Large Loops of Magnetic Current and Confinement in Four Dimensional $U(1)$ Lattice Gauge Theory

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Abstract

We calculate the heavy quark potential from the magnetic current due to monopoles in four dimensional $U(1)$ lattice gauge theory. The magnetic current is found from link angle configurations using the DeGrand-Toussaint identification method. The link angle configurations are generated in a cosine action simulation on a $24^4$ lattice. The magnetic current is resolved into large loops which wrap around the lattice and simple loops which do not. Wrapping loops are found only in the confined phase. It is shown that the long range part of the heavy quark potential, in particular the string tension, can be calculated solely from the large, wrapping loops of magnetic current.
In this paper, we report new results on confinement via monopoles for $U(1)$ lattice gauge theory in four dimensions. Our main result is that the confining part of the heavy quark potential, in particular the string tension, is determined solely by large loops of magnetic current. It has been established for some time that large loops which extend over the entire lattice are present only in the confined phase of the theory [1, 2]. Their presence can now be quantitatively tied to the string tension. Our work is carried out on a $24^4$ lattice, near the deconfining transition.

The role of monopoles in $U(1)$ lattice gauge theory is seen most clearly using the Villain [3] or periodic gaussian form of the $U(1)$ theory. Under a dual transformation, the usual link angle description goes over into one involving an integer-valued magnetic current $m_\mu(x)$, defined on the links of the dual lattice [4]. The link angle path integral becomes a sum over all possible configurations of magnetic current. In this monopole representation, the system can be visualized as a plasma of magnetic monopoles moving on Euclidean world lines, interacting via photon exchange.

In either representation, a Wilson loop calculation is needed to determine the heavy quark potential. In the link angle representation, a Wilson loop is specified by the exponential of a line integral:

$$W(R, T) = \left\langle \exp\left(i \sum_x \theta_\mu(x) J_\mu(x)\right) \right\rangle_{\theta}^\theta,$$

where the integer-valued electric current $J_\mu$ is non-vanishing on the rectangular $R \times T$ loop contour, and $\langle \cdot \rangle_{\theta}$ denotes the expected value taken over configurations of link angles $\theta_\mu(x)$.

In the monopole representation, the determination of a Wilson loop involves the exponential of an area integral over a surface with the loop contour as its boundary [4, 5]. The electric current $J_\mu$ is first expressed as the curl of a Dirac sheet variable [6]; where $\partial_\nu$ denotes a discrete derivative. The sheet variable $D_{\mu\nu}$ is nonunique. For $|J_\mu| = 1$, a specific choice is made by setting $D_{\mu\nu} = 1$ on the plaquettes of an (arbitrary) open surface.
with boundary $J_\mu$, and $D_{\mu\nu} = 0$ on all other plaquettes. The area integral represents the
dual flux set up by the magnetic current through this surface. To compute it, we define
the magnetic vector potential

$$A_\mu(x) = \sum_y v(x - y)m_\mu(y), \tag{2}$$

where $v$ satisfies $-\partial \cdot \partial v(x - y) = \delta_{x,y}$. The field strength is given by $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$. 

In terms of $D_{\mu\nu}$ and the dual field strength $F^{*}_{\mu\nu}(x) = \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} F_{\alpha\beta}(x)$, the monopole representation of a Wilson loop is finally given by:

$$W(R, T) = W_{ph}(R, T) \cdot \left\langle \exp \left( \frac{i2\pi}{2} \sum_x D_{\mu\nu}(x) F^{*}_{\mu\nu}(x) \right) \right\rangle_m, \tag{3}$$

where $\langle \cdot \rangle_m$ denotes the sum over configurations of magnetic current. The factor
$2\pi$ which appears in the exponent of Eq. (3) arises from the Dirac condition on the
product of electric and magnetic charge, and guarantees that the value of a Wilson loop
is independent of the surface chosen to define $D_{\mu\nu}$. The prefactor in Eq. (3) describes one
photon exchange between the quark and anti-quark:

$$W_{ph}(R, T) = \exp \left( -\frac{e^2}{2} \sum_{x,y} J_\mu(x)v(x - y)J_\mu(y) \right), \tag{4}$$

where the electric coupling $e^2$ is related to the coupling $\beta_V$ in the Villain action by
$e^2 = 1/\beta_V$. The factor $W_{ph}(R, T)$ contributes a purely perturbative Coulomb term to the
heavy quark potential.

