AFFINE ALGEBRAIC GROUPS WITH PERIODIC COMPONENTS

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Abstract. A connected component of an affine algebraic group is called periodic if all its elements have finite order. We give a characterization of periodic components in terms of automorphisms with finite number of fixed points. It is also discussed which connected groups have finite extensions with periodic components. The results are applied to the study of the normalizer of a maximal torus in a simple algebraic group.

1. Introduction

It is well known that every connected affine complex algebraic group of positive dimension contains elements of infinite order. On the other hand, for a non-connected group some of connected components may consist of elements of finite order. Such components we call periodic. For example, one of the connected components of the orthogonal group $O_2(\mathbb{C})$ consists of reflections and therefore is periodic. This work is devoted to the study of affine algebraic groups with periodic components.

Let $G$ be an affine algebraic group over an algebraic closed field $k$ of characteristic zero and $G^0 = H$ be its connected component of unity. In Section 2 we formulate equivalent conditions for a component $gH$ of the group $G$ to be periodic in terms of automorphisms induced by the action of the elements of $gH$ on $H$ by conjugation. Namely, a component $gH$ is periodic if and only if the automorphism $\varphi_g : H \to H$, $h \mapsto g^{-1}hg$ has only finitely many fixed points. Also we prove that for a periodic component $gH$ the action of $H$ on $gH$ by conjugation is transitive and, consequently, all elements of $gH$ have same orders. In Section 3 we investigate which connected algebraic groups have finite extensions with periodic components and show that all such groups are solvable. It turns out that solvability is only a necessary condition: we give an example of a series of connected solvable groups having no extensions with periodic components. Section 4
is devoted to the study of torus extensions and, in particular, to estimates of order of elements in periodic components of such extensions. Namely, if there is an extension $G = T \cup gT \cup \ldots \cup g^{m-1}T$ of a torus $T$ with a periodic component $gT$, and $\text{ord}(\varphi_g) = k$, then $\text{ord}(g)$ divides $mk$. Using the obtained results in Section 5 we investigate periodic components of the normalizer of a maximal torus in simple groups and find the order of elements in these components. In particular, we give a formula for the number of periodic components in the normalizer and show that the components defined by the Coxeter elements of the Weyl group are periodic.

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2. Periodic components and automorphisms

The following theorem suggests several characterizations of periodic components.

**Theorem 1.** Let $G$ be an affine algebraic group and $H = G^0$ be its connected component of unity. For each $g \in G$ define the automorphism $\varphi_g : H \to H; h \mapsto g^{-1}hg$. Then the following conditions are equivalent:

1. the connected component $gH$ is periodic;
2. the subgroup of fixed points of the automorphism $\varphi_g$ is finite;
3. the action of $H$ on $gH$ by conjugation is transitive.

**Proof.** First of all note that $H^{\varphi_g}$ as a subgroup of fixed points of a regular automorphism is an algebraic subgroup.

$(1) \Rightarrow (2)$ Let the component $gH$ consist of elements of finite order with the group $H^{\varphi_g}$ being infinite. Then the latter has positive dimension and therefore contains an algebraic subgroup of dimension 1. But such a subgroup may only be either a one-dimensional torus or an additive group of the ground field [6, Th. 3.2.8] and therefore contains an element of infinite order. Denote it by $h_0$.

But $h_0g = gh_0$ and so $\forall n \in \mathbb{N} : (gh_0)^n = g^n h_0^n \neq e$, since $g^n$, as an element of finite order, can not be inverse to $h_0^n$. This yields a contradiction.
(2) ⇒ (3) Let $|H^{\varphi g}| < \infty$. The quotient group $G/H$ is finite and, consequently, $\exists m \in \mathbb{N}$ such, that $g^m \in H$. But then $g^m \in H^{\varphi g}$, implying that $g$ has finite order.

Consider the morphism $f_g : H \to gH, h \mapsto hgh^{-1}$ (it is well-defined, because $\forall h \in H : hgh^{-1} = g\varphi_g(h)h^{-1} \in gH$).

**Lemma 1.** If the subgroup $H^{\varphi g}$ is finite, then the fibers of the morphism $f_g$ are finite.

**Proof.** Fix an arbitrary $h_0 \in H$. Then $\forall h \in H : hgh^{-1} = h_0gh_0^{-1} \iff h_0^{-1}hgh^{-1}h_0 = g \iff h_0^{-1}h = gh_0^{-1}h^{-1} \iff h^{-1}h \in H^{\varphi g}$, implying $h \in h_0H^{\varphi g}$. But the subgroup $H^{\varphi g}$ is finite.

Denote by $C(g)$ the image of the morphism $f_g$. By the theorem on the dimension of fibers of a morphism $\dim C(g) = \dim H$ [9, Ch. I § 6.3]. But, as a connected component of an algebraic group, $gH$ is irreducible, and therefore $C(g)$ is dense in $gH$. This yields that the component $gH$ contains dense subset, consisting of the elements conjugate to $g$ and, consequently, of the same order.

Let $N$ be the order of the element $g$. Note that the subset $M = \{x \in gH | (gh)^N = e\}$ is closed in $gH$ in Zariski topology. On the other hand, $M$ contains the subset $C(g)$ that is dense in $gH$, and therefore $M$ is equal to $gH$.

So $\forall s \in gH$ the order of $s$ is finite and, consequently, the image $C(s)$ of the morphism $f_s$ is dense in $gH$ and is opened in $gH$ as an orbit of $H : gH$. But for different $s \in gH$ the sets $C(s)$ either are disjoint or are equal, and as opened subsets, they must have common points. This means that $gH = C(s) \forall s \in gH$.

(3) ⇒ (1) Consider the Jordan decomposition $g = tu$ with $t$ being semi-simple, $u$ being unipotent and $tu = ut$. Then $t, u \in G$ [9, Th. 3.4.6] and $G(u) \subset G$ where $G(u)$ is the minimal algebraic subgroups containing $u$. But $G(u)$ is unipotent and therefore connected implying $G(u) \subset H$, so $u \in H$. Consequently, $t = gu^{-1} \in gH$ and, as all elements in $gH$ are conjugate, $g$ is also semi-simple.
Assume that \( g \) has infinite order. Then the component \( gH \) contains infinitely many elements that are its powers. Consider the faithful linear representation \( \rho : G \longrightarrow GL(V) \) [6, Th. 3.1.8]. As \( g \) is semi-simple, it and all its powers are represented by diagonal matrices in an appropriate basis of \( V \). But those of them lying in \( \rho(gH) \) are conjugate (by matrices of \( \rho(H) \)) and consequently may differ only by a permutation of diagonal elements. So among the degrees of the matrix \( \rho(g) \) that lie in \( \rho(gH) \) there are only finitely many different ones. But the representation \( \rho \) is faithful, thus yielding the contradiction.

\[
\square
\]

**Corollary 1.** If the component \( gH \) is periodic, then all its elements have same orders.

**Corollary 2.** If the component \( gH \) is periodic, then \( H \) lies in the commutant \([G,G]\) of the group \( G \).

