EULER CHARACTERISTICS AND $p$-SINGULAR ELEMENTS IN FINITE GROUPS

JESPER M. MØLLER

Abstract. We use the Euler characteristic of the orbit category of a finite group to establish equivalences between theorems of Frobenius and K.S. Brown and between theorems of Steinberg and L. Solomon.

1. Introduction

Let $G$ be a finite group with unit element $e$, $p$ a prime number and $|G|_p$ the $p$-part of the group order. An element of $G$ is $p$-singular if its order is a power of $p$ [6, Definition 40.2, §82.1]. Write $G_p = \{g \in G \mid g|G|_p = e\} = \bigcup \text{Syl}_p(G)$ for the set of all $p$-singular elements in $G$. In other words, $G_p$ is the solution set in $G$ to the equation $X|G|_p = 1$ or the union of all the Sylow $p$-subgroups of $G$. A theorem of Frobenius from 1907, or even earlier, contains as a special case a basic fact about the number, $|G_p|$, of $p$-singular group elements [7, 12] [6, Corollary 41.11] [18, 11.2, Corollary 2].

Theorem 1.1 (Frobenius 1907). $|G|_p \mid |G_p|

The number of $p$-singular elements is known for the symmetric groups and for the finite groups of Lie type in defining characteristic $p$:

- The exponential generating function for the number of $p$-singular permutations in the symmetric groups $\Sigma_n$ is [21, Example 5.2.10]
  \[
  \sum_{n=1}^{\infty} \frac{|(\Sigma_n)_p| x^n}{n!} = \exp(x + \frac{x^p}{p} + \frac{x^{p^2}}{p^2} + \cdots + \frac{x^{p^n}}{p^n} + \cdots)
  \]

- $|K_p| = |K|^2_p$ for a finite group $K$ of Lie type in defining characteristic $p$ (Theorem 4.1)

The aim of this note is to relate Frobenius’ theorem (Theorem 1.1) to a theorem of K.S. Brown (Theorem 3.1) and Steinberg’s theorem (Theorem 4.1) to a theorem of L. Solomon (Theorem 4.2). These two pairs of theorems are linked by the Euler characteristic of the orbit category discussed in Proposition 2.3.(1).

The following notation will be used in this note:

- $G$ a finite group
- $\mathcal{S}_G$ the poset of subgroups of $G$ ordered by inclusion, $H \leq K \iff H \subseteq K$
- $\mathcal{O}_G$ the orbit category of subgroups of $G$, $\mathcal{O}_G(H, K) = \{g \in G \mid H^g \subseteq K\}/K$, $\mathcal{O}_G(H) = N_G(H)/H$
- $p$ a prime number
- $O_p(G)$ the biggest normal $p$-subgroup of $G$
- $n_p$ the $p$-part of the natural number $n$
- $\mathcal{C}(a, b)$ set of morphisms from $a$ to $b$ in category $\mathcal{C}$
- $\mathcal{C}(a)$ monoid $\mathcal{C}(a, a)$ of endomorphisms of $a$ in category $\mathcal{C}$
- $q$ a prime power
- $\mathbb{F}_q$ the finite field with $q$ elements

If $\mathcal{C}_G$ is a category whose objects are all subgroups of $G$, then $\mathcal{C}_G^p$, $\mathcal{C}_G^{p+}$, $\mathcal{C}_G^{p+\text{rad}}$ denotes the full subcategory of $\mathcal{C}_G$ on all $p$-subgroups, non-trivial $p$-subgroups, $p$-radical $p$-subgroups, respectively. (A $p$-subgroup $H$ of $G$ is $p$-radical if $H = O_pN_G(H)$.)

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2. Using Euler characteristics to count $p$-singular elements

We apply Tom Leinster’s theory of Euler characteristics of finite categories [14] to the orbit category $O^p_G$. Let $ζ$ be a square matrix with rational coefficients. A weighting for $ζ$ is a vector $k$ such that all coordinates of $ζk$ equal 1. A coweighting is a weighting for the transpose of $ζ$. The matrix $ζ$ has Euler characteristic if it admits both a weighting and a coweighting, and the Euler characteristic of $ζ$, $χ(ζ)$, is then the coordinate sum of a weighting or a coweighting [14, Lemma 2.1, Definition 2.2]. The Euler characteristic of an invertible matrix is the sum of the entries of the inverse.

Let $C$ be a finite category. The $ζ$-matrix of $C$ is the square matrix $ζ(C) = (ζ(a,b))_{a,b \in \text{Obj}(C)}$ recording the cardinalities of all the morphism sets in $C$. A weighting or coweighting for $C$ is a weighting or coweighting for $ζ(C)$. The Euler characteristic of $C$ is $χ(C) = χ(ζ(C))$ when $C$ has a weighting and a coweighting. The reduced Euler characteristic of $C$ is $\tilde{χ}(C) = χ(C) − 1$. The Euler characteristic of any finite category with an initial or terminal object is 1.