Fortunately, the numerical evaluation of Wilson loops via Eq. (3) does not require a
direct simulation in terms of the magnetic current $m_\mu(x)$. This is impractical owing to
the long-range interactions generated by photon exchange between the monopole currents.
DeGrand and Toussaint \[7\] showed how to locate monopoles directly in configurations of
link angles. In their procedure, the plaquette angle $\theta_{\mu\nu}(x) = \partial_\mu \theta_\nu - \partial_\nu \theta_\mu$ is resolved into
a fluctuating part $\tilde{\theta}_{\mu\nu}(x)$, and an integer-valued Dirac sheet variable $m^*_{\mu\nu}(x)$:

$$\theta_{\mu\nu}(x) = \tilde{\theta}_{\mu\nu}(x) + 2\pi m^*_{\mu\nu}(x), \tag{5}$$
where \( m^*_{\mu\nu}(x) = \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} m_{\alpha\beta}(x) \) and \( \tilde{\theta}_{\mu\nu}(x) \in (-\pi, \pi) \). The magnetic current is then given by \( m_\mu(x) = \partial_\nu m_{\mu\nu}(x) \). This procedure allows only values of \( m_\mu \in [\pm 2, \pm 1, 0] \), whereas in principle all integer values are allowed. However, at values of the coupling near the deconfining transition, the values \( m_\mu = \pm 1 \) are overwhelmingly dominant; even \( m_\mu = \pm 2 \) occurs only a small fraction of the time. Thus negligible error is caused by omitting higher values of \( m_\mu \).

The monopole current can be used to calculate physical quantities, as well as merely to count monopoles. In our previous work in four dimensions \( [5] \), we used the current \( m_\mu(x) \) to evaluate Wilson loops from Eq. \((3)\). Similarly in three dimensions \( [8] \), we used the monopole density \( m(x) \) to evaluate Wilson loops using the \( d = 3 \) analog of Eq. \((3)\). In both cases the resulting heavy quark potential agreed with that extracted from the link angles and Eq. \((1)\), to within statistical errors. In the present work on a \( 24^4 \) lattice, we have again checked that potentials deduced directly from link angles and Eq. \((1)\) agree with those obtained from the magnetic current and Eq. \((3)\). These calculations show that quantitative results on confinement can be obtained using topological objects.

The derivation of Eq. \((3)\) as an exact formula is only possible for Villain’s form of the \( U(1) \) theory. On the other hand, Wilson’s cosine form \( [9] \) of the \( U(1) \) action can be simulated much more efficiently. In our previous work \( [5, 8] \), we have shown that Villain action results can be extracted from a cosine action simulation, if a simple coupling constant mapping is used. More precisely, a simulation using the cosine action at coupling \( \beta_W \) is equivalent to a Villain action simulation at coupling \( \beta_V \), with \( \beta_V \) related to \( \beta_W \) by \( [3, 10] \)

\[
1/\beta_V = -2 \ln \left( \frac{I_0(\beta_W)}{I_1(\beta_W)} \right),
\]

where \( I_0 \) and \( I_1 \) are modified Bessel functions. Eq. \((3)\) determines the value of \( 1/\beta_V \), and hence \( e^2 \), which result from a cosine action simulation at a given value of \( \beta_W \). The factor \( W_{ph} \) in Eq. \((3)\) is then completely determined. The magnetic current is identified from the cosine action link angle configurations and the result used to calculate the second factor
of Eq. (3). The Wilson loops calculated in this manner using the cosine action and Eq. (6) differ from pure Villain action results by a harmless perimeter term. The R-dependent terms in the potentials agree within statistical errors \[8, 11\].

To summarize, our simulations use the cosine form of the $U(1)$ action, identify $m_\mu(x)$ using the DeGrand-Toussaint method, and evaluate Wilson loops in the monopole representation. The link angle configurations were generated using a heatbath algorithm \[12\]. The calculation of $A_\mu(x)$ from $m_\mu(x)$ in Eq. (2) was done using a four dimensional vectorized FFT \[13, 14\]. The $R \times T$ rectangle lying in the Wilson loop plane was used as the defining surface for $D_{\mu\nu}$. Magnetic current configurations were saved every 10 sweeps. After Wilson loops were obtained from these configurations using Eq. (3), potentials were extracted using standard methods. The heavy quark potential $V(R)$ was obtained from a straight line fit of $\ln W(R, T)$ vs. $T$, over an interval $T_{\text{min}}(R)$ to $T_{\text{max}}$, where $T_{\text{min}}(R) = R + 2$ for $R = 2, 3$, and $R + 1$ otherwise, while $T_{\text{max}} = 16$. To determine the string tension $\sigma$ and Coulomb coupling $\alpha$, the potentials were then fitted to a linear plus Coulomb form, $V(R) = \sigma R - \alpha/R + V_0$, over the interval $R = 2$ to $R = 7$. The large string tension present in $U(1)$ makes it difficult to work at larger values of $R$. Errors in physical quantities were estimated using both the jacknife method and binning the data into bins of various size.

