LECH’S INEQUALITY FOR THE BUCHSBAUM-RIM MULTIPLETILITY 
AND MIXED MULTIPLETILITY 

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ABSTRACT. We generalize an improved Lech bound, due to Huneke, Smirnov, and Validashti, for the Buchsbaum-Rim multiplicity and mixed multiplicity. We reduce the problem to the graded case and then to the polynomial ring case. There we use complete reductions, studied by Rees, to prove sharper bounds for the mixed multiplicity in low dimensions.

1. INTRODUCTION 

For a Noetherian local ring \((R, m)\) of dimension \(d\) and an \(m\)-primary ideal \(I\), Lech [5, 3] proved the following bound on the Hilbert-Samuel multiplicity, \(e(I) \leq d!\lambda(R/I)e(R)\). Recently, Huneke, Smirnov, and Validashti [4, 6.1] improved the bound to \(e(mI) \leq d!\lambda(R/I/e(R))\) for \(d \geq 4\).

For a submodule \(E \subseteq F = R^r\) of a free module with \(\lambda(F/E) < \infty\) one can consider the colengths of \(E^n \subseteq F^n = \text{Sym}_n(F)\) which for \(n \gg 0\) is a polynomial of degree \(d + r - 1\) if \(E \subseteq F\). Then the Buchsbaum-Rim multiplicity of \(E\) is defined as the normalized leading coefficient of the polynomial:

\[
br(E) = \lim_{n \to \infty} \frac{(d + r - 1)!\lambda(F^n/E^n)}{n^{d+r-1}}.
\]

The Buchsbaum-Rim multiplicity generalizes the Hilbert-Samuel multiplicity. It plays a key role in the theory of equisingularities in complex-analytic geometry, see [3]. When \(E\) is a direct sum of ideals then the Buchsbaum-Rim multiplicity of \(E\) can be expressed as a sum of mixed multiplicities [1, 4.9]. If \(I_1, \ldots, I_r\) are \(m\)-primary ideals of \(R\), then for \(n_1, \ldots, n_r \gg 0\), \(\lambda(R/(I_1^{n_1} \cdots I_r^{n_r}))\) is a polynomial of degree \(d\) in \(r\) variables, \(x_1, \ldots, x_r\). For non-negative integers \(a_1, \ldots, a_r\) with \(a_1 + \ldots + a_r = d\), the coefficient of the term \(x_1^{a_1} \cdots x_r^{a_r}\) is

\[
\frac{1}{a_1! \cdots a_r!} e(I_1^{[a_1]}, \ldots, I_r^{[a_r]}).
\]

We call \(e(I_1^{[a_1]}, \ldots, I_r^{[a_r]})\) the mixed multiplicity of \(I_1, \ldots, I_r\) of type \((a_1, \ldots, a_r)\). With each \(I_i\) listed \(a_i\) times, \(e(I_1, \ldots, I_1, \ldots, I_r, \ldots, I_r)\) also denotes the mixed multiplicity of \(I_1, \ldots, I_r\) of type \((a_1, \ldots, a_r)\) [7, 17.4.3].

In this paper we generalize the improved Lech Bound to the Buchsbaum-Rim multiplicity and also to the mixed multiplicity. When \(r = 1\) then the Buchsbaum-Rim multiplicity is the Hilbert-Samuel multiplicity, hence our result generalizes that of Huneke, Smirnov, and Validashti [4, 6.1].

**Theorem 1.1.** Let \((R, m)\) be a Noetherian local ring with \(\dim R = d \geq 4\), and \(E \subseteq F = R^r\) a submodule of a free module with \(\lambda(F/E) < \infty\) and \(E \subseteq mF\). Then

\[
br(mE) < \frac{(d + r - 1)!}{r!} \lambda(F/E)e(R).
\]
We prove the theorem in several steps, using the techniques of Lech in [5] and Huneke, Smirnov, and Validashti in [3]. First we pass to the associated graded ring, as in [5], to reduce to the case where $R$ is a standard graded ring over $R/m = k$ and $E$ is a homogeneous submodule. Passing to a Noether normalization of $R$, we reduce to the case where $R$ is a polynomial ring over $k$. Then by passing to the initial module, we may assume $E$ is a direct sum of ideals. We prove bounds for the mixed multiplicity of ideals in a polynomial ring. Finally we use a formula relating the Buchsbaum-Rim multiplicity and the mixed multiplicity from [1] to obtain the desired bound.

Acknowledgement: The authors would like to thank Bernd Ulrich for suggesting the problem and for his insights while we were working on the problem. We are also grateful for his thorough reading of our drafts and for his various corrections.