*Proof.* The condition \( gH = Hg = \{hgh^{-1} \mid h \in H\} \) implies \( H = \{hgh^{-1}g^{-1} \mid h \in H\} \subseteq [G,G] \).

\[
\square
\]

So if the group \( G \) contains a periodic component, then computation of its commutant can be reduced to computation of the commutant of the finite group of its components.

**Corollary 3.** If the group \( G \) has a periodic component, then for each its linear representation \( \rho : G \longrightarrow GL(V) \) the image of \( \rho(G^0) \) lies in \( SL(V) \).

Now let \( H \) be a connected group and \( \varphi \) be its automorphism of order \( m < \infty \) with finitely many fixed points. Consider the semi-direct product \( H \rtimes \langle a \rangle_m \) defined by this automorphism, where \( \langle a \rangle_m \) is a cyclic group of order \( m \) with generator \( a \). By Theorem 1 the coset \( \{(h, a) \mid h \in H\} \) is a periodic component of the resulting group. Thus we get the following criterion: a connected group has a finite extension with a periodic component if and only if it has an automorphism of finite order with finitely many fixed points.

Let \( G \) be a finite extension of a connected group \( H \). Note that \( \forall g \in G \) the automorphism \( \varphi_g \) has finite order in a quotient group \( Aut(H)/Int(H) \). In fact, in the previously described criterion we can replace the condition for
an automorphism to have finite order by the condition to have finite order modulo the group \( \text{Int}(H) \).

**Proposition 1.** Let \( H \) be a connected algebraic group, \( \varphi \) be its automorphism, of order \( k < \infty \) modulo inner automorphisms of \( H \) and such, that \( |H^s| < \infty \). Then the order of automorphism \( \varphi \) is finite.

**Proof.** We have \( \varphi^k \in \text{Int} H \) and, consequently, \( \exists x \in H \) such, that \( \varphi^k(h) = xhx^{-1} \forall h \in H \). For an arbitrary \( h \in H \) compute \( \varphi^{k+1}(h) \) in two ways: on one hand

\[
\varphi^{k+1}(h) = \varphi^k(\varphi(h)) = x\varphi(h)x^{-1},
\]

and on the other hand

\[
\varphi^{k+1}(h) = \varphi(\varphi^k(h)) = (xhx^{-1}) = \varphi(x)\varphi(h)(\varphi(x))^{-1}.
\]

But \( \varphi \) is an automorphism, therefore \( \varphi(H) = H \) and \( \forall h \in H \) we have \( x\varphi(h)x^{-1} = \varphi(x)\varphi(h)\varphi(x)^{-1} \), which means that \( x^{-1}\varphi(x) \) lies in the center \( Z(H) \) of the group \( H \). So we get \( x^{-1}\varphi(x) = xx^{-1}\varphi(x)x^{-1} = \varphi(x)x^{-1} \). Consequently, \( \forall s \in \mathbb{N} : \)

\[
x^{-s}\varphi(x^s) = (x^{-s}\varphi(x))^s = (x^{-1}\varphi(x))^s \in Z(G).
\]

Consider the morphism \( f : H \to H, h \mapsto h^{-1}\varphi(h) \).

**Lemma 2.** Fibers of the morphism \( f \) are finite.

**Proof.** Fix an element \( h_0 \in H \). Then \( \forall g \in H \) one has:

\[
h_0^{-1}\varphi(h_0) = g^{-1}\varphi(g) \iff gh_0^{-1} = \varphi(g)\varphi(h_0^{-1}) = \varphi(gh_0^{-1}) \iff gh_0^{-1} \in H^s \iff g \in h_0H^s.
\]

But the group \( H^s \) is finite.

Consider morphisms \( f_s : x^sZ(H) \to Z(H), h \mapsto h^{-1}\varphi(h) \), \( s = 1, 2, \ldots \). As \( f_s = f|_{x^sZ(H)} \) fibers of \( f_s \) are finite and contain less than \( |H^s| \) points. On the other hand by the theorem on the dimension of fibers of a morphism \( \dim \text{Im} f_s = \dim x^sZ(H) = \dim Z(H) \). Hence for all \( s \) the set \( \text{Im} f_s \) is a union of several connected components of the group \( Z(H) \). Therefore, by the theorem on the image of a dominant morphism \( \text{Im} f_s \) contains an open in Zariski topology subset of this union. If the number of different cosets \( x^sZ(H) \) is sufficiently big, then we get the contradiction with a limitation on the number of points in a fiber of the morphism \( f \). Namely, let \( d \) be a number of connected components of \( Z(H) \), \( m \) be less than \( d \cdot |H^s| \) and all \( x^sZ(H), s = 1, 2, \ldots, m \) be different. Then among the sets \( \text{Im} f_s (s = 1, 2, \ldots m) \) some \( |H^s| + 1 \)
have non-empty intersection. Let \( g_0 \) be an element lying in \( |H^\varphi| + 1 \) of the sets. Then \( f^{-1}(g_0) \supseteq \bigcup_{s=1}^{m} f^{-1}(g_0) \) and consequently \( |f^{-1}(g_0)| \geq m/d > |H^\varphi| \), which is a contradiction.

Hence among the cosets \( x^sZ(H), s = 1, 2, \ldots \) only finitely many of them are different and therefore \( \exists q : x^q \in Z(H) \) (it was shown that \( q \leq d \cdot |H^\varphi| \) where \( d \) is a number of connected components of \( Z(H) \)). But then \( \varphi^{kq} = \text{id} \).

\[ \square \]

### 3. Extensions with periodic components

Now we turn to the question, whether a given connected algebraic group has extensions with periodic components.

**Proposition 2.** If a reductive algebraic group has a periodic component, then its connected component of unity is a torus.

