EVALUATION OF THE CONVOLUTION SUMS
\[ \sum_{l+15m=n} \sigma(l)\sigma(m) \text{ AND } \sum_{3l+5m=n} \sigma(l)\sigma(m) \text{ AND AN APPLICATION} \]

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Dedicated to Srinivasa Ramanujan on the occasion of his 125th birth anniversary

Abstract. We evaluate the convolution sums
\[ \sum_{l,m \in \mathbb{N}, l+15m=n} \sigma(l)\sigma(m) \text{ and } \sum_{l,m \in \mathbb{N}, 3l+5m=n} \sigma(l)\sigma(m) \]
for all \( n \in \mathbb{N} \) using the theory of quasimodular forms and use these convolution sums to determine the number of representations of a positive integer \( n \) by the form
\[ x_1^2 + x_2^2 + x_3^2 + x_4^2 + 5(x_5^2 + x_5x_6 + x_6^2 + x_7^2 + x_7x_8 + x_8^2). \]
We also determine the number of representations of positive integers by the quadratic form
\[ x_1^2 + x_2^2 + x_3^2 + x_4^2 + 6(x_5^2 + x_6^2 + x_7^2 + x_8^2), \]
by using the convolution sums obtained earlier by Alaca, Alaca and Williams [6, 3].

1. Introduction

Following [33, 29], for \( n, N \in \mathbb{N} \), we define \( W_N(n) \) as follows.
\[ W_N(n) = \sum_{m<n/N} \sigma(m)\sigma(n-Nm), \]
where \( \sigma_r(n) \) is the sum of the \( r \)-th powers of the divisors of \( n \). We write \( \sigma_1(n) = \sigma(n) \). Also, following [1], we define \( W_{a,b}(n) \) for \( a, b \in \mathbb{N} \) by
\[ W_{a,b}(n) := \sum_{l,m} \sigma(l)\sigma(m). \]
Note that \( W_{1,N}(n) = W_{N,1}(n) = W_N(n) \). These type of sums were evaluated as early as the 19th century. For example, the sum \( W_1(n) \) was evaluated by Besge, Glaisher and Ramanujan [9, 15, 28].

The convolution sums \( W_N(n) \) (for \( 1 \leq N \leq 24 \) with a few exceptions) and \( W_{a,b}(n) \) for \( (a, b) \in \{(2, 3), (3, 4), (3, 8), (2, 9)\} \) have been evaluated by using either elementary methods or analytic methods (which use ideas of Ramanujan) or algebraic methods (using quasimodular forms) (cf. [9, 15].

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Evaluation of these convolution sums has been applied to find the number of representations of integers by certain quadratic forms (cf. [18, 11, 2, 6, 3, 4, 33, 34]). In [29], Royer used the theory of quasimodular forms, especially the structure of the space of quasimodular forms (see Eq. (6) below), to evaluate the convolution sums $W_N(n)$ for $1 \leq N \leq 14$, except for $N = 12$. For a list of evaluation of the convolution sums $W_N(n)$, we refer the reader to Table 1 in [29]. In this article, following the method of Royer, we evaluate the convolution sums $W_{15}(n)$ and $W_{3,5}(n)$ by using the theory of quasimodular forms. The evaluation of these convolution sums is then used to determine the number of representations of a positive integer by a certain quadratic form. More precisely, we use these convolution sums to determine the number of representations of integers by the quadratic form $x_1^2 + x_1x_2 + x_2^2 + x_3^2 + x_3x_4 + x_4^2 + 5(x_5^2 + x_5x_6 + x_6^2 + x_7^2 + x_7x_8 + x_8^2)$. We also give a formula for the number of representations of integers by the quadratic forms $x_1^2 + x_2^2 + x_3^2 + x_4^2 + k(x_5^2 + x_6^2 + x_7^2 + x_8^2)$, $k = 3, 6$, by using the convolution sums $W_3(n), W_6(n), W_{12}(n), W_{24}(n), W_{23}(n)$ and $W_{3,4}(n)$ evaluated by K. S. Williams and his co-authors [11, 6, 3, 18]. The formula for $k = 3$ was obtained by Alaca-Williams [7], where the terms corresponding to the cusp forms are different from our formula. The referee has informed us that the evaluation when $k = 6$ has been carried out in a similar manner by Köklüce.

