Optimal measurements in phase estimation: simple examples

Tomasz Wasak1 · Augusto Smerzi2 · Luca Pezzé2 · Jan Chwedeńczuk1

Abstract We identify optimal measurement strategies for phase estimation in different scenarios in which the interferometer acts on two-mode symmetric states. For pure states of a single qubit, we show that optimal measurements form a broad set parametrized with a continuous variable. When the state is mixed, this set reduces to merely two possible measurements. For two-qubit symmetric Werner state, we find the optimal measurement and show that estimation from the population imbalance is optimal only if the state is pure. We also determine the optimal measurements for a wide class of symmetric $N$-qubit Werner-like states. Finally, for a pure symmetric state of $N$ qubits, we find under which conditions the estimation from the full $N$-body correlation and from the population imbalance is optimal.

Keywords Estimation theory · Quantum interferometry · Optimal measurements · Entanglement

1 Introduction

Quantum interferometry aims at estimating an unknown phase value $\theta$ with the smallest possible uncertainty $\Delta \theta$, taking advantage of quantum correlations in the probe state. The phase estimation protocol generally consists of three steps (for recent reviews on quantum-enhanced phase estimation see [1–3]). First is the preparation of $N$ particles...
in an input (probe) state $\hat{\rho}_0$, which then undergoes the phase-dependent transformation (determined by the choice of the interferometer). Finally, a measurement, which is in general described by the quantum-mechanical positive-operator-valued measures (POVMs), is carried out at the output. Usually, the protocol is repeated $m$ times and the phase shift is inferred using an estimator $\theta_{\text{est}}$, which is a properly chosen function of the collected results. There are good and obvious bad choices of the estimator. The good ones are those which are unbiased and minimize the phase sensitivity $\Delta\theta$, defined here as the statistical mean square fluctuation of the estimator. The Cramér–Rao theorem sets a lower bound on the sensitivity of any estimator $[4,5]$, i.e., $\Delta\theta \geq \Delta\theta_{\text{CR}}$. For unbiased estimators and independent measurements, we have

$$\Delta\theta_{\text{CR}} = \frac{1}{\sqrt{mF}},$$

(1)

where $F$ is the Fisher information (FI), which depends on all the three steps of the interferometric sequence $[5]$. Therefore, the Fisher information tells how information about $\theta$ can be deduced from the particular measurement. The Cramér–Rao lower bound (CRLB) from Eq. (1) is saturated by the maximum-likelihood estimator when the number of measurements tends to infinity $[5]$.

A particularly important value of the FI is expressed in terms of the number of particles $N$ passing through the interferometer. For separable states, the FI is bounded by the shot-noise limit (SNL), i.e., $F \leq N$. In such case, according to the CRLB the phase sensitivity is not better then $\Delta\theta \geq \frac{1}{\sqrt{mN}}$ $[6]$. The SNL is not fundamental $[3,6–10]$ and can be surpassed when the particles are prepared in an entangled state $[10]$. Moreover, for a fixed $N$, the value of the FI may raise by increasing the amount of particle entanglement in the probe $[11,12]$. The ultimate bound is the Heisenberg limit (HL) $F = N^2$ giving $\Delta\theta \geq \frac{1}{\sqrt{mN}}$, reached when the phase is imprinted on the maximally entangled NOON state $|\psi\rangle = \frac{1}{\sqrt{2}}(|N0\rangle + |0N\rangle)$ $[6]$. Recent experiments achieve sensitivities below the SNL with entangled ions $[13–16]$, photons $[17–20]$, cold $[21–23]$ and ultracold $[24–29]$ atoms.

Note that once the input state and the interferometric transformation are fixed, the value of the FI can be optimized with respect to all possible POVMs. This maximal value is called the quantum Fisher information (QFI) $[4,30]$ and will be denoted by $F_Q$ throughout this manuscript. In some situations, especially when the input state is pure, the QFI has a simple form $[30,31]$. In such case the optimal measurements can be identified: They are related to the projections over the output state and over the orthogonal subspace of the Hilbert space. On the other hand, for mixed input states it is usually very difficult to tell which is the optimal measurement.

In this work we identify the optimal measurements in different two-mode interferometric systems. We start in Sect. 2 with the simplest possible two-mode object: The single qubit rotated in the Bloch sphere by an unknown angle $\theta$. Interestingly, while for pure states we find a continuous class of optimal POVMs, they boil down to only two phase-dependent possibilities when $\hat{\rho}_0$ represents a mixed state. For symmetric states of two qubits, general considerations are not possible and, in Sect. 3.1, we focus on a particular example, namely the symmetric Werner state. We find the
optimal POVM and discuss the precision reached when the phase is estimated from the measurement of the population imbalance between the two modes. In Sect. 3.2 we consider an extension of the Werner-like states to higher $N$ and in this case find the optimal measurements. Finally, in Sect. 4 we show under which conditions the estimation from the population imbalance or the $N$-th-order correlation function is optimal, using symmetric $N$-qubit pure states as input. We conclude in Sect. 5.

2 Single qubit

We begin our analysis with the basic two-mode object, which is a single qubit. Its density matrix is represented by a combination of Pauli matrices $\hat{\sigma}$ (hereafter bold symbols will indicate vectors)

$$\hat{\rho}_0 = \frac{1}{2} (\hat{1} + s_0 \cdot \hat{\sigma}).$$  \hspace{1cm} (2)

Depending on the length of the vector $s_0$, the state is either pure (if $|s_0| = 1$) or mixed ($|s_0| < 1$). We now consider a rotation of the density matrix $\hat{\rho}_0$ by an angle $\theta$ around the y-axis

$$\hat{U}_{\text{mzi}}(\theta) = e^{-i\theta \hat{\sigma}_y}.$$  \hspace{1cm} (3)

The choice of such transformation is dictated by the fact that it is commonly implemented in experiments. Namely, when the two modes are linked with the external degrees of freedom, the operator (3) represents the Mach–Zehnder interferometer (MZI) \cite{7,8} (i.e., two beam splitters separated by the phase imprint). For manipulations between internal degrees of freedom, such transformation is called the Ramsey interferometer (where the beam splitters are realized by means of $\frac{\pi}{2}$-pulses). The output density matrix reads

$$\hat{\rho} = \hat{U}_{\text{mzi}}(\theta) \hat{\rho}_0 \hat{U}_{\text{mzi}}^\dagger(\theta) = \frac{1}{2} (\hat{1} + s \cdot \hat{\sigma}),$$  \hspace{1cm} (4)

where the three components of the rotated vector are

$$s_x = s_{0,x} \cos \theta + s_{0,z} \sin \theta$$ \hspace{1cm} (5a)  

$$s_y = s_{0,y}$$ \hspace{1cm} (5b)  

$$s_z = s_{0,z} \cos \theta - s_{0,x} \sin \theta.$$ \hspace{1cm} (5c)

Note that we have omitted the explicit dependence of $\hat{\rho}$ and $s$ on the phase $\theta$—in order to simplify the notation. The task is to determine the value of the angle $\theta$ with the precision maximized with respect to all possible measurements. As we show below, for a single qubit a full family of optimal measurements can be identified, both for pure and mixed states.
2.1 Classical and quantum Fisher information

The broadest family of measurement allowed by quantum mechanics is related to POVMs, which include the commonly used projective measurements \cite{4}. A POVM is formed by a complete set of self-adjoint operators with nonnegative eigenvalues. For a single qubit, any POVM is represented by the following family of operators

\[ \hat{E}_q = \gamma_q \left( \mathbb{1} + q \cdot \hat{\sigma} \right), \]  

parametrized by \( q \). The conditions

\[ |q| \leq 1, \quad \text{and} \quad \gamma_q \geq 0, \]  

guarantee that \( \hat{E}_q \) is nonnegative definite, while

\[ \int dq \gamma_q = 1, \quad \text{and} \quad \int dq \gamma_q q = 0, \]  

ensures the completeness of the POVM set, i.e., \( \int dq \hat{E}_q = \mathbb{1} \). The following trace

\[ p(q|\theta) = \text{Tr}[\hat{\rho}, \hat{E}_q] \]  

is in fact the probability for finding the qubit aligned along the direction \( q \).

