Legendrian Contact Homology and Topological Entropy

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Abstract

We study a homotopical growth rate for a version of Legendrian contact homology, called strip Legendrian contact homology, in 3-dimensional contact manifolds and its relation to the topological entropy of Reeb flows. We show that if, for a pair of Legendrian knots in a contact 3-manifold, the strip Legendrian contact homology is defined and has exponential homotopical growth with respect to the action, then every Reeb flow on this contact manifold has positive topological entropy. This implies that on such a contact manifold for all Reeb flows, even degenerate ones, the number of hyperbolic periodic orbits grows exponentially with respect to the period. We show that for an infinite family of different 3-manifolds, infinitely many different contact structures exist which present exponential growth rate of strip Legendrian contact homology for certain pairs of Legendrian knots.

1 Introduction

The objective of this paper is study the growth rate of Legendrian contact homology and its implications to the dynamics of Reeb flows on contact 3-manifolds. It is part of a larger project of the author which aims at understanding the relationship between SFT-invariants of a contact structure and dynamical invariants of Reeb flows. The first study of such a relation was done by Macarini and Schlenk in [31]; they used the exponential growth rate of Lagrangian Floer homology to show that the topological entropy of Reeb flows on the unit tangent bundle \((T_1Q, \xi_{can})\) of an energy hyperbolic manifold \(Q\) (where \(\xi_{can}\) is the contact structure associated to geodesic flows) is always positive. As for more general contact manifolds Lagrangian Floer homology cannot usually be defined, we use here Legendrian Floer homology to obtain entropy estimates for more general families of contact 3-manifolds. We construct many examples of contact 3-manifolds (most of which are non-symplectically fillable) that have pairs of Legendrian knots with exponential homotopical growth rate of strip Legendrian contact homology and show that this implies positivity of topological entropy for all Reeb flows on this contact manifold; we refer the reader to section 2 for the definition of the strip Legendrian contact homology and a discussion on the conditions needed to guarantee its existence, and to section 3 for the definition of homotopical growth rate. If for a contact manifold, all associated Reeb flows
have positive topological entropy we will say that this contact manifold has the positive
topological entropy property.

We first recall some definitions from contact geometry. A 1-form \( \alpha \) on a \((2n + 1)\)-
dimensional manifold \( Y \) is said to be a contact form if \( \alpha \wedge (d\alpha)^n \) is a volume form on \( Y \).
The hyperplane \( \xi = \ker(\alpha) \) is called the contact structure. For us a contact manifold will
be a pair \((Y, \xi)\) such that \( \xi \) is the kernel of some contact form \( \alpha \) in \( Y \) (we point out that in
the literature these are sometimes called co-oriented contact manifolds). When \( \alpha \) satisfies
\( \xi = \ker(\alpha) \), we will say that \( \alpha \) is a contact form associated to \((Y, \xi)\). Notice that for a
given contact manifold, there exist infinitely many different contact forms associated to
it. Given a contact form \( \alpha \), its Reeb vector field is the unique vector field \( X_\alpha \) satisfying
\( \alpha(X_\alpha) = 1 \) and \( i_{X_\alpha}d\alpha = 0 \); the Reeb flow of \( \alpha \) is the flow of the vector field \( X_\alpha \). We will
refer to the periodic orbits of the Reeb flow as Reeb orbits. A contact form \( \alpha \) is called
hypertight if it doesn’t have any contractible Reeb orbits. For a Reeb orbit \( \gamma \) of the Reeb
flow of \( \alpha \) its action \( A(\gamma) \) is the period of \( \gamma \). A Reeb orbit \( \gamma \) is said to be non-degenerate
when 1 is not an eigenvalue of the linearisation \( D\phi^{A(e)}_{X_\alpha} \big|_\xi \) of the Poincare return map
associated to the \( \gamma \).

An isotropic submanifold of \((Y, \xi)\) is a submanifold of \( Y \) whose tangent space is always
contained \( \xi \); when this submanifold \( \Lambda \) is of maximal possible dimension, it is called a
Legendrian submanifold of \((Y, \xi)\). It turns out that this maximal possible dimension is \( n \), i.e \( \dim(\Lambda) = n \). Given a contact form \( \alpha \) and a pair of Legendrian submanifolds \((\Lambda, \hat{\Lambda})\), a Reeb chord from \( \Lambda \) to \( \hat{\Lambda} \) is a trajectory \( c \) of the Reeb flow of \( \alpha \) that starts in \( \Lambda \) and
ends at \( \hat{\Lambda} \) (notice that here \( \Lambda \) and \( \hat{\Lambda} \) might coincide). We define the action \( A(c) \) of a Reeb
chord \( c \) as \( A(c) = \int_c \alpha \). Lastly, a Reeb chord \( c \) is said to be transverse, if the intersection
\( \phi^{A(e)}_{X_\alpha}(\Lambda) \cap \hat{\Lambda} \) is transverse at the endpoint of \( c \).

It is an important result in the theory of geodesic flows that if a manifold \( Q \) has a
"complicated" topology then for every Riemannian metric on this manifold the geodesic
flow on the unit tangent bundle \( T_1Q \) of this manifold has positive topological entropy; see Paternain’s book [33] for more on this subject. The geodesic flow on the unit tangent
bundle is an important example of a Reeb flow, and thus defines a contact structure \( \xi_{\text{can}} \)
on \( T_1Q \). The work of Macarini and Schlenk can be seen as an extension of these results
on geodesic flows to the Reeb flows associated to \((T_1X, \xi_{\text{can}})\). In their work, they use
the Lagrangian Floer homology of the Lagrangian fibers of \( TQ \). As more general contact
manifolds are not symplectically fillable, we substitute the Lagrangian Floer homology by
strip Legendrian contact homology which we define and show to be well-defined.

Our main results are the following:

**Theorem 1.** Let \((Y, \xi = \ker(\lambda_0))\) be a contact 3-manifold with a hypertight contact form
\( \lambda_0 \) adapted to the pair of Legendrian knots \((\Lambda, \hat{\Lambda})\). Assume that \( \text{LCH}_{\text{st}}(\lambda_0, \Lambda \to \hat{\Lambda}) \) has
exponential homotopical growth rate (with respect to the action) with exponential weight
\( a > 0 \). For any contact form \( \lambda \) associated to \((Y, \xi)\), let \( f_\lambda \) be the unique function such that
\( \lambda = f_\lambda \lambda_0 \). Then, the Reeb flow of \( X_\lambda \) has positive topological entropy, and moreover:
We explain in section 2.2, what it means for a contact form to be adapted to a pair of Legendrian curves. The notion of homotopical growth rate is used in theorem 1 to avoid dealing with transversality problems that arise from the appearance of multiply covered holomorphic curves. However, if one accepts that these transversality problems can be solved (as it is believed by many, through the use of the Polyfold technology being developed by Hofer, Wysocki and Zehnder) then we can substitute “exponential homotopical growth rate” by “exponential growth rate”. It is an interesting fact, that in all 3-dimensional examples known to the author where one has exponential growth rate of Legendrian contact homology, one also has exponential homotopical growth rate of Legendrian contact homology. Although I believe that this should indeed always be the case, a proof of this fact seems completely beyond current technology.

Very importantly for our point view, we present also many new examples which satisfy the hypothesis of theorem 1 above:

**Theorem 2.** Let $M$ be a closed oriented connected 3-manifold which can be cut along a nonempty family of incompressible tori into a family $\{M_i, 0 \leq i \leq k\}$ of irreducible manifolds with boundary such that the component $M_0$ satisfies:

- $M_0$ is the mapping torus of a punctured torus $S$ by a diffeomorphism $h : S \to S$ such that the homology map $h_* : H_1(S) \iso \Z \oplus \Z$ is a hyperbolic automorphism.

Then $M$ can be given infinitely many different tight contact structures $\xi_k$, such that there exists Legendrian links $\Lambda_k$, $\Lambda'_k$ and contact forms $\tau_k$ associated to $(M, \xi_k)$ and a adapted to the pair $\Lambda_k$, $\Lambda'_k$ for which $LC_{\mathbb{H}_{st}}(\tau_k, \Lambda_k \to \Lambda'_k)$ has exponential homotopical growth rate.

The contact manifolds of this theorem fall under the umbrella of the examples constructed in [12] and studied in [35] and in [3]. In particular, the theorem above implies that the contact 3-manifolds above have the positive topological entropy property, a result that also follows from Theorem 3 in [3].

The structure of the paper is the following: in section 2 we define the strip Legendrian contact homology and show it to be well defined under certain conditions; in section 3 we define the exponential growth rate of linearized Legendrian contact homology and use it to make estimates on the number of Reeb chords between 2 Legendrian knots. In subsection 3.2, we apply these estimates to prove Theorem 1. In section 4 we construct examples of contact 3-manifolds that have a pair Legendrian knots for which the strip Legendrian contact homology grows exponentially and prove Theorem 2. In section 5 we recall the consequences of positivity of topological entropy for flows in 3-manifolds, make some concluding remarks on future developments and pose some open questions.

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2 The strip Legendrian contact homology

The strip Legendrian contact homology is defined in the spirit of the SFT invariants introduced in [19]. To define it we use pseudoholomorphic curves in symplectizations of contact manifolds. Pseudoholomorphic curves were introduced in symplectic manifolds by Gromov in [23] and adapted to symplectizations of contact manifolds by Hofer [25]. We refer the reader to [9] for a rather complete discussion on pseudoholomorphic curves in symplectic cobordisms.

2.1 Almost complex structures in symplectizations and symplectic cobordisms

We start by reviewing the basic facts about pseudoholomorphic curves in symplectizations and symplectic cobordisms.

2.1.1 Cylindrical almost complex structures

Let \((Y, \xi)\) be a contact manifold and \(\lambda\) an associated contact form. The symplectization of \((Y, \xi)\) is the product \(\mathbb{R} \times Y\) with the symplectic form \(d(e^s \lambda)\) (where \(s\) denotes the \(\mathbb{R}\) coordinate in \(\mathbb{R} \times Y\)). \(d\lambda\) restricts to a symplectic form on the vector bundle \(\xi\) and it is well known that the set \(j(\lambda)\) of \(d\lambda\)-compatible almost complex structures on the symplectic vector bundle \(\xi\) is non-empty and contractible. Notice that if \(Y\) is 3-dimensional the set \(j(\lambda)\) doesn’t depend on the contact form \(\lambda\) associated to \((Y, \xi)\).

For \(j \in j(\lambda)\) we can define an \(\mathbb{R}\)-invariant almost complex structure \(J\) on \(\mathbb{R} \times Y\) by demanding that:

\[
J \partial_s = X_\lambda, \quad J |_\xi = j
\]  

We will denote by \(\mathcal{J}(\lambda)\) the set of almost complex structures in \(\mathbb{R} \times Y\) that are \(\mathbb{R}\)-invariant, \(d(e^s \lambda)\)-compatible and satisfy equation (1) above.

2.1.2 Exact symplectic cobordisms with cylindrical ends

Let \(\varpi = d\sigma\) be an exact symplectic form on \(\mathbb{R} \times Y\) for which there exist contact forms \(\lambda^+\) and \(\lambda^-\) in \(Y\), real numbers \(R^+ > R^-\) such that:

\[
\]
\[ \sigma = e^{s-R^+} \lambda^+ \text{ in } [R^+, +\infty) \times Y \] (3)

\[ \sigma = e^{s-R^-} \lambda^- \text{ in } (-\infty, R^-] \times Y \] (4)

We call \((W = \mathbb{R} \times Y, \varpi)\) an exact symplectic cobordism from \(\lambda^+\) to \(\lambda^-\). Notice that the symplectization of a \((Y, \lambda)\) is a special case of an exact symplectic cobordism from \(\lambda\) to \((1-\varepsilon)\lambda\), with \(1 > \varepsilon > 0\). We divide \((W, \varpi)\) in 3 pieces and denote \(W(\lambda^+) = [R^+, +\infty) \times Y\), \(W(\lambda^+, \lambda^-) = [R^-, R^+] \times Y\) and \(W(\lambda^-) = (-\infty, R^-] \times Y\). In such a cobordism we say that an almost complex structure \(J\) is cylindrical if:

\[ \mathcal{J} \text{ coincides with } J^+ \in \mathcal{J}(\lambda^+) \text{ in the region } [R^+, +\infty) \times Y \] (5)

\[ \mathcal{J} \text{ coincides with } J^- \in \mathcal{J}(\lambda^-) \text{ in the region } (-\infty, R^-] \times Y \] (6)

\[ \mathcal{J} \text{ is compatible with } \varpi \text{ in } [R^-, R^+] \times Y \] (7)

The set of cylindrical almost complex structures in \((\mathbb{R} \times Y, \varpi)\) is denoted by \(\mathcal{J}(\lambda^+, \lambda^-)\), and it is well known to be contractible. We will write \(\lambda^+ \succ_{\text{ex}} \lambda^-\) when there exists an exact symplectic from \(\lambda^+\) to \(\lambda^-\) as above. We remind the reader that \(\lambda^+ \succ_{\text{ex}} \lambda\) and \(\lambda \succ_{\text{ex}} \lambda^-\) implies \(\lambda^+ \succ_{\text{ex}} \lambda^-\); or in other words that the relation \(\succ_{\text{ex}}\) is transitive; see [9].

### 2.1.3 Splitting symplectic cobordisms

Let \(\lambda^+, \lambda\) and \(\lambda^-\) be contact forms associated to \((Y, \xi)\) such that \(\lambda^+ \succ_{\text{ex}} \lambda\), \(\lambda \succ_{\text{ex}} \lambda^-\). In this case, there exists an \(\varepsilon > 0\) such that \(\lambda^+ \succ_{\text{ex}} (1+\varepsilon)\lambda\) and \((1-\varepsilon)\lambda \succ_{\text{ex}} \lambda^-\). It is then possible to construct, for each \(R > 0\), an exact symplectic form \(\varpi_R = d\sigma_R\) on \(W = \mathbb{R} \times Y\) satisfying:

\[ \sigma_R = e^{s-R^+} \lambda^+ \text{ in } [R + 2, +\infty) \times Y \] (8)

\[ \sigma_R = f(s)\lambda \text{ in } [-R, R] \times Y \] (9)

\[ \sigma_R = e^{s-R^-} \lambda^- \text{ in } (-\infty, -R - 2] \times Y \] (10)

where \(f : [-R, R] \to [1-\varepsilon, 1+\varepsilon]\), \(f(-R) = 1-\varepsilon\), \(f(R) = 1+\varepsilon\) and \(f' > 0\). In \((\mathbb{R} \times Y, \varpi_R)\) we consider a compatible cylindrical almost complex structure \(\tilde{J}_R\); but we demand an extra condition on \(\tilde{J}_R\):

\[ \tilde{J}_R \text{ coincides with } J \in \mathcal{J}(\lambda) \text{ in } [-R, R] \times Y \] (11)

Again we divide \(W\) in regions: \(W(\lambda^+) = ([R+2, +\infty) \times Y)\), \(W(\lambda^+, \lambda) = ([R, R+2] \times Y)\), \(W(\lambda) = ([R-2, R] \times Y)\), \(W(\lambda, \lambda^-) = ([R-2, -R] \times Y)\) and \(W(\lambda^-) = ((-\infty, -R-2] \times Y)\).
Let $\lambda^+, \lambda$ and $\lambda^-$ be contact forms in $Y$. Suppose now that $\lambda^+ = \zeta^+ \lambda^-$ and $\lambda = \zeta \lambda^-$ for functions $\zeta : Y \to \mathbb{R}$ and $\zeta^+ : Y \to \mathbb{R}$ satisfying $\zeta^+ > \zeta + \epsilon > \zeta - \epsilon > 1$ for some $\epsilon > 0$. For each $R > 1$, let $\chi_R : \mathbb{R} \times Y \to \mathbb{R}$ such that:

\begin{align*}
\partial_s \chi_R &> 0 \text{ in } [-R, R] \times Y \quad (12) \\
\chi_R(s) &= e^{s-R-2} \zeta^+ \text{ in } [R+2, +\infty) \times Y \quad (13) \\
\chi_R(s) &= (1 + \frac{s}{R}) \zeta \text{ in } [-R, R] \times Y \quad (14) \\
\chi_R(s) &= e^{s+R+2} \text{ in } (-\infty, -R-2] \times Y \quad (15)
\end{align*}

Then $(\mathbb{R} \times Y, d(\chi R \lambda^-))$ is an exact symplectic cobordism with cylindrical ends from $\lambda^+$ to $\lambda^-$. As $R \to +\infty$ the region where the symplectic form $d(\chi R \lambda^-)$ becomes similar to the symplectization of $(Y, \lambda)$ becomes arbitrarily large.

To gain an intuition about this construction, one can initially think that in the limit as $R \to +\infty$ the sequence $(\mathbb{R} \times Y, d(\chi R \lambda^-))$ splits into two exact symplectic cobordisms, $V(\lambda^+, \lambda)$ from $\lambda^+$ to $\lambda$, followed by $V(\lambda, \lambda^-)$ from $\lambda$ to $\lambda^-$. Actually when one studies pseudoholomorphic curves in such cobordisms the limiting object is more complicated than just the pair of two cobordisms we mentioned; levels of symplectizations have to be inserted above $V(\lambda^+, \lambda)$, between $V(\lambda^+, \lambda)$ and $V(\lambda, \lambda^-)$, and below $V(\lambda, \lambda^-)$ to complete the picture. We refer again to the paper [9] for a complete discussion about this topic.

### 2.1.4 Exact Lagrangian cobordisms

Let $\Lambda^+$ and $\Lambda^-$ be Legendrian submanifolds in, respectively, $(Y, \ker(\lambda^+))$ and $(Y, \ker(\lambda^-))$ and let $(\mathbb{R} \times Y, \omega = d\kappa)$ be an exact symplectic cobordism from $\lambda^+$ to $\lambda^-$. We call a Lagrangian submanifold in $(\mathbb{R} \times Y, \omega)$ a Lagrangian cobordism if there exists Legendrian submanifolds $\Lambda^+$ in $(Y, \ker(\lambda^+))$ and $\Lambda^-$ in $(Y, \ker(\lambda^-))$, and $N > 0$ such that:

\begin{align*}
L \cap ([N, +\infty) \times Y) &= ([N, +\infty] \times \Lambda^+), \quad (16) \\
L \cap ((-\infty, -N] \times Y) &= ((-\infty, -N] \times \Lambda^-). \quad (17)
\end{align*}

In this case we say that $L$ is a Lagrangian cobordism from $\Lambda^+$ to $\Lambda^-$. If such an $L$ is an exact Lagrangian submanifold of $(\mathbb{R} \times Y, d\kappa)$, we call it an exact Lagrangian cobordism $\Lambda^+$ to $\Lambda^-$. 

**Example:** if we take a Legendrian submanifold $\Lambda$ in $(Y, \ker(\lambda^-))$ then $\mathbb{R} \times \Lambda$ is an exact Lagrangian submanifold in the symplectization of $(Y, \lambda^-)$. It is also an exact Lagrangian cobordism in $(\mathbb{R} \times Y, d(\chi R \lambda^-))$ from $\Lambda$ to itself.
2.1.5 Pseudoholomorphic curves

Let \((S, i)\) be a closed Riemann surface with boundary, with a finite set \(\Gamma \subset S\). We denote \(\Gamma_\partial = \partial(S) \cap \Gamma\).

