To Maria Pop on her 60th birthday

On the number of simple modules of Iwahori–Hecke algebras of finite Weyl groups

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Abstract

Let \( H_k(W, q) \) be the Iwahori–Hecke algebra associated with a finite Weyl group \( W \), where \( k \) is a field and \( 0 \neq q \in k \). Assume that the characteristic of \( k \) is not “bad” for \( W \) and let \( e \) be the smallest \( i \geq 2 \) such that \( 1 + q + q^2 + \cdots + q^{i-1} = 0 \). We show that the number of simple \( H_k(H, q) \)-modules is “generic”, i.e., it only depends on \( e \). The proof uses some computations in the CHEVIE package of GAP and known results due to Dipper–James, Ariki–Mathas, Rouquier and the author.

1 Introduction

Let \( \Gamma \) be one of the graphs in Table 1. Let \( S \) be the set of nodes of \( \Gamma \). For \( s, t \in S \), \( s \neq t \), we define an integer \( m(s, t) \) as follows. If \( s, t \) are not joined in the graph, then \( m(s, t) = 2 \); if \( s, t \) are joined by an unlabelled edge, then \( m(s, t) = 3 \). Finally, if \( s, t \) are joined by an edge with label \( m \), we set \( m(s, t) = m \). We then define a group \( W = W(\Gamma) \) by the following presentation:

\[
W = \langle s \in S \mid s^2 = 1 \text{ for } s \in S \text{ and } (st)^{m(s,t)} = 1 \text{ for } s, t \in S, s \neq t \rangle.
\]

It is known that the groups defined in this way are all finite and that they are precisely the Weyl groups arising in the theory of complex simple Lie algebras or linear algebraic groups; see [2]. For example, if \( \Gamma \) is the graph \( A_n \), then \( W \) is isomorphic to the symmetric group \( S_{n+1} \), where the generator \( s_i \in S \) attached to the node labelled by \( i \) corresponds to the basic transposition \( (i, i + 1) \) for \( 1 \leq i \leq n \).

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Now let $k$ be any field and $q \in k$ be any non-zero element which has a square root in $k$. Then we define an associative $k$-algebra $H = H_k(W, q)$ (with identity $1_H$) by a presentation with

- **Generators:** $T_w$, $w \in W$,
- **Relations:**
  - $T_s^2 = q 1_H + (q - 1)T_s$ for all $s \in S$,
  - $T_u T_{w'} = T_{uw'}$ for $w, w' \in W$ with $l(ww') = l(w) + l(w')$.

Here, the length function $l: W \to \mathbb{N}_0$ is defined as follows. Any element $w \in W$ can be written in the form $w = s_1 \cdots s_m$ with $s_1, \ldots, s_m \in S$; if $m$ is as small as possible, we set $l(w) := m$. It is known that $\{T_w \mid w \in W\}$ is in fact a $k$-basis of $H$ and that we have the following multiplication rules. Let $s \in S$ and $w \in W$. Then we have $l(sw) = l(w) \pm 1$ and

$$T_sT_w = \begin{cases} T_{sw} & \text{if } l(sw) = l(w) + 1, \\ qT_{sw} + (q - 1)T_w & \text{if } l(sw) = l(w) - 1; \end{cases}$$

see [10] Chapter 4. These algebras play an important role, for example, in the representation theory of finite groups of Lie type (see [2] and [4]) or in the theory of knots and links (see [10] Chapter 4).

Now the structure of $H$ is relatively well-understood in the case where $H_k(W, q)$ is split semisimple. Then Lusztig (see [12] and the references there) has constructed a canonical isomorphism from $H$ onto the so-called asymptotic algebra which itself is isomorphic to the group algebra of $W$. 

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**Table 1: Graphs of finite Weyl groups**

| $A_n$ | $B_n$ | $D_n$ | $E_7$ | $E_8$ |
|------|------|------|------|------|
| $n \geq 1$ | $n \geq 2$ | $n \geq 4$ | 1 3 4 5 6 7 8 | 1 3 4 5 6 7 8 9 |

- **$A_n$**
- **$B_n$**
- **$D_n$**
- **$E_7$**
- **$E_8$**
over $k$ (if the characteristic of $k$ does not divide the order of $W$). The case where $H_k(W, q)$ is not semisimple is much more difficult and far from being solved. Let $\text{Irr}(H_k(W, q))$ be the set of simple $H_k(W, q)$-modules modulo isomorphism. The purpose of this paper is to establish the following result.

