Pfaffian Structure with an Integrality Condition

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ABSTRACT

Some Pfaffian manifolds admit the construction of an associated Weyl line-bundle in which the lift $\alpha$ of the Pfaffian structure defines a 2-form $\omega = d\alpha$ which is basic. We identify the conditions under which this construction is possible, implement it, and investigate some properties of the foliated structure of these special manifolds and of their canonical flows.

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1. Introduction

In a Pfaffian structure, the 2-form $\omega = d\alpha$ is exact. However, in general, it is not basic-exact. A 1-form $\beta$ is said to be basic if $i_\xi \beta = 0$ and $L_\xi \beta = 0$, where $\xi$ is the Liouville vector field (Molino 1988, Craiveranu & Puta 1987). We shall say that a two form $\phi$ is basic-exact when it can be written as $\phi = d\beta$, where $\beta$ is basic.

In certain special circumstances, a Pfaffian manifold permits the construction of an associated Weyl bundle (Woodhouse 1980) in which the lift $\overline{\alpha}$ of the Pfaffian structure defines a 2-form $\overline{\omega} = d\overline{\alpha}$ which is basic-exact. In this case, the construction of the Weyl bundle is analogous to that of the standard one, and the success of the construction relies on the existence of a particular type of Čech cocycle.

In this paper, we implement this construction for this special case of Pfaffian structure. In Section 2, we consider the concept of a Čech basic 2-cocycle for a Pfaffian Structure. In Section 3, we identify the conditions under which a Pfaffian manifold admits the construction of an associated Weyl bundle. The conditions include an integrality condition on the basic 2-cocycle. We investigate the canonical objects admitted by this Weyl bundle and list some of their properties.

2. Čech Basic 2-cocycle of a Pfaffian Structure

Let $\{U\}$ be a simple covering of $M$ by canonical charts, and denote the coordinates of a canonical chart by $\{x^0, x^1, \ldots, x^n, p_1, \ldots, p_n\}$. Then in each chart $U$ of this covering we have

$$\alpha = dx^0 + p_i dx^i$$

and

$$\omega = dp_i \wedge dx^i$$

where, in accordance with the Einstein summation convention, a Latin index repeated once in superscript position and once in subscript position indicates a summation on the index range $\{1, 2, \ldots, n\}$. Since $\omega$ is exact, its restriction to $U$ must possess a local potential $\beta_U$ such that on $U$$

$$\omega = d\beta_U$$
In fact, we can always choose $\beta_U$ to be a locally basic 1-form by putting
\[ \beta_U = p_i dx^i \]
Thus, on each $U$ of the covering, there exists at least one basic 1-form $\beta_U$ such that
\[ \omega = d\beta_U \]
First, consider all intersections $U \cap V$ of the charts of this covering. In each $U \cap V \neq \emptyset$ we have
\[ \omega = d\beta_U = d\beta_V \]
so that
\[ d(\beta_U - \beta_V) = 0 \]
Therefore there exists a function $\beta_{UV}$ on $U \cap V$ such that
\[ \beta_U - \beta_V = d\beta_{UV} \]
with each $\beta_{UV}$ being a basic function on $U \cap V$.

Now consider all threefold intersections $U \cap V \cap W$ of the charts of this covering. In each $U \cap V \cap W \neq \emptyset$ we have
\[ \beta_U - \beta_V = d\beta_{UV} \]
\[ \beta_V - \beta_W = d\beta_{VW} \]
\[ \beta_W - \beta_U = d\beta_{WU} \]
so that
\[ d(\beta_{UV} + \beta_{VW} + \beta_{WU}) = 0 \]
and hence on each $U \cap V \cap W \neq \emptyset$ there exists a constant function $c_{UVW}$ such that
\[ \beta_{UV} + \beta_{VW} + \beta_{WU} = c_{UVW} \]
The constant functions $c_{UVW}$ define a basic 2-cocycle in a basic Čech cohomology [Woodhouse 1980, Craiveau & Puta 1987].

3. The Integrality Condition

In the general case there is little more that can be said about the existence of the basic 2-cocycle described above. However, in the special case when the cocycle is cohomological to an integer basic cocycle, the manifold permits the construction of an associated Weyl bundle.

Assume therefore that the basic 2-cocycle defined by the functions $\{\beta_{UV}\}$ is cohomological to an integer basic cocycle. We call this assumption the integrality condition. We can suppose without loss of generality that the functions $\{\beta_{UV}\}$ have been chosen in such a way that the constants $\{c_{UVW}\}$ are integers. Put
\[ a_{UV} = e^{2\pi i \beta_{UV}} \]
Then on each non-trivial $U \cap V \cap W$, the functions $a_{UV}$ satisfy the relation
\[ a_{UV} a_{VW} a_{WU} = 1 \]
and thus define in a natural way a principal $U(1)$ bundle $P$ over $M$,
\[ \pi : P \to M \]
Since the form $\omega$ is exact, this bundle is trivial. However, it is not necessarily basic-trivial.

