Empirical distribution of good channel codes with non-vanishing error probability

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Abstract

This paper studies several properties of channel codes that approach the fundamental limits of a given (discrete or Gaussian) memoryless channel with a non-vanishing probability of error. The output distribution induced by an \( \epsilon \)-capacity-achieving code is shown to be close in a strong sense to the capacity achieving output distribution. Relying on the concentration of measure (isoperimetry) property enjoyed by the latter, it is shown that regular (Lipschitz) functions of channel outputs can be precisely estimated and turn out to be essentially non-random and independent of the actual code. It is also shown that the output distribution of a good code and the capacity achieving one cannot be distinguished with exponential reliability. The random process produced at the output of the channel is shown to satisfy the asymptotic equipartition property. Using related methods it is shown that quadratic forms and sums of \( q \)-th powers when evaluated at codewords of good AWGN codes approach the values obtained from a randomly generated Gaussian codeword.

Index Terms

Shannon theory, discrete memoryless channels, additive white Gaussian noise, relative entropy, empirical output statistics, asymptotic equipartition property, concentration of measure.

I. INTRODUCTION

A reliable channel codebook (or code, for the purposes of this paper) is a collection of codewords of fixed duration distinguishable with small probability of error when observed through a noisy channel. Such a code is optimal if it possesses the maximal cardinality among all codebooks of equal duration and probability of error. In this paper, we characterize several properties of optimal and close-to-optimal channel codes indirectly, i.e. without identifying the best code explicitly. This characterization provides theoretical insight and ultimately may facilitate the search for new good code families by providing a necessary condition they must satisfy.

Shannon \(^1\) was the first to recognize, in the context of the additive white Gaussian noise channel, that to maximize information transfer across a memoryless channel codewords must be “noise-like”, i.e. resemble a typical sample of a memoryless random process with marginal distribution that maximizes mutual information. Specifically, in \(^1\) Section 25\] Shannon states:

> To approximate this limiting rate of transmission the transmitted signals must approximate, in statistical properties, a white noise.

A general and formal statement of this property of optimal codes was put forward by Shamai and Verdú \(^2\) who showed that a capacity-achieving sequence of codes with vanishing probability of error must satisfy \(^2\) Theorem 2\]

\[
\frac{1}{n} D(P_{Y^n} || P_{Y^n}^*) \to 0, \quad (1)
\]

where \( P_{Y^n} \) denotes the output distribution induced by the codebook (assuming equiprobable codewords) and

\[
P_{Y^n}^* = P_Y^* \times \cdots \times P_Y^* \quad (2)
\]

is the \( n \)-th power of the single-letter capacity achieving output distribution \( P_Y^* \) and \( D(\cdot || \cdot) \) is the relative entropy. Furthermore, \(^2\) shows that (under regularity conditions) the empirical frequency of input letters (or sequential \( k \)-letter blocks) inside the codebook approaches the capacity achieving input distribution (or its \( k \)-th power) in the sense of vanishing relative entropy.

In this paper, we extend the result in \(^2\) to the case of non-vanishing probability of error. Studying this regime as opposed to vanishing probability of error has recently proved to be fruitful for the non-asymptotic characterization of the maximal achievable rate \(^3\). Although for the memoryless channels considered in this paper the \( \epsilon \)-capacity \( C_\epsilon \) is independent of the probability of error \( \epsilon \), it does not immediately follow that a \( C_\epsilon \)-achieving code necessarily satisfies the empirical distribution property \(1\). In fact, we will show that \(1\) fails to be necessary under the average probability of error criterion.

\(^1\) In \(^1\) “white noise” means white Gaussian noise.
To illustrate the delicacy of the question of approximating $P_{Y^n}$ with $P_{Y^n}^*$, consider a good, capacity-achieving $k$-to-$n$ code for the binary symmetric channel (BSC) with crossover probability $\delta < \frac{1}{2}$ and capacity $C$. The probability of the codebook under $P_{Y^n}$ is larger than the probability that no errors occur: $(1 - \delta)^n$. Under $P_{Y^n}^*$, the probability of the codebook is $2^{k-n}$ —which is exponentially smaller asymptotically since for a reliable code $k \leq n - nh(\delta) < \log 2(1 - \delta)$. On the other hand, consider a set $E$ consisting of a union of small Hamming balls surrounding each codeword, whose radius $\approx \delta n$ is chosen such that $P_{Y^n}[E] = \frac{1}{2}$, say. Assuming that the code is decodable with small probability of error, the union will be almost disjoint and hence $P_{Y^n}^*[E] \approx 2^{k-nC}$ —the two becoming exponentially comparable (provided $k \approx nC$). Thus, for certain events, $P_{Y^n}$ and $P_{Y^n}^*$ differ exponentially, while on other, less delicate, events they behave similarly. We will show that as long as the error probability is strictly less than one, the normalized relative entropy in (1) is upper bounded by the difference between capacity and code rate.

Studying the output distribution $P_{Y^n}$ also becomes important in the context of secure communication, where the output due to the code is required to resemble white noise; and in the problem of asynchronous communication where the output statistics of the code imposes limits on the quality of synchronization [4]. For example, in a multi-terminal communication problem, the channel output of one user may create interference for another. Assessing the average impairment caused by such interference involves the analysis of the expectation of a certain function of the channel output $\mathbb{E}[F(Y^n)]$. We show that under certain regularity assumptions on $F$ not only can one approximate the expectation of $F$ by substituting the unknown $P_{Y^n}$ with $P_{Y^n}^*$, as in

$$\int F(y^n) dP_{Y^n} \approx \int F(y^n) dP_{Y^n}^* ,$$

(3)

but one can also prove that in fact the distribution of $F(Y^n)$ will be tightly concentrated around its expectation. Thus, we are able to predict with overwhelming probability the random value of $F(Y^n)$ without any knowledge of the code used to produce $Y^n$ (but assuming the code is $\epsilon$-capacity-achieving).

Besides (1) and (3) we will show that

1) the hypothesis testing problem between $P_{Y^n}$ and $P_{Y^n}^*$ has zero Stein exponent;
2) a convenient inequality holds for the conditional relative entropy for the channel output in terms of the cardinality of the employed code;
3) codewords of good codes for the additive white Gaussian noise (AWGN) channel become more and more isotropically distributed (in the sense of evaluating quadratic forms) and resemble white Gaussian noise (in the sense of $\ell_q$ norms) as the code approaches the fundamental limits;
4) the output process $Y^n$ enjoys an asymptotic equipartition property.

Throughout the paper we will observe a number of connections with the concentration of measure (isoperimetry) and optimal transportation, which were introduced into the information theory by the seminal works [5]–[7]. Although some key results are stated for general channels, most of the discussion is specialized to discrete memoryless channels (DMC) (possibly with a (separable) input cost constraint) and to the AWGN channel.

The organization of the paper is as follows. Section II contains the main definitions and notation. Section III proves a sharp upper bound on the relative entropy $D(P_{Y^n}||P_{Y^n}^*)$. In Section IV we discuss various implications of the bounds on relative entropy and in particular prove approximation (3). Section V considers the hypothesis testing problem of discriminating between $P_{Y^n}$ and $P_{Y^n}^*$. The asymptotic equipartition property of the channel output process is established in Section VI. Section VII discusses results for the quadratic forms and $\ell_p$ norms of the codewords of good Gaussian codes.

II. DEFINITIONS AND NOTATION

A. Codes and channels

A random transformation $P_{Y|X} : \mathcal{X} \to \mathcal{Y}$ is a Markov kernel acting between a pair of measurable spaces. An $(M, \epsilon)_{avg}$ code for the random transformation $P_{Y|X}$ is a pair of random transformations $f : \{1, \ldots, M\} \to \mathcal{X}$ and $g : \mathcal{Y} \to \{1, \ldots, M\}$ such that

$$P[\hat{W} \neq W] \leq \epsilon ,$$

(4)

where in the underlying probability space $X = f(W)$ and $\hat{W} = g(Y)$ with $W$ equiprobable on $\{1, \ldots, M\}$, and $W, X, Y, \hat{W}$ forming a Markov chain:

$$W \xrightarrow{f} X \xrightarrow{P_{Y|X}} Y \xrightarrow{g} \hat{W} .$$

(5)

In particular, we say that $P_X$ (resp., $P_Y$) is the input (resp., output) distribution induced by the encoder $f$. An $(M, \epsilon)_{max}$ code is defined similarly except that (4) is replaced with the more stringent maximal probability of error criterion:

$$\max_{1 \leq j \leq M} P[\hat{W} \neq W | W = j] \leq \epsilon .$$

(6)

A code is deterministic if the encoder $f$ is a functional (non-random) mapping. We will frequently specify that a code is deterministic with the notation $(M, \epsilon)_{max, det}$ or $(M, \epsilon)_{avg, det}$.
A channel is a sequence of random transformations, \( \{P_{Y^n|X^n}, n = 1, \ldots\} \) indexed by the parameter \( n \), referred to as the blocklength. An \((M, \epsilon)\) code for the \( n\)-th random transformation is called an \((n, M, \epsilon)\) code, and the foregoing notation specifying average/maximal error probability and deterministic encoder will also be applied to that case. The non-asymptotic fundamental limit of communication is defined as

\[
M^*(n, \epsilon) = \max \{M : \exists (n, M, \epsilon)\text{-code}\}.
\]

(B. Capacity-achieving output distribution)

To the three types of channels considered below we also associate a special sequence of output distributions \( P^*_Y \), defined as the \( n \)-th power of a certain single-letter distribution \( P^*_Y \)

\[
P^*_Y \triangleq (P^*_Y)^n = P^*_Y \times \cdots \times P^*_Y,
\]

where \( P^*_Y \) is a distribution on the output alphabet defined as follows:

1) A DMC (without feedback) is built from a single letter transformation \( P_Y|X : \mathcal{X} \to \mathcal{Y} \) acting between finite spaces by extending the latter to all \( n \geq 1 \) in a memoryless way. Namely, the input space of the \( n \)-th random transformation \( P^*_{Y|X^n} \) is given by

\[
\mathcal{X}_n = \mathcal{X}^n \triangleq \mathcal{X} \times \cdots \times \mathcal{X}
\]

and similarly for the output space \( \mathcal{Y} = \mathcal{Y} \times \cdots \times \mathcal{Y} \), while the transition kernel is set to be

\[
P^*_{Y^n|X^n}(y^n|x^n) = \prod_{j=1}^{n} P_Y(y_j|x_j).
\]

The capacity \( C \) and \( P^*_Y \), the unique capacity-achieving output distribution (caod), are found by solving

\[
C = \max_{P_X} I(X;Y).
\]

2) A DMC with input constraint \((c, P)\) is a generalization of the previous construction with an additional restriction on the input space \( \mathcal{X}_n \):

\[
\mathcal{X}_n = \left\{ x^n \in \mathcal{X}^n : \sum_{j=1}^{n} c(x_j) \leq nP \right\}
\]

In this case the capacity \( C \) and the caod \( P^*_Y \) are found by restricting the maximization in (11) to those \( P_X \) that satisfy

\[
E[c(X)] \leq P.
\]

3) The AWGN(\( P) \) channel has an input space

\[
\mathcal{X}_n = \{ x \in \mathbb{R}^n : \|x\|_2 \leq \sqrt{nP} \}
\]

the output space \( \mathcal{Y} = \mathbb{R}^n \) and the transition kernel

\[
P^*_{Y^n|X^n=x} = \mathcal{N}(x, I_n),
\]

where \( \mathcal{N}(x, \Sigma) \) denotes a (multidimensional) normal distribution with mean \( x \) and covariance matrix \( \Sigma \) and \( I_n \) is the \( n \times n \) identity matrix. Then

\[
C = \frac{1}{2} \log(1 + P)
\]

\[
P^*_Y = \mathcal{N}(0, 1 + P).
\]

As shown in [9], [10] in all three cases \( P^*_Y \) is unique and \( P^*_{Y^n} \) satisfies the key property:

\[
D(P Y^n|X^n=x || P^*_{Y^n}) \leq nC,
\]

2Additionally, one should also specify which probability of error criterion, \( \epsilon \) or \( \epsilon_x \), is used.
3For general channels, the sequence \( \{P^*_Y\} \) is required to satisfy a quasi-caod property, see [8] Section IV.
4To unify notation we denote the input space as \( \mathcal{X}_n \) (instead of the more natural \( \mathcal{X}^n \)) even in the absence of cost constraints.
5For convenience we denote the elements of \( \mathbb{R}^n \) as \( x, y \) (for non-random vectors) and \( X^n, Y^n \) (for the random vectors).
6As usual, all logarithms \( \log \) and exponents \( \exp \) are taken to arbitrary fixed base, which also specifies the information units.
for all $x \in \mathcal{X}_n$. Since $I(U; V) = D(P_{Y|U} \| Q_{Y|U}) - D(P_{Y} \| Q)$, Property (18) implies that for every input distribution $P_X$ the induced output distribution $P_Y^n$ satisfies
\begin{align}
D(P_{Y^n} || P^n_Y) & \leq nC - I(X^n; Y^n), \\
P_{Y} & \ll P^n_Y, \\
P_{Y^n | X^n = x^n} & \ll P^n_Y, \quad \forall x^n \in \mathcal{X}_n.
\end{align}
As a consequence of (21) the information density is well defined:
\begin{align}
i^*_X^n; Y^n(x^n; y^n) & \triangleq \log \frac{dP^n_{Y^n | X^n = x^n}}{dP^n_Y}(y^n). 
\end{align}
Moreover, for every channel considered here there is a constant $a_1 > 0$ such that
\begin{align}
\sup_{x^n \in \mathcal{X}_n} \text{Var}[i^*_X^n; Y^n(X^n; Y^n) \mid X^n = x^n] \leq na_1.
\end{align}
In all three cases, the $\epsilon$-capacity $C_\epsilon$ equals $C$ for all $0 < \epsilon < 1$, i.e.
\begin{align}
\log M^*(n, \epsilon) = nC + o(n), \quad n \to \infty.
\end{align}
In fact, see [3]
\begin{align}
\log M^*(n, \epsilon) = nC - \sqrt{nVQ^{-1}(\epsilon)} + O(\log n), \quad n \to \infty,
\end{align}
for any $0 < \epsilon < \frac{1}{2}$, a certain constant $V \geq 0$, called the channel dispersion, and $Q^{-1}$ is the inverse of the standard complementary normal cdf.

C. Good codes

We introduce the following increasing degrees of optimality for sequences of $(n, M_n, \epsilon)$ codes. A code sequence is called:
1) $o(n)$-achieving or $\epsilon$-capacity-achieving if
\begin{align}
\frac{1}{n} \log M_n \to C.
\end{align}
2) $O(\sqrt{n})$-achieving if
\begin{align}
\log M_n = nC + O(\sqrt{n}).
\end{align}
3) $o(\sqrt{n})$-achieving or dispersion-achieving if
\begin{align}
\log M_n = nC - \sqrt{nVQ^{-1}(\epsilon)} + o(\sqrt{n}).
\end{align}
4) $O(\log n)$-achieving if
\begin{align}
\log M_n = nC - \sqrt{nVQ^{-1}(\epsilon)} + O(\log n).
\end{align}

D. Binary hypothesis testing

We also need to introduce the performance of an optimal binary hypothesis test, which is one of the main tools in [3]. Consider an $A$-valued random variable $B$ which can take probability measures $P$ or $Q$. A randomized test between those two distributions is defined by a random transformation $P_{Z|B}: A \mapsto \{0, 1\}$ where 0 indicates that the test chooses $Q$. The best performance achievable among those randomized tests is given by
\begin{align}
\beta_\alpha(P, Q) = \min \sum_{a \in A} Q(a)P_{Z|B}(1|a),
\end{align}
where the minimum is over all probability distributions $P_{Z|B}$ satisfying
\begin{align}
P_{Z|B} : \sum_{a \in A} P(a)P_{Z|B}(1|a) \geq \alpha.
\end{align}
The minimum in (30) is guaranteed to be achieved by the Neyman-Pearson lemma. Thus, $\beta_\alpha(P, Q)$ gives the minimum probability of error under hypothesis $Q$ if the probability of error under hypothesis $P$ is no larger than $1 - \alpha$.

\footnote{For discrete channels [23] is shown, e.g., in [3] Appendix E.}
\footnote{We sometimes write summations over alphabets for simplicity of exposition. For arbitrary measurable spaces $\beta_\alpha(P, Q)$ is defined by replacing the summation in (30) by an expectation.}
III. UPPER BOUND ON THE OUTPUT RELATIVE ENTROPY

The main goal of this section is to establish (for each of the three types of memoryless channels introduced in Section sec:notation) that

\[ D(P_{Y^n}||P_{Y^n}^*) \leq nC - \log M_n + o(n), \]

where \( P_{Y^n} \) is the sequence of output distributions induced by a sequence of \((n, M_n, \epsilon)\) codes, and \( o(n) \) depends on \( \epsilon \).

Furthermore, for all channels except DMCs with zeros in the transition matrix \( P_{Y|X} \), \( o(n) \) in (32) can be replaced by \( O(\sqrt{n}) \).

We start by giving a one-shot converse due to Augustin [11] in Section III-A. Then, we prove (32) for DMCs in Section III-B and for the AWGN in Section III-C.

A. Augustin’s converse

The following result first appeared as part of the proofs in [11] Satz 7.3 and 8.2 by Augustin and formally stated in [12] Section 2]. Note that particularizing Theorem 1 to a constant function \( \rho \) recovers the nonasymptotic converse bound that can be derived from Wolfowitz’s proof of the strong converse [13].

