Abstract

For any integer $q \geq 1$, let $T_{q+1}$ denote the $(q + 1)$-regular tree with discrete Laplacian associated to the adjacency matrix. Let $K_{T_{q+1}}(x, x_0, t)$ be the associated discrete time heat kernel, i.e. the fundamental solution to the discrete-time diffusion equation defined below. We derive an explicit formula for $K_{T_{q+1}}(x, x_0, t)$ in terms of the discrete $I$-Bessel function. The expression is analogous to the corresponding real time heat kernel, as proved in [CJK14]. However, the formula obtained here for $K_{T_{q+1}}(x, x_0, t)$ has the significant feature of being a finite sum, rather than a convergent infinite series, which is what occurs in the continuous time setting. We then derive an explicit expression for the discrete time heat kernel $K_X(x, x_0, t)$ on any $(q + 1)$-regular graph. Going further, we develop a general trace formula which relates the spectral data on $X$ to its topological data. Though we emphasize the results in the case if $X$ is finite, our method applies when $X$ has a countably infinite number of vertices. As an application, we derive the recurrent formula which expresses the number of closed geodesics on $X$, in the case when $X$ is finite, in terms of moments of the spectrum of the adjacency matrix, complementing the results of [Mn07]. Finally, by relating $K_X(x, x_0, t)$ to the random walk on a $(q + 1)$-regular graph, we obtain a closed-form expressions for the return time probability distribution of the universal random walk on any $(q + 1)$-regular graph. In particular, we show that if $\{X_h\}$ is a sequence of $(q + 1)$-regular graphs whose number of vertices goes to infinity, and which satisfy a certain natural geometric condition, then the limit of the return time probability distribution on $X_h$ is equal to the return time probability distribution on the tree $T_{q+1}$.

1 Introduction

We let $X$ denote an undirected and connected graph with a finite or countably infinite set of vertices. Let $V_X$ denote the set of vertices of $X$, and let $E_X$ denote the set of the edges of $X$. We allow an edge to connect a vertex to itself; such an edge is sometimes called a self-loop. We assume that $X$ is a $(q + 1)$-regular graph, which means each vertex $x \in V_X$ has precisely $(q + 1)$ edges that emanate from $x$. If $x, y \in V_X$, we will write $x \sim y$ if there is an edge $E_X$ which connects $x$ and $y$. The universal cover of every $(q + 1)$-regular graph is a $(q + 1)$-regular tree, which is a connected $(q + 1)$-regular graph without closed paths. We let $T_{q+1}$ denote a $(q + 1)$-regular tree.

*The second named author acknowledges grant support from several PSC-CUNY Awards, which are jointly funded by the Professional Staff Congress and The City University of New York.
Let \( f : V_X \to \mathbb{C} \) be any complex-valued function on \( V_X \). Let \( L^2(V_X) \) denote the Hilbert space of \( L^2 \) functions defined on \( V_X \) equipped with the inner product
\[
\langle f, g \rangle = \sum_{x \in V_X} f(x)\overline{g(x)} \quad \text{for } f, g \in L^2(V_X).
\]

The Laplacian \( \Delta_X \) on \( X \) is a non-negative, self-adjoint operator on \( L^2(V_X) \) which is defined through its action on \( L^2(V_X) \)-functions, namely that
\[
\Delta_X f(x) = (q + 1)f(x) - \sum_{y \sim x} f(y) \quad \text{for } f \in L^2(V_X).
\]

In other symbols, we can write that \( \Delta_X = (q + 1)\text{Id} - \mathcal{A}_X \) where \( \text{Id} \) is the identity operator and \( \mathcal{A}_X \) is the adjacency operator on the graph \( X \) defined by
\[
(\mathcal{A}_X f)(x) = \sum_{y \sim x} f(y).
\]

If \( V_X \) is finite, so \( X \) has a finite number of vertices, say \( M \), then we let \( \lambda_0 = q + 1 > \lambda_1 \geq \ldots \geq \lambda_{M-1} \geq -(q + 1) \) denote the eigenvalues of \( \mathcal{A}_X \), counted with multiplicities. Let \( \psi_j \in \mathbb{R}^M, \ j = 0, \ldots, M - 1 \) be the corresponding orthonormal eigenvectors of \( \mathcal{A}_X \).

Let \( t \in T \) denote a time variable; we will consider either \( T = \mathbb{R}_{\geq 0} \), the set of non-negative real numbers, or \( T = \mathbb{N}_0 \), the set of non-negative integers. In the case \( T = \mathbb{R}_{\geq 0} \), \( \partial_{t,T} \) denotes the partial derivative with respect to the real variable \( t \). If \( T = \mathbb{N}_0 \), then \( \partial_{t,T} \) denotes the forward difference operator, meaning
\[
\partial_{t,T} f(t) = f(t + 1) - f(t) \quad \text{if } T = \mathbb{N}_0.
\]

When \( T = \mathbb{N}_0 \), the heat kernel on \( X \) is defined as the function
\[
K_{X,T}(x, y; t) : V_X \times V_X \times T \to \mathbb{R}
\]
which satisfies the equation
\[
\Delta_X K_{X,T}(x_0, x; t) + \partial_{t,T} K_{X,T}(x_0, x; t) = 0 \quad (1)
\]
subject to the initial condition
\[
K_{X,T}(x_0, x; 0) = \begin{cases} 1, & \text{if } x = x_0 \\ 0, & \text{otherwise.} \end{cases} \quad (2)
\]

