Traces for star products on the dual of a Lie algebra

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Abstract

In this paper, we describe all traces for the BCH star-product on the dual of a Lie algebra. First we show by an elementary argument that the BCH as well as the Kontsevich star-product are strongly closed if and only if the Lie algebra is unimodular. In a next step we show that the traces of the BCH star-product are given by the ad-invariant functionals. Particular examples are the integration over coadjoint orbits. We show that for a compact Lie group and a regular orbit one can even achieve that this integration becomes a positive trace functional. In this case we explicitly describe the corresponding GNS representation. Finally we discuss how invariant deformations on a group can be used to induce deformations of spaces where the group acts on.

1 Introduction

Trace functionals play an important role in deformation quantization [4] (for recent reviews on deformation quantization we refer to [19, 24, 35, 39], existence and classification results can be found in [5, 20, 28, 31, 32, 41]).

Physically, traces correspond to states of thermodynamical equilibrium characterized by the KMS condition at infinite temperature [3, 10]. Note however, that for reasonable physical interpretation one has to impose an additional positivity condition on the traces [11, 8].
On the mathematical side traces are one half of the index theorem, namely the part of cyclic cohomology. The other half comes from the $K$-theory part. Having a trace functional $\text{tr} : A \to \mathbb{C}$ of an associative algebra $A$ over some commutative ring $\mathbb{C}$ and having a projection $P = P^2 \in M_n(A)$ representing an element $[P] \in K_0(A)$ the value $\text{tr}(P) \in \mathbb{C}$ does not depend on $P$ but only on its class $[P]$. This is just the usual natural pairing of cyclic cohomology with $K$-theory, see e.g. [17, Chap. III.3], and the value $\text{ind}([P]) = \text{tr}(P)$ is called the index of $[P]$ with respect to the chosen trace.

In the case of deformation quantization the situation is as follows. The starting point is a star-product $\star$ for a Poisson manifold $(M, \pi)$ whence the algebra of interest is $A = (C^\infty(M)[[\nu]], \star)$ viewed as an algebra over $\mathbb{C}[[\nu]]$. Then a trace is a $\mathbb{C}[[\nu]]$-linear functional $\text{tr} : C^\infty(M)[[\nu]] \to \mathbb{C}[[\nu]]$ such that

$$\text{tr}(f \star g) = \text{tr}(g \star f),$$

whenever one function has compact support. For the $K$-theory part of the index theorem one knows that $K$-theory is stable under deformation, see e.g. [34]: any projection $P_0$ of the undeformed algebra $M_n(C^\infty(M))$ can be deformed into a projection

$$P = \frac{1}{2} + \left(P_0 - \frac{1}{2}\right) \star \frac{1}{\sqrt{1 + 4(P_0 \star P_0 - P_0)}}$$

(1.2)

with respect to $\star$, see [22, Eq. (6.1.4)]. Moreover, this deformation is unique up to equivalence of projections and any projection of the deformed algebra arises this way. It follows that $\text{ind}([P])$ only depends on $[P_0] \in K_0(C^\infty(M))$, which is the isomorphism class of the vector bundle defined by $P_0$, see also [12] for a more detailed discussion.

Now let $\tilde{\star}$ be an equivalent star product with equivalence transformation $T(f \star g) = Tf \tilde{\star} Tg$. Then clearly $\tilde{\text{tr}} = \text{tr} \circ T^{-1}$ defines a trace functional with respect to $\tilde{\star}$. From (1.2) we see that $\text{ind}([\tilde{P}]) = \tilde{\text{tr}} \circ T^{-1}$ defines a trace. Thus the index transforms well under equivalences of star products provided one uses the ‘correct’ corresponding trace. It happens that in the symplectic case there is only one trace up to normalization [31]. So suppose that $M$ is compact and that for each star product $\star$ we have chosen a trace $\text{tr}_\star$ normalized such that

$$\text{tr}_\star(1) = c$$

where $c$ does not depend on $\star$. Then $T1 = 1$ implies $\text{tr}_\star = \text{tr}_\star \circ T^{-1}$ and thus the index does not depend on the choice of $\star$ but only on the equivalence class $[\star]$. This simple reasoning already explains the structure of Fedosov’s index formula [21, Thm. 6.1.6], [31]. Nevertheless we would like to mention that the computation of $\text{ind}([\tilde{P}])$ in geometrical terms is a quite non-trivial task.

For a formulation of the index theorem in the general Poisson case we refer to [37]. Here the situation is far more non-trivial as in general there is no longer a unique trace. In [21] it is shown that integration over $M$ with respect to some smooth density $\Omega$ is a trace for Kontsevich’s star product provided the Poisson tensor is $\Omega$-divergence free. However, there are much more traces, typically involving integrations over the symplectic leaves.

An elementary proof that in the symplectic case one has a unique trace is presented in [21]. This approach uses the canonical way of normalization of the trace, introduced by Karabegov [23] using local $\nu$-Euler derivations, see [29] and the elementary proof of the uniqueness up to scaling of a trace as given in [11]: Here one uses the fact that in the whole algebraic dual of $C^\infty(M)$ there is only one Poisson trace

$$\tau_0(\{f, g\}) = 0,$$

(1.3)
namely the integration with respect to the Liouville measure.

In this article we shall now consider the most simple case of a Poisson manifold: the dual of a Lie algebra. Here we shall determine all the traces for the BCH star product on \( \mathfrak{g}^* \) by very elementary arguments.

The paper is organized as follows. In Sect. 2 we recall the construction of various star products on the dual of a Lie algebra \( \mathfrak{g}^* \) as well as their relation to star products on \( \mathfrak{g}^* \) by elementary computations in Sect. 3 and in Sect. 4 we show that any ad-invariant functional is a trace for the BCH star product. In Sect. 5 we prove the positivity of a trace \( \tau_0 \) associated to a regular orbit \( \emptyset \subseteq \mathfrak{g}^* \) for compact \( G \) by a BRST construction of a star product on \( \emptyset \). Sect. 6 contains a characterization of the GNS representation obtained from the positive trace \( \tau_0 \). Finally, Sect. 7 is devoted to a construction of trace functionals by a group action using a 'universal deformation' on the group, inspired by techniques developed in [1,22].

### 2 Star products on \( \mathfrak{g}^* \) and \( T^*G \)

In this section we shall recall the construction of several star products on the dual \( \mathfrak{g}^* \) of a Lie algebra \( \mathfrak{g} \) and on \( T^*G \) where \( G \) is a Lie group with Lie algebra \( \mathfrak{g} \). First we shall establish some notation.

By \( e_1, \ldots, e_n \) we denote a basis of \( \mathfrak{g} \) with dual basis \( e^1, \ldots, e^n \in \mathfrak{g}^* \). Such a basis gives rise to linear coordinates \( x = x^ie_i \) on \( \mathfrak{g} \) and \( \xi = \xi^ie_i \) on \( \mathfrak{g}^* \). Here and in the following we shall use Einstein’s summation convention. With a capital letter \( X \) we shall denote the left-invariant vector field \( X \in \Gamma^\infty(TG) \) corresponding to \( x \in \mathfrak{g} \), i.e. \( X_x = x \). A vector \( x \in \mathfrak{g} \) determines a linear function \( \hat{x} \in \text{Pol}^1(\mathfrak{g}^*) \) by \( \hat{x}(\xi) = \xi(x) \). Analogously, \( X \in \Gamma^\infty(TG) \) determines a function \( \hat{X} \in \text{Pol}^1(T^*G) \), linear in the fibers, by setting \( \hat{X}(\alpha_g) = \alpha_g(X_g) \), where \( \alpha_g \in T_g^*G \) and \( g \in G \). We shall use the same symbol \( ^\ast \) for the corresponding graded algebra isomorphism between the symmetric algebra \( \bigwedge^\ast \mathfrak{g} \) of \( \mathfrak{g} \) and all polynomials \( \text{Pol}^\ast(\mathfrak{g}^*) \) on \( \mathfrak{g}^* \). Similar we have a graded algebra isomorphism between \( \Gamma(\bigwedge^\ast TG) \) and \( \text{Pol}^\ast(T^*G) \). By use of left-invariant vector fields and one-forms, \( TG \) and \( T^*G \) trivialize canonically. This yields \( TG \cong G \times \mathfrak{g} \) and \( T^*G \cong G \times \mathfrak{g}^* \). The corresponding projections are denoted by

\[
G \xrightarrow{\pi} G \times \mathfrak{g}^* \xrightarrow{\vartheta} \mathfrak{g}^*,
\]

whence in particular \( \hat{X} = \vartheta \hat{x} \) for a left-invariant vector field \( X \). More generally, \( \text{Pol}^\ast(T^*G)^G = \vartheta^\ast \text{Pol}^\ast(\mathfrak{g}^*) \). For the symplectic Poisson bracket on \( T^*G \) we use the sign convention such that the map \( ^\ast : \Gamma^\infty(TG) \to \text{Pol}^1(T^*G) \) becomes an isomorphism of Lie algebras (and not an anti-isomorphism as in [8]). Then the canonical linear Poisson bracket on \( \mathfrak{g}^* \) can be obtained by the observation that left-invariant functions on \( T^*G \) (with respect to the lifted action) are a Poisson sub-algebra which is in linear bijection with \( C^\infty(\mathfrak{g}^*) \) via \( \vartheta^\ast \). Thus it is meaningful to require \( \vartheta^\ast \) to be a morphism of Poisson algebras. In the global coordinates \( \xi_1, \ldots, \xi_n \) the resulting Poisson bracket on \( \mathfrak{g}^* \) reads as

\[
\{f, g\} = \xi_k e^k_{ij} \frac{\partial f}{\partial \xi_i} \frac{\partial g}{\partial \xi_j},
\]

where \( e^k_{ij} = e^k([e_i, e_j]) \) are the structure constants of \( \mathfrak{g} \) and \( f, g \in C^\infty(\mathfrak{g}^*) \).