Theorem 1 ([11], [12]): Consider a random transformation \( P_{Y|X} \), a distribution \( P_X \) induced by an \((M, \epsilon)_{max, det}\) code, a distribution \( Q_Y \) on the output alphabet and a function \( \rho: \mathcal{X} \rightarrow \mathbb{R} \). Then, provided the denominator is positive, \( \rho \)

\[ M \leq \inf_x P_{Y|X=x} \left\{ \log \frac{dP_{Y|X=x}}{dQ_Y}(Y) \leq \rho(x) \right\} - \epsilon \],

with the infimum taken over the support of \( P_X \).

Proof: Fix a \((M, \epsilon)_{max, det}\) code, \( Q_Y \), and the function \( \rho \). Denoting by \( c_i \) the \( i \)-th codeword, we have

\[ Q_Y[\hat{W}(Y) = i] \geq \beta_{1-\epsilon}(P_{Y|X=c_i}, Q_Y), \quad i = 1, \ldots, M, \]

since \( \hat{W}(Y) = i \) is a suboptimal test to decide between \( P_{Y|X=c_i} \) and \( Q_Y \), which achieves error probability no larger than \( \epsilon \) when \( P_{Y|X=c_i} \) is true. Therefore,

\[ \frac{1}{M} \geq \frac{1}{M} \sum_{i=1}^{M} \beta_{1-\epsilon}(P_{Y|X=c_i}, Q_Y) \]

\[ \geq \frac{1}{M} \sum_{i=1}^{M} \left( P_{Y|X=c_i} \left[ \log \frac{dP_{Y|X=c_i}}{dQ_Y}(Y) \leq \rho(c_i) \right] - \epsilon \right) \exp\{ -\rho(c_i) \} \]

\[ \geq \left( \inf_x P_{Y|X=x} \left[ \log \frac{dP_{Y|X=x}}{dQ_Y}(Y) \leq \rho(x) \right] - \epsilon \right) \frac{1}{M} \sum_{i=1}^{M} \exp\{ -\rho(c_i) \} \]

\[ \geq \left( \inf_x P_{Y|X=x} \left[ \log \frac{dP_{Y|X=x}}{dQ_Y}(Y) \leq \rho(x) \right] - \epsilon \right) \exp\{ -\mathbb{E}[\rho(X)] \}, \]

where (35) is by taking the arithmetic average of (34) over \( i \), (38) is by Jensen’s inequality, and (36) is by the standard estimate of \( \beta_\alpha \), e.g. [3] (102)],

\[ \beta_{1-\epsilon}(P, Q) \geq \left( \mathbb{P} \left[ \log \frac{dP}{dQ}(Z) \leq \rho \right] - \epsilon \right) \exp\{ -\rho \}, \]

with \( Z \) distributed according to \( P \).

Remark 1: Following an idea of Poor and Verdú [14] we may further strengthen Theorem 1 in the special case of \( Q_Y = P_Y \): The maximal probability of error \( \epsilon \) for any test of \( M \) hypotheses \( \{P_j, j = 1, \ldots, M\} \) satisfies:

\[ \epsilon \geq \left( 1 - \frac{\exp\{ \tilde{\beta} \}}{M} \right) \inf_{1 \leq j \leq M} \mathbb{P}[w_Y : Y = j \leq \rho_j | W = j], \]

where the information density is as defined in [22], \( \rho_j \in \mathbb{R} \) are arbitrary, \( \tilde{\beta} = \frac{1}{M} \sum_{j=1}^{M} \rho_j \) is equiprobable on \( \{1, \ldots, M\} \) and \( P_{Y|W=j} = P_j \). Indeed, since \( w_Y : Y = j \leq \log M \) we get from [15] Lemma 35]

\[ \exp\{ \rho_j \} \beta_{1-\epsilon}(P_{Y|W=j}, P_Y) + \left( 1 - \frac{\exp\{ \rho_j \}}{M} \right) \mathbb{P}[w_Y : Y > \rho_j | W = j] \geq 1 - \epsilon. \]

Multiplying by \( \exp\{ -\rho_j \} \) and using resulting bound in place of (39) we repeat steps (35)-(38) to obtain

\[ \frac{1}{M} \inf_{1 \leq j \leq M} \mathbb{P}[w_Y : Y \leq \rho_j | W = j] \geq \left( \inf_{1 \leq j \leq M} \mathbb{P}[w_Y : Y \leq \rho_j | W = j] - \epsilon \right) \exp\{ -\tilde{\beta} \}. \]
which in turn is equivalent to (40).

Choosing \( \rho(x) = D(P_{Y|X=x}||Q_Y) + \Delta \) we can specialize Theorem 1 in the following convenient form:

**Theorem 2:** Consider a random transformation \( P_{Y|X} \), a distribution \( P_X \) induced by an \((M, \epsilon)_{\text{max, det}}\) code and an auxiliary output distribution \( Q_Y \). Assume that for all \( x \in \mathcal{X} \) we have

\[
d(x) \triangleq D(P_{Y|X=x}||Q_Y) < \infty
\]  

and

\[
\sup_x P_{Y|X=x} \left[ \log \frac{dP_{Y|X=x}}{dQ_Y}(Y) \geq d(x) + \Delta \right] \leq \delta', \tag{44}
\]

for some pair of constants \( \Delta \geq 0 \) and \( 0 \leq \delta' < 1 - \epsilon \). Then, we have

\[
D(P_{Y|X}||Q_Y|P_X) \geq \log M - \Delta + \log(1 - \epsilon - \delta'). \tag{45}
\]

**Remark 2:** Note that (44) holding with a small \( \delta' \) is a natural non-asymptotic embodiment of information stability of the underlying channel, cf. [9, Section IV].

A simple way to estimate the upper deviations in (44) is by using Chebyshev’s inequality. As an example, we obtain

**Corollary 3:** If in the conditions of Theorem 2 we replace (44) with

\[
\sup_x \text{Var} \left[ \log \frac{dP_{Y|X=x}}{dQ_Y}(Y) | X = x \right] \leq S_m
\]

for some constant \( S_m \geq 0 \), then we have

\[
D(P_{Y|X}||Q_Y|P_X) \geq \log M - \sqrt{2S_m} + \log \frac{1 - \epsilon}{2}. \tag{47}
\]

**B. DMC**

Notice that when \( Q_Y \) is chosen to be a product distribution, such as \( P_{Y^n}^* \), \( \log dP_{Y|X=x} \) becomes a sum of independent random variables. In particular, (25) leads to a necessary and sufficient condition for (1):

**Theorem 4:** Consider a memoryless channel belonging to one of the three classes in Section II. Then for any \( 0 < \epsilon < 1 \) and any sequence of \((n, M_n, \epsilon)_{\text{max, det}}\) capacity-achieving codes we have

\[
\frac{1}{n} I(X^n; Y^n) \to C \iff \frac{1}{n} D(P_{Y^n}||P_{Y^n}^*) \to 0, \tag{48}
\]

where \( X^n \) is the output of the encoder.

**Proof:** The direction \( \Rightarrow \) is trivial from property (19) of \( P_{Y^n}^* \). For the direction \( \Leftarrow \) we only need to lower-bound \( I(X^n; Y^n) \) since, asymptotically, it cannot exceed \( nC \). To that end, we have from (25) and Corollary 3

\[
D(P_{Y^n}|X^n||P_{Y^n}^*|P_{X^n}) \geq \log M_n + O(\sqrt{n}). \tag{49}
\]

Then the conclusion follows from (26) and the following identity applied with \( Q_Y^n = P_{Y^n}^* \):

\[
I(X^n; Y^n) = D(P_{Y^n|X^n}||Q_Y^n|P_{X^n}) - D(P_{Y^n}||Q_Y^n), \tag{50}
\]

which holds for all \( Q_Y^n \) such that the unconditional relative entropy is finite.

We remark that Theorem 4 can also be derived from a simple extension of the Wolfowitz converse [13] to an arbitrary output distribution \( Q_Y \), e.g. [15, Theorem 10], and then choosing \( Q_Y^n = P_{Y^n}^* \). Note that Theorem 4 implies the result in [2] since the capacity-achieving codes with vanishing error probability are a subclass of those considered in Theorem 4.

Fano’s inequality only guarantees the left side of (48) for code sequences with vanishing error probability. If there was a strong converse showing that the left side of (48) must hold for any sequence of \((n, M_n, \epsilon)\) codes, then the desired result would follow. In the absence of such a result we will consider three separate cases in order to show (1), and, therefore, through Theorem 4 the left side of (48).

1) **DMC with \( C_1 < \infty \):** For a given DMC denote the parameter introduced by Burnashev [16]

\[
C_1 = \max_{\alpha, \alpha'} D(P_{Y|X=x}||P_{Y|X=x'}). \tag{51}
\]

Note that \( C_1 < \infty \) if and only if the transition matrix does not contain any zeros. In this section we show (52) for a (regular) class of DMCs with \( C_1 < \infty \) by an application of the main inequality (45). We also demonstrate that (1) may not hold for codes with non-deterministic encoders or unconstrained maximal probability of error.

**Theorem 5:** Consider a DMC \( P_{Y|X} \) with \( C_1 < \infty \) and capacity \( C > 0 \) (with or without an input constraint). Then for any \( 0 \leq \epsilon < 1 \) there exists a constant \( a = a(\epsilon) > 0 \) such that any \((n, M_n, \epsilon)_{\text{max, det}}\) code satisfies

\[
D(P_{Y^n}||P_{Y^n}^*) \leq nC - \log M_n + a\sqrt{n}, \tag{52}
\]
where \( P_{Y^n} \) is the output distribution induced by the code. In particular, for any capacity-achieving sequence of such codes we have
\[
\frac{1}{n} D(P_{Y^n} \| P_{Y^n}') \to 0 ,
\] (53)

**Proof:** Fix \( y^n, \bar{y}^n \in \mathcal{Y}^n \) which differ in the \( j \)-th letter only. Then, denoting \( y_{\cdot j} = \{ y_k, k \neq j \} \) we have
\[
| \log P_{Y^n}(y^n) - \log P_{Y^n}(\bar{y}^n) | = \left| \log \frac{P_{Y^n}(y^n)}{P_{Y^n}(\bar{y}^n)} \right| \leq \max_{a,b,\bar{b}} \frac{P_{Y|X}(b|a)}{P_{Y|X}(\bar{b}|a)} \leq a_1 < \infty ,
\] (54)

where (55) follows from
\[
P_{Y_j|Y_{\cdot j}}(b|y_{\cdot j}) = \sum_{a \in \mathcal{X}} P_{Y|X}(b|a) P_{X|Y_{\cdot j}}(a|y_{\cdot j}).
\] (57)

Thus, the function \( y^n \mapsto \log P_{Y^n}(y^n) \) is \( a_1 \)-Lipschitz in Hamming metric on \( \mathcal{Y}^n \). Its discrete gradient (absolute difference of values taken at consecutive integers) is bounded by \( n(a_1)^2 \) and thus by the discrete Poincaré inequality (the variance of a function with countable support is upper bounded by (a multiple of) the second moment of its discrete gradient) [17, Theorem 4.1f] we have
\[
\text{Var} [ \log P_{Y^n}(Y^n)|X^n = x^n ] \leq n|a_1|^2.
\] (58)

Therefore, for some \( 0 < a_2 < \infty \) and all \( x^n \in \mathcal{X}_n \) we have
\[
\text{Var} [ x^n, y^n(X^n; Y^n)|X^n = x^n ] \leq 2 \text{Var} [ \log P_{Y^n|X^n}(Y^n|X^n)|X^n = x^n ] + 2 \text{Var} [ \log P_{Y^n}(Y^n)|X^n = x^n ] \leq 2na_2 + 2n|a_1|^2 ,
\] (59)

where (59) follows from
\[
\text{Var} \left[ \sum_{i=1}^{K} Y_i \right] \leq K \text{Var}[Y_i]
\] (61)

and (60) follows from (58) and the fact that the random variable in the first variance in (59) is a sum of \( n \) independent terms. Applying Corollary 3 with \( S_m = 2a_2 + 2n|a_1|^2 \) and \( Q_Y = P_{Y^n} \) we obtain:
\[
D( P_{Y^n|X^n} \| P_{Y^n}|P_{X^n} ) \geq \log M_n + O(\sqrt{n} ) .
\] (62)

We can now complete the proof:
\[
D( P_{Y^n} \| P_{Y'n} ) = D( P_{Y^n|X^n} \| P_{Y'n|X^n} ) \geq \log M_n + O(\sqrt{n} ) \] (63)

where (64) is because \( P_{Y'n} \) satisfies (18) and (65) follows from (62). This completes the proof of (52).

**Remark 3:** As we will see in Section V-A, (53) implies
\[
H(Y^n) = nH(Y^n') + o(n)
\] (66)

(by (13) applied to \( f(y) = \log P_{Y'}(y) \)). Note also that traditional combinatorial methods, e.g. (18), are not helpful in dealing with quantities like \( H(Y^n), D( P_{Y^n} \| P_{Y'n} ) \) or \( P_{Y^n} \)-expectations of functions that are not of the form of cumulative average.

**Remark 4:** Note that any \((n, M, \epsilon)\) code is also an \((n, M, \epsilon')\) code for all \( \epsilon' \geq \epsilon \). Thus \( a(\epsilon) \), the constant in (53), is a non-decreasing function of \( \epsilon \). In particular, (52) holds uniformly in \( \epsilon \) on compact subsets of [0, 1). In their follow-up to the present paper, Raginsky and Sason [19] use McDiarmid’s inequality to derive a tighter estimate for \( a \).

**Remark 5:** (53) need not hold if the maximal probability of error is replaced with the average or if the encoder is allowed to be random. Indeed, for any \( 0 < \epsilon < 1 \) we construct a sequence of \((n, M_n, \epsilon')\) capacity-achieving codes which do not satisfy (53) can be constructed as follows. Consider a sequence of \((n, M_n, \epsilon'_n)_{\text{max,det}}\) codes with \( \epsilon'_n \to 0 \) and
\[
\frac{1}{n} \log M'_n \to C .
\] (67)

For all \( n \) such that \( \epsilon'_n < \frac{1}{2} \) this code cannot have repeated codewords and we can additionally assume (perhaps by reducing \( M'_n \) by one) that there is no codeword equal to \((x_0, \ldots, x_0) \in \mathcal{X}_n\), where \( x_0 \) is some fixed letter in \( \mathcal{X} \) such that
\[
D( P_{Y|X=x_0} \| P_{Y'} ) > 0
\] (68)
(the existence of such \(x_0\) relies on the assumption \(C > 0\)). Denote the output distribution induced by this code by \(P_{Y^n}\). Next, extend this code by adding \(\frac{\log 2}{\log (1 - \epsilon)} M'_n\) identical codewords: \((x_0, \ldots, x_0) \in X_n\). Then the minimal average probability of error achievable with the extended codebook of size
\[
M_n = \frac{1 - \epsilon_n}{1 - \epsilon} M'_n
\]
is easily seen to be not larger than \(\epsilon\). Denote the output distribution induced by the extended code by \(P_{Y^n}\) and define a binary random variable
\[
S = 1\{X^n = (x_0, \ldots, x_0)\}
\]
with distribution
\[
P_S(1) = 1 - P_S(0) = \frac{\epsilon - \epsilon_n}{1 - \epsilon_n},
\]
which satisfies \(P_S(1) \to \epsilon\). We have then
\[
D(P_{Y^n}||P_{Y^n}^*) = D(P_{Y^n|S}||P_{Y^n}^*|P_S) - I(S; Y^n)
\]
\[
\geq D(P_{Y^n|S}||P_{Y^n}^*|P_S) - \log 2
\]
\[
= n D(P_{Y|X=x_0}||P_{Y}^*) P_S(1) + D(P_{Y^n}||P_{Y^n}^*) P_S(0) - \log 2
\]
\[
= n D(P_{Y|X=x_0}||P_{Y}^*) P_S(1) + o(n),
\]
where \(72\) is by \(50\), \(73\) follows since \(S\) is binary, \(74\) is by noticing that \(P_{Y^n|S=0} = P_{Y^n}^*\), and \(75\) is by \(53\). It is clear that \(68\) and \(75\) show the impossibility of \(53\) for this code.

Similarly, one shows that \(53\) cannot hold if the assumption of the deterministic encoder is dropped. Indeed, then we can again take the very same \((n, M_n', \epsilon_n')\) code and make its encoder randomized so that with probability \(\frac{\epsilon - \epsilon_n}{1 - \epsilon_n}\) it outputs \((x_0, \ldots, x_0) \in X_n\) and otherwise it outputs the original codeword. The same analysis shows that \(75\) holds again and thus \(53\) fails.

The counterexamples constructed above can also be used to demonstrate that in Theorem 2 (and hence Theorem 1) the assumptions of maximal probability of error and deterministic encoders are not superfluous, contrary to what is claimed by Ahlswede [12, Remark 1].

2) DMC with \(C_1 = \infty\): Next, we show an estimate for \(D(P_{Y^n}||P_{Y^n}^*)\) differing by a \(\log^2 n\) factor from \(52\) for the DMCs with \(C_1 = \infty\).