2. Reduction Steps

The reduction steps are essentially due to Lech [5], but we will follow the more modern approach in [3, 3.1]. Let $(R, m, k)$ be a Noetherian local ring of dimension $d$ and $E \subseteq F = R^e$ with $\lambda(F/E) < \infty$. First we use a standard technique to assume $k$ is infinite. Let $R(\lambda) = \lambda[R[X]_{\lambda R[X]}]$. We know that $R \to R(\lambda)$ is a faithfully flat extension of rings with the same dimensions and multiplicities. Moreover, for any $R$-module $M$, $\lambda_R(M) = \lambda_{R(\lambda)}(M \otimes R(X))$. Hence, replacing $R$ with $R(\lambda)$, we may assume that $k$ is infinite.

Now let $G = \text{gr}_m(R)$. Define

$$E^* = \bigoplus_{i \geq 0} \left( E \cap m^i F + m^{i+1} F \right) / m^{i+1} F \subseteq \text{gr}_m(F)$$

which is a homogeneous submodule of $\text{gr}_m(F)$. Similarly, define

$$(E^n)^* = \bigoplus_{i \geq 0} \left( E^n \cap m^i F^n + m^{i+1} F^n \right) / m^{i+1} F^n \subseteq \text{gr}_m(\text{Sym}_n(F)) = \text{Sym}_n(\text{gr}_m(F)).$$

Comparing elements, we see that $(E^*)^n \subseteq (E^n)^*$, and $G + E^* \subseteq (mE)^*$. Now

$$br_R(mE) \leq br_G(G + E^*)$$

because for any $n$

$$\lambda_R \left( \frac{F^n}{(mE)^n} \right) = \lambda_G \left( \text{gr}_m \left( \frac{F^n}{(mE)^n} \right) \right) \leq \lambda_G \left( \frac{\text{Sym}_n(\text{gr}_m(F))}{((mE)^*)^n} \right) \leq \lambda_G \left( \frac{\text{Sym}_n(\text{gr}_m(F))}{(G + E^*)^n} \right).$$

Since $\lambda_G(\text{gr}_m(F/E)) = \lambda_G(\text{gr}_m(F/E)) = \lambda_R(F/E)$ and $e(R) = e(G)$, if the inequality in Theorem 1.1 holds for $E^* \subseteq G^* = \text{gr}_m(F)$ then it also holds for $E \subseteq F$. Therefore we may replace $E \subseteq F$ with $E^* \subseteq G^*$ and $R$ with $G$ to assume $R$ is a standard graded ring over an infinite field and $E$ is a graded submodule of $F$.

Now let $S = k[t_1, ..., t_d]$ be a homogeneous Noether normalization of $R$ with $(t_1, ..., t_d)R$ a reduction of $m$. Denote by $n = (t_1, ..., t_d)$ the maximal homogeneous ideal of $S$. Using the associativity formula and the fact that $S$ is a domain, we have $\text{rank}_S(R) = e(n, R) = e(R)$. 


Let \( \mathcal{F} = S^r \subseteq F \) and \( \mathcal{E} = E \cap \mathcal{F} \). Then we have \( \frac{E}{F} \hookrightarrow \frac{E}{E} \) and hence \( \lambda_S(\mathcal{F}/\mathcal{E}) \leq \lambda_S(F/E) = \lambda_R(F/E) \).

To establish notation, for \( N \) a finite \( S \)-module and \( M \subseteq \mathcal{F} \) a submodule with finite colength, define

\[
br_S(M, N) = \lim_{n \to \infty} (d + r - 1)! \frac{\lambda_S(\frac{F^n}{1 \cdot \cdots \cdot 1} \otimes_S N)}{n^{d+r-1}}.\]

This limit was shown to exist for the \( j \)-multiplicity in section 6.2 of [8]. Then, because the Buchsbaum-Rim multiplicity is a special case of the \( j \)-multiplicity, the limit defining \( br_S(M, N) \) exists. With this notation we will show that \( br_S(n\mathcal{E}, R) = br_R(n\mathcal{E}R) \).

Notice \( \dim R = \dim S \) and \( \text{rank}_R(F) = \text{rank}_S(\mathcal{F}) \). Hence it is enough to show that \( \lambda_S(\frac{F^n}{1 \cdot \cdots \cdot 1} \otimes_S R) = \lambda_R(\frac{E^n}{1 \cdot \cdots \cdot 1} \otimes R) \). But this follows as the two modules are isomorphic as \( R \)-modules and lengths of modules over \( S \) and \( R \) coincide.

Next, since \( \mathcal{E}R \subseteq E \) and \( nR \subseteq m \), we have \( br_R(mE) \leq br_R(n\mathcal{E}R) = br_S(n\mathcal{E}, R) \). We now use the associativity formula for \( br_S(-) \) which follows from the associativity formula for the \( j \)-multiplicity [8, 6.5.1]. Because \( S \) is a domain, the formula reduces to

\[
br_S(n\mathcal{E}, R) = br_S(n\mathcal{E})\text{rank}_S(R) = br_S(n\mathcal{E})e(R).
\]

Now if the bound holds for \( \mathcal{E} \subseteq \mathcal{F} \) then, because \( \lambda_S(\mathcal{F}/\mathcal{E}) \leq \lambda_R(F/E) \),

\[
br_R(mE) \leq br_S(n\mathcal{E})e(R) < \frac{(d + r - 1)!}{r!} \lambda_S(\mathcal{F}/\mathcal{E})e(R) \leq \frac{(d + r - 1)!}{r!} \lambda_R(F/E)e(R).
\]

This shows that we can replace \( R \) with \( S \) and \( E \subseteq F \) to assume \( R \) is a polynomial ring over an infinite field.

