Quantum moment maps and symmetric bounded domains quantizations

Stéphane Korvers

Université du Luxembourg (2015-2017)
FSTC, Unité de Recherche en Mathématiques
rue Richard Coudenhove-Kalergi, 6
L-1359 Luxembourg, Grand Duchy of Luxembourg

E-mail: korvers.s@gmail.com

January 2017
reviewed in June 2018

Abstract

We introduce an explicit construction for realizing of the space of invariant deformation quantizations on an arbitrary symmetric bounded domain of $\mathbb{C}^n$.

1 Introduction and notations

Generally speaking, in the context of mathematical physics and quantum mechanics, the terminology of quantization is used to allude to the expression at a quantum level of facts related to a classical system. This problem starts with the data of a symplectic manifold $(M, \omega)$, or more generally a Poisson manifold $(M, \{-,-\})$, modeling the phase space of the classical system. Usually, by quantizing $(M, \omega)$, one asks for a way to link some classical objects to potential quantum analogs. For example, with the symplectic manifold $(M, \omega)$ and the algebra of smooth functions on $M$ representing the classical observables, we can respectively associate a Hilbert space $\mathcal{H}$ and an algebra of linear operators on $\mathcal{H}$. Many methods exist to approach this problem. Among them, the deformation quantization promotes the idea of an understanding of this quantization problem as a deformation of the commutative structure of the algebra of classical observables $C^\infty(M) := C^\infty(M, \mathbb{C})$ into a noncommutative direction given by the Poisson bracket $\{-,-\}$ associated with the symplectic form $\omega$. 

1
At a formal level, this notion is encoded in the data of a star-product on $M$ which is an associative $\mathbb{C}[\nu]$-linear product on the space of formal power series in the formal parameter $\nu$ with coefficients in $\mathcal{C}^\infty (M)$

$$*_{\nu} : \mathcal{C}^\infty (M)[[\nu]] \times \mathcal{C}^\infty (M)[[\nu]] \rightarrow \mathcal{C}^\infty (M)[[\nu]] : (f_1, f_2) \mapsto f_1 *_{\nu} f_2 := \sum_{k \in \mathbb{N}} \nu^k C_k (f_1, f_2)$$

where $\{C_k : \mathcal{C}^\infty (M) \times \mathcal{C}^\infty (M) \rightarrow \mathcal{C}^\infty (M)\}_{k \in \mathbb{N}}$ are bi-differential operators such that

$$C_0 (f_1, f_2) = f_1 f_2, \quad C_1 (f_1, f_2) - C_1 (f_2, f_1) = 2 \{f_1, f_2\} \quad \text{and} \quad C_l (1, 1) = C_l (f_1, 1) = 0$$

for each $l \in \mathbb{N} \setminus \{0\}$ and $f_1, f_2 \in \mathcal{C}^\infty (M)$. This was introduced by Bayen, Flato, Fronsdal, Lichnerowicz and Sternheimer in 1978; [B+78a], [B+78b]. This approach has the property to be universal in the sense that there exists a star-product on each Poisson manifold; [Ko03].

Although one does not worry about the convergence of the formal series appearing in the previous definition, under adapted functional hypothesis, it may happen that a new function on $M$ can be defined from the star-product of two functions on $M$. In this case, one talks about non-formal deformation quantization on $M$. More specifically, we are interested in such quantization described by an explicit 3-point kernel $K_{\nu} (-, -, -)$ through the formula

$$(f *_{\nu} g)(x) = \int_{M \times M} K_{\nu} (x, y, z) f (y) g (z) \, dy \, dz$$

when $f$ and $g$ belongs to an adapted space of functions, with $dx$ the Liouville measure on $M$.

In some situations, it is relevant to look for specific deformation quantizations that take account of symmetries of a classical system through the quantization process. If $G$ is a Lie group which acts by symplectomorphisms on the symplectic manifold $(M, \omega)$ through the action map

$$\tau : G \times M \rightarrow M : (g, x) \mapsto \tau_g (x),$$

a star-product $*_{\nu}$ on $M$ will be said to be $G$-invariant if

$$\tau_g^* (f_1 *_{\nu} f_2) = \tau_g^* f_1 *_{\nu} \tau_g^* f_2 \quad \text{(1)}$$

for each $g \in G$ and $f_1, f_2 \in \mathcal{C}^\infty (M)$. When $G$ preserves a symplectic connexion on $M$, then there always exists a $G$-invariant star-product on $M$. It is a consequence of the well known Fedosov construction of star-products on symplectic manifolds; [Fe94].

All along this text, we will consider $\mathbb{D} \subset \mathbb{C}^N$ an arbitrary symmetric bounded domain of $\mathbb{C}^N$ for $N \in \mathbb{N} \setminus \{0\}$, i.e. an open connected bounded subset of $\mathbb{C}^N$ endowed with a structure of symmetric space for which the symmetries are biholomorphisms. Such domain is connected simply connected; [He01] Ch. 8, thm. 4.6]. When it is endowed with its Bergman metric, it has a structure of an Hermitian symmetric space of non compact type; [He01] Ch. 8, thm. 7.1]. In addition, every Hermitian symmetric space of non compact type can be realized as a symmetric bounded domain; [He01] Ch. 8, thm. 7.1]. As before, we will denote by $\omega$ and $\{-,-\}$ respectively the symplectic structure on $\mathbb{D}$ and the Poisson bracket on $\mathcal{C}^\infty (\mathbb{D})$ associated with $\omega$. 
Let $G$ be the identity component of the automorphism group of $D$ and $\mathfrak{g}$ its Lie algebra. It is well known that $G$ is a semi-simple Lie group of transformations of $D$ which acts holomorphically and transitively on $D$; [He01 Ch. 4 & 8], [Ko14 Ch. 1, §2]. We will denote by

$$\tau : G \times D \to D : (g, x) \mapsto \tau_{g}(x)$$

the action of $G$ on $D$. For $X \in \mathfrak{g}$, the notation $X^{\ast} \in \Gamma (T^{\ast}D)$ will refers to the fundamental vector field associated with $X$ which is defined at point $x \in D$ by

$$X^{\ast}_{x} := \left. \frac{d}{dt} \right|_{t=0} \tau_{\exp(-tX)}(x).$$

The action of $G$ on $D$ is Hamiltonian and admits a unique (co-)moment map

$$\lambda : \mathfrak{g} \to C^{\infty}(D) : X \mapsto \lambda_{X}$$

defined by the equality $\iota_{X^{\ast}}\omega = -d\lambda_{X}$ for each $X \in \mathfrak{g}$; [So97 Ch. 2, thm.11.8], [Ca08 Ch. 26]. In particular, for each $X,Y \in \mathfrak{g}$, we have

$$X^{\ast} = \{\lambda_{X}, -\} : f \in C^{\infty}(D) \mapsto \{\lambda_{X}, f\} \quad \text{and} \quad X^{\ast}(\lambda_{Y}) = \lambda_{[X,Y]}.$$

In the present work, we develop a method unifying constructions of $G$-invariant star-products on $D$. We present a characterization of the space of all these invariant star-products as solutions to an explicit hierarchy of partial differential equations and we explicit how to write these equations.

The method used in this work combines modern mathematics of various research fields in an innovative way, and is based on the retract method initiated by Bieliavsky and his collaborators in the 2000s. It have already proven its power in the obtention of similar descriptions for the particular cases of the Poincaré disk and the unit ball of $\mathbb{C}^{N}$; [B+09], [Ko14]. In the following sections, we show that a similar approach can be performed under hypothesis that we describe. We also develop tools for simplifying computations underlined by practical applications of this method.

Acknowledgement

This work is supported by the Fonds National de la Recherche FNR/AFR-Postdoc grant no.8960322.

The author thanks the Fonds National de la Recherche, the University of Luxembourg, and Martin Schlichenmaier for supporting him and giving him the opportunity to pursue his research in a stimulating international working environment.

The author thanks Pierre Bieliavsky for introducing him to this field, and for inspiring this quantization method through our collaboration in [Ko14].
2 Structure of the automorphism group of $\mathbb{D}$

In this section, we describe the structure of the automorphism group of $\mathbb{D}$ and its Lie algebra $\mathfrak{g}$. In particular, we explicit the restricted root space decomposition and the Pyatetskii-Shapiro decomposition of $\mathfrak{g}$. We show that the domain $\mathbb{D}$ can be identify with the Iwasawa group of $G$.

2.1 Root space decomposition

Let's fix $o \in \mathbb{D}$. Then, the subgroup $K := \{g \in G \mid \tau_g(o) = o\} \subset G$ is compact and the map

$$G/K \to \mathbb{D} : gK \mapsto \tau_g(o)$$

is a diffeomorphism; [He01, Ch.4, thm.3.3]. As the domain $\mathbb{D}$ has a structure of Hermitian symmetric space of non compact type, the Lie algebra $\mathfrak{g}$ admits a Cartan decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

where $\mathfrak{k}$ is the Lie algebra of $K$ and $\mathfrak{p}$ is invariant under the adjoint action of $K$; [He01, Ch.8, §4]. Let’s denote by

$$\sigma = \text{Id}_{\mathfrak{k}} \oplus -\text{Id}_{\mathfrak{p}} : \mathfrak{g} \to \mathfrak{g}$$

the associated Cartan involution and $\beta$ the Killing form of $\mathfrak{g}$. Then, the symmetric bilinear form

$$\beta_\sigma : (X,Y) \in \mathfrak{g} \times \mathfrak{g} \mapsto -\beta(X,\sigma(Y))$$

is positive definite and $\beta(X,Y) = 0$ for each $X \in \mathfrak{k}$ and $Y \in \mathfrak{p}$. Let’s consider a an abelian Lie subalgebra of $\mathfrak{g}$ contained in $\mathfrak{p}$ and maximal for this property. We set

$$r := \dim(\mathfrak{a})$$

to be the rank of $\mathbb{D}$. This number is independent from the choice of $\mathfrak{a}$; [Kn02, Ch.6, thm.6.51]. For each linear form $[\lambda : \mathfrak{a} \to \mathbb{R}] \in \mathfrak{a}^*$, we can define

$$\mathfrak{g}_\lambda := \{X \in \mathfrak{g} : [H,X] = \lambda(H)X \text{ for each } H \in \mathfrak{a}\} \subset \mathfrak{g}.$$  

Definition 1. A linear form $\lambda \in \mathfrak{a}^* \setminus \{0\}$ such that $\mathfrak{g}_\lambda$ is non trivial will be called (restricted) root of $\mathfrak{g}$. The set of all these roots will be denoted by $\Sigma \subset \mathfrak{a}^*$. For $\lambda \in \Sigma$, the subspace $\mathfrak{g}_\lambda$ is called (restricted) root space of $\mathfrak{g}$.

Proposition 2. [Kn02, Ch.6, prop.6.40] The Lie algebra $\mathfrak{g}$ admits a root space decomposition

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \left( \bigoplus_{\lambda \in \Sigma} \mathfrak{g}_\lambda \right).$$

For each $\lambda, \mu \in \mathfrak{a}^*$, we have $[\mathfrak{g}_\lambda, \mathfrak{g}_\mu] \subset \mathfrak{g}_{\lambda+\mu}$ and $\mathfrak{g}_{-\lambda} = \sigma(\mathfrak{g}_\lambda)$. In addition, the subspace $\mathfrak{g}_0$ is a Lie subalgebra of $\mathfrak{g}$ which admits a decomposition

$$\mathfrak{g}_0 = \mathfrak{a} \oplus \mathfrak{m} \quad \text{with} \quad \mathfrak{m} := \{X \in \mathfrak{t} : [H,X] = 0 \text{ for each } H \in \mathfrak{a}\}.$$
The root space decomposition of \( g \) is an orthogonal direct sum for the inner product \( \beta_\sigma \) given that
\[
\beta (X, Y) = 0
\]
for each \( X \in g_\lambda \) and \( Y \in g_\mu \) if \( \lambda, \mu \in \mathfrak{a}^\ast \) satisfy \( \lambda + \mu \neq 0 \).

The Lie algebra \( m \) admits the decomposition
\[
m = [m, m] \oplus Z(m)
\]
where \( Z(m) \) denotes the center of \( m \). \cite[Ch.1, cor.1.56 & Ch.7, prop.7.48]{Kn02}. As \( \beta \) is positive definite on \( \mathfrak{a} \times \mathfrak{a} \), for \( \lambda \in \mathfrak{a}^\ast \), we can define \( H_\lambda \in \mathfrak{a} \) as the unique element in \( \mathfrak{a} \) such that
\[
\beta (H_\lambda, H) = \lambda (H)
\]
for each \( H \in \mathfrak{a} \). The set \( \{ H_\lambda : \lambda \in \Sigma \} \) spans \( \mathfrak{a} \); \cite[Ch.6, cor.6.53]{Kn02}. For all \( \lambda \in \Sigma \) and \( X \in g_\lambda \), we have
\[
[X, \sigma(X)] = \beta (X, \sigma(X)) H_\lambda;
\]
\cite[Ch.6, prop.6.52]{Kn02}. Let’s notice that \( \beta (X, \sigma(X)) < 0 \) if \( X \neq 0 \) in the previous equality, as \( \beta_\sigma \) is positive definite.

We conduct this section with the following technical lemma.

**Lemma 3.** \cite[thm.1]{Ko16} Let’s consider \( \lambda \in \Sigma \) and \( X \in g_\lambda \setminus \{0\} \). Then, we have
\[
[m, X] = X^\perp(\lambda) := \{ Y \in g_\lambda : \beta_\sigma (X, Y) = 0 \}.
\]
In particular, the root space \( g_\lambda \) admits the decomposition \( g_\lambda = RX \oplus [m, X] \).

As a consequence of this lemma, if \( \lambda \in \Sigma \) is such that \( \dim (g_\lambda) = 1 \), then \( [m, X] = 0 \). In addition, if \( m = 0 \), all the root spaces of \( g \) are one-dimensional.

### 2.2 Iwasawa decomposition

Let’s fix \( \{ \varphi_1, ..., \varphi_r \} \) a basis of \( \mathfrak{a}^\ast \). We will say that the root \( \lambda \in \Sigma \) is positive if there exists \( 1 \leq k_0 \leq r \) such that \( \varphi_{k_0}(H_\lambda) > 0 \) and \( \varphi_k(H_\lambda) = 0 \) for each \( k < k_0 \). We will denote by \( \Sigma^+ \) the set of positive roots of \( g \). Now, we can introduce the Iwasawa decomposition of \( g \) and \( G \).

