Spanning tree with lower bound on the degrees

Zoltán Király
Eötvös Loránd University
Department of Computer Science and
MTA-ELTE Egerváry Research Group
Pázmány Péter sétány 1/C
Budapest, Hungary, H-1117
kiraly@cs.elte.hu

April 5, 2018

Abstract

We concentrate on some recent results of Egawa and Ozeki [1, 2], and He et al. [5]. We give shorter proofs and polynomial time algorithms as well.

We present two new proofs for the sufficient condition for having a spanning tree with prescribed lower bounds on the degrees, achieved recently by Egawa and Ozeki [1]. The first one is a natural proof using induction, and the second one is a simple reduction to the theorem of Lovász [9]. Using an algorithm of Frank [4] we show that the condition of the theorem can be checked in time $O(m\sqrt{n})$, and moreover, in the same running time – if the condition is satisfied – we can also generate the spanning tree required. This gives the first polynomial time algorithm for this problem.

Next we show a nice application of this theorem for the simplest case of the Weak Nine Dragon Tree Conjecture, and for the game coloring number of planar graphs, first discovered by He et al. [5].

Finally, we give a shorter proof and a polynomial time algorithm for a good characterization of having a spanning tree with prescribed degree lower bounds, for the special case when $G[S]$ is a cograph, where $S$ is the set of the vertices having degree lower bound prescription at least two. This theorem was proved by Egawa and Ozeki [2] in 2014 while they did not give a polynomial time algorithm.

1 Introduction

Let $G = (V, E)$ be a simple undirected graph, $S \subseteq V$ and $f : S \rightarrow \{2, 3, 4, \ldots\}$ be an integer-valued function on $S$. For a subset $X$ of vertices let $f(X) = \sum_{x \in X} f(x)$. For disjoint sets of vertices $X$ and $Y$, $d_G(X, Y)$ denotes the number of edges between $X$ and $Y$, $d_G(X) = d_G(X, V - X)$ and $d_G(u) = d_G(\{u\})$. When the graph $G$ is clear from the context, we omit it from the notation.

The open neighborhood is denoted by $\Gamma_G(X) = \{u \in V - X | \exists x \in X, ux \in E\}$, and the closed neighborhood is denoted by $\Gamma^+_G(X) = \Gamma_G(X) \cup X$. A subgraph induced by a vertex set $X \subseteq V$ is denoted by $G[X]$, the number of its edges by
i_G(X), and the number of its components by \( c(G[X]) \) or \( c_G(X) \). We will use the convention that \( \Gamma_G(\emptyset) = \emptyset \) and \( c(G[\emptyset]) = 0 \).

Egawa and Ozeki proved the following sufficient condition for having a forest (or spanning tree) with prescribed lower bounds on the degrees.

**Theorem 1 ([1])** If for all nonempty subsets \( X \subseteq S \) we have \( |\Gamma^*_G(X)| > f(X) \) then there is a forest subgraph \( F \) of \( G \), such that for all vertices \( v \in S \) we have \( d_F(v) \geq f(v) \).

**Corollary 2 ([1])** If for all nonempty subsets \( X \subseteq S \) we have \( |\Gamma^*_G(X)| > f(X) \) and \( G \) is connected, then there is a spanning tree \( T \) of \( G \), such that for all vertices \( v \in S \) we have \( d_T(v) \geq f(v) \).

Special cases of this theorem appeared in the literature as follows. When \( G \) is bipartite and \( S \) is one of the classes, it was proved by Lovász in 1970 [9]. For general \( G \), if \( S \) is a stable set, it was proved by Frank in 1976 [4] in a stronger form, as in this case the condition above is also necessary, giving a special case of Theorem 9 in Section 6. For a not necessary stable \( S \) a stronger condition is proved to be sufficient by Singh and Lau [11], namely: \( |\Gamma^*_G(X)| > f(X) + c_G(X) \).

Deciding whether for a triplet \( (G, S, f) \) there is a spanning tree \( T \) with degree lower bounds, i.e., \( d_T(v) \geq f(v) \) for all \( v \in S \), is NP-complete (let \( S = V - \{u, v\} \) and \( f(x) = 2 \) for each \( x \in S \); any appropriate spanning tree is a Hamiltonian path). However, consider the following algorithmic problem. For given \( (G, S, f) \) check, whether the condition of Corollary 2 is satisfied, and if yes, then construct the appropriate spanning tree \( T \). We show that this problem is polynomially solvable, namely in time \( O(m\sqrt{n}) \), where \( n = |V| \) and \( m = |E| \).

In the next section we give a simpler proof than that of Egawa and Ozeki, using induction. In Section 3 we give another proof, that is a simple reduction to the theorem of Lovász, yielding also a fast algorithm, detailed in Section 4. In Section 5 we show an application (as an example) of Theorem 1 for the game coloring number of some planar graphs. Finally, in Section 6 we show how we can use these ideas to prove a good characterization of Egawa and Ozeki [2] for a special case. Our proof is not only shorter but also yields the first polynomial time algorithm for this case.

## 2 First proof – by induction

We prove Theorem 1 by induction on the number of edges. If \( G \) is a forest or \( S = \emptyset \) then the theorem is obviously true.

We call a set \( X \subseteq S \) tight if it satisfies the condition \( |\Gamma^*_G(X)| \geq f(X) + 1 \) with equality.

