Ultra-Slow Vacancy-Mediated Tracer Diffusion in Two Dimensions: The Einstein Relation Verified.

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Abstract

We study the dynamics of a charged tracer particle (TP) on a two-dimensional lattice all sites of which except one (a vacancy) are filled with identical neutral, hard-core particles. The particles move randomly by exchanging their positions with the vacancy, subject to the hard-core exclusion. In case when the charged TP experiences a bias due to external electric field $E$, (which favors its jumps in the preferential direction), we determine exactly the limiting probability distribution of the TP position in terms of appropriate scaling variables and the leading large-$n$ ($n$ being the discrete time) behavior of the TP mean displacement $\mathbf{x}_n$; the latter is shown to obey an anomalous, logarithmic law $|\mathbf{x}_n| = \alpha_0(|E|) \ln(n)$. On comparing our results with earlier predictions by Brummelhuis and Hilhorst (J. Stat. Phys. \textbf{53}, 249 (1988)) for the TP diffusivity $D_n$ in the unbiased case, we infer that the Einstein relation $\mu_n = \beta D_n$ between the TP diffusivity and the mobility $\mu_n = \lim_{|E| \to 0} (|\mathbf{x}_n|/|E|n)$ holds in the leading in $n$ order, despite the fact that both $D_n$ and $\mu_n$ are not constant but vanish as $n \to \infty$. We also generalize our approach to the situation with very small but finite vacancy concentration $\rho$, in which case we find a ballistic-type law $|\mathbf{x}_n| = \pi \alpha(0)(|E|) \rho n$. We demonstrate that here, again, both $D_n$ and $\mu_n$, calculated in the linear in $\rho$ approximation, do obey the Einstein relation.
Consider a square lattice of which each site except one is filled with a hard-core particle. The empty site is referred to as a "vacancy". The particles move randomly on the lattice, their random walks being constrained by the condition that each site can be at most singly occupied. More specifically, at each moment of time \( n = 1, 2, 3, \ldots \) one particle selected with probability \( 1/4 \) among the four particles surrounding the vacancy will exchange its position with the vacancy. Suppose next that one selects one of the particles, "tags" it and follows its trajectory \( X_n \). Evidently, dynamics of the tagged - the tracer particle (TP) will be quite complicated, in contrast to the standard, by definition, lattice random walk executed by the vacancy: The TP can move only when encountered by the vacancy and its successive moves will be correlated, since the vacancy will always have a greater probability to return for its next encounter from the direction it has left than from a perpendicular or opposite direction. On the other hand, it is clear that on a two-dimensional lattice the TP will make infinitely long excursions as \( n \to \infty \) even in the presence of a single vacancy, since its random walk is recursive in 2D and the vacancy is certain to encounter the tracer particle many times. A natural question is, of course, what are the statistical properties of the TP random walk, its mean-square displacement \( \overline{X^2_n} \) from its initial position at time moment \( n \), and the probability \( P_n^{(tr)}(X) \) that at time \( n \) the TP appears at position \( X = (x_1, x_2) \)?

The just described model, which represents, in fact, one of the simplest cases of the so-called "slaved diffusion processes", has been studied over the years in various guises, ranging from the "constrained dynamics" model of Palmer \cite{Palmer}, vacancy-mediated bulk diffusion in metals and crystals (see, e.g. \cite{Palmer,Palmer2}), frictional properties of dynamical percolative environments \cite{Palmer3,Palmer4} or dynamics of impure atoms in close-packed surfaces of metal crystals, such as, e.g., a copper \cite{Palmer5,Palmer6}. Brummelhuis and Hilhorst \cite{Palmer7} were first to present an exact solution of this model in the lattice formulation. It has been shown that in the presence of a single vacancy the TP trajectories are remarkably confined; the mean-square displacement shows an unbounded growth, but it does grow only logarithmically with time,

\[
\overline{X^2_n} \sim \frac{\ln(n)}{\pi(\pi - 1)}, \quad \text{as } n \to \infty, \tag{1}
\]

which implies that the TP diffusivity \( D_n \), defined as

\[
D_n = \frac{\overline{X^2_n}}{4n} \sim \frac{\ln(n)}{4\pi(\pi - 1)n}, \tag{2}
\]

is not constant but rather vanishes as time \( n \) progresses.

Moreover, it has been found \cite{Palmer8} that at sufficiently large times \( P_n^{(tr)}(X) \) converges to a limiting form as a function of the scaling variable \( \eta = |X|/\sqrt{\ln(n)} \). Still striking, this limiting distribution is not a Gaussian but a modified Bessel function \( K_0(\eta) \), which signifies that the successive steps of the TP, although separated by long time intervals, are effectively correlated. These results have been subsequently reproduced by means of different analytical techniques in Refs.\cite{Palmer9} and \cite{Palmer10,Palmer11}. 

1 Introduction.
Brummellhuis and Hilhorst have also generalized their analytical approach to the case of a very small but finite vacancy concentration \( \rho \), in which case a conventional diffusive-type behavior

\[
\mathbf{X}_n^x = \frac{\rho n}{(\pi - 1)}, \quad \rho \ll 1, \quad n \to \infty,
\]  

has been recovered. Note that Eq.(3) coincides with the earlier result of Nakazato and Kitahara \[2\] in the limit \( \rho \ll 1 \), and is well confirmed by numerical simulations \[4, 18\].

This paper is devoted to the following, rather fundamental to our point, problem: Suppose that we charge the tracer particle, (while the rest are kept neutral), and switch on an electric field \( E \). In such a situation, the TP will have asymmetric hopping probabilities and in its exchanges with the vacancy, depending on the TP and vacancy relative orientation, the TP will have a preference (or, on contrary, a reduction of the rate) for exchanging its position with the vacancy compared to other three neighboring particles. One might expect that in this case the TP mean displacement \( \mathbf{X}_n \) will not be exactly equal to zero and might define the TP mobility as

\[
\mu_n = \lim_{|E| \to 0} \frac{|\mathbf{X}_n|}{|E|n}.
\]  

Now, the question is whether the mobility \( \mu_n \), calculated from the TP mean displacement in the presence of an external electric field, and the diffusivity \( D_n \), Eq.(2), deduced from the TP mean-square displacement in the absence of the field, obey the generalized Einstein relation of the form

\[
\mu_n = \beta D_n,
\]  

where \( \beta \) denotes the reciprocal temperature?

Note that this question has been already addressed within the context of the TP diffusion in one-dimensional hard-core lattice gases with arbitrary finite vacancy concentration \[3, 6, 7, 19, 20\]. It has been found that Eq.(3) holds not only for the TP diffusion in a 1D hard-core gas on a finite lattice \[3\], but also for infinite 1D lattices with non-conserved \[6\] and conserved particles number \[7, 19, 20\]. Remarkably, in the latter case Eq.(3) holds for \( n \) sufficiently large despite the fact that both the TP mobility and the diffusivity are not constant as \( n \to \infty \) but all vanish in proportion to \( 1/\sqrt{n} \) \[6, 19, 20\]. On the other hand, it is well known that the Einstein relation is violated in some physical situations; for instance, it is not fulfilled for Sinai diffusion \[29\] or diffusion on percolation clusters, due to effects of strong temporal trapping in the dangling ends (see also Refs.\[27\] and \[28\] for some other examples). Hence, in principle, it is not a priori clear whether Eq.(3) should be valid for the model under study; here, the TP walk proceeds only due to encounters with a single vacancy, its mean-square displacement grows only logarithmically with time and the diffusivity follows much faster decay law in Eq.(2), compared to the \( D_n \sim n^{-1/2} \) law obtained for the one-dimensional systems with finite vacancy concentrations.

The paper is structured as follows: In section 2 we present more precise formulation of the problem and introduce basic notations. In Section 3 we discuss our general approach to computation of the probability \( P_n^{(tr)}(\mathbf{X}) \) of finding the TP at position \( \mathbf{X} \) at time moment \( n \) and to evaluate
$P_n^{(tr)}(X)$ in the general form as a function of some return probabilities describing the random walk executed by the vacancy. The Section 4 is devoted to calculation of these return probabilities in the general case, as well as to the derivation of explicit expressions determining their asymptotical behavior. In Section 5, we present explicit asymptotical results for both the probability distribution and the TP mean displacement. We show that as $n \to \infty$, $P_n^{(tr)}(X)$ written in terms of two appropriate scaling variables, converges to a rather unusual limiting distribution. We also demonstrate here that the TP mobility, which is obtained in the present work in the leading in $n$ order, and the TP diffusivity in the unbiased case, calculated earlier by Brummelhuis and Hilhorst [13], do obey the Einstein relation. Further on, in Section 6 we extend our approach to the situation with very small but finite vacancy concentration and determine, in the leading in $n$ order, the TP mobility. We show that also in this case the TP mobility and the TP diffusivity in the unbiased case do obey the Einstein relation, in the linear in $\rho$ approximation and in the leading in $n$ order. Finally, in Section 7, we conclude with a brief summary and discussion of our results.