**Theorem 1.1** Let $\ell \geq 0$ be the characteristic of $k$. Assume that

\[
\ell \neq 2 \quad \text{if } W \text{ is of type } B_n \text{ or } D_n;
\]

\[
\ell \neq 2, 3 \quad \text{if } W \text{ is of type } G_2, F_4, E_6 \text{ or } E_6;
\]

\[
\ell \neq 2, 3, 5 \quad \text{if } W \text{ is of type } E_8.
\]

Let $e$ be the smallest $i \geq 2$ such that $1 + q + q^2 + \cdots + q^{i-1} = 0$. (We set $e = \infty$ if no such $i$ exists.) Then we have

\[
|\text{Irr}(H_k(W, q))| = |\text{Irr}(H_C(W, \zeta_e))|,
\]

where $\zeta_e \in \mathbb{C}$ is a primitive $e$th root of unity. (We set $\zeta_\infty = 1$.) In particular, $|\text{Irr}(H_k(W, q))|$ only depends on $e$, but not on the particular choice of $k$ or $q$.

The above result is already known in the following cases.

**A$_n$:** In this case, $|\text{Irr}(H_k(W, q))|$ is the number of $e$-regular partitions of $n+1$; see Dipper–James [5]. A partition of $n+1$, written in exponential form $(1^{n_1}, 2^{n_2}, \ldots)$, is called $e$-regular if $n_i < e$ for all $i$.

**B$_n$:** In this case, $|\text{Irr}(H_k(W, q))|$ is the number of so-called Kleshchev bi-partitions of $n$; see Ariki–Mathas [1] for the proof and the precise definition.

**D$_n$:** In this case, the problem of computing $|\text{Irr}(H_k(W, q))|$ can be reduced to the analogous problem for type $B_n$ (but one has to consider an algebra of type $B_n$ with unequal parameters); see Geck [8].

For the exceptional types $G_2$, $F_4$ and $E_6$, the numbers $|\text{Irr}(H_k(W, q))|$ are explicitly known by work of Geck and Lux without any restriction on the characteristic, but assuming that $\ell = 0$ or that $q$ lies in the prime field of $k$; see [4, §3.4] and the references there.

Consequently, we will only have to prove Theorem 1.1 for the exceptional types $G_2$, $F_4$, $E_6$, $E_7$ and $E_8$. In Section 2, we present a result (originally due to Geck–Rouquier [11]), which reduces that proof to the verification of a finite number of cases. These finitely many cases will be settled by some explicit computations in Section 3, using the CHEVIE package [9] of the computer algebra system GAP [13]. The numbers $|\text{Irr}(H_C(W, \zeta_e))|$ (for $W$ of exceptional type) are printed in the table in [10, 11.5.13].
Remark 1.2 The conditions on \( \ell \) in Theorem 1.1 mean that \( \ell \) is a “bad prime” for \( W \) in the sense of [2] p. 28. Simple examples show that, in general, we will have \(|\text{Irr}(H_k(W,q))| < |\text{Irr}(H_C(W,\zeta))|\) if the characteristic of \( k \) is bad for \( W \); see, for example, type \( G_2 \) in [4, Ex. 3.11] and the remarks at the end of this paper.

2 The generic algebra and its specializations

We keep the setting and the notation of Section 1. In order to get hold of \(|\text{Irr}(H_k(W,q))|\), we will work with the generic Iwahori–Hecke algebra associated with \( W \) and use specialization arguments. For this purpose, we have to introduce some notation. We use [10] as a standard reference for general facts about finite Weyl groups and Iwahori–Hecke algebras.

Let \( A = \mathbb{Z}[v, v^{-1}] \) be the ring of Laurent polynomials in an indeterminate \( v \); let \( u = v^2 \). Then we define the generic Iwahori–Hecke algebra \( H = H_A(W, u) \) in a similar way as \( H_k(W, q) \) was defined in Section 1. Although now we are not working over a field, it is still true that \( H \) is free as an \( A \)-module, with basis \( \{ T_w \mid w \in W \} \); see [10, Chapter 4]. Let \( K \) be the field of fractions of \( A \) and \( H_K = K \otimes_A H \) be the \( K \)-algebra obtained by extending scalars from \( A \) to \( K \); we have \( H_K = H_K(W, u) \) canonically. It is known that the algebra \( H_K \) is split semisimple; see [10, 9.3.5].