If \( uv \) is an edge and \( G - uv \) satisfies the condition then we are done by induction. So we may assume that for every edge \( uv \) the graph \( G - uv \) has a set \( X \subseteq S \) violating the condition (a violating set). This implies that there are no edges outside \( S \), and also that for each edge \( uv \) either \( u \) or \( v \) is contained in a tight set \( X \), where the other one is connected to \( X \) by exactly one edge. If \( u \) is contained in tight set \( X \) with \( d(v, X) = 1 \) then we orient edge \( uv \) from \( v \) to \( u \), otherwise, from \( u \) to \( v \). (If
both \( u \) and \( v \) is contained in such a tight set, we choose arbitrarily.) This oriented graph \( \overrightarrow{G} \) has the property that no arc leaves \( S \). (The word \emph{arc} will always refer to a directed edge, in this section a directed edge of \( \overrightarrow{G} \).) The in-degree of a vertex \( u \) (set \( X \)) is denoted by \( \varrho(u) \) (or \( \varrho(X) \) resp.).

**Claim 3** For each \( u \in S \) we have \( f(u) \geq \varrho(u) \).

**Proof of the Claim.** If \( \varrho(u) > 0 \) then \( u \) is contained in a tight set. As \( |\Gamma^*_G(X)| \) is a submodular set function, the intersection and the union of two intersecting tight sets are both tight. Thus the intersection \( I(u) \) of all tight sets containing \( u \) is also a tight set. Every arc \( vu \) of \( \overrightarrow{G} \) was oriented this way because it entered a tight set containing \( u \), consequently, it must enter \( I(u) \) as well.

If \( |I(u)| = 1 \) then, by the tightness, we have \( f(u) = d_G(u) \geq \varrho(u) \). Otherwise, as \( I(u) - u \) is not a violating set, if \( vu \) is an arc of \( \overrightarrow{G} \), then the vertex \( v \) does not have any neighbors in \( I(u) - u \). Thus we have \( f(I(u)) + 1 - f(u) = f(I(u) - u) + 1 \leq |\Gamma^*_G(I(u) - u)| \leq |\Gamma^*_G(I(u))| - \varrho(u) = f(I(u)) + 1 - \varrho(u) \), giving the claim.

To finish the proof of the theorem it is enough to prove that \( G \) is a forest. Suppose this is not the case. Choose a cycle \( C \) which minimizes \( |V(C) - S| \). Let \( X = V(C) \cap S \) and let \( \overline{X} \) be the closure of \( X \) relative to \( S \): \( \overline{X} = \{v \in S \mid \exists x \in X, \text{ such that } v \text{ and } x \text{ are in the same component of } G[S]\} \). Clearly \( c(G[\overline{X}]) \leq c_G(X) \) and, by the observation made above, no arc leaves \( \overline{X} \).

If \( V(C) \subseteq S \) then, using Claim 3 and the fact that \( G[\overline{X}] \) is now connected and contains a cycle, \( f(\overline{X}) \geq i_G(\overline{X}) + \varrho(\overline{X}) \geq |\overline{X}| + \varrho(\overline{X}) \geq |\Gamma^*_G(\overline{X})| \), and this contradicts to the assumption of the theorem.

Otherwise, \( G[S] \) is a forest and \( |V(C) - S| \geq c(G[\overline{X}]) \). Now, by Claim 3, \( f(\overline{X}) \geq i_G(\overline{X}) + \varrho(\overline{X}) \geq |\overline{X}| - c(G[\overline{X}]) + \varrho(\overline{X}) \). As \( V - S \) is an independent set and no arc leaves \( S \), at least two arcs go from any vertex of \( V(C) - S \) to \( \overline{X} \), that is \( \overline{X} \) has at most \( \varrho(\overline{X}) - |V(C) - S| \leq \varrho(\overline{X}) - c(G[\overline{X}]) \) different neighbors in \( V - S \). Thus \( |\Gamma^*_G(\overline{X})| \leq |\overline{X}| + \varrho(\overline{X}) - c(G[\overline{X}]) \leq f(\overline{X}) \), a contradiction again.

\[ \Box \Box \]

### 3 Second proof – reduction to Lovász’ theorem

In this section we prove Theorem 1 using a theorem of Lovász [9]. We quote this old theorem reformulated for fitting the notation used in this paper. We denote by \( f^+ \) the function \( f + 1 \), i.e., \( f^+(x') = f(x') + 1 \) for \( x' \in S' \).

**Theorem 4 (\cite{9})** Let \( B = (S' \cup V, E') \) be a bipartite graph and \( f : S' \to \{2, 3, 4, \ldots \} \) be a function. \( B \) has a forest subgraph \( F_0 \) with the property \( d_{F_0}(x') = f^+(x') \) for every \( x' \in S' \), if and only if for all nonempty \( X' \subseteq S' \) we have \( |\Gamma_B(X')| > f(X') \).

**Proof of Theorem 1.** We have \( (G, S, f) \) given, and let \( S' \) be a set disjoint from \( V \) with elements \( S' = \{u' \mid u \in S\} \), and extend \( f \) to \( S' \) in the obvious way: \( f(u') := f(u) \) for each \( u \in S \).

