QUANTITATIVE CHARACTERISTICS OF CYCLES AND THEIR RELATIONS WITH STRETCH AND SPANNING TREE CONGESTION

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Abstract. The main goal of this article is to introduce new quantitative characteristics of cycles in finite simple connected graphs and to establish relations of these characteristics with the stretch and spanning tree congestion of graphs. The main new parameter is named the support number. We give a polynomial approximation algorithm for the support number with the aid of yet another characteristic we introduce, named the cycle width of the graph.

1. Introduction

This paper is devoted to connections between several characteristics of finite connected simple graphs. Two such characteristics are the well-known notion of stretch (introduced in [PU89], see surveys in [CC95, FK01, LW08, Pel00]) and that of spanning tree congestion (introduced in [Ost04], see also [LO10, Ost10, Ota11, Ota20]). These two notions are closely related for plane graphs (see Lemma 2.5 below).

It is known that the problem of finding the minimum stretch spanning tree in a graph $G$ is NP-hard [CC95, FK01]. An approximation algorithm with approximations ratio $O(\log n)$ was found for it in [EP09].

The main goal of this paper is to introduce and to study a new related characteristic, namely that of support number. We establish relations between this characteristic, stretch, and spanning tree congestion. The main result of the paper is an approximation algorithm with constant approximation ratio for computing the support number. This is done by introducing yet another characteristic the cycle width of a graph (denoted $W(G)$) which is computable in polynomial time and is equivalent to the support number $k$ in the sense

$$W(G)/3 \leq k \leq W(G) + 4.$$  

We conjecture that finding the support number of $G$ is an NP-hard problem.
2. Congestion and Stretch

Our Graph Theory terminology and notations are standard (see [BM08, CH91]). Let $G$ be a finite, simple, connected graph. We denote by $V(G)$ the vertex set of $G$, and by $E(G)$ the edge set of $G$. Recall that a spanning subgraph of $G$ is a subgraph that contains all the vertices $V(G)$ of the original graph. A spanning tree of $G$ is a connected, spanning subgraph, which does not have any cycles.

We use the following definitions introduced in [Ost04] (some of them were known before, see [Sim87]). Let $T$ be a spanning tree in $G$.

**Definition 2.1** ([Ost04]). For an edge $e$ of $T$ let $A_e$ and $B_e$ be the vertex sets of the two components of $T - e$. Denote by $e_G(A_e, B_e)$ the number of edges in $G$ that connect a vertex in $A_e$ to a vertex in $B_e$.

![Diagram](image)

(A) Spanning tree consists of blue edges  
(B) $A_e = $ blue vertices, $B_e = $ pink vertices.

**Figure 1.** Red edges show $e_G(A_e, B_e) = 6$.

The edge congestion of $G$ in $T$ is

$$ec(G : T) = \max_{e \in E(T)} e_G(A_e, B_e).$$

The spanning tree congestion of $G$ is

$$s(G) = \min_T ec(G : T).$$

In short, the spanning tree congestion of $G$ is the min-max characteristic

$$s(G) = \min_T \max_{e \in E(T)} e_G(A_e, B_e).$$
Definition 2.2 ([PU89]). If $H$ is a connected spanning subgraph in $G$, then the stretch of $G$ in $H$ is defined by

$$\text{Stretch}(G : H) = \max_{u,v \in V(G)} \frac{d_H(u,v)}{d_G(u,v)}$$

It suffices to test the stretch only for adjacent vertices $u,v$ in $G$. That is,

$$\text{Stretch}(G : H) = \max_{uv \in E(G)} d_H(u,v).$$

The stretch of $G$ is defined by minimizing with respect to the spanning trees $T$ in $G$

$$\sigma(G) = \min_T \text{Stretch}(G : T).$$

In short, the stretch of $G$ is the min-max characteristic

$$\sigma(G) = \min_T \max_{uv \in E(G)} d_T(u,v).$$

Definition 2.3. The dual graph $G^*$ of a plane graph $G$ is the multigraph whose vertices correspond to the faces of $G$, including the exterior face. Every edge $e^*$ joining two vertices of $G^*$ corresponds to an edge $e$ of $G$ in the common boundary of the corresponding faces of $G$, and vice versa.