**Proposition 4.** \cite[Ch.6, prop.6.43 & thm.6.46]{Kn02} The Lie algebra \( g \) admits the following vector space decomposition
\[
g = a \oplus n \oplus \mathfrak{t} \quad \text{with} \quad n := \bigoplus_{\lambda \in \Sigma^+} g_\lambda.
\]
The connected Lie subgroup \( A \subset G \) (resp. \( N \subset G \)) which Lie algebra \( a \) (resp. \( n \)) is abelian (resp. nilpotent) an simply connected. The group
\[
S := AN
\]
is a connected simply connected solvable Lie subgroup of \( G \) called Iwasawa group of \( G \). In addition, the maps
\[
A \times N \to S : (a, n) \mapsto an \quad \text{and} \quad S \times K \to G : (s, k) \mapsto sk
\]
are global diffeomorphisms between smooth manifolds.
As a consequence, we get a diffeomorphism $S \to G/K : s \mapsto sK$. In particular, the action of the Iwasawa group $S$ on the symmetric bounded domain $D$ is simply transitive and we have the identification $S \simeq D$. Let’s extend the notation $\tau$ to denote the $G$-equivariant transport of this action on $S \simeq G/K$. It is easy to notice that

$$\tau_s (s') = ss' =: L_s (s')$$

for each $s, s' \in S$. In particular, through its identification with $D$, the group $S$ becomes a left-invariant Kählerian Lie group.

In this text, we will denote by $s$ the Lie algebra of $S$. We have the following vector space isomorphisms:

$$s \simeq a \oplus n \simeq p \simeq \text{To} (D).$$

We can notice the identities

$$[s, s] = n \quad \text{and} \quad N (n) = s \oplus m$$

where $N (n)$ is the normalizer of $n$ in $g$. The first equality and the inclusion $[g_0 \oplus n, n] \subset n$ are direct from the properties of root space decomposition of $g$. As a consequence, the second equality follows from (3).

The Iwasawa decompositions of $g$ and $G$ can be written

$$g = s \oplus t \quad \text{and} \quad G = SK \simeq S \times K$$

respectively. We will denote the associated decompositions of $X \in g$ and $g \in G$ respectively by

$$X = [X]_s + [X]_t \quad \text{and} \quad g = [g]_S [g]_K$$

with $[X]_s \in s$, $[X]_t \in t$, $[g]_S \in S$ and $[g]_K \in K$. With these notations, we can remark that

$$\tau_g (s) = [gs]_S \quad \text{and} \quad [X]_s = \frac{d}{dt} \bigg|_{t=0} \left[ \exp (tX) \right]_S$$

for each $s \in S$, $g \in G$ and $X \in g$.

**Remark 5.** The Lie algebra $s$ is endowed with a scalar product $(-|-) -$ induced by the Kählerian structure of $S \simeq D$. Up to a constant $C_D \in \mathbb{R}$, we have

$$( [X]_a | [Y]_a ) = C_D \beta (X, Y) = C_D \beta (X, Y)$$

for each $X, Y \in p \simeq s$; [Ko14 Ch.1, rem.1.5.9]. In addition, lemma [3] the ad-invariance of the Killing form $\beta$, and the equality $[Y, [X]_a] = [[Y, X]]_a$ for each $X \in p$ and $Y \in m \subset t$, allow us to show that

$$X^{-1} (\lambda) = [m, X] = \{ Y \in g_\lambda : (X | Y) = 0 \}$$

for all $\lambda \in \Sigma$ and $X \in g_\lambda \setminus \{0\}$.

### 2.3 Pyatetskii-Shapiro decomposition

The following proposition explicits the so-called Pyatetskii-Shapiro decomposition of the Lie group $S \simeq D$ into elementary bricks. It is obtained by combining results from the reference [Py69 Ch.2, §3] as well as [Kn02 Ch.1, thm.1.125] and [Ko14 Ch.1, lem.1.3.10, lem.1.4.12 & prop.1.5.10].
Proposition 6. There exists \( n_1, \ldots, n_r \in \mathbb{N} \setminus \{0\} \) such that the Lie group \( S \) admits the decomposition

\[
S = (\cdots (S_r \ltimes S_{r-1}) \ltimes \cdots \ltimes S_2) \ltimes S_1
\]

where \( S_j \) is a Lie subgroup of \( S \) which is isomorphic to the Iwasawa group of \( G_j := SU(1, n_j) \) for each \( 1 \leq j \leq r \). In addition, the group \( S_j \) acts simply transitively on the complex unit ball of \( \mathbb{C}^{n_j} \) and this space admits a structure of symmetric bounded domain with automorphism group \( G_j \).

In some sense, the complex unit ball of \( \mathbb{C}^N \) is part of the building blocks of every symmetric bounded domain. We will further explicit our quantization method for this elementary case.

Originally, this decomposition was written at the infinitesimal level from [Py69, Ch.2, lem.1] where the previous proposition finds its root. We can formulate its Lie algebraic version in the following way.

Lemma 7. The Lie algebra \( s \) can be decomposed as

\[
s = (\cdots (s_r \ltimes s_{r-1}) \ltimes \cdots \ltimes s_2) \ltimes s_1
\]

where, for each \( 1 \leq j \leq r \), the factor \( s_j \) is a Lie subalgebra of \( s \) which contains:

- a generator \( E_j \) of a one-dimensional ideal of \( s_j \),
- a vector subspace \( V_j \subset s_j \) of dimension \( 2(n_j - 1) \in \mathbb{N} \) endowed with a symplectic form \( \Omega_j \in V_j^* \otimes V_j^* \),
- an element \( H_j \notin V_j \oplus \mathbb{R}E_j \),

such that

\[
s_j = \mathbb{R}H_j \ltimes (V_j \oplus \mathbb{R}E_j)
\]

with the Lie bracket described by the equalities

\[
[v_j, E_j] = 0, \quad [v_j, v_j'] = \Omega_j (v_j, v_j') E_j \quad \text{and} \quad [H_j, v_j + zE_j] = v_j + 2zE_j
\]

for all \( v_j, v_j' \in V_j \) and \( z \in \mathbb{R} \). The Lie algebra structure of \( s \) satisfies

\[
[X, H_j] = [X, E_j] = 0 \quad \text{and} \quad \text{ad}_X \in \text{sp}(V_j, \Omega_j)
\]

for each \( 1 \leq j \leq r-1 \) and \( X \in (s_r \ltimes \cdots) \ltimes s_{j+1} \).

Remark 8. For each \( 1 \leq j \leq r \), the Lie subalgebra \( s_j \subset s \) is isomorphic to the Lie algebra of the Iwasawa group of \( SU(1, n_j) \); [Ko14, Ch.1, prop.1.5.10].

Let’s point out that both the number of Lie subalgebras \( s_j \) and the number of Lie subgroups \( S_j \) in these Pyatetskii-Shapiro decompositions correspond to the rank of the domain \( \mathbb{D} \). This fact is not completely obvious in the statement [Py69, Ch.2, lem.1] but it can be deduced from the relations

\[
\mathfrak{n} = [s, s] = \bigoplus_{j=1}^r (V_j \oplus \mathbb{R}E_j) \quad \text{and} \quad \alpha \simeq \bigoplus_{j=1}^r \mathbb{R}H_j
\]

further in the reference [Py69, Ch.2, §3].
3 Intertwining invariant deformation quantizations

Till the end of this article, we will work through the identification $\mathbb{D} \simeq S$.

We now introduce the premises of a strategy leading to a realization of the space of the $G$-invariant star-products on the domain $\mathbb{D}$. This strategy is based on the retract method initiated in \cite{B+09} and further extended by Bieliavsky both in for formal and non-formal deformation quantizations; \cite{Ko14, Ch. 2, § 5 & 8}, \cite{Bi17}. Roughly speaking, in this context, this method can be described by two steps:

(i) computing a set of invariant deformation quantizations on a curvature contraction of $\mathbb{D}$ sharing a common symmetry group with $\mathbb{D}$;

(ii) intertwining these deformation quantizations with equivariant operators reversing the contraction process.

This approach is intuitively motivated by the fact that it should be easier to compute invariant deformation theory on a curvature contraction of $\mathbb{D}$. Once step (i) is completed, the difficulty consists in reversing the contraction process. In the case of formal deformation quantizations, i.e. star-products, the intertwiners are expressed as formal differential operators called equivalence of invariant star-products. In the case of non-formal deformation quantizations, intertwiners calculus involves equivalence of Lie group representations.

3.1 Equivalence of invariant star-products

Our starting point is the recent memoir \cite{BG15} in which Bieliavsky and Gayral developped a formal and non-formal left-invariant deformation theory on every negatively curved left-invariant Kählerian Lie group. In particular, their work yields an explicit infinite dimensional parameter family of $S$-invariant star-products on $S$, each of them underlying a non-formal deformation quantization. With the objective of exploiting this major result, we are going to use well-established properties of star-products in order to transform such $S$-invariant star-products into $G$-invariant ones.

**Definition 9.** Let $G_1$ be a Lie subgroup of $G$. Two $G_1$-invariant star-products $\ast_\nu$ and $\ast'_\nu$ on $\mathbb{D}$ are said to be $G_1$-equivalent if there exists a sequence $\{T_k : k \in \mathbb{N}\}$ of $C[[\nu]]$-linear differential operators on $C^\infty (\mathbb{D})[[\nu]]$ that vanish on constants, commute with the action $\tau$ of $G_1$ and are such that the operator

$$T = \text{Id} + \sum_{k=1}^{\infty} \nu^k T_k$$

satisfies $T (f_1 \ast_\nu f_2) = T(f_1) \ast'_\nu T(f_2)$ (5)

for each $f_1, f_2 \in C^\infty (\mathbb{D})$. In this case, the operator $T$ is called a $G_1$-equivalence and this relation between $\ast_\nu$ and $\ast'_\nu$ is denoted by $\ast'_\nu = T(\ast_\nu)$.

In the present text, if $\ast_\nu$ is a $S$-invariant star-product on $\mathbb{D}$, the notation $\text{Op}^S (\ast_\nu)$ will designate the collection of $S$-equivalences between $\ast_\nu$ and any other $S$-equivalent $S$-invariant star-product on $\mathbb{D}$. The following remark from harmonic analysis is quite important in a non-formal perspective.
Remark 10. Let $T$ be a $S$-equivalence of star-products of the form (5). Through the identification $D \simeq S$, given that $\tau_s = I_s$ for all $s \in S$, the operator $T_k$ has to commute with the left-invariant translations on $S$ for each $k \in \mathbb{N} \setminus \{0\}$. As a consequence, the $S$-equivalence $T$ should necessarily be an invertible linear convolution operators on $C^\infty (S) [[\nu]]$. If $ds$ denotes the left-invariant Haar measure on $S$, we then have

$$T : f \in D (S) \mapsto \left[ T (f) : s_0 \in S \mapsto \int_S u_T (s^{-1} s_0) f (s) \, ds \right]$$

(6)

where $u_T \in D' (S) [[\nu]]$ is a formal distribution on $S$ associated with $T$.

In view of this remark, it would be legitimate to express some of these $S$-equivalences within a functional framework allowing to compute explicitly $G$-invariant non-formal deformation quantizations on $D$ in further work.

3.2 Classification results

For every explicit $S$-invariant star-product $*_\nu$, obtained in [BG15], a natural approach to our quantization problem would be to determine the set of $S$-equivalences of star-products $T \in \text{Op}^S (*_\nu)$ such that $T (*_\nu)$ is $G$-invariant.

Nevertheless, in order to justify such method, we have to prove that every $G$-invariant star-product on $D$ can be reached in this way. For this, we need the following classification result.

Proposition 11. [B+98 thm.4.1] For every Lie subgroup $G_1 \subset G$, the $G_1$-equivalence classes of $G_1$-invariant star-products on $D$ are parametrized by the space of formal power series with coefficients in the second cohomology space of the $G_1$-invariant de Rham complex on $D$.

As a consequence of this proposition, the $S$-equivalence classes of $S$-invariant star-products on $D \simeq S$ are parametrized by the space of formal power series with coefficients in the second Chevalley-Eilenberg cohomology space $H^{2}_{CE}(s)$ for the trivial representation of $s$ on $C$. Let’s compute explicitly this cohomology space by using the Pyatetskii-Shapiro decomposition of $s$.

Lemma 12. In the notations of lemma 7 an anti-symmetric bilinear map $c : s \times s \to C$ defines a Chevalley-Eilenberg 2-cocycle for the trivial representation of $s$ on $C$ if and only if it satisfies the following conditions:

1. $c(v_j, E_j) = 0$ and $2c(v_j, v'_j) = \Omega_j (v_j, v'_j) c(H_j, E_j)$ for each $1 \leq j \leq r$ and $v_j, v'_j \in V_j$;
2. $c(X, E_j) = 0$ for each $1 \leq j < r$ and $X \in (s_r \times ...) \ltimes s_{j+1}$;
3. $c(X, v_j) = c(H_j, [X, v_j])$ for each $1 \leq j < r$, $X \in (s_r \times ...) \ltimes s_{j+1}$ and $v_j \in V_j$;
4. $c(v_k, H_j) = 0$ and $c(E_k, H_j) = 0$ for each $1 \leq j < k \leq r$ and $v_k \in V_k$.

In particular, the data of such a Chevalley-Eilenberg 2-cocycle $c$ is completely determined by an arbitrary choice of:

- linear maps $c_j : V_j \oplus \mathbb{R} E_j \to C : X \mapsto c_j (X) := c(H_j, X)$ for $1 \leq j \leq r$;
- constants $c_{jk} := c(H_j, H_k) \in C$ for $1 \leq j < k \leq r$. 

9
Proof. Let \( c : s \times s \to \mathbb{C} \) be an anti-symmetric bilinear map. By definition, it is a Chevalley-Eilenberg 2-cocycle for the trivial representation of \( s \) on \( \mathbb{C} \) if and only if it satisfies

\[
\delta c(X,Y,Z) := c([X,Y],Z) + c([Y,Z],X) + c([Z,X],Y) = 0
\]

for all \( X,Y,Z \in s \). We are going to use the properties of \( c \) and the Lie algebra structure of \( s \) described in lemma \( \ref{lem:properties} \) in order to implement explicitly this condition on \( c \). We proceed by induction on the rank \( r \) of the domain \( \mathbb{D} \simeq S \).

