Low-Rank Matrix Estimation From Rank-One Projections by Unlifted Convex Optimization

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Abstract

We study an estimator with a convex formulation for recovery of low-rank matrices from rank-one projections. Using initial estimates of the factors of the target $d_1 \times d_2$ matrix of rank-$r$, the estimator operates as a standard quadratic program in a space of dimension $r(d_1 + d_2)$. This property makes the estimator significantly more scalable than the convex estimators based on lifting and semidefinite programming. Furthermore, we present a streamlined analysis for exact recovery under the real Gaussian measurement model, as well as the partially derandomized measurement model by using the spherical 2-design. We show that under both models the estimator succeeds, with high probability, if the number of measurements exceeds $r^2(d_1 + d_2)$ up to some logarithmic factors. This sample complexity improves on the existing results for nonconvex iterative algorithms.

1 Introduction

We consider the problem of estimating a matrix $M_0 \in \mathbb{C}^{d_1 \times d_2}$ of rank $r \ll \min\{d_1, d_2\}$ from rank-one “sketches” of the form

$$m_i = a_i^* M_0 b_i, \quad i = 1, \ldots, n,$$

for random vectors $a_i \in \mathbb{C}^{d_1}$ and $b_i \in \mathbb{C}^{d_2}$ drawn from certain distributions. More specifically, given the observations $\{(a_i, b_i, m_i)\}_{i=1}^n$, our goal is to estimate factors $X_0 \in \mathbb{C}^{d_1 \times r}$ and $Y_0 \in \mathbb{C}^{d_2 \times r}$ of $M_0$ (i.e., $M_0 = X_0 Y_0^*$).

1.1 Anchored regression

Let $\tilde{X}_0$ and $\tilde{Y}_0$ be a pair of matrices for which $\tilde{X}_0 \tilde{Y}_0^*$ approximates the ground truth matrix $M_0$. Our proposed estimator is formulated as

$$(\tilde{X}, \tilde{Y}) \in \arg\max_{X,Y} \langle \tilde{X}_0, X \rangle + \langle \tilde{Y}_0, Y \rangle - \frac{2}{n} \sum_{i=1}^n \ell_i(X,Y),$$

where

$$\ell_i(X,Y) \equiv \frac{1}{2} \|X^* a_i\|^2 + \frac{1}{2} \|Y^* b_i\|^2 + |a_i^* X Y^* b_i - m_i|.$$

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Here and throughout, the inner product \( \langle U, V \rangle \) is real-valued and is defined as
\[
\langle U, V \rangle \overset{\text{def}}{=} \text{Re} \left( \text{tr} \left( U^\ast V \right) \right).
\]

The optimization in (2) is effectively a convex program and can be solved efficiently. To clarify this fact, observe that the functions \( \ell_i(X, Y) \) can be written equivalently as
\[
\ell_i(X, Y) = \sup_{\phi : |\phi| = 1} \left\{ \frac{1}{4} \| X^\ast a_i \|_2^2 + \frac{1}{2} \| Y^\ast b_i \|_2^2 + |\text{Re}(\phi (a_i^\ast XY^\ast b_i - m_i))| \right\}
\]
\[
= \sup_{\phi : |\phi| = 1} 2 \max \left\{ \frac{1}{4} \| X^\ast a_i + \phi Y^\ast b_i \|_2^2 - \text{Re}(\phi m_i), \frac{1}{4} \| X^\ast a_i - \phi Y^\ast b_i \|_2^2 \right\}.
\]

For any fixed \( \phi \), the argument of the supremum is clearly convex in \([X; Y] \in \mathbb{C}^{(d_1 + d_2) \times r}\). Therefore, \( \ell_i(X, Y) \) is also a convex function of \([X; Y] \), meaning that (2) is a convex program.

Because of the specific form of the loss functions \( \ell_i(X, Y) \), the estimator in (2) can be viewed as a “convexification” of the (nonconvex) least absolute deviation (LAD) estimator by quadratic regularization. Previously, [1] has studied similar estimators for observations in the form of difference of convex functions, with the bilinear observations for rank-1 matrices as a special case. In this paper we provide a streamlined analysis of the estimator tailored to desketching of a low-rank matrix from its rank-one measurements in (1). The following provides the high-level description of the sample complexity we have established for the anchored regression estimator. The precise statements are provided in Theorems 1 and 2.

**Theorem.** The anchored regression estimator reconstructs the unknown rank-\( r \) matrix \( M_0 \) exactly from \( O(r(d_1 + d_2) \text{polylog}(d_1 + d_2)) \) random rank-one measurements with high probability starting from a certain neighborhood of \( M_0 \).

The sufficient number of samples for the exact recovery provided by the theorem is near optimal compared to the degrees of freedom of the rank-\( r \) matrix model. The random sketching models and the size of the neighborhood will be specified in the following section.

### 1.2 Sketching Models

We study the sample complexity of our proposed estimator under two different random measurements models. The first model, which we refer to as the real Gaussian sketching, simply uses the outer product of two independent Gaussian vectors as the rank-one sketching matrix. The second model, called the partially derandomized sketching, mimics the behavior of the first model but the random rank-one sketching matrix takes realizations from a finite set. More precisely, the first few moments of the sketching matrix of the second model are designed to coincide with those of the real Gaussian model.

#### 1.2.1 Real Gaussian sketching

This model simply considers the random vectors \([a_i; b_i] \in \mathbb{R}^{d_1 + d_2}\) to be independent copies of a random vector \([a; b] \in \mathbb{R}^{d_1 + d_2}\) satisfying
\[
[a; b] \sim \mathcal{N}(0, I_{d_1 + d_2}).
\]
1.2.2 Partially derandomized sketching

Next we consider a measurement distribution supported on a special finite set that mimics the following property of the real Gaussian model. Let \( g \) denote the concatenated measurement vector \([a; b]\) in the real Gaussian model. Then \( g \) is isotropic (i.e., \( \mathbb{E}(gg^\top) = I_{d_1+d_2} \)), and the second moment of its outer product with itself satisfies

\[
\mathbb{E}(gg^\top \otimes gg^\top)(x \otimes y) = x \otimes y + y \otimes x + x^\top y I_{d_1+d_2}, \quad \forall x, y \in \mathbb{R}^{d_1+d_2},
\]

where for simplicity of the notation we treated the tensor product \( x \otimes y \) (resp. \( y \otimes x \)) as the matrix \( xy^\top \) (resp. \( yx^\top \)). The symmetry of the right-hand side in (4), together with the mentioned isotropy, are the properties that the partially derandomized model mimics. A variant of this model has been previously introduced for the phase retrieval in the context of quantum tomography \([8, 11]\). Below we describe the version of the model that is relevant for our purposes.

Let \( P_{\text{Sym}} \) denote the totally symmetric subspace of \((\mathbb{C}^d)^\otimes t\) such that all elements are invariant under every possible permutation of \( t \) factors (see, e.g., \([12]\)). Then a weighted \( t \)-design is defined as follows \([8]\).

**Definition 1.** Let \( t \in \mathbb{N} \) and \( w_1, \ldots, w_N \in \mathbb{C}^d \) be unit vectors. The set \( \{w_i\}_i \) with corresponding weights \( \{p_i\}_i \) such that \( p_i \geq 0 \) for all \( i = 1, \ldots, N \), and \( \sum_{i=1}^N p_i = 1 \) is a weighted complex projective \( t \)-design of dimension \( n \) and cardinality \( N \), if

\[
\sum_{i=1}^N p_i (w_i w_i^*)^\otimes t = \binom{d + t - 1}{t} P_{\text{Sym}^t},
\]

where \( P_{\text{Sym}^t} \) denotes the projector onto the totally symmetric subspace \( \text{Sym}^t \) of \((\mathbb{C}^d)^\otimes t\).

Our second sketching model is given by the concatenation of two independent random vectors \( a \) and \( b \) in the following construction: Given a weighted 2-design \( \{(w_i, p_i)\}_i \) in \( \mathbb{C}^d \), let \( a \) be a random vector given by

\[
P\left\{ a = \sqrt{d_1} w_i \right\} = p_i, \quad i = 1, \ldots, N_1.
\]

Then \( a \) satisfies

\[
\mathbb{E}a a^* = I_{d_1}
\]

and

\[
\mathbb{E}a a^* \otimes a a^* = \frac{2d_1}{d_1+1} \cdot P_{\text{Sym}^2}.
\]

Similarly, given a weighted 2-design \( \{(w'_i, p'_i)\}_i \) in \( \mathbb{C}^d \), let \( b \) be a random vector given by

\[
P\left\{ b = \sqrt{d_2} w'_i \right\} = p'_i, \quad i = 1, \ldots, N_2.
\]

Then \( b \) satisfies

\[
\mathbb{E}b b^* = I_{d_2}
\]

and

\[
\mathbb{E}b b^* \otimes b b^* = \frac{2d_2}{d_2+1} \cdot P_{\text{Sym}^2}.
\]
1.3 Spectral initialization

Our main results rely on the availability of $\tilde{X}_0$ and $\tilde{Y}_0^*$ such that $\tilde{X}_0\tilde{Y}_0^*$ is close to the ground truth matrix $M_0$. To provide a stand-alone theory that does not require any oracle information, we also analyze a specific method to obtain such matrices $\tilde{X}_0$ and $\tilde{Y}_0^*$ described below.

Let $A: \mathbb{C}^{d_1 \times d_2} \to \mathbb{C}^n$ denote the linear operator that obtain the rank-one measurements in (1), i.e., it is defined by

$$A(M) = \left(\frac{1}{\sqrt{n}} a_i^* M b_i\right)_{i=1}^n.$$  

Then its adjoint operator, denoted by $A^*$, is given by

$$A^*(y) = \frac{1}{n} \sum_{i=1}^n y_i a_i b_i^*.$$  

The spectral method computes an estimate $\tilde{M}_0$ of the unknown matrix $M_0$ as the best rank-$r$ approximation of $A^*A(M_0)$ with respect to the Frobenius norm. Under the two random sketching models, we will provide upper bounds on the estimation error by matrix concentration inequalities.