The first star-product on \( \mathfrak{g}^* \) is essentially given by the Baker-Campbell-Hausdorff series of \( \mathfrak{g} \). One uses the total symmetrization map \( \sigma_\nu : \text{Pol}^\ast(\mathfrak{g}^*)[\nu] \to \mathfrak{u}(\mathfrak{g})[\nu] \) into the universal enveloping
algebra of $\mathfrak{g}$, defined by

$$\sigma_\nu(x_1 \vee \cdots \vee x_k) = \frac{\nu^k}{k!} \sum_{\tau \in S_k} x_{\tau(1)} \cdots \cdot x_{\tau(k)}, \quad (2.3)$$

where we have built in the formal parameter $\nu$ already at this stage. Then

$$\sigma_\nu(f \ast_{BCH} g) = \sigma_\nu(f) \cdot \sigma_\nu(g) \quad (2.4)$$
yields indeed a deformed product $\ast_{BCH}$ for $f, g \in \text{Pol}^\ast(\mathfrak{g}^\ast)[[\nu]]$, which turns out to extend to a differential star-product for $C^\infty(\mathfrak{g}^\ast)[[\nu]]$, see [23] for a detailed discussion. Here we shall just mention a few properties of $\ast_{BCH}$. First, $\ast_{BCH}$ is strongly $\mathfrak{g}$-invariant, i.e. for $f \in C^\infty(\mathfrak{g}^\ast)[[\lambda]]$ and $x \in \mathfrak{g}$ we have

$$\dot{x} \ast_{BCH} f - f \ast_{BCH} \dot{x} = \nu \{x, f\}. \quad (2.5)$$

Moreover, $\ast_{BCH}$ is homogeneous: this means that the operator $\mathcal{H} = \nu \frac{\partial}{\partial \nu} + \mathcal{L}_E$, where $E = \xi_t \frac{\partial}{\partial \xi}$ is the Euler vector field, is a derivation of $\ast_{BCH}$, i.e.

$$\mathcal{H}(f \ast_{BCH} g) = \mathcal{H}f \ast_{BCH} g + f \ast_{BCH} \mathcal{H}g \quad (2.6)$$

for all $f, g \in C^\infty(\mathfrak{g}^\ast)[[\nu]]$. It follows immediately that $\text{Pol}^\ast(\mathfrak{g}^\ast)[[\nu]]$ is a ‘convergent’ sub-algebra generated by the constant and linear polynomials. The relation to the BCH series can be seen as follows: Consider the exponential functions $e_x(\xi) := e^{\xi(x)}$. Then for all $x, y \in \mathfrak{g}$ one has

$$e_x \ast_{BCH} e_y = e_{1/2 H(\nu_x, \nu_y)}. \quad (2.7)$$

where $H(\cdot, \cdot)$ is the BCH series of $\mathfrak{g}$. Since bidifferential operators on $\mathfrak{g}^\ast$ are already determined by their values on the exponential functions $e_x, x \in \mathfrak{g}$, the star-product $\ast_{BCH}$ is already determined by $(2.7)$. For a more detailed analysis and proofs of the above statements we refer to [8, 23].

The other star-product we shall mention is the Kontsevich star-product $\ast_{K}$ for $\mathfrak{g}^\ast$. His general construction of a star product for arbitrary Poisson structures on $\mathbb{R}^n$ simplifies drastically in the case of a linear Poisson structure $(2.2)$. We shall not enter the general construction but refer to [1, 2, 18, 24] for more details and just mention a few properties of $\ast_{K}$. First, $\ast_{K}$ is $\mathfrak{g}$-covariant, i.e. one has

$$\dot{x} \ast_{K} \dot{y} - \dot{y} \ast_{K} \dot{x} = \nu \{x, y\} = \nu [x, y]. \quad (2.8)$$

for all $x, y \in \mathfrak{g}$. This is a weaker compatibility with the (classical) $\mathfrak{g}$-action than $(2.5)$. Moreover, $\ast_{K}$ is homogeneous, too, but in general $\ast_{K}$ and $\ast_{BCH}$ do not coincide but are only equivalent, see [15].

Let us now recall how the star-product $\ast_{BCH}$ on $\mathfrak{g}^\ast$ is related to star-products on $T^*G$. The main idea is to make the Poisson morphism $\varphi^*$ into an algebra morphism of star-product algebras. This requirement does not determine the star-product on $T^*G$ completely and the remaining freedom (essentially the choice of an ‘ordering prescription’ between functions depending only on $G$ and on $\mathfrak{g}^*$, respectively) can be used to impose further properties. In [23] a star-product $\ast_{G}$ of Weyl-type was constructed by inserting additional derivatives in $G$-direction into the bidifferential operators of $\ast_{BCH}$. In [8] a star product $\ast_{s}$ of standard-ordered type was obtained by a (standard-ordered) Fedosov construction using the lift of the half-commutator connection on $G$ to a symplectic connection on $T^*G$. The star-product $\ast_{s}$ can also be understood as the resulting composition law of symbols from the standard-ordered symbol and differential operator calculus induced by the half-commutator connection. A further ‘Weyl-symmetrization’ yields a star-product $\ast_{w}$ of Weyl-type.
which does not coincide in general with the original Fedosov star-product $\ast_F$ built out of the half-commutator connection directly. However, it was shown in [3, Sect 8] that $\ast_W$ coincides with $\ast_G$. Moreover, the pull-back $\varrho^*$ is indeed an algebra morphism for both star-products $\ast_G$ and $\ast_S$, i.e. one has

$$\varrho^* f \ast_{G/S} \varrho^* g = \varrho^*(f \ast_{BCH} g)$$

(2.9)

for all $f, g \in C^\infty(g^*)[[\nu]]$. All the star products $\ast_G$, $\ast_S$, and $\ast_F$ are homogeneous in the sense of star-products on cotangent bundles whence it follows that they are all strongly closed: integration over $T^*G$ with respect to the Liouville form defines a trace on the functions with compact support, see [3, Sect. 8].

3 Strong closedness of $\ast_{BCH}$ and $\ast_K$

We shall now discuss an elementary proof of the fact that $\ast_{BCH}$ as well as $\ast_K$ are strongly closed with respect to the constant volume form $d^n\xi$ on $g^*$ if and only if the Lie algebra $g$ is unimodular, i.e. $\text{tr} \text{ad}(x) = 0$ for all $x \in g$, or, equivalently, $c_{ij}^k = 0$. The unimodularity of $g$ is easily seen to be necessary since it is exactly the condition that the integration is a Poisson trace, see also [10, Sect. 4] for the Poisson case and [21] for a different and more general proof for Kontsevich’s star product on $\mathbb{R}^n$.

Before we discuss the general case let us consider the case where $G$ is compact. In this case $g$ is known to be in particular unimodular.

**Proposition 3.1** Let $G$ be compact. Then $\ast_{BCH}$ is strongly closed.

**Proof:** Let $f, g \in C^\infty_0(g^*)$. Since $G$ is compact, $\varrho^* f, \varrho^* g \in C^\infty_0(T^*G)$ and thus the strong closedness of $\ast_G$ and (2.9) implies

$$0 = \int_{T^*G} (\varrho^* f \ast_G \varrho^* g - \varrho^* g \ast_G \varrho^* f) \Omega = \text{vol}(G) \int_{g^*} (f \ast_{BCH} g - g \ast_{BCH} f) d^n\xi,$$

where $\Omega$ is the (suitably normalized) Liouville measure on $T^*G$. $\square$

Clearly the above proof relies on the compactness of $G$, otherwise the integration would not be defined. As an amusing observation we remark that one can also use the above proposition to obtain the well-known fact that compact Lie groups have unimodular Lie algebras.