2 The model.

Consider a two-dimensional, infinite in both $x_1$ and $x_2$ directions, square lattice every site of which except one (a vacancy) is filled by identical hard-core particles (see Fig.1). All particles except one are electrically neutral. The charged particle, which is initially at the origin, will be referred to in what follows as the tracer particle - the TP. Its position at the lattice at time $n$ will be denoted by $X_n$. Electric field $\mathbf{E}$ of strength $E = |\mathbf{E}|$ is oriented in the positive $x_1$ direction. For simplicity, the charge of the TP is set equal to unity.

Next, we suppose that at each tick of the clock, $n = 1, 2, 3, \ldots$, each particle selects at random a jump direction and attempts to hop onto the target site. Evidently, the jump event can be only successful for four particles adjacent to the vacancy.

The form of the jump direction probabilities depends on whether the particle is charged or not. For uncharged particles all hopping directions are equally probable and hence, all jump direction probabilities are equal to $1/4$. On the other hand, the charged particle - the TP, "prefers" to jump in the direction of the applied electric field; the normalized jump direction probabilities of the TP are given, in a usual fashion, by

$$p_\nu = Z^{-1} \exp \left[ \frac{\beta}{2} (E \cdot e_\nu) \right],$$

(6)

where $Z$ is the normalization constant, $e_\nu$ is the unit vector denoting the jump direction, $\nu \in \{\pm 1, \pm 2\}$, and $(E \cdot e_\nu)$ stands for the scalar product. We adopt the notations $e_{\pm 1} = (\pm 1, 0)$ and $e_{\pm 2} = (0, \pm 1)$, which means that $e_1$ ($e_{-1}$) is the unit vector in the positive (negative) $x_1$-direction, while $e_2$ ($e_{-2}$) is the unit vector in the positive (negative) $x_2$-direction. Consequently, the normalization constant $Z$ is

$$Z = \sum_\mu \exp \left[ \frac{\beta}{2} (E \cdot e_\mu) \right],$$

(7)
where the sum with the subscript $\mu$ denotes summation over all possible orientations of the vector $e_\mu$; that is, $\mu = \{\pm 1, \pm 2\}$. Note that the jump direction probabilities defined by Eqs. (6) and (7) do preserve the detailed balance condition.

Figure 1: Two-dimensional, infinite in both directions, square lattice in which all sites except one are filled with identical hard-core particles (grey spheres). The black sphere denotes a single tracer particle, which is subject to external field $E$, oriented in the positive $x_1$ direction, and thus has asymmetric hopping probabilities.

Next, it is expedient to reformulate the dynamics between two consecutive jumps of the TP in terms of the random walk executed by the vacancy and its jump direction probabilities $q_\nu$. From the viewpoint of the vacancy, the jump direction probabilities depend on whether the TP is one of four surrounding particles or not. Evidently, in case when all four surrounding particles are electrically neutral, we still have that on the next time step the vacancy will change its position with one of four neighboring particles selected at random with equal probabilities. Hence, in case when the TP is not adjacent to the vacancy, all four jump directions for the vacancy are equally probable, i.e. $q_\nu = 1/4$. On the other hand, the situation is a bit more complex when one of these four particles is the TP, which has asymmetric jump direction probabilities, Eq. (6). A natural choice of the normalized jump direction probabilities of the vacancy in this case is as follows: Suppose that at time moment $n$ the tracer particle is at position $X_n$ and the vacancy occupies $X_{n+1}$.

1Note, that normalization here insures that the vacancy performs one jump each time step. Otherwise, we will introduce artificial "temporal trapping" probability, which would definitely lead to the violation of Eq. (5).
an adjacent site \( \mathbf{X}_n + \mathbf{e}_\nu \). Then, an exchange of the positions between the TP and the vacancy, which implies that the TP is moved one step in the \( \mathbf{e}_\nu \)-direction, takes place with the probability

\[
q_{-\nu} = Z^* p_\nu
\]  

(8)

while the probability of the exchange of positions with any of other three adjacent particles is given by

\[
q_{\mu \neq -\nu} = \frac{1}{4} Z^*
\]  

(9)

The normalization constant \( Z^* \) in this case is, evidently,

\[
Z^* = \frac{3}{4} + p_\nu,
\]  

(10)

where \( p_\nu \) has been defined previously in Eq.(6).

Consequently, apart of four sites in the immediate vicinity of the tracer particle, the vacancy performs a standard, symmetric random walk. In the vicinity of the TP, the vacancy jump direction probabilities are perturbed by the TP asymmetric hopping rules. Hence, the random walk executed by the vacancy can be thought off as a particular case of the so-called “random walk with defective sites” (see Ref.[5] for more details), or as a realization of the “random walk with a hop-over site”[16].

3 Probability distribution function \( P_n^{(tr)}(\mathbf{X}) \).

A standard approach to define the properties of the TP random walk would be to start with a master equation determining the evolution of the whole configuration of particles. In doing so, similarly to the analysis of the tracer diffusion on 2D lattices in the presence of a finite vacancy concentration (see, e.g. Ref.[26]), one obtains the evolution of the joint distribution \( P_n(\mathbf{X}, \mathbf{Y}) \) of the TP position \( \mathbf{X} \) and of the vacancy position \( \mathbf{Y} \) at time moment \( n \). The property of interest, i.e. the reduced distribution function of the TP alone will then be found from \( P_n(\mathbf{X}, \mathbf{Y}) \) by performing lattice summation over all possible values of the variable \( \mathbf{Y} \).

Here we pursue, however, a different approach, which has been first put forward in the original work of Brummellhuis and Hilhorst[13]; that is, we construct the distribution function of the TP position at time \( n \) directly in terms of the return probabilities of the random walk performed by the vacancy. The only complication, compared to the unbiased case considered by Brummellhuis and Hilhorst[13], is that in our case ten different return probabilities would be involved, instead of three different ones appearing in the unbiased case. Hence, the analysis will be slightly more involved.

We begin by introducing some basic notations. Let

- \( P_n^{(tr)}(\mathbf{X}) \) be the probability that the TP, which starts its random walk at the origin, appears at the site \( \mathbf{X} \) at time moment \( n \), given that the vacancy is initially at site \( \mathbf{Y}_0 \).
• \( F_n^*(0 \mid Y_0) \) be the probability that the vacancy, which starts its random walk at the site \( Y_0 \), arrives at the origin \( 0 \) for the first time at the time step \( n \).

• \( F_n^*(0 \mid e_\nu \mid Y_0) \) be the conditional probability that the vacancy, which starts its random walk at the site \( Y_0 \), appears at the origin for the first time at the time step \( n \), being at time moment \( n - 1 \) at the site \( e_\nu \).

Further on, for any time-dependent quantity \( L_n \) we define the generating function of the form:

\[
L(\xi) = \sum_{n=0}^{+\infty} L_n \xi^n
\]

and for any space-dependent quantity \( Y(X) \) the discrete Fourier transform

\[
\tilde{Y}(k) = \sum_X \exp\left(i(k \cdot X)\right) Y(X),
\]

where the sum runs over all lattice sites.

Now, following Brummelhuis and Hilhorst [13], we write down directly the equation obeyed by the reduced probability distribution \( P_n^{(tr)}(X) \) (cf Ref.[15] for a study of the joint probability of the TP position and of the vacancy position in the unbiased case):

\[
P_n^{(tr)}(X) = \delta_{X,0} \left( 1 - \sum_{j=0}^{n} F_j^*(0 \mid Y_0) \right) + \sum_{p=1}^{+\infty} \sum_{m_1=1}^{+\infty} \cdots \sum_{m_p=1}^{+\infty} \sum_{j=0}^{+\infty} \sum_{m_{p+1}=0}^{+\infty} \sum_{\nu_1} \cdots \sum_{\nu_p} \delta_{e_{\nu_1} + \cdots + e_{\nu_p}, X} \times \left( 1 - \sum_{j=0}^{m_{p+1}} F_j^*(0 \mid - e_{\nu_p}) \right) \times F_{m_p}^*(0 \mid e_{\nu_p}) \cdots F_{m_2}^*(0 \mid e_{\nu_2}) \cdots F_{m_1}^*(0 \mid e_{\nu_1} \mid Y_0).
\]

Next, using the definition of the generating functions and of the discrete Fourier transforms, Eqs.(11) and (12), we obtain the following matricial representation of the generating function of the TP probability distribution:

\[
\tilde{P}^{(tr)}(k; \xi) = \frac{1}{1 - D^{-1}(k; \xi)} \left( 1 + D^{-1}(k; \xi) \sum_\mu U_\mu(k; \xi) F^*(0 \mid e_\mu \mid Y_0; \xi) \right).
\]