Next we factorize the estimated matrix into $\tilde{M}_0 = \tilde{X}_0 \tilde{Y}_0^*$ through the singular value decomposition. Let $\tilde{M}_0 = \tilde{U}_0 \Sigma_0 \tilde{V}_0^*$ denotes the compact singular value decomposition. Then we choose $\tilde{X}_0$ and $\tilde{Y}_0$ by

$$\tilde{X}_0 = \tilde{U}_0 \Sigma_0^{1/2} \quad \text{and} \quad \tilde{Y}_0 = \tilde{V}_0 \Sigma_0^{1/2},$$

so that they have the same singular values. This particular decomposition provides a set of useful properties utilized in the proof of our main results.

1.4 Discussion and Related work

Under the real Gaussian sketching model and given an initial estimate satisfying (9), we demonstrate that, with high probability, the estimator in (2) recovers $M_0$ exactly, provided the number of measurements scales as $n \geq Cdr$, where $d = \max(d_1, d_2)$. This sample complexity coincides with the sample complexity achieved by the estimators based on lifting and semidefinite relaxation [4, 3]. On the other hand, our estimator is formulated through an explicit factorization only with $r(d_1 + d_2)$ variables while the lifted convex estimator over $d_1 d_2$ variables [4, 3]. Furthermore, because the methods based on semidefinite relaxation do not operate in the factorized domain, they often need singular value calculations which further complicates their scalability.

However, computationally inexpensive methods used to find the initial estimates obeying (9), often lead to a suboptimal overall sample complexity. In fact, we show that, with high probability, the spectral initialization succeeds if $n \geq Cdr^2$ which dominates the sample complexity $n \geq Cdr$ for the “oracle-assisted” estimator mentioned above.

Several iterative algorithms have been proposed and analyzed under the real Gaussian sketching model. Earlier methods used resampling to draw fresh measurements per iteration. Therefore, these methods need to terminate after finitely many iterations, which only allows for approximate recovery up to a prescribed accuracy $\epsilon$. Prior work on this approach achieve the sample complexities $O(dr^3 \log^2 d \log(1/\epsilon))$ [24], $O(dr^3 \log(1/\epsilon))$ [15], and $O(dr^2 \log^3 d \log(1/\epsilon))$ [21]. In more recent work, [19] and [14] studied performance of the nonconvex gradient descent and established the sample complexities $O(dr^6 \log^2 d)$ and $O(dr^4 \log d)$, respectively. Our estimator outperforms these results for nonconvex approaches. In fact, our estimator would have achieved the ideal sample complexity should there be an initialization with the sample complexity $O(dr)$. 
2 Main results

Our main results demonstrate how many observations suffice for the estimator (2) to reconstruct the unknown matrix $X_0Y_0^*$. Our first theorem, provides a sample complexity that guarantees accuracy of the estimator (2) under the real Gaussian sketching model.

**Theorem 1** (Real Gaussian desketching). Let $([\mathbf{a}_i; \mathbf{b}_i])_{i=1}^n$ be independent copies of $[\mathbf{a}; \mathbf{b}] \sim \mathcal{N}(0, I_{d_1+d_2})$. Let $X_0 \in \mathbb{C}^{d_1 \times r}$ and $Y_0 \in \mathbb{C}^{d_2 \times r}$ be matrices that satisfy $X_0^*X_0 = Y_0^*Y_0$ and

$$\|X_0Y_0^* - M_0\| \lesssim r^{-1/2}\kappa^{-2}\|M_0\|,$$

(9)

where $\kappa$ denotes the condition number of $M_0$. If the number of measurements $n$ obeys

$$n \gtrsim \max\{\kappa r(d_1 + d_2), \log(1/\delta)\},$$

(10)

then with probability at least $1 - \delta$ the estimates $\hat{X}$ and $\hat{Y}$ obtained by the anchored regression satisfy $\hat{X}\hat{Y}^* = M_0$.

The result by Theorem 1 is comparable to the analogous result for the lifted convex optimization by nuclear norm minimization [4, 3]. However, the dependence on the condition number, which does not appear in the lifted case, is the cost we need to pay to save the computation through explicit factorization.

We also present the sample complexity for the success of the spectral initialization under the same model.

**Proposition 1.** Let $([\mathbf{a}_i; \mathbf{b}_i])_{i=1}^n$ be independent copies of $[\mathbf{a}; \mathbf{b}] \sim \mathcal{N}(0, I_{d_1+d_2})$. Then the estimate $(\tilde{X}_0, \tilde{Y}_0)$ by the spectral initialization satisfies (9) with probability at least $1 - n^{-1}$, if

$$\frac{n}{\log^5 n} \gtrsim r^2\kappa^4(d_1 + d_2).$$

As shown in Theorem 1 and Proposition 1, the number of samples enough for the success of the spectral initialization dominates that for the estimator. Although the spectral initialization is just one approach to obtain an initial estimate satisfying (9), it has not been shown any alternative practical method providing the same accuracy from fewer measurements.

Next we present the corresponding results for partially derandomized sketching below.

**Theorem 2** (Partially derandomized desketching). Let $\mathbf{a}$ and $\mathbf{b}$ are independent random vectors uniformly distributed over the corresponding 2-design sets according to (5) and (7). Let $([\mathbf{a}_i; \mathbf{b}_i])_{i=1}^n$ be independent copies of $[\mathbf{a}; \mathbf{b}]$. Let $\tilde{M}_0$, $\tilde{X}_0$, and $\tilde{Y}_0$ as in Theorem 1 satisfying (9). If the number of measurements $n$ satisfies

$$n \gtrsim \kappa r(d_1 + d_2) \max\{\log(d_1 + d_2), \log(1/\delta)\},$$

(11)

then the anchored regression exactly recovers $M_0$ with probability at least $1 - \delta$.

**Proposition 2.** Under the sketching model in Theorem 2, the spectral initialization provides $(\tilde{X}_0, \tilde{Y}_0)$ satisfying (9) with probability at least $1 - \delta$ provided

$$n \gtrsim r^2(d_1 + d_2) \max\{\log(d_1 + d_2), \log(1/\delta)\}.$$

Compared to the real Gaussian sketching model, the derandomized case is guaranteed by slightly more measurements (larger by a logarithmic factor).
3 Proof of the Main Theorems

We prove Theorems 1 and 2 in two steps. First, we derive a set of deterministic conditions to guarantee exact recovery for the proposed estimator. Then, we show that these conditions hold, with high probability, for the sketching models introduced in Section 1.2.

3.1 A deterministic sufficient condition for exact recovery

Recall that $M_0 = U_0 \Sigma_0 V_0^*$ is the compact singular decomposition of the ground truth matrix $M_0$, where both $U_0 \in \mathbb{C}^{d_1 \times r}$ and $V_0 \in \mathbb{C}^{d_2 \times r}$ have orthonormal columns, and $\Sigma_0 \in \mathbb{R}^{r \times r}$ is the diagonal matrix of the singular values. The support space of $M_0$, denoted by $T$, is defined as

$$T \overset{\text{def}}{=} \left\{ \Delta_1 V_0^* + U_0 \Delta_2^* : \Delta_i \in \mathbb{C}^{d_i \times r}, i = 1, 2 \right\}.$$ 

With these notations, the following proposition provides a set of deterministic sufficient conditions for the estimator $(2)$ to exactly recover $M_0$. The proof is deferred to Section 4.

**Proposition 3.** Suppose that there exists $\rho \in (0, 1]$ such that

$$\frac{1}{n} \sum_{i=1}^{n} |\langle a_i b_i^*, H \rangle| \geq \rho \|H\|_F, \quad \text{for all } H \in T,$$

(12)

and $(\tilde{X}_0, \tilde{Y}_0)$ satisfies $\tilde{X}_0^* \tilde{X}_0 = \tilde{Y}_0^* \tilde{Y}_0$. Let $r$ and $\kappa$ denote the rank and the condition number of $M_0$ respectively. Then there exist numerical constants $C_1, C_2 \geq 1$ such that if

$$\left\| I_{d_1} - \frac{1}{n} \sum_{k=1}^{n} a_k a_k^* \right\| \leq \frac{\rho}{C_1 \sqrt{r \kappa}},$$

(13)

$$\left\| I_{d_2} - \frac{1}{n} \sum_{k=1}^{n} b_k b_k^* \right\| \leq \frac{\rho}{C_1 \sqrt{r \kappa}},$$

(14)

and

$$\left\| \tilde{X}_0 \tilde{Y}_0^* - M_0 \right\| \leq \frac{\rho \|M_0\|}{C_2 \sqrt{r \kappa^2}},$$

(15)

then the maximizer $(\hat{X}, \hat{Y})$ to $(2)$ is unique and satisfies $\hat{X} \hat{Y}^* = M_0$.

3.2 Verifying the sufficient condition under the random models

We demonstrate that under the random measurement models introduced in Section 2 the assumptions made in Proposition 3 hold with high probability.

3.2.1 Small-ball method

Using the small-ball method [10, 17], we first show in the following proposition that (12) is satisfied with high probability.

**Proposition 4** (Lower-tail via the small-ball method). Let $T$ be a subset of $\mathbb{C}^{d_1 \times d_2}$ that is invariant under multiplication by unit-modulus scalars. For i.i.d. and isotropic random vectors...
\[ [a; b], [a_1; b_1], \ldots, [a_n; b_n] \in \mathbb{C}^{d_1 \times d_2} \text{ define} \]
\[ p_\tau(T) \overset{\text{def}}{=} \inf_{H \in T \setminus \{0\}} \mathbb{P} \{ |a^* H b| \geq \tau \| H \|_F \}, \]
\[ \mathcal{C}_n(T) \overset{\text{def}}{=} \mathbb{E} \sup_{H \in T \setminus \{0\}} \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \varepsilon_i (a_i b_i^*, H), \]
where \( \varepsilon_1, \ldots, \varepsilon_n \) are i.i.d. Rademacher random variables independent of everything else. Then, for any \( \tau > 0 \) and \( \delta \in (0, 1) \), with probability at least \( 1 - \delta \), we have
\[ \inf_{H \in T \setminus \{0\}} \frac{1}{n} \sum_{i=1}^{n} |a_i^* H b_i| \geq \frac{1}{n} \sum_{i=1}^{n} \|a_i^* H b_i\| \leq \tau \| H \|_F, \]
(16)

Proof. Using \( [z]_{\leq t} \overset{\text{def}}{=} \min\{z, t\} \) to denote the “saturation” at \( t \), for any \( \tau > 0 \), we have
\[ \frac{1}{n} \sum_{i=1}^{n} |a_i^* H b_i| \geq n \frac{1}{n} \sum_{i=1}^{n} |a_i^* H b_i| \leq \tau \| H \|_F, \]
(17)

for every \( H \in \mathbb{R}^{d_1 \times d_2} \). By normalizing by \( \| H \|_F \), it suffices to find a lower bound for the right-hand side of (17) for all \( H \in T \cap \mathbb{S} \), where \( \mathbb{S} \) denotes the unit sphere of the Frobenius norm in \( \mathbb{C}^{d_1 \times d_2} \).