So, we may assume that \( R = k[x_1, \ldots, x_d] \), with \( k \) infinite, and \( E \subseteq F \) is a homogeneous submodule. Now we pass to \( in(E) \) with the lex order on \( R \) extended to \( F \) by considering term over position. From a direct generalization of [2, 15.3], we have \( \lambda(F/E) = \lambda(F/in(E)) \) and \( \lambda(F^n/E^n) = \lambda(F^n/in(E^n)) \). Further, \( \lambda(F^n/in(E^n)) \leq \lambda(F^n/in(E^n)) \) since \( in(E)^n \subseteq in(E^n) \). Hence \( br(E) \leq br(in(E)) \) and we can replace \( E \) with \( in(E) \) to assume that \( E \) is a direct sum of ideals.

In section 4 we will prove Lech type bounds for the mixed multiplicity of ideals in a polynomial ring. To obtain our results in the polynomial ring case we employ extensively Rees’s notion of complete reductions [6]. In the next section we will define complete reductions and the related notion, joint reductions. In addition, we state the various results that we will use in later sections.

3. Complete Reductions and Joint Reductions

For this section, \( (R, m, k) \) is a Noetherian local ring with infinite residue field and \( d = \dim R \).

**Definition 3.1.** [7, 17.1.3] Let \( I_1, \ldots, I_r \) be ideals of \( R \). If \( x_i \in I_i \) and \( \sum_{i=1}^r x_i I_1 \cdots \hat{I}_i \cdots I_r \) is a reduction of \( I_1 \cdots I_r \), then \( x_1, \ldots, x_r \) is a joint reduction of \( I_1, \ldots, I_r \).

As with the Hilbert-Samuel multiplicity, the mixed multiplicity is invariant under integral closure. This can be seen through joint reduction. Since \( I_1 \cdots I_r \) is a reduction of \( \overline{I}_1 \cdots \overline{I}_r \), it follows that if \( x_1, \ldots, x_r \) is a joint reduction of \( I_1, \ldots, I_r \) then it is also a joint reduction of \( \overline{I}_1, \ldots, \overline{I}_r \). It follows from [6, 2.4] that \( e(I_1^{[a_1]} \cdots \hat{I}_i \cdots I_r^{[a_r]}) = e(\overline{I}_1^{[a_1]} \cdots \overline{I}_i^{[a_i]} \cdots \overline{I}_r^{[a_r]}) \). This allows us to replace ideals with their integral closures when dealing with mixed multiplicities.
Definition 3.2. [6] Let \( U = (I_1, \ldots, I_r) \) be a set of, not necessarily distinct, ideals of \( R \). The set of elements \( \{x_{ij} : 1 \leq i \leq r, 1 \leq j \leq d, x_{ij} \in I_i \} \), with \( y_j = x_{1j} \cdots x_{rj} \), is a complete reduction of \( U \) if \((y_1, \ldots, y_d) \) is a reduction of \( I_1 \cdots I_r \).

Rees showed that complete reductions exist if \( k \) is infinite [6, 1.3]. Furthermore complete reductions are related to joint reductions by the following corollary.

Corollary 3.3. [6 (i)] Let \( r \geq d \) and \( I_1, \ldots, I_r \) be, not necessarily distinct, ideals of \( R \) such that \( I_1 \cdots I_r \) is not contained in any minimal prime ideal of \( R \). Let \( I_{n_1}, \ldots, I_{n_d} \) be a subset of \( I_1, \ldots, I_r \). Let \( (x_{ij}) \) be a complete reduction of \((I_1, \ldots, I_r)\). Set \( x_j = x_{n_jj} \), then \( x_1, ..., x_d \) is a joint reduction of \( I_{n_1}, ..., I_{n_d} \).

The next theorem shows that going modulo part of a joint reduction preserves mixed multiplicity. We will use this result often in the next section to reduce the dimension of the ring.

Theorem 3.4. [7, 17.4.6] Suppose \( d \geq 2 \) and \( |k| = \infty \). Let \( I_1, ..., I_d \) be \( m \)-primary ideals of \( R \) and let \( (x_1, ..., x_d) \) be a joint reduction of \( I_1, ..., I_d \) where \( x_1 \) is a general element. Denote by \( \rightarrow^r \) images in \( R/(x_1) \), then

\[
e(I_1, ..., I_d) = e(I'_2, ..., I'_d).
\]

The last tool that we will use is a formula relating the Buchsbaum-Rim multiplicity of a direct sum of ideals to the mixed multiplicities of the ideals.

