Mesoscopic fluctuations of nonlinear conductance of chaotic quantum dots

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The nonlinear dc conductance of a two-terminal chaotic cavity is investigated. The fluctuations of the conductance (anti)symmetric with respect to magnetic flux inversion through multichannel cavities are found analytically for arbitrary temperature, magnetic field, and interaction strength. For few-channel dots the effect of dephasing is investigated numerically. A comparison with recent experimental data is provided.

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Introduction. Recently the non-linear dc conductance of mesoscopic structures and, more specifically, their dependence on magnetic flux \( \Phi \) has found considerable attention and. Application of large voltages induces a rearrangement of the charge distribution. The charge re-distribution is subject to Coulomb interactions. Consequently investigation of non-linear transport reveals information on interaction parameters. This is in marked contrast to linear transport where the dc conductance \( G_{\alpha\beta} = dI_\alpha / dV_\beta \) for large cavities can with high accuracy be treated within a theory of non-interacting electrons. However to extract this information the role of temperature and dephasing need to be known with precision.

Interaction constants are extracted by investigating the magnetic field symmetry of non-linear transport. Under flux \( \Phi \) reversal, in the linear regime, the Onsager-Casimir relations dictate that the conductance matrix has the symmetry \( G_{\alpha\beta}(\Phi) = G_{\beta\alpha}(-\Phi) \). In particular, the conductance of a two probe conductor is an even function of flux. However, away from equilibrium, the non-linear conductance lacks such a symmetry. Importantly, the deviations from Onsager symmetry are entirely due to interactions. Therefore by investigating the departure from the Onsager-Casimir relations, information on the interaction properties can be obtained.

Our work is motivated by very recent experiments on non-linear transport in various open systems: carbon nanotubes, quantum dots, ballistic billiards, and quantum rings. Because of quantum interference, the samples exhibit strong mesoscopic (sample-to-sample) fluctuations, and a theory has thus to predict statistical values. Only recently two theories have explored such statistics for two-terminal open samples: Sánchez and Büttiker considered chaotic quantum dots in the universal regime with arbitrary interaction strength and high magnetic fields at \( T = 0 \), and Spivak and Zyuzin concentrated on weakly interacting open diffusive samples at low fields and temperatures. Although different aspects of these theories found good agreement with experiment in quantum dots, a more general theory that accounts for the effects of temperature and dephasing at arbitrary fields and interaction strength remains to be developed. Such a theory is our main goal.

This Letter presents a theory of fluctuations of conductance nonlinearity in open chaotic dots within Random Matrix Theory (RMT). A key result of our work is illustrated in Fig. 1. The (anti) symmetric parts of the non-linear conductance strongly fluctuate from sample to sample due to quantum effects. These fluctuations are sensitive to the flux \( \Phi \) through the dot. The fluctuations of the symmetric part \( G_s(\Phi) \) decrease as the magnetic flux increases, while the anti-symmetric part \( G_a(\Phi) \) has a stronger (linear) dependence at low fields. As the flux grows, the values of fluctuations reach their saturation values. The asymptotic values are equal for \( G_{a,s} \) as expected from the linear combination of uncorrelated contributions of random sign.

The correct definition of the crossover scale \( \Phi_c \) is important for the quantitative comparison of the theory with experiment. It determines the slope of \( G_a \) at small flux \( \Phi \) (see Fig. 1). One might naively expect \( \Phi_c \sim \Phi_0 \) but importantly we find \( \Phi_c \ll \Phi_0 \) in agreement with experiment. The scale of the crossover flux \( \Phi_c \) corresponds to a flux quantum \( \Phi_0 = \hbar / c \) through a typical trajectory of an electron and not through the area of the dot. In a chaotic quantum dot the time \( \tau_{\text{erg}} \) an electron typically spends inside the dot is usually much larger than the ergodic time necessary to explore its phase space.
During $\tau_d/\tau_{\text{erg}}$ random attempts to explore the dot with flux $\Phi$ through its area, $a$, the flux penetrating the electronic trajectory scales with $\Phi/(\tau_d/\tau_{\text{erg}})^{1/2}$. Therefore the crossover universally occurs at $\Phi_c \sim \Phi_0(\tau_{\text{erg}}/\tau_d)^{1/2} \ll \Phi_0$ for ballistic or diffusive dots (the diffusive approach ignores $\tau_d \gg \tau_{\text{erg}}$, so $\Phi_c \sim \Phi_0$ of Ref. 4 corresponds to the flux quantum through the area of a diffusive dot).