The finite categories of this note all admit weightings and coweightings. By the weighting for e.g. $O^p_G$ we mean the weighting that is constant on isomorphism classes of objects [8, p 3035].

Lemma 2.1. The number of $p$-singular elements in $G$ is

$$|G_p| = p^{-1} + \sum_{1 \leq C \subseteq G} (1 − p^{-1})|C|$$

where the sum is over all cyclic $p$-subgroups $C$ of $G$.

Proof. Declare two $p$-singular elements to be equivalent of they generate the same cyclic subgroup. The set of equivalence classes is the set of cyclic $p$-subgroups $C$ of $G$. The number of elements in the equivalence class $C$ is the number of generators of $C$: $p^{-1} + (1 − p^{-1})|C|$ if $|C| = 1$ and $(1 − p^{-1})|C|$ if $|C| > 1$. □

The weightings for the poset $S^p_G$ and the category $O^p_G$, $k^G_S = −\tilde{χ}(S^p_{O^G_G(K)})$ and $k^K_S = −\frac{1}{|K|}\tilde{χ}(S^p_{O^G_G(K)})|K|$ [13, Theorem 1.3], vanish off the $p$-radical subgroups by Quillen’s [17, Proposition 2.4]. Thus the weightings for $S^p_G$, $O^p_G$ restrict to weightings for the full subcategories $S^{p+\text{rad}}_G$, $O^{p+\text{rad}}_G$ and $χ(S^{p+\text{rad}}_G) = χ(S^p_G) = 1$, $χ(O^{p+\text{rad}}_G) = χ(O^p_G)$ [13, Lemma 2.9].

It can be more convenient to work with conjugacy classes of subgroups rather than the subgroups themselves. Let $[S^{p+\text{rad}}_G]$ be the set of conjugacy classes $[K]$ of $p$-radical subgroups $K$ of $G$. Since $|O^G_G(H,K)| = |(K \cap G)^H|$ is the mark of $H$ on the transitive right $G$-set $K \cap G$, the square matrix $[O^{p+\text{rad}}_G]$ with entries

$$[O^{p+\text{rad}}_G](H, [K]) = |O^G_G(H,K)|, \quad [H], [K] \in [S^{p+\text{rad}}_G]$$

is Burnside’s table of marks [3] for the $p$-radical subgroup classes. It is easy to see that the category $O^{p+\text{rad}}_G$ and the table of marks $[O^{p+\text{rad}}_G]$ have the same Euler characteristic [13, §2.4].

Proposition 2.3. Let $G$ be a finite group and $p$ a prime number.

$$\begin{align*}
(1) \sum_{K \in S^{p+\text{rad}}_G} -\tilde{χ}(S^{p+\text{rad}}_{O^G_G(K)})|K| &= |G_p| \\
(2) \sum_{K \in S^{p+\text{rad}}_G} -\tilde{χ}(S^{p+\text{rad}}_{O^G_G(K)}) &= 1 \\
(3) \sum_{[K] \in [S^{p+\text{rad}}_G]} -\tilde{χ}(S^{p+\text{rad}}_{O^G_G(K)})|O^G_G(H,K)| &= 1 \text{ for any } p\text{-radical subgroup } H
\end{align*}$$

Proof. Lemma 2.1 combined with [13, Theorem 1.3.(4)] show that

$$|G_p| = |G|χ(O^{p+\text{rad}}_G) = \sum_{K \in S^{p+\text{rad}}_G} -\tilde{χ}(S^{p+\text{rad}}_{O^G_G(K)})|K|$$

where the sum ranges over all $p$-radical subgroups $K$ of $G$. This proves (1). Item (2) simply expresses that $S^{p+\text{rad}}_G$, with $O^G_G$ as its least element [8, Proposition 6.3], has Euler characteristic equal to 1.

The weighting, $k^{\bullet}_G : [S^{p+\text{rad}}_G] \to \mathbb{Q}$, for the table of marks of the $p$-radical subgroup classes (2.2) satisfies

$$\sum_{[K] \in [S^{p+\text{rad}}_G]} |O^G_G(H,K)||K^K_G| = 1$$
for all \( p \)-radical subgroups \( H \). The weightings, \( k_{G}^{K} \) and \( k_{S}^{K} \), for \( \mathcal{O}_{G}^{p+\text{rad}} \) and \( \mathcal{S}_{G}^{p+\text{rad}} \) are [13, Proposition 2.14]

\[
k_{G}^{K} = \frac{|N_{G}(K)|}{|G|} k_{O}^{K}, \quad k_{S}^{K} = \frac{|G|}{|K|} k_{G}^{K} = \frac{|G|}{|K|} \frac{|N_{G}(K)|}{|G|} k_{O}^{K} = |\mathcal{O}_{G}(K)| k_{O}^{K}
\]

and therefore

\[
\sum_{[K] \in [\mathcal{S}_{G}^{p+\text{rad}}]} \frac{|\mathcal{O}_{G}(H,K)|}{|\mathcal{O}_{G}(K)|} k_{S}^{K} = 1
\]

which is the third item.