If two faces have several common edges in their boundaries, the corresponding edges are multiple edges in $G^*$.

Definition 2.4. If $T$ is a spanning tree of a plane graph $G$, then the dual spanning tree $T^\sharp$ is defined as the spanning subgraph of $G^*$ such that $e^* \in E(T^\sharp)$ if and only if $e \notin E(T)$.

The connection between stretch and spanning tree congestion is revealed in the case of planar graphs by the following lemma (Lemma 8 in [LLO14]).

Lemma 2.5. Let $G$ be a connected planar graph.

(a) If $T$ is a spanning tree in $G$ and $T^\sharp$ is its dual spanning tree, then

$$\text{ec}(G : T) = \text{Stretch}(G^* : T^\sharp) + 1.$$ 

(b) $s(G) = \sigma(G^*) + 1$.

Note 2.6. It is proved in [FK01] that determination of the least $t$ for which a planar graph has a spanning tree $T$ with $\text{Stretch}(G : T) = t$ is NP-hard. Combining this with Lemma 2.5 we get that the problem of computing $s(G)$ for planar graphs is also NP-hard. This fact was also observed in [BFGOV12, Low10]. The paper [BFGOV12] contains many interesting results on the complexity of the spanning tree congestion.
3. Statements of the main results

Since there are no efficient algorithms for calculating stretch and congestion for general graphs, it is highly desirable to find polynomially computable parameters which can be used to estimate the two characteristics. This is the main motivation for our study.

Presence of a cycle in a graph implies that $\sigma(G) \geq 2$. Without additional restrictions on the cycle, it does not imply more; complete graph has stretch 2 and contains cycles of all possible lengths. To get stronger estimates for stretch from below, we need cycles which, in some coarse metric sense, resemble circles rather than trees. The strongest notion of this type is that of a “long” isometric cycle in a graph.

**Definition 3.1.** Let $C$ be a cycle in a graph $G$. We say that $C$ is an isometric cycle in $G$ if for any two vertices $u, v$ of $C$,

$$d_C(u, v) = d_G(u, v).$$

We will be interested also in the following weakening of the notion of isometric cycle:

**Definition 3.2.** Let $\alpha \in (0, 1)$. We say that $C$ is a $(\alpha, 1)$-bilipschitz cycle in $G$ if

$$\forall u, v \in C \quad \alpha d_C(u, v) \leq d_G(u, v) \leq d_C(u, v).$$

Isometric and bilipschitz classes of cycles are proper subclasses of the class of supported cycles introduced in the next definition. The class of $k$-supported cycles seems to be the most relevant in the study of the stretch of graphs.

**Definition 3.3.** If a cycle $C$ in a graph $G$ can be partitioned into three edge disjoint paths $I_1, I_2$, and $I_3$ with $I_1 \cap I_2$, $I_2 \cap I_3$, and $I_3 \cap I_1$ containing one vertex each, and in such a way that for every triple $(u_1, u_2, u_3)$ of vertices satisfying $u_i \in I_i$, $i = 1, 2, 3$, we have

$$\max_{i,j \in \{1,2,3\}} d_G(u_i, u_j) \geq k,$$

we say that $C$ is a $k$-supported cycle contained in $G$.

**Definition 3.4.** For a finite, simple, connected graph $G$, define its support number as the largest integer $k$ for which $G$ contains a $k$-supported cycle. If $G$ is a tree, the support number is defined to be zero.

**Remark 3.5.** If $G$ is not a tree, then any of its cycles is 1-supported. This can be seen by considering any two consecutive edges of $C$ as $I_1$ and $I_2$, while the remaining part of the cycle is the path $I_3$. Therefore the support number of any non-tree is some $k \geq 1$.

