Fredholm determinant representation of the homogeneous Painlevé II $\tau$-function

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Abstract

We formulate the generic $\tau$-function of the homogeneous Painlevé II equation as a Fredholm determinant of an integrable (Its-Izergin-Korepin-Slavnov) operator. The $\tau$-function depends on the isomonodromic time $t$ and two Stokes parameters. The vanishing locus of the $\tau$-function, called the Malgrange divisor is then determined by the zeros of the Fredholm determinant.

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1 Introduction

Painlevé equations describe isomonodromic deformations of certain meromorphic linear ordinary differential equations on \( \mathbb{CP}^1 \). In the theory of isomonodromic deformations, the Jimbo–Miwa–Ueno \( \tau \)-function is defined in terms of a closed 1–form, \( \omega_{JMU} \) \[^{32}\] by the formula

\[
\delta \log \tau_{JMU} := \omega_{JMU},
\]

where \( \delta \) denotes the total differential with respect to the 'deformation' parameters. In this paper, we study the isomonodromic \( \tau \)-function of the second order scalar nonlinear ordinary differential equation (ODE) in the complex domain of the form

\[
\frac{d^2}{dx^2} u(x) = xu(x) - 2u(x)^3, \quad x \in \mathbb{R},
\]

called the homogeneous Painlevé II equation. It arises as a consistency (zero-curvature) condition for the following set of linear ODEs for the \( 2 \times 2 \) complex valued matrix \( \Phi(\lambda, x) \)

\[
\frac{d\Phi}{d\lambda} = \left[-i \left(4\lambda^2 + x + 2u^2\right) \sigma_3 + 4\lambda u \sigma_1 - 2v \sigma_2\right] \Phi,
\]

\[
\frac{d\Phi}{dx} = [-i\lambda \sigma_3 + u \sigma_1] \Phi,
\]

where \( v(x) = u_x \), and the Pauli matrices

\[
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

The set of ODEs \[^{12}\] are called the Flaschka-Newell (FN) Lax pair \[^{21}\]. The ODE \[^{12}\] has an irregular singularity at \( \lambda = \infty \) of Poincaré rank 3 and thus exhibits Stokes phenomenon on six rays in the complex plane. The generalized monodromy data described by the Stokes matrices is encoded in the jump matrix \( G(\lambda, x) \in GL(2, \mathbb{C}), x \in \mathbb{R}, \lambda \in \Sigma \) on the contour \( \Sigma \in \mathbb{C} \). The inverse problem consists of reconstructing the function

\[
\Psi(\lambda, x) = \Phi(\lambda, x)e^{-\frac{1}{3}\lambda^3 - x\lambda}, \quad \lambda = \mathbb{C}\setminus\Sigma, \quad x \in \mathbb{R}
\]

from the generalized monodromy data. Such a solution \( \Psi \) is piece-wise defined on the Stokes sectors with jumps on the Stokes rays specified by Stokes matrices. This is achieved by solving the following Riemann–Hilbert problem (RHP)

\[
\Psi_+(\lambda, x) = \Psi_-(\lambda, x)G(\lambda, x), \quad \lambda \in \Sigma
\]

\[
\Psi(\lambda, x) = 1 + O\left(\frac{1}{\lambda}\right), \text{ as } \lambda \to \infty,
\]

where \( \Psi_\pm \) indicate the boundary value of \( \Psi \) from the left side and the right side respectively of the oriented contour \( \Sigma \), and \( G(\lambda, x) \) is piece-wise defined on each of the Stokes rays with appropriate constraints on the Stokes matrices (see section \[^{2}\] for a detailed description). With this data, the Malgrange form is defined as follows.

**Definition 1.** The Malgrange form associated with the RHP \[^{15}\] is defined as \[^{36}\]

\[
\omega_{\Sigma} = \int_{\Sigma} \frac{d\lambda}{2\pi i} \text{Tr} \left[ \Psi_\pm^{-1} \frac{\partial \Psi_\pm}{\partial \lambda} \delta GG^{-1} \right],
\]

where \( \delta \) denotes the total differential with respect to the isomonodromic parameter \( x \).
Since $\omega_\Sigma$, is a closed one form in the space of isomonodromic parameters, one can define (locally) the corresponding $\tau_\Sigma$ function as

$$\delta \log \tau_\Sigma = \omega_\Sigma.$$ 

The Malgrange form is a logarithmic form in the sense that it has only simple poles with integer residues. The locus in the parameter space where the RHP problem (1.5) becomes unsolvable is called the Malgrange divisor because (in the language of algebraic geometry) it can be described locally as the zero level set of a locally analytic function.

The general gist is that this local expression can be represented (in abstract terms) as a Fredholm determinant (see for example [39]). A concrete realization of this local function (a $\tau$-function) as a Fredholm determinant (possibly globally defined on an open dense set of parameters) is of practical interest since it potentially allows for numerical investigation of the Malgrange divisor.

This paper treads this line of approach by providing a concrete representation for the $\tau$-function of Painlevé II in terms of a Fredholm determinant expressed via an explicit (albeit complicated) kernel. According with this general framework, the zeros of the $\tau$-function indicate the points where the RHP (i.e, the inverse monodromy problem) is not solvable.

It is well known that certain special solutions of Painlevé equations have a Fredholm determinant representation [8, 9, 31, 42, 44, 45]. The recent works of Lisovyy, Cafasso, Gavrylenko [14, 26] provide a method to formulate the isomonodromic $\tau$-functions of general solutions of PIII, PV, PVI as Fredholm determinants. There are two key aspects to their construction. One is the property that the RHPs of these Painlevé equations can be reduced on to a RHP on the unit circle. The second feature is that the jump on the unit circle enables the formulation of the $\tau$-function as "Widom constant". An important feature of their construction is that the local parametrices of the RHP of the Painlevé equations are described by known special functions which in turn act as 'building blocks' of the $\tau$-function. For example, the local parametrices of the Painlevé VI RHP are given by hypergeometric functions and the $\tau$-function is expressed as a Fredholm determinant of a hypergeometric kernel.

A natural question then is whether the $\tau$-functions of Painlevé I, II, IV admit a Fredholm determinant representation. In a first step to answer this question in the case of Painlevé II, the present author recently showed that the RHP corresponding to the special 1-parameter (Ablowitz-Segur) family of solutions to the Painlevé II equation [41] can be recast as a RHP on the imaginary axis as opposed to the unit circle in [14], hinting at a similar structure for the general RHP of Painlevé II [19]. As a consequence, the corresponding $\tau$-function (which is known to be the determinant of the Airy kernel [42]), can be formulated as a Widom constant.

In the case of the RHP of Painlevé II, it is known that under particular transformations that facilitate asymptotic analysis pf the Painlevé II transcendent at $x \to -\infty$, the local parametrices are described by parabolic cylinder functions $D_\nu(z)$ which we recall in Section 2. We then reduce the RHP to a RHP with a discontinuity on the imaginary axis in Section 3. However, we will see that the jump on the imaginary axis does not admit a Birkhoff factorization and hence the technique to construct Fredholm determinants in [19] is not applicable to our case. Instead, we use a variation of the formalism in [3] namely, a lower, diagonal, upper triangular (LDU) factorization of the jump matrix to construct the $\tau$-function as a Fredholm determinant of an integrable (IIKS) [16, 31] operator with the parabolic cylinder functions acting as the 'building blocks'. In order to formulate our
result, let
\[ t = (-x)^{3/2} > 0. \] (1.7)

We encode the Stokes parameters \( s_1 \) and \( s_3 \) that define the Painlevé II RHP, (see (2.28), (2.32) below) by:
\[ \nu = -\frac{1}{2\pi i} \log(1 - s_1 s_3), \quad h = -\frac{\sqrt{2\pi}}{\Gamma(-\nu)s_3} e^{i\pi\nu}, \] (1.8)

with \( 1 - s_1 s_3 \neq 0, \ \arg(1 - s_1 s_3) \in (-\pi, \pi) \).

**Theorem 1.** The \( \tau \)-function of Painlevé II equation can be expressed in terms of a Fredholm determinant of an integrable operator \( \widetilde{K} \) as follows
\[ d_t \log \tau_{\text{P}II} = d_t \log \det \left[ I_{L^2(\mathbb{R})} - \widetilde{K} \right] - \left[ \frac{4i\nu}{3} + \frac{2\nu^2}{t} \right] + F(t, \nu, h), \] (1.9)

where \( F(t, \nu, h) \) is a regular function of the parameters \( t, h \) and \( \nu \) defined in (1.7), (1.8). The kernel of \( \widetilde{K} \) takes the form
\[ \widetilde{K}(z, w) = \frac{C(z, t)}{A(z, t)} \varphi_+^2(w) \int_{i\mathbb{R}+c} \frac{\varphi_-^2(\bar{w})}{2\pi i (z - \bar{w})(\bar{w} - w)} A(\bar{w}, t) B(\bar{w}, t) \] (1.10)

with the functions (see (3.14))
\[ A(z, t) = \zeta^\nu \xi^\nu e^{\frac{2i}{3}t} (e^{-\pi\nu} D_{-\nu}(i\zeta) D_{-\nu}(i\xi) + \nu^2 h^4 e^{2\pi\nu} D_{-\nu-1}(\zeta) D_{-\nu-1}(\xi)) \] (1.11)
\[ B(z, t) = \left( \frac{z + 1/2}{z - 1/2} \right)^{2\nu} \zeta^\nu \xi^{-\nu} \left( i h^2 e^{-i\pi\nu} D_{-\nu}(i\zeta) D_{-\nu-1}(i\xi) + \nu h^{-2} e^{2\pi\nu} D_{-\nu-1}(\zeta) D_{-\nu}(\xi) \right) \] (1.12)

where \( D_{\nu} \) is the parabolic cylinder function, \( \zeta = \zeta(z, t) \), \( \xi = \xi(z, t) \) are given by
\[ \zeta \equiv \zeta(z, t) = 2t^{1/2} \sqrt{-\frac{4i}{3} z^3 + i z - \frac{i}{3}}; \quad \xi = \zeta(-z, t) \]
with branch cuts \((-\infty, -1] \) and \([1, \infty) \) respectively, and the function \( \left( \frac{z - z_-}{z - z_+} \right)^\nu \) is defined on \( \mathbb{C} \setminus [z_-, z_+] \) and the branch is fixed by the following asymptotic condition for \( z \to \infty \)
\[ \left( \frac{z - z_-}{z - z_+} \right)^\nu \to 1. \] (1.13)

Moreover,
\[ C(z, t) = B(-z, t); \quad \varphi_+(w, t) = \frac{1}{\pi i} \int_{i\mathbb{R}+c} \frac{d\omega'}{2\pi i} \log A(w', t) \] (1.14)

From (3.14), one can see that the functions \( A, B, C \) are analytic on a strip on the imaginary axis and
\[ \lim_{z \to \pm \infty} A(z, t) = 1, \quad \lim_{z \to \pm \infty} B(z, t) = 0, \quad \lim_{z \to \pm \infty} C(z, t) = 0. \] (1.15)

Some comments are in order.
1. The \( \tau \)-function (1.9) is defined on \( \mathbb{C} \setminus \{0\} \) and is analytic in \( t \). Refer to Remark 2 for the details.

2. The important point of theorem 1 is that the zeros of \( \tau_{\text{PII}} \), called the Malgrange divisor are determined solely by the zero locus of the Fredholm determinant. The Malgrange divisor is in one to one correspondence with the poles of the Painlevé II transcendent.

3. The Malgrange divisor could be calculated numerically by computing the Fredholm determinant in (1.9) employing the algorithm developed in [7].