Below we find the fluctuations of $G_a$ for arbitrary flux $\Phi$, temperature $T$ and capacitance $C$ and compare in detail with previously considered limits 8 9. We numerically investigate the effect of dephasing on $G_a$ at high magnetic fields, low temperatures, and strong interaction, which is relevant for experiments. Although the dephasing diminishes the conductance fluctuations 11 12, the uniform (locally weak) dephasing is found to have a stronger effect. We conclude with a comparison of theory and experiment.

System. The 2D quantum dot, see Fig. 1, is biased with dc voltages $V_{1,2}$ at contacts with $N_{1,2}$ ballistic channels, and by the voltage $V_a$ at the gates with capacitance $C$. This capacitance defines the strength of the Coulomb interaction in the dot ($C \to 0$ corresponds to strong repulsion) 10. The dot is in the universal regime 11, when the Thouless energy $E_T = h/\tau_{\text{erg}}$ is large, so that the results are applicable to dots with area $A = \pi L^2$ (taken circular), either diffusive with mean free path $l \ll L$, or ballistic, with $l \gg L$ and chaotic classical dynamics (in the latter case the substitution $l \to \pi L/4$ is used 12). The mean level spacing $\Delta = 2\pi\hbar^2/(m^* A)$ and the total number of conducting channels $N$ together define the dwell time $\tau_d = h/(N\Delta) \gg \tau_{\text{erg}}$. A dephasing with rate $\gamma_{\text{ph}} = N_c\Gamma\Delta/2\pi$ is introduced with the dephasing probe model: a fictitious probe with $N_{\gamma}$ channels of transparency $\Gamma$ is attached to the dot 13. We also require that $eV \ll N\Delta$ and treat the nonlinearity only in $(eV)^2$. Scattering is spin-independent and this spin degeneracy is accounted for by the coefficient $n_s$. We use RMT for the energy-dependent scattering matrix $S(\varepsilon)$ and refer a reader to reviews 11 14 15 for details.

The electric potential $U$ in the dot is taken uniform. If the screening length is much larger than the Fermi wavelength, WKB can be applied. As a consequence, electrons with kinetic energy $\varepsilon$ have a well-defined electrochemical potential $\tilde{\varepsilon}_{\alpha} = \varepsilon - eV_{\alpha}$ in the contact $\alpha$ and $\tilde{\varepsilon} = \varepsilon - eU$ in the dot. Therefore, transport depends on the Fermi-distributions $f(\tilde{\varepsilon}_{\alpha})$ and the scattering matrix $S(\tilde{\varepsilon})$. The current in the contact $\alpha$ is $I_{\alpha} = \int d\varepsilon f(\varepsilon)\sigma_{\alpha\beta}(\varepsilon)$ and for $eV \ll N\Delta$ the spectral current $I_{\alpha}(\varepsilon)$ can be expanded in powers of $eV$:

\[ I_{\alpha}(\varepsilon) = \frac{\nu_{\alpha} e^2}{\hbar} \sum_{\delta=1}^{2} f(\tilde{\varepsilon}_{\delta}) \text{tr} \left[ \mathbb{1}_{\alpha\delta} - \mathbb{1}_{\alpha\delta} S(\tilde{\varepsilon}) \right] \]

\[ \approx \frac{-f'(\varepsilon)\nu_{\alpha} e^2}{\hbar} \sum_{\beta} V_{\beta} \left( g_{\alpha\beta}(\varepsilon) + \sum_{\gamma} g_{\alpha\beta\gamma}(\varepsilon)eV_{\gamma} \right) \]  