**Proof.** Let \( G \) be a reductive group and \( gG^0 \) be its periodic component. Since every connected reductive group \( G^0 \) is an almost direct product of a central torus \( T \) and a semi-simple subgroup \( S \) [6, Ch. 6]. But \( S = [G^0, G^0] \) and consequently it is stable under the action of the automorphism \( \varphi_g \). Hence it is sufficient to prove the following lemma.

**Lemma 3.** The subgroup of fixed points of an automorphism of a semi-simple group is infinite.

**Proof.** Let \( H \) be a connected semi-simple group and \( \mathfrak{h} = \text{Lie} \ H \) be its Lie algebra. Consider a map \( D : \text{Aut} \ H \rightarrow \text{Aut} \mathfrak{h}, f \mapsto d_e f \). As \( D(\text{Int} \ H) = \text{Int} \mathfrak{h} \), it is evident that \( D \) defines correctly a mapping from \( \text{Out} \ H = \text{Aut} \ H/\text{Int} \ H \) to \( \text{Out} \mathfrak{h} = \text{Aut} \mathfrak{h}/\text{Int} \mathfrak{h} \). For semi-simple Lie algebras the following fact is well known.

**Proposition 3.** The group \( \text{Int} \mathfrak{h} \) is a connected component of unity in \( \text{Aut} \mathfrak{h} \); diverse connected components of the group \( \text{Aut} \mathfrak{h} \) are the sets \((\text{Int} \mathfrak{h})\hat{\tau}\) for diverse \( \tau \in \text{Aut} \Pi \) (\( \Pi \) is a positive roots system of algebra \( \mathfrak{h} \)), where \( \hat{\tau} \) are defined in the following way:

\[ \hat{\tau}(h_\alpha) = h_{\tau^{-1}(\alpha)}, \hat{\tau}(e_\alpha) = e_{\tau^{-1}(\alpha)}, \hat{\tau}(e_{-\alpha}) = e_{-\tau^{-1}(\alpha)} \quad (\alpha \in \Pi). \]

(Here \( h_\alpha, e_\alpha, e_{-\alpha} \) are the canonic generators of the algebra \( \mathfrak{h} \); for more information see [6, § 4, Ch. 4].)
Hence for an appropriate choice of a maximal torus and a positive system $\Pi$ the automorphism $d_e \phi$ acts as $\hat{\tau}$ for a certain $\tau \in \text{Aut} \, \Pi$. But then consider $\lambda = \sum_{\alpha \in \Pi} h_\alpha$. By definition of $\hat{\tau}$ it is evident that $d_e \phi$ acts identically on the line spanned by $\lambda$ and therefore $\phi$ acts identically on the corresponding one-dimensional subgroup. This implies $|H^\phi| = \infty$. □

This proves Proposition 2. □

**Theorem 2.** If an affine algebraic group $G$ has a periodic component, then $G^0$ is solvable.

**Proof.** Consider the Levi decomposition: $G^0 = L \ltimes U$, where $U$ is the unipotent radical, and $L$ is reductive (i.e. a Levi subgroup) [6, § 4, Ch. 6]. Let $gG^0$ be a periodic component. Then $gLg^{-1}$ is also a Levi subgroup and by Mal’tsev’s Theorem $\exists h \in U : gLg^{-1} = hLh^{-1}$; hence the automorphism $\phi_{h^{-1}g}$ stabilizes the subgroup $L$. But as $L$ is reductive, Proposition 2 implies that the restriction of the automorphism $\phi_{h^{-1}g}$ on $L$ may have finitely many fixed points if and only if $L$ is a torus. This means that $G^0 = L \ltimes U$ is solvable. □

The automorphism of a Lie algebra is called *regular* provided it fixes no point except the zero. Note that an automorphism of an algebraic group having finitely many fixed points induces a regular automorphism of its Lie algebra. After having proved Theorem 2 the author found the work [1] where it is proved that a finite dimensional Lie algebra over the field of characteristic zero possessing a regular automorphism of finite order is solvable. It gives an alternative proof for Theorem 2. In [5] one can find some other properties of Lie algebras with regular automorphism.

**Example 1.** Consider $G = U_n \cup gU_n \cup \ldots \cup g^n U_n$ where $U_n \subset GL_n(k)$ as the group of all uni-triangular matrices and $g$ is a diagonal matrix with eigenvalues $1, \xi, \xi^2, \ldots, \xi^{n-1}$, where $\xi$ is a primitive root of unity of degree $n$. All elements of the connected components $gU_n$ are upper-triangular matrices with different roots of unity on the diagonal, hence they are semi-simple and have finite order.
Now the problem arises that is if every unipotent group has extensions with periodic components or, just the same, if every nilpotent Lie algebra possesses a periodic regular automorphism. By improved Ado’s Theorem [4, Ch. 1, §5.3] every finite dimensional nilpotent Lie algebra is isomorphic to a subalgebra of the algebra of all nilpotent triangular matrices of a certain dimension \( n \in \mathbb{N} \). By the Campbell-Hausdorff Formula [8, Part 1, Ch. IV, §7.8] the image of this subalgebra under the exponential mapping is a subgroup in \( U_n \) and, furthermore, the exponential mapping establishes an isomorphism of algebraic manifolds. So if there exists a nilpotent Lie algebra \( g \) with only unipotent automorphisms, then the corresponding unipotent group will have no extensions with periodic components. An example of such an algebra can be found in [3].

The following proposition shows that if both torus \( T \) and unipotent group \( U \) have extensions with periodic components, then their semidirect product may have no such extensions.

**Proposition 4.** Among the algebraic groups of type \( \mathbb{k}^* \rtimes \mathbb{k} \) only \( \mathbb{k}^* \times \mathbb{k} \) has a finite extension with periodic components.

**Proof.** In case if the product is not direct it is sufficient to note that a non-commutative two-dimensional Lie algebra has no regular automorphisms of finite order.

For the group \( \mathbb{k}^* \times \mathbb{k} \) there is an extension with periodic components. It can be constructed as follows:

\[
G = \left\{ \begin{pmatrix} t & 0 & 0 \\ 0 & t^{-1} & 0 \\ 0 & 1 & s \end{pmatrix} \right\} \cup \left\{ \begin{pmatrix} 0 & t^{-1} & 0 \\ t & 0 & 0 \\ 0 & 1 & s \end{pmatrix} \right\}, \quad t \in \mathbb{k}^*, s \in \mathbb{k}.
\]

One may suggest that only the groups of type \( T \times U \), where \( T \) is a torus and \( U \) is a unipotent group, have automorphisms with finitely many fixed points. But it is not true.