Proposition 2.3.(1) expresses that \( |G_{p}| = |G| \chi(\mathcal{O}_{G}^{p+\text{rad}}) = |G| \chi(\mathcal{O}_{G}^{p+\text{rad}}) \) can be computed from the table of marks for the \( p \)-radical subgroups (2.2).

The content of Proposition 2.3.(3) is that the vector \( (k_{S}^{K})_{K \in [\mathcal{S}_{G}^{p+\text{rad}}]} \) is a weighting for \( [[\mathcal{O}_{G}^{p+\text{rad}}]], \) the modified table of marks, defined to be the square matrix with entries

\[
[[\mathcal{O}_{G}^{p+\text{rad}}]](H,[K]) = \frac{|\mathcal{O}_{G}(H,K)|}{|\mathcal{O}_{G}(K)|}, \quad [H],[K] \in [\mathcal{S}_{G}^{p+\text{rad}}]
\]

In other words, \( k_{S}^{K} = k_{O}^{K} \) for all \( p \)-radical subgroups \( K \) where \( k_{O}^{K} \) is the weighting for \( [[\mathcal{O}_{G}^{p+\text{rad}}]] \).

The normalizer \( N_{G}(K) \) acts on the transporter set \( N_{G}(H,K) = \{ g \in G \mid Hg \subseteq K \} \) and the orbit set corresponds bijectively via the map \( g \mapsto K^{g^{-1}} \) to the set \( \{ L \in [K] \mid H \supseteq L \} \) of conjugates of \( K \) containing \( H \). Thus the modified mark

\[
\frac{|\mathcal{O}_{G}(H,K)|}{|\mathcal{O}_{G}(K)|} = |N_{G}(H,K)/N_{G}(K)| = |\{ L \in [K] \mid H \supseteq L \}|
\]

is the number of \( H \)-supergroups conjugate to \( K \).

**Example 2.5** \((G = \Sigma_{4}, p = 2)\). The 2-radical subgroup classes of the symmetric group \( \Sigma_{4} \) are the Sylow 2-subgroup \( D_{8} \) and \( O_{2}(\Sigma_{4}) = C_{2} \times C_{2} \) of order 4. The table of marks (2.2) and the modified table of marks (2.4) for the 2-radical subgroup classes in \( \Sigma_{4} \) are

\[
[\mathcal{O}_{\Sigma_{4}}^{2+\text{rad}}] = \begin{pmatrix} 1 & 0 \\ 3 & 6 \end{pmatrix}, \quad [[\mathcal{O}_{\Sigma_{4}}^{2+\text{rad}}]] = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}
\]

The weighting for the table of marks is \( k_{[O]} = (1,-1/3) \) and \( |\Sigma_{4}| \chi([[\mathcal{O}_{\Sigma_{4}}^{2+\text{rad}}]]) = 24(1 - 1/3) = 16 \) is the number of \( 2 \)-singular elements in \( \Sigma_{4} \) in agreement with Proposition 2.3.(1). The modified table of marks has weighting \( k_{[O]} = (1,-2) \) which by Proposition 2.3.(3) means that \( k_{S}^{D_{8}} = 1 \) and \( k_{S}^{C_{2} \times C_{2}} = -2 \). Since the subgroups \( D_{8} \), \( C_{2} \times C_{2} \) have lengths 3, 1, the Euler characteristic of the Brown poset \( \mathcal{S}_{\Sigma_{4}}^{2+\text{rad}} \) is \( \chi(\mathcal{S}_{\Sigma_{4}}^{2+\text{rad}}) = 3 \cdot 1 + 1 \cdot (-2) = 1 \) in agreement with Proposition 2.3.(2).

3. **The theorems of Frobenius and Brown for finite groups are equivalent**

The following theorem was proved by K.S. Brown [1] (and reproved by Quillen [17, Corollary 4.2], Webb [23, Theorem 8.1] and others).

**Theorem 3.1** (Brown 1975). \(|G_{p}| \mid \chi(\mathcal{O}_{G}^{p+\ast})\)

It was observed in [2, 11] that Möbius functions link the theorems of Frobenius and Brown. We here note that also Proposition 2.3.(1) connects the two theorems.

