Long range spatial correlation between two Brownian particles under external driving

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Abstract

We study the large distance behavior of a steady distribution of two Brownian particles under external driving in a two-dimensional space. Employing a method of perturbative system reduction, we analyze a Fokker-Planck equation that describes the time evolution of the probability density for the two particles. The expression we obtain shows that there exists a long range correlation between the two particles, of $1/r^2$ type.

1 Introduction

The statistical properties of fluctuations at equilibrium are described by equilibrium statistical mechanics. This has been established through experimental measurements carried out to test the theoretical predictions of statistical mechanics. In contrast to the equilibrium case, there is no known general principle determining the statistical properties of fluctuations under nonequilibrium conditions. Indeed, it might be thought that it is quite difficult to obtain a universal theoretical framework on nonequilibrium fluctuations.

There are many nonequilibrium steady states (NESSs) that settled down into an equilibrium state if one condition, such as the strength of an external driving force or the chemical potential at a boundary, is controlled. In such NESSs, the statistical properties of fluctuations can be elucidated through an approach that seeks to determine how equilibrium fluctuations are modified under the influence of nonequilibrium conditions.

In a pioneering work in this context, Kuramoto studied open chemical systems in 1974 and pointed out that large scale fluctuations should be considered separately from thermodynamic fluctuations occurring locally in space [1]. What is referred to in Ref. [1] as “long range coherence” is found more explicitly to appear in the form of long range correlations of fluctuations for conserved quantities [2]. Here, there are two classes of systems that exhibit long range correlation. One class consists of systems driven by nonequilibrium boundary conditions. It includes laminar flow systems [3], temperature gradient systems [4] and density gradient systems [5]. In such systems, anomalous fluctuations originate from the spatial inhomogeneity of averaged quantities. The power law decay exponents of the spatial correlation for conserved quantities are basically of $1/r^{(d-2)}$ type, but different exponents can also be realized through composite effects [6]. The other type of systems exhibiting long range correlation are locally driven systems. In such systems, the statistical properties of local fluctuations differ substantially from those of equilibrium systems. This modification yields
long range correlation of the form $1/r^d$ in $d \geq 2$ dimensional systems [7]. This behavior can be easily understood if we model the time development of a conserved quantity with a phenomenological linear Langevin equation [8, 9] in which the anisotropy of both the current noise intensity and the transportation coefficient without detailed balance is assumed.

In this paper, we inquire whether long range correlation of $1/r^d$ type is peculiar to the fluctuations of macroscopic variables in driven systems. In general, a chain of correlated two-body interactions among many particles provides a contribution to the correlation function for the density field. Thus, when we consider the system of a microscopic level (at which particle motion is described), it is reasonable to conjecture that long range correlation appears only in the macroscopic limit. However, if anisotropy with a local violation of detailed balance is the essence of long range correlation in driven systems, it may not be necessary to have a many body system in order to observe such correlation. As an extreme case, a system consisting of two particles under external driving may exhibit long range correlation.

With this motivation, quite recently, in a calculation of the steady probability for the positions of two interacting random walkers in a $d \geq 2$ dimensional lattice under external driving, it has been found that the large distance behavior of the probability, including the existence or non-existence of long range correlation, depends on the choice of the transition rules satisfying the condition of local detailed balance [10]. It is surprising that there is such a dependence, considering the fact that universal relations in the linear response regime do not depend on the precise nature of these rules. Given this situation, we are led to ask, Which rule is physically meaningful? However, it is difficult to answer this question by considering such transition rules themselves. For this reason, we investigate a physical model that corresponds directly to an experimental system.

In the present paper, we study the large distance behavior of two Brownian particles with a local interaction under nonequilibrium conditions. Specifically, we consider two Brownian particles in a two dimensional space of temperature $T$. Their interaction is characterized by an interaction length $\xi$. They are driven in one direction, which we choose as the $x$ direction, by a constant external force $f$, under the influence of a periodic potential with period $\ell$. Let $p_s^{(2)}(r_1, r_2)$ be the steady probability density for the positions of the two particles, $(r_1, r_2)$. Calculating $p_s^{(2)}(r_1, r_2)$ to leading order of in the asymptotic limit $|r_1 - r_2| \to \infty$ under the assumption that $\xi \gg \ell$, we find that

$$p_s^{(2)}(r_1, r_2) - p_s(r_1)p_s(r_2) \simeq \frac{c(r_1, r_2)}{|r_1 - r_2|^2},$$  

(1)

where $c(r_1, r_2)$ does not depend on $|r_1 - r_2|$, but on the direction of the vector $r_1 - r_2$, and $p_s(r)$ is the one body steady probability density for a system without an interaction. Thus, it is found that a long range correlation of $1/r^d$ type appears in a system consisting of two locally interacting particles under external driving.

