EUCLID’S THEOREM ON THE INFINITUDE OF PRIMES: A HISTORICAL SURVEY OF ITS 200 PROOFS (300 B.C.–2022)

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“The laws of nature are but the mathematical thoughts of God.”

Euclid (circa 300 B.C.)

“If Euclid failed to kindle your youthful enthusiasm, then you were not born to be a scientific thinker.”

Albert Einstein

Abstract. In the fourth extended version of this article, we provide a comprehensive historical survey of 200 different proofs of famous Euclid’s theorem on the infinitude of prime numbers (300 B.C.–2022). The author is trying to collect almost all the known proofs on infinitude of primes, including some proofs that can be easily obtained as consequences of some known problems or divisibility properties. Furthermore, here are listed numerous elementary proofs of the infinitude of primes in different arithmetic progressions.

All the references concerning the proofs of Euclid’s theorem that use similar methods and ideas are exposed subsequently. Namely, presented proofs are divided into the first five subsections of Section 2 in dependence of the methods that are used in them. 14 proofs which are proved from 2012 to 2017 are given in Subsection 2.9, and 18 recent proofs from 2018 to 2022 are presented in Subsection 2.10.

In Section 3, we mainly survey elementary proofs of the infinitude of primes in different arithmetic progressions. Presented proofs are special cases of Dirichlet’s theorem. In Section 4, we give a new simple “Euclidean’s proof” of the infinitude of primes.
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1. Euclid’s theorem on the infinitude of primes

1.1. Primes and the infinitude of primes. prime number (or briefly in the sequel, a prime) is an integer greater than 1 that is divisible only by 1 and itself. Starting from the beginning, prime numbers have always been around but the concepts and uniqueness was thought to be first considered during Egyptian times. However, mathematicians have been studying primes and their properties for over twenty-three centuries. Ancient Greek mathematicians knew that there are infinitely many primes. Namely, circa 300 B.C., Euclid of Alexandria, from the Pythagorean School proved (Elements, Book IX, Proposition 20) the following celebrated result as rendered into modern language from the Greek ([91], [327]):

If a number be the least that is measured by prime numbers, it will not be measured by any other prime number except those originally measuring it.

Euclid’s “Elements” are one of the most popular and most widely printed mathematicians books and they are been translated into many languages. Elements presents a remarkable collection of 13 books that contained much of the mathematical known at the time. Books VII, VIII and IX deal with properties of the integers and contain the early beginnings of number theory, a body of knowledge that has flourished ever since.

Recall that during Euclid’s time, integers were understood as lengths of line segments and divisibility was spoken of as measuring. According to G. H. Hardy [126], “Euclid’s theorem which states that the number of primes is infinite is vital for the whole structure of arithmetic. The primes are the raw material out of which we have to build arithmetic, and Euclid’s theorem assures us that we have plenty of material for the task.” Hardy [126] also remarks that this proof is “as fresh and significant as when it was discovered—two thousand years have not written a wrinkle on it”. A. Weil [316] also called “the proof for the existence of infinitely many primes represents undoubtedly a major advance, but there is no compelling reason either for attributing it to Euclid or for dating back to earlier times. What matters for our purposes is that the very broad diffusion of Euclid in latter centuries, while driving out all earlier texts, made them widely available to mathematicians from then on”.

Sir Michael Atyah remarked during an interview [243]: Any good theorem should have several proofs, more the better. For two reasons: usually, different proofs have different strenghts and weaknesses, and they generalize in different directions - they are not just repetitions...
of each other. For example, the *Pythagorean theorem* has received more than 360 proofs [171] of all sorts as algebraic, geometric, dynamic and so on. The *irrationality of \( \sqrt{2} \) is another famous example of a theorem which has been proved in many ways ([295]; on the web page [31] fourteen different proofs appear). C. F. Gauss himself had 10 different proofs for the *law of quadratic reciprocity* [103, Sections 112–114]. Surprisingly, here we present 183 different proofs of *Euclid’s theorem on the infinitude of primes*, including 44 proofs of the infinitude of primes in special arithmetic progressions.

1.2. Euclid’s proof of Euclid’s theorem. Even after almost two and a half millennia ago Euclid’s theorem on the infinitude of primes stands as an excellent model of reasoning. Below we follow Ribenboim’s statement of Euclid’s proof [248, p. 3]. Namely, in Book IX of his celebrated *Elements* (see [91]) we find Proposition 20, which states:

**Euclid’s theorem.** There are infinitely many primes.

Elegant proof of Euclid’s theorem runs as follows. Suppose that \( p_1 = 2 < p_2 = 3 < \cdots < p_k \) are all the primes. Take \( n = p_1p_2\cdots p_k + 1 \) and let \( p \) be a prime dividing \( n \). Then \( p \) cannot be any of \( p_1, p_2, \ldots, p_k \), otherwise \( p \) would divide the difference \( n - p_1p_2\cdots p_k = 1 \). □

The above proof is actually quite a bit different from what Euclid wrote. Since ancient Greeks did not have our modern notion of infinity, Euclid could not have written “there are infinitely many primes”, rather he wrote: “prime numbers are more than any assigned multitude of prime numbers.” Below is a proof closer to that which Euclid wrote, but still using our modern concepts of numbers and proof. An English translation of Euclid’s actual proof given by D. Joyce in his webpages [145] also can be found in [http://primes.utm.edu/notes/proofs/infinite/euclids.html](http://primes.utm.edu/notes/proofs/infinite/euclids.html). It is a most elegant proof by *contradiction* (reduction ad absurdum) that goes as follows.

**Euclid’s theorem.** There are more primes than found in any finite list of primes.

*Proof.* Call the primes in our finite list \( p_1, p_2, \ldots, p_k \). Let \( P \) be any common multiple of these primes plus one (for example \( P = p_1p_2\cdots p_k + 1 \)). Now \( P \) is either prime or it is not. If it is prime, then \( P \) is a prime that was not in our list. If \( P \) is not prime, then it is divisible by some prime, call it \( p \). Notice \( p \) cannot be any of \( p_1, p_2, \ldots, p_k \), otherwise \( p \) would divide 1, which is impossible. So this prime \( p \) is some prime that was not in our original list. Either way, the original list was incomplete. □
The statement of Euclid’s theorem together with its proof is given by B. Mazur in 2005 [180, p. 230, Section 3] as follows.

“If you give me any finite (non-empty, of course!) collection of prime numbers, I will form the number \( N \) that is 1 more than the product of all the primes in the collection, so that every prime in your collection has the property that when \( N \) is divided by it, there is a remainder of 1. There exists at least one prime number dividing this number \( N \) and any prime number dividing \( N \) is new in the sense that it is not in your initial collection.”

Remarks. Euclid’s proof is often said to be “indirect” or “by contradiction”, but this is unwarranted: given any finite set of primes \( p_1, \ldots, p_n \), it gives a perfectly definite procedure for constructing a new prime. Indeed, if we define \( E_1 = 2 \), and having defined \( E_1, \ldots, E_n \), we define \( E_{n+1} \) to be the smallest prime divisor of \( E_1E_2 \cdots E_{n+1} \), we get a sequence of distinct primes, nowadays called the Euclid-Mullin sequence (of course, we could get a different sequence by taking \( p_1 \) to be a prime different from 2). This is Sloane’s sequence A000945 whose first few terms are 2, 3, 7, 43, 13, 53, 5, 6221671, 38709183810571, 139, . . . . The natural question - does every prime occur eventually in the Euclid-Mullin sequence remains unanswered. Note that D. Shanks [267] conjectured on probabilistic grounds that this sequence contains every prime. This conjecture was supported by computational results up to 43rd term of the sequence \( (E_n) \) given in 1993 by S. S. Wagstaff, Jr. [309]. For a discussion on this conjecture, see [37, Section 2], where it was noticed that N. Kurokawa and T. Satoh [157] have shown that an analogue of this conjecture for the Euclidean domains \( \mathbb{F}_p[x] \) is false in general. Notice that the sequence \( (E_n) \) and several related sequences were studied in [123].

Moreover, Mullin [201] constructed the second sequence of primes, say \( (P_n) \) similarly as the above sequence \( (E_n) \), except that we replace the words “smallest prime divisor” by “largest prime divisor”. This is the sequence A000946 in [278]. It was proved in 2013 by A. R. Booker [37, Theorem 1] that the sequence \( (P_n) \) omits infinitely many primes, confirming a conjecture of C. D. Cox and A. J. Van der Poorten [68]. Notice that in 2014 P. Pollack and E. Treviño [236] gave a completely elementary proof of this conjecture.

Notice also that Euclid’s proof actually uses the fact that there is a prime dividing given positive integer greater than 1. This follows from Proposition 31 in Book VII of his Elements [91, 20, 128, p.2, Theorem 1]) which asserts that “any composite number is measured by some prime number”, or in terms of modern arithmetic, that every
integer $n > 1$ has at least one representation as a product of primes. Of course, he also used an unexpressed axiom which states that if $a$ divides $b$ and $a$ divides $c$, $a$ will divide the difference between $b$ and $c$. □

The unique factorization theorem, otherwise known as the “fundamental theorem of arithmetic,” states that any integer greater than 1 can, except for the order of the factors, be expressed as a product of primes in one and only one way. This theorem does not appear in Euclid’s Elements ([91]; also see [20]). However, as noticed in [20, page 208], in fact, the unique factorization theorem follows from Propositions 30-31 in Book VII (given in Remarks of Section 4). More generally, in 1976 W. Knorr [152] gave a reasonable discussion of the position of unique factorization in Euclid’s theory of numbers. Nevertheless, as noticed in [20], Euclid played a significant role in the history of this theorem (specifically, this concerns to some propositions of Books VII and IX). However, the first explicit and clear statement and the proof of the unique factorization theorem seems to be in C. F. Gauss’ masterpiece Disquisitiones Arithmeticae [103, Section II, Article 16]. His Article 16 is given as the following theorem: A composite number can be resolved into prime factors in only one way. After Gauss, many mathematicians provided different proofs of this theorem in their work (these proofs are presented and classified in [2]). In particular, the unique factorization theorem was used in numerous proofs of the infinitude of primes provided below.

Notice also that for any field $F$, Euclid’s argument works to show that there are infinitely many irreducible polynomials over $F$. This follows inductively taking $p_1(t) = t$, and having produced $p_1(t), \ldots, p_k(t)$, consider the irreducible factors of $p_1(t) \cdots p_k(t) + 1$.

1.3. Sequences arising from Euclid’s proof of IP. As usually, for each prime $p$, $p^#$ denotes the product of all the primes less than or equal to $p$ and it is called the primorial number (Sloane’s sequence A002110; also see A034386 for the second definition of primorial number as a product of primes in the range 2 to $n$). The expressions $p^# + 1$ and $p^# - 1$ have been considered in connection with variants of the Euclid’s proof of the infinitude of primes.

Further, $n$th Euclid’s number $E_n$ (see e.g., [303]) is defined as a product of first $n$ consecutive primes plus one (Sloane’s sequence A006862). Similarly, Kummer’s number is defined as a product of first $n$ consecutive primes minus one (Sloane’s sequence A057588). Euclid’s numbers were tested for primality in 1972 by A. Borning [40], in 1980 by M. Templer [294], in 1982 by J. P. Buhler, R. E. Crandall and M. A. Penk
and in 1995 by C. K. Caldwell. Recall also that two interesting conjectures involving the numbers $E_n$ are quite recently proposed by Z.-W. Sun. Namely, for any given $n$, if $w_1(n)$ is defined as the least integer $m > 1$ such that $m$ divides none of those $E_i - E_j$ with $1 \leq i < j \leq n$, then Sun conjectured that $w_1(n)$ is a prime less than $n^2$ for all $n = 2, 3, 4, \ldots$. The same conjecture is proposed in relation to the sums $E_i + E_j - 2$ instead of $E_i - E_j$ (cf. Sloane’s sequences A210144 and A210186).

The numbers $p^# \pm 1$ (in accordance to the first definition given above) and $n! \pm 1$ have been frequently checked for primality (see [49], [115], [279] and [248, pp. 4–5]). The numbers $p^# \pm 1$ have been tested for all $p < 120000$ in 2002 by C. Caldwell and Y. Gallot. They were reported that in the tested range there are exactly 19 primes of the form $p^# + 1$ and 18 primes of the form $p^# - 1$ (these are in fact Sloane’s sequences A005234 extended with three new terms and A006794, respectively). It is pointed out in [248, p. 4] that the answers to the following questions are unknown: 1) Are there infinitely many primes $p$ for which $p^# + 1$ is prime?). Are there infinitely many primes $p$ for which $p^# + 1$ is composite?

In terms of the second definition of primorial numbers given above, similarly are defined Sloane’s sequences A014545 and A057704 (they also called primorial primes).

Other Sloane’s sequences related to Euclid’s proof and Euclid numbers are: A018239 (primorial primes), A057705, A057713, A065314, A065315, A065316, A065317, A006794, A068488, A068489, A103514, A0166266, A066267, A066268, A066269, A088054, A093804, A103319, A104350, A002981, A002982, A038507, A088332, A005235, A000945 and A000946.

1.4. Proofs of Euclid’s theorem: a brief history. Euclid’s theorem on the infinitude of primes has fascinated generations of mathematicians since its first and famous demonstration given by Euclid (300 B.C.). Many great mathematicians of the eighteenth and nineteenth century established different proofs of this theorem (for instance, Goldbach (1730), Euler (1736, 1737), Lebesgue (1843, 1856, 1859, 1862), Sylvester (1871, 1888 (4)), Kronecker (1875/6), Hensel (1875/6), Lucas (1878, 1891, 1899), Kummer (1878/9), Stieltjes (1890) and Hermite (1915). Furthermore, in the last hundred years various interesting proofs of the infinitude of primes, including the infinitude of primes in different arithmetic progressions, were obtained by I. Schur (1912/13), K. Hensel (1913), G. Pólya (1921), G. Pólya and G. Szegő (1925), P.
Erdős (1934 (2), 1938 (2)), G. H. Hardy and E. M. Wright (1938 (2)), L.
G. Schnirelman (published posthumously in 1940), R. Bellman (1943,
1947), H. Furstenberg (1955), J. Lambek and L. Moser (1957), S. W.
Golomb (1963), A. W. F. Edwards (1964), A. A. Mullin (1964), W.
Sierpiński (1964, 1970 (4)), S. P. Mohanty (1978 (3)), A. Weil (1979),
L. Washington (1980), S. Srinivasan (1984 (2)), M. Deaconescu and
J. Sándor (1986), J. B. Paris, A. J. Wilkie and A. R. Woods (1988),
M. Rubinstein (1993), N. Robbins (1994 (2)), R. Goldblatt (1998),
M. Aigner and G. M. Ziegler (2001 (2)), Š. Porubsky (2001), D. Cass
and G. Wildenberg (2003), T. Ishikawa, N. Ishida and Y. Yukimoto
(2004), R. Crandall and C. Pomerance (2005), A. Granville (2007 (2),
2009), J. P. Whang (2010), R. Cooke (2011), P. Pollack (2011) and
by several other authors. We also point out that in numerous proofs
of Euclid’s theorem were used methods and arguments due to Euclid
(“Euclidean’s proofs”), Goldbach (proofs based on elementary divisibility
properties of integers) or Euler (analytic proofs based on Euler’s
product). Moreover, numerous proofs of Euclid’s theorem are based
on some of the following methods or results: algebraic number
theory arguments (Euler’s totient function, Euler theorem, Fermat little
theorem, arithmetic functions, Theory of Finite Abelian Groups etc.),
Euler’s formula for the Riemann zeta function, Euler’s factorization,
elementary counting methods (enumerative arguments), Furstenberg’s
topological proof of the infinitude of primes and its combinatorial and
algebraic modifications etc. All the proofs of the infinitude of primes
exposed in this article are divided into 8 subsections of Section 2 in
dependence of used methods in them. In the next section we mainly
survey elementary proofs of the infinitude of primes in different arith-
metic progressions. These proofs are also based on some of mentioned
methods and ideas. Finally, in Section 4, we give a new simple proof of
the infinitude of primes. The first step of our proof is based on Euclid’s
idea. The remaining of the proof is quite simple and elementary and
it does not use the notion of divisibility.

In Dickson’s History of the Theory of Numbers [76, pp. 413–415] and
the books by Ribenboim [246, pp. 3–11], [248, Chapter 1, pp. 3–13],
Pollack [233, pp. 2–19], Hardy and Wright [128, pp. 12–17], [129, pp.
14–18], Aigner and Ziegler [6, pp. 3–6], and in Narkiewicz’s monograph
[210, pp. 1–10] can be found many different proofs of Euclid’s theorem.
Several proofs of this theorem were also explored by P. L. Clark [59,
Ch. 10, pp. 115–121] and T. Yamada [324, Sections 1-6, 10-12]. In
Appendix C) of this article we give a list of all 168 different proofs of
Euclid’s theorem presented here (including elementary proofs related
to the infinitude of primes in special arithmetic progressions), together
with the corresponding reference(s), the name(s) of his (their) author(s) and the main method(s) and/or idea(s) used in it (them). This list is arranged by year of publication. We also give a comprehensive (Subject and Author) Index to this article.

The Bibliography of this article contains 291 references, consisting mainly of articles (including 47 Notes and Articles published in Amer. Math. Monthly) and mathematical textbooks and monographs. It also includes a few unpublished works or problems that are available on Internet Websites, especially on http://arxiv.org/, one Ph.D. thesis, an interview, one private correspondence, one Course Notes and Sloane’s On-Line Encyclopedia of Integer Sequences. Some of these references does not concern directly to proofs of the infinitude of primes, but results of each of them that are cited here give possibilities to simplify some of these known proofs.