Let \( \tau: H \to A \) be the \( A \)-linear map defined by \( \tau(T_1) = 1 \) and \( \tau(T_w) = 0 \) for \( 1 \neq w \in W \). Then, by [10, 8.1.1], \( \tau \) is a symmetrizing trace on \( H \) and \( H \) is a symmetric algebra. Let \( \text{Irr}(H_K) \) be the set of simple \( H_K \)-modules, up to isomorphism. Let \( c_V \in K \) be the Schur element corresponding to \( V \in \text{Irr}(H_K) \); see [10, §7.2]. Since \( H_K \) is split semisimple, we have \( c_V \neq 0 \) for all \( V \). Since \( A \) is integrally closed in \( K \), we have \( c_V \in A \); see [10, 7.3.8]. We shall need the following fact, which is a combination of [2, p. 75], [10, 9.3.6] and the remarks in [12, 3.4], [7, 2.4].

(2.1) Let \( P_W = \sum_{w \in W} u^{l(w)} \). Then we have a factorization

\[
(u - 1)^{|S|} P_W = \prod_{i=1}^{|S|} (u^{d_i} - 1), \quad \text{where } d_i \geq 1.
\]

The \( d_i \geq 1 \) are called the degrees of \( W \); we have \(|W| = d_1 \cdots d_{|S|} \). Each Schur element \( c_V \) lies in \( \mathbb{Q}[u, u^{-1}] \) and divides \( P_W \). The denominators in the coefficients of \( c_V \) are divisible by bad primes only.
Now let $k$ be a field and $0 \neq q \in k$ be such that $q$ has a square root in $k$. Then there is a unique ring homomorphism $\theta: A \to k$ such that $\theta(u) = q$ (and $v$ is mapped to a chosen square root of $q$ in $k$). Regarding $k$ as an $A$-module via $\theta$, we can extend scalars from $A$ to $k$ and obtain a $k$-algebra $H_k = k \otimes_A H$ which is canonically isomorphic to $H_k(W, q)$. Thus, it will be sufficient to work with $H$ and its specializations. We can now settle a large number of cases occurring in Theorem 1.1.

**Lemma 2.2** Assume that the characteristic of $k$ satisfies the conditions in Theorem 1.1. Then, if $q^{d_i} \neq 1$ for all $i$ (where the $d_i$ are the degrees of $W$), the algebra $H_k$ is split semisimple and we have $|\text{Irr}(H_k)| = |\text{Irr}(H_K)| = |\text{Irr}(H_{C}(W, 1))|$. 

**Proof.** Using (2.1), the above conditions on $k$ and $q$ imply that $\theta(c_V) \neq 0$ for all $V \in \text{Irr}(H_K)$. Consequently, by [10, 9.3.9], the algebra $H_k$ is split semisimple. Then Tits' Deformation Theorem (see [10, 7.4.6]) applies and we have $|\text{Irr}(H_k)| = |\text{Irr}(H_K)|$. Applying the same argument to the specialization $A \to C$, $u \mapsto 1$, we obtain $|\text{Irr}(H_K)| = |\text{Irr}(H_{C}(W, 1))|$. 

Thus, from now on, we assume that the conditions in Theorem 1.1 on the characteristic of $k$ are satisfied and that $q$ is a root of unity. Let $k_0 \subseteq k$ be the field of fractions of the image of $\theta$. Then, by [7, 3.6], the algebra $H_{k_0}$ is split. So the scalar extension from $k_0$ to $k$ defines a bijection between $\text{Irr}(H_{k_0})$ and $\text{Irr}(H_k)$; see [10, Lemma 7.3.4]. Thus, we may assume without loss of generality that $k = k_0$.

It will be further convenient to take the following point of view. Let $p \subseteq A$ be the kernel of $\theta$. Then $p \subseteq A$ is a prime ideal and, identifying $k$ with the field of fractions of $A/p$, we may regard $\theta$ as the natural map $A \to A/p \subseteq k$. Let $e$ be the smallest $i \geq 2$ such that $1 + q + q^2 + \cdots + q^{i-1} = 0$ and $\Phi_e(u) \in \mathbb{Z}[u]$ be the $e$th cyclotomic polynomial. Then we have $\Phi_e(q) = 0$ in $k$ and so $\Phi_e(u) = \Phi_e(v^2) \in p$. Now we have

$$\Phi_e(v^2) = \Phi_{2e}(v) \quad \text{if } e \text{ is even},$$
$$\Phi_e(v^2) = \Phi_e(v)\Phi_{2e}(v) \quad \text{if } e \text{ is odd}.$$ 

Thus, choosing a suitable square root of $q$ in $k$, we can assume that $\Phi_{2e}(v) \in p$. Let $q \subseteq A$ be the prime ideal generated by $\Phi_{2e}(v)$; we have $0 \neq q \subseteq p \subseteq A$ and $A/q = \mathbb{Z}[\zeta_{2e}]$, the ring of algebraic integers in the field $F = \mathbb{Q}[\zeta_{2e}] \subseteq C$; see [3, (4.5)]. Then, as above, we see that $H_F = H_F(W, \zeta_e)$ is split and the scalar extension from $F$ to $C$ defines a bijection $\text{Irr}(H_F) \leftrightarrow \text{Irr}(H_{C}(W, q))$.