The equation (1) also can be viewed as a diffusion equation
\[
\partial_{t,T} K_{X,T}(x_0, x; t) = \sum_{y \sim x} K_{X,T}(x_0, y; t) - (q + 1)K_{X,T}(x_0, x; t)
\]
on the discrete space \( V_X \). When \( T = \mathbb{R}_{\geq 0} \), one replaces (2) by
\[
\lim_{t \to 0^+} K_{X,T}(x_0, x; t) = \begin{cases} 1, & \text{if } x = x_0 \\ 0, & \text{otherwise.} \end{cases} \quad (3)
\]

In that case, the solution to equation (1) subject to the initial condition (3) is called the heat kernel on the graph \( X \), or the continuous time heat kernel. We will refer to the solution of
subject to the initial condition \( u(x; 0) = 1 \) if \( x = 0 \) and \( u(x; 0) = 0 \) if \( x \neq 0 \). For \( d \neq 1/2 \), it is proved in \( \text{[SIL18]} \) that the solution to (5), with the above stated initial condition, is given by

\[
\partial_{t_{N_{0}}} u(x; t) = d(u(x + 1; t) - 2u(x; t) + u(x - 1; t)) \quad \text{for } x \in \mathbb{Z}
\]

subject to the initial condition \( u(x; 0) = 1 \) if \( x = 0 \) and \( u(x; 0) = 0 \) if \( x \neq 0 \). For \( d \neq 1/2 \), it is proved in \( \text{[SIL18]} \) that the solution to (5), with the above stated initial condition, is given by

\[
u(n, t) = (1 - 2d)^t I_{2d/(1 - 2d)}^2 \quad \text{for } n \in \mathbb{Z} \text{ and } t \in \mathbb{N}_0.
\]

When comparing the continuous time heat kernel on \( \mathbb{Z} \) from \( \text{[KN06]} \) and the discrete time heat kernel on \( \mathbb{Z} \) from \( \text{[SIL18]} \), one is led to consider the discrete \( I \)-Bessel functions from \( \text{[SIL18]} \) to form the “building blocks” for the construction of the discrete time heat kernel on a \((q + 1)\)-regular tree \( T_{q+1} \), and subsequently for the discrete time heat kernel on any \((q + 1)\)-regular graph. Let \( r \in \mathbb{N}_0 \) represent the graph distance of the point \( x \) to the base point \( x_0 \). We prove that the building blocks for the discrete time heat kernel are functions of the form

\[
(-q)^t q^{-r/2} I_{r-2/\sqrt{q}}^2 (t) \quad \text{for } t \in \mathbb{N}_0.
\]

Indeed, the function \((-q)^t\) is the discrete time analogue of the exponential function \( e^{-(q+1)t} \); see Example 2.50 of \( \text{[BP01]} \). As such, there is a clear analogy between (4) and (7).

For the remainder of this article, we only will consider the discrete time heat kernel only. Hence, we will omit the distinction in the time variable, and we will use the notation \( K_X(x_0, x; t) \) for the discrete time heat kernel.

Our first main result is the following theorem; we refer to section 2.1, as well as \( \text{[Se03]} \), for a detailed description of the notation and terminology which we are using.

**Theorem 1.** Let \( X \) be a \((q + 1)\)-regular graph with a fixed base point \( x_0 \). For any point \( x \in V_X \), let \( c_m(x) \) be the number of closed geodesics \( c_m(x) \) of length \( m \) from \( x_0 \) to \( x \). Let

\[
b_m(x) = c_m(x) - (q - 1)(c_{m-2}(x) + c_{m-4}(x) + \ldots + c_0(x)),
\]

where \( r \) is the radial or distance variable between \( x \) and \( x_0 \), and as well as the set of their pre-images on \( T_{q+1} \), and \( J_r \) is the classical \( I \)-Bessel function of order \( r \). As discussed in \( \text{[CJIK14]} \), the building block (4) is closely related to the heat kernel \( e^{-2t} J_r(2t) \) on the graph \( \mathbb{Z} \) when viewed as a 2-regular Cayley graph, which itself is comparable to the classical real time heat kernel on the continuous space \( \mathbb{R} \); see \( \text{[KN06]} \) and \( \text{[CGRTV17]} \).

The purpose of this article is to derive explicit formulas for the discrete time heat kernel on any \((q + 1)\)-regular graph. Given the results of \( \text{[CJIK14]} \), we will consider “building blocks” which are analogous to (4) obtained by replacing the continuous time \( I \)-Bessel function \( J_r \) with a similar discrete time version which we now describe.