Theorem 6: For any DMC \(P_{Y|X}\) with capacity \(C > 0\) (with or without input constraints), \(C_1 = \infty\), and \(0 \leq \epsilon < 1\) there exists a constant \(b > 0\) with the property that for any sequence of \((n, M_n, \epsilon)_{\text{max}, \text{det}}\) codes we have for all \(n \geq 1\)
\[
D(P_{Y^n}||P_{Y^n}^*) \leq n C - \log M_n + b \sqrt{n} \log^2 n.
\]

In particular, for any such sequence achieving capacity we have
\[
\frac{1}{n} D(P_{Y^n}||P_{Y^n}^*) \to 0.
\]

Proof: Let \(c_i\) and \(D_i, i = 1, \ldots, M_n\) denote the codewords and the decoding regions of the code. Denote the sequence
\[
\ell_n = b_1 \sqrt{n \log n}
\]
with \(b_1 > 0\) to be further constrained shortly. According to the isoperimetric inequality for Hamming space [18, Corollary I.5.3], there is a constant \(a > 0\) such that for every \(i = 1, \ldots, M_n\)
\[
1 - P_{Y^n|X^n = c_i}[\Gamma^{\ell_n} D_i] \leq Q \left( Q^{-1}(\epsilon) + \frac{\ell_n}{\sqrt{n}} \right)
\]
\[
\leq \exp \left\{ -b_2 \frac{\ell_n^2}{n} \right\}
\]
\[
= n^{-b_2}
\]
\[
\leq \frac{1}{n},
\]
where the \(\ell\)-blowup of \(D\) is defined as
\[
\Gamma^{\ell}D = \{\bar{y}^n \in \mathcal{Y}^n : \exists y^n \in D \text{ s.t. } |\{j : y_j \neq \bar{y}_j\}| \leq \ell\}
\]
denotes the \(\ell\)-th Hamming neighborhood of a set \(D\) and we assumed that \(b_1\) was chosen large enough so there is \(b_2 \geq 1\) satisfying \(82\).

Let
\[
M'_n = \frac{M_n}{n(\ell_n)} \mathcal{Y}^{\ell_n}
\]
and consider a subcode $F = (F_1, \ldots, F_{M_n^*})$, where $F_i \in C = \{c_1, \ldots, c_M\}$ and note that we allow repetition of codewords. Then for every possible choice of the subcode $F$ we denote by $P_{X^n(F)}$ and $P_{Y^n(F)}$ the input/output distribution induced by $F$, so that for example:

$$P_{Y^n(F)} = \frac{1}{M_n^*} \sum_{j=1}^{M_n^*} P_{Y^n|X^n=F_j}.$$  \hfill (85)

We aim to apply the random coding argument over all equally likely $M_n^{M_n^*}$ choices of a subcode $F$. Random coding among subcodes was originally invoked in [6] to demonstrate the existence of a good subcode. The expected (over the choice of $F$), so that for example:

$$\mathbb{E}[P_{Y^n(F)}] \triangleq \frac{1}{M_n^{M_n^*}} \sum_{F_i \in C} \cdots \sum_{F_{M_n^*} \in C} P_{Y^n(F)}$$ \hfill (86)

$$= \frac{1}{M_n^{M_n^*}} \frac{1}{M_n^*} \sum_{j=1}^{M_n^*} \cdots \sum_{F_{M_n^*} \in C} P_{Y^n|X^n=F_j}$$ \hfill (87)

$$= \frac{M_n^{M_n^*} - 1}{M_n^{M_n^*}} \sum_{c \in C} P_{Y^n|X^n=c}$$ \hfill (88)

$$= P_{Y^n}.$$ \hfill (89)

Next, for every $F$ we denote by $\epsilon'(F)$ the minimal possible average probability of error achieved by an appropriately chosen decoder. With this notation we have, for every possible value of $F$:

$$D(P_{Y^n(F)}||P_{Y^n}^*) = D(P_{Y^n|X^n}|P_{X^n}^*) - I(X^n(F);Y^n(F)) \leq nC - I(X^n(F);Y^n(F))$$ \hfill (90)

$$\leq nC - (1 - \epsilon'(F)) \log M_n^* + \log 2$$ \hfill (91)

$$\leq nC - \log M_n^* + n\epsilon'(F) \log |X| + \log 2$$ \hfill (92)

$$\leq nC - \log M_n^* + n\epsilon'(F) \log |X| + b_3 \sqrt{n} \log \frac{3}{n}$$ \hfill (93)

where (90) is by (50), (91) is by (18), (92) is by Fano’s inequality, (93) is because $\log M_n^* \leq n \log |X|$ and (94) holds for some $b_3 > 0$ by the choice of $M_n^*$ in (84) and by

$$\log \left( \frac{n}{\ell_n} \right) \leq \ell_n \log n.$$ \hfill (95)

Taking the expectation of both sides of (94), applying convexity of relative entropy and (89) we get

$$D(P_{Y^n}||P_{Y^n}^*) \leq nC - \log M_n + n \mathbb{E}[\epsilon'(F)] \log |X| + b_3 \sqrt{n} \log \frac{3}{n}.$$ \hfill (96)

Accordingly, it remains to show that

$$n \mathbb{E}[\epsilon'(F)] \leq 2.$$ \hfill (97)

To that end, for every subcode $F$ define the suboptimal randomized decoder:

$$\hat{W}(y) = F_j \quad \forall F_j \in L(y, F) \quad \text{(with probability } \frac{1}{|L(y, F)|}),$$ \hfill (98)

where $L(y, F)$ is a list of those indices $i \in F$ for which $y \in \Gamma^{\ell_n} D_i$. Since the transmitted codeword $F_W$ is equiprobable on $F$, averaging over the selection of $F$ we have

$$\mathbb{E}[|L(Y^n, F)| | F_W \in L(Y^n, F)] \leq 1 + \frac{n}{M_n^*} \left\lfloor \frac{|Y|^\ell_n}{M_n^*} \right\rfloor (M_n^* - 1),$$ \hfill (99)

because each $y \in Y^n$ can belong to at most $\left\lfloor \frac{n}{\ell_n} \right\rfloor |Y|^\ell_n$ enlarged decoding regions $\Gamma^{\ell_n} D_i$ and each $F_j$ is chosen independently and equiprobably among all possible $M_n^*$ alternatives. The average (over random decoder, $F$, and channel) probability of error
for can be upper-bounded as
\[ \mathbb{E}[\epsilon'(F)] = \mathbb{P}[F_W \not\in L(Y^n, F)] + \mathbb{E}
\left[ \frac{1}{|L(Y^n, F)|} - 1 \{ F_W \in L(Y^n, F) \} \right] \]
\[ \leq \mathbb{P}[F_W \not\in L(Y^n, F)] + \frac{\left( \frac{\epsilon}{\delta} \right) \left( |F|^2 \right)^{t_n} M_n'}{M_n} \quad (101) \]
\[ \leq \frac{1}{n} + \frac{\left( \frac{\epsilon}{\delta} \right) \left( |F|^2 \right)^{t_n} M_n'}{M_n} \quad (102) \]
\[ \leq \frac{2}{n}, \quad (103) \]

where \((100)\) reflects the fact that a correct decision requires that the true codeword not only belong to \(L(Y^n, F)\) but that it be the one chosen from the list; \((101)\) is by Jensen’s inequality applied to \(\frac{1}{\delta} \) and \((99)\); \((102)\) is by \((92)\); and \((103)\) is by \((84)\). Since \((103)\) also serves as an upper bound to \(\mathbb{E}[\epsilon'(F)]\) the proof of \((97)\) is complete. □

Remark 6: Claim \((77)\) fails to hold if either the maximal probability of error is replaced with the average, or if we allow the encoder to be stochastic. Counterexamples are constructed exactly as in Remark 5.

Remark 7: Raginsky and Sason \cite{19} give a sharpened version of \((76)\) with explicitly computed constants but with the same \(O(\sqrt{n} \log^2 n)\) remainder term behavior.

C. Gaussian channel

Theorem 7: For any \(0 < \epsilon < 1\) and \(P > 0\) there exists \(a = a(\epsilon, P) > 0\) such that the output distribution \(P_{Y_n}\) of any \((n, M_n, \epsilon)_{\max, \det}\) code for the AWGN \((P)\) channel satisfies
\[ D(P_{Y_n} \| P_{Y_n}^*) \leq nC - \log M + a \sqrt{n}, \quad (104) \]
where \(P_{Y_n}^* = N(0, 1 + P)^n\). In particular for any capacity-achieving sequence of such codes we have
\[ \frac{1}{n} D(P_{Y_n} \| P_{Y_n}^*) \to 0. \quad (105) \]

Proof: Denote by \(p_{Y^n_{\mid X^n=x}}\) and \(p_{Y^n}\) the densities of \(P_{Y^n_{\mid X^n=x}}\) and \(P_{Y^n}\), respectively. The argument proceeds step by step as in the proof of Theorem \(5\) with \((106)\) taking the place of \((58)\) and recalling that property \((18)\) holds for the AWGN channel too. Therefore, the objective is to show
\[ \text{Var}[\log p_{Y^n}(Y^n) \mid X^n] \leq a_1 n \quad (106) \]
for some \(a_1 > 0\). Poincaré’s inequality for the Gaussian measure, e.g. \cite{20} (2.16)] states that if \(Y\) is an \(N\)-dimensional Gaussian measure, then
\[ \text{Var}[f(Y)] \leq \mathbb{E}[\| \nabla f(Y) \|^2] \quad (107) \]
Since conditioned on \(X^n\), the random vector \(Y^n\) is Gaussian, the Poincaré inequality ensures that the left side of \((106)\) is bounded by
\[ \text{Var}[\log p_{Y^n}(Y^n) \mid X^n] \leq \mathbb{E} [\| \nabla \log p_{Y^n} \|^2 \mid X^n] \quad (108) \]
Therefore, the reminder of the proof is devoted to showing that the right side of \((108)\) is bounded by \(a_2 n\) for some \(a_2 > 0\). An elementary computation shows
\[ \nabla \log p_{Y^n}(y) = \frac{\log e}{p_{Y^n}(y)} \nabla p_{Y^n}(y) \]
\[ = \frac{\log e}{p_{Y^n}(y)} \sum_{j=1}^{M} \frac{1}{M(2\pi)^{\frac{N}{2}}} e^{-\frac{1}{2} || y - c_j ||^2} \]
\[ = \frac{\log e}{p_{Y^n}(y)} \sum_{j=1}^{M} \frac{1}{M(2\pi)^{\frac{N}{2}}} (c_j - y) e^{-\frac{1}{2} || y - c_j ||^2} \]
\[ = (\mathbb{E}[X^n | Y^n = y] - y) \log e. \quad (112) \]
For convenience denote
\[ \hat{X}^n = \mathbb{E}[X^n | Y^n] \quad (113) \]
and notice that since \(\| X^n \| \leq \sqrt{n} P\) we have also
\[ \| \hat{X}^n \| \leq \sqrt{n} P. \quad (114) \]

More precisely, our proof yields a bound \(nC - \log M + \sqrt{6n(3 + 4P) \log e + \log \frac{1}{12}}\).
Then
\[
\frac{1}{\log^2 e} \mathbb{E} [\| \nabla \log p_{Y^n}(Y^n) \|^2 | X^n] = \mathbb{E} \left[ \| Y^n - \hat{X}^n \|^2 | X^n \right] \leq 2 \mathbb{E} \left[ \| Y^n \|^2 | X^n \right] + 2 \mathbb{E} \left[ \| \hat{X}^n \|^2 | X^n \right] \leq 2 \mathbb{E} \left[ \| Y^n \|^2 | X^n \right] + 2nP = 2 \mathbb{E} \left[ \| X^n + Z^n \|^2 | X^n \right] + 2nP \leq 4\| X^n \|^2 + 4n + 2nP \leq (6P + 4)n,
\]
where (116) is by (107), (117) is by (114), in (118) we introduced $Z^n \sim \mathcal{N}(0, I_n)$ which is independent of $X^n$, (119) is by (121) and (120) is by the power-constraint imposed on the codebook. In view of (108), we have succeeded in identifying a constant $a_1$ such that (106) holds.

Remark 8: (105) need not hold if the maximal probability of error is replaced with the average or if the encoder is allowed to be stochastic. Counterexamples are constructed similarly to those for Remark 5 with $x_0 = 0$. Note also that Theorem 7 need not hold if the power-constraint is in the average-over-the-codebook sense; see [15, Section 4.3.3].

IV. Implications

We have shown that there is a constant $a = a(\epsilon)$ independent of $n$ and $M$ such that
\[
D(P_{Y^n} || P_{Y^n}^\ast) \leq nC - \log M + a\sqrt{n},
\]
where $P_{Y^n}$ is the output distribution induced by an arbitrary $(n, M, \epsilon)_{\max, \text{det}}$ code. Therefore, any $(n, M, \epsilon)_{\max, \text{det}}$ necessarily satisfies
\[
\log M \leq nC + a(\epsilon)\sqrt{n}
\]
as is classically known [21]. In particular, (122) implies that any $\epsilon$-capacity-achieving code must satisfy (1). In this section we discuss this and other implications of this result, such as:

1) (122) implies that the empirical marginal output distribution
\[
\tilde{P}_n \triangleq \frac{1}{n} \sum_{j=1}^{n} P_{Y_j}
\]
converges to $P_Y^\ast$ in a strong sense (Section IV-A);

2) (122) guarantees estimates of the precision in the approximation (3) (Sections IV-B and IV-C);

3) (122) provides estimates for the deviations of $f(Y^n)$ from its average (Sections IV-D);

4) relation to optimal transportation (Section IV-D);

5) implications of (1) for the empirical input distribution of the code (Sections IV-C and IV-H).

A. Empirical distributions and empirical averages

Considering the empirical marginal distributions, the convexity of relative entropy and (1) result in
\[
D(\tilde{P}_n || P_Y^\ast) \leq \frac{1}{n} D(P_{Y^n} || P_{Y^n}^\ast) \to 0,
\]
where $\tilde{P}_n$ is the empirical marginal output distribution (124).

More generally, we have [2] (41)]
\[
D(\tilde{P}_n^{(k)} || P_Y^\ast) \leq \frac{k}{n-k+1} D(P_{Y^n} || P_{Y^n}^\ast) \to 0,
\]
where $\tilde{P}_n^{(k)}$ is a $k$-th order empirical output distribution
\[
\tilde{P}_n^{(k)} = \frac{1}{n-k+1} \sum_{j=1}^{n-k+1} P_{Y_j}.
\]
Knowing that a sequence of distributions $P_n$ converges in relative entropy to a distribution $P$, i.e.
\[
D(P_n||P) \to 0
\]
(128)
implies convergence properties for the expectations of functions
\[
\int f dP_n \to \int f dP
\]
(129)
1) For bounded functions, (129) follows from the Csiszár-Kemperman-Kullback-Pinsker inequality (e.g. \[22\]):
\[
||P_n - P||^2_{TV} \leq \frac{1}{2 \log e} D(P_n||P),
\]
where
\[
||P - Q||_{TV} \triangleq \sup_A |P(A) - Q(A)|
\]
(130)
2) For unbounded $f$, (129) holds as long as $f$ satisfies Cramer’s condition under $P$, i.e.
\[
\int e^{tf} dP < \infty
\]
for all $t$ in some neighborhood of 0; see \[23\] Lemma 3.1.
Together (129) and (125) show that for a wide class of functions $f: \mathcal{Y} \to \mathbb{R}$ empirical averages over distributions induced by good codes converge to the average over the capacity achieving output distribution (caod):
\[
\mathbb{E} \left[ \frac{1}{n} \sum_{j=1}^{n} f(Y_j) \right] \to \int f dP^*Y
\]
(133)
From (126) a similar conclusion holds for $k$-th order empirical averages.

B. Averages of functions of $Y^n$

To go beyond empirical averages, we need to provide some definitions and properties (see \[20\])

**Definition 1:**
The function $F: \mathcal{Y}^n \to \mathbb{R}$ is called $(b, c)$-concentrated with respect to measure $\mu$ on $\mathcal{Y}^n$ if for all $t \in \mathbb{R}$
\[
\int \exp \{ t(F(Y^n) - \mathbb{F}) \} d\mu \leq b \exp \{ ct^2 \}, \quad \mathbb{F} = \int F d\mu.
\]
(134)
A function $F$ is called $(b, c)$-concentrated for the channel if it is $(b, c)$-concentrated with respect to every $P_{Y^n|X^n=x}$ and $P^*_Y$ and all $n$.

A couple of simple properties of $(b, c)$-concentrated functions:
1) Gaussian concentration around the mean:
\[
\mathbb{P}[|F(Y^n) - \mathbb{E}[F(Y^n)]| > t] \leq b \exp \left\{ -\frac{t^2}{4c} \right\}.
\]
(135)
2) Small variance:
\[
\text{Var}[F(Y^n)] = \int_0^\infty \mathbb{P}[|F(Y^n) - \mathbb{E}[F(Y^n)]|^2 > t] dt
\]
\[
\leq \int_0^\infty \min \left\{ b \exp \left\{ -\frac{t}{4c} \right\}, 1 \right\} dt
\]
\[
= 4c \log(2be).
\]
(137)
Some examples of concentrated functions include:
- A bounded function $F$ with $\|F\|_\infty \leq A$ is $(\exp\{A^2(4c)^{-1}\}, c)$-concentrated for any $c$ and any measure $\mu$. Moreover, for a fixed $\mu$ and a sufficiently large $c$ any bounded function is $(1, c)$-concentrated.
- If $F$ is $(b, c)$-concentrated then $\lambda F$ is $(b, \lambda^2 c)$-concentrated.
- Let $f: \mathcal{Y} \to \mathbb{R}$ be $(1, c)$-concentrated with respect to $\mu$. Then so is
\[
F(y^n) = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} f(y_j)
\]
(139)
with respect to $\mu^n$. In particular, any $F$ defined in this way from a bounded $f$ is $(1, c)$-concentrated for a memoryless channel (for a sufficiently large $c$ independent of $n$).
• If \( \mu = \mathcal{N}(0, 1)^n \) and \( F \) is a Lipschitz function on \( \mathbb{R}^n \) with Lipschitz constant \( \| F \|_{Lip} \), then \( F \) is \( (1, \frac{\| F \|_{Lip}^2}{2 \log e}) \)-concentrated with respect to \( \mu \), e.g. \[24\] Proposition 2.1:

\[
\int_{\mathbb{R}^n} \exp \left\{ t(F(y^n) - \bar{F}) \right\} d\mu(y^n) \leq \exp \left\{ \frac{\| F \|_{Lip}^2 t^2}{2 \log e} \right\}.
\]

Therefore any Lipschitz function is \( (1, \frac{\| F \|_{Lip}^2}{2 \log e}) \)-concentrated for the AWGN channel.

