The Vlasov-Poisson equation in $\mathbb{R}^3$
with infinite charge and velocities

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Abstract

We consider the Vlasov-Poisson equation in $\mathbb{R}^3$ with initial data which are not $L^1$ in space and have unbounded support in the velocities. Assuming for the density a slight decay in space and a strong decay in velocities, we prove existence and uniqueness of the solution, thus generalizing the analogous result given in [5] for data compactly supported in the velocities.

Keywords: Vlasov-Poisson equation; infinitely extended plasma; unbounded velocities; local energy.

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1 Introduction

We consider a one-species, positively charged plasma, under the influence of the auto-induced electric field. The equation describing the time evolution of this system is the following Vlasov-Poisson equation:

\[
\begin{aligned}
\partial_t f(x, v, t) + v \cdot \nabla_x f(x, v, t) + E(x, t) \cdot \nabla_v f(x, v, t) &= 0 \\
E(x, t) &= \int \frac{x - y}{|x - y|^3} \rho(y, t) \, dy \\
\rho(x, t) &= \int f(x, v, t) \, dv \\
f(x, v, 0) &= f_0(x, v) \geq 0
\end{aligned}
\]

where $f(x, v, t)$ is the distribution of charged particles at the point of the phase space $(x, v)$ at time $t$, $\rho$ is the spatial density and $E$ is the electric

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field. Equation (1.1) shows that \( f \) is time-invariant along the solutions of the so called characteristics equations:

\[
\begin{align*}
\dot{X}(t) &= V(t) \\
V(t) &= E(X(t), t) \\
(X(0), V(0)) &= (x, v)
\end{align*}
\]

(1.2)

where \((X(t), V(t)) = (X(x, v, t), V(x, v, t))\) denote position and velocity of a particle starting at time \( t = 0 \) from \((x, v)\) and the initial datum \((x, v)\) is distributed according to \( f_0 \). It is well known that along (1.2) the partial differential equation (1.1) transforms into an ordinary differential equation, hence a result of existence and uniqueness of solutions to (1.2) implies the same result for solutions to (1.1), with regularity properties depending on the regularity of \( f_0 \). Since \( f \) is time-invariant along this motion, \( f(X(t), V(t), t) = f_0(x, v) \), and

\[
\|f(t)\|_{L^\infty} = \|f_0\|_{L^\infty}.
\]

(1.3)

This equation has been widely studied and the problem of existence and uniqueness of the solution in \( \mathbb{R}^3 \), for \( L^1 \) data, is completely solved, as it can be seen in many papers. We quote \[10\ [12\ [13\] and a nice review of many such results \[8\] for all. The subsequent problem of a spatial density not belonging to \( L^1 \) has been investigated since many years, for instance in papers \[1\ [2\] and more recently in \[3\ [4\ [5\]. Other papers related to this problem, with many different optics, are, to quote some of them, \[7\ [9\ [11\ [14\ [15\ [16\]. In particular, in \[4\] and \[5\] we have considered an infinitely extended plasma, confined in a cylinder by an external magnetic field and in the whole space respectively. In both cases the spatial density is not supposed to be in \( L^1 \) and in both cases it is of primary relevance to achieve a good position of the equations, since the electric field could be infinite. To avoid this problem, we have assumed that the spatial density, even if not integrable, is slightly decaying at infinity. Another problem coming from the infinite charge of the plasma is that the total energy of the system is infinite. We bypass this difficulty by introducing the local energy, that is a sort of energy of a bounded region, which however takes into account the whole interaction with the rest of the particles. By our hypotheses at \( t = 0 \) it turns out that the local energy is finite and has good properties, such to enable us to prove, in both papers \[4\] and \[5\], the existence and uniqueness of the solution globally in time, for initial data which are not \( L^1 \) in space. These results are proved in case that the initial distribution \( f_0 \) has compact support in the velocities.

In the present paper we extend the analysis to the unbounded velocities case, which appears physically more relevant. The strategy of the proof is
the following: we start from the already known case with a cutoff \( N \) on the maximal velocity and we study the limit \( N \to \infty \). Assuming a slight decay in space and a strong decay in velocity of the initial distribution \( f_0(x,v) \), we prove that the limit \( N \to \infty \) does exist, satisfies the Vlasov-Poisson equation, preserves the decay behavior and it is unique in this class. We remark that this decay law includes the important Maxwell-Boltzmann distribution.

The main point for the present generalization is a sharp estimate on the electric field which refines the preceding one given in [5]. We observe that in the present paper the plasma can move in the whole space \( \mathbb{R}^3 \). Recently an unbounded velocity case has been studied for a plasma confined in an unbounded cylinder by a magnetic mirror [6], where the plasma moves in a quasi one-dimensional region, and this allow for a slighter spatial decay at infinity.

We refer to the characteristics equations (1.2) since, as it is well known, the existence of a unique solution to (1.2) would imply the same result for (1.1), in case of smooth data. The main results of the present paper are stated in the following Theorems:

**Theorem 1.** Let us fix an arbitrary positive time \( T \). Let \( f_0 \) satisfy the following hypotheses:

\[
0 \leq f_0(x,v) \leq C_0e^{-\lambda|v|^2}g(|x|) \tag{1.4}
\]

where \( g \) is a positive, bounded, continuous function satisfying, for any \( i \in \mathbb{Z}^3 \setminus \{0\} \),

\[
\int_{|i-x|\leq 1} g(|x|) \, dx \leq C_1 \frac{1}{|i|^{2+\epsilon}} \tag{1.5}
\]

for any fixed \( 1/15 < \epsilon < 1 \), being \( \lambda, C_0 \) and \( C_1 \) positive constants. Then there exists a solution to equations (1.2) on \([0,T] \) and positive constants \( C \) and \( \bar{\lambda} \) such that

\[
0 \leq f(x,v,t) \leq Ce^{-\bar{\lambda}|v|^2}, \tag{1.6}
\]

and for any \( i \in \mathbb{Z}^3 \setminus \{0\} \)

\[
\int_{|i-x|\leq 1} \rho(x,t) \, dx \leq C\frac{\log^2(1+|i|)}{|i|^{2+\epsilon}}. \tag{1.7}
\]

This solution is unique in the class of those satisfying (1.6) and (1.7).

**Remark 1.** The assumption (1.5) can be satisfied in case that the spatial density \( \rho(x,0) \), even being not integrable, has a suitable decay at infinity, but also whenever \( \rho(x,0) \) is piecewise constant (or has an oscillatory character) over suitable sparse sets. Hence hypothesis (1.5) allows for spatial densities which possibly do not belong to any \( L^p \) space. If we assume that the spatial
density is point-wise decreasing for large $|x|$, then Theorem 1 of course remains valid, but the thesis can be improved, since we are able to show that at any time $t \in [0, T]$ the spatial density keeps the same decreasing property. This is the object of the next Theorem 2.

Note that the upper bound for $\epsilon$ is due to allow infinite mass. We do not study the cases $\epsilon = 1$ (a border case with infinite mass) and $\epsilon > 1$ (finite mass), in which the proof is simpler.

**Theorem 2.** Let us fix an arbitrary positive time $T$. Let $f_0$ satisfy the following hypotheses:

$$0 \leq f_0(x, v) \leq C_0 e^{-\lambda |v|^2} g(|x|)$$  \hspace{1cm} (1.8)

where $g$ is a bounded, continuous, not increasing function such that, for $|x| \geq 1$,

$$g(|x|) = C \frac{1}{|x|^{2+\epsilon}}$$  \hspace{1cm} (1.9)

for any fixed $1/15 < \epsilon < 1$, being $\lambda$, $C_0$ and $C$ positive constants. Then there exists a solution to equations (1.2) on $[0, T]$ and positive constants $C'$ and $\lambda'$ such that

$$0 \leq f(x, v, t) \leq C' e^{-\lambda' |v|^2} g(|x|).$$  \hspace{1cm} (1.10)

This solution is unique in the class of those satisfying (1.10).

The paper is devoted to the proofs of Theorems 1 and 2 and is planned in the following way. In Section 2 we define the partial dynamics, which is an evolution equation for a system regularized with respect to our purpose, that is with a cutoff in the velocities. For this system we introduce the local energy and we state many properties, in particular the bound on the auto-induced electric field and the bound on the local energy. The proof of Theorems 1 and 2 are given in Section 3, where we show that the estimates proved in the previous section can be made uniform with respect to the cutoff. Finally in Section 4 we give the proof of the main estimate on the electric field, which is fundamental in the proofs of both theorems, and the Appendix is devoted to the proofs of some technical lemmas.

**2 The partial dynamics**

Beside system (1.2) we introduce a modified differential system, called the partial dynamics, in which the initial density has compact support in the velocities. More precisely, for any positive integer $N$, we consider the following equations:

$$
\begin{cases}
\dot{X}^N(t) = V^N(t) \\
\dot{V}^N(t) = E^N(X^N(t), t) \\
(X^N(0), V^N(0)) = (x, v),
\end{cases}
$$

(2.1)
where

\((X^N(t), V^N(t)) = (X^N(x, v, t), V^N(x, v, t))\)

\[E^N(x, t) = \int \frac{x - y}{|x - y|^3} \rho^N(y, t) \, dy\]

\[\rho^N(x, t) = \int f^N(x, v, t) \, dv\]

and

\[f^N(X^N(t), V^N(t), t) = f^N_0(x, v),\]

being the initial distribution \(f^N_0\) defined as

\[f^N_0(x, v) = f_0(x, v) \chi(|v| \leq N) \quad (2.2)\]

where \(\chi(\cdot)\) is the characteristic function of the set \((\cdot)\). Since in the equations (2.1) the initial datum \(f^N_0\) has compact support in the velocities, for any fixed \(N\) the solution does exist unique, as it is proved in [5]. Our purpose, in order to prove both Theorems [1] and [2], is to show that this solution converges, as \(N \to \infty\), to a solution to (1.2).

Along the paper some positive constants will appear, generally denoted by \(C\), except some which will be numbered in order to be quoted in the sequel. All of them will depend exclusively on \(\|f_0\|_{L^\infty}\) and an arbitrarily fixed, once for ever, time \(T\), but not on \(N\), while any dependence on \(N\) in the estimates will be clearly stressed.

### 2.1 The local energy

We define the local energy, already introduced in our previous papers, as a fundamental tool to deal with the infinite charge of the plasma.

For any vector \(\mu \in \mathbb{R}^3\) and any \(R > 0\) we define the function:

\[\varphi^{\mu, R}(x) = \varphi \left( \frac{|x - \mu|}{R} \right) \quad (2.3)\]

with \(\varphi\) a smooth function such that:

\[\varphi(r) = 1 \text{ if } r \in [0, 1] \quad (2.4)\]

\[\varphi(r) = 0 \text{ if } r \in [2, +\infty) \quad (2.5)\]

\[-2 \leq \varphi'(r) \leq 0. \quad (2.6)\]

We define the local energy as:

\[W^N(\mu, R, t) = \frac{1}{2} \int dx \varphi^{\mu, R}(x) \left[ \int dv |v|^2 f^N(x, v, t) + \rho^N(x, t) \int dy \frac{\rho^N(y, t)}{|x - y|} \right]. \quad (2.7)\]
It depends on the property of \( f^N \) whether or not \( W^N \) is bounded. For the moment, we stress that the local energy takes into account the complete interaction with the rest of the plasma out of the sphere of center \( \mu \) and radius \( 2R \).