The following theorem states that the existence of a $k$-supported cycle in a graph $G$ leads to an estimate from below for the stretch of $G$.

**Theorem 3.6.** If a graph $G$ contains a $k$-supported cycle, then $\sigma(G) \geq k$.

If the graph contains large isometrically embedded cycles, the result above implies the following estimate from below for the stretch.
Figure 2. The cycle obtained by concatenating the green, red, and blue paths is 2-supported in $G$.

**Corollary 3.7.** If a graph $G$ contains an isometrically embedded cycle $C$ of length $n$, then $\sigma(G) \geq \lceil \frac{n}{3} \rceil$.

**Remark 3.8.** Similarly, if a graph $G$ contains an $(\alpha, 1)$-bilipschitz embedded cycle $C$, then $C$ is $k$-supported with $k = \alpha \lceil \frac{n}{3} \rceil$ and hence $\sigma(G) \geq \alpha \lceil \frac{n}{3} \rceil$.

We find also an approximate algorithm for computing the support number. Namely, for a graph $G$ we describe below (Definition 6.1) a polynomially computable numerical parameter $W(G)$ and prove two theorems.

**Theorem 3.9.** Each graph $G$ contains a $W(G)/3$-supported cycle.

**Theorem 3.10.** Each graph $G$ containing a $k$-supported cycle satisfies $W(G) \geq k - 4$.

4. $k$-supported cycles

Before proving Theorem 3.6, we estimate the support number for rectangular and triangular grids.

**Proposition 4.1.** For the rectangular grid $G = P_m \times P_n$ with $2 \leq m \leq n$, the exterior rectangle is an $(m - 1)$-supported cycle.

**Proof.** We color the left side (this is path $I_1$ with $(m - 1)$ vertical edges) in red, the top side (this is path $I_2$ with $(n - 1)$ horizontal edges) in green, and the remaining two sides (this is path $I_3$) in blue.
The blue vertex $u_3$ is either on the bottom horizontal path and therefore at distance at least $(m - 1)$ to the green vertex $u_2$, or on the right vertical path and therefore at distance at least $(n - 1)$ to the red vertex $u_1$. In either case,

$$\max_{i,j \in \{1,2,3\}} d_G(u_i, u_j) \geq (m - 1).$$

\[\square\]

**Proposition 4.2.** For the triangular grid $G = T_n$ with $n \geq 2$, the exterior triangle is an $\lceil \frac{n-1}{2} \rceil$-supported cycle.

**Proof.** We prove in the Euclidean case that an equilateral triangle of side $\ell$ has the property that for any three points $u_1, u_2, u_3$, one on each side, it holds that

$$\max_{i,j \in \{1,2,3\}} d(u_i, u_j) \geq \frac{\ell}{2}.$$

Here, $d(u_i, u_j)$ is the Euclidean distance in the plane. Since the graph distance on the triangular grid dominates the Euclidean distance, inequality [1] implies that for any three vertices $u_1, u_2, u_3$ of the grid, one on each side, it holds that

$$\max_{i,j \in \{1,2,3\}} d_G(u_i, u_j) \geq \frac{n - 1}{2}.$$

Therefore, let us consider the Euclidean case of an equilateral triangle of side $\ell$. Assume by contradiction that there exist $u_1, u_2, u_3$, one on each side, such that

$$\max_{i,j \in \{1,2,3\}} d(u_i, u_j) < \frac{\ell}{2}.$$
Clearly, none of the points can be in the red region because the distance from a red wedge to the opposite side is $\ell/2$.

Also, none of the points can be a midpoint on its side, for if for example $u_1$ is midpoint of its side, then $u_2$ has to be in the black region and $u_3$ in the blue, therefore $d(u_2, u_3) \geq \ell/2$.

