AMPLE SUBVARIETIES AND RATIONALLY CONNECTED FIBRATIONS

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Abstract. Under some positivity assumptions, extension properties of rationally connected fibrations from a submanifold to its ambient variety are studied. Given a family of rational curves on a complex projective manifold $X$ inducing a covering family on a submanifold $Y$ with ample normal bundle in $X$, the main results relate, under suitable conditions, the associated rational connected fiber structures on $X$ and on $Y$. Applications of these results include an extension theorem for Mori contractions of fiber type and a classification theorem in the case $Y$ has a structure of projective bundle or quadric fibration.

Introduction

Both from a biregular and a birational standpoint, the geometry of algebraic varieties is often studied in terms of their fibrations. Given a smooth complex projective variety $X$, there are two parallel theories able to detect on $X$ fibrations whose fibers are varieties of negative Kodaira dimension. The first one, initiated by Mori, associates a morphism $X \to S$, whose general positive dimensional fiber is a Fano variety, to any $K_X$-negative extremal ray of the cone of effective curves of $X$. The second theory, introduced in works of Campana and of Kollár, Miyaoka and Mori, produces a rational map $X \to S$, with proper and rationally chain connected fibers within its domain of definition, from any family of rational curves on $X$, the most studied case being the one in which the family of all rational curves of $X$ is considered.

The purpose of this paper is to study extension properties of fibrations of the types described above, when these fibrations are defined on a smooth subvariety $Y$ of $X$ with ample normal bundle: our goal is to determine additional conditions that guarantee that given fiber structures on $Y$ extend to analogous structures on $X$, and to compare the corresponding fibrations. Following the terminology of [13], we will refer to such a $Y$ as an “ample subvariety” of $X$. We remark that, in the special case of codimension one, this setting is more general than the classical setting of ample divisors.

Our first result is an extension property for rationally connected fibrations. The precise formulation requires some additional notation, and is given in Theorems 3.1 and 3.6.
Roughly speaking, Theorem 3.1 says that if $Y$ is a submanifold with ample normal bundle, the inclusion inducing a surjection $N^1(X) \rightarrow N^1(Y)$, and $V$ is a family of rational curves on $X$ inducing a covering family $V_Y$ on $Y$, then there is a commutative diagram

$$
\begin{array}{ccc}
Y & \xymatrix{\ar[r] & X} & Y/Y \\
\ar[u]^{\pi} & \ar[u]_{\phi} & \ar[u] \\
Y//V_Y & \xymatrix{\ar[r]^{\delta} & X//V} & 
\end{array}
$$

where $\phi$ is the rational map associated to the family $V$, $\pi$ is the map associated to $V_Y$, and $\delta$ is a surjective morphism of normal varieties; here $X//V$ and $Y//V_Y$ denote the respective rational quotients.

With the second theorem, we determine sufficient conditions to ensure that the morphism $\delta$ in the above diagram is generically finite; this is one of the core results of the paper (see Theorem 3.6).

**Theorem A.** Assuming that the normal bundle of $Y$ in $X$ is ample and that the inclusion of $Y$ induces a surjection $N^1(X) \rightarrow N^1(Y)$, the morphism $\delta$ is generically finite if one of the following two conditions is satisfied:

(a) $V$ is an unsplit family; or
(b) $\text{codim}_X Y < \dim Y - \dim Y//V_Y$.

When the rational quotient $Y//V_Y$ is one-dimensional, it turns out from a general fact that $\pi$ is a morphism (see Proposition 3.12); using this fact, the previous result can be improved in this case, and we obtain a commutative diagram as the one above where now the vertical arrows are morphisms and $\delta$ is a finite morphism between smooth curves (see Corollary 3.13). The situation when $Y$ is the zero scheme of a regular section of an ample vector bundle on $X$ and $\pi$ is the MRC-fibrations over a base of positive geometric genus was also treated in [25] (see Remark 3.10).

Next, we address the problem of extending extremal Mori contractions of fiber-type. Using the previous result, we prove the following theorem (see Theorem 4.1).

**Theorem B.** Let $Y$ be a submanifold of a projective manifold $X$ with ample normal bundle, and assume that the inclusion induces an isomorphism $N^1(X) \cong N^1(Y)$. Let $\pi: Y \rightarrow Z$ be an extremal Mori contraction of fiber-type, and let $W$ be an irreducible covering family of rational curves on $Y$ that are contracted by $\pi$. If these curves do not break in $X$ under deformation (that is, if the family they generate in $X$ is unsplit), then $\pi$ extends to an extremal Mori contraction $\phi: X \rightarrow S$, and there is a commutative diagram

$$
\begin{array}{ccc}
Y & \xymatrix{\ar[r] & X} & Y \\
\ar[u]^{\pi} & \ar[u]_{\phi} & \ar[u] \\
Z & \xymatrix{\ar[r]^{\delta} & S} & 
\end{array}
$$

where $\delta$ is a finite surjective morphism.

In the special case when $Y$ is defined by a regular section of an ample vector bundle on $X$, related results were obtained in [11, 25] (see Remark 4.3).
In the last section of the paper, the above results are finally applied to classify, under suitable conditions, projective manifolds containing “ample subvarieties” that admit a structure of projective bundles or quadric fibrations. Beginning with classical results on hyperplane sections, there has been given evidence to the fact that projective manifolds are, so to speak, at least as “special” as their ample divisors (cf. [29]). The study of projective bundles and quadric fibrations embedded in projective manifolds as zero schemes of regular sections of ample vector bundles was already undertaken in \[18, 19, 20, 8, 25\], where classification results were obtained when the base of the fibration, if positive dimensional, has positive geometric genus. Under the additional assumption of a polarization on the ambient variety inducing a relatively linear polarization on the fibration, classification results were obtained in \[7, 1, 21\]. With some dimensional restrictions on the fiber structure of the subvariety in terms of its codimension, we extend such results in two ways.

In our first generalization, we consider fibrations over curves and drop the hypothesis that the subvariety is defined by a regular section of an ample vector bundle, only assuming that the normal bundle is ample. In this case, since the positivity condition is local near \(Y\), we additionally require that the inclusion induces an isomorphism on the Picard groups. In the second generalization, we allow the base of the projective fibration to have arbitrary dimension. In either case, we do not put any additional restriction on the base of the fibration and do not require \(a \text{ priori}\) any global polarization (such a polarization will turn out to exist \(a \text{ posteriori}\)). We state here the extension property that we obtain in the case of projective bundles.

**Theorem C.** Let \(Y\) be a submanifold of a projective manifold \(X\) with ample normal bundle, of codimension

\[\text{codim}_X Y < \dim Y - \dim Z.\]

Assume that \(Y\) admits a projective bundle structure \(\pi: Y \to Z\) over a smooth projective variety \(Z\), and that one of the following two situations occurs:

(a) \(Z\) is a curve and the inclusion of \(Y\) in \(X\) induces an isomorphism \(\text{Pic}(X) \cong \text{Pic}(Y)\); 

or

(b) \(Y\) is the zero scheme of a regular section of an ample vector bundle on \(X\).

Then \(\pi\) extends to a projective bundle structure \(\tilde{\pi}: X \to Z\) on \(X\), giving a commutative diagram

\[
\begin{array}{ccc}
  Y & \xrightarrow{\pi} & X \\
  \downarrow \pi & & \downarrow \tilde{\pi} \\
  Z & &
\end{array}
\]

and the fibers of \(\pi\) are linearly embedded in the fibers of \(\tilde{\pi}\).

A similar statement is also proven in the case of quadric fibrations with irreducible reduced fibers and relative Picard number one. For the full result, see Theorem 5.8.

The proofs of the above results rely on various properties concerning families of rational curves on projective manifolds and their associated rational quotients, which are collected in Section 2. General facts from deformation theory of rational curves needed in the sequel are recalled in this section, and several new notions are introduced and studied.
Among other things, we study numerical properties of families of rational curves, introducing in particular the notion of numerically covering family (see Definition 2.18), which turns out to be very useful in our context. Appropriate notions for the extension and restriction of families of rational curves are also introduced in relation to given submanifolds of the ambient variety.

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1. Conventions and basic notation

We work over the complex numbers, and use standard notation in algebraic geometry, although tensor product between line bundles is often denoted additively. For $n \geq 1$, we denote by $Q^n$ the smooth quadric hypersurface of $\mathbb{P}^{n+1}$. We use the word \textit{general} to mean that the choice is made outside a properly contained (Zariski) closed subset, and the word \textit{very general} to mean that the choice is made outside a countable union of such subsets. For any projective algebraic variety $X$, we denote by $N^1(X) := (\text{Pic}(X)/\equiv) \otimes \mathbb{R}$ the Néron–Severi space of $X$, and by $N_1(X) := (Z_1(X)/\equiv) \otimes \mathbb{R}$ the space generated by numerical classes of curves on $X$. The dimension of these spaces, which is equal to the Picard number of $X$, is denoted by $\rho(X)$. For a morphism of algebraic varieties $X \to S$, we set $N_1(X/S) := (Z_1(X/S)/\equiv) \otimes \mathbb{R}$, and we denote by $\rho(X/S)$ the dimension of this space. The numerical class of a line bundle $L$ (resp., of a curve $C$) on $X$ is denoted by $[L]$ (resp., $[C]$). We denote by $\overline{\text{NE}}(X)$ the closure of the cone in $N^1(X)$ spanned by numerical classes of effective curves, and by $\text{Nef}(X)$ its dual cone, namely, the cone in $N^1(X)$ spanned by classes of nef divisors. In the case $X$ is a projective variety with $\text{Pic}(X) \cong \mathbb{Z}$, we will denote by $\mathcal{O}_X(1)$ the ample generator of this group.

2. Families of rational curves on varieties

Throughout this section we consider morphisms from the projective line $\mathbb{P}^1$ to a smooth projective variety $X$; for convenience, we fix once for all two distinct points 0 and $\infty$ in $\mathbb{P}^1$. We will use notation as in [15], some of which we recall below. For general facts on the theory of deformation of rational curves on varieties, we refer to [15, 6].

Let $\text{Hom}(\mathbb{P}^1, X)$ be the scheme parameterizing morphisms from $\mathbb{P}^1$ to $X$. We will denote a morphism $f : \mathbb{P}^1 \to X$ by $[f]$ whenever we think of it as a point of $\text{Hom}(\mathbb{P}^1, X)$. If $\Sigma \subseteq \mathbb{P}^1$ and $\Theta \subseteq X$ are closed subschemes, then we denote by

$$\text{Hom}(\mathbb{P}^1, X; \Sigma \to \Theta)$$

the subscheme of $\text{Hom}(\mathbb{P}^1, X)$ parameterizing those morphisms $f$ such that $f(\Sigma) \subseteq \Theta$ (the image $f(\Sigma)$ of $\Sigma$ is here intended scheme-theoretically). In the particular case $\Sigma = \{0\}$ and $\Theta = \{x\}$ for some $x \in X$, we also use the notation $\text{Hom}(\mathbb{P}^1, X; 0 \to x)$. Note that, for any closed subscheme $j : \Sigma \hookrightarrow \mathbb{P}^1$, there is a natural morphism

$$j^* : \text{Hom}(\mathbb{P}^1, X) \to \text{Hom}(\Sigma, X)$$

defined by $j^*([f]) := [f \circ j]$. 

Definition 2.1. A morphism $f: \mathbb{P}^1 \to X$ is free if it is non-constant and $f^*T_X$ is nef.

We will use the following well-known properties (see [15, Theorem II.1.7]).

Proposition 2.2. Let $f: \mathbb{P}^1 \to X$ be a non-constant morphism, and let $x = f(0)$. Let $m_0 \subset \mathcal{O}_{\mathbb{P}^1}$ be the maximal ideal sheaf of $0 \in \mathbb{P}^1$.

(a) The Zariski tangent space of $\text{Hom}(\mathbb{P}^1, X)$ at $[f]$ is isomorphic to $H^0(\mathbb{P}^1, f^*T_X)$, and $\text{Hom}(\mathbb{P}^1, X)$ is smooth at $[f]$ if $h^1(\mathbb{P}^1, f^*T_X) = 0$.