In \( \text{[BC18]} \) the authors defined and studied a discrete analogue of the \( J \)-Bessel function. Continuing in this direction, A. Slavík in \( \text{[SIL18]} \) introduced the discrete \( I \)-Bessel function \( I_r^c(t) \) which depends on a space variable \( x \in \mathbb{N}_0 \), a time variable \( t \in \mathbb{N}_0 \), and a complex parameter \( c \); see \( \text{[16]} \) below. In Theorem 4.1 of \( \text{[SIL18]} \) the author considers the discrete time diffusion equation

\[
\partial_{t_{N_{0}}} u(x; t) = d(u(x + 1; t) - 2u(x; t) + u(x - 1; t)) \quad \text{for } x \in \mathbb{Z}
\]

subject to the initial condition \( u(x; 0) = 1 \) if \( x = 0 \) and \( u(x; 0) = 0 \) if \( x \neq 0 \). For \( d \neq 1/2 \), it is proved in \( \text{[SIL18]} \) that the solution to (5), with the above stated initial condition, is given by

\[
u(n, t) = (1 - 2d)^t I_{2d/(1 - 2d)}^2 (t) \quad \text{for } n \in \mathbb{Z} \text{ and } t \in \mathbb{N}_0.
\]
where
\[ c_\ast(x) = \begin{cases} c_0(x) & \text{if } m \text{ is even} \\ c_1(x) & \text{if } m \text{ is odd} \end{cases} \] (9)

Then the discrete time heat kernel \( K_X(x_0, x; t) \) on \( X \) is given by
\[ K_X(x_0, x; t) = (-q)^t \sum_{m=0}^t b_m(x) q^{-m/2} I_m^{-2/\sqrt{q}}(t) \quad \text{for } x \in V_X \text{ and } t \in \mathbb{N}_0. \] (10)

Furthermore, if we take \( x = x_0 \), then for \( x \in V_X \) and \( t \in \mathbb{N}_0 \) we have that
\[ K_X(x_0, x_0; t) = (-q)^t \left( \sum_{m=0}^t N_m(x_0) q^{-m/2} I_m^{-2/\sqrt{q}}(t) + (1 - q) \sum_{j=1}^{[t/2]} q^{-j} I_{2j}^{-2/\sqrt{q}}(t) \right) \] (11)

where \( N_m(x_0) \) is the number of distinct geodesic paths from \( x_0 \) to itself without tails.

The right-hand side of (10) and of (11) are finite sums because \( I_m^{-2/\sqrt{q}}(t) = 0 \) if \( t > m \). Hence, for any fixed \( x \in V_X \), one obtains closed-form expressions for \( b_m(x) \) as a linear combination of values of \( K_X(x, 0; t) \) at times \( 0 \leq t \leq m \). Also, one can obtain explicit, closed-form formulas for \( c_m(x) \) in terms of \( K_X(x_0, x; t) \) for \( 0 < t \leq \ell \).

Theorem 2 implies the following result, which can be viewed as a pre-trace formula.

**Corollary 2.** Let \( X \) be a \((q + 1)\)-regular graph with a fixed base point \( x_0 \). Let \( \mu_x(\lambda) \) be the spectral measure at the point \( x \in V_X \) associated to the adjacency operator \( A_X \) and the base point \( x_0 \). Then, with the notation from Theorem 7, we have that
\[ \int_{-(q+1)}^{q+1} (\lambda - q)^t d\mu_x(\lambda) = (-q)^t \sum_{m=0}^t b_m(x) q^{-m/2} I_m^{-2/\sqrt{q}}(t) \quad \text{for } x \in V_X \text{ and } t \in \mathbb{N}_0. \]

When the graph \( X \) is finite with \( M \) vertices, then
\[ \sum_{j=0}^{M-1} (\lambda_j - q)^t \psi_j(x) \psi_j(x_0) = (-q)^t \sum_{m=0}^t b_m(x) q^{-m/2} I_m^{-2/\sqrt{q}}(t) \quad \text{for } x \in V_X \text{ and } t \in \mathbb{N}_0. \] (12)

When \( X \) is finite, we can set \( x = x_0 \) in (12) and sum over all \( x \in V_X \) to obtain the following trace formula.

**Corollary 3.** Let \( X \) be a finite, undirected, \((q + 1)\)-regular graph with \( M \) vertices. Let \( N_m \) denotes the number of distinct closed geodesics of length \( m \) from any fixed starting point, with a distinct direction. Then, with the notation as above, for \( t \in \mathbb{N}_0 \) we have that
\[ \sum_{j=0}^{M-1} \left( 1 - \frac{\lambda_j}{q} \right)^t = \sum_{m=0}^t N_m q^{-m/2} I_m^{-2/\sqrt{q}}(t) + M (1 - q) \sum_{j=1}^{[t/2]} q^{-j} I_{2j}^{-2/\sqrt{q}}(t). \] (13)

Formula (13) can be viewed as a discrete time counterpart of the continuous time trace formula developed in [Mn07]. Note that it holds true for any \((q + 1)\)-regular graph with \( M \) vertices, while the trace formula proved in [Mn07] is proved for finite connected regular graphs of degree \( q + 1 \geq 2 \), without multiple edges and loops. In effect, the left-hand side of (13) is the
trace of \((Id - \Delta_X)^t\), normalized by multiplying with \((-q)^{-t}\), as opposed to normalizing factor \(\exp(-q + 1)t\) when time is continuous.

In general terms, one can interpret Corollary 3 as showing that the discrete time heat kernel carries the “same amount” of spectral information about the graph as the continuous time kernel. The advantage of the discrete time approach is that the “geometric side” of the expression of the heat kernel is always a finite sum. By comparison, we note that Theorem 1.1 of [CJK14] which gives the continuous time heat kernel in terms of continuous time \(I\)-Bessel functions is an infinite sum; see also [Ah87] and [TW03] for further results, again in continuous time.