We set

\[
Q^N(R,t) = \sup_{\mu \in \mathbb{R}^3} W^N(\mu, R, t).
\] (2.8)

First of all we observe that the hypotheses in Theorem 1 (and hence those in Theorem 2 which are more strict) ensure that the local energy is bounded at time \( t = 0 \) and that the following holds:

**Lemma 1.** In the hypotheses of Theorem 1, \( \forall R \geq 1 \) it holds

\[
Q^N(R,0) \leq CR^{1-\epsilon}.
\] (2.9)

**Proof.** We take any \( \mu \in \mathbb{R}^3 \) and \( R \geq 1 \), assumed integer for simplicity, and we estimate \( W^N(\mu, R, 0) \). By assumption (1.4) we have

\[
W^N(\mu, R, 0) \leq C \int dx \, \varphi^{\mu,R}(x)g(|x|) \left[ 1 + \int dy \, \frac{g(|y|)}{|x-y|} \right].
\] (2.10)

By the definition of the function \( \varphi^{\mu,R} \) it is

\[
\int \varphi^{\mu,R}(x)g(|x|) \, dx \leq \int_{|\mu-x| \leq 2R} g(|x|) \, dx.
\] (2.11)

We estimate the integral on the right in both cases, \(|\mu| \leq 3R \) and \(|\mu| > 3R \). Consider the case \(|\mu| \leq 3R \). By assumption (1.5) we have:

\[
\int_{|\mu-x| \leq 3R} g(|x|) \, dx \leq \int_{|x| \leq 5R} g(|x|) \, dx \leq \int_{|x| \leq 1} g(|x|) \, dx + \sum_{1 \leq |i| \leq 5R} \int_{|i-x| \leq 1} g(|x|) \, dx \leq C \left[ 1 + \sum_{i \in \mathbb{Z}^3 \atop 1 < |i| \leq 5R} \frac{1}{|i|^{2+\epsilon}} \right] \leq CR^{1-\epsilon}.
\]
If on the contrary it is \( |\mu| > 3R \), then

\[
\int_{|\mu - x| \leq 2R} g(|x|) \, dx \leq C \left[ 1 + \sum_{1 < |i| \leq 2R} \int_{|\mu + i - x| \leq 1} g(|x|) \, dx \right] \leq C \left[ 1 + \sum_{1 < |i| \leq 2R} \frac{1}{|\mu + i|^{2+\epsilon}} \right]
\]

Since in this case it is \( |\mu + i| \geq \frac{|i|}{2} \), we get

\[
\int_{|\mu - x| \leq 2R} g(|x|) \, dx \leq C \left[ 1 + \sum_{i \in \mathbb{Z}^3} \frac{1}{|i|^{2+\epsilon}} \right] \leq CR^{1-\epsilon}. \tag{2.12}
\]

Hence, in both cases we have

\[
\int_{|\mu - x| \leq 2R} g(|x|) \, dx \leq CR^{1-\epsilon}
\]

which implies, by (2.11),

\[
\int \varphi^{\mu,R}(x) g(|x|) \, dx \leq CR^{1-\epsilon}. \tag{2.13}
\]

Let us now estimate the potential energy of a single particle.

\[
\int \frac{g(|y|)}{|x - y|} \, dy \leq C \left[ \int_{|x - y| \leq 1} \frac{g(|y|)}{|x - y|} \, dy + \sum_{i \in \mathbb{Z}^3, |i| \geq 1} \frac{1}{|i|} \int_{|x + i - y| \leq 1} g(|y|) \, dy \right] \leq C \left[ 1 + \sum_{i \in \mathbb{Z}^3, |i| > 1} \frac{1}{|i||x + i|^{2+\epsilon}} \right]. \tag{2.14}
\]

Now, considering in the sum in (2.14) the two subsets of indices \( \{ i : |i| \leq |x + i| \} \) and \( \{ i : |i| > |x + i| \} \), we get

\[
\sum_{i \in \mathbb{Z}^3, |i| > 1, |x + i| \geq 1} \frac{1}{|i||x + i|^{2+\epsilon}} \leq \sum_{i \in \mathbb{Z}^3, |i| \geq 1} \frac{1}{|i|^{2+\epsilon}} + \sum_{i \in \mathbb{Z}^3, |x + i| \geq 1} \frac{1}{|x + i|^{2+\epsilon}} \leq C.
\]
Hence we have proved that
\[
\int \frac{g(|y|)}{|x-y|} \, dy \leq C
\]  
(2.15)

By (2.10), estimates (2.13) and (2.15) prove the thesis.

We fix once for ever an arbitrary time \( T > 0 \), and all the estimates in the sequel are to be intended to hold on the interval \([0, T]\).

Let us introduce the following functions, for any \( t \in [0, T] \):
\[
V_N(t) = \max \left\{ \tilde{C}, \sup_{s \in [0,t]} \sup_{(x,v)\in \mathbb{R}^3 \times B(0,N)} |V^N(x,v,s)| \right\}
\]  
(2.16)
\[
R_N(t) = 1 + \int_0^t V_N(s) \, ds
\]  
(2.17)

where \( \tilde{C} > 1 \) is a positive constant chosen suitably large for further technical purposes (see for example Lemma 6 in the Appendix), and \( B(0,N) \) is the ball in \( \mathbb{R}^3 \) of center 0 and radius \( N \).

The following statement is our main result on the local energy. Its proof has already been done in previous works, as for instance [5], and we will not repeat it here. We only stress that it is strongly based on the positivity of the local potential energy.

**Proposition 1.** For any \( t \in [0, T] \) it holds
\[
Q^N(R_N(t), t) \leq CQ^N(R_N(t), 0).
\]

**Remark 2.** In the hypotheses of Theorems 1 and 2 the local energy, defined in (2.7), satisfies at time \( t \), in the limit \( N \to \infty \), the same properties which hold initially, that is it remains bounded and it grows with \( R \) as stated in Lemma 4.

2.2 Some preliminary estimates

We give now some results on system (2.1). Hereafter we assume that hypotheses of Theorem 1 are satisfied, since the same results will follow from the hypotheses of Theorem 2, by an a fortiori argument.

The use of the local energy and its properties give us the following estimate:

**Lemma 2.** For any \( \mu \in \mathbb{R}^3 \) and \( R \geq 1 \) it holds
\[
\int_{1 \leq |\mu-x| \leq R} \frac{p^N(x,t)}{|\mu-x|^2} \, dx \leq CQ^N(R, t)^{\frac{2}{3}}.
\]
Proof. Notice that it is, for any $a > 0$

$$
\rho^N(x, t) = \int f^N(x, v, t) \, dv \leq \int_{|v| \leq a} f^N(x, v, t) \, dv + \frac{1}{a^2} \int_{|v| > a} |v|^2 f^N(x, v, t) \, dv
$$

so that, by (1.3), we have

$$
\rho^N(x, t) \leq Ca^3 + \frac{1}{a^2} k(x, t)
$$

(2.18)

where

$$
k(x, t) = \int |v|^2 f^N(x, v, t) \, dv.
$$

By minimizing over $a$ and taking the power $\frac{5}{3}$ of both members we get

$$
\rho^N(x, t)^{\frac{5}{3}} \leq Ck(x, t)
$$

which implies, by integrating over the set $\{x : |\mu - x| \leq R\}$

$$
\int_{|\mu - x| \leq R} \rho^N(x, t)^{\frac{5}{3}} \, dx \leq CW^N(\mu, R, t) \leq CQ^N(R, t). \tag{2.19}
$$

Notice that we have bounded the local kinetic energy by $W^N$, which is allowed by the fact that the local potential energy is positive. Now we have

$$
\int_{1 \leq |\mu - x| \leq R} \frac{\rho^N(x, t)}{|\mu - x|^2} \, dx \leq \left( \int_{|\mu - x| \leq R} \rho^N(x, t)^{\frac{5}{3}} \, dx \right)^{\frac{2}{5}} \left( \int_{1 \leq |\mu - x|} \frac{1}{|\mu - x|^5} \, dx \right)^{\frac{2}{5}} \tag{2.20}
$$

and (2.19) implies

$$
\int_{1 \leq |\mu - x| \leq R} \frac{\rho^N(x, t)}{|\mu - x|^2} \, dx \leq CQ^N(R, t)^{\frac{2}{5}}. \tag{2.21}
$$

Lemma 3. Let $R^N(t)$ be defined in (2.17). Then for any $\mu \in \mathbb{R}^3$ it holds

$$
\int_{3R^N(t) \leq |\mu - x|} \frac{\rho^N(x, t)}{|\mu - x|^2} \, dx \leq C. \tag{2.22}
$$
Proof. We notice that if $|\mu - x| \geq 3R^N(t)$ then
\[
|\mu - X^N(t)| \geq |\mu - x| - R^N(t) \geq 2R^N(t)
\] (2.23)
and also
\[
|\mu - X^N(t)| \geq |\mu - x| - \frac{|\mu - x|}{3} = \frac{2}{3}|\mu - x|.
\] (2.24)
By the invariance of $f$ along the characteristics and by the change of variables $(X^N(x, v, t), V^N(x, v, t)) \rightarrow (x, v)$ we get
\[
\int_{|\mu - x|} \frac{\rho^N(x, t)}{|\mu - x|^2} dy = \int_{|\mu - x|} \frac{f^N(x, v, t)}{|\mu - x|^2} dx dv \leq \int_{|\mu - x|} \frac{f^N(x, v, t)}{|\mu - X^N(t)|^2} dx dv
\] (2.25)
so that, by (2.24) and using the assumption (12), we have
\[
\int_{|\mu - x|} \frac{\rho^N(x, t)}{|\mu - x|^2} dx \leq \frac{9}{4} \int_{|\mu - x|} \frac{f^N(x, v, t)}{|\mu - x|^2} dx dv \leq C \int_{|\mu - x|} \frac{g(|x|)}{|\mu - x|^2} dx.
\] (2.26)
Now it is
\[
\int_{|\mu - x|} \frac{g(|x|)}{|\mu - x|^2} dx \leq \sum_{i \in \mathbb{Z}^1 \setminus \{0\}} \int_{|\mu - i| \leq 1} \frac{g(|x|)}{|\mu - x|^2} dx.
\]
Since
\[
|\mu - x| \geq |\mu - i| - 1 \geq |\mu - i| - R^N(t) \geq \frac{|\mu - i|}{2}
\]
from hypothesis (1.5) it follows
\[
\int_{|\mu - x|} \frac{g(|x|)}{|\mu - x|^2} dx \leq \sum_{i \in \mathbb{Z}^1 \setminus \{0\}} \frac{4}{|\mu - i|^2} \int_{|x| \leq 1} g(|x|) dx \leq 4C_1 \sum_{i \in \mathbb{Z}^1 \setminus \{0\}} \frac{1}{|\mu - i|^2 |i|^{2+\epsilon}}.
\] (2.27)
By considering in the sum the two subsets of indices $\{i : |\mu - i| \geq |i|\}$ and $\{i : |\mu - i| < |i|\}$, we get:
\[
\sum_{i \in \mathbb{Z}^1 : |i| \geq 1, |\mu - i| \geq 1} \frac{1}{|\mu - i|^2 |i|^{2+\epsilon}} \leq \sum_{i \in \mathbb{Z}^1 : |i| \geq 1} \frac{1}{|i|^{4+\epsilon}} + \sum_{i \in \mathbb{Z}^1 : |i| \geq 1} \frac{1}{|\mu - i|^{4+\epsilon}} \leq C
\] (2.28)
which proves the thesis. \qed
Now we give a first bound on the electric field $E$.