(b) Similarly, the Zariski tangent space of $\text{Hom}(\mathbb{P}^1, X; 0 \mapsto x)$ at $[f]$ is isomorphic to $H^0(\mathbb{P}^1, f^*T_X \otimes m_0)$, and $\text{Hom}(\mathbb{P}^1, X; 0 \mapsto x)$ is smooth at $[f]$ if $h^1(\mathbb{P}^1, f^*T_X \otimes m_0) = 0$.

In particular, if $f$ is free, then $\text{Hom}(\mathbb{P}^1, X)$ and $\text{Hom}(\mathbb{P}^1, X; 0 \mapsto x)$ are smooth at $[f]$.

For closed subschemes $\Theta \subseteq X$ and $\Sigma \subseteq \mathbb{P}^1$, and any subset $S \subseteq \text{Hom}(\mathbb{P}^1, X)$, we denote

$$S(\Sigma \to \Theta) := S \cap \text{Hom}(\mathbb{P}^1, X; \Sigma \to \Theta),$$

and, in a similar way, we define $S(0 \mapsto x)$ for any point $x \in X$.

Definition 2.3. Given a subset $S \subseteq \text{Hom}(\mathbb{P}^1, X)$, the image of the universal map $\mathbb{P}^1 \times S \to X$ is called locus of $S$, and is denoted by $\text{Locus}(S)$. Analogous definitions are given for the loci of $S(\Sigma \to \Theta)$ and $S(0 \mapsto x)$, which are respectively denoted by $\text{Locus}(S; \Sigma \to \Theta)$ and $\text{Locus}(S; 0 \mapsto x)$.

We now restrict our attention to the open subscheme $\text{Hom}_{\text{bir}}(\mathbb{P}^1, X)$ of $\text{Hom}(\mathbb{P}^1, X)$ parameterizing morphisms that are birational to their images. The previous notation is adapted to $\text{Hom}_{\text{bir}}(\mathbb{P}^1, X)$ in the obvious way.

Definition 2.4. Any union $V = \bigcup_{\alpha \in A} V_\alpha$ of irreducible components $V_\alpha$ of $\text{Hom}_{\text{bir}}(\mathbb{P}^1, X)$ is said to be a family of (parameterized) rational curves on $X$. If $V$ consists of only one irreducible component of $\text{Hom}_{\text{bir}}(\mathbb{P}^1, X)$, then $V$ is said to be an irreducible family. If $\text{Locus}(V_\alpha)$ is dense in $X$ for every $\alpha \in A$, then $V$ is said to be a covering family.

Remark 2.5. The scheme $\text{Hom}_{\text{bir}}(\mathbb{P}^1, X)$ has at most countably many irreducible components. In particular, the same holds for every family of rational curves on $X$.

Proposition 2.6. Let $V$ be an irreducible family of rational curves on $X$. Then the locus $\text{Locus}(V)$ of $V$ is dense in $X$ if and only if $f$ is free for a general $[f] \in V$.

Proof. $\text{Locus}(V)$ is dense in $X$ if and only if the differential of the universal map $\mathbb{P}^1 \times V \to X$ has rank equal to $\dim X$ at a general point $(p, [f]) \in \mathbb{P}^1 \times V$. By [15, Proposition II.3.10], this occurs if and only if $f$ is free for a general $[f] \in V$. □

Definition 2.7. Given any closed subset $S \subseteq \text{Hom}_{\text{bir}}(\mathbb{P}^1, X)$, we denote by $\langle S \rangle$ the union of all irreducible components of $\text{Hom}_{\text{bir}}(\mathbb{P}^1, X)$ that contain at least one irreducible component of $S$, and by $\langle S \rangle$ the union of all irreducible components of $\text{Hom}_{\text{bir}}(\mathbb{P}^1, X)$ that are irreducible components of $S$. We call $\langle S \rangle$ the minimal family generated by $S$, and $\langle S \rangle$ the maximal family contained in $S$.

Analogous notation and definitions will be adopted for $S \subseteq \text{Hom}(\mathbb{P}^1, X)$. 
Definition 2.8. Let \( V \) be a family of rational curves on \( X \). We say that a curve \( C \subset X \) is a \( V \)-curve if \( C = f(\mathbb{P}^1) \) for some \([f] \in V\). A \( V \)-chain of length \( \ell \) is the union of \( \ell \) distinct curves \( f_i(\mathbb{P}^1) \subset X \) (\( 1 \leq i \leq \ell \)) parameterized by elements \([f_i] \in V\) such that \( f_{i+1}(0) = f_i(\infty) \) for every \( 1 \leq i \leq \ell - 1 \).

Consider a family \( V = \cup V_\alpha \) of rational curves on \( X \). Let \( V'_\alpha \) be the irreducible components of the normalization \( \text{Hom}^n_{\text{bir}}(\mathbb{P}^1, X) \) of \( \text{Hom}_{\text{bir}}(\mathbb{P}^1, X) \) that map to \( V_\alpha \), and let \( \overline{V} \) be the closure in Chow\(_1\)(\( X \)) of the image of \( \cup V'_\alpha \) via the natural morphism

\[
\text{Hom}^n_{\text{bir}}(\mathbb{P}^1, X) \to \text{Chow}_1(X)
\]

(see [15, Comment II.2.7]). By [15, Theorem I.5.10], \( \overline{V} \) is proper. Associated to \( \overline{V} \) (hence to \( V \)), one can define a proper prolongebraic relation on \( X \), as explained in [15, Section IV.4] (see in particular [15, Example IV.4.10]). We denote this relation by \( \text{RC}_V \). We have that \((x, y) \in \text{RC}_V \) for two very general points \( x, y \in X \) if and only if there is a \( V \)-chain containing \( x \) and \( y \).

Definition 2.9. \( X \) is said to be \( \text{RC}_V \)-connected if there is only one \( \text{RC}_V \)-equivalence class. We say that \( X \) is \( V \)-chain connected if every two points of \( X \) lie on a \( V \)-chain. More generally, a closed subset \( T \subset X \) is said to be \( \text{V-chain connected} \) if any two points of \( T \) lie on a \( V \)-chain supported on \( T \).

Clearly if \( X \) is \( V \)-chain connected, then it is \( \text{RC}_V \)-connected.

Definition 2.10. A family of rational curves \( V \) is said to be an unsplit family if it is irreducible and, after normalization, its image in Chow\(_1\)(\( X \)) via the map (2) is a proper scheme.

Remark 2.11. If \( V \) is an irreducible family of rational curves on \( X \), and there is an ample vector bundle \( \mathcal{E} \) on \( X \) such that \( \det \mathcal{E} \cdot C < 2 \text{rk} \mathcal{E} \) for a \( V \)-curve \( C \), then \( V \) is unsplit. Indeed, if \( \sum_{i=1}^{k} |C_i| \) is any degeneration of the cycle \([C]\), then \( \det \mathcal{E} \cdot C = \sum_{i=1}^{k} \det \mathcal{E} \cdot C_i \geq k \text{rk} \mathcal{E} \) by the amleness of \( \mathcal{E} \). Thus \( k = 1 \) and \([C_1]\) lies in the image of the normalization of \( V \) in Chow\(_1\)(\( X \)) via the map (2).

Proposition 2.12. If \( X \) is \( \text{RC}_V \)-connected by a family \( V = \cup V_\alpha \) with each \( V_\alpha \) unsplit, then it is \( \text{V-chain connected} \) and \( N_1(X) \) is generated by numerical classes of \( V \)-curves. In particular, if \( V \) is unsplit, then \( X \) has Picard number \( \rho(X) = 1 \).

Proof. Fix \( \ell \gg 0 \) such that every two very general points of \( X \) are connected by a \( V \)-chain of length \( \ell \), and let \( S \subset \text{Chow}_1(X) \) be the subset parameterizing connected 1-cycles with \( \ell \) components (counting multiplicities) supported on \( V \)-curves. Since each component \( V_\alpha \) of \( V \) is unsplit, it follows that \( S \) is proper. By the choice of \( \ell \), there is an irreducible component \( T \) of \( S \) such that \( u(\ell) : U \times_T U \to X \times X \) is dominant, where \( U \to T \) is the universal family and \( u : U \to X \) is the cycle map. Note that \( T \) is a closed subvariety of \( \text{Chow}_1(X) \), and in particular it is proper. Therefore the image of \( u(\ell) \) is proper, and thus \( u(\ell) \) is surjective. This implies that \( X \) is \( V \)-chain connected. The second assertion follows from [15, Proposition IV.3.13.3]. \( \square \)

If \( V = \text{Hom}_{\text{bir}}(\mathbb{P}^1, X) \), then it is a result of Campana [5] and Kollár–Miyaoka–Mori [16] that one can “pass to the quotient”. The natural generalization of this result when \( V \) is
an arbitrary family of rational curves was later given by Kollár (see [15, Theorem IV.4.17]), and can be stated as follows.

**Theorem 2.13.** Let $V$ be a family of rational curves on $X$. Then there is an open set $X^0 \subseteq X$, a normal projective variety $X//V$, and a dominant morphism $X^0 \to X//V$ with connected fibers and proper over the image, whose very general fibers are $RC_V$-equivalence classes in $X$.

**Proof.** We refer the reader to [15, Theorem IV.4.17] for the existence of a proper morphism $X^0 \to Y^0$ onto a variety $Y^0$ with connected fibers and whose very general fibers are $RC_V$-equivalence classes in $X$. Since $X^0$ is normal and the fibers are connected, $Y^0$ is normal. It follows by construction that $Y^0$ is quasi-projective. Then we can take $X//V$ to be the normalization of the closure of $Y^0$ in some projective embedding. $\square$

**Definition 2.14.** With the notation as in the previous theorem, the resulting rational map $X \dashrightarrow X//V$ is called the $RC_V$-fibration of $X$, and $X//V$ is the $RC_V$-quotient. We also say that $X//V$ (resp., $X \dashrightarrow X//V$) is the rational quotient (resp., the rational fibration) defined by $V$. We denote by

$$\dim RC_V := \dim X - \dim X//V$$

the dimension of a very general $RC_V$-equivalence class (which is the same as the dimension of a general fiber of $X \dashrightarrow X//V$).

**Remark 2.15.** We have $\dim RC_V > 0$ if and only if $V$ is a covering family. We would like to stress that $X//V$ is well defined only up to birational equivalence; throughout the paper, we will often make suitable choices of rational quotients. We remark that, after possibly shrinking further $X^0$, one can always take a smooth projective model for the rational quotient $X//V$.

**Remark 2.16.** A case of particular interest is when $V = \text{Hom}_{\text{bir}}(\mathbb{P}^1, X)$. In this case $X//V$ is called the MRC-quotient (maximal rational quotient) of $X$. It follows from a result of Graber, Harris and Starr [12] that in this situation $X//V$ is not uniruled.

Every connected subset $S \subseteq \text{Hom}_{\text{bir}}(\mathbb{P}^1, X)$ determines in a natural way a numerical class $[S] \in N_1(X)$, by taking the class of the curve $f(\mathbb{P}^1)$ for an arbitrary $[f] \in S$. More generally, we give the following definition.

**Definition 2.17.** For any subset $S \subseteq \text{Hom}_{\text{bir}}(\mathbb{P}^1, X)$, we denote

$$\mathbb{R}_{\geq 0}[S] := \sum_{[f] \in S} \mathbb{R}_{\geq 0}[f(\mathbb{P}^1)] \subset N_1(X).$$

We call $\mathbb{R}_{\geq 0}[S]$ the cone numerically spanned by $S$. If $S$ is connected, then we call $[S]$ the numerical class of $S$.