The local basic potentials $\beta_U$ on $M$ define on $P$ a global 1-form $\beta_U$ given by
\[ \beta_U = ds + \pi^* \beta_U \]
where $\pi^* \beta_U$ is the pullback by the projection $\pi$ of $\beta_U$, $s$ is the fibre parameter along the $S^1$ fibres of $P$, and $ds$ is its differential. We then have
\[ d\beta = \overline{\omega} \]
where
\[ \overline{\omega} = \pi^* \omega \]
Put
\[ \overline{\alpha} = \pi^* \alpha \]
\[ E = \frac{\partial}{\partial s} \]
Thus $E$ is the vector field which defines the free action of the group $U(1)$ on $P$. Clearly $\overline{\beta}$ is a connection 1-form on $P$, with curvature $\overline{\omega}$.

Denote by $X$ the horizontal lift of $\xi$. We thus have the following objects on $P$: $\overline{\beta}$, $\overline{\alpha}$, $\overline{\omega}$, $E$, $X$. In canonical coordinates $\{s, x^0, x^i, p_i\}$ on $\pi^{-1}(U)$ for each $U \subset M$, we have
\[ \overline{\beta} = ds + \beta_U \] (1)
where
\[ \beta_U = A^i(x^1, ..., x^n, p_1, ..., p_n) \, dp_i \]
\[ + B_i(x^1, ..., x^n, p_1, ..., p_n) \, dx^i \]
and

\[ d\overline{\beta} = \overline{\omega} \]

Also

\[ \overline{\alpha} = dx^0 + p_i \, dx^i \quad (2) \]
\[ \overline{\omega} = dp_i \wedge dx^i \quad (3) \]
\[ E = \frac{\partial}{\partial s} \quad (4) \]
\[ X = \frac{\partial}{\partial x^0} \quad (5) \]

The quantities \( \overline{\beta}, \overline{\omega}, \) and \( \overline{\alpha} \) generate a closed, nowhere vanishing 1-form

\[ \overline{\theta} = \overline{\alpha} - \overline{\beta} \quad (6) \]

and a pre-symplectic form

\[ \overline{\Omega} = \overline{\pi} \wedge \overline{\beta} + \overline{\omega} \quad (7) \]

with

\[ d\overline{\Omega} = \overline{\theta} \wedge \overline{\omega} \quad (8) \]

Denote by \( ^*_{\overline{\Omega}} \) the raising operator for \( \overline{\Omega} \). Then,

\( E = ^*_{\overline{\Omega}} \overline{\alpha} \)

and

\( X = ^*_{\overline{\Omega}} \overline{\beta} \)

These quantities therefore do not generate a new vector field on \( P \) independent of \( E \) and \( X \).

Using the representation of \( \overline{\beta}, \overline{\pi}, \overline{\omega}, E \) and \( X \) in canonical coordinates, we easily derive the following relations:

1. Properties of \( E \),

\[ \begin{align*}
E|_{\overline{\pi}} &= 0 & \mathcal{L}_E \overline{\pi} &= 0 \\
E|_{\overline{\beta}} &= 1 & \mathcal{L}_E \overline{\beta} &= 0 \\
E|_{\overline{\theta}} &= -1 & \mathcal{L}_E \overline{\theta} &= 0 \\
E|_{\overline{\Omega}} &= 0 & \mathcal{L}_E \overline{\Omega} &= 0 \\
[E, X] &= 0
\end{align*} \quad (9) \]

2. Properties of \( X \),

\[ \begin{align*}
X|_{\overline{\pi}} &= 1 & \mathcal{L}_X \overline{\pi} &= 0 \\
X|_{\overline{\beta}} &= 0 & \mathcal{L}_X \overline{\beta} &= 0 \\
X|_{\overline{\theta}} &= 0 & \mathcal{L}_X \overline{\theta} &= 0 \\
X|_{\overline{\Omega}} &= \beta & \mathcal{L}_X \overline{\Omega} &= 0 \\
\end{align*} \quad (10) \]

3. Properties of \( Y = X + E \),

\[ \begin{align*}
Y|_{\overline{\pi}} &= 1 & \mathcal{L}_Y \overline{\pi} &= 0 \\
Y|_{\overline{\beta}} &= 1 & \mathcal{L}_Y \overline{\beta} &= 0 \\
Y|_{\overline{\theta}} &= 0 & \mathcal{L}_Y \overline{\theta} &= 0 \\
Y|_{\overline{\Omega}} &= \theta & \mathcal{L}_Y \overline{\Omega} &= 0 \\
\end{align*} \quad (11) \]

We now assume furthermore that the Liouville field \( \xi \) is complete. Then \( X \) is also complete.

The \( U(1) \times \mathbb{R} \) action on \( P \): With this assumption, the fields \( E \) and \( X \) generate a \( U(1) \times \mathbb{R} \) action on \( P \). Since \( E \) and \( X \) are everywhere linearly independent, the orbits of this action are \( S^1 \times \mathbb{R} \), that is, cylinders, or \( T^2 = S^1 \times S^1 \). Each orbit projects onto a corresponding orbit of \( \xi \) in \( M \).

