PRODUCT STRUCTURES AND FRACTIONAL INTEGRATION ALONG CURVES IN THE SPACE

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Abstract. In this paper we establish $L^p$ boundedness ($1 < p < \infty$) for a double analytic family of fractional integrals $S^\gamma_z$, $\gamma, z \in \mathbb{C}$, when $\Re z = 0$. Our proof is based on product-type kernels arguments. More precisely, we prove that the convolution kernel of $S^\gamma_z$ is a product kernel on $\mathbb{R}^3$, adapted to the polynomial curve $x_1 \mapsto (x_1^n, x_2^n)$ (here $m, n \in \mathbb{N}$, $m \geq 1$, $n > m$).

1. Introduction. In this paper we start the study of $L^p - L^q$ boundedness properties for an analytic family of fractional integrals along curves in the space. More specifically, the family under attention is

\[
(S^\gamma_z f)(x_1, x_2, x_3) := C_z \int \int_{\mathbb{R}^2} \rho(u_1 - 1, u_2 - 1)^z \psi(u_1 - 1, u_2 - 1)
\]

\[
f(x_1 - t, x_2 - u_1 t^m, x_3 - u_2 t^n) \, du_1 \, du_2 \, |t|^{\gamma} \frac{dt}{t},
\]

where $\psi$ is a suitable bump function in $\mathbb{R}^2$ compactly supported near the origin, $\rho$ is the polynomial $\rho(u_1, u_2) := u_1^{2n} + u_2^{2m}$, $m \geq 1$, $n > m$, and $Q = \frac{1}{2m} + \frac{1}{2n}$ (the reason of the choice of $Q$ will be soon clarified). Here $f$ belongs to $C_c(\mathbb{R}^3)$ and $z, \gamma \in \mathbb{C}$, with $\Re \gamma > 0$. A similar problem in the plane was studied in [5] and [2].

In the space, this family arises as a generalization of the fractional integration operator of order $\gamma$ along the curve $t \mapsto (t, t^m, t^n)$. Indeed, for $z = 0$ and $\Re \gamma > 0$, $S^\gamma_0$ coincides with this operator.

We are interested, in particular, in the behaviour of the family for $\Re z = 0$ and $\Re \gamma = 0$. Recall that a necessary condition for $S^\gamma_z$ to be a bounded operator from $L^p(\mathbb{R}^3)$ to $L^q(\mathbb{R}^3)$ is that $\frac{1}{p} - \frac{1}{q} = \frac{\Re \gamma}{1 + m + n}$, so that in the case $\Re \gamma = 0$ boundedness is only possible from $L^p(\mathbb{R}^3)$ to $L^p(\mathbb{R}^3)$. This particular case is interesting, since it fills into the more general problem of the connection between the curvature properties of the singular support of a distribution and the $L^p$ boundedness properties of the corresponding convolution operator. Anyway, in spite of its apparent simplicity,
the case corresponding to $\Re z = 0$ and $\Re \gamma = 0$ turns out to be technically quite involved. We treat it here by means of the theory of product–type kernels; more precisely, we prove that for $\Re z = 0$ and $\Re \gamma = 0$ the operator $S_\gamma^z$ is bounded on $L^p(\mathbb{R}^3)$ for all $1 < p < \infty$, by showing that the convolution kernel of $S_\gamma^z$ is a product kernel adapted to a polynomial curve.

The subject of product kernels has been drawing increasing attention in the last thirty years. The first case which was considered is that of a convolution operator $Tf = K * f$ on $\mathbb{R}^n = \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, where $K(x, y) = K_1(x)K_2(y)$, $x \in \mathbb{R}^{n_1}$, $y \in \mathbb{R}^{n_2}$, and $K_1$ and $K_2$ are of Calderón-Zygmund type. In this case the $L^p$ boundedness of $T$ may be easily obtained by an argument that iterates the one-dimensional theory. In [3] Calderón-Zygmund singular integral operators with convolution kernels which cannot be decomposed into a product were studied. Such convolution kernels are usually called product kernels and are defined as tempered distributions, satisfying suitable differential inequalities and certain cancellation conditions, expressible in terms of normalized bump functions (we refer to [8] for more details).

Many applications of the product theory in $\mathbb{R}^n$ to operators arising in boundary value problems have been studied [8], [10]. Moreover, the euclidean spaces $\mathbb{R}^{n_j}$, $j = 1, 2$, have been replaced by nilpotent Lie groups [7], [8] and by subriemannian manifolds [9].

Recently the authors considered product kernels adapted to curves in the space $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ whose singularities are supported along a coordinate plane and a transversal curve of polynomial type.

We shall now briefly describe the kind of kernels we have in mind. For the sake of simplicity, fix $n = 3$ and denote an element in $\mathbb{R}^{3} = \mathbb{R} \times \mathbb{R}^2$ by the pair $(x_1, x)$, where $x = (x_2, x_3) \in \mathbb{R}^2$. On $\mathbb{R}$ we consider the usual dilations by $\delta > 0$, while on $\mathbb{R}^2$ we introduce the non isotropic dilation given by

$$
\delta o x = (\delta^{\frac{1}{m}} x_2, \delta^{\frac{1}{n}} x_3), \quad \delta > 0, \ m, n \in \mathbb{N}, \ m < n.
$$

(2)

The homogeneous dimension of $\mathbb{R}^2$ with respect to the dilations (2) is given by

$$
Q = \frac{1}{2n} + \frac{1}{2m}.
$$

(3)

We denote by $\rho(x) = x_2^{2n} + x_3^{2m}$ a smooth homogeneous norm on $\mathbb{R}^2$.

In this context the prototype of a product kernel in $\mathbb{R}^3$ is given by the distribution

$$
H(x_1, x) = C_\mu p.v. \frac{1}{x_1} \rho(x)^{-Q+i\mu}, \quad \mu \in \mathbb{R} \setminus \{0\}.
$$

(4)

Now consider the curve $\Gamma(x_1) = (x_1^m, x_1^n)$, $x_1 \in \mathbb{R}$, $m, n \in \mathbb{N}$, $2 \leq m < n$. A rather simple example of a product-type kernel in $\mathbb{R}^3$ whose singularities are supported on the coordinate plane $x_1 = 0$ and along the transversal curve $\Gamma$, is given by

$$
K(x_1, x) = C_\mu p.v. \frac{1}{x_1} \rho(x - \Gamma(x_1))^{-Q+i\mu}, \quad \mu \in \mathbb{R} \setminus \{0\}.
$$

(5)

More generally, in [1] we introduced the following class of product-type kernels.

**Definition 1.1.** Assume that $K_0$ is a product kernel on $\mathbb{R}^3$ and consider the curve $x = \Gamma(x_1)$ with $\Gamma(x_1) = (x_1^m, x_1^n)$, $x_1 \in \mathbb{R}$, $m, n \in \mathbb{N}$, $2 \leq m < n$. We define a distribution $K$ by

$$
\int K(x_1, x)f(x_1, x) \, dx_1 \, dx := \int K_0(x_1, x)f(x_1, x + \Gamma(x_1)) \, dx_1 \, dx
$$

(5)
for a Schwartz function $f$ on $\mathbb{R}^3$. $K$ will be called an adapted kernel.

Here with an abuse of notation we write pairings between distributions and test functions as integrals.

The main result in [1] was the following theorem.

**Theorem 1.2.** Let $K$ be the tempered distribution defined by the formula (5). Then the convolution operator $T : f \mapsto f * K$, initially defined on the Schwartz space $S(\mathbb{R}^3)$, extends to a bounded operator on $L^p(\mathbb{R}^3)$ for $1 < p < \infty$.

The $L^p$ bounds follow from the decomposition of the adapted kernel $K$ into the sum of a kernel $K_1$ with singularities concentrated on the coordinate plane $x_1 = 0$ and of a kernel $K_2$ singular along the curve $x = \Gamma(x_1)$, $x_1 \in \mathbb{R}$. Then the operator given by the convolution by $K_1$ is bounded on $L^p(\mathbb{R}^3)$, since the multiplier associated with $K_1$ is of Marcinkiewicz-type. The kernel $K_2$ is treated instead by means of analytic interpolation.

In order to study $S_2^\gamma$, it is essential to know the location of the singularities of the distribution

$$I^z(u_1, u_2) := (\rho(u_1, u_2))^{z-1}, \quad z \in \mathbb{C},$$

which may be considered as a non-isotropic version of the Riesz potentials. In particular, in [1] we proved that $I^z$ may be analytically continued to a meromorphic distribution-valued function of $z$ by means of the algebraic theory of the Bernstein-Sato polynomials. The basic properties of $I^z$ are recalled in Section 2.