**Proposition 2.**

\[
\|E^N(t)\|_{L^\infty} \leq C_2 V^N(t)\frac{4}{3}Q^N(R^N(t), t)\frac{1}{\xi}.
\] (2.29)

**Proof.** We have:

\[
|E^N(x, t)| \leq \sum_{k=1}^3 J_k
\] (2.30)

where

\[
J_1 = \int_{0 <|x-y| \leq a} \frac{\rho^N(y, t)}{|x-y|^2} dy
\]

with $a < 1$ to be suitably chosen in what follows,

\[
J_2 = \int_{a <|x-y| \leq 3R^N(t)} \frac{\rho^N(y, t)}{|x-y|^2} dy
\]

and

\[
J_3 = \int_{|x-y| > 3R^N(t)} \frac{\rho^N(y, t)}{|x-y|^2} dy.
\]

Let us estimate $J_1$. We have:

\[
J_1 \leq 4\pi \|\rho^N(t)\|_{L^\infty} a.
\]

Since

\[
\rho^N(x, t) = \int dv f^N(x, v, t) \leq CV^N(t)^3
\] (2.31)

we get

\[
J_1 \leq CV^N(t)^3 a.
\] (2.32)

By estimate (2.19) we get:

\[
J_2(x, t) \leq C \left( \int_{|x-y| \leq 3R^N(t)} \rho^N(y, t)\frac{1}{|x-y|^\xi} dy \right) \left( \int_{a <|x-y| \leq 3R^N(t)} \frac{1}{|x-y|^2} dy \right) \frac{1}{\xi}
\]

\[
\leq CQ^N(3R^N(t), t)^\frac{2}{\xi} \left[ a^{-\frac{4}{\xi}} + R^N(t)^{-\frac{4}{\xi}} \right].
\] (2.33)

Now, it is easy to see that

\[
Q^N(3R^N(t), t) \leq CQ^N(R^N(t), t).
\] (2.34)
Indeed, recalling the definition of the function \( \varphi^{\mu,R} \) in the local energy (2.7), for any positive number \( R' > R \) and for any \( \mu \in \mathbb{R}^3 \) it is

\[
\varphi^{\mu,R'}(x) = \varphi \left( \frac{|x - \mu|}{R'} \right) \leq \sum_{i \in \mathbb{Z}^3 : |i| \leq \frac{R'}{R}} \varphi \left( \frac{|x - (\mu + iR)|}{R} \right).
\]

Hence, since both terms in the function \( W^N \) are positive, by monotony we have:

\[
W^N(\mu, R', t) \leq \sum_{i \in \mathbb{Z}^3 : |i| \leq \frac{R'}{R}} W^N(\mu + iR, R, t) \leq \left( \frac{R'}{R} \right)^3 Q^N(R, t).
\]

This implies (2.34) which, used in (2.33), gives

\[
J_2(x, t) \leq CQ^N(R^N(t), t)^{\frac{2}{3}} a^{-\frac{5}{6}}.
\]

The minimum value of \( J_1(x, t) + J_2(x, t) \) is attained at

\[
a = CV^N(t)^{-\frac{2}{3}} Q^N(R^N(t), t)^{\frac{1}{2}}
\]

so that we get

\[
J_1(x, t) + J_2(x, t) \leq CV^N(t)^{\frac{2}{3}} Q^N(R^N(t), t)^{\frac{1}{2}}.
\]

Finally by Lemma 3 we have

\[
J_3(x, t) \leq C.
\]

By (2.30), formulas (2.37) and (2.38) imply the thesis.

The main estimate on \( E^N \) is stated in the following proposition. Its proof, rather lengthy, is given in Section 4.

**Proposition 3.** There exists a positive constant \( C_3 \) such that, for any \( t \in [0, T] \):

\[
\int_0^t |E^N(X^N(s), s)| ds \leq C_3 \left[ V^N(t) \right]^\alpha \quad \forall \alpha \in \left[ \frac{5 - \epsilon}{9}, \frac{2}{3} \right)
\]

being \( \epsilon \) the one in (1.5).

This bound is finer than the analogous one in [5], in which it was \( \alpha < 1 \). The presence of plasma particles having infinite velocities forces us to get a better estimate on the electric field, hence the upper bound 2/3 for the parameter \( \alpha \) is a key tool in the proof of the convergence of the iterative scheme in Section 3 (see (3.8)), while the lower bound \( (5 - \epsilon)/9 \) is used in eq. (2.35).

As a consequence of this result we have the following
Corollary 1.

\[ V^N(T) \leq C_4N \quad (2.39) \]
\[ \rho^N(x, t) \leq C N^{3\alpha} \quad (2.40) \]
\[ Q^N(R^N(t), t) \leq C N^{1-\epsilon}. \quad (2.41) \]

Proof. The bound on \( V^N(T) \) is obvious, and comes from Proposition 3, being

\[ |V^N(t)| \leq |v| + \int_0^t |E^N(X^N(s), s)| \, ds \quad (2.42) \]

and recalling that \(|v| \leq N \) and \( \alpha < 1 \).

Now we prove (2.40). Setting

\[ (\bar{x}, \bar{v}) = (X^N(x, v, t), V^N(x, v, t)) \]

by the invariance of the density \( f^N \) along the characteristics we have

\[ \rho^N(\bar{x}, t) = \int f^N(\bar{x}, \bar{v}, t) \, d\bar{v} = \int f^N_0(x, v) \, d\bar{v}. \]

We consider the set \( A_1 = \{ \bar{v} : |\bar{v}| \leq 2C_3(C_4N)^{\alpha} \} \) and its complementary set \( A_1^c \). By assumption (1.4) we have:

\[ \rho^N(\bar{x}, t) \leq \int_{A_1} f^N_0(x, v) \, d\bar{v} + C \int_{A_1^c} e^{-\lambda|v|^2} \, d\bar{v} \]
\[ \leq C N^{3\alpha} + C \int_{A_1^c} e^{-\lambda|v|^2} \, d\bar{v}. \quad (2.43) \]

By (2.42), Proposition 3 and (2.39) it is

\[ |v| \geq |\bar{v}| - C_3(C_4N)^{\alpha} \]

and hence for any \( \bar{v} \in A_1^c \) it is \(|v| \geq |\bar{v}|/2\). Hence from (2.43) it follows

\[ \rho^N(\bar{x}, t) \leq C N^{3\alpha} + C \int e^{-\frac{\lambda}{2}|v|^2} \, d\bar{v} \leq C N^{3\alpha}. \quad (2.44) \]

Finally (2.41) comes from Proposition 1, Lemma 1, definition (2.17) and (2.39). \[ \square \]
2.3 Estimate of the term $|X^N(t) - X^{N+1}(t)|$

Let $f_0$ satisfy the assumptions of Theorem 4 and let us fix an initial condition $(x, v)$ in the support of $f_0^N$. We consider the time evolved of the point in the phase space $(x, v)$ in the $N$-th and in the $(N+1)$-th dynamics, that is we consider $(X^N(t), V^N(t))$ and $(X^{N+1}(t), V^{N+1}(t))$, the solutions to eq. (2.1) with initial condition $f_0^N$ and $f_{0}^{N+1}$ respectively. We set

$$\delta^N(x, v, t) = \sup_{(x,v) \in \mathbb{R}^3 \times B(0,N)} \delta^N(x, v, t),$$

$$\delta^N(x, v, t) = |X^N(t) - X^{N+1}(t)|.$$

We prove the following result:

**Proposition 4.** For any $t \in [0, T]$ it holds

$$\delta^N(t) \leq C \int_0^t dt_1 \int_0^{t_1} dt_2 \left[ N^{3\alpha} \delta^N(t_2) \left( |\log \delta^N(t_2)| + 1 \right) + e^{-\frac{\lambda}{2}N^2} \right]. \quad (2.45)$$

**Proof.** We have

$$\delta^N(x, v, t) = 
\left| \int_0^t dt_1 \int_0^{t_1} dt_2 \left[ E^N(X^N(t_2), t_2) - E^{N+1}(X^{N+1}(t_2), t_2) \right] \right| \leq
\int_0^t dt_1 \int_0^{t_1} dt_2 \left[ \mathcal{F}_1(X^N(t_2), X^{N+1}(t_2), t_2) + \mathcal{F}_2(X^N(t_2), X^{N+1}(t_2), t_2) \right], \quad (2.46)$$

where

$$\mathcal{F}_1(X^N(t), X^{N+1}(t), t) = |E^N(X^N(t), t) - E^{N+1}(X^{N+1}(t), t)| \quad (2.47)$$

and

$$\mathcal{F}_2(X^{N+1}(t), t) = |E^N(X^{N+1}(t), t) - E^{N+1}(X^{N+1}(t), t)|. \quad (2.48)$$

**Estimate of the term $\mathcal{F}_1$**

We prove a quasi-Lipschitz property for $E^N$. We consider the difference $|E^N(x, t) - E^N(y, t)|$ at two generic points $x$ and $y$, and set:

$$\mathcal{F}_1(x, y, t) = |E^N(x, t) - E^N(y, t)| \chi(|x - y| \geq 1)$$

and

$$\mathcal{F}_1'(x, y, t) = |E^N(x, t) - E^N(y, t)| \chi(|x - y| < 1)$$

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with \( \chi \) the characteristic function. Hence it is
\[
\mathcal{F}_1(X^N(t), X^{N+1}(t), t) = \mathcal{F}'_1(X^N(t), X^{N+1}(t), t) + \mathcal{F}''_1(X^N(t), X^{N+1}(t), t).
\]

(2.49)

By Proposition 2 and Corollary 1 we have
\[
\mathcal{F}'_1(x, y, t) \leq |E^N(x, t)| + |E^N(y, t)| \leq CN^{-\frac{5-\varepsilon}{3}}
\]

(2.50)

By the range of the parameter \( \alpha \) it is \( 5-\varepsilon < 3\alpha \), so that we get
\[
\mathcal{F}'_1(x, y, t) \leq CN^{3\alpha} \leq CN^{3\alpha}|x-y|.
\]

(2.51)

Let us now estimate the term \( \mathcal{F}''_1(x, y, t) \).

We put \( \zeta = \frac{x+y}{2} \) and consider the sets:
\[
S_1 = \{ z : |\zeta - z| \leq 2|x-y| \},
\]
\[
S_2 = \left\{ z : 2|x-y| \leq |\zeta - z| \leq \frac{4}{|x-y|} \right\}
\]
\[
S_3 = \left\{ z : |\zeta - z| \geq \frac{4}{|x-y|} \right\}.
\]

(2.52)

We have:
\[
\mathcal{F}''_1(x, y, t) \leq \int_{S_1 \cup S_2 \cup S_3} \frac{1}{|x-z|^2} - \frac{1}{|y-z|^2} |\rho^N(z, t)| dz.
\]

(2.53)

By (2.40) and the definition of \( \zeta \) we have
\[
\int_{S_1} \frac{1}{|x-z|^2} - \frac{1}{|y-z|^2} |\rho^N(z, t)| dz \leq CN^{3\alpha} \int_{S_1} \left( \frac{1}{|x-z|^2} + \frac{1}{|y-z|^2} \right) dz \leq CN^{3\alpha}|x-y|.
\]

(2.54)

Let us pass to the integral over the set \( S_2 \). By the Lagrange theorem, there exists \( \xi_z \) such that \( \xi_z = \theta x + (1-\theta)y \) and \( \theta \in [0, 1] \) (depending on \( z \)), for which
\[
\int_{S_2} \frac{1}{|x-z|^2} - \frac{1}{|y-z|^2} |\rho^N(z, t)| dz \leq C|x-y| \int_{S_2} \frac{\rho^N(z, t)}{|\xi_z - z|^3} dz \leq C N^{3\alpha}|x-y| \int_{S_2} \frac{1}{|\xi_z - z|^3} dz
\]

(2.55)

Since in \( S_2 \) it results \( |\xi_z - z| \geq \frac{|x-y|}{2} \), we have
\[
\int_{S_2} \frac{1}{|\xi_z - z|^3} dz \leq 8 \int_{S_2} \frac{1}{|z|^3} dz \leq C(\log |x-y| + 1)
\]

(2.56)
and
\[
\int_{S_2} \left| \frac{1}{|x-z|^2} - \frac{1}{|y-z|^2} \right| \rho^N(z, t) \, dz \leq CN^{3\alpha}|x-y|(|\log|x-y||+1). \quad (2.57)
\]

For the last integral over $S_3$, again by the Lagrange theorem, we have for some $\xi = \theta x + (1-\theta)y$ and $\theta \in [0,1],$
\[
\int_{S_3} \left| \frac{1}{|x-z|^2} - \frac{1}{|y-z|^2} \right| \rho^N(z, t) \, dz \leq C|x-y| \int_{S_3} \frac{\rho^N(z, t)}{|\xi_z - z|^3} \, dz. \quad (2.58)
\]
Notice that, since $z \in S_3$ and $|x-y| < 1$ by definition of $F''_1$, it is $|\xi_z - z| \geq |\zeta - z| - |x-y|,$
and
\[
\frac{1}{|\xi_z - z|} \leq \frac{1}{|\zeta - z| - |x-y|} = \frac{1}{\frac{3}{4}|\zeta - z| + \frac{1}{4}|\zeta - z| - |x-y|} \leq \frac{4}{3}|\zeta - z|,
\]
then we have
\[
\int_{S_3} \rho^N(z, t) \, dz \leq C \int_{|\zeta - z|\geq 4} \frac{\rho^N(z, t)}{|\zeta - z|^3} \, dz. \quad (2.59)
\]