We claim that $u_1, u_2, u_3$ are all either in the blue intervals or all in the black intervals on their respective sides (see Fig 4). This is because if, say $u_1$ is in the blue and $u_2$ is in the black, then if $u_3$ is in the black, its at distance to $u_1$ is at least $\ell/2$, otherwise if $u_3$ is in the blue then its at distance to $u_2$ is at least $\ell/2$, in either case contradicting the assumption (2).

Let us therefore assume that all three points are in the blue intervals. For simplicity denote $a = \ell/2$, and by $x_1, x_2, x_3$ the distances from $u_1, u_2, u_3$ to the midpoints of their respective sides. We claim that $d(u_1, u_2) < \frac{\ell}{2}$ implies $x_2 < x_1$. Indeed, the law of cosines, applied to the triangle with sides $a - x_1, a + x_2$, angle of size $\frac{\pi}{3}$ between them, and the remaining side of length $d(u_1, u_2)$, leads to

$$(a - x_1)^2 + (a + x_2)^2 - (a - x_1)(a + x_2) < a^2.$$ 

Therefore

$$x_1^2 + x_2^2 + x_1x_2 + ax_2 < ax_1.$$ 

Dropping the positive terms $x_1^2 + x_2^2 + x_1x_2$ this leaves $ax_2 < ax_1$, i.e. $x_2 < x_1$.

Similarly we obtain $x_3 < x_2$ and $x_1 < x_3$. Since last three inequalities cannot hold simultaneously, it means that our assumption (2) is false. \hfill \Box

\textbf{Proof of Theorem 3.6.} Let $C$ be a $k$-supported cycle and let $T$ be an arbitrary spanning tree in $G$. We view the union of $T$ and $C$ as a metric space consisting of unit length intervals, one interval for each edge. We denote this space by $U$. It is important to keep in mind that although we use the same notation, when the embedding of $C$ in $G$ is considered, the
cycle $C$ is viewed only as a subgraph, while in the metric space $U$, $C$ is viewed as a curve (topologically a circle).

In $U$, consider the mapping that takes each edge of $C$ continuously onto its detour in $T$, i.e. the unique path in the tree $T$ joining the ends of the edge. We define this mapping so that points on an edge of $C$ are mapped on the detour of that edge in such a way that the ratio of the distances from a point $t$ inside an edge to the endpoints of the edge and from its image $t'$ in the detour of the edge to the ends of the detour path are equal.

If the edge is already in $T$, it is therefore mapped onto itself identically. Combining these maps we get a continuous map of $C$ into $T$. We emphasize that this happens in the metric space $U$.

Since $C$ is a $k$-supported cycle, it can be partitioned into paths $I_1, I_2, I_3$ satisfying the conditions of Definition 3.3. We color the intervals $I_1, I_2, I_3$ into blue, green, and red, respectively. To each vertex that is common to two of the intervals we associate two colors, one from each interval. Except for these three points, any other point on $C$ has a unique color associated to it.

![Diagram](attachment:diagram.png)

To the continuous map described above from $C$ (thus colored) to $T$ we apply the following proposition.

**Proposition 4.3** ([RR98] Proposition 5.2). Let $f : S^1 \rightarrow T$ be a continuous map, and let $\{I_1, I_2, I_3\}$ be an arbitrary partition of $S^1$ into three intervals with mutually disjoint interiors. Then there exists $c \in T$ such that $f^{-1}(c)$ has a representative in each of these intervals.

Let $x, y, z$ be three points, one in each of the $I_1, I_2, I_3$, mapped onto the same point $c$ in $T$. We note that it is possible that two of the points coincide, for example $x = y$ if they are at the intersection of $I_1$ and $I_2$.

If $x$ is on a blue edge of $C$, and this edge is also in $T$, then $x = c$. Otherwise, if $x$ is on a blue edge $e$ of $C$ and $e$ is not in $T$, then the cycle formed by $e$ and its detour in $T$ has length at most $\text{Stretch}(G:T) + 1$.

