THE BOREL SUBGROUP AND
BRANES ON THE HIGGS MODULI SPACE

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Abstract. We consider two families of branes supported on the singular locus of the moduli space of Higgs bundles over a smooth projective curve $X$. On the one hand, a (BBB)-brane $\text{Car}(L)$ constructed from the Cartan subgroup and a topologically trivial line bundle $L$ on $\text{Jac}^0(X)$. On the other hand, a (BAA)-brane $\text{Uni}(L)$ associated to the unipotent radical of the Borel subgroup and the previous line bundle $L$. We give evidence of both branes being dual under mirror symmetry, in the sense that an ad-hoc Fourier–Mukai integral functor relates the restriction of the hyperholomorphic bundle of the (BBB)-brane to a generic Hitchin fibre, with the support of the (BAA)-brane. We provide analogous constructions of (BBB)-branes and (BAA)-branes associated to a choice of a parabolic subgroup $P$ with Levi subgroup $L$, obtaining families of branes which cover the whole singular locus of the moduli space.

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1. Introduction

Hitchin introduced in [H1] Higgs bundles over a smooth projective curve \(X\) and soon it was noted that their moduli space \(\mathcal{M}_n\) carries a very interesting geometry. In particular \(\mathcal{M}_n\) can be endowed with a hyperkähler structure \((g, \Gamma_1, \Gamma_2, \Gamma_3)\) [H1, Si1, Si2, N] and fibres over a vector space \(M_n \to H\) with Lagrangian tori as generic fibres [H2]. A natural generalization is to consider Higgs bundles for complex reductive Lie groups other than \(GL(n, \mathbb{C})\). After the work of [HT, DG, DP], the moduli spaces of Higgs bundles for two Langlands dual groups equipped with the afore mentioned fibrations become SYZ mirror partners (as defined by [HT] based on work by [SYZ]). In this paper we focus in the case of \(GL(n, \mathbb{C})\), which is Langlands self-dual.

Branes in the Higgs moduli space were introduced in [KW] and have since attracted great attention. A (BBB)-brane in the the moduli space \(\mathcal{M}_n\) is given by a pair \((N, \mathcal{F})\), where \(N \subset \mathcal{M}_n\) is a hyperholomorphic subvariety and \(\mathcal{F}\) a hyperholomorphic bundle on \(N\). Additionally, a (BAA)-brane is a pair \((S, W)\) where \(S \subset \mathcal{M}_n\) is a subvariety which is complex Lagrangian with respect to the holomorphic symplectic form in complex structure \(\Gamma_1\), and \(W = (W, \nabla_W)\) is a flat bundle over \(S\). As stated in [KW], mirror symmetry is expected to interchange (BBB)-branes with (BAA)-branes. This is explored in [H4, BS1, BG, HS, GW, BCFG, H5, Ga, FJ, BS2], where (BBB) and (BAA)-branes are constructed and studied. The first obstacle one encounters to prove a duality statement is the difficulty to construct a hyperholomorphic bundle. Hitchin [H4] already provides a description of a non-trivial hyperholomorphic bundle arising from the Dirac–Higgs bundle. This appears also in recent work [H5, Ga, FJ], where a duality between (BBB) and (BAA)-branes is proven in the smooth locus of the Hitchin fibration.

In this paper we construct a family of (BBB) and (BAA)-branes indexed by a topologically trivial line bundle \(L \to \text{Jac}^0(X)\) and supported on the singular locus of the Hitchin fibration. Both constructions involve the Borel subgroup \(B < GL(n, \mathbb{C})\) and the Cartan subgroup \(C < B\). We denote by \(\text{Car}\) and \(\text{Bor}\) the locus of Higgs bundles reducing its structure group to \(C\) and \(B\) respectively. Since \(C\) is a complex reductive subgroup of \(GL(n, \mathbb{C})\), we observe that \(\text{Car}\) is a hyperkähler subvariety of \(\mathcal{M}_n\). We construct a hyperholomorphic vector bundle on \(\text{Car}\) out of our line bundle \(L \to \text{Jac}^0(X)\) and this constitutes our (BBB)-brane, that we denote by \(\text{Car}(L)\). We also define a subvariety \(\text{Uni}(L) \subset \mathcal{M}_n\) given by those Higgs bundles in \(\text{Bor}\) whose underlying vector bundle has a fixed reduction to \(C\), defined in terms of \(L\). We prove that \(\text{Uni}(L)\) is complex Lagrangian, and define \(\text{Uni}(L)\) to be the (BAA)-brane given by the trivial line bundle on \(\text{Uni}(L)\) with the trivial connection.

To study the behaviour of \(\text{Car}(L)\) and \(\text{Uni}(L)\) under mirror symmetry one would like to transform them under a fibrewise Fourier–Mukai transform. Since these branes are supported on the singular locus, the Hitchin fibers are not fine compactified Jacobians, and therefore a full Fourier–Mukai transform is not known to exist, not even after restricting ourselves to the open subset of the Cartan locus whose associated spectral curves are nodal. We can however define an ad-hoc Fourier–Mukai transform relating the generic loci of both branes. We expect that the weaker form of duality proven here would be induced from the global duality if a full Fourier–Mukai transform were to exist. In view of these results we conjecture that \(\text{Car}(L)\) and \(\text{Uni}(L)\) are dual branes under mirror symmetry.

We finish by discussing how this construction can be generalized to a large class of branes in the moduli space \(\mathcal{M}_n\) of rank \(n\) Higgs bundles covering the whole singular locus. In the (BBB)-case, the support of these branes correspond to the image of \(\mathcal{M}_r \times \cdots \times \mathcal{M}_s\), or equivalently, the locus of those Higgs bundles reducing its structure group to the Levi subgroup \(GL(r_1, \mathbb{C}) \times \cdots \times GL(r_s, \mathbb{C})\). We observe...
that these subvarieties cover the singular locus of $M_n$. The (BAA)-brane is given by a complex Lagrangian subvariety constructed in a similar way as before, but substituting the Borel subgroup with the parabolic subgroup associated to the partition $n = r_1 + \cdots + r_s$.

This paper is organized as follows. Section 2 gives the necessary background on the Hitchin system, torsion-free rank one sheaves on reducible curves, and Fourier–Mukai transforms over fine compactified Jacobians on reduced curves.

Section 3 studies the (BBB)-brane $\text{Car}(L)$. Its construction is addressed in Section 3.1. We consider the Cartan locus, $\text{Car}$, given by those Higgs bundles whose structure group reduces to the Cartan subgroup $C \cong (\mathbb{C}^*)^n < \text{GL}(n, \mathbb{C})$. The Cartan locus is given by the image of $c : \text{Sym}^n(M_1) \hookrightarrow M_n$, where $M_1$ is the rank one Higgs moduli space. Since the projection $M_1 \to \text{Jac}^0(X)$ is compatible with the three complex structures $\Gamma_1$, $\Gamma_2$ and $\Gamma_3$, we prove that the choice of a holomorphic bundle on $\text{Jac}^0(X)$ yields a hyperholomorphic bundle on $\text{Car}$ (Proposition 3.1, Lemma 3.2 and Remark 3.3). For the remainder of the paper we focus on the case of a line bundle $L$ on $\text{Jac}^0(X)$, which produces the (BBB)-brane $\text{Car}(L)$. In Section 3.2 we analyze the spectral data for the Higgs bundles in $\text{Car}(L)$ (Proposition 3.7), which is crucial to study the behaviour of $\text{Car}(L)$ under mirror symmetry.

Section 4 addresses the construction and description of the (BAA)-brane $\text{Uni}(L)$. In Section 4.1 we consider $\text{Bor}$, the locus of all the Higgs bundles reducing to the Borel subgroup $B$. Taking those whose underlying vector bundle project to a certain $C$-bundle determined by $L$, we obtain a subvariety $\text{Uni}(L)$ which is isotropic by gauge considerations (Proposition 4.2). In Section 4.2 we study the generic Hitchin fibers corresponding to completely reducible spectral curves with nodal singularities, obtaining that they are fully contained in $\text{Bor}$ (Theorem 4.4). We next compute the spectral data of the points of $\text{Uni}(L)$ in Proposition 4.5, what allows us to show that it is mid-dimensional and, therefore, Lagrangian (Theorem 4.7).

After Sections 3.2 and 4.2 we have a description of the generic restriction of $\text{Car}(L)$ and $\text{Uni}(L)$ to a Hitchin fibre, which is isomorphic to the coarse compactified Jacobian of a reduced but reducible curve. We study in Section 5 the transformation under Fourier–Mukai of the restriction of $\text{Car}(L)$, in spite of the lack of literature on the construction of Poincaré sheaves over coarse compactified Jacobians. To overcome this problem, we imitate in (5.1) the construction of the Poincaré bundle for fine compactified Jacobians that we reviewed in Section 2.4. In this case, instead of starting from the universal sheaf bundle for the classification of rank one torsion free sheaves on our spectral curve, we use a universal sheaf for the Cartan locus of the Hitchin fibre. As discussed in Remark 5.1 one expects that a Poincaré sheaf on the whole Hitchin fibre would restrict to this ad-hoc Poincaré sheaf. Hence, it is natural to study the behaviour of $\text{Car}(L)$ under a Fourier–Mukai integral functor constructed with it, which we do. We obtain that the generic restriction of $\text{Car}(L)$ to a Hitchin fibre is sent to corresponding restriction of $\text{Uni}(L)$ (Corollary 5.4). This lead us to conjecture that $\text{Car}(L)$ and $\text{Uni}(L)$ are dual branes under mirror symmetry.

In Section 6 we adapt the above results to arbitrary parabolic subgroups. Given a partition $n = r_1 + \cdots + r_s$ we consider the associated parabolic subgroup $P_\mathbf{r} < \text{GL}(n, \mathbb{C})$ with Levi subgroup $L_\mathbf{r} < P_\mathbf{r}$. In Section 6.1 we consider the subvariety $M_\mathbf{r}$ of $M_n$, consisting of Higgs bundles whose structure group reduces to $L_\mathbf{r}$, and describe the intersection with generic Hitchin fibers (Proposition 6.3). The variety $M_\mathbf{r}$ is a complex subscheme for $\Gamma_1$, $\Gamma_2$ and $\Gamma_3$, hence the support of a (BBB)-brane. By varying the partition $\mathbf{r}$, we produce families of branes covering the
strictly semistable locus of $M_n$. On the other hand, in Section 6.2 we consider $\text{Uni}_P$, consisting of Higgs bundles with structure group reducing to $P$ and fixed associated graded bundle. We prove that this is a Lagrangian submanifold (Theorem 6.7), and so a choice of flat bundle on it produces a (BAA)-brane. A look at the spectral data of both $M_P$ and $\text{Uni}_P$, as well as the comparison with the case $P_{(1,\ldots,1)}$, indicates the existence of a duality.

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2. Preliminaries

2.1. Non-abelian Hodge theory. Let $X$ be a smooth projective curve over $\mathbb{C}$. A Higgs bundle over $X$ is a pair $(E, \varphi)$ given by a holomorphic vector bundle $E \to X$ and a Higgs field $\varphi \in H^0(X, \text{End}(E) \otimes K)$, a holomorphic section of the endomorphisms bundle twisted by the canonical bundle $K$ of $X$.

The moduli space of rank $n$ and degree 0 semistable Higgs bundles on $X$, was constructed in [H1, Si1, Si2, N]. This is a quasi-projective variety $M_n$ of dimension

$$\dim M_n = 2n^2(g - 1) + 2,$$

where $g$ is the genus of $X$. It can be constructed as a GIT quotient as follows: fix a topological bundle $E$ of degree 0 on $X$. Consider $A$ the space of holomorphic structures on $E$. This is an affine space modelled on $\Omega^1(X, \text{ad}(E))$ whose cotangent bundle is $T^*A = A \times \Omega^0(X, \text{ad}(E) \otimes K)$ where we have identified $\text{ad}(E)$ and its dual by means of the Killing form (rather, a non degenerate extension of it to the center, to which we will henceforth refer as Killing form). Inside $T^*A = A \times \Omega^0(X, \text{ad}(E))$, we consider stable pairs $(\mathcal{D}_A, \varphi) \in T^*A$ satisfying that there exists a hermitian metric $h$ on $E$ such that the Chern connection $\nabla_h$ associated with $(E, \mathcal{D}_A)$ and $h$ satisfies:

1) $\nabla_h^2 + [\varphi, \varphi^* h] = 0$

2) $\mathcal{D}_A(\varphi) = 0$.

Now, the complex gauge group

$$G^c = \Omega^0(X, \text{Aut}(E))$$

acts on $(T^*\mathcal{A})^s$, and we may identify

$$M_n \cong T^*\mathcal{A}/G^c = (T^*\mathcal{A})^s/G^c,$$

where the double quotient denotes the GIT quotient. This is a complex manifold with complex structure $\Gamma_1$.