**Example 2.** Consider the group \( H = T \rtimes U \), where \( T \cong (\mathbb{k}^*)^{n-1}, U \cong \mathbb{k}^n \), and multiplication is defined as follows: \( (t, u) \cdot (s, v) = (ts, \psi(s)(u) + \ldots \)
where
\[ \psi(s)(u) = (s_1^{-1}u_1, \ldots, s_{n-1}^{-1}u_{n-1}, s_1 \ldots s_{n-1}u_n). \]

Fix a natural number \( k \), that divides \( n \), and define the mapping \( \varphi : H \to H \) as follows: \( \varphi((t, u)) = (\beta(t), \alpha(u)) \), where \( \beta((t_1, t_2, \ldots, t_{n-1})) = (t_2, \ldots, t_{n-1}, t_1^{-1}t_2^{-1} \ldots t_{n-1}^{-1}) \),
\[
\alpha(u) = \begin{pmatrix}
0 & 1 & 0 & 1 & \cdots & 0 \\
0 & 1 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 1 & 0 & 1 & \cdots & 0 \\
\xi & 0 & 1 & 0 & \cdots & 0
\end{pmatrix} u,
\]
where \( \xi \) is a primitive root of unity of degree \( k \). We should now prove that \( \varphi \) is an automorphism:
\[
\varphi((t, u)(s, v)) = \varphi((ts, \psi(s)(u) + v)) = (\beta(ts), \alpha(\psi(s)(u) + \alpha(v));
\]
\[
\varphi((t, u))\varphi((s, v)) = (\beta(t, \alpha(u))(\beta(s), \alpha(v)) = (\beta(t)\beta(s), \psi(\beta(s))(\alpha(u)) + \alpha(v)).
\]
But it is easy to see, that \( \forall t \in T, u \in U : \)
\[
\alpha(\psi(t)(u)) =
\]
\[
= \begin{pmatrix}
0 & 1 & 0 & 1 & \cdots & 0 \\
0 & 1 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 1 & 0 & 1 & \cdots & 0 \\
\xi & 0 & 1 & 0 & \cdots & 0
\end{pmatrix} u =
\]
\[
= \begin{pmatrix}
0 & 1 & 0 & 1 & \cdots & 0 \\
0 & 1 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 1 & 0 & 1 & \cdots & 0 \\
\xi & 0 & 1 & 0 & \cdots & 0
\end{pmatrix} u =
\]
\[
= \psi(\beta(t))(\alpha(u)),
\]
so \( \varphi \) is an automorphism. Furthermore, \( \varphi \) has order \( nk \) and as many as \( n \) fixed points \((\eta, \ldots, \eta, 0)\), where \( \eta \) is a root of unity of degree \( n \).
We now construct the matrix realization of the group $H$, that realizes the automorphism $\varphi$: 

$$
H = \left\{ \begin{pmatrix} T & K \\ 0 & E \end{pmatrix} \middle| \begin{array}{l}
T = \text{diag}(t_1, \ldots, t_{n-1}, t_n), \ t_i \in k^* \\
K = \text{diag}(t_1a_1, \ldots, t_{n-1}a_{n-1}, t_n a_n), \ a_i \in k \\
t_1 \ldots t_n = 1
\end{array} \right\}
$$

$$
G = H \cup g_0 H \cup \ldots \cup g_0^{n-1} H,
$$

$$
g_0 = \begin{pmatrix} \Sigma & 0 \\ 0 & \Sigma D \end{pmatrix}, \ \Sigma = \begin{pmatrix}
0 & 0 & & & \xi^{n-1} \\
1 & 0 & & & \\
& \ddots & \ddots & \ddots & \ddots \\
& & 1 & 0 & \\
& & & 1 & 0
\end{pmatrix}, \ D = \text{diag}(\xi, 1, \ldots, 1).
$$

It is only left to calculate the order of elements in the periodic component $g_0 H$. As we know, $\forall g \in g_0 H : \text{ord}(g) = \text{ord}(g_0)$ equals the least common multiple of $\text{ord}(\Sigma)$ and $\text{ord}(\Sigma D)$, but since $\Sigma^n = \xi^{n-1} E$, we have $\text{ord}(\Sigma) = nk$, and

$$
(\Sigma D)^n = \begin{pmatrix} 0 & \xi^{n-1} \\
\xi & 0 & & & \\
& \ddots & \ddots & \ddots & \ddots \\
& & 1 & 0 & \\
& & & 1 & 0
\end{pmatrix} = \xi^n E = E,
$$

so we receive that $\text{ord}(g_0) = nk$.

### 4. Torus extensions

Let the connected component of unity of the group $G$ be a torus $T$. In this case the criterion of the existence of periodic components may be formulated in a more convenient way. Note that the automorphism $\varphi_g$ induces the automorphism $A_{\varphi_g}$ of character lattice of torus $T$:

$$(A_{\varphi_g} \circ \lambda)(t) = \lambda(\varphi_g(t)).$$

Further, the character lattice of $T$ is a free abelian group with generators $\chi_s : T \cong (k^*)^m \longrightarrow k^*$, $(t_1, \ldots, t_m) \mapsto t_s$. Using the additive notation for the group of characters we may associate the matrix $A_g \in GL_n(Z)$ with the automorphism $A_{\varphi_g}$.

**Proposition 5.** If $G^0 = T$, then the component $gT$ is periodic if and only if the matrix $A_g$ has no eigenvalue equal to 1.
Proof. Let the component \( gT \) be periodic and \( g^n = e \). Then by Corollary 1 \( \forall h \in T : (gh)^n = e \). But evidently \((gh)^n = g^n \cdot \varphi_{g^{-1}}(h) \cdot \ldots \cdot \varphi_g(h) \cdot h = e\) or in terms of the matrix \(A_g\):

\[
A_g^{n-1} + \ldots + A_g + E = 0.
\]

Consequently the matrix \(A_g\) has no eigenvalue equal to 1.

Conversely, assume that \( g \) is an element of infinite order and that the matrix \(A_g\) has no eigenvalue equal to 1. Since the number of connected components of the group \( G \) is finite, we have that \( \exists k \in \mathbb{N} : g^k \in T \) and hence \( \varphi_g^k = id \). Consequently, \( 0 = A_g^k - E = (A_g - E)(A_g^{k-1} + \ldots + A_g + E) \). But by our assumption the matrix \(A_g - E\) is non-singular, and so \( A_g^{k-1} + \ldots + A_g + E = 0 \) yielding \( \forall h \in T : \varphi_g^{k-1}(h) \cdot \ldots \cdot \varphi_g(h) \cdot h = e \).