In order to obtain this result, we apply a method of perturbative system reduction to the Fokker-Planck equation describing the time development of the probability density in the system described above. Here, what we refer to as “perturbative system reduction” consists of a perturbative calculation designed to obtain a simpler representation of a dynamical
system by restricting what we wish to describe. The first application of perturbative system reduction to reaction diffusion systems was carried out by Kuramoto and Tsuzuki \[12\]. They obtained a complex Ginzburg-Landau equation near a Hopf bifurcation and subsequently derived the simplest partial differential equation exhibiting spatially extended chaos \[13\], which is now called the Kuramoto-Sivashinsky equation \[11\]. These methods of derivation have matured since that time, and the universal structure underlying the calculations has been elucidated \[14\] \[15\].

This paper is organized as follows. In Section 2, we present the stochastic model we study and describe the basic features of the model. In Section 3, we formulate a perturbative system reduction by first reviewing the basic ideas introduced by Kuramoto. In Section 4, performing a perturbative expansion of the model, we obtain a reduced model describing the large scale behavior of the system in question. Using this result, in Section 5, we derive the asymptotic form of the large distance behavior of the steady probability density. Section 6 is devoted to concluding remarks.

2 Model

We study the motion of two small particles (on the order of micro-meters in radius) interacting with each other in a fluid of temperature $T$. The particles are confined to a two dimensional square of length $L$ and are subject to a periodic potential $U$ of period $\ell$ in a single direction, which we chose as the $x$ direction. Typical systems with such properties can be realized experimentally \[16\] \[17\]. Further, a flow with constant velocity can be used to apply a constant driving force $f$ to the particles in the $x$ direction. In this way, it is possible to experimentally realize NESSs for such a particle system.

Let $(r_1, r_2)$ represent the positions of the particles. We assume that their motion is described by the Langevin equation

$$\gamma \dot{r}_i = f - \frac{\partial U(r_i)}{\partial r_i} - \frac{\partial V(r_1 - r_2)}{\partial r_i} + \sqrt{2\gamma T} \eta_i(t), \quad (2)$$

where $r = (x, y), V(r_1 - r_2)$ is an interaction potential with interaction length $\xi$, (e.g. $V(r) = \text{const. for } |r| \geq \xi.),$ and $U(r)$ is a periodic potential satisfying

$$U(r + \ell e_x) = U(r). \quad (3)$$

Further, $\eta_i(t)$ represents Gaussian white noise with zero mean and unit dispersion. Here, the Boltzmann constant is set to unity. For simplicity, we assume periodic boundary conditions in both directions and that the system size $L$ is sufficiently larger than $\xi$ and $\ell$.

In this system, the probability density of the particle positions, $p(r_1, r_2, t)$, obeys the Fokker-Planck equation

$$\frac{\partial p}{\partial t} = \frac{1}{\gamma} \sum_{i=1}^{2} \frac{\partial}{\partial r_i} \left[ -f e_x p + \frac{\partial U(r_i)}{\partial r_i} p + \frac{\partial V(r_1 - r_2)}{\partial r_i} p + T \frac{\partial p}{\partial r_i} \right]. \quad (4)$$
Although we wish to obtain the steady solution of (11), it seems unfeasible to derive such a solution in exact form. For this reason, we formulate a perturbation method to extract the large scale behavior of the steady state solution under some assumptions. Before presenting the analysis, some preparation is needed.

We first note that when \( V(r_1 - r_2) = 0 \), the steady state solution of (11) can be derived easily in the form \( p_s(r_1; f)p_s(r_2; f) \), where

\[
p_s(x, y; f) = \frac{1}{Z} I_-(x),
\]

with

\[
I_-(x) = \int_0^\ell dx' e^{-\beta U(x) + \beta U(x' + x) - \beta fx'}.\]

Here, \( \beta = 1/T \) and \( Z \) is a normalization factor that is chosen so that we have

\[
\int_0^\ell dx p_s(x, y; f) = \ell.
\]

For later convenience, we define the following quantities:

\[
J_s(f) = \frac{1}{\gamma} \left( f - \frac{\partial U(r)}{\partial x} \right) p_s(r; f) - \frac{T}{\gamma} \frac{\partial p_s(r; f)}{\partial x},
\]

\[
\phi_s(r; f) = \log p_s(r; f).
\]

Note that \( J_s(f) \) does not depend on \( r \), because \( J_s(f) \) corresponds to the one particle steady state probability current for the case \( V = 0 \).