**Definition 2.18.** A family of rational curves $V = \cup_{a \in A} V_a$ is said to be numerically covering if there is a subset $A' \subseteq A$ such that

$$\mathbb{R}_{\geq 0}[V] = \sum_{a \in A'} \mathbb{R}_{\geq 0}[V_a]$$

Every element of $\mathbb{R}_{\geq 0}[V]$ is a numerical class of a curve on $X$. If $V$ is numerically covering, then $\mathbb{R}_{\geq 0}[V]$ is a conic linear set.
and Locus($V_\alpha$) is dense in $X$ for every $\alpha \in A'$. If $V$ is a numerically covering family, then we denote by $V_{\text{cov}}$ the subfamily of $V$ consisting of all the covering irreducible components of $V$.

**Remark 2.19.** Note that by definition $\mathbb{R}_{\geq 0}[V_{\text{cov}}] = \mathbb{R}_{\geq 0}[V]$ for every numerically covering family $V$ of rational curves on $X$. Clearly, in the notation of the definition, we have $\bigcup_{\alpha \in A'} V_\alpha \subseteq V_{\text{cov}}$.

**Proposition 2.20.** Let $V = \bigcup_{\alpha \in A'} V_\alpha$ and $W = \bigcup_{\beta \in B} W_\beta$ be two numerically covering families of rational curves on $X$, and suppose that $\mathbb{R}_{\geq 0}[V] \subseteq \mathbb{R}_{\geq 0}[W]$.

Then the $\text{RC}_W$-fibration $\psi: X \rightarrow X/\text{w}$ factors through the $\text{RC}_V$-quotient of $X$, i.e., we have a commutative diagram

In particular, if $\mathbb{R}_{\geq 0}[V] = \mathbb{R}_{\geq 0}[W]$, then $V$ and $W$ define the same rational fibration.

**Proof.** Consider a resolution of the indeterminacies of $\psi$

and let $E \subset X'$ be the exceptional locus of $\sigma$. Note that the locus of indeterminacies of $\psi$ is contained in $X \setminus X^o$, and hence, by Theorem 2.13, it does not dominate $X/\text{w}$, since it does not meet the very general $\text{RC}_V$-equivalence class. Therefore we can assume that $\psi'(E)$ is a proper closed subset of $X/\text{w}$.

**Lemma 2.21.** If $C \subset X$ is a $V$-curve not contained in $\sigma(E)$, then its proper transform $C' \subset X'$ is mapped to a point in $X/\text{w}$ by $\psi'$.

**Proof of the lemma.** Let $B' \subset B$ be the subset parameterizing the irreducible components of $W$ that are covering, so that $W_{\text{cov}} = \bigcup_{\beta \in B} W_\beta$. Since $\mathbb{R}_{\geq 0}[V] \subseteq \mathbb{R}_{\geq 0}[W] = \mathbb{R}_{\geq 0}[W_{\text{cov}}]$, we can pick general elements $[g_\beta] \in W_\beta$ for $\beta \in B'$, and numbers $\lambda_\beta \geq 0$, such that, denoting $\Gamma_\beta = g_\beta(\mathbb{P}^1)$, we have

$$[C] = \sum_{\beta \in B'} \lambda_\beta [\Gamma_\beta] \quad \text{in} \quad N_1(X).$$

By the definition of $\psi$ and Theorem 2.13 and the fact that the $W_\beta$ are covering families for every $\beta \in B'$, we can assume that $\Gamma_\beta \cap \sigma(E) = \emptyset$ for every $\beta$. Let $C'$ and $\Gamma_\beta'$ be the proper transforms of $C$ and $\Gamma_\beta$ on $X'$, and let $D \subset X'$ be the pull-back, via $\psi'$, of an
ample divisor on $X//W$. Then $\psi'_*[\Gamma'_f] = 0$, so we have $\sigma_*D \cdot \Gamma_\beta = D \cdot \Gamma'_\beta = 0$ for all $\beta$, and thus

$$D \cdot C' \leq \sigma_*D \cdot C = \sum_{\beta \in B'} \lambda_\beta (\sigma_*D \cdot \Gamma_\beta) = 0.$$ 

Since $D$ is the pull-back of an ample divisor on $X//W$, this implies that $\psi'_*[C'] = 0$. \qed

We can now conclude the proof of the proposition. We consider a very general fiber $G$ of $\phi$, and fix two very general points $x, y \in G$ that are not contained in $\sigma(E)$. By the generality of the choices, we can assume that both $x$ and $y$ are outside the locus of indeterminacies of $\psi$, hence $\psi(x), \psi(y) \notin \psi'(E)$. Furthermore, we can assume that $x$ and $y$ are connected by a chain of $V$-curves. By Lemma \ref{lem:covering} every irreducible component of this chain that is not contained in $\sigma(E)$ is mapped to a point by $\psi$. Therefore, since $\psi(x) \notin \psi'(E)$, we deduce that this chain is in fact disjoint from $\sigma(E)$. Thus we can apply the lemma to each one of its components. Since the chain is connected, this implies

$$\psi(x) = \psi(y).$$

In view of the generality of the choice of $x$ and $y$ in $G$, we conclude that there exists a natural rational map $X//V \dashrightarrow X//W$ commuting with the respective projections. \qed

**Definition 2.22.** Let $V \subseteq \text{Hom}_{\text{rat}}(\mathbb{P}^1, X)$ be any family of rational curves, and let $a > 0$ be an integer. We denote by $\|V\|$ the largest family of rational curves on $X$ such that $\mathbb{R}_{\geq 0}[\|V\|] = \mathbb{R}_{\geq 0}[V]$. We call $\|V\|$ the family numerically generated by $V$.

**Remark 2.23.** Quite obviously, $V$ is a subfamily of $\|V\|$, and if $V$ is a numerically covering family then so is $\|V\|$.

The previous proposition implies the following useful property.

**Corollary 2.24.** Let $V$ be a numerically covering family of rational curves on $X$. Then the families $V, V_{\text{cov}},$ and $\|V\|$ define the same rational fibration.

**Proof.** Since $\mathbb{R}_{\geq 0}[\|V\|] = \mathbb{R}_{\geq 0}[V] = \mathbb{R}_{\geq 0}[V_{\text{cov}}]$, it follows from Proposition \ref{prop:covering} \qed

An important theorem, due to Kollár, Miyaoka and Mori, says that a smooth projective manifold is maximally rationally chain connected if and only if it is maximally rationally connected (see \cite[Theorem IV.3.10]{KollarMiyaokaMori}). The proof of this property can be adapted to the situation in which an arbitrary family of rational curves $V$ is considered. More precisely, one can prove that if $X$ is $V$-chain connected, then every two very general points of $X$ are connected by a $\|V\|$-curve. In the sequel, we will need the following slightly different version of this property (the proofs of the two properties are almost the same).

**Proposition 2.25.** Let $V$ be a family of rational curves on $X$, and assume that $X$ is $V$-chain connected. Let $y \in X$ be any point that is connected by a $V$-chain to a very general point of $X$. Then $\text{Locus}([\|V\|], 0 \rightarrow y)$ is dense in $X$.

**Proof.** Fix a very general point $x \in X$, and let $C = C_1 + \cdots + C_n$ be a $V$-chain connecting $x$ to $y$, with $C_i = f_i(\mathbb{P}^1)$. Let $p_0 = x, p_n = y,$ and $p_i = f_i(\infty) = f_{i+1}(0)$ for $1 \leq i \leq n - 1$. From here, we follow step by step the proof of \cite[Complement IV.3.10.1]{KollarMiyaokaMori}, proving inductively that there is a free rational curve $g_i: \mathbb{P}^1 \rightarrow X$ which connects $p_{i-1}$ to $p_i$; the resulting chain of free rational curves is then smoothed into a free rational
Consider now a submanifold $Y$ of $X$. The inclusion $i: Y \hookrightarrow X$ naturally induces an injective morphism

$$i_*: \text{Hom}(\mathbb{P}^1, Y) \hookrightarrow \text{Hom}(\mathbb{P}^1, X),$$

which is defined by $i_*([g]) := [i \circ g]$. Similar notation will be used for the injection $\text{Hom}_{\text{bir}}(\mathbb{P}^1, Y) \hookrightarrow \text{Hom}_{\text{bir}}(\mathbb{P}^1, X)$. Statements analogous to the following two propositions also hold for $\text{Hom}_{\text{bir}}(\ )$ in place of $\text{Hom}(\ )$.

**Proposition 2.26.** With the above notation, assume that the normal bundle of $Y$ in $X$ is ample. Then, for every $[g] \in \text{Hom}(\mathbb{P}^1, Y)$ with $g$ free, the morphism $f := i \circ g$ is free on $X$, and the schemes $\text{Hom}(\mathbb{P}^1, X)$ and $\text{Hom}(\mathbb{P}^1, X; 0 \mapsto g(0))$ are smooth at $[f]$.

**Proof.** The bundle $g^*T_Y$ is nef, since $[g]$ is free. Thus, by the ampleness of $N_{Y/X}$ and the exact sequence

$$0 \rightarrow g^*T_Y \rightarrow f^*T_X \rightarrow g^*N_{Y/X} \rightarrow 0,$$

we conclude that $f$ is free. Then the last two assertions follow from Proposition 2.2. 

We will need the following generalization of Proposition 2.2.

**Proposition 2.27.** Let $Y$ be a submanifold of a smooth projective variety $X$ and let $i: Y \hookrightarrow X$ be the inclusion. Let $g: \mathbb{P}^1 \rightarrow Y$ be a free morphism, and let $f := i \circ g$. Then the Zariski tangent space of $\text{Hom}(\mathbb{P}^1, X; \{0\} \rightarrow Y)$ at $[f]$ sits naturally in an exact sequence

$$0 \rightarrow H^0(\mathbb{P}^1, g^*T_Y) \rightarrow T_{[f]} \text{Hom}(\mathbb{P}^1, X; \{0\} \rightarrow Y) \rightarrow H^0(\mathbb{P}^1, g^*N_{Y/X} \otimes m_0).$$

Moreover, if the normal bundle of $Y$ in $X$ is ample, then the sequence completes to a short exact sequence

$$0 \rightarrow H^0(\mathbb{P}^1, g^*T_Y) \rightarrow T_{[f]} \text{Hom}(\mathbb{P}^1, X; \{0\} \rightarrow Y) \rightarrow H^0(\mathbb{P}^1, g^*N_{Y/X} \otimes m_0) \rightarrow 0$$

and $\text{Hom}(\mathbb{P}^1, X; \{0\} \rightarrow Y)$ is smooth at $[f]$.

**Proof.** The natural maps $f^*T_X \rightarrow g^*N_{Y/X} \rightarrow g^*N_{Y/X}|_{\{0\}}$, passing to cohomology and taking into account the isomorphism $T_{[f]} \text{Hom}(\mathbb{P}^1, X) \cong H^0(\mathbb{P}^1, f^*T_X)$ given by Proposition 2.2 yield a natural homomorphism

$$r: T_{[f]} \text{Hom}(\mathbb{P}^1, X) \rightarrow (g^*N_{Y/X})|_{\{0\}}.$$ 

We have a Cartesian square

$$\begin{array}{ccc}
\text{Hom}(\mathbb{P}^1, X; \{0\} \rightarrow Y) & \rightarrow \text{Hom}(\mathbb{P}^1, X) \\
\downarrow & \downarrow \\
\text{Hom}(\{0\}, Y) & \rightarrow \text{Hom}(\{0\}, X).
\end{array}$$

Note that there are natural identifications $\text{Hom}(\{0\}, Y) = Y$ and $\text{Hom}(\{0\}, X) = X$. The Zariski tangent space of $\text{Hom}(\mathbb{P}^1, X; \{0\} \rightarrow Y)$ at $[f]$, viewed as a subspace of
Thus, by (3) and Proposition 2.26 (b), there exists a dominant morphism $h$ of dimension $h^0(P^1, T_Y)$. Moreover, there is a dominant morphism $V(\{0\} \to Y)$, defined by $[h] \mapsto h(0)$, whose fiber over a point $y \in Y$ is $V(0 \to y)$. This implies that $\dim V(\{0\} \to Y) = \dim V(0 \to y) + \dim Y$ for a general $y \in Y$, and therefore

$$\dim V(\{0\} \to Y) = h^0(P^1, T_Y) + \dim Y$$

by Proposition 2.22 (b). On the other hand, we have

$$h^0(P^1, N_{Y/X} \otimes m_0) = h^0(P^1, f^*T_X \otimes m_0) - h^0(P^1, g^*T_Y)$$

$$= h^0(P^1, T_Y) - h^0(P^1, g^*T_Y) + h^0(g^*T_Y|_{\{0\}}).$$

Therefore, by (3) and $h^0(g^*T_Y|_{\{0\}}) = \dim Y$, we get

$$\dim \ker(r) = h^0(P^1, T_X \otimes m_0) + \dim Y.$$

Then, comparing (5) with (4), we conclude at once that $T_{[f]} \Hom(P^1, X; \{0\} \to Y) = \ker(r)$ and that $V(\{0\} \to Y)$ is smooth at $[f]$. \hfill \qed

**Definition 2.28.** If $W \subseteq \Hom_{\text{bir}}(P^1, Y)$ is a family of rational curves on $Y$, then we call $\langle i_*(W) \rangle \subseteq \Hom_{\text{bir}}(P^1, X)$ the extension of $W$ to $X$. Conversely, for every family of rational curves $V$ on $X$, we call $\langle i_*^{-1}(V) \rangle \subseteq \Hom_{\text{bir}}(P^1, Y)$ the restriction of $V$ to $Y$.

