Absolutely continuous copulas with prescribed support constructed by differential equations, with an application in toxicology

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ABSTRACT
A new method for constructing absolutely continuous two-dimensional copulas by differential equations is presented. The copulas are symmetric with respect to reflection in the opposite diagonal. The support of the copula density may be prescribed to arbitrary opposite symmetric hypographs of invertible functions, containing the diagonal. The method is applied to toxicological probit modeling, where new compatibility conditions for the probit parameters are derived.

1. Introduction and main results

This article is motivated by the following result, which is probably well known, although we have not been able to find any explicit statement or proof:

**Proposition 1.1.** Suppose that \( a, \Delta \in \mathbb{R}, a > 0 \). Then, there exists random variables \( X, Y \) satisfying

\[
Y \leq aX + \Delta \quad \text{and} \quad X, Y \text{ standard normal}
\]

if and only if \( a = 1 \) and \( \Delta \geq 0 \), and then if \( \Delta > 0 \), there exists \( X, Y \) with absolutely continuous joint distribution satisfying (1.1), with density \( p(x, y) \) that is continuous except for \( y = x + \Delta \).

A proof is given at the end of this section. **Proposition 1.1** is cast with a generic formulation, presumably allowing it to be utilized in a variety of problems. More specifically, the application that triggered this study originated from toxicology modeling and is accounted for in **Section 7** where we prove new compatibility conditions for toxicological probit models, derived from **Proposition 1.1**. Briefly, in this application, \( X \) and \( Y \) represent random probit value thresholds \( c_i, c_{i+1} \) that represent the transition into consecutive injury levels \( i \) and \( i+1 \), e.g., “severe injury” and “death,” for individuals in a population exposed to toxic airborne substances. The probit values for an individual exposed to a possibly time-dependent concentration \( c(t) \) are of the form \( \Gamma_i(t) = \)
\[ x_i + \beta_i \log \left( \int_0^t c(t)^n dt \right), \] where \( x_i, \beta_i, n_i \) are parameters depending on toxic substance and injury level. In a simulation, as time proceeds, each exposed individual yield increasing probit values, and when threshold values are reached, individuals are moved to the subsequent injury level. To guarantee that the injury levels are passed in the correct order, a condition of the form \( (1.1) \) must be satisfied, leading to compatibility conditions for the probit parameters \( x_i, \beta_i, n_i \) by Proposition 1.1. These conditions are stated in Theorem 7.2 and Theorem 7.3, which are the main results for the toxicology part of the article, presented in more details in Section 7. For such simulation and model purposes, we are also interested in constructing absolute continuous distributions of Proposition 1.1:

**Problem 1.2.** Given a number \( \Delta > 0 \), construct a pair of standard normal random variables \( X, Y \) with absolutely continuous joint distribution supported on \( y \leq x + \Delta \), with density \( p(x, y) \) that is continuous except for \( y = x + \Delta \).

A solution is provided by our main result Theorem 1.4. Combined with the compatibility conditions on the probit parameters in the toxicology part, this article provides a complete reference for an agent–based probit model.

Problem 1.2 seems to be a very simple and basic problem in probability theory, but to our surprise we could not find any simple constructions in the literature. Independent standard normal \( X, Y \) have absolutely continuous joint distribution but do not fulfill the support condition, and truncating to \( y \leq x + \Delta \) yields non normal marginals. It is easy to construct singular solutions to the problem, the simplest being \( X=Y \). We reduce Problem 1.2 to a problem of the dependence structure, or copula (Nelsen 2006, Lemma 3.1), one can construct absolutely continuous copulas fulfilling the requirements of Problem 1.2, except the required continuity of the density. Before we state our main result, let us briefly review the main facts about copulas.

A function \( C: [0,1]^2 \rightarrow [0,1] \) is said to be a copula if \( C(u,0) = C(0,v) = 0, C(u,1) = u, C(1,v) = v \) and \( C(u_2, v_2) - C(u_2, v_1) - C(u_1, v_2) + C(u_1, v_1) \geq 0 \) for all \( u, v, u_1, v_1, u_2, v_2 \in [0,1] \) such that \( u_1 \leq u_2, v_1 \leq v_2 \), cf. Nelsen (2006, Definition 2.2.2). By Sklar’s theorem (Nelsen (2006, Theorem 2.3.3)), the cumulative distribution function (CDF) \( F_{X,Y} \) of any bivariate random variable \( (X, Y) \) is representable by the marginal CDF’s \( F_X, F_Y \), and a copula \( C \) as

\[
F_{X,Y}(x,y) = C(F_X(x), F_Y(y)) \tag{1.2}
\]

This may be regarded as a change of variables \( X = F_X^{-1}(U), Y = F_Y^{-1}(V) \) such that \( (U, V) \) has uniform marginals, where \( F^{-1} \) denotes the generalized inverse, so called quantile function. The copula \( C \) is uniquely defined on \( \text{Range}(F_X) \times \text{Range}(F_Y) \) for all bivariate random variables \( (X, Y) \), and if \( F_X, F_Y \) are continuous, \( C \) is uniquely defined on \( [0,1]^2 \). Moreover, the partial derivatives \( C_u, C_v \) of a copula \( C(u, v) \) are defined almost everywhere on \( [0,1]^2 \) (Nelsen 2006, Theorem 2.2.7)) and the function (where defined) \( u \mapsto C_u(u,v) \) and \( v \mapsto C_v(u,v) \) are non decreasing. If the mixed derivative is defined almost everywhere and \( \int \int C''_{uv} dudv = 1 \), \( C \) is an absolutely continuous copula. Copulas are common in statistical modeling, in particular mathematical finance. The main benefit of copulas is that by Sklar’s theorem, the marginal statistics and dependence
structure can be modeled separately. For an introduction to copulas we refer to Nelsen (2006), for a recent review see Flores et al. (2017).

Returning to Problem 1.2, the half-plane \( \{ (x, y) : y \leq x + \Delta \} \) is symmetric with respect to reflection \( (x, y) \mapsto (-y, -x) \) through the line \( x + y = 0 \). Therefore, we assume that \( (X, Y) \) and \( (-Y, -X) \) are equal in distribution. Moreover, \( X, -X, Y, -Y \) are all identically distributed so it follows (from Theorem 2.4 below) that the copula \( C(u, v) \) of \( (X, Y) \) is opposite symmetric, according to the following definition.

**Definition 1.3.** A copula \( C \) is said to be opposite symmetric if

\[
C(u, v) = C(1 - v, 1 - u) + u + v - 1
\]

for all \((u, v) \in [0, 1]^2\).

Opposite symmetry means symmetry with respect to reflection \( (u, v) \mapsto (1 - v, 1 - u) \) in the opposite diagonal \( u + v = 1 \) and was introduced in De Baets, De Meyer, and Ubeda-Flores (2009). Applying the copula transformation, using the standard normal CDF \( \Phi \):

\[
u = \Phi(x), \quad v = \Phi(y), \quad F_{X,Y}(x, y) = C(u, v)
\]

Problem 1.2 reduces to finding an absolutely continuous opposite symmetric copula \( C(u, v) \) with density supported on \( \{ (u, v) \in [0, 1]^2 : v \leq H(u) \} \) where

\[
H(u) = \Phi(\Phi^{-1}(u) + \Delta)
\]

Our main result is the construction of \( C(u, v) \) in the following Theorem 1.4. We want to emphasize its simplicity, involving \( H \) and its inverse explicitly. The crucial part is the evaluation of the integral in (1.10), which is suitable for numerical integration if not analytically integrable.