Let $\eta$ be a Hermitian metric on the topological bundle $E \to X$. Let

$$G = \Omega^0(X, \text{Aut}(E, \eta)),$$

be the unitary gauge group of automorphisms of $E$ preserving the metric $\eta$. Also, one can naturally define three complex structures $\bar{\Gamma}_1$, $\bar{\Gamma}_2$ and $\bar{\Gamma}_3$ on $T^*\mathcal{A}$ satisfying the quaternionic relations, together with a hyperkähler metric preserved by $G$. This action defines a moment map $\mu$, associated to each of the complex structures $\Gamma_1$, and one can define the hyperkähler quotient by the action of $G$. The complex structures $\bar{\Gamma}_1$, $\bar{\Gamma}_2$ and $\bar{\Gamma}_3$ on $T^*\mathcal{A}$ descend naturally to complex structures $\Gamma_1$, $\Gamma_2$ and $\Gamma_3$ on the quotient. It is proven in [H1, Si1, Si2] that the moduli space of
Higgs bundles is identified with the hyperholomorphic quotient equipped with the complex structure $\Gamma_1$,
\[
M_n \cong \mu_1^{-1}(0) \cap \mu_2^{-1}(0) \cap \mu_3^{-1}(0) / \mathcal{G}.
\]
Additionally, [D3] [C] proved that the moduli space of rank $n$ flat connections on the $C^\infty$ vector bundle $E$ over $X$ of degree 0 is isomorphic to the above hyperkähler quotient equipped with the complex structure $\Gamma_2$.

The hyperkähler structure defined on $M_n$ induces a holomorphic 2-form $\Omega_1 = \omega_2 + i\omega_3$ on $M_n$, where $\omega_2$ and $\omega_3$ are the Kähler forms associated to $\Gamma_2$ and $\Gamma_3$.

We next give the expression of $\Omega_1$ by means of the gauge theoretic construction of $M_n$. Let $(\partial_A, \varphi) \in T^*\mathcal{A}^s$, and consider two tangent vectors
\[
(A_i, \hat{\varphi}_i) \in T_{(\partial_A, \varphi)}T^*\mathcal{A} \quad i = 1, 2
\]
we have
\[
(2.2) \quad \Omega_1 \left((A_1, \hat{\varphi}_1), (A_2, \hat{\varphi}_2)\right) = \int_X A_1 \hat{\lambda}\hat{\varphi}_2 - A_2 \hat{\lambda}\hat{\varphi}_1.
\]
where to define the wedge product $\hat{\lambda}$, we identify $\Omega^{(0,1)}(X, \text{ad}(\mathbb{E})) \cong (\Omega^0(X, \text{ad}(\mathbb{E})) \otimes \Omega_X^{1,0})$ and $\Omega^0(\text{ad}(\mathbb{E}) \otimes K) \cong (\Omega^0(\text{ad}(\mathbb{E}) \otimes \Omega_X^{1,0})$, and for $Z_i \otimes \omega_i$, $i = 1, 2$, $Z_i \in \Omega^0(X, \text{ad}(\mathbb{E}))$, $\omega_i \in \Omega^1(X)$, we set
\[
(2.3) \quad (Z_1 \otimes \omega_1) \hat{\lambda}(Z_2 \otimes \omega_2) = \langle Z_1, Z_2 \rangle \otimes \omega_1 \otimes \omega_2
\]
with $\langle \ , \ \rangle$ the Killing form.

2.2. The Hitchin fibration. We recall here the spectral construction given in [H2] [BNR]. Let $(q_1, \ldots, q_n)$ be a base of the algebra $\mathbb{C}[gl(n, \mathbb{C})]^{GL(n, \mathbb{C})}$ of regular functions on $gl(n, \mathbb{C})$ invariant under the adjoint action of $GL(n, \mathbb{C})$. We choose them so that $\deg(q_i) = i$. The Hitchin map is defined by
\[
h : \quad M_n \rightarrow H := \bigoplus_{i=1}^n H^0(X, K^i)
\]
\[
(E, \varphi) \mapsto (q_1(\varphi), \ldots, q_n(\varphi)).
\]
It is a surjective proper morphism $[H2] [N]$ endowing the moduli space with the structure of an algebraically completely integrable system. In particular, its generic fibers are abelian varieties and every fiber is a compactified Jacobian.

To describe the Hitchin fibration, we consider the total space $|K|$ of the canonical bundle and the obvious algebraic surjection $\pi : |K| \rightarrow X$. We note that the pullback bundle $\pi^*K \rightarrow |K|$ admits a tautological section $\lambda$. Given an element $b \in H$, with $b = (b_1, \ldots, b_n)$, we construct the spectral curve $X_b \subset |K|$ by considering the vanishing locus of the section of $\pi^*K^n$
\[
\lambda^n + \pi^*b_1\lambda^{n-1} + \cdots + \pi^*b_{n-1}\lambda + \pi^*b_n.
\]
The restriction of $\pi : |K| \rightarrow X$ to $X_b$ is a ramified degree $n$ cover that which by abuse of notation we also denote by
\[
\pi : X_b \rightarrow X.
\]
Since the canonical divisor of the symplectic surface $|K|$ is zero and $X_b$ belongs to the linear system $|nK|$, one can compute the arithmetic genus of $X_b$,
\[
(2.3) \quad g(X_b) = 1 + n^2(g - 1).
\]
By Riemann-Roch, the rank $n$ bundle $\pi_*\mathcal{O}_{X_b}$ is has degree
\[
\deg(\pi_*\mathcal{O}_{X_b}) = -(n^2 - n)(g - 1).
\]
Given a torsion-free rank one sheaf $F$ over $X_b$ of degree $\delta$, where
\[
(2.4) \quad \delta := n(n-1)(g-1),
\]
we have that $E_{\pi} := \pi_* F$ is a vector bundle on $X$ of rank $n$ and degree $0$. Since $\pi$ is an affine morphism, the natural $O_X$-module structure on $F$, given by understanding $F$ as a sheaf supported on $|K|$, corresponds to a $\pi_* O_X[K] = \text{Sym}^*(K^n)$-module structure on $E_{\pi}$. Such structure on $E_{\pi}$ is equivalent to a Higgs field

$$\varphi_{\pi} : E_{\pi} \rightarrow E_{\pi} \otimes K.$$ 

This establishes a one-to-one correspondence between torsion-free rank one sheaves on $\mathbb{X}_b$ and Higgs bundles $(E_{\pi}, \varphi_{\pi})$ such that

$$h((E_{\pi}, \varphi_{\pi})) = b.$$ 

In fact, stability is preserved under the spectral correspondence as we see in the following theorem. We refer to the moduli space of torsion-free rank one sheaves on $\mathbb{X}_b$ as the compactified Jacobian $\overline{\text{Jac}}^\delta (\mathbb{X}_b)$. We denote by $\overline{\text{Jac}}^\delta (\mathbb{X}_b) \subset \overline{\text{Jac}}^\delta (\mathbb{X}_b)$ the open subset of invertible sheaves.

**Theorem 2.1** ([Se, Section 7]). A torsion-free rank one sheaf $F$ on the spectral curve $\mathbb{X}_b$ is stable (resp. semistable, polystable) if and only if the corresponding Higgs bundle $(E_{\pi}, \varphi_{\pi})$ on $X$ is stable (resp. semistable, polystable). Hence, the Hitchin fibre over $b \in H$ is isomorphic to the moduli space of torsion-free rank one sheaves of degree $\delta = (n^2 - n)(g - 1)$ over $\mathbb{X}_b$,

$$h^{-1}(b) \cong \overline{\text{Jac}}^\delta (\mathbb{X}_b).$$

For non-integral curves, [Sch, Théorème 3.1] gives an easy characterization of semistability, modulo some corrections pointed out in [CL, Remark 4.2] and [dC, Section 2.4]. Assuming the spectral curve is reduced, a torsion-free rank one sheaf $F \rightarrow \mathbb{X}_b$ is stable (resp. semi-stable) if and only if for every closed sub-scheme $Z \subset \mathbb{X}_b$ pure of dimension one, and every rank one torsion-free quotient sheaf $F|_Z \rightarrow \mathcal{F}_Z$, one has that

$$(2.5) \quad \deg_Z \mathcal{F}_Z > (n_Z^2 - n_Z)(g - 1) \quad (\text{resp.} \geq),$$

where $n_Z = \text{rk}(\pi_* O_Z)$.

### 2.3. Rank one torsion free sheaves on connected nodal reducible curves.

Motivated by Section 2.2, we recall in this section some well-known facts about rank one torsion free sheaves on connected reducible nodal curves whose singularities always lie on two (and only two) irreducible components. Let $\mathbb{X}$ be such a curve, and let $X_1, \ldots, X_n$ be its irreducible components. The normalization

$$(2.6) \quad \nu : \tilde{X} \rightarrow \mathbb{X}$$

satisifies $\tilde{X} \cong \bigsqcup X_i$. Let $D_{ij} = X_i \cap X_j$ for $1 \leq i < j \leq n$, and $D = \bigsqcup_{i,j} D_{ij}$. By assumption, $D_{ij}$ consists of simple points, which are nodal singularities of $\mathbb{X}$.

**Definition 2.2.** Let $F$ be a coherent sheaf on $X$. We say that $F$ is torsion free if $\text{Tor}(F_x) = 0$ for all $x \in \mathbb{X}$, where $\text{Tor}(F_x)$ denotes the torsion submodule of $F_x$, defined as

$$\text{Tor}(F_x) = \left\{ f \in F_x : \exists a \in O_{\mathbb{X},x} \setminus \text{Div}_0(O_{\mathbb{X},x}), \text{ such that } a \cdot f = 0 \right\}$$

with $\text{Div}_0(O_{\mathbb{X},x})$ the divisors of zero of $O_{\mathbb{X},x}$.

Since $\mathbb{X}$ is nodal, then torsion free sheaves are precisely sheaves of depth one (cf. [Sc, Section 7]).

**Definition 2.3.** Let $F$ be a torsion free sheaf. We define its degree as the integer $d$ appearing in the Euler characteristic

$$\chi(\mathbb{X}, F) = d + r(1-g).$$
Definition 2.4. A torsion free sheaf $\mathcal{F} \to X$ is of rank $r$ if
\[ \text{rk} (\mathcal{F}|_X / \text{Tor} (\mathcal{F}|_X)) = r. \]

Remark 2.5. The above is the definition given in [Sch]; in other sources, such a sheaf is called of multirank $r$. Note that the rank is not always well defined, but it ensures that the Higgs bundle obtained has the right characteristic polynomial (see discussion at the beginning of [HC Section 2.4]).

A particular example of rank one torsion free sheaves are line bundles on $\tilde{\mathbb{C}}$. The variety of all such bundles is denoted by $\text{Jac}(\mathbb{C})$. Line bundles admit a simple description in terms of their pullback to the normalization, as shown in the following lemma due to Grothendieck [Gr, Proposition 21.8.5], that we reproduce adapted to our notation.

Lemma 2.6. Let $R \subset D$, and let $\nu_D : \tilde{X}_D \to X$ be the partial normalization at $R$. Note that $\nu_D : \tilde{X}_D \to X$ is just the normalization map $\tilde{\mathbb{C}}$. The pullback map
\[ \nu_R : \text{Jac}(\mathbb{C}) \to \text{Jac}(\tilde{X}_D) \]
\[ L \quad \mapsto \quad \nu_R^* L \]
is a smooth fibration with fiber $(\mathbb{C}^*)^{R| - n_R + 1}$ where $n_R$ is the number of connected components of $\tilde{X}_R$.

Proof. Since both $\text{Jac}(\mathbb{C})$ and $\text{Jac}(\tilde{X}_D)$ are torsors for the groups $\text{Jac}^0(\mathbb{C})$ and $\text{Jac}^0(\tilde{X}_D)$, the tangent space at any point of them is isomorphic to the tangent space at fixed points, $L$ and $\nu_R^* L$, of each connected component. Then, both varieties are clearly smooth and it is enough to prove smoothness of $\nu_R$ at these fixed points. Since both are stable, the tangent space of $\text{Jac}(\mathbb{C})$ and $\text{Jac}(\tilde{X}_D)$ at $L$ and $\nu_R^* L$ are, respectively, $\text{Ext}^1(L, L) \cong H^1(\mathbb{C}, \mathcal{O})$ and $\text{Ext}^1(\nu_R^* L, \nu_R^* L) \cong H^1(\tilde{X}_R, \mathcal{O}_{\tilde{X}_R})$ (see [HL] Corollary 4.5.2) for instance) and the differential of $\nu_R$ is given by the pullback under $\nu_R$. Taking the short exact sequence
\[ 0 \to \mathcal{O}_X \to \nu_R^* \mathcal{O}_{\tilde{X}_R} \to \mathcal{O}_R \to 0. \]
with associated long exact sequence
\[ 0 \to \mathbb{C} \to \mathbb{C}^n \to \mathbb{C}^{|R|} \to H^1(\mathbb{C}, \mathcal{O}) \xrightarrow{\nu_R^*} H^1(\tilde{X}_R, \mathcal{O}_{\tilde{X}_R}) \to 0, \]
we can easily check that $\nu_R^* : H^1(\mathbb{C}, \mathcal{O}) \to H^1(\tilde{X}_R, \mathcal{O}_{\tilde{X}_R})$ has maximal rank, so it is a smooth morphism.

The rest of the statement follows naturally from the the short exact sequence
\[ 0 \to \mathcal{O}_X^\times \to \nu_R^* \mathcal{O}_{\tilde{X}_R}^\times \to \mathcal{O}_R^\times \to 0, \]
whose associated long exact sequence reads
\[ 0 \to \mathbb{C}^\times \to (\mathbb{C}^*)^{n_R} \to (\mathbb{C}^*)^{|R|} \to \text{Jac}(\mathbb{C}) \xrightarrow{\nu_R^*} \text{Jac}(\tilde{X}_R) \to 0. \]

We define
\[ \hat{\nu} : \text{Jac}(\mathbb{C}) \to \text{Jac}(\tilde{X}) \]
\[ L \quad \mapsto \quad \nu^* L, \]
which by Lemma 2.6 is a smooth fibration with fiber $(\mathbb{C}^*)^{d-n+1}$.