**Remark 2.29.** If $V$ is an irreducible family on $X$, then $\langle i_*^{-1}(V) \rangle \subseteq \Hom_{\text{bir}}(P^1, Y)$ need not be irreducible. In fact, the example of the restriction to a smooth quadric $Q^2$ of the family of lines in $\mathbb{P}^3$ shows that, in general, different elements in $\langle i_*^{-1}(V) \rangle$ may even define linearly independent numerical classes in $N_1(Y)$. In particular, if $V$ is an unsplit family on $X$, then its restriction to $Y$ may not be an unsplit family. Although we do not have examples, it seems likely that $i_*^{-1}(V)$ might fail to be a family on $Y$ even if $V$ is a family on $X$.

We close this section with the following “relative” version of Proposition 2.22. The proof, which can be found within the proof of [4, Lemma 1.4.5], uses a “non-breaking lemma” due to Wiśniewski [28].

**Proposition 2.30.** Let $V$ be an unsplit family on $X$. Let $Y \subseteq X$ be a subvariety, and assume that $\text{Locus}(V, \{0\} \to Y)$ is dense in $X$. Then, for every curve $\Gamma \subseteq X$, we have

$$[\Gamma] = a[\Gamma_Y] + b[C] \quad \text{in} \quad N_1(X),$$

where $a$ and $b$ are integers.
where \( C \) is a \( V \)-curve, \( \Gamma_Y \) is a curve contained in \( Y \), and \( a \geq 0 \).

3. Extension of rationally connected fibrations

In this section we study the relationship between rational connected fibrations of a projective manifold \( X \) and those of an ample submanifold \( Y \subset X \). Our main interest is in the case when it is given on \( X \) an irreducible family of rational curves \( V \) whose restriction \( V_Y \) to \( Y \) contains a covering component. Given this situation, the goal is to compare the associated rationally connected fibrations \( X \to X//V \) and \( Y \to Y//V_Y \). However, for technical reasons that will be evident in the proof of Theorem 3.6, it is convenient to consider from the beginning a more general setting, allowing \( V \) to be reducible. The conditions given in items (i) and (ii) in the following two theorems capture the essential properties of \( V \) needed in our arguments.

**Theorem 3.1.** Let \( X \) be a smooth projective variety, and assume that \( Y \subset X \) is a smooth subvariety with ample normal bundle. Let \( i: Y \hookrightarrow X \) be the inclusion, and suppose that the induced map \( i^*: N^1(X) \to N^1(Y) \) is surjective. Let \( V = \bigcup_{\alpha \in A} V_\alpha \) be a family of rational curves on \( X \), and assume that there is a subset \( B \subseteq A \) such that

(i) \( \mathbb{R}_{\geq 0}[V] = \sum_{\beta \in B} \mathbb{R}_{\geq 0}[V_\beta] \), and

(ii) Locus \( (i^{-1}_*V_\beta) \) is dense in \( Y \) for every \( \beta \in B \).

Let \( V_Y := i^{-1}_*V \) be the restriction of \( V \) to \( Y \). Then both \( V \) and \( V_Y \) are numerically covering families (respectively, on \( X \) and on \( Y \)), and, for suitable choices of the rational quotients, there is a commutative diagram

\[
\begin{array}{ccc}
Y & \xrightarrow{\pi} & X \\
\downarrow & & \downarrow \phi \\
Y//V_Y & \xrightarrow{\delta} & X//V
\end{array}
\]

where \( \pi \) and \( \phi \) are the projections to the respective rational quotients and \( \delta \) is a surjective morphism.

**Remark 3.2.** If one assumes that \( V \) is irreducible, then the hypothesis that \( i^*: N^1(X) \to N^1(Y) \) be surjective is unnecessary.

**Remark 3.3.** An analogous property has been independently observed to hold in the case \( V = \text{Hom}_{\text{bir}}(\mathbb{P}^1, X) \) by A.J. de Jong and J. Starr.

The proof of Theorem 3.1 is based upon the following two lemmas.

**Lemma 3.4.** With the same notation and assumptions as in Theorem 3.1, Locus \( \text{Locus}(V_\beta; \{0\} \to Y) \) is dense in \( X \) for every \( \beta \in B \).

**Proof.** To keep the notation light, we suppose throughout the proof that \( V \) is irreducible, so that \( V_\beta = V \). By hypothesis, Locus \( (V_\beta) \) is dense in \( Y \). Fix a general element \( [g] \) of an irreducible component of \( V_Y \) with dense locus in \( Y \), and let \( f := i \circ g \). Note that \( [f] \in V(\{0\} \to Y) \). Since the chosen component of \( V_Y \) has dense locus in \( Y \), we can assume that \( y := f(0) \) is a general point of \( Y \). We know that \( g \) is free by Proposition 2.6.
Thus, since $V$ is an irreducible component of $\text{Hom}(\mathbb{P}^1, X)$ containing $[f]$, Proposition 2.27 applies to say that $V(\{0\} \to Y)$ is smooth at $[f]$, with Zariski tangent space

$$T[\beta]V(\{0\} \to Y) \cong H^0(\mathbb{P}^1, g^*T_Y) \oplus H^0(\mathbb{P}^1, g^*N_{Y/X} \otimes m_0)).$$

We can view the right hand side as a vector subspace $\Lambda \subseteq H^0(\mathbb{P}^1, f^*T_X)$ via the isomorphism (3) given in the proof of Proposition 2.27. It follows from the ampleness of $g^*N_{Y/X}$ that $f^*T_X$ is spanned by the sections in $\Lambda$ at every point $q \in \mathbb{P}^1 \setminus \{0\}$. By [15, Proposition I.2.19], we conclude that the differential of the universal map

$$\mathbb{P}^1 \times V(\{0\} \to Y) \to X$$

has rank equal to $\dim X$ at $(q, [f])$ for every $q \in \mathbb{P}^1 \setminus \{0\}$. Since $V(\{0\} \to Y)$ is smooth at $[f]$, this implies that its locus in $X$ is dense. □

**Lemma 3.5.** With the same notation and assumptions as in Theorem 3.1, for every $\beta \in B$ the family $V_\beta$ is covering (in $X$) and every irreducible component of $i_\ast^{-1}(V_\beta)$ with dense locus in $Y$ is a family (on $Y$).

**Proof.** By the hypothesis (ii) of Theorem 3.1, we can fix an arbitrary irreducible component $T$ of $i_\ast^{-1}(V_\beta)$ with dense locus in $Y$. Let $[g] \in T$ be a general element. We know by Proposition 2.6 that $g$ is free. Then by Proposition 2.26 $f := i \circ g$ is free and $\text{Hom}(\mathbb{P}^1, X)$ is smooth at $[f]$, and thus, in particular, $V_\beta$ is a covering family by Proposition 2.6 again. Moreover, we deduce by the smoothness of $\text{Hom}(\mathbb{P}^1, X)$ at $[f]$ that for every irreducible one-parameter family $[g_t]$ in $\text{Hom}_{\text{bir}}(\mathbb{P}^1, Y)$ specializing to $[g]$, the corresponding family $[f_t] := [i \circ g_t]$ is contained in $V$. Therefore $[g_t] \in i_\ast^{-1}(V)$, and in fact $[g_t] \in T$ by the generality of the choice of $[g]$ in the component $T$. We conclude that $T$ is an irreducible component of $\text{Hom}_{\text{bir}}(\mathbb{P}^1, Y)$, and hence, in particular, of $V_Y$. □

**Proof of Theorem 3.7.** It follows from Lemma 3.5 and the hypothesis (i) that $V$ is a numerically covering family on $X$ and similarly, using the fact that the $i_\ast : N_1(Y) \to N_1(X)$ is injective, it follows that $V_Y$ is a numerically covering family on $Y$. Let $\phi : X \to Z//V$ and $\pi : Y \to Y//V_Y$ be the rational quotients, as in Theorem 2.13. Let $X^0 \subseteq X$, $S^0 \subseteq X//V$, $Y^0 \subseteq Y$, and $Z^0 \subseteq Y//V_Y$ be open subsets such that the maps $\phi$ and $\pi$ restrict to proper surjective morphisms

$$\phi^0 : X^0 \to S^0 \quad \text{and} \quad \pi^0 : Y^0 \to Z^0.$$

Let $G$ be a very general fiber of $\phi^0$, and let $x$ be a general point of $G$. By Lemma 3.4 we can assume that $x \in \text{Locus}(V; \{0\} \to Y)$, and hence we can find a point $y \in Y$ such that $x \in \text{Locus}(V; 0 \to y)$ by Theorem 2.13. In particular, $x$ and $y$ are $V$-chain connected. This implies that $y \in G$, and therefore that $G \cap Y \neq \emptyset$. Since $G$ is a general fiber of $\phi^0$, this means that $\phi^0(Y \cap X^0) = S^0$. Moreover, by the generality of $G$, we can in fact assume that $G$ has non-empty intersection with a very general fiber of $\pi^0$. On the other hand, recalling that $\pi$ is defined by the restriction $V_Y$ of the family $V$ defining $\phi$, we see that if $F$ is a very general fiber of $\pi^0$ meeting $G$, then necessarily $F \subseteq G$. In conclusion, after possibly further
shrinking $Z$ (and consequently $Y$), we may assume to have a commutative diagram

\[
\begin{array}{ccc}
Y \overset{\pi}{\longrightarrow} X \\
\downarrow & & \downarrow \phi \\
Z \overset{\delta}{\longrightarrow} S
\end{array}
\]

where $\delta$ is a dominant morphism.

We need to show that, after possibly changing the birational model for $Y//V$, the map $\delta$ extends to a surjective morphism $\delta: Y//V \rightarrow X//V$. We first take the normalization $Z$ of a projective compactification of $Z$. Note that $\delta$ extends to a rational map $Z//V$, which is defined by some linear system $H$ on $Z$. Then we take $Z'$ to be the normalization of the blow-up of the base scheme of $H$. Since the proper transform of $H$ on $Z'$ is base-point free, the rational map $\delta$ lifts to a well defined morphism $\delta: Z' \rightarrow X//V$. This is a morphism of projective varieties, and $\delta$ is dominant, so $\delta$ is surjective. Moreover, the morphism $Z' \rightarrow Z$ is an isomorphism over $Z$, and thus we can identify the latter with a subset of $Z'$. Then we take $Z'$ as the $V$-quotient $Y//V$ of $Y$. This proves the theorem.

Theorem 3.1 can be viewed as a general property of rationally connected fibrations. Aiming also at applications in extension problems of specific fibrations, that will be addressed in the following sections, we are interested in determining sufficient conditions to ensure that the map $\delta$, whose existence was proven in the previous theorem, is generically finite. This is the content of the next theorem, which is the main result of this section.