Theorem 3.5. [11, 4.9] Let \( I_1, ..., I_r \) be \( m \)-primary ideals of \( R \). Then

\[
br(\bigoplus_{i=1}^r I_i) = \sum_{a_1 + \ldots + a_r = d} e(I_1^{[a_1]}, ..., I_r^{[a_r]}).
\]

4. Mixed Multiplicity Bounds in Polynomial Rings

In this section we prove a Lech type bound for the mixed multiplicities of ideals in a polynomial ring. We first prove technical bounds for dimensions 2 and 3. For the higher dimensional case we use complete reductions to reduce to the low dimensional case. Throughout this section \( k \) is assumed to be infinite and \( m \) denotes the homogeneous maximal ideal.

Proposition 4.1. Let \( R = k[x_1, x_2] \) and \( I_1, ..., I_r \) be \( m \)-primary ideals with \( r \geq 2 \). Let \( x \) be a general linear form and denote \( \rightarrow^r \) to be images in \( R^r = R/(x) \). Then

\[
2 \sum_{1 \leq i < j \leq r} e(I_i, I_j) + (r - 1) \sum_{i=1}^r e(I'_i) \leq 2(r - 1) \sum_{i=1}^r \lambda(R/I_i).
\]

Proof. First, notice that \( 2e(I_i, I_j) \leq e(I_i) + e(I_j) \), see [7, p. 365]. Using this inequality, we have reduced to showing

\[
(r - 1) \sum_{i=1}^r (e(I_i) + e(I'_i)) \leq 2(r - 1) \sum_{i=1}^r \lambda(R/I_i).
\]

From [11, 4.5], we have \( e(I_i) + e(I'_i) \leq 2\lambda(R/I_i) \), which gives the result. \( \square \)

When all of the ideals are powers of the maximal ideal the two bounds in the proof are sharp, hence our bound is also sharp in this case. Next we show a Lech bound for mixed multiplicity that is valid in any dimension. We will need this result to deal with a particular case in the proof of 4.3.
Proposition 4.2. Let \( R = k[x_1, ..., x_d] \) and \( I_1, ..., I_d \) be \( m \)-primary ideals. Then

\[ e(I_1, ..., I_d) \leq (d - 1)! \sum_{i=1}^{d} \lambda(R/I_i). \]

Proof. We may assume \( I_1, ..., I_d \) are integrally closed. We induct on \( d \). The base case where \( d = 1 \) is clear. Let \( d > 1 \) and we now induct on \( \sum_{i=1}^{d} \lambda(R/I_i) \). The base case is when each \( I_i = m \), which holds since \( e(m, ..., m) = 1 \). For the induction, after rearranging we may assume that \( \lambda(R/I_i) = \max\{\lambda(R/I_j) : 1 \leq i \leq d\} \). Let \( x \in m \) be a general linear form not contained in any \( I_i \neq m \). We may choose \( x \) to be part of a joint reduction of \( m, I_2, ..., I_d \). Set \( \tilde{I}_1 = I_1 : x \). Denote \(-'\) to be images in \( R' = R/(x) \). By [3] 2.3 \( m\tilde{I}_1 \subseteq I_1 \), hence \( e(I_1, ..., I_d) \leq e(m\tilde{I}_1, ..., I_d) \). Expanding the mixed multiplicity using [6, 2.5], applying 3.4, and induction we have

\[
e(I_1, ..., I_d) \leq e(m\tilde{I}_1, ..., I_d) = e(m, I_2, ..., I_d) + e(\tilde{I}_1, ..., I_d) \\
= e(I'_2, ..., I'_d) + (d - 1)! \left( \lambda(R/\tilde{I}_1) + \sum_{i=2}^{d} \lambda(R/I_i) \right) \\
\leq (d - 2)! \sum_{i=2}^{d} \lambda(R'/I'_i) + (d - 1)! \left( \lambda(R/\tilde{I}_1) + \sum_{i=2}^{d} \lambda(R/I_i) \right) \\
\leq (d - 1)! \lambda(R'/I'_1) + (d - 1)! \left( \lambda(R/\tilde{I}_1) + \sum_{i=2}^{d} \lambda(R/I_i) \right) \\
= (d - 1)! \sum_{i=1}^{d} \lambda(R/I_i).
\]

For the last equality we use [3, 2.6] to get \( \lambda(R'/I'_1) + \lambda(R/\tilde{I}_1) = \lambda(R/I_1) \). \( \square \)

Now we can use the above inequality and the dimension two result to prove a Lech type bound for a polynomial ring in three variables.