The FI, which measures the amount of information about \( \theta \) contained in \( p(q|\theta) \), reads

\[ F = \int dq \frac{1}{p(q|\theta)} \left( \frac{\partial p(q|\theta)}{\partial \theta} \right)^2. \]  

Clearly, through Eq. (9), it depends on the choice of the POVM. For a given probe state and interferometric transformation, some output measurements are better than others as they give a larger FI (for an experimental demonstration see Ref. \cite{32}). The optimal measurements are those which give the maximal value of the FI. The maximization procedure of (10) can be performed analytically \cite{30} and the resulting QFI for unitary transformations is

\[ F_Q = 2 \sum_{j,k} \left(\frac{p_j - p_k}{p_j + p_k}\right)^2 |\langle j|\hat{\sigma}_y|k\rangle|^2. \]  

The ket \( |k\rangle \) denotes the eigenvector of the density matrix \( \hat{\rho} \) with eigenvalue \( p_k \), i.e., \( \hat{\rho} = \sum_k p_k |k\rangle\langle k| \), with \( p_k \geq 0, \sum_k |k\rangle\langle k| = \mathbb{1} \). The sum in Eq. (11) extends over values \( p_j + p_k \neq 0 \). For the matrix \( \hat{\rho} \) from Eq. (4) we obtain

\[ F_Q = s_x^2 + s_z^2, \]  

\( \otimes \) Springer
which does not depend on $\theta$. The maximum $F_Q = 1$ is for pure states ($|s| = 1$ giving $F_Q = 4(\Delta \hat{\sigma}_y)^2$) with a vanishing component along the rotation, $s_y = 0$.

### 2.2 Optimal measurements

Next, we determine the optimal measurements for which the FI from Eq. (10) equals the QFI from Eq. (12). As shown in [30], a POVM is optimal if and only if it satisfies the condition

$$\hat{E}_q \hat{\rho} = \lambda_q \hat{E}_q \hat{L}_\hat{\rho} \hat{\rho}$$

with $\lambda_q \in \mathbb{R}$. The super-operator $\hat{L}_\hat{\rho}$ is the symmetric logarithmic derivative, defined by the relation

$$\frac{\partial \hat{\rho}}{\partial \theta} = \hat{\rho} \hat{L}_\hat{\rho} + \hat{\rho} \hat{L}_\hat{\rho} \hat{\rho}.$$  

When $\hat{\rho}$ can be inverted (which is not the case of pure states), we multiply Eq. (13) by $\hat{\rho}^{-1}$ and get

$$\hat{E}_q \left( \hat{1} - \lambda_q \hat{L}_\hat{\rho} \right) = 0,$$

which is fulfilled if the POVM $\hat{E}_q$ is constructed from the projectors over the eigenstates of $\hat{L}_\hat{\rho}$ with $\lambda_q$ equal to the inverse of the corresponding real eigenvalue.

In all cases, the optimal measurements are found in two steps. First, we write down the symmetric logarithmic derivative (SLD) in its general form $\hat{L}_\hat{\rho} = \hat{1} \ell + \hat{n} \cdot \hat{\sigma}$. Here $\ell$ is a real number and $\hat{n}$ is a three-dimensional vector. Using Eqs. (4) and (14), we obtain the expressions for $\ell$ and $\hat{n}$. The result is that $\ell = 0$, while $\hat{n} = s_x e_y - s_y e_x \equiv s_\perp$. Here $e_x, e_y$ and $e_z$ are the three orthogonal unit vectors and $s \cdot s_\perp = 0$. Therefore, the SLD reads

$$\hat{L}_\hat{\rho} = s_\perp \cdot \hat{\sigma}.$$  

In the second step, we insert $\hat{L}_\hat{\rho}$ into Eq. (13) and use the general parametrization of the POVM, Eq. (6). By comparing the scalar and vector parts and then the real and imaginary parts, we obtain the set of equations

\begin{align}
q \cdot (s_\perp \times s) &= 0 \quad (17a) \\
\lambda_q &= \frac{1 + q \cdot s}{q \cdot s_\perp} \quad (17b) \\
q + s &= \lambda_q \left[ s_\perp - q \times (s_\perp \times s) \right] \quad (17c) \\
q \times s &= \lambda_q \left[ s_\perp \times s + q \times s_\perp \right]. \quad (17d)
\end{align}

From Eq. (17a) we deduce that $q$ lies in the plane spanned by vectors $s$ and $s_\perp$, so it can be written as $q = q_1 e_s + q_2 e_{s_\perp}$. Here, $e_s$ and $e_{s_\perp}$ are unit vectors pointing into
Fig. 1  Schematic representation of the directions of vectors $q$ for which the POVM set (6) is optimal. The vector $s_\perp$ is orthogonal to $s$ and both lie in the $x-z$ plane. For pure states the elements of the optimal POVM can be arbitrary chosen in the unit (gray) circle. For mixed states the optimal POVM set is made only by the two vectors on the circle, $q = \pm e_{s_\perp}$, indicated by the black dots.

directions $s$ and $s_\perp$, respectively. Both vectors are orthogonal to the rotation direction $e_y$. These observations reduce the set of eight Equations (17) to

$$\lambda_q = \frac{1 + sq_1}{s_\perp q_2} \quad (18a)$$

$$q_1 + s = \lambda_q s_\perp q_2 \quad (18b)$$

$$q_2 s = \lambda_q s_\perp (s - q_1) \quad (18c)$$

$$q_2 = \lambda_q s_\perp (1 - sq_1), \quad (18d)$$

where $s = |s|$ and $s_\perp = |s_\perp|$. This set of four equations for the three variables $q_1$, $q_2$ and $\lambda_q$ is non-contradictory when two of these equations are linearly dependent.

If the state is pure ($s = 1$), Eqs. (18c) and (18d) are equivalent and the solution is $q_1^2 + q_2^2 = 1$. Thus for pure states there are several choices of optimal POVM. The obvious choice is to take the projection over the output state ($q_1 = 1$, $q_2 = 0$) and over the orthogonal vector ($q_1 = 0$, $q_2 = 1$), with $\gamma_q = 1/2$. However, we can also consider POVMs with two elements along different directions (still satisfying $q_1^2 + q_2^2 = 1$), or even a continuous set of POVMs parametrized by a vector $q$ which lies on a circle of unit radius.

If the state is mixed $(s < 1)$, Eqs. (18c) and (18d) are non-contradictory only if $q_1 = 0$ and the other two equations give $q_2 = \pm 1$. There is only one optimal POVM and its elements are $(\hat{1} \pm e_{s_\perp} \cdot \hat{\sigma})/2$. This optimal POVM corresponds to the projection over pure states pointing along the $\pm e_{s_\perp}$ direction in the Bloch sphere. These are the eigenstates of Eq. (16) with (real) eigenvalues $\pm s_\perp$. In other words, the POVM with elements $(\hat{1} \pm e_{s_\perp} \cdot \hat{\sigma})/2$ is necessary and sufficient to saturate the QFI for mixed state, while it is only sufficient for pure states (for which the density matrix is non invertible).