For the general unimodular case we use a different argument which is essentially the same as for homogeneous star-products on a cotangent bundle [3, Sect. 8]. A differential operator $D$ on $g^*$ is called homogeneous of degree $r \in \mathbb{Z}$ if $[\mathcal{L}_E, D] = r D$, where $\mathcal{L}_E$ is the Lie derivative with respect to the Euler vector field.

**Lemma 3.2** Let $D$ be a homogeneous differential operator of degree $-r$ with $r \geq 1$. Then for all $f \in C^\infty_0(g^*)$ one has

$$\int_{g^*} Df \ d^n\xi = 0.$$  

(3.1)

From here we can follow [3] almost literally: If $f \in \text{Pol}^{k}(g^*)$ and $g \in C^\infty_0(g^*)$ then for every homogeneous star product $\ast$ on $g^*$ one has

$$\int_{g^*} f \ast g \ d^n\xi = \sum_{r=0}^{k} \nu^r \int_{g^*} C_r(f, g) \ d^n\xi,$$

(3.2)
where \( C_r \) is the \( r \)-th bidifferential operator of \( * \). This follows from Lemma 3.2 since \( C_r(f, \cdot) \) is homogeneous of degree \( k - r \). The analogous statement holds for the integral over \( g * f \). From this we conclude the following lemma:

**Lemma 3.3** Let \( * \) be a homogeneous star-product for \( g^* \), \( f \in \text{Pol}^*(g^*) \), and \( g \in C_0^\infty(g^*) \). Then

\[
\int_{g^*} (f * g - g * f) \, d^n\xi = 0 \tag{3.3}
\]

if and only if \( g \) is unimodular.

**Proof:** The proof is done by induction on the polynomial degree \( k \) of \( f \). For \( k = 0 \) the statement (3.3) is true by (3.2). For \( k = 1 \) we obtain (3.3) by (3.2) if and only if the integral vanishes on Poisson brackets, i.e. if and only if \( g \) is unimodular. For \( k \geq 2 \) we can write \( f \) as a \( * \)-polynomial in at most linear polynomials since these polynomials generate \( \text{Pol}^*(g^*)[\nu] \) by the homogeneity of \( * \). Then we can use the cases \( k = 0, 1 \) to prove (3.3). \( \square \)

Having the trace property for polynomials and compactly supported functions, we only have to use a density argument, i.e. the Stone-Weierstraß theorem, to conclude the trace property in general:

**Theorem 3.4** Let \( * \) be a homogeneous star-product for \( g^* \). Then the integration over \( g^* \) with respect to the constant volume \( d^n\xi \) is a trace if and only if \( g \) is unimodular.

Since \( *_{\text{BCH}} \) as well as \( *_{\text{K}} \) are homogeneous this theorem proves in an elementary way that they are strongly closed in the sense of [17].

### 4 Trace properties of \( g \)-invariant functionals

Quite contrary to the symplectic case it turns out that in the Poisson case traces are no longer unique in general.

Before we give an elementary proof in the case of \( g^* \) we shall make a few comments on the general situation. As we have seen already before, the trace functionals are typically not defined on the whole algebra but on a certain subspace, as e.g. the functions with compact support. On the other hand, the property of being a trace only becomes interesting if this subspace is not only a sub-algebra but even an ideal. This motivates the following terminology: For an associative algebra \( A \) we call a functional \( \tau \) defined on \( J \subseteq A \) a trace on \( J \) if \( J \) is a two-sided ideal and for all \( A \in A \) and \( B \in J \) one has \( \tau([A,B]) = 0 \). Similarly we define a Poisson trace on a Poisson ideal of a Poisson algebra.

With this notation the traces which are given by integrations are traces on the ideals \( C_0^\infty(g^*) \) and \( C_0^\infty(g^*)[[\nu]] \), respectively. However, there will be some interesting traces with a slightly different domain. If we want to integrate over a sub-manifold \( \iota : N \hookrightarrow M \) then the following space becomes important. Here and in the following we shall only consider the case where \( \iota \) is an embedding. We define

\[
C_N^\infty(M) := \{ f \in C^\infty(M) \mid \iota(N) \cap \text{supp} \, f \text{ is compact} \}. \tag{4.1}
\]

If \( N \) is a closed embedded sub-manifold then \( C_0^\infty(N) \subseteq C_N^\infty(M) \). Moreover, the locality of a star-product ensures that \( C_N^\infty(M)[[\nu]] \) is a two-sided ideal of \( C^\infty(M)[[\nu]] \).

Taking such a subspace as example we consider more generally domains of the form \( \mathcal{D}[[\nu]] \) where \( \mathcal{D} \subseteq C^\infty(M) \). In this case \( \mathcal{D} \) is necessarily a Poisson ideal which follows immediately from the ideal
properties of $\mathcal{D}[[\nu]]$. Moreover, if $\tau : \mathcal{D}[[\nu]] \to \mathbb{R}[[\nu]]$ is a trace for a local star-product $*$ on $M$ with domain $\mathcal{D}[[\nu]]$ then $\tau = \sum_{r=0}^{\infty} \nu^r \tau_r$ with linear functionals $\tau_r : \mathcal{D} \to \mathbb{R}$. For the following we shall assume that all $\tau_r$ have some reasonable continuity property, e.g. with respect to the locally convex topology of smooth functions. This requirement seems to be reasonable as long as we are dealing with star products having at least continuous cochains in every order of $\nu$.

Now let us come back to the case of $g^*$ with the star product $*_{BCH}$. As a first observation we remark that the strong $g$-invariance of $*_{BCH}$ implies that for a two-sided ideal $\mathcal{D}[[\nu]]$ the space $\mathcal{D}$ is $g$-invariant. Moreover, we have the following theorem:

**Theorem 4.1** Let $\mathcal{D} \subseteq C^\infty(g^*)$ be a subspace such that $\mathcal{D}[[\nu]]$ is a two-sided ideal with respect to $*_{BCH}$ and let $\tau = \sum_{r=0}^{\infty} \nu^r \tau_r$ be a $\mathbb{R}[[\nu]]$-linear functional on $\mathcal{D}[[\nu]]$ with the following continuity property: For a given $f \in C^\infty(g^*)$ and $g \in \mathcal{D}$ and a sequence $p_n \in \text{Pol}^\star(g^*)$ such that $p_n \to f$ in the locally convex topology of smooth functions we have $\tau_r([p_n,g]_{*_{BCH}}) \to \tau_r([f,g]_{*_{BCH}})$ (in each order of $\nu$).

Then $\tau$ is a $*_{BCH}$-trace on $\mathcal{D}[[\nu]]$ if and only if $\tau$ is a Poisson trace on $\mathcal{D}$ which is the case if and only if $\tau$ is $g$-invariant.

**Proof:** The continuity ensures that $g$-invariance coincides with the property of being a Poisson trace. Now let $\tau_0$ be a Poisson trace and let $g \in \mathcal{D}$. Then for all $x \in g$ we have $\tau_0([\hat{x},g]) = \nu \tau_0(\{\hat{x},g\}) = 0$ by the strong invariance of $*_{BCH}$. But since $\text{Pol}^\star(g^*)[[\nu]]$ together with the constants generates $\text{Pol}^\star(g^*)[[\nu]]$ we have $\tau_0([p,g]) = 0$ for every polynomial $p$. Together with the fact that the polynomials are dense in $C^\infty(g^*)$ and $\tau_0$ has the above continuity it follows that $\tau_0$ is a $*_{BCH}$-trace. Now if $\tau$ is a $*_{BCH}$-trace then $\tau_0$ is a Poisson trace and hence a $*_{BCH}$-trace itself. Thus $\tau - \tau_0$ is still a $*_{BCH}$-trace and a simple induction proves the theorem. \(\square\)

The somehow technical continuity property needed above turns out to be rather mild. In the main example it is trivially fulfilled:

**Example 4.2**

i.) Let $\iota : \mathcal{O} \hookrightarrow g^*$ be a not necessarily closed but embedded coadjoint orbit and consider $\mathcal{D} = C^\infty_{\mathcal{O}}(g^*)$. Then the integration with respect to the Liouville measure $\Omega_{\mathcal{O}}$ on $\mathcal{O}$,

$$\tau_{\mathcal{O}}(f) := \int_{\mathcal{O}} \iota^* f \Omega_{\mathcal{O}},$$

is a $*_{BCH}$-trace on $C^\infty_{\mathcal{O}}(g^*)[[\nu]]$.

ii.) If in addition $\Delta$ is a $g$-invariant differential operator on $g^*$ then $\tau_{\mathcal{O}}^\Delta$, defined by

$$\tau_{\mathcal{O}}^\Delta(f) := \tau_{\mathcal{O}}(\Delta f) = \int_{\mathcal{O}} \iota^* (\Delta f) \Omega_{\mathcal{O}},$$

is still a trace on $C^\infty_{\mathcal{O}}(g^*)[[\nu]]$.