**Theorem 1.4.** Suppose that \( 0 < u_0 < 1/2 \) and that \( H \) is a strictly increasing continuous function defined on \([0, 1]\), continuously differentiable on \((0, u_0)\), satisfying

\[
H(u_0) = 1 - u_0
\]

and the symmetry condition

\[
H(u) + H^{-1}(1 - u) = 1
\]

Furthermore, suppose that

\[
H(u) > u, \quad u \in (0, 1)
\]

and

\[
\lim_{u \to 1} \int_{u_0}^{u} \frac{dz}{H(z) - z} = \infty
\]

Let

\[
G(v) = \exp \left( - \int_{u_0}^{1-v} \frac{dz}{H(z) - z} \right), \quad v \in [0, 1 - u_0]
\]
\[ K(u) = \int_{u_0}^{u} \frac{H'(z)dz}{G(1-z)}, \quad u \in [u_0, 1] \]  
(1.11)

and

\[ F(u) = (1 - 2u_0)(1 - G(1 - u)), \quad u \in [u_0, 1] \]  
(1.12)

Let \( C(u, v) \) be given by

\[ C(u, v) = \int_{u_0}^{u} \int_{v_0}^{v} p(w, z)dwdz \]  
(1.13)

where

\[ p(u, v) = \begin{cases} 
G'(v)/G(H(u)) & \text{if } 0 < u \leq u_0, \quad 0 < v \leq H(u) \\
0 & \text{if } 0 < u \leq u_0, \quad H(u) < v \leq 1 - u \\
F'(u)G'(v) & \text{if } u_0 < u < 1, \quad 0 < v \leq 1 - u \\
p(1 - v, 1 - u) & \text{if } 0 < u < 1, \quad 1 - u < v < 1 
\end{cases} \]  
(1.14)

Then, \( C(u, v) \) is an absolutely continuous opposite symmetric copula with probability density \( p \) supported on \( v \leq H(u) \), and

1. If \( 0 \leq u \leq u_0 \) and \( 0 \leq v \leq H(u) \), then

\[ C(u, v) = H^{-1}(v) + (K(1 - v) - K(H^{-1}(1 - u)))G(v) \]  
(1.15)

2. If \( 0 \leq u \leq u_0 \) and \( H(u) \leq v \leq 1 - u \) then

\[ C(u, v) = u \]  
(1.16)

3. If \( u_0 \leq u \leq 1 \) and \( 0 \leq v \leq 1 - u \) then

\[ C(u, v) = H^{-1}(v) + (K(1 - v) + F(u))G(v) \]  
(1.17)

4. If \( 0 \leq u \leq 1 \) and \( u + v > 1 \) then \( C(u, v) \) is given by (1.3).

Note that the hypograph \( v \leq H(u) \) is opposite symmetric if and only if (1.7) holds true. The copula is piecewisely defined, on parts of the unit square depicted in Figure 1. Theorem 1.4 is proved at the end of Section 5. Before that, we develop a theory for construction of opposite symmetric copulas by differential equations in Sections 3 and 5, which we believe is of interest in its own right, and gives in fact a much larger class of copulas than Theorem 1.4. In Section 4, we compare our method to two other methods in the literature, Durantes and Jaworski’s construction of absolutely continuous copulas with given diagonal section (Durante and Jaworski 2008), and Jaworski’s characterization of copulas using differential equations (Jaworski 2014). In Section 6, we adapt our differential equation method to sampling from the copula. We conclude the article with Section 7, an application in toxicological probit modeling, where new compatibility conditions for the probit coefficients are derived.

**Example 1.5.** In this example, we construct a solution to Problem 1.2 using Theorem 1.4. Let \( \Phi \) be the standard normal CDF, \( \phi(x) = \Phi'(x) \) the standard normal probability density function (PDF), \( \Delta > 0 \) and \( H \) given by (1.5). Then, \( H^{-1}(v) = \Phi(\Phi^{-1}(v) - \Delta) \) and because of the symmetries \( \Phi(x) + \Phi(-x) = 1, \Phi^{-1}(u) + \Phi^{-1}(1 - u) = 0 \), condition (1.7) is satisfied, and
Moreover, with the change of variables $z = U(w)$ and the mean value theorem we obtain

$$
\int_{u_0}^{u} \frac{dz}{H(z) - z} = \int_{-\Delta/2}^{\Phi^{-1}(u)} \frac{\phi(w)dw}{\Phi(w + \Delta) - \Phi(w)}
$$

$$
= \int_{-\Delta/2}^{\Phi^{-1}(u)} \frac{\phi(w)dw}{\Phi(w + \theta(w)\Delta)} = \frac{1}{\Delta} \int_{-\Delta/2}^{\Phi^{-1}(u)} \exp\left(\frac{w\Delta\theta(w) + \frac{\Delta^2\theta(w)^2}{2}}{2}\right)dw
$$

for some function $\theta(w)$ with $0 \leq \theta(w) \leq 1$. The integrand in the last integral is estimated from below by $\exp\left(-\Delta^2/2\right)$ for $w \geq -\Delta/2$, so integration yields

$$
\int_{u_0}^{u} \frac{dz}{H(z) - z} \geq \frac{e^{-\Delta^2/2}}{\Delta} \left(\Phi^{-1}(u) + \frac{\Delta}{2}\right)
$$

which proves that condition (1.9) is satisfied. The function $G$ defined by Equation (1.10) cannot be expressed in terms of special functions (to our knowledge), but can be determined by numerical integration, and $C(u, v)$ is then determined by Equations (1.3) and (1.15)–(1.17). The density of $C$ is illustrated in Figure 2. The joint PDF of $(X, Y)$ is given by

$$
p(x, y) = C_{uv}(\Phi(x), \Phi(y)) \phi(x) \phi(y)
$$

and is illustrated in Figure 3. Here, $G(v)$ is computed with the MATLAB® function integral at 400 uniformly distributed grid points on $[\epsilon, 1 - u_0]$, and computed at

\begin{align*}
u_0 &= \Phi(-\Delta/2) \\
\end{align*}

Figure 1. Parts of the unit square for piecewise definition of the copula in Theorem 1.4.
intermediate points on \([\epsilon, 1 - u_0]\) by spline interpolation, where \(\epsilon = 10^{-11}\). Consequently, the copula and its density is computed on \([\epsilon, 1 - \epsilon]^2\).

**Proof of Proposition 1.1.** If (1.1) is satisfied then \(\Phi((y - \Delta)/a) = P\{aX + \Delta \leq y\} \leq P\{Y \leq y\} = \Phi(y)\) for all \(y \in \mathbb{R}\), which is possible only if \(a = 1\) and \(\Delta \geq 0\). For \(\Delta \geq 0\), we can take \(X = Y\), which gives a singular distribution supported on \(x = y\). If \(\Delta > 0\), Example 1.5 shows that \(X, Y\) with absolutely continuous joint distribution exists. \(\square\)

### 2. Symmetries and copulas

Several notions of bivariate symmetries are considered in Nelsen (1993). A pair of random variables \((X, Y)\) are said to be **exchangeable** if \((X, Y)\) and \((Y, X)\) are equal in distribution, and \((X, Y)\) is exchangeable if and only if its copula \(C(u, v)\) is a symmetric function, i.e., \(C(u, v) = C(v, u)\). Moreover, \((X, Y)\) is said to be **radially symmetric** about \((a, b) \in \mathbb{R}^2\) if \((X - a, Y - b)\) and \((a - X, b - Y)\) are equal in distribution, or equivalently,
Also, \((X, Y)\) is said to be \(\text{marginally symmetric}\) about \(\left(\frac{a}{C_0}, \frac{b}{C_0}\right)\) if
\[
F_X(\frac{a+x}{C_0}) = \frac{1}{C_0} F_X(\frac{a}{C_0}) \quad \text{and} \quad F_Y(\frac{b+y}{C_0}) = \frac{1}{C_0} F_Y(\frac{b}{C_0}) \quad (2.2)
\]

The following theorem is proved in Nelsen (1993, Theorem 3.2):

**Theorem 2.1.** Suppose \((X, Y)\) is marginally symmetric about \((a, b)\) with copula \(C\). Then, \((X, Y)\) is radially symmetric about \((a, b)\) if and only if \(C\) satisfies the functional equation
\[
C(u, v) = C(1 - u, 1 - v) + u + v - 1 \quad (2.3)
\]

There is a corresponding class of bivariate random variables associated to opposite symmetric copulas, which we propose to call \(\text{opposite radially symmetric variables}\), in accordance with the terminology in De Baets, De Meyer, and Ubeda-Flores (2009), and analogous to the radially symmetric variables of Nelsen (1993).

**Definition 2.2.** The bivariate random variable \((X, Y)\) is said to be \(\text{opposite radially symmetric}\) about \((a, b)\) if \((a + X, b + Y)\) and \((b - Y, a - X)\) are equal in distribution, or, equivalently,
\[
F_{X,Y}(a+x, b+y) = 1 - F_X(a - y) - F_Y(b - x) + F_{X,Y}(a - y, b - x) \quad (2.4)
\]

We need to replace marginal symmetry with the following analog of (2.1):

**Definition 2.3.** The bivariate random variable \((X, Y)\) is said to be \(\text{opposite marginally symmetric}\) about \((a, b)\) if \(F_X, F_Y\) satisfy
\[
F_X(a + x) = 1 - F_Y(b - x) \quad \text{and} \quad F_Y(b + y) = 1 - F_X(a - y) \quad (2.5)
\]

for all \(x, y\).
Remark. If \(X, Y\) are identically distributed and marginally symmetric about \((a, a) \in \mathbb{R}^2\), then \((X, Y)\) is opposite marginally symmetric about \((a, a)\). There are no identically distributed opposite marginally symmetric \((X, Y)\) about \((a, b)\) if \(b \neq a\), since then the common CDF \(F_X = F_Y = F\) would satisfy \(F(x) = F(x + b - a)\) for all \(x\).