Let \(\lambda\) be a contact form in \(Y\) and \(J \in \mathcal{J}(\lambda)\). A finite energy pseudoholomorphic curve in the symplectization \((\mathbb{R} \times Y, J)\) with boundary in a Lagrangian submanifold \(L\) is a map \(\tilde{w} : (S \setminus \Gamma; \partial(S) \setminus \Gamma_\partial) \to (\mathbb{R} \times Y; L)\) satisfying:

\[
d\tilde{w} \circ i = J \circ d\tilde{w},
\]

and

\[
0 < E(\tilde{w}) = \sup_{q \in \mathcal{E}} \int_{S \setminus \Gamma} \tilde{w}^* d(q\lambda)
\]

where \(\mathcal{E} = \{q : \mathbb{R} \to [0, 1]; q' \geq 0\}\). The quantity \(E(\tilde{w})\) is called the Hofer energy, and was introduced in [25]. We write \(\tilde{w} = (s, w) \in \mathbb{R} \times Y\).

For us, it will be particularly important the case where \((S \setminus \Gamma, i)\) is biholomorphic \((\mathbb{R} \times [0, 1], i_0)\) (where \(i_0\) is the complex structure in \(\mathbb{C}\)) and \(L = (\mathbb{R} \times \Lambda) \cup (\mathbb{R} \times \hat{\Lambda})\), with \(\tilde{w}(\{0\} \times \mathbb{R}) \subset (\mathbb{R} \times \Lambda)\) and \(\tilde{w}(\{1\} \times \mathbb{R}) \subset (\mathbb{R} \times \hat{\Lambda})\). In this case \(\tilde{w}\) is called a pseudoholomorphic strip. By using a bi-holomorphism \(\varphi : (\mathcal{D} \setminus \{-1, 1\}, i_0) \to (\mathbb{R} \times [0, 1], i_0)\) satisfying \(\varphi(H^+) = \{0\} \times \mathbb{R}\) (where \(H^+ \subset (S^1 \setminus \{-1, 1\})\) is the northern hemisphere of \(S^1\)) and \(\varphi(H^-) = \{0\} \times \mathbb{R}\) (where \(H^- \subset (S^1 \setminus \{-1, 1\})\) is the southern hemisphere) we can also view pseudoholomorphic strips as maps having as domain the disc with two punctures on the boundary.

For an exact symplectic cobordism \((W = \mathbb{R} \times Y, \omega)\) from \(\lambda^+\) to \(\lambda^-\), and \(\overline{\mathcal{J}} \in \mathcal{J}(J^-, J^+)\) a finite energy pseudoholomorphic curve with boundary in a Lagrangian submanifold \(L\) is again a map \(\tilde{w} : (S \setminus \Gamma, \partial(S) \setminus \Gamma_\partial) \to (\mathbb{R} \times Y, L)\) satisfying:

\[
d\tilde{w} \circ i = \overline{\mathcal{J}} \circ d\tilde{w},
\]

and

\[
0 < E_{\lambda^-}(\tilde{w}) + E_c(\tilde{w}) + E_{\lambda^+}(\tilde{w}) < +\infty,
\]

where:

\[
E_{\lambda^-}(\tilde{w}) = \sup_{q \in \mathcal{E}} \int_{\mathcal{W}^{-1}(\lambda^+)} \tilde{w}^* d(q\lambda^+),
\]

\[
E_{\lambda^+}(\tilde{w}) = \sup_{q \in \mathcal{E}} \int_{\mathcal{W}^{-1}(\lambda^+)} \tilde{w}^* d(q\lambda^+),
\]

\[
E_c(\tilde{w}) = \int_{\mathcal{W}^{-1}(\lambda^-)} \tilde{w}^* \omega.
\]

These energies were also introduced in [25].

In splitting symplectic cobordisms the definition of finite energy pseudoholomorphic map is the same, except that we consider a slightly modified version of energy. Instead of demanding \(0 < E_{\lambda^-}(\tilde{w}) + E_c(\tilde{w}) + E_{\lambda^+}(\tilde{w}) < +\infty\) we demand:

\[
0 < E_{\lambda^-}(\tilde{w}) + E_{\lambda^-}(\tilde{w}) + E_{\lambda}(\tilde{w}) + E_{\lambda^+}(\tilde{w}) + E_{\lambda^+}(\tilde{w}) < +\infty
\]
where:
\[ E_\lambda(\tilde{w}) = \sup_{q \in M} \int_{\tilde{w}^{-1}W(\lambda)} \tilde{w}^* d(q\lambda), \]
\[ E_{\lambda-,\lambda}(\tilde{w}) = \int_{\tilde{w}^{-1}W(\lambda-,\lambda)} \tilde{w}^* \omega, \]
\[ E_{\lambda+,\lambda}(\tilde{w}) = \int_{\tilde{w}^{-1}W(\lambda+,\lambda)} \tilde{w}^* \omega, \]
and \( E_{\lambda-}(\tilde{w}) \) and \( E_{\lambda+}(\tilde{w}) \) are as above.

2.1.6 Asymptotic behaviour of pseudoholomorphic curves

The elements of the set \( \Gamma \subset S \) are called punctures of the pseudoholomorphic \( \tilde{w} \). We first divide \( \Gamma \) in two classes: we call the elements of \( \Gamma_\partial \) boundary punctures and the elements in \( \Gamma \setminus \Gamma_\partial \) interior punctures. The work of Hofer [25], Hofer et al. [26] and Abbas [1] allows us do classify the punctures in four different types.

Before presenting this classification we introduce some notation: we let \( B_\delta(z) \) be the ball of radius \( \delta \) centered at the puncture \( z \), and denote by \( b_\delta(z) \) the set defined as the closure \( \overline{\partial(B_\delta(z)) \cap \text{int}(S)} \) of the intersection of the boundary of \( B_\delta(z) \) with the interior of \( S \). Notice that \( b_\delta(z) \) is a circle or an interval, depending on whether \( z \) is an interior or a boundary puncture. With this in hand, we can describe the types of punctures as follows:

- \( z \in \Gamma \) is called positive boundary puncture when \( z \in \Gamma_\partial \) and \( \lim_{z' \to z} s(z') = +\infty \), and in this case, there exists a sequence \( \delta_n \to 0 \) and a Reeb chord \( c^+ \) of \( X^+ \) to itself, such that \( w(b_{\delta_n}(z)) \) converges in \( C^\infty \) to \( c^+ \) as \( n \to +\infty \);
- \( z \in \Gamma \) is called negative boundary puncture when \( z \in \Gamma_\partial \) and \( \lim_{z' \to z} s(z') = -\infty \), and in this case there exists a sequence \( \delta_n \to 0 \) and Reeb chord \( c^- \) of \( X^- \) to itself, such that \( w(b_{\delta_n}(z)) \) converges in \( C^\infty \) to \( c^- \) as \( n \to +\infty \);
- \( z \in \Gamma \) is called positive interior puncture when \( z \in \Gamma \setminus \Gamma_\partial \) and \( \lim_{z' \to z} s(z') = +\infty \), and in this case there exists a sequence \( \delta_n \to 0 \) and Reeb orbit \( \gamma^+ \) of \( X^+ \), such that \( w(b_{\delta_n}(z)) \) converges in \( C^\infty \) to \( \gamma^+ \) as \( n \to +\infty \);
- \( z \in \Gamma \) is called negative interior puncture when \( z \in \Gamma \setminus \Gamma_\partial \) and \( \lim_{z' \to z} s(z') = -\infty \), and in this case there exists a sequence \( \delta_n \to 0 \) and Reeb orbit \( \gamma^- \) of \( X^- \), such that \( w(b_{\delta_n}(z)) \) converges in \( C^\infty \) to \( \gamma^- \) as \( n \to +\infty \).

The results in [25], [26] and [1] imply that these are indeed the only real possibilities we need to consider for the behaviour of the \( \tilde{w} \) near punctures. Intuitively, we have that at the punctures, the pseudoholomorphic curve \( \tilde{w} \) detects Reeb chords and Reeb orbits. For a boundary (interior) puncture \( z \), if there is a subsequence \( \delta_n \) such that \( w(b_{\delta_n}(z)) \) converges to a given Reeb chord \( c \) (orbit \( \gamma \)), we will say that \( \tilde{w} \) is asymptotic to this Reeb chord \( c \) (orbit \( \gamma \)) at the puncture \( z \).

If a pseudoholomorphic curve is asymptotic to a transverse Reeb chord or a non-degenerate Reeb orbit at a puncture, more can be said about its asymptotic behaviour in neighbourhoods of this puncture. In order to describe this behaviour for a boundary puncture \( z \), we take a neighbourhood \( U \) of \( z \) that admits a holomorphic chart \( \psi_U : (U \setminus \{z\}) \to \mathbb{R}^+ \times [0,1] \subset \mathbb{C} \), such that \( \psi_U((U \cap \partial(S)) \setminus \{z\}) = \mathbb{R}^+ \times \{0\} \cup \mathbb{R}^+ \times \{1\} \).

In coordinates \((r,t) \in \mathbb{R} \times [0,1]\) we have \( r(x) \to +\infty \) when \( x \) tends to the puncture,

\[ \lim_{x \to \text{puncture}} r(x) = +\infty. \]

In coordinates \((r,t) \in \mathbb{R} \times [0,1]\) we have \( r(x) \to +\infty \) when \( x \) tends to the puncture,

\[ \lim_{x \to \text{puncture}} r(x) = +\infty. \]
2.2 Strip Legendrian contact homology

We are now ready to define the strip Legendrian contact homology. First, we introduce the following notation: for a given contact form $\lambda$ we denote by $\mathcal{T}_{A \to \hat{\Lambda}}(\lambda)$ the set of Reeb chords of $X_\lambda$ starting at $\Lambda$ and ending at $\hat{\Lambda}$. 

$\overline{z}$. With this notation, it is shown in [1], that if $z$ is a positive boundary puncture on which $\tilde{w}$ is asymptotic to a transverse Reeb chord $c^+$ of $X_{\lambda^+}$ from $\overline{\Lambda}^+$ to itself, then $\tilde{w} \circ \psi^{-1}_w(r, t) = (s(r, t), w(r, t))$ satisfies:

- $w^r(t) = w(r, t)$ converges uniformly in $C^\infty$ to the Reeb chord $c^+$ of $X_{\lambda^+}$ from $\Lambda^+$ to $\hat{\Lambda}^+$ as $r \to +\infty$ (where $\Lambda^+$ and $\hat{\Lambda}^+$ denote connected components of $\overline{\Lambda}^+$); moreover the convergence rate is exponential.

Similarly, if $z$ is a negative boundary puncture on which $\tilde{w}$ is asymptotic to a transverse Reeb chord $c^-$ of $X_{\lambda^-}$ from $\overline{\Lambda}^-$ to itself, then $\tilde{w} \circ \psi^{-1}_w(r, t) = (s(r, t), w(r, t))$ satisfies:

- $w^r(t) = w(r, t)$ converges uniformly in $C^\infty$ to the inverse parametrization of Reeb chord $c$ of $X_{\lambda^-}$ from $\Lambda^-$ to $\hat{\Lambda}^-$ as $r \to +\infty$ (where $\Lambda^-$ and $\hat{\Lambda}^-$ denote connected components of $\overline{\Lambda}^-$); moreover the convergence rate is exponential.

We discuss now the case of $z$ being an interior puncture; we pick a neighbourhood $U$ of $z$ and a holomorphic chart $\psi_U : (U, z) \to (D, 0)$. Using polar coordinates $(r, t) \in (0, +\infty) \times S^1$ we can write $x \in (D \setminus 0)$ as $x = e^{-t}$. With this notation, it is shown in [20][25], that if $z$ is a positive interior puncture on which $\tilde{w}$ is asymptotic to a non-degenerate Reeb orbit $\gamma^+$ of $X_{\lambda^+}$, then $\tilde{w} \circ \psi^{-1}_w(r, t) = (s(r, t), w(r, t))$ satisfies:

- $w^r(t) = w(r, t)$ converges uniformly in $C^\infty$ to a Reeb orbit $\gamma$ of $X_{\lambda^+}$, and the convergence rate is exponential.

Similarly, if $z$ is a negative interior puncture on which $\tilde{w}$ is asymptotic to a non-degenerate Reeb orbit $\gamma^-$ of $X_{\lambda^-}$, then $\tilde{w} \circ \psi^{-1}_w(r, t) = (s(r, t), w(r, t))$ satisfies:

- $w^r(t) = w(r, t)$ converges uniformly in $C^\infty$ to a Reeb orbit $\gamma$ of $-X_{\lambda^-}$ as $r \to +\infty$, and the convergence rate is exponential.

Remark: the exponential rate of convergence of pseudoholomorphic curves near punctures to Reeb orbits and Reeb chords is of crucial importance for the Fredholm theory that gives the dimension of the space of pseudoholomorphic curves with fixed asymptotic data, and is a consequence of asymptotic formulas obtained in [20] and [4].

The discussion above can be summarised by saying that near punctures the finite pseudoholomorphic curves detect Reeb orbits and Reeb chords. It is exactly this behavior that makes these objects useful for the study of dynamics of Reeb vector fields.

Fact: as a consequence of the exactness of the symplectic cobordisms and the Lagrangian submanifolds considered above we obtain that the energy $E(\tilde{w})$ of $\tilde{w}$ satisfies $E(\tilde{w}) \leq 5A(\tilde{w})$ where $A(\tilde{w})$ is the sum of the action of the Reeb orbits and Reeb chords detected by the punctures of $\tilde{w}$ counted with multiplicity.
Suppose that there is a contact form \( \lambda_0 \) associated to \((Y, \xi)\), and disjoint Legendrian knots \( \Lambda \) and \( \hat{\Lambda} \) in \((Y, \xi)\), such that the Reeb flow of \( \lambda_0 \):

- (a) has no contractible orbits,
- (b) has no Reeb chords of from \( \Lambda \) to itself that vanish in \( \pi_1(Y, \Lambda) \),
- (c) has no Reeb chords of from \( \hat{\Lambda} \) to itself that vanish in \( \pi_1(Y, \hat{\Lambda}) \),
- (d) and all Reeb chords in \( T_{\Lambda \rightarrow \hat{\Lambda}}(\lambda_0) \) are transverse.

We will call such a contact form \textit{adapted} to the pair \((\Lambda, \hat{\Lambda})\). In order to assign a \( \mathbb{Z}_2 \)-grading to the Reeb chords in \( T_{\Lambda \rightarrow \hat{\Lambda}}(\lambda_0) \), we use the Conley-Zehnder index. For the definition, we first fix, once and for all, orientations for \( \Lambda \) and \( \hat{\Lambda} \). Then, for each Reeb chord \( c \in T_{\Lambda \rightarrow \hat{\Lambda}}(\lambda_0) \), let \( \Psi_c \) be a nowhere vanishing section of the vector bundle \( \xi|_c \) that:

- is tangent to \( \Lambda \) on the initial point of \( c \) and, furthermore, coincides with the orientation we fixed for \( \Lambda \) at this initial point;
- is tangent to \( \hat{\Lambda} \) on the final point of \( c \) and, furthermore, coincides with the orientation we fixed for \( \hat{\Lambda} \) at this final point.

The section \( \Psi_c \) induces a \textit{(unique up to homotopy) symplectic trivialisation of} \((\xi|_c, d\lambda_0)\), which we also denote by \( \Psi_c \).

Using the Reeb flow \( \phi_{X_{\lambda_0}} \) we define a path of Lagrangian subspaces \( \mathcal{Z} \) of \((\xi|_c, d\lambda_0)\). We consider the parametrisation \( c : [0, T_c) \rightarrow Y \) of the Reeb chord \( c \) given by the Reeb flow. Letting \( D\phi_{X_{\lambda_0}} \) denote linearisation of the Reeb flow, we define \( \mathcal{Z}(t) \) to be the unique Lagragian subspace of \((\xi|_{c(t)}, d\lambda_0)\) that contains \( D\phi_{X_{\lambda_0}}(c(0))(q) \), where \( q \in \xi|_{c(0)} \) is a vector tangent to \( \Lambda \) and giving the orientation we chose for \( \Lambda \). After arriving at the endpoint \( c(T_c) \), we complete \( \mathcal{Z} \) to obtain a Lagrangian loop by making a continuous left-rotation of \( \mathcal{Z}(T_c) \) (among Lagrangian subspaces of \( \xi|_{c(T_c)} \)) till it meets the tangent space to \( \hat{\Lambda} \). With this completion and using our trivialisation \( \Psi_c \), we associate to \( \mathcal{Z} \) a path of Lagrangian subspaces of the standard symplectic plane; the Conley-Zehnder \( \mu_{CZ}^\psi(c) \) is defined to be the Maslov index of this path.

It is clear from the constructions above that, because we fixed the orientations of \( \Lambda \) and \( \hat{\Lambda} \), the parity of \( \mu_{CZ}^\psi(c) \) is independent of the trivialisation \( \Psi_c \). This allows us to define, for each \( c \in T_{\Lambda \rightarrow \hat{\Lambda}}(\lambda_0) \), its \( \mathbb{Z}_2 \)-grading by \( |c| = (\mu_{CZ}^\psi(c) + 1) \mod 2 \). We call chords with grading 0 even chords, and chords with grading 1 odd chords.

Let \( LCH_{st}(\lambda_0, \Lambda \rightarrow \hat{\Lambda}) \) be the \( \mathbb{Z}_2 \) vector-space generated by \( T_{\Lambda \rightarrow \hat{\Lambda}}(\lambda_0) \). We denote by \( LCH_{st,odd}(\lambda_0, \Lambda \rightarrow \hat{\Lambda}) \) to be the subspace of \( LCH_{st}(\lambda_0, \Lambda \rightarrow \hat{\Lambda}) \) generated by odd chords, and \( LCH_{st,even}(\lambda_0, \Lambda \rightarrow \hat{\Lambda}) \) to be the subspace generated by even chords. Given two Reeb chords \( c_1 \) and \( c_2 \), and an almost complex structure \( J \in J(\lambda_0) \), we know from the previous section that it makes sense to consider the moduli space \( M(c_1, c_2; J) \) of finite energy pseudoholomorphic strips, modulo reparametrizations, \( \tilde{w} : (\overline{D} \setminus \{-1, 1\}, i_0) \rightarrow (\mathbb{R} \times Y, J) \) satisfying:

- 1 is a positive boundary puncture, and \( \tilde{w} \) is asymptotic to \( c_1 \) at 1,
• \(-1\) is a negative boundary puncture, and \(\tilde{w}\) is asymptotic to \(c_2\) at \(-1\),
• \(\tilde{w}(H_+) \subset \mathbb{R} \times \Lambda\),
• \(\tilde{w}(H_-) \subset \mathbb{R} \times \hat{\Lambda}\).