We believe that our exposition of different proofs of Euclid’s theorem may be useful for establishing proofs of many new and old results in Number Theory via elementary methods.

2. A survey of different proofs of Euclid’s theorem

To save the space, in the sequel we will often denote by “IP” “the infinitude of primes”.

2.1. Proofs of IP based on Euclid’s idea. Ever since Euclid of Alexandria, sometimes before 300 B.C., first proved that the number of primes is infinite (see Proposition 20 in Book IX of his legendary Elements in [91] (also see [128, p. 4, Theorem 4]) where this result is called Euclid’s second theorem), mathematicians have amused themselves by coming up with alternate proofs. For more information about the Euclid’s proof of the infinitude of primes see e.g., [67], [76, p. 414, Ch. XVIII], [77], [80, pp. 73–75], [127] and [180, Section 3].

Euclid’s proof of IP is a paragon of simplicity: given a finite list of primes, multiply them together and add one. The resulting number, say $N$, is not divisible by any prime on the list, so any prime factor of $N$ is a new prime. There are several variants of Euclid’s proof of IP. The simplest of them, which according to H. Brocard [42] is due in 1915 to C. Hermite, immediately follows from the obvious fact that the smallest prime divisor of $n! + 1$ is greater than $n$. Another of these proof, due to E. E. Kummer in 1878/9 [156] (also see [248, page 4] and [324]) is in fact an elegant variant of Euclid’s proof. In a long paper published in two installments 120 years ago ([228], [229]) J. Perott noticed that Euclid’s proof works if we consider $p_1 p_2 \cdots p_k − 1$ instead of $p_1 p_2 \cdots p_k + 1$. Stieltjes’ proof in 1890 given in his work
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(10 ROMEO MEŠTROVIĆ) C. O. Boije af Gennäs’ proof in 1893
(10 ROMEO MEŠTROVIĆ) Braun’s proof in 1899
(10 ROMEO MEŠTROVIĆ) Lévi’s proof in 1909/10
(10 ROMEO MEŠTROVIĆ) Métrod’s proof in 1917
(10 ROMEO MEŠTROVIĆ) Thompson’s proof in 1953
(10 ROMEO MEŠTROVIĆ) Mullin’s proof of 1964
(10 ROMEO MEŠTROVIĆ) Trigg’s proof in 1974
(10 ROMEO MEŠTROVIĆ) and Aldaz and Bravo’s proof (10 ROMEO MEŠTROVIĆ) in 2003

present refinements of Euclid’s proof on IP. For example, supposing
that the set of all primes is a finite \(\{p_1, p_2, \ldots, p_k\}\) with their product

\[ P = p_1 p_2 \cdots p_k \]

we find that

\[ a/P > 1/2 + 1/3 + 1/5 = 31/30 > 1. \]

Therefore, Braun (10 ROMEO MEŠTROVIĆ) concluded that

\[ a \] must have a prime divisor, say \(p_j\), but then \(p_j\) must divide

\[ P/p_j, \]

which is not possible.

Using algebraic number theory, in 1985 R. W. K. Odoni (10 ROMEO MEŠTROVIĆ) investigated the sequence \((w_n)\) recursively defined by R. K. Guy and R. Nowakowski (10 ROMEO MEŠTROVIĆ) as

\[ w_1 = 2, \quad w_{n+1} = 1 + w_1 \cdots w_n \quad (n \geq 1) \]

and observed that \(w_n \to \infty\) as \(n \to \infty\) and the \(w_n\) are pairwise relatively prime. Clearly, this yields IP.

Furthermore, Problem 62 of (10 ROMEO MEŠTROVIĆ) whose solution uses Euclid’s idea, asserts that if \(a, b\) and \(m\) are positive integers such that \(a\) and \(b\) are relatively prime, then the arithmetic progression \(\{ak + b : k = 0, 1, 2, \ldots\}\) contains infinitely many terms relatively prime to \(m\). This together with Euclid’s argument (i.e., assuming \(m\) to be a product of consecutive primes) immediately yields IP. A proof of IP quite similar to those of Braun is given in 2008 by A. Scimone (10 ROMEO MEŠTROVIĆ).

If \(p_n\) denotes the \(n\)th prime, then by (10 ROMEO MEŠTROVIĆ) (Problem 47; pages 8 and 55, Problem 92) solved by A. Mąkowski, \(p_{n+1} + p_{n+2} \leq p_1 p_2 \cdots p_n\) for each \(n \geq 3\). This shows that for each \(n \geq 3\) there are at least two primes between the \(n\)th prime and the product of the first \(n\) primes. This estimate is in 1998 improved by J. Sándor (10 ROMEO MEŠTROVIĆ) who showed that \(p_n + p_{n-2} + p_1 p_2 \cdots p_{n-1} \leq p_1 p_2 \cdots p_n\) for all \(n \geq 3\).

In 2008 B. Joyal (10 ROMEO MEŠTROVIĆ) proved IP using the sieve of Eratosthenes, devised about 200 B.C., which is a beautiful and efficient algorithm for finding all the primes less than a given number \(x\).

Recently, using Euclid’s idea and a representation of a rational number in a positive integer base, in (10 ROMEO MEŠTROVIĆ) the author of this article obtained
an elementary proof of $IP$. The second similar author’s proof of $IP$ is given here in Section 4.

We see from Euclid’s proof that $p_{n+1} < p_1p_2 \cdots p_n$ for each $n \geq 2$, where $p_k$ is the $k$th prime. In 1907 H. Bonse [36] gave an elementary proof of a stronger inequality, now called Bonse’s inequality [300, p. 87]: if $n \geq 4$, then $p_{n+1}^2 < p_1p_2 \cdots p_n$. In 2000 M. Dalezman [70, Theorem 1] gave an elementary proof of stronger inequality $p_{n+1}p_{n+2} < p_1p_2 \cdots p_n$ with $n \geq 4$. J. Sondow [282, Theorem 1] exposed a simple proof based on the Euler formula $\zeta(2) := \sum_{n=1}^{\infty} 1/n^2 = \pi^2/6$ (suggested by P. Ribenboim in 2005), that for all sufficiently large $n$, $p_{n+1} < (p_1p_2 \cdots p_n)^{2\mu}$, where $\mu$ is the irrationality measure for $6/\pi^2$ (for this concept and related estimates see e.g., [247, pp. 298–309]). Recall also that Bonse’s inequality is refined in 1960 by L. Pósa [242], in 1962 by S. E. Mamangakis [179], in 1971 by S. Reich [244] and in 1988 by J. Sándor [257].

Remarks. Euclid’s proof of $IP$ may be used to generate a sequence $(a_n)$ of primes as follows: put $a_1 = 2$ and if $a_1, a_2, \ldots, a_{n-1}$ are already defined then let $a_n$ be the largest prime divisor of $P_n := a_1a_2 \cdots a_{n-1} + 1$ (Sloane’s sequence A002585). This sequence was considered by A. A. Mullin in 1963 [201] who asked whether it contains all primes and is monotonic. After a few terms of this sequence were computed (in 1964 by R. R. Korfhage [153], in 1975 by R. K. Guy and R. Nowakowski [123] and in 1984 by T. Naur [212]) it turned out that $a_{10} < a_9$. It is still unknown whether a sequence $(a_n)$ contains all sufficiently large primes. Moreover, it can be constructed the second sequence of primes, similarly as the above sequence $(a_n)$, except that we replace the expression “$P_n := a_1a_2 \cdots a_{n-1} + 1$” by “$Q_n := a_1a_2 \cdots a_{n-1} - 1$”. This is the sequence A002584 in [278].

2.2. Proofs of $IP$ based on Goldbach’s idea on mutually prime integers. Goldbach’s idea consists in the obvious fact that any infinite sequence of pairwise relatively prime positive integers leads to a proof of Euclid’s theorem. C. Goldbach’s proof presented in a letter to L. Euler in July 20, 1730 (see Fuss [101, pp. 32–34, I], [248, p. 6], [98, pp. 40–41], [233, p. 4] or [9, pp. 85–86]) is based on the fact that the Fermat numbers $F_n := 2^n + 1, n = 0, 1, 2, \ldots$ are mutually prime (that is, pairwise relatively prime). Indeed, it is easy to see by induction that $F_{m-2} = F_0F_1 \cdots F_{m-1}$. This shows that if $n < m$, then $F_n$ divides $F_{m-2}$. Therefore, any prime dividing both $F_m$ and $F_n$ ($n < m$) must divide the difference $2 = F_m - (F_{m-2})$. But this is impossible since $F_n$ is odd, and this shows that Fermat numbers are pairwise relatively
Finally, assuming a prime factor of each of integers $F_n$, we obtain an infinite sequence of distinct prime numbers.

It seems that this was the first proof of IP which essentially differed from that of Euclid. In 1994 P. Ribenboim [245] wrote that the previous proof appears in an unpublished list of exercises of A. Hurwitz preserved in ETH in Zürich. A quite similar proof was published in the well known collections of exercises of G. Pólya and G. Szegő [240, p. 322, Problem 94] in 1925 (see also [128, p. 14, Theorem 16]).

Clearly, Goldbach’s idea is based on the fact that, in general the prime divisors of a sequence of integers greater than 1 form an infinite sequence of distinct primes if the integers in the sequence are pairwise relatively prime. In other words, Goldbach’s proof of IP will work with any sequence of positive integers for which any two distinct terms of the sequence are relatively prime.

Notice that Fermat numbers $F_n$ are Sloane’s sequence A000215; other sequences related to Fermat numbers are A019434, A094358, A050922, A023394 and A057755 and A080176. Today, the Fermat and Mersenne numbers $M_n := 2^n − 1$ which are considered in the next subsection, are important topics of discussion in many courses devoted to elementary number theory. For more information on classical and alternative approaches to the Fermat and Mersenne numbers see the article [143].

In 1880 J. J. Sylvester (see e.g., [305] and Wikipedia) generalized Fermat numbers via a recursively defined sequence of positive integers in which every term of the sequence is the product of the previous terms, plus one. This sequence is called Sylvester’s sequence and it is recursively defined as $a_{n+1} = a_n^2 − a_n + 1$ with $a_1 = 2$ (this is Sloane’s sequence A000058) and generalized by Sloane’s sequences A001543 and A001544. Clearly, choosing a prime factor of each term of Sylvester’s sequence yields IP.

Goldbach’s idea is later used by many authors to prove Euclid’s theorem by a construction of an infinite sequence of positive integers $1 < a_1 < a_2 < a_3 < \cdots$ that are pairwise relatively prime (i.e., without a common prime factor). In particular, in 1956 V. C. Harris [130] (see also [233, p. 6, Exercise 1.2.5], [321]) inductively defined an increasing sequence of pairwise relatively prime positive integers (cf. Sloane’s sequence A001685). This is the sequence $(A_n)$ recursively defined as $A_n = A_0A_1 \cdots A_{n−3}A_{n−1} + A_{n−2}$, for $n ≥ 3$ ($A_0, A_1$ and $A_2$ are given pairwise coprime positive integers, and $A_n$ is the numerator of approximants of some regular infinite continued fraction).

Euclid’s argument and Goldbach’s idea are applied in solution of Problem 52 [275, pages 5 and 40] to show that there exist arbitrarily long arithmetic progressions formed of different positive integers such
that every two terms of these progressions are relatively prime; namely, for any fixed integer \( m \geq 1 \) the numbers \((m!k + 1)\) for \( k = 1, 2, \ldots, m \) are relatively prime (cf. Sloane’s sequence A104189). This yields IP. This proof was later communicated to P. Ribenboim by P. Schorn [248 pp. 7–8].

Several other sequences leading to proofs of IP were established in 1957 by J. Lambek and L. Moser [160] and in 1966 by M. V. Subbarao [286]. Furthermore, in 1964 A. W. F. Edwards ([82], [248, page 7]) indicated various sequences, defined recursively, having this property (two related sequences are Sloane’s sequences A002715 and A002716). Similarly, in 2003 M. Somos and R. Haas [281] proved IP using an integer sequence defined recursively whose terms are pairwise relatively prime (cf. Sloane’s sequences A064526, A000324 and A007996). All these sequences (excluding one defined by Harris) and several other sequences of pairwise relatively prime positive integers are presented quite recently by A. Nowicki in his monograph [221, pp. 50–53, Section 3.5]. For example, if \( f(x) = x^2 - x + 1 \), then for any fixed \( n \in \mathbb{N} \), a sequence \( n, f(n), f(f(n)), f(f(f(n))), \ldots \) has this property [221, p. 51, Problem 3.5.4]. This is also satisfied for the following sequences \((a_n)\) defined recursively as:

\[
a_{n+1} = a_n^3 - a_n + 1; \quad a_1 = a, a_2 = a_1 + b, \ldots, a_{n+1} = a_1 a_2 \cdots a_n + b, \ldots
\]

with any fixed \( a, b \in \mathbb{N} \) such that \( b > a \geq 1; \quad a_1 = 2, a_{n+1} = 2^{a_n} - 1 \), and also for the sequence \( a_n := 1 + 3^n + 9^n \) given in [221 pp. 51–52, Problems 3.5.5, 3.5.6, 3.5.7, 3.5.10 and 3.5.15, respectively]. Furthermore, by a problem of 1997 Romanian IMO Team Selection Test [10, p. 149, Problem 7.2.3]), for any fixed integer \( a > 1 \), the sequence \((a^{n+1} + a^n + 1)\) \((n = 1, 2, \ldots)\) contains an infinite subsequence consisting of pairwise relatively prime positive integers. By a problem of the training of the German IMO team [85, pp. 121–122, Problem E3], using the factorization \( 2^{2n+1} + 2^{2n} + 1 = (2^n - 2^n - 1)(2^n + 2^n - 1) \), it was proved that \( 2^{2n+1} + 2^{2n} + 1 \) has at least \( n \) different prime factors for each positive integer \( n \).

In 1965 M. Wunderlich [321] (also see [210 p. 9, eleventh proof of Theorem 1.1]) indicated that every sequence \((a_n)\) of distinct positive integers having the property that \((m, n) = 1\) implies \((a_m, a_n) = 1\) leads to the proof of IP ((\(m, n\)) denotes the greatest common divisor of \(m\) and \(n\)). In particular, M. Wunderlich [321] noticed that Fibonacci’s sequence \((f_n)\) (defined by conditions \( f_1 = f_2 = 1, f_{n+2} = f_{n+1} + f_n \) with \( n = 1, 2, \ldots \); Sloane’s sequence A000045) has this property (proved in 1846 by H. Siebeck [272; also see 307 p. 30]). Notice that the sequence \((2^n - 1)\) also satisfies this property because of the well known fact that
\((2^n - 1, 2^m - 1) = 2^{(m,n)} - 1\) for all \(n, m \in \mathbb{N}\) (cf. \[233\] 5]). Using Wunderlich's argument indicated above, in 1966 R. L. Hemminiger \[134\] established IP by proving that the terms of the sequence \((a_n)\) defined recursively as \(a_1 = 2, a_{n+1} = 1 + \prod_{i=1}^{n} a_i\), are mutually prime. However, it is easy by induction to show that \(a_{n+1} = a_2^n - a_n + 1\) for each \(n \in \mathbb{N}\) (cf. Granville's proof in \[233\] p. 5, Exercise 1.2.3), i.e., \((a_n)\) coincides with Sylvester's sequence.

Further, IP obviously follows from Problem 51 of \[275\] pages 4 and 39 solved by A. Rotkiewicz which asserts that Fibonacci's sequence contains an infinite increasing subsequence such that every two terms of this sequence are relatively prime. This means that the set of all prime divisors of Fibonacci sequence is infinite. It was shown in 1921 by G. Pólya \[239\] that the same happens for a large class of linear recurrences (also cf. related results of H. Hasse \[133\] in 1966, J. C. Lagarias \[159\] in 1985, P. J. Stephens \[285\] in 1976, M. Ward \[310\] and \[311\] in 1954 and 1961, and H. R. Morton \[197\] in 1995).

Proof of IP due to S. P. Mohanty \[194\] Theorem 1 and Corollary 1; also see \[195\], \[233\] pp. 5–6, Exercise 1.2.4) in 1978, uses sequences that generalized Sylvester's sequence. By a problem of Polish Mathematical Olympiad in 2001/02 \[232\] Problem 6], see also \[221\] p. 51, Problem 3.5.3]), for any fixed positive integer \(k\), all the terms of a sequence \((a_n)\) defined by \(a_1 = k + 1, a_{n+1} = a_n^2 - ka_n + k\), are pairwise relatively prime. Notice that this sequence is a generalization of Sylvester's sequence and a particular case of a sequence from mentioned Mohanty's proof. Motivated by the same idea, in 1947 R. Bellman \[31\] (see also \[248\] page 7]) gave a simple “polynomial method” to produce infinite sequences with the mentioned property. In 1978 S. P. Mohanty \[194\] Theorem 3] proved that for any prime \(p > 5\), every prime divisor of Fibonacci number \(f_p\) is greater than \(p\). This immediately yields IP. IP also follows from Problem 42 of \[275\] pages 4, 35 and 362 which asserts that there exists an increasing infinite sequence of pairwise relatively prime triangular numbers \(t_n := n(n + 1)/2\), with \(n = 1, 2, \ldots\) (Sloane's sequence A000217). The same statement related to the tetrahedral numbers \(T_n := n(n + 1)(n + 2)/6\), with \(n = 1, 2, \ldots\), was given by Problem 43 of \[275\] pages 4 and 362 (Sloane's sequence A000292).