• For discrete \( Y^n \) endowed with the Hamming distance

\[
d(y^n, z^n) = |\{ i : y_i \neq z_i \}|
\]

(141)
define Lipschitz functions in the usual way. In this case, a simpler criterion is: \( F : Y^n \to \mathbb{R} \) is Lipschitz with constant \( \ell \) if and only if

\[
\max_{y^n, b, j} |F(y_1, \ldots, y_j, \ldots, y_n) - F(y_1, \ldots, b, \ldots, y_n)| \leq \ell.
\]

(142)

Let \( \mu \) be any product probability measure \( P_1 \times \ldots \times P_n \) on \( Y^n \), then the standard Azuma-Hoeffding estimate shows that

\[
\sum_{y^n \in Y^n} \exp \left\{ t(F(y^n) - \bar{F}) \right\} \mu(y^n) \leq \exp \left\{ \frac{n\| F \|_{Lip}^2 t^2}{2 \log e} \right\}
\]

(143)

and thus any Lipschitz function \( F \) is \( (1, \frac{n\| F \|_{Lip}^2}{2 \log e}) \)-concentrated with respect to any product measure on \( Y^n \).

Note that unlike the Gaussian case, the constant of concentration \( c \) worsens linearly with dimension \( n \). Generally, this growth cannot be avoided by the coefficient \( \frac{1}{\sqrt{n}} \) in the exact solution of the Hamming isoperimetric problem [25]. At the same time, this growth does not mean that (143) is “weaker” than (140); for example, \( F = \sum_{j=1}^{n} \phi(y_j) \) has Lipschitz constant \( O(\sqrt{n}) \) in Euclidean space and \( O(1) \) in Hamming. However, for convex functions the concentration (140) holds for product measures even under Euclidean distance [26].

We now show how to approximate expectations of concentrated functions:

**Proposition 8:** Suppose that \( F : Y^n \to \mathbb{R} \) is \((b, c)\)-concentrated with respect to \( P_{Y^n} \). Then

\[
| \mathbb{E}[F(Y^n)] - \mathbb{E}[F(Y^{**})] | \leq 2 \sqrt{cD(P_{Y^n} \| P_{Y^n}^*) + c \log b},
\]

(144)

where

\[
\mathbb{E}[F(Y^{**})] = \int F(y^n) dP_{Y^n}^*.
\]

(145)

**Proof:** Recall the Donsker-Varadhan inequality [27] Lemma 2.1: For any probability measures \( P \) and \( Q \) with \( D(P \| Q) < \infty \) and a measurable function \( g \) such that \( \int \exp \{g\} dQ < \infty \) we have that \( \int gdP \) exists (but perhaps is \(-\infty\)) and moreover

\[
\int gdP - \log \int \exp \{g\} dQ \leq D(P \| Q).
\]

(146)

Since by (134) the moment generating function of \( F \) exists under \( P_{Y^n}^* \), applying (146) to \( tF \) we get

\[
t \mathbb{E}[F(Y^n)] - \mathbb{E}[\exp \{tF(Y^{**})\}] \leq D(P_{Y^n} \| P_{Y^n}^*)
\]

(147)

From (134) we have

\[
ct^2 - t \mathbb{E}[F(Y^n)] + t \mathbb{E}[F(Y^{**})] + D(P_{Y^n} \| P_{Y^n}^*) + \log b \geq 0
\]

(148)

for all \( t \). Thus the discriminant of the parabola in (148) must be non-positive which is precisely (144).

Note that for empirical averages \( F(y^n) = \frac{1}{n} \sum_{j=1}^{n} f(y_j) \) we may either apply the estimate for concentration in the example (139) and then use Proposition 8 or directly apply Proposition 8 to (125); the result is the same:

\[
\left| \frac{1}{n} \sum_{j=1}^{n} \mathbb{E}[f(Y_j)] - \mathbb{E}[f(Y^*)] \right| \leq 2 \sqrt{\frac{2}{n} D(P_{Y^n} \| P_{Y^n}^*)} \to 0,
\]

(149)

for any \( f \) which is \((1, c)\)-concentrated with respect to \( P_{Y^n}^* \).

For the Gaussian channel, Proposition 8 and (140) yield:

**Corollary 9:** For any \( 0 < \epsilon < 1 \) there exist two constants \( a_1, a_2 > 0 \) such that for any \((n, M, \epsilon)_{\text{max, det}}\) code for the AWGN(\( P \)) channel and for any Lipschitz function \( F : \mathbb{R}^n \to \mathbb{R} \) we have

\[
| \mathbb{E}[F(Y^n)] - \mathbb{E}[F(Y^{**})] | \leq a_1 \| F \|_{Lip} \sqrt{nC - \log M_n + a_2 \sqrt{n}},
\]

(150)

where \( C = \frac{1}{2} \log(1 + P) \) is the capacity.

Note that in the proof of Corollary 9 concentration of measure is used twice: once for \( P_{Y^n | X^n} \) in the form of Poincaré’s inequality (proof of Theorem 7) and once in the form of (134) (proof of Proposition 8).
C. Concentration of functions of \( Y^n \)

Not only can we estimate expectations of \( F(Y^n) \) by replacing the unwieldy \( P_Y^n \) with the simple \( P_{Y^n}^* \), but in fact the distribution of \( F(Y^n) \) exhibits a sharp peak at its expectation:

**Proposition 10**: Consider a channel for which (122) holds. Then for any \( F \) which is \((b,c)\)-concentrated for such channel, we have for every \((n,M,\epsilon)_{\max,\det} \) code:

\[
\Pr[|F(Y^n) - \mathbb{E}[F(Y^n)]| > t] \leq 3b \exp \left\{ nC - \log M + a\sqrt{n} - \frac{t^2}{16c} \right\} \quad (151)
\]

and,

\[
\text{Var}[F(Y^n)] \leq 16c \left( nC - \log M + a\sqrt{n} + \log(6bc) \right). \quad (152)
\]

**Proof**: Denote for convenience:

\[
\bar{F} \triangleq \mathbb{E}[F(Y^n)], \quad \phi(x^n) = \mathbb{E}[F(Y^n)|X^n = x^n]. \quad (153, 154)
\]

Then as a consequence of \( F \) being \((b,c)\)-concentrated for \( P_{Y^n|X^n=x^n} \) we have

\[
\Pr[|F(Y^n) - \phi(x^n)| > t|X^n = x^n] \leq b \exp \left\{ -\frac{t^2}{4c} \right\}. \quad (155)
\]

Consider now a subcode \( C_1 \) consisting of all codewords such that \( \phi(x^n) > \bar{F} + t \) for \( t > 0 \). The number \( M_1 = |C_1| \) of codewords in this subcode is

\[
M_1 = M \Pr[\phi(X^n) > \bar{F} + t]. \quad (156)
\]

Let \( Q_{Y^n} \) be the output distribution induced by \( C_1 \). We have the following chain:

\[
\bar{F} + t \leq \frac{1}{M_1} \sum_{x \in C_1} \phi(x^n) \quad (157)
\]

\[
= \mathbb{E}[F(Y^n)]dQ_{Y^n} \quad (158)
\]

\[
\leq \bar{F} + 2\sqrt{cD(Q_{Y^n}||P_{Y^n}^*)} + c\log b \quad (159)
\]

\[
\leq \bar{F} + 2\sqrt{c(nC - \log M + a\sqrt{n})} + c\log b \quad (160)
\]

where (157) is by the definition of \( C_1 \), (158) is by (154), (159) is by Proposition 8 and the assumption of \((b,c)\)-concentration of \( F \) under \( P_{Y^n}^* \), and (160) is by (122).

Together (156) and (160) imply:

\[
\Pr[\phi(X^n) > \bar{F} + t] \leq b \exp \left\{ nC - \log M + a\sqrt{n} - \frac{t^2}{4c} \right\}. \quad (161)
\]

Applying the same argument to \(-F\) we obtain a similar bound on \( \Pr[|\phi(X^n) - \bar{F}| > t] \) and thus

\[
\Pr[|F(Y^n) - \bar{F}| > t] \leq \Pr[|F(Y^n) - \phi(X^n)| > t/2] + \Pr[|\phi(X^n) - \bar{F}| > t/2] \quad (162)
\]

\[
\leq b \exp \left\{ -\frac{t^2}{16c} \left( 1 + 2\exp\{nC - \log M + a\sqrt{n}\} \right) \right\} \quad (163)
\]

\[
\leq 3b \exp \left\{ -\frac{t^2}{16c} + nC - \log M + a\sqrt{n} \right\}, \quad (164)
\]

where (163) is by (155) and (161) and (164) is by (122). Thus (151) is proven. Moreover, (152) follows by (138). \( \blacksquare \)

Following up on Proposition 10, [19] gives a bound, which in contrast to (151), shows explicit dependence on \( \epsilon \).

D. Relation to optimal transportation

Since the seminal work of Marton [7], [28], optimal transportation theory has emerged as one of the major tools for proving \((b,c)\)-concentration of Lipschitz functions. Marton demonstrated that if a probability measure \( \mu \) on a metric space satisfies a \( T_1 \) inequality

\[
W_1(\nu, \mu) \leq \sqrt{cD(\nu||\mu)} \quad \forall \nu
\]

(165)
then any Lipschitz $f$ is $(b, \|f\|_{Lip}^2 c)$-concentrated with respect to $\mu$ for some $b = b(c, c')$ and any $0 < c < \frac{1}{4\theta}$. In \[(165)\]

$W_1(\nu, \mu)$ denotes the linear-cost transportation distance, or Wasserstein-1 distance, defined as

$$W_1(\nu, \mu) \overset{\Delta}{=} \inf_{P_{Y'}} E[d(Y, Y')] ,$$

(166)

where $d(\cdot, \cdot)$ is the distance on the underlying metric space and the infimum is taken over all couplings $P_{YY'}$ with fixed marginals $P_Y = \mu$, $P_{Y'} = \nu$. Note that according to \[29\] we have $\|\nu - \mu\|_{TV} = W_1(\nu, \mu)$ when the underlying distance on $\mathcal{Y}$ is $d(y, y') = 1\{y \neq y'\}$.

In this section we show that \[(165)\] in fact directly implies the estimate of Proposition \[8\] without invoking either Marton’s argument or Donsker-Varadhan inequality. Indeed, assume that $F: \mathcal{Y}^n \to \mathbb{R}$ is a Lipschitz function and observe that for any coupling $P_{Y^n,Y'^n}$ we have

$$|E[F(Y^n)] - E[F(Y'^n)]| \leq \|F\|_{lip} \cdot d(Y^n, Y'^n) ,$$

(167)

where the distance $d$ is either Hamming or Euclidean depending on the nature of $\mathcal{Y}$. Now taking the infimum in the right-hand side of \[(167)\] with respect to all couplings we observe

$$|E[F(Y^n)] - E[F(Y'^n)]| \leq \|F\|_{lip} W_1(P_{Y^n}, P_{Y'^n})$$

(168)

and therefore by the transportation inequality \[(165)\] we get

$$|E[F(Y^n)] - E[F(Y'^n)]| \leq \sqrt{c^2\|F\|_{lip}^2 D(P_{Y^n}||P_{Y'^n})}$$

(169)

which is precisely what Proposition \[8\] yields for $(1, \frac{c^2\|F\|_{lip}^2}{\theta})$-concentrated functions.

Our argument can be turned around and used to prove linear-cost transportation $T_1$ inequalities \[(165)\]. Indeed, by the Kantorovich-Rubinstein duality \[30\] Chapter 1 we have

$$\sup_{P'} |E[F(Y^n)] - E[F(Y'^n)]| = W_1(P_{Y^n}, P_{Y'^n}) ,$$

(170)

where the supremum is over all $F$ with $\|F\|_{lip} \leq 1$. Thus the argument in the proof of Proposition \[8\] shows that \[(165)\] must hold for any $\mu$ for which every 1-Lipschitz $F$ is $(1, c')$-concentrated, demonstrating an equivalence between $T_1$ transportation and Gaussian-like concentration—a result reported in \[31\] Theorem 3.1.

We also mention that unlike general iid measures, an iid Gaussian $\mu = \mathcal{N}(0, 1)^n$ satisfies a much stronger $T_2$-transportation inequality \[32\]

$$W_2(\nu, \mu) \leq c'(D(\nu||\mu))^\frac{3}{2} \quad \forall \nu \ll \mu ,$$

(171)

where remarkably $c'$ does not depend on $n$ and the Wasserstein-2 distance $W_2$ is defined as

$$W_2(\nu, \mu) \overset{\Delta}{=} \inf_{P_{YY'}} \sqrt{E[d^2(Y, Y')]},$$

(172)

the infimum being over all couplings as in \[(166)\].

**E. Empirical averages of non-Lipschitz functions**

One drawback of relying on the transportation inequality \[(165)\] in the proof of Proposition \[8\] is that it does not show anything for non-Lipschitz functions. In this section we demonstrate how the proof of Proposition \[8\] can be extended to functions that do not satisfy the strong concentration assumptions.

**Proposition 11**: Let $f: \mathcal{Y} \to \mathbb{R}$ be a (single-letter) function such that for some $\theta > 0$ we have $m_1 \overset{\Delta}{=} \mathbb{E}[\exp\{\theta f(Y^*)\}] < \infty$ (one-sided Cramer condition) and $m_2 = \mathbb{E}[f^2(Y^*)] < \infty$. Then there exists $b = b(m_1, m_2, \theta) > 0$ such that for all $n \geq \frac{16}{\theta^2}$ we have

$$\frac{1}{n} \sum_{j=1}^n E[f(Y_j)] \leq E[f(Y^*)] + \frac{1}{n^\frac{3}{2}} D(P_{Y^n}||P_{Y^*}^n) + \frac{b}{n^{\frac{1}{2}}}$$

(173)

**Proof**: It is clear that if the moment-generating function $t \mapsto \mathbb{E}[\exp\{tf(Y^*)\}]$ exists for $t = \theta > 0$ then it also exists for all $0 \leq t \leq \theta$. Notice that since

$$x^2 \exp\{-x\} \leq 4e^{-2 \log e} , \quad \forall x \geq 0$$

(174)
we have for all $0 \leq t \leq \frac{\theta}{2}$:
\[
\mathbb{E}[f^2(Y^*) \exp\{tf(Y^*)\}] \leq \mathbb{E}[f^2(Y^*)1\{f < 0\}] + \frac{16e^{-2}\log e}{(\theta - t)^2} \mathbb{E}[\exp\{\theta f(Y^*)\}1\{f \geq 0\}] 
\]
(175)
\[
\leq m_2 + \frac{e^{-2}m_1 \log e}{(\theta - t)^2} 
\]
(176)
\[
\leq m_2 + \frac{4e^{-2}m_1 \log e}{\theta^2} 
\]
(177)
\[
\triangleq b(m_1, m_2, \theta) \cdot 2 \log e. 
\]
(178)

Then a simple estimate
\[
\log \mathbb{E}[\exp\{tf(Y^*)\}] \leq t \mathbb{E}[f(Y^*)] + bt^2, \quad 0 \leq t \leq \frac{\theta}{2}, 
\]
(179)
can be obtained by taking the logarithm of the identity
\[
\mathbb{E}[\exp\{tf(Y^*)\}] = 1 + \frac{t^2}{2} \log \mathbb{E}[f(Y^*)] + \frac{1}{2} \log \mathbb{E} \left[ \int_0^t ds \int_0^s \mathbb{E}[f(Y^*) \exp\{tf(Y^*)\}] du \right] 
\]
(180)
and invoking (178) and $\log x \leq (x - 1) \log e$.

Next, we define $F(y^n) = \frac{1}{n} \sum_{i=1}^n f(y_i)$ and consider the chain:
\[
t \mathbb{E}[F(Y^n)] \leq \log \mathbb{E}[\exp\{tF(Y^n)\}] + D(P_{Y^n}||P_{Y^n}) 
\]
(181)
\[
= n \log \mathbb{E}[\exp\{\frac{t}{n} f(Y^*)\}] + D(P_{Y^n}||P_{Y^n}) 
\]
(182)
\[
\leq t \mathbb{E}[f(Y^*)] + \frac{bt^2}{n} + D(P_{Y^n}||P_{Y^n}), 
\]
(183)
where (181), (182), (183) follow from (147), $P_{Y^n} = (P_Y^n)^n$ and (179) assuming $\frac{t}{n} \leq \frac{\theta}{2}$. The proof concludes by letting $t = n^{\frac{3}{4}}$ in (183).

A natural extension of Proposition 11 to functions such as
\[
F(y^n) = \frac{1}{n - r + 1} \sum_{j=1}^{n-r+1} f(g_j^{y+r}) 
\]
(184)
is made by replacing the step (182) with an estimate
\[
\log \mathbb{E}[\exp\{tf(Y^*)\}] \leq \frac{n - r + 1}{r} \log \mathbb{E} \left[ \exp\left\{ \frac{tr}{n} f(Y^*) \right\} \right], 
\]
(185)
which in turn is shown by splitting the sum into $r$ subsums with independent terms and then applying Holder’s inequality:
\[
\mathbb{E}[X_1 \cdots X_r] \leq (\mathbb{E}[|X_1|^r]) \cdots (\mathbb{E}[|X_1|^r])^{\frac{1}{r}} 
\]
(186)

F. Functions of degraded channel outputs

Notice that if the same code is used over a channel $Q_{Y|X}$ which is stochastically degraded with respect to $P_{Y|X}$ then by the data-processing for relative entropy, the upper bound (122) holds for $D(Q_{Y^n}||Q_{Y^n}^*)$, where $Q_{Y^n}$ is the output of the $Q_{Y|X}$ channel and $Q_{Y^n}^*$ is the output of $Q_{Y|X}$ when the input is distributed according to a capacity-achieving distribution of $P_{Y|X}$. Thus, in all the discussions the pair $(P_{Y^n}, P_{Y^n}^*)$ can be replaced with $(Q_{Y^n}, Q_{Y^n}^*)$ without any change in arguments or constants. This observation can be useful in questions of information theoretic security, where the wiretapper has access to a degraded copy of the channel output.