In Eq.(14) the function \( D(k; \xi) \) stands for the determinant of the following 4 \( \times 4 \) matrix,

\[
D(k; \xi) \equiv \det(I - T(k; \xi)),
\]

where the matrix \( T(k; \xi) \) has the elements \( (T(k; \xi))_{\nu,\mu} \) defined by

\[
(T(k; \xi))_{\nu,\mu} = \exp \left( i(k \cdot e_\nu) \right) A_{\nu,-\mu}(\xi).
\]
Explicitly, the matrix $T(k; \xi)$ is given by

$$
T(k; \xi) \equiv \begin{pmatrix}
e^{ik_1 A_{1,-1}(\xi)} & e^{ik_1 A_{1,1}(\xi)} & e^{ik_1 A_{1,-2}(\xi)} & e^{ik_1 A_{1,2}(\xi)} \\
e^{-ik_1 A_{1,-1}(\xi)} & e^{-ik_1 A_{1,1}(\xi)} & e^{-ik_1 A_{1,-2}(\xi)} & e^{-ik_1 A_{1,2}(\xi)} \\
e^{ik_2 A_{2,-1}(\xi)} & e^{ik_2 A_{2,1}(\xi)} & e^{ik_2 A_{2,-2}(\xi)} & e^{ik_2 A_{2,2}(\xi)} \\
e^{-ik_2 A_{2,-1}(\xi)} & e^{-ik_2 A_{2,1}(\xi)} & e^{-ik_2 A_{2,-2}(\xi)} & e^{-ik_2 A_{2,2}(\xi)}
\end{pmatrix}, \tag{17}
$$

where the coefficients $A_{\nu,\mu}(\xi)$, $\nu, \mu = \pm 1, \pm 2$, stand for

$$
A_{\nu,\mu}(\xi) \equiv F^*(0 \mid e_{\nu} \mid e_{\mu}; \xi) = \sum_{n=0}^{+\infty} F^n(0 \mid e_{\nu} \mid e_{\mu}) \xi^n, \tag{18}
$$
i.e. are the generating functions of the conditional probabilities for the first time visit of the origin by the vacancy, conditioned by constraint of the passage through a specified site on the previous step. Note that, by symmetry,

$$
A_{2,\nu}(\xi) = A_{-2,\nu}(\xi),
A_{\nu,2}(\xi) = A_{\nu,-2}(\xi) \tag{19}
$$

for $\nu = \pm 1$ and

$$
A_{2,2}(\xi) = A_{-2,-2}(\xi),
A_{2,-2}(\xi) = A_{-2,2}(\xi). \tag{20}
$$

As a result of such a symmetry, we have to consider just ten independent functions $A_{\mu,\nu}(\xi)$ (note that in the unbiased case one has to deal with only three such functions \cite{13}). Explicit expression of the determinant in Eq.\cite{13} in terms of these generating function is presented in the Appendix. Lastly, the matrix $U_{\mu}(k; \xi)$ in Eq.\cite{14} is given by

$$
U_{\mu}(k; \xi) \equiv \mathcal{D}(k; \xi) \sum_{\nu} (1 - e^{-i(k \cdot e_\nu)})(1 - T(k; \xi))^{-1}_{\nu,\mu} e^{i(k \cdot e_\nu)}. \tag{21}
$$
The property of interest - the TP probability distribution function, will be then obtained by inverting $\tilde{P}^{(tr)}(k; \xi)$ with respect to the wave-vector $k$ and to the variable $\xi$:

$$
P_{n}^{(tr)}(X) = \frac{1}{2\pi} \oint_{\mathcal{C}} \frac{d\xi}{\xi^{n+1}} \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} dk_1 \int_{-\pi}^{\pi} dk_2 e^{i(k \cdot X)} \tilde{P}^{(tr)}(k; \xi), \tag{22}
$$
where the contour of integration $\mathcal{C}$ encircles the origin counterclockwise.

Finally, we remark that as far as we are interested in the leading large-$n$ behavior of the probability distribution $P_{n}^{(tr)}(X)$ only, we may constrain ourselves here to the study of the asymptotic behavior of the generating function $\tilde{P}^{(tr)}(k; \xi)$ in the vicinity of its singular point nearest to $\xi = 0$. We notice that similarly to the unbiased case, this point is $\xi = 1$ when $k = 0$. As a matter of fact, such a behavior stems from the \textit{a priori} non-evident fact that the vacancy, starting from a given neighbouring site to the origin, is certain to eventually reach the origin. This will be demonstrated explicitly in section 4 (cf. Eq.\cite{17}); as a matter of fact, one can see from Eq.\cite{17} and the explicit representation of $\mathcal{D}(0; \xi)$ presented in the Appendix that $\mathcal{D}(0; \xi = 1) \equiv 0$. In consequence, expansion in powers of a small deviation $(1 - \xi)$ has to be accompanied by a small-$k$ expansion, exactly as it has been performed in Ref.\cite{13}.
4 The return probabilities $F^*_n(0 \mid e_\mu \mid e_\nu)$.

As we have already remarked, the vacancy random walk between two successive visits of the lattice site occupied by the TP can be viewed as a standard, two-dimensional, symmetric random walk with some boundary conditions imposed on the four sites adjacent to the site occupied by the TP.

In order to compute the return probabilities $F^*_n(0 \mid e_\mu \mid e_\nu)$ for such a random walk, we add, in a usual fashion [5, 22], an additional constraint that the site at the lattice origin is in absorbing state. Then, the vacancy random walk can be formally represented as a lattice random walk with site-dependent probabilities of the form $p^+(s \mid s') = 1/4 + q(s \mid s')$, where $s$ is the site occupied by the vacancy at the time moment $n$, $s'$ denotes the target, nearest-neighboring to $s$ site,

$$q(s \mid s') \equiv \begin{cases} 0 & \text{if } s' \notin \{0, e_\mu, e_\nu, e_{\mu \pm 1}, e_{\nu \pm 1}\}, \\ \delta_{s,0} - 1/4 & \text{if } s' = 0, \\ \delta_{q_\nu} & \text{if } s' = e_\nu \text{ and } s = 0, \\ -\delta_{q_\nu}/3 & \text{if } s' = e_\nu \text{ and } s' \neq 0, \end{cases} \quad (23)$$

where $\delta_{q_\nu}$ is defined, according to Eqs. (8), (9) and (10), by

$$\delta_{q_\nu} \equiv \frac{p_\nu}{p_\nu + 3/4} - \frac{1}{4} \quad (24)$$

Further on, we define $P^+_n(s \mid s_0)$ as the probability distribution associated with such a random walk starting at site $s_0$ at step $n = 0$.

Now, let the symbols $E$, $A$ and $B$ define the following three events:

- the event $E$: the vacancy, which has started its random walk at the site $e_\nu$, visits the origin $0$ for the first time at the $n$-th step exactly, being at the site $e_\mu$ at the previous step $n - 1$;
- the event $A$: the vacancy, which started its random walk at the site $e_\nu$, is at the site $e_\mu$ at the time moment $n - 1$ and the origin $0$ has not been visited during the $n - 1$ first steps of its walk;
- the event $B$: the vacancy jumps from the neighboring to the origin site $e_\mu$ to the site $0$ at the $n$-th step exactly.

Evidently, by definition, the desired first visit probability $F^*_n(0 \mid e_\mu \mid e_\nu)$ is just the probability of the $E$ event

$$F^*_n(0 \mid e_\mu \mid e_\nu) = \text{Prob}(E). \quad (25)$$

To calculate $\text{Prob}(E)$ we note first that the probabilities of such three events obey:

$$\text{Prob}(E) = \text{Prob}(A \cap B) = \text{Prob}(A) \text{ Prob}(B). \quad (26)$$

On the other hand, we have that

$$\text{Prob}(A) = P^+_{n-1}(e_\mu \mid e_\nu), \quad (27)$$
\[ \text{Prob}(B) = \frac{p\mu}{3/4 + p\mu}. \]  

Hence, in virtue of Eqs. (25), (26), (27) and (28), the return probability \( F_n^+(0 \mid e_\mu \mid e_\nu) \) is given explicitly by

\[ F_n^+(0 \mid e_\mu) = \xi \left( \frac{p\mu}{3/4 + p\mu} \right) F_n^+(e_\mu, e_\nu, \xi). \]  

Therefore, calculation of the return probabilities \( F_n^+(0 \mid e_\mu \mid e_\nu) \) amounts to the evaluation of the probability distribution \( P_n^+(s \mid s_0) \) of the vacancy random walk in the presence of an absorbing site placed at the lattice origin. Such a probability distribution will be determined in the next subsection.

### 4.1 The generating function of the probability distribution \( P_n^+(s \mid s_0) \).

Making use of the generating function technique adapted to random walks on lattices with defective sites [5] and [23], we obtain

\[ P_n^+(s \mid s_0) = \sum_{s_0} P(s \mid s_0) = \sum_{s_0} P(s \mid s_0, \xi) = \sum_{s_0} P(s \mid s_0, \xi). \]  

where

\[ s_i \equiv \begin{cases} e_i, & \text{for } i \in \{ \pm 1, \pm 2 \}, \\ 0, & \text{for } i = 0, \end{cases} \]  

and

\[ A(s_i \mid s_j; \xi) = \sum_{s'} P(s_i \mid s_j, \xi) q(s' \mid s_i), \]  

\( P(s_i \mid s_j; \xi) \) being the generating function of the unperturbed associated random walk (that is, symmetric random walk with no defective sites).