Adding and subtracting \( \mathbb{E} \left( |a_i^* H b_i| \leq \tau \right) \), and using the fact that \( \mathbb{E} \left( |a_i^* H b_i| \leq \tau \right) \geq \tau \mathbb{P} (|a_i^* H b_i| \geq \tau) \), we can write
\[ \frac{1}{n} \sum_{i=1}^{n} \mathbb{E} \left( |a_i^* H b_i| \leq \tau \right) - \frac{1}{n} \sum_{i=1}^{n} \mathbb{P} (|a_i^* H b_i| \geq \tau) - \frac{1}{n} \sum_{i=1}^{n} \mathbb{E} \left( |a_i^* H b_i| \leq \tau \right). \]
(18)

The function \( F : (\mathbb{C})^n \rightarrow \mathbb{R}_{\geq 0} \) defined as
\[ F([a_1; b_1], \ldots, [a_n; b_n]) \overset{\text{def}}{=} \sup_{H \in T \cap \mathbb{S}} \frac{1}{n} \sum_{i=1}^{n} \mathbb{E} \left( |a_i^* H b_i| \leq \tau \right) - \frac{1}{n} \sum_{i=1}^{n} \mathbb{E} \left( |a_i^* H b_i| \leq \tau \right) \]
has the bounded difference property. Therefore, invoking the bounded difference inequality [16], with probability at least \( 1 - \delta \), we have
\[ F([a_1; b_1], \ldots, [a_n; b_n]) \leq E F([a_1; b_1], \ldots, [a_n; b_n]) + \tau \sqrt{\frac{\log(1/\delta)}{2n}}. \]
(19)

Using the standard Giné-Zinn symmetrization argument (e.g., see [23, Lemma 2.3.1]), the expectation on the right-hand side of (19) can be bounded as
\[ E F([a_1; b_1], \ldots, [a_n; b_n]) \leq \frac{2}{\sqrt{n}} E \left( \sup_{H \in T \cap \mathbb{S}} \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \varepsilon_i |a_i^* H b_i| \leq \tau \right), \]
where the expectation on the right-hand side is with respect to \([a_1; b_1], \ldots, [a_n; b_n]\) as well as the i.i.d. Rademacher random variables \( \varepsilon_1, \ldots, \varepsilon_n \). Since the function \( z \mapsto |z| \leq \tau \) is 1-Lipschitz, invoking the Rademacher contraction principle [13, Theorem 4.12] yields
\[ E \left( \sup_{H \in T \cap \mathbb{S}} \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \varepsilon_i |a_i^* H b_i| \leq \tau \right) \leq E \left( \sup_{H \in T \cap \mathbb{S}} \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \varepsilon_i |a_i^* H b_i| \right). \]
(20)
Let $\phi$ be a unit-modulus scalar in $\mathbb{C}$ that is selected uniformly at random, and $\mathbb{E}_\phi$ denote the expectation with respect to $\phi$ conditioned on everything else. Straightforward calculus shows that for any $z \in \mathbb{C}$ we have $|z| = (\pi/2)\mathbb{E}_\phi(|\text{Re}(\phi^*z)|)$. Applying this identity in (20) then yields

$$
\mathbb{E} \left( \sup_{H \in \mathcal{T} \cap \mathbb{S}} \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \varepsilon_i |a_i^* H b_i| \right) \leq \frac{\pi}{2} \mathbb{E} \left( \sup_{H \in \mathcal{T} \cap \mathbb{S}} \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \varepsilon_i |\langle a_i^*, \phi H \rangle| \right)
$$

$$
\leq \frac{\pi}{2} \mathbb{E} \mathbb{E}_\phi \left( \sup_{H \in \mathcal{T} \cap \mathbb{S}} \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \varepsilon_i |\langle a_i^*, \phi H \rangle| \right)
$$

$$
\leq \frac{\pi}{2} \mathbb{E} \left( \sup_{H \in \mathcal{T} \cap \mathbb{S}} \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \varepsilon_i |\langle a_i^*, H \rangle| \right)
$$

$$
\leq \frac{\pi}{2} \mathcal{C}_n(\mathcal{T}),
$$

where the second, third, and fourth lines follow respectively from the Jensen’s inequality, the assumption that $\mathcal{T}$ is invariant under multiplication by unit-modulus scalars, and applying the Rademacher contraction principle once more.

Furthermore, since $[a_1; b_1], \ldots, [a_n; b_n] \in \mathbb{C}^{d_1+d_2}$ are identically distributed, we have

$$
\frac{1}{n} \sum_{i=1}^{n} \mathbb{P}(|a_i^* H b_i| \geq \tau) = \mathbb{P}(|a^* H b| \geq \tau).
$$

(21)

Therefore, in view of (18), (19), and (21), with probability at least $1 - \delta$, for all $H \in \mathcal{T} \cap \mathbb{S}$ we have

$$
\frac{1}{n} \sum_{i=1}^{n} |a_i^* H b_i| \leq \tau \mathbb{P}(|a^* H b| \geq \tau) - \frac{\pi \mathcal{C}_n(\mathcal{T})}{\sqrt{n}} - \tau \sqrt{\frac{\log(1/\delta)}{2n}}.
$$

Recalling the definition of $p_r(\mathcal{T})$ is enough to complete the proof. □

We apply Proposition 4 under the assumptions in either of Theorems 1 and 2. Then (12) is satisfied with high probability provided that the right-hand side of (16) is lower bounded by a numerical constant $\rho$. The following lemmas provides a lower (resp. upper) bound on $p_r(\mathcal{T})$ (resp. $\mathcal{C}_n(\mathcal{T})$).

**Lemma 1** (Lower bound on probability). Let $[a; b]$ to be a random vector drawn either according to (3), or the pair (5) and (7). Then

$$
p_r(\mathcal{T}) \geq c(1 - \tau^2)^2
$$

for a numerical constant $c > 0$.

**Lemma 2** (Upper bound on Rademacher complexity). Let $[a; b]$ satisfy that i) $a$ and $b$ are independent; ii) each of $a$ and $b$ is isotropic. Then

$$
\mathcal{C}_n(\mathcal{T}) \leq \sqrt{(d_1 + d_2)r},
$$

By plugging in the results of the above lemmas, a sufficient condition for satisfying (12) with probability $1 - \delta$ is given by

$$
c(1 - \tau^2)^2 - \frac{4\sqrt{(d_1 + d_2)r}}{\sqrt{n}} - \tau \sqrt{\frac{\log(1/\delta)}{2n}} \geq \rho.
$$

(22)

Given $\rho$, by choosing $C$ in the assumption $n \geq C \max\{r(d_1 + d_2), \log(1/\delta)\}$ large enough and by choosing $\tau$ small we obtain that (22) holds with probability $1 - \delta$. 

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3.2.2 Near Isotropy

Next we show that (13) and (14) are satisfied with high probability for both the Gaussian and 2-design cases. To simplify the notation, let \( \eta \) denote the right-hand side of (13), which coincides with that of (14), i.e., \( \eta = \rho / C_1 \sqrt{n} \). We are interested in the regime where \( 0 < \eta < 1 \).

In the Gaussian case, the concentration of extreme singular values of a Wishart matrix has been well studied in the literature (e.g., see [5, Theorem II.13], which is summarized as Theorem 3 in Appendix). If \([a; b]\) is a standard Gaussian random vector, then we have

\[
\mathbb{P} \left\{ \frac{1}{n} \sum_{i=1}^{n} a_i a_i^* - I_{d_1} \right\} > 3 \max \left( \frac{4d_1}{n}, \frac{4d_1}{n} \right) \leq 2 \exp(-d_1/2)
\]

and

\[
\mathbb{P} \left\{ \frac{1}{n} \sum_{i=1}^{n} b_i b_i^* - I_{d_2} \right\} > 3 \max \left( \frac{4d_2}{n}, \frac{4d_2}{n} \right) \leq 2 \exp(-d_2/2).
\]

Therefore, (13) and (14) are satisfied with probability \( 1 - \delta \) if

\[
n \geq \max \{36 \eta^2 (d_1 + d_2), 2 \log(4/\delta)\} = \max \{C_1^2 \rho^{-2} \kappa r (d_1 + d_2), 2 \log(4/\delta)\},
\]

which is implied by (10) in Theorem 1.

In the 2-design case, we obtain a similar result via the matrix Bernstein inequality [22, Theorem 1.6], summarized as Theorem 4 in Appendix. If \([a; b]\) satisfy (5) and (7), then we have

\[
\mathbb{P} \left\{ \frac{1}{n} \sum_{i=1}^{n} a_i a_i^* - I_{d_1} \right\} > \eta \leq 2d_1 \exp \left( -\frac{\eta^2 n}{4d_1} \right)
\]

and

\[
\mathbb{P} \left\{ \frac{1}{n} \sum_{i=1}^{n} b_i b_i^* - I_{d_2} \right\} > \eta \leq 2d_2 \exp \left( -\frac{\eta^2 n}{4d_2} \right).
\]

Therefore, (13) and (14) are satisfied with probability \( 1 - \delta \) if

\[
n \geq \frac{4(d_1 + d_2)}{\eta^2} \cdot \log \left( \frac{d_1 + d_2}{\delta} \right) = \frac{4C_1^2 \rho^{-2} \kappa r (d_1 + d_2)}{\rho^2} \cdot \log \left( \frac{d_1 + d_2}{\delta} \right),
\]

which is implied by (11) in Theorem 2.