Proposition 4.3. Let \( R = k[x_1, x_2, x_3] \) and \( I_1, ..., I_4 \) be \( m \)-primary ideals. Let \( x, y \) be general linear forms. Denote \(-'\) and \(-''\) to be images in \( R' = R/(x) \) and \( R'' = R/(x, y) \) respectively. Then

\[
\sum_{1 \leq i < j < k \leq 4} e(I_i, I_j, I_k) + \sum_{1 \leq i < j \leq 4} e(I'_i, I'_j) + \sum_{i=1}^{4} e(I''_i) + 1 \leq 6 \sum_{i=1}^{4} \lambda(R/I_i).
\]

Proof. We may assume each \( I_i \) is integrally closed. We induct on \( \sum_{i=1}^{4} \lambda(R/I_i) \). In the base case, each \( I_i = m \), and the result holds. We now have two cases. In the first case, where all of the ideals are not \( m \), we will continue the induction. In the later case, where at least one ideal is \( m \), we will show the inequality directly.

(1) Suppose all \( I_i \neq m \). Let \( \tilde{I}_i = I_i : x \). As each \( I_i \) is integrally closed, by [4, 2.3] \( m\tilde{I}_i \subseteq I_i \). Hence the mixed multiplicity cannot decrease whenever we replace \( I_i \) with \( m\tilde{I}_i \). We will use this to estimate the terms in the first summation. First we replace \( I_1, I_2 \) with \( m\tilde{I}_1, m\tilde{I}_2 \) and then apply [6, 2.5] and 3.4. Afterwards we replace \( I_3, I_4 \) with \( m\tilde{I}_3, m\tilde{I}_4 \) and apply the two results again.
\[
\sum_{1 \leq i < j < k \leq 4} e(I_i, I_j, I_k) \leq e(m \tilde{I}_1, m \tilde{I}_2, I_3) + e(m \tilde{I}_1, m \tilde{I}_2, I_4) + e(m \tilde{I}_1, I_3, I_4) + e(m \tilde{I}_2, I_3, I_4) \\
\leq e(\tilde{I}_1, \tilde{I}_2, I_3) + e(\tilde{I}_1, \tilde{I}_2, I_4) + e(\tilde{I}_1, I_3, I_4) + e(\tilde{I}_2, I_3, I_4) + e(\tilde{I}_1', \tilde{I}_2) + e(\tilde{I}_2', I_3) + e(\tilde{I}_2', I_4) + 2e(I_3', I_4') + e(I_3'') + e(I_4'') \\
\leq e(\tilde{I}_1, \tilde{I}_2, m \tilde{I}_3) + e(\tilde{I}_1, \tilde{I}_2, m \tilde{I}_4) + e(\tilde{I}_1, m \tilde{I}_3, m \tilde{I}_4) + e(\tilde{I}_2, m \tilde{I}_3, m \tilde{I}_4) \\
+ e(\tilde{I}_1', \tilde{I}_3') + e(\tilde{I}_1', \tilde{I}_4') + e(\tilde{I}_2', \tilde{I}_3') + e(\tilde{I}_2', \tilde{I}_4') + e(\tilde{I}_1'', \tilde{I}_2) + e(\tilde{I}_2'', \tilde{I}_2') + e(\tilde{I}_1', \tilde{I}_3') + e(\tilde{I}_3', \tilde{I}_4') + e(\tilde{I}_1'', \tilde{I}_3') + e(\tilde{I}_2', \tilde{I}_3') + e(\tilde{I}_1'', \tilde{I}_4') + e(\tilde{I}_2', \tilde{I}_4') + 2e(I_3', I_4') + e(I_3'') + e(I_4'').
\]

There are a couple of terms missing for us to use induction on \( \tilde{I}_i \), however we can bring in the missing terms by estimating the term \( 2e(I_3', I_4') \).

\[
2e(I_3', I_4') \leq e(m' \tilde{I}_3', I_4') + e(I_3', m' \tilde{I}_4') \\
\leq e(\tilde{I}_3', I_4') + e(I_3''') + e(I_3'') + e(I_4'') \\
\leq e(\tilde{I}_3', m' \tilde{I}_4') + e(I_3'') + e(m' \tilde{I}_3', \tilde{I}_4') + e(I_4'') \\
\leq 2e(\tilde{I}_3', \tilde{I}_4') + e(\tilde{I}_3'') + e(\tilde{I}_4'') + e(I_3'') + e(I_4'').
\]

Now, because \( I_i \subseteq \tilde{I}_i \), we have \( e(\tilde{I}_i, I_i') \leq e(I_i, I_i') \) and \( e(\tilde{I}_i', \tilde{I}_i) \leq e(I_i', I_i') \). Using this and putting together the estimations above we get

\[
\sum_{1 \leq i < j < k \leq 4} e(I_i, I_j, I_k) \leq \sum_{1 \leq i < j < k \leq 4} e(\tilde{I}_i, \tilde{I}_j, \tilde{I}_k) + \sum_{1 \leq i < j < k \leq 4} e(\tilde{I}_i', \tilde{I}_j') + \sum_{i=1}^4 e(\tilde{I}_i'') \\
+ \sum_{1 \leq i < j \leq 4} e(I_i', I_j') + 2 \sum_{i=1}^4 e(I_i''') \\
\leq 6 \sum_{i=1}^4 \lambda(R/\tilde{I}_i) - 1 + \sum_{1 \leq i < j \leq 4} e(I_i', I_j') + 2 \sum_{i=1}^4 e(I_i''').
\]