By starting with equation (13) and using the properties of the discrete \(I\)-Bessel functions, we will prove a general trace formula for any finite \((q + 1)\)-regular graphs. This theorem, which we now state, is our second main result.

**Theorem 4.** Let \(X\) be a finite, undirected, \((q + 1)\)-regular graph with \(M\) vertices. For any real number \(a > 3 + 2/q\), let \(h(z) : \mathbb{C} \rightarrow \mathbb{C}\) be a complex valued function, holomorphic for \(|z| > 1/a\). For \(n \in \mathbb{N}_0\) and a complex number \(c\), set

\[
f_c^n(z) = \frac{1}{\sqrt{(1 - z)^2 - c^2z^2}} \left(1 - \sqrt{(1 - z)^2 - c^2z^2} \right)^n,
\]

where the square root defined by taking the principal branch in the right half-plane. Then

\[
\sum_{j=0}^{M-1} h \left(\frac{q}{q - \lambda_j}\right) = \frac{1}{2\pi i} \sum_{m=0}^{\infty} \frac{N_m q^m}{m} \int_{c(0,b)} f_m^{-2/\sqrt{q}}(z) h(z) \frac{dz}{z} \\
+ \frac{M(1-q)}{2\pi i} \sum_{j=1}^{\infty} q^{-j} \int_{c(0,b)} f_j^{-2/\sqrt{q}}(z) h(z) \frac{dz}{z},
\]

where complex integrals above are taken along the circle centered at zero and radius \(b\) such that \(1/a < b < q/(3q + 2)\).

The heat kernel \(K_X(x_0, x; t)\) is closely related to the uniform random-walk on the graph \(X\). By a uniform random-walk, we imagine a particle starting at an initial vertex \(x_0\) at time \(t = 0\) and at each time step moves along one of the \((q + 1)\) edges to another vertex, with all edges being selected with equal probability. Let \(K_{rw}^X(x_0, x; t)\) denote the probability that the uniform random-walk which starts at \(x_0\) at time \(t = 0\) ends up at the vertex \(x\) at time \(t\). It is proved in Section 4 below that

\[
K_{rw}^X(x_0, x; t) = \left(\frac{q}{q + 1}\right) \sum_{k=0}^{t} \binom{t}{k} q^{-k} K_X(x_0, x; k).
\]

Combining this result with Theorem 4 we obtain the following corollary.

**Corollary 5.** Let \(X\) be a \((q + 1)\)-regular graph. For \(t \in \mathbb{Z}_{\geq 0}\), let \(K_{rw}^X(x_0, x; t)\) denote the probability that the uniform random-walk on the graph \(X\) starts at \(x_0\) and ends at \(x\) at time \(t\). Then

\[
K_{rw}^X(x_0, x; t) = \left(\frac{q}{q + 1}\right) \sum_{k=0}^{t} \binom{t}{k} (-1)^k \sum_{m=0}^{k} b_m(x) q^{-m/2} I_m^{-2/\sqrt{q}}(k).
\]
When \( x_0 = x \), the probability distribution \( K_X^{RW}(x_0, x_0; t) \) equals that of the return to \( x_0 \) of a random walk on \( X \) after \( t \) steps. When the graph is finite, we will compute in Section 5 the limiting distribution of \( K_X^{RW}(x_0, x; t) \) as the number of vertices tends to infinity. We will show that the limiting probability distribution depends only upon \( t \) and \( q + 1 \); see [TBK21] for a related result on the first return times on finite graphs. Further related results on \((q + 1)\)-regular graphs which are Bethe lattices with coordination number \((q + 1)\) are given in [HSS2] and [Gi95], while [SRb-A05] treats the case of Erdős-Rényi random graphs.

The paper is organized as follows. In Section 2 we recall basic notation and we prove necessary results associated to the discrete \( I \)-Bessel function, including explicit formulas and asymptotic expansions. Section 3 is devoted to proofs of our main results, except Corollary 5. The proof of Corollary 5 is completed in Section 4. Random walks on a sequence of finite \((q + 1)\)-regular graphs are further studied in Section 5. We prove that the limiting distribution of return probabilities of a uniform random walk on a sequence of finite \((q + 1)\)-regular graphs (satisfying certain natural geometric condition), as their number of vertices tends to infinity, approaches the distribution of return probabilities on the tree \( T_{q+1} \), see Proposition 21. In Section 6 we derive explicit and finite expressions for \( N_m \), the number of geodesics of length \( m \), in terms of the spectrum of the Laplacian and special values of discrete \( I \)-Bessel functions. If \( X \) is finite, we show that our results complement main results of [Mn07]. Finally, in Section 7 we show how one can solve certain other diffusion equation using the discrete \( I \)-Bessel function.

2 Preliminaries

2.1 Regular graphs

We refer the reader to [Se03] for standard notation and definitions.

As stated above, \( X \) denotes an undirected and connected graph with vertices \( V_X \) and edges \( E_X \). The set of vertices \( V_X \) is either finite or countably infinite. We allow edges which connect a vertex to itself; such an edge is sometimes called a self-loop. The degree \( d_x \) of a vertex \( x \in V_X \) is the number of vertices \( y \in V_X \) such that the pair \( \langle x, y \rangle \) is an edge in \( E_X \).

The graph \( X \) is \( d \)-regular if \( d = d_x \), for all \( x \in V_X \). Throughout this article, we only consider regular graphs. We set \( d = q + 1 \) and use the parameter \( q \) in our formulas. We will choose and fix a base vertex \( x_0 \) in \( X \).

