Primal and dual conic representable sets: a fresh view on multiparametric analysis *

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Abstract

This paper introduces the concepts of the primal and dual conic (linear inequality) representable sets and applies them to explore a novel kind of duality in multiparametric conic linear optimization. Such a kind of duality may be described by the set-valued mappings between the primal and dual conic representable sets, which allows us to generalize as well as treat previous results for multiparametric analysis in a unified framework. We then discuss the behaviour of the optimal partition of a conic representable set and investigate the multiparametric analysis of conic linear optimization problems. This leads to the invariant region decomposition of a conic representable set that is more general than the known results in the literatures. We also study the properties of the optimal objective values as a function of that parametric vectors. All results are corroborated by examples having correlation.

Keywords: Multiparametric conic linear optimization, conic representable set, optimal partition, multiparametric analysis, duality, multiparametric KKT property

AMS subject classifications. Primary: 90C31; Secondary: 90C25, 90C05, 90C22, 90C46

1 Introduction

We are interested in the following multiparametric conic linear optimization (mp-CLO) problems with two independent vectors of parameters $u, v \in \mathbb{R}^r$

$$p^*(u) = \min \langle c + M^T u, x \rangle$$
$$s.t. \quad Ax = b,$$
$$x \in K$$

(1)

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and

\[ d^*(v) = \min_{s.t. \ By = a, \ y \in K^*} \langle d + M^Tv, y \rangle \]

for given vectors \( a \in \mathbb{R}^l, b \in \mathbb{R}^m, c, d \in \mathbb{R}^q \) and matrices \( A \in \mathbb{R}^{m \times q}, B \in \mathbb{R}^{l \times q}, \ M \in \mathbb{R}^{r \times q} \), where \( K \subset \mathbb{R}^q \) is a pointed, closed, convex, solid (with non-empty interior) cone (for this formulation of a primal-dual pair and its properties, Ref. [35]); and \( K^* \) is the dual of \( K \) under the standard inner-product, that is,

\[ K^* = \{ y \in \mathbb{R}^q | \langle y, x \rangle \geq 0, \forall x \in K \}. \]

The minimum values of the objective functions are denoted by \( p^*(u) \) and \( d^*(v) \), respectively. Such two problems are multiparametric linear programming (mpLP) if \( K \) is the nonnegative orthant, and multiparametric semidefinite programming (mpSDP) if \( K \) is the cone of symmetric positive semidefinite matrices (e.g., Refs. [2, 5, 8, 36]).

The following two technical claims are assumed to hold throughout this paper.

**Assumption 1.** Three range spaces \( R(A^T), R(B^T), R(M^T) \) are orthogonal to each other, and their direct sum is equal to the whole space \( \mathbb{R}^q \).

**Assumption 2.** \( MM^T = I_r \), a \( r \times r \) unit matrix.

Assumption 1 ensures that the perturbation of the objective function is independent of the constraints in mpCLO problems (1) and (2). A detailed explanation for the assumption will be given in the next section.

### 1.1 Motivations

The original motivation of this paper comes from the inductive proof of the strong duality theorem of conic linear optimization, see Theorem 2.3 and Subsection 3.3. This proof requires that the complement slackness property remains the same when the number of constraints changes. More precisely, for the problems (1) and (2), their optimal solution pair \( (x^*(u), y^*(v)) \) satisfies the complement slackness property

\[ \langle x^*(u), y^*(v) \rangle = 0, \]

see Theorem 3.6 below. The surprising aspect of this result is that it holds for some vector pairs \( (u, v) \in \mathbb{R}^r \times \mathbb{R}^r \), under one Slater type condition.

We exploit the special structure of a CLO problem to obtain a new algebraic representation of the Lagrangian dual, see Section 2. Such a dual, called the nonstandard dual by us, results in a unified representation of the feasible regions of a pair of primal-dual CLO problems. Surprisingly, two nonstandard dual programs of the mpCLO problems (1) and (2) also satisfy the complement slackness property

\[ \langle \bar{x}^*(v), \bar{y}^*(u) \rangle = 0 \]

under another Slater type condition, where \( (\bar{x}^*(v), \bar{y}^*(u)) \) denotes the corresponding optimal solution pair, see Theorem 3.6 below.
These characterization results directly motivate us to define set-valued mappings between the following conic (linear inequality) representable sets:

\[
\Theta_P = \{ v \in \mathbb{R}^r | d + M^T v + B^T w^1 \in K \text{ for some } w^1 \in \mathbb{R}^l \}, \quad (5a)
\]

\[
\Theta_D = \{ u \in \mathbb{R}^r | c + M^T u + A^T w^2 \in K^* \text{ for some } w^2 \in \mathbb{R}^m \}, \quad (5b)
\]

see Section 3.1. The definition helps us to present a geometric framework that unifies and extends some of the properties of parametric LPs and SDPs to the case of mpCLOs. In our terminology, \( \Theta_P \) and \( \Theta_D \) denote the primal and dual conic representable sets, respectively.

### 1.2 Related works

Prior to the study of parametric analysis in recent years, the actual invariancy region plays an important role in the development of parametric LPs and SDPs. Adler and Monteiro [1] investigated the parametric analysis of LPs first using the optimal partition approach, in which they identify the range of the single-parameter where the optimal partition remains invariant. Other treatments of parametric analysis for LPs based on the optimal partition approach reference the papers [7, 14, 22, 25, 28, 41, 30] and etc. The actual invariancy region has been studied extensively in the setting of SDP (Refs. [24, 33]), the second-order conic optimization (Refs. [34]) and more generally in CLO (Refs. [47]).

The first method for solving parametric LPs was proposed by Gass and Saaty [21], whereas the first method for solving multiparametric LPs was presented by Gal and Nedoma [20]. Recently there has been growing interest in multiparametric optimizations arising from especially the area of process engineering such as process design, optimization and control. The survey by Pistikopoulos et al. [37] contributes recent theoretical and algorithmic advances, and applications in the areas of multi-parametric programming and specifically explicit/multi-parametric model predictive control. So far, various kinds of invariances in parametric/multiparametric problems were used mainly in single-parametric or bi-parametric analysis. To the best of our knowledge, almost all approaches to multiparametric LPs that have appeared in the literature (Refs [11, 16, 17, 18, 19, 42, 45]) use bases to get a description of the invariancy regions.

### 1.3 Contributions

This paper studies multiparametric optimization in general CLO problems in which either the objective function or the right hand side is perturbed along many fixed directions. We first establish the connection by showing that the conic representable set defined by (5a) or (5b) is viewed as the feasible region of a appropriately defined mpCLO problem, although this connection does not seem to be easily identifiable. We then develop the classical duality theory by means of the set-valued mappings to relate the two conic representable sets. This treatment makes it possible to tie together some known, yet scattered, results and to derive new ones. All these results
can be used to develop the optimal partition approach given in [1] for parametric LPs and in [22] for parametric SDPs.

Roughly speaking, our main contributions are
(1) Characterization of the relationship between the primal and dual conic representable sets.
(2) Presentation of a novel kind of duality in CLO.
(3) Identification of the optimal partitions and development of parametric analysis technique.

In the first category, we provide a useful tool—the set-valued mapping. Such a tool will play a critical role in our analysis.

In the second category, we develop the classical duality theory of CLO. Besides the complementary slackness properties mentioned earlier, we present the weak and the strong duality properties for the pair of problems (1) and (2). We show that the sum of their objective optimal values is a bilinear function with respect to the vectors of parameters $u$ and $v$, see Theorem 3.8 below. The corresponding multiparametric KKT (mpKKT) conditions are also given, see Theorem 3.10 below.

In the third category, along with the invariant region decomposition, our result also captures and generalizes the SDP cases from Mohammad-Nezhad and Terlaky in [33]. We develop the concepts of the nonlinearity region and the transition point for the optimal partition to conic representable sets by means of the set-valued mappings, and provide some sufficient conditions for the existence of a nonlinearity region and a transition point. Such concepts are very useful for the analysis of a mpCLO problem since the nonlinearity region can be regarded as a stability region and its identification has a great influence on the post-optimal analysis of SDPs.

1.4 Organization of the paper

In the next section we review some useful results from convex analysis and the duality theory of cone linear optimization. In particular, we give a new algebraic representation of dual programs and obtain the nonstandard dual forms of the problems (1) and (2). In Section 3, we use the set-valued mappings to establish the connection between the aforementioned primal and dual conic representable sets and develop the duality theory in CLO. In Section 4, some fundamental concepts are introduced and several examples are presented to illustrate the concepts. The extension of the corresponding approach to multiparametric analysis of cone linear optimization is the topic of Section 5. The identification of the optimal partitions of a conic representable set is discussed and the behavior of the optimal value function on its domain is studied in this section. Finally, we conclude the paper with some remarks in Section 6.

2 Preliminaries

The aim of this section is two twofold. On the one hand, we want to introduce the notation that will be used throughout this paper. On the other hand, we state some
useful results which facilitate some of the subsequent proofs.

2.1 Notation

We first review some useful facts about convex sets and convex cones. A standard reference for convex analysis is the book by Rockafellar [38].

The (topological) boundary of a set \( C \in \mathbb{R}^q \) is denoted \( \partial C \) and defined as \( \partial C = \text{cl}(C) \setminus \text{int}(C) \), where \( \text{cl}(C) \) and \( \text{int}(C) \) denote closure and interior of \( C \), respectively. The notation \( \dim(C) \) denotes the affine dimensional of \( C \). A set \( C \) is called simply connected if for any two points \( x, y \in C \), there is a continuous curve \( \Gamma \subset C \) connecting \( x \) and \( y \). In particular, a singleton set is called a simply connected set. A set \( C \) is called convex if for any \( x, y \in C \), the linear segment \( [x, y] = \{ \alpha x + (1 - \alpha) y | \alpha \in [0, 1] \} \) is contained in \( C \). The convex hull of \( C \), denoted by \( \text{conv}(C) \), is a set of all convex combinations from \( C \).

Let \( C \) be a nonempty convex set in \( \mathbb{R}^q \). A vector \( 0 \neq h \in \mathbb{R}^q \) is called a recession direction of \( C \) if \( c + \lambda h \in C \) for all \( c \in C \) and \( \lambda > 0 \). The set of all recession directions of \( C \) is called the recession cone of \( C \) and denoted by \( 0^+(C) \). Given a boundary point \( \bar{x} \) of \( C \), its normal cone \( \text{Normal}(C, \bar{x}) \) is defined by

\[
\text{Normal}(C, \bar{x}) = \{ c \in \mathbb{R}^q | \langle c, \bar{x} \rangle \geq \langle c, x \rangle \text{ for all } x \in C \}.
\]

A point \( \bar{x} \) is a vertex of \( C \) if its normal cone is full-dimensional.

A mapping \( \Phi(\xi) : \mathbb{R}^r \rightrightarrows \mathbb{R}^r \) is called a set-valued mapping if it assigns a subset of \( \mathbb{R}^r \) to each element of \( \mathbb{R}^r \). For every set \( C \in \mathbb{R}^r \), the image of a set-valued mapping \( \Phi \) is

\[
\Phi(C) = \{ \Phi(\xi) | \xi \in C \}.
\]

The notation \( R(A) = \{ Ax | x \in \mathbb{R}^q \} \) denotes the range space of the matrix \( A \).

Let \( X \) and \( Y \) be two vector spaces and \( U \) be an open subset of \( X \). We say that the mapping \( f : U \to Y \) is Gâteaux differentiable at a point \( x \in U \) in a direction \( h \in X \) if there exists \( f'(x, h) \in Y \) such that

\[
f'(x, h) = \lim_{t \to 0^+} \frac{1}{t}(f(x + th) - f(x)),
\]

where \( f'(x, d) \) is called the directional derivative of \( f \) at \( x \) in the direction \( h \). The mapping \( f \) is called Gâteaux differentiable at \( x \), provided that \( f'(x, h) \) exists for all \( h \in X \) and the mapping \( f'(x) : h \to f'(x, h) \) is a continuous linear operator from \( X \) into \( Y \).

2.2 The nonstandard dual

Note that the mpCLO problem [1], without perturbation is the typical form of CLO

\[
\begin{aligned}
\text{min} & \quad \langle c, x \rangle \\
\text{s.t.} & \quad Ax = b, \\
& \quad x \in K,
\end{aligned}
\]
and the corresponding (Lagrangian) dual is as follows

$$\begin{align*}
    \max & \sum_{i=1}^{m} b_i w_i \\
    \text{s.t.} & \quad A^T w + y = c, \\
    & \quad w \in \mathbb{R}^m, \ y \in K^*.
\end{align*}$$

If such two programs are feasible, and if $b = Ad$, then

$$\langle d, c - y \rangle = \langle d, A^T w \rangle = \langle Ad, w \rangle = b^T w.$$  \hfill (8)

For reasons of clarity and elegance, we will use an equivalent form instead of the dual program (7). That is, we rewrite the dual program (7) as a new objective function and new constraints. This can be implemented through the following process.