We have the following analog of Theorem 2.1:

**Theorem 2.4.** Suppose that \((X, Y)\) is opposite marginally symmetric about \((a, b) \in \mathbb{R}^2\) with copula \(C\), and suppose that \(F_X, F_Y\) are continuous. Then, \((X, Y)\) is opposite radially symmetric about \((a, b)\) if and only if \(C\) is opposite symmetric.

**Proof.** It follows from Equations (2.4) and (2.5) that \((X, Y)\) is opposite radially symmetric if and only if
\[
C(1 - F_Y(b - x), 1 - F_X(a - y)) = C(F_X(a + x), F_Y(b + y))
\]
\[
= 1 - F_X(a - y) - F_Y(b - x) + C(F_X(a - y), F_Y(b - x))
\]  
Since the range of \(F_X\) and \(F_Y\) is \([0, 1]\), this proves the theorem. \(\square\)

**Remark.** There is an erroneous statement in De Baets, De Meyer, and Ubeda-Flores (2009, Remark 1) that if \(C\) is opposite symmetric, then \((X, Y)\) and \((1 - Y, 1 - X)\) are equal in distribution, i.e., \((X, Y)\) is opposite radially symmetric about \((1/2, 1/2)\), but additional assumptions like opposite marginal symmetry in Theorem 2.4 is needed to draw that conclusion.

### 3. Differential equations for copulas with opposite symmetry

The following theorem provides a characterization of absolutely continuous copulas with opposite symmetry and constitutes the basis for deriving the differential equations.

**Theorem 3.1.** Assume that \(p\) is an integrable function on \([0, 1]^2\) satisfying
\[
p(u, v) = p(1 - v, 1 - u)
\]  
and let \(C(u, v)\) be given by (1.13). Then,
\[ C(u, v) = C(1 - v, 1 - u) + C(u, 1) + C(1, v) - C(1, 1) \]  

and the following two conditions are equivalent:

1. \( C(u, 1) = u \) for all \( u \in [0, 1] \).
2. \( C(1, v) = v \) for all \( v \in [0, 1] \).

Furthermore, if \( p \geq 0 \) these conditions are equivalent to

3. \( C \) is an absolutely continuous opposite symmetric copula.

Proof of Theorem 3.1. By the inclusion-exclusion principle for integrals, we have

\[
\int_0^1 \int_0^1 p(w, z) dz dw = C(u, v) + C(1, 1) - C(u, 1) - C(1, v) \tag{3.3}
\]

By change of variables and symmetry (3.1), we also have

\[
\int_0^1 \int_0^1 p(w, z) dz dw = \int_0^{1-u} \int_0^{1-v} p(1-z, 1-w) dz dw \\
= \int_0^{1-v} \int_0^{1-u} p(w, z) dz dw = C(1 - v, 1 - u) \tag{3.4}
\]

which proves (3.2). Assume that \( C(u, 1) = u \) for \( u \in [0, 1] \). Then, (3.2) with \( u = 0 \) simplifies to \( 0 = C(1, v) - v \). Similarly, \( C(1,v) \equiv v \Rightarrow C(u, 1) \equiv u \). If these conditions hold, \( C \) is a copula, which is absolutely continuous by Equation (1.13), and Equation (3.2) implies Equation (1.3), i.e., opposite symmetry. Conversely, if \( C \) is a copula, \( C(u, 1) \equiv u \) and \( C(1,v) \equiv v \) by definition. \( \square \)

We will now show that copulas satisfying the assumptions in Theorem 2.4, with the additional assumption of being conditionally independent on \( u + v \leq 1 \) can be characterized by differential equations. This method is reminiscent of the well-known method of separation of variables for construction of solutions to partial differential equations. This will also give a construction method for absolutely continuous copulas with given opposite diagonal section, a problem considered in De Baets, De Meyer, and Ubeda-Flores (2009), cf. Theorem 3.7 below. Later, we will modify the construction, restricting the copula density support to \( v \leq H(u) \), which is required to solve Problem 1.2.

Theorem 3.2. Assume that

\[
p(u,v) = \begin{cases} 
F'(u)G(v) & \text{if } u + v \leq 1 \\
F'(1-v)G'(1-u) & \text{if } u + v > 1 
\end{cases} \tag{3.5}
\]

where \( F, G \in C^1(0,1), F(0) = G(0) = 0, G' \geq 0 \), and \( C \) is given by (1.13). Then,

\[
C'(u,v) = \begin{cases} 
F'(u)G(v) & \text{if } u + v \leq 1 \\
G(1-u)F'(u) + G'(1-u)(F(u) - F(1-v)) & \text{if } u + v > 1 
\end{cases} \tag{3.6}
\]

and the following are equivalent:

1. \( F' \geq 0 \) and

\[
G(1-u)F'(u) + G'(1-u)F(u) = 1, u \in [0, 1] \tag{3.7}
\]
2. \( C(u, v) \) is an absolutely continuous copula, and then

\[
C(u, v) = \begin{cases} 
F(u)G(v) & \text{if } u + v \leq 1 \\
F(1 - v)G(1 - u) + u + v - 1 & \text{if } u + v > 1 
\end{cases}
\]

(3.8)

**Proof.** Integration \( C'_u(u, v) = \int_0^v C'_{uv}(u, z)dz \) of the piecewise defined function \( p = C''_{uv} \) yields \( C'_u(u, v) = F'(u)G(v) \) for \( u + v \leq 1 \) and \( C'_u(u, v) = G(1 - u)F'(u) + G'(1 - u)(F(u) - F(1 - v)) \) for \( u + v > 1 \), so \( C'_u(u, 1) = G(1 - u)F'(u) + G'(1 - u)F(u) \).

Suppose that \( F' \geq 0 \) and (3.7) holds true. Then, \( p \geq 0 \) and \( C'_u(u, 1) \equiv 1 \) so \( C \) is an absolutely continuous copula by Theorem 3.1. Conversely, suppose that \( C \) is an absolutely continuous copula. Then, \( C''_{uv} = p \geq 0 \) so \( F' \geq 0 \) by (3.5), and (3.7) holds since \( C'_u(u, 1) \equiv 1 \). Moreover, integration \( C(u, v) = \int_0^u C'_u(z, v)dz \) yields (3.8) for \( u + v \leq 1 \), and (3.8) for \( u + v > 1 \) follows from Theorem 3.1.

The differential Equation (3.7) can be solved with the integrating factor method. Moreover, a condition for \( F'(u) \geq 0 \) can be derived.

**Theorem 3.3.** Assume that \( G \) satisfies the assumptions of Theorem 3.2. Then, \( F(u) \) satisfy (3.7) and \( F(0) = 0 \) if and only if

\[
F(u) = G(1 - u)\int_0^u \frac{dz}{G(1 - z)^2}, \quad u \in (0, 1)
\]

(3.9)

Moreover, if \( F(u) \) is given by (3.9), and \( G \) is twice differentiable, leftside derivative \( G'(1^-) \) exists and is positive, \( G'(u) > 0 \) for \( u \in (0, 1) \) then

\[
F'(u) = G'(1 - u) \left( \frac{L(0)}{G(1)} + \int_0^u \frac{1 + L'(z)}{G(1 - z)^2}dz \right), \quad u \in (0, 1)
\]

(3.10)

where

\[
L(u) = \frac{G(1 - u)}{G'(1 - u)}, \quad u \in (0, 1), \quad \text{and} \quad L(0) = \frac{G(1)}{G'(1^-)}
\]

(3.11)

Finally, if there exists \( u^* \in [0, 1] \) such that \( L'(u) \leq -1 \) for \( u \in (0, u^*) \) and \( L'(u) \geq -1 \) for \( u \in (u^*, 1) \), and if

\[
-\int_0^{u^*} \frac{1 + L'(z)}{G(1 - z)^2}dz \leq \frac{L(0)}{G(1)^2}
\]

(3.12)

then \( F'(u) \geq 0 \) for \( u \in (0, 1) \).