Heuristically, a path in \( X \) is a sequence of adjacent edges which connect a sequence of vertices. We refer to page 14 of [Se03] for a precise definition of a path in \( X \). A particular type path of importance is a geodesic, which simply means a path without backtracking. Also, a closed geodesic is a closed path without a tail and backtracking. Again, we refer to [Se03], in this instance pages 15 and 18, for a detailed discussion of these terms.

For the convenience of the reader, let us summarize the notation we will use.

**Definition 6.** Let \( X \) be a \((q + 1)\)-regular graph, and let \( x_0 \in X \) be a fixed vertex.

1. For any \( x \in X \) and integer \( k \geq 0 \), let \( c_k(x) \) be the number of distinct geodesic paths of length \( k \) from \( x_0 \) to \( x \). In the case \( x = x_0 \), we allow geodesics with and without tails.

2. For any integer \( k \geq 0 \), let

\[
c_k = \sum_{x \in X} c_k(x).
\]
3. For any \( x \in X \) let \( b_0(x) = c_0(x) = 1, b_1(x) = c_1(x) \), and for \( k \geq 2 \), let
\[
b_m(x) = c_m(x) + (1 - q) \left( c_{m-2}(x) + c_{m-4}(x) + \cdots + c_*(x) \right),
\]
where \( c_*(x) \) is defined by (9).

4. For any \( x \in X \) and integer \( k \geq 0 \), let \( N_k(x) \) be the number of distinct geodesic paths of length \( k \) without tails from \( x \) to \( x \).

5. For any integer \( k \geq 0 \), let
\[
N_k = \sum_{x \in X} N_k(x).
\]
Note that we count closed geodesics as having a distinguished orientation.

Following the discussion in Section 2.2 of [CJK14], we obtain the following formulas which relate the sets of values \( \{N_m(x)\} \) and \( \{c_m(x)\} \).

**Lemma 7.** Let \( X \) be a \((q+1)\)-regular graph, and let \( x_0 \in X \) be a fixed vertex. Assume the notation from above.

1. For \( m \leq 2 \), set \( c_m(x) = N_m(x) \). Then for any \( m \geq 3 \), we have that
\[
N_m(x) = c_m(x) + (1 - q) \left( c_{m-2}(x) + c_{m-4}(x) + \cdots + c_*(x) \right)
\]
where
\[
c_*(x) = \begin{cases} c_2(x) & \text{if } m \text{ is even} \\ c_1(x) & \text{if } m \text{ is odd.} \end{cases}
\]

2. For any \( m \geq 3 \), we have that
\[
c_m(x_0) = \begin{cases} N_m(x_0) + (q - 1) \sum_{j=1}^{\ell-1} q^{j-1} N_{m-2j}(x_0) & \text{if } m = 2\ell \text{ is even} \\ N_m(x_0) + (q - 1) \sum_{j=1}^{\ell} q^{j-1} N_{m-2j}(x_0) & \text{if } m = 2\ell + 1 \text{ is odd.} \end{cases}
\]

3. For any \( m \geq 2 \), we have that
\[
b_m(x_0) = \begin{cases} N_m(x_0) + (1 - q) & \text{if } m \text{ is even} \\ N_m(x_0) & \text{if } m \text{ is odd.} \end{cases}
\]

We refer to [CJK14] and references therein, specifically [Se03], for a proof of these assertions.

### 2.2 Discrete \( I \)-Bessel functions

Let \( c \in \mathbb{C} \) be an arbitrary complex number, and let \( t, n \in \mathbb{N}_0 \) be non-negative integers. By expanding ideas from [BC18], the author of [Sl18] defines the discrete modified \( I \)-Bessel function \( I_c^n(t) \) as follows. Let
\[
(-t)_0 = 1 \quad \text{and} \quad (-t)_n = (-t)(-t+1) \cdots (-t+n-1) \quad \text{for } n > 0
\]
be the Pochhammer symbol. Recall that the (Gauss) hypergeometric series is defined by
\[
F(\alpha, \beta; \gamma; z) := \sum_{k=0}^{\infty} \frac{(\alpha)_k (\beta)_k}{k! (\gamma)_k} z^k;
\]
Lemma 8. With the notation as above, the discrete $I$-Bessel function $I_n^c(t)$ satisfies the following properties.

(i) $I_n^c(t) = 0$ for $n > t$.

(ii) $I_0^c(0) = 1$.

(iii) $\partial_t I_n^c(0) = c/2$ and $\partial_t I_0^c(t) = cI_1^c(t)$ for $t \geq 1$.

(iv) $\partial_t I_x^c(t) = \frac{c}{2} (I_{x-1}^c(t) + I_{x+1}^c(t))$ for any $x \geq 1$ and $t \geq 0$.

If $n \leq t$, one of numbers $\frac{n-t}{2}$ and $\frac{n-t}{2} + \frac{1}{2}$ must be a non-positive integer. As a result, by (L7), the hypergeometric series in formula (16) yields a polynomial in variable $c^2$ whenever $n \leq t$. Therefore, $I_n^c(t)$ is a polynomial in variable $c$. The following proposition gives an explicit evaluation of $I_n^c(t)$ in this instance.