But \( g^k \) lies in \( T \) and has infinite order and therefore \( \varphi_g^{k-1}(g^k) \cdot \ldots \cdot \varphi_g(g^k) \cdot g^k = g^{k^2} \neq e \) which is a contradiction.

\[\square\]

**Remark 1.** Let \( T \) be a torus, \( G = T \cup gT \cup \ldots \cup g^{m-1}T \) be its cyclic extension with periodic component \( gT \). By Theorem 1 in this case \( gT = \{ tgt^{-1} | t \in T \} \), and hence \((gT)^n = \{tg^n t^{-1} | t \in T \}\) is a conjugacy class of the element \( g^n \in T \). But a torus is commutative yielding \((gT)^n = \{g^n\}\).

However for general solvable groups it is not true. Return to Example 2. Since

\[
g_0^n = \begin{pmatrix} \xi^{n-1}E & 0 \\ 0 & E \end{pmatrix} \notin Z(H(U)),
\]

we have \((g_0 H)^n = \{hg_0^n h^{-1} | h \in H \} \supset \{g_0^n\}\). Hence Example 2 illustrates the difference between finite extensions of tori and general solvable groups.

The future examples will show that the eigenvalues of the matrix \(A_g\) do not define the order of elements in the component \(gT\). However, we may give some estimates. For this we formulate a slightly more general proposition.

**Proposition 6.** Let \( T \) be a torus, \( G = T \cup gT \cup \ldots \cup g^{m-1}T \) be its cyclic extension of order \( m \) and \( k \) be the order of the automorphism \( \varphi_g \) where \( \varphi_g(t) = g^{-1}tg \). Then there exists an element \( g_0 \in gT \) such, that \( g_0^{mk} = e \).
Proof. Let $P_g(t) = \varphi_g^{m-1}(t) \cdot \varphi_g(t) \cdot t$, $Q_g(t) = \varphi_g^{k-1}(t) \cdot \varphi_g(t) \cdot t$. Then $\forall t \in T : (gt)^m = g^m \cdot P_g(t)$ $(gt)^k = g^m \cdot Q_g(t)$. Since $\varphi_g$ is a homomorphism and torus is commutative, $P_g$ and $Q_g$ are homomorphisms of torus $T$. Also $\varphi_g^k = id \Rightarrow P_g(t) = (Q_g(t))^r$, where $m = kr$.

**Lemma 4.** The following equations are satisfied:
1. $Q_g(Q_g(t)) = (Q_g(t))^k$;
2. $P_g(P_g(t)) = (P_g(t))^m$.

Proof. Using that $\varphi_g$ is a homomorphism and $\varphi_g^k = id$:

$$Q_g(Q_g(t)) = Q_g(\varphi_g^{k-1}(t) \cdot \varphi_g(t) \cdot t) =$$

$$= \prod_{i=0}^{k-1} \varphi_g^i(\varphi_g^{k-1}(t) \cdot \varphi_g(t) \cdot t) = \prod_{i=0}^{k-1} \varphi_g^i(\varphi_g^{k-1}(t)) \cdot \varphi_g(t) =$$

$$= \prod_{i=0}^{k-1} \varphi_g^{k+i-1}(t) \cdot \varphi_g^{i+(k-i+1)}(t) \cdot \varphi_g^{i+(k-i)}(t) \cdot \varphi_g^{i+(k-i-1)}(t) \cdot \varphi_g^{i-1}(t) =$$

$$= \prod_{i=0}^{k-1} \varphi_g^{i-1}(t) \cdot \varphi_g(t) \cdot t \cdot \varphi_g^{k-1}(t) \cdot \varphi_g^{i+1}(t) \cdot \varphi_g^i(t) = (Q_g(t))^k.$$  

To prove the second statement note that

$$P_g(P_g(t)) = (Q_g((Q_g(t))^r))^r = (Q_g(Q_g(t)))^r =$$

$$= ((Q_g(t))^r)^r = ((Q_g(t))^r)^{kr} = (P_g(t))^m.$$  

$\square$

**Corollary 4.** The groups $\text{Ker } P_g \cap \text{Im } P_g$ and $\text{Ker } Q_g \cap \text{Im } Q_g$ are finite.

Proof. If $s \in \text{Im } P_g$, then $\exists t \in T : s = P_g(t)$. Further, $s \in \text{Ker } P_g \Rightarrow e = P_g(s) = P_g(P_g(t)) = (P_g(t))^m = s^m$. But in a torus there are only finitely many elements of order not greater then $m$. For $Q_g$ the proof is just the same. $\square$

We have $T/\text{Ker } P_g \cong \text{Im } P_g$ and, consequently, $\dim \text{Ker } P_g + \dim \text{Im } P_g = \dim T$. Consider now the homomorphism

$$\gamma : \text{Ker } P_g \times \text{Im } P_g \to T, \quad (t_1, t_2) \mapsto t_1 \cdot t_2.$$  

Since $\text{Ker } \gamma = \{(t, t^{-1})\} \subset \text{Ker } P_g \times \text{Im } P_g$ and is by Corollary 4 a finite subgroup we have $T = \text{Ker } P_g \cdot \text{Im } P_g$. 
Lemma 5. \( \text{Im} Q_g = \text{Im} P_g \)

\begin{proof}
If \( t \in \text{Im} P_g \), then \( \exists s \in T : t = P_g(s) \). Therefore \( Q_g(s^r) = (Q_g(s))^r = P_g(s) = t \), i.e. \( t \in \text{Im} Q_g \). Conversely, let \( t \in \text{Im} Q_g \), i.e. \( \exists s \in T : t = Q_g(s) \). We should prove that \( \exists s_1 \in T : t = P_g(s_1) \). But the ground field is algebraically closed, therefore \( \exists s' \in T : (s')^r = s \), implying \( P_g(s') = (Q_g(s'))^r = Q_g((s')^r) = Q_g(s) = t \). This means that \( t \in \text{Im} P_g \).
\end{proof}

Hence \( T / \ker Q_g \cong \text{Im} Q_g = \text{Im} P_g \), and repeating the previous speculations we get that \( T = \ker Q_g \cdot \text{Im} P_g \).