**2.3 Factorization of decomposition maps.** In the above set-up, the natural map $A \to A/p \subseteq k$ induces a well-defined decomposition map
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$d_p: R_0(H_K) \to R_0(H_k)$ between the Grothendieck groups of $H_K$ and $H_k$; see [10] 7.4.3. Similarly, the map $A \to A/q \subseteq F$ induces a decomposition map $d_e: R_0(H_K) \to R_0(H_F)$. Since $A/q$ is integrally closed in $F$, we have the following factorization (see [6] 2.6):

\[
\begin{array}{ccc}
R_0(H_K) & \xrightarrow{d_p} & R_0(H_k) \\
\xrightarrow{d_e} & R_0(H_F) & \\
\end{array}
\]

Here, $d_p$ is the decomposition map induced by the natural map $A/q \to A/p$. Now, since $d_p$ and $d_e$ are surjective by [7] 3.3, we conclude that $|\text{Irr}(H_k)|$ equals the rank of $d_p$ and the above factorization implies that

\[|\text{Irr}(H_k)| \leq |\text{Irr}(H_F)| = |\text{Irr}(H_C(W, \zeta))|.
\]

Note that, if $q = p$, then Theorem 1.1 is now clear since $H_k = H_F$.

Thus, we are finally reduced to the situation where $q \neq p$. In this case, $p$ is a maximal ideal containing $\Phi_{2e}(v)$ and a prime number $\ell > 0$. (See the description of all prime ideals of $A$ in [10] Exercise 7.9, for example.)

In order to proceed, we shall need some facts about the center of $H$, which we denote by $Z(H)$. Let $\text{Cl}(W)$ be the set of conjugacy classes of $W$. For each $C \in \text{Cl}(W)$, consider the element

\[z_C = \sum_{w \in W} u^{-l(w)} f_{w,C} T_w \in H,
\]

where $f_{w,C}(u) \in \mathbb{Z}[u]$ are the class polynomials defined in [10] §8.2. By [10] 8.2.4 (see also [11]), we have $z_C \in Z(H)$ and \{ $z_C \mid C \in \text{Cl}(W)$ \} is an $A$-basis of $Z(H)$.

**Definition 2.4 (Geck–Rouquier)** Consider the natural map $A \to A/q \subseteq F$ and the corresponding decomposition map $d_e: R_0(H_K) \to R_0(H_F)$. For $M \in \text{Irr}(H_F)$, we define $P_M: Z(H_K) \to K$ by

\[P_M(z_C) = \sum_{V \in \text{Irr}(H_K)} \frac{d_{V,M}}{c_V} \omega_V(z_C) \quad \text{for } C \in \text{Cl}(W),
\]

where $d_{V,M}$ are the decomposition numbers and $\omega_V$ is the central character associated with $V$. By [10] 7.5.3, we have

\[P_M(z_C) \in A_q \quad \text{for all } C \in \text{Cl}(W).
\]
Note that, since \( q \) is a principal ideal, the localization \( A_q \subset K \) is a discrete valuation ring, with residue field \( F \).

For \( C \in \text{Cl}(W) \), let \( \tau_C := \sum_{w \in W} \zeta_C^{-\ell(w)}f_{w,C}(\zeta_w)T_w \in \mathbf{H}_F \). Then \( \tau_C \in Z(\mathbf{H}_F) \) and \( \{\tau_C \mid C \in \text{Cl}(W)\} \) is an \( F \)-basis of \( Z(\mathbf{H}_F) \); see \([10, 8.2.5]\). Thus, we obtain an induced function

\[
\overline{P}_M: Z(\mathbf{H}_F) \to F, \quad \overline{P}_M(\tau_C) = P_M(z_C) \text{ mod } q.
\]

In a slightly different form, the following results appeared in \([11, 3.3]\).

**Lemma 2.5** Assume that each Schur element \( c_V \) can be expressed as

\[
(*) \quad c_V = \Phi_{2e}(v)^{d_V}f_V(v) \quad \text{where } d_V \geq 0 \text{ and } f_V(v) \in A \setminus \mathfrak{p}.
\]

Then we have \( P_M(z_C) \in A_\mathfrak{p} \) for all \( M \in \text{Irr}(\mathbf{H}_F) \) and \( C \in \text{Cl}(W) \).