Left multiplying both sides of the equality constraint in the dual program (7) by the matrix $B$ and $M$, respectively, one has $By = Bc$ and $My = Mc$ under Assumptions 1 and 2. Conversely, if $y \in K^* satisifies By = Bc$ and $My = Mc$, then there is $w \in \mathbb{R}^m$ such that $y = c - A^T w$ based on the same assumptions. Therefore, the dual program (7) can be equivalently expressed as the following form

$$\begin{align*}
    \max & \langle d, c - y \rangle \\
    \text{s.t.} & \quad By = a, \\
    & \quad My = Mc, \\
    & \quad y \in K^*,
\end{align*}$$

where $a = Bc$. Since it has the same form as the primal program (6), the above dual program will be very convenient to be used to consistently represent the feasible regions of the primal-dual problems. To distinguish them, the problems (7) and (9) are called the standard dual and the nonstandard dual of the primal program (6), respectively. Of course, there’s no difference between such two dual programs except in forms.

For the primal and dual programs (6) and (7), the weak duality property holds. Then from the equality (8), the following result holds.

**Corollary 2.1.** If $b = Ad$ and $a = Bc$, then for any primal feasible solution $x$ of (6) and any dual feasible solution $y$ of (9), the weak duality property holds, i.e.,

$$\langle c, x \rangle \geq \langle d, c - y \rangle.$$ \hfill (10)

Equality holds if and only if $(x, y)$ is a pair of optimal solutions.

If the primal and dual programs have optimal solutions and the duality gap is zero, i.e., the equality (10) holds, then the Karush-Kuhn-Tucker (KKT) conditions for the primal-dual CLO pair are

$$\begin{align*}
    Ax &= b, \ x \in K, \quad \text{(11a)} \\
    By &= a, \ My = Mc, \ y \in K^*, \quad \text{(11b)} \\
    \langle x, y \rangle &= 0. \quad \text{(11c)}
\end{align*}$$

Conversely, if $(x^*, y^*) \in \mathbb{R}^q \times \mathbb{R}^q$ is a pair of solutions of the system (11a)-(11c), then $(x^*, y^*)$ is a pair of optimal solutions of the primal-dual program pair (6) and (9).
Corollary 2.2. The optimal solutions of the primal-dual CLO pair (6) and (9) are independent of the choice of $c$ and $d$ only if the pair $(d, c)$ satisfies $Ad = b$ and $Bc = a$.

**Proof.** It is a direct consequence of Corollary 2.1 □

In the following statements, we always assume that $b = Ad$ and $a = Bc$. Furthermore, if $d \in K$ and $c \in K^*$, then the problems (1) and (2) are feasible.

If $L = R(A)$, then we can rewrite the primal program (6) in the geometric form

$$\min \langle c, x \rangle, \quad s.t. \ x \in d + L, \quad x \in K.$$  \hspace{1cm} (12)

Correspondingly, its dual is in the geometric form

$$\min \langle d, y \rangle, \quad s.t. \ y \in c + L^\perp, \quad y \in K^*,$$  \hspace{1cm} (13)

where $L^\perp$ denotes the orthogonal complement of $L$ in $\mathbb{R}^q$, e.g., see Nesterov and Nemirovski [35], and Todd [43]. Under Assumption 1, $L^\perp = R(M^T) \oplus R(B^T)$ implies that the above dual program (13) is the same as the nonstandard dual (9). Of course, the choice of the matrices $B$ and $M$ in (9) is not unique for a given matrix $A$.

It should be noted that in the inductive proof of Theorem 2.3, the matrix $M$ denotes the change of constraints, see Subsection 3.3. When some constraints change, $R(M^T)$ is transplanted from $L^\perp$ to $L$, i.e., if $L^\perp = R(B^T)$, then $L = R(A^T) \oplus R(M^T)$. This motivates us to propose Assumption 1.

We say that a CLO problem is strictly feasible or the Slater condition holding if there is a feasible interior. For example, for the primal program (6), a feasible solution $x$ is called to be strictly feasible if $x \in \text{int}(K)$; and for the nonstandard dual program (9), a feasible solution $y$ is called to be strictly feasible if $y \in \text{int}(K^*)$. The following strong duality theorem is fundamental (e.g., Refs. [4, 9, 12, 13, 31, 44]).

**Theorem 2.3.** Consider the primal-dual CLO pair (6)-(9).

(1) If the dual problem is bounded from above and if it is strictly feasible, then the primal problem attains its minimum and there is no duality gap.

(2) If the primal problem is bounded from above and if it is strictly feasible, then the dual problem attains its maximum and there is no duality gap.

### 2.3 Multiparametric optimizations

Under Assumptions 1 and 2, as discussed in the previous subsection, the nonstandard dual of the problem (1) can be defined by

$$\bar{d}^*(u) = \max_{s.t.} \langle d, c + M^T u - y \rangle$$

$$By = a,$$

$$My = Mc + u,$$

$$y \in K^*,$$  \hspace{1cm} (14)
and the nonstandard dual of the problem (2) can be described as

\[
\bar{p}^*(v) = \max_{s.t.} \langle c, d + MTv - x \rangle \\
Ax = b, \\
Mx = Md + v, \\
x \in K,
\]

where the maximum values of the objective functions are denoted \(d^*(u)\) and \(p^*(v)\), respectively. If the objective functions in problems (14) and (15) are replaced by \(\langle d, c - y \rangle\) and \(\langle c, d - x \rangle\), respectively, then their optimal solutions remain unchanged. So the perturbations in (14) and (15) occur in the right hand side, not in the objective function data. The main duality studied in this paper are given in Figure 1.

**Figure 1: Dual and almost dual**

**Corollary 2.4.** (1) There is a vector \(w^1 \in \mathbb{R}^l\) such that the primal slackness vector

\[
x = d + M^Tw + B^Tw^1
\]

is feasible for the problem (15) if and only if \(v \in \Theta_P\). Moreover, when \(d \in \text{int}(K)\), it is strictly feasible if and only if \(v \in \text{int}(\Theta_P)\);

(2) There is a vector \(w^2 \in \mathbb{R}^m\) such that dual slackness vector

\[
y = c + M^Tu + A^Tw^2
\]

is feasible for the problem (14) if and only if \(u \in \Theta_D\). Moreover, when \(c \in \text{int}(K^*)\), is strictly feasible if and only if \(u \in \text{int}(\Theta_D)\).

**Proof.** The first part of the first claim follows from Assumption 1. It is obvious that \(u \in \text{int}(\Theta_D)\) if \(x\) is strictly feasible for the problem (15). Conversely, if \(d \in \text{int}(K)\), then, for every \(u \in \text{int}(\Theta_D)\), \(x\) is strictly feasible. The proof is completed. \(\square\)

As mentioned in the previous section, the problems (1) and (2) cover mpLP and mpSDP problems. Analogously, \(\Theta_P\) and \(\Theta_D\) cover their feasible regions that are called the polyhedron and the spectrahedron, respectively.
\(d \in \text{int}(K)\) means that the problem (1) is strictly feasible. Such a condition in first argument of Corollary 2.3 is necessary. We illustrate it with a semidefinite system, with \(\mathbb{R}^{\frac{n(n+1)}{2}} \simeq \mathbb{S}^n\) the set of order \(n\) symmetric matrices and \(K = K^* = \mathbb{S}^n_+\) as the set of order \(n\) symmetric positive semidefinite matrices. The inner product of \(\mathbb{S}^n\) is \(c \cdot d = \langle c, d \rangle = \text{tr}(cd)\). Note that we denote the elements of \(\mathbb{S}^n\) by small letters.

The row vectors of the matrix \(A\) are denoted by \(a^1, a^2, \ldots, a^m\). For any \(x \in \mathbb{S}^n\), \(x \succeq 0\) means \(x \in \mathbb{S}^n_+\). In our all examples, only Assumption 1 holds.

**Example 2.5.** Consider the following parametric SDP problem

\[
\min_{x \in \mathbb{S}^n_+} (c + m^1 u) \cdot x \\
\text{s.t.} \quad a^1 \cdot x = 0, \\
\quad a^2 \cdot x = 1,
\]

where \(c = 0 \in \mathbb{R}^{3\times 3}\),

\[
m^1 = \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & -0.5 \\
0 & -0.5 & 0
\end{pmatrix}, \quad a^1 = \begin{pmatrix}
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad a^2 = \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{pmatrix}.
\]

Then

\[
c + m^1 u + a^1 w_1 + a^2 w_2 = \begin{pmatrix}
u + w_2 & 0 & 0 \\
0 & w_1 & -0.5u + w_2 \\
0 & -0.5u + w_2 & 0
\end{pmatrix} \succeq 0
\]

for some \(w_1, w_2 \in \mathbb{R}\) if and only if

\[
u + w_2 \geq 0, \quad w_1 \geq 0, \quad -0.5u + w_2 = 0
\]

for some \(w_1, w_2 \in \mathbb{R}\). From the first inequality and the last equality, one has \(1.5u \geq 0\), which means that \(\Theta_D = [0, +\infty)\). It has at least an interior point.

There is no any feasible interior in the primal program, but

\[
d = \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

is primal feasible. Then the nonstandard dual program

\[
\max_{y \in \mathbb{S}^n_+} d \cdot (c + m^1 u - y) \\
\text{s.t.} \quad b^i \cdot y = 0, \quad i = 1, 2, 3, \\
\quad m^1 \cdot y = 1.5u
\]

has no any feasible interior, where

\[
b^1 = \begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad b^2 = \begin{pmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
1 & 0 & 0
\end{pmatrix}, \quad b^3 = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix}.
\]
Let $X$ and $Y$ denote the feasible regions of the problems (1) and (2), respectively. Such two sets do not depend on the given parametric vectors $u$ and $v$. Let us assume that they are nonempty throughout this paper. Without causing confusion, the optimal solutions of the problems (1) and (2) are denoted by $x^*(u)$ and $y^*(v)$, respectively, if they exist. That is,

$$x^*(u) \in X^*(u) = \arg\min\{\langle c + M^T u, x \rangle | x \in X\},$$

$$y^*(v) \in Y^*(v) = \arg\min\{\langle d + M^T v, y \rangle | y \in Y\}.$$ 

Analogously, the optimal solutions of the corresponding nonstandard dual programs (14) and (15) are denoted by $\bar{y}^*(u)$ and $\bar{x}^*(v)$, respectively, if they exist. By Corollary 2.4 one has

$$\bar{y}^*(u) \in \bar{Y}^*(u) = \arg\min\{\langle d, y \rangle | y \in \bar{Y}(u)\},$$

$$\bar{x}^*(v) \in \bar{X}^*(v) = \arg\min\{\langle c, x \rangle | x \in \bar{X}(v)\},$$

where $\bar{X}(v)$ and $\bar{Y}(u)$ denote the feasible regions of the problems (15) and (14), respectively.

### 3 A modern duality

This section is devote to present a novel kind of duality in CLO, namely the duality between either the problems (1) and (2) or the problems (15) and (14).

#### 3.1 Set-valued mappings

The application of the set-valued mapping theory in optimization date back to 1960’s (Refs. [6, 11, 29]). In this subsection we will introduce a new set-valued mapping between the conic representable sets $\Theta_P$ and $\Theta_D$.

Let us start with some rather technical lemmas.

**Lemma 3.1.** (1) If $Mx = Md + v$ for some $x \in \mathcal{X}$, then

$$\langle M^T u, x \rangle = \langle M^T u, d + M^T v \rangle.$$ 

(18)

(2) If $My = Mc + u$ for some $y \in \mathcal{Y}$, then

$$\langle M^T v, y \rangle = \langle M^T v, c + M^T u \rangle.$$ 

(19)

**Proof.** If $Mx = Md + v$ for some $x \in \mathcal{X}$, then

$$\langle M^T u, x \rangle = \langle u, Mx \rangle = \langle u, Md + v \rangle = \langle M^T u, d + M^T v \rangle,$$

in which in the last step $MM^T = I_r$ (Assumption 2) is used. The proof is finished. \(\square\)
Lemma 3.2. (1) For any $x \in \mathcal{X}(v)$, the sum of the objective value of the problems (1) and (15) is equal to $\langle c + M^Tu, d + M^Tv \rangle$, i.e.,

$$\langle c + M^Tu, x \rangle + \langle c, d + M^Tv - x \rangle = \langle c + M^Tu, d + M^Tv \rangle. \quad (20)$$

(2) For any $y \in \mathcal{F}(u)$, the sum of the objective value of the problems (2) and (14) is equal to $\langle c + M^Tu, d + M^Tv \rangle$, i.e.,

$$\langle d + M^Tv, y \rangle + \langle d, c + M^Tu - y \rangle = \langle c + M^Tu, d + M^Tv \rangle. \quad (21)$$

\textbf{Proof.} If $x \in \mathcal{X}(v)$, then $x \in \mathcal{X}$ and $Mx = Md + v$. Furthermore, it follows from the equality (18) that the equality (20) holds. The proof is completed. \hfill \square

It should be noted that Assumption 2 in Lemma 3.1 is not necessary. Indeed, we can replace $u$ and $v$ by $MM^Tu$ and $MM^Tv$, respectively, if the assumption does not hold.

The following result plays an important role in our analysis.