In this paper we prove that for $\Re\gamma \leq 0$ and $\Re\gamma = 0$ the convolution kernel of $S_2^\gamma$ is a product kernel adapted to the curve $\Gamma(x_1)$. More precisely, we first prove that the distribution $H^{\mu,\nu}$, where $\mu, \nu \in \mathbb{R}$, $\mu \neq 0$, defined by

$$\langle H^{\mu,\nu}, f \rangle := \lim_{\epsilon \to 0^+} \int \rho(x_2, x_3)^{-(Q+i\mu)} \psi \left( \frac{x_2}{x_1^m}, \frac{x_3}{x_1^n} \right) |x_1|^{-1+\epsilon+i\nu} \text{sgn} x_1 f(x_1, x_2, x_3) dx_1 dx_2 dx_3$$

for every $f \in S(\mathbb{R}^3)$, is a product kernel in $\mathbb{R}^3$. Then it is straightforward to check that the convolution kernel of $S_2^\gamma$, for $\Re\gamma = 0$ and $\Re\gamma = 0$, coincides, for suitable values of $\mu$ and $\nu$, with the kernel $H^{\mu,\nu}$ adapted to the curve $\Gamma(x_1) = (x_2^\mu, x_3^\nu)$, $x_1 \in \mathbb{R}$. Thus, as a consequence of Theorem 1.2, we prove in Theorem 4.7 that the operator $S_2^\gamma$ is bounded on $L^p(\mathbb{R}^3)$, $1 < p < \infty$, for $\Re\gamma = 0$ and $\Re\gamma = 0$.

This paper is an intermediate step in the study of the $L^p - L^q$ boundedness of the family of operators $S_2^\gamma$. We hope to give a complete picture of the characteristic set of this double analytic family of operators (that is, of the set of all $(p, q, \Re\gamma)$ such that $S_2^\gamma$ is bounded from $L^p$ to $L^q$) in the near future.

2. **A family of Riesz-type kernels.** Consider the polynomial

$$\rho(u_1, u_2) := u_1^{2n} + u_2^{2m}, \quad (6)$$

with $m, n \in \mathbb{N}$, $m \geq 1$, $n > m$. Observe that $\rho$ is homogeneous with respect to the one-parameter family of non isotropic dilations given by

$$\delta \circ (u_1, u_2) := \left( \delta \frac{x_1}{u_1}, \delta \frac{x_2}{u_2} \right), \quad (7)$$

where $\delta > 0$.

Define now the distribution

$$I^z(u_1, u_2) := (\rho(u_1, u_2))^{z-1}, \quad (8)$$
where \( z \in \mathbb{C} \) and \( Q := \frac{1}{2n} + \frac{1}{2m} \) represents the homogeneous dimension of \( \mathbb{R}^2 \) with respect to the dilations defined by (7).

It is easy to prove that for \( \Re z > 0 \) \( \mathcal{I}^z \) is a well defined tempered and locally integrable distribution, and that \( \mathcal{I}^z \) is an analytic family of tempered distributions.

In [1] we proved, by means of the Bernstein-Sato polynomials, that \( \mathcal{I}^z \) admits a meromorphic extension, with poles in an at most countable set, consisting of rational negative points. For further details about the Bernstein-Sato polynomials, we refer the reader to [6] and [1]. Here we just recall the definition of this algebraic tool.

It is well-known in algebra that, given a non-zero polynomial \( p(u_1, u_2) \) with complex coefficients, there exist a non-zero polynomial \( b_p(s) \in \mathbb{C}[s] \) and a differential operator \( L(s) \) whose coefficients are polynomials in \( s, u_1, u_2 \), such that formally

\[
L(s) \left( p(u_1, u_2) \right)^{s+1} = b_p(s) \left( p(u_1, u_2) \right)^s \quad \text{for all } s \in \mathbb{C}.
\]

The set of all polynomials \( b_p(s) \in \mathbb{C}[s] \) satisfying this formal identity (for some operator \( L \)) is an ideal, and the unique monic generator of this ideal is called the Bernstein-Sato polynomial of \( p \).

In [1] we proved the following result, concerning the meromorphic continuation of \( \mathcal{I}^z \).

**Proposition 2.1.** \( \mathcal{I}^z \) may be analytically continued to a meromorphic distribution-valued function of \( z \), also denoted by \( \mathcal{I}^z \), with poles in a set \( A \), consisting of rational negative points. More precisely,

\[
A = \{ \zeta_{j,k} := Q + s_j - k : k \in \mathbb{N}, j = 1, \ldots, h \},
\]

where \( s_j, j = 1, \ldots, h \), denote the zeros of the Bernstein-Sato polynomial \( b_p \) in \((-Q-1, -Q)\), each listed as many times as its multiplicity. Each pole has order one, with the exception of the points \(-1 + Q - k, k \in \mathbb{N}, \) which have order two.

Set \( \zeta_j := \zeta_{j,0} \). Observe that \( \zeta_1 = 0 \) is a pole of order 1 for \( \mathcal{I}^z \).

Consider now the function \( G \), given by

\[
G(z) := \Gamma \left( z + 1 - Q \right) \cdot \prod_{j=2}^h \Gamma(z - \zeta_j).
\]

If \( S \) denotes the sphere

\[
S := \{(u_1, u_2) \in \mathbb{R}^2 : \rho(u_1, u_2) = 1\}
\]

with surface measure \( \sigma(S) \), set

\[
\mathcal{I}^z(u_1, u_2) := \frac{G(0) \left( u_1^{2n} + u_2^{2m} \right)^{z-Q}}{\sigma(S) \Gamma(z) G(z)}.
\]

In the sequel we will denote by \( \mathcal{S}(\mathbb{R}^s) \), \( s = 2, 3 \), the Schwartz space on \( \mathbb{R}^s \) endowed with a denumerable family of norms \( \| \cdot \|_{(N)} \) given by

\[
\| \Phi \|_{(N)} = \sum_{|\alpha| \leq N} \sup_{u \in \mathbb{R}^s} (1 + |u|)^N |\partial_\alpha^u \Phi(u)|.
\]

In the following, given a function \( g \) on \( \mathbb{R}^s \), we will denote by \( \partial_i g, i = 1, \ldots, s \), the derivative of \( g \) with respect to its \( i \)-th argument.

Moreover, we shall use the conventional notation

\[
\partial_u^\alpha = \frac{\partial^{\alpha_1}}{\partial u_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_s}}{\partial u_2^{\alpha_s}},
\]
with \( \alpha = (\alpha_1, \ldots, \alpha_s) \) s-tuple of natural numbers and \( |\alpha| = \alpha_1 + \cdots + \alpha_s \).

We shall need the following properties of \( I^z \), proved in [1].

**Proposition 2.2.** The distribution \( I^z \) satisfies

\[
I^0 = \delta_0.
\]

**Proposition 2.3.** \( I^z \) is a homogeneous tempered distribution of degree \(-Q+z\).

We recall that this means that for all \( \varphi \in \mathcal{S}(\mathbb{R}^2) \) the following equality is satisfied

\[
\langle I^z, \varphi \rangle = \delta^{-Q} (I^z, \varphi),
\]

where

\[
\varphi \delta(u_1, u_2) := \delta^{-Q} \varphi (\delta^{-1} \circ u) = \delta^{-Q} \varphi \left( \delta^{-\frac{1}{z}} u_1, \delta^{-\frac{1}{z}} u_2 \right).
\]

Thus the Fourier transform of the (tempered and homogeneous) distribution \( I^z \) is a well-defined distribution, homogeneous of degree \(-Q-z\). Finally, we recall the regularity and decay properties of \( \hat{I}^z \), exploited in Proposition 2.8 in [1].

**Proposition 2.4.** \( \hat{I}^z \) agrees with a function \( C^\infty(\mathbb{R}^2 \setminus \{(0,0)\}) \) away from \((0,0)\).

Moreover,

\[
|\hat{I}^z(\xi)| \leq C \rho(\xi)^{-Re z},
\]

for all \( \xi \in \mathbb{R}^2 \setminus \{(0,0)\} \).

Finally, we shall often use, in the following, the relations between \( \rho \) and the euclidean norm \( | \cdot | \) in \( \mathbb{R}^2 \)

\[
A \rho(u)^{\frac{1}{2}} \leq |u| \leq B \rho(u)^{\frac{1}{2}} \quad \text{if} \ \rho(u) > 1
\]

and

\[
A' \rho(u)^{\frac{1}{2}} \leq |u| \leq B' \rho(u)^{\frac{1}{2}} \quad \text{if} \ \rho(u) \leq 1
\]

for some \( A, B, A', B' > 0 \).

3. A double analytic family of fractional integrals. For any \( r > 0 \) denote by \( B_r \) the open ball in \( \mathbb{R}^2 \)

\[
B_r := \{(u_1, u_2) \in \mathbb{R}^2 : u_1^{2n} + u_2^{2m} < r \}.
\]

Let \( \psi \) be a smooth function on \( \mathbb{R}^2 \), such that \( \psi(u_1, -u_2) = \psi(u_1, u_2) \) for every \((u_1, u_2) \in \mathbb{R}^2, \psi \equiv 1 \) on \( B_r^2 \) and \( \psi \equiv 0 \) outside \( B_r \), with \( 0 \leq \psi \leq 1 \) on \( \mathbb{R}^2 \).

