Large genus asymptotics for intersection numbers and principal strata volumes of quadratic differentials

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Abstract In this paper we analyze the large genus asymptotics for intersection numbers between $\psi$-classes, also called correlators, on the moduli space of stable curves. Our proofs proceed through a combinatorial analysis of the recursive relations (Virasoro constraints) that uniquely determine these correlators, together with a comparison between the coefficients in these relations with the jump probabilities of a certain asymmetric simple random walk. As an application of this result, we provide the large genus limits for Masur–Veech volumes and area Siegel–Veech constants associated with principal strata in the moduli space of quadratic differentials. These confirm predictions of Delecroix–Goujard–Zograf–Zorich from 2019.

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1 Introduction

In this paper we analyze the large genus asymptotics for intersection numbers of $\psi$-classes on the moduli space of stable curves, and for principal strata volumes in the moduli space of quadratic differentials. After describing the context for these results in Sects. 1.1, 1.2, and 1.3, we state them more precisely in Sect. 1.4.

1.1 Asymptotics for intersection numbers

Fix integers $g, n \geq 0$ such that $2g + n \geq 3$; let $\mathcal{M}_{g,n}$ denote the moduli space of smooth, complex, genus $g$ curves with $n$ marked points; and let $\overline{\mathcal{M}}_{g,n}$ denote its Deligne–Mumford compactification. Equivalently, $\overline{\mathcal{M}}_{g,n}$ is the moduli space of tuples $(C; x_1, x_2, \ldots, x_n)$, where $C$ is a stable curve of genus $g$ and $(x_1, x_2, \ldots, x_n)$ is an ordered collection of nonsingular points on $C$. For each index $i \in \{1, 2, \ldots, n\}$, let $L_i$ denote the line bundle on $\overline{\mathcal{M}}_{g,n}$ whose fiber over any point $(C; x_1, x_2, \ldots, x_n) \in \overline{\mathcal{M}}_{g,n}$ is the cotangent space $T_{x_i}^* C$, and let $\psi_i = c_1(L_i)$ denote the first Chern class of this bundle; these are referred to as $\psi$-classes on $\overline{\mathcal{M}}_{g,n}$.

For any $n$-tuple $d = (d_1, d_2, \ldots, d_n) \in \mathbb{Z}_{\geq 0}^n$, define the correlator $\langle \prod_{i=1}^n \tau^{d_i} \rangle$ as the intersection of $\psi$-classes given by

$$\langle \prod_{i=1}^n \tau^{d_i} \rangle = \int_{\overline{\mathcal{M}}_{g,n}} \prod_{i=1}^n \psi_i^{d_i}. \quad (1.1)$$

This quantity is nonzero only if $|d| = 3g + n - 3$, where we have denoted $|d| = \sum_{i=1}^n d_i$.

Over the past several decades, it has been understood that numerous fundamental invariants in physics and geometric topology can be expressed in terms of the intersection numbers (1.1). Perhaps the earliest result in this direction can be attributed to Witten [47], who realized them as correlation functions for a model of two-dimensional quantum gravity. Later appearances of these correlators include the identity of Mirzakhani [35] for Weil–Petersson volumes of moduli spaces of bordered Riemann surfaces and the formulas of Mirzakhani [33] and Delecroix–Goujard–Zograf–Zorich [11] for frequencies of geodesic multi-curves in hyperbolic and random flat surfaces, respectively. In an analogous but slightly different direction, the works of Chen et al. [7–9]...
and Sauvaget [41] write Masur–Veech volumes and Siegel–Veech constants
for strata of holomorphic and quadratic differentials also through intersec-
tion numbers, but between certain classes on compactified projectivizations of
Hodge bundles [39], instead of on $\overline{\mathcal{M}}_{g,n}$.

It is therefore of interest to evaluate the intersection numbers (1.1), which can
be done through the Kontsevich–Witten theorem [26,47]; the latter amounts to
a family of recursions, widely referred to as Virasoro constraints after the works
of Dijkgraaf [13] and Verlinde–Verlinde [45], that explicitly determines the
correlators. A different set of recursions for these quantities was later found by
Liu and Xu [28]. Alternative, non-recursive routes towards evaluating these
intersection numbers include exact formulas for their generating series by
Okounkov [38] and Zhou [49].

Although the methods above all in principle enable one to explicitly evaluate
intersection numbers (1.1), it is generally understood that these quantities are
quite intricate. Except in certain exceptional cases, they are not believed to
admit simple closed form expressions.

However, one might still hope that they are accessible in an asymptotic sense,
in the large genus limit as $g$ tends to $\infty$. Indeed, it was predicted by Delecroix–
Goujard–Zograf–Zorich as Conjecture E.6 of [11] that the correlators (1.1)
simplify considerably under this limit, namely, that

$$\left\langle \prod_{i=1}^{n} \tau_{d_i} \right\rangle = \frac{(6g + 2n - 5)!!}{24^g g! \prod_{i=1}^{n} (2d_i + 1)!!} \left(1 + o(1)\right)$$

as $g$ tends to $\infty$, uniformly in $d$, if $n \leq 2 \log g$. (1.2)

The asymptotic (1.2) was shown to hold by Liu and Xu [27] in the special case when the numbers $n$ and $d_2, d_3, \ldots, d_n$ are all uniformly bounded.
Furthermore, the predicted lower bound in (1.2) was proven by Delecroix et
al. [12] in the case when $d_1 + d_2 = 3g - o(g)$ (that is, when $d$ is “mostly
concentrated” on $d_1$ and $d_2$) and $n = o(g)$. Along a similar direction, the
large genus limits for a different, but related, family of intersection numbers
on $\overline{\mathcal{M}}_{g,n}$ that arise in the context of Weil–Petersson volumes were predicted
by Zograf [51] and later analyzed by Mirzakhani [34] and Mirzakhani and
Zograf [37]. The proofs of these results were all primarily based on combi-
natorial analyses of the Virasoro recursions that determine the corresponding
intersection numbers.

In this paper we establish as Theorem 1.5 below the limit (1.2) under the
slightly weaker constraint that $n = o(g^{1/2})$. As we explain in Remark 1.6
(and was observed earlier, in Appendix E of [11]), this condition is essentially
optimal.
To establish this result, we also proceed through a study of the Virasoro constraints. However, in addition to implementing a combinatorial analysis, our framework also involves a probabilistic aspect that does not seem to have been present in the previous works [12, 27, 34, 37] on large genus asymptotics. In particular, by comparing the coefficients in the Virasoro constraints with the jump probabilities of an asymmetric simple random walk, we show that the correlators (1.1) can be bounded above and below by certain functionals associated with this walk; see Propositions 4.8 and 5.5 below. The latter can be analyzed directly, leading to the asymptotic (1.2).

1.2 Masur–Veech volumes and Siegel–Veech constants

After establishing (1.2), we proceed to an application concerning large genus asymptotics for Masur–Veech volumes and Siegel–Veech constants associated with principal strata of moduli spaces of quadratic differentials. So, in this section we recall the definitions of these quantities.

As previously, fix integers \( g, n \geq 0 \) with \( 2g + n \geq 3 \). Let \( Q_{g,n} \) denote the moduli space of pairs \((X, q)\), where \( X \) is a Riemann surface of genus \( g \) and \( q \) is a meromorphic quadratic differential on \( X \) with \( n \) poles, each of which is simple. Under this notation, \( q \) has \( 4g + n - 4 \) zeroes (counted with multiplicity) on \( X \). The moduli space \( Q_{g,n} \) can be decomposed as a disjoint union of orbifolds called strata, prescribing how many distinct zeroes \( q \) has, along with their multiplicities. In this paper we will only be concerned with the principal stratum \( Q(1^{4g+n-4}, -1^n) \subseteq Q_{g,n} \), consisting of those quadratic differentials \((X, q) \in Q_{g,n}\) with \( 4g + n - 4 \) distinct simple zeroes and \( n \) simple poles. This stratum is both open and dense in \( Q_{g,n} \), and its complement has positive codimension.

The moduli space \( Q_{g,n} \) and each of its strata admit an \( SL_2(\mathbb{R}) \)-action, under which an element \( A \in SL_2(\mathbb{R}) \) acts on a quadratic differential \((X, q) \in Q_{g,n}\) by composing the local coordinate charts on \( X \) with \( A \). This action is closely related to billiard flow on rational polygons; dynamics on translation surfaces; the theory of interval exchange maps; enumeration of square-tiled surfaces; and Teichmüller geodesic flow. We will not explain these topics further here and instead refer to the surveys of Masur and Tabachnikov [31], Wright [48] and Zorich [52] for more information.

In any case, there exists a measure on \( Q_{g,n} \) (or equivalently, on each of its strata) that is invariant with respect to this \( SL_2(\mathbb{R}) \)-action. We recall the definition of this measure in the case of the principal stratum here, although it is entirely analogous for the remaining strata.

For any quadratic differential \((X, q) \in Q(1^{4g+n-4}, -1^n)\), there exists a double cover \( f = f_{X,q} : \tilde{X} \to X \) such that the pullback \( f^* q = \omega^2 \) is the square of some holomorphic one-form \( \omega = \omega_{X,q} \) on \( \tilde{X} \). In this way, \( X \) is the
quotient of \( \tilde{X} \) by the involution \( \sigma = \sigma_{X,q} : \tilde{X} \to \tilde{X} \) that interchanges the two preimages above any regular point of \( f \) in \( X \). Then since \( \sigma \) fixes each zero of \( \omega \), the set of which we denote by \( Z = Z_\omega = \{z_1, z_2, \ldots, z_m\} \subset \tilde{X} \), it induces an involution \( \sigma_* \) on the relative homology group \( H = H_{X,q} = H_1(\tilde{X}, \{z_1, z_2, \ldots, z_m\}, \mathbb{Z}) \). Let \( H^+ = H_{X,q}^+ \subseteq H \) and \( H^- = H_{X,q}^- \subseteq H \) denote the subspaces of \( H \) that are invariant and anti-invariant with respect to \( \sigma_* \), respectively (that is, they are the eigenspaces of \( H \) whose eigenvalues with respect to \( \sigma_* \) are 1 and \(-1\), respectively). Further let \( \gamma_1, \gamma_2, \ldots, \gamma_k \) denote a basis of the anti-invariant subspace \( H^- \subseteq H \).

Define the period map \( \Phi : \mathcal{Q}(1^{4g+n-4}, -1^n) \to \mathbb{C}^k \) by setting \( \Phi(X, q) = (\int_{\gamma_1} \omega, \int_{\gamma_2} \omega, \ldots, \int_{\gamma_k} \omega) \), for any quadratic differential \((X, q) \in \mathcal{Q}(1^{4g+n-4}, -1^n) \). It can be shown that the map \( \Phi \) defines a local coordinate chart, called period coordinates, for the stratum \( \mathcal{Q}(1^{4g+n-4}, -1^n) \). Pulling back the Lebesgue measure on \( \mathbb{C}^k \) with respect to \( \Phi \) yields a measure \( \nu \) on \( \mathcal{Q}(1^{4g+n-4}, -1^n) \), which is quickly verified to be independent of the basis \( \{\gamma_i\} \) and invariant under the action of \( \text{SL}_2(\mathbb{R}) \).

As stated, the volume \( \nu(\mathcal{Q}(1^{4g+n-4}, -1^n)) \) will be infinite since \((X, cq) \in \mathcal{Q}(1^{4g+n-4}, -1^n) \) for any \((X, \omega) \in \mathcal{Q}(1^{4g+n-4}, -1^n) \) and constant \( c \in \mathbb{C} \). To remedy this issue, let \( \mathcal{Q}_0(1^{4g+n-4}, -1^n) \subset \mathcal{Q}(1^{4g+n-4}, -1^n) \) denote the moduli space of pairs \((X, q) \in \mathcal{Q}(1^{4g+n-4}, -1^n) \) such that \( \int_X |q| = \frac{1}{2} \); this is the hypersurface of the stratum \( \mathcal{Q}(1^{4g+n-4}, -1^n) \) consisting of those differentials \((X, q) \), for which \( q \) has area \( \frac{1}{2} \). Let \( \nu_0 \) denote the measure induced by \( \nu \) on \( \mathcal{Q}_0(1^{4g+n-4}, -1^n) \).

It was shown independently by Masur [29] and Veech [43] that \( \nu_0 \) is ergodic on \( \mathcal{Q}_0(1^{4g+n-4}, -1^n) \) under the action of \( \text{SL}_2(\mathbb{R}) \), and that the volume \( \nu_0(\mathcal{Q}_0(1^{4g+n-4}, -1^n)) \) is finite. The latter quantity is called the Masur–Veech volume of the principal stratum \( \mathcal{Q}(1^{4g+n-4}, -1^n) \). Since the complement of this stratum in \( \mathcal{Q}_{g,n} \) has positive codimension, we also write \( \text{Vol} \mathcal{Q}_{g,n} = \nu_0(\mathcal{Q}_0(1^{4g+n-4}, -1^n)) \).

Next we recall the definition of the Siegel–Veech constant that will be relevant to us in this paper. For any \((X, q) \in \mathcal{Q}_{g,n}, |q| \) defines metric on \( X \) that is flat away from a finite set of conical singularities, also called saddles, which constitute the zeros and poles of \( q \); this makes \((X, q) \) into a flat surface. A saddle connection on \((X, q) \) is a geodesic on \( X \) connecting two saddles with no saddle in its interior, and a maximal cylinder on \((X, q) \) is a Euclidean cylinder isometrically embedded in \( X \) whose two boundaries are both unions of saddle connections.

The area Siegel–Veech constant concerns the enumeration of maximal cylinders, weighted by area, on a typical flat surface in \( \mathcal{Q}_{g,n} \). More specifically, for any real number \( L > 0 \), let
\[ N_{\text{area}}(L) = N_{\text{area}}(L; (X, q)) = \sum_{w(C) \leq L} A(C), \]

where \( C \) ranges over all maximal cylinders of \((X, q)\) of circumference at most \( L \), and \( A(X) \) and \( A(C) \) denote the areas of \( X \) and \( C \), respectively. Viewing maximal cylinders as “thickenings” of closed geodesics, one might interpret \( N_{\text{area}}(L) \) as a count for closed geodesics weighted by “thickness.”

It was shown by Eskin and Masur [15] that, for a typical (by which we mean full measure subset with respect to the Masur–Veech volume) flat surface \((X, q) \in Q_{g,n}\), the quantity \( N_{\text{area}}(L) \) grows quadratically in \( L \) with asymptotics

\[
N_{\text{area}}(L; (X, q)) \approx \frac{\pi L}{2 c_{\text{area}}(Q_{g,n})},
\]

where the constant \( c_{\text{area}}(Q_{g,n}) \) is independent of the choice typical flat surface \((X, q) \in Q_{g,n}\). This constant falls into a class of quantities known as Siegel–Veech constants, which had been studied in the earlier work [44] of Veech; \( c_{\text{area}} \) is specifically known as an area Siegel–Veech constant.

In addition to enumerating geometric phenomena, area Siegel–Veech constants also contain information about dynamics on the moduli space \( Q_{g,n} \). One example is through the Teichmüller geodesic flow on \( Q_{g,n} \), which is the action of the diagonal one-parameter subgroup \([e^t \begin{smallmatrix} \gamma & 0 \\ 0 & e^{-t} \end{smallmatrix}] \subset SL_2(\mathbb{R})\) on this moduli space. This flow lifts to the Hodge bundle over \( Q(1^{4g+n-4}, -1^n) \) and, by Oseledets theorem, one can associate this flow with \( 2g \) Lyapunov exponents, denoted by \( \lambda(1(Q_{g,n}) \geq \lambda(2(Q_{g,n}) \geq \cdots \geq \lambda(2g(Q_{g,n}) \). These exponents are symmetric with respect to 0, that is, \( \lambda_i + \lambda_{2g-i+1} = 0 \) for each integer \( i \in [1, 2g] \).

It was shown as part (a) of Theorem 2 in the work [14] of Eskin–Kontsevich–Zorich that the sum of the first \( g \) (namely, the nonnegative) Lyapunov exponents associated with the principal stratum can be expressed explicitly in terms of the area Siegel–Veech constant \( c_{\text{area}}(Q_{g,n}) \) through

\[
\sum_{i=1}^{g} \lambda_i(Q_{g,n}) = \frac{1}{24} \left( \frac{20g}{3} - \frac{4n}{3} - \frac{20}{3} \right) + \frac{\pi^2}{3} c_{\text{area}}(Q_{g,n}).
\]

Thus, knowledge of the area Siegel–Veech constant \( c_{\text{area}}(Q_{g,n}) \) enables one to evaluate the sum of the associated positive Lyapunov exponents of the Teichmüller geodesic flow. An analog of this result holds for all connected components of any stratum of \( Q_{g,n} \) [14].
1.3 Volume asymptotics

Although the finiteness of the Masur–Veech volumes was established in 1982 [29,43], it was nearly two decades until mathematicians produced general ways of determining them explicitly. In the apparently simpler setting of moduli spaces of holomorphic differentials, one of the earlier exact volume evaluations was due to Zorich [53], who found them for certain strata of low genus. Later, Eskin and Okounkov [17] and Eskin et al. [19] proposed a way of evaluating the volume of any (connected component of a) stratum in the moduli space of holomorphic differentials, through an algorithm based on asymptotic Hurwitz theory. The more recent work of Chen et al. [9] provides an alternative way to access these strata volumes through a recursion.

Similarly, although the work of Eskin and Masur [15] showed that the limits (1.3) defining the area Siegel–Veech constants exist, it did not indicate how to determine them. Again in the case of holomorphic differentials, this was done by Eskin et al. [16] (combined with a result of Vorobets [46]), who expressed these constants as combinatorial sums involving the Masur–Veech strata volumes.

As with the intersection numbers (1.1), it is widely believed that these volumes and constants are quite intricate, and that they do not in general admit simple closed form expressions. Still, based on numerical data tabulated from implementing the above results on explicit strata, Eskin and Zorich [20] posed precise predictions for how the Masur–Veech volumes and Siegel–Veech constants associated with arbitrary strata of holomorphic differentials should simplify in the large genus limit. These were first established for the principal and minimal strata by Chen et al. [8] and Sauvaget [41], respectively. They were then confirmed in general in [1,2] through a combinatorial analysis of the Eskin–Okounkov algorithm [17] and of the expressions of Eskin et al. [16]. Soon later, an independent and very different, algebro-geometric proof of these predictions was posed by Chen et al. [9]. More recently, through both combinatorial and algebro-geometric methods, Sauvaget [40] provided an all-order genus expansion of the Masur–Veech volumes of any stratum of holomorphic differentials.

Our understanding of these limiting phenomena in the context of quadratic differentials, which will be of interest to us in this paper, is more limited. In this setting, Goujard [23] provided expressions for Siegel–Veech constants in terms of strata volumes of quadratic differentials. Earlier work by Eskin and Okounkov [18] also proposed an algorithm (again based on asymptotic Hurwitz theory) that finds the volume of any such stratum, but this algorithm is considerably more intricate than its counterpart for holomorphic differentials. Still, it was effectively implemented by Goujard [24] to obtain numerical data for volumes of strata of dimension at most 11.
For the principal stratum, other (and sometimes more efficient) methods of volume evaluation exist. These include through expressions in terms of correlators by Mirzakhani [32,35]; lattice point enumeration of Strebel–Jenkins differentials by Athreya et al. [6] and Delecroix et al. [11]; topological recursions of Andersen et al. [4]; and intersection numbers on compactified Hodge bundles by Chen et al. [7]. The latter framework through intersection numbers was reformulated as an explicit, bivariate recursion for these principal strata volumes by Kazarian [25].

Based on empirical data obtained by implementing these results, as well as more elaborate geometrical, analytical, and dynamical considerations, precise predictions for the large genus asymptotic behavior of the Masur–Veech volume and Siegel–Veech constant associated with any stratum of quadratic differentials were proposed recently in [3]. Until the present paper, these predictions had not been established for any stratum. Let us state them in the case of the principal stratum.

For the volumes, the specialization of Conjecture 1 of [3] to the principal stratum (see also Conjecture 1.10 of [11] for the $n = 0$ case) states

$$\text{Vol} \, Q_{g,n} = \pi^{-1} 2^{n+2} \left( \frac{8}{3} \right)^{4g+n-4} \left( 1 + o(1) \right)$$

as $g$ tends to $\infty$, if $n \leq \log g$. \hspace{1cm} (1.5)

For the Siegel–Veech constant, the specialization of Conjecture 2 of [3] to the principal stratum states that

$$c_{\text{area}}(Q_{g,n}) = \frac{1}{4} + o(1), \quad \text{as } g \text{ tends to } \infty, \text{ if } n \leq \log g. \hspace{1cm} (1.6)$$

As an application of (1.2), we establish (1.5) for $20n \leq \log g$ as Theorem 1.7 below and (1.6) for $n$ fixed as Theorem 1.8 below; with further effort, these constraints on $n$ can likely be improved through similar methods to allow $n = g^c$ for some constant $c > 0$, but we decided not to pursue this here. This essentially confirms the predictions of [3] in the case of the principal stratum.

To establish the volume asymptotic (1.5), we begin with an expression of Delecroix et al. [11] for $\text{Vol} \, Q_{g,n}$ as a sum involving the correlators (1.1). Although (1.2) enables one to approximate many of these intersection numbers explicitly, the sum remains quite intricate; it is indexed by stable graphs of genus $g$ with $n$ marked points, the number of which grows exponentially in $g$. It was predicted in [11] that the dominant contribution to this sum arises from graphs with one vertex. A similar phenomenon was observed in [1,2] for Masur–Veech volumes and Siegel–Veech constants in strata of holomorphic differentials, where analogously large sums were dominated by a single term.
However, the situation here appears to be considerably more elaborate than in the holomorphic setting. Indeed, even the leading order contribution to $\text{Vol} \mathcal{Q}_{g,n}$ is not quite immediate to evaluate, as it involves an infinite sum of special functions (namely, particular deformations of multi-variate harmonic sums given by Definition 6.1 below). A detailed, but heuristic, analysis of this sum was performed in Appendix D of [11], leading to an exact prediction for its value (and to the constant prefactor on the right side of (1.5)). Thus, we must first establish this prediction, which we do in Sects. 6 and 7 (see Propositions 6.4 and 7.2 below) using the complex analytic saddle point method on certain generating series. This yields the leading order contribution to $\text{Vol} \mathcal{Q}_{g,n}$ coming from stable graphs with one vertex.

We then show that graphs on two or more vertices do not asymptotically contribute to this volume, which is partly facilitated by the combinatorial analysis implemented in [1,2] to establish similar phenomena in the holomorphic setting; this leads to the proof of (1.5). Given this, (1.6) is deduced using an identity of Goujard [23] for $c_{\text{area}}(\mathcal{Q}_{g,n})$ in terms of principal strata volumes.

Let us conclude this section by mentioning that volume asymptotics of moduli spaces can often be used to deduce quantitative geometric properties for random surfaces of high genus. For instance, in the context of random hyperbolic surfaces, large genus asymptotics for Weil–Petersson volumes were used by Mirzakhani [34] to estimate Cheeger constants, diameters, and systole lengths; by Mirzakhani and Petri [36] to establish Poisson limiting results for extrema of the length spectrum; and by Gilmore et al. [22] and Thomas [42] to prove delocalization results for Laplacian eigenfunctions. Similarly, large genus asymptotics for Masur–Veech strata volumes of holomorphic differentials were used by Masur et al. [30] to bound the covering radius of a typical translation surface.

In our context, each summand in the expression of $\text{Vol} \mathcal{Q}_{g,n}$ as a weighted sum over stable graphs can be interpreted geometrically, as a multiple of the probability of a random flat surface exhibiting a certain cylinder decomposition or alternatively as a count of closed geodesics on a typical hyperbolic surface with given multi-curve type (see the works of Delecroix et al. [11] and Arana-Herrera [5] for independent and different proofs of this equivalence). Using this, the recent work of Delecroix et al. [10] combines our results with detailed geometric, combinatorial, and analytic considerations to establish precise probabilistic limit theorems for multi-curve statistics in random flat and hyperbolic surfaces.

1.4 Results

We begin by stating our results on the intersection numbers (1.1). For any integer $n \geq 1$ and $n$-tuple $d = (d_1, d_2, \ldots, d_n) \in \mathbb{Z}_{\geq 0}^n$ of nonnegative inte-
gers, recall that we set $|d| = \sum_{i=1}^{n} d_i$. The following definition provides a normalization for the correlators (1.1) according to (1.2).

**Definition 1.1** Let $g \geq 0$ and $n \geq 1$ denote integers. For any $d = (d_1, d_2, \ldots, d_n) \in \mathbb{Z}_{\geq 0}^n$ such that $|d| = 3g + n - 3$, define the normalized intersection number $\langle d \rangle = \langle d \rangle_{g,n} = \langle d_1, d_2, \ldots, d_n \rangle = \langle d_1, d_2, \ldots, d_n \rangle_{g,n}$ by

$$
\langle d \rangle = \frac{24^g g! \prod_{i=1}^{n} (2d_i + 1)!!}{(2|d| + 1)!!} \int_{\mathcal{M}_{g,n}} \prod_{i=1}^{n} \psi_{d_i}^d.
$$

(1.7)

Before providing an asymptotic result for these normalized intersection numbers, we first state the following exponential bound on $\langle d \rangle_{g,n}$ that holds uniformly in the genus $g$; in particular, it also holds for small values of $g$. This bound will be established in Sect. 3 below.

**Proposition 1.2** Let $n \in \mathbb{Z}_{\geq 1}$ and $d \in \mathbb{Z}_{\geq 0}^n$ satisfy $|d| = 3g + n - 3$, for some $g \in \mathbb{Z}_{\geq 0}$. Then

$$
\langle d \rangle_{g,n} \leq \left( \frac{3}{2} \right)^{n-1}.
$$

(1.8)

**Remark 1.3** Up to a factor of $n^{-1/2}$ (which we have not attempted to optimize), the exponential estimate (1.8) is sharp. Indeed, letting $(1^{n-3}, 0^3)$ denote the $n$-tuple consisting of $n - 3$ parts equal to one and 3 parts equal to zero, it can quickly be seen as a consequence of (2.11), (2.1), and (2.7) below that

$$
\langle 1^{n-3}, 0^3 \rangle_{0,n} = 3^{n-3} \frac{(n-3)!}{(2n-5)!!} = \frac{2\pi^{1/2}}{9n^{1/2}} \left( \frac{3}{2} \right)^{n-1} \left( 1 + o(1) \right) \text{ as } n \text{ tends to } \infty.
$$

Although Proposition 1.2 will follow from the Virasoro constraints by a reasonably direct induction, it has to the best of our knowledge not appeared before in the literature. In addition to being useful in our proof of the asymptotic (1.2), Proposition 1.2 provides a general estimate on intersection numbers in situations where (1.2) is no longer valid. The latter point will be beneficial in our application to the asymptotic analysis of $\mathrm{Vol}_{Q_{g,n}}$. Indeed, although the dominant contribution to this volume comes from intersection numbers satisfying (1.2), the exact expression for it also involves correlators outside this regime. The uniform bound given by Proposition 1.2 is helpful in showing that such terms are indeed negligible in the large genus limit.
Next, we state an asymptotic result for the intersection numbers (1.1), which is essentially (1.2) under the restriction $n = o(g^{1/2})$. The latter is made more precise through the following definition.

**Definition 1.4** For any real number $\varepsilon > 0$ and integer $g > \varepsilon^{-2}$, define

$$\Delta(g; \varepsilon) = \{d = (d_1, d_2, \ldots, d_n) \in \mathbb{Z}_{\geq 0}^n : |d| = 3g + n - 3, n < \varepsilon g^{1/2}\}.$$

The following theorem, which will be established in Sect. 4.1 below, implies (1.2) and essentially states $\langle d \rangle_{g,n} \approx 1$, as $g$ tends to $\infty$, if $n = o(g^{1/2})$.

**Theorem 1.5** We have that

$$\lim_{\varepsilon \to 0} \left( \lim_{g \to \infty} \max_{d \in \Delta(g; \varepsilon)} |\langle d \rangle - 1| \right) = 0.$$

**Remark 1.6** The approximation $\langle d \rangle_{g,n} \approx 1$ is no longer true for large $g$ if $n = \mathcal{O}(g^{1/2})$. Indeed, letting $(3g - 2, 1^{n-1})$ denote the $n$-tuple consisting of one part equal to $3g - 2$ and $n - 1$ parts equal to 1, it can be quickly seen as a consequence of (2.11) and Lemma 2.7 below that

$$\langle 3g - 2, 1^{n-1} \rangle_{g,n} = \exp \left( \frac{n^2}{12g} \right) (1 + o(1)) \text{ if } n = \mathcal{O}(g^{1/2}) \text{ and } g \text{ tends to } \infty.$$

We next describe our results concerning the large genus limits for Masur–Veech volumes and Siegel–Veech constants in the principal stratum of quadratic differentials. The following theorem, which will be established in Sect. 8.1 below, provides the volumes asymptotics. In particular, it establishes (1.5) under the slightly stronger hypothesis that $20n \leq \log g$; it is likely that one can further improve this dependence between $n$ and $g$, but we will not pursue this here.

**Theorem 1.7** We have that

$$\lim_{g \to \infty} \left( \max_{20n \leq \log g} \left| 2^{-n} \left( \frac{8}{3} \right)^{4-4g-n} \frac{\text{Vol } \mathcal{Q}_{g,n} - 4}{\pi} \right| \right) = 0.$$

The next theorem, which will be established in Sect. 11.1 below, verifies the large genus asymptotic (1.6) for the Siegel–Veech constant $c_{\text{area}}(\mathcal{Q}_{g,n})$, assuming that $n$ is fixed as $g$ tends to $\infty$. As before, it is likely that one can strengthen this statement using similar methods to allow $n$ to grow slowly with $g$, but we will not address this improvement here.
**Theorem 1.8** For any fixed integer \( n \geq 0 \), we have that

\[
\lim_{g \to \infty} c_{\text{area}}(Q_{g,n}) = \frac{1}{4}.
\]

We conclude with the following corollary that approximates for the sum of Lyapunov exponents of the Teichmüller geodesic flow on \( Q_{g,n} \); it follows as a consequence of Lemma 1.8 and (1.4) (see also equation (3) of [3]).

**Corollary 1.9** For any fixed integer \( n \geq 0 \), we have that

\[
\lim_{g \to \infty} \left( \sum_{i=1}^{g} \lambda_i(Q_{g,n}) - \frac{5g}{18} \right) = \frac{\pi^2}{12} - \frac{n}{18} - \frac{5}{18}.
\]

The remainder of this paper is organized as follows. After recalling several preliminary results in Sect. 2, we establish general bound on intersection numbers given by Proposition 1.2 in Sect. 3. Next, we establish the limiting result Theorem 1.5 for these intersection numbers in Sects. 4 and 5. In Sects. 6 and 7, we introduce and provide asymptotics for certain types of multi-variate harmonic sums that appear in the Masur–Veech volumes of principal strata. We then analyze the large genus asymptotics for these volumes, proving Theorem 1.7 in Sects. 8, 9 and 10. We conclude with the proof of the Siegel–Veech asymptotic result given by Theorem 1.8 in Sect. 11.

## 2 Miscellaneous preliminaries

In this section we state several (mostly known) results that will be used throughout this paper. We begin in Sect. 2.1 with several combinatorial estimates, and next we will recall certain recursions and bounds on intersection numbers in Sect. 2.2. Then, we recall from [11] an expression for \( \text{Vol} Q_{g,n} \) through a weighted sum over stable graphs.

### 2.1 Estimates

In this section we recall some notation, identities, and estimates that will be useful later in this paper. Throughout this article, for any two functions \( F_1, F_2 : \mathbb{Z} \to \mathbb{R} \) such that \( F_2(k) \) is nonzero for sufficiently large \( k \), we write \( F_1 \sim F_2 \) if \( \lim_{k \to \infty} F_1(k)F_2(k)^{-1} = 1 \).
First recall for any integers $A, B \geq 0$ that

$$(2A + 1)! = \frac{(2A + 1)!}{2^A A!}; \quad \sum_{k=0}^{A} \binom{2A + 2}{2k + 1} = 2^{2A+1};$$

$$\frac{(2A + 2B + 1)!}{(2A + 1)!(2B + 1)!} = \frac{1}{2(A + B + 1)} \binom{2A + 2B + 2}{2A + 1}.$$  \hspace{1cm} (2.1)

Next, for any integers $m \geq 1$ and $N \geq 0$, let $C_m(N)$ denote the set of compositions of $N$ of length $k$, that is, the set of $m$-tuples $a = (a_1, a_2, \ldots, a_m) \in \mathbb{Z}_{\geq 1}^m$ of positive integers such that $|a| = N$. Further let $K_m(N)$ denote the set of nonnegative compositions of $N$ of length $k$, that is, the set of $m$-tuples $a = (a_1, a_2, \ldots, a_m) \in \mathbb{Z}_{\geq 0}^m$ of nonnegative integers such that $|a| = N$. Observe in particular that

$$|C_m(N)| = \binom{N - 1}{m - 1}; \quad |K_m(N)| = \binom{N + m - 1}{m - 1}.$$  \hspace{1cm} (2.2)

The following estimate on products of factorials will be used in Sect. 10.2 below.

**Lemma 2.1** Fix integers $m \geq 1$ and $A, B, C \geq 0$. Then

$$\sum_{i=1}^{m} \prod_{i=1}^{m} (a_i + b_i + c_i - 2)! \leq 2^{12m+9} (A + B + C - 3m + 1)!,$$

where on the left side of (2.3), we sum over all triples $(a, b, c)$ of nonnegative compositions $a = (a_1, a_2, \ldots, a_m) \in K_m(A), b = (b_1, b_2, \ldots, b_m) \in K_m(B),$ and $c = (c_1, c_2, \ldots, c_m) \in K_m(C),$ such that $a_i + b_i + c_i \geq 3$, for each $i \in [1, m]$.

Observe here that the factor $(A + B + C - 3m + 1)!$ appearing on the right side of (2.3) occurs if $a_1 + b_1 + c_1 = A + B + C - 3m + 3$ and $a_i + b_i + c_i = 3,$ for each $i \in [2, m]$. Thus, Lemma 2.1 indicates that the sum over all compositions of $\prod_{i=1}^{m} (a_i + b_i + c_i - 2)!$ bounded by this single summand, up to a factor $2^{12m+9}$.

To establish Lemma 2.1, we will use the following lemma from [1] that estimates a similar quantity as on the left side of (2.3), but without the constraint that each $a_i + b_i + c_i \geq 3$.

**Lemma 2.2** ([1, Lemma 2.7]) Fix integers $m \geq 1$ and $A, B, C \geq 0$. Then

$$\sum_{a \in K_m(A)} \sum_{a \in K_m(A)} \sum_{a \in K_m(A)} \prod_{i=1}^{m} (a_i + b_i + c_i + 1)! \leq 2^{8m+9} (A + B + C + 1)!,$$
where we have denoted \( a = (a_1, a_2, \ldots, a_m), \ b = (b_1, b_2, \ldots, b_m), \) and \( c = (c_1, c_2, \ldots, c_m). \)

Given Lemma 2.2, we can now establish Lemma 2.1.

**Proof of Lemma 2.1** For any integer \( n \geq 1 \) and sequences \( y = (y_1, y_2, \ldots, y_n) = \mathbb{Z}^n \) and \( z = (z_1, z_2, \ldots, z_n) \in \mathbb{Z}^n \), we write \( y \geq z \) if \( y_i \geq z_i \) for each \( i \in [1, n] \). Then, since each \( a_i + b_i + c_i \geq 3 \) on the left side of (2.3), there must exist for each \( i \in [1, m] \) some nonnegative composition \( x(i) \in K_3(3) \) such that \( (a_i, b_i, c_i) \geq x(i). \) For any \( m \)-tuple of nonnegative compositions \( X = (x(1), x(2), \ldots, x(m)) \in K_3(3)^m \), let \( \mathcal{Y}(X) = \mathcal{Y}(A, B, C(X)) \subseteq K_m(A) \times K_m(B) \times K_m(C) \) denote the subset of triples \( (a, b, c) \in K_m(A) \times K_m(B) \times K_m(C) \) such that \( (a_i, b_i, c_i) \geq x(i), \) for each \( i \in [1, m]. \)

Under this notation, the left side of (2.3) is bounded above by

\[
\sum_{X \in K_3(3)^m} \sum_{(a, b, c) \in \mathcal{Y}(X)} \prod_{i=1}^{m} (a_i + b_i + c_i - 2)! \leq 2^{4m} \max_{X \in K_3(3)^m} \sum_{(a, b, c) \in \mathcal{Y}(X)} \prod_{i=1}^{m} (a_i + b_i + c_i - 2)!, \tag{2.4}
\]

where in the last inequality we used the fact that \( |K_3(3)| = (\frac{5}{2}) = 10 < 24. \)

So, fix some \( X = (x(1), x(2), \ldots, x(m)) \in K_3(3)^m; \) denote \( x(j) = (x_1(j), x_2(j), x_3(j)) \) for each integer \( j \in [1, m]; \) and set \( X_i = \sum_{j=1}^{m} x_i(j), \) for each \( i \in \{1, 2, 3\}. \) Then, defining

\[
a'_i = a_i - x_1(i); \quad b'_i = b_i - x_2(i); \quad c'_i = c_i - x_3(i),
\]

and the nonnegative compositions \( a' = (a'_1, a'_2, \ldots, a'_m) \in K_m(A - X_1), \ b' = (b'_1, b'_2, \ldots, b'_m) \in K_m(B - X_2), \) and \( c' = (c'_1, c'_2, \ldots, c'_m) \in K_m(C - X_3), \) we obtain

\[
\sum_{(a, b, c) \in \mathcal{Y}(X)} \prod_{i=1}^{m} (a_i + b_i + c_i - 2)! = \sum_{a' \in K_m(A - X_1)} \sum_{b' \in K_m(B - X_2)} \sum_{c' \in K_m(C - X_3)} \prod_{i=1}^{m} (a'_i + b'_i + c'_i + 1)!. \tag{2.5}
\]
By Lemma 2.2 and the fact that $X_1 + X_2 + X_3 = 3m$, we deduce that

$$\sum_{a' \in K_m(A-X_1)} \sum_{b' \in K_m(B-X_2)} \sum_{c' \in K_m(C-X_3)} \prod_{i=1}^m (a'_i + b'_i + c'_i + 1)! \leq 2^{8m+9} (A + B + C - 3m + 1)!,$$

which together with (2.4) and (2.5) implies the lemma. $\square$

We will also require the following estimate on products of multinomial coefficients. This result is known, but we will provide its proof for completeness.