**Theorem 3.6.** Let $X$ be a smooth projective variety, and assume that $Y \subset X$ is a smooth subvariety with ample normal bundle. Let $i: Y \hookrightarrow X$ be the inclusion, and suppose that the induced map $i^*: N^1(X) \rightarrow N^1(Y)$ is surjective. Let $V = \bigcup_{\alpha \in A} V_\alpha$ be a family of rational curves on $X$, and assume that there is a subset $B \subseteq A$ such that

(i) $\mathbb{R}_{\geq 0}[V] = \sum_{\beta \in B} \mathbb{R}_{\geq 0}[V_\beta]$, and

(ii) Locus $(i_\beta^{-1}(V_\beta))$ is dense in $Y$ for every $\beta \in B$.

Let $V_Y := i_\beta^{-1}(V)$ be the restriction of $V$ to $Y$, and let

\[
\begin{array}{ccc}
Y^c \overset{\pi}{\longrightarrow} X \\
\downarrow & & \downarrow \phi \\
Y//V \overset{\delta}{\longrightarrow} X//V
\end{array}
\]

be the commutative diagram given by Theorem 3.1. Suppose that one of the following two conditions holds:

(a) $V$ is an unsplit family; or
(b) $\text{codim}_X Y < \dim \text{RC}_{V_Y}$.

Then the morphism $\delta$ is generically finite.
Proof. We first prove that $\delta$ is generically finite when the condition (a) of the statement is satisfied. If $Y//_{V_Y}$ is a point, then the statement is obvious, so we can suppose that $Y//_{V_Y}$ is positive dimensional. We suppose by contradiction that $\delta$ is not generically finite.

Keeping the notation introduced in the proof of Theorem 3.1, let $G$ be a very general fiber of $\phi^\circ$, and consider $Y_G := Y \cap G$. Note that $G$ is smooth and rationally connected by the restriction $V_G$ of $V$ to $G$ in the sense of Definition 2.28. So, since we are assuming that $V$ is unsplit, every irreducible component of $V_G$ is an unsplit family on $G$, and hence $N_1(G)$ is generated by classes of $V_G$-curves by Proposition 2.12. Since such curves are all numerically equivalent in $X$, it follows that the image of the restriction map $\text{Pic}(X) \to \text{Pic}(G)$ has rank 1. Consider the inclusions $j_X : G \hookrightarrow X$, $j_Y : Y_G \hookrightarrow Y$ and $i_G : Y_G \subseteq G$. In order to reach a contradiction, we consider the commutative diagram

$$
\begin{array}{ccc}
\text{Pic}(X) & \overset{j_X^*}{\longrightarrow} & \text{Pic}(G) \\
\downarrow i^* & & \downarrow i_G^* \\
\text{Pic}(Y) & \overset{j_Y^*}{\longrightarrow} & \text{Pic}(Y_G).
\end{array}
$$

The plan is to give a lower-bound for the rank of the map $\text{Pic}(X) \to \text{Pic}(Y_G)$ by factoring it through $\text{Pic}(Y)$.

Note that $Y_G^\circ := Y^\circ \cap G$ is a non-empty open subset of $Y_G$. Since we are supposing that $\delta^\circ$ is not generically finite, we have $\dim (\pi^\circ(Y_G^\circ)) \geq 1$. Therefore we can find a curve $\Gamma \subseteq Y_G$ such that $\Gamma \cap Y_G^\circ \neq \emptyset$ and $\dim (\pi^\circ(\Gamma \cap Y_G^\circ)) \geq 1$. Since $\text{Locus}(V_Y)$ is dense in $Y$, every fiber of $\pi^\circ$ has positive dimension and, since these fibers are proper, we can fix a curve $C \subset Y_G$ lying inside a fiber of $\pi^\circ|_{Y_G^\circ}$.

Now, fix a projective model $Y//_{V_Y}$ for the rational quotient of $Y$ containing $Z^\circ$ as an open subset, and let $L \subset Y//_{V_Y}$ be a general hyperplane section passing through a point in $\pi^\circ(\Gamma \cap Y_G^\circ)$ but not containing the point $\pi^\circ(C)$. Let $L^\circ := L|_{Z^\circ}$ be the restriction of $L$ to $Z^\circ$, and let $D \subset Y$ be the closure (in $Y$) of $(\pi^\circ)^{-1}(L^\circ)$. If $U \subset Z^\circ \setminus L^\circ$ is an open neighborhood of $\pi^\circ(C)$, then $(\pi^\circ)^{-1}(U)$ is an open neighborhood of $C$ in $X^\circ$ (hence in $X$) and is disjoint from $(\pi^\circ)^{-1}(L^\circ)$; this shows that $D \cap C = \emptyset$. Let then $D_G$ and $H_G$ be the restrictions of $D$ and of a general hyperplane section $H$ of $Y$ to $Y_G$. By construction, we have $D_G \cdot \Gamma > 0$ and $D_G \cdot C = 0$, whereas $H_G$ is ample; in particular, $D_G$ and $H_G$ induce numerically independent elements of $\text{Pic}(Y_G)$. Since both of them are restrictions of divisors on $Y$, this implies that $\text{rk} \text{Im}(j_Y^* i^*) \geq 2$. Therefore, observing that the cokernel of $i^* : \text{Pic}(X) \to \text{Pic}(Y)$ is torsion due to the surjectivity of $N^1(X) \to N^1(Y)$, we conclude that

$$\text{rk} \text{Im}(j_Y^* i^*) \geq 2.$$

On the other hand, we have $\text{rk} \text{Im}(i_G^* j_Y^*) \leq 1$, and therefore we have a contradiction by the commutativity of the above diagram. This proves that $\delta$ is generically finite if the condition (a) of the statement of the theorem is satisfied.

It remains to prove that $\delta$ is generically finite under the hypothesis (b). Let $G$ be a very general (smooth and connected) fiber of $\phi^\circ$, and let $F$ be a very general fiber of $\pi$ among those that are contained in $G$. Choosing $G$ sufficiently general, we can also ensure that $F$ is general among the fibers of $\pi^\circ$. Then, to conclude the proof of the theorem, we
need to show that
\[
\dim G = \dim F + \operatorname{codim}_X Y.
\]
Note indeed that, since \(\dim G = \dim \operatorname{RC}_V\) and \(\dim F = \dim \operatorname{RC}_{V', \gamma}\), (6) implies that \(\delta\) is generically finite.

In order to prove (6), we fix a very general point \(y \in F\). By Lemma 3.4, \(\operatorname{Locus}(V, \{0\} \mapsto Y)\) is dense in \(X\). Moreover, we know that \(\operatorname{Locus}(V, 0 \mapsto y)\) is contained in \(G\) by Theorem 2.13. Thus \(\operatorname{Locus}(V, 0 \mapsto y)\) sweeps a dense subset of \(G\), if we let \(y\) vary in \(F\). Therefore, by our choices of \(F\) and \(y\), we can assume that \(\operatorname{Locus}(V, 0 \mapsto y)\) contains a very general point of \(G\). Note that any point of \(G\) is connected by a \(V_G\)-chain to the point \(y\), where \(V_G\) denotes the restriction of \(V\) to \(G\) in the sense of Definition 2.28. Therefore Proposition 2.27 applied to \(G\) and \(V_G\) implies that \(\operatorname{Locus}(\|V\|, 0 \mapsto y)\) is dense in \(G\). Note that the two families \(\|V\|(0 \mapsto y)\) and \(\|V_G\|(0 \mapsto y)\) are the same since \(G\) is an \(\operatorname{RC}_{\|V\|}\) equivalence class. In particular, we have
\[
\dim G = \dim \operatorname{Locus}(\|V\|, 0 \mapsto y).
\]

At this point the idea is to replace \(V\) with \(\|V\|\), so that, by the above argument, we can directly assume, without loss of generality, that the two sets \(\operatorname{Locus}(V, 0 \mapsto y)\) and \(G\) have the same dimension (and, in fact, that the first set is dense in the second one). In order to make this step, we need to show, first of all, that \(\|V\|\) satisfies hypotheses analogous to (i) and (ii) imposed to \(V\) in the statement of the theorem, and moreover that replacing \(V\) with \(\|V\|\) does not affect the quotient maps \(\phi\) and \(\pi\) and the respective rational quotients, which implies condition (b) for \(\|V\|\).

Lemma 3.7. The family \(\|V\|\) satisfies the hypotheses (i) and (ii) imposed to \(V\) in Theorem 3.6. Moreover, the families \(V_Y = i_*^{-1}(V)\langle\) and \(\|V\|_Y := i_*^{-1}(\|V\|)\langle\) define the same rational fibration on \(Y\). In particular, \(\operatorname{codim}_X Y < \dim \operatorname{RC}_{\|V\|_Y}\).

Proof of the lemma. First recall that \(V_Y\) is a numerically covering family by Theorem 3.1, hence so is \(\|V\|_Y\). We compare \(\|V\|\) with
\[
W := (i_*(\|V\|_Y))\langle.
\]
Note that \(\|V\|_Y\) is non-empty and \(W\) is a subfamily of \(\|V\|\). We claim that
\[
\mathbb{R}_{\geq 0}[\|V\|] = \mathbb{R}_{\geq 0}[W] \quad \text{in} \quad N_1(X).
\]
By the definition of \(\|V\|\), this is equivalent to \(\mathbb{R}_{\geq 0}[V] = \mathbb{R}_{\geq 0}[W]\). The inclusion \(\mathbb{R}_{\geq 0}[V] \supseteq \mathbb{R}_{\geq 0}[W]\) is obvious, so we need to show that the reverse inclusion holds. In fact, by the hypothesis (i) on \(V\), it suffices to show that \(\mathbb{R}_{\geq 0}[V_\beta] \subseteq \mathbb{R}_{\geq 0}[W]\) for every \(\beta \in B\). Since \(\operatorname{Locus}(i_*^{-1}(V_\beta))\) is dense in \(Y\), at least one of the irreducible components of \(i_*^{-1}(V_\beta)\langle\) is a covering family on \(Y\), hence a subfamily of \(\|V\|_Y\) by Definition 2.18. This implies that \(\mathbb{R}_{\geq 0}[V_\beta] \subseteq \mathbb{R}_{\geq 0}[W]\), hence (7) is proven.

On the other hand, the restriction to \(Y\) of any irreducible component of \(W\) is a union of irreducible components of \(\|V\|_Y\), which are covering by Definition 2.18. Therefore their loci are dense in \(Y\). Combining this with (7) proves the first assertion.

We are currently assuming that \(V\) satisfies the condition (b) in Theorem 3.6. Therefore we get the desired inequality \(\operatorname{codim}_X Y < \dim \operatorname{RC}_{\|V\|_Y}\) as soon as we show
that the families $V_Y$ and $\|V\|_Y$ define the same rational quotient of $Y$, because then $\dim RC_{V_Y} = \dim RC_{\|V\|_Y}$. By using Proposition 2.20 it is enough to show that
\begin{equation}
\mathbb{R}_{\geq 0}[\|V\|_Y] = \mathbb{R}_{\geq 0}[V_Y] \quad \text{in} \quad N_1(Y).
\end{equation}

The inclusion $\mathbb{R}_{\geq 0}[\|V\|_Y] \supseteq \mathbb{R}_{\geq 0}[V_Y]$ is obvious. To prove the other, note that the embedding of $Y$ in $X$ induces an inclusion

$$i: N_1(Y) \hookrightarrow N_1(X),$$

which follows from the hypothesis that $i^*: N^1(X) \rightarrow N^1(Y)$ is surjective made in Theorem 3.4. Observe that

$$i\left(\mathbb{R}_{\geq 0}[\|V\|_Y]\right) \subseteq \mathbb{R}_{\geq 0}[\|V\|] = \mathbb{R}_{\geq 0}[V] = \sum_{\beta \in B} \mathbb{R}_{\geq 0}[V^\beta] = i\left(\mathbb{R}_{\geq 0}[V_Y]\right) \quad \text{in} \quad N_1(X).$$

Since $i$ is injective, the inclusion holds in $N_1(Y)$, before taking images. This proves equality (8) and completes the proof of the lemma. \qed

We now come back to the proof of the theorem. By the previous lemma, we are allowed to replace $V$ with $\|V\|$. Therefore we can directly assume without loss of generality that

\begin{equation}
\dim \text{Locus}(V, 0 \mapsto y) = \dim G.
\end{equation}

Then, in order to show (6) and hence to conclude the proof, it suffices to show that $\dim \text{Locus}(V, 0 \mapsto y) = \dim X - \dim Y//V_Y$.

