Global dynamics of a general competition diffusion system in spatially heterogeneous environments

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Abstract

In this paper, we study a diffusive Lotka-Volterra competition model under homogeneous Dirichlet boundary conditions. We shall discuss the effects of dispersal rate and spatial heterogeneity on population dynamics. More precisely, we establish the main results about the global asymptotic stability of semitrivial as well as coexistence steady states.

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1 Introduction

The evolution of dispersal has been one of the central topics in recent theoretical studies of population dynamics [22]. In particular, spatially heterogeneous reaction-diffusion models for interacting biological species in a bounded isolated habitat patch are especially well-suited for capturing biogeographic features of species’ interactions or movements. The joint action of dispersal and spatial heterogeneity often give rise to some interesting phenomenon. In the past a few decades, the phenomenon of spatial heterogeneity of resources has attracted the attention of many researchers from both biology and mathematics, see [1, 21, 23, 28], for example. The dynamical properties (existence, bifurcation and local/global stability of nonconstant steady state) of mathematical models with spatial heterogeneity are more complicated. It is shown in [5] that for a spatially heterogeneous Lotka-Volterra competition model with the same resources, the slower diffuser always win. For another Lotka-Volterra competition system [10] with the total resource being fixed exactly at the same level, the environmental heterogeneity is usually superior to its homogeneous counterpart in the present of dispersal. There has been considerable interest, by both mathematicians and ecologists, in two-species Lotka-Volterra competition models with spatially heterogeneous interactions, see [3, 4, 6, 17, 19, 22, 24, 29] and references therein.

In this paper, we study a diffusive Lotka-Volterra competition model in a nonhomogeneous environment with homogeneous Dirichlet boundary conditions

\[
\begin{aligned}
& u_t = d_1 \Delta u + u(r_1(x) - b_1(x)u - c_1(x)v), \quad \text{in } \Omega \times (0,T), \\
& v_t = d_2 \Delta v + v(r_2(x) - b_2(x)u - c_2(x)v), \quad \text{in } \Omega \times (0,T), \\
& u = v = 0, \quad \text{on } \partial \Omega \times (0,T), \\
& u(x,0) = u_0(x) \geq 0, \quad v(x,0) = v_0(x) \geq 0, \quad \text{in } \Omega,
\end{aligned}
\]

where \( u(x,t) \) and \( v(x,t) \) represent the population densities of competing species 1 and 2, and are therefore assumed to be nonnegative, with corresponding dispersal rates \( d_1 > 0 \) and \( d_2 > 0 \). \( r_i(x) > 0, i = 1, 2 \), represent the densities of non-uniform resources, and the positive function \( b_i(x), c_i(x) \) are the strength of competition for species \( u \) and \( v \) at location \( x \). The habitat \( \Omega \) is a bounded region in \( \mathbb{R}^N \) with smooth

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boundary \( \partial \Omega \). The homogeneous Dirichlet boundary conditions mean the exterior region of \( \Omega \) is a completely hostile world for both species. Throughout this paper, we shall assume that

\[ r_i(x), b_i(x), c_i(x) > 0, u_0(x), v_0(x) \geq 0, u_0 = v_0 = 0 \text{ on } \partial \Omega, \text{ and } r_i, b_i, c_i, u_0, v_0 \in C^1(\Omega). \]  

(H1)

There are many results concerning the Lotka-Volterra competition system

\[
\begin{aligned}
    u_t &= \mu \Delta u + u(m(x) - u - bv), & \quad & \text{in } \Omega \times (0,T), \\
    v_t &= \nu \Delta v + v(m(x) - cu - v), & \quad & \text{in } \Omega \times (0,T), \\
    \frac{\partial u}{\partial n} &= \frac{\partial v}{\partial n} = 0, & \quad & \text{on } \partial \Omega \times (0,T), \\
    u(x,0) &= u_0(x) \geq 0, v(x,0) = v_0(x) \geq 0, & \quad & \text{in } \Omega.
\end{aligned}
\]