**Lemma 2.3** Let $n$ and $r$ be positive integers; also let $\{A_i\}$ and $\{A_i,j\}$, for $1 \leq i \leq n$ and $1 \leq j \leq r$, be sets of nonnegative integers such that $\sum_{j=1}^r A_{i,j} = A_i$ for each $i$. Then

$$\prod_{i=1}^n \left( A_i, A_{i,2}, \ldots, A_{i,r} \right) \leq \left( \sum_{i=1}^n A_{i,1}, \sum_{i=1}^n A_{i,2}, \ldots, \sum_{i=1}^n A_{i,r} \right).$$

(2.6)

**Proof** For each $0 \leq k \leq n$, let $T_k = \sum_{j=1}^k A_j$ (with $T_0 = 0$). Define the sets $S = \{1, 2, \ldots, T_n\}$ and $S_i = \{T_{i-1} + 1, T_{i-1} + 2, \ldots, T_i\}$ for each $1 \leq i \leq n$. Then, the left side of (2.6) counts the number of ways to partition each of the $S_i$ into $r$ mutually disjoint subsets $S_{i,1}, S_{i,2}, \ldots, S_{i,r}$ consisting of $A_{i,1}, A_{i,2}, \ldots, A_{i,r}$ elements, respectively. Similarly, the right side of (2.6) counts the number of ways to partition $S$ into $r$ mutually disjoint subsets $S^{(1)}, S^{(2)}, \ldots, S^{(r)}$ consisting of $\sum_{i=1}^n A_{i,1}, \sum_{i=1}^n A_{i,2}, \ldots, \sum_{i=1}^n A_{i,r}$ elements, respectively.

Any partition of the former type injectively gives rise to a partition of the latter type by setting $S^{(k)} = \bigcup_{i=1}^n S_{i,k}$ for each $1 \leq k \leq r$. Thus, the left side of (2.6) is at most equal to the right side of (2.6). $\square$

Next we state and establish estimates on exponentials and factorials that will later be useful to us. First, we have that the limit

$$k! \sim (2\pi k)^{1/2} \left( \frac{k}{e} \right)^k \quad \text{as } k \text{ tends to } \infty,$$

(2.7)

and also the finite $k \geq 1$ bound

$$2e^{-k}k^{k+1/2} \leq k! \leq 3e^{-k}k^{k+1/2}.$$  

(2.8)

We also have the following lemma, which estimates the error in a truncation for the Taylor expansion of $e^R$. 

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Lemma 2.4 Let $R, \delta > 0$ denote two real numbers, and let $K \in \mathbb{Z}$ denote an integer such that $K > (1 + 2\delta)R$. Then

$$\left| e^R - \sum_{j=0}^{K} \frac{R^j}{j!} \right| < \delta^{-1}(1 + \delta)^{-\delta R} e^R.$$ 

Proof By a Taylor expansion and the fact that $j! \geq \left(\frac{j}{e}\right)^j$ (by (2.8)), we have

$$\left| e^R - \sum_{j=0}^{K} \frac{R^j}{j!} \right| = \sum_{j=K+1}^{\infty} \frac{R^j}{j!} \leq \sum_{j=K+1}^{\infty} \left( \frac{e^R}{j} \right)^j. \quad (2.9)$$

Next, observe that $(\frac{e^R}{j+1})^{j+1} \leq (1 + \delta)^{-1}(\frac{e^R}{j})^j$ for $j \geq (1 + \delta)R$, since $(1 - \frac{1}{j+1})^{j+1} e \leq 1$. Thus, since $(\frac{e^R}{j})^j \leq e^R$ for any $j > 0$ and $K > (1 + 2\delta)R$, it follows that

$$\left( \frac{e^R}{j} \right)^j \leq \left( \frac{e^R}{j - R(1 + \delta)} \right)^{j-R(1+\delta)} (1 + \delta)^{R(1+\delta)-j} \leq e^R(1 + \delta)^{R(1+\delta)-j} \leq e^R(1 + \delta)^{K-j-\delta R}$$

whenever $j \geq K$.

Hence,

$$\sum_{j=K+1}^{\infty} \left( \frac{e^R}{j} \right)^j \leq e^R(1 + \delta)^{-\delta R} \sum_{j=K+1}^{\infty} (1 + \delta)^{K-j} \leq \delta^{-1}(1 + \delta)^{-\delta R} e^R,$$

which with (2.9) yields the lemma. \hfill $\Box$

2.2 Recursions and identities for intersection numbers

The proofs of Proposition 1.2 and Theorem 1.5 are based on an analysis of recursive relations (resulting from the Witten–Kontsevich theorem [26,47]) determining the $\langle d \rangle$. These are summarized through the following lemma.

Lemma 2.5 ([13,45,47]) Fix integers $g \geq 0$ and $n \geq 1$ such that $3g + n \geq 3$, and an $n$-tuple $d = (d_1, d_2, \ldots, d_n) \in \mathbb{Z}_{\geq 0}^n$.

(1) Initial data: We have $\langle 0, 0, 0 \rangle_{0,3} = 1 = \langle 1 \rangle_{0,1}.$

(2) String equation: If $|d| = 3g + n - 3$, then

$$\langle d, 0 \rangle_{g,n+1} = \frac{1}{6g + 2n - 3} \sum_{j=1}^{n} (2d_j + 1)(d_j - 1, d \setminus \{d_j\})_{g,n}. \quad (2.10)$$
(3) Dilation equation: If $|\mathbf{d}| = 3g + n - 3$, then

$$
\langle \mathbf{d}, 1 \rangle_{g,n+1} = \frac{6g + 3n - 6}{6g + 2n - 3} \langle \mathbf{d} \rangle_{g,n}.
$$

(2.11)

(4) Virasoro constraints: Fix an integer $k \geq -1$. If $|\mathbf{d}| = 3g + n - k - 3$, then

$$
\langle k + 1, \mathbf{d} \rangle_{g,n+1} = \frac{1}{6g + 2n - 3} \sum_{j=1}^{n} (2d_j + 1) \langle d_j + k, \mathbf{d} \setminus \{d_j\} \rangle_{g,n}
$$

$$
+ \frac{12g}{(6g + 2n - 3)(6g + 2n - 5)} \sum_{r+s=k-1, r,s \geq 0} \langle r, s, \mathbf{d} \rangle_{g-1,n+2}
$$

$$
+ \frac{1}{2} \sum_{r+s=k-1} \sum_{|I \cup J|=|\{1,2,\ldots,n\}|} \frac{g!}{g'!g''!} \times \frac{(6g' + 2n' - 3)!!(6g'' + 2n'' - 3)!!}{(6g + 2n - 3)!!}
$$

$$
\times \langle r, \mathbf{d}|_I \rangle_{g',n'+1} \langle s, \mathbf{d}|_J \rangle_{g'',n''+1},
$$

(2.12)

where in (2.12) we have set $|I| = n'$, $|J| = n''$, and

$$
\mathbf{d}_S = (d_s)_{s \in S}; \quad |\mathbf{d}_I| + r = 3g' + n' - 2; \quad |\mathbf{d}_J| + s = 3g'' + n'' - 2,
$$

(2.13)

which in particular satisfy $n' + n'' = n$ and $g' + g'' = g$.

Moreover, all normalized intersection numbers $\langle \mathbf{d} \rangle$ are determined by the above four properties.

**Remark 2.6** It is more common to phrase the results of Lemma 2.5 in terms of the correlators from (1.1). Under this notation, the initial data takes the form $\langle \tau_0^3 \rangle = 1$ and $\langle \tau_1 \rangle = \frac{1}{24}$; see equations (2.42) and (2.46) of [47] for the former and latter, respectively. The analog of (2.12), sometimes referred to as the Dijkgraaf–Verlinde–Verlinde formulation of the Virasoro constraints given originally as equation (7.13) of [45] (and, in our form, as equation (7.27) of [13]), states.
\[
\langle \tau_{k+1} \prod_{i=1}^{n} \tau_{d_i} \rangle = \frac{1}{(2k+3)!!} \left( \sum_{j=1}^{n} \frac{(2k+2d_j+1)!!}{(2d_j-1)!!} \right) \langle \tau_{d_j+k} \prod_{1 \leq i \leq n, i \neq j} \tau_{d_i} \rangle \\
+ \frac{1}{2} \sum_{r+s=k-1}^{r,s \geq 0} (2r+1)!!(2s+1)!! \langle \tau_{r} \tau_{s} \prod_{i=1}^{n} \tau_{d_i} \rangle \\
+ \frac{1}{2} \sum_{r+s=k-1}^{r,s \geq 0} (2r+1)!!(2s+1)!! \\
\times \sum_{I \cup J = \{1, 2, \ldots, n\}} \left( \langle \prod_{i \in I} \tau_{d_i} \rangle \langle \tau_{s} \prod_{j \in J} \tau_{d_j} \rangle \right) \). 
\]

The string and dilation equations (which are also the \( k = -1 \) and \( k = 0 \) special cases of the Virasoro constraints, respectively) are given by

\[
\langle \tau_{0} \prod_{i=1}^{n} \tau_{d_i} \rangle = \sum_{j=1}^{n} \langle \tau_{d_j-1} \prod_{1 \leq i \leq n, i \neq j} \tau_{d_i} \rangle; \quad \langle \tau_{1} \prod_{i=1}^{n} \tau_{d_i} \rangle = (2g + n - 2) \langle \prod_{i=1}^{n} \tau_{d_i} \rangle.
\]

We refer to equations (2.40) and (2.41) of [47] for the former and latter, respectively.

Next we recall certain identities and asymptotics for the intersection numbers \( \langle d \rangle \) in the cases \( n \in \{1, 2\} \). In the case \( n = 1 \), an exact identity for these numbers was first predicted below equation (2.26) in [47] and then proven as result of Theorem 1.2 of [26].

**Lemma 2.7** ([26, Theorem 1.2]) For any integer \( g \geq 1 \), we have \( \langle 3g-2 \rangle_{g,1} = 1 \).

In the case \( n = 2 \), the large genus asymptotics for the intersection numbers \( \langle k, 3g-k-1 \rangle_{g,2} \) was provided by Proposition 4.1 of [11] (based on an exact identity given below equation (8) of [50]).

**Lemma 2.8** ([11, Proposition 4.1]) For any integers \( g \geq 1 \) and \( 0 \leq k \leq 3g-1 \), we have

\[
\frac{6g-3}{6g-1} \leq \langle k, 3g-k-1 \rangle_{g,2} \leq 1.
\]
2.3 Stable graphs and volumes

For the proof of Theorem 1.7, we will require an expression from [11] for \( \text{Vol} \mathcal{Q}_{g,n} \) in terms of the normalized intersection numbers \( \langle d \rangle \) from Definition 1.1, through a weighted sum over stable graphs. So, we begin by recalling the latter.

**Definition 2.9** Fix integers \( g, n \geq 0 \). A graph \( \Gamma \) with \( n \) legs and a genus decoration consists of the following.

1. A finite, nonempty set \( V = V(\Gamma) \) of vertices
2. A finite set \( H = H(\Gamma) \) of half-edges
3. A map \( \alpha : H \to V \) associating each half-edge with a vertex
4. An \( n \)-element subset \( L = L(\Gamma) \subseteq H(\Gamma) \) of legs
5. A bijection \( \lambda : L \to \{1, 2, \ldots, n\} \) that labels each leg of \( \Gamma \)
6. An involution \( i : H \to H \), which fixes each element in \( L \) but none in \( H \setminus L \)
7. A genus decoration, which is a set \( g = (g_v)_{v \in V} \) of nonnegative integers, one associated with each vertex \( v \in V \)

We view any pair \( e = (h, h') \in (H \setminus L) \times (H \setminus L) \) of distinct half-edges (which are not legs) such that \( i(h) = h' \) as an edge of \( \Gamma \) connecting \( \alpha(h) \) with \( \alpha(h') \). We say that \( e \) is a self-edge if \( \alpha(h) = \alpha(h') \), and we say that \( e \) is a simple edge if \( \alpha(h) \neq \alpha(h') \). Let \( \mathcal{E} = \mathcal{E}(\Gamma) \) denote the set of edges of \( \Gamma \), where we identify \( (h, h') \) and \( (h', h) \). Observe that \( |\mathcal{E}| = 2|\mathcal{E}| + n \), since \( i : \mathcal{H} \to \mathcal{H} \) is an involution, whose \( n \) fixed points are given by the elements of \( L \).

We say that \( \Gamma \) is a *stable graph of genus \( g \)* if the following three conditions are satisfied.

1. **Connectivity**: The graph \( \Gamma \) is connected, meaning that for any distinct vertices \( u, v \in V \), there exists a sequence of vertices \( u = v_0, v_1, \ldots, v_k = v \in V \) such that \( v_j \) and \( v_{j+1} \) are connected by an edge \( (h_j, h'_j) \in \mathcal{E} \), for each \( j \in [0, k - 1] \).
2. **Genus condition**: Denoting \( V = V_\Gamma = |V(\Gamma)| \) and \( E = E_\Gamma = |\mathcal{E}(\Gamma)| \), we have
   \[
   \sum_{v \in V} g_v = g - E + V - 1. \tag{2.14}
   \]
3. **Stability condition**: Letting \( m_v = |\alpha^{-1}(v)| \) denote the number of half-edges incident to any vertex \( v \in V(\Gamma) \), we have
   \[
   2g_v + m_v \geq 3. \tag{2.15}
   \]

An *isomorphism* between two graphs with genus decoration, \( \Gamma = (V, \mathcal{H}, \mathcal{L}, \lambda, i, g) \) and \( \Gamma' = (V', \mathcal{H}', \mathcal{L}', \lambda', i', g') \), consists of two bijections \( \mu : V \to V \) and \( \nu : \mathcal{H} \to \mathcal{H}' \) such that \( \alpha'(\nu(h)) = \mu(\alpha(h)) \) and

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\[ i'(v(h)) = v(i(h)), \text{ for each half-edge } h \in \mathcal{H}; \text{ such that } \lambda'(v(h)) = \lambda(h), \text{ for each leg } h \in \mathcal{L}; \text{ and such that } g'_{\mu(v)} = g_v, \text{ for each vertex } v \in \mathcal{V}. \]

For any integer \( g \geq 0 \), we further denote by \( \mathcal{G}_{g,n} \) the set of stable graphs with \( n \) legs of genus \( g \), up to isomorphism (so, in particular, the vertices and edges of a stable graph in \( \mathcal{G}_{g,n} \) are unlabeled, but the legs are). For any \( \Gamma \in \mathcal{G}_{g,n} \), an isomorphism \((\mu, \nu)\) from \( \Gamma \) to itself is an called an automorphism of \( \Gamma \); let \( \text{Aut}(\Gamma) \) denote the set of automorphisms of \( \Gamma \).

**Remark 2.10** Fix \( g \in \mathbb{Z}_{\geq 0} \) and \( \Gamma \in \mathcal{G}_{g,n} \), and adopt the notation from Definition 2.9. Then,

\[
\sum_{v \in \mathcal{V}} m_v = 2E + n. \tag{2.16}
\]

Moreover, by summing (2.15) over all \( v \in \mathcal{V}(\Gamma) \) and using (2.14) and (2.16) yields

\[
2g + n \geq V + 2. \tag{2.17}
\]

Since \( \Gamma \) is connected, we must also have \( E \geq V - 1 \). Furthermore, (2.14), the fact that each \( g_v \geq 0 \), and (2.17) together imply

\[
E \leq g + V - 1 \leq 3g + n - 3. \tag{2.18}
\]

Following equation (1.12) of [11], we will next define a polynomial associated with any stable graph, which will be used to evaluate \( \text{Vol} \mathcal{Q}_{g,n} \). However, we first require the following multi-variate polynomial, whose coefficients are given in terms of the normalized intersection numbers from Definition 1.1, and linear map that was originally introduced as equation (1.11) of [11]. In the below, for any integer \( k > 1 \), \( \zeta(k) = \sum_{j=1}^{\infty} j^{-k} \) denotes the Riemann zeta function.

**Definition 2.11** For any integers \( g \geq 0 \) and \( n \geq 1 \), and set of \( n \) variables \( b = (b_1, b_2, \ldots, b_n) \), define the polynomial

\[
N_{g,n}(b) = \frac{(6g + 2n - 5)!!}{2^{5g+n-3}3^g g!} \sum_{\mathbf{d} \in \mathcal{K}_n(3g+n-3)} (\mathbf{d})_{g,n} \prod_{j=1}^{n} \frac{b_j^{2d_j}}{(2d_j + 1)!}, \tag{2.19}
\]

where we have recalled the normalized intersection number \((\mathbf{d})_{g,n}\) from Definition 1.1 and denoted \( \mathbf{d} = (d_1, d_2, \ldots, d_n) \). We further define the map

\( \zeta \) Springer
$Z : \mathbb{C}[b] \to \mathbb{C}$ by setting it on monomials to be

$$Z\left(\prod_{j=1}^{k} b_{ij}^{r_j}\right) = \prod_{j=1}^{k} r_j! \zeta(r_j + 1),$$

for any mutually distinct indices $i_1, i_2, \ldots, i_k \in [1, n]$ and integers $r_1, r_2, \ldots, r_k \geq 1$, and then extending it by linearity to all of $\mathbb{C}[b]$.

Now we can define the following polynomial associated with any stable graph $\Gamma$, given by equation (1.12) of [11].

**Definition 2.12** Fix integers $g, n \geq 0$ and a stable graph $\Gamma \in \mathcal{G}_{g,n}$; adopt the notation for $\Gamma = (\mathcal{V}, \mathcal{E}, \alpha, \mathcal{L}, \lambda, \iota, g)$ from Definition 2.9. For each edge $e \in \mathcal{E}(\Gamma)$, let $b_e$ denote a variable; for each half-edge $h \in \mathcal{E}(\Gamma) \setminus \mathcal{L}(\Gamma)$ that is not a leg, set $b_h = b_e$, where $e = (h, h')$ is the edge containing $h$; and for each leg $h \in \mathcal{L}(\Gamma)$, set $b_h = 0$. For each vertex $v \in \mathcal{V}(\Gamma)$, let $b^{(v)} = (b_h)$ denote the set of variables $b_h$, where $h$ ranges over all half-edges that are incident to $v$. In particular if $e$ is a simple edge incident to $v$, then $b_e$ is counted with multiplicity one in $b^{(v)}$; if $e$ is a self-edge incident to $v$, then $b_e$ is counted with multiplicity two in $b^{(v)}$; and $b^{(v)}$ also contains $n = \lvert \mathcal{L}(\Gamma) \rvert$ entries equal to 0. Therefore, $b^{(v)}$ consists of $\lvert \alpha^{-1}(v) \rvert = m_v$ elements. Define the polynomial $P(\Gamma) \in \mathbb{C}[(b_e)_{e \in \mathcal{E}(\Gamma)}]$ by setting

$$P(\Gamma) = 2^{6g + 2n - 4} \frac{(4g + n - 4)!}{(6g + 2n - 7)! \cdot 2^{\lvert \mathcal{V}(\Gamma) \rvert} \cdot \lvert \text{Aut}(\Gamma) \rvert} \times \prod_{e \in \mathcal{E}(\Gamma)} b_e \prod_{v \in \mathcal{V}(\Gamma)} N_{g_v, m_v}(b^{(v)}).$$

Now we can state the following expression for $\text{Vol} \mathcal{Q}_{g,n}$ from [11].

**Proposition 2.13** ([11, Theorem 1.6]) For any integers $g, n \geq 0$, we have

$$\text{Vol} \mathcal{Q}_{g,n} = \sum_{\Gamma \in \mathcal{G}_{g,n}} Z(P(\Gamma)). \tag{2.20}$$

### 3 Exponential upper bound on $\langle d \rangle_{g,n}$

In this section we establish Proposition 1.2. We begin in Sect. 3.1 by bounding the sum of coefficients in the last term of (2.12), and then we prove Proposition 1.2 in Sect. 3.2.
3.1 Estimating a sum of coefficients

In this section we establish the following lemma.

**Lemma 3.1** Fix integers $k \geq 1$ and $n \geq 2$, and an $n$-tuple $d = (d_1, d_2, \ldots, d_n) \in \mathbb{Z}_{\geq 0}^n$ of nonnegative integers. Assume that $d_j \geq k + 1$ for each $j \in [1, n]$ and that there exists an integer $g \geq 0$ such that $|d| + k = 3g + n - 3$. Then,

\[
S = \frac{1}{2} \sum_{r+s=k-1} \sum_{I \cup J=[1,2,\ldots,n]} g! \frac{(6g' + 2n' - 3)!!(6g'' + 2n'' - 3)!!}{g''!!}(6g + 2n - 3)!!
\leq \frac{n + 1}{8g^2},
\]

(3.1)

where we have set $|I| = n'$, $|J| = n''$, and have adopted the notation from (2.13).

To that end, we begin by bounding summands appearing in the quantity $S$ from Lemma 3.1.

**Lemma 3.2** Let $g', g'' \geq 1$ and $n'$, $n'' \geq 0$ be integers, and set $g = g' + g''$ and $n = n' + n''$. Then,

\[
\binom{n}{n'} \binom{g}{g'} \frac{(6g' + 2n' - 3)!!(6g'' + 2n'' - 3)!!}{(6g + 2n - 3)!!} \leq \frac{1}{6g + 2n - 3} \left(\frac{2g - 2}{2g' - 1}\right)^{-1}.
\]

**Proof** Since, $n' + n'' = n$ and $g' + g'' = g$, we have

\[
(6g + 2n - 3)!! = (6g + 2n - 3)\frac{(6g + 2n - 4)!}{2^{3g+n-2}(3g + n - 2)!};
\]

\[
(6g' + 2n' - 3)!!(6g'' + 2n'' - 3)!! = \frac{(6g' + 2n' - 2)!!(6g'' + 2n'' - 2)!!}{2^{3g+n-2}(3g' + n' - 1)!(3g'' + n'' - 1)!},
\]

and so

\[
\frac{(6g' + 2n' - 3)!!(6g'' + 2n'' - 3)!!}{(6g + 2n - 3)!!} = \frac{1}{6g + 2n - 3} \left(\frac{3g + n - 2}{3g' + n' - 1}\right) \left(\frac{6g + 2n - 4}{6g' + 2n' - 2}\right)^{-1}.
\]

(3.2)
Moreover, since \( g', g'' \geq 1 \), we have

\[
\binom{n}{n'} \binom{g}{g'} \binom{3g + n - 2}{3g' + n' - 1} \binom{6g + 2n - 4}{6g' + 2n' - 2}^{-1} = \left(\frac{6g' + 2n' - 2}{g', n', 3g' + n' - 1, 2g' - 1}\right) \left(\frac{6g'' + 2n'' - 2}{g'', n'', 3g'' + n'' - 1, 2g'' - 1}\right) \times \left(\frac{6g + 2n - 4}{g, n, 3g + n - 2, 2g - 2}\right)^{-1}.
\]

Now applying the \((n, r) = (2, 4)\) case of Lemma 2.3 (with the facts that \( g' + g'' = g \) and \( n' + n'' = n \)), we obtain that

\[
\left(\frac{6g' + 2n' - 2}{g', n', 3g' + n' - 1, 2g' - 1}\right) \left(\frac{6g'' + 2n'' - 2}{g'', n'', 3g'' + n'' - 1, 2g'' - 1}\right) \leq \left(\frac{6g + 2n - 4}{g, n, 3g + n - 2, 2g - 2}\right).
\]

Inserting (3.4) into (3.3), we obtain that

\[
\binom{n}{n'} \binom{g}{g'} \binom{3g + n - 2}{3g' + n' - 1} \binom{6g + 2n - 4}{6g' + 2n' - 2}^{-1} \leq \left(\frac{2g - 2}{2g' - 1}\right)^{-1},
\]

which upon insertion into (3.2) yields the lemma. \(\Box\)

Now we can establish Lemma 3.1.

**Proof of Lemma 3.1** First observe that, since \( d_j \geq k + 1 \geq 2 \) for each \( j \in [1, n] \), we have \( 2n' + r \leq |d_I| + r = 3g' + n' - 2; \) so, \( 3g' - n' - 2 \geq r \geq 0 \), and similarly \( 3g'' - n'' - 2 \geq r \geq 0 \). Hence, \( g', g'' \geq 1 \). Since \( g' + g'' = g \), it follows that \( g', g'' \in [1, g - 1] \).

Now, for a fixed pair of nonnegative integers \((n', n'')\) with \( n' + n'' = n \), there are \( \binom{n}{n'} \) choices for a pair of nonempty, disjoint sets \((I, J)\) such that \( I \cup J = \{1, 2, \ldots, n\}; |I| = n' \); and \( |J| = n'' \). Fixing these two subsets and a pair of nonnegative integers \((r, s)\) with \( r + s = k - 1 \) also fixes \( g', g'' \in [1, g - 1] \) through (2.13). Thus,

\[
S \leq \frac{1}{2} \sum_{g' = 1}^{g-1} \sum_{n' = 0}^{n} \binom{n}{n'} \binom{g}{g'} \binom{6g' + 2n' - 3}{6g' + 2n' - 3}!! \binom{6g'' + 2n'' - 3}{6g + 2n - 3}!!.
\]
which by Lemma 3.2 yields
\[ S \leq \frac{1}{2(6g + 2n - 3)} \sum_{g' = 1}^{g-1} (n + 1) \left( \frac{2g - 2}{2g' - 1} \right)^{-1}. \] (3.5)

Thus, for \( g = 2 \), we have since \( n \geq 2 \) that
\[ S \leq \frac{n + 1}{2(6g + 2n - 3)(2g - 2)} = \frac{n + 1}{8n + 36} < \frac{n + 1}{32} = \frac{n + 1}{8g^2}, \]
which verifies (3.1) if \( g = 2 \). So, since \( g = g' + g'' \geq 2 \), we may assume that \( g \geq 3 \). In this case, since \( \left( \frac{2g - 2}{2g' - 1} \right) \geq \left( \frac{2g - 2}{3} \right) \) for \( g' \in [2, g - 2] \), it follows from (3.5) and the bounds \( n \geq 2 \) and \( g \geq 3 \) that
\[ S \leq \frac{n + 1}{12g + 2} \left( \frac{1}{g - 1} + \sum_{g' = 2}^{g-2} \left( \frac{2g - 2}{2g' - 1} \right)^{-1} \right) \leq \frac{n + 1}{12g + 2} \left( \frac{1}{g - 1} + (g - 3) \left( \frac{2g - 2}{3} \right)^{-1} \right) \leq \frac{n + 1}{8g^2}, \]
from which we deduce the lemma. \( \square \)

### 3.2 Proof of Proposition 1.2

In this section we establish Proposition 1.2. To that end, we will bound weighted sums of the coefficients appearing on the right side of (2.12). This is provided by the following lemma, where in the below \( U \) denotes the sum of the coefficients in the first term there; \( V \) denotes the sum of those in the second; and \( S \) denotes the sum of those in the third.

**Lemma 3.3** Adopt the notation and assumptions of Lemma 3.1, and further set
\[ U = \frac{1}{6g + 2n - 3} \sum_{j=1}^{n} (2d_i + 1); \quad V = \frac{12gk}{(6g + 2n - 3)(6g + 2n - 5)}. \] (3.6)

Then
\[ \frac{2U}{3} + \frac{3V}{2} + S \leq 1. \] (3.7)
Proof First observe, since $|d| + k = 3g + n - 3$, we have

$$U = \frac{2|d| + n}{6g + 2n - 3} = 1 + \frac{n - 2k - 3}{6g + 2n - 3}.$$ 

Applying this, the definition (3.6) of $V$, and Lemma 3.1, we deduce that

$$\frac{2U}{3} + \frac{3V}{2} + S \leq \frac{2}{3} + \frac{2n - 6}{18g + 6n - 9} - \frac{4k}{3(6g + 2n - 3)} + \frac{18gk}{(6g + 2n - 3)(6g + 2n - 5)} + \frac{n + 1}{8g^2}$$

$$= \frac{2}{3} + \frac{1}{3} \frac{2n - 6}{18g + 6n - 9} + \frac{k}{8g^2}.$$ 

(3.8)

Now, since $d_j \geq k + 1$ for each $j \in [1, n]$, we have $n(k + 1) + k \leq |d| + k = 3g + n - 3$, and so

$$(n + 1)k \leq 3g - 3. \quad (3.9)$$

Inserting (3.9) into (3.8), we obtain that

$$\frac{2U}{3} + \frac{3V}{2} + S \leq \frac{2}{3} + \frac{2n - 6}{18g + 6n - 9} + \frac{n + 1}{8g^2} + \frac{3g - 3}{(n + 1)(6g + 2n - 3)(6g + 2n - 5)} \left(10g - \frac{8n}{3} + \frac{20}{3}\right). \quad (3.10)$$

Since $n \geq 2$, we have $\frac{5}{3}(6g + 2n - 3) \geq 10g - \frac{8n}{3} + \frac{20}{3}$ and $3g - 3 \leq \frac{1}{2}(6g + 2n - 5)$, so

$$\frac{3g - 3}{(n + 1)(6g + 2n - 3)(6g + 2n - 5)} \left(10g - \frac{8n}{3} + \frac{20}{3}\right) \leq \frac{5}{6(n + 1)}.$$ 

Hence, (3.10) yields

$$\frac{2U}{3} + \frac{3V}{2} + S \leq \frac{2}{3} + \frac{2n - 6}{18g + 6n - 9} + \frac{5}{6(n + 1)} + \frac{n + 1}{8g^2} + \frac{5}{12n + 15} + \frac{5}{6(n + 1)} + \frac{5}{8(n + 4)^2}, \quad (3.11)$$

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where in the second bound we used the fact that $3g \geq n + 4$ (which follows from (3.9) and the fact that $k \geq 1$). If $n = 2$, then the right side of (3.11) is equal to $\frac{3695}{3744} < 1$, in which case (3.7) holds.

If instead $n \geq 3$, then $\frac{2n-6}{12n+15} \leq \frac{n-3}{6(n+1)}$, and so (3.11) implies

$$\frac{2U}{3} + \frac{3U}{2} + S \leq \frac{2}{3} + \frac{n+2}{6(n+1)} + \frac{9(n+1)}{8(n+4)^2} = \frac{5}{6} + \frac{1}{6(n+1)} + \frac{9(n+1)}{8(n+4)^2},$$

(3.12)

It is quickly verified that the right side of (3.12) is decreasing for $n \geq 2$ and is equal to $\frac{283}{288} < 1$ for $n = 2$. Thus, we deduce that (3.7) holds for any $n \geq 3$, which by the above verifies the lemma. \(\square\)

Now we can establish Proposition 1.2.

**Proof of Proposition 1.2** We induct on $|d| = 3g + n - 3$. The result holds if $3g + n - 3 = 0$, since then $\langle d \rangle_{g,n} = (0, 0, 0)_{0,3} = 1$, by Lemma 2.5. Thus, we fix some integer $B \geq 1$ and assume that (1.8) holds whenever $|d| \leq B - 1$; we will show the same bound holds for any $d$ with $|d| = B$. To that end, fix some $d = (d_1, d_2, \ldots, d_n)$ with $|d| = B$. In view of Lemmas 2.7 and 2.8, (1.8) holds when $n \in \{1, 2\}$. Thus, we will assume in what follows that $n \geq 3$.

Let $k \in \mathbb{Z}_{\geq -1}$ be such that $k + 1 = \min_{j \in [1, n+1]} d_j$. Set $d' = d \setminus \{k + 1\} = (d'_1, d'_2, \ldots, d'_{n-1}) \in \mathbb{Z}_{\geq 0}^{n-1}$, so that $d = (k + 1, d')$. We consider the three cases $k = -1, k = 0, k \geq 1$ separately.

So, let us first assume that $k = -1$, in which case (2.10) implies

$$\langle d \rangle_{g,n} = \langle d', 0 \rangle_{g,n} = \frac{1}{6g + 2n - 5} \sum_{j=1}^{n-1} (2d'_j + 1)(d'_j - 1, d' \setminus \{d'_j\})_{g,n-1}. \quad (3.13)$$

Applying (1.8) on each summand on the right side of (3.13) gives

$$\langle d \rangle_{g,n} = \langle d', 0 \rangle_{g,n} \leq \frac{1}{6g + 2n - 5} \left( \frac{3}{2} \right)^{n-2} \sum_{j=1}^{n-1} (2d'_j + 1) = \frac{2|d'| + n - 1}{6g + 2n - 5} \left( \frac{3}{2} \right)^{n-2}. \quad (3.14)$$

Since $k = -1$ implies that $|d'| = |d| = 3g + n - 3$, we have for $g \geq 1$ that

$$\frac{2|d'| + n - 1}{6g + 2n - 5} = \frac{6g + 3n - 7}{6g + 2n - 5} \leq \frac{3}{2}. \quad (3.15)$$
Together, (3.14) and (3.15) yield (1.8) when \( k = -1 \) and \( g \geq 1 \). To address the case \((k, g) = (-1, 0)\), observe that \( g = 0 \) implies \(|d'| = n - 3\), so \( n \geq 3 \) and at least two of the \( d'_j \) are equal to 0. Then, at least two summands on the right side of (3.13) are equal to 0, and so

\[
\langle d \rangle_{g,n} = \langle d', 0 \rangle_{g,n} \\
\leq \frac{1}{2n-5} \left( \sum_{j=1}^{n-1} (2d'_j + 1) - 2 \right) \max_{j \in [1,n-1]} \langle d'_j - 1, d' \setminus \{d_j\} \rangle_{g,n-1} \\
= \frac{3n - 9}{2n-5} \max_{j \in [1,n-1]} \langle d'_j - 1, d' \setminus \{d_j\} \rangle_{g,n-1} \\
\leq \frac{3n - 9}{2n-5} \left( \frac{3}{2} \right)^{n-2} \leq \left( \frac{3}{2} \right)^{n-1},
\]

where in the second statement we used the fact that \( 2|d'| + n - 1 = 3n - 7 \) when \( g = 0 \), in the third statement we used (1.8), and in the fourth statement we used the bound \( \frac{3n - 9}{2n-5} \leq \frac{3}{2} \) that holds for each \( n \geq 3 \). This verifies (1.8) if \( k = -1 \).

Next we assume that \( k = 0 \). In this case, (2.11) and (1.8) together give

\[
\langle d \rangle_{g,n} = \langle d', 1 \rangle_{g,n} = \frac{6g + 3n - 9}{6g + 2n - 5} \langle d' \rangle_{g,n-1} \\
\leq \frac{6g + 3n - 9}{6g + 2n - 5} \left( \frac{3}{2} \right)^{n-2} \leq \left( \frac{3}{2} \right)^{n-1},
\]

where in the last bound we used the fact that \( \frac{6g + 3n - 9}{6g + 2n - 5} \leq \frac{3}{2} \) holds whenever \( g \geq 0 \) and \( 6g + 2n - 5 \geq 0 \). This verifies (1.8) if \( k = 0 \).

Now let us assume that \( k \geq 1 \). Then, the assumptions of Lemmas 3.1 and 3.3 apply. Adopting the notation from those statements, applying (2.12) (with the \( n \) there equal to \( n - 1 \) here), using (1.8) on each summand there, and using the fact that \( n' + n'' \) in (2.12) is equal to \( n - 1 \) here then gives

\[
\langle d \rangle_{g,n} = \langle k + 1, d' \rangle_{g,n} \leq \left( \frac{3}{2} \right)^{n-2} U + \left( \frac{3}{2} \right)^n V + \left( \frac{3}{2} \right)^{n-1} S \\
= \left( \frac{2U}{3} + \frac{3V}{2} + S \right) \left( \frac{3}{2} \right)^{n-1}. \tag{3.16}
\]

Then, (1.8) follows from Lemma 3.3 and (3.16); this establishes the proposition. \( \square \)
4 Asymptotics for $(d)_{g,n}$

In Sect. 4.1 we prove Theorem 1.5, assuming an asymptotic lower and upper bound, given by Propositions 4.1 and 4.2, respectively. The remainder of this section is devoted to the proof of the lower bound Proposition 4.1. We begin in Sect. 4.2 by providing a recursive lower bound for the normalized intersection number $(d)_{g,n}$ (given by Lemma 4.5). By interpreting this estimate through a random walk, we then in Sect. 4.3 bound $(d)_{g,n}$ by a certain probability for this random walk (given by Proposition 4.8), which we analyze in Sect. 4.4 to establish Proposition 4.1.

4.1 Proof of Theorem 1.5

The proof of Theorem 1.5 will consist in a lower and upper bound on $(d)_{g,n}$, given by the following two propositions. The first will be established in Sect. 4.4 and the second in Sect. 5.2. In the below, we recall $\mathcal{K}_n(N)$ from Sect. 2.1.

**Proposition 4.1** Let $g > 2^{15}$ and $n \geq 1$ be integers such that $g > 30n$, and let $d \in \mathcal{K}_n(3g + n - 3)$ be a nonnegative composition. Then, we have

$$(d)_{g,n} \geq 1 - 20g^{-1}(n + 4 \log g).$$

**Proposition 4.2** Let $g > 2^{30}$ and $n \geq 1$ be integers such that $g > 800n^2$, and let $d \in \mathcal{K}_n(3g + n - 3)$ be a nonnegative composition. Then, we have

$$(d)_{g,n} \leq \exp \left(625g^{-1}(n + 2 \log g)^2\right).$$

Assuming Propositions 4.1 and 4.2, we can now quickly establish Theorem 1.5.

