,t) + w_t^2(x,t)) dx > \frac{1}{4} \int_0^1 (w_x^2(x,t) + w_t^2(x,t)) dx.
\]
The inequality (7) follows.

We calculate
\[
\dot{E}_0(t) = \int_0^1 w_x(x,t) w_{xt}(x,t) dx + \int_0^1 (w_t(x,t) + \alpha w(x,t))(w_{tt}(x,t) + \alpha w_t(x,t)) dx \\
+ \alpha^2 \int_0^1 w(x,t) w_t(x,t) dx \\
= w_x(1,t) w_t(1,t) + \alpha \int_0^1 w(x,t) w_{xt}(x,t) dx - \alpha \int_0^1 w_t^2(x,t) dx \\
= w_x(1,t) w_t(1,t) + \alpha w_x(1,t) w(1,t) - \alpha \int_0^1 (w_x^2(x,t) + w_t^2(x,t)) dx.
\]
The desired result follows. \( \square \)

We define
\[
E_1(t) = \int_0^1 x w_x(x,t) w_t(x,t) dx.
\]
Then we have the following result.

**Lemma 3.2.** Let \( E_1(t) \) be defined as (9). Then
\[
|E_1(t)| \leq \frac{1}{2} \int_0^1 \left| x(w_x^2(x,t) + w_t^2(x,t)) \right| dx,
\]
and
\[
\dot{E}_1(t) = \frac{1}{2} w_x^2(1,t) + \frac{1}{2} w_t^2(1,t) - \frac{1}{2} \int_0^1 (w_x^2(x,t) + w_t^2(x,t)) dx.
\]
Proof. We calculate

\[
-2\alpha \int_0^1 xw_x(x,t)w_t(x,t)dx.
\] (11)

\[
E_1(t) = \int_0^1 xw_x(x,t)w_t(x,t)dx + \int_0^1 xw_x(x,t)w_{tt}(x,t)dx
\]

\[
= \int_0^1 xw_x(x,t)w_t(x,t)dx + \int_0^1 xw_x(x,t)(w_{xx}(x,t) - 2\alpha w_t(x,t))dx
\]

\[
= \frac{1}{2} w_x^2(1,t) + \frac{1}{2} w_x^2(0,t) - \frac{1}{2} \int_0^1 (w_x^2(x,t) + w_t^2(x,t))dx
\]

\[
-2\alpha \int_0^1 xw_x(x,t)w_t(x,t)dx.
\]

The desired equality follows.

Now we define a function \(V_0(t) = E_0(t) - \rho E_1(t)\) with \(\rho \in \left(0, \frac{\sqrt{1+\alpha^2}}{\alpha + \sqrt{1+\alpha^2}}\right)\), i.e.,

\[
V_0(t) = \frac{1}{2} \int_0^1 \left[w_x^2(x,t) + (w_t(x,t) + \alpha w(x,t))^2\right] dx + \frac{\alpha^2}{2} \int_0^1 w^2(x,t)dx
\]

\[
-\rho \int_0^1 xw_x(x,t)w_t(x,t)dx
\] (12)

Lemma 3.3. Let \(V_0(t)\) be defined as (14). Then

\[
V_0(t) > \frac{1}{2} \left(\frac{\sqrt{1+\alpha^2}}{\alpha + \sqrt{1+\alpha^2}} - \rho\right) \int_0^1 (w_x^2(x,t) + w_t^2(x,t))dx
\] (13)

and

\[
\dot{V}_0(t) = -\frac{\rho}{2} (w_t^2(1,t) + w_x^2(1,t)) + w_x(1,t)w_t(1,t) + \alpha w_x(1,t)w(1,t)
\]

\[-\left(\alpha - \frac{\rho}{2}\right) \int_0^1 (w_x^2(x,t) + w_t^2(x,t))dx + 2\alpha \rho \int_0^1 xw_x(x,t)w_t(x,t)dx.
\] (14)

Proof. From (7) and (10) we see that

\[
V_0(t) \geq \frac{1}{2} \left(\frac{\sqrt{1+\alpha^2}}{\alpha + \sqrt{1+\alpha^2}} - \rho\right) \int_0^1 (w_x^2(x,t) + w_t^2(x,t))dx
\]

holds.

Using (8) and (11), we get

\[
\dot{V}_0(t) = -\frac{\rho}{2} (w_t^2(1,t) + w_x^2(1,t)) + w_x(1,t)w_t(1,t) + \alpha w_x(1,t)w(1,t)
\]

\[-\left(\alpha - \frac{\rho}{2}\right) \int_0^1 (w_x^2(x,t) + w_t^2(x,t))dx + 2\alpha \rho \int_0^1 xw_x(x,t)w_t(x,t)dx.
\]

In order to study the stability of system (11), we define

\[
V_1(t) = e^{2\lambda t} V_0(t)
\]

\[
= \frac{1}{2} e^{2\lambda t} \int_0^1 \left[w_x^2(x,t) + (w_t(x,t) + \alpha w(x,t))^2\right] dx + \frac{\alpha^2}{2} e^{2\lambda t} \int_0^1 w^2(x,t)dx
\]

\[-\rho e^{2\lambda t} \int_0^1 xw_x(x,t)w_t(x,t)dx
\] (15)
Again using the inequality

\[
\frac{\lambda}{2} \leq \frac{\lambda + (\alpha - \lambda - a)}{2} e^{2\lambda t} \int_0^1 w_x^2(x, t) dx
\]

Proof. Using (14), we deduce

\[
\dot{V}_1(t) \leq \left[ \frac{\lambda}{2} + \lambda + \frac{\alpha \lambda}{\varepsilon_0} + (\alpha - \lambda) \rho - \alpha \right] e^{2\lambda t} \int_0^1 w_x^2(x, t) dx
\]

where \(\varepsilon_0\) is a positive constant. Under the boundary condition \(w_x(1, t) = kw_t(1, t - \tau)\), we have

\[
\dot{V}_1(t) \leq H(\alpha, \tau, k, \varepsilon_0, \rho) e^{2\lambda t} \int_0^1 (w_x^2(x, t) + w_t^2(x, t)) dx + \left[ \frac{|k|}{2} e^{\lambda t} - \frac{\rho}{2} \right] w_t^2(1, t) e^{2\lambda t}
\]

where

\[
H(\alpha, \lambda, k, \tau, \rho) = \frac{\alpha}{2} \left[ \sqrt{2\lambda^2 + \left( \lambda \alpha + |k| e^{\frac{1}{2} \lambda t} \right)^2 + \left( \lambda \alpha + |k| e^{\frac{1}{2} \lambda t} \right) e^{2\lambda t} + \lambda + (\alpha - \lambda) \rho - \alpha.}
\]

\[
\dot{V}_1(t) = e^{2\lambda t} \int_0^1 \left[ w_x^2(x, t) + \left( w_t(x, t) + \alpha w(x, t) \right)^2 \right] dx + \lambda \alpha^2 e^{2\lambda t} \int_0^1 w^2(x, t) dx
\]

\[
-2\lambda \rho e^{2\lambda t} \int_0^1 x w_x(x, t) w_t(x, t) dx
\]

\[
-\frac{\rho}{2} \left( w_t^2(1, t) + w_x^2(1, t) \right) e^{2\lambda t} + \left( w_x(1, t) w_t(1, t) + \alpha w_x(1, t) w(1, t) \right) e^{2\lambda t}
\]