2. Evaluation of $W_{a,b}(n)$ and some applications

2.1. Evaluation of $W_{15}(n)$ and $W_{3,5}(n)$. In this section, following Royer [29], we evaluate the convolution sums $W_{15}(n)$ and $W_{3,5}(n)$ by using the theory of quasimodular forms. As an application, we use these convolution sums together with the convolution sum $W_5(n)$ derived by Lemire and Williams [22] to obtain a formula for the number of representations of a positive integer $n$ by the quadratic form $Q$ given by:

$$Q : x_1^2 + x_1x_2 + x_2^2 + x_3^2 + x_3x_4 + x_4^2 + 5(x_5^2 + x_5x_6 + x_6^2 + x_7^2 + x_7x_8 + x_8^2). \quad (3)$$

Let

$$\Delta_{4,5}(z) = [\Delta(z)\Delta(5z)]^{1/6} = \eta^4(z)\eta^4(5z) = q \prod_{n=1}^{\infty} (1 - q^n)^4(1 - q^{5n})^4$$

$$= \sum_{n \geq 1} \tau_{4,5}(n)q^n \quad (4)$$

be the normalized newform of weight 4 on $\Gamma_0(5)$ (see [29]), where $q = e^{2\pi i z}$. In the above, $\eta(z) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n)$ is the Dedekind eta-function and the Ramanujan function $\Delta(z) = \eta^{24}(z)$ is the normalized cusp form of weight 12 on the full modular group $SL_2(\mathbb{Z})$. The following theorem was proved by Lemire and Williams [22]:
Theorem 2.1.

\[ W_5(n) = \frac{5}{312} \sigma_3(n) + \frac{125}{132} \sigma_3 \left( \frac{n}{5} \right) + \frac{5 - 6n}{120} \sigma(n) + \frac{1 - 6n}{24} \sigma \left( \frac{n}{5} \right) - \frac{1}{130} \tau_{4,5}(n). \]  

In order to evaluate \( W_{15}(n) \) and \( W_{3,5}(n) \), we use the structure theorem on quasimodular forms of weight \( k \) and depth \( \leq k/2 \). Let \( k \geq 2 \) and \( N \geq 1 \) be natural numbers. Let \( M_k(\Gamma_0(N)) \) denote the \( \mathbb{C} \)-vector space of modular forms of weight \( k \) on the congruence subgroup \( \Gamma_0(N) \). For details on modular forms of integral weight we refer the reader to [30, 31, 10]. We now define quasimodular forms. A complex valued holomorphic function \( f \) defined on the upper half-plane \( \mathcal{H} \) is called a quasimodular form of weight \( k \), depth \( s \) (\( s \) is a non-negative integer), if there exist holomorphic functions \( f_0, f_1, \ldots, f_s \) on \( \mathcal{H} \) such that

\[
(cz + d)^{-k} f \left( \frac{az + b}{cz + d} \right) = \sum_{i=0}^{s} f_i(z) \left( \frac{c}{cz + d} \right)^i,
\]

for all \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) \) and such that \( f_s \) is holomorphic at the cusps and not identically vanishing. It is a fact that the depth of a quasimodular form of weight \( k \) is less than or equal to \( k/2 \). For details on quasimodular forms we refer to [19, 25, 10]. The Eisenstein series \( E_2 \), which is a quasimodular form of weight 2, depth 1 on \( SL_2(\mathbb{Z}) \) is given by

\[ E_2(z) = 1 - 24 \sum_{n \geq 1} \sigma(n) e^{2\pi i nz} \]

and this fundamental quasimodular form will be used in our results. The space of quasimodular forms of weight \( k \), depth \( \leq k/2 \) on \( \Gamma_0(N) \) is denoted by \( \tilde{M}_k^{\leq k/2}(\Gamma_0(N)) \). We need the following structure theorem (see [19, 25]). For an even integer \( k \) with \( k \geq 2 \), we have