**Proof of Theorem 1.5 Assuming Proposition 4.1 and Proposition 4.2** Since for sufficiently large $g$ and any $d = (d_1, d_2, \ldots, d_n) \in \Delta(g; \varepsilon)$ we have $n + 4 \log g < 2\varepsilon g^{1/2}$, Propositions 4.1 and 4.2 together imply

$$\lim_{\varepsilon \to 0} \left(\lim_{g \to \infty} \max_{d \in \Delta(g; \varepsilon)} |(d) - 1|\right)$$

$$\leq \lim_{\varepsilon \to 0} \left(\lim_{g \to \infty} \max \{40\varepsilon g^{-1/2}, \exp(2500\varepsilon^2) - 1\}\right) = 0,$$

from which we deduce the theorem. \qed

Before proceeding, it will be useful to introduce the following notation.
**Definition 4.3** For any integers \( g \geq 0 \) and \( n \geq 2 \), let

\[
\Omega(g, n) = \bigcup_{G=g}^{\infty} \bigcup_{m=2}^{n} K_m(3G + m - 3),
\]

and set

\[
\theta_{g,n} = \inf_{d \in \Omega_{g,n}} \langle d \rangle_{g,n}; \quad \Theta_{g,n} = \sup_{d \in \Omega_{g,n}} \langle d \rangle_{g,n}.
\]

**Remark 4.4** Observe in particular that

\[
\theta_{g,n} \geq \theta_{g',n'}, \quad \text{and} \quad \Theta_{g,n} \leq \Theta_{g',n'}, \quad \text{if} \quad g \geq g' \text{ and } n \leq n',
\]

since then \( \Omega(g, n) \subseteq \Omega(g', n') \).

As such, if \( n \geq 2 \) (the case \( n = 1 \) is addressed separately by Lemma 2.7), Proposition 4.1 corresponds to a lower bound on \( \theta_{g,n} \) and Proposition 4.2 to an upper bound on \( \Theta_{g,n} \).

### 4.2 Recursive Estimate for \( \theta_{g,n} \)

Let us briefly (and heuristically) explain how we will use the linear recursion (2.12). One might initially hope that, after applying (2.12) to some \( \langle k + 1, d \rangle_{g,n+1} \), all resulting terms will correspond to intersection numbers \( \langle d' \rangle \) of length at most \( n \). If this were true, then by using (2.12) \( n - 1 \) times, we would obtain an expression for \( \langle d \rangle_{g,n} \) as a linear combination of two-point intersection numbers \( \langle d' \rangle_{g',2} \). Since the large genus asymptotics for the latter are known by Lemma 2.8, this would yield those for \( \langle d \rangle_{g,n} \).

Unfortunately, it is not quite true that the length of \( d \) decreases after applying (2.12). Indeed, although this is indeed the case for the first term on the right side of that equation, it is not true for the second term. However, we will show that the length of \( d \) has a “decreasing drift,” through the following lemma. In the below, we recall the quantity \( \theta_{g,n} \) from Definition 4.3.

**Lemma 4.5** For any integers \( g \geq 1 \) and \( n \geq 2 \), we have

\[
\theta_{g,n+1} \geq \left( \frac{2\theta_{g,n}}{3} + \frac{\theta_{g-1,n+2}}{3} \right) \left( 1 - \frac{1}{4g} \right).
\]

**Proof** Fix some \( d = (d_1, d_2, \ldots, d_m+1) \in \Omega(g, n+1) \), for some integer \( m \in [2, n] \), and define \( G \in \mathbb{Z}_{\geq 0} \) so that \( |d| = 3G + m - 2 \). Let \( k \geq -1 \)
be such that \( k + 1 = \min_{j \in [1, m+1]} d_j \), and let \( d = (k + 1, d') \), for some \( d' = (d'_1, d'_2, \ldots, d'_m) \in \mathbb{Z}^m_{\geq 0} \). We consider three cases.

The first is if \( k = -1 \), in which case (2.10) yields

\[
\langle d \rangle_{G, m+1} = \langle d', 0 \rangle_{G, m+1} = \frac{1}{6G + 2m - 3} \sum_{j=1}^{m} (2d'_j + 1)(d'_j - 1) \langle d' \setminus \{d'_j\} \rangle_{G, m} \geq \left( \frac{2|d'| + 1}{6G + 2m - 3} \right) \theta_{g,n} = \theta_{g,n},
\]

where in the last equality we used the identity \( |d'| = |d| = 3G + m - 2 \). Since (4.1) gives \( \theta_{g,n} \geq \theta_{g-1,n+2} \), by ranging over \( d \in \Omega(g, n + 1) \) in (4.3) we deduce

\[
\theta_{g,n+1} \geq \theta_{g,n} \geq \frac{2\theta_{g,n}}{3} + \frac{\theta_{g-1,n+2}}{3},
\]

which implies (4.2).

The second is if \( k = 0 \), in which case (2.11) yields (using the fact that \( m \geq 2 \))

\[
\langle d \rangle_{G, m+1} = \langle d', 1 \rangle_{G, m+1} = \left( \frac{6G + 3m - 6}{6G + 2m - 3} \right) \langle d' \rangle_{G, m} \geq \left( 1 - \frac{1}{6G} \right) \theta_{g,n} \geq \left( \frac{2\theta_{g,n}}{3} + \frac{\theta_{g-1,n+2}}{3} \right) \left( 1 - \frac{1}{6g} \right),
\]

where in the last inequality we used the fact that \( G \geq g \) and again the bound \( \theta_{g,n} \geq \theta_{g-1,n+2} \). Ranging over \( d \in \Omega(g, n + 1) \) in (4.4) then gives (4.2).

The third is if \( k \geq 1 \), in which case (2.12) yields

\[
\langle d \rangle_{G, m+1} = \langle k + 1, d' \rangle_{G, m+1} \geq \frac{\theta_{g,n}}{6G + 2m - 3} \sum_{j=1}^{m} (2d'_j + 1) + \frac{12Gk\theta_{g-1,n+2}}{(6G + 2m - 3)(6G + 2m - 5)},
\]

where we have used the nonnegativity of the intersection numbers (1.7) to omit the last term on the right side of (2.12). Therefore, using the fact that \( |d'| = |d| - k - 1 = 3G + m - k - 3 \), we obtain
\[
\langle d \rangle_{G,m+1} \geq \frac{2|d'| + m}{6G + 2m - 3} \theta_{g,n} + \frac{12Gk\theta_{g-1,n+2}}{(6G + 2m - 3)(6G + 2m - 5)} \\
\quad \quad = \frac{(6G + 3m - 2k - 6)\theta_{g,n}}{6G + 2m - 3} + \frac{12Gk\theta_{g-1,n+2}}{(6G + 2m - 3)(6G + 2m - 5)}.
\]

(4.5)

Now, letting
\[
U = \frac{6G + 3m - 2k - 6}{6G + 2m - 3}; \quad V = \frac{12Gk}{(6G + 2m - 3)(6G + 2m - 5)},
\]
we claim that
\[
U + V \geq 1 - \frac{1}{6g}; \quad V \leq \frac{1}{3}.
\]

(4.6)

Assuming (4.6), it follows by ranging over \(d \in \Omega(g,n+1)\) in (4.5) and the bound \(\theta_{g,n} \geq \theta_{g-1,n+2}\) that
\[
\theta_{g,n+1} \geq U\theta_{g,n} + V\theta_{g-1,n+2} \geq \left(U + V - \frac{1}{3}\right)\theta_{g,n} + \frac{\theta_{g-1,n+2}}{3}
\]
\[
\quad \geq \left(\frac{2}{3} - \frac{1}{6g}\right)\theta_{g,n} + \frac{\theta_{g-1,n+2}}{3},
\]
which implies (4.2).

So, it remains to verify both bounds in (4.6). We begin with the latter. To that end, observe that since \(k + 1 = \min_{j \in [1,m+1]} d_j\), we have \((m + 1)(k + 1) \leq |d| = 3G + m - 2\). Thus,
\[
k \leq \frac{3G - 3}{m + 1} \leq G - 1,
\]
where in the last bound we used the fact that \(m \geq 2\). Hence (4.7); the fact that \(G \geq k + 1 \geq 2\), which follows from (4.7) and the bound \(k \geq 1\); and the fact that \(m \geq 2\) together imply
\[
V \leq \frac{12Gk}{(6G + 1)(6G - 1)} \leq \frac{12G(G - 1)}{36G^2 - 1} \leq \frac{1}{3}.
\]

This establishes the second estimate in (4.6). To verify the first, observe that
\[
U + V - 1 = \frac{(6G + 2m - 5)(m - 3) - 2k(2m - 5)}{(6G + 2m - 3)(6G + 2m - 5)}.
\]

(4.8)
If \( m \geq 4 \), then (4.7) implies

\[
(6G + 2m - 5)(m - 3) - 2k(2m - 5) \geq 6G(m - 3) - 2k(2m - 5) \\
\geq (6G - 6k)(m - 3) \geq 0,
\]

and so by (4.8) the first bound in (4.6) holds. If \( m = 3 \), then (4.8) and (4.7) together yield

\[
U + V - 1 = \frac{-2k}{(6G + 3)(6G + 1)} \geq \frac{2 - 2G}{(6G + 3)(6G + 1)} \geq -\frac{1}{18G} \geq -\frac{1}{18g},
\]

which again verifies the first bound in (4.6).

If \( m = 2 \), then (4.8) implies that

\[
U + V - 1 = \frac{1 - 6G + 2k}{36G^2 - 1} \geq -\frac{1}{6G} \geq -\frac{1}{6g},
\]

which verifies (4.6) and therefore (4.2) in the case when \( k \geq 1 \). This confirms (4.2) in all cases, thereby establishing the lemma.

\[\square\]

### 4.3 Comparison to a random walk

In view of (4.2) (and omitting the factor of \( 1 - \frac{1}{4g} \), which should tend to 1 as \( g \) tends to \( \infty \)), one might view the \( n \)-parameter in \( \theta_{g,n} \) as performing an asymmetric simple random walk with left jump probability \( \frac{2}{3} \) and right jump probability \( \frac{1}{3} \). The following provides notation for this walk, that starts at some integer \( n \geq 3 \) is absorbed at 2.

**Definition 4.6** Fix an integer \( n \geq 3 \), and define the random function \( w_n(t) : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 2} \) as follows. Set \( w_n(0) = n \) and, for \( t \geq 1 \), define \( w_n(t) \) through the following recursive procedure.

1. If \( w_n(t - 1) > 2 \), then set \( w_n(t) = w_n(t - 1) - 1 \) with probability \( \frac{2}{3} \) and \( w(t) = w_n(t - 1) + 1 \) with probability \( \frac{1}{3} \).
2. If \( w_n(t - 1) = 2 \), then set \( w_n(t) = 2 \).

We further let \( \mathbb{P} = \mathbb{P}_n \) and \( \mathbb{E} = \mathbb{E}_n \) denote the probability measure and expectation with respect to \( w_n \), respectively.

Next we define a quantity associated with this walk \( w_n(t) \) that we will use to lower bound \( \theta_{g,n} \) in Proposition 4.8 below.

**Definition 4.7** For any integers \( g > t \geq 0 \) and \( n \geq 2 \), define \( f_{g,n}(t) \) as follows.
(1) If \( n = 2 \), then set \( f_{g,n}(t) = \frac{6g-3}{6g-1} \).

(2) If \( n \geq 3 \), then define the absorbing random walk \( w_{n}(t) \) as in Definition 4.6 and set

\[
f_{g,n}(t) = \left(1 - \frac{1}{4g - 4t}\right)^{t} \left(1 - \frac{2}{6g - 6t - 1}\right) \mathbb{P}[w_{n}(t) = 2].
\]

**Proposition 4.8** For any integers \( g > t \geq 0 \) and \( n \geq 2 \), we have \( \theta_{g,n} \geq f_{g,n}(t) \).

**Proof** If \( n = 2 \), then Lemma 2.8 implies that \( \theta_{g,n} \geq \frac{6g-3}{6g-1} \geq f_{g,n}(t) \), so we may assume \( n \geq 3 \).

In this case, we induct on \( t \). If \( t = 0 \), then since \( n \geq 3 \) we have \( f_{g,n}(0) = 0 \) (as \( w_{n}(0) = n \neq 2 \)), and so \( \theta_{g,n} \geq f_{g,n}(t) \) by the nonnegativity of the intersection numbers (1.7). Thus, let us suppose for some integer \( T \geq 1 \) that \( \theta_{g,n} \geq f_{g,n}(t) \) holds for any \( t \in [0, T - 1] \) when \( g > t \), and we will show that \( \theta_{g,n} \geq f_{g,n}(T) \) when \( g > T \).

To that end, observe by Lemma 4.5 (with the \( n + 1 \) there equal to \( n \) here) that

\[
\theta_{g,n} \geq \left(\frac{2\theta_{g,n-1}}{3} + \frac{\theta_{g-1,n+1}}{3}\right) \left(1 - \frac{1}{4g}\right) \\
\geq \left(\frac{2f_{g,n-1}(T - 1)}{3} + \frac{f_{g-1,n+1}(T - 1)}{3}\right) \left(1 - \frac{1}{4g}\right).
\]

Hence, by Definition 4.7,

\[
\theta_{g,n+1} \geq \frac{2}{3} \left(1 - \frac{1}{4g}\right) \left(1 - \frac{1}{4g - 4T + 4}\right)^{T-1} \left(1 - \frac{2}{6g - 6T + 5}\right) \\
+ \frac{1}{3} \left(1 - \frac{1}{4g}\right) \left(1 - \frac{1}{4g - 4T}\right)^{T-1} \left(1 - \frac{2}{6g - 6T - 1}\right) \\
\geq \left(1 - \frac{1}{4g}\right) \left(1 - \frac{1}{4g - 4T}\right)^{T-1} \left(1 - \frac{2}{6g - 6T - 1}\right) \\
\times \left(\frac{2}{3} \mathbb{P}[w_{n-1}(T - 1) = 2] + \frac{1}{3} \mathbb{P}[w_{n+1}(T - 1) = 2]\right).\tag{4.9}
\]
Now, observe that
\[
\mathbb{P}[w_n(T) = 2] = \mathbb{P}[w_n(1) = n - 1] \mathbb{P}[w_{n-1}(T - 1) = 2] + \mathbb{P}[w_n(1) = n + 1] \mathbb{P}[w_{n+1}(T - 1) = 2] = \frac{2}{3} \mathbb{P}[w_{n-1}(T - 1) = 2] + \frac{1}{3} \mathbb{P}[w_{n+1}(T - 1) = 2],
\]
which upon insertion into (4.9) yields
\[
\theta_{g,n+1} \geq \left(1 - \frac{1}{4g}\right) \left(1 - \frac{1}{4g - 4T}\right)^{T-1} \left(1 - \frac{2}{6g - 6T - 1}\right) \mathbb{P}[w_n(T) = 2]\]
\[
\geq \left(1 - \frac{1}{4g - 4T}\right)^T \left(1 - \frac{2}{6g - 6T - 1}\right) \mathbb{P}[w_n(T) = 2] = f_{g,n}(T),
\]
from which we deduce the proposition.

4.4 Proof of Proposition 4.1

By Proposition 4.8, to show Proposition 4.1 it remains to lower bound \(f_{g,n} \geq 1 - o(1)\), to which end by Definition 4.7 it suffices to show that \(\mathbb{P}[w_n(t) = 2] \approx 1\) for some choice of \(t = o(g)\). Since \(w_n(t)\) has a drift of \(\frac{1}{3}\) to the left, by the law of large numbers, one expects for \(w_n(t) = 2\) to hold for \(n\) substantially larger than \(3n\). Although this can be quickly made precise, we will proceed slightly differently through an exponential weighting estimate given by the following lemma, as this will also be useful in the proof of the upper bound Proposition 4.2 in Sect. 5.2 below.

**Lemma 4.9** Recalling \(w_n(t)\) from Definition 4.6, we have for any integers \(n \geq 3\) and \(t \geq 0\) that

\[
\mathbb{E}\left[\left(\frac{3}{2}\right)^{w_n(t)} \mathbf{1}_{w_n(t) \geq 2}\right] \leq \left(\frac{2}{3}\right)^{t/10 - n}.\]

**Proof** For any integers \(n \geq 3\) and \(t \geq 0\), define

\[
\mathcal{W}_n(t) = \mathbb{E}\left[\left(\frac{3}{2}\right)^{w_n(t)} \mathbf{1}_{w_n(t) \geq 2}\right].
\]

We claim for any integer \(t \geq 1\) that

\[
\mathcal{W}_n(t) \leq \frac{17}{18} \mathcal{W}_n(t - 1). \quad (4.10)
\]
Given (4.10), we can quickly establish the lemma. Indeed, by induction on $t$ and the fact that $\mathcal{W}_n(0) = \left(\frac{3}{2}\right)^n$ (since $w_n(0) = n$ and $n \geq 3$), we have

$$\mathcal{W}_n(t) \leq \left(\frac{17}{18}\right)^t \mathcal{W}_n(0) \leq \left(\frac{17}{18}\right)^t \left(\frac{3}{2}\right)^n,$$

from which the lemma follows since $\frac{17}{18} \leq \left(\frac{3}{2}\right)^{1/10}$.

It therefore remains to establish (4.10). To that end, since $P[w_n(t) = w_n(t - 1) - 1] = \frac{2}{3}$ and $P[w_n(t) = w_n(t - 1) + 1] = \frac{1}{3}$ whenever $w_n(t - 1) > 2$, we have

$$\mathcal{W}_n(t) = \mathbb{E}\left[\left(\frac{3}{2}\right)^{w_n(t)} 1_{w_n(t) > 2}\right]$$

$$\leq \frac{2}{3} \mathbb{E}\left[\left(\frac{3}{2}\right)^{w_n(t-1)} 1_{w_n(t-1) > 2}\right] P[w_n(t) = w_n(t - 1) - 1]$$

$$+ \frac{3}{2} \mathbb{E}\left[\left(\frac{3}{2}\right)^{w_n(t-1)} 1_{w_n(t-1) > 2}\right] P[w_n(t) = w_n(t - 1) + 1]$$

$$= \frac{17}{18} \mathbb{E}\left[\left(\frac{3}{2}\right)^{w_n(t-1)} 1_{w_n(t-1) > 2}\right] = \frac{17}{18} \mathcal{W}_n(t - 1),$$

which verifies (4.10) and therefore establishes the lemma. $\square$

We can then deduce as a corollary that $w_n(t) = 2$ likely holds for $t$ much larger than $10n$.

**Corollary 4.10** For any integers $n \geq 3$ and $t \geq 0$, we have

$$P[w_n(t) \neq 2] < \left(\frac{2}{3}\right)^{t/10-n}.$$

**Proof** Since $w_n(t) \geq 2$ and $(\frac{3}{2})^{w_n(t)} \geq 1$ whenever $w_n(t) \geq 3$, we have

$$P[w_n(t) \neq 2] = P[w_n(t) > 2] \leq \mathbb{E}\left[\left(\frac{3}{2}\right)^{w_n(t)} 1_{w_n(t) > 2}\right] \leq \left(\frac{3}{2}\right)^{t/10-n},$$

where in the last inequality we applied Lemma 4.9; this yields the corollary. $\square$

Now we can quickly establish Proposition 4.1.

**Proof of Proposition 4.1** If $n = 1$, then the proposition follows from Lemma 2.7, so we may assume that $n \geq 2$. Throughout the remainder of this proof, we set $t = 10n + 30\lceil\log g\rceil$. 

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Since $10n < \frac{g}{3}$, we have for $g > 2^{15}$ that $\frac{g}{2} > t$. Thus, for $g > 2^{15}$, Proposition 4.8 and Definition 4.7 together give

$$\theta_{g,n} \geq \left(1 - \frac{1}{4g - 4t}\right)^t \left(1 - \frac{2}{6g - 6t - 1}\right) \mathbb{P}[w_n(t) = 2]$$

$$\geq \left(1 - \frac{1}{2g}\right)^{t+3} \left(1 - \mathbb{P}[w_n(t) > 2]\right) \geq \left(1 - \frac{t}{g}\right) \left(1 - \mathbb{P}[w_n(t) \neq 2]\right).$$

(4.11)

By Corollary 4.10, and the facts that $\frac{t}{10} - n \geq 3 \log g$ and $\left(\frac{2}{3}\right)^3 < e^{-1}$, we have

$$\mathbb{P}[w_n(t) \neq 2] \leq \left(\frac{2}{3}\right)^{t/10 - n} \leq \left(\frac{2}{3}\right)^{3 \log g} < \frac{1}{g}.$$

Hence, it follows from (4.11) that

$$\langle d \rangle_{g,n} \geq \theta_{g,n} \geq \left(1 - \frac{t}{g}\right) \left(1 - \frac{1}{g}\right) \geq 1 - \frac{2t}{g},$$

for any $d = (d_1, d_2, \ldots, d_n) \in \mathbb{Z}_{\geq 0}^n$ with $n \geq 2$ and $|d| = 3g + n - 3$. Together with the fact that $t = 10n + 30[\log g] \leq 10n + 40 \log g$ for $g > 2^{15}$, this yields the proposition.

\[\square\]

5 Upper bound on $\langle d \rangle$

In this section we establish Proposition 4.2. As in the proof of Proposition 4.1, we begin with a recursive upper bound on $\langle d \rangle_{g,n}$ in Sect. 5.1. Then, in Sect. 5.2, we interpret this recursion through a random walk, which we then analyze to prove Proposition 4.2.

5.1 Recursive estimate for $\Theta_{g,n}$

To establish Proposition 4.2, we proceed as in the proof of Proposition 4.1. More specifically, we first establish a recursive estimate for the quantity $\Theta_{g,n}$ from Definition 4.3 and then bound it by a function associated with the random walk $w_n(t)$ from Definition 4.6. The following lemma implements the former task.
Lemma 5.1 For any integers \( g, n \geq 2 \) with \( g \geq 2n \), we have

\[
\Theta_{g,n+1} \leq \left( \frac{2\Theta_{g,n}}{3} + \frac{\Theta_{g-n,n+2}}{3} \right) \left( 1 + \frac{n + 10}{4g} \right). \tag{5.1}
\]

Unlike in the proof of Lemma 4.5, providing a recursive estimate for the lower bound \( \theta_{g,n} \), to establish Lemma 5.1 we must now estimate the third term on the right side of (2.12). This bound is given by Lemma 5.3 below but, to prove it, we first require the following binomial coefficient estimate.

Lemma 5.2 Let \( g, g' \geq 1 \) denote integers such that \( g \geq 2g' \). Then,

\[
\binom{2g - 2}{2g' - 1} \geq 2^{2g' - 1}(g - 1).
\]

Proof This follows from the fact that

\[
\binom{2g - 2}{2g' - 1} = (2g - 2) \prod_{j=1}^{2g' - 2} \frac{2g - j - 2}{j + 1} \leq (2g - 2)2^{2g' - 2},
\]

where the last bound holds since \( 2g \geq 2j + 3 \) for each \( j \in [1, 2g' - 2] \) (as \( g \geq 2g' \)).

Lemma 5.3 Fix integers \( g \geq 0, k \geq 1, \) and \( n \geq 2 \) such that \( g \geq 2n \); let \( d = (d_1, d_2, \ldots, d_n) \in \mathbb{Z}_{\geq 0}^n \) denote an \( n \)-tuple of nonnegative integers such that \( |d| = 3g + n - k - 3 \); and assume that \( k + 1 \leq \min_{j \in [1,n]} d_j \). Define \( T = T(k + 1; d) \) by

\[
T = \frac{1}{2} \sum_{r+s=k-1} \sum_{r,s \geq 0, |I\cup J|=[1,2,...,n]} \frac{g! (6g' + 2n' - 3)!!(6g'' + 2n'' - 3)!!}{g'^! g'^!} \frac{(6g + 2n - 3)!!}{(6g + 2n - 3)!!} \times \langle r, d | I \rangle_{g',n'+1} \langle s, d | J \rangle_{g'',n''+1},
\]

where we have set \( |I| = n', |J| = n'' \), and have adopted the notation from (2.13). Then, we have

\[
T \leq \frac{\Theta_{g-n,n+1}}{2g} + \frac{1}{2g}.
\]

Proof First observe that (2.13) and the fact that \( r+s = k-1 \) together imply that \( g' + g'' = g \) and \( n' + n'' = n \). Moreover, (2.13) and the fact that \( d_j \geq k+1 \geq 2 \)
for each \( i \in [1, n] \) together imply that \( 3g' \geq |d_I| - n' + 2 \geq n' + 2 \) and \( 3g'' \geq |d_J| - n'' + 2 \geq n'' + 2 \). Defining
\[
g_0(m) = \left\lfloor \frac{m + 2}{3} \right\rfloor, \quad \text{for any integer } m \geq 0,
\]
these imply that
\[
g' \geq g_0(n') \geq 1; \quad g'' \geq g_0(n'') \geq 1; \quad g = g' + g'' \geq \left\lfloor \frac{n + 4}{3} \right\rfloor \geq 2.
\]

Next, observe (as in the proof of Lemma 3.1) that for a fixed pair of non-
negative integers \((n', n'')\) with \(n' + n'' = n\), there are \(\binom{n}{n'}\) choices for a pair of
nonempty, disjoint sets \((I, J)\) such that \(I \cup J = \{1, 2, \ldots, n\}; |I| = n'; \) and
\(|J| = n''\). Fixing these two subsets and a pair of nonnegative integers \((r, s)\) with
\(r + s = k - 1\) also fixes \(g' \in [g_0(n'), g - g_0(n'')]\) and \(g'' \in [g_0(n''), g - g_0(n')]\)
through (2.13). Thus,
\[
T \leq \frac{1}{2} \sum_{g' = g_0(n')} \sum_{n' = 0}^{n} \binom{n}{n'} \left( g \right) \left( g' \right) (6g' + 2n' - 3)!!(6g'' + 2n'' - 3)!!
\]
\[
\times \langle r, d|I \rangle g',n'+1 \langle s, d|J \rangle g'',n''+1
\]
\[
\leq \frac{1}{2(6g + 2n - 3)} \sum_{n' = 0}^{n} \sum_{g' = g_0(n')} g - g_0(n') \left( 2g - 2 \right) \left( 2g' - 1 \right)^{-1} \Theta_{g',n'+1} \Theta_{g-g',n-n'+1},
\]
(5.2)

where in the last inequality we used Lemma 3.2 and the facts that
\(\langle r, d|I \rangle g',n'+1 \leq \Theta_{g',n'+1}\) and \(\langle s, d|J \rangle g'',n''+1 \leq \Theta_{g'',n''+1} = \Theta_{g-g',n-n'+1}\).

By the symmetry on the right side of (5.2) between \((n', g')\) and \((n - n', g - g')\),
it follows that
\[
T \leq \frac{1}{6g + 2n - 3} \sum_{n' = 0}^{n} \binom{g/2}{g' - 1} \left( 2g - 2 \right) \left( 2g' - 1 \right)^{-1} \Theta_{g',n'+1} \Theta_{g-g',n-n'+1}
\]
(5.3)

where \(X(n') = X(n'; g, n)\) and \(Y(n') = Y(n'; g, n)\) are defined by
\[
X(n') = \sum_{g' = g_0(n')} \left( 2g' - 2 \right) \left( 2g' - 1 \right)^{-1} \Theta_{g',n'+1} \Theta_{g-g',n-n'+1};
\]

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\[ Y(n') = \sum_{g'=n+1}^{\lfloor g/2 \rfloor} \left( \frac{2g-2}{2g'-1} \right)^{-1} \Theta_{g',n'+1} \Theta_{g-g',n-n'+1}. \]

We claim that

\[ X(n') \leq \frac{3\Theta_{g-n,n+1}}{g-1}; \quad Y(n') \leq \frac{1}{2(g-1)}. \tag{5.4} \]

To establish these estimates, first observe by the bound \( g \geq 2n \), Lemma 5.2, and Proposition 1.2 that

\[ \left( \frac{2g-2}{2g'-1} \right)^{-1} \Theta_{g',n'+1} \Theta_{g-g',n-n'+1} \leq \frac{1}{2^{2g'-1}(g-1)} \Theta_{g',n'+1} \Theta_{g-g',n-n'+1} \]

\[ \leq \frac{1}{2^{2g'+1}(g-1)} \left( \frac{3}{2} \right)^{n'} \Theta_{g-g',n-n'+1}. \tag{5.5} \]

Now, to verify the first estimate in (5.4), observe by (5.5) that

\[ X(n') = \sum_{g'=g_0(n')}^{n} \left( \frac{2g-2}{2g'-1} \right)^{-1} \Theta_{g',n'+1} \Theta_{g-g',n-n'+1} \]

\[ \leq \frac{2}{g-1} \left( \frac{3}{2} \right)^{n'} \sum_{g'=g_0(n')}^{n} 2^{-2g'} \Theta_{g-g',n-n'+1} \]

\[ \leq \frac{2}{g-1} \left( \frac{3}{2} \right)^{n'} 2^{-2n'/3} \sum_{j=0}^{n} 2^{-2j} \Theta_{g-j,n-n'+1}, \]

where in the last bound we changed variables \( g' = g_0(n') + j \) and used the fact that \( g_0(n') \geq \frac{n'}{3} \). Since \( 2^{-2/3} < \frac{2}{3} \), it follows that

\[ X(n') \leq \frac{2}{g-1} \sum_{j=0}^{n} 2^{-2j} \Theta_{g-j,n-n'+1} \leq \frac{2\Theta_{g-n,n+1}}{g-1} \sum_{j=0}^{\infty} 2^{-2j} \leq \frac{3\Theta_{g-n,n+1}}{g-1}, \]

where in the second inequality we used the fact that \( \Theta_{g-n,n+1} \geq \Theta_{g-j,n-n'+1} \) for \( j \leq n \) (which holds by (4.1)). This yields the first bound in (5.4).

To prove the second, we again apply (5.5) and Proposition 1.2 to obtain that

\[ Y(n') \leq \frac{2}{g-1} \sum_{g'=n+1}^{\lfloor g/2 \rfloor} 2^{-2g'} \left( \frac{3}{2} \right)^{n'} \Theta_{g-g',n-n'+1} \leq \frac{2}{g-1} \sum_{g'=n+1}^{\lfloor g/2 \rfloor} 2^{-2g'} \left( \frac{3}{2} \right)^{n}, \]
\[
\frac{1}{2(g-1)} \left( \frac{3}{4} \right)^n \sum_{j=0}^{\infty} 2^{-2j-n} \leq \frac{1}{2(g-1)},
\]

where in the third statement we changed variables \( g' = n + j + 1 \) and in the fourth we used the fact that \( n \geq 1 \). This establishes the second bound in (5.4). Now, inserting (5.4) into (5.3) yields

\[
T \leq \frac{n+1}{2(6g + 2n - 3)(g-1)} (6\Theta_{g,n,n+1} + 1) \leq \frac{\Theta_{g,n,n+1}}{2g} + \frac{1}{2g},
\]

where in the last bound we used the facts that \( 6g + 2n - 3 \geq 6g \) (since \( n \geq 2 \)) and that \( n + 1 \leq g - 1 \) (since \( g \geq 2n \geq 4 \)). This establishes the lemma.

Now we can establish Lemma 5.1.

**Proof of Lemma 5.1** To show (5.1), we proceed as in the proof of Lemma 4.5. To that end, fix some \((m+1)\)-tuple \( d = (d_1, d_2, \ldots, d_{m+1}) \in \Omega(g, n+1)\), for some integer \( m \in [2, n] \); define \( G \in \mathbb{Z}_{\geq 0} \) by \( |d| = 3G + m - 2 \). Let \( k \geq -1 \) be such that \( k + 1 = \min_{j \in [1, m+1]} d_j \), and let \( d' = (k + 1, d') \), for some \( d' = (d'_1, d'_2, \ldots, d'_m) \in \mathbb{Z}_0^m \). We consider three cases.

The first is if \( k = -1 \), in which case (2.10) yields

\[
\langle d \rangle_{G,m+1} = \langle d' \rangle_{G,m+1} = \frac{1}{6G + 2m - 3} \sum_{j=1}^{m} (2d'_j + 1)(d'_j - 1, d' \setminus \{d'_j\})_{G,m} \leq \left( \frac{2|d'| + m}{6G + 2m - 3} \right) \Theta_{g,n} = \left( 1 + \frac{m - 1}{6G + 2m - 3} \right) \Theta_{g,n},
\]

where in the last equality we used the identity \( |d'| = |d| = 3G + m - 2 \). Hence, since (4.1) gives \( \Theta_{g,n} \leq \Theta_{g,n,n+2} \), by we obtain

\[
\langle d \rangle_{G,m+1} \leq \left( \frac{2\Theta_{g,n}}{3} + \frac{\Theta_{g,n,n+2}}{3} \right) \left( 1 + \frac{m - 1}{6G + 2m - 3} \right) \leq \left( \frac{2\Theta_{g,n}}{3} + \frac{\Theta_{g,n,n+2}}{3} \right) \left( 1 + \frac{n}{6g} \right),
\]

where in the last bound we used the facts that \( m \in [2, n] \) and \( G \geq g \). Now (5.1) follows from ranging over \( d \in \Omega(g, n+1) \) in (5.6).
The second is if $k = 0$, in which case (2.11) yields

$$\langle d \rangle_{G,m+1} = \langle d', 1 \rangle_{G,m+1} = \left(1 + \frac{m - 3}{6G + 2m - 3}\right) \langle d' \rangle_{G,m} \leq \left(1 + \frac{n}{6g}\right) \Theta_{g,n} \leq \left(\frac{2\Theta_{g,n}}{3} + \frac{\Theta_{g,n,n+2}}{3}\right)
\left(1 + \frac{n}{6g}\right),$$

(5.7)

where in the third statement we used the facts that $m \in [2, n]$ and $G \geq g$ and last inequality we again used the fact that $\Theta_{g,n} \leq \Theta_{g,n,n+2}$. Ranging over $d \in \Omega(g, n + 1)$ in (5.7) then gives (5.1).

The third is if $k \geq 1$, in which case (2.12) yields (recalling $T = T(k + 1; d)$ from Lemma 5.3)

$$\langle d \rangle_{G,m+1} = \langle k + 1, d' \rangle_{G,m+1} \leq \frac{\Theta_{g,n}}{6G + 2m - 3} \sum_{j=1}^{m} (2d'_{j} + 1) + \frac{12Gk\Theta_{g-1,n+2}}{(6G + 2m - 3)(6G + 2m - 5)} + T.$$

Therefore, using the fact that $|d'| = |d| - k - 1 = 3G + m - k - 3$ and Lemma 5.3, we obtain

$$\langle d \rangle_{G,m+1} \leq \frac{(6G + 3m - 2k - 6)\Theta_{g,n}}{6G + 2m - 3} + \frac{12Gk\Theta_{g-1,n+2}}{(6G + 2m - 3)(6G + 2m - 5)} + T \leq U \Theta_{g,n} + V \Theta_{g-1,n+2} + \frac{\Theta_{g,n,n+1}}{2g} + \frac{1}{2g},$$

(5.8)

where we have set

$$U = \frac{6G + 3m - 2k - 6}{6G + 2m - 3}; \quad V = \frac{12Gk}{(6G + 2m - 3)(6G + 2m - 5)}.$$

Now, similarly to (4.6), we claim that

$$U + V \leq 1 + \frac{n}{6g}; \quad V \leq \frac{1}{3}.$$

(5.9)

Assuming (5.9), we can quickly establish (5.1). Indeed, (5.8), (5.9), and the bounds $\Theta_{g,n} \leq \Theta_{g-1,n+2} \leq \Theta_{g,n,n+2}$ and $\Theta_{g,n,n+1} \leq \Theta_{g,n,n+2}$ (which
follow from (4.1)) together imply that

\[(d)_{G,m+1} \leq U \Theta_{g,n} + V \Theta_{g-1,n+2} + \frac{\Theta_{g-n,n+1}}{2g} + \frac{1}{2g}\]

\[\leq \left( U + V - \frac{1}{3} \right) \Theta_{g,n} + \frac{\Theta_{g-1,n+2}}{3} + \frac{\Theta_{g-n,n+1}}{2g} + \frac{1}{2g}\]

\[\leq \left( 1 + \frac{n + 1}{4g} \right) \frac{2 \Theta_{g,n}}{3} + \left( 1 + \frac{3}{2g} \right) \frac{\Theta_{g-n,n+2}}{3} + \frac{1}{2g}\]

\[\leq \left( \frac{2 \Theta_{g,n}}{3} + \frac{\Theta_{g-n,n+2}}{3} \right) \left( 1 + \frac{n + 10}{4g} \right),\]

where the last bound follows from the fact that \( \Theta_{g,n} \geq \frac{1}{2} \) (which is a consequence of Lemma 2.8). So, (5.1) follows from again ranging (5.10) over \( d \in \Omega(g, n + 1) \).

Hence, it remains to prove (5.9). The second estimate there was shown in (4.6), so we must verify the first. To that end, observe that

\[U + V = 1 + \frac{(6G + 2m - 5)(m - 3) - 2k(2m - 5)}{(6G + 2m - 3)(6G + 2m - 5)}\]

\[\leq 1 + \frac{m - 3}{6G + 2m - 3} + \frac{2k}{(6G + 2m - 3)(6G + 2m - 5)}\]

\[\leq 1 + \frac{m - 2}{6G + 2m - 3} \leq 1 + \frac{m}{6G} \leq 1 + \frac{n}{6g}.\]

Here, in the second statement we used the fact that \( 2m - 5 \geq -1 \) (since \( m \geq 2 \)); in the third we used the fact that \( 2k \leq 2(|d| - 1) = 6G + 2m - 6 < 6G + 2m - 5 \); in the fourth we used the fact that \( 6G + 2m - 3 \geq 6G \) (since \( m \geq 2 \)); and in the fifth we used the facts that \( m \leq n \) and that \( G \geq g \). This verifies the first bound in (5.9) and therefore establishes (5.1) in all cases. \( \square \)

5.2 Proof of Proposition 4.2

Analogously to Definition 4.7, the following definition provides a quantity associated with the random walk \( w_n(t) \) from Definition 4.6 that we will compare to \( \Theta_{g,n} \) in Proposition 5.5 below.