It follows from Abbas’ asymptotic analysis seen in the previous section, that all the elements of the moduli space \(\mathcal{M}(c_1, c_2; J)\) are somewhere injective pseudoholomorphic curves. It is well known that the linearization \(D\overline{\partial}_J\) at any element \(\mathcal{M}(c_1, c_2; J)\) is a Fredholm map (we remark that this property is valid for more general moduli spaces of curves with prescribed asymptotic behaviour). Moreover, Dragnev [10] and Abbas [2] showed that for a generic set \(\mathcal{J}_{\text{reg}}(\lambda_0) \subset \mathcal{J}(\lambda_0)\) all the elements in \(\mathcal{M}(c_1, c_2; J)\) are transverse in the sense that the linearization \(D\overline{\partial}_J\) of the Cauchy-Riemann operator \(\partial_J\) at the elements of \(\mathcal{M}(c_1, c_2; J)\) is surjective; this being valid for every Reeb chords \(c_1\) and \(c_2\).

Thus, in the case where \(J \in \mathcal{J}_{\text{reg}}(\lambda_0)\) one can use the implicit function theorem, and obtain that any connected component of the moduli space \(\mathcal{M}(c_1, c_2; J)\) is a finite dimensional manifold, and its dimensions is given by the Fredholm index \(I_F\) of \(D\overline{\partial}_J\) computed in any element of this connected component of \(\mathcal{M}(c_1, c_2; J)\). We let \(\mathcal{M}^k(c_1, c_2; J) \subset \mathcal{M}(c_1, c_2; J)\) be the moduli space of pseudoholomorphic strips with Fredholm index \(k\).

It follows from the formula in [2], for the Fredholm index \(I_F\) of the linearised \(D\overline{\partial}_J\) operator over a strip in \(\mathcal{M}(c_1, c_2; J)\), that \(I_F\) has the same parity of the \(|c_1 + c_2|\). Notice also that, as \(0 < \int_{\mathcal{D}_{\mathcal{A}}(-1,1)} \tilde{w}^*(d\lambda_0) = A(c_1) - A(c_2)\), \(\mathcal{M}(c_1, c_2; J)\) can only be non-empty if \(A(c_1) \geq A(c_2)\). Because of the \(\mathbb{R}\)-invariance of the almost complex structure \(J\) there is an \(\mathbb{R}\)-action on the spaces \(\mathcal{M}(c_1, c_2; J)\), and we let \(\tilde{\mathcal{M}}(c_1, c_2; J) = \mathcal{M}(c_1, c_2; J)/\mathbb{R}\).

We are now ready to define a differential \(d_J\) in \(LCH_{st}(\lambda_0, \Lambda \to \hat{\Lambda})\).

**Definition 1.** Let \(c \in T_{\Lambda \to \hat{\Lambda}}(\lambda_0)\) and \(J \in \mathcal{J}_{\text{reg}}(\lambda_0) \subset \mathcal{J}(\lambda_0)\). We define:

\[
d_J(c) = \sum_{c' \in T_{\Lambda \to \hat{\Lambda}}(\lambda_0)} |n_{c,c'} \mod 2| c'
\]  

(23)

where \(n_{c,c'}\) is the cardinality of the moduli space \(\mathcal{M}^1(c, c'; J)\) of pseudoholomorphic strips of Fredholm index 1 modulo the \(\mathbb{R}\)-action.

The differential is extended to \(LCH_{st}(\lambda_0, \Lambda \to \hat{\Lambda})\) by linearity.

To complete the construction of the strip Legendrian contact homology, we must prove that \(d_J\) is well-defined and that \(d_J \circ d_J = 0\). Before proceeding to give proofs of these results we will discuss the intuition behind the definition of this homology theory. The strip Legendrian contact homology can be seen as a relative version of the cylindrical contact homology (see [7] and [19]). For cylindrical contact homology to be well-defined for a contact form, this contact form has to have some special property; for example, for a hypertight contact form (i.e. one that doesn’t have contractible periodic orbits) cylindrical contact homology is well-defined. As we will see later, the non-existence of contractible Reeb orbits precludes the “bubbling” of pseudoholomorphic planes. This, together with SFT-compactness, implies that if its asymptotic orbits are in a primitive homotopy class, a
sequence of pseudoholomorphic cylinders of Fredholm index 2 can only break in a pseudoholomorphic building of 2 levels, each containing a cylinder of Fredholm index 1; thus only such buildings can appear in the boundary of the compactified moduli space of pseudoholomorphic cylinders of Fredholm index 2. This description of the compactified moduli spaces of pseudoholomorphic cylinders of index 2, is the crucial step that allows us to define cylindrical contact homology with coefficients in \( \mathbb{Z}_2 \).

The strip Legendrian contact homology is the natural adaptation of cylindrical contact homology to the relative case. This time the differential involves pseudoholomorphic strips with boundary conditions on Lagrangian submanifolds. For such a theory to be well-defined we have to preclude not only “bubbling” of planes but also of pseudoholomorphic half-planes. The conditions (b) and (c) above serve exactly to make impossible such “bubbling” phenomena, and the condition (d) is a non-degeneracy condition. Under these hypothesis for it is possible to define the strip Legendrian contact homology, and to carry this constructions one uses results on the analytical properties of pseudoholomorphic strips and discs. For these results we refer to: \[1\] for the necessary results on the asymptotic behaviour of punctures, \[2\] for the necessary results on Fredholm theory, \[2\] and \[16\] for the necessary transversality results, \[18\] for (essentially) the necessary techniques to perform gluing. We now proceed to prove:

**Lemma 1.** For \( J \in J_{reg}(\lambda_0) \subset J(\lambda_0) \), and \( d_J \) defined before we have:

- \( d_J \) is well defined,
- for each \( c \in T_{\Lambda \to \hat{\Lambda}}(\lambda_0) \), \( d_J(c) \) is a finite sum,
- \( d_J \) decreases the action of Reeb chords,
- \( d_J : \text{LCH}_{st, odd}(\lambda_0, \Lambda \to \hat{\Lambda}) \to \text{LCH}_{st, even}(\lambda_0, \Lambda \to \hat{\Lambda}) \) and \( d_J : \text{LCH}_{st, even}(\lambda_0, \Lambda \to \hat{\Lambda}) \to \text{LCH}_{st, odd}(\lambda_0, \Lambda \to \hat{\Lambda}) \)

**Proof:** in order for \( d_J \) to be well-defined we have to prove that \( \tilde{M}^1(c, c'; J) \) is finite for every \( c \) and \( c' \). Because of \( J \in J_{reg}(\lambda_0) \subset J(\lambda_0) \), \( \tilde{M}^1(c, c'; J) \) is a 0-dimensional manifold.

If we show that it is compact then it has to be a finite set.

To obtain the compactness we will apply the standard “bubbling of” analysis for pseudoholomorphic curves (see \[23\]) and the SFT compactness results of \[9\]. Let \( \tilde{w}_n \) be a sequence of elements of \( \tilde{M}^1(c, c'; J) \). Because of the assumptions we made on the contact form \( \lambda_0 \), the sequence \( \tilde{w}_n \) cannot have interior bubbling points: an interior bubbling point would imply the existence of a finite energy plane and thus of a contractible periodic orbit of \( X_{\lambda_0} \) which contradicts (a) above. Boundary bubbling points are also forbidden: they would give rise to either a pseudoholomorphic disc with boundary in \( \mathbb{R} \times \Lambda \), a pseudoholomorphic disc with boundary on \( \mathbb{R} \times \hat{\Lambda} \), a pseudoholomorphic disc with only one puncture asymptotic to a Reeb chord from \( \Lambda \) to itself, or a pseudoholomorphic disc with only one puncture asymptotic to a Reeb chord from \( \hat{\Lambda} \) to itself. The first two possibilities are impossible because \( \mathbb{R} \times \Lambda \) and \( \mathbb{R} \times \hat{\Lambda} \) are exact Lagrangian submanifolds; the later two because they would contradict conditions (b) and (c) above. Combining this information with the SFT-compactness results of \[9\] we have that \( \tilde{w}_n \) converges in the SFT sense to a
pseudoholomorphic building $\tilde{w}$ with $k$-levels $\tilde{w}^l$, and all levels $\tilde{w}^l$ are pseudoholomorphic strips satisfying:

- 1 is a positive boundary puncture, and $\tilde{w}^l$ is asymptotic to $c_l \in T_{A \to \Lambda}(\lambda_0)$ at 1
- $-1$ is a negative boundary puncture, and $\tilde{w}$ is asymptotic to $c_{l+1} \in T_{A \to \Lambda}(\lambda_0)$ at $-1$
- $\tilde{w}(H_+) \subset \mathbb{R} \times \Lambda$ where $H_- \subset (S^1 \setminus \{-1, 1\})$ is the northern hemisphere
- $\tilde{w}(H_-) \subset \mathbb{R} \times \hat{\Lambda}$ where $H_- \subset (S^1 \setminus \{-1, 1\})$ is the southern hemisphere

where $c_1 = c$ and $c_{k+1} = c'$. Because every $\tilde{w}^l$ is somewhere injective we have that the Fredholm indexes satisfy $I_F(\tilde{w}^l) \geq 1$. Thus we have $I_F(\tilde{w}) = \sum I_F(\tilde{w}^l) \geq l$; on the other hand as $\tilde{w}$ is the limit of a sequence of pseudoholomorphic strips of Fredholm index 1, it has to satisfy $I_F(\tilde{w}) = 1$. Therefore $l = 1$, and $\tilde{w} \in \tilde{M}^1(c, c'; J)$, which implies the desired compactness. This proves that $n_{c, c'}$ is finite for every $c, c' \in T_{A \to \Lambda}(\lambda_0)$, and thus that $d_J$ is well defined.

To check the third claim, notice that given $c \in T_{A \to \Lambda}(\tilde{\lambda}_0)$, $n_{c, c'}$ can only be non-zero for Reeb chords $c'$ such that $A(c') < A(c)$ and so $d_J$ decreases the action of Reeb chords. By condition (d) above one obtains that the set of Reeb chords with action smaller then $A(c)$ is finite, and so $n_{c, c'}$ is non-zero only for a finite number of $c'$; this finishes the proof that $d_J(c)$ is a finite sum.

The fourth claim follows easily from the fact mentioned above that the Fredholm index of a strip connecting two chords $c$ and $c'$ has the same parity as $|c| + |c'|$, as this implies $\tilde{M}^1(c, c'; J)$ can be non-empty only if $c$ and $c'$ have different parity.

As is seen in [9] the moduli spaces $\tilde{M}^k(c, c'; J)$ admit a compactification. The compactified moduli space is composed not only of pseudoholomorphic curves, but also of pseudoholomorphic buildings. We will denote this compactification of $\tilde{M}^1(c, c'; J)$, as constructed in [9], by $\tilde{M}^1(c, c'; J)$

**Lemma 2.** For $J \in \mathcal{J}_{reg}(\lambda_0) \subset \mathcal{J}(\lambda_0)$, and $d_J$ as defined before we have: $d_J \circ d_J = 0$

**Proof:** the lemma will be a consequence of the description we will give of the compactified moduli space $\tilde{M}^2(c, c'; J)$ of pseudoholomorphic strips with Fredholm index 2. Because of regularity of $J$, it will follow that for all $c, c' \in T_{A \to \Lambda}(\alpha_0)$, $\tilde{M}^2(c, c'; J)$ is either empty, or the finite union of disjoint circles and closed intervals. We summarise that in the following claim:

*Compactness Claim:* suppose $\tilde{M}^2(c, c'; J)$ is non-empty. Then, each connected component $I$ of $\tilde{M}^2(c, c'; J)$ is either a circle or a closed interval. Moreover, when $I$ is diffeomorphic to a closed interval, its boundary is composed by pseudoholomorphic buildings $\tilde{w}$ with 2 levels $\tilde{w}_1$ and $\tilde{w}_2$ satisfying:

$\tilde{w} \in \tilde{M}^1(c, \hat{c}; J)$ and $\tilde{w}_2 \in \tilde{M}^1(\hat{c}, c'; J)$ for some $\hat{c} \in T_{A \to \Lambda}(\alpha_0)$.

Before proving the claim above we will use it to prove the lemma. For this, we write:
\[ d_J \circ d_J(c) = \sum_{r^e \in \mathcal{T}_{\Lambda \rightarrow \hat{\Lambda}}(\alpha_0)} m_{c,c'} e' \]

(24)

It is clear that the lemma follows if we can prove that \( m_{c,c'} \) is even. On one hand, notice that it follows from our definition of \( d_J \), that \( m_{c,c'} \) counts the number of 2-level pseudoholomorphic buildings whose levels \( \tilde{w}_1 \) and \( \tilde{w}_2 \) satisfy: \( \tilde{w}_1 \in \mathcal{M}^2(c, \hat{c}; J) \) and \( \tilde{w}_2 \in \mathcal{M}^2(\hat{c}, c'; J) \) for some \( \hat{c} \in \mathcal{T}_{\Lambda \rightarrow \hat{\Lambda}}(\alpha_0) \). On the other hand, because of the regularity of \( J \in \mathcal{J}_{\text{reg}}(\alpha_0) \), we can apply the gluing theorem: this theorem implies that each such 2-level pseudoholomorphic building \( \tilde{w} \) appears exactly once in the boundary of \( \mathcal{M}^2(\hat{c}, c'; J) \), more precisely in boundary of exactly one connected component \( I_{\tilde{w}} \subset \mathcal{M}^2(c, \hat{c}; J) \). Thus, it follows from the compactness claim, that \( I_{\tilde{w}} \) is diffeomorphic to a closed interval, and that the other boundary component of \( I_{\tilde{w}} \) is also a 2 level building \( \tilde{w}' \) (distinct from \( \tilde{w} \)) satisfying:

\[ \tilde{w}'_1 \in \mathcal{M}^1(c, \hat{c}; J) \text{ and } \tilde{w}'_2 \in \mathcal{M}^1(\hat{c}', c'; J) \text{ for some } \hat{c}' \in \mathcal{T}_{\Lambda \rightarrow \hat{\Lambda}}(\alpha_0) \]

Summarising, the combination of the Compactness Claim and the gluing theorem allows us to conclude that the number \( m_{c,c'} \) is exactly the number of boundary components of the moduli space \( \mathcal{M}^2(c, c'; J) \). Because \( \mathcal{M}^2(c, c'; J) \) is a finite union of disjoint intervals and circles, this number is even. This finishes the proof of the lemma modulo the Compactness claim.

This combination of the Compactness Claim and the gluing, imply that the number \( m_{c,c'} \) counts the number of boundary components of \( \mathcal{M}^2(c, c'; J) \). Because \( \mathcal{M}^2(c, c'; J) \) is a union of intervals and circles, this number is even. This finishes the proof of the lemma modulo the Compactness claim.

**Proof of Compactness Claim:** suppose \( \mathcal{M}^2(c, c'; J) \) is non-empty. It follows from the regularity of \( J \) that \( \mathcal{M}^2(c, c'; J) \) is a 1-dimensional manifold and consequently has infinitely many distinct elements. We have now two possibilities:

If for any sequence \( \tilde{w}_n \) of elements of \( \mathcal{M}^2(c, c'; J) \), all its converging subsequences converge to elements of \( \mathcal{M}^2(c, c'; J) \) then \( \mathcal{M}^2(c, c'; J) \) is a closed 1-dimensional manifold without boundary. As we know that all sequences of elements in \( \mathcal{M}^2(c, c'; J) \) have converging subsequences, we conclude that \( \mathcal{M}^2(c, c'; J) \) is a compact 1-dimensional manifold without boundary; i.e a circle.

If that is not the case, let \( \tilde{w}_n \) be a sequence of elements of \( \mathcal{M}^2(c, c'; J) \) converging to the boundary of \( \mathcal{M}^2(c, c'; J) \). As we remarked in the proof of the previous lemma no “bubbling” can occur. Thus the SFT compactness theorem of implies that \( \tilde{w}_n \) converges to a pseudoholomorphic building \( \tilde{w} \) with k-levels \( \tilde{w}^j \) such that all levels \( \tilde{w}^j \) are pseudoholomorphic strips satisfying:

- 1 is a positive boundary puncture, and \( \tilde{w}^j \) is asymptotic to \( c_1 \in \mathcal{T}_{\Lambda \rightarrow \hat{\Lambda}}(\alpha_0) \) at 1
- \(-1\) is a negative boundary puncture, and \( \tilde{w} \) is asymptotic to \( c_{l+1} \in \mathcal{T}_{\Lambda \rightarrow \hat{\Lambda}}(\alpha_0) \) at \(-1\)
\[\hat{w}(H^+) \subset \mathbb{R} \times \Lambda \text{ where } H_- \subset (S^1 \setminus \{-1, 1\}) \text{ is the northern hemisphere}\]
\[\hat{w}(H_-) \subset \mathbb{R} \times \hat{\Lambda} \text{ where } H_- \subset (S^1 \setminus \{-1, 1\}) \text{ is the southern hemisphere}\]

where \(c_1 = c\) and \(c_{k+1} = c'\). Again because every \(\hat{w}\) is somewhere injective we have that the Fredholm index \(F(\hat{w}) \geq 1\) and thus \(I_F(\hat{w}) = \sum I_F(\hat{w}) \geq l\). On the other hand as \(\hat{w}\) is the limit of a sequence of pseudoholomorphic strips of Fredholm index 2, it has to satisfy \(I_F(\hat{w}) = 2\).

We have then 2 possibilities: either \(l = 1\) and \(\hat{w} \in \overline{\mathcal{M}}^2(c, c'; J)\); or \(l = 2\) which forces \(I_F(\hat{w}^1) = I_F(\hat{w}^2) = 2\), \(\hat{w}^1 \in \overline{\mathcal{M}}^1(c, c_2; J)\) and \(\hat{w}^2 \in \overline{\mathcal{M}}^1(c_2, c'; J)\). The first case is ruled out because we supposed that \(\hat{w}_n\) is converging to the boundary of \(\overline{\mathcal{M}}^2(c, c'; J)\).

We have obtained that all the elements on the boundary of \(\overline{\mathcal{M}}^2(c, c'; J)\) are 2-level pseudoholomorphic buildings with the properties claimed. This implies that the boundary of \(\overline{\mathcal{M}}^2(c, c'; J)\) is 0-dimensional manifold.

On the other hand, the gluing theorem gives the description of a neighbourhood of the 2-level pseudoholomorphic buildings appearing in the boundary \(\overline{\mathcal{M}}^2(c, c'; J)\). This neighbourhood admits a diffeomorphism to the infinite interval \([0, +\infty)\), that takes 0 to the 2-level building and all other values to pseudoholomorphic strips in \(\overline{\mathcal{M}}^2(c, c'; J)\).

Summing up, the compactified moduli \(\overline{\mathcal{M}}^2(c, c'; J)\) has the structure of a manifold with boundary, and in our particular case \(\overline{\mathcal{M}}^2(c, c'; J)\) it must be a 1-dimensional with 0-dimensional boundary; i.e a closed interval. This finishes the proof of the compactness claim.

We will denote by \(LCH_{st}(\lambda_0, \Lambda \rightarrow \hat{\Lambda})\) the homology associated to the chain-complex \((LCH_{st}(\lambda_0, \Lambda \rightarrow \hat{\Lambda}), d_J)\).