Further on, Eq. (30) can be recast into the following matricial form:

\[ P^+ = (1 - A)^{-1} P, \]  

in which equation \( P, P^+, A \) stand for the 5 \times 5 matrices with the elements defined by

\[ P_{i,j} = P(s_i \mid s_j; \xi), \quad P^+_{i,j} = P^+(s_i \mid s_j; \xi), \quad A_{i,j} = A(s_i \mid s_j; \xi), \]  

where \( i, j = 0, +1, -1, +2, -2 \). Using next an evident relation [3]:

\[ P(s_k \mid s_i; \xi) = \delta_{k,i} + \frac{\xi}{4} \sum_{\nu} P(s_k \mid s_i + e_\nu; \xi), \]  

and the symmetry properties of a standard random walk, one can readily show that:
• for $s_l \neq s_0$ and $s_k \neq s_0$,
  \[ A(s_k \mid s_l; \xi) = \frac{4}{3} \delta_{l1} \left( P(0 \mid 0; \xi) - 1 - P(s_k \mid s_l; \xi) + \delta_{l,k} \right), \]
  \[ A(s_0 \mid s_l; \xi) = \frac{4}{3} \xi \delta_{l1} \left( P(0 \mid 0; \xi) - \frac{1}{\xi^2} (P(0 \mid 0; \xi) - 1) \right), \]
  \[ A(s_k \mid s_0; \xi) = \delta_{k,0} - (1 - \xi) P(s_k \mid 0; \xi), \]

Consequently, the matrices $A$ and $P$ in Eq. (33) are given by

\[
A = \begin{pmatrix}
    a & \delta q_1 f & \delta q_{-1} f & \delta q_2 f & \delta q_3 f \\
    b & 0 & \delta q_{-1} e & \delta q_2 c & \delta q_3 c \\
    b & \delta q_1 e & 0 & \delta q_2 c & \delta q_3 c \\
    b & \delta q_1 e & \delta q_{-1} c & 0 & \delta q_3 c \\
    b & \delta q_1 e & \delta q_{-1} c & \delta q_2 c & 0,
\end{pmatrix},
\]

where

\[
a \equiv 1 - (1 - \xi) G(\xi), \quad b \equiv \frac{1 - \xi}{\xi} (1 - G(\xi)), \quad e \equiv \frac{4}{3}(2g(\xi) - 1),
\]
\[
c \equiv \frac{4}{3} \left( -1 + \frac{2}{\xi^2} + 2G(\xi) \left( 1 - \frac{1}{\xi^2} \right) - g(\xi) \right),
\]

and

\[
P = \begin{pmatrix}
    G(\xi) & (G(\xi) - 1)/\xi & (G(\xi) - 1)/\xi & (G(\xi) - 1)/\xi & (G(\xi) - 1)/\xi \\
    (G(\xi) - 1)/\xi & G(\xi) & G(\xi) - 2g(\xi) & \tau(\xi) & \tau(\xi) \\
    (G(\xi) - 1)/\xi & G(\xi) - 2g(\xi) & G(\xi) & \tau(\xi) & \tau(\xi) \\
    (G(\xi) - 1)/\xi & \tau(\xi) & \tau(\xi) & G(\xi) & G(\xi) - 2g(\xi) \\
    (G(\xi) - 1)/\xi & \tau(\xi) & \tau(\xi) & G(\xi) - 2g(\xi) & G(\xi)
\end{pmatrix},
\]

with

\[
G(\xi) \equiv P(0 \mid 0; \xi), \quad g(\xi) = \frac{1}{2} (P(e_1 \mid -e_1; \xi) - P(0 \mid 0; \xi)),
\]
\[
\tau(\xi) \equiv \left( \frac{2}{\xi^2} - 1 \right) G(\xi) - \frac{2}{\xi^2} + g(\xi).
\]

Note that Eqs. (39) and (41) now define the $P^+$ matrix explicitly, and hence, define the generating function of the probability distribution $P^+(s \mid s_0)$.

### 4.2 Asymptotic behavior of the generating functions of the return probabilities in the vicinity of $\xi = 1$.

As we have already remarked, here we constrain our consideration to the analysis of the leading in $n$ behavior; this amounts to consideration of the leading in the limit $\xi \to 1^-$ behavior of the
corresponding generating functions. Expanding $G(\xi)$ and $g(\xi)$ in the vicinity of the singular point $\xi = 1$, (cf Refs.\cite{3} and \cite{13, 24, 25}), we have
\[
G(\xi) = \frac{1}{\pi} \ln \frac{8}{1-\xi} - \frac{1}{2\pi} (1-\xi) \ln(1-\xi) + \mathcal{O}(1-\xi), \quad \xi \to 1^-,
\]
(43)
and
\[
g(\xi) = \left(2 - \frac{4}{\pi}\right) + \frac{2}{\pi} (1-\xi) \ln(1-\xi) + \mathcal{O}((1-\xi)), \quad \xi \to 1^-.
\]
(44)
Consequently, we find by solving the matricial equation (33), that the generating functions of the return probabilities obey
\[
A_{\nu,\mu}(\xi) = \frac{A^{(1)}_{\nu,\mu}(u)}{S(u)} - \frac{A^{(2)}_{\nu,\mu}(u)}{S^2(u)} \left(\ln(1-\xi)\right)^{-1} + \mathcal{O}(1-\xi),
\]
(45)
where $u \equiv \exp(\beta E/2)$, $A^{(1)}_{\nu,\mu}(u)$ and $A^{(2)}_{\nu,\mu}(u)$ are some rational fractions (all listed explicitly in the Appendix), while
\[
S(u) \equiv \left\{ (\pi - 2) u^6 + (2 \pi^2 - 6 \pi + 12) u^5 + (8 \pi^2 - 25 \pi + 34) u^4 -
\right.
\]
\[
- (4 \pi^2 - 60 \pi + 88) u^3 + (8 \pi^2 - 25 \pi + 34) u^2 + (2 \pi^2 - 6 \pi + 12) u + \pi - 2 \right\}.
\]
(46)
It follows from Eqs.\cite{13} and explicit expressions for $A_{\nu,\mu}(\xi)$ presented in the Appendix, that, in particular, the generating functions of the return probabilities fulfil:
\[
A_{1,-1}(1^-) + A_{-1,-1}(1^-) + 2A_{2,-1}(1^-) = 1
\]
\[
A_{1,1}(1^-) + A_{-1,1}(1^-) + 2A_{2,1}(1^-) = 1
\]
\[
A_{1,2}(1^-) + A_{-1,2}(1^-) + A_{-2,2}(1^-) + A_{2,2}(1^-) = 1,
\]
(47)
which relations imply that the vacancy, starting its random walk from a given, neighbouring to the origin site, is certain to return eventually to the origin.

5 The TP mean displacement and the probability distribution.

In this section, we proceed as follows: Taking advantage of the asymptotical expansion obtained in the previous section, we first determine the small $(1 - \xi)$ behavior of the generating function $\tilde{P}^{(tr)}(\mathbf{k}; \xi)$, accompanied by the small-$\mathbf{k}$ expansion. Next, we evaluate the generating function of the TP mean displacement, by differentiating the obtained asymptotical expression for $\tilde{P}^{(tr)}(\mathbf{k}; \xi)$ with respect to the components of the wave-vector, and analyse its large-$n$ behavior. Lastly, we invert the asymptotical expansion of the generating function $\tilde{P}^{(tr)}(\mathbf{k}; \xi)$ and obtain the corresponding probability distribution $P_n^{(tr)}(\mathbf{X})$ in a certain scaling limit.
5.1 Asymptotic expansion of the generating function \( \bar{P}^{(tr)}(k; \xi) \).

Using the explicit representation of the determinant \( D(k; \xi) \) in Eq.(54) in terms of the generating functions of the return probabilities \( A_{\nu,\mu}(k) \), presented in the Appendix, as well as the asymptotical expansions in Eq.(43), we find that in the vicinity of \( \xi = 1 \) and for small values of the wave-vector \( k \), \( D(k; \xi) \) is given by

\[
D(k; \xi) = iF_1(u)k_1 + F_2(u)k_1^2 + F_3(u)k_1^3 - F_4(u) \ln^{-1}(1 - \xi) + \ldots ,
\]

where we have used the shortenings

\[
F_1(u) \equiv \frac{\pi - 2(u - 1)(1 + u)^5(u^2 + 2(2\pi - 3)u + 1)}{(u^2 + 2(\pi - 1)u + 1)S(u)},
\]

\[
F_2(u) \equiv \frac{\pi - 2}{2}(1 + u)^4(u^2 + 1)(u^2 + 2(2\pi - 3)u + 1)}{2(u^2 + 2(\pi - 1)u + 1)S(u)},
\]

\[
F_3(u) \equiv \frac{u(\pi - 2)(1 + u)^4((2\pi - 3)u^2 + 2u + 2\pi - 3)}{(u^2 + 2(\pi - 1)u + 1)S(u)},
\]

and

\[
F_4(u) \equiv \frac{\pi(\pi - 2)(1 + u)^4((2\pi - 3)u^2 + 2u + 2\pi - 3)(u^2 + 2(2\pi - 3)u + 1)}{(u^2 + 2(\pi - 1)u + 1)S(u)},
\]

and assumed, for simplicity, that the starting point \( Y_0 \) of the vacancy random walk is \( Y_0 = e_{-1} \).

