Klauder-Bargmann Integral Representation of
Gaussian Symmetries and Generating Functions of
Gaussian States

Tiju Cherian John and K. R. Parthasarathy

Indian Statistical Institute, Delhi Centre, 7, SJSS Marg, New Delhi, India, 110059
tijucherian@gmail.com, krp@isid.ac.in

In memory of V. S. Varadarajan.

Abstract

Let $\mathcal{H}$ be a finite dimensional complex Hilbert space of dimension $n$ and let $\Gamma(\mathcal{H})$ be the boson Fock space over $\mathcal{H}$. A unitary operator $U$ in $\Gamma(\mathcal{H})$ is called a $n$-mode gaussian symmetry if, for every gaussian state $\rho$ in $\Gamma(\mathcal{H})$, the transformed state $U\rho U^\dagger$ is also gaussian. It is shown that every gaussian symmetry admits a Klauder-Bargmann integral representation in terms of coherent states. This construction provides an explicit strongly continuous, irreducible, and projective unitary representation of the Lie group which is the semidirect product of the additive group $\mathcal{H}$ and the group $Sp(\mathcal{H})$ of all real linear symplectic transformations of $\mathcal{H}$.

For any bounded operator $Z$ in $\Gamma(\mathcal{H})$, the notion of a generating function $G_Z(u, v)$ with $u, v$ in $\mathcal{H}$ is introduced by using the matrix entries of $Z$ in the overcomplete basis of exponential vectors. The generating functions of gaussian symmetries and gaussian states are explicitly computed. It is shown that such symmetries and states belong to a semigroup $E_2(\mathcal{H})$ of operators in $\Gamma(\mathcal{H})$, in which every element $Z$ is completely determined by its matrix entries at 0, 1 and 2-particle vectors in a complete orthonormal basis of $\Gamma(\mathcal{H})$ consisting of finite particle vectors. This leads to some remarks on the role of generating functions in the tomography of gaussian states.

The notion of generating function yields an algorithm for constructing all mean zero pure gaussian states in a finite particle basis. The same algorithm leads to the construction of a hierarchy of mean zero, pure gaussian states which are completely entangled and symmetric under the action of the permutation group on the set of modes.

Keywords: Quantum gaussian states, coherent states, particle basis, completely entangled states, tomography of gaussian states, nonlinear information channel, symplectic group, generating function, semigroup of operators.

2010 Mathematics Subject classification: 81P40, 81R30, 81S05, 46L53.

1 Introduction

The principal aim of this paper is an analysis of gaussian states and their symmetries through a new scheme of parametrization. It replaces the customary mean values
and covariances of position and momentum observables which assume all values on the real line. To this end we consider the \( n \)-mode boson Fock space \( \Gamma(\mathcal{H}) \) over an \( n \)-dimensional complex Hilbert space \( \mathcal{H} \) with a chosen and fixed orthonormal basis \( \{ e_j, 1 \leq j \leq n \} \), where the index \( j \) stands for the \( j \)-th mode and \( n \) for the total number of modes. The study is based on three already well-known tools and a fourth one which is not widely used. We shall repeatedly use the gaussian integral formula, properties of exponential vectors and coherent states, the Weyl displacement operator and the quantum Fourier transform and, finally, the elementary idea of generating function of a bounded operator on \( \Gamma(\mathcal{H}) \).

Section 2 contains a brief summary of some well-known properties of exponential vectors and coherent states as well as the definition of generating function of a bounded operator on \( \Gamma(\mathcal{H}) \). In order to make the exposition fairly self-contained the Klauder-Bargmann isometry from the Hilbert space \( \Gamma(\mathcal{H}) \) into \( L^2(\mathbb{C}^n) \) is described along with a short proof. The Klauder-Bargmann formula for the resolution of the identity operator on \( \Gamma(\mathcal{H}) \) as an integral of coherent states with respect to a suitably normalized Lebesgue measure in \( \Gamma(\mathbb{C}^n) \) is given. This is repeatedly used in our analysis.

In Section 3, the Weyl displacement operators are presented as a projective unitary representation of the additive group \( \mathcal{H} \) in the Hilbert space \( \Gamma(\mathcal{H}) \) and the associated quantum Fourier transform of a state in \( \Gamma(\mathcal{H}) \) is defined. The Wigner isomorphism between the Hilbert space \( \mathcal{B}_2(\Gamma(\mathcal{H})) \) of all Hilbert-Schmidt operators on \( \Gamma(\mathcal{H}) \) and \( L^2(\mathbb{C}^n) \) is established.

A unitary operator \( U \) in \( \Gamma(\mathcal{H}) \) is called a gaussian symmetry if \( U\rho U^\dagger \) is a gaussian state whenever \( \rho \) is a gaussian state. In Section 4 every gaussian symmetry \( U \) is realized as a Klauder-Bargmann integral in terms of coherent states with respect to the Lebesgue measure in \( \mathbb{C}^n \). This construction yields a strongly continuous, projective unitary and irreducible representation of the Lie group which is the semidirect product of the additive group \( \mathcal{H} \) and the group \( Sp(\mathcal{H}) \) of all symplectic linear transforms of \( \mathcal{H} \). It is also shown that the generating function of a gaussian symmetry admits an exponential formula.

It is in Section 5 we construct the central object of our paper, namely, the operator semigroup \( \mathcal{E}_2(\mathcal{H}) \) contained in the algebra \( \mathcal{B}(\Gamma(\mathcal{H})) \) of all bounded operators on \( \Gamma(\mathcal{H}) \) by using the idea of generating function. To this end, we identify \( \mathcal{H} \) with \( \mathbb{C}^n \) through the mode-basis mentioned at the beginning of Section 2. We say that a bounded operator \( Z \) on \( \Gamma(\mathcal{H}) \) is in the class \( \mathcal{E}_2(\mathcal{H}) \) if, for all \( \mathbf{u}, \mathbf{v} \) in \( \mathcal{H} \), the following holds:

\[
\langle e(\mathbf{u})|Z|e(\mathbf{v}) \rangle = c \exp \{ \mathbf{u}^T \mathbf{\alpha} + \mathbf{\beta}^T \mathbf{v} + \mathbf{u}^T A \mathbf{u} + \mathbf{u}^T \Lambda \mathbf{v} + \mathbf{v}^T B \mathbf{v} \},
\]

for some ordered 6-tuple \( (c, \mathbf{\alpha}, \mathbf{\beta}, A, \Lambda, B) \) consisting of a scalar \( c \neq 0 \), vectors \( \mathbf{\alpha}, \mathbf{\beta} \) in \( \mathbb{C}^n \) and \( n \times n \) complex matrices \( A, \Lambda, B \) with \( A \) and \( B \) being symmetric. Here \( e(\mathbf{u}) \) and \( e(\mathbf{v}) \) are the exponential vectors in \( \Gamma(\mathcal{H}) \) associated with \( \mathbf{u} \) and \( \mathbf{v} \) respectively, bar indicating complex conjugation. We say that this 6-tuple are the \( \mathcal{E}_2(\mathcal{H}) \)-parameters of the operator \( Z \). By the properties of the exponential vectors in \( \Gamma(\mathcal{H}) \) summarised in Section 2 this parametrisation is unambiguous. If \( Z \) is a selfadjoint element of \( \mathcal{E}_2(\mathcal{H}) \), \( c \) is real, \( \mathbf{\beta} = \mathbf{\bar{\alpha}}, \mathbf{B} = \mathbf{\bar{A}} \) and \( \Lambda \) is hermitian. Thus the \( \mathcal{E}_2(\mathcal{H}) \)-parameters of a selfadjoint element in \( \mathcal{E}_2(\mathcal{H}) \) reduce to a quadruple \( (c, \mathbf{\alpha}, A, \Lambda) \). The class \( \mathcal{E}_2(\mathcal{H}) \) is shown to enjoy the following properties:

1. \( \mathcal{E}_2(\mathcal{H}) \) is a \( \dagger \)-closed multiplicative semigroup.
2. A unitary operator $U$ is in $E_2(\mathcal{H})$ if and only if it is a gaussian symmetry.

3. A density operator $\rho$ is in $E_2(\mathcal{H})$ if and only if $\rho$ is a gaussian state. The element $\Lambda$ in the quadruple of $E_2(\mathcal{H})$-parameters of a gaussian state is a positive matrix operator in $\mathcal{H}$. The real dimension of the $E_2(\mathcal{H})$ parameter set of the set of all gaussian states is $2n^2 + 3n + 1$, one more than the dimension involved in the conventional parametrization using means and covariances of the position and momentum observables in the $n$-mode Fock space $\Gamma(\mathbb{C}^n)$, or equivalently, $L^2(\mathbb{R}^n)$. The extra dimension arises from a normalization factor in the condition $\text{Tr} \rho = 1$ for a state.

Some of the proofs spill over to Section 6.

Section 6 is devoted entirely to gaussian states. The $E_2(\mathcal{H})$-parameters of a gaussian state $\rho$ can all be derived from the matrix entries $\langle s|\rho|t \rangle$, where $s, t \in \mathbb{Z}_+^n$ with $|s| + |t| \leq 2$ and $|s\rangle, |t\rangle$ are particle states occurring in the particle basis of $\Gamma(\mathbb{C}^n)$. Thus a gaussian state $\rho$ is completely determined by its matrix entries \{ $\langle s|\rho|t \rangle | s, t \in \mathbb{Z}_+^n, |s| + |t| \leq 2$ \}. This reduces the tomography of an unknown gaussian state $\rho$ essentially to that of a finite level state $(\text{Tr} \rho \mathcal{P})^{-1} \mathcal{P} \rho \mathcal{P}$, where $\mathcal{P}$ is the orthogonal projection onto the subspace $\mathbb{C} \oplus \mathcal{H} \oplus \mathcal{H} \otimes \mathcal{H}$ in $\Gamma(\mathcal{H})$. Furthermore, we derive all the relations between the $E_2(\mathcal{H})$-parameters and the mean-covariance parameters for the position and momentum observables in this section.

Section 7 is devoted to a study of pure gaussian states in $\mathcal{H}$ by examining its particle basis expansion. This becomes possible because the $E_2(\mathcal{H})$-parameters for such a state reduces to $(c, A)$ where $c$ is a positive scalar and $A$ is an $n \times n$ complex symmetric matrix. By making judicious choices for $A$, we reach a hierarchy of interesting examples of $n$-mode pure gaussian states which are completely entangled and symmetric under the action of the permutation group $S_n$ on the set of modes.

## 2 Exponential vectors, Coherent States and the Klauder-Bargmann Isometry

Let $\Gamma(\mathcal{H})$ denote the boson (symmetric) Fock space over a finite dimensional complex Hilbert space $\mathcal{H}$ of dimension $n$. Choose and fix an orthonormal basis, $\{ e_j \}_{1 \leq j \leq n}$ in $\mathcal{H}$ and identify $\mathcal{H}$ with $\mathbb{C}^n$, such that $z = \sum_{j=1}^n z_je_j \in \mathcal{H}$ is identified with $z = (z_1, z_2, \ldots, z_n)^T \in \mathbb{C}^n$. Then $\Gamma(\mathcal{H}) = \Gamma(\mathbb{C}^n) = \otimes_{j=1}^n \Gamma(\mathbb{C}e_j)$. Let $\mathbf{k} = (k_1, k_2, \ldots, k_n)$, with nonnegative symmetric tensor product $k_j, 1 \leq j \leq n$. Denote by $|k_1, k_2, \ldots, k_n\rangle$, the normalized symmetric tensor product of $e_r$ taken $k_r$ times, $r = 1, 2, \ldots, n$ so that $|k_1, k_2, \ldots, k_n\rangle$ is a unit vector in the subspace $\mathcal{H} \otimes^{k_1+k_2+\cdots+k_n}$ of $\Gamma(\mathcal{H})$. Write $|\mathbf{k}\rangle = |k_1, k_2, \ldots, k_n\rangle$.

The set $\{|\mathbf{k}\rangle | \mathbf{k} \in \mathbb{Z}_+^n \}$ is a complete orthonormal basis for $\Gamma(\mathcal{H})$, called a particle basis with reference to the choice $\{ e_j \}_{1 \leq j \leq n}$ in $\mathcal{H}$. When $\mathcal{H} = \mathbb{C}^n$ we choose its canonical basis so that $e_j$ is the column vector with 1 in the position $j$ and 0 elsewhere. The Fock space $\Gamma(\mathcal{H})$ as well as $\Gamma(\mathbb{C}^n)$ is also called an $n$-mode Fock space describing a quantum system of an arbitrary finite number of boson particles but in $n$ different modes. Any unit vector $\phi$ as well as the corresponding 1-dimensional projection $|\phi\rangle\langle\phi|$ is called a pure state. The pure state $|\mathbf{k}\rangle$, thus defined is understood
to be a state in which there are \( k_r \) particles in the \( r \)-th mode for \( r = 1, 2, \ldots, n \). For any \( z \in \Gamma(\mathcal{H}) \), define \( |e(z)\rangle, |\psi(z)\rangle \) in \( \Gamma(\mathcal{H}) \) respectively by

\[
|e(z)\rangle = \bigoplus_{k=0}^{\infty} \frac{z^k}{\sqrt{k!}},
\]

\[
|\psi(z)\rangle = e^{-\|z\|^2/2} |e(z)\rangle.
\]

Then

\[
\langle e(z)|e(z')\rangle = e^{\langle z|z'\rangle},
\]

\[
||\psi(z)|| = 1.
\]

We call \( |e(z)\rangle \) and \( |\psi(z)\rangle \) respectively the exponential vector and coherent state with parameter \( z; |e(0)\rangle \) is known as the vacuum state and is denoted by \( |\Omega\rangle \). Then \( |\psi(z)\rangle \) is a superposition of finite particle states \( \{|k\rangle | k \in \mathbb{Z}_+^n \} \), \( |k\rangle \) having probability amplitude \( \Pi_r e^{k_r}/\sqrt{\Pi_r k_r!} \). To simplify notations in the \( \Gamma(\mathbb{C}^n) \) representation, we shall follow the multi-index convention \( k! = k_1!k_2!\cdots k_n! \) and \( z^k = z_1^{k_1}z_2^{k_2}\cdots z_n^{k_n} \), \( k_j \in \mathbb{Z}_+^n, z_j \in \mathbb{C}, 1 \leq j \leq n \). Then

\[
|e(z)\rangle = \sum_{k\in\mathbb{Z}_+^n} \frac{z^k}{\sqrt{k!}} |k\rangle \tag{2.1}
\]

and for the particle basis measurement in the coherent state \( |\psi(z)\rangle \), the probability of observing \( k_r \) particles in the \( r \)-th mode for \( 1 \leq r \leq n \) is

\[
e^{-|z|^2} \frac{|z^k|^2}{k!} = \Pi_r e^{-|z_r|^2} \frac{|z_r|^2}{k_r!}.
\]

In other words, number of particles in different modes have independent Poisson distributions with mean values \( |z_r|^2, r = 1, 2, \ldots, n \). Later, in our exposition, we shall meet \( |\psi(z)\rangle \) as an example of a pure gaussian state.

Any bounded operator \( Z \) in \( \Gamma(\mathcal{H}) \), admits the following matrix representation in the particle basis:

\[
Z = \sum_{j,k\in\mathbb{Z}_+^n} Z_{jk} |j\rangle\langle k|, \tag{2.2}
\]

where \( Z_{jk} = \langle j|Z|k\rangle \). Define the generating function of the operator \( Z \),

\[
G_Z(u, v) := \langle e(\bar{u})|Z|e(v)\rangle = \sum_{j,k\in\mathbb{Z}_+^n} Z_{jk} \frac{u^j v^k}{\sqrt{j!k!}}, \tag{2.3}
\]

where \( \bar{u} \) is understood using the identification of \( \mathcal{H} \) with \( \mathbb{C}^n \). If \( u = (u_1, u_2, \ldots, u_n)^T \) and \( v = (v_1, v_2, \ldots, v_n)^T \), observe that \( G_Z(u, v) \) is a power series in the \( 2n \) variables \( u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n \).

We now recall some of the well-known properties of exponential vectors and coherent states:

1. The map \( z \mapsto |e(z)\rangle \) from \( \mathbb{C}^n \) into \( \Gamma(\mathbb{C}^n) \) is analytic where as \( z \mapsto |\psi(z)\rangle \) is real analytic.
2. For any nonempty finite set $F \subset H$, the set $\{|e(z)| z \in F\}$ is linearly independent and the linear span of all coherent states is dense in $\Gamma(H)$. In other words, the coherent states form a linearly independent and total set.

3. Let $H = \bigoplus_j S_j$, where $S_j$'s are mutually orthogonal subspaces. Furthermore, let $P_j$ denote the orthogonal projection of $H$ onto $S_j$. Then the map

$$|e(z)\rangle \mapsto \otimes_j |e(P_j z)\rangle$$

extends to an isomorphism between $\Gamma(H)$ and $\otimes_j \Gamma(S_j)$. In particular, if $\{v_1, v_2, \ldots, v_n\}$ is any orthonormal basis of $H$ and $z = \sum_j z_j v_j$, then $|\psi(z)\rangle = \otimes_j |\psi(z_j v_j)\rangle$. In other words, in any orthonormal basis of $H$, any coherent state $|\psi(z)\rangle$ can be viewed as a product state.