(1.1)

It is demonstrated in [17] that if \( \mu, \nu \) are sufficiently small, the function \( m(x) \) is positive, and \( 0 < b, c < 1 \), then (1.1) admits a unique, globally asymptotically stable positive steady state \((u^*, v^*)\). Moreover, \((u^*, v^*)\) converges to \((1 - b)m \quad (1 - c)m \quad 1 - bc \quad 1 - bc) \) in \( L^\infty(\Omega) \) as \( \mu, \nu \to 0 \). In [11], He and Ni provided a complete classification on the global dynamics of system (1.1), which says that either one of the two semi-trivial steady states is globally asymptotically stable, or there is a unique co-existence steady state which is globally asymptotically stable, or the system is degenerate in the sense that there is a compact global attractor consisting of a continuum of steady states which connect the two semi-trivial steady states. We refer the interested readers to [11] for some developments of (1.1).

As for system (1.1) with spatially nonhomogeneous coefficients, it was studied in [20]. The global stability of non-constant equilibrium solutions in heterogeneous environment was proved in [20] by a new Lyapunov functional method and a new integral inequality.

In this paper, we would like to investigate how dispersal rate and spatial heterogeneity affect the dynamics of (1.1). Thus we shall always assume that the function \( r_i(x), b_i(x), c_i(x) > 0, i = 1, 2 \) are non-constant. It is well known in [20] that the following logistic equation

\[
\begin{aligned}
    d\Delta \theta + \theta(r(x) - b(x)\theta) &= 0, & \quad & \text{in } \Omega, \\
    \theta &= 0, & \quad & \text{on } \partial \Omega,
\end{aligned}
\]

(1.3)

has a unique positive solution for every \( b(x) > 0, r(x) > d\lambda_1(1), \) denoted by \( \theta(x; d) \), where \( \lambda_1(1) \) is uniquely determined by the eigenvalue problem (2.1) in Section 2. For convenience, we frequently write \( \theta(x; d) \) as \( \theta_d \). Clearly, (2.1) has two semi-trivial nonnegative steady states for

\[ r_1(x) > d_1 \lambda_1(1) \text{ and } r_2(x) > d_2 \lambda_1(1), \]

(H2)

i.e., \((\theta_d, 0)\) and \((0, \theta_d)\) respectively. In fact, by the same methods in [9] Theorem 3.1, we can get that for the steady states for system (2.1), if either

\[
\max_{\Omega} \frac{c_1}{b_1} \max_{\Omega} \frac{b_2}{c_2} \left( \max_{\Omega} \frac{b_1}{c_2} \right)^\# \leq 1 \text{ or } \max_{\Omega} \frac{b_1}{b_2} \max_{\Omega} \frac{b_2}{c_1} \left( \max_{\Omega} \frac{b_1}{c_2} \right)^\# \leq 1,
\]

(H3)
then one of the following four alternatives holds:

\[
\begin{align*}
&\text{(A)} \quad \mu_1(d_2, r_2 - 2b\theta_1) \leq 0 \text{ and } \mu_1(d_1, r_1 - c_1\theta_2) > 0; \\
&\text{(B)} \quad \mu_1(d_2, r_2 - 2b\theta_1) > 0 \text{ and } \mu_1(d_1, r_1 - c_1\theta_2) \leq 0; \\
&\text{(C)} \quad \mu_1(d_2, r_2 - 2b\theta_1) > 0 \text{ and } \mu_1(d_1, r_1 - c_1\theta_2) > 0; \\
&\text{(D)} \quad \mu_1(d_2, r_2 - 2b\theta_1) = 0 \text{ and } \mu_1(d_1, r_1 - c_1\theta_2) = 0,
\end{align*}
\]

(1.4)

where \(\mu_1(d_2, r_2 - 2b\theta_1)\) and \(\mu_1(d_1, r_1 - c_1\theta_2)\) are defined in Lemma 2.3. Then we shall consider the dynamics of (P) during (A), (B), (C).