Construct a bipartite graph \( B = (V \cup S', E') \) as follows. For each ordered vertex pair \( (u \in S, v \in V) \) we put an edge \( uu'v \) into \( E' \) if \( uv \in E \), and we also put the ‘vertical’ edges \( u'u \) for each \( u \in S \). Observe that for each \( X \subseteq S \) (if \( X' \) is
denotes the corresponding subset of $S'$ we have $\Gamma_B(X') = \Gamma_G^*(X)$. Therefore the condition of Theorem 4 is satisfied and thus we have a forest subgraph $F_0$ of $B$ with $d_{F_0}(x') = f^+(x')$ for every $x' \in S'$.

First we claim that we can modify $F_0$ to get another forest subgraph $F_1$ of $B$, so that $d_{F_1}(x') \geq f^+(x')$ for every $x' \in S'$, with the additional property that $F_1$ contains every vertical edge. Suppose $u'u \not\in E'(F_0)$. If $u'$ and $u$ are in different components of $F_0$, then we add the edge $u'u$. Otherwise, there is unique path $u'vu_1 \ldots u'_1u$ in $F_0$, in this case we delete the edge $u'v$ and add the edge $u'u$, still resulting in a forest (with the same degrees inside $S'$).

![Figure 1: An example for the proof, $F_0$, $F_1$ and $F$ are shown in blue and bold.](image)

Finally, we construct the desired forest $F$ by contracting each vertical edge (we contract $u'$ to vertex $u$). It is easy to see, that in this way $F$ becomes a forest subgraph of the graph $G$, and $d_F(x) \geq d_{F_1}(x') - 1 \geq f(x)$ for every $x \in S$. See Figure 1 for an example, where vertices if $S \cup S'$ are black, and $f(u) = 2$ for every $u \in S$.

Remark: Theorem 4 remains true if we allow $f(x') = 1$ for some $x' \in S'$.

4 A polynomial time algorithm for checking the condition and constructing the tree

In this section we first describe the algorithmic proof of Frank [4] for Theorem 4. After cloning each vertex $u' \in S'$ into $f(u')$ copies and running e.g., the algorithm of Hopcroft and Karp for maximum bipartite matching, we either get a forest $F'$ with degrees $d_{F'}(u') = f(u')$ for every $u' \in S'$ (and $d_{F'}(v) \leq 1$ for each $v \in V$), or we get a subset $X' \subseteq S'$ that violates the Hall condition, namely $|\Gamma_B(X')| < f(X')$. In this latter case the corresponding $X \subseteq S$ clearly violates the condition of the theorem as well.

We make an auxiliary digraph $D = (U, A)$, where $U = V \cup S' \cup \{r\}$. We orient edges of $F'$ from $S'$ to $V$, other edges of $B$ from $V$ to $S'$, and finally add arcs $rv$ for each $v \in V$ uncovered by $F'$. We run a BFS from vertex $r$ in digraph $D$. This gives an arborescence $T$ rooted at $r$ which spans all vertices reachable from $r$. If every vertex in $S'$ is reachable from $r$, then for each arc $vu'$ of $T$ leading from $V$ to $S'$ we add the corresponding edge to $F'$ resulting in the desired forest $F_0$ in $B$ (these are not edges of $F'$, so they increase the degree of every $u' \in S'$). Observe that we did not create any cycle because every vertex $u' \in S'$ has in-degree one in $T$ and the arborescence $T$ does not contain any directed cycle.) See Figure 2 for an example.
Figure 2: An example for Frank’s algorithm, $F'$ and $F$ are shown in blue and bold, $T$ is shown in red and bold.

Otherwise, if $X'$ denotes the set of vertices of $S'$ that are not reachable from $r$, then we claim that $X'$ violates the condition of Theorem 4. If not, then there exist $u \in V$ and $x' \in X'$ such that $ux' \in A$ but either $u$ is uncovered by $F'$ or $u$ is a leaf of $F'$, and its unique $F'$-neighbor $y'$ is in $S' - X'$. In both cases $u$ is reachable from $r$ (in the first case $ru$ is an arc, in the second case $y'$ is reachable from $r$ and $y'u$ is an arc); consequently, $x'$ is also reachable from $r$, a contradiction.

**Final algorithm and running time** We are ready to give an algorithm running in time $O(m\sqrt{n})$ for deciding whether the condition of Corollary 2 is satisfied or not. Moreover, – if the condition holds – we can also generate the spanning tree required in the same running time. This gives the first polynomial time algorithm for this problem.

First we construct the bipartite graph $B$ in time $O(m)$. Then we follow the steps of Frank’s algorithm. Observe that for running the algorithm of Hopcroft and Karp, we do not need to make the cloning in reality; it is enough to do it imaginarily. Doing so keeps the running time $O(m\sqrt{n})$. Constructing $D$, running BFS and constructing $F_0$ can be done in time $O(m)$.

Next we make $F_1$ from $F_0$ and then we construct $F$ by contracting the vertical edges as in the proof presented in Section 3, these are algorithmically easy jobs, they can be done in time $O(m)$.

Finally we make the desired spanning tree from forest $F$, it can also be done in time $O(m)$.