Proposition 9. Let $t, n \in \mathbb{N}_0$ such that $n \leq t$. Set $\ell = \lfloor (t - n)/2 \rfloor$. Then for any $c \in \mathbb{C}$, we have that

$$I_n^c(t) = \sum_{j=0}^{\ell} \frac{t!}{j!(t - 2j - n)!((n + j)!(c/2)^{2j+n}}. \tag{18}$$

Proof. For any $d \in \mathbb{R} \setminus \{1/2\}$, consider the discrete time diffusion equation (5) subject to the initial condition $u(x; 0) = 1$ if $x = 0$ and $u(x; 0) = 0$ for $x \neq 0$. From Theorem 4.1 of [S11], we have that the solution for this equation, with the restriction that $d \neq 1/2$, is given by (6). On the other hand, from Example 3.3 of [SSL], we also have that

$$u(n, t) = (1 - 2d)^t \sum_{j=0}^{\ell} \binom{t}{j, t - 2j - n, j + n} \left(\frac{d}{1 - 2d}\right)^{2j+n};$$

see in addition Remark 4.2 of [S11]. (Note: We follow the common convention that multinomial coefficients with negative terms are equal to zero.) Therefore, when comparing the two expressions for $u(n, t)$, we deduce that

$$I_n^c(t) = \sum_{j=0}^{\ell} \binom{t}{j, t - 2j - n, j + n} \left(\frac{c}{2}\right)^{2j+n},$$

for $n \in \mathbb{N}_0$ and any real number $c \neq -1$. Since $t - 2j - n < 0$ for $j > \lfloor (t - n)/2 \rfloor$, the assertion is proved for all $c \in \mathbb{R} \setminus \{-1\}$. From this, note that one can view $I_n^c(t)$ as a function of the complex variable $c$. As such, the proposition follows for all $c \in \mathbb{C}$ by analytic continuation. □
From Proposition 9 we have that $I_n^c(t)$ is a polynomial in the variable $c$ of degree $2\lfloor \frac{t-n}{2} \rfloor + n$. In other words, $I_n^c(t)$ is a polynomial in variable $c$ of degree $t$ when $t - n$ is even and of degree $t - 1$ when $t - n$ is odd.

Going further, it is necessary to determine the asymptotic behavior of the "building block" $(-q)^{t-r/2}I_r^{-2/\sqrt{q}}(t)$ as $q \to \infty$. To do so, one takes $c = -2/\sqrt{q}$ and employs (13), from which we obtain the following corollary.

**Corollary 10.** For positive integers $q,r,t$ such that $r \leq t$ we have that

$$(-q)^{t-r/2}I_r^{-2/\sqrt{q}}(t) \sim (-1)^{t-r} \left( \frac{t}{r} \right) q^{t-r} \text{ as } q \to \infty.$$ 

**Remark 11.** Let $t,n \in \mathbb{N}_0$ such that $n \leq t$ and put $\ell = \lfloor (t - n)/2 \rfloor$. Recall that the Jacobi polynomial $P_{\alpha,\beta}^n(x)$ can be expressed in terms of the hypergeometric function, and also we have that $P_{\alpha,\beta}^n(x)$ has a precise expression whose coefficients are given in terms of combinatorial coefficients; see 8.962.1 and 8.960 of [GR07], respectively. By combining those expressions, we can derive another formula for the discrete $I$-Bessel function. Namely, for any $c \in \mathbb{C}$, we have that

$$I_n^c(t) = \left( \frac{c}{2} \right)^n \sum_{m=0}^{\ell} \frac{\binom{n}{\ell}}{\binom{n+\ell}{m}} \binom{\ell - (-1)^{t-n}/2}{m} \binom{\ell + n}{\ell - m} c^{2m}. \quad (19)$$

We find the representation (18) to be more desirable since it is reminiscent of the series representation of the classical $I$-Bessel function, which is

$$J_n(z) = \sum_{k=0}^{\infty} \frac{1}{k!(n+k)!} \left( \frac{z}{2} \right)^{2k+n}.$$ 

### 2.3 Asymptotic behavior of discrete $I$-Bessel functions

In this section, we will determine the asymptotic behavior of the discrete $I$-Bessel function $I_n^c(t)$ for a fixed real parameter $c$ and as $t \to \infty$. For our analysis, we will use the representation (16) of the function $I_n^c(t)$, recalling that the asymptotics of the hypergeometric function for the large values of parameters is well studied; see for example [Lo03]. Specifically, we will use the transformations of the hypergeometric function given in formulas 9.122, 9.131, and 9.132 from [GR07]. From these formulas, the asymptotic behavior of the discrete $I$-Bessel function is obtained by considering the different cases studied in [Wa18] and [Jo01] where the authors proved certain asymptotic and uniform asymptotic results.

**Proposition 12.** For any real, nonzero parameter $c$ and a fixed $n \in \mathbb{N}$, we have that

$$I_n^c(t) \sim \frac{\text{sgn}(c)^n}{\sqrt{2\pi|c|t}} (1 + |c|)^{t+1/2} \quad \text{as } t \to \infty, \quad (20)$$

where $\text{sgn}(c)$ denotes the sign of $c$.