4 Proof of Proposition 3

For conciseness, we introduce the following shorthand notations. Let

\[
A = \begin{bmatrix} a_1 & a_2 & \ldots & a_n \end{bmatrix}, \quad B = \begin{bmatrix} b_1 & b_2 & \ldots & b_n \end{bmatrix},
\]

and define

\[
\eta \overset{\text{def}}{=} \frac{\rho}{C_1 \sqrt{\kappa r}}.
\]

Then, (13) and (14) are equivalent to

\[
\left\| I_{d_1} - \frac{1}{n} AA^* \right\| \leq \eta \quad \text{and} \quad \left\| I_{d_2} - \frac{1}{n} BB^* \right\| \leq \eta.
\]

First, through the following lemma we establish a sufficient optimality condition needed to prove Proposition 3.
Lemma 3. Let $X_0 \in \mathbb{C}^{n_1 \times r}$ and $Y_0 \in \mathbb{C}^{n_2 \times r}$ satisfy $X_0 Y_0^* = M_0$. Then $[X_0; Y_0]$ is the unique maximizer of (2), if for any $\Delta_1 \in \mathbb{C}^{d_1 \times r}$ and $\Delta_2 \in \mathbb{C}^{d_2 \times r}$ we have
\[
\langle \tilde{X}_0 - \frac{1}{n}AA^* X_0, \Delta_1 \rangle + \langle \tilde{Y}_0 - \frac{1}{n}BB^* Y_0, \Delta_2 \rangle \leq \frac{1}{n} \sum_{i=1}^{n} |\langle a_i b_i^*, X_0 \Delta_2^* + \Delta_1 Y_0^* \rangle|\]
(25)
with equality occurring only when both $\Delta_1$ and $\Delta_2$ are zero.

For any $X_0 \in \mathbb{C}^{d_1 \times r}$ and $Y_0 \in \mathbb{C}^{d_2 \times r}$ that satisfy $X_0 Y_0^* = M_0$, we have $X_0 \Delta_2^* + \Delta_1 Y_0^* \in T$. Therefore, by Lemma 3 and (12), it suffices to show that
\[
\langle \tilde{X}_0 - \frac{1}{n}AA^* X_0, \Delta_1 \rangle + \langle \tilde{Y}_0 - \frac{1}{n}BB^* Y_0, \Delta_2 \rangle \leq \rho \|X_0 \Delta_2^* + \Delta_1 Y_0^*\|_F
\]
(26)
for all $\Delta_1 \in \mathbb{C}^{d_1 \times r}$ and $\Delta_2 \in \mathbb{C}^{d_2 \times r}$ with the equality only when $[\Delta_1; \Delta_2] = 0$.

Define the linear operator $L : \mathbb{C}^{(d_1 + d_2) \times r} \to \mathbb{C}^{d_1 \times d_2}$ by
\[
L \left( \left[ \Delta_1; \Delta_2 \right] \right) = X_0 \Delta_2^* + \Delta_1 Y_0^*, \quad \text{for all } \Delta_1 \in \mathbb{C}^{d_1 \times r}, \Delta_2 \in \mathbb{C}^{d_2 \times r},
\]
whose adjoint operator is
\[
L^*(Z) = \left[ Z Y_0; Z^T \tilde{X}_0 \right], \quad \text{for all } Z \in \mathbb{C}^{d_1 \times d_2}.
\]
With
\[
\Delta = \left[ \Delta_1; \Delta_2 \right],
\]
and
\[
E = \left[ \tilde{X}_0 - \frac{1}{n}AA^* X_0; \tilde{Y}_0 - \frac{1}{n}BB^* Y_0 \right],
\]
(27)
we can rewrite (26) as
\[
\langle E, \Delta \rangle \leq \rho \|L(\Delta)\|_F.
\]

Note that $L$ generally has a nontrivial nullspace, particularly, if $(d_1 + d_2)r < d_1 d_2$. Therefore, in view of the inequality above, it is necessary to have $\langle E, \Delta \rangle = 0$ for all $\Delta$ in the nullspace of $L$. Fortunately, for a certain choice of $(X_0, Y_0)$ the corresponding matrix $E$ satisfies the required condition, as shown by the following lemma.

Lemma 4. Let $(X_0, Y_0)$ be the solution to
\[
\text{maximize } \langle \tilde{X}_0, X \rangle + \langle \tilde{Y}_0, Y \rangle - \frac{1}{2n} \|X^* A\|_F^2 - \frac{1}{2n} \|Y^* B\|_F^2
\]
subject to $X Y^* = M_0$.
(28)
For the operator $L$ and the matrix $E$, defined by (27), defined with respect to this particular solution, we have
\[
E \in \mathcal{V} \overset{\text{def}}{=} \text{range}(L^*)
\]
Furthermore, if (24) holds, then
\[
\left\| \begin{bmatrix} X_0 \\ Y_0 \end{bmatrix} - \begin{bmatrix} \tilde{X}_0 \\ \tilde{Y}_0 \end{bmatrix} \right\|_F^2 \leq \frac{1 + \eta}{1 - \eta} \cdot \left\| \begin{bmatrix} X \\ Y \end{bmatrix} - \begin{bmatrix} \tilde{X}_0 \\ \tilde{Y}_0 \end{bmatrix} \right\|_F^2 + \frac{2 \eta}{1 - \eta} \cdot \left\| \begin{bmatrix} X_0 \\ Y_0 \end{bmatrix} - \begin{bmatrix} X \\ Y \end{bmatrix} \right\|_F \left\| \begin{bmatrix} \tilde{X}_0 \\ \tilde{Y}_0 \end{bmatrix} \right\|_F
\]
(29)
for all $X \in \mathbb{C}^{d_1 \times r}$ and $Y \in \mathbb{C}^{d_2 \times r}$ satisfying $X Y^* = M_0$. 

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Hereafter, the pair \((X_0, Y_0)\) is chosen as in Lemma 4. The subspace \(\mathcal{V}\) can be described explicitly as
\[
\mathcal{V} = \text{range}(\mathcal{L}^*) = \left\{ [ZY_0; Z^* X_0] : Z \in \mathbb{C}^{d_1 \times d_2} \right\}.
\] (30)
By the fundamental theorem of linear algebra, we also have \(\mathcal{V} = \text{null}(\mathcal{L}^{-1})\). Thus, with \(P_{\mathcal{V}}\) denoting the orthogonal projection onto the subspace \(\mathcal{V}\), Lemma 4 implies that \(E = P_{\mathcal{V}} E\). Consequently, to guarantee (26), it suffices to have
\[
\|E\|_F \|P_{\mathcal{V}} \Delta\|_F \leq \rho \|\mathcal{L}(P_{\mathcal{V}} \Delta)\|_F,
\] (31)
because by the Cauchy-Schwartz inequality
\[
\langle E, \Delta \rangle = \langle P_{\mathcal{V}} E, \Delta \rangle = \langle E, P_{\mathcal{V}} \Delta \rangle \leq \|E\|_F \|P_{\mathcal{V}} \Delta\|_F.
\]
Furthermore, the following technical lemma provides a lower bound for \(\|\mathcal{L}(P_{\mathcal{V}} \Delta)\|_F/\|P_{\mathcal{V}} \Delta\|_F\).

**Lemma 5.** The linear operator \(\mathcal{L}\) satisfies
\[
\|\mathcal{L}(P_{\mathcal{V}} \Delta)\|_F \geq \min\{\sigma_{\min}(X_0), \sigma_{\min}(Y_0)\} \|P_{\mathcal{V}} \Delta\|_F, \quad \forall \Delta \in \mathbb{C}^{(d_1 + d_2) \times r}.
\] (32)

**Proof.** Let \([\Delta_1; \Delta_2]\) belong to \(\mathcal{V} = \text{range}(\mathcal{L}^*)\). Then there exists \(Z \in \mathbb{C}^{d_1 \times d_2}\) such that \(\Delta_1 = ZY_0\) and \(\Delta_2 = Z^* X_0\). Thus
\[
\|\mathcal{L}([\Delta_1; \Delta_2])\|_F^2 = \|X_0 \Delta_2 + \Delta_1 Y_0^*\|_F^2
\]
\[
= \|X_0 X_0^* Z + ZY_0 Y_0^*\|_F^2
\]
\[
= \|X_0 X_0^* Z\|_F^2 + \|Z Y_0 Y_0^*\|_F^2 + 2 \langle X_0 X_0^* Z, Z Y_0 Y_0^* \rangle
\]
\[
= \|X_0 X_0^* Z\|_F^2 + \|Z Y_0 Y_0^*\|_F^2 + 2 \|X_0^* Z Y_0\|_F^2
\]
\[
\geq \sigma_{\min}^2(X_0) \|X_0^* Z\|_F^2 + \sigma_{\min}^2(Y_0) \|Z Y_0\|_F^2
\]
\[
\geq \min\{\sigma_{\min}^2(X_0), \sigma_{\min}^2(Y_0)\} \|\Delta_1; \Delta_2\|_F^2.
\]
\[ \square \]

Note that
\[
\max_{X, Y : XY^* = M_0} \min\{\sigma_{\min}(X), \sigma_{\min}(Y)\} = \sqrt{\sigma_r(M_0)}.
\]
Indeed, the assumptions of the proposition implies that \(\min\{\sigma_{\min}(X_0), \sigma_{\min}(Y_0)\}\) is larger than \(\sqrt{\sigma_r(M_0)}\) divided by a numerical constant. In order to show this, we introduce another pair \((X_1, Y_1)\) with \(X_1 Y_1^* = M_0\) so that \([X_1; Y_1]\) approximates \([X_0; Y_0]\). The following lemma provides an upper bound on the approximation error.