\[ \tilde{M}_k^{\leq k/2}(\Gamma_0(N)) = \bigoplus_{j=0}^{k/2-1} D^j M_{k-2j}(\Gamma_0(N)) \oplus C D^{k/2-1} E_2, \]

where the differential operator \( D \) is defined by \( D := \frac{1}{2\pi i} \frac{d}{dz} \). Using this one can express each quasimodular form of weight \( k \) and depth \( \leq k/2 \) as a linear combination of \( j \)-th derivatives of modular forms of weight \( k - 2j \) on \( \Gamma_0(N) \), \( 0 \leq j \leq k/2 - 1 \) and the \( (k/2-1) \)-th derivative of the quasimodular form \( E_2 \).

We need the following newforms of weights 2 and 4 on \( \Gamma_0(15) \) in order to use the structure of the space \( \tilde{M}_4^{\leq 2}(\Gamma_0(15)) \) to prove our theorem. These newforms are either eta-products or eta-quotients or linear combinations of
Theorem 2.2. Let \( \Delta_{2,15}(z) \) be a cusp form of weight 2 on \( \Gamma_0(15) \). We now show that these cusp forms are newforms in the respective spaces of cusp forms. A theorem of J. Sturm [32] states that the Fourier coefficients of newforms are given in order to determine the modularity of an eta-quotient (with weight, level and character). Using these conditions, it follows that the functions \( \Delta_{4,15;1}(z) \) and \( \Delta_{4,15;2}(z) \) are newforms.

The following are the main theorems of this section.

Theorem 2.2. Let \( n \in \mathbb{N} \), then

\[
W_{15}(n) = \frac{1}{624} \sigma_3(n) + \frac{3}{208} \sigma_3 \left( \frac{n}{3} \right) + \frac{25}{624} \sigma_3 \left( \frac{n}{5} \right) + \frac{75}{208} \sigma_3 \left( \frac{n}{15} \right) + \frac{5 - 2n}{120} \sigma(n) + \frac{1 - 6n}{24} \sigma \left( \frac{n}{15} \right) - \frac{1}{455} \tau_{4,15}(n) - \frac{9}{455} \tau_{4,5} \left( \frac{n}{3} \right) - \frac{1}{84} \tau_{4,15;1}(n) - \frac{1}{80} \tau_{4,15;2}(n),
\]

\[
W_{3,5}(n) = \frac{1}{624} \sigma_3(n) + \frac{3}{208} \sigma_3 \left( \frac{n}{3} \right) + \frac{25}{624} \sigma_3 \left( \frac{n}{5} \right) + \frac{75}{208} \sigma_3 \left( \frac{n}{15} \right) + \frac{5 - 6n}{120} \sigma \left( \frac{n}{3} \right) + \frac{1 - 2n}{24} \sigma \left( \frac{n}{5} \right) - \frac{1}{455} \tau_{4,15}(n) - \frac{9}{455} \tau_{4,5} \left( \frac{n}{3} \right) - \frac{1}{84} \tau_{4,15;1}(n) + \frac{1}{80} \tau_{4,15;2}(n).
\]

Proof. Let \( E_k \) denote the normalized Eisenstein series of weight \( k \) on \( SL_2(\mathbb{Z}) \) (see [30] for details). When \( k = 4 \), the Eisenstein series \( E_4 \) has the following Fourier expansion.

\[
E_4(z) = 1 + 240 \sum_{n \geq 1} \sigma_3(n) q^n.
\]
Using the structure of $\tilde{M}_{2}^{\leq 2}(15)$ from (10), we get

$$\tilde{M}_{4}^{\leq 2}(\Gamma_{0}(15)) = M_{4}(\Gamma_{0}(15)) \oplus DM_{2}(\Gamma_{0}(15)) \oplus CD E_{2}. \quad (10)$$