G. Input distribution: DMC

As shown in Section IV-A we have for every $\epsilon$-capacity-achieving code:
\[
P_n = \frac{1}{n} \sum_{j=1}^n P_{Y_j} \rightarrow P_{Y}. 
\]
(187)
As noted in [2], convergence of output distributions can be propagated to statements about the input distributions. This is obvious for the case of a DMC with a non-singular (more generally, injective) matrix $P_{Y|X}$. Even if the capacity-achieving input distribution is not unique, the following argument extends that of [2] Theorem 4. By Theorem 4 and 5 we know that
\[
\frac{1}{n} I(X^n; Y^n) \rightarrow C. 
\]
(188)
Denote the single-letter empirical input distribution by $P_X = \frac{1}{n} \sum_{j=1}^{n} P_{X_j}$. Naturally, $I(\hat{X}; \hat{Y}) \leq C$. However, in view of (188) and the concavity of mutual information, we must necessarily have
\[
I(\hat{X}; \hat{Y}) \to C,
\]
(189)

By compactness of the simplex of input distributions and continuity of the mutual information on that simplex the distance to the (compact) set of capacity achieving distributions $\Pi$ must vanish:
\[
d(P_X; \Pi) \to 0.
\]
(190)

If the capacity achieving distribution $P_X^*$ is unique, then (190) shows the convergence of $P_X \to P_X^*$ in the (strong) sense of total variation.

H. Input distribution: AWGN

In the case of the AWGN, just like in the discrete case, (48) implies that for any capacity achieving sequence of codes we have
\[
P_X^{(n)} = \frac{1}{n} \sum_{j=1}^{n} P_{X_j} \xrightarrow{w} P_X^* \triangleq \mathcal{N}(0, P),
\]
(191)

however, in the sense of weak convergence of distributions only. Indeed, the induced empirical output distributions satisfy
\[
P_Y^{(n)} = P_X^{(n)} \ast \mathcal{N}(0, 1),
\]
(192)

where $\ast$ denotes convolution. By (48), (192) converges in relative entropy and thus weakly. Consequently, characteristic functions of $P_Y^{(n)}$ converge pointwise to that of $\mathcal{N}(0, 1 + P)$. By dividing out the characteristic function of $\mathcal{N}(0, 1)$ (which is strictly positive), so do characteristic functions of $P_X^{(n)}$. Then Levy’s criterion establishes (191).

We now discuss whether (191) can be claimed in a stronger topology than the weak one. Since $P_X$ is purely atomic and $P_X^*$ is purely diffuse, we have
\[
\|P_X - P_X^*\|_{TV} = 1,
\]
(193)

and convergence in total variation (let alone in relative entropy) cannot hold.

On the other hand, it is quite clear that the second moment of $\frac{1}{n} \sum P_{X_j}$ necessarily converges to that of $\mathcal{N}(0, P)$. Together weak convergence and control of second moments imply [30] (12), p.7]
\[
W_2^2 \left( \frac{1}{n} \sum_{j=1}^{n} P_{X_j}, P_X^* \right) \to 0.
\]
(194)

Therefore (191) holds in the sense of topology metrized by the $W_2$-distance.

Note that convexity properties of $W_2^2(\cdot, \cdot)$ imply
\[
W_2^2 \left( \frac{1}{n} \sum_{j=1}^{n} P_{X_j}, P_X^* \right) \leq \frac{1}{n} \sum_{j=1}^{n} W_2^2 \left( P_{X_j}, P_X^* \right)
\]
(195)
\[
\leq \frac{1}{n} W_2^2 \left( P_X^n, P_X^n^* \right),
\]
(196)

where we denoted
\[
P_X^n \triangleq (P_X^*)^n = \mathcal{N}(0, PI_n).
\]
(197)

Comparing (194) and (196), it is natural to conjecture a stronger result: For any capacity-achieving sequence of codes
\[
\frac{1}{\sqrt{n}} W_2(P_X^n, P_X^n^*) \to 0.
\]
(198)

Another reason to conjecture (198) arises from considering the behavior of Wasserstein distance under convolutions. Indeed from the $T_2$-transportation inequality [171] and the relative entropy bound [122] we have
\[
\frac{1}{n} W_2^2 \left( P_X^n \ast \mathcal{N}(0, I_n), P_X^n \ast \mathcal{N}(0, I_n) \right) \to 0,
\]
(199)

since by definition
\[
P_Y^n = P_X^n \ast \mathcal{N}(0, I_n)
\]
(200)
\[
P_Y^n = P_X^n \ast \mathcal{N}(0, I_n),
\]
(201)
where * denotes convolution of distributions on \( \mathbb{R}^n \). Trivially, for any \( P, Q \) and \( \mathcal{N} \) — probability measures on \( \mathbb{R}^n \) it is true that (e.g. [30, Proposition 7.17])

\[
W_2(P \ast \mathcal{N}, Q \ast \mathcal{N}) \leq W_2(P, Q).
\] (202)

Thus, overall we have

\[
0 \leq \frac{1}{\sqrt{n}} W_2(P_{X^n} \ast \mathcal{N}(0, I_n), P_{X^n} \ast \mathcal{N}(0, I_n)) \leq \frac{1}{\sqrt{n}} W_2(P_{X^n}, P_{X^n}'),
\] (203)

and (198) implies that the convolution with the Gaussian kernel is unable to significantly decrease \( W_2 \).

Despite the foregoing intuitive considerations, conjecture (198) is false. Indeed, define \( D^*(M, n) \) to be the minimum achievable average square distortion among all vector quantizers of the memoryless Gaussian source \( \mathcal{N}(0, P) \) for blocklength \( n \) and cardinality \( M \). In other words,

\[
D^*(M, n) = \frac{1}{n} \inf_Q W_2^2(P_{X^n}, Q),
\] (204)

where the infimum is over all probability measures \( Q \) supported on \( M \) equiprobable atoms in \( \mathbb{R}^n \). The standard rate-distortion (converse) lower bound dictates

\[
\frac{1}{n} \log M \geq \frac{1}{2} \log \frac{P}{D^*(M, n)}
\] (205)

and hence

\[
W_2^2(P_{X^n}, P_{X^n}') \geq nD^*(n, M)
\] (206)

\[
\geq nP \exp \left\{ -\frac{2}{n} \log M \right\},
\] (207)

which shows that for any sequence of codes with \( \log M_n = O(n) \), the normalized transportation distance stays strictly bounded away from zero:

\[
\liminf_{n \to \infty} \frac{1}{\sqrt{n}} W_2(P_{X^n}, P_{X^n}') > 0.
\] (208)

Nevertheless, assertion (191) may be strengthened in several ways, see Section VII.

I. Extension to other channels: tilting

Let us review the scheme of investigating functions of the output \( F(Y^n) \) that was employed in this paper so far. First, an inequality (122) was shown by verifying that \( Q_Y = P_{Y^n}^* \) satisfies the conditions of Theorem 2. Then an approximation of the form

\[
F(Y^n) \approx \mathbb{E}[F(Y^n)] \approx \mathbb{E}[F(Y^{*n})]
\] (209)

follows by Propositions 5 and 10 simultaneously for all concentrated (e.g. Lipschitz) functions. In this way, all the channel-specific work is isolated in proving (122). On the other hand, verifying conditions of Theorem 2 for \( Q_Y = P_{Y^n}^* \) may be quite challenging even for memoryless channels. In this section we show how Theorem 2 can be used to show (209) for a given function \( F \) in the absence of the universal estimate in (122).

Let \( P_{Y|X}: \mathcal{X} \to \mathcal{Y} \) be a random transformation, \( Y' \) distributed according to auxiliary distribution \( Q_Y \) and \( F: \mathcal{Y} \to \mathbb{R} \) a function such that

\[
Z_F = \log \mathbb{E}[\exp\{F(Y')\}] < \infty,
\] (210)

Let \( Q_Y^{(F)} \) an \( F \)-tilting of \( Q_Y \), namely

\[
dQ_Y^{(F)} = \exp\{F - Z_F\}dQ_Y
\] (211)

The core idea of our technique is that if \( F \) is sufficiently regular and \( Q_Y \) satisfies conditions of Theorem 2 then \( Q_Y^{(F)} \) also does. Consequently, the expectation of \( F \) under \( P_Y \) (induced by the code) can be investigated in terms of the moment-generating function of \( F \) under \( Q_Y \). For brevity we only present a variance-based version (similar to Corollary 3):

**Theorem 12:** Let \( Q_Y \) and \( F \) be such that (210) holds and

\[
S = \sup_x \text{Var}\left[ \log \frac{dP_{Y|X=x}}{Q_Y}(Y) \middle| X = x \right] < \infty,
\] (212)

\[
S_F = \sup_x \text{Var}[F(Y)|X=x].
\] (213)

Then there exists a constant \( a = a(\epsilon, S) > 0 \) such that for any \( (M, \epsilon)_{\text{max}, \text{det}} \) code we have for all \( 0 \leq t \leq 1 \)

\[
t \mathbb{E}[F(Y)] - \log \mathbb{E}[\exp\{tF(Y')\}] \leq D(P_{Y|X}||Q_Y|P_X) - \log M + aS + t^2S_F
\] (214)
Proof: Note that since
\[ \log \frac{dP_{Y|X}}{dQ_Y} = \log \frac{dP_{Y|X}}{dQ_Y} - \frac{X}{F} \] (215)
we have for any \( 0 \leq t \leq 1 \):
\[ D(P_{Y|X}||Q_Y^{(tF)}) = D(P_{Y|X}||P_X) - tE[F(Y)] + \log \mathbb{E}[\exp\{tF(Y)\}] \] (216)
and from (61)
\[ \text{Var} \left[ \log \frac{dP_{Y|X}}{dQ_Y} \right| X = x \leq 2(S + t^2S_F) \] (217)
We conclude by invoking Corollary 3 with \( Q_Y \) and \( S \) replaced by \( Q_Y^{(tF)} \) and \( 2S + 2t^2S_F \), respectively.

For example, Corollary 9 is recovered from (214) by taking \( Q_Y = P_{Y^n} \), applying (19), estimating the moment-generating function via (140) and bounding \( S_F \) via Poincaré inequality:
\[ S_F \leq b\|F\|_{\text{Lip}}^2. \] (218)

V. BINARY HYPOTHESIS TESTING \( P_{Y^n} \) VS. \( P_{Y^n}^* \)

We now turn to the question of distinguishing \( P_{Y^n} \) from \( P_{Y^n}^* \) in the sense of binary hypothesis testing. First, a simple data-processing reasoning yields for any \( 0 < \alpha \leq 1 \),
\[ d(\alpha || \beta_n(P_{Y^n}, P_{Y^n}^*)) \leq D(P_{Y^n}||P_{Y^n}^*), \] (219)
where we have denoted the binary relative entropy
\[ d(x||y) = x \log \frac{x}{y} + (1-x) \log \frac{1-x}{1-y}. \] (220)
From (122) and (219) we conclude: Every \( (n, M, \epsilon)_{\text{max,det}} \) code must satisfy
\[ \beta_n(P_{Y^n}, P_{Y^n}^*) \geq \left( \frac{M}{2} \right)^{\frac{1}{\alpha}} \exp \left\{ -\frac{C}{\alpha} + \sqrt{n\frac{a_1}{\alpha}} \right\} \] (221)
for all \( 0 < \alpha \leq 1 \). Therefore, in particular we see that the hypothesis testing problem for discriminating \( P_{Y^n} \) from \( P_{Y^n}^* \) has zero Stein’s exponent \( -\frac{1}{n} \log \beta_n(P_{Y^n}, P_{Y^n}^*) \), provided that the sequence of \( (n, M, \epsilon)_{\text{max,det}} \) codes with output distribution \( P_{Y^n} \), is capacity achieving.

The main result in this section gives a better bound than (224):

**Theorem 13:** Consider one of the three types of channels introduced in Section II. Then every \( (n, M, \epsilon)_{\text{avg}} \) code must satisfy
\[ \beta_n(P_{Y^n}, P_{Y^n}^*) \geq M \exp\{-nC - a_2\sqrt{n}\} \quad \epsilon \leq \alpha \leq 1, \] (222)
where \( a_2 = a_2(\epsilon, a_1) > 0 \) depends only on \( \epsilon \) and the constant \( a_1 \) from (23). To prove Theorem 13 we introduce the following converse whose particular case \( \alpha = 1 \) is (3) Theorem 27:

**Theorem 14:** Consider an \( (M, \epsilon)_{\text{avg}} \) code for an arbitrary random transformation \( P_{Y^n}|X \). Let \( P_X \) be equiprobable on the codebook \( C \) and \( P_Y \) be the induced output distribution. Then for any \( Q_Y \) and \( \epsilon \leq \alpha \leq 1 \) we have
\[ \beta_n(P_Y, Q_Y) \geq M \beta_{n-\epsilon}(P_{XY}, P_{X^n}Q_Y). \] (223)
If the code is \( (M, \epsilon)_{\text{max,det}} \) then additionally
\[ \beta_n(P_Y, Q_Y) \geq \frac{\delta}{1 - \alpha + \delta} M \inf_{x \in C} \beta_{n-\epsilon}(P_{Y^n}|X=x, Q_Y) \quad \epsilon + \delta \leq \alpha \leq 1, \] (224)

**Proof:** For a given \( (M, \epsilon)_{\text{avg}} \) code, define
\[ Z = 1\{W(Y) = W, Y \in E\}, \] (225)
where \( W \) is the message and \( E \) is an arbitrary event of the output space satisfying
\[ P_Y[E] \geq \alpha. \] (226)
As in the original meta-converse (3) Theorem 26 the main idea is to use \( Z \) as a suboptimal hypothesis test for discriminating \( P_{XY} \) against \( P_{X^n}Q_Y \). Following the same reasoning as in (3) Theorem 27 one notices that
\[ (P_{X^n}Q_Y)|Z = 1| \leq \frac{Q_Y[E]}{M}, \] (227)
and
\[ P_{XY}[Z = 1] \geq \alpha - \epsilon. \]  
(228)

Therefore, by definition of \( \beta_\alpha \) we must have
\[ \beta_\alpha(P_{XY}, P_{X|Y}) \leq \frac{Q_Y[E]}{M}. \]  
(229)

To complete the proof of (223) we take the infimum in (229) over all \( E \) satisfying (226).

To prove (224), we again consider any set \( E \) satisfying (226). Denote the codebook \( C = \{c_1, \ldots, c_M\} \) and for \( i = 1, \ldots, M \)
\[ p_i = P_{Y|X = c_i}[E] \]  
(230)
\[ q_i = Q_Y[\hat{W} = i, E]. \]  
(231)

Since the sets \( \{\hat{W} = i\} \) are disjoint, the (arithmetic) average of \( q_i \) is upper-bounded by
\[ E[q_W] \leq \frac{1}{M} Q_Y[E], \]  
(232)
whereas because of (226) we have
\[ E[p_W] \geq \alpha. \]  
(233)

Thus, the following lower bound holds:
\[ E \left[ \frac{Q_Y[E]}{M} p_W - \delta q_W \right] \geq \frac{Q_Y[E]}{M} (\alpha - \delta), \]  
(234)

implying that there must exist \( i \in \{1, \ldots, M\} \) such that
\[ \frac{Q_Y[E]}{M} p_i - \delta q_i \geq \frac{Q_Y[E]}{M} (\alpha - \delta). \]  
(235)

For such \( i \) we clearly have
\[ P_{Y|X = c_i}[E] \geq \alpha - \delta \]  
(236)
\[ Q_Y[\hat{W} = i, E] \leq \frac{Q_Y[E]}{M} \frac{1 - \alpha - \delta}{\delta}. \]  
(237)

By the maximal probability of error constraint we deduce
\[ P_{Y|X = c_i}[E, \hat{W} = i] \geq \alpha - \epsilon - \delta \]  
(238)
and thus by the definition of \( \beta_\alpha \):
\[ \beta_\alpha(\hat{W} = i, E) \leq \frac{Q_Y[E]}{M} \frac{1 - \alpha - \delta}{\delta}. \]  
(239)

Taking the infimum in (239) over all \( E \) satisfying (226) completes the proof of (224).

Proof of Theorem 13: To show (222) we first notice that as a consequence of (18), (23) and [3, Lemma 59] (see also [15, (2.71)]) we have for any \( x^n \in \mathcal{X}_n \):
\[ \beta_\alpha(P_{Y^n|X^n = x^n}, P_{Y^n}^*) \geq \frac{\alpha}{2} \exp \left\{ -nC - \sqrt{\frac{2a_1 n}{\alpha}} \right\}. \]  
(240)

From [15, Lemma 32] and the fact that the function of \( \alpha \) in the right-hand side of (240) is convex we obtain that for any \( P_{X^n} \)
\[ \beta_\alpha(P_{X^n Y^n}, P_{X^n} P_{Y^n}^*) \geq \frac{\alpha}{2} \exp \left\{ -nC - \sqrt{\frac{2a_1 n}{\alpha}} \right\}. \]  
(241)

Finally, (241) and (223) imply (222).