In either case, denote by $u_1$ the endpoint of $e$ closest to $c$ (select arbitrarily in the case the endpoints of $e$ are equidistant to $c$ in $U$). Therefore, $u_1$ is a vertex of $C$ colored in blue and $d_U(u_1,c) \leq \text{Stretch}(G:T)/2$.

Similarly, we have $d_U(u_2,c) \leq \text{Stretch}(G:T)/2$ and $d_U(u_3,c) \leq \text{Stretch}(G:T)/2$ for vertices $u_2$ and $u_3$ colored in green and red, respectively. Therefore by triangle inequality
each of the distances satisfies
\[ d_C(u_i, u_j) = d_U(u_i, u_j) \leq d_U(u_i, c) + d_U(u_j, c) \leq \text{Stretch}(G : T). \]

Since
\[ k \leq \max_{i,j} d_G(u_i, u_j) \leq \max_{i,j} d_C(u_i, u_j) \leq \text{Stretch}(G : T), \]

and \( T \) was arbitrary, the theorem follows. \( \square \)

5. ISOMETRIC CYCLES

A somewhat surprising result is obtained by Lokshtanov [Lok09, Theorem 3.8]: the length of the longest isometric cycle in a finite simple graph can be found in polynomial time. From our perspective, the result is interesting because of Corollary 3.7 which connects the NP-hard notion of stretch with that of polynomially computable maximal length of isometrically embedded cycles. We now present a part of Lokshtanov’s construction and make a correction in one of the steps in his argument.

In Lokshtanov’s construction, to a graph \( G = (V, E) \) and to an integer \( k \geq 3 \) one associates an auxiliary graph \( G_k \) as follows. The set of vertices of \( G_k \) is the set of ordered pairs
\[ \left\{(u, v) \in V \times V : d(u, v) = \left\lfloor \frac{k}{2} \right\rfloor \right\}. \]

There is an edge in \( G_k \) between vertices \((u, v)\) and \((w, x)\) if and only if \((u, w) \in E\) and \((v, x) \in E\).

For a vertex \((u, v) \in V(G_k)\), define the set
\[ M_k(u, v) = \{(u, v)\} \quad \text{if } k \text{ is even}, \]

and
\[ M_k(u, v) = \{(u, x) \in V \times V : (u, x) \in V(G_k) \text{ and } (v, x) \in E\} \quad \text{if } k \text{ is odd}. \]

Lokshtanov stated the following theorem.

**Theorem 5.1** ([Lok09]). A graph \( G \) has an isometric cycle of length \( k \) if and only if there are vertices \( u \) and \( v \) and \( x \) in \( V(G) \) so that \((v, x) \in M_k(v, u)\) and \( d_{G_k}[(u, v), (v, x)] = \left\lfloor \frac{k}{2} \right\rfloor \).

The theorem is proved in two cases, for \( k \) even and for \( k \) odd. For odd \( k \) the proof is based on the following lemma.

**Lemma 5.2** ([Lok09, Lemma 3.6]). If \( k \) is odd and the distance in \( G_k \) between \((u, v)\) and a vertex \((v, x) \in M_k(v, u)\) is \( \left\lfloor \frac{k}{2} \right\rfloor \), then there is an isometric cycle of length \( k \) in \( G \), going through \( u \) and \( v \).

We should remark that Lemma 3.6 in [Lok09] is erroneous, as the graph in Figure 5 shows. In fact, the graph has no isometric cycle of length 7, while all conditions of the Lemma 5.2 are satisfied.

The case of even \( k \) in Theorem 5.1 (whose proof in [Lok09] is correct) is contained in [Lok09, Corollary 3.4]. We restate it as
Theorem 5.3 ([Lok09]). If $k$ is even, there is an isometric cycle of length $k$ in $G$ if and only if there is a pair of vertices $u$ and $v$ with $d_{G_k}[(u,v),(v,u)] = k/2$.

Our goal now is to prove:

Observation 5.4. Lokshtanov’s theorem for odd $k$ can be derived from Theorem 5.3.