5 **Positivity of traces**

If one replaces the formal parameter $\nu$ by a new formal parameter $\lambda$ such that $\nu = i \lambda$ and if one treats $\lambda$ as a real quantity, i.e. $\overline{\lambda} = \lambda$, then it is well-known that the complex conjugation of functions in $C^\infty(g^*)[[\lambda]]$ becomes a $^*$-involution for $*_{BCH}$. One has

$$\overline{f *_{BCH} g} = g *_{BCH} \overline{f}$$

(5.1)
for all \( f, g \in C^\infty(g)[[\lambda]] \). Such a star product is also called a Hermitian star-product, see e.g. [13] for a detailed discussion. In particular one can ask whether the traces for \( \ast_{\mathrm{BCH}} \) are positive linear functionals, i.e. satisfy 
\[
\tau(f \ast_{\mathrm{BCH}} g) \geq 0
\]
in the sense of formal power series, if the corresponding classical functional \( \tau_0 \) comes from a positive Borel measure on \( g \). In general a classically positive linear functional is no longer positive for a deformed product, see e.g. [11, Sect. 2] for a simple example and [13]. But sometimes one can deform the functional as well in order to make it positive again: in the case of star-products on symplectic manifolds this is always possible [13, Prop. 5.1]. Such deformations are called positive deformations. In our case we are faced with the question whether we can deform the traces \( \tau_O \) such that on one hand they are still traces and on the other hand they are positive.

One strategy could be the following: First prove that the trace can be deformed into a positive functional perhaps loosing the trace property. Secondly average over the group in order to obtain a \( g \)-invariant functional and hence a trace. This would require to have a compact group. However, we shall follow another idea giving some additional insight in the problem. Nevertheless we shall first ask the following question as a general problem in deformation quantization of Poisson manifolds:

**Question 5.1** Is every Hermitian star-product on a Poisson manifold a positive deformation?

We shall now consider the following more particular case. We assume the group \( G \) to be compact and \( \iota : \mathcal{O} \hookrightarrow g^\ast \) to be a regular coadjoint orbit. Then we want to find a positive trace for \( \ast_{\mathrm{BCH}} \) with zeroth order given by \( \tau_O \) as in (4.2). The construction is based on the following theorem which is of independent interest:

**Theorem 5.2** Let \( G \) be compact and let \( \iota : \mathcal{O} \hookrightarrow g^\ast \) be a regular coadjoint orbit. Then there exists a star-product \( \ast_o \) on the symplectic manifold \( \mathcal{O} \) and a series of \( g \)-invariant differential operators 
\[
S = \text{id} + \sum_{r=1}^\infty \lambda^r S_r
\]
on \( g^\ast \) such that the deformed restriction map 
\[
\iota^* = \iota^* \circ S : C^\infty(g^\ast)[[\lambda]] \to C^\infty(\mathcal{O})[[\lambda]]
\]
becomes a real surjective homomorphism of star-products, i.e. 
\[
\iota^* f \ast_o \iota^* g = \iota^*(f \ast_{\mathrm{BCH}} g) \quad \text{and} \quad (\iota^* f)^* = \iota^* f^*
\]
for all \( f, g \in C^\infty(g^\ast)[[\lambda]] \). Hence \( \ast_o \) becomes a Hermitian deformation.

One can view this theorem as a certain ‘deformed tangentiality property’ of the star product \( \ast_{\mathrm{BCH}} \): Though \( \ast_{\mathrm{BCH}} \) is not tangential, i.e. restricts to all orbits, for a particular orbit it can be arranged such that it restricts by deforming the restriction map, see [13] for a more detailed discussion.

From this theorem and [11, Lem. 2] we immediately obtain a positive trace deforming \( \tau_O \):

**Corollary 5.3** Let \( G \) be compact and \( \iota : \mathcal{O} \hookrightarrow g^\ast \) a regular orbit with deformed restriction map \( \iota^* \) as in (5.2). Then the functional 
\[
\tau_O(f) := \int_\mathcal{O} \iota^* f \, \Omega_O
\]
is a positive trace with classical limit \( \tau_O \). In particular, \( \ast_o \) is strongly closed.

Thus it remains to prove Theorem 5.2. We shall use here arguments from phase space reduction of star-products via the BRST formalism as discussed in detail in [16]. In order to make this article self-contained we shall recall the basic steps of [16] adapted to the case of Poisson manifolds.
Proof of Theorem 5.2: Since $\mathcal{O}$ is assumed to be a regular orbit there are real-valued Casimir polynomials $J_1, \ldots, J_k \in \text{Pol}^*(\mathfrak{g}^*)$ such that $\mathcal{O}$ can be written as level surface $\mathcal{O} = J^{-1}(\{0\})$ for the map $J = (J_1, \ldots, J_k): \mathfrak{g}^* \to \mathbb{R}^k$, where $0$ is a regular value. Since the components of $J$ commute with respect to the Poisson bracket this can be viewed as a moment map $J: \mathfrak{g}^* \to \mathfrak{t}^*$ where $\mathfrak{t}^*$ is the dual of the $k$-dimensional Abelian Lie algebra. Moreover, the $J$’s are in the Poisson center whence the corresponding torus action is trivial.

Since the differential operators $S_r$ will only be needed near $\mathcal{O}$ it will be sufficient to construct them in a tubular neighborhood around $\mathcal{O}$. In fact, a globalization beyond is also easily obtained, see [6, Lem. 6]. As $0$ is a regular value of $J$ we can use $J$ for the transversal coordinates and find a $G$-invariant tubular neighbourhood $U$ of $\mathcal{O}$. On $U$ we can define the following maps: First we need a prolongation map $\text{prol}: C^\infty(\mathcal{O}) \hookrightarrow C^\infty(U)$ given by

$$ (\text{prol}\phi)(a, \mu) = \phi(a), \quad (5.5) $$

where $a \in \mathcal{O}$ and $\mu \in \mathfrak{t}^*$ is the transversal coordinate in $U$. Next we consider $\bigwedge^*(t) \otimes C^\infty(\mathfrak{g}^*)$ and define the Koszul coboundary operator $\partial$ by the (left-)insertion of $J$, i.e. $\partial(t \otimes f) = \sum_{i} i(e^t) t \otimes J_i f$, where $J = \sum_{i} e^t J_i$. Clearly $\partial$ is $G$-invariant with respect to the $G$ action $g^* (t \otimes f) = t \otimes g^* f$ and $\partial^2 = 0$. We shall denote the homogeneous components of $\partial$ by $\partial_l : \bigwedge^l(t) \otimes C^\infty(\mathfrak{g}^*) \to \bigwedge^{l+1}(t) \otimes C^\infty(\mathfrak{g}^*)$ for $l \geq 1$. In the case $l = 0$ we set $\partial_0 = 0$.

Finally, we define the chain homotopy $h$ on $\bigwedge^*(t) \otimes C^\infty(U)$ by

$$ h(t \otimes f)(a, \mu) = \sum_{l=1}^{k} e_l \wedge t \otimes \int_{0}^{1} \frac{\partial f}{\partial \mu_l}(a, s \mu)s^k ds, \quad (5.6) $$

and denote the corresponding homogeneous components by $h_l$. For convenience we set $h_{-1} = \text{prol}$. Then $h$ is obviously $G$-invariant and it is indeed a chain homotopy for $\partial$, i.e. for all $l = 0, \ldots, k$ we have

$$ h_{l-1}\partial_l + \partial_{l+1}h_l = \text{id}_{\bigwedge^l(t) \otimes C^\infty(U)}. \quad (5.7) $$

Moreover, one has the obvious identities

$$ \iota^* \text{prol} = \text{id}_{C^\infty(\mathcal{O})}, \quad \text{and} \quad h_0\text{prol} = 0. \quad (5.8) $$

In a next step we quantize the above chain complex and it’s homotopy. The first easy observation is that the star-product $*_\text{BCH}$ is strongly $\iota$-invariant, i.e. the components of $J$ are in the center of $*_\text{BCH}$, too. Thus we can define a deformed Koszul operator $\partial$ on the space $(\bigwedge^*(t) \otimes C^\infty(\mathfrak{g}^*))[\lambda]$ by

$$ \partial(t \otimes f) = \sum_{i} i(e^t) t \otimes f *_{\text{BCH}} J_i. \quad (5.9) $$

Then we still have $\partial^2 = 0$ as well as $\partial(t \otimes f) = \partial(t \otimes f)$ since the $J_i$ commute and are real. Moreover, $\partial$ is still $G$-invariant. In a next step one constructs the deformations of $h$ and $\iota^*$ as follows. We define $h_{-1} = \text{prol}$ without deformation and set

$$ \partial_0 := \iota^* := \iota^*(\text{id} - (\partial_1 - \partial_1) h_0)^{-1} \text{ and } h_l := h_l (h_{l-1}\partial_l + \partial_{l+1}h_l)^{-1}. \quad (5.10) $$

Clearly the used inverse operators exist as formal power series thanks to (5.7). The proof of the following lemma is completely analogously to the proofs of [6, Prop. 25 and 26]. The $G$-invariance is obvious.