4. Suppose for some function $f \in L^2(\mathbb{R})$, $\int_{\mathbb{R}} \bar{f}(x) e^{-\frac{1}{2}x^2 + ux} dx = 0, \forall u \in \mathbb{C}$, then this implies, in particular, that the Fourier transform of the function $\bar{f} e^{-\frac{1}{2}x^2}$ vanishes and so does $f$. Hence the set of functions $\{e^{-\frac{1}{2}x^2 + ux} | u \in \mathbb{C}\}$ is total in $L^2(\mathbb{R})$. Putting

$$g_u(x) = \pi^{-1/4} e^{-\frac{1}{2}(x^2 + u^2) + \sqrt{2}u x}, \forall x \in \mathbb{R},$$

we conclude that the set $\{g_u | u \in \mathbb{C}\}$ is total in $L^2(\mathbb{R})$.

When $H$ is one dimensional, it may be identified with $\mathbb{C}$ and then the map

$$e(u) \mapsto g_u$$

is scalar product preserving between total sets in $\Gamma(H)$ and $L^2(\mathbb{R})$. So it extends uniquely to an isomorphism between $\Gamma(H)$ and $L^2(\mathbb{R})$. In general, when $H$ is $n$-dimensional, by Property 3 above, we see that $\Gamma(H)$ is isomorphic to $L^2(\mathbb{R}^n)$ via the mapping

$$e(u) \mapsto \otimes_j g_{u_j}$$

where $u = \oplus_{j=1}^n u_j e_j$.

5. Klauder-Bargmann Isometry. The map $\phi \mapsto \pi^{-n/2} \langle \psi(z)|\phi\rangle, z \in \mathbb{C}^n$ is an isometry from $\Gamma(H)$ into $L^2(\mathbb{C}^n)$, where $\mathbb{C}^n$ is equipped with the $2n$-dimensional Lebesgue measure.

We indicate a proof of Property [5]:

Proof. Since $\Gamma(\mathbb{C}^n)$ is the $n$-fold tensor product of copies of $\Gamma(\mathbb{C})$, which, by definition, is $\ell_2(\mathbb{Z}_+)$ and by Property [3] $|\psi(z)\rangle = \otimes_{j=1}^n |\psi(z_j)\rangle$, it is enough to prove Property [5] when $n = 1$. In this case,

$$\psi(z) = \{e^{-|z|^2/2} \frac{z^k}{\sqrt{k!}}, k = 0, 1, 2, \ldots \}, z \in \mathbb{C}.$$
Let $\phi$ and $\phi'$ be two sequences given by $\{a_j\}$ and $\{b_j\}$, $j \in \mathbb{Z}_+$. Putting $z = re^{i\theta}$, and using polar co-ordinates along with Parseval’s identity we get

$$
\frac{1}{\pi} \int_{\mathbb{C}} |\psi(z)|\langle \psi(z)|\phi'\rangle \, dz = 2 \int_0^\infty e^{-r^2} \int_0^{2\pi} \sum_{j=0}^\infty \frac{a_j r^j e^{ij\theta}}{\sqrt{j!}} \left( \sum_{j=0}^\infty \frac{b_j r^j e^{ij\theta}}{\sqrt{j!}} \right) \frac{d\theta}{2\pi} \, dr.
$$

$$
= \int_0^\infty \sum_{j=0}^\infty \bar{a}_j b_j r^{2j} e^{-r^2} \, 2\pi r \, dr.
$$

$$
= \sum_{j=0}^\infty \bar{a}_j b_j = \langle \phi|\phi'\rangle.
$$

We have a few important corollaries from the Klauder-Bargmann isometry.

**Corollary 2.1.** (i) [Klauder-Bargmann formula] The coherent states yield a resolution of identity $I$ into 1-dimensional projections:

$$
1 \pi^{-n} \int_{\mathbb{C}^n} |\psi(z)|\langle \psi(z)|\phi\rangle \, dz = I,
$$

where the left hand side integral is a weak operator integral with respect to the $2n$-dimensional Lebesgue measure on $\mathbb{C}^n$. In particular, for any element $|\phi\rangle$ in $\Gamma(\mathcal{H})$ the following holds:

$$
|\phi\rangle = \frac{1}{\pi^{-n}} \int_{\mathbb{C}^n} \langle \psi(z)|\phi\rangle |\psi(z)\rangle \, dz,
$$

which has the interpretation that $\{|\psi(z)\rangle | z \in \mathcal{H}\}$ is an ‘overcomplete basis’ for $\Gamma(\mathcal{H})$.

(ii) Any bounded operator $Z$ admits the representation

$$
Z = \int_{\mathbb{C}^n} Z |\psi(z)|\langle \psi(z)| \, dz.
$$

(iii) The positive operator valued measure (POVM) $m$ defined by

$$
m(E) := \frac{1}{\pi^{-n}} \int_{E} |\psi(z)|\langle \psi(z)| \, dz,
$$

$E$ a Borel set, yields a $\mathbb{R}^{2n}$-valued continuous measurement in $\Gamma(\mathcal{H})$.

**Remark 2.2.** The formula in (2.7) was first discovered in the present form by Klauder in Page 125-126 of [Kla60], with a heuristic proof. A rigorous proof of this first appeared in Page 194 of [Bar61], where he proved it for a slightly different version of exponential vectors called principal vectors in the Segal-Bargmann space. Equation (2.7) later appeared separately in the works of Glauber (again with a heuristic proof [Gla63b, Gla63a]) and Sudarshan (who refers to Bargmann [Snd63, KS68]) and was used by them to prove various results in quantum optics including the well-known Glauber-Sudarshan P representation. We call (2.7), Klauder-Bargmann formula.
3 Weyl Operators, Quantum Fourier Transform and the Wigner Isomorphism

The correspondence $|\psi(z)\rangle \mapsto e^{-i\text{Im}(u|z)}|\psi(u+z)\rangle, z \in \mathcal{H}$ is a scalar product preserving map for any fixed $u \in \mathcal{H}$. Since the coherent states constitute a total set (Property 2, Section 2) in $\Gamma(\mathcal{H})$, it follows that there exists a unique unitary operator $W(u)$ on $\Gamma(\mathcal{H})$ satisfying the relation

$$W(u)|\psi(z)\rangle = e^{-i\text{Im}(u|z)}|\psi(u+z)\rangle, z \in \mathcal{H}$$

(3.1)

We call $W(u)$ the Weyl operator at $u \in \mathcal{H}$. It is also known as the displacement operator at $u$. The Weyl operators obey the multiplication relations

$$W(u)W(v) = \exp(-i\text{Im}(u|v))W(u+v), \forall u, v \in \mathcal{H}$$

$$W(u)W(v) = W(v)W(u)e^{-2i\text{Im}(u|v)}, u, v \in \mathcal{H}.$$  

(3.2)

Equations in (3.2) are known as Weyl commutation relations or canonical commutation relations (CCR) and the $C^*$-algebra generated by the Weyl operators denoted $CCR(\mathcal{H})$ is called the CCR-algebra. We recall a few basic properties of the Weyl operators.

1. The map $u \mapsto W(u)$ is a strongly continuous, projective, unitary and irreducible representation of the additive group $\mathcal{H}$ known as the Weyl representation in $\Gamma(\mathcal{H})$. Furthermore, it follows from the irreducibility that the von Neumann algebra generated by Weyl operators is all of $\mathcal{B}(\Gamma(\mathcal{H}))$, i.e.,

$$CCR(\mathcal{H})^\text{sot} = \mathcal{B}(\Gamma(\mathcal{H})),$$  

(3.3)

where $\text{sot}$ indicates the closure in the strong operator topology.

2. The Weyl representation enjoys the factorizability property: if $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \cdots \oplus \mathcal{H}_k$, $u = u_1 \oplus u_2 \oplus \cdots \oplus u_k, u_j \in \mathcal{H}_j$ for each $j$, then $W(u) = W(u_1) \otimes W(u_2) \otimes \cdots \otimes W(u_k)$.

3. For every fixed $u \in \mathcal{H}$, the set $\{W(tu) : t \in \mathbb{R}\}$ is a strongly continuous, one parameter unitary group and hence has the form

$$W(tu) = e^{-it\sqrt{2}p(u)}, t \in \mathbb{R}, u \in \mathcal{H},$$

(3.4)

where $p(u)$ is a self-adjoint operaor in $\Gamma(\mathcal{H})$. Define

$$q(u) = p(-iu),$$

$$a(u) = \frac{1}{\sqrt{2}}(q(u) + ip(u)),$$

$$a^\dagger(u) = \frac{1}{\sqrt{2}}(q(u) - ip(u)).$$

Then $a(u)$ and $a^\dagger(u)$ are the well-known annihilation and creation operators at $u,$

$$q(u) = \frac{a(u) + a^\dagger(u)}{\sqrt{2}},$$

$$p(u) = \frac{a(u) - a^\dagger(u)}{i\sqrt{2}}.$$
When \( \mathcal{H} = \mathbb{C}^n = \mathbb{R}^n \oplus i\mathbb{R}^n \) the families \( \{ q(x)|x \in \mathbb{R}^n \} \) and \( \{ p(x)|x \in \mathbb{R}^n \} \) are commuting families of self-adjoint operators or observables and the CCR in (3.2) becomes

\[
[q(x), p(y)] = iX^T y, \forall x, y \in \mathbb{R}^n.
\]

These are the well-known Heisenberg commutation relations, again called CCR. It is also expressed as

\[
[a(u), a(v)] = 0,
\]
\[
[a^\dagger(u), a^\dagger(v)] = 0,
\]
\[
[a(u), a^\dagger(v)] = \langle u|v \rangle, \forall u, v \in \mathcal{H} \text{ or } \mathbb{C}^n.
\]

It may also be noted that the map \( u \mapsto a(u) \) and \( u \mapsto a^\dagger(u) \) are respectively antilinear and linear in the variable \( u \). Going back to Weyl operators we have

\[
W(u) = e^{(a^\dagger(u) - a(u))}, \forall u \in \mathcal{H} \text{ or } \mathbb{C}^n.
\]

In all these relations we are dealing with unbounded operators and we have been silent on matters concerning their domains. For details we refer to [Par92]. Write

\[
p_j = p(e_j),
\]
\[
q_j = q(e_j) = -p(ie_j) \quad (3.5)
\]
\[
a_j = a(e_j) = \frac{1}{\sqrt{2}} (q_j + ip_j)
\]
\[
a_j^\dagger = a^\dagger(e_j) = \frac{1}{\sqrt{2}} (q_j - ip_j) \quad (3.6)
\]

for each \( 1 \leq j \leq n \). The operators \( p_j, q_j, a_j \) and \( a_j^\dagger \) are respectively called the momentum, position, annihilation and creation operators of the \( j \)-th mode. In particular, the observable \( a_j^\dagger a_j \) is the number operator describing the number of particles in the \( j \)-th mode.

4. **Stone-von Neumann Theorem.** If \( \mathcal{K} \) is a complex separable Hilbert space and \( u \mapsto W'(u) \) is a strongly continuous, projective and unitary representation of \( \mathcal{H} \) in \( \mathcal{K} \) satisfying the relations (3.2) with \( W \) replaced by \( W' \), then there exists a Hilbert space \( \mathcal{k} \) and a unitary isomorphism \( \Gamma : \mathcal{K} \to \Gamma(\mathcal{H}) \otimes \mathcal{k} \) such that

\[
\Gamma W'(u) \Gamma^{-1} = W(u) \otimes I_k, \forall u \in \mathcal{H},
\]

where \( I_k \) is the identity operator in \( \mathcal{k} \). In particular, if \( W' \) is also irreducible then \( \mathcal{k} = \mathbb{C} \), the 1-dimensional Hilbert space and \( \Gamma \) is a unitary isomorphism from \( \mathcal{K} \) to \( \Gamma(\mathcal{H}) \).

5. Let \( L \) be a real linear transformation of \( \mathcal{H} \) satisfying

\[
\text{Im} \langle Lu|Lv \rangle = \text{Im} \langle u|v \rangle, \forall u, v \in \mathcal{H}.
\]

Such a transformation is said to be symplectic. Define

\[
W_L(u) = W(Lu), u \in \mathcal{H}.
\]

The map \( u \mapsto W_L(u) \) is a strongly continuous, projective, unitary and irreducible representation of \( \mathcal{H} \) in \( \Gamma(\mathcal{H}) \) obeying (3.2). Hence by the Stone-von
Neumann theorem in Property 4 there exists a unitary operator $\Gamma(L)$ in $\Gamma(H)$ satisfying

$$\Gamma(L)W(u)\Gamma(L)^{-1} = W(Lu), \forall u \in H.$$  

(3.7)

Such a unitary operator $\Gamma(L)$ is unique up to multiplication by a scalar of modulus unity. The operator $\Gamma(L)$ is said to intertwine the representations $W$ and $W_L$.

Let $B_j(\Gamma(H)) \subset B(\Gamma(H))$, for $j = 1$ and $2$ denote the ideal of trace class operators and Hilbert-Schmidt operators respectively on $\Gamma(H)$. Then $B_1(\Gamma(H))$ is a Banach space with $\|\rho\|_1 = \text{Tr} \sqrt{\rho^\dagger \rho}$, $\rho \in B_1(\Gamma(H))$, $B_2(\Gamma(H))$ is a Hilbert space with scalar product $\langle \rho_1 | \rho_2 \rangle_2 = \text{Tr} \rho_1^\dagger \rho_2$ and $B_1(\Gamma(H)) \subset B_2(\Gamma(H))$ as a linear manifold.

**Definition 3.1.** If $\rho \in B_1(\Gamma(H))$, then the complex valued function

$$\hat{\rho}(z) := \text{Tr} \rho W(z), z \in H$$

is called the quantum Fourier transform (or Wigner transform) of $\rho$.

We summarize a few properties of the quantum Fourier transform:

1. The function $\hat{\rho}$ is bounded and continuous on $H$.

   Since $B_1(\Gamma(H))$ is the predual of $B(\Gamma(H))$ and $W(z)$ is a unitary operator,

   $$|\text{Tr} \rho W(z)| \leq \|\rho\|_1.$$  

   The continuity of $\hat{\rho}$ follows from the strong continuity of the Weyl representation.

2. The correspondence $\rho \rightarrow \hat{\rho}$ is bijective.

   Let $\rho_1, \rho_2 \in B_1(\Gamma(H))$. The equation $\hat{\rho}_1 = \hat{\rho}_2$ implies that $\text{Tr}(\rho_1 - \rho_2)W(z) = 0$ for all $z \in H$ and by (3.3) $\text{Tr}(\rho_1 - \rho_2)X = 0$ for any $X \in B(H)$.

3. The quantum Fourier transform is factorizable.

   Indeed, for $\rho_j \in B_1(\Gamma(H_j)), j = 1, 2$, by property 2 of Weyl operators,

   $$(\rho_1 \otimes \rho_2)\hat{}(u \oplus v) = \hat{\rho}_1(u)\hat{\rho}_2(v).$$

4. A positive operator $\rho$ of unit trace in $\Gamma(H)$ is called an $n$-mode state. For such a state $\rho$, by Property 3 of Weyl operators, the function $\hat{\rho}(t\mathbf{z}), t \in \mathbb{R}$ is the characteristic function of the probability distribution of the observable $-\sqrt{2}p(\mathbf{z}) = i(a(\mathbf{z}) - a^\dagger(\mathbf{z})) = \sqrt{2}(q(\mathbf{y}) - p(\mathbf{x}))$ for any fixed $\mathbf{z} = \mathbf{x} + i\mathbf{y}$.

5. **Quantum Bochner Theorem** [SW75, Par10]. A complex valued function $f$ defined on $H$ is the quantum Fourier transform of an $n$-mode state if and only if the following are satisfied:

   (a) $f(0) = 1$ and $f$ is continuous at 0.

   (b) The kernel $K(z, w) = e^{i\text{Im}(z\overline{w})}f(w - z)$ is positive definite.

We now state the gaussian integral in a form which we frequently use in the rest of this paper.
Proposition 3.2 (Gaussian integral formula). Let $A$ be an $n \times n$ complex matrix such that $A$ is symmetric ($A = A^T$) and $\text{Re} A$ is a positive definite matrix. Then for any $m \in \mathbb{C}^n$,

$$\int_{\mathbb{R}^n} \exp \{-x^TAx + m^Tx\} dx = \sqrt{\frac{\pi^n}{\det A}} \exp \left\{ \frac{1}{4} m^T A^{-1} m \right\}, \quad (3.8)$$

where the branch of the square root is determined in such a way that $\det^{-1/2} A > 0$ when $A$ is real and positive definite.

We defined the quantum Fourier transform on the trace class ideal. Now we proceed to extend this definition to the Hilbert-Schmidt class in the same spirit as in the classical theory of Fourier transforms. Let $\mathcal{F} = \{ |e(u)\rangle \langle e(v)| : u, v \in \mathcal{H} \}$, then $\mathcal{F} \subset \mathcal{B}_1(\Gamma(\mathcal{H})) \subset \mathcal{B}_2(\Gamma(\mathcal{H}))$. Since exponential vectors form a total subset (Property 2 Section 2) of $\Gamma(\mathcal{H})$, $\mathcal{F}$ is a total set in $\mathcal{B}_2(\Gamma(\mathcal{H}))$. The following example illustrates an important property of the quantum Fourier transform of elements of $\mathcal{F}$.