Proof. We show that the existence of an isometric cycle of odd length $k$ in $G$ is equivalent to the existence of an isometric cycle of length $2k$ in an auxiliary bipartite graph $G'$.

We derive the auxiliary graph $G'$ from a graph $G$ as follows. Denote $|V(G)| = n$ and $|E(G)| = m$ the number of vertices and edges in the connected simple graph $G$. For visualization purposes consider the vertices of $G$ colored in blue. Define $G'$ from $G$ by dividing every edge in $E(G)$ into two edges in $E(G')$ by the introduction of a new red vertex. The set $V(G')$ thus consists of $n + m$ vertices, $n$ of which are blue and $m$ are red. There are $2m$ edges in $E(G')$, all joining a blue and a red vertex, therefore $G'$ is bipartite. Note that for any two blue vertices $u$ and $v$, we have

$$d_{G'}(u,v) = 2d_G(u,v).$$

We use Theorem 5.3 to find isometric cycles of length $2k$ in the graph $G'$. Since in such a cycle $C_{2k}$ blue and red vertices alternate, we can label the vertices of the cycle as

$$b_0, r_1, b_1, r_2, b_2, \ldots, b_{k-1}, r_k, b_k = b_0.$$ 

By dropping the red vertices, we obtain the cycle $C_k$ in $G$ with vertices $b_0, b_1, b_2, \ldots, b_k = b_0$. If for two vertices $u, v$ in $C_k$ we had

$$d_G(u,v) < d_{C_k}(u,v),$$

Figure 5. Counterexample to Lemma 3.6 in [Lok09]
this would imply
\[ d_{G'}(u,v) < 2d_{G''}(u,v) = d_{G'''}(u,v). \]
But this contradicts the fact that \( C_{2k} \) is isometric in \( G' \). Therefore the cycle \( C_k \) embeds isometrically in \( G \). \( \square \)

Our next goal is the following observation.

**Observation 5.5.** In some cases Corollary [3.7] provides very weak estimates for stretch.

In fact, by Theorem [3.6] Proposition [4.1] implies that \( \sigma(P_n \times P_n) \geq n - 1 \) and Proposition [4.2] implies that \( \sigma(T_n) \geq \lfloor \frac{n-1}{2} \rfloor \). On the other hand one can show that the maximal sizes of isometric cycles in \( P_n \times P_n \) and \( T_n \) are 4 and 3 respectively.

For \( P_n \times P_n \) it is proved as follows: Clearly, the cycle of length four is isometrically embedded. This is the only case when two consecutive turns of either 90° or -90° are allowed for edges \( x_0x_1, x_1x_2, \) and \( x_2x_3 \). Assume \( x_0, x_1, \ldots, x_k = x_0 \) is an isometrically embedded cycle of length \( k > 4 \). For any \( 3 \leq i \leq k \), if there are turns in the path between \( x_0 \) and \( x_i \), they have to alternate so that after a turn by 90° the next turn will be by -90° and vice versa. However, such alternating turns only force the distance \( d(x_0, x_i) \) to increase and therefore the path cannot close into a cycle.

For \( T_n \) it is proved as follows: Clearly, the cycle of length three is isometrically embedded. This is the only case when a turn of either 60° or -60° is allowed for edges \( x_0x_1, x_1x_2 \). Assume \( x_0, x_1, \ldots, x_k = x_0 \) is an isometrically embedded cycle of length \( k > 3 \). For any \( 2 \leq i \leq k \), if there are turns in the path between \( x_0 \) and \( x_i \), they have to alternate so that after a turn by 120° the next turn will be by -120° and vice versa. However, such alternating turns only force the distance \( d(x_0, x_i) \) to increase and therefore the path cannot close into a cycle.

### 6. Approximation algorithm for \( k \)-supported cycles

Our goal in this section is to construct an approximation algorithm (in the sense of [Vaz01], [WS11]) for computing the support number, i.e. the maximal \( k \) such that a given graph \( G \) contains a \( k \)-supported cycle.