**5 Some applications: WNDT Conjecture and game coloring number of planar graphs**

We show an interesting application of Theorem 1 as an example. For a subset $X$ of vertices, if $|X| > 1$, then we define $\lambda_G(X) = \frac{i(X)}{|X| - 1}$ and $\text{Arb}(G) = \max\{\lambda_G(X) \mid X \subseteq V, |X| > 1\}$. The Weak Nine Dragon Tree (WNDT for short) Conjecture is the following: if for integers $k$ and $d$ we have $\text{Arb}(G) \leq k + \frac{d+1}{d+e}$, then there are $k$ forests $F_1, \ldots, F_k$, such that the maximum degree in $G - F_1 - \ldots - F_k$ is at most $d$. The conjecture was proved by Kim et al. [7] for the case of $d > k$.

Here we show that if we further restrict ourselves to the special case of $k = 1$, then this results in a simple consequence of Theorem 1.
Theorem 5 [7] If \( d \geq 2 \) is an integer and \( \text{Arb}(G) \leq 1 + \frac{d}{d+2} \), then there is a forest subgraph \( F \) of \( G \), such that for every vertex \( v \) we have \( d_G(v) - d_F(v) \leq d \).

Actually we prove a stronger form (also proved in [7]).

Theorem 6 [7] If \( d \geq 2 \) is an integer and for each nonempty subset \( X \) of the vertices we have \( 2(d+1) \cdot |X| > (d+2) \cdot i(X) \), then there is a forest subgraph \( F \) of \( G \), such that for every vertex \( v \) we have \( d_G(v) - d_F(v) \leq d \).

Proof. Let \( S = \{v \in V \mid d(v) \geq d+2\} \) and let \( f \) be defined on \( S \) by \( f(v) = d(v) - d \). For a subset \( X \subseteq S \) let \( \Gamma_j(X) = \{v \in V - X \mid d(v, X) = j\} \), and let \( \bar{X} = X \cup \bigcup_{j=2}^{|X|} \Gamma_j(X) \). By the condition of the theorem \( (d+1) \cdot |\bar{X}| > \frac{d+2}{2} \cdot i(\bar{X}) \), i.e., \( (d+1) \cdot |X| + (d+1) \cdot \sum_{j=2}^{|X|} \text{\textbf{}} |\Gamma_j(X)| > \frac{d+2}{2} \cdot (i(X) + \sum_{j=2}^{|X|} j \cdot |\Gamma_j(X)|) \). Realigned we get
\[
(d+1) \cdot |X| > \frac{d+2}{2} \cdot i(X) + \sum_{j=2}^{|X|} \left[(j \cdot \frac{d+2}{2} - (d+1)) \cdot |\Gamma_j(X)|\right] \geq 2 \cdot i(X) + \sum_{j=2}^{|X|} \left[\left((j-2) \cdot \frac{d}{2} + j - 1\right) \cdot |\Gamma_j(X)|\right] \geq 2 \cdot i(X) + \sum_{j=2}^{|X|} \left[(j-1) \cdot |\Gamma_j(X)|\right],
\]

as \( d \geq 2 \). We have \( f(X) = 2 \cdot i(X) + \sum_{j=2}^{|X|} j \cdot |\Gamma_j(X)| - d \cdot |X| = 2 \cdot i(X) + \sum_{j=2}^{|X|} \left[(j-1) \cdot |\Gamma_j(X)|\right] + \left(\left|\Gamma(X)\right| - d \cdot |X|\right) < (d+1) \cdot |X| + \left|\Gamma(X)\right| = \left|\Gamma^*(X)\right| ,
\]
thus the condition of Theorem 1 is satisfied and the forest \( F \) produced fulfills the statement of our theorem. \( \Box \)

Of course, we can apply the algorithm described in the previous section and efficiently make this decomposition.

Let \( G \) be a simple connected planar graph with girth \( g \geq 5 \). We know by Euler’s formula that \( i(X) < \frac{g-2}{g-4} \cdot |X| \) for every subset \( X \) of the vertices, and \( \frac{g-2}{g-4} \leq \frac{2d+2}{d+2} \) if \( d \geq \frac{4}{g-4} \). We get the following corollary (which is a strengthening of a theorem proved first by He et al. in [5], the improvement was reported to be proved in [8]).

Corollary 7 [5, 8] If \( G \) is a simple connected planar graph with girth at least \( g \) (where \( g = 5 \) or \( g = 6 \)), then there is a spanning tree \( T \) of \( G \), such that for every vertex \( v \) we have \( d_G(v) - d_T(v) \leq \frac{4}{g-4} \).

The game coloring number was defined by Zhu [12] via a two-person game (for upper bounding the so-called “game chromatic number”). Alice and Bob remove vertices of \( G \) in turns. The back-degree of a vertex is the number of its previously removed neighbors. The game coloring number \( \text{col}_g(G) \) is the smallest \( k \geq 1 \), where Alice can achieve that every vertex has back-degree at most \( k \). An easy observation of Zhu [12] states that if the edges of \( G \) can be partitioned into graphs \( G' \) and \( H \), then \( \text{col}_g(G) \leq \text{col}_g(G') + \Delta(H) \), where \( \Delta \) denotes the maximum degree. Faigle et al. [3] proved that the game coloring number of a tree is at most 4. Consequently, we get the following result, that is also a strengthening of a theorem proved by He et al. in [5]). We also note, that by our algorithmic results we also provide a simple polynomial time algorithm for Alice for winning the game, as the proofs in [3] and [12] are algorithmic.