We use induction for the last inequality. Finally we have

\[
\sum_{1 \leq i < j < k \leq 4} e(I_i, I_j, I_k) + \sum_{1 \leq i < j < k \leq 4} e(I_i', I_j') + \sum_{i=1}^4 e(I_i''') + 1 \\
\leq 6 \sum_{i=1}^4 \lambda(R/\tilde{I}_i) + 2 \sum_{1 \leq i < j \leq 4} e(I_i', I_j') + 3 \sum_{i=1}^4 e(I_i''').
\]

The result follows by applying 4.1 and [H 2.6].

(2) For the second case, assume that \( I_4 = m \). Then the inequality we need to show is

\[
e(I_1, I_2, I_3) + 2 \sum_{1 \leq i < j \leq 3} e(I_i', I_j') + 2 \sum_{i=1}^3 e(I_i''') + 2 \leq 6 \sum_{i=1}^3 \lambda(R/I_i) + 6.
\]
Theorem 4.4. We may assume that each 

\[ \sum_{i=1}^{3} e(I'_i, I'_j) + 2 \sum_{i=1}^{3} e(I''_i) \leq 4 \sum_{i=1}^{3} \lambda(R/I_i). \]

It remains to show that \( e(I_1, I_2, I_3) \leq 2 \sum_{i=1}^{3} \lambda(R/I_i) \), which is given by 4.2. \( \square \)

The next result is a generalization of [4, 6.1] for mixed multiplicities of ideals in a polynomial ring.

Theorem 4.4. Let \( R = k[x_1, ..., x_d] \) with \( d \geq 4 \). For \( m \)-primary ideals \( I_1, ..., I_d \),

\[ e(mI_1, ..., mI_d) < (d-1)! \sum_{i=1}^{d} \lambda(R/I_i). \]

Proof. We may assume that each \( I_i \) is integrally closed. We proceed by induction on the dimension. The base case of \( d = 4 \) will be handled last. Now assume \( d > 4 \). Next, we induct on \( \sum_{i=1}^{d} \lambda(R/I_i) \). The base case is where \( \sum_{i=1}^{d} \lambda(R/I_i) = d \). Then every \( I_i = m \), so \( e(mI_1, ..., mI_d) = e(m^2) = 2^d < d! \) when \( d \geq 4 \).

For the induction, take \( x \in m \) to be a general linear form not contained in any \( I_i \neq m \).

We may assume that \( I_1 \) is not \( m \) and that \( \lambda(R'/I'_1) = \max \{ \lambda(R'/I'_i) : 1 \leq i \leq d \} \). Define \( \tilde{I}_1 = I_1 : x \). By induction,

\[ e(mI_1, ..., mI_d) = e(I_1, mI_2, ..., mI_d) + e(m, mI_2, ..., mI_d) \]
\[ \leq e(m\tilde{I}_1, mI_2, ..., mI_d) + e(mI'_2, ..., mI'_d) \]
\[ < (d-1)! \sum_{i=2}^{d} \lambda(R/I_i) + (d-1)! \lambda(R/\tilde{I}_1) + (d-2)! \sum_{i=2}^{d} \lambda(R'/I'_i) \]
\[ \leq (d-1)! \sum_{i=2}^{d} \lambda(R/I_i) + (d-1)! \lambda(R/\tilde{I}_1) + (d-2)! \sum_{i=2}^{d} \lambda(R'/I'_i) \]
\[ \leq (d-1)! \sum_{i=2}^{d} \lambda(R/I_i) + (d-1)! \lambda(R/\tilde{I}_1) + (d-1)! \lambda(R'/I'_i) \]
\[ = (d-1)! \sum_{i=1}^{d} \lambda(R/I_i). \]

It remains to show the case for \( d = 4 \). We will use complete reductions to reduce to the result in dimension 3.

For \( d = 4 \), we induct on \( \sum_{i=1}^{4} \lambda(R/I_i) \). First, if \( \sum_{i=1}^{4} \lambda(R/I_i) = 4 \), then each \( I_i = m \), so the result holds. Again, we may assume that \( I_1 \) is not \( m \) and that the image of \( I_1 \) has maximum colength in \( R' \). Now, let \( (x_{ij}) \) be a complete reduction of \( (m, m, m, I_1, I_2, I_3, I_4) \) where \( x_{11}, x_{22}, \) and \( x_{33} \) are general linear forms. Furthermore, we may choose \( x_{11} \) not contained in any \( I_i \neq m \). Let \( -', -'' \), and \( -''' \) denote images in \( R' = R/(x_{11}), R'' = R/(x_{11}, x_{22}), \) and \( R''' = R/(x_{11}, x_{22}, x_{33}) \), respectively. Using [3, 2.5] and 3.4 we expand
the mixed multiplicity as
\[
e(mI_1, mI_2, mI_3, mI_4) = e(I_1, I_2, I_3, I_4) + \sum_{1 \leq i < j < k \leq 4} e(I'_i, I'_j, I'_k) + \sum_{1 \leq i < j \leq 4} e(I''_i, I''_j) + \sum_{i=1}^4 e(I'''_i) + 1.
\]