Initial step. In the case \( r = 1 \), we can remark that relation \( (i) \) is equivalent to the equations

\[
\delta c(H_1,E_1,v_1) = 0 \quad \text{and} \quad \delta c(H_1,v_1,v_1') = 0 \quad \text{for} \quad v_1,v_1' \in V_1.
\]

An easy computation shows that these equalities implies \( \delta c = 0 \). As a consequence, the result follows given that relations \( (ii) \), \( (ii') \) and \( (iii) \) are trivially satisfied.

Inductive step. Let’s assume that the statement of the lemma is true for \( r = r_0 \in \mathbb{N} \setminus \{0\} \) and let’s prove it for \( r = r_0 + 1 \). We set

\[
\mathfrak{S} := (...) \mathfrak{s}_{r_0+1} \ltimes \mathfrak{s}_{r_0} \ltimes \cdots \ltimes \mathfrak{s}_3 \ltimes \mathfrak{s}_2, \quad c_{\mathfrak{S}} := c|_{\mathfrak{S} \times \mathfrak{S}} \quad \text{and} \quad c_{\mathfrak{s}_1} := c|_{\mathfrak{s}_1 \times \mathfrak{s}_1}.
\]

We notice that \( c \) is a Chevalley-Eilenberg 2-cocycle if and only if \( c_{\mathfrak{S}} \) and \( c_{\mathfrak{s}_1} \) are Chevalley-Eilenberg 2-cocycles and \( \delta c \) vanishes on \( \mathfrak{S} \times \mathfrak{S} \times s_1 \) and \( \mathfrak{S} \times s_1 \times s_1 \). For each \( X,Y \in \mathfrak{S} \) and \( v_1,v_1' \in V_1 \), the Lie algebra structure of \( s \) yields the equalities

\[
\begin{align*}
(ii) \quad & \delta c(H_1,E_1,X) = 2 c(E_1,X), \\
(ii') \quad & \delta c(H_1,v_1,X) = c(v_1,X) + c(H_1,[X,v_1]), \\
(0) \quad & \delta c(v_1,E_1,X) = c([X,v_1],E_1) \quad \text{and} \quad \delta c(v_1,v_1',X) = \Omega_1 (v_1,v_1') c(E_1,X) \quad \text{as} \quad \text{ad}_X \in \text{sp}(V_1,\Omega_1), \\
(iii) \quad & \delta c(H_1,X,Y) = c([X,Y],H_1) \quad \text{and} \quad \delta c(E_1,X,Y) = c([X,Y],E_1), \\
(0') \quad & \delta c(v_1,X,Y) = c([v_1,X],Y) + c([X,Y],v_1) + c([Y,v_1],X).
\end{align*}
\]

• Necessary condition. If \( c \) is a Chevalley-Eilenberg 2-cocycle, as \( c_{\mathfrak{S}} \) and \( c_{\mathfrak{s}_1} \) satisfy our induction hypothesis, the necessary condition will be proven if we have relations \( (ii) \), \( (ii') \) and \( (iii) \) for \( j = 1 \). These relations can be respectively deduced from \( (ii) \), \( (ii') \) and \( (iii) \) given that

\[
\delta c = 0 \quad \text{and} \quad [\mathfrak{S},\mathfrak{S}] = \bigoplus_{j=2}^{r_{0}+1} (V_j \oplus \mathbb{R}E_j).
\]

• Sufficient condition. Let’s assume that relations \( (i) \), \( (ii) \), \( (ii') \) and \( (iii) \) are satisfied. Then, by using relation \( (ii') \) and the Jacobi identity, the equation \( (0') \) can be written

\[
\delta c(v_1,X,Y) = c(H_1,[[v_1,X],Y]) + c(H_1,[[X,Y],v_1]) + c(H_1,[[Y,v_1],X]) = 0
\]

for each \( X,Y \in \mathfrak{S} \) and \( v_1 \in V_1 \). From the combinaison of this last equality with relations \( (ii)-(ii'), \quad (ii)-(ii') \), \( (0)-(i)-(ii) \) and \( (iii)-(iii) \), it is clear that \( \delta c \) vanishes on \( \mathfrak{S} \times \mathfrak{S} \times s_1 \) and \( \mathfrak{S} \times s_1 \times s_1 \). Since the maps \( c_{\mathfrak{S}} \) and \( c_{\mathfrak{s}_1} \) are Chevalley-Eilenberg 2-cocycles by induction hypothesis, the proof of the sufficient condition is complete. \( \square \)
Lemma 13. Let \( c : s \times s \to \mathbb{C} \) be a Chevalley-Eilenberg 2-cocycle for the trivial representation of \( s \) on \( \mathbb{C} \). The following assertions are equivalent:

1. the 2-cocycle \( c \) is a Chevalley-Eilenberg 2-coboundary;
2. in the notations of lemma 7 we have \( c(H_j, H_k) = 0 \) for each \( 1 \leq j \leq r \) and \( 1 \leq k \leq r \);
3. the 2-cocycle \( c \) vanishes on \( a \times a \).

Proof. We prove separately the implications (1) \( \Rightarrow \) (2) \( \land \) (3), (2) \( \Rightarrow \) (1) and (3) \( \Rightarrow \) (1).

- By definition, if \( c \) is a Chevalley-Eilenberg 2-coboundary, there exists a linear map \( \alpha : s \to \mathbb{C} \) such that \( c(X, Y) = \alpha([X, Y]) \) for each \( X, Y \in s \). Given that \( \mathbb{R} H_1 \oplus \ldots \oplus \mathbb{R} H_r \) and \( a \) are two isomorphic abelian Lie subalgebras of \( s \), the implication (1) \( \Rightarrow \) (2) \( \land \) (3) follows trivially.

- If the 2-cocycle \( c \) satisfies \( c(H_j, H_k) = 0 \) for each \( j, k \in \{1, \ldots, r\} \), we can easily use lemmas 7 and 12 for checking that the linear map \( \alpha \), defined on \( s \) by

\[
\alpha(H_j) := 0, \quad \alpha(v_j) := c(H_j, v_j) \quad \text{and} \quad \alpha(E_j) := \frac{c(H_j, E_j)}{2} \quad \text{for all} \quad 1 \leq j \leq r \quad \text{and} \quad v_j \in V_j,
\]

is such that \( c(X, Y) = \alpha([X, Y]) \) for each \( X, Y \in s \). This proves the implication (2) \( \Rightarrow \) (1).

- Let’s assume that the 2-cocycle \( c \) vanishes on \( a \times a \). In the notations of the root space decomposition of \( g \), we define the linear map \( \alpha' : s \to \mathbb{C} \) by

\[
\alpha'(H_\lambda) := 0 \quad \text{and} \quad \alpha'(X) := \frac{c(H_\lambda, X)}{\lambda(H_\lambda)} \quad \text{for all} \quad \lambda \in \Sigma^+ \quad \text{and} \quad X \in g_\lambda.
\]

Then, for each \( \lambda, \mu \in \Sigma^+, \ X \in g_\lambda \) and \( Y \in g_\mu \), the properties of \( c \) as 2-cocycle yields

\[
c(H_\lambda, Y) = \frac{c(H_\lambda, [H_\mu, Y])}{\mu(H_\mu)} = \frac{c(H_\mu, [H_\lambda, Y]) + c(Y, [H_\mu, H_\lambda])}{\mu(H_\mu)} = \frac{\mu(H_\lambda)}{\mu(H_\mu)} c(H_\mu, Y) = \mu(H_\lambda) \alpha'(Y) = \alpha'([H_\lambda, Y])
\]

and

\[
c(X, Y) = \frac{c([H_\lambda+\mu, X], Y) + c(X, [H_\lambda+\mu, Y])}{(\lambda + \mu)(H_{\lambda+\mu})} = \frac{c(H_{\lambda+\mu}, [X, Y])}{(\lambda + \mu)(H_{\lambda+\mu})} = \alpha'([X, Y]),
\]

where the last equality comes from the relation \( [X, Y] \in [g_\lambda, g_\mu] \subset g_{\lambda+\mu} \). As a consequence, we obtain \( c(X', Y') = \alpha'([X', Y']) \) for each \( X', Y' \in s \) and the proof of the lemma is complete.

In the rest of the text, the notation \([c]\) will refer to the Chevalley-Eilenberg cohomology class of a Chevalley-Eilenberg 2-cocycle \( c \). The following corollary is direct from proposition 11 and lemmas 12 and 13.

Corollary 14. The map

\[
[c] \in H^2_{CE}(s) \mapsto (c(H_j, H_k))_{1 \leq j < k \leq r}
\]

induces an isomorphism between \( H^2_{CE}(s) \) and \( \mathbb{C}^{r(r-1)/2} \). In particular, the \( \mathbb{S} \)-equivalence classes of \( \mathbb{S} \)-invariant star-products on \( \mathbb{D} \) are parametrized by

\[
\mathbb{C}^{r(r-1)/2} \langle [\nu] \rangle.
\]
We are now going to prove that all the $G$-invariant star-products on $\mathbb{D}$ belong to the same $\mathbb{S}$-equivalence class of $\mathbb{S}$-invariant star-products. For the case $r > 1$, this result is very important in order to certify that the natural approach via intertwiners we described above is not so naive and can really be efficient. In what follows, we will denote by $Z^2(\mathbb{D})^G$ the set of $G$-invariant closed differential 2-form on $\mathbb{D}$.

**Lemma 15.** Every $G$-invariant closed differential 2-form on $\mathbb{D}$ is the exterior derivative of a $\mathbb{S}$-invariant differential 1-form on $\mathbb{D}$.

**Proof.** It is well known that the data of any $G$-invariant closed differential 2-form on $\mathbb{S} \simeq \mathbb{D}$ is given by its evaluation at the base point $\text{Id} \in \mathbb{S}$ which defines a Chevalley-Eilenberg 2-cocycle for the trivial representation of $s$ on $\mathbb{C}$. In particular, for such an arbitrary 2-cocycle $c$, the proof will be complete if we show that $c$ is a Chevalley-Eilenberg 2-coboundary. Let’s notice that our $G$-invariance hypothesis yields
\[
c((\text{Ad}_k(X))_s, [\text{Ad}_k(Y)]_s) = c((\tau_k)_*\text{Id}(X), (\tau_k)_*\text{Id}(Y)) = c(X, Y)
\]
for each $k \in K$ and $X, Y \in s$. The infinitesimal version of this last relation can be written
\[
c([Z, X]_s, Y) + c(X, [Z, Y]_s) = 0
\]
for each $Z \in \mathfrak{k}$ and $X, Y \in s$. Let’s now consider $\lambda, \mu \in \Sigma^+$ and $X \in \mathfrak{g}_\lambda$. Since $X + \sigma(X) \in \mathfrak{k}$, we can use (3) and (7) in order to obtain
\[
\beta(X, \sigma(X)) c(H_\lambda, H_\mu) = c([X, X + \sigma(X)], H_\mu) = c(X, [X + \sigma(X), H_\mu]_s)
\]
\[
= c(X, [H_\mu(\sigma(X) - X)]_s)
\]
\[
= c(X, -2\lambda(\mu) X) = 0.
\]
Therefore, the 2-cocycle $c$ vanishes on $\alpha \times \alpha$ and lemma 13 allows us to conclude the proof. \[\blacksquare\]

**Remark 16.** From the previous proof, the evaluation operator of differential forms on $\mathbb{D}$ at the base point $\text{Id} \in \mathbb{S} \simeq \mathbb{D}$ induces a linear isomorphism between $Z^2(\mathbb{D})^G$ and the space of linear maps $\alpha : n = [s, s] \rightarrow \mathbb{C}$ satisfying
\[
\alpha([\text{Ad}_k(X)]_s, [\text{Ad}_k(Y)]_s) = \alpha([X, Y])
\]
for each $k \in K$ and $X, Y \in s$. In particular, for $Z \in m \subset \mathfrak{k}$ and $X, Y \in s$, such a linear map $\alpha : n \rightarrow \mathbb{C}$ has to satisfy the relation
\[
\alpha([Z, [X, Y]]) = \alpha([Z, X, Y]) + \alpha([X, [Z, Y]]) = 0,
\]
given that $m \subset N(n)$ and $[m, a] = 0$. As a consequence, we deduce from condition (3) and lemma 3 that such a map $\alpha$ has to vanish on every root space $\mathfrak{g}_\lambda$ with $\lambda \in \Sigma^+$ and $\dim(\mathfrak{g}_\lambda) > 1$.

As a Kählerian manifold, the domain $\mathbb{D}$ is naturally endowed with a $G$-invariant symplectic connection $\nabla$. Then, given an arbitrary formal power series $\omega_\nu \in Z^2(\mathbb{D})^G[[\nu]]$, the Fedosov’s construction of star-products provides us with a $G$-invariant star-product $*_\nu(\nabla, \omega_\nu)$ on $\mathbb{D}$; [Fe94]. In addition, every $G$-invariant star-product on $\mathbb{D}$ is $G$-equivalent to a Fedosov star-product of this form; [B+98 §4, prop. 4.1]. If $G_1$ is a
Lie subgroup of $G$, for each $\omega_\nu, \eta_\nu \in Z^2(\mathbb{D})^G[\nu]$, the $G$-invariant star-products $*_{\nu}(\nabla, \omega_\nu)$ and $*_{\nu}(\nabla, \eta_\nu)$ are $G_1$-equivalent if and only if $\omega_\nu - \eta_\nu$ is a formal power series in $\nu$ which coefficients are exterior derivative of $G_1$-invariant differential 1-forms on $\mathbb{D}$; [BG15] § 3, thm. 3.1 & thm. 3.2. As a consequence, we get the following proposition from lemma [15]

**Proposition 17.** Every $G$-invariant star-product on $\mathbb{D}$ is $S$-equivalent to the Fedosov star-product $*_{\nu}(\nabla, 0)$.

Among the set of $S$-invariant star-products explicitly described in the work of Bieliavsky and Gayral [BG15], let’s choose a star-product $*_\nu^0$, which is $S$-equivalent to the Fedosov star-product $*_{\nu}(\nabla, 0)$. Then, the previous proposition allows us to refine a method for realizing the space of $G$-invariant star-products on $\mathbb{D}$ as the description of the set of operators $T \in \text{Op}^S(*_\nu^0)$ such that $T(*_{\nu^0})$ is $G$-invariant. Let’s point out that it is enough to intertwine the only one initial $S$-invariant star-product $*_{\nu^0}$ in this way for obtaining the full set of $G$-invariant star-products on $\mathbb{D}$.