\[
-\left( \alpha - \frac{\rho}{2} e^{2\lambda t} \int_0^1 (w_x^2(x, t) + w_t^2(x, t)) dx + 2\alpha \rho e^{2\lambda t} \int_0^1 x w_x(x, t) w_t(x, t) dx
\]

\[
= \left[ \frac{\lambda}{2} + \lambda - \alpha \right] e^{2\lambda t} \int_0^1 w_x^2(x, t) dx + 2\alpha \lambda e^{2\lambda t} \int_0^1 w(x, t) w_t(x, t) dx
\]

\[
+ \left( \frac{\lambda}{2} + \lambda - \alpha \right) e^{2\lambda t} \int_0^1 w_t^2(x, t) dx + 2\alpha \lambda e^{2\lambda t} \int_0^1 w^2(x, t) dx
\]

\[
+ (2\alpha - 2\lambda) \rho e^{2\lambda t} \int_0^1 x w_x(x, t) w_t(x, t) dx
\]

\[
-\frac{\rho}{2} \left( w_t^2(1, t) + w_x^2(1, t) \right) e^{2\lambda t} + \left( w_x(1, t) w_t(1, t) + \alpha w_x(1, t) w(1, t) \right) e^{2\lambda t}
\]

Again using the inequality

\[
\int_0^1 w^2(x, t) dx \leq \frac{1}{2} \int_0^1 w_x^2(x, t) dx,
\]
we get

\[ \dot{V}_1(t) \leq \left( \frac{\rho}{2} + \lambda - \alpha \right) e^{2\lambda t} \int_0^1 w^2_2(x,t) dx + \left( \frac{\rho}{2} + \lambda - \alpha \right) e^{2\lambda t} \int_0^1 w^2_1(x,t) dx \\
+ \alpha \lambda e^{2\lambda t} \left( \frac{\varepsilon_0}{2} \int_0^1 w^2_2(x,t) dx + \frac{1}{\varepsilon_0} \int_0^1 w^2_1(x,t) dx \right) \\
+ \alpha^2 \lambda e^{2\lambda t} \int_0^1 w^2_2(x,t) dx + (\alpha - \lambda) \rho e^{2\lambda t} \int_0^1 \left( w^2_2(x,t) + w^2_2(x,t) \right) dx \\
- \frac{\rho}{2} (w^2_2(1,t) + w^2_2(1,t)) e^{2\lambda t} + (w_x(1,t)w_t(1,t) + \alpha w_x(1,t)w_v(1,t)) e^{2\lambda t} \]

This is (16).

Applying to the boundary condition \( w_x(1,t) = kw_t(1,t - \tau) \) and

\[ w_t(1,t)w(1,t - \tau) \leq \frac{1}{2} \left( \varepsilon_1 w^2_2(1,t - \tau) + \frac{1}{\varepsilon_1} w^2_1(1,t) \right), \]

\[ w(1,t)w(1,t - \tau) \leq \frac{1}{2} \left( \varepsilon_2 w^2_2(1,t - \tau) + \frac{1}{\varepsilon_2} w^2_2(1,t) \right) \\
\leq \frac{1}{2} \left( \varepsilon_2 w^2_2(1,t - \tau) + \frac{1}{\varepsilon_2} \int_0^1 w^2_2(x,t) dx \right), \]

we get

\[ \dot{V}_1(t) \leq \left[ \frac{\rho}{2} + \lambda + \alpha \lambda \varepsilon_0 \frac{1}{2} + (\alpha - \lambda) \rho - \alpha \right] e^{2\lambda t} \int_0^1 w^2_2(x,t) dx \\
+ \left[ \frac{\rho}{2} + \lambda + \alpha \lambda \varepsilon_0 \frac{1}{2} + (\alpha - \lambda) \rho - \alpha \right] e^{2\lambda t} \int_0^1 w^2_1(x,t) dx \\
- \frac{\rho}{2} (w^2_2(1,t) + k^2 w^2_2(1,t - \tau)) e^{2\lambda t} + \left( \frac{|k| \varepsilon_1}{2} w^2_1(1,t - \tau) + \frac{|k|}{\varepsilon_1} w^2_1(1,t) \right) e^{2\lambda t} \\
+ \left( \frac{|k| \varepsilon_2}{2} w^2_2(1,t - \tau) + \frac{|k|}{\varepsilon_2} \int_0^1 w^2_2(x,t) dx \right) e^{2\lambda t} \\
= \left[ \frac{\rho}{2} + \lambda + \alpha \lambda \varepsilon_0 \frac{1}{2} + (\alpha - \lambda) \rho + \frac{|k|}{\varepsilon_1} - \alpha \right] e^{2\lambda t} \int_0^1 w^2_2(x,t) dx \\
+ \left[ \frac{\rho}{2} + \lambda + \alpha \lambda \varepsilon_0 \frac{1}{2} + (\alpha - \lambda) \rho - \alpha \right] e^{2\lambda t} \int_0^1 w^2_1(x,t) dx \\
+ \left( \frac{|k|}{\varepsilon_1} - \frac{\rho}{2} \right) w^2_1(1,t) e^{2\lambda t} + \left( \frac{|k| \varepsilon_1}{2} + \frac{|k| \varepsilon_2}{2} - \frac{k^2}{2} \right) w^2_1(1,t - \tau) e^{2\lambda t} \]

namely,

\[ \dot{V}_1(t) \leq \left[ \frac{\rho}{2} + \lambda + \alpha \lambda \varepsilon_0 \frac{1}{2} + (\alpha - \lambda) \rho + \frac{|k|}{\varepsilon_2} - \alpha \right] e^{2\lambda t} \int_0^1 w^2_2(x,t) dx \]
we take

where \( \varepsilon_j, j = 0, 1, 2 \) are positive constants.

Obviously, inequality (19) has too many parameters, namely \( \varepsilon_j, j = 0, 1, 2 \). We must optimize these parameters in order to discuss the sign of \( \dot{V}_1(t) \).

At first, we optimize \( \varepsilon_1 \) and \( \varepsilon_2 \). Let us consider the boundary part of inequality (19). Set

\[
B(w) = \left( \frac{|k|}{2\varepsilon_1} - \frac{\rho}{2} \right) e^{2\lambda t} w_{1t}^2(1, t) + \left( \frac{|k|\varepsilon_1}{2} + \frac{\alpha|k|\varepsilon_2}{2} - \frac{k^2\rho}{2} \right) e^{2\lambda t} w_{1t}^2(1, t - \tau).
\]

We rewrite it as

\[
B(w) = \left( \frac{|k|}{2\varepsilon_1} - \frac{\rho}{2} \right) e^{2\lambda t} w_{1t}^2(1, t) + \left( \frac{|k|\varepsilon_1}{2} + \frac{\alpha|k|\varepsilon_2}{2} - \frac{k^2\rho}{2} \right) e^{2\lambda t} w_{1t}^2(1, t - \tau)
\]

\[
= \left[ \frac{|k|}{2\varepsilon_1} - \frac{\rho}{2} + \left( \frac{|k|\varepsilon_1}{2} + \frac{\alpha|k|\varepsilon_2}{2} - \frac{k^2\rho}{2} \right) e^{2\lambda t} \right] e^{2\lambda t} w_{1t}^2(1, t)
\]

\[
- \left( \frac{|k|\varepsilon_1}{2} + \frac{\alpha|k|\varepsilon_2}{2} - \frac{k^2\rho}{2} \right) \left[ e^{2\lambda(t+\tau)} w_{1t}^2(1, t) - e^{2\lambda t} w_{1t}^2(1, t - \tau) \right].
\]