**Proof.** Equation (3.9) is obtained by multiplying (3.7) with the integrating factor \( 1/G(1 - u)^2 \). Equation (3.7) yields

\[
F'(u) = \frac{1}{G(1 - u)} - \frac{1}{L(u)}F(u)
\]

(3.13)

and substituting (3.9) in (3.13) using (3.11) yields

\[
F'(u) = G'(1 - u) \left( \frac{L(u)}{G(1 - u)^2} - \int_0^u \frac{dz}{G(1 - z)^2} \right)
\]

(3.14)

and the identity...
\[
\frac{d}{du} \left( \frac{L(u)}{G(1-u)^2} \right) = \frac{2 + L'(u)}{G(1-u)^2}
\]  \hfill (3.15)

yields

\[
\frac{L(u)}{G(1-u)^2} = \frac{L(0)}{G(1)^2} + \int_0^u \frac{2 + L'(z)}{G(1-z)^2} \, dz
\]  \hfill (3.16)

which proves (3.10). Moreover, by the assumptions, \( u \to -\int_0^u (1 + L'(z))/G(1-z)^2 \, dz \) has its maximum for \( u = u^* \), so it follows from (3.12) that \( F'(u) \geq F'(u^*) \geq 0 \) for \( u \in [0,1] \).

\[ \square \]

**Example 3.4.** \( G(v) = v, L(u) = 1 - u, 1 + L'(u) = 0, F'(u) = G'(1-u)/G(1) \), yields the independence copula \( C(u,v) = uv \).

**Example 3.5.** If \( k \geq 1 \) and \( G(v) = v^k \), then (3.7) has solution
\[
F(u) = \frac{(1-u)^{1-k} - (1-u)^k}{2k-1}
\]  \hfill (3.17)

and \( F'(u) \geq 0 \) for \( u \in [0,1] \), so

\[
C(u,v) = \begin{cases} 
(1-u)^{1-k} - (1-u)^k v^k / (2k-1) & \text{if} \quad u + v \leq 1 \\
(1-u)^k (v^{1-k} - v^k) / (2k-1) + u + v - 1 & \text{if} \quad u + v > 1
\end{cases}
\]  \hfill (3.18)

is a one-parameter family of absolutely continuous copulas. In particular, for \( k = 1 \) we obtain the independence copula \( uv \). For \( k > 1 \), \( \lim_{u \to 1} F(u) = \infty \).

**Example 3.6.** If \( G(v) = \sin (\pi v / 2) \), then (3.7) has solution
\[
F(u) = 2 \sin (\pi u / 2) / \pi
\]  \hfill (3.19)

and \( F'(u) \geq 0 \) for \( u \in [0,1] \), so

\[
C(u,v) = \begin{cases} 
2 \sin (\pi u / 2) \sin (\pi v / 2) / \pi & \text{if} \quad u + v \leq 1 \\
2 \cos (\pi u / 2) \cos (\pi v / 2) / \pi + u + v - 1 & \text{if} \quad u + v > 1
\end{cases}
\]  \hfill (3.20)

is an absolutely continuous copula.

Since the positivity conditions in **Theorem 3.3** is formulated in terms of the function \( L \), it is natural to start by specifying \( L \) satisfying (3.12). This is also related to the problem of constructing copulas with prescribed opposite diagonal section \( \omega(u) = C(u, 1-u) \) considered in De Baets, De Meyer, and Ubeda-Flores (2009). In fact, given \( \omega \), the function \( L \) is given by the explicit formula (3.25) below. This is formulated in **Theorem 3.7** below.

**Theorem 3.7.** Suppose that \( L \) is a positive real-valued function defined on \([0,1]\) such that
\[
\int_0^u \frac{dz}{L(z)} < \infty
\]  \hfill (3.21)

for \( u \in [0,1] \) and
\[
\lim_{u \to 1^-} \int_0^u \frac{dz}{L(z)} = \infty \quad (3.22)
\]

Let
\[
G(v) = \exp \left( - \int_0^{1-v} \frac{dz}{L(z)} \right) \quad (3.23)
\]
and suppose that (3.12) holds true. Moreover, let \( F(u) \) be given by (3.9). Then, \( C \) given by (3.8) is an absolutely continuous copula. Moreover, the opposite diagonal section
\[
\omega(u) \equiv C(u, 1 - u) \quad (3.24)
\]
satisfies
\[
L(u) = \frac{2\omega(u)}{1 - \omega'(u)} \quad (3.25)
\]

**Proof.** Clearly, because \( L \) is positive and satisfies (3.21) and (3.22), \( G \) defined by (3.23) is positive, \( G \) is increasing (in fact strictly increasing) and \( G(0) = 0 \). Moreover, it follows from (3.23) that (3.11) holds true. By Theorem 3.3, \( F'(u) \geq 0 \) and by Theorem 3.2, \( C \) is an absolutely continuous copula. Differentiation of \( F(u)G(1 - u) = \omega(u) \) yields \( F'(u)G(1 - u) - F(u)G'(1 - u) = \omega'(u) \), so in view of (3.7) we get
\[
F'(u)G(1 - u) = \frac{1 + \omega'(u)}{2} \quad (3.26)
\]
and
\[
F(u)G'(1 - u) = \frac{1 - \omega'(u)}{2} \quad (3.27)
\]

Solving for \( F(u) \) in (3.27), differentiating and substituting \( F'(u) \) in the left-hand side of (3.26) yields
\[
(1 - \omega'(u)) \frac{G''(1 - u)G(1 - u) - \omega''(u) G(1 - u)}{G'(1 - u)^2} = 1 + \omega'(u). \quad (3.28)
\]

Using (3.11) and the identity
\[
\frac{G''(1 - u)G(1 - u)}{G'(1 - u)^2} = 1 + L'(u) \quad (3.29)
\]
we get
\[
(1 - \omega'(u))L'(u) - \omega''(u)L(u) = 2\omega'(u) \quad (3.30)
\]
which is integrated to \( (1 - \omega'(u))L(u) = 2\omega(u) + \text{constant} \). Since \( \omega(1) = C(1, 0) = 0 \) and \( L(1) = 0 \) in view of (3.22), the integration constant is zero, which proves (3.25).

**Example 3.8.** Assume that \( k \geq 1 \) and let \( L(u) = (1 - u)/k \). Then, we get \( G(1 - u) = (1 - u)^k \) so we recover Example 3.5. Also, \( L'(u) = -1/k \geq -1 \) so \( u' = 0 \) and since \( 0 \leq L(0) = 1/k \) we infer from Theorem 3.3 that an absolutely continuous copula is obtained.
**Example 3.9.** Assume that \( a \in [0, 1) \) and let \( L(u) = (1 - u)(1 - au) \). Then,

\[
G(1 - u) = \left( \frac{1 - u}{1 - au} \right)^{1/(1-a)}
\]

and \( u^* = 1/2 \): \( L'(u) = -1 + a(-1 + 2u) \leq -1 \) if \( u \leq 1/2, L'(u) \geq -1 \) if \( u \geq 1/2 \). We obtain

\[
\int_0^{u^*} \frac{1 + L'(z)}{G(1-z)^2} \, dz = \int_0^{1/2} \left( \frac{1 - au}{1 - u} \right)^{2/(1-a)} a(1 - 2u) \, du
\]

\[
= \frac{1}{2} \, F_1 \left( 1, \frac{2}{1 - a}, -\frac{2}{1 - a}; 3; \frac{1 - a}{2} \right)
\]

Here, \( F_1 \) is the Appell series (see Gradshteyn and Ryzhik (2014, 1027) for a definition), which may be represented by Picard’s integral formula, cf. Cuyt et al. (1999):

\[
F_1(a, b, b'; c; x, y) = \frac{\Gamma(c)}{\Gamma(a) \Gamma(c - a)} \int_0^1 t^{a-1}(1-t)^{c-a-1}(1 - tx)^{-b}(1 - ty)^{-b'} \, dt
\]

Here, \( \Gamma \) denotes Euler’s gamma function (Gradshteyn and Ryzhik 2014, 901). The function \( F_1 \) is available in computer algebra systems like Maple® and Mathematica®, and numerical investigation reveals that the right-hand side is an increasing function of \( a \) and approaches the value 0.861485 as \( a \to 1^- \). Therefore, condition (3.12) is satisfied, so Theorem 3.3 yields an absolutely continuous copula, and (3.9) can be evaluated to

\[
F(u) = uG(1 - u)F_1 \left( 1, \frac{2}{1 - a}, -\frac{2}{1 - a}; 2; au, u \right).
\]

When \( 2/(1 - a) \) is integer, this expression can be simplified to a finite sum of powers and logarithms, cf. Cuyt et al. (1999).

**4. Comparison with other methods**

A method by Durante and Jaworski is found in Durante and Jaworski (2008), where absolutely continuous copulas \( C(u, v) \) with given diagonal section \( C(t, t) \) are constructed, in terms of convex combinations of singular diagonal copulas

\[
C_\delta(u, v) = \min \left( u, v, \frac{\delta(u) + \delta(v)}{2} \right)
\]

(satisfying \( C_\delta(t, t) = \delta(t) \)). The problem with this approach for our purposes is that the constraint \( v \leq H(u) \) imposes functional inequalities \( \delta(H(u)) + \delta(u) \leq 2u \) that must be fulfilled for the \( \delta \)'s used in the construction. In comparison, the advantage of our differential equation method is that \( H \) is used explicitly, using only elementary calculus.