Set \( \tilde{I}_i = I_i : x \) where again \( x \in m \) is a general linear form not contained in any \( I_i \neq m \). We have four cases to consider.

1. Suppose \( I_i \neq m \) for all \( i \). Then \( \tilde{I}_i \neq R \) for all \( i \). Using the induction hypothesis, we have
\[
e(mI_1, mI_2, mI_3, mI_4)
\leq e(m\tilde{I}_1, m\tilde{I}_2, m\tilde{I}_3, m\tilde{I}_4) + \sum_{1 \leq i < j < k \leq 4} e(I'_i, I'_j, I'_k) + \sum_{1 \leq i < j \leq 4} e(I''_i, I''_j) + \sum_{i=1}^4 e(I'''_i) + 1
\leq 6 \sum_{i=1}^4 \lambda(R/\tilde{I}_i) + \sum_{1 \leq i < j < k \leq 4} e(I'_i, I'_j, I'_k) + \sum_{1 \leq i < j \leq 4} e(I''_i, I''_j) + \sum_{i=1}^4 e(I'''_i) + 1.
\]
We have the result, after applying \( [1] \ 2.6 \), if the following holds
\[
\sum_{1 \leq i < j < k \leq 4} e(I'_i, I'_j, I'_k) + \sum_{1 \leq i < j \leq 4} e(I''_i, I''_j) + \sum_{i=1}^4 e(I'''_i) + 1 \leq 6 \sum_{i=1}^4 \lambda(R'/I'_i).
\]
This is the result of 4.3 and completes this case.

2. Suppose \( I_i = m \) for exactly one \( i \). We may assume \( I_4 = m \), so then \( \tilde{I}_4 = R \). Then we have
\[
e(mI_1, mI_2, mI_3, mI_4)
\leq e(m\tilde{I}_1, m\tilde{I}_2, m\tilde{I}_3, m) + \sum_{1 \leq i < j < k \leq 4} e(I'_i, I'_j, I'_k) + \sum_{1 \leq i < j \leq 4} e(I''_i, I''_j) + \sum_{i=1}^4 e(I'''_i) + 1
\leq 6 \sum_{i=1}^4 \lambda(R/\tilde{I}_i).
\]
As in the first case, by using \( [1] \ 2.6 \), it suffices to show
\[
e(m\tilde{I}_1, m\tilde{I}_2, m\tilde{I}_3, m) < 6 \sum_{i=1}^4 \lambda(R/\tilde{I}_i).
\]
Now, by \( [3] \ 2.5 \) and 3.4, we have
\[
e(m\tilde{I}_1, m\tilde{I}_2, m\tilde{I}_3, m) = e(\tilde{I}_1', \tilde{I}_2', \tilde{I}_3') + \sum_{1 \leq i < j \leq 3} e(\tilde{I}_i'', \tilde{I}_j'') + \sum_{i=1}^3 e(\tilde{I}_i''') + 1.
\]
Define \( J'_i = \tilde{I}_i' \) for \( i = 1, 2, 3 \). Define \( J'_4 = mR' \). Then, by 4.3, we have
\[
\sum_{1 \leq i < j < k \leq 4} e(J'_i, J'_j, J'_k) + \sum_{1 \leq i < j \leq 4} e(J''_i, J''_j) + \sum_{i=1}^4 e(J'''_i) + 1 \leq 6 \sum_{i=1}^4 \lambda(R'/J'_i).
\]
Now, because \( J'_4 = m \), we have the following two inequalities
\[
6 \sum_{i=1}^4 \lambda(R'/J'_i) \leq 6 + 6 \sum_{i=1}^3 \lambda(R/\tilde{I}_i),
\]
we have

\[ e(\tilde{I}_1', \tilde{I}_2', \tilde{I}_3') + \sum_{1 \leq i < j < k \leq 3} e(\tilde{I}_i'', \tilde{I}_j'') + \sum_{i=1}^{3} e(\tilde{I}_i''') + 8 \]

\[ \leq \sum_{1 \leq i < j < k \leq 4} e(J'_i, J'_j, J'_k) + \sum_{1 \leq i < j \leq 4} e(J''_i, J''_j) + \sum_{i=1}^{4} e(J'''_i) + 1. \]