In Eq. 1 the total current $I_{\alpha}$ is expressed in terms of the dimensionless linear conductance at energy $\varepsilon$, $g_{\alpha\beta}(\varepsilon) = \text{tr}(\mathbb{1}_{\alpha\delta} - \mathbb{1}_{\alpha\delta} S(\varepsilon))$ and the nonlinear conductance $g_{\alpha\beta\gamma}(\varepsilon)$ related to $\partial^2 I_{\alpha}/\partial \beta\partial \gamma V_{\varepsilon}$, which depends on $U$. To this accuracy $U$ needs to be known only up to the first order derivatives, the characteristic potentials $u_\delta = \partial U/\partial \delta$ 16. The characteristic potentials $u_\delta \in (0, 1)$ are found self-consistently from current conservation and gauge-invariance requirements and expressed in terms of the Wigner-Smith matrix $Q = S^\dagger \partial S/(2\pi i)$ 17:

\[ u_\delta = -\frac{\int d\varepsilon f'(\varepsilon) \text{tr} \Lambda_{\delta} Q}{C/e^2 u_\delta - \int d\varepsilon f'(\varepsilon) \text{tr} Q}, \quad u_0 = 1 - \frac{2}{\sum_{\delta=1}^2 u_\delta}. \]  

To leading order in $N$ the mesoscopic average of Eq. 2 is flux-insensitive. However, the fluctuations of $u_\delta$ are strongly dependent on $\Phi$, and determine the asymmetry of the non-linear conductance. These derivatives are used to express the conductances $g_{\alpha\beta\gamma}(\varepsilon)$:

\[ g_{\alpha\beta\gamma}(\varepsilon) = [\delta_{\beta\gamma}g'_{\alpha\beta}(\varepsilon) - u_\beta g'_{\alpha\gamma}(\varepsilon) - u_\gamma g'_{\alpha\beta}(\varepsilon)]/2, \]  

where the prime stands for energy derivative. The matrix $S$ depends on magnetic field, $S(\Phi) = S^T(-\Phi)$, so that $u_0(-\Phi) = u_0(\Phi)$, but importantly $u_\delta(\Phi)$ lacks such symmetry. Therefore quite generally $g_{\alpha\beta\gamma}(\Phi) \neq g_{\alpha\beta\gamma}(-\Phi)$. However some symmetries still hold for $g_{\alpha\beta\gamma}$, which is revealed in the (anti)symmetric to $\Phi \to -\Phi$ components of conductance ($G_a$) 18 (in units of inverse energy),

\[ \left( G_a \right)_{\alpha\beta\gamma} = \left( G_a \right)_{\alpha\beta\gamma} \equiv -\int d\varepsilon f'(\varepsilon)g_{\alpha\beta\gamma}(\varepsilon, \Phi) \pm g_{\alpha\beta\gamma}(\varepsilon, -\Phi) \right] / 2, \]  

where we investigate now in detail. First we derive their dependence on temperature $T$, magnetic flux $\Phi$ and capacitance $C$ for coherent multi-channel dots, $N \gg 1$, and later investigate partially coherent dots at $T, C \to 0$ and high $\Phi$.

Coherent dot at arbitrary $T$, $\Phi$ and $C$. In a two-terminal dot without dephasing, we use gauge invariance and set $V_2 = 0$ and consider derivatives with respect to $V_1$ only. We define $G_{a(\varepsilon)} = G_a(\varepsilon + \Delta)$ and introduce a traceless matrix $\Lambda = (N_c/N) \mathbb{1} - (N_1/N) \mathbb{1}$ such that

\[ G_a = \frac{\pi}{\Delta^2} \int \int d\varepsilon d\varepsilon' f'(\varepsilon)f'(\varepsilon')\chi_1(\varepsilon)\chi_2(\varepsilon') \frac{C}{(e^2 u_\delta) - \int d\varepsilon f'(\varepsilon) \text{tr} \Lambda [S, \partial_{\delta} S]}, \]

with fluctuating $\chi_1(\varepsilon) = (\Delta/2\pi)^2 \partial \varepsilon \Lambda S^\dagger \Delta S$ and $\chi_2(\varepsilon) = (\Delta/2\pi)^2 \partial \varepsilon \Lambda S^\dagger \Delta S$. The mesoscopically averaged $G_a$ vanishes, $(\bar{G}_a) = 0$, and we need to find correlations of $\bar{G}_a$ to leading order in $N$. To this end the products of $S(\varepsilon, \Phi)$ and $S^\dagger(\varepsilon', \Phi')$ are averaged, and the pair correlators, Cooperon $C_{E-E'}$ and Diffuson $D_{E-E'}$, are introduced as $X_{\varepsilon} = (N_c - 2\pi\varepsilon/\Delta)^{-1} X = C, D$, with the flux-dependent effective number of channels $N_X$ 16:

\[ \left( N_c \right)_{N_D} = N + \frac{(\Phi + \Phi')^2}{4\Phi_0^2} \frac{\hbar v_F l}{L^2 \Delta}. \]
The denominator of Eq. (5) is a self-averaging quantity, \((\langle...\rangle)^2 = \langle(...)^2\rangle = (C/(C_\nu \Delta))^2\), with the electrochemical capacitance \(C_\nu = C/(1 + C\Delta/(\nu_\nu e^2))\). The functions \(\chi_1(\epsilon, \Phi)\) and \(\chi_2(\epsilon', \Phi')\) are uncorrelated, and their auto-correlations

\[
\left(\frac{\chi_1\chi_1}{\chi_2\chi_2}\right) = \frac{2N_1N_2}{N^4} \left(\frac{N_1N_2}{N^2}\right) \left(\left|D_{\epsilon'-\epsilon}| \pm |C_{\epsilon'-\epsilon}\right|\right)
\]

readily allow one to find correlations of \(G_{\alpha, s}\) at different magnetic fields and/or temperatures. In what follows we present the results for the variance of various experimentally measurable quantities. Calculated at the same \(T\) and \(\Phi\), they are given by very similar expressions. Most important are the mesoscopic fluctuations of \(G_{\alpha, s}\):

\[
\langle G_{\alpha, s}^2 \rangle = \frac{\left(2\pi C_\alpha \Delta\right)^2}{\Delta C} \frac{N_1^2N_2^2}{N^{10}} \left(\frac{\Phi}{\Phi_c}\right)^2 \equiv \frac{\Phi_0 L}{2\sqrt{\gamma_\Delta V_{\text{ext}}}} \tag{8}
\]

The variance \(\langle G_{\alpha, s}^2 \rangle\) for a realistic dot \(\chi\) with \(N_1 = N_2\) is given by Eq. (8) with \(F_\pm \to F_+\). As a consequence, high magnetic field reduces the fluctuations of \(G_{\alpha, s}\) by a factor two, see Fig. 1. The Eqs. (8) at high and low magnetic fields and weak interaction are of special interest and considered below in detail, since they allow us to compare with the results of Refs. 8, 9.

Low magnetic fields. When \(N_C \approx N\) the magnetic field is low and can not destroy time-reversal symmetry (TRS). In the limit \(T = 0\)

\[
\langle G_{\alpha, s}^2 \rangle = \frac{\left(2\pi C_\alpha \Delta\right)^2}{\Delta C} \frac{N_1^2N_2^2}{N^{10}} \left(\frac{\Phi}{\Phi_c}\right)^2 \tag{9}
\]

At small \(T \ll N\Delta/\pi\), we have \(G_{\alpha, s}^2(T)/G_{\alpha, s}^2(0) \approx 1 - 8(e^2T/\pi N\Delta)^2\). For \(T \gg N\Delta/2\pi\), Eq. (10) is multiplied by \(2(N\Delta/24T)^2\) and later used to compare with experiment. We point out that the TRS-breaking flux that destroys weak localization correction in open dots \(\chi\), obtained by other methods \(\chi\) for chaotic and disordered dots has the same dependence \(\gamma_\Delta\) on \(\tau_d\) as \(\Phi_c\) in Eq. (11).

For weak interaction, \(C/e^2\Delta \to \infty\), we find

\[
\langle G_{\alpha, s}^2 \rangle = \frac{\left(e^2 \tau_d \Phi \right)^2}{2\pi C h^2 \Phi_0} \left(\frac{4N_1N_2}{N^3}\right)^2 \times \frac{\tau_d}{4L^2} \tag{11}
\]

We underline that Eq. (11) holds for chaotic dots independently of the nature of scattering, diffusive or ballistic. The first term, rewritten via \(2e^2/C \to \beta/\nu A\), reproduces the result of Ref. 8 if \(h/\tau_d\) were substituted by the escape rate \(h/\tau_{\text{erg}}\) of an open diffusive sample (as is common for the crossover from ballistic to diffusive systems).