**Proof.** First consider the case when $|c| = 1$. From formula 9.122 of [GR07] we have that

$$F\left( \frac{n-t}{2}, \frac{n-t}{2} + \frac{1}{2}, n+1; 1 \right) = \frac{\Gamma(n+1)\Gamma\left( t + \frac{1}{2} \right)}{\Gamma\left( \frac{n+t+1}{2} + \frac{1}{2} \right) \Gamma\left( \frac{n+t+1}{2} \right)}.$$
The classical duplication formula for the gamma function and definition \[16\] yields that
\[
I_n^\pm(t) = (\pm1)^n \cdot 2^{-n} \frac{t(t-1) \ldots (t-(n-1))}{n!} \cdot \frac{n\Gamma\left(t + \frac{1}{2}\right) 2^{t+n}}{\sqrt{\pi} \Gamma(t+n+1)}.
\]

Using the functional equation for the gamma function we then get that
\[
I_n^\pm(t) = (\pm1)^n \cdot \frac{t(t-1) \ldots (t-(n-1))}{(t+n)(t+n-1) \ldots (t+1)} \cdot \frac{\Gamma\left(t + \frac{1}{2}\right) 2^t}{\sqrt{\pi} \Gamma(t+1)} \sim (\pm1)^n \frac{2^t}{\sqrt{\pi} t}
\]
as \(t \to \infty\). This proves (20) when \(c \in \{-1, 1\}\).

Next, assume that \(|c| < 1\). To begin, we apply formula 9.131 of \([GR07]\) which gives that
\[
F\left(\frac{n-t}{2}, \frac{n-t}{2} + \frac{1}{2}; n+1; c^2\right) = (1 - c^2)^{t/2-n/2} F\left(\frac{t+n+1}{2}, \frac{n-t}{2}; n+1; \frac{c^2}{c^2-1}\right).
\]

Since \(|c| < 1\), we can write \(\frac{c^2}{c^2-1} = \frac{1}{2}(1-z)\) for \(z = \frac{1+c}{1-c} > 1\). Hence, \(z + \sqrt{z^2 - 1} = \frac{1+|c|}{1-|c|} = e^\zeta\), for \(\zeta > 0\), in the notation of page 289 of \([Wa18]\). By applying the asymptotic formula from \([Wa18]\), we get that
\[
F\left(\frac{t+n+1}{2}, \frac{n-t}{2}; n+1; \frac{c^2}{c^2-1}\right) \sim \frac{2^n n!\Gamma\left(\frac{t-n}{2} + 1\right)}{\sqrt{\pi t} \Gamma\left(\frac{t+n+1}{2}\right)} \cdot \left(\frac{2|c|}{1+|c|}\right)^{n-\frac{1}{2}} \frac{1+|c|}{1-|c|} \frac{t-n}{2}
\]
as \(t \to \infty\). Now, let us multiply the above asymptotic formula by \(\frac{(-c/2)^n(-t)n}{n!} (1 - c^2)^{t/2-n/2}\) and use that
\[
\frac{(-1)^n(-t)n}{n!} \Gamma\left(\frac{t-n}{2} + 1\right) \sim 1
\]
as \(t \to \infty\). In doing so, we deduce the claimed asymptotic formula for \(|c| < 1\), thus proving (20) for \(|c| < 1\).

Finally, assume that \(|c| > 1\). In this case, we proceed analogously as above. In the application of the asymptotic formulas from \([Wa18]\), the lead term in the absolute value is \(\frac{1+|c|}{1-|c|} = e^{-\zeta}\) where \(\zeta = x + i\pi\) with \(x < 0\). Therefore, we have that
\[
F\left(\frac{t+n+1}{2}, \frac{n-t}{2}; n+1; \frac{c^2}{c^2-1}\right)
\sim \frac{2^n n!\Gamma\left(\frac{t-n}{2} + 1\right)}{\sqrt{\pi t} \Gamma\left(\frac{t+n+1}{2}\right)} \cdot \left(\frac{2|c|}{1+|c|}\right)^{-n+\frac{1}{2}} \frac{-1}{2} \frac{1}{1-|c|} \frac{n+1}{2}
\]
as \(t \to \infty\). One then multiplies this asymptotic formula by \(\frac{(-c/2)^n(-t)n}{n!} (1 - c^2)^{t/2-n/2}\) and, by arguing as above, we obtain (20). With all this, the proof of the assertion is complete.

The above proposition is very useful in finding the limiting behavior of solutions to the heat equation when the time variable tends to infinity. A simple consequence of formulas \([6]\) and \([20]\) is the following corollary:

**Corollary 13.** For \(d > 0\), consider the solution \([6]\) to the discrete diffusion equation \([5]\) subject to the initial condition \(u(x;0) = 1\) if \(x = 0\) and \(u(x;0) = 0\) if \(x \neq 0\). Then, one has that
\[
u(n,t) \sim \frac{1}{2\sqrt{\pi dt}}\text{ for }d \in (0,1/2)\text{ and as } t \to \infty
\]
and
\[
u(n,t) \sim \frac{(-1)^{n+t}(4d-1)^{t+1/2}}{2\sqrt{\pi dt}}\text{ for }d \in (1/2, \infty)\text{ and as } t \to \infty.
\]
2.4 A generating function of discrete $I$-Bessel functions

For now, let $c \in \mathbb{C} \setminus \{0\}$ and $n \in \mathbb{N}_0$. Since $|I_n^c(t)| \leq |I_n^1(t)|$ for all integers $n, t \geq 0$, the asymptotic formula \[20\] implies that the radius of convergence of the power series

$$f_n^c(z) := \sum_{t=0}^{\infty} I_n^c(t) z^t$$

equals $(1 + |c|)^{-1}$. Therefore, the series \[21\] is a holomorphic function of $z \in \mathbb{C}$ in the disc $|z| < (1 + |c|)^{-1}$. We will now derive a closed expression for the generating function \[21\] when $|z| < (1 + |c|)^{-1}$.