**Lemma 6.** Suppose that the rank-\(r\) matrices \(M_0\) and \(\tilde{M}_0\), whose compact SVDs are respectively \(U_0 \Sigma_0 V_0^*\) and \(\tilde{U}_0 \tilde{\Sigma}_0 \tilde{V}_0^*\), satisfy \(\|\tilde{M}_0 - M_0\| < \sigma_r(M_0)\). Then
\[
\min_{Q : Q^{-1} = Q^*} \left\| \begin{bmatrix} U_0 & \tilde{U}_0 \end{bmatrix} \Sigma_0^{1/2} Q - \begin{bmatrix} \Sigma_0^{1/2} \tilde{V}_0 \end{bmatrix} \right\|_F \leq \frac{8 \sqrt{2} \sigma_1(\Sigma_0)}{\sigma_r(\Sigma_0)} \cdot \frac{\|\tilde{M}_0 - M_0\|_F}{\sigma_r(\Sigma_0) - \|\tilde{M}_0 - M_0\|_F}.
\]

Let \(\tilde{U}_0, \tilde{\Sigma}_0\) and \(\tilde{V}_0\) be as in Lemma 6. Let \(Q\) be the minimizer in Lemma 6. Let
\[
X_1 = U_0 \Sigma_0^{1/2} Q \quad \text{and} \quad Y_1 = V_0 \Sigma_0^{1/2} Q.
\]
Then (15) implies

\[
\frac{\|[X_1; Y_1] - [\tilde{X}_0; \tilde{Y}_0]\|_F}{\sigma_r(M_0)} \leq \frac{8\sqrt{2}\rho}{C_2 - \rho/\sqrt{r_1}} \leq \frac{8\sqrt{2}\rho}{C_2 - 1}.
\] (33)

Choosing \(C_2 \geq 8\sqrt{2} + 1\) yields

\[
\|[X_1; Y_1] - [\tilde{X}_0; \tilde{Y}_0]\|_F \leq \rho\sqrt{\sigma_r(M_0)}.
\] (34)

It follows from (34) via the triangle inequality that

\[
\||X_0; Y_0\|_F \leq \||X_1; Y_1\|_F + \||\tilde{X}_0; \tilde{Y}_0\|_F \leq \rho\sqrt{\sigma_r(M_0)} + \sqrt{r\sigma_1(M_0)} \leq 2\sqrt{r\sigma_1(M_0)}.
\] (35)

Plugging in (35) to (29) with \(X = X_1\) and \(Y = Y_1\) gives

\[
\begin{align*}
\||X_0; Y_0\|_F &\leq \left(\frac{1 + \eta}{1 - \eta}\right) \left(\||X_1; Y_1\|_F + \|\tilde{X}_0; \tilde{Y}_0\|_F\right)
&\quad + \frac{4\eta\sqrt{r\sigma_1(M_0)}}{1 - \eta}
&\quad \left(\||X_0; Y_0\|_F + \||\tilde{X}_0; \tilde{Y}_0\|_F + \||X_1; Y_1\|_F + \|\tilde{X}_0; \tilde{Y}_0\|_F\right).
\end{align*}
\]

After some simplification, the above inequality and (33) imply

\[
\begin{align*}
\frac{\|[X_0; Y_0] - [\tilde{X}_0; \tilde{Y}_0]\|_F}{\rho\sqrt{\sigma_r(M_0)}} &\leq \frac{1}{C_1 - 1} \left(2 + \frac{128(C_1 + 1)}{(C_1 - 1)(C_2 - 1)^2} + \frac{32\sqrt{2}}{(C_1 - 1)(C_2 - 1)} + \frac{4}{(C_1 - 1)^2}\right).
\end{align*}
\]

By choosing both \(C_1\) and \(C_2\) large enough, we obtain

\[
\left\|\begin{bmatrix} X_1 \\ Y_1 \end{bmatrix} - \begin{bmatrix} \tilde{X}_0 \\ \tilde{Y}_0 \end{bmatrix}\right\|_F \leq \frac{\rho\sqrt{\sigma_r(M_0)}}{10}
\]

and

\[
\left\|\begin{bmatrix} X_0 \\ Y_0 \end{bmatrix} - \begin{bmatrix} \tilde{X}_0 \\ \tilde{Y}_0 \end{bmatrix}\right\|_F \leq \frac{\rho\sqrt{\sigma_r(M_0)}}{10}.
\]

Then by the triangle inequality we have

\[
\left\|\begin{bmatrix} X_0 \\ Y_0 \end{bmatrix} - \begin{bmatrix} X_1 \\ Y_1 \end{bmatrix}\right\|_F \leq \frac{\rho\sqrt{\sigma_r(M_0)}}{5}.
\]

By the singular value perturbation theory by Weyl [2] and \(\rho \in (0, 1)\), it follows that

\[
\sigma_r(X_0) \geq \sigma_r(X_1) - \|X_0 - X_1\| \geq \frac{4\sqrt{\sigma_r(M_0)}}{5}
\]

and

\[
\sigma_r(Y_0) \geq \sigma_r(Y_1) - \|Y_0 - Y_1\| \geq \frac{4\sqrt{\sigma_r(M_0)}}{5}.
\]

Then (32) is implied by

\[
\|\mathcal{L}(P_V \Delta)\|_2 \geq \frac{\sqrt{\sigma_r(M_0)}}{2} \cdot \|P_V \Delta\|_F.
\]
Therefore, by combining the above estimates, we obtain a sufficient condition for (31) given by
\[
\|E\|_F \leq \frac{4\rho\sqrt{\sigma_r(M_0)}}{5}.
\] (36)

Next the following lemma provides an upper bound on the left-hand side of (36).

**Lemma 7.** Let \((X_0, Y_0)\) and \(E\) be as in Lemma 4. Suppose that (24) holds. Then for all \((X, Y)\) satisfying \(XY^* = M_0\), we have
\[
\|E\|_F \leq \frac{3 + \eta}{1 - \eta} \left( \|\widetilde{X}_0 - X; \widetilde{Y}_0 - Y\|_F + \eta\|X; Y\|_F \right).
\]

By applying Lemma 7 with \(X = X_1\) and \(Y = Y_1\), and by (23), we obtain
\[
\|E\|_F \leq \frac{3C_1 + 1}{C_1 - 1} \left( \frac{1}{10} + \frac{1}{C_1} \right) \rho\sqrt{\sigma_r(M_0)}.
\]

Finally we choose \(C_1\) large enough so that
\[
\frac{3C_1 + 1}{C_1 - 1} \left( \frac{1}{10} + \frac{1}{C_1} \right) \leq \frac{4}{5}.
\]
This completes the proof.

5 **Analysis of Spectral Initialization**

Note that
\[
A^*A(M_0) = \frac{1}{n} \sum_{i=1}^{n} a_i a_i^* M_0 b_i b_i^*.
\]

Then by the independence between \(a\) and \(b\) together with the isotropy of each of them implies
\[
\mathbb{E} A^* A(M_0) = M_0.
\]

Since the spectral initialization computes \(\tilde{M}_0\) as the best rank-\(r\) approximation of \(A^* A(M_0)\), by the optimality, we have
\[
\|\tilde{M}_0 - M_0\| \leq 2\|(A^* A - \text{Id})M_0\|.
\]

Our goal is to show that
\[
\|(A^* A - \text{Id})M_0\| \leq \frac{C\|M_0\|}{\sqrt{rK^2}}
\]
for a numerical constant \(C\). Then it will imply (9).

Let
\[
Y_i = a_i a_i^* M_0 b_i b_i^* - M_0, \quad i = 1, \ldots, n.
\]

Then we will show that
\[
\left\| \frac{1}{n} \sum_{i=1}^{n} Y_i \right\| \leq \frac{C\|M_0\|}{\sqrt{rK^2}}
\] (37)

holds with high probability respectively in the cases of Gaussian and 2-design measurements.
5.1 Proof of Proposition 1

We use the noncommutative Rosenthal inequality [9, Theorem 0.4]. We have
\[ \mathbb{E} Y_i Y_i^* = (d_2 + 2) \| M_0 \|_F^2 I_{d_1} + (2d_2 + 3) M_0 M_0^* \]
and
\[ \mathbb{E} Y_i^* Y_i = (d_1 + 2) \| M_0 \|_F^2 I_{d_2} + (2d_1 + 3) M_0^* M_0. \]

On the other hand, since
\[ \| Y_i \| \leq \| (a_i a_i^* - I_{d_1}) M_0 (b_i b_i^* - I_{d_2}) \| + \| (a_i a_i^* - I_{d_1}) M_0 \| + \| M_0 (b_i b_i^* - I_{d_2}) \|, \]
by the moment bound on the $\chi^2$-distribution (e.g., by the Rosenthal inequality), we have
\[ (\mathbb{E} \| Y_i \|^p)^{1/p} \lesssim (\sqrt{pd} + p d^{1/p}) \| M_0 \|, \quad \forall p \geq 2, \]
where $d = d_1 + d_2$.

Then by the noncommutative Rosenthal inequality we obtain
\[ \left( \mathbb{E} \left\| \frac{1}{n} \sum_{i=1}^n Y_i \right\|^p \right)^{1/p} \lesssim \sqrt{pnd} \| M_0 \|_F + pn^{1/p}(pd + p d^{2/p}) \| M_0 \|, \]
which, by the Markov inequality, implies
\[ \mathbb{P} \left( \left\| \frac{1}{n} \sum_{i=1}^n Y_i \right\| \geq t \right) \leq \frac{\mathbb{E} \left\| \sum_{i=1}^n Y_i \right\|^p}{n^p t^p} \leq C_1 \left( \frac{\sqrt{pnd} \| M_0 \|_F + pn^{1/p}(pd + p d^{2/p}) \| M_0 \|}{nt} \right)^p \]
for a numerical constant $C_1$.

We choose $p = \log n$ and $t = C \| M_0 \| / \sqrt{\kappa^2}$. We further assume that $n \geq d$. Then (37) holds with probability $1 - n^{-1}$ if
\[ \left( \frac{\sqrt{\kappa^2}}{C_2} \frac{\sqrt{nr d \log n + d \log^2 n + \log^5 n}}{n} \right)^{\log n} \leq \frac{1}{C_1 n}. \]
A sufficient condition is given by
\[ \frac{n}{\log^5 n} \geq C_3 r^2 \kappa^4 d. \]

Here we did not attempt to optimize the dependence on the logarithmic term.