### 2.2.1 Strip Legendrian contact homology in special homotopy classes

Just as in the case of cylindrical contact homology, the free homotopy classes of paths starting at \(\Lambda\) and ending at \(\hat{\Lambda}\) generated subcomplexes of \(LCH_{st}(\lambda_0, \Lambda \rightarrow \hat{\Lambda})\). To formalize this we denote by \(\Sigma_{\Lambda \rightarrow \hat{\Lambda}}\) the set of homotopy classes of paths starting at \(\Lambda\) and ending at \(\hat{\Lambda}\). For our contact form \(\lambda_0\) and an element \(\rho \in \Sigma_{\Lambda \rightarrow \hat{\Lambda}}\) we denote by \(\mathcal{T}^\rho_{\Lambda \rightarrow \hat{\Lambda}}(\lambda_0)\) the set of Reeb chords from \(\Lambda\) to \(\hat{\Lambda}\) that belong to \(\rho\).

It is clear, for all \(c \in \mathcal{T}^\rho_{\Lambda \rightarrow \hat{\Lambda}}(\lambda_0)\), that the terms \([n_{c, c'} \mod 2]\), appearing in the differential \(d_J(c) = \sum_{c' \in \mathcal{T}_{\Lambda \rightarrow \hat{\Lambda}}(\lambda_0)} [n_{c, c'} \mod 2] c'\), can be non-zero only when \(c' \in \mathcal{T}^\rho_{\Lambda \rightarrow \hat{\Lambda}}(\lambda_0)\). This implies that the vector spaces \(LCH^\rho_{st}(\lambda_0, \Lambda \rightarrow \hat{\Lambda})\) are subcomplexes in the chain complex \((LCH_{st}(\lambda_0, \Lambda \rightarrow \hat{\Lambda}), d_J)\) and that:

\[
LCH_{st}(\lambda_0, \Lambda \rightarrow \hat{\Lambda}) = \bigoplus_{\rho \in \Sigma_{\Lambda \rightarrow \hat{\Lambda}}} LCH^\rho_{st}(\lambda_0, \Lambda \rightarrow \hat{\Lambda})
\]  

(25)
2.2.2 Cobordism maps

Symplectic cobordisms play a crucial role in SFT because they induce maps for the SFT invariants. Under suitable conditions this is also true for the strip Legendrian contact homology.

**Proposition 1.** Let \((Y^+, \lambda^+)\) be contact manifolds, with \(\lambda^+\) an adapted contact form for a pair of Legendrian curves \(\Lambda^+\) and \(\hat{\Lambda}^+\); and \((Y^-, \lambda^-)\) be a contact manifold, with \(\lambda^-\) an adapted contact form for a pair of Legendrian curves \(\Lambda^-\) and \(\hat{\Lambda}^-\). Suppose that there exist an exact symplectic cobordism \((V, \varpi)\) from \((Y^+, \lambda^+)\) to \((Y^-, o\lambda^-)\) for some constant \(o > 0\), and exact Lagrangian cobordisms \(L\), from \(\Lambda^+\) to \(\Lambda^-\), and \(\hat{L}\) from \(\hat{\Lambda}^+\) to \(\hat{\Lambda}^-\). Then these cobordisms induce map \(\Phi_{V,\varpi,L,\hat{L}}\) from \(LCH_{st}(\lambda^+, \Lambda^+ \to \hat{\Lambda}^+)\) to \(LCH_{st}(\lambda^-, \Lambda^- \to \hat{\Lambda}^-)\).

**Proof:** taking \(J^+ \in J_{reg}(\lambda^+)\) and \(J^- \in J_{reg}(\lambda^-)\), we can define the homologies \(LCH_{st}(\lambda^+, \Lambda^+ \to \hat{\Lambda}^+)\) and \(LCH_{st}(\lambda^-, \Lambda^- \to \hat{\Lambda}^-)\). We then take \(J_V \in J(J^+, J^-)\) as the almost complex structure in the cobordism \((V, \varpi)\). The map \(\Phi_{V,\varpi,L,\hat{L}}\) will count pseudoholomorphic strips \(\tilde{w} : (\mathcal{D} \setminus \{-1,1\}, i_0) \to (V, J)\) with Fredholm index 0, having 1 as a positive boundary puncture asymptotic to a Reeb chord \(c^+ \in T_{\Lambda^+ \to \hat{\Lambda}^+}(\lambda^+)\), and \(-1\) as a negative boundary puncture asymptotic to a Reeb chord \(c^- \in T_{\Lambda^- \to \hat{\Lambda}^-}(\lambda^-)\) and with boundary conditions \(\tilde{w}(H^+) \in L\) and \(\tilde{w}(H^-) \in \hat{L}\). For the pair \(c^+\) and \(c^-\), the moduli space of pseudoholomorphic strips with Fredholm index \(k\) will be denoted by \(\mathcal{M}^k(c^+, c^-; J_V)\). As all the strips in these moduli spaces are somewhere injective curves, we have for a generic set \(J_{reg}(J^+, J^-)\) we have for a generic set \(J_{reg}(J^+, J^-)\) all the pseudoholomorphic curves in \(\mathcal{M}^k(c^+, c^-; J_V)\), for every \(c^+\) and \(c^-\), are Fredholm regular. We assume from now on that \(J_V \in J_{reg}(J^+, J^-)\).

Initially, the map \(\Phi_{V,\varpi,L,\hat{L}}\) is obtained by counting elements in \(\mathcal{M}^0(c^+, c^-; J_V)\), and is the defined from \(LCH_{st}(\lambda^+, \Lambda^+ \to \hat{\Lambda}^+)\) to \(LCH_{st}(\lambda^-, \Lambda^- \to \hat{\Lambda}^-)\). To see that it actually induces a map on the homology level one has to check that \(d_{J^-} \circ \Phi_{V,\varpi,L,\hat{L}} = \Phi_{V,\varpi,L,\hat{L}} \circ d_{J^+}\).

The proof of this fact consists of a combination of compactness and gluing results, and is identical to a similar statement for cylindrical contact homology (see [2]); thus we will just sketch it here.

Because of the regularity for all pseudoholomorphic curves involved in \(d_{J^+}\), \(d_{J^-}\) and \(\Phi_{V,\varpi,L,\hat{L}}\), it possible to perform gluing for the pseudoholomorphic curves involved in these maps. This means that the two level buildings involved in the composed maps \(d_{J^-} \circ \Phi_{V,\varpi,L,\hat{L}}\) and \(\Phi_{V,\varpi,L,\hat{L}} \circ d_{J^+}\) always appear in the boundary of a compactified moduli space \(\overline{\mathcal{M}}^1(c^+, c^-; J_V)\). On the other hand because of the exactness of \((V, \varpi)\), and of the Lagrangians \(L\) and \(\hat{L}\), and because \(\lambda^+\) is an adapted contact form for the pairs of Legendrian curves \(\Lambda^+\) and \(\hat{\Lambda}^+\) and \(\lambda^-\) is an adapted contact form for the pairs of Legendrian curves \(\Lambda^-\) and \(\hat{\Lambda}^-\), a sequence of elements in \(\mathcal{M}^1(c^+, c^-; J_V)\) can only break in two level buildings that are counted either for \(d_{J^-} \circ \Phi_{V,\varpi,L,\hat{L}}\) or for \(\Phi_{V,\varpi,L,\hat{L}} \circ d_{J^+}\). Combining compactness and gluing as we did in the proof of the lemma 2 above, we conclude that \(\overline{\mathcal{M}}^1(c^+, c^-; J_V)\) must be a finite union of circles and intervals.

Writing for each \(c^+ \in T_{\Lambda^+ \to \hat{\Lambda}^+}(\lambda^+)\) we obtain:
\[(d_{J^+} \circ \Phi_{V,\overline{\varpi},L,\overline{\Lambda}} - \Phi_{V,\overline{\varpi},L,\overline{\Lambda}} \circ d_{J^-})(c^+) = \sum_{c^- \in \mathcal{C}_{\Lambda^- \to \Lambda^-}(\lambda^-)} [a(c^+, c^-) \mod 2] c^- \quad (26)\]

where \(a(c^+, c^-)\) is the number of pseudoholomorphic buildings of Fredholm index 1 appearing on the boundary of \(\overline{\mathcal{M}}(c^+, c^-; J_V)\). As \(\overline{\mathcal{M}}(c^+, c^-; J_V)\) is a 1-dimensional manifold the number of its boundary components is even which implies that \([a(c^-) \mod 2] = 0\).

It is clear that if \(V\) is just the symplectization of a contact manifold with an \(\mathbb{R}\)-invariant regular almost complex structure the induced cobordism map is the identity; this is because the only pseudoholomorphic strips with Fredholm index 0 in this situation are trivial strips over Reeb chords.

2.2.3 A special type of cobordisms

For us, the case of most importance is when the contact manifolds \((Y^+, \lambda^+)\) and \((Y^-, \lambda^-)\) are to a same contact manifold \((Y, \lambda_0)\), \(\Lambda^+ = \Lambda^- := \Lambda\) and \(\hat{\Lambda}^+ = \hat{\Lambda}^- := \hat{\Lambda}\), and the exact symplectic cobordism \((V, \overline{\varpi})\) and the exact Lagrangian cobordisms \(L\) and \(\overline{L}\) can be deformed, respectively, to the symplectization of \((Y, \lambda_0)\), and the Lagrangian cylinders over \(\Lambda\) and \(\hat{\Lambda}\). In this case the cobordism actually induces a map that respect the homotopy classes of the chords; so we can define (for each \(\rho \in \Sigma_{\Lambda \to \hat{\Lambda}}\)) a map \(\Phi_{V,\overline{\varpi},L,\overline{\Lambda}}\) from \(\text{LC}_{\Lambda_{st}}^\rho(\lambda, \Lambda \to \hat{\Lambda})\) to \(\text{LC}_{\hat{\Lambda}_{st}}^\rho(\lambda, \Lambda \to \hat{\Lambda})\). In order to define the cobordism maps for these subcomplexes \(\text{LC}_{\Lambda_{st}}^\rho(\lambda^+, \Lambda^+ \to \hat{\Lambda}^+)\) the assumptions of regularity on the almost complex structure \(J_V\) are slightly weaker.

Fix \(\rho \in \Sigma_{\Lambda \to \hat{\Lambda}}\). For any pair \(c\) and \(c'\) of Reeb chords in \(\mathcal{T}_{\Lambda \to \hat{\Lambda}}^\rho(\lambda_0)\), the moduli space of pseudoholomorphic strips with Fredholm index \(k\) will be denoted by \(\mathcal{M}^k(c, c'; J_V)\). As all the strips in these moduli spaces are somewhere injective curves, we have for a generic set \(\mathcal{J}_{\text{reg}, \rho}(J, J) \in \mathcal{J}(J, J)\), all the pseudoholomorphic curves in all the moduli spaces \(\mathcal{M}^k(c, c'; J_V)\), for every \(c\) and \(c'\), are Fredholm regular. An argument identical to the proof of the above proposition shows that for \(J_V \in \mathcal{J}_{\text{reg}, \rho}(J, J)\) we have a map \(\Phi_{V,\overline{\varpi},L,\overline{\Lambda}}\) from \(\text{LC}_{\Lambda_{st}}^\rho(\lambda, \Lambda \to \hat{\Lambda})\) to \(\text{LC}_{\hat{\Lambda}_{st}}^\rho(\lambda, \Lambda \to \hat{\Lambda})\); notice that \(\mathcal{J}_{\text{reg}, \rho}(J, J)\) contains elements that are not in \(\mathcal{J}_{\text{reg}}(J, J)\).

We study the cobordism map in the following situation: let \((V = \mathbb{R} \times Y, \overline{\varpi})\) be an exact symplectic cobordism from \((Y, \lambda_0)\) to \((Y, o\lambda_0)\) where \(0 < o < 1\), and assume that \(\lambda_0\) is adapted to a pair of Legendrians \(\Lambda\) and \(\hat{\Lambda}\) and that there exists exact Lagrangian cobordisms \(L \subset V\) from \(\Lambda\) to \(\Lambda\), and \(\hat{L} \subset V\) from \(\hat{\Lambda}\) to \(\hat{\Lambda}\). Suppose that one can make an isotopy \((\mathbb{R} \times Y, \varpi_t)\) of exact symplectic cobordisms from \((Y, \lambda_0)\) to \((Y, o\lambda_0)\), where \(\varpi_t\) satisfies \(\varpi_0 = \overline{\varpi}\) and \(\varpi_1 = d(e^s \lambda_0)\), and:

- (A) \(L_t \in (\mathbb{R} \times Y, \varpi_t)\) is an exact Lagrangian cobordisms from \(\Lambda\) to \(\Lambda\), with \(L_1 = \mathbb{R} \times \Lambda\);
- (B) \(\hat{L}_t \in (V, \varpi_t)\) is exact Lagrangian cobordisms from \(\hat{\Lambda}\) to \(\hat{\Lambda}\), with \(\hat{L}_1 = \mathbb{R} \times \hat{\Lambda}\).
We consider the space $\tilde{J}(J, J)$ of smooth homotopies:

$$t \in [0, 1]; J_t \in J(J, J)$$

with $J_0 = J\nu \in J_{reg, \rho}(J, J)$ and $J_1 \in J_{reg}(\lambda_0)$, and $J_t$ is compatible with $\nu_t$. For Reeb chords $c, c' \in T_{\Lambda \to \hat{\Lambda}}(\lambda_0)$ we consider the moduli space:

$$\tilde{M}^k(c, c'; J_t) = \{(t, \tilde{w}) \mid t \in [0, 1] \text{ and } \tilde{w} \in M^k(c, c'; J_t)\}$$

By the results in [16] and [2], we know that there is a generic subset $\tilde{J}_{reg}(J^+, J^-) = \tilde{J}(J^+, J^-)$ such that $\tilde{M}^k(c, c'; J_t)$ is a smooth $k + 1$-dimensional manifold with boundary. The crucial condition that makes this valid is again the fact that the all the pseudoholomorphic curves that make part of this moduli space are somewhere injective.

**Proposition 2.** Let $(V = \mathbb{R} \times Y, \omega)$ be an exact symplectic cobordism from $(Y, \lambda_0)$ to $(Y, o\lambda_0)$ where $o < 1$, and $\lambda_0$ is adapted to a pair of Legendrians $\Lambda$ and $\hat{\Lambda}$. Assume also that there exists in $(\mathbb{R} \times Y, \omega)$ exact Lagrangian cobordisms $L \subset \mathbb{R} \times Y$ from $\Lambda$ to $\hat{\Lambda}$, and $\hat{\tilde{L}} \subset V$ from $\hat{\Lambda}$ to $\hat{\Lambda}$. Suppose that one can make an isotopy of exact symplectic cobordisms $(Y, \lambda_0)$ to $(Y, o\lambda_0)$, where $\nu_t$ satisfies $\nu_0 = \nu$ and $\nu_1 = d(e^s \lambda_0)$, and such that $L_t \in (\mathbb{R} \times Y, \nu_t)$ and $\hat{\tilde{L}}_t \in (\mathbb{R} \times Y, \nu_t)$ are exact Lagrangian cobordisms satisfying conditions (A) and (B) above. Then for all $J_t \in J_{reg, \rho}(J, J)$ the map $\Phi_{V, \omega, L, \tilde{L}}$ from $LC_{reg}^p(\lambda, \Lambda \to \hat{\Lambda})$ to itself is chain homotopic to the identity.

**Outline of the proof:** the proof is a standard argument in SFT, and we direct the reader to the original source [19] for an exposition of this argument for general SFT invariants and [7, 6] where the very similar case of cylindrical contact homology is treated. We first take an almost complex structure $J \in J_{reg}(\lambda_0)$ and choose an almost complex structure $J_\nu \in J_{reg}(J, J)$ compatible with $\omega$. The map $\Phi_{V, \omega, L, \tilde{L}}$ will count pseudoholomorphic strips in $(V, J_\nu)$ satisfying boundary conditions as stated in the previous proposition. For the deformation $\nu_t$, we can take an homotopy $J_t$ of almost complex structures in $\tilde{J}_{reg}(J, J)$.

In this case we can define a map $\mathcal{K}: LCH^p_{reg}(\lambda_0, \Lambda \to \hat{\Lambda}) \to LCH^p_{reg}(\lambda_0, \Lambda \to \hat{\Lambda})$ that counts finite energy, Fredholm index $-1$ pseudoholomorphic strips in $(V, \omega_t)$, with one boundary component in $L_t$ and one in $\tilde{L}_t$. Because of the regularity of our homotopy, the set of index $-1$ strips whose positive puncture detects a fixed chord $c$ is finite, and therefore the map $\mathcal{K}$ is really well defined.

Notice that for $t = 0$ the cobordism map is the identity, and the pseudoholomorphic curves that define it are trivial strips over Reeb chords. As we increase $t$ the strip might break; however because $\lambda_0$ is adapted to the pair $(\Lambda, \hat{\Lambda})$, $L_t$ and $\tilde{L}_t$ are exact and our homotopy $J_t$ is regular, we can describe precisely this breaking. A sequence of strips with Fredholm index 0 can only break in a 2-level building formed by two strips, one of them with Fredholm index $-1$, with one boundary component in $L_t$ and other in $\tilde{L}_t$, and another in the symplectization of $(Y, \lambda_0)$ and Fredholm index 1. These two level buildings are exactly maps that appear in the counting of $\mathcal{K} \circ d_f + d_f \circ \mathcal{K}$; in other words the 2-level buildings that appear in the definition of $\mathcal{K} \circ d_f + d_f \circ \mathcal{K}$ appear also in the boundary
of the SFT compactification of $\widehat{\mathcal{M}}^0(c,c';J_t)$. By applying the gluing theorem to the 2-level buildings that appear in the counting of $\mathcal{K} \circ d_J$ and $d_J \circ \mathcal{K}$ we conclude that they form the boundary of $\widehat{\mathcal{M}}^0(c,c';J_t)$. This boundary of $\widehat{\mathcal{M}}^0(c,c';J_t)$ measures exactly the “difference” of the initial map $I$ and $\Phi_{V,\pi,L,L_t}$ and therefore is given by $\mathcal{K} \circ d_J + d_J \circ \mathcal{K}$, as in the statement of the proposition.

One very important consequence of the above proposition is that the strip Legendrian contact homology $LC_{\mathbb{H}}^{\rho}_{st}(\lambda_0, \Lambda \to \tilde{\Lambda})$ doesn’t depend on the almost complex structure $J \in \mathcal{J}_{\text{reg}}(\lambda_0)$ used to define $d_J$, something which is not at all obvious from the definition of $LC_{\mathbb{H}}^{\rho}_{st}(\lambda_0, \Lambda \to \tilde{\Lambda})$.

### 3 Growth rate of $LC_{\mathbb{H}}^{\rho}_{st}(\lambda_0, \Lambda \to \tilde{\Lambda})$ and topological entropy

Given a contact manifold $(Y, \xi)$, for a contact form $\lambda$ and a pair of Legendrian knots $\Lambda$ and $\tilde{\Lambda}$ we define the exponential homotopical growth of the linearized Legendrian contact homology $LC_{\mathbb{H}}^{\rho}_{st}(\lambda_0, \Lambda \to \tilde{\Lambda})$ with respect to the action. We then it to estimate the growth of the number of Reeb chords from $\Lambda$ to $\Lambda'$ in other contact forms associated to $(Y, \xi)$ when $\Lambda'$ is sufficiently close to $\tilde{\Lambda}$.