Example 3.3. For $u, v \in \mathbb{C}$, consider $\rho = |e(u)\rangle \langle e(v)| \in \mathcal{B}(\Gamma(\mathcal{C}))$. Then

$$\hat{\rho}(z) = \text{Tr} |e(u)\rangle \langle e(v)| W(z) = \langle e(v)| W(z) |e(u)\rangle = e^{vu} e^{-\frac{1}{2} |z|^2 + vz - zu}. \quad (3.9)$$

Thus $\hat{\rho} \in L^1(\mathbb{C}) \cap L^2(\mathbb{C})$. Since $L^2(\mathbb{C}) = L^2(\mathbb{R}) \otimes L^2(\mathbb{R})$ and (by Property 4 in Section 2) the set $\{ e^{-\frac{1}{2} z^2 + cz} : z \in \mathbb{C} \}$ is total in $L^2(\mathbb{R})$ it follows that $\{ \rho | \rho \in \mathcal{F} \}$ is a total set in $L^2(\mathbb{C})$.

Theorem 3.4 (Wigner isomorphism). For $\rho \in \mathcal{B}_1(\Gamma(\mathcal{H}))$, let $\mathcal{F}_n(\rho)$ be the function defined on $\mathbb{C}^n$ such that

$$\mathcal{F}_n(\rho)(z) = \pi^{-n/2} \hat{\rho}(z), z \in \mathbb{C}^n. \quad (3.10)$$

Then $\mathcal{F}_n$ extends uniquely to a Hilbert space isomorphism from $\mathcal{B}_2(\Gamma(\mathcal{H}))$ onto $L^2(\mathbb{C}^n)$.

Proof. First we prove the theorem when $n = 1$, i.e., $\mathcal{H} = \mathbb{C}$. Let $\rho_j = |e(u_j)\rangle \langle e(v_j)|, j = 1, 2$. Using (3.9) and the gaussian integral formula (Proposition 3.2), we get

$$\frac{1}{\pi} \int_{\mathbb{C}} \hat{\rho}_1(z) \hat{\rho}_2(z) dz = e^{\langle u_1 | v_1 \rangle + \langle v_2 | u_2 \rangle} \frac{1}{\pi} \int_{\mathbb{C}} \exp \{-|z|^2 + \langle v_2 - u_1 | z \rangle + \langle z | v_1 - u_2 \rangle\} dz$$

$$= e^{\langle u_1 | v_1 \rangle + \langle v_2 | u_2 \rangle} e^{\langle v_2 - u_1 | v_1 - u_2 \rangle}$$

$$= e^{\langle u_1 | u_2 \rangle + \langle v_2 | v_1 \rangle}$$

$$= \text{Tr} \rho_1^1 \rho_2.$$ 

Now Example 3.3 shows that $\mathcal{F}_1$ is a scalar product preserving map between total sets and thus extends uniquely to a Hilbert space isomorphism. Hence $\mathcal{F}_1^{\otimes n}$ is an isomorphism from $\mathcal{B}_2(\Gamma(\mathcal{H}))$ onto $L^2(\mathbb{C}^n)$. Observe that $\mathcal{F}_1^{\otimes n}$ coincides with $\mathcal{F}_n$ on $\mathcal{B}_1(\Gamma(\mathcal{H}))$, hence $\mathcal{F}_1^{\otimes n}$ is the extension we sought of $\mathcal{F}_n$ on $\mathcal{B}_1(\Gamma(\mathcal{H}))$. \qed
Corollary 3.5. (i) The map $e(u \oplus v) \mapsto |e(u)\rangle\langle e(v)|$ extends as an isomorphism $\eta_1$ from $\Gamma(\mathcal{H} \oplus \mathcal{H})$ onto $\mathcal{B}_2(\Gamma(\mathcal{H}))$.

(ii) Let $u = \sum_j u_j e_j \in \mathcal{H}$ define $g_u = \otimes g_{u_j}$, where $g_{u_j} \in L^2(\mathbb{R})$ is as defined by equation (2.4). The map $|e(u)\rangle\langle e(v)| \mapsto g_u \otimes g_v$ extends as an isomorphism $\eta_2$ from $\mathcal{B}_2(\Gamma(\mathcal{H}))$ onto $L^2(\mathbb{C}^n) = L^2(\mathbb{R}^n) \otimes L^2(\mathbb{R}^n)$.

(iii) Let $v = \sum_j v_j e_j \in \mathcal{H}$, define by $\bar{v} := \sum_j \bar{v}_j e_j$. The Wigner isomorphism satisfies $\bar{F}_n(|e(u)\rangle\langle e(\bar{v})|) = g_u' \otimes g_v'$, where $\begin{pmatrix} u' \\ v' \end{pmatrix} = S \begin{pmatrix} u \\ v \end{pmatrix}$ with the unitary matrix

$$S = \frac{1}{\sqrt{2}} \begin{bmatrix} -i & I \\ i & iI \end{bmatrix}.$$  \hfill (3.11)

Proof. (i) This follows from a direct computation showing that $\eta_1$ is scalar product preserving.

(ii) We know from Property 3 in Section 2 that the map $e(u) \mapsto g_u$ extends to a Hilbert space isomorphism, the required result follows from (i).

(iii) Again it is enough to prove this when $\mathcal{H} = \mathbb{C}$. We have

$$\bar{F}_1(|e(u)\rangle\langle e(\bar{v})|)(x, y) = \text{Tr} |e(u)\rangle\langle e(\bar{v})| W(x + iy)$$

$$= \langle e(\bar{v}) | W(x + iy) | e(u) \rangle$$

$$= \langle e(\bar{v}) | e^{-\frac{1}{2}(x^2 + y^2) - (x - iy)u} e(u + x + iy) \rangle$$

$$= e^{-\frac{1}{2}(x^2 + y^2) - (x - iy)u + v(u + x + iy)}$$

$$= e^{-\frac{1}{2}(x^2 + y^2) + (v - u)x + i(v + u)y + vu}$$

$$= e^{-\frac{1}{2}x^2 + \left(\frac{v - u}{\sqrt{2}}\right)x - \frac{1}{2}\left(\frac{v - u}{\sqrt{2}}\right)^2} e^{\frac{1}{4}\left(\frac{v + u}{\sqrt{2}}\right)^2}$$

$$\times e^{-\frac{1}{2}y^2 + \left(\frac{v + u}{\sqrt{2}}\right)y - \frac{1}{2}\left(\frac{v + u}{\sqrt{2}}\right)^2} e^{\frac{1}{4}\left(\frac{v + u}{\sqrt{2}}\right)^2} \times e^{vu}$$

$$= g_u(x)g_v(y),$$

where

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 \\ i & iI \end{bmatrix} \begin{pmatrix} u \\ v \end{pmatrix}. \hfill (3.12)$$

$\square$

Remarks 3.6. 1. All the isomorphisms above are described by the Figure 1 via the mappings in Figure 2.

2. The quantum Fourier transform on $\mathcal{B}_2(\Gamma(\mathcal{H}))$ can be viewed as the second quantization $\Gamma(S)$ on $\Gamma(\mathcal{H}\oplus\mathcal{H})$, where $S$ is the unitary matrix given by (3.11). The eigenvalues of $S$ are $\lambda_1 = ie^{-in/12}$ and $\lambda_2 = e^{in/12}$ with multiplicities $n$ each. Then $\lambda_j^2 = -1, j = 1, 2$. Write $\bar{F}_n' = \bar{\Gamma}(S) = \eta_2^{-1} F_n$, then $\bar{F}_n' : \mathcal{B}_2(\Gamma(\mathcal{H})) \to \mathcal{B}_2(\Gamma(\mathcal{H}))$ is an isomorphism and satisfies the property

$$\bar{F}_n'^{12} = -I. \hfill (3.13)$$

In the classical theory, the Fourier transform defined on $L^1(\mathbb{R}^n)$ extends to a unitary $F$ on $L^2(\mathbb{R}^n)$, furthermore $F^2 = -I$. Equation (3.13) can be viewed as a noncommutative analogue of this fact.
\[ \Gamma(\mathcal{H} \oplus \mathcal{H}) \xrightarrow{\eta_1} \mathcal{B}_2(\Gamma(\mathcal{H})) \xrightarrow{\eta_2} L^2(\mathbb{C}^n) \]
\[ \Gamma(\mathcal{H} \oplus \mathcal{H}) \xrightarrow{\eta_1} \mathcal{B}_2(\Gamma(\mathcal{H})) \xrightarrow{\eta_2} L^2(\mathbb{C}^n) \]

Figure 1: The maps \( \tilde{\Gamma}(S) \) and \( \hat{\Gamma}(S) \) are the corresponding compositions.

\[ e(u \oplus v) \xrightarrow{\eta_1} |e(u)\rangle|e(\bar{v})\rangle \xrightarrow{\eta_2} \rho_u \otimes \rho_v \]
\[ e(u' \oplus v') \xrightarrow{\eta_1} |e(u')\rangle|e(\bar{v}')\rangle \xrightarrow{\eta_2} \rho_{u'} \otimes \rho_{v'} \]

Figure 2: \( u' \oplus v' = S(u \oplus v) \) where \( S \) as in equation (3.11).

**Theorem 3.7** (Quantum Fourier Inversion). If \( \rho \in \mathcal{B}_1(\Gamma(\mathcal{H})) \) then,

\[ \rho = \frac{1}{\pi^n} \int_{\mathcal{H}} \hat{\rho}(z)W(-z)dz, \tag{3.14} \]

where the integral is a weak operator integral with respect to the 2n-dimensional Lebesgue measure on \( \mathcal{H} \) inherited from \( \mathbb{C}^n \).

**Proof.** If \( \phi, \psi \in \Gamma(\mathcal{H}) \), by Theorem 3.4

\[ \langle \phi | \rho | \psi \rangle = \text{Tr} |\phi\rangle\langle \psi| \rho = \int_{\mathbb{C}^n} F_n(|\phi\rangle\langle \psi|)(z)F_n(\rho)(z)dz \]

\[ = \frac{1}{\pi^n} \int_{\mathcal{H}} \langle \phi | W(-z) | \psi \rangle \hat{\rho}(z)dz \]

\[ = \langle \phi | \frac{1}{\pi^n} \int_{\mathcal{H}} \hat{\rho}(z)W(-z)dz | \psi \rangle. \]

This is same as (3.14). \( \square \)

4 Klauder-Bargmann representation for Gaussian Symmetries

We continue our discussions with a finite dimensional Hilbert space \( \mathcal{H} \) with an orthonormal basis \( \{e_j | 1 \leq j \leq n \} \) and the identification of \( z = \sum_{j=1}^n z_je_j \in \mathcal{H} \) with \( z = (z_1, z_2, \ldots, z_n)^T \in \mathbb{C}^n \). Let \( \mathcal{H}_R \) be the real linear span of the orthonormal basis, then \( \mathcal{H} = \mathcal{H}_R + i\mathcal{H}_R \), i.e., if \( z \in \mathcal{H}, z = x + iy, x, y \in \mathcal{H}_R \). Furthermore, let \( \mathcal{L}_R(\mathcal{H}) \) be the real algebra of real linear operators on \( \mathcal{H} \). Then for \( L \in \mathcal{L}_R(\mathcal{H}) \), \( Lz = (Ax + By) + i(Cx + Dy) \), where \( A, B, C, D \) are operators in \( \mathcal{H}_R \) with respective real matrices denoted again by \( A, B, C, D \). Write

\[ L_0 = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \]
where $L_0 \in M_{2n}(\mathbb{R})$, i.e., a $2n \times 2n$ real matrix. We now have

$$Lz = \begin{bmatrix} I & iI \end{bmatrix} L_0 \begin{bmatrix} x \\ y \end{bmatrix}, \quad z \in \mathbb{C}^n, x, y \in \mathbb{R}^n.$$  \hspace{1cm} (4.1)

where $I$ is the identity matrix of order $n$.

**Lemma 4.1.** Let $L, M \in \mathfrak{L}_{\mathbb{R}}(\mathcal{H})$ and $z = x + iy, z' = x' + iy'$, where $x, x', y, y' \in \mathcal{H}_{\mathbb{R}}$.

Then

$$\langle Lz | Mz' \rangle = \begin{bmatrix} x^T & y^T \end{bmatrix} L_0^T (I + iJ) M_0 \begin{bmatrix} x' \\ y' \end{bmatrix},$$  \hspace{1cm} (4.2)

where $J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$. In particular,

$$|Lz|^2 = \begin{bmatrix} x^T & y^T \end{bmatrix} L_0^T L_0 \begin{bmatrix} x \\ y \end{bmatrix}. \hspace{1cm} (4.3)$$

**Proof.** By (4.1)

$$\langle Lz | Mz' \rangle = \begin{bmatrix} x^T & y^T \end{bmatrix} L_0^T \begin{bmatrix} I & iI \\ -iI & I \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x^T & y^T \end{bmatrix} L_0^T (I + iJ) M_0 \begin{bmatrix} x' \\ y' \end{bmatrix}. \hspace{1cm} \square$$

**Definition 4.2.** A real linear operator $L \in \mathfrak{L}_{\mathbb{R}}(\mathcal{H})$ is called a symplectic transformation of $\mathcal{H}$ if

$$\text{Im} \langle Lz | Lz' \rangle = \text{Im} \langle z | z' \rangle, \forall z, z' \in \mathcal{H}.\hspace{1cm}$$

By Lemma 4.1 this is equivalent to

$$\begin{bmatrix} x^T & y^T \end{bmatrix} L_0^T J L_0 \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x^T & y^T \end{bmatrix} J \begin{bmatrix} x' \\ y' \end{bmatrix}, \forall x, y, x', y' \in \mathbb{R}^n,$$

or equivalently,

$$L_0^T J L_0 = J. \hspace{1cm} (4.4)$$

Any $L_0 \in M_{2n}(\mathbb{R})$ satisfying (4.4) is called a symplectic matrix.

Suppose $L_0$ is any symplectic matrix. Since $J$ is an orthogonal matrix, taking determinants on both sides of (4.4) we get $(\det L_0)^2 = 1$. Thus $L_0$ is nonsingular. Furthermore, (4.4) shows that $L_0$ and $(L_0^{-1})^T$ are orthogonally equivalent through $J$. Thus $a$ is an eigenvalue of $L_0$ if and only if $a^{-1}$ is so, hence $\det(L) = 1$. Multiplying by $(L_0^{-1})^T$ on the left and by $L_0^{-1}$ on the right on both sides of (4.4) shows that $L_0^{-1}$ is symplectic. Thus symplectic matrices form a group under multiplication. Indeed, it is a unimodular Lie group, denoted $Sp(2n, \mathbb{R})$ and known as the symplectic real matrix group of order $2n$. From our discussions it is clear that all symplectic transformations of $\mathcal{H}$ constitute a group, denoted $Sp(\mathcal{H})$, isomorphic to the Lie group $Sp(2n, \mathbb{R})$.

We now make a detailed analysis of the unitary operators $\Gamma(L), L \in Sp(\mathcal{H})$ occurring in Property 5 of Weyl operators in Section 3.
Proposition 4.3. Let \( L \in Sp(\mathcal{H}) \). Then
\[
| \langle \Omega | \Gamma(L) | \Omega \rangle | = \alpha(L)^{-1/4}
\]
where
\[
\alpha(L) = \det \frac{1}{2}(I + L^T_0 L_0).
\]

Proof. Let \( \rho = |\Omega\rangle \langle \Omega| \). Its quantum Fourier transform (Definition 3.1) is given by,
\[
\hat{\rho}(z) = \langle \Omega | W(z) | \Omega \rangle = e^{-\frac{1}{2} |z|^2}, z \in \mathcal{H}.
\]
By quantum Fourier inversion formula (3.14),
\[
|\Omega\rangle \langle \Omega| = \frac{1}{\pi^n \sqrt{n}} \int_{C^n} e^{-\frac{1}{2} |z|^2} W(-z)dz.
\]
Conjugation by \( \Gamma(L) \) gives
\[
\Gamma(L) |\Omega\rangle \langle \Omega| \Gamma(L)^{-1} = \frac{1}{\pi^n \sqrt{n}} \int_{C^n} e^{-\frac{1}{2} |z|^2} W(-Lz)dz.
\]
Multiplying by \( |\Omega\rangle \) on the left and \( | \Omega | \) on the right in both sides of this equation, using the unitarity of \( \Gamma(L) \), Lemma 4.1, and gaussian integral formula (Proposition 3.2), we get
\[
| \langle \Omega | \Gamma(L) | \Omega \rangle |^2 = \frac{1}{\pi^n} \int_{C^n} \exp \left\{ -\frac{1}{2} (|z|^2 + |Lz|^2) \right\} dz
\]
\[
= \frac{1}{\pi^n} \int_{\mathbb{R}^{2n}} \exp \left\{ -\frac{1}{2} \begin{bmatrix} x^T & y^T \end{bmatrix} (I + L_0^T L_0) \begin{bmatrix} x \\ y \end{bmatrix} \right\} dx dy
\]
\[
= \left( \det \frac{1}{2}(I + L_0^T L_0) \right)^{-1/2}.
\]

\[\square\]

Theorem 4.4. For \( L \in Sp(\mathcal{H}) \), there exists a unique unitary operator \( \Gamma_0(L) \) in \( \Gamma(\mathcal{H}) \) satisfying the following:

1. \( \langle \Omega | \Gamma_0(L) | \Omega \rangle = \alpha(L)^{-1/4} \), where
\[
\alpha(L) = \det \frac{1}{2}(I + L^T_0 L_0).
\]

2. \( \Gamma_0(L) W(u) \Gamma_0(L)^{-1} = W(Lu), \forall u \in \mathcal{H} \).