The main results of this paper are listed as follows.

**Theorem 1.1.** Assume that (H1) and (H2) hold. Suppose that \(\max_{\Omega} r_1(x) < +\infty, \max_{\Omega} r_2(x) < +\infty\). Once \(\mu_1(d_2, r_2 - 2b\theta_1) > 0\) and \(\mu_1(d_1, r_1 - c_1\theta_2) > 0\), (P) admits a unique coexistence steady-state \((\bar{u}, \bar{v})\) of (P). Moreover it is globally asymptotically stable, i.e., every solution \((u, v)\) of (P) converges to \((\bar{u}, \bar{v})\) as \(t \to +\infty\) regardless of initial values \((u_0, v_0)\).

**Theorem 1.2.** Assume that (H1) and (H2) hold. Suppose that \(\max_{\Omega} r_1(x) < +\infty, \max_{\Omega} r_2(x) < +\infty\) and the following conditions hold

\[
\min_{\Omega} \frac{r_1(x) - d_1\lambda_1(1)}{c_1(x)\max_{\Omega} \frac{b_2}{b_1}} > 1, \quad c_1 + \frac{d_1}{d_2} b_2 \geq 2b_1, \quad \max_{\Omega} \frac{r_1(x)}{r_2(x)} \leq \frac{d_1}{d_2}. \tag{1.5}
\]

If \(\mu_1(d_2, r_2 - 2b\theta_1) < 0\), then the semi-trivial steady state \((\theta_d, 0)\) is globally asymptotically stable, i.e., every solution \((u, v)\) of (P) converges to \((\theta_d, 0)\) as \(t \to +\infty\) regardless of initial values \((u_0, v_0)\).

**Theorem 1.3.** Assume that (H1) and (H2) hold. Suppose that \(\max_{\Omega} r_2(x) < +\infty, \max_{\Omega} r_1(x) < +\infty\) and the following conditions hold

\[
\min_{\Omega} \frac{r_2(x) - d_2\lambda_1(1)}{b_2(x)\max_{\Omega} \frac{b_1}{b_2}} > 1, \quad b_2 + \frac{d_2}{d_1} c_1 \geq 2c_2, \quad \max_{\Omega} \frac{r_2(x)}{r_1(x)} \leq \frac{d_2}{d_1}. \tag{1.6}
\]

If \(\mu_1(d_1, r_1 - c_1\theta_2) < 0\), then the semi-trivial steady state \((0, \theta_d)\) is globally asymptotically stable, i.e., every solution \((u, v)\) of (P) converges to \((0, \theta_d)\) as \(t \to +\infty\) regardless of initial values \((u_0, v_0)\).

This paper is organized as follows. In Section 2 we present some preliminary results. Theorem 1.1, 1.2 will be proved in Section 3-4, respectively.

## 2 Preliminaries

In this section, we will recall some useful lemmas which can be obtained in [30] directly.

**Lemma 2.1.** Assume that \(r(x)\) is non-constant, Hölder continuous and \(|\{x \in \Omega : r(x) > 0\}| \neq 0\). The following eigenvalue problem

\[
\begin{align*}
-\Delta \phi &= \lambda r(x)\phi, \text{ in } \Omega, \\
\phi &= 0, \text{ on } \partial\Omega
\end{align*}
\]

(2.1)

admits a positive principal eigenvalue \(\lambda_1(r)\) which is given by

\[
\frac{1}{\lambda_1(m)} = \max_{\phi \in H^1_0(\Omega), \phi \neq 0} \frac{\int_{\Omega} \lambda r(x)\phi^2 dx}{\int_{\Omega} |\nabla \phi|^2 dx} \tag{2.2}
\]