Let $\Lambda$ and $\tilde{\Lambda}$ be disjoint Legendrian curves in the contact manifold $(Y, \xi$) and $\lambda_0$ be a hypertight contact form associated to $\xi$ and adapted to $\Lambda$, $\tilde{\Lambda}$.

We define for each number $C \geq 0$ the set $\Sigma^C_{\Lambda \to \tilde{\Lambda}}(\lambda_0)$ of homotopy classes $\rho$ satisfying:

- all the chords in $T^\rho_{\Lambda \to \tilde{\Lambda}}(\lambda_0)$ have action smaller then $C$
- $LC_{\mathbb{H}}^{\rho}_{st}(\lambda_0, \Lambda \to \tilde{\Lambda}) \neq 0$

**Definition 2.** We say that a hypertight contact form $\lambda$ (associated to a contact manifold $(Y, \xi)$) presents exponential homotopical growth of strip Legendrian contact homology if $\lambda$ is adapted to a Legendrian link $\Lambda$, $\tilde{\Lambda}$, and there are constants $a > 0$ and $d$ such that:

$$
\#(\Sigma^C_{\Lambda \to \tilde{\Lambda}}(\lambda_0)) > e^{ac^d} \quad (29)
$$

for all $C > 0$.

In that case we will say that $LC_{\mathbb{H}}^{\rho}_{st}(\lambda_0, \Lambda \to \tilde{\Lambda})$ has exponential homotopical growth with exponential weight $a > 0$.

### 3.1 The growth of the number of Reeb chords

Let $\Lambda$ and $\Lambda'$ be Legendrian submanifolds of $(Y, \xi = \ker(\lambda))$: we will say that $\Lambda'$ is $(\lambda, \Lambda)$-transverse if all the Reeb chords in $T_{\Lambda \to \Lambda'}(\lambda)$ are transverse. We will denote by $N_C(\lambda, \Lambda, \Lambda')$ the number of Reeb chords in $T_{\Lambda \to \Lambda'}(\lambda)$ with action $\leq C$.

**Proposition 3.** Let $(Y, \xi)$ be a contact manifold and $\Lambda$ and $\tilde{\Lambda}$ be two disjoint Legendrian submanifolds, such that $\lambda$ is associated to $(Y, \xi)$ and adapted to the pair $(\Lambda, \tilde{\Lambda})$. Suppose
that the strip contact homology $LC^\mathbb{H}_{st}(\lambda, \Lambda \to \hat{\Lambda})$ has exponential homotopical growth with exponential weight $a > 0$. Let $\lambda'$ be another contact form associated to $(Y, \xi)$, and take $g > 0$ to be the unique function such that $\lambda' = g\lambda$. Then given $\epsilon > 0$ there exists $\delta > 0$ such that:

for any smooth Legendrian $\lambda'$ which is $\delta(\epsilon)$ close to $\hat{\lambda}$ in the $C^4$ topology and $(\lambda', \Lambda)$ transverse the numbers $N_C(\lambda', \Lambda, \lambda')$ satisfy:

$$e^{(1+\frac{a}{\epsilon}) \max(g)} < N_C(\lambda', \Lambda, \lambda')$$  \hspace{1cm} (30)

Remark: notice that we don’t ask the contact form $\lambda'$ to be non-degenerate.

Proof: we divide the proof in steps.

Step 1: reduction to the non-degenerate case. We first show that it suffices to prove the estimate to non-degenerate contact forms; as the general case follows from this one by an approximation procedure.

Let $j$ be a natural number. As $\lambda'$ is $(\lambda', \Lambda)$ transverse, it is possible to make a $C^\infty$ small perturbation of the the contact form $\lambda'$ to a non-degenerate contact form $\lambda'(j)$ which generates the same contact structure as $\lambda'$ and such that:

- $N_C(\lambda', \Lambda, \lambda') = N_C(\lambda'_j, \Lambda, \lambda')$, for all $C \leq j$
- $\lambda'$ is $(\lambda'(j), \Lambda)$ transverse

We demand that the perturbations $\lambda'(j)$ are taken to be close enough to $\lambda'$ to guarantee that there is are exact symplectic cobordisms from $\lambda'$ to $\frac{1}{1+j}\lambda'(j)$ and from $\lambda'(j)$ to $\frac{1}{1+j}\lambda'$. Supposing now that the proposition is true for all $\lambda'(j)$, we get, because of the conditions on the $\lambda'(j)$, that for a given $C > 0$ if we have for all $j \geq C$:

$$N_C(\lambda', \Lambda, \lambda') = N_C(\lambda'_j, \Lambda, \lambda') > e^{\frac{aC(1-\epsilon)}{\max(g_j)}}$$ \hspace{1cm} (31)

where $g_j$ is the unique function such that $g_j\lambda = \lambda'(j)$.

Letting $j$ go to $+\infty$ we have that $g_j \to g$, what implies the desired inequality for $\lambda'$.

Step 2: we construct a series of different symplectic cobordisms.

We construct a symplectic cobordism $W^u$ from $(Y, (\max(g) + \mu)\lambda)$ to $(Y, \lambda')$; for any $\mu > 0$ we pick a function $f_\mu : \mathbb{R} \times M \to \mathbb{R}$ such that $\frac{\partial f}{\partial t} \geq 0$, $f(t, x) = g(x); t \leq 0$ and $f(t, x) = (\max(g) + \mu)$ for $t \geq 1$; notice that as $\max(g)$ is a positive constant, the Reeb flow of $\max(g)\lambda$ is just a reparametrization of the Reeb flow of $\lambda$. By an analogous construction we can define a symplectic cobordism $W^d$ from $(Y, \lambda')$ to $(Y, b\lambda)$ for a sufficiently small constant $b > 0$. Notice that in both constructions $\hat{L} = \mathbb{R} \times \hat{\Lambda}$ is an exact Lagrangian submanifold.

Now, if we take $\delta$ small enough, $\lambda'$ is Legendrian isotopic to $\hat{\lambda}$ by a path $\Lambda_t$ $(t \in \mathbb{R})$ with $\Lambda_t \to \hat{\lambda}; t \leq 0$ and $\Lambda_t = \lambda'; t \geq 1$. Taking an even smaller $\delta(\epsilon)$ if necessary, it is then possible to take a path of diffeomorphisms $F_t : Y \to Y$, which equals the identity outside

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a small neighbourhood of \( \hat{\Lambda} \) that doesn’t intersect \( \Lambda \) and such that \( F_t(\hat{\Lambda}) = \Lambda_t \). Making \( \delta(\epsilon) > 0 \) even smaller if necessary, we can also demand that our \( F_t \) satisfies, \( \| F_t \|_{C^{1,\alpha}} < \epsilon \).

We then define the smooth family of contact forms \( \lambda^t = F_t^* \lambda' \). Let \( h : \mathbb{R} \to \mathbb{R} \), be an increasing function satisfying:

\[-h(t) = (1 - 2\epsilon) \text{ if } t \leq 0 \text{ and } h(t) = 1 \text{ for } t \geq 1.\]

\[-h'(t) > 0 \text{ for all } t \in \mathbb{R} \text{ and } h'(t) = (1 + 2\epsilon) \text{ for } t \in [0, 1].\]

For \( \epsilon \) (and consequently also \( \delta(\epsilon) \)) sufficiently small, we have that \( d(h(t)\lambda_t) \) is an exact symplectic form in the manifold \([0, 1] \times Y\). This defines an exact symplectic cobordism from \( \lambda^t \) to \((1 + 2\epsilon)\lambda'\), with the property that the \( \hat{L} = \mathbb{R} \times \hat{\Lambda} \) is an exact Lagrangian submanifold. In an analogous way one construct an exact symplectic cobordism from \( \lambda' \) to \((1 - 2\epsilon)\lambda^t\) where \( \hat{L} = \mathbb{R} \times \hat{\Lambda} \) is an exact Lagrangian submanifold.

**Step 3: the exact Lagrangian cobordism and the chain map on the strip Legendrian contact homology of the ends.**

By gluing the exact symplectic cobordisms (see [19]) above we can produce an exact symplectic cobordism \((V = \mathbb{R} \times Y, \omega)\) such that:

- \( \omega = d(e^{-5}(\max(g) + \mu)\lambda) \) in \([5, +\infty) \times Y\) and \( \omega = d(\frac{h(t)\lambda}{2}) \) in \([-5, -\infty) \times Y\)
- \( \omega = dt\Lambda^t \) in \([-\epsilon, \epsilon] \times Y\)
- \( \hat{\hat{L}} = \mathbb{R} \times \hat{\Lambda} \) and \( \hat{L} = \mathbb{R} \times \Lambda \) are exact Lagrangian submanifolds

Following section 2.1.3 we produce the splitting family of symplectic cobordisms \((V, \omega_R)\) for \( R > 0 \). It is clear from our construction that for all \( R \) the symplectic cobordisms \((V, \omega_R)\) can be deformed through exact symplectic cobordisms to a symplectization, in such a way that \( L = \mathbb{R} \times \Lambda \) and \( \hat{\hat{L}} = \mathbb{R} \times \hat{\Lambda} \) are exact Lagrangian submanifolds at every stage of the isotopy. As a consequence, we can apply Proposition 2, to obtain that for regular almost complex structures \( \hat{J} \in \mathcal{J}_{reg, \rho}(J, J) \) in the cobordisms \((V, \omega_R)\) we get an induced isomorphism \( \Phi_{V, \omega, L, \hat{\hat{L}}} \) for \( LCH_{st}^o(\lambda, \Lambda \to \hat{\Lambda}) \).

**Step 4: proof of the theorem for \( \lambda' \) non-degenerate.**

We first pick for \((V, \omega_R)\) an almost complex structure \( J_V \) as in section 2.1.3 and take \( \rho \in \Sigma^{C^0}_{\Lambda \to \hat{\Lambda}} \). We claim that for such an almost complex structure, there exist chords \( c, c' \in T^{\rho}_{\Lambda \to \hat{\Lambda}}(\lambda) \) such that \( \mathcal{M}(c, c'; J_V) \) is non-empty.

We argue by contradiction; if no such strip existed we have that \( J_V \in \mathcal{J}_{reg, \rho}(J, J) \) and therefore induces an isomorphism on \( LCH_{st}^o(\lambda, \Lambda \to \hat{\Lambda}) \). Because \( \mathcal{M}(c, c'; J_V) \) is empty we now that the cobordism map \( \Phi_{V, \omega, L, \hat{\hat{L}}} \) is the zero map; on the other hand, as \( LCH_{st}^o(\lambda, \Lambda \to \hat{\Lambda}) \neq 0 \) and \( \Phi_{V, \omega, L, \hat{\hat{L}}} \) is an isomorphism it cannot be the zero map. Thus we have reached a contradiction and therefore \( \mathcal{M}(c, c'; J_V) \) is non-empty.

We now send \( R \to +\infty \) and take a sequence of elements \( \hat{\hat{u}}_R \in \mathcal{M}(c, c'; J_V) \) invoke the SFT compactness results of [9]. Because there is a global bound on the energy of all elements of \( \mathcal{M}(c, c'; J_V) \), the results in [9] imply that \( \hat{\hat{u}}_R \) converges to a holomorphic
building \( \tilde{w} \). Because of the stretching the neck process, we have that one of the levels of this building lives in the symplectization of \( \lambda'^{1} \).

We will see that for topological reasons one of the punctures of this level has to detect a Reeb chord \( \hat{c} \in T^{\rho}_{\Lambda \rightarrow \widehat{\Lambda}}(\lambda'^{1}) \); with action smaller than \((1 + 2\epsilon)(1 + \mu)\max(g)C\). Let \( \tilde{w}^{k} \) for \( k \in \{1, \ldots, m\} \) be the levels of the pseudoholomorphic building \( \tilde{w} \). Because the topology of our pseudoholomorphic curve doesn’t change on the breaking we must have the following picture:

- the upper level \( \tilde{w}^{1} \) is composed of one pseudoholomorphic disc, with has one positive puncture, which is asymptotic to an orbit \( c_{0} \in T^{\rho}_{\Lambda \rightarrow \widehat{\Lambda}}(\lambda) \), and several negative boundary and interior punctures. All of the negative punctures detect contractible Reeb orbits or contractible self Reeb chords of either \( \Lambda \) or \( \Lambda' \), excepting one negative boundary puncture that detects a Reeb chord \( c_{1} \) in either \( T^{\rho}_{\Lambda \rightarrow \widehat{\Lambda}}(\lambda) \) in case this level lives in the symplectization of \( \lambda \) or in \( T^{\rho}_{\Lambda \rightarrow \widehat{\Lambda}}(\lambda'^{1}) \) in case this level lives in a cobordism from \( \lambda \) to \( \lambda'^{1} \).
- on every other level \( \tilde{w}^{k} \) there is a special curve which has one positive puncture, which is asymptotic to a Reeb chord \( c_{k-1} \) in \( \rho \) and possibly several interior and boundary negative punctures. Of the negative boundary punctures there is one that is asymptotic to an orbit \( c_{k} \) in \( \rho \) and all the others are contractible.

As a consequence we obtain that the level \( \tilde{w}^{k} \) living in the symplectization of \( \lambda'^{1} \) contains a curve with one positive puncture asymptotic to a Reeb chord \( \hat{c} \in T^{\rho}_{\Lambda \rightarrow \widehat{\Lambda}}(\lambda'^{1}) \). The action bound follows from the fact that all punctures in the building detect Reeb orbits and chords with action smaller than the action of \( c_{0} \).

As a consequence of this existence result for all our homotopy classes \( \rho \in \Sigma^{C}_{\Lambda \rightarrow \widehat{\Lambda}} \), we obtain that \( e^{[1+\mu]((1+2\epsilon)\max(g))} < N_{C}(\lambda'^{1}, \Lambda, \widehat{\Lambda}) \). As we have an obvious action preserving bijection between elements in \( T^{\rho}_{\Lambda \rightarrow \widehat{\Lambda}}(\lambda'^{1}) \) and elements in \( T^{\rho}_{\Lambda \rightarrow \Lambda'}(\lambda') \) it follows that \( e^{[1+\mu]((1+2\epsilon)\max(g))} < N_{C}(\lambda', \Lambda, \Lambda') \). As \( \mu \) can be chosen arbitrarily small, we obtain the promised estimate for \( \lambda' \).

\[ \square \]

3.2 Positivity of the topological entropy

Using the results of the previous subsection, we will now prove Theorem 1 which says that given a contact 3-manifold \( (Y, \xi) \), if there exists a contact form \( \lambda \) (with \( \xi = \ker(\lambda) \)) adapted a pair of Legendrian knots \( \Lambda \) and \( \widehat{\Lambda} \) for which the strip Legendrian contact homology \( LC^{\text{H}}(\lambda, \Lambda \rightarrow \widehat{\Lambda}) \) has exponential homotopical growth rate with respect to the action, then every Reeb flow for this contact manifold has positive topological entropy.

By the Weinstein tubular neighbourhood theorem the Legendrian knot \( \widehat{\Lambda} \) has a tubular neighbourhood \( N_{\delta(\epsilon)} \) contactomorphic to the local model given by:

\[ dx + \theta dy \] (32)

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where \((\theta, z) \in S^1 \times D_v(e)\), \(D_v(e) = \{z \in \mathbb{C}; |z| \leq v(e)\}\) and \(\hat{\Lambda}\) is identified with the circle \((\theta, 0, 0)\). We denote by \(\Lambda^z\) the Legendrian submanifolds \((\theta, z)\) where \(z\) is fixed. The \(N_{\delta(e)}\) is taken to small enough, so that all \(\Lambda^z\) are \(\delta(e)\)-close to \(\hat{\Lambda}\). As the \(\Lambda^z\) form a fibration of \(N\) by Legendrian knots we will refer to them as Legendrian fibers.

We will now study the Reeb chords from a fixed Legendrian knot \(\Lambda\) (disjoint from \(N\)) to \(\Lambda^z\). Our first result is:

**Proposition 4.** Let \(\lambda\) be a contact form associated to a contact 3-manifold \((Y, \xi)\), with \(\Lambda\) and \(\hat{\Lambda}\) being Legendrian curves. We consider a neighbourhood \(N_{\delta(e)}\) of \(\hat{\Lambda}\) with coordinates \((\theta, z)\) as explained above. Then, for almost every \(z \in D_v(e)\) (with respect to the Lebesgue measure in \(D_v(e)\)), all chords from \(\Lambda\) to \(\Lambda^z\) are \((\lambda, \Lambda)\) transverse.

**Proof:** taking a parametrization \(\rho : S^1 \to \Lambda\), we use the flow to define the following map from the cylinder \(S^1 \times \mathbb{R}\):

\[
F(s, t) = \phi^s_{\lambda}(\rho(s))
\]

\(F^{-1}(N_{\delta(e)})\) is an open subset \(U\) of \(S^1 \times \mathbb{R}\). Let \(\pi : N_{\delta(e)} \to D_v(e)\) be the projection in the two last coordinates. The restriction \(F|_U\) can be composed with \(\pi\) to obtain a differentiable map:

\[
\pi \circ F|_U : U \to D_v(e).
\]

We remark that the critical points of \(\pi \circ F|_U\) are exactly points \((s, t)\) where \(\phi^t(\rho(s))\) is tangent to some fiber \(\Lambda^z\). To see this note that the \(\partial_t(F|_U) = X_\lambda\) and \(\partial_s(F|_U) \neq 0\). This implies that \(\partial_t(\pi \circ F|_U)\) is always non-zero for \(X_\lambda\) is never tangent to the Legendrian fibers \(\Lambda^z\), and \(\partial_s(\pi \circ F|_U)\) is zero if and only if \(\partial_s(F|_U)\) is tangent to a Legendrian fiber. This implies that the regular values \(z\) are of \(\pi \circ F|_U\) in \(D_v(e)\) are in bijective correspondence with the \(\Lambda^z\) satisfying that every Reeb chord from \(\Lambda\) to \(\Lambda^z\) is transverse.

Applying the finite dimensional Sard’s theorem to \(\pi \circ F|_U\) we have that almost every element of \(D_v(e)\) is a regular value of \(\pi \circ F|_U\), completing the proof of the proposition.

With this in hand, we are ready to prove the main theorem of this section.