**Definition 5.4** For any integers \( t \geq 0 \) and \( g, n \geq 2 \) such that \( g > tn + t^2 \), define \( F_{g,n}(t) \) as follows.

1. If \( n = 2 \), then set \( F_{g,n}(t) = 1 \).
2. If \( n \geq 3 \), then define the absorbing random walk \( w_n(t) \) as in Definition 4.6 and set

\[
F_{g,n}(t) = \left( 1 + \frac{n + 2t + 9}{4(g - tn - t^2)} \right)^t \times \left( \mathbb{P}[w_n(t) = 2] + \mathbb{E}\left[ \left( \frac{3}{2} \right)^{w_n(t)} 1_{w_n(t) > 2} \right] \right).
\]

**Proposition 5.5** For any integers \( t \geq 0 \) and \( g, n \geq 2 \) such that \( g > (t + 2)n + t^2 \), we have \( \Theta_{g,n} \leq F_{g,n}(t) \).

**Proof** If \( n = 2 \), then Lemma 2.8 implies that \( \Theta_{g,n} \leq 1 = F_{g,n}(t) \), so we may assume \( n \geq 3 \).

In this case, upon setting

\[
H_{g,n}(t) = \left( 1 + \frac{n + 2t + 9}{4(g - tn - t^2)} \right)^t \mathbb{E}\left[ \Theta_{g-tn-t^2,w_n(t)} \right],
\]

it suffices to show that

\[
\Theta_{g,n} \leq H_{g,n}(t), \quad \text{for } g > (t + 2)n + t^2.
\]

Indeed, given (5.12), the bound \( \Theta_{g,n} \leq F_{g,n}(t) \) follows from the fact that \( F_{g,n}(t) \leq H_{g,n}(t) \), which holds since

\[
\mathbb{E}\left[ \Theta_{g-tn-t^2,w_n(t)} \right] = \mathbb{E}\left[ \Theta_{g-tn-t^2,w_n(t)} 1_{w_n(t) = 2} \right] + \mathbb{E}\left[ \Theta_{g-tn-t^2,w_n(t)} 1_{w_n(t) > 2} \right] \\
\leq \mathbb{E}[1_{w_n(t) = 2}] + \mathbb{E}\left[ \left( \frac{3}{2} \right)^{w_n(t)} 1_{w_n(t) > 2} \right] \\
= \mathbb{P}[w_n(t) = 2] + \mathbb{E}\left[ \left( \frac{3}{2} \right)^{w_n(t)} 1_{w_n(t) > 2} \right].
\]

Here, the first statement follows from the fact that \( w_n(t) \geq 2 \), and the second follows from the facts that \( \Theta_{g,2} \leq 1 \) (by Lemma 2.8) and \( \Theta_{g,n} < \left( \frac{3}{2} \right)^n \) (by Proposition 1.2).

Therefore, it remains to show (5.12), to which end we induct on \( t \), similarly to as in the proof of Proposition 4.8. If \( t = 0 \), then (5.12) holds since \( H_{g,n}(0) = \Theta_{g,n} \) (as \( w_n(0) = n \)). Thus, let us suppose for some integer \( T \geq 1 \) that \( \Theta_{g,n} \leq H_{g,n}(t) \) holds for any \( t \in [0, T - 1] \) when \( g > (t + 2)n + t^2 \), and we will show that \( \Theta_{g,n} \leq H_{g,n}(T) \) when \( g > (T + 2)n + T^2 \).
To that end, observe that

\[
\Theta_{g,n} \leq \left( \frac{2\Theta_{g,n-1}}{3} + \frac{\Theta_{g-n+1,n+1}}{3} \right) \left( 1 + \frac{n + 9}{4g} \right) \\
\leq \left( \frac{2H_{g,n-1}(T-1)}{3} + \frac{H_{g-n+1,n+1}(T-1)}{3} \right) \left( 1 + \frac{n + 9}{4g} \right).
\]

Here, in the first inequality we used Lemma 5.1 (with the \( n + 1 \) there equal to \( n \) here), which applies since \( g > (T + 2)n + T^2 \geq 2(n - 1) \), and in the third we used the facts that \( \Theta_{g,n-1} \leq H_{g,n-1}(T - 1) \) and \( \Theta_{g-n+1,n+1} \leq H_{g-n+1,n+1}(T - 1) \), which apply since \( g \geq (T + 1)(n - 1) + (T - 1)^2 \) and \( g - n + 1 \geq (T + 1)(n + 1) + (T - 1)^2 \) both hold if \( g > (T + 2)n + T^2 \) (as \( T \geq 1 \)). Hence, by (5.11)

\[
\Theta_{g,n+1} \leq \left( 1 + \frac{n + 9}{4g} \right) \left( \frac{2}{3} \left( 1 + \frac{n + 2T + 6}{4(g - (T - 1)(n - 1) - (T - 1)^2)} \right) \right) T^{-1} \\
\times \mathbb{E}\left[ \Theta_{g-(T-1)(n-1)-(T-1)^2,w_{n-1}(T-1)} \right] \\
+ \frac{1}{3} \left( 1 + \frac{n + 2T + 8}{4(g - n + 1 - (T - 1)(n + 1) - (T - 1)^2)} \right) T^{-1} \\
\times \mathbb{E}\left[ \Theta_{g-n+1-(T-1)(n+1)-(T-1)^2,w_{n+1}(T-1)} \right].
\]

Using the facts that \( g - (T - 1)(n - 1) - (T - 1)^2 \geq g - Tn - T^2 \) and \( g - n + 1 - (T - 1)(n + 1) - (T - 1)^2 \geq g - Tn - T^2 \) and applying (4.1), we therefore deduce that

\[
\Theta_{g,n+1} \leq \left( 1 + \frac{n + 9}{4g} \right) \left( \frac{2}{3} \left( 1 + \frac{n + 2T + 6}{4(g - Tn - T^2)} \right) \right) T^{-1} \\
\times \mathbb{E}\left[ \Theta_{g-Tn-T^2,w_{n-1}(T-1)} \right] \\
+ \frac{1}{3} \left( 1 + \frac{n + 2T + 8}{4(g - Tn - T^2)} \right) T^{-1} \mathbb{E}\left[ \Theta_{g-Tn-T^2,w_{n+1}(T-1)} \right] \\
\leq \left( 1 + \frac{n + 9}{4g} \right) \left( 1 + \frac{n + 2T + 8}{4(g - Tn - T^2)} \right) T^{-1} \\
\times \left( \frac{2}{3} \mathbb{E}\left[ \Theta_{g-Tn-T^2,w_{n-1}(T-1)} \right] + \frac{1}{3} \mathbb{E}\left[ \Theta_{g-Tn-T^2,w_{n+1}(T-1)} \right] \right).
\]

(5.13)
Now, observe for any integer $G \geq 0$ that
\[
\mathbb{E} \left[ \Theta_{G, w_n(T)} \right] = \mathbb{P} \left[ w_n(1) = n - 1 \right] \mathbb{E} \left[ \Theta_{G, w_{n-1}(T-1)} \right] + \mathbb{P} \left[ w_n(1) = n + 1 \right] \mathbb{E} \left[ \Theta_{G, w_{n+1}(T-1)} \right] = \frac{2}{3} \mathbb{E} \left[ \Theta_{G, w_{n-1}(T-1)} \right] + \frac{1}{3} \mathbb{E} \left[ \Theta_{G, w_{n+1}(T-1)} \right],
\]
(5.14)
Inserting the $G = g - Tn - T^2$ case of (5.14) into (5.13) yields
\[
\Theta_{g, n+1} \leq \left( 1 + \frac{n + 9}{4g} \right) \left( 1 - \frac{n + 2T + 8}{4(g - Tn - T^2)} \right)^{T-1} \mathbb{E} \left[ \Theta_{g-nT-T^2, w_n(T)} \right] \leq \left( 1 + \frac{n + 2T + 9}{4(g - Tn - T^2)} \right)^T \mathbb{E} \left[ \Theta_{g-nT-T^2, w_n(T)} \right] = H_{g, n}(T),
\]
from which we deduce (5.12) and therefore the proposition.

Now we can establish Proposition 4.2.

**Proof of Proposition 4.2** If $n \in \{1, 2\}$, then Lemmas 2.7 and 2.8 imply that $\langle d \rangle_{g, n} \leq 1$, from which the proposition follows. So, we may assume in the below that $n \geq 3$. Throughout the remainder of this proof, we set $t = 10n + 30\lceil \log g \rceil$. Now, observe that $g > (t + 2)n + t^2$, since
\[
(t + 2)n + t^2 = 10n^2 + 2n + 30n\lceil \log g \rceil + (10n + 30\lceil \log g \rceil)^2 \\
\leq 227n^2 + 1815\lceil \log g \rceil^2 \\
< 400n^2 + \frac{g}{2} < g,
\]
where in the second statement we used the facts that $30n\lceil \log g \rceil \leq 15n^2 + 15\lceil \log g \rceil^2$, $n \geq 1$, and $(a + b)^2 \leq 2a^2 + 2b^2$ (with $a = 10n$ and $b = 30\lceil \log g \rceil$); in the third, we used the fact that $g > 2^{30}$; and, in the fourth, we used the fact that $g > 800n^2$.

So Proposition 5.5 applies, which together with Definition 5.4, yields (recalling the random walk $w_n(t)$ from Definition 4.6)
\[
\langle d \rangle_{g, n} \leq F_{g, n}(t) \leq \left( 1 + \frac{n + 2t + 9}{4(g - tn - t^2)} \right)^t \left( 1 + \mathbb{E} \left[ \left( \frac{3}{2} \right)^{w_n(t)} 1_{w_n(t) > 2} \right] \right).
\]
In view of Lemma 4.9; the fact that $\frac{t}{10} = n + 3\lceil \log g \rceil \leq n + 4 \log g$ for $g > 2^{30}$; the fact that $g \geq 2(tn + t^2)$ for $g > 2^{30}$ (following the proof
of (5.15), using the bound $g \geq 800n^2$; and the estimates $\left(\frac{2}{3}\right)^3 < e^{-1}$ and $x + 1 \leq e^x$ for any $x \geq 0$, it follows that

$$
\langle d \rangle_{g,n} \leq F_{g,n}(t) \leq \left(1 + \frac{20n + 20 \log g}{g - tn - t^2}\right)^t \left(1 + \left(\frac{2}{3}\right)^{-3 \log g}\right)^t \\
\leq \left(1 + \frac{40n + 40 \log g}{g}\right)^{10n + 40 \log g} \left(1 + \frac{1}{g}\right) \\
\leq \exp \left(\frac{(40n + 40 \log g)(10n + 40 \log g)}{g} + \frac{1}{g}\right) \\
\leq \exp \left(g^{-1}(20n + 40 \log g)^2\right),
$$

which implies the proposition. \qed

6 Multi-variate harmonic sums

In this section we provide estimates and asymptotics for certain types of multi-variate harmonic sums that will appear in the proof of Theorem 1.7. After stating these results in Sect. 6.1, we introduce generating series for these sums in Sect. 6.2. Using these series, we will establish one of the results (Lemma 6.3) from Sect. 6.1 in Sect. 6.3; the other will be proven in Sect. 7 below.

6.1 Estimates on multi-variate harmonic sums

Recalling $C_k(m)$ from Sect. 2, the following definition recalls two types of multi-variate harmonic sums from equations (1.45) and (1.46) of [11].

**Definition 6.1** For any integers $m \geq k \geq 1$, define

$$
H_k(m) = \sum_{a \in C_k(m)} \prod_{i=1}^{k} \frac{1}{a_i}; \quad Z_k(m) = \sum_{a \in C_k(m)} \prod_{i=1}^{k} \frac{\zeta(2a_i)}{a_i},
$$

(6.1)

where we have denoted $a = (a_1, a_2, \ldots, a_k) \in \mathbb{Z}_{\geq 1}^k$. We also set $Z_0(0) = 1$; $Z_0(m) = 0$ for any integer $m \geq 1$ and $Z_k(m) = 0$ if $m < k$.

The series $Z_k(m)$ will appear in Sects. 8, 9 and 10 below when analyzing the limiting behavior of $\text{Vol} \mathcal{Q}_{g,n}$. Although the function $H_k(m)$ will not be relevant for the purposes of this paper, it was originally intended on being used in the work [10] to be established more geometric information about random surfaces in $\mathcal{Q}_{g,n}$. Since the analyses of $Z_k(m)$ and $H_k(m)$ are in any case entirely analogous, we provide the asymptotics for both here.
We will require several estimates and asymptotics on these sums. The first is given by the following lemma.

**Lemma 6.2** Fix integers \( m, j \geq 0 \) and \( r \geq 1 \), and a composition \( j = (j_1, j_2, \ldots, j_r) \in \mathcal{K}_r(j) \). Then

\[
\sum_{m \in \mathcal{K}_r(m)} \prod_{k=1}^{r} Z_{j_k}(m_k) \leq Z_j(m),
\]

where we have denoted \( m = (m_1, m_2, \ldots, m_r) \in \mathcal{K}_r(m) \).

**Proof** Since \( Z_0(0) = 1 \) and \( Z_0(m) = 0 \) for \( m > 0 \), it suffices to address the situation when \( j_k \geq 1 \) for each \( k \), namely, that \( j \in \mathcal{C}_r(j) \). In this case, we have

\[
\sum_{m \in \mathcal{K}_r(m)} \prod_{k=1}^{r} Z_{j_k}(m_k) = \sum_{m \in \mathcal{C}_r(m)} \sum_{a^{(1)} \in \mathcal{C}_{j_1}(m_1)} \cdots \sum_{a^{(r)} \in \mathcal{C}_{j_r}(m_r)} \prod_{k=1}^{r} \prod_{i=1}^{j_k} \frac{1}{a^{(k)}_i}, \tag{6.2}
\]

where we have denoted the compositions \( a^{(k)} = (a^{(j)}_1, a^{(j)}_2, \ldots, a^{(j)}_{j_k}) \in \mathcal{C}_{j_k}(m_k) \) (and used the fact that \( Z_{j_k}(0) = 0 \) to restrict the sum over \( m \in \mathcal{K}_r(m) \) on the left side of (6.2) to one over \( m \in \mathcal{C}_r(m) \) on the right side). The facts that \( \sum_{k=1}^{r} j_k = j \) and \( \sum_{k=1}^{r} m_k = m \) together imply for any \( m \in \mathcal{C}_r(m) \) that the (ordered) union \( a = \bigcup_{k=1}^{r} a^{(k)} \in \mathcal{C}_j(m) \). Since any \( a \) can be obtained in this way from at most one family of compositions \( (m, a^{(1)}, a^{(2)}, \ldots, a^{(r)}) \), (6.2) yields

\[
\sum_{m \in \mathcal{K}_r(m)} \prod_{k=1}^{r} Z_{j_k}(m_j) \leq \sum_{a \in \mathcal{C}_r(m)} \prod_{k=1}^{r} \frac{1}{a_j},
\]

from which we deduce the lemma. \( \square \)

Next, we have the following bound on \( Z_k(N) \) that will be established in Sect. 6.3 below.

**Lemma 6.3** For any integers \( N \geq k \geq 1 \), we have

\[
Z_k(N) \leq \frac{2k(\log N + 5)^{k-1}}{N}.
\]

The following proposition provides asymptotics for certain linear combinations of the series \( H_k(N) \) and \( Z_k(N) \). It was predicted as Conditional Theorem D.9 of [11] and will be established in Sect. 7.1 below. Subsequent to the appearance of this article, an alternative proof of (a generalization of) the below proposition was estabished as Theorem 3.7 of [10].
Proposition 6.4 For any fixed real number $\omega > \frac{1}{2}$, we have

$$\lim_{N \to \infty} N^{1/2} \sum_{k=1}^{\lfloor \omega \log N \rfloor} \frac{H_k(N)}{2^{k-1}k!} = 2\pi^{-1/2};$$

$$\lim_{N \to \infty} N^{1/2} \sum_{k=1}^{\lfloor \omega \log N \rfloor} \frac{Z_k(N)}{2^{k-1}k!} = 2^{3/2}\pi^{-1/2}. \tag{6.3}$$

6.2 Contour integral representations

In this section we provide generating series, which lead to exact contour integral representations, for the sums $H_k(N)$ and $Z_k(N)$ (given by (6.9) below); we will then asymptotically analyze these integrals later, in Sect. 7. To that end, we begin with the following lemma.

In the below, we define $f, g, F_k, G_k : \mathbb{C} \setminus \mathbb{Z} \to \mathbb{C}$ by

$$f(w) = -\log(1 - w); \quad g(w) = -\sum_{j=1}^{\infty} \log \left(1 - \frac{w}{j^2}\right);$$

$$F_k(w) = f(w)^k; \quad G_k(w) = g(w)^k, \tag{6.4}$$

where here are using the principal branch of the logarithm. Observe that the series for $g(w)$ converges since $|\log \left(1 - \frac{w}{j^2}\right)| < \frac{2|w|}{j^2}$, for sufficiently large $j$, and $\sum_{j=1}^{\infty} \frac{|w|}{j^2} < \infty$.

Lemma 6.5 For any $w \in \mathbb{C}$ with $|w| < 1$, we have

$$F_k(w) = \sum_{m=k}^{\infty} H_k(m)w^m; \quad G_k(w) = \sum_{m=k}^{\infty} Z_k(m)w^m. \tag{6.5}$$

Proof By Definition 6.1, we have

$$\sum_{m=k}^{\infty} H_k(m)w^m = \left(\sum_{m=1}^{\infty} \frac{w^m}{m}\right)^k; \quad \sum_{m=k}^{\infty} Z_k(m)w^m = \left(\sum_{m=1}^{\infty} \frac{\zeta(2m)w^m}{m}\right)^k. \tag{6.6}$$
The first statement of (6.5) then follows from (6.6) and the fact that \( f(w) = \sum_{m=1}^\infty \frac{w^m}{m} \) for \(|w| < 1\). To establish the second, observe that

\[
\sum_{m=1}^\infty \frac{\zeta(2m)w^m}{m} = \sum_{m=1}^\infty \sum_{j=1}^\infty \frac{w^m}{j^{2m}m} = \sum_{j=1}^\infty \sum_{m=1}^\infty \frac{1}{m} \left(\frac{w}{j^2}\right)^m
\]

\[
= -\sum_{j=1}^\infty \log \left(1 - \frac{w}{j^2}\right) = g(w),
\]

which by the second identity in (6.6) yields the second statement of (6.5). \(\square\)

By Lemma 6.5, we have

\[
H_k(N) = \frac{1}{2\pi i} \oint_C w^{-N-1} F_k(w)dw; \quad Z_k(N) = \frac{1}{2\pi i} \oint_C w^{-N-1} G_k(w)dw,
\]

(6.7)

where the contour \( C \) is a positively oriented circle centered at 0 with radius \( \frac{1}{2} \) (see the left side of Fig. 1). It will next be useful to deform the contour \( \mathcal{C} \) to the contour \( \gamma \), defined as follows.

**Definition 6.6** Fix the real numbers

\[
R = R_N = 1 + \frac{(\log N)^3}{N}; \quad \kappa = \sqrt{R^2 - N^2},
\]

(6.8)

and define the subsets \( \gamma_0, \gamma_1, \gamma_2, \gamma_3 \subset \mathbb{C} \) by

\[
\gamma_0 = \left\{ z \in \mathbb{C} : |z| = R, |\Im z| > \frac{1}{N} \right\} \cup \left\{ z \in \mathbb{C} : |z| = R, \Re z < 0 \right\};
\]

\[
\gamma_1 = \left\{ z \in \mathbb{C} : \Re z \in [1, \kappa], \Im z = \frac{1}{N} \right\};
\]

\[
\gamma_3 = \left\{ z \in \mathbb{C} : \Re z \in [1, \kappa], \Im z = -\frac{1}{N} \right\};
\]

\[
\gamma_2 = \left\{ z \in \mathbb{C} : |z - 1| = \frac{1}{N}, \Re z < 1 \right\}.
\]

Define the contour \( \gamma = \gamma_0 \cup \gamma_1 \cup \gamma_2 \cup \gamma_3 \), oriented counterclockwise. We refer to the right side of Fig. 1 for a depiction.
Since $F$ and $G$ are analytic away from $\mathbb{Z}$, we may deform the contour $C$ in (6.7) to $\gamma$, if $N > 2^{15}$ (so that $R < \frac{3}{2}$). Thus, (6.7) yields

\begin{align*}
H_k(N) &= \frac{1}{2\pi i} \int_{\gamma_0} w^{-N-1} F_k(w) dw + \frac{1}{2\pi i} \int_{\gamma_1 \cup \gamma_2 \cup \gamma_3} w^{-N-1} F_k(w) dw; \\
Z_k(N) &= \frac{1}{2\pi i} \int_{\gamma_0} w^{-N-1} G_k(w) dw + \frac{1}{2\pi i} \int_{\gamma_1 \cup \gamma_2 \cup \gamma_3} w^{-N-1} G_k(w) dw.
\end{align*}

(6.9)

### 6.3 Proof of Lemma 6.3

In this section we establish Lemma 6.3 using the generating series given by Lemma 6.5. To that end, we begin with the following two bounds.

**Lemma 6.7** For any integer $N \geq 1$, we have

$$
\sum_{j=1}^{N} \frac{\zeta(2j)}{j} \leq \log N + 5.
$$

**Proof** Observe for any integer $k \geq 2$ that

$$
\zeta(k) - 1 = \sum_{m=2}^{\infty} m^{-k} = 2^{-k} \sum_{m=2}^{\infty} \left( \frac{2}{m} \right)^k \leq 2^{-k} \sum_{m=2}^{\infty} 4m^{-2} = 2^{-k} \left( 4 \zeta(2) - 4 \right) \leq 3 \cdot 2^{-k}.
$$
Thus,
\[
\sum_{j=1}^{N} \frac{\zeta(2j)}{j} \leq \sum_{j=1}^{N} \frac{1}{j} + 3 \sum_{j=1}^{\infty} 2^{-2j} \leq \log N + 5,
\]
from which we deduce the lemma. \Box

**Corollary 6.8** For any integers \( N \geq k \geq 1 \), we have
\[
\sum_{m=1}^{N} Z_k(m) \leq (\log N + 5)^k. \tag{6.10}
\]

**Proof** First observe that
\[
\sum_{m=1}^{N} Z_k(m) \leq \left( \sum_{j=1}^{N} \frac{\zeta(2j)}{j} \right)^k, \tag{6.11}
\]
since the expansion of the right side of (6.11) contains each summand appearing in the definition (6.1) of \( Z_k(m) \), for every \( m \in [1, N] \). Thus, the corollary follows from (6.11) and Lemma 6.7. \Box

Now we can establish Lemma 6.3.

**Proof of Lemma 6.3** Since Lemma 6.5 implies that \( Z_k(N) \) is the coefficient of \( w^N \) in \( G_k(w) = g(w)^k \), it follows that \( N Z_k(N) \) is the coefficient of \( w^{N-1} \) in
\[
\frac{\partial}{\partial w} (g(w)^k) = k g'(w) g(w)^{k-1} = k \left( \sum_{j=1}^{\infty} \frac{1}{j^2 - w} \right) G_{k-1}(w)
= k \left( \sum_{j=1}^{\infty} \frac{1}{j^2} \sum_{i=0}^{\infty} w^i \right) \sum_{m=k-1}^{\infty} Z_{k-1}(m) w^m.
\]

Hence,
\[
Z_k(N) = \frac{k}{N} \sum_{j=1}^{\infty} \frac{1}{j^2} \sum_{m=k-1}^{N-1} Z_{k-1}(m) \leq \frac{k}{N} \sum_{j=1}^{\infty} \frac{1}{j^2} \sum_{m=k-1}^{N-1} Z_{k-1}(m)
\leq \frac{k \zeta(2)}{N} \sum_{m=1}^{N} Z_{k-1}(m)
\leq \frac{2k(\log N + 5)^{k-1}}{N},
\]
where in the last estimate we applied Corollary 6.8 and the bound $\zeta(2) \leq 2$. \square

7 Asymptotics for $H_k(N)$ and $Z_k(N)$

In Sect. 7.1 we establish Proposition 6.4, assuming an asymptotic expansion (given by Proposition 7.2 below) of the sums $H_k(N)$ and $Z_k(N)$, as $N$ tends to $\infty$. The remainder of this section is directed toward the proof of this expansion, which proceeds through an asymptotic analysis of the contour integral representation given by (6.9).

7.1 Proof of Proposition 6.4

We will establish Proposition 6.4 using asymptotic expansions for $H_k(N)$ and $Z_k(N)$ (as $N$ tends to $\infty$). To state this expansion, we first require the following coefficients $\varphi_j$.

**Definition 7.1** For each $j \in \mathbb{Z}_{\geq 0}$ and $s \in \mathbb{C} \setminus \mathbb{Z}_{\geq 1}$, set

$$
\Phi_j(s) = \frac{1}{\pi} \frac{\partial^j}{\partial s^j} \left( \Gamma(1 - s) \sin(\pi s) \right); \quad \varphi_j = \Phi_j(0).
$$

Now we can state the following proposition, which provides asymptotic expansions for $H_k(N)$ and $Z_k(N)$, with explicit estimates on the respective errors $\varepsilon^H_k(N)$ and $\varepsilon^Z_k(N)$. It was predicted as Conjecture 1.30 of [11] and will be proven in Sect. 7.5 below.

**Proposition 7.2** There exists a constant $C > 1$ such that the following holds. Let $k$ and $N$ be positive integers satisfying $k \leq (\log N)^2$. Setting

$$
\varepsilon^H_k(N) = \left| N H_k(N) - \sum_{j=1}^{k} \binom{k}{j} \varphi_j (\log N)^{k-j} \right|;
$$

$$
\varepsilon^Z_k(N) = \left| N Z_k(N) - \sum_{j=1}^{k} \binom{k}{j} \varphi_j (\log N + \log 2)^{k-j} \right|
$$

we have

$$
\varepsilon^H_k(N) \leq C 2^k k! (\log N)^{14} N^{-1/2}; \quad \varepsilon^Z_k(N) \leq C 2^k k! (\log N)^{14} N^{-1/2}.
$$

(7.1)

Theorem 6.2 of [21] establishes the same expansion for $H_k(N)$, but does not make the $k$-dependence of the error bound explicit or address the series $Z_k(N)$. 

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Still, the proof of Proposition 7.2 will closely follow that of Theorem 6.2 of [21].

Assuming Proposition 7.2, we can now prove Proposition 6.4. To that end, we first require the following lemma that provides a generating series for the coefficients \( \varphi_j \) through the function \( \Phi_0 \).

**Lemma 7.3** For any \( z \in \mathbb{C} \) with \( |z| < 1 \), we have

\[
\sum_{j=0}^{\infty} \frac{|\varphi_j z^j|}{j!} < \infty; \quad \Phi_0(z) = \sum_{j=0}^{\infty} \frac{\varphi_j z^j}{j!}. \tag{7.2}
\]

**Proof** The first statement of (7.2) follows from the facts that \( \Phi_0 \) is analytic on \( \{z \in \mathbb{C} : |z| < 1\} \) and that \( \varphi_j = \frac{\partial^j}{\partial z^j} \Phi_0(0) \). The second follows from these two facts and a Taylor expansion. \( \square \)

Now we can establish Proposition 6.4.

**Proof of Proposition 6.4** Assuming Proposition 7.2 We only establish the second identity in (6.3), as the proof of the first is entirely analogous. To that end, set

\[
\delta = \delta_\omega = \frac{1}{4}(\omega - \frac{1}{2}); \quad \mu = \mu_\omega(N) = \lfloor \omega \log N \rfloor; \\
\nu = \nu_\omega(N) = \lfloor \delta \log N \rfloor.
\]

By Proposition 7.2; the fact that \( \mu = O(\log N) \); and the fact that \( \varphi_0 = 0 \), we have

\[
N^{1/2} \sum_{k=1}^{\mu} \frac{Z_k(N)}{2^{k-1}k!} = N^{-1/2} \sum_{k=1}^{\mu} \frac{1}{2^{k-1}k!} \sum_{j=1}^{k} \binom{k}{j} \varphi_j (\log N + \log 2)^{k-j} \\
+ O \left( N^{-1/2} \sum_{k=1}^{\mu} \frac{\epsilon_k^H(N)}{2^k k!} \right) \\
= N^{-1/2} \sum_{k=1}^{\mu} \sum_{j=1}^{k} \frac{(\log N + \log 2)^{k-j}}{2^{k-j}(k-j)!} \varphi_j 2^{j-1} j! \\
+ O \left( \frac{\mu (\log N)^{14}}{N} \right) \\
= 2N^{-1/2} \sum_{j=0}^{\mu} \varphi_j 2^j j! \sum_{r=0}^{\mu-j} \frac{(\log N + \log 2)^r}{2^r r!} \\
+ O \left( \frac{(\log N)^{15}}{N} \right),
\]

\( \square \)
where we have set \( r = k - j \), and the implicit constant is uniform in \( N \) (but might depend on \( \omega \)).

Next, let us replace the sum over \( j \in [0, \mu] \) in the right side of (7.3) with a sum over \( j \in [0, \nu] \). To that end, observe that

\[
2N^{-1/2} \left| \sum_{j=0}^{\mu} \frac{\varphi_j}{2^j j!} \sum_{r=0}^{\mu-j} \frac{(\log N + \log 2)^r}{2^r r!} - \sum_{j=0}^{\nu} \frac{\varphi_j}{2^j j!} \sum_{r=0}^{\mu-j} \frac{(\log N + \log 2)^r}{2^r r!} \right| \\
 \leq 2N^{-1/2} \sum_{j=\nu+1}^{\mu} \frac{|\varphi_j|}{2^j j!} \sum_{r=0}^{\infty} \frac{(\log N + \log 2)^r}{2^r r!} \\
 = 2N^{-1/2} \exp \left( \frac{\log N + \log 2}{2} \right) \sum_{j=\nu+1}^{\mu} \frac{|\varphi_j|}{2^j j!} \\
 \leq 2^{3/2} \sum_{j=\nu+1}^{\infty} \frac{|\varphi_j|}{2^j j!} = o(1), \tag{7.4}
\]

where for the last estimate we used the first statement of (7.2) and the fact that \( \lim_{N \to \infty} \nu_{\omega}(N) = \infty \).

Inserting (7.4) into (7.3) yields

\[
N^{1/2} \sum_{k=1}^{\mu} \frac{Z_k(N)}{k!2^{k-1}} = 2N^{-1/2} \sum_{j=0}^{\nu} \frac{\varphi_j}{2^j j!} \sum_{r=0}^{\mu-j} \frac{(\log N + \log 2)^r}{2^r r!} + o(1). \tag{7.5}
\]

Now, observe from the \( R = \frac{\log(2N)}{2} \) case of Lemma 2.4 that

\[
\left| \sum_{r=0}^{\mu-j} \frac{(\log N + \log 2)^r}{2^r r!} - (2N)^{1/2} \right| \\
 \leq \delta^{-1} (1 + \delta)^{-\delta} \log N/2 (2N)^{1/2} = o(N^{1/2}) \text{ for } j \leq \nu,
\]

which upon insertion into (7.5) (using the first statement of (7.2)) yields

\[
N^{1/2} \sum_{k=1}^{\mu} \frac{Z_k(N)}{k!2^{k-1}} = 2^{3/2} \sum_{j=0}^{\nu} \frac{\varphi_j}{2^j j!} + o(1).
\]
Together with the $z = \frac{1}{2}$ case of Lemma 7.3 and the fact that $\lim_{N \to \infty} \nu_\omega(N) = \infty$, this yields

$$N^{1/2} \sum_{k=1}^{\mu} \frac{Z_k(N)}{k! 2^{k-1}} = 2^{3/2} \Phi_0 \left( \frac{1}{2} \right) + o(1),$$

from which we deduce the theorem since $\Phi_0 \left( \frac{1}{2} \right) = \pi^{-1} \Gamma \left( \frac{1}{2} \right) = \pi^{-1/2}$. \hfill $\Box$

### 7.2 Scaling around $w = 1$

In this section we first bound the contributions of the first terms (integrals along $\gamma_0$) on the right sides of (6.9); we then change variables in the remaining integrals (over $\gamma_1 \cup \gamma_2 \cup \gamma_3$ in (6.9) by scaling $w$ by a factor of $N$ around 1. The following lemma implements the former task.

Throughout the remainder of this paper, we assume that $N \geq 2^90$ and omit the parameter $k$ from our notation, abbreviating $F = F_k$; $G = G_k$; $H(m) = H_k(m)$; and $Z(m) = Z_k(m)$.

**Lemma 7.4** We have that

$$\left| \int_{\gamma_0} w^{-N-1} F(w) dw \right| \leq \frac{2\pi (\log N)^k}{N^2};$$

$$\left| \int_{\gamma_0} w^{-N-1} G(w) dw \right| \leq \frac{2\pi (\log N + 2)^k}{N^2}. \tag{7.6}$$

**Proof** Recalling the functions $f$ and $g$ from (6.4), observe since $N \geq 3$ that

$$\sup_{w \in \gamma_0} |f(w)| \leq \max \left\{ |\log(R - 1)|, |\log(R + 1)| \right\} \leq \max \{ \log N, \log 3 \} = \log N. \tag{7.7}$$

Together with the facts that $|w| = R$ on $\gamma_0$ and that the length of $\gamma_0$ is less than $2\pi R$, (7.7) yields

$$\left| \int_{\gamma_0} w^{-N-1} F(w) dw \right| \leq 2\pi (\log N)^k R^{-N}. \tag{7.8}$$

Now the first estimate in (7.6) follows from (7.8) and the fact that $R^{-N} < N^{-2}$ (which holds since $R = 1 + \frac{(\log N)^3}{N}$ and $N \geq 2^90$).
To establish the second, observe since $N \geq 2^{90}$ that $R < \frac{3}{2}$. Therefore,

$$
\sup_{w \in \gamma_0} \left| g(w) \right| \leq \sup_{w \in \gamma_0} \left| f(w) \right| + \sum_{j=2}^{\infty} \sup_{w \in \gamma_0} \left| \log \left( 1 - \frac{w^2}{j^2} \right) \right| 
\leq \log N + \sum_{j=2}^{\infty} \log \left( 1 + \frac{9}{4j^2} \right) \leq \log N + \frac{9}{4} \sum_{j=2}^{\infty} \frac{1}{j^2} \quad (7.9)
$$

$$
< \log N + 2,
$$

where we have used the facts that $\log(1 + z) \leq z$ for $z \geq 0$ and that $\sum_{j=2}^{\infty} \frac{1}{j^2} < \frac{2}{3}$. Now the proof of the second estimate in (7.6) is entirely analogous to that of the former. \(\square\)

The remaining quantities on the right sides of (6.9) involve integration over $\gamma_1 \cup \gamma_2 \cup \gamma_3$. Since this contour lies in (slightly larger than) a $N^{-1}$-neighborhood of $w = 1$, we change variables

$$
w = 1 + \frac{t}{N},
$$

(7.10)

**Definition 7.5** Define the contours $\Xi_1 = \Xi_1(N)$, $\Xi_2 = \Xi_2(N)$, and $\Xi_3 = \Xi_3(N)$ to be the images of $\gamma_1, \gamma_2, \gamma_3$ under (7.10); also set $\Xi = \Xi(N) = \Xi_1 \cup \Xi_2 \cup \Xi_3$. We refer to Fig. 2 for a depiction, where there $\rho = N(\kappa - 1)$ is the image of $\kappa$ under the change of variables (7.10).

**Remark 7.6** By (6.8) and (7.10), $\rho$ is quickly seen to satisfy $(\log N)^3 - 1 < \rho < (\log N)^3$.

Denoting the function

$$
\Psi(t) = \Psi(t; N) = \log N - \log(-t) - \sum_{j=2}^{\infty} \log \left( 1 - \frac{1}{j^2} - \frac{t}{Nj^2} \right), \quad (7.11)
$$

(6.9) and (7.6) together imply that

$$
H(N) = \frac{1}{2\pi iN} \int_{\Xi} \left( 1 + \frac{t}{N} \right)^{-N-1} \left( \log N - \log(-t) \right)^k dt + O\left( \frac{\log N}{N^2} \right);
$$

$$
Z(N) = \frac{1}{2\pi iN} \int_{\Xi} \left( 1 + \frac{t}{N} \right)^{-N-1} \Psi(t)^k dt + O\left( \frac{(\log N + 2)^k}{N^2} \right),
$$

(7.12)
where the implicit constants are uniform in $k$ and $N$. Observe that $-t$ does not intersect the branch cut (which is $\mathbb{R}_{\leq 0}$) for the principal logarithm, as $t$ ranges over $\Xi$.

### 7.3 The limiting integrands

In this section we establish the following proposition, which approximates the integrands on the right sides of (7.12) by simpler quantities.

**Proposition 7.7** We have that

\[
H(N) = \frac{1}{2\pi iN} \int_\Xi e^{-t} \left( \log N - \log(-t) \right)^k dt + O\left( \frac{(\log N + 4 \log \log N)^k + 9}{N^2} \right),
\]

\[
Z(N) = \frac{1}{2\pi iN} \int_\Xi e^{-t} \left( \log N - \log(-t) + \log 2 \right)^k dt + O\left( \frac{k(\log N + 5 \log \log N)^k + 9}{N^2} \right),
\]

(7.13)

where the implicit constants are uniform in $k$ and $N$.

