Approximate optimal synthesis of operational control systems for dynamic objects based on quasilinearization and sufficient optimality conditions

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Abstract. A new approach to the analytical design of linear and nonlinear integrated automatic control systems (ACS) of a real (accelerated) time scale, based on the joint use of dynamic programming technologies and the quasilinearization method, is presented. For continuous dynamic systems, the foundations of the theory of nonlinear synthesis in a degenerate (synergetic) formulation are given.

1. Introduction

Control is the creation at each current moment of time of targeted influences on the controlled object, depending on the available information about the behavior of the object and the disturbances acting on it. In control theory, three principles of control are considered: 1) in an open loop, 2) in a closed loop, 3) in real (accelerated) time. When using the first principle, before the start of the control process, a program (program control) is built using a priori information, which is not corrected during the control process. In the second control principle, current control actions (positional controls) are created according to pre-drawn (before the start of the control process) rules defined on all kinds of information that may appear about the behavior of the object and the disturbances acting on it in the control process. These rules are implemented in the form of direct, reverse and combined links. When using the third control principle, the listed connections are not created in advance, their current (required future) values are calculated in real (accelerated) time scale during the operation of the object.

An approach to the problem of synthesis of ordinary dynamical systems, focused on the principle of optimal control (OC) in real (accelerated) time, was proposed in the early 70s by V.S. Shendrick (at the initiative of B.N.Petrov) and developed by A.A. Krasovsky and his students [1]. The greatest contribution to the development of this direction of the theory of control was made by V.N. Bukov [2]. In the early 90s, the principle of real-time control was “rediscovered” by R. Gabasov and F.M. Kirillova and is successfully developing in the Belarusian school of mathematicians [3].

It is known that rather stringent conditions are imposed on traditional algorithms of successive improvements in terms of convergence and choice of initial approximations. On the way of using only sufficient optimality conditions or the theory of quasilinearization of simple and reliable (guaranteeing pointwise convergence) methods, as noted by R. Bellman [4], it was not possible to create. To overcome these difficulties, the article develops a multi-method technology based on a combination of the quasilinearization method with sufficient conditions for optimality. It is proposed to apply quasilinearization for local optimization in the vicinity of stationarity points, and sufficient conditions for optimality for interval optimization. The main idea of the proposed two-method technology: by means of
interval optimization, to carry out a rough search for the initial approximation according to sufficient conditions, and then iteratively refine the resulting approximation according to the conditions of local optimality: stationarity or in the form of the minimum principle.

To organize minimizing sequences of a weak minimum, an auxiliary (formulation-degenerate) problem of approximate optimal synthesis is formulated. The degeneracy here is embedded in the very formulation of the control problem and manifests itself in a special way: the original (non-degenerate) synthesis problem is redefined to be singular in order to include the limiting control functions in the set of admissible ones, but in such a way that the transformed problem contains an optimal solution. If in the traditional formulations of degenerate control problems the singular curve is to be determined, then in the transformed problem it is known: it is the optimal trajectory of the original problem.

Thus, in contrast to the known approaches to solving the problem of approximately optimal synthesis of controllers, when the solution method uses the existing ambiguity in the choice of generating functions with the properties of the Lyapunov function, in this case there is another continuation of the theory of sufficient conditions: by means of phase linearization of the equations of the dynamic system and the integrand of the functional of the initial problem of nonlinear synthesis relatively unknown in advance, but determined in the process of the object's operation, the optimal values of the vector control functions (quasi-linearization) that are constant at small time intervals and through the formation of the gradient strategy of the Newtonian type, based on variations of controls on the same intervals, the stationarity points of the required minimum are successively determined and refined. In order to fix the limiting elements of minimizing sequences of the search for the optimal solution under the stationarity conditions, it is proposed to use the functional of the generalized work by A.A. Krasovsky.

An effective method of combined synthesis and a procedure for solving a two-point boundary value problem according to the differential dynamic programming (DP) scheme have been developed, which ensure interval monotonically decreasing (relaxation) convergence of control processes according to the necessary conditions of local optimality. The main theorems are formulated and one of the variants of the algorithmic implementation of the method is presented. The need for such a development is due to the factual absence of reliable methods for nonlinear synthesis of digital controllers, which guarantee high accuracy and stability of the solution at acceptable computational costs.

2. Statement of the problem of approximately optimal control synthesis

By optimization of continuous control processes we mean the solution of the choice problem on the time interval $T = [t_0, t_f]$ of positional control

$$u = u(t_0, x(t_0), t, x(t))$$

for a dynamic system

$$\dot{x} = f(t, x, u)$$

such that on the trajectory of motion of an object that satisfies the given constraints on the sets of initial and final states

$$\mu(t_0, x(t_0), t_f, x(t_f)) = 0, \quad \mu \in \mathbb{R}^p,$$

functional

$$I = V_g(t_0, x(t_0), t_f, x(t_f)) + \int_{t_0}^{t_f} f_0(t, x(t), u(t))dt, \quad I \in \mathbb{R},$$

reached the minimum (maximum) or the smallest (largest) exact edge (infimum or supremum). Here the functions $f, \mu, V_g$ and $f_0$ are fixed piecewise continuous in $t$ and continuous and sufficiently smooth in $x, u$ (differentiable or absolutely continuous) vector and scalar functions of the indicated arguments.

In what follows, we will consider a less general formulation of the optimization problem – the formulation of a nonlinear synthesis problem for which condition (3) can be taken into account without loss of generality in the construction of the modified Lagrangian; and the scalar function $V_g(x(t_f)) = V_g(t_f, x(t_f))$ defines the boundary conditions only at the right end of the trajectory (terminal
The boundary conditions at the left end of the trajectory $x(t_0) = x^0 \in \mathbb{R}^n$ are arbitrary. The final constraints on the boundary conditions and on the values of the control functions and the trajectory of the process (3) will be written as

$$\{(x(t), u(t)) \in F(t), \quad (5)$$

where $F(t) \subseteq G_x \times G_u, G_x = X, G_u = U$ is the time-dependent Cartesian product of sets of topological degree $(n+m)$.

The set of pairs of vector functions $\{x(t), u(t)\}$ that satisfy differential constraints (2) and finite constraints (5) is called the set $D$ of admissible ones. It is assumed that $D \neq \emptyset$.

A pair of functions $\{x_{opt}(t), u_{opt}(t)\} \in D$ will be called the optimal process (minimum) for $I$ on $D$ if

$$I(x_{opt}(t), u_{opt}(t)) = d. \quad (6)$$

Here $d = \inf_{D} I(x(t), u(t))$ is the lower exact bound for functional (4).