Lemma 5.4 The operators $\iota^*$ and $h$ are $G$-invariant and fulfill the relations

$$ h_{l-1}\partial_l + \partial_{l+1}h_l = \text{id}_{\bigwedge^l(t) \otimes C^\infty(U)[[\lambda]]} \quad (5.11) $$

as well as

$$ \iota^* \partial_1 = 0 \quad \text{and} \quad \iota^* \text{prol} = \text{id}_{C^\infty(\mathcal{O})[[\lambda]]}. \quad (5.12) $$

Having the deformed restriction map and the chain homotopy it is quite easy to characterize the ideal generated by the ‘constraints’ $J$:
Lemma 5.5 Let $\mathcal{J}(J)$ be the (automatically two-sided) ideal generated by $J_1, \ldots, J_k$. Then the map $\iota^*: C^\infty(U)[[\lambda]] \to C^\infty(\mathcal{O})[[\lambda]]$ is surjective and

$$\ker \iota^* = \text{im} \partial_1 = \mathcal{J}(J).$$

Thus we can simply define $*_\omega$ by (5.3) which gives a well-defined star-product on the quotient. It is an easy computation that the first order commutator of $*_\omega$ gives indeed the desired Poisson bracket. Moreover, since the $J$’s are real the ideal generated by them is automatically a *-ideal. Since $h_0$ as well as $\partial$ and $\partial_1$ are real operators, it follows that $\iota^*$ is real, too.

Remark 5.6 It seems that the above statement is not the most general one can obtain: There are certainly more general orbits and also non-compact groups where one can find such deformed restriction maps. We leave this as an open question for future projects.

6 GNS representation of the positive traces

Throughout this section we shall assume that $G$ is compact and $\iota: \mathcal{O} \hookrightarrow g^*$ is a regular orbit. Then we shall investigate the GNS representation of the positive trace $\tau_\mathcal{O}$ as constructed in the last section.

Let us briefly recall the basic steps of the GNS construction, see [1]. Having a *-algebra $\mathcal{A}$ over $\mathbb{C}[[\lambda]]$ with a positive linear functional $\omega: \mathcal{A} \to \mathbb{C}[[\lambda]]$ one finds that $\mathfrak{J}_\omega = \{ A \in \mathcal{A} | \omega(A^*A) = 0 \}$ is a left ideal of $\mathcal{A}$, the so-called Gel’fand ideal of $\omega$. Then $\mathfrak{J}_\omega := \mathcal{A}/\mathfrak{J}_\omega$ becomes a pre-Hilbert space over $\mathbb{C}[[\lambda]]$ via $\langle \psi_A, \psi_B \rangle_\omega := \omega(A^*B)$, where $\psi_A \in \mathfrak{J}_\omega$ denotes the equivalence class of $A$. Finally, the left representation $\pi_\omega(A)\psi_B = \psi_{AB}$ of $\mathcal{A}$ on $\mathfrak{J}_\omega$ turns out to be a *-representation, i.e. one has $\langle \psi_B, \pi_\omega(A)\psi_C \rangle_\omega = \langle \pi_\omega(A)\psi_B, \psi_C \rangle_\omega$.

According to Theorem 5.2 we have in our case a surjective *-homomorphism

$$\iota^*: C^\infty(g^*)[[\lambda]] \to C^\infty(\mathcal{O})[[\lambda]]$$

and a positive linear functional $\tau_\mathcal{O}$ which is the pull back of a positive linear functional on $C^\infty(\mathcal{O})[[\lambda]]$ under $\iota^*$, namely the trace $\text{tr}_\mathcal{O}$ on $\mathcal{O}$. Thus we can use the functoriality properties of the GNS construction, see [1], Prop. 5.1 and Cor. 5.2 in order to relate the GNS construction for $\tau_\mathcal{O}$ with the one for $\text{tr}_\mathcal{O}$, which is well-known, see [2], Sect. 5] and [1], Lem. 4.3. Since $\text{tr}_\mathcal{O}$ is a faithful functional the GNS representation of $C^\infty(\mathcal{O})[[\lambda]]$ with respect to $\text{tr}_\mathcal{O}$ is simply given by left multiplication $L$ with respect to $*_\omega$, where $\mathfrak{J}_{\text{tr}_\mathcal{O}} = C^\infty(\mathcal{O})[[\lambda]]$. Thus we arrive at the following theorem which can also be checked directly:

Theorem 6.1 Let $G$ be compact, $\iota: \mathcal{O} \hookrightarrow g^*$ a regular orbit, and $\tau_\mathcal{O} = \text{tr}_\mathcal{O} \circ \iota^*$ the positive trace as in (6.1).

i.) $\text{supp} \tau_\mathcal{O} = \iota(\mathcal{O})$.

ii.) The Gel’fand ideal $\mathfrak{J}_{\tau_\mathcal{O}}$ of $\tau_\mathcal{O}$ coincides with $\ker \iota^*$.
iii.) The GNS pre-Hilbert space $\mathcal{H}_\tau_\theta$ is unitarily isomorphic to $C^\infty(\mathcal{O})[[\lambda]]$ endowed with the inner product $\langle \phi, \chi \rangle_\mathcal{O} := \text{tr}_\mathcal{O}(\overline{\phi} \star_\theta \chi)$ via

$$U : \mathcal{H}_\tau_\theta \ni \psi_f \mapsto \psi \in C^\infty(\mathcal{O})[[\lambda]]$$

with inverse $U^{-1} : \phi \mapsto \psi_{\text{prop}}$.

iv.) For the GNS representation $\pi_\mathcal{O}$ one obtains

$$\pi_\mathcal{O}(f)\phi := U\pi_\mathcal{O}(f)U^{-1}\phi = \psi^{\dagger}(f \ast_{\text{BCH}} \text{prol}\phi) = \mathcal{L}_{\psi^{\dagger}}\phi.$$

Since the group $G$ acts on $\mathcal{O}$ and since all relevant maps are $G$-invariant/equivariant we arrive at the following $G$-invariance of the representation. This can be checked either directly or follows again from [4, Prop. 5.1 and Cor. 5.2].

**Lemma 6.2** The GNS representation $\pi_\mathcal{O}$ is $G$-equivariant in the sense that

$$\pi_\mathcal{O}(g^{\dagger}f)g^{\dagger}\phi = g^{\dagger}(\pi_\mathcal{O}(f)\phi)$$

for all $\phi \in C^\infty(\mathcal{O})[[\lambda]]$, $f \in C^\infty(\mathfrak{g}^*)[[\lambda]]$ and $g \in G$. Moreover, the $G$-representation on $C^\infty(\mathcal{O})[[\lambda]]$ is unitary.

Let us finally mention a few properties of the commutant of $\pi_\mathcal{O}$ and the ‘baby-version’ of the Tomita-Takesaki theory arising from this representation. The following statements follow almost directly from the considerations in [38, Sect. 7]. We consider the anti-linear map

$$J : \phi \mapsto \overline{\phi},$$

where $\phi \in C^\infty(\mathcal{O})[[\lambda]]$, which is clearly anti-unitary with respect to the inner product $\langle \cdot, \cdot \rangle_\mathcal{O}$ and involutive. This map plays the role of the modular conjugation. The modular operator $\Delta$ is just the identity map since in our case the linear functional is a trace, i.e. a KMS functional for inverse temperature $\beta = 0$. Then we can characterize the commutant of the representation $\pi_\mathcal{O}$ as follows:

**Proposition 6.3** For $f \in C^\infty(\mathfrak{g}^*)[[\lambda]]$ we denote by $R_{\psi^{\dagger}}f$ the right multiplication with $\psi^{\dagger}f$ with respect to the star-product $\ast_\mathcal{O}$. Then the map

$$\pi_\mathcal{O}(f) = \mathcal{L}_{\psi^{\dagger}}f \mapsto J\mathcal{L}_{\psi^{\dagger}}fJ = R_{\psi^{\dagger}}\mathcal{T}$$

is an anti-linear bijection onto the commutant $\pi_\mathcal{O}'$ of $\pi_\mathcal{O}$.