We define a family of analytic distributions \( D_z, \Re z > 0 \), as

\[
\langle D_z, h \rangle := (\psi(\cdot, \cdot) I^z(\cdot, \cdot), h(\cdot + 1, \cdot + 1))
\]

\[
= C \int_{B^2} \rho(u_1 - 1, u_2 - 1)^{-Q} \psi(u_1 - 1, u_2 - 1) h(u_1, u_2) \, du_1, du_2,
\]

where \( C := \frac{C(0)}{\sigma(\mathcal{S}^{(1)}(\mathbb{R}^2 \setminus \mathbb{C}))} \) and \( h \in \mathcal{C}_c^\infty(\mathbb{R}^2) \). It is straightforwad to check that \( D_z \) may be extended to all \( z \in \mathbb{C} \).

As a consequence of Proposition 2.2 we have

\[
\langle D_0, h \rangle = h(1,1).
\]

Define now a double analytic family of distributions \( K^z_\gamma \), for \( \gamma \) and \( z \) in \( \mathbb{C}, \Re \gamma \geq 0 \), in the following way

\[
\langle K^z_\gamma, f \rangle := \int \langle D_z(u_1, u_2), f(t, u_1 t^m, u_2 t^n) \rangle |t|^{-\gamma} \frac{dt}{t}
\]
We remark that, if \( \Re \gamma = 0 \), then
\[
\langle K_z^\gamma, f \rangle := \lim_{\varepsilon \to 0} \int (D_z(u_1, u_2), f(t, u_1 t^m, u_2 t^n)) |t|^{\Re m \gamma + \varepsilon} \frac{dt}{t},
\]
for every \( f \in \mathcal{C}_c^\infty(\mathbb{R}^2) \). Observe moreover that \( K_z^\gamma \) depends analytically on both \( \gamma \) and \( z \).

At this point we may introduce the family of convolution operators with kernel \( K_z^\gamma \) defined by (15), that is
\[
(S^\gamma f) (x_1, x_2, x_3) := (K_z^\gamma * f) (x_1, x_2, x_3)
= \int (D_z(u_1, u_2), f(x_1 - t, x_2 - u_1 t^m, x_3 - u_2 t^n)) |t|^{\gamma} \frac{dt}{t}.
\]

It is worth noticing that, at the light of (14), we have
\[
(S_0^\gamma f) (x_1, x_2, x_3) := C_0 \int f(x_1 - t, x_2 - t^m, x_3 - t^n) |t|^{\gamma} \frac{dt}{t},
\]
that is, for \( z = 0 \) we recover the fractional integration operator along the curve \( t \mapsto (t, t^m, t^n) \) in the space.

4. A product kernel. Let \( \psi \) be a bump function on \( \mathbb{R}^2 \), as defined at the beginning of Section 3. Take \( \mu, \nu \in \mathbb{R} \), \( \mu \neq 0 \). Define
\[
\langle H^{\mu, \nu}, f \rangle := \lim_{\varepsilon \to 0} \int \rho(x_2, x_3)^{-Q + i \nu} \psi \left( \frac{x_2}{x_1^m}, \frac{x_3}{x_1^n} \right) |x_1|^{-1 + \varepsilon + i \nu} \text{sgn} x_1 f(x_1, x_2, x_3) \, dx_1 dx_2 dx_3
\]
for every \( f \in \mathcal{S}(\mathbb{R}^3) \).

We shall prove in Theorem 4.6 that \( H^{\mu, \nu} \) defines a product kernel on \( \mathbb{R} \times \mathbb{R}^2 \). The proof of this result, which may be of independent interest, will be divided into some propositions.

For the sake of completeness, we recall the definition of product kernels, given, in a greater generality, in Def. 2.1.1 in [8], by adapting it to this particular case.

We begin recalling that a normalized bump function on \( \mathbb{R}^m \) is a \( C^1 \) function supported on the unit ball in \( \mathbb{R}^m \), with \( C^1 \) norm bounded by 1. Moreover, if \( \varphi_1 \) is a function of \( x_1 \) and \( \varphi \) is a function of \( (x_2, x_3) \), by the symbol \( \varphi_1 \otimes \varphi \) we will denote the function on \( \mathbb{R}^3 \) defined by \( (\varphi_1 \otimes \varphi)(x_1, x_2, x_3) := \varphi_1(x_1) \varphi(x_2, x_3) \).

**Definition 4.1.** A product kernel on \( \mathbb{R} \times \mathbb{R}^2 \) is a distribution \( H \) on \( \mathbb{R}^3 \), which coincides with a \( C^\infty \) function away from the coordinate subspaces \( \{ x_1 = 0 \} \) and \( \{ x_2 = x_3 = 0 \} \) and which satisfies the following

1. differential inequalities: For any multi-index \( \alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}^3 \) there exists a constant \( C_\alpha \) such that
\[
|\partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \partial_{x_3}^{\alpha_3} H(x_1, x_2, x_3)| \leq C_\alpha |x_1|^{-1 - \alpha_1} \rho(x_2, x_3)^{-Q - \frac{\alpha_2}{m} - \frac{\alpha_3}{m}}
\]
on \( \mathbb{R}^3 \setminus (\{ x_1 = 0 \} \cup \{ x_2 = x_3 = 0 \}) \)

2. cancellation conditions:
   - For any normalized bump function \( \varphi_1(x_1) \) in \( C^1(\mathbb{R}_{x_1}) \) and any \( R_1 > 0 \), set \( \varphi_{1, R_1}(x_1) := \varphi_1(x_1/R_1) \). Then the distribution \( H_{\varphi_{1, R_1}} \), defined on on \( \mathbb{R}^2_{(x_2, x_3)} \) by
\[
(H_{\varphi_{1, R_1}}, \varphi) := (H, \varphi_{1, R_1} \otimes \varphi)
\]
for any test function \( \varphi \) on \( \mathbb{R}^2_{(x_2, x_3)} \) is a product kernel on \( \mathbb{R}^2_{x_2, x_3} \).
• For any normalized bump function $\varphi$ in $C^1(\mathbb{R}^2_{x_2,x_3})$ and any $R > 0$, set $\varphi_R(x_2,x_3) := \varphi \left( \frac{x_2}{R}, \frac{x_3}{R} \right)$. Then the distribution $H_{\varphi,R}$, defined on $\mathbb{R}_{x_1}$ by

$$
\langle H_{\varphi,R}, \varphi_1 \rangle := \langle H, \varphi_1 \otimes \varphi_R \rangle
$$

for any test function $\varphi_1$ on $\mathbb{R}_{x_1}$, is a product kernel on $\mathbb{R}_{x_1}$.

In the following, the symbols $C$ and $C_\sigma$ will denote constants which may vary from one formula to the other and that grow at most exponentially in $|\sigma|$ when $|\sigma|$ tends to $+\infty$. Here $\sigma$ may denote a set of indices, like, e.g., $\sigma = (\beta, \mu, \nu)$; in this case we require that $C_\sigma$ grows at most exponentially in $|\beta|, |\mu|, |\nu|$ when $|\beta|, |\mu|, |\nu|$ tend to $+\infty$. These constants will be called of admissible growth. We start now proving that $H^{\mu,\nu}$ is a product kernel on $\mathbb{R} \times \mathbb{R}^2$.

**Proposition 4.2.** $H^{\mu,\nu}$, defined by (17), is a tempered distribution.

**Proof.** For any $f \in S(\mathbb{R}^3)$ we write

$$
\langle H^{\mu,\nu}, f \rangle = \lim_{\varepsilon \to 0^+} \int \rho(x_2, x_3)^{-Q+i\mu} \psi \left( \frac{x_2}{x_1^m}, \frac{x_3}{x_1^n} \right) |x_1|^{-1+\varepsilon+i\nu} \sgn x_1^1 f(x_1, x_2, x_3) dx_1 dx_2 dx_3
$$

$$
= \lim_{\varepsilon \to 0^+} \left( \int_{|x_1| \leq 1, \rho(x_2, x_3) \leq x_1^{2m}} + \int_{|x_1| > 1, \rho(x_2, x_3) \leq x_1^{2m}} \right) \rho(x_2, x_3)^{-Q+i\mu} \psi \left( \frac{x_2}{x_1^m}, \frac{x_3}{x_1^n} \right) |x_1|^{-1+\varepsilon+i\nu} \sgn x_1^1 f(x_1, x_2, x_3) dx_1 dx_2 dx_3
$$

$$
= \lim_{\varepsilon \to 0^+} (J_{1,\varepsilon} + J_{2,\varepsilon}).
$$

By the change of variables $y_2 = x_2/x_1^m$ and $y_3 = x_3/x_1^n$, we can write

$$
\int_{|x_1| \leq 1, \rho(x_2, x_3) \leq x_1^{2m}} \rho(x_2, x_3)^{-Q+i\mu} \psi \left( \frac{x_2}{x_1^m}, \frac{x_3}{x_1^n} \right) |x_1|^{-1+\varepsilon+i\nu} \sgn x_1^1 dx_1 dx_2 dx_3
$$

$$
= \left( \int_{|y_1| \leq 1} |y_1|^{-1+\varepsilon+i(\nu+2mn\mu)} \sgn y_1 dy_1 \right) \times \left( \int_{\rho(y_2, y_3) \leq 1} \rho(y_2, y_3)^{-Q+i\mu} \psi(y_2, y_3) dy_2 dy_3 \right) := J'_{\varepsilon} \cdot J''.
$$

Since $|y_1|^{-1+\varepsilon+i(\nu+2mn\mu)} \sgn y_1 \in L^1(\{ y_1 \in \mathbb{R} : |y_1| \leq 1 \})$ for every $\varepsilon > 0$ and it is odd, we have $J'_{\varepsilon} = 0$.