One can give the following geometrical interpretation of Lemma 2.6. Given $R = \bigcup_{j>1} D_{1j}$, a line bundle $L \to X$ consists of a line bundle $L_1 \to X_1$ and a line bundle $L_2 \to X_R$ where $X_R = \bigcup_{j>1} X_i$ (namely, a line bundle on $\tilde{X}_R = X_1 \sqcup \bigcup_{j>1} X_i$)
together with an identification $z_x : (L_1)_{x_1} \to (L_j)_{x_j}$ at each point $x \in D_{1j}$ with preimages $x_i \in X_i$. To recover $L$ from these data, tensoring (2.7) with $L$, we obtain

$$
0 \longrightarrow L \longrightarrow L_1 \oplus L_2 \longrightarrow L|_R \longrightarrow 0
$$

$s \longmapsto (s|_{X_1}, s|_{X_2}) (a, b) \longmapsto (a(x) - z_x b(x))_{x \in R}.$

One sees that two tuples $(z_x)_{x \in R}$ and $(z'_x)_{x \in R}$ induce isomorphic bundles $L \cong L'$ if and only if they differ by a non zero factor. Repeating the process with $L|_Y$, one obtains a description of line bundles on reducible curves in terms of line bundles on each of the irreducible components, together with gluing data.

We next explain how to compute the degrees of line bundles on $\overline{X}$ from the degrees on each connected component.

**Lemma 2.7.** Let $L \to \overline{X}$ be a line bundle on a connected nodal curve with irreducible components $X_i$, $i = 1, \ldots, n$ such that the only singularities lie on two components. For any $L \in \text{Jac}(\overline{X})$, $\deg L = \sum_i \deg L|_{X_i}.$

**Proof.** We proceed by induction on the number of irreducible components of $\overline{X}$. If there is just one such component, the statement is trivial. Assume there are $n$. Let $Y = \bigcup_{i=1}^n X_i$.

Let $R = \bigcup_{j \leq n-1} D_j$, and consider $\nu_R : \tilde{X}_R \to \overline{X}$ the partial normalization of $\overline{X}$ along $R$, that is

$$\tilde{X}_R = Y \cup X_n.$$

By the induction hypothesis, it is enough to prove that $\deg(L) = \deg(L|_Y) + \deg(L|_{X_n}).$ Tensoring (2.7) by $L$ we get

$$0 \longrightarrow L \longrightarrow \nu_R^* \mathcal{O}_{\tilde{X}_R} \otimes L \longrightarrow L|_R \longrightarrow 0.$$

Now, given a coherent sheaf $\mathcal{F}$ on $\overline{X}$, let $\chi(\overline{X}, \mathcal{F}(m)) = \sum_i (-1)^i h^i(\overline{X}, \mathcal{F}(m))$ be its Hilbert polynomial. We have that

$$\chi(\overline{X}, L(m)) - \chi(\overline{X}, L|_R) = \chi(\tilde{X}_R, \nu_R^* L).$$

Note also that

$$\chi(\overline{X}, L(m)) = m + \deg(L) - g(\overline{X}) + 1,$$

and

$$\chi(\overline{X}, L|_R) = |R|.$$

Taking an ample divisor so that half of the points belong to $Y$ and half to $X$,

$$\chi(\tilde{X}_R, \nu_R^* L) = \chi(Y, L(m/2)|_Y) + \chi(X_n, L(m/2)|_{X_n})$$

$$= m + \deg(L|_Y) + \deg(L|_{X_n}) + 1 - g(Y) + 1 - g(X_n).$$

Therefore,

$$\deg(L) - g(\overline{X}) + 1 - |R| = \deg(L|_Y) + \deg(L|_{X_n}) + 1 - g(Y) + 1 - g(X_n).$$

From the long exact sequence induced from (2.7), we have that

$$g(\tilde{X}_R) + |R| - 1 = h^1(\tilde{X}_R, \mathcal{O}_{\tilde{X}_R}) + |R| - 1 = h^1(\overline{X}, \mathcal{O}_{\overline{X}}) = g(\overline{X})$$

which together with the fact that $g(\tilde{X}_R) = g(Y) + g(X_n)$ concludes the proof. \qed

**Definition 2.8.** We say that the multidegree of a line bundle $L \to \overline{X}_v$ is the multidegree of $\nu(L) = \nu^* L \to \tilde{X}_v$, that is, the degree on each of the connected components of $\tilde{X}$. In the above $\nu$ is defined in (2.8).
A rank one torsion free sheaf on $\overline{X}$ is either a line bundle or a pushforward of a line bundle on a partial normalization of $\overline{X}$ [Se]. Geometrically, rank one torsion free coherent sheaves on $\overline{X}$ which are a pushforward from an element in $\text{Jac}(\hat{X}_R)$ are obtained as $n$-uples of line bundles $L_i \rightarrow X_{R,i}$ on each connected component of $\hat{X}_R$, together with identifications at all points $x \in D \setminus R$.

**Lemma 2.9.** With the same notation as in Lemma 2.6, for any $F \in \text{Jac}(\hat{X}_R)$, one has

$$\deg(\nu_{R,*}F) = \deg(F) + |R|.$$  

**Proof.** We have

$$\deg(\nu,F) + 1 - g_{\overline{X}} = \chi(\overline{X}, \nu_{R,*}F) = \chi(\hat{X}_R, F) = \deg(F) + n_R - g_{\overline{X}_R}.$$  

From (2.7),

$$\chi(\overline{X}, \mathcal{O}_{\overline{X}}) - \chi(\overline{X}, \nu_{R,*}\mathcal{O}_{\overline{X}_R}) + |R| = 0,$$

which implies $g_{\overline{X}_R} = n_R - 1 + g_{\overline{X}} - |R|$. Substitution in the above equation proves the statement. 

□

2.4. Fourier–Mukai on fine compactified Jacobians. In this Section we recall the results from [MRV1] [MRV2], where a Poincaré sheaf is built over the product of a fine compactified Jacobian of a curve with nodal singularities and its dual, yielding a Fourier–Mukai transform between these spaces.

Given a flat morphism $f : Y \rightarrow S$ whose geometric fibres are curves, for any $S$-flat sheaf $\mathcal{F}$ on $Y$, we can construct the determinant of cohomology $\mathcal{D}_f(\mathcal{E})$ (see for instance [KM], [Es, Section 6.1]), which is an invertible sheaf on $S$ constructed locally as the determinant of complexes of free sheaves locally quasi-isomorphic to $Rf_*\mathcal{E}$.

Since $\text{Jac}^\delta(\overline{X})$ is a fine moduli space by hypothesis, one has a universal sheaf $\mathcal{U} \rightarrow \overline{X} \times \text{Jac}^\delta(\overline{X})$. Denote by $\mathcal{U}^\delta$ its restriction to $\overline{X} \times \text{Jac}^\delta(\overline{X})$. Consider the triple product $\overline{X} \times \text{Jac}^\delta(\overline{X}) \times \text{Jac}^\delta(\overline{X})$ and denote by $f_{ij}$ the projection to the product of the $i$-th and $j$-th factors. We define the Poincaré bundle $\mathcal{P} \rightarrow \text{Jac}^\delta(\overline{X}) \times \text{Jac}^\delta(\overline{X})$ as the invertible sheaf

$$\mathcal{P} = \mathcal{D}_{f_{ij}} \left( f_{ij}^*\mathcal{U} \otimes f_{ij}^*\mathcal{U}^\delta \right)^{-1} \otimes \mathcal{D}_{f_{ij}} \left( f_{ij}^*\mathcal{U}^\delta \right) \otimes \mathcal{D}_{f_{ij}} \left( f_{ij}^*\mathcal{U} \right).$$

Given $J \in \text{Jac}^\delta(\overline{X})$, we have that the restriction $\mathcal{P}_J := \mathcal{P}|_{\text{Jac}^\delta(\overline{X}) \times \{J\}}$ is a line bundle over $\text{Jac}^\delta(\overline{X})$. In fact, if we consider the obvious projections $f_1 : \overline{X} \times \text{Jac}^\delta(\overline{X}) \rightarrow \overline{X}$ and $f_2 : \overline{X} \times \text{Jac}^\delta(\overline{X}) \rightarrow \text{Jac}^\delta(\overline{X})$, one has [MRV1] Lemma 5.1

$$\mathcal{P}_J = \mathcal{D}_{f_2} (\mathcal{U} \otimes f_1^*J)^{-1} \otimes \mathcal{D}_{f_2} (f_1^*J) \otimes \mathcal{D}_{f_2}(\mathcal{U}).$$

Using $\mathcal{P}$ one can construct an associated integral functor. We set $\pi_1$ and $\pi_2$ to be, respectively, the projection from $\text{Jac}^\delta(\overline{X}) \times \text{Jac}^\delta(\overline{X})$ to the first and second factors. Then, define

$$\Phi : D^b \left( \text{Jac}^\delta(\overline{X}) \right) \otimes \mathcal{E} \rightarrow R\pi_{2,*}(\pi_1^*\mathcal{E} \otimes \mathcal{P}).$$

By [MRV2], one can extend the Poincaré bundle $\mathcal{P} \rightarrow \text{Jac}^\delta(\overline{X}) \times \text{Jac}^\delta(\overline{X})$ to a Cohen-Macaulay sheaf $\overline{\mathcal{P}} \rightarrow \text{Jac}^\delta(\overline{X}) \times \text{Jac}^\delta(\overline{X})$. Denote by $\pi_1$ (resp. $\pi_2$) the projection $\text{Jac}^\delta(\overline{X}) \times \text{Jac}^\delta(\overline{X}) \rightarrow \text{Jac}^\delta(\overline{X})$ to the first (resp. second) factor. Using $\overline{\mathcal{P}}$ as a kernel, one can consider the integral functor

$$\overline{\Phi} : D^b \left( \text{Jac}^\delta(\overline{X}) \right) \otimes \mathcal{E} \rightarrow R\pi_{2,*}(\pi_1^*\mathcal{E} \otimes \overline{\mathcal{P}}),$$
3. A (BBB)-brane from the Cartan subgroup

In this section we construct a (BBB)-brane of $M_n$, namely, a pair $(N, F)$ given by:

1) A hyperholomorphic subvariety $N \subset M_n$, i.e. a subvariety which is holomorphic with respect to the three complex structures $\Gamma_1$, $\Gamma_2$ and $\Gamma_3$.

2) A hyperholomorphic bundle $F$ on $N$, i.e. a vector bundle with a connection whose curvature is of type $(1, 1)$ in the complex structures $\Gamma_1$, $\Gamma_2$ and $\Gamma_3$.

3.1. A hyperholomorphic bundle on the Cartan locus. The embedding of the Cartan subgroup $C \cong (\mathbb{C}^*)^n$ into $GL(n, \mathbb{C})$ induces the Cartan locus of the moduli space of Higgs bundles $Car = \{(E, \varphi) \in M_n \mid \exists s \in H^0(X, E/C), \varphi \in H^0(X, E_s(\xi) \otimes K)\}$, where $c = \text{Lie}(C)$ and $E_s$ is the principal $C$-bundle on $X$ constructed from the section $s$. Observe that $Car$ is the image of the injective morphism $c : \text{Sym}^n(M_1) \to M_n$.

Note also that $Car$ is a hyperholomorphic subvariety, since in the complex structure $(M_n, \Gamma_2)$, it corresponds with the locus of the moduli space of flat connections given by those reducing its structure group to $C$.

Note that both, the moduli space of topologically trivial rank 1 Higgs bundles $(M_1, \Gamma_1) \cong T^* \text{Jac}^0(X)$ and the moduli space of rank 1 flat connections $(M_1, \Gamma_2) \cong \text{Loc}_1(X)$ fiber algebraically over $\text{Jac}^0(X)$. In fact, this projection $M_1 \to \text{Jac}^0(X)$ is hyperholomorphic. As a immediate consequence, the induced map $p : \text{Sym}^n(M_1) \to \text{Sym}^n(\text{Jac}^0(X))$ is hyperholomorphic as well.

These remarks imply the following proposition.

**Proposition 3.1.** Suppose that $F$ is a vector bundle on $\text{Sym}^n(\text{Jac}^0(X))$, then $p^* F$ defines a hyperholomorphic vector bundle on the hyperholomorphic manifold $\text{Sym}^n(M_1)$.

**Proof.** This follows from the fact that the Chern connection of a holomorphic bundle is of type $(1, 1)$, that curvature commutes with pullbacks and that holomorphic maps respect types under pullback. \qed

In view of Proposition 3.1, one has that $F := c_* p^* F$ is a hyperholomorphic vector bundle on $Car$, the Cartan locus of the moduli space. The pair $(Car, F)$ constitutes a (BBB)-brane on the Higgs moduli space $M_n$.

Associated to a line bundle $E \to \text{Jac}^0(X)$ one can define a line bundle on $\text{Sym}^n(\text{Jac}^0(X))$ as we explain in the following lemma.

**Lemma 3.2.** Consider $\pi_i : (\text{Jac}^0(X))^\times_n \to \text{Jac}^0(X)$ the projection onto the $i$-th factor. Let $\mathcal{L}^{\otimes n} := \bigotimes_{i=1}^n \pi_i^* \mathcal{L}$. Then $\mathcal{L}^{\otimes n}$ descends to a bundle $\mathcal{L}^{(n)}$ on $\text{Sym}^n(\text{Jac}^0(X))$. 