**Proof.** Let \( M \in \text{Irr}(\mathbf{H}_F) \). Consider the ideal \( I \) of all \( g(v) \in \mathbb{Q}[v, v^{-1}] \) such that \( g(v)P_M(z_C) \in \mathbb{Q}[v, v^{-1}] \) for all \( C \in \text{Cl}(W) \). Since \( \mathbb{Q}[v, v^{-1}] \) is a principal ideal domain, \( I \) is generated by a polynomial \( g_0(v) \in \mathbb{Q}[v, v^{-1}] \). Now, since \( P_M(z_C) \in A_\mathfrak{q} \) for all \( C \in \text{Cl}(W) \), there exists a polynomial \( h(v) \in I \) which is not divisible by \( \Phi_{2e}(v) \). Thus, \( \Phi_{2e}(v) \) does not divide \( g_0(v) \).

On the other hand, Definition 2.4 shows that the product of all \( c_V \) lies in \( I \) (note that \( \omega_V(z_C) \in A \) since \( A \) is integrally closed in \( K \)). So, using (\( * \)) and setting \( d := \sum_V d_V \geq 0 \), we find that

\[
\Phi_{2e}(v)^d f(v) \in I \quad \text{where } f(v) = \prod_{V \in \text{Irr}(\mathbf{H}_K)} f_V(v) \in A \setminus \mathfrak{p}.
\]

Since \( \mathbb{Q}[v, v^{-1}] \) is a factorial ring, the fact that \( \Phi_{2e}(v) \) does not divide \( g_0(v) \) implies that \( g_0(v) \) divides \( f(v) \). Consequently, we have \( f(v)P_M(z_C) \in \mathbb{Q}[v, v^{-1}] \) for all \( C \in \text{Cl}(W) \).

Now Definition 2.3 actually shows that \( \Phi_{2e}(v)^d f(v)P_M(z_C) \in A \) for all \( C \in \text{Cl}(W) \). Therefore, since \( f(v) \in A \) and \( \Phi_{2e}(v) \) is monic, we conclude that

\[
f(v)P_M(z_C) \in A \quad \text{for all } C \in \text{Cl}(W).
\]

Thus, since \( f(v) \in A \setminus \mathfrak{p} \), we have \( P_M(z_C) \in A_\mathfrak{p} \) for all \( C \in \text{Cl}(W) \).

**Proposition 2.6 (Geck–Rouquier)** In the above set-up, assume that the condition (\( * \)) in Lemma 2.5 holds. Then we have \( |\text{Irr}(\mathbf{H}_F)| = |\text{Irr}(\mathbf{H}_K)| \).
Proof. Consider the natural map $A_{\mathfrak{p}} \to k$ and the corresponding decomposition map $d_p: R_0(H_K) \to R_0(H_k)$. For $V \in \text{Irr}(H_K)$, let $\chi_V: H_K \to K$ be the character afforded by $V$. Since $A$ is integrally closed in $K$, we have $\chi_V(T_w) \in A$ for all $w \in W$. Thus, we obtain an induced function $\tilde{\chi}_V: H_k \to K$ such that $\tilde{\chi}_V(T_w) = \chi_V(T_w) \mod \mathfrak{p}$. By the definition of $d_p$, the function $\tilde{\chi}_V$ is a $k$-linear combination of the characters afforded by the simple $H_k$-modules. Hence, we certainly have

$$|\text{Irr}(H_k)| \geq \dim_k \langle \tilde{\chi}_V \mid V \in \text{Irr}(H_k) \rangle_k.$$  

On the other hand, consider the decomposition map $d_e$ induced by $A_q \to F$. Let $B \subseteq \text{Irr}(H_K)$ be such that the matrix of decomposition numbers $d_{V,M}$, where $V \in B$ and $M \in \text{Irr}(H_F)$, is square and invertible over $\mathbb{Z}$. Such a subset $B$ exists by [7] Theorem 3.3. We have already seen in (2.3) that $|B| = |\text{Irr}(H_F)| \geq |\text{Irr}(H_k)|$. Hence it is enough to show that

\[
\{\tilde{\chi}_V \mid V \in B\} \text{ is linearly independent.}
\]

To prove (1), assume that we have a linear relation $\sum_{V \in B} \lambda_V \tilde{\chi}_V = 0$ with $\lambda_V \in k$.