Clearly

\[
\min_{\varepsilon_1 > 0, \varepsilon_2 > 0} \left[ \frac{|k|}{2\varepsilon_1} - \frac{\rho}{2} + \left( \frac{|k|\varepsilon_1}{2} + \frac{\alpha|k|\varepsilon_2}{2} - \frac{k^2\rho}{2} \right) e^{2\lambda t} + \frac{\alpha|k|}{2\varepsilon_2} \right] - \frac{\rho}{2} - \frac{k^2\rho e^{2\lambda t}}{2}
\]

\[
= \left( |k| e^{\lambda t} + \frac{\alpha|k| e^{\lambda t} - \rho}{2} - \frac{k^2\rho e^{2\lambda t}}{2} \right).
\]

So \( \varepsilon_1 = \varepsilon_2 = e^{-\lambda t} \). In this case, inequality (19) becomes

\[
\dot{V}_1(t) \leq \left[ \frac{\rho}{2} + \lambda + \lambda^2 + \frac{\alpha\lambda\varepsilon_0}{2} + (\alpha - \lambda)\rho + \frac{\alpha|k| e^{\lambda t} - \alpha}{2} \right] e^{2\lambda t} \int_0^1 w_{1x}^2(x,t) dx
\]

\[
+ \left[ \frac{\rho}{2} + \lambda + \frac{\alpha\lambda}{\varepsilon_0} + (\alpha - \lambda)\rho - \alpha \right] e^{2\lambda t} \int_0^1 w_{1x}^2(x,t) dx
\]

\[
+ \left( \frac{|k|}{2} e^{-\lambda t} \frac{\rho}{2} \right) \left| w_{1t}^2(1, t) e^{2\lambda t} + \left( |k| \frac{e^{-\lambda t} + \frac{\alpha|k| e^{-\lambda t} - \frac{k^2\rho}{2}}{2} \right) e^{2\lambda t} \right|
\]

with only one uncertain parameter \( \varepsilon_0 \). Set

\[
T(\varepsilon_0) = \left[ \frac{\rho}{2} + \lambda + \frac{\alpha\lambda\varepsilon_0}{2} + (\alpha - \lambda)\rho + \frac{\alpha|k| e^{\lambda t} - \alpha}{2} \right],
\]

\[
S(\varepsilon_0) = \left[ \frac{\rho}{2} + \lambda + \frac{\alpha\lambda}{\varepsilon_0} + (\alpha - \lambda)\rho - \alpha \right].
\]

we take \( \varepsilon_0 \) such that \( S(\varepsilon_0) = T(\varepsilon_0) \), i.e.,

\[
\frac{\alpha\lambda^2}{2} + \frac{\alpha\lambda\varepsilon_0}{2} + \frac{\alpha|k|}{2} e^{\lambda t} = \frac{\alpha\lambda}{\varepsilon_0}.
\]
Solving the above equation, we get
\[
\varepsilon_0 = \sqrt{8\lambda^2 + (2\lambda\alpha + |k|e^{\lambda\tau})^2} - (2\lambda\alpha + |k|e^{\lambda\tau})
\]
\[
\frac{2\lambda}{4\lambda}
\]
\[
\frac{2\lambda}{4\lambda} = \sqrt{8\lambda^2 + (2\lambda\alpha + |k|e^{\lambda\tau})^2} + (2\lambda\alpha + |k|e^{\lambda\tau})
\]

Denote that
\[
H(\alpha, \lambda, k, \tau, \rho) = \frac{\alpha}{4} \left[ \sqrt{8\lambda^2 + (2\lambda\alpha + |k|e^{\lambda\tau})^2} + (2\lambda\alpha + |k|e^{\lambda\tau}) \right] + \frac{\rho}{2} + \lambda + (\alpha - \lambda)\rho - \alpha,
\]
thus
\[
\dot{V}_1(t) \leq H(\alpha, \lambda, k, \tau, \rho)e^{2\lambda t} \int_0^1 \left( w^2(x, t) + w^2_t(x, t) \right) dx
\]
\[
+ \left( \frac{|k|}{2}e^{\lambda\tau} - \frac{\rho}{2} \right) w^2_t(1, t)e^{2\lambda t} + \left( \frac{|k|}{2}e^{-\lambda\tau} + \frac{\alpha|k|}{2}e^{-\lambda\tau} - \frac{k^2\rho}{2} \right) w^2_t(1, t - \tau)e^{2\lambda t}.
\]
(17) follows.

Next we consider the solvability of \((\alpha, \rho, k, \lambda)\) with respect to the following inequalities
\[
\begin{cases}
0 < \rho < \frac{\sqrt{1 + \alpha^2}}{\alpha + \sqrt{1 + \alpha^2}}, \\
\frac{|k|}{2}e^{\lambda\tau} - \frac{\rho}{2} \leq 0, \\
\frac{\alpha}{4} \left[ \sqrt{8\lambda^2 + (2\lambda\alpha + |k|e^{\lambda\tau})^2} + (2\lambda\alpha + |k|e^{\lambda\tau}) \right] + \frac{\rho}{2} + \lambda + (\alpha - \lambda)\rho - \alpha \leq 0.
\end{cases}
\]
They are equivalent to
\[
\begin{cases}
0 < \rho < \frac{1 + \alpha^2}{\alpha + \sqrt{1 + \alpha^2}}, \\
\frac{1 + \alpha}{|k|}e^{\lambda\tau} \leq \rho, \\
\frac{\alpha}{4} \left[ \sqrt{8\lambda^2 + (2\lambda\alpha + |k|e^{\lambda\tau})^2} + (2\lambda\alpha + |k|e^{\lambda\tau}) \right] + \frac{\rho}{2} + \lambda + (\alpha - \lambda)\rho \leq \alpha.
\end{cases}
\]
It is necessary that
\[
\begin{cases}
0 < \rho < \frac{\sqrt{1 + \alpha^2}}{\alpha + \sqrt{1 + \alpha^2}}, \\
\frac{1 + \alpha}{|k|} \leq \rho, \\
|k| \leq \rho, \\
\alpha|k| + (1 + 2\alpha)\rho \leq 2\alpha.
\end{cases}
\]
The last inequality implies \(|k| < 2\) and
\[
\rho < \frac{\alpha(2 - |k|)}{1 + 2\alpha}.
\]
Combining the second and the third inequalities, we get
\[
\frac{1 + \alpha}{|k|} < \frac{\alpha(2 - |k|)}{1 + 2\alpha}, \quad |k| < \frac{\alpha(2 - |k|)}{1 + 2\alpha}.
\]
Thus
\[ |k| < \frac{2\alpha}{1 + 3\alpha} < \frac{2}{3}, \]
and
\[ (|k| - 1)^2 < -\frac{1 + 2\alpha + 2\alpha^2}{\alpha}. \]
This is impossible. So there is no solution of inequalities \((20)\).