On the other hand, we find that

\[
\sum_{\nu} U_{\nu}(k, \xi)F^*(0 | e_{\nu} | -e_1; \xi) = -iF_1(u)k_1 - F_2(u)k_1^2 - F_3(u)k_1^3 + \ldots .
\]

Consequently, in the small-\( k \) limit and \( \xi \to 1^- \), the generating function \( \bar{P}^{(tr)}(k; \xi) \) obeys

\[
\bar{P}^{(tr)}(k; \xi) = \frac{1}{1 - \xi}\left(1 - \left(-i\alpha_0 k_1 + \frac{1}{2}\alpha_1 k_1^2 + \frac{1}{2}\alpha_2 k_1^3 \right) \ln (1 - \xi) \right)^{-1},
\]

where the coefficients

\[
\begin{align*}
\alpha_0(E) & \equiv \pi^{-1} \sinh(\beta E/2)((2\pi - 3) \cosh(\beta E/2) + 1)^{-1}, \\
\alpha_1(E) & \equiv \pi^{-1} \cosh(\beta E/2)((2\pi - 3) \cosh(\beta E/2) + 1)^{-1}, \\
\alpha_2(E) & \equiv \pi^{-1} (\cosh(\beta E/2) + 2\pi - 3)^{-1},
\end{align*}
\]

are all functions of the field strength \( E \) and of the temperature only.

5.2 The TP mean displacement for arbitrary field strength \( E \).

As a matter of fact, the leading large-\( n \) asymptotical behavior of the TP mean displacement can be obtained directly from Eq.(56), since the generating function of the TP mean displacement, i.e.

\[
\mathbf{\mathbf{\Xi}}(\xi) \equiv \sum_{n=0}^{+\infty} \mathbf{\mathbf{\Xi}_n} \xi^n,
\]

Theorem 5.5.

\[
\mathbf{\mathbf{\Xi}}(\xi) \equiv \sum_{n=0}^{+\infty} \mathbf{\mathbf{\Xi}_n} \xi^n,
\]

are all functions of the field strength \( E \) and of the temperature only.
obeys (see, e.g. Ref.[4]):

$$\mathbf{X}(\xi) = -i \left( \frac{\partial \tilde{P}_{1}^{(tr)}(0; \xi)}{\partial k_1} \mathbf{e}_1 + \frac{\partial \tilde{P}_{2}^{(tr)}(0; \xi)}{\partial k_2} \mathbf{e}_2 \right).$$

(57)

Consequently, differentiating the expression on the right-hand-side of Eq.(54) with respect to the components of the wave-vector $\mathbf{k}$, we find that the asymptotical behavior of the generating function of the TP mean displacement in the vicinity of $\xi = 1^-$ follows

$$\mathbf{X}(\xi) \sim \left( \frac{a_0(E)}{1 - \xi} \ln \frac{1}{1 - \xi} \right) \mathbf{e}_1,$$

(58)

Further on, using the discrete Tauberian theorem (cf. Ref.[5]) and Eq.(55), we find the following general force-velocity relation for the system under study

$$\mathbf{X}_n \sim \left( a_0(E) \ln n \right) \mathbf{e}_1 = \left( \frac{1}{\pi} \frac{\sinh(\beta E/2)}{2(\pi - 3) \cosh(\beta E/2) + 1} \ln n \right) \mathbf{e}_1, \quad \text{as } n \to \infty,$$

(59)

which shows that the TP mean displacement grows logarithmically with $n$. In consequence, one may claim that the typical displacement along the $x_1$ direction scales as $\ln(n)$ as $n \to \infty$. On the other hand, typical displacement in the $x_2$-direction is expected to grow only in proportion to $\sqrt{\ln(n)}$, as in the unbiased case [13]. These claims will be confirmed in what follows by the form of the scaling variables involved in the limiting distribution.

Consider next behavior of the coefficient $a_0(E)$ in the limit $E \to 0$. Here, we find from Eq.(53) that

$$a_0(E) = \frac{\beta E}{4\pi(\pi - 1)} + O(E^3),$$

(60)

and hence, the mobility $\mu_n$, defined in Eq.(4), follows

$$\mu_n \sim \frac{\beta}{4\pi(\pi - 1)} \ln \left( \frac{n}{n} \right), \quad \text{as } n \to \infty.$$

(61)

Comparing next the result in Eq.(61) with that for the diffusivity $D_n$, Eq.(2), derived by Brummelhuis and Hilhorst [13] in the unbiased case, we infer that the TP mobility and diffusivity do obey, at least in the leading in $n$ order, the generalized Einstein relation of the form $\mu_n = \beta D_n$. Note, that this can not be, of course, an a priori expected result, in view of an intricate nature of the random walks involved and anomalous, logarithmic confinement of the random walk trajectories.

5.3 Probability distribution $P_n^{(tr)}(X)$.

We turn next to calculation of the asymptotic forms of the probability distribution $P_n^{(tr)}(X)$. Inverting $\tilde{P}_{1}^{(tr)}(k; \xi)$ with respect to $k$, we notice first that in the limit $\xi \to 1^-$ the integrand is sharply peaked around $k = 0$, such that the bulk contribution to the integral comes from the values $k_1 = 0$ and $k_2 = 0$; this implies that we can extend the limits of integration from $\pm \pi$ to
±∞, which yields in the limit ξ → 1−:

\[ P^{(tr)}(X; \xi) \sim \frac{1}{(1 - \xi)(2\pi)^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} dk_1 dk_2 \exp\left(-ik_1 x_1 - ik_2 x_2\right) \times \left\{ 1 - \left(-i\alpha_0(E)k_1 + \frac{1}{2}\alpha_1(E)k_1^2 + \frac{1}{2}\alpha_2(E)k_2^2\right) \ln(1 - \xi) \right\}^{-1} \] (62)

Further on, using the integral equality

\[ \int_{0}^{+\infty} dv \exp\left(-v\left\{ 1 - \left(-i\alpha_0(E)k_1 + \frac{1}{2}\alpha_1(E)k_1^2 + \frac{1}{2}\alpha_2(E)k_2^2\right) \ln(1 - \xi) \right\} \right) \] (63)

we cast the integral in Eq.(62) into the form:

\[ P^{(tr)}(X; \xi) \sim \frac{1}{(1 - \xi)(2\pi)^2} \int_{0}^{+\infty} dv \exp(-v) \int_{-\infty}^{+\infty} dk_1 \int_{-\infty}^{+\infty} dk_2 \exp(-ik_2 x_2) \times \exp\left(\frac{v}{2} \left(\alpha_1(E)k_1^2 + \alpha_2(E)k_2^2\right) \ln(1 - \xi) - ik_1 (x_1 + v\alpha_0(E)\ln(1 - \xi))\right) \] (64)

Note now that in order to evaluate explicitly the Gaussian integral in Eq.(64), we have to consider separately two cases: when (a) the external field in infinitely strong, \( E = \infty \) (which implies \( \alpha_2 = 0 \)), such that the TP performs a totally directed walk, and (b) - when \( E \) is bounded, \( E < \infty \) (and hence, \( \alpha_2 > 0 \)).

5.3.1 Directed walk, \( E = \infty \).

We start with the simplest case when the TP performs a totally directed walk under the influence of an infinitely strong field. In this case, the probability distribution is defined for non negative \( x_1 \) values only, and the equation (64) reduces to:

\[ P^{(tr)}(X; \xi) \sim \frac{\delta(x_2) \theta(x_1)}{2\pi(1 - \xi)} \int_{0}^{+\infty} dv \exp(-v) \int_{-\infty}^{+\infty} dk_1 \times \exp\left(\frac{v}{2} \alpha_1(E)k_1^2 \ln(1 - \xi) - ik_1 (x_1 + v\alpha_0(E)\ln(1 - \xi))\right) \] (65)

where \( \theta(x_1) \) denotes the Heaviside theta-function. Performing the integrals, we find that, in the limit \( \xi \rightarrow 1^- \), the generating function \( P^{(tr)}(X; \xi) \) obeys:

\[ P^{(tr)}(X; \xi) \sim -\delta(x_2)\theta(x_1) \frac{\pi(2\pi - 3)}{\ln(1 - \xi)} \exp\left(\frac{\pi(2\pi - 3)}{\ln(1 - \xi)} x_1\right). \] (66)

Applying next the discrete Tauberian theorem, we find eventually,

\[ P^{(tr)}_n(X) \sim \delta(x_2)\theta(x_1) \frac{\pi(2\pi - 3)}{\ln(n)} \exp\left(-\frac{\pi(2\pi - 3)}{\ln(n)} x_1\right), \] (67)

which means that in the totally directed case, in the large-\( n \) and large-\( x_1 \) limit, the scaled variable \( \eta_\infty \equiv \pi(2\pi - 3)x_1/\ln(n) \) is asymptotically distributed according to

\[ P(\eta_\infty) = \theta(\eta_\infty) \exp(-\eta_\infty), \] (68)

i.e. has an exponential scaling function.
5.3.2 Arbitrary bounded field $E < \infty$.