Functional (4) in the general theory of extremal problems is called the support functional (support) [9].

The sequence $\{x(t), u(t)\} \in D$ in which

$$I(x(t), u(t)) \rightarrow d,$$

is minimizing for the functional $I$ on the set $D$.

3. Quasilinearization and sufficient conditions for optimality

Let us introduce a continuous and sufficiently smooth (differentiable or absolutely continuous) function $\varphi(t, x) \in \Phi$ and consider the following constructions [5, p. 263]

$$R(t, x, u) = \frac{\partial \varphi(t, x)}{\partial \lambda} + \frac{\partial \varphi(t, x)}{\partial \lambda} f(t, x, u) + f_0(t, x, u),$$

$$\Phi(x(t_0), x(t_f)) = V_g(x(t_f)) + \varphi(t_f, x(t_f)) + \varphi(t_0, x(t_0)).$$

In order for a pair $(x_{opt}, u_{opt}) \in D$ to be a minimum in problem (1)–(5), it is sufficient that such a smooth function exists for the conditions [6]

$$\mu(t) = R(t, x_{opt}, u_{opt}) = \inf_{(t, x, u) \in F(t)} R(t, x, u) \quad \text{for anyone } t \in [t_0, t_f],$$

$$\Phi(x_{opt}(t_0), x_{opt}(t_f)) = \inf_{x(t_0) \in F_x(t_0)} \Phi(x(t_0), x(t_f)), \quad (7)$$

where the inclusion $x(t) \in F_x(t)$ determines the restriction on the values of the state vector of system (2), $F_x(t)$ is the projection of the set $F(t)$ onto the space $X$.

In the original constructions (7), (8), we take into account the Taylor expansion of functions $f, f_0$ in a small neighborhood of the local minimum $(x(t_0), u(t_0)) = (x_{opt}(t), u_{opt}(t, \tau)_{\tau=\varepsilon})$

$$f(t, x, u) = f(t, x_0, u_0) + \frac{\partial f(t, x_0, u_0)}{\partial x} \delta x + \frac{\partial f(t, x_0, u_0)}{\partial u} \delta u + o_1(\delta x, \delta u),$$

$$f_0(t, x, u) = f_0(t, x_0, u_0) + \frac{\partial f_0(t, x_0, u_0)}{\partial x} \delta x + \frac{\partial f_0(t, x_0, u_0)}{\partial u} \delta u + o_1(\delta x, \delta u), \quad (9)$$

where for brevity we denote

$$\frac{\partial f(t, x_0, u_0)}{\partial x} = \left[ \frac{\partial f(t, x, u)}{\partial x} \right]_{x=x_0}, \quad \frac{\partial f(t, x_0, u_0)}{\partial u} = \left[ \frac{\partial f(t, x, u)}{\partial u} \right]_{x=x_0},$$

$$\frac{\partial f_0(t, x_0, u_0)}{\partial x} = \left[ \frac{\partial f_0(t, x, u)}{\partial x} \right]_{x=x_0}, \quad \frac{\partial f_0(t, x_0, u_0)}{\partial u} = \left[ \frac{\partial f_0(t, x, u)}{\partial u} \right]_{x=x_0}. \quad (10)$$
It is assumed that the expansion terms above the second order are negligible, and it is assumed that the insignificant function of time \( \mu(t) = R(t, x_{opt}, u_{opt}) = \inf_{(x,u) \in F(t)} R(t, x, u) = 0 \), and sufficient conditions (7), (8), taking into account (9), (10), will be rewritten in the form

\[
\inf_{x \in F_f} \left( \frac{\partial \phi(t, x)}{\partial x} + f(t, x_0, u_0) + f_0(t, x_0, u_0) \right) + \inf_{\phi \in F_{\phi}} \left[ \frac{\partial \phi(t, x)}{\partial x} \frac{\partial f}{\partial x} + \frac{\partial f_0}{\partial x} \right] \delta x + \inf_{u \in U} \left( \frac{\partial \phi(t, x)}{\partial x} + \frac{\partial f_0}{\partial x} \right) \delta u = 0, \tag{11}
\]

\[
\inf_{x(t_f) = x_{opt}(t_f)} \left( \frac{\partial \phi(t, x)}{\partial x} + \frac{\partial \phi(t, x)}{\partial x} f(t, x_0, u_0) + f_0(t, x_0, u_0) \right) + \inf_{\phi \in F_{\phi}} \left[ \frac{\partial \phi(t, x)}{\partial x} \frac{\partial f}{\partial x} + \frac{\partial f_0}{\partial x} \right] \delta x + \inf_{u \in U} \left( \frac{\partial \phi(t, x)}{\partial x} + \frac{\partial f_0}{\partial x} \right) \delta u = 0. \tag{12}
\]

Expressions in square brackets of formula (11) can be written through a scalar function \( H(t, x, \phi, u) = \frac{\partial \phi(t, x)}{\partial x} f + f_0 \).

Then formula (12) can be represented as

\[
\inf_{x \in F_f} \left( \frac{\partial \phi(t, x)}{\partial x} + f(t, x_0, u_0) + f_0(t, x_0, u_0) \right) + \inf_{\phi \in F_{\phi}} \left[ \frac{\partial H(t, x, \phi, u)}{\partial x} \delta x \right] + \inf_{u \in U} \left( \frac{\partial H(t, x, \phi, u)}{\partial u} \delta u \right) = 0. \tag{13}
\]

Relation (13) will characterize four different situations, each of which has its own design of optimal control algorithms.

The first situation is typical when solving control problems based on the minimum principle, where the very fact of the existence of Pontryagin’s extremal is postulated: \( x = x_0(t), u_{opt}(t, t) = u_0(t), \phi_x = \psi^T(t) \) Then, with fixed initial conditions \( (\delta x(0) = 0) \) from (13), one can formally write out the equation of the conjugate system (the equation of momenta), and from formula (11), determine the conditions for its transversality.

The second situation takes place when solving problems of CO synthesis by the method of differential DP [7], when the optimality of the trajectory \( x = x_0(t) \) can be indirectly judged by the optimality conditions of its individual sections (intervals \([t, t_f]\)) with nonzero variation of control \( (\delta u \neq 0) \) on these sections. This method of computation is not associated with direct variation of control and trajectory and, as follows from formula (13), is reduced to finding a minimizing sequence (to organizing a weak local improvement procedure) \( u_{opt}(t, \tau) \to u_{opt}(t, t) = u_0(t) \), where the local optimal control \( u_0(t) \) is determined by the stationary condition. When \( x = x_0(t) \) the function \( \phi(t, x) \) is the Bellman function \( S(t, x) (S_x = \psi^T(t)) \).