4. Consequences of the Integrality Condition

The 1-form \( \overline{\theta} \) is closed and nowhere zero. It therefore defines a foliation \{\( F \)\} on \( P \). Let \( F \) be a leaf of \{\( F \)\}. The \( F \) clearly is a Pfaffian manifold with \( \overline{\pi}_F = \overline{\pi}_F \), and \( \overline{\omega}_F = d(\overline{\pi}_F) \). The characteristic field on \( F \) is \( Y_F = Y|_F \). For each \( p \in F \), the tangent space \( T_pF \) to the leaf projects one-to-one and onto \( T_{\pi(p)}M \).

Given an orbit \( O \) of the fields \( E \) and \( X \), the restriction \( \overline{\theta}_O \) defines a 1-dimensional foliation on \( O \) generated by the restriction \( Y_O \) of \( Y \) to \( O \).

We now apply Tischler's structural theorem (Tischler 1970).

Case 1: \( \overline{\theta} = df \) and the leaves of the foliation are given by \( f = \text{constant} \). Furthermore, the leaves define a trivial fibration of \( P \) over \( \mathbb{R} \), and the flow of \( E \) maps leaves of the foliation \{\( F \)\} onto leaves. \( E \) therefore defines a flow on \( \mathbb{R} \). Since the flow for \( E \) is periodic, so also is the induced flow on \( \mathbb{R} \). But the only periodic flow on \( \mathbb{R} \) is the constant flow. So the flow of \( E \) must keep the leaves of \{\( F \)\} fixed. But this is impossible, because \( E|_{\overline{\theta}} = -1 \). We have therefore proved the following proposition.

**Proposition 1:** The foliation defined by \( \overline{\theta} \) cannot be simple, or \( \overline{\theta} \) is not an exact 1-form.
Case 2: The foliation \{F\} defines a locally trivial fibration \( p : P \to S^1 \) over \( S' \), all \( S' \to S^1 \). Since \( E|_\theta = -1 \) and \( X|_\theta = 1 \), the fields \( E \) and \( X \) induce flows on \( S' \) with no stationary points. So, the flows of \( E \) and \( X \) act transitively on the leaves of the foliation \{F\}, and all its leaves are diffeomorphic closed submanifolds in \( P \).

Consider first an orbit \( O \) generated by \( E \) and \( X \) (or, \( E \) and \( Y \)). From (1) and (2), the restrictions to \( O \) of \( \overline{\alpha} \) and \( \overline{\beta} \) are given respectively by \( \overline{\alpha}|_O = ds \) and \( \overline{\beta}|_O = dx^0 \). The 2-form
\[
ds \otimes dx^0 + dx^0 \otimes ds
\]
defines a flat Riemannian metric on \( O \), and the flows of \( E \), \( X \) and \( Y \) restricted to \( O \) are isometries of this metric, with \( E \) and \( X \) orthogonal. The orbits of \( E \) are circles, so the orthogonal orbits of \( X \) are circles in the case of a toral \( O \) and generating lines in the case of a cylinder. Since \( Y = E + X \), the orbits of \( Y \) are closed if \( O \) is a torus and helices if \( O \) is a cylinder. From the construction of the Weyl bundle, all orbits of \( E \) have period 1.

The intersection of the leaves of \{F\} with \( O \) are precisely the orbits of \( Y \) on \( O \). Let \( T \) be the basic period of the flow induced by \( X \) on the base space \( S' \), and let \( T_O \) be the period of the flow of \( X \) on \( O \). Then \( T_O \) is a multiple of \( T \). This proves the following proposition.

**Proposition 2:** The periods of periodic trajectories of \( X \) are multiples of \( T \).

Case 3: All the leaves of \{F\} are dense in \( P \). In this case, the consideration of Case 2 apply to \( O \), but the conclusion stated in Proposition 2 does not follow.

Consider now the leaves of \{F\} in cases 2 and 3 above. The flow of \( E \) has period 1, and \( T_O \) denotes the the basic period of the flow of \( X \) on the base space \( S' \). So, in Case 2, according to Proposition 2, there must be some positive integer \( \ell \) such that \( \ell T_O = 1 \). This means that each leaf \( F \) intersects \( P_x = \pi^{-1}(x) \), \( x \in M \), \( \ell \) times. We have thus proved the following proposition.

**Proposition 3:** In Case 2, each leaf \( F \) is an \( \ell \)-fold covering of \( M \).

In Case 3, each leaf of \( F \) is dense in \( P \). Consider \( F \cap P_x \) for a given \( U(1) \) orbit \( P_x \) over \( x \in M \). Suppose that, for some interval \((t_1,t_2)\) of \( P_x \), \( F \cap (t_1,t_2) = \emptyset \). Since \( F \) projects locally one-to-one onto \( M \), this implies that an open set of \( P \) is disjoint from \( F \). This is a contradiction. We have therefore proved the following proposition.

**Proposition 4:** In Case 3, the intersection \( P_x \cap F \) is a dense subset of \( P_x \).

These results has consequences for the flow of \( \xi \). Since the orbits of \( X \) intersect those of \( E \), we arrive at the following conclusion from Proposition 2.

**Proposition 5:** The periods of periodic trajectories of \( \xi \) are multiples of \( T \).

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