4. The limit from the general \( \tau \)-function of Painlevé II in (1.9) to the \( \tau \)-function of the Hastings-McLeod family of solutions (determinant of the Airy kernel) [12] is singular because \( s_3s_1 = 1 \) (\( \nu \) in (1.8) goes to infinity).

Theorem 1 is the first step in deriving the Fourier series representation of the Painlevé II tau-function obtained in [30]. A series representation of the \( \tau \)-function (1.9) can be obtained from the minor expansion of the Fredholm determinant on an appropriate basis. A similar computation for the case of the Ablowitz-Segur solutions of Painlevé II is worked out in [19]. Furthermore, we expect that the methods developed in this manuscript can be applied to some solutions of the Painlevé I and IV equations. Finally, the study of the inhomogeneous Painlevé II equation (1.16) requires a significant extension of the techniques developed for the homogeneous Painlevé II equation (1.1). Indeed the parameter \( \alpha \) which we set to zero in (1.1) induces a monodromy at the origin and the ideas developed in this manuscript need a non-trivial generalization.

The \( \tau \)-function of the general solution of (1.16) has been expressed as a ratio of Hankel determinants in [33], [34]. Finally, we remark that the \( \tau \)-functions of rational solutions of Painlevé equations have an interpretation not as Fredholm determinants, but as determinants of some special polynomials [15].

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2 Setup

We recall the RHP associated to the Flaschka-Newell Lax pair (1.2) from [22] (see also [10]). The matrix \( \Psi(\lambda, x) \in GL(2, \mathbb{C}) , x \in \mathbb{R} \) satisfies the following RHP on the contour in fig. 1.

**Riemann-Hilbert problem 1.**

- \( \Psi(\lambda, x) \) is piece-wise analytic for \( \lambda \in \mathbb{C} \setminus \cup_{k=1}^{6} \gamma_k \),

\[
\gamma_k = \left\{ \lambda \in \mathbb{C} : \arg \lambda = \frac{\pi}{6} + \frac{\pi}{3}(k - 1) \right\} , \quad k = 1, ..., 6.
\]

(2.1)
We define $\Psi_k := \Psi(\lambda, x)|_{\Omega_k}$ with the Stokes sector $\Omega_k$ defined by

$$\Omega_k = \left\{ \lambda \in \mathbb{C} : \frac{\pi}{6}(2k-3) < \arg \lambda < \frac{\pi}{6}(2k-1) \right\}, \ k = 1, ..., 6. \quad (2.2)$$

- For $\lambda \in \gamma_k$, the following boundary condition is satisfied.

$$\Psi_{k+1} = \Psi_k S_k, \quad \Psi_7 = \Psi_1, \quad (2.3)$$

where the matrices $S_k$ are alternatively lower or upper triangular

$$S_k = \begin{pmatrix} 1 & 0 \\ s_k e^{2i\theta(\lambda,x)} & 1 \end{pmatrix} \quad \text{for } k \equiv 1 \mod 2, \quad S_k = \begin{pmatrix} 1 & s_k e^{-2i\theta(\lambda,x)} \\ 0 & 1 \end{pmatrix} \quad \text{for } k \equiv 0 \mod 2, \quad (2.4)$$

and the exponent $\theta(\lambda, x) = \frac{4}{3} \lambda^3 + x\lambda$. The Stokes parameters $s_k$ are constants satisfying the constraint

$$s_{k+3} = -s_k, \ s_1 - s_2 + s_3 + s_1 s_2 s_3 = 0, \quad (2.5)$$

and the jump matrices satisfy the following identity

$$S_1 S_2 ... S_6 = 1. \quad (2.6)$$

- In the asymptotic limit of $\lambda$,

$$\lim_{\lambda \to \infty} \Psi(\lambda, x) = 1. \quad (2.7)$$

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{stokes_rays.png}
\caption{Stokes rays}
\end{figure}
The constraint on Stokes data (2.5) implies that the solution \( \Psi(z, t) \) depends only on two Stokes parameters. We will see in the next subsection that all the functions depend on \( s_1, s_3 \). In this paper, we are concerned with the generic 2-parameter solutions that correspond to the following constraints on the Stokes parameters

\[
s_1 s_3 \neq 1; \quad \arg(1 - s_1 s_3) \in (-\pi, \pi).
\] (2.8)

In order to modify the Riemann–Hilbert contour of Painlevé II, we perform the change of variables \((x \in \mathbb{R})\)

\[
\lambda = (-x)^{1/2} z, \quad t = (-x)^{3/2} > 0.
\] (2.9)

The characteristic exponent \( e^{i\theta(z)} \) in (2.4) is then replaced by

\[
e^{it\theta(z)}, \quad \theta(z) = \frac{4}{3} z^3 - z.
\] (2.10)

The stationary points are then \( z_{\pm} = \pm 1/2 \). The contour in fig. 1 can be deformed into fig. 2.

![Figure 2: Deforming the contour in fig. 1](image)

Noticing that the product of Stokes matrices \((S_3 S_4 S_5)^{-1}\) can be written as a product of lower triangular, diagonal and upper triangular matrices (LDU)

\[
(S_3 S_4 S_5)^{-1} = \begin{pmatrix} 1 & -s_1 s_3 & s_1 e^{-2it\theta(z)} \\ s_1 e^{2it\theta(z)} & 1 + s_1 s_2 \\ s_1 (1 - s_1 s_3)^{-1} e^{2it\theta(z)} & 1 \\ 0 & 0 & (1 - s_1 s_3)^{-1} \\ 0 & 0 & 1 \\ 1 & 0 & 1
\end{pmatrix} = S_L S_D S_U
\]

(2.11)

the contour in fig. 1 can be transformed into the contour \( \Sigma \) in fig. 3. One can easily check there is no monodromy around the points \( z = \pm 1/2 \).

\[\text{2} \]The Ablowitz-Segur family of solutions correspond to the case \( s_2 = 0 \).
In fig. 3, $S_D = (1 - s_1s_3)^{\sigma_3}$. Defining
\[ \nu = -\frac{1}{2\pi i} \log(1 - s_1s_3), \] one can verify that the function
\[ \Psi_D(z, t) = \left( \frac{z - z_-}{z - z_+} \right)^\nu \] satisfies the RHP on the segment $[z_-, z_+]$ where $z_\pm = \pm 1/2$ and the branch is fixed by the following asymptotic condition for $z \to \infty$
\[ \left( \frac{z - z_-}{z - z_+} \right)^\nu \to 1. \] On the contour $\Sigma$ in fig. 3, the function $\Psi(\lambda(z), x) \equiv \Psi(z, t)$ in (1.2) solves the following RHP

**Riemann-Hilbert problem 2.**

- $\Psi(z, t)$ is analytic on $z \in \mathbb{C}\setminus\Sigma, t \in \mathbb{R}$.
- For $z \in \Sigma$, on each of the Stokes rays
  \[ \Psi_-(z, t) \Psi_+(z, t) = G(z, t), \] where $G(z, t)$ is piece-wise defined on each of the rays of the contour $\Sigma$, $\Psi_\pm$ are the boundary values of $\Psi$ from the left side and the right side of the oriented contour $\Sigma$ respectively, and the constraint (2.6) holds.
- $\lim_{z \to \infty} \Psi(z, t) = 1$

In terms of the RHP (2.15), the $\tau$-function (1.6) is
\[ \delta \log \tau_{\text{P1I}} \equiv \delta \log \tau_{\Sigma}(t) := \int \frac{dz}{2\pi i} \text{Tr} \left[ \Psi^{-1} \Psi' \hat{G} G^{-1} \right], \] where $\hat{G}$ means the total derivative with respect to $t$ and $\Psi'$ means partial derivative with respect to $z$. We will convert the above expression to a Fredholm determinant in theorem 1.
2.1 Parametrices

To express the $\tau$-function (2.16) in terms of a Fredholm determinant we need to construct a “parametrix” solutions, namely “local solutions” of the RHP. These local solutions are patched together and the actual problem can be recast as the solution of a compact (trace-class) perturbation of the identity.

The effectiveness of the idea relies entirely upon the level of simplicity of these parametrices; the simpler (or rather, more explicit) these reference parametrices are, the more practical the approach is in studying the final problem.

Keeping this in mind, in this section we construct an explicit solution to a Riemann–Hilbert problem to be used as parametrix for the final one. To this end we recall from ([22], Ch.9 pg.318) the construction of the local parametrices of Painlevé II RHP in fig. 3 (the left and right parametrices around the points $z = \pm 1/2$ respectively), in terms of parabolic cylinder functions [22].

2.1.1 Model problem

Let $Z(\zeta)$ be a $2 \times 2$ matrix valued function that solves the following RHP.

Riemann-Hilbert problem 3.

- $Z(\zeta)$ is a piece-wise holomorphic function defined as follows in each sector shown in fig. 3

\[
Z(\zeta) = \begin{cases} 
Z_0(\zeta), & \arg \zeta \in (-\frac{\pi}{4}, 0) \\
Z_1(\zeta), & \arg \zeta \in (0, \frac{\pi}{2}) \\
Z_2(\zeta), & \arg \zeta \in (\frac{\pi}{2}, \pi) \\
Z_3(\zeta), & \arg \zeta \in (\pi, \frac{3\pi}{2}) \\
Z_4(\zeta), & \arg \zeta \in (\frac{3\pi}{2}, \frac{7\pi}{4}) .
\end{cases}
\]  

(2.17)

Under the transformation $\zeta \rightarrow -\zeta$ the following symmetry relation holds

\[
\sigma_3 Z_{k+2} (e^{i\pi \zeta}) \sigma_3 = Z_k(\zeta) e^{-i\pi (\nu + 1) \sigma_3} .
\]

(2.18)

Figure 4: Riemann–Hilbert contour of parabolic cylinder function.
In each sector, the following jump conditions are satisfied
\[
Z_{k+1}(\zeta) = Z_k(\zeta)H_k, \quad \arg \zeta = \frac{\pi}{2}k, \quad k = 0, 1, 2, 3, 4, \quad (2.19)
\]
and \(Z_5 = Z_0\). The jump matrices
\[
H_0 = \begin{pmatrix} 1 & 0 \\ h_0 & 1 \end{pmatrix}, \quad H_1 = \begin{pmatrix} 1 & h_1 \\ 0 & 1 \end{pmatrix}, \quad H_4 \equiv H_D = e^{2\pi i\nu\sigma_3};
\]
\[
H_{k+2} = e^{i\pi(\nu+\frac{1}{2})\sigma_3}H_k e^{-i\pi(\nu+\frac{1}{2})\sigma_3}, \quad \text{for } k = 0, 1.
\]

The Stokes parameters \(h_0\) and \(h_1\) are defined as follows
\[
h_0 = -i\frac{\sqrt{2\pi}}{\Gamma(\nu + 1)} e^{-\nu\sigma_3/2} \left( \begin{array}{cc} 1 & 1 \\ 1 & -1 \end{array} \right) \left( 1 + O(\zeta^{-2}) \right) e^{\left( \frac{\zeta^2}{\tau} \right) (-\nu + \frac{1}{2}) \log \zeta} \sigma_3.
\]
\[
(2.20)
\]
and the identity \(e^{2\pi i\nu\sigma_3}H_0H_1H_2H_3 = I\) implies the triviality of the monodromy at the origin.