Let’s now conclude this section by some considerations on the classification of the $G$-invariant star-products on $\mathbb{D}$.

**Lemma 18.** The $G$-equivalence classes of $G$-invariant star-products on $\mathbb{D}$ are parametrized by the space $Z^2(\mathbb{D})^G[\nu]$.

**Proof.** For each $\lambda \in \Sigma^+$ and $X \in g_\lambda \setminus \{0\}$, we have $X + \sigma(X) \in \mathfrak{k}$ and the properties of the root space decomposition of $g$ give

$$X = \left[ X - \frac{X + \sigma(X)}{2} \right]_{s} = \left[ \frac{X - \sigma(X)}{2} \right]_{s} = \left( \frac{1}{\lambda(H_\lambda)} \right) \left[ H_\lambda, \frac{X + \sigma(X)}{2} \right]_{s}$$

and

$$H_\lambda = \left( \frac{1}{\beta(X, \sigma(X))} \right) [X, \sigma(X)] = \left( \frac{1}{\beta(X, \sigma(X))} \right) [[X, X + \sigma(X)]_{s}],$$

Therefore, we obtain the relation $s = \{ [[X, Y]]_{s} : X \in s, Y \in \mathfrak{k} \}$. A similar argument as the one used for the proof of lemma [15] shows that a $G$-invariant differential 1-form on $\mathbb{D}$ is completely determine by a linear map $\alpha : s \to \mathbb{C}$ which satisfies $\alpha([[X, Y]]_{s}) = 0$ for each $X \in s$ and $Y \in \mathfrak{k}$. As a consequence, such a $G$-invariant 1-form is identically zero and the proof is then complete in view of proposition [11].

We can notice that there exist $G$-invariant star-products on $\mathbb{D}$ that are not $G$-equivalent since $\omega$ is a $G$-invariant symplectic form on $\mathbb{D}$. Nevertheless, it happens that $\omega$ is the only one generator of the space $Z^2(\mathbb{D})^G$. In fact, let’s assume that the domain $\mathbb{D}$ is an irreducible Hermitian symmetric space of non compact type as introduced in the reference [He01], Ch. 8. § 5. In the notations of the previous section, this hypothesis implies that the adjoint action of $K$ on $p$ is irreducible. If we denote by $\iota$ the vector space isomorphism $p \simeq s$ which associates $[X]_{s}$ with $X \in p$, it follows that the map

$$K \times s \to s : (k, X) \mapsto (\iota \circ \text{Ad}_k \circ \iota^{-1})(X) = [\text{Ad}_k(X)]_{s}$$

defines an irreducible action of $K$ on $s$. Let’s now consider $\nu$ a $G$-invariant closed differential 2-form on $\mathbb{D}$ and the linear map $\alpha : n \to \mathbb{C}$ which is associated with $\nu$ via remark [16]. As the map $\alpha$ satisfies the equality $[9]$ for each $k \in K$, the set $E_\alpha := \{ X \in s : \alpha \circ \text{ad}_X = 0 \}$ defines an invariant subspace of $s$ for
the action \([\mathcal{G}]\). Since this action is irreducible, we conclude that either \(E_{\alpha} = \{0\}\) or \(E_{\alpha} = s\). Given that \(\nu\) is non-degenerate if and only if \(E_{\alpha} = \{0\}\), then either \(\nu\) is a \(G\)-invariant symplectic form on \(\mathbb{D}\) or \(\nu = 0\). As a consequence, the structure theory of irreducible Hermitian symmetric spaces of non compact type allows us to deduce the existence of a constant \(c_{\alpha} \in \mathbb{C}\) and a non-zero element \(Z_0\) generating the center of the Lie algebra \(\mathfrak{t}\) such that

\[
\alpha ([X, Y]) = c_{\alpha} \beta (Z_0, [\nu^{-1}(X), \nu^{-1}(Y)])
\]

for all \(X, Y \in \mathfrak{s}; \ [5M01] \ §1, prop. 1.1\). We then obtain the following proposition.

**Proposition 19.** As a Hermitian symmetric space of non compact type, if the symmetric bounded domain \(\mathbb{D}\) is irreducible, then the symplectic form \(\omega\) generates completely the space \(\mathcal{Z}^2 (\mathbb{D})^G\). In this case, the \(G\)-equivalence classes of \(G\)-invariant star-products on \(\mathbb{D}\) are parametrized by \(\mathbb{C}[\nu]\).

## 4 Derivations and quantum moment maps

Let \(*_{\nu}\) be an arbitrary star-product on \(\mathbb{D}\). If \(\{D_k : k \in \mathbb{N}\}\) is a sequence of \(\mathbb{C}[\nu]\)-linear differential operators on \(C^\infty (\mathbb{D})[\nu]\) such that the operator

\[
D = \sum_{k \in \mathbb{N}} \nu^k D_k \text{ satisfies } D (f_1 *_{\nu} f_2) = D (f_1) *_{\nu} f_2 + f_1 *_{\nu} D (f_2)
\]

for each \(f_1, f_2 \in C^\infty (\mathbb{D})[\nu]\), then we say that \(D\) is a derivation of \(*_{\nu}\). We define \(\text{Der} (*_{\nu})\) to be the set of derivations of \(*_{\nu}\).

**Remark 20.** Let \(G_1\) be a connected Lie subgroup of \(G\) with Lie algebra \(\mathfrak{g}_1\). The star-product \(*_{\nu}\) satisfies condition \([\mathcal{H}]\) for each \(g \in G_1\) if and only if \(X^* \in \text{Der} (*_{\nu})\) for each \(X \in \mathfrak{g}_1\). In particular, this last relation is the infinitesimal version of the \(G_1\)-invariance condition for \(*_{\nu}\).

As the domain \(\mathbb{D}\) is a connected simply connected symplectic manifold, the derivations of \(*_{\nu}\) satisfy the following lemma based on the result \([GR03] \ §6, lem. 6.1\).

**Lemma 21.** For each \(D \in \text{Der} (*_{\nu})\), there exists \(f_D \in C^\infty (\mathbb{D})[\nu]\) such that

\[
D = \frac{1}{2\nu} [f_D, -]_{*_{\nu}} : f \in C^\infty (\mathbb{D})[\nu] \mapsto \frac{1}{2\nu} [f_D, f]_{*_{\nu}} := \frac{1}{2\nu} (f_D *_{\nu} f - f *_{\nu} f_D).
\]

In addition, for all \(f \in C^\infty (\mathbb{D})[\nu]\), the operator \(D_f := \frac{1}{2\nu} [f, -]_{*_{\nu}}\) defines a derivation of \(*_{\nu}\) and we have \(D_f = 0\) if and only if \(f \in \mathbb{C}[\nu]\).

For all \(f_1, f_2 \in C^\infty (\mathbb{D})[\nu]\), it is standard to notice that

\[
\left( \frac{1}{2\nu} [f_1, -]_{*_{\nu}}, \frac{1}{2\nu} [f_2, -]_{*_{\nu}} \right) = \frac{1}{2\nu} \left( \frac{1}{2\nu} [f_1, f_2]_{*_{\nu}}, - \right)_{*_{\nu}}.
\]

The following result is inspired by \([Xu98] \ §6\) and \([Ko14] \ Ch. 2, \ §3\).
Lemma 22. Let $D : g → \text{Der}(*_{\nu}) : X → D_X$ be a Lie algebra homomorphism. Then, there exists a unique linear map

$$Λ : g → C^∞(\mathbb{D})[[\nu]] : X → Λ_X$$

such that

$$D_X = \frac{1}{2\nu} [Λ_X, -]_{\nu} \quad \text{and} \quad Λ_{[X,Y]} = \frac{1}{2\nu} [Λ_X, Λ_Y]_{\nu}$$

(11)

for each $X, Y ∈ g$. In addition, the $0^{th}$ order term of $Λ$ in $ν$ coincides with the moment map $λ$ if and only if the $0^{th}$ order term of $D_X$ in $ν$ coincides with $X^*$ for each $X ∈ g$.

Proof. The existence of a linear map $\hat{Λ} : g → C^∞(\mathbb{D})[[\nu]] : X → \hat{Λ}_X$ such that

$$\hat{Λ}_X = \frac{1}{2\nu} [\hat{Λ}_X, -]_{\nu}$$

for any $X ∈ g$

is clear from the linearity of $D$ and lemma 21. Let’s consider such a map $\hat{Λ}$. As $D$ is a Lie algebra homomorphism, we deduce from relation (10) and lemma 21 the existence of an antisymmetric bilinear map $c : g × g → \mathbb{C}[[\nu]]$ such that

$$\hat{Λ}_{[X,Y]} = \frac{1}{2\nu} [\hat{Λ}_X, \hat{Λ}_Y]_{\nu} + c(X,Y)$$

for each $X, Y ∈ g$. The Jacobi identity allows us to remark that $c$ is a Chevalley-Eilenberg 2-cocycle for the trivial representation of $g$ on $\mathbb{C}$. As the Lie algebra $g$ is semi-simple, an application of the Whitehead lemma provides us with a linear map $α : g → \mathbb{C}[[\nu]]$ such that

$$c(X,Y) = α([X,Y]) \quad \text{for all} \quad X, Y ∈ g.$$

Then, the map $Λ := \hat{Λ} - α$ satisfies (11) for each $X, Y ∈ g$. This proves the existence result of this lemma. In order to prove the unicity of $Λ$, let’s consider an arbitrary linear map $Λ' : g → C^∞(\mathbb{D})[[\nu]]$ such that the analog of conditions (11) hold for all $X, Y ∈ g$. In view of lemma 21, for each $X ∈ g$, we have $Λ_X - Λ'_X ∈ \mathbb{C}[[\nu]]$, and then

$$Λ_{[X,Y]} = \frac{1}{2\nu} [Λ_X, Λ_Y]_{\nu} = \frac{1}{2\nu} [Λ'_X, Λ'_Y]_{\nu} = Λ'_{[X,Y]}$$

(12)

for all $X, Y ∈ g$. As the Lie algebra $g$ is semi-simple, it coincides with its derived algebra $[g, g]$ and we get $Λ = Λ'$ from (12). This leads us to the unicity of $Λ$. For each $X ∈ g$, let’s define $Λ_X^\nu ∈ C^∞(\mathbb{D})$ to be the $0^{th}$ order term of $Λ_X$ in $ν$. From the definition of star-product on a symplectic manifold, we deduce

$$D_X = \{Λ_X^\nu, -\} + o(ν) \quad \text{for all} \quad X ∈ g.$$

As a consequence, if the $0^{th}$ order term of $D_X$ in $ν$ coincides with $X^*$ for each $X ∈ g$, the map

$$Λ^0 : g → C^∞(\mathbb{D}) : X → Λ^0_X$$

satisfies the same properties as the moment map $λ$ and the unicity of such a map implies $Λ^0 = λ$. Reciprocally, if $Λ^0 = λ$, then $D_X = \{λ_X, -\} + o(ν) = X^* + o(ν)$ for each $X ∈ g$. The proof is complete. ■
We notice that the semi-simplicity of $g$ is crucial in the previous statement.

**Definition 23.** In the notations of lemma 22 such a linear map $\Lambda = \lambda + o(\nu)$ will be called *quantum moment map* associated with $D$.

Similarly to the previous section, we denote by $Z^2(\mathbb{D})^G$ the set of $S$-invariant closed differential 2-form on $\mathbb{D} \cong S$. Let’s recall the notation $\nabla$ for the $G$-invariant symplectic connection associated with the Kählerian structure of $\mathbb{D}$. From proposition 17, we know that the $S$-invariant star-products on $\mathbb{D}$ which are $S$-equivalent to the Fedosov star-product $\ast_{(\nu,0)}$ play a particular role in our work. In the spirit of this section on quantum moment maps, we are now going to introduce an alternative definition of these star-products.

**Proposition 24.** The star-product $\ast_{\nu}$ is $S$-invariant and $S$-equivalent to the Fedosov star-product $\ast_{(\nu,0)}$ if and only if there exists a linear map $\Lambda : s \to C^\infty(\mathbb{D})[[\nu]] : X \mapsto \Lambda_X$ such that (11) hold with $D_X = X^*$ for each $X, Y \in s$.

**Proof.** It is clear from remark 20 and lemma 21 that the $S$-invariance property of $\ast_{\nu}$ is expressed both in the necessary condition and in the sufficient condition of this proposition. As a consequence, we can choose an operator $T \in \text{Op}^S(\ast_{\nu})$ and a formal power series $\omega_\nu \in Z^2(\mathbb{D})^G[[\nu]]$ such that the star-product $\ast_{(\nu,\omega_\nu)}$ provided by the Fedosov’s construction [Fe94] satisfies the relation $T(\ast_{\nu}) = \ast_{(\nu,\omega_\nu)}$; [B+98 §4, prop. 4.1]. In view of remark 20, the vector field $X^*$ is a derivation of $\ast_{(\nu,\omega_\nu)}$ for each $X \in s$. The combinaison of this fact with well known results on quantum moment maps for Fedosov star-products produces a linear map $\hat{\Lambda} : s \to C^\infty(\mathbb{D})[[\nu]] : X \mapsto \hat{\Lambda}_X$ such that

$$i_{X^*}(\omega + \nu \omega_\nu) = -d\hat{\Lambda}_X \quad \text{and} \quad X^* = \frac{1}{2\nu} \left[ \hat{\Lambda}_X, - \right]_{\ast_{(\nu,\omega_\nu)}}$$

(13)

for each $X \in s$; [GR03 §7, thm. 7.2]. After these preliminary considerations, let’s prove separately the necessary condition and the sufficient condition of this proposition.