The third situation corresponds to the case \( x \neq x_0(t), u_{opt}(t, t) = u_0(t) \), allows and actually recommends a type of approximation, called the approximation in the policy space [4], which is also absent in classical analysis. The policy (the procedure for strong local improvement) \( x(\tau) \to x_0(t) \) according to (11), (12) is formed according to the condition

\[
\inf_{x \in F_f} \left( \frac{\partial \phi(t, x)}{\partial x} \frac{\partial f}{\partial x} + \frac{\partial f_0}{\partial x} \right) \delta x = \inf_{x \in F_f} \left( \frac{\partial H(t, x_0, \phi, u_0)}{\partial x} \delta x \right) = 0,
\]

from which, due to the stationarity of the points \( x_0(t) \) (the function \( H \) does not depend on \( x \)) for a small nonzero variation of the trajectory \( \delta x \), the identity
and the vector \( x_0 \) is determined. By organizing the improvement procedure \( x(\tau) \to x_0(\tau) \), an approximate calculation of the Bellman function \( S(t, x_0) \) is provided through a function \( \varphi(t, x) \) that can be chosen arbitrarily here, that is, in fact, it is a Lyapunov function.

The fourth situation formally resembles the classical formulation of the solution of variational problems, since it is supposed to use nonzero variations of the trajectory and control in it due to quasilinearization: \( x \neq x_0(t), \ u \neq u_0(t) \). Through the relations
\[
\frac{\partial H(t, x_0, \varphi, u_0)}{\partial x} = 0,
\]

\[
\frac{\partial H(t, x_0, \varphi, u)}{\partial u} = 0,
\]

\[
\frac{\partial H(t, x_0, \varphi, u)}{\partial \varphi} = 0,
\]

iterative improvement procedures \( u_{\text{opt}}(t, \tau) \to u_{\text{opt}}(t, t) = u_0(t) \), \( x(\tau) \to x_0(\tau) \) are organized here, ensuring the fulfillment of the necessary condition for the absolute local minimum of functional (4): \( \delta x \to 0, \ \delta u \to 0 \) convergence is expected at
\[
\frac{\partial H(t, x, \varphi, u)}{\partial x} \delta x + \frac{\partial H(t, x, \varphi, u)}{\partial u} \delta u \to 0.
\]

We define the sets
\[
D_1(\varepsilon) = \{(x(t), u(t)) \in D : |x(t) - x_0(t)| < \varepsilon, t \in [t_0, t_f] \},
\]

\[
D_2(\varepsilon) = \{(x(t), u(t)) \in D : |u(t) - u_0(t)| < \varepsilon, t \in [t_0, t_f] \}.
\]

**Definition 1** [8]. A pair of functions \( (x_0(t), u_0(t)) \in D \) is called a strong (weak) local minimum if there is a number \( \varepsilon > 0 \) such that \( I(x_0, u_0) \leq I(x, u) \) for all \( (x(t), u(t)) \in D_1(\varepsilon)(D_2(\varepsilon)) \).

Let us now formulate a number of theoretical positions on the weak minimum, which formally follow from the analysis of formulas (12), (13).

**Theorem 1.** (conditions of local optimality in the form of the minimum principle). If in problem (1)–(5) there is a local minimum \( (x_0, u_0) \), then at each point of stationarity the following conditions are satisfied:

1) \[
\frac{\partial H(t, x_0, \varphi)}{\partial x} = 0, \quad \frac{\partial H(t, x_0, \psi)}{\partial x} = \psi^T(t),
\]

2) \[
V_g(x_0(t_f)) = \varphi(t_f, x_0(t_f)) - \varphi(t_0, x_0(t_0)),
\]

3) \[
H(t, x_0, \psi) = H(t, x_0, \psi, u_0) = \inf_{u \in D} H(t, x_0, \psi, u).
\]

Here conditions 1, 2) correspond to the canonically conjugate system of equations, which forms the two-point boundary value problem
\[
\dot{x}_0 = \frac{\partial H^T(t, x_0, \psi)}{\partial \psi} = f(t, x_0, u_0), \quad x_0(t_0) = x^0,
\]

\[
\psi = - \frac{\partial H^T(t, x_0, \psi)}{\partial x} = - \frac{\partial^T(t, x_0, u_0)}{\partial x} + \frac{\partial H^T(t, x_0, \psi)}{\partial x}, \quad \psi^T(t_f) = \frac{\partial N_g(x_0(t_f))}{\partial \psi(t_f)}.
\]

Condition 3) determines the control vector \( u_0(t) = u_{\text{opt}}(t, t) = \arg \min_{u \in D} H(t, x_0, \psi, u) \) which in the local sense provides a minimum to functional (4)
Thus, in the first situation considered above, the local minimum (optimal program) and the support functional $I_*$ are calculated through the solution of the two-point boundary value problem (14), (15).

Note that the assumption of Theorem 1 that a pair $(x_0,u_0)$ is a local minimum in problem (1)–(5) is somewhat heuristic until the fact of its existence has been proved [10, p. 24 – 26]. This fact is established by such a reformulation of the original CO problem, for which it is possible to organize procedures for finding minimizing sequences that monotonically converge with respect $u$ to a local minimum.