Regarding copulas and differential equations, there is a characterization of all copulas by Jaworski, in terms of a certain type of weak solutions to differential equations in Jaworski (2014). For comparison, we give here a simplified account of his method in
the special case of absolutely continuous copulas with differentiable density and sectional inverse. For fixed \( u \in [0, 1] \), let \( C(u, \cdot)^{-1}(z) \) denote the assumed unique solution \( v \) to the equation \( C(u, v) = z \), i.e., \( C(u, C(u, \cdot)^{-1}(z)) = z \) for all \( z \in [0, 1] \), and define
\[
C_{[u]}(t, z) = u^{-1}C(ut, C(u, \cdot)^{-1}(uz))
\] (4.2)

Moreover, define
\[
F_C(u, z) = \frac{\partial}{\partial t} C_{[u]}(t, z) \big|_{t=1} - z = C_u'(u, C(u, \cdot)^{-1}(uz)) - z
\] (4.3)

Now suppose that for each \( v \in [0, 1] \), \( g_v(u) \) is solution to the terminal value problem
\[
ug'_v(u) = F_C(u, g_v(u)), u \in (0, 1)\] (4.4)
\[
g_v(1) = v
\] (4.5)

Then, \( C \) can be characterized in terms of \( g_v(u) \) as
\[
C(u, v) = ug_v(u)
\] (4.6)

To see this, note that by the definition of \( F_C \) and the product rule of differentiation, (4.4) is equivalent to
\[
\frac{d}{du} (ug_v(u)) = C_u'(u, C(u, \cdot)^{-1}(ug_v(u)))
\] (4.7)

and this ODE for \( g_v(u) \) is satisfied for \( g_v(u) = C(u, v)/u \), so by uniqueness of solution to (4.4)–(4.5), (4.6) must hold. The general result (valid for all copulas) can be found in Jaworski (2014, Theorems 3.1 and 3.2). Now, applying Jaworski’s characterization theorem to a copula of the form (3.8), we need to compute \( C(u, \cdot)^{-1}(z) \) to obtain \( F_C \). For \( z \leq 1 - u \), we get \( F(u)G(v) = z \), which can be solved explicitly, yielding \( v = C(u, \cdot)^{-1}(z) = G^{-1}(z/F(u)) \). However, for \( z > 1 - u, v = C(u, \cdot)^{-1}(z) \) is implicitly defined by \( F(1 - v)G(1 - u) + u + v - 1 = z \), which cannot be solved for \( v \) in terms of \( F, G \), and their inverses. Therefore, we have not been able to use Jaworski’s method to obtain equations for \( F, G \) for copulas of the type (3.8).

5. Absolutely continuous copulas with prescribed support

Here, we construct absolutely continuous opposite symmetric copulas with the support of the probability measure prescribed by a constraint \( v \leq H(v) \). The construction is simple, using elementary calculus and a piecewise definition of the copula density, similar to Theorem 3.2.

**Theorem 5.1.** Suppose that \( 0 < u_0 < 1/2 \) and that \( H \) is a strictly increasing continuous function defined on \([0, 1]\), continuously differentiable on \((0, u_0)\), satisfying (1.6) and (1.7). Furthermore, suppose that \( F \) is a differentiable function defined on \([u_0, 1]\) such that \( F(u_0) = 0 \), \( G \) is a differentiable function defined on \([0, 1 - u_0]\) such that \( G(0) = 0, G' \geq 0 \) and \( C(u, v) \) given by (1.13), (1.14). Furthermore, let \( K \) be defined by (1.11). Then, the following are equivalent:
- $F' \geq 0$ and

$$F'(u)G(1-u) + G'(1-u)(F(u) + K(u)) = 1 \quad (5.1)$$

for $u \in [u_0, 1)$.

- $C(u, v)$ is an absolutely continuous copula, fulfilling (1.3) and (1.15)–(1.17).

**Proof.** The basic idea of the proof is similar to Theorem 3.2: integrate the given piecewise defined ansatz for the copula density $C''_{uv}$ to derive $C'_u$ and use Theorem 3.1. By definition, $p(u, v) = C''_{uv}(u, v)$ and piecewisely defined on the regions 1–7 depicted in Figure 4 as follows; region 1: $C''_{uv} = G'(v)/G(H(u))$, region 2, 3, 7: $C''_{uv} = 0$, region 4: $C''_{uv} = F'(u)G'(v)$, region 5: $C''_{uv} = F'(1-v)G'(1-u)$, and region 6: $C''_{uv} = G'(1-u)/G(H(1-v))$. Integration yields $C'_u(u, v) = \int_0^v C''_{uv}(u, z)dz$, piecewisely defined as follows; region 1: $C'_u = G(v)/G(H(u))$, region 2, 3: $C'_u = 1$, region 4: $C'_u = F'(u)G(v)$, region 5: $C'_u = F'(u)G(1-u) + (F(u) - F(1-v))G'(1-u)$, region 6: $C'_u = F'(u)G(1-u) + (F(u) + K(H^{-1}(v)))G'(1-u)$, and region 7: $C'_u = F'(u)G(1-u) + (F(u) + K(u))G'(1-u)$. To derive the expression in region 6, write $K$ on the alternate form

$$K(u) = \int_{H^{-1}(1-u)}^{u_0} \frac{dw}{G(H(w))} \quad (5.2)$$

(derived by the change of variables $z = 1 - H(w) = H^{-1}(1-w)$) and note that

$$\int_{1-u_0}^v \frac{dz}{G(H(1-z))} = \int_{1-v}^{u_0} \frac{dw}{G(H(w))} = \int_{H^{-1}(1-H^{-1}(v))}^{u_0} \frac{dw}{G(H(w))} = K(H^{-1}(v))$$

in view of (1.7). If $F' \geq 0$ and (5.1) holds true, then $p \geq 0$ by (1.14) and $C''_{uv}(u, 1) \equiv 1$ by (5.1) since the left-hand side of (5.1) is the expression for $C'_u$ in region 7. Thus, by theorem 3.1, $C$ is an absolutely continuous copula. Conversely, if $C$ is an absolutely continuous copula, then $p = C''_{uv} \geq 0$ so $F' \geq 0$ and by (1.14), and $C''_{uv}(u, 1) \equiv 1$ which proves (5.1). The conditions $C'_u(u, 1) \equiv 1$ and $C'_v(1, v) \equiv 1$ are equivalent by Theorem 3.1. Assume now that $C$ is an absolutely continuous copula, then $C'_u = 1$ in region 7 by (5.1). Integration $C(u, v) = \int_0^u C'_u(z, v)dz$ yields the following piecewise defined function $C(u, v)$; region 2, 3, 7: $C = u$ which proves (1.16), region 1: $C = H^{-1}(v) + (K(1-v) - K(H^{-1}(1-u)))G(v)$ which proves (1.15), and region 4: $C = H^{-1}(v) + (K(1-v) + F(u))G(v)$ which proves (1.17). The final statement for $u + v > 1$ follows from Theorem 3.1. \hfill $\Box$

**Remark.** Note that the symmetry condition (1.7) and the continuity and strict monotonicity of $H$ implies that $H(0) = 0$ and $H(1) = 1$. Indeed, if $H(0) = a > 0$ then $H(u) = 1$ for $u \in (1-a, 1)$, which contradicts the strict monotonicity of $H$. Note also that $H(u) = u$ is not allowed since $H(u_0) = 1 - u_0$ and $u_0 \in (0, 1/2)$.

Equation (5.1) can be solved with the integrating factor method, and a positivity condition can be derived, analogous to Theorem 3.3:

**Theorem 5.2.** Assume that $H$, $G$, $K$, and $F$ satisfy the assumptions in Theorem 5.1. Then, $F(u)$ satisfies (5.1) if and only if
\[ F(u) = G(1 - u) \int_{u_0}^{u} 1 + \frac{H'(z)}{G(1 - z)^2} \, dz \] (5.3)

Moreover, if \( F(u) \) is given by (5.3), then
\[ F'(u) = G'(1 - u) \left( \frac{L(u_0)}{G(1 - u_0)^2} + \int_{u_0}^{u} \frac{1 + L'(z) - H'(z)}{G(1 - z)^2} \, dz \right) \] (5.4)

where \( L \) is given by (3.11). Finally, if there exists \( u^* \in [u_0, 1] \) such that \( L'(u) - H'(u) \leq -1 \) for \( u \in (u_0, u^*) \) and \( L'(u) - H'(u) \geq -1 \) for \( u \in (u^*, 1) \), and if
\[ -\int_{u_0}^{u^*} \frac{1 + L'(z) - H'(z)}{G(1 - z)^2} \, dz \leq \frac{L(u_0)}{G(1 - u_0)^2} \] (5.5)

then \( F'(u) \geq 0 \) for \( u \in (u_0, 1) \).