Hence,

\[ e(\tilde{I}_1', \tilde{I}_2', \tilde{I}_3') + \sum_{1 \leq i < j < k \leq 3} e(\tilde{I}_i'', \tilde{I}_j'') + \sum_{i=1}^{3} e(\tilde{I}_i''') + 1 < 6 \sum_{i=1}^{4} \lambda(\mathcal{R}/\tilde{I}_i). \]

(3) Suppose \( I_i = m \) for exactly two values of \( i \). We may assume \( I_3 = I_4 = m \). As previously, it suffices to show

\[ e(m\tilde{I}_1, m\tilde{I}_2, m, m) < 6 \sum_{i=1}^{4} \lambda(\mathcal{R}/\tilde{I}_i). \]

By [6] 2.5 and 3.4, we have

\[ e(m\tilde{I}_1, m\tilde{I}_2, m, m) = e(\tilde{I}_1'', \tilde{I}_2'') + e(\tilde{I}_1''') + e(\tilde{I}_2''') + 1. \]

As in the proof of 4.1, using [7] p. 365 and [4] 4.5, we have

\[ e(\tilde{I}_1'', \tilde{I}_2'') + e(\tilde{I}_1''') + e(\tilde{I}_2''') + 1 \leq 2 \sum_{i=1}^{2} \lambda(\mathcal{R}/\tilde{I}_i) + 1. \]

Hence,

\[ e(m\tilde{I}_1, m\tilde{I}_2, m, m) < 6 \sum_{i=1}^{4} \lambda(\mathcal{R}/\tilde{I}_i). \]

(4) Last, suppose \( I_i \subseteq m \) and \( I_i = m \) for \( i = 2, 3, 4 \). Once more, it suffices to show

\[ e(m\tilde{I}_1, m, m, m) < 6 \sum_{i=1}^{4} \lambda(\mathcal{R}/\tilde{I}_i). \]

Now, \( e(m\tilde{I}_1, m, m, m) = 1 + e(\tilde{I}_1, m, m, m) = 1 + e(\tilde{I}_1''') \) and \( e(\tilde{I}_1''') \leq \lambda(\mathcal{R}''/\tilde{I}_1''') \). Hence,

\[ e(m\tilde{I}_1, m, m, m) \leq 2 \sum_{i=1}^{4} \lambda(\mathcal{R}/\tilde{I}_i), \]

which gives the desired result. \( \square \)

5. PROOF OF THE MAIN THEOREM

**Theorem 5.1.** Let \( (R, m) \) be a Noetherian local ring with \( \text{dim} \ R = d \geq 4 \), and \( E \subseteq F = R^r \) a submodule of a free module with \( \lambda(F/E) < \infty \) and \( E \subseteq mF \). Then

\[ br(mE) < \frac{(d + r - 1)!}{r!} \lambda(F/E)e(R). \]

**Proof.** As shown in section two we reduce to the case where \( R \) is a polynomial ring over an infinite field, and \( E = \oplus_{i=1}^{r} I_i \) is a direct sum of \( m \)-primary ideals. Applying 3.5 and 4.4 we have

\[ br(mE) = \sum_{a_1 + \ldots + a_r = d} e(mI_1^{[a_1]}, \ldots, mI_r^{[a_r]}) < (d - 1)! \sum_{a_1 + \ldots + a_r = d} \left( \sum_{i=1}^{r} a_i \lambda(R/I_i) \right). \]
Writing the colengths as a vector \( \lambda = \langle \lambda(R/I_1), ..., \lambda(R/I_r) \rangle \), we rewrite the sum as

\[
\sum_{a_1 + \ldots + a_r = d} \left( \sum_{i=1}^{r} a_i \lambda(R/I_i) \right) = \sum_{a_1 + \ldots + a_r = d} \langle a_1, ..., a_r \rangle \cdot \lambda = \lambda \cdot \sum_{a_1 + \ldots + a_r = d} \langle a_1, ..., a_r \rangle.
\]

By symmetry each component of the sum of the vectors are the same

\[
\sum_{a_1 + \ldots + a_r = d} \langle a_1, ..., a_r \rangle = \langle c, ..., c \rangle.
\]

We will compute \( c \). Summing up the components of the above equation gives

\[
rc = \sum_{a_1 + \ldots + a_r = d} a_1 + \ldots + a_r = \sum_{a_1 + \ldots + a_r = d} d = \binom{d + r - 1}{r - 1} d
\]

\[
c = \frac{(d + r - 1)!}{r!(d - 1)!}.
\]

Putting everything together yields

\[
br(mE) < (d - 1)! \sum_{a_1 + \ldots + a_r = d} \left( \sum_{i=1}^{r} a_i \lambda(R/I_i) \right) = (d - 1)! \sum_{i=1}^{r} \lambda(R/I_i) c
\]

\[
= (d - 1)! \lambda(F/E) \frac{(d + r - 1)!}{r!(d - 1)!} = \frac{(d + r - 1)!}{r!} \lambda(F/E).
\]

\( \square \)

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