**Theorem 2** (first-order conditions for a local minimum). For a pair $(x_0,u_0)$ to be a weak local minimum of problem (1)–(6), it is necessary and sufficient to satisfy the following conditions:

1) \[ \nabla \phi(t,x_0) + \frac{\partial \phi(t,x_0)}{\partial x} f(t,x_0,u_0) + f_0(t,x_0,u_0) = 0, \]
2) \[ V_\gamma(x_0(t_f)) = \phi(t_f,x_0(t_f)) - \phi(t_0,x_0(t_0)), \]
3) \[ \frac{\partial H(t,x_0,\psi,u_0)}{\partial u} \equiv 0 \text{ for } u \in \text{int } U \text{ or } U = \mathbb{R}^m \text{ and for nonzero admissible control variation } \partial u. \]

**Remark 1.** Apparently, on the boundaries of the set $U$, the equality \[ \inf_{u \in \partial U} \left( \frac{\partial H(t,x,\phi,\psi,u)}{\partial u} \right) \partial u = 0 \]
following from the analysis of (13) will give way to the relation \[ \inf_{u \in \partial U} H(t,x,\phi,\psi,u) \xi(t) = 0, \]
where $\xi$ is a small number. $\xi(t)$ is a continuous and piecewise-smooth function on a set of small measure. Then condition 3) of Theorem 3 can be replaced by the condition: \[ \inf_{u \in \partial U} H(t,x,\phi,\psi,u) = 0. \]

Theorem 2 corresponds to the case of solving the problem of locally optimal synthesis of CO according to the differential DP scheme [7]. Here, the local improvement of control is carried out through the quasilinearization of the differential connection (3) and the integrand of the quality functional (4) in the neighborhood $u_0(t)$, i.e.

\[ \dot{x} = f(t,x_0,u_0) + \frac{\partial f}{\partial u} \partial u, \]
\[ I(u(\cdot)) = V_\gamma(x_0(t_f)) + \int_{t_0}^{t_f} \left( f_0(t,x_0(t),u_0(t)) + \frac{\partial f_0}{\partial u} \partial u \right) dt = I_* + \int_{t_0}^{t_f} \left( \frac{\partial f_0}{\partial u} \partial u \right) dt. \]

It can be seen directly from formulas (16), (17) that when organizing the approximation procedure $u_{\text{opt}}(t,\varepsilon) \rightarrow u_{\text{opt}}(t,\varepsilon) = u_0(t)$, the values of functional (17) tend to the lower exact boundary of the functional of the original problem (1)–(5).

Assertions similar to Theorem 2 were formulated [11, 12] for an approximation scheme in the policy space and an analogue of a variational scheme in the problem of localization and improvement.

**4. A method for solving two-point boundary value problems for continuous dynamic systems according to the differential DP scheme**

**A. Relaxation expansion of the state space.** The next constructive step towards the practical implementation of the above schemes for solving problem (1)–(5) is to determine the strategy for the approximate synthesis of locally optimal controls through the relaxation extension of the state space. Relaxation extension is associated with the study of the properties of the limiting elements of minimizing control search sequences that determine the initial formulation of the synthesis problem for a differential system (3).
The organization of the search for limiting elements is based on [13]:

1. The idea of quasi-linearization is the phase linearization of the process (3) and the integrand of the functional (4) with respect to the optimal ones determined during the operation of the object and constant on a finite number of small lengths of optimization \( \Delta t \) parameters \( u_0 = u^* \) according to formulas (16), (17).

2. The assumption about the admissibility of the choice of controls and \( I \) or trajectories that differ little from the optimal ones at a finite number of lengths \( \Delta t \), which makes it possible to organize an approximate synthesis strategy on a pair \( (u, u_0) \) according to the differential DP scheme

\[
\frac{d \delta u}{d t} = \mathcal{G}, \quad \delta u = u - u_0,
\]

where \( \mathcal{G} \) is "new" \( m \)-control vector.

If the optimization lengths \( \Delta t \) are small, then the derivatives in (18) with a sufficient degree of accuracy are described by the relation

\[
u(t) = u_0(t) + \mathcal{G}\Delta t,
\]

which can be implemented as an iterative procedure of Newtonian type for determining the local minimum: \( \mathcal{G} \rightarrow 0 \) at each point of stationarity, the following condition is met: \( u(t) \rightarrow u_0(t) \). Therefore, the choice of a gradient strategy of type (18) is natural, corresponds to the ideology of quasilinearization and the idea of nonlinear synthesis in the process of object functioning (3) (combined synthesis) at short lengths \( \Delta t \). Thus, the synthesis problem itself is represented in a linearized form: the set of all points \( x(t_1) \) of the terminal term of functional (5) becomes close to convex [14].

B. Application of the functional of generalized work in the problem of approximately-optimal synthesis of controllers. The essence of the gradient strategy (18) is a relaxation extension of the state space (3): \( y = (x, \delta u) \) for a differential DP scheme. Therefore, it is required in this way to reformulate the original formulation of the optimization problem (1)–(5) in order to be able to fix the limiting elements of minimizing sequences at the points of stationarity \( u = u_0 \). For this, it is proposed to apply the functional of generalized work (FGW) [1]

\[
I(y) = S_g(y(t_f)) + \int_{t_0}^{t_f} \left[ Q_p(y, \delta u) + L_{g_1}(\delta \mathcal{G}) + L_{g_1}(\delta \mathcal{G}) \right] d\theta.
\]

the result of minimizing which is the cost \( I \) of the original synthesis problem. Here

\[
L_{g_1}(\delta \mathcal{G}) = 0.5 \mathcal{G}^T r^{-1} \delta \mathcal{G}, \quad L_{g_1}(\mathcal{G}) = 0.5 \mathcal{G}^T r^{-1} \mathcal{G}
\]

are some quadratic forms from the "new" controls. The integrant of the functional (4) linearized in the FGW (19) for the differential DP scheme is given in the form

\[
Q_p(t, y) = f_0(t, x_0, u_0) + \frac{\partial Q_p}{\partial \mathcal{G}} \delta u.\]

The variable \( \mathcal{G}_0 \) at lengths \( \Delta t \) is a constant, non-variable parameter: \( \mathcal{G}_0 = \mathcal{G}^* \).

The problem of obtaining the optimal solution for the optimization process (3) linearized at small lengths \( \Delta t \) is formulated as follows: to organize iterative procedures for finding the weak (Theorem 2), which provide the infimum of the FGW (19) under differential constraints (3).

The formulated problem of approximately optimal synthesis is solved by the method of characteristic strips [1]. The main result is formulated as the following theorem.