Lemmas 2, 3 and (2.34) imply
\[
\int_{S_3} \frac{\rho^N(z, t)}{|\xi_z - z|^3} \, dz \leq C \int_{|\zeta - z|\geq 4} \frac{\rho^N(z, t)}{|\zeta - z|^3} \, dz \leq CQ^N(R^N(t), t) + C \quad (2.60)
\]
and by (2.41) we get
\[
\int_{S_3} \frac{\rho^N(z, t)}{|\xi_z - z|^3} \, dz \leq CN^\frac{3}{2}(1-\epsilon). \quad (2.61)
\]

Using this estimate in (2.58) and going back to (2.53), by (2.61) and (2.57) we have proved that
\[
F''_1(x, y, t) \leq CN^{3\alpha}|x-y|(|\log|x-y||+1). \quad (2.62)
\]

In conclusion, recalling (2.49), estimates (2.51) and (2.62) show that
\[
F_1(X^N(t), X^{N+1}(t), t) \leq CN^{3\alpha} \delta^N(t) (1 + |\log \delta^N(t)|). \quad (2.63)
\]

**Estimate of the term $F_2$**

Putting $\tilde{X} = X^{N+1}(t)$, we have:
\[
F_2(\tilde{X}, t) \leq F'_2(\tilde{X}, t) + F''_2(\tilde{X}, t), \quad (2.64)
\]
where

\[
\mathcal{F}'_2(\bar{X}, t) = \left| \int_{|\bar{X} - y| \leq 2\delta^N(t)} \frac{\rho^N(y, t) - \rho^{N+1}(y, t)}{|X - y|^2} \, dy \right| \quad (2.65)
\]

\[
\mathcal{F}''_2(\bar{X}, t) = \left| \int_{2\delta^N(t) \leq |\bar{X} - y|} \frac{\rho^N(y, t) - \rho^{N+1}(y, t)}{|X - y|^2} \, dy \right| \quad (2.66)
\]

By estimate (2.40) we get

\[
\mathcal{F}'_2(\bar{X}, t) \leq \int_{|\bar{X} - y| \leq 2\delta^N(t)} \frac{\rho^N(y, t) + \rho^{N+1}(y, t)}{|X - y|^2} \, dy \leq C N^{3\alpha} \delta^N(t). \quad (2.67)
\]

Now we estimate the term \( \mathcal{F}''_2 \). By the invariance of \( f^N(t) \) along the characteristics, making a change of variables, we decompose the integral as follows:

\[
\mathcal{F}''_2(\bar{X}, t) = \left| \int_{S^N(t)} dy \, dw \frac{f_0^N(y, w)}{|X - Y^N(t)|^2} - \int_{S^{N+1}(t)} dy \, dw \frac{f_0^{N+1}(y, w)}{|X - Y^{N+1}(t)|^2} \right| \quad (2.68)
\]

where we put, for \( i = N, N + 1 \),

\[
(Y^i(t), W^i(t)) = (X^i(y, w, t), V^i(y, w, t))
\]

\[
S^i(t) = \{(y, w) : 2\delta^N(t) \leq |X - Y^i(t)|\}.
\]

We notice that it is

\[
\mathcal{F}''_2(\bar{X}, t) \leq \mathcal{I}_1(\bar{X}, t) + \mathcal{I}_2(\bar{X}, t) + \mathcal{I}_3(\bar{X}, t) \quad (2.69)
\]

where

\[
\mathcal{I}_1(\bar{X}, t) = \int_{S^N(t)} dy \int dw \frac{1}{|X - Y^N(t)|^2} - \frac{1}{|X - Y^{N+1}(t)|^2} \left| f_0^N(y, w) \right|,
\]

\[
\mathcal{I}_2(\bar{X}, t) = \int_{S^{N+1}(t)} dy \int dw \frac{|f_0^N(y, w) - f_0^{N+1}(y, w)|}{|X - Y^{N+1}(t)|^2} \quad (2.70)
\]

\[
\mathcal{I}_3(\bar{X}, t) = \int_{S^N(t) \setminus S^{N+1}(t)} dy \int dw \frac{f_0^N(y, w)}{|X - Y^{N+1}(t)|^2} \quad (2.71)
\]

We start by estimating \( \mathcal{I}_1(\bar{X}, t) \). By the Lagrange theorem

\[
\mathcal{I}_1(\bar{X}, t) \leq C |Y^N(t) - Y^{N+1}(t)| \int_{S^N(t)} dy \int dw \frac{|f_0^N(y, w)|}{|X - \xi^N(t)|^2} \quad (2.72)
\]

where \( \xi^N(t) = \theta Y^N(t) + (1 - \theta) Y^{N+1}(t) \) for \( \theta \in [0, 1] \). By putting

\[
(\bar{y}, \bar{w}) = (Y^N(t), W^N(t))
\]
and
\[ S^N(t) = \{ (\bar{y}, \bar{w}) : (y, w) \in S^N(t) \}, \]
we get
\[ \mathcal{I}_1(\bar{X}, t) \leq C\delta^N(t) \int_{S^N(t)} d\bar{y} \int d\bar{w} \frac{f^N(\bar{y}, \bar{w}, t)}{|\bar{X} - \bar{y}|^3}. \quad (2.74) \]

If \((y, w) \in S^N(t)\) then
\[ |\bar{X} - \bar{N}(t)| > |\bar{X} - \bar{y}| - |\bar{y} - Y^{N+1}(t)| \geq |\bar{X} - \bar{y}| - \delta^N(t) \geq \frac{|\bar{X} - \bar{y}|}{2} \]
which, by (2.74), implies
\[ \mathcal{I}_1(\bar{X}, t) \leq C\delta^N(t) \int_{S^N(t)} d\bar{y} \int d\bar{w} f^N(\bar{y}, \bar{w}, t)|\bar{X} - \bar{y}|^3. \quad (2.75) \]

Now we consider the sets
\[ \begin{align*}
A_1 &= \left\{ (\bar{y}, \bar{w}) : 2\delta^N(t) \leq |\bar{X} - \bar{y}| \leq \frac{4}{\delta^N(t)} \right\} \\
A_2 &= \left\{ (\bar{y}, \bar{w}) : 1 \leq |\bar{X} - \bar{y}| \leq 3R^N(t) \right\} \\
A_3 &= \left\{ (\bar{y}, \bar{w}) : 3R^N(t) \leq |\bar{X} - \bar{y}| \right\}.
\end{align*} \]

Then it is \(S^N(t) \subset \bigcup_{i=1,2,3} A_i\), and
\[ \mathcal{I}_1(\bar{X}, t) \leq C\delta^N(t) \sum_{i=1}^3 \int_{A_i} d\bar{y} \frac{\rho^N(\bar{y}, t)}{|\bar{X} - \bar{y}|^3}. \quad (2.76) \]

We estimate the integral over \(A_1\) as we did in (2.55), the one over \(A_2\) by means of Lemma 2 and the last one by means of Lemma 3, yielding
\[ \mathcal{I}_1(\bar{X}, t) \leq C\delta^N(t) \left[ N^{3\alpha} |\log \delta^N(t)| + Q^N (3R^N(t), t) \right] + 1 \quad (2.77) \]

Equation (2.34) implies that
\[ \mathcal{I}_1(\bar{X}, t) \leq C\delta^N(t) \left[ N^{3\alpha} |\log \delta^N(t)| + Q^N (R^N(t), t) \right] + 1 \quad (2.78) \]
and by (2.41) in Corollary 1 we get
\[ \mathcal{I}_1(\bar{X}, t) \leq CN^{3\alpha} \delta^N(t) \left[ |\log \delta^N(t)| + 1 \right]. \quad (2.79) \]
We estimate now the term \( I_2(\bar{X}, t) \). By the definition of \( f_0^i \), for \( i = N, N + 1 \), and by (1.4) it follows

\[
I_2(\bar{X}, t) = \int_{S^{N+1}(t)} dy \int_{\mathbb{R}} \frac{f_0^{N+1}(y, w)}{|X - Y^{N+1}(t)|^2} \chi (|w| \leq N + 1) \leq C_0 e^{-\lambda N^2} \int_{S^{N+1}(t)} dy \int_{\mathbb{R}} \frac{g(|y|)}{|X - Y^{N+1}(t)|^2} \chi (|w| \leq N + 1). \tag{2.80}
\]

We evaluate the integral over \( S^{N+1}(t) \) by considering the sets

\[
B_1 = \{ (y, w) : |\bar{X} - Y^{N+1}(t)| \leq 4R^{N+1}(t) \}
\]

\[
B_2 = \{ (y, w) : 4R^{N+1}(t) \leq |\bar{X} - Y^{N+1}(t)| \},
\]

so that

\[
S^{N+1}(t) \subset \bigcup_{i=1,2} B_i. \tag{2.81}
\]

By putting

\[
(\bar{y}, \bar{w}) = (Y^{N+1}(t), W^{N+1}(t)),
\]

by (2.39) it is \(|\bar{w}| \leq CN\), so that we have

\[
\int_{B_1} dy \int_{\mathbb{R}} \frac{g(|y|)}{|X - Y^{N+1}(t)|^2} \chi (|w| \leq N + 1) \leq C \int_{|y| \leq CN} d\bar{w} \int_{|X - y| \leq 4R^{N+1}(t)} dy \frac{1}{|X - y|^2} \leq CN^3 R^{N+1}(t) \leq CN^4. \tag{2.82}
\]

For the integral over the set \( B_2 \), we observe that, if \(|\bar{X} - Y^{N+1}(t)| \geq 4R^{N+1}(t)\), then \(|\bar{X} - y| \geq 3R^{N+1}(t)\) and

\[
|\bar{X} - Y^{N+1}(t)| \geq |\bar{X} - y| - R^{N+1}(t) \geq \frac{2}{3}|\bar{X} - y|.
\]

Hence

\[
\int_{B_2} dy \int_{\mathbb{R}} \frac{g(|y|)}{|X - Y^{N+1}(t)|^2} \chi (|w| \leq N + 1) \leq \frac{9}{4} \int_{|X - y| \geq 3R^{N+1}(t)} dy \int_{|w| \leq N+1} dw \frac{g(|y|)}{|X - y|^2} \leq CN^3 \int_{|X - y| \geq 3R^{N+1}(t)} dy \frac{g(|y|)}{|X - y|^2}. \tag{2.83}
\]

The last integral is smaller than the analogous one already estimated in (2.27) and (2.28), and is bounded by a constant. Hence we can conclude that

\[
\int_{B_2} dy \int_{\mathbb{R}} \frac{g(|y|)}{|X - Y^{N+1}(t)|^2} \chi (|w| \leq N + 1) \leq CN^3. \tag{2.84}
\]

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Going back to (2.80), by (2.81), (2.82) and (2.84) we have
\[ I_2(\bar{x}, t) \leq C e^{-\lambda N^2} \left( N^4 + N^3 \right) \leq C e^{-\frac{1}{2} N^2}. \]  
(2.85)

Finally, we estimate the term \( I_3(\bar{x}, t) \). If \((y, w) \in S^N(t)\), then
\[ |\bar{x} - Y^{N+1}(t)| \geq |\bar{x} - Y^N(t)| - \delta^N(t) \geq \delta^N(t), \]
so that
\[
I_3(\bar{x}, t) = \int_{S^N(t) \setminus S^{N+1}(t)} dy \int dw \frac{f_0^N(y, w)}{|\bar{x} - Y^{N+1}(t)|^2} \leq \int_{(S^{N+1}(t))^c} dy \int dw f_0^N(y, w). 
\]  
(2.86)