Based on the above discussion, we know
\[ |k| e^{-\frac{\lambda\tau}{2}} + \frac{\alpha |k| e^{-\frac{\lambda\tau}{2}}}{2} - \frac{\rho}{2} k^2 \geq 0. \]

So we define
\[ V_2(t) = \left( \frac{|k| e^{-\frac{\lambda\tau}{2}}}{2} + \frac{\alpha |k| e^{-\frac{\lambda\tau}{2}}}{2} - \frac{\rho}{2} k^2 \right) \int_{t-\tau}^{t} e^{2\lambda(s+\tau)} w_t^2(1,s) ds \]
and
\[ V(t) = V_1(t) + V_2(t). \]
Thus using \(17\) we have
\[
\dot{V}(t) = \dot{V}_1(t) + \dot{V}_2(t) \\
\leq \frac{H(\alpha, \lambda, k, \tau, \rho) e^{2\lambda t}}{2} \int_{0}^{1} (w_t^2(x,t) + w_1^2(x,t)) dx \\
+ \left( \frac{2 + \alpha}{2} |k| e^{\lambda \tau} - \frac{\rho}{2} - k^2 \rho e^{2\lambda \tau} \right) w_1^2(1,t) e^{2\lambda t}. \]

**Theorem 3.5.** Let \( V_0, V_1, V_2(t) \) and \( V(t) \) be defined as above. If \( \alpha, k, \tau, \rho \) and \( \lambda \) satisfy
\[
\begin{align*}
0 < \rho &< \frac{\sqrt{1 + \alpha^2}}{\alpha + \sqrt{1 + \alpha^2}}, \\
\frac{|k| e^{-\frac{\lambda\tau}{2}}}{2} + \frac{\alpha |k| e^{-\frac{\lambda\tau}{2}}}{2} - \frac{\rho}{2} k^2 &\geq 0, \\
\frac{(2 + \alpha)}{2} |k| e^{\lambda \tau} - \frac{\rho}{2} - k^2 \rho e^{2\lambda \tau} &\leq 0, \\
\alpha &\geq \frac{\sqrt{32}}{3} \left( 2\lambda^2 + \left( 2\lambda \alpha + |k| e^{\lambda \tau} \right)^2 + \left( 2\lambda \alpha + |k| e^{\lambda \tau} \right) \right) \\
&\geq \frac{\sqrt{8\lambda^2 + (2\lambda \alpha + |k| e^{\lambda \tau})^2 + (2\lambda \alpha + |k| e^{\lambda \tau})}}{2} + \frac{\alpha}{2} + \lambda + (\alpha - \lambda) \rho - \alpha \leq 0,
\end{align*}
\]
then \( V_0(t) \) decays exponentially at the rate of \( 2\lambda \).

**Proof.** Suppose that \( \alpha, k, \tau, \rho \) and \( \lambda \) satisfy inequalities \(21\). Then
\[
\dot{V}(t) = \dot{V}_1(t) + \dot{V}_2(t) \\
\leq \frac{H(\alpha, \lambda, k, \tau, \rho) e^{2\lambda t}}{2} \int_{0}^{1} (w_t^2(x,t) + w_1^2(x,t)) dx \\
+ \left( \frac{2 + \alpha}{2} |k| e^{\lambda \tau} - \frac{\rho}{2} - k^2 \rho e^{2\lambda \tau} \right) w_1^2(1,t) e^{2\lambda t} \leq 0,
\]
which implies \( V(t) \leq V(0) \). Note that \( V_2(t) \geq 0 \) and
\[ e^{2\lambda t} V_0(t) = V_1(t) \leq V(t) \leq V(0), \]
so \( V_0(t) \leq V(0)e^{-2\lambda t}. \) \qed
Further, the solvability of inequalities \((21)\) is considered. Note that inequalities \((21)\) is equivalent to

\[
\begin{cases}
0 < \rho < \frac{\sqrt{1+\alpha^2}}{\alpha + \sqrt{1+\alpha^2}}, \\
\frac{1+\alpha}{|k|} e^{-\lambda x} \geq \rho, \\
\frac{(2+\alpha)|k| e^{\lambda x}}{(1+k^2)} \leq \rho, \\
\frac{2}{\sqrt{8}} \sqrt{8 \lambda^2 + (2\lambda \alpha + |k| e^{\lambda x})^2 + (2\lambda \alpha + |k| e^{\lambda x})^2} + \frac{\alpha}{2} + \lambda + (\alpha - \lambda) \rho \leq \alpha.
\end{cases}
\]

Obviously, the necessary conditions on the solvability of inequalities \((21)\) are that \(\alpha, k, \rho\) satisfy

\[
\begin{cases}
0 < \rho < \frac{\sqrt{1+\alpha^2}}{\alpha + \sqrt{1+\alpha^2}}, \\
\frac{1+\alpha}{|k|} > \rho, \\
\frac{(2+\alpha)|k|}{(1+k^2)} < \rho, \\
\alpha|k| + (1 + \alpha) \rho \leq 2\alpha.
\end{cases}
\]

We need to establish the relationship among \(\alpha, k, \rho\). We observe that the last inequality in \((22)\) implies \(|k| < 2\) and

\[
\rho < \frac{\alpha(2 - |k|)}{1 + 2\alpha}.
\]

Directly comparing both terms \(\frac{(1+\alpha)}{|k|}\) and \(\frac{\alpha(2 - |k|)}{1 + 2\alpha}\), we have

\[
\frac{\alpha(2 - |k|)}{1 + 2\alpha} < \frac{(1 + \alpha)}{|k|}, \quad \forall k \in (0, 2), \forall \alpha > 0.
\]

Next we compare the terms \(\frac{\sqrt{1+\alpha^2}}{\alpha + \sqrt{1+\alpha^2}}\) and \(\frac{2\alpha}{1+2\alpha}\). Note that \(\alpha \leq \frac{1}{4} \sqrt{2 + 2\lambda^2}\) is equivalent to \(2\alpha^2 \leq \sqrt{1 + \alpha^2}\), and

\[
\begin{cases}
\frac{2\alpha}{1+2\alpha} \leq \frac{\sqrt{1+\alpha^2}}{\alpha + \sqrt{1+\alpha^2}}, & \text{if } \alpha \leq \frac{1}{4} \sqrt{2 + 2\lambda^2}, \\
\frac{2\alpha}{1+2\alpha} \geq \frac{\sqrt{1+\alpha^2}}{\alpha + \sqrt{1+\alpha^2}}, & \text{if } \alpha \geq \frac{1}{4} \sqrt{2 + 2\lambda^2}.
\end{cases}
\]

So

\[
\begin{cases}
\rho < \frac{\alpha(2-x)}{1+2\alpha} \leq \frac{\sqrt{1+\alpha^2}}{\alpha + \sqrt{1+\alpha^2}}, & \text{if } \alpha \leq \frac{1}{4} \sqrt{2 + 2\lambda^2}, \quad x \in (0, 2), \\
\rho < \frac{(2+\alpha)|k|}{(1+k^2)} \leq \frac{\sqrt{1+\alpha^2}}{\alpha + \sqrt{1+\alpha^2}}, & \text{if } \alpha \geq \frac{1}{4} \sqrt{2 + 2\lambda^2}, \quad 0 \leq x \leq \frac{2\alpha^2 - \sqrt{1+\alpha^2}}{\alpha(\alpha + \sqrt{1+\alpha^2})}, \\
\rho < \frac{(2-x)}{1+2\alpha} \leq \frac{\sqrt{1+\alpha^2}}{\alpha + \sqrt{1+\alpha^2}}, & \text{if } \alpha \geq \frac{1}{4} \sqrt{2 + 2\lambda^2}, \quad \frac{2\alpha^2 - \sqrt{1+\alpha^2}}{\alpha(\alpha + \sqrt{1+\alpha^2})} < x < 2.
\end{cases}
\]