For short, let $d = \dim X$ and $k = \dim Y//V_Y$. Since $y$ is a general point of $Y$, we can assume that $V_Y(0 \mapsto y)$ contains a sufficiently general point $[g]$ of a covering component of $V_Y$, so that $g$ is free and $C := g(\mathbb{P}^1)$ is a smooth curve (the smoothness of $C$ is not essential for the proof, but simplifies the notation).

We observe that $N_{Y/X}|_F$ is an ample vector bundle over $F$, and that $\text{rk} N_{Y/X}|_F < \dim F$ by condition (b), and therefore $H^1((N_{Y/X}|_F)^*) = 0$ by the Le Potier vanishing theorem [23]. This implies that the short exact sequence

$$0 \rightarrow N_{F/Y} \rightarrow N_{F/X} \rightarrow N_{Y/X}|_F \rightarrow 0$$

splits since $N_{F/Y} \cong \mathcal{O}_F^{\oplus k}$, and therefore we get a surjection $N_{F/X} \rightarrow N_{F/Y}$. Restricting to $C$ and composing with the natural surjections $T_X|_C \rightarrow N_{C/X}$ and $N_{C/X} \rightarrow N_{F/X}|_C$, we obtain a chain of surjections

$$T_X|_C \rightarrow N_{C/X} \rightarrow N_{F/X}|_C \rightarrow N_{F/Y}|_C \cong \mathcal{O}_{\mathbb{P}^1}^{\oplus k}.$$

Since $g$ is free, so is $f := i \circ g$ by Proposition 2.20 and therefore $T_X|_C$ is nef. Writing $T_X|_C \cong \oplus_i \mathcal{O}_{\mathbb{P}^1}(b_i)$, we have $b_i \geq 0$, and thus the existence of a quotient of $T_X|_C$ isomorphic to $\mathcal{O}_{\mathbb{P}^1}^{\oplus k}$ implies that

\begin{equation}
\#\{b_i \mid b_i = 0\} \geq k.
\end{equation}

Note that $(V, 0 \mapsto y)$ is smooth at $[f]$ by Proposition 2.2. Let $u: \mathbb{P}^1 \times (V, 0 \mapsto y) \rightarrow X$ be the universal map, and let $q \in \mathbb{P}^1$ be a point different from 0. By [15] Proposition II.3.10 and (10), we have

$$\text{rk} du(q, [f]) \leq d - k.$$
Since \((V, 0 \mapsto y)\) is smooth at \([f]\), we obtain \(\dim \text{Locus}(V, 0 \mapsto y) \leq d - k\). On the other hand, we have \(\dim G \geq d - k\) by Theorem 3.1. Thus by (9) we conclude that \(\dim \text{Locus}(V, 0 \mapsto y) = d - k\), which proves (6). This concludes the proof of the theorem. □

Remark 3.8. It is interesting to compare the result of Theorem 3.6 when the condition in case (b) is satisfied with a result of Sommese [26, Proposition III] (see also [3, Theorem (5.3.1)]) in which the case of an ample divisor endowed with a fibration is considered. Indeed, one notices that if \(\text{codim}_X Y = 1\), then the inequality in (b) reduces exactly to the hypothesis imposed in the theorem of Sommese.

Remark 3.9. Hartshorne’s conjecture [13, Conjecture 4.5] on the intersection of ample submanifolds with complementary dimensions, if true, would imply that the morphism \(\delta\) in Theorem 3.6 is in fact an isomorphism whenever the condition given in (b) is satisfied. Indeed, let \(G\) be a general fiber of \(\phi\). Then \(Y \cap G\) is a disjoint union of \(\deg \delta\) fibers \(F_j\) of \(\pi\), and by using standard exact sequences one easily sees that \(N_{Y \cap G/G} \cong N_{Y/X}|_{Y \cap G}\). In particular, we have \(N_{F_j/G} \cong N_{Y \cap G/G}|_{F_j} \cong N_{Y/X}|_{F_j}\), which is ample. Note that condition (b) implies that \(2 \dim F_j > \dim G\). Then, assuming that \(\deg \delta \geq 2\), Hartshorne’s conjecture would give the contradiction \(F_1 \cap F_2 \neq \emptyset\); therefore \(\delta\) can only be birational. In fact, modulo suitable choices of the rational quotients, one could even assume that \(\delta\) is an isomorphism. We recall that this conjecture of Hartshorne is known to be true, for instance, for homogeneous spaces [24]; we will use this fact in the proof of Theorem 5.8.

Remark 3.10. The previous results are related to a result of Occhetta [25, Proposition 4], where an analogous extension property is obtained under more restrictive hypotheses. In the setting of [25], \(Y\) is supposed to be the zero locus of a regular section of an ample vector bundle on \(X\), \(\pi\) is the MRC-fibration of \(Y\), and the rational quotient of \(Y\) is assumed to have positive geometric genus.

In applications, it might be useful to start with families of rational curves on \(Y\), rather than only considering restrictions of families on \(X\). The following elementary property will be useful. Let \(X\) and \(Y\) be as above, with the inclusion inducing a surjection \(N^1(X) \to N^1(Y)\), and consider a numerically covering family \(W\) of rational curves on \(Y\). Let then \(V = (i_*(W))\) be the family on \(X\) obtained as the extension of \(W\), and let \(V_Y = (i_*(W))^{-1}(V)\) be its restriction back to \(Y\). Note that \(V_Y\) is numerically covering since \(W\) is so. We clearly have \(W \subseteq V_Y\), but in general the inclusion may be strict (one can take as an example the case of a quadric \(Y = \mathbb{Q}^2\) in \(X = \mathbb{P}^3\), taking \(W\) to be the family of lines on \(Y\) corresponding to one of the two rulings). By this inclusion and Theorem 3.1, we have a commutative diagram

\[
\begin{array}{ccc}
Y & \xrightarrow{\gamma} & Y/V_Y \\
\downarrow \tau & & \downarrow \phi \\
X & \xrightarrow{\delta} & X/V
\end{array}
\]

for a suitable choice of the rational quotients, and some surjective morphisms \(\gamma\) and \(\delta\).
Proposition 3.11. Keeping the notation introduced above, both families $W$ and $V_Y$ define the same rational quotient (i.e., $Y//W = Y//V_Y$ and $\tau = \pi$) if either one of the following two situations occurs:

(a) $\dim Y//W = \dim X//V$, or
(b) $i^*: N_1(X) \to N_1(Y)$ is surjective.

Proof. We observe that the general fiber of $\gamma$ is rationally connected, since it is dominated by the general fiber of $\pi$, and in particular it is connected. Then the conclusion in case (a) follows immediately from the commutativity of the above diagram. We now focus on case (b), and thus assume that $N_1(X) \to N_1(Y)$ is surjective. By duality, we have an injection $N_1(Y) \hookrightarrow N_1(X)$ and, by the construction of $V_Y$, we deduce that $\mathbb{R}_{\geq 0}[V_Y] = \mathbb{R}_{\geq 0}[W]$. Then the assertion follows from Proposition 2.20.

The case when the rational quotient is one-dimensional is particularly well suited because of the following well-known property. We are grateful to the referee for suggesting this proof, which simplifies our original argument.

Proposition 3.12. Let $X$ be a normal projective variety, let $\phi^o: X^o \to C$ be a dominant morphism from a non-empty open subset $X^o \subseteq X$ to a smooth projective curve $C$, and assume that $\phi^o$ is proper over its image. Then $\phi^o$ extends to a (surjective) morphism $\phi: X \to C$.

Proof. We consider a projection $C \to \mathbb{P}^1$. Composing with $\phi^o$, we have a rational map $X \dashrightarrow \mathbb{P}^1$ which is defined by some linear pencil. Because of the properness of $\phi^o$, this pencil is free, hence the map $X \to \mathbb{P}^1$ is regular and we can lift it via Stein factorization to a regular morphism $X \to C$ which extends $\phi^o$.

Coming back to the extension of rationally connected fibrations, we immediately obtain the following result.

Corollary 3.13. With the same notation and assumptions as in Theorem 3.6, suppose furthermore that $\dim Y//V_Y = 1$. Then the RC$_V$-fibration $\pi$ of $Y$ and the RC$_Y$-fibration $\phi$ of $X$ are surjective morphisms fitting in a commutative diagram

$$
\begin{array}{ccc}
Y & \xrightarrow{\pi} & X \\
\downarrow & & \downarrow \phi \\
Y//V_Y & \xrightarrow{\delta} & X//V
\end{array}
$$

where $\delta$ is a finite morphism of smooth curves.

Proof. It follows from Theorems 3.1 and 3.6 and Proposition 3.12.

4. Extension of Mori contractions

In this section we consider the problem to extend Mori contractions. Let $Y$ be a smooth projective variety with Picard number $\rho := \rho(Y) \geq 2$, and suppose that $R \subset N_1(Y)$ is a $K_Y$-negative extremal ray of $\overline{\text{NE}}(Y)$ such that the associated Mori contraction $\pi: Y \to Z$ is of fiber type, namely, that $\dim Z < \dim Y$. 
We fix a rational curve $C$ in a general fiber of $\pi$ passing through a general point of $Y$, whose numerical class $[C]$ generates $R$. Then we fix a normalization morphism $f : \mathbb{P}^1 \to C \subset Y$, and let

$$W = \langle [f] \rangle \subset \text{Hom}_{\text{bir}}(\mathbb{P}^1, Y)$$

be the minimal family of rational curves on $Y$ generated by $[f]$. Note that, by construction,

$$R_{\geq 0}[W] = R_{\geq 0}[C] = R \quad \text{in } N_1(Y),$$

and $W$ is a covering family on $Y$, since $C$ passes through a general point of $Y$. Moreover, $\pi$ can be considered as the RC$_W$-fibration of $Y$.

**Theorem 4.1.** With the above notation, assume that $Y$ is a smooth subvariety, with ample normal bundle, of a smooth projective variety $X$, such that the inclusion $i : Y \hookrightarrow X$ induces an isomorphism $i^* : N^1(X) \to N^1(Y)$. Let

$$V := \langle i_* (W) \rangle \subset \text{Hom}_{\text{bir}}(\mathbb{P}^1, X)$$

be the extension of the family $W$ to $X$, and assume that $V$ is an unsplit family on $X$.

Then $R_{\geq 0}[V]$ is a $K_X$-negative extremal ray of $X$. Moreover, denoting by $\phi : X \to S$ the Mori contraction of this ray, there is a finite surjective morphism $\delta : Z \to S$ giving a commutative diagram

$$\begin{array}{ccc}
Y & \xrightarrow{\pi} & X \\
\downarrow & & \downarrow \phi \\
Z & \xrightarrow{\delta} & S.
\end{array}$$

**Proof.** By the hypothesis and duality, the inclusion $i : Y \hookrightarrow X$ induces a natural isomorphism $\iota : N_1(Y) \cong N_1(X)$. Let $R_X := \iota(R)$. This is a ray in $N_1(X)$, and is contained in $\overline{\text{NE}}(X)$, because of the inclusion $\iota(\overline{\text{NE}}(Y)) \subseteq \overline{\text{NE}}(X)$. Moreover, if $C$ is a curve as above, then adjunction formula gives

$$K_X \cdot C = (K_Y - \det N_{Y/X}) \cdot C < 0,$$

and thus $R_X$ is a $K_X$-negative ray. The main point here is to prove that $R_X$ is an extremal ray of $\overline{\text{NE}}(X)$.

