CORRECTION OF OVERFITTING BIAS IN REGRESSION MODELS

BY EMANUELE MASSA\textsuperscript{1}, MARIANNE A. JONKER\textsuperscript{2,†}, KIT C.B. ROES\textsuperscript{2,‡}, AND ANTHONY C.C. COOLEN\textsuperscript{1,3,*}

\textsuperscript{1}Department of Biophysics, Donders Institute, Radboud University, The Netherlands emanuele.massa@donders.ru.nl; \textsuperscript{*}a.coolen@science.ru.nl
\textsuperscript{2}Section Biostatistics, Department for Health Evidence, Radboud UMC, The Netherlands \textsuperscript{†}marianne.jonker@radboudumc.nl; \textsuperscript{‡}kit.roes@radboudumc.nl
\textsuperscript{3}Saddle Point Science Europe BV, Nijmegen, The Netherlands

Regression analysis based on many covariates is becoming increasingly common. When the number \( p \) of covariates is of the same order as the number \( N \) of observations, statistical inference like maximum likelihood estimation of regression and nuisance parameters in the model becomes unreliable due to overfitting. This overfitting most often leads to systematic biases in (some of) the estimators. In the literature, several methods to overcome overfitting bias or to adjust estimates have been proposed. The vast majority of these focus on the regression parameters only, either via empirical regularization methods or by expansion for small ratios \( p/N \). This failure to correctly estimate also the nuisance parameters may lead to significant errors in outcome predictions. In this paper we study the overfitting bias of maximum likelihood estimators for regression and the nuisance parameters in parametric regression models in the overfitting regime (\( p/N < 1 \)). We compute the asymptotic characteristic function of the maximum likelihood estimators, and show how it can be used to estimate their overfitting biases by solving a small set of non-linear equations. These estimated biases enable us to correct the estimators and make them asymptotically unbiased. To illustrate the theory we performed simulation studies for multiple parametric regression models. In all cases we find excellent agreement between theory and simulations.

1. Introduction.

When the number of parameters included in a statistical model is large compared to the number of observations, the noise (or residual model variance) is wrongly interpreted as part of the underlying model. This is called overfitting \cite{2, 19, 24}. An overfitting model may appear to perform well, but will fail when used to predict outcomes for new data. In fact, this happens already when the number of parameters is a moderate fraction of the number of observations \cite{7, 9, 33}. In modern applications, regression models frequently include a large number of regression parameter, which is why correction for overfitting in regression models has become an important research topic \cite{5, 7, 9, 20, 24, 36}.

In this paper we study the relationship between a set of \( p \) covariates or predictor variables \( X \in \mathbb{R}^p \) and a response variable \( T \in \mathbb{R} \), described by a parametric density of the form
\begin{equation}
 p(t|X^T\beta, \sigma).
\end{equation}
In densities of this form, the regression parameters \( \beta \in \mathbb{R}^p \) appear as coefficients in the linear predictor \( X^T\beta \), and \( \sigma \in \mathbb{R}_+^s \) are the nuisance parameters. To estimate the relationship we have a random sample of \( N \) independently drawn observations \((T_1, X_1), \ldots, (T_N, X_N)\) with

\textit{MSC2020 subject classifications:} Primary overfitting bias, high dimensional statistics; secondary statistical physics.

\textit{Keywords and phrases:} overfitting bias, High Dimensional statistics.
Fig 1: Scatter plot of ML-inferred regression parameters $\{\hat{\beta}_k\}_{k=1}^p$ versus the true regression parameters $\{\beta_{0,k}\}_{k=1}^p$ for the Weibull model. Left panel: $\zeta = p/N = 0.01$; right panel: $\zeta = 0.45$. For sufficiently large values of $\zeta$, the overfitting bias is visible in a deviation between the least squares regression line (in blue) and the diagonal (in red), as in the right panel. To ensure that the scatter plots for different values of $p$ contain the same number of points, the number of simulations is chosen to be $M_p = 100N/p$.
CORRECTION OF OVERFITTING BIAS IN REGRESSION MODELS

Fig 2: Figures (a) and (b): slope $\kappa$ and width $\Delta$ computed via least squares from the scatter plots as in Figure 1, plotted against $\zeta$. Figures (c) and (d): ratios of average inferred and true values of the nuisance parameters $\lambda$ and $\rho$, shown as functions of the ratio $\zeta$. In panels (c) and (d) the error bars represent 95% confidence intervals, based on the normal approximation for the distribution of the mean. Error bars in (a) and (b) represent 95% confidence intervals, based on the asymptotic normality of the ML estimator. Averages are computed over $M_p = N100/p$ datasets each of sample size $N = 200$.

In Figure 1 we present the results for $\zeta = p/N = 0.01$ (i.e. $p = 2$) and for $\zeta = 0.45$ (i.e. $p = 90$). The red lines in each panel denote the diagonal, and the blue lines denote the least squares regression lines. These two lines are seen to coincide in Figure 1(a), which indicates that on average for $\zeta = 0.01$ the estimates of the regression parameters are roughly unbiased. In contrast, in Figure 1(b) the (blue) least squares regression line for $\zeta = 0.45$ clearly has a much larger slope than the perfect regression line (red). This discrepancy signals that when $\zeta$ is sufficiently large, the ML estimator is seriously biased.

In Figures 2 (a,b) we plot the slope $\kappa$ and the width $\Delta$ of the clouds such as those shown in Figure 1 against $\zeta = p/N$, but now for data generated for different values of $p$. We conclude that both the bias and the width of the distribution of inferred parameters $\hat{\beta}_{N,k}$ increase with the ratio $\zeta$. In Figures 2 (c,d) we show the ratios $\langle \hat{\lambda}_N \rangle / \lambda$ and $\langle \hat{\rho}_N \rangle / \rho$ of the average and true values of the nuisance parameters, where the averages $\langle \ldots \rangle$ are computed over regression outcomes from $M_p$ independently generated data sets. We deduce that also the ML estimators $\hat{\rho}_N$ and $\hat{\lambda}_N$ are biased, and that this bias increases with the ratio $\zeta$.

In literature one finds a variety of rules of thumb for the maximum number of covariates to prevent overfitting. For instance, for Cox regression the guideline of a maximum of one covariate per 10 to 15 events is often advised [19, 33]. However, simulation studies have
shown that one should be careful in applying such rules [34, 36, 37]. Moreover, in some research fields, like medicine, the number of observations is often small and models may be complex, which makes such restriction on the number of covariates unacceptable. One way to address the overfitting bias observed in Figure 1 would be to shrink the regression coefficients after estimation, i.e. by multiplying them by a uniform shrinkage factor [10, 36]. The shrinkage factor may be based on heuristic formulae [10] or can be estimated via bootstrapping [36]. Shrinkage can also be performed during the estimation procedure by a penalized maximum likelihood method [5, 19, 36]; a penalty function is added to the log likelihood before maximization. This is mathematically equivalent to Bayesian Maximum A Posteriori (MAP) regression [16].

There exists an interesting analogy between statistical physics and optimization [26, 27, 28, 31] which enables tools from statistical physics to be applied to problems in statistical inference [3, 13, 15, 22, 38]. Unfortunately, in the overfitting regime also most of the latter papers focus on estimators for the regression parameters, assuming the nuisance parameters to be known. In contrast, the studies [7, 8, 9] used statistical physics methods to estimate the statistical relation between ML-inferred and true parameters, including regression and nuisance parameters, for the Cox model, logistic regression, and generalized linear models (GLM), in the overfitting regime.

In this paper we derive the general formalism of [9] for regression models of the form (1), but here we follow an alternative and more statistical route. In addition we apply the theory to several new regression models. We find that overfitting always biases the estimates of the nuisance parameters, even when the regression parameters are asymptotically unbiased. With our theory we can correct all ML parameter estimators so that they will be asymptotically unbiased. The beauty of the theory in [9] lies in its simplicity when it comes to applications: its end result takes the form of a finite set of nonlinear equations, valid in the limit \( N \to \infty \) with \( \zeta = p/N \) finite. One needs only specify the statistical model assumed responsible for the available data, and solve these equations. For linear models with arbitrary noise distributions the ML estimators for the model parameters \((\beta, \sigma)\) are computed by maximizing the log-

\[
\log p(T_i|X_i^T \beta, \sigma)
\]

In Section 2 we present the general theory for estimating the asymptotic bias in GLMs and show how it simplifies in the case of linear models with arbitrary additive noise distributions. Section 3 deals with novel applications of the theory to three specific models: the Weibull proportional hazards model, the Log-Logistic model and the Exponential model with a Gamma random effect term [14, 25]. We show how our theory can be used to correct the ML estimates for the overfitting bias, and we confirm our results with simulations. We end the paper with a discussion in Section 4 of our results and future directions of investigation. All mathematical derivations are delegated to appendices.

2. Mathematical analysis of overfitting.

The ML estimates for the model parameters \((\beta, \sigma)\) are computed by maximizing the log-

\[
l_N(\beta, \sigma) := \sum_{i=1}^{N} \log p(T_i|X_i^T \beta, \sigma)
\]
We have seen that for $\zeta = p/N > 0$ overfitting leads to biases in these estimates. Here we show that these biases can be estimated and hence removed. To do so, we compute the characteristic function of the ML estimators in the regime where $p, N \to \infty$ with fixed ratio $0 \leq \zeta < 1$. In order to retain finite response variables in the above regime, the data generating model must have $\|\beta_0\|_2 = O(1)$. We will therefore use

$$\hat{\theta}_N = \sqrt{p} \hat{\beta}_N$$

which insures that typically $\hat{\theta}_{N,k} = O(1)$ for all $k$. Henceforth we use the convention that the limits $N \to \infty$ and $p \to \infty$ always are taken simultaneously, subject to $p/N = \zeta$.

2.1. Theory of overfitting in regression models.

**THEOREM 2.1** (Asymptotic characteristic function of the ML estimator). Let

$$\Phi_{\hat{\theta}_N, \hat{\sigma}_N}(q, s) := E_{(T, X)^N}\left[e^{i q^T \hat{\theta}_N + i s^T \hat{\sigma}_N}\right]$$

be the characteristic function of $(\hat{\theta}_N, \hat{\sigma}_N) = (\sqrt{p} \hat{\beta}_N, \hat{\sigma}_N)$, where $(\hat{\beta}_N, \hat{\sigma}_N)$ is the ML estimator computed by maximizing the log-likelihood $l_N(\beta, \sigma)$ in (3) with respect to $\beta$ and $\sigma$, and were the covariates are independently distributed with finite variance. Suppose that $l_N(\beta, \sigma)$ is twice continuously differentiable with respect to $\beta$ and $\sigma$, and convex on $\mathbb{R}^p \times \mathbb{R}^s$.

Assume furthermore that the vector $q$ is sparse (i.e. with only a finite number $d'$ of non-zero components, such that $d'/N \to 0$) then

$$\Phi_{\hat{\theta}, \hat{\sigma}}(q, s) := \lim_{N \to \infty} \Phi_{\hat{\theta}_N, \hat{\sigma}_N}(q, s) = e^{iw^T \sigma_\ast + iw^T \theta_0 + \frac{1}{2}w^T S w}$$

where $S^2 := \lim_{p \to \infty} \theta_0^T \theta_0/p = \lim_{p \to \infty} \beta_0^T \beta_0$, and $(w_\ast, v_\ast, \sigma_\ast, u_\ast)$ is the solution of the so-called Replica Symmetric (RS) equations

$$\zeta v^2 = E_{T, Z_0, Q}\left[(\xi^2 - vQ - wZ_0)^2\right]$$

$$v(1 - \zeta) = E_{T, Z_0, Q}\left[\frac{\partial}{\partial Q}\xi^\ast\right]$$

$$w \zeta = E_{T, Z_0, Q}\left[\xi^\ast \frac{\partial}{\partial Z_0} \log p(T|SZ_0, \sigma_0)\right]$$

$$0 = E_{T, Z_0, Q}\left[\nabla_\sigma \log p(T|\xi_\ast, \sigma)\right]$$

with $Z_0, Q \sim N(0, 1)$, $Z_0 \perp Q$, $T|Z_0 \sim p(.|SZ_0, \sigma_0)$ and $\xi_\ast := \xi(vQ + wZ_0, u, \sigma, T)$, where we use the short-hand

$$\xi_\ast(v, u, \sigma, T) = \arg\min_{\xi} \left\{\frac{1}{2} \left(\frac{\xi - v}{u}\right)^2 - \log p(T|\xi, \sigma)\right\}.$$

The proof of this theorem, which is based on methods from statistical physics [7, 9, 29], is presented in appendix A.

From the asymptotic characteristic function of the ML estimator in (6), the overfitting bias of the ML estimator can be easily computed. This follows from

**COROLLARY 2.2.** Under the assumptions of **Theorem 2.1**

$$\lim_{N \to \infty} E[\hat{\sigma}_N] = \sigma_\ast$$

$$\lim_{N \to \infty} E[\hat{\theta}_{N,k}] = \frac{w_\ast}{S} \theta_{0,k} \quad \forall k$$
and hence the asymptotic biases in $\hat{\theta}_N$ and $\hat{\sigma}_N$ equal

\begin{align}
\lim_{N \to \infty} \text{Bias}[\hat{\theta}_{N,k}] &:= \lim_{N \to \infty} \mathbb{E}[\hat{\theta}_{N,k} - \theta_{0,k}] = \theta_{0,k}(w_*/S - 1) \\
\lim_{N \to \infty} \text{Bias}[\hat{\sigma}_N] &:= \lim_{N \to \infty} \mathbb{E}[\hat{\sigma}_N - \sigma_0] = \sigma_* - \sigma_0
\end{align}

with $\sigma_*, w_*, S$ as defined in Theorem 2.1.

**Proof.** By the definition of characteristic function of the ML estimators in (6), we have

\begin{align}
\lim_{N \to \infty} \mathbb{E}[\hat{\theta}_{N,k}] &= -i \lim_{q, s \to 0} \frac{\partial}{\partial q_k} \Phi_{\hat{\theta}, \hat{\sigma}}(q, s) = w_*/S \theta_{0,k} \\
\lim_{N \to \infty} \mathbb{E}[\hat{\sigma}_N] &= -i \lim_{q, s \to 0} \nabla_s \Phi_{\hat{\theta}, \hat{\sigma}}(q, s) = \sigma_*
\end{align}

We can also compute the asymptotic variance of each component of $\hat{\theta}_N$. The latter is finite because of the scaling factor introduced in (4).

**Corollary 2.3.** Under the assumptions in Theorem 2.1

\begin{align}
\lim_{N \to \infty} \text{Var}[\hat{\theta}_{N,k}] = v_2^2
\end{align}

**Proof.** By definition of characteristic function one may write

\begin{align}
\lim_{N \to \infty} \text{Var}[\hat{\theta}_{N,k}] &= -\lim_{q, s \to 0} \frac{\partial^2}{\partial q_k^2} \Phi_{\hat{\theta}, \hat{\sigma}}(q, s) = v_2^2
\end{align}

Combining Corollary 2.3 with definition (4) shows that in the overfitting regime $p = O(N)$ both the sample-to-sample fluctuations and the average of $\hat{\beta}_{N,k}$ will be of order $p^{-1/2}$. In Section 1 we illustrated and quantified the relationship between $\hat{\beta}_N$ and $\beta_0$ by means of a simple linear regression. This amounts to fitting the relationship

\begin{align}
\hat{\beta}_{N,k} = \kappa \beta_{0,k} + \epsilon \\
\epsilon \sim \mathcal{N}(0, \Delta^2)
\end{align}

which implies

\begin{align}
\frac{1}{p} \text{Var}[\hat{\theta}_{N,k}] = \text{Var}[\hat{\beta}_{N,k}] = \Delta^2
\end{align}

By corollaries 2.2 and 2.3 it can be seen that when the number of observations is large but finite, then the link between the quantities measured in Figure 1 and the main macroscopic quantities of the theory is given by

\begin{align}
\kappa \approx w_*/S, \quad \Delta^2 \approx v_2^2/p
\end{align}

2.2. **Construction of unbiased estimators.**

Given a regression model, equations (7–10) provide explicit formulae for the asymptotic relations between properties of the true and estimated regression parameters (expressed at the level of the macroscopic quantities $v_*$ and $w_*$) and the true and estimated nuisance parameters. Theorem 2.1 moreover implies that any finite set of regression coefficients will have a joint Gaussian distribution, and gives the statistical interpretation of $(v_*, w_*)$. In combination, we now seek to use these results in order to construct unbiased estimators for the regression and nuisance parameters in the overfitting regime.
Note that solving the equations (7–10) for \((w_\star, v_\star, \sigma_\star, u_\star)\) would require knowledge of the a priori unknown values of \(S\) and \(\sigma_0\). For some models, as we are going to argue later, \(w_\star/S\) and \(v_\star\) are independent from the values of \(S\). Alternatively one can estimate \(S\) and plug in the estimated value. In this case though, the correction factor is estimated. The situation with \(\sigma_0\) is more subtle. We will use the fact that for \(N, p \to \infty\) (with fixed ratio) the quantity \(\sigma\) in equation (10) actually equals the ML-inferred nuisance parameters (12) with probability one, and is hence observable. We therefore do not need to know \(\sigma_0\) since it can be solved from (10) simultaneously with (7–9).