Now we consider once more the function $P_M: Z(H_K) \to K$ introduced in Definition 2.4. By Lemma 2.5, we have $P_M(z_C) \in A_{\mathfrak{p}}$ for all $C \in \text{Cl}(W)$. To simplify notation, for $h = \sum_{w \in W} a_w(v)T_w \in H$ with $a_w(v) \in A$, we write $\tilde{h} := \sum_{w \in W} a_w(q^{1/2})T_w \in H_k$, where $q^{1/2}$ is the chosen square root of $q$. Then, again by [10] 8.2.5, we have $\tilde{z}_C \in Z(H_k)$ and $\{\tilde{z}_C \mid C \in \text{Cl}(W)\}$ is a $k$-basis of $Z(H_k)$. Thus, we obtain an induced function

$$\tilde{P}_M: Z(H_k) \to k, \quad \tilde{P}_M(\tilde{z}_C) = P_M(z_C) \mod \mathfrak{p},$$

For each $V \in \text{Irr}(H_K)$, let $\chi^*_V$ be the unique element in $Z(H_K)$ such that $\omega_V(\chi^*_V) = c_{V'}\delta_{V,V'}$ for all $V' \in \text{Irr}(H_K)$. Such an element exists by [10] 7.2.6; explicitly, we have $\chi^*_V = \sum_{w \in W} w^{-l(w)}\chi_V(T_w)T_{w^{-1}}$. Since $A$ is integrally closed in $K$, we have $\chi^*_V \in Z(H)$. Thus, we obtain a corresponding element $\chi^*_V \in Z(H_k)$. The defining equation in (2.4) now yields that

$$P_M(\chi^*_V) = \sum_{V' \in \text{Irr}(H_K)} \frac{d_{V',M}}{c_{V'}} \omega_{V'}(\chi^*_V) = d_{V,M} \quad \text{for all } M \in \text{Irr}(H_F).$$

Applying this to the above linear relation, we obtain

$$0 = \sum_{V \in B} \lambda_V \tilde{P}_M(\tilde{z}_V) = \sum_{V \in B} \lambda_V \tilde{d}_{V,M}, \quad \text{where } \tilde{d}_{V,M} = d_{V,M} \mod \mathfrak{p}.$$  

But $B$ was chosen such that the matrix $(d_{V,M})$ is invertible over $\mathbb{Z}$. Hence $(\tilde{d}_{V,M})$ is invertible over $k$ and so $\lambda_V = 0$ for all $V \in B$, proving (1).
3 Proof of Theorem 1.1

By the discussion in Section 2, in order to complete the proof of Theorem 1.1, it remains to consider the following situation. The field $k$ is the field of fractions of $A/\mathfrak{p}$, where $\mathfrak{p}$ is a maximal ideal containing a prime number $\ell > 0$ which is not bad for $W$. In particular, $k$ is a finite field of characteristic $\ell$. Furthermore, $\mathfrak{p}$ contains the cyclotomic polynomial $\Phi_{2e}(v) \in \mathbb{Z}[v]$, where $e \geq 2$ divides some $d_i$ as in (2.1). The number $e$ is minimal such that $1 + q + q^2 + \cdots + q^{e-1} = 0$ in $k$ and there is a square root of $q$ which is also a root of $\Phi_{2e}(v)$ in $k$. Under these conditions, we must show:

(E) $|\text{Irr}(H_k)| = |\text{Irr}(H_F(W, \zeta_e))|$, where $F = \mathbb{Q}[\zeta_{2e}] \subset \mathbb{C}$.

We shall need the following well-known property of cyclotomic polynomials.

**Lemma 3.1** Assume that $\Phi_d(q) = 0$ for some $d \geq 2$. If $q = 1$, then $d = \ell^n$ for some $n \geq 1$. If $q \neq 1$, then $e$ is the order of $q$ in $k^\times$ and $d = e\ell^n$ for some $n \geq 0$.

**Proof.** Let us write $d = d'\ell^n$ where $\ell$ does not divide $d'$. Now $\Phi_d(u)$ certainly divides $\Phi_{d'}(u^{\ell^n})$. Since, furthermore, taking $\ell$th powers is an automorphism of $k$, we conclude that $\Phi_{d'}(q) = 0$. If $d' = 1$, this means that $q = 1$ and $e = \ell$. So the assertion is true in this case. Now assume that $q \neq 1$. Then $e$ is minimal such that $q^e - 1 = 0$, i.e., $e$ is the multiplicative order of $q$. Hence $e$ divides the order of $k^\times$ and so is coprime to $\ell$. Then $\Phi_{d'}(q) = 0$ implies $q^{d'} = 1$ and so $e$ divides $d'$. Assume, if possible, that $e \neq d'$. Then we can write

$$\frac{u^{d'} - 1}{u - 1} = a(u)\Phi_e(u)\Phi_{d'}(u) \quad \text{where } a(u) \in \mathbb{Z}[u].$$

Differentiating with respect to $u$ and applying the natural map $A \to A/\mathfrak{p} \subseteq k$ yields that $d' = 0$ in $k$, contradicting the fact that $d'$ is coprime to $\ell$. 