We start by observing two things. First of all, we have $R_X = R_{\geq 0}[V]$ in $N_1(X)$, and we can consider $\phi$ as the RC$_W$-fibration of $X$. Moreover, if $V_Y := \iota^{-1}(V)(\phi)$ is the restriction of $V$ to $Y$, then we have

$$R_{\geq 0}[V_Y] = \iota^{-1}(R_{\geq 0}[V]) = R_{\geq 0}[W] \quad \text{in } N_1(Y),$$

by the isomorphism $\iota : N_1(Y) \cong N_1(X)$. Therefore $V_Y$ and $W$ define the same rational quotient, by Proposition 2.20.

Note that $V$ satisfies the conditions (i) and (ii) of Theorem 3.1. Then, by Lemma 3.4 and the fact that $V$ is unsplit, we have

$$\text{Locus}(V, \{0\} \to Y) = X.$$ 

Indeed the lemma implies that Locus$(V, \{0\} \to Y)$ is dense in $X$, and the unsplitness of $V$ implies that such locus is proper, hence closed in $X$. 

Let $D$ be any divisor on $X$ whose restriction $D|_Y$ is a good supporting divisor for $R$. Note that $D \cdot C = 0$. Let $\Gamma$ be any curve on $X$. Then, by Proposition 2.30 there is a curve $\Gamma_Y \subset Y$ such that $[\Gamma] = a[\Gamma_Y] + b[C]$ in $N_1(X)$, with $a \geq 0$. This implies that $D \cdot \Gamma = aD \cdot \Gamma_Y + bD \cdot C = aD|_Y \cdot \Gamma_Y \geq 0$, since $D \cdot C = 0$ and $D|_Y$ is nef. This proves that $D$ is nef.

Since $R$ is an extremal ray of $\overline{\text{NE}}(Y)$, by duality it corresponds to an extremal face of $\text{Nef}(Y)$ of maximal dimension $\rho - 1$, and therefore we can find $\rho - 1$ good supporting divisors of $R$ whose numerical classes are linearly independent in $N^1(Y)$. By the previous argument, this implies that such good supporting divisors, up to numerically equivalence, extend to divisors on $X$ that are nef and trivial on $R_X$, and whose numerical classes are linearly independent in $N^1(X)$. Since there are $\rho - 1$ of them, and $\rho = \rho(X)$ by the isomorphism $N^1(X) \cong N^1(Y)$, this implies that $R_X$ is an extremal ray of $\overline{\text{NE}}(X)$.

So, let $\phi: X \to S$ be the Mori contraction of $R_X$. By taking the Stein factorization of the restricted morphism $\phi|_Y: Y \to S$, we obtain a commutative diagram

$$
\begin{array}{ccc}
Y & \xrightarrow{\phi} & X \\
\downarrow{\psi} & & \downarrow{\phi} \\
T & \xrightarrow{\delta} & S,
\end{array}
$$

where $T$ is normal, $\psi$ is surjective with connected fibers, and $\delta$ is finite. We observe that an irreducible curve on $Y$ is contracted by $\psi$ if and only if it is numerically proportional (in $X$, and hence in $Y$) to $C$, and therefore if and only if it is contracted by $\pi$. This implies that $T = Z$ and $\psi = \pi$. Moreover, by (11), we see that $Y$ meets every fiber of $\phi$, and therefore $\delta$ is surjective. This completes the proof of the theorem. \hfill \Box

Remark 4.2. Theorem 4.1 can be generalized, with minor changes, to the case of contractions of $K_Y$-negative extremal faces of $\text{NE}(Y)$.

Remark 4.3. A result analogous to Theorem 4.1 is obtained in [25, Proposition 5] in a more restrictive context (in the setting of [25], $Y$ is assumed to be the zero locus of a regular section of an ample vector bundle on $X$, with $\dim X \geq 4$, and $\delta$ is proven to be an isomorphism). Let us note that the proof of [25, Proposition 5] makes use in an essential way of the fact that the divisor $D$ (in our and in his notation as well) is an adjoint divisor, i.e., a divisor of the form $D = K_X + A$ for some ample line bundle $A$ on $X$. This circumstance seems not to be true in general. The gap was pointed out by Paltin Ionescu. Our arguments fill up that gap. A similar discussion is given in [2] in the case $Y$ is an ample divisor on $X$. In connection with Theorem 4.1 we should also mention [1, Theorem 3.4] and [9, Lemma (1.4)].

We close this section with the following elementary property on extensions of relative polarizations, which applies in particular to the setting of Theorem 4.1.

Proposition 4.4. Let

$$
\begin{array}{ccc}
Y & \xrightarrow{i} & X \\
\downarrow{\pi} & & \downarrow{\phi} \\
Z & \xrightarrow{\phi} & S
\end{array}
$$
be a commutative diagram of morphisms of projective varieties, and assume that \( i \) is an embedding, that \( \pi \) is not a finite morphism, and that \( \rho(X/S) = 1 \). Let \( M \) be a line bundle on \( X \) whose restriction \( M|_Y \) to \( Y \) is \( \pi \)-ample. Then there exists an ample line bundle \( H \) on \( X \) such that \( H|_F \cong M|_F \) for every fiber \( F \) of \( \pi \).

**Proof.** Since the numerical class of a curve in a positive dimensional fiber of \( \pi \) spans \( N_1(X/S) \), it follows from the relative Kleiman criterion of ampleness (see, e.g., [17, Theorem 1.4]) that \( M \) is \( \phi \)-ample. Then the line bundle \( H := M + \phi^*(kA) \) is ample if \( A \) is an ample line bundle on \( S \) and \( k \gg 0 \) (see, e.g., [22, Proposition 1.7.10]), and \( H|_F \cong M|_F \) for every fiber of \( \pi \). \( \Box \)

5. Projective bundles and quadric fibrations as ample subvarieties

The results obtained in the previous sections are here applied to classify, under suitable conditions, projective manifolds containing “ample subvarieties” that admit a structure of projective bundles or quadric fibrations.

Here we will adopt the following definitions of projective bundles and quadric fibrations. We remark that there are in fact several slightly different notions of quadric fibrations in the literature, and the one adopted here is a little more restrictive than others (see Remark 5.2 below).

**Definition 5.1.** A surjective morphism \( \pi: Y \to Z \) between smooth projective varieties is said to be a \( \mathbb{P}^m \)-bundle (resp., a \( \mathbb{Q}^m \)-fibration) if \( \pi \) is an extremal Mori contraction and there is a line bundle \( L \) on \( Y \) such that every fiber \( F \) of \( \pi \) is mapped isomorphically to \( \mathbb{P}^m \) (resp., is embedded as an irreducible and reduced quadric hypersurface of \( \mathbb{P}^{m+1} \)) via the complete linear system \( |L|_F \).

**Remark 5.2.** With the above definition, a morphism \( \pi: Y \to Z \) is a \( \mathbb{P}^m \)-bundle if and only if \( Y \cong \mathbb{P}_Z(F) \) for some vector bundle \( F \) of rank \( m+1 \) on \( Z \); we can take as \( L \) the tautological line bundle of \( F \) on \( Y \). On the other hand, not all scrolls (resp., quadric fibrations) in the adjunction theoretic sense (cf. [3, Sections 3.3, 14.1, 14.2]) are \( \mathbb{P}^m \)-bundles (resp., \( \mathbb{Q}^m \)-fibrations) in our sense. In fact, our definition of \( \mathbb{Q}^m \)-fibration does not include conic bundles with singular fibers, since we require all fibers to be irreducible; moreover, the assumption that \( \pi \) is an extremal Mori contraction implies that \( \rho(Y/Z) = 1 \), which excludes (as we will see in the proof of Lemma 5.3) those fibrations with all fibers isomorphic to \( \mathbb{Q}^2 \) for which the fundamental group of the base \( Z \) acts with trivial monodromy. Note that the general fiber of a \( \mathbb{Q}^m \)-fibration is isomorphic to \( \mathbb{Q}^m \).

**Lemma 5.3.** Let \( \pi: Y \to Z \) be either a \( \mathbb{P}^m \)-bundle or a \( \mathbb{Q}^m \)-fibration, and let \( W \) be the family of rational curves on \( Y \) generated by the lines in the fibers of \( \pi \). Then \( W \) is an irreducible family.

**Proof.** The assertion is clear in all cases except when \( \pi \) is a \( \mathbb{Q}^2 \)-fibration. In this case, a straightforward extension of the arguments in the proof of [8, Proposition 1.3.1]) shows two things. First of all, \( \rho(Y/Z) \neq 1 \) if and only if all fibers of \( \pi \) are isomorphic to \( \mathbb{Q}^2 \) and the fundamental group of \( Z \) acts with trivial monodromy. Moreover, in all remaining cases, the lines in the fibers generate an irreducible family. The fact that the base is one-dimensional in the setting considered in [8] is not essential in the arguments. \( \Box \)
**Definition 5.4.** An embedding $\mathbb{P}^m \hookrightarrow \mathbb{P}^n$ is said to be linear if the image of $\mathbb{P}^m$ is a linear subspace of $\mathbb{P}^n$. Similarly, an embedding of irreducible varieties $F \subset G$, with $F$ isomorphic to a quadric of $\mathbb{P}^{m+1}$ and $G$ isomorphic either to $\mathbb{P}^n$ or to a quadric in $\mathbb{P}^{n+1}$, is said to be linear if $\mathcal{O}_G(1)|_F \cong \mathcal{O}_F(1)$.

For the reminder of this section, we consider the following setting. Let $X$ be a smooth projective variety, and assume that $Y \subset X$ is a positive dimensional submanifold with ample normal bundle $N_{Y/X}$. An easy application of a well-known theorem of Kobayashi and Ochiai gives us the following property, in which the case when $Y$ is a projective space or a smooth quadric of dimension $\geq 3$ is considered.

**Proposition 5.5.** With the above notation, assume that the inclusion $i: Y \hookrightarrow X$ induces an isomorphism $i^*: \text{Pic}(X) \to \text{Pic}(Y)$, and denote $n := \dim X$. If $Y \cong \mathbb{P}^m$ for some $m \geq 1$, then $X \cong \mathbb{P}^n$ and $Y$ is linearly embedded in $X$. If $Y \cong \mathbb{Q}^m$ for some $m \geq 3$, then either $X \cong \mathbb{P}^n$ or $X \cong \mathbb{Q}^n$, and $Y$ is linearly embedded in $X$.

**Proof.** Suppose first that $Y \cong \mathbb{P}^m$. By the assumption, we have $\rho(X) = 1$, and the ample generator $\mathcal{O}_X(1)$ of $\text{Pic}(X)$ satisfies $\mathcal{O}_X(1)|_Y \cong \mathcal{O}_Y(1)$. Observe that

$$d := \deg N_{Y/X} \geq \text{rk} N_{Y/X} = n - m.$$ 

By adjunction, we have $\mathcal{O}_X(-K_X) \cong \mathcal{O}_X(m + 1 + d)$, and therefore $X$ is a Fano manifold of index $\geq n + 1$. Then the assertion follows from Kobayashi and Ochiai’s theorem (see e.g., [13], (3.1.6)). The argument for the case when $Y \cong \mathbb{Q}^m$ with $m \geq 3$ is analogous. \(\square\)

**Remark 5.6.** The case when $Y \cong \mathbb{Q}^2$ is harder to deal with under such weak hypotheses. However, if we additionally assume the existence of an ample line bundle $H$ on $X$ such that $H|_Y \cong \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, 1)$, then this case cannot occur. If at the same time we relax the hypothesis on the Picard groups, and only assume that $i^*: \text{Pic}(X) \to \text{Pic}(Y)$ is injective, then it is easy to see that necessarily $\rho(X) = 1$, either $X \cong \mathbb{P}^n$ or $X \cong \mathbb{Q}^n$, and $Y$ is linearly embedded in $X$. Indeed, to exclude the case $\rho(X) = 2$, it suffices to apply Theorem [4.1] with the family $W$ being either one of the two rulings of $\mathbb{Q}^2$. In this way one obtains two distinct Mori contractions of $X$ whose relative dimensions, added together, exceed the dimension of $X$, but this is impossible.