As a consequence of Corollary 2.2 and the discussion above, we can now define

\[
\tilde{\theta}_{N,k} := \frac{S}{w_*} \tilde{\theta}_{N,k} \quad k = 1, \ldots, p.
\]

which are (asymptotically) unbiased estimators if \(S\) is known, otherwise the factor \(S/w_*\) is to be intended as estimated. Similarly, unbiased estimators \(\tilde{\sigma}_N\) for the nuisance parameters are defined by the solution of equation (10), when solved for \(\sigma_0\) and upon substituting \(\hat{\sigma}_N\) for \(\sigma\).

Given the assumptions made on the covariates and the parameters \(\beta_0\), the above theory is completely general and applicable to (almost) any regression model of the generic form \(p(T|X^T\beta, \sigma)\).

2.3. Overfitting in linear regression models.

To illustrate the theory in the most transparent setting, we first turn to linear regression models with arbitrary additive noise. Here our formalism is found to simplify greatly.

**Lemma 2.4 (RS equations for linear regression models with arbitrary noise term).** For regression models of the form

\[
T = X^T\beta + Z \quad Z \sim p_Z(\cdot|\sigma)
\]

where \(p_Z(\cdot|\sigma)\) is a parametric density, the RS equations (7,8,9,10) and (11) reduce to

\[
\zeta v^2 = \mathbb{E}_{T',Q}[\left(\eta_* - vQ\right)^2]
\]

\[
v(1 - \zeta) = \mathbb{E}_{T',Q}[\frac{\partial}{\partial Q}\eta_*]
\]

\[
w_* = S
\]

\[
o = \mathbb{E}_{T',Q}[\nabla_\sigma \log p_Z(T' - \eta_*|\sigma)]
\]

with \(Q \sim \mathcal{N}(0, 1), T' \sim p_Z(\cdot|\sigma_0)\) and

\[
\eta_* = \arg \max_\eta \left\{\frac{1}{2} \left(\frac{\eta - vQ}{w}\right)^2 - \log p_Z(T' - \eta|\sigma)\right\}
\]

**Proof.** The idea behind the proof is that for linear models, the RS equation (9) simplifies. Here the conditional density of the response equals

\[
p(t|X^T\beta, \sigma) = p_Z(t - X^T\beta|\sigma)
\]

thus

\[
\frac{\partial}{\partial Z_0} \log p(T|SZ_0, \sigma_0) = \frac{\partial}{\partial Z_0} \log p_Z(T - SZ_0|\sigma_0) = -S \frac{\partial}{\partial T} \log p_Z(T - SZ_0|\sigma_0)
\]
and the RS equation (9) reduces to

\[ w\zeta = -SE_{T,Z_0,Q}\left[\xi_\star \frac{\partial}{\partial T}\log p_Z(T - SZ_0|\sigma_0)\right] \]

By partial integration in \( T \) and applying the equation (29) we have that

\[ E_{T|Z_0}\left[\xi_\star \frac{\partial}{\partial T}\log p_Z(T - SZ_0|\sigma_0)\right] = \int_{-\infty}^{\infty} \xi_\star \frac{\partial}{\partial t} p(t|SZ_0,\sigma_0) \, dt \]

\[ = \left[\xi_\star p(t|SZ_0,\sigma_0)\right]_{-\infty}^{\infty} - E_{T|Z_0}\left[\frac{\partial}{\partial T}\xi_\star\right] \]

The first term is zero and thus equation (31) becomes

\[ w\zeta = SE_{T,Z_0,Q}\left[\frac{\partial}{\partial T}\xi_\star\right] \]

We now compute explicitly the derivative of \( \xi_\star \) with respect to \( T \). For compactness of the calculations, we define

\[ f(x|\sigma) := \frac{\partial}{\partial x}\log p_Z(x|\sigma) \]

Then equation (11), defining \( \xi_\star \) in Theorem 2.1, is equal to

\[ \xi_\star = vQ + wZ_0 + u^2 \frac{\partial}{\partial T}\log p_Z(T - \xi_\star|\sigma_\star)|_{\xi_\star=\xi} \]

\[ = vQ + wZ_0 - u^2 f(T - \xi_\star|\sigma_\star) \]

and we obtain that for \( x := T - \xi_\star \)

\[ \frac{\partial}{\partial T}\xi_\star = -u^2 \frac{\partial}{\partial x} f(x|\sigma) \frac{\partial}{\partial T} x = -u^2 \frac{\partial}{\partial x} f(x|\sigma) \left(1 - \frac{\partial}{\partial T}\xi_\star\right) \]

The equation above can be solved for \( \frac{\partial}{\partial T}\xi_\star \), giving

\[ \frac{\partial}{\partial T}\xi_\star = \frac{-u^2 \frac{\partial}{\partial x} f(x|\sigma)}{1 - u^2 \frac{\partial}{\partial x} f(x|\sigma)} = 1 - \frac{1}{1 - u^2 \frac{\partial}{\partial x} f(x|\sigma)} \]

Since, similarly

\[ \frac{\partial}{\partial Q}\xi_\star = \frac{v}{1 - u^2 \frac{\partial}{\partial x} f(x|\sigma)} \]

we also conclude that

\[ \frac{\partial}{\partial T}\xi_\star = 1 - \frac{1}{v} \frac{\partial}{\partial Q}\xi_\star \]

Inserting the last equality (39) into equation (33) and using the RS equation (8):

\[ v(1 - \zeta) = E_{T,Z_0,Q}\left[\frac{\partial}{\partial Q}\xi_\star\right] \]

we find \( w_\star = S \). With the transformations

\[ \eta_\star = \xi_\star - SZ_0 \quad T' = T - SZ_0 \]

and the fact that \( w_\star = S \), we obtain the RS equations in (24–27).

This lemma implies the following:
RESULT 2.5. In a linear regression model of the form
\[ T = X^T \beta + Z \quad Z \sim p_Z(\cdot, \sigma) \]
for any well behaved parametric density \( p_Z(\cdot, \sigma) \), the ML estimator for the regression parameters \( \hat{\beta}_N \) is asymptotically unbiased also in the overfitting regime \((0 < \zeta < 1)\). In contrast, the ML estimator for the nuisance parameters \( \hat{\sigma}_N \) is generally asymptotically biased in this regime.

\[
\text{PROOF. From Corollary 2.2 we know that the asymptotic bias in } \hat{\beta}_N \text{ is equal to}\]
\[
\beta_0 \left( \frac{w_\star}{S} - 1 \right).
\]
For linear regression models as in Lemma 2.4, \( w_\star = S \). Hence this bias is zero. For the vector \( \sigma \) of nuisance parameters this does not need to be the case. \( \square \)

The distribution \( p_Z \) of the noise is unspecified in Result 2.5, which holds therefore not only for standard models with normally distributed noise, but for all models with arbitrary noise distributions.

3. Application to parametric regression models.

The theory of overfitting in regression models is general (see Theorem 2.1); it applies to generalized linear models and beyond ([7, 9]). The aim of this section is to compare the theoretical bias computed in Section 2 to the bias computed in simulation studies for multiple regression models (as in the introduction of this paper). Specifically, we will consider 1) the Weibull proportional hazards regression model, 2) the Log-Logistic AFT model and 3) the Gamma frailty (random effect) model with an Exponentially distributed time-to-event. These models have been chosen based on their popularity in applied statistics, the fact that their overfitting phenomenology has not yet been analyzed in previous studies, and also because of their analytical properties. In the subsections below we show the solution of the RS equations for each of the three specific models.

3.1. The Weibull model.

Let \( T \) be the event time; i.e. the time between a well defined initialisation and the moment the event took place. The most popular model for \( T \) is the Proportional Hazards (PH) model, which assumes the conditional density of \( T \) given the covariates \( X \) to be of the form
\[ p(t|X^T \beta, h_0) = h_0(t)e^{X^T \beta - H_0(t)\exp(X^T \beta)} \]
where \( h_0(t) : \mathbb{R}^+ \to \mathbb{R}^+ \) is the baseline hazard rate, i.e. the hazard rate in case \( X = 0 \), and \( H_0(t) = \int_0^t h_0(s)ds \). If \( h_0 \) is assumed to have a parametric form, the model is known as the parametric PH regression model. Here we will consider the parametric form
\[ h_0(t) = \rho \lambda^t t^{\rho-1} \]
for the baseline hazard function with \( \rho > 0 \). The corresponding conditional density equals
\[ p(t|X^T \beta, \lambda, \rho) = \rho \lambda^t t^{\rho-1} e^{X^T \beta - (\lambda t)^\rho \exp(X^T \beta)} \]
It follows that the event time \( T \) has a Weibull distribution [14, 25]. This so-called Weibull model (45) is one of the most popular parametric PH models for time-to-event data.

The Weibull PH model can be transformed to a Log-Linear model via
\[ Y := -\log(T) = X^T \phi + Z \quad Z \sim p_Z = \frac{1}{\sigma} e^{-(z-\phi)/\sigma - \exp(-(z-\phi)/\sigma)} \]
There is a one-to-one relation between the parameters of (45) and (46) [14, 25]:
\[ \lambda = \exp(\phi) \quad \beta = \sigma^{-1} \phi \quad \rho = \sigma^{-1} \]
Solution of RS equations. After transforming the Weibull PH model (45) to the linear model (46), the vector \((v_*, u_*, \phi_*, \sigma_*)\) can be obtained by solving the RS equations (24-27). Thanks to Lemma 2.4 in the previous section, we know that because the Log-Linear Weibull model is indeed a linear one we have

\[
w_* = S
\]

Here (27) translates into two equations for the nuisance parameters \(\phi\) and \(\sigma\). We now have

\[
p(y|\eta, \phi, \sigma) = p_Z(y - \eta|\phi, \sigma) = \frac{1}{\sigma} \exp \left( - \left( \frac{y - \eta - \phi}{\sigma} + \exp \left( - \frac{y - \eta - \phi}{\sigma} \right) \right) \right)
\]

and equation (28) for \(\eta_*\) becomes

\[
\eta_* - \left( vQ + \frac{u^2}{\sigma} \right) + \frac{u^2}{\sigma} \exp \left( \frac{\eta + \phi - y}{\sigma} \right) = 0
\]

The solution can be written in terms of the Lambert W-function [11] (which is defined as the inverse of \(f(z) = ze^z\)):

\[
\eta_* = vQ + \frac{u^2}{\sigma} - \sigma W \left( \frac{u^2}{\sigma^2} \exp \left( \frac{u^2}{\sigma^2} + \frac{\phi + vQ - Y}{\sigma} \right) \right)
\]

After some additional algebraic manipulations (details are given in Appendix B.1), the RS equations equal

\[
\zeta = \frac{\frac{u^2}{\sigma^2}}{\frac{u^2}{\sigma^2} - W \left( \frac{u^2}{\sigma^2} \exp \left( \frac{u^2}{\sigma^2} + \frac{\phi - \phi_0 + vQ}{\sigma} \right) \right)^2} = \mathbb{E}_{F,Q} \left[ \frac{W \left( \frac{\phi - \phi_0 + vQ}{\sigma} \right) F^{\sigma_0/\sigma}}{1 + W \left( \frac{\phi - \phi_0 + vQ}{\sigma} \right) F^{\sigma_0/\sigma}} \right]
\]

\[
\frac{\sigma}{\sigma_0} = \gamma_E + \frac{\sigma^2}{u^2} \mathbb{E}_{F,Q} \left[ \log(F)W \left( \frac{u^2}{\sigma^2} \exp \left( \frac{u^2}{\sigma^2} + \frac{\phi - \phi_0 + vQ}{\sigma} \right) \right) \right]
\]

\[
\frac{u^2}{\sigma^2} = \mathbb{E}_{F,Q} \left[ W \left( \frac{\phi - \phi_0 + vQ}{\sigma} \right) F^{\sigma_0/\sigma} \right]
\]

where

\[
F \sim \text{Exp}(1) \quad Q \sim \mathcal{N}(0,1), \quad \text{with} \quad F \perp Q
\]

and \(\gamma_E\) is the Euler-Mascheroni constant. Note that, as expected because of Lemma 2.4, the above equations no longer depend on \(S\). At this point, instead of solving (52–55) for \(\phi_*, \sigma_*, u_*, v_*\), it is easier to solve for certain combinations of the latter, as suggested by the form of the equations. For instance, all models considered in this article are such that one can solve them more easily for \((\phi_0 - \phi_0)/\sigma_*, \phi_*/\sigma_0\) and \(u^2_*/\sigma^2_*, \nu^2_*/\sigma^2_2\). An important feature of these four ratios is that they depend only on \(\zeta\) and not on the true values \(\phi_0\) and \(\sigma_0\) of the nuisance parameters.

In Figure 3 we compare the theoretical results to those of ML regression on simulated data, for \(\rho_0 = 1\) and \(\phi_0 = -\log 3\). For each value of \(\zeta\) we estimate the biases by computing averages over \(M = 100/\zeta\) data-sets\(^1\), of sizes \(N = 200, 300, 400\). We find excellent agreement

\(^1\)This number \(M\) of data-sets is chosen such that the number of points in scatter plots as in Section 1 is identical for all values of \(\zeta\).
Fig 3: Theory versus simulated data for the log-transformed Weibull PH model (46) in panels (a)-(d) and the untransformed Weibull model (45) in (e) and (f): (a) slope $\kappa(\zeta)$ of the least squares regression line in the scatterplot of true versus inferred regression parameters, (b) width $\Delta(\zeta)$ of this scatterplot, (c,d) rescaled deviations between inferred and true values of the nuisance parameters as in (58). In all cases the expectations over the data sets are estimated as sample average over $M = 100/\zeta$ sets, each of sizes $N = 400, 300, 200$. Error bars are omitted for ease of visualization as more values of $N$ are displayed. The bottom two panels show slope and width of the scatterplot as in (a) and (b), but now for the untransformed Weibull model (45) (here $N = 200$). Here the bias is clearly evident.

between theory and simulations, in spite of the fact that the theory refers to the asymptotic regime $N, p \to \infty$. We can summarize the effects of overfitting in the present transformed model as follows: absent bias in the regression parameters $\varphi$ (as expected from Result 2.5), an increase of the standard deviation of $\varphi$, an overestimation of the intercept $\phi$ (as appearing in $pZ$), and an underestimation of the noise width $\sigma$. 
Bias correction. We now use the theory to compute bias corrections for the nuisance parameters in the model (46). These corrections no longer depend on S. Defining

\[
\frac{\phi_\star - \hat{\phi}_0}{\sigma_0} = g(\zeta) \quad \frac{\sigma_\star}{\sigma_0} = f(\zeta),
\]

where \(f(\zeta)\) and \(g(\zeta)\) follow from numerical solution of (52–55), and given ML estimated nuisance parameters \(\hat{\phi}_N\) and \(\hat{\sigma}_N\), we can solve (57) for \(\hat{\phi}_0\) and \(\sigma_0\). By Corollary 2.2 we obtain the following asymptotically unbiased estimators:

\[
\tilde{\phi}_N = \hat{\phi}_N - \hat{\sigma}_N \frac{g(\zeta)}{f(\zeta)} \quad \tilde{\sigma}_N = \frac{\hat{\sigma}_N}{f(\zeta)}
\]

We can now compute the corrections to the ML estimators of the parameters of original untransformed PH model (45), using the mapping (47). For these parameters we obtain

\[
\tilde{\beta}_N = f(\zeta) \hat{\beta}_N \quad \tilde{\lambda}_N = \hat{\lambda}_N e^{-g(\zeta)/(f(\zeta)\hat{\rho}_N)} \quad \tilde{\rho}_N = f(\zeta)\hat{\rho}_N
\]

There is a systematic bias in the ML estimator \(\hat{\beta}_N\) due to the fact that \(\sigma_0\) is progressively under-estimated as \(\zeta\) increases. This is visible in Figure 3e as \(\kappa\) gets larger than one.