Goldbach's idea is later also applied by some authors. Firstly, notice that IP is indirectly proved by S. W. Golomb in 1963 (\[112\] the sequence (1)), also see \[3\] Section 2.5) which was constructed a recursive sequence whose terms are pairwise relatively prime and it present a generalization of Fermat numbers. (cf. Sloane's sequence A000289).
Analyzing the prime factors of $a^n - 1$ for given integer $a > 1$ and different integer values $n \geq 1$, in 2004 T. Ishikawa, N. Ishida and Y. Yuki-moto [138, Corollary 3] proved that there are infinitely many primes. Further, in 2007, for given $n \geq 2$ M. Gilchrist [108] constructed the so-called *-set of positive integers $a_1, a_2, \ldots, a_n$ satisfying $a_j \mid a_i - a_j$ for all distinct $i$ and $j$ with $1 \leq i, j \leq n$, and showed that the numbers $b_k := 2^{a_k} + 1$, $k = 1, 2, \ldots, n$ are mutually prime. Consequently, the set of primes is infinite. In a similar way, using the fact that for any integer $n > 1$, $n$ and $n + 1$ are mutually prime, and repeating this to $n(n + 1)$ and $n(n + 1) + 1$ etc., in 2006 F. Saidak [254] (for a generalization of this proof, see [196, pp. 26–27]) proved the infinitude of primes. Recently, J. M. Ash and T. K. Petersen [19, Examples 4a)-4e)] proved IP by presenting similar recursively defined sequences of positive integers. For a construction of some infinite coprime sequence see the paper [172] of N. Lord in 2008.

2.3. Proofs of IP based on algebraic number theory arguments. In 1736 L. Euler was derived second proof of Euclid’s theorem (published posthumously in 1862 [93] (also see [96, Sect. 135] and [76, p. 413]) by using the totient function $\phi(n)$, defined as the number of positive integers not exceeding $n$ and relatively prime to $n$ (Sloane’s sequence A000010); for a proof also see [46, pp. 134–135], [233, page 3]. As noticed by Dickson [76, p. 413] (see also [258, page 80]), this proof is also attributed in 1878/9 by Kummer [156] who gave essentially Euler’s argument. The proof is based on the multiplicativity of the $\phi$-function. Namely, if $p_1, p_2, \ldots, p_n$ is a list of distinct $n \geq 2$ primes with product $P$, then

$$\phi(P) = (p_1 - 1)(p_2 - 1) \cdots (p_n - 1) \geq 2^{n-1} \geq 2.$$ 

This inequality says there exists an integer in the range $[2, P]$ that is relatively prime to $P$, but such an integer has a prime factor necessarily different from any of the $p_k$ with $k = 1, 2, \ldots, n$. This yields IP.

Euler’s idea is in 2009 applied by J. P. Pinasco [230]. Assuming that $p_1, p_2, \ldots, p_n$ are all the primes and using the Inclusion-Exclusion Principle, Pinasco derived the formula for number of integers in the interval $[1, x]$ that are divisible by at least of one of primes $p_i$, which yields

$$[x] - 1 = \sum_i \left[ \frac{x}{p_i} \right] - \sum_{i<j} \left[ \frac{x}{p_ip_j} \right] + \sum_{i<j<k} \left[ \frac{x}{p_ip_jp_k} \right] - \cdots + (-1)^{n+1} \left[ \frac{x}{p_1p_2 \cdots p_n} \right]$$

($[\cdot]$ denotes the greatest integer function), whence letting $x \to \infty$ easily follows that $1 > 1$; a contradiction. Using the identity $\sum_{n=1}^{\infty} \mu(n) \left[ x/n \right] =$
1 established in 1854 by E. Meissel \[182\] (cf. also \[270\], the formula (3.5.14)), in 2012 the author of this article \[186\] presented a very short “Pinasco’s revisted” proof of IP. Furthermore, the author \[186, Remark\] noticed that a quite similar proof of IP also follows using Legendre’s formula stated in the modern form \[210\] p. 33, Theorem 1.17 as

\[
\pi(n) - \pi(\sqrt{n}) = \sum_{d \mid \Delta} \mu(d) \left\lfloor \frac{x}{d} \right\rfloor - 1
\]

(\(\pi(n)\) denotes the number of primes not exceeding \(n\)).

Using Theory of Commutative Groups, in 1888 J. Perott \([228\), \([229, pp. 303–305]\); also cf. \([65]\) showed that, if \(p_1, p_2, \ldots, p_n\) are primes, then there exist at least \(n - 1\) primes between \(p_n\) and \(p_1 p_2 \cdots p_n\).

Using Euler theorem which asserts that \(a^{\varphi(n)} \equiv 1 \pmod{n}\) with relatively prime integers \(a\) and \(n \geq 1\), in 1921 G. Pólya \([239\) pp. 19–21] (also see \([240\) pp. 131, 324, Problem 107]) proved that the set of primes dividing the integer values of the exponential function \(ab^x + c\) (\(x = 0, 1, 2, \ldots\)) with integer coefficients \(a \neq 0, c \neq 0\) and \(b \geq 2\) is infinite.

Another proof of IP, based on the divisibility property \(n \mid \varphi(a^n - 1)\) (\(a, n > 1\) are integers), is given in 1986 by M. Deaconescu and J. Sándor \([71\] (see also \([237]\)). Notice that the \(\varphi\)-function is applied by G. E. Andrews \([12\) p. 102, Theorem 8-4] to give an elementary proof that \(\lim_{x \to \infty} \pi(x)/x = 0\), where \(\pi(x)\) is the prime-counting function defined as the number of primes not exceeding \(x\) (\(x\) is any real number). In other words, the “probability” that a randomly chosen positive integer is prime is 0. Using the Inclusion-Exclusion Principle, this result is by an elementary way also proved by A. M. Yaglom and I. M. Yaglom \([322\) pp. 34, 209–211, Problem 94]

It was noticed in \([233\) p. 4, Exercise 1.2.1] that adapting Euclid’s proof of IP, it can be proved that for every integer \(m \geq 3\), there exist infinitely many primes \(p\) such that \(p - 1\) is not divisible by \(m\). This result is generalized by A. Granville \([233\) p. 4, Exercise 1.2.2], \([132\) p. 168]; also cf. \([118\) p. 4, Exercises 1.3 a\]) to prove that if \(H\) is a proper subgroup of the multiplicative group \(\mathbb{Z}/m\mathbb{Z}^*\) of elements \((\mod m)\), then there exist infinitely many primes \(p\) with \(p \pmod{m} \notin H\).

Similarly, considering order of \(a\pmod{p}\) in the multiplicative group modulo \(p\), in 1979 A. Weil \([315\) p. 36, Exercise VIII.3] proved that if \(p\) is an odd prime divisor of \(a^{2n} + 1\), with \(a \geq 2\) and \(n \geq 1\), then \(p - 1\) is divisible by \(2^{n+1}\). This immediately yields IP.

Using Euler’s theorem, it can be proved by induction that the sequence \(2^n - 3, n = 1, 2, \ldots\) contains an infinite subsequence whose
terms are pairwise relatively prime (Problem 3 proposed on International Mathematical Olympiad (IMO) 1971 [79, pages 70 and 392–393]). Another less known proof is based on Lagrange theorem on order of subgroup of a finite group and Mersenne number $2^p - 1$ with a prime $p$ as follows. Namely, using Lagrange theorem it can be shown that each prime divisor $q$ of $2^p - 1$ divides $q - 1$, and so $p < q$, which implies $IP$; using this fact, we can inductively obtain an infinite increasing sequence $(p_n)$ of primes assuming that $p_{n+1} | 2^{p_n} - 1$ for each $n = 1, 2, \ldots$. This proof can be found in [6, p. 3, Second proof], [11, p. 32, Proposition 1.30 and p. 72, Theorem 1.50] and at webpage [72]. Mersenne numbers ([248, pp. 75–87, Ch. VII], [115, pp. 109–110]) $2^n - 1$, $n = 1, 2, \ldots$ and the numbers $2^p - 1$ with $p$ prime form Sloane’s sequences A000225 and A001348, respectively; also see related sequences A000668, A000043, A046051 and A028335).

Similarly, in 1978 Mohanty [194, Theorem 2] proved that for any prime $p > 3$, every prime divisor of $(2^p + 1)/3$ is greater than $p$, and this together with the previous argument yields $IP$.

Using the Theory of periodic continued fractions (cf. related Sloane’s sequence A003285) and the Theory of negative Pell’s equations $x^2 - dy^2 = -1$, in 1976 C. W. Barnes [25] proved $IP$. Namely, supposing that $p_1 = 2, p_2, \ldots, p_k$ are all the primes with a product $2Q$, Barnes proved that $Q^2 + 1$ cannot be a power of two; but T. Yamada [324, p. 8] noticed that this fact is obvious since $Q^2 + 1 \equiv 2 \pmod{4}$.

A proof of D. P. Wegener [314] of 1981 based on a study of the sums of the legs of primitive Pythagorean triples also contains Euclid’s idea (these triples are triples $(x, y, z)$ of positive integers such that $x^2 + y^2 = z^2$ and $x$ and $y$ are relatively prime; cf. [59, Ch. 2, pp. 31–34]).

We also point out an interesting result established as a solution of advanced problem in [11] pp. 110–111, Problem 37 (a)]; namely, this result (with two solutions) asserts that if $a$ and $b$ are relatively prime positive integers, then in the arithmetic progression $a + nb, n = 1, 2, \ldots$ there are infinitely many pairwise relatively prime terms, which yields $IP$.

Washington’s proof of Euclid’s theorem from 1980 ([312], [248 pp. 11–12]) is via commutative algebra, applying elementary facts of the Theory of principal ideal domains, unique factorization domains, Dedekind domains and algebraic numbers, may be found in [256]. Namely, using the fact that $(1 + \sqrt{-5})(1 - \sqrt{-5}) = 2 \times 3$ in the ring of algebraic integers $a + b\sqrt{-5}$ ($a, b \in \mathbb{Z}$) (i.e., in the field of numbers $a + b\sqrt{-5}$ ($a, b \in \mathbb{Q}$)), it follows that this ring is not a unique factorization domain. Hence, it is not a principal domain, whence Washington deduced
The algebraic arguments applied in this proof are exposed and well studied in 2001 by B. Chastek [54].

Quite recently in 2011, applying two simple lemmas in the Theory of Finite Abelian Groups related to the product of some cyclic groups $\mathbb{Z}_m$, R. Cooke [65] modified Perott’s proof noticed above, to establish that there are at least $n - 1$ primes between the $n$th prime and the product of the first $n$ primes.

A “dynamical systems proof” due to S. Srinivasan ([283], also see [324]) in 1984 uses a polynomial method and Fermat little theorem. Srinivasan constructed the sequence $(a_n)$ of positive integers satisfying $a_i | a_{i+1}$ and $a_i | a_{i+1}/a_i$ for each $i = 1, 2, \ldots$. Then we immediately see that the sequence $(a_{n+1}/a_n)$ contains no two integers which has a nontrivial common divisor. This yields IP.

In 2011 P. Pollack [235] considered a M"obius pair of arithmetic functions $(f, g)$; that is, functions satisfying $f(n) = \sum_{d|n} g(d)$ for all $n = 1, 2, \ldots$, and hence, one can express $g$ in terms of $f$ by the M"obius inversion formula [218]. Then Pollack deduced IP by proving the uncertainty principle for the M"obius transform which asserts that the functions $f$ and $g$ that become M"obius pair cannot both be of finite support unless they both vanish identically. The strategy of Pollack proof goes back to J. J. Sylvester [288] in 1871, who using certain identities between rational functions, gave an argument in the same spirit for IP of the form $p \equiv 3(\text{mod } 4)$ and $p \equiv 5(\text{mod } 6)$ (cf. Remarks (ii) in [235]).

In 2011 R. M. Abrarov and S. M. Abrarov [4, p. 9] deduced IP applying Euclid’s idea to the identity $\mu(n) = -\sum_{i,j=1}^{\sqrt{n}} \mu(i)\mu(j)\delta\left(\frac{n}{ij}\right)$ ($n \geq 2$) obtained in their earlier paper [3, the identity (11)] (also see [4, p. 2, the identity (3)]), involving the M"obius function $\mu(n)$ (defined so that $\mu(1) = 1$, $\mu(n) = (-1)^k$ if $n$ is a product of $k$ distinct primes, and $\mu(n) = 0$ if $n$ is divisible by the square of a prime), and the delta function $\delta(x)$ (defined as $\delta(x) = 1$ if $x \in \mathbb{N}_0 := \{0, 1, 2, \ldots\}$, and $\delta(x) = 0$ if $x \not\in \mathbb{N}_0$). In the same paper, the authors proved IP [4, p. 9] as an immediate consequence of [4, the formula (26)] for the asymptotic density of prime numbers. Their third proof [4, pp. 9–10] follows from [4, p. 2, the formula (4)] related to the prime detecting function.

2.4. Proof of IP based on Euler’s idea on the divergence of the sum of prime reciprocals and Euler’s formula. Notice that the proofs of Euclid’s theorem presented in the previous subsections are mainly elementary. On the other hand, there are certain proofs of Euclid’s theorem that are based on ideas from Analytic Number Theory. A more sophisticated proof of Euclid’s theorem was given...
many centuries later by the Swiss mathematician Leonhard Euler. In
1737 Euler in his work [94, pp. 172–174] (also see [92]) showed that
by adding the reciprocals of successive prime numbers you can attain
a sum greater than any prescribed number; that is, in terms of modern
Analysis, the sum of the reciprocals of all the primes is divergent (cf.
[248, page 8], [98, pp. 8–9]). For more information on Euler’s work on
infinite series see [304]. Briefly, Euler considered the possibly infinite
product \( \prod_{p \leq x} \left( 1 - \frac{1}{p} \right)^{-1} \), where the index \( p \) runs over all primes. He
expanded the product to obtain the divergent infinite harmonic series
\( \sum_{n=1}^{\infty} \frac{1}{n} \), concluded the infinite product was also divergent, and from
this concluded that the infinite series \( \sum \frac{1}{p} \) also diverges. This can be
written symbolically as
\[
\frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \frac{1}{13} + \frac{1}{17} + \frac{1}{19} + \cdots = +\infty.
\]
A result related to this divergence was refined in 1874 by F. Mertens
[184] (see also [128, p. 351, Theorem 427]); namely, by Mertens’ second theorem,
as \( n \to \infty \) the sum \( \sum_{p \leq n} 1/p - \log \log n \) (taken over all
primes \( p \) not exceeding \( n \)) converges to the Meissel-Mertens constant
\( M = 0.261497\ldots \) (also known as the Hadamard-de la Vallée-Poussin constant).

Using the Euler’s idea, in 1888 J. J. Sylvester [290] (also cf. [210, p.
7, Sixth proof of Theorem 1.1]) observed that
\[
\prod_{p \leq x} \left( 1 - \frac{1}{p} \right)^{-1} = \prod_{p \leq x} \left( 1 + \frac{1}{p} + \frac{1}{p^2} + \cdots \right) \geq \sum_{n \leq x} \frac{1}{n} \geq \log x,
\]
(where the product runs over all primes \( p \) not exceeding \( x \)), and since
\( x \) may be arbitrarily large, the set of primes must be infinite. Using
the above estimate and the convergence of the series \( \sum_{n=1}^{\infty} 1/n^2 \), in the
same paper J. J. Sylvester [290] (also cf. [210, pp. 11–12, Second proof
of Theorem 1.4]) easily proved that the product \( \prod_{p \leq x} (1 + 1/p) \) tends
to infinity as \( x \to \infty \). This implies IP.

A correct realization of Euler’s idea was presented by L. Kronecker
in his lectures in 1875/76 ([154]; also see [132, pp.269–273] and [76, p.
413, Ch. XVIII]). Kronecker noted that “Euler’s” proof also follows
from the Euler’s formula
\[
\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod \left( 1 - \frac{1}{p^s} \right)^{-1} (s > 1),
\]
where the product on the right is taken over all primes \( p \) (the first
formula in the next subsection), and the fact that the series \( \sum_{n=1}^{\infty} 1/n^s \)
diverges for each \( s > 1 \). For some discussion of the history of this
formula in relation to the infinitude of primes, see [64]. As noticed by Dickson [76, p. 413], in 1887/8 L. Gegenbauer [104] proved $\sum_{n=1}^{\infty} 1/n^s$. Dickson [76, p. 413] remarked that in 1876 R. Jaensch [141] repeated Euler’s argument, also ignoring convergency.

Other elementary proofs of the fact that the sum of reciprocals of all the primes diverges were given in 1943 by R. Bellman [29], in 1956 by E. Dux [81], in 1958 by L. Moser [199], in 1966 by J. A. Clarkson [61] and in 1995 by D. Treiber [298]. A survey of some these proofs was given in 1965 by T. Salát [292]. Furthermore, in 1980 C. Vanden Eynden [303] considered Euler’s type product of all expressions of the form

$$
\left(1 + \frac{1}{p}\right) \sum_{k=0}^{\infty} \frac{1}{p^{2k}} = \sum_{j=0}^{\infty} \frac{1}{p^j},
$$

where $p$ ranges over the set of all primes not exceeding $x$. This equality together with the divergence of the series $\sum_{n=1}^{\infty} 1/n$ and the convergence of the series $\sum_{n=1}^{\infty} 1/n^2$ easily yields the divergence of the sum of the reciprocals of all the primes.