The most important is the third term, large as \(\tau_d/\tau_{\text{erg}} \gg 1\) for chaotic dots. It universally predicts the only relevant scale \(\Phi_c \ll \Phi_0\) rather then \(\Phi_c \sim \Phi_0\) as stated in Ref. 8. Data of Ref. 8, where a nonlinearity with \(\Phi\) sets in at flux \(\Phi \ll \Phi_0\), substantiate our estimate.

High magnetic fields. When \(N_C \gg N\) the TRS is fully broken, \(C/D \to 0\), and at \(T = 0\) the functions \(F_\pm = 1/N^2\) the result of Ref. 8. As a consequence, high magnetic field reduces the fluctuations of \(G_{\alpha, s}\) by a factor two, see Fig. 1. The Eqs. (8) at high and low magnetic fields and weak interaction are of special interest and considered below in detail, since they allow us to compare with the results of Refs. 8, 9.

Partially coherent dot. Dephasing with rate \(\gamma_{\varphi}\) is treated using the dephasing probe model \(\chi\): the current into the probe \(\varphi = 0\) is zero, \(I_{\varphi}(\epsilon) = 0\) at every energy \(\epsilon\). The probe generates currents in the leads \(\alpha\) due to a "voltage" \(V_{\varphi}(\epsilon)\) at the probe. \(V_{\varphi}(\epsilon)\) defines the distribution \(f(\epsilon - eV_{\varphi}(\epsilon))\) at the probe). For simplicity we take here a strongly-interacting dot, \(C = 0\), and find \(v_{\varphi}(\epsilon) = \partial V_{\varphi}(\epsilon)/\partial \Phi\), \(\gamma_{\varphi}(\epsilon) = 0\) for \(\varphi\) and \(u_{\delta}(\epsilon) = \int d\epsilon' f'(\epsilon')\), \(\tau_{\text{erg}}\), \(\Phi_c\) and \(\Phi_0\) coincide with that of Ref. 8 up to a numerical coefficient.

For a dot at \(T = 0\) and \(\Phi \gg \Phi_c\) we numerically consider the antisymmetric component \(G_{\alpha, s}\):

\[
G_{\alpha, s} = \frac{u_+ - u_-}{2} \left(\frac{g_{11} + g_{11}v_+ + g_{11}v_+ + g_{11}v_+}{4\gamma_{\varphi}v_+ + v_+} \right), \tag{14}
\]

with \(u_+ \equiv u_+ (\pm \Phi), v_+ \equiv v_+ (\pm \Phi)\). If the dimensionless dephasing rate \(\gamma_{\varphi}\), normalized by \(2\pi/\Delta\), is fixed, one can vary the transmission \(\Gamma\) together with the number of channels \(N_{\varphi}\) to go from uniform dephasing, \(N_{\varphi} \gg 1\), to non-uniform dephasing with a small number of perfectly conducting channels in the probe \(\mathcal{P}\). We follow Refs. 22, 23 to convolve \(Q\) and \(S\) for the broken TRS for a non-ideal coupling with the probe. The results for \(\gamma_{\varphi}\) as a function of \(N_{\varphi}\) for \(N = 2\), \(N_1 = 1\) are presented in Fig. 2. We expect that uniform dephasing damps \(G_{\alpha, s}\) stronger, which is clearly seen in Fig. 2 (the uniform limit \(N_{\varphi} \gg 1\) is reached at \(N_{\varphi} \sim 10\)). In the limit \(\gamma_{\varphi} \gg N\) we have...
$G_a \to 0$. This behavior of $G(\gamma_{\varphi})$ could also be explored in experiments with a real additional probe (in the limit $eV, T, \Delta \ll N \lambda$ the results of the dephasing probe \cite{18} and the inelastic probe \cite{24} models coincide).