Using the dimension formula (see for example [26][31]), it follows that the vector space $M_{4}(\Gamma_{0}(15))$ has dimension 8 (a basis of this vector space contains 4 non-cusp forms and 4 cusp forms) and the vector space $M_{2}(\Gamma_{0}(15))$ has dimension 4 (a basis of this space contains 3 non-cusp forms and 1 cusp form). Now, it is easy to see that the set

$$\{E_{4}(z), E_{4}(3z), E_{4}(5z), E_{4}(15z), \Delta_{4,5}(z), \Delta_{4,5}(3z), \Delta_{4,15;1}(z), \Delta_{4,15;2}(z)\}$$

forms a basis of the space $M_{4}(\Gamma_{0}(15))$ and the set

$$\{\Phi_{1,15}(z), \Phi_{5,15}(z), \Phi_{1,3}(z), \Delta_{2,15}(z)\}$$

forms a basis of the space $M_{2}(\Gamma_{0}(15))$, where

$$\Phi_{a,b}(z) := \frac{1}{b-a}(bE_{2}(bz) - aE_{2}(az)). \quad (11)$$

Consider the quasimodular form $E_{2}(z)E_{2}(15z)$ which belongs to $\tilde{M}_{4}^{\leq 2}(\Gamma_{0}(15))$. Therefore, using (10) and the bases mentioned above, we have

$$E_{2}(z)E_{2}(15z) = \frac{1}{260}E_{4}(z) + \frac{9}{260}E_{4}(3z) + \frac{5}{52}E_{4}(5z) + \frac{45}{52}E_{4}(15z)$$

$$- \frac{576}{455}\Delta_{4,5}(z) - \frac{5184}{455}\Delta_{4,5}(3z) - \frac{48}{7}\Delta_{4,15;1}(z)$$

$$- \frac{36}{5}\Delta_{4,15;2}(z) + \frac{28}{5}D\Phi_{1,15}(z) + \frac{4}{5}DE_{2}(z).$$

Similarly, considering $E_{2}(3z)E_{2}(5z)$, which is a quasimodular form of weight 4, depth 2 and level 15, we get

$$E_{2}(3z)E_{2}(5z) = \frac{1}{260}E_{4}(z) + \frac{9}{260}E_{4}(3z) + \frac{5}{52}E_{4}(5z) + \frac{45}{52}E_{4}(15z)$$

$$- \frac{576}{455}\Delta_{4,5}(z) - \frac{5184}{455}\Delta_{4,5}(3z) - \frac{48}{7}\Delta_{4,15;1}(z) + \frac{36}{5}\Delta_{4,15;2}(z)$$

$$+ \frac{28}{5}D\Phi_{1,15}(z) - 4D\Phi_{5,15}(z) + \frac{4}{5}D\Phi_{1,3}(z) + \frac{4}{5}DE_{2}(z).$$

By comparing the $n$-th Fourier coefficients, we get the required convolution sums. \qed

2.2. Application to the number of representations. In this section we apply the convolution sums $W_{15}(n)$ and $W_{3,5}(n)$ to derive the following theorem. Our method of proof is similar to that used by Alaca-Alaca-Williams (see for example [6][1][2]).

**Theorem 2.3.** The number of representations of a positive integer $n$ by the quadratic form $x_{1}^{2}+x_{1}x_{2}+x_{2}^{2}+x_{3}^{2}+x_{3}x_{4}+x_{4}^{2}+5(x_{5}^{2}+x_{5}x_{6}+x_{6}^{2}+x_{7}^{2}+x_{7}x_{8}+x_{8}^{2})$
is equal to

\[
\frac{12}{13} \sigma_3(n) + \frac{108}{13} \sigma_3 \left( \frac{n}{3} \right) + \frac{300}{13} \sigma_3 \left( \frac{n}{5} \right) + \frac{2700}{13} \sigma_3 \left( \frac{n}{15} \right) + \frac{72}{91} r_{4,5}(n)
\]
\[
+ \frac{648}{91} r_{4,5} \left( \frac{n}{3} \right) + \frac{72}{7} r_{4,15,1}(n).
\]