It is interesting to notice that in actual reality, Euler never presented his work as a proof of Euclid’s theorem, though that conclusion is clearly implicit in what he did. Euler’s remarkable proof of $\sum_{n=1}^{\infty} 1/n$ amounts to unique factorization, and it is also discussed at length by R. Honsberger in his book [137] and modified in 2003 by C. W. Neville [213, Theorem 1(a)]. In 1938 P. Erdős ([88]; also see [6, pp. 5–6, Sixth proof], [128, p. 17, Theorem 19] and [233, pp. 12–13]) gave an elementary “counting” proof of the divergence of the sum of reciprocals of primes, and consequently, the set of all primes is infinite. P. Pollack [233, p. 11] pointed out that it is remarkable that this method of proving $\sum_{n=1}^{\infty} 1/n$ (in contrast with Euclid’s proof, for instance) is independent of the additive structure of the integers.

Remarks. Notice that the asymptotic behavior of the product of

$$
\prod_{p \leq n} (1 - 1/p) \sim e^{-\gamma}/\log n,
$$

where the product runs over all primes $p$ not exceeding $n$, and $\gamma = 0.577216 \ldots$ is Euler-Mascheroni constant. An elementary geometrical proof of Mertens’ third theorem with another constant $c$ instead of $e^{-\gamma}$, was given in 1954 by A. M. Yaglom and I. M. Yaglom [323, pp. 41; 194–196, Problem 174]. Using Mertens’ third theorem (with the constant $e^{-\gamma}$), in [323, p. 42] the authors also derived the curious formula

$$
\prod_{p \leq n} (1 + 1/p) \sim (e^\gamma \log n)/\pi^2,
$$

as $n \to \infty$.

Furthermore, using the classical Chebyshev’s argument based on the well known de Polignac’s formula (attributed by Dickson[p. 263, Ch. IX]d to A.-M. Legendre [167, p. 8] in 1808) for the exponent $\nu_p(n!)$
of prime $p$ dividing the factorial $n!$ given as $\nu_p(n!) = \sum_{k=1}^{\infty} \left\lfloor \frac{n}{p^k} \right\rfloor$, a short proof that the sum $\sum_p \log p/p$ diverges due to P. Erdős is presented in [324, 8th proof] and this yields $IP$. Similarly, using de Polignac’s formula, in 1969 [62, pp. 613–614, Remark 6] (cf. also [69, p. 54, Exercise 1.21]) E. Cohen gave a short simple proof that the series $\sum_p \log p/p$ diverges (the sum ranges over all the primes), which yields $IP$. This result also follows from Mertens’ first theorem obtained in 1874 by F. Mertens [184], which asserts that the quantity $|\sum_{p \leq n} \log p/p - \log n|$ is bounded, in fact $< 4$ (for an elementary proof, see [323, pp. 171, 183–186, Problem 171]). Notice that this result immediately follows from Mertens’ second theorem.

Further, combining the Euler’s idea with the geometrical interpretation of definite integral $\int_1^x \left(1/t\right) \, dt = \log x$ with $n \leq x < n + 1$ in their Problems book [6, p. 4, Fourth proof] A. M. Yaglom and I. M. Yaglom proved the inequality $\log x \leq \pi(x) + 1$, where $\pi(x)$ is the prime-counting function. This inequality immediately yields $IP$.

Another modification of Euler’s proof, involving the logarithmic complex function, can be found in book [69, p. 35] of R. Crandall and C. Pomerance.

Remarks. Notice that from Euclid’s proof (see e.g., [128, p. 12, Theorem 10]) easily follows that $\pi(x) \geq \log_2 \log_2 x$ for each $x > 1$, and the same bound follows more readily from the Fermat numbers proof. Of course, this is a horrible bound. From the Erdős’s proof [88] given above it can be easily deduced the bound $\pi(x) \geq \log x/(2 \log 2) = \log_3 x/2$ for each $x \geq 1$ [128, p. 17, Theorem 20]. This estimate can be improved using Bonse’s inequality presented above. Namely, applying induction, it follows from this inequality that $p_n \leq 2^n$; so, given $x \geq 2$, taking $x = 2^n + y$ with $0 \leq y < 2^n$, we find that $\pi(x) \geq \pi(2^n) \geq n \geq \log_2 x - 1$. □

Remarks. Recall that an extremely difficult problem in Number Theory is the distribution of the primes among the natural numbers. This problem involves the study of the asymptotic behavior of the counting function $\pi(x)$ which is one of the more intriguing functions in Number Theory. For elementary methods in the study of the distribution of prime numbers, see [74]. Studying tables of primes, C. F. Gauss in the late 1700s and A.-M. Legendre in the early 1800s conjectured the celebrated Prime Number Theorem: $\pi(x) = |\{p \leq x : p \text{ prime}\}| \sim x/\log x$ ($|S|$ denotes the cardinality of a set $S$). This theorem was proved much later ([69, p. 10, Theorem 1.1.4]; for its simple analytic proof see [215] and [326], and for its history see [26] and [110]). Briefly, $\pi(x) \sim x/\log x$ as $x \to \infty$, or in other words, the density of primes $p \leq x$ is $1/\log x;$
that is, the ratio $\pi(x) : (x/\log x)$ converges to 1 as $x$ grows without bound. Using L'Hôpital's rule, Gauss showed that the logarithmic integral $\int_2^x dt \log t$, denoted by $\text{Li}(x)$, is asymptotically equivalent to $x/\log x$. Recall that Gauss felt that $\text{Li}(x)$ gave better approximations to $\pi(x)$ than $x/\log x$ for large values of $x$. Though unable to prove the Prime Number Theorem, several significant contributions to the proof of Prime Number Theorem were given by P. L. Chebyshev in his two important 1851–1852 papers ([55] and [56]). Chebyshev proved that there exist positive constants $c_1$ and $c_2$ and a real number $x_0$ such that $c_1 x/\log x \leq \pi(x) \leq c_1 x/\log x$ for $x > x_0$. In other words, $\pi(x)$ increases as $x \log x$.

Using methods of complex analysis and the ingenious ideas of Riemann (forty years prior), this theorem was first proved in 1896, independently by J. Hadamard and C. de la Vallée-Poussin (see e.g., [233, Section 4.1]).

2.5. Proof of IP based on Euler's product for the Riemann zeta function and the irrationality of $\pi^2$ and $e$. Proofs of IP presented in this subsection involve the Riemann zeta function (for $\Re(s) > 1$, to ensure convergence) defined as $\zeta(s) := \sum_{n=1}^{\infty} 1/n^s$. Riemann introduced the study of $\zeta(s)$ as a function of a complex variable in an 1859 memoir on the distribution of primes [249]. However, the connection between the zeta function and the primes goes back earlier. Over a hundred years prior, Euler had looked at the same series for real $s$ and had shown that [94, Theorema 8]

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \frac{1}{1 - \frac{1}{p^s}} \quad (s > 1).$$

This is the Euler's factorization which is often called an analytic statement of unique factorization (this is a consequence of a well known standard uniqueness theorem for Dirichlet series [15, Theorem 11.3]).

Dickson [76, p. 414] (also see [233, p. 10]) noticed that in 1899 J. Braun attributed a proof of IP by means of the Euler’s formula $\sum_{n=1}^{\infty} 1/n^2 = \pi^2/6$ (for elementary proofs of this formula see [58], [107] and [170]) and the Euler’s factorization $\prod 1/(1 - p^{-s}) = \sum_{n=1}^{\infty} 1/n^2$ (Sloane’s sequence A013661) and the irrationality of $\pi^2$ proved in 1794 by Legendre [166] (also see [128, p. 47, Theorem 49], [247, p. 285]). Namely, if there were only finitely many primes, then $\zeta(2)$ would be rational; a contradiction. Notice also that this proof was reported in 1967 in the reminiscences of Luzin’s Moscow school of mathematics 100 years ago by L. A. Lyusternik [177, p. 176] (also cf. [64, p. 466]) which ascribed this proof to A. Y. Khinchin. Such proofs attract interest because they make unexpected connections. According
to Lyusternik, “exotic” proofs of IP were a routine challenge among Luzin’s students, and many such proofs were found. But apparently no one thought of publishing them. The previous equality is in fact, the well known Euler’s formula (or Euler’s product) [128, p. 245] for the Riemann zeta function \( \zeta(2) := \sum_{n=1}^{\infty} \frac{1}{n^2} \) [128, p. 246, Theorem 280]. The same proof of IP was also presented in 2007 by J. Sondow [282]. Notice that, applying the same argument for the product formula \( \prod_{1 < p} \left( \frac{1}{1 - \frac{1}{p^2}} \right) = \sum_{n=1}^{\infty} \frac{1}{n^3} := \zeta(3) \) together with a result of R. Apéry in 1979 [13] that \( \zeta(3) \) is irrational, we obtain IP.

Further, using the Euler’s formulas for \( \zeta(2) \) and \( \zeta(4) = \sum_{n=1}^{\infty} \frac{1}{n^4} = \pi^4/90 \) [128, p. 245] (Sloane’s sequence A0013662), it can be easily obtained that 5/2 = \( \prod_{p} \left( \frac{1}{1 - \frac{1}{p^2}} \right) \), where the product is taken over all the primes [233, p. 11]. In 2009 P. Pollack [233, p. 11] observed that if the set of all primes is finite, then the numerator of the ratio on the right of this formula is not divisible by 3, but its denominator is divisible by 3. This contradiction yields IP. We recall Wagstaff’s (open) question [122, B48] as to whether there exists an elementary proof of the previous formula.

Notice that \( \zeta(2) \) and \( \zeta(4) \) are two special cases of the following classic formula discovered by Euler in 1734/35 [92], which express \( \zeta(2n) \) as a rational multiple of \( \pi^{2n} \) involving Bernoulli number \( B_{2n} \): \( \zeta(2n) = (-4)^{n-1}B_{2n}\pi^{2n}/(2 \cdot (2k)!) \) \((n = 1, 2, 3, \ldots)\). An elementary proof of this formula for \( n = 1 \) is given by I. Papadimitriou [224] in 1973 and for arbitrary \( n \) by T. M. Apostol [14] in the same year (for another elementary evaluations of \( \zeta(2n) \) see [32] and [223]). For instance, since \( B_2 = 1/6 \) and \( B_4 = 1/30 \), we find that \( \zeta(2) = \pi^2/6 \) and \( \zeta(4) = \pi^4/90 \), respectively. □

2.6. Combinatorial proofs of IP based on enumerative arguments. Several combinatorial proofs of IP involve simple counting arguments. More precisely, these proofs are mainly based on counting methods which are used in them to count the cardinality of integers
less than a given integer \( N \) and which satisfy certain divisibility properties. The first such proof, given by J. Perott in 1881 ([227], [248, p. 10] and [210, p. 8]) is based on the facts that the series \( \sum_{n=1}^{\infty} 1/n^2 \) is convergent with the sum smaller than 2 and that there exist exactly \( 2^n \) divisors of the product of \( n \) distinct primes. In his proof Perott also established the estimate \( \pi(n) > \log_2(n/3) \), where \( \pi(n) \) is the number of primes less than or equal to \( n \). Perott’s proof was modified in [233, pp. 11–12] by eliminating use of the formula \( \zeta(2) = \pi^2/6 \). Using Perott’s method, in 2006 L. J. P. Kilford [148] presented a quite similar proof of IP based on the fact that for any given \( k \geq 2 \), the sum \( \sum_{n=1}^{\infty} 1/n^k \) converges to a real number which is strictly between 1 and 2.

A classical proof of IP which is combinatorial in spirit and entirely elementary, was given by Thue in 1897 in his work [297] (also see [76, 414] and [248, page 9]). This proof uses a “counting method” and the fundamental theorem of unique factorization of positive integers as a product of prime numbers as follows.

Choose integers \( n, k \geq 1 \) such that \( (n + 1)^k < 2^n \) and set \( m = 2^{e_1} \cdot 3^{e_2} \cdots p_r^{e_r} \), where we assume that \( 2 < 3 < \cdots < p_r \) is a set of all the primes and \( 1 \leq m \leq 2^n \). Suppose that \( m \leq 2^n \). Since \( m \leq 2^n \), we have \( 0 \leq e_i \leq n \) for each \( i = 1, 2, \ldots, r \). Then counting all the possibilities, it follows that \( 2^n \leq (n + 1)n^{r-1} < (n + 1)^r \leq (n + 1)^k < 2^n \). This contradiction yields \( r \geq k + 1 \). Now taking \( n = 2k^2 \), then since \( 1 + 2k^2 < 2^{2k} \) for each \( k \geq 1 \), it follows that \( (1 + 2k^2)^k \leq 2^{2k^2} = 4^k \), and so there at least \( k + 1 \) primes \( p \) such that \( p < 4^k \). Thus, letting \( k \to \infty \) yields IP.

Applying a formula for the number of positive integers less than \( N \) given in [47, Ch. XI], in 1890 J. Hacks [124] (see also [76, p. 414]) proved IP.

In order to prove IP, similar enumerating arguments to those of Thue were used in a simple Auric’s proof, which appeared in 1915 [21, p. 252] (also see [76, 414], [248, page 11]), as well by P. R. Chernoff in 1965 [57], M. Rubinstein [252] in 1993 and M. D. Hirschorn [136] in 2002. A proof of IP similar to that of Auric is given in 2010 by M. Coons [66].

Using a combinatorial argument, the unique factorization theorem and the pigeonhole principle, IP is recently proved by D. G. Mixon [193].

A less known elementary result of P. Erdős [87, p. 283] (also see [86]) in 1934, based on de Polignac’s formula (actually due to A.-M. Legendre), asserts that there is a prime between \( \sqrt{n} \) and \( n \) for each positive integer \( n > 2 \). In the same paper Erdős proved that if \( n \geq 2k \), then \( n \choose k \) contains a prime divisor greater than \( k \). In particular, this fact for \( n = 2k \) obviously yields IP. Notice also that IP follows by two
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results of W. Sierpiński from his monograph in 1964 [273]. Namely, if we suppose that there are a total of \( k \) primes, then by [273, page 132–133, Lemmas 1 and 4], we have \( 4^n/2\sqrt{n} \leq (2n)^k \) for each positive integer \( n > 1 \). This contradicts the fact that \( 4^n/(2\sqrt{n}) \geq (2n)^k \) for sufficiently large \( n \).

In 2010 J. P. Whang [318] gave a short proof of IP by using de Polignac’s formula.

2.7. Furstenberg’s topological proof of IP and its modifications. A proof of Euclid’s theorem due to H. Furstenberg in 1955 ([100]; also see [248, pp. 12–13], [233, p. 12] or [6, p. 5]) is a short ingenious proof based on topological ideas. In order to achieve a contradiction, Furstenberg introduced a topology on the set of all integers, namely the smallest topology in which any set of all terms of a non-constant arithmetic progression is open. Here we quote this proof in its entirety: “In this note we would like to offer an elementary “topological” proof of the infinitude of prime numbers. We introduce a topology into the space of integers \( S \), by using the arithmetic progressions (from \(-\infty \) to \(+\infty \)) as a basis. It is not difficult to verify that this actually yields a topological space. In fact under this topology \( S \) may be shown to be normal and hence metrizable. Each arithmetic progression is closed as well as open, since its complement is the union of other arithmetic progressions (having the same difference). As a result the union of any finite number of arithmetic progressions is closed. Consider now the set \( A = \bigcup A_p \), where \( A_p \) consists of all multiples of \( p \), and \( p \) runs though the set of primes \( \geq 2 \). The only numbers not belonging to \( A \) are \(-1 \) and \( 1 \), and since the set \( \{-1, 1\} \) is clearly not an open set, \( A \) cannot be closed. Hence \( A \) is not a finite union of closed sets which proves that there are an infinite of primes.”

In 1959 S. W. Golomb [111] developed further the idea of Furstenberg and gave another proof of Euclid’s theorem using a topology \( D \) on the set \( \mathbb{N} \) of natural numbers with the base \( B = \{\{an + b\} : (a, b) = 1\} \) \((a, b)\) denotes the greatest common divisor of \( a \) and \( b \), defined in 1953 by M. Brown [43]. In the same paper Golomb proved that the topology \( D \) is Hausdorff, connected and not regular; \( \mathbb{N} \) is \( D \)-connected, and the Dirichlet’s theorem (on primes in arithmetic progressions) is equivalent to the \( D \)-density of the set of primes in \( \mathbb{N} \). Moreover, in 1969 A. M. Kirch [149] proved that the topological space \((\mathbb{N}, D)\) is not locally connected.

In 2003 D. Cass and G. Wildenberg [51] (also cf. [150]) have shown that Furstenberg’s proof can be reformulated in the language of periodic functions on integers, without reference to topology. This is in
fact, a beautiful combinatorial version of Furstenberg’s proof. Studying arithmetic properties of the multiplicative structure of commutative rings and related topologies, in 2001 Š. Porubsky \[241\] established new variants of Furstenberg’s topological proof. Notice also that Furstenberg’s proof of IP is well analyzed in 2009 by A. Arana \[17\], in 2008 by M. Baaz, S. Hetzl, A. Leitsch, C. Richter and H. Spohr \[22\], and also discussed in greater detail in 2011 by M. Detlefsen and A. Arana \[73\]. Furthermore, C. W. Neville \[213, Theorem 1(a)\] pointed out that this proof has been extended in various directions, for example, to the setting of Abstract Ideal Theory see \[151\] and \[241\].