In conclusion, there is a substantial difference between the sets of optimal measurements for pure and mixed states. The possible choice of POVM in the continuous set for pure states reduces to just two projection operators for mixed states, as schematically shown in Fig. 1.
2.3 Estimation from the population imbalance

Population imbalance is a measurement, which is commonly used in experiments to determine the value of the interferometric phase. For a single qubit, this measurement is realized by verifying whether the particle is in the \(|+\rangle\) or in the \(|-\rangle\) state. Since the outcome of such measurement can depend on \(\theta\), it might be used to estimate the value of the interferometric phase. Here we show under which conditions this estimation strategy is optimal in the one-qubit case, when it corresponds to the projection over the eigenstates \(|\pm\rangle\) of \(\hat{\sigma}_z (\hat{\sigma}_z |\pm\rangle = \pm |\pm\rangle)\):

\[
\hat{E}_+ = |+\rangle\langle+| \quad \text{and} \quad \hat{E}_- = |-\rangle\langle-|.
\]  

(19)

The probability for detecting the qubit in \(|\pm\rangle\) is

\[
p_{\pm} = \text{Tr}[\hat{\rho}, \hat{E}_{\pm}] = \frac{1}{2} (1 \pm s_z).
\]  

(20)

According to Eq. (10) and using the \(\theta\)-dependence of the vector \(s\) from Eq. (4), we obtain that the FI for the estimation from the population imbalance is

\[
F_{\text{imb}} = \frac{1}{p_+} \left( \frac{\partial p_+}{\partial \theta} \right)^2 + \frac{1}{p_-} \left( \frac{\partial p_-}{\partial \theta} \right)^2 = \frac{s_x^2}{1 - s_z^2}.
\]  

(21)

This equation should be compared with Eq. (12). We obtain that \(F_{\text{imb}} = F_Q\) when either (a) \(s_x^2 + s_z^2 = 1\) or (b) \(s_z = 0\). The condition (a) is satisfied with pure states only. Moreover, for any initial pure state \(\hat{\rho}_0\) lying in the \(x-z\) plane and rotated along the \(y\) direction according to Eq. (3), the population imbalance is not only optimal but it also gives the maximum possible value \(F_{\text{imb}} = 1\), independently from \(\theta\). The condition (b) can be fulfilled both by mixed and pure states. However, \(F_{\text{imb}} = s_x^2\) depends on \(\theta\) and \(F_{\text{imb}} = 1\) only for pure states and a specific rotation angle \(\tan \theta = s_{0,z}/s_{0,x}\). When the states are mixed, \(F_{\text{imb}} < 1\) and for every \(\theta\) there is only one orientation of \(\hat{\rho}_0\), which gives \(s_z = 0\).

This once again shows how the optimal estimation strategies change abruptly when we switch between mixed and pure states. There is a continuum of pure states, which used for the estimation from the population imbalance saturate the QFI, but there is only one such mixed state for each \(\theta\).

3 Symmetric Werner states

In this section, we extend the investigation of the optimal POVM to higher \(N\). However, since general considerations are no longer possible, we restrict our analysis to a particular family of states. We begin with the two-qubit symmetric Werner state. Then, we extrapolate these results to higher number of qubits. Finally, basing on the structure of the Werner states, we construct a model for a a noisy NOON state, for which we determine the optimal measurements.
3.1 Two qubits: symmetric Werner states

A general symmetric density matrix of two qubits has eight independent real parameters and is too difficult to treat. We thus restrict our analysis to the “Werner” states [33], which are described with one real coefficient $0 \leq \alpha \leq 1$:

$$\hat{\rho}_w = \frac{1 - \alpha}{3} \mathbb{1} + \alpha \hat{N}_{TF}. \quad (22)$$

Here $\hat{N}_{TF}$ is the projection over the twin-Fock (TF) state $|1, 1\rangle$, where each mode occupied by one particle. When $\alpha$ varies from 0 to 1, $\hat{\rho}_w$ changes from a complete mixture which is useless for parameter estimation, to a strongly entangled pure TF state, which is known to provide a sub-shot-noise sensitivity (SSN) in the MZI [34].

The Werner state written in the mode occupation basis ($|j, 2-j\rangle$, with $j = 0, 1, 2$) is diagonal

$$\hat{\rho}_w = \begin{pmatrix}
\frac{1 - \alpha}{3} & 0 & 0 \\
0 & \frac{1 + 2\alpha}{3} & 0 \\
0 & 0 & \frac{1 - \alpha}{3}
\end{pmatrix}. \quad (23)$$

This matrix is always invertible except from the case when the state is pure. Werner states (22) are a narrow subset of all possible two spin-$\frac{1}{2}$ symmetric states, nevertheless—as we show below—they provide valuable insight into the optimal estimation strategies in quantum metrology. We take a generic linear interferometric transformation

$$\hat{U}(\theta) = e^{-i\theta \hat{n} \cdot \hat{J}}, \quad (24)$$

where $\hat{J} = \frac{1}{2} \hat{\sigma}^{(1)} + \frac{1}{2} \hat{\sigma}^{(2)}$ is the “composite” pseudo-angular momentum operator, given by the sum of corresponding single-particle Pauli matrices (the upper index labels the particle).

First, we calculate the QFI, using the expression from Eq. (11), keeping in mind that the previously used generator of the interferometric transformation $\hat{\sigma}_y$ must be replaced by $\hat{n} \cdot \hat{J}$. The outcome is

$$F_Q(\alpha) = \frac{12\alpha^2}{2 + \alpha} \left( n_x^2 + n_y^2 \right) \quad (25)$$

Note that the $z$-component of the generator does not contribute to the QFI, because $\hat{\rho}_w$ is invariant upon rotation around the $z$ axis. The QFI depends only on the length of the projection of the vector $\hat{n}$ onto the $x-y$ plane. Thus, without any loss of generality, in the remaining of this section we will restrict to the MZI transformation $\hat{U}_{\text{mzi}}(\theta) = e^{-i\theta \hat{J}_y}$. The output density matrix $\hat{\rho}_{\text{mzi}}(\theta) = \hat{U}_{\text{mzi}}(\theta) \hat{\rho}_w \hat{U}_{\text{mzi}}^\dagger(\theta)$ reads
\[
\hat{\rho}_w(\theta) = \begin{pmatrix}
\frac{2-\alpha(2-3\sin^2\theta)}{6} & -\frac{\alpha\sin2\theta}{2\sqrt{2}} & -\frac{\alpha\sin^2\theta}{2} \\
-\frac{\alpha\sin2\theta}{2\sqrt{2}} & \frac{1-\alpha(1-3\cos^2\theta)}{3} & \frac{\alpha\sin2\theta}{2\sqrt{2}} \\
-\frac{\alpha\sin^2\theta}{2} & \frac{\alpha\sin2\theta}{2\sqrt{2}} & \frac{2-\alpha(2-3\sin^2\theta)}{6}
\end{pmatrix},
\]

and the QFI reduces to
\[
F_Q(\alpha) = \frac{12\alpha^2}{2+\alpha}.
\] (26)

As anticipated, if \( \alpha = 0 \) the state (22) is a complete mixture useless for phase estimation and \( F_Q(0) = 0 \). The SSN sensitivity \( F_Q(\alpha) > 2 \) is reached when \( \alpha > 2/3 \). In the extreme case \( \alpha = 1 \), we obtain the Heisenberg scaling, i.e., \( F_Q(1) = 4 \). We compare this result with the concurrence [35], the entanglement measure for two qubits, which tells that the state (22) is entangled already when \( \alpha > \frac{1}{4} \). This example confirms the known fact [10,36] that not all entangled states provide a SSN sensitivity.

Note, that the value of the QFI can be optimized by allowing independent rotations of both particles. In such case, the non-symmetric part of the Hilbert space must be included. We replace the transformation (24) with
\[
\hat{U}(\theta) = e^{-i\theta/2}(n_1\hat{\sigma}_1 + n_2\hat{\sigma}_2).
\] (27)

For the Werner state (22), the QFI can be calculated analytically for any \( \alpha \) and \( n_1, n_2 \). In a particular case when \( n_1 = e_z \) and \( n_2 = -e_z \) we obtain that
\[
F_Q(\alpha) = \frac{4}{3}(2\alpha + 1),
\] (28)

which exceeds the SNL \( F_Q(\alpha) > 2 \), when \( \alpha > \frac{1}{4} \). Therefore, if the physically allowed operations distinguish the particles, all entangled two-qubit states of the form (22) become useful for SSN metrology.