**Proposition 3.2.** Theorems 1.1 and 3.1 are equivalent given Proposition 2.3.(1).

**Proof.** Proposition 2.3.(1) may be rewritten on the form

\[
|G_{p}| + \chi(\mathcal{O}_{G}^{p+\ast}) + \sum_{[H] \neq 1} \frac{\chi(\mathcal{S}_{\mathcal{O}_{G}(H)}^{p+\ast})}{|\mathcal{O}_{G}(H)|_{p}} \frac{|G|}{|\mathcal{O}_{G}(H)|_{p'}} = 0
\]

where we have isolated the contribution from the trivial subgroup and the sum is over classes of non-trivial \( p \)-radical subgroups of \( G \).

Assume first that Theorem 1.1 holds. In Equation (3.3), we may assume that

- \( \chi(\mathcal{S}_{\mathcal{O}_{G}(H)}^{p+\ast})/|\mathcal{O}_{G}(H)|_{p} \) is an integer when \( H \) is nontrivial (as part of an induction argument)
- \(|G|/|\mathcal{O}_{G}(H)|_{p'}\) is an integer divisible by \(|G|_{p} \) (as \(|\mathcal{O}_{G}(H)| \) divides \(|G|)
Thus every term in the sum is divisible by $|G|_p$ and so is $|G_p|$ by assumption. We conclude that $\overline{\chi}(S_{G}^{p+*})$ is divisible by $|G|_p$ and we have arrived at Theorem 3.1.

Assume next that Theorem 3.1 holds. In Equation (3.3)

- $|G|/|O_G(H)|_p'$ is an integer divisible by $|G|_p$
- $\overline{\chi}(S_{G}^{p+*})/|O_G(H)|_p$ is an integer
- $\overline{\chi}(S_{G}^{p+*})$ is divisible by $|G|_p$

and thus $|G|_p$ divides for $|G_p|$. This is Theorem 1.1.

4. The theorems of Solomon and Steinberg for finite groups of Lie type are equivalent

Let $\Sigma$ be a reduced and crystallographic root system with fundamental and positive roots $\Pi, \Sigma^+ \subseteq \Sigma$ [10, Definition 1.8.1]. Suppose $\overline{K}(\Sigma)$ is a semisimple $\overline{F}_p$-algebraic group with root system $\Sigma$ [10, Theorem 1.10.4] equipped with a (standard form) Steinberg endomorphism $\sigma$ [10, Definition 1.15.(b), Remarks 2.2.5.(e)]. Assuming $\Sigma$ to be also irreducible [10, Definition 1.8.4], let $K = O'P\overline{C}_K(\Sigma)(\sigma)$ be the finite group in $\text{Lie}(p)$ with $\sigma$-setup $(\overline{K}(\Sigma), \sigma)$ [10, Definition 2.2.2].

The number of $p$-singular elements in $K$ was determined by Steinberg [22, 15.2].

**Theorem 4.1** (Steinberg 1968). $|K_p| = |K|_p^2$

The surjections $\Sigma \to \overline{\Sigma} \to \overline{\Sigma}$ of [10, (2.3.1)] induce surjections $\Pi \to \overline{\Pi} \to \overline{\Pi}$ of sets. Here, $\overline{\Sigma}$ is the twisted root system of $K$ [10, p 41], and $\overline{\Sigma} = \overline{\Sigma}/\sim$ is the set of equivalence classes of twisted roots pointing in the same direction. If $K$ is an un twisted group of Lie type [10, Definition 2.2.4], $\Sigma = \overline{\Sigma}$.

For every subset $J \subseteq \overline{\Pi}$ we have associated subgroups $P_J, U_J, L_J \subseteq K$ such that $U_J = O_pP_J, P_J = N_K(U_J)$ and $P_J = U_J \rtimes L_J$ [10, Theorem 2.6.5]. The $P_J$ are parabolic subgroups, the $U_J$ are unipotent $p$-radical subgroups and the $L_J$ are Levi complements [10, Definition 2.6.4, Definition 2.6.6]. The extreme cases where $J = \emptyset, \overline{\Pi}$ are, $P_0 = U_0 \rtimes L_0$, where $U_0 = B$ is a Borel subgroup of $K, U_0 = U$ a Sylow $p$-subgroup [10, p 41, Theorems 2.3.4, 2.3.7] and $L_0 = H$ is a maximal torus or Cartan subgroup [10, Theorem 2.4.7, Definition 2.4.12], and $P_{\overline{\Pi}} = K = L_{\overline{\Pi}}, U_{\overline{\Pi}} = 1$.

The following polynomial identity dates back to L. Solomon [19, Corollary 1.1] [4, Theorem 9.4.5, §14].