In particular, this approximates $(1 + \frac{i}{N})^{-N-1} \approx e^{-t}$ and $\Psi(t) \approx \log N - \log(-t) + \log 2$ (recall (7.11)) in (7.12). To implement the former, observe from a Taylor expansion of $\log(1 + z)$ and the fact that $|t| \leq (\log N)^3 < N^{1/3}$.
for any $t \in \Xi$ (since $N > 2^{90}$) that

$$\sup_{t \in \Xi} \left| e^{t \left(1 + \frac{t}{N}\right)} - 1 \right| = O\left(\sup_{t \in \Xi} \frac{|t|^2}{N}\right) = O\left(\frac{(|t|^2)}{N}\right),$$  \quad (7.14)$$

where the implicit constants are uniform in $t$ and $N$.

The following lemma approximates $\Psi(t) \approx \log N - \log(-t) + \log 2$.

**Lemma 7.8** For any integers $N \geq 2^{90}$ and $k \geq 1$, we have

$$\sup_{t \in \Xi} \left| \Psi(t) \right| < \log N + 5 \log \log N;$$

$$\sup_{t \in \Xi} \left| \Psi(t)^k - (\log N - \log(-t) + \log 2)^k \right| \leq \frac{2k(\log N + 5 \log \log N)^{k+2}}{N}. \quad (7.15)$$

**Proof** By a Taylor expansion of $\log(1 + z)$ and the facts that $N \geq 2^{90}$ and $|t| \leq (\log N)^3 < N^{1/3}$ whenever $t \in \Xi$, we have

$$\sup_{t \in \Xi} \left| \log \left(1 - \frac{1}{j^2} - \frac{t}{Nj^2}\right) - \log \left(1 - \frac{1}{j^2}\right) \right| < \frac{2|t|}{Nj^2},$$

for any $j \geq 2$. Summing over $j$ and using the facts that $\sum_{j=2}^{\infty} \log \left(1 - \frac{1}{j^2}\right) = -\log 2$ and $\sum_{j=2}^{\infty} \frac{1}{j^2} < 1$, we obtain

$$\sup_{t \in \Xi} \left| \sum_{j=2}^{\infty} \log \left(1 - \frac{1}{j^2} - \frac{t}{Nj^2}\right) - \log 2 \right| < \frac{2|t|}{N} \leq \frac{2(\rho + 1)}{N}. \quad (7.16)$$

Thus, the first bound in (7.15) follows from (7.11); (7.16); Remark 7.6; and the facts that $N \geq 2^{90}$ and that $|\log(-t)| \leq \log |t| + 2\pi \leq 4 \log \log N$ (which holds by Remark 7.6). The second bound in (7.15) follows from the first bound there; (7.16); Remark 7.6; and the facts that $N \geq 2^{90}$ and that $|A^k - B^k| \leq k(|A| - |B|) \max\{|A|^{k-1}, |B|^{k-1}\}$ for any $A, B \in \mathbb{C}$. \hfill \qed

Now we can establish Proposition 7.7.

**Proof of Proposition 7.7** Let us only establish the second bound in (7.13), as the proof of the first is entirely analogous. To that end, (7.14); the fact that the length of the contour $\Xi$ is at most $2\rho + \pi \leq 3(\log N)^3$ (by Remark 7.6); and
the first bound in (7.15) together imply that
\[
\left| \int_{\Xi} \left(1 + \frac{t}{N}\right)^{-N-1} \Psi(t) dt - \int_{\Xi} e^{-t} \Psi(t) dt \right| = O\left(\frac{(\log N)^9}{N} \sup_{t \in \Xi} |\Psi(t)|^k \right) = O\left(\frac{(\log N + 5 \log \log N)^{k+9}}{N}\right), \tag{7.17}
\]
where the implicit constants are uniform in $N$ and $k$. Moreover, from the second bound of (7.15) (and from the fact that the length of $\Xi$ is at most $3(\log N)^3$), we deduce that
\[
\left| \int_{\Xi} e^{-t} \Psi(t) dt - \int_{\Xi} e^{-t} \left(\log N - \log(-t) + 2\right)^k dt \right| < \frac{6k(\log N + 5 \log \log N)^{k+5}}{N} \sup_{t \in \Xi} |e^{-t}|, \tag{7.18}
\]
where to establish the second estimate we used the fact that $\inf_{t \in \Xi} \Re t = -1$. Now the second bound in (7.13) follows from the second bound in (7.12), (7.17), and (7.18).

7.4 The limiting contour

Observe that the contour $\Xi$ in (7.13) depends on $N$. In this section we will estimate the error in replacing $\Xi$ by a contour that is independent of $N$, given by the following definition.

**Definition 7.9** Define the contours
\[
\Omega_1 = \{t \in \mathbb{C} : \Re t \geq 0, \Im t = 1\}; \quad \Omega_2 = \left\{t = e^{i\theta} : \theta \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]\right\}; \quad \Omega_3 = \{t \in \mathbb{C} : \Re t \geq 0, \Im t = -1\},
\]
and set $\Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3$.

Observe that $\Xi_2 = \Omega_2$ and, since $\lim_{N \to \infty} N(R_N - 1) = \infty$, the limits of the contours $\Xi_1(N)$ and $\Xi_3(N)$ are $\Omega_1$ and $\Omega_3$, respectively, as $N$ tends to $\infty$. The difference between $\Xi$ and $\Omega$ is depicted as dashed in Fig. 2. The following proposition estimates the error in replacing the integration over $\Xi$ in (7.13) with integration over $\Omega$. 

\begin{figure}[h]  
\centering  
\includegraphics[width=\textwidth]{fig2.png}  
\caption{The limiting contour $\Omega$.}  
\end{figure}
Proposition 7.10  For any integers \( N > 1 \) and \( k \leq (\log N)^2 \), we have

\[
\begin{align*}
\int_{\Omega \setminus \Xi} e^{-t\left(\log N - \log(-t)\right)^k} dt &< \frac{2}{N^2}; \\
\int_{\Omega \setminus \Xi} e^{-t\left(\log N - \log(-t) + \log 2\right)^k} dt &< \frac{2}{N^2},
\end{align*}
\]  

where the implicit constants are independent of \( k \) and \( N \).

Proof  Let us only establish the second estimate in (7.19), as the proof of the first is entirely analogous. By Remark 7.6 and the facts that \( N \geq 2^{90} \) and \( |\log(-t)| \leq \log |t| + 2\pi \) for \( t \in (\Omega_1 \setminus \Xi_1) \cup (\Omega_3 \setminus \Xi_3) = \Omega \setminus \Xi \), we obtain

\[
\left| \int_{\Omega \setminus \Xi} e^{-t\left(\log N - \log(-t) + \log 2\right)^k} dt \right| < 2 \int_{(\log N)^3 - 1}^{\infty} e^{-s\left(\log N + \log s + 7\right)^k} ds. 
\]

(7.20)

Now, since \( N \geq 2^{90} \), it is quickly verified that

\( (\log N + \log s + 7)^k \leq e^{s/2} \) whenever \( k \leq (\log N)^2 \) and \( s \geq (\log N)^3 - 1 \),

and

\[
\int_{(\log N)^3 - 1}^{\infty} e^{-s/2} ds < \frac{1}{N^2}.
\]

Upon insertion into (7.20), these estimates yield the second bound in (7.19). \( \square \)

7.5 Proof of Proposition 7.2

In this section we establish Proposition 7.2. However, before doing so, we require the following two lemmas. The first bounds the second terms appearing on the right sides of (7.13); the second provides a contour integral representation for the coefficients \( \varphi_j \) from Definition 7.1.

Lemma 7.11  For each integer \( K \geq 1 \), we have

\[
\frac{(\log N + 5 \log \log N)^K}{N} \leq 2^K \frac{K!(\log N)^{5/2}}{N^{1/2}}. 
\]

(7.21)
Proof Recall from (2.8) that $K! \geq \left( \frac{K}{e} \right)^K$, and observe for fixed $v > 0$ that the function $\left( \frac{K}{e} \right)^K$ is minimized at $K = v$. Thus,

$$\frac{2^K K!}{(\log N + 5 \log \log N)^K} \geq \left( \frac{2K}{e(\log N + 5 \log \log N)} \right)^K \geq e^{-(\log N + 5 \log \log N)/2} = N^{-1/2}(\log N)^{-5/2},$$

which verifies (7.21).

Lemma 7.12 For each integer $j \geq 0$, we have

$$\frac{1}{2\pi i} \int_{\Omega} (-\log(-t))^j e^{-t} dt = \varphi_j. \quad (7.22)$$

Proof By Theorem B.1 of [21], we have

$$\frac{1}{2\pi i} \int_{\Omega} (-t)^{-s} e^{-t} dt = \pi^{-1} \Gamma(1-s) \sin(\pi s), \quad (7.23)$$

for each $s \in \mathbb{R}_{<1}$. Thus, (7.22) follows from differentiating (7.23) $j$ times with respect to $s$ and then setting $s = 0$.

Now we can establish Proposition 7.2.

Proof of Proposition 7.2 Let us only establish the second bound in (7.1), as the proof of the first is entirely analogous. Together, Propositions 7.7, 7.10, Lemmas 7.11, 7.12, and the fact that $k \leq (\log N)^2$ yield

$$NZ(N) = \frac{1}{2\pi i} \int_{\Omega} e^{-t}(\log N - \log(-t) + \log 2)^k dt$$

$$+ O\left( \frac{2^k(k+1)! (\log N)^{12}}{N^{1/2}} \right)$$

$$= \frac{1}{2\pi i} \sum_{j=1}^{k} \binom{k}{j} (\log N + \log 2)^{k-j} \int_{\Omega} (-\log(-t))^j e^{-t} dt$$

$$+ O\left( \frac{2^k k! (\log N)^{14}}{N^{1/2}} \right)$$

$$= \sum_{j=1}^{k} \binom{k}{j} \varphi_j (\log N + \log 2)^{k-j} + O\left( \frac{2^k k! (\log N)^{14}}{N^{1/2}} \right),$$
where the implicit constants are uniform in \( k \) and \( N \); this establishes the second bound in (7.1).

\[ \square \]

8 Volume asymptotics for the principal stratum

In this section we analyze the large \( g \) limit for \( \text{Vol} \ Q_{g,n} \), by considering the stable graph contributions to this volume coming from (2.20). In Sect. 8.1 we establish Theorem 1.7, assuming three asymptotic estimates for these contributions arising from graphs with one vertex, two vertices, and at least three vertices; these are given by Propositions 8.3, 8.4 and 8.5 respectively. Then, in Sect. 8.2 we prove the single-vertex asymptotic result (Proposition 8.3), which provides the leading order contribution to the volume. The remaining two results are established in Sects. 9 and 10. Throughout the remainder of this paper, we recall the notation from Sect. 2.3.

8.1 Proof of Theorem 1.7

We begin with the following notation for sets and quantities associated with stable graphs in \( G_{g,n} \) with a prescribed number of vertices.

**Definition 8.1** Fix integers \( g,n \geq 0 \) with \( 2g + n \geq 3 \), and \( V \in [1,2g+n-2] \). Let \( G_{g,n}(V) \subseteq G_{g,n} \) denote the set of stable graphs \( \Gamma \in G_{g,n} \) with \( V \) vertices, that is, such that \( |\mathcal{Q}(\Gamma)| = V \). Further let

\[
\Upsilon^{(V)}_{g,n} = \sum_{\Gamma \in G_{g,n}(V)} \mathcal{Z}(P(\Gamma)). \tag{8.1}
\]

**Remark 8.2** By Proposition 2.13 and (2.17), we have

\[
\text{Vol} \ Q_{g,n} = \sum_{V=1}^{2g+n-2} \Upsilon^{(V)}_{g,n}. \tag{8.2}
\]

Now we can state the following three results that provide asymptotic estimates for \( \Upsilon^{(V)}_{g,n} \) in the cases \( V = 1, \ V = 2, \) and \( V \geq 3 \). We will establish Proposition 8.3 in Sect. 8.2, Proposition 8.4 in Sect. 9.3, and Proposition 8.5 in Sect. 10.3. In the below we recall that, for any two functions \( F_1, F_2 : \mathbb{Z} \to \mathbb{R} \) such that \( F_2(k) \) is nonzero for sufficiently large \( k \), we write \( F_1 \sim F_2 \) if \( \lim_{k \to \infty} F_1(k) F_2(k)^{-1} = 1 \).

**Proposition 8.3** As \( g \) tends to \( \infty \), we have for \( 20n \leq \log g \) that

\[
\Upsilon^{(1)}_{g,n} \sim \pi^{-1} 2^{n+2} \left( \frac{8}{3} \right)^{4g+n-4}. 
\]
Proposition 8.4 Fix integers \( g > 2^{120} \) and \( n \geq 0 \) such that \( 20n < \log g \). Then,
\[
2^{-n} \left( \frac{8}{3} \right)^{-4g-n} \gamma_{g,n}^{(2)} \leq 2^{140} (\log g)^{14} g^{-1/4}.
\]

Proposition 8.5 Fix integers \( g > 2^{500} \) and \( n \geq 0 \) such that \( 20n \leq \log g \). Then,
\[
\sum_{V=3}^{2g-2} 2^{-n} \left( \frac{8}{3} \right)^{-4g-n} \gamma_{g,n}^{(V)} \leq 2^{243} (\log g)^{24} g^{-1/8}.
\]

Given these three results, we can quickly establish Theorem 1.7.

Proof of Theorem 1.7 Assuming Propositions 8.3, 8.4, and 8.5 Due to the identity (8.2), this follows from Propositions 8.3, 8.4 and 8.5. □

Before proceeding, it will be useful introduce the following notation for sets and quantities associated with stable graphs with prescribed total numbers of vertices, self-edges, and simple edges.

Definition 8.6 Fix integers \( g, n \geq 0 \) with \( 2g + n \geq 3 \); \( V \in [1, 2g + n - 2] \); \( S \geq 0 \); and \( T \geq V - 1 \). Set \( E = S + T \), and assume \( E \leq 3g + n - 3 \). Let \( \mathcal{G}_{g,n}(V; S, T) \subseteq \mathcal{G}_{g,n}(V) \) denote the set of stable graphs \( \Gamma \in \mathcal{G}_{g,n}(V) \) with \( V \) vertices, \( S \) self-edges, and \( T \) simple edges (we must have \( T \geq V - 1 \) for \( \Gamma \) to be connected). In particular, \(|\mathcal{E}(\Gamma)| = S + T = E \) holds for any \( \Gamma \in \mathcal{G}_{g,n}(V; S, T) \). Analogously to (8.1), define
\[
\gamma_{g,n}^{(V; S, T)} = \sum_{\Gamma \in \mathcal{G}_{g,n}(V; S, T)} Z(P(\Gamma)),
\]
so that (by (2.18))
\[
\gamma_{g,n}^{(V)} = \sum_{T=0}^{3g+n-3} \sum_{S=0}^{3g+n-3-T} \gamma_{g,n}^{(V; S, T)}.
\]

8.2 Proof of Proposition 8.3

Here we analyze the large genus asymptotics for \( \gamma_{g,n}^{(1)} \) (recall Definition 8.1), which provides the contribution to the right side of (2.20) over all stable graphs \( \Gamma \in \mathcal{G}_{g,n} \) with one vertex. The following definition provides notation for the unique stable genus \( g \) graph with \( n \) legs, one vertex, and \( E \) edges.
Definition 8.7 For any integers $g, n \geq 0$ and $E \in [0, g]$, we define $\Gamma_{g,n}(E) \in G_{g,n}(1)$ to be the stable genus $g$ graph with $n$ legs, one vertex, $E$ self-edges, and genus decoration $g = (g - E)$.

We now have the following lemma that explicitly evaluates $\mathcal{Z}(P(\Gamma_{g,n}(E)))$.

Lemma 8.8 For any integers $g \geq 2$, $n \geq 0$, and $E \in [0, g]$, we have

$$
\mathcal{Z}(P(\Gamma_{g,n}(E))) = \frac{12^E}{2^{2g-1}3^g} \frac{(6g + 2n - 2E - 5)!}{(6g + 2n - 7)!} \frac{(4g + n - 4)!}{(3g + n - E - 3)!(g - E)!} \frac{1}{E!} 
$$

$$
\times \sum_{d \in \mathcal{K}(3g + n - E - 3)} \langle d, 0^n \rangle_{g-E,2E+n} \prod_{j=1}^{E} \left( \frac{2d_{2j-1} + 2d_{2j}}{2d_{2j} + 1} \right) 
$$

$$
\times \zeta(2d_{2j-1}2d_{2j} + 2) \frac{d_{2j-1} + d_{2j} + 1}{d_{2j-1} + d_{2j} + 1},
$$

where we have denoted $\mathbf{d} = (d_1, d_2, \ldots, d_{2E})$ and $(\mathbf{d}, 0^n) = (d_1, d_2, \ldots, d_{2E}, 0, 0, \ldots, 0)$ (containing $n$ zeroes at the end).

Proof Let $b_1, b_2, \ldots, b_E$ denote $E$ variables, and define the set

$$
\mathbf{b} = (b_1, b_1, b_2, b_2, \ldots, b_E, b_E, 0, 0, \ldots, 0),
$$

where each $b_j$ appears with multiplicity two and 0 appears with multiplicity $n$ (as prescribed by Definition 2.12, since all $E$ edges of $\Gamma_{g,n}(E)$ are self-edges and $\Gamma_{g,n}(E)$ has $n$ legs). Then, $|Aut(\Gamma_{g,n}(E))| = 2^E E!$, since there are $E!$ permutations of the edges in $\Gamma_{g,n}(E)$ and 2 ways to permute the half-edges in each edge. Thus, by Definition 2.12 implies that

$$
P(\Gamma_{g,n}(E)) = 2^{6g+2n-E-5} \frac{(4g + n - 4)!}{(6g + 2n - 7)!E!} N_{g-E,2E+n}(\mathbf{b}) \prod_{i=1}^{E} b_i.
$$

Next, observe from Definition 2.11 that

$$
N_{g-E,2E+n}(\mathbf{b}) = \frac{(6g + 2n - 2E - 5)!!}{2^{5g+n-3E-3}3^gE(g - E)!} 
$$

$$
\times \sum_{d \in \mathcal{K}(3g + n - E - 3)} \langle d, 0^n \rangle_{g-E,2E+n} \prod_{j=1}^{E} \frac{b_j^{2d_{2j-1}+2d_{2j}}}{(2d_{2j-1} + 1)!(2d_{2j} + 1)!}. \tag{8.5}
$$
since the last \( n \) entries of any \( \mathbf{d}' \in K_{2E+n}(3g + n - E - 3) \) contributing to \( N_{g,n}(\mathbf{b}) \) on the right side of (2.19) in our setting must be equal to 0, since the last \( n \) entries of \( \mathbf{b} \) are.

By (8.4), (8.5), and the first statement of (2.1), we obtain

\[
Z\left(P\left(\Gamma_{g,n}(E)\right)\right) = \frac{2^{g+n-2}(4g + n - 4)!}{3^g (6g + 2n - 7)!} \frac{12^E (6g + 2n - 2E - 5)!}{E!(g - E)!} \times \sum_{\mathbf{d} \in K_{2E}(3g+n-E-3)} Z\left(\prod_{j=1}^{E} b_j^{2d_{j-1}+2d_j+1} \right) \langle \mathbf{d}, 0^n \rangle_{g-E,2E+n} \\
= \frac{24^E}{2^{2g-13g}} \frac{(6g + 2n - 2E - 5)!}{(6g + 2n - 7)!} \frac{(4g + n - 4)!}{(3g + n - E - 3)!(g - E)! E!} \times \sum_{\mathbf{d} \in K_{2E}(3g+n-E-3)} \langle \mathbf{d}, 0^n \rangle_{g-E,2E+n} \\
\times \prod_{j=1}^{E} \frac{(2d_{j-1} + 2d_j + 1)! \zeta(2d_{j-1} + 2d_j + 2)}{(2d_{j-1} + 1)(2d_j + 1)!}.
\]

Now the lemma follows from the last statement of (2.1).

Now we must analyze the sum of \( Z\left(P\left(\Gamma_{g,n}(E)\right)\right) \) over \( E \in [0, g] \). The following proposition does this in the case \( 0 \leq E \leq 9 \log g \), which provides the leading order contribution. It is a minor generalization of Conditional Theorem F.4 of [11] to the case \( n \neq 0 \), which was established there assuming Theorem 1.5 and Proposition 6.4 above.

**Proposition 8.9** As \( g \) tends to \( \infty \), we have for \( 20n \leq \log g \) that

\[
\sum_{E=0}^{\lfloor 9 \log g \rfloor} Z\left(P\left(\Gamma_{g,n}(E)\right)\right) \sim \pi^{-12n+2} \left(\frac{8}{3}\right)^{4g+n-4}. 
\]

**Proof** In view of Lemma 8.8, Theorem 1.5, and the fact that \( 20n \leq \log g \), we have for \( E \leq 9 \log g \) that

\[
Z\left(P\left(\Gamma_{g,n}(E)\right)\right) \sim \frac{12^E}{2^{2g-13g}} \frac{(6g + 2n - 2E - 5)!}{(6g + 2n - 7)!} \frac{(4g + n - 4)!}{(3g + n - E - 3)!(g - E)! E!} \times \sum_{\mathbf{d} \in K_{2E}(3g+n-E-3)} \langle \mathbf{d}, 0^n \rangle_{g-E,2E+n} \\
\times \prod_{j=1}^{E} \frac{(2d_{j-1} + 2d_j + 1)! \zeta(2d_{j-1} + 2d_j + 2)}{(2d_{j-1} + 1)(2d_j + 1)!}.
\]
Denote $d_{2j-1} + d_{2j} = D_j$ and set $D = (D_1, D_2, \ldots, D_E)$; since $d \in K_{2E}(3g + n - E - 3)$, we have $D \in K_E(3g + n - E - 3)$. So, for $E \leq 9 \log g$,

\[
\mathcal{Z} \left( P \left( \Gamma_{g,n}(E) \right) \right)
\sim \frac{12^E}{2^{2g-1}3^g 6^E} \frac{(6g + 2n - 2E - 5)!}{(6g + 2n - 7)!} \frac{(4g + n - 4)!}{(3g + n - E - 3)!(g - E)!} \times \frac{1}{E!} \sum_{D \in K_E(3g+n-E-3)} \frac{\zeta(2D_j + 2)}{D_j + 1} \prod_{j=1}^{E} \sum_{d_{2j-1} + d_{2j} = D_j} \frac{2D_j + 2}{2d_j + 1}.
\]

By the second statement of (2.1) and the identity $\sum_{j=1}^{E}(2D_j + 1) = 6g + 2n - E - 6$, we deduce for $E \leq 9 \log g$ that

\[
\mathcal{Z} \left( P \left( \Gamma_{g,n}(E) \right) \right)
\sim \frac{2^{4g+2n-5}6^E}{3^g} \frac{(6g + 2n - 2E - 5)!}{(6g + 2n - 7)!} \frac{(4g + n - 4)!}{(3g + n - E - 3)!(g - E)!} \times \frac{1}{E!} \sum_{D \in K_E(3g+n-E-3)} \frac{\zeta(2D_j + 2)}{D_j + 1}.
\]  

Next observe since $20n \leq \log g$ we have for $E \leq 9 \log g$ that

\[
\frac{(6g + 2n - 2E - 5)!}{(6g + 2n - 7)!} \sim (6g)^{2-2E}; \quad \frac{(3g + n - 3)!}{(3g + n - E - 3)!} \sim (3g)^E; \quad \frac{(g - 1)!}{(g - E)!} \sim g^{E-1}.
\]

This, together with (8.6), implies for $E \leq 9 \log g$ that

\[
\mathcal{Z} \left( P \left( \Gamma_{g,n}(E) \right) \right) \sim \frac{2^{4g+2n-3}g}{3^g} \frac{(4g + n - 4)!}{(3g + n - 3)!(g - 1)!} \frac{1}{2^E} \times \sum_{D \in K_E(3g+n-3)} \frac{\zeta(2D_j)}{D_j},
\]  

(8.7)
where we have also replaced the $D_j + 1$ in (8.6) with $D_j$ here. Again using the bound $20n \leq \log g$ gives

$$(3g + n - 3)! \sim (3g - 3)! (3g - 3)^n;$$

$$(4g + n - 4)! \sim (4g - 4)! (4g - 4)^n,$$

which upon insertion into (8.7) yields for $E \leq 9 \log g$ that

$$Z \left( P \left( \Gamma_{g,n}(E) \right) \right) \sim \frac{2^{4g-3}g}{3g-2} \left( \frac{16}{3} \right)^n \frac{(4g - 4)!}{(3g - 3)!(g - 1)! 2^E E!} \sum_{D \in C_E(3g+n-3)} \frac{\zeta(2D_j)}{D_j} \frac{1}{2^E E!},$$

(8.8)

Further using (2.7) gives

$$\frac{(4g - 4)!}{(3g - 3)!(g - 1)!} \sim \left( \frac{2}{3\pi g} \right)^{1/2} 4^{4g-4} \frac{1}{3^{3g-3}},$$

which upon insertion into (8.8) and recalling the definition of $Z_k(m)$ from Definition 6.1 yields for $E \leq 9 \log g$ that

$$Z \left( P \left( \Gamma_{g,n}(E) \right) \right) \sim \frac{2^{12g-12}}{3^{4g-5}} \left( \frac{16}{3} \right)^n \frac{2g}{3\pi} \frac{1/2}{2^{E-1} E!} \frac{Z_E(3g+n-3)}{\pi(3g+n-3)} \frac{1}{2^{E-1} E!}.$$

(8.9)

By the second limit in (6.3), we have

$$\sum_{E=0}^{[9 \log g]} \frac{Z_E(3g+n-3)}{2^{E-1} E!} \sim \left( \frac{8}{\pi(3g+n-3)} \right)^{1/2},$$

which upon insertion into (8.9) (again using the fact that $20n \leq \log g$) yields the proposition.

Before providing an asymptotic for the sum of $Z \left( P \left( \Gamma_{g,n}(E) \right) \right)$ over $E > 9 \log g$, we require the following bound for a term that will appear in these quantities. For the purposes of establishing Proposition 8.3 in this section, only the $V = 1$ case of this lemma will be necessary. However, the $V \geq 2$ case will be useful for the proofs of Propositions 8.4 and 8.5 in Sects. 9 and 10, respectively.
Lemma 8.10 For any integers \( g \geq 2 \), \( n \in [0, g] \), \( V \geq 1 \), and \( E \in [V - 1, 3g + n - 3] \), we have

\[
\frac{(6g + 2n - 2E - 5)!}{(6g + 2n - 7)!} \frac{(4g + n - 4)!}{(3g + n - E - 3)!(g - E + V - 1)!} \leq 128g^{3/2-V}3^{2V}12^{-E} \left(\frac{4}{3}\right)^n \left(\frac{256}{27}\right)^{g-1}.
\]

Proof Observe using the first identity in (2.1) that

\[
\frac{(6g + 2n - 2E - 5)!}{(3g + n - E - 3)!} \frac{(4g + n - 4)!}{(6g + 2n - 7)!} \frac{(g - 1)!}{(g - E + V - 1)!} = 2^{-E}(6g + 2n - 6)(6g + 2n - 5) \frac{(6g + 2n - 2E - 5)!!}{(6g + 2n - 5)!!},
\]

and so

\[
\frac{(6g + 2n - 2E - 5)!}{(3g + n - E - 3)!} \frac{(g - 1)!}{(6g + 2n - 7)!} \frac{(g - 1)!}{(g - E + V - 1)!} = 2^{-E}g^{-1}(6g + 2n - 6)(6g + 2n - 5) \prod_{k=0}^{V-2} \frac{1}{6g + 2n - 2k - 5} \times \prod_{j=0}^{E-V} \frac{g - j}{6g + 2n - 2V - 2j - 3}.
\]

Using the facts that \( 6(g - j) \leq 6g + 2n - 2j - 2V - 3 \) for \( j \geq V + 1 \); that \( 2(g - j) \leq 6g + 2n - 2j - 2V - 3 \) for any \( j \geq 0 \) (since \( V \leq 2g + n - 2 \) by (2.17)); that \( 6g + 2n - 2V - 1 \geq 2g \) (again since \( V \leq 2g + n - 2 \)); and that \( (6g + 2n - 6)(6g + 2n - 5) \leq 64g^2 \) (since \( n \leq g \)), it follows that

\[
\frac{(6g + 2n - 2E - 5)!}{(3g + n - E - 3)!} \frac{(g - 1)!}{(6g + 2n - 7)!} \frac{(g - 1)!}{(g - E)!} \leq 64g^{2-V}3^{2V}12^{-E}.
\]

Hence,

\[
\frac{(6g + 2n - 2E - 5)!}{(6g + 2n - 7)!} \frac{(4g + n - 4)!}{(3g + n - E - 3)!(g - E)!} \leq 64g^{2-V}3^{2V}12^{-E} \frac{(4g + n - 4)!}{(3g + n - 3)!(g - 1)!}.
\]
Since
\[
\frac{(4g + n - 4)!}{(3g + n - 3)!} = \frac{(4g - 4)!}{(3g - 3)!} \prod_{j=1}^{n} \frac{4g + j - 4}{3g + j - 3} \leq \left(\frac{4}{3}\right)^n \frac{(4g - 4)!}{(3g - 3)!},
\]
it follows that
\[
\frac{(6g + 2n - 2E - 5)!}{(6g + 2n - 7)!} \frac{(4g + n - 4)!}{(3g + n - E - 3)!(g - E)!} \leq 64g^{2-V} 3^{2V} 12^{-E} \left(\frac{4}{3}\right)^n \frac{(4g - 4)!}{(3g - 3)!} \frac{1}{(g - 1)!}. \tag{8.10}
\]
We additionally have by (2.8) that
\[
\frac{(4g - 4)!}{(3g - 3)!(g - 1)!} \leq (g - 1)^{-1/2} 4^{4g-4} 3^{3g-3g} \leq 2g^{-1/2} \left(\frac{256}{27}\right)^{g-1},
\]
which upon insertion into (8.10) implies the lemma.

Now we can bound the sum of \( \mathcal{Z}(P(\Gamma_{g,n}(E))) \) over \( E > 9 \log g \).

**Proposition 8.11** For any integers \( g \geq 2^{75} \) and \( n \geq 0 \) with \( 20n \leq \log g \), we have
\[
\sum_{E=\lfloor 9 \log g \rfloor + 1}^{3g+n-3} \mathcal{Z}(P(\Gamma_{g,n}(E))) \leq \frac{3^n}{g} \left(\frac{8}{3}\right)^{4g+n}.
\]

**Proof** In view of Lemma 8.8 and Proposition 1.2, we have
\[
\sum_{E=\lfloor 9 \log g \rfloor + 1}^{3g+n-3} \mathcal{Z}(P(\Gamma_{g,n}(E))) \leq \sum_{E=\lfloor 9 \log g \rfloor + 1}^{3g+n-3} \frac{12^E}{2^{2g-13g}} \left(\frac{3}{2}\right)^{2E+n} \frac{1}{E!} \\
\times \frac{(6g + 2n - 2E - 5)!}{(6g + 2n - 7)!} \frac{(4g + n - 4)!}{(3g + n - E - 3)!(g - E)!} \\
\times \sum_{d \in \mathcal{K}_{2E}(3g+n-E-3)} \prod_{j=1}^{E} \frac{2d_{2j-1} + 2d_{2j} + 2}{2d_{2j} + 1} \frac{\zeta(2d_{2j-1} + 2d_{2j} + 2)}{d_{2j-1} + d_{2j} + 1}.
\]
As in the proof of Proposition 8.9, we set \(d_{2j-1} + d_{2j} = D_j\) and \(D = (D_1, D_2, \ldots, D_E)\); use the second statement of (2.1); and the identity \(\sum_{j=1}^{E}(2D_j + 1) = 6g + 2n - E - 6\) to find that

\[
\sum_{E=[9 \log g] + 1}^{3g+n-3} \mathcal{Z}(P(\Gamma_{g,n}(E))) \\
\leq \sum_{E=[9 \log g] + 1}^{3g+n-3} \frac{(6g + 2n - 2E - 5)!}{(6g + 2n - 7)!} \frac{(4g + n - 4)!}{(3g + n - E - 3)!(g - E)!} \\
\times \frac{2^4g + 2n - 56E}{3g} \left(\frac{3}{2}\right)^{2E+n} \mathcal{Z}_E(3g + n - 3) \frac{E}{E!}.
\]

This, together with the \(V = 1\) case of (8.10) and the bound \(3^5 < 2^8\), implies that

\[
\sum_{E=[9 \log g] + 1}^{3g+n-3} \mathcal{Z}(P(\Gamma_{g,n}(E))) \\
\leq 4g^{1/2}8^n \left(\frac{8}{3}\right)^{4g} \sum_{E=[9 \log g] + 1}^{3g+n-3} \left(\frac{9}{8}\right)^E \mathcal{Z}_E(3g + n - 3) \frac{E}{E!}.
\]

By Lemma 6.3 and the bounds \(3g + n - 3 \geq g\) (which holds since \(g \geq 2^{75}\)) and \(\log(3g + n - 3) \leq \log g + 2\) (since \(n \leq \log g \leq g\), we find that

\[
\sum_{E=[9 \log g] + 1}^{3g+n-3} \mathcal{Z}(P(\Gamma_{g,n}(E))) \\
\leq 9g^{-1/2}3^n \left(\frac{8}{3}\right)^{4g+n} \sum_{E=[9 \log g] + 1}^{3g+n-3} \left(\frac{9}{8}\right)^E \frac{(\log g + 7)^{E-1}}{(E - 1)!} \quad (8.11)
\]

Now set \(R = \frac{9}{8}(\log g + 7)\), and apply Lemma 2.4, with the \(\delta\) there equal to 3 here. Since \([9 \log g] \geq 7R\) (as \(g \geq 2^{75}\)), this gives

\[
\sum_{E=[9 \log g] + 1}^{\infty} \left(\frac{9}{8}\right)^E \frac{(\log g + 7)^{E-1}}{(E - 1)!} \leq e^{-2R} \leq \frac{1}{9g^2}, \quad (8.12)
\]

Combining (8.11) and (8.12) then yields the proposition.
Given the above bounds, we can quickly establish Proposition 8.3.

Proof of Proposition 8.3 This follows from the definition (8.1) of \( \Upsilon_{g,n}^{(1)} \), combined with Propositions 8.9 and 8.11 (using the fact that \( \left( \frac{3}{2} \right)^n < g^{1/2} \), since \( 20n < \log g \)).  

\[ \square \]

9 Bounds for \( \Upsilon_{g,n}^{(2)} \)

In this section we establish Proposition 8.4, which bounds the contribution to the right side of (2.20) arising from stable graphs with two vertices. Although these graphs will be addressed slightly differently from those with \( V \geq 3 \) vertices in Sect. 10 below (essentially due to the fact that the decaying factor of \( g^{3/2-V} \) on the right side of (8.10) is not yet small enough when \( V = 2 \) for the method in Sect. 10 to directly apply), both cases will still exhibit several common aspects. So, we hope that the analysis presented here will provide some indication for how the contributions coming from graphs with more vertices can be bounded.

In Sect. 9.1 we estimate the contribution to the volume coming from a single graph; in Sect. 9.2 we then simplify this bound, which we use to prove Proposition 8.4 in Sect. 9.3.

9.1 Estimates for an individual graph

We begin with the following notation for a graph on two vertices with a prescribed number of legs, self-edges, and simple edges at each vertex, and also a prescribed genus decoration.

Definition 9.1 Fix integers \( g \geq 2, n \geq 0, T \geq 1, \) and \( S \geq 0; \) set \( E = S + T, \) and assume that \( E \leq 3g + n - 3. \) Further fix nonnegative compositions \( n = (n_1, n_2) \in K_2(n), \) \( g = (g_1, g_2) \in K_2(g - E + 1), \) and \( s = (s_1, s_2) \in K_2(S) \) such that \( 2g_i + 2s_i + n_i + T \geq 3, \) for each \( i \in \{1, 2\}. \) Then let \( G_{g,n}(n; s, T; g) \subseteq G_{g,n}(2; S, T) \) denote the set of stable graphs on two vertices with \( T \) simple edges connecting these vertices, such that the following holds.

There exists a labeling of its two vertices with a (distinct) integer in \( \{1, 2\}, \) such that vertex \( i \in \{1, 2\} \) has \( n_i \) legs, \( s_i \) self-edges, and is assigned \( g_i \) under its genus decoration.

Under this notation, \( m_i = 2s_i + n_i + T \) half-edges are incident to vertex \( i \in [1, 2]. \)

The following lemma bounds \( Z(P(\Gamma)) \) for any \( \Gamma \in G_{g,n}(n; s, T; g). \) Here, we recall \( Z_k(m) \) from Definition 6.1.
Lemma 9.2  Adopt the notation of Definition 9.1, and set

$$
\xi_i = \max_{d \in K_{m_i}(3g_i + m_i + 3)} (d)_{g_i, m_i}, \quad \text{for each } i \in \{1, 2\}. \quad (9.1)
$$

Then, for any $\Gamma \in G_{g,n}(n; s; T; g)$, we have that

$$
Z(P(\Gamma)) \leq \frac{2^{4g+2n+2E-S-8}}{3^{g-E+1}} \frac{(4g + n - 4)!}{(6g + 2n - 7)!} \frac{Z_E(3g + n - 3)}{T!} \times \prod_{i=1}^{2} \xi_i \left(\frac{(6g_i + 4s_i + 2n_i + 2T - 5)!}{(3g_i + 2s_i + n_i + T - 3)!g_i!s_i!}\right). \quad (9.2)
$$

Proof. We first assign variables to each edge of $\Gamma$, following Definition 2.12. To that end, label each vertex of $\Gamma$ with an index in $\{1, 2\}$, such that the conditions in Definition 9.1 hold. For any $i \in \{1, 2\}$, let $b_1^{(i)}, b_2^{(i)}, \ldots, b_s^{(i)}$ be variables (associated with the $s_i$ self-edges at vertex $i$), and let $b_1^{(1,2)}, b_2^{(1,2)}, \ldots, b_T^{(1,2)}$ be variables (associated with the $T$ simple edges between the two vertices of $\Gamma$). For each $i \in \{1, 2\}$, define the variable set

$$
b^{(i)} = (b_1^{(i)}, b_1^{(i)}, b_2^{(i)}, b_2^{(i)}, \ldots, b_s^{(i)}, b_s^{(i)})
\cup (b_1^{(1,2)}, b_2^{(1,2)}, \ldots, b_T^{(1,2)}) \cup \{0, 0, \ldots, 0\},
$$

As prescribed in Definition 2.12, here the variables $b_k^{(i)}$ associated with self-edges appear with multiplicity two; the variables $b_k^{(1,2)}$ associated with simple edges appear with multiplicity one; and zero appears with multiplicity $n_i$.