\section{Formulation}

Next, we consider the effect of the interaction potential \( V \) by first writing

\[
p(r_1, r_2, t) = p_s(r_1; f_1)p_s(r_2; f_2)q(r_1, r_2, t),
\]

where \( f_1 \) and \( f_2 \) are configuration dependent forces defined by

\[
f_i(r_1, r_2) = f - \frac{\partial V(r_1 - r_2)}{\partial x_i}.
\]

Then, substituting (10) into (11), we obtain the evolution equation of \( q(r_1, r_2, t) \) as

\[
\frac{\partial q}{\partial t} = \sum_{i=1}^2 \left[ \hat{M}_i q - \frac{1}{p_s(r_i; f_i)} \frac{dJ_s(f_i)}{df_i} \frac{\partial f_i}{\partial x_i} q + \frac{1}{\gamma} \frac{\partial}{\partial y_i} \left( \frac{\partial V}{\partial y_i} q \right) \right],
\]

where the operator \( \hat{M}_i \) is defined as

\[
\hat{M}_i \equiv -\frac{J_s(f_i)}{p_s(r_i; f_i)} \frac{\partial}{\partial x_i} + \frac{T}{\gamma} \frac{\partial \phi_s(r_i; f_i)}{\partial x_i} \frac{\partial}{\partial x_i} + \frac{T}{\gamma} \left( \frac{\partial^2}{\partial x_i^2} + \frac{\partial^2}{\partial y_i^2} \right).
\]
We study a simple situation where we fix the period ℓ of the periodic potential \( U(r) \) and make the range \( \xi \) of the interaction longer and longer. We then introduce a small parameter \( \varepsilon \) representing the extent of the scale separation: \( \varepsilon \equiv \ell/\xi \ll 1 \). The existence of two typical length scales is accounted for explicitly by introducing the large scale coordinate \( R_i = (X_i, Y_i) \equiv \varepsilon r_i \) and the periodic coordinate \( \theta_i \equiv \mod(x_i, \ell) \), which expresses the \( i \)-th particle position in terms of the phase of the periodic potential. Using these coordinates, we rewrite \( U \) and \( V \) as
\[
U(r_i) = \tilde{U}(\theta_i) \quad \text{(14)}
\]
\[
V(r_1 - r_2) = \tilde{V}(R_1 - R_2) \quad \text{(15)}
\]
In a similar way, we further define \( \tilde{p}_s(\theta_i; \tilde{f}_i) \) and \( \tilde{\phi}_s(\theta_i; \tilde{f}_i) \), where
\[
\tilde{f}_i(R_1, R_2) = f - \varepsilon \frac{\partial \tilde{V}(R_1 - R_2)}{\partial X_i} \quad \text{(16)}
\]
Now, we introduce a slowly varying field \( Q(R_1, R_2, t) \) in such a way that
\[
q(r_1, r_2, t) = Q(R_1, R_2, t) + \varepsilon \rho^{(1)}(\theta_1, \theta_2; [Q]) + \varepsilon^2 \rho^{(2)}(\theta_1, \theta_2; [Q]) + \cdots \quad \text{(17)}
\]
where \( g([Q]) \) represents the functional dependence of \( g \) on \( Q(R_1, R_2, t) \). At this stage, \( Q \) is not determined. According to Ref. [15] written by Kuramoto in 1989, \( Q \) can be regarded as the coordinate of a point on the slow manifold in the functional space \{\( q \)\}. With this interpretation, (17) provides a representation of the slow manifold in terms of \( Q \). Here, obviously, such a representation can be chosen arbitrarily. We therefore choose the following convenient form of the time evolution of \( Q \) is expressed by
\[
\frac{\partial Q}{\partial t} = \varepsilon \Omega^{(1)}([Q]) + \varepsilon^2 \Omega^{(2)}([Q]) + \cdots \quad \text{(18)}
\]
If \( Q \) can be determined uniquely with this requirement, then (18) represents the system reduction we seek. Equations (17) and (18) constitute the basic assumptions of the perturbative system reduction for (12), with the replacements \( U \) by \( \tilde{U} \), and so on. Thus, the problem we face is to determine whether \( \rho^{(n)}(\theta_1, \theta_2; [Q]) \) and \( \Omega^{(n)}([Q]) \) \( (n = 1, 2, \cdots) \) can be determined in an essentially unique way. In the next section, we see that indeed this can be done.