More than 50 years later, in 2009 using Furstenberg’s ideas but rephrased without topological language, I. D. Mercer \[183\] provided a new short proof that the number of primes is infinite. Finally, notice that Furstenberg’s proof is an important beginning example in the Theory of profinite groups (see book reviews by A. Lubotzky \[173\] in 2001).

2.8. Another proofs of IP. Euclid’s proof of IP was revisited in 1912/13 by I. Schur \[260\] (see also \[240\] pp. 131, 324, Problem 108]) who showed that the set of primes dividing the integer values of a nonconstant integer polynomial is infinite. Suppose that $Q$ is a polynomial with integer coefficients such that \{p_1, p_2, \ldots, p_k\} is a set of all primes with this property is finite. Then assuming that $Q(a) = b \neq 0$, we will consider the integer value $c = (Q(a + bp_1 p_2 \cdots p_k))/b$. Then obviously $c \equiv 1 \pmod{p_1 p_2 \cdots p_k}$ and therefore, $c$ has at least one prime divisor, say $p$, distinct from every element of the set \{p_1, p_2, \ldots, p_k\}. It follows that the value $Q(a + bp_1 p_2 \cdots p_k) = bc$ is also divisible by $p$; a contradiction. In particular, for $Q(x) = x + 1$ the previous proof is a copy of Euclid’s proof of IP. If $Q(x) = \Phi_m(x)$ is the $m$th cyclotomic polynomial, then the above proof yields that there are infinitely many primes which are congruent to 1(mod $m$) (cf. Section 3).

Remarks. In 1990 P. Morton \[198\] considered a related problem for an integer sequence $(a_n)$ for which there is an integer constant $c$ such that for all $i \in \mathbb{Z} = \{\ldots -2, -1, 0, 1, 2, \ldots\}$ $a_n = i$ holds for almost $c$ values of $n$. If for such an integer sequence $(a_n)$, the so called almost-injective, define the set $S(a_n) = \{p$ prime : $p \mid a_n$ for at least one $n \in \mathbb{N}\}$, then Morton \[198\] proved that $S(a_n)$ is infinite if $(a_n)$ has at most polynomial growth, i.e., $|a_n| \leq an^d$ for some positive constants $a$ and $d$. This result is extended quite recently in 2012 by C. Elsholtz \[83\] for almost-injective integer sequences of subexponential growth, i.e., for almost-injective integer sequences $(a_n)$ for which $a_n = o(\log n)$. As noticed in \[83\] p. 333, another way to look at this theorem is to study
“primitive divisors” of integer sequences. Given an integer sequence \((a_n)\), a divisor \(d\) is called primitive if \(a_i\) is divisible by \(d\), but \(a_j\) is not divisible by \(d\) for any \(j < i\). For a good survey of this topic, see Chapter 6 of the book [97].

However, it is not known whether there are polynomials of degree greater than 1 with integer coefficients representing infinitely many primes for integer argument. Using Chebyshev’s estimate \(\pi(x) \geq x/\log x\) and a simple counting argument, in 1964 W. Sierpiński [274] (also see [233, p. 35, Theorem 1.6.1]) proved that for every \(N\) there exists an integer \(k\) for which there are more than \(N\) primes represented by \(x^2 + k\) with \(x = 0, 1, 2, \ldots\). In 1990 B. Garrison [102] (cf. [233, p. 36, Exercise 1.6.2]) generalized Sierpiński’s result to polynomials \(x^d + k\) of degree \(d \geq 2\) and proved that for any such \(d\) and any \(N\) there exists a positive integer \(k\) such that \(x^d + k\) \((x = 0, 1, 2, \ldots)\) assumes more than \(N\) prime values. P. Pollack [233, p. 36, Exercise 1.6.2 b)] noticed that the previous assertion remains true if “positive” is replaced by “negative”. This obviously implies IP. Modifying Garrison’s proof, in 1992 R. Forman [99] extended Garrison’s result to a large class of sequences. Forman [99, Proposition] proved that if \(f(x)\) is a nonconstant polynomial with positive leading coefficient (the coefficients need not be integers), then for any \(N\) there are infinitely many nonnegative integers \(k\) such that the sequence \([f(n)] + k\) \((n = 0, 1, 2, \ldots)\) contains at least \(N\) primes ([\(\cdot\)] denotes the greatest integer function). Furthermore, in 1993 U. Abel and H. Siebert [1] also extended Garrison’s result. They proved that if \(f(x) \in \mathbb{Z}[x]\) is a polynomial of degree \(d \geq 2\) with positive leading coefficient, then for every \(N\) there exists an integer \(k\) for which \(f(x) + k\) \((x = 0, 1, 2, \ldots)\) assumes more than \(N\) prime values. Their argument of proof depends on counting the number of solutions of certain inequalities and shows that no arithmetical properties of polynomials are needed other than rate of growth. In particular, in [1] p. 167, proof of Theorem] it was applied the well known Sylvestre’s version of the Chebyshev inequalities \(0.9 \leq \pi(x) \log x/x \leq 1.1\) (for sufficiently large \(x\) \([289],\) see also [74, p. 555, (1.7)]).

However, the problem of characterizing the prime divisors of a polynomial of degree \(> 2\) is still unsolved, except in certain special cases. We see that if \(p\) is any prime that does not divide \(a\), then \(p\) divides each polynomial \(Q_1(x) = ax + b\) with arbitrary \(b \in \mathbb{Z}\). Similarly, the set of all prime divisors of \(Q_2(x) = x^2 - a\) can be determined by using law of quadratic reciprocity. Some known and new related results for various classes of integer polynomials were presented by I. Gerst and J. Brillhart [106].
By Problem 3 proposed on International Mathematical Olympiad (IMO) 2008 [79, pages 336 and 776], there exist infinitely many positive integers \( n \) such that \( n^2 + 1 \) has a prime divisor greater than \( 2n + \sqrt{2n} \). This immediately yields \( IP \).

In [52] (see also [53] and [59, page 118, Section 10.1.5]), in 1979 the computer scientist G. J. Chaitin gave a proof of \( IP \) using algorithmic information theory. If \( p_1, p_2, \ldots, p_k \) are all the primes, then for a fixed \( N = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k} \) Chaitin defines algorithmic entropy
\[
H(N) := \sum_{i=1}^{k} a_i \log p_i
\]
of \( N \), and uses various properties, such as subaddivity of algorithmic entropy expressed as
\[
H(N) \leq \sum_{i=1}^{k} H(n_i) + O(1).
\]
In order to prove this property, Chaitin estimates how many integers \( n \) with \( 1 \leq n \leq N \), could possibly be expressed in the form \( p_1^{b_1} p_2^{b_2} \cdots p_k^{b_k} \). In order for this expression to be at most \( N \), every exponent has to be much smaller than \( N \); precisely, we need \( 0 \leq b_i \leq \log_{p_i} N \); the latter quantity is at most \( \log_2 N + 1 \) choices for each exponent, or \((\log_2 N + 1)^k\) choices overall. However, this latter quantity is much smaller than \( N \) for sufficiently large \( N \); a contradiction which implies \( IP \).

We also notice that Chaitin’s proof is quite similar to those of \( IP \) due to L. G. Schnirelman’s book [259, pp. 44–45] published posthumously in 1940. Moreover, a more sophisticated version of Chaitin’s proof which uses an obvious representation \( n = m^2k \) of a positive integer \( n \) where \( k \) is squarefree, can be found in the book [128, pp. 16–17] of Hardy and Wright (which was first written in 1938). A similar idea is used in 2008 by E. Baronov [139, pp. 12–13, Problem 6] to show that if a sequence of positive integers \((a_n)\) satisfies \( a_n < a_{n+1} \leq a_n + c \), with a fixed \( c \in \mathbb{N} \) and for each \( n \in \mathbb{N} \), then the set of prime divisors of this sequence is infinite. This immediately yields \( IP \). Similarly, the same author [139, pp. 12–13, Problem 6] proved that if \( m \) and \( n \) are positive integers such that \( m > n^{a-1} \), then there exist distinct primes \( p_i, i = 1, 2, \ldots, n \) such that \( p_i \mid m + i \) for each \( i = 1, 2, \ldots, n \). This also implies \( IP \).

**Remarks.** The argument in Chaitin’s proof also shows that the percentage of nonnegative integers up to \( N \) which we can express as a product of any \( k \) primes tends to 0 as \( N \) approaches infinity. Notice that this proof gives a lower bound on \( \pi(x) \) which is between \( \log \log x \) and \( \log x \) (but much closer to \( \log x \)). Using the same method, the lower bound \( \pi(x) \geq (1 + o(1)) \log x / (\log \log x) \) was established in [233, p. 15, Proof of Lemma 1.2.5] (cf. also [135, pp. 15–17, Lemma 0.3 and Exercise 0.5]). In revisited Chaitin’s proof H. N. Shapiro [270, pp.
34–35, Theorem 2.8.1] obtained the estimate $\pi(x) > \log x/(3 \log \log x)$ for each $x > e^2$.

In his dissertation, in 1981 A. R. Woods \[320\] proved $IP$ by adding $PHP\Delta_0$ to a *weak system of arithmetic* $I\Delta_0$, where $PHP\Delta_0$ stands for the pigeonhole principle formulated for functions defined by $\Delta_0$-formulas. ($I\Delta_0$ is the theory over the vocabulary $0, 1, +, \cdot, <$ that is axiomatized by basic properties of this vocabulary and induction axioms for all bounded formulas). In 1988 J. B. Paris, A. J. Wilkie and A. R. Woods (\[226\]; also see \[16, pp. 162–164\] and \[225\]) replaced Woods’ earlier proof with one using an even weaker version of the pigeonhole principle. They showed that a considerable part of elementary number theory, including $IP$, is provable in a weak system of arithmetic $I\Delta_0$ with the *weak pigeonhole principle* for $\Delta_0$-definable functions added as an axiom scheme. It is a longstanding open question \[319\] whether or not one can dispense with the weak pigeonhole principle, by proving the existence of infinitely many primes within $I\Delta_0$. Studying the problem of proving in *weak theories of Bounded Arithmetic* that there are infinitely many of primes, in 2008 P. Nguyen \[216\] showed that $IP$ can be proved by some “minimal” reasoning (i.e., in the theory $<\text{Emphasis Type="Bold"} I\Delta < /\text{Emphasis}> <\text{Subscript} i\text{Emphasis Type="Bold"} > 0 < /\text{Emphasis}> <\text{Subscript} >$) using concepts such as (the logarithm) of a binomial coefficient.

Euclid’s revised proof of $IP$ via methods of nonstandard Analysis was given by R. Goldblatt \[109\] in 1998 (also see \[233\] p. 16, Section 1.2.6)).

2.9. **14 proofs of $IP$ (2012–2017).**

1) In 2012 the author of this article by \[187\] Theorem 1] improved Cooke’s result \[65\] Theorem] (see page 17 of this article), refining the Euler’s proof of $IP$ by the following result: “Let $\alpha$ be a real number such that $1 < \alpha < 2$ and let $x_0 = x_0(\alpha)$ be a (unique) positive solution of the equation

$$x^{\alpha-1} - \frac{\pi}{e^2 \sqrt{3}} x + 1 = 0.$$ 

Then for each positive integer $n > x_0$ there exist at least \([n^\alpha]\) primes between the $(n+1)$th prime and the product of the first $n+1$ primes, where \([a]\) denotes the greatest integer less than or equal to $a$.

Moreover, for each positive integer $n$ there are at least $n$ primes between the $(n+1)$th prime and the product of the first $n+1$ primes.”

2) In 2015 L. Alpoge \[8\] establihed $IP$ as the amusing consequence of the following (called by Khinchin \[147\] beautiful) theorem of van der Waerden \(302); also see \[8\] Theorem 1\]): “Suppose the positive integers
are colored with finitely many colors. Then there are arbitrarily many arithmetic progressions containing integers all of the same color."

More formally, if \( f : \mathbb{Z}^+ \to S \) is any function to a finite set \( S \), then for each \( k > 0 \), there are \( n \) and \( d \) for which

\[
f(n) = f(n + d) = \cdots = f(n + kd).
\]

3) Motivated by the previous Alpoge’s proof of \( IP \), in 2017 A. Granville \[120, \text{Theorem 1}\] proved \( IP \) combining van der Waerden’s theorem with a famous result of Fermat which asserts that \( \text{there are no four-term arithmetic progressions of distinct integer squares} \) (see, e.g., \[277\]).

4) Proceeding in a similar way as in Saidak’s proof of \( IP \) (see Subsection 22, p. 14 of this article), in 2015 B. Maji \[178\] constructed an infinite sequence of pairwise relatively prime positive integers. This fact immediately yields \( IP \).

5) Assuming that the set of all primes is finite, in 2015 S. Northshield \[219\] proved \( IP \) by considering the product

\[
\prod_p \sin \left( \frac{\pi}{p} \right),
\]

where \( p \) runs over all primes (“a one-line proof”).

6) In 2016 A. R. Booker \[38\] considered a generalization of Euclid’s proof of \( IP \) and showed that it leads to variants of the Euclid-Mullin sequence that provably contain every prime number. Namely, given a finite set \( \{p_1, \ldots, p_k\} \) of primes, let \( p_{k+1} \) be a prime factor of \( 1 + p_1 \cdots p_k \). Then, as Euclid showed, \( p_{k+1} \) is necessarily distinct from \( p_1, \ldots, p_k \). Iterating this procedure, we thus obtain an infinite sequence of distinct primes. For instance, beginning with \( k = 0 \) (with the convention that the empty product is 1) and choosing \( p_{k+1} \) as small as possible at each step, one obtains the Euclid-Mullin sequence given as the Sloane’s sequence A000945 in \[278\] (cf. Remarks on pages 5 and 11 of this paper). Following \[38\], any sequence resulting from this construction is called a \emph{generalized Euclid sequence with seed} \( \{p_1, \ldots, p_k\} \) (for such a particular sequence, see the sequence A167604 in \[278\]; for related sequences, see \[37\] and \[39\]). More precisely, Booker in \[38\] considered a generalization of Euclid’s construction described as follows. If \( \{p_1, \ldots, p_k\} \) is a set of primes, then for any \( I \subseteq \{1, \ldots, k\} \), the number \( N_I := \prod_{i \in I} p_i + \prod_{i \in \{1, \ldots, k\} \setminus I} p_i \) is coprime to \( p_1 \cdots p_k \) and has at least one prime factor. Iteratively choosing a set \( I \) and a prime \( p_{k+1} | N_I \),
we obtain an infinite sequence $p_1, p_2, \ldots$ of distinct primes, as in Euclid’s proof. It was proved in [38, Theorem 1] that for any finite set $P$ of primes, there is a generalized Euclid sequence with seed $P$ containing every prime. Notice that in 2016 A. R. Booker and S. A. Irvine [39] introduced the so-called the Euclid-Mullin graph which encodes all instances of Euclid's proof of $IP$.

7) In 2016 P. L. Clark [60] recast Euclid’s proof of $IP$ as a Euclidean Criterion for a domain to have infinitely many atoms. It is showed that there is a connection with Furstenberg’s topological proof of $IP$ (see Subsection 2.7 of this article, p. 25) and that the presented criterion applies even in certain domains in which not all nonzero nonunits factor into products of irreducibles.

8), 9) In 2017 A. Sadhukhan [253] introduced a partition of the positive integers and used it to give two proofs of the infinitude of primes. The first proof is a slight variant of the various known combinatorial proofs. The second is similar to Euler’s proof but it makes no use of Euler’s product formula.

10), 11) In 2017 S.-I. Seki [263] gave two proofs of $IP$ via valuation theory and gave a new proof of the divergence of the sum of prime reciprocals by Roth’s theorem and Euler-Legendre’s theorem for arithmetic progressions.

12), 13) In 2017 S. Northshield [220] presented two new proofs of $IP$. The first proof uses the basic idea of Furstenberg’s celebrated topological proof of $IP$ (see Subsection 2.7 of this article, p. 24) but without using topology. Namely, while Furstenberg’s proof is in terms of topological space, this proof is in terms of the continuous functions on the space. The second proof in [220] uses probability theory. Namely, this proof is built on the difficulty of defining a random integer.

14) Finally, in 2017 the author of this article in the short note [188] supposed that $\{p_1, p_2, p_3, \ldots, p_k\}$ is a set of all primes with $p_1 = 2$. Then by considering the set of all positive integers that are relatively prime to the product $p_2 p_3 \cdots p_k$, we easily obtain a contradiction which implies $IP$.

2.10. 18 recent proofs of $IP$ (2018–2022). 1) In 2018 S. Silwal [277, Theorem 1] proved that the following inequality holds for sufficiently large $n$:

$$\sum_{p \leq n} \frac{1}{\log p} > \frac{1}{3} \log n,$$

where the summation ranges over all primes $p$ such that $p \leq n$. Clearly, the above inequality implies $IP$. 
2) In 2018 K. Saito [255] gives a short proof of $IP$ by using the upper box dimension, which is one of fractal dimensions.

3) In 2020 V.J.-Vera and C.S.-Ávila [306, Theorem 2] gave a new proof of the divergence of the sum of the reciprocals of primes using the number of distinct prime divisors of a positive integer $n$, and the placement of lattice points on a hyperbola given by $n = pr$ with a prime $p$. This immediately yields $IP$.