Now note that the image of the mapping
\[ \alpha : T \longrightarrow T, t \mapsto (gt)^m = g^m \cdot P_g(t) \]
equals to the coset \( g^m \text{Im} P_g \subset T \). But \( T = \ker Q_g \cdot \text{Im} P_g \), and consequently any such coset contains an element of \( \ker Q_g \), i.e. \( \exists t_0 \in T \) such that \( (gt_0)^m = g^m \cdot P_g(t) = s \in \ker Q_g \). Without loss of generality we may assume that \( g^m \in \ker Q_g \).

Lemma 6. If \( g^m \in \ker Q_g \), then \( g^{mk} = e \)

\begin{proof}
By the definition \( \forall l = 1, \ldots, m - 1 : \varphi_g^l(g^m) = g^{-l}g^mg^l = g^m \).
But \( g^m \in \ker Q_g \) and therefore \( e = Q_g(g^m) = \varphi_{g^{k-1}}(g^m) \cdot \ldots \cdot \varphi_g(g^m) \cdot g^m = (g^m)^k \), yielding \( g^{mk} = e \).
\end{proof}

Proposition 6 is proved.

Corollary 5. If the component \( gT \) is periodic, then the order of any its element divides \( mk \), where \( m = \text{ord}(\varphi_g) \) and \( k \) is the order of \( gT \) in \( G/T \).

In Example 4 will be shown that there exists a torus \( T \) and its automorphism \( \varphi \) such, that any extension \( G = T \cup gT \cup \ldots \cup g^{m-1}T \), in which \( \forall t \in T : g^{-1}tg = \varphi(t) \), satisfies the inequality \( \text{ord}(g^r) < m \cdot \text{ord}(\varphi) \forall g^r \in gT \). But in many situations the estimate proves to be precise.
Example 3. Consider $G = T \cup gT \cup \ldots \cup g^{r-1}T$, where $k$ and $r$ are natural numbers and $T$ is as follows:

$$T = \begin{cases} \begin{pmatrix} t_1 & 0 \\ t_2 & 0 \\ \vdots & \vdots \\ t_{k-1} & 0 \\ 0 & t_k \end{pmatrix} & t_1t_2\ldots t_{k-1}t_k = 1 \end{cases} \subset GL_k(k),$$

$$g = \begin{pmatrix} 0 & \xi \\ 1 & 0 & 0 \\ \xi \end{pmatrix},$$

where $\xi$ is a primitive root of unity of degree $k$. In this case

$$A_g = \begin{pmatrix} 0 & -1 \\ 1 & 0 & -1 \\ \vdots & \vdots & \vdots \\ 1 & 0 & -1 \\ 1 & -1 \end{pmatrix} \in GL_{k-1}(\mathbb{Z}).$$

It is easy to see that the eigenvalues of the matrix $A_g$ are the following: $\eta, \eta^2, \ldots, \eta^{k-1}$, where $\eta$ is a primitive root of unity of degree $k$, and therefore the connected component $gT$ is periodic, and the order of its elements are equal to $k^2r$.

Let now the component $gT$ of cyclic extension $G = T \cup gT \cup \ldots \cup g^{m-1}T$ be periodic. Since $g^m \in T^{\varphi\sigma}$, we have $\text{ord}(g^m) \leq \max\{\text{ord}(t) \mid t \in T^{\varphi\sigma}\}$. Using this consideration, one can get the following estimate on the order of $g$.

**Proposition 7.** For the periodic component $gT$ of cyclic extension $G = T \cup gT \cup \ldots \cup g^{m-1}T$ the inequality is satisfied: $\text{ord}(g) \leq m|\chi_g(1)|$, where $\chi_g(\lambda)$ is the characteristic polynomial of the matrix $A_g$. 
Proof. Let \( \dim T = n \). Note that fixed points of the automorphism \( \varphi_g \) can be found from the following simultaneous equations:

\[
\begin{align*}
\begin{cases}
t_1^{a_{11}} t_2^{a_{21}} \cdots t_n^{a_{n1}} &= t_1, \\
t_1^{a_{11}} t_2^{a_{22}} \cdots t_n^{a_{n2}} &= t_2, \\
&\quad \cdots \\
t_1^{a_{1n}} t_2^{a_{2n}} \cdots t_n^{a_{nn}} &= t_n,
\end{cases}
\end{align*}
\]

where \( A_g = (a_{ij}) \). This means that

\[
\begin{align*}
\begin{cases}
t_1^{a_{11}-1} t_2^{a_{21}} \cdots t_n^{a_{n1}} &= 1, \\
t_1^{a_{12}} t_2^{a_{22}-1} \cdots t_n^{a_{n2}} &= 1, \\
&\quad \cdots \\
t_1^{a_{1n}} t_2^{a_{2n}} \cdots t_n^{a_{nn}-1} &= 1.
\end{cases}
\end{align*}
\]

(1)

By multiplying one equation on another and changing their order we will be performing the integer elementary transformations upon the rows of the matrix \( B = (a_{ij} - \delta_{ij}) \). Such transformations on one hand do not change the absolute value of its determinant and on the other hand using them we shall make the matrix \( B \) upper-triangular. System (1) will turn to

\[
\begin{align*}
\begin{cases}
t_1^{a'_{11}} t_2^{a'_{12}} t_3^{a'_{13}} \cdots t_n^{a'_{1n}} &= 1, \\
t_2^{a'_{22}} t_3^{a'_{23}} \cdots t_n^{a'_{2n}} &= 1, \\
&\quad \cdots \\
t_n^{a'_{nn}} &= 1.
\end{cases}
\end{align*}
\]

So we get that \( t_n^{a'_{nn}} = 1, t_{n-1}^{a'_{n-1,n-1}} = t_n^{-a'_{n-1,n}}, \ldots, t_1^{a'_{11}} = t_2^{-a'_{12}} \cdots t_n^{-a'_{1n}} \). Hence while solving the equations we shall at first extract a root of degree \( |a'_{nn}| \) from unity, then a root of degree \( |a'_{n-1,n-1}| \) from a certain root of unity of degree \( |a'_{nn}| \) and so on, but we shall never get more than a root of degree \( |a'_{11} \cdots a'_{nn}| \) of unity. The order of the fixed point of the automorphism \( \varphi_g \) that we have found is equal to the lesser common multiply of \( \text{ord}(t_1), \ldots, \text{ord}(t_n) \), but \( \text{ord}(t_i) \) divides \( |a'_{ii} \cdots a'_{nn}| \), and therefore their lesser common multiply divides \( |a'_{11} \cdots a'_{nn}| = | \det(a_{ij}')| = | \det(B)| = | \det(A - E)| = | \chi_g(1)|. \]

\( \square \)

In some situations this estimate is stronger than the estimate of Corollary 5.
Example 4. For any extension $G = T \cup gT \cup \ldots \cup g^{m-1}T$ with $\chi_g(\lambda) = \frac{\lambda^{r+1}}{\lambda+1}$, where $r = \dim T + 1$, we have $\chi_g(1) = 1$. This means that $\text{ord}(g) = m$ yielding $G \simeq T \rtimes \langle g \rangle_m$.