Corollary 3.3. (1) Suppose that $\mathcal{X}^*(u) \neq \emptyset$ for some $u \in \Theta_D$. If $v = M(x^*(u) - d)$, then $x^*(u) \in \mathcal{X}^*(v)$;

(2) Suppose that $\mathcal{Y}^*(v) \neq \emptyset$ for some $v \in \Theta_P$. If $u = M(y^*(v) - c)$, then $y^*(v) \in \mathcal{Y}^*(u)$.

\textbf{Proof.} Suppose that $\mathcal{X}^*(u) \neq \emptyset$ for some $u \in \Theta_D$. If $v = M(x^*(u) - d)$ for some $x^*(u) \in \mathcal{X}^*(u)$, then $x^*(u) \in \mathcal{X}(v)$. On the other hand, if $x \in \mathcal{X}(v)$, then $x \in \mathcal{X}^*$. From Lemma 3.2, one has

$$\langle c + M^Tu, x \rangle + \langle c, d + M^Tv - x \rangle = \langle c + M^Tu, x^*(u) \rangle + \langle c, d + M^Tv - x^*(u) \rangle.$$ 

Therefore,

$$\langle c, d + M^Tv - x^*(u) \rangle - \langle c, d + M^Tv - x \rangle = \langle c + M^Tu, x \rangle - \langle c + M^Tu, x^*(u) \rangle \geq 0,$$

in which the last inequality holds since $x^*(u) \in \mathcal{X}^*(u)$. Therefore, the left-hand expression is greater than or equal to zero, that is, $x^*(u) \in \mathcal{X}^*(v)$. \hfill \square

The crucial conditions of the above corollary derive us to define two set-valued mappings as follow

$$\Phi(u) = \{M(x^*(u) - d)|x^*(u) \in \mathcal{X}^*(u)\}, \quad \forall u \in \Theta_D \tag{22}$$

and

$$\Psi(v) = \{M(y^*(v) - c)|y^*(v) \in \mathcal{Y}^*(v)\}, \quad \forall v \in \Theta_P. \tag{23}$$

Here the value of every mapping could be a set if the optimal solution is not unique corresponding to the fixed parametric vector. We refer to [3, 39, 40] for a detailed introduction to set-valued mappings.

The following theorem shows that the above set-valued mappings can be well defined.
Theorem 3.4. (1) If \((d, c) \in K \times \text{int}(K^*)\), then for every \(u \in \text{int}(\Theta_D)\), \(\Phi(u) \neq \emptyset\); (2) If \((d, c) \in \text{int}(K) \times K^*\), then for every \(v \in \text{int}(\Theta_P)\), \(\Psi(v) \neq \emptyset\).

Proof. If \((d, c) \in K \times \text{int}(K^*)\), then the primal problem (1) is feasible. By Corollary 2.4, \(\Theta_D\) has an interior at least. The latter implies that the nonstandard dual problem (14) for every \(u \in \text{int}(\Theta_D)\) satisfies the Slater condition. By Theorem 2.3, the primal problem (1) is solvable, i.e., \(\mathcal{X}^*(u) \neq \emptyset\) for every \(u \in \text{int}(\Theta_D)\). Then \(\Phi(u)\) is well defined. □

In LP both \(\Phi(u)\) on \(\Theta_D\) and \(\Psi(v)\) on \(\Theta_P\) are always well defined. However, in SDP they could be undefined on the boundary of the conic representable sets. Here is a counterexample.

Example 3.5. Consider the following parametric SDP problem

\[
\min_{x \in S_2^+} \quad (c + m^1 u) \cdot x \\
s.t. \quad a^1 \cdot x = 2,
\]

where

\[
c = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad m^1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad a^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\]

Then

\[
c + m^1 u + a^1 w = \begin{pmatrix} 1 & w \\ w & u \end{pmatrix} \succeq 0
\]

for some \(w \in \mathbb{R}\) means that \(u \geq 0\), i.e., \(\Theta_D = [0, +\infty)\).

Since

\[
d = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}
\]

is primal feasible, the nonstandard dual program is

\[
\max_{y \in S_2^+} \quad d \cdot (c + m^1 u - y) \\
s.t. \quad b^1 \cdot y = 1, \\
\quad m^1 \cdot y = u,
\]

where

\[
b^1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.
\]

If \(u > 0\), then the primal and dual programs have the optimal solutions

\[
x^*(u) = \begin{pmatrix} \sqrt{u} & 1 \\ 1 & \sqrt{u} \end{pmatrix}, \quad \bar{y}^*(u) = \begin{pmatrix} 1 & -\sqrt{u} \\ -\sqrt{u} & u \end{pmatrix},
\]

respectively, and there is no duality gap. However, if \(u = 0\), the primal program is not solvable although the dual program has 0 maximum at

\[
y^*(0) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}
\]
and there is no duality gap. Therefore, for any \( u \in (0, +\infty) \),

\[
\Phi(u) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{u} & 1 \\ 1 & \frac{1}{\sqrt{u}} \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} = \frac{1}{\sqrt{u}} - 2
\]

is well defined and \( \Phi(0) \) is undefined. In addition, it is easy to verify that \( \Psi(v) = \frac{1}{(v+2)^2} \) for any \( v = (-2, +\infty) \) and \( \Psi(-2) \) is undefined.

### 3.2 Duality properties

In this subsection we present the main duality properties between either the problems (1) and (2) or the problems (15) and (14). These properties include the complement slackness properties, the strong and the weak duality properties, and the mpKKT conditions.

**Theorem 3.6.** Suppose that \((d, c) \in \text{int}(K \times K^*)\). If either \( v \in \Phi(u) \) for some \( u \in \text{int}(\Theta_D) \) or \( u \in \Psi(v) \) for some \( v \in \text{int}(\Theta_P) \), then the complement slackness properties (3) and (4) hold.

**Proof.** This result follows immediately from Corollary 3.3, Theorem 3.4 and the KKT conditions (11a)-(11c). □

In the classical Lagrangian dual approach, the complement slackness property is in demand in the primal and dual optimization pair such that the zero duality gap holds. Theorem 3.6 shows that the complement slackness property (4) holds for the mpCLO problems (15) and (14), although the sum of their objective optimal values is not equal to zero (actually the sum is a bilinear function of two parametric vectors), see Theorem 3.8 below. So the mpCLO problems (15) and (14) are called one pair of almost primal and dual programs. Meanwhile, the mpCLO problems (1) and (2) are called another pair of almost primal and dual programs based on the same reason. The relationships between the four problems is shown in Figure 1. In addition, by Corollary 2.3 each element of \( \Theta_P \) and \( \Theta_D \) is associated with the feasible region of the problems (15) and (14), respectively. To distinguish them, \( \Theta_P \) and \( \Theta_D \) are called the primal and dual conic representable sets, respectively.

**Corollary 3.7.** Suppose that \((d, c) \in K \times K^*\), and either \( u \in \Theta_D \) and \( v \in \Phi(u) \) or \( v \in \Theta_P \) and \( u \in \Psi(v) \).

1. For any \((x, y) \in \mathcal{X}(v) \times \mathcal{Y}(u)\), one has

\[
\langle c, d + M^T x - x \rangle + \langle d, c + M^T u - y \rangle \leq \langle c + M^T x, d + M^T v \rangle.
\]

Equality holds if and only if \((x, y) \in \mathcal{X}^*(v) \times \mathcal{Y}^*(u)\).

2. For any \((x, y) \in \mathcal{X}^*(v) \times \mathcal{Y}^*(u)\), one has

\[
\langle c + M^T u, x \rangle + \langle d + M^T v, y \rangle \geq \langle c + M^T x, d + M^T v \rangle.
\]

Equality holds if and only if \((x, y) \in \mathcal{X}^*(u) \times \mathcal{Y}^*(v)\).

**Proof.** This result follows from Lemma 3.2 and Corollaries 2.1 and 3.3 □
Theorem 3.8. Consider either the almost primal-dual mpCLO pair \((1)\) and \((2)\) or \((13)\) and \((14)\).

(1) If \((d,c) \in K \times \text{int}(K^*)\), then for every \(u \in \text{int}(\Theta_D)\) and \(v \in \Phi(u)\), the primal problem is solvable and the following equality holds:

\[
p^*(u) + d^*(v) = \bar{d}^*(u) + \bar{p}^*(v) = \langle c + M^T u, d + M^T v \rangle.
\]

(2) If \((d,c) \in \text{int}(K) \times K^*\), then for every \(v \in \text{int}(\Theta_P)\) and \(u \in \Psi(v)\), the dual problem is solvable and the equality \((26)\) holds.

Proof. Suppose that \((d,c) \in K \times \text{int}(K^*)\) and \(u \in \text{int}(\Theta_D)\). Applying the strong duality theorem 2.3 to the primal and dual pair \((1)\) and \((14)\), the problem \((1)\) attains its minimum at \(x^*(u)\) and there is no duality gap, i.e., \(p^*(u) = \bar{d}^*(u)\).

Applying the strong duality theorem 2.3 to the primal and dual pair \((15)\) and \((2)\), the problem \((15)\) attains its minimum and there is no duality gap, i.e., \(\bar{p}^*(v) = d^*(v)\).

If \(v \in \Phi(u)\), then, by Corollary 3.3, \(x^*(u)\) is an optimal solution of the problem \((15)\). It follows from \((20)\) that

\[
p^*(u) + \bar{p}^*(v) = \langle c + M^T u, d + M^T v \rangle,
\]

which implies the equality \((26)\). \(\square\)

For a pair of the almost primal-dual problems \((15)\) and \((14)\), the equality \((26)\) and the inequality \((24)\) represent the strong duality and the weak duality properties, respectively. Similarly, the inequality \((25)\) can be viewed as the weak duality property of the problem pair \((1)\) and \((2)\).

Corollaries 2.4 and 3.3 and Theorem 3.8 provide an interesting geometric relationship between the conic representable sets \(\Theta_P\) and \(\Theta_D\). We illustrate this relationship with the problems \((1)\) and \((15)\): the perturbation of the former occurs in the objective function, whereas the perturbation of the latter occurs in the right hand side, respectively, in which their relationship is established by the set-valued mapping \(\Phi(u)\) from \(\Theta_D\) to \(\Theta_P\). Geometrical, \(\Theta_D\) can denote the objective perturbation set of the problem \((1)\) (Corollary 3.3(2) and Theorem 3.8(2)), in which the hyperplane \(H_u = \{x \in \mathbb{R}^q | \langle c + M^T u, x \rangle = \langle c + M^T u, x^*(u) \rangle \}\) passing through the boundary point \(x^*(u)\) supports the feasible region \(\mathcal{X}\); and \(\Theta_P\) denotes the right side hand perturbation set of the problem \((15)\) (Corollary 2.4(1)), in which the affine hyperplane \(C_v = \{x \in \mathbb{R}^q | Mx = Md + v \}\) passing through the point \(x^*(u)\) cuts the feasible region \(\mathcal{X}\) if \(v \in \Phi(u)\). For a given \(u \in \Theta_D\), if \(H_u \cap \mathcal{X}\) is a set, then there is more than one cutting hyperplane \(C_v\), which implies that \(\Phi(u)\) is a set; if \(x^*(u)\) is a vertex of \(\mathcal{X}\), then there is more than one supporting hyperplane \(H_u\), which implies that there is a set \(U \subset \Theta_D\) such that \(\Phi(u)\) takes the same value for all \(u \in U\).

When neither of the preceding conditions occurs, there is only one hyperplane \(H_u\) supporting the set \(\mathcal{X}\), and there is only one hyperplane \(C_v\) cutting the set \(\mathcal{X}\). All of these cases will be examined in more detail later in the next section.

From the above geometric interpretation, the inverse of Corollary 3.3 is true. Corollary 3.3 and its inverse is contained in the following result.

Theorem 3.9. (1) Suppose that \(\mathcal{X}^*(u) \neq \emptyset\) for some \(u \in \Theta_D\). Then \(x^*(u) \in \mathcal{X}^*(v)\) if and only if \(v \in \Phi(u)\);
(2) Suppose that \( \mathcal{Y}^*(v) \neq \emptyset \) for some \( v \in \Theta_P \). Then \( y^*(v) \in \mathcal{Y}^*(u) \) if and only if \( u \in \Psi(v) \).

Now we present the mpKKT conditions for the almost primal-dual program pairs.

Theorem 3.10. Let \( u \in \Theta_D \) and \( v \in \Theta_P \) be arbitrary. Then there are a pair of vectors \((\bar{x}, \bar{y}) \in \mathbb{R}^q \times \mathbb{R}^q\) such that the mpKKT conditions hold

\[
A\bar{x} = b, \quad M\bar{x} = Md + v, \quad \bar{x} \in K, \tag{27a}
\]
\[
B\bar{y} = a, \quad M\bar{y} = Mc + u, \quad \bar{y} \in K^*, \tag{27b}
\]
\[
\langle \bar{x}, \bar{y} \rangle = 0, \tag{27c}
\]
if and only if both \( v \in \Phi(u) \) and \( u \in \Psi(v) \) hold. Furthermore, \( \bar{x} \in \mathcal{X}^*(u) \cap \mathcal{X}^*(v) \) and \( \bar{y} \in \mathcal{Y}^*(v) \cap \mathcal{Y}^*(u) \).