As \(\zeta \to \infty\),
\[
Z(\zeta) = \zeta^{-\sigma_3/2} \frac{1}{\sqrt{2}} \left( \begin{array}{cc} 1 & 1 \\ 1 & -1 \end{array} \right) \left( 1 + O(\zeta^{-2}) \right) e^{\left( \frac{\zeta^2}{\tau} \right) (-\nu + \frac{1}{2}) \log \zeta} \sigma_3.
\]
\[
(2.22)
\]
In the zeroth sector, \(Z(\zeta)\) is expressed in terms of the Wronskian of the parabolic cylinder functions as
\[
Z_0(\zeta) = 2^{-\sigma_3/2} \left( \begin{array}{cc} D_{-\nu-1}(i\zeta) & D_\nu(\zeta) \\ \frac{d}{d\zeta}D_{-\nu}(i\zeta) & \frac{d}{d\zeta}D_\nu(\zeta) \end{array} \right) \left( e^{i\frac{\pi}{4}(\nu+1)} 0 \\ 0 1 \right).
\]
\[
(2.23)
\]
The parabolic cylinder functions \(D_\nu(z), D_{-\nu-1}(iz)\) are independent solutions to the differential equation
\[
\frac{d^2y(z)}{dz^2} + \left( \nu + \frac{1}{2} - \frac{1}{4}z^2 \right) y(z) = 0,
\]
\[
(2.24)
\]
and \(\lim_{z \to \infty} D_\alpha(z)e^{z^2/4}z^{-a} = 1\) for \(|\arg(z)| < \frac{\pi}{2}\). Refer to Ch. 19, [1] for details.

### 2.1.2 Local parametrices

Under the map
\[
\zeta(z) = 2\sqrt{-\frac{4it}{3}z^3 + itz - \frac{it}{3}} = \frac{4}{\sqrt{3}} e^{-3i\pi/4\sqrt{t}} \left( z - \frac{1}{2} \right) \sqrt{z + 1},
\]
\[
(2.25)
\]
where \(\zeta(z)\) has a branch cut on \((-\infty, -1]\), we define the right parametrix around \(z_+ = 1/2\) as
\[
\psi_t(z, t) = \left( \zeta(z) \frac{z - z_+}{z - z_+} \right)^{-\sigma_3/2} \left( -\frac{h_1}{s_3} \right)^{-\sigma_3/2} \left( \frac{\zeta(z)}{1} \right) Z(\zeta(z)) \left( -\frac{h_1}{s_3} \right)^{\sigma_3/2} e^{it\theta(z)}\sigma_3,
\]
\[
(2.26)
\]
and the left parametrix around $z = -1/2$ is determined through the symmetry relation
\[ \psi_l(z, t) = \sigma_2 \psi_r(-z, t) \sigma_2. \]  
(2.27)

In (2.26), the parameter $\nu$ and $h_1$ are determined by the Stokes parameters $s_1, s_3$. Recall from (2.12) and (2.21),
\[ \nu = -\frac{1}{2\pi i} \log(1 - s_1 s_3); \quad h_1 = \frac{\sqrt{2\pi}}{\Gamma(-\nu)} e^{i\pi\nu}. \]  
(2.28)

\[\begin{array}{c}
\text{Figure 5: Mapping the } \zeta \text{-plane to the right-half of } z \text{-plane}
\end{array}\]

In each sector,\[
\psi_r^{(k)}(z, t) = \left( \frac{z - z}{z + z^\ast} \right)^{\nu\sigma_3} \left( -\frac{h_1}{s_3} \right)^{-\sigma_3/2} e^{i\sigma_3/2} \left( \begin{array}{cc}
\zeta(z) - e^{it\theta(z)} \sigma_3 \\
1
\end{array} \right) Z_k(\zeta(z)) \left( -\frac{h_1}{s_3} \right)^{-\sigma_3/2} e^{i\sigma_3/2}.
\]  
(2.29)

The jumps on Stokes rays in the right and left half planes are denoted by
\[ G_r := G(z, t)|_{\Re(z)>0}; \quad G_l := G(z, t)|_{\Re(z)<0}. \]  
(2.30)

As a consequence of (2.27),
\[ G_l(z, t) = \sigma_2 G_r(-z, t) \sigma_2 \]  
(2.31)

We now establish the relation between the Stokes matrices of the parabolic cylinder functions $H_i$ in (2.20) and $G_r \equiv G_r^{(i)}$ in (2.30). Introducing the notation
\[ h = \left( -\frac{h_1}{s_3} \right)^{1/2}, \]  
(2.32)

in a sector $k$ on the right half-plane in fig. 3, $\psi_r$ satisfies the following jump condition
\[ \psi_r^{k+1}(z, t) = \psi_r^k(z, t) e^{-it\theta(z)\sigma_3} \left( -\frac{h_1}{s_3} \right)^{-\sigma_3/2} Z_k^{-1} Z_{k+1} \left( -\frac{h_1}{s_3} \right)^{\sigma_3/2} e^{it\theta(z)\sigma_3} \]  
(2.33)
Note that $Z_5 = Z_0$ implies that $\psi_5^r(z, t) = \psi_0^r(z, t)$. Therefore, in terms of $H_k$, $G_r$ is

$$G_r^{(k)}(z, t) = e^{-it\sigma_3}h^{-\sigma_3}H_k\sigma_3 e^{it\sigma_3}. \quad (2.34)$$

We define the variable

$$\xi(z, t) := \zeta(-z, t) \quad (2.35)$$

with a branch cut on $[+1, \infty)$, that maps the $\xi$-plane to the left half-plane of fig. 3 and a similar computation follows for the left parametrix due to the symmetry relation (2.27). We denote the jump condition in each sector on the respective half-planes in fig. 3 by

$$\psi_{r,l}^{+,}(z, t) = \psi_{r,l}^{-,}(z, t)G_{r,l}(z, t). \quad (2.36)$$

**Remark 1.** It is easy to check that the map in fig. 2.1.2 holds not just locally but asymptotically in the right half plane, and similarly, the transformation $\xi(z, t)$ is valid in the left half plane. Therefore the functions $\psi_r(z, t)$ and $\psi_l(z, t)$ can indeed be defined on the right and left half planes respectively.

**Remark 2.** The transformation (2.25) is not valid at the point $t = 0$. This implies that $\tau_{P_{II}}$ in (1.9) is valid for $t \in \mathbb{C}\{0\}$.

### 3 Reduction to a RHP along the imaginary axis

Define a matrix function $\Theta(z, t)$ as a ratio of the global solution $\Psi$ on $\Sigma$ in (2.15) and the local parametrices $\psi_r$ in (2.26), $\psi_l$ in (2.27).

$$\Theta(z, t) := \begin{cases} \Psi(z, t)\psi_r^{-1}(z, t); & \Re(z) > 0 \\ \Psi(z, t)\psi_l^{-1}(z, t); & \Re(z) < 0. \end{cases} \quad (3.1)$$

Note that the local parametrices cancel the jump of the global parametrix on $\Sigma$, ensuring that the function $\Theta(z, t)$ has a jump only on the imaginary axis, solving the following RHP.

**Riemann-Hilbert problem 4.**

- $\Theta(z, t)$ is analytic on $z \in \mathbb{C}\{i\mathbb{R}\}$
- For $z \in i\mathbb{R}$,
  $$\Theta_+(z, t) = \Theta_-(z, t)J(z, t) \quad (3.2)$$
  where $J(z, t) = \psi_{r}^{(0)}(z, t)\left[\psi_{l}^{(3)}(z, t)\right]^{-1}$.
- As $z \to \infty$, $\Theta(z, t) = 1 + O\left(z^{-1}\right)$. 

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Remark 3. The solution of the RHP $\text{(3.1)}$ defines, via $\text{(3.1)}$ a solution of the RHP $\text{(4)}$. Vice versa any solution of the RHP $\text{(4)}$ provides a solution to the RHP $\text{(3.1)}$ by means of the inverse of the transformation $\text{(3.1)}$. Thus we regard these two problems as equivalent in the sense that the solvability of one of them is necessary and sufficient condition for the solvability of the other.

For later use we compute the expression of the jump matrix $J$ in $\text{(3.2)}$.

**Lemma 1.** The jump on the imaginary axis equals

$$J(z, t) = \Theta_-(z, t)^{-1}\Theta_+(z, t) = \psi_r^{(0)}(z, t) \left[\psi_l^{(4)}(z, t)\right]^{-1} = \begin{pmatrix} A(z, t) & B(z, t) \\ C(z, t) & D(z, t) \end{pmatrix} \tag{3.3}$$

where

$$A(z, t) = \frac{1}{\pi} \zeta^{\nu} \xi^{-\nu} e^{\frac{2i}{\nu}t} \left( -e^{-\pi i\nu} h^4 D_{-\nu}(i\zeta) D_{-\nu}(i\xi) - \nu^2 e^{2\pi i\nu} D_{\nu-1}(\overline{\zeta}) D_{\nu-1}(\overline{\xi}) \right)$$

$$B(z, t) = -\frac{1}{\pi^2} \left( \frac{z-z_-}{z-z_+} \right)^{2\nu} \zeta^{\nu} \xi^{-\nu} \left( -ie^{-\pi i\nu} h^4 D_{-\nu}(i\zeta) D_{-\nu-1}(i\xi) - \nu e^{2\pi i\nu} D_{\nu-1}(\overline{\zeta}) D_{\nu}(\overline{\xi}) \right)$$

$$C(z, t) = B(-z, t) \quad \text{det} J = 1. \tag{3.4}$$

The variables $\zeta \equiv \zeta(z, t)$, $\xi \equiv \xi(z, t)$ are defined in $\text{(2.25)}$, $\text{(2.35)}$; and $h$ is defined in terms of Stokes parameters in $\text{(2.32)}$.

**Proof.** Since $\Psi(z, t)$ has no jump on $i\mathbb{R}$, $J(z, t)$ can be determined solely in terms of $\psi_r^{(0)}(z, t)$ and $\psi_l^{(4)}(z, t)$. One can check the no monodromy condition at the origin,

$$\psi_r^{(0)}(z, t) \left[\psi_l^{(4)}(z, t)\right]^{-1} = \psi_r^{(4)}(z, t) \left[\psi_l^{(0)}(z, t)\right]^{-1}. \tag{3.5}$$

To ease the notation, we define

$$m(z) := \frac{z-z_-}{z-z_+}, \tag{3.6}$$

Figure 6: Reducing the Painlevé II RHP on to the imaginary axis.
and observe that the following identities hold
\[
\theta(z) = \frac{4}{3}z^3 - z = i\frac{\xi^2}{\theta} - \frac{1}{3} = -i\frac{\xi^2}{\theta} + \frac{1}{3},
\]
(3.7)
\[
(\xi^2 + \zeta^2) = -\frac{8it}{3}.
\]

The function \(\psi_r^0(z, t)\) is computed by substituting the zeroth sector solution of the parabolic cylinder function (2.23) in (2.29),
\[
\psi_r^0(z) = \left(\frac{z - z_0}{z - z_+} \right)^{\frac{\nu\sigma_3}{8\pi}} \left( -\frac{h_1}{s_3} \right)^{-\sigma_3/2} e^{\frac{\nu}{2}\sigma_3} 2^{-\sigma_3/2} \left( \begin{array}{c} \zeta(z) \\ 1 \end{array} \right) Z_0(\zeta(z)) \left( \frac{h_1}{s_3} \right)^{\sigma_3/2} \times e^{it\theta(z)\sigma_3}
\]
(3.8)

The last line is obtained by using (3.7), and the following identity of parabolic cylinder functions
\[
\frac{z}{2}D_\nu(z) + D_\nu'(z) = \nu D_{\nu-1}(z).
\]
(3.9)

The left parametrix \(\psi_r^4(z, t)\) can be obtained in a similar fashion, first by substituting \(Z_4 = Z_0 e^{-2\pi i\nu\sigma_3}\) from (2.19) in (2.29) to obtain \(\psi_r^4\) and using the relation (2.27) to obtain \(\psi_r^4\) as follows
\[
\psi_r^4(z, t) = \sigma_2 \psi_r^4(-z, t)\sigma_2 = \sigma_2 \psi_r^0(-z, t) H_D^{-1} \sigma_2
\]
(3.10)

To obtain the last line, we substitute the expression for \(Z_0\) (2.23) and simplify the resulting expression using (3.7). Furthermore,
\[
\det \left[ \psi_r^0(z, t) \right] = 1; \quad \det \left[ \psi_r^4(z, t) \right] = 1
\]
(3.11)
due to the following identity for the Wronskian determinant of parabolic cylinder functions
\[
\mathcal{W}[D_{\nu-1}(i\xi), D_\nu(\zeta)] = ie^{-i\pi\nu/2}.
\]
(3.12)
The jump \(J(z, t)\) is then obtained by a straightforward substitution of (3.8) and (3.10) in (3.3), and using (3.7).
\[
J(z, t) = \psi_r^0(z, t) \left[ \psi_r^4(z, t) \right]^{-1} = \begin{pmatrix} A(z, t) & B(z, t) \\ C(z, t) & D(z, t) \end{pmatrix}
\]
(3.13)
Proof. (3.41) see F where

Recalling the Malgrange form of Painlevé II on

in fact do) differ, but only by a non-vanishing term which we now set up to compute.