**Proof.** Multiplying (5.1) with the integrating factor \( 1/G(1 - u)^2 \) and integrating by parts (using \( K(u_0) = 0 \)) yields
\[ F(u) = G(1 - u) \int_{u_0}^{u} \frac{1 - G'(1 - z)K(z)}{G(1 - z)^2} \, dz \]
\[ = G(1 - u) \left( \int_{u_0}^{u} \frac{dz}{G(1 - z)^2} - \frac{K(u)}{G(1 - u)} + \int_{u_0}^{u} \frac{K'(z)dz}{G(1 - z)} \right) \]
so substituting
\[ K'(z) = \frac{H'(z)}{G(1 - z)} \] (5.6)

according to (1.11) yields (5.3). Solving for \( F' \) in (5.1):
\[ F'(u) = \frac{1}{G(1 - u)} - \frac{1}{L(u)} \left( K(u) + F(u) \right) \] (5.7)

and substituting
\[ K(u) + F(u) = \int_{u_0}^{u} \frac{1 + H'(z)}{G(1 - z)^2} \] (5.8)

according to (5.3) yields
\[ F'(u) = G'(1 - u) \left( \frac{L(u)}{G(1 - u)^2} - \int_{u_0}^{u} \frac{1 + H'(z)}{G(1 - z)^2} \, dz \right) \] (5.9)

The identity (3.15) yields
\[ \frac{L(u)}{G(1 - u)^2} = \frac{L(u_0)}{G(1 - u_0)^2} + \int_{u_0}^{u} \frac{2 + L'(z)}{G(1 - z)^2} \, dz \] (5.10)

which proves (5.4). Finally, by the assumptions, \( u \mapsto -\int_{u_0}^{u} (1 + L'(z) - H'(z))/G(1 - z)^2 \, dz \) has its maximum for \( u = u^* \), so it follows from (5.5) that \( F'(u) \geq F'(u^*) \geq 0 \) for \( u \in [u_0, 1] \). \( \square \)
Example 5.3. If
\[ H(u) = \begin{cases} \frac{(1-u_0)u}{u_0} & \text{if } u \leq u_0 \\ 1-u_0(1-u)/(1-u_0) & \text{if } u > u_0 \end{cases} \]
(5.11)

Then, we obtain a two-parameter family of absolutely continuous copulas (with parameters \(0<\leftarrow(u_0)\)) evaluated to
\[ G(v) = v^k \text{ and } k \geq (1-u_0)/(1-2u_0), \]
then (1.11) yields
\[ K(u) = \frac{(1-u)^{1-k} - (1-u_0)^{1-k})u_0}{(1-u_0)(k-1)} \]
(5.12)

(5.3) evaluates to
\[ F(u) = \frac{(1-2u_0)k - (1-u_0)}{(2k-1)(k-1)(1-u_0)}(1-u)^{1-k} - \frac{(1-u_0)^{1-2k}}{(2k-1)(1-u_0)}(1-u)^k + \frac{(1-u_0)^{1-k}u_0}{(k-1)(1-u_0)} \]
(5.13)

Moreover, \(L(u) = (1-u)/k\), so \(L'(-H'(u)) = -1/k - u_0/(1-u_0) \geq -1\) if and only if \(k \geq (1-u_0)/(1-2u_0)\), in which case \(F'(u)\) is positive. By Theorem 5.1 we obtain a two-parameter family of absolutely continuous copulas (with parameters \(0<\leftarrow(u_0)\)) with probability density supported on \(v \leq H(u)\). Indeed, in this example \(F'(u)\) can be computed explicitly:
\[ F'(u) = \frac{((1-2u_0)k - (1-u_0))(1-u)^{-k} + k(1-u_0)^{1-2k}(1-u)^{k-1}}{(2k-1)(1-u_0)} \]
(5.14)

and is strictly positive on \([u_0,1]\) if and only if the coefficient for \((1-u)^{-k}\) is positive, which is equivalent to \(k \geq (1-u_0)/(1-2u_0)\).

Example 5.4. In this example, we construct more solutions to Problem 1.2, using Theorem 5.2. Let \(k \in \mathbb{R}, k > 1\) and \(L(u) = (1-u)/k\). Then, we obtain \(G(v) = v^k/(1-u_0)^k\) and
\[ K(u) = (1-u_0)^k \int_{u_0}^{u} \frac{H'(z)}{(1-z)^{k}} dz \]
(5.15)

and
\[ F(u) = -K(u) + (1-u_0)^k(1-u)^k \int_{u_0}^{u} \frac{1+H'(z)}{(1-z)^{2k}} dz \]
(5.16)

where \(H\) is given by (1.5) and
\[ H'(z) = \frac{1}{\sqrt{2\pi}} \exp \left(-\Delta \left( \Phi^{-1}(u) + \frac{\Lambda}{2} \right) \right) \]
(5.17)

Since \(L'(u) = -1/k\) and \(H'\) decreasing we have \(u^*\) satisfying the assumptions in Theorem 5.2 and determined by \(H'(u^*) = 1 - 1/k\). Solving this equation yields
Thus, \( u^* = \Phi\left( -\sqrt{\frac{2\pi}{\Delta}} \left( 1 - \frac{1}{k} \right) - \frac{\Delta}{2} \right) \) (5.18)

so if \( k \) satisfies this condition, an absolutely continuous copula is obtained. The joint PDF of \((U,V)\) is illustrated in Figure 5, and the joint PDF of \((X,Y)\) is illustrated in Figure 6.

We have the following analogue of Theorem 3.7. Here, given the opposite diagonal section \( \omega \), the function \( L \) is given by an integral Equation (5.23), (5.24) below.

**Theorem 5.5.** Suppose that \( H, u_0 \) satisfies (1.7) and (1.6). Suppose also that \( L \) is a positive real–valued function defined on \([u_0, 1]\) such that

\[
\int_{u_0}^{u} \frac{H'(z)}{(1 - z)^{2k}} dz \leq \frac{(1 - u_0)^{1-2k}}{2k - 1} + \left( 1 - \frac{1}{k} \right) (1 - u^*)^{1-2k}
\]

(5.19)

Moreover, let \( K(u) \) and \( F(u) \) be given by (1.11) and (5.3) and suppose that (5.5) holds true. Then, \( C \) given by (1.3) and (1.15)–(1.17) is an absolutely continuous copula.

Moreover, the opposite diagonal section (3.24) satisfies

\[
\omega(u) = u \quad \text{for} \quad u \in [0, u_0]
\]

(5.20)

and

\[
\lim_{u \to 1} \int_{u_0}^{u} \frac{dz}{L(z)} = \infty
\]

(5.21)

Let

\[
G(v) = \exp\left( - \int_{u_0}^{1-v} \frac{dz}{L(z)} \right)
\]

(5.22)

Figure 6. Probability density function \( p(x, y) \) for Example 5.4, \( \Delta = 1, k = 2 \). The wiggles in the level curves at the upper right and lower left corners of right plot are numerical artifacts.
for \( u \in [u_0, 1] \), where
\[
G(1 - u)K(u) = \int_{u_0}^{u} \exp\left(-\int_{z}^{u} \frac{dw}{L(w)}\right)H'(z)dz 
\] (5.24)

**Proof.** The proof is similar to the proof of Theorem 3.7, with some additional terms involving \( K \). More precisely, (3.26) and (3.27) are replaced by
\[
G(1/C_0u)F(z) = \frac{1 + \omega'(u) - G'(1 - u)K(u)}{2} 
\] (5.25)
and
\[
G'(1 - u)F(u) = \frac{1 - \omega'(u) - G'(1 - u)K(u)}{2} 
\] (5.26)
Solving for \( F \) in (5.26), differentiating and substituting for \( F_0 \) in the left-hand side of (5.25) yields
\[
(1 - \omega'(u))(1 + L'(u)) = \omega''(u)L(u) 
\] (5.27)
which is integrated to \((1 - \omega'(u))L(u) = 2\omega(u) + G(1 - u)K(u) + \text{constant.}\) The Equation (5.24) follows from (1.11) and (5.22). For each fixed \( z \), the integrand in (5.24) is decreasing toward 0 as \( u \to 1^{-} \) in view of (5.20) and (5.21), so by the monotone convergence theorem, \( \lim_{u \to 1^{-}} G(1 - u)K(u) = 0 \). Hence, the constant of integration is zero, which proves (5.23).