Proof. We first show that if the mpKKT conditions (27a)-(27c) hold, then both \( v \in \Phi(u) \) and \( u \in \Psi(v) \) hold. Clearly, the above mpKKT conditions imply that

\[
M\bar{y} = Mc + u \tag{28}
\]

and

\[
A\bar{x} = b, \quad M\bar{x} = Md + v, \quad \bar{x} \in K,
\]
\[
B\bar{y} = a, \quad \bar{y} \in K^*,
\]
\[
\langle \bar{x}, \bar{y} \rangle = 0.
\]

From the KKT conditions (11a)-(11c), the latter implies that \((\bar{x}, \bar{y})\) is a pair of optimal solutions of the primal-dual problems (15) and (2). Therefore, \( \bar{y} \in \mathcal{Y}^*(v) \), i.e., there is \( y^*(v) \in \mathcal{Y}^*(v) \) such that \( y^*(v) = \bar{y} \). Finally, the equality (28) implies that \( u = M(y^*(v) - c) \in \Psi(v) \). Analogously, one has \( v \in \Phi(u) \).

The converse is easy to prove. If both \( v \in \Phi(u) \) and \( u \in \Psi(v) \) hold, then \( \Phi(u) \) and \( \Psi(v) \) are well defined, which means the almost primal-dual programs (1) and (2) have a pair of optimal solutions \((x^*(u), y^*(v))\). Furthermore, by Corollary 3.3, \((x^*(u), y^*(v))\) is also a pair of optimal solutions of the almost primal-dual program pair (15) and (14). Thus, from the KKT conditions (11a)-(11c), \((\bar{x}, \bar{y}) = (x^*(u), y^*(v))\) satisfies the mpKKT conditions (27a)-(27c).

The final claim follows from the KKT conditions (11a)-(11c) trivially. □

Our duality inherits many useful properties of classical duality, including the complement slackness properties (Theorem 3.6), the strong and the weak duality properties (Theorem 3.8 and Corollary 3.7) and the mpKKT conditions (Theorem 3.10), although some of them are different from the classical forms. In Theorem 3.8 the first argument implies the fact that the dual programs (2) and (14) are bounded and strictly feasible. This fact is the same as the classical strong duality theorem. Since it depends on the perturbation parameters, our duality characterizes more features of a CLO problem. If you put two kinds of the primal-dual program pairs together, then these features, well connected by set-valued mappings, have a very intuitive geometric explanation, as described in the previous paragraph. In short, our duality embodies a dynamic process, whereas the classical duality is a static result.
3.3 Other related results

The following result shows that the set-valued mapping \( \Phi(u) \) iterates over every value in the set \( \text{int}(\Theta_P) \) if \( u \) goes through every value in the set \( \Theta_D \), although \( \Phi(u) \) could be undefined for some boundary point of \( \Theta_D \).

**Theorem 3.11.** Suppose that \((d,c) \in \text{int}(K \times K^*)\), then

\[
\begin{align*}
\text{int}(\Theta_P) & \subset \bigcup_{u \in \Theta_D} \Phi(u), \quad (29a) \\
\text{int}(\Theta_D) & \subset \bigcup_{v \in \Theta_P} \Psi(v). \quad (29b)
\end{align*}
\]

**Proof.** If \( v \in \text{int}(\Theta_P) \), then the primal-dual optimization pair (15) and (2) satisfy the Slater conditions. By Theorem 2.3, there are \( \bar{x}^*(v) \in \bar{X}^*(v) \) and \( y^*(v) \in \bar{Y}^*(v) \) such that

\[
\langle c, d + M^Tv - \bar{x}^*(v) \rangle = \bar{p}^*(v) = d^*(v) = \langle d + M^Tv, y^*(v) \rangle.
\]

From Corollary 3.3, there is \( u \in \Theta_D \) such that \( u \in \Psi(v) \). It follows from (26) and (20) that

\[
p^*(u) = \langle c + M^Tu, d + M^Tv \rangle - d^*(v) = \langle c + M^Tu, d + M^Tv \rangle - \bar{p}^*(v) = \langle c + M^Tu, \bar{x}^*(v) \rangle.
\]

Since it is feasible for the problem (1), \( \bar{x}^*(v) \) is optimal for the problem (1), i.e., \( \bar{x}^*(v) \in \bar{X}^*(u) \). Or equivalently, \( v \in \Phi(u) \). The proof is completed. \( \square \)

Sometimes, we need a version of Theorem 3.11. This version shows that the set-valued mapping \( \Psi(v) \) is almost the inverse of the set-valued mapping \( \Phi(u) \).

**Corollary 3.12.** Suppose that \((d,c) \in \text{int}(K \times K^*)\).

1. If \( u \in \Psi(v) \) for some \( v \in \text{int}(\Theta_P) \), then \( v \in \Phi(u) \).
2. If \( v \in \Phi(u) \) for some \( u \in \text{int}(\Theta_D) \), then \( u \in \Psi(v) \).

**Corollary 3.13.** Suppose that \((d,c) \in \text{int}(K \times K^*)\).

1. For every \( u \in \text{int}(\Theta_D) \), \( \Phi(u) \) is a closed convex set. And it can be described as

\[
\Phi(u) = \{ v | \exists (\bar{x}, \bar{y}) \text{ s.t. the mpKKT conditions } (27a) - (27c) \text{ holds} \}; \quad (30)
\]

2. For every \( v \in \text{int}(\Theta_P) \), \( \Psi(v) \) is a closed convex set. And it can be identified by

\[
\Psi(v) = \{ u | \exists (\bar{x}, \bar{y}) \text{ s.t. the mpKKT conditions } (27a) - (27c) \text{ holds} \} \quad (31)
\]

**Proof.** Firstly, the equalities (30) and (31) follow from Theorem 3.10 and Corollary 3.12.
Now we prove that $\Phi(u)$ for every $u \in \text{int}(\Theta_D)$ is a closed convex set. If $v^1, v^2 \in \Phi(u)$, then from (30) and (11a)-(11c), there are $\bar{x}^1, \bar{x}^2$ and $y^*$ such that
\[
Ax^1 = b, \quad M\bar{x}^1 = M\bar{c} + v^1, \quad \bar{x}^1 \in K, \\
B\bar{y} = a, \quad M\bar{y} = M\bar{c} + u, \quad \bar{y} \in K^*, \\
\langle \bar{x}^1, \bar{y} \rangle = 0
\]
and
\[
Ax^2 = b, \quad M\bar{x}^2 = M\bar{c} + v^2, \quad \bar{x}^2 \in K, \\
B\bar{y} = a, \quad M\bar{y} = M\bar{c} + u, \quad \bar{y} \in K^*, \\
\langle \bar{x}^2, \bar{y} \rangle = 0.
\]
Therefore, for any $\alpha \in [0, 1]$, $\bar{x}_\alpha = \alpha \bar{x}^1 + (1 - \alpha)\bar{x}^2$ satisfies
\[
Ax_\alpha = b, \quad M\bar{x}_\alpha = M\bar{c} + \alpha v^1 + (1 - \alpha) v^2, \quad \bar{x}_\alpha \in K, \\
B\bar{y} = a, \quad M\bar{y} = M\bar{c} + u, \quad \bar{y} \in K^*, \\
\langle \bar{x}_\alpha, \bar{y} \rangle = 0.
\]
Applying Theorem 3.10 again, one has $v = \alpha v^1 + (1 - \alpha) v^2 \in \Phi(u)$. That is, $\Phi(u)$ is convex.

Finally, the intersection of the supporting hyperplane
\[
H = \{ x \in \mathbb{R}^q | \langle c + MTu, x \rangle = \langle c + MTu, x^*(u) \rangle \}
\]
and $X = \{ x \in K | Ax = b \}$ is closed, where $x^*(u)$ is equal to $\bar{x}^1$ or $\bar{x}^2$. This means that $\Phi(u)$ is closed. The proof is finished. $\square$

Corollary 3.13 has an interesting geometric interpretation. Let us assume that for a given parameter $u \in \Theta_D$, $\Phi(u)$ is neither empty nor a singleton set. It is easy to verify that for any $v^1, v^2 \in \Phi(u)$, all points of line segment $[\bar{x}^1, \bar{x}^2]$ are feasible solutions of the program (15) and have the same objective value, which implies that $\Phi(u)$ is convex for a given parameter $u$. On the other hand, the optimal solution of the program (2) with the perturbed objection function remains unchanged, which means that $y^*(v) = \bar{y}$ is a vertex of $Y$ for any $v \in \Phi(u)$.

**Corollary 3.14.** Let $u \in \Theta_D$ and $v \in \Theta_P$ be both arbitrary.

1. Suppose that $(d, c) \in K \times \text{int}(K^*)$. The problem (15) is solvable if and only if $v \in \Phi(u)$;
2. Suppose that $(d, c) \in \text{int}(K) \times K^*$. The problem (14) is solvable if and only if $u \in \Phi(v)$.

**Proof.** Let us prove the first claim. By Theorem 2.3, the problem (1) and (15) are solvable, in which the cutting hyperplane $C_v = \{ x \in \mathbb{R}^q | Mx = Md + v \}$ must intersect the feasible region $X = \{ x \in K | Ax = b \}$. Then by Theorem 3.9 the problem (15) is solvable if and only if $u \in \Phi(v)$. $\square$

To conclude this subsection, we offer a new proof of Theorem 2.3 by the use of Corollary 3.14.

**Proof of Theorem 2.3.** Let’s assume that the dual problem (9) is bounded and is strictly feasible, i.e., $(d, c) = K \times \text{int}(K^*)$. We prove the first claim by the mathematical induction for $k = q - m$. 

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Footnotes:

1. Theorem 2.3
2. Corollary 3.13
3. Corollary 3.14
4. Proof of Theorem 2.3
Initial step: $k = 0$. Here the feasible region of the primal program (3) is a singleton set. It has a minimum and there is no duality gap. The first claim holds.

Inductive step: Assume that the problem (15) is solvable for some $v \in \Theta_P$. By Corollaries 3.3 and 3.14, the problem (1) for all $u \in \text{int}(\Theta_P)$ is solvable and there is no duality gap. Since $0 \in \text{int}(\Theta_D)$, the primal program (6) is solvable and there is no duality gap for $k := k - l$. Then the first claim holds by induction. □

As mentioned in the previous section, for a pair of primal-dual problems, the row vectors of the primal and the nonstandard dual constraint matrices form an orthogonal basis of the whole space. When the above induction is used, the decrease in the row vectors of the primal constraint matrix occurs simultaneously with the increase in the row vectors of the dual constraint matrix. This inevitably leads to the perturbation of the right hand side whenever the complement slackness property remains the same. If the row vectors of $M, A, B$ denote the row vectors of the changed, the old primal and the new dual constraint matrices, respectively, then $R(M^T)$ is not only orthogonal to the old primal space $R(A^T)$, but also to the new dual space $R(B^T)$. This provides a theoretical foundation for us to propose Assumption 1.

4 Invariancy sets and illustrative examples

The concept of the optimal partition was introduced for parametric LP in [1, 30], in which the given optimal basic partition is invariant. Later this concept was used in [26, 46] for quadratic programming problems, in [7] for linear complementarity problems and in [24, 33] for SDPs. In this context we use the set-valued mappings described in the previous section to define the optimal partition of conic representable sets. The mappings may have practical significance in investigating the behaviour of the optimal partition under perturbation.

Definition 4.1. Let $\mathcal{V}$ be a simply connected subset of $\Theta_P$. Then

(1) $\mathcal{V}$ is called a linearity set if either $\mathcal{V}$ is not a singleton set, and $\Psi(v^1) = \Psi(v^2)$ for all $v^1, v^2 \in \mathcal{V}$ or $\Psi(\mathcal{V})$ is not a singleton set when $\mathcal{V}$ is a singleton set.

(2) $\mathcal{V}$ is called a nonlinearity set if $\mathcal{V}$ is not a singleton set, and for any $v^2 \neq v^1 \in \mathcal{V}$, $\Psi(v^2) \neq \Psi(v^1)$, and $\Psi(v^1)$ is a singleton set.

Both a linearity set and a nonlinearity set are called an invariancy set. For the dual conic representable set $\Theta_D$, the definitions of the invariant set and the linear/nonlinearity set are similar.

From Definition 4.1, a conic representable set includes two types of invariancy sets: linearity sets and nonlinearity sets. On every linearity set, the primal optimal solution remains unchanged if the first claim holds and $\Psi(\mathcal{V})$ is a singleton set; and the dual optimal solution remains unchanged if the second claim holds. In LP, there are only linearity sets. In SDP, a nonlinearity set could exist. This type of invariancy sets for SDP was carefully studied by Mohammad-Nezhad and Terlaky [33]. As indicated in [22, 33] and also demonstrated by Example 3.5, the optimal partition for $\Theta_P$ may vary with the parameter $v$ on a subinterval of $[-2, +\infty)$; and the optimal partition for $\Theta_D$ may vary with the parameter $u$ on a subinterval of $[0, +\infty)$. It is
easy to see that $\Phi(u)$ on $(0, +\infty)$ and $\Psi(v)$ on $(-2, +\infty)$ are inverse functions of each other; and by this, $(-2, +\infty)$ and $(0, +\infty)$ the nonlinearity intervals of $\Theta_P$ and $\Theta_D$, respectively.