5. The normalizer of a maximal torus of a simple group

In this section we study the normalizers of maximal torus of classical simple groups as well as of the exceptional group $G_2$ and partially of $F_4$ and $E_8$, finding their periodic components and the orders of elements in these components.

First of all we recall a few general facts. Let $T$ be a maximal torus of a simply connected simple algebraic group $G$, $N_G(T)$ be its normalizer in $G$, $W = N_G(T)/T$ be its Weyl group, and $\{\alpha_1, \ldots, \alpha_n\}$ be a system of simple roots. A product $c = r_1r_1\ldots r_n$, where $r_i$ are the reflections associated with simple roots, is called a Coxeter element of group $W$, and the order $h$ of $c$ is called the Coxeter number of $W$. Note that all Coxeter elements are conjugate in $W$, and that every element of the Weyl group conjugate to a Coxeter element is also a Coxeter element for a certain system of simple roots. In particular $c$ is conjugate to $c^{-1}$.

As it is shown in [7, Ch. 3.16], a Coxeter element has no eigenvalue equal to 1. Hence its eigenvalues are as follows: $\zeta^{m_1}, \ldots, \zeta^{m_n}$, where $\zeta$ is a primitive root of unity of degree $h$ and $m_1 \leq m_2 \leq \ldots \leq m_n$. The numbers $m_i$ are called the exponents of $G$. The values of the exponents for each simple group can be found, for example, in [6]. All that was said above immediately proves the following proposition.

**Proposition 8.** The component $gT$ of the normalizer $N_G(T)$, corresponding to a Coxeter element of the Weyl group, is periodic.

As it is shown in [2, Prop. 30], a Coxeter element commutes only with its powers, so the number $N_c$ of Coxeter elements in $W$ equals to $|W|/h$.

However in many simple groups the normalizer of maximal torus has periodic components corresponding neither to Coxeter elements nor to their powers. Consider a natural action of the group $W$ in the rational vector space spanned by the lattice of characters of $T$. To find the number of all periodic components, we use the following proposition, see [10].

**Proposition 9.** If $g_k$ is the number of elements of Weyl group $W$, whose dimension of the space of fixed points is equal to $n - k$, then $\sum_{k=0}^{n} g_k z^k = \prod_{i=1}^{n} (1 + m_i z)$.
By Theorem 1 the number of periodic components in $N_G(T)$ equals to $g_n = m_1 \ldots m_n$. The results are collected in the table:

| $SL_n$ | $SO_{2n}$ | $SO_{2n+1}, Sp_{2n}$ | $G_2$ | $F_4$ |
|--------|-----------|----------------------|-------|-------|
| $|W|$   | $n!$      | $2^{n-1}n!$          | $2^n n!$ | 12    | 1525  |
| $g_n$  | $(n-1)!$  | $(2n-3)!!(n-1)$     | $(2n-1)!!$ | 5     | 385   |
| $N_c$  | $(n-1)!$  | $2^{n-2}(n-2)!n$    | $2^{n-1}(n-1)!$ | 2     | 96    |

| $E_6$  | $E_7$     | $E_8$    |
|--------|-----------|----------|
| $|W|$   | $2^4 \cdot 3^4 \cdot 5$ | $2^{10} \cdot 3^4 \cdot 5 \cdot 7$ | $2^{14} \cdot 3^6 \cdot 5^2 \cdot 7$ |
| $g_n$  | $2^5 \cdot 5 \cdot 7 \cdot 11$ | $2^3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17$ | $7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 29$ |
| $N_c$  | $2^5 \cdot 3^3 \cdot 5$ | $2^9 \cdot 3^2 \cdot 5 \cdot 7$ | $2^{13} \cdot 3^4 \cdot 5 \cdot 7$ |

Now we find the order of elements in periodic components.

1. The group $SL_n$.

   It is easy to see that Coxeter elements in the Weyl group consisting of permutations on $n$ elements are precisely the cycles of length $n$. Their number is equal to $(n-1)!$ and it is the number of periodic components in $N(T)$.

   To calculate the order of elements in the component corresponding to a cycle of length $n$, consider the monomial $(0,1,-1)$-matrix $A$, that lies in it. Here

   $\text{ord}(A) = \begin{cases} n & \text{for odd } n; \\ 2n & \text{for even } n. \end{cases}$

2. The group $SO_{2n}$.

   Consider the following matrix representation: $SO_{2n} \cong \{ g \in GL_{2n} \mid g^T \Omega g = \Omega \}$, where $\Omega$ is a monomial matrix with unities on the secondary diagonal.

   The normalizer of a maximal torus $T$ is

   $\left\{ A = (a_{i,j}) \in SL_{2n} \mid A \text{ is a monomial matrix; } a_{2n-i+1,2n-j+1} = a_{i,j}^{-1} a_{i,j} \neq 0 \right\}$.

   Consider a component $gT$ of the normalizer. It defines a homomorphism $\varphi_g : t \mapsto gtg^{-1}$, and the latter has a corresponding matrix $A_g$ of the mapping induced in the character lattice. It is easy to see that it is a monomial $(0,1,-1)$-matrix. We call a monomial matrix $A \in GL_n(\mathbb{Z})$ corresponding to a permutation $\tau \in S_n$ (we denote it by $A = A_\tau$), if $\forall j = 1, \ldots, n \exists \alpha_j \in \mathbb{k}^*: Ae_j = \alpha_j e_{\tau(j)}$. Then the matrix $A_g$ corresponds to a certain permutation $\sigma \in S_n$. 