Finally we consider

\[
\frac{(2 + \alpha)|k|}{(1+k^2)} < \rho < \min \left\{ \frac{\alpha(2 - |k|)}{1 + 2\alpha}, \frac{\sqrt{1+\alpha^2}}{\alpha + \sqrt{1+\alpha^2}} \right\}.
\]

**Lemma 3.6.** Set

\[
f(x) = \frac{(2-x)(1+x^2)}{x}, \quad x \in (0, 2),
\]

then \(f(x)\) is a decreasing function. For each \(\alpha > 0\), there exists a unique \(x(\alpha)\) such that \(f(x(\alpha)) = \frac{(2+\alpha)(1+2\alpha)}{\alpha}\). In particular, we have

\[
\begin{cases}
0 < x(\alpha) < \frac{1}{2}, \\
x'(\alpha) = \frac{x'(1-\alpha^2)}{\alpha^2(2x^2-1)}.
\end{cases}
\]
Hence $\max_{\alpha > 0} x(\alpha) = x(1) > \frac{1}{5}$.

Proof. Set

$$f(x) = \frac{(2 - x)(1 + x^2)}{x}, \quad x \in (0, 2).$$

Then $f(2) = 0$, $\lim_{x \to 0} f(x) = +\infty$ and

$$f'(x) = \frac{(4x - 1 - 3x^2)x - (2 + 2x^2 - x - x^3)}{x^2} = -\frac{2(x^3 - x^2 + 1)}{x^2} < 0, \quad x \in (0, 2),$$

i.e., $f(x)$ is a decreasing function. So for each $\alpha > 0$, there exists a unique $x(\alpha)$ such that $f(x(\alpha)) = \frac{(2 + \alpha)(1 + 2\alpha)}{\alpha}$.

Since $f(x(\alpha)) = \frac{(2 + \alpha)(1 + 2\alpha)}{\alpha}$ is equivalent to

$$0 = \alpha(2 - x(\alpha) + 2x^2(\alpha) - x^3(\alpha)) - x(\alpha)(2 + 5\alpha + 2\alpha^2),$$

we get

$$0 = x\alpha(2 - x + 2x^2 - x^3) - x^2\alpha(5 + 4\alpha) + x\alpha^2(-1 + 4x - 3x^2)x'(\alpha) - x'(\alpha)x\alpha(2 + 5\alpha + 2\alpha^2) = x\alpha(2 - x + 2x^2 - x^3) - x\alpha(2 + 5\alpha + 2\alpha^2) + x\alpha'\alpha(-1 + 4x - 3x^2)x'(\alpha) - x\alpha'(\alpha) - x(\alpha)(2 + 5\alpha + 2\alpha^2) = x\alpha(2 - x + 2x^2 - x^3) - x\alpha(2 + 5\alpha + 2\alpha^2) + x\alpha'(\alpha) - x(\alpha)(2 + 5\alpha + 2\alpha^2) = 2x^2(1 - \alpha^2) - \alpha^2x'(\alpha)(2 - 2x^2 + 2x^3).$$

So it holds that

$$x'(\alpha) = \frac{x^2(1 - \alpha^2)}{\alpha^2(x^3 - x^2 + 1)}.$$

Since $x'(\alpha) > 0$ as $\alpha < 1$, and $x'(\alpha) < 0$ as $\alpha > 1$, we have $\max_{\alpha > 0} x(\alpha) = x(1)$.

Note that $\frac{(2 + \alpha)(1 + 2\alpha)}{\alpha} \geq 9$, so $x(\alpha) \leq x(1)$, and $f(x(\alpha)) \geq f(x(1)) = 9$. We calculate

$$f\left(\frac{1}{4}\right) = \frac{(2 - \frac{1}{4})\left(1 + \frac{1}{16}\right)}{\frac{1}{4}} = \frac{7 \times 17}{16} < 9,$$

$$f\left(\frac{1}{5}\right) = \frac{(2 - \frac{1}{5})\left(1 + \frac{1}{25}\right)}{\frac{1}{5}} = \frac{9 \times 26}{25} > 9,$$

hence $\frac{1}{5} < x(1) < \frac{1}{4}$.  

\[\square\]

Corollary 1. Let $x(\alpha)$ be determined by Lemma 3.6. Then for any $\alpha > 0$, $|k| \in (0, x(\alpha)) \subset (0, \frac{1}{4})$, we have

$$\frac{(2 + \alpha)|k|}{1 + k^2} < \frac{(2 + \alpha)x(\alpha)}{1 + x(\alpha)^2} = \frac{\alpha(2 - x(\alpha))}{(1 + 2\alpha)} < \frac{\alpha(2 - |k|)}{1 + 2\alpha}.$$

Since $|k| \in (0, x(\alpha))$, it holds that $f(|k|) > f(x(\alpha))$, i.e.,

$$\frac{(2 - |k|)(1 + k^2)}{|k|} > \frac{(1 + 2\alpha)(2 + \alpha)}{\alpha},$$
or equivalently
\[
\frac{\alpha(2 - |k|)}{(1 + 2\alpha)} > \frac{(2 + \alpha)|k|}{(1 + k^2)}
\]
So the result is obvious.

Next we consider
\[
\frac{(2 + \alpha)|k|}{1 + k^2} < \frac{\sqrt{1 + \alpha^2}}{\alpha + \sqrt{1 + \alpha^2}}. \quad 2\alpha^2 > \sqrt{1 + \alpha^2}.
\tag{27}
\]

**Lemma 3.7.** Let \(\alpha_0\) be the solution of \(2\alpha_0^2 = \sqrt{1 + \alpha_0^2}\), and \(y(\alpha)\) be the solution of the following equation
\[
y = \frac{\sqrt{1 + \alpha^2}}{(2 + \alpha)(\alpha + \sqrt{1 + \alpha^2})}, \quad \forall \alpha \geq \alpha_0.
\tag{28}
\]
Then \(y(\alpha)\) is a decreasing function as regards \(\alpha\).

Given \(\alpha > 0\), when \(|k| \in (0, y(\alpha))\), we have
\[
\frac{(2 + \alpha)|k|}{1 + k^2} < \frac{\sqrt{1 + \alpha^2}}{\alpha + \sqrt{1 + \alpha^2}}.
\]

**Proof.** Set
\[
g(\alpha) = \frac{(2 + \alpha)(\alpha + \sqrt{1 + \alpha^2})}{\sqrt{1 + \alpha^2}}, \quad \alpha \geq 0.
\]
Because
\[
g'(\alpha) = \frac{(\alpha + \sqrt{1 + \alpha^2}) + (2 + \alpha)\left(\frac{\alpha}{\sqrt{1 + \alpha^2}}\right)}{\sqrt{1 + \alpha^2}} - \frac{(2 + \alpha)(\alpha + \sqrt{1 + \alpha^2})}{(\sqrt{1 + \alpha^2})^2} \frac{\alpha}{\sqrt{1 + \alpha^2}}
\]
\[
= \frac{(1 + \alpha^2)(\alpha + \sqrt{1 + \alpha^2}) + \sqrt{1 + \alpha^2}(2 + \alpha)(\alpha + \sqrt{1 + \alpha^2})}{(1 + \alpha^2)\sqrt{1 + \alpha^2}} - \frac{(2 + \alpha)(\alpha + \sqrt{1 + \alpha^2})}{(1 + \alpha^2)} \frac{\alpha}{\sqrt{1 + \alpha^2}}
\]
\[
= \frac{(\alpha + \sqrt{1 + \alpha^2})}{(1 + \alpha^2)\sqrt{1 + \alpha^2}} \left[ (1 + \alpha^2) + (2 + \alpha)(\sqrt{1 + \alpha^2} - \alpha) \right] > 0,
\]
g(\alpha) is an increasing function.