**Lemma 2.2.** Suppose that \(r(x) \in L^\infty(\Omega)\). The principal eigenvalue \(\mu_1(d, r)\) of

\[
\begin{align*}
\{d\Delta \varphi + r(x)\varphi &= \mu \varphi, \text{ in } \Omega, \\
\varphi &= 0, \text{ on } \partial\Omega
\end{align*}
\]

(2.3)
is uniquely determined by

\[
\mu_1(d, r) = \max_{\varphi \in H^1_0(\Omega), \varphi \neq 0} \frac{-d \int_{\Omega} |\nabla \varphi|^2 \, dx + \int_{\Omega} r(x) \varphi^2 \, dx}{\int_{\Omega} \varphi^2 \, dx},
\]

Moreover, \(\mu_1(d, r)\) is strictly increasing in \(r\) in the sense that if \(r_1 \leq r_2\), then \(\mu_1(d, r_1) \leq \mu_1(d, r_2)\), and if \(r_1 < r_2\) on a subset of positive measure then \(\mu_1(d, r_1) < \mu_1(d, r_2)\).

**Lemma 2.3.** The semi-trivial steady state \((\theta_{d_1}, 0)\) and \((0, \theta_{d_2})\) are stable/unstable if and only if the following eigenvalue problem has a negative/positive principal eigenvalue (denoted as \(\mu_1(d_2, r_2 - b_2\theta_{d_1})\) and \(\mu_1(d_1, r_1 - b_1\theta_{d_2})\)):

\[
\begin{aligned}
&d_2 \Delta \psi + (r_2 - b_2\theta_{d_1}) \psi = \mu \psi, \quad \text{in } \Omega, \\
&\psi = 0, \quad \text{on } \partial \Omega,
\end{aligned}
\]

and

\[
\begin{aligned}
&d_1 \Delta \varphi + (r_1 - c_1\theta_{d_2}) \varphi = \mu \varphi, \quad \text{in } \Omega, \\
&\varphi = 0, \quad \text{on } \partial \Omega.
\end{aligned}
\]

**3 Global stability of coexistence steady state**

In this section, our goal is to prove Theorem 1.1. More precisely, if both semi-trivial steady states of \(\text{[P]}\) are unstable, the existence and globally asymptotic stability of coexistence steady state will be proved. Indeed, by the theory of monotone dynamical system [16, Proposition 9.1], it is sufficient to verify the uniqueness of coexistence steady state of \(\text{[P]}\), if exists.

**Theorem 3.1.** Assume that [H1] and [H2] hold. Suppose that \(\max_{i \in I} \frac{r_1(x)}{b_1(x)} < +\infty\), \(\max_{i \in I} \frac{r_2(x)}{c_2(x)} < +\infty\). If there exists a coexistence steady-state \((\tilde{u}, \tilde{v})\) of \(\text{[P]}\), then it is unique.

**Proof:** We shall modify some ideas in [26] to show this theorem. In order to show the uniqueness of \((\tilde{u}, \tilde{v})\), we only need to linearize the steady state problem of \(\text{[P]}\) at \((\theta_{d_1}, 0)\) and \((0, \theta_{d_2})\), we have

\[
\begin{aligned}
&d_2 \Delta \psi + (r_2 - b_2\theta_{d_1}) \psi = \mu \psi, \quad \text{in } \Omega, \\
&\psi = 0, \quad \text{on } \partial \Omega,
\end{aligned}
\]

and

\[
\begin{aligned}
&d_1 \Delta \varphi + (r_1 - b_1\theta_{d_2}) \varphi = \mu \varphi, \quad \text{in } \Omega, \\
&\varphi = 0, \quad \text{on } \partial \Omega.
\end{aligned}
\]