3.2. The Log-logistic model.

The second model we study is the Log-logistic regression model, one of the most common among the Accelerated Failure Time (AFT) models. AFT models are often used if the proportional hazards assumption does not seem to hold [19, 25]. Moreover, their interpretation is easier than for the Cox proportional hazard model or a parametric version of it [19]. The Log-Linear model assumes that the conditional density of the response, given the covariates, is of a specific form such that \(\text{log}\text{-logistic}\) has a linear relationship with the covariates \(X\) [25]. For the specific choice the conditional density of \(T\) is given by

\[
p(t|X^T\beta,\lambda,\rho) = \frac{\rho(teX^T\beta/\lambda)^{\rho-1}}{\lambda(1+(teX^T\beta/\lambda)^{\rho})^2}
\]

and one has

\[
Y = X^T\varphi + Z, \quad Z \sim p_Z(z) = \frac{\exp\left(-\frac{z-\phi}{\sigma}\right)}{\sigma(1+\exp\left(-\frac{z-\phi}{\sigma}\right))^2}
\]

with the one-to-one mapping for the model parameters

\[
\rho = 1/\sigma, \quad \lambda = e^{\phi}, \quad \beta = \varphi
\]

Solution of RS equations. We follow the same steps as for the Weibull PH model in Subsection 3.1. We solve \(\eta_\star\) from equation (28), which here takes the form

\[
\eta_\star = vQ + \frac{u^2}{2\sigma} \tanh \left(\frac{Y - \eta_\star - \phi}{2\sigma}\right)
\]

After a simple transformation we obtain

\[
\eta_\star = Y - \phi + 2\sigma x_\star
\]

with \(x_\star\) defined as the solution of the self-consistent equation

\[
x = \frac{vQ - (Y - \phi)}{2\sigma} - \frac{u^2}{2\sigma^2} \tanh(x)
\]
Introducing $F := (Y - \phi_0)/\sigma_0$ with density $p_F(f) = e^{-f}(1 + e^{-f})^{-2}$, we have

$$x_\star = \frac{v}{2\sigma}Q + \frac{\phi - \phi_0}{2\sigma} - \frac{\sigma_0}{2\sigma}F - \frac{u^2}{2\sigma^2}\tanh(x_\star)$$

Equation (66) is of the form $x = a - b\tanh(x)$ with $a \in \mathbb{R}$ and $b \in \mathbb{R}^+$, which always has a unique solution. After the by now usual algebraic steps (see appendix B.2), the system of RS equations (24–27) can then be written as

$$v^2 \zeta = \frac{u^4}{\sigma^2} E_{F,Q}[\tanh^2(x_\star)]$$

$$\zeta = E_{F,Q}\left[\frac{u^2}{2\sigma^2\cosh^2(x_\star)} + u^2\right]$$

$$\frac{\phi - \phi_0}{2\sigma} = E_{F,Q}[x_\star]$$

$$\frac{\sigma}{\sigma_0} = -E_{F,Q}[F \tanh(x_\star)]$$

where $Q \sim \mathcal{N}(0, 1)$

In Figure 4 we again compare the analytical predictions of the theory, obtained by solving the above equations numerically, with the results of regressions with simulated data. Once more we find excellent agreement. As with the transformed Weibull model of Subsection 3.1 the linear form of the transformed model ensures the regression parameters to be asymptotically unbiased, while the nuisance parameters are seriously biased due to overfitting.

**Bias correction.** Once we have computed our ML estimates for the nuisance parameters, i.e. $\hat{\phi}_N, \hat{\sigma}_N$, we can use the estimated asymptotic relations that follow from the RS equations,

$$\frac{\phi_\star - \phi_0}{\sigma_0} = g(\zeta) \quad \frac{\sigma_\star}{\sigma_0} = f(\zeta)$$

and obtain the asymptotically unbiased estimators

$$\tilde{\phi}_N = \hat{\phi}_N - \hat{\sigma}_N \frac{g(\zeta)}{f(\zeta)} \quad \tilde{\sigma}_N = \frac{\hat{\sigma}_N}{f(\zeta)}$$

For the Log-logistic (transformed) model one has $g(\zeta) = 0$ (found by numerically solving the RS equations) and, as a consequence, the ML estimator $\hat{\phi}_N$ is asymptotically unbiased. This also holds for $\hat{\phi}_N$ by Result 2.5.

Since $\beta = \varphi$ in (62) the ML estimator of the regression parameters in the original untransformed AFT model, $\hat{\beta}_N = \hat{\varphi}_N$, is asymptotically unbiased. The (corrected) ML estimators of the nuisance parameters in this model equal $\tilde{\rho}_N = f(\zeta)\hat{\rho}_N$ and $\tilde{\lambda}_N = \hat{\lambda}_N = \exp(\tilde{\phi}_N)$.

### 3.3. The Exponential model with Gamma Frailty

The third model that will be considered is the Gamma frailty model with an exponentially distributed failure time. A frailty variable (or random variable) is often included in the model to account for unobserved heterogeneity among the subjects. In this sense, frailty models [14] can be regarded as a generalization of survival analysis models. In medical applications, unobserved heterogeneity can be present for several reasons. For instance, because a number of possibly relevant characteristics are not included as covariates in the model. Frailty models account for these situations by introducing a (positive) random variable $U$ which multiplies the base hazard rate of a survival model. If one assumes, for simplicity, a constant base hazard rate, then the “marginal” density that is used to construct the likelihood of the data, reads

$$p(t|X^T\beta, \lambda) = \mathbb{E}_U \left[U \lambda e^{X^T\beta - U\lambda t} \exp(X^T\beta)\right]$$
For analytical convenience, a Gamma distribution for the frailty variable \( U \) is usually chosen. This leads to

\[
p(t|X^T\beta, \lambda, \theta) = \lambda e^{X^T\beta} (1 + \theta t e^{X^T\beta})^{-(1+1/\theta)}
\]

in which \( \theta \) is the variance of the Gamma distribution for the frailty. By considering the transformed response variable \( Y = -\log(T) \), the model (74) can be rewritten in the linear form

\[
Y = X^T\varphi + Z \quad Z \sim p_Z(z) = e^{-(z-\phi)} (1 + \theta e^{-(z-\phi)})^{-(1+1/\theta)}
\]

where \( \varphi = \beta, \phi = \log \lambda \). As with the previous examples in subsections 3.1 and 3.2, since the transformed model is linear we already know that \( w_\star = S \). The self consistent RS equation (28) for \( \eta_\star \) reads

\[
\eta_\star = vQ + \frac{u^2}{2} \left( 1 - \frac{1}{\theta} \right) - \frac{u^2}{2} \left( 1 + \frac{1}{\theta} \right) \tanh \left( \frac{\eta_\star - (Y - \phi) + \log(\theta)}{2} \right)
\]

Defining

\[
x = \frac{\eta_\star - (Y - \phi) + \log(\theta)}{2}
\]
the latter condition (76) can be reduced to

\[(78) \quad x = a - b \tanh(x)\]

with

\[(79) \quad a = vQ - (Y - \phi) + \log(\theta) + \frac{u^2}{4} \left(1 - \frac{1}{\theta}\right)\]

\[(80) \quad b = \frac{u^2}{4} \left(1 + \frac{1}{\theta}\right) > 0\]

The equation above (78) has a unique solution \(x^\star(Y, Q)\). At this point we change variable

\[(81) \quad F := \left(1 + \theta_0 e^{-(Y - \phi_0)}\right)^{-1/\theta_0} \sim Unif[0, 1]\]

and after the by now standard manipulations (see appendix B.3), we arrive at

\[(82) \quad \zeta v^2 = E_{F, Q} \left[\frac{1}{2} \left(1 - \frac{1}{\theta}\right) - \frac{u^2}{2} \left(1 + \frac{1}{\theta}\right) \tanh(x^\star)\right]^2\]

\[(83) \quad \zeta = E_{F, Q} \left[\frac{u^2(\theta + 1)}{u^2(\theta + 1) + 4\theta \cosh^2(x^\star)}\right]\]

\[(84) \quad \phi - \phi_0 = E_{F, Q} \left[2x^\star - \log \left(F^{\theta_0} - 1\right)\right] - \log(\theta/\theta_0)\]

\[(85) \quad \theta = E_{F, Q} \left[x^\star + \log \left(2 \cosh(x^\star)\right)\right]\]

In Figure 5 we compare our theoretical predictions for the transformed model, obtained by solving the RS equations, with the results of regressions with simulated data. The curves \(\theta(\zeta)\) and \(\phi - \phi_0(\zeta)\) together with \(\Delta\) depend on the value of \(\theta_0\). This is shown in our plots, where we present solutions of the RS equations for different values of \(\theta_0\).

We notice once again that the theory is in excellent agreement with the regression data. The effects of overfitting are qualitatively similar to those in the previous two example models, but here we observe a new phenomenon: the variance \(\theta\) of the frailty density is progressively underestimated as \(\zeta\) increases. Furthermore, there exists a critical value of \(\zeta\) at which the inferred variance \(\theta\) becomes virtually zero. The latent heterogeneity represented by the frailty variable \(U\) in the untransformed model (73) is in the overfitting regime erroneously ‘explained’ in terms of the regression parameters \(\beta\). This again leads to an increased variance \(\Delta\) in the estimates of these parameters.

**Bias correction.** Compared to the overfitting corrections of nuisance parameters in the previous subsections, correcting the estimate for the nuisance parameter \(\theta\) in the present frailty model is somewhat more complicated. Upon solving our RS equations for \(\theta\) with different values of \(\theta_0\), we find that the curves \(\theta^\star(\zeta)\) as shown in Figure 5(d) do not intersect. This allows us, in principle, to compute a corrected unbiased estimator \(\hat{\theta}_N\) by following the relevant curves in Figure 5(d), starting from \((\zeta, \hat{\theta}_N)\) down to the \(\theta^\star\)-value (on the vertical axis) at \(\zeta = 0\) on the same curve, provided we are in the regime where \(\hat{\theta}_N > 0\).

This procedure clearly leaves room for future improvements. One would like to have directly an estimate of \(\theta_0\) once the \(\hat{\theta}_N\) obtained from an actual regression is supplied as for the other models previously analyzed. In principle, one could do this by solving the RS equations for \(\theta_0\), given \(\theta = \hat{\theta}_N\) (which holds with probability one asymptotically in \(N\)). We will pursue this and several related ideas in future studies with the aim of applying our theoretical results to the analysis of real data.
Fig 5: Theory versus simulated data for the log-linear exponential model with Gamma frailty (75) for different values of the ratio $\zeta$ and $\theta_0$: (a) slope $\kappa(\zeta)$ of the least squares regression line in the scatterplot of true versus inferred regression parameters, (b) width $\Delta(\zeta)$ of this scatterplot, (c,d) rescaled deviations between inferred and true values of the nuisance parameters $\hat{\phi}_N - \phi_0$ and $\hat{\theta}_N$. The expectations over the data sets are estimated as sample average over $M = 100/\zeta$ sets, each of size $N = 400$. Error bars are omitted for ease of visualization.

4. Discussion.

In this paper we derived within a statistical framework the overfitting theory for regression with generalized linear models and proportional hazards survival analysis models, as originally proposed in [7, 9], and we added three new applications to relevant survival analysis models. More specifically, we described how to estimate the bias in the ML estimators of the regression and nuisance parameters in parametric regression models in the overfitting regime, and correct these estimators afterwards. Having a proper quantitative theory of the overfitting phenomenon in regression models is a prerequisite for achieving reliable estimates of the effects of covariates on risk, as measured by association parameters. Moreover, estimating both association and nuisance parameters without overfitting bias is certainly needed for improving outcome predictions. In contrast to the vast majority of other correction methods, the proposed theory estimates also the bias in the ML estimators for the nuisance parameters. This inclusion of overfitting corrections for the nuisance parameters is crucial in order to achieve correct outcome predictions for future covariate data. We compared our theoretical findings with the results of numerical simulations, for multiple models and different parameter settings, and we found excellent agreement.

In the present study we assumed the covariates to be distributed independently, and that the individual predictors $X_i^T\beta$ are asymptotically normally distributed (with zero mean). The
covariate independence assumption is largely harmless, since any covariate correlations can be transformed away (see [7]). The normality assumption on the predictors is motivated by the central limit theorem (CLT), but the validity of the latter requires certain conditions to be met. There will indeed be values of $\beta_0$ such that the CLT cannot be applied and this is left for future investigation.

The derivation of our result is based on the replica method from the statistical physics of disordered systems [35]. Most of the results derived in the literature using the replica method have later been confirmed rigorously using alternative mathematical approaches (see e.g. [18, 30] and more recently [4, 13, 23]). Our simulation studies confirmed convincingly that also for the problem of modelling overfitting in regression models the replica method gives reliable estimates. It would nevertheless be instructive to benchmark our findings by re-deriving our results without making use of the replica method2.

We have applied the general overfitting theory to three specific models, for which accurate overfitting correction methods had not yet been developed. In these applications we did not yet include censoring, which is a distinctive feature of Survival data. The general theory is conjectured to be still applicable to survival analysis models with censoring, but its equations would be more involved, and hence this is left for future work. An alternative route would be to use the so called pseudo observations from Andersen et al [1]. These are generate using a jacknife procedure and allow to analyze survival data with GLMs. This is an interesting direction which requires further investigation and will be the subject of future studies. There are further interesting questions that could be addressed within the present formalism. These include for instance the effects of model mis-specification, and application to models with regularizers to probe the high-dimensional regime $p \gg N$, which is of interest in many modern fields of research.

APPENDIX A: DERIVATION OF THE ASYMPTOTIC CHARACTERISTIC FUNCTION

In this appendix we describe the main steps in the derivation of the general results in Section 2. We first rewrite the characteristic function of the ML estimator in a form that enables its calculation for $N \to \infty$ via the steepest descent method. We then analyse the steepest descent saddle point equations, and take the so-called replica limit.

A.1. The characteristic function of the ML estimator as a steepest descent integral.

**Proposition A.1.** For $N$ observations $(T_1, X_1), \ldots, (T_N, X_N)$ each generated independently from the density $p(t, x|\varphi_0)$, we define the log-likelihood and the ML estimator of the model parameters as, respectively,

$$l_N(\varphi) := \sum_{i=1}^{N} \log p(T_i|X_i, \varphi) \tag{86}$$

$$\hat{\varphi}_N := \arg\max_{\varphi} l_N(\varphi) \tag{87}$$

Assume that $l_N(\varphi)$ is twice differentiable with respect to $\varphi$, with continuous second partial derivatives and a unique absolute maximum. Then the characteristic function of the ML estimator $\Phi_{\hat{\varphi}_N}(q)$ is given by the following expression:

$$\Phi_{\hat{\varphi}_N}(q) = \lim_{\gamma \to \infty} \lim_{n \to 0} \frac{\int \left( \mathbb{E}_{T, X} \left[ \prod_{\alpha=1}^{n} p^\gamma(T|X, \varphi_\alpha) \right] \right)^N e^{i q^T \varphi} \prod_{\alpha=1}^{n} d\varphi_\alpha}{\int \left( \mathbb{E}_{T, X} \left[ \prod_{\alpha=1}^{n} p^\gamma(T|X, \varphi_\alpha) \right] \right)^N \prod_{\alpha=1}^{n} d\varphi_\alpha} \tag{88}$$

2Indeed a similarity between regression models and the Sherbina-Tirozzi model of spin-glasses has already been noticed and exploited to rigorously assess the performance of a Bayesian Linear Regression models with mismatch [4]. (Still this analysis assumes the nuisance parameters to be fixed).
To prove this proposition we will need Laplace integration, according to which for any function \( f : \mathbb{R}^d \rightarrow \mathbb{R} \) with a unique absolute maximum at \( \psi_0 \) and any bounded function \( g : \mathbb{R}^d \rightarrow \mathbb{R} \), both with continuous second order partial derivatives, the following holds:

\[
\lim_{\gamma \to \infty} \frac{\int_{\mathbb{R}^d} g(\psi) e^{\gamma f(\psi)} \, d\psi}{\int_{\mathbb{R}^d} e^{\gamma f(\psi)} \, d\psi} = g(\psi_0)
\]

We now present the proof of Proposition A.1.