### 3.1.1 Optimal measurements

In the following, we go back to the symmetric case and find the optimal measurements for which the FI equals the QFI (26). In analogy to the single-qubit case, we should now parametrize the POVM similarly as in Eq. (6) and find the parameters from the condition Eq. (13). However, this procedure gives equations which cannot be solved analytically. Therefore, we restrict to the optimal POVMs obtained from the diagonalization of the SLD, which using Eq. (14) is
\[
\hat{L}_{\hat{\rho}_w(\theta)} = -\frac{6i\alpha}{2+\alpha}[\hat{J}_y, \hat{\rho}_w(\theta)].
\] (29)
We rewrite Eq. (29) in the matrix form
\[
\hat{L}_{\hat{\rho}_w(\theta)} = C_\alpha \begin{pmatrix}
\frac{1}{\sqrt{2}} \sin 2\theta & -\cos 2\theta & -\frac{1}{\sqrt{2}} \sin 2\theta \\
-\cos 2\theta & -\sqrt{2} \sin 2\theta & \cos 2\theta \\
-\frac{1}{\sqrt{2}} \sin 2\theta & \cos 2\theta & \frac{1}{\sqrt{2}} \sin 2\theta
\end{pmatrix}
\] (30)
where \( C_\alpha = \frac{6\alpha}{\sqrt{2(2+\alpha)}} \). A direct diagonalization provides the three (normalized) eigenstates
\[
|\Psi_1\rangle = \frac{(\cos \theta - \sin \theta)}{\sqrt{2}} |\psi_-\rangle - \frac{(\cos \theta + \sin \theta)}{\sqrt{2}} |1, 1\rangle
\]
(31a)
\[
|\Psi_2\rangle = \frac{(\cos \theta + \sin \theta)}{\sqrt{2}} |\psi_-\rangle + \frac{(\cos \theta - \sin \theta)}{\sqrt{2}} |1, 1\rangle
\]
(31b)
\[
|\Psi_3\rangle = |\psi_+\rangle,
\]
(31c)
where \(|\psi_\pm\rangle = \frac{|2, 0\rangle \pm |0, 2\rangle}{\sqrt{2}} \). The set of projectors over the eigenstates of \( \hat{L}_{\hat{\rho}_w(\theta)} \) forms an optimal POVM. They depend on \( \theta \) and have a complicated form\(^1\) but nevertheless do not depend on \( \alpha \). This last fact is a consequence of the structure of the Werner state (22). In this case the SLD is proportional to the commutator of the projection operator with the generator of the interferometric transformation, see Eq. (29). Since \( \alpha \) enters in Eq. (30) through the prefactor \( C_\alpha \), only the eigenvalues of the SLD—contrary to the eigenvectors—depend on \( \alpha \).

It is interesting to see explicitly that the FI calculated with the projectors on the states (31) saturates the QFI. We first calculate
\[
P_i(\phi) = \langle \Psi_i | e^{-i\phi \hat{J}_y} \hat{\rho}_w e^{+i\phi \hat{J}_y} |\Psi_i\rangle
\]
(32)
with \( i = 1, 2, 3 \) and obtain
\[
P_1(\phi) = \beta + \alpha \left[ \sin \left( \theta - \frac{\pi}{4} \right) \sin \phi - \sin \left( \theta + \frac{\pi}{4} \right) \cos \phi \right]^2,
\]
\[
P_2(\phi) = \beta + \alpha \left[ \sin \left( \theta + \frac{\pi}{4} \right) \sin \phi + \sin \left( \theta - \frac{\pi}{4} \right) \cos \phi \right]^2,
\]
\[
P_3(\phi) = \beta,
\]
\(^1\) Note that at \( \theta = 0 \), there is a simple way to reduce the optimal POVM to the simple measurement of population imbalance. Indeed, the following transformation
\[
\hat{V} = \exp \left( i \frac{\pi}{2} \hat{J}_x \hat{J}_y + \frac{\pi}{4} \hat{J}_x \hat{J}_y \right)
\]
applied to the states (31) gives \( \hat{V}|\Psi_1\rangle = |0, 2\rangle, \hat{V}|\Psi_2\rangle = |1, 1\rangle \) and \( \hat{V}|\Psi_3\rangle = |2, 0\rangle \). In this way, we obtain the eigenstates of the \( \hat{J}_z \) operator, and the optimal measurement is based on the determination of the population imbalance. Nevertheless, to accomplish this we needed an additional operation \( \hat{V} \) acting on the state. This transformation is non-local—it correlates the particles, since the product of two angular momentum operators cannot be written as a sum of operators acting on each qubit independently.
where $\beta = (1 - \alpha)/3$. For the specific phase value $\phi = \theta$, we have

$$F(\theta) = \sum_{i=1}^{3} \frac{1}{P_i(\phi = \theta)} \left( \frac{\partial P_i(\phi)}{\partial \phi} \bigg|_{\phi = \theta} \right)^2 = \frac{12\alpha^2}{2 + \alpha}, \quad (33)$$

as expected. It should be noticed that for $\alpha < 1$, when the density matrix is invertible, the projection over the states (31) is the unique optimal POVM. For $\alpha = 1$, there are other POVMs, besides the projection over the states (31), for which $F = F_Q$. An example is shown in the following section.

3.1.2 Estimation from the population imbalance

To this end we again focus on estimation from the measurement of the population imbalance between the two modes. The FI is

$$F_{imb} = \sum_{j=0}^{2} \frac{1}{p(j|\theta)} \left( \frac{\partial p(j|\theta)}{\partial \theta} \right)^2, \quad (34)$$

where $p(j|\theta)$ is a probability for finding $j$ particles in one of the modes and $2 - j$ in the other and is given by

$$p(j|\theta) = \langle j, 2 - j | \hat{\rho}_\alpha(\theta) | j, 2 - j \rangle. \quad (35)$$

A direct calculation shows that $p(0|\theta) = p(2|\theta) = \frac{1-\alpha}{3} + \frac{\alpha}{2} \sin^2 \theta$ and $p(1|\theta) = \frac{1-\alpha}{3} + \alpha \cos^2 \theta$. These, put into Eq. (34), give

$$F_{imb} = \frac{36 \alpha^2 \sin^2(2\theta)}{[4 - \alpha(1 + 3 \cos(2\theta))][2 + \alpha(1 + 3 \cos(2\theta))]} \quad (36)$$

In Fig. 2 we plot Eq. (36) as a function of $\theta$ and for different values of $\alpha$. When $\alpha = 1$ the Werner state is pure and we find $F_{imb} = 4$, independently from $\theta$. In this case
$F_{\text{imb}}$ saturates the QFI and reaches the maximum value attainable for two qubits. For $\alpha < 1$, the estimation from the population imbalance is generally non-optimal and depends strongly on $\theta$.

This is another example, after the single-qubit case, of how the estimation strategy, which is optimal for a pure state, immediately deteriorates as soon as the state becomes mixed. It also shows that, for pure states there are measurements that allow to saturate the QFI for any value of $\theta$. This highly desirable property is lost as soon as the state becomes mixed.

### 3.2 Generalization to higher N

Note that the above considerations can be generalized to higher $N$ as follows. First, basing on Eq. (22), we introduce a symmetric state of $N$ qubits

$$\hat{\rho} = \frac{1 - \alpha}{N + 1} \hat{1} + \alpha \hat{\Pi}_N. \quad (37)$$

Here, the unity operator acts in the $N + 1$-dimensional symmetrized space of $N$ qubits. Quite generally, $\hat{\Pi}_N$ projects onto any state within this space. The density matrix (37) undergoes a linear interferometric transformation governed by the evolution operator $\hat{U}(\theta) = e^{-i\theta \hat{h}}$,

$$\hat{\rho}(\theta) = \hat{U}(\theta) \hat{\rho} \hat{U}^\dagger(\theta). \quad (38)$$

Owing to the simple structure of the state (37), the QFI is easily evaluated and the outcome is

$$F_Q = \frac{\alpha^2(N + 1)}{2 + \alpha(N - 1)} 4(\Delta \hat{h})^2. \quad (39)$$

Here the variance of the generator is calculated using the state onto which $\hat{\Pi}_N$ projects. We now consider two distinct cases, one when $\hat{\Pi}_N$ projects onto the twin-Fock state, and the other when $\hat{\Pi}_N$ projects onto the NOON state. For these two examples, we determine the set of optimal measurements using the SLD, analogically to the approach presented in the previous Section.