Let \(y_1(\alpha)\) and \(y_2(\alpha)\) be the solutions of the following equation
\[
y = \frac{\sqrt{1 + \alpha^2}}{(2 + \alpha)(\alpha + \sqrt{1 + \alpha^2})} = \frac{1}{g(\alpha)}.
\]
It is equivalent to
\[
0 = 1 - g(\alpha)y + y^2 = \left(y - \frac{g(\alpha)}{2}\right)^2 + 1 - \frac{g^2(\alpha)}{4}.
\]
Solving the above equation, we get
\[
y_1(\alpha) = \frac{g(\alpha) - \sqrt{g^2(\alpha) - 4}}{2}, \quad y_2(\alpha) = \frac{g(\alpha) + \sqrt{g^2(\alpha) - 4}}{2}.
\]
Since \(y_1(\alpha) < y_2(\alpha)\) and \(y_1(\alpha)y_2(\alpha) = 1\),
\[
y_1(\alpha) < 1, \quad y_2(\alpha) > 1.
\]
When $1 \leq y_2(\alpha) \leq 2$, we have $2 < g(\alpha) = \frac{2 + \alpha \sqrt{1 + \alpha^2}}{\sqrt{1 + \alpha^2}} \leq \frac{5}{2}$, or
\[ 4\sqrt{1 + \alpha^2} < 2(2 + \alpha)(\alpha + \sqrt{1 + \alpha^2}) \leq 5\sqrt{1 + \alpha^2}. \]

It is equivalent to
\[ 2\alpha(2 + \alpha) \leq (1 - 2\alpha)\sqrt{1 + \alpha^2}, \]
which implies $\alpha < \frac{1}{2} < \frac{1}{4}\sqrt{2 + 2\sqrt{17}} = \alpha_0$. Therefore, equation (28) has a unique solution $0 < y(\alpha) < 1$ as $\alpha \geq \alpha_0$.

Let $y(\alpha)$ be the solution of equation (28). Then $y(\alpha)$ satisfies
\[ (2 + \alpha) \left( \alpha + \sqrt{1 + \alpha^2} \right) y(\alpha) - (1 + y^2(\alpha))\sqrt{1 + \alpha^2} = 0. \]

We calculate
\[
0 = (2 + \alpha) \left( \alpha + \sqrt{1 + \alpha^2} \right) y'(\alpha) - 2y(\alpha)y'(\alpha)\sqrt{1 + \alpha^2} + \alpha \left( y'(\alpha) \left( 2 + \frac{2\alpha}{\sqrt{1 + \alpha^2}} + 2\alpha + \sqrt{1 + \alpha^2} + \frac{\alpha^2}{\sqrt{1 + \alpha^2}} \right) y(\alpha) - (1 + y^2(\alpha))\frac{\alpha}{\sqrt{1 + \alpha^2}} \right.
\]
\[ = y'(\alpha) \left[ (2 + \alpha) \left( \alpha + \sqrt{1 + \alpha^2} \right) - 2y(\alpha)\sqrt{1 + \alpha^2} \right)
\]
\[ + \left( 2\left( \alpha^2 + \alpha + (1 + \alpha)\sqrt{1 + \alpha^2} \right) + \frac{1}{\sqrt{1 + \alpha^2}} \right) y(\alpha) - (1 + y^2(\alpha))\frac{\alpha}{\sqrt{1 + \alpha^2}} \right]
\[ = \frac{y'(\alpha)}{y(\alpha)} \left[ (1 + y^2(\alpha))\sqrt{1 + \alpha^2} - 2y^2(\alpha)\sqrt{1 + \alpha^2} \right] - \frac{\alpha}{1 + \alpha^2} (2 + \alpha)(\alpha + \sqrt{1 + \alpha^2}) y(\alpha)
\]
\[ + \left( 2\left( \alpha^2 + \alpha + (1 + \alpha)\sqrt{1 + \alpha^2} \right) + \frac{1}{\sqrt{1 + \alpha^2}} \right) y(\alpha) \right]
\[ = \frac{y'(\alpha)}{y(\alpha)} (1 - y^2(\alpha))\sqrt{1 + \alpha^2} + \frac{y(\alpha)}{1 + \alpha^2} \left[ (\alpha^2 + 1)\sqrt{1 + \alpha^2} + \alpha^2 + 2\alpha + 2 \right]. \]

So
\[
y'(\alpha) = -\frac{y^2(\alpha)}{(1 - y^2(\alpha))(1 + \alpha^2)\sqrt{1 + \alpha^2}} \left[ \alpha^2(\sqrt{1 + \alpha^2} - \alpha) + 2 + 2\alpha^2 + 2\alpha^4 + \sqrt{1 + \alpha^2} \right].
\]

Obviously, if $y(\alpha) < 1$, we have $y'(\alpha) < 0$.

Clearly, if $|k| < y(\alpha)$, it holds that
\[ \frac{|k|}{1 + k^2} < \frac{y(\alpha)}{1 + y^2(\alpha)} = \frac{\sqrt{1 + \alpha^2}}{2 + \alpha}(\alpha + \sqrt{1 + \alpha^2}) \]

and
\[ \frac{(2 + \alpha)|k|}{1 + k^2} < \frac{\sqrt{1 + \alpha^2}}{\alpha + \sqrt{1 + \alpha^2}}. \]

The desired result is proved. \[\square\]

Now we compare $y(\alpha)$ and $x(\alpha)$.  

Lemma 3.8. Let \( x(\alpha) \) and \( y(\alpha) \) be the solutions of equations \((25)\) and \((28)\), respectively. Let \( \alpha_0 \) be the solution of \( 2\alpha^2 = \sqrt{1 + \alpha^2} \), and \( \alpha^* \) be the solution of \( x(\alpha) = \frac{2\alpha^2 - \sqrt{1 + \alpha^2}}{\alpha(\alpha + \sqrt{1 + \alpha^2})} \). If \( \alpha_0 < \alpha^* < 1 \), then \( x(\alpha) \) and \( y(\alpha) \) satisfy

\[
\begin{align*}
&\begin{cases}
x(\alpha) < y(\alpha), & 0 \leq \alpha < \alpha^*, \\
x(\alpha^*) = y(\alpha^*), & \alpha = \alpha^*, \\
y(\alpha) < x(\alpha), & \alpha > \alpha^*.
\end{cases}
\end{align*}
\]

\((29)\)

Proof. Let \( x(\alpha) \) be the solution of the following equation

\[
\frac{(2 + \alpha)x}{1 + x^2} = \frac{\alpha(2 - x)}{(1 + 2\alpha)}, \quad \alpha \geq 0.
\]

According to Lemma 3.6, \( x(\alpha) \) is increasing as \( \alpha < 1 \), and decreasing as \( \alpha > 1 \).