The following result (first established in [11]) provides a general proof of (E), but under a condition on $\ell$ which is more restrictive than that in Theorem 1.1.

**Theorem 3.2 (Geck–Rouquier)** Assume that $\ell \neq 2$ and that $\ell\ell$ does not divide any degree $d_i$ of $W$. (This condition is satisfied, for example, if $\ell$ does not divide $|W|$.) Then condition (*) in Lemma 2.5 is satisfied and (E) holds.
Proof. By Proposition 2.6 it is enough to verify that condition (*) in Lemma 2.5 is satisfied. Using (2.1), we can write
\[ c_{V} = \Phi_{2e}(v)^{d_{V}} f_{V}(v), \]
where \( d_{V} \geq 0 \) and \( f_{V}(v) \in \mathbb{Z}[v, v^{-1}] \) is a product of an integer divisible by bad primes only, an integral power of \( u, \Phi_{e}(v) \) (if \( e \) is odd) and various cyclotomic polynomials \( \Phi_{d}(u) \), where \( d \neq e \) and \( d \geq 2 \) divides some degree \( d_{i} \).

Now consider the natural map \( \theta: A \to A/p \subseteq k \). Assume, if possible, that \( \theta(f_{V}(u)) = 0 \) for some \( V \in \text{Irr}(H_{K}) \). Since the characteristic of \( k \) is not a bad prime, we must have \( \theta(\Phi_{e}(v)) = 0 \) (if \( e \) is odd) or \( \theta(\Phi_{d}(u)) = 0 \) for some \( d \neq e \) where \( d \geq 2 \) divides a degree \( d_{i} \). The second possibility cannot occur by Lemma 3.1 and our assumption. Hence \( e \) must be odd and \( \theta(\Phi_{e}(v)) = 0 \). As in the proof of Lemma 3.1, we write \( (v^{2e} - 1)/(v - 1) = b(v)\Phi_{e}(v)\Phi_{2e}(v) \) with some \( b(v) \in \mathbb{Z}[v] \), differentiate with respect to \( v \) and obtain the conclusion that \( \theta(2e) = 0 \), i.e., \( \ell \) divides \( 2e \). This contradicts the fact that \( \ell \neq 2 \) and \( e \) is the order of \( q \).

In Section 1, we have already remarked that Theorem 1.1 is known to hold for \( W \) of type \( A_{n}, B_{n} \) and \( D_{n} \). Thus, it remains to consider the finitely many exceptional types. For each of these types, we need to know the prime divisors of \( |W| \), the bad primes and the degrees \( d_{i} \). This information is provided in Table 2. (The information in this table is obtained by inspection of the tables in [10, Appendix E] and [2, p. 75].)

Table 2: Bad primes and degrees for the exceptional types.

| Type  | \( |W| \)       | \( \ell \) bad | degrees \( d_{i} \) |
|-------|----------------|--------------|-------------------|
| \( G_{2} \) | \( 2^{2} \cdot 3 \) | 2, 3         | 2, 6              |
| \( E_{6} \) | \( 2^{27} \cdot 3^{2} \cdot 5 \) | 2, 3         | 2, 6, 8, 12       |
| \( E_{7} \) | \( 2^{10} \cdot 3^{4} \cdot 5 \cdot 7 \) | 2, 3         | 2, 6, 8, 10, 12, 14, 18 |
| \( E_{8} \) | \( 2^{14} \cdot 3^{6} \cdot 5^{2} \cdot 7 \) | 2, 3, 5      | 2, 8, 12, 14, 18, 20, 24, 30 |

Now, given \( W, e \) and \( \ell \), the CHEVIE function in Table 3 computes the number of simple \( H_{K} \)-modules. This is done by computing the rank of the character table of \( H_{K} \) under the specialization \( A \to A/p \subseteq k \). That this rank indeed equals the cardinality of \( |\text{Irr}(H_{K})| \) follows from [10, 7.5.7 and 8.2.9] and the fact that \( d_{p} \) is surjective (see [7, 3.3]). Note also that the character tables for all \( H_{K} \) are explicitly known and available in CHEVIE.