Note that in this particularly simple case the modular one-parameter group $U_t$ is just the identity $U_t = \text{id}_{C^\infty(\mathcal{O})[[\lambda]]}$, since we have a trace. More generally, one could also consider KMS functionals of the form $f \mapsto \pi_\mathcal{O}(\text{Exp}(-\beta H) \ast_{\text{BCH}} f)$ where $H \in C^\infty(\mathfrak{g}^*)[[\lambda]]$ and Exp denotes the star exponential with respect to $\ast_{\text{BCH}}$ and $\beta \in \mathbb{R}$ is the ‘inverse temperature’.

From the above proposition we immediately have the following result on the relation between the $\mathfrak{g}$-representations on $C^\infty(\mathcal{O})[[\lambda]]$ arising from the GNS construction.

**Lemma 6.4** For $x, y \in \mathfrak{g}$ we have

$$\pi_\mathcal{O}(\hat{x})\pi_\mathcal{O}(\hat{y}) - \pi_\mathcal{O}(\hat{y})\pi_\mathcal{O}(\hat{x}) = i\lambda \pi_\mathcal{O}([x, y])$$

$$R_{\psi^{\dagger}}\hat{x}R_{\psi^{\dagger}}\hat{y} - R_{\psi^{\dagger}}\hat{y}R_{\psi^{\dagger}}\hat{x} = -i\lambda R_{\psi^{\dagger}}[\hat{x}, \hat{y}]$$

and

$$\pi_\mathcal{O}(\hat{x}) - R_{\psi^{\dagger}}\hat{x} = i\lambda \mathcal{L}_{x_\mathcal{O}},$$

where $\mathcal{L}_{x_\mathcal{O}}$ denotes the Lie derivative in direction of the fundamental vector field of $x$. 

11
7 Traces for deformations via group actions

Let us now describe a quite general mechanism for constructing deformations and traces via group actions. We first consider the algebraic part of the construction. Let $G$ be a group and denote the right translations by $R_g: h \mapsto hg$, where $g, h \in G$. The left translations are denoted by $L_g$, respectively. Moreover, let $A_G \subseteq \text{Fun}(G)$ be a sub-algebra of the complex-valued functions on $G$, closed under complex conjugation. We require $R_g^*A_G \subseteq A_G$ for all $g \in G$. Then an associative formal deformation $(A_G[[\lambda]], \ast_G)$ of $A_G$ is called (right) universal deformation if it is right-invariant, i.e.

$$R_g^*(f_1 \ast_G f_2) = R_g^*f_1 \ast_G R_g^*f_2 \quad (7.1)$$

for all $g \in G$ and $f_1, f_2 \in A_G[[\lambda]]$. Thus the right translations act as automorphisms of $\ast_G$. In the sequel we shall always assume that $1 \in A_G$ and $1 \ast_G f = f = f \ast_G 1$.

Remark 7.1 If $G$ is a Lie group and $A_G$ are all smooth functions on $G$ then the existence of a right-invariant deformation gives quite strong conditions on $G$. However, in typical examples one may only deform a smaller class of functions. For instance the data of a $G$-invariant star product on a homogeneous symplectic space $G \xrightarrow{\pi} H \backslash G$ determines a right deformation of $A_G := \pi^*C^\infty(H \backslash G)$. In the extreme case where $H = \{e\}$, the pair $(A_G, \ast_G)$ becomes a star product algebra $(C^\infty(G[[\lambda]], \ast_\lambda)$. The Poisson structure on $G$ associated to the first order term of $\ast_\lambda$ is then right-invariant. Its characteristic distribution (generated by Hamiltonian vector fields)—being integrable and right-invariant—determines a Lie subalgebra $S$ of $g = \text{Lie}(G)$ endowed with a non-degenerate Chevalley 2-cocycle $\Omega$ with respect to the trivial representation of $S$ on $\mathbb{R}$. This type of Lie algebras $(S, \Omega)$ (or rather their associated Lie groups) has been studied by Lichnerowicz et al.

When unimodular such a Lie algebra is solvable [30].

Now consider a set $X$ with a left action $\tau: G \times X \to X$ of $G$. For abbreviation we shall sometimes write $g.x$ instead of $\tau(g, x)$. We shall use the universal deformation $\ast_G$ in order to induce a deformation of a certain sub-algebra of $\text{Fun}(X)$. First we define $\alpha^x: \text{Fun}(X) \to \text{Fun}(G)$ by

$$(\alpha^x f)(g) = (\tau^*_g f)(x) \quad (7.2)$$

for $x \in X$ and $g \in G$. Having specified $A_G$ we define the space

$$A_X = \{ f \in \text{Fun}(X) \mid \alpha^x f \in A_G \quad \text{for all} \quad x \in X \}, \quad (7.3)$$

which is clearly a sub-algebra of $\text{Fun}(X)$ stable under complex conjugation. Let us remark that $A_X$ contains at least those functions on $X$ which are constant along the orbits of $\tau$. Indeed, let $f \in \text{Fun}(X)$ satisfy $f(g.x) = f(x)$ for all $x \in X$ and $g \in G$. Then $(\alpha^x f)(g) = f(g.x) = f(x)$ is constant (not depending on $g$).

The deformation $\ast_G$ induces canonically an associative deformation $\ast_X$ of $A_X$, thereby justifying the name ‘universal deformation’. Indeed, define

$$(f_1 \ast_X f_2)(x) = (\alpha^x f_1 \ast_G \alpha^x f_2)(e), \quad (7.4)$$

where $e \in G$ denotes the unit element. Then we have the following proposition:

Proposition 7.2 Let $(A_G[[\lambda]], \ast_G)$ be a universal deformation and $(A_X[[\lambda]], \ast_X)$ as above.
i.) Then \((A_\lambda, \star_X)\) is an associative formal deformation of \(A_X\) which is Hermitian if \(\star_G\) is Hermitian. Moreover, \(\alpha^\star : (A_\lambda, \star_X) \to (A_G, \star_G)\) is a homomorphism of associative algebras.

ii.) If \(f_1\) is constant on some orbit \(G.x_0\) then

\[
(f_1 \star_X f_2)(g.x_0) = f_1(g.x_0)f_2(g.x_0) = (f_2 \star_X f_1)(g.x_0)
\]

(7.5)

for all functions \(f_2 \in A_\lambda\). In particular, the \(\star_X\)-product with a function, which is constant along all orbits, is the undeformed product. Thus \(\star_X\) is ‘tangential’ to the orbits in a very strong sense.

Proof: Let us first recall a few basic properties of \(\alpha^\star\), \(\tau\), \(R\), and \(L\). The following relations are straightforward computations:

\[
R_g^* \alpha^\star = \alpha^{\star_R} \quad \text{and} \quad L_g^* \alpha^\star = \alpha^\star \tau_g^*.
\]

(7.6)

Using the right invariance of \(\star_G\) and the above rules we find the following relation

\[
\alpha^\star(f_1 \star_X f_2) = \alpha^\star f_1 \star_G \alpha^\star f_2
\]

(7.7)

for \(f_1, f_2 \in A_\lambda\). This implies on one hand that \(A_\lambda\) is indeed closed under the multiplication law \(\star_X\). On the other hand it follows that \(\alpha^\star\) is a homomorphism. With (7.7) the associativity of \(\star_X\) is a straightforward computation. Finally, if \(\star_G\) is Hermitian then \(\star_X\) is Hermitian, too, since all involved maps are real, i.e. commute with complex conjugation. For the second part one computes

\[
(f_1 \star_X f_2)(g.x_0) = (\alpha^{\tau_0} f_1 \star_G \alpha^{\tau_0} f_2)(g).
\]

(7.8)

Now \(\alpha^{\tau_0} f_1\) is constant whence the \(\star_G\)-product is the pointwise product. Thus the claim easily follows. If this holds even for all orbits and not just for \(G.x_0\) then the \(\star_X\)-product with \(f_1\) is the pointwise product globally.

\[\square\]

Remark 7.3 From (7.3) we conclude that, heuristically speaking, the deformation \(\star_X\) becomes more non-trivial the larger the orbits of \(\tau\) are.