Let \( y(\alpha) \) be the minimal solution of the following equation

\[
\frac{y}{1 + y^2} = \frac{\sqrt{1 + \alpha^2}}{(2 + \alpha)(\alpha + \sqrt{1 + \alpha^2})}, \quad \alpha \geq 0.
\]

Then

\[
\frac{(2 + \alpha)y(\alpha)}{1 + y^2(\alpha)} - \frac{(2 + \alpha)x(\alpha)}{1 + x^2(\alpha)} = \frac{\sqrt{1 + \alpha^2}}{(\alpha + \sqrt{1 + \alpha^2})} - \frac{\alpha(2 - x(\alpha))}{1 + 2\alpha}
\]

\[
= \frac{\sqrt{1 + \alpha^2} - 2\alpha^2 + \alpha (\alpha + \sqrt{1 + \alpha^2}) x(\alpha)}{(1 + 2\alpha)(\alpha + \sqrt{1 + \alpha^2})}
\]

\[
= \frac{\alpha}{(1 + 2\alpha)} \left( x(\alpha) - \frac{2\alpha^2 - \sqrt{1 + \alpha^2}}{\alpha (\alpha + \sqrt{1 + \alpha^2})} \right).
\]

Obviously, if \( x(\alpha) - \frac{2\alpha^2 - \sqrt{1 + \alpha^2}}{\alpha (\alpha + \sqrt{1 + \alpha^2})} > 0 \), then \( y(\alpha) > x(\alpha) \) and \( x(\alpha) - \frac{2\alpha^2 - \sqrt{1 + \alpha^2}}{\alpha (\alpha + \sqrt{1 + \alpha^2})} < 0 \), we have \( y(\alpha) < x(\alpha) \).

Set \( \alpha^* \) be the solution of the following equation

\[
\frac{(2 + \alpha)y(\alpha)}{1 + y^2(\alpha)} = \frac{(2 + \alpha)x(\alpha)}{1 + x^2(\alpha)} = \frac{\sqrt{1 + \alpha^2}}{(\alpha + \sqrt{1 + \alpha^2})} = \frac{\alpha(2 - x(\alpha))}{1 + 2\alpha}
\]

or equivalently

\[
x(\alpha) = \frac{2\alpha^2 - \sqrt{1 + \alpha^2}}{\alpha (\alpha + \sqrt{1 + \alpha^2})}.
\]

We calculate as follows

\[
\begin{align*}
\alpha &= 0, \quad x(0) = 0, \quad y(0) = 1, \\
\alpha &= \alpha_0, \quad \frac{(2 + \alpha_0)x(\alpha_0)}{1 + x^2(\alpha_0)} = \frac{\alpha_0(2 - x(\alpha_0))}{1 + 2\alpha_0}, \quad \frac{(2 + \alpha_0)y(\alpha_0)}{1 + y^2(\alpha_0)} = \frac{\sqrt{1 + \alpha_0^2}}{\alpha_0 + \sqrt{1 + \alpha_0^2}}, \\
\alpha &= 1, \quad \frac{3x(1)}{1 + x^2(1)} = \frac{2 - x(1)}{3}, \quad \frac{3y(1)}{1 + y^2(1)} = \frac{\sqrt{2}}{1 + \sqrt{2}}.
\end{align*}
\]

Since

\[
\frac{\sqrt{1 + \alpha_0^2}}{\alpha_0 + \sqrt{1 + \alpha_0^2}} = \frac{2\alpha_0}{1 + 2\alpha_0} > \frac{\alpha_0(2 - x(\alpha_0))}{1 + 2\alpha_0},
\]

\[
\frac{2 - x(1)}{3} - \frac{\sqrt{2}}{1 + \sqrt{2}} = \frac{3\sqrt{2} - 4 - x(1)}{3} > 0,
\]

we have \( x(\alpha_0) < y(\alpha_0) \) and \( y(1) < x(1) \). Therefore, \( \alpha_0 < \alpha^* < 1 \).
Since $y(\alpha)$ is a decreasing function, and $x(\alpha)$ is increasing in $\alpha \in (0,1)$ and decreasing as $\alpha > 1$, we have

\[
\begin{align*}
y(\alpha) > x(\alpha), & \quad 0 < \alpha < \alpha^*, \\
y(\alpha^*) = x(\alpha^*), & \quad \alpha = \alpha^*, \\
y(\alpha) < x(\alpha), & \quad \alpha^* < \alpha. 
\end{align*}
\]

The desired result follows. \(\square\)

**Corollary 2.** Let $x(\alpha)$ and $y(\alpha)$ be the solutions of equations (25) and (28), respectively. If $|k| < \min\{x(\alpha), y(\alpha)\}$, then we have

\[
\frac{(2 + \alpha)|k|}{1 + k^2} < \min \left\{ \frac{\alpha(2 - |k|)}{1 + 2\alpha}, \frac{\sqrt{1 + \alpha^2}}{1 + \sqrt{1 + \alpha^2}} \right\}. \tag{30}
\]

**Corollary 3.** Let $x(\alpha)$ and $y(\alpha)$ be the solutions of equations (25) and (28), respectively. If $|k| < \min\{x(\alpha), y(\alpha)\}$, and

\[
\frac{(2 + \alpha)|k|}{(1+k^2)} < \rho < \min \left\{ \frac{\alpha(2 - |k|)}{1 + 2\alpha}, \frac{\sqrt{1 + \alpha^2}}{1 + \sqrt{1 + \alpha^2}} \right\}, \tag{31}
\]

then the inequalities (22) hold.

The following Lemma gives the solvability of inequalities (21).

**Theorem 3.9.** Let $\alpha$, $k$ and $\rho$ satisfy inequality (31). Then

\[
\frac{\alpha}{4} \left[ \sqrt{8\lambda^2 + (2\alpha - |k|e^{|\lambda\tau}|)^2} + (2\alpha - |k|e^{|\lambda\tau}|) \right] + \frac{\rho}{2} + \lambda_0 + (\alpha - \lambda_0)\rho = \alpha. \tag{32}
\]

has a unique solution $\lambda_0$. For any $0 < \lambda \leq \min \left\{ \lambda_0, \frac{1}{\tau} \ln \left( \frac{2\rho + \lambda_0 + (\alpha - \lambda_0)\rho}{|k|} \right) \right\}$, $\lambda$ and $\tau$ satisfy inequalities (21).

**Proof.** Suppose that $\alpha$, $k$ and $\rho$ satisfy inequality (31). Then we have $2\alpha - (1 + 2\alpha)\rho - \alpha |k| > 0$.