**Proof.** Let \( N_0 = \mathbb{N} \cup \{0\}. \) For \( l \in N_0, \) let

\[
r(l) = \# \left\{ (x_1, x_2, x_3, x_4) \in \mathbb{Z}^4 \mid x_1^2 + x_1 x_2 + x_2^2 + x_3 x_4 + x_4^2 = l \right\}
\]

so that \( r(0) = 1. \) For \( l \in \mathbb{N}, \) we know that (see [18])

\[
r(l) = 12 \sum_{d|l, \ 3 \nmid d} d = 12 \sigma(l) - 36 \sigma \left( \frac{l}{3} \right).
\]

Let \( N(n) \) be the number of representations of the given quadratic form \( Q \) defined by (3). Then \( N(n) \) is given by

\[
N(n) = \sum_{l, m \in N_0, \ l + 5m = n} \left( \sum_{(x_1, x_2, x_3, x_4) \in \mathbb{Z}^4} 1 \right) \left( \sum_{(x_5, x_6, x_7, x_8) \in \mathbb{Z}^4} 1 \right)
\]
\[
= r(0) r \left( \frac{n}{3} \right) + r(n) r(0) + \sum_{l, m \in \mathbb{N}, \ l + 5m = n} r(l) r(m)
\]
\[
= 12 \sigma \left( \frac{n}{5} \right) - 36 \sigma \left( \frac{n}{15} \right) + 12 \sigma(n) - 36 \sigma \left( \frac{n}{3} \right)
\]
\[
+ \sum_{l, m \in \mathbb{N}, \ l + 5m = n} \left( 12 \sigma(l) - 36 \sigma \left( \frac{l}{3} \right) \right) \left( 12 \sigma(m) - 36 \sigma \left( \frac{m}{3} \right) \right)
\]
\[
= 12 \sigma \left( \frac{n}{5} \right) - 36 \sigma \left( \frac{n}{15} \right) + 12 \sigma(n) - 36 \sigma \left( \frac{n}{3} \right) + 144 \sum_{l, m \in \mathbb{N}, \ l + 5m = n} \sigma(l) \sigma(m)
\]
\[
- 432 \sum_{l, m \in \mathbb{N}, \ l + 5m = n} \sigma(l) \sigma \left( \frac{m}{3} \right) - 432 \sum_{l, m \in \mathbb{N}, \ l + 5m = n} \sigma \left( \frac{l}{3} \right) \sigma(m)
\]
\[
+ 1296 \sum_{l, m \in \mathbb{N}, \ l + 5m = n} \sigma \left( \frac{l}{3} \right) \sigma \left( \frac{m}{3} \right)
\]
\[
= 12 \sigma \left( \frac{n}{5} \right) - 36 \sigma \left( \frac{n}{15} \right) + 12 \sigma(n) - 36 \sigma \left( \frac{n}{3} \right)
\]
\[
+ 144 W_5(n) - 432 W_{15}(n) - 432 W_{3,5}(n) + 1296 W_5 \left( \frac{n}{3} \right).
\]

Substituting the convolution sums using Theorem 2.1 and Theorem 2.2, we get the required formula for \( N(n). \)
2.3. More applications. Let \( Q_k \) be the quadratic form \( x_1^2 + x_2^2 + x_3^2 + x_4^2 + k(x_5^2 + x_6^2 + x_7^2 + x_8^2) \) and \( N_k(n) \) be the number of representations of integers \( n \geq 1 \) by \( Q_k \). In this section we use the convolution sums derived in \( [6, 3] \) to derive a formula for \( N_k(n) \). We note that for \( k = 2, 3, 4 \) similar formulas were obtained earlier by Williams \( [34] \), Alaca-Williams \( [7] \) and Alaca-Alaca-Williams \( [4] \) respectively. As mentioned in the introduction, we learnt from the referee that the evaluation of \( N_6(n) \) has also been derived recently by Köklıce. To find \( N_6(n) \) using our method, we need the convolution sums \( W_6(n), W_{2,3} \) and \( W_{24}(n) \) which were derived by Alaca-Alaca-Williams and they are given in the following theorem.