Remark 7.4 Given a right universal deformation \((A_G, \star_R)\), one gets a left universal deformation \((A_G, \star_L)\) via the formula

\[
a \star_L b = \iota^* (\iota^* a \star_R \iota^* b)
\]

(7.9)

provided \(A_G\) is a bi-invariant subspace. Here \(\iota : G \to G\) denotes the inversion map \(g \to g^{-1}\). Starting with a left invariant deformation \((A_G, \star_G)\) of \(G\) and an action \(\tau : G \times X \to X\), the associated deformation of \(A_X\) is then defined by the formula

\[
(f_1 \star f_2)(x) = (\iota^* \alpha^\star f_1 \star_G \iota^* \alpha^\star f_2)(e).
\]

(7.10)

In some interesting cases, in particular in the Abelian case, the universal deformation \(\star_G\) is also left invariant, i.e. the left translations \(L_g^*\) acts as automorphisms of \(\star_G\), too. In this situation the induced deformation \(\star_X\) is invariant under \(\tau_g^*\):

Lemma 7.5 Let \(A_G\) be in addition left invariant and let \(\star_G\) be a bi-invariant universal deformation. Then \(A_X\) is invariant under \(\tau_g^*\) for all \(g \in G\) and

\[
\tau_g^*(f_1 \star_X f_2) = \tau_g^* f_1 \star_X \tau_g^* f_2.
\]

(7.11)
Our main interest in the universal deformations comes from the following simple observation:

**Theorem 7.6** Let \((A_G[[\lambda]], \star_G)\) be a right universal deformation and let \(\text{tr}_G : A_G[[\lambda]] \to \mathbb{C}[[\lambda]]\) be a trace with respect to \(\star_G\). Let \(\Phi : \text{Fun}(X)[[\lambda]] \to \mathbb{C}[[\lambda]]\) be an arbitrary \(\mathbb{C}[[\lambda]]\)-linear functional. Then \(\text{tr}_\Phi : A_X[[\lambda]] \to \mathbb{C}[[\lambda]]\) defined by

\[
\text{tr}_\Phi(f) = \Phi(x \mapsto \text{tr}_G(\alpha^x f))
\]

is a trace with respect to \(\star_X\).

**Proof:** This follows directly from the homomorphism property of \(\alpha^x\) and the trace property of \(\text{tr}_G\). \(\square\)

In particular the trace \(\text{tr}_G\) combined with the evaluation functionals at some point \(x \in X\)

\[
\text{tr}_x : f \mapsto \text{tr}_G(\alpha^x f)
\]

yields a trace for \(\star_X\). Thus the only difficult task is to find traces for \(\star_G\).

As a last remark we shall discuss the positivity of the traces \(\text{tr}_\Phi\). We assume that \(\text{tr}_G\) is a positive trace whence \(\text{tr}_G(f \star_G f) \geq 0\) in the sense of formal power series for all \(f \in A_G[[\lambda]]\).

**Lemma 7.7** Assume \(\text{tr}_G\) is a positive trace and \(\Phi\) takes non-negative values on non-negative valued functions on \(X\). Then \(\text{tr}_\Phi\) is positive. In particular \(\text{tr}_x\) is always positive.

**Remark 7.8** The above construction has the big advantage that it can be transferred to the framework of topological deformations instead of formal deformations. This has indeed been done by Rieffel \[33\] in a \(C^*\)-algebraic framework for actions of \(\mathbb{R}^d\). For a class of non-abelian groups this has been done in \[6\].

Let us finally mention two examples. The first one is the well-known example of the Weyl-Moyal product for \(\mathbb{R}^{2n}\) and the second is obtained as the asymptotic version of \[6\] for rank one Iwasawa subgroups of \(SU(1,n)\).

**Example 7.9** Let \(\star_W\) be the Weyl-Moyal star product on \(\mathbb{R}^{2n}\), explicitly given by

\[
f \star_W g = \mu \circ e^{\frac{X}{2n} \sum_k (\partial_q^k \otimes \partial_p^k - \partial_p^k \otimes \partial_q^k)} f \otimes g,
\]

where \(\mu(f \otimes g) = fg\) is the pointwise product and \(q^1, \ldots, p_n\) are the canonical Darboux coordinates on \(\mathbb{R}^{2n}\). Clearly \(\star_W\) is invariant under translations whence it is a bi-invariant universal deformation of \(C^\infty(\mathbb{R}^{2n})[[\lambda]]\). Moreover, it is well-known that \(\star_W\) is strongly closed, whence the integration with respect to the Liouville measure provides a trace, which is positive. Thus one can apply the above general results to this situation.

**Example 7.10** This example is the asymptotic version of \[6\]. The groups we consider are Iwasawa subgroups \(G = AN\) of \(SU(1,n)\), where \(SU(1,n) = ANK\) is an Iwasawa decomposition. One has the obvious \(G\)-equivariant diffeomorphism \(G \to SU(1,n)/K\) (here \(K = U(n)\)). The group \(G\) therefore inherits a left-invariant symplectic (Kähler) structure coming from the one on the rank one Hermitian symmetric space \(SU(1,n)/U(n)\). The symplectic group may then be described as follows. As a manifold, one has

\[
G = \mathbb{R} \times \mathbb{R}^{2n} \times \mathbb{R}.
\]
In these coordinates the group multiplication law reads

\[ L_{(a,x,z)}(a',x',z') = \left( a + a', e^{-a' x} x' + e^{-2a' z} + z' + \frac{1}{2} \Omega(x, x')e^{-a'} \right), \] (7.16)

where \( \Omega \) is a constant symplectic structure on the vector space \( \mathbb{R}^{2n} \). The 2-form

\[ \omega = \Omega + da \wedge dz \] (7.17)

then defines a left-invariant symplectic structure on \( G \). The universal deformation \( \ast_{BM} \) we are looking for is a star product for this symplectic structure. Since on \( \mathbb{R}^{2n+2} \) all symplectic star products are equivalent, it will be sufficient to describe \( \ast_{BM} \) be means of an equivalence transformation \( T = \text{id} + \sum_{r=1}^{\infty} \lambda^r T_r \) relating \( \ast_{BM} \) and \( \ast_{\text{Weyl}} \). In [6] an explicit integral formula for \( T \) has been given, which is defined on the Schwartz space \( S(\mathbb{R}^{2n+2}) \). It allows for an asymptotic expansion in \( \hbar \) and gives indeed the desired equivalence transformation \( T \). Then \( \ast_{BM} \) defined by

\[ f \ast_{BM} g = T^{-1}(Tf \ast_{\text{Weyl}} Tg) \] (7.18)

is a left-invariant universal deformation of \( G \) and again we can use this to apply the above results on universal deformations. Moreover, since \( \ast_{\text{Weyl}} \) is strongly closed, the functional

\[ \text{tr}^G(f) := \int_G T(f) \omega^{n+1} \] (7.19)

defines a trace functional for \( \ast_{BM} \) on \( C_0^\infty(G)[[\lambda]] \). This is again positive since that \( T \) is real i.e. \( \overline{Tf} = T\overline{f} \).

In what follows we give a precise description of the star product \( \ast_{BM} \) in the two dimensional case i.e. on the group \( ax + b \). The higher dimensional case is similar but more intricate. The non-formal deformed product in the \( ax + b \) case is obtained by transforming Weyl’s product on \( (\mathbb{R}^2, da \wedge d\ell) \) under the equivalence

\[ T = F^{-1} \circ \phi^*_\hbar \circ F \] (7.20)

where

\[ Fu(a, \alpha) = \int e^{-ia\ell} u(a, \ell) \, d\ell \quad \text{with} \quad u \in S(\mathbb{R}^2) \] (7.21)

is the partial Fourier transform in the second variable and where \( \phi_\hbar : \mathbb{R}^2 \to \mathbb{R}^2 \) is the one-parameter family of diffeomorphisms given by

\[ \phi_\hbar(a, \alpha) = (a, \frac{1}{\hbar} \sinh(\alpha h)) \quad (h \in \mathbb{R}). \] (7.22)

One has

\[ Tu(a, \ell) = c \int e^{i\ell q} e^{i\frac{1}{\hbar} \sinh(\alpha q)} u(a, q) \, dq \, d\alpha = c \int e^{i\alpha(\ell - q)} e^{-iv_\hbar(\alpha) q} u(a, q) \, dq \, d\alpha \] (7.23)

with

\[ v_\hbar(\alpha) = \sum_{k \geq 1} \frac{\hbar^{2k} \alpha^{2k+1}}{(2k+1)!}. \] (7.24)
Setting $p = \hbar \alpha$, one gets

\[ Tu(a, \ell) = \frac{\hbar}{i} \int e^{i\frac{\hbar}{\alpha} p(\ell - q)} e^{i\frac{\hbar}{\alpha} \psi_1(\alpha \ell)} u(a, q) \, dq \, dp \quad (7.25) \]

which precisely coincides with

\[ \text{id} \otimes O_{\hbar,1}(e^{i\frac{\hbar}{\alpha} \psi_1(\alpha \ell)}) u(a, \ell) \quad (7.26) \]

where $O_{\hbar,1}f(p,q)$ denotes the anti-normally ordered quantization of the function $f(q,p)$. Recall that the $\kappa$-ordered pseudodifferential quantization rule on $(\mathbb{R}^2, dq \wedge dp)$ is defined (at the level of test functions) by $O_{\hbar,\kappa} : \mathcal{D}(\mathbb{R}^2) \longrightarrow \text{End}(L^2(\mathbb{R}))$ with

\[ O_{\hbar,\kappa}(f) \varphi(q) = \frac{\hbar}{i} \int e^{i\frac{\hbar}{\alpha} p(q - \xi)} f(\kappa \xi + (1 - \kappa) q, p) \varphi(\xi) \, d\xi \, dp \quad (\kappa \in [0,1]). \quad (7.27) \]

The explicit asymptotic expansion formula for $O_{\hbar,\kappa}(f)$ is well known, see e.g. [39, Sect. 1.2, p. 231 and Eq. (58), p. 258]. It yields an expression for the equivalence $T$ at the formal level which we write, with natural delicacy, as

\[ T = \text{id} \otimes \exp \left( \frac{i}{\hbar} \psi_1 \left( \frac{\hbar}{i} \partial_\ell \right) \ell \right), \quad (7.28) \]

where the operator $T(\ell) := \exp \left( \frac{i}{\hbar} \psi_1 \left( \frac{\hbar}{i} \partial_\ell \right) \ell \right)$ is to be understood as anti-normally ordered ($\kappa = 1$). Observe the reality of the equivalence, which may be directly checked using the fact that the function $\psi_1$ is odd. Moreover, for every right-invariant vector field $X$ on $G = ax + b$, one checks [6] that $T \circ X \circ T^{-1}$ is an inner derivation of the Moyal-Weyl product $*_{\text{Weyl}}$. In other words, the star product $*_{\text{BM}}$ is left-invariant on $G$.