3. \( \Gamma_0(L) |e(v)\rangle = \alpha(L)^{1/4} \frac{1}{\pi^n} \int_{C^n} \exp \left\{ -\frac{1}{2} (|z|^2 + |Lz|^2) + (z | v) \right\} |e(Lz)\rangle \, dz, \forall v \in \mathcal{H} \).
Proof. Choose $\Gamma(L)$ to be any unitary operator satisfying $\Gamma(L)W(u)\Gamma(L)^{-1} = W(u)$. Define

$$
\Gamma_0(L) = \frac{|\langle \Omega | \Gamma(L) | \Omega \rangle|}{\langle \Omega | \Gamma(L) | \Omega \rangle} \Gamma(L).
$$

By Proposition 4.3, $\Gamma_0(L)$ is a well defined unitary operator differing from $\Gamma(L)$ by a scalar multiple of modulus unity and satisfying properties (1) and (2) of the theorem. To prove (3) we look at the rank one operator $|e(v)\rangle\langle \Omega|$ as the inverse of its quantum Fourier transform,

$$
|e(v)\rangle\langle \Omega| = \frac{1}{\pi^n} \int_{\mathbb{C}^n} \exp\left\{-\frac{1}{2}|z|^2 - \langle z|v \rangle\right\}W(-z)dz.
$$

Conjugation by $\Gamma_0(L)$ on both sides yields

$$
\Gamma_0(L)|e(v)\rangle\langle \Omega|\Gamma_0(L)^{-1} = \frac{1}{\pi^n} \int_{\mathbb{C}^n} \exp\left\{-\frac{1}{2}|z|^2 - \langle z|v \rangle\right\}W(-Lz)dz.
$$

Now right multiplication by $|\Omega\rangle$ on both sides followed by a change of variable $z \mapsto -z$ in the integral on the right hand side completes the proof. \qed

**Corollary 4.5.** The generating function (equation (2.3)) of $\Gamma_0(L)$ is given by

$$
G_{\Gamma_0(L)}(u, v) = \alpha(L)^{-1/4} \exp\{u^T A_L u + u^T \Lambda_L v + v^T B_L v\},
$$

where $\alpha(L)$ is as in (4.6), $A_L, \Lambda_L, B_L$ are $n \times n$ matrices given by

$$
A_L = \frac{1}{2} \begin{bmatrix} I & iI \end{bmatrix} L_0(I + L_0^T L_0)^{-1} L_0^T \begin{bmatrix} I \\
-iI \end{bmatrix},
$$

$$
\Lambda_L = \begin{bmatrix} I & iI \end{bmatrix} L_0(I + L_0^T L_0)^{-1} \begin{bmatrix} I \\
-iI \end{bmatrix},
$$

$$
B_L = \frac{1}{2} \begin{bmatrix} I & -iI \end{bmatrix} (I + L_0^T L_0)^{-1} \begin{bmatrix} I \\
-iI \end{bmatrix}.
$$

(4.7)

Proof. Left multiplying by $\langle e(\bar{u})|$ on both sides of the identity (4) in Theorem 4.4 we get

$$
G_{\Gamma_0(L)}(u, v) = \alpha(L)^{1/4} \frac{1}{\pi^n} \int_{\mathbb{C}^n} \exp\left\{-\frac{1}{2}(|z|^2 + |Lz|^2) + \langle z|v \rangle + \langle \bar{u}|Lz \rangle\right\}dz
$$

(4.8)

$$
= \alpha(L)^{1/4} \frac{1}{\pi^n} \int_{\mathbb{R}^{2n}} \exp\left\{-\begin{bmatrix} x \\
y \end{bmatrix}^T \begin{bmatrix} I & -iI \end{bmatrix} \frac{1}{2} \begin{bmatrix} x \\
y \end{bmatrix} + \begin{bmatrix} m \\
0 \end{bmatrix}^T \begin{bmatrix} x \\
y \end{bmatrix}\right\}dx dy,
$$

where $m^T = v^T [I & -iI] + u^T [I & iI] L_0$. By gaussian integral formula now it follows that

$$
G_{\Gamma_0(L)}(u, v) = \alpha(L)^{-1/4} \exp\left\{\frac{1}{2} m^T (I + L_0^T L_0) m\right\}.
$$

Expanding the exponent in the right side after substituting the expression for $m$ in terms of $u, v$ we get the required result. \qed
Corollary 4.6 (Klauder-Bargmann representation for $\Gamma_0(L)$). Let $|\psi(z)\rangle$ denote the coherent state at $z \in \mathcal{H}$. Then the unitary operator $\Gamma_0(L)$ has the following weak operator integral representation

$$\Gamma_0(L) = \alpha(L)^{1/4} \frac{1}{\pi^n} \int_{\mathcal{H}} |\psi(Lz)\rangle\langle\psi(z)| \, dz.$$  \hspace{1cm} (4.9)

In particular, the map $L \mapsto \Gamma_0(L)$ from $Sp(\mathcal{H})$ into $\mathcal{U}(\mathcal{H})$ is strongly continuous.

Proof. For $\mathcal{H} = \mathbb{C}^n$ we have

$$\langle e(\bar{u})|\psi(Lz)\rangle \langle \psi(z)|e(v)\rangle = \exp\left\{-\frac{1}{2}(|z|^2 + |Lz|^2) + \langle \bar{u}|Lz \rangle + \langle z|v \rangle\right\}.$$

Now equation (4.8) implies that

$$G_{\Gamma_0(L)}(u, v) = \alpha(L)^{1/4} \frac{1}{\pi^n} \int_{\mathbb{C}^n} \langle e(\bar{u})|\psi(Lz)\rangle \langle \psi(z)|e(v)\rangle \, dz, \forall u, v \in \mathcal{H}$$

which yields (4.9). \hfill \Box

Remarks 4.7. For any $L \in Sp(\mathcal{H})$, a unitary operator $\Gamma$ in $\Gamma(\mathcal{H})$ is said to intertwine $W(u)$ and $W(Lu)$ for all $u \in \mathcal{H}$ if

$$\Gamma W(u)\Gamma^{-1} = W(Lu).$$

Let $\mathcal{G}_L$ denote the set of all such intertwiners. We know that there exists a unique element $\Gamma_0(L)$ which satisfies the condition $\langle \Omega|\Gamma_0(L)|\Omega \rangle > 0$.

1. If $U$ is a unitary operator in $\mathcal{H}$, i.e., $U \in \mathcal{U}(\mathcal{H})$ and $\Gamma(U)$ is the associated second quantization operator satisfying $\Gamma(U)e(u) = e(Uu), \forall u \in \mathcal{H}$, then $\langle \Omega|\Gamma(U)|\Omega \rangle = 1$ and hence $\Gamma_0(U) = \Gamma(U)$.

2. For any $U, V$ in $\mathcal{U}(\mathcal{H})$ and $L \in Sp(\mathcal{H})$, $\Gamma_0(U)\Gamma_0(L)\Gamma_0(V) = \Gamma_0(ULV)$ and $\langle \Omega|\Gamma_0(U)\Gamma_0(L)\Gamma_0(V)|\Omega \rangle = \langle \Omega|\Gamma_0(ULV)|\Omega \rangle > 0$. Hence

$$\Gamma_0(U)\Gamma_0(L)\Gamma_0(V) = \Gamma_0(ULV).$$

3. For any $L \in Sp(\mathcal{H})$, $\langle \Omega|\Gamma_0(L)^{-1}|\Omega \rangle = \langle \Omega|\Gamma_0(\overline{L})|\Omega \rangle > 0$, $\Gamma_0(L)^{-1}$ and $\Gamma_0(L^{-1})$ lie in $\mathcal{G}_{L^{-1}}$ and hence $\Gamma_0(L^{-1}) = \Gamma_0(L)^{-1}$.

Definition 4.8. An $n$-mode state $\rho \in \mathcal{B}(\Gamma(\mathcal{H}))$ is called a gaussian state if there exists an $m \in \mathbb{C}^n$ and a $2n \times 2n$ real, symmetric matrix $S$ such that

$$\hat{\rho}(z) = \exp\left\{-2i\text{Im} \langle z|m \rangle - [x^T \ y^T] S \begin{bmatrix} x \\ y \end{bmatrix}\right\},$$

where $z \mapsto z = x + iy$ is an identification of $\mathcal{H}$ with $\mathbb{C}^n$. In this case, we write $\rho = \rho_{m,s}$, $m$ is the mean annihilation vector, simply called the mean of the state and $S$ is the position-momentum covariance matrix.
Remarks 4.9.  1. A $2n \times 2n$ real, symmetric matrix $S$ is the
position-momentum covariance of a gaussian state if and only if

$$S + \frac{i}{2} J \geq 0,$$  \hspace{1cm} (4.11)

where the positive definiteness is that in $M_{2n}(\mathbb{C})$ \cite{Par10}. More generally, $S$ is the position-momentum covariance matrix of an $n$-mode state $\rho$ if and it satisfies (4.11).

2. We say that a unitary operator $U$ on $\Gamma(\mathcal{H})$ is a gaussian symmetry if for any gaussian state $\rho$ in $\Gamma(\mathcal{H})$, the state $U\rho U^\dagger$ is also gaussian. It is a theorem that any such gaussian symmetry $U$ is equal to $\lambda W(u) \Gamma_0(L)$, where $\lambda$ is a scalar of modulus unity, $u \in \mathcal{H}$, $L \in Sp(\mathcal{H})$ (See \cite{Par13, BJS19}). Now Corollary 4.10 below says that every gaussian symmetry has a Klauder-Bargmann integral representation (4.12).

Corollary 4.10 (Klauder-Bargmann representation for symmetries of gaussian states). For $u \in \mathcal{H}, L \in Sp(\mathcal{H})$,

$$W(u) \Gamma_0(L) = \frac{\alpha(L)^{1/4}}{\pi^n} \int_{\mathcal{H}} \exp \{-i \text{Im} \langle u|Lz \rangle\} |\psi(u + Lz)\rangle \langle \psi(z)| dz.$$ \hspace{1cm} (4.12)

Proof. This is immediate from (4.9).

Theorem 4.11. Denote by $\mathcal{H} \overbrace{\otimes}^{\Delta} Sp(\mathcal{H})$, the Lie group which is the semidirect product of the additive group $\mathcal{H}$ and the Lie group $Sp(\mathcal{H})$ acting on $\mathcal{H}$ so that the multiplication in $\mathcal{H} \overbrace{\otimes}^{\Delta} Sp(\mathcal{H})$ is defined by

$$(u, L)(v, M) = (u + Lv, LM) \text{ for all } u, v \in \mathcal{H}, L, M \in Sp(\mathcal{H}).$$

Then the map $(u, L) \mapsto W(u) \Gamma_0(L)$ is a strongly continuous, projective unitary representation of $\mathcal{H} \overbrace{\otimes}^{\Delta} Sp(\mathcal{H})$ in $\Gamma(\mathcal{H})$.

Proof. Only the strong continuity of the map remains to be proved. To this end we consider the Lebesgue measure preserving group action $(u, L) : z \mapsto u + Lz, (u, L) \in \mathcal{H} \overbrace{\otimes}^{\Delta} Sp(\mathcal{H})$. This yields a strongly continuous unitary representation

$$(U_{(u, L)} f)(z) = f((u, L)^{-1} z), f \in L^2(\mathcal{H})$$

with Lebesgue measure on $\mathcal{H}$ by identification of $\mathcal{H}$ with $\mathbb{C}^n$. This implies that the map

$$\phi \mapsto \langle \phi|\psi(u + Lz)\rangle, z \in \mathcal{H}, \phi \in \Gamma(\mathcal{H})$$

is continuous as a map from $\Gamma(\mathcal{H})$ into $L^2(\mathcal{H})$, thanks to the Klauder-Bargmann isometry. Now an application of Klauder-Bargmann representation implies that

$$(u, L) \mapsto \langle \phi|W(u) \Gamma_0(L)|\phi'\rangle$$

is continuous in $(u, L)$ for any $\phi, \phi'$ in $\Gamma(\mathcal{H})$. In other words, $(u, L) \mapsto W(u) \Gamma_0(L)$ is weakly continuous. Unitarity implies strong continuity. \qed
Remark 4.12. Writing
\[ W(u, L) = W(u)\Gamma_0(L) \]
we conclude from Theorem 4.11 that the map \((u, L) \mapsto W(u, L)\) is a strongly continuous irreducible unitary representation of the semidirect product group \(\mathcal{H} \oplus \text{Sp}(\mathcal{H})\) in \(\Gamma(\mathcal{H})\) and

\[ W(u, L)W(v, M) = e^{-i\text{Im}(u|Lv)}\sigma_0(L, M)W(u + Lv, LM) \quad \forall u, v \in \mathcal{H}, L, M \in \text{Sp}(\mathcal{H}) \]

where \(\sigma_0(L, M)\) is a continuous function of \((L, M)\).

5 The Semigroup \(\mathcal{E}_2(\mathcal{H})\)

Definition 5.1. An operator \(Z\) on \(\Gamma(\mathcal{H})\) is said to be in the class \(\mathcal{E}_2(\mathcal{H})\) if there exists \(c \in \mathbb{C}, \alpha, \beta \in \mathbb{C}^n, A, B, \Lambda \in M_n(\mathbb{C})\), with \(A\) and \(B\) symmetric, such that the generating function of \(Z\) is of the form

\[
G_Z(u, v) = c\exp\{u^T\alpha + \beta^T v + u^T A u + u^T \Lambda v + v^T B v\}, \forall u, v \in \mathbb{C}^n. \tag{5.1}
\]

The ordered 6-tuple \((c, \alpha, \beta, A, \Lambda, B)\) completely characterizes \(Z \in \mathcal{E}_2(\mathcal{H})\) and we call them the \(\mathcal{E}_2(\mathcal{H})\)-parameters of \(Z\).

Examples 5.2.
1. Let \(C\) be any contraction operator on \(\mathcal{H}\), then the second quantization contraction \(\Gamma(C)\) satisfies \(\Gamma(C)\{e(v)\} = \{e(Cv)\}, v \in \mathcal{H}\). So,

\[
G_{\Gamma(C)}(u, v) = \exp\{u^T C v\}. \tag{5.2}
\]

Hence \(\Gamma(C) \in \mathcal{E}_2(\mathcal{H})\) with parameters \((1, 0, 0, 0, C, 0)\).

2. Let \(z \in \mathcal{H}\), then the associated Weyl displacement operator satisfies

\[
W(z)e(v) = \exp\left\{-\frac{1}{2}|z|^2 - \langle z|v \rangle\right\}e(z + v).
\]

So

\[
G_{W(z)}(u, v) = \exp\left\{-\frac{1}{2}|z|^2 - \langle z|v \rangle + \langle \bar{u}|z + v \rangle\right\}. \tag{5.3}
\]

Hence \(W(z) \in \mathcal{E}_2(\mathcal{H})\) with parameters \((e^{-\frac{1}{2}|z|^2}, z, -\bar{z}, 0, I, 0)\).

3. Let \(L \in \text{Sp}(\mathcal{H})\). By Corollary 4.1.5 \(\Gamma_0(L) \in \mathcal{E}_2(\mathcal{H})\), with parameters

\((\alpha(L)^{-1/4}, 0, 0, A_L, \Lambda_L, B_L)\), where \(\alpha(L)\) is as in (4.6) and \(A_L, \Lambda_L, B_L\) are as in (4.4).