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Malgrange form (1.6) is

analytic in a strip on the imaginary axis and

Proposition 1. It is obvious that

(2.25),

h

C

(2.25), h = \left( \frac{h_1}{h_3} \right)^{1/2} : (2.32) where h_1, \nu are determined by the Stokes parameters s_1, s_3 as in (2.21), (2.28) respectively. One can further verify that the functions \( A, B, C, D \) are analytic in a strip on the imaginary axis and

\[
\lim_{z \to \pm i \infty} J(z, t) = 1. \tag{3.15}
\]

The two equivalent RHPs 2, 4 give rise to two corresponding Malgrange forms. Although the two problems are equivalent, the two corresponding tau function may (and in fact do) differ, but only by a non-vanishing term which we now set up to compute. Recalling the Malgrange form of Painlevé II on \( \Sigma \) in (2.16):

\[
d_t \log \tau_\infty = \int_{\Sigma} \frac{dz}{\Sigma 2\pi i} \text{Tr} \left[ \Psi_-^{-1} \Psi'_- \tilde{G} \tilde{G}^{-1} \right]. \tag{3.16}
\]

Similarly on \( i\mathbb{R} \), the RHP 4 satisfies the jump condition \( \Theta_+ = \Theta_- J \) and the corresponding Malgrange form (1.6) is

\[
d_t \log \tau_{i\mathbb{R}} = \int_{i\mathbb{R}} \frac{dz}{\Sigma 2\pi i} \text{Tr} \left[ \Theta_-^{-1} \Theta'_- \tilde{J} \tilde{J}^{-1} \right]. \tag{3.17}
\]

**Proposition 1.** The Malgrange forms corresponding to the RHPs on the contours \( \Sigma \) and \( i\mathbb{R} \) are related as

\[
d_t \log \tau_\Sigma = d_t \log \tau_{i\mathbb{R}} - \int_{i\mathbb{R}} \frac{dz}{\Sigma 2\pi i} F(z; t; \nu, h) = \left[ \frac{4iv}{3} + \frac{\nu^2}{t} \right] - 2g(\nu, h; t), \tag{3.18}
\]

where \( F(z; t; \nu, h) \) is a regular function explicit in terms of parabolic cylinder functions, see (3.41).

**Proof.** We begin by computing the following trace on \( i\mathbb{R} \)

\[
\text{Tr} \left\{ \Theta_-^{-1} \Theta'_- \tilde{J} \tilde{J}^{-1} \right\}. \tag{3.19}
\]

\footnote{In the proof, we drop the \( z, t \) dependence for the ease of writing. All the functions here on depend on \( z, t \) unless specified. One should pay attention to the total derivative w.r.t \( t \) as it involves a derivative w.r.t \( z \) due to the transformation (2.9). We thank Nikolai Iorgov and Yuri Zhuravlev for bringing this subtlety with the \( t \) derivative to our attention.}
Computing \((3.19)\) term by term using \((3.1)\): \(\Theta_- = \Psi \psi_{-1}^{-1}\),

\[
\Theta_-^{-1} \Theta_-^\prime = (\Psi \psi_{-1}^{-1})^{-1}(\Psi \psi_{-1}^{-1})' = \psi_r \Psi^{-1} (\Psi' \psi_{-1}^{-1} - \Psi \psi_{-1}^{-1} \psi_r' \psi_{-1}^{-1}) = \psi_r (\Psi^{-1} \Psi' - \psi_{-1}^{-1} \psi_r') \psi_{-1}^{-1}.
\]  

(3.20)

Since \((3.3)\): \(J = \psi_r \psi_{-1}^{-1}\),

\[
J J^{-1} = \frac{\partial}{\partial t} (\psi_r \psi_{-1}^{-1}) (\psi_r \psi_{-1}^{-1})^{-1} = \left( \dot{\psi}_r \psi_{-1}^{-1} - \psi_r \dot{\psi}_r' \psi_{-1}^{-1} \right) \psi_r \psi_{-1}^{-1} = -\psi_r \Delta \left( \psi_{-1}^{-1} \dot{\psi}_r \right) \psi_{-1}^{-1},
\]

(3.21)

where

\[
\Delta(\psi_{-1}^{-1} \dot{\psi}_r) = \psi_{-1}^{-1} \dot{\psi}_l - \psi_{-1}^{-1} \dot{\psi}_r.
\]

Substituting \((3.20)\) and \((3.21)\) in \((3.19)\) and using cyclicity of trace,

\[
\text{Tr} \left\{ \Theta_-^{-1} \Theta_-^\prime J J^{-1} \right\} = \text{Tr} \left\{ (-\Psi^{-1} \Psi' + \psi_{-1}^{-1} \psi_r') \Delta \left( \psi_{-1}^{-1} \dot{\psi}_r \right) \right\}.
\]

(3.22)

Since the term \(\psi_{-1}^{-1} \psi_r' \Delta \left( \psi_{-1}^{-1} \dot{\psi}_r \right)\) is integrated on \(iR\) in \((3.17)\),

\[
\int_{iR} \frac{dz}{2\pi i} \text{Tr} \left[ \psi_{-1}^{-1} \psi_r' \Delta \left( \psi_{-1}^{-1} \dot{\psi}_r \right) \right] = \int_{iR} \frac{dz}{2\pi i} \text{Tr} \left[ (\psi_r(0))^{-1} (\psi_l(0))' \left( (\psi_l(4))^{-1} \psi_l(4) - (\psi_r(0))^{-1} \psi_r(0) \right) \right]
\]

(3.23)

with \(\psi_l(0)\) defined in \((3.8)\), \(\psi_l(4)\) in \((3.10)\). We collect the explicit terms and compute them in the end. Since \(\Psi\) has no jump on \(iR\), using Cauchy theorem

\[
- \int_{iR} \frac{dz}{2\pi i} \text{Tr} \left\{ \Psi^{-1} \Psi' \Delta \left( \psi_{-1}^{-1} \dot{\psi}_r \right) \right\} = - \int_{iR} \frac{dz}{2\pi i} \text{Tr} \Delta \left( \Psi^{-1} \Psi' \left( \psi_{-1}^{-1} \dot{\psi}_r \right) \right)
\]

\[
= \int_{\Sigma_L} \frac{dz}{2\pi i} \text{Tr} \Delta \left( \Psi^{-1} \Psi' \left( \psi_{-1}^{-1} \dot{\psi}_r \right) \right) + \int_{\Sigma_R} \frac{dz}{2\pi i} \text{Tr} \Delta \left( \Psi^{-1} \Psi' \left( \psi_{-1}^{-1} \dot{\psi}_r \right) \right),
\]

(3.24)

where \(\Sigma_{L,R}\) are \(\Sigma\) restricted to the left and right half-planes respectively. Since \(\Psi\) has jumps on \(\Sigma_L\),

\[
\int_{\Sigma_L} \frac{dz}{2\pi i} \text{Tr} \Delta \left( \Psi^{-1} \Psi' \left( \psi_{-1}^{-1} \dot{\psi}_r \right) \right) = \int_{\Sigma_L} \frac{dz}{2\pi i} \text{Tr} \left\{ \Psi^{-1} \Psi' \left( \psi_{-1}^{-1} \psi_l \right) - \Psi^{-1} \Psi' \left( \psi_{-1}^{-1} \psi_l \right) \right\}.
\]

(3.25)

Similarly on \(\Sigma_R\)

\[
\int_{\Sigma_R} \frac{dz}{2\pi i} \text{Tr} \Delta \left( \Psi^{-1} \Psi' \left( \psi_{-1}^{-1} \dot{\psi}_r \right) \right) = \int_{\Sigma_R} \frac{dz}{2\pi i} \text{Tr} \left\{ \Psi^{-1} \Psi' \left( \psi_{-1}^{-1} \psi_r \right) - \Psi^{-1} \Psi' \left( \psi_{-1}^{-1} \psi_r \right) \right\}.
\]

(3.26)

In order to estimate \((3.24)\), we begin by computing the integrand on \(\Sigma_L\). Computing \((3.25)\) term by term using \((2.15)\): \(\Psi_+ = \Psi_- G_l\),

\[
\Psi^{-1} \Psi_+ = (\Psi_- G_l)^{-1} (\Psi_- G_l)' = G_l^{-1} \Psi^{-1} (\Psi_+ G_l + \Psi_- G_l') = G_l^{-1} (\Psi_- \Psi_+ + G_l G_l^{-1} G_l) G_l.
\]

(3.27)
Since (2.36): $\psi_{l+} = \psi_{l-} G_l$,
\[
\psi_{l+}^{-1} \psi_{l+} = G_l^{-1} \psi_{l-}^{-1} (\psi_{l-} G_l + \psi_{l-} \dot{G}_l) = G_l^{-1} \left( \psi_{l-}^{-1} \psi_{l-} + \dot{G}_l G_l^{-1} \right) G_l.
\] (3.28)

The product of (3.27) and (3.28) under the trace reads
\[
\text{Tr} \left\{ \Psi_{l+}^{-1} \Psi'_{l+} \right\} = \text{Tr} \left[ (\Psi_{l-}^{-1} G_l^{-1} + G_l') (\psi_{l-}^{-1} \psi_{l-} + \dot{G}_l G_l^{-1}) \right].
\] (3.29)

Substituting (3.29) in (3.25),
\[
\text{Tr} \Delta \left( \Psi^{-1} \Psi' \right) = \text{Tr} \left[ \Psi_{l-}^{-1} G_l^{-1} G_l' + G_l' G_l^{-1} (\psi_{l-}^{-1} \psi_{l-} + \dot{G}_l G_l^{-1}) \right].
\] (3.30)

A parallel computation for $\Sigma_R$ gives
\[
\text{Tr} \Delta \left( \Psi^{-1} \Psi' \right) = \text{Tr} \left[ \Psi_{l-}^{-1} G_r G_r^{-1} + G_r' G_r^{-1} (\psi_{l-}^{-1} \psi_{l-} + \dot{G}_r G_r^{-1}) \right].
\] (3.31)