Let $\sigma = \sigma_1 \ldots \sigma_k$ be the decomposition in a product of independent cycles (and of length 1 as well). Then $A$ is conjugate to the matrix
\[
A' = \begin{pmatrix}
S_{\sigma_1} & & \\
& S_{\sigma_2} & \\
& & \ddots \\
& & & S_{\sigma_k}
\end{pmatrix},
\]
where $S_{\sigma_j}$ is a monomial matrix corresponding to $\sigma_j$. But if $\tau \in S_m$ is a cycle of length $m$, then $(-1)^\tau = (-1)^{m-1}$ and the characteristic polynomial of $S_\tau$ is equal to
\[
\chi_{S_\tau}(\lambda) = \lambda^m + (-1)^\tau(-1)^k P = \lambda^m - P,
\]
where $P$ is the product of non-zero elements of $S$. Hence the characteristic polynomial of the matrix $A$ equals to
\[
\chi_A = \chi_{S_{\sigma_1}} \ldots \chi_{S_{\sigma_k}} = (\lambda^{s_1} - P_1) \ldots (\lambda^{s_k} - P_k),
\]
where $s_j$ is the order of $\sigma_j$ and $P_j$ is the product of non-zero elements of $S_{\sigma_j}$.

It is clear now that the matrix $A$ has no eigenvalue equal to 1 if and only if all $P_j \neq 1$, and since non-zero elements of $A$ may be only equal to $\pm 1$ (and of $A'$ as well, because it has been made from $A$ by permuting rows and columns), we get that the matrix $A$ has no eigenvalue equal to 1 if and only if in each block $S_{\sigma_j}$ of the matrix $A'$ the number of elements -1 is odd.

Finally, periodic components correspond to the following elements of the Weyl group: $(\sigma, \theta)$, where $\sigma \in S_n$, $\theta = (\varepsilon_1, \ldots, \varepsilon_n) \in \{\pm 1\}^n$, and if $\sigma = (i_{1,1}, \ldots, i_{1,s_1}) \ldots (i_{k,1}, \ldots, i_{k,s_k})$ is the decomposition in a product of independent cycles (of length 1 as well), then
\[
\forall j = 1, \ldots, k : \prod_{q=1}^{s_j} \varepsilon_{i_{j,q}} = -1. \tag{2}
\]
(Note that the number of minus unities among the components of the vector $\theta$ is even).

Now we calculate the order of elements in the corresponding component. For that consider the monomial $(0, 1)$-matrix $g$ lying in it. Direct calculations show that the order of $g$, and with it the order of every element of the component equals to double least common multiple of $s_1, \ldots, s_k$. 
3. The group $SO_{2n+1}$.

The results are the same as in the paragraph 2. The only difference is that the number of minus unities among the components of the vector $\theta$ need not be even.

4. The group $Sp_{2n}$.

Consider the following matrix representation: $Sp_{2n} \cong \{ g \in GL_{2n} \mid g^T \Lambda g = \Lambda \}$, where

$$\Lambda = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

(here $I$ is a monomial matrix with unities on the secondary diagonal).

The normalizer of maximal torus $T$ is

$$A = (a_{i,j}) \in GL_{2n} \begin{cases} A \text{ is a monomial matrix;} \\ a_{2n-i+1,2n-j+1} = a_{i,j}^{-1}, \\ \text{if } a_{i,j} \neq 0 \text{ and } i,j \leq n \text{ or } i,j > n; \\ a_{2n-i+1,2n-j+1} = -a_{i,j}^{-1}, \\ \text{if } a_{i,j} \neq 0 \text{ and } i \leq n < j \text{ or } j \leq n < i \end{cases}.$$ 

It is easy to see that the Weyl group here is the same as for $SO_{2n+1}$, so we can use the results of the paragraph 2. Periodic components correspond to the following elements of the Weyl group: $(\sigma, \theta)$, where $\sigma \in S_n$, $\theta = (\varepsilon_1, \ldots, \varepsilon_n) \in \{\pm 1\}^n$, so that if $\sigma = (i_1, \ldots, i_{s_1}) \ldots (i_k, \ldots, i_{s_k})$ is the decomposition in a product of independent cycles (of length 1 as well), then the equations (2) are satisfied.

Direct calculations show that the order of the monomial $(0, 1, -1)$-matrix lying in $gT$ equals to the least common multiple of $s_1, \ldots, s_k$ multiplied by 4.

5. The group $G_2$.

The corresponding Weyl group consists of 12 elements: the identity, six reflections and five rotations through angles multiple to $\pi/6$. The rotations are the powers of Coxeter element $c$ and have no eigenvalue equal to 1. Hence periodic components correspond to the elements $c, \ldots, c^5$.

To find the orders of elements in these components consider the characteristic polynomial $\chi_c(\lambda)$ of $A_c$. It is quadratic, its coefficients are integers and the roots are equal to $\zeta$ and $\zeta^5$, where $\zeta$ is a primitive root of unity of degree 6. One can easily understand that $\chi_c(\lambda) = \lambda^2 - \lambda + 1$. Proposition 7 yields then that for every element $g$ of the component corresponding to
The inequality \( \text{ord}(g) \leq 6|\chi_c(1)| = 6 \) is satisfied, implying \( \text{ord}(g) = 6 \).

The orders of elements in the components corresponding to \( c^2, \ldots, c^5 \) are equal to 3, 2, 3 and 6.

6. The group \( F_4 \).

Here we find the order of elements in the components corresponding to Coxeter element and its powers.

Note that the exponents of \( F_4 \) are equal to 1, 5, 7 and 11 and hence are coprime with the Coxeter number \( h = 12 \). Therefore all the degrees of Coxeter elements correspond to periodic components.

Let the component \( gT \) correspond to Coxeter element \( c \). By Corollary 5 we have \( \text{ord}(g) = \text{ord}(c)a = 12a \), where \( a|12 \). But then \( \text{ord}(g^6) = 2a \), and by Corollary 5 we get \( a|2 \). Considering \( \text{ord}(g^4) \), we get \( a|3 \), which means that \( a = 1 \). Finally the orders of elements in the components corresponding to \( c, c^2, c^3, c^4, c^5, c^6 \), are equal to 12, 6, 4, 3, 12, 2. It is easy to see that in the normalizer of maximal torus of the group \( F_4 \) there are periodic components corresponding neither to Coxeter elements nor to their powers.

Note that similar speculations may be performed for Coxeter elements of the group \( E_8 \). But as for the elements of periodic components of the normalizer of maximal torus of the groups \( E_6 \) and \( E_7 \), and of periodic components of the normalizer of maximal torus of \( F_4 \) and \( E_8 \), not corresponding to powers of Coxeter elements, calculation of their order requires further investigation.

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