**Theorem 4.2** (Solomon 1966). $\sum_{J \subseteq \overline{\Pi}} (-1)^{|J|} |P_J| = |K|_p$

Solomon’s theorem, essentially a statement about reflection groups, generalizes to the following two identities that are Möbius inverses to each other.

**Corollary 4.3.** For any subset $I$ of $\overline{\Pi}$, $\sum_{J \supseteq I} (-1)^{|J|} |P_I : P_J| = |L_I|_p$ and $\sum_{I \supseteq J} (-1)^{|J|} |P_I : P_J||L_J|_p = 1$.

We use a consequence of the Borel–Tits theorem [10, Theorem 3.1.3] to determine the modified table of marks for the $p$-radical subgroups of $K$.

**Lemma 4.4.** The modified table of marks (2.4) for the $p$-radical subgroups, $U_I, I \subseteq \overline{\Pi}$, of $K$ has entries

$$[\mathcal{O}_K^{p+\text{rad}}]^0(U_I, U_J) = \begin{cases} |P_I : P_J| & I \supseteq J \\ 0 & \text{otherwise} \end{cases}$$

for all subsets $I, J \subseteq \overline{\Pi}$.

**Proof.** By [10, Corollary 3.16], assisted by [10, Theorem 2.6.7] to show that any parabolic subgroup containing $U$ contains $B$, the set $\{U_I \mid I \subseteq \overline{\Pi}\}$ is the Alperin–Goldschmidt conjugation family [9, Theorem 16.1] controlling fusion in $U$. It follows that if $g \in K$ conjugates $U_I$ into some $U_I'$ then $U_I'' = U_I$ so that $I \supseteq J$ and $g \in N_K(U_I) = P_I$ [10, Theorem 2.6.5]. This shows that the transporter set $N_K(U_I, U_J)$ equals $N_K(U_I)$ in case $I$ contains $J$ and is empty otherwise.

This leads to a weak version of the Solomon–Tits theorem [5, Corollary 7.3].

**Corollary 4.5.** $-\overline{\chi}(S_{L_I}^{p+*}) = (-1)^{|I|} |L_I|_p$ for all $I \subseteq \overline{\Pi}$.

**Proof.** This follows immediately from Proposition 2.3.(3) and the second identity of Corollary 4.3 as the coefficients $|P_I : P_J|$ are the the modified marks by Lemma 4.4. □
**Example 4.6.** The group $GL_3(F_2) \in Lie(2)$ has fundamental roots $\Pi = \{\alpha_1, \alpha_2\}$. Its parabolic subgroups, $P_0$, $P_{(\alpha_1)}$, $P_{(\alpha_2)}$, $P_1$, have orders $8, 8, 3, 3, 8, 3, \cdot 21$ and Levi complements, $1, GL_2(F_2), GL_2(F_2), GL_2(F_2)$, with Sylow 2-subgroup orders $1, 2, 2, 8$. The signed vector $(1, -2, -2)$ of Corollary 4.5 is indeed the weighting for the modified table of marks of Lemma 4.4 as

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
3 & 1 & 0 & 0 \\
3 & 0 & 1 & 0 \\
21 & 7 & 7 & 1
\end{pmatrix}
\begin{pmatrix}
1 \\
-2 \\
-2 \\
8
\end{pmatrix}
= \begin{pmatrix}
1 \\
1 \\
1 \\
1
\end{pmatrix}
\]

By Proposition 2.3.1, this weighting for the modified table of marks records the negative reduced Euler characteristics, $-\chi(S^{2+}_L)$, of the Brown posets for the Levi complements.

We are now prepared to prove a version of Steinberg’s theorem valid for all parabolic subgroups of $K$.

**Theorem 4.7.** $|P_p|O_p(P)| = |K|_{p^2}$ for any parabolic subgroup $P$ of $K$.

**Proof.** There is always a bijection between the $p$-radical subgroups of a finite group $G$ and those of $G/O_p(G)$ [8, Proposition 6.3]. In particular, $\{U_J \mid I \supseteq J\}$ is a complete set of representatives for the $p$-radical subgroups of $P_I$ corresponding to the $p$-radical subgroup classes of $L_I$. Obviously, $NP_I(U_J) = P_I \cap NK(U_J) = P_I \cap P_J = P_J = N_K(U_J)$, $\mathcal{O}_P(U_J) = \mathcal{P}_J \cup \mathcal{U}_I(U_J)$ and $|P_I: NP_I(U_J)| = |P_I: P_J|$ is the length of $U_J$ in $P_I$ and $K$. By Proposition 2.3.1, the number of $p$-singular elements in $P_I$ is

\[
|P_I| = \sum_{I \supseteq J}(-1)^{|J|}P_I: P_J|L_J|_{p^2}|U_J| = \sum_{I \supseteq J}(-1)^{|J|}P_I: P_J|L_J|_{p^2}|U_J| = |K|_{p^2}\sum_{I \supseteq J}(-1)^{|J|}P_I: P_J|
\]

Cor 4.3 $|K|_{p^2}|L_I|_{p} = |K|_{p}|P_I: U_I|_{p} = |K|_{p^2}/|U_I|

This finishes the proof as $U_I = O_P(P_I)$.