Suppose that \((\tilde{u}_1, \tilde{v}_1), (\tilde{u}_2, \tilde{v}_2)\) are two coexistence steady state of system \(\text{[P]}\), then \(\tilde{u}_1, \tilde{v}_1, \tilde{u}_2, \tilde{v}_2 > 0\) and satisfy

\[
\begin{aligned}
&d_1 \Delta \tilde{u}_i + \tilde{u}_i(r_1(x) - b_1(x)\tilde{u}_i - c_1(x)\tilde{v}_i) = 0, \quad \text{in } \Omega, \\
&d_2 \Delta \tilde{v}_i + \tilde{v}_i(r_2(x) - b_2(x)\tilde{u}_i - c_2(x)\tilde{v}_i) = 0, \quad \text{in } \Omega,
\end{aligned}
\]

\(i = 1, 2\).

Let \(w(x) = \tilde{u}_1 - \tilde{u}_2, z(x) = \tilde{v}_1 - \tilde{v}_2\). By subtracting the first two equations in \(\text{[E]}\), we have that

\[
\begin{aligned}
&d_1 \Delta w + w(r_1(x) - b_1(x)\tilde{u}_1 - b_1(x)\tilde{u}_2 - c_1(x)\tilde{v}_1) = c_1(x)\tilde{u}_2, \quad \text{in } \Omega, \\
&d_2 \Delta z + z(r_2(x) - b_2(x)\tilde{u}_2 - c_2(x)\tilde{v}_2) = b_2(x)\tilde{v}_1, \quad \text{in } \Omega,
\end{aligned}
\]

\(w = z = 0, \quad \text{on } \partial \Omega\).

Set

\[
L_1 = d_1 \Delta + (r_1(x) - b_1(x)\tilde{u}_1 - b_1(x)\tilde{u}_2 - c_1(x)\tilde{v}_1) \quad \text{and} \quad L_2 = d_2 \Delta + (r_2(x) - b_2(x)\tilde{u}_2 - c_2(x)\tilde{v}_1 - c_2(x)\tilde{v}_2). \quad (3.4)
\]
Then (3.3) is equivalent to
\[
\begin{align*}
\begin{cases}
L_1 w &= c_1(x) \tilde{u}_2 z, \text{ in } \Omega, \\
L_2 z &= b_2(x) \tilde{v}_1 w, \text{ in } \Omega, \\
w &= z = 0, & \text{on } \partial\Omega.
\end{cases}
\end{align*}
\]
Moreover, by (4.1), (4.2) and Lemma 2.2, it is easy to check that the principal eigenvalues
\[
\begin{align*}
\mu_1(d_1, r_1(x) - b_1(x) \tilde{u}_1 - b_1(x) \tilde{u}_2 - c_1(x) \tilde{v}_1) < 
\mu_1(d_1, r_1(x) - b_1(x) \tilde{u}_i - c_1(x) \tilde{v}_i) = 0, \\
\mu_1(d_2, r_2(x) - b_2(x) \tilde{u}_2 - c_2(x) \tilde{v}_1 - c_2(x) \tilde{v}_2) < 
\mu_1(d_2, r_2(x) - b_2(x) \tilde{u}_2 - c_2(x) \tilde{v}_i) = 0.
\end{align*}
\]

We first claim that both \(w\) and \(z\) must change sign in \(\Omega\). Suppose \(z > 0\) in \(\Omega\) without loss of generality. Then it follows from the first equation of (3.3) that \(L_1 w > 0\) in \(\Omega\). By the strong maximum principle, we have \(w < 0\) in \(\Omega\). Multiplying the first equation of (3.3) by \(\tilde{u}_1\) and \(\tilde{E}(i = 1)\) by \(w\), integrating over \(\Omega\), we have
\[
-\int_{\Omega} b_1(x) \tilde{u}_1 \tilde{u}_2 w dx = \int_{\Omega} c_1(x) \tilde{u}_1 \tilde{u}_2 z dx,
\]
a contradiction. Assume \(w > 0\) in \(\Omega\). Similar arguments as above lead to
\[
-\int_{\Omega} c_2(x) \tilde{v}_1 \tilde{v}_2 z dx = \int_{\Omega} b_2(x) \tilde{v}_1 \tilde{v}_2 w dx,
\]
a contradiction. Hence, both \(w, z\) must change sign in \(\Omega\).