Corollary 8 [5, 8] If \( G \) is a simple planar graph with girth at least 5, then \( \text{col}_g(G) \leq 8 \). If \( G \) is a simple planar graph with girth at least 6, then \( \text{col}_g(G) \leq 6 \).
6 Good characterization for a special case

In [2] Egawa and Ozeki proved the following theorem stating a good characterization if $G[S]$ is a cograph, i.e., it does not contain an induced $P_4$. By the definition, an induced subgraph of a cograph is a cograph, and for any two different vertices of the same connected component of a cograph, they are either adjacent or have a common neighbor. The latter property is equivalent to saying that every component has diameter at most 2.

Egawa and Ozeki also showed by a simple example, that this characterization does not remain true if $G[S] = P_4$: let the vertices of the $P_4$ be $v_1, v_2, v_3, v_4$ and let $G$ have two more vertices, $a$ and $b$, such that $a$ is connected to $v_1$ and $v_4$ while $b$ connected to $v_2$ and $v_3$; and let $f(v_1) = f(v_2) = f(v_3) = f(v_4) = 2$.

**Theorem 9 ([2])** If $G[S]$ is a cograph, then $G$ has a forest subgraph with degree lower bounds $f : S \to \{2, 3, 4, \ldots\}$ on $S$ if and only if for all nonempty subsets $X \subseteq S$ we have

$$|\Gamma_G(X)| + 2|X| - c_G(X) > f(X).$$

**Proof.** We follow the outline of the proof in [2] but we make some simplifications resulting in a significantly shorter proof. We also give a polynomial time algorithm for finding the appropriate forest.

It is not hard to see that the condition above is necessary (even for the case when $G[S]$ is not a cograph). Let $F$ be a forest with $d_F(u) \geq f(u)$ for each $u \in S$, and let $X \subseteq S$. Now $i_G(F[X]) = |X| - c(F[X])$ and $f(X) \leq 2i_G(F[X]) + d_F(X) = 2|X| - 2c(F[X]) + d_F(X)$ and $d_F(X) \leq |\Gamma_G(X)| - 1 \leq |\Gamma_G(X)| - 1$, because $F$ is a forest subgraph of $G$.

For $Z, X \subseteq V$, let $\Gamma_Z(X) = \Gamma(X) \cap Z$. (For notational symmetry we will use the notation also for $\Gamma_Z(Z)$, though this set is always empty.) We denote the cograph $G[S]$ by $H$.

**Claim 10** If $A, B \subseteq S$, then

$$|\Gamma_{A \cup B}(A \cup B)| - c_H(A \cup B) + |\Gamma_{A \cup B}(A \cap B)| - c_H(A \cap B) \leq$$

$$\leq |\Gamma_{A \cup B}(A)| - c_H(A) + |\Gamma_{A \cup B}(B)| - c_H(B).$$

**Proof.** We first prove the claim for the case when $G[A \cup B]$ is a connected cograph and $A \cap B \neq \emptyset$. We use the well-known observation of Erdős and Rado stating that a graph or its complement is connected. As for any $x, y \in A \cap B$ they are either in the same component of $G[A]$ or in the same component of $G[B]$, (if they are not connected, then they have a common neighbor in $A \cup B$), we may assume that $G[A \cap B]$ is inside a component $K$ of $G[A]$. Let $K_1, \ldots, K_a$ denote the other components of $G[A]$, and $L_1, \ldots, L_b, I_1, \ldots, I_c$ denote the components of $G[B]$, where $I_j$ are the components intersecting $A \cap B$. As $c_H(A \cap B) \geq c$, it is enough to prove

$$0 + |\Gamma_{A \cup B}(A \cap B)| - 1 - c \leq |\Gamma_{A \cup B}(A)| + |\Gamma_{A \cup B}(B)| - (a + 1) - (b + c),$$

i.e., $|\Gamma_{A \cup B}(A)| + |\Gamma_{A \cup B}(B)| - |\Gamma_{A \cup B}(A \cap B)| \geq a + b$. This can be easily seen, as $\Gamma_{A \cup B}(A \cap B) = (A \cup B) - (\bigcup_{i=1}^b K_i \cup \bigcup_{j=1}^c L_j)$ and – as we assumed that $G[A \cup B]$
Suppose the condition of the theorem, i.e.,

\[ |\left| \Gamma_{A\cup B}(A) \right| + \left| \Gamma_{A\cup B}(B) \right| \geq a + b - 1, \]

however, as we assumed \( G[A\cup B] \) to be connected, this is always satisfied with a strict inequality.

If \( G[A\cup B] \) has several components, then it is enough to prove the claim for each component separately, thus the same proof works. We remark that for the case \( A \cap B = \emptyset \) we still have strict inequality if there is an edge between \( A \) and \( B \).