The state (37) has a similar structure to (22)—it is a mixture of identity and the projection operator. Therefore, the derivation of the SLD is also analogous, and we obtain that

$$\hat{L} = \beta \left[ \hat{h}, \hat{\Pi}_N(\theta) \right], \quad (40)$$

with $\beta = \frac{-2i\alpha(N+1)}{2+\alpha(N-1)}$ and $\hat{\Pi}_N(\theta) = \hat{U}(\theta) \hat{\Pi}_N \hat{U}^\dagger(\theta)$. Since the evolution is generated by the $\hat{h}$ operator, we write the SLD as

$$\hat{L} = \hat{U}(\theta) \left( \beta \left[ \hat{h}, \hat{\Pi}_N \right] \right) \hat{U}^\dagger(\theta). \quad (41)$$
The optimal measurements are found in two steps. First, we evaluate the eigenstates of the matrix, which is the outcome of the above commutator $[\hat{h}, \hat{\Pi}_N]$. Then, we rotate the resulting states using the evolution operator. Clearly, this procedure yields the optimal projectors, which do not depend on $\alpha$, just as was the case of the two-particle Werner state.

The optimal POVM is constructed from those rotated eigenstates and projections onto all the remaining orthogonal states from the $(N + 1)$-dimensional space. Nevertheless, these remaining states do not contribute to the Fisher information, since their trace with the density matrix $\hat{\rho}(\theta)$ vanishes. Therefore, to find the maximal Fisher information, it is sufficient to determine only the projection operators onto the eigenstates of the SLD.

### 3.2.1 Twin-Fock state

First, we apply this procedure to find the optimal measurements for the “noisy” twin-Fock state. In this case, the operator $\hat{\Pi}_N$ projects onto the state $|\frac{N}{2}, \frac{N}{2}\rangle$. If the generator is equal to $\hat{h} = \hat{J}_y$, the commutator $[\hat{h}, \hat{\Pi}_N]$ in Eq. (41) is proportional to

$$\hat{C} = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad (42)$$

expressed in the basis $|\frac{N}{2} - 1, \frac{N}{2} + 1\rangle$, $|\frac{N}{2}, \frac{N}{2}\rangle$ and $|\frac{N}{2} + 1, \frac{N}{2} - 1\rangle$. We immediately obtain the three eigenvectors, which are $v_1 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, -\frac{1}{2}\right)$, $v_2 = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, -\frac{1}{2}\right)$ and $v_3 = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$. The optimal measurements are the projections onto $v_1$, $v_2$ and $v_3$ and the remaining $N - 2$ orthogonal states, rotated by the operator $e^{-i\theta \hat{J}_y}$.

### 3.3 Noisy NOON state

We now exploit the algebraic properties of the Werner states to construct a simple model of a symmetric NOON state of $N$ particles subject to decoherence. A pure NOON state is $|\psi\rangle_{\text{NOON}} = \frac{1}{\sqrt{2}}(|N0\rangle + |0N\rangle)$. Upon the influence of the environment, it might deteriorate in a complicated way, depending on the type of decoherence. Here, we will assume that the interaction with the environment transforms it into the mixture (37), where the projection is onto $|\psi\rangle_{\text{NOON}}$. Choosing the generator to be the pure phase imprint, i.e., $\hat{h} = \hat{J}_z$, we obtain that the commutator $[\hat{h}, \hat{\Pi}_N]$ is proportional to

$$\hat{C} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad (43)$$

expressed in the basis $|N0\rangle$ and $|0N\rangle$. Optimal measurements are projections onto $|\psi_\pm\rangle = \frac{1}{\sqrt{2}} (1, \pm i)$ plus all the other states which do not contribute to the Fisher information, rotated by the operator $e^{-i\theta \hat{J}_z}$. 
4 Optimal measurements for symmetric N-qubit pure states

We now show, that for generic pure symmetric states of \( N \) qubits, some optimal estimation strategies can be found using the notion of the statistical distance.

4.1 QFI and the statistical distance

In [30] it is shown how the QFI from Eq. (11) is related to the statistical distance between two neighboring states [37]. To be more precise, the QFI emerges as the coefficient of the leading (second order) term in the Taylor expansion of the Bures distance between the probe state and the phase-shifted one. This result is valid for pure and mixed states and for any parameter-dependent transformation. In the following, we use the concept of the statistical distance to provide a general condition under which the FI obtained from the population imbalance measurement saturates the QFI. To this end, we follow the scheme pictured in Fig. 3. First, we consider a pure two-mode input state \(|\psi_0\rangle\), which is transformed by a unitary evolution operator \( \hat{U}(\theta) = e^{-i\theta \hat{J}_n} \), where \( n \) is an arbitrary direction in the three-dimensional space. As a result, we obtain a \( \theta \)-dependent output state, which expanded in some orthogonal basis \( \{|j\rangle\} \) reads

\[
|\psi\rangle = e^{-i\theta \hat{J}_n} |\psi_0\rangle = \sum_{j=0}^{N} \sqrt{p_j} e^{i\varphi_j} |j\rangle = \sum_{j=0}^{N} C_j |j\rangle ,
\]

(44)

with \( C_j \equiv \langle j |\psi \rangle \). The neighboring state is found by applying a further infinitesimal transformation

\[
|\tilde{\psi}\rangle = e^{-i d\theta \hat{J}_n} |\psi\rangle
\]

(45)

and is approximated as

\[
|\tilde{\psi}\rangle \simeq \sum_{j=0}^{N} (1 - i d\theta \hat{J}_n) C_j |j\rangle = \sum_{j=0}^{N} (1 - i \theta \eta(n)_j) C_j |j\rangle \equiv \sum_{j=0}^{N} \tilde{C}_j |j\rangle .
\]

(46)

Fig. 3 (Color online) Schematic representation of the two steps performed to calculate the QFI. The input state \(|\psi_0\rangle\), which enters the interferometer, is transformed by a unitary evolution operator \( \hat{U}(\theta) \), giving the output state \(|\psi\rangle\). To calculate the speed at which the state changes, and thus the statistical distance, we make a further infinitesimal rotation \( \hat{U}(d\theta) \) to obtain \(|\tilde{\psi}\rangle\).
where

\[ \eta_{j}^{(n)} C_j \equiv \langle j | \hat{J}_n | \psi \rangle. \]  

(47)

Notice that this equation unequivocally defines \( \eta_{j}^{(n)} \) only if \( C_j \neq 0 \). The state (45) can be alternatively written as

\[ \lvert \tilde{\psi} \rangle = \sum_{j=0}^{N} \sqrt{p_j + dp_j} e^{i(\varphi_j + dp_j)} \lvert j \rangle, \]  

(48)

where the probability and phase increments are

\[ dp_j = |\tilde{C}_j|^2 - |C_j|^2 = 2 \text{Im} \eta_{j}^{(n)} |C_j|^2 d\theta \]  

(49a)

\[ e^{id\varphi_j} = \frac{\tilde{C}_j}{|\tilde{C}_j|} \frac{|C_j|}{C_j} = e^{-i \text{Re} \eta_{j}^{(n)} d\theta}. \]  

(49b)

The distance between two neighboring states is equal to

\[ ds_{\text{ps}}^2 = 1 - \left| \langle \psi | \tilde{\psi} \rangle \right|^2 \]  

(50a)

\[ = \sum_{j=0}^{N} \frac{dp_j^2}{p_j} + 4 \left[ \sum_{j=0}^{N} p_j dp_j^2 - \left( \sum_{j=0}^{N} p_j dp_j \right)^2 \right] \]  