It is well established for $U(1)$ that appreciable correlation lengths occur only in the immediate vicinity of the deconfining phase transition. The location of the phase transition moves to larger values of $\beta_W$ as the lattice size increases, in a manner roughly consistent with finite size scaling theory \[2, 15\]. Since only lattices of size up to $16^4$ were available in the published literature when we began our work, it was first necessary for us to locate the transition for a $24^4$ lattice. To do this, we performed a series of runs with various initial configurations for $1.0100 < \beta_W < 1.0120$, and monitored the value of the $1 \times 1$ Wilson loop, $W(1, 1)$. For $\beta_W \geq 1.0114$, the system always reached a state with $W(1, 1) \sim 0.65$. For $\beta_W \leq 1.0112$, the system always reached a state with
$W(1, 1) \approx 0.63$. Subsequent analysis of the heavy quark potential showed these two states to be deconfined and confined, respectively. While we have not precisely located the deconfining phase transition, consistency with our results requires that the transition be in the interval $1.0112 \leq \beta_W < 1.0114$ for a $24^4$ lattice.

To avoid problems associated with long autocorrelation times that occur near the transition, we chose to use a run of 20,000 sweeps at $\beta_W = 1.0103$ for the results to be presented below. At this value of $\beta_W$, the correlation length is large enough to observe the beginnings of continuum behavior, but small enough to avoid problems with long autocorrelation times. The autocorrelation time $\tau$ measured from the $1 \times 1$ Wilson loop was approximately 100 sweeps. In Fig.(1), we show the heavy quark potential determined from Eq. (3) for $\beta_W = 1.0103$ using 936 configurations of magnetic current. A linear-plus-Coulomb fit gave a string tension of $\sigma = 0.56(1)$, and a Coulomb coupling of $\alpha = 0.30(2)$. The total number of links carrying magnetic current at this $\beta_W$ was 98,400(800). For comparison, we also show in Fig.(1), the potential determined from Eq. (3) for the deconfined $\beta_W$ value, $\beta_W = 1.0114$, where 400 configurations of magnetic current were analyzed. At this value of $\beta_W$, the string tension was statistically zero, while the Coulomb coupling was $\alpha = 0.22(4)$. The total number of links carrying magnetic current was 40,000(300).

We now turn to the resolution of the magnetic current into loops. For every other configuration of magnetic current, or every 20 lattice sweeps, magnetic current loops were individually identified and catalogued. The loop-finding algorithm proceeded by choosing a non-zero current link $m_\mu(x_0)$ and following the current it carried through the lattice until a loop was completed by a return to the site $x_0$. This process was carried out repeatedly from different starting points and ended when the entire configuration of current had been resolved into loops, with each current-carrying link belonging to a specific loop. The algorithm was deterministic: when looking for an outgoing current link at a particular lattice site, the direction $\mu = 1$ was chosen first, followed by $\mu = 2, 3, 4$. Intersections
of loops did occur (i.e., more than one outgoing link associated with a site), so the set of loops identified was not unique. However, self-intersections of loops were relatively rare, occurring with approximately the same probability as self-intersections of a purely random walk in $d = 4$ \cite{16}.

Each loop identified as described above forms a closed path of links and automatically satisfies current conservation. Two different topologies are possible, depending on whether tracing out a loop also involves winding or wrapping around the lattice. A lattice with periodic boundary conditions is a hypertoroid, so topologically nontrivial loops which wrap around the lattice are permitted. To distinguish the two possibilities, the net current was measured for each loop:

$$\Lambda_\mu = \sum_{x \in \text{loop}} m_\mu(x).$$

It is easy to show that only loops which wrap around the lattice have a non-vanishing $\Lambda_\mu$, and that the components of $\Lambda_\mu$ must be integer multiples of the lattice size along an axis; $\Lambda_\mu = n_\mu \cdot N$, for a cubic lattice of size $N^4$. The integer $n_\mu$ is the net current of the loop in the $\mu$th direction. For example, for a loop with $\Lambda_4 \neq 0$, the sum of the charge density $m_4$ over each spacial cube or “time slice” will equal the net charge $n_4$. Likewise for other directions. While an individual loop can have a non-vanishing $\Lambda_\mu$, a net current cannot actually occur on a finite lattice, so the sum of $\Lambda_\mu$ over all loops vanishes identically. In our data, it was typical for a loop with non-vanishing $\Lambda_\mu$ to be wrapped around the lattice several times in more than one direction. Values of $|n_\mu|$ up to 10 were observed. In what follows we will use the terms “wrapped” for loops with non-vanishing $\Lambda_\mu$ and “simple” for loops with vanishing $\Lambda_\mu$.