4 Analysis

We first substitute (17) and (18) into (12) and extract all terms proportional to \( \varepsilon \). We then obtain
\[
\Omega^{(1)}([Q]) = \sum_{i=1}^{2} [\hat{M}_i^{(0)} \rho^{(1)} + \hat{M}_i^{(1)} Q], \quad \text{(19)}
\]
where the operators $\hat{M}_i^{(0)}$ and $\hat{M}_i^{(1)}$ are given by

$$\hat{M}_i^{(0)} \equiv -\frac{J_s(f_i)}{\tilde{p}_s(\theta_i; \tilde{f}_i)} \frac{\partial}{\partial \theta_i} + \frac{T}{\gamma} \frac{\partial \phi_s(\theta_i; \tilde{f}_i)}{\partial \theta_i} + \frac{T}{\gamma} \frac{\partial^2 \phi_s(\theta_i; \tilde{f}_i)}{\partial \theta_i^2},$$

$$\hat{M}_i^{(1)} \equiv -\frac{J_s(f_i)}{\tilde{p}_s(\theta_i; \tilde{f}_i)} \frac{\partial}{\partial X_i} + \frac{T}{\gamma} \frac{\partial \phi_s(\theta_i; \tilde{f}_i)}{\partial \theta_i} + \frac{2T}{\gamma} \frac{\partial^2 \phi_s(\theta_i; \tilde{f}_i)}{\partial \theta_i \partial X_i}.$$  

Because $\hat{M}_i^{(0)} \cdot 1 = 0$, $\rho^{(1)}$ can be obtained only when the solvability condition is satisfied. Thus, we need to find an explicit form of the solvability condition.

Let us define an operator $\hat{L}_i$ as

$$\hat{L}_i \cdot \equiv \frac{1}{\gamma} \frac{\partial}{\partial \theta_i} \left( \frac{\partial U}{\partial \theta_i} + T \frac{\partial \phi}{\partial \theta_i} \right).$$

Then, using the relation

$$\hat{L}_i(\tilde{p}_s(\phi(\theta_i))) = \tilde{p}_s(\phi(\theta_i)),$$

for an arbitrary square integrable periodic function $\phi(\theta_i)$, we obtain

$$\int d\theta_i \tilde{p}_s(\phi(\theta_i)) = 0.$$  

From this, the solvability condition for (19) turns out to be

$$\int d\theta_1 d\theta_2 \tilde{p}_s(\phi(\theta_1)) \left[ \Omega^{(1)}([Q]) - \sum_{i=1}^{2} \hat{M}_i^{(1)} Q \right] = 0,$$

which yields

$$\Omega^{(1)}([Q]) = -\sum_{i=1}^{2} J_s(f_i) \frac{\partial Q}{\partial X_i}.$$  

Under this condition, we can derive $\rho^{(1)}$ of (19) in the form

$$\rho^{(1)} = \sum_{i=1}^{2} a(\theta_i; f_i) \frac{\partial Q}{\partial X_i} + \chi^{(1)}(R_1, R_2),$$

where $a(\theta_i; f_i)$ can be calculated explicitly as in Ref. 19, and $\chi^{(1)}(R_1, R_2)$ is an arbitrary function of $(R_1, R_2)$. (The choice of this function does not influence the result for $\Omega^{(n)}.$)