By Theorem 3.2 we still have to consider all primes \( \ell \) which divide \( |W| \) but are not bad and all \( e \) such that \( e \) and \( e \ell \) divide some degree \( d_{i} \). Thus,
if \( W \) is of type \( G_2 \) or \( F_4 \), then all prime divisors of \(|W|\) are bad and there is nothing more to prove. If \( W \) is of type \( E_6 \), we only have to consider the prime \( \ell = 5 \). But the only degree divisible by 5 is 5 itself, hence we are done. If \( W \) is of type \( E_7 \) or \( E_8 \), then 2 is the only possible value for \( e \) such that \( e \) and \( e\ell \) (with \( \ell \) bad) divide some \( d_i \). Hence it remains to consider the following cases:

- **\( E_7 \):** \( e = 2 \) and \( \ell = 5, 7 \), where we obtain \(|\text{Irr}(H_k)| = 12.\)
- **\( E_8 \):** \( e = 2 \) and \( \ell = 7 \), where we obtain \(|\text{Irr}(H_k)| = 23.\)

These results are in accordance with the corresponding values in characteristic 0 printed in [10, 11.5.13]. Thus, Theorem 1.1 is proved.

**Table 3:** A CHEVIE program for computing the number of simple modules.

```gap
gap> RequirePackage("chevie");
gap> NumberSimples := function( W, e, ell )
gap>   local H, i, z;
>   if Gcd( ell, e ) = 1 then
>     # find i such that GF(ell^i) has an element z
>     # of order 2e
>     i:=1;
>     while not IsInt( ( ell^i-1 )/( 2*e ) ) do
>       i := i+1;
>     od;
>     z := Z( ell^i )^( ( ell^i-1 )/( 2*e ) );
>   elif e = ell and ell>2 then
>     z := Z( ell )^( ( ell-1 )/2 );
>   elif e = ell and ell=2 then
>     z := Z( ell );
>   fi;
>   # define the Iwahori-Hecke algebra with parameter z^2
>   H := Hecke( W, z^2, z );
>   # return the rank of the specialized character table
>   return RankMat( CharTable( H ).irreducibles );
> end;
> NumberSimples( CoxeterGroup( "E", 8 ), 2, 5 ); # an example
```

**Remark 3.3** Using the above program we find that if \( e = \ell \) is a bad prime, then \(|\text{Irr}(H_k)| < |\text{Irr}(H_F)|\). So, in general, the assertion in Theorem 1.1 will not hold if \( \ell \) is a bad prime. (Another example is given in Table 3.)
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References

[1] S. Ariki and A. Mathas, The number of simple modules of the Hecke algebras of type $G(r,1,n)$, Math. Z. 233, 601–623.

[2] R. W. Carter, Finite groups of Lie type: Conjugacy classes and complex characters, Wiley, New York (1985).

[3] C. W. Curtis and I. Reiner, Methods of representation theory Vol. I and II, Wiley, New York, 1981 and 1987.

[4] R. Dipper, M. Geck, G. Hiss and G. Malle, Representations of Hecke algebras and finite groups of Lie type. In: Algorithmic algebra and number theory (Heidelberg, 1997), pp. 331–378, Springer Verlag, Berlin/Heidelberg, 1998.

[5] R. Dipper and G. D. James, Representations of Hecke algebras of the general linear groups, Proc. London Math. Soc. 52 (1986), 20–52.

[6] M. Geck, Representations of Hecke algebras at roots of unity, Séminaire Bourbaki, 50ème année, 1997-98, Astérisque No. 252 (1998), Exp. 83 6, 33–55.

[7] M. Geck, Kazhdan–Lusztig cells and decomposition numbers, Represent. Theory 2 (1998), 264–277 (electronic).

[8] M. Geck, On the representation theory of Iwahori–Hecke algebras of extended finite Weyl groups, Represent. Theory 4 (2000), 370–397 (electronic).

[9] M. Geck, G. Hiss, F. Lübeck, G. Malle, and G. Pfeiffer, CHEVIE – A system for computing and processing generic character tables for finite groups of Lie type, Weyl groups and Hecke algebras, AAECC 7 (1996), 175–210.

[10] M. Geck and G. Pfeiffer, Characters of finite Coxeter groups and Iwahori–Hecke algebras, London Math. Soc. Monographs, New Series 21, Oxford University Press, New York 2000. xvi+446 pp.

[11] M. Geck and R. Rouquier, Centers and simple modules for Iwahori-Hecke algebras. In: Finite reductive groups: Related structures and representations (ed. M. Cabanes), pp. 251–272. Birkhäuser, Basel, 1997.

[12] G. Lusztig, Leading coefficients of character values of Hecke algebras, Proc. Symp. Pure Math. 47, Amer. Math. Soc., Providence, RI, 1987, pp. 235–262.

[13] M. Schönert et al., GAP – Groups, Algorithms, and Programming, Lehrstuhl D für Mathematik, RWTH Aachen, Germany, fourth ed., (1994).

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