**Theorem 3.** For process (3), the optimal control in the sense of achieving a local minimum of functional (3) and FGW (19) is determined by the weak improvement procedure \( u_{opt}(t, \tau) \rightarrow u_{opt}(t, t) = u_0(t) \)

obtained from the canonically conjugate system: differential constraint (16) and the equations
\[
\dot{p}_s = -\frac{\partial f^T_0(t, x_0, u_0)}{\partial x} - \frac{\partial f^T_0(t, x_0, u_0)}{\partial x} p_x + \frac{\partial (\delta u)^T \hat{p}_\delta u}{\partial x}, \tag{20}
\]
\[
\dot{p}_\delta u = -\frac{\partial f^T_0(t, x_0, u_0)}{\partial u} - \frac{\partial f^T_0(t, x_0, u_0)}{\partial u} p_x, \tag{21}
\]
\[
\delta = -r p_\delta u, \tag{22}
\]
\[
\dot{S}(t, x, \delta u) = -f_0(t, x_0, u_0) - \frac{\partial f_0(t, x_0, u_0)}{\partial u} \delta u, \tag{23}
\]

where \( S \) is the Bellman function for a problem with an extended state vector,

\[
p_x = \frac{\partial S^T}{\partial x} = \psi_x(t), \quad p_\delta u = \frac{\partial S^T}{\partial \delta u} = \psi_{\delta u}(t) \text{ is co-trajectories of state and control variations.}
\]

The proof of Theorem 3 is carried out through a direct transformation of the optimality conditions to simpler conditions in the form of the Lyapunov equation for the extended state space, followed by its solution by the method of characteristics.

The method of proving Theorem 3 is reduced to the following.

1) We introduce an extended state vector \( y = (x, \delta u) \) and transform equations (16), (17) to the form

\[
\dot{y} = f(t, y) + \Gamma_1 \delta \mathcal{H}, \tag{24}
\]

where \( f(t, y) = (f(t, x_0, u_0) + \frac{\partial f}{\partial u} \delta u, 0) \) is the vector function obtained in the case of quasi-linearization according to the differential DP scheme; \( \Gamma_1 = \begin{bmatrix} 0 \\ E_1 \end{bmatrix} \) is the rectangular matrix with a “new” control vector \( \delta \); \( E_1 \) is the unit matrix of \( m \times m \) dimensions, respectively; \( y(t_0) = (x(t_0), 0). \)

Then we form the FGW for the extended state and control space \( X \times U \times T \) in the form (19)

\[
(S_y(y(t_0))) = S_y(x(t_0))).
\]

2) Writing out sufficient conditions for optimality

\[
\inf_{\delta \in \mathbb{R}^n} \left( \frac{\partial \phi(t, y)}{\partial t} + \frac{\partial \phi(t, y)}{\partial y} (f(t, y) + \Gamma_1 \delta \mathcal{H}) + Q_p(t, y) + 0.5 \delta_0^T r^{-1} \delta_0 \right) = 0, \tag{25}
\]

from which we determine the “new” controls \( \delta \mathcal{H} \) that are optimal in the local sense

\[
\delta \mathcal{H} = \delta_0 = -r \Gamma_1 \frac{\partial \phi^T(t, y)}{\partial \delta u} = -r \frac{\partial \phi^T(t, y)}{\partial \delta u}. \tag{26}
\]

If we introduce the notation: \( p_\delta u = \frac{\partial \phi^T}{\partial \delta u} \) in the last expressions, then condition (22) of the theorem is fulfilled. Formula (26) can be obtained differently through the stationarity condition: \( \frac{\partial H}{\partial \delta u} = 0 \),

where \( H(t, y, \phi) = \frac{\partial \phi(t, y)}{\partial y} (f(t, y) + \Gamma_1 \delta \mathcal{H}) + Q_p(t, y) + 0.5 \delta_0^T r^{-1} \delta_0 \) is the Hamiltonian of system (24), is the vector is the row of dimension \( 1 \times (n + m) \).

3) We formulate formula (26) into expression (25), as a result of which the sufficient optimality conditions are written in the form of the equation

\[
\frac{\partial \phi(t, y)}{\partial t} + \frac{\partial \phi(t, y)}{\partial y} f(t, y) + Q_p(t, y) = 0, \tag{27}
\]
where the function $\phi(t,y)$ has the meaning of the Lyapunov function in stability theory.

Formula (27) defines the “free” motion of the system (24). The total derivative calculated on the “free” motion is determined by the expression

$$\dot{\phi}(t,y) = \frac{\partial \phi(t,y)}{\partial t} + \frac{\partial \phi(t,y)}{\partial y} f(t,y).$$

(28)

Equation (27) taking into account expression (28) takes the form

$$\dot{\phi}(t,y) = -Q_p(t,y),$$

from which condition (23) of Theorem 3 follows.

4) Using the method of characteristics, we determine the solution of the partial differential equation (27) in the form of a canonically conjugate system [1]

$$\dot{y} = \mathcal{H}^T(t,y,p) = f(t,y), \quad \dot{p} = -\frac{\partial \mathcal{H}^T(t,y,p)}{\partial y} = -\frac{\partial f^T(t,y)}{\partial y} p - \frac{\partial Q_p^T(t,y)}{\partial y},$$

(29)

where $\mathcal{H}(t,y,p) = \frac{\partial \phi(t,y)}{\partial t} + \frac{\partial \phi(t,y)}{\partial y} f(t,y)$ is the Hamiltonian of the “free” motion of system (24), $p = \frac{\partial \phi^T(t,y)}{\partial y}$ is the column vector of partial derivatives of dimension.

Expanding vectors $y, p$ through subvectors $x, \partial u$ and $p_x, p_{\partial u}$ as a result of decomposition of relations (29), we obtain formulas corresponding to conditions (16), (20)–(21) of Theorem 3.

5) Using the procedure for finding a weak local minimum $u_{op}(t, \tau) \rightarrow u_{op}(t, t) = u_0(t)$, we find the optimal one in the sense of achieving a local minimum of functional (4) of process (3).

Thus, all conditions of Theorem 3 are satisfied. The theorem is proved.

5. Algorithms with a predictive model

Various variants and editions of algorithms with a predictive model are known and developed [1, 4]. The basic ones are the modified algorithm and the algorithm with the sensitivity matrix, the software implementation of the first of which will be shown using the example of using the differential DP scheme (Theorem 3).

The algorithm is based on direct computation of gradients $p_x, p_{\partial u}$ by formulas (20), (21) and implements the procedure of weak local improvement through the sequence of the following operations.

Step 1. The state of the object on the interval $[t_u, t_f]$ is predicted using the model

$$\dot{x}_u = F_u(\tau, x_u, u_u),$$

(30)

where $F_u = f(\tau, x_u, u_0) + \frac{\partial f}{\partial u} \partial u$, $x_u(\tau_u) = x(t_u)$, $u_u(\tau_u) = u(t_u)$; $t_u$ are the moments of time corresponding to the end of the next optimization cycle of length $\Delta t$; the index "M" denotes belonging to a model (predictive) movement. The forecast of the movement of the model is carried out by integrating equations (30) in accelerated direct time $\tau$ with different initial conditions $x_u(\tau_u)$ ($i = 1, \cdots, m + 1$) lying in the vicinity of the current state of the object $x(t_u)$.