First we define a polynomial algorithm for computing the integer characteristic \( W(G) \) of a graph \( G \) introduced in Definition 6.1 below. Then we prove Theorems [3.9] and [3.10].

For each vertex \( r \) in a simple connected graph \( G = (V(G), E(G)) \) we build the breadth-first search tree (see [BM08] pp. 137–139) rooted at \( r \). For \( x \in V(G) \) denote the level of \( x \) by \( \ell(x) = d_G(r, x) \). For each \( n \in \mathbb{N} \) we consider the subset \( R(n) \) of edges in \( G \) consisting of edges \( xy \) for which
\[ \max\{\ell(x), \ell(y)\} > n. \]

Thus, \( R(n) \) is the set of edges of \( G \) with at least one endpoint outside the ball centered at the root \( r \) of radius \( n \).
Definition 6.1. Write \( x \simeq_n y \) is \( x \) and \( y \) in the same component of \((V(G), R(n))\). Define the cycle width of \( G \) as \( W(G) = \max_{r \in V(G)} W(r) \) where

\[
W(r) = \max_n \max \{d_G(x, y) : \ell(x) = \ell(y) = n, x \simeq_n y\}.
\]

Remark 6.2. Since the distance matrix for \( V(G) \) is polynomially computable, it is straightforward that \( W(G) \) is also a polynomially computable quantity.

Proof of Theorem 3.9. Suppose that we have computed \( W(G) \) and found a root \( r \), a radius \( n \), and corresponding vertices \( x \) and \( y \) such that \( \ell(x) = \ell(y) = n \), and \( d_G(x, y) = W(G) \).

Denote by \( T \) the breadth-first tree rooted at \( r \). To produce a \( k \)-supported cycle with \( k \geq W(G)/3 \), we start at the point \( w \) where the paths \( rx \) and \( ry \) in \( T \) depart from each other (it can be the root \( r \)), and consider the cycle consisting of

- path from \( x \) to \( w \)
- path from \( w \) to \( y \)
- path consisting of edges in \( R(n) \) joining \( x \) and \( y \).

The paths above form the partition \( I_1, I_2, \) and \( I_3 \) of the cycle under consideration. Consider arbitrary vertices \( u_1 \in I_1, u_2 \in I_2, \) and \( u_3 \in I_3 \). Because \( \ell(u_3) \geq n \), we have that

\[
d_G(u_1, u_3) \geq d_G(u_1, x) \quad \text{and} \quad d_G(u_2, u_3) \geq d_G(u_2, y).
\]

Since

\[
d_G(x, u_1) + d_G(u_1, u_2) + d_G(u_2, y) \geq d_G(x, y) = W(G),
\]

we obtain that

\[
d_G(u_1, u_3) + d_G(u_1, u_2) + d_G(u_2, u_3) \geq W(G).
\]

Therefore the maximum of the three distances has to be at least \( W(G)/3 \) and thus the cycle is \( W(G)/3 \)-supported. \( \square \)

We now bound the support number of \( G \) from above in terms of \( W(G) \).

Proof of Theorem 3.10. We need to show that the existence of a \( k \)-supported cycle in \( G \) implies \( W(G) \geq k - 4 \).

Let \( C \) be a \( k \)-supported cycle in \( G \), and let \( x, y, z \) be the vertices where \( C \) is partitioned into \( I_1, I_2, \) and \( I_3 \) showing the situation. For an arbitrarily selected root \( r \) we prove that the presence of the cycle \( C \) implies \( W(r) + 4 \geq k \) (in the estimates we will use however the larger \( W(G) \) instead of \( W(r) \)).

We introduce additional structure on the vertex set of \( G \), relative to the chosen root \( r \).