We prove that $J''$ is finite. By introducing polar coordinates we obtain

$$
J'' = \int_0^1 \int_{S^1} r^{-1+i\mu} \psi(r^{1/2n} v_1, r^{1/2m} v_2) dr d\sigma(v_1, v_2)
$$
Hence, we obtain that

\[ |J''| = \left| \int_{S} \int_{0}^{1} \frac{r^{i\mu}}{i\mu} \left( \frac{1}{2n} r^{(\frac{1}{2} - 1)} v_1 (\partial_1 \psi) (r^{1/2n} v_1, r^{1/2m} v_2) + \frac{1}{2m} r^{(\frac{1}{2} - 1)} v_2 (\partial_2 \psi) (r^{1/2n} v_1, r^{1/2m} v_2) \right) \, dr \, d\sigma (v_1, v_2) \right| \]

\[ \leq \frac{1}{|\mu|} \| \nabla \psi \|_{\infty} \int_{S} \int_{0}^{1} \left( \frac{1}{2n} r^{(\frac{1}{2} - 1)} |v_1| + \frac{1}{2m} r^{(\frac{1}{2} - 1)} |v_2| \right) \, dr \, d\sigma (v_1, v_2) \]

\[ = \frac{1}{|\mu|} \| \nabla \psi \|_{\infty} \int_{S} (|v_1| + |v_2|) \, d\sigma (v_1, v_2) \]

\[ \leq \frac{1}{|\mu|} \| \nabla \psi \|_{\infty} \int_{S} (\rho (v_1, v_2) \frac{i\mu}{2n} + \rho (v_1, v_2) \frac{i\mu}{2m}) \, d\sigma (v_1, v_2) \]

\[ = \frac{2}{|\mu|} \| \nabla \psi \|_{\infty} \sigma (S). \]

Hence, we obtain that

\[ \int_{|x_1| \leq 1, \rho (x_2, x_3) \leq 1, \mu} \rho (x_2, x_3)^{-Q+i\mu} \varphi \left( \frac{x_2}{x_1^m}, \frac{x_3}{x_1^n} \right) |x_1|^{-1+\epsilon+i\nu} \mathrm{sgn} x_1 \, dx_1 \, dx_2 \, dx_3 = 0. \]

Then introducing polar coordinates and integrating by parts with respect to \( r \), we may write \( J_{1,\epsilon} \) as

\[ J_{1,\epsilon} = \int_{|x_1| \leq 1, \rho (x_2, x_3) \leq 1, \mu} \rho (x_2, x_3)^{-Q+i\mu} \varphi \left( \frac{x_2}{x_1^m}, \frac{x_3}{x_1^n} \right) |x_1|^{-1+\epsilon+i\nu} \mathrm{sgn} x_1 \]

\[ = \int_{|x_1| \leq 1} \int_{S} \int_{0}^{2\pi} r^{-1+i\nu} \psi (r^{1/2n} x_1^{-m} v_1, r^{1/2m} x_1^{-n} v_2) |x_1|^{-1+\epsilon+i\nu} \mathrm{sgn} x_1 \]

\[ \left( f (x_1, r^{1/2n} v_1, r^{1/2m} v_2) - f (0, 0, 0) \right) \, dr \, d\sigma (v_1, v_2) \, dx_1 \]

\[ = - \int_{|x_1| \leq 1} \int_{S} \int_{0}^{2\pi} \frac{r^{1/2n-1}}{x_1^m} v_1 (\partial_1 \psi) \left( r^{1/2n} x_1^{-m} v_1, r^{1/2m} x_1^{-n} v_2 \right) \frac{r^{1/2m-1}}{x_1^n} v_2 (\partial_2 \psi) \left( r^{1/2n} x_1^{-m} v_1, r^{1/2m} x_1^{-n} v_2 \right) \}

\[ \left( f (x_1, r^{1/2n} v_1, r^{1/2m} v_2) - f (0, 0, 0) \right) \, dr \, d\sigma (v_1, v_2) \, dx_1 \]

\[ \times \left( \frac{1}{2n} r^{(\frac{1}{2} - 1)} v_1 (\partial_2 f) (x_1, r^{1/2n} v_1, r^{1/2m} v_2) + \frac{1}{2m} r^{(\frac{1}{2} - 1)} v_2 (\partial_3 f) (x_1, r^{1/2n} v_1, r^{1/2m} v_2) \right) \, dr \, d\sigma (v_1, v_2) \, dx_1. \]

As a consequence of the Mean Value Theorem and of the fact that
\[ |(x_1, x_2^{1/2}v_1, x_3^{1/2}v_2)| \leq |x_1| + r^{2\varepsilon} |(v_1, v_2)| \leq |x_1| \left( 1 + \sqrt{\rho(v_1, v_2)^{1/n} + \rho(v_1, v_2)^{1/m}} \right), \]

since \( 0 < r \leq x_1^{2mn} \leq 1 \), we find that

\[
|J_{1,\varepsilon}| \leq \frac{1}{|\mu|} \|
abla \psi\|_\infty \|f\|(1) \int_{|x_1| \leq 1} \int_{S} \int_{0}^{x_1^{2mn}} |x_1|^\varepsilon \left( 1 + \sqrt{\rho(v_1, v_2)^{1/n} + \rho(v_1, v_2)^{1/m}} \right) \left( \frac{1}{2n} r^{(\frac{1}{2n} - 1)} |x_1|^{n} |v_1| + \frac{1}{2m} r^{(\frac{1}{2m} - 1)} |x_1|^{-m} |v_2| \right) dr d\sigma(v_1, v_2) dx_1 \\
+ \frac{1}{|\mu|} \|
abla \psi\|_\infty \|f\|(1) \int_{|x_1| \leq 1} \int_{S} \int_{0}^{x_1^{2mn}} |x_1|^{-1+\varepsilon} \left( \frac{1}{2n} r^{(\frac{1}{2n} - 1)} |x_1| + \frac{1}{2m} r^{(\frac{1}{2m} - 1)} |v_2| \right) dr d\sigma(v_1, v_2) dx_1 \\
\leq \frac{C}{|\mu|} \sigma(S)(\|
abla \psi\|_\infty + \|
abla \psi\|_\infty) \|f\|(1),
\]

uniformly with respect to \( \varepsilon \in (0, 1/2) \). With analogous computations we easily prove that

\[
|J_{2,\varepsilon}| \leq \frac{C}{|\mu|} \sigma(S)(\|
abla \psi\|_\infty + \|
abla \psi\|_\infty) \|f\|_{(2n)},
\]

uniformly with respect to \( \varepsilon \in (0, 1/2) \), yielding, together with the previous estimate,

\[
|(H^{\mu,\nu}, f)| \leq \frac{C}{|\mu|} \sigma(S)(\|
abla \psi\|_\infty + \|
abla \psi\|_\infty) \|f\|_{(2n)} = C_\mu \|f\|_{(2n)},
\]

where \( C_\mu \) is a positive constant of admissible growth.

The kernel \( H^{\mu,\nu} \) defined by (17) coincides, as is easy to check, with the function

\[
H^{\mu,\nu}(x_1, x_2, x_3) = \rho(x_2, x_3)^{-Q+\mu+\nu} \left( \frac{x_2}{x_1^m}, \frac{x_3}{x_1^n} \right) |x_1|^{-1+i\nu} \text{sgn} x_1 \quad (19)
\]
on \( \mathbb{R}^3 \setminus \{(x_1 = 0) \cup \{x_2 = x_3 = 0\}\} \), so that we shall now prove that \( H^{\mu,\nu} \) satisfies the expected differential inequalities.