**Definition 4.2.** Let $\mathcal{V}$ be an invariancy set of $\Theta_P$. Then $\mathcal{V}$ is called a transition face if $\dim(\mathcal{V}) < r$. In particular, if $\dim(\mathcal{V}) = 0$, then $\mathcal{V}$ is called a transition point; and if $\dim(\mathcal{V}) = 1$, then $\mathcal{V}$ is called a transition line, and etc.

We refer to an invariancy set as a nontrivial invariancy set if it is not a transition face. In contrast, a transition face is called a trivial invariancy set.

Now we state the main result of this section.

**Theorem 4.3.** Any two different invariancy regions of a conic representable set do not intersect.

**Proof.** This result follows immediately from Definition 4.1.

By Corollary 3.3 and Theorem 4.3, we can obtain the invariant set decomposition of a conic representable set. To build intuition, we present the following two examples: one of them is a LP problem and another is a SDP problem.

**Example 4.4.** Taking $c = (-1, -1, 0, 0, 0)^T$, $A = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix}$, $M = (-0.5, 0.5, 0, -0.5, 0.5)$, $b = (3, 2, 2.5)^T$, the following parametric LP program

$$\min_{x \in \mathbb{R}^5} (c + M^Tu)^T x$$

s.t. $Ax = b, \quad x \geq 0$

has a feasible solution $d = (1, 1, 1, 1.5)^T$. Its nonstandard dual program can be expressed as follows

$$\max_{y \in \mathbb{R}^5} d^T(c + M^Tu - y)$$

s.t. $By = -2, My = u, y \geq 0,$

where $B = (1, 1, -2, -1, -1)$. If the slackness variables $x_3, x_4, x_5$ are omitted, then the primal feasible region reduces a convex pentagon

$$\{x \in \mathbb{R}^2 | x_1 + x_2 \leq 3, 0 \leq x_1 \leq 2.5, 0 \leq x_2 \leq 2\}$$

in the two-dimensional plane. This convex pentagon has five vertices

$$(0, 0)^T, (0, 2)^T, (2.5, 0)^T, (2.5, 0.5)^T, (1, 2)^T.$$  

When $0 \leq u \leq 1$, the optimal pair $(x^*(u), \bar{y}^*(u))$ is as follows

$$x^*(u) = (2.5, 0.5, 0, 1.5, 0)^T, \quad \bar{y}^*(u) = (0, 0, 1 - u, 0, 2u)^T.$$
Geometrical, the trajectory of \( y^* (u) \) in the interval \([0, 1]\) is an edge of the polyhedral
\[
\{ y \in \mathbb{R}^5 | y_1 + y_2 - 2y_3 - y_4 - y_5 = -2; y_1, y_2, y_3, y_4, y_5 \geq 0 \},
\]
in which the edge connects two vertices \( y^1 = (0, 0, 1, 0, 0)^T \) and \( y^2 = (0, 0, 0, 0, 2)^T \). Then for every \( u \in (0, 1) \), \( \Phi(u) = -2 \). If \( v = M x^* (u) - Md = -2 \), then \( \Psi(v) = [0, 1] \) is an interval. When \( u \) is equal to either 0 or 1, \( \Phi(u) \) is also an interval. The following parallel table lists the values of the set-valued mappings \( \Phi(u) \) and \( \Psi(v) \).

| \( \bar{x}^*(v) \)         | \( v \) | \( u \) | \( \bar{y}^*(u) \) | \( \bar{y}^*(u) \) |
|-----------------------------|--------|--------|-------------------|-------------------|
| \((0, 2, 1, 0, 2.5)^T\)    | 2      | \((-\infty, -1)\) | \((-1 - u, 0, 0, 1 - u, 0)^T\) | \((-1 - u, 0, 0, 1 - u, 0)^T\) |
| \((2 - v, 2, v - 1, 0, 0.5 + v)^T\) | (1, 2) | -1     | \((0, 0, 0, 2, 0)^T\) | \((0, 0, 0, 2, 0)^T\) |
| \((1,2,0,0,1.5)\)         | 1      | \((-1, 0)\) | \((0,0,1+u,-2u,0)^T\) | \((0,0,1+u,-2u,0)^T\) |
| \((0.5(3 - v, 3 + v, 0, 1 - v, 2 + v)\) | \((-2, 1)\) | 0      | \((0, 0, 1, 0)^T\) | \((0, 0, 1, 0)^T\) |
| \((2.5, 0.5, 0.15, 0)^T\) | -2     | \((0, 1)\) | \((0, 0, 1 - u, 0, 2u)^T\) | \((0, 0, 1 - u, 0, 2u)^T\) |
| \((2.5, v + 2.5, -2 - v, -0.5 - v, 0)^T\) | \((-2.5, -2)\) | 1      | \((0, 0, 0, 2)^T\) | \((0, 0, 0, 2)^T\) |
| \((2.5, 0.05, 2, 0)^T\)   | -2.5   | \((1, +\infty)\) | \((0, 0, u - 1, 0, u + 1)\) | \((0, 0, u - 1, 0, u + 1)\) |

The following observations can be understand from Table 1.

1. The primal linear representable set \( \Theta_P \) is equal to \([-2.5, 2]\). It contains three open invariancy intervals \((-2.5, -2), (-2, -1), (1, 2)\) and four transition points \(-2.5, -2, 1, 2\). The dual linear representable set \( \Theta_D \) is equal to \((-\infty, +\infty)\). It contains four open invariancy intervals \((-\infty, -1), (-1, 0), (0, 1), (1, +\infty)\) and three transition points \(-1, 0, 1\). For each invariancy interval, the trajectory of the optimal solution is an edge of the polyhedron that connects two adjacent vertices. And for each transition point, the corresponding optimal solution is a vertex of the polyhedron.

2. All invariancy sets are linearity. They are either open invariancy intervals or endpoints of invariancy intervals, in which the endpoints are transition points. The image of the set-valued mapping is a closed interval if and only if the primage is a transition point, and the image is a transition point if and only if the primage is an open interval.

3. The intersection of the images of the set-valued mapping at any two transition points is either empty or a transition point associated with its dual representable set.

4. Either \( \Phi((-\infty, -1)) = 2 \) or \( \Psi(2) = (-\infty, -1) \) implies that the primal objective function with perturbations takes the maxima and minima in the primal feasible region. And either \( \Phi((1, +\infty)) = -2.5 \) or \( \Psi(-2.5) = [1, +\infty) \) implies that the perturbed objective function takes the maxima and minima in the dual feasible region.

In this example, the given optimal basic partition is invariant for single-parametric LPs, which is similar to the past definition, e.g., see [I] and etc.
Example 4.5. Take
\[ c = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \quad m^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad m^2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \]
\[ a^1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad a^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad a^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \]
and
\[ d = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad b^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \]

Consider the following mpSDP pair
\[
\begin{align*}
\min_{x \in S^3_+} & \quad (c + m^1 u_1 + m^2 u_2) \cdot x \\
\text{s.t.} & \quad a^i \cdot x = 1, \quad i = 1, 2, 3
\end{align*}
\tag{32}
\]
and
\[
\begin{align*}
\min_{y \in S^3_+} & \quad (d + m^1 v_1 + m^2 v_2) \cdot y \\
\text{s.t.} & \quad b^1 \cdot y = -2.
\end{align*}
\tag{33}
\]

The primal feasible region is a 3-elliptope whose image is given by Figure 2.

Figure 2: The primal feasible region is a 3-elliptope

Since the nonstandard dual problem of (32) is
\[
\begin{align*}
\max_{y \in S^3_+} & \quad d \cdot (c + m^1 u_1 + m^2 u_2 - y) \\
\text{s.t.} & \quad b^1 \cdot y = -2, \\
& \quad m^1 \cdot y = 2u_1, \\
& \quad m^2 \cdot y = 2u_2,
\end{align*}
\]
the optimal solution pair \((x^*(u), \bar{y}^*(u))\) has six indeterminate entries:

\[
x^*(u) = \begin{pmatrix} 1 & x_{12} & x_{13} \\ x_{12} & 1 & x_{23} \\ x_{13} & x_{23} & 1 \end{pmatrix}, \quad \bar{y}^*(u) = \begin{pmatrix} \bar{y}_{11} & u_1 & -u_2 \\ u_1 & \bar{y}_{22} & -1 \\ -u_2 & -1 & \bar{y}_{33} \end{pmatrix},
\]

in which we are interested in the six indeterminate entries \(x_{11}, x_{12}, x_{23}\) and \(\bar{y}_{11}, \bar{y}_{22}, \bar{y}_{33}\) as a function of \((u_1, u_2) \in \mathbb{R}^2\). Let us consider the following two cases.

Case I. The rank of \(x^*(u)\) is equal to 1. This condition implies that the indeterminate entries \(x_{11}, x_{12}, x_{23}\) must satisfy

\[
\frac{1}{x_{12}} = \frac{x_{13}}{1}, \quad \frac{1}{x_{13}} = \frac{x_{23}}{1},
\]

which results in \(x^*(u)\) is one of four matrices

\[
\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}.
\]

Such matrices are vertices on the surface of the primal feasible region, see Figure 2 and the paper [3]. Applying the complement slackness property (3) to the first vertex, one has

\[
\begin{pmatrix} \bar{y}_{11} & u_1 & -u_2 \\ u_1 & \bar{y}_{22} & -1 \\ -u_2 & -1 & \bar{y}_{33} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.
\]

Solving this system of linear equations, \(\bar{y}_{11}, \bar{y}_{22}, \bar{y}_{33}\) can be expressed as a function of \((u_1, u_2) \in \mathbb{R}^2\). That is,

\[
x^*(u) = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{pmatrix}, \quad \bar{y}^*(u) = \begin{pmatrix} -u_1 - u_2 & u_1 & -u_2 \\ u_1 & -u_1 - 1 & -1 \\ -u_2 & -1 & -u_2 - 1 \end{pmatrix}.
\]

Since \(\bar{y}^*(u)\) is positive semidefinite, it is easy to verify that \(u = (u_1, u_2)^T\) belongs to the following set

\[
\Theta_D^1 = \{(u_1, u_2)^T|u_1 < -1, u_2 < -1, u_1 + u_2 + u_1 u_2 \geq 0\}.
\]

This region is bounded by one branch of a hyperbola in the plane and is a convex set. Hence for any \(u = (u_1, u_2)^T \in \Theta_D^1\), one has

\[
\Phi(u) = \frac{1}{2} \begin{pmatrix} m^1 \cdot x^*(u) \\ m^2 \cdot x^*(u) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}
\]

since \(m^i \cdot (x^*(u) - d) = 2v_i\) for \(i = 1, 2\). Moreover, one has \(\Psi((1, 1)^T) = \Theta_D^1\).

Analogously, for the second vertex, one has

\[
\begin{pmatrix} \bar{y}_{11} & u_1 & -u_2 \\ u_1 & \bar{y}_{22} & -1 \\ -u_2 & -1 & \bar{y}_{33} \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
\]
Then
\[
x^*(u) = \begin{pmatrix}
1 & -1 & 1 \\
-1 & 1 & -1 \\
1 & -1 & 1 \\
\end{pmatrix}, \quad \bar{y}^*(u) = \begin{pmatrix}
u_1 + u_2 & u_1 & -u_2 \\
u_1 & u_1 - 1 & -1 \\
-u_2 & -1 & u_2 - 1 \\
\end{pmatrix},
\]
in which \(u = (u_1, u_2)^T\) belongs to the following two-dimensional region
\[
\Theta_D^2 = \{(u_1, u_2)^T | u_1 > 1, u_2 > 1, -u_1 - u_2 + u_1 u_2 \geq 0\}.
\]
It is bounded by one branch of a hyperbola in the plane and is a convex set. Then for any \(u \in \Theta_D^2\), one has \(\Phi(u) = (-1, -1)^T\) and \(\Psi((-1, -1)^T) = \Theta_D^2\).

For the third vertex, one has
\[
\left(\begin{array}{ccc}
\bar{y}_{11} & u_1 & -u_2 \\
u_1 & \bar{y}_{22} & -1 \\
u_2 & -1 & \bar{y}_{33}
\end{array}\right) \left(\begin{array}{c}1 \\
1 \\
1
\end{array}\right) = \left(\begin{array}{c}0 \\
0 \\
0
\end{array}\right),
\]
then
\[
x^*(u) = \begin{pmatrix}1 & 1 & 1 \\1 & 1 & 1 \\1 & 1 & 1\end{pmatrix}, \quad \bar{y}^*(u) = \begin{pmatrix}u_2 - u_1 & u_1 & -u_2 \\
u_1 & 1 - u_1 & -1 \\
u_2 & -1 & u_2 + 1\end{pmatrix},
\]
in which \(u = (u_1, u_2)^T\) belongs to the following region
\[
\Theta_D^3 = \{(u_1, u_2)^T | u_1 < 1, u_2 > -1, u_2 \geq u_1, u_2 - u_1 - u_1 u_2 \geq 0\}.
\]
Therefore, for any \(u \in \Theta_D^3\), one has \(\Phi(u) = (1, -1)^T\) and \(\Psi((1, -1)^T) = \Theta_D^3\).