Theorems 4.1 and 4.7 apply e.g. to $\Omega^\pm_{2m}(F_q)$, $P^\Omega_{2m}(F_q)$, $Spin^\pm_{2m}(F_q)$ for all prime powers $q$, $SO^\pm_{2m}(F_q)$ for odd $q$, but not to $SO_{2m}^+(F_{2^e})$ [10, §2.7]. $(SO^+_2(F_{2^e}) \cong \Sigma_5$ of order $8 \cdot 15$ contains 56 < $8^3$-2-singular elements.) They also apply to $\Omega^\pm_{2m+1}(F_q)$, $SO_{2m+1}(F_q)$ and $Spin^\pm_{2m+1}(F_q)$ for all $q$.

The $q$-bracket of the natural number $d$ is the polynomial $[d](q) = q^{d-1} + \cdots + q + 1 \in \mathbb{Z}[q]$ of degree $d - 1$ with value $[d](1) = d$ at $q = 1$. For a reflection group $W$, put $W(q) = \prod_{i \in I}[d](q)$ where the product is over the degrees $d$ of the basic polynomial invariants [4, Proposition 10.2.5]. The two identities of Corollary 4.3 with $I = \Pi$ for the Chevalley (untwisted) groups associated to $K(\Sigma)$ with Weyl group $W$,

\[
(4.8) \sum_{\Pi \supseteq J}(-1)^J W_{\Pi}(q) W_{J}(q) = q^{[\Sigma^+]} , \quad \sum_{\Pi \supseteq J}(-1)^J W_{\Pi}(q) W_{J}(q)^q = 1
\]

are two $q$-analogues of Witt’s identity [24] [19, (5)] $\sum_{\Pi \supseteq J}(-1)^J W_{\Pi} = W_J = 1$.

Let $OP(m) = \{(m_1, \ldots, m_k) \mid k \geq 1, m_i \geq 1, \sum m_i = m\}$ denote the set of all the $2^{m-1}$ ordered partitions of $m$ [20, p 14].

**Example 4.9.** Subsystems of the root systems $A_{m-1}$ or $B_{m-1}$ are indexed by $OP(m)$ via the bijection taking $(m_1, \ldots, m_k) \in OP(m)$ to $A_{m-1} \times \cdots \times A_{m-1}$ or $A_{m-1} \times \cdots \times A_{m-1} \times B_{m-1}$ (where $A_0$ is the empty root system and $B_0 = A_0, B_1 = A_1$). The incarnations of equations (4.8) for the Chevalley groups $SL^+_m(F_q)$ and $SO_{2m-1}(F_q)$ of rank $m - 1$ with root systems $\Sigma = A_{m-1}, B_{m-1}$ are the polynomial identities

\[
\sum_{\Pi \supseteq J}(-1)^J W_{\Pi}(q) W_{J}(q) = q^{[\Sigma^+]} , \quad \sum_{\Pi \supseteq J}(-1)^J W_{\Pi}(q) W_{J}(q)^q = 1
\]

are two $q$-analogs of Witt’s identity [24] [19, (5)] $\sum_{\Pi \supseteq J}(-1)^J W_{\Pi} = W_J = 1$.
Example 4.10 (SL_{m}^-(F_q)). The two identities of Corollary 4.3 with \( I = \hat{\Pi} \) for the Steinberg \( \text{SL}_{2m}^{-}(F_q) \) of rank \( 2m - 1 \) and twisted rank \( m \) are

\[
\sum_{d=1}^{2m} \frac{(-1)^k}{d} \prod_{d=1}^{[d]} |d|((-1)^d q) - \sum_{d=2m+2}^{2m} \frac{(-1)^k}{d} \prod_{d=2m+2}^{[d]} |d|((-1)^d q) = (-1)^m q^{(2m)}
\]

and for the Steinberg group \( \text{SL}_{2m+1}^{-}(F_q) \) of rank \( 2m \) and twisted rank \( m \) they are

\[
\sum_{d=1}^{2m+1} \frac{(-1)^k}{d} \prod_{d=1}^{[d]} |d|((-1)^d q) - \sum_{d=2m+2}^{2m+1} \frac{(-1)^k}{d} \prod_{d=2m+2}^{[d]} |d|((-1)^d q) = (-1)^m q^{(2m+1)}
\]