Second, we claim that if both \(w, z\) change sign in \(\Omega\), we also get a contradiction. Clearly, \(\max(\tilde{u}_1, \tilde{u}_2) \geq 0, \max(\tilde{v}_1, \tilde{v}_2) \geq 0\). By the comparison principle, one can get that \(\tilde{u}_i < \max \frac{r_1(x)}{b_1(x)}\) and \(\tilde{v}_i < \max \frac{r_2(x)}{c_2(x)}\). Take \(M := \max \left(\max \frac{r_1(x)}{b_1(x)}, \max \frac{r_2(x)}{c_2(x)}\right)\), \(\epsilon\) small enough. Then some calculations read that \((M, \epsilon \phi), (\min(\tilde{u}_1, \tilde{u}_2), \max(\tilde{v}_1, \tilde{v}_2))\) are super-solutions of (4). Meanwhile, \((\epsilon \phi, M), (\max(\tilde{u}_1, \tilde{u}_2), \min(\tilde{v}_1, \tilde{v}_2))\) are sub-solutions of (E). Here \(\phi\) is the principal eigenfunction of (4.1) with principal eigenvalue \(\lambda_1(1)\). It follows from [27, Theorem 8.4.2] that there exist two solution \((\tilde{u}, \tilde{v})\) and \((\bar{u}, \bar{v})\) for (4). Moreover they satisfy \(\bar{u} < \tilde{u}\) and \(\bar{v} < \tilde{v}\) which contradict to our first claim. Then \(w \equiv 0\) and \(z \equiv 0\). The proof of Theorem 3.1 is therefore completed. \(\square\)

Since \(\mu_1(d_2, r_2 - b_2 \theta_{d_1}) > 0\) and \(\mu_1(d_1, r_1 - c_1 \theta_{d_1}) > 0\) indicate that both two semi-trivial steady states \((\theta_{d_1}, 0), (0, 0, \theta_{d_2})\) are unstable, we then finish the proof of Theorem 1.1 by means of Theorem 3.1 and the theory of monotone dynamical system [16, Proposition 9.1].

4 Global stability of semi-trivial steady states

This section is devoted to prove Theorem 1.2 and Theorem 1.3 can be derived similarly.

Proof of Theorem 1.2 Since \(\mu_1(d_2, r_2 - b_2 \theta_{d_1}) < 0\), we see from (4.1) that \(\mu_1(d_1, r_1 - c_1 \theta_{d_1}) > 0\). Let \((u, v)\) be the solution of (4). We next verify this theorem in two steps.

Step 1 We prove the theorem under the additional conditions
\[
v_0(x) \leq \bar{k} \theta_{d_2} \text{ for all } x \in \Omega,
\]
where \(\bar{k} \in (1, \min_{\Omega} \left(\frac{r_1(x) - d_1 \lambda_1(1)}{c_1(x) \max_{\Omega} \frac{r_2(x)}{c_2(x)}}\right))\). Consider the functions \(U_1(x,t), U_2(x,t), V_1(x,t), V_2(x,t)\), which satisfy
\[
\begin{align*}
\begin{cases}
\frac{\partial U_1}{\partial t} &= d_1 \Delta U_1 + U_1(r_1(x) - b_1(x) U_1 - c_1(x) V_1), \text{ in } \Omega \times (0, T), \\
\frac{\partial V_1}{\partial t} &= d_2 \Delta V_1 + V_1(r_2(x) - b_2(x) U_1 - c_2(x) V_1), \text{ in } \Omega \times (0, T), \\
U_1 &= V_1 = 0, & \text{on } \partial \Omega \times (0, T), \\
U_1(x, 0) &= \epsilon \varphi, V_1(x, 0) = \bar{k} \theta_{d_2}(x), & \text{in } \Omega, \\
U_2(x, 0) &= k \theta_{d_2}(x), V_2(x, 0) = 0, & \text{in } \Omega,
\end{cases}
\end{align*}
\]
where \( k \geq 1 \), \( \varphi \) is the principal eigenfunction of \( \mathcal{L} \) with principal eigenvalue \( \mu_1(d_1, r_1 - c_1k\theta_{d_1}) \).