Comparison with experiment. Zumbühl et al.\cite{24} measure the statistics of the (anti)symmetrized respect with magnetic field $B$ conductance $(g_{B+})g_{B+}$ at various $V, B$. At low $eV$ they correspond to $(2\nu e^2/h)V G_a$ and $(\nu e^2/h)(g_{11} + 2eV G_a)$. Of particular interest are their $\delta g_{B+}$ and the coefficient $\delta \alpha = \delta g_{B+}/(VB)$ at $eV, B \to 0$. The measurements of $\delta \alpha$ are performed at $\Phi \ll \Phi_0$ and $T = 4\mu eV$ for samples with $\Delta = 7\mu eV$ for $N = 2, 4, 8$. Theory (see \cite{18} or Eq. \cite{31} for $\Phi = \Phi_0$) predicts that the scaled coefficient $\delta \alpha'(N) = \delta \alpha N^2 \Phi_0 \hbar \Delta/(2e^3 A)$ is independent of $N$. Experiment indeed finds a reasonable agreement with this prediction only at $N = 8$ where the measured value is $\delta \alpha'(8) = 1.1$ versus the predicted $\delta \alpha'_8 = \pi C_\mu/2C$. Instead of a constant $\delta \alpha'$ the experiment revealed a strong dependence of $\delta \alpha'(2) \approx 0.015, \delta \alpha'(4) \approx 0.36$ on $N$. Our theory proposes an explanation of this unexpected growth. Interestingly, it turns out that, since $T \tau_d \sim \hbar$, we can use the low-field high-temperature asymptote of Eq. \cite{31}. It provides a behavior similar to the experiment: $\delta \alpha'(N) = (\pi^2 C_\mu/24CT)(N\Delta \hbar v_F/L)^{1/2}$. At $\hbar v_F/L \sim 250\mu eV$, one finds $\delta \alpha'(N) \approx 4.5\sqrt{N C_\mu}/C$ growing with $N$ but not nearly as steeply as observed.

Experiment \cite{24} derives dephasing rate from weak localization measurements, $\gamma_\varphi = 0.3$. Numerics performed at $\Phi \gg \Phi_c$ shows that uniform dephasing $\gamma_\varphi = 0.3$ diminishes $\delta \alpha'(N)$ only by a factor 0.4, 0.65 and 0.85 for $N = 2, 4, 8$ respectively. This is close to the result one would obtain with the semi-empirical substitution $N \to N + \gamma_\varphi$ in Eqs. \cite{31}, which would strongly diminish fluctuations in the few-channel dots. As a result, assuming non-ideal screening, $C_\mu/C < 1$, and strong dephasing one can fit our results to the experimental data of Ref. \cite{24}. If the experimental estimate of $\gamma_\varphi = 0.3$ and the assumption that $C_\mu/C = 1$ are used, our analytical results and experimental data disagree. Therefore both an accurate determination of the dephasing rate and an independent measurement of the capacitance are needed.

We would like to point out that the symmetrized part of the non-linear conductance $g_{B+}$ permits just such an independent determination of the capacitance ratio $C_\mu/C$. For symmetric ($N_1 = N_2$) coherent realistic dots \cite{24} at small $eV$ it holds

$$\left(C_\mu/C\right)^2 = \frac{\text{var } g_B g_{B+}}{(\text{var } g_{B+})^2} \left(\frac{\nu e^2 \Delta}{2\pi \hbar}\right)^2.$$ \hfill (15)

at arbitrary flux $\Phi$ and temperature $T$. Since $C_\mu/C$ is apriori unknown, Eq. \cite{31} presents an independent way to measure screening. Independent capacitance measurements and in particular also experiments for asymmetric dots ($N_1 \neq N_2$), or measurements of the nonequilibrium distribution \cite{31} are needed for a more detailed comparison of theory and experiment.

Conclusions. The Onsager-Casimir relations are a cornerstone of irreversible linear transport. We go beyond the linear regime and consider the quantum fluctuations of transport properties due to finite voltage and interactions by the example of open chaotic dots. Our key universal result, confirmed by experiment \cite{24}, is that the only relevant magnetic-flux scale is the flux quantum through a dot $\Phi$ but rather $\Phi_c \ll \Phi_0$ determined by a long dwell time of electrons inside the dot. Since our theory accounts for a wide range of parameters (temperature, flux, dephasing, contact widths), it provides a basis for further experimental investigation of the nonequilibrium transport.

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