5.2 Proof of Proposition 2

In the 2-design case, random vectors $a$ and $b$ take unit-vectors of the corresponding length almost surely. Therefore,
\[ \| Y_i \| \leq \| M_0 \| (1 + \| a_i \|^2 \| b_i \|^2) = 2 \| M_0 \|. \]

Since the spectral norm of each summand is bounded almost surely, we can use the matrix Bernstein inequality.
Similarly to the real Gaussian case, we have

\[ \mathbb{E}Y_i^*Y_i = \frac{d_1d_2\|M_0\|^2I_{d_1}}{d_1 + 1} + \left(\frac{d_1d_2}{d_1 + 1} - 1\right)M_0M_0^* \]

and

\[ \mathbb{E}Y_i^*Y_i = \frac{d_1d_2\|M_0\|^2I_{d_2}}{d_2 + 1} + \left(\frac{d_1d_2}{d_2 + 1} - 1\right)M_0^*M_0. \]

Then

\[ \sigma^2 = \max \left( \left\| \sum_{i=1}^n Y_iY_i^* \right\|, \left\| \sum_{i=1}^n Y_i^*Y_i \right\| \right) \leq 4nr(d_1 + d_2)\|M_0\|^2. \]

By the matrix Bernstein inequality \cite[Theorem 1.6]{22}, we obtain

\[ \mathbb{P} \left( \left\| \frac{1}{n} \sum_{i=1}^n Y_i \right\| \geq t \right) \leq (d_1 + d_2) \exp \left( \frac{-n^2t^2/2}{4nr(d_1 + d_2)\|M_0\|^2 + 2\|M_0\|nt/3} \right). \]

We choose \( t = C\|M_0\|/\sqrt{7}\kappa^2 \).

Therefore (37) holds with probability \( 1 - \delta \) if

\[ n \geq C_1r^2\kappa^4d\log(d/\delta) . \]

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**A Tools from Random Matrix Theory**

**Theorem 3** ([5, Theorem II.13]). Let \( G \in \mathbb{R}^{m \times n} \) be a random matrix whose entries are independent copies of \( g \sim \mathcal{N}(0, 1) \). Then

\[ \mathbb{P} \left\{ \left\| \frac{1}{m}G^TG - I_n \right\| > 3 \max \left( \sqrt{\frac{4n}{m}}, \frac{4n}{m} \right) \right\} \leq 2 \exp(-n/2). \]

**Theorem 4** (Matrix Bernstein inequality \cite[Theorem 1.6]{22}). Let \( (Y_k) \subset \mathbb{C}^{m \times n} \) be a finite sequence of independent zero-mean random matrices such that \( \|Y_k\| \leq R \) almost surely for all \( k \). Let

\[ \sigma^2 = \max \left\{ \left\| \sum_k \mathbb{E}Y_kY_k^* \right\|, \left\| \sum_k \mathbb{E}Y_k^*Y_k \right\| \right\}. \]  

(38)

Then for all \( t > 0 \)

\[ \mathbb{P} \left\{ \left\| \sum_k Y_k \right\| \geq t \right\} \leq (m + n) \cdot \exp \left( \frac{-t^2/2}{\sigma^2 + Rt/3} \right). \]

**Theorem 5** (Noncommutative Rosenthal inequality \cite[Theorem 0.4]{9}). Let \( (Y_k) \) be a finite sequence of independent zero-mean random matrices. Let \( \sigma^2 \) be defined by (38). Then there exists a numerical constant \( C > 0 \) such that for all \( 1 \leq p < \infty \)

\[ \left( \mathbb{E}\left\| \sum_k Y_k \right\|^p \right)^{1/p} \leq C \left[ \sqrt[p]{\sigma} \vee p \left( \sum_k \mathbb{E}\|Y_k\|^p \right)^{1/p} \right]. \]
Proof of Lemma 1

Let $\tau > 0$ be fixed. Since $a$ and $b$ are independent and isotropic, they satisfy

$$\mathbb{E} aa^* \otimes bb^* = I_{d_1 d_2}, \quad (39)$$

which implies

$$\mathbb{E} |a^* H b|^2 = \|H\|_F^2, \quad \forall H \in \mathbb{R}^{d_1 \times d_2}. \quad (40)$$

Therefore, the Paley-Zygmund inequality [18] (see also [6, Corollary 3.3.2]), we have

$$\mathbb{P} (|a^* H b| \geq \tau \|H\|_F) = \mathbb{P} \left( |a^* H b|^2 \geq \tau^2 \mathbb{E} |a^* H b|^2 \right) \geq \frac{(1 - \tau^2)^2}{\mathbb{E} |a^* H b|^4}. \quad (41)$$

Then it suffices to show that the fourth order moment $\mathbb{E} |a^* H b|^4$ is upper-bounded by $\left( \mathbb{E} |a^* H b|^2 \right)^2 = \|H\|_F^4$ within a constant factor. We first show this for the real Gaussian case. Since $[a; b] \sim \mathcal{N}(0, I_{d_1 + d_2})$, we have

$$\mathbb{E} |a^* H b|^4 = \mathbb{E}_b (b^* H^* \otimes b^* H^*) \mathbb{E}_a (aa^* \otimes aa^*)(H b \otimes H b)$$

$$= \mathbb{E}_b \|b^* H^*\|_F^4 + 2 \|b^* H^*\|_{S_4}^4$$

$$= 3 \mathbb{E}_b \|b^* H^*\|_2^4$$

$$= 3 \text{tr} \left( [H \otimes H] \mathbb{E}_b (bb^* \otimes bb^*)(H^* \otimes H^*) \right)$$

$$= 3 \|H\|_F^4 + 6 \|H\|_{S_4}^4$$

$$\leq 9 \|H\|_F^4, \quad (42)$$

where $\|\cdot\|_{S_4}$ denotes the Schatten-4 norm.

By plugging in (40) and (42) to (41), we obtain

$$\mathbb{P} (|a^* H b| \geq \tau \|H\|_F) \geq \frac{(1 - \tau^2)^2}{18}. \quad (43)$$

We obtain an analogous upper bound in the 2-design case. With the isotropic normalization, $a$ and $b$ satisfy (6) and (8). Therefore,

$$\mathbb{E} |a^* H b|^4 = \mathbb{E}_b (b^* H^* \otimes b^* H^*) \mathbb{E}_a (aa^* \otimes aa^*)(H b \otimes H b)$$

$$= \frac{d_1}{d_1 + 1} \left( \mathbb{E}_b \|b^* H^*\|_F^4 + \|b^* H^*\|_{S_4}^4 \right)$$

$$= \frac{2d_1}{d_1 + 1} \cdot \mathbb{E}_b \|b^* H^*\|_2^4$$

$$= \frac{2d_1}{d_1 + 1} \cdot \text{tr} \left( [H \otimes H] \mathbb{E}_b (bb^* \otimes bb^*)(H^* \otimes H^*) \right)$$

$$= \frac{2d_1 d_2}{(d_1 + 1)(d_2 + 1)} \cdot \left( \|H\|_F^4 + \|H\|_{S_4}^4 \right)$$

$$\leq \frac{4d_1 d_2}{(d_1 + 1)(d_2 + 1)} \cdot \|H\|_F^4. \quad (43)$$

By plugging in (40) and (43) to (41), we obtain

$$\mathbb{P} (|a^* H b| \geq \tau \|H\|_F) \geq \frac{(1 - \tau^2)^2}{8}. \quad (44)$$
C  Proof of Lemma 2

The independence and isotropy assumptions imply (39). Note that any $H \in \mathcal{T}$ can be written as $H = \Delta_1 V_0^* + U_0 \Delta_2^*$. Furthermore, without loss of generality, we may assume

$$\langle \Delta_1, U_0 \rangle = 0,$$  \hfill (44)

which implies

$$\|H\|_F^2 = \|\Delta_1\|_F^2 + \|\Delta_2\|_F^2.$$  

Since

$$a_i^* H b_i = \text{tr} \left( (a_i b_i^* V_0)^* \Delta_1 \right) + \text{tr} \left( (U_0^* a_i b_i^*)^* \Delta_2^* \right),$$

by (44) we have

$$\mathcal{E}_n(\mathcal{T}) \leq \mathbb{E} \sup_{H \in \mathcal{T} \cap \mathbb{S}} \left| \frac{1}{\sqrt{n}} \sum_{i=1}^n \varepsilon_i a_i^* H b_i \right|$$

$$\leq \mathbb{E} \sup_{\|\Delta_1\|_F^2 + \|\Delta_2\|_F^2 = 1} \left| \frac{1}{\sqrt{n}} \sum_{i=1}^n \varepsilon_i a_i b_i^* V_0 \right| \left| \frac{1}{\sqrt{n}} \sum_{i=1}^n \varepsilon_i U_0^* a_i b_i^* \right|$$

$$= \mathbb{E} \left( \left| \frac{1}{\sqrt{n}} \sum_{i=1}^n \varepsilon_i a_i b_i^* V_0 \right|^2 + \left| \frac{1}{\sqrt{n}} \sum_{i=1}^n \varepsilon_i U_0^* a_i b_i^* \right|^2 \right)^{1/2}$$

$$\leq \sqrt{\left( \mathbb{E} \left| \frac{1}{\sqrt{n}} \sum_{i=1}^n \varepsilon_i a_i b_i^* V_0 \right|^2 + \mathbb{E} \left| \frac{1}{\sqrt{n}} \sum_{i=1}^n \varepsilon_i U_0^* a_i b_i^* \right|^2 \right)^{1/2}}$$

where the second step follows by the Cauchy-Schwartz inequality, the fourth step is obtained by Jensen’s inequality, and the last step holds since $(\varepsilon_i)_{i=1}^n$ is a Rademacher sequence and $(a, b)$ satisfies (39).