References

[1] Arnal, D., Ben Amar, N.: Kontsevich’s Wheels and Invariant Polynomial Functions on the Dual of Lie Algebras. Lett. Math. Phys. 52 (2000), 291–300.

[2] Arnal, D., Ben Amar, N., Masmoudi, M.: Cohomology of Good Graphs and Kontsevich Linear Star Products. Lett. Math. Phys. 48 (1999), 291–306.

[3] Basart, H., Flato, M., Lichnerowicz, A., Sternheimer, D.: Deformation Theory applied to Quantization and Statistical Mechanics. Lett. Math. Phys. 8 (1984), 483–394.

[4] Bayen, F., Flato, M., Fronsdal, C., Lichnerowicz, A., Sternheimer, D.: Deformation Theory and Quantization. Ann. Phys. 111 (1978), 61–151.

[5] Bertelson, M., Cahen, M., Gutt, S.: Equivalence of Star Products. Class. Quantum Grav. 14 (1997), A93–A107.

[6] Bieliavsky, P., Massar, M.: Strict Deformation Quantizations for Actions of a Class of Symplectic Lie Groups. Preprint [math.QA/0011144] (November 2000).

[7] Bordemann, M., Herbig, H.-C., Waldmann, S.: BRST Cohomology and Phase Space Reduction in Deformation Quantization. Commun. Math. Phys. 210 (2000), 107–144.
[8] Bordemann, M., Neumaier, N., Waldmann, S.: Homogeneous Fedosov Star Products on Cotangent Bundles I: Weyl and Standard Ordering with Differential Operator Representation. Commun. Math. Phys. 198 (1998), 363–396.

[9] Bordemann, M., Neumaier, N., Waldmann, S.: Homogeneous Fedosov star products on cotangent bundles II: GNS representations, the WKB expansion, traces, and applications. J. Geom. Phys. 29 (1999), 199–234.

[10] Bordemann, M., Römer, H., Waldmann, S.: A Remark on Formal KMS States in Deformation Quantization. Lett. Math. Phys. 45 (1998), 49–61.

[11] Bordemann, M., Waldmann, S.: Formal GNS Construction and States in Deformation Quantization. Commun. Math. Phys. 195 (1998), 549–583.

[12] Bursztyn, H., Waldmann, S.: Deformation Quantization of Hermitian Vector Bundles. Lett. Math. Phys. 53 (2000), 349–365.

[13] Bursztyn, H., Waldmann, S.: On Positive Deformations of ∗-Algebras. In: Dito, G., Sternheimer, D. (eds.): Conférence Moshé Flato 1999. Quantization, Deformations, and Symmetries, Mathematical Physics Studies no. 22, 69–80. Kluwer Academic Publishers, Dordrecht, Boston, London, 2000.

[14] Bursztyn, H., Waldmann, S.: Algebraic Rieffel Induction, Formal Morita Equivalence and Applications to Deformation Quantization. J. Geom. Phys. 37 (2001), 307–364.

[15] Cahen, M., Gutt, S., Rawnsley, J.: On Tangential Star Products for the Coadjoint Poisson Structure. Commun. Math. Phys. 180 (1996), 99–108.

[16] Connes, A.: Noncommutative Geometry. Academic Press, San Diego, New York, London, 1994.

[17] Connes, A., Flato, M., Sternheimer, D.: Closed Star Products and Cyclic Cohomology. Lett. Math. Phys. 24 (1992), 1–12.

[18] Dito, G.: Kontsevich Star Product on the Dual of a Lie Algebra. Lett. Math. Phys. 48 (1999), 307–322.

[19] Dito, G., Sternheimer, D.: Deformation Quantization: Genesis, Developments and Metamorphoses. To appear in the Proceedings of the meeting between mathematicians and theoretical physicists, Strasbourg, 2001. IRMA Lectures in Math. Theoret. Phys., vol. 1, Walter De Gruyter, Berlin 2002, pp. 9–54.

[20] Fedosov, B. V.: Deformation Quantization and Index Theory. Akademie Verlag, Berlin, 1996.

[21] Felder, G., Shoikhet, B.: Deformation Quantization with Traces. Lett. Math. Phys. 53 (2000), 75–86.

[22] Giaquinto, A., Zhang, J. J.: Bialgebra actions, twists, and universal deformation formulas. J. Pure Appl. Algebra 128.2 (1998), 133–152.

[23] Gutt, S.: An Explicit ∗-Product on the Cotangent Bundle of a Lie Group. Lett. Math. Phys. 7 (1983), 249–258.
[24] Gutt, S.: Variations on deformation quantization. In: Dito, G., Sternheimer, D. (eds.): Conference Moshé Flato 1999. Quantization, Deformations, and Symmetries, Mathematical Physics Studies no. 21, 217–254. Kluwer Academic Publishers, Dordrecht, Boston, London, 2000.

[25] Gutt, S., Rawnsley, J.: Equivalence of star products on a symplectic manifold; an introduction to Deligne’s Čech cohomology classes. J. Geom. Phys. 29 (1999), 347–392.

[26] Gutt, S., Rawnsley, J.: Traces for star products on symplectic manifolds. Preprint math.QA/0105089 (May 2001).

[27] Karabegov, A. V.: On the Canonical Normalization of a Trace Density of Deformation Quantization. Lett. Math. Phys. 45 (1998), 217–228.

[28] Kontsevich, M.: Deformation Quantization of Poisson Manifolds, I. Preprint math.q-alg/9709040 (September 1997).

[29] Kontsevich, M.: Operads and Motives in Deformation Quantization. Lett. Math. Phys. 48 (1999), 35–72.

[30] Lichnerowicz, A., Medina, A.: Groupes a structures symplectiques ou kahleriennes invariantes. C. R. Acad. Sci., Paris, Ser. I 306, No.3 (1988), 133–138.

[31] Nest, R., Tsygan, B.: Algebraic Index Theorem. Commun. Math. Phys. 172 (1995), 223–262.

[32] Omori, H., Maeda, Y., Yoshioka, A.: Weyl Manifolds and Deformation Quantization. Adv. Math. 85 (1991), 224–255.

[33] Rieffel, M. A.: Deformation quantization for actions of $\mathbb{R}^d$. Mem. Amer. Math. Soc. 106.506 (1993).

[34] Rosenberg, J.: Rigidity of K-theory under deformation quantization. Preprint math.q-alg/9607021 (July 1996).

[35] Sternheimer, D.: Deformation Quantization: Twenty Years After. In: Rembieliński, J. (ed.): Particles, Fields, and Gravitation. AIP Press, New York 1998.

[36] Stein, E.M.: Harmonic Analysis Real-Variable Methods, Orthogonality, & Oscillatory Integrals. Princeton Mathematical Series, Princeton University Press (1993).

[37] Tamarkin, D., Tsygan, B.: Cyclic Formality and Index Theorems. Lett. Math. Phys. 56 (2001), 85–97.

[38] Waldmann, S.: Locality in GNS Representations of Deformation Quantization. Commun. Math. Phys. 210 (2000), 467–495.

[39] Weinstein, A.: Deformation Quantization. Séminaire Bourbaki 46ème année 789 (1994).

[40] Weinstein, A.: The modular automorphism group of a Poisson manifold. J. Geom. Phys. 23 (1997), 379–394.
[41] Weinstein, A., Xu, P.: Hochschild cohomology and characteristic classes for star-products. In: Khovanskij, A., Varchenko, A., Vassiliev, V. (eds.): Geometry of differential equations. Dedicated to V. I. Arnold on the occasion of his 60th birthday, 177–194. American Mathematical Society, Providence, 1998.