where the sums run over all \((m_1, \ldots, m_k) \in \text{OP}(m)\). These identities are obtained by analyzing the \( C_2 \)-subsystems of the \( C_2 \)-root system \( A_{m-1} \) [4, 13.3.8]. Write \( S(A_{m-1}) \) for the multisets of all \( C_2 \)-subsystems of \( A_{m-1} \). One subsystem of \( A_{2m-1} \) is \( a_{2m-1} \) defined to be the free part of \( A_{2m-1} \), i.e. the subsystem obtained by deleting the middle root \( \alpha_m \). The fundamental roots of the \( C_2 \)-root systems \( a_1, a_3, a_5, a_7 \) are

\[
a_1, a_3, a_5, a_7
\]

The first multisets of subsystems are \( S(A_1) = \{a_1, A_1\}, S(A_2) = \{a_1, A_2\}, S(A_3) = \{a_1, a_1, a_3, A_3\} = a_1 \times S(A_1) \cup \{a_3, A_3\}, S(A_4) = \{a_1, a_2, a_3, A_4\} = a_1 \times S(A_2) \cup \{a_3, A_4\}\). In general, the \( 2^m \) subsystems of \( A_{2m-1} \) and \( A_{2m} \), \( m \geq 2 \), are the multisets

\[
S(A_{2m-1}) = a_1 \times S(A_{2m-3}) \cup \cdots \cup a_{2i-1} \times S(A_{2(m-i)-1}) \cup \cdots \cup a_{2m-3} \times S(A_1) \cup \{a_{2m-1}, A_{2m-1}\}
\]

\[
S(A_{2m}) = a_1 \times S(A_{2m-2}) \cup \cdots \cup a_{2i-1} \times S(A_{2(m-i)-2}) \cup \cdots \cup a_{2m-3} \times S(A_2) \cup \{a_{2m-1}, A_{2m}\}
\]

For each subsystem \( a \) of \( A_m \), let \( P(a)(q) = |P : B| \in Z[q] \) be the index of the Borel subgroup \( B \) in the parabolic subgroup of \( \text{SL}_{m+1}^{-}(F_q) \) corresponding to \( a \). In particular, \( P(A_m)(q) \) and \( P(a_{2m-1})(q) \) are the polynomials

\[
P(A_m)(q) = \prod_{1 \leq d \leq m+1} |d|((-1)^d q), \quad P(a_{2m-1}) = \prod_{1 \leq d \leq m} |d|(q^2) = [m]!(q^2) \quad m \geq 1
\]

of degrees \( \binom{m+1}{2} \) and \( \binom{m}{2} \). Consider the multiset of signed polynomials associated to all subsystems of \( A_m \)

\[
P(S(A_m)) = \{(\sigma(I(a))C_2) \mid a \in S(A_m)\}
\]

where \( \Pi(a) \) is the set of fundamental roots and \( \Pi(a)/C_2 \) the orbit set. Then \( P(S(A_1)) = \{1, -P(A_1)\}, P(S(A_2)) = \{1, -P(A_2)\} \) and one may now determine the multisets of polynomials for all the \( C_2 \)-root systems \( A_{2m-1} \) and \( A_{2m} \), \( m \geq 2 \). This leads to the above polynomial identities.

Steinberg’s theorem applies in the equicharacteristic case and does not hold in the cross-characteristic case. However, it is known that the number of \( p \)-singular classes in \( \text{GL}_n(F_q) \), \( p \nmid q \), is

\[
|\text{GL}_n(F_q)/\text{GL}_n(F_q)| = \frac{1}{n!} \sum_{\lambda \in n} T(\lambda) \prod_{b \in \lambda} (q^b - 1)_p
\]

where \( \lambda \) ranges over all partitions of \( n \) and \( T(\lambda) \) is the number of permutations of cycle type \( \lambda \) in \( \Sigma_n \) [15, Corollary 4.22]. The number of \( p \)-singular classes, but not the number of \( p \)-singular elements, in \( \text{GL}_n(F_q) \), \( p \nmid q \), depends only on the \( p \)-fusion system.
EULER CHARACTERISTICS AND $p$-SINGULAR ELEMENTS IN FINITE GROUPS