In this case the coefficient $\alpha_2 > 0$ and the probability distribution is defined also for negative values of $x_1$; as well, $P^{(tr)}(X; \xi)$ is defined also for non-zero values of $x_2$. In this general case, we find, performing integrations over the components of the wave-vector, that $P^{(tr)}(X; \xi)$ attains, as $\xi \to 1^-$, the following form:

$$
P^{(tr)}(X; \xi) \sim - \left(2\pi(1 - \xi) \ln (1 - \xi) \sqrt{\alpha_1(E)\alpha_2(E)}\right)^{-1} \int_0^{+\infty} dv \exp(-v) \times$$

$$
\exp \left(\frac{1}{2v \ln (1 - \xi)} \left(\frac{x_1}{\alpha_1(E)} + v \frac{\alpha_0(E)}{\alpha_1(E)} \ln (1 - \xi) + \left(\frac{x_2}{\alpha_2(E)}\right)^2\right)\right)$$

(69)

The integral in the latter equation can be calculated exactly, which yields

$$
P^{(tr)}(X; \xi) \sim - \left(\pi(1 - \xi) \ln (1 - \xi) \sqrt{\alpha_1(E)\alpha_2(E)}\right)^{-1} \exp \left(\frac{\alpha_0(E)}{\alpha_1(E)} x_1\right) K_0 \left(\frac{1}{1 - \xi}\right),$$

(70)

where $K_0$ is the modified Bessel (McDonald) function of zeroth order, and

$$
\eta_E(\lambda) \equiv \sqrt{\frac{2}{\ln (\lambda)}} + \frac{\alpha_0^2(E)}{\alpha_1(E)} \sqrt{\frac{x_1^2}{\alpha_1(E)} + \frac{x_2^2}{\alpha_2(E)}}.
$$

(71)

Finally, using the discrete Tauberian theorem [8, 21], we find from Eq. (71) that in the large-$n$ and large-$X$ limits, the probability distribution $P_n^{(tr)}(X)$ obeys

$$
P_n^{(tr)}(X) \sim \left(\pi \sqrt{\alpha_1(E)\alpha_2(E)} \ln (n)\right)^{-1} \exp \left(\frac{\alpha_0(E)}{\alpha_1(E)} x_1\right) K_0(\eta_E(n)).$$

(72)

Note that in the unbiased case, i.e. when $E = 0$, the probability distribution $P_n^{(tr)}(X)$ defined by Eqs. (71) and (72) reduces to the form predicted earlier by Brummelhuis and Hilhorst [13].

5.3.3 Limiting probability distribution function.

Now, we recollect that the scaling behavior expected is $x_1 \sim \ln (n)$ (for $E > 0$) and $x_2 \sim \sqrt{\ln (n)}$.

In order to obtain from Eqs. (71) and (72) the limiting probability distribution, we introduce two scaling variables:

$$
\begin{cases}
\eta_1 = x_1/\alpha_0(E) \ln (n), \\
\eta_2 = x_2/\sqrt{2\alpha_2(E)\ln (n)}.
\end{cases}
$$

(73)

Note that $\eta_1$ becomes $\eta_\infty$ in the special case $E = \infty$. In terms of these scaling variables $\eta_E(n)$ in Eq. (71) takes the form:

$$
\eta_E(n) = \frac{\alpha_0^2(E)}{\alpha_1(E)} \ln (n) |\eta_1| \left(1 + \frac{\alpha_1(E)}{\alpha_0(E)\ln (n)} \left(1 + \frac{\eta_2^2}{\eta_1^2}\right) + O(1/\ln^2 (n))\right)
$$

(74)

Note now that for arbitrary fixed $\eta_1$ and $\eta_2$, the argument of the Bessel function $\eta_E(n)$ written in terms of the scaling variables tends to infinity as $n \to \infty$. Consequently, using the limiting behavior of the modified Bessel function

$$
K_0(y) = \left(\frac{\pi}{2y}\right)^{1/2} \exp(-y)(1 + O(1/y)),
$$

(75)
we find that the probability distribution \( P_n^{(tr)}(X) \), written in terms of the scaling variables, converges as \( n \to \infty \) to the limiting form

\[
P_n^{(tr)}(X) \sim_{n \to \infty} \begin{cases} 
(2\pi \alpha_2(E)\alpha_3^2(E)\eta \ln^3(n))^{-1/2} \exp(-\eta_1 - \eta_2^2/\eta_1), & \text{for } \eta_1 \geq 0, \\
0, & \text{for } \eta_1 < 0,
\end{cases}
\tag{76}
\]

or, equivalently, that the scaling variables \( \eta_1 \) and \( \eta_2 \) have the following, rather unusual limiting joint distribution function:

\[
P(\eta_1, \eta_2) = \frac{\theta(\eta_1)}{\sqrt{\pi \eta_1}} \exp\left(-\eta_1 - \frac{\eta_2^2}{\eta_1}\right)
\tag{77}
\]

We note that this distribution is properly normalized and yields, of course, the same result for the TP mean displacement as the approach based on differentiation of the asymptotical expansion of the generating function. We also remark that the reduced distributions \( P(\eta_1) = \int d\eta_2 P(\eta_1, \eta_2) \) and \( P(\eta_2) = \int d\eta_1 P(\eta_1, \eta_2) \) take the form:

\[
P(\eta_1) = \frac{\theta(\eta_1)}{\sqrt{\pi \eta_1}} \exp(-\eta_1),
\]

\[
P(\eta_2) = \exp(-2|\eta_2|),
\tag{78}
\]

and hence, the reduced distribution \( P(\eta_1) \) appears to be exactly the same as in the case \( E = \infty \).

6 Finite vacancy concentration.

In this last section, we generalize our analysis of the biased TP mean displacement to the case when vacancies are present at a very small, but finite concentration \( \rho \). In our approach, we follow closely that of Brummelhuis and Hilhorst [17], who pointed out that for the unbiased case in the limit of low vacancy concentration, the many-vacancy problem can be interpreted in terms of the one-vacancy solution, which entails meaningful results to the leading order in the concentration of vacancies for \( \rho \ll 1 \). We thus just extend here their consideration over the biased case.

Following Ref. [17], we begin by considering a finite lattice of size \( L \times L \), containing \( M \) vacancies. The mean concentration of the vacancies is thus \( \rho = M/L^2 \ll 1 \). We suppose that the charged TP is initially at the origin and initial positions of the vacancies are \( Y^{(1)}_0, Y^{(2)}_0, \ldots, Y^{(M)}_0 \), which all are different from each other and from \( 0 \). All other sites are filled with neutral hard-core particles. The field \( E \) is again supposed to be oriented in the positive \( x_1 \)-direction.

Similar to the single vacancy case, we stipulate that at each time step, all vacancies exchange their positions with either of neighboring particles, such that each vacancy makes a step each time step. Exchanges with the charged TP are governed by the same rules as described in Section 2. Note that, of course, when many vacancies are present, it may appear that two or more vacancies occupy adjacent sites or have common neighboring particles, in which case their random walks will interfere. However, as noticed in [17], these cases contribute only to \( O(\rho^2) \) and thus go beyond our approximation; hence, we discard such possibilities here.
Now, let $\mathcal{P}_{n}(X|Y_{0}^{(1)}, Y_{0}^{(2)}, \ldots, Y_{0}^{(M)})$ denote the probability of finding at time moment $n$ the TP at position $X$ as a result of its interaction with all $M$ vacancies collectively. Further on, let $P_{n}(X^{(j)}|Y_{0}^{(j)})$ denote the probability of finding the TP at site $X^{(j)}$ at time moment $n$ due to interactions with a vacancy initially at $Y_{0}^{(j)}$ in a system with a single vacancy. Then, assuming that the vacancies contribute independently to the TP displacement, one has, following Ref.\[17] :

$$\mathcal{P}_{n}(X|Y_{0}^{(1)}, Y_{0}^{(2)}, \ldots, Y_{0}^{(M)}) \approx \sum_{Y_{0}^{(1)}} \cdots \sum_{Y_{0}^{(M)}} \delta_{X,Y_{0}^{(1)}+\ldots+Y_{0}^{(M)}} P_{n}(X^{(j)}|Y_{0}^{(j)})$$  

Upon averaging over all initial vacancy configurations, and denoting this average by the angular brackets, we have

$$\langle \mathcal{P}_{n}(X|Y_{0}^{(1)}, Y_{0}^{(2)}, \ldots, Y_{0}^{(M)}) \rangle \approx \sum_{Y_{0}^{(1)}} \cdots \sum_{Y_{0}^{(M)}} \delta_{X,Y_{0}^{(1)}+\ldots+Y_{0}^{(M)}} \langle \prod_{j=1}^{M} P_{n}(X^{(j)}|Y_{0}^{(j)}) \rangle,$$

which, in the limit of very small vacancy concentration, simplifies to \[17] :

$$\langle \mathcal{P}_{n}(X|Y_{0}^{(1)}, Y_{0}^{(2)}, \ldots, Y_{0}^{(M)}) \rangle \approx \sum_{Y_{0}^{(1)}} \cdots \sum_{Y_{0}^{(M)}} \delta_{X,Y_{0}^{(1)}+\ldots+Y_{0}^{(M)}} \prod_{j=1}^{M} \langle P_{n}(X^{(j)}|Y_{0}^{(j)}) \rangle$$  