**Proposition 4.3.** For any multi-index \( \alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}^3 \) we have

\[
|\partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \partial_{x_3}^{\alpha_3} H^{\mu,\nu}(x_1, x_2, x_3) | \leq C_{\alpha,\mu,\nu} |x_1|^{-1-\alpha_1} \rho(x_2, x_3)^{-Q - \frac{\alpha_2}{n} - \frac{\alpha_3}{m}} \quad (20)
\]
on \( \mathbb{R}^3 \setminus \{(x_1 = 0) \cup \{x_2 = x_3 = 0\}\} \), for some constant \( C_{\alpha,\mu,\nu} \) of admissible growth.

**Proof.** Let \( (x_1, x_2, x_3) \in \mathbb{R}^3 \setminus \{(x_1 = 0) \cup \{x_2 = x_3 = 0\}\} \). If \( (\alpha_1, \alpha_2, \alpha_3) = (0, 0, 0) \), then

\[
|H^{\mu,\nu}(x_1, x_2, x_3)| \leq \|\psi\|_\infty |x_1|^{-1} \rho(x_2, x_3)^{-Q} \leq |x_1|^{-1} \rho(x_2, x_3)^{-Q}.
\]

We give the proof of the differential inequalities (20) only in some situations, the other cases being essentially the same, with the extra disadvantages of more complicated notation and computations.
First, let $\alpha = (1, 0, 0)$. Then
\[
|\partial_{x_1} H^{\mu,\nu}(x_1, x_2, x_3)| = \left| \rho(x_2, x_3)^{-Q+i\mu} \left( -1 + i\nu \right) |x_1|^{-2+i\nu} \left( \text{sgn} x_1 \right)^2 \psi \left( \frac{x_2}{x_1^{m}}, \frac{x_3}{x_1^{n}} \right) \right|
\]
\[
+ \left( -m \frac{x_2}{x_1^{m+1}} (\partial_1 \psi) \left( \frac{x_2}{x_1^{m}}, \frac{x_3}{x_1^{n}} \right) - n \frac{x_3}{x_1^{n+1}} (\partial_2 \psi) \left( \frac{x_2}{x_1^{m}}, \frac{x_3}{x_1^{n}} \right) \right) |x_1|^{-1+i\nu} \text{sgn} x_1 \right|.
\]
On the set where $\psi$ does not vanish the inequality $(\frac{x_2}{x_1})^{2n} + (\frac{x_3}{x_1})^{2m} \leq 1$ holds, so that in particular $|\frac{x_2}{x_1}| \leq 1$ and $|\frac{x_3}{x_1}| \leq 1$. Hence
\[
|\partial_{x_1} H^{\mu,\nu}(x_1, x_2, x_3)| \leq \rho(x_2, x_3)^{-Q} \left( -Q + i\nu \|\psi\|_{\infty} |x_1|^{-2} + (m + n) \|\nabla \psi\|_{\infty} |x_1|^{-2} \right)
\]
\[
= C_{\alpha,\nu} |x_1|^{-1} \rho(x_2, x_3)^{-Q}.
\]
Then assume that $\alpha = (0, 1, 0)$. Thus
\[
|\partial_{x_2} H^{\mu,\nu}(x_1, x_2, x_3)| = \left| x_1^{-1+i\nu} \text{sgn} x_1 \right( 2n (-Q + i\mu) x_2^{2n-1}
\]
\[
\rho(x_2, x_3)^{-Q-1+i\nu} \psi \left( \frac{x_2}{x_1^{m}}, \frac{x_3}{x_1^{n}} \right) + \rho(x_2, x_3)^{-Q+i\nu} x_1^{-m} (\partial_1 \psi) \left( \frac{x_2}{x_1^{m}}, \frac{x_3}{x_1^{n}} \right) \right|.
\]
On the set where $\psi$ does not vanish we have $\rho(\frac{x_2}{x_1}, \frac{x_3}{x_1}) \leq 1$, so that $|x_1|^{-m} \leq \rho(x_2, x_3)^{-1/2n}$ and $|x_2|^{2n-1} \leq \rho(x_2, x_3)^{1-\frac{1}{2n}}$. Therefore
\[
|\partial_{x_2} H^{\mu,\nu}(x_1, x_2, x_3)| \leq |x_1|^{-1} \left( 2n | -Q + i\mu | \|\psi\|_{\infty} \rho(x_2, x_3)^{-Q-\frac{1}{2n}}
\]
\[
+ \|\nabla \psi\|_{\infty} \rho(x_2, x_3)^{-Q-\frac{1}{2n}} \right) = C_{\alpha,\mu} |x_1|^{-1} \rho(x_2, x_3)^{-Q-\frac{1}{2n}}.
\]

We shall now to prove some essential cancellation properties.

Let $\varphi_1(x_1)$ be any normalized bump function in $C^1(\mathbb{R}_{x_1})$ (that is $\varphi_1$ is a $C^1$ function on $\mathbb{R}$ supported on $(-1, 1)$, with $C^1$-norm bounded by 1). Let $R_1 > 0$. Set $\varphi_{1,R_1}(x_1) := \varphi_1(x_1/R_1)$. Then define the distribution $H^{\mu,\nu}_{\varphi_1,R_1}$ on $\mathbb{R}^2_{(x_2,x_3)}$ by
\[
\langle H^{\mu,\nu}_{\varphi_1,R_1}, \varphi \rangle := \langle H^{\mu,\nu}, \varphi_{1,R_1} \otimes \varphi \rangle
\]
for any test function $\varphi$ on $\mathbb{R}^2_{(x_2,x_3)}$.

The following result holds.

**Proposition 4.4.**

i) The distribution $H^{\mu,\nu}_{\varphi_1,R_1}$ coincides with the smooth function
\[
H^{\mu,\nu}_{\varphi_1,R_1}(x_2, x_3) = \rho(x_2, x_3)^{-Q+i\mu} \int \psi \left( \frac{x_2}{R_1}, \frac{x_3}{R_1} \right) |x_1|^{-1+i\nu} \text{sgn} x_1 \varphi_1 \left( \frac{x_1}{R_1} \right) dx_1
\]
on $\mathbb{R}^2_{(x_2,x_3)} \setminus \{(0,0)\}$. Moreover, for any multi-index $\beta = (\beta_1, \beta_2) \in \mathbb{N}^2$ there exists a constant $C_{\beta,\mu}$ of admissible growth such that
\[
|\partial_2^{\beta_2} H^{\mu,\nu}_{\varphi_1,R_1}(x_2, x_3)| \leq C_{\beta,\mu} \rho(x_2, x_3)^{-Q-\frac{1}{2n}-\frac{\beta_1}{2m}} \text{ for all } (x_2, x_3) \in \mathbb{R}^2 \setminus \{(0,0)\},
\]
uniformly in $\varphi_1, R_1$.
ii) For any normalized bump function \( \varphi \) of class \( C^1(\mathbb{R}_2^2) \) and any \( R > 0 \) there exists a constant \( C_\mu \) of admissible growth such that

\[
|\{H_{\mu,\nu}^{\varphi_1, R_1}, \varphi_2, R_2\}| \leq C_\mu
\]

independently of \( \varphi_1, R_1, \varphi, R \), where \( \varphi_2, R_2(x_2, x_3) := \varphi\left(\frac{x_2}{R_1^{1/2}}, \frac{x_3}{R_1^{1/2}}\right) \).

Proof. Some easy computations show that \( H_{\mu,\nu}^{\varphi_1, R_1} \) coincides with a \( C^\infty \) function on \( \mathbb{R}_2^2 \setminus \{(0,0)\} \), so that we shall only prove that \( H_{\mu,\nu}^{\varphi_1, R_1} \) satisfies the differential inequalities (21) and cancellation condition (22).

Take \( \beta = (0,0) \) and \((x_2, x_3) \in \mathbb{R}^2 \setminus \{(0,0)\}\).