4) By applying a geometric approach, in 2020 H. Göral [113] provided a proof of $IP$ via $p$-adic metrics. Notice that this is a novel approach to a well-known and quite old result.

5), 6, 7) In 2020 H. Göral and H.B. Öczan [114] provided three proofs of $IP$ by considering the properties of the Jacobson radical of the ring of integers $\mathbb{Z}$. In all of these proofs, the authors supposed that the set $\mathbb{P}$ of all primes is finite, i.e., $\mathbb{P} = \{p_1, p_2, \ldots, p_n\}$. Let $P = p_1p_2\cdots p_n$ be the product of all primes. Then by considering the sum $aP^2 + P$ with a fixed integer $a \geq 0$, it is proved in [114, Theorem 2.2] that the fundamental theorem of arithmetic implies that $aP^2 + P = P$, i.e., $a = 0$. This contradiction implies $IP$.

Recall that the Jacobson radical of a commutative ring $R$, denoted by $J(R)$, is defined as the intersection of all maximal ideals of $R$. Since all maximal ideals of the ring of integers are of the form $p\mathbb{Z}$ for a prime $p$, the Jacobson radical $J(\mathbb{Z})$ is the intersection of maximal ideals $p_1\mathbb{Z}, \ldots, p_n\mathbb{Z}$, and hence,

$$J(\mathbb{Z}) = \prod_{i=1}^{n} p_i\mathbb{Z} = (p_1p_2\cdots p_n)\mathbb{Z}.$$ 

The second proof and the third proof of $IP$ given in [114, Theorem 2.2] are based on consideration of Jacobson radical $J(\mathbb{Z})$. The authors noticed that there are similarities between Furstenberg’s topological proof of $IP$ [100] and their second proof and third proof of $IP$.

8) In 2020 F. Lemmermeyer [168] provided a short simple proof of $IP$. This proof is a simplification of the proof of $IP$ using continued fractions given in 1976 by Barnes [25]. Assume that there are only infinitely many primes, namely, 2, $p_1 = 3, \ldots, p_n$. Let $q = p_1\cdots p_n$ be the product of all odd primes. Then $q^2 + 1$ is not divisible by any odd prime, hence must be a power of two. Since $q^2 + 1 \equiv 2 \pmod{4}$, must be $q^2 + 1 = 2$ and therefore, $q = 1$, which is a contradiction. Since no odd prime $p \equiv 3 \pmod{4}$ can divide $q^2 + 1 = 2$, the proof actually shows that there are infinitely many primes $p \equiv 1 \pmod{4}$.

9), 10) By considering the notion of a realizable integer sequence, in 2020 P. Moss and T. Ward [200, Lemma 1] proved that the set of primes
dividing a denominator of $\frac{1}{n} \sum_{d|n} \mu \left( \frac{n}{d} \right) f_d$ for some positive integer $n$ is infinite, where $\mu(n)$ is the classical Möbius arithmetic function and $(f_n)$ is the well known Fibonacci’s sequence defined by the conditions $f_1 = f_2 = 1$, $f_{n+2} = f_{n+1} + f_n$ with $n = 1, 2, \ldots$. This implies IP.

Furthermore, P. Moss and T. Ward [200, Corollary 3] proved that if $j$ is an arbitrary odd positive integer, then the set of primes dividing denominators of $\frac{1}{n} \sum_{d|n} \mu \left( \frac{n}{d} \right) f_d$ $j$. This implies IP.

11) Let $p_1, p_2, \ldots, p_{\pi(n)}$ be all the primes less than or equal to $n$. Using the inclusion-exclusion principle, in 2020 S. Laad [158] proved the inequality

$$n \prod_{i=1}^{\pi(n)} \left( 1 - \frac{1}{p_i} \right) < 1 + 2^{\pi(n) - 1}.$$ 

Assuming that there are only $k$ primes, then clearly, the left hand side of the above inequality is unbounded, while the right hand side is a constant. This contradiction impies IP.

12), 13, 14) In 2021 C. Elsholtz [84, Theorem 1] showed that Fermat’s last theorem and a combinatorial theorem of Schur on monochromatic solutions of $a + b = c$ (Lemma 1 in [84]) implies IP. In particular, since there exist elementary proofs of Fermat’s last theorem for $n = 3$ $n = 4$ and $n = 5$ (concerning the Diophantine equations in positive integers $x^3 + y^3 = z^3$, $x^4 + y^4 = z^4$ and $x^5 + y^5 = z^5$, respectively; (see [247]), Theorem 1 in [84] implies the elementary proof of IP.

It follows from Theorem 2 of [84] that Roth’s theorem (Lemma 2 in [84]) implies IP.

It was also proved in [84, Theorem 3] that Hindman’s theorem (Lemma 4 in [84]) implies IP.

15) In 2021 L. Haddad [125] simplified the above mentioned proof of IP due to C. Elsholtz [84, Theorem 1]. Namely, this proof is greatly simplified, making no use at all Fermat’s last theorem, and using only a weak form of the theorem of Schur on monochromatic solutions of $a + b = c$.

16) In 2022 J. Mehta [181, Theorem 1] generalized Métrod’s proof of IP given in 1917 [192] (also see [248, p. 11]). Assume that $p_1, \ldots, p_n$ are distinct primes whose product is $P$, and choose any factorization of $P$ into $k \geq 2$ terms, say $P = d_1 \cdots d_k$, and put [181, Proof of Theorem 1]

$$M = \frac{P}{d_1} + \cdots + \frac{P}{d_k}.$$ 

Then it is easy to show that there exists a prime $p \not\in \{p_1, \ldots, p_n\}$ dividing $M$. This implies IP.
Notice that Stieltjes’ proof of IP given in his work in 1890 ([285, p. 14]; also see [210, p. 4]) is a particular case of Theorem 1 of [181] with \( k = 2 \). Furthermore, Métrod’s proof of IP is recovered by taking \( k = n \) and \( d_i \) for \( i = 1, 2, \ldots, n \), i.e., by considering the divisors of \( M = p_1 p_2 \cdots p_n \left( \frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_n} \right) \).

17) Using prime factorization theorem of a positive integer, in 2022 R. Meštrović [190] gave a short proof by contradiction of IP (this is in fact the proof in Section 4 of this article, pp. 40–41).

18) Using Möbius inversion formula [218], in 2023 R. Meštrović [191] gave a very short proof of the formula due in 2009 to J. Pinasco [230] which is applied in his proof of IP. Consequently, using a simpler argument than those of Pinasco’s proof, it follows IP.

3. Proofs of IP in arithmetic progressions: special cases of Dirichlet’s theorem

3.1. Dirichlet’s theorem. In 1775 L. Euler [95] (also cf. [76, p. 415], [293, p. 108, Section 3.6]) stated that an arithmetic progression with the first term equals 1 and the difference \( a \) to be a positive integer, contains infinitely many primes. More generally, in 1798 in the second edition of his book A.-M. Legendre [167] (cf. [76, p. 415] and [293, p. 108, Section 3.6]) conjectured that for relatively prime positive integers \( a \) and \( m \) there are infinitely many primes which leave a remainder of \( m \) when divided by \( a \). In other words, if \( a \) and \( m \) are relatively prime positive integers, then the arithmetic progression \( a, a + m, a + 2m, a + 3m, \ldots \) contains infinitely many primes. The condition that \( a \) and \( m \) are relatively prime is essential, for otherwise there would be no primes at all in the progression. However, Legendre gave a proof that was faulty. In 1837 Peter Gustav Lejeune Dirichlet, Gauss’s successor of Göttingen and father of analytic number theory, gave a correct proof. Namely, Dirichlet [78] proved the following theorem which is a far-reaching extension of Euclid’s theorem on the infinitude of primes and is one of the most beautiful results in all of Number Theory. It can be stated as follows.

**Dirichlet’s theorem.** Suppose \( a \) and \( m \) are relatively prime positive integers. Then there are infinitely many primes of the form \( mk + a \) with \( k \in \mathbb{N} \cup \{0\} \).

Dirichlet’s proof is derived by means of \( L \)-functions and analysis. The main strategy is, as in Euler’s proof of IP (which in fact shows
that the sum of reciprocals of primes diverges), to consider the function

\[ P_m(s) := \sum_{p \equiv a \pmod{m}} \frac{1}{p^s}, \]

(where the sum is only over those primes \( p \) that are congruent to \( a \pmod{m} \)) which is defined say for real numbers \( s > 1 \), and to show that \( \lim_{s \to 1^+} P_m(s) = +\infty \). Of course this suffices, because a divergent series must have infinitely many terms. The function \( P_m(s) \) will in turn be related to a finite linear combination of logarithms of Dirichlet \( L \)-series, and the differing behavior of the Dirichlet series for principal and non-principal characters is a key aspect of the proof. Dirichlet used an ingenious argument to show that the sum \( \sum_{p \equiv a \pmod{m}} 1/p \) diverges, where the sum ranges over all primes \( p \) that are congruent to \( a \pmod{m} \).

Remarks. As it is pointed out by P. Pollack [234], there exist proofs of Dirichlet’s theorem which minimize analytic prerequisites (e.g., those of A. Selberg [264] in 1949, A. Granville [116] in 1989 and H. N. Shapiro ([268] and [269]) in 1950). For example, Selberg [264] gave a proof that is, he wrote “more elementary in the respect that we do not use the complex characters mod \( k \), and also in that we consider only finite sums.” An “elementary proof” of Dirichlet’s theorem in the sense that it does not use complex analysis is given by M. B. Nathanson [211, Ch. 10]. Nevertheless, all these “elementary” proofs exhibit at least as complicated a structure as Dirichlet’s original argument. This is well discussed and considered in 2010 by A. Granville in his expository article [119, Sections 2 and 3].

3.2. A survey of elementary proofs of IP in special arithmetic progressions. For many arithmetic progressions with small differences one can obtain simple elementary (i.e. not using analytic means) proofs of Dirichlet’s theorem. Several of them are listed by Dickson [76 pp. 418–420, Chapter XVIII] and Narkiewicz [210, pp. 87–96, Section 2.5]. In [206] M. R. Murty and N. Thain asked “how far Euclid’s proof can be pushed to yield Dirichlet’s theorem”. The existence of such a “Euclidean proof” (precised in [206]) for certain arithmetic progressions is well known. For example, considering the product \( k(2 \cdot 3 \cdot \cdots \cdot p_n) \), Euclid’s elementary proof can be used to prove that for any fixed positive integer \( k > 2 \) there are infinitely many primes which are not congruent to \( 1 \pmod{k} \). This result was proved in 1911 by H. C. Pocklington [231] (also see [59 p. 116, Theorem 114] and [76 p. 419]).
Further, we expose other proofs of $IP$ in special arithmetic progressions of the form $1(\text{mod } k)$ and $-1(\text{mod } k)$. An excellent source for this subject is Narkiewicz’s monograph [210, pp. 87–93, Section 2.5]. An elementary proof of $IP$ in every progression $1(\text{mod } 2^p)$, where $p$ is any prime, was established in 1843 by V. A. Lebesgue ([162, p. 51], [76, p. 418]) who showed the fact that $x^{p-1} - x^{p-2}y + \cdots + y^{p-1}$ has besides the possible factor $p$ only prime factors of the form $2kp+1$ ($k = 1, 2, \ldots$). Using a quite similar method, in 1853 F. Landry ([161], [76, p. 418]) considered prime divisors of $(n^p + 1)/(n + 1)$ to prove $IP$ for the same progressions. This proof can be found in [18, p. 121, Ch. 24, Exercise 24.1]. By a quite similar method, the same result can be obtained using the fact that for any prime $q$ every prime divisor $p$ of $(n^q - 1)/(n - 1)$ coprime with $q$ satisfies $p \equiv 1(\text{mod } q)$ (see e.g. [140, p. 34, Section 2.3] or [10, pp. 151–152, Problem 7.3.3]). The analogous method is also applied by Lebesgue in 1862 ([165], [76, p. 418]) for the progression $-1(\text{mod } 2^p)$ with a prime $p$. Using the rational and irrational parts of $(a + \sqrt{b})^k$, in 1868/9 A. Genocchi ([105], [76, p. 418]) proved $IP$ in both progressions $1(\text{mod } 2^p)$ and $-1(\text{mod } 2^p)$, where $p$ is an arbitrary prime. Furthermore, in lectures of 1875/6 L. Kronecker ([154], [132, pp. 440–442]) gave another proof of $IP$ in the progression $1(\text{mod } 2p)$ with a prime $p$. Another simple proof of the same result based on Euler’s totient function and Fermat little theorem is recently given in [189].

Using the fact that $(2^{mp} - 1)/(2^m - 1)$ ($p$ a prime and $m$ a positive integer) has at least one prime divisor of the form $p^n k + 1$ ([266, p. 107, proof of Theorem 47]; also cf. [89, pp. 178–179, Theorem 11] or [204, p. 209, Exercise 1.5.28]), in 1978 D. Shanks [266] proved that for every prime power $p^n$ there are infinitely primes $\equiv 1(\text{mod } p^n)$. Another elementary proof of $IP$ in the progression $1(\text{mod } p^n)$ for any prime $p$ and $n = 1, 2, \ldots$ was given in 1931 by F. Hartmann [131].

Using divisibility properties of cyclotomic polynomials, in 1888 J. J. Sylvester [291] proved $IP$ in the progressions $-1(\text{mod } p^n)$, where $p^n$ is any prime power. In 1896 R. D. von Sterneck [308] (cf. [210, p. 90]) considered a product $F(n) := \prod_{d|n} f(n/d)^{\mu(d)}$, where $\mu$ is the Möbius function, $f(n)$ is an integer-valued function satisfying $f(1) = 1$ and two divisibility properties. Then every prime dividing $F(n)$ divides $f(n)$ but does not divide $f(i)$ for each $i = 1, 2, \ldots, n-1$. Von Sterneck remarked that a recursive sequence $f(n)$ defined as $f(n) = f(n-1) + cf(n-2)$ with $f(1) = 1$ and a positive integer $c$, satisfies these conditions, and used this it can be obtained an elementary proof of infinitely many primes $\equiv -1(\text{mod } p^n)$ for any fixed prime power $p^n$. 
The same result for powers of odd primes and the infinitude of primes \( \equiv -1(\mod 3 \cdot 2^n) \) were proved in 1913 by R. D. Carmichael [50].

As remarked by Dickson [76, p. 418], using cyclotomic polynomials \( \Phi_m(x) \), in 1886 A. S. Bang ([23], [76, p. 418]) and in 1888 Sylvester ([291], also cf. [76, p. 418]) obtained proofs of IP in arithmetic progressions \( 1(\mod k) \), where \( k \) is any integer \( \geq 2 \). Both these proofs are based on the fact that if \( p \) is a prime not dividing \( m \), then \( p \) divides \( \Phi_m(a) \) if and only if the order of \( a(\mod p) \) is \( m \). (Here \( \Phi_m(x) \) is the \( m \)th cyclotomic polynomial). Such a simple classical proof of IP in arithmetic progressions \( 1(\mod k) \) which is in spirit “Euclidean” can be found in ([121] and [59, pp. 116–117]; also cf. [146, pp. 97–99] and [313, pp. 12–13]). Considering the least common multiple of polynomials \( \{x^d - 1 : d | n\} \), in 1895 E. Wendt [317] (cf. [210, p. 89]) gave a simple proof of the same result. Moreover, Narkiewicz [210, p. 88] noticed that, according to a theorem of Kummer [155] (also see [209, Theorem 4.16]), a rational prime \( p \) splits in the \( k \)th cyclotomic field \( \mathbb{Q}(\zeta_k) \) (where \( \zeta_k \) denotes a primitive \( k \)th root of unity) if and only if it is congruent to \( 1(\mod k) \). Using this and the fact that in any given finite extension of \( \mathbb{Q} \) there are infinitely many splitting primes, we obtain IP in every arithmetic progression \( 1(\mod k) \) with \( k \geq 2 \). Studying the existence of primitive prime divisors of integers \( a^n - b^n \), where \( n \in \mathbb{N} \) and \( a \) and \( b \) are relatively prime integers, in 1903/04 G. D. Birkhoff and H. S. Vandiver [33] gave an elementary proof of this result. A variation of this proof has been given in 1961 by A. Rotkiewicz [251], whose proof was simplified in 1962/3 by T. Estermann [90] and in 1976 by I. Niven and B. Powell [217]. In their proof Niven and Powell use only elementary divisibility properties and the fact that the number of roots of a non-zero polynomial cannot exceed its degree. Applying Birkhoff-Vandiver theorem (see e.g., [210, p. 88]), the same result was proved in 1981 by R. A. Smith [280] (see also [208, Chapter 1] and [210, pp. 88–89]). Another two elementary proofs were given in 1984 by S. Srinivasan [283] and in 1998 by N. Sedrakian and J. Steinig [262]. An elementary proof of this assertion was provided in 2004 by J. Yoo [325] without using cyclotomic polynomials. Another two old proofs of this result are due to K. Th. Vahlen [301] in 1897 by using Gauss’ periods of roots of unity and É. Lucas [176, p. 291, Ch. XVII] in 1899 applying his (Lucas) sequence \( u_n \).

A short but not quite elementary proof of IP in the progressions \( -1(\mod k) \) for each \( k \geq 2 \) was given by M. Bauer [28] in 1905/6. In 1951 T. Nagell [207, pp. 170–173] gives an elementary proof of IP in arithmetic progression \( -1(\mod k) \) with \( k \geq 2 \).
Applying a similar argument to those of Niven and Powell for IP in the progressions \( \equiv 1(\text{mod } k) \), in 1950 by M. Hasse \[132\] proved IP in the progressions \(-1(\text{mod } k)\) for each \( k \geq 2 \).