For the fourth vertex, one has
\[
\left(\begin{array}{ccc}
\bar{y}_{11} & u_1 & -u_2 \\
u_1 & \bar{y}_{22} & -1 \\
u_2 & -1 & \bar{y}_{33}
\end{array}\right) \left(\begin{array}{c}1 \\
1 \\
1
\end{array}\right) = \left(\begin{array}{c}0 \\
0 \\
0
\end{array}\right),
\]
then
\[
x^*(u) = \begin{pmatrix}1 & -1 & -1 \\1 & 1 & 1 \\1 & 1 & 1\end{pmatrix}, \quad \bar{y}^*(u) = \begin{pmatrix}u_1 - u_2 & u_1 & -u_2 \\
u_1 & u_1 + 1 & -1 \\
u_2 & -1 & 1 - u_2\end{pmatrix},
\]
in which \(u = (u_1, u_2)^T\) belongs to the following region
\[
\Theta_D^4 = \{(u_1, u_2)^T | u_1 > -1, u_2 < 1, u_1 \geq u_2, u_1 - u_2 - u_1 u_2 \geq 0\}.
\]
Then for any \(u \in \Theta_D^4\), one has \(\Phi(u) = (-1, 1)^T\) and \(\Psi((-1, 1)^T) = \Theta_D^4\).

The sets \(\Theta_D^1, \Theta_D^2, \Theta_D^3\) and \(\Theta_D^4\) are four different invariancy regions of \(\Theta_D\), and four points \((-1, 1)^T, (-1, -1)^T, (1, 1)^T, (-1, -1)^T\) are transition points of \(\Theta_P\).

It should be noted that \(\Theta_D^3 \cap \Theta_D^4 = \{(0, 0)^T\}\). If \(u^0 = ((0, 0)^T)\), then
\[
x^*(u^0) = \begin{pmatrix}
1 & 2\alpha - 1 & 2\alpha - 1 \\
2\alpha - 1 & 1 & 1 \\
2\alpha - 1 & 1 & 1
\end{pmatrix}, \quad \bar{y}^*(u^0) = \begin{pmatrix}0 & 0 & 0 \\
0 & 1 & -1 \\
0 & -1 & 1
\end{pmatrix}.
\]
They are the convex combinations of the third and fourth optimal solution pairs. Then
\[ \Phi(u^0) = \{(v_1, -v_1)^T | -1 \leq v_1 \leq 1\} \]
is a line segment connecting two points \((1, -1)^T\) and \((-1, 1)^T\). Therefore, \(u^0\) is a transition point of the dual conic representable set \(\Theta_D\) and \(\text{int}(\Phi(u^0))\) is a transition line of the primal conic representable set \(\Theta_P\). And for every \(v \in \text{int}(\Phi(u^0))\), one has \(\Psi(v) = u^0\). However, at the two endpoints of the transition line, one has \(\Psi[(1, -1)^T] = \Theta_D^3\) and \(\Psi[(-1, 1)^T] = \Theta_D^4\).

Case II: The rank of \(\bar{y}^*(u)\) is equal to 1. It follows from \(\bar{y}^* = (u_1, u_1 - u_2, u_2, -u_1, -u_2)\), one can express \(\bar{y}^*(u)\) in terms of the parameters \(u_1\) and \(u_2\) as follows:

\[
\bar{y}^*(u) = \begin{pmatrix} u_1u_2 & u_1 & -u_2 \\ u_1 & u_1 & -1 \\ -u_2 & -1 & u_2 \\ u_1 & -u_2 \\ -u_2 \\ -u_2 & u_1 \\ u_1 & -u_2 \\ u_2 \\ u_1u_2 & 0 & -u_2 \\ 0 & u_1u_2 & u_1 \\ 0 & u_1u_2 & 0 \\ x_{12} & x_{13} & x_{23} \\ x_{13} & x_{12} & x_{23} \\ x_{23} & x_{12} & x_{13} \\ x_{12} & x_{13} & x_{23} \\ x_{13} & x_{23} & x_{12} \\ x_{23} & x_{23} & x_{12} \\ x_{12} & x_{13} & x_{23} \\ x_{13} & x_{23} & x_{12} \\ x_{23} & x_{23} & x_{12} \\ x_{12} & x_{13} & x_{23} \\ x_{13} & x_{23} & x_{12} \\ x_{23} & x_{23} & x_{12} \\ x_{12} & x_{13} & x_{23} \\ x_{13} & x_{23} & x_{12} \\ x_{23} & x_{23} & x_{12} \\ x_{12} & x_{13} & x_{23} \\ x_{13} & x_{23} & x_{12} \\ x_{23} & x_{23} & x_{12} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},
\]

or equivalently,
\[
\begin{pmatrix} u_1 & -u_2 & 0 \\ u_1u_2 & 0 & -u_2 \\ 0 & u_1u_2 & u_1 \\ x_{12} & x_{13} & x_{23} \\ x_{12} & x_{13} & x_{23} \\ x_{12} & x_{13} & x_{23} \end{pmatrix} = \begin{pmatrix} -u_1u_2 \\ -u_1 \\ -u_2 \end{pmatrix}.
\]

Solve the system of linear equations to get
\[
x_{12} = \frac{u_2}{2u_1} - \frac{u_2}{2} - \frac{1}{2u_2},
\]
\[
x_{13} = \frac{1}{2u_1} + \frac{u_1}{2} - \frac{u_1}{2u_2},
\]
\[
x_{23} = \frac{u_2}{2u_1} + \frac{u_1}{2u_2} - \frac{u_1u_2}{2}.
\]

Hence six indeterminate entries are established. In particular, if \(u_1 = u_2\), then
\[
x^*(u) = \begin{pmatrix} 1 & -u_2 & u_2 \\ -u_2 & 1 & 1 - \frac{u_2^2}{2} \\ \frac{u_2}{2} & 1 - \frac{u_2^2}{2} & 1 \end{pmatrix},
\]
\[
\bar{y}^*(u) = \begin{pmatrix} u_2^2 & u_2 & -u_2 \\ u_2 & 1 & -1 \\ -u_2 & -1 & 1 \end{pmatrix}.
\]

This case was discussed by Mohammad-Nezhad and Terlaky in \([33]\).

Note that \(x^*(u)\) is positive semidefinite if and only if \(|x_{23}| \leq 1\), i.e.,
\[
-2 \leq \frac{u_2}{u_1} + \frac{u_1}{u_2} - u_1u_2 \leq 2.
\]
Then $u_1 u_2 > 0$ yields that

$$-2u_1 u_2 \leq u_1^2 + u_2^2 - u_1^2 u_2^2 \leq 2u_1 u_2,$$

or equivalently,

$$0 \leq (u_1 + u_2)^2 - (u_1 u_2)^2, \quad \text{and} \quad (u_1 - u_2)^2 - (u_1 u_2)^2 \leq 0.$$  

This concludes the following inequalities

\begin{align}
(u_1 + u_2 + u_1 u_2)(u_1 + u_2 - u_1 u_2) & \geq 0, \quad (34) \\
(u_1 - u_2 + u_1 u_2)(u_1 - u_2 - u_1 u_2) & \leq 0. \quad (35)
\end{align}

**Lemma 4.6.** Define a set

$$\Theta_D^0 = \{(u_1, u_2)^T \in \mathbb{R}^2 | u_1 u_2 > 0, \text{the strictly inequalities (34) and (35) hold}\}.$$  

Then $\Theta_D^0 = \mathbb{R}^2 - \bigcup_{i=1}^4 \Theta_D^i$.

**Proof.** Define four curves as follow

\begin{align*}
l_1 &: \quad u_1 + u_2 + u_1 u_2 = 0, \quad u_1 < -1, u_2 < -1, \\
l_2 &: \quad -u_1 - u_2 + u_1 u_2 = 0, \quad u_1 > 1, u_2 > 1, \\
l_3 &: \quad u_2 - u_1 - u_1 u_2 = 0, \quad u_1 < 1, u_2 > -1, \\
l_4 &: \quad u_1 - u_2 - u_1 u_2 = 0, \quad u_1 > -1, u_2 < 1.
\end{align*}

Each of them represents a unilateral branch of the hyperbola, in which $l_1$ and $l_2$ are symmetric about the line $u_1 + u_2 = 0$, and $l_3$ and $l_4$ are symmetric about the line $u_1 - u_2 = 0$. Such four curves define some areas in $\mathbb{R}^2$. Instead of a formal and tedious proof we simply plot these areas in Figure 3. The four outer closed regions are $\Theta_D^1, \Theta_D^2, \Theta_D^3$ and $\Theta_D^4$. Intuitively, $\Theta_D^0$ lies in the two curved triangles inside (the colored part in Figure 3), in which the two strictly inequalities (34) and (35) hold for any $(u_1, u_2) \in \{(u_1, u_2) \in \mathbb{R}^2 | u_1 u_2 > 0\}$.

**Lemma 4.7.** For any $u \in \Theta_D^0$, one has $|x_{12}| < 1$ and $|x_{13}| < 1$.  

Figure 3: Primal conic representable set separated by four curves
Proof. Assume that \( u_1 > 0 \) and \( u_2 > 0 \). Then \( \Theta^0_D \) in the first quadrant is bounded by three curves \( l_2, l_3, l_4 \). That is,

\[
\begin{align*}
    u_1 + u_2 - u_1 u_2 &> 0, \\
    u_2 - u_1 - u_1 u_2 &< 0, \\
    u_1 - u_2 - u_1 u_2 &> 0, \\
    u_1 > 0, \ u_2 > 0.
\end{align*}
\]

Or equivalently,

\[
\begin{align*}
    0 < \frac{1}{u_1} + \frac{1}{u_2} < 1, \\
    -1 < \frac{1}{u_1} - \frac{1}{u_2} < 1.
\end{align*}
\]

Therefore, one has

\[
\frac{1}{u_1^2} - \frac{1}{u_2^2} < 1 < 1 + \frac{2}{u_2},
\]

i.e., \( x_{12} < 1 \). On the other hand, one has

\[
2x_{12} = \left( \frac{1}{u_1} + \frac{u_2}{u_1^2} - \frac{u_2}{u_1} \right) + \left( 1 - \frac{1}{u_1} - \frac{1}{u_2} \right) + \left( \frac{u_2}{u_1} - u_2 + 1 \right) - 2 > -2.
\]

The first claim for \( u_1 < 0 \) and \( u_2 < 0 \) and the second claim are proved in an analogous fashion. \( \square \)

From Lemma 4.6, for any \( u = (u_1, u_2)^T \in \Theta^0_D \), there is a pair of optimal solution \((x^*(u), y^*(u))\) such that the ranks of \( x^*(u) \) and \( y^*(u) \) are equal to two and one, respectively. Moreover, one has

\[
\Phi(u) = \frac{1}{2} \begin{pmatrix} m_1 & (x^*(u) - d) \\ m_2 & (x^*(u) - d) \end{pmatrix} = \begin{pmatrix} \frac{u_2}{2u_1^2} - \frac{u_2}{2} - \frac{1}{2u_2} \\ -\frac{1}{2u_1} - \frac{u_1}{2} + \frac{u_1}{2u_2} \end{pmatrix}.
\]

Similarly, let

\[
\bar{x}^*(v) = \begin{pmatrix} 1 & v_1 & -v_2 \\ v_1 & 1 & \bar{x}_{23} \\ -v_2 & \bar{x}_{23} & 1 \end{pmatrix}, \quad y^*(v) = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}^T \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}
\]

denote the optimal solutions of the problem (33) and its nonstandard dual, in which \( \det(x^*(v)) = 0 \) and \( y_2 y_3 = -1 \) are assumed. The first assumption implies that

\[
1 - 2v_1 v_2 \bar{x}_{23} - v_1^2 - v_2^2 - \bar{x}_{23}^2 = 0.
\]

Solving the quadratic equation with one variable \( \bar{x}_{23} \) to get

\[
\bar{x}_{23} = v_1 v_2 \pm \sqrt{(1 - v_1^2)(1 - v_2^2)}.
\]
On the other hand, it follows from the complement slackness property \(4\) that
\[
\begin{pmatrix}
1 & v_1 & -v_2 \\
v_1 & 1 & \bar{x}_{23} \\
-v_2 & \bar{x}_{23} & 1
\end{pmatrix}
\begin{pmatrix}
y_1 \\
y_2 \\
y_3
\end{pmatrix} = 0,
\]
to yield
\[
(1 - v_1^2)y_2 = (v_1v_2 + \bar{x}_{23})y_3.
\]
Then the second assumption implies that
\[
\bar{x}_{23} = v_1v_2 - \sqrt{(1 - v_1^2)(1 - v_2^2)}.
\]
Therefore, one has
\[
\Psi(v) = \frac{1}{2} \begin{pmatrix}
m^1 \bullet (y^*(v) - c) \\
m^2 \bullet (y^*(v) - c)
\end{pmatrix}
\]
\[
\begin{pmatrix}
-v_2 - v_1 \sqrt{\frac{1-v_2^2}{1-v_1^2}} \\
-v_1 - v_2 \sqrt{\frac{1-v_1^2}{1-v_2^2}}
\end{pmatrix}.
\]