Let \( b_0(X) \) and \( b(X) \) be set-functions on the subsets of \( S \) defined by \( b_0(X) = |\Gamma(X)| - c_H(X) \), and \( b(X) = b_0(X) + 2|X| - f(X) \). Let \( A, B \subseteq S \), and denote by \( U(A, B) \) the set of vertices in \( V-(A\cup B) \) connected to both \( A-B \) and \( B-A \) but not to \( A\cap B \) (in other words \( U(A, B) = \Gamma(A) \cap \Gamma(B) - \Gamma(A\cap B) \)). We claim that \( b_0 \) and \( b \) are submodular, moreover, \( b_0(A\cup B) + b_0(A\cap B) + |U(A, B)| \leq b_0(A) + b_0(B) \). As \( b_0(A) = |\Gamma_{A\cup B}(A)| + |\Gamma_{-(A\cup B)}(A)| - c_H(A) \), using Claim 10, it is enough to prove that

\[ |\Gamma_{-(A\cup B)}(A\cup B)| + |\Gamma_{-(A\cup B)}(A\cap B)| + |U(A, B)| \leq |\Gamma_{-(A\cup B)}(A)| + |\Gamma_{-(A\cup B)}(B)|. \]

However, this is obvious by the definition of \( U(A, B) \). As \( b(X) \) is the sum of \( b_0(X) \) and the modular function \( 2|X| - f(X) \), the same statement holds for \( b \) as well.

We call a nonempty subset \( X \subseteq S \) **tight** if \( b(X) = 1 \).

**Corollary 11** Suppose the condition of the theorem, i.e., \( b(X) \geq 1 \) for all \( \emptyset \neq X \subseteq S \) holds. The intersection and union of two intersecting tight sets \( A \) and \( B \) is tight, and \( |U(A, B)| = 0 \). If \( A \) and \( B \) are disjoint tight sets, and either \( U(A, B) \neq \emptyset \) or there is an edge connecting \( A \) and \( B \), then \( A\cup B \) is tight, moreover either \( |U(A, B)| = 1 \) and no edge connects \( A \) to \( B \); or \( U(A, B) = \emptyset \).

Let \( W \) denote the set \( V-S \). We may assume that every vertex \( v \in S \) is contained in a tight set, otherwise, we can increase \( f(v) \) without violating the condition. By Corollary 11, \( I(v) \), the intersection of all tight sets containing \( v \) is also a tight set. We also suppose that for every edge \( wv \) (where \( w \in W \) the graph \( G-wv \) would violate the condition, i.e., \( v \in S \) and \( d(w, I(v)) = 1 \). (If this is not the case, then we simply delete the edge \( wv \).) Unfortunately, we may not assume similar condition about edges induced by \( S \) because deleting such an edge can introduce an induced \( P_4 \).

We denote the components of \( G[S] \) by \( Z_1, \ldots, Z_t \). Call a vertex \( u \in Z_i \) **proper** if \( I(u) \subseteq Z_i \). Our first goal is to prove that every vertex in \( S \) is proper. We need some preliminary observations.

**Claim 12** Suppose tight sets \( A \) and \( B \) intersect the same component \( Z_i \) of \( G[S] \). Then \( A\cup B \) is tight. Consequently, if \( u, v \in Z_i \) and \( u \neq v \), then \( I(u) \cup I(v) \) is tight, moreover, a vertex \( w \in W \) cannot be connected to both \( u \) and \( v \).

**Proof.** Either the sets \( A \) and \( B \) are intersecting, or connected by an edge, or otherwise – using that \( G[Z_i] \) is a cograph – they have a common neighbor \( x \in Z_i \).
which is not in $A \cup B$, so $x \in U(A, B)$. Thus the first statement is a consequence of Corollary 11.

Suppose $wu$ and $wv$ are edges. As $wu$ does not enter $I(v)$ (because $wv$ is the unique edge entering $I(v)$), and $wv$ does not enter $I(u)$, we have $w \in U(I(u), I(v))$, so using Corollary 11 again, we get a contradiction. □

Lemma 13 Every vertex $u \in S$ is proper.

Proof. Suppose this is not the case and let $u \in S$ be an unproper vertex for which $I(u)$ is minimal. Let $I = I(u)$, and denote the components of $G[S]$ intersected by $I$ by $Z_1, \ldots, Z_r$. Take any vertex $v \in I$. If $I(v) \neq I$, then $I(v) \subset I$ by Corollary 11, and thus $v$ is proper by the minimality of $I$.

Let $E'' = \{wv \in E \mid w \in \Gamma_W(I), v \in I\}$. For $i = 1, \ldots, r$, let $A_i = I \cap Z_i$. First we claim that if we consider the subgraph defined by $E''$, and thereafter we contract each $A_i$ to a vertex $a_i$, then the resulting bipartite graph is a forest.

Suppose not, i.e., it contains a cycle, wlog $C'' = w_1, a_1, w_2, a_2, \ldots, w_k, a_k, w_1$ in cyclic order. Let $C$ be its “pre-image” in $G$, a cycle $w_1, v_1, x_1, u_2, w_2, v_2, x_2, u_3, w_3, \ldots, u_k, w_k, v_k, x_k, u_{k+1} = u_1, w_1,$

where $w_i \in \Gamma_W(I), v_i, u_{i+1} \in A_i$ and $x_i \in Z_i$ (the vertices $v_i$ and $u_{i+1}$ are well defined by the edges of $C''$, and we connect them with a shortest path inside $Z_i$). Note that $v_i, x_i, u_{i+1}$ are not necessarily distinct vertices, so two subsequent vertices of this sequence are either identical or connected by an edge. As every $w_i$ is connected to two contracted vertices, both $v_i$ and $u_{i+1}$ are proper vertices, otherwise, e.g., $I(v_i) = I$ and the edge $w_iv_i$ is not a unique edge from $w_i$ that enters $I(v_i)$.