D  Proof of Lemma 3

Let $X_0 \in \mathbb{C}^{d_1 \times r}$ and $Y_0 \in \mathbb{C}^{d_2 \times r}$ satisfy $X_0 Y_0^* = M_0$. Note that $(X_0, Y_0)$ is the unique maximizer of (2) if for any $\Delta_1 \in \mathbb{R}^{d_1 \times r}$ and $\Delta_2 \in \mathbb{R}^{d_2 \times r}$

$$\langle \Delta_1, \Delta_1 \rangle + \langle \Delta_2, \Delta_2 \rangle \leq \frac{2}{n} \sum_{i=1}^n \ell_i(X_0 + \Delta_1, Y_0 + \Delta_2) - \ell_i(X_0, Y_0),$$  \hfill (45)

with equality holding only for $\Delta_1 = 0$ and $\Delta_2 = 0$. Since $a_i^* X_0 Y_0^* b_i = m_i$, for each $i$, we obtain

$$\ell_i(X_0 + \Delta_1, Y_0 + \Delta_2) - \ell_i(X_0, Y_0)$$

$$= \frac{1}{2} \|\Delta_1 a_i\|^2 + \langle X_0^* a_i, \Delta_1^* a_i \rangle + \frac{1}{2} \|Y_0^* b_i\|^2 + \langle Y_0^* b_i, \Delta_2^* b_i \rangle$$

$$+ |a_i^* (X \Delta_1^* + \Delta_1 Y^* + \Delta_1 \Delta_2^*) b_i|$$

$$\geq \langle a_i a_i^* X_0, \Delta_1 \rangle + \langle b_i b_i^* Y_0, \Delta_2 \rangle + |a_i^* (X_0 \Delta_1^* + \Delta_1 Y_0^*) b_i|$$

$$- |a_i^* \Delta_1 \Delta_2^* b_i| + \frac{1}{2} \|\Delta_1 a_i\|^2 + \frac{1}{2} \|Y_0^* b_i\|^2$$

$$\geq \langle a_i a_i^* X_0, \Delta_1 \rangle + \langle b_i b_i^* Y_0, \Delta_2 \rangle + |a_i^* (X_0 \Delta_1^* + \Delta_1 Y_0^*) b_i|$$,  \hfill (46)
where the first lower bound is obtained by the triangle inequality and the next lower bound follows from the Cauchy-Schwartz inequality.

Therefore, by plugging in (46) to (45) for all \( i = 1, \ldots, n \), we conclude that (45) holds if (25) holds with equality occurring only at \( |\Delta_1; \Delta_2| = 0 \), and the claim is proved.

**E  Proof of Lemma 4**

This lemma basically follows from requiring stationarity at \( Q = I \) which is away from singularities of the objective. However, to avoid complications arising from derivatives with respect to complex-valued variables, we provide a “lower level” proof. To show this claim, first observe that any \((X, Y)\) satisfying \( XY^* = M_0 \) can be parameterized by an invertible \( r \times r \) matrix \( Q \) as \((X, Y) = (X_0Q, Y_0Q^{-*})\). Let \((X_0, Y_0)\) be the maximizer considered in the statement of the lemma. Therefore, for any invertible \( Q \in \text{GL}_r(\mathbb{C}) \), we should necessarily have

\[
\langle \widetilde{X}_0, X_0Q \rangle + \langle \widetilde{Y}_0, Y_0Q^{-*} \rangle - \frac{1}{2n} \| Q^*X_0^*A \|_F^2 - \frac{1}{2n} \| Q^{-1}Y_0^*B \|_F^2 \\
\leq \langle \widetilde{X}_0, X_0 \rangle + \langle \widetilde{Y}_0, Y_0 \rangle - \frac{1}{2n} \| X_0^*A \|_F^2 - \frac{1}{2n} \| Y_0^*B \|_F^2,
\]

which is equivalent to

\[
\langle \widetilde{X}_0, X_0Q - X_0 \rangle + \langle \widetilde{Y}_0, Y_0Q^{-*} - Y_0 \rangle - \frac{1}{n} \langle (Q - I_r)^*X_0^*A, X_0^*A \rangle - \frac{1}{n} \langle (Q^{-1} - I_r)Y_0^*B, Y_0^*B \rangle \\
- \frac{1}{2n} \| (Q - I_r)^*X_0^*A \|_F^2 - \frac{1}{2n} \| (Q^{-1} - I_r)Y_0^*B \|_F^2 \leq 0
\]

To simplify the notation let us define the short-hands

\[
\Theta_X = X_0^* \left( \widetilde{X}_0 - \frac{1}{n} AA^*X_0 \right),
\]

and

\[
\Theta_Y = Y_0^* \left( \widetilde{Y}_0 - \frac{1}{n} BB^*Y_0 \right).
\]

The necessary inequality can be expressed as

\[
\langle \Theta_X, Q - I_r \rangle + \langle \Theta_Y, (Q^{-1} - I_r)^* \rangle \\
\leq \frac{1}{2n} \| (Q - I_r)^*X_0^*A \|_F^2 + \frac{1}{2n} \| (Q^{-1} - I_r)Y_0^*B \|_F^2.
\]

Choosing \( Q \) within an arbitrarily small neighborhood of \( I_r \) allows us to use the identity

\[
Q^{-1} - I_r = \sum_{k=1}^{\infty} (-1)^k (Q - I_r)^k.
\]

Applying this identity in (47) yields

\[
\langle \Theta_X - \Theta_Y^*, Q - I_r \rangle + \sum_{k=2}^{\infty} (-1)^k \langle \Theta_Y^*, (Q - I_r)^k \rangle \\
\leq \frac{1}{2n} \| (Q - I_r)^*X_0^*A \|_F^2 + \frac{1}{2n} \| (Q^{-1} - I_r)Y_0^*B \|_F^2 \\
\leq \frac{1}{2n} \| Q - I_r \|_F^2 \| X_0^*A \|_F^2 + \frac{1}{2n} \| Q - I_r \|_F^2 \| Q^{-1} \|_F^2 \| Y_0^*B \|_F^2.
\]
As $Q \rightarrow I_r$, the linear terms in $Q - I_r$ dominate and the terms with superlinear dependence on $\|Q - I_r\|$ vanish faster. Therefore, the inequality above implies

$$\Theta_X = \Theta_Y^r. \quad (48)$$

Recall that $E_X = \tilde{X}_0 - \frac{1}{n} AA^* X_0$ and $E_Y = \tilde{Y}_0 - \frac{1}{n} BB^* Y_0$. Then (48) implies that $[E_X; E_Y]$ belongs to a subspace defined by

$$\mathbb{W} = \left\{ [E_1; E_2] : \tilde{X}_0^* E_1 = E_2^* \tilde{Y}_0, E_1 \in \mathbb{C}^{d_1 \times r}, E_2 \in \mathbb{C}^{d_2 \times r} \right\}.$$

It remains to show that $\mathbb{W}$ coincides with $\mathbb{V} = \text{null}(\mathcal{L})^\perp$, which has been identified in (30). Indeed, by the definition of $\mathcal{T}$, $\mathbb{V}$ is equivalently rewritten as

$$\mathbb{V} = \left\{ \left[ ZY_0; Z^\top \tilde{X}_0 \right] : Z \in \mathcal{T} \right\}.$$

We first verify that $\mathbb{V} \subset \mathbb{W}$. Suppose that $[E_1; E_2]$ belongs to $\mathbb{V}$. Then there exists $Z \in \mathcal{T}$ such that $E_1 = ZY_0$ and $E_2 = Z^\top \tilde{X}_0$. Therefore,

$$\tilde{X}_0^* E_1 = \tilde{X}_0^* ZY_0 = (Z^\top \tilde{X}_0)^\top Y_0 = E_2^* Y_0$$

which implies $[E_1; E_2]$ in contained in $\mathbb{W}$. Since $[E_1; E_2]$ was arbitrary, we have shown $\mathbb{V} \subset \mathbb{W}$.

The dimension of $\mathbb{V}$ is no larger than that of $\mathcal{T}$, which is $r(d_1 + d_2 - r)$. Indeed, the subspace $\mathbb{W}$ is given the collection of zeros to given homogeneous linear equations. Since each of $X_0$ and $Y_0$ has full column rank (otherwise, it violates $X_0Y_0^* = M_0$ where rank($M_0$) = $r$), these equations correspond to a set of linearly independent vectors in $\mathbb{C}^{(d_1 + d_2) \times r}$. Therefore, the dimension of $\mathbb{W}$ is no larger than $r(d_1 + d_2 - r)$. Then we deduce that $\mathbb{V} = \mathbb{W}$. Particularly, we have

$$[E_1; E_2] \in \mathbb{W} = \mathbb{V}.$$

Next we prove the second part of the lemma. By (24), we have

$$(1 - \eta) \left\| X_0 - \tilde{X}_0 \right\|_F^2 + (1 - \eta) \left\| Y_0 - \tilde{Y}_0 \right\|_F^2$$

$$\leq \frac{1}{n} \left\| A^* (X_0 - \tilde{X}_0) \right\|_F^2 + \frac{1}{n} \left\| B^* (Y_0 - \tilde{Y}_0) \right\|_F^2$$

$$= \frac{1}{n} \left\| A^* X_0 \right\|_F^2 - 2 \left\langle X_0, \frac{1}{n} AA^* X_0 \right\rangle + \frac{1}{n} \left\| A^* \tilde{X}_0 \right\|_F^2$$

$$+ \frac{1}{n} \left\| B^* Y_0 \right\|_F^2 - 2 \left\langle Y_0, \frac{1}{n} BB^* Y_0 \right\rangle + \frac{1}{n} \left\| B^* \tilde{Y}_0 \right\|_F^2$$

$$= \frac{1}{n} \left\| A^* X_0 \right\|_F^2 - 2 \left\langle X_0, \tilde{X}_0 \right\rangle + 2 \left\langle X_0, \left( I_{d_1} - \frac{1}{n} AA^* \right) \tilde{X}_0 \right\rangle + \frac{1}{n} \left\| A^* \tilde{X}_0 \right\|_F^2$$

$$+ \frac{1}{n} \left\| B^* Y_0 \right\|_F^2 - 2 \left\langle Y_0, \tilde{Y}_0 \right\rangle + 2 \left\langle Y_0, \left( I_{d_2} - \frac{1}{n} BB^* \right) \tilde{Y}_0 \right\rangle + \frac{1}{n} \left\| B^* \tilde{Y}_0 \right\|_F^2.$$

Since $(X_0, Y_0)$ is the maximizer, for any $X \in \mathbb{C}^{d_1 \times r}$ and $Y \in \mathbb{C}^{d_2 \times r}$ satisfying $XY^* = M_0$, we can
where the last step follows from (24). Finally note that
\[
\|X - \overline{X}_0\|_F \|\overline{X}_0\|_F + \|Y - \overline{Y}_0\|_F \|\overline{Y}_0\|_F \leq \left(\|X - \overline{X}_0\|_F^2 + \|Y - \overline{Y}_0\|_F^2\right)^{1/2} \left(\|\overline{X}_0\|_F^2 + \|\overline{Y}_0\|_F^2\right)^{1/2}.
\]
This completes the proof.