Take \(\mathcal{H} = \mathbb{C}^n\). We proceed to identify the parameters of elements in \(\mathcal{E}_2(\mathcal{H})\). To this end we shall adopt the following notations.

\[
|\chi_i\rangle := |0, \cdots, 0, 1, 0, \cdots, 0\rangle, \quad |\chi_i\rangle := |0, \cdots, 0, 1, 0, \cdots, 0, 1, 0, \cdots, 0\rangle,
\]

\[
|\chi_{ij}\rangle := |0, \cdots, 0, 2, 0, \cdots, 0\rangle, \quad 1 \leq i, j \leq n, i \neq j.
\]
Suppose \( Z \in \mathcal{E}_2(\mathcal{H}) \) with parameters \((c, \alpha, \beta, A, \Lambda, B)\), by equations (2.5) and (5.1),

\[
\sum_{r,s} \langle r|Z|s \rangle \frac{u^r v^s}{\sqrt{r!s!}} = c\{1 + (u^T \alpha + \beta^T v + u^T A u + u^T \Lambda v + v^T B v)
\]

\[
+ \frac{1}{2!}(u^T \alpha + \beta^T v + u^T A u + u^T \Lambda v + v^T B v)^2 + \cdots \}. \tag{5.4}
\]

Let \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)^T, \beta = (\beta_1, \beta_2, \ldots, \beta_n)^T, A = [a_{ij}], B = [b_{ij}], \Lambda = [\lambda_{ij}] \). Comparing the coefficients of \( u^r v^s \) up to degree 2 on both sides of equation (5.4) we get,

\[
c = \langle \Omega|Z|\Omega \rangle, \quad c\left(\frac{a_{ij}^2}{2} + a_{jj}\right) = \frac{1}{\sqrt{2}} \langle \chi_{jj}|Z|\Omega \rangle,
\]

\[
ca_j = \langle \chi_j|Z|\Omega \rangle, \quad c\left(\frac{b_{jj}^2}{2} + b_{jj}\right) = \frac{1}{\sqrt{2}} \langle \Omega|Z|\chi_{jj} \rangle, \tag{5.5}
\]

\[
c\beta_j = \langle \Omega|Z|\chi_j \rangle, \quad c(\alpha_i \alpha_j + 2a_{ij}) = \langle \chi_{ij}|Z|\Omega \rangle,
\]

\[
c(\alpha_i \beta_j + \lambda_{ij}) = \langle \chi_i|Z|\chi_j \rangle, \quad c(\beta_i \beta_j + 2b_{ij}) = \langle \Omega|Z|\chi_{ij} \rangle,
\]

where \(1 \leq i, j \leq n\) and the equations involving \( \chi_{ij} \) are valid only for \( i \neq j \). We call \( c \) the vacuum parameter of \( Z \), \( \alpha \) and \( \beta \) the 1-particle annihilation and creation vectors, and \( A, \Lambda, B \) the 2-particle annihilation, exchange and creation matrices respectively.

**Remark 5.3.** The most notable feature of an operator \( Z \in \mathcal{E}_2(\mathcal{H}) \) is the property that all the matrix entries of \( Z \) in the particle basis are completely determined by the entries \( \{ \langle k|Z|\ell \rangle ||k| + |\ell| \leq 2 \} \) which is a finite set of cardinality \( 2n^2 + 3n + 1 \). Indeed, if \( Z \) is hermitian then it is determined by \((n + 1)^2\) entries out of which \( n \) are real, \((n + 1)^2 - n \) may be complex entries. Later we shall prove that a state \( \rho \) in \( \Gamma(\mathcal{H}) \) is gaussian if and only if it is in \( \mathcal{E}_2(\mathcal{H}) \). Thus tomography of a gaussian state in \( \Gamma(\mathcal{H}) \) requires the estimation of atmost \( 2(n + 1)^2 - n \) events which are one dimensional projections in the subspace spanned by 0, 1 and 2-particle vectors.

**Proposition 5.4.** If \( \rho \) is an \( n \)-mode gaussian state then \( \rho \in \mathcal{E}_2(\mathcal{H}) \).

**Proof.** Take \( \mathcal{H} = \mathbb{C}^n \). Let \( \rho = \rho_{m,s} \) be a gaussian state on \( \Gamma(\mathbb{C}^n) \). By (4.10), (5.3) and (4.2),

\[
\hat{\rho}(x) = \exp \left\{ \left[ x^T \ y^T \right] \left[ \frac{(\bar{m} - m)}{i(\bar{m} + m)} \right] - \left[ x^T \ y^T \right] S \left[ x \ y \right] \right\} \tag{5.6}
\]

\[
G_{W(x)}(u, v) = e^{u^T v} \exp \left\{ -\frac{1}{2} \left[ x^T \ y^T \right] I \left[ x \ y \right] + \left[ x^T \ y^T \right] \left( \left[ I \ I \right] u - \left[ I \ -I \right] v \right) \right\} \tag{5.7}
\]
Now by the Wigner isomorphism Theorem 3.4, and equations (5.6) and (5.7),
\[
G_\rho(u, v) = \text{Tr} \rho |e(v)\rangle \langle e(\bar{v})| = \langle \rho| e(v)\rangle \langle e(\bar{v})| \rangle \equiv B_2(\Gamma(H))
\]
\[
= \frac{1}{\pi^n} \int_{\mathbb{C}^n} \bar{\rho}(z) \langle e(\bar{u})|W(z)|e(v)\rangle \, dz
\]
\[
= \frac{1}{\pi^n} \int_{\mathbb{C}^n} \bar{\rho}(z) G_{W(z)}(u, v) \, dz
\]
\[
= \frac{e^{u^T v}}{\pi^n} \int_{\mathbb{R}^{2n}} \exp \left\{ - \left[ \begin{array}{c} x^T \\ y^T \end{array} \right] \left( \frac{1}{2} + S \right) \left[ \begin{array}{c} x \\ y \end{array} \right] + q^T \left[ \begin{array}{c} x \\ y \end{array} \right] \right\} \quad (5.8)
\]
where \( q = \left[ \begin{array}{c} I \\ if \end{array} \right] u - \left[ \begin{array}{c} I \\ -if \end{array} \right] v + \left[ \begin{array}{c} (\bar{m} - m) \\ i(m + \bar{m}) \end{array} \right] \). The right hand side of (5.8) being a gaussian integral in \( \mathbb{R}^{2n} \) implies that \( \rho_{m, S} \in E_2(H) \).

**Proposition 5.5.** Let \( \rho \in B_1(\Gamma(H)) \), then the quantum Fourier transform of \( \rho \) can be expressed in terms of the generator of \( \rho \) as
\[
\hat{\rho}(u) = e^{-\frac{1}{2}|u|^2} \int_{\mathcal{H}} \exp\left\{-|z|^2 - \langle u |z \rangle \right\} G_\rho(z, u + z) \, dz \quad (5.9)
\]

**Proof.** By the Klauder-Bargmann formula,
\[
\hat{\rho}(u) = \text{Tr} \rho W(u) = \frac{1}{\pi^n} \int_{\mathcal{H}} \langle \psi(z) | \rho e^{-i \text{Im} \langle u |z \rangle} | \psi(u + z) \rangle \, dz
\]
\[
= \frac{1}{\pi^n} \int_{\mathcal{H}} \exp\left\{-|z|^2 - \frac{1}{2}|u|^2 - \langle u |z \rangle \right\} \langle e(z) | \rho | e(u + z) \rangle \, dz
\]
This is same as (5.9).

**Proposition 5.6.** Suppose \( Z \in E_2(H) \) with parameters \( (c, \lambda, \mu, A, \Lambda, B) \). If \( Z = Z^\dagger \), then \( c \in \mathbb{R}, \lambda = \bar{\mu}, A = \bar{B}, \text{ and } \Lambda = \Lambda^\dagger \). Furthermore, if \( Z \geq 0 \) and \( Z \neq 0 \), then \( c > 0 \) and \( \Lambda \geq 0 \).

**Proof.** 1. Take \( \mathcal{H} = \mathbb{C}^n \) without loss of generality. By definition,
\[
G_{Z^\dagger}(u, v) = \overline{G_Z(\bar{v}, \bar{u})}.
\]
Therefore self-adjointness of \( Z \) implies,
\[
G_Z(u, v) = \overline{G_Z(\bar{v}, \bar{u})}.
\]
Furthermore, if \( Z \in E_2(H) \),
\[
c \exp\left\{ u^T \alpha + \beta^T v + u^T A u + u^T \Lambda v + v^T B v \right\}
\]
\[
= \bar{c} \exp\left\{ v^T \bar{\alpha} + \bar{\beta}^T u + v^T \bar{A} v + v^T \bar{\Lambda} u + u^T \bar{B} u \right\}, \forall u, v \in \mathbb{C}^n. \quad (5.10)
\]
This is possible only if \( c = \bar{c}, \alpha = \bar{\beta}, A = \bar{B} \) and \( \Lambda = \Lambda^\dagger \). If \( Z \geq 0 \) and \( Z \neq 0 \), then \( c \neq 0 \) and \( c = G_Z(0,0) > 0 \). Furthermore the kernel \( (u,v) \mapsto G_Z(u,v) \) is positive definite and is of the form

\[
G_Z(u,v) = c f(u)f(v)e^{(u|\Lambda|v)},
\]

where \( f(v) = \exp\{v^T \bar{A} v + \bar{\beta}^T v\} \). Since \( c f(u)f(v) \) is already a positive definite kernel, \( e^{(u|\Lambda|v)} \) has to be positive definite. Hence \( (u|\Lambda|v) \) is positive semidefinite, in other words \( \Lambda \geq 0 \).

**Theorem 5.7.** The class \( \mathcal{E}_2(\mathcal{H}) \) is a semigroup.

**Proof.** Let \( Z_j \in \mathcal{E}_2(\mathcal{H}) \), with parameters \( (a_j, \alpha_j, \beta_j, A_j, A_j, B_j), j = 1, 2 \). Assume without loss of generality that \( \mathcal{H} = \mathbb{C}^n \). By the Klauder-Bargmann isometry and formula,

\[
G_{Z_1Z_2}(u,v) = \langle e(\bar{u})|Z_1Z_2|e(v)\rangle = \frac{1}{\pi^n} \int_{\mathcal{H}} \langle e(\bar{u})|Z_1|\psi(z)\rangle \langle \psi(z)|Z_2|e(v)\rangle = \frac{1}{\pi^n} \int_{\mathcal{H}} G_{Z_1}(u,z)G_{Z_2}(\bar{z},v)e^{-|z|^2}dz,
\]

where the integrand is an \( L_1 \) function of \( z \) of the form

\[
f(u,v)g(u,v,z),
\]

with

\[
f(u,v) = \frac{a_1a_2}{\pi^n} \exp\{u^T \alpha_1 + \beta_2^T v + u^T A_1 u + v^T B_2 v\}
\]

\[
g(u,v,z) = \exp\{-|z|^2 + z^T B_1 z + \bar{z}^T A_2 \bar{z} + (\beta_1^T + u^T A_1)z + (\alpha_2^T + v^T A_2^T)\bar{z}\}
\]

which is an \( L^1 \) function of \( z \). If we write \( z = x + i y \) then \( g(u,v,z) \) assumes the form

\[
g(u,v,x + iy) = \exp\left\{ -\begin{bmatrix} x^T & y^T \end{bmatrix} R \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} u^T \\ v^T \end{bmatrix} S + \begin{bmatrix} \mu^T \\ \nu^T \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \right\}
\]

where \( R \) is a symmetric \( 2n \times 2n \) matrix, \( S \) is a \( 2n \times 2n \) complex matrix and \( \mu, \nu \) are vectors in \( \mathbb{C}^n \). Because \( g(u,v,z) \) is integrable, the real part of \( R \) is strictly positive definite. Now an application of gaussian integral formula shows that \( G_{Z_1Z_2}(u,v) \) as a function of \( (u,v) \) has the structure of the generating function of an operator in \( \mathcal{E}_2(\mathcal{H}) \). This completes the proof.

The following is a corollary of Klauder-Bargmann formula.

**Lemma 5.8.** Let \( \rho \in \mathcal{B}_1(\mathcal{H}) \) then

\[
\frac{1}{\pi^n} \int_{\mathcal{H}} \text{Tr} \rho |\psi(z)\rangle \langle \psi(z)|dz = \text{Tr} \rho.
\]

21
Proof. If ρ = |φ⟩⟨φ| then (5.12) is immediate from the Klauder-Bargmann isometry. If ρ is a positive trace class operator then (5.12) follows from the preceding case by an application of the spectral theorem. Now the Lemma follows from the fact that any trace class operator is a linear combination of four positive trace class operators. \[\square\]

Theorem 5.9. Let \( \mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \) with \( \dim \mathcal{H}_0 = n_0 \) and \( \dim \mathcal{H}_1 = n_1 \) so that \( \Gamma(\mathcal{H}) = \mathfrak{h}_0 \otimes \mathfrak{h}_1 \) where \( \mathfrak{h}_i = \Gamma(\mathcal{H}_i), i = 0, 1 \). Let ρ be any state on \( \Gamma(\mathcal{H}) \) and let \( \rho_{1-i} := \text{Tr}_i \rho \in \mathcal{B}(\mathfrak{h}_{1-i}) \) denote the \( \mathfrak{h}_{1-i} \) marginal of \( \rho \), \( i = 0, 1 \). For any \( u_0, v_0 \in \mathcal{H}_0 \),

\[
\frac{1}{\pi^{n_1}} \int_{\mathcal{H}_1} \text{Tr} \rho |\psi(u_0 \oplus z)\rangle \langle \psi(v_0 \oplus z)| dz = G_{\rho_{0i}}(v_0, u_0)e^{-\frac{1}{2}(\|u_0\|^2 + \|v_0\|^2)}.
\]

(5.13)

Proof. Take \( \mathcal{H} = \mathbb{C}^n \) and \( \mathcal{H}_i = \mathbb{C}^{n_i}, i = 0, 1 \). Let \( I_i \) denote the identity operator in \( \mathfrak{h}_i, i = 0, 1 \). Using Lemma 5.8 and the general property that, \( \text{Tr} A(I_0 \otimes B) = \text{Tr}(\text{Tr}_0 A)B \) and \( \text{Tr} C(D \otimes I_1) = \text{Tr}(\text{Tr}_1 C)D \) for operators \( B, D \in \mathcal{B}(\mathfrak{h}_1), \mathcal{B}(\mathfrak{h}_0) \) respectively, we have

\[
\frac{1}{\pi^{n_1}} \int_{\mathbb{C}^{n_1}} \text{Tr} \rho |\psi(u_0 \oplus z)\rangle \langle \psi(v_0 \oplus z)| dz
\]

\[= \frac{1}{\pi^{n_1}} \int_{\mathbb{C}^{n_1}} \text{Tr} \rho |\psi(u_0)\rangle \langle \psi(v_0)| \otimes |\psi(z)\rangle \langle \psi(z)| dz
\]

\[= \frac{1}{\pi^{n_1}} \int_{\mathbb{C}^{n_1}} \text{Tr} \rho(|\psi(u_0)\rangle \langle \psi(v_0)| \otimes I_1)(I_0 \otimes |\psi(z)\rangle \langle \psi(z)|) dz
\]

\[= \frac{1}{\pi^{n_1}} \int_{\mathbb{C}^{n_1}} \text{Tr}(\text{Tr}_0 \rho |\psi(u_0)\rangle \langle \psi(v_0)| \otimes I_1) |\psi(z)\rangle \langle \psi(z)|) dz
\]

\[= \text{Tr} \rho(|\psi(u_0)\rangle \langle \psi(v_0)| \otimes I_1)
\]

\[= \text{Tr}(\text{Tr}_1 \rho) |\psi(u_0)\rangle \langle \psi(v_0)|
\]

\[= \text{Tr} \rho_0 |\psi(u_0)\rangle \langle \psi(v_0)|
\]

\[= G_{\rho_{0i}}(v_0, u_0)e^{-\frac{1}{2}(\|u_0\|^2 + \|v_0\|^2)}.
\]

\[\square\]

Corollary 5.10. Let \( \mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \) with \( \dim \mathcal{H}_0 = n_0 \) and \( \dim \mathcal{H}_1 = n_1 \) so that \( \Gamma(\mathcal{H}) = \mathfrak{h}_0 \otimes \mathfrak{h}_1 \) where \( \mathfrak{h}_i = \Gamma(\mathcal{H}_i), i = 0, 1 \). Let ρ be any state on \( \Gamma(\mathcal{H}) \) and let \( \rho_{1-i} := \text{Tr}_i \rho \in \mathcal{B}(\mathfrak{h}_{1-i}) \) denote the \( \mathfrak{h}_{1-i} \) marginal of \( \rho \), \( i = 0, 1 \). For any \( u_0, v_0 \in \mathcal{H}_0 \),

\[
G_{\rho_{0i}}(u_0, v_0) = \frac{1}{\pi^{n_1}} \int_{\mathcal{H}_1} G_{\rho}(u_0 + \bar{z}, v_0 + \bar{z}) e^{-\frac{1}{2} \|\bar{z}\|^2} dz.
\]

(5.14)

Proof. Equation (5.14) is a direct consequence of (5.13). \[\square\]

6 Gaussian States

In Proposition 5.4 of the previous section we proved that every gaussian state is an element of the semigroup \( \mathcal{E}_2(\mathcal{H}) \). Here we shall prove that any state belonging to \( \mathcal{E}_2(\mathcal{H}) \) is a gaussian state and examine some consequences of this result.
**Remark 6.3.** Proposition 5.6 allows a reduction in the number of parameters required to describe a state $\rho$ belonging to $\mathcal{E}_2(\mathcal{H})$. We now parametrize such a state $\rho$ by the quadruple $(c, \alpha, A, \Lambda)$, where $c = |\langle \Omega | \rho | \Omega \rangle| > 0$, $\alpha \in \mathbb{C}^n$, $A$ is a complex symmetric matrix and $\Lambda$ is a positive matrix, so that the generating function of $\rho$ takes the form
\[
G_\rho(u, v) = c \exp \left\{ u^T \alpha + \alpha^T v + u^T A u + u^T \Lambda v + v^T \bar{A} v \right\}.
\]

**Theorem 6.1.** A state $\rho$ in $\Gamma(\mathcal{H})$ is gaussian if and only if $\rho \in \mathcal{E}_2(\mathcal{H})$.