**Proof of Theorem 1.4.** Clearly \( H, G, K, \) and \( F \) satisfy the assumptions of Theorem 5.1. Differentiation of (1.12) yields \( F'(u) = (1 - 2u_0)G'(1 - u) \), which shows that (5.4) is satisfied, since \( 1 + L'(z) - H'(z) = 0, L(u_0) = H(u_0) - u_0 = 1 - 2u_0, \) and \( G(1 - u_0) = 1 \). Thus, \( F(u) \) satisfies \( F' \geq 0 \) and (5.1) by Theorem 5.2. By Theorem 5.1, \( C \) is an absolutely continuous copula with density \( p \), having the stated form according to Theorem 5.1.

**6. Sampling**

To sample from a two-dimensional copula \( C(u, v) \), we use the conditional density \( C'_u \) of Corollary 6.1 in the following way (cf. Nelsen (2006, Chap. 2.9)): First sample \( U, T \), independently from \( U(0, 1) \). Then for each \( T_i, T_i \) let \( V_i \) satisfy \( T_i = C'_u(U_i, V_i) \). Then, \( (U_i, V_i) \) is distributed according to \( C(u, v) \). For sampling from the copula, the following corollary is useful:

**Corollary 6.1.** Suppose that \( C(u, v) \) is an absolutely continuous copula given by Theorem 5.1 and \( F, G, K \) defined accordingly. Then, \( C'_u(u, v) \) is given by the following formulas:

1. If \( 0 \leq u \leq u_0 \) and \( 0 < v < H(u) \) then
\[
C'_u(u, v) = \frac{G(v)}{G(H(u))} 
\] (6.1)
2. If $0 \leq u \leq 1$ and $H(u) \leq v \leq 1$ then
   \[ C_u'(u, v) = 1 \] (6.2)

3. If $u_0 < u < 1$ and $0 < v \leq 1 - u$ then
   \[ C_u'(u, v) = (1 - G'(1 - u))(K(u) + F(u))G(v)/G(1 - u) \] (6.3)

4. If $u_0 < u < 1$ and $1 - u < v \leq 1 - u_0$ then
   \[ C_u'(u, v) = 1 - G'(1 - u)(K(u) + F(1 - v)) \] (6.4)

5. If $u_0 < u < 1$ and $1 - u_0 < v \leq H(u)$ then
   \[ C_u'(u, v) = 1 - G'(1 - u)(K(u) - F(1 - H(1 - v))) \] (6.5)

6. If $u_0 < u < 1$ and $H(u) < v < 1$ then $C_u'(u, v) = 1$.

**Proof.** Follows from the equations for $C_u'$ in the proof of Theorem 5.1, and Equations (1.7), (5.1).

Figure 7 illustrates sampling in Example 1.5.

### 7. Application to toxicological probit models

The probit model is the standard statistical method for estimating the injury outcome of a population exposed to a toxic substance. It originates from an analysis on the effect of pesticides conducted by Bliss in 1934 (Bliss 1934). The methodology was later cast in a more rigid mathematic formulation by Finney and Tattersfield (1952). It has since then been used frequently in toxicological assessments of the injury outcome when a population has been exposed to dangerous chemicals (Björnham et al. 2017; Burman and Jonsson 2015; Hauptmanns 2005; Lovreglio et al. 2016; Stage 2004). In short, the probit model operates as follows. The exposure concentration $c(t)$ is integrated over time to yield *probit values*

\[ \Gamma_i(t) = z_i + \beta_i \log \left( \int_0^t c(t)^{\alpha_i} dt \right) \] (7.1)
The fraction of the population that has attained the injury at time $t$ is then estimated by
\[ \Phi(\Gamma_i(t)) \tag{7.2} \]
where $\alpha_i, \beta_i, n_i$ are model parameters associated with the substance, and $\Phi$ is the CDF for a standard normal variable. There are often several levels of injury outcome used in toxicology, e.g., light injury, severe injury, and death. These different injury levels are indexed by $i = 1, 2, \ldots$ in Equations (7.1)–(7.2). The fraction of the population that obtains an injury increases continuously with growing exposure due to the individual variation of the toxic susceptibility within the population. It is believed that modeling this variation improves the quantitative toxicological risk assessment, cf. Hattis, Banati, and Goble (1999).

A population that is not resolved on an individual level is referred to as a macroscopic population and can be described as a density field. In contrast, a population can be described as a set of discrete individuals, referred to as agents. A model that uses this type of population representation is called a microscale model or an agent-based model. In an agent-based toxicological model, see for example Lovreglio et al. (2016), the overall population statistics is obtained from the set of agents that are exposed to the toxic substance. In such a setting, individual probit values $\Gamma_i(t)$, acquired by exposure to individual model concentrations $c(t)$, are computed for each agent. In the transition from a macroscopic population to an agent-based population, it is convenient to distribute individual threshold values, $\gamma_i$, for the probit values to all agents representing their susceptibilities. Thus, when an agent has been exposed to a concentration yielding a probit value exceeding the corresponding threshold value, the agent has acquired that injury. Every agent is attributed one threshold value for each injury level. These thresholds are drawn from a standard normal distribution to maintain the overall probability distribution for the entire population. This method implies that the injury outcome of the agent-based population approaches asymptotically that of the macroscopic population (with static populations) when the number of agents increases. An advantage with an agent-based population is that the agents may have individual properties including their movement patterns. In a dynamic simulation, each agent follows its individual spacetime path, passing through concentration fields, and thereby proceeds through some or all of the injury stages, transiting successive injury stages when the agent’s increasing probit functions $\Gamma_i(t)$ pass their threshold values $\gamma_i$. As mentioned, the individual toxic susceptibility thresholds $\gamma_i$ are random variables and must obey the requirement
\[ P(\gamma_i \leq \Gamma) = \Phi(\Gamma) \tag{7.3} \]

We propose that the $\gamma_1, \gamma_2, \ldots$ are modeled as a discrete time Markov process with absolutely continuous transition densities $p_{i+1|i}$, so by the Markov property, the joint density $p$ is
\[ p(\gamma_1, \ldots, \gamma_n) = p_1(\gamma_1)p_{2|1}(\gamma_2|\gamma_1)p_{3|2}(\gamma_3|\gamma_2)\ldots p_{n|n-1}(\gamma_n|\gamma_{n-1}) \tag{7.4} \]

However, there is a potential pitfall: the injury stages must be passed in the correct order. Therefore, it must be true with probability one that if an injury level is acquired, then also the previous injury level is acquired, i.e.,
\[ \gamma_{i+1} \leq \Gamma_{i+1}(t) \Rightarrow \gamma_i \leq \Gamma_i(t) \]  

(7.5)

Therefore, the transition densities \( p_{i+1|i} \) must satisfy

\[ p_{i+1|i}(\gamma_{i+1}|\gamma_i) = 0 \text{ if } \gamma_{i+1} \leq \Gamma_{i+1}(t) \text{ and } \gamma_i > \Gamma_i(t) \]  

(7.6)

This imposes a restriction on the support of the joint probability density of \((\gamma_i, \gamma_{i+1})\), which we need to investigate in order to ensure that the model is consistent. To this end, we need to relate possible values of \( \Gamma_i(t), \Gamma_{i+1}(t) \) for all possible exposures \( c(t), t \geq 0 \). This can be done in terms of

\[ \frac{\Gamma_i(t) - z_i}{\beta_i} = \log \left( \int_0^t c^n dt \right) \]  

(7.7)

according to the following lemma:

**Lemma 7.1.** Assume that \( n \geq m > 0 \) and \( c \geq 0, t > 0 \). Then,

\[ \log \left( \int_0^t c^m dt \right) \leq \frac{m}{n} \log \left( \int_0^t c^n dt \right) + \left( 1 - \frac{m}{n} \right) \log (t) \]  

(7.8)

and

\[ \log \left( \int_0^t c^n dt \right) \leq \log \left( \int_0^t c^m dt \right) + (n - m) \log \left( \max c \right) \]  

(7.9)

Moreover, the inequalities are sharp: if \( c(t) = \text{constant} \), then equalities holds in the inequalities above.

**Proof.** Apply Hölder’s inequality \( \int fdg \leq (\int f^p dt)^{1/p} (\int g^q dt)^{1/q} \) and the elementary estimate \( \int f^p dt \leq (\text{max} f)^{p-1} \int f dt \) with \( f = c^m, g = 1 \) and \( p = n/m \). \( \square \)

The following theorems provide sufficient conditions for (7.5), and necessary compatibility conditions for the probit parameters \( \alpha, \beta, n \).