- **Necessary condition.** In view of our hypothesis, we can choose $\omega_\nu = 0$. Therefore, the definition of the moment map $\lambda$, relation (13) and lemma 21 and allow us to deduce that $\hat{\Lambda}_X - \lambda_X \in C[[\nu]]$ and $X^* = \frac{1}{2\nu} \left[ \lambda_X, - \right]_{\ast_{(\nu,\omega_\nu)}}$ for each $X \in s$. In particular, as the operator $T$ is a $S$-equivalence, we obtain

$$X^* = T^{-1} \circ \left( \frac{1}{2\nu} \left[ \lambda_X, - \right]_{\ast_{(\nu,\omega_\nu)}} \right) \circ T = \frac{1}{2\nu} \left[ T^{-1}(\lambda_X), - \right]_{\ast_{\nu}}$$

and

$$\frac{1}{2\nu} \left[ T^{-1}(\lambda_X), T^{-1}(\lambda_Y) \right]_{\ast_{\nu}} = T^{-1} \left( \frac{1}{2\nu} \left[ \lambda_X, \lambda_Y \right]_{\ast_{(\nu,\omega_\nu)}} \right) = T^{-1}(X^*(\lambda_Y)) = T^{-1}(\hat{\Lambda}_{(X,Y)}) ,$$

for each $X, Y \in s$. As a consequence, the linear map $T^{-1} \circ \lambda$ satisfies (11) with $D_X = X^*$ for each $X, Y \in s$. This proves the necessary condition.

- **Sufficient condition.** Let $\Lambda : s \to C^\infty(\mathbb{D})[[\nu]] : X \mapsto \Lambda_X$ be a linear map such that (11) hold with $D_X = X^*$ for each $X, Y \in s$. By using the properties of $T \in \text{Op}^S(\ast_{\nu})$ and the definition of $\hat{\Lambda}$, we get

$$\frac{1}{2\nu} \left[ \hat{\Lambda}_X, - \right]_{\ast_{(\nu,\omega_\nu)}} = X^* = T \circ X^* \circ T^{-1} = T \circ \left( \frac{1}{2\nu} \left[ \lambda_X, - \right]_{\ast_{\nu}} \right) \circ T^{-1} = \frac{1}{2\nu} \left[ T(\Lambda_X), - \right]_{\ast_{(\nu,\omega_\nu)}}$$

for all $X \in s$. In particular, we deduce from lemma 21 and relation (13) that $\hat{\Lambda}_X - T(\Lambda_X) \in C[[\nu]]$ and
then \( \iota_X \cdot (\omega + \nu \omega_\nu) = -d(T(\Lambda_X)) \) for all \( X \in \mathfrak{s} \). As a result, for each \( X, Y \in \mathfrak{s} \), we have
\[
(\omega + \nu \omega_\nu)(X^*, Y^*) = -d(T(\Lambda_X))(Y^*) = -\frac{1}{2\nu} [T(\Lambda_Y), T(\Lambda_X)]_{s(\nu, \omega_\nu)}
= T \left( \frac{1}{2\nu} [\Lambda_X, \Lambda_Y]_{s_\nu} \right) = T(\Lambda_{[X,Y]}) .
\]

The evaluation of \( \omega + \nu \omega_\nu \) at the base point \( \text{Id} \in \mathbb{S} \simeq \mathbb{D} \) defines a formal power series \( c \) in \( \nu \) which coefficients are Chevalley-Eilenberg 2-cocycles for the trivial representation of \( \mathfrak{s} \) on \( \mathbb{C} \). As we have the equality \( c(X, Y) = T(\Lambda_{[X,Y]})(\text{Id}) \) for each \( X, Y \in \mathfrak{s} \), it is clear that these coefficients are Chevalley-Eilenberg 2-cocycles. It follows that \( \omega_\nu \) is a formal power series in \( \nu \) which coefficients are exterior derivative of \( \mathbb{S} \)-invariant differential 1-forms on \( \mathbb{D} \). The star-product \( \ast_\nu(\nabla, \omega_\nu) \) is then \( \mathbb{S} \)-equivalent to \( \ast_\nu(\nabla, 0) \) in virtue of the theorem \([B+98] \S 3, \text{thm.} 3.1 \& \text{thm.} 3.2 \) mentioned in the previous section. This concludes the proof of the sufficient condition.

Let’s point out that the key of this proof lies in the fact that we can choose the map \( \hat{\Lambda} \) such that
\[
T^{-1} \left( \hat{\Lambda}_{[X,Y]} \right) = \frac{1}{2\nu} \left[ T^{-1} \left( \hat{\Lambda}_X \right), T^{-1} \left( \hat{\Lambda}_Y \right) \right]_{s_\nu}
\]
for each \( X, Y \in \mathfrak{s} \) if and only if \( \ast_\nu \) is \( \mathbb{S} \)-equivalent to \( \ast_\nu(\nabla, 0) \). The non-triviality of this statement comes from the fact that the Lie algebra \( \mathfrak{s} \) is not semi-simple.

**Remark 25.** In the context of proposition 24, we deduce from lemma 21 and relation 12 that the map \( \Lambda_X \) is uniquely determined for \( X \in \mathfrak{n} = [\mathfrak{s}, \mathfrak{s}] \) and uniquely determined up to a formal constant for \( X \in \mathfrak{a} \). Moreover, the same argument as the one used in the proof of lemma 22 shows that \( \Lambda = \lambda + o(\nu) \).

Let’s conclude this section by introducing a particular case of proposition 24.

**Definition 26.** We say that the star-product \( \ast_\nu \) is \( \mathfrak{s} \)-covariant if the equation
\[
\frac{1}{2\nu} [\lambda_X, \lambda_Y]_{s_\nu} = \lambda_{[X,Y]}
\]
is satisfied for each \( X, Y \in \mathfrak{s} \).

**Proposition 27.** The star-product \( \ast_\nu \) is \( \mathbb{S} \)-invariant, \( \mathfrak{s} \)-covariant and \( \mathbb{S} \)-equivalent to the Fedosov star-product \( \ast_{\nu}(\nabla, 0) \) if and only if \( X^* = \{ \lambda_X, -\} = \frac{1}{2\nu} [\lambda_X, -]_{s_\nu} \) for each \( X \in \mathfrak{s} \).

**Proof.** The sufficient condition is clear in view of properties of the moment map \( \lambda \), definition 26 and proposition 24. Let’s assume that \( \ast_\nu \) is \( \mathbb{S} \)-invariant, \( \mathfrak{s} \)-covariant and \( \mathbb{S} \)-equivalent to \( \ast_{\nu}(\nabla, 0) \) and let’s prove the necessary condition. In virtue of proposition 24, we consider a linear map \( \Lambda : \mathfrak{s} \to C^\infty(\mathbb{D})[[\nu]] : X \mapsto \Lambda_X \) satisfying 11 with \( D_X = X^* \) for each \( X, Y \in \mathfrak{s} \). For each \( X \in \mathfrak{s} \), we set
\[
\Lambda'_X := \frac{1}{\nu} (\Lambda_X - \lambda_X) \in C^\infty(\mathbb{D})[[\nu]].
\]
For all \( X, Y \in \mathfrak{s} \), our hypotheses and the properties of the moment map \( \lambda \) allow us to deduce the equality
\[
\frac{1}{2\nu} [\nu \Lambda'_X, \lambda_Y]_{s_\nu} = \frac{1}{2\nu} [\Lambda_X - \lambda_X, \lambda_Y]_{s_\nu} = X^*(\lambda_Y) - \lambda_{[X,Y]} = 0.
\]
As a consequence, for all \( X, Y \in \mathfrak{s} \), we obtain
\[
\frac{1}{2\nu} [\nu \Lambda'_{X}, \nu \Lambda'_{Y}]_{s_{\nu}} = \frac{1}{2\nu} [\Lambda_{X}, \Lambda_{Y}]_{s_{\nu}} - \frac{1}{2\nu} [\lambda_{X}, \lambda_{Y}]_{s_{\nu}} = \Lambda_{[X,Y]} - \lambda_{[X,Y]} = \nu \Lambda'_{[X,Y]}.
\]
For all \( X \in \mathfrak{n} \), by using the properties of root space decomposition of \( \mathfrak{g} \) and a recursive argument on the order terms of \( \Lambda' \) in \( \nu \) in the relation
\[
\frac{1}{2\nu} [\nu \Lambda'_{X}, \nu \Lambda'_{H}]_{s_{\nu}} = \Lambda'_{[X,H]} - \lambda_{[X,H]} = \nu \Lambda'_{[X,H]}
\]
for \( H \in \mathfrak{a} \), we deduce easily the equality \( \Lambda'_{X} = 0 \). Therefore, as \( [s,s] = \mathfrak{n} \), we get
\[
X^* (\nu \Lambda'_{H}) = X^* (\Lambda_{H} - \lambda_{H}) = \frac{1}{2\nu} [\Lambda_{X}, \Lambda_{H}]_{s_{\nu}} - \lambda_{[X,H]} = \nu \Lambda'_{[X,H]} = 0
\]
for all \( X \in s \) and \( H \in a \). It follows that \( \Lambda'_{H} \in \mathbb{C}[\nu] \) for each \( H \in a \). As a conclusion, we have
\[
\frac{1}{2\nu} [\Lambda_{X}, -]_{s_{\nu}} = 0 \quad \text{for each} \quad X \in s.
\]
This completes the proof of the proposition.

5 Deformation quantization method

As suggested above in view of reference [BG15] and proposition 17, we fix an explicit \( S \)-invariant star-product \( *_{\nu} \) on \( D \) which is \( S \)-equivalent to the Fedosov star-product \( *_{0}^{(\nabla,0)} \) where \( \nabla \) is the \( G \)-invariant symplectic connection associated to the Kählerian structure of \( D \). The data of a \( G \)-invariant star-products on \( D \) is then equivalent to the data of an invertible linear convolution operator \( T \in \text{Op}^{\mathbb{S}} (*_{\nu}) \) of the form (6) such that \( T (*_{\nu}) \) is \( G \)-invariant. We will work through the identification \( D \simeq \mathbb{S} \).

5.1 Equivariant automorphisms of star-products

In this subsection, we develop some basic results about the following class of operators
\[
\text{Aut} (*_{\nu}) := \left\{ T \in \text{Op}^{\mathbb{S}} (*_{\nu}) : T (*_{\nu}) = *_{\nu} \right\}.
\]
A element of \( \text{Aut} (*_{\nu}) \) we be called \( S \)-automorphism of \( *_{\nu} \).

**Lemma 28.** For each operators \( T, T' \in \text{Op}^{\mathbb{S}} (*_{\nu}) \) such that \( T (*_{\nu}) \) and \( T' (*_{\nu}) \) are two \( G \)-equivariant \( G \)-invariant star-products on \( D \), there exist \( S \in \text{Aut} (*_{\nu}) \) and \( U \) a \( G \)-equivalence of \( G \)-invariant star-products on \( D \) satisfying
\[
T' = U \circ T \circ S.
\]
This lemma is direct. In fact, if \( U \) denotes a \( G \)-equivalence of \( G \)-invariant star-products on \( \mathbb{S} \) such that \( (U \circ T) (*_{\nu}) = T' (*_{\nu}) \), we can set \( S := (U \circ T)^{-1} \circ T' \in \text{Aut} (*_{\nu}) \).

Let \( S \) be a \( S \)-automorphism of \( *_{\nu} \). Given that \( \mathbb{S} \simeq D \) is a connected simply connected Lie group, there exists \( \Lambda \in C^{\infty} (\mathbb{S}) [\nu] \) such that
\[
S = \exp \left( \frac{1}{2\nu} [\nu \Lambda, -]_{s_{\nu}} \right); \quad (14)
\]
[Gu11 § 4, prop. 23]. Let’s consider $X \in \mathfrak{g}$ and $\Lambda_X \in \mathcal{C}^\infty (\mathbb{S}) \llbracket \nu \rrbracket$ such that $\frac{1}{2\nu} [\Lambda_X, -]_{*_\nu} = X^* \in \text{Der} (*_\nu)$. As $S$ commutes with the left translations on $\mathbb{S}$, we have

$$0 = \left[ X^*, \frac{1}{2\nu} [\nu \Lambda, -]_{*_\nu} \right] = \frac{1}{2\nu} \left[ \left[ \Lambda_X, \nu \Lambda \right]_{*_\nu}, - \right]_{*_\nu} = \frac{1}{2\nu} [\nu X^*(\Lambda), -]_{*_\nu}$$

where relation (10) is used in the second equality. From lemma 21, we deduce that $X^*(\Lambda)$ lies in $\mathbb{C}[\nu]$. In particular, as $X$ is arbitrary, we have $X^*(\Lambda) = 0$ for each $X \in [\mathfrak{s}, \mathfrak{s}] = \mathfrak{n}$. In addition, by using the relation $\text{Ad}_{a_\nu}(n) = n$, we get

$$0 = (\text{Ad}_{a_\nu}(X))^*_n(\Lambda) = X^*_n(\tau_\nu \Lambda)$$

for each $a \in A$, $n \in N$ and $X \in \mathfrak{n}$. This leads us to the following proposition.

**Lemma 29.** Each $S \in \text{Aut} (*_\nu)$ is of the form (14) where $\Lambda \in \mathcal{C}^\infty (\mathbb{S}) \llbracket \nu \rrbracket$ satisfies

$$\Lambda(an) = \Lambda(a) \quad \text{and} \quad \gamma^S_H := H^*(\Lambda) \in \mathbb{C}[\nu]$$

for each $a \in A$, $n \in N$ and $H \in \mathfrak{a}$. In particular, the linear map $\gamma^S : H \in \mathfrak{a} \mapsto \gamma^S_H$ encodes univocally the data of $S$ and the space $\text{Aut} (*_\nu)$ is parametrized by $\mathbb{C}[\nu]$.

Let’s point out that the second assertion follows from the equality

$$S(\Lambda_H) = \Lambda_H - \nu \gamma^S_H$$

which is valid for each $S \in \text{Aut} (*_\nu)$, $H \in \mathfrak{a}$ and $\Lambda_H \in \mathcal{C}^\infty (\mathbb{S}) \llbracket \nu \rrbracket$ such that $\frac{1}{2\nu} [\Lambda_H, -]_{*_\nu} = H^*$.

### 5.2 Intertwiners and quantum moment maps

For any operator $T \in \text{Op}^S (*_\nu)$ such that $T(*_\nu)$ is a $G$-invariant star-product on $\mathbb{D}$, let’s define the map

$$D^T : X \in \mathfrak{g} \mapsto D^T_X := T^{-1} \circ X^* \circ T.$$ 

We deduce from remark 20 that $D^T$ is a Lie algebra homomorphism of which the image lies in $\text{Der} (*_\nu)$; [Ko14 Ch. 2, lem. 2.3.3]. As the $0$th order term of $T$ in $\nu$ coincides with $\text{Id}$, we have

$$D^T_X = X^* + o(\nu) \quad \text{for any} \quad X \in \mathfrak{g}.$$ 

As a consequence, lemma 22 provides us with a unique quantum moment map $\Lambda^T = \lambda + o(\nu)$ associated with $D^T$. The data of this quantum moment map is equivalent to the data of $D^T$.