Proof. The bundle $\mathcal{L} \boxtimes n$ is invariant by the action of $\mathfrak{S}$ and moreover the natural linearization action derived from the one on the bundle $\boxplus_{i=1}^{n} \mathcal{L}$ satisfies that over point $p \in (\text{Jac}^0(X))^\times n$ with non trivial centraliser $Z_p \subset \mathfrak{S}$, the centraliser $Z_p$ acts trivially on $\mathcal{L} \boxtimes n$. It follows from Kempf’s descent lemma that $\mathcal{L} \boxtimes n$ descends to a bundle $\mathcal{L}(n)$ on $\text{Sym}^n(\text{Jac}^0(X))$. □

Remark 3.3. The same argument yields a hyperholomorphic bundle on $\text{Car}$ for any choice of a holomorphic bundle $\mathcal{F} \to \text{Jac}^0(X)$.

Given $L \to \text{Jac}^0(X)$ topologically trivial, consider the associated hyperholomorphic line bundle $L = c_1^\star p^\star \mathcal{L}(n)$ on the Cartan locus $\text{Car}$ and denote by $\text{Car}(L) := (\text{Car}, L)$ the associated (BBB)-brane on $M_n$, which we call Cartan (BBB)-brane associated to $L$.

3.2. Spectral data for the Cartan locus. In this section we compute the fibers of the Hitchin map restricted to the hyperholomorphic subvariety $\text{Car} \subset M_n$.

We call $h(\text{Car})$, the image of $\text{Car}$ under the Hitchin map, the Cartan locus of the Hitchin base. Since the polystable Higgs bundles contained in $\text{Car}$ decompose as direct sums of line bundles, we have that the Cartan locus of the Hitchin base is the image of

$$V := \text{Sym}^n(H^0(X, K)),$$

under the injection

$$\begin{array}{rcl}
(\alpha_1, \ldots, \alpha_n)_{\mathfrak{S}} & \mapsto & (q_1(\alpha_1, \ldots, \alpha_n), \ldots, q_n(\alpha_1, \ldots, \alpha_n)).
\end{array}$$

Hence

$$\dim V = ng.$$

Let $u \in V$, and denote by $\overline{X}_u$ the corresponding spectral curve. We define $\Delta$ to be the big diagonal of $V$,

$$\Delta := \{(\alpha_1, \ldots, \alpha_n)_{\mathfrak{S}} \in V \text{ such that } \alpha_i = \alpha_j \text{ for some } i, j\}.$$ 

Clearly $V \setminus \Delta$ is dense inside $V$. For any two $\alpha_i$ and $\alpha_j$ with $i \neq j$, denote the divisor $D_{ij} = \alpha_i(X) \cap \alpha_j(X)$. Consider also the Cartan nodal locus of the Hitchin base to be subset of $V \setminus \Delta$

$$V^{\text{nod}} := \left\{ (\alpha_1, \ldots, \alpha_n)_{\mathfrak{S}} \in V \setminus \Delta \text{ such that for every } i < j < k \right.\Bigg\}.$$

(3.3) $V^{\text{nod}}$ is a dense open subset of $V$.

Remark 3.4. Since conditions (a) and (b) are generic, $V^{\text{nod}}$ is a dense open subset of $V$.

Lemma 3.5. For any $v \in V^{\text{nod}}$, with $v = (\alpha_1, \ldots, \alpha_n)_{\mathfrak{S}}$, the spectral curve,

$$\overline{X}_v = \bigcup_{i=1}^{n} X_i,$$

is a reduced curve, with $n$ irreducible components $X_i := \alpha_i(X)$ isomorphic to $X$ and only nodal singularities at the points

$$I := \bigcup_{i,j} D_{ij}$$

where $|I| = (n^2 - n)(g - 1) = \delta$. 

Proof. The spectral curve $\overline{X}$ is given by the equation
\[ \lambda^n + \pi^* b_1 \lambda^{n-1} + \cdots + \pi^* b_n = 0, \]
where $b_i = q_i(\alpha_1, \ldots, \alpha_n)$. By the properties of the invariant polynomials $q_i$, one can rewrite this equation as
\[ \prod_{i=1}^n (\lambda - \pi^* \alpha_i) = 0, \]
and (3.4) follows. We only have nodal points at most by the construction of $V^{\text{nod}}$.

The two curves, $X_i$ and $X_j$, intersect each other at $D_{ij}$, which by definition of $V^{\text{nod}}$ is a set of $2g - 2$ distinct points. There are $n$ irreducible components, so the number of intersection points is
\[ |I| = 2(g - 1) \binom{n}{2} = (n^2 - n)(g - 1) = \delta, \]
where we have used the condition $D_{ij} \cap D_{ik} = \emptyset$ if $j \neq k$ in the definition of $V^{\text{nod}}$. \qed

After Theorem 2.1, we are interested in the moduli space $\text{Jac}^\delta(X_v)$ of torsion-free rank one sheaves on $X_v$ of degree $\delta$.

Since $X_v = \bigcup X_i$, then the normalization $\nu : \tilde{X}_v \to X_v$ is isomorphic to $\tilde{X}_v \cong X \sqcup \ldots \sqcup X$.

Consider the following morphisms
\[ (3.5) \]

\[ \xymatrix{ \tilde{X}_v \ar[rr]^\nu \ar@/_/[dd]_p \ar@/^/[dd]^\delta_j & & X_j \ar[dl]_{\alpha_j} \ar@/^/[dd]^\iota_j \ar@/_/[dd]_{\tilde{\pi}} \ar@/^/[dd]^\iota_j \ar@/_/[dd]_{\tilde{\alpha}_j} \\ & X_v \ar[rr]^\pi & & X. } \]

Lemma 3.6. Set
\[ (3.6) \]
\[ \nu : \text{Jac}^{(d_1, \ldots, d_n)}(\tilde{X}_v) \to \text{Jac}^{\delta}(X_v) \]
where $(d_1, \ldots, d_n)$ is the multidegree of the line bundle on $\tilde{X}$ (cf. Definition 2.8). Then, the map is well defined and an injection if and only if $d_i = 0$ for all $i = 1, \ldots, n$.

Proof. By Lemma 2.9 ii), a necessary condition is for $\sum_{i=1}^n d_i = 0$. We need to check which multidegrees yield semistable bundles. But since for any $F \in \text{Jac}(\tilde{X})$, $\tilde{\pi}_* F = (\bigoplus_{i=1}^n F|_{X_i} \oplus \alpha_i)$ (where we identify $X \cong X_i$), and the only semistable such bundles must satisfy $d_i = 0$, Theorem 2.1 allows us to conclude. \qed
Proposition 3.7. For any \( v \in V^{\text{nod}} \), one has
\[
h^{-1}(v) \cap \text{Car} = \hat{\nu} \left( \Jac^0(\tilde{X}_v) \right).
\]
Moreover, under the isomorphism
\[
m : \Jac^0(\tilde{X}_v) \cong \Jac^0(X)^{\times n}
\]
induced by the ordering \((X_1, X_2, \ldots, X_n)\) of the connected components of \( \tilde{X} \).

i) the spectral datum \( L \in \hat{\nu} \left( \Jac^0(\tilde{X}_v) \right) \) corresponding to \( \bigoplus_{i=1}^n (L_i, \alpha_i) \in \text{Car} \) is taken to \((L_1, \ldots, L_n) \in \Jac^0(\tilde{X})^{\times n}\). Namely: \( L = \nu_* F = \bigoplus_j (\iota_j)_* L_j \) where \( \iota_j \) is as in \((3.5)\) and \( F \in \Jac(\tilde{X}) \) restricts to \( F|_{X_i} = L_j \).

ii) the restriction of \( \mathcal{L} \to \text{Car} \) to \( h^{-1}(v) \cap \text{Car} \) corresponds to \( \mathcal{L}^{\otimes n} \to \Jac^0(X)^{\times n} \) defined in Lemma 3.6.

Proof. i) By construction, a Higgs bundle in \( \text{Car} \) decomposes as a direct sum of line bundles,
\[
(E, \varphi) \cong \bigoplus_{i=1}^n (L_i, \alpha_i).
\]
By the argument in the proof of Lemma 3.6, \( \hat{\nu}(\Jac^0) \subset h^{-1}(v) \cap \text{Car} \). Now, let \( L \in \Jac^0(X) \) be the spectral datum corresponding to and element \((E, \varphi) \in h^{-1}(v) \cap \text{Car}\). It is easy to see that the Higgs bundle is totally decomposable if and only if its \( \pi_* \mathcal{O}_X \)-module structure factors through a \( \pi_* \nu_* \mathcal{O}_{\tilde{X}} \cong \mathcal{O}_X^{\otimes n} \)-module structure.

Hence \( L = \nu_* F \) for some \( F \in \Jac(\tilde{X}) \). Lemma 3.6 finishes the proof, as the only possible multidegree is \((0, \ldots, 0)\).

ii) In order to prove the second statement, note that the isomorphism \((3.7)\) is totally determined by a choice of an ordering of the connected components of \( X \), in this case \((X_1, \ldots, X_n)\). Now, the choice of such an ordering induces an embedding \( j : (\Jac^0(X))^{\times n} \hookrightarrow \text{Sym}^n(\Jac^0(X)) \) making the following diagram commute:
\[
\begin{array}{ccc}
\Jac^0(X)^{\times n} & \xrightarrow{m} & h^{-1}(u) \\
\downarrow{j} & & \downarrow{i} \\
\text{Sym}^n(M_1) & \xrightarrow{c} & \text{Car} \\
\downarrow{p} & & \downarrow{} \\
\text{Sym}^n(\Jac^0(X)) & &
\end{array}
\]
with \( q = p \circ j \) being the usual quotient map. We need to check that
\[
m^* i^* \mathcal{L} \cong \mathcal{L}^{\otimes n}.
\]
But, since the above diagram commutes and \( c \) is an injection, the LHS is equal to \( j^* c^* \mathcal{L} = j^* c^* p^* \mathcal{L}(n) \cong j^* p^* \mathcal{L}(n) \cong q^* \mathcal{L}(n) \) and the statement follows by the construction of \( \mathcal{L}(n) \).

\[\square\]

4. (BAA)-branes from the unipotent radical of the Borel

4.1. An isotropic subvariety. Starting from the line bundle \( \mathcal{L} \to \Jac^0(X) \), we construct in this section a complex Lagrangian subvariety \( \text{Uni}(\mathcal{L}) \) of the moduli space of Higgs bundles, mapping to the Cartan locus \( V \subset H \) of the Hitchin base.

Fix a Borel subgroup \( B < \text{GL}_n(\mathbb{C}) \) containing \( C \), so that \( B = C \ltimes U \) where \( U = [B, B] \) is the unipotent radical of \( B \). Denote by \( \text{Bor} \) the subvariety of the
moduli space $M_n$ given by those Higgs bundles whose structure group reduces to $B$,

$$\text{Bor} = \left\{ (E, \varphi) \in M_n \middle| \exists \sigma \in H^0(X, E/B), \varphi \in H^0(X, E_{\sigma}(b) \otimes K) \right\},$$

(4.1)

where $E_{\sigma} := \sigma^*E$ is the principal $B$-bundle on $X$ associated to the section $\sigma \in H^0(X, E/B)$.

After fixing a point $x_0 \in X$, we have an embedding $j_{x_0} : X \hookrightarrow \text{Jac}^0(X)$. Denote by $\hat{L}$ the restriction of $L$ to $X \subset \text{Jac}^0(X)$ tensored $\delta/n = (n-1)(g-1)$ times by $O_X(x_0)$,

$$\hat{L} := j_{x_0}^*L \otimes O_X(x_0)^{(n-1)(g-1)}.$$  

(4.2)

Now, define the closed subvariety of Bor given by those Higgs bundles $(E, \varphi)$ whose underlying vector bundle $E$ has associated graded piece totally determined by $\hat{L}$:

$$\text{Uni}(L) = \left\{ (E, \varphi) \in \text{Bor} \middle| E_C := E_{\sigma}/U \cong \bigoplus_{i=1}^{n}(\hat{L} \otimes K^{\otimes i-n}) \right\}. $$

(4.3)

In order to prove that $\text{Uni}(L)$ is an isotropic submanifold of $(M_n, \Omega_1)$ we first give a description of it in gauge theoretic terms. Let $E$ denote the topologically trivial rank $n$ vector bundle; choose a reduction of the structure group to $B$ (which always exists), and let $E_B$ be the corresponding principal $B$-bundle, so that $E \cong E_B(\text{GL}(n, \mathbb{C}))$. Define $E_C = E_B/U$. It follows from (4.3) that

$$\text{Uni}(L) = \left\{ (\overline{\sigma}_B, \varphi) \in M_n : \exists \sigma \in \mathcal{G}^c \middle| \begin{aligned} &1) \ g \cdot \overline{\sigma} = \overline{\sigma}_B + N, \\
&2) \ g \cdot \varphi \in \Omega^0(X, E_B(n)), \\
&\quad \quad \quad \quad \quad \quad (E_C, \overline{\sigma}_C) = \bigoplus_{i=1}^{n}(\hat{L} \otimes K^{\otimes i-n}); \\
&\quad \quad \quad \quad \quad \quad \Omega^0(X, E_B(b) \otimes K). 
\end{aligned} \right\}, $$

(4.4)

Remark 4.1. Both Car and $\text{Uni}(L)$ are subvarieties of Bor, but they do not intersect, as the elements of $\text{Car} \cap \text{Uni}(L)$ would have underlying bundle of the form $E_C$ in (4.3), which is unstable, and totally decomposable Higgs field, conditions which yield unstable Higgs bundles.