Step 2. The values of the vector $x_M(\tau_f)$ are determined, and on the basis of the previously differentiated by $x_u$ and by $u_u$ the function $S$, its gradients are calculated at the final moment of time $\tau_f = t_f$

$$p_x(\tau_f) = \frac{\partial S^T(x_M(\tau_f), \partial u(\tau_f))}{\partial x_M(\tau_f)}; \quad p_{\partial u}(\tau_f) = \frac{\partial S^T(x_M(\tau_f), \partial u(\tau_f))}{\partial \partial u(\tau_f)}.$$
Step 3. With initial conditions (31) on an interval \([t_0, \tau_f] \) with a step \( \Delta \tau \) at \( j = 0 \), the system of equations (16), (20)–(23) is integrated in the accelerated reciprocal time
\[
\dot{x}_m = -T^F_m (\tau, x_m, u_{mj}),
\]
\[
\dot{p}_x = \frac{\partial f^T_0 (\tau, x_m, u_{mj})}{\partial x_m} + \frac{\partial f^T (\tau, x_m, u_{mj})}{\partial x_m} p_x - \frac{\partial (\dot{u}_m^T \dot{p}_{su})}{\partial x_m},
\]
\[
\dot{p}_{su} = \frac{\partial F^T_0 (\tau, x_m, u_{mj})}{\partial u_m} + \frac{\partial F^T (\tau, x_m, u_{mj})}{\partial u_m} p_x, \tag{32}
\]
\[
\varphi_{mj} = r p_{su},
\]
\[
u_{mj+1} = u_{mj} - \varphi_{mj} (T_p - n \Delta \tau),
\]
\[
T_p = t_x - t_0, \quad \Delta \tau = k \Delta t, \quad k = \frac{T_p}{\Delta t}, \quad n = 1, k. \]

Remark 2. When organizing the iterative procedure, the model control of the previous iteration \( u_0 \) is considered instead of the vector \( u_{mj} \) in the PM equations.

Additionally, together with relations (32), the Bellman function is calculated for the problem with an extended state vector
\[
\dot{S}(\tau, x_m, \delta u) = \dot{G}_m (\tau, x_m, \delta u), \quad \delta u = u_{mj+1} - u_{mj}. \tag{33}
\]

The partial differential matrices of the vector and scalar functions \( F, \dot{G} \) with respect to components \( x_m, \delta u \) are calculated based on predictions (30).

Step 4. At the optimization lengths \( \Delta t \), an iterative procedure for refining the points of stationarity of the local minimum is organized (figure 1) at \( j = 1, N_g \) according to formulas (32), (33). The condition for stopping the iterative procedure is the fulfillment of the inequality
\[
\delta u \leq \varepsilon_g
\]
or with a coarser estimate for the accuracy of calculations: \( j \leq N_g \).

Step 5. When the stop condition is met, the newly obtained control is taken as optimal \( u_{mj+1} = u_0 \) and fed to the object (3). After a period of time \( \Delta t \), steps 1–4 of the algorithm are repeated (figure 1). In this case, at each section \( \Delta t \) in model (32), the condition is satisfied with a given accuracy: \( \varphi_{mj} = \delta_0 = 0 \).

6. Properties of convergence of algorithms with a predictive model

A drawback of all modern numerical methods used in direct methods (for example, the method of inverse problems of dynamics [15]) and in indirect methods of interval optimization in a precise formulation is the lack of a strictly mathematical proof of their convergence. The procedures of direct methods are gradient and allow determining only a local extremum, and not minimizing functional (4) in the entire domain of its definition. For the methods of interval optimization considered above, a “ravine” situation is often characteristic: the process gets stuck in the vicinity of one of the local extremum points, due to which problem (1)–(5) either has a solution far from true, or in general divergence of control processes can be observed.

To overcome these difficulties, it is proposed to apply the differential DP procedure to obtain a rough initial approximation of the points of stationarity of the local minimum, and the gradient quasi-linearization procedure for the subsequent refinement of these points. Therefore, the proof of the convergence of algorithms constructed on the basis of formulas (16), (20), (21) of Theorem 3 should consist of two stages: proof of the convergence of processes under the stationarity conditions to the
solution of the original problem (1)–(5), proof of the convergence to the local minimum in the vicinity of the stationarity points by the method of quasilinearization. Let's consider both of these stages.

A. Properties of convergence of algorithms with predictive models for interval optimization of control processes. The proof of the convergence according to the above-considered differential DP scheme is based on considering the properties of the limiting elements of minimizing sequences for finding a local minimum \((x_0, u_0)\). To reveal these properties, we introduce the concept of an attainable set [16, 17].

**Definition 2.** The set of attainability \(X(t, X_0, \tau)\) at the time \(\tau\) instant of systems (16), (20), (21) is the set generated at the time instant \(t\) by the set of initial states \(X_0\) of these systems.

The reachability set at \(\tau \to t\) is estimated by the result of minimizing functional (19), which is approximately calculated through an iterative improvement procedure by \(u_{opt}(t, \tau) \to u_{opt}(t, t) = u_0(t)\) the value of the support functional \(I_*\) at the time instant \(t\) \((t \in [t_0, t_f])\) equal to the value of the local optimal functional (4) of the original synthesis problem (1)–(5).

**Definition 3.** The limit set of reachability \(X^*(t, X_0, \tau)\) of systems (16), (20), (21) is a set fixed at a time \(\tau = t\) instant generated by the set of initial states \(X_0\) of these systems.

![Image](image.jpg)
The limiting reachability set for $u_{\text{opt}}(t) = u_0(t)$ defines the same result of minimizing the FGW (19) in the extended formulation (Theorem 3) and functional (4) in the original formulation of the nonlinear synthesis problem (1)–(5).