Euclidean’s proofs of IP in various arithmetic progressions can be found in Problems book of Murty and Esmonde \[205\] Section 7.5] in 2005. For example, the known facts that every prime divisor of the Fermat number \( F_n := 2^{2^n} + 1 \) is of the form \( 2^{2^n} + 1 \) (see e.g., \[205\] p. 8, Exercise 1.2.8) and that \( F_n \) and \( F_m \) are relatively prime if \( m \neq n \) (see Subsection 2.2) yield that there are infinitely primes \( \equiv 1(\text{mod } 2^n) \) for any given \( n \) (\[205\] p. 11, Exercise 1.4.13, also cf. \[10\] p. 151, Problem 7.3.2)).

As noticed by K. Conrad \[63\], a Euclidean proof of Dirichlet’s theorem for \( m(\text{mod } a) \) involves, at the very least, the construction of a nonconstant polynomial \( h(T) \in \mathbb{Z}[T] \) for which any prime factor \( p \) of any integer \( h(n) \) satisfies, with finitely many exceptions, either \( p \equiv 1(\text{mod } a) \) or \( p \equiv m(\text{mod } a) \), and infinitely many primes of the latter type occur. For example \[63\], Euclidean proofs of Dirichlet’s theorem exist for arithmetic progressions \( 1(\text{mod } a) \) with any \( a \geq 2 \), \( 3(\text{mod } 8) \), \( 4(\text{mod } 5) \) and \( 6(\text{mod } 7) \).

A characterization of arithmetic progressions for which Euclidean proof exist is given by I. Schur \[260\] and M. R. Murty \[203\]. In 1912/13 I. Schur \[260\] proved that if \( m^2 \equiv 1(\text{mod } a) \), then a Euclidean proof of Dirichlet’s theorem exists for the arithmetic progression \( m(\text{mod } a) \). In particular, Schur extended Serret’s approach based on law of quadratic reciprocity to establish proofs of IP for the progressions \( 2^{m-1} + 1(\text{mod } 2^m) \), \( 2^{m-1} - 1(\text{mod } 2^m) \) \( (m \geq 1) \), and \( l(\text{mod } k) \) for \( k = 8m \) (with \( m \) being an odd positive squarefree integer) and \( l = 2m + 1 \), \( l = 4m + 1 \) or \( l = 6m + 1 \) (cf. \[210\] p. 91). A similar method was used in 1937 by A. S. Bang \[24\] (cf. \[210\] p. 91) who proved IP in the progressions \( 2p^m + 1(\text{mod } 4p^m) \) with prime \( p \equiv 3(\text{mod } 4) \), \( 2p^{2n+1} + 1(\text{mod } 6p^{2n+1}) \) with prime \( p \equiv 2(\text{mod } 3) \), and \( 4p^{2n} + 1(\text{mod } 6p^{2n}) \) with prime \( p \equiv 2(\text{mod } 3) \).

**Remarks.** In 1988 Murty \( \{203; \text{ also see } 206\} \) proved the converse of Schur’s result, i.e., he showed that a Euclidean proof exists for the arithmetic progression \( m(\text{mod } a) \) only if \( m^2 \equiv 1(\text{mod } a) \). This means that it is impossible to prove Dirichlet’s theorem for certain arithmetic progression by Euclid’s method. The proof due to Murty is not difficult, but involves some Galois Theory. For example, since \( 2^2 \equiv 4 \not\equiv 1(\text{mod } 5) \), there is no proof of Dirichlet’s theorem for \( 2(\text{mod } 5) \) which can mimic Euclid’s proof of IP. Notice also that Dirichlet’s theorem can be proved by Euclidean’s methods for all the possibilities modulo \( a = 24 \).
(cf. 27). Recently, P. Pollack 234 discussed Murty’s definition of a “Euclidean proof” and Murty’s converse of Schur’s result. Finally, we point out an interesting expository article of A. Granville 117 in 2007 in which are compared numbers of primes in different arithmetic progressions with the same small difference.

3.3. Elementary proofs of IP in arithmetic progressions with small differences. In this subsection, we expose several Euclidean proofs of IP in different arithmetic progressions with small differences. Dickson’s History records several further attempts at giving Euclidean proofs for particular progressions (see the listing on 76 pp. 418–420)). Considering the product $2^2 \cdot 3 \cdot 5 \cdot p_n - 1$, Euclid’s idea is used by V. A. Lebesgue 163 in 1856 (also cf. 128, p. 13, Theorem 11)) for the progression $3(\text{mod } 4)$. A. Granville 118, p. 3, Section 1.3] remarked that a similar proof works for primes $\equiv 2(\text{mod } 3)$. The same idea that involves the product $2 \cdot 3 \cdot 5 \cdot p_n - 1$ was also used by V. A. Lebesgue in 1859 (164; also see 128 p. 13, Theorem 13 and 76 p. 419) for the proof of IP in the progressions $5(\text{mod } 6)$ and $1(\text{mod } 2^n)$ with a fixed $n = 1, 2, \ldots$. The situation is more complicated for the progression $1(\text{mod } 4)$ and related proof is based on the consideration of the product $N := (5 \cdot 13 \cdot 17 \cdots p_n)^2 + 1$ and the fact that if integers $a$ and $b$ have no common factor, then any odd prime divisor of $a^2 + b^2$ is congruent to $1(\text{mod } 4)$ 128, Theorem 13]. In fact, using this property of quadratic residues and Euclid’s idea, Hardy and Wright proved in his book (128, Theorem 14] which was first written in 1938 that the progression $5(\text{mod } 8)$ contains infinitely many primes. Dickson 76 p. 419] noticed that this result and proofs of IP in progressions $1(\text{mod } 8)$, $3(\text{mod } 8)$ and $7(\text{mod } 8)$ were firstly proved in 1856 also by A. V. Lebesgue 165. Using some properties of Fermat and Fibonacci numbers, two constructive proofs of IP in progression $1(\text{mod } 4)$ were presented in 1994 by N. Robbins 250.

Dickson 76 p. 419] pointed out the proofs of IP also in the following arithmetic progressions: $9(\text{mod } 10)$ due to J. A. Serret 265 in 1852, $2(\text{mod } 5)$ and $7(\text{mod } 8)$ due to É. Lucas 171, p. 309] in 1878, $1(\text{mod } 4)$, $5(\text{mod } 6)$ and $5(\text{mod } 8)$ due to É. Lucas 175, pp. 353–354] in 1891, $1(\text{mod } 4)$, $1(\text{mod } 6)$ and $5(\text{mod } 8)$ due to E. Cahen 174, pp. 318–319] in 1900 and also 132 pp. 438–439] in 1875/6, $1(\text{mod } 4)$, $1(\text{mod } 6)$, $3(\text{mod } 8)$, $7(\text{mod } 8)$, $9(\text{mod } 10)$ and $11(\text{mod } 12)$, due to K. Hensel 132] in 1913. Furthermore, using law of quadratic reciprocity 103, Sections 112–114], in 1852 J. A. Serret 265 (also cf. 210, pp. 90–91, Theorem 2.19)] proved IP in the progressions $3(\text{mod } 8)$, $5(\text{mod } 8)$ and $7(\text{mod } 8)$.
Considering divisors of integer \((11 \cdot 31 \cdot 41 \cdot 61 \cdots p_n)^5 - 1\), it was proved in 1962 [271, pages 60, 371–373, Problem 254(c)] \(IP\) in the progression \(1 \pmod{10}\). The analogous idea was used in 2007 by A. Granville [118, p. 4] to show \(IP\) in the progression \(1 \pmod{3}\).

There are also elementary arguments in spirit of Euclid’s idea showing that there are infinitely many primes in other arithmetic progressions with small differences, such as \(4 \pmod{5}\), \(1 \pmod{8}\) and \(3 \pmod{8}\). In 1965 P. Bateman and M. E. Low [27] give a proof similar to Euclid’s that for every coprime residue class \(a \pmod{24}\) there are infinitely many primes in progression \(a \pmod{24}\). Their proof makes use of the interesting fact that every integer \(a\) relatively prime to 24 has the property \(a^2 \equiv 1 \pmod{24}\). Using a couple of observations about the polynomial \(f(x) = x^4 - x^3 + 2x^2 + x + 1\) and the law of quadratic reciprocity, a Euclid-type proof for the progression \(4 \pmod{15}\) is presented in 2005 by M. R. Murty and J. Esmonde [205, pp. 92–64, Example 7.5.4].

When considering the linear second order recurrence \(u_n = u_{n-1} + 3u_{n-2}\) with \(u_0 = 1\), \(u_1 = 1\), in 2005 R. Neville [214] gave a simple proof of \(IP\) in progression \(1 \pmod{3}\). The author [214, Remarks] also noticed that if \(q \geq 5\) is a given prime, then considering the Lucas sequence \(u_n = u_{n-1} + 3u_{n-3}\) with \(u_0 = 0\), \(u_1 = 1\), similarly one can prove that there are infinitely many primes \(p\) such that \((\frac{-q}{p}) = 1\) (\((\frac{\cdot}{p})\) denotes the Legendre symbol). In particular, for \(q = 5\) this yields \(IP\) in all progressions \(a \pmod{20}\) with \(a \in \{1, 3, 7, 9\}\). In book [247, p. 15] P. Ribenboim noticed that in 1958 D. Jarden [142] proved \(IP\) in the progression \(1 \pmod{20}\).

4. Another simple Euclidean’s proof of Euclid’s theorem

Proof of Euclid’s theorem. Suppose that \(p_1 = 2 < p_2 = 3 < \cdots < p_k\) are all the primes. Take \(n = p_1p_2 \cdots p_k+1\) and let \(p\) be a prime dividing \(n\).

The first step is a “shifted” first step of Euclid’s proof. Suppose that \(p_1 = 2 < p_2 = 3 < \cdots < p_k\) are all the primes. Take \(n = p_1p_2 \cdots p_k\). Then \(n-1 = p_1^{e_1}p_2^{e_2} \cdots p_k^{e_k} (\geq 5)\) for some \(k\)-tuple of nonnegative integers \((e_1, e_2, \ldots, e_k)\), and so taking \(s = \max\{e_1, e_2, \ldots, e_k\}\), we find that

\[
n - 1 = p_1^{e_1}p_2^{e_2} \cdots p_k^{e_k} = \frac{p_1^s p_2^s \cdots p_k^s}{p_1^{s-e_1} p_2^{s-e_2} \cdots p_k^{s-e_k}} = \frac{n^s}{a}.
\]
where $a = p_1^{s_1-1}p_2^{s_2-1} \cdots p_k^{s_k-1}$ and $s$ are positive integers. The above equality yields
\[
a = \frac{n^s}{n-1} = \frac{(n^s-1)+1}{n-1} = \sum_{i=0}^{s-1} n^i + \frac{1}{n-1},
\]
whence it follows that $1/(n-1) = a - \sum_{i=0}^{s-1} n^i$ is a positive integer. This contradicts the fact that $n - 1 \geq 4$, and the proof is completed. \( \square \)

**Remarks.** Unlike most other proofs of the Euclid’s theorem, Euclid’s proof and our proof does not require Proposition 30 in Book VII of *Elements* (see [327], [128], where this result is called “Euclid’s first theorem”; sometimes called “Euclid’s Lemma”) that states into modern language from the Greek [91]: *that if two numbers, multiplied by one another make some number, and any prime number measures the product, then it also measures one of the original numbers*, or in terms of modern Arithmetic: if $p$ is a prime such that $p \mid ab$ then either $p \mid m$ or $p \mid b$. It was also pointed in [128, page 10, Notes on Chapter 1] that this result does not seem to have been stated explicitly before Gauss of 1801 who gave the first correct proof of this assertion [103, Sections 13–14]. The only divisibility property used in our proof and Euclid’s proof is the fact that every integer $n > 1$ has at least one representation as a product of primes. This is in fact, Proposition 31 in Book VII of *Elements* (see above Remarks).

In order to achieved a contradiction, in the second step of his proof Euclid take a prime that divides a product $P$ of all the primes plus one, and further consider two cases in dependence on whether $P$ is prime or not. But in the second step of our proof we obtain directly a contradiction dividing $n^s$ by $n - 1$. \( \square \)

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APPENDICES

A) External Links on Euclid's theorem and its proofs

Wikipedia [http://en.wikipedia.org/wiki/Euclid’s_theorem](http://en.wikipedia.org/wiki/Euclid’s_theorem)
http://mathworld.wolfram.com/EuclidsTheorems.html from MathWorld.

http://primes.utm.edu/notes/proofs/infinite/eulids.html
http://mathforum.org/
http://aleph0.clarku.edu/~djoyce/java/elements/elements.html
http://planetmath.org/encyclopedia/
http://mathoverflow.net
http://tech.groups.yahoo.com/group/primenumber/

B) Sloane's sequences related to proofs of Euclid's theorem

A000040, A002110, A034386, A210144, A210186, A006862, A005234, A006794, A014545, A057704, A057713, A065314, A065315, A065316, A065317, A018239, A057588, A057705, A06794, A002584, A002585, A051342, A068488, A068489, A103514, A066266, A066267, A066268, A066269, A088054, A093804, A103319, A104350, A002981, A002982, A038507, A007917, A007918, A088332, A05235, A000945, A000946, A005265, A005266, A0084598, A0084599, A005266; A000215, A019434, A094358, A050922, A023394, A057755, A080176, A002715; A000668, A001348, A000225, A000043, A046051, A028335; A002716; A104189; A001685; A000045; A000217; A000292; A064526, A000324, A007996; A000289; A000058, A001543, A001544, A126263; A005267; A0013661, A0013662; A003285; A000010; A000984; A167604.

In “The On-Line Encyclopedia of Integer Sequences.” (published electronically at www.research.att.com/~njas/sequences/) [278].
C) List of papers and their authors arranged by year of publication followed by the main argument(s) of related proof given into round brackets

1 For brevity, into round brackets after a reference in the following list we denote the method(s) and/or idea(s) that are used in related proof by:

- AP—an arithmetic progression/arithmetic progressions;
- C—a combinatorial method;
- CM—a counting method, based on some combinatorial enumerating arguments;
- CS—an idea based on a convergence of sums $\sum_{n=1}^{\infty} 1/n^s$ with $s > 1$ etc;
- DS—Euler idea, that is an idea based on the divergence of reciprocals of primes and related series;
- E—Euclid’s idea of the proof of the infinitude of primes, that is, a consideration of product $P := p_1p_2\cdots p_k+1$ or some analogous product;
- FT—a factorization (not necessarily to be unique) of a positive integer as a product of prime powers;
- MPI—the idea based on a construction of sequences consisting of mutually prime positive integers;
- T—a topological method;
- UFT—the unique factorization theorem of a positive integer as a product of prime powers.