(50b)

\[ \equiv \sum_{j=0}^{N} \frac{dp_j^2}{p_j} + 4 \Delta^2 d\varphi. \]  

(50c)

Finally, the QFI can be interpreted as the speed at which the state changes upon the infinitesimal increment of the parameter \( \theta \). By using Eqs. (50c) with \( dp_j \) and \( dp_j^2 \) obtained from Eqs. (49a) and (49b), respectively, we find

\[ F_Q = \frac{ds_{\text{ps}}^2}{d\theta^2} = 4 \sum_{j=0}^{N} |C_j|^2 \left( \text{Im} \eta_{j}^{(n)} \right)^2 + \frac{4}{N} \sum_{j=0}^{N} |C_j|^2 \left( \text{Re} \eta_{j}^{(n)} \right)^2 - 4 \left( \sum_{j=0}^{N} |C_j|^2 \text{Re} \eta_{j}^{(n)} \right)^2. \]  

(51b)

The above result is equivalent to the well-known expression for the QFI for pure states

\[ F_Q = 4(\Delta \hat{J}_n)^2. \]  

(52)

However, as will become evident below, for the purpose of finding the optimal measurements, it is more convenient to keep the QFI in the form of Eq. (51), i.e., as a
sum of two nonnegative parts—the change of the probability $p_j$ and the variance of the phase increment $d\phi_j$. We thus arrive at the main result of this section: The FI calculated from projective measurements on the basis $\{|j\rangle\}$ saturates the QFI if and only line (51b) is equal to zero. In this case, no information about $\theta$ is carried by the phases $\phi_j$, which are not witnessed by the projection measurement $|j\rangle\langle j|$ and thus do not contribute to probabilities $p_j$. The line (51b) vanishes if and only if

$$\text{Re} \eta_j^{(n)} = \langle \psi | \hat{J}_n | \psi \rangle, \quad \forall j \text{ such that } C_j \neq 0. \quad (53)$$

To demonstrate this result, we use the Cauchy–Schwarz inequality

$$\left( \sum_{j=0}^{N} |C_j|^2 \text{Re} \eta_j^{(n)} \right)^2 \leq \sum_{j=0}^{N} |C_j|^2 (\text{Re} \eta_j^{(n)})^2. \quad (54)$$

When this inequality is saturated, the line (51b) vanishes. This happens if and only if $\text{Re} \eta_j^{(n)} = c$, where $c$ is a real number which does not depend on $j$. To find $c$, we multiply both sides of Eq. (47) by $C_j^*$, sum over $j$ and obtain $c = \langle \psi | \hat{J}_n | \psi \rangle$. In particular, if one of the states in the basis $\{|j\rangle\}$ is equal to $|\psi\rangle$ (and thus $C_j = 1$ for $|j\rangle = |\psi\rangle$ and $C_j = 0$ otherwise), then Eq. (53) is always satisfied. In this case, we recover a well-known result [31]: for pure states, any POVM consisting of projectors over the probe state and over any basis on the orthogonal subspace allows to saturate the QFI.

4.2 “In situ” measurements: localized modes

In this section, we focus on the measurement of the number of particles in the two output modes. In this case, it is most convenient to identify the states $\{|j\rangle\}$ introduced in Eq. (44) with the mode occupation basis, where $|j\rangle$ denotes a Fock state with $j$ particles in the left and $N - j$ in the right arm. The coefficient $\eta_j^{(n)}$ is a result of acting with $\hat{J}_n$ on a ket $|j\rangle$ and is equal to

$$\eta_j^{(x)} = \frac{1}{2} \frac{\alpha_j C_{j+1} + \alpha_{j-1} C_{j-1}}{C_j} \quad (55a)$$

$$\eta_j^{(y)} = \frac{1}{2i} \frac{\alpha_j C_{j+1} - \alpha_{j-1} C_{j-1}}{C_j} \quad (55b)$$

$$\eta_j^{(z)} = j - \frac{N}{2}, \quad (55c)$$

where $\alpha_j = \sqrt{(j+1)(N-j)}$. By referring to Eq. (51), we identify two optimal measurements performed “in situ,” when the particles remain trapped in the two arms of the interferometer and their spatial mode functions do not overlap.

© Springer
4.2.1 Estimation from the full correlation

As a first example, we consider the phase estimation from the full $N$-body probability

$$p_N(r|\theta) = \frac{1}{N!} \langle \psi | \hat{\Psi}(x_1) \ldots \hat{\Psi}(x_N) \hat{\Psi}(x_N) \ldots \hat{\Psi}(x_1) | \psi \rangle$$

$$\equiv \langle \psi | \hat{G}(r) | \psi \rangle.$$  \hspace{1cm} (56)

of finding particles at positions $r = (x_1 \ldots x_N)$. The two-mode field operator is

$$\hat{\Psi}(x) = \psi_a(x) \hat{a} + \psi_b(x) \hat{b}$$

and the wave-packets are separated in two arms of the interferometer, for instance by imposing $\psi_a(x) = 0$ for $x < 0$ and $\psi_b(x) = 0$ for $x > 0$. The $\theta$-dependence of the probability $p_N(r|\theta)$ comes from the state $|\psi\rangle$ from Eq. (44), which is used to calculate the average value of the operator $\hat{G}(r)$.

The estimation sequence relies upon detecting positions of $N$ atoms in $m \gg 1$ experiments. If the phase is obtained from the maximum likelihood estimator, according to the Fisher theorem, its sensitivity is given by

$$\Delta^2 \theta = \frac{1}{m} \frac{1}{F_N}.$$  \hspace{1cm} (57)

where $F_N$ is the FI equal to

$$F_N = \int d\mathbf{r} \frac{1}{p_N(r|\theta)} \left( \frac{\partial p_N(r|\theta)}{\partial \theta} \right)^2.$$  \hspace{1cm} (58)

In order to calculate $F_N$, we first evaluate the derivative of the probability (56),

$$\partial_\theta p_N(r|\theta) = i \langle \psi | \hat{J}_n \hat{G}(r) | \psi \rangle - i \langle \psi | \hat{G}(r) \hat{J}_n | \psi \rangle = 2 \text{Im} \langle \psi | \hat{G}(r) \hat{J}_k | \psi \rangle = 2 \text{Im} \sum_{j,j'=0}^N C_j^* C_{j'} \eta_{n,j'} \langle j | \hat{G}(r) | j' \rangle.$$  \hspace{1cm} (59)

The FI is therefore equal to

$$F_N = 4 \int d\mathbf{r} \frac{\left[ \text{Im} \sum_{j,j'=0}^N C_j^* C_{j'} \eta_{n,j'} \langle j | \hat{G}(r) | j' \rangle \right]^2}{\text{Im} \sum_{j,j'=0}^N C_j^* C_{j'} \langle j | \hat{G}(r) | j' \rangle}.$$  \hspace{1cm} (60)

We now define $\Omega_\mu$ by saying that $r \in \Omega_\mu$ when $x_1 \ldots x_\mu < 0$ and $x_{\mu+1} \ldots x_N > 0$. Using this definition we obtain

$$F_N = 4 \sum_{\mu=0}^N \binom{N}{\mu} \int_{r \in \Omega_\mu} d\mathbf{r} \frac{\left[ \text{Im} \sum_{j,j'=0}^N C_j^* C_{j'} \eta_{n,j'} \langle j | \hat{G}(r) | j' \rangle \right]^2}{\text{Im} \sum_{j,j'=0}^N C_j^* C_{j'} \langle j | \hat{G}(r) | j' \rangle}.$$
where the combinatory factor is due to the symmetry of the state and stands for all possible choices of \( \mu \) particles out of a set of \( N \). When \( r \in \Omega_\mu \), then for separated wave-packets \( \hat{G}(r) |n\rangle \propto \frac{\mu!(N-\mu)!}{N!} |n\rangle \delta_{\eta\mu} \) and the above integral gives

\[
F_N = 4 \sum_{j=0}^{N} |C_j|^2 \left( \text{Im} \eta_j^{(n)} \right)^2.
\]  

(61)

We notice that this expression is equal to the first line of the QFI, see (51a).