**Proposition 30.** The derivation $D^T_X$ does not depend on $T$ for $X \in \mathfrak{s} \oplus \mathfrak{m}$ and it coincides with $X^*$ if $X \in \mathfrak{s}$. In particular, for each operators $T, T' \in \text{Op}^S (*_\nu)$ such that $T(*_\nu)$ and $T'(*_\nu)$ are $G$-invariant star-products on $\mathbb{D}$, we have

$$\Lambda^T_X - \Lambda^{T'}_X \in \mathbb{C}[\nu] \quad \text{when} \quad X \in \mathfrak{a} \oplus \mathbb{Z}(\mathfrak{m}) \quad \text{and} \quad \Lambda^T_X = \Lambda^{T'}_X \quad \text{when} \quad X \in \mathfrak{n} \oplus [\mathfrak{m}, \mathfrak{m}].$$
Corollary 32. for each

\[
\frac{1}{2\nu} [\Lambda_X^T, -]_{*\nu} = D_X^T = X^* = D_X^{T'} = \frac{1}{2\nu} [\Lambda_X^{T'}, -]_{*\nu}
\]

for each \(X \in \mathfrak{s}\). Then, lemma \[21\] and a similar argument to (12) give us

\[\Lambda_X^T = \Lambda_X^{T'} \text{ for each } X \in \mathfrak{n} = \{s, s\} \quad \text{and} \quad \Lambda_H^T - \Lambda_H^{T'} \in \nu \mathbb{C}[\nu]\]

for each \(H \in \mathfrak{a}\).

Let's consider \(Y \in \mathfrak{m}\). If we combine this result and the inclusion \(\mathfrak{m} \subset N(\mathfrak{n}) \cap \mathfrak{g}_0\) with the quantum moment map property of \(\Lambda^T\) and \(\Lambda^{T'}\), we obtain

\[X^* \left( \Lambda_T^T - \Lambda_T^{T'} \right) = \Lambda_{[X,Y]}^T - \Lambda_{[X,Y]}^{T'} = 0\]

for each \(X \in \mathfrak{s}\). As \(\{X^*_s : X \in \mathfrak{s}\}\) spans \(T_s(\mathbb{S})\) for \(s \in \mathbb{S}\), these last relations give us \(\Lambda_T^T - \Lambda_T^{T'} \in \nu \mathbb{C}[\nu]\). In particular, we deduce from lemma \[21\] that \(D_Y^T = D_Y^{T'}\). Equality (2) and the adaptation of expression (12) in this framework allow us to conclude the proof.

Remark 31. The hypothesis made on the star-product \(*_{\nu}\) is crucial. We remark that the existence of such a map \(\Lambda_T^T|_s = \lambda + o(\nu)\) uniquely determined on \(\mathfrak{n}\) was already mentioned in proposition \[24\]

Let's notice that the previous proposition is specific to the Lie subalgebra \(\mathfrak{s} \oplus \mathfrak{m} \subset \mathfrak{g}\) as it coincides with \(N(\mathfrak{n})\). It shows that the derivation \(D_X^T\) depends only on the star-product \(*_{\nu}\) for \(X \in \mathfrak{s} \oplus \mathfrak{m}\). In particular, if this star-product \(*_{\nu}\) is \(s\)-covariant, we have the following result.

Corollary 32. The star-product \(*_{\nu}\) is \(s\)-covariant if and only if

\[\Lambda_X^T = \lambda_X \quad \text{and} \quad \Lambda_H^T - \lambda_H \in \nu \mathbb{C}[\nu]\]

for each \(X \in \mathfrak{n}, H \in \mathfrak{a}\) and \(T \in \text{Op}^S(\ast_{\nu})\) such that \(T(\ast_{\nu})\) is \(G\)-invariant. In this case, if \(T \in \text{Op}^S(\ast_{\nu})\) is any operator such that \(T(\ast_{\nu})\) is \(G\)-invariant, we have \(\Lambda_T^T - \lambda_Y \in \nu \mathbb{C}[\nu]\) for each \(Y \in \mathfrak{m}\).

The first part of this corollary is clear in view of propositions \[27\] and \[30\]. The proof of the second assertion follows from a similar argument to the one used in the proof of proposition \[30\]. In fact, we have

\[X^*(\Lambda_Y^T) = \frac{1}{2\nu} [\Lambda_X^T, \Lambda_Y^T]_{*\nu} = \Lambda_{[X,Y]}^T = \lambda_{[X,Y]} \in C^\infty(\mathbb{S})\]

for all \(X \in \mathfrak{s}, Y \in \mathfrak{m}\) and \(T \in \text{Op}^S(\ast_{\nu})\) such that \(T(\ast_{\nu})\) is \(G\)-invariant. The next results follow directly from remark \[20\] and proposition \[30\].

Corollary 33. Let's denote by \(M \subset K\) the connected Lie subgroup of \(G\) which Lie algebra \(\mathfrak{m} \subset \mathfrak{t}\). Then, all the \(G\)-invariant star-products on \(\mathbb{S}\) which are \(s\)-equivalent to \(\ast_{\nu}\) are \(SM\)-equivalent. In addition, the following assertions are equivalent:

- there exists \(T \in \text{Op}^S(\ast_{\nu})\) such that \(T(\ast_{\nu})\) is \(G\)-invariant and such that \(D_Y^T = Y^*\) for each \(Y \in \mathfrak{m}\);
- for each \(T \in \text{Op}^S(\ast_{\nu})\) such that \(T(\ast_{\nu})\) is \(G\)-invariant and for each \(Y \in \mathfrak{m}\), we have \(D_Y^T = Y^*\);
- the star-product \(\ast_{\nu}\) is \(SM\)-invariant and there exists \(T \in \text{Op}^S(\ast_{\nu})\) such that \(T(\ast_{\nu})\) is \(G\)-invariant and \(SM\)-equivalent to \(\ast_{\nu}\).
5.3 Retractable homomorphisms

We are now going to look at the considerations of the previous subsection from a more general point of view. Let’s denote by $\mathbb{H}(\ast_\nu)$ the space of Lie algebra homomorphisms of the form

$$D : g \to \text{Der}(\ast_\nu) : X \mapsto D_X = X^* + o(\nu)$$

such that $D_X = X^*$ for each $X \in s$. From lemma 22 it is clear that it can be identified to the space of quantum moment maps associated with elements of $\mathbb{H}(\ast_\nu)$.

Remark 34. Let $\Lambda$ and $\Lambda'$ be the quantum moment maps associated with $D \in \mathbb{H}(\ast_\nu)$ and $D' \in \mathbb{H}(\ast_\nu)$ respectively. An obvious adaptation of the proof of proposition 30 shows that $\Lambda_X - \Lambda'_X \in \nu C[[\nu]]$ for each $X \in s \oplus m$. In addition, given that $\Lambda_{\sigma(Y)}$ satisfies the equations

$$X^* (\Lambda_{\sigma(Y)}) = \Lambda_{[X,\sigma(Y)]_s} + \Lambda_{[X,\sigma(Y)]_m}$$

with $[X,\sigma(Y)]_s \in s \oplus m$ and $[X,\sigma(Y)]_m \in \sigma(n)$ for all $X \in s$ and $Y \in n$, we deduce that the space $\mathbb{H}(\ast_\nu)$ has a structure of finite dimensional vector space over the field $\mathbb{C}[[\nu]]$.

Lemma 35. Let’s consider $D \in \mathbb{H}(\ast_\nu)$ and $S \in \text{Aut}(\ast_\nu)$. Then we have

$$D^S := S^{-1} \circ D \circ S \in \mathbb{H}(\ast_\nu) .$$

In addition, the homomorphisms $D$ and $D^S$ coincide if and only if $S = \text{Id}$.

**Proof.** The first assertion is clear as $S = \text{Id} + o(\nu) \in \text{Aut}(\ast_\nu)$ commutes with the left translations on $\mathfrak{s}$; [Ko14 Ch. 2, lem. 2.3.3]. Let $\Lambda$ be the quantum moment map associated with $D$ and let’s consider $X \in g_\lambda$ for $\lambda \in \Sigma^+$. As $S$ is a $\mathfrak{s}$-automorphism of $\ast_\nu$, we have

$$\left( D_{\sigma(X)} - D_{\sigma(X)}^S \right) (\Lambda_X) = \frac{1}{2\nu} \left[ \Lambda_{\sigma(X)} - S^{-1} (\Lambda_{\sigma(X)}) , \Lambda_X \right]_{\ast_\nu} = -X^* (\Lambda_{\sigma(X)} - S^{-1} (\Lambda_{\sigma(X)})) .$$

Lemma 29 allows us to choose $\Lambda' \in C^\infty (\mathfrak{s})[[\nu]]$ which is induced by the data of $S^{-1}$ through the relation

$$S^{-1} = \exp \left( \frac{1}{2\nu} [\nu \Lambda' , -]_{\ast_\nu} \right) .$$

In particular, we have $X^* (\Lambda') = 0$ and $\gamma_H^{S^{-1}} := H^* (\Lambda') \in \mathbb{C}[[\nu]]$ for each $H \in \mathfrak{a}$. By combining these relations with the Jacobi identity and the expression (3), we obtain

$$X^* \left( \frac{1}{2\nu} [\nu \Lambda', \Lambda_{\sigma(X)}]_{\ast_\nu} \right) = -\frac{1}{2\nu} \left[ \nu \Lambda' , \frac{1}{2\nu} \Lambda_{\sigma(X)} , \Lambda_X \right]_{\ast_\nu} - \frac{1}{2\nu} \Lambda_{\sigma(X)} , \frac{1}{2\nu} \Lambda_{\sigma(X)} , \nu \Lambda' \right]_{\ast_\nu} \right)_{\ast_\nu} = -\frac{1}{2\nu} \left[ \nu \Lambda' , \Lambda_{[\sigma(X),X]} \right]_{\ast_\nu} - \frac{1}{2\nu} \Lambda_{\sigma(X)} , \nu X^* (\Lambda') \right]_{\ast_\nu} = -\frac{1}{2\nu} \left( X , \sigma (X) \right) \gamma_H^{S^{-1}} \in \mathbb{C}[[\nu]] .$$

An inductive approach shows that

$$X^* \left( \left( \frac{1}{2\nu} [\nu \Lambda' , -]_{\ast_\nu} \right)^k \right) (\Lambda_{\sigma(X)}) = 0$$

21
for each integer \(k > 1\). As a consequence, we get
\[
\left( D_{\sigma(X)} - D_{\sigma(X)}^S \right) (\Lambda x) = -\nu \beta(X, \sigma(X)) \gamma_{H_\lambda}^{S^{-1}}.
\]
Since \(\{H_\lambda : \lambda \in \Sigma^+\}\) spans \(a\) and \(\beta(X, \sigma(X)) < 0\), it is clear that the map \(\gamma_{S^{-1}} : a \mapsto \mathbb{C}[\nu]\) vanishes identically if \(D = D^S\). As \(S = \text{Id}\) if and only if \(\gamma_{S^{-1}} = 0\), the proof is complete. \(\blacksquare\)

In this section, we will be interested in a particular class of elements in \(\mathbb{H}(\ast_\nu)\) which appeared previously.

**Definition 36.** For \(D \in \mathbb{H}(\ast_\nu)\), an operator \(T \in \text{Op}^S(\ast_\nu)\) such that \(T(\ast_\nu)\) is a \(G\)-invariant star-product on \(\mathbb{D}\) is called a \(D\)-retract if \(D = D^T\). We say that \(D \in \mathbb{H}(\ast_\nu)\) is retractable when \(D\) admits a \(D\)-retract.

**Proposition 37.** The space of retractable homomorphisms of \(\mathbb{H}(\ast_\nu)\) is parametrized by the space of formal power series with coefficients in \(\mathbb{C}^r \oplus Z^2(\mathbb{D})^G\).

**Proof.** Let’s fix \(T \in \text{Op}^S(\ast_\nu)\) such that \(T(\ast_\nu)\) is a \(G\)-invariant star-product on \(\mathbb{D} \simeq \mathbb{S}\). Let’s consider \(\mathbb{H}^T(\ast_\nu)\) the space of retractable homomorphisms \(D \in \mathbb{H}(\ast_\nu)\) of the form \(D = D^T\) for \(T' \in \text{Op}^S(\ast_\nu)\) such that \(T'(\ast_\nu)\) is \(G\)-invariant and \(G\)-equivalent to \(T(\ast_\nu)\). From lemma \ref{lem:equivalence} we know that \(\mathbb{H}^T(\ast_\nu)\) coincides with the space of homomorphisms \(D \in \mathbb{H}(\ast_\nu)\) defined by
\[
D : X \in \mathfrak{g} \longrightarrow D_X = S^{-1} \circ T^{-1} \circ U^{-1} \circ X^* \circ U \circ T \circ S
\]
where \(U\) is a \(G\)-equivalence of \(G\)-invariant star-products on \(\mathbb{S}\) and \(S\) a \(\mathbb{S}\)-automorphism of \(\ast_\nu\). Given that the operator \(U\) commutes with the action of \(G\) on \(\mathbb{S}\) in this last expression, we get
\[
\mathbb{H}^T(\ast_\nu) = \{ S^{-1} \circ D^T \circ S : S \in \text{Aut}(\ast_\nu) \}.
\]
Therefore, we deduce from lemmas \ref{lem:equivalence} and \ref{lem:retractable} that \(\mathbb{H}^T(\ast_\nu)\) is parametrized by \(\mathbb{C}^r[\nu]\). The proof is then complete in view of lemma \ref{lem:equivalence} \(\blacksquare\).