**Theorem 1:** Let \((Y, \xi = \ker(\lambda_0))\) be a contact 3-manifold with a hypertight contact form \(\lambda_0\) adapted to the pair of Legendrian knots \((\Lambda, \hat{\Lambda})\). Assume that \(LC_{\mathbb{H}_{st}}(\lambda_0, \Lambda \to \hat{\Lambda})\) has exponential homotopical growth rate (with respect to the action) with exponential weight \(a > 0\). For any contact form \(\lambda\) associated to \((Y, \xi)\), let \(f_\lambda\) be the unique function such that \(\lambda = f_\lambda\lambda_0\). Then, the Reeb flow of \(X_\lambda\) has positive topological entropy, and moreover:

\[
h_{\text{top}}(\phi_{X_\lambda}) \geq \frac{a}{\max(f_\lambda)} \tag{33}
\]

**Proof:** our idea is to mimic the use of Yomdin’s theorem which is done for geodesic flows (see [31] and [33 ]).
Step 1:
Let $\lambda_0$ be our contact form satisfying the hypothesis of the theorem. Taking the tubular neighbourhood $N_\delta(\epsilon)$ as above. As all $\Lambda^z$ are $\delta(\epsilon)$-close to $\hat{\Lambda}$ (where $\delta(\epsilon)$ is as in Proposition 3), we can use the Proposition 3 in combination with Proposition 4 to obtain that, for almost every $z \in \mathbb{D}_v(\epsilon)$, the number $N_C(\lambda, \Lambda, \Lambda^z)$ of Reeb chords of $X_\lambda$ from $\Lambda$ to $\Lambda^z$ satisfies:

$$e^{(1+2\epsilon)} a_C \max(f_{\lambda^z}) < N_C(\lambda, \Lambda, \Lambda^z)$$ (34)

Step 2:
We first introduce a Riemannian metric on the manifold $Y$ which, at $N_\delta(\epsilon)$ is the euclidean metric for the coordinates $(\theta, z)$. This metric induces a measure of area $\text{Area}(\Sigma)$ for all surfaces $\Sigma$ immersed in $Y$. We want to estimate the area $\text{Area}(\pi(F_{C,\Lambda}(U)))$ of the immersed surface $\{\phi^t_{\lambda}(\Lambda), t \in [0, C]\}$, where $\phi^t_{\lambda}$ is the Reeb flow of $\lambda$. The surface $\{\phi^t_{\lambda}(\Lambda), t \in [0, C]\}$, can be seen as the image of the map $F_{C,\Lambda} : \Lambda \times [0, C] \to Y$, where $F_{C,\Lambda}(p, t) = \phi^t_{\lambda}(p)$. Denoting $U = F_{C,\Lambda}^{-1}(N_\delta(\epsilon))$, we have:

$$\text{Area}^C(\Lambda) \geq \text{Area}(F_{C,\Lambda}(U)) \geq \text{Area}(\pi(F_{C,\Lambda}(U)))$$

where the last area is taken with respect to the Lebesgue measure in $\mathbb{D}_v$. Using the estimate for counting function $N_C(\lambda, \Lambda, \Lambda^z)$ (that counts the number of chords from $\Lambda$ to $\Lambda^z$) we made in step 1 gives us the inequality:

$$\text{Area}(\pi(F_{C,\Lambda}(U))) = \int_{\mathbb{D}_v} N_C(\lambda, \Lambda, \hat{\Lambda}) dx \wedge dy \geq \int_{\mathbb{D}_v} e^{(1+2\epsilon)} a_C \max(f_{\lambda^z}) dx \wedge dy$$

It follows that:

$$\text{Area}(\pi(F_{C,\Lambda}(U))) \geq (v(\epsilon))^2 e^{(1+2\epsilon)} a_C \max(f_{\lambda^z})$$ (35)

Combining this inequality with Fubini’s theorem, we have:

$$\limsup_{C \to +\infty} \frac{1}{C} \log(\text{Vol}(\phi^C(\Lambda))) \geq \frac{a}{(1+2\epsilon) \max(f_{\lambda})}$$ (36)

Yomdin’s theorem now implies that $h_{\text{top}}(\phi_{X_\lambda}) \geq \frac{a}{(1+2\epsilon) \max(f_{\lambda})}$. As in Proposition 3, the constant $\epsilon$ can be taken arbitrarily small, we obtain the desired estimate.

Remark: it is expected that by using the Polyfold technology which is being developed by Hofer, Wysocki and Zehnder one will be able to replace the condition “exponential homotopical growth rate” by the weaker condition “exponential growth rate” on the statement Theorem 1 above. It is also expected that by unpublished work of Bourgeois, Ekholm and Eliashberg one could obtain a similar estimate on the topological entropy from the exponential growth of the linearized Legendrian contact homology, which includes the strip Legendrian contact homology as a special case. As these results would depend on technologies which are still being developed we opted for the use of our less general versions which are, however, sufficient do deal for all the examples that we are aware of.
4 Proof of Theorem 2

In this section we construct examples of contact manifolds which have pairs of Legendrians with exponential homotopical growth of the linearized Legendrian contact homology.

We denote by $S$ the surface with boundary obtained by taking the two-dimensional torus and taking out a small open disc and $\omega$ a symplectic form on $S$. The first homology group $H_1(S)$ is equal to $\mathbb{Z} \oplus \mathbb{Z}$. Let $h$ be a symplectomorphism of $(S, \omega)$ to itself, such that:

- the map $h$ is the identity in a small neighbourhood $V$ of the boundary circle $\partial S$;
- the induced map $h^*: H_1(S) \to H_1(S)$ is given by a hyperbolic automorphism of $\mathbb{Z} \oplus \mathbb{Z}$.

We follow [14] to construct a special contact form on the mapping torus of $(S, h)$. The following lemma of Eliashberg (see [14]) will be important for our construction:

**Lemma 3.** Let $h$ be a diffeomorphism of a surface $S$ with nonempty boundary which preserves a symplectic form $\omega$. If 1 is not an eigenvalue of the $h^*$, then $\omega$ admits a primitive $\beta$ such that $[h^* \beta - \beta] = 0$ in $H^1(S; \mathbb{R})$ and such that the characteristic vector field of $\beta$ is transverse to $\partial S$.

Remark: we reproduce here the proof of Lemma 3.4 from [14] only because we want to obtain a primitive $\beta$ of $\omega$ with the extra assumption that the characteristic vector field $v_\beta$ of the 1-form $\beta$ (defined by the equation $i_{v_\beta} \omega = \beta$) is transverse to $\partial S$.

Proof: because $\omega$ is an area-form on a punctured torus, there exists a primitive $\beta_0$ of $\omega$ such that its characteristic vector field $v_{\beta_0}$ is transverse to $\partial S$. By our hypothesis, the map $h^*: H^1(S, \mathbb{R}) \to H^1(S, \mathbb{R})$ is surjective. As $d(h^* \beta_0 - \beta_0) = 0$, $(h^* \beta_0 - \beta_0)$ represents a cohomology class. Thus, one can find $\theta \in H^1(S; \mathbb{R})$ such that $[h^* \beta_0 - \beta_0] = (h^* - id)[\theta]$.

As $\partial S$ is null-homologous, $[\theta]$ evaluates to 0 over $\partial S$. We can thus pick a representative $\theta_0$ of the cohomology class $[\theta]$ that vanishes on $V$. Setting $\beta = \beta_0 - \theta$ gives the desired primitive. The primitive $\beta$ satisfies that $v_\beta$ is transverse to $\partial S$, as $\beta$ coincides with $\beta_0$ at $V$.

Remark: we can parametrize $V$ using coordinates $(r, \vartheta) \in [\delta_0, 0]$. We can take these coordinates so that the base $(\partial_r, \partial_\vartheta)$ is positively oriented with respect to $\omega$. We will assume without loss of generality that $\beta_0 = H(r)d\vartheta$ in $V$, where $H > 0$ and $H' > 0$.

We note that the characteristic vector field $v_\beta$ is also a Liouville vector field for the symplectic form $\omega$; therefore $v_\beta$ has to be pointing in the outward direction over $\partial S$.

Over the manifold $\mathbb{R} \times S$ (with coordinates $(t, p) \in \mathbb{R} \times S$ consider the 1-form $\alpha = dt + \beta$, which is easily seen to be a contact form. The Reeb vector field of $\alpha$ is $\partial_t$. The following construction of Giroux (presented in [14] Lemma 2.3) gives us a special contact form on the mapping torus of $(S, h)$. As $[h^* \beta - \beta] = 0$ we can find a positive function $f$, which is constant in $V$, such that $df = h^* \beta - \beta$. $\alpha$ is invariant by the diffeomorphism:
from $\mathbb{R} \times S$ to itself, and therefore it induces a contact form $\tilde{\alpha}$ on the mapping torus $\Omega(S,h) = (\mathbb{R}) \times S/((t,p) \sim F(t,p))$. We denote by $p_H : \mathbb{R} \times S \to \Omega(S,h)$ the covering map associated to the above construction.

In the covering $\mathbb{R} \times S$ the Reeb vector field of $X_{\tilde{\alpha}}$ lifts to the simple form $X_{\alpha} = \partial_t$. This will be useful in making some of our subsequent arguments simpler.

### 4.1 A special Legendrian knot in $\Omega(S,h)$

We begin with the following lemma.

**Lemma 4.** There is an embedded curve $\eta$ in $\{0\} \times S$ such that $\int_\eta \beta = 0$

**Proof:** by the fact that the characteristic vector field $\zeta_{\beta_0}$ is transverse to the boundary Peixoto’s theorem [21][page 172] is valid for the vector field $\zeta_{\beta_0}$ on $\{0\} \times S$. We can thus apply the arguments of [21][Proposition 4.6.1] to make a $C^\infty$ small perturbation $S'$ of $\{0\} \times S$, that makes the characteristic vector field $\zeta'$ (defined by $i_{\zeta'}(d\tilde{\alpha} |_{S'}) = \tilde{\alpha} |_{S'}$) induced by the contact structure $\ker(\alpha |_{S'})$ is a Morse-Smale vector field, and $S'$ is a graph of $S$ on which we have $d\tilde{\alpha} |_{S'}$ as an area form in $S'$ (see [15] and [34] for properties Morse-Smale vector fields). In the remark after the proof of the Lemma, we give another way of constructing the perturbation $S'$. From the $C^\infty$ proximity of $S$ and $S'$, it follows that $v'$ points outward on the boundary of $S'$. Notice that as $S'$ comes with an area form $d\alpha |_{S'}$, it comes endowed with an orientation.

As $d\tilde{\alpha} |_{S'}$ is an area form, the condition $i_{\zeta'}(d\tilde{\alpha} |_{S'}) = \tilde{\alpha} |_{S'}$ means that the vector field $-\zeta'$ contracts the area form $d\tilde{\alpha} |_{S'}$. This implies that $-\zeta'$ has no singularities of source type.

Because of the Morse-Smale condition for the flow generated by $-\zeta'$, its $\omega$-limit is the union of periodic orbits and singularities of the flow. If this flow has a periodic orbit $P$ we take our $\eta$ to be the projection of $P$ on $\{0\} \times S$ and we are done.

If this is not the case, let the 1-skeleton $\Delta$ be the union of the singularities of the flow of $-\zeta'$ and the unstable manifolds of its saddle singularities. Because $-\zeta'$ is of Morse-Smale type it has no source singularities, and as it is directed inward along $\partial(S')$ the flow of $-v'$ retracts the surface $S'$ to the the 1-skeleton $\Delta$, as the time goes to $+\infty$. This means that $\Delta$ is a deformation retract of $S'$ by the flow of $-\zeta'$. The topology of $S'$ forces $\Delta$ to contain a piecewise smooth simple curve $\gamma$ tangent to the $\ker(\alpha)$. The vertices of $\gamma$ are located at sink singularities of the characteristic foliation of $S'$. Observe that because $\int_\gamma \alpha = 0$ and $d\alpha |_{S'}$ is an area-form, $\gamma$ cannot be null-homologous in $S'$. We pick an orientation for $\gamma$.

Fix a point $p_0$ in $\gamma$ which is not a vertex. We can then smoothen the vertices of $\gamma$ in a small neighbourhood of the vertices disjoint from $p_0$ and produce a smooth embedded curve $\gamma'$ on $S'$. As $\gamma'$ coincides with $\gamma$ in a neighbourhood of $p_0$, we pick the orientation in $\gamma'$ which coincide with the orientation chosen for $\gamma$ in the region the two curves overlap.
Given $\delta > 0$, one can ensure, by doing the smoothing in a sufficiently small neighbourhood of the vertices, that $q = \int_{\gamma'} \alpha$ satisfies $q \in (-\delta, \delta)$.

Using that near the point $p_0$ the curve $\gamma'$ is tangent to $\ker(\alpha)$, it is possible, for sufficiently small $\delta$, to make a small perturbation $\tilde{\gamma}$ of $\gamma'$ supported near $p_0$ such that, for the small region $U$ bounded by $\tilde{\gamma}$ of $\gamma'$, and oriented such that $\partial U = \gamma' - \tilde{\gamma}$, we have:

$$\int_U d\alpha = q$$

(38)

Now Stokes’ theorem and $q = \int_{\gamma'} \alpha$ implies that $\int_{\tilde{\gamma}} \alpha = 0$. This perturbation $\tilde{\gamma}$ which aims at correcting the change in the integral after the smoothing of the corners, can be made explicitly if one uses Darboux coordinates in a neighbourhood of $p_0$.

Since $S'$ is a graph of $\{0\} \times S$ and $\tilde{\gamma}$ is an embedded curve in $S'$, by projecting $\tilde{\gamma}$ on $\{0\} \times S$ one obtains an embedded curve $\eta$ on $\{0\} \times S$. Notice that $\int_{\tilde{\gamma}} \alpha = \int_{\tilde{\gamma}} (dt + \beta) = \int_{\tilde{\gamma}} \beta$ and thus $\int_{\eta} \beta = 0$.

Remark: maybe the easiest way to obtain the surface $S'$ used in the proof above is to construct a contact manifold $(M, \lambda)$ which contains $(\Omega(S, h), \alpha)$ as a component (see the next subsection) and where $\{0\} \times S$ can be extended to an embedded surface $\tilde{S}$ in $M$ (this is the case, for example, if all the other components $M$ are also mapping tori glued in an appropriate manner to $\Omega(S, h)$). In this case one can apply [21][Proposition 4.6.1] to $\tilde{S}$ in $(M, \lambda)$ to obtain a $C^\infty$ perturbation $\tilde{S}'$ of $\tilde{S}$ with a Morse-Smale characteristic foliation. The restriction of $\tilde{S}'$ to $\Omega(S, h)$ is the desired $S'$; observe that the $C^\infty$ proximity of $S'$ with $\{0\} \times S$ implies that the characteristic foliation of $S'$ is transverse to $\partial(S')$ and that $S'$ is a graph of $\{0\} \times S$ in $\mathbb{R} \times S$.

Because $\int_{\eta} \beta = 0$ we have that the curve $\eta$ obtained in the lemma is the Lagrangian projection of a Legendrian curve $\Lambda_0$ in $(\Omega(S, h), \alpha)$.

Using that $X_\alpha = \partial_t$, one sees that a Legendrian knot which is the graph of an embedded curve in $\{0\} \times S$ has the remarkable property that they have no Reeb chords for the Reeb flow of $X_\alpha$. Thus all Legendrians close to $\Lambda_0$ in the $C^\infty$ topology do not have any Reeb chords for the Reeb flow of $X_\alpha$.

By making a vertical translation of $\Lambda_0$ if necessary, there exists $N > 0$ such that $\Lambda_0 \subset [1, N - 1] \times S$. By summing a sufficiently large constant, we can demand that the function $f$ used to construct the mapping torus satisfies $f > 2N$. Remember that $f$ is constant in $V$, and let $K = f(p), \forall p \in V$. We denote by $\Lambda$ the Legendrian submanifold of $\Omega(S, h)$, which is the image by $p_H$ of $\Lambda_0$.

**Lemma 5.** For the Reeb vector field $X_\alpha$ there is no Reeb chord $c$ from $\Lambda$ to itself such that $[c]$ is the trivial element in $\pi_1(\Omega(S, h), \Lambda)$. The same is true for Legendrians which are sufficiently close to $\Lambda$ in the $C^\infty$ topology.

The lemma follows from the remark above about the non-existence of Reeb chords from $\Lambda_0$ to itself for the Reeb vector field $X_\alpha$ in $\mathbb{R} \times S$ (which is the lift of $X_\alpha$) for $\Lambda_0$ equal or sufficiently close to $\Lambda_0$. 

$\square$
4.2 Contact 3-manifolds containing \((Ω(S, h), \alpha)\) as a component

Let \(W\) be a compact oriented irreducible 3-manifold such that \(\partial(W)\) is a union of incompressible tori. The following theorem of Colin and Honda \cite{13} tells us that \(W\) admits a hypertight contact form tangent to the boundary:

**Theorem 3. (Colin-Honda \cite{13})** Let \(W\) be a compact, oriented, irreducible 3-manifold such that \(\partial(W)\) is a union of incompressible tori. Then there exists a hypertight contact form \(\zeta\) such that, in a neighbourhood \((\mathbb{R}/K\mathbb{Z}) \times S^1 \times I\) with coordinates \((\bar{t}, \bar{r}, \bar{\vartheta})\), \(\bar{r}\) of each component of \(\partial(W)\), \(\zeta = \cos(\bar{r})d\bar{t} - \sin(\bar{r})d\bar{\vartheta}\).

Now suppose we are given a finite collection compact oriented irreducible 3-manifolds \(\{W_i, 0 \leq i \leq N\}\), such that \(W_0 = Ω(S, h)\), and can be glued along their boundaries to give an oriented 3-manifold \(M\). This means that \(\{W_i, 0 \leq i \leq N\}\) is the JSJ decomposition of the 3-manifold \(M\). Using the above theorem of Colin and Honda we put hypertight contact forms tangent to the boundary on the manifolds \(W_i\) for \(i \geq 1\); on the special piece \(W_0\) we consider the the contact form \(\tilde{\alpha}\) constructed above (which is also tangent to the boundary). Our objective now is to glue these contact forms to obtain a contact form on \(M\). The details on how to make the gluing process are presented in \cite{35} and \cite{13}; we sketch it here for the convenience of the reader. From the remark following the proof of Lemma 3, we know that in a neighbourhood of \(\partial(Ω(S, h))\) diffeomorphic to \((\mathbb{R}/K\mathbb{Z}) \times V\), with coordinates \((t, r, \vartheta)\), we have \(\tilde{\alpha} = dt + H(r)d\vartheta\) where \(H > 0\) and \(H' > 0\).

For a natural number \(n > 3\) we consider a neck \(T\) of the form \((\mathbb{R}/K\mathbb{Z}) \times [0,1] \times S^1\) with coordinates \((t', r', \vartheta')\). Let \(g_1\) and \(g_2\) be functions from \(\times [0,1]\) to \(\mathbb{R}\) satisfying:

- \(g_1(0) = 1\) and \(g_1^{(j)}(0) = 0\) for all positive integer \(j\)
- \(g_2^{(j)}(0) = H^{(j)}(0)\) for all non-negative integer \(j\)
- \((g_1g_2 - g_1'g_2')(r') > 0\) for all \(r' \in [0,n]\)
- \(g_1(r') = \cos(2\pi r')\) and \(g_2(r') = \sin(2\pi r')\) if \(r' \in [1,n]\)

Then \(\nu_n = g_1(r)d\bar{t} + g_2(r)d\bar{\vartheta}\) is a contact form in \(T_n\). The idea is that \(\nu_n\) interpolates between the contact form on a neighbourhood of the boundary of \(\partial(Ω(S, h))\) to the contact form \(\cos(\bar{r})d\bar{t} - \sin(\bar{r})d\bar{\vartheta}\) which appears in the boundary of \(W_i\) for \(i \geq 1\) by the above theorem of Colin and Honda. Thus by introducing the necks \(T_n\) we can interpolate the contact forms in the boundaries of the components \(W_i\) to obtain a contact form \(\tau\) on \(M\). The hypertightness of \(\tau\) comes from the hypertightness of \(\tilde{\alpha}\) and of the contact forms on the components \(W_i\) for \(i \geq 1\), combined with the fact that all the periodic orbits in the neck \(T_n\) represent non-trivial homology classes in the incompressible tori; these boundary tori of the components \(W_i\) remain incompressible in \(M\).