If \((y, w) \in (S^{N+1}(t))^c\), then
\[ |\bar{x} - Y^N(t)| \leq |\bar{x} - Y^{N+1}(t)| + \delta^N(t) \leq 3\delta^N(t). \]

Hence, by putting
\[ (\bar{y}, \bar{w}) = (Y^N(t), W^N(t)) \]
we get
\[
I_3(\bar{x}, t) \leq \frac{1}{[\delta^N(t)]^2} \int_{|\bar{x} - \bar{y}| \leq 3\delta^N(t)} d\bar{y} \int d\bar{w} f^N(\bar{y}, \bar{w}, t) \leq \frac{1}{[\delta^N(t)]^2} \int_{|\bar{x} - \bar{y}| \leq 3\delta^N(t)} d\bar{y} \rho(\bar{y}, t) \leq C N^{3\alpha} \delta^N(t). 
\]  
(2.87)

Going back to (2.69), by (2.79), (2.85) and (2.87) we have
\[ F''(\bar{x}, t) \leq C \left[ N^{3\alpha} \delta^N(t) (|\log \delta^N(t)| + 1) + e^{-\frac{1}{2} N^2} \right], \]  
(2.88)
so that, by (2.64), (2.67) and (2.88), we get
\[ F_2(\bar{x}, t) \leq C \left[ N^{3\alpha} \delta^N(t) (|\log \delta^N(t)| + 1) + e^{-\frac{1}{2} N^2} \right]. \]  
(2.89)

Finally, (2.65), (2.63) and (2.89) conclude the proof of the proposition.
3 Proofs of Theorems 1 and 2

3.1 Proof of Theorem 1

We put \( V^i(t) = V^i(x,v,t) \) for \( i = N, N+1 \) and define

\[
\eta^N(t) = \sup_{(x,v) \in \mathbb{R}^3 \times B(0,N)} |V^N(t) - V^{N+1}(t)|
\]

and

\[
\sigma^N(t) = \max\{\delta^N(t), \eta^N(t)\}.
\]

Proposition 4 enables us to state the following result:

**Proposition 5.** There exists a constant \( C > 1 \) such that

\[
\sup_{t \in [0,T]} \sigma^N(t) \leq C^{-CN}.
\]  \( (3.1) \)

**Proof.** We start the proof for \( \delta^N(t) \). To prove this proposition we need to iterate the estimate \((2.45)\), inserting into the integral the same estimate for \( \delta^N(t_2) \) with \( t_2 \leq t \). Unfortunately, being \( E^N \) not Lipschitz continuous, estimate \((2.45)\) is not linear in \( \delta^N(t_2) \). However, it is easily seen that for any \( r \in (0,1) \) and \( a \in (0,1) \) the following inequality holds by convexity:

\[
r(|\log r| + 1) \leq r |\log a| + a.
\]

Hence, for \( \delta^N(t_2) < 1 \), Proposition 4 gives us, for any \( a < 1 \),

\[
\delta^N(t) \leq C \int_0^t dt_1 \int_0^{t_1} dt_2 \left[ N^{3\alpha} \left( \delta^N(t_2) |\log a| + a \right) + e^{-\frac{3}{2} N^2} \right].
\]  \( (3.2) \)

On the other side, were \( \delta^N(t_2) \geq 1 \), Corollary 1 and Proposition 3 would provide a bound like

\[
\sup_{t \in [0,T]} \delta^N(t) \leq C N
\]  \( (3.3) \)

and in that case estimate \((2.45)\) would be simply

\[
\delta^N(t) \leq C \int_0^t dt_1 \int_0^{t_1} dt_2 \left[ N^{3\alpha} \log N \delta^N(t_2) + e^{-\frac{3}{2} N^2} \right].
\]  \( (3.4) \)

In case \( \delta^N(t_2) < 1 \), we choose \( a = e^{-\frac{3}{2} N^2} \) in \((3.2)\), yielding

\[
\delta^N(t) \leq C \int_0^t dt_1 \int_0^{t_1} dt_2 \left[ N^{3\alpha+2} \delta^N(t_2) + N^{3\alpha} e^{-\frac{3}{2} N^2} \right] \leq C \int_0^t dt_1 \int_0^{t_1} dt_2 \left[ N^{3\alpha+2} \delta^N(t_2) + e^{-\frac{3}{2} N^2} \right].
\]  \( (3.5) \)
Note that the bound for $\delta^N(t_2) \geq 1$ is included in that for $\delta^N(t_2) < 1$.
Now we are in condition to iterate inequality (2.45), by inserting in the integral the same estimate for $\delta^N(t_2)$. We make $k$ iterations up to time $t_{2k+2}$ and at the last step we use the estimate (3.3). Notice that at any step we get a double factorial at the denominator, due to the double time integration, obtaining
\[
\delta^N(t) \leq CN \left[ e^{-\frac{1}{4}N^2} \sum_{i=1}^{k} \frac{C^i N^{(3\alpha+2)i} T^{2i}}{(2i)!} + \frac{C^k N^{(3\alpha+2)k} T^{2k}}{(2k)!} \right].
\] (3.6)
The latter term is exponentially vanishing as $N \to \infty$, provided $k$ has been chosen sufficiently large in function of $N$, (for instance $k > N^{(3\alpha+2)}$).
Hence
\[
\delta^N(t) \leq CNe^{-\frac{1}{4}N^2} e^{\sqrt{CT} N (\frac{3\alpha+1}{2})} + C^{-N}.
\] (3.8)
We stress the fact that the double factorial, coming from the fact that the characteristics equation is a second order equation, allows us to halve the exponent of the series. On the other hand, by Proposition 3, it is \(\frac{3\alpha+1}{2} < 2\), so that we have
\[
\delta^N(t) \leq C^{-CN}.
\] (3.9)

The same arguments used in Proposition 4 allow us to prove, in analogy with (3.5), that
\[
\eta^N(t) \leq C \int_0^t dt_1 \left[ N^{3\alpha+2} \delta^N(t_1) + e^{-\frac{1}{4}N^2} \right] \leq CN^{3\alpha+2} \int_0^t dt_1 \int_0^{t_1} dt_2 \eta^N(t_2) + \int_0^t dt_1 e^{-\frac{1}{4}N^2},
\] (3.10)
so that we can proceed as above in proving that
\[
\eta^N(t) \leq C^{-CN},
\] (3.11)
which concludes the proof of the proposition.

In the previous Proposition we have shown that
\[
\max \left\{ \sum_{N=1}^{\infty} \delta^N(t), \sum_{N=1}^{\infty} \eta^N(t) \right\} = C.
\]
This implies that the sequences $\{X^N(x, v, t)\}$ and $\{V^N(x, v, t)\}$ are Cauchy sequences and then, for any fixed $(x, v)$, they are bounded uniformly in $N$. 22
Indeed, let us fix an initial datum \((x, v)\) and choose a positive integer \(N_0\) such that \(N_0 = \text{intg}(|v| + C)\), where \(\text{intg}(a)\) represents the integer part of the number \(a\) and \(C\) is assumed sufficiently large. Then, for any \(N > N_0\), by Proposition 5 and (2.39) in Corollary 1 we have

\[
|V^N(t) - v| \leq |V^{N_0}(t) - v| + \sum_{k=N_0}^{N-1} \eta^k(t) \leq |V^{N_0}(t) - v| + C N_0
\]

which implies, by the choice of \(N_0\),

\[
|V^N(t)| \leq C(|v| + 1)
\]

and hence

\[
|X^N(t)| \leq |x| + C(|v| + 1).
\]

Since for any fixed \((x, v)\) the sequences \(\{X^N(t)\}\), \(\{V^N(t)\}\), and consequently \(\{f^N(t)\}\), are Cauchy sequences, they converge, as \(N \to \infty\), to some limit functions, which we call \(X(x, v, t)\), \(V(x, v, t)\), \(f(x, v, t)\).

Properties (3.13) and (3.14) allow to show that the functions \(f^N(t)\) enjoy properties (1.6) and (1.7), with constants \(C\) and \(\bar{\lambda}\) independent of \(N\). Indeed,

\[
f^N(X^N(t), V^N(t), t) = f_0^N(x, v) \leq C_0 e^{-\lambda |v|^2} g(x) \leq C e^{-\bar{\lambda}|v^N(t)|^2}.
\]

To prove that \(\rho^N(t)\) satisfies (1.7) uniformly in \(N\), we fix \(i \in \mathbb{Z}^3 \setminus \{0\}\) and we decompose the velocity space into the sets \(S_i\) and \(S_i^c\), where

\[
S_i = \left\{ v \in \mathbb{R}^3 : |v| \geq a_i = \sqrt{\frac{2(2 + \epsilon) \log |i|}{\lambda}} \right\}.
\]

Hence for any such \(i\) it is

\[
\int_{|v| \leq 1} \rho^N(x, t) \, dx = \int_{|v| \leq 1} dx dv f^N(x, v, t) = \int_{|v| \leq 1} \left[ \int_{S_i} f^N(x, v, t) \, dv + \int_{S_i^c} f^N(x, v, t) \, dv \right].
\]

By (3.15) and the choice of \(a_i\) we get

\[
\int_{|v| \leq 1} dx \int_{S_i} dv f^N(x, v, t) \leq C \int_{|v| \leq 1} dx \int_{S_i} e^{-\lambda |v|^2} dv \leq C \int_{|v| \leq 1} e^{-\bar{\lambda}|v|^2} dv \leq \frac{C}{|v|^{2+\epsilon}}.
\]
On the other side, by (3.14) we have

\[ \int_{|i-x| \leq 1} \sum_{k \in \mathbb{Z}^3 \atop |k| \leq C_{a_i}} \int_{S_i^c} g(x) \, dx \leq C \sum_{k \in \mathbb{Z}^3 \atop |k| \leq C_{a_i}} \frac{1}{|i+k|^{2+\epsilon}}. \]  

(3.18)

Since \(|k| \leq C_{a_i}\), by the definition of \(a_i\) it follows that for large \(|i|\) it is \(|i+k| \geq \frac{|i|}{2}\), and then

\[ \sum_{k \in \mathbb{Z}^3 \atop |k| \leq C_{a_i}} \frac{1}{|i+k|^{2+\epsilon}} \leq 2^{2+\epsilon} \sum_{k \in \mathbb{Z}^3 \atop |k| \leq C_{a_i}} \frac{1}{|i|^{2+\epsilon}} \leq C \left(\frac{\log |i|}{|i|^{2+\epsilon}}\right)^2. \]

so that we get

\[ \int_{|i-x| \leq 1} \sum_{k \in \mathbb{Z}^3 \atop |k| \leq C_{a_i}} \int_{S_i^c} g(x) \, dx \leq C \left(\frac{\log |i|}{|i|^{2+\epsilon}}\right)^2 \]  

(3.19)

which, together with (3.16) and (3.17), proves (1.7). This implies that also the limit function \(f(x,v,t)\) satisfies the same properties (1.6) and (1.7).

It remains to prove that the couple \((X(x,v,t),V(x,v,t))\) satisfies the right equation, that is we have to show that \(|E^N(x,t) - E(x,t)| \to 0\) as \(N \to \infty\), with \(E(x,t)\) defined in (1.1). To prove this, we note first that from estimate (3.15) it follows

\[ \|\rho^N(t)\|_{L^\infty} \leq C. \]  

(3.20)

The term \(|E^N(x,t) - E(x,t)|\) can be handled as the term \(\mathcal{F}_2\) in the previous section, by using the bound (3.20), yielding

\[ |E^N(x,t) - E(x,t)| \leq C \left[ \delta^N(t) \left(1 + |\log \delta^N(t)|\right) + e^{-\frac{\delta^N(t)}{2}} \right] \]  

(3.21)

which is infinitesimal as \(N \to \infty\), by Proposition 5. Uniqueness could be proved along the same lines and this concludes the proof of the Theorem.