**Derivation of Proposition A.1.** The characteristic function of the ML estimator \( \hat{\phi}_N \) is defined as

\[
\Phi_{\hat{\phi}_N}(q) := \mathbb{E}_{\{(T_i, X_i)\}_{i=1}^N}[e^{iq^T \hat{\phi}_N}] = \int e^{iq^T \hat{\phi}_N} \prod_{1 \leq i \leq N} p(t_i, x_i | \varphi_0) dt_i dx_i
\]

Using the Laplace identity (89) with \( g(\varphi) = e^{iq^T \varphi} \) and \( f(\varphi) = l_N(\varphi) \), we obtain

\[
e^{iq^T \hat{\phi}_N} = \lim_{\gamma \to \infty} \frac{\int e^{iq^T \varphi} e^{\gamma l_N(\varphi)} \, d\varphi}{\int e^{\gamma l_N(\varphi)} \, d\varphi}
\]

Since \(|e^{iq^T \varphi}| \leq 1\), by Lebesgue dominated convergence theorem we can write

\[
\Phi_{\hat{\phi}_N}(q) = \lim_{\gamma \to \infty} \lim_{n \to 0} \mathbb{E}_{\{(T_i, X_i)\}_{i=1}^N}[e^{iq^T \hat{\phi}_N} e^{\gamma l_N(\varphi)} \, d\varphi] \quad \text{as} \quad n \to 0
\]

We next employ the identity \( \lim_{n \to 0} x^{n-1} = x^{-1} \), and rewrite the previous expression as

\[
\Phi_{\hat{\phi}_N}(q) = \lim_{\gamma \to \infty} \lim_{n \to 0} \mathbb{E}_{\{(T_i, X_i)\}_{i=1}^N}[\lim_{n \to 0} \int e^{\gamma l_N(\varphi)} e^{iq^T \varphi} \, d\varphi \left( \int e^{\gamma l_N(\varphi)} \, d\varphi \right)^{n-1}]
\]

Upon exchanging the operations of expectation and limit,

\[
\Phi_{\hat{\phi}_N}(q) = \lim_{\gamma \to \infty} \lim_{n \to 0} \mathbb{E}_{\{(T_i, X_i)\}_{i=1}^N}[\int e^{\gamma l_N(\varphi)} e^{iq^T \varphi} \, d\varphi \left( \int e^{\gamma l_N(\varphi)} \, d\varphi \right)^{n-1}]
\]

If \( n \in \mathbb{N} \) we can write

\[
\int e^{\gamma l_N(\varphi)} e^{iq^T \varphi} \, d\varphi \left( \int e^{\gamma l_N(\varphi)} \, d\varphi \right)^{n-1} = \int e^{\gamma \sum_{\alpha=1}^n l_N(\varphi_\alpha)} e^{iq^T \varphi_1} \prod_{\alpha=1}^n d\varphi_\alpha
\]

where the \( n \) newly introduced integration variables \( \{\varphi_\alpha\} \) are called replicas of the original model parameter \( \varphi \). As a consequence

\[
\phi_{\hat{\phi}_N, n}(q) := \mathbb{E}_{\{(T_i, X_i)\}_{i=1}^N}[\int e^{\gamma \sum_{\alpha=1}^n l_N(\varphi_\alpha)} e^{iq^T \varphi_1} \prod_{\alpha=1}^n d\varphi_\alpha]
\]

The replica formalism [7, 9, 17, 27, 35] assumes that analytical continuation allows us to take the limit \( n \to 0 \) using expression (97), in spite of the fact that the latter was derived for integer \( n \). One thus obtains

\[
\Phi_{\hat{\phi}_N}(q) = \lim_{\gamma \to \infty} \lim_{n \to 0} \mathbb{E}_{\{(T_i, X_i)\}_{i=1}^N}[\int e^{\gamma \sum_{\alpha=1}^n l_N(\varphi_\alpha)} e^{iq^T \varphi_1} \prod_{\alpha=1}^n d\varphi_\alpha]
\]
For reasons which will become clear later, we will use the property
\begin{equation}
\lim_{n \to 0} \mathbb{E}_{\{(T_i, X_i)\}_{i=1}^n} \left[ \int e^{\gamma \sum_{\alpha=1}^n l_N(\varphi_\alpha)} \prod_{\alpha=1}^n d\varphi_\alpha \right] = \lim_{n \to 0} \mathbb{E}_{\{(T_i, X_i)\}_{i=1}^n} \left[ \left( \int e^{l_N(\varphi)} d\varphi \right)^n \right] = 1
\end{equation}
to rewrite expression (98) as
\begin{equation}
\Phi_{\varphi_N}(q) = \lim_{\gamma \to \infty} \lim_{n \to 0} \frac{\mathbb{E}_{\{(T_i, X_i)\}_{i=1}^n} \left[ \int e^{\gamma \sum_{\alpha=1}^n l_N(\varphi_\alpha)} e^{i q^T \varphi_\alpha} \prod_{\alpha=1}^n d\varphi_\alpha \right]}{\mathbb{E}_{\{(T_i, X_i)\}_{i=1}^n} \left[ \int e^{\gamma \sum_{\alpha=1}^n l_N(\varphi_\alpha)} \prod_{\alpha=1}^n d\varphi_\alpha \right]}
\end{equation}
Furthermore
\begin{equation}
e^{\gamma \sum_{\alpha=1}^n l_N(\varphi_\alpha)} = \prod_{i=1}^N \left( \prod_{\alpha=1}^n p(T_i | X_i, \varphi_\alpha) \right) \gamma
\end{equation}
and since the observations are i.i.d.
\begin{equation}
\mathbb{E}_{\{(T_i, X_i)\}_{i=1}^n} \left[ e^{\gamma \sum_{\alpha=1}^n l_N(\varphi_\alpha)} \right] = \left( \mathbb{E}_{T; X} \left[ \prod_{\alpha=1}^n p^\gamma(T | X, \varphi_\alpha) \right] \right)^N
\end{equation}
which directly leads us to expression (88).

Upon differentiating between association and nuisance parameters, as these will be treated differently, the characteristic function of the ML estimator \( \hat{\varphi}_N = (\hat{\beta}_N, \hat{\sigma}_N) \) becomes
\begin{equation}
\Phi_{\hat{\beta}_N, \hat{\sigma}_N}(q, s) := \mathbb{E}_{\{(T_i, X_i)\}_{i=1}^n} \left[ e^{i(q^T \hat{\beta}_N + s^T \hat{\sigma}_N)} \right] = \lim_{\gamma \to \infty} \lim_{n \to 0} \phi_{N,n}(\gamma, q, s)
\end{equation}
where
\begin{equation}
\phi_{N,n}(\gamma, q, s) := \int \left( \mathbb{E}_{T; X} \left[ \prod_{\alpha=1}^n p^\gamma(T | X^T \beta_\alpha, \sigma_\alpha) \right] \right)^N e^{i(q^T \beta_1 + s^T \sigma_1)} \prod_{\alpha=1}^n d\beta_\alpha d\sigma_\alpha
\end{equation}
\begin{equation}
\left( \int \left( \mathbb{E}_{T; X} \left[ \prod_{\alpha=1}^n p^\gamma(T | X^T \beta_\alpha, \sigma_\alpha) \right] \right)^N \prod_{\alpha=1}^n d\beta_\alpha d\sigma_\alpha \right)
\end{equation}
Note that \( T \sim p(. | X^T \beta_0, \sigma_0) \), where \( \beta_0, \sigma_0 \) are the true values of the parameters that were responsible for the data. We next try to compute the asymptotic value \( \lim_{n \to \infty} \phi_{\hat{\beta}_N, \hat{\sigma}_N}(q, s) \) of the characteristic function. First, however, we will do the integrals over the association parameters \( \beta_1 \leq \alpha \leq n \in \mathbb{R}^p \) analytically.

Before proceeding we notice the following.

**Proposition A.2.** Suppose \( X \sim \mathcal{N}(0, I_p) \), then
\begin{equation}
\mathbb{E}_{T; X} \left[ \prod_{\alpha=1}^n p^\gamma(T | X^T \beta_\alpha, \sigma_\alpha) \right] = f(\gamma, \Sigma(\beta), \{\sigma\})
\end{equation}
with
\begin{equation}
f(\gamma, \Sigma, \{\sigma\}) = \int \frac{e^{-\frac{1}{2}y^T \Sigma^{-1}y}}{\sqrt{(2\pi)^{n+1} \det(\Sigma)}} \mathbb{E}_{T|y_0} \left[ \prod_{\alpha=1}^n p^\gamma(T | y_\alpha, \sigma_\alpha) \right] dy
\end{equation}
Here \( y = (y_0, \ldots, y_n) \) and the expectation \( \mathbb{E}_{T|y_0} \) refers to the distribution \( p(T | Y_0, \sigma) \), and the \((n+1) \times (n+1)\) matrix \( \Sigma(\beta) \) has entries \( \Sigma_{\alpha \rho} = \beta_\alpha^T \beta_\rho \).
Proof of Proposition A.2. By definition we may write

\[
\mathbb{E}_{T,X} \left[ \prod_{\alpha=1}^{n} p^\gamma(T|X^T \beta_\alpha, \sigma_\alpha) \right] = \mathbb{E}_X \mathbb{E}_{T|X} \left[ \prod_{\alpha=1}^{n} p^\gamma(T|X^T \beta_\alpha, \sigma_\alpha) \right] = \mathbb{E}_Y \mathbb{E}_{T|Y} \left[ \prod_{\alpha=1}^{n} p^\gamma(T|Y_\alpha, \sigma_\alpha) \right]
\]

(107)

in which the \( n+1 \) variables \( Y_\alpha \) are defined as \( Y_\alpha = X^T \beta_\alpha \). Since \( X \) are zero-average Gaussian variables, the same holds for \( Y = (Y_0, \ldots, Y_n) \), with the \( (n+1) \times (n+1) \) covariance matrix \( \Sigma(\beta) \) with entries \( \Sigma_{\alpha\rho} = \beta_\alpha^T \beta_\rho \). Writing the expectation over \( Y \) in explicit form as an integral immediately leads to (105).

Before we can evaluate the integrals over \{\( \beta_\alpha \)\} in (104) we first need to transform \( \beta_\alpha = \theta_\alpha/\sqrt{p} \), in which all components of each \( \theta_\alpha \) scale as \( O(1) \) for \( p, N \to \infty \). Now also \( \Sigma_{\alpha\rho} = \theta_\alpha^T \theta_\rho/p = O(1) \) for all \( (\alpha, \rho) \). With the short-hand (105) we can then write

\[
\phi_{N,n}(\gamma, q, s) = \frac{\int f^N(\gamma, \Sigma(\theta/\sqrt{p}), \{\sigma\}) e^{i(q^T \theta/\sqrt{p} + s^T \sigma)} \prod_{\alpha=1}^{n} d\theta_\alpha d\sigma_\alpha}{\int f^N(\gamma, \Sigma(\theta/\sqrt{p}), \{\sigma\}) \prod_{\alpha=1}^{n} d\theta_\alpha d\sigma_\alpha}
\]

(108)

Second, to make the \{\( \theta_\alpha \)\} integrals analytically doable, we need the following proposition which effectively enables us to relocate the matrix \( \Sigma(\theta/\sqrt{p}) \) to a more convenient place.

Proposition A.3. For any sufficiently smooth function \( h(\Sigma) \) we may write

\[
h(\Sigma) = (2\pi)^{-(n+1)/2} \int e^{i[\text{Tr}(\Omega \Sigma') - \text{Tr}(\Omega \Sigma)]} h(\Sigma') d\Omega d\Sigma'
\]

(109)

Proof of Proposition A.3. We write the Fourier transform \( \tilde{h}(\Omega) \) of \( h(\Sigma) \) in the standard manner as

\[
\tilde{h}(\Omega) = (2\pi)^{-\frac{1}{2}(n+1)^2} \int h(\Sigma') e^{i \sum_{\alpha, \beta=0}^{n} \Omega_{\alpha, \beta} \Sigma'_{\alpha, \beta}} d\Sigma'
\]

(110)

Application of the inverse Fourier transform now gives

\[
h(\Sigma) = (2\pi)^{-\frac{1}{2}(n+1)^2} \int \tilde{h}(\Omega) e^{-i \sum_{\alpha, \beta=0}^{n} \Omega_{\alpha, \beta} \Sigma_{\alpha, \beta}} d\Omega
\]

\[
= (2\pi)^{-(n+1)^2} \int e^{i \sum_{\alpha, \beta=0}^{n} \Omega_{\alpha, \beta} \Sigma'_{\alpha, \beta}} h(\Sigma') d\Omega d\Sigma'
\]

(111)

which immediately implies (109).

Proposition A.4. If \( f(\gamma, \Sigma(\theta/\sqrt{p}), \{\sigma\}) \) depends sufficiently smoothly on \( \Sigma(\theta/\sqrt{p}) \), then

\[
\phi_{N,n}(\gamma, q, s) = \frac{\int e^{ip \text{Tr}(\Omega \Sigma)} e^{is^T \sigma} f^N(\gamma, \Sigma, \{\sigma\}) J(\Omega, \theta_0, q) d\Sigma d\Omega \prod_{\alpha=1}^{n} d\sigma_\alpha}{\int e^{ip \text{Tr}(\Omega \Sigma)} f^N(\gamma, \Sigma, \{\sigma\}) J(\Omega, \theta_0, 0) d\Sigma d\Omega \prod_{\alpha=1}^{n} d\sigma_\alpha}
\]

with definition (106) and where

\[
J(\Omega, \theta_0, q) = \int e^{-ip \text{Tr}(\Omega \Sigma(\theta/\sqrt{p})) + iq^T \theta/\sqrt{p}} \prod_{\alpha=1}^{n} d\theta_\alpha
\]

(112)
PROOF OF PROPOSITION A.4. One simply substitutes the Fourier representation (109) of $f$ into (108), and one rescales the matrices $\Omega$ by a factor $p$. This immediately gives (112). \hfill \Box

In order to compute the Gaussian integrals $J(\Omega, \theta_0, q)$ in (113) we introduce the $n \times n$ block $K$ of the $(n+1) \times (n+1)$ matrix $\Omega$

$$\Omega := \begin{pmatrix} K_{00} & k^T \\ k & K \end{pmatrix}, \quad \text{with} \quad k := \begin{bmatrix} k_1 \\ \vdots \\ k_n \end{bmatrix}, \quad K := \begin{pmatrix} K_{11} & \cdots & K_{1n} \\ \vdots & \ddots & \vdots \\ K_{n1} & \cdots & K_{nn} \end{pmatrix}$$

With these definitions, and upon denoting the $\nu$-th unit vector in $\mathbb{R}^n$ as $e_\nu$, we can evaluate the integrals (113) further and obtain the following:

**LEMMA A.5.**

$$\phi_{N,n}(\gamma, q, s) = \frac{\int e^{NF(\gamma, \Sigma, \Omega; \sigma)} + \frac{1}{2} q^T (K^{-1})_{11} - i(q^T \beta_0) (e_1^T K^{-1} k) + i s^T \sigma_1 d \Sigma d \Omega \prod_{\alpha=1}^n d \sigma_\alpha}{\int e^{NF(\gamma, \Sigma, \Omega; \sigma)} d \Sigma d \Omega \prod_{\alpha=1}^n d \sigma_\alpha}$$