In conclusion, the dual conic representable set \(\Theta_D\) is the whole two-dimensional space and the primal conic representable set \(\Theta_P\) is a rectangular given by
\[
\Theta_P = \{(v_1, v_2) \mid -1 \leq v_1, -1 \leq v_2 \leq 1\}.
\]
For the set \(\Theta_D\), two open sets contained in \(\Theta_D^0\) are nonlinearity sets and four angular domains \(\Theta_D^1, \Theta_D^2, \Theta_D^3, \Theta_D^4\) are nontrivial invariancy regions, the origin is only one transition point that is not associated with vertices. For the set \(\Theta_P\), two open triangle in the two-dimensional plane given by
\[
\Theta_P^0 = \{(v_1, v_2) \mid -1 < v_1 \neq v_2 < 1\}
\]
are the nonlinearity sets and the four angular points
\[
(1, 1)^T, (-1, -1)^T, (1, -1)^T, (-1, 1)^T
\]
are transition points that are associated with vertices. The diagonal of the rectangular region \(\Theta_P\)
\[
\Theta_P^1 = \{(v_1, -v_1) \mid -1 < v_1 < 1\}
\]
is a transition line of \(\Theta_P\), and its closure is a line segment that joins the two diagonal angular points \((1, -1)^T\) and \((-1, 1)^T\). However, for any
\[
v \in \partial(\Theta_P) - \{(1, 1)^T, (-1, -1)^T, (1, -1)^T, (-1, 1)^T\},
\]
one has \(\Psi(v) = \emptyset\). The comparison table of the set-valued mappings is shown in Table 2,

| \(v\)   | (1, 1)^T | (-1, -1)^T | (1, -1)^T | \(\Theta_P^1\) | (1, -1)^T | \(\Theta_P^2\) | \(\Phi(u)\) |
|--------|----------|------------|----------|---------------|----------|------------|--------|
| \(u\)  | \(\Theta_D^0\) | \(\Theta_D^1\) | \(\Theta_D^2\) | \(\Theta_D^3\) | \(\Theta_D^4\) | \(\Theta_D^5\) | \(\Psi(v)\) |

in which either \(\Theta_P^0\) or \(\Theta_D^0\) contains two nontrivial invariancy sets.
5 Multiparametric analysis

In this section we present some applications of our duality properties on the multi-parametric analysis. We discuss the recession directions and the identification of the optimal partitions of a conic representable set, and the existence of a nonlinearity set in a conic representable set. Finally, we study the behavior of the optimal value function on its domain.

5.1 Recession directions

In this subsection we study the problem of finding the range of parametric vector $u$ (or $v$) for which some given primal (or dual) solution remains optimal. In other words, we study the multiparametric analysis of primal (or dual) optimal solutions. The following corollary provides a theoretical foundation for the shadow vertex algorithm, see, e.g., [10, 21].

Corollary 5.1. Suppose that $(d, c) \in \text{int}(K \times K^*)$.

1. If $h \in 0^+(\Theta_D)$ is in a linearity set, then there is a vector $v_0 \in \partial(\Theta_P)$ such that $h \in 0^+ (\Psi(v_0))$;
2. If $h \in 0^+(\Theta_P)$ is in a linearity set, then there is a vector $u_0 \in \partial(\Theta_D)$ such that $h \in 0^+ (\Psi(u_0))$.

Proof. If $h \in 0^+(\Theta_D)$, then the dual program (14) with $u = u^0 + \lambda h$ is strictly feasible for any $u^0 \in \Theta_D$ and $\lambda > 0$. Applying Theorem 2.3, the primal problem (1) is solvable for $u = u^0 + \lambda h$ attain a minimum at the same point $x^*(h)$ for some $u^0$. Then the problem (15) with $v^0 = M x^*(h) - Md \in \Phi(u^0 + \lambda h)$ attain a minimum at $\bar{x}^*(u^0) = x^*(h)$. By Corollary 3.12, for any $\lambda > \Lambda$, $u = u^0 + \lambda h \in \Psi(v^0)$. From Corollary 3.13 one has $h \in 0^+(\Psi(v^0))$. Finally, from the optimality of $\bar{x}^*(u^0)$ we get $v^0 \in \partial(\Theta_P)$.

In Example 4.4, the two invariancy intervals $(-\infty, -1]$ and $[1, +\infty)$ corresponding to the dual conic representable set contain the recession directions. And in Example 4.5, the four invariancy regions $\Theta^1_D, \Theta^2_D, \Theta^3_D, \Theta^4_D$ corresponding to the dual conic representable set contain the recession directions. There is no any recession direction in the respective primal conic representable sets since they are bounded.

Finally, in Corollary 5.1, the assumption that the recession directions belong to linearity sets is necessary. For example, see Example 3.5.

5.2 Identification of the optimal partitions

When the parametric vectors $u$ and $v$ are a scalar, the nontrivial invariancy set reduces an invariancy interval, and the transition face reduces an transition point. At the endpoints of the invariancy interval, optimal partitions change on transiting to the adjacent invariancy interval. The transition point for parametric LPs, as a separation point of invariancy intervals, was mentioned many times, e.g., see [22, 23, 11]. The concepts of the transition point and the nonlinearity invariant
set for parametric SDPs is formally defined by Mohammad-Nezhad and Terlaky [33].

**Theorem 5.2.** (1) If \((d, c) \in K \times \text{int}(K^*)\), then every nonlinearity region \(U\) of \(\Theta_D\) is open and \(\Phi(u)\) is continuous in the region \(U\);

(2) If \((d, c) \in \text{int}(K) \times K^*\), then every nonlinearity region \(V\) of \(\Theta_P\) is open and \(\Phi(v)\) is continuous in the region \(V\).

**Proof.** We first show that the nonlinearity region is open. We assume that for every \(\bar{u} \in U\), \(\Phi(\bar{u})\) is a singleton set. Geometrically, the supporting hyperplane

\[
H_{\bar{u}} = \{ x \in \mathbb{R}^q | \langle c + M^T \bar{u}, x \rangle = \langle c + M^T \bar{u}, x^*(\bar{u}) \rangle \}
\]

is tangent to the primal feasible region \(X = \{ x \in K | Ax = b \}\) at the unique point \(x^*(\bar{u})\), which implies that there is a neighbourhood \(U(\bar{u})\) such that for any \(u \in U(\bar{u})\), \(x^*(u)\) lies in the local smooth surface of \(X\). Then for any \(u \in U(\bar{u})\), the supporting hyperplane

\[
H_u = \{ x \in \mathbb{R}^q | \langle c + M^T u, x \rangle = \langle c + M^T u, x^*(u) \rangle \}
\]

is tangent to the primal feasible region \(X\). Furthermore, for any \(u \in U(\bar{u})\), \(\Phi(u)\) is a singleton set, which means that \(U\) is open.

We now show that \(\Phi(u)\) is continuous in the region \(U\). If for every \(u \in U\), \(\Phi(u)\) is a singleton set, then the set-valued map \(\Phi(u)\) on \(U\) degrades into a single-valued map. Then for two different vectors \(u^1 \in U\) and \(u^2 \in U\), \(\Phi(u^1) \neq \Phi(u^2)\). By the connectivity of \(U\), if \(\Gamma\) denotes a continuous curve connecting two different points \(u^1 \in U\) and \(u^2 \in U\), then the trajectory of the optimal solution \(x^*(u) (u \in \Gamma)\) is a continuous curve along the boundary of the primal feasible region. Therefore, \(\Phi(\Gamma)\) is a continuous curve connecting two points \(v^1 = \Phi(u^1)\) and \(v^2 = \Phi(u^2)\), which implies the continuity of \(\Phi(u)\) over the set \(U\). The proof is finished. \(\Box\)

Every set-valued mapping in a nonlinearity set reduces to an ordinary single-valued mapping. By Corollaries 3.12 and 5.2, \(\Psi(v)\) and \(\Phi(u)\) together form a pair of reversible single-valued mappings, e.g., see Example 4.5.

It should be noted that our definition of the nonlinearity set is different from that given by Mohammad-Nezhad and Terlaky [33]. In our definition, a nonlinearity set can not contain a transition point; however, in [33], a nonlinearity set could contain a transition point. For instance, if \(u_1 = u_2\) in Example 4.5, then the origin belongs to the nonlinearity interval, see [33] Example 3.1. Recently, Hauenstein et al. [27] went on analyze the continuity of the optimal set mapping and showed that continuity may fail on a nonlinearity interval under the old definition. This does not contradict Theorem 5.2.

A direct consequence of Theorem 5.2 is as follows.

**Corollary 5.3.** Let \(U\) and \(V\) be invariancy sets of \(\Theta_D\) and \(\Theta_P\), respectively.

1. Suppose that \((d, c) \in K \times \text{int}(K^*)\). Then \(U\) is nonlinearity if and only if \(\Phi(U)\) is nonlinearity;
2. Suppose that \((d, c) \in \text{int}(K) \times K^*\). Then \(V\) is nonlinearity if and only if \(\Psi(V)\) is nonlinearity.
In Example 4.5, for the primal conic representable set $\Theta_P$, four vertices are transition points and the two open sets consisting of interiors of $\Theta_P$ separated by the diagonal $\{(v_1,-v_1)^T | v_1 \in (-2, 2)\}$ are nonlinearity sets, in which the diagonal is a transition line of $\Theta_P$. For the dual conic representable set $\Theta_D$, the origin is only one transition point and the regions contained in the two curved triangles in the first and third quadrants are two different nonlinearity sets.

**Theorem 5.4.** Suppose that $(d,c) \in \text{int}(K \times K^*)$.

1. If $U$ is a nontrivial linearity set of $\Theta_D$, then $U$ is convex, and for all $u \in \text{cl}(U)$ and $v \in \Phi(U)$, one has $u \in \Psi(v)$.
2. If $V$ is a nontrivial linearity set of $\Theta_P$, then $U$ is convex, and for all $v \in \text{cl}(V)$ and $u \in \Psi(V)$, one has $v \in \Phi(u)$.

**Proof.** Let us prove the second claim. By Definition 4.1 for any $v^1, v^2 \in V$, one has $\Psi(v^1) = \Psi(v^2)$. From Corollary 3.12 for any $u \in \Psi(v^1)$, one has $v^1, v^2 \in \Phi(u)$. By Corollary 3.13 for any $\alpha \in [0, 1]$, one has $v^\alpha = \alpha v^1 + (1 - \alpha) v^2 \in \Phi(u)$ and $u \in \Psi(v^\alpha)$. That is, $\Psi(v^\alpha) = \Psi(v^1)$. Then $V$ is convex. The rest of the second claim follows from the above proof trivially. □

In LP, the actual invariancy region is convex, see Ghaffari-Hadigheh [22]. However, in SDP, the actual invariancy region could not be convex. For instance, there are six different invariancy sets of $\Theta_D$ in Example 4.5. Among them, two open nonlinearity sets are not convex, although other four linearity sets $\Theta^1_D, \Theta^2_D, \Theta^3_D$ and $\Theta^4_D$ are convex.

**Corollary 5.5.** Suppose that $(d,c) \in \text{int}(K \times K^*)$.

1. Let $U_1$ and $U_2$ be two different nontrivial linearity sets of $\Theta_D$. If $U = \text{cl}(U_1) \cap \text{cl}(U_2) \neq \emptyset$, then $U$ is a transition face of $\Theta_D$ and

$$\Phi(U) = \text{conv}(\Phi(U_1) \cup \Phi(U_2)).$$

2. Let $V_1$ and $V_2$ be two different nontrivial linearity sets of $\Theta_P$. If $V = \text{cl}(V_1) \cap \text{cl}(V_2) \neq \emptyset$, then $V$ is a transition face of $\Theta_P$ and

$$\Phi(V) = \text{conv}(\Psi(V_1) \cup \Psi(V_2)).$$

**Proof.** By Theorem 4.3, the affine dimensional of the set $U$ is less than $r$. From Corollary 5.4 for any $v^1 \in \Phi(U_1), v^2 \in \Phi(U_2)$, and for any $u \in U$, one has

$$u \in \Psi(v^1) \cap \Psi(v^2) \quad \text{and} \quad v^1, v^2 \in \Phi(u).$$

Then by Corollary 3.13 for any $\alpha \in [0, 1]$, one has $v^\alpha = \alpha v^1 + (1 - \alpha) v^2 \in \Phi(u)$ and $u \in \Psi(v^\alpha)$. The the first claim is proved. □

**Corollary 5.6.** Suppose that $(d,c) \in \text{int}(K \times K^*)$.

1. Let $U_1$ and $U_2$ be two different nontrivial linearity sets of $\Theta_D$. If $V = \Phi(U_1) \cap \Phi(U_2) \neq \emptyset$, then $V$ is a transition face of $\Theta_P$ and

$$\Psi(V) = \text{cl}(\Phi(U_1 \cup U_2)).$$
(2) Let $V_1$ and $V_2$ be two different nontrivial linearity sets of $\Theta_P$. If $U = \Psi(V_1) \cap \Psi(V_2) \neq \emptyset$, then $U$ is a transition face of $\Theta_D$ and
\[
\Phi(U) = \text{cl}(\text{conv}(V_1 \cup V_2)).
\]

**Corollary 5.7.** Suppose that $(d, c) \in \text{int}(K \times K^*)$.

(1) Let $U$ be a nontrivial linearity set of $\Theta_P$. If $U$ contains a recession direction, then $\Phi(U)$ is a transition face of $\Theta_D$.