**Remark 38.** In view of this proof, it becomes clear that this statement describes a parametrization of the space of retractable homomorphisms of \(\mathbb{H}(\ast_\nu)\) by a choice of \(\mathbb{S}\)-automorphism of \(\ast_\nu\) and a choice of \(G\)-equivalence class of \(G\)-invariant star-products on \(\mathbb{D}\). In particular, we can easily deduce from proposition \ref{prop:retractable} and expression \ref{eq:retractable} that this parametrization of \(\mathbb{H}^T(\ast_\nu)\) by a choice of \(\mathbb{S}\)-automorphism of \(\ast_\nu\) is associated to a parametrization by a choice of a linear map \(\Lambda : s \mapsto \mathcal{C}^\infty(\mathcal{D})[\nu]\) given in proposition \ref{prop:homomorphism}.

The following corollaries can be deduced easily from proposition \ref{prop:isomorphism}, remark \ref{rem:equivalence} and proposition \ref{prop:retractable}.

**Corollary 39.** As a Hermitian symmetric space of non compact type, if the symmetric bounded domain \(\mathbb{D}\) is irreducible, then the space of retractable homomorphisms of \(\mathbb{H}(\ast_\nu)\) is parametrized by \(\mathbb{C}^r+1[\nu]\).

**Corollary 40.** The dimension of \(\mathbb{H}(\ast_\nu)\) as vector space over \(\mathbb{C}[\nu]\) is greater than \(r + \dim\left( Z^2(\mathbb{D})^G \right) \) and this lower bound is reached if and only if each homomorphism of \(\mathbb{H}(\ast_\nu)\) is retractable.
5.4 PDE’s for the retract

We now describe our quantization method based on the above-mentioned retract method, as well as its interaction with tools that we developed for simplifying underlied computations.

(i) The first step is to fix explicitly a linear map \( \Lambda : s \rightarrow C^\infty (\mathbb{D}) \) given by proposition 24. In virtue of proposition 27, if the star-product \( *_\nu \) is \( s \)-covariant, we can simply choose \( \Lambda = \lambda \).

(ii) Then, we compute the set of quantum moment maps that coincide on \( s \) with the map \( \Lambda \) chosen in (i) and that are associated with retractable homomorphisms of \( \mathbb{H}(*_\nu) \). The best way for doing that is to solve the equations of remark 34. The corollaries 32 and 33 can be helpful if the star-product \( *_\nu \) is either \( s \)-covariant or \( SM \)-invariant.

Remark 41. From remark 38, we note that the space of such quantum moment maps is parametrized by \( Z^2 (\mathbb{D})^G [\nu] \). In particular, a choice of parameter \( \omega_\nu \in Z^2 (\mathbb{D})^G [\nu] \) will correspond to a choice of \( G \)-equivalence class of \( G \)-invariant star-products comprising the Fedosov star-product \( *_{(\nabla, \omega_\nu)} \).

(iii) For each retractable homomorphism \( D \in \mathbb{H}(*_\nu) \) associated to a quantum moment map obtained in (ii), we compute the set of \( D \)-retract, ie. the set of invertible linear convolution operator \( T \in \text{Op}^S (*_\nu) \) of the form (6) such that \( T \circ D_X \circ T^{-1} = X^* \) for all \( X \in \mathfrak{g} \).

Remark 42. If the operator \( T \) satisfies this last equation for two elements in \( \mathfrak{g} \), then it satisfies also this equation for any multiple of these two elements and for the Lie bracket of these two elements.

Proposition 17, lemma 28, lemma 35, proof of proposition 37 and, more generally, the whole material developed through this section, lead us to the following major result.

Theorem 43. This method yields the set of \( G \)-invariant star-products on \( \mathbb{D} \). In particular, the set of \( D \)-retract for a retractable homomorphism \( D \in \mathbb{H}(*_\nu) \) associated to a quantum moment map from (ii) generates exactly a \( G \)-equivalence class of \( G \)-invariant star-products on \( \mathbb{D} \).

In order to reach the objectives stated in our introduction, we need to develop tools for the resolution of the equations expressed in point (iii) of the method. An important step in this direction lies in the following theorem which was firstly proved for \( r = 1 \) in collaboration with Bieliavsky ; [Ko14, Ch. 2, thm. 2.5.10].

Theorem 44. Let’s consider a retractable homomorphism \( D \in \mathbb{H}(*_\nu) \). An operator \( T \in \text{Op}^S (*_\nu) \) is a \( D \)-retract if and only if it is the inverse of a convolution operator of the form (6) such that its kernel \( v_T := u_{T^{-1}} \in D'(S) [\nu] \) satisfies the partial differential equation

\[
D_X (v_T) = ([X]_s)^* (v_T) \tag{16}
\]

for each \( X \in \mathfrak{g} \).

Remark 45. Equation (16) is trivially satisfied for each \( X \in s \). In particular, we have

\[
D_X (v_T) = ([X]_s)^* (v_T) \quad \text{for all} \ X \in \mathfrak{g} \iff D_X (v_T) = 0 \quad \text{for all} \ X \in \mathfrak{k}.
\]
Proof. Let $\Lambda : g \to C^\infty (\mathbb{D})[\nu] : X \mapsto \Lambda_X = \lambda_X + o(\nu)$ be a quantum moment map associated to $D$ by lemma \ref{lm:quantum_moment_map}. Let $T \in \text{Op}^S(\ast_\nu)$ be a $S$-equivalence of star-products on $\mathbb{D} \simeq S$. In particular, its inverse $T^{-1}$ is an invertible linear convolution operators on $C^\infty (S)[\nu]$ of the form \ref{eq:linear_convolution} with kernel

$$v_T := u_{T^{-1}} \in D'(S)[\nu].$$

Let's denote by $ds$ the left-invariant Haar measure on $S$. For $X \in g$, functional analysis theory on the left-invariant Lie group $(S, ds)$ allows us to obtain formally the equivalences:

$$D_X \circ T^{-1} = T^{-1} \circ X^* \iff (D_X)_{s_0} \left( \int_S v_T (s^{-1} s_0) f(s) \, ds \right) = \int_S v_T (s^{-1} s_0) X^*_s (f) \, ds$$

for all $s_0 \in S$ and $f \in D(S)$

$$\iff \int_S (D_X)_{s_0} (v_T (s^{-1} s_0)) f(s) \, ds = - \int_S X^*_s (v_T (s^{-1} s_0)) f(s) \, ds$$

for all $s_0 \in S$ and $f \in D(S)$

$$\iff (D_X)_{s_0} (v_T (s^{-1} s_0)) = - X^*_s (v_T (s^{-1} s_0)) \quad \text{for all } s_0, s \in S. \quad \text{(17)}$$

For each $s_0, s \in S$ and $X \in g$, given that the star-product $\ast_\nu$ is $S$-invariant, we have

$$(D_X)_{s_0} (v_T (s^{-1} s_0)) = \frac{1}{2\nu} [\Lambda_X, L^*_{s^{-1}} \ast_{\nu} (v_T)]_{s_0} (s_0) = \frac{1}{2\nu} \left( L^*_{s^{-1}} \left( [L^*_s (\Lambda_X), v_T]_{s_0} \right) \right) (s_0) = \frac{1}{2\nu} \left[ \Lambda_{\text{Ad}_{s^{-1}}(X)}, v_T \right]_{s_0} (s^{-1} s_0) = \left( D_{\text{Ad}_{s^{-1}}(X)} \right)_{s^{-1} s_0} (v_T) \quad \text{(18)}$$

where the third equality comes from the relation $L^*_{s^{-1}} \circ \Lambda = \Lambda \circ \text{Ad}_s$ which is the integral version of equality $Y^* \circ \Lambda = \Lambda \circ \text{ad}_Y$ for $Y \in g$; \cite[Ch. 2, lem. 2.5.8]{Ko14}.

For $s_0 \in S$, let $i_\mathbb{S}$ and $R_{s_0}$ be the maps defined on $\mathbb{S}$ by $i_\mathbb{S} (s) := s^{-1}$ and $R_{s_0} (s) := ss_0$ respectively. For each $s_0, s \in S$ and $X \in g$, by using relations \ref{eq:quantum_moment_map} we get the equalities

$$- X^*_s (v_T (s^{-1} s_0)) = - \frac{d}{dt} \bigg|_{t=0} v_T \left( \left( L_{\exp(-tX)} (s) \right)^{-1} s_0 \right)$$

$$= \frac{d}{dt} \bigg|_{t=0} \left( (R_{s_0} \circ i_\mathbb{S} \circ L_s)^* v_T \right) \left( \left[ \exp (t \text{Ad}_{s^{-1}} (X)) \right]_S \right)$$

$$= \frac{d}{dt} \bigg|_{t=0} \left( (R_{s_0} \circ i_\mathbb{S} \circ L_s)^* v_T \right) \left( \exp (t \text{Ad}_{s^{-1}} (X)) \left[ L_s \right]_S \right)$$

$$= \frac{d}{dt} \bigg|_{t=0} v_T \left( \exp \left( - t \text{Ad}_{s^{-1}} (X) \left[ L_s \right]_S \right) s^{-1} s_0 \right)$$

$$= \frac{d}{dt} \bigg|_{t=0} v_T \left( \left( L_{\exp(-t\text{Ad}_{s^{-1}}(X))} \right) \left( s^{-1} s_0 \right) \right) = \left( \left[ \text{Ad}_{s^{-1}} (X) \right]_S \right)^* \left( s^{-1} s_0 \right) (v_T) \quad \text{(19)}$$

If we combine relations \ref{eq:quantum_moment_map}, \ref{eq:quantum_moment_map} and \ref{eq:quantum_moment_map}, we obtain the equivalence

$$D_X \circ T^{-1} = T^{-1} \circ X^* \iff \left( D_{\text{Ad}_{s^{-1}}(X)} \right) (v_T) = \left( \left[ \text{Ad}_{s^{-1}} (X) \right]_S \ast_\nu \right) (v_T) \quad \text{for all } s \in S. \quad \text{(20)}$$

As the operator $T \in \text{Op}^S(\ast_\nu)$ is a $D$-retract if and only if $D_X \circ T^{-1} = T^{-1} \circ X^*$ for all $X \in g$, the proof is complete. \hfill \blacksquare
As we will see in the next section, it can be hard to solve such a hierarchy of partial differential equations on the kernel of an invertible linear convolution operator. The fact that the homomorphism $D \in H(\ast_{\nu})$ is retractable is crucial to ensure that there exist solutions to these equations.

The following results are also helpful tools for the resolution of these equations on specific examples. The first proposition can be deduced from lemma 3, remark 5 and corollary 33 [Ko16, cor. 3].

**Proposition 46.** Let’s assume that the star-product $\ast_{\nu}$ is $SM$-invariant and $SM$-equivalent to a $G$-invariant star-product on $\mathbb{D}$. Then, equation (16) is satisfied for each $X \in s \oplus m$ if and only if $Y^\ast (v_T) = 0$ for all $Y \in m$. In this case, the map

$$X \in g_\lambda \mapsto v_T (\exp (X))$$

is radial in the Euclidian vector spaces $(g_\lambda, \beta_\sigma)$ and $(g_\lambda, (-|-))$ for any $\lambda \in \Sigma^+$. This first proposition can advantageously be used to reduce the number of variables in the equations.

The second proposition is a consequence of remark 42 and relation (20).

**Proposition 47.** Let $W$ be a vector space such that $s \subseteq W \subseteq g$ and $\{ Ad_s^{-1} (X) : X \in W, s \in S \} = W$. If equation (16) is satisfied for all $X \in W$, then it is also satisfied for all $X \in [W, W]$.

The major advantage of this second proposition is to reduce the number of equations to be solved. We choose this formulation for this proposition because of its correspondence with an efficient computation method based on the root space decomposition of $g$. Otherwise, it can be expressed obviously from remark 45 as $D|_{\mathfrak{t}}$ is a Lie algebra homomorphism on $\mathfrak{t}$.

## 6 Example: the complex unit ball

For $n \geq 1$, the goal of this section is to develop the above-mentioned method on the complex unit ball in $\mathbb{C}^N$ as symmetric bounded domain. We will denote it by $\mathbb{D}_N$. This section is entirely based on the reference [Ko14] in which the reader will find the computational details of the results described hereafter. We will refer continuously to the notations introduced in this text.

Let’s describe the geometry of the domain $\mathbb{D}_N$ and the Lie algebra structure of its automorphism group from propositions 4, 4, 6 and references [Ko14] Ch. 1, § 1.5, [He01] Ch. 10, § 6.3.

**Proposition 48.**

(a) The complex unit ball $\mathbb{D}_N$ admits a structure of rank one irreducible Hermitian symmetric space of non compact type with automorphism group $G = SU(1, N)$.  

25
(b) The Iwasawa group $\mathbb{S}$ of $G$ is an elementary Pyatetskii-Shapiro group acting simply transitively on $\mathbb{D}_N$.

(c) The structure of its Lie algebra

$$\mathbb{R}^{2N} \simeq \mathfrak{s} = \mathfrak{a} \ltimes \mathfrak{n} = \mathbb{R}H \ltimes (V \oplus \mathbb{R}E)$$

is described in lemma 7 and the root space decomposition of the Lie algebra $\mathfrak{g}$ is of the form

$$\mathfrak{g} = \mathfrak{g}_{2\lambda} \oplus \mathfrak{g}_\lambda \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{-\lambda} \oplus \mathfrak{g}_{-2\lambda} \text{ for } \lambda \in \Sigma^+$$

with $\mathfrak{g}_{2\lambda} = \mathbb{R}E$, $\mathfrak{g}_\lambda = V$, $\mathfrak{g}_0 = \mathbb{R}H \oplus \mathfrak{m} \simeq \mathbb{R}H \oplus \mathfrak{u}(N-1)$, $\mathfrak{g}_{-\lambda} = \sigma(V)$ and $\mathfrak{g}_{-2\lambda} = \mathbb{R}\sigma(E)$.

(d) The map

$$\mathbb{R}^{2N} \simeq \mathfrak{s} \rightarrow \mathbb{S} : (a, v_1^1, ..., v_{N-1}^1, v_1^i, ..., v_{N-1}^i, z) \simeq aH + v + zE \mapsto \exp(aH) \exp(v) \exp(zE)$$

defines a global Darboux chart on $\mathbb{S} \simeq \mathbb{D}_N$, where the vector $(v_1^i, ..., v_{N-1}^i)$ is the decomposition of $v$ in a symplectic basis of $(V, \Omega)$.