Using Claim 12, the sets $B_i = I(v_i) \cup I(u_{i+1}) \subset A_i$ are tight sets. By repeatedly using Corollary 11 we get that $D_j = \cup_{i=1}^j B_i$ are tight sets for $j = 1, 2, \ldots, k - 1$; note that $D_{k-1} \subset \cup A_1 \cup \ldots \cup A_{k-1}$. Finally, we get a contradiction to Corollary 11 for disjoint tight sets $D_{k-1}$ and $B_k$ as $w_k, v_1 \in U(D_{k-1}, B_k)$.

Now we are able to finish the proof of Lemma 13. By our assumptions, $b(A_i) \geq 1$ for $i = 1, \ldots, r$, and there is a $j$ such that $u \in A_j$, we have $b(A_j) \geq 2$ as $A_j$ is not tight. Thus we have $\sum_{i=1}^r b(A_i) \geq r + 1$. We claim that $|\Gamma_W(I)| + r \leq |\Gamma_W(I)| + \sum_{i=1}^r b(A_i) - b(I) = |E''|$, this would give the required contradiction. We have $b(I) = 1$ because $I$ is tight. The sets $\Gamma_S(A_i) \subset Z_i$ are pairwise disjoint and $I = \cup A_i$; therefore $\Gamma_S(I)$ is the disjoint union of sets $\Gamma_S(A_i)$. Moreover $c_G(I) = \sum_{i=1}^r c_G(A_i)$.

Hence $\sum_{i=1}^r b(A_i) - b(I) = \sum_{i=1}^r |\Gamma_S(A_i)| + \sum_{i=1}^r |\Gamma_W(A_i)| - |\Gamma_S(I)| - |\Gamma_W(I)| = \sum_{i=1}^r |\Gamma_W(A_i)| - |\Gamma_W(I)| = \sum_{i=1}^r d(A_i, \Gamma_W(I)) - |\Gamma_W(I)|$ by the second statement of Claim 12, and $\sum_{i=1}^r d(A_i, \Gamma_W(I)) = |E''|$, so our last claim is proved, finishing the proof of the lemma. □

For any component $Z_i$ of $G[S]$, we have $Z_i = \cup_{u \in Z_i} I(u)$ by Lemma 13; thus repeated usage of Claim 12 shows that $\cup_{u \in Z_i} I(u)$ is a tight set.

Corollary 14 For every component $Z_i$ of $G[S]$, the set $Z_i$ is tight.

We make an auxiliary bipartite graph $G'$ by contracting each component of $G[S]$ (we delete the loops arising). By Claim 12 no parallel edges can arise. The
components of \( G[S] \) are \( Z_1, Z_2, \ldots, Z_t \), hence the corresponding contracted vertices of \( G' \) will be denoted by \( z_1, z_2, \ldots, z_t \).

We prove Theorem 9 by induction on the number of vertices. If \( G' \) has an isolated vertex, then this vertex is either \( w \in W \) (we simply delete \( w \) and use induction), or a vertex \( z_j \). In this latter case \( G[Z_j] \) is a component of \( G \), so we can use induction separately for \( G[Z_j] \) and for \( G - Z_j \).

If \( G' \) has a vertex of degree one, then it is either \( w \in W \) or a vertex \( z_j \). If \( w \) has degree one we take its neighbor \( u \in S \), delete \( w \) and reset \( f(u) = f(u) - 1 \). Now we can use induction, the assumption of the theorem is not violated (it may be the case that \( u \) gets outside of \( S - u \) remains a cograph). Suppose \( z_j \) has degree one in \( G' \) and \( uv \) is the unique edge leaving \( Z_j \) in \( G \), where \( u \in Z_j \) and \( v \in W \). We delete edge \( uv \), reset \( f(u) = f(u) - 1 \), and then we can use induction separately for \( G[Z_j] \) and for \( G - Z_j \), finally the edge \( uv \) can be put back safely to the union of the two resulting forests.

Otherwise, we have a cycle in \( G' \) with vertices \( w_1, z_1, w_2, z_2, \ldots, w_k, z_k \). We repeat the arguments as in the proof of Lemma 13 in order to show that this assumption leads to a contradiction. Now let \( D_j = \bigcup_{i=1}^{j} Z_i \), these are tight sets for \( j = 1, 2, \ldots, k - 1 \) by Corollaries 14 and 11, and finally we get a contradiction for tight sets \( D_{k-1} \) and \( Z_k \) as \( w_k, w_1 \in U(D_{k-1}, Z_k) \).

\[ \square \square \]

6.1 Algorithmic aspects

Egawa and Ozeki already observed that their proof is “almost” algorithmic but they were not able to give a polynomial time algorithm. They wrote: ‘we believe that there is a polynomial time algorithm to find an \((X, f)\)-tree in a graph satisfying condition (1).’