Observing that
\[
\begin{align*}
\begin{cases}
  d_1 \Delta (k\theta_{d_1}) + k\theta_{d_1}(r_1 - b_1k\theta_{d_1}) \leq k(d_1 \Delta \theta_{d_1} + \theta_{d_1}(r_1 - b_1\theta_{d_1})), \\
  d_1 \Delta (\epsilon \varphi) + \epsilon \varphi(r_1 - b_1\epsilon \varphi - c_1k\theta_{d_1}) = \epsilon(d_1 \Delta \varphi + \varphi(r_1 - c_1\theta_{d_1}) - b_1\epsilon \varphi^2) = \epsilon(\mu_1(d_1, r_1 - c_1k\theta_{d_1}) - b_1\epsilon \varphi^2), \\
  \l(\hat{d}(d_2 \Delta \theta_{d_2} + \theta_{d_2}(r_2 - b_2\epsilon \varphi - c_2k\theta_{d_2})) \leq k\hat{b}_2\epsilon \varphi \theta_{d_2},
\end{cases}
\end{align*}
\]
Then \((k\theta_{d_1}, 0)\) is a super-solution of \((\mathcal{L})\) and \((\epsilon \varphi, \hat{\theta}_{d_2})\) is a sub-solution of \((\mathcal{L})\) for sufficiently small \( \epsilon \).

By the comparison principle, we get that
\[
\begin{align*}
\frac{\partial U_1}{\partial t} > 0, \quad \frac{\partial U_2}{\partial t} \leq 0, \quad \frac{\partial V_1}{\partial t} < 0, \quad \frac{\partial V_2}{\partial t} \geq 0,
\end{align*}
\]
and moreover
\[
\epsilon \varphi < U_1 < U_2 < k\theta_{d_1}, \quad 0 \leq U_2 < V_1 < k\theta_{d_2}.\]  

Hence,
\[
e \varphi < \lim_{t \to +\infty} U_1 \leq \lim_{t \to +\infty} U_2 < k\theta_{d_1}, \quad 0 \leq \lim_{t \to +\infty} V_2 \leq \lim_{t \to +\infty} V_1 < k\theta_{d_2}.\]

From Theorem 3.1 there hold
\[
\lim_{t \to +\infty} U_1 = \lim_{t \to +\infty} U_2, \quad \lim_{t \to +\infty} V_2 = \lim_{t \to +\infty} V_1.
\]

Set now \( w_1(x, t) = \theta_{d_1} - a \varphi \psi, w_2(x, t) = a \varphi \psi \), where \( \psi \) is the eigenfunction in \((\mathcal{L})\), \( \lambda < 0 \) will be determined later. It then follows from some calculations that
\[
\begin{align*}
\begin{cases}
  \frac{\partial w_1}{\partial t} \geq d_1 \Delta w_1 + w_1(r_1 - b_1w_1 - c_1w_2), \\
  w_1(x, 0) = \theta_{d_1} - a \psi < \theta_{d_1} \leq k\theta_{d_1} = U_2(x, 0), \\
  \frac{\partial w_2}{\partial t} \geq d_2 \Delta w_2 + w_2(r_2 - b_2w_1 - c_2w_2), \\
  w_2(x, 0) = a \psi \geq 0 = V_2(x, 0),
\end{cases}
\end{align*}
\]
provided that \( c_1 + \frac{d_1}{d_2}b_2 \geq 2b_1, \max_{\Omega} \frac{r_1(x)}{r_2(x)} \leq \frac{d_1}{d_2}, \max_{\Omega}(\frac{d_1 \mu_1(d_2, r_2 - b_2\theta_{d_1})) < \lambda < 0, \)
\[
0 < a < \min \left\{ \frac{\lambda - \mu_1(d_2, r_2 - b_2\theta_{d_1}))}{\|b_2 - c_2\|_\infty}, \frac{\mu_1(d_2, r_2 - b_2\theta_{d_1}))}{\|b_1 - c_1\|_\infty} \right\}.
\]