Summing the terms (3.30) and (3.31), we obtain that
\[
d_t \log \tau_R = \int_{iR} \frac{dz}{2\pi i} \text{Tr} \left[ \Theta^{-1} \Theta' J J^{-1} \right] + \int_{i\Sigma} \frac{dz}{2\pi i} \text{Tr} \left[ \Psi_{l-}^{-1} G_{l-} G_{l-}^{-1} \right] + \int_{i\Sigma} \frac{dz}{2\pi i} \text{Tr} \left[ \Psi_{l-}^{-1} G_{l-} G_{l-}^{-1} \right] + \int_{i\Sigma} \frac{dz}{2\pi i} \text{Tr} \left[ \Psi_{l-}^{-1} G_{l-} G_{l-}^{-1} \right]
\]
\[
+ \int_{i\Sigma_L} \frac{dz}{2\pi i} \text{Tr} \left[ G_{l-}^{-1} (\psi_{l-}^{-1} \psi_{l-} + G_l' G_l^{-1}) \right] + \int_{i\Sigma_R} \frac{dz}{2\pi i} \text{Tr} \left[ G_{l-}^{-1} (\psi_{l-}^{-1} \psi_{l-} + G_l' G_l^{-1}) \right]
\]
\[
= d_t \log \tau_R + \int_{iR} \frac{dz}{2\pi i} \text{Tr} \left[ \psi_{l-}^{-1} G_{l-} G_{l-}^{-1} \right] + \int_{i\Sigma} \frac{dz}{2\pi i} \text{Tr} \left[ \psi_{l-}^{-1} G_{l-} G_{l-}^{-1} \right] + \int_{i\Sigma} \frac{dz}{2\pi i} \text{Tr} \left[ \psi_{l-}^{-1} G_{l-} G_{l-}^{-1} \right]
\]
\[
+ \int_{i\Sigma_L} \frac{dz}{2\pi i} \text{Tr} \left[ G_{l-}^{-1} (\psi_{l-}^{-1} \psi_{l-} + G_l' G_l^{-1}) \right] + \int_{i\Sigma_R} \frac{dz}{2\pi i} \text{Tr} \left[ G_{l-}^{-1} (\psi_{l-}^{-1} \psi_{l-} + G_l' G_l^{-1}) \right] + \int_{i\Sigma} \frac{dz}{2\pi i} \text{Tr} \left[ \psi_{l-}^{-1} G_{l-} G_{l-}^{-1} \right].
\] (3.32)

Notice that $\psi_{l,R}$ and $G_{l,R}$ are completely determined in terms of parabolic cylinder functions. The final expression is
\[
d_t \log \tau_R = d_t \log \tau_R - \int_{i\Sigma_L} \frac{dz}{2\pi i} \text{Tr} \left[ \psi_{l-}^{-1} G_{l-} G_{l-}^{-1} \right] - \int_{i\Sigma_R} \frac{dz}{2\pi i} \text{Tr} \left[ G_{l-}^{-1} (\psi_{l-}^{-1} \psi_{l-} + G_l' G_l^{-1}) \right] - \int_{i\Sigma} \frac{dz}{2\pi i} \text{Tr} \left[ G_{l-}^{-1} (\psi_{l-}^{-1} \psi_{l-} + G_l' G_l^{-1}) \right].
\] (3.33)

The following can be said about the explicit terms in (3.33),

- We can completely determine the integrals on $\Sigma_{R,L}$.
- The symmetry relations (2.27), (2.31) imply that
\[
\int_{i\Sigma_L} \frac{dz}{2\pi i} \text{Tr} \left[ G_{l-}^{-1} (\psi_{l-}^{-1} \psi_{l-} + G_l' G_l^{-1}) \right] = \int_{i\Sigma_R} \frac{dz}{2\pi i} \text{Tr} \left[ G_{l-}^{-1} (\psi_{l-}^{-1} \psi_{l-} + G_l' G_l^{-1}) \right].
\] (3.34)
Furthermore, (2.21) implies that the jump $G_r^{(k)}$ in (2.34) is lower triangular for $k = 0, 2$; upper triangular for $k = 1, 3$; diagonal and constant for $k = 4$. Therefore,

$$\text{Tr} \left[ G_r' G_r^{-1} \hat{G}_r G_r^{-1} \right] = \text{Tr} \left[ G_l' G_l^{-1} \hat{G}_l G_l^{-1} \right] = 0.$$  \hspace{1cm} (3.35)

We now proceed to compute the following term in (3.33)

$$\int_{\Sigma_R} \frac{dz}{2\pi i} \text{Tr} \left[ G_r' G_r^{-1} \psi_{r-}^{-1} \psi_{r-} \right] = \sum_{k=1}^{5} \int_{\Sigma_k} \frac{dz}{2\pi i} \text{Tr} \left[ (G_r^{(k)})'(G_r^{(k)})^{-1}(\psi_{r-}^{(k-1)})^{-1}\psi_{r-}^{(k-1)} \right].$$ \hspace{1cm} (3.36)

In each sector, $\psi_r$ and $G_r$ can be computed starting from $\psi_r^{(0)}$ in (3.8), and the jumps in (2.34). A lengthy but straightforward computation yields

$$\int_{\Sigma_R} \frac{dz}{2\pi i} \text{Tr} \left[ G_r' G_r^{-1} \left( \psi_{r-}^{-1} \psi_{r-} + \hat{G}_r G_r^{-1} \right) \right] = \left[ \frac{2i}{3} + \frac{\nu^2}{2t} \right] + g(\nu, h; t),$$ \hspace{1cm} (3.37)

where $g(.)$ is a function depending on $t$ with the following behaviour for $t \to -\infty$

$$g(\nu, h; t) = -\frac{2i}{81t^2} \left( 3\nu^4 + 33\nu^2 - 1 \right) + O((-t)^{-3}).$$

The relation (3.34) then implies,

$$\int_{\Sigma_L} \frac{dz}{2\pi i} \text{Tr} \left[ G_l' G_l^{-1} \left( \psi_{l-}^{-1} \psi_{l-} + \hat{G}_l G_l^{-1} \right) \right] + \int_{\Sigma_R} \frac{dz}{2\pi i} \text{Tr} \left[ G_r' G_r^{-1} \left( \psi_{r-}^{-1} \psi_{r-} + \hat{G}_r G_r^{-1} \right) \right] = \left[ \frac{4i\nu}{3} + \frac{\nu^2}{t} \right] + 2g(\nu, h; t).$$ \hspace{1cm} (3.38)

• The remaining explicit term in (3.33)

$$\int_{\Sigma_R} \frac{dz}{2\pi i} \text{Tr} \left[ \psi_{r-}^{-1} \psi_r^{-1} \Delta \left( \psi_{r-}^{-1} \psi_r^{-1} \right) \right] = \int_{\Sigma_R} \frac{dz}{2\pi i} \text{Tr} \left[ \left( \psi_r^{(0)} \right)^{-1} \left( \psi_r^{(4)} \right)^{-1} \left( \psi_{r-}^{(4)} \right)^{-1} \left( \psi_r^{(0)} \right)^{-1} \psi_r^{-1} \psi_r^{-1} \Delta \left( \psi_{r-}^{-1} \psi_r^{-1} \right) \right].$$ \hspace{1cm} (3.39)

The functions $\psi_r^{(0)}$ and $\psi_r^{(4)}$ depend on $z$ through $\zeta(z, t)$ in (2.25) and $\xi(z, t)$ in (2.35) respectively. In order to solve the integral, we need to compute integrals of the form

$$\int \frac{dz}{2\pi i} D(\zeta) D_{\rho}(\zeta) D_{-\rho}(i\xi) D_{-\sigma}(i\xi),$$ \hspace{1cm} (3.40)

which is not exactly solvable. The expression (3.39) is however, explicit. Defining a function $F$ as

$$F(z, t; \nu, h) := \text{Tr} \left[ \psi_{r-}^{-1} \psi_r^{-1} \Delta \left( \psi_{r-}^{-1} \psi_r^{-1} \right) \right].$$ \hspace{1cm} (3.41)

The final expression in (3.33) reads

$$d_t \log \tau_{\Sigma} = d_t \log \tau_{\Sigma_r} - \int_{\partial R} \frac{dz}{2\pi i} F(z, t; \nu, h) = \left[ \frac{4i\nu}{3} + \frac{\nu^2}{t} \right] - 2g(\nu, h; t).$$ \hspace{1cm} (3.42)
4 Integrable kernel and Fredholm determinant

Up to this point, we started with the RHP of Painlevé II in fig. 3, used the description of the local parametrices in terms of parabolic cylinder functions in the subsection 2.1 to define a RHP on \( i\mathbb{R} \) (3.2) in section 3. We then showed that the corresponding Malgrange forms are related in proposition 1. Our goal now reduces to expressing \( \tau_{iR} \) as a Fredholm determinant.

It is known that a jump \( J(z, t) \in SL(2, \mathbb{C}) \) on non-intersecting contours can be expressed in terms of lower and upper triangular matrices called the LULU decomposition and the corresponding \( \tau \)-function can then be written as a Fredholm determinant of an integrable operator [3]. Here, we modify the construction in [3] by using LDU decomposition instead, which then gives us a simpler kernel. In this section, we

1. transform RHP on \( i\mathbb{R} \) on to a set of two parallel lines with lower and upper triangular jumps using the LDU decomposition,

2. formulate the \( \tau \)-function on the set of parallel lines, call it \( \tau_{LU} \) as a Fredholm determinant of an integrable operator, and

3. prove that the Malgrange forms on the contours LU and \( i\mathbb{R} \) coincide.

4.1 LU decomposition

The RHP on \( i\mathbb{R} \) can be transformed on to a set of two parallel lines with jumps that are upper and lower triangular respectively. We decompose the jump \( J(z, t) \) (3.3) into lower, diagonal and upper triangular matrices, called the LDU decomposition [17], which recasts the RHP on \( i\mathbb{R} \) on to a set of three parallel lines.

\[
J(z, t) = \begin{pmatrix} \mathcal{A}(z, t) & \mathcal{B}(z, t) \\ \mathcal{C}(z, t) & \mathcal{D}(z, t) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{\mathcal{C}(z, t)}{\mathcal{A}(z, t)} & 1 \end{pmatrix} \begin{pmatrix} \mathcal{A}(z, t) & 0 \\ 0 & \frac{1}{\mathcal{A}(z, t)} \end{pmatrix} \begin{pmatrix} 1 & \mathcal{B}(z, t) \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{\mathcal{C}(z, t)}{\mathcal{A}(z, t)} & 1 \end{pmatrix} \begin{pmatrix} \mathcal{A}(z, t) & \mathcal{B}(z, t) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\]

\[= F_1(z, t)F_2(z, t)F_3(z, t). \tag{4.1} \]

\[
\begin{pmatrix} 1 & \frac{\mathcal{C}}{\mathcal{A}} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathcal{A}^\sigma_3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{\mathcal{C}}{\mathcal{A}} & 1 \end{pmatrix}
\]

\[Y_4 = \mathcal{L} \quad Y_3 \quad Y_2 \quad Y_1 = \mathcal{R} \]

\[l_3 \quad l_2 \quad l_1 \]

Figure 7: LDU decomposition

The function \( Y(z, t) \) then solves the following RHP.

Riemann-Hilbert problem 5.

\[\text{The author thanks A.Its for suggesting LDU decomposition.}\]
• $Y(z, t)$ is a piecewise analytic in $\mathbb{C} \setminus (\cup_{i=1}^{3} l_i)$.

• On each line $l_i$ in fig. [7] the following jump condition holds

$$Y_{i+1}(z, t) = Y_i(z, t)F_i(z, t),$$

(4.2)

with the identification

$$Y_4(z, t) = \Theta_+(z, t) ; \quad Y_1(z, t) = \Theta_-(z, t).$$

(4.3)

$\Theta_\pm$ are defined in (3.1).

• $\lim_{z \to \infty} Y(z, t) = 1$.