**Theorem 2.4.** (cf. \( [6, 3] \))

\[
W_6(n) = \frac{1}{120} \sigma_3(n) + \frac{1}{30} \sigma_3 \left( \frac{n}{2} \right) + \frac{3}{40} \sigma_3 \left( \frac{n}{3} \right) + \frac{3}{10} \sigma_3 \left( \frac{n}{6} \right) + \frac{1-n}{24} \sigma(n) + \frac{1-6n}{24} \sigma \left( \frac{n}{6} \right) - \frac{1}{120} c_6(n),
\]

\[
W_{2,3}(n) = \frac{1}{120} \sigma_3(n) + \frac{1}{30} \sigma_3 \left( \frac{n}{2} \right) + \frac{3}{40} \sigma_3 \left( \frac{n}{3} \right) + \frac{3}{10} \sigma_3 \left( \frac{n}{6} \right) + \frac{1-2n}{24} \sigma \left( \frac{n}{2} \right) + \frac{1-3n}{24} \sigma \left( \frac{n}{3} \right) - \frac{1}{120} c_6(n),
\]

\[
W_{24}(n) = \frac{1}{1920} \sigma_3(n) + \frac{1}{640} \sigma_3 \left( \frac{n}{2} \right) + \frac{3}{640} \sigma_3 \left( \frac{n}{3} \right) + \frac{1}{160} \sigma_3 \left( \frac{n}{4} \right) + \frac{9}{640} \sigma_3 \left( \frac{n}{6} \right) + \frac{1}{30} \sigma_3 \left( \frac{n}{8} \right) + \frac{9}{160} \sigma_3 \left( \frac{n}{12} \right) + \frac{3}{10} \sigma_3 \left( \frac{n}{24} \right) + \frac{4-n}{96} \sigma(n) + \frac{1-6n}{24} \sigma \left( \frac{n}{24} \right) - \frac{61}{1920} c_{1,24}(n),
\]

where \( c_6(n) \) and \( c_{1,24}(n) \) are the \( n \)-th Fourier coefficients of weight 4 normalized newforms which are given in \( [6, \text{p. } 492] \) and \( [3, \text{p. } 94] \) respectively.

In the following we use Theorem 2.4 to derive a formula for \( N_6(n) \).

**Theorem 2.5.** The number of representations of a positive integer \( n \) by the quadratic form \( Q_6 \) is given by

\[
N_6(n) = \frac{2}{5} \sigma_3(n) - \frac{2}{5} \sigma_3 \left( \frac{n}{2} \right) + \frac{18}{5} \sigma_3 \left( \frac{n}{3} \right) - \frac{8}{5} \sigma_3 \left( \frac{n}{4} \right) - \frac{18}{5} \sigma_3 \left( \frac{n}{6} \right) + \frac{128}{5} \sigma_3 \left( \frac{n}{8} \right) - \frac{72}{5} \sigma_3 \left( \frac{n}{12} \right) + \frac{1152}{5} \sigma_3 \left( \frac{n}{24} \right) - \frac{8}{15} c_6(n) + \frac{32}{15} c_6 \left( \frac{n}{2} \right) - \frac{128}{15} c_6 \left( \frac{n}{4} \right) + \frac{122}{15} c_{1,24}(n).
\]

**Proof.** For \( l \in \mathbb{N}_0 \), let

\[
r_4(l) = \# \{ (x_1, x_2, x_3, x_4) \in \mathbb{Z}^4 | x_1^2 + x_2^2 + x_3^2 + x_4^2 = l \}
\]
so that \( r(0) = 1 \). For \( l \in \mathbb{N} \), we know the formula due to Jacobi (see [17])

\[
r_4(l) = 8 \sum_{d|l, \ 4|d} \sigma(d) - 32 \sigma \left( \frac{l}{4} \right).
\]