F Proof of Lemma 6

It suffices to show the upper bound for some \( Q \in \mathcal{O}_r \). Let \( Q \) be given by
\[
Q \in \arg\min_{R \in \mathbb{R}^{r \times r}} \left\{ \left\| \begin{bmatrix} U_0 \\ V_0 \end{bmatrix} R - \begin{bmatrix} U_0 \\ V_0 \end{bmatrix} \right\|_F | R^* R = I_r \right\}.
\]
Then it follows that
\[
\left\| \begin{bmatrix} U_0 \\ V_0 \end{bmatrix} \Sigma_0^{1/2} - \begin{bmatrix} U_0 \\ V_0 \end{bmatrix} \Sigma_0^{1/2} Q \right\|_F \leq \left\| \left( \begin{bmatrix} U_0 \\ V_0 \end{bmatrix} - \begin{bmatrix} U_0 \\ V_0 \end{bmatrix} Q \right) \Sigma_0^{1/2} \right\|_F + \left\| \begin{bmatrix} U_0 \\ V_0 \end{bmatrix} Q \Sigma_0^{1/2} - Q \Sigma_0^{1/2} \right\|_F \leq \sqrt{\sigma_1(\Sigma_0)} \left\| \begin{bmatrix} U_0 \\ V_0 \end{bmatrix} - \begin{bmatrix} U_0 \\ V_0 \end{bmatrix} Q \right\|_F + \left\| \Sigma_0^{1/2} - Q \Sigma_0^{1/2} \right\|_F.
\]

Since \( \Sigma_0^{1/2} \) (resp. \( Q^* \Sigma_0^{1/2} \)) is the matrix square root of \( \tilde{\Sigma}_0 \) (resp. \( Q^* \Sigma_0 Q \)), by the perturbation bound by Schmitt [20, equation (1.3)], we obtain
\[
\left\| \Sigma_0^{1/2} - Q^* \Sigma_0^{1/2} \right\|_F \leq \frac{\left\| \Sigma_0 - Q^* \Sigma_0 Q \right\|_F}{\sqrt{\sigma_1(\Sigma_0) + \sqrt{\sigma_r(\Sigma_0)}}} \leq \frac{\left\| \Sigma_0 - Q^* \Sigma_0 Q \right\|_F}{\sqrt{\sigma_r(\Sigma_0)}}.
\]
On the other hand, by the triangle inequality, we have
\[
\| \tilde{M}_0 - M_0 \|_F = \| \tilde{U}_0 \tilde{\Sigma}_0 \tilde{V}_0^* - U_0 Q Q^* \Sigma_0 Q Q^* V_0^* \|_F \geq \| U_0 Q (\tilde{\Sigma}_0 - Q^* \Sigma_0 Q) \tilde{V}_0^* \|_F - \| (\tilde{U}_0 - U_0 Q) \tilde{\Sigma}_0 \tilde{V}_0^* \|_F - \| U_0 Q Q^* \Sigma_0 Q (\tilde{V}_0 - V_0 Q)^* \|_F \geq \| \tilde{\Sigma}_0 - Q^* \Sigma_0 Q \|_F - \sigma_1(\tilde{\Sigma}_0) \| \tilde{U}_0 - U_0 Q \|_F - \sigma_1(\Sigma_0) \| \tilde{V}_0 - V_0 Q \|_F.
\]
By rearranging the above inequality, we obtain
\[ \|\tilde{\Sigma}_0 - Q^*\Sigma_0Q\|_F \leq \|\tilde{M}_0 - M_0\|_F + \sigma_1(\tilde{\Sigma}_0) \|\tilde{U}_0 - U_0Q\|_F + \sigma_1(\Sigma_0) \|\tilde{V}_0 - V_0Q\|_F. \]

By combining the results with Weyl’s inequality, we obtain
\[
\left\| \begin{bmatrix} U_0 \\ V_0 \end{bmatrix} \right\|_{\Sigma_0^{1/2}} - \left\| \begin{bmatrix} \tilde{U}_0 \\ \tilde{V}_0 \end{bmatrix} \right\|_{\tilde{\Sigma}_0^{1/2}} \leq \sqrt{\sigma_1(\Sigma_0)} \left\| \begin{bmatrix} U_0 \\ V_0 \end{bmatrix} \right\|_F - \left\| \begin{bmatrix} U_0 \\ V_0 \end{bmatrix} \right\|_F Q
\]
\[ + \frac{\|\tilde{M}_0 - M_0\|_F + \sigma_1(\tilde{\Sigma}_0) \|\tilde{U}_0 - U_0Q\|_F + \sigma_1(\Sigma_0) \|\tilde{V}_0 - V_0Q\|_F}{\sqrt{\sigma_r(\Sigma_0)}} \leq \frac{\|\tilde{M}_0 - M_0\|_F}{\sqrt{\sigma_r(\Sigma_0)}} + \sqrt{2\sigma_1(\Sigma_0)} \left(1 + 2 \frac{\sigma_1(\tilde{\Sigma}_0)}{\sigma_r(\Sigma_0)}\right) \left\| \begin{bmatrix} \tilde{U}_0 \\ \tilde{V}_0 \end{bmatrix} \right\|_F - \left\| \begin{bmatrix} U_0 \\ V_0 \end{bmatrix} \right\|_F Q. \]

Finally, (2) is bounded by a variant of the Davis-Kahan theorem by Dopico [7, Theorem 2.1] as follows:
\[
\left\| \begin{bmatrix} \tilde{U}_0 \\ \tilde{V}_0 \end{bmatrix} - \begin{bmatrix} U_0 \\ V_0 \end{bmatrix} \right\|_F \leq \frac{2(\|\tilde{M}_0 - M_0\|_F^2 + \|\tilde{U}_0^* (\tilde{M}_0 - M_0)\|_F^2)}{\sigma_r(\Sigma_0) - \|\tilde{M}_0 - M_0\|}.
\]

G Proof of Lemma 7

Since \((X_0, Y_0)\) is a maximizer to (28), we have
\[
\langle X_0, X_0 - X \rangle + \langle Y, Y_0 - Y \rangle \geq \frac{1}{2n} \left( \|A^*X_0\|_F^2 - \|A^*X\|_F^2 + \|B^*Y_0\|_F^2 - \|B^*Y\|_F^2 \right)
\]
\[ \geq \frac{1}{n} \langle AA^*X, X_0 - X \rangle + \frac{1 - \eta}{2} \|X_0 - X\|_F^2 + \frac{1}{n} \langle BB^*Y, Y_0 - Y \rangle + \frac{1 - \eta}{2} \|Y_0 - Y\|_F^2, \tag{49} \]
where the second inequality follows from the strong convexity of \(\|A^*X_0\|_F^2\) and \(\|B^*Y_0\|_F^2\), which are quadratic functions of \(X_0\) and \(Y_0\), respectively. Then (49) is rearranged to
\[
\frac{1 - \eta}{2} \|X_0 - X\|_F^2 + \frac{1 - \eta}{2} \|Y_0 - Y\|_F^2 \leq \langle X - \frac{1}{n} AA^*X, X_0 - X \rangle + \langle Y_0 - \frac{1}{n} BB^*Y, Y_0 - Y \rangle
\]
\[ \leq \sqrt{\|X_0 - X\|_F^2 + \|Y_0 - Y\|_F^2} \cdot \sqrt{\|X - \frac{1}{n} AA^*X\|_F^2 + \|Y_0 - \frac{1}{n} BB^*Y\|_F^2}, \]
where the last step follows from the Cauchy-Schwartz inequality.

Therefore it follows that
\[
\|\left[ \begin{array}{c} \tilde{X}_0 - \frac{1}{n} AA^*X \\ \tilde{Y}_0 - \frac{1}{n} BB^*Y \end{array} \right]\|_F \geq \frac{1 - \eta}{2} \|[X_0 - X; Y_0 - Y]\|_F. \]
Finally by the triangle inequality we obtain
\[
\|\left[ \tilde{X}_0 - \frac{1}{n} AA^* X_0; \tilde{Y}_0 - \frac{1}{n} BB^* Y_0 \right]\|_F \\
\leq \left| \left| \left[ \tilde{X}_0 - \frac{1}{n} AA^* X_0; \tilde{Y}_0 - \frac{1}{n} BB^* Y_0 \right] \right|\|_F \\
+ \left| \left| \left[ \frac{1}{n} AA^* (X_0 - X); \frac{1}{n} BB^* (Y_0 - Y) \right] \right|\|_F \\
\leq \left| \left| \left[ \tilde{X}_0 - \frac{1}{n} AA^* X; \tilde{Y}_0 - \frac{1}{n} BB^* Y \right] \right|\|_F \\
+ \left(1 + \eta\right) \left| \left| \left[ \tilde{X}_0 - \frac{1}{n} AA^* X; \tilde{Y}_0 - \frac{1}{n} BB^* Y \right] \right|\|_F \\
\leq \left(1 + \frac{2(1 + \eta)}{1 - \eta}\right) \left| \left| \left[ \tilde{X}_0 - X; \tilde{Y}_0 - Y \right] \right|\|_F \\
+ \left| \left| \left[ X - \frac{1}{n} AA^* X; Y - \frac{1}{n} BB^* Y \right] \right|\|_F \right). \\
\]

Finally note that
\[
\left| \left| \left[ X - \frac{1}{n} AA^* X; Y - \frac{1}{n} BB^* Y \right] \right|\|_F \leq \eta \| [X; Y]\|_F ,
\]

which completes the proof.

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