The RHP on $l_2$ with the diagonal jump $A^{\sigma_3}$ locally with $A$ defined in (3.4). The ratio

$$e^Y(z, t) := \frac{Y_{i+1}(z, t)}{Y_i(z, t)},$$

(4.6)

where

$$\tilde{F}_i(z, t) = \begin{pmatrix} 1 & 1 \varphi(z, t)^2 \\ \frac{1}{A(z, t)} \end{pmatrix} ; \quad \tilde{F}_3(z, t) = \begin{pmatrix} 1 & 0 \\ 0 & \varphi(z, t)^{-2} \end{pmatrix} \frac{B(z, t)}{\varphi(z, t)^2} \frac{1}{A(z, t)}.$$

(4.7)

Note that $\tilde{Y}(z, t)$ has no jump on $l_2$, implying that $\tilde{Y}_3(z, t) = \tilde{Y}_2(z, t)$.

The RHP in fig. [8] is of the ‘integrable’ type and its solvability is determined by the invertibility of an integrable operator i.e, its $\tau$-function is the Fredholm determinant of an integrable operator.
4.2 Integrable kernel

**Proposition 2.** The \( \tau \)-function on \( l_1 \cup l_3 \) denoted by \( \tau_{LU} \) is a Fredholm determinant of an integrable operator

\[
\tau_{LU} = \det \left[ \frac{\mathbb{1}}{A(z,t)} - \tilde{K} \right]
\]

where

\[
(\tilde{K}h)(z) = \frac{C(z,t)}{A(z,t)} \int_{iR} \frac{dw}{2\pi i} \int_{iR+i\epsilon} \frac{d\bar{w}}{2\pi i} \frac{\varphi_+^2(w)\varphi_-^2(\bar{w})}{(z - \bar{w})(\bar{w} - w)} A(\bar{w},t)B(\bar{w},t)h(\bar{w}).
\]

The functions \( A, B, C \) are defined in (3.4) and \( \varphi_+ \) is the positive (left of the imaginary axis) boundary value of \( \varphi \).

**Proof.** Let us recall the jumps in (4.7)

\[
\tilde{F}(z,t) = \begin{cases} \tilde{F}_1(z,t) = \left( \begin{array}{cc} 1 & 0 \\ \frac{C(z,t)}{A(z,t)} \varphi(z,t) & 1 \end{array} \right); & \text{on } l_1 \\ \tilde{F}_3(z,t) = \left( \begin{array}{cc} 1 & 0 \\ \frac{B(z,t)}{A(z,t)} \varphi(z,t)^2 & 1 \end{array} \right); & \text{on } l_3 \end{cases}
\]

We define the functions

\[
f(z,t) = \frac{1}{2\pi i} \left( \begin{array}{c} \frac{B(z,t)}{A(z,t)} \chi_3(z) \\ \frac{C(z,t)}{A(z,t)} \chi_1(z) \end{array} \right); \quad g(z,t) = \left( \begin{array}{c} \varphi(z,t)^2 \chi_1(z) \\ \varphi(z,t)^2 \chi_1(z) \end{array} \right)
\]

where \( \chi_1(z), \chi_3(z) \) denote the characteristic functions on the contours \( l_1, l_3 \) respectively. The jump \( \tilde{F}(z,t) \) can be written in terms of (4.11) as

\[
\tilde{F} = 1 - 2\pi i f(z)g^T(z),
\]

and clearly \( f^T(z)g(z) = 0 \). The associated integrable kernel is then

\[
K(z,w) = \frac{f^T(z)g(w)}{z - w} = \\
\frac{1}{2\pi i(z - w)} \left( \begin{array}{cc} \chi_1(z) & \chi_3(z) \\ \chi_1(z) & \chi_3(z) \end{array} \right) \left( \begin{array}{cc} 0 & \frac{C(z,t)}{A(z,t)} \varphi(z,t)^2 \chi_1(z) \\ \frac{B(z,t)}{A(z,t)} \varphi(z,t)^2 \chi_1(z) & 0 \end{array} \right) \left( \begin{array}{c} \chi_1(w) \\ \chi_3(w) \end{array} \right) \\
\equiv \left( \begin{array}{cc} \chi_1(z) & \chi_3(z) \\ \chi_1(z) & \chi_3(z) \end{array} \right) \left( \begin{array}{cc} 0 & K_{31}(z,w) \\ K_{13}(z,w) & 0 \end{array} \right) \left( \begin{array}{c} \chi_1(w) \\ \chi_3(w) \end{array} \right).
\]

The kernels \( K_{13}(z,w) \) and \( K_{31}(z,w) \) in (4.13) take the form

\[
K_{13}(z,w) = \frac{B(z,t)\varphi(z,t)^2}{(2\pi i)A(z,t)(z - w)}
\]

\[
K_{31}(z,w) = \frac{C(z,t)\varphi(z,t)^2}{(2\pi i)A(z,t)(z - w)}.
\]

We introduce the operators

\[
\mathcal{K}_{31} : L^2(l_3) \to L^2(l_1)
\]

\[
\mathcal{K}_{13} : L^2(l_1) \to L^2(l_3),
\]

(4.14)
We can therefore move the integration in \( z \) as
\[
K_{31}(z, w) = \int_{l_3} K_{31}(z, w) h(w) dw,
\]
and
\[
K_{13}(z, w) = \int_{l_1} K_{13}(z, w) \tilde{h}(w) dw.
\]
The \( \tau \)-function corresponding to RHP: \( \square \) is then
\[
\tau_{L,U}(t) = \det \left[ 1_{L^2(l_1 \cup l_3)} - \left( \begin{array}{c c}
K_{31} & 0 \\
K_{13} & 0
\end{array} \right) \right].
\] (4.16)

Since \( \varphi(w, t) \) is analytic in \( \Re(w) > 0 \) and \( \lim_{w \to \infty} \varphi(w, t) = 1 \), \( K_{31}, K_{31} \) are Trace-class. Therefore we can write \( \tau_{L,U}(t) \) in the form
\[
\tau_{L,U}(t) = \det \left[ 1_{L^2(l_3)} - K_{13} \circ K_{31} \right].
\] (4.17)

The form of the \( \tau \)-function (4.17) can be further modified such that the operator acts on \( L^2(i\Re) \) instead of \( L^2(l_3) \). We begin by splitting the function \( h(z) \) as
\[
h(z) = h_L(z) + h_R(z)
\] (4.18)
where \( h_{L,R}(z) \) are analytic to the left and right of \( l_3 \) respectively, and \( h_{L,R}(z) = O(z^{-1}) \) as \( z \to \infty \). The integrable operator (4.15) acts on \( h(z) \) as
\[
(K_{13}K_{31}h_R)(z) \equiv 0 \Rightarrow (K_{13}K_{31}h)(z) = (K_{13}K_{31}h_L)(z).
\] (4.19)
We can therefore move the integration in \( w \) from \( l_3 \) to \( i\Re \) in (4.15) and identify the space of functions \( (K_{31}h)(z) \) with \( H_R(i\Re) \), the Hardy space on the right half-plane. So, the operator
\[
(\tilde{K}h)(z) := (K_{13}K_{31}h)(z) = \frac{C(z, t)}{A(z, t)} \varphi^2_+(w) \int_{l_1} \frac{d\tilde{w}}{2\pi i} \frac{\varphi^{-2}(\tilde{w})}{z - \tilde{w}} \frac{B(\tilde{w}, t)}{A(\tilde{w}, t)} \frac{\varphi^2_+}{\varphi^{-2}}(w, t) h(w)
\] (4.20)
The kernel, \( \tilde{K}(z, w) \) is
\[
\tilde{K}(z, w) = \frac{C(z, t)}{A(z, t)} \varphi^2_+(w) \int_{l_1} \frac{d\tilde{w}}{2\pi i} \frac{\varphi^{-2}(\tilde{w})}{z - \tilde{w}} \frac{B(\tilde{w}, t)}{A(\tilde{w}, t)}.
\] (4.21)
We can now move \( l_1 \) to \( i\Re + \epsilon \) from the right without changing the kernel \( \tilde{K} \)
\[
\tilde{K}(z, w) = \frac{C(z, t)}{A(z, t)} \varphi^2_+(w) \int_{i\Re+\epsilon} \frac{d\tilde{w}}{2\pi i} \frac{\varphi^{-2}(\tilde{w})}{z - \tilde{w}} \frac{B(\tilde{w}, t)}{A(\tilde{w}, t)}.
\] (4.22)
\[
= \frac{C(z, t)}{A(z, t)} \varphi^2_+(w) \int_{i\Re+\epsilon} \frac{d\tilde{w}}{2\pi i} \frac{\varphi^{-2}(\tilde{w})}{z - \tilde{w}} \frac{B(\tilde{w}, t)}{A(\tilde{w}, t)}.
\] (4.23)
where in the last identity we use the relation from (4.2), (4.4): \( \varphi_+(\tilde{w}) = \varphi_-(\tilde{w})A(\tilde{w}, t) \).
Therefore we conclude from (4.17) and the above discussion that
\[
\tau_{L,U}(t) = \det \left[ 1_{L^2(i\Re)} - K_{13} \circ K_{31} \right] = \det \left[ 1_{L^2(i\Re)} - \tilde{K} \right]
\] (4.24)
4.3 Malgrange forms

In (4.24), we expressed the \( \tau \)-function on LU as a Fredholm determinant. To relate \( \tau_{LU} \) to \( \tau_\Sigma \) in (3.16), we will first prove that the \( \tau \)-function corresponding to the RHP (5) call it \( \tau_{LDU} \), is equal to \( \tau_{i\mathbb{R}} \) plus non-vanishing explicit factors as in proposition 1 and then show that \( \tau_{LU} \) is related to \( \tau_{LDU} \) up to explicit terms. We know that the Malgrange form for the RHP on \( i\mathbb{R} \) (3.17) is:

\[
d_t \log \tau_{i\mathbb{R}} = \int_{i\mathbb{R}} \frac{dz}{2\pi i} \text{Tr} \left[ \Theta^{-1} \Theta' \hat{J} J^{-1} \right].
\]  

(4.25)

Similarly, the Malgrange form of the RHP on LDU (4.2): \( Y_{i+1} = Y_i \) is

\[
d_t \log \tau_{LDU} = \sum_{i=1}^{3} \int_{i\mathbb{R}} \frac{dz}{2\pi i} \text{Tr} \left[ Y_i^{-1} Y_i' \hat{F}_i \hat{F}_i^{-1} \right].
\]  

(4.26)

**Proposition 3.** The Malgrange forms for the RHPs on the contours \( i\mathbb{R} \) (RHP4) and on LDU (RHP5) are related as

\[
d_t \log \tau_{i\mathbb{R}} = d_t \log \tau_{LDU} - \int_{i\mathbb{R}} \frac{dz}{2\pi i} \left( \frac{\hat{B}}{\hat{A}} \right) (AC' - A'C).
\]  

(4.27)

The functions \( A, B, C \) are defined in (3.4).