Proposition 4.2. The complex subvariety $\text{Uni}(L)$ of $M_n$ is isotropic with respect to the symplectic form $\Omega_1$ defined in (2.2).

Proof. It is enough to prove the statement for open subset of stable points in $\text{Uni}(L)$. We will check this subset is non empty in Proposition 4.5.

So let $(E, \varphi) \in \text{Uni}(L)$ be a stable point. By (4.4), a vector $(\dot{A}, \dot{\varphi}) \in T_{(E, \varphi)}M_n$ satisfies that, up to the adjoint action of the gauge Lie algebra,

$$\dot{(A, \varphi)} \in \Omega^{0,1}(X, E_B(n)) \times \Omega^0(X, E_B(b) \otimes K).$$

The result follows from gauge invariance of the symplectic form $\Omega_1$ and the fact that $n \subset b^+$, where orthogonality is taken with respect to the Killing form. \hfill $\Box$

4.2. Spectral data for $\text{Uni}(L)$. In this section we give a description of the spectral data of the Higgs bundles corresponding to the points of $\text{Uni}(L)$. This will allow us to show that this subvariety is mid-dimensional, and, after Proposition 4.2, Lagrangian.

We begin by studying the spectral data over $V^{nod}$. We recall that this is the subset of $V$ whose corresponding spectral curves $X_v$ are nodal curves. We will use the notation from (3.5).
Let us fix some notation. See also Figure 4.2. Let $v \in V^{\text{nod}}$, and let $R \subset D$. Consider the partial normalization along $R$,

\[ \tilde{X}_R \xrightarrow{\nu_R} X_v \]

\[ \xrightarrow{p_R} \pi \]

\[ \xrightarrow{} X. \]

Assume that $R = R_1 \sqcup \cdots \sqcup R_n + R_s$ with $\tilde{X}_R = \bigsqcup_{i=1}^n \tilde{X}_{R,i}$ being the decomposition into connected components such that

\[ \nu_{R,i} : \tilde{X}_{R,i} \longrightarrow X_{R_i} \]

is a partial normalization onto its image $X_{R_i}$ along a non-separating divisor $R_i$, and $R_s$ is the separating divisor in $R$ (i.e. the divisor along which connected components are to appear after normalization). Let $D_i$ be the ramification divisor of

\[ p_i : \tilde{X}_{R,i} \longrightarrow X \]

and set

\[ D_i = \sum_{j,k \in C_i} D_{jk} - R_i. \]

Note that

\[ D = \sum_i (D_i + R_i) + R_s. \]
Choose an ordering \((\tilde{X}_{R,1}, \ldots, \tilde{X}_{R,n_R})\) of the connected components of \(\tilde{X}_R\) inducing an isomorphism
\[
\text{Jac}(\tilde{X}_R) \cong \text{Jac}^{n_1}(\tilde{X}_{R,1}) \times \cdots \times \text{Jac}^{n_{n_R}}(\tilde{X}_{R,n_R}).
\]
Consider the decomposition
\[
(4.9) \quad \text{Jac}^{\eta}(\tilde{X}_R) \cong \bigcup_{i=1}^{n_R} \text{Jac}^{n_i}(\tilde{X}_{R,i}) \times \cdots \times \text{Jac}^{n_{n_R}}(\tilde{X}_{R,n_R}).
\]

Let also
\[
\text{Jac}^{n_i}(\tilde{X}_{R,i}) = \bigcup_{\sum_i d_i = n_i} \text{Jac}(d_i^{c_1}, \ldots, d_i^{c_{n_R}})(\tilde{X}_{R,i}).
\]

**Lemma 4.3.** i) If \(R_s \neq \emptyset\), the Higgs bundles whose corresponding spectral data is in \(\nu_{R,*} \text{Pic}(\tilde{X}_R)\) are strictly semistable.

ii) The pushforward map
\[
\nu_R : \text{Jac}(d_1^{c_1}, \ldots, d_i^{c_{n_R}})(\tilde{X}_{R,1}) \times \cdots \times \text{Jac}(d_R^{c_1}, \ldots, d_R^{c_{n_R}})(\tilde{X}_{R,n_R}) \to \text{Jac}(\tilde{X})
\]
is well defined and an injection only if
\[
\eta = \sum_{i=1}^{n_R} d_i^{c_i} = |D_i|.
\]

**Proof.** i) Let \(F \in \text{Pic}(\tilde{X}_R)\) be such that \(\nu_{R,*} F\) is the spectral datum for a Higgs bundle in \(M_n\). Assume \(F\) satisfies \(\tilde{\nu}_k F = F_k\). Then it follows that
\[
(4.10) \quad \pi_{v,*} \nu_{R,*} F = p_{R,*} F = \bigoplus_{i=1}^{n_R} p_i F_i,
\]
where the notation is as in [4.6]. Note that the direct sum is invariant by the Higgs field, since the Higgs field is equivalent to a \(\pi_{v,*} \mathcal{O}_{\tilde{X}_R}\) module structure on \(\pi_{v,*} \nu_{R,*} F\), and the latter factors through a \(\pi_{v,*} \nu_{R,*} \mathcal{O}_{\tilde{X}_R}\) module structure. This proves point i), as \(R_s \neq \emptyset \iff n_R \geq 2\).

ii) Let \(F \in \text{Jac}(\tilde{X}_{R,1}) \times \cdots \times \text{Jac}(\tilde{X}_{R,n_R}) \subset \text{Jac}(\tilde{X}_R)\), and assume that \(\tilde{\nu}_k F = F_k\), where the notation is as in [4.6].

Assume first that \(R_s = \emptyset\). We first compute the value of \(\eta\) for \(\tilde{\nu}_R(F)\) to have degree \(\delta = |D|\).

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1}
\caption{Partial normalization along \(R\)}
\end{figure}
From \( (2.7) \), we compute
\[
g(\Delta_R) = g\Delta - |R| + nR - 1.
\]
Since
\[
\chi(\chi, \nu_R(F)) = \chi(\Delta_R, F),
\]
we find that \( \nu_R(F) = \delta \) if and only if \( \deg F = \delta - |R| \). This proves the statement, as in this case \( \eta_R = 1 \) and \( D_1 = D \setminus R \), so \( \eta = \eta_R \).

Now, if \( R_\ast \neq 0 \), from (4.10), it must happen that \( \deg p_i\ast F_i = 0 \) for the Higgs bundle to be semistable.

Given that \( p_i = \pi_i \circ \nu_{R,i} \), and that \( \Delta_R \) is a totally reducible nodal spectral curve with \( |C_i| \) irreducible components, arguing as in Lemma 3.5 (compare with (2.4)) we find that
\[
\deg p_i\ast F_i = 0 \iff \deg \nu_{R,i}\ast F_i = \deg F_i + |R_i|,
\]
we have that
\[
0 \rightarrow \nu_{R,i}\ast \mathcal{O}_{\Delta_{R,i}} \rightarrow \mathcal{O}_{\Delta_R} \rightarrow \mathcal{O}_{R_i} \rightarrow 0,
\]
which together with (4.7) implies the statement.

\section*{4.4. Theorem 4.4.} One has the following: i) The image under the Hitchin map of \( \text{Bor} \) in (1.1) is \( h(\text{Bor}) = V \).

ii) There is an equality of subvarieties \( \text{Bor} \times \mathbb{H} = \mathbb{H} \times \mathbb{H} \).

iii) For every \( v \in \mathbb{H} \), identify \( \text{Jac}\Delta(\Delta_R) \) (for suitable multidegree \( \mathfrak{d} \)) with the corresponding subset of the Hitchin fibre \( h^{-1}(v) \cong \text{Jac}\mathfrak{d}(\mathbb{X}_v) \). Then, if \( R_\ast \neq 0 \), the Higgs bundles corresponding to spectral data in \( \text{Jac}\mathfrak{d}(\Delta_R) \) admit a reduction of their structure group to \( \text{B}_\mathfrak{d} \subset \text{B} \) where \( \text{B}_\mathfrak{d} \) is the Borel subgroup of \( \text{GL}(C_i, \mathbb{C}) \).

iv) Let \( v = (\alpha_1, \ldots, \alpha_n) \in \mathbb{H} \) and let \( (E, \varphi) \in h^{-1}(v) \) be a Higgs bundle whose spectral datum \( L \rightarrow \mathbb{X}_v \) lies in \( \text{Jac}\mathfrak{d}(\mathbb{X}_v) \). For any ordering \( J = (j_1, \ldots, j_n) \) of \( \{1, \ldots, n\} \), and associated ordering \( \mathfrak{c}_J = (\alpha_{j_1}, \ldots, \alpha_{j_n}) \) of \( \{\alpha_1, \ldots, \alpha_n\} \), there exists a filtration
\[
(E_J)_0 : 0 \subset (E_{j_1}, \varphi_{j_1}) \subset \cdots \subset (E_j, \varphi_j) = (E, \varphi),
\]
such that
\[
(E_{j_1}, \varphi_{j_1})/(E_{j_1-1}, \varphi_{j_1-1}) = (\alpha_{j_1}^*, \alpha_{j_1}^* L \otimes K^{i-n}, \alpha_{j_1}),
\]
where the notation is as in (3.5).

v) Given \( v = (\alpha_1, \ldots, \alpha_n) \in \mathbb{H} \), let \( (E, \varphi) = \bigoplus_{i=1}^n (E_i, \varphi_i) \in h^{-1}(v) \cap \text{Jac}\mathfrak{d}(\Delta_R) \) correspond to the line bundle \( L \rightarrow \Delta_R \). Then, for any ordering \( J_k = (j_1, \ldots, j_{|C_k|}) \) of \( C_k \), and associated ordering \( \alpha_{j_k} = (\alpha_{j_1}, \ldots, \alpha_{j_{|C_k|}}) \) of \( \{\alpha_j\}_{j \in C_k} \), \( k = 1, \ldots, n_R \) there exists a filtration
\[
(E_{J_k})_0 : 0 \subset (E_{j_1}, \varphi_{j_1}) \subset \cdots \subset (E_{|C_k|}, \varphi_{j_{|C_k|}}) = (E_k, \varphi_k)
\]
such that
\[
(E_{j_1}, \varphi_{j_1})/(E_{j_1-1}, \varphi_{j_1-1}) = (L|_{\Delta_{j_1}} \otimes \mathcal{O}(- \sum_{k \geq i+1} \Delta_{j_k} \cap \Delta_{j_k}, \alpha_{j_1})
\]
where we abuse notation by identifying the subdivisors \( \Delta_{j_k} \cap \Delta_{j_k} \subset \Delta_k \) and their images under \( p_k \), and \( L|_{\Delta_{j_1}} \) with its pullback under \( \alpha_{j_1} \circ (\nu_{R,i})^{-1} \).
Proof. i) This is a consequence of the following fact: given the Jordan–Chevalley decomposition of \( x = x_s + x_n \in \mathfrak{gl}_n(\mathbb{C}) \) into a semisimple \( x_s \) and a nilpotent piece \( x_n \), the invariant polynomials \( q \), defining the Hitchin fibration evaluate independently of the nilpotent part, namely \( q_i(x) = q_i(x_s) \).

ii) By the universal property of fibered products, we need to find a morphism \( M_n \times \mathbb{V}^{\text{mod}} \rightarrow \text{Bor} \times \mathbb{V}^{\text{mod}} \) making the following diagram commute:

\[
\begin{array}{ccc}
M_n \times \mathbb{V}^{\text{mod}} & \xrightarrow{\pi_2} & \text{Bor} \times \mathbb{V}^{\text{mod}} \\
& \downarrow{\pi_1} & \downarrow{h} \\
M_n & \rightarrow & B.
\end{array}
\]

In other words, it is enough to prove that any Higgs bundle \((E, \varphi) \in M_n \times \mathbb{V}^{\text{mod}}\) admits a full flag decomposition. This is clear from an analysis of the spectral data, as the spectral curve is totally reducible. Indeed, let \( v \in \mathbb{V}^{\text{mod}} \), and assume first that \( L \in \text{Jac}(\overline{X}_v) \) is the spectral datum for \((E, \varphi)\).

Define

\[
(4.11) \quad Y_i := \bigcup_{j=1}^i X_j, \quad Z_i := \bigcup_{k=i+1}^n X_k.
\]

We consider the restriction of \( L \) to \( L|_{Z_i} \), and denote its kernel by \( L_i \),

\[
(4.12) \quad 0 \rightarrow L_i \rightarrow L \rightarrow L|_{Z_i} \rightarrow 0.
\]

Since \( L_i \) is a subsheaf of \( L \), it gives the Higgs subbundle \((E_i, \varphi_i) \subset (E, \varphi)\) under the spectral correspondence. Since \( L_{i-1} \) is a subsheaf of \( L_i \), we have that \((E_{i-1}, \varphi_{i-1}) \subset (E_i, \varphi_i)\) and the existence of the filtration follows for Higgs bundles in \( h^{-1}(v) \cap \text{Jac}^j(\overline{X}_v) \).