Recalling the polynomial $N_{g,n}$ from Definition 2.11, we then have that

$$
Z(P(\Gamma)) = 2^{6g+2n-6} \frac{(4g + n - 4)!}{(6g + 2n - 7)!} \frac{1}{|\text{Aut}(\Gamma)|} \frac{1}{Z} \times \left( \prod_{k=1}^{T} b_{k}^{(1,2)} \prod_{i=1}^{s_i} b_{k}^{(i)} \prod_{i=1}^{2} N_{g_i,m_i}(b^{(i)}) \right). \quad (9.3)
$$
Now, observe using the first identity in (2.1) that

\[ N_{g_i,m_i}(b^{(i)}) = \sum_{d^{(i)} \in \mathcal{K}_{2g_i+T}(3g_i+m_i-3)} \frac{(6g_i + 2m_i - 5)!}{2^{8g_i+2m_i-6}(3g_i + m_i - 3)!g_i!} \binom{d^{(i)}}{0^{n_i}} g_i, m_i \]

\[ \times \prod_{k=1}^{T} \frac{(b_k^{(1,2)})^{2d_{2s_i+k}}}{(2d_{2s_i+k}+1)!} \prod_{i=1}^{s_i} \frac{(b_k^{(i)})^{2d_{2k-1}+2d_{2k}}}{(2d_{2j-1}+1)!(2d_{2j}+1)!}, \]

(9.4)

where for each \( i \in \{1, 2\} \) we have denoted

\[ d^{(i)} = (d_1^{(i)}, d_2^{(i)}, \ldots, d_{2n_i+T}^{(i)}); \]

\[ (d^{(i)}, 0^{n_i}) = (d_1^{(i)}, d_2^{(i)}, \ldots, d_{2n_i+T}^{(i)}, 0, 0, \ldots, 0), \]

where \((d^{(i)}, 0^{n_i})\) contains \( n_i \) zeroes at the end. Indeed, this is due to the fact that the last \( n_i \) entries of any \( d' \in \mathcal{K}_{m_i}(3g_i + m_i - 3) \) contributing to \( N_{g_i,m_i}(b^{(i)}) \) in the right side of (2.19) in our setting must be equal to 0, since the last \( n_i \) entries of \( b^{(i)} \) are.

Together (2.14), the fact that

\[ (8g_1 + 2m_1 - 6) + (8g_2 + 2m_2 - 6) = 8(g - E + 1) + 4E + 2n - 12 = 8g + 2n - 4E - 4, \]

and inserting (9.4) into (9.3) yield

\[ \mathcal{Z}(P(\Gamma)) = \frac{2^{4E-2g-2}}{3^{g-E+1}} \frac{(4g + n - 4)!}{(6g + 2n - 7)!} \left| \text{Aut}(\Gamma) \right| \]

\[ \times \sum_{d^{(1)} \in \mathcal{K}_{2g_1+T}(3g_1+m_1-3)} \sum_{d^{(2)} \in \mathcal{K}_{2g_2+T}(3g_2+m_2-3)} \]

\[ \times \prod_{i=1}^{2} \binom{d^{(i)}}{0^{n_i}} g_i, m_i \frac{(6g_i + 2m_i - 5)!}{(3g_i + m_i - 3)!g_i!} \]

\[ \times \mathcal{Z} \left( \prod_{i=1}^{s_i} \prod_{k=1}^{d_{2s_i+k}} \frac{(b_k^{(i)})^{2d_{2k-1}+2d_{2k}}}{(2d_{2j-1}+1)!(2d_{2j}+1)!} \right) \]

\[ \times \prod_{k=1}^{T} \frac{(b_k^{(1,2)})^{2d_{2s_i+k}}}{(2d_{2s_i+k}+1)!} \frac{(b_k^{(2)})^{2d_{2s_i+k}}}{(2d_{2s_i+k}+1)!}, \]

(9.5)
Next, let us bound $|\text{Aut}(\Gamma)|$. At each vertex $i \in \{1, 2\}$, there are $s_i$ permutations of the self-edges at vertex $i$ and 2 ways to permute the half-edges in constituting them edge; moreover, there are $T!$ permutations of the $T$ simple edges connecting vertices 1 and 2. Thus, $|\text{Aut}(\Gamma)| \geq 2^{s_1 + s_2} s_1 s_2 T! = 2^s s_1! s_2! T!$. Inserting this, the definition of $\mathcal{Z}$ (from Definition 2.11), the fact that $m_i = 2s_i + n_i + T$, and the definition (9.1) of the $\xi_i$ into (9.5) we obtain

$$
\mathcal{Z}(P(\Gamma)) \leq \frac{2^{4E - 2g - S - 2}(4g + n - 4)!}{3g - E + 1} \frac{1}{(6g + 2n - 7)! T!} \times \prod_{i=1}^2 \left( \frac{6g_i + 4s_i + 2n_i + 2T - 5)!}{(3g_i + 2s_i + n_i + T - 3)!g_i!s_i!} \times \sum_{d^{(1)} \in \mathcal{K}_{2s_1 + T}(3g_1 + m_1 - 3)} \sum_{d^{(2)} \in \mathcal{K}_{2s_2 + T}(3g_2 + m_2 - 3)} \xi_1 \xi_2 \times \prod_{i=1}^2 \prod_{k=1}^{s_i} \left( \frac{2d_{2k-1}^{(i)} + 2d_{2k}^{(i)} + 1}{2d_{2k}^{(i)} + 1}\right) \xi \left( \frac{2d_{2k-1}^{(i)} + 2d_{2k}^{(i)} + 2}{d_{2k-1}^{(i)} + d_{2k}^{(i)} + 1}\right) \times \prod_{k=1}^T \left( \frac{2d_{2s_1+k}^{(1)} + 2d_{2s_2+k}^{(2)} + 1}{2d_{2s_2+k}^{(2)} + 1}\right) \xi \left( \frac{2d_{2s_1+k}^{(1)} + 2d_{2s_2+k}^{(2)} + 2}{d_{2s_1+k}^{(1)} + d_{2s_2+k}^{(2)} + 1}\right).$$

(9.6)

Together with the last statement of (2.1) and the identity $s_1 + s_2 + T = E$, (9.6) gives

$$
\mathcal{Z}(P(\Gamma)) \leq \frac{2^{3E - 2g - 2S + 2}(4g + n - 4)!}{3g - E + 1} \frac{1}{(6g + 2n - 5)! T!} \times \sum_{d^{(1)} \in \mathcal{K}_{2s_1 + T}(3g_1 + m_1 - 3)} \sum_{d^{(2)} \in \mathcal{K}_{2s_2 + T}(3g_2 + m_2 - 3)} \xi \left( \frac{2d_{2k-1}^{(i)} + 2d_{2k}^{(i)} + 1}{2d_{2k}^{(i)} + 1}\right) \xi \left( \frac{2d_{2k-1}^{(i)} + 2d_{2k}^{(i)} + 2}{d_{2k-1}^{(i)} + d_{2k}^{(i)} + 1}\right) \times \prod_{i=1}^2 \prod_{k=1}^{s_i} \left( \frac{2d_{2s_1+k}^{(1)} + 2d_{2s_2+k}^{(2)} + 1}{2d_{2s_2+k}^{(2)} + 1}\right) \xi \left( \frac{2d_{2s_1+k}^{(1)} + 2d_{2s_2+k}^{(2)} + 2}{d_{2s_1+k}^{(1)} + d_{2s_2+k}^{(2)} + 1}\right).$$

(9.7)
Now for each \( i \in \{1, 2\} \) and \( k \in [1, s_i] \), define

\[
D_k^{(i)} = d_{2k-1}^{(i)} + d_{2k}^{(i)}; \quad U_i = 3g_i + m_i - 3 - \sum_{k=1}^{s_i} D_k^{(i)}; \quad U = U_1 + U_2;
\]

\[
D^{(i)} = (D_1^{(i)}, D_2^{(i)}, \ldots, D_{s_i}^{(i)}) \in \mathcal{K}_{s_i}(3g_i + m_i - U_i - 3);
\]

\[
U = (U_1, U_2) \in \mathcal{K}_2(U),
\]

and, for each \( k \in [1, T] \), define

\[
u_k = d_{2s_1+k}^{(1)} + d_{2s_2+k}^{(2)}; \quad \nu = (u_1, u_2, \ldots, u_T) \in \mathcal{K}_T(U).
\]

Under this notation, instead of summing over the \( d^{(i)} \) in (9.7), we may sum over \( U \in [0, 3g + n - E - 3] \); \( \nu \in \mathcal{K}_T(U) \); the \( d_{2s_1+k}^{(1)} \in [0, u_k] \); \( \nu \in \mathcal{K}_2(U) \); the \( D^{(i)} \in \mathcal{K}_{s_i}(3g_i + m_i - U_i - 3) \); and the \( d_{2k}^{(i)} \in [0, D_k^{(i)}] \). This yields

\[
\mathcal{Z}(P(\Gamma)) \leq \frac{2^{3E-2g-S-2}}{3g-E+1} \left( \frac{4g + n - 4)!}{(6g + 2n - 7)! \cdot T!} \right)
\]

\[
\times \prod_{i=1}^{2} \frac{\xi_i}{(3g_i + 2s_i + n_i + T - 3)!g_i!s_i!}
\]

\[
\times \sum_{U=0}^{3g+n-E-3} \sum_{\nu \in \mathcal{K}_T(U)} \prod_{k=1}^{T} \frac{\xi(2u_k + 2)}{u_k + 1} \sum_{d_{2s_1+k}^{(1)}=0}^{u_k} \left( \frac{2u_k + 2}{2d_{2s_1+k}^{(1)} + 1} \right)
\]

\[
\times \sum_{\nu \in \mathcal{K}_2(U)} \prod_{i=1}^{2} \sum_{D^{(i)} \in \mathcal{K}_{s_i}(3g_i + m_i - U_i - 3)} \prod_{k=1}^{s_i} \frac{\xi(D_k^{(i)} + 2)}{D_k^{(i)} + 1}
\]

\[
\times \sum_{d_{2k}^{(i)}=0}^{D_k^{(i)}} \left( \frac{2D_k^{(i)} + 2}{2d_{2k}^{(i)} + 1} \right).
\]  

(9.8)

Using the second statement of (2.1) and the fact (which is a consequence of the identities (2.14), (2.16), and \( S + T = E \)) that

\[
\sum_{k=1}^{T} (2u_k + 1) + \sum_{i=1}^{2} \sum_{k=1}^{s_i} (2D_k^{(i)} + 1)
\]

\[
= 2U + T + 2 \sum_{i=1}^{2} (3g_i + m_i - U_i - 3) + S = 6g + 2n - E - 6,
\]
it follows from (9.8) that

\[
\mathcal{Z}(P(\Gamma)) \leq \frac{2^{4g+2n+2E-S-8}}{3^{g-E+1}} \left( \frac{4g+n-4)!}{(6g+2n-7)!} \right) \times \prod_{i=1}^{2} \xi_i \left( \frac{(6g_i + 4s_i + 2n_i + 2T - 5)!}{(3g_i + 2s_i + n_i + T - 3)!g_i!s_i!} \right)
\]

\[
\times \sum_{U=0}^{3g+n-E-3} \sum_{\mathcal{U} \in \mathcal{K}(U)} \frac{T}{(2u_k + 2)} \frac{\zeta(D_k^{(i)} + 2)}{D_k^{(i)} + 1}.
\]

(9.9)

Recalling the definition of \(Z_k(m)\) from Definition 6.1, we deduce

\[
\mathcal{Z}(P(\Gamma)) \leq \frac{2^{4g+2n+2E-S-8}}{3^{g-E+1}} \left( \frac{4g+n-4)!}{(6g+2n-7)!} \right) \times \prod_{i=1}^{2} \xi_i \left( \frac{(6g_i + 4s_i + 2n_i + 2T - 5)!}{(3g_i + 2s_i + n_i + T - 3)!g_i!s_i!} \right)
\]

\[
\times \sum_{U=0}^{3g+n-E-3} \sum_{\mathcal{U} \in \mathcal{K}(U)} \frac{T}{(2u_k + 2)} \frac{\zeta(D_k^{(i)} + 2)}{D_k^{(i)} + 1} \cdot Z_T(U + T)
\]

\[
\times \sum_{\mathcal{U} \in \mathcal{K}_2(U)} \prod_{i=1}^{2} \frac{\zeta(D_k^{(i)} + 2)}{D_k^{(i)} + 1}.
\]

(9.10)

Then, the identities \(s_1 + s_2 = S\), \(E = S + T\), and

\[
\sum_{i=1}^{2} (3g_i + s_i + m_i - U_i - 3) = 3(g - E + 1) + S + 2E + n - U - 6
\]

\[
= 3g + n + S - E - U - 3,
\]

together with repeated application of Lemma 6.2 gives

\[
\sum_{U=0}^{3g+n-E-3} \mathcal{Z}_T(U + T) \sum_{\mathcal{U} \in \mathcal{K}_2(U)} \prod_{i=1}^{2} \frac{\zeta(D_k^{(i)} + 2)}{D_k^{(i)} + 1}.
\]

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\[ \leq \sum_{U=0}^{3g+n-E-3} Z_T(U + T) Z_S(3g + n + S - E - U - 3) \]
\[ \leq Z_E(3g + n - 3), \]

which upon insertion into (9.10) yields the proposition. \( \square \)

9.2 Simplification of the bound

In this section we simplify the bound appearing on the right side of (9.2). We begin with the following lemma that bounds the product there (without the factors of \( \xi_i \)); its first part provides a general bound, and its second provides an improvement under a certain assumption. Let us mention that the latter improvement will be useful in the proof of Proposition 8.4 to analyze the contribution of graphs on two vertices but unnecessary to analyze those of graphs with \( V \geq 3 \) vertices in Sect. 10 below.

**Lemma 9.3** Fix integers \( g \geq 2, n \geq 0, S \geq 0, \) and \( T \geq 1; \) set \( E = S + T, \) and assume that \( E \leq 3g + n - 3. \) Further let \( g_1, g_2, s_1, s_2 \geq 0 \) denote integers with \( g_1 + g_2 = g - E + 1 \) and \( s_1 + s_2 = S, \) such that \( 2g_i + 2s_i + n_i + T \geq 3, \) for each \( i \in \{1, 2\}. \) Then the following two statements hold.

(1) We have that

\[ \prod_{i=1}^{2} \frac{(6g_i + 4s_i + 2n_i + 2T - 5)!}{(3g_i + 2s_i + n_i + T - 3)!g_i!s_i!} \leq 64(E + 1) \frac{(6g + 2n - 2E - 5)!}{(g - E + 1)!S!}. \] (9.11)

(2) Denote \( r = \min\{2g_1 + s_1 + n_1 + T - 3, 2g_2 + s_2 + n_2 + T - 3\}. \) If \( r \geq r_0 \geq 0 \) for some \( r_0 \in \mathbb{Z}, \) then

\[ \prod_{i=1}^{2} \frac{(6g_i + 4s_i + 2n_i + 2T - 5)!}{(3g_i + 2s_i + n_i + T - 3)!g_i!s_i!} \leq 2^{r_0+7} \frac{(E + 1)(2g + n - S - 2)^{-r_0}}{(r_0 + 1)!(6g + 2n - 2E - 5)!} \times \frac{(r_0 + 1)!}{(3g + n - E - 3)!S!}. \] (9.12)
\textbf{Proof} We begin with the proof of the second statement. To that end, first observe that

\[ \prod_{i=1}^{2} \frac{(6g_i + 4s_i + 2n_i + 2T - 5)!}{(3g_i + 2s_i + n_i + T - 3)!g_i!s_i!} \]
\[ = \prod_{i=1}^{2} \left( \frac{6g_i + 4s_i + 2n_i + 2T - 6}{3g_i + 2s_i + n_i + T - 3, g_i, s_i, 2g_i + s_i + n_i + T - 3} \right) \times (6g_i + 4s_i + 2n_i + 2T - 5)(2g_i + s_i + n_i + T - 3)! \]
\[ \leq \left( \frac{6g + 2n - 2E - 6}{3g + n - E - 3, g - E + 1, S, 2g + n - S - 4} \right) \times 64(2g_1 + s_1 + n_1 + T - 2)!(2g_2 + s_2 + n_2 + T - 2)!. \]

where in the last inequality we used Lemma 2.3 and the identities \( s_1 + s_2 = S, S + T = E, \) and \( g_1 + g_2 = g - E + 1. \) Since \( 6g_i + 4s_i + 2n_i + 2T - 5 \leq 8(2g_i + s_i + n_i + T - 2) \) \( \) (as \( 4(2g_i + s_i + n_i + T) \geq 12 \) for each \( i \in \{1, 2\}, \) since \( r \geq 0 \) and \( S + T = E, \) it follows that

\[ \prod_{i=1}^{2} \frac{(6g_i + 4s_i + 2n_i + 2T - 5)!}{(3g_i + 2s_i + n_i + T - 3)!g_i!s_i!} \]
\[ \leq \left( \frac{6g + 2n - 2E - 6}{3g + n - E - 3, g - E + 1, S, 2g + n - S - 4} \right) \times 64(2g_1 + s_1 + n_1 + T - 2)!(2g_2 + s_2 + n_2 + T - 2)!. \]

Combining this with the fact that

\[ \frac{(2g_1 + s_1 + n_1 + T - 2)!(2g_2 + s_2 + n_2 + T - 2)!}{(2g + n - S - 4)!} \]
\[ = (2g + n - S - 2)(2g + n - S - 3)^{-1}, \]

and the bounds \( \binom{2g+n-S-2}{r+1} \geq \binom{2g+n-S-2}{r_0+1} \) \( \) \( (as \ r \geq r_0 \geq 0 \) and \( 2r + 2 \leq 2g+n-S-2 \) and \( (E+1)(6g+2n-2E-5) \geq 6g+2n-E-5 \geq 2g+n-S-3 \)
(\( \) where \( \) the first statement holds since \( 6g + 2n - 2E - 5 \geq 1 \) and the second
holds since \( S + T = E \leq 3g + n - 3 \) we obtain

\[
\prod_{i=1}^{2} \frac{(6g_i + 4s_i + 2n_i + 2T - 5)!}{(3g_i + 2s_i + n_i + T - 3)!g_i!s_i!} \leq 64(2g + n - S - 2)(2g + n - S - 2)^{-1}(E + 1) \\
\times \frac{(6g + 2n - 2E - 5)!}{(3g + n - E - 3)!(g - E + 1)!S!}.
\] (9.13)

So, (9.12) follows from the bound \( \binom{k}{m} \geq \frac{k^m}{2^m m!} \) if \( 2m \leq k \) (used with \( k = 2g - S - 2 \) and \( m = r_0 + 1 \)).

To establish (9.11), first observe that if \( r = \min\{2g_1 + s_1 + n_1 + T - 3, 2g_2 + s_2 + n_2 + T - 3\} \geq 0 \) then it holds by the \( r_0 = 0 \) case of (9.13), since \( \binom{2g-S-2}{r+1} \geq 2g - S - 2 \). So, let us assume that \( r < 0 \), in which case we may suppose that \( 2g_1 + s_1 + n_1 + T \leq 2 \). Since \( T \geq 1 \), this can only occur if \( (g_1, s_1, n_1, T) \in \{(0, 0, 0, 1), (0, 0, 0, 2), (0, 0, 1, 1), (0, 1, 0, 1)\} \).

The first three cases contradict the bound \( 2g_1 + 2s_1 + n_1 + T \geq 3 \). So, we must have \( (g_1, s_1, n_1, T) = (0, 1, 0, 1) \), meaning that \( (g_2, s_2, n_2, T) = (g - E + 1, E - 2, n, 1) \) (since \( s_2 = S - s_1 = E - T - 1 = E - 2 \). Then,

\[
\prod_{i=1}^{2} \frac{(6g_i + 4s_i + 2n_i + 2T - 5)!}{(3g_i + 2s_i + n_i + T - 3)!g_i!s_i!} = \frac{(6g + 2n - 2E - 5)!}{(3g + n - E - 2)!(g - E + 1)!(E - 2)!},
\]

which verifies (9.11) (as \( \frac{E+1}{S!} \geq \frac{1}{(E-2)!} \), since \( S = E - 1 \)).

Now we can deduce the following simplified bound on \( \mathcal{Z}(P(\Gamma)) \).

**Corollary 9.4** Adopt the notation of Lemma 9.2, and assume that \( 20n \leq \log g \).

1. We have

\[
2^{-n} \left( \frac{8}{3} \right)^{-4g-n} \mathcal{Z}(P(\Gamma)) \leq 2^{10}(S^2 + T^2)g^{-3/2} \xi_1 \xi_2 \log g + 1 S^{S+T-1} 2^S S!T!.
\] (9.14)

2. If \( r = \min\{2g_1 + s_1 + n_1 + T - 3, 2g_2 + s_2 + n_2 + T - 3\} \geq 10 \) and \( S \leq g - 2 \), then

\[
2^{-n} \left( \frac{8}{3} \right)^{-4g-n} \mathcal{Z}(P(\Gamma))
\]
\[ \leq 2^{47} (S^2 + T^2) g^{-11} \left( \frac{3}{2} \right)^{2S+2T+n} \frac{\log g + 7}{2^S S! T!} \].

**Proof** We begin by establishing (9.14). To that end, by Lemma 9.2 and (9.11), we obtain

\[
\mathcal{Z}(P(\Gamma)) \leq (E + 1) 12 E \frac{2^{4g+2n-2}}{3^{g+1}} \xi_1 \xi_2 \frac{Z_E(3g + n - 3)}{2^S S! T!} \times \frac{(6g + 2n - 2E - 5)!}{(3g + n - E - 3)! (g - E + 1)!},
\]

where we have recalled the \( \xi_i \) from (9.1). Then applying the \( V = 2 \) case of Lemma 8.10 and using the facts that \( 3^6 < 2^{10} \) and \( E = S + T \), we deduce

\[
2^{-n} \left( \frac{8}{3} \right)^{-4g-n} \mathcal{Z}(P(\Gamma)) \leq 2^7 (S + T + 1) g^{-1/2} \xi_1 \xi_2 \frac{Z_{S+T}(3g + n - 3)}{2^S S! T!}.
\]

Thus, by Lemma 6.3

\[
2^{-n} \left( \frac{8}{3} \right)^{-4g-n} \mathcal{Z}(P(\Gamma)) \leq 2^8 (S + T)(S + T + 1) g^{-1/2} (3g + n - 3)^{-1} \xi_1 \xi_2 \times \frac{\log(3g + n - 3) + 5}{2^S S! T!},
\]

which implies (9.14) in view of the bounds \( 3g + n - 3 \geq g \), \( \log(3g + n - 3) \leq \log g + 2 \) (since \( n \leq g \)), and \( (S + T)(S + T + 1) \leq 3S^2 + 3T^2 \).

To establish (9.15), observe that if \( r \geq 10 \) then Lemma 9.2, (9.12) (with the \( r_0 \) there equal to 10 here), and the facts that \( 2^{17} 11! \leq 2^{43} \) and \( 2g + n - S - 2 \geq g \) for \( S \leq g - 2 \) together yield

\[
\mathcal{Z}(P(\Gamma)) \leq g^{-10} (E + 1) 12 E \frac{2^{4g+2n+35}}{3^{g+1}} \xi_1 \xi_2 \frac{Z_{S+T}(3g + n - 3)}{2^S S! T!} \times \frac{(6g + 2n - 2E - 5)!}{(6g + 2n - 7)!} \frac{(4g + n - 4)!}{(3g + n - E - 3)! (g - E + 1)!},
\]

Again applying the \( V = 2 \) case of Lemmas 8.10, 6.3, and the facts that \( E = S + T \), \( 3g + n - 3 \geq g \), \( \log(3g - 3) \leq \log g + 2 \), and \( 3^6 < 2^{10} \), we deduce

\[
2^{-n} \left( \frac{8}{3} \right)^{-4g-n} \mathcal{Z}(P(\Gamma))
\]
\[
\leq 2^{45} (S + T)(S + T + 1) g^{-23/2} \xi_1 \xi_2 \frac{(\log g + 7)^{S+T-1}}{2^S S! T!}.
\]

This, together with the bound \(\xi_i \geq \left( \frac{3}{2} \right)^{2s_1 + n_1 + T} \) (which follows from Proposition 1.2); the identities \(s_1 + s_2 = S\) and \(n_1 + n_2 = n\); and the fact that \((S + T)(S + T + 1) \leq 3(S^2 + T^2)\), imply the corollary. \(\square\)

### 9.3 Proof of Proposition 8.4

In this section we establish Proposition 8.4. To that end, we begin with the following lemma that bounds \(\Upsilon_{g,n}^{(2;S,T)}\) (recall Definition 8.6).

**Lemma 9.5** Fix integers \(g \geq 2120, n \geq 0, T \geq 1, \text{ and } S \geq 0\) with \(S + T \leq \frac{3}{2} g + 3\). If \(20n \leq \log g\), then the following three bounds hold.

1. If \(S \geq g - 1\), then
   \[
   2^{-n} \left( \frac{8}{3} \right)^{-4g-n} \Upsilon_{g,n}^{(2;S,T)} \leq g^{-10}. \tag{9.16}
   \]

2. If \(T > 13\) and \(S \leq g - 2\), then
   \[
   2^{-n} \left( \frac{8}{3} \right)^{-4g-n} \Upsilon_{g,n}^{(2;S,T)} \leq 2^{55} g^{-7} 3^n \left( \frac{9}{4} \right)^{S+T} \frac{(\log g + 7)^{S+T-1}}{2^S S! T!}. \tag{9.17}
   \]

3. If \(T \leq 13\) and \(S \leq g - 2\), then
   \[
   2^{-n} \left( \frac{8}{3} \right)^{-4g} \Upsilon_{g,n}^{(2;S,T)} \leq 2723^n \left( \frac{\log g + 7}{S!} \right)^{S+12} \left( g^{-3/2} (S^2 + 1) + g^{-7} \left( \frac{9}{4} \right)^S \right). \tag{9.18}
   \]

**Proof** Set \(E = S + T\), and observe that

\[
2^{-n} \left( \frac{8}{3} \right)^{-4g-n} \Upsilon_{g,n}^{(2;S,T)} \leq \sum_{s \in K_2(S)} \sum_{g \in K_2(g-E+1)} \left( \frac{8}{3} \right)^{-4g-n} \max_{n \in K_2(n)} \max_{\Gamma \in G_{g,n}(\mathbf{n};S,T:g)} \mathcal{Z}(P(\Gamma)), \tag{9.19}
\]

since there are at most \(2^n\) ways of labeling the legs of any stable graph \(\Gamma \in G_{g,n}(2)\) on two vertices (as there are two ways to assign a vertex to each index.
in \( \{1, 2, \ldots, n\} \). Further observe that since \( S + T \leq 3g + n - 3 \leq 4g - 3 \), we have

\[
S^2 + T^2 \leq 32g^2; \quad |\mathcal{K}_2(S)| = S + 1 \leq 4g; \\
|\mathcal{K}_2(g - E + 1)| \leq g + 1 \leq 2g,
\]

(9.20)

To establish (9.16), observe that (9.14) and the fact that the \( \xi_1 \xi_2 \) there is at most equal to \( \left( \frac{3}{2} \right)^{2S+2T+n} \) here (by Proposition 1.2, (2.16), and the fact that \( E = S + T \)) together yield

\[
\left( \frac{8}{3} \right)^{-4g-n} \mathcal{Z}(P(\Gamma)) \leq 2^{n+10} (S^2 + T^2) g^{-3/2} \\
\times \left( \frac{8}{3} \right)^{2S+2T+n} \frac{(\log g + 7)^{S+T+1}}{2^S S! T!}.
\]

Since \( E \geq g - 1 \) and \( S + T = E \leq 3g + n - 3 \leq 4g \), it follows that

\[
\left( \frac{8}{3} \right)^{-4g-n} \mathcal{Z}(P(\Gamma)) \leq 2^{n+15} g^{1/2} \left( \frac{8}{3} \right)^{8g} \frac{(\log g + 7)^{4g+1}}{(g - 1)!}.
\]

Upon insertion into (9.19) this gives, together with the last two bounds of (9.20) and the fact that \( 2^n \leq \left( \frac{8}{3} \right)^g \), that

\[
2^{-n} \left( \frac{8}{3} \right)^{-4g-n} \mathcal{Z}(P(\Gamma)) \leq 2^{18} g^{7/2} \left( \frac{8}{3} \right)^{9g} \frac{(\log g + 7)^{5g}}{g!}.
\]

(9.21)

Since (2.8) implies \( g! \geq \left( \frac{8}{e} \right)^g \) and the fact that \( g > 2^{120} \) implies

\[
g^{-1} \left( \frac{8}{3} \right)^9 e (\log g + 7)^5 \leq 2^{20} (\log g)^5 g^{-1} \leq g^{-1/2},
\]

it follows from (9.21) that

\[
2^{-n} \left( \frac{8}{3} \right)^{-4g-n} \mathcal{Z}(P(\Gamma)) \leq 2^{18} g^{7/2 - g/2},
\]

which implies (9.16) since \( g > 2^{120} \).
To establish (9.17) observe that, if \( T \geq 13 \) and \( S \leq g - 2 \), then (9.15) applies, which by (9.19) yields

\[
2^{-n} \left( \frac{8}{3} \right)^{-4g-n} \gamma_{g,n}^{(2:S,T)} \leq 2^{n+47} (S^2 + T^2) g^{-11} \left( \frac{3}{2} \right)^{2S+2T+n} \frac{(\log g + 7)^{S+T-1}}{2^S S!} \times |\mathcal{K}_2(S)| |\mathcal{K}_2(g - E + 1)|. 
\]

Thus, inserting (9.20) into (9.22) yields (9.17).

Next we establish (9.17). For any integer \( T \geq 1 \) and nonnegative compositions \( n = (n_1, n_2) \in \mathcal{K}_2(n) \), \( s = (s_1, s_2) \in \mathcal{K}_2(S) \), and \( g = (g_1, g_2) \in \mathcal{K}_2(g - E + 1) \), set

\[
r_{n:T}(s; g) = \min \{2g_1 + s_1 + n_1 + T - 3, 2g_2 + s_2 + n_2 + T - 3\},
\]

and \( m_i = 2s_i + n_i + T \) for each \( i \in \{1, 2\} \). Let \( \mathcal{R} = \mathcal{R}_n = \mathcal{R}_{g,n}(S, T) \) denote the set of pairs \( (s, g) \in \mathcal{K}_2(S) \times \mathcal{K}_2(g - E + 1) \) for which \( r_{n:T}(s; g) \leq 10 \).

Let us bound \( |\mathcal{R}| \). There are at most two choices for the index \( i \in \{1, 2\} \) for which \( r_{n:T}(s; g) = 2g_i + s_i + T - 3 \). Then, given such an \( i \), there are at most 7 choices for \( g_i \in [0, 6] \) and at most 14 choices for \( s_i \in [0, 13] \). Hence, \( |\mathcal{R}| \leq 196 < 2^8 \). Moreover, \( |\mathcal{K}_2(S)| |\mathcal{K}_2(g - E + 1)| \leq 8g^2 \), by the second and third bounds in (9.20).

Thus, applying (9.19); (9.15) for \( (s, g) \in \mathcal{R}; (9.14) \) for \( (s, g) \in \mathcal{K}_2(S) \times \mathcal{K}_2(g - E + 1) \); the first bound in (9.20); and the fact that \( T \leq 13 \), we find that

\[
2^{-n} \left( \frac{8}{3} \right)^{-4g-n} \gamma_{g,n}^{(2:S,T)} \leq 2^{n+18} (S^2 + 169) g^{-3/2} \frac{\eta (\log g + 7)^{S+12}}{2^S S!} \\
+ 2^{n+55} g^{-7} \left( \frac{3}{2} \right)^{2S+26+n} \frac{(\log g + 7)^{S+12}}{2^S S!},
\]

where we have defined

\[
\eta = \eta_{g,n}(S, T) = \max_{n \in \mathcal{K}_2(n)} \max_{(s, g) \in \mathcal{R}_n} \max_{d^{(1)} \in \mathcal{K}_{m_1}(3g_1 + m_1 - 3)} \max_{d^{(2)} \in \mathcal{K}_{m_2}(3g_2 + m_2 - 3)} \left\{ d^{(1)} \right\}_{g_1, m_2} \left\{ d^{(2)} \right\}_{g_2, m_2}.
\]

Let us estimate \( \eta \). To that end, let \( n \in \mathcal{K}_2(n) \) and \( (s, g) \in \mathcal{R} \) maximize the right side of (9.24). We may assume that \( r_{n:T}(s; g) = 2g_1 + s_1 + n_1 + T - 3 \).
Then \( m_1 = 2s_1 + n_1 + T \leq 2(r_n,T(s,g) + 3) \leq 26 \), and so Proposition 1.2 implies that

\[
\eta \leq \left( \frac{3}{2} \right)^{26} \max_{d \in \mathcal{K}_{m_2}(3g_2 + m_2 - 3)} \langle d \rangle_{g_2,m_2}. \tag{9.25}
\]

Now we consider two cases depending on \( S \). The first is if \( S \geq 15 \log g \), in which case \( 2^S \geq g^{10} \) and so Proposition 1.2, (9.25), and the fact that \( m_2 \leq 2S + T + n \leq 2S + n + 13 \) together imply that

\[
\frac{\eta}{2^S} \leq g^{-10} \left( \frac{3}{2} \right)^{2S+n+39}. \tag{9.26}
\]

The second is if \( S < 15 \log g \), in which case \( m_2 \leq 2S + T + n \leq 30 \log g + n + 13 \). Moreover, since \( E = S + T, g_1 + T \leq r_n,T(s,g) + 3 \leq 13, 20n \leq \log g \), and \( g > 2^{120} \), we have

\[
g_2 = g - E + 1 - g_1 \geq g - S - 12 \geq g - 15 \log g - 12 > \frac{g}{2}
\]
\[
> 800(33 \log g + 13)^2 \geq 800(m_2 + 2 \log g_2)^2.
\]

Therefore, Proposition 4.2 applies, which together with the facts that \( g_2 > \frac{g}{2} \), \( m_2 \leq 30 \log g + n + 10 \), \( g > 2^{120} \), and \( 20n \leq \log g \) gives

\[
\max_{d \in \mathcal{K}_{m_2}(3g_2 + m_2 - 3)} \langle d \rangle_{g_2,m_2} \leq \exp \left( 2500g^{-1}(33 \log g + 13)^2 \right) \leq e. \tag{9.27}
\]

Combining (9.25), (9.26), and (9.27) yields

\[
\frac{\eta}{2^S} \leq \left( \frac{3}{2} \right)^{30} + g^{-10} \left( \frac{3}{2} \right)^{2S+n+39},
\]

which upon insertion into (9.23) (with the bounds \( S \leq E \leq 3g + n - 3 < 4g \), \( 169 < 2^8 \) and \( \left( \frac{3}{2} \right)^{26} < 2^{16} \)) yields (9.18).

Now we can establish Proposition 8.4.
Proof of Proposition 8.4 By (8.3) and Lemma 9.5, we have

\[
2^{-n} \left( \frac{8}{3} \right)^{-4g-n} \gamma^{(2)}_{g,n} \\
= \sum_{S=0}^{g-2} 2^{-n} \left( \frac{8}{3} \right)^{-4g-n} \left( \sum_{T=1}^{12} \gamma^{(2;S,T)}_{g,n} + \sum_{T=13}^{3g+n-S-3} \gamma^{(2;S,T)}_{g,n} \right) \\
+ \sum_{S=g-1}^{3g+n-3} \sum_{T=1}^{3g+n-S-3} 2^{-n} \left( \frac{8}{3} \right)^{-4g-n} \gamma^{(2;S,T)}_{g,n},
\]

(9.28)

\[
\leq 2^{76} 3^n (\log g + 7)^{12} \sum_{S=0}^{\infty} \left( g^{-3/2} \frac{(S^2 + 1)}{S!} (\log g + 7)^S \right) \\
+ \frac{1}{g^7 3^S} \left( \frac{9(\log g + 7)}{4} \right)^S \\
+ 2^{55} 3^n g^{-7} \sum_{S=0}^{\infty} \sum_{T=0}^{\infty} \frac{1}{S!T!} \left( \frac{9(\log g + 7)}{4} \right)^{S+T} + (4g)^2 g^{-10},
\]

where in the last bound we used the fact that \((3g + n - 3)^2 \leq (4g)^2\) (as \(20n \leq \log g \leq g\)).