Therefore, estimation from the \( N \)-body probability of trapped particles is optimal only if the other terms in line (51b) vanish, which requires \( \text{Re} \eta_j^{(n)} \equiv 0 \) for all \( j \). According to Eq. (55), this condition is satisfied only for the rotations around \( x \) and \( y \) axes. For the rotations around \( x \)-axis, \( \text{Re} \eta_j^{(x)} = 0 \) if \( C_j = i^j a_j \), while for rotations around the \( y \)-axis the (necessary and sufficient) condition is \( C_j = e^{i\phi} a_j \), where \( a_j \in \mathbb{R} \) and \( \phi \) is a common phase. In particular for \( \phi = 0 \), the measurement is optimal, when all \( C_j \)'s are real. Since the elements of the Wigner rotation matrix—which transforms the input state \( |\psi_0\rangle \) into the output state \( |\psi\rangle \)—are all real \([38]^2\), we conclude that if the input state of the MZI has real coefficients, the estimation from \( p_N \) is optimal. For rotations around the \( z \) axis, \( \text{Im} \eta_j^{(z)} = 0 \) and thus \( F_N = 0 \), because the simple phase imprint \( e^{-i\theta}\hat{J}_z \) requires further mode mixing to provide information about \( \theta \).

### 4.2.2 Estimation from the population imbalance

Although phase estimation from the \( N \)-body probability is optimal for a wide class of states and rotations around \( x \) and \( y \), it has one major flaw—it is impractical, since it requires sampling of a vast configurational space. We now show, that the same value of the FI as in Eq. (61) is obtained, when the phase is estimated from a simple population imbalance measurement. The probability of having \( j \) atoms in the mode \( a \) and \( N - j \) in \( b \) is

\[
p(j|\theta) = |\langle j|\psi\rangle|^2 = |C_j|^2.
\]  

(62)

Similarly as in Eq. (59), its derivative reads

\[
\partial_\theta p(j|\theta) = 2|C_j|^2 \text{Im} \eta_j^{(n)}.
\]  

(63)

\(^2\) The elements of the angular momentum matrix, \( d_{jk}(\theta) \equiv \langle j|e^{-i\theta}\hat{J}_y|k\rangle \) are

\[
d_{jk}(\theta) = \sqrt{\frac{j!(N-j)!}{k!(N-k)!}} \left[ \begin{array}{c} \sin \theta \\ \cos \theta \end{array} \right]^{j-k} \left[ \begin{array}{c} j+k-N \\ \frac{\sin \theta}{2} \end{array} \right] \times P_{j+k-N}^{j-k,n-j}(\cos \theta).
\]

where \( P_{n}^{\alpha,\beta}(x) \) is the Jacobi polynomial.
Therefore, the FI calculated with (62)

\[ F_{\text{imb}} = \sum_{j=0}^{N} \frac{1}{p(j|\theta)} \left( \frac{\partial p(j|\theta)}{\partial \theta} \right)^2 = 4 \sum_{j=0}^{N} |C_j|^2 \left( \text{Im} \eta^{(m)}_j \right)^2 \]

(64)
is equal to (61). In consequence, the QFI from Eq. (51) is saturated with the same family of states for the \( x \) and \( y \) rotations as in the case of the estimation from \( p_N(r|\theta) \). Note that this result has been obtained independently in Ref. [39] (see also considerations in Ref. [40] for a specific class of probe mixed states and Ref. [41] for the discussion of the population imbalance measurements with squeezed states). Furthermore, if the input state of the Mach–Zehnder interferometer (i.e., a rotation around the \( y \) axis) has real coefficients, the estimation from the population imbalance is optimal for all values of \( \theta \). If the coefficients \( C_{j'}^{(0)} \) are imaginary, it is still possible to saturate the QFI, at specific phase values, provided that the condition \( \text{Re} \eta^{(y)}_j = 0 \) for all \( j \) is satisfied. Note that less general conditions for the saturation of the QFI, independently if the phase shift, and with population imbalance measurement, has been given in Ref. [42] (see also [43]).

4.3 Measurement after expansion

As argued above, when the interferometer rotates the state around the \( z \)-axis, giving a sole phase imprint, further manipulation is necessary to exchange the information about the phase between the two modes. Here we assume, that this operation is realized by letting \( \psi_a(x) \) and \( \psi_b(x) \) expand and form an interference pattern. In such situation, the two modes cannot be distinguished anymore, and it is not possible to define a proper population imbalance operator. Instead, one must estimate \( \theta \) in some different way. For instance, estimation from the least-squares fit of the one-body probability

\[ p_1(x|\theta) = \frac{1}{N} \langle \psi | \hat{\Psi}^\dagger(x) \hat{\Psi}(x) | \psi \rangle \]

to the interference pattern, although gives sub-shot-noise sensitivity when the input state \( |\psi_0\rangle \) is phase-squeezed [46], is never optimal [44]. Nevertheless, the optimal measurement can be identified, and it is the \( N \)-body FI from Eq. (58) which saturates the bound of the QFI under following additional assumptions [45]. First, the information between the two modes must be fully exchanged. This means, that the envelopes of \( \psi_a(x) \) and \( \psi_b(x) \) fully overlap and the functions only differ by the phase. This is true if initially \( \psi_a(x) \) and \( \psi_b(x) \) are of the same shape but are separated in space and then expand to reach the far-field regime. Another requirement is that the coefficients of the initial state \( C_{j'}^{(0)} \) are real and posses the symmetry \( C_{j'}^{(0)} = C_{N-j}^{(0)} \). States having these properties naturally appear in the context of quantum interferometry with ultra-cold gas trapped in the double-well potential. Namely, the ground state of the symmetric two-mode Bose–Hubbard Hamiltonian for every ratio of the interaction strength \( U \) to the tunneling rate \( J \) has real and symmetric coefficients \( C_{j'}^{(0)} \).

According to Eq. (44), the rotation around the \( z \) axis transforms the state into

\[ |\psi\rangle = e^{-i\theta \hat{J}_z} |\psi_0\rangle = \sum_{j=0}^{N} C_j |j\rangle \]

(65)
where $C_j = C_j^{(0)} e^{-i\theta (j - N/2)}$. As argued in detail in [45], the FI from Eq. (58), calculated under the aforementioned assumptions, is

$$F = 4 \sum_j |C_j|^2 \eta_j^{(c)} = 4(\Delta \hat{J}_z)^2 = F_Q,$$

(66)

where $\eta_j^{(c)}$ was defined in Eq. (55c). This shows that the estimation from the $N$-th body correlation in the far field is optimal.

5 Conclusions

In this work we have identified optimal measurements in various two-mode interferometric systems. We have first considered the qubit, which is the simplest—but nevertheless illustrative—testbed for the theory. There is a clear picture emerging from these considerations: While for pure probe states there is a continuous set of optimal estimation strategies (i.e., optimal POVMs for which the FI saturates the QFI), they reduce to a specific POVM when the probe state is mixed. As a direct consequence, while for pure states it is possible to saturate the QFI independently of the phase $\theta$, this is not possible for mixed states. Indeed, in this case, the optimal POVM strongly depends on the parameter $\theta$. This overall picture has been confirmed in the case of two qubits forming the Werner state. In addition, in the case of single pure qubit state and two-qubit pure Werner states, we have shown that the measurement of the population imbalance among the two output modes of the interferometer is optimal and saturates the QFI independently from $\theta$. We have further extended the analysis to the general case of pure $N$-qubit states. We have found the broad conditions on the state and interferometric transformation for which the FI obtained from the population imbalance measurement or $N$-th-order correlation function can saturate the QFI, independently on the phase shift.