Let us now recall the following extension of the Lefschetz hyperplane theorem, essentially due to Sommese [27, Proposition 1.16] (see also [19, Theorem 1.1]), which will be used in the proof of case (b) of Theorem 5.8 below.

**Proposition 5.7.** Let $X$ be a projective manifold, and suppose that $Y \subset X$ is a submanifold defined by a regular section of an ample vector bundle on $X$. Suppose that $\dim Y \geq 3$. Then the inclusion $i: Y \hookrightarrow X$ induces an isomorphism $i^*: \text{Pic}(X) \to \text{Pic}(Y)$.

We now address the case when $Y$ is a $\mathbb{P}^m$-bundle or a $\mathbb{Q}^m$-fibration over a smooth projective variety. The following theorem (which includes Proposition 5.5 as a very special case) is the main result of this section.

**Theorem 5.8.** Let $Y$ be a submanifold of a projective manifold $X$ with ample normal bundle. Assume that $Y$ admits either a $\mathbb{P}^m$-bundle or a $\mathbb{Q}^m$-fibration structure $\pi: Y \to Z$ over a smooth projective variety $Z$ (see Definition 5.1), and that $Y$ has codimension

$$\text{codim}_X Y < \dim Y - \dim Z.$$
Further suppose that one of the following two situations occurs:

(a) $Z$ is a curve and the inclusion $i: Y \hookrightarrow X$ induces an isomorphism $\text{Pic}(X) \cong \text{Pic}(Y)$; or

(b) $Y$ is the zero scheme of a regular section of an ample vector bundle $E$ on $X$.

Then $\pi$ extends to a morphism $\tilde{\pi}: X \to Z$, that is, there is a commutative diagram

\[
\begin{array}{ccc}
Y & \xrightarrow{i} & X \\
\downarrow{\pi} & & \downarrow{\tilde{\pi}} \\
Z & \xrightarrow{\delta} & S,
\end{array}
\]

Moreover, letting $n := m + \text{codim}_X Y$, the following holds:

- if $\pi$ is a $\mathbb{P}^m$-bundle, then $\tilde{\pi}$ is a $\mathbb{P}^n$-bundle; and
- if $\pi$ is a $\mathbb{Q}^m$-fibration, then $\tilde{\pi}$ is either a $\mathbb{P}^n$-bundle or a $\mathbb{Q}^n$-fibration.

In either case, the fibers of $\pi$ are linearly embedded in the fibers of $\tilde{\pi}$.

**Proof.** For clarity of exposition, we discuss case (a) and case (b) separately, as certain steps require different arguments.

Case (a). Let $W$ be the family of rational curves on $Y$ generated by the lines in the fibers of $\pi$. Note that $W$ is an irreducible family by Lemma 5.3, and that $\pi$ is the $\text{RC}_W$-fibration. By Propositions 2.6 and 2.26, $\text{Hom}_{\text{bir}}(\mathbb{P}^1, X)$ is smooth at the generic point of $i_*(W)$. Let $V = \langle i_*(W) \rangle$ (note that $V$ is irreducible). By Proposition 2.20, the rational fibration defined by the restriction $V_Y$ of $V$ to $Y$ coincides with $\pi$. Then, by Corollary 3.13, we obtain the commutative diagram

\[
\begin{array}{ccc}
Y & \xrightarrow{i} & X \\
\downarrow{\pi} & & \downarrow{\phi} \\
Z & \xrightarrow{\delta} & S,
\end{array}
\]

where $\phi$ is the $\text{RC}_Y$-fibration and $\delta$ is a finite morphism of smooth projective curves.

By definition, there is a line bundle $L$ on $Y$ such that $L|_F \cong \mathcal{O}_F(1)$ for every fiber $F$ of $\pi$. Since $L$ extends to a line bundle on $X$ and $\rho(X/S) = 1$, Proposition 4.4 implies that there exists an ample line bundle $H$ on $X$ such that $H|_F \cong \mathcal{O}_F(1)$ for every $F$.

If $C$ is a line in a fiber of $\pi$ and we set $b = n$ if $Y$ is a $\mathbb{P}^m$-bundle and $b = n - 1$ if $Y$ is a $\mathbb{Q}^m$-fibration, then we have

\[(K_X + bH) \cdot C = (K_Y + bH|_Y) \cdot C - \det N_{Y/X} \cdot C < 0.\]

It thus follows from the classification of polarized manifolds with large nef-value due to Fujita and Ionescu (see [11, Section 1] or [14, Section 1]) that $\phi: X \to S$ is either a $\mathbb{P}^n$-bundle or a fibration with fibers isomorphic to hyperquadrics in $\mathbb{P}^n+1$ and relative Picard number 1, and in fact, since $\dim X \geq 3$ and $\dim S = 1$, in the second case $X$ is a $\mathbb{Q}^n$-fibration.

To conclude, observing that the general fiber of $\phi$ is a homogeneous space, we deduce from Remark 3.9 that $\delta$ is in fact an isomorphism, and therefore we get a diagram as in the statement by setting $\tilde{\pi} := \delta^{-1} \circ \phi$. 
Case (b). The first step is to prove the existence of a diagram as in the statement. We will use different arguments according to the codimension of $Y$. For short, let $r := \text{rk} \mathcal{E}$. Note that $\text{codim}_{X} Y = r = n - m$.

If $r = 1$, then (12) implies that $\pi$ has relative dimension $m \geq 2$. Therefore in this case we can apply a general result of Sommese [26 Proposition III], which says that $\pi$ extends to a morphism $\overline{\pi}: X \to Z$.

We now assume that $r \geq 2$. Note that in this case $m \geq 3$ by (12). To extend $\pi$ in this situation, we will use the results from the previous sections. As in the proof of case (a), let $W$ be the irreducible family of rational curves on $Y$ generated by the lines in the fibers of $\pi$, and let $V$ be the unique irreducible component of $\text{Hom}_{bH}(\mathbb{P}^{1},X)$ containing the generic point of $i_{*}(W)$. By Proposition 2.20 the rational fibration defined by the restriction $V_{Y}$ of $V$ to $Y$ coincides with the one defined by $W$, that is, with $\pi$. Then, by Theorems 3.1 and 3.6 we obtain a commutative diagram

\[
\begin{array}{ccc}
Y & \xrightarrow{\phi} & X \\
\pi \downarrow & & \downarrow \phi \\
Z & \dashrightarrow & S,
\end{array}
\]

where $\phi$ is the rational quotient defined by $V$ and $\delta$ is a dominant and generically finite map. Let $G$ be a general fiber of $\phi$, let $F$ be one of the fibers of $\pi$ that is contained in $G$, and fix a line $C \subset F$ (we can assume that $G$ and $F$ are smooth). Note that $\dim G = n$.

**Lemma 5.9.** Assuming $r \geq 2$, the family $V$ is unsplit.

**Proof of the lemma.** Since the section of $\mathcal{E}$ defining $Y$ in $X$ restricts to a regular section of $\mathcal{E}|_{G}$ whose zero scheme is $F$, we can apply Proposition 5.7 to this setting. This implies that the inclusion of $F$ in $G$ induces an isomorphism $\text{Pic}(G) \cong \text{Pic}(F)$. Note that $\rho(F) = 1$, since $F$ is either a projective space or a quadric of dimension $m \geq 3$. Therefore we have $\rho(G) = 1$. Let $L$ be a line bundle on $Y$ such that $L|_{F} \cong \mathcal{O}_{F}(1)$, and let $\overline{L}$ be the extension of $L$ to $X$. We observe that $\overline{L}|_{G}$ is ample and $-K_{G} \equiv a\overline{L}|_{G}$ for some integer $a$, which is easily seen to be positive. Then, since $\overline{L}|_{G} \cdot C = 1$, we see that $G$ is a Fano manifold of index $a$. In particular, we have $-K_{G} \cdot C = a \leq n + 1$. Note that, if $X^{o} \subset X$ is as in Theorem 2.13 then we can assume that $G$ is a fiber of the morphism $X^{o} \to Z$, and hence $K_{X}|_{G} = K_{X^{o}}|_{G} = K_{G}$. Therefore

\[
\det \mathcal{E} \cdot C = K_{Y} \cdot C - K_{X} \cdot C = K_{F} \cdot C - K_{G} \cdot C \leq -m + n + 1 = r + 1.
\]

Since we are assuming that $r \geq 2$, this implies that $\det \mathcal{E} \cdot C < 2r$. We conclude that the family $V$ is unsplit by Remark 2.11. \qed

We come back to the proof of the theorem, still assuming for the moment that $r \geq 2$. Since $V$ is unsplit by Lemma 5.9, we can apply Theorem 4.11. This implies that, in the above diagram, both $\phi$ and $\delta$ are morphisms and $\delta$ is finite, and moreover $\rho(X/S) = 1$. Note in particular that $\text{rk} \text{Im} \left( \text{Pic}(X) \to \text{Pic}(G) \right) = 1$. We have $\dim G = n$. We set $b = n$ if $F \cong \mathbb{P}^{m}$, and $b = n - 1$ if $F \cong \mathbb{Q}^{m}$. Denoting by $C$ a line in a general fiber of $\pi$, we obtain

\[
(K_{G} + bH|_{G}) \cdot C = (K_{F} + bH|_{F}) \cdot C - \text{det} N_{F/G} \cdot C < 0,
\]
since \( N_{F/G} \cong N_{Y/X} |_{F} \) is ample of rank \( n - m \). By applying again [11, 14] and taking into account that \( \rho(X/S) = 1 \), we deduce that either \( (G, H |_{G}) \cong (\mathbb{P}^{n}, \mathbb{O}_{\mathbb{P}^{n}}(1)) \) or \( (G, H |_{G}) \cong (\mathbb{Q}^{m}, \mathbb{O}_{\mathbb{Q}^{m}}(1)) \), with the second case occurring only if \( F \cong \mathbb{Q}^{m} \). Since the general fiber \( G \) of \( \pi \) is a homogeneous space, we deduce from Remark 3.9 that \( \pi \) is equidimensional with irreducible and reduced fibers. To see this, let \( G_{z} := \pi^{-1}(z) \) for an arbitrary \( z \in Z \), and let \( \Delta \) be any component of \( G_{z} \). Note that \( \Delta \cap Y \) is contained in the fiber \( F_{z} := \pi^{-1}(z) \) of \( \pi \), which is irreducible and reduced of dimension \( m \); in particular, we have \( \dim(\Delta \cap Y) \leq m \). Since \( \dim \Delta \geq n \) and \( \Delta \cap Y \) is defined by a section of \( E|_{\Delta} \), we have \( \dim(\Delta \cap Y) \geq \dim \Delta - r \). We conclude that \( \dim \Delta = n \) so that \( \pi \) is equidimensional, and \( F_{z} \subset \Delta \). Recalling that \( F_{z} \) is irreducible and reduced, and cut out on \( Y \) by \( G_{z} \) (scheme theoretically), we see that \( G_{z} \) is also irreducible and reduced. Since the fibers of \( \pi \) are irreducible and reduced, we can apply the semi-continuity of the \( \Delta \)-genus [10 Section 5 and (2.1)]. We conclude that \( \pi \) is either a \( \mathbb{P}^{n} \)-bundle or a \( \mathbb{Q}^{m} \)-fibration over \( Z \). \( \square \)

We note that all cases in Theorem 5.8 are effective.

Remark 5.10. We already listed in the introduction a series of works related to Theorem 5.8. The classical setting when \( Y \) is an ample divisor is widely studied in the literature; for a survey and complete references we refer to [3, Chapter 5] and [2]. In the case when \( Y \) has codimension \( r \geq 2 \), we are not aware of other results of this type in which the base of the fibration is allowed to be arbitrary.

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1 Alternatively, one can deduce this fact from [27 Proposition 1.16]. Indeed this result, applied to \( G \cap Y \) once this is thought of as a subvariety of \( G \) defined by a regular section of \( E|_{G} \), gives the isomorphism \( H^{0}(G \cap Y, \mathbb{Z}) \cong H^{0}(G, \mathbb{Z}) \), and this implies that \( G \cap Y \) is connected.
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