Since

\[
|\psi\left(\frac{x_2}{x_1^m}, \frac{x_3}{x_1^n}\right)| \leq \|\psi\|_{(1)} x_2^{2n} \rho(x_2, x_3)^{-\frac{1}{m}},
\]

the map \( x_1 \mapsto \psi\left(\frac{x_2}{x_1^m}, \frac{x_3}{x_1^n}\right) x_1^{-1+i\nu} \)sgn\( x_1 \) is absolutely integrable on the set \( \{x_1 \in \mathbb{R} : |x_1| \leq R_1\} \) and is odd, so that

\[
\int_{|x_1| \leq R_1} |x_1|^{-1+i\nu} \)sgn\( x_1 \) \psi\left(\frac{x_2}{x_1^m}, \frac{x_3}{x_1^n}\right) dx_1 = 0.
\]

Thus we can write

\[
H_{\mu,\nu, \varphi_1, R_1}^{\varphi_2, R_2}(x_2, x_3) = \rho(x_2, x_3)^{-Q+i\mu} \int_{|x_1| \leq R_1} \psi\left(\frac{x_2}{x_1^m}, \frac{x_3}{x_1^n}\right)|x_1|^{-1+i\nu}
\times \left(\varphi_1\left(\frac{x_1}{R_1}\right) - \varphi_1(0)\right) \)sgn\( x_1 dx_1
\]

and a standard application of the Mean Value Theorem implies

\[
|H_{\mu,\nu, \varphi_1, R_1}^{\varphi_2, R_2}(x_2, x_3)| = \rho(x_2, x_3)^{-Q} \int_{|x_1| \leq R_1} \psi\left(\frac{x_2}{x_1^m}, \frac{x_3}{x_1^n}\right)|x_1|^{-1+i\nu}
\times \left(\varphi_1\left(\frac{x_1}{R_1}\right) - \varphi_1(0)\right) \)sgn\( x_1 dx_1
\]

\[
\leq \|\varphi_1\|_{C^1} \rho(x_2, x_3)^{-Q} \int_{|x_1| \leq R_1} \psi\left(\frac{x_2}{x_1^m}, \frac{x_3}{x_1^n}\right)|x_1|^{-1} \frac{|x_1|}{R_1} dx_1
\]

\[
\leq 2\|\psi\|_{C^\infty} \rho(x_2, x_3)^{-Q} = C_\beta \rho(x_2, x_3)^{-Q},
\]

uniformly with respect to \( R_1 \) and \( \varphi_1 \). In the last inequality we used in particular the fact that \( \|\varphi_1\|_{C^1} \leq 1 \).

We shall estimate the derivatives of \( H_{\mu,\nu, \varphi_1, R_1}^{\varphi_2, R_2}(x_2, x_3) \) only in the case \( \beta = (1,0) \), since this already contains all the ideas of the proof. Assume therefore that \( H_{\mu,\nu, \varphi_1, R_1}^{\varphi_2, R_2}(x_2, x_3) \) is differentiated only in \( x_2 \). Then

\[
\partial_{x_2} H_{\mu,\nu, \varphi_1, R_1}^{\varphi_2, R_2}(x_2, x_3) = 2n(-Q + i\mu)x_2^{2n-1} \rho(x_2, x_3)^{-Q-1+i\mu} \int_{|x_1| \leq R_1} \psi\left(\frac{x_2}{x_1^m}, \frac{x_3}{x_1^n}\right)
\times \left(\varphi_1\left(\frac{x_1}{R_1}\right) - \varphi_1(0)\right) \)sgn\( x_1 dx_1
\]

\[
+ \rho(x_2, x_3)^{-Q+i\mu} \int_{|x_1| \leq R_1} \partial_{x_2} \psi\left(\frac{x_2}{x_1^m}, \frac{x_3}{x_1^n}\right)|x_1|^{-m+i\nu}\left(\varphi_1\left(\frac{x_1}{R_1}\right) - \varphi_1(0)\right) \)sgn\( x_1 dx_1
\]
Now, we may proceed as in the previous case. Since $|x_2|^{2n-1} \leq \rho(x_2, x_3)^{1+\frac{m}{n}}$ and $|x_1|^{-m} \leq \rho(x_2, x_3)^{-\frac{m}{n}}$ on the support of $\psi\left(\frac{x_2}{x_1}, \frac{x_3}{x_1}\right)$, we obtain that

$$\left|\partial_{x_2} H_{\varphi_1, R_1}^{\mu, \nu}(x_2, x_3)\right| \leq 2n\left|Q + i\mu\right|\|\psi\|_{\infty} \|\varphi_1\|_{C^1} \frac{1}{R_1} \rho(x_2, x_3)^{-Q-\frac{m}{n}} \int_{|x_1| \leq R_1} dx_1 + \left|\nabla \psi\right|_{\infty} \frac{1}{R_1} \rho(x_2, x_3)^{-Q-\frac{m}{n}} \int_{|x_1| \leq R_1} dx_1 = C_{\beta, \mu} \rho(x_2, x_3)^{-Q-\frac{m}{n}}.$$

We shall now prove that $H_{\varphi_1, R_1}^{\mu, \nu}$ satisfies the right cancellation conditions as well. Let $\varphi$ be any normalized bump function of class $C^1(\mathbb{R}_2^n)$ (that is $\varphi$ is a $C^1$ function on $\mathbb{R}^2$ supported in the ball $B_1$, with $C^1$-norm bounded by 1), take $R > 0$ and set $\varphi_R(x_2, x_3) := \varphi\left(\frac{x_2}{R^m}, \frac{x_3}{R^m}\right)$.

After a suitable change of variables (more precisely, we set $y_2 := \frac{R^m x_2}{R^{2m}}$ and $y_3 := \frac{R^m x_3}{R^{2m}}$) we obtain

$$\left\langle H_{\varphi_1, R_1}^{\mu, \nu}, \varphi_R \right\rangle = R_1^{\mu} R_1^{\nu} \lim_{\varepsilon \to 0^+} I_\varepsilon,$$

where

$$I_\varepsilon := \int \rho(x_2, x_3)^{-Q + i\mu} \psi\left(\frac{R_1^{1/2n} x_2}{(R_1 x_1)^m}, \frac{R_1^{1/2m} x_3}{(R_1 x_1)^n}\right) |x_1|^{-1+\varepsilon + i\nu} \text{sgn} x_1 \varphi_1(x_1) \varphi(x_2, x_3) dx_1 dx_2 dx_3.$$

In order to estimate $I_\varepsilon$, we consider separately the case $R \geq R_1^{2mn}$ and $R < R_1^{2mn}$. Assume that $R \geq R_1^{2mn}$ and set

$$\mathcal{A} := \{(x_1, x_2, x_3) \in \mathbb{R}^3 : |x_1| \leq 1, \rho(x_2, x_3) \leq \frac{(R_1 |x_1|)^{2mn}}{R}\}.$$

By reasoning in a similar way as in the first part of Proposition 4.2, we show that

$$\int_{\mathcal{A}} \rho(x_2, x_3)^{-Q + i\mu} \psi\left(\frac{R_1^{1/2n} x_2}{(R_1 x_1)^m}, \frac{R_1^{1/2m} x_3}{(R_1 x_1)^n}\right) |x_1|^{-1+\varepsilon + i\nu} \text{sgn} x_1 \rho(x_2, x_3) dx_1 dx_2 dx_3 = 0$$

for every $\varepsilon > 0$, so that

$$I_\varepsilon = \int_{\mathcal{A}} \rho(x_2, x_3)^{-Q + i\mu} \psi\left(\frac{R_1^{1/2n} x_2}{(R_1 x_1)^m}, \frac{R_1^{1/2m} x_3}{(R_1 x_1)^n}\right) |x_1|^{-1+\varepsilon + i\nu} \text{sgn} x_1 (\varphi_1(x_1) \varphi(x_2, x_3) - \varphi_1(0) \varphi(0, 0)) dx_1 dx_2 dx_3.$$
Then a change to polar coordinates and an integration by parts with respect to the variable $r$ yield

$$I_ε' = \int_{|x_1| \leq 1} \int_{S_0} (r_{x_1}^{2mn}/R) r^{-1+i\mu} \psi \left( \frac{(rR)^{1/2m}}{(R_{x_1}^{2mn})} v_1, \frac{(rR)^{1/2m}}{(R_{x_1}^{2mn})} v_2 \right) \left| x_1 \right|^{-1+\varepsilon+i\nu sgn x_1} \left( \varphi_1(x_1) \varphi(r^{1/2m} v_1, r^{1/2m} v_2) - \varphi_1(0) \varphi(0,0) \right) \, dr \, d\sigma(v_1, v_2) \, dx_1$$

$$= -\int_{|x_1| \leq 1} \int_{S_0} (r_{x_1}^{2mn}/R) r^{i\mu} \left( \varphi_1(x_1) \varphi(r^{1/2m} v_1, r^{1/2m} v_2) - \varphi_1(0) \varphi(0,0) \right) \times |x_1|^{-1+i\varepsilon+i\nu sgn x_1} \cdot \left( \frac{1}{2m} \frac{(rR)^{1/2m}}{(R_{x_1}^{2mn})} v_1 (\partial_1 \psi) \left( \frac{(rR)^{1/2m}}{(R_{x_1}^{2mn})} v_1, \frac{(rR)^{1/2m}}{(R_{x_1}^{2mn})} v_2 \right) \right) \frac{d\sigma(v_1, v_2)}{dx_1}$$

$$+ \frac{1}{2m} \frac{(rR)^{1/2m}}{(R_{x_1}^{2mn})} v_2 (\partial_2 \psi) \left( \frac{(rR)^{1/2m}}{(R_{x_1}^{2mn})} v_1, \frac{(rR)^{1/2m}}{(R_{x_1}^{2mn})} v_2 \right) \frac{d\sigma(v_1, v_2)}{dx_1}$$