Assume next that \((E, \varphi) \in h^{-1}(v) \cap \text{Jac}(\overline{X}_v)\). Since, with the notation of \((4.5)\), \( pr = \pi_v \circ \nu_R \), the same argument as before allows us to conclude.

iii) Follows from ii) above and Lemma \[1.3\]

iv) Let \( J \) be an ordering of \( \{1, \ldots, n\} \), and set \( Y_{J_i} = \bigcup_{k=1}^i X_{j_k}, Z_{J_i} = \bigcup_{k=i+1}^n X_{j_k} \).

Let

\[
0 \rightarrow L_{J_i} \rightarrow L \rightarrow L|_{Z_{J_i}} \rightarrow 0.
\]

Then, reasoning as in Point ii) above we may conclude that the filtration exists.

Note that \( L_{J_i} = L \otimes \mathcal{I}_{X_{J_i},Z_{J_i}} \) where \( \mathcal{I}_{X_{J_i},Z_{J_i}} \) denotes the ideal defining the subscheme \( Z_{J_i} \subset \overline{X} \). Now, \( \mathcal{I}_{X_{J_i},Z_{J_i}} \cong \mathcal{O}_{Y_{J_i}} \otimes \mathcal{I}_{Y_{J_i},Z_{J_i},\cap Y_{J_i}} \), thus

\[
L_{J_i} \cong L|_{Y_{J_i}} \otimes \mathcal{I}_{Y_{J_i},Z_{J_i},\cap Y_{J_i}}.
\]

Note that

\[
0 \rightarrow L_{J_i} \rightarrow L_{J_{i-1}} \rightarrow L|_{Z_{J_{i-1}}} \rightarrow L|_{Z_{J_i}} \rightarrow 0
\]

is exact, so that

\[
L_{J_i}/L_{J_{i-1}} \cong L|_{Z_{J_i}} \otimes \mathcal{I}_{Z_{J_{i-1}},Z_{J_i}} \cong L|_{Z_{J_i}} \otimes \mathcal{O}_{X_{J_i}} \otimes \mathcal{I}_{X_{J_i},Z_{J_i},\cap X_{J_i}} \cong L|_{X_{J_i}} (- \sum_{r=i+1}^n D_{J_i,j_r}).
\]
Now, the pushforward of 
\[ 0 \longrightarrow L_{j_{i-1}} \longrightarrow L_{j_i} \longrightarrow \frac{L_{j_i}}{L_{j_{i-1}}} \longrightarrow 0 \]
gives under the spectral correspondence
\[ (E_i, \varphi_i) / (E_{i-1}, \varphi_{i-1}) \cong \left( \alpha_j^* t_j^* L \left( - \sum_{r=i+1}^{n} D_{j_{r-1}} \right), \alpha_i \right), \]
where we abuse notation by identifying the divisor \( D_{jk} \) and its image under \( \pi \). Naturally, \( K \cong \mathcal{O}_X(D_{jk}) \), which yields the result.

\( v \) To simplify notation, take the orderings \( ((\alpha_1, \ldots, \alpha_{[G]_{C_{ij}}}), \ldots, (\alpha_{[G]_{C_{ik}}}, \ldots, \alpha_n)) \).

The reasoning that follows adapts just the same way to any other choice of orderings. The statement is proven as \( iv \) above, taking the following remarks into account:

1. The subscheme \( Z_i \subset \tilde{X} \) is the image of its partial normalization \( \tilde{Z}_i \subset \tilde{X}_R \), on which the filtration will be given on each of the connected components.
2. This restricts the proof to line bundles over connected curves \( \tilde{X}_R \).
3. So we may assume \( \tilde{X}_R \) is connected and the ordering is \( (\alpha_1, \ldots, \alpha_n) \). We obtain a full flag in the same way, the difference with \( iv \) being that the ideal
   \[ \mathcal{I}_{\tilde{Z}_{i-1}, \tilde{Z}_i} \cong \mathcal{O}_{\tilde{X}_i}(-\tilde{X}_i \cap \tilde{Z}_i) \]
   depends on the ordering (and \( R \)) and so does
   \[ \tilde{X}_i \cap \tilde{Z}_i = \sum_{k \geq i+1} \tilde{X}_i \cap \tilde{X}_k. \]

As a corollary of Theorem 4.4 we obtain the description of the Hitchin fibers intersected with \( \text{Uni}(L) \). Before we state the result we need some extra definitions.

Given \( b \in V \), let \( D \subset \tilde{X}_b \) be the singular divisor, and let \( R \subset D \) be a subdivisor. For each ordering \( J \) of \( \{1, \ldots, n\} \), and each \( i \in \{1, \ldots, n\} \), define the divisors
\[
B_{j_i} = \sum_{j \geq i+1} D_{j, j_k} \cap R, \quad B_{k}^{i,i} = \sum_{j \geq i+1} D_{j, j_k} \cap R.
\]

**Proposition 4.5.** Let \( \tilde{\mathcal{L}} \) be defined as in (4.2). For every \( v \in V^{\text{odd}}, \tilde{\mathcal{J}} \subset \mathbb{Z}^n \), identify \( \text{Jac}^\delta(\tilde{X}_R) \) with the corresponding subset of the Hitchin fibre \( h^{-1}(v) \cong \text{Jac}^\delta(\tilde{X}_v) \).

Then
1. For \( R = \emptyset \)
   
   \[ \text{Uni}(L) \cap \text{Jac}^\delta(\tilde{X}_v) = \left\{ L \in \text{Jac}^\delta(\tilde{X}_v) \text{ such that } \nu^* L = p^* \tilde{\mathcal{L}} \cong \left( \mathcal{L}, \ldots, \mathcal{L} \right) \right\}. \]

   Furthermore, Higgs bundles described in (4.14) are stable.
   
   ii. Suppose that \( R \neq \emptyset \). Then \( \text{Uni}(L) \cap \text{Jac}^\delta(\tilde{X}_R) \) is a subset of
   
   \[
   \left\{ L \in \text{Jac}^\delta(\tilde{X}_R) \left| \left\{ L|_{\tilde{X}_i} : 1 \leq i \leq n \right\} = \left\{ \mathcal{L} \otimes \mathcal{O}(B_{j_i}^{i,i}) : j_i \in J \right\} \text{ for some ordering } J \text{ of } \{1, \ldots, n\} \right\}. \]

   **Proof.**
   
   i. The inclusion \( \subset \) in (4.14) follows from Theorem 4.4 which implies that the spectral datum \( L \) of any \( (E, \varphi) \in \text{Uni}(L) \cap \text{Jac}^\delta(\tilde{X}_v) \) satisfies
   
   \[ \tilde{\mathcal{L}} = \alpha_i^* t_i^* L. \]

Now, since any line bundle on \( \tilde{X}_v \) is totally determined by its restriction to all the connected components, it is enough to check that \( j_i^* p^* \tilde{\mathcal{L}} = j_i^* \nu^* L \), which follows from commutativity of the arrows in (3.5) and the fact that \( \alpha_i : X \to X_i \) is an isomorphism.
All there is left to check is that all line bundles are in the RHS of (4.14) are stable, as by Theorem 4.4 this implies automatically that they are inside Uni(L).

Let \( L \in \text{Jac}^\delta(X_v) \), which by Theorem 2.1 is the spectral data of \((E, \varphi)\), semistable or unstable. We will check that all such line bundles satisfy the strict inequality in (2.5). First of all, any subscheme pure of dimension 1 of \( X \) is of the form

\[
Z_I = \bigcup_{i \in I} X_i, \quad Y_I = \bigcup_{i \in I^c} X_i.
\]

for some \( I \subset \{1, \ldots, n\} \).

Now, \( L \) being a line bundle, it follows that any rank one torsion free quotient of \( L|_{Z_I} \) must be isomorphic to \( L|_{Z_I} \), so it is enough to check that \( L|_{Z_I} \) satisfies the strict inequalities in (2.5), and therefore is stable. This is an easy computation.

\( ii \) With the notation of (4.1), note that \((E, \varphi) \in \text{Uni}(L)\) if and only if for some reduction \( E = E_\sigma(C^n) \) of its structure group to a Borel, then the associated graded bundle \( E_C \cong \bigoplus_{i=1}^n \hat{L} \otimes K^{i-1} \).

Now, once the image under the Hitchin map has been fixed, the above statement is equivalent to the following: \((E, \varphi) \in \text{Uni}(L) \cap h^{-1}(v), v = (\alpha_1, \ldots, \alpha_n) \in \mathbb{A}_n\), if and only if for some ordering \( J \) of \( \{1, \ldots, n\} \), there exists a filtration

\[
0 = (E_0, \varphi_0) \subset (E_1, \varphi_1) \subset \cdots \subset (E_n, \varphi_n) = (E, \varphi)
\]

such that

\[
(E_i, \varphi_i) / (E_{i-1}, \varphi_{i-1}) \cong (\mathcal{L} \otimes K^{i-1}, \alpha_{j_i}).
\]

The statement then follows from Theorem 4.4 noting that

\[
K^{i-1} = \mathcal{O}(B_{J,i}) \otimes \mathcal{O}(B^{J,i})
\]

and that

\[
\nu^*_E \mathcal{O}(B_{J,i}) = \sum_{k \geq i+1} \mathcal{O}(\tilde{X}_{j_k} \cap \tilde{X}_{j_i}).
\]

The description of the spectral data given in Proposition 4.5 allows us to study the dimension of \( \text{Uni}(L) \), which turns up to be one half of \( \dim M_n \).

**Proposition 4.6.** The complex subvariety \( \text{Uni}(L) \) of \( M_n \) has dimension

\[
\dim \text{Uni}(L) = n^2(g-1) + 1 = \frac{1}{2} \dim M_n.
\]

**Proof.** First, we observe that \( \text{Uni}(L) \) is a fibration over \( V \) and recall that \( \dim V = ng \). By Proposition 4.5, over the dense open subset \( V^{\text{nod}} \subset V \), the fibre of \( \text{Uni}(L)|_{V^{\text{nod}}} \to V^{\text{nod}} \) at \( v \) has a dense open subset

\[
\hat{v}^{-1}(\hat{L}, \ldots, \hat{L}) \subset \text{Jac}^\delta(X_v) \cong h^{-1}(v),
\]

where we recall (2.8). Now, by Lemma 2.6

\[
\hat{v}^{-1}(\hat{L}, \ldots, \hat{L}) \cong (\mathbb{C}^\times)^{\delta-n+1}.
\]

By smoothness of the point, the Hitchin fiber is transverse to the (local) Hitchin section, so

\[
\dim \text{Uni}(L)|_{V^{\text{nod}}} = \dim V^{\text{nod}} + \dim \hat{v}^{-1}(\hat{L}, \ldots, \hat{L})
\]

\[
= ng + \delta - n + 1
\]

\[
= ng + (n^2 - n)(g-1) - n + 1
\]

\[
= n^2(g-1) + 1.
\]
which is half of the dimension of $M_n$, as we recall from (2.1). This finishes the proof since by Proposition 4.2 $\text{Uni}(L)$ is isotropic, so its dimension can not be greater than $\frac{1}{2} \dim M_n$. □

Finally, we can state the main result of the section.

**Theorem 4.7.** The complex subvariety $\text{Uni}(L)$ of $M_n$ is complex Lagrangian with respect to $\Omega_1$.

**Proof.** This is clear after Propositions 4.2 and 4.6. □

By Theorem 4.7 we can define the Borel anipotent (BAA)-brane associated to $L$ as the triple given by

$$\begin{align*}
\text{Uni}(L) &:= (\text{Uni}(L), \mathcal{O}_{\text{Uni}(L)}, \nabla),
\end{align*}$$

where $\nabla$ is the trivial connection on $\mathcal{O}_{\text{Uni}(L)}$.

5. Duality

In this section we discuss about the duality under mirror symmetry of the (BBB)-brane $C_{\text{ar}}(L)$, and the (BAA)-brane $\text{Uni}(L)$. Ideally, we would like to transform them under a Fourier–Mukai transform, but such a tool is currently unavailable for coarse compactified Jacobians of reducible curves, which is the situation that we face in this case. We can however define an ad-hoc Fourier–Mukai transform relating the generic loci of both branes. We expect for the weaker form of duality proven here to be induced from the global duality if a full Fourier–Mukai transform relating the generic loci of both branes. We expect for the weaker form of duality proven here to be induced from the global duality if a full Fourier–Mukai transform were to exist, and so it is a hint of the existence of the latter.

Fourier–Mukai transforms for fine compactified Jacobians of reducible curves are studied in [MRV1, MRV2], which is the case that we review in Section 2.4. The construction of the Poincaré sheaf given in (2.9) does not apply to the case of coarse compactified Jacobians, since there is no universal bundle over $[X_\nu \times \text{Jac}^\delta(X_\nu)]$ in this case. We can consider the following bundle parametrizing the sheaves in the Cartan locus,

$$U_{\text{Car}} := (\nu \times \hat{\nu})_* \tilde{U} \rightarrow X_\nu \times \hat{\nu} \left( \text{Jac}_{\hat{\nu}}(X_\nu) \right),$$

where $\tilde{U}$ is the Universal line bundle of multidegree $U$ on $X_\nu$. Also, one can consider the universal line bundle of degree $\delta$ over $X_\nu$, $U^0 \rightarrow X_\nu \times \text{Jac}^\delta(X_\nu)$. Since $\text{Jac}^\delta(X_\nu)$ lies in the stable locus of $[X_\nu \times \text{Jac}^\delta(X_\nu)]$, the existence of $U^0$ follows from the existence of a universal bundle over the stable locus [SII, Theorem 1.21 (4)]. We already have all the ingredients for the following definition, analogous to (2.9), of a Poincaré sheaf over $\hat{\nu} \left( \text{Jac}_{\hat{\nu}}(X_\nu) \right) \times \text{Jac}^\delta(X_\nu)$,

$$\begin{align*}
P_{\text{Car}} := D_{f_{23}} \left( f_{13}^* U_{\text{Car}} \otimes f_{12}^* U^0 \right)^{-1} \otimes D_{f_{23}} \left( f_{13}^* U^0 \right) \otimes D_{f_{23}} \left( f_{12}^* U_{\text{Car}} \right),
\end{align*}$$

where the $f_{ij}$ are the corresponding projections from $X_\nu \times \hat{\nu} \left( \text{Jac}_{\hat{\nu}}(X_\nu) \right) \times \text{Jac}^\delta(X_\nu)$ to the product of the $i$-th and $j$-th factors.