### 3.2 Proof of Theorem 2

To prove Theorem 2 we only need to prove that (1.10) holds for any \(t \in [0,T]\), under the hypotheses (1.8) and (1.9), since existence and uniqueness of the solution has already been proved in the preceding subsection. To this purpose, we write again:

\[ f^N(X^N(t),V^N(t),t) = f_0^N(x,v) \leq f_0(x,v) \leq C_0 e^{-\lambda|x|^2} g(|x|). \]  

(3.22)
Calling $C^*$ the constant appearing in (3.14), we see that
\[ |x| \geq |X^N(t)| - C^* (|v| + 1). \] (3.23)
Now we consider the two possible cases:
\[ C^* (|v| + 1) \leq \frac{1}{2} |X^N(t)| \] (3.24)
and
\[ C^* (|v| + 1) \geq \frac{1}{2} |X^N(t)|. \] (3.25)
In the first case, by (3.23) it is
\[ |x| \geq \frac{1}{2} |X^N(t)| \]
and consequently, by the properties of $g$ defined in (1.9) we have
\[ g(|x|) \leq g \left( \frac{1}{2} |X^N(t)| \right) \leq C g (|X^N(t)|). \] (3.26)
From here, going back to (3.22) and recalling (3.13), there exists $\lambda' > 0$ such that
\[ f^N (X^N(t), V^N(t), t) \leq C e^{-\lambda' |V^N(t)|^2} g (|X^N(t)|) . \] (3.27)
Let us now consider the second case. By using again (3.13) in (3.22), we have
\[ f^N (X^N(t), V^N(t), t) \leq C e^{-\frac{1}{2} |v|^2} e^{-\lambda' |V^N(t)|^2} g(|x|) \leq C e^{-\frac{1}{2} |v|^2} e^{-\lambda' |V^N(t)|^2} . \]
Notice that in the case at hand it is $|v| \geq C |X^N(t)|$ and hence
\[ f^N (X^N(t), V^N(t), t) \leq C e^{-C |X^N(t)|^2} e^{-\lambda' |V^N(t)|^2} . \]
Since for any positive $r$ it results $e^{-r^2} \leq \frac{C}{r^{2r}},$ we have also in this case
\[ f^N (X^N(t), V^N(t), t) \leq C e^{-\lambda' |V^N(t)|^2} g (|X^N(t)|) \] (3.28)
provided the constant in (1.9) is sufficiently large.
Estimates (3.27) and (3.28) conclude the proof of Theorem 2.

4 Proof of Proposition 3

4.1 Proof of Proposition 3 for $\epsilon > 13/19$
This proposition will be proved in complete analogy to what done in [5], with the only effort to lower the exponent of the estimate, from 1 to 2/3.
We repeat here the proof, choosing accurately some parameters, and from now on we simplify the notation skipping the index $N$. We also put

$$P = V^N(T)$$

$$Q = \sup_{t \in [0,T]} Q^N(R^N(t), t).$$

We define

$$\beta = 1 - \epsilon,$$

being $\epsilon > 1/15$ the parameter in (1.5), consequently it is $\beta < 14/15$. We divide the proof in a first part in which $\beta < 6/19$ (that is, $\epsilon > 13/19$), which does not require an iterative method on a suitable time average, and after we improve the result up to $\beta < 14/15$ (that is, $\epsilon > 1/15$) by an iterative method.

We fix a time interval

$$\Delta = \frac{1}{4C_2P^{\frac{\beta}{4}}Q^{\frac{1}{4}}}$$

where $C_2$ is the constant in (2.29) and $\gamma$ is any number satisfying

$$\frac{4}{3} \beta < \gamma < \frac{2 - \beta}{4}$$

(the reason for this range of the parameter $\gamma$ will be clear in the following; we remark that when $\beta \to \frac{6}{19}$ the lower bound tends to the upper bound). We note that such choice assures that $\Delta \ll T$ (taking the constant $\tilde{C}$ in (2.16) suitably large); indeed it is $\gamma < \frac{4}{7}$ and $P$ is defined to be greater than $\tilde{C}$.

Let us consider two solutions to the $N$-truncated system, $(X(t), V(t))$ and $(Y(t), W(t))$. The following results, whose proofs are given in the Appendix, hold.

**Lemma 4.** Let $t' \in [0, T]$.

If $|V(t') - W(t')| \leq P^\gamma$

then

$$\sup_{t \in [t', t' + \Delta]} |V(t) - W(t)| \leq 2P^T.$$

(4.6)

If $|V(t') - W(t')| \geq P^\gamma$

then

$$\inf_{t \in [t', t' + \Delta]} |V(t) - W(t)| \geq \frac{1}{2}P^\gamma$$

(4.7)
Lemma 5. Let $t' \in [0, T]$ and assume that $|V(t') - W(t')| \geq P^\gamma$. Then there exists $t_0 \in [t', t' + \Delta]$ such that for any $t \in [t', t' + \Delta]$:

$$|X(t) - Y(t)| \geq \frac{P^\gamma}{4}|t - t_0|.$$ 

Let us divide the interval $[0, T]$ into $n$ sub-intervals $[t_i, t_{i+1}]$, $i = 0, 1, \ldots, n-1$, with $t_0 = 0$, $t_n = T$ and $\frac{1}{2}\Delta \leq t_{i+1} - t_i \leq \Delta$. Hence it is:

$$\int_0^t |E(X(s), s)| \, ds = \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} |E(X(s), s)| \, ds. \quad (4.8)$$

Fixing the index $i$, we consider the time evolution of the system over the time interval $[t_i, t_{i+1}]$. We have

$$|E(X(t), t)| \leq \int dy dw \frac{f(y, w, t)}{|X(t) - y|^2} \quad (4.9)$$

and we decompose the phase space $(y, w) \in \mathbb{R}^3 \times \mathbb{R}^3$ in the following way, being $\gamma$ the positive parameter introduced in (4.4), (4.5):

$$A = \{(y, w) : |X(t_i) - y| \leq 4R(T)\} \quad (4.10)$$

$$B_1 = \{(y, w) : |V(t_i) - w| \leq P^\gamma\} \quad (4.11)$$

$$B_2 = \{(y, w) : |V(t_i) - w| > P^\gamma\}. \quad (4.12)$$

Hence we have, for any $t \in [t_i, t_{i+1}]$,

$$|E(X(t), t)| \leq \sum_{j=1}^{3} \mathcal{I}_j(X(t)) \quad (4.13)$$

where for $j = 1, 2$

$$\mathcal{I}_j(X(t)) = \int_{A \cap B_j} dy dw \frac{f(y, w, t)}{|X(t) - y|^2}$$

and

$$\mathcal{I}_3(X(t)) = \int_{A_c} dy dw \frac{f(y, w, t)}{|X(t) - y|^2}$$

being

$$A_c = \{(y, w) : |X(t_i) - y| > 4R(T)\}.$$

Let us start by the first integral. Proceeding as in the proof of (2.37), with $\mathcal{I}_1(X(t))$ in place of $\mathcal{J}_1(X(t), t) + \mathcal{J}_2(X(t), t)$, and enlarging the region of integration to $A' = \{(y, w) : |X(t) - y| \leq 5R(T)\}$, we get

$$\mathcal{I}_1(X(t)) \leq CP^\frac{3}{\gamma}Q^{\frac{3}{2}}. \quad (4.14)$$
Now we analyse the term $I_2$. We introduce the sets $C_k(t)$ and $D_k(t)$, $t \in [t_i, t_{i+1}]$, with $k = 0, 1, 2, \ldots, m$, which constitute a finer decomposition of the set $B_2$ and are defined in the following way:

$$C_k(t) = \{ (y, w) : \alpha_{k+1} < |V(t_i) - w| \leq \alpha_k, |X(t) - y| \leq l_k \} \quad (4.15)$$

$$D_k(t) = \{ (y, w) : \alpha_{k+1} < |V(t_i) - w| \leq \alpha_k, |X(t) - y| > l_k \} \quad (4.16)$$

and we choose the parameters $\alpha_k$ and $l_k$ as:

$$\alpha_k = \frac{P}{2^k} \quad l_k = \frac{2^{3k} Q^{\frac{3}{4}}}{P^{\frac{5}{4}} + \eta} \quad (4.17)$$

with $\eta$ any number satisfying

$$\frac{3 + \beta}{3} < \eta < 1 + \gamma - \beta \quad (4.18)$$

(the reason for this range of the parameter $\eta$ will be clear in the following; we remark that, due to the choice of $\gamma$ (4.5), this interval is well defined, and when $\beta \to \frac{6}{19}$ the lower bound tends to the upper bound). Consequently we put

$$I_2(X(t)) \leq \sum_{k=0}^{m} (I'_2(k) + I''_2(k)) \quad (4.19)$$

being

$$I'_2(k) = \int_{A \cap C_k(t)} f(y, w, t) \frac{1}{|X(t) - y|^2} \, dydw \quad (4.20)$$

and

$$I''_2(k) = \int_{A \cap D_k(t)} f(y, w, t) \frac{1}{|X(t) - y|^2} \, dydw. \quad (4.21)$$

By the choice of the parameters $\alpha_k$ and $l_k$ made in (4.17), we have:

$$I'_2(k) \leq C l_k \int_{\alpha_{k+1} < |V(t_i) - w| \leq \alpha_k} dw \leq C l_k \alpha_k^3 \leq C P^{\frac{3}{4} - \eta} Q^{\frac{1}{4}}. \quad (4.22)$$

Hence

$$\sum_{k=0}^{m} I'_2(k) \leq C P^{\frac{3}{4} - \eta} Q^{\frac{1}{4}} \log P, \quad (4.23)$$

since, considering we are in the set $B_2$, it is

$$m \leq (1 - \gamma) \log_2 P. \quad (4.24)$$

Now we pass to $I''_2(k)$, for which we need to make the time integration over the interval $[t_i, t_{i+1}]$. We set briefly

$$(Y(t), W(t)) := (Y(t, t_i, \bar{y}, \bar{w}), W(t, t_i, \bar{y}, \bar{w}))$$
being

\[(Y(t_i), W(t_i)) = (\bar{y}, \bar{w}),\]

hence we have

\[
\int_{t_i}^{t_{i+1}} I''_2(k) \, dt \leq \int_{t_i}^{t_{i+1}} \int_{A \cap D_k(t)} \frac{f(y, w, t)}{|X(t) - y|^2} \, dy \, dw =
\]

\[
\int_{t_i}^{t_{i+1}} \int_{A \cap D_k(t)} \frac{f(\bar{y}, \bar{w}, t_i)}{|X(t) - Y(t)|^2} \, \chi(|X(t) - Y(t)| > l_k) \, d\bar{y} \, d\bar{w}.
\]

(4.25)

under the change of variables \((y, w) = (Y, W)(t, t_i, \bar{y}, \bar{w})\), being \(A \cap D_k(t)\) the backward in time evolved set of \(A \cap D_k(t)\) at time \(t_i\), recalling the invariance of \(f\) along the characteristics and the conservation of the measure of the phase space.