$$- \int_{|x_1| \leq 1} \int_{S_0} (r_{x_1}^{2mn}/R) r^{i\mu} \left( \varphi_1(x_1) \varphi(r^{1/2m} v_1, r^{1/2m} v_2) \right) |x_1|^{-1+i\varepsilon+i\nu sgn x_1} \times \left( \frac{1}{2m} \frac{(rR)^{1/2m}}{(R_{x_1}^{2mn})} v_1 (\partial_1 \varphi) (r^{1/2m} v_1, r^{1/2m} v_2) + \frac{1}{2m} \frac{(rR)^{1/2m}}{(R_{x_1}^{2mn})} v_2 (\partial_2 \varphi) (r^{1/2m} v_1, r^{1/2m} v_2) \right) \frac{d\sigma(v_1, v_2)}{dx_1}$$

$$=: I_ε' + I_ε''.$$

As a consequence of the Mean Value Theorem we obtain that

$$\left| \varphi_1(x_1) \varphi(r^{1/2m} v_1, r^{1/2m} v_2) - \varphi_1(0) \varphi(0,0) \right| \leq \left| \varphi_1(x_1) - \varphi_1(0) \right| \left| \varphi(r^{1/2m} v_1, r^{1/2m} v_2) \right|$$

$$+ \left| \varphi_1(0) \right| \left| \varphi(r^{1/2m} v_1, r^{1/2m} v_2) - \varphi(0,0) \right|$$

$$\leq \left( |x_1| + r^{1/2m} |v_1, v_2| \right) \| \varphi_1 \|_{C^1} \| \varphi \|_{C^1}$$

$$\leq |x_1| \left( 1 + \sqrt{\rho(v_1, v_2)\frac{1}{m} + \rho(v_1, v_2)\frac{1}{m}} \right),$$

where we used in particular the fact that $0 < r \leq \frac{(x_1 R_{x_1}^{2mn})}{R} \leq x_1^{2mn}$ and the fact that both $\| \varphi_1 \|_{C^1}$ and $\| \varphi \|_{C^1}$ are bounded by 1. Thus

$$|I_ε'| \leq \frac{1}{|\mu|} \| \nabla \psi \|_{\infty} \int_{|x_1| \leq 1} \int_{S_0} (r_{x_1}^{2mn}/R) \left( \frac{1}{2n} \frac{(rR)^{1/2m}}{(R_{x_1}^{2mn})} v_1 \right)$$

$$+ \frac{1}{2m} \frac{(rR)^{1/2m}}{(R_{x_1}^{2mn})} v_2) |x_1|^\varepsilon \left( 1 + \sqrt{\rho(v_1, v_2)\frac{1}{m} + \rho(v_1, v_2)\frac{1}{m}} \right) \frac{d\sigma(v_1, v_2)}{dx_1}$$

$$= \frac{1}{|\mu|} \| \nabla \psi \|_{\infty} \int_{|x_1| \leq 1} \int_{S_0} |x_1|^\varepsilon (|v_1| + |v_2|) \left( 1 + \sqrt{\rho(v_1, v_2)\frac{1}{m} + \rho(v_1, v_2)\frac{1}{m}} \right)$$

$$\leq \frac{C}{|\mu|} \| \nabla \psi \|_{\infty} \sigma(S),$$

uniformly with respect to $\varphi_1$, $\varphi$, $R_1$, $R$, $\varepsilon$.  

(23)
A similar estimate holds for $|I'_{\varepsilon}|$, since

$$|I'_{\varepsilon}| \leq \frac{1}{|\mu|} \|\psi\|_\infty \|\varphi_1\|_1 \|\varphi\|_1 \int_{|x_1| \leq 1} \int_S \rho(R_1 x_1)^{2mn}/R \left( \frac{1}{2m} \varepsilon^{1+\frac{2n}{m}} |v_1| \right. \left. + \frac{1}{2m} \varepsilon^{1+\frac{2n}{m}} |v_2| \right) |x_1|^{1+\varepsilon} \, dr \, d\sigma(v_1, v_2) \, dx_1$$

$$\leq \frac{1}{|\mu|} \|\psi\|_\infty \int_{|x_1| \leq 1} \int_S \left( \frac{R_1 |x_1|^m}{R^{1/2n}} |v_1| + \frac{R_1 |x_1|^m}{R^{1/2m}} |v_2| \right) |x_1|^{1+\varepsilon} \, d\sigma(v_1, v_2) \, dx_1$$

$$\leq \frac{1}{|\mu|} \|\psi\|_\infty \int_{|x_1| \leq 1} \int_S \left( \frac{R_1 |x_1|^m}{R^{1/2n}} |v_1| + \frac{R_1 |x_1|^m}{R^{1/2m}} |v_2| \right) |x_1|^{1+\varepsilon} \, d\sigma(v_1, v_2) \, dx_1$$

$$\leq \frac{1}{|\mu|} \|\psi\|_\infty \int_{|x_1| \leq 1} \int_S \left( \frac{R_1 |x_1|^m}{R^{1/2n}} |v_1| + \frac{R_1 |x_1|^m}{R^{1/2m}} |v_2| \right) |x_1|^{1+\varepsilon} \, d\sigma(v_1, v_2) \, dx_1$$

$$\leq \frac{2}{|\mu|} \|\psi\|_\infty \sigma(S) \left( \frac{R_1^m}{R^{1/2n}} + \frac{R_1^m}{R^{1/2m}} \right)$$

$$\leq C \frac{1}{|\mu|} \|\psi\|_\infty \sigma(S),$$

(24)

uniformly with respect to $\varphi_1, \varphi, R_1, R, \varepsilon$.

It follows from (23) and (24) that

$$|I_{\varepsilon}| \leq |I'_{\varepsilon}| + |I''_{\varepsilon}| \leq C_\mu,$$

uniformly with respect to $\varphi_1, \varphi, R_1, R, \varepsilon$.

We shall now consider the case $R < R_1^{2mn}$. Set

$$B_1 := \{(x_1, x_2, x_3) \in \mathbb{R}^3 : |x_1| \leq \frac{1}{R_1} \frac{1}{R^{1/2n}} , \rho(x_2, x_3) \leq \frac{1}{R}(R_1 |x_1|)^{2mn} \}$$

and

$$B_2 := \{(x_1, x_2, x_3) \in \mathbb{R}^3 : \frac{1}{R_1} \frac{1}{R^{1/2n}} \leq |x_1| \leq 1 , \rho(x_2, x_3) \leq 1 \}.$$

$I_\varepsilon$ may be written as

$$I_\varepsilon = \left( \int_{B_1} + \int_{B_2} \right) \rho(x_2, x_3) \psi Q + \mu \psi \left( \frac{R^{1/2n} x_2^m}{(R_1 x_1)^m} , \frac{R^{1/2m} x_2^m}{(R_1 x_1)^m} \right) |x_1|^{1+\varepsilon+i\nu} \, dx_1,$$

$$\varphi_1(x_1) \varphi_2(x_3) \, dx_1 \, dx_2 \, dx_3 \leq I_{\varepsilon, B_1} + I_{\varepsilon, B_2}.$$

Now a change to polar coordinates, integration by parts with respect to $r$ and the Mean value theorem yield an estimate for $I_{\varepsilon, B_1}$. Since it is similar to the bound of $|I'_{\varepsilon}|$, we omit it and we only claim that

$$|I_{\varepsilon, B_1}| \leq C_\mu,$$

uniformly with respect to $\varphi_1, \varphi, R_1, R, \varepsilon$. 
In order to estimate $I_{\varepsilon,B_2}$, we rewrite it after a change to polar coordinates and an integration by parts with respect to $H$ as

\[
I_{\varepsilon,B_2} = -\int_{\frac{R_1^{1/mn}}{R_1^{1/mn}}} |x_1|^{-1+\varepsilon+i\omega} \text{sgn} x_1 \varphi_1(x_1) \varphi(r^{\frac{1}{m}+\frac{1}{n}} v_1, r^{\frac{1}{m}+\frac{1}{n}} v_2) \cdot \left( \frac{1}{2n} \frac{r^{1+\frac{1}{m}+\frac{1}{n}}}{(R_1 x_1)^m} \varphi_1 \left( \frac{(R_1)^{\frac{1}{m}} v_1, (R_1)^{\frac{1}{m}} v_2} \right) \right) \cdot \left( \frac{1}{2m} \frac{r^{1+\frac{1}{m}+\frac{1}{n}} v_2(\partial_2 \varphi)}{(R_1 x_1)^{n}} \right) dr d\sigma(v_1, v_2) dx_1 + \frac{1}{2m} r^{-1+\frac{1}{m}+\frac{1}{n}} v_2(\partial_2 \varphi)(r^{1/2n} v_1, r^{1/2m} v_2)
\]

\[
=: I_1' + I_2'.
\]