Now, defining the Fourier transformed distributions

$$\mathcal{P}_{n}(k,M,L) = \sum_{X} \exp(i(k \cdot X)) \langle \mathcal{P}_{n}(X|Y_{0}^{(1)}, Y_{0}^{(2)}, \ldots, Y_{0}^{(M)}) \rangle,$$

and performing the corresponding summations, we find that

$$\mathcal{P}_{n}(k,M,L) \approx \left( \mathcal{P}_{n}(k) \right)^{M}$$

Turning next to the limit $L,M \to \infty$ (while the ratio $M/L^2 = \rho$ is kept fixed), we obtain

$$\mathcal{P}_{n}(k,\rho) = \lim_{L,M \to \infty} \mathcal{P}_{n}(k,M,L) \approx \exp \left( -\rho \Omega_{n}(k) \right),$$

where

$$\Omega_{n}(k) \equiv \sum_{j=0}^{n} \sum_{\nu} \Delta_{n-j}(k|e_{\nu}) \sum_{Y \neq 0} F_{j}^{*}(0|e_{\nu}|Y),$$

$$\Delta_{n}(k|e_{\nu}) = 1 - \mathcal{P}_{n}(k|e_{\nu}) \exp(i(k \cdot e_{\nu})).$$

$F_{j}^{*}(0|e_{\nu}|Y)$ are conditional return probabilities, defined in Section 3, and $\mathcal{P}_{n}(k|e_{\nu})$ is the Fourier transformed single-vacancy probability distribution $P_{n}(X|e_{\nu})$. The latter can be readily obtained by applying the discrete Fourier transformation to the relation

$$P_{n}(X|Y) = \delta_{X,0} \left( 1 - \sum_{j=0}^{n} F_{j}^{*}(0|Y) \right) +$$

$$+ \sum_{j=0}^{n} \sum_{\nu} P_{n-j}^{(tr)}(X - e_{\nu} | -e_{\nu}) F_{j}^{*}(0|e_{\nu}|Y).$$

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and choosing $Y = -e_\nu$.

Further on, using the results of the previous section, we find that in the limit $\xi \to 1^-$, $k \to 0$, the generating function of $\Omega_n(k)$ is given by

$$\Omega(k; \xi) = \sum_{\nu} \Delta(k|e_\nu; \xi) \sum_{Y \neq 0} F^*(0|e_\nu|Y; \xi),$$

with

$$\Delta(k|e_\nu; \xi) \equiv \frac{1}{1 - \xi} \left\{ 1 - \exp(i(k \cdot e_\nu)) \right\} \left\{ 1 - \ln(1 - \xi) \right\} \left( -i\alpha_0(E)k_1 + \frac{1}{2}\alpha_1(E)k_1^2 + \frac{1}{2}\alpha_2(E)k_2^2 \right) \cdots$$

We turn next to calculation of $\sum_{Y \neq 0} F^*(0|e_\nu|Y; \xi)$ in the limit $\xi \to 1^-$, $k \to 0$, which can be done rather straightforwardly by taking advantage of the results of Section 3. We have then

$$\sum_{Y \neq 0} F^*(0|e_\nu|Y; \xi) = \xi \left( \frac{p_\nu}{3/4 + p_\nu} \right) \sum_{Y \neq 0} P^+(e_\nu|Y; \xi) = \xi \left( \frac{p_\nu}{3/4 + p_\nu} \right) B_\nu (1 - A)^{-1} \sum_{Y \neq 0} B(Y; \xi),$$

where $B_\nu$ is the $\nu$-th basis vector, $B_\nu^*$ denotes the transposition of $B_\nu$, and $B(Y; \xi)$ is the vector, whose elements are $P(s_i|Y; \xi)_i$, $i = 0, 1, -1, 2, -2$. Explicitly,

$$B_0 \equiv \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad B_1 \equiv \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad B_{-1} \equiv \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad B_2 \equiv \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad B_{-2} \equiv \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Further on, using an evident symmetry relation

$$P(s_i|Y; \xi) = P(Y_i|s_i; \xi),$$

as well as the relation in Eq. (93), we obtain

$$\sum_{Y \neq 0} B(Y; \xi) = \left( \frac{1}{1 - \xi} - G(\xi) \right) B_0 + \left( \frac{1}{1 - \xi} - \frac{1}{\xi}(G(\xi) - 1) \right) (B_1 + B_{-1} + B_2 + B_{-2}).$$

Then, combining Eqs. (91), (94) and (93), (93), (i), (i3), and performing some straightforward but cumbersome calculations, we find that in the limit $\xi \to 1^-$ and $k \to 0$, the sum $\sum_{Y \neq 0} F^*(0|e_\nu|Y; \xi)$ is given by

$$\sum_{Y \neq 0} F^*(0|e_\nu|Y; \xi) = -\frac{\pi}{(1 - \xi) \ln(1 - \xi)} + \cdots,$$

which is, remarkably, independent of $u$ and $\nu$ in the leading in $\xi$ order. Consequently, in the limit $\xi \to 1^-$ and $k \to 0$, the generating function $\Omega(k; \xi)$ obeys:

$$\Omega(k; \xi) \approx \frac{\pi}{(1 - \xi)^2} \frac{-i\alpha_0(E)k_1 + \frac{1}{2}\alpha_1(E)k_1^2 + \frac{1}{2}\alpha_2(E)k_2^2}{1 - \ln(1 - \xi)} \cdots$$
Next, using the discrete Tauberian theorem, we obtain from the latter equation that in the limit 
\( n \to \infty \) and \( k \to 0 \),
\[
\Omega_n(k) \approx \frac{\pi}{1 + \ln(n)} \left( -i\alpha_0(E)k_1 + \frac{1}{2}\alpha_1(E)k_1^2 + \frac{1}{2}\alpha_2(E)k_2^2 \right) n. \tag{97}
\]
Finally, inverting Eq.\((85)\) with respect to the wave-vector,
\[
P^{(tr)}_n(X,\rho) \approx \frac{1}{4\pi^2} \int_{-\pi}^{\pi} dk_1 \int_{-\pi}^{\pi} dk_2 \exp \left( -i (k \cdot X) - \rho \Omega_n(k) \right), \tag{98}
\]
and taking advantage of Eq.\((87)\), we find that the leading, large-\( n \) behavior of the TP mean 
displacement is given by
\[
\bar{X}_n \sim \left( \pi \alpha_0(E) \rho n \right) e_1 = \frac{\sinh(\beta E/2)}{(2\pi - 3) \cosh(\beta E/2) + 1} \rho n e_1, \tag{99}
\]
i.e. grows linearly with time. This signifies that the TP mobility attains a constant value at sufficiently large times \( n \),
\[
\mu_n = \lim_{|E|\to 0} \frac{\bar{X}_n}{|E|n} = \frac{\beta \rho}{4(\pi - 1)} \tag{100}
\]
Lastly, on comparing Eq.\((100)\) and the result of Brummelhuis and Hilhorst \([17]\) for the TP diffusivity in absence of the field, Eq.\((3)\), we notice that again the Einstein relation is fulfilled!

### 7 Conclusion

In conclusion, we have studied the dynamics of a charged tracer particle diffusing on a two-dimensional lattice, all sites of which except one (a vacancy) are filled with identical neutral, hard-core particles. The system evolves in discrete time \( n, n = 0, 1, 2, \ldots \), by particles exchanging their positions with the vacancy, subject to the condition that each site can be at most singly occupied. The charged TP experiences a bias due to external field \( E \), which favors its jumps in the preferential direction. We determine exactly, for arbitrary strength of the field \( E = |E| \),
the leading large-\( n \) behavior of the TP mean displacement \( \bar{X}_n \), which is not zero here due to external bias, and the limiting probability distribution of the TP position. We have shown that the TP trajectories are anomalously confined and its mean displacement grows with time only logarithmically, \( \bar{X}_n = (\alpha_0(E) \ln(n)) e_1 \) as \( n \to \infty \). On comparing our results with the earlier analysis of the TP diffusivity \( D_n \) in the unbiased case by Brummelhuis and Hilhorst \([13]\), we have demonstrated that, remarkably, the Einstein relation \( \mu_n = \beta D_n \) between the diffusivity and the mobility \( \mu_n \) of the TP holds in the leading in \( n \) order, despite the fact that both \( D_n \) and \( \mu_n \) tend to zero as \( n \to \infty \). Note, however, that validity of the Einstein relation for the system under study relies heavily on the proper normalization of the vacancy transition probabilities (see, Eqs.\((8),(9)\) and \([10]\)). In absence of such a normalization, artificial "temporal trapping" effects may emerge, which will result ultimately in the violation of the Einstein relation for the system under study.
also Refs. [27] and [28] for physical situations in which such type of effects is observed). Further on, we have also generalized our approach to the situation with small but finite vacancy concentration $\rho$, in which case we have found a ballistic-type law of the form $X_n = (\pi \alpha_0(E) \rho \ n) \ e_1$. We have shown that here, again, both $D_n$ and $\mu_n$ calculated in the linear in $\rho$ approximation do obey the Einstein relation.