**Theorem 4** (on the definition of the elements of the limit reachability set). If the control $u = u_0(t)$ corresponds to the local minimum of the functional (4) and determines the local minimum of the process (3) on non-closed sets $u \in R^n$, $x \in R^n$, then there exist such vector, nonzero, and continuous functions $p_x$ and scalar functions $\varphi$ that satisfy the following conditions:

1) a controlled predictive model: a system of canonically conjugate equations is solved in accelerated time: a differential connection (3) and a vector equation

$$\dot{p}_s = - \partial f^T_0(t, x_0, u_0) \partial x - \partial f^T(t, x_0, u_0) \partial x p_s, \quad p_s(t_f) = \partial V^T_f(x_0(t_f)),$$

2) the condition for the points of the Hamilton function to be stationary with respect to

$$\frac{\partial H^T}{\partial \dot{u}} = \dot{p}_\delta = - \frac{\partial f^T_0(t, x_0, u_0)}{\partial x} \dot{u}, \quad \frac{\partial f^T(t, x_0, u_0)}{\partial x} p_s = 0,$$

3) the condition for the vanishing of the limiting elements of the minimizing sequence with respect to in the gradient procedure (22)

$$\vartheta = \vartheta_0 = -rp_\delta = 0,$$

4) the condition for the minimum of the local functional determined by calculating the Lyapunov function

$$\dot{I}(t) = \dot{\varphi}(t, x_0) = - f_0(t, x_0, u_0) \varphi(t_f, x_0(t_f)) = V^T_f(x_0(t_f)).$$

Let us carry out the proof of the theorem for the differential DP scheme (Theorem 3).

The equations of the canonically conjugate system (3), (34) and formula (37) are obtained from formulas (16), (20), (23) at $u = u_0(t)$.

We obtain stationarity conditions (35) and condition 3) of the theorem following the scheme of Bliss’s proof [18]. To do this, consider the control variations determined by the gradient procedure (18), which at small optimization lengths $\Delta t$ is represented by the relation $u(t) = u_0(t) + \vartheta \Delta t$.

From the last expression it can be seen that a weak local minimum $(x_0, u_0)$ is formed through the fulfillment of the condition $u = u_0(t)$ at each point of stationarity: exactly by ensuring the equality of the limiting elements of the minimizing sequences to zero in: $u : \vartheta = \vartheta_0 = 0$; approximately by decreasing the optimization lengths: $\Delta t \rightarrow 0$.

Suppose the opposite that there is a control $\bar{u}(t) = u_0(t) + \Delta u(t)$ in which the minimum of the local criterion is less than the minimum of the local functional: $\bar{I}(t) < I(t)$.

Thus, $d(\bar{u}(t) - u_0(t)) = \frac{d\Delta u(t)}{dt}$, which contradicts the conditions of local control optimality in (36):

$$\frac{d\Delta u}{dt} = 0.$$

Then $\Delta u(t) = 0$ and $\vartheta = \vartheta_0 = 0$.

Further, from formula (22) in Theorem 3 it follows that in the absence of left zero divisors, the subvector $p_\delta$ of the extended skew-state vector is: $p_\delta = 0$.

The derivative of this subvector will also be equal to zero: $p_\delta = 0$, whence, by virtue of the fulfillment of the relation $\dot{p}_\delta = \frac{\partial H^T}{\partial \dot{u}}$, equation (35) of condition 2) of Theorem 4 turns out to be valid. The theorem is proved.
Thus, Theorem 4 completes the proof of the convergence of algorithms with a predictive model in terms of stationarity. The main property of a weak local minimum is that in the absence of constraints on control, it is determined by the stationary points of the Hamilton function, which satisfy the vector equation:
\[
\frac{\partial H}{\partial \delta u} = 0.
\]

Note that the same result of minimizing the FGW (19) in the statement of the problem of synthesizing the CO with the relaxation extension of the state space and functional (4) in the original statement allows us to conclude that the limit reachability sets for the extended problem coincides with the set the reachability of problem (1)–(5). The main difference between the extended formulation of the problem of nonlinear synthesis is that it defines a minimizing sequence for \( u \) finding a weak local minimum, while in the original formulation only the fact of its existence is stated.

If the control can be uniquely determined (synthesized) in the form of feedbacks \( u = u_0(t, x, p_x) \), then the solution of the original problem (1)–(5) is reduced to the solution of a two-point boundary value problem for the differential system (3), (34), (35).

Apparently, a weak local minimum exists only when, when the necessary condition (stationarity or the minimum principle) is satisfied, the sufficient condition for optimality is simultaneously satisfied (as in stability theory), but not vice versa. In the problem of nonlinear synthesis, this is not entirely obvious.

**B. Properties of convergence of algorithms with a predictive model in the vicinity of stationarity points of the local minimum.** The differential DP procedure allows obtaining a rough initial approximation of the stationarity points of the local minimum at the optimization lengths \( \Delta t \) (figure 1). To refine the obtained solution, quasi-linearization is used where the gradient procedure (18) for sequential improvement of the stationarity points. Therefore, it can be argued [4, 19] that in algorithms with a predictive model, the quasi-linearization method is simply an application of the Newton – Raphson – Kantorovich method in a functional space. Since the convergence of the method takes place for initial approximations from some neighborhood of the stationarity points of a weak local minimum, the gradient procedure (18) is locally convergent.

It is known [4] that the quasi-linearization method has the basic properties – monotonicity and quadratic convergence. Let us prove similar properties of local convergence of predictive algorithms, following the Bellman method.

1) Monotonicity property.

**Definition 4 [20].** If, during the implementation of the gradient procedure (18), the values of functional (19) at iterations \( u_j \) decrease monotonically, then the sequence \( u_j \rightarrow u_0 \) is called relaxation, and the parameter \( \mathcal{G} \) is called a relaxation parameter.

There are two types of convergence of gradient procedures: functional convergence and strong convergence (convergence).

**Definition 5.** Gradient procedure (18) is called convergent if
\[
\lim_{j \rightarrow \infty} \| u_j - u_0 \| = 0.
\]

Obviously, the convergence of procedures implies convergence in terms of the functional. The converse is generally not true.

The property of monotony in algorithms with a predictive model implies the convergence of the gradient procedure (18), which, in turn, depends on the initial approximations in the vicinity of the stationarity points of the weak local minimum and the chosen relaxation parameter. Since the initial approximations are calculated according to the differential DP scheme, to prove the monotonicity of iterative procedures, it remains to determine the property of the relaxation parameter \( \mathcal{G} \).

This parameter has a functional meaning and is calculated by the formula (22)
\[
\mathcal{G} = -r \frac{\partial \mathcal{F}^T}{\partial u} = -rp_{\delta u}.
\]
At the optimization lengths $\Delta t$, at each point of stationarity of the weak local minimum, the parameter $\mathcal{G}$ vanishes (formula (40)), due to which it can be argued that for $r > 0$, the gradient $p_{\mathcal{G}}$ is a decreasing positive function of time, and the variable $u$ in steps of iterations obeys the condition

$$u_0(t) < u_0'(t) < \cdots < u_0^2(t) < u_0^i(t).$$

This condition characterizes the property of monotonicity of process (18) and is illustrated in figure 2 for the quasi-linearization procedure for the calculated initial approximation of local optimal control.