It is not difficult to prove that both integrals above are uniformly bounded by some $C_\mu$, so that

\[
|I_1| \leq |I_{\varepsilon,B_1}| + |I_{\varepsilon,B_2}| \leq C_\mu,
\]

independently of $\varphi_1, \varphi, R_1, \varepsilon$, yielding the thesis. \qed

Let $\varphi(x_2, x_3)$ be, as in the Proposition 4.4, any normalized bump function in $C^1(\mathbb{R}^2_{x_2,x_3})$. Take $R > 0$ and set, as above, $\varphi_R(x_2, x_3) := \varphi \left( \frac{x_2}{R^{1/2m}}, \frac{x_3}{R^{1/2m}} \right)$. Denote by the symbol $H_{\varphi,R}^{\mu,\nu}$ the distribution on $\mathbb{R}^2_{x_2,x_3}$ given by

\[
\langle H_{\varphi,R}^{\mu,\nu}, \varphi_1 \rangle := \langle H_{\varphi,R}^{\mu,\nu}, \varphi_1 \otimes \varphi_R \rangle
\]

\[
= \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^3} \varphi_1(x_1) \varphi \left( \frac{x_2}{R^{1/2m}}, \frac{x_3}{R^{1/2m}} \right) dx_1 dx_2 dx_3
\]

for any test function $\varphi_1$ on $\mathbb{R}_{x_1}$.

To conclude our proof that $H_{\varphi,R}^{\mu,\nu}$ defines a product kernel on $\mathbb{R}^2$ we only need the following cancellation property.

**Proposition 4.5.**

i) The distribution $H_{\varphi,R}^{\mu,\nu}$ coincides with the function

\[
H_{\varphi,R}^{\mu,\nu}(x) = |x_1|^{-1+\varepsilon+i\omega} \text{sgn} x_1 \rho(x_2, x_3)^{-Q+i\mu} \psi \left( \frac{x_2}{x_1^m}, \frac{x_3}{x_1^n} \right) \varphi \left( \frac{x_2}{R^{1/2m}}, \frac{x_3}{R^{1/2m}} \right) dx_2 dx_3
\]

(25)

on $\mathbb{R}_{x_1} \setminus \{0\}$. Moreover, for any nonnegative integer $\beta$ there exists a constant $C_{\beta,\mu,\nu}$ of admissible growth such that

\[
|\partial_1^\beta H_{\varphi,R}^{\mu,\nu}(x)| \leq C_{\beta,\mu,\nu} |x_1|^{-1-\beta} \text{ for all } x_1 \in \mathbb{R} \setminus \{0\},
\]

uniformly in $\varphi$, $R$. 

ii) For any normalized bump function $\varphi_1$ of class $C^1(\mathbb{R}_{x_1})$ and any $R_1 > 0$ there exists a constant $C_\mu$ of admissible growth such that
\[
|\langle H^{\mu,\nu}_{\varphi,R}, \varphi_1, R_1 \rangle| \leq C_\mu
\] (27) independently of $\varphi_1$, $R_1$, $\varphi$, $R$.

Proof. First of all, by a change to polar coordinates and an integration by parts with respect to the variable $r$, we observe that the integral on the right-hand side of (25) is absolutely convergent. Thus (25) follows from a routine application of Fubini’s theorem.

It will now be shown that $H^{\mu,\nu}_{\varphi,R}$ satisfies the right differential inequalities and cancellation conditions.

If $x_1 \in \mathbb{R} \setminus \{0\}$ and $\beta = 0$, introducing polar coordinates and integrating by parts with respect to $r$, we obtain
\[
|H^{\mu,\nu}_{\varphi,R}(x_1)| = |x_1|^{-1} \left| \int_{S_0} \min\{x_1^{2m}, R\} \frac{r^{i\mu}}{i\mu} \left( \frac{r^{1/2n}}{R^{1/2n} v_1}, \frac{x_1^{1/2n}}{R^{1/2n} v_1} v_2 \right) \right.
\]
\[
\times \left( \frac{r^{-1+\frac{i\mu}{2n}}}{2n|x_1|^n} v_1 (\partial_1 \varphi) \left( \frac{r^{1/2n}}{R^{1/2n} v_1}, \frac{x_1^{1/2n}}{R^{1/2n} v_1} v_2 \right) + \frac{1}{2n} \frac{r^{-1+\frac{i\mu}{2n}}}{R^{1/2n} v_2} (\partial_2 \varphi) \left( \frac{r^{1/2n}}{R^{1/2n} v_1}, \frac{x_1^{1/2n}}{R^{1/2n} v_1} v_2 \right) \right) \right| \right.
\]
\[
\left. \left. dr \, d\sigma(v_1, v_2) \right| \right.
\]
\[
\leq \frac{1}{\mu} (\|\varphi\|_\infty + \|\nabla \varphi\|_\infty) |x_1|^{-1} \left( \int_{S_0} \min\{x_1^{2m}, R\} \left( \frac{1}{2n} \frac{r^{-1+\frac{i\mu}{2n}}}{R^{1/2n} v_1} |v_1| + \frac{1}{2n} \frac{r^{-1+\frac{i\mu}{2n}}}{R^{1/2n} v_2} |v_2| \right) \right.
\]
\[
l \left. \left. dr \, d\sigma(v_1, v_2) + \int_0^R \left( \frac{1}{2n} \frac{r^{-1+\frac{i\mu}{2n}}}{R^{1/2n} v_1} |v_1| + \frac{1}{2n} \frac{r^{-1+\frac{i\mu}{2n}}}{R^{1/2n} v_2} |v_2| \right) dr \, d\sigma(v_1, v_2) \right| \right.
\]
\[
= \frac{2}{\mu} (\|\varphi\|_\infty + \|\nabla \varphi\|_\infty) |x_1|^{-1} \int_S (|v_1| + |v_2|) \, d\sigma(v_1, v_2)
\]
\[
\leq C_0|x_1|^{-1},
\]
uniformly with respect to $R$ and $\varphi$.

By the same arguments we can prove the estimates for the higher-order derivatives of $H^{\mu,\nu}_{\varphi,R}(x_1)$ and we omit the details.

Finally, we should prove ii). Anyway, this inequality coincides essentially with (22) and it has been proved in Proposition 4.4.

As a consequence of the previous propositions, we obtain the following

**Theorem 4.6.** The distribution $H^{\mu,\nu}$, defined by (17), is a product kernel on $\mathbb{R} \times \mathbb{R}^2$.

Finally, we may prove some strong $L^p$-bounds for the operator $S^2_{\gamma}$. 

**Theorem 4.7.** The operator $S^2_{\gamma}$ maps $L^p(\mathbb{R}^3)$ to $L^p(\mathbb{R}^3)$ for all $1 < p < +\infty$ if $\Re \gamma = 0$ and $\Re z = 0$. 
Proof. Write $z = i\theta$, where $\theta$ is a real number.

If $\theta \neq 0$ and $\Re \gamma = 0$, then, starting from (15), the convolution kernel $K^\gamma_{i\theta}$ may be written, by means of the change of variables $x_1 = t$, $x_2 = u_1 t^m$, $x_3 = u_2 t^n$, as

$$< K^\gamma_{i\theta}, f > = C_\theta \lim_{\varepsilon \to 0} \int_{\mathbb{R}^3} \rho(x_2 - x_1^m, x_3 - x_1^n)^{-Q+i\theta} \psi \left( \frac{x_2}{x_1^m} - 1, \frac{x_3}{x_1^n} - 1 \right) |x_1|^{-1+z+i(3m\gamma-2mn\theta)} \text{sgn} x_1 f(x_1, x_2, x_3) dx_1 dx_2 dx_3$$

where $C_\theta = G(0)/\left(2\sigma(S)\Gamma(i\theta)G(i\theta)\right)$ and $G$ has been defined in (9), for all $f \in S(\mathbb{R}^3)$. This kernel essentially coincides with the kernel $H^\mu,\nu$, defined by (17), with $\mu = \theta$, $\nu = i\Im \gamma - 2mn\theta$, adapted to the curve $x_1 \mapsto (x_1^m, x_1^n)$. Then Theorem 4.6 and Theorem 1.2 imply that the operator $S^\gamma_{i\theta}$ maps $L^p(\mathbb{R}^3)$ to $L^p(\mathbb{R}^3)$ for all $1 < p < +\infty$.

If $\theta = 0$ and $\Re \gamma = 0$, then

$$S^\gamma_0 f(x_1, x_2, x_3) = \lim_{\varepsilon \to 0} \int f(x_1 - t, x_2 - t^m, x_3 - t^n)|t|^{i3m\gamma+\varepsilon} dt$$

It follows from some well-known results [11] that $S^\gamma_0$ maps $L^p$ to $L^p$ for all $1 < p < +\infty$. \qed

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Received March 2010; revised February 2012.

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