Next, we sum up all terms proportional to $\varepsilon^2$. This yields

$$\Omega^{(2)}([Q]) + \frac{\delta \rho^{(1)}}{\delta Q} \cdot \Omega^{(1)}([Q]) = \sum_{i=1}^{2} \left[ \hat{M}_i^{(0)} \rho^{(2)} + \hat{M}_i^{(1)} \rho^{(1)} + \hat{M}_i^{(2)} Q \right] + \frac{\mu a(f_i)}{\tilde{p}_s(\theta_i; \tilde{f}_i)} \frac{\partial^2 \tilde{V}}{\partial X_i^2} Q + \frac{1}{\gamma} \frac{\partial}{\partial Y_i} \left( \frac{\partial \tilde{V}}{\partial Y_i} Q \right).$$
where
\[
\hat{M}_i^{(2)} = \frac{T}{\gamma} \left( \frac{\partial^2}{\partial X_i^2} + \frac{\partial^2}{\partial Y_i^2} \right),
\] (29)
and we have introduced the differential mobility \( \mu_a(f) \equiv dJ_s/df \), which is found to be
\[
\mu_a(f) = \frac{1}{\gamma} \frac{1}{\frac{T}{\gamma} \int_0^t dx I_-(x) I_+(x)} \frac{dI_+}{df},
\] (30)
with
\[
I_+(x) = \int_0^x dx' e^{\beta U(x) - \beta U(x-x') - \beta f x'}. \] (31)
The solvability condition for \( \rho^{(2)} \) in (28), which is obtained by multiplying both sides of (28) by \( \int d\theta_1 \int d\theta_2 \tilde{p}_s(\theta_1) \tilde{p}_s(\theta_2) \), yields
\[
\Omega^{(2)} = \sum_{i=1}^2 \left[ \mu_a(\tilde{f}_i) \frac{\partial^2 \tilde{V}}{\partial X_i^2} + D(f) \frac{\partial^2 Q}{\partial X_i^2} + T \frac{\partial^2 Q}{\partial Y_i^2} + 1 \frac{\partial}{\partial Y_i} \left( \frac{\partial \tilde{V}}{\partial y_i} Q \right) \right], \] (32)
where \( D \) is obtained as
\[
D(f) = \frac{T}{\gamma} \frac{1}{\frac{T}{\gamma} \int_0^t dx I_-(x)^2 I_+(x)} \frac{dI_+}{df}. \] (33)
This quantity clearly represents the diffusion constant in the \( x \) direction \[18\]. The derivation here is parallel to that given in Ref. \[19\]. Under the solvability condition (32), we can obtain \( \rho^{(2)} \).

Finally, carrying out a similar calculation, we can determine \( \rho^{(n)}(\theta_1, \theta_2; [Q]) \) and \( \Omega^{(n)}([Q]) \) from the terms proportional to \( \varepsilon^n \) in (12), with (17) and (18). This iterative procedure constitutes the perturbative system reduction.

### 5 Long range correlation

Recall that our goal is to obtain the steady state solution of the Fokker-Planck equation (4) under the assumption \( \varepsilon \ll 1 \). Let us express this solution as
\[
p_s^{(2)}(r_1, r_2) = p_s(r_1)p_s(r_2)q_s(r_1, r_2). \] (34)
Then, as far as we focus on the large distance behavior of \( q_s(r_1, r_2) \), from (10), (17), (18), (20) and (32), we can assume that \( q_s(r_1, r_2) \) satisfies the equation
\[
\sum_{i=1}^2 \left[ -J_s(f) \frac{\partial q_s}{\partial x_i} + \mu_a(f) \frac{\partial}{\partial x_i} \left( \frac{\partial \tilde{V}}{\partial x_i} q_s \right) + D(f) \frac{\partial^2 q_s}{\partial x_i^2} + T \frac{\partial^2 q_s}{\partial y_i^2} + 1 \frac{\partial}{\partial y_i} \left( \frac{\partial \tilde{V}}{\partial y_i} q_s \right) \right] = 0. \] (35)
Now, we define $\psi(r_1, r_2)$, which represents the non-equilibrium contribution of $q_s$, through the relation

$$q_s(r_1, r_2) = e^{-\beta V(r_1-r_2)+\psi(r_1, r_2)}. \tag{36}$$

Furthermore, assuming that $V$ is sufficiently small, we linearize (35) with respect to $\psi$ and $V$. This yields

$$\sum_{i=1}^{2} \left[ \left\{ -J_s(f) + (D(f) - \mu_d(f)) T \right\} \left( -\beta \frac{\partial V}{\partial x_i} + \frac{\partial \psi}{\partial x_i} \right) + T \gamma \left( \frac{\partial^2 }{\partial x_i^2} + \frac{\partial^2 }{\partial y_i^2} \right) \psi \right] = 0. \tag{37}$$