**Proof.** We begin by substituting (4.1): \( J = F_1 F_2 F_3 \) in the term

\[
jJ^{-1} = \left( \hat{F}_1 F_2 F_3 + F_1 \hat{F}_2 F_3 + F_1 F_2 \hat{F}_3 \right) (F_3^{-1} F_2^{-1} F_1^{-1}) = (\hat{F}_1 F_1^{-1} + F_1 \hat{F}_2 F_2^{-1} F_1^{-1} + F_1 F_2 \hat{F}_3 F_3^{-1} F_1^{-1}) \cdot
\]  

(4.28)

Substituting in (4.28) in the integrand of (4.25),

\[
\text{Tr} \left[ \Theta^{-1} \Theta' \hat{J} J^{-1} \right] = \text{Tr} \left[ \Theta^{-1} \Theta' \left( \hat{F}_1 F_1^{-1} + F_1 \hat{F}_2 F_2^{-1} F_1^{-1} + F_1 F_2 \hat{F}_3 F_3^{-1} F_1^{-1} \right) \right].
\]  

(4.29)

The equivalence (4.3) along with the jump condition (4.2) imply that

\[
\Theta_- = Y_1, \quad \Theta_- F_1 = Y_2, \quad \Theta_- F_1 F_2 = Y_3.
\]  

(4.30)

Substituting (4.30) in (4.29),

\[
\text{Tr} \left[ \Theta^{-1} \Theta' \left( \hat{F}_1 F_1^{-1} + F_1 \hat{F}_2 F_2^{-1} F_1^{-1} + F_1 F_2 \hat{F}_3 F_3^{-1} F_1^{-1} \right) \right] = \text{Tr} \left[ \Theta^{-1} \Theta' \hat{F}_1 F_1^{-1} + F_1^{-1} \Theta^{-1} \Theta' \hat{F}_2 F_2^{-1} + F_2^{-1} \Theta^{-1} \Theta' \hat{F}_3 F_3^{-1} \right]
\]

\[
= \text{Tr} \left[ Y^{-1}_1 Y'_1 \hat{F}_1 F_1^{-1} + Y_2^{-1} Y'_2 \hat{F}_2 F_2^{-1} - F_1^{-1} F_1' \hat{F}_2 F_2^{-1} + Y_3^{-1} Y'_3 \hat{F}_3 F_3^{-1} - (F_1 F_2)^{-1} (F_1 F_2)' \hat{F}_3 F_3^{-1} \right]
\]

\[
= \sum_{i=1}^{3} \text{Tr} \left[ Y^{-1}_i Y'_i \hat{F}_i F_i^{-1} \right] - \text{Tr} \left[ (F_1 F_2)^{-1} (F_1 F_2)' \hat{F}_3 F_3^{-1} \right].
\]  

(4.31)

Therefore,

\[
\int_{i\mathbb{R}} \frac{dz}{2\pi i} \text{Tr} \left[ \Theta^{-1} \Theta' \hat{J} J^{-1} \right] = \sum_{i=1}^{3} \int_{i\mathbb{R}} \frac{dz}{2\pi i} \text{Tr} \left[ Y^{-1}_i Y'_i \hat{F}_i F_i^{-1} \right] - \int_{i\mathbb{R}} \frac{dz}{2\pi i} \text{Tr} \left[ (F_1 F_2)^{-1} (F_1 F_2)' \hat{F}_3 F_3^{-1} \right].
\]  

(4.32)

Let us analyze the explicit terms.
\begin{itemize}
\item Since $F_1$ is upper triangular with constant diagonal entries, and $F_2$ is diagonal as defined in (4.1),
\begin{equation}
\text{Tr} \left[ F_1^{-1} F_1' F_2 F_2^{-1} \right] = 0. \tag{4.33}
\end{equation}
\item Substituting $F_{1,2,3}$ in the last term in (4.32),
\begin{equation}
\text{Tr} \left[ (F_1 F_2)^{-1} (F_1 F_2') F_3 F_3^{-1} \right] = \left( \frac{B}{A} \right) (AC' - A'C) \tag{4.34}
\end{equation}
where $A, B, C$ are explicit in terms of parabolic cylinder functions (3.4).
\end{itemize}
Therefore,
\begin{equation}
d_i \log \tau_{ir} = d_i \log \tau_{LDU} - \int_{iR} \frac{dz}{2\pi i} \left( \frac{B}{A} \right) (AC' - A'C). \tag{4.35}
\end{equation}

Recall from proposition 3 the Malgrange form of the RHP on LDU (4.26):
\begin{equation}
d_i \log \tau_{LDU} = \sum_{i=1}^{3} \int_{l_i} \frac{dz}{2\pi i} \text{Tr} \left[ Y_i^{-1} Y_i' F_i F_i^{-1} \right]. \tag{4.36}
\end{equation}
For the RHP on LU (RHP 6) with the jump condition (4.6): $\tilde{Y}_{i+1} = \tilde{Y}_i \tilde{F}_i$ where $i = 1, 3$, the Malgrange form reads
\begin{equation}
d_i \log \tau_{LU} = \sum_{i=1,3} \int_{l_i} \frac{dz}{2\pi i} \text{Tr} \left[ \tilde{Y}_i^{-1} \tilde{Y}_i' \tilde{F}_i \tilde{F}_i^{-1} \right]. \tag{4.37}
\end{equation}
Refer to Appendix A [4] for a proof of the integral form of the logarithmic derivative of the Fredholm determinant of the IIKS kernel.

**Proposition 4.** The Malgrange forms of the RHPs on contours LDU (RHP 5) and LU (RHP 6) are related as
\begin{equation}
d_i \log \tau_{LDU} = d_i \log \tau_{LU} + 2 \int_{iR} \frac{dz}{2\pi i} \frac{\tilde{A}(z, t)}{A(z, t)} \int_{\bar{iR}} \frac{dw}{2\pi i} \frac{A'(w, t)}{A(w, t)(z - w)}. \tag{4.38}
\end{equation}

**Proof.** We will first simplify the integrals on $l_1$ and $l_3$ in (4.36). Given that (4.5): $Y_i = \tilde{Y}_i \varphi^3$ and (4.10): $F_i = \varphi^{-\sigma_3} \tilde{F}_i \varphi^{\sigma_3}$,
\begin{align}
\sum_{i=1,3} \int_{l_i} \frac{dz}{2\pi i} \text{Tr} \left[ Y_i^{-1} Y_i' \tilde{F}_i \tilde{F}_i^{-1} \right] = & \sum_{i=1,3} \int_{l_i} \frac{dz}{2\pi i} \text{Tr} \left[ \left( \tilde{Y}_i \varphi^3 \right)^{-1} \left( \tilde{Y}_i \varphi^3 \right)' \partial_t \left( \varphi^{-\sigma_3} \tilde{F}_i \varphi^{\sigma_3} \right) \left( \varphi^{-\sigma_3} \tilde{F}_i^{-1} \varphi^{\sigma_3} \right) \right] \\
= & \sum_{i=1,3} \int_{l_i} \frac{dz}{2\pi i} \text{Tr} \left[ \left( \tilde{Y}_i^{-1} \tilde{Y}_i' \sigma_3 \varphi^{-1} \right) \left( -\sigma_3 \varphi^{-1} + \tilde{F}_i \tilde{F}_i^{-1} + \tilde{F}_i \sigma_3 \varphi^{-1} \tilde{F}_i^{-1} \right) \right] \\
= & \sum_{i=1,3} \int_{l_i} \frac{dz}{2\pi i} \text{Tr} \left[ \left( \tilde{Y}_i^{-1} \tilde{Y}_i' \tilde{F}_i \tilde{F}_i^{-1} \right) \right] + \sum_{i=1,3} \int_{l_i} \frac{dz}{2\pi i} \text{Tr} \left[ \left( \tilde{Y}_i^{-1} \tilde{Y}_i' \right) \left( -\sigma_3 \varphi^{-1} + \tilde{F}_i \sigma_3 \varphi^{-1} \tilde{F}_i^{-1} \right) \right]. \tag{4.39}
\end{align}
In (4.39), \( \tilde{F}_i \) are either lower or upper triangular with constant diagonal entries as in (4.10). Therefore,

\[
\int_{l_{1,2,3}} \frac{dz}{2\pi i} \text{Tr} \left[ \frac{\sigma_3 \varphi' \varphi^{-1}}{-\sigma_3 \varphi \varphi^{-1} + \tilde{F}_i \tilde{F}_i^{-1} + \tilde{F}_i \sigma_3 \varphi \varphi^{-1} \tilde{F}_i^{-1}} \right] = 0. \tag{4.40}
\]

Therefore, given (4.37), (4.39) reads

\[
\sum_{i=1,3} \int_{l_i} \frac{dz}{2\pi i} \text{Tr} \left[ Y_{i-1} Y_i Y_i F_i F_i^{-1} \right] = \sum_{i=1,3} \int_{l_i} \frac{dz}{2\pi i} \text{Tr} \left[ \tilde{Y}_{i-1} \tilde{Y}_i \tilde{F}_i \tilde{F}_i^{-1} \right] \\
+ \sum_{i=1,3} \int_{l_i} \frac{dz}{2\pi i} \text{Tr} \left[ \left( \tilde{Y}_{i-1} \tilde{Y}_i \right) \left( -\sigma_3 \varphi \varphi^{-1} + \tilde{F}_i \sigma_3 \varphi \varphi^{-1} \tilde{F}_i^{-1} \right) \right] \\
= d_t \log \tau_{lu} + \sum_{i=1,3} \int_{l_i} \frac{dz}{2\pi i} \text{Tr} \left[ \left( \tilde{Y}_{i-1} \tilde{Y}_i \right) \left( -\sigma_3 \varphi \varphi^{-1} + \tilde{F}_i \sigma_3 \varphi \varphi^{-1} \tilde{F}_i^{-1} \right) \right]. \tag{4.41}
\]

Recalling (4.6), \( \tilde{Y}_{i+1} = \tilde{Y}_i \tilde{F}_i \), the second term in (4.41) can be further simplified

\[
\sum_{i=1,3} \int_{l_i} \frac{dz}{2\pi i} \text{Tr} \left[ \left( \tilde{Y}_{i-1} \tilde{Y}_i \right) \left( -\sigma_3 \varphi \varphi^{-1} + \tilde{F}_i \sigma_3 \varphi \varphi^{-1} \tilde{F}_i^{-1} \right) \right] \\
= \sum_{i=1,3} \int_{l_i} \frac{dz}{2\pi i} \text{Tr} \left[ -\tilde{Y}_{i-1} \tilde{Y}_i \sigma_3 \varphi \varphi^{-1} + \tilde{Y}_{i-1} \tilde{Y}_i \tilde{F}_i \sigma_3 \varphi \varphi^{-1} \tilde{F}_i^{-1} \right] \\
= \sum_{i=1,3} \int_{l_i} \frac{dz}{2\pi i} \text{Tr} \left[ \Delta \left( \tilde{Y}_{i-1} \tilde{Y}_i \right) \sigma_3 \varphi \varphi^{-1} - \tilde{F}_i \sigma_3 \varphi \varphi^{-1} \tilde{F}_i^{-1} \right] \\
= \sum_{i=1,3} \int_{l_i} \frac{dz}{2\pi i} \text{Tr} \left[ \Delta \left( \tilde{Y}_{i-1} \tilde{Y}_i \right) \sigma_3 \varphi \varphi^{-1} \right], \tag{4.42}
\]

where now \( \Delta \left( \tilde{Y}_{i-1} \tilde{Y}_i \right) = \tilde{Y}_{i+1} \tilde{Y}_{i+1} - \tilde{Y}_{i} \tilde{Y}_{i} \). The last line is obtained using the fact that \( \text{Tr} \left[ \tilde{F}_i^{-1} \sigma_3 \varphi \varphi^{-1} \right] = 0 \) since \( \tilde{F}_i \) is either lower or upper triangular with constant diagonal entries, and \( \varphi \) is scalar.