VI. AEP FOR THE OUTPUT PROCESS \( Y^n \)

Conventionally, we say that a sequence of distributions \( P_{Y^n} \) on \( \mathcal{Y}^n \) (with \( \mathcal{Y} \) a countable set) satisfies the asymptotic equipartition property (AEP) if
\[ \frac{1}{n} \log \frac{1}{P_{Y^n}(Y^n)} - H(Y^n) \to 0 \]  
(242)
in probability. In this section, we will take the AEP to mean convergence of (242) in the stronger sense of \( L_2 \), namely,
\[ \text{Var} \left[ \log P_{Y^n}(Y^n) \right] = o(n^2), \quad n \to \infty. \]  
(243)
A. DMC

Although the sequence of output distributions induced by a code is far from being (a finite chunk of) a stationary ergodic process, we will show that (242) is satisfied for $\epsilon$-capacity-achieving codes (and other codes). Thus, in particular, if the channel outputs are to be almost-losslessly compressed and stored for later decoding, $\frac{1}{n} H(Y^n)$ bits per sample would suffice (cf. (66)). In fact, $\log \frac{1}{P(Y^n)}$ concentrates up to $\sqrt{n}$ around the entropy $H(Y^n)$. Such questions are also interesting in other contexts and for other types of distributions, see [33], [34].

**Theorem 15:** Consider a DMC $P_{Y|X}$ with $C_1 < \infty$ (with or without input constraints) and a capacity achieving sequence of $(n, M_n, c)_{\max,det}$ codes. Then the output AEP (242) holds.

**Proof:** In the proof of Theorem 5 it was shown that $\log P_{Y^n}(y^n)$ is Lipschitz with Lipschitz constant upper bounded by $a_1$. Thus by (143) and Proposition 10 we find that for any capacity-achieving sequence of codes (243) holds. For many practically interesting DMCs (such as those with additive noise in a finite group), the estimate (243) can be improved to $O(n)$ even without assuming the code to be capacity-achieving.

**Theorem 16:** Consider a DMC $P_{Y|X}$ with $C_1 < \infty$ (with or without input constraints) and such that $H(Y|X=x)$ is constant on $X$. Then for any sequence of $(n, M_n, c)_{\max,det}$ codes there exists a constant $a = a(\epsilon)$ such that for all $n$ sufficiently large

$$\Var \left[ \log P_{Y^n}(Y^n) \right] \leq an. \tag{244}$$

In particular, the output AEP (243) holds.

**Proof:** First, let $X$ be a random variable and $A$ some event (think $P[A^c] \ll 1$) such that

$$|X - E[X]| \leq L \tag{245}$$

if $X \notin A$. Then, denoting $\Var[X|A] = E[X^2|A] - E^2[X|A],$$

$$\Var[X] = E[(X - E[X])^2]_A + E[(X - E[X])^2]_A^c \tag{246}$$

$$\leq E[(X - E[X])^2]_A + \Var[A^c]L^2 \tag{247}$$

$$= P[A]\left(\Var[X|A] + \left(\frac{P[A^c]}{P[A]}\right)^2 (E[X] - E[X|A])^2\right) + \Var[A^c]L^2 \tag{248}$$

$$\leq \Var[X|A] + \frac{\Var[A^c]}{P[A]}L^2, \tag{249}$$

where (247) is by (245), (248) is because

$$E[(X - E[X])^2|A] = \Var[X|A] + (E[X|A] - E[X])^2 \tag{250}$$

$$= \Var[X|A] + \left(\frac{P[A^c]}{P[A]}\right)^2 (E[X] - E[X|A])^2$$

which in turn follows from identity

$$E[X|A] = \frac{E[X] - P[A^c]E[X|A^c]}{P[A]} \tag{252}$$

and (249) is because (243) implies $|E[X|A^c] - E[X]| \leq L$.

Next, fix $n$ and for any codeword $x^n \in X^n$, denote for brevity

$$d(x^n) = D(P_{Y^n|X^n=x^n}||P_{Y^n}) \tag{253}$$

$$v(x^n) = E \log \frac{1}{P_{Y^n}(Y^n)} | X^n = x^n \tag{254}$$

$$= d(x^n) + H(Y^n|X^n = x^n). \tag{255}$$

If we could show that for some $a_1 > 0$

$$\Var[d(X^n)] \leq a_1 n \tag{256}$$

the proof would be completed as follows:

$$\Var \left[ \log \frac{1}{P_{Y^n}(Y^n)} \right] = \Var \left[ \log \frac{1}{P_{Y^n}(Y^n)} \right]_{X^n} + \Var[v(X^n)] \tag{257}$$

$$\leq a_2 n + \Var[v(X^n)] \tag{258}$$

$$= a_2 n + \Var[d(X^n)] \tag{259}$$

$$\leq (a_1 + a_2)n, \tag{260}$$
where (258) follows for an appropriate constant $a_2 > 0$ from (58), (259) is by (255) and $H(Y^n|X^n = x^n)$ does not depend on $x^n$ by assumption\(^\text{10}\) and (260) is by (256).

To show (260), first note the bound on the information density
\[
\iota_{X^n,Y^n}(x^n;y^n) = \log \frac{P_{Y^n|X^n}(y^n|x^n)}{P_{X^n}(x^n)} \leq \log M_n.
\] (261)

Second, as shown in (64) one may take $S_m = a_3 n$ in Corollary \(\text{8}\). In turn, this implies that one can take $\Delta = \sqrt{2a_3 n / (1 - \epsilon)}$ and $\delta' = \frac{1}{2n}$ in Theorem \(\text{2}\) that is:
\[
\inf_{x^n} \mathbb{P} \left[ \log \frac{P_{Y^n|X^n=x^n}(y^n)}{P_{Y^n}(y^n)} < d(x^n) + \Delta \right] \geq \frac{1 + \epsilon}{2}.
\] (262)

Then applying Theorem \(\text{1}\) with $\rho(x^n) = d(x^n) + \Delta$ to the $(M'_n, \epsilon)_{\text{max, det}}$ subcode consisting of all codewords with $\{d(x^n) \leq \log M_n - 2\Delta \}$ we get
\[
\mathbb{P}[d(X^n) \leq \log M_n - 2\Delta] = \frac{2}{1 - \epsilon} \exp\{-\Delta\},
\] (263)
since $M'_n = M_n \mathbb{P}[d(X^n) \leq \log M_n - 2\Delta]$ and
\[
\mathbb{E}[\exp(\rho(X^n))] d(X^n) \leq \log M_n - 2\Delta \leq M_n \exp\{-\Delta\}.
\] (264)

Now, we apply (249) to $d(X^n)$ with $L = \log M_n$ and $A = \{d(X^n) > \log M_n - 2\Delta\}$. Since $\text{Var}[X|A] \leq \Delta^2$ this yields
\[
\text{Var}[d(X^n)] \leq \Delta^2 + \frac{2 \log^2 M_n}{1 - \epsilon} \exp\{-\Delta\}
\] (265)
for all $n$ such that $\frac{2}{1 - \epsilon} \exp\{-\Delta\} \leq \frac{1}{2}$. Since $\Delta = O(\sqrt{n})$ and $\log M_n = O(n)$ we conclude from (265) that there must be a constant $a_1$ such that (256) holds.

\[\text{B. AWGN}\]

Following the argument of Theorem \(\text{16}\) step by step with (106) used in place of (58), we arrive at a similar AEP for the AWGN channel.

\textbf{Theorem 17}: Consider the AWGN channel. Then for any sequence of $(n, M_n, \epsilon)_{\text{max, det}}$ codes there exists a constant $a = o(\epsilon)$ such that for all $n$ sufficiently large
\[
\text{Var}\left[\log p_{Y^n}(Y^n)\right] \leq an,
\] (266)
where $p_{Y^n}$ is the density of $Y^n$.

\textbf{Corollary 18}: If in the setting of Theorem \(\text{17}\), the codes are spherical (i.e., the energies of all codewords $X^n$ are equal) or, more generally,
\[
\text{Var}[||X^n||^2] = o(n^2),
\] (267)
then
\[
\frac{1}{n} \left| \log \frac{dP_{Y^n}}{dP_{Y^n}^*}(y^n) - D(P_{Y^n}\|P_{Y^n}^*) \right| \to 0
\] (268)
in $P_{Y^n}$-probability.

\textbf{Proof}: To apply Chebyshev’s inequality to $\log \frac{dP_{Y^n}}{dP_{Y^n}^*}(y^n)$ we need, in addition to (266), to show
\[
\text{Var}[\log p_{Y^n}^*(Y^n)] = o(n^2),
\] (269)
where $p_{Y^n}^*(y^n) = (2\pi(1 + P))^n \mathrm{e}^{-\|y^n\|^2/(2(1 + P)^2)}$. Introducing i.i.d. $Z_j \sim \mathcal{N}(0, 1)$ we have
\[
\text{Var}[\log p_{Y^n}^*(Y^n)] = \frac{\log^2 e}{4(1 + P)^2} \text{Var} \left[ ||X^n||^2 + 2 \sum_{j=1}^{n} X_j Z_j + ||Z^n||^2 \right].
\] (270)

The variances of the second and third terms are clearly $O(n)$, while the variance of the first term is $o(n^2)$ by assumption (267). Then (270) implies (269) via (61).

\(^\text{10}\)This argument also shows how to construct a counterexample when $H(Y|X = x)$ is non-constant: merge two constant composition subcodes of types $P_1$ and $P_2$ such that $H(W|P_1) \neq H(W|P_2)$ where $W = P_{Y|X}$ is the channel matrix. In this case one clearly has $\text{Var}[\log P_{Y^n}(y^n)] \geq \text{Var}[v(X^n)] = \text{const} \cdot n^2$.\]
VII. EXPECTATIONS OF NON-LINEAR POLYNOMIALS OF GAUSSIAN CODES

This section contains results special to the AWGN channel. Because of the algebraic structure available on \( \mathbb{R}^n \) it is natural to ask whether we can provide approximations for polynomials. Since Theorem 7 shows the validity of (122), all the results for Lipschitz (in particular linear) functions from Section IV follow. Polynomials of higher degree, however, do not admit bounded Lipschitz constants. In this section we discuss the case of quadratic polynomials (Section VII-A) and polynomials of higher degree (Section VII-B). We present results directly in terms of the polynomials in \((X_1, \ldots, X_n)\) on the input space. This is strictly stronger than considering polynomials on the output space, since \( E[q(Y^n)] = E[q(X^n + Z^n)] \) and thus by taking integrating over distribution of \( Z^n \) problem reduces to computing the expectation of a (different) polynomial of \( X^n \). The reverse reduction is not possible, clearly.

A. Quadratic forms

We denote the canonical inner product on \( \mathbb{R}^n \) as

\[
(a, b) = \sum_{j=1}^{n} a_j b_j ,
\]

and write the quadratic form corresponding to matrix \( A \) as

\[
(Ax, x) = \sum_{j=1}^{n} \sum_{i=1}^{n} a_{i,j} x_i x_j .
\]

Note that when \( X^n \sim \mathcal{N}(0, P) \) we have trivially

\[
E[(AX^n, X^n)] = P \text{ tr } A ,
\]

where \( \text{tr} \) is the trace operator. Therefore, the next result shows that the distribution of good codes must be close to isotropic Gaussian distribution, at least in the sense of evaluating quadratic forms:

**Theorem 19:** For any \( P > 0 \) and \( 0 < \epsilon < 1 \) there exists a constant \( b = b(P, \epsilon) > 0 \) such that for all \( (n, M, \epsilon)_{\text{max, det}} \) codes and all quadratic forms \( A \) such that

\(-I_n \leq A \leq I_n\)

we have

\[
|E[(AX^n, X^n)] - P \text{ tr } A| \leq \frac{2(1 + P)\sqrt{n}e}{\sqrt{\log e}} \sqrt{nC - \log M + b\sqrt{n}} \]

(275)

and (a refinement for \( A = I_n \))

\[
|\sum_{j=1}^{n} E[X_j^2] - nP| \leq \frac{2(1 + P)}{\log e} (nC - \log M + b\sqrt{n}) .
\]

(276)

**Remark 9:** By using the same method as in the proof of Proposition 10 one can also show that the estimate (275) holds on a per-codeword basis for an overwhelming majority of codewords.

**Proof:** Denote

\[
\Sigma = E[xx^T],
\]

(277)

\[
V = (I_n + \Sigma)^{-1},
\]

(278)

\[
Q_{Y^n} = \mathcal{N}(0, I_n + \Sigma),
\]

(279)

\[
R(y|x) = \frac{dP_{Y^n|x=x}}{dQ_{Y^n}}(y),
\]

(280)

\[
= \frac{\log e}{2} (\ln \det(I_n + \Sigma) + (Vy, y) - ||y - x||^2) ,
\]

(281)

\[
d(x) = E[R(Y^n|x)|X^n = x] ,
\]

(282)

\[
v(x) = \text{Var}[R(Y^n|x)|X^n = x] .
\]

(283)

Denote also the spectrum of \( \Sigma \) by \( \{\lambda_i, i = 1, \ldots, n\} \) and its eigenvectors by \( \{v_i, i = 1, \ldots, n\} \). We have then

\[
|E[(AX^n, X^n)] - P \text{ tr } A| = |\text{tr}(\Sigma - PI_n)A|
\]

(285)

\[
= \left| \sum_{i=1}^{n} (\lambda_i - P)(Av_i, v_i) \right| \leq \sum_{i=1}^{n} |\lambda_i - P| ,
\]

(286)

(287)
where (286) follows by computing the trace in the eigenbasis of $\Sigma$ and (287) is by (274). From (283), it is straightforward to check that

$$D(P_{Y^n|X^n}|Q_{Y^n|P_{X^n}}) = \mathbb{E}[d(X^n)]$$

$$= \frac{1}{2} \log \det(I_n + \Sigma)$$

$$= \frac{1}{2} \sum_{j=1}^{n} \log(1 + \lambda_j).$$

By using (61) we estimate

$$v(x) \leq 3 \log e \left( \frac{1}{4} \text{Var}[||Z^n||^2] + \frac{1}{4} \text{Var}[(VZ^n, Z^n)] + \text{Var}[(VX, Z^n)] \right)$$

$$\leq n \left( \frac{9}{4} + 3P \right) \log e \cong nb^2$$

where (292) results from applying the following identities and bounds for $Z^n \sim N(0, I_n)$:

$$\text{Var}[||Z^n||^2] = 3n,$$

$$\text{Var}[a, Z^n] = ||a||^2,$$

$$\text{Var}[(VZ^n, Z^n)] = 3 \text{tr} V^2 \leq 3n$$

$$||Vx||^2 \leq ||x||^2 \leq nP.$$ 

Finally from Corollary 3 applied with $S_m = b^2/n$ and (290) we have

$$\frac{1}{2} \sum_{j=1}^{n} \log(1 + \lambda_j) \geq \log M - b_1 \sqrt{n} - \log \frac{2}{1 - \epsilon}$$

$$\geq \log M - b \sqrt{n}$$

$$= \frac{n}{2} (\log(1 + P) - \delta_n),$$

where we abbreviated

$$b = \sqrt{\frac{2 (\frac{9}{4} + 3P)}{1 - \epsilon} \log e + \log \frac{2}{1 - \epsilon}},$$

$$\delta_n = 2(nC + b \sqrt{n} - \log M).$$

To derive (276) consider the chain:

$$- \delta_n \leq \frac{1}{n} \sum_{j=1}^{n} \log \frac{1 + \lambda_i}{1 + P}$$

$$\leq \log \left( \frac{1}{n} \sum_{j=1}^{n} \frac{1 + \lambda_i}{1 + P} \right)$$

$$\leq \frac{\log e}{n(1 + P)} \sum_{j=1}^{n} (\lambda_i - P)$$

$$= \frac{\log e}{n(1 + P)} (\mathbb{E}[||X^n||^2] - nP)$$

where (302) is (299), (303) is by Jensen’s inequality, (304) is by $\log x \leq (x - 1) \log e$. Note that (296) is equivalent to (305).

Finally, (275) follows from (287), (302) and the next Lemma applied with $X$ equiprobable on $\{\frac{i + \lambda_i}{1 + P}, i = 1, \ldots, n\}$. □

**Lemma 20:** Let $X > 0$ and $\mathbb{E}[X] \leq 1$, then

$$\mathbb{E}[||X - 1||] \leq 2 \mathbb{E} \left[ \frac{1}{X} \right].$$

**Proof:** Define two distributions on $\mathbb{R}_+$:

$$P[E] \triangleq \mathbb{P}[X \in E]$$

$$Q[E] \triangleq \mathbb{E}[X \cdot 1\{X \in E\}] + (1 - \mathbb{E}[X])1\{0 \in E\}.$$
Then, we have
\[2\| P - Q \|_{TV} = 1 - \mathbb{E}[X] + \mathbb{E}[|X - 1|] \]  
(309)
\[D(P||Q) = \mathbb{E} \left[ \log \frac{1}{X} \right]. \]  
(310)
and (306) follows by (130).