We note that we enlarge the set \(A \cap D_k(t)\) by integrating over the set \(A' \cap D'_k\), where

\[
A' = \{(\bar{y}, \bar{w}) : |X(t_i) - \bar{y}| \leq 5R(T)\},
\]

\[
D'_k = \{(\bar{y}, \bar{w}) : \frac{\alpha_{k+1}}{4} < |V(t_i) - \bar{w}| \leq 2\alpha_k\},
\]

since by the definition of the maximum displacement \(R(T)\) and by Lemma 4 (slightly adapted), particles belonging to \(A\) and \(D_k(t)\) at time \(t_i\) come necessarily from \(A'\) and \(D'_k\) at time \(t_i\). This observation allows us to change the order of integration in (4.25), arriving at

\[
\int_{t_i}^{t_{i+1}} I''_2(k) \, dt \leq \int_{A' \cap D'_k} f(\bar{y}, \bar{w}, t_i) \left(\int_{t_i}^{t_{i+1}} \frac{\chi(|X(t) - Y(t)| > l_k)}{|X(t) - Y(t)|^2} \, dt\right) \, d\bar{y} \, d\bar{w}.
\]

(4.26)

By rephrasing Lemma 4 in this case it is easily seen that it holds:

\[
\forall (\bar{y}, \bar{w}) \in D'_k \quad |V(t) - W(t)| \geq \frac{\alpha_{k+1}}{4}
\]

(4.27)

and consequently, by Lemma 5 there exists \(t_0 \in [t_i, t_{i+1}]\) such that, for any \(t \in [t_i, t_{i+1}]\):

\[
|X(t) - Y(t)| \geq \frac{\alpha_{k+1}}{16} |t - t_0|.
\]

(4.28)
Hence, putting \( a = l_k/\alpha_k \), we have:

\[
\int_{t_i}^{t_{i+1}} \frac{\chi([X(t) - Y(t)] > l_k)}{|X(t) - Y(t)|^2} \, dt \leq 0
\]

\[
\int_{\{t:|t-t_0| \leq a\}} \frac{\chi([X(t) - Y(t)] > l_k)}{|X(t) - Y(t)|^2} \, dt + \int_{\{t:|t-t_0| > a\}} \frac{\chi([X(t) - Y(t)] > l_k)}{|X(t) - Y(t)|^2} \, dt \leq (4.29)
\]

\[
\frac{1}{l_k^2} \int_{\{t:|t-t_0| \leq a\}} \frac{1}{t^2} \, dt + \frac{C}{\alpha_{k+1}^2} \int_{\{t:|t-t_0| > a\}} \frac{1}{|t-t_0|^2} \, dt \leq \frac{2a}{l_k^2} + \frac{C}{\alpha_{k+1}^2} \int_{a}^{\infty} \frac{1}{t^2} \, dt = \frac{C}{\alpha_k l_k}.
\]

Moreover, defining \( \tilde{y} = \hat{y}, \tilde{w} = \hat{w} - V(t_i) \), and

\[
D_k'' = \{(\tilde{y}, \tilde{w}) : \frac{\alpha_{k+1}}{2} < |\tilde{w}| \leq 2\alpha_k\},
\]

\[
\int_{A \cap D_k''} f(\tilde{y}, \tilde{w}, t_i) \, d\tilde{y}d\tilde{w} \leq \frac{C}{\alpha_k^2} \int_{A \cap D_k''} \tilde{w}^2 f(\tilde{y}, \tilde{w}, t_i) \, d\tilde{y}d\tilde{w}
\]

(4.30)

so that by (4.25), (4.26), (4.29) and (4.30), taking into account (4.17), we get:

\[
\sum_{k=0}^{m} \int_{t_i}^{t_{i+1}} I_k''(k) \leq \sum_{k=0}^{m} \frac{C}{\alpha_k l_k} \int_{A \cap D_k''} \tilde{w}^2 f(\tilde{y}, \tilde{w}, t_i) \, d\tilde{y}d\tilde{w} \leq \frac{C}{P_{\frac{1}{2} - \eta} Q_{\frac{1}{2}}} \sum_{k=0}^{m} \int_{A \cap D_k''} \tilde{w}^2 f(\tilde{y}, \tilde{w}, t_i) \, d\tilde{y}d\tilde{w}.
\]

(4.31)

Now notice that:

\[
\sum_{k=0}^{m} \int_{A \cap D_k''} \tilde{w}^2 f(\tilde{y}, \tilde{w}, t_i) \, d\tilde{y}d\tilde{w} \leq \int_{A'} \tilde{w}^2 f(\tilde{y}, \tilde{w}, t_i) \, d\tilde{y}d\tilde{w}
\]

\[
W(X(t_i),5R(T),t_i) \leq CQ(R(T),t_i) \leq CQ,
\]

as it follows from (2.33), and this implies:

\[
\sum_{k=0}^{m} \int_{t_i}^{t_{i+1}} I_k''(k) \, dt \leq \frac{CQ^2}{P_{\frac{1}{2} - \eta}}.
\]

(4.33)

From eqns. (4.14), (4.19), (4.23) and (4.33) it follows:

\[
\sum_{j=1}^{2} \int_{t_j}^{t_{j+1}} I_j(X(t)) \, dt \leq CP_{\frac{1}{2} + \eta} Q_{\frac{1}{2}} \Delta + CP_{\frac{1}{2} - \eta} Q_{\frac{1}{2}} \log P \Delta + \frac{CQ^2}{P_{\frac{1}{2} - \eta}}.
\]

(4.34)
For the last term $I_3(X(t))$, we observe that if $|X(t_i) - y| \geq 4R(T)$, then $|X(t_i) - y| - R(T) \geq 3R(T)$. Hence

$$I_3(X(t)) = \int_{A^c} dy dw \frac{f(y, w, t)}{|X(t) - y|^2} \leq \int_{|X(t) - y| \geq 3R(T)} dy dw \frac{f(y, w, t)}{|X(t) - y|^2}.$$ 

Along the same lines of the estimate of $J_3(x, t)$ in Proposition 2 we can proceed with the estimate of $I_3(X(t))$, obtaining:

$$I_3(X(t)) \leq C.$$ (4.35)

By (4.13), (4.34) and (4.35), recalling the definition (4.4) of the time interval $\Delta$, we have:

$$\int_{t_i}^{t_{i+1}} |E(X(t), t)| dt \leq C \Delta \left( P^{4\gamma - \eta} Q + CP^{\frac{b}{3} - \eta} \log P + \frac{CQ}{P^{\frac{b}{3} - \eta}} \right).$$ (4.36)

Proposition 1, Lemma 1, and the definition of $R(t)$ imply that

$$Q \leq CP^{\beta}.$$ (4.37)

so that from (4.36) it follows:

$$\int_{t_i}^{t_{i+1}} |E(X(t), t)| dt \leq C \Delta \left( P^{4\gamma - \eta} + P^{\frac{b}{3} - \eta} \log P + P^{\beta - \frac{b}{3} - \gamma + \eta} \right).$$ (4.38)

Hence, since $n\Delta = T$, by (4.38) we get:

$$\int_0^t |E(X(s), s)| ds \leq \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} |E(X(t), t)| dt \leq C T \left( P^{4\gamma + \frac{\beta}{3}} + P^{\frac{b}{3} + \frac{\beta}{3}} \log P + P^{\beta - \frac{b}{3} - \gamma + \eta} \right).$$ (4.39)

By the assumption on $\beta < \frac{6}{19}$ and the choices of the parameters made in (4.3) and (4.18) we finally have:

$$\int_0^t |E(X(s), s)| ds \leq CP^{\alpha},$$ (4.40)

and the thesis is proved.

### 4.2 Proof of Proposition 3 for $\frac{6}{19} \leq \beta < \frac{14}{15}$

In the case $\frac{6}{19} \leq \beta < \frac{14}{15}$ it is not possible to choose the parameters $\gamma$ and $\eta$ in (4.39) in order to prove (4.40). What we do then is performing further iterations of the estimate of the time average of the electric field (4.36), by using at each step a time interval larger than the previous one; in such a
way we obtain at any step an improved estimate, up to the achievement of the searched bound. More precisely, for a positive integer $\ell$ we define
\[
\Delta_{\ell} = \Delta_{\ell-1} G = \Delta_{\ell-2} G^2 = \ldots = \Delta_1 G^{\ell-1}
\] (4.41)
where $\Delta_1 := \Delta$ (defined in (4.41)),
\[
G = \text{intg} \left( P^\delta \right),
\] (4.42)
being $\text{intg}(a)$ the integer part of $a$, and $\delta$ is any number satisfying
\[
0 < \delta < \frac{7}{6} - \frac{5}{4} \beta.
\] (4.43)
The reason for such bounds relies in the proof of Lemma 6 in the Appendix. The integer part in (4.42) is taken in order to use well known properties of the average over intervals which increase by an integer factor (see Remark 3 in the Appendix).

We succeed in the control of the time average of the electric field over a suitable time interval, as stated in the following proposition:

**Proposition 6.** There exists a positive constant $\bar{\Delta}$ such that
\[
\langle E \rangle_{\bar{\Delta}} := \frac{1}{\bar{\Delta}} \int_t^{t+\bar{\Delta}} |E(X(s), s)| ds \leq CP^\alpha \quad \forall \alpha \in \left( \frac{5 - \epsilon}{9}, \frac{2}{3} \right)
\] (4.44)
for any $t \in [0, T]$ such that $t \leq T - \bar{\Delta}$.

**Proof.** We claim that the following estimate holds for any positive integer $\ell$ (its proof is given in the following subsection):
\[
\langle E \rangle_{\Delta_{\ell}} \leq C \left[ P^{\frac{2}{3} + \eta} Q^{\frac{1}{3}} + P^{\frac{2}{3} - \eta} Q^{\frac{1}{3}} \log P + \frac{Q}{P^{\frac{1}{4} + \gamma + (\ell-1)\delta}} \right].
\] (4.45)
Notice that here we get an improved bound by a factor $P^{-(\ell-1)\delta}$ with respect to (4.36). From this, by (4.37) it follows
\[
\langle E \rangle_{\Delta_{\ell}} \leq C \left[ P^{\frac{2}{3} + \eta} Q^{\frac{1}{3}} + P^{\frac{2}{3} + \eta - \delta} \log P + P^{\delta - \frac{1}{4} + \gamma - (\ell-1)\delta} \right].
\] (4.46)
A different choice of the parameters $\gamma$ and $\eta$ (with respect (4.3) and (4.18)) is necessary for the case $\frac{6}{19} \leq \beta < \frac{14}{15}$, in order to prove also Lemmas 6 and 7 (later on):
\[
\max \left\{ 0, \beta - \frac{2}{3} + \delta \right\} < \gamma < \frac{2 - \beta}{4},
\] (4.47)
and
\[
\frac{3 + \beta}{3} < \eta < \frac{5}{3} + \gamma - \frac{2}{3} \beta - \delta.
\] (4.48)
Defining $\bar{\ell}$ as the smallest integer such that
\[
\beta - \frac{1}{3} + \eta - \gamma - (\bar{\ell} - 1)\delta < \frac{2}{3},
\] (4.49)
estimate (4.46) implies (4.44) with $\bar{\Delta} = \Delta_{\bar{\ell}}$. \qed
It can be easily seen that
\[ C \frac{P^{\beta + \eta - \frac{3}{2}}}{Q^{\frac{1}{3}}} < \Delta < C \frac{P^{\beta + \eta + \delta - \frac{3}{2}}}{Q^{\frac{1}{3}}} \]
and by the choice of the parameters both the exponents of \( P \) are negative, in order to be \( \Delta \ll T \) (taking the constant \( \tilde{C} \) in (2.16) suitably large).

### 4.3 Proof of (4.45).

By an inductive method we give now the proof of (4.45).

We premise the following Lemmas, direct generalizations of Lemmas 4 and 5, whose statements hold also at the \( \ell \)-th iterative step. Their proofs are given in [5], but for completeness we repeat them in the Appendix:

**Lemma 6.** Let \( t \in [0, T] \) such that \( t + \Delta_{\ell} \in [0, T] \).

If \( |V(t) - W(t)| \leq P^{\gamma} \), then
\[
\sup_{s \in [t, t + \Delta_{\ell}]} |V(s) - W(s)| \leq 2P^{\gamma}. \tag{4.50}
\]

If \( |V(t) - W(t)| \geq P^{\gamma} \), then
\[
\inf_{s \in [t, t + \Delta_{\ell}]} |V(s) - W(s)| \geq \frac{1}{2} P^{\gamma}. \tag{4.51}
\]

**Lemma 7.** Let \( t \in [0, T] \) such that \( t + \Delta_{\ell} \in [0, T] \) and assume that \( |V(t) - W(t)| \geq P^{\gamma} \). Then there exists \( t_0 \in [t, t + \Delta_{\ell}] \) such that for any \( s \in [t, t + \Delta_{\ell}] \) it holds:
\[
|X(s) - Y(s)| \geq \frac{P^{\gamma}}{4} |s - t_0|.
\]

Formula (4.45) is already proved in the case \( \ell = 1 \) (it is in fact formula (4.36)). We show now that, assuming true (4.45) for \( \ell - 1 \), then it holds also for \( \ell \). We note that (see Remark 3 in the Appendix) if estimate (4.45) holds for \( \langle E \rangle_{\Delta_{\ell - 1}} \), then the same estimate holds for \( \langle E \rangle_{\Delta_{\ell}} \), since both the intervals \( \Delta_{\ell} \) and the bound (4.45) are uniform in time. From what done before, the only term in (4.45) which is affected by the time average is the last one (see (4.33)). Then, by using the estimate on the time average \( \langle E \rangle_{\Delta_{\ell - 1}} \) on the larger time interval \( \Delta_{\ell} \) we get for this term:
\[
\frac{Q}{P^{1 + \gamma - \eta + (\ell - 2)\delta}} \Delta_{\ell} \frac{\Delta_{\ell - 1}}{\Delta_{\ell}} \leq \frac{Q}{P^{1 + \gamma - \eta + (\ell - 1)\delta}} \Delta_{\ell}, \tag{4.52}
\]
which proves (4.45).