Then, using the Fourier expansion, we can solve for $\psi$ in (37) as

$$\psi(r_1, r_2) = (D - \mu_d T) \beta \int \frac{d^2 k}{(2\pi)^2} e^{-i k (r_1 - r_2)} \frac{k_x^2}{D k_x^2 + T k_y^2 / \gamma} \hat{V}(k), \tag{38}$$

where

$$\hat{V}(k) = \int d^2 r e^{i k r} V(r). \tag{39}$$

From this result, it is straightforward to derive the asymptotic form

$$\psi(r_1, r_2) \simeq \frac{1}{2\pi} \frac{D - \mu_d T}{(DT)^{3/2}} \frac{1}{\gamma^{1/2}} \hat{V}(0) \frac{((x_1 - x_2)^2 / D - (y_1 - y_2)^2 \gamma / T)}{((x_1 - x_2)^2 / D + (y_1 - y_2)^2 \gamma / T)^2} \tag{40}$$

in the region $\xi \ll |r_1 - r_2| \ll L$. As shown in Ref. [19], the Einstein relation $D = \mu_d T$ is violated in general NESSs. Thus, we conclude that there is long range correlation of $1/r^2$ type.

### 6 Concluding remarks

We have demonstrated that there exists long-range spatial correlation between two interacting Brownian particles under external driving. We have found that this long range correlation is proportional to $D - \mu_d T$, which represents the degree of the breakdown of detailed balance. It is quite reasonable to expect that the long range correlation found for this two particle system exists also in many particle systems, with a quantitative correction arising from many body effects. It is a future project to study many body effects by extending approach in the present paper.

The existence of long range correlation of $1/r^d$ type makes it difficult to construct a universal framework for a statistical theory. Let us explain the reason for this by considering $N$ particles in a twodimensional box of length $L$. We write the $N$-body steady state distribution for this system as

$$p\left(\{r_i\}\right) = e^{-\beta \Phi(\{r_i\})}. \tag{41}$$

One may naively interpret $\Phi(\{r_i\})$ as "effective energy" of the particles under nonequilibrium conditions, because $\Phi(\{r_i\})$ corresponds to the Hamiltonian of the particle configuration $\{r_i\}$ at equilibrium. With this interpretation, there appears an effective long range...
interaction potential of $1/r^2$ type. Then, for such systems, simple considerations yield the estimation

$$\langle \Phi \rangle \sim L^2 \log L$$

(42)

in the limit that $L \to \infty$, with fixed $L^2/N$. This implies that extensivity, which is the most essential property of thermodynamic systems, does not hold. Such statistical systems are pathological for the following reasons. First, a statistical distribution located in the central region of the system depends sensitively on the nature of the boundary conditions \[20\]. That is, it is difficult to define a bulk region for the system. Second, if $\Phi$ can be measured as “energy” using some experimental method, (42) implies that a significant amount of energy can be extracted by merely splitting one system or combining two systems. Because such a situation seems to be unphysical, it is reasonable to conjecture that $\Phi$ does not represent an “energy”. These conclusions cast doubt on the possibility of realizing a unified statistical framework.

Despite these seemingly intractable properties, we wish to seek a universal statistical framework for NESS by separating the problem in the following way. First, we propose to check the possibility that large scale fluctuations can be distinguished from small scale fluctuations in some way. If this can be done, we hope to determine whether small scale fluctuations, which may deviate substantially from those of an equilibrium system, can be characterized in terms of an energetic quantity. Then, finally, we hope to study large scale fluctuations on the basis of the characterization of small scale fluctuations. Recently, we have made some progress in the characterization of small scale fluctuations through an extension of thermodynamic functions \[21, 22\]. We now propose to attempt unifying large scale anomalous fluctuations with our thermodynamic framework, going beyond the result of the present study.

In closing this paper, we would like to return to Ref. \[1\] written by Kuramoto in 1974. The observation made there that large scale fluctuations should be considered separately led Kuramoto to focus on dynamical behavior of macroscopic variables. In particular, when solutions of deterministic equations for macroscopic variables describe a rich variety of phenomena including oscillations and chaos, the understanding of such phenomena from a dynamical system point of view may be most important. With this realization, Kuramoto naturally was led to study dynamical systems. This is regarded as the genesis of nonlinear dynamics as a method for studying nonequilibrium statistical phenomena. The most important message here seems to be that to formulate questions that do not conform to the contemporary mainstream can lead to new fields of research. Today, the study of nonlinear dynamics has been fully developed. Following Kuramoto, we should consider to seek the formulation of precise and deep questions regarding nonlinear and nonequilibrium systems that do not conform to current trends.
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