**Remark 5.1.** The Poincaré bundle over fine compactified Jacobians is defined in (2.9) using the universal sheaf $U \rightarrow X_\nu \times \text{Jac}^\delta(X_\nu)$. In the case of coarse compactified Jacobians there is no universal sheaf, but, locally, one can repeat the construction in (2.9) using local universal sheaves $U' \rightarrow U \times \text{Jac}^\delta(X_\nu)$ giving $P' \rightarrow U \times \text{Jac}^\delta(X_\nu)$. It is reasonable to expect that, if a Poincaré bundle $P \rightarrow \text{Jac}^\delta(X_\nu) \times \text{Jac}^\delta(X_\nu)$ over coarse compactified Jacobians were to exist, the restriction of it to $U \times \text{Jac}^\delta(X_\nu)$ would coincide with $P'$. The restriction of any local
universal sheaf to $X_v \times \hat{\nu} \left( \text{Jac}^\delta(\tilde{X}_v) \right) \subset X_v \times \text{Jac}^\delta(\tilde{X}_v)$ would be isomorphic to $U^{\text{Car}}$ and this justifies the construction of $P^{\text{Car}}$.

Using $P^{\text{Car}}$ we construct a Fourier–Mukai integral functor analogous to (2.11),

\begin{equation}
(5.2) \quad \Phi^{\text{Car}} : D^b\left( \hat{\nu} \left( \text{Jac}^\delta(\tilde{X}_v) \right) \right) \rightarrow D^b\left( \text{Jac}^\delta(\tilde{X}_v) \right)
\end{equation}

where $\pi_1$ and $\pi_2$ to be, respectively, the projection from $\hat{\nu} \left( \text{Jac}^\delta(\tilde{X}_v) \right) \times \text{Jac}^\delta(\tilde{X}_v)$ to the first and second factors.

Recall that our (BBB)-brane $\text{Car}(L)$ is given by the hyperholomorphic bundle $L$ supported on $\text{Car}$. By Proposition 3.7, over the dense open subset $V^{\text{nod}}$ of the Cartan locus of the Hitchin base $V = h(\text{Car}) \subset H$, the hyperholomorphic sheaf $L$ restricted to a certain Hitchin fibre $\text{Jac}^\delta(\tilde{X}_v)$ is $\hat{\nu}_*L_{\tilde{\mathcal{E}}_n}$, supported on $\hat{\nu}(\text{Jac}^\delta(\tilde{X}_v))$. The main result of this section is the study of the behaviour of $\hat{\nu}_*(L_{\tilde{\mathcal{E}}_n})$ under $\varphi^{\text{Car}}$, but first we need some technical results.

Fix $x_0$ and take the line bundle $O(x_0)^{(n-1)(g-1)}$. Denote

$$\tau : \text{Jac}^\delta(\tilde{X}) \xrightarrow{\cong} \text{Jac}^\delta(\tilde{X})$$

the isomorphism given, on each of the components, by tensorization by the previous line bundle. We can define a Poincaré bundle $\tilde{P} \rightarrow \text{Jac}^\delta(\tilde{X}_v) \times \text{Jac}^\delta(\tilde{X}_v)$.

Consider the projections to the first and second factors

$$\text{Jac}^\delta(\tilde{X}_v) \times \text{Jac}^\delta(\tilde{X}_v)$$

and, using $\tilde{P}$, one can construct another Fourier–Mukai integral functor

$$\tilde{\Phi} : D^b(\text{Jac}^\delta(\tilde{X}_v)) \rightarrow D^b(\text{Jac}^\delta(\tilde{X}_v))$$

$$\mathcal{E}^\bullet \quad \rightarrow \quad R\pi_{2,*}(\tilde{\pi}_1^*\mathcal{E}^\bullet \otimes \tilde{P}).$$

Note that $\tilde{\Phi}$ is governed by the usual Fourier–Mukai transform on each of the $\text{Jac}^\delta(X_v)$. We need the following lemma in order to describe the interplay between $\Phi^{\text{Car}}$ and $\tilde{\Phi}$.

**Lemma 5.2.** One has that

$$(\hat{\nu} \times 1_{\text{Jac}})^* P^{\text{Car}} \cong (1_{\text{Jac}} \times \hat{\nu})^* \tilde{P}.$$
and consider the following commuting diagram

\[
\begin{array}{ccc}
\tilde{X}_v \times \text{Jac}^\delta(\tilde{X}_v) & \xrightarrow{\nu \times f^\delta} & \tilde{X}_v \times \text{Jac}^\delta(\tilde{X}_v) \\
\downarrow f_2 & & \downarrow f_2 \\
\text{Jac}^\delta(\tilde{X}_v) & \xrightarrow{\nu} & \text{Jac}^\delta(\tilde{X}_v)
\end{array}
\]

We know from [ES, Proposition 44 (1)] that the determinant of cohomology \(\nu \times f^\delta \cong f_2\), i.e.

\[
\tilde{U}^\nu \cong \tilde{U}^\nu \cong \tilde{U}^\nu.
\]

The additive property of the determinant of cohomology [ES, Proposition 44 (4)], says that, whenever we have an exact sequence as in (5.3),

\[
0 \rightarrow \mathcal{U}_0^\text{Car} \rightarrow \mathcal{U}^\text{Car} \rightarrow (\mathcal{U}^\text{Car} / \mathcal{U}_0^\text{Car}) \rightarrow 0,
\]

where \((\mathcal{U}^\text{Car} / \mathcal{U}_0^\text{Car}) \rightarrow \tilde{X}_v \times \tilde{\nu} \left(\text{Jac}^\delta(\tilde{X}_v)\right)\) is a family of sky-scraper sheaves on \(\tilde{X}_v\) supported on \(\text{sing}(\tilde{X}_v)\), and

\[
(\nu \times \tilde{\nu})^\text{Car} \cong \tilde{U}.
\]

Using these properties, we can show that

\[
\tilde{P}_{\nu(\tilde{\nu})} \cong \nu^* \mathcal{P}^\text{Car}_{f_2}
\]

\[
\cong \nu^* \left(\mathcal{D}_{f_2} (\mathcal{U}^\text{Car} \otimes f_1^* J)^{-1} \otimes \mathcal{D}_{f_2} (f_1^* J) \otimes \mathcal{D}_{f_2} (\mathcal{U}^\text{Car})\right)
\]

\[
\cong \nu^* \mathcal{D}_{f_2} (\mathcal{U}^\text{Car} \otimes f_1^* J)^{-1} \otimes \nu^* \mathcal{D}_{f_2} (f_1^* J) \otimes \nu^* \mathcal{D}_{f_2} (\mathcal{U}^\text{Car})
\]

\[
\cong \nu^* \mathcal{D}_{f_2} (\mathcal{U}^\text{Car} \otimes f_1^* J)^{-1} \otimes \nu^* \mathcal{D}_{f_2} (\mathcal{U}^\text{Car} / \mathcal{U}_0^\text{Car})^{-1} \otimes \nu^* \mathcal{D}_{f_2} (f_1^* J) \otimes \nu^* \mathcal{D}_{f_2} (\mathcal{U}^\text{Car} / \mathcal{U}_0^\text{Car})
\]

\[
\cong \mathcal{D}_{f_2} (\mathcal{U}_0^\text{Car} \otimes f_1^* J)^{-1} \otimes \mathcal{D}_{f_2} (\mathcal{U}^\text{Car} / \mathcal{U}_0^\text{Car})^{-1} \otimes \mathcal{D}_{f_2} (f_1^* J) \otimes \mathcal{D}_{f_2} (\mathcal{U}^\text{Car} / \mathcal{U}_0^\text{Car})
\]

Then, \(\nu = \tilde{\nu}\) and this finish the proof. \(\square\)

We can now study the image of \(\tilde{\nu}_*(\mathcal{P}^\text{Bar})\) under (5.2).
Proposition 5.3. One has the isomorphism

\[ \Phi^{\text{Car}}(\tilde{\nu}_*(\mathcal{L}^{2g})) \cong \tilde{\nu}^* \tilde{\Phi}(\mathcal{L}^{2g}), \]

and furthermore, \( \tilde{\nu}^* \tilde{\Phi}(\mathcal{L}^{2g}) \) is a complex supported on degree \( g \) given by \( \tilde{\nu}^* \mathcal{O}(\mathcal{L}^{2g}) \).

Proof. Let us also consider the following maps

\[
\begin{align*}
\text{Jac}^\pi(X_v) & \xrightarrow{\pi_1'} \text{Jac}^\pi(X_v) \\
\text{Jac}^\pi(X_v) & \xrightarrow{\pi_2'} \text{Jac}^\pi(X_v),
\end{align*}
\]

and observe that

\begin{itemize}
  \item \( \pi_1' = \pi_2 \circ (\tilde{\nu} \times 1_{\text{Jac}}) \),
  \item \( \pi_1' = \tilde{\pi}_1 \circ (1_{\text{Jac}} \times \tilde{\nu}) \),
  \item \( \pi_1' \circ (\tilde{\nu} \times 1_{\text{Jac}}) = \tilde{\nu} \circ \pi_1' \), and
  \item \( \tilde{\pi}_2 \circ (1_{\text{Jac}} \times \tilde{\nu}) = \tilde{\nu} \circ \pi_2' \).
\end{itemize}

Recalling Lemma 5.2 that \( \tilde{\nu} \) is an injection and that \( \tilde{\nu} \) is flat by Lemma 2.6 one has the following,

\[
\begin{align*}
\Phi^{\text{Car}}(\tilde{\nu}_*(\mathcal{L}^{2g})) & = R\pi_{2,*} \left( \pi_1^* \tilde{\nu}_*(\mathcal{L}^{2g}) \otimes \mathcal{P}^{\text{Car}} \right) \\
& \cong R\pi_{2,*} \left( (\tilde{\nu} \times 1_{\text{Jac}})_*(\pi_1')^*(\mathcal{L}^{2g}) \otimes \mathcal{P}^{\text{Car}} \right) \\
& \cong R\pi_{2,*} R(\tilde{\nu} \times 1_{\text{Jac}})_*(\pi_1')^*(\mathcal{L}^{2g}) \otimes (\tilde{\nu} \times 1_{\text{Jac}})^* \mathcal{P}^{\text{Car}} \\
& \cong R\pi_{2,*} R(\tilde{\nu} \times 1_{\text{Jac}})_* \left( (\pi_1')^*(\mathcal{L}^{2g}) \otimes (1_{\text{Jac}} \times \tilde{\nu})^* \mathcal{P} \right) \\
& \cong R\pi_{2,*} \left( (\pi_1')^*(\mathcal{L}^{2g}) \otimes (1_{\text{Jac}} \times \tilde{\nu})^* \mathcal{P} \right) \\
& \cong R\pi_{2,*} \left( (1_{\text{Jac}} \times \tilde{\nu})^* \tilde{\pi}_1^* (\mathcal{L}^{2g}) \otimes (1_{\text{Jac}} \times \tilde{\nu})^* \mathcal{P} \right) \\
& \cong R\pi_{2,*} \left( (1_{\text{Jac}} \times \tilde{\nu})^* \tilde{\pi}_1^* \mathcal{P} \right) \\
& \cong \tilde{\nu}^* \tilde{\Phi}(\mathcal{L}^{2g}).
\end{align*}
\]

Finally, recalling that the usual Fourier–Mukai transform on \( \text{Jac}^0(X) \times \text{Jac}^{\delta/n}(X) \) sends the line bundle \( \mathcal{L} \) to the (complex supported on degree \( g \) given by) sky-scraper sheaf \( \mathcal{O}_\mathcal{L} \), we have that \( \Phi^{\text{Car}}(\tilde{\nu}_*,\mathcal{L}^{2g}) \) is (the complex supported on degree \( g \) given by)

\[
\tilde{\nu}^* \tilde{\Phi}(\mathcal{L}^{2g}) \cong \tilde{\nu}^* \mathcal{O}(\mathcal{L}^{2g}),
\]

and the proof is complete. \( \square \)

Recalling Proposition 5.5, we arrive to the main result of the section, which shows that our (BBB)-brane \( \text{Car}(\mathcal{L}) \) and our (BAA)-brane \( \text{Uni}(\mathcal{L}) \) are related under the Fourier–Mukai integral functor \( \Phi^{\text{Car}} \).