A contact 3-manifold is said to have positive Girou torsion if there is a contact embedding \((T; ξ_1)\) where: \(T = S^1 \times S^1 \times [0,1]\) and \(ξ_1\) is the kernel of the contact form \(\cos(2\pi \varpi)dq_1 + \sin(2\pi \varpi)dq_2\) for \((q_1, q_2, \varpi) \in S^1 \times S^1 \times [0,1]\). It is immediate that the \((M, \tau)\) we just constructed in this example has positive Girou torsion.
It is thus clear that the above construction includes manifolds with positive Giroux torsion. By a theorem of Gay [22] (see also [36]) manifolds with positive Giroux torsion are not strongly fillable. An interesting feature of these examples is that they are not strongly fillable, while the unit tangent bundles studied in [31] are. The examples constructed above are coincide with the ones studied by Colin in [11] and Colin and Honda [13] for the manifold \( M \) with the JSJ decomposition given by \( \{ W_i, 0 \leq i \leq N \} \). As a consequence of their work we can produce infinitely many different contact structures on \( M \) by the recipe above.

**Proposition 5.** For the Reeb vector field \( X_\tau \) there is no Reeb chord \( c \) from \( \Lambda \) to itself such that \([ c ]\) is the trivial element in \( \pi_1( M, \Lambda) \).

**Proof:** we will show that Lemma 5 implies the proposition.

By contradiction suppose there is a smooth disc \( D \) such that \( \partial D \) is the concatenation of a Reeb chord \( c \) with a path \( \gamma \subset \Lambda \). By genericity, we can suppose that \( D \) intersects \( \partial W_0 \) transversely. This implies that \( D \cap \partial W_0 \) is a collection of embedded circles \( w_1, \ldots, w_n \) in \( \partial W_0 \); these circles need not be disjoint, they might intersect each other. As all \( w_i \) are contractible in \( M \) and \( \partial W_0 \) is an incompressible torus in \( M \), this implies that the \( w_i \) are also contractible in \( \partial W_0 \).

Let \( u_i \) be the disc in \( \partial W_0 \) whose boundary is \( w_i \), and \( v_i \) be the disc in \( D \) whose boundary \( w_i \). Select from the set \( \{ v_i, 1 \leq i \leq n \} \) a subset \( K = \{ v_{i_1}, \ldots, v_{i_k} \} \) such that each \( v_i \) is contained in at least one \( v_{i_j} \), and such that no \( v_{i_j} \) is contained in a \( v_{i_l} \) for \( l \neq j \). Then by cutting of the discs \( v_{i_j} \) and gluing in their place the discs \( u_i \) we get a disc \( D' \) in \( \Omega(S, h) \) whose boundary is the concatenation of the Reeb chord \( c \) with the path \( \gamma \subset \Lambda \). The existence of such a disc contradicts Lemma 5; and this finishes the proof of the proposition.

\( \square \)

It is clear that, for \( \epsilon > 0 \) sufficiently small, the above proposition is valid also for Legendrians \( \epsilon \) close to \( \Lambda \) in the \( C^\infty \) topology. As a consequence of the above proposition, we have the following corollary:

**Corollary 1.** Let \( \hat{\Lambda} \) be a generic Legendrian \( \epsilon \)-close to \( \Lambda \) in the \( C^\infty \) topology and disjoint from \( \Lambda \). Then:

the strip Legendrian contact homology \( LCH_{st}(M, \tau, \Lambda \rightarrow \hat{\Lambda}) \) is defined.

We now fix \( \hat{\Lambda} \) satisfying the hypothesis of the corollary above and proceed to show that the homotopical growth rate of \( LCH_{st}(M, \tau, \Lambda \rightarrow \hat{\Lambda}) \) is exponential.

### 4.3 Exponential homotopical growth of \( LCH_{st}(M, \tau, \Lambda \rightarrow \hat{\Lambda}) \)

To study the growth rate of \( LCH_{st}(M, \tau, \Lambda \rightarrow \hat{\Lambda}) \) we will consider some special relative homotopy classes of paths from \( \Lambda \) to \( \hat{\Lambda} \).

**Definition 3.** Let \( c_1 \) and \( c_2 \) be Reeb chords from \( \Lambda \) to \( \hat{\Lambda} \). We say that \( c_1 \) and \( c_2 \) are in the same **Relative Nielsen class** if, and only if, there exists a smooth strip \( u : [0,1] \times [0,1] \rightarrow \Omega(S, h) \) such that:
- \( u(0 \times [0,1]) \) is a path in \( \Lambda \) and \( u(1 \times [0,1]) \) is a path in \( \hat{\Lambda} \)
- \( u([0,1] \times 0) = c_1 \) and \( u([0,1] \times 1) = c_2 \)

It is immediate to check that the relative Nielsen classes are equivalence classes, because relative Nielsen classes are just homotopy classes of paths from \( \Lambda \) to \( \hat{\Lambda} \) in the mapping torus \( \Omega(S,h) \). Our first step is to prove that the Relative Nielsen classes generate a partition of \( LCH_{st}(M,\tau,\Lambda \rightarrow \hat{\Lambda}) \) in subcomplexes because they can be regarded as elements in the set \( \Sigma_{\Lambda \rightarrow \hat{\Lambda}} \) of homotopy classes of paths from \( \Lambda \) to \( \hat{\Lambda} \) in \( M \).

**Lemma 6.** Let \( c_1 \) and \( c_2 \) be Reeb chords from \( \Lambda \) to \( \hat{\Lambda} \), and \( u : [0,1] \times [0,1] \rightarrow M \) such that:

- \( u(0 \times [0,1]) \) is a path in \( \Lambda \) and \( u(1 \times [0,1]) \) is a path in \( \hat{\Lambda} \)
- \( u([0,1] \times 0) = c_1 \) and \( u([0,1] \times 1) = c_2 \)

Then, there exists a strip \( u' : [0,1] \times [0,1] \rightarrow \Omega(S,h) \) such that \( u'((\partial([0,1] \times [0,1])) = u(\partial([0,1] \times [0,1])) \).

**Proof:** the proof is very similar to the one of proposition 5 above, so we will only give an outline of it.

By genericity we can assume that the image of \( u \) intersects \( \partial(\Omega(S,h)) \) transversely. The intersection consists of a finite collection of circles \( w_1,\ldots,w_k \) which are contractible in \( M \). The assumption that \( \partial(\Omega(S,h)) \) is incompressible implies that \( w_1,\ldots,w_k \) are also contractible in \( \partial(\Omega(S,h)) \). The intersection of \( u([0,1] \times [0,1]) \) with \( W \) is composed by discs \( d_i \) with boundary \( w_i \). We can cut out these discs and replace them by discs \( d_i' \) contained in \( \partial(\Omega(S,h)) \) and whose boundary is \( w_i \). This cut and paste procedure gives the desired \( u' \).

Remember from section 2.1 that the differential \( \partial_{st} \) of the strip Legendrian contact homology \( LCH_{st}(M,\tau,\Lambda \rightarrow \hat{\Lambda}) \) count index 1 holomorphic strips \( \tilde{u} : \mathbb{R} \times [0,1] \rightarrow \mathbb{R} \times M \) in the symplectization of \( (M,\tau) \) with the boundary conditions:

- \( \tilde{u}(\mathbb{R} \times \{0\}) \subset \mathbb{R} \times \Lambda \)
- \( \tilde{u}(\mathbb{R} \times \{1\}) \subset \mathbb{R} \times \hat{\Lambda} \).

As mentioned earlier, it is a consequence of the above lemma that relative Nielsen classes can be seen as elements in \( \Sigma_{\Lambda \rightarrow \hat{\Lambda}} \). More precisely, denoting by \( \Ree \) the set of relative Nielsen classes, we have a map \( I : \Ree \rightarrow \Sigma_{\Lambda \rightarrow \hat{\Lambda}} \), defined as follows: given a relative Nielsen class \( \rho \), we pick a Reeb chord \( c \in \rho \) and define \( I(\rho) \) to be the class of \( [c] \in \Sigma_{\Lambda \rightarrow \hat{\Lambda}} \). It is easy to see that \( I \) is well defined and the above lemma implies that \( I \) is injective.

**Remark:** notice that because of the way we constructed the contact form \( \tau \), we have that all the Reeb chords in \( T_{\Lambda \rightarrow \hat{\Lambda}}(\tau) \) are contained in the component \( M_0 = \partial(\Omega(S,h)) \), and therefore belong to elements in \( \Sigma_{\Lambda \rightarrow \hat{\Lambda}} \) which are in the image of our map \( I \).

It is therefore possible to write \( LCH_{st}(M,\tau,\Lambda \rightarrow \hat{\Lambda}) \) as a direct sum:
\[ LC^I_{\mathcal{H}_d}(M, \tau, \Lambda \to \hat{\Lambda}) = \bigoplus_{\varphi \in \mathcal{R}} LC^I_{\mathcal{H}_d}(M, \tau, \Lambda \to \hat{\Lambda})_{\varphi} \] (39)

### 4.3.1 The Relative Nielsen classes

We will use the covering \((\mathbb{R} \times S, \alpha)\) of \((\Omega(S, h), \tilde{\alpha})\) to obtain information about the Relative Nielsen classes. We begin by fixing in \((\mathbb{R} \times S, \alpha)\) the lift \(\Lambda_0\) of \(\Lambda\) that is contained in \([0, N] \times S\). The lifts of \(\hat{\Lambda}\) to \((\mathbb{R} \times S, \alpha)\) can be ordered in the following way: letting \(\tilde{\Lambda}_0\) be the lift of \(\tilde{\Lambda}\) contained in \([0, N] \times S\), \(\tilde{\Lambda}_n = F^{-n}(\tilde{\Lambda})\).

Given a Reeb chord \(c\) from \(\Lambda\) to \(\tilde{\Lambda}\) we take the lift \(\tilde{c}\) which has its starting point in \(\Lambda_0\). It is not difficult to see that if \(c_1\) and \(c_2\) are Reeb chords from \(\Lambda\) to \(\tilde{\Lambda}\) that are in the same Relative Nielsen class, then \(\tilde{c}_1\) and \(\tilde{c}_2\) have to have endpoints in the same lift \(\tilde{\Lambda}_n\) of \(\tilde{\Lambda}\). We will see, however, that this condition of \(\tilde{c}_1\) and \(\tilde{c}_2\) having the endpoints in the same lift \(\tilde{\Lambda}_n\) is far from sufficient to guarantee that \(c_1\) and \(c_2\) are in the same Relative Nielsen class.

Let \(\pi_S : \mathbb{R} \times S \to S\) be the projection in the second coordinate. Remembering our construction in section 5.3, we know that \(\eta = \pi_S(\Lambda_0)\) and \(\tilde{\eta} := \pi_S(\tilde{\Lambda}_0)\) are embedded curves in \(S\). From the definition of the map \(F\), we have that \(\pi_S(\tilde{\Lambda}_n) = \pi_S \circ F^{-n}(\tilde{\Lambda}) = h^{-n}(\tilde{\eta})\), and as \(h\) is a diffeomorphism, \(\pi_S(\tilde{\Lambda}_n)\) is an embedded curve in \(S\). Observe that the Reeb chords from \(\Lambda_0\) to \(\tilde{\Lambda}_n\) are in one-to-one correspondence with the intersection points of \(\eta\) and \(h^{-n}(\tilde{\eta})\). Notice that because \(\partial_t\) is the pull-back of the Reeb vector field in this covering space, the transversality of all the Reeb chords from \(\Lambda\) to \(\tilde{\Lambda}\) is equivalent to the transversality of \(\eta\) and \(h^{-n}(\tilde{\eta})\) for every natural number \(n\). We now proceed for the following characterization of the Relative Nielsen classes.

**Proposition 6.** Let \(c_1\) and \(c_2\) be Reeb chords in \(\mathcal{T}_{\Lambda \rightarrow \tilde{\Lambda}}(\tilde{\alpha})\) with \(p_1 := \pi_S(c_1)\) and \(p_2 := \pi_S(c_2)\). Then \(c_1\) and \(c_2\) are in the same Relative Nielsen class if, and only if, \(\tilde{c}_1\) and \(\tilde{c}_2\) have end points in the same \(\tilde{\Lambda}_n\), and there exists a map \(v : [0, 1] \times [0, 1] \to S\), such that:

\[
\begin{align*}
v([0, 1] \times \{0\}) &= p_1, \quad (40) \\
v([0, 1] \times \{1\}) &= p_2, \quad (41) \\
v(\{0\} \times [0, 1]) &\subset \eta, \quad (42) \\
v(\{1\} \times [0, 1]) &\subset h^{-n}(\tilde{\eta}) \quad (43)
\end{align*}
\]

**Proof:** suppose \(c_1\) and \(c_2\) are in the same relative Nielsen class. We take the map \(u : [0, 1] \times [0, 1] \to \Omega(S, h)\) given in Definition 4, and consider its lift \(\tilde{u} : [0, 1] \times [0, 1] \to \mathbb{R} \times S\), such that \(\tilde{u}([0, 1] \times \{0\}) = \tilde{c}_1\) and \(\tilde{u}([0, 1] \times \{1\}) = \tilde{c}_2\). It is easy to see that taking \(v = \pi_S \circ \tilde{u}\) gives a strip in \(S\) satisfying the conditions in the proposition; this plus the remark above finishes one implication.

To prove the reverse implication take a \(v\) satisfying the conditions in the statement proposition. By taking the path \(v(\{0\} \times [0, 1]) \subset \eta\) there exists a unique function \(g_0 : [0, 1] \to \mathbb{R}\) such that the path \(\gamma_0(s)\) of the form: \(\gamma_0(s) = (v(0, s), g_0(s))\) is a path in \(\Lambda_0\).
Analogously there exists a function $g_1 : [0, 1] \to \mathbb{R}$ such that $\gamma_1(s) = (v(1,s), g_1(s))$ is a path in $\tilde{\Lambda}_n$. Take $f : [0, 1] \times [0, 1] \to \mathbb{R}$ to be an homotopy between $g_0$ and $g_1$ that is, $f(0,s) = g_0(s)$ and $f(1,s) = g_1(s)$. Now we can define the strip $u(r,s) = (v(r,s), f(r,s))$ in $\mathbb{R} \times S$, and considering $p_H \circ u$ we get a strip in $\Omega(S,h)$ which satisfies the conditions of the definition of Relative Nielsen classes for $\tilde{c}_1$ and $\tilde{c}_2$; this finishes the second implication and the proof of the proposition.

The above proposition gives a complete description of the Relative Nielsen classes. It also shows how to identify different Relative Nielsen classes of Reeb chords by looking at properties of intersection points of the curves $h^{-n}(\tilde{\eta})$ and $\eta$. This is the crucial link that will allow us to use the hyperbolicity of $h^*$ to estimate the growth of the number of Relative Nielsen classes. Among the relative Nielsen classes, the subset of relative Nielsen classes with an odd number of chords will be of special importance to us: we will call them fundamental Relative Nielsen classes and denote their set by $\mathcal{R}_f$.

From the discussion above we can partition the set $\mathcal{R}_f$, in subsets $\mathcal{R}_n^f$ defined by: an element $c \in \mathcal{R}_n^f$, if, and only if, for every Reeb chord $c \in c$, the lift $\tilde{c}$ has its endpoint in $\tilde{\Lambda}_n$. Our next step will be to estimate the cardinality of $\mathcal{R}_n^f$. As $\eta$ is an embedded closed curve in the oriented surface $S$ it is possible to take another embedded curve $\nu$ such that $\{[\eta], [\nu]\}$ is an oriented basis of $H_1(S)$. It is well known that the intersection number of the pair $\{\eta, \nu\}$ is 1.

By the assumption made on the map $h : S \to S$ on the beginning of the chapter, the matrix $P \in PSL(2, \mathbb{Z})$ representing $h^* : H_1(S) \to H_1(S)$ in the basis $\{\eta, \nu\}$ is hyperbolic. The homology class of curve $h^{-n}(\eta)$ can be written as a linear combination of $[\eta]$ and $[\nu]$; let $(a_n, b_n)$ be the unique pair of integers such that $[h^{-n}(\eta)] = a_n[\eta] + b_n[\nu]$. It is immediate from the well known description of the dynamics of hyperbolic linear automorphisms of the 2-torus (see [34]), that the hyperbolicity of the matrix $P$ implies that there exist a constant $d > 0$ such that:

$$|a_n|, |b_n| > e^{dn}$$

or, in other words, $a_n$ and $b_n$ grow exponentially. We remind the reader of the well known fact that $b_n$ equals the intersection number of $h^{-n}(\eta)$ and $\eta$. We are now ready to prove the main result of this subsection:

**Theorem 4.** $\sharp(\mathcal{R}_n^f) \geq b_n$, and consequently $\sharp(\mathcal{R}_n^f)$ grows exponentially with respect to $n$.

**Proof:** We endow $S$ with a hyperbolic metric $g$ having $\partial S$ as a geodesic boundary. Notice that as $\eta$ and $\nu$ are simple closed curves, and the number of intersection of $\eta$ and $\nu$ equals the intersection number $i([\eta], [\nu])$ then Lemma 2.6 on page 28 of [10], implies that there is a homeomorphism $\psi : S \to S$, homotopic to the identity and such that $\psi(\eta)$ and $\psi(\nu)$ are geodesics of the metric the hyperbolic metric $g$.

As $\psi(h^{-n}(\tilde{\eta}))$ is an embedded closed curve in $S$ it is possible to isotopy it to an embedded hyperbolic geodesic $\gamma$. Such a geodesic $\gamma$ has intersection number $b_n$ with $\psi(\eta)$. 

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We denote by \( \{ p_1^n, \ldots, p_{z_n}^n \} \) the set of the intersection points of \( \gamma \) and \( \psi(\eta) \), and it is clear that \( z_n \geq b_n \).

We consider the Poincaré disc \( (\mathbb{D}, g_{-1}) \) as the universal cover of \( (S, g) \), and denote \( \pi : \mathbb{D} \to S \) the covering map. Given an embedded closed curve \( q \) in \( S \) which is not homologous to \( \psi(\eta) \), let \( \overline{q} \) be a lift of \( q \) in \( \mathbb{D} \) and take a closed subinterval \( I_q \) of \( \overline{q} \) such that \( (\pi(\partial I)) = p_0 \notin \psi(\eta) \) and that covers every point \( x \neq p_0 \) of \( \gamma \) exactly once (the intersection of \( \pi^{-1}(x) \) and \( I \) has one element). We call \( I_q \) a fundamental interval of \( q \).