(115)

with, using $\zeta = p/N$,

$$F(\gamma, \Sigma, \Omega, \{\sigma\}) = \log f(\gamma, \Sigma, \{\sigma\}) + \zeta i \text{Tr} (\Omega \Sigma)$$

+ $\zeta \left( i S^2 k^T K^{-1} k - i K_{00} S^2 + \log \int e^{-iv^T K v} dv \right)$

**PROOF OF LEMMA A.5.** We define the short-hand $S^2 := \theta_0^T \theta_0/p$ and evaluate the function $J(\Omega, \theta_0, q)$ in (113). Since the integration in (113) is over all $\theta_{\alpha}$ with $1 \leq \alpha \leq n$, we rewrite the trace $\text{Tr} (\Omega \Sigma (\theta/\sqrt{p}))$ as follows

$$\text{Tr} (\Omega \Sigma (\theta/\sqrt{p})) = \sum_{\alpha, \rho=1}^n K_{\alpha \rho} \theta^T_\alpha \theta_\rho + 2 \sum_{\rho=1}^n k^T_\rho \theta^T_0 \theta_\rho + K_{00} \theta_0^T \theta_0$$

(117)

Insertion into (113) gives

$$J(\Omega, \theta_0, q) = e^{-i K_{00} \theta_0^T \theta_0} \prod_{j=1}^p \int e^{-i \left( \sum_{\alpha, \rho=1}^n K_{\alpha \rho} \theta^T_\alpha \theta_\rho + 2 \sum_{\rho=1}^n k^T_\rho \theta^T_0 \theta_\rho \right) + i q^T \theta_0 \theta_0 + i T \theta_0 \theta_0 + i T \theta_0 \theta_0} \prod_{\alpha=1}^n d \theta_\alpha$$

(118)

The above integral factorizes over the $p$ components of each vector $\theta_\alpha$. We indicate with $(\theta_\nu)_k$ the $k$-th entry of the vector $\theta_\nu$, so that

$$J(\Omega, \theta_0, q) = e^{-i K_{00} \theta_0^T \theta_0} \prod_{j=1}^p \int e^{-i \left( \sum_{\alpha, \rho=1}^n K_{\alpha \rho} \theta^T_\alpha \theta_\rho + 2 \sum_{\rho=1}^n k^T_\rho \theta^T_0 \theta_\rho \right) + i q^T \theta_0 \theta_0} \prod_{\alpha=1}^n d \theta_\alpha$$

(119)

With the short-hand $a = (\theta_0)_k - \frac{1}{2} j q, e_1/\sqrt{p}$ the integral in the above expression becomes

$$\int_{\mathbb{R}^n} e^{-i v^T K v - 2 a^T v} dv = \int_{\mathbb{R}^n} e^{-i \left( (v + K^{-1} a)^T K (v + K^{-1} a) \right) + i a^T K^{-1} a} dv$$

$$= e^{ia^T K^{-1} a} \int_{\mathbb{R}^n} e^{-i v^T K v} dv$$

$$= e^{i ((\theta_0)_k - \frac{1}{2} q, e_1/\sqrt{p})^T K^{-1} ((\theta_0)_k - \frac{1}{2} q, e_1/\sqrt{p})} \int_{\mathbb{R}^n} e^{-i v^T K v} dv$$

(120)
Insertion into (119), followed by rearranging of terms and transforming back to \( \beta_0 = \theta_0 / \sqrt{N} \), then leads directly to (115).

If the vector \( q \) is sparse, i.e. has only a finite number of nonzero components, the integral (115) can for \( N \to \infty \) be evaluated by the steepest descent method [6, 21]. If in line with the scaling demand \( \beta = \theta / \sqrt{N} \) we subsequently define the new asymptotic generating function

\[
\phi_n(\gamma, q, s) = \lim_{N \to \infty} \Phi_{N,n}(\gamma, \sqrt{N} q, s)
\]

then we obtain

\[
\phi_n(\gamma, q, s) = e^{\gamma(s)} q^2 \left( K^{-1} \right)_{11} e^{-i(q^T \theta_0)} (e^2 \sqrt{\gamma} k_i + i \sigma_k)
\]

where the quantities \( K, k \) and \( \sigma_k \) are now those values for which the function \( F(\gamma, \Sigma, \Omega, \{ \sigma \}) \) is extremized (these will therefore depend on \( \gamma \)).

A.2. Saddle point equations. It will prove useful to write the matrix \( \Sigma \) as

\[
\Sigma = \begin{pmatrix} C_{00} & c^T \\ c & C \end{pmatrix}
\]

with

\[
c = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}, \quad C = \begin{pmatrix} C_{11} & \cdots & C_{1n} \\ \vdots & \ddots & \vdots \\ C_{n1} & \cdots & C_{nn} \end{pmatrix}
\]

With these definitions, and upon inserting definition (105), we find that (116) takes the form

\[
F(\gamma, \Sigma, \Omega, \{ \sigma \}) = \log \mathbb{E} \left[ \prod_{\alpha=1}^{n} p^\gamma(T[Y_\alpha, \sigma_\alpha]) \right] + \zeta iK_{00}(\Sigma_{00} - S^2) + 2\zeta ic^T k + \zeta \text{Tr}(KC)
\]

\[
\quad + \zeta iS^2 k^T K^{-1} k + \zeta \log \int e^{-iv^T K v} dv
\]

Lemma A.6. Let \( F(\gamma, \Sigma, \Omega, \{ \sigma \}) \) be defined in (124) above, let \( K \) be defined by the identity \( K = -\frac{1}{2}iK \), and let the \((n+1) \times (n+1)\) dimensional matrix \( R := (R_{\alpha\beta})_{0 \leq \alpha, \beta \leq n} \) be defined by its entries

\[
R_{\alpha\beta} := \frac{\mathbb{E} \left[ Y_\alpha Y_\beta \prod_{\alpha=1}^{n} p^\gamma(T[Y_\alpha, \sigma_\alpha]) \right]}{\mathbb{E} \left[ \prod_{\alpha=1}^{n} p^\gamma(T[Y_\alpha, \sigma_\alpha]) \right]}
\]

Then the saddle point equations, obtained by setting the gradient of \( F(\cdot) \) with respect to its arguments equal to zero, take the following form for \( \alpha, \beta = 1 \ldots n; \)

\[
\frac{\partial F}{\partial K_{00}} = 0 : \quad \Sigma_{00} = S^2
\]

\[
\frac{\partial F}{\partial k_\alpha} = 0 : \quad c - \frac{1}{2} S^2 K^{-1} k = 0, \quad \forall \alpha
\]

\[
\frac{\partial F}{\partial \sigma_{\alpha j}} = 0 : \quad \frac{\mathbb{E} \left[ \left( \frac{\partial}{\partial \sigma_{\alpha j}} \log p(T[Y_\alpha, \sigma_\alpha]) \right) \prod_{\rho=1}^{n} p^\gamma(T[Y_\rho, \sigma_\rho]) \right]}{\mathbb{E} \left[ \prod_{\rho=1}^{n} p^\gamma(T[Y_\rho, \sigma_\rho]) \right]} = 0, \quad \forall \alpha, j
\]

\[
\frac{\partial F}{\partial K_{\alpha\beta}} = 0 : \quad C_{\alpha\beta} + 4S^2 (K^{-1} k)_\alpha (K^{-1} k)_\beta - (K^{-1})_{\alpha\beta} = 0. \quad \forall \alpha, \beta
\]

\[
\frac{\partial F}{\partial C_\alpha} = 0 : \quad \frac{1}{2} (\Sigma^{-1} R \Sigma^{-1} - \Sigma^{-1})_{0\alpha} + 2\zeta ik_\alpha = 0, \quad \forall \alpha
\]

\[
\frac{\partial F}{\partial C_{\alpha\beta}} = 0 : \quad \frac{1}{2} (\Sigma^{-1} R \Sigma^{-1} - \Sigma^{-1})_{\alpha\beta} + \zeta K_{\alpha\beta} = 0, \quad \forall \alpha, \beta
\]
PROOF OF LEMMA A.6. Equations (126-128) follow immediately. We only need to derive in more detail equations (129-131). To derive (129) we can use the general identity $\frac{\partial}{\partial k_{\alpha\beta}}(K^{-1})_{\gamma\rho} = -(K^{-1})_{\alpha\gamma}(K^{-1})_{\beta\rho}$, and the fact that $K$ has to be purely imaginary for the integral in (124) to exist:

\begin{equation}
0 = \frac{\partial F}{\partial K_{\alpha\beta}} = \zeta i \left\{ C_{\alpha\beta} - S^2 (K^{-1})_{\alpha}(K^{-1})_{\beta} - \int e^{-iv^T K v_{\alpha\beta}} dv \right\}
\end{equation}

Putting $K = -\frac{1}{2} i \tilde{K}$, in which $K$ is symmetric, real-valued and positive definite, gives

\begin{equation}
C_{\alpha\beta} = -4S^2 (K^{-1})_{\alpha}(K^{-1})_{\beta} + \int e^{-\frac{1}{2} v^T K v_{\alpha\beta}} dv
\end{equation}

(133)

To derive equations (130,131) we first work out, using (106) and the general identity

$\frac{\partial}{\partial \Sigma_{\alpha\beta}} \log \det(\Sigma) = (\Sigma^{-1})_{\alpha\beta}$.

\begin{align}
\Xi_{\alpha\beta} &= \frac{\partial}{\partial \Sigma_{\alpha\beta}} \log \mathbb{E}_{\Sigma, T} \left[ \prod_{\rho=1}^{n} p^\gamma(T | y_\rho, \sigma_\rho) \right] \\
&= \frac{\partial}{\partial \Sigma_{\alpha\beta}} \left\{ \log \int e^{-\frac{1}{2} y^T \Sigma^{-1} y} \mathbb{E}_{T|y_\rho} \left[ \prod_{\rho=1}^{n} p^\gamma(T | y_\rho, \sigma_\rho) \right] dy - \frac{1}{2} \log \det(\Sigma) \right\} \\
&= -\frac{1}{2} \sum_{\lambda\nu} \frac{\partial (\Sigma^{-1})_{\lambda\nu}}{\partial \Sigma_{\alpha\beta}} \int e^{-\frac{1}{2} y^T \Sigma^{-1} y} \mathbb{E}_{T|y_\rho} \left[ \prod_{\rho=1}^{n} p^\gamma(T | y_\rho, \sigma_\rho) \right] dy - \frac{1}{2} (\Sigma^{-1})_{\alpha\beta} \\
&= \frac{1}{2} \sum_{\lambda\nu} (\Sigma^{-1})_{\alpha\lambda} (\Sigma^{-1})_{\beta\nu} R_{\lambda\nu} - \frac{1}{2} (\Sigma^{-1})_{\alpha\beta}
\end{align}

(134)

The extremization of (124) over $c_\alpha$ (with $\alpha = 1 \ldots n$) gives $\Xi_{0\alpha} + \zeta i k_{\alpha} = 0$, and hence via (134) we arrive at (130). Similarly, the extremization of (124) over $C_{\alpha\beta}$ (with $\alpha, \beta = 1 \ldots n$) gives $\Xi_{\alpha\beta} + \zeta i K_{\alpha\beta} = 0$, and hence via (134) we arrive at (131). This concludes the proof. □

We can simplify the equations (126-131) further. First, (126) enables us to remove $\Sigma_{00}$. Second, we transform the elements of the $n \times n$ matrices: $K_{\alpha\beta} = -\frac{1}{2} i \tilde{K}_{\alpha\beta}$ (introduced earlier) and $\tilde{C}_{\alpha\beta} = C_{\alpha\beta} - S^{-2} c_\alpha c_\beta$. Third, since in contrast to the parameters $\{k_\alpha, \tilde{K}_{\alpha\beta}\}$, the quantities $\{c_\alpha, \tilde{C}_{\alpha\beta}\}$ have clear interpretations, we use (127) and (131) to eliminate the former. Equation (127) gives $k = -S^{-2} \tilde{K} c = \frac{1}{2} i S^{-2} \tilde{K} c$, which simplifies (129) to $C_{\alpha\beta} - S^{-2} c_\alpha c_\beta - (\tilde{K}^{-1})_{\alpha\beta} = 0$. Equivalently we may write

\begin{equation}
\tilde{K} = \tilde{C}^{-1}, \quad k = \frac{1}{2} i S^{-2} \tilde{C}^{-1} c
\end{equation}

(135)

The final result is a set of three coupled equations, namely (128) and the following two equations which result upon elimination $\tilde{K}$ and $k$ from (130,131) via (135):

\begin{align}
S^2 (\Sigma^{-1} \mathbf{R} \Sigma^{-1} - \Sigma^{-1})_{0\alpha} - \zeta (\tilde{C}^{-1} c)_{\alpha} &= 0 \\
(\Sigma^{-1} \mathbf{R} \Sigma^{-1} - \Sigma^{-1})_{\alpha\beta} + \zeta (\tilde{C}^{-1})_{\alpha\beta} &= 0
\end{align}

(136) (137)
with $\alpha, \beta = 1 \ldots n$. Once $\tilde{C}$ and $c$ are solved from (136,137), one finds $K$ and $k$ via (135), $\Sigma_{00} = S^2$, and the matrix $C$ via $C_{\alpha \beta} = \tilde{C}_{\alpha \beta} + S^{-2} c_{\alpha} c_{\beta}$. From these relations, in turn, follow the matrices $\Sigma$ and $\Sigma^{-1}$. Finally, the auxiliary parameters $\sigma_{\alpha j}$ are solved from (128).

It is known from the general theory behind the replica method [27, 35] that for ergodic systems the relevant solution of the saddle point equations will be invariant under all permutations of the replica labels $\{1, \ldots, n\}$. Such solutions are called replica symmetric (RS). In contrast, systems with complex ergodicity breaking would exhibit solutions with broken replica symmetry (RSB) [17, 32]. In the survival analysis context there is no evidence for RSB [7], and we will validate a posteriori the results of the RS assumption via simulations.

The assumption of RS saddle points implies that

$$\forall \alpha, \beta = 1 \ldots n: \quad C_{\alpha \beta} = C_{\delta \alpha \beta} + c(1 - \delta_{\alpha \beta}), \quad c_\alpha = c_0, \quad \sigma_{\alpha k} = \sigma_{1k}$$

with the Kronecker symbol (i.e. $\delta_{\alpha \beta} = 1$ for $\alpha = \beta$ and $\delta_{\alpha \beta} = 0$ for $\alpha \neq \beta$). The replica permutation symmetries in (138) induce similar symmetries in $K$, $C$ and $k$, and imply the following form of the $(n+1) \times (n+1)$ matrix $\Sigma^{-1}$:

$$\Sigma^{-1} = \begin{pmatrix}
d_{00} & d_0 & \ldots & d_0 \\
d_0 & D & d & \ldots & d \\
: & d & D & \ddots & : \\
: & : & \ddots & \ddots & d \\
d_0 & d & \ldots & d & D
\end{pmatrix}$$

A similar claim is true for the matrix $R$:

**Proposition A.7.** Given the RS assumption (138), the matrix $R$ will have the form:

$$R = \begin{pmatrix}
r_{00} & r_0 & \ldots & r_0 \\
r_0 & R & r & \ldots & r \\
: & r & R & \ddots & : \\
: & : & \ddots & \ddots & r \\
r_0 & r & \ldots & r & R
\end{pmatrix}$$

where

$$r = \frac{\mathbb{E}_{Q,T,Z_0} \left[ \left( \mathbb{E}_{Z} \left[ Z_0 - \frac{1}{d-a} (Q\sqrt{-d - \frac{Z_{d0}}{\sqrt{d_{00}}}}) \right] \right)^2 A^n(\gamma, T, Q, Z_0) \right]}{\mathbb{E}_{Q,T,Z_0} \left[ A^n(\gamma, T, Q, Z_0) \right]}$$

$$R = \frac{\mathbb{E}_{Q,T,Z_0} \left[ \mathbb{E}_{Z} \left[ Z_0 - \frac{1}{d-a} (Q\sqrt{-d - \frac{Z_{d0}}{\sqrt{d_{00}}}}) \right] \right] A^n(\gamma, T, Q, Z_0)}{\mathbb{E}_{Q,T,Z_0} \left[ A^n(\gamma, T, Q, Z_0) \right]}$$

$$r_0 = \frac{\mathbb{E}_{Q,T,Z_0} \left[ Z_0^2 A^n(\gamma, T, Q, Z_0) \right]}{\sqrt{d_{00}} \mathbb{E}_{Q,T,Z_0} \left[ A^n(\gamma, T, Q, Z_0) \right]}$$

$$r_{00} = \frac{\mathbb{E}_{Q,T,Z_0} \left[ A^n(\gamma, T, Q, Z_0) \right]}{d_{00} \mathbb{E}_{Q,T,Z_0} \left[ A^n(\gamma, T, Q, Z_0) \right]}$$
with

\begin{align*}
A(\gamma, T, Q, Z_0) &:= \mathbb{E}_Z \left[ p^\gamma (T \left| \frac{Z}{\sqrt{D-d}} + \frac{1}{D-d} (Q \sqrt{-d} - \frac{Z_0 d_0}{\sqrt{d_0}}), \sigma \right. \right] \\
\mathbb{E}^\prime \left[ f(Z) \right] &:= \frac{\mathbb{E}_Z \left[ f(Z) p^\gamma (T \left| \frac{Z}{\sqrt{D-d}} + \frac{1}{D-d} (Q \sqrt{-d} - \frac{Z_0 d_0}{\sqrt{d_0}}), \sigma \right. \right] }{\mathbb{E}_Z \left[ p^\gamma (T \left| \frac{Z}{\sqrt{D-d}} + \frac{1}{D-d} (Q \sqrt{-d} - \frac{Z_0 d_0}{\sqrt{d_0}}), \sigma \right. \right] }
\end{align*}

and where $T, Q, Z, Z_0 \sim \mathcal{N}(0, 1)$ are independent Gaussian random variables.