The proof of Theorem 19 relied on a direct application of the main inequality (in the form of Corollary 3) and is independent of the previous estimate (122). At the expense of a more technical proof we could derive an order-optimal form of Theorem 19 starting from (122) using concentration properties of Lipschitz functions. Indeed, notice that because \(\mathbb{E}[Z^n] = 0\) we have
\[\mathbb{E}[(AY^n, Y^n)] = \mathbb{E}[(AX^n, X^n)] + trA. \]  
(311)
Thus, (275) follows from (122) if we can show
\[\| \mathbb{E}[(AY^n, Y^n)] - \mathbb{E}[(AY^n, Y^n)] \| \leq b\sqrt{nD(P_{Y^n}||P_{Y^n}^*)} . \]  
(312)
This is precisely what Corollary 7 would imply if the function \(y \mapsto (Ay, y)\) were Lipschitz with constant \(O(\sqrt{n})\). However, \((Ay, y)\) is generally not Lipschitz when considered on the entire of \(\mathbb{R}^n\). On the other hand, it is clear that from the point of view of evaluation of both the \(\mathbb{E}[(AY^n, Y^n)]\) and \(\mathbb{E}[(AY^n, Y^n)]\) only vectors of norm \(O(\sqrt{n})\) are important, and when restricted to the ball \(S = \{ y : \|y\|_2 \leq b/\sqrt{n} \}\) quadratic form \((Ay, y)\) does have a required Lipschitz constant of \(O(\sqrt{n})\). This approximation idea can be made precise using Kirzbraun’s theorem (see 35 for a short proof) to extend \((Ay, y)\) beyond the ball \(S\) preserving the maximum absolute value and the Lipschitz constant \(O(\sqrt{n})\). Another method of showing (312) is by using the Bobkov-Götze extension of Gaussian concentration (140) to non-Lipschitz functions (31 Theorem 1.2) to estimate the moment generating function of \((AY^n, Y^n)\) and apply (127) with \(t = \sqrt{nD(P_{Y^n}||P_{Y^n}^*)}\). Both methods yield (312), and hence (275), but with less sharp constants than those in Theorem 19.

B. Behavior of \(\| x \|_q\)

The next natural question is to consider polynomials of higher degree. The simplest example of such polynomials are \(F(x) = \sum_{j=1}^n x_j^q\) for some power \(q\), to analysis of which we proceed now. To formalize the problem, consider \(1 \leq q \leq \infty\) and define the \(q\)-th norm of the input vector in the usual way
\[\| x \|_q \triangleq \left( \sum_{i=1}^n |x_i|^q \right)^{\frac{1}{q}}. \]  
(313)

The aim of this section is to investigate the values of \(\| x \|_q\) for the codewords of good codes for the AWGN channel. Notice that when the coordinates of \(x\) are independent Gaussians we expect to have
\[\sum_{i=1}^n |x_i|^q \approx n \mathbb{E}[|Z|^q], \]  
(314)
where \(Z \sim \mathcal{N}(0,1)\). In fact it can be shown that there exists a sequence of capacity achieving codes and constants \(B_q, 1 \leq q \leq \infty\) such that every codeword \(x\) at every blocklength \(n\) satisfies
\[\| x \|_q \leq B_q n^{\frac{q}{q+1}} = O(n^{\frac{q}{q+1}}) \]  
(315)
\[\| x \|_\infty \leq B_\infty \sqrt{\log n} = O(\sqrt{\log n}). \]  
(316)

But do (315)-(316) hold (possibly with different constants) for any good code? It turns out that the answer depends on the range of \(q\) and on the degree of optimality of the code. Our findings are summarized in Table I. The precise meaning of each entry will be clear from Theorems 21 and 24 and their corollaries. The main observation is that the closer the code size comes to \(M(n, \epsilon)\), the better \(\ell_q\)-norms reflect those of random Gaussian codewords (315), (316). Loosely speaking, very little can be said about \(\ell_q\)-norms of capacity-achieving codes, while \(O(\log n)\)-achieving codes are almost indistinguishable from the random Gaussian ones. In particular, we see that, for example, for capacity-achieving codes it is not possible to approximate expectations of polynomials of degrees higher than 2 (or 4 for dispersion-achieving codes) by assuming Gaussian inputs, since even the asymptotic growth rate with \(n\) can be dramatically different. The question of whether we can approximate expectations of arbitrary polynomials for \(O(\log n)\)-achieving codes remains open.

We proceed to support the statements made in Table I.

\[\text{This does not follow from a simple random coding argument since we want the property to hold for every codeword, which constitutes exponentially many constraints. However, the claim can indeed be shown by invoking the } \kappa\beta\text{-bound [3 Theorem 25] with a suitably chosen constraint set } F.\]
In fact, each estimate in Table I, except \( n^{\frac{1}{q}} \log \frac{n^q}{q} n \), is tight in the following sense: if the entry is \( n^\alpha \), then there exists a constant \( B_q \) and a sequence of \( O(\log n) \)-, dispersion-, \( O(\sqrt{\alpha}) \)-, or capacity-achieving \( (n, M, 1, \epsilon)_{max,det} \) codes such that each codeword \( x \in \mathbb{R}^n \) satisfies for all \( n \geq 1 \)

\[
\|x\|_q \geq B_q n^\alpha. \tag{317}
\]

If the entry in the table states \( o(n^\alpha) \) then there is \( B_q \) such that for any sequence \( \tau_n \to 0 \) there exists a sequence of \( O(\log n) \)-, dispersion-, \( O(\sqrt{\alpha}) \)-, or capacity-achieving \( (n, M, 1, \epsilon)_{max,det} \) codes such that each codeword satisfies for all \( n \geq 1 \)

\[
\|x\|_q \geq B_q \tau_n n^\alpha. \tag{318}
\]

First, notice that a code from any row is an example of a code for the next row, so we only need to consider each entry which is worse than the one directly above it. Thus it suffices to show the tightness of \( o(n^\ast), n^\ast, o(n^{\frac{1}{q}}) \) and \( n^{\frac{1}{q}} \).

To that end recall that by [3, Theorem 54] the maximum number of codewords \( M^*, (n, \epsilon) \) at a fixed probability of error \( \epsilon \) for the AWGN channel satisfies

\[
\log M^*(n, \epsilon) = nC - \sqrt{nVQ^{-1}(\epsilon)} + O(\log n), \tag{319}
\]

where \( V(P) = \frac{\log^2 e P^2}{(P+2)^2} \) is the channel dispersion. Next, we fix a sequence \( \delta_n \to 0 \), such that \( n\delta_n \to \infty \) and construct the following sequence of codes. The first coordinate \( x_1 = \sqrt{n\delta_n P} \) for every codeword and the rest \((x_2, \ldots, x_n)\) are chosen as coordinates of an optimal AWGN code for blocklength \( n-1 \) and power-constraint \((1-\delta_n)P\). Following the argument of [3, Theorem 67] the number of codewords \( M_n \) in such a code will be at least

\[
\log M_n = (n-1)C(P - \delta_n) - \sqrt{(n-1)V(P-\delta_n)Q^{-1}(\epsilon)} + O(1) \tag{320}
\]

\[
= nC(P) - \sqrt{nV(P)Q^{-1}(\epsilon)} + O(n\delta_n). \tag{321}
\]

At the same time, because \( x_1 \) of each codeword \( x \) is abnormally high we have

\[
\|x\|_q \geq \sqrt{n\delta_n P}. \tag{322}
\]

So all the examples are constructed by choosing a suitable \( \delta_n \) as follows:

- **Row 1:** see (315)–(316).
- **Row 2:** nothing to prove.
- **Row 3:** for entries \( o(n^{\frac{1}{q}}) \) taking \( \delta_n = \frac{\tau^2}{\sqrt{n}} \) yields a dispersion-achieving code according to (321); the estimate (318) follows from (322).
- **Row 4:** for entries \( n^{\frac{1}{q}} \) taking \( \delta_n = \frac{\tau^2}{\sqrt{n}} \) yields an \( O(\sqrt{\alpha}) \)-achieving code according to (321); the estimate (317) follows from (322).
- **Row 5:** for entries \( o(n^{\frac{1}{q}}) \) taking \( \delta_n = \tau^2 \sqrt{n} \) yields a capacity-achieving code according to (321); the estimate (318) follows from (322).
- **Row 6:** for entries \( n^{\frac{1}{q}} \) we can take a codebook with one codeword \((\sqrt{n^2 P}, 0, \ldots, 0)\).

**Remark 10:** The proof can be modified to show that in each case there are codes that simultaneously achieve all entries in the respective row of Table I (except \( n^{\frac{1}{q}} \log \frac{n^q}{q} n \)).

We proceed to proving upper bounds. First, we recall some simple relations between the \( \ell_q \) norms of vectors in \( \mathbb{R}^n \). To estimate a lower-\( q \) norm in terms of a higher one, we invoke Holder’s inequality:

\[
\|x\|_q \leq n^{\frac{1}{p} - \frac{1}{q}} \|x\|_p, \quad 1 \leq q \leq p \leq \infty. \tag{323}
\]

To provide estimates for \( q > p \), notice that obviously

\[
\|x\|_\infty \leq \|x\|_p. \tag{324}
\]
Then, we can extend to \( q < \infty \) via the following chain:

\[
\|x\|_q \leq \|x\|_\infty^{1 - \frac{q}{p}} \|x\|_p^\frac{q}{p} \leq \|x\|_p, \quad q \geq p
\]  

(325)

(326)

Trivially, for \( q = 2 \) the answer is given by the power constraint

\[
\|x\|_2 \leq \sqrt{nP}
\]  

(327)

Thus by (325) and (326) we get: Each codeword of any code for the AWGN(\( P \)) channel must satisfy

\[
\|x\|_q \leq \sqrt{P} \cdot \begin{cases} 
\frac{n}{q}, & 1 \leq q \leq 2, \\
\frac{n}{\sqrt{q}}, & 2 < q \leq \infty.
\end{cases}
\]  

(328)

This proves the entries in the first column and the last row of Table I.

Before proceeding to justify the upper bounds for \( q > 2 \) we point out an obvious problem with trying to estimate \( \|x\|_q \) for each codeword. Given any code whose codewords lie exactly on the power sphere, we can always apply an orthogonal transformation to it so that one of the codewords becomes \((\sqrt{nP}, 0, 0, \ldots, 0)\). For such a codeword we have

\[
\|x\|_q = \sqrt{nP}
\]  

(329)

and the upper-bound (328) is tight. Therefore, to improve upon the (328) we necessarily consider subsets of codewords of a given code. For simplicity below we show estimates for the half of all codewords.

The following result, proven in the Appendix, takes care of the sup-norm:

**Theorem 21** \((q = \infty)\): For any \( 0 < \epsilon < 1 \) and \( P > 0 \) there exists a constant \( b = b(P, \epsilon) \) such that for any \( \epsilon \)

\[
\|n, M, \epsilon\)_{max, det}-code for the AWGN(\( P \)) channel at least half of the codewords satisfy

\[
\|x\|_\infty \leq \frac{4(b + P)}{\log e} \left( nC - \sqrt{nVQ^{-1}(\epsilon)} + 2 \log n + \log b - \log \frac{M}{2} \right),
\]  

(330)

where \( C \) and \( V \) are the capacity and the dispersion. In particular, the expression in \( \cdot \) is non-negative for all codes and blocklengths.

**Remark 11**: What sets Theorem 21 apart from other results is its sensitivity to whether the code achieves the dispersion term. This is unlike estimates of the form (122), which only sense whether the code is \( O(\sqrt{n}) \)-achieving or not.

From Theorem 21 the explanation of the entries in the last column of Table I becomes obvious: the more the code achieves in the asymptotic expansion of \( \log M^*(n, \epsilon) \) the closer \( \|x\|_\infty \) becomes to the \( O(\sqrt{\log n}) \), which arises from a random Gaussian codeword (310). To be specific, we give the following exact statements:

**Corollary 22** \((q = \infty \) for \( O(\log n)\)-codes): For any \( 0 < \epsilon < 1 \) and \( P > 0 \) there exists a constant \( b = b(P, \epsilon) \) such that for any \( (n, M, \epsilon)_{max, det}\)-code for the AWGN(\( P \)) with

\[
\log M_n \geq nC - \sqrt{nVQ^{-1}(\epsilon)} - K \log n
\]  

(331)

for some \( K > 0 \) we have that at least half of the codewords satisfy

\[
\|x\|_\infty \leq \sqrt{(b + K) \log n} + b.
\]  

(332)

**Corollary 23** \((q = \infty \) for capacity-achieving codes): For any capacity-achieving sequence of \( (n, M_n, \epsilon)_{max, det}\)-codes there exists a sequence \( \tau_n \to 0 \) such that for at least half of the codewords we have

\[
\|x\|_\infty \leq \tau_n^{\frac{1}{2}}.
\]  

(333)

Similarly, for any dispersion-achieving sequence of \( (n, M_n, \epsilon)_{max, det}\)-codes there exists a sequence \( \tau_n \to 0 \) such that for at least half of the codewords we have

\[
\|x\|_\infty \leq \tau_n^{\frac{1}{2}}.
\]  

(334)

**Remark 12**: By (310), the sequences \( \tau_n \) in Corollary 23 are necessarily code-dependent.

For the \( q = 4 \) we have the following estimate (see Appendix for the proof):

**Theorem 24** \((q = 4)\): For any \( 0 < \epsilon < \frac{1}{2} \) and \( P > 0 \) there exist constants \( b_1 > 0 \) and \( b_2 > 0 \), depending on \( P \) and \( \epsilon \), such that for any \( (n, M, \epsilon)_{max, det}\)-code for the AWGN(\( P \)) channel at least half of the codewords satisfy

\[
\|x\|_4^2 \leq \frac{2}{b_1} \left( nC + b_2 \sqrt{n} - \log \frac{M}{2} \right),
\]  

(335)

\(^{12} N(P, \epsilon) = 8(1 + 2P^{-1})(Q^{-1}(\epsilon))^2 \) for \( \epsilon < \frac{1}{2} \) and \( N(P, \epsilon) = 1 \) for \( \epsilon \geq \frac{1}{2} \).
where $C$ is the capacity of the channel. In fact, we also have a lower bound

$$\mathbb{E}[\|x\|_4^4] \geq 3nP^2 - (nC - \log M + b_3 \sqrt{n})n^\frac{1}{2},$$  
(336)

for some $b_3 = b_3(P, \epsilon) > 0$.

Remark 13: Note that $\mathbb{E}[\|z\|_4^4] = 3nP^2$ for $z \sim N(0, P)^n$.

We can now complete the proof of the results in Table 1:

1) Row 2: $q = 4$ is Theorem 24; $2 < q \leq 4$ follows by (323) with $p = 4$; $q = \infty$ is Corollary 22

2) Row 3: $q \leq 4$ is treated as in Row 2; $q = \infty$ is Corollary 23; for $4 < q < \infty$ apply interpolation (323) with $p = 4$.

3) Row 4: $q \leq 4$ is treated as in Row 2; $q \geq 4$ follows from (326) with $p = 4$.

4) Row 5: $q = \infty$ is Theorem 23 for $2 < q < \infty$ we apply interpolation (325) with $p = 2$.

The upshot of this section is that we cannot approximate values of non-quadratic polynomials in $x$ (or $y$) by assuming iid Gaussian entries, unless the code is $O(\sqrt{n})$-achieving, in which case we can go up to degree 4 but still will have to be content with one-sided (lower) bounds only, cf. (336).13

Before closing this discussion we demonstrate the sharpness of the arguments in this section by considering the following example. Suppose that a power of a codeword $x$ from a capacity-dispersion optimal code is measured by an imperfect tool, such that its reading is described by

$$\mathcal{E} = \frac{1}{n} \sum_{i=1}^{n} (x_i)^2 H_i,$$
(337)

where $H_i$’s are i.i.d bounded random variables with expectation and variance both equal to 1. For large blocklengths $n$ we expect $\mathcal{E}$ to be Gaussian with mean $P$ and variance $\frac{1}{n} \mathbb{E}[\|x\|_4^4]$. On the one hand, Theorem 24 shows that the variance will not explode; (336) shows that it will be at least as large as that of a Gaussian codebook. Finally, to establish the asymptotic normality rigorously, the usual approach based on checking Lyapunov condition will fail as shown by (318), but the Lindenberg condition does hold as a consequence of Theorem 23. If in addition, the code is $O(\log n)$-achieving then

$$\mathbb{P}[|\mathcal{E} - \mathbb{E}[\mathcal{E}]| > \delta] \leq e^{-\frac{n \delta^2}{2C(\lambda - \exp(\lambda n)\sigma^2)}},$$
(338)

APPENDIX

In this appendix we prove results from Section VII-B.

To prove Theorem 21 the basic intuition is that any codeword which is abnormally peaky (i.e., has a high value of $\|x\|_\infty$) is wasteful in terms of allocating its power budget. Thus a good capacity- or dispersion-achieving codebook cannot have too many of such wasteful codewords. A non-asymptotic formalization of this intuitive argument is as follows:

**Lemma 25:** For any $\epsilon \leq \frac{1}{2}$ and $P > 0$ there exists a constant $b = b(P, \epsilon)$ such that given any $(n, M, \epsilon)_{\max, \det}$ code for the AWGN($P$) channel, we have for any $0 \leq \lambda \leq P$14

$$\mathbb{P}[\|X^n\|_\infty \geq \sqrt{\lambda n}] \leq \frac{b}{M} \exp \left\{ nC(P - \lambda) - \sqrt{nV(P - \lambda)} Q^{-1}(\epsilon) + 2 \log n \right\},$$
(339)

where $C(P)$ and $V(P)$ are the capacity and the dispersion of the AWGN($P$) channel, and $X^n$ is the output of the encoder assuming equiprobable messages.