Given that $\text{rank}(\mathbb{D}_N) = 1$, it is clear from corollary 14 that all the $\mathbb{S}$-invariant star-products on $\mathbb{D}_N$ are $\mathbb{S}$-equivalent. In particular, in this case, the statement of proposition 17 is trivial. From proposition 19, we deduce that the $G$-equivalence classes of $G$-invariant star products on $\mathbb{D}_N$ are parametrized by $\mathbb{Z}^2(\mathbb{D}_N)$ $G$-equivariant and $G$-equivalent up to a reparametrization of the formal parameter $\nu$; [BB03].

Let’s consider the $\mathbb{S}$-invariant star-product $\ast_{\nu}$ on $\mathbb{S} \simeq \mathbb{D}_N$ obtained from the explicit deformation quantization described in [BG15 Ch.4, thm.4.5], [BM01 Ch.3, thm.3.1] and [Ko14 Ch.2, thm.2.4.7].

From computations made in reference [Ko14 Ch.2, §2.3.4 and §2.4.1], it is proven that the Moyal-Weyl star-product on $\mathbb{S} \simeq \mathbb{R}^{2N}$ is $\mathbb{S}$-covariant and equivalent to $\ast_{\nu}$ by an operator $T_0$ preserving the moment map on $\mathbb{S}$; [BG15], [BM01]. As a consequence, the star-product $\ast_{\nu}$ is $\mathbb{S}$-covariant. In addition, elementary computations from reference [Ko14 Ch.2, §2.6.3] give us the relation

$$\frac{1}{2\nu} [\lambda Y, -]_{\ast_{\nu}} = Y^*$$

for each $Y \in \mathfrak{m}$. As a consequence, corollaries 32, 33 and 39 lead us to the following result.

**Lemma 49.** The star-product $\ast_{\nu}$ is $\mathbb{S}$-covariant, $SM$-invariant and $SM$-equivalent to any $G$-invariant star-product on $\mathbb{D}_N$. If $D \in \mathbb{H}(\ast_{\nu})$ is a retractable homomorphism and $\Lambda$ the quantum moment map associated to $D$ by lemma 22, then

$$DX = X^* \text{ and } \Lambda X - \lambda X \in \nu \mathbb{C}[\nu] \text{ for each } X \in \mathfrak{s} \oplus \mathfrak{m}.$$  

In addition, the space of retractable homomorphisms of $\mathbb{H}(\ast_{\nu})$ is parametrized by $\mathbb{C}^2[\nu]$.

As the star-product $\ast_{\nu}$ is $\mathbb{S}$-covariant, we can fix the moment map $\lambda$ as the linear map on $\mathfrak{s}$ given by proposition 24. After some computations, we obtain the following expressions in the global Darboux coordinate system $(a, v, z)$.


Lemma 50. Let $\Lambda : g \rightarrow C^\infty (S) [\nu]$ be the quantum moment map associated to a homomorphism of $\mathbb{H} (\ast_\nu)$ by lemma 22. If $\Lambda|_s = \lambda$, then there exists a formal constant $\alpha \in C J_\nu K$ such that

$$T_0 \Lambda Y = \lambda Y$$

for all $Y \in [m, m]$ and

$$T_0 \Lambda Z = \lambda Z + \alpha$$

for all $Z \in Z (m)$;

$$T_0 \Lambda \sigma (v_0) (a, v, z) = e^{a} [4 (v_0 | v) z - ((v | v) + \alpha) \Omega (v_0, v)]$$

for all $v_0 \in V$;

$$T_0 \Lambda \sigma (E) (a, v, z) = e^{2a} [4 z^2 + ((v | v) + \alpha)^2 + (N - 1) \nu^2]$$

where $(-|-)$ is the scalar product on $s$ induced by the Kählerian structure of $S \simeq D_N$ and described in remark 5.

In view of this result, corollary 40 and remark 41 lead us to the following statement.

Corollary 51. Every homomorphism of $\mathbb{H} (\ast_\nu)$ is retractable.

Let’s consider a homomorphism $D \in \mathbb{H} (\ast_\nu)$ associated to a quantum moment map $\Lambda$ obtained in lemma 50. In particular, the previous corollary ensures the existence of solutions to the hierarchy of partial differential equations (16) characterizing the kernel $v_T \in D' (S) [\nu]$ of the inverse of any $D$-retract.

Let’s consider the vector space

$$W = s \oplus m \oplus g - \lambda.$$

As it satisfies the hypothesis of proposition 47, the kernel $v_T$ is solution to PDE (16) for all $X \in W$ if and only if it is solution to PDE (16) for all $X \in g = [W, W]$; [Ko14, Ch. 2, lem. 2.6.5]. Therefore, it is not necessary to write and to solve the PDE (16) for $X \in g - 2\lambda$ if we solve it for all $X \in W$. This result is very important because it reduces the number of equations that we have to consider. This case is discussed in [Ko14, Appendix B & Ch. 2, §2.7].

Moreover, an application of proposition 46 with our choice Darboux coordinate system proves that $v_T$ depends only on $a$, $z$ and the radial component $r$ of $v$ in the Euclidian vector space

$$(g_\lambda = V, (-|-)).$$

It is easy to check that any such solution of the form $v_T (a, r, z)$ satisfies equation (16) for $X \in s \oplus m$; [Ko14, Ch. 2, lem. 2.6.9]. As a consequence, we obtain the following lemma.

Lemma 52. An operator $T \in \text{Op}^\infty S (\ast_\nu)$ is a $D$-retract if and only if it is the inverse of a convolution operator with a kernel of the form $v_T (a, r, z)$ satisfying the partial differential equation

$$D_{\sigma (v_0)} (v_T) + (v_0)^* v_T = 0$$

(21)

for each $v_0 \in V$.

In order to simplify the resolution of this equation, we can fix $\alpha = 1$ in quantum moment map $\Lambda$. In fact, the computation of the set of $D$-retract for only one such homomorphism $D$ is enough to determine the set of $G$-invariant star-products on $\mathbb{D}$ up a reparametrization of the formal parameter $\nu$. 

27
For computational reasons and without loss of generality, equation (21) was intertwined by a partial Fourier transform $F$ in the $z$ variable and written with the notation $\sigma(v_0) = [\sigma(E), w]$ for $w \in V$; [Ko14, Ch. 2, §2.6]. We obtained

$$0 = \left[ i \xi e^a \left( 1 + \sqrt{1 - \nu^2 \xi^2} \right) r^2 + 2 + 2 i \xi e^{-a} \right] (w|v) \vartheta$$

$$- \left[ e^a \left( 1 + \sqrt{1 - \nu^2 \xi^2} \right) \Omega(w, v) \right] \vartheta$$

$$- \left[ e^a \left( 1 + \sqrt{1 - \nu^2 \xi^2} \right) \Omega(w, v) \right] \partial_a (\vartheta)$$

$$+ \left[ e^a \left( -1 + \sqrt{1 - \nu^2 \xi^2} \right) - \frac{e^{-a}}{r} \right] \Omega(w, v) r \partial_r (\vartheta)$$

$$+ \left[ e^a \left( 1 + \sqrt{1 - \nu^2 \xi^2} \right) r^2 + 2 \right] \Omega(w, v) \frac{\Omega(w, v)}{2r} \partial_r (\vartheta)$$

$$- \left[ 2 e^a \xi \sqrt{1 - \nu^2 \xi^2} \Omega(w, v) \right] \partial_\xi (\vartheta)$$

$$- \left[ \frac{2 i e^a}{\xi} \left( -1 + \sqrt{1 - \nu^2 \xi^2} \right) (w|v) \right] \left( \vartheta_\vartheta \right) + \frac{2n - 3}{r} \partial_r (\vartheta)$$

$$+ \left[ \frac{2 i e^a}{\xi} \left( -1 + \sqrt{1 - \nu^2 \xi^2} \right) \right] (w|v) \vartheta_\vartheta (\vartheta)$$

$$- \left[ \frac{2 i e^a}{\xi} \left( -1 + \sqrt{1 - \nu^2 \xi^2} \right) \right] \frac{(w|v)}{r} \vartheta (\vartheta)$$

$$- \left[ \frac{2 i e^a}{\xi} \left( -1 + \sqrt{1 - \nu^2 \xi^2} \right) \right] \frac{(w|v)}{r} \partial_a (\vartheta)$$

$$- \left[ \frac{4 i e^a}{\xi^2} \left( -1 + \sqrt{1 - \nu^2 \xi^2} \right) \right] \frac{(w|v)}{r} \partial_\xi (\vartheta)$$

$$- \left[ \frac{e^a}{\xi^2} \left( -1 + \sqrt{1 - \nu^2 \xi^2} \right) \right] \frac{\Omega(w, v)}{2r} \left[ \frac{2n - 3}{r} \vartheta_\vartheta (\vartheta) - \frac{2n - 3}{r^2} \partial_r (\vartheta) \right]$$

$$- \left[ \frac{e^a}{\xi^2} \left( -1 + \sqrt{1 - \nu^2 \xi^2} \right) \right] \frac{\Omega(w, v)}{2r} \vartheta_\vartheta (\vartheta).$$

where $\vartheta = F(v_T)(a, r, \xi)$. The resolution of this equation is widely discussed in reference [Ko14, Ch. 2, §2.6.5] where various changes of variables and integral transforms are used.

References

[B+78a] BAYEN François, FLATO Moshé, FRONSDAL Christian, LICHNEROWICZ André, STERNHEIMER Daniel, *Deformation theory and quantization I: Deformations of symplectic structures*, Annals of Physics, volume 111, issue 1, pp 61-110, Academic Press, Inc., 1978.

[B+78b] BAYEN François, FLATO Moshé, FRONSDAL Christian, LICHNEROWICZ André, STERNHEIMER Daniel, *Deformation theory and quantization II: Physical applications*, Annals of Physics, volume 111, issue 1, pp 111-151, Academic Press, Inc., 1978.

[B+97] BERTELSON Mélanie, CAHEN Michel & GUTT Simone, *Equivalence of star-products*, Classical and Quantum Gravity, volume 14, issue 1A, A93-A107, 1997.
Quantum moment maps and symmetric bounded domains quantizations

S. Korvers

[B+98] BERTELSON Mélanie, BIELIAVSKY Pierre & GUTT Simone, Parametrizing equivalence classes of invariant star-products, Letters in Mathematical Physics, volume 46, issue 4, pp 339-345, Kluwer Academic Publishers, 1998.

[B+09] BIELIAVSKY Pierre, DETOURNAY Stéphane & SPINDEL Philippe, The deformation quantizations of the hyperbolic plane, Communications in mathematical physics, volume 289, issue 2, pp 529-559, Springer-Verlag, 2009.

[Bi17] BIELIAVSKY Pierre, Quantum differential surfaces of higher genera, arXiv preprint no.1712.06367v1, 2017.

[BB03] BIELIAVSKY Pierre & BONNEAU Philippe, On the geometry of the characteristic class of star product on a symplectic manifold, Reviews in Mathematical Physics, volume 15, issue 2, pp 199-215, 2003.

[BG15] BIELIAVSKY Pierre & GAYRAL Victor, Deformation quantization for actions of Kählerian Lie groups, Memoirs of the American Mathematical Society, volume 236, number 1115, 2015, ISBN 978-1-4704-1491-7.

[BM01] BIELIAVSKY Pierre & MASSAR Marc, Oscillatory integral formulae for left-invariant star-products on a class of Lie groups, Letters in Mathematical Physics, volume 58, issue 2, pp 115-128, Kluwer Academic Publishers, 2001.

[Ca08] CANNAS DA SILVA Ana, Lectures on Symplectic Geometry, Lecture Notes in Mathematics, volume 1764, Springer-Verlag, 2008, originally published in 2001, ISBN 978-3-540-42195-5.

[De95] DELIGNE Pierre, Déformations de l’algèbre des fonctions d’une variété symplectique : comparaison entre Fedosov et De Wilde, Lecomte, Selecta Mathematica, volume 1, issue 4, pp 667-697, Birkhäuser-Verlag, 1995.

[Fe94] FEDOSOV Boris V., A simple geometrical construction of deformation quantization, Journal of Differential Geometry, volume 40, number 2, pp 213-238, International Press of Boston, Inc., 1994.

[GR03] GUTT Simone & RAWNSLEY John, Natural Star Products on Symplectic Manifolds and Quantum Moment Maps, Letters in Mathematical Physics, volume 66, issue 1-2, pp 123-139, Kluwer Academic Publishers, 2003.

[Gu11] GUTT Simone, Deformation quantisation of Poisson manifolds, Geometry & Topology Monographs, volume 17, pp 171-220, Mathematical Sciences Publishers, 2011.

[Fe94] FEDOSOV Boris V., A simple geometrical construction of deformation quantization, Journal of Differential Geometry, volume 40, number 2, pp 213-238, International Press of Boston, Inc., 1994.

[He01] HELGASON Sigurdur, Differential geometry, Lie groups, and symmetric spaces, Graduate Studies in Mathematics, volume 34, American Mathematical Society, 2001, originally published in 1978, ISBN 0-8218-2848-7.

[Kn02] KNAPP Anthony William Lie groups beyond and introduction, second edition, Progress in Mathematics, volume 140, Birkhäuser, Springer, 2002, ISBN 0-8176-4259-5.

[Ko03] KONTSEVITCH Maxim, Deformation quantization of Poisson manifolds, Letters in Mathematical Physics, volume 66, issue 3, pp 157-216, Kluwer Academic Publishers, 2003.

[Ko14] KORVERS Stéphane, Quantification par déformation formelles et non formelles de la boule unité de C^n, PhD thesis, Université catholique de Louvain, 2014.
[Ko16] KORVERS Stéphane, *Notes on a Lie algebraic relation*, preprint available online, Université du Luxembourg, 2016.

[NT95] NEST Ryszard & TSYGAN Boris, *Algebraic index theorem for families*, Advances in Mathematics, volume 113, issue 2, pp 151-205, Elsevier, 1995.

[Py69] PYATETSKII-SHAPIRO Ilya Iosífovich, *Automorphic functions and the geometry of classical domains*, Mathematics and its applications, volume 8, Gordon and Breach Science Publishers, 1969, originally published in 1961.

[So97] SOURIAU Jean-Marie, *Structure of Dynamical Systems*, Progress in Mathematics, volume 149, Birkhäuser, Springer, 1997, originally published in 1970, ISBN 0-8176-3695-1.

[Xu98] XU Ping, *Fedosov star-products and quantum momentum maps*, Communications in mathematical physics, volume 197, issue 1, pp 167-197, Springer-Verlag, 1998.