The final expression in (4.42) can be further simplified by noting that the function \( \varphi \) has no jumps on \( l_1 \) and \( l_3 \). Beginning with the integral on \( l_1 \),

\[
\int_{l_1} \frac{dz}{2\pi i} \text{Tr} \left[ \Delta \left( \tilde{Y}_{1-1} \tilde{Y}_1 \right) \sigma_3 \varphi \varphi^{-1} \right] = \int_{l_1} \frac{dz}{2\pi i} \text{Tr} \left[ \left( \tilde{Y}_2^{-1} \tilde{Y}_2 - \tilde{Y}_2^{-1} \tilde{Y}_1 \right) \sigma_3 \varphi \varphi^{-1} \right] \\
= \int_{l_1} \frac{dz}{2\pi i} \text{Tr} \left[ \tilde{Y}_2^{-1} \tilde{Y}_2 \sigma_3 \varphi \varphi^{-1} \right]. \tag{4.43}
\]

To obtain the last line, we notice from fig. 8 that \( \int_{l_1} \frac{dz}{2\pi i} \text{Tr} \left[ \tilde{Y}_{i-1} \tilde{Y}_i \sigma_3 \varphi \varphi^{-1} \right] = 0 \) by closing the contour on the right. A similar computation follows for the integral on \( l_3 \) in (4.42)

\[
\int_{l_3} \frac{dz}{2\pi i} \text{Tr} \left[ \Delta \left( \tilde{Y}_{3-1} \tilde{Y}_3 \right) \sigma_3 \varphi \varphi^{-1} \right] = \int_{l_3} \frac{dz}{2\pi i} \text{Tr} \left[ \left( \tilde{Y}_4^{-1} \tilde{Y}_4 - \tilde{Y}_3^{-1} \tilde{Y}_3 \right) \sigma_3 \varphi \varphi^{-1} \right] \\
= - \int_{l_3} \frac{dz}{2\pi i} \text{Tr} \left[ \tilde{Y}_3^{-1} \tilde{Y}_3 \sigma_3 \varphi \varphi^{-1} \right]. \tag{4.44}
\]

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To obtain the last line, we note that \( \int_{l_3} \frac{dz}{2\pi i} \text{Tr} \left[ \hat{Y}_4^{-1} \hat{Y}'_4 \sigma_3 \phi \right] = 0 \) by closing the contour on the left (see fig. 3).

Gathering the terms (4.43), (4.44), and using (4.6) \( \hat{Y}_2 = \hat{Y}_3, (4.42) \) reads

\[
\sum_{i=1,3} \int_{l_i} \frac{dz}{2\pi i} \text{Tr} \left[ \Delta \left( \hat{Y}_i^{-1} \hat{Y}'_i \right) \sigma_3 \phi \right] = \int_{l_1} \frac{dz}{2\pi i} \text{Tr} \left[ \hat{Y}_2^{-1} \hat{Y}'_2 \sigma_3 \phi \right] - \int_{l_2} \frac{dz}{2\pi i} \text{Tr} \left[ \hat{Y}_3^{-1} \hat{Y}'_3 \sigma_3 \phi \right] = 0. \tag{4.45}
\]

Substituting (4.45) in (4.41),

\[
\sum_{i=1,3} \int_{l_i} \frac{dz}{2\pi i} \text{Tr} \left[ Y^{-1}_i Y'_i F_i^{-1} \right] = \sum_{i=1,3} \int_{l_i} \frac{dz}{2\pi i} \text{Tr} \left[ \hat{Y}_i^{-1} \hat{Y}'_i \hat{F}_i \right] = \partial_t \log \tau_{LU}. \tag{4.46}
\]

We now compute the integral on \( l_2 \) in (4.39)

\[
\int \frac{dz}{2\pi i} \text{Tr} \left[ Y_2^{-1} Y'_2 \hat{F}_2 F_2^{-1} \right] = \int \frac{dz}{2\pi i} \text{Tr} \left[ \left( \hat{Y}_2 \sigma_3 \right)^{-1} \left( \hat{Y}'_2 \sigma_3 + \hat{Y}_2 \left( \sigma_3 \right)' \right) \hat{F}_2 F_2^{-1} \right]
= \int \frac{dz}{2\pi i} \text{Tr} \left[ \sigma_3 \hat{Y}_2^{-1} \left( \hat{Y}'_2 \sigma_3 + \hat{Y}_2 \left( \sigma_3 \right)' \right) \hat{F}_2 F_2^{-1} \right]
= \int \frac{dz}{2\pi i} \text{Tr} \left[ Y_2^{-1} Y'_2 \hat{F}_2 F_2^{-1} + \sigma_3 \phi^{-1}_- \hat{F}_2 F_2^{-1} \right]. \tag{4.47}
\]

Since \( \hat{Y}_2 \) does not jump on \( l_2 \), Liouville theorem implies that

\[
\text{Tr} \left[ Y_2^{-1} Y'_2 \hat{F}_2 F_2^{-1} \right] = 0. \tag{4.48}
\]

The term

\[
\text{Tr} \left[ \sigma_3 \phi^{-1}_- \hat{F}_2 F_2^{-1} \right] \tag{4.49}
\]

in (4.47) is an explicit function of \( A(z, w) \) in (3.4). From (4.1),

\[
F_2 = A^{-1} \Rightarrow \hat{F}_2 F_2^{-1} = \frac{\hat{A}}{A} \sigma_3. \tag{4.50}
\]

The function \( \phi_- \) is the boundary value of \( \phi \) defined in (4.4)

\[
\phi_- = \exp \left[ \int_{iR-\epsilon} \frac{dw}{2\pi i} \log A(w, t) \right] \Rightarrow \phi_-^{-1} \phi' = \int_{iR-\epsilon} \frac{dw}{2\pi i} \frac{A'(w, t)}{A(w, t)(z-w)}. \tag{4.51}
\]

The expression (4.47) simplifies as follows due to (4.50), (4.51)

\[
\int \frac{dz}{2\pi i} \text{Tr} \left[ Y_2^{-1} Y'_2 \hat{F}_2 F_2^{-1} \right] = \int \frac{dz}{2\pi i} \text{Tr} \left[ \sigma_3 \phi_-^{-1} \phi' \hat{F}_2 F_2^{-1} \right]
= \int_{iR} \frac{dz}{2\pi i} \left( 2 \frac{\hat{A}(z, t)}{A(z, t)} \right) \int \frac{dw}{2\pi i} \frac{A'(w, t)}{A(w, t)(z-w)}. \tag{4.52}
\]

26
Substituting (4.52) and (4.46), in (4.36)

\[ d_t \log \tau_{LU} = \sum_{i=1}^{3} \int_{2\pi i} \frac{dz}{2\pi i} \mathrm{Tr} \left[ Y_i^{-1} Y'_i \hat{F}_i F_i^{-1} \right] \]

\[ = \sum_{i=1,3} \int_{2\pi i} \frac{dz}{2\pi i} \mathrm{Tr} \left[ Y_i^{-1} Y'_i \hat{F}_i F_i^{-1} \right] + \int_{2\pi i} \frac{dz}{2\pi i} \mathrm{Tr} \left[ \sigma_3 \phi^{-1} \phi' \hat{F}_2 F_2^{-1} \right] \]

\[ = d_t \log \tau_{LU} + 2 \int_{2\pi i} \frac{dz}{2\pi i} \hat{A}(z,t) \int_{2\pi i} \frac{dw}{2\pi i} \frac{A'(w,t)}{A(w,t)(z-w)} \] (4.53)
5 Proof of theorem 1

Proof. The propositions 1, 3, 4 imply that the \( \tau \)-functions \( \tau_\Sigma \) and \( \tau_{LU} \) are related through explicit factors, and the proposition 2 expresses \( \tau_{LU} \) as a Fredholm determinant. Therefore, the \( \tau \)-function of Painlevé II equation defined in (2.16)

\[
d_t \log \tau_{pII} \equiv d_t \log \tau_\Sigma - \int_{i\mathbb{R}} \frac{dz}{2\pi i} \hat{F}(z, t; \nu, h) - \left[ \frac{4i\nu}{3} + \frac{2\nu^2}{t} \right] - 2g(\nu, h; t)
\]

\[
d_t \log \tau_{pII} \equiv d_t \log \tau_{LU} - \int_{i\mathbb{R}} \frac{dz}{2\pi i} \left\{ \left( \frac{\hat{B}}{\hat{A}} \right) \left( AC' - A'C \right) + \hat{F}(\zeta, \xi, t; \nu, h) \right\} - \left[ \frac{4i\nu}{3} + \frac{2\nu^2}{t} \right] - 2g(\nu, h; t)
\]

\[
d_t \log \tau_{pII} \equiv d_t \log \tau_{LU} + \int_{i\mathbb{R}} \frac{dz}{2\pi i} \left\{ \frac{2\hat{A}(z, t)}{\hat{A}(z, t)} \left( \int_{i\mathbb{R}-\epsilon} \frac{dw}{2\pi i} \frac{A'(w, t)}{A(w, t)(z-w)} \right) - \left( \frac{\hat{B}}{\hat{A}} \right) \left( AC' - A'C \right) - \hat{F}(z, t; \nu, h) \right\}
\]

\[
t - \left[ \frac{4i\nu}{3} + \frac{2\nu^2}{t} \right] - 2g(\nu, h; t)
\]

\[
+ \int_{i\mathbb{R}} \frac{dz}{2\pi i} \left\{ \frac{2\hat{A}(z, t)}{\hat{A}(z, t)} \left( \int_{i\mathbb{R}-\epsilon} \frac{dw}{2\pi i} \frac{A'(w, t)}{A(w, t)(z-w)} \right) - \left( \frac{\hat{B}}{\hat{A}} \right) \left( AC' - A'C \right) - \hat{F}(z, t; \nu, h) \right\}.
\]

In (5.1), the functions \( \mathcal{A}, \mathcal{B}, \mathcal{C} \) defined in (3.4) are explicit in terms of parabolic cylinder functions, \( \mathcal{F} \) is defined in (3.41), and the term

\[
\int_{i\mathbb{R}} \frac{dz}{2\pi i} \left\{ \frac{2\hat{A}(z, t)}{\hat{A}(z, t)} \left( \int_{i\mathbb{R}-\epsilon} \frac{dw}{2\pi i} \frac{A'(w, t)}{A(w, t)(z-w)} \right) - \left( \frac{\hat{B}}{\hat{A}} \right) \left( AC' - A'C \right) - \hat{F}(z, t; \nu, h) \right\}
\]

depends only on \( h, \nu \) and \( t \). We then define

\[
\mathcal{F}(t, \nu, h) := \int_{i\mathbb{R}} \frac{dz}{2\pi i} \left\{ \frac{2\hat{A}(z, t)}{\hat{A}(z, t)} \left( \int_{i\mathbb{R}-\epsilon} \frac{dw}{2\pi i} \frac{A'(w, t)}{A(w, t)(z-w)} \right) - \left( \frac{\hat{B}}{\hat{A}} \right) \left( AC' - A'C \right) - \hat{F}(z, t; \nu, h) \right\}
\]

\[- 2g(\nu, h; t).
\]

In terms of \( \mathcal{F}(t, \nu, h) \), (5.1) reads

\[
d_t \log \tau_{pII} = d_t \log \det \left[ 1_{L^2(\mathbb{R}^3)} - \tilde{\mathcal{K}} \right] + \mathcal{F}(t, \nu, h) - \left[ \frac{4i\nu}{3} + \frac{2\nu^2}{t} \right].
\]

Therefore, the \( \tau \)-function of Painlevé II can be expressed as a Fredholm determinant of an integrable operator up to explicit factors. Furthermore, solving the RHP 4 is equivalent to solving the RHP 2 which in turn is tantamount to solving the RHP 2. Therefore, the zeros of \( \tau_{pII} \) (solvability condition of RHP 2) are completely determined by the zeros of the Fredholm determinant (4.24).
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