Set

\[
G(\lambda) = \frac{\alpha}{4} \left[ \sqrt{8\lambda^2 + (2\alpha - |k|e^{|\lambda\tau}|)^2} + (2\alpha - |k|e^{|\lambda\tau}|) \right] + \alpha^2 \lambda - \frac{\alpha |k|}{2} (e^{\lambda \tau} - 1) + \lambda (1 - \rho).
\]

We see that $G(0) = 0$ and

\[
G'(\lambda) = \frac{\alpha}{4} \left[ \frac{8\lambda + (2\alpha - |k|e^{|\lambda\tau}|)(2\alpha - |k|e^{|\lambda\tau}|) - (2\alpha + |k|e^{|\lambda\tau}|)}{\sqrt{8\lambda^2 + (2\alpha + |k|e^{|\lambda\tau}|)^2}} + (\alpha^2 - (1 - \rho) + \frac{\alpha |k|}{2} e^{\lambda \tau}) \right] = \frac{\alpha}{4} \frac{8\lambda + (2\alpha + |k|e^{|\lambda\tau}|)(2\alpha + |k|e^{|\lambda\tau}|) + \alpha}{\sqrt{8\lambda^2 + (2\alpha + |k|e^{|\lambda\tau}|)^2}} + \frac{\alpha}{4} (2\alpha + |k|e^{|\lambda\tau}| + (1 - \rho) > 0.
\]

So $G(\lambda)$ is a monotone increasing function with $\lim_{\lambda \to \infty} G(\lambda) = \infty$. Hence if given $\alpha - \alpha \rho - \frac{\rho}{2} - \frac{\alpha |k|}{2} > 0$, then there exists a unique $\lambda_0$ such that

\[
G(\lambda_0) = \alpha - \alpha \rho - \frac{\rho}{2} - \frac{\alpha |k|}{2}.
\]
That is just the equation (32).

From (32) we see
\[ \frac{\alpha |k|}{2} e^{\lambda_0 \tau} + \rho > 2 \alpha \rho < \alpha, \]
or equivalently,
\[ \rho < \frac{\alpha (2 - |k| e^{\lambda_0 \tau})}{1 + 2 \alpha} < \frac{\alpha (2 - |k|)}{1 + 2 \alpha}. \]

According to \( \frac{\alpha (2 - |k| e^{\lambda_0 \tau})}{1 + 2 \alpha} < (1 + \alpha) \frac{|k| e^{\lambda_0 \tau}}{|k| e^{\lambda_0 \tau}}, \) we deduce the second inequality in (21). In particular, \( |k| e^{\lambda_0 \tau} < 2. \)

Set
\[ \lambda_1 = \frac{1}{\tau} \ln |k| \left( \frac{2 \rho}{(2 + \alpha) + \sqrt{(2 + \alpha)^2 - 4 \rho^2}} \right). \]
We have
\[ e^{\lambda_1 \tau} = \frac{(2 + \alpha) - \sqrt{(2 + \alpha)^2 - 4 \rho^2}}{2 \rho |k|} = \frac{2 \rho |k|}{|k| \left( (2 + \alpha) + \sqrt{(2 + \alpha)^2 - 4 \rho^2} \right)}, \]
or equivalently,
\[ \frac{(2 + \alpha)|k| e^{\lambda_1 \tau}}{(1 + k^2 e^{2 \lambda_1 \tau})} = \rho. \]

Now we suppose that \( 0 < \lambda \leq \frac{1}{\tau} \ln \frac{2 \rho}{|k| \left( (2 + \alpha) + \sqrt{(2 + \alpha)^2 - 4 \rho^2} \right)}, \)
\[ \frac{x}{1 + x} \] is an increasing function as \( x \in (0, 1), \) the inequality \( \lambda \leq \lambda_1 \) implies
\[ \frac{(2 + \alpha)|k| e^{\lambda \tau}}{(1 + k^2 e^{2 \lambda \tau})} < \frac{(2 + \alpha)|k| e^{\lambda_1 \tau}}{(1 + k^2 e^{2 \lambda_1 \tau})} = \rho < \frac{2 \alpha (1 - |k| e^{\lambda_0 \tau})}{1 + 2 \alpha} < \frac{2 \alpha (1 - |k| e^{\lambda_0 \tau})}{1 + 2 \alpha}. \]

The third inequality in (21) follows.

Again using the monotonicity of \( G(\lambda), \) we have \( G(\lambda) \leq G(\lambda_0) = \alpha - \alpha \rho - \frac{\rho}{2} - \frac{\alpha |k|}{2}, \)
\[ \frac{\alpha}{4} \left[ \sqrt{8 \lambda^2 + (2 \lambda + |k| e^{\lambda \tau})^2 + (2 \lambda + |k| e^{\lambda \tau})} \right] + \frac{\rho}{2} + \lambda + (\alpha - \lambda) \rho - \alpha \leq 0. \]

All inequalities in (21) are verified. \( \square \)

Observe that the stability result in Theorem 3.9 depends strongly upon the additional parameter \( \rho. \) However, in equation (1), we only have three parameters \( \alpha, \) \( k \) and \( \tau. \) Hence, we shall further consider how to give a stability result by balancing the relationship between \( \alpha \) and \( k. \)

**Theorem 3.10.** Let \( \alpha \) and \( k \) be given as in (7). If \( \alpha \) and \( |k| \) satisfy
\[ \frac{(2 + \alpha)|k|}{(1 + k^2)} < \min \left\{ \frac{2 \alpha (1 - |k|)}{1 + 2 \alpha}, \frac{\sqrt{1 + \alpha^2}}{\alpha + \sqrt{1 + \alpha^2}} \right\}, \]
then (7) is exponentially stable.

**Proof.** Suppose that \( \alpha \) and \( |k| \) satisfy
\[ \frac{(2 + \alpha)|k|}{(1 + k^2)} < \min \left\{ \frac{2 \alpha (1 - |k|)}{1 + 2 \alpha}, \frac{\sqrt{1 + \alpha^2}}{\alpha + \sqrt{1 + \alpha^2}} \right\}. \]

Set
\[ \rho = \frac{1}{2} \left( \frac{(2 + \alpha)|k|}{(1 + k^2)} + \min \left\{ \frac{2 \alpha (1 - |k|)}{1 + 2 \alpha}, \frac{\sqrt{1 + \alpha^2}}{\alpha + \sqrt{1 + \alpha^2}} \right\} \right), \]
then \(\alpha, k\) and \(\rho\) satisfy
\[
\frac{(2 + \alpha)|k|}{(1 + k^2)} < \rho < \min \left\{ \frac{2\alpha(1 - |k|)}{1 + 2\alpha}, \frac{\sqrt{1 + \alpha^2}}{\alpha + \sqrt{1 + \alpha^2}} \right\}.
\]

According to Theorem 3.9 if \(0 < \lambda < \min\{\lambda_0, \lambda_1\}\), then \(\alpha, k, \rho, \lambda\) and \(\tau\) satisfy (21). Theorem 3.5 asserts that \(V_0(t)\) is exponentially stable. Hence system (1) is exponentially stable.

4. Conclusions. In this paper, we use the Lyapunov function of the system to discuss the stability of an wave equation with the interior control and the boundary delay. Here we mainly discuss the relationship between \(\alpha\) and \(k\) which makes the system exponentially stable.

Our main contribution in this paper includes the following aspects:

1) We give the condition on the exponential stability of system (1) by discussing the relationship between \(\alpha\) and \(k\).

2) We obtain the estimate of the exponential decay rate of system (1), i.e. \(0 < 2\lambda < 2\min\{\lambda_0, \lambda_1\}\).

However, there are still some open problems to be solved. For example, if \(\alpha\) is large enough, how can we obtain the exponential decay rate? Especially, if the condition in (33) is not true, how can we explore the stability? We will seriously consider such problems in the future.

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