**Theorem 7.2.** Assume that \( \Gamma_i(t), \Gamma_{i+1}(t) \) are probit functions defined by (7.1), and \( n_{i+1} \leq n_i \). Also assume that \((\gamma_i, \gamma_{i+1})\) is a bivariate random variable such that

\[ \frac{\gamma_{i+1} - \alpha_{i+1}}{\beta_{i+1}} \geq \frac{n_{i+1}}{n_i} \frac{\gamma_i - \alpha_i}{\beta_i} + \left( 1 - \frac{n_{i+1}}{n_i} \right) \log (t) \]  

(7.10)

almost surely. Then, \( \gamma_{i+1} \leq \Gamma_{i+1}(t) \Rightarrow \gamma_i \leq \Gamma_i(t) \) almost surely. Moreover, there exists standard normal \( \gamma_i, \gamma_{i+1} \) satisfying (7.10) if and only if

\[ n_{i+1} \beta_{i+1} = n_i \beta_i \]  

(7.11)

and

\[ \Delta_i \equiv z_i - \alpha_{i+1} - \beta_{i+1} \left( 1 - \frac{n_{i+1}}{n_i} \right) \log t \geq 0 \]  

(7.12)

and then if \( \Delta_i > 0 \) there exists \((\gamma_i, \gamma_{i+1})\) with absolutely continuous joint density.
**Proof.** Assume that $\gamma_{i+1} \leq \Gamma_{i+1}(t)$. Then, we get by (7.7), (7.8) with $m = n_{i+1}$, $n = n_i$, and (7.10) that

$$\frac{n_{i+1} \Gamma_i(t) - x_i}{n_i} + \left(1 - \frac{n_{i+1}}{n_i}\right) \log(t)$$

$$\geq \frac{\Gamma_{i+1}(t) - x_{i+1}}{\beta_{i+1}} \geq \frac{\gamma_{i+1} - x_{i+1}}{\beta_{i+1}} \geq \frac{n_{i+1} \gamma_i - x_i}{n_i} + \left(1 - \frac{n_{i+1}}{n_i}\right) \log(t) \tag{7.13}$$

i.e., $\Gamma_i(t) \geq \gamma_i$, which proves the first part. The second part follows from Proposition 1.1, since Equation (7.10) is equivalent to Equation (1.1) with $X = -\gamma_i, Y = -\gamma_{i+1}, a = (\beta_{i+1}n_{i+1})/\beta_i n_i$ and

$$\Delta = \frac{\beta_{i+1} n_{i+1}}{\beta_i n_i} x_i - x_{i+1} - \beta_{i+1} \left(1 - \frac{n_{i+1}}{n_i}\right) \log(t),$$

and $a = 1, \Delta \geq 0$ is equivalent to Equations (7.11), (7.12).

**Theorem 7.3.** Assume that $\Gamma_i(t), \Gamma_{i+1}(t)$ are probit functions defined by (7.1), and $n_{i+1} \geq n_i$. Also assume that $(\gamma_i, \gamma_{i+1})$ is a bivariate random variable such that

$$\frac{\gamma_{i+1} - x_{i+1}}{\beta_{i+1}} \geq \frac{\gamma_i - x_i}{\beta_i} \geq \frac{(n_{i+1} - n_i)}{\beta_i} \log \left(\text{max} \, c \right) \tag{7.14}$$

almost surely. Then, $\gamma_{i+1} \leq \Gamma_{i+1}(t) \Rightarrow \gamma_i \leq \Gamma_i(t)$ almost surely. Moreover, there exist standard normal $\gamma_i, \gamma_{i+1}$ satisfying (7.14) if and only if

$$\beta_{i+1} = \beta_i \tag{7.15}$$

and

$$\Delta_i \equiv x_i - x_{i+1} - \beta_i (n_{i+1} - n_i) \log \left(\text{max} \, c \right) \geq 0 \tag{7.16}$$

and then if $\Delta_i > 0$ there exists $(\gamma_i, \gamma_{i+1})$ with absolutely continuous joint density.

**Proof of Theorem 7.3.** Assume that $\gamma_{i+1} \leq \Gamma_{i+1}(t)$. Then, we get by (7.7), (7.9) with $m = n_i, n = n_{i+1}$, and (7.14) that

$$\frac{\Gamma_i(t) - x_i}{\beta_i} + (n_{i+1} - n_i) \log \left(\text{max} \, c \right)$$

$$\geq \frac{\Gamma_{i+1}(t) - x_{i+1}}{\beta_{i+1}} \geq \frac{\gamma_{i+1} - x_{i+1}}{\beta_{i+1}} \geq \frac{\gamma_i - x_i}{\beta_i} + (n_{i+1} - n_i) \log \left(\text{max} \, c \right) \tag{7.17}$$

i.e., $\Gamma_i(t) \geq \gamma_i$, which proves the first part. The second part follows from Proposition 1.1, since Equation (7.14) is equivalent to Equation (1.1) with $X = -\gamma_i, Y = -\gamma_{i+1}, a = \beta_{i+1}/\beta_i$ and
\[ \Delta = \frac{\beta_{i+1}}{\beta_i} z_i - x_{i+1} - \beta_{i+1}(n_{i+1} - n_i) \log \left( \max_{[0, t]} c(t) \right), \]

and \( a = 1, \Delta \geq 0 \) is equivalent to Equations (7.15), (7.16).

**Remark.** Note that if \( n_{i+1} = n_i \), then the compatibility conditions (7.11), (7.12) and (7.15), (7.16) in the preceding theorems involve only the probit coefficients \( \alpha, \beta, n \), not \( t \) or \( \max_t c \).

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**References**

Björnham, O., H. Grahn, P. von Schoenberg, B. Liljedahl, A. Waleij, and N. Brännström. 2017. The 2016 Al-Mishraq sulphur plant fire: Source and health risk area estimation. *Atmospheric Environment* 169:287–96. doi:10.1016/j.atmosenv.2017.09.025.

Bliss, C. I. 1934. The method of probits. *Science (New York, N.Y.)* 79 (2037):38–39. doi:10.1126/science.79.2037.38.

Burman, J., and L. Jonsson. 2015. Issues when linking computational fluid dynamics for urban modeling to toxic load models: The need for further research. *Atmospheric Environment* 104:112–24. doi:10.1016/j.atmosenv.2014.12.068.

Cuyt, A., K. Driver, J. Tan, and B. Verdonk. 1999. A finite sum representation of the Appell series \( F_1(a, b, b^*; c; x, y) \). *Journal of Computational and Applied Mathematics* 105 (1):213–19.

De Baets, B., H. De Meyer, and M. Ubeda-Flores. 2009. Opposite diagonal sections of quasi-copulas and copulas. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems* 17 (04):481–90. doi:10.1142/S0218488509006108.

Durante, F., and P. Jaworski. 2008. Absolutely continuous copulas with given diagonal sections. *Communications in Statistics - Theory and Methods* 37 (18):2924–42. doi:10.1080/03610920802050927.

Finney, D. J., and F. Tattersfield. 1952. *Probit analysis*. New York: Cambridge University Press.

Flores, M. U., E. de Amo, A. F. Durante, and J. F. Sanchez. 2017. *Copulas and dependence models with applications*. Springer International Publishing AG.

Gradshteyn, I. S., and I. M. Ryzhik. 2014. *Table of integrals, series, and products*. 8th ed. Amsterdam: Elsevier/Academic Press.

Hattis, D., P. Banati, and R. Goble. 1999. Distributions of individual susceptibility among humans for toxic effects. How much protection does the traditional tenfold factor provide for what fraction of which kinds of chemicals and effects? *Annals of the New York Academy of Sciences* 895 (1):286–316. doi:10.1111/j.1749-6632.1999.tb08092.x.

Hauptmanns, U. 2005. A risk-based approach to land-use planning. *Journal of Hazardous Materials* 125 (1-3):1–9. doi:10.1016/j.jhazmat.2005.05.015.

Jaworski, P. 2014. On the characterization of copulas by differential equations. *Communications in Statistics - Theory and Methods* 43 (16):3402–28. doi:10.1080/03610926.2012.700375.
Lovreglio, R., E. Ronchi, G. Maragkos, T. Beji, and B. Merci. 2016. A dynamic approach for the impact of a toxic gas dispersion hazard considering human behaviour and dispersion modeling. *Journal of Hazardous Materials* 318:758–71. doi:10.1016/j.jhazmat.2016.06.015.

Nelsen, R. B. 1993. Some concepts of bivariate symmetry. *Journal of Nonparametric Statistics* 3 (1):95–101. doi:10.1080/10485259308832574.

Nelsen, R. B. 2006. *An introduction to copulas*. 2nd ed. New York: Springer.

Stage, S. A. 2004. Determination of acute exposure guideline levels in a dispersion model. *Journal of the Air & Waste Management Association (1995)* 54 (1):49–59. doi:10.1080/10473289.2004.10470885.