(2) Let $V$ be a nontrivial linearity set of $\Theta_P$. If $V$ contains a recession direction, then $\Psi(V)$ is a transition face of $\Theta_D$.

The proofs of the above two results are similar and omitted.

As in Example 4.4, if two vertices $x^*(u_1)$ and $x^*(u_2)$ corresponding to two transition points $u_1 \in \Theta_P$ and $u_2 \in \Theta_P$ are adjacent, then $\Phi(u_1) \cap \Phi(u_2)$ is a transition point of $\Theta_D$. The same results to transition points of $\Theta_D$ hold. In Example 4.5 if $v^1 = (1, -1)^T$ and $v^2 = (-1, 1)^T$, then $\Psi(v^1) \cap \Psi(v^2) = \Theta^3_D \cap \Theta^4_D = \{(0, 0)^T\}$ is a singleton set. It is easy to verify that $\Phi((0, 0)^T) = \text{cl}(\Theta^1_P) = \text{cl}(\text{conv}\{v^1, v^2\})$ and the origin is only one transition point of the primal conic representable set $\Theta_P$.

### 5.3 On the existence of a nonlinearity set

In LP, every slackness vector corresponding to a transition point is a vertex of the polyhedron, e.g., see Example 4.4. In SDP, however, the slackness matrix corresponding to a transition point could not be a vertex of the spectrahedron. For instance, in Example 4.5 the origin is only one transition point of the primal conic representable set $\Theta_P$, but it is not vertex. Even so, the slackness matrix of a transition point is closely related to vertices. Since the number of the vertices of the polyhedron is finite, we make the following conjecture:

**Conjecture:** the number of the vertices of the feasible region of a CLO problem is finite.

If this conjecture is true, then the number of transition points could be finite. Let $x^*(u_1), x^*(u_2), \cdots, x^*(u_k)$ denote all vertices of the feasible region of the problem \[1\] corresponding to the transition points $u_1, u^2, \cdots, u^k \in \Theta_D$. Now we assume that the linear segments
\[
[x^*(u_1), x^*(u_2)], [x^*(u_2), x^*(u_3)], \cdots, [x^*(u_{k-1}), x^*(u_k)]
\]
do not lie in the boundary of the feasible region of the problem \[1\], that is, $\Phi(u_i) \cap \Phi(u_i) = \emptyset$, $i = 1, 2, \cdots, k - 1$. By Corollaries 3.3 and 3.13, the following set
\[
\Theta_D - \bigcup_{i=1}^{k} \Phi(u_i)
\]
is a nonempty open set. Therefore, by Theorem 4.3, there is a nonlinearity region of $\Theta_D$. In a word, if there is not any linear segment connecting $x^*(u_k)$ and $x^*(u_j)$ ($j = 1, 2, \cdots, k - 1$) on the boundary of the feasible region, then a nonlinearity region exists. Of course, if there is no any vertex on the boundary of the feasible region, then a nonlinearity region exists. This discussion yields the following result.
Theorem 5.8. Suppose that \((d, c) \in \text{int}(K \times K^*)\). A nonlinearity region exists if one of the following holds:

1. There is no any vertex on the boundary of the feasible region.
2. None of the supporting hyperplanes of the feasible region passing through the point \(x^*(\bar{u})\) contains other vertex except for \(x^*(\bar{u})\), where \(\bar{u}\) denotes a transition point.

5.4 The first analysis of multiparametric objective functions

In this subsection we discuss the behaviour of multiparametric objective function values of the mpCLO problems (1) and (2).

Consider for some \(u \in \text{int}(\Theta_D)\) and any \(h \in \mathbb{R}^r\), \(x^*(u + th)\) as \(t \to 0^+\). Since the points \(x^*(u + th)\) for \(t \in (0, \delta)\), where \(\delta\) is some positive number, lie in a compact set, \(x^*(u + th)\) has a limit point as \(t \to 0^+\). Let \(x^*_h(u)\) be such a limit point. We should point out that due to the possible multiplicity of the solutions in \(X^*(u + th)\) and in \(X^*_h(u)\), we shall assume that \(x^*(u + th) \to x^*_h(u)\), since if this is not the case, we choose an appropriate sequence that converges. It is easy to see that \(x^*_h(u) \in \mathscr{X}^*(u)\).

Analogously, we may define a limits point \(y^*_h(v)\) of \(y^*(v + th)\) as \(t \to 0^+\) and assume that \(y^*(v + th) \to y^*_h(v)\) as \(t \to 0^+\).

Lemma 5.9. Let \(h \in \mathbb{R}^r\) be arbitrary. Then for any \(u \in \text{int}(\Theta_D)\)

\[
\lim_{t \to 0^+} \frac{1}{t} \langle c + M^Tu, x^*(u + th) - x^*_h(u) \rangle = 0
\]

and for any \(v \in \text{int}(\Theta_P)\)

\[
\lim_{t \to 0^+} \frac{1}{t} \langle d + M^Tv, y^*(u + th) - y^*_h(v) \rangle = 0.
\]

Proof. We follow the proof of Lemma 3.1 in [24]. Let us assume that

\[
\liminf_{t \to 0^+} \frac{1}{t} \langle c + M^Tu, x^*(u + th) - x^*_h(u) \rangle \leq \varepsilon < 0
\]

(including the case \(\liminf_{t \to 0^+} \cdot = -\infty\)). Then there exists a sequence \(t_k \to 0^+\) such that

\[
\langle c + M^Tu, x^*(u + t_kh) \rangle \\
\leq \langle c + M^Tu, x^*_h(u) \rangle + \varepsilon t_k + o(t_k) \\
< \langle c + M^Tu, x^*_h(u) \rangle
\]

for \(t_k\) sufficiently small which contradicts the fact that \(x^*_h(u) \in \mathscr{X}^*(u)\).

Similarly, assume that

\[
\limsup_{t \to 0^+} \frac{1}{t} \langle c + M^Tu, x^*(u + th) - x^*_h(u) \rangle \geq \varepsilon > 0
\]
which contradicts the fact

\[ \liminf_{t \to 0^+} \] (including the case \( \liminf_{t \to 0^+} = +\infty \)). Then there exists a sequence \( t_k \to 0^+ \) such that

\[
\langle d + MT(u + t_k h), x^*(u + t_k h) \rangle \geq \langle d + MT(u + t_k h), x_h^*(u) \rangle + \varepsilon t_k + t_k \langle MTv, x^*(u + th) - x_h^*(u) \rangle + o(t_k).
\]

Since \( x^*(u + th) - x_h^*(u) \to 0 \) as \( t_k \to 0^+ \), it follows from that for \( t_k \) sufficiently small,

\[
\langle d + MT(u + t_k h), x^*(u + t_k h) \rangle > \langle d + MT(u + t_k h), x_h^*(u) \rangle,
\]

which contradicts the fact \( x_h^*(u) \in \mathcal{R}^*(u + t_k h) \).

Consequently,

\[
\liminf_{t \to 0^+} \frac{1}{t} \langle c + MTu, x^*(u + th) - x_h^*(u) \rangle = \limsup_{t \to 0^+} \frac{1}{t} \langle c + MTu, x^*(u + th) - x_h^*(u) \rangle = 0
\]

and it follows that in the statement of the lemma holds. The second limit is proved in an analogous fashion. \( \Box \)

**Theorem 5.10.** The solution of the following problem produces the directional derivative of \( p^*(\cdot) \) at \( u \in \text{int}(\Theta_D) \) in a direction \( h \in \mathbb{R}^r \)

\[
p''(u, h) = \min_v \{ \langle h, Md + v \rangle | v \in \Phi(u) \}.
\]

The solution of the following problem produces the directional derivative of \( d^*(\cdot) \) at \( v \in \text{int}(\Theta_P) \) in a direction \( h \in \mathbb{R}^r \)

\[
d''(v, h) = \min_u \{ \langle h, Mc + u \rangle | u \in \Psi(v) \}.
\]

**Proof.** Let us now consider the directional derivative of the objective value function \( p^*(\cdot) \) at \( u \in \text{int}(\Theta_D) \) in a direction \( h \in \mathbb{R}^r \):

\[
\frac{1}{t} (p^*(u + th) - p^*(u))
\]

\[
= \frac{1}{t} (\langle c + MT(u + th), x^*(u + th) \rangle - \langle c + MTu, x^*(u) \rangle)
\]

\[
= \langle MTu + x^*(u + th) \rangle + \frac{1}{t} \langle c + MTu, x^*(u + th) - x^*(u) \rangle.
\]

Then from Lemma 5.9 and from the fact that \( \langle c + MTu, x^*(u) \rangle = \langle c + MTu, x_h^* \rangle \), we have

\[
p''(u, h) = \langle MTu, x_h^*(u) \rangle.
\]

If \( v \in \Phi(u) \), then

\[
\langle MTu, x_h^*(u) \rangle = \langle h, Mx_h^*(u) \rangle = \langle h, Md + v \rangle
\]

such that

\[
p''(u, h) = \min_x \{ \langle MTu, x \rangle | Ax = Ad, Mx = Md + v, x \in \Phi(u) \} = \min_v \{ \langle h, Md + v \rangle | v \in \Phi(u) \}.
\]

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The proof is completed. □

It should be noted that the formulas (36) and (37) hold only if Φ(\(u\)) and Ψ(\(v\)) are well defined.

**Example 5.11.** Example 4.5 (continue)

(1) For the transition point \(v^1 = (1, 1)^T\) of \(\Theta_P\), one has

\[
\left\langle h, \left( \begin{array}{c} c \cdot m_1 \\ c \cdot m_2 \end{array} \right) + u \right\rangle = u_1 h_1 + u_2 h_2,
\]

in which \((u_1, u_2)^T \in \Psi(v^1) = \Theta^1_D\). Then

\[
d^*(v^1, h) = \begin{cases} -h_1 - h_2, & \text{if } h_1 \leq 0 \text{ and } h_2 \leq 0, \\ -\infty, & \text{if } h_1 > 0 \text{ or } h_2 > 0. \end{cases}
\]

For other three the transition points of \(\Theta_P\), the similar results also hold.

For every \(v \in \text{int}(\Theta_P)\), the Gâteaux derivative of \(d^*(\cdot)\) at \(v\) is equal to

\[
d^*(v) = \Psi(v).
\]

(2) For the transition point \(u^0 = (0, 0)^T\) of \(\Theta_D\), one has

\[
\left\langle h, \left( \begin{array}{c} d \cdot m_1 \\ d \cdot m_2 \end{array} \right) + v \right\rangle = v_1 h_1 + v_2 h_2,
\]

in which \((v_1, v_2)^T \in \Phi(u^0) = \{(v_1, -v_2)^T | -1 \leq v_1 \neq v_2 \leq 1\}\). Then

\[
p^*(u^0, h) = \begin{cases} h_1 - h_2, & \text{if } h_1 \geq 0 \text{ and } h_2 \leq 0, \\ -h_1 + h_2, & \text{if } h_1 \leq 0 \text{ and } h_2 \geq 0, \\ -\infty, & \text{otherwise.} \end{cases}
\]

For every \(u \in \Theta^0_D\), the Gâteaux derivative of \(p^*(\cdot)\) at \(u\) is equal to

\[
p^*(u) = \Phi(u).
\]

### 6 Conclusions

The paper started with the discussion of almost primal and dual mpCLOs and used them to define the appropriate primal and dual conic representable sets. We then established our duality theory in CLO by the definition of the set-valued mappings and investigated the parametric analysis of convex CLO problems. Our results characterizes a nonlinearity region of the optimal partition by generalizing the concept of the optimal partition to arbitrary conic representable sets. This efficient procedure leads to the invariant region decomposition of a conic representable set, which extends the optimal partition approach to parametric analysis in convex CLO. Similar to the special cases for LP and SDP, it is possible to perform better parametric
analysis based on the optimal partition for perturbations of both the right-hand side and the objective function.

This paper presented some existence results for a transition point and a nonlinear region, partially having answered the open question proposed Hauenstein et al. These results depend entirely on the conjecture presented in Section 5. The conjecture is also helpful to understand the geometry of a conic representable set.

Several examples in this paper confirmed our results. Currently, we are investigating more theoretical results for CLO by the use of the parametric analysis technique presented in this paper.

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