**Proof of Proposition A.7.** To compute the elements of the matrix $\mathbf{R}$ we first evaluate

\begin{align}
\sum_{\alpha, \rho=0}^{n} y_{\alpha} y_{\rho} (\Sigma^{-1})_{\alpha \rho} &= d_0 y_0^2 + 2d_0 y_0 \sum_{\alpha=1}^{n} y_{\alpha} + (D-d) \sum_{\alpha=1}^{n} y_{\alpha}^2 + d \left( \sum_{\alpha=1}^{n} y_{\alpha} \right)^2 \\
\text{This gives} &\quad e^{-\frac{1}{2} y^T \Sigma^{-1} y} = e^{-\frac{1}{2} d_0 y_0^2 - d_0 y_0 \sum_{\alpha=1}^{n} y_{\alpha} - \frac{1}{2} (D-d) \sum_{\alpha=1}^{n} y_{\alpha}^2 - \frac{1}{2} d \left( \sum_{\alpha=1}^{n} y_{\alpha} \right)^2 } \\
&\quad = e^{-\frac{1}{2} d_0 y_0^2} \mathbb{E}_Q \left[ \prod_{\alpha=1}^{n} e^{-d_0 y_0 y_{\alpha} - \frac{1}{2} (D-d) y_{\alpha}^2 + Q y_{\alpha} \sqrt{-d}} \right]
\end{align}

and hence (we will find later that $d \leq 0$):

\begin{align}
R_{\alpha \rho} &= \frac{\int e^{-\frac{1}{2} d_0 y_0^2} y_{\alpha} y_{\rho} \mathbb{E}_Q \left[ \prod_{\alpha=1}^{n} e^{-\frac{1}{2} (D-d) y_{\alpha}^2 + (Q \sqrt{-d} - d_0 y_0) y_{\alpha} p^\gamma (T \left| y_{\alpha}, \sigma_{\alpha} \right. \right) \right] dy}{\int e^{-\frac{1}{2} d_0 y_0^2} \mathbb{E}_Q \left[ \prod_{\alpha=1}^{n} e^{-\frac{1}{2} (D-d) y_{\alpha}^2 + (Q \sqrt{-d} - d_0 y_0) y_{\alpha} p^\gamma (T \left| y_{\alpha}, \sigma_{\alpha} \right. \right) \right] dy} \\
&= \frac{\int e^{-\frac{1}{2} d_0 y_0^2} y_{\alpha} y_{\rho} \mathbb{E}_Q \left[ \prod_{\alpha=1}^{n} e^{-\frac{1}{2} (D-d) y_{\alpha}^2 + (Q \sqrt{-d} - d_0 y_0) y_{\alpha} - \gamma \frac{\sqrt{-d} y_{\alpha} - d_0 y_0}{\sqrt{d_0}})^2 p^\gamma (T \left| y_{\alpha}, \sigma_{\alpha} \right. \right) \right] dy}{\int e^{-\frac{1}{2} d_0 y_0^2} \mathbb{E}_Q \left[ \prod_{\alpha=1}^{n} e^{-\frac{1}{2} (D-d) y_{\alpha}^2 + (Q \sqrt{-d} - d_0 y_0) y_{\alpha} - \gamma \frac{\sqrt{-d} y_{\alpha} - d_0 y_0}{\sqrt{d_0}})^2 p^\gamma (T \left| y_{\alpha}, \sigma_{\alpha} \right. \right) \right] dy}
\end{align}

We transform the integration variables, writing $y_{\alpha} = y_{\alpha}(z_{\alpha}, z_0)$, where

\begin{align}
y_0 &= \frac{z_0}{\sqrt{d_0}} \quad , \quad y_{\alpha > 0} = \frac{Q \sqrt{-d} - d_0 z_0 \sqrt{-d_0}}{D-d} + \frac{z_0}{\sqrt{D-d}}
\end{align}

giving

\begin{align}
R_{\alpha \rho} &= \frac{\int e^{-\frac{1}{2} \gamma z_{\alpha}^2} y_{\alpha}(z_{\alpha}, z_0) y_{\rho}(z_{\rho}, z_0) \mathbb{E}_Q \left[ \prod_{\alpha=1}^{n} e^{-\frac{1}{2} \gamma z_{\alpha}^2} p^\gamma (T \left| y_{\alpha}(z_{\alpha}, z_0), \sigma_{\alpha} \right. \right) \right] dz}{\int e^{-\frac{1}{2} \gamma z_{\alpha}^2} \mathbb{E}_Q \left[ \prod_{\alpha=1}^{n} e^{-\frac{1}{2} \gamma z_{\alpha}^2} p^\gamma (T \left| y_{\alpha}(z_{\alpha}, z_0), \sigma_{\alpha} \right. \right) \right] dz}
\end{align}

The various permutation symmetries for replicas $\alpha = 1 \ldots n$ guarantee that the matrix $\mathbf{R}$ indeed has the form (140). We can now compute the relevant entries, and arrive after some straightforward calculations at the formulae in the proposition.

**Proposition A.8 (Elements of $\Sigma^{-1}$).** Given the RS forms of the matrices $\Sigma$ and $\Sigma^{-1}$, see (139), the components of the latter read

\begin{align}
d_0 &= \frac{c_0}{nc_0^2 - S^2(C + c(n-1))} \\
d_{00} &= \frac{C + c(n-1)}{S^2(C + c(n-1)) - nc_0^2}
\end{align}
\[
\begin{align*}
  d &= \frac{1}{C-c} \frac{c_0^2 - cS^2}{S^2(C+c(n-1)) - nc_0^2} \\
  D &= \frac{1}{C-c} \left( 1 + \frac{c_0^2 - cS^2}{S^2(C+c(n-1)) - nc_0^2} \right)
\end{align*}
\]

**Proof of Proposition A.8.** One can simply substitute the above expressions into (139) and confirm that they give \( \Sigma \Sigma^{-1} = \Sigma^{-1} \Sigma = I_{n+1} \), which completes the proof. \( \square \)

**Proposition A.9.** Given the RS form of \( \Sigma \) as defined in (123), which implies that \( \tilde{C}_{\alpha\beta} = c - (c_0/S)^2 + \delta_{\alpha\beta}(C-c) \), the elements of the \( n \times n \) matrix \( \tilde{C}^{-1} \) with entries \( (\tilde{C}^{-1})_{\alpha\beta} = b_0 \delta_{\alpha\beta} + b_1 (1 - \delta_{\alpha\beta}) \) are given by

\[
  b_0 = \frac{1}{C-c} + b_1, \quad b_1 = \frac{(c_0/S)^2 - c}{C-c+n(c-c_0^2/S^2)}
\]

**Proof of Proposition A.9.** Straightforward substitution of the above expressions for \( b_0 \) and \( b_1 \) confirms that \( \Sigma \Sigma^{-1} = \tilde{C}^{-1} \tilde{C} = I_n \), which completes the proof. \( \square \)

Finally we can write down the Replica Symmetric saddle point equations, obtained by substitution of the various replica symmetric forms into the three closed equations (128,136,137).

**Lemma A.10 (Replica Symmetric Saddle Point equations).** Given the RS form of \( \Sigma \) as defined in (123), the remaining coupled saddle point equations (128,136,137) translate into

\[
\begin{align*}
  \mathbb{E}_{Q,T,Z \alpha} \left[ \mathbb{E}_Z \left[ \frac{\partial}{\partial \sigma_j} \log p(T \frac{Z}{\sqrt{D-d}} + \frac{1}{D-d} (Q \sqrt{\frac{1}{D-d} - \frac{Z d_0}{\sqrt{d_0}}}) \right] A^n(\gamma, T, Q, Z_0) \right] &= 0 \forall j \\
  d_0 d_0 r_{00} + (d_0 [D+(n-1)d] + nd_0^2) r_0 + d_0 [D+(n-1)d][R+(n-1)r] \\
  &= d_0 + \frac{\zeta c_0}{S^2} [b_0 + (n-1)b_1] \\
  nd_0^2 r_{00} + 2nd_0 [D+(n-1)d] r_0 + [D+(n-1)d]^2 [R+(n-1)r] \\
  &= D + (n-1)d - \zeta [b_0 + (n-1)b_1] \\
  d_0^2 r_{00} + 2d_0 [D+(n-1)d] r_0 + [D^2+(n-1)d^2] R + d[2D+(n-2)d](n-1)r \\
  &= D - \zeta b_0 \end{align*}
\]

**Proof of Lemma A.10.** Equation (157) follows directly upon substitution of (150) into (128). To derive (158,159,160) we use the properties that in RS solutions all components of (136) are identical, and that all \( n^2 \) components of (137) depend solely on whether or not they are diagonal entries. Thus we may carry out suitable index summations and simplify (136,137) to the following three scalar equations, where we used \( c = c_0(1, \ldots, 1) \):

\[
\begin{align*}
  \frac{1}{n} \sum_{\alpha=1}^n (\Sigma^{-1} R \Sigma^{-1})_{\alpha\alpha} - \frac{1}{n} \sum_{\alpha=1}^n (\Sigma^{-1})_{\alpha\alpha} &= \frac{\zeta c_0}{nS^2} \sum_{\alpha=1}^n (\tilde{C}^{-1})_{\alpha\alpha} \\
  0 &= \frac{1}{n} \sum_{\alpha=1}^n (\Sigma^{-1} R \Sigma^{-1})_{\alpha\beta} - \frac{1}{n} \sum_{\alpha=1}^n (\Sigma^{-1})_{\alpha\beta} + \frac{\zeta}{n} \sum_{\alpha=1}^n (\tilde{C}^{-1})_{\alpha\beta} \\
  0 &= \frac{1}{n} \sum_{\alpha=1}^n (\Sigma^{-1} R \Sigma^{-1})_{\alpha\alpha} - \frac{1}{n} \sum_{\alpha=1}^n (\Sigma^{-1})_{\alpha\alpha} + \frac{\zeta}{n} \sum_{\alpha=1}^n (\tilde{C}^{-1})_{\alpha\alpha}
\end{align*}
\]
From the RS form of $\hat{C}^{-1}$ (see Proposition A.9) we infer that $n^{-1} \sum_{\alpha=1}^{n} (\hat{C}^{-1})_{\alpha\alpha} = b_0$ and $n^{-1} \sum_{\alpha=1}^{n} (\hat{C}^{-1})_{\alpha\beta} = b_0 + (n-1)b_1$. Similarly we have $n^{-1} \sum_{\alpha=1}^{n} (\Sigma^{-1})_{\alpha\alpha} = D$, $n^{-1} \sum_{\alpha=1}^{n} (\Sigma^{-1})_{\alpha\beta} = D + (n-1)d$, and $n^{-1} \sum_{\alpha=1}^{n} (\Sigma^{-1})_{\alpha\beta} = d_0$. The tricky terms are those involving the matrix product $\Sigma^{-1}R\Sigma^{-1}$. Upon defining $\tau_\alpha = n^{-1} \sum_{\beta=1}^{n} (\Sigma^{-1})_{\alpha\beta}$, $\tau = (\tau_0, \ldots, \tau_n)$, $\rho = ((\Sigma^{-1})_{00}, \ldots, (\Sigma^{-1})_{0n})$, and $H_{\lambda\rho} = \sum_{\alpha=1}^{n} (\Sigma^{-1})_{\lambda\alpha} (\Sigma^{-1})_{\alpha\rho}$, our three equations above become
\begin{align}
(164) & \quad \rho^T R \tau = d_0 + \frac{\zeta c_0}{S^2} \left[ b_0 + (n-1)b_1 \right] \\
(165) & \quad n\tau^T R \tau = D + (n-1)d - \zeta [b_0 + (n-1)b_1] \\
(166) & \quad \frac{1}{n} \sum_{\lambda\rho=0}^{n} H_{\lambda\rho} R_{\mu\lambda} = D - \zeta b_0
\end{align}
Working out the remaining quantities, one finds that $\tau_\alpha = d_0 \delta_{\alpha0} + [d + n^{-1}(D-d)](1-\delta_{\alpha0})$ and $\rho_\alpha = d_{00}\delta_{\alpha0} + d_0(1-\delta_{\alpha0})$. It follows from (139) that the $(n+1)^2$ entries $\{H_{\lambda\rho}\}$ are
\begin{align}
(167) & \quad \lambda = \rho = 0 : \quad H_{\lambda\rho} = nd_0^2 \\
(168) & \quad \lambda = 0, \rho \neq 0 \text{ or } \rho = 0, \lambda \neq 0 : \quad H_{\lambda\rho} = d_0[D + (n-1)d] \\
(169) & \quad \lambda, \rho \neq 0 : \quad H_{\lambda\rho} = nd_0^2 + 2d(D-d) + (D-d)^2\delta_{\lambda\rho}
\end{align}
Thus, upon using also identities such as $\sum_{\beta=1}^{n} R_{\alpha\beta} = nrr_0\delta_{\alpha0} + [R + (n-1)r](1-\delta_{\alpha0})$, the equations (164,165,166) are then indeed seen to become (158,159,160), as claimed.