**Proof.** If $\rho$ is a gaussian state then by Proposition 5.3, $\rho \in \mathcal{E}_2(\mathcal{H})$. Conversely, let $\rho \in \mathcal{E}_2(\mathcal{H})$ be a state with parameters $(c, \alpha, A, \Lambda)$. Now by Proposition 5.3, $\hat{\rho}(u)$ takes the form
\[
\hat{\rho}(u) = e^{-\frac{1}{2} |u|^2 + u^T A u + \alpha^T u} \frac{C}{\pi^{2n}} \int_{\mathbb{R}^{2n}} \exp \left\{ - \begin{bmatrix} x^T & y^T \end{bmatrix} M \begin{bmatrix} x \\ y \end{bmatrix} + \ell^T \begin{bmatrix} x \\ y \end{bmatrix} \right\} dx dy, \tag{6.1}
\]
where $M$ is an $n \times n$ complex symmetric matrix and $\ell \in \mathbb{C}^{2n}$ is of the form
\[
\ell^T = u^T M_1 + u^T M_2 + p^T,
\]
where $M_1$ and $M_2$ are $n \times 2n$ constant complex matrices and $p \in \mathbb{C}^{2n}$ is a constant complex vector. Since the function under the integral sign is integrable, $M$ has strictly positive real part. If $u = \xi + i \eta$, an application of the gaussian integral formula in $\mathbb{R}^{2n}$ shows that
\[
\hat{\rho}^{\xi + i \eta} = c' \exp \left\{ -Q(\xi, \eta) + q_1^T \xi + q_2^T \eta \right\}, \forall \xi, \eta \in \mathbb{R}^n, \tag{6.2}
\]
where $c'$ is a constant scalar, $Q$ is a quadratic form in $2n$ real variables with complex coefficients and $q_1, q_2$ are some elements in $\mathbb{C}^n$. Furthermore, by Property 4 of quantum Fourier transform, the map $t \mapsto \hat{\rho}(t(\xi + i \eta)), t \in \mathbb{R}$ is the characteristic function of a probability distribution $\mu_{\xi, \eta}$ on the real line for any fixed $\xi, \eta$. Hence $\mu_{\xi, \eta}$ is a normal distribution on $\mathbb{R}$, $Q(\xi, \eta) \geq 0, \forall \xi, \eta$ and $q_j = i \gamma_j, j = 1, 2$ for some real vectors $\gamma_1, \gamma_2 \in \mathbb{R}^n$. Thus $\hat{\rho}$ is the quantum Fourier transform of a gaussian state $\rho \in \Gamma(\mathcal{H})$. \hfill $\Box$

**Corollary 6.2.** Let $\rho$ be a gaussian state, $Z \in \mathcal{E}_2(\mathcal{H})$. Then for any $t > 0$, $\frac{Z^{\rho^t} Z^{\dagger}}{t^{1/2} e^{\rho^t} Z^{\dagger} Z}$ is a gaussian state.

**Proof.** By the structure theorem for gaussian states (Theorem 4, Par92), $\frac{Z^{\rho^t} Z^{\dagger}}{t^{1/2} e^{\rho^t} Z^{\dagger} Z}$ is again a gaussian state, and hence $\rho^t \in \mathcal{E}_2(\mathcal{H})$. Since $\mathcal{E}_2(\mathcal{H})$ is a semigroup, the Corollary follows immediately. \hfill $\Box$

**Remark 6.3.** For any $Z \in \mathcal{E}_2(\mathcal{H})$ the map $\rho \mapsto \frac{Z^{\rho^t} Z^{\dagger}}{t^{1/2} e^{\rho^t} Z^{\dagger} Z}$ on the set of all states yields a nonlinear gaussian state preserving information channel.

**Proposition 6.4.**

1. Any unitary operator in $\mathcal{E}_2(\mathcal{H})$ is a gaussian symmetry.

2. Any finite rank projection operator in $\mathcal{E}_2(\mathcal{H})$ is conjugate to the vacuum projection $|\Omega \rangle \langle \Omega|$ by a gaussian symmetry.

23
Proof. Let $U \in \mathcal{E}_2(\mathcal{H})$ be unitary. If $\rho$ is any gaussian state, then $\rho, U\rho U^\dagger \in \mathcal{E}_2(\mathcal{H})$. Thus $U\rho U^\dagger$ is a gaussian state. Hence $U$ is a gaussian symmetry.

Let $P$ be a finite rank projection operator in $\mathcal{E}_2(\mathcal{H})$. Then $P$ is a constant multiple of a gaussian state. By the structure theorem for gaussian states (Theorem 4, [Par92]), it is a one dimensional projection onto the span of $W(z)\Gamma_0(L)\ket{\Omega}$ for some $z \in \mathcal{H}, L \in \text{Sp}(\mathcal{H})$.

Our investigations show that there are two different ways of parametrizing the set of all gaussian states in $\Gamma(\mathcal{H})$, one obtained from the quantum Fourier transform and the other from generating functions. In the first approach, a gaussian state $\rho$ is described by the pair $(m, S)$, where $m$ is the mean annihilation vector and $S$ is the position-momentum covariance matrix. Such a description involves $2n^2 + 3n$ real parameters. The parametrization $(c, \alpha, A, \Lambda)$ arising from the generating function has $2n^2 + 3n + 1$ real parameters, the extra one being a normalization factor arising from the relation $\text{Tr} \rho = 1$. It is natural to explore the relationship between the two parametrizations, particularly, in the context of tomography of gaussian states as well as quantum information theory in infinite dimensions. Our next two propositions describe the exact relationship between the two parametrizations.

Proposition 6.5. Let $\rho = \rho_{m, S}$ be a gaussian state with quantum Fourier transform given by (4.10). Then the $\mathcal{E}_2(\mathcal{H})$-parameters of $\rho$, $(c, \alpha, A, \Lambda)$ are expressed in terms of $m$ and $S$ by

$$c = \det \left( \frac{1}{2} + S \right)^{-1/2} \exp \left\{ \left[ \text{Re}\{m\}^T \quad \text{Im}\{m\}^T \right] J \left( \frac{1}{2} + S \right)^{-1} J \left[ \begin{array}{c} \text{Re}\{m\} \\ \text{Im}\{m\} \end{array} \right] \right\},$$

$$\alpha = i \left[ I \quad iI \right] \left( \frac{1}{2} + S \right)^{-1} \left[ \begin{array}{c} \text{Re}\{m\} \\ \text{Im}\{m\} \end{array} \right],$$

$$A = \frac{1}{4} \left[ I \quad iI \right] \left( \frac{1}{2} + S \right)^{-1} \left[ I \quad iI \right], \quad \Lambda = I - \frac{1}{2} \left[ I \quad iI \right] \left( \frac{1}{2} + S \right)^{-1} \left[ I \quad -iI \right].$$

(6.3)

Proof. The expressions in (6.3) are obtained by applying gaussian integral formula to equation (5.8).\]

Proposition 6.6. Let $(c, \alpha, A, \Lambda)$ be the $\mathcal{E}_2(\mathcal{H})$-parameters of a gaussian state $\rho$. Then the covariance matrix $S$ and the mean annihilation vector $m$ of $\rho$ can be expressed in terms of the parameters $c, \alpha, A$ and $\Lambda$ by the following relations:

1. \begin{align*}
\left( \frac{1}{2} + S \right)^{-1} &= I + 2 \begin{bmatrix}
\text{Re}\{A\} & \text{Im}\{A\} \\
\text{Im}\{A\} & -\text{Re}\{A\}
\end{bmatrix} + \begin{bmatrix}
-\text{Re}\{\Lambda\} & \text{Im}\{\Lambda\} \\
-\text{Im}\{\Lambda\} & -\text{Re}\{\Lambda\}
\end{bmatrix}.
\end{align*}

(6.4)

2. \begin{align*}
\begin{bmatrix}
\text{Im}\{\alpha\} \\
-\text{Re}\{\alpha\}
\end{bmatrix} &= \left( \frac{1}{2} + S \right)^{-1} \begin{bmatrix}
\text{Im}\{m\} \\
-\text{Re}\{m\}
\end{bmatrix}.
\end{align*}

(6.5)

In particular, $\alpha = 0$ if and only if $m = 0$.\]
Proof. Write \((\frac{1}{2}I + S)^{-1} = \begin{bmatrix} P & Q \\ Q^T & R \end{bmatrix}\) as a block matrix where \(P, Q, R\) are of order \(n \times n\). We now solve for \(P, Q\) and \(R\) from (6.3). From the expression for \(A\) we get

\[ 4A = (P - R) + i(Q + Q^T). \]

Hence

\[ P - R = 2(A + \bar{A}). \]  \hspace{1cm} (6.6)

From the expression for \(A\) we get \(2(\Lambda - I) = -(P + R) + i(Q - Q^T)\). Hence

\[ P + R = 2I - (\Lambda + \Lambda^T). \]  \hspace{1cm} (6.7)

We get \(P\) and \(R\) from equations (6.6) and (6.7). The matrix \(Q\) is obtained by substitution. Finally we have

\[ P = I + (A + \bar{A}) - \frac{\Lambda + \Lambda^T}{2}, \]

\[ R = I - (A + \bar{A}) - \frac{\Lambda + \Lambda^T}{2}, \]

\[ Q = -i \left[(A - \bar{A}) + \frac{\Lambda - \Lambda^T}{2}\right]. \]

\[ \square \]

Remarks 6.7. 1. It would be interesting to find a simple necessary and sufficient condition on the pair \((A, \Lambda)\) consisting of a complex symmetric matrix \(A\) and a non negative definite matrix \(\Lambda\) of order \(n \times n\) which ensures the existence of a gaussian state \(\rho\) with \(E_2(H)\)-parameters \((c, \alpha, A, \Lambda)\) for some \(c, \alpha\). One may take \(\alpha\) to be 0. An obvious necessary condition is the positivity of the \(2n \times 2n\) matrix on the right hand side of (6.4). This necessary condition implies that

\[ I - \text{Re}\{A\} > 0 \quad \text{and} \quad -\frac{1}{2}(I - \text{Re}\{\Lambda\}) \leq \text{Re}\{A\} \leq \frac{1}{2}(I - \text{Re}\{\Lambda\}). \]

2. If \(\rho = \rho_{m,S}\) is a gaussian state, then \(W(-m)\rho W(-m)^\dagger = \rho_{0,S}\). If \((c, \alpha, A, \Lambda)\) are the \(E_2(H)\)-parameters of \(\rho_{m,S}\) then the transformed state

\[ W(-m)\rho W(-m)^\dagger \]

has its parameters \((c', 0, A, \Lambda)\) for some \(c' > 0\).

Proposition 6.8. Let \(H = H_0 \oplus H_1\). Let \(\rho\) be a gaussian state with \(E_2(H)\)-parameters \((c, 0, A, \Lambda)\). If \(A = \begin{bmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{bmatrix}\) and \(\Lambda = \begin{bmatrix} \Lambda_{00} & \Lambda_{01} \\ \Lambda_{10} & \Lambda_{11} \end{bmatrix}\) in the representation \(H = H_0 \oplus H_1\), then the marginal state \(\rho_0 = \text{Tr}_1 \rho\) has the \(E_2(H_0)\)-parameters \((c_0, 0, A_0, \Lambda_0)\) where

\[ c_0 = \frac{c}{\sqrt{\det M}}, \]

\[ A_0 = A_{00} + \frac{1}{4} \{2A_{01} [I - iI] + \Lambda_{01} [I + iI]\} M^{-1} \{2A_{01} [I - iI] + \Lambda_{01} [I + iI]\}^T, \]

\[ \Lambda_0 = \Lambda_{00} + \frac{1}{4} \{2A_{01} [I - iI] + \Lambda_{01} [I + iI]\} M^{-1} \{2A_{01} [I - iI] + \Lambda_{01} [I + iI]\}^\dagger, \]  \hspace{1cm} (6.8)

with

\[ M = I - 2 \begin{bmatrix} \text{Re}\{A_{11}\} & \text{Im}\{A_{11}\} \\ \text{Im}\{A_{11}\} & -\text{Re}\{A_{11}\} \end{bmatrix} + \begin{bmatrix} -\text{Re}\{\Lambda_{11}\} & \text{Im}\{\Lambda_{11}\} \\ -\text{Im}\{\Lambda_{11}\} & -\text{Re}\{\Lambda_{11}\} \end{bmatrix}. \]  \hspace{1cm} (6.9)
Proof. The proof follows from a routine computation using (5.14) and the gaussian integral formula.

Remark 6.9. It may be noted by comparing equations (6.4) and (6.9) that the matrix $M$ appearing in (6.9) is same as $(\frac{1}{2} I + S_1)^{-1}$, where $S_1$ is the covariance matrix of the gaussian state $(Tr \rho \Gamma(P_1))^{-1} \Gamma(-iP_1)\rho \Gamma(iP_1)|_{\mathcal{H}_1}$ in $\mathcal{H}_1$, with $P_1$ denoting the projection of $\mathcal{H}$ onto $\mathcal{H}_1$.

We conclude this section with some remarks on the tomography of an unknown gaussian state in $\Gamma(\mathbb{C}^n)$ through the estimation of its $\mathcal{E}_2(\mathcal{H})$-parameters $(c, \alpha, A, \Lambda)$ by using finite set valued-measurements. For tomography based on the estimation of the mean annihilation and position-momentum covariance matrix parameters with countable set-valued measurements, we refer to [PS15]. Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)^T$, $A = ((a_{jk}))$, $\Lambda = ((\lambda_{jk}))$. We have the following relations from (5.5)

$$
\langle \Omega | \rho | \Omega \rangle = c, \quad \langle x_{jj} | \rho | \Omega \rangle = \sqrt{2}c(\alpha_j^2 + a_{jj}),
$$
$$
\langle x_{j} | \rho | \Omega \rangle = c\alpha_j, \quad \langle x_{j} | \rho | \Omega \rangle = 2c(\alpha_j\alpha_k + a_{jk}),
$$
$$
\langle x_{j} | \rho | x_{k} \rangle = c(\alpha_j\alpha_k + \lambda_{jk}),
$$

where $1 \leq j, k \leq n$ and the equations involving $x_{jk}$ are valid only for $j \neq k$. Our aim is to estimate the $\mathcal{E}_2(\mathcal{H})$-parameters by making measurements in the state $\rho$. To estimate $c$, consider the projection $P_0 = |\Omega\rangle \langle \Omega|$ and the yes-no measurement

$$
\mathcal{M}_0 = \{P_0, I - P_0\}.
$$

Measurement of $\mathcal{M}_0$ in the state $\rho$ yields a classical random variable $X_0$ on the two point set $\mathcal{M}_0$ with values in $\{0, 1\}$ and

$$
\text{Pr}(X_0 = 1) = \text{Tr} \rho P_0 = c.
$$

Hence by the law of large numbers, estimates of $c$ can be obtained by making measurements $\mathcal{M}_{0}^{\otimes k}$ in $\rho^{\otimes k}$, $k \in \mathbb{N}$. The parameters $\alpha, A$ and $\Lambda$ are functions of the scalars $\langle u | \rho | v \rangle$, where $u, v$ vary over the set

$$
\mathcal{B} = \{|\Omega\rangle, |x_j\rangle, |x_{jk}\rangle | 1 \leq j \leq k \leq n \}.
$$

Towards estimating these parameters, recall the polarisation formula

$$
\langle u | \rho | v \rangle = (\frac{u + v}{\sqrt{2}} | \rho | \frac{u + v}{\sqrt{2}}) - i (\frac{u + iv}{\sqrt{2}} | \rho | \frac{u + iv}{\sqrt{2}}) - \frac{1 - i}{2}(\langle u | \rho | u \rangle + \langle v | \rho | v \rangle). \quad (6.11)
$$

For each $j, 1 \leq j \leq n$, let $\mathcal{P}_j = |x_j\rangle \langle x_j|$, $\mathcal{P}_{j0} = |\frac{x_{j} + \Omega}{\sqrt{2}}\rangle \langle \frac{x_{j} + \Omega}{\sqrt{2}}|$, $\mathcal{P}_j' = |\frac{x_{j} + \Omega}{\sqrt{2}}\rangle \langle \frac{x_{j} + \Omega}{\sqrt{2}}|$. Consider the yes-no measurements

$$
\mathcal{M}_j = \{\mathcal{P}_j, I - \mathcal{P}_j\}, \quad \mathcal{M}_{j0} = \{\mathcal{P}_{j0}, I - \mathcal{P}_{j0}\}, \quad \mathcal{M}_{j0}' = \{\mathcal{P}_{j0}', I - \mathcal{P}_{j0}'\}. \quad (6.12)
$$

To estimate $\alpha_j$, take $u = x_j$ and $v = \Omega$ in (6.11). Each term on the right hand side of (6.11) can be estimated using the procedure described to estimate $c$ but using the
measurements $\mathcal{M}_{j0}, \mathcal{M}'_{j0}, \mathcal{M}_j$ and $\mathcal{M}_0$ respectively. Thus $\alpha$ can be estimated using the measurements in (6.12).