From now on we suppose \( n \geq 1 \) so that \( \gamma \) and \( \psi(h^{-n}(\hat{\eta})) \) are not homologous to \( \psi(\eta) \). Consider a lift \( \overline{\psi} \) in \( \mathbb{D} \), and take a fundamental interval \( I_\gamma \) of \( \overline{\psi} \). Because of the convexity of the hyperbolic metric, we know that a \( \overline{\psi} \) cannot intersect one lift of \( \psi(\eta) \) more than once. Therefore, \( I_\gamma \) intersects exactly \( z_n \) different lifts \( \{ \kappa_1, \ldots, \kappa_{z_n} \} \) of \( \psi(\eta) \).

Denote by \( \gamma_t \) isotopy for \( t \in [0, 1) \) between \( \gamma \) and \( \psi(h^{-n}(\hat{\eta})) \). Because of the hyperbolicity of \( h^* \), and as \( n \geq 1 \), \( \psi(h^{-n}(\hat{\eta})) \), and thus all \( \gamma_t \), are not homologous to \( \psi(\eta) \). Thus using the isotopy \( \gamma_t \), we can construct a path \( I_t \) of fundamental intervals of \( \gamma_t \). This generates an isotopy of \( I_\gamma \) to a fundamental interval \( I_{\psi(h^{-n}(\hat{\eta}))} \) of \( \psi(h^{-n}(\hat{\eta})) \) through fundamental intervals of \( \gamma_t \). From the properties of fundamental intervals we have that \( \pi(\partial I_t) \) is disjoint from \( \psi(\eta) \) for all \( t \in [0, 1] \). It is then clear that \( \psi(h^{-n}(\hat{\eta})) \) must also intersect the same \( z_n \) different lifts \( \{ \kappa_1, \ldots, \kappa_{z_n} \} \) of \( \psi(\eta) \) intersected by \( I_\gamma \), though it can in theory intersect also others lifts of \( \psi(h^{-n}(\hat{\eta})) \).

The set \( A \) of intersection points of \( \eta \) and \( h^{-n}(\hat{\eta}) \) is in bijective correspondence with the set \( O \) of intersection points of \( \psi(\eta) \) and \( \psi(h^{-n}(\hat{\eta})) \). Because of the properties of a fundamental interval, there also exists a bijection between the set \( O \) of intersection points of \( \psi(\eta) \) and \( \psi(h^{-n}(\hat{\eta})) \), and the set \( B \) of intersection points of \( I_{\psi(h^{-n}(\hat{\eta}))} \) with the geodesics \( \{ \kappa_1, \ldots, \kappa_{z_n} \} \). There exists then a bijection map \( \varphi : A \to B \). We remind the reader that as we mentioned above, \( A \) is in bijective correspondence with the set of Reeb chords from \( \Lambda_0 \) to \( \hat{\Lambda}_n \).

Taking now \( p_1, p_2 \in A \), we claim that there is a strip \( v \) satisfying the four conditions of Proposition 6 above if, and only if, \( \varphi(p_1) \) and \( \varphi(p_2) \) lie in the same \( \kappa_j \). To prove one direction of the claim notice that if there exists such a strip \( v \) then we can take a lift \( \overline{v} \) of \( v \) in the universal cover \( \mathbb{D} \). By looking at the boundary conditions that are satisfied by \( \overline{v} \) and using that \( \psi(\eta) \) and \( \psi(h^{-n}(\hat{\eta})) \) are embedded in \( S \), it is easy to see that \( \varphi(p_1) \) and \( \varphi(p_2) \) have to lie in the same \( \kappa_j \). For the other direction if \( \varphi(p_1) \) and \( \varphi(p_2) \) lie in the same \( \kappa_j \) we can construct a strip \( \overline{v} \) satisfying \( \overline{v}([0, 1] \times \{0\}) = \varphi(p_1) \), \( \overline{v}([0, 1] \times \{1\}) = \varphi(p_2) \), \( \overline{v}(\{0 \} \times [0, 1]) \subset \kappa_j \) and \( \overline{v}([1] \times [0, 1]) \subset \ell \) (\( \ell \) being the lift of \( \psi(h^{-n}(\hat{\eta})) \) that contains \( I_{\psi(h^{-n}(\hat{\eta}))} \)), and taking \( v = \pi(\overline{v}) \) we obtain the desired strip satisfying the conditions of Proposition 6.

As a consequence of the previous claim and Proposition 6, we have that for each \( \kappa_j \) is associated a different Relative Nielsen class \( \varrho_j \) in \( \mathcal{R}_n \), and the intersections between \( I_{\psi(h^{-n}(\hat{\eta}))} \) and \( \kappa_j \) are in bijective correspondence with the Reeb chords in \( \varrho_j \). An immediate consequence is that there are at least \( z_n \) different Relative Nielsen classes in \( \mathcal{R}_n \).
To conclude the proof of the theorem, we have to prove that $\varrho_j$ is a fundamental Relative Nielsen class. But $I_t$ intersects $\kappa_j$ an odd number of times, and we have an isotopy $I_t$ between $I_\gamma$ and $I_{\psi(h^{-n}(\eta))}$ such that $\partial(I_t)$ never intersects $\kappa_j$. As $I_\gamma$ and $I_{\psi(h^{-n}(\eta))}$ are both transversal to $\kappa_j$, $I_{\psi(h^{-n}(\eta))}$ also has to intersect $\kappa_j$ an odd number of times, which proves that $\varrho_j$ is a fundamental Relative Nielsen class. Thus, there are in fact at least $z_n$ different fundamental relative Nielsen classes in $\mathcal{R}_n$, and the theorem is proved.

We have now, all the ingredients needed to obtain the exponential homotopical growth rate of $LCH_{st}(M, \tau, \Lambda \to \hat{\Lambda})$.

The inverse of the diffeomorphism $F : \mathbb{R} \times S \to \mathbb{R} \times S$ is $F^{-1}(t, p) = (t + f(h^{-1}), h^{-1}(p))$. Let $K > 0$ be a constant such that $\max(f) < K$. Then $\hat{\Lambda}_n = F^{-1}(\Lambda_0) \subset [0, (n+1)K] \times S$. This implies that for all Relative Nielsen classes $\varrho \in \mathcal{R}_k$ with $0 \leq k \leq n$ all the Reeb chords $c \in \varrho$ satisfy $A(c) \leq (n + 1)K$.

We are now ready to prove our main theorem:

**Theorem 5.** The linearized Legendrian contact homology $LCH_{st}(M, \tau, \Lambda \to \hat{\Lambda})$ has exponential homotopical growth rate with exponential weight $\frac{d'}{K}$.

**Proof:** the strategy is to use the growth rate of the number of different fundamental relative Nielsen classes, to estimate the set $\#(\Sigma^{K(n+1)}_{\Lambda \to \hat{\Lambda}}(\tau))$ defined in Definition 2 (see page 16).

**Step 1:** for every $\varrho \in \mathcal{R}_k^f$ with $0 \leq k \leq n$, we have $\mathcal{I}(\varrho) \in \Sigma^{K(n+1)}_{\Lambda \to \hat{\Lambda}}(\tau)$ (for the constant $K > 0$ as above).

From the defining property of fundamental relative Nielsen classes we know that:
\[
\dim(LCH^\tau_{st}(M, \tau, \Lambda \to \hat{\Lambda})) = \dim(\ker(\partial_{st})) + \dim(\ker(\partial_{st})) = \dim(LCH^\tau_{st}(M, \tau, \Lambda \to \hat{\Lambda})) - 2(\dim(\ker(\partial_{st})))
\]
implies that the numbers $\dim(LCH^\tau_{st}(M, \tau, \Lambda \to \hat{\Lambda}))$ and $\dim(LCH^\tau_{st}(M, \tau, \Lambda \to \hat{\Lambda}))$ cannot be zero, and has to be a positive number.

This combined with the fact that for all Relative Nielsen classes $\varrho \in \mathcal{R}_k^f$ with $0 \leq k \leq n$, all the Reeb chords $c \in \varrho$ satisfy $A(c) \leq (n + 1)C$, imply that:

for all $\varrho \in \mathcal{R}_k^f$ with $0 \leq k \leq n$, we have $\mathcal{I}(\varrho) \in \Sigma^{K(n+1)}_{\Lambda \to \hat{\Lambda}}(\tau)$.

**Step 2:**
From step 1 above we know that $\#(\Sigma^{K(n+1)}_{\Lambda \to \hat{\Lambda}}(\tau)) \geq b_n$.
Taking $d' = \frac{d}{K}$ and $a = e^{-d'}$ it follows that:

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\[ a e^{d(n+1)K} = e^{dn} \leq b_n \leq \#(\Sigma_{\Lambda \to \hat{\Lambda}}^{K(n+1)}(\tau)) \]  

which establishes the theorem. \qed

**Proof of Theorem 2:**

As we mentioned before, Colin showed in [11] that the recipe we used to produce the contact form \( \tau \) in \( M \) can generate infinitely many different contact structures in \( M \); this is because the construction of \( \tau \) depended on the necks \( T_n \) we glued to make the interpolation of the forms, and depending on the \( T_n \) used one gets different contact structures. With this, we have finished the proof of theorem.

\[ \square \]

As a consequence of this theorem and Theorem 1 in section 4, we have that for every contact form \( \tau' \) associated to \((M, \xi = \ker(\tau))\), the Reeb flow of \( X_{\tau'} \) have positive topological entropy. As we will see in the next section this has important dynamical consequences; in particular it implies a very strong form of the Weinstein conjecture for the contact manifolds \((M, \ker(\tau))\) constructed in this section.

### 5 Dynamical consequences of positive topological entropy and concluding remarks

By the work of Katok [28] [29] we know that for a flow in a 3-manifold positivity of topological entropy is equivalent to the existence of a Smale “horseshoe” in the flow. For a flow \( \phi_X \), a “horseshoe” is an invariant set \( \mathcal{H} \), such that the dynamics of the restriction of \( \phi_X \) to \( \mathcal{H} \) is topologically conjugated to a subshift of finite type and \( h_{\text{top}}(\phi_X | \mathcal{H}) > 0 \). “Horseshoes” play a very important role in the theory of dynamical systems because they are the best understood dynamical systems that present chaotic behaviour; see [34] for example. In the recent paper [30] of Lima and Sarig, these authors obtained a more detailed description of the symbolic dynamics for “horseshoes” appearing in 3-dimensional flows; with this they proved that the number \( P_{\text{hyp}}^{T}(\phi_X) \) of hyperbolic periodic orbits of \( \phi_X \) of period \( \leq T \) is bounded from below by \( e^{h_{\text{top}}(\phi_X)T} \), refining a previous estimate due to Katok. As a consequence, for a contact 3-manifold \((M, \xi)\) satisfying the hypothesis of Theorem 1, we have that for any Reeb flow associated to \((M, \xi)\) the number of hyperbolic Reeb orbits grows exponentially with the action.

An interesting property of the entropy estimate obtained in this paper, in [2] and [31], is that it gives estimates on the number of Reeb orbits also for degenerate contact forms. Another point of interest, is that the method used in the present paper and [31] allows us to prove existence of Reeb orbits by studying contact topological invariants which are defined through Reeb chords. This suggests that the growth rate of the number of Reeb chords and the growth rate of the number of Reeb orbits should be deeply connected.
We mention that the methods developed in this present paper can be used to obtain estimates for the topological entropy in other families of contact manifolds. One other class of examples is obtained by applying the Foulon-Hasselblatt integral surgery (introduced in [20]) on the Legendrian lift of a separating a separating geodesic of a hyperbolic surface; we refer the reader to [20] and [3] for the precise definition of the Foulon-Hasselblatt surgery. By the analysis of [20] and [24], for most integral surgeries the resulting contact 3-manifold $M'$ is not a Seifert fibre space, but an "exotic" graph manifold. In the author's PhD thesis, [4] the following theorem is proved:

**Theorem 6.** Let $(M, \lambda_F)$ be the contact manifold endowed with the hypertight contact form obtained via the integral Foulon-Hasselblatt surgery on the Legendrian $L_{\rho} \subset T_1 S$; where $\rho$ is a separating geodesic in the closed hyperbolic surface $S$ and $L_c$ is its Legendrian lift. Then there exist disjoint Legendrian curves $\Lambda$ and $\Lambda'$ of $(M, \ker(\lambda_F))$ such that: $\lambda_F$ is adapted to the pair $\Lambda, \Lambda'$ and $LCH_{\text{st}}(\lambda_F, \Lambda \to \Lambda')$ has exponential homotopical growth rate.

We also mention that a combination of the methods of the current paper with those with the ideas of Momin [32] and [27], can be used to prove positivity of topological entropy for a tight Reeb flow in $S^3$, provided that this Reeb flow has a particular transverse knot as a Reeb orbit. This is done in [5].

Lastly we mention 3 questions that we consider interesting for future investigations:

- does the positivity of topological entropy also holds for all non-identity contactomorphisms associated to the contact manifolds of Theorem 1?
- for a contact 3-manifold admitting an Anosov Reeb flow do there always exist a pair of Legendrian knots such that the linearized Legendrian contact homology is well defined and grows exponentially?
- does the exponential growth rate of linearized Legendrian contact homology for a pair of Legendrian submanifolds also implies positivity of topological entropy in dimensions bigger then 3? Notice that in this case the normal bundle of a Legendrian submanifold need not be trivial and thus the argument of section 4 does not adapt directly to this case.

It would also be interesting to find examples of tight contact structures on hyperbolic 3-manifolds for which there is exponential growth rate of linearized Legendrian contact homology for some Legendrians.

**References**

[1] C. Abbas. Finite energy surfaces and the chord problem, *Duke Math. J.*, Vol 96, No.2, (1999), pp. 241–316

[2] C. Abbas. Pseudoholomorphic strips in symplectisations II: Fredholm theory and Transversality, *Comm. Pure Appl. Math.*, Vol. 57 (2004), no. 1, 1–58
[3] M. Alves. Cylindrical contact homology and topological entropy. preprint
[4] M. Alves. Contact topological invariants and chaotic dynamics. PhD thesis. Universite Libre de Bruxelles, to appear in the fall of 2014
[5] M. Alves and P. Salomao. Legendrian contact homology on the complement of Reeb orbits and topological entropy. preprint in preparation
[6] F. Bourgeois. A Morse-Bott approach to Contact Homology. PhD thesis, Stanford University, 2002
[7] F. Bourgeois, A survey of contact homology, in “New perspectives and challenges in symplectic field theory”, 45–71, CRM Proc. Lecture Notes, 49, Amer. Math. Soc., RI, 2009.
[8] F. Bourgeois, T. Ekholm et Y. Eliashberg, Effect of Legendrian surgery, Geometry and Topology 16 (2012) 301–389
[9] F. Bourgeois, Y. Eliashberg, H. Hofer, K. Wysocki, E. Zehnder. Compactness results in symplectic field theory. Geometry and Topology, 7:799–888, 2003
[10] S. Bleiler and A. Casson. Automorphisms of surfaces after Nielsen and Thurston, volume 9 of London Mathematical Society Student Texts. Cambridge University Press, 1988.
[11] Y. Chekanov. Differential algebra of Legendrian links; Inventiones mathematicae, Volume 150, Issue 3, pp 441-483; 2002.
[12] V. Colin. Une infinité de structures de contact tendues sur les varietes toroidales, Commentarii Mathematici Helvetici, 76(2): 353-372, 2001.
[13] V. Colin, K. Honda. Construction controles des champs de Reeb et applications. Geometry and Topology 9:2193-2226, 2005.
[14] V. Colin, K. Honda, F. Laudenbach. On the flux of pseudo-Anosov homeomorphisms, Algebr. Geom. Topol. 8, 2147-2160, 2008.
[15] W. de Melo, J. Palis. Geometric theory of dynamical systems, an introduction. Springer-Verlag, New York, 1982.
[16] D. Dragnev, Fredholm theory and transversality for non compact pseudoholomorphic curves in symplecticisations. Communications on pure and applied mathematics, 57:726–763, 2004.
[17] T. Ekholm. Rational SFT, linearized Legendrian contact homology, and Lagrangian Floer cohomology, in Perspectives in Analysis, Geometry, and Topology, Progress in Mathematics 296, 109–146, Birkhauser, 2012.
[18] T. Ekholm, J. Etnyre, M. Sullivan, The contact homology of Legendrian submanifolds in IR2”+1, J. Differential Geom. 71 (2005), no. 2, 177-305.
[19] Y. Eliashberg, A. Givental, H. Hofer, Introduction to symplectic field theory, Geom. Funct. Anal. (2000), Special Volume, Part II, 560–673.
[20] P. Foulon and B. Hasselblatt. Contact Anosov flows on hyperbolic 3–manifolds. Geometry and Topology 17 (2013) 1225–1252
[21] H. Geiges, An Introduction to Contact Topology. Cambridge University Press, 2008
[22] D. T. Gay, Four-dimensional symplectic cobordisms containing three-handles, Geom. Topol. 10, 1749–1759 (electronic), 2006.
[23] M. Gromov. pseudoholomorphic curves in symplectic manifolds. Inventiones Mathematicae, 82:307–347, 1985.
[24] M. Handel and W. P. Thurston. Anosov flows on new three manifolds. Invent. Math. 59 (1980), no. 2, 95–103.
[25] H. Hofer. Pseudoholomorphic curves in symplectization with applications to the Weinstein conjecture in dimension three. Inventiones mathematicae, 114:515–563, 1993.
[26] H. Hofer, K. Wysocki, E. Zehnder. Properties of pseudoholomorphic curves in symplectisations I: Asymptotics. Ann. Inst. H. Poincare Anal. Nonlin., 13 (1996)
[27] U. Hryniewicz, A. Momin, P. A. S. Salomao. A Poincare-Birkhoff theorem for tight Reeb flows on $S^3$, arXiv:1110.3782v2
[28] A. Katok, Lyapunov exponents, entropy and periodic orbits for diffeomorphisms, Inst. Hautes Études Sci. Publ. Math. 51, 137–173, 1980.
[29] A. Katok. Entropy and closed geodesics. Ergodic Theory Dynam. Systems 2 339–365 (1983)
[30] Y. Lima, O. Sarig. Symbolic dynamics for three dimensional flows with positive topological entropy. arXiv:1408.3427 2014.
[31] L. Macarini, F. Schlenk. Positive topological entropy of Reeb flows on spherizations, preprint arXiv:1008.4566 (2010).
[32] A. Momin. Contact Homology of Orbit Complements and Imp lied Existence. Journal of Modern Dynamics, Pages: 409 - 472, Issue 3, July 2011
[33] G. Paternain. Geodesic Flows (Progress in Mathematics) Birkhauser, 1999.
[34] C. Robinson. Dynamical Systems: Stability, Symbolic Dynamics, and Chaos (Studies in Advanced Mathematics), CRC Press, 1998.
[35] A. Vaugon. On growth rate and contact homology, arXiv:1203.5589 2012.
[36] C. Wendl. Strongly fillable contact manifolds and J-holomorphic foliations, Duke Math. J. 151, no. 3, 337–384, 2010.