**LEMMA A.11 (Replica Symmetric moment generating function).** Given the RS form of $\Sigma$ as defined in (123), the characteristic function (122) for sparse vectors $q$ takes the form
\begin{equation}
\phi_n(\gamma, q, s) = e^{-\frac{1}{2}((C-c_0^2)/S^2)q^2 + i(q^T\theta_0)(c_0/S^2) + is^T\sigma}
\end{equation}

**PROOF OF LEMMA A.11.** In RS saddle points we can evaluate and simplify the quantities in (135) further, and find
\begin{align}
(171) & \quad (K^{-1})_{\alpha\beta} = \left[ (-\frac{1}{2}i\hat{C}^{-1})^{-1} \right]_{\alpha\beta} = 2i\hat{C}_{\alpha\beta} = 2i[C + c(1-\delta_{\alpha\beta}) - c_0^2/S^2] \\
(172) & \quad k_\alpha = \left[ \frac{1}{2}iS^{-2}C^{-1} \right]_{\alpha\beta} = \frac{1}{2}ic_0S^{-2}[b_0 + (n-1)b_1]
\end{align}
Thus at the RS saddle point one has $(K^{-1})_{11} = 2i(C-c_0^2/S^2)$ and $e_1^T K^{-1} k = -c_0/S^2$. Inserting these expressions into (135) reproduces (170).
\[\square\]

**A.3. The limit $n \to 0$.** Our theory demands that we take the limit $n \to 0$. For RS saddle points this has become easy, and one indeed finds that all RS steepest descent parameters have well-defined limits. For $n \to 0$ the relations (152-155) and (156) reduce to
\begin{align}
(173) & \quad d_0 = -\frac{c_0}{S^2(C-c)}, \quad d_{00} = \frac{1}{S^2}, \quad d = -\frac{c-c_0^2/S^2}{(C-c)^2}, \quad D = d + \frac{1}{C-c} \\
(174) & \quad b_0 = \frac{1}{C-c} + b_1, \quad b_1 = \frac{(c_0/S)^2-c}{(C-c)^2}
\end{align}
Upon taking the limit \( n \to 0 \) in our saddle point equations (157,158,159,160), and using the relations above to eliminate the parameters \((d_0, d_{00}, d, D, b_0, b_1)\) in favour of \((C, c, c_0)\), our equations are found to simplify to

\[
\begin{align*}
E_{Q,T,Z_0} \left[ \frac{\partial}{\partial \sigma} \mathbb{E}_Z \left[ p(T|Z) \sqrt{C-c} + Q\sqrt{c-c_0^2/S^2} + (c_0/S)Z_0, \sigma \right] \right] = 0 \quad \forall j \\
\frac{r_0}{c_0} - \frac{r_{00}}{S^2} - \frac{R-r}{C-c} = \zeta - 1 \\
\frac{R-r}{C-c} = 1 - \zeta \\
\frac{c_0^2 r_{00}}{S^4} - \frac{2c_0 r_0}{S^2} + R - \frac{2(R-r)(c-c_0^2/S^2)}{C-c} = (1-\zeta) \left( C - 2c + \frac{c_0^2}{S^2} \right)
\end{align*}
\]

At this stage it is advantageous to introduced transformed variables \((\bar{u}, v, w)\), via

\[
c_0 = wS, \quad C-c = \bar{u}^2, \quad c-c_0^2/S^2 = v^2
\]

Our four equations (175,176,177,178) then become (and after some simple manipulations):

\[
\begin{align*}
0 &= E_{Q,T,Z_0} \left[ \mathbb{E}_Z' \left[ \frac{\partial}{\partial \sigma} \log p(T|\bar{u}Z + vQ + wZ_0, \sigma) \right] \right] \quad \forall j \\
r_0 &= r_{00}w/S \\
r &= w^2r_{00}/S^2 + (1-\zeta)v^2 \\
R &= w^2r_{00}/S^2 + (1-\zeta)(\bar{u}^2 + v^2)
\end{align*}
\]

Similarly, we can take the limit \( n \to 0 \) in the entries of (140). This gives \( r_{00} = 1/d_{00} = S^2 \), and upon eliminating also \((r, R, r_0)\) our RS equations (181,182,183) become

\[
\begin{align*}
w^2 + (1-\zeta)v^2 &= E_{Q,T,Z_0} \left[ \left( \mathbb{E}_Z' [\bar{u}Z + vQ + wZ_0] \right)^2 \right] \\
w^2 + (1-\zeta)(\bar{u}^2 + v^2) &= E_{Q,T,Z_0} \left[ \mathbb{E}_Z' [(\bar{u}Z + vQ + wZ_0)^2] \right] \\
w &= E_{Q,T,Z_0} \left[ Z_0 \mathbb{E}_Z' [\bar{u}Z + vQ + wZ_0] \right]
\end{align*}
\]

in which now

\[
\mathbb{E}_Z' [f(Z)] = \frac{E_Z \left[ f(Z)p^\gamma(T|\bar{u}Z + vQ + wZ_0, \sigma) \right]}{E_Z \left[ p^\gamma(T|\bar{u}Z + vQ + wZ_0, \sigma) \right]}
\]

In terms of the parameters \((\bar{u}, v, w)\) we obtain the following expression for the characteristic function (170) in the limit \( n \to 0 \):

\[
\phi(\gamma, q, s) = \lim_{n \to 0} \phi_n(\gamma, q, s) = e^{-\frac{1}{2}(\bar{u}^2 + v^2)q^2 + i((w/S)q^T \theta_0 + s^T \sigma)\bar{u}}
\]

For subsequent manipulations, especially the limit \( \gamma \to \infty \), it will be helpful to write our equations in terms of averages over the random variable \( \xi = \bar{u}Z + vQ + wZ_0 \) instead of \( Z \).

**Lemma A.12.** Our four coupled RS saddle point equations (180,181,182,183) in the limit \( n \to 0 \) are equivalent to

\[
(1-\zeta)v = E_{Q,T,Z_0} \left[ Q\mathbb{E}_Z [\xi] \right]
\]
\[ w = E_{Q,T,Z_0}[Z_0 E_{\xi}[\xi]] \]

\[ (1 - \zeta)v^2 + w^2 = E_{Q,T,Z_0}[E_{\xi}[\xi]^2] \]

\[ 0 = E_{Q,T,Z_0}\left[ E_{\xi} \left[ \frac{\partial}{\partial \sigma_j} \log p(T|\xi, \sigma) \right] \right] \quad \forall j \]

in which

\[ E_{\xi}[f(\xi)] = \frac{\int f(\xi)e^{-\frac{1}{2}(\xi-vQ-wZ_0)^2/\bar{u}^2} p^\gamma(T|\xi, \sigma) \, d\xi}{\int e^{-\frac{1}{2}(\xi-vQ-wZ_0)^2/\bar{u}^2} p^\gamma(T|\xi, \sigma) \, d\xi} \]

PROOF. Equations (190,191,192) follow immediately upon writing their corresponding precursors (186), (184) and (180), respectively, in terms of averages over \( \xi \). To confirm the validity of expression (189) one carries out partial integration over the zero-average and unit-variance Gaussian variable \( Q \), using (184) and (185):

\[ E_{Q,T,Z_0}[Q E_{\xi}[\xi]] = E_{Q,T,Z_0}\left[ \frac{\partial}{\partial Q} E_{\xi}[\xi] \right] = \frac{v}{u^2} E_{Q,T,Z_0}[E_{\xi}[\xi^2] - E_{\xi}[\xi]^2] \]

\[ = \frac{v}{u^2} \left( w^2 + (1 - \zeta)(\bar{u}^2 + v^2) - w^2 - (1 - \zeta)v^2 \right) = (1 - \zeta)v \]

This completes the proof. \( \square \)

A.4. Limit \( \gamma \to \infty \). The final stage of the theory is taking the limit \( \gamma \to \infty \) in the saddle point equations. Careful inspection shows that \( (v, w) \) and \( \sigma \) have well-defined limits, but that \( \bar{u} = O(\gamma^{-\frac{1}{2}}) \) as \( \gamma \to \infty \). Hence we must first transform \( \bar{u} = u/\sqrt{\gamma} \), so that

\[ \lim_{\gamma \to \infty} E_{\xi}[f(\xi)] = \lim_{\gamma \to \infty} \frac{\int f(\xi)\gamma^{-\frac{1}{2}}(\xi-vQ-wZ_0)^2/u^2 + \log p(T|\xi, \sigma) \, d\xi}{\int \gamma^{-\frac{1}{2}}(\xi-vQ-wZ_0)^2/u^2 + \log p(T|\xi, \sigma) \, d\xi} = f(\xi_*) \]

in which

\[ \xi_* = \arg \min_{\xi} \left\{ \frac{1}{2} \left( \xi - vQ - wZ_0 \right)^2/u^2 - \log p(T|\xi, \sigma) \right\} \]

Hence, upon application to (189,190,191,192) we conclude that the \( \gamma \to \infty \) limits of the RS parameters \( (u, v, w, \sigma) \) are to be solved from

\[ (1 - \zeta)v = E_{Q,T,Z_0}[Q \xi_*] \]

\[ w = E_{Q,T,Z_0}[Z_0 \xi_*] \]

\[ (1 - \zeta)v^2 + w^2 = E_{Q,T,Z_0}[\xi_*^2] \]

\[ 0 = E_{Q,T,Z_0}\left[ \frac{\partial}{\partial \sigma_j} \log p(T|\xi_*, \sigma) \right] \quad \forall j \]

After some simple final manipulations, such as integration by parts over \( Q \) in (197) and integration by parts over \( Z_0 \) in (198), we then arrive at equations (7,8,9,10) in the main text. The characteristic function (188) simplifies for \( \gamma \to \infty \) to

\[ \phi(q,s) = \lim_{\gamma \to \infty} \phi(\gamma, q, s) = e^{-\frac{1}{2}v^2q^2 + i[w/S]q^r \theta_0 + s^r \sigma} \]
APPENDIX B: DETAILED CALCULATIONS FOR THE MODELS ANALYZED

We present here for completeness additional details on how to derive from the generic RS equations (7,8,9,10) the final model-specific forms in the main text. Each subsection deals explicitly with one model. Since after transforming the response to a logarithmic scale, these specific models become linear, we will always be able start from equations (24,25,26,27)\(^3\).

B.1. The Weibull model. The self consistent equation of the general theory (28) admits for the Weibull model the (formal) solution (51). We note that specific models become linear, we will always be able start from equations (24,25,26,27)

\begin{equation}
F := e^{-\frac{(Y - X_0)}{\sigma}} \sim Exp(1)
\end{equation}

Hence

\begin{equation}
\eta_\ast = vQ + \frac{u^2}{\sigma} - \sigma W_0 \left( \frac{u^2}{\sigma^2} F \frac{\sigma_0}{\sigma} \exp \left( \frac{u^2}{\sigma^2} \phi - \frac{vQ}{\sigma} \right) \right)
\end{equation}

We then find RS equation (24) becoming

\begin{equation}
\zeta v^2 = E_{F,Q} \left( (\eta_\ast - vQ)^2 \right) = E_{F,Q} \left( \left( \frac{u^2}{\sigma} - \sigma W_0 \left( \frac{u^2}{\sigma^2} F \frac{\sigma_0}{\sigma} \exp \left( \frac{u^2}{\sigma^2} \phi - \frac{vQ}{\sigma} \right) \right) \right)^2 \right)
\end{equation}

Upon dividing both sides of this expression by \( \sigma^2 \) we obtain (52). To simplify RS equation (25), we compute

\begin{equation}
\frac{\partial}{\partial Q} \eta_\ast = v \left( 1 - \frac{\partial X}{\partial Q} \frac{d}{dX} W_0(X) \right) = v \left( 1 - \frac{W_0(X)}{1 + W_0(X)} \right)
\end{equation}

\begin{equation}
X := \frac{u^2}{\sigma^2} \exp \left( \frac{u^2}{\sigma^2} - \frac{Y - \phi - vQ}{\sigma} \right)
\end{equation}

where we have used the identity

\begin{equation}
\frac{d}{dX} W_0(X) = \frac{W_0(X)}{X(1 + W_0(X))}
\end{equation}

Substituting (205) into (25) then gives (53). Next we turn to equation (26). We compute

\begin{equation}
\frac{\partial}{\partial \phi} \log p(Y | \eta_\ast, \phi, \sigma) = \frac{1}{\sigma} - \frac{1}{\sigma} e^{-\left( Y - \phi - \xi \right)} = \frac{1}{\sigma} \left( 1 - \frac{W_0(X)}{u^2} \right)
\end{equation}

where we used the identity \( \exp[-W_0(X)] = X^{-1} W_0(X) \). RS equation (26) now becomes

\begin{equation}
\frac{1}{\sigma} \left\{ 1 - \frac{\sigma^2}{u^2} E_{F,Q} \left[ W_0 \left( \frac{u^2}{\sigma^2} F \frac{\sigma_0}{\sigma} \exp \left( \frac{u^2}{\sigma^2} \phi - \frac{vQ}{\sigma} \right) \right) \right] \right\} = 0
\end{equation}

which implies (54). Finally, RS equation (27) requires more work. The relevant derivative is

\begin{equation}
\frac{\partial}{\partial \sigma} \log p(Y | \eta_\ast, \phi, \sigma) = \frac{1}{\sigma} \left\{ \left( 1 - \frac{\sigma^2}{u^2} W_0(X) \right) \frac{Y - \phi - vQ}{\sigma} - \frac{\sigma^2}{u^2} \left( \frac{u^2}{\sigma^2} - W_0(X) \right)^2 - 1 \right\}
\end{equation}

We now obtain

\begin{equation}
\sigma E_{Y,Q} \left[ \frac{\partial}{\partial \sigma} \log p(Y | \eta_\ast, \phi, \sigma) \right] = E_{F,Q} \left[ \left( 1 - \frac{\sigma^2}{u^2} W_0 \left( \frac{u^2}{\sigma^2} F \frac{\sigma_0}{\sigma} \exp \left( \frac{u^2}{\sigma^2} \phi - \frac{vQ}{\sigma} \right) \right) \right) \times \left( \frac{\phi - \phi_0}{\sigma} - \frac{\sigma_0}{\sigma} \log F - \frac{vQ}{\sigma} \right) \right] - \frac{\sigma^2}{u^2} \left( \frac{vQ}{\sigma} \right) - 1
\end{equation}

\(^3\)Note that the general theory applies also to models that do not allow for such a linearization transformation.
Demanding this to be to zero, and rearranging terms, one obtains

\[ \frac{u^2}{\sigma^2} + \zeta \frac{v^2}{\sigma^2} = -\mathbb{E}_{F,Q} \left[ \left( \frac{u^2}{\sigma^2} - W_0 \left( \frac{u^2}{\sigma^2} F^\frac{\sigma_0}{\sigma} \exp \left( \frac{u^2}{\sigma^2} + \frac{\phi - \phi_0}{\sigma} + \frac{v}{\sigma} Q \right) \right) \right) \left( \frac{\sigma_0}{\sigma} \log F + \frac{v}{\sigma} Q \right) \right] \]

where we have used that

\[ \mathbb{E}_{F,Q} \left[ \left( \frac{u^2}{\sigma^2} - W_0 \left( \frac{u^2}{\sigma^2} F^\frac{\sigma_0}{\sigma} \exp \left( \frac{u^2}{\sigma^2} + \frac{\phi - \phi_0}{\sigma} + \frac{v}{\sigma} Q \right) \right) \right) \phi - \phi_0 \right] = 0 \]

(213)

We note that

\[ \frac{u^2}{\sigma^2} \mathbb{E}_{F,Q} \left[ \left( \frac{u^2}{\sigma^2} - W_0 \left( \frac{u^2}{\sigma^2} F^\frac{\sigma_0}{\sigma} \exp \left( \frac{u^2}{\sigma^2} + \frac{\phi - \phi_0}{\sigma} + \frac{v}{\sigma} Q \right) \right) \right) \right] Q \]

\[ = \mathbb{E}_{F,Q} \left[ \frac{\partial}{\partial Q} \left( \frac{u^2}{\sigma^2} - W_0 \left( \frac{u^2}{\sigma^2} F^\frac{\sigma_0}{\sigma} \exp \left( \frac{u^2}{\sigma^2} + \frac{\phi - \phi_0}{\sigma} + \frac{v}{\sigma} Q \right) \right) \right) \right] \phi - \phi_0 = 0 \]

(214)

which leads us to

\[ \frac{u^2}{\sigma^2} = \frac{u^2}{\sigma^2} \gamma_E + \frac{\sigma_0}{\sigma} \mathbb{E} \left[ \log(F) W_0 \left( \frac{u^2}{\sigma^2} F^\frac{\sigma_0}{\sigma} \exp \left( \frac{u^2}{\sigma^2} + \frac{\phi - \phi_0}{\sigma} + \frac{v}{\sigma} Q \right) \right) \right] \]

(215)

where we used the integral definition of Euler-Mascheroni constant

\[ \gamma_E = -\int_0^{\infty} e^{-f} \log f df = -\mathbb{E}_F \left[ \log F \right] \quad F \sim \text{Exp}(1) \]

Rearranging this result recovers expression (55).