For each $j, k, 1 \leq j \leq k \leq n$, let $\mathcal{P}_{j,k} = |\chi_{jk}+\Omega\rangle\langle \chi_{jk}+\Omega|$, $\mathcal{P}'_{j,k} = |\chi_{jk}+\Omega\rangle\langle \chi_{jk}+\Omega|$. Now consider the measurements

$$\mathcal{M}_{j,k} = \{\mathcal{P}_{j,k}, I - \mathcal{P}_{j,k}\}, \quad \mathcal{M}'_{j,k} = \{\mathcal{P}'_{j,k}, I - \mathcal{P}'_{j,k}\}.$$ (6.13)

Assume that we have already estimated $c$ and $\alpha$. Now by (6.10), to estimate $A$ and $\Lambda$, it is enough to estimate the scalars $\langle \chi_{jk} | \rho | \Omega \rangle$ and $\langle \chi_j | \rho | \chi_k \rangle$, $1 \leq j \leq k \leq n$. To this end, let $\mathcal{P}$ denote the projection onto the subspace spanned by $\mathcal{B}$ and $\mathcal{M}$ be the von-Neumann measurement

$$\mathcal{M} = \{|\zeta\rangle\langle \zeta| \mid \zeta \in \mathcal{B}\} \cup \{I - \mathcal{P}\},$$

which contains $N := \frac{(n+1)(n+2)}{2} + 1$ mutually orthogonal projections. Now we label the elements of $\mathcal{M}$ using numbers from 0 to $N - 1$, as follows

$$|\Omega\rangle\langle \Omega| \mapsto 0, \quad |\chi_j\rangle\langle \chi_j| \mapsto j, 1 \leq r \leq n,$$

$$|\chi_{jk}\rangle\langle \chi_{jk}| \mapsto n + \frac{(2n-j)(j-1)}{2} + k, 1 \leq j \leq k \leq n,$$ (6.14)

$$I - \mathcal{P} \mapsto N - 1.$$

Thus we get a classical random variable $X$ on the $N$ point sample space $\mathcal{M}$, taking values $0, 1, 2, \ldots, N - 1$ with respective probabilities

$$\Pr(X = 0) = \langle \Omega | \rho | \Omega \rangle, \quad \Pr(X = r) = \langle \chi_r | \rho | \chi_r \rangle, 1 \leq r \leq n,$$

$$\Pr(X = n + \frac{(2n-j)(j-1)}{2} + k) = \langle \chi_{jk} | \rho | \chi_{jk} \rangle, 1 \leq j \leq k \leq n,$$ (6.15)

$$\Pr(X = N - 1) = 1 - \sum_{t=0}^{N-2} \Pr(X = t).$$

Hence the scalars $\langle \chi_j | \rho | \Omega \rangle$ and $\langle \chi_j | \rho | \chi_k \rangle$ can be approximated by using the polarisation formula (6.11) and making the measurements $\mathcal{M}^{\otimes k}$ in $\rho^{\otimes k}$, $k \in \mathbb{N}$.

Denoting by $Q$ the projection $\mathcal{P}_0 + \mathcal{P}_1 + \mathcal{P}_2$ where $\mathcal{P}_j$ is the projection on the $j$-th particle subspace and tomographing the finite dimensional density operator $(\text{Tr} \rho Q)^{-1}Q \rho Q$, it is possible to reduce the number of measurements. We leave this problem open for the present.

## 7 Pure Gaussian States in the Particle Basis

In this section, we obtain the complete description of a mean zero pure gaussian state in the particle basis. The following lemma shows that, for such a state, the exchange matrix $\Lambda$ vanishes identically.

**Lemma 7.1.** Let $\rho = |\psi\rangle\langle \psi|$ be a mean zero pure gaussian state such that $\langle \psi | \Omega \rangle > 0$. Then there exists $L \in \mathfrak{sp}(\mathcal{H})$ such that the $\mathcal{E}_2(\mathcal{H})$-parameters of $\rho$ are given by $(\alpha(L)^{-1/2}, 0, A_L, 0)$, where $\alpha(L)$ and $A_L$ are as in Corollary 4.5.
Proof. By the structure theorem for gaussian states, there exists an \( L \in Sp(\mathcal{H}) \) such that \(|\psi\rangle = \Gamma_0(L)|\psi\rangle\). Now
\[
G_\rho(u, v) = \langle e(\bar{u})|\Gamma_0(L)|\Omega\rangle \langle \Omega|\Gamma_0(L)^\dagger|e(v)\rangle
= G_{\Gamma_0(L)}(u, 0)G_{\Gamma_0(L)}(0, v)
= \alpha(L)^{-1/2}\exp\{u^T A_L u + v^T \bar{A}_L v\}
\]
where the last line follows from Corollary 4.3.

**Theorem 7.2.** Let \( \rho \) be a gaussian state with covariance matrix \( S \) and \( \mathcal{E}_2(\mathcal{H}) \)-parameters \((e, \alpha, \Lambda, \Lambda)\), then \( \rho \) is a pure state if and only if one of the following holds:

1. The two particle exchange matrix \( \Lambda = 0 \).
2. The covariance matrix \( S \) satisfies the relation
\[
(\frac{1}{2} + S)^{-1} = \begin{bmatrix} P & Q \\ Q^T & 2I - P \end{bmatrix}
\]
for some real symmetric matrices \( P, Q \) of order \( n \).

**Proof.** Write \((\frac{1}{2} + S)^{-1} = \begin{bmatrix} P & Q \\ Q^T & R \end{bmatrix}\) as a \( 2 \times 2 \) block matrix. Then the expression for \( \Lambda \) in (6.3) shows that \( \Lambda \) vanishes if and only if condition \( 2 \) of the theorem holds. The necessity of condition \( 1 \) follows from Lemma 7.1. To prove sufficiency, take \( \mathcal{H} = \mathbb{C}^n \) and note that the condition \( \Lambda = 0 \) implies
\[
G_\rho(\bar{u}, v) = F(u)F(v),
\]
where \( F(x) = \sqrt{\det} \exp^{T \bar{A}x}, x \in \mathbb{C}^n \). Expanding the left side of (7.2) in the particle basis and comparing coefficients we get the matrix elements in the particle basis as \( \langle k|\rho|\ell \rangle = \beta(k)\beta(\ell) \) for some function \( \beta \) with \( \sum_{k \in \mathbb{Z}_+} |\beta(k)|^2 = \text{Tr} \rho = 1 \). Hence \( \rho \) is a rank one operator and thus a pure state. \( \square \)

**Corollary 7.3.** Let \( S \) be the covariance matrix of a pure gaussian state, i.e., \( S = \frac{1}{2}L^T L \) for some symplectic matrix \( L \in Sp(2n, \mathbb{R}) \). Let \( \mathcal{P} = \frac{1}{2}(I + iJ), \mathcal{P}^\perp = \frac{1}{2}(I - iJ) \). Then \( \mathcal{P} \) and \( \mathcal{P}^\perp \) are mutually orthogonal projections with the property that in the direct sum decomposition \( \mathbb{C}^{2n} = \text{Ran} \mathcal{P} \oplus \text{Ran} \mathcal{P}^\perp \), the operator \((\frac{1}{2} + S)^{-1}\) admits the block representation
\[
\begin{bmatrix} I_\mathcal{P} & Q \\ Q^* & I_{\mathcal{P}^\perp} \end{bmatrix},
\]
where \( I_\mathcal{P} \) and \( I_{\mathcal{P}^\perp} \) are identity operators on \( \text{Ran} \mathcal{P} \) and \( \text{Ran} \mathcal{P}^\perp \) respectively and \( Q : \text{Ran} \mathcal{P}^\perp \to \text{Ran} \mathcal{P} \) is the operator \( \mathcal{P}(\frac{1}{2} + S)^{-1}|\text{Ran} \mathcal{P}^\perp\rangle \).

**Proof.** Since \( S \) is the covariance operator of a pure gaussian state, the exchange matrix \( \Lambda = 0 \) in its \( \mathcal{E}_2(\mathcal{H}) \) representation. Hence from the expression for \( \Lambda \) in (6.3),
\[
\frac{1}{2} \begin{bmatrix} I & iI \\ iI & -iI \end{bmatrix} \left(\frac{1}{2} + S\right)^{-1} \begin{bmatrix} I & iI \\ iI & -iI \end{bmatrix} = I
\]
\[
= \frac{1}{2} \begin{bmatrix} I & iI \\ -iI & I \end{bmatrix} \left(\frac{1}{2} + S\right)^{-1} \begin{bmatrix} I & iI \\ iI & -iI \end{bmatrix} = \begin{bmatrix} I & iI \\ -iI & I \end{bmatrix} \begin{bmatrix} I & iI \end{bmatrix}.
\]
Since \( \begin{bmatrix} I & iI \\ -iI & I \end{bmatrix} \begin{bmatrix} I & iI \\ i & -iI \end{bmatrix} = I + iJ \), we get
\[
P(\frac{1}{2} + S)^{-1}P = P.
\] (7.5)

By doing a similar computation after taking transpose on both sides of (7.4) we get
\[
\frac{1}{2}(I - iJ)(\frac{1}{2} + S)^{-1}\frac{1}{2}(I - iJ) = \frac{1}{2}(I - iJ).
\] (7.6)

But \( P^\perp = I - P = \frac{1}{2}(I - iJ) \). Hence (7.6) is same as
\[
P^\perp(\frac{1}{2} + S)^{-1}P^\perp = P^\perp.
\]

This together with (7.5) completes the proof. \( \square \)

**Notation.** Let \( \text{Sym}_n(\mathbb{Z}_+^n) \) denote the set of all symmetric \( n \times n \) matrices with entries from \( \mathbb{Z}_+ \). For each symmetric matrix \( A \in M_n(\mathbb{C}) \), \( k \in \mathbb{Z}_+ \), and \( t = (t_1, t_2, \ldots, t_n)^T \in \mathbb{Z}_+^n \) with \( t_1 + \cdots + t_n = 2k \), define
\[
I(A, t, k) = \left\{ [r_{ij}] \in \text{Sym}_n(\mathbb{Z}_+^n) \mid \sum_{i,j=1}^n r_{ij} = k, r_{ii} + r_{i} = t_i, r_{ij} = 0 \text{ if } a_{ij} = 0 \right\}.
\] (7.7)

where \( r_i = \sum_j r_{ij} \). Furthermore, we use the convention \( 0^0 = 1 \) in what follows.

**Theorem 7.4.** Let \( |\psi\rangle \) be a pure gaussian state in \( \Gamma(\mathbb{C}^n) \) with mean annihilation \( 0 \) and \( \langle \psi|\Omega \rangle > 0 \). Then there exists a unique \( n \times n \) symmetric matrix, \( A = [a_{ij}], a_{ij} \in \mathbb{C} \) such that

1. The equation
\[
\langle \psi|e(\mathbf{v})\rangle = \langle \psi|\Omega \rangle \exp\{\mathbf{v}^T \bar{A} \mathbf{v}\}.
\] (7.8)
is satisfied for all \( \mathbf{v} \in \mathbb{C}^n \).

2. For \( 1 \leq i, j \leq n \), define
\[
\epsilon_{ij} = |a_{ij}|, \quad w_{ij} = \begin{cases} \frac{a_{ij}}{|a_{ij}|} & \text{if } a_{ij} \neq 0 \\ 1 & \text{otherwise.} \end{cases}
\] (7.9)

Now for \( R = [r_{ij}] \in I(A, t, k) \), define \( \omega(A, R) = \Pi_i w_{ij}^{r_{ij}} \) so that \( |\omega(A, R)| = 1 \). Then in the particle basis \( \{ t \} = \{ t_1, t_2, \ldots, t_n | t_j \in \mathbb{Z}_+, 1 \leq j \leq n \} \),
\[
\langle \psi|t_1, t_2, \ldots, t_n \rangle = \begin{cases} 0 & \text{if } t_1 + t_2 + \cdots + t_n \text{ is odd} \\ \nu_k(A, t) & \text{if } t_1 + t_2 + \cdots + t_n = 2k, \end{cases}
\] (7.10)

where
\[
\nu_k(A, t) = \sum_{R \in I(A, t, k)} 2^{(k-\text{Tr} R)} \omega(A, R) \prod_i \left[ t_i^{\frac{r_{ii}}{r_{ii}}} \prod_j \frac{\epsilon_{ij}^{r_{ij}}}{r_{ij}!} \right]^{1/2}
\] (7.11)

and
\[
\sum_{k=0}^{\infty} \sum_{t_1 + t_2 + \cdots + t_n = 2k} |\nu_k(A, t)|^2 < \infty.
\] (7.12)
Conversely, given an $n \times n$ matrix $A$ satisfying (7.12), $|\psi\rangle$ defined as

$$|\psi\rangle = c \sum_{k=0}^{\infty} \sum_{t_1,\ldots,t_n} \nu_k(A,t) |t_1,\ldots,t_n\rangle,$$

(7.13)

where $c = \left( \sum_{k=0}^{\infty} \sum_{t_1,\ldots,t_n=2k} |\nu_k(A,t)|^2 \right)^{-1/2}$, is a mean zero pure gaussian state satisfying (7.8).

**Proof.** By Lemma (7.1) there exists a symmetric matrix $A$ and a positive number $c$ such that

$$\langle e(\bar{u})|\psi\rangle \langle \psi|e(v)\rangle = c^2 \exp\left\{ u^T A u + v^T A v \right\}, \forall u, v \in \mathbb{C}^n.$$

(7.14)

Taking $u = v = 0$, we see that $c = \langle \Omega|\psi\rangle$. Furthermore, by taking $u = 0$ in (7.14), we get (7.8).

Expanding both sides of equation (7.8) in particle basis with $v = \sum_{j=1}^{n} x_j e_j$ yields

$$\sum_{k=0}^{\infty} \left\langle \psi \left| \frac{(\sum x_j e_j)^k}{\sqrt{2k!}} \right. \right\rangle = c \sum_{k=0}^{\infty} \frac{\left( \sum a_{ij} x_i x_j \right)^k}{k!}.$$

(7.15)

In the right hand side of equation (7.13), only even degree monomials in $x_1,\ldots,x_n$ occur. So $\left\langle \psi \left| v \otimes^k \right. \right\rangle = 0, \forall v$ when $k$ is odd. Now

$$\sum_{k=0}^{\infty} \left\langle \psi \left| \frac{(\sum x_j e_j)^{2k}}{\sqrt{2k!}} \right. \right\rangle = \sum_{k=0}^{\infty} \sum_{t_1,\ldots,t_n=2k} \left\langle \psi \left| t_1,\ldots,t_n \right. \right\rangle \frac{x_1^{t_1} x_2^{t_2} \cdots x_n^{t_n}}{\sqrt{t!}}.$$

(7.16)

Since $A$ is a symmetric matrix, we use multinomial expansion in $\frac{n(n+1)}{2}$ summands in the right side of equation (7.15), to get

$$c \sum_{k=0}^{\infty} \frac{\left( \sum a_{ij} x_i x_j \right)^k}{k!} = c \sum_{k=0}^{\infty} \frac{\left( \sum a_{ii} x_i^2 + 2 \sum_{i<j} a_{ij} x_i x_j \right)^k}{k!} = c \sum_{k=0}^{\infty} \sum_{r_{ij}=k} \frac{\Pi x_i^{2r_{ii}} \prod_{i<j} (2a_{ij} x_i x_j)^{r_{ij}}}{r_{ij}!} \prod_{i<j} r_{ij}!,$$

(7.17)

where $r_{ij} = 0$ whenever $a_{ij} = 0$. Then by using the matrix index set $I(A,t,k)$ defined by (7.7), the second sum in equation (7.17) can be written as

$$\sum_{r_{ij}=k} \frac{\Pi x_i^{2r_{ii}} \prod_{i<j} (2a_{ij} x_i x_j)^{r_{ij}}}{r_{ij}!} \prod_{i<j} r_{ij}! = c \sum_{t_1,\ldots,t_n=2k} \left( \sum_{I(A,t,k)} \frac{\sum_{r_{ij}=k} \prod_{i<j} r_{ij}! \prod_{i<j} (2a_{ij} x_i x_j)^{r_{ij}}}{r_{ij}!} \right).$$

(7.18)
Thus, comparing (7.16) and (7.17) and equating coefficients of $x_1^{t_1}x_2^{t_2} \cdots x_n^{t_n}$ we get from (7.15),

$$
\langle \psi | t_1, t_2, \ldots, t_n \rangle = \sum_{(r_{ij}) \in I(A, t, k)} \sum_{i \leq j} \frac{\prod_{i \leq j} r_{ij}!}{\prod_{i \leq j} r_{ij}!} \sqrt{t_1! \cdots t_n!}
$$

where $t_1 + \cdots + t_n = 2k$. This proves (7.10). The converse follows from retracing the same computations backwards. \(\square\)

**Example 7.5.** Let $n = 2$ and $A = A_{\theta, 2} := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ for some $\theta \in \mathbb{C}$. In this case, the set

$$
I(A, t, k) = \begin{cases} \{ \begin{bmatrix} 0 & k \\ k & 0 \end{bmatrix} \}, & \text{if } t_1 = t_2 = k, \\
\phi, & \text{if } t_1 \neq t_2,
\end{cases}
$$

where $\phi$ denotes the empty set. Now by (7.18)

$$
\nu_k(A, t) = \begin{cases} 2^k \theta^k, & \text{if } t_1 = t_2 = k, \\
0, & \text{if } t_1 \neq t_2.
\end{cases}
$$

Hence

$$
\sum_{k=0}^{\infty} \sum_{t_1+t_2=2k} |\nu_k(A, t)|^2 = \sum_{k=0}^{\infty} (2|\theta|)^{2k}
$$

which converges if and only if $|\theta| < \frac{1}{2}$. Thus for $A = A_{\theta, 2}$, $|\psi_{\theta, 2} := |\psi\rangle$ defined by (7.13) exists if and only if $|\theta| < \frac{1}{2}$ and in this case

$$
|\psi_{\theta, 2} \rangle = \sqrt{1 - |2\theta|^2} \sum_{k=0}^{\infty} (2\theta)^k |k, k\rangle.
$$

Thus we have obtained a two-mode, mean zero, pure gaussian state which is invariant under the permutation of modes and has the mixed one-mode marginal gaussian state

$$
(1 - 4|\theta|^2) \sum_{k=0}^{\infty} (4|\theta|^2)^k |k\rangle |k\rangle.
$$

This is a well known example of an entangled gaussian state which is called a photon number entangled state (PNES) by some authors [DGCZ00, EP03, SIS11]. We shall meet a heirarchy of such arbitrary mode gaussian states in what follows. To achieve this goal we go back first to the general case.