Now observe since \(g > 2^{120}\) that

\[
\sum_{S=0}^{\infty} \frac{(S^2 + 1)}{S!} (\log g + 7)^S \\
= \sum_{S=2}^{\infty} \frac{(\log g + 7)^S}{(S - 2)!} + \sum_{S=1}^{\infty} \frac{(\log g + 7)^S}{(S - 1)!} + \sum_{S=0}^{\infty} \frac{(\log g + 7)^S}{S!} \\
= ((\log g + 7)^2 + \log g + 8) g \leq 4(\log g)^2 g
\]

and

\[
\sum_{S=0}^{\infty} \sum_{T=0}^{\infty} \frac{1}{S!T!} \left( \frac{9(\log g + 7)}{4} \right)^{S+T} = \left( \sum_{S=0}^{\infty} \sum_{T=0}^{\infty} \frac{1}{S!} \left( \frac{9(\log g + 7)}{4} \right)^S \right)^2 \\
= \exp \left( \frac{9(\log g + 7)}{2} \right) = e^{63/2} g^{9/2}.
\]
Upon insertion into (9.28), these estimates give

\[
2^{-n} \left( \frac{8}{3} \right)^{-4g-n} \gamma^{(2)}_{g,n} = 2^{76} 3^n (\log g + 7)^{12} \\
	imes (4(\log g)^2 g^{-1/2} + 2e^{63/2} g^{-5/2} + g^{-8}).
\]

Together with the bounds \(\log g + 7 \leq 2(\log g)\), \(e^{63/2} < e^{32} < 2^{48}\), and \(3^n < g^{1/4}\) (since \(20n \leq \log g\)) this implies the proposition. \(\Box\)

10 Bounds for \(\gamma^{(V)}_{g,n}\) if \(V > 2\)

In this section we establish Proposition 8.5, which bounds the contribution to the right side of (2.20) arising from graphs with at least three vertices. We begin in Sect. 10.1 by analyzing the contribution from a fixed graph. Then, in Sect. 10.2 we sum this bound to estimate \(\gamma^{(V; S, T)}_{g,n}\) from Sect. 8.6, which we use to prove Proposition 8.5 in Sect. 10.3.

10.1 General estimates for an individual graph

We begin with the following definition that specifies the set of stable graphs with a given genus decoration and prescribed numbers of legs, self-edges, and simple edges incident to each vertex.

**Definition 10.1** Fix integers \(g, n \geq 0\) with \(2g + n \geq 4\); \(V \in [2, 2g + n - 2]\); \(S \geq 0\); and \(T \geq V - 1\). Set \(E = S + T\), and assume \(E \leq 3g + n - 3\). Further fix (nonnegative) compositions \(n \in K_{V}(n), g \in K_{V}(g - E + V - 1), T \in C_{V}(2T)\), and \(s \in K_{V}(S)\). Denote

\[
\begin{align*}
  n &= (n_1, n_2, \ldots, n_V); \quad g = (g_1, g_2, \ldots, g_V), \quad T = (T_1, T_2, \ldots, T_V), \\
  s &= (s_1, s_2, \ldots, s_V).
\end{align*}
\]

Then let \(G_{g,n}(n; s, T; g) = G_{g,n}(V; n; s, T; g) \subseteq G_{g,n}(V; S, T)\) denote the set of stable graphs \(\Gamma \in G_{g,n}(V; S, T)\) such that there exists a labeling of each of the \(V\) vertices of \(\Gamma\) with a (distinct) integer in \(\{1, 2, \ldots, V\}\) satisfying the following two properties.

- Vertex \(i \in [1, V]\) is incident to \(n_i\) legs, \(s_i\) self-edges, and \(T_i\) simple edges.
- The genus decoration of \(\Gamma\) assigns \(g_i\) to vertex \(i \in [1, V]\).

Observe under this notation that \(m_i = 2s_i + n_i + T_i\) half-edges are incident to any vertex \(i \in [1, V]\). Further observe that, since \(V \geq 2\), we must have \(T \geq V - 1\) and each \(T_i \geq 1\) to allow for the connectivity of any graph \(\Gamma \in G_{g,n}(n; s, T; g)\).
We then have the following proposition, analogous to Lemma 9.2, that bounds \( Z(P(\Gamma)) \) for any fixed \( \Gamma \in G_{g,n}(n; s, T; g) \).

**Proposition 10.2** Adopt the notation of Definition 10.1, and fix \( \Gamma \in G_{g,n}(n; s, T; g) \). Then,

\[
Z(P(\Gamma)) \leq \frac{2^{4g+2n+2E-3V-2}}{3^{g-E+V-1}} \left( \frac{3}{2} \right)^{2E+n} \frac{(4g+n-4)!}{(6g+2n-7)!} \frac{Z_E(3g+n-3)}{|Aut(\Gamma)|} \times \prod_{i=1}^{V} \frac{(6g_i + 4s_i + 2n_i + 2T_i - 5)!}{(3g_i + 2s_i + n_i + T_i - 3)!g_i!}.
\]

**Proof** The proof of this proposition will be similar to that of Lemma 9.2. We first define the collection \( t = t(\Gamma) = (t_{i,j})_{1 \leq i \neq j \leq V} \subset \mathbb{Z}_{\geq 0} \) of \( V^2 - V \) nonnegative integers as follows. Under the vertex labeling of \( \Gamma \) as prescribed in Definition 10.1, let \( t_{i,j} \) denote the number of (simple) edges connecting vertices \( i \) and \( j \), for any \( 1 \leq i \neq j \leq V \). Then observe that

\[
\sum_{1 \leq i < j \leq V} t_{i,j} = T \quad \text{and} \quad t_{i,j} = t_{j,i} \quad \text{for each} \quad 1 \leq i \neq j \leq V.
\]

We next assign variables to each edge of \( \Gamma \), following Definition 2.12. For any index \( i \in [1, V] \), let \( b_1^{(i)}, b_2^{(i)}, \ldots, b_{s_i}^{(i)} \) be variables (associated with the \( s_i \) self-edges at vertex \( i \)) and, for any distinct indices \( 1 \leq i < j \leq V \), let \( b_1^{(i,j)}, b_2^{(i,j)}, \ldots, b_{t_{i,j}}^{(i,j)} \) be variables (associated with the \( t_{i,j} \) simple edges between vertices \( i \) and \( j \)); also set \( b_k^{(i,j)} = b_k^{(j,i)} \) for any \( 1 \leq j < i \leq V \) and \( 1 \leq k \leq t_{i,j} \). For each index \( i \in [1, V] \), define the variable set

\[
\mathbf{b}^{(i)} = (b_1^{(i)}, b_1^{(i)}, b_2^{(i)}, b_2^{(i)}, \ldots, b_{s_i}^{(i)}, b_{s_i}^{(i)}) \cup \bigcup_{j \neq i} (b_1^{(i,j)}, b_2^{(i,j)}, \ldots, b_{t_{i,j}}^{(i,j)})
\]

\[
\cup (0, 0, \ldots, 0).
\]

As prescribed in Definition 2.12, here the variables \( b_k^{(i,j)} \) associated with self-edges appear with multiplicity two; the variables \( b_k^{(i,j)} \) associated with simple edges appear with multiplicity one; and zero appears with multiplicity \( n_i \).
Recalling the polynomial $N_{g,n}$ from Definition 2.11, we then have

\[
\mathcal{Z}(P(\Gamma)) = 2^{6g+2n-V-4} \frac{(4g + n - 4)!}{(6g + 2n - 7)! \left| \text{Aut}(\Gamma) \right|} \times \mathcal{Z}\left( \prod_{1 \leq i < j \leq V} l_{i,j} \prod_{k=1}^{l_{i,j}} b_k^{(i,j)} \prod_{i=1}^{V} s_i \prod_{k=1}^{s_i} b_k^{(i)} \prod_{i=1}^{V} N_{g_i,m_i}(b^{(i)}) \right). \tag{10.1}
\]

Now, observe using the first identity in (2.1) that

\[
N_{g_i,m_i}(b^{(i)}) = \frac{(6g_i + 2m_i - 5)!}{2^{8g_i+2m_i-6} \cdot 3^{g_i} \cdot (g_i + m_i - 3)! g_i!} \times \sum_{\mathbf{d}^{(i)} \in \mathcal{K}_{2s_i+7_1} \cdot (3g_i + m_i - 3)} \left( \mathbf{d}^{(i)} , 0^{n_i} \right)_{g_i,m_i}
\]

\[
\times \prod_{j \neq i} \prod_{k=1}^{l_{i,j}} \left( b_k^{(i,j)} \right)^{2d_{j,k}^{(i,j)}} \prod_{k=1}^{s_i} \left( b_k^{(i)} \right)^{2d_k^{(i)} + 2d_{2k}^{(i)}}, \tag{10.2}
\]

where we have denoted

\[
\mathbf{d}^{(i)} = (d_1^{(i)}, d_2^{(i)}, \ldots, d_{2s_i}^{(i)}) \cup \bigcup_{j \neq i} (d_{j;1}^{(i)}, d_{j;2}^{(i)}, \ldots, d_{j;\ell_{i,j}}^{(i)});
\]

\[
(\mathbf{d}^{(i)}, 0^{n_i}) = (d_1^{(i)}, d_2^{(i)}, \ldots, d_{2s_i}^{(i)}) \cup \bigcup_{j \neq i} (d_{j;1}^{(i)}, d_{j;2}^{(i)}, \ldots, d_{j;\ell_{i,j}}^{(i)}) \cup (0, 0, \ldots, 0),
\]

where $(\mathbf{d}^{(i)}, 0^{n_i})$ contains $n_i$ zeroes at the end. Indeed, the last $n_i$ entries of any $\mathbf{d}' \in \mathcal{K}_{m_i} \cdot (3g_i + m_i - 3)$ contributing to $N_{g_i,m_i}(b^{(i)})$ in the right side of (2.19) in our setting must be equal to 0, since the last $n_i$ entries of $b^{(i)}$ are.

By (2.14) and the identity

\[
\sum_{i=1}^{V} (8g_i + 2m_i - 6) = 8(g - E + V - 1) + 4E + 2n - 6V
\]

\[
= 8g + 2n - 4E + 2V - 8,
\]

inserting (10.2) into (10.1) yields

\[
\mathcal{Z}(P(\Gamma)) = 2^{4E - 3V - 2g + 4} \frac{(4g + n - 4)!}{3^{g-E+V-1} \cdot (6g + 2n - 7)! \left| \text{Aut}(\Gamma) \right|} \prod_{i=1}^{V} \frac{(6g_i + 2m_i - 5)!}{(3g_i + m_i - 3)! g_i!}.
\]
\[
\times \sum_{d^{(i)} \in K_{2s_1 + t_1}(3g_1 + m_1 - 3)} \ldots \sum_{d^{(V)} \in K_{2s_V + t_V}(3g_V + m_V - 3)} \prod_{i=1}^{V} \left( d^{(i)}, 0^{n_i} \right)_{g_i, m_i}
\]

\[
\times \mathcal{Z} \left( \prod_{i=1}^{V} \prod_{k=1}^{s_i} \left( b^{(i)}_k \right)^{2d^{(i)}_{2k-1} + 2d^{(i)}_{2k} + 1} \frac{(2d^{(i)}_{2k} + 1)!}{(2d^{(i)}_{2k-1} + 1)!} \right) \prod_{1 \leq i < j \leq V} \prod_{k=1}^{t_{i,j}} \left( b^{(i)}_{j,k} \right)^{2d^{(j)}_{i,k} + 2d^{(i)}_{i,k} + 1} \frac{(2d^{(i)}_{i,k} + 1)!}{(2d^{(j)}_{i,k} + 1)!}.
\]

Recalling the definition Definition 2.11 of \( \mathcal{Z} \), and applying Proposition 1.2 and (2.16), it follows that

\[
\mathcal{Z}(P(\Gamma)) \leq \frac{2^{4E-3V-2g+4}}{3g-E+V-1} \left( \frac{3}{2} \right)^{2E+n} \frac{(4g + n - 4)!}{(6g + 2n - 7)!} \frac{1}{|\text{Aut}(\Gamma)|}
\times \sum_{d^{(i)} \in K_{2s_1 + t_1}(3g_1 + m_1 - 3)} \ldots \sum_{d^{(V)} \in K_{2s_V + t_V}(3g_V + m_V - 3)} \prod_{i=1}^{V} \frac{(6g_i + 2m_i - 5)!}{(3g_i + m_i - 3)!g_i!}
\times \prod_{i=1}^{V} \prod_{k=1}^{s_i} \left( 2d^{(i)}_{2k-1} + 2d^{(i)}_{2k} + 1 \right)! \frac{(2d^{(i)}_{2k-1} + 1)!}{(2d^{(i)}_{2k} + 1)!} \prod_{1 \leq i < j \leq V} \prod_{k=1}^{t_{i,j}} \left( 2d^{(i)}_{j,k} + 2d^{(j)}_{i,k} + 1 \right)! \frac{(2d^{(i)}_{j,k} + 1)!}{(2d^{(j)}_{i,k} + 1)!}.
\]

Together with the last statement of (2.1) and the fact that

\[
\sum_{i=1}^{V} s_i + \sum_{1 \leq i < j \leq V} t_{i,j} = E,
\tag{10.3}
\]
this gives

\[ Z(P(\Gamma)) \leq \frac{2^{3E-3V-2g+4}}{3^g-E+V-1} \left( \frac{3}{2} \right)^{2E+n} \frac{(4g + n - 4)!}{(6g + 2n - 7)! |Aut(\Gamma)|} \]
\[ \times \sum_{d^{(1)} \in K_{2g_1 + t_1}(3g_1 + m_1 - 3)} \cdots \sum_{d^{(V)} \in K_{2g_V + t_V}(3g_V + m_V - 3)} \]
\[ \times \prod_{i=1}^{V} \frac{(6g_i + 2m_i - 5)!}{(3g_i + m_i - 3)! g_i!} \]
\[ \times \prod_{i=1}^{V} \prod_{k=1}^{s_i} \left( \frac{2d_{2k-1}^{(i)} + 2d_{2k}^{(i)} + 2}{2d_{2k}^{(i)} + 1} \right) \frac{\zeta(2d_{2k-1}^{(i)} + 2d_{2k}^{(i)} + 2)}{d_{2k-1}^{(i)} + d_{2k}^{(i)} + 1} \]
\[ \times \prod_{1 \leq i < j \leq V \ k=1} \prod_{k=1}^{t_{i,j}} \left( \frac{2d_{j;k}^{(i)} + 2d_{i;k}^{(j)} + 2}{2d_{j;k}^{(i)} + 1} \right) \frac{\zeta(2d_{j;k}^{(i)} + 2d_{i;k}^{(j)} + 2)}{d_{j;k}^{(i)} + d_{i;k}^{(j)} + 1} . \]

(10.4)

Now for each \( 1 \leq i \neq j \leq V \) and \( k \) (either in \([1, s_i]\) or \([1, t_{i,j}]\)), define

\[ D_k^{(i)} = d_{2k-1}^{(i)} + d_{2k}^{(i)}; \quad u_{k}^{(i,j)} = d_{j;k}^{(i)} + d_{i;k}^{(j)}; \]
\[ U_i = 3g_i + m_i - 3 - \sum_{i=1}^{s_i} D_k^{(i)}; \quad V = \sum_{i=1}^{V} U_i. \]

Further set

\[ D^{(i)} = (D_1^{(i)}, D_2^{(i)}, \ldots, D_{s_i}^{(i)}) \in K_{s_i}(3g_i + m_i - U_i - 3); \]
\[ U = (U_1, U_2, \ldots, U_V) \in K_V(U), \]

and define the \( T \)-tuple of nonnegative integers \( u = (u_k^{(i,j)}) \), where \( i, j \) range over pairs of distinct indices in \([1, V]\) and \( k \) ranges over all indices in \([1, t_{i,j}]\). In particular, \( u \in K_T(U) \).

Under this notation, instead of summing over the \( d^{(i)} \) in (10.4), we may sum over \( U \in [0, 3g + n - E - 3]; U \in K_T(U); \) the \( d_{j;k}^{(i)} \in [0, u_k^{(i,j)}]; U \in K_V(U); \) the \( D^{(i)} \in K_{s_i}(3g_i + m_i - U_i - 3); \) and the \( d_{2k}^{(i)} \in [0, D_k^{(i)}] \). This yields

\[ Z(P(\Gamma)) \leq \frac{2^{3E-3V-2g+4}}{3^g-E+V-1} \left( \frac{3}{2} \right)^{2E+n} \frac{(4g + n - 4)!}{(6g + 2n - 7)! |Aut(\Gamma)|} \]

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\[ \times \prod_{i=1}^{V} \frac{(6g_i + 2m_i - 5)!}{(3g_i + m_i - 3)!g_i!} \]

\[ \times \sum_{U=0}^{3g+n-E-3} \sum_{u \in K_T(U)} \prod_{1 \leq i < j \leq V} \prod_{k=1}^{l_{i,j}} \zeta \left( \frac{2u_k^{(i,j)} + 2}{u_k^{(i,j)} + 1} \right) \]

\[ \times \sum_{d_{j,k}^{(i)} = 0}^{u_k^{(i,j)}} \left( \frac{2u_k^{(i,j)} + 2}{2d_{j,k}^{(i)} + 1} \right) \]

\[ \times \sum_{U \in K_V(U)} \prod_{i=1}^{V} \sum_{D_k^{(i)} \in K_{s_i}(3g_i + m_i - U_i - 3) k=1}^{s_i} \zeta \left( \frac{D_k^{(i)} + 2}{D_k^{(i)} + 1} \right) \]

\[ \times \sum_{d_{2k}^{(i)} = 0}^{D_k^{(i)}} \left( \frac{2D_k^{(i)} + 2}{2d_{2k}^{(i)} + 1} \right). \quad (10.5) \]

Using the second statement of (2.1) and the fact (which is a consequence of the identities (2.14), (2.16), and \( E = S + T \)) that

\[ \sum_{1 \leq i < j \leq V} \sum_{k=1}^{l_{i,j}} (2u_k^{(i,j)} + 1) + \sum_{i=1}^{V} \sum_{k=1}^{s_i} (2D_k^{(i)} + 1) \]

\[ = 2U + T + 2 \sum_{i=1}^{V} (3g_i + m_i - U_i - 3) + S \]

\[ = 6g + 2n - E - 6, \]

it follows from (10.5) that

\[ \mathcal{Z}(P(\Gamma)) \leq \frac{2^{4g+2n+E-3V-2} \left( \frac{3}{2} \right)^{2E+n} (4g + n - 4)!}{(6g + 2n - 7)! |\text{Aut}(\Gamma)|} \]

\[ \times \prod_{i=1}^{V} \frac{(6g_i + 2m_i - 5)!}{(3g_i + m_i - 3)!g_i!} \]

\[ \times \sum_{U=0}^{3g+n-E-3} \sum_{u \in K_T(U)} \prod_{1 \leq i < j \leq V} \prod_{k=1}^{l_{i,j}} \zeta \left( \frac{2u_k^{(i,j)} + 2}{u_k^{(i,j)} + 1} \right) \]
\[
\times \sum_{U \in \mathcal{K}_V(U)} \prod_{i=1}^{V} \sum_{D(i) \in \mathcal{K}_{s_i}(3g_i+m_i-U_i-3)} \prod_{k=1}^{s_i} \frac{\zeta(D_k(i)+2)}{D_k(i)+1}.
\]

(10.6)

Now let us define

\[
w_{i,j} = \sum_{k=1}^{t_{i,j}} u_k^{(i,j)}; \quad u^{(i,j)} = (u_1^{(i,j)}, u_2^{(i,j)}, \ldots, u_{t_{i,j}}^{(i,j)}) \in \mathcal{K}_{s_{i,j}}(w_{i,j}),
\]

and define the \((\binom{V}{2})\)-tuple of integers \(w = (w_{i,j})\), where \(i\) and \(j\) range over all indices in \([1, V]\), with \(i < j\). In particular, we have \(w \in \mathcal{K}_V(w_{i,j})\).

So, instead of summing over \(u\) in (10.6), we may sum over \(w\) and \(u^{(i,j)}\) to obtain that

\[
\mathcal{Z}(P(\Gamma)) \leq \frac{2^{4g+2n+2E-3V-2}}{3g-E+V-1} \left( \frac{3}{2} \right)^{2E+n} \frac{(4g + n - 4)!}{(6g + 2n - 7)!} \frac{1}{|\text{Aut}(\Gamma)|} \\
\times \prod_{i=1}^{V} \frac{(6g_i + 2m_i - 5)!}{(3g_i + m_i - 3)!g_i!} \\
\times \sum_{U=0}^{V} \sum_{w \in \mathcal{K}_V(U)} \prod_{1 \leq i < j \leq V} \\
\times \sum_{u^{(i,j)} \in \mathcal{K}_{s_{i,j}}(w_{i,j})} \prod_{k=1}^{t_{i,j}} \frac{\zeta(2u_k^{(i,j)}+2)}{u_k^{(i,j)}+1} \\
\times \sum_{U \in \mathcal{K}_V(U)} \prod_{i=1}^{V} \sum_{D(i) \in \mathcal{K}_{s_i}(3g_i+m_i-U_i-3)} \prod_{k=1}^{s_i} \frac{\zeta(D_k(i)+2)}{D_k(i)+1}.
\]

Recalling the definition of \(Z_k(m)\) from Definition 6.1, it follows that

\[
\mathcal{Z}(P(\Gamma)) \leq \frac{2^{4g+2n+2E-3V-2}}{3g-E+V-1} \left( \frac{3}{2} \right)^{2E+n} \frac{(4g + n - 4)!}{(6g + 2n - 7)!} \frac{1}{|\text{Aut}(\Gamma)|} \\
\times \prod_{i=1}^{V} \frac{(6g_i + 2m_i - 5)!}{(3g_i + m_i - 3)!g_i!}
\]
\[
\sum_{U=0}^{3g+n-E-3} \sum_{w \in \mathcal{K}_V(U)} \prod_{1 \leq i < j \leq V} Z_{t_{i,j}}(w_{i,j} + t_{i,j})
\times \sum_{U \in \mathcal{K}_V(U)} \prod_{i=1}^{V} Z_{s_i}(3g_i + s_i + m_i - U_i - 3). \tag{10.7}
\]

Then, the identities

\[
\sum_{i=1}^{V} (3g_i + s_i + m_i - U_i - 3)
= 3(g - E + V - 1) + S + 2E + n - U - 3V
= 3g + n + S - E - U - 3,
\]

and

\[
\sum_{1 \leq i < j \leq V} (w_{i,j} + t_{i,j}) = U + T; \quad \sum_{1 \leq i < j \leq V} t_{i,j} = T;
\]
\[
\sum_{i=1}^{V} s_i = S; \quad S + T = E,
\]

and repeated application of Lemma 6.2 gives

\[
\sum_{U=0}^{3g+n-E-3} \sum_{w \in \mathcal{K}_V(U)} \prod_{1 \leq i < j \leq V} Z_{t_{i,j}}(w_{i,j} + t_{i,j})
\times \sum_{U \in \mathcal{K}_V(U)} \prod_{i=1}^{V} Z_{s_i}(3g_i + s_i + m_i - U_i - 3)
\leq \sum_{U=0}^{3g+n-E-3} Z_T(U + T)Z_S(3g + n + S - E - U - 3)
\leq Z_E(3g + n - 3),
\]

which upon insertion into (10.7) (and recalling that \(m_i = 2s_i + n_i + T_i\)) yields the proposition. \(\square\)
10.2 Bounds for $\Upsilon_{g,n}^{(V;S,T)}$

In this section we estimate the quantity $\Upsilon_{g,n}^{(V;S,T)}$ from Definition 8.6. To that end, we begin with the following lemma that bounds this quantity by a certain sum.

Lemma 10.3 Fix integers $g \geq 2; n \geq 0; V \in [2, 2g + n + 2]; S \geq 0; $ and $T \geq V - 1$. Set $E = S + T$, and assume $E \leq 3g + n - 3$. Then,

$$\Upsilon_{g,n}^{(V;S,T)} \leq \frac{2^{4g+2n+2E-3V-S-2}}{3^{g-E+V-1}} \left(\frac{3}{2}\right)^{2E+n} \frac{(4g + n - 4)!}{(6g + 2n - 7)!}$$

$$\times \frac{n!(2T - 1)!!}{V!} \sum_{n \in \mathcal{K}_V(n) \ s \in \mathcal{K}_V(S) \ T \in \mathcal{C}_V(2T) \ g \in \mathcal{K}_V(g - E + V - 1)}$$

$$\times \prod_{i=1}^{V} \frac{(6g_i + 4s_i + 2n_i + 2T_i - 5)!}{(3g_i + 2s_i + n_i + T_i - 3)!g_i!s_i!T_i!n_i!},$$

where we have denoted $n = (n_1, n_2, \ldots, n_V)$, $g = (g_1, g_2, \ldots, g_V)$, $s = (s_1, s_2, \ldots, s_V)$, and $T = (T_1, T_2, \ldots, T_V)$.

Proof For any compositions $n \in \mathcal{K}_V(n)$, $g \in \mathcal{K}_V(g - E + V - 1)$, $s \in \mathcal{K}_V(S)$, and $T \in \mathcal{C}_V(2T)$, define

$$F_{g,n}(n; s, T; g) = \frac{2^{4g+2n+2E-3V-S-2}}{3^{g-E+V-1}} \left(\frac{3}{2}\right)^{2E+n} \frac{(4g + n - 4)!}{(6g + 2n - 7)!} \sum_{n \in \mathcal{K}_V(n) \ s \in \mathcal{K}_V(S) \ T \in \mathcal{C}_V(2T) \ g \in \mathcal{K}_V(g - E + V - 1)}$$

$$\times \prod_{i=1}^{V} \frac{(6g_i + 4s_i + 2n_i + 2T_i - 5)!}{(3g_i + 2s_i + n_i + T_i - 3)!g_i!s_i!T_i!n_i!}.$$
By Definition 10.1, Proposition 10.2, the identity $(2A)! = 2^A A!(2A - 1)!$, and the fact that $\sum_{i=1}^V s_i = S$, it suffices to show that

$$\sum_{\Gamma \in G_{g,n}(V;S,T)} \left| \text{Aut}(\Gamma) \right|^{-1} F_{g,n}(n(\Gamma); s(\Gamma), T(\Gamma); g(\Gamma))$$

$$\leq \frac{n!(2T - 1)!!}{V!} \sum_{n \in \mathcal{K}_V(n)} \sum_{s \in \mathcal{K}_V(S)} \sum_{T \in \mathcal{C}_V(2T)} \sum_{g \in \mathcal{K}_V(g-E+V-1)} (2s_i - 1)!! \prod_{i=1}^V \frac{(2s_i)!T_i!n_i!}{(2s_i)!T_i!n_i!},$$

(10.9)

where $n(\Gamma)$, $s(\Gamma)$, $T(\Gamma)$, and $g(\Gamma)$ are defined so that $\Gamma \in G_{g,n}(n(\Gamma); s(\Gamma), T(\Gamma); g(\Gamma))$.

To that end, for any integer $k \geq 1$, let $\mathcal{S}_k$ denote the symmetric group on $k$ elements. Observe that $\mathcal{S}_V$ acts on $\mathbb{Z}_{\geq 0}^V$ by setting $\sigma(a) = (a_{\sigma(1)}, a_{\sigma(2)}, \ldots, a_{\sigma(V)})$, for any $\sigma \in \mathcal{S}_V$ and $a = (a_1, a_2, \ldots, a_V) \in \mathbb{Z}_{\geq 0}^V$. In this way, $\mathcal{S}_V$ acts on a quadruple of compositions in $\mathcal{K}_V(n) \times \mathcal{K}_V(S) \times \mathcal{C}_V(2T) \times \mathcal{K}_V(g-E+V-1)$ diagonally. Let $\text{Aut}(n, s, T, g) \subseteq \mathcal{S}_V$ denote the stabilizer of a quadruple $(n, s, T, g) \in \mathcal{K}_V(n) \times \mathcal{K}_V(S) \times \mathcal{C}_V(2T) \times \mathcal{K}_V(g-E+V-1)$ under this action.

In particular, the orbit under this action of a quadruple $(n, s, T, g)$ has size $V! \left| \text{Aut}(n, s, T, g) \right|^{-1}$. So, since the vertices of any stable graph are unlabeled, we obtain that

$$\sum_{\Gamma \in G_{g,n}(V;S,T)} \left| \text{Aut}(\Gamma) \right|^{-1} F(n(\Gamma); s(\Gamma), T(\Gamma); g(\Gamma))$$

$$\leq \sum_{n \in \mathcal{K}_V(n)} \sum_{s \in \mathcal{K}_V(S)} \sum_{T \in \mathcal{C}_V(2T)} \sum_{g \in \mathcal{K}_V(g-E+V-1)} V!^{-1} \left| \text{Aut}(n, s, T, g) \right|$$

$$\times \sum_{\Gamma \in G_{g,n}(n,s,T;g)} \left| \text{Aut}(\Gamma) \right|^{-1} F(n; s, T; g).$$

Therefore, to establish (10.9), it suffices to show for any fixed $(n, s, t, g) \in \mathcal{K}_V(n) \times \mathcal{K}_V(S) \times \mathcal{C}_V(2T) \times \mathcal{K}_V(g-E+V-1)$ that

$$\prod_{i=1}^V (2s_i + T_i)! \sum_{\Gamma \in G_{g,n}(n,s,T;g)} \left| \text{Aut}(\Gamma) \right|^{-1} \left| \text{Aut}(n, s, T, g) \right|$$

$$\leq \binom{n}{n_1, n_2, \ldots, n_V} (2T - 1)!! \prod_{i=1}^V \frac{(2s_i + T_i)!}{(2s_i)!} (2s_i - 1)!!.$$

(10.10)
To establish (10.10), we will compare both sides (10.10) to the number of labeled stable graphs, which constitute a genus $g$ stable graph $\Gamma$ with $n$ legs together with bijections $v : \mathcal{V}(\Gamma) \to \{1, 2, \ldots, V\}$ and $h : \mathcal{H}(\Gamma) \setminus \mathcal{L}(\Gamma) \to \{1, 2, \ldots, 2E\}$ (that is, a labeling of each vertex and each half-edge that is not a leg of $\Gamma$ with a distinct index in $[1, V]$ and $[1, 2E]$, respectively), subject to the following three conditions. First, if $h, h' \in \mathcal{H}$ are two distinct half-edges such that $\alpha(h) < \alpha(h')$, then $\mathcal{H}(h) < \mathcal{H}(h')$; stated alternatively, any half-edge incident to vertex $i \in [1, V]$ has label in the interval $[W_{i-1} - 1, W_i]$, where $W_0 = 0$ and $W_k = \sum_{j=1}^{k} (2s_j + T_j)$, for each $k \in [1, V]$. Second, vertex $i \in [1, V]$ is incident to $n_i$ legs, $s_i$ self-edges, and $T_i$ simple edges. Third, the genus decoration of $\Gamma$ assigns the integer $g_i$ to vertex $i$, for each $i \in [1, V]$.

As for stable graphs, an isomorphism between two labeled graphs, $\Gamma = (\mathcal{V}, \mathcal{H}, \alpha, \mathcal{L}, \lambda, \iota, v, h, g)$ and $\Gamma' = (\mathcal{V}', \mathcal{H}', \alpha', \mathcal{L}', \lambda', \iota', v', h', g')$, consists of two bijections $\mu : \mathcal{V} \to \mathcal{V}'$ and $\nu : \mathcal{H} \to \mathcal{H}'$ such that $\alpha'(\nu(h)) = \mu(\alpha(h))$ and $\iota'(\nu(h)) = \nu(\iota(h))$, for each half-edge $h \in \mathcal{H}$; such that $\lambda'(\nu(h)) = \lambda(h)$, for each leg $h \in \mathcal{L}$; such that $h'(\nu(h)) = h(h)$, for each half-edge $h \in \mathcal{H} \setminus \mathcal{L}$ that is not a leg; and such that $\nu'(\mu(\nu)(v)) = \nu(v)$ and $g'_{\mu(\nu)}(v) = g_{v}$, for each vertex $v \in \mathcal{V}$. An isomorphism from a labeled stable graph to itself is called an automorphism of the labeled stable graph.

To see that the number of labeled stable graphs (up to automorphism) is bounded above by the right side of (10.10), observe that any such graph can be produced as follows. First, fix the set of leg labels at each vertex in $\Gamma$; this can be done in $\binom{n_1, n_2, \ldots, n_V}{n}$ ways. Then, for each vertex $i \in [1, V]$, fix which pairs of half-edges combine to form simple edges; for each $i$, this can be done in $\binom{2s_i + T_i}{2s_i}$ ways. Next, at each vertex $i \in [1, V]$, fix $s_i$ pairs of half-edges that combine to form self-edges; for each $i$, this can be done in $(2s_i - 1)!!$ ways. Then, fix which pairs of half-edges combine to form simple edges; this can be done in at most $(2T - 1)!!$ ways. The number of labeled stable graphs is therefore at most equal to the product of these quantities, which is the right side of (10.10).

To see that the number of labeled stable graphs (up to automorphism) is equal to the left side of (10.10), observe that the product of symmetric groups $\mathcal{G} = \prod_{i=1}^{V} \mathcal{G}_{2s_i + T_i}$ acts on the set of such graphs by having $\mathcal{G}_{2s_i + T_i}$ permute the $2s_i + T_i$ labels for half-edges, which are not legs, incident to vertex $i$ (in the interval $[W_{i-1} + 1, W_i]$), for each $i \in [1, V]$. This group action fixes any stable map, but might change the labeling of a labeled stable graph. Let $\text{Aut}^{(\text{lab})}(\Gamma)$ denote the automorphism group of any labeled stable graph $\Gamma$, which only depends on the (unlabeled) graph $\Gamma$ and not on the labeling.
Then, \( \text{Aut}(\Gamma) \) is isomorphic to \( \text{Aut}^{\text{lab}}(\Gamma) \times \text{Aut}(n, s, T, g) \) for any \( \Gamma \in \mathcal{G}_{g,n}(n; s, T; g) \). So,

\[
\prod_{i=1}^{V}(2s_i + T_i)! \sum_{\Gamma \in \mathcal{G}_{g,n}(n; s, T; g)} |\text{Aut}(\Gamma)|^{-1} |\text{Aut}(n, s, T, g)| = \sum_{\Gamma \in \mathcal{G}_{g,n}(n; s, T; g)} |\text{Aut}^{\text{lab}}(\Gamma)|^{-1} |\mathcal{S}|,
\]

which counts the number of labeled stable graphs, since every labeled stable graph is obtained by applying an element of \( \mathcal{S} \) to a stable (unlabeled) graph in \( \mathcal{G}_{g,n}(n; s, T; g) \).

It follows that the number of labeled stable graphs is equal to the left side of (10.10) and bounded above by the right side of (10.10). This establishes (10.10) and therefore the lemma. \( \square \)

The following proposition estimates the sum on the right side of (10.8), thereby providing a simplified bound on \( \gamma_{g,n}^{(V, S, T)} \).

**Proposition 10.4** Fix integers \( g \geq 2; n \geq 0; V \in [2, 2g + n - 2]; S \geq 0; \) and \( T \geq V - 1 \). Set \( E = S + T \), and assume \( E \leq 3g + n - 3 \). Denoting \( X = \min\{S, V\} \) and \( Y = \min\{2T, 3V\} \), we have

\[
2^{-n} \left( \frac{8}{3} \right)^{-4g-n-1} \gamma_{g,n}^{(V, S, T)} \leq (S + T)^{1/2-V} 2^{20V+11} \left( \frac{3V}{2} \right)^{n} \left( \frac{9}{8} \right)^{S} \left( \frac{9}{4} \right)^{T} \left( \frac{T}{V} \right)^{2V} \times \frac{(\log g + 7)^{S+T-1}(2T - 1)!!}{V^{V}(S - X)!(2T - Y)!}.
\]

**Proof** We begin by estimating the sum on the right side of (10.8) over \( (s, T, g) \in \mathcal{K}_{V}(S) \times \mathcal{C}_{V}(2T) \times \mathcal{K}_{V}(g) \), for some fixed \( n \in \mathcal{K}_{V}(n) \). To that end, fix \( n = (n_1, n_2, \ldots, n_V) \) and define for each \( i \in [1, V] \) and any \( (s, T, g) \in \mathcal{K}_{V}(S) \times \mathcal{C}_{V}(2T) \times \mathcal{K}_{V}(g) \) the quantities

\[
A_i = \min\{s_i, 1\}; \quad B_i = \min\{T_i, 3\}; \quad A = \sum_{i=1}^{V} A_i; \quad B = \sum_{i=1}^{V} B_i.
\]

Then, we claim that

\[
2g_i + s_i + n_i + A_i + B_i \geq 3, \quad \text{for each } i. \tag{10.11}
\]
Indeed, if $T_i \geq 3$, then $B_i \geq 3$ and (10.11) holds. Therefore, let us assume that $T_i < 3$, in which case $B_i = T_i$, and so $2g_i + s_i + n_i + A_i + B_i = 2g_i + s_i + n_i + A_i + T_i$. If $s_i \leq 1$, then $A_i = s_i$, and so (2.15) implies $2g_i + s_i + n_i + A_i + B_i = 2g_i + 2s_i + n_i + T_i \geq 3$, which verifies (10.11). If instead $s_i \geq 2$, then $A_i = 1$ and so $2g_i + s_i + n_i + A_i + B_i \geq s_i + A_i \geq 3$, which again confirms (10.11).