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8 Appendix.

In this Appendix, we list some explicit expressions skipped in the body of the manuscript. First of all, explicit form of the determinant $\mathcal{D}(k; \xi)$ in Eq. (17) in terms of the generating functions of the return probabilities $A_{\nu,\mu} = A_{\nu,\mu}(\xi)$ reads

$$\mathcal{D}(k; \xi) = 1 - A_{2,2}^2 + 2 A_{-1,1} A_{2,1} A_{-2,2} + 2 A_{1,-1} A_{2,1} A_{-1,2} A_{2,2} -
- 2 A_{2,-1} A_{-2,2} A_{1,2} A_{-1,1} - A_{-1,-1} A_{1,1} A_{2,2}^2 + A_{1,-1} A_{-1,1} A_{2,2}^2 +
+ A_{1,-1} A_{-1,1} - A_{-1,-1} A_{1,1} - A_{-1,-1} A_{1,1} A_{2,2}^2 + A_{1,-1} A_{-1,1} A_{2,2}^2 -
- 2 A_{-1,-1} A_{2,1} A_{1,2} A_{2,2} + 2 A_{2,-1} A_{2,2} A_{1,2} A_{-1,1} - 2 A_{-1,-1} A_{2,1} A_{-1,2} A_{2,2} -
- 2 A_{2,-1} A_{2,2} A_{1,1} A_{-1,2} + 2 A_{2,-1} A_{-2,2} A_{1,1} A_{1,2} + A_{2,2}^2 +
+ 2 \left( A_{2,-1} A_{1,2} A_{-1,1} + A_{1,-1} A_{2,1} A_{-1,2} - A_{-1,-1} A_{2,1} A_{1,2} -
- A_{1,-1} A_{-1,1} A_{-2,2} - A_{2,-1} A_{1,1} A_{-2,2} - A_{2,-1} A_{1,1} A_{-1,2} \right) \cos k_2 +
+ \left( A_{-1,1} A_{2,2}^2 - A_{-1,1} + 2 A_{2,1} A_{-1,2} A_{-2,2} - 2 A_{2,1} A_{-1,2} A_{2,2} -
- A_{-1,1} A_{2,2}^2 \right) e^{-ik_1} + \left( 2 A_{2,-1} A_{-2,2} A_{1,2} - 2 A_{2,-1} A_{2,2} A_{1,2} +
+ A_{1,-1} A_{2,2} - A_{1,-1} - A_{1,-1} A_{-2,2} \right) e^{ik_1} + 2 \left( A_{1,-1} A_{-2,2} -
- A_{2,-1} A_{1,2} \right) e^{ik_1} \cos k_2 + 2 \left( - A_{2,1} A_{-1,2} + A_{-1,1} A_{-2,2} \right) e^{-ik_1} \cos k_2$$

Next, the coefficients in Eq. (45) defining asymptotical behavior of the generating functions of the return probabilities are given explicitly by:

$$A_{1,-1}^{(1)}(u) = u^2 (u + 1)^2 ((\pi - 2)u^2 - 2(\pi^2 - 3\pi - 2)u + \pi - 2),$$
$$A_{1,-1}^{(2)}(u) = -\pi u^2 (u + 1)^2 (u^2 + 2u + 2\pi - 3) \times
\times \left( (2\pi - 3)u^2 + 2u + 1 \right)((\pi - 2)u^2 + 4u + \pi - 2)^2,$$
$$A_{-1,-1}^{(1)}(u) = (4\pi^2 - 15\pi + 14)u^4 - (6\pi^2 - 56\pi + 80)u^3 +
+ (8\pi^2 - 34\pi + 52)u^2 + (2\pi^2 - 8\pi + 16)u + \pi - 2,$$
$$A_{-1,-1}^{(2)}(u) = -\pi (u + 1)^2 ((2\pi - 3)u^2 + 2u + 1)^2 ((\pi - 2)u^2 + 4u + \pi - 2)^2,$$
$$A_{2,-1}^{(1)}(u) = u(\pi - 2)(u + 1)^2 ((2\pi - 3)u^2 + 2u + 1),$$
$$A_{2,-1}^{(2)}(u) = -\pi u(u + 1)^2 ((2\pi - 3)u^2 + 2u + 1)((\pi - 2)u^2 + 4u + \pi - 2) \times
\times ((\pi^2 - 4\pi + 6)u^4 + (2\pi^2 - 6\pi + 8)u^3 - (2\pi^2 -
- 20\pi + 28)u^2 + (2\pi^2 - 6\pi + 8)u + \pi^2 - 4\pi + 6),$$
\[ A_{1,1}^{(1)}(u) = u^2((\pi - 2)u^4 + (2\pi^2 - 8\pi + 16)u^3 + (8\pi^2 - 34\pi + 52)u^2 - \\
- (6\pi^2 - 56\pi + 80)u + 4\pi^2 - 15\pi + 14), \]
\[ A_{1,1}^{(2)}(u) = -u^2\pi (u + 1)^2(u^2 + 2u + 2\pi - 3)^2((\pi - 2)u^2 + 4u + \pi - 2)^2, \]
\[ A_{1,1}^{(1)}(u) = (u + 1)^2((\pi - 2)u^2 - 2(\pi^2 - 3\pi - 2)u + \pi - 2), \]
\[ A_{1,1}^{(2)}(u) = -\pi(u + 1)^2(u^2 + 2u + 2\pi - 3) \times \\
\quad ((2\pi - 3)u^2 + 2u + 1)((\pi - 2)u^2 + 4u + \pi - 2)^2, \]
\[ A_{2,1}^{(1)}(u) = (\pi - 2)u(u + 1)^2(u^2 + 2u + 2\pi - 3), \]
\[ A_{2,1}^{(2)}(u) = -\pi u(u + 1)^2(u^2 + 2u + 2\pi - 3)((\pi - 2)u^2 + 4u + \pi - 2) \times \\
\quad ((\pi^2 - 4\pi + 6)u^4 + (2\pi^2 - 6\pi + 8)u^3 - \\
- (2\pi^2 - 20\pi + 28)u^2 + (2\pi^2 - 6\pi + 8)u + \pi^2 - 4\pi + 6), \]
\[ A_{1,2}^{(1)}(u) = u^2(\pi - 2)(u + 1)^2(u^2 + 2u + 2\pi - 3), \]
\[ A_{1,2}^{(2)}(u) = -\pi u^2(u + 1)^2(u^2 + 2u + 2\pi - 3)((\pi - 2)u^2 + 4u + \pi - 2) \times \\
\quad ((\pi^2 - 4\pi + 6)u^4 + (2\pi^2 - 6\pi + 8)u^3 - \\
- (2\pi^2 - 20\pi + 28)u^2 + (2\pi^2 - 6\pi + 8)u + \pi^2 - 4\pi + 6), \]
\[ A_{1,1}^{(1)}(u) = (\pi - 2)(u + 1)^2((2\pi - 3)u^2 + 2u + 1), \]
\[ A_{1,1}^{(2)}(u) = -\pi(u + 1)^2((2\pi - 3)u^2 + 2u + 1)((\pi - 2)u^2 + 4u + \pi - 2) \times \\
\quad ((\pi^2 - 4\pi + 6)u^4 + (2\pi^2 - 6\pi + 8)u^3 - \\
+ (2\pi^2 - 6\pi + 8)u + \pi^2 - 4\pi + 6), \]
\[ A_{2,2}^{(1)}(u) = -u(u^2 + (2\pi - 2)u + 1)^{-1}(u + 1)^2 \times \\
\quad ((\pi - 6)u^4 - 8u^3 + (4\pi^3 - 16\pi^2 - 2\pi + 28)u^2 - 8u + \pi - 6), \]
\[ A_{2,2}^{(2)}(u) = -\pi u(u + 1)^2(\pi^2 - 4\pi + 6)u^4 + \\
\quad + (2\pi^2 - 6\pi + 8)u^3 - (2\pi^2 - 20\pi + 28)u^2 + (2\pi^2 - 6\pi + 8)u + \pi^2 - 4\pi + 6)^2, \]
\[ A_{2,2}^{(1)}(u) = u(u^2 + (2\pi - 2)u + 1)^{-1}((2\pi^2 - 9\pi + 14)u^6 + \\
\quad + (4\pi^3 - 20\pi^2 + 46\pi - 28)u^5 + (12\pi^3 - 66\pi^2 + 169\pi - 142)u^4 - \\
\quad - (16\pi^3 - 168\pi^2 + 412\pi - 312)u^3 + (12\pi^3 - 66\pi^2 + 169\pi - 142)u^2 + \\
\quad + (4\pi^3 - 20\pi^2 + 46\pi - 28)u + 2\pi^2 - 9\pi + 14), \]
\[ A_{2,2}^{(2)}(u) = -\pi u(u + 1)^2((\pi^2 - 4\pi + 6)u^4 + (2\pi^2 - 6\pi + 8)u^3 - \\
\quad - (2\pi^2 - 20\pi + 28)u^2 + (2\pi^2 - 6\pi + 8)u + \pi^2 - 4\pi + 6)^2.
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