**Theorem 7.6.** Let $A = [a_{ij}]$ be any $n \times n$ complex symmetric matrix such that

$$
\|A\|_{\text{max}} := \max_{i,j} |a_{ij}| < \frac{1}{2(n+1)}.
$$

Then there exists a mean 0 pure gaussian state $|\psi\rangle$ in $\Gamma(\mathbb{C}^n)$, with the particle basis expansion (7.13).
**Proof.** Define \( \nu_k(A,t) \) as in (7.11). For each \( i, \ 1 \leq i \leq n \), we have

\[
\left( \epsilon_{ii} + \sum_j \epsilon_{ij} \right)^{t_i} = \sum_{R_{ii}^j \Pi_j r_{ij}^j \epsilon_{ij}} \frac{t_i!}{r_{ii}^j r_{ij}^j} \prod_j r_{ij}^j \epsilon_{ij}.
\]

Hence

\[
\left[ t_i \left( \epsilon_{ii} + \sum_j \epsilon_{ij} \right)^{t_i} \right]^{1/2} \leq \left( \epsilon_{ii} + \sum_j \epsilon_{ij} \right)^{t_i/2} \leq (n+1)^{t_i/2} \|A\|^{t_i/2} \|A\|^{t_i/2}. \tag{7.21}
\]

Recall that \( |\omega(A,R)| = 1 \) and \( t_1 + \cdots + t_n = 2k \) in (7.11), then

\[
|\nu_k(A,t)| \leq \sum_{R \in I(A,t,k)} 2^k \prod_i [(n+1)\|A\|_\text{max}]^{t_i/2} = \sum_{R \in I(A,t,k)} 2^k (n+1)^k \|A\|_\text{max}^k. \tag{7.22}
\]

But by definition, the cardinality of \( I(A,t,k) \) does not exceed the number of partitions of \( k \) into \( \frac{n(n+1)}{2} \) non-negative integers, i.e.,

\[
|I(A,t,k)| \leq \binom{k + \frac{n(n+1)}{2} - 1}{\frac{n(n+1)}{2} - 1} = \frac{(k+1)(k+2)\cdots(n+1)}{n(n+1)} - 1,
\]

which is a polynomial \( P_n(k) \) in the variable \( k \) of degree \( \frac{n(n+1)}{2} - 1 \). Hence from (7.22),

\[
|\nu_k(A,t)|^2 \leq P_n(k)^2 (2(n+1)\|A\|_\text{max})^{2k}. \tag{7.23}
\]

Since the number of partitions of \( 2k \) into \( n \) non-negative integers is the polynomial

\[
Q_n(k) = \binom{2k + n - 1}{n - 1}
\]

of degree \( n-1 \), (7.23) implies

\[
\sum_{k=0}^{\infty} \sum_{t_1 + \cdots + t_n = 2k} |\nu_k(A,t)|^2 \leq \sum_{k=0}^{\infty} Q_n(k) P_n(k)^2 (2(n+1)\|A\|_\text{max})^{2k}
\]

which converges when \( \|A\|_\text{max} < \frac{1}{2(n+1)} \). Thus (7.12) holds and an application of Theorem (7.4) completes the proof. \( \square \)

**Corollary 7.7.** Let \( A = [a_{ij}] \) be any \( n \times n \) complex symmetric matrix such that \( a_{ii} = 0, \forall \ i \). Then there exists a mean 0 pure gaussian state in \( \Gamma(C^n) \), with particle basis expansion (7.13) if

\[
\|A\|_\text{max} < \frac{1}{2(n-1)}. \tag{7.24}
\]

**Proof.** The proof follows in a similar manner as that of Theorem (7.6). \( \square \)

**Example 7.8.** Let \( n = 3 \), and

\[
A = \begin{bmatrix}
0 & a_{12} & a_{13} \\
 a_{12} & 0 & a_{23} \\
 a_{13} & a_{23} & 0
\end{bmatrix}
\]

32
Hence with the permutation group acting on the modes. Our next theorem generalizes this marginal states
\[ s = \begin{pmatrix} t & \ldots & t \end{pmatrix} \]
As a special case of the preceding example, choose
\[ \psi = \begin{pmatrix} 1 \ldots 1 \end{pmatrix} \]
where \( j \in \mathbb{Z}_+^3 \). Now we compute a marginal of this state. Let \( \text{Tr}_3 \rho_{\theta,3} \) denote the marginal state obtained by ‘tracing out’ the third space in \( \Gamma(\mathbb{C}^3) = \Gamma(\mathbb{C}) \otimes \Gamma(\mathbb{C}) \otimes \Gamma(\mathbb{C}) \). We have
\[ \text{Tr}_3 \rho_{\theta,3} = c^2 \sum_{j,k=0}^\infty 2^{j+k} \theta^j \bar{\theta}^k \sum_{t_1+t_2+t_3=2j}^\infty \sum_{t_1 \leq t_2} C_j(t) \sum_{r_1+r_2+r_3=2k}^\infty C_k(r) |t\rangle |t\rangle , \]
where \( t = (t_1, t_2, t_3), r = (r_1, r_2, r_3) \in \mathbb{Z}_+^3 \), and \( C_i(s) = \frac{\sqrt{s_1!s_2!s_3!}}{(s_1+s_2+s_3)!} \) for \( i \in \mathbb{Z}_+^3 \). Similar formulæ can be obtained for the other marginal states \( \text{Tr}_1 \rho_{\theta,3} \) and \( \text{Tr}_2 \rho_{\theta,3} \) also. All these marginals are invariant under the permutation group acting on the modes. Our next theorem generalizes this example and proves that such states are completely entangled.
Theorem 7.10. Let $0 < \theta < \frac{1}{2(n-1)}$ and $A_{\theta,n} = [a_{ij}]$ be the $n \times n$ matrix with $a_{ij} = \theta(1 - \delta_{ij}), 1 \leq i, j \leq n$, $\delta$ being the Kronecker-$\delta$ function. Then the mean 0 pure gaussian state $|\psi\rangle = |\psi_{\theta,n}\rangle$ in $\Gamma(\mathbb{C}^n)$ given by (7.13), with $A$ replaced by $A_{\theta,n}$ is a completely entangled state which is invariant under the action of the permutation group $S_n$ on the $n$-modes.

Proof. For $k, r, s \in \{1, 2, \ldots, n\}$ let $A_{\theta,k} = [a_{ij}] \in M_k(\mathbb{R})$ be such that $a_{ij} = \theta(1 - \delta_{ij}), 1 \leq i, j \leq k$, and $B_{r,s}$ be the $r \times s$ matrix with all the entries equal to 1. By Corollary 7.7, there exists a mean 0 pure gaussian state $|\psi\rangle = |\psi_{\theta,n}\rangle$ in $\Gamma(\mathbb{C}^n)$ given by (7.13), with $A$ replaced by $A_{\theta,n}$. We have to prove that $|\psi_{\theta,n}\rangle$ is a completely entangled state, i.e., if we take $\mathbb{C}^n = \mathbb{C}^{k_0} \oplus \mathbb{C}^{k_1}$ with $k_0 + k_1 = n$ and write

$$
A_{\theta,n} = \begin{bmatrix}
A_{\theta,k_0} & B_{k_0,k_1} \\
B_{k_1,k_0} & A_{\theta,k_1}
\end{bmatrix}
$$

then the $\mathbb{C}^{k_0}$-marginal, $\rho_0 = \text{Tr}_1 \rho$ of the state $\rho = |\psi_{\theta,n}\rangle \langle \psi_{\theta,n}|$ is not a pure state. By Theorem 7.10, it is enough to prove that the two particle exchange matrix $\Lambda_0 \neq 0$ in the $E_2(\mathbb{C}^{k_0})$ representation of $\rho_0$. By Proposition 6.8

$$
\Lambda_0 = B_{k_0,k_1} \left[ I - iI \right] M^{-1} \left[ I \right] B_{k_1,k_0}
$$

(7.28)

where $M = \begin{bmatrix}
I - 2A_{\theta,k_1} & 0 \\
0 & I + 2A_{\theta,k_1}
\end{bmatrix}$. Hence

$$
\left[ I - iI \right] M^{-1} \left[ I \right] = (I - 2A_{\theta,k_1})^{-1} + (I + 2A_{\theta,k_1})^{-1}.
$$

Now from (7.28) and the equation above we have

$$
\text{Tr} \Lambda_0 = \text{Tr} \left\{ \left[ I - 2A_{\theta,k_1} \right]^{-1} + \left[ I + 2A_{\theta,k_1} \right]^{-1} \right\} B_{k_1,k_0} B_{k_0,k_1}
$$

$$
k_0 \text{Tr} \left\{ \left[ I - 2A_{\theta,k_1} \right]^{-1} + \left[ I + 2A_{\theta,k_1} \right]^{-1} \right\} B_{k_1,k_1}
$$

$$
= 2k_0 \text{Tr} \left\{ \left[ I - (2A_{\theta,k_1})^2 \right]^{-1} B_{k_1,k_1} \right\}
$$

(7.29)

where $\sigma$ is the sum of the entries of the matrix $[I - (2A_{\theta,k_1})^2]^{-1}$. Since $\|2A_{\theta,k_1}\|^2 < 1$, $[I - (2A_{\theta,k_1})^2]^{-1} = I + (2A_{\theta,k_1})^2 + (2A_{\theta,k_1})^4 + (2A_{\theta,k_1})^8 + \cdots$ is a matrix with positive entries. Hence $\text{Tr} \Lambda_0 > 0$ and thus $\Lambda_0 \neq 0$. \hfill \Box

Remark 7.11. Theorem 7.10 suggests that the quantity $\text{Tr} \Lambda_0$ occuring in the proof can be viewed as a measure of entanglement when the pure state $|\psi_{\theta,n}\rangle$ is viewed as a bipartite state in the factorization $\Gamma(\mathbb{C}^n) = \Gamma(\mathbb{C}^{k_0}) \otimes \Gamma(\mathbb{C}^{k_1})$ where the first factor is the Hilbert space for the first $k_0$ modes and the second factor for the last $k_1 = n - k_0$ modes. The permutation invariance property for $|\psi_{\theta,n}\rangle$ shows that the same entanglement property holds when $\Gamma(\mathbb{C}^n) = \mathcal{H}_0 \otimes \mathcal{H}_1$ where $\mathcal{H}_0$ and $\mathcal{H}_1$ are the Fock spaces corresponding to arbitrary partition of the set $\{1, 2, \ldots, n\}$ into any pair of nonempty disjoint subsets.

This raises the following question: if $\mathbb{C}^n$ is decomposed as an arbitrary direct sum $\mathbb{C}^n = S \oplus S^\perp$ and $\Gamma(\mathbb{C}^n) = \Gamma(S) \otimes \Gamma(S^\perp)$, does the entanglement property continue to be valid for the state $|\psi_{\theta,n}\rangle$?
Conclusions

1. A Klauder-Bargmann integral representation of all gaussian symmetries in an \( n \)-mode boson Fock space is obtained.

2. The notion of generating function of a bounded operator in the boson Fock space \( \Gamma(\mathbb{C}^n) \) over the \( n \)-dimensional Hilbert space \( \mathbb{C}^n \) is introduced and a \( \dagger \)-closed multiplicative semigroup \( \mathcal{E}_2(\mathcal{H}) \) with \( \mathcal{H} = \mathbb{C}^n \) is constructed. The semigroup \( \mathcal{E}_2(\mathcal{H}) \) is shown to contain all the gaussian states and their symmetries in \( \Gamma(\mathbb{C}^n) \).

3. Using the properties of the semigroup \( \mathcal{E}_2(\mathcal{H}) \), the set of all \( n \)-mode gaussian states is parametrized by a set of scalars derived from the matrix entries of the gaussian state at 0, 1, and 2-particle vectors of a particle basis in \( \Gamma(\mathbb{C}^n) \). The exact relations between these new parameters and the conventional set of means and covariances of position and momentum observables are obtained.

4. Mean zero pure gaussian states are parametrized by pairs \((c, A)\) where \( c \) is a positive scalar and \( A \) is a complex symmetric matrix of order \( n \). An explicit particle basis expansion of the mean zero pure gaussian state is obtained in terms of its parameters \((c, A)\).

5. Making appropriate choices of the matrix parameter \( A \), explicit examples of mean zero pure gaussian states exhibiting the property of complete entanglement and invariance under the action of the permutation group \( S_n \) on the set of all the \( n \) modes, are given.

Acknowledgements: The first author thanks Gayathri Varma for several fruitful discussions and the Indian Statistical Institute, Delhi centre for the postdoctoral fellowship.

The second author thanks the Indian Statistical Institute for providing a friendly environment and all the facilities for research in the preparation of this article under an emeritus professorship.

References

[Bar61] V. Bargmann, On a hilbert space of analytic functions and an associated integral transform part i, Communications on Pure and Applied Mathematics 14 (1961), no. 3, 187–214.

[BJS19] B. V. Rajarama Bhat, Tiju Cherian John, and R. Srinivasan, Infinite Mode Quantum Gaussian States, Reviews in Mathematical Physics 31 (2019), 1950030.

[DGCZ00] Lu-Ming Duan, G. Giedke, J. I. Cirac, and P. Zoller, Entanglement purification of gaussian continuous variable quantum states, Phys. Rev. Lett. 84 (2000), 4002–4005.

[EP03] J. Eisert and M. B. Plenio, Introduction to the basics of entanglement theory in continuous-variable systems, International Journal of Quantum Information 01 (2003), no. 04, 479–506.
[Gla63a] Roy J. Glauber, *Coherent and incoherent states of the radiation field*, Phys. Rev. **131** (1963), 2766–2788.

[Gla63b] ———, *Photon correlations*, Phys. Rev. Lett. **10** (1963), 84–86.

[Kla60] John R Klauder, *The action option and a feynman quantization of spinor fields in terms of ordinary c-numbers*, Annals of Physics **11** (1960), no. 2, 123 – 168.

[KS68] J. R. Klauder and E. C. G. Sudarshan, *Fundamentals of quantum optics*, The Mathematical Physics Monographs Series, Benjamin, 1968.

[Par92] K. R. Parthasarathy, *An introduction to quantum stochastic calculus*, Modern Birkhäuser Classics, Birkhäuser/Springer Basel AG, Basel, 1992, [2012 reprint of the 1992 original] [MR1164866]. MR 3012668

[Par10] ———, *What is a Gaussian state?*, Commun. Stoch. Anal. **4** (2010), no. 2, 143–160. MR 2662722

[Par13] Kalyanapuram R. Parthasarathy, *The symmetry group of Gaussian states in $L^2(\mathbb{R}^n)$*, Prokhorov and contemporary probability theory, Springer Proc. Math. Stat., vol. 33, Springer, Heidelberg, 2013, pp. 349–369. MR 3070484

[PS15] K. R. Parthasarathy and Ritabrata Sengupta, *From particle counting to gaussian tomography*, Infinite Dimensional Analysis, Quantum Probability and Related Topics **18** (2015), no. 04, 1550023.

[SIS11] Krishna Kumar Sabapathy, J. Solomon Ivan, and R. Simon, *Robustness of non-gaussian entanglement against noisy amplifier and attenuator environments*, Phys. Rev. Lett. **107** (2011), 130501.

[Sud63] E. C. G. Sudarshan, *Equivalence of semiclassical and quantum mechanical descriptions of statistical light beams*, Phys. Rev. Lett. **10** (1963), 277–279.

[SW75] M. D. Srinivas and E. Wolf, *Some nonclassical features of phase-space representations of quantum mechanics*, Phys. Rev. D **11** (1975), 1477–1485.