An indication that wrapped loops are crucial for confinement is that they are present only in the confined state and never are observed to occur in the deconfined state. As mentioned earlier, there are almost twice as many links carrying magnetic current in the confined phase. When the current is broken up into loops, it is found that this excess in the confined phase consists of a small number of wrapped loops. At $\beta = 1.0103$, out of the
total of 98,400(800) links carrying magnetic current, 51,000(400) are in wrapped loops, the remainder in simple loops. The average number of wrapped loops is only 4.6(1), so that the typical wrapped loop contains thousands of links. In contrast the average total number of simple loops is 6,210(6), of which 3,507(4) are in the form of elementary one-plaquette current loops composed of four links. The number of simple loops with a given number of links decreases rapidly as the number of links increases. Over 90% of the links in simple loops are included in loops with 60 links or less.

Since wrapped loops occur only in the confined phase, it is natural to ask if they can explain the long range, confining part of the heavy quark potential. To investigate this, we computed the heavy quark potential again using Eq. (3), but for each configuration, including only the magnetic current from wrapped loops. The results are shown (omitting the photon factor $W_{ph}$) in Fig. (2). A linear plus Coulomb fit to the resulting potential gave a string tension $\sigma_w = 0.56(2)$, and a Coulomb term $\alpha_w = 0.09(1)$. The string tension is within statistical errors of the value 0.58(2) found earlier from the heavy quark potential calculated using the full magnetic current. Next, we carried out a similar calculation using only the magnetic current from the simple loops. This produced the rather flat potential also shown in Fig. (2). A linear plus Coulomb fit to this potential gives zero string tension within statistical errors ($\sigma_s = 0.0000(6)$), and a Coulomb term $\alpha_s = 0.06(1)$. The result of these two fits gives strong evidence that in the long distance region, there is a clean separation between the contributions of the two classes of loops. Only the large, wrapped loops containing thousands of links contribute to the confining part of the potential. This has been demonstrated here only within certain error bars, but it may well be an exact statement.

In the fits described above for $\beta_W = 1.0103$, the wrapped loops required a Coulomb term with coupling $\alpha_w = 0.09(1)$, while the simple loops required a Coulomb term with coupling $\alpha_s = 0.06(1)$. In addition, there is a Coulomb term coming from the $W_{ph}$ factor in Eq. (3). Using Eq. (3) to evaluate $\beta_V(1.0103)$, gives $\alpha_{ph} = 0.13$ as the Coulomb coupling
arising from $W_{ph}$. Simply adding the various terms, we obtain $\alpha_{ph} + \alpha_s + \alpha_w = 0.28(2)$, consistent with our previous result of 0.30(2) obtained with the full magnetic current.

The results on the string tension and Coulomb coupling are consistent with additivity of the potential over the various contributions. In Fig.(3), we compare the potential determined from the full magnetic current and $W_{ph}$ (shown previously in Fig.(1)), with the potential obtained by summing the contributions from $W_{ph}$, wrapped loops, and simple loops, plus a constant. A glance at Fig.(3) shows that the agreement is quite good. Additivity of the potential over the various contributions would imply that the contributions from wrapped and simple loops factor in the average over configurations. To check factorization, we performed a fit to the “potential” extracted from the ratio of Wilson loops assuming factorization to Wilson loops calculated with the full magnetic current:

$$\langle W_w(R,T) \rangle_m \cdot \langle W_s(R,T) \rangle_m / \langle W_w(R,T) \cdot W_s(R,T) \rangle_m .$$ (7)

The string tension and Coulomb coupling resulting from this were both zero to within statistical errors. This shows that factorization and therefore additivity of the potential is consistent with our data. This is not surprising at large R, where the R-dependence comes predominantly from the wrapped loops. However, in the small R region, both wrapped and simple loops produce Coulomb terms, and additivity is not expected to hold as an exact statement. Nevertheless it appears to be a good approximation and holds within the accuracy of our data.

We have shown that confinement in $U(1)$ comes about through a particular component of the magnetic current, the large wrapping loops. The small, simple loops contribute only to the Coulombic part of the potential. While the latter is intuitively reasonable, still lacking is a physical picture of how the large, wrapping loops of current disorder the vacuum and produce the string tension. The fact that these loops extend over the whole lattice suggest that there are low mass (perhaps massless) magnetically charged excitations present in the confined phase. We plan to report elsewhere on this question.
as well as how the magnetic current screens itself. The results obtained in our work are likely to have an impact on the monopole approach to confinement in non-Abelian gauge theories.

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Figure 1: The quark potential calculated using the magnetic current configurations for $\beta_W = 1.0103$ (triangles) and $\beta_W = 1.0114$ (squares). The solid lines are linear-plus-Coulomb fits to the potentials.
Figure 2: The potential calculated using only the wrapping monopole loops (triangles) and using only the simple monopole loops (squares). The photon contribution from $W_{ph}$ has not been included.
Figure 3: Comparison of the quark potential calculated using the full magnetic current and photons (triangles) with the potential obtained by summing the potentials determined separately from photons, wrapped loops, and simple loops (squares). The solid line is the linear-plus-Coulomb fit from Fig. (1) for $\beta_W = 1.0103$. 