The author of the present paper can only guess at the reason of this. We think of two possibilities. Actually, they also used induction but they used the induction hypothesis for every tight set \( I \neq S \), for the same graph with \( S' = I \), thus exponentially many times. However, they did not really need the forests arising from the hypothesis, only their existence. On the other hand, they wrote: ‘To find an appropriate vertex or edge ...’ Probably they did not realize that they do not need to find an appropriate edge. It is enough to check the condition for graph \( G - wv \) for every edge \( wv \) where \( w \in W \) and \( v \in S \). If the condition still holds for \( G - wv \), then the edge \( wv \) can be deleted. If for none of the edges it holds, then their Claim 8 applies, and so they could make the recursion for \( G - wv \) for any edge between \( S \) and \( W \). However, this train of thought is not obvious at all, one must check their long proof thoroughly.

Our proof uses the inductive hypothesis only once, so we may call the forest-construction procedure for two graphs with total number \(|V|\) of the vertices. So from this proof, it is easy to conclude that by using general strongly polynomial submodular function minimization (SPSFM hereafter) of either Iwata, Fleischer and Fujishige [6] or of Schrijver [10] we have a polynomial time algorithm for constructing the desired forest if the condition of the theorem holds (Egawa and Ozeki already showed that checking the condition can be done by one call of SPSFM).

To be more precise, our algorithm is as follows. First we delete edges inside \( W \)
and check the condition for $G, S, f$. Then for each $v \in S$ we check whether the condition still holds if we increase $f(v)$ by one. If yes, then we increase $f(v)$ and continue. While the condition holds, we have $f(S) < 2|V|$, so this process can be done by at most $2|V|$ calls of SPSFM.

When none of the $f$ values can be increased, then every vertex $v \in S$ contained in a tight set. We claim that we can also get the sets $I(v)$ for all $v \in S$. This can be done in many ways, for example minimizing the submodular function $b'(X)$, where $b'(X) = |V| \cdot b(X) + |X|$ if $v \in X$ and $b'(X) = 3|V| + |V| \cdot b(X) + |X|$ otherwise. In short, after $O(|V|)$ calls of SPSFM we ensured that every vertex $v \in S$ is in a tight set and we calculated $I(v)$.

Next we check that for every edge $wv$ (where $w \in W$ and $v \in S$) $d(w, I(v)) = 1$ or not. We do not need any further call of SPSFM, this can be done in $O(|V||E|)$ steps. If $d(w, I(v)) > 1$, then we delete the edge $wv$.

Finding an isolated vertex or a leaf of $G'$ is easy. The main point for the remaining part is that we do not need to recalculate anything when making recursive calls for graphs $G[Z_i]$ and $G - Z_i$ or when we delete a vertex $w \in W$. This is because the same sets $I(v)$ do the job.

In conclusion, we can find the appropriate forest by $O(|V|)$ calls of SPSFM and by $O(|V||E|)$ simple graph operations.

## 7 Acknowledgment

Special thanks to András Frank, who, (after I wrote down my first simple proof and gave the first polynomial time algorithm), suggested the bipartite graph construction used here, in order for simpler checking the condition; and who taught me his nice algorithm.

Research is supported by a grant (no. K 109240) from the National Development Agency of Hungary, based on a source from the Research and Technology Innovation Fund.

## References

[1] Y. Egawa, K. Ozeki, Spanning trees with vertices having large degrees, *J. Graph Theory*, 79 (2015), pp. 213-221.

[2] Y. Egawa, K. Ozeki, A sufficient and necessary condition for the existence of a spanning tree with specified vertices having large degrees, *Combinatorica*, 34 (2014), pp. 47-60.

[3] U. Faigle, U. Kern, A. Kierstad, W.T. Trotter, On the game chromatic number of some classes of graphs, *Ars Combin.*, 35 (1993), pp. 143-150.

[4] A. Frank, Kombinatorikus algoritmusok, algoritmusik bizonyítások, (Combinatorial algorithms, algorithmic proofs), in Hungarian, Doctoral Thesis, Eötvös University, Budapest, 1975.
[5] W. HE, X. HOU, K.W. LIH, J. SHAO, W. WANG and X. ZHU, Edge-partitions of planar graphs and their game coloring numbers, *J. Graph Theory*, 41 (2002), pp. 307-317.

[6] S. IWATA, L. FLEISCHER, S. FUJISHIGE, A combinatorial, strongly polynomial-time algorithm for minimizing submodular functions, *J. ACM*, 48 (2001), pp. 761-777.

[7] S-J. KIM, A.V. KOSTOCHKA, H. WU, D.B. WEST and X. ZHU, Decomposition of sparse graphs into forests and a graph with bounded degree, *J. Graph Theory*, 74 (2003), pp. 369-391.

[8] D.J. KLEITMAN, Partitioning the edges of a girth 6 planar graph into those of a forest and those of a set of disjoint paths and cycles, Unpublished manuscript (2006).

[9] L. LOVÁSZ, A generalization of König’s theorem, *Acta Math. Hung.* 21 (1970), pp. 443-446.

[10] A. SCHRIJVER, A combinatorial algorithm for minimizing submodular functions in strongly polynomial time, *J. Combin. Theory Ser. B*, 80 (2000), pp. 346-355.

[11] M. SINGH and L.C. LAU, Approximating Minimum Bounded Degree Spanning Trees to within One of Optimal, in Proceedings of the 47th Annual IEEE Symposium on Foundations of Computer Science (2007), pp. 661-670.

[12] X. ZHU, The game coloring number of planar graphs, *J. Combin. Theory Ser. B*, 75 (1999), pp. 245-258.