2). Quadratic convergence.

The second important and not so obvious property of the gradient procedure (18) is quadratic convergence.

**Figure 2.** Graph explaining the property of monotone convergence gradient procedures

**Statement 1.** For the gradient procedure (18), the following estimate holds:

$$\left| u_0^i - u_0 \right| \leq k_1 \left| u_0' - u_0 \right|^2, \quad \left| u_0^i - u_0' \right| \leq k_2 \left| u_0' - u_0^{-1} \right|^2,$$

where $k_1 = \max_{u_0 \leq \theta \leq u_0'} \frac{\tilde{\mathcal{G}}(\theta)}{2}$, $k_2 = \max_{u_0 \leq \theta \leq u_0'} \left( \frac{\tilde{\mathcal{G}}(\theta)}{2} + \frac{1}{\Delta t} E_2 \right)$, $E_2$ is the unit size matrix.

The proof of the validity of estimate (38) is reduced to the following.

Let $u_0^i$ be the initial approximation of the locally optimal control vector $u_0$, and the general recurrence relation is written out from (18)

$$u_0^{i+1} = u_0' + \mathcal{G}(u_0') \Delta t,$$

where $\mathcal{G}(u_0') = -r p_{\mathcal{G}}(u_0')$.

To verify the validity of the estimate in (39) for the $i$ components of vectors $u_0^{i+1}, u_0, \mathcal{G}(u_0')$, we write

$$u_0^{i+1} - u_0 = u_0' + \mathcal{G}(u_0') \Delta t - u_0 = u_0' + \mathcal{G}(u_0') \Delta t - (u_0 + \mathcal{G}(u_0') \Delta t) = \xi(u_0') - \xi(u_0),$$

where $\xi(u) = u + \mathcal{G}(u) \Delta t$,

$$\xi(u_0) = -r p_{\mathcal{G}}(u_0) = 0 \text{ (see Theorem 4).}$$

Expression (40) is represented by a Taylor series, in which we take into account the first three terms, including the remainder.
Here, by the mean value theorem (Lagrange’s theorem [4]), the remainder is
\[
\int_{u_0}^{u_i} (u_{0i} - u_i) \tilde{z}_i^1 (\theta) d\theta = \frac{(u_{0i} - u_0)^2}{2} \tilde{z}_i^1 (\theta),
\]
where \( \theta \) is some value of the independent variable, intermediate between \( u_{0i} \), \( u_0 \), \( u_i \) \( \leq \theta \leq u_{0i} \).

Since \( \tilde{z}_i^1 (u) = \hat{u}_i + \hat{\theta}_i (u) \Delta t \), then at the points of stationarity: \( u_i = u_{0i} : \tilde{z}_i^1 (u_0) = 0 \), since \( u_{0i} \) is a parameter not varying over the optimization lengths, and the condition \( \hat{p}_{\delta u} (u_0) = 0 \) follows from Theorem 4.

Therefore, the estimate
\[
|u_{0i}^{j+1} - u_0| \leq k_{u_i} |u_{0i}^j - u_0|^2,
\]
where \( k_{u_i} = \max_{u_0 \leq u \leq u_{0i}} \frac{\tilde{z}_i^1 (\theta)}{2} \).

Formula (41), written in vector form, is the first sought-for estimate in (38).

The second estimate in formula (38) is obtained by direct transformation of the recurrence relation (40)
\[
u_{0i}^{j+1} - u_0^j = \tilde{z}_i^1 (u_0^j) - s_i^1 (u^{j-1}) = (u_{0i}^j - u_0^{j-1}) \tilde{z}_i^1 (u_0^{j-1}) + \frac{(u_i^j - u_0^{j-1})^2}{2} \tilde{z}_i^1 (\theta),
\]
where \( u_i^{j-1} \leq \theta \leq u_0^j \). \( \tilde{z}_i^1 (u^{j-1}) = \hat{u}_i^{j-1} + \hat{\theta}_i (u_0^{j-1}) \Delta t \).

On the other hand, it follows from formula (40) that \( \hat{u}_i^j = \hat{u}_i^{j-1} + \hat{\theta}_i (u_0^{j-1}) \Delta t \). Then
\[
\tilde{z}_i^1 (u_0^{j-1}) = \hat{u}_i^{j-1} \frac{u_0^j - u_0^{j-1}}{\Delta t} = \hat{u}_i^{j-1} + \hat{\theta}_i (u_0^{j-1}) \Delta t.
\]

From which follows the estimate
\[
|u_{0i}^{j+1} - u_0^j| \leq k_{z_i} |u_{0i}^{j-1} - u_0^j|^2,
\]
where \( k_{z_i} = \max_{u_0 \leq u \leq u_{0i}} \frac{\tilde{z}_i^2 (\theta)}{2} + \frac{1}{\Delta t} \).

Equation (43), written in vector form, is the second sought-for estimate in formulas (38). The property expressed by the second estimates in expressions (38) is called quadratic convergence [4].

Obviously, the convergence of the quasilinearization method is significantly accelerated as it approaches \((x_0^i, u_0^i) \rightarrow (x_0, u_0)\). It turns out that each next step asymptotically doubles the number of correct signs in this approximation. Therefore, the property of quadratic convergence turns out to be especially useful in solving multidimensional problems, and not only because the calculation time is directly proportional to the number of iterations, but also due to the increase in round-off errors during these iterations.

Thus, the combined use of interval and local optimization procedures can significantly simplify control algorithms. Instead of simultaneously using the characteristics of the first and second orders (the equations of the adjoint system and the matrix nonlinear equation of the Riccati type) to ensure the expected quadratic convergence, it is proposed to use only the characteristics of the first order, the stationarity points of which are refined by quasilinearization at the lengths \( \Delta t \), and the rough initial approximation
of the local minimum is determined from the procedure according to the differential dynamic programming scheme.

7. Conclusion
On the basis of the conditions of Theorems 3 – 4, algorithmic support of the integrated ACS has been developed, stratified by the levels of aircraft control (for example, [21, 22]). The efficiency of the non-linear synthesis algorithms has been tested on a number of test examples [23–24] and on model problems of the dynamics of promising automated collision avoidance and wind shear mitigation systems when approaching a middle class aircraft.

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