(a) We split the vertex set of \( G \) into the set of levels, that is, for each \( n \in \mathbb{N} \), the level \( n \) is the set of vertices \( u \) satisfying \( d_G(u, r) = n \).
(b) For each \( n \in \mathbb{N} \) we split the level \( n \) into subsets which are equivalence classes of the following relation: \( x \sim y \) if \( x = y \) or if \( x \) and \( y \) are joined by a path in \( R(n) \). In such a way we get pairwise disjoint sets of vertices. Let us denote them \( \{C(n, j)\}_{j=1}^{c(n)} \) and call them blocks of level \( n \).

(c) We also introduce bunches of level \( n \) as unions of blocks of level \( n \) defined by the following condition: If there is an edge joining two blocks of level \( n \), they should be in the same bunch. More formally: Blocks \( C(n, j) \) and \( C(n, i) \) are in the same bunch if and only if there exists a sequence of blocks \( \{C(n, j(t))\}_{t=0}^{N(j, i)} \) such that \( C(n, j(0)) = C(n, j) \), \( C(n, j(N(j, i))) = C(n, i) \) and there is an edge between \( C(n, j(t)) \) and \( C(n, j(t+1)) \) for each \( t = 0, \ldots, N(j, i) - 1 \).

We denote bunches of level \( n \) by \( \{B(n, s)\}_{s=1}^{b(n)} \).

**Lemma 6.3.** All edges of \( G \) going from a bunch \( B(n, s) \) to the level \( (n - 1) \) have their other ends in the same block of level \( (n - 1) \).

**Proof.** If \( u \) and \( v \) are such ends in level \( (n - 1) \) we can construct a path in \( R(n - 1) \) joining them. \( \square \)

**Corollary 6.4.** The diameter of a bunch is at most \( W(G) + 2 \).

**Proof.** By the definition of cycle width \( W(G) \), any block of any level has diameter at most \( W(G) \). Since by Lemma 6.3 the predecessors of a bunch at level \( n \) are all in a block at level \( (n - 1) \), we obtain that the diameter of the bunch is no more than \( W(G) + 2 \). \( \square \)

Given a bunch or a block, by the part of \( G \) “above” the bunch or block we mean the set of vertices of \( G \) which are disconnected from the root by the removal of the bunch or block.

Assume that the graph contains a \( k \)-supported cycle \( C \), and color vertices of \( C \) blue, green, and red according to the partition of \( C \) into \( I_1, I_2 \) and \( I_3 \) (following Definition 3.3). Vertices where two of the intervals meet are considered to have both colors.

Corollary 6.4 implies that if there is a bunch which contains points of all three colors, then \( k \leq W(G) + 2 \).

We prove that there is a bunch containing vertices of two colors and having distance 1 to a vertex of the third color. We start by proving the following lemma.

**Lemma 6.5.** Either there exists a bunch containing all three colors, or there exists a bunch such that one of the colors is present only “above” the bunch.

**Proof.** Consider three cases:

(1) The cycle contains the root.

It is easy to see that in this case the cycle intersects exactly one of the bunches of level 1. If the bunch does not contain all three colors, then one of the colors is only “above” the bunch.

(2) The cycle does not contain the root, but contains a vertex of level 1.
It is easy to see that in this case the cycle intersects only one of the bunches of level 1. Unless the intersection contains all three colors, one of the colors is only “above” the bunch.

(3) The lowest-level vertices in the cycle have level 2 or higher.

In this case all three colors are “above” the block immediately preceding the cycle (we mean the block of the previous level through which the cycle is connected with the root).

□

To conclude the proof of the Theorem 3.10 we only have to consider the case when no bunch contains all three colors.

Let bunch \( B(n, s) \) be the bunch with the largest \( n \) such that one of the colors is present only “above” the bunch. In such a case \( B(n, s) \) together with those vertices of level \( (n + 1) \) which are “above” \( B(n, s) \) should contain all three colors. In fact, since any two colors should “meet”, otherwise there are bunches “above” \( B(n, s) \) for which one of the colors is present only “above” the bunch.

Since the diameter of \( B(n, s) \) is at most \( W(G) + 2 \), we conclude that there is a triple of points of different colors with pairwise distances \( \leq W(G) + 4 \).

□

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