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REFERENCES

[1] Kenneth S. Brown, Euler characteristics of groups: the $p$-fractional part, Invent. Math. 29 (1975), no. 1, 1–5. MR 035008 (52 #5878)
[2] Kenneth S. Brown and Jacques Thévenaz, A generalization of Sylow’s third theorem, J. Algebra 115 (1988), no. 2, 414–430. MR 943266
[3] W. Burnside, Theory of groups of finite order, Dover Publications Inc., New York, 1955, 2d ed. MR 0069818 (16,1086c)
[4] Roger W. Carter, Simple groups of Lie type, Wiley Classics Library, John Wiley & Sons Inc., New York, 1989, Reprint of the 1972 original, A Wiley-Interscience Publication. MR 90g:20001
[5] C. W. Curtis, G. I. Lehrer, and J. Tits, Spherical buildings and the character of the Steinberg representation, Invent. Math. 58 (1980), no. 3, 201–210. MR 571572
[6] Charles W. Curtis and Irving Reiner, Representation theory of finite groups and associative algebras, AMS Chelsea Publishing, Providence, RI, 2006. Reprint of the 1962 original. MR 2215618 (2006m:16001)
[7] G. Frobenius, Über einen Fundamentalsatz der Gruppentheorie, II, Sitzungsberichte der Preussischen Akademie Weissenstein (1907), 428–437.
[8] Matthew Gelvin and Jesper M. Møller, Homotopy equivalences between $p$-subgroup categories, J. Pure Appl. Algebra 219 (2015), no. 7, 3030–3052. MR 3313517
[9] Daniel Gorenstein, Richard Lyons, and Ronald Solomon, The classification of the finite simple groups. Number 2. Part I. Chapter G, Mathematical Surveys and Monographs, vol. 40, American Mathematical Society, Providence, RI, 1996, General group theory. MR MR1358135 (96h:20032)
[10] ______, The classification of the finite simple groups. Number 3. Part I. Chapter A, Mathematical Surveys and Monographs, vol. 40, American Mathematical Society, Providence, RI, 1998, Almost simple $K$-groups. MR MR1490581 (98j:20011)
[11] T. Hawkes, I. M. Isaacs, and M. Özaydin, On the Möbius function of a finite group, Rocky Mountain J. Math. 19 (1989), no. 4, 1003–1034. MR MR1039540 (90k:20046)
[12] I. M. Isaacs and G. R. Robinson, On a theorem of Frobenius: solutions of $x^n = 1$ in finite groups, Amer. Math. Monthly 99 (1992), no. 4, 352–354. MR 1157226 (93a:20034)
[13] Martin Wedel Jacobsen and Jesper M. Møller, Euler characteristics and Möbius algebras of $p$-subgroup categories, J. Pure Appl. Algebra 216 (2012), no. 12, 2665–2696. MR 2943749
[14] Tom Leinster, The Euler characteristic of a category, Doc. Math. 13 (2008), 21–49, Doc. Math. MR MR2393085
[15] Jesper M. Møller, Equivariant Euler characteristics of subspace posets, ArXiv e-prints (2017).
[16] T. Kyle Petersen, Eulerian numbers, Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks], Birkhäuser/Springer, New York, 2015, With a foreword by Richard Stanley. MR 3408615
[17] Daniel Quillen, Homotopy properties of the poset of nontrivial $p$-subgroups of a group, Adv. in Math. 28 (1978), no. 2, 101–128. MR MR049016 (80k:20049)
[18] Jean-Pierre Serre, Linear representations of finite groups, Springer-Verlag, New York, 1977, Translated from the second French edition by Leonard L. Scott, Graduate Texts in Mathematics, Vol. 42. MR MR0450380 (56 #8675)
[19] Louis Solomon, The orders of the finite Chevalley groups, J. Algebra 3 (1966), 376–393. MR 0199275
[20] Richard P. Stanley, Enumerative combinatorics. Vol. 1, Cambridge Studies in Advanced Mathematics, vol. 49, Cambridge University Press, Cambridge, 1997, With a foreword by Gian-Carlo Rota, Corrected reprint of the 1986 original. MR MR1442260 (98b:05001)
[21] ______, Enumerative combinatorics. Vol. 2, Vol. 62, Cambridge Studies in Advanced Mathematics, vol. 62, Cambridge University Press, Cambridge, 1999, With a foreword by Gian-Carlo Rota and appendix 1 by Sergey Fomin. MR 1676282 (2000k:05026)
[22] Robert Steinberg, Endomorphisms of linear algebraic groups, Memoirs of the American Mathematical Society, No. 80, American Mathematical Society, Providence, R.I., 1968. MR MR0230728 (37 #6288)
[23] P. J. Webb, A local method in group cohomology, Comment. Math. Helv. 62 (1987), no. 1, 135–167. MR 882969 (88h:20065)
[24] Ernst Witt, Spiegelungsgruppen und Aufzählung halbeinfacher Liescher Ringe, Abh. Math. Sem. Hansischen Univ. 14 (1941), 289–322. MR 005099

INSTITUT FOR MATHEMATISKE FAG, UNIVERSITETSPARKEN 5, DK–2100 KØBENHAVN
E-mail address: moller@math.ku.dk
URL: http://www.math.ku.dk/~moller