Acknowledgments J. Ch. acknowledges the Foundation for Polish Science International TEAM Programme co-financed by the EU European Regional Development Fund and the support of the Polish NCBiR under the ERA-NET CHIST-ERA project QUASAR. T.W. acknowledges the Foundation for Polish Science International Ph.D. Projects Programme co-financed by the EU European Regional Development Fund and the National Science Center Grant No. DEC-2011/03/D/ST2/00200. L.P. acknowledges financial support by MIUR through FIRB Project No. RBFR08H058. This research was partially supported by the EU-STREP Project QIBEC.

References

1. Paris, M.G.A.: Quantum estimation for quantum technology. Int. J. Quantum Inf. 7, 125–137 (2009)
2. Giovannetti, V., Lloyd, S., Maccone, M.: Advances in quantum metrology. Nat. Photonics 5, 222–229 (2011)
3. Pezzé, L., Smerzi, A.: Quantum theory of phase estimation. In Proceedings of the International School of Physics “Enrico Fermi”, Course 188, Societá Italiana di Fisica, Bologna and IOS Press, Amsterdam (2014)
4. Helstrom, C.W.: Quantum Detection and Estimation Theory. Academic Press, London (1976)
5. Cramér, H.: Mathematical Methods of Statistics. Princeton University Press, Princeton (1946)
6. Giovannetti, V., Lloyd, S., Maccone, M.: Quantum metrology. Phys. Rev. Lett. 96, 010401–010401-5 (2006)
7. Wineland, D.J., Bollinger, J.J., Itano, W.M., Heinzen, D.J.: Squeezed atomic states and projection noise in spectroscopy, Phys Rev. A 50, 67–88 (1994)
8. Yurke, B., McCall, S.L., Klauder, J.R.: SU(2) and SU(1,1) interferometers. Phys. Rev. A 33, 4033–4054 (1986)
9. Sørensen, A.S., Duan, L.-M., Cirac, J.I., Zoller, P.: Many-particle entanglement with Bose–Einstein condensates. Nature 409, 63–66 (2001)
10. Pezzé, L., Smerzi, A.: Entanglement, nonlinear dynamics, and the Heisenberg limit. Phys. Rev. Lett. 102, 100401-1–100401-4 (2009)
11. Hyllus, P., et al.: Fisher information and multipartite entanglement. Phys. Rev. A 85, 022321-1–022321-10 (2012)
12. Tóth, G.: Multipartite entanglement and high-precision metrology. Phys. Rev. A 85, 022322-1–022322-8 (2012)
13. Leibfried, D., et al.: Toward Heisenberg-limited spectroscopy with multiparticle entangled states. Science 304, 1476–1478 (2004)
14. Leibfried, D., et al.: Creation of a six-atom Schrödinger cat state. Nature 438, 639–642 (2006)
15. Roos, C.F., et al.: Designer atoms for quantum metrology. Nature 443, 316–319 (2006)
16. Monz, T., et al.: 14-Qubit entanglement: creation and coherence. Phys. Rev. Lett. 106, 130506-1–130506-4 (2011)
17. Nagata, T., et al.: Beating the standard quantum limit with four-entangled photons. Science 316, 726–729 (2007)
18. Kacprowicz, M., et al.: Experimental quantum-enhanced estimation of a lossy phase shift. Nat. Photonics 4, 357–360 (2010)
19. Xiang, G.Y., et al.: Entanglement-enhanced measurement of a completely unknown optical phase. Nat. Photonics 5, 43–47 (2011)
20. Krischek, R., et al.: Useful multiparticle entanglement and sub-shot-noise sensitivity in experimental phase estimation. Phys. Rev. Lett. 107, 080504-1–080504-5 (2011)
21. Appel, J., et al.: Mesoscopic atomic entanglement for precision measurements beyond the standard quantum limit. PNAS 106, 10960–10965 (2009)
22. Schleier-Smith, M.H., Leroux, I.D., Vuletic, V.: States of an ensemble of two-level atoms with reduced quantum uncertainty. Phys. Rev. Lett. 104, 073604-1–073604-4 (2010)
23. Chen, Z., et al.: Conditional spin squeezing of a large ensemble via the vacuum Rabi splitting. Phys. Rev. Lett. 106, 133601-1–133601-4 (2011)
24. Estève, J., et al.: Squeezing and entanglement in a Bose–Einstein condensate. Nature 455, 1216–1219 (2008)
25. Riedel, M.F., et al.: Atom-chip-based generation of entanglement for quantum metrology. Nature 464, 1170–1173 (2010)
26. Ockeloen, C.F., et al.: Quantum metrology with a scanning probe atom interferometer. Phys. Rev. Lett. 111, 143001-1–143001-5 (2013)
27. Gross, C., et al.: Nonlinear atom interferometer surpasses classical precision limit. Nature 464, 1165–1169 (2010)
28. Lücke, B., et al.: Twin matter waves for interferometry beyond the classical limit. Science 11, 773–776 (2011)
29. Berrada, T., et al.: Integrated MachZehnder interferometer for BoseEinstein condensates. Nat. Commun. 4 (2013), 1–8 (2013)
30. Braunstein, S.L., Caves, C.M.: Statistical distance and the geometry of quantum states. Phys. Rev. Lett. 72, 3439–3443 (1994)
31. Braunstein, S.L., Caves, C.M., Milburn, G.J.: Generalized uncertainty relations: theory, examples, and Lorentz invariance. Ann. Phys. 247, 135–173 (1996)
32. Pezzé, L., et al.: Phase detection at the quantum limit with multiphoton Mach–Zehnder interferometry. Phys. Rev. Lett. 99, 223602-1–223602-4 (2007)
33. Werner, R.F.: Quantum states with Einstein–Podolsky–Rosen correlations admitting a hidden-variable model. Phys. Rev. A 40, 4277–4281 (1989)
34. Holland, M.J.: Interferometric detection of optical phase shifts at the Heisenberg limit. Phys. Rev. Lett. 71, 1355–1358 (1993)
35. Horodecki, R., Horodecki, P., Horodecki, M., Horodecki, K.: Quantum entanglement. Rev. Mod. Phys. 81, 865–942 (2009)
36. Hyllus, P., Gühne, O., Smerzi, A.: Not all pure entangled states are useful for sub-shot-noise interferometry. Phys. Rev. A 82, 012337-1–012337-12 (2010)
37. Wootters, W.K.: Statistical distance and Hilbert space. Phys. Rev. D 23, 357–362 (1981)
38. Biedenharn, L.C., Louck, J.D.: Angular Momentum in Quantum Physics: Theory and Applications. Cambridge University Press, Cambridge (1984)
39. Lang, M.D., Caves, C.M.: Optimal quantum-enhanced interferometry using a laser power source. Phys. Rev. Lett. 111, 173601-1–173601-5 (2013)
40. Pezzé, L., Smerzi, A.: Ultrasensitive two-mode interferometry with single-mode number squeezing. Phys. Rev. Lett. 110, 163604-1–163604-5 (2013)
41. Wineland, D.J., Bollinger, J.J., Itano, W.M., Moore, F.L., Heinzen, D.J.: Spin squeezing and reduced quantum noise in spectroscopy. Phys. Rev. A 46, R6797–R6800 (1992)
42. Hofmann, H.F.: All path-symmetric pure states achieve their maximal phase sensitivity in conventional two-path interferometry. Phys. Rev. A 79, 033822-1–033822-4 (2009)
43. Seshadreesan, K.P., Kim, S., Dowling, J.P., Lee, H.: Phase estimation at the quantum Cramér–Rao bound via parity detection. Phys. Rev. A 87, 043833-1–043833-6 (2013)
44. Chwedeńczuk, J., Hyllus, P., Piazza, F., Smerzi, A.: Sub-shot-noise interferometry from measurements of the one-body density. New J. Phys. 14(093001), 1–19 (2012)
45. Chwedeńczuk, J., Piazza, F., Smerzi, A.: Phase estimation from atom position measurements. New J. Phys. 13(065023), 1–18 (2011)
46. Grond, J., Hohenester, U., Mazets, I., Schmiedmayer, J.: Atom interferometry with trapped BoseEinstein condensates: impact of atom–atom interactions. New J. Phys. 12(065036), 1–29 (2010)