Thus, since $(s_i - A_i)! \leq s_i!$ and $(T_i - B_i)! \leq T_i!$, we have

$$\frac{(6g_i + 4s_i + 2n_i + 2T_i - 5)!}{(3g_i + 2s_i + n_i + T_i - 3)!g_i!s_i!T_i!} \leq \left(\frac{6g_i + 4s_i + 2n_i + 2T_i - 6}{(3g_i + 2s_i + n_i + T_i - 3, g_i, s_i - A_i, T_i - B_i, 2g_i + s_i + n_i + A_i + B_i - 3)}\right) \times \left(\frac{6g_i + 4s_i + 2n_i + 2T_i - 5)(2g_i + s_i + n_i + A_i + B_i - 3)!}{(3g_i + 2s_i + n_i + T_i - 3)!g_i!s_i!T_i!}\right).$$

So, using Lemma 2.3 together with (2.14), (2.16), and the facts that $\sum_{i=1}^{V} s_i = S$ and $\sum_{i=1}^{V} T_i = 2T$, we obtain

$$\prod_{i=1}^{V} \frac{(6g_i + 4s_i + 2n_i + 2T_i - 5)!}{(3g_i + 2s_i + n_i + T_i - 3)!g_i!s_i!T_i!} \leq \left(\frac{6g + 2n - 2E - 6}{(3g + n - E - 3, g - E + V - 1, S - A, 2T - B, 2g + n - 2E - V + S + A + B - 2)}\right) \times \left(\frac{6g_i + 4s_i + 2n_i + 2T_i - 5)(2g_i + s_i + n_i + A_i + B_i - 3)!}{(3g_i + 2s_i + n_i + T_i - 3)!g_i!s_i!T_i!}\right).$$

(10.12)

Now, observe that

$$6g_i + 4s_i + 2n_i + 2T_i - 5 \leq 4(2g_i + s_i + n_i + A_i + B_i - 2) + 2T_i \leq 2(T_i + 2)(2g_i + s_i + n_i + A_i + B_i - 2),$$

where the first bound follows from the fact that $B_i \geq 1$ and second from the fact that $2g_i + s_i + n_i + A_i + B_i - 2 \geq 1$. Hence,

$$\prod_{i=1}^{V} (6g_i + 4s_i + 2n_i + 2T_i - 5)(2g_i + s_i + n_i + A_i + B_i - 3)! \leq 2^V \prod_{i=1}^{V} (T_i + 2) \prod_{i=1}^{V} (2g_i + s_i + n_i + A_i + B_i - 2)! \leq 2^V \left(\frac{2T + 2V}{V}\right)^V \prod_{i=1}^{V} (2g_i + s_i + n_i + A_i + B_i - 2)!$$
\[
\leq 12^V \left( \frac{T}{V} \right)^V \prod_{i=1}^{V} \frac{(2g_i + s_i + n_i + A_i + B_i - 2)!}{(3g_i + 2s_i + n_i + T_i - 3)!g_i!s_i!T_i!}.
\] (10.13)

Here, in the second inequality, we used the fact that \( \sum_{i=1}^{V} (T_i + 2) = 2T + 2V \) and, in the third, we used the fact that \( V \leq T + 1 \leq 2T \). Moreover, since each \( A_i \leq 1 \) and each \( B_i \leq 3 \), we have \( A \leq \min\{S, V\} = X \) and \( B \leq \min\{2T, 3V\} \leq Y \).

Thus, \( (S - A)! \geq (S - X)! \) and \( (2T - B)! \geq (2T - Y)! \), and so (10.12) and (10.13) together imply

\[
\prod_{i=1}^{V} \frac{(6g_i + 4s_i + 2n_i + 2T_i - 5)!}{(3g_i + 2s_i + n_i + T_i - 3)!g_i!s_i!T_i!} \leq \frac{(6g + 2n - 2E - 6)!}{(3g + n - E - 3)!} \frac{1}{(g - E + V - 1)!} \frac{1}{(S - X)!} (2T - Y)! \times 12^V \left( \frac{T}{V} \right)^V \prod_{i=1}^{V} \frac{(2g_i + s_i + n_i + A_i + B_i - 2)!}{(2g_i + M_i + B_i - 2)!}.
\] (10.14)

Denoting for each \( i \in [1, V] \) the quantities

\[
M_i = s_i + n_i + A_i, \quad B = (B_1, B_2, \ldots, B_V); \quad M = (M_1, M_2, \ldots, M_V),
\]

we find that \( B \in C_V(B) \) and \( M \in K_V(S + n + A) \). So,

\[
\sum_{s \in K_V(S)} \sum_{T \in C_V(2T)} \sum_{g \in K_V(g - E + V - 1)} \prod_{i=1}^{V} (2g_i + s_i + n_i + A_i + B_i - 2)! \leq |C_V(2T)| \sum_{M \in K_V(S + n + A)} \sum_{B \in C_V(B)} \sum_{g \in K_V(g - E + V - 1)} \prod_{i=1}^{V} (2g_i + M_i + B_i - 2)! \leq 2^{12V+9} \left( \frac{2T - 1}{V - 1} \right) (2g + n - 2E - V + S + A + B - 1)!,
\]

where in the last inequality we applied the first identity in (2.2) and Lemma 2.1 (with the \( m \) there equal to \( V \) here; the \( A_i, B_i, \) and \( C_i \) there equal to \( 2g_i, B_i, \) and \( M_i \) here, respectively; and the \( A, B, \) and \( C \) there equal to \( 2g - 2E + 2V - 2, B, \) and \( S + n + A \) here, respectively). Together with (10.14) and the fact that \( 2g + n - 2E - V + S + A + B - 1 \leq 2g + n - E - 1 \leq 6g + 2n - 2E - 5 \) (where in the first bound we used the facts that \( A \leq V \) and \( S + B \leq S + T = E \), and in the second we used (2.18) and the fact that \( g \geq 2 \)), this implies

\[
n! \sum_{n \in K_V(n)} \sum_{s \in K_V(S)} \sum_{T \in C_V(2T)} \sum_{g \in K_V(g - E + V - 1)} \sum \sum \sum \sum.
\]

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\[
\times \prod_{i=1}^{V} \frac{(6g_i + 4s_i + 2n_i + 2T_i - 5)!}{(3g_i + 2s_i + n_i + T_i - 3)!g_i!s_i!T_i!n_i!}
\leq \frac{2^{16^V+9}}{V}
\left(\frac{T}{V}\right)^{V}
\left(\frac{2T - 1}{V - 1}\right)
\frac{(6g + 2n - 2E - 5)!}{(3g + n - E - 3)!}
\times \frac{1}{(g - E + V - 1)! (S - X)! (2T - Y)!}
\sum_{\mathbf{n} \in \mathcal{K}_V(n)} \binom{n}{n_{11}, n_{12}, \ldots, n_{V}}
\]
\[= \frac{2^{16^V+9}}{V}
\left(\frac{T}{V}\right)^{V^n}
\left(\frac{2T - 1}{V - 1}\right)
\frac{(6g + 2n - 2E - 5)!}{(3g + n - E - 3)!}
\times \frac{1}{(g - E + V - 1)! (S - X)! (2T - Y)!},
\]

where in the last equality we used the identity
\[
\sum_{\mathbf{n} \in \mathcal{K}_V(n)} \binom{n}{n_{11}, n_{12}, \ldots, n_{V}} = V^n.
\]

This, together with Lemma 10.3 yields
\[
Y^{(V; S, T)}_{g, n} \leq 12^E \frac{(6g + 2n - 2E - 5)!}{(3g + n - E - 3)! (g - E + V - 1)!}
\times Z_E(3g + n - 3)
\times \frac{2^{4g+2n+13V-S+7}}{3^{g+V-1}}
\times \frac{V^n}{(S - X)! (2T - Y)!}
\]
\[\frac{(3g + n - 3)}{(3g + n - 3)! (g - E + V - 1)!}
\times \frac{2^{4g+2n+13V-S+7}}{3^{g+V-1}}
\times \frac{V^n}{(S - X)! (2T - Y)!}.
\]

This with Lemmas 8.10, 6.3, and the facts that $3g + n - 3 \geq g$ and $\log(3g + n - 3) \leq \log g + 2$ (since $20n \leq \log g$) together imply that
\[
Y^{(V; S, T)}_{g, n} \leq g^{1/2-V} 2^n \left(\frac{8}{3}\right)^{4g+n} E \log g + 7)^{E-1}
\times \frac{2^{13V-S+7} 3^{V+2}}{V^n}
\times \frac{E}{(S - X)! (2T - Y)!}
\]

\(\gamma^*(V; S, T)\)
Together with the identity \( E = S + T \) and the bounds \( \binom{2T-1}{V-1} \leq \frac{(2T)^{V-1}}{V!} \leq \frac{(2T)^{V}}{V!} \), \( V! \leq 4^{-V} V^V \) (recall (2.8)), \( 3^V \leq 2^{2V} \), and \( 3^2 < 2^4 \), this implies the proposition. \( \square \)

### 10.3 Proof of Proposition 8.5

In section we establish Proposition 8.5 by summing the bound from Proposition 10.4 over \( S \), \( T \), and \( V \). The following two lemmas implement the sums over \( S \) and \( T \).

**Lemma 10.5** Fix integers \( g \geq 2 \), \( n \geq 0 \), \( V \in [2, 2g + n - 2] \), and \( T \in [V - 1, 3g + n - 3] \). Denoting \( Y = \min\{2T, 3V\} \), we have

\[
\sum_{S=0}^{3g-T-3} 2^{-n} \left( \frac{8}{3} \right)^{-4g-n} \gamma_{g,n}^{(V;S,T)} \leq g^{13/8-V} \left( \frac{3V}{2} \right)^n 2^{2V+25} \\
\times \left( \frac{9}{8} \right)^T \frac{T^{2V+Y+1}}{V^3 V} \frac{\log g + 7)^T V^V}{T!}.
\]

**Proof** Denoting \( X_S = \min\{S, V\} \) for each \( S \geq 0 \), we have by Proposition 10.4 that

\[
\sum_{S=0}^{3g+n-T-3} 2^{-n} \left( \frac{8}{3} \right)^{-4g-n} \gamma_{g,n}^{(V;S,T)} \\
\leq (T + 1) g^{1/2-V} 2^{20V+11} \left( \frac{3V}{2} \right)^n \left( \frac{9}{4} \right)^T \left( \frac{T}{V} \right)^{2V} \\
\times \frac{(\log g + 7)^T - 1 (2T - 1)!!}{V^V (2T - Y)!} \\
\times \left( \sum_{S=0}^{V-1} S \left( \frac{9}{8} \right)^S \frac{(\log g + 7)^S}{(S - X_S)!} + \sum_{S=V}^{3g+n-T-3} S \left( \frac{9}{8} \right)^S \frac{(\log g + 7)^S}{(S - X_S)!} \right) \\
\leq (T + 1) g^{1/2-V} 2^{20V+11} \left( \frac{3V}{2} \right)^n \left( \frac{9}{4} \right)^T \left( \frac{T}{V} \right)^{2V} \\
\times \frac{(\log g + 7)^T - 1 (2T - 1)!!}{V^V (2T - Y)!} \\
\times \left( V^2 \left( \frac{9}{8} \right)^V (\log g + 7)^V + \sum_{S=V}^{\infty} S \left( \frac{9}{8} \right)^S \frac{(\log g + 7)^S}{(S - V)!} \right).
\]

(10.16)
Moreover, by changing variables from $S - V$ to $S$, we have

\[
\sum_{S=V}^{\infty} S \left( \frac{9}{8} \right)^S \frac{(\log g + 7)^S}{(S - V)!}
\leq (\log g + 7)^V \left( \frac{9}{8} \right)^V \sum_{S=0}^{\infty} \frac{S + V}{S!} \left( \frac{9(\log g + 7)}{8} \right)^S
\leq (\log g + 7)^V \left( \frac{9}{8} \right)^V \left( V + \frac{9(\log g + 7)}{8} \right) \exp \left( \frac{9(\log g + 7)}{8} \right)
\leq 2^{12} (\log g + 7)^{V+1} \left( \frac{9}{4} \right)^V g^{9/8},
\]

(10.17)

where in the last bound we used the fact that $V \leq 2^{V-1}$ and that $e^{63/8} < e^{8} < 2^{12}$.

Inserting (10.17) into (10.16) and using the bounds $V^2 \leq 2^V$ and $T + 1 \leq 2T$, we obtain

\[
\sum_{S=0}^{3g+n-T-3} 2^{-n} \left( \frac{8}{3} \right)^{-4g-n} \gamma_{g,n}^{(V;S,T)}
\leq T g^{13/8-V} 2^{23V+25} \left( \frac{3V}{2} \right)^n \left( \frac{9}{4} \right)^T \left( \frac{T}{V} \right)^{2V} \frac{(\log g + 7)^T+V(2T - 1)!!}{V^V(2T - Y)!}.
\]

Now the lemma follows from the bounds $(2T)^V (2T - Y)! \geq (2T)! = 2^T T!(2T - 1)!!$ and $Y \leq 3V$. \( \square \)

**Lemma 10.6** Fix integers $g \geq 2$, $n \geq 0$, and $V \in [2, 2g + n - 2]$. Then,

\[
2^{-n} \left( \frac{8}{3} \right)^{-4g-n} \gamma_{g,n}^{(V)} \leq 2^{26} g^{11/4} \left( \frac{3V}{2} \right)^n \left( \frac{2^{61} V^{1/2}(\log g + 7)^8}{g} \right)^V.
\]
Proof By (8.3) and Lemma 10.5, we have

\[ 2^{-n} \left( \frac{8}{3} \right)^{-4g-n} \gamma_{g,n}^{(V)} \leq g^{13/8-V} \left( \frac{3V}{2} \right)^n 2^{26V+25} V^{-3V} (\log g + 7)^V \]

By (8.3) and Lemma 10.5, we have

\[ 2^{-n} \left( \frac{8}{3} \right)^{-4g-n} \gamma_{g,n}^{(V)} \leq g^{13/8-V} \left( \frac{3V}{2} \right)^n 2^{26V+25} V^{-3V} (\log g + 7)^V \]

\[ \times \left( \sum_{T=V-1}^{[3V/2]} \frac{T^{2V+2T+1}}{T!} \left( \frac{9(\log g + 7)}{8} \right)^T \right) \]

\[ + \sum_{T=\lceil 3V/2 \rceil}^{\infty} \frac{T^{5V+1}}{T!} \left( \frac{9(\log g + 7)}{8} \right)^T. \]

(10.18)

We will analyze the two sums on the right side of (10.18) individually. To bound the first, observe since \( T! \geq e^{-T} T^T \) (by (2.8)) and \( V \geq 2 \) that

\[ \sum_{T=V-1}^{[3V/2]} \frac{T^{2V+2T+1}}{T!} \left( \frac{9(\log g + 7)}{8} \right)^T \leq \left( \frac{9e}{8} \right)^{3V/2} \left( \frac{3}{2} \right)^{7V/2+1} \sum_{T=V-1}^{[3V/2]} V^{2V+T+1} (\log g + 7)^T \]

\[ \leq 2^{7V} V^{7V/2+2} (\log g + 7)^{3V/2}. \]

To analyze the second, observe using the bounds \( T! \geq e^{-T} T^T \), \( (T-5V-1)! \leq 6^{V+1} T^{-5V-1} T^T \) for \( T > 6V + 1 \) (since \( T - 5V - 1 \geq \frac{T}{6} \)), and \( V \leq 2^V \) that

\[ \sum_{T=\lceil 3V/2 \rceil}^{\infty} \frac{T^{5V+1}}{T!} \left( \frac{9(\log g + 7)}{8} \right)^T \]

\[ \leq \sum_{T=\lceil 3V/2 \rceil}^{6V+1} T^{5V-T+1} \left( \frac{9e(\log g + 7)}{8} \right)^T + 6^{5V+1} \]

\[ \times \sum_{T=6V+2}^{\infty} \frac{1}{(T-5V-1)!} \left( \frac{9(\log g + 7)}{8} \right)^T \]

\[ \leq (13V)^{7V/2+2} \left( \frac{9e(\log g + 7)}{8} \right)^{6V+1} + \left( \frac{27(\log g + 7)}{4} \right)^{5V+1} \]

\[ \times \sum_{T=V+1}^{\infty} \frac{1}{T!} \left( \frac{9(\log g + 7)}{8} \right)^T \]

\[ \leq 13^{5V} V^{7V/2+2} 2^{12V} (\log g + 7)^{6V+1} + e^{8g^{9/8}76V} (\log g + 7)^{5V+1} \]
where to deduce the second inequality we changed variables from \( T + 5V + 1 \) to \( T \) in the second sum. Inserting (10.19) and (10.20) into (10.18) yields

\[
2^{-n} \left( \frac{8}{3} \right)^{-4g-n} \gamma_{g,n}^{(V)} \leq g^{11/4-V} \left( \frac{3V}{2} \right)^n 2^{59V+26} V^{V/2+2} (\log g + 7)^8 V,
\]

from which we deduce the lemma, since \( V^2 \leq 2^2 V \).

We now deduce Proposition 8.5 by summing the bound from Lemma 10.6 over \( V \).

**Proof of Proposition 8.5** Observe for \( g > 2^{500} \) that \( 2^{n+63} V^{1/2} (\log g + 7)^8 \leq g^{3/4} \), whenever \( V \leq 3g \) and \( 20n \leq \log g \). Thus, since \( (\frac{3V}{2})^n \leq 2^V \), Lemma 10.6 implies that

\[
\sum_{V=3}^{2g-2} \left( \frac{8}{3} \right)^{-4g} \gamma_{g,n}^{(V)} \leq 2^{26} g^{11/4} \sum_{V=3}^{2g-2} \left( \frac{3V}{2} \right)^n \left( \frac{2^{61} V^{1/2} (\log g + 7)^8}{g} \right)^V
\]

\[
\leq 2^{26} g^{11/4} \sum_{V=3}^{2g-2} \left( \frac{2^{n+63} V^{1/2} (\log g + 7)^8}{g} \right)^V
\]

\[
\leq 2^{26} g^{11/4} \left( \sum_{V=3}^{11} \left( \frac{2^{n+63} (\log g + 7)^8}{g} \right)^V \right)
\]

\[
+ \left( \sum_{V=12}^{2g-2} \left( \frac{2^{n+63} V^{1/2} (\log g + 7)^8}{g} \right)^V \right)
\]

\[
\leq 2^{26} g^{11/4} \left( g \left( \frac{2^{n+63} (\log g + 7)^8}{g} \right)^3 + \sum_{V=12}^{2g-2} g^{-V/4} \right)
\]

\[
= 2^{3n+219} (\log g + 7)^{24} g^{-1/4} \leq 2^{243} (\log g)^{24} g^{-1/8},
\]

where in the last inequality we used the bounds \( \log g + 7 \leq 2 \log g \) (as \( g \geq 2^{500} \)) and \( 20n \leq \log g \). This yields the proposition. \( \square \)
10.4 Relative contributions of single-vertex graphs

In this section we establish the following proposition, which is not directly related to Theorem 1.7 but was also predicted in [11] (see, in particular, Conjecture 1.33 there). It essentially implies that the dominant contribution to the sum of $Z(P(\Gamma))$ over all stable graphs $\Gamma \in \mathcal{G}_{g,n}$ with a fixed number $E$ of edges is dominated by the single-vertex graph $\Gamma_{g,n}(E)$ from Definition 8.7.

**Proposition 10.7** As $g$ tends to $\infty$, we have for $12E \leq \log g$ and $20n \leq \log g$ that

$$
\sum_{\Gamma \in \mathcal{G}_{g,n}} \mathbb{Z}(P(\Gamma)) \sim \mathbb{Z}\left(P\left(\Gamma_{g,n}(E)\right)\right).
$$

**Proof** Observe that

$$
\sum_{\Gamma \in \mathcal{G}_{g,n}} \mathbb{Z}(P(\Gamma)) = \mathbb{Z}\left(P\left(\Gamma_{g,n}(E)\right)\right) + \sum_{V=2}^{2g+n-2} \sum_{T=V-1}^{E} \Upsilon_{g,n}^{(V;E-T,T)},
$$

and so it suffices to show for sufficiently large $g$ that

$$
\sum_{V=2}^{2g+n-2} \sum_{T=V-1}^{E} \Upsilon_{g,n}^{(V;E-T,T)} \leq g^{-1/2} \mathbb{Z}\left(P\left(\Gamma_{g,n}(E)\right)\right)). \quad (10.21)
$$

To lower bound the right side of (10.21), observe by (8.9) that, for sufficiently large $g$, 

$$
\mathbb{Z}\left(P\left(\Gamma_{g,n}(E)\right)\right) \geq g^{1/2} 2^{n-6} \left(\frac{8}{3}\right)^{4g+n} \frac{Z_E(3g + n - 3)}{2^E E!}. \quad (10.22)
$$

To upper bound the left side of (10.21), observe for $V \geq 2$ that, by (10.15) and Lemma 8.10,

$$
\Upsilon_{g,n}^{(V;E-T,T)} \leq g^{3/2-V} 2^n \left(\frac{8}{3}\right)^{4g+n} Z_E(3g + n - 3) \times 2^{13V+6} 3^V 2^{E+n} \left(\frac{3}{2}\right)^{2E+n} \left(\frac{T}{V}\right)^V \left(\frac{2T - 1}{V - 1}\right)^{2T-E(2T - 1)!!} \times \frac{1}{V! (E - T - X)!(2T - Y)!}.
$$
where we have set $X = X_T = \min\{E - T, V\}$ and $Y = Y_T = \min\{2T, 3V\}$. Using the facts that $(E - T)! \leq E^V (E - T - X)!$; the identity $2^T T! (2T - 1)! = (2T)!$; and the bounds $3^V < 2^V$, $3^2 < 2^4$, $T \leq E$, and

$$\left(\frac{2T - 1}{V - 1}\right) \leq \frac{(2T)^V}{V!}; \quad \frac{1}{(E - T)! T!} = \left(\frac{E}{T}\right) \frac{1}{E!} \leq \frac{2E}{E!};$$

it follows that

$$\Upsilon_{g,n}(V; E - T, T) \leq g^{3/2 - V} 2^n \left(\frac{8}{3}\right)^{4g+n} Z_E (3g + n - 3) 2^{19V+10} \left(\frac{3V}{2}\right)^n \left(\frac{9}{4}\right)^E \times \left(\frac{E^6}{V}\right)^V \frac{1}{V!^2 E!}.$$

This, together with (10.22) (and the fact that $V! \geq 1$), implies for $V \geq 2$ and sufficiently large $g$ that

$$\left(\mathcal{Z}(\Gamma_{g,n}(E))\right)^{-1} \Upsilon_{g,n}(V; E - T, T) \leq 2^{19V+16} g^{1-V} \left(\frac{3V}{2}\right)^n \left(\frac{9}{2}\right)^E \left(\frac{E^6}{V}\right)^V.$$

Summing over $T \in [V - 1, E]$ and $V \in [2, 2g + n - 2]$, we deduce using the fact that $\left(\frac{3V}{2}\right)^n < 2^V$ and $E \leq 2^E$ that, for sufficiently large $g$,

$$\left(\mathcal{Z}(\Gamma_{g,n}(E))\right)^{-1} \sum_{V=2}^{2g+n-2} \sum_{T=V-1}^{E} \Upsilon_{g,n}(V; E - T, T) \leq 2^{16} E \left(\frac{9}{2}\right)^E g^{2g+n-2} \sum_{V=2}^{E} \left(\frac{2^{19} E^6}{g V}\right)^V \left(\frac{3V}{2}\right)^n \left(\frac{E^6}{V}\right)^V \left(\frac{9}{2}\right)^E \left(\frac{E^6}{V}\right)^V.$$

Since $20n \leq \log g$ and $12E \leq \log g$, we have for sufficiently large $g$ that $2^{16} E < g^{1/4}$ and $E^{2n+19} \leq g^{1/8}$. Therefore, (10.23) implies that

$$\left(\mathcal{Z}(\Gamma_{g,n}(E))\right)^{-1} \sum_{V=2}^{\infty} \sum_{T=V-1}^{E} \Upsilon_{g,n}(V; E - T, T) \leq g^{5/4} \sum_{V=2}^{\infty} \left(\frac{1}{g^{7/8} V}\right)^V \leq g^{-1/2} \sum_{V=2}^{\infty} V^{-V} < g^{-1/2}.$$
for sufficiently large $g$. This verifies (10.21) and therefore the proposition. □

11 Asymptotics for Siegel–Veech constants

In this section we establish Theorem 1.8 that provides the large genus asymptotics for $c_{\text{area}}(Q_{g,n})$. To that end, we begin in Sect. 11.1 by first recalling a result from [23] that expresses this Siegel–Veech constant in terms of (other) principal strata volumes, and then establishing Theorem 1.8 assuming certain estimates; the latter estimates are proven in Sect. 11.2.

11.1 Proof of Theorem 1.8

We first state an identity from [23] for the area Siegel–Veech constant $c_{\text{area}}(Q_{g,n})$ in terms of the principal strata volumes. In what follows, for any integers $g \geq 2$ and $n \geq 0$, we define the set of pairs of compositions

$$\mathcal{X}(g,n) = \left\{ (g,n) \in K_2(g) \times C_2(n+2) : 3g_i + n_i \geq 4 \right\}.$$

where we have denoted $g = (g_1, g_2) \in K_2(g)$ and $n = (n_1, n_2) \in C_2(n+2)$.

**Proposition 11.1** ([23, Corollary 1]) For any integers $g \geq 2$ and $n \geq 0$, we have that

$$c_{\text{area}}(Q_{g,n}) = \kappa_1 + \kappa_2 + \kappa_3,$$

where

$$\kappa_1 = \frac{1}{8 \text{Vol } Q_{g,n}} \sum_{(g,n) \in \mathcal{X}(g,n)} \frac{(4g + n - 4)!}{(4g_1 + n_1 - 4)!(4g_2 + n_2 - 4)!} \times \frac{n!}{(n_1 - 1)!(n_2 - 1)!} \times \frac{(6g_1 + 2n_1 - 7)!(6g_2 + 2n_2 - 7)!}{(6g + 2n - 7)!} \text{Vol } Q_{g_1,n_1} \text{Vol } Q_{g_2,n_2}; \quad (11.1)$$

$$\kappa_2 = \frac{n(n-1)(4g + n - 4)}{(6g + 2n - 7)(6g + 2n - 8)} \frac{\text{Vol } Q_{g,n-1}}{4 \text{Vol } Q_{g,n}};$$

$$\kappa_3 = \frac{(4g + n - 4)(4g + n - 5)}{(6g + 2n - 7)(6g + 2n - 8)} \frac{\text{Vol } Q_{g-1,n+2}}{\text{Vol } Q_{g,n}}.$$

Thus, it remains to understand the limiting behaviors of $\kappa_1$, $\kappa_2$, and $\kappa_3$ from (11.1). This is done through the following three lemmas; the first two will be established in Sect. 11.2 below.
Lemma 11.2 For any fixed integer $n \geq 0$, there exists a constant $C = C(n) > 1$ such that $\kappa_1 < C g^{-1}$ holds for each integer $g \geq 2$.

Lemma 11.3 For any fixed integer $n \geq 0$, there exists a constant $C = C(n) > 1$ such that $\kappa_2 < C g^{-1}$ holds for each integer $g \geq 2$.

Lemma 11.4 For fixed integer $n \geq 0$, we have as $g$ tends to $\infty$ that $\kappa_3 \sim \frac{1}{4}$.

Proof By (1.7), we have as $g$ tends to $\infty$ that

$$\frac{\text{Vol } Q_{g^{-1},n+2}}{\text{Vol } Q_{g,n}} \sim 4 \left( \frac{8}{3} \right)^{-2}.$$ Combining this with the fact that as $g$ tends to $\infty$ we have

$$\frac{(4g + n - 4)(4g + n - 5)}{(6g + 2n - 7)(6g + 2n - 8)} \sim \left( \frac{2}{3} \right)^2,$$

we deduce the lemma. \qed

Assuming Lemmas 11.2 and 11.3, we can quickly establish Theorem 1.8.

Proof of Theorem 1.8 Assuming Lemma 11.2 and Lemma 11.3 This follows from Proposition 11.1 and summing the results of Lemmas 11.2, 11.3 and 11.4. \qed

11.2 Proofs of Lemma 11.2 and Lemma 11.3

In this section we establish Lemmas 11.2 and 11.3. To that end, we let $n \geq 0$ denote an integer. Throughout this section, we further let $R = R(n) > 1$ denote an $n$-dependent constant (whose existence is guaranteed by Theorem 1.7), such that

$$R^{-1} \left( \frac{8}{3} \right)^{4G} \leq \text{Vol } Q_{G,m} \leq R \left( \frac{8}{3} \right)^{4G}$$ holds for any integers $G \geq 0$ and $m \in [0, n + 2]$. \hspace{0.5cm} (11.2)

Given this notation, we can quickly establish Lemma 11.3.

Proof of Lemma 11.3 First observe that since $4g + n - 4 \leq 4g(n + 1)$ for $g \geq 2$; $n(n - 1)(n + 1) \leq n^3$ for $n \geq 0$; and $(6g + 2n - 7)(6g + 2n - 8) \geq g^2$ for $g \geq 2$, we have that

$$\frac{n(n - 1)(4g + n - 4)}{(6g + 2n - 7)(6g + 2n - 8)} \leq \frac{4n^3}{g}.$$ \hspace{0.5cm} (11.3)
Now, applying \((11.2)\) with the \((G, m)\) there first equal to \((g, n)\) here; then equal to \((g - 1, n + 2)\) here; and next dividing yields

\[
\frac{\text{Vol } Q_{g,n-1}}{4 \text{Vol } Q_{g,n}} \leq \frac{R^2}{4}.
\]  

(11.4)

We now deduce the lemma from combining \((11.3)\) and \((11.4)\).

To establish Lemma 11.2, we begin with the following lemma that bounds the summands appearing in the definition \((11.1)\) of \(\chi_1\).

**Lemma 11.5** For any fixed integer \(n \geq 0\), there exists a constant \(C = C(n) > 1\) such that the following holds. Let \(n_1, n_2 \geq 1\) and \(g_1, g_2 \geq 0\) be integers such that \(n = n_1 + n_2 - 2\). Set \(g = g_1 + g_2\); assume that \(g \geq 2\) and that \(3g_i + n_i \geq 4\) for each \(i \in \{1, 2\}\). Then,

\[
\frac{1}{(4g + n - 4)!} \frac{n!}{(4g_1 + n_1 - 4)! (4g_2 + n_2 - 4)! (n_1 - 1)! (n_2 - 1)!} \times \frac{(6g_1 + 2n_1 - 7)! (6g_2 + 2n_2 - 7)!}{(6g + 2n - 7)!} \times \text{Vol } Q_{g_1,n_1} \text{ Vol } Q_{g_2,n_2} \leq C \left( \frac{4g + n - 4}{4g_1 + n_1 - 3} \right) \left( \frac{6g + 2n - 8}{6g_1 + 2n_1 - 6} \right)^{-1}.
\]

(11.5)

**Proof** Applying \((11.2)\) with \((G, m) \in \{(g, n), (g_1, n_1), (g_2, n_2)\}\) and using the fact that \(g_1 + g_2 = g\), we deduce that

\[
\frac{\text{Vol } Q_{g_1,n_1} \text{ Vol } Q_{g_2,n_2}}{\text{Vol } Q_{g,n}} \leq R^3.
\]

(11.6)

Next, in view of the facts that \(n_1 + n_2 = n + 2\) and \(g = g_1 + g_2\), we have

\[
\frac{(4g + n - 4)!}{(4g_1 + n_1 - 4)! (4g_2 + n_2 - 4)! (n_1 - 1)! (n_2 - 1)!} \times \frac{n!}{(6g_1 + 2n_1 - 7)! (6g_2 + 2n_2 - 7)!} \times \frac{1}{(6g + 2n - 7)!} \times \frac{(4g_1 + n_1 - 3)(4g_2 + n_2 - 3)}{(6g_1 + 2n_1 - 6)(6g_2 + 2n_2 - 6)} \times \left( \frac{n}{n_1 - 1} \right) \left( \frac{4g + n - 4}{4g_1 + n_1 - 3} \right) \left( \frac{6g + 2n - 8}{6g_1 + 2n_1 - 6} \right)^{-1}.
\]
Since \( n_i \geq 1, g_i \geq 0, \) and \( 3g_i + n_i \geq 4, \) we have \( 6g_i + 2n_i - 6 \geq 4g_i + n_i - 3 \) for each \( i \in \{1, 2\}, \) and so it follows that

\[
\frac{(4g + n - 4)!}{(4g_1 + n_1 - 4)!(4g_2 + n_2 - 4)! (n_1 - 1)!(n_2 - 1)!} \times \frac{n!}{(6g_1 + 2n_1 - 7)!(6g_2 + 2n_2 - 7)!} \leq \frac{1}{6g + 2n - 7} \left( \frac{n}{n_1 - 1} \right) \left( \frac{4g + n - 4}{4g_1 + n_1 - 3} \right) \left( \frac{6g + 2n - 8}{6g_1 + 2n_1 - 6} \right)^{-1}. 
\]

Moreover, since \( 6g + 2n - 7 \geq g \) (as \( g = g_1 + g_2 \geq 4 \)) and \( \binom{n}{n_1 - 1} \leq 2^n, \) we deduce that

\[
\frac{(4g + n - 4)!}{(4g_1 + n_1 - 4)!(4g_2 + n_2 - 4)! (n_1 - 1)!(n_2 - 1)!} \times \frac{n!}{(6g_1 + 2n_1 - 7)!(6g_2 + 2n_2 - 7)!} \leq \frac{2^n}{g} \left( \frac{4g + n - 4}{4g_1 + n_1 - 3} \right) \left( \frac{6g + 2n - 8}{6g_1 + 2n_1 - 6} \right)^{-1}. 
\]

This, together with (11.6), implies the lemma. \( \square \)

We next have the following lemma that bounds the right side of (11.5) if \( g_1, g_2 \geq 2. \)

**Lemma 11.6** Fix integers \( n_1, n_2 \geq 1 \) and \( g_1, g_2 \geq 2; \) set \( n = n_1 + n_2 - 2 \) and \( g = g_1 + g_2. \) Then,

\[
\left( \frac{4g + n - 4}{4g_1 + n_1 - 3} \right) \left( \frac{6g + 2n - 8}{6g_1 + 2n_1 - 6} \right)^{-1} \leq \frac{1}{g}. 
\]

**Proof** Observe that \( 2g_i + n_i \geq 4, \) since \( g_i \geq 2, \) for each \( i \in \{1, 2\}. \) Then, Lemma 2.3 gives

\[
\left( \frac{4g + n - 4}{4g_1 + n_1 - 3} \right) \left( \frac{6g + 2n - 8}{6g_1 + 2n_1 - 6} \right)^{-1} \leq \left( \frac{2g + n - 4}{2g_1 + n_1 - 3} \right)^{-1} \leq \frac{1}{2g + n - 4},
\]

where in the second statement we used the fact that \( \binom{a}{b} \geq a \) if \( b \in [1, a - 1] \) (applied with the \( (a, b) \) there equal to \( (2g + n - 4, 2g_1 + n_1 - 3) \) here, where the condition \( b \in [1, a - 1] \) is verified since \( \min\{2g_1 + n_1, 2g_2 + n_2\} \geq 4, n_1 + n_2 = n + 2, \) and \( g_1 + g_2 = g \)). This, together with the fact that \( g + n \geq g_1 + g_2 \geq 4, \) implies the lemma. \( \square \)
We can now establish Lemma 11.2.

**Proof of Lemma 11.2** Observe by Lemma 11.5 that there exists a constant \( C_1 = C_1(n) > 1 \) such that

\[
\chi_1 \leq \frac{C_1}{g} \sum_{(g,n) \in \mathcal{X}(g,n)} \left( \frac{4g + n - 4}{4g_1 + n_1 - 3} \right) \left( \frac{6g + 2n - 8}{6g_1 + 2n_1 - 6} \right)^{-1},
\]

where we have denoted \( g = (g_1, g_2) \) and \( n = (n_1, n_2) \). Since \( |C_2(n + 2)| \leq n + 1 \), it follows that there exists a constant \( C_2 = C_2(n) > 1 \) such that

\[
\chi_1 \leq \frac{C_2}{g} \sum_{g \in \mathcal{K}_2(g), (g,n) \in \mathcal{X}(g,n)} \frac{\max_{n \in \mathcal{C}_2(n+2)} \left( \frac{4g + n - 4}{4g_1 + n_1 - 3} \right) \left( \frac{6g + 2n - 8}{6g_1 + 2n_1 - 6} \right)^{-1}}{2C_2 g}. \tag{11.8}
\]

In view of Lemma 11.6 and the fact that \( \mathcal{K}_2(g) \leq g + 1 \leq 2g \) (where the last bound holds since \( g \geq 2 \)), we have that

\[
\sum_{g_1 + g_2 = g, g_1, g_2 \geq 2} \max_{n \in \mathcal{C}_2(n+2)} \left( \frac{4g + n - 4}{4g_1 + n_1 - 3} \right) \left( \frac{6g + 2n - 8}{6g_1 + 2n_1 - 6} \right)^{-1} \leq 2.
\]

This, together with (11.8), gives

\[
\chi_1 \leq \frac{C_2}{g} \sum_{g \in \mathcal{K}_2(g), (g,n) \in \mathcal{X}(g,n)} \frac{\max_{n \in \mathcal{C}_2(n+2)} \left( \frac{4g + n - 4}{4g_1 + n_1 - 3} \right) \left( \frac{6g + 2n - 8}{6g_1 + 2n_1 - 6} \right)^{-1}}{2C_2 g} + \frac{2C_2}{g}, \tag{11.9}
\]

where in the last inequality we used the fact that there are at most 4 nonnegative compositions \( g = (g_1, g_2) \in \mathcal{K}_2(g) \) such that \( \min\{g_1, g_2\} \leq 1 \).

Now observe that \( 4g + n - 4 \leq 6g + 2n - 8 \) (as \( g \geq 2 \)) and \( 6g_1 + 2n_1 - 6 \geq 4g_i + n_i - 3 \) if \( g_i \leq 1 \) and \( 3g_i + n_i \geq 4 \) (as \( 2g_i + n_i - 3 \geq 3g_i + n_i - 4 \geq 0 \)). Therefore, since \( \left( \frac{a}{b} \right) \leq \left( \frac{a'}{b'} \right) \) whenever \( a \leq a', b \leq b', 2b \leq a, \) and \( 2b' \leq a' \), it follows that

\[
\max_{g \in \mathcal{K}_2(g), (g,n) \in \mathcal{X}(g,n)} \frac{\max_{n \in \mathcal{C}_2(n+2)} \left( \frac{4g + n - 4}{4g_1 + n_1 - 3} \right) \left( \frac{6g + 2n - 8}{6g_1 + 2n_1 - 6} \right)^{-1}}{2C_2 g} \leq 1,
\]

and so the lemma follows from (11.9). \( \square \)
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