In order to solve the set (52,53,54,55) numerically we regard these coupled RS equations as fixed-point equations \( v = f(v) \) of a mapping for rescaled variables \( v = (a, b, c, \zeta) \), with fixed parameter \( d \), in which

\[ a := \frac{u}{\sigma}, \quad b := \frac{v}{\sigma}, \quad c := \frac{\sigma}{\sigma_0}, \quad d := \frac{\phi - \phi_0}{\sigma} \]

(217)

The various integrals in \( f \) are evaluated via Gauss-Hermite quadrature (for the Gaussian averages) and Gauss-Laguerre quadrature (for the Exponential ones).

**B.2. The Log-Logistic AFT model.** For this model equation (28) takes the form

\[ \frac{\eta - vQ}{u^2} = \frac{1}{\sigma} \tanh \left( \frac{Y - \phi - \eta}{2\sigma} \right) \]

(218)

Setting \( x = [\eta - (Y - \phi)]/(2\sigma) \) transforms this to

\[ x = \frac{vQ - (Y - \phi)}{2\sigma} - \frac{u^2}{2\sigma^2} \tanh(x) \]

(219)

This equation has a unique solution \( x_* \), since the right hand side is a monotonically decreasing function of \( x \). We then have

\[ \eta_* = (Y - \phi) + 2\sigma x_* = vQ - \frac{u^2}{2\sigma} \tanh \left( x_* \right) \]

(220)
Note that

\[(221)\quad F := \frac{Y - \phi_0}{\sigma_0} \sim \frac{e^{-f}}{(1 + e^{-f})^2}\]

which is the density of a standard Logistic random variable. Now (219) becomes

\[(222)\quad x_* = \frac{v}{2\sigma} Q - \frac{\phi - \phi_0}{2\sigma} + \frac{\sigma_0}{2\sigma} F - \frac{u^2}{2\sigma^2} \tanh(x_*)\]

From this the RS equation (67) follows immediately. Next, to derive equation (68) we compute the derivative of $\eta_*$

\[(223)\quad \frac{\partial}{\partial Q} \eta_* = 2\sigma \frac{\partial}{\partial Q} x_* = \frac{v \cosh^2(x_*)}{\cosh^2(x_*) + \frac{u^2}{2\sigma^2}} = v \left( 1 - \frac{u^2}{2\sigma^2 \cosh^2(x_*) + u^2} \right)\]

Substitution into (25) gives

\[(224)\quad v(1 - \zeta) = \mathbb{E}_{Y,Q} \left[ \frac{\partial}{\partial Q} \eta_* \right] = v \mathbb{E}_{F,Q} \left[ 1 - \frac{u^2}{2\sigma^2 \cosh^2(x_*) + u^2} \right]\]

dividing both sides by $v$ and simplifying we indeed obtain (68). To proceed to the remaining equations we need the following derivatives:

\[(225)\quad \frac{\partial}{\partial \phi} \log p(Y|\eta_*, \phi, \sigma) = -\frac{1}{\sigma} \tanh(x_*) = -\left( \frac{\sigma_0}{2\sigma} F + \frac{\phi - \phi_0}{2\sigma} - \frac{v}{2\sigma} Q + x_* \right) \frac{2\sigma}{u^2}\]

\[(226)\quad \frac{\partial}{\partial \sigma} \log p(Y|\eta_*, \phi, \sigma) = -\frac{1}{\sigma} + \frac{1}{\sigma} x_* \tanh(x_*)\]

Equations (69) now follows upon substituting

\[(227)\quad \frac{u^2}{\sigma} \mathbb{E}_{Y,Q} \left[ \frac{\partial}{\partial \phi} \log p(Y|\eta_*, \phi, \sigma) \right] = \frac{\phi - \phi_0}{2\sigma} - \mathbb{E}_{F,Q} [x_*] = 0\]

This can also be rewritten as

\[(228)\quad \mathbb{E}_{F,Q} [\tanh(x_*)] = 0\]

Deriving equation (70) requires more work. We note that the relevant average can be separated into three contributions

\[(229)\quad \sigma \mathbb{E}_{Y,Q} \left[ \frac{\partial}{\partial \sigma} \log p(Y|\eta_*, \phi, \sigma) \right] = -1 - \frac{\sigma_0}{\sigma} \mathbb{E}_{F,Q} \left[ F \tanh(x_*) \right] + \mathbb{E}_{F,Q} \left[ \frac{v}{2\sigma} Q \tanh(x_*) \right] - \frac{v^2}{2\sigma^2} \zeta \frac{2\sigma^2}{u^2}\]

where we used (228) to get rid of the term linear in $(\phi - \phi_0)/\sigma$. The middle term can be simplified via partial integration,

\[(230)\quad \mathbb{E}_{F,Q} \left[ Q \tanh(x_*) \right] = \frac{v}{2\sigma} \zeta \frac{2\sigma^2}{u^2}\]

because

\[(231)\quad \frac{\partial}{\partial Q} \tanh(x_*) = \frac{1}{\cosh^2(x_*)} \frac{\partial}{\partial Q} x_* = \frac{v}{2\sigma} \frac{1}{\cosh^2(x_*) + \frac{u^2}{2\sigma^2}}\]
After substitution of these identities the middle and last term simplify. Rearranging the remaining terms then leads us to (70).

Our strategy to solve the resulting system of RS equations numerically is the same as that used for the Weibull model. We define the re scaled variables

\[
\begin{align*}
  a &:= \frac{u}{\sigma}, \\
  b &:= \frac{v}{\sigma}, \\
  c &:= \frac{\sigma}{\sigma_0}, \\
  d &:= \frac{\phi - \phi_0}{\sigma}
\end{align*}
\]

and again regard the RS equations as fixed point conditions \(v = f(v)\) of a dynamical map, here with \(v := (b, c, d, \zeta)\) and fixed parameter \(a\). Numerical integrations are done via Gauss-Hermite quadrature (for the Gaussian averages) and Gauss-Laguerre quadrature (for averages over the Logistic distribution, after suitable transformations of integrals).

**B.3. The Exponential model with Gamma-distributed random effect.** For this model, RS equation (28) takes the form

\[
\eta_* = vQ + \frac{u^2}{2}(1 - 1/\theta) - \frac{u^2}{2}(1 + 1/\theta) \tanh\left(\frac{\eta_* - (Y - \phi) + \log(\theta)}{2}\right)
\]

We define

\[
x_* = \frac{1}{2}[\eta_* - (Y - \phi) + \log(\theta)]
\]

so that \(\eta_* = (Y - \phi) - \log \theta + 2x_*,\) in which \(x_*\) is now to be solved from

\[
x_* = \frac{vQ - (Y - \phi) + \log(\theta)}{2} + \frac{u^2}{4}\left(1 - \frac{1}{\theta}\right) - \frac{u^2}{4}\left(1 + \frac{1}{\theta}\right) \tanh(x_*)
\]

Note that

\[
F := (1 + \theta_0 e^{-(Y - \phi_0)})^{-1/\theta_0} \sim \text{Unif}[0, 1]
\]

with which the equation for \(x_*\) becomes

\[
x_* = \frac{1}{2}vQ + \frac{1}{2}\log(\theta/\theta_0) + \frac{1}{2}\log(F^{-\theta_0} - 1) + \frac{u^2}{4}\left(1 - \frac{1}{\theta}\right) - \frac{u^2}{4}\left(1 + \frac{1}{\theta}\right) \tanh(x_*)
\]

Equation (82) then follows by substitution of \(\eta_*\) into (24). To derive equation (83) we need

\[
\frac{\partial}{\partial Q} \eta_* = 2\frac{\partial}{\partial Q}x = v\left(1 - \frac{u^2(1 + \frac{1}{\theta})}{u^2(1 + \frac{1}{\theta}) + 4 \cosh^2(x)}\right)
\]

Substitution into (25) then indeed reproduces (83). Next, to derive equation (84) we compute

\[
\frac{\partial}{\partial \phi} \log p(Y|\eta_*, \phi, \theta) = -\frac{1}{2}\left(1 - \frac{1}{\theta}\right) - \frac{1}{2}\left(1 + \frac{1}{\theta}\right) \tanh(x_*)
\]

using the self consistent equation for \(x_*\), we obtain

\[
\frac{\partial}{\partial \phi} \log p(Y|\eta_*, \phi, \theta) = \frac{1}{u^2} \left(2x - vQ - \log(\theta/\theta_0) - \log(F^{-\theta_0} - 1)\right)
\]

Equation (84) follows by taking the expectation, demanding the result to be to zero, and rearranging terms. Using the self consistent equation for \(x_*\), this implies also that

\[
\mathbb{E}_{F, Q}[\tanh(x_*)] = \frac{1 - \frac{1}{\theta}}{1 + \frac{1}{\theta}} = \frac{\theta - 1}{\theta + 1}
\]
Finally, to derive the final equation (85) we need

\[
\frac{\partial}{\partial \theta} \log p(Y|\eta, \phi, \theta) = \frac{1}{\theta^2} \left( x_* + \log \left( 2 \cosh(x_*) \right) - (1 + \theta) \frac{e^{x_*}}{2 \cosh(x_*)} \right)
\]

Using (241) and simple identities such as \( e^{-x[2 \cosh(x)]^{-1}} = \frac{1}{2} [1 + \tanh(x)] \), and demanding that the expectation of (242) must be zero, we obtain (85).

Our strategy to solve the resulting system of RS equations numerically is the same as that used for the previous models. We define the rescaled variables

\[
a := \frac{u}{\sigma}, \quad b := \frac{v}{\sigma}, \quad c := \phi - \phi_0, \quad d := \theta
\]

and again regard the RS equations as fixed point conditions here with \( v := (b, c, d, \zeta) \) and fixed parameters \( a \) and \( \theta_0 \). Numerical integrations are done via Gauss-Hermite quadrature (for the Gaussian averages) and Gauss-Legendre quadrature (for averages over the unit interval Uniform distribution, after suitable transformations).

REFERENCES

[1] Andersen, P. K., Klein, J. P. and Rosthoj, S. (2003). Generalised Linear Models for Correlated Pseudo-Observations, with Applications to Multi-State Models. Biometrika 90 15–27.
[2] Babyak, M. A. (2004). What you see may not be what you get: a brief, nontechnical introduction to overfitting in regression-type models. Psychosomatic medicine 66 411–421.
[3] Barbieri, J. (2020). High-dimensional inference: a statistical mechanics perspective.
[4] Barbieri, J., Chen, W.-K., Panchenko, D. and Saenz, M. (2021). Performance of Bayesian linear regression in a model with mismatch.
[5] Buhlman, P. and Van de Geer, S. (2011). Statistics for High Dimensional Data: Methodology, Theory, and Applications. Springer series in Statistics. Springer.
[6] Goutis, C. and Casella, G. (1999). Explaining the saddlepoint approximation. The American Statistician 53.
[7] Coelen, A. C. C., Barrett, J. E., Paga, P. and Perez-Vicente, C. J. (2017). Replica analysis of overfitting in regression models for time-to-event data. Journal of Physics A: Mathematical and Theoretical 50.
[8] Coelen, A. C. C. and Sheikh, M. (2019). Analysis of over-fitting in the regularized Cox model. Journal of Physics A: Mathematical and Theoretical 52 384002.
[9] Coelen, A. C. C., Sheikh, M., Mozeika, A., Aguirre-Lopez, F. and Antenucci, F. (2020). Replica analysis of overfitting in generalized linear regression models. Journal of Physics A: Mathematical and Theoretical 53.
[10] Copas, J. B. (1983). Regression, Prediction and Shrinkage. Journal of the Royal Statistical Society: Series B (Methodological) 45 311–335.
[11] Corless, R., Gonnet, G., Jeffrey, D. H. and Knuth, D. (1996). On the Lambert W-function.
[12] Cox, D. R. and Oakes, D. (1984). Analysis of Survival Data. monographs on Statistics and Applied Probability. Chapman and Hall.
[13] Donoho, D. and Montanari, A. (2013). High dimensional robust M-estimation: asymptotic variance via approximate message passing. Proceedings of the National Academy of Sciences of the United States of America.
[14] Duchateau, L. and Janssen, P. (2008). The Frailty Model. Statistics for Biology and Health. Springer.
[15] El Karoui, N., Bean, D., Bickel, P. J., Lim, C. and Yu, B. (2013). On robust regression with high-dimensional predictors. Proceedings of the National Academy of Sciences of the United States of America 110 14557-14562.
[16] Gelman, A., Carlin, J. B., Stern, H. S., Dunson, D. B., Vehtari, A. and Rubin, D. B. (2013). Bayesian Data Analysis (3rd ed.). Chapman and Hall/CRC.
[17] Parisi, G. (2009). The Mean Field Theory of Spin Glasses: The Heuristic Replica Approach and Recent Rigorous Results.
[18] Guerra, F. and Toninelli, F. L. (2002). The Thermodynamic Limit in Mean Field Spin Glass Models. Communications in Mathematical Physics 230 71–79.
[19] Harrell, F. E. (2001). Regression Modelling Strategies: With Applications to Linear Models, Logistic Regression and Survival Analysis. Springer series in Statistics. Springer.
[20] Hastie, T., Tibshirani, R. and Friedman, J. (2016). *The Elements of Statistical Learning*. Springer series in Statistics. Springer.
[21] Daniels, H. E. (1954). Saddle point approximation in statistics. *The Annals of Mathematical statistics* **25** 631-650.
[22] Huber, P. J. (1972). The 1972 Wald Lecture Robust Statistics: A Review. *The Annals of Mathematical Statistics* **43** 1041 – 1067.
[23] Barbier, J. and Macris, N. (2019). The adaptive interpolation method for proving replica formulas. Applications to the Curie–Weiss and Wigner spike models.
[24] Johnstone, I. M. and Titterington, D. M. (2009). Statistical challenges of high-dimensional data. *Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences* **367** 4237-4253.
[25] Kalbfleisch, J. D. and Prentice, R. L. (2002). *Statistical Analysis of Failure time Data*. John Wiley and sons.
[26] Kirkpatricle, S., Gelatt, C. D. and Vecchi, M. P. (1983). Optimization by Simulated Annealing. *Science* **220** 671-680.
[27] Mezard, M., Parisi, G. and Virasoro, M. (1986). *Spin Glass Theory and Beyond*. WORLD SCIENTIFIC.
[28] Mezard, M. and Parisi, G. (1985). Replicas and Optimization. *Journal de Physique Lettres* **46** 771-778.
[29] Mozeika, A., Sheikh, M., Aguirre-Lopez, F., Antenucci, F. and Coolen, A. C. C. (2021). Exact results on high-dimensional linear regression via statistical physics. *Phys. Rev. E* **103**.
[30] Talangrad, M. (2010). *Mean field models for spin glasses: Vol 1*. Springer.
[31] Nishimori, H. (2001). Statistical Physics of Spin Glasses and Information Processing: An introduction. *International series of monographs in Physics*. Oxford Science Publications.
[32] Parisi, G. (1979). Toward a mean field theory for spin glasses. *Physics Letters A* **73** 203-205.
[33] Peduzzi, P., Concato, J., Feinstein, A. R. and Holmford, T. R. (1995). Importance of events per independent variable in proportional hazards regression analysis II. Accuracy and precision of regression estimates. *Journal of Clinical Epidemiology* **48** 1503-1510.
[34] Riley, R. D., Snell, K. I. E., Ensor, J., Burke, D. L., Harrell Jr, F. E., Moons, K. G. M. and Collins, G. S. (2019). Minimum sample size for developing a multivariable prediction model: Part I – Continuous outcomes. *Statistics in Medicine* **38** 1262-1275.
[35] Sherrington, D. and Kirkpatrick, S. (1975). Solvable Model of a Spin Glass. *Physical Reviews Letters* **35** 1792-1796.
[36] Steyerberg, E. W. (2009). *Clinical Predictions Models. Statistics for biology and health*. Springer.
[37] van Smeden, M., de Groot, J. A. H., Moons, K. G. M., Collins, G. S., Altman, D. G., Eijkemans, M. J. C. and Reitsma, J. B. (2016). No rationale for 1 variable per 10 events criterion for binary logistic regression analysis. *BMC Medical Research Methodology*.
[38] Zdeborová, L. and Krzakala, F. (2016). Statistical physics of inference: thresholds and algorithms. *Advances in Physics* **65** 453-552.