The proof of Proposition 3 follows immediately, as in the passage from (4.39) to (4.40).
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Appendix

Proof of Lemma 4

We have by (2.29), for any \( t \in [t', t' + \Delta] \subset [0, T] \),

\[
|V(t) - W(t)| \leq |V(t') - W(t')| + \int_{t'}^{t' + \Delta} \left[ |E(X(s), s)| + |E(Y(s), s)| \right] ds \leq P\gamma + 2C_2P^\frac{4}{3}Q^\frac{1}{3} \Delta \leq 2P\gamma.
\]

Analogously we prove the second statement:

\[
|V(t) - W(t)| \geq |V(t') - W(t')| - \int_{t'}^{t' + \Delta} \left[ |E(X(s), s)| + |E(Y(s), s)| \right] ds \geq P\gamma - 2C_2P^\frac{4}{3}Q^\frac{1}{3} \Delta \geq \frac{1}{2}P\gamma.
\]

Proof of Lemma 5

Let \( t_0 \in [t', t' + \Delta] \subset [0, T] \) be the time at which \( |X(s) - Y(s)| \) has the minimum value, for \( s \in [t', t' + \Delta] \). We put \( \Gamma(s) = X(s) - Y(s) \). Moreover we define the function

\[
\bar{\Gamma}(s) = \Gamma(t_0) + \dot{\Gamma}(t_0)(s - t_0).
\]

It results

\[
\dot{\Gamma}(s) - \dot{\bar{\Gamma}}(s) = E(X(s), s) - E(Y(s), s)
\]

\[
\Gamma(t_0) = \bar{\Gamma}(t_0), \quad \dot{\Gamma}(t_0) = \dot{\bar{\Gamma}}(t_0)
\]

from which it follows

\[
\Gamma(s) = \bar{\Gamma}(s) + \int_{t_0}^{s} d\tau \int_{t_0}^{\tau} d\xi \left[ E(X(\xi), \xi) - E(Y(\xi), \xi) \right].
\]

By (2.29)

\[
\int_{t_0}^{s} d\tau \int_{t_0}^{\tau} d\xi |E(X(\xi), \xi) - E(Y(\xi), \xi)| \leq 2C_2P^\frac{4}{3}Q^\frac{1}{3}\frac{|s - t_0|^2}{2} \leq C_2P^\frac{4}{3}Q^\frac{1}{3} \Delta |s - t_0| \leq P\gamma \frac{|s - t_0|}{4}.
\]
Hence,
\[ |\Gamma(s)| \geq |\overline{\Gamma}(s)| - P^2 |s-t_0| \quad \text{(A.2)} \]
Now we have:
\[ |\overline{\Gamma}(s)|^2 = |\Gamma(t_0)|^2 + 2\Gamma(t_0) \cdot \dot{\Gamma}(t_0)(s-t_0) + |\dot{\Gamma}(t_0)|^2 |s-t_0|^2. \]
We observe that \( \Gamma(t_0) \cdot \dot{\Gamma}(t_0)(s-t_0) \geq 0 \). Indeed, if \( t_0 \in (t', t' + \Delta) \) then \( \dot{\Gamma}(t_0) = 0 \) while if \( t_0 = t' \) or \( t_0 = t' + \Delta \) the product \( \Gamma(t_0) \cdot \dot{\Gamma}(t_0)(s-t_0) \geq 0 \).
Hence
\[ |\overline{\Gamma}(s)|^2 \geq |\dot{\Gamma}(t_0)|^2 |s-t_0|^2. \]
By Lemma \[ \text{Lemma 4} \] since \( t_0 \in [t', t' + \Delta] \) it is
\[ |\dot{\Gamma}(t_0)| \geq \frac{P^2}{2} \]
hence
\[ |\overline{\Gamma}(s)| \geq \frac{P^2}{2} |s-t_0| \]
and finally by (A.2),
\[ |\Gamma(s)| \geq \frac{P^2}{4} |s-t_0|, \]
which concludes the proof.

Proof of Lemma \[ \text{Lemma 6} \]

Remark 3. Before giving the proofs of Lemmas \[ \text{Lemma 6} \] and \[ \text{Lemma 7} \] we observe that it holds:
\[ \sup_{t \in [0, T-\Delta_\ell]} \langle E \rangle_{\Delta_\ell} \leq \sup_{t \in [0, T-\Delta_{\ell-1}]} \langle E \rangle_{\Delta_{\ell-1}} \quad \forall \ell \leq \overline{\ell}. \quad \text{(A.3)} \]
In fact, \( \Delta_\ell = \mathcal{G}\Delta_{\ell-1} \) with \( \mathcal{G} \) given in (4.42), so that:
\[ [t, t + \Delta_\ell] = \bigcup_{i=1}^{\mathcal{G}} [t + (i-1)\Delta_{\ell-1}, t + i\Delta_{\ell-1}], \]
hence
\[ \frac{1}{\Delta_\ell} \int_t^{t+\Delta_\ell} |E(X(s), s)| ds \leq \max_{i} \frac{1}{\Delta_{\ell-1}} \int_{t+(i-1)\Delta_{\ell-1}}^{t+i\Delta_{\ell-1}} |E(X(s), s)| ds, \quad \text{(A.4)} \]
whence we get (A.3).

In order to prove Lemma \[ \text{Lemma 6} \] we show now that Lemma \[ \text{Lemma 6} \] holds true also over a time interval \( \Delta_\ell, 2 \leq \ell < \overline{\ell}, \) under the assumption that estimate
(4.45) at level \( \ell - 1 \) holds. Indeed, by the use of (4.45) we get, for any \( s \in [t, t + \Delta t] \),

\[
|V(s) - W(s)| \leq |V(t) - W(t)| + \int_t^{t+\Delta t} \left[ |E(X(\tau), \tau)| + |E(Y(\tau), \tau)| \right] d\tau \leq P^\gamma + C \left[ P^{\frac{2}{3}} \gamma Q_{\frac{1}{3}} + P^{\frac{2}{3} - \eta} Q_{\frac{1}{3}} \log P + \frac{Q}{P^{\frac{2}{3} + \eta - (\ell - 2)\delta}} \right] \Delta t := P^\gamma + C[a_1 + a_2 + a_3] \Delta t,
\]

where

\[
a_1 = P^{\frac{2}{3}} \gamma Q_{\frac{1}{3}}, \quad a_2 = P^{\frac{2}{3} - \eta} Q_{\frac{1}{3}} \log P, \quad a_3 = \frac{Q}{P^{\frac{2}{3} + \eta - (\ell - 2)\delta}}.
\]

Since \( Q \leq R(T)^\beta \leq CP^\beta \), recalling that

\[
\Delta t = \frac{[\text{intg} (P^\delta)]^{\ell - 1}}{4C_2 P^{\frac{2}{3} - \eta} Q_{\frac{1}{3}}},
\]

we have:

\[
a_1 \Delta t \leq P^{\frac{2}{3}} \gamma - \frac{4}{3} + \gamma + (\ell - 1)\delta
\]

(A.5)

\[
a_2 \Delta t \leq P^{\frac{1}{3} - \eta + \gamma + (\ell - 1)\delta} \log P
\]

(A.6)

\[
a_3 \Delta t \leq P^{\frac{2}{3} - \eta - \frac{2}{3} + \delta}.
\]

(A.7)

Being \( \ell \leq \bar{\ell} \), from the definition of \( \bar{\ell} \) given in (4.49) it follows that

\[
\beta - \frac{1}{3} + \eta - \gamma - (\bar{\ell} - 2)\delta \geq \frac{2}{3},
\]

(A.8)

from which

\[
(\ell - 1)\delta \leq (\bar{\ell} - 1)\delta \leq \beta - \frac{1}{3} - \gamma - \frac{2}{3} + \delta
\]

(A.9)

(note that the right hand side of (A.10) is positive for \( \beta \geq \frac{6}{19} \)). Therefore, by the choices (4.43), (4.47) and (4.48) made on the parameters, it follows

\[
a_i \Delta t \leq P^\theta \quad \theta < \gamma, \quad i = 1, 2, 3.
\]

In fact, for the term \( a_2 \Delta t \), it is sufficient to prove

\[
\frac{1}{3} - \eta + \gamma + (\ell - 1)\delta < \gamma,
\]

which holds true since, by (A.10) and (4.47),

\[
\frac{1}{3} - \eta + (\ell - 1)\delta \leq \beta - \gamma - \frac{2}{3} + \delta < 0.
\]
Let us observe that the interval in (4.47) is well defined if
\[ \beta - \frac{2}{3} + \delta < \frac{2 - \beta}{4}, \]
which gives condition (4.48)
\[ 0 < \delta < \frac{7}{6} - \frac{5}{4}\beta. \]

Let us consider the term \(a_3 \Delta_\ell\), for which it is sufficient to prove
\[ \frac{2}{3} \beta - \frac{5}{3} + \eta + \delta < \gamma, \]
that is
\[ \eta < \frac{5}{3} + \gamma - \frac{2}{3}\beta - \delta, \]
which is implied by (4.48). The interval in (4.48) is well defined, as it follows by the condition on \(\gamma\) (4.47).

Finally we examine the term \(a_1 \Delta_\ell\), for which we require
\[ \frac{4}{3} \gamma - \frac{4}{3} + \gamma + (\ell - 1)\delta < \gamma, \]
which holds true since
\[ \frac{4}{3} \gamma - \frac{4}{3} + (\ell - 1)\delta < \frac{4}{3} \gamma - \frac{4}{3} + \beta - \frac{1}{3} - \gamma - \frac{2}{3} + \delta = \gamma - \frac{7}{3} + \beta + \eta + \delta < 0, \]
by taking
\[ \eta < \frac{7}{3} - \frac{2}{3} - \beta - \delta, \]
condition which is automatically fulfilled by (4.48), since
\[ \frac{5}{3} + \gamma - \frac{2}{3}\beta - \delta < \frac{7}{3} - \gamma - \frac{2}{3} - \beta - \delta, \]
as it is evident by using (4.47).

Hence, provided that \(P\) is sufficiently large (as the constant \(\tilde{C}\) in (2.16) assures), we have:
\[ C[a_1 + a_2 + a_3] \Delta_\ell \leq P^\gamma \]
which proves the thesis.

We proceed analogously for the lower bound.
Proof of Lemma 7

We note that the same arguments used in the proof of Lemma 5 work also for Lemma 7 considering the interval \([t, t + \Delta]\), \(\ell > 1\), and for the electric field the estimate (4.45) at level \(\ell - 1\). In fact, going back to (A.1), we have for any \(s \in [t', t' + \Delta]\)

\[
\int_{t_0}^{s} d\tau \int_{t_0}^{\tau} d\xi |E(X(\xi), \xi) - E(Y(\xi), \xi)| \leq 2\langle E \rangle_{\Delta_{\ell - 1}} \Delta_{\ell} \int_{t_0}^{s} d\tau \leq \frac{P\gamma}{4} |s - t_0|,
\]

(A.11)

where we treat the term \(\langle E \rangle_{\Delta_{\ell - 1}} \Delta_{\ell}\) in the same way as in (A.5). Hereafter the proof proceeds in the same way as the proof of Lemma 5.

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