Then \( N_6 \) is given by

\[
N_6(n) = \sum_{l,m \in \mathbb{N}} \left( \sum_{(x_1, x_2, x_3, x_4) \in \mathbb{Z}^4} 1 \right) \left( \sum_{(x_5, x_6, x_7, x_8) \in \mathbb{Z}^4} 1 \right)
\]

\[
= r_4(0) r_4 \left( \frac{n}{6} \right) + r_4(n) r_4(0) + \sum_{l,m \in \mathbb{N}} r_4(l) r_4(m)
\]

\[
= 8 \sigma(n) - 32 \sigma \left( \frac{n}{4} \right) + 8 \sigma \left( \frac{n}{6} \right) - 32 \sigma \left( \frac{n}{24} \right)
\]

\[
+ \sum_{l,m \in \mathbb{N}} \left( 8 \sigma(l) - 32 \sigma \left( \frac{l}{4} \right) \right) \left( 8 \sigma(m) - 32 \sigma \left( \frac{m}{4} \right) \right)
\]

\[
= 8 \sigma(n) - 32 \sigma \left( \frac{n}{4} \right) + 8 \sigma \left( \frac{n}{6} \right) - 32 \sigma \left( \frac{n}{24} \right) + 64 \sum_{l,m \in \mathbb{N}} \sigma(l) \sigma(m)
\]

\[
- 256 \sum_{l,m \in \mathbb{N}} \sigma(l) \sigma \left( \frac{m}{4} \right) - 256 \sum_{l \in \mathbb{N}} \sigma \left( \frac{l}{4} \right) \sigma(m)
\]

\[
+ 1024 \sum_{l \in \mathbb{N}} \sigma \left( \frac{l}{4} \right) \sigma \left( \frac{m}{4} \right)
\]

\[
= 8 \sigma(n) - 32 \sigma \left( \frac{n}{4} \right) + 8 \sigma \left( \frac{n}{6} \right) - 32 \sigma \left( \frac{n}{24} \right) + 64 W_6(n)
\]

\[
+ 1024 W_6 \left( \frac{n}{4} \right) - 256 W_{24}(n) - 256 W_{2,3} \left( \frac{n}{2} \right).
\]

Substituting the convolution sums from Theorem 2.4 in the above gives the required formula for \( N_6(n) \). \( \square \)

**Remark 2.1.** The representation numbers \( N_k(n) \) for \( k = 2, 4 \) were obtained by Williams [34] and by Alaca-Alaca-Williams [4] using the convolution sums \( W_{1,8}(n) \), \( W_{1,16}(n) \) and for \( k = 3 \) it was derived by Alaca-Williams [7] as a consequence of the representation of positive integers by certain octonary quadratic forms. Note that \( N_3(n) \) can also be obtained in a similar way as done in the cases \( k = 2, 4 \). In fact,

\[
N_3(n) = 8 \sigma(n) - 32 \sigma \left( \frac{n}{4} \right) + 8 \sigma \left( \frac{n}{6} \right) - 32 \sigma \left( \frac{n}{12} \right)
\]

\[
+ 64 W_3(n) + 1024 W_3 \left( \frac{n}{4} \right) - 256 W_{12}(n) - 256 W_{3,4}(n).
\]
Using the convolution sums $W_3(n)$, $W_{3,4}(n)$ and $W_{12}(n)$ obtained in [18], we have the following formula for $N_3(n)$:

$$N_3(n) = \frac{8}{5} \sigma_3(n) - \frac{16}{5} \sigma_3\left(\frac{n}{2}\right) + \frac{72}{5} \sigma_3\left(\frac{n}{3}\right) + \frac{128}{5} \sigma_3\left(\frac{n}{4}\right) - \frac{144}{5} \sigma_3\left(\frac{n}{6}\right) + \frac{1152}{5} \sigma_3\left(\frac{n}{12}\right) + \frac{88}{15} c_{1,12}(n) + \frac{8}{15} c_{3,4}(n).$$

(13)

The difference between the formula given in [7] Theorem 1.1 (ii) and (13) is due to different cusp forms used. In [7] coefficients of the newform of weight 4 and level 6 appear while in the above formula Fourier coefficients of two cusp forms of weight 4 and level 12 appear.

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