We then obtain from (4.9) and the comparison principle that
\[
\theta_{d_1} = \lim_{t \to +\infty} w_1 \leq \lim_{t \to +\infty} U_2 = \lim_{t \to +\infty} U_1, 0 \leq \lim_{t \to +\infty} w_2 \geq \lim_{t \to +\infty} V_2 = \lim_{t \to +\infty} V_1 \geq 0,
\]
which imply \( \lim_{t \to +\infty} V_2 = \lim_{t \to +\infty} V_1 = 0 \) and \( \lim_{t \to +\infty} U_2 = \lim_{t \to +\infty} U_1 = \theta_{d_1} \).

Notice that for any \( u_0(x) > 0 \) there exists \( k \geq 1, 0 < \epsilon < 1 \) such that \( \epsilon \varphi < u_0(x) \leq k\theta_{d_1} \). Therefore the comparison principle yields that the solution \((u, v)\) for \((\mathcal{L})\) satisfies \( U_1 < u(x, t) < U_2, V_2 < v(x, t) < V_1 \). Furthermore, \( \lim_{t \to +\infty} u = \theta_{d_1}, \lim_{t \to +\infty} v = 0 \).

**Step 2.** We now remove condition (4.1) on the initial function \( v_0 \). It is clear that there exists a large \( K > 1 \) such that \( v_0(x) < K\theta_{d_2} \). Consider the following system
\[
\begin{align*}
\begin{cases}
  V_t = d_2 \Delta V + V(r_2(x) - c_2(x)V), \quad &\text{in } \Omega \times (0, T), \\
  V = 0, \quad &\text{on } \partial \Omega \times (0, T), \\
  V(x, 0) = K\theta_{d_2} \geq 0, \quad &\text{in } \Omega
\end{cases}
\end{align*}
\]
Then (4.3) guarantees that \( \lim_{t \to +\infty} V(x, t) = \theta_{d_2} \). Since
\[
d_2 \Delta (K\theta_{d_2}) + K\theta_{d_2}(r_2 - c_2K\theta_{d_2}) = K(d_2 \Delta \theta_{d_2} + \theta_{d_2}(r_2 - c_2K\theta_{d_2})) < K(d_2 \Delta \theta_{d_2} + \theta_{d_2}(r_2 - c_2\theta_{d_2})) = 0
\]
reads $V_t(x,0) < 0$, it then follows that $V_t < 0$. By the equation of $v, V$, we arrive at

\[
\begin{cases}
(V - v)_t = d_2 \Delta (V - v) + (V - v)(r_2(x) - c_2(x)V - c_2(x)v) + b_2(x)u, & \text{in } \Omega \times (0,T), \\
(V - v) = 0, & \text{on } \partial \Omega \times (0,T), \\
(V - v)(x,0) = K\theta_{d_2} - v(x,0) > 0, & \text{in } \Omega.
\end{cases}
\]  

(4.13)

Therefore combined with the comparison principle we reach that $\lim_{t \to +\infty} v(x,t) \leq \lim_{t \to +\infty} V(x,t) = \theta_{d_2}$. Hence one can choose $s \gg 1$ such that $v(x,s) < \bar{k}\theta_{d_2}$. Then we obtain the conclusion of this theorem by using Step 1 of the proof.

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