1* denotes that a related proof of IP concerns a particular arithmetic progression

[91, ~ 300 B.C.], [128, p. 4, Theorem 4], Euclid of Alexandria (E)
[101, 1730, pp. 32–34, I], [248, p. 6], [98, pp. 40–41], [233, p. 4] C. Goldbach, (MPI, especially Fermat numbers $F_n := 2^{2^n} + 1$)
[93, 1736] (posthumous paper), [46, pp. 134–135], [76, p. 413], [233, p. 3] L. Euler (multiplicativity of Euler’s totient function $\varphi$)
[94, 1737, pp. 172–174], [92, [248, p. 8], [98, pp. 8–9], L. Euler (UFT, DS; especially, the series of the reciprocals of the primes is divergent)
*[162, 1843], [76, p. 418] V. A. Lebesgue (prime factors of $x^{p-1} - x^{p-2} y + \cdots + y^{p-1}$, Fermat little theorem, and IP in AP 1 (mod 2p) with a $p$ a prime)
*[265, 1852] J. A. Serret (E, and IP in AP 9 (mod 10))
*[265, 1852] [210, pp. 90–91, Theorem 2.19] J. A. Serret, (law of quadratic reciprocity, and IP in AP 3 (mod 8), 5 (mod 8) and 7 (mod 8))
*[161, 1853], [76, p. 418], [140, p. 34, Section 2.3] F. Landry (prime divisors of $(n^p + 1)/(n + 1)$, Fermat little theorem, and IP in AP 1 (mod 2p) with a prime $p$)
*[163, 1856], [128, p. 13] V. A. Lebesgue (E, and IP in AP 3 (mod 4))
EUCLID’S THEOREM ON THE INFINITUDE OF PRIMES... 59

[164], 1859], [76], p. 418], [128], p. 13], V. A. Lebesgue (E, IP in AP 5(mod 6) and IP in AP 1(mod 2^n k) with some fixed k, n ∈ N)

∗ [165], 1862], [76], p. 418] V. A. Lebesgue, (prime factors of an integer polynomial in two variables, Fermat little theorem, and IP in AP −1(mod 2p) with a p a prime)

∗ [105], 1868/9], [76], p. 418] A. Genocchi (rational and irrational parts of (a + √b)^k, and IP in AP ±1(mod 2^p), with a p a prime)

∗ [184], 1874], [323], pp. 171, 183–186] F. Mertens (DS, the boundedness of the quantity \( |\sum_{p \leq n} \log p/p - \log n| \) as n → ∞)

[132], 1875/6, pp. 438–439] K. Hensel (E, and IP in AP 1(mod 4), 1(mod 6) and 5(mod 8))

∗ [132], 1875/6, pp. 438–439] K. Hensel (E, and IP in AP 1(mod 4), 1(mod 6) and 5(mod 8))

∗ [228], 1878] J. J. Sylvester (E, Lucas sequences, and IP in AP 2(mod 5) and 7(mod 8))

[156], 1878/9], [248], p. 4], [324] E. E. Kummer (E and Euclid’s proof revisited with \( p_1p_2 \cdots p_n - 1 \) instead of \( p_1p_2 \cdots p_n + 1 \))

[227], 1881], [324] J. Perott (CS, UFT, CM, the fact that \( \sum_{n=1}^{\infty} 1/n^2 < 2 \), the estimate of upper bound of number of integers ≤ N by some square)

∗ [23], 1886], [76], p. 418] A. S. Bang (E, cyclotomic polynomials, and IP in AP 1(mod k) with k ≥ 2)

[104], 1887/8], [76], p. 413] L. Gegenbauer (CS and the the convergent series \( \sum_{n=1}^{\infty} 1/n^s \))

[228], 1888], [76], p. 414] J. Perott (Theory of Commutative Groups)

[290], 1888], [210], p. 7] J. J. Sylvester, (evaluation of Euler’s product \( \prod_{p \leq x} (1 - 1/p)^{-1} \) and the estimate \( \sum_{n \leq x} 1/n \geq \log x \))

[290], 1888], [210], pp. 11–12] J. J. Sylvester (DS, the series \( \sum_{n=1}^{\infty} 1/n \) is divergent and the series \( \sum_{n=1}^{\infty} 1/n \) is convergent)

* [291], 1888], [76], p. 418] J. J. Sylvester (E, cyclotomic polynomials, and IP in AP 1(mod k) with k ≥ 2)

* [291], 1888] J. J. Sylvester (E, and IP in AP −1(mod p^n) with p any fixed prime)

[228], 1889], [229] J. Perott (E, Euclid’s proof revisited, with \( p_1p_2 \cdots p_k - 1 \) instead of \( p_1p_2 \cdots p_k + 1 \))

[124], 1890], [76], p. 414] J. Hacks (formula for the number of positive integers less than N from [47], Ch. XI)

[285], 1890, p. 14], [76], p. 414], [246], [324] T. J. Stieltjes (E and the fact that the sum \( p_1p_2 \cdots p_k + p_{k+1}p_{k+2} \cdots p_{k+r} \) is not divisible by any \( p_i \) (i = 1, 2, ..., k + r))

* [175], 1891] É. Lucas (E, Lucas sequences, IP in AP 1(mod 4), 5(mod 6) and in AP 5(mod 8))
The representation $Q = P/a - a > 1$, where $a$ and $P/a$ are relatively prime factors of $P := p_1^{e_1}p_2^{e_2} \cdots p_n^{e_n}$.

* 1896, p. 89] E. Wendt (the factorization $x^n - 1 = f(x)g(x)$, where $g(x)$ is the least common multiple of polynomials $\{x^d - 1 : d \mid n\}$, common divisors of integers $f(x)$ and $g(x)$ with $x \in \mathbb{Z}$, and $IP$ in $AP 1(\mod k)$)

* 1896] R. D. von Sterneck (E, and $IP$ in $AP -1(\mod k)$ with $k = 2, 3, \ldots$)

* 1897, p. 9] A. Thue ($CM$ and $UFT$)

* 1897] K. Th. Vahlen (Gauss’ periods of roots of unity, and $IP$ in $AP 1(\mod k)$ with $k \geq 2$)

* 1899, p. 414] 233, p. 3], 324 J. Braun (E and a prime divisor of $\sum_{i=1}^{k} (p_1p_2 \cdots p_k)/p_i$)

* 1899, p. 414], 252 J. Hacks (Euler’s formula $\prod 1/(1 - p^{-2}) = \sum_{n=1}^{\infty} 1/n^2 = \pi^2/6$ and the irrationality of $\pi^2$)

* 1899, p. 291] É. Lucas (Lucas sequence, and $IP$ in $AP 1(\mod k)$ with $k \geq 2$)

* 1900, pp. 318–319] E. Cahen (E, and $IP$ in $AP 1(\mod 4), 1(\mod 6)$ and $5(\mod 8)$)

* 1903/04] G. D. Birkhoff and H. S. Vandiver (the existence of primitive prime divisors of integers $a^n - b^n$, where $n \in \mathbb{N}$ and $a$ and $b$ are relatively prime integers)

* 1905/6] M. Bauer (E, and $IP$ in $AP -1(\mod k)$ with $k \geq 2$)

* 1907], 300 p. 87] H. Bonse (E)

* 1909/10], 76 p. 414] A. Lévy (E)

* 1911], 59 p. 116, Theorem 114], 76 p. 419]) H. C. Pocklington (E, and $IP$ which are not congruent to $1(\mod k)$)

* 1912/13], 240 pp. 131, 324, Problem 108] I. Schur (E, and $IP$ of primes dividing the integer values of a nonconstant integer polynomial)

* 1913] R. D. Carmichael ($IP$ in $AP -1(\mod p^n)$ with $p$ any fixed odd prime, and $IP$ in $AP -1(\mod 3 \cdot 2^n)$)

* 1913] K. Hensel (E, and $IP$ in $AP 1(\mod 4), 1(\mod 6)$ and $7(\mod 8), 3(\mod 8), 9(\mod 10)$ and $11(\mod 12)$)

* 1915], 76 p. 414], 248 p. 11] A. Auric ($CM$, $FT$ and the estimate of number of positive integers $m = p_1^{e_1}p_2^{e_2} \cdots p_r^{e_r}$ less than $N$)

* 1917], 76 p. 415], 248 p. 11] G. Métrod (E and a prime divisor of $\sum_{i=1}^{n} N/p_i$, where $N = p_1p_2 \cdots p_n$)
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G. Pólya and G. Szegő (1921), pp. 131, 324, Problem 107

F. Hartmann (1931) F. Hartmann (IP in AP \(1 \equiv 1 (mod p^n)\))

G. Pólya and G. Szegő (1934, p. 283), P. Erdős (C and de Polignac’s formula)

F. Hartmann (1931) F. Hartmann (IP \(\equiv 1 (mod p^n)\))

P. Erdős (1934), P. Erdős (de Polignac’s formula and inequalities for central binomial coefficients)

A. S. Bang, (1937), A. S. Bang, \(E\), and IP in AP \(2p^n + 1 (mod 4p^n)\) with prime \(p \equiv 3 (mod 4)\), \(2p^{2n+1} + 1 \mod 6p^{2n+1}\) with prime \(p \equiv 2 (mod 3)\), and \(4p^{2n} + 1 \mod 6p^{2n}\) with prime \(p \equiv 2 (mod 3)\)

P. Erdős (1938), 8th proof P. Erdős (Chebyshev’s argument, de Polignac’s formula and DS)

G. H. Hardy and E. M. Wright (1938, pp. 16–17) G. H. Hardy and E. M. Wright \(FT\), a representation \(n = m^2 k\) where \(k\) is squarefree and CM

G. H. Hardy and E. M. Wright (1938, p. 13) G. H. Hardy and E. M. Wright \(E\), prime divisor of \(a^2 + b^2\), and IP in AP \(5 (mod 8)\)

L. G. Schnirelman (1940, pp. 44–45) (published posthumously) L. Schnirelman (the estimates \(\lim_{x \to \infty} (\sum_{p \text{ prime}} 1) x^{-k} = 0\) for \(a > 1\) and \(k > 0\) and an enumerative argument)

R. Bellman (1943) R. Bellman \(DS\) and the sum of prime reciprocals

R. Bellman (1947), p. 7 R. Bellman \(MPI\) and a polynomial method

T. Nagell, (1951) \(IP\) in AP \(1 (mod k)\) for all \(k \geq 2\)

J. G. Thompson (1953) J. G. Thompson \(E\)

E. Dux (1956) E. Dux \(DS\) and the sum of prime reciprocals

V. C. Harris (1956), \(\mu(d)\) with relatively primes integers \(n\) and \(m\), the order of \(k\) modulo a prime \(p\), and IP in AP \(1 (mod 10)\)

S. W. Golomb (1957) S. W. Golomb \(MPI\)

D. Jarden (1958) D. Jarden (recurring sequences and IP in AP \(1 (mod 20)\))

S. W. Golomb (1959) S. W. Golomb \(T\)

A. Rotkiewicz (1961) A. Rotkiewicz (Birkhoff-Vandiver theorem, the order of \(k\) modulo a prime \(p\), and IP in AP \(1 (mod k)\))

D. O. Shklarsky, N. N. Chentzov and I. M. Yaglom (1962, pp. 60, 371–373) D. O. Shklarsky, N. N. Chentzov and I. M. Yaglom \(E\), divisors of \(a^5 - 1\), Fermat little theorem, and IP in AP \(1 (mod 10)\)

T. Estermann (1962/3) T. Estermann (prime divisors of \(nm\), where \(n/m := \prod_{d|n} (k^{d/n} - 1)^{\mu(d)}\) with relatively primes integers \(n\) and \(m\), the order of \(k\) modulo a prime \(p\), and IP in AP \(1 (mod 10)\)

S. W. Golomb (1963) S. W. Golomb \(MPI\)
Inequality for central binomial coefficient, mathematical induction, \textit{UFT}, de Polignac's formula

* \cite{27} 1965 P. Bateman and M. E. Low (E, law of quadratic reciprocity, and IP in AP \(1 \pmod{24}\))

\cite{57} 1965 P. R. Chernoff (CM, FT, the estimate of upper bound of number of \(k\)-tuples \((e_1, e_2, \ldots, e_k)\) satisfying \(p_1^{e_1}p_2^{e_2}\cdots p_k^{e_k} \leq N\))

\cite{321} 1965, \cite{210} p. 9, M. Wunderlich (MPI, Fibonacci sequence \((f_n)\), the property \((m, n) = 1 \implies (f_m, f_n) = 1 \) and the factorization \(f_{19} = 113 \cdot 37\))

\cite{134} 1966 R. L. Hemminiger (MPI, a sequence \((a_n)\) with the property: \((m, n) = 1 \implies (f_m, f_n) = 1\), the sequence \((a_n)\) defined recursively as \(a_1 = 2\), \(a_{n+1} = 1 + \prod_{i=1}^{n} a_i\))

\cite{286} 1966 M. V. Subbarao (MPI)

\cite{62} 1969 E. Cohen (de Polignac's formula and DS)

\cite{275} 1970, Problems 47 and 92] A. Mąkowski (E and relatively prime numbers)

\cite{275} 1970, Problem 50] A. Rotkiewicz (MPI and Fibonacci numbers)

\cite{275} 1970, Problem 52] W. Sierpiński (attributed to P. Schorn by P. Ribenboim [248 pp. 7–8]) (E, MPI and AP \((m!)k + 1\) for a fixed \(k = 1, 2, \ldots, m\))

\cite{275} 1970, Problem 62] W. Sierpiński (E, MPI and AP)

\cite{275} 1970, Problem 36] W. Sierpiński (MPI and triangular numbers)

\cite{275} 1970, Problem 36] W. Sierpiński (MPI and tetrahedral numbers)

\cite{79} 1971 Problem 3 on IMO 1971 (FT and Euler’s theorem)

\cite{299} 1974] C. W. Trigg (E)

\cite{25} 1976 C. W. Barnes (E, Theory of periodic continued fractions and Theory of negative Pell’s equations \(x^2 - dy^2 = -1\))

* \cite{214} 1976 I. Niven and B. Powell (the induction, the order of \(k\) modulo a prime \(p\), a polynomial equation, and IP in AP \(1 \pmod{k}\))

\cite{194} 1978, Theorem 1], \cite{195}, \cite{233} pp. 5–6] S. P. Mohanty (MPI and the induction)

\cite{194} 1978, Theorem 2], S. P. Mohanty (MPI and Fermat little theorem)

\cite{194} 1978, Theorem 3], S. P. Mohanty (MPI and prime divisors of Fibonacci numbers \(f_p\))

* \cite{266} 1978, p. 107], \cite{89} pp. 178–179], \cite{204} p. 209], D. Shanks (a prime divisor of \((2^{mp} - 1)/(2^m - 1)\) of the form \(p^\alpha k + 1\), and IP in AP \(\equiv 1 \pmod{p^\alpha}\))

\cite{13} 1979 R. Apéry (Euler’s formula \(\prod 1/(1 - p^{-3}) = \sum_{n=1}^{\infty} 1/n^3 \equiv \zeta(3)\) and the irrationality of \(\zeta(3)\))

\cite{52} 1979], \cite{59} p. 118, Section 10.1.5] G. Chaitin (algorithmic information theory and an enumerative argument)

\cite{315} 1979, p. 36], A. Weil (E and Group Theory)
[303] 1980] C. Vanden Eynden (DS, the divergence of the series $\sum_{n=1}^{\infty} 1/n$ and the convergence of the series $\sum_{n=1}^{\infty} 1/n^2$)
[312] 1980], [238] pp. 11–12], [54] L. C. Washington (Theory of principal ideal domains, and the factorizations $(1 + \sqrt{-5})(1 - \sqrt{-5}) = 2 \times 3$ of 6 in the ring $\mathbb{Z}[a + b\sqrt{-5}]$)
* [250] 1981] R. A. Smith (Birkhoff-Vandiver idea, the solvability of the congruence $x^k \equiv 1 \pmod{p}$ with an integer of order $k$ modulo a prime $p$, and IP in AP $1 \pmod{k}$)

D. P. Wegener (E and primitive Pythagorean triples)

[280] 1981] R. A. Smith (Birkhoff-Vandiver idea, the solvability of the congruence $x^k \equiv 1 \pmod{p}$ with an integer of order $k$ modulo a prime $p$, and IP in AP $1 \pmod{k}$)

[314] 1981] D. P. Wegener (E and primitive Pythagorean triples)

[320] 1981] A. R. Woods (weak system of arithmetic $I \Delta_0$, $\Delta_0$-definable functions, the pigeonhole principle PHP$\Delta_0$ formulated for functions defined by $\Delta_0$-formulas)

[283] 1984], [324] S. Srinivasan (MPI, “dynamical systems proof” and the sequence $\left(\frac{2^{2^{n+1}} + 2^{2^n} + 1}{(2^{2^n} + 2^{2^n-1} + 1)}\right)$)

[283] 1984], [324] S. Srinivasan (MPI, ”dynamical systems proof”, Fermat little theorem and the sequence $\left(\frac{2^{p^{n+1}} - 1}{(2^{p^n} - 1)}\right)$)

[222] 1985] R. W. K. Odoni (E, MPI and a sequence $w_n$ recursively defined as $w_1 = 2$, $w_{n+1} = 1 + w_1 \cdots w_n$ ($n \geq 1$))

[71] 1986], [257] M. Deaconescu and J. Sándor (divisibility property $n | \varphi(a^n - 1)$, $a, n > 1$)

[226] 1988], [225] J. B. Paris, A. J. Wilkie and A. R. Woods (weak system of arithmetic $I \Delta_0$, weak pigeonhole principle, $\Delta_0$-definable functions)

[252] 1993] M. Rubinstein (CM, UFT and the asymptotic formula for the cardinality of a set $\{(e_1, \ldots, e_k) \in \mathbb{N}^k : x_1 \log p_1 + x_2 \log p_2 + \cdots + x_k \log p_k \leq \log x\}$)

* [250] 1994] N. Robbins (MPI, prime divisors of Fermat numbers, and IP in AP $1 \pmod{4}$)

* [250] 1994] N. Robbins (MPI, prime divisors of Fibonacci numbers, and IP in AP $1 \pmod{4}$)

[298] 1995] D. Treiber (DS and the sum of prime reciprocals)

[10] 1997, Problem 7.2.3] Problem on 1997 Romanian IMO Team Selection Test, (MPI, the induction, Euler theorem and a subsequence of the sequence $(a^{n+1} + a^n + 1)$ for a fixed integer $a > 1$)

[85] 1998, Problem E3] Problem of the training of the German IMO team, (MPI, the induction, the factorization $2^{2^{n+1}} + 2^{2^n} + 1 = (2^{2^n} - 2^{2^{n-1}} + 1)(2^{2^n} + 2^{2^{n-1}} + 1)$ and $2^{2^{n+1}} + 2^{2^n} + 1$ has at least $n$ different prime factors for each $n = 0, 1, 2, \ldots$)

[109] 1998], [233] p. 16] R. Goldblatt (E and nonstandard Analysis)

* [262] 1998] N. Sedrakian and J. Steinig (a prime divisor of $(k^k-1)/[k^k/p_1 - 1, \ldots, k^k/p_s - 1]$, where $p_1, \ldots, p_s$ are all distinct prime divisors of $k$ and $[a_1, \ldots, a_s]$ denotes the greatest common divisor of $a_1, \ldots, a_s$, and IP in AP $1 \pmod{k}$)

[70] 2000] M. Dalezman (E, CM)
2001, p. 4] M. Aigner and G. M. Ziegler (CM, definite integral of the function $1/t$, DS, UFT)
2001, p. 3, [11] p. 72, [72] M. Aigner and G. M. Ziegler, (Lagrange’s theorem of Group Theory and Mersenne numbers)
2001] Š. Porubský (T and Theory of commutative rings)
2001/2, Problem 6], [221] p. 51, Problem 3.5.3] Problem on Polish Mathematical Olympiad (MPI and recursive sequence)
2001, [241] M. D. Hirschorn (CM and FT)
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