**Proof:** Our method is to apply the meta-converse in the form of [3 Theorem 30] to the subcode that satisfies $\|X^n\|_\infty \geq \sqrt{\lambda n}$. Application of a meta-converse requires selecting a suitable auxiliary channel $Q_{Y^n|X^n}$. We specify this channel now. For any $x \in \mathbb{R}^n$ let $j^*(x)$ be the first index s.t. $|x_j| = \|x\|_\infty$, then we set

$$Q_{Y^n|X^n} (y^n|x) = P_{Y^n|X^n} (y^n|x) \prod_{j \neq j^*(x)} P^*_{j} (y_j)$$
(340)

We will show below that for some $b_1 = b_1(P)$ any $M$-code over the $Q$-channel [350] has average probability of error $\epsilon'$ satisfying:

$$1 - \epsilon' \leq \frac{b_1 n^{\frac{1}{2}}}{M}.$$  
(341)

On the other hand, writing the expression for $\log \frac{dP_{Y^n|X=x}}{dP_{Y^n|X=x}}(Y^n)$ we see that it coincides with the expression for $\log \frac{dP_{Y^n|X=x}}{dP_{Y^n}}$ except that the term corresponding to $j^*(x)$ will be missing; compare with [15 (4.29)]. Thus, one can repeat step by step the

13 Using quite similar methods, [336] can be extended to certain bi-quadratic forms, i.e. 4-th degree polynomials $\sum_{i,j} a_{i,j} x_i^2 x_j^2$, where $A = (a_{i,j})$ is a Toeplitz positive semi-definite matrix.

14 For $\epsilon > \frac{1}{2}$ one must replace $V(P - \lambda)$ with $V(P)$ in (339). This does not modify any of the arguments required to prove Theorem 21.
analysis in the proof of [3, Theorem 65] with the only difference that \( nP \) should be replaced by \( nP - \|x\|_\infty^2 \) reflecting the reduction in the energy due to skipping of \( j^* \). Then, we obtain for some \( b_2 = b_2(\alpha, P) \):

\[
\log \beta_{1-\epsilon}(P_{Y^n|X^n=x}, Q_{Y^n|X^n=x}) \geq -nC \left( P - \frac{\|x\|_\infty^2}{n} \right) + \sqrt{nV \left( P - \frac{\|x\|_\infty^2}{n} \right) Q^{-1}(\epsilon)} - \frac{1}{2} \log n - b_2, \tag{342}
\]

which holds simultaneously for all \( x \) with \( \|x\| \leq \sqrt{nP} \). Two remarks are in order: first, the analysis in [3, Theorem 64] must be done replacing \( n \) with \( n - 1 \), but this difference is absorbed into \( b_2 \). Second, to see that \( b_2 \) can be chosen independent of \( x \) notice that \( B(P) \) in [3, (620)] tends to 0 with \( P \to 0 \) and hence can be bounded uniformly for all \( P \in [0, P_{\text{max}}] \).

Denote the cardinality of the subcode \( \{\|x\|_\infty \geq \sqrt{\lambda n}\} \) by

\[
M_\lambda = |\mathbb{P}[\|X\|_\infty \geq \sqrt{\lambda n}]|. \tag{343}
\]

Then according to [3, Theorem 30], we get

\[
\inf_{x} \beta_{1-\epsilon}(P_{Y^n|X^n=x}, Q_{Y^n|X^n=x}) \leq 1 - \epsilon', \tag{344}
\]

where the infimum is over the codewords of \( M_\lambda \)-subcode. Applying both (341) and (342) we get

\[
\inf_{x} \left( -nC \left( P - \frac{\|x\|_\infty^2}{n} \right) + \sqrt{nV \left( P - \frac{\|x\|_\infty^2}{n} \right) Q^{-1}(\epsilon)} \right) - \frac{1}{2} \log n - b_2 \leq -\log M_\lambda + \log b_1 + \frac{3}{2} \log n \tag{345}
\]

and, further, since the function of \( \|x\|_\infty \) in left-hand side of (345) is monotone in \( \|x\|_\infty \):

\[
-nC(P - \lambda) + \sqrt{nV(P - \lambda)Q^{-1}(\epsilon)} - \frac{1}{2} \log n - b_2 \leq -\log M_\lambda + \log b_1 + \frac{3}{2} \log n. \tag{346}
\]

Thus, overall

\[
\log M_\lambda \leq nC(P - \lambda) - \sqrt{nV(P - \lambda)Q^{-1}(\epsilon)} + 2 \log n + b_2 + \log b_1, \tag{347}
\]

which is equivalent to (339) with \( b = b_1 \exp\{b_2\} \).

It remains to show (341). Consider an \((n, M', \epsilon')_{\text{avg, det}}\)-code for the \( Q \)-channel and let \( M_j, j = 1, \ldots, n \) denote the cardinality of the set of all codewords with \( j^*(x) = j \). Let \( \epsilon'_j \) denote the minimum possible average probability of error of each such codebook achievable with the maximum likelihood (ML) decoder. Since

\[
1 - \epsilon' \leq \frac{1}{M} \sum_{j=1}^{n} M_j (1 - \epsilon'_j) \tag{348}
\]

it suffices to prove

\[
1 - \epsilon'_j \leq \frac{\sqrt{2nP} + 2}{M_j} \tag{349}
\]

for all \( j \). Without loss of generality assume \( j = 1 \) in which case the observations \( Y_{2n}^n \) are useless for determining the value of the true codeword. Moreover, the ML decoding regions \( D_i, i = 1, \ldots, M_j \) for each codeword are disjoint intervals in \( \mathbb{R} \) (so that the decoder outputs message estimate \( i \) whenever \( Y_1 \in D_i \)). Note that for \( M_j \leq 2 \) there is nothing to prove, so assume otherwise. Denote the first coordinates of the \( M_j \) codewords by \( x_{i_1} \), \( i = 1, \ldots, M_j \) and assume (without loss of generality) that
\(-\sqrt{nP} \leq x_1 \leq x_2 \leq \cdots \leq x_{M_j} \leq \sqrt{nP}\) and that \(D_2, \ldots, D_{M_j-1}\) are finite intervals. We have the following chain then

\[
1 - \epsilon'_j = \frac{1}{M_j} \sum_{i=1}^{M_j} P_{Y|X}(D_i|x_i) \tag{350}
\]

\[
\leq \frac{2}{M_j} + \frac{1}{M_j} \sum_{i=2}^{M_j-1} P_{Y|X}(D_i|x_i) \tag{351}
\]

\[
\leq \frac{2}{M_j} + \frac{1}{M_j} \sum_{j=2}^{M_j-1} \left(1 - 2Q\left(\frac{\text{Leb}(D_i)}{2}\right)\right) \tag{352}
\]

\[
\leq \frac{2}{M_j} + \frac{M_j - 2}{M_j} \left(1 - 2Q\left(\frac{1}{2M_j - 4} \sum_{i=2}^{M_j-1} \text{Leb}(D_i)\right)\right) \tag{353}
\]

\[
\leq \frac{2}{M_j} + \frac{M_j - 2}{M_j} \left(1 - 2Q\left(\sqrt{nP}\right)\right) \tag{354}
\]

\[
\leq \frac{2}{M_j} + \frac{2nP}{M_j}, \quad \tag{355}
\]

where in (350) \(P_{Y|X=x} = N(x, 1), \tag{351}\) follows by upper-bounding probability of successful decoding for \(i = 1\) and \(i = M_j\) by 1, \(352\) follows since clearly for a fixed value of the length \(\text{Leb}(D_i)\) the optimal location of the interval \(D_i\), maximizing the value \(P_{Y|X}(D_i|x_i)\), is centered at \(x_i\), \(353\) is by Jensen’s inequality applied to \(x \to 1 - 2Q(x)\) concave for \(x \geq 0\), \(354\) is because

\[
\bigcup_{i=2}^{M_j-1} D_i \subset [-\sqrt{nP}, \sqrt{nP}] \tag{356}
\]

and \(D_i\) are disjoint, and \(355\) is by

\[
1 - 2Q(x) \leq \sqrt{\frac{2}{\pi} x}, \quad x \geq 0. \tag{357}
\]

Thus, \(355\) completes the proof of \(349\), \(341\) and the theorem. \(\square\)

**Proof of Theorem 21.** Notice that for any \(0 \leq \lambda \leq P\) we have

\[
C(P - \lambda) \leq C(P) - \frac{\lambda \log e}{2(1 + P)}. \tag{358}
\]

On the other hand, by concavity of \(\sqrt{V(P)}\) and since \(V(0) = 0\) we have for any \(0 \leq \lambda \leq P\)

\[
\sqrt{V(P - \lambda)} \geq \sqrt{V(P)} - \frac{\sqrt{V(P)}}{P} \lambda. \tag{359}
\]

Thus, taking \(s = \lambda n\) in Lemma 25 we get with the help of \(358\) and \(359\):

\[
\mathbb{P}[\|x\|_\infty^2 \geq s] \leq \exp \left\{ \Delta_n - (b_1 - b_2 n^{-\frac{1}{2}}) s \right\}, \tag{360}
\]

where we denoted for convenience

\[
b_1 = \frac{\log e}{2(1 + P)}, \tag{361}
\]

\[
b_2 = \frac{\sqrt{V(P)}}{P} Q^{-1}(e), \tag{362}
\]

\[
\Delta_n = nC(P) - \sqrt{nV(P)} Q^{-1}(e) + 2 \log n - \log M + \log b. \tag{363}
\]

Note that Lemma 25 only shows validity of \(360\) for \(0 \leq s \leq nP\), but since for \(s > nP\) the left-hand side is zero, the statement actually holds for all \(s \geq 0\). Then for \(n \geq N(P, e)\) we have

\[
(b_1 - b_2 n^{-\frac{1}{2}}) \geq \frac{b_1}{2} \tag{364}
\]

and thus further upper-bounding \(360\) we get

\[
\mathbb{P}[\|x\|_\infty^2 \geq s] \leq \exp \left\{ \Delta_n - \frac{b_1 s}{2} \right\}. \tag{365}
\]
Finally, if the code is so large that $\Delta_n < 0$, then (365) would imply that $P[\|x\|_\infty^2 \geq s] < 1$ for all $s \geq 0$, which is clearly impossible. Thus we must have $\Delta_n \geq 0$ for any $(n, M, \epsilon)_{\max, \det}$ code. The proof concludes by taking $s = \frac{2(\log 2 + \Delta_n)}{b_1}$ in (365).

Proof of Theorem 24 To prove (338) we will show the following statement: There exist two constants $b_0$ and $b_1$ such that for any $(n, M, \epsilon)$ code for the AWGN($P$) channel with codewords $x$ satisfying

$$\|x\|_4 \geq b n^{\Delta}$$

we have an upper bound on the cardinality:

$$M_1 \leq \frac{4}{1-\epsilon} \exp \left\{ nC + 2 \sqrt{\frac{nV}{1-\epsilon}} - b_1(b-b_0)^2 \sqrt{n} \right\} ,$$

(367)

provided $b \geq b_0(P, \epsilon)$. From here (335) follows by first upper-bounding $(b-b_0)^2 \geq \frac{b^2}{2} - b_0^2$ and then verifying easily that the choice

$$b^2 = \frac{2}{b_1 \sqrt{n}} (nC + b_2 \sqrt{n} - \log \frac{M}{2})$$

(368)

with $b_2 = b_0^2 b_1 + 2 \sqrt{\frac{V}{1-\epsilon} + \log \frac{4}{1-\epsilon}}$ takes the right-hand side of (367) below $\log \frac{M}{2}$.

To prove (367), denote

$$S = b - \left(\frac{6}{1+\epsilon}\right)^{\frac{1}{\epsilon}}$$

(369)

and choose $b$ large enough so that

$$\delta \geq S - 6^{\frac{1}{2}} \sqrt{1+P} > 0.$$  

(370)

Then, on one hand we have

$$P_{Y^n}[\|Y^n\|_4 \geq Sn^{\frac{1}{2}}] = P[\|X^n + Z^n\|_4 \geq Sn^{\frac{1}{2}}]$$

(371)

$$\geq P[\|X^n\|_4 - \|Z^n\|_4 \geq Sn^{\frac{1}{2}}]$$

(372)

$$\geq P[\|Z^n\|_4 \leq n^{\frac{1}{2}} (S-b)]$$

(373)

$$\geq \frac{1+\epsilon}{2},$$

(374)

where (372) is by the triangle inequality for $\|\cdot\|_4$. (373) is by the constraint (366) and (374) is by the Chebyshev inequality applied to $\|Z^n\|_4 = \sum_{j=1}^n Z^n_j$. On the other hand, we have

$$P_{Y^n}[\|Y^n\|_4 \leq Sn^{\frac{1}{2}}] = P_{Y^n}[\|Y^n\|_4 \leq (6^{\frac{1}{2}} \sqrt{1+P} + \delta)n^{\frac{1}{2}}]$$

(375)

$$\geq P_{Y^n}[\|Y^n\|_4 \leq 6^{\frac{1}{2}} \sqrt{1+Pn^{\frac{1}{2}}} + \|Z^n\|_4 \leq \delta n^{\frac{1}{2}}]$$

(376)

$$\geq P_{Y^n}[\|Y^n\|_4 \leq 6^{\frac{1}{2}} \sqrt{1+Pn^{\frac{1}{2}}} + \|Y^n\|_2 \leq \delta n^{\frac{1}{2}}]$$

(377)

$$\geq 1 - \exp\{-b_1 \delta^2 \sqrt{n}\},$$

(378)

where (375) is by the definition of $\delta$ in (370), (376) is by the triangle inequality for $\|\cdot\|_4$ which implies the inclusion

$$\{y: \|y\|_4 \leq a + b\} \supset \{y: \|y\|_4 \leq a\} + \{y: \|y\|_4 \leq b\}$$

(379)

with $+$ denoting the Minkowski sum of sets. (377) is by (326) with $p = 2, q = 4$; and (378) holds for some $b_1 = b_1(P) > 0$ by the Gaussian isoperimetric inequality [36] which is applicable since

$$P_{Y^n}[\|Y^n\|_4 \leq 6^{\frac{1}{2}} \sqrt{1+Pn^{\frac{1}{2}}} + \|Z^n\|_4 \leq \frac{1}{2}$$

(380)

by the Chebyshev inequality applied to $\sum_{j=1}^n Y^n_j$ (note: $Y^n \sim \mathcal{N}(0, 1+P)^n$ under $P_{Y^n}$). As a side remark, we add that the estimate of the large-deviations of the sum of $4\text{-}th$ powers of iid Gaussians as $\exp\{-O(\sqrt{n})\}$ is order-optimal.

Together (374) and (378) imply

$$\beta_1(P_{Y^n}, P_{Y^n}) \leq \exp\{-b_1 \delta^2 \sqrt{n}\}. $$

(381)

On the other hand, by [3] Lemma 59] we have for any $x$ with $\|x\|_2 \leq \sqrt{nP}$ and any $0 < \alpha < 1:

$$\beta_{\alpha}(P_{Y^n|X^n=x}, P_{Y^n}) \geq \alpha \frac{\alpha}{2} \exp\left\{-nC - \sqrt{\frac{2nV}{\alpha}}\right\},$$

(382)
where $C$ and $V$ are the capacity and the dispersion of the AWGN($P$) channel. Then, by convexity in $\alpha$ of the right-hand side of (382) and [15, Lemma 32] we have for any input distribution $P_X$:

$$\beta_{\alpha}(P_{X^nY^n}, P_{X^n}P_{Y^n}) \geq \frac{\alpha}{2} \exp \left\{ -nC - \sqrt{\frac{2nV}{\alpha}} \right\}. \quad (383)$$

We complete the proof of (367) by invoking Theorem [14] in the form (223) with $Q_Y = P_{Y^n}$ and $\alpha = \frac{1+\epsilon}{2}$:

$$\beta_{1+\epsilon}(P_{Y^n}, P_{Y^n}^\star) \geq M_1 \beta_{\epsilon/2}(P_{X^nY^n}, P_{X^n}P_{Y^n}^\star). \quad (384)$$

Applying bounds (381) and (383) to (384) we conclude that (367) holds with

$$b_0 = \left( \frac{6}{1+\epsilon} \right) + 6^\epsilon \sqrt{1+P}. \quad (385)$$

Next, we proceed to the proof of (366). On one hand, we have

$$\sum_{j=1}^{n} E\left[ Y_j^4 \right] = \sum_{j=1}^{n} E\left[ (X_j + Z_j)^4 \right] \geq \sum_{j=1}^{n} E\left[ X_j^4 + 6X_j^2Z_j^2 + Z_j^4 \right] \geq E\left[ \|x\|_4^4 \right] + 6nP + 3n, \quad (387)$$

where (386) is by the definition of the AWGN channel, (387) is because $X^n$ and $Z^n$ are independent and thus odd terms vanish, (388) is by the power-constraint $\sum X_j^2 \leq nP$. On the other hand, applying Proposition [11] with $f(y) = -y^4$, $\theta = 2$ and using (122) we obtain:

$$\sum_{j=1}^{n} E\left[ Y_j^4 \right] \geq 3n(1+P)^2 - (nC - \log M + b_3 \sqrt{n})n^\star, \quad (389)$$

for some $b_3 = b_3(P, \epsilon) > 0$. Comparing (389) and (386) statement (366) follows.

We remark that by extending Proposition [11] to expectations like $\frac{1}{n} \sum_{j=1}^{n-1} E[|X_j^2Y_j^2|]$, cf. (185), we could provide a lower bound similar to (386) for more general 4-th degree polynomials in $x$. For example, it is possible to treat the case of $p(x) = \sum_{i,j} a_{i,j} x_i^2 x_j^2$, where $A = (a_{i,j})$ is a Toeplitz positive semi-definite matrix. We would proceed as in (388), computing $E[p(Y^n)]$ in two ways, with the only difference that the peeled off quadratic polynomial would require application of Theorem [19] instead of the simple power constraint. Finally, we also mention that the method (388) does not work for estimating $E[\|x\|_4^4]$ because we would need an upper bound $E[\|x\|_4^4] \leq 3nP^2$, which is not possible to obtain in the context of $O(\sqrt{n})$-achieving codes as the counterexamples (317) show.

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