Corollary 5.4. For every \( v \in V^{\text{mod}} \), the support of the image under \( \Phi^{\text{Car}} \) of the (BBB)-brane \( \text{Car}(\mathcal{L}) \) restricted to a Hitchin fibre \( h^{-1}(v) \), is the support of our (BAA)-brane \( \text{Uni}(\mathcal{L}) \) restricted to the open subset of the (dual) Hitchin fibre given by the locus of invertible sheaves,

\[
\text{supp}(\Phi^{\text{Car}}(\tilde{\nu}_*(\mathcal{L}^{2g}))) = \text{Uni}(\mathcal{L}) \cap \text{Jac}^\delta(\bar{X}_v).
\]

In view of Corollary 5.4 we conjecture the following.
Conjecture 1. The branes $\text{Car}(\mathcal{L})$ and $\text{Uni}(\mathcal{L})$ are dual under mirror symmetry.

6. Parabolic subgroups and branes on the singular locus

Cartan branes are the simplest example of branes supported on the singular locus of the moduli space $M_n^{\text{sing}}$. In this section we construct hyperholomorphic and Lagrangian subvarieties covering the singular locus, and study their spectral data.

6.1. Levi subgroups and the singular locus. Let $L < \text{GL}(n, \mathbb{C})$ be a maximal rank reductive subgroup. Then $L$ is conjugate to $L_r := \text{GL}(r_1, \mathbb{C}) \times \cdots \times \text{GL}(r_s, \mathbb{C})$ where $\sum_{i=1}^{s} r_i = n$, $0 < r_1 \leq \cdots \leq r_s$. Denote by $M_r \subset M_n$ the image of the moduli space $M_{1,r}$ of $L_r$-Higgs bundles. The same arguments as in the case of Cartan subgroups shows that this is a complex subscheme in all three complex structures.

Remark 6.1. In particular, $\text{Car} = M_1(1,\ldots,1)$.

Note that $M_n^{\text{sing}} \subset \bigcup_{\sum r_i = n} M_r,$ as the singular locus lies in the locus of strictly semistable bundles [Si2, Section 11].

Now, if $(r_1, \ldots, r_s)$ and $(l_1, \ldots, l_m)$ are such that for all $j = 1, \ldots, m$ there is an $n_j$ such that $\sum_{i=1}^{j} r_i = \sum_{k=1}^{l_j} l_j$, then $M_r \subset M_l$. In particular $M_n^{\text{sing}} \subset \bigcup_{\sum r_i = n} M_r = \bigsqcup_{r_1 \leq r_2, r_1 + r_2 = n} M_{(r_1, r_2)}$.

Fix $r$, and consider $M_r \subset M_n$. Observe that this manifold is complex in all three complex structures $\Gamma_1, \Gamma_2, \Gamma_3$, and therefore hyperholomorphic.

Consider the restriction of the Hitchin map $h_r : M_r \to H_r,$

where $H_r \subset H =: H_n$ is the image under the Hitchin map of $M_r$. Note that $H_r$ is the image of the morphism $H_r = H_{r_1} \times \cdots \times H_{r_s}$

$(b_1, \ldots, b_s) \mapsto b_i,$

where $H_r$ denotes the Hitchin base for $\text{GL}(r, \mathbb{C})$-Higgs bundles, and if $b_i = (b_{i1}, \ldots, b_{ir_i})$, $b_{ij} \in H^0(X, K^j)$, then

$$b = \left( \sum_{i=1}^{s} b_{i1}, \ldots, \sum_{\sum_{k=1}^{l_i} k = l} b_{ik}, \ldots, \prod_{l} b_{i,r_i} \right).$$

This implies that for $b = (b_1, \ldots, b_s) \in H_r$, the corresponding spectral curve $\overline{X}_b$ has at least $s$ irreducible components $\overline{X}_{b_{i1}}, \ldots, \overline{X}_{b_{is}}$, which are in turn spectral curves for $b_i \in H_{r_i}$. Denote by

$$\overline{X}_{b_{i1}} \overset{i_{b_i}}{\leftarrow} \overline{X}_b \overset{\pi_b}{\rightarrow} \overline{X}.$$

We consider the nodal locus $H_n^{\text{nod}} \subset H_r$, consisting of generic points corresponding to spectral curves with exactly $s$ irreducible components and nodal singularities.
Let $b \in H_{\Gamma}$. Then $D_{ij} = X_{b_i} \cap X_{b_j}$ is a divisor of degree $2r_ir_j(g-1)$. If moreover $b \in H_{\Gamma}^{\text{red}}$, the divisor consists of simple points and the normalization of $X_b$ is $\nu_b : \tilde{X}_b = \overline{X_{b_1} \sqcup \cdots \sqcup X_{b_s}} \to X_b$.

Proof. To see the first statement, deform the plane curve $X_{b_j}$ to $X^{r_i} = 0$. Then, the intersection with $X_{b_j}$ is the vanishing locus of a section of $\pi_{b_i}^*K^{r_j}$.

The second statement is obvious.

The following proposition is proved as Proposition 3.7

**Proposition 6.3.** Let $b \in H_{\Gamma}^{\text{red}}$, and let $\delta_i = (r_i^2 - r_i)(g-1)$. Then

$$h^{-1} = \nu_* \text{Jac}^\delta(\tilde{X}_b),$$

where $\delta = (\delta_1, \ldots, \delta_s)$.

### 6.2 Parabolic subgroups and complex Lagrangian subvarieties

In this section we define another submanifold $\text{Uni}^\gamma(E) \subset M_n$ associated to the choice of a parabolic subgroup $P$ whose Levi subgroup is $L$, as well as some vector bundles $E = (E_1, \ldots, E_s)$ on $X$ with $\text{rk} E_i = r_i$.

Let $P = L \ltimes U$ be a parabolic subgroup containing $L$, where $U = [P, P]$ is its unipotent radical. Denote by

$$\text{Par} = \left\{ (E, \varphi) \in M_n \mid \exists \sigma \in H^0(X, E/P), \varphi \in H^0(X, E_{\sigma}(p) \otimes K) \right\},$$

and let

$$\text{Uni}^\gamma(E) = \left\{ (E, \varphi) \mid \exists \sigma \in H^0(X, E/P) : \varphi \in H^0(X, E_{\sigma}(p) \otimes K) : E_{\sigma}/U = E_1 \cong \bigoplus_{i=1}^s E_i. \right\}$$

In what follows we prove that under Assumption 1 below, $\text{Uni}^\gamma(E)$ is a Lagrangian submanifold. This is proven through the study of the associated spectral data.

Consider restriction of $h_\gamma$ to $\text{Uni}^\gamma(E)$,

$$h^\gamma(E) : \text{Uni}^\gamma(E) \to H_{\Gamma}.$$

The fact that the image is contained in $H_{\Gamma}$ can be argued as in Theorem 4.3.

**Assumption 1.** Let $\deg E_i = e_i$. Define $d_i = e_i + (r_i^2 - r_i)(g-1)$. Given an ordering $J = (j_1, \ldots, j_s)$ of $\{1, \ldots, s\}$, let

$$d_i^J = d_i + 2R_i^J(g-1)$$

where $R_i^J = \sum_{k \geq i+1} r_j r_{jk}$.

Assume that the following holds:

i) There exists an ordering $J$ such that for all $I \subset \{1, \ldots, s\}$, there are inequalities

$$\sum_{i \in I} d_i^J > (r_i^2 - r_i)(g-1), \quad \sum_{i=1}^s d_i^J = (n^2 - n)(g-1),$$

where $r_I = \sum_{i \in I} r_i$.

ii) For all $(b_1, \ldots, b_s) \in H_{\Gamma}^{\text{red}}$, there exists

$$\mathcal{L}_{b_i} \in \text{Pic}^{d_i}(X_{b_i})$$

such that $\pi_{b_i}^* \mathcal{L}_{b_i} = E_i$. 


Remark 6.4. As an example of a situation in which Assumption \[1\] is satisfied, consider an \(s\)-tuple of natural numbers \(m_1, \ldots, m_s\) such that for all \(I \subseteq \{1, \ldots, s\}\),

\[
\sum_{i \in I} r_i m_i > (r_I^2 - r_I)(g - 1), \quad \sum_{i=1}^s r_i m_i = (n^2 - n)(g - 1).
\]

Let \(J = \{1, \ldots, s\}\), and let

\[
E_i = K^{-R_i} O(m_i)(O \oplus K^{-1} \oplus \ldots K^{-r_i+1}).
\]

Then, \(d_i^J = r_i m_i\), and so by (6.6) we see that Assumption \[1\(i\)\] holds.

On the other hand, the bundle

\[
\hat{L}_{b_i} = \pi^*_b(K^{-R_i} \otimes O(m_i)) \in \text{Pic}^{d_i}(\mathcal{X}_n)
\]

pushes forward to \(E_i\), so Assumption \[1\(ii\)\] also holds.

Note that if \(r_i = r\) for all for all \(i = 1, \ldots, s\), (6.5) holds, and so Assumption \[1\] is non-empty.

In order to state the equivalent to Theorem 4.4, some extra care is needed, as the fact that the integers \(r_i\) are different, breaks the symmetry we have in the case of Borel groups.

Proposition 6.5. One has the following: i) The image under the Hitchin map of \(\text{Par in } \overset{[1]}{\text{Assumption}}\) \(\overset{[2]}{\text{is } h(\text{Par}) = H^r}\).

ii) There is an equality of subvarieties \(\text{Par} \times_H H^\text{mod} = M_n \times_H H^\text{mod}\).

iii) Let \(b = (b_1, \ldots, b_s) \in H^\text{mod}\) and let \((E, \varphi) \in h^{-1}(\nu)\) be a Higgs bundle whose spectral data \(L \to \mathcal{X}_n\) lies in \(\text{Jac}^0(\mathcal{X}_n)\). For any ordering \(J = (j_1, \ldots, j_s)\) of \(\{1, \ldots, s\}\) there exists a filtration

\[
(E_J)_*: 0 \subsetneq (E_{j_1}, \varphi_{j_1}) \subsetneq \cdots \subsetneq (E_{j_s}, \varphi_{j_s}) = (E, \varphi),
\]

such that

\[
(E_{j_i}, \varphi_{j_i})/(E_{j_{i-1}}, \varphi_{j_{i-1}}) = (\pi^*_{b_{j_i}}L|_{\mathcal{X}_{j_i}} \otimes K^{-R_i', \pi^*_{b_{j_i}}/\varphi_{j_{i-1}}}),
\]

where \(R_i' = \sum_{k \geq i+1} r_{j_k} v_{j_k}\).

Before we can prove the analogues to Proposition 4.5, we need a lemma.

Assume that \(\mathcal{E}\) satisfies Assumption \[1\]. Let \(b \in H^\text{mod}\), and let \(\tilde{L} = (\hat{L}_b, \ldots, \hat{L}_b)\) be as in Assumption \[1\(ii\)\]. Given an ordering \(J\), set

\[
\tilde{L}_J = \hat{L} \otimes (\pi^*_b K^{R_i'}, \ldots, \pi^*_b K^{R_i'}).
\]

Proposition 6.6. Assume that \(\mathcal{E}\) satisfies Assumption \[4\(i\)\]. Let \(b \in H^\text{mod}\), and let \(\text{Ord}_n\) denote the set of orderings of \(\{1, \ldots, s\}\). For each \(J \in \text{Ord}_n\), let \(\tilde{J}\) be as in (6.6). Then, \(h^r(\mathcal{E})^{-1}(b) \cap \text{Jac}^0(\mathcal{X}_b)\) is either empty or

\[
h^r(\mathcal{E})^{-1}(b) \cap \text{Jac}^0(\mathcal{X}_b) = \bigcup_{J \in \text{Ord}_n} \nu^{-1}(\tilde{L}_J)
\]

where we identify \(\text{Jac}^0(\mathcal{X}_b)\) with an open subset of \(h^{-1}(b)\) and define

\[
\nu : \text{Jac}^0(\mathcal{X}_b) \to \text{Jac}^0(\mathcal{X}_b)
\]

to be the pullback map.

Proof. After checking that (6.3) ensures the stability of the points of \(h^r(\mathcal{E})^{-1}(b)\), the proof follows as in Proposition 4.5. □
Theorem 6.7. Under Assumption 1, the subscheme $\text{Uni}^r(E)$ is Lagrangian.

Proof. By Proposition 6.6, there are smooth points in $\text{Uni}^r(E)$, after which isotropy is proved as we did in Proposition 4.2.

By Lemma 6.2, there is an exact sequence

$$0 \rightarrow C^s \rightarrow (C^s)^s \rightarrow (C^s)^s \rightarrow \text{Jac}(X_b) \rightarrow \text{Jac}(%$$

where $\delta_r = \sum_{1 \leq i < j \leq s} 2r_ir_j(g-1)$. By Assumption 1

$H_{\text{mod}} \subset H^r(\text{Uni}^r(E))$, and since by Proposition 6.6 there are smooth points in $\text{Uni}^r(E)$, it follows that the dimension is

$$\dim \text{Uni}^r(E) = \delta_r - s + 1 + \dim H_r = \delta_r - s + 1 + \sum_i (r_i^2(g-1)+1),$$

which is half of the dimension of $\text{M}_n$.

Remark 6.8. For the sake of clarity, we have chosen to work with the moduli space of degree 0 Higgs bundles. Note however that the subvarieties $M_r$ and $\text{Uni}^r(E)$ also make sense in the context of $\text{M}_{X,n,sd}$, the moduli space of rank $n$, degree $sd$ Higgs bundles, for any integer $d$, and where $s$ is the length of the tuple $\mathfrak{r}$. The (semi)stability condition for torsion free sheaves should then be modified:

$$\deg Z F > sdn_Z - (n^2n_Z - n_Z^2n)(g-1) \quad (\text{resp. } \geq),$$

where $n_Z = \text{rk}(\pi_* O_Z)$.

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