**Proposition 14.** For any $c \in \mathbb{C} \setminus \{0\}$, $n \in \mathbb{N}_0$, and $|z| < (1 + |c|)^{-1}$, the power series \[21\] is equal to

$$f_n^c(z) = \frac{1}{\sqrt{(1 - z)^2 - c^2 z^2}} \left( (1 - z) - \sqrt{(1 - z)^2 - c^2 z^2} \right)^n$$

where we are using the principal value of the square root, which has values in a right half-plane.

**Proof.** Using Lemma \[8\] (iii) and (iv), one can immediately deduce that the power series \[21\] satisfies the difference equation

$$(1 - z)f_n^c(z) = \begin{cases} 1 + cz f_n^c(z), & \text{if } n = 0; \\ \frac{c}{n} (f_{n-1}^c(z) + f_{n+1}^c(z)), & \text{if } n \geq 1. \end{cases} \quad (22)$$

To solve \[22\], let us set the Ansatz that $f_n^c(z) = a(z)(b(z))^n$, where we omitted $c$ from the notation on the right-hand side. When $n = 0$, \[22\] yields that $a(z) = (1 - z - czb(z))^{-1}$. In the case $n \geq 1$, equation \[22\] implies that

$$b(z)^2 - \frac{2(1 - z)}{cz} b(z) + 1 = 0$$

for $|z| < (1 + |c|)^{-1}$ and $z \neq 0$. There are two solutions to this equation, namely

$$b_1(z) = \frac{1 - z}{cz} + \sqrt{\frac{(1 - z)^2 - c^2 z^2}{c^2 z^2}}, \quad \text{and} \quad b_2(z) = \frac{1 - z}{cz} - \sqrt{\frac{(1 - z)^2 - c^2 z^2}{c^2 z^2}}.$$

In order to choose the solution which yields the generating function, recall that $f_n^c(0) = I_n^c(0)$. Furthermore, $I_0^c(0) = 1$ and $I_n^c(0) = 0$ for $n \geq 1$. Therefore, the function $b(z)$ must be such that $\lim_{z \to 0} b(z) = 0$, and this is true when

$$b(z) = b_2(z) = \frac{(1 - z) - \sqrt{(1 - z)^2 - c^2 z^2}}{cz}.$$

Therefore, we have that $a(z) = (\sqrt{(1 - z)^2 - c^2 z^2})^{-1}$, and the proof of the assertion is complete. \[11\]

3 Proofs of main results

In this section, we prove our main results. We start by proving an expression for the heat kernel on a $(q + 1)$-regular tree in terms of the discrete $I$-Bessel function. As stated above, the formula is analogous to the result from \[CJK14\] which gives the continuous time heat kernel on the $(q + 1)$-regular tree in terms of the continuous $I$-Bessel function.
3.1 The discrete time heat kernel on a \((q + 1)\)-regular tree

Let us fix a base point \(x_0 \in X\). Let \(T_{q+1}\) be a \((q + 1)\)-regular tree, which is a universal cover of the graph \(X\) with base point \(x_0\). Choose a lift of \(x_0 \in X\) to \(T_{q+1}\) to give a base point of \(T_{q+1}\) which in a slight abuse of notation we also denote by \(x_0\). Let \(r \in \mathbb{N}_0\) be the radial coordinate on \(T_{q+1}\) with the chosen base point \(x_0\); recall that the radial coordinate of a point \(x\) is the graph distance from \(x\) to the base point \(x_0\).

As before, \(t \in \mathbb{N}_0\) and \(\partial_t\) is the forward difference operator with respect to the variable \(t\). Let \(K_{q+1}(x_0, x; t)\) be the discrete time heat kernel on \(T_{q+1}\). There is an automorphism of \(T_{q+1}\) to itself, which fixes \(x_0\) and sends any point \(x'\) of with radial coordinate \(r\) to any other point \(x\) of radial coordinate \(r\). Therefore, the heat kernel \(K_{q+1}(x_0, x; t)\) can be viewed as a function of \(r\) and \(t\), meaning

\[
K_{q+1}(x_0, x; t) = K_{q+1}(r; t) \quad \text{where} \quad r = r(x).
\]

In this form, the heat kernel \(K_{q+1}(r; t)\) satisfies the equation

\[
\partial_t K_{q+1}(r; t) = \begin{cases} 
-(q + 1)K_{q+1}(0; t) + (q + 1)K_{q+1}(1; t), & \text{if } r = 0 \\
-(q + 1)K_{q+1}(r; t) + qK_{q+1}(r + 1; t) + K_{q+1}(r - 1; t), & \text{if } r \geq 1,
\end{cases} \tag{23}
\]

with initial condition

\[
K_{q+1}(r; 0) = \begin{cases} 
1, & \text{if } r = 0 \\
0, & \text{if } r \geq 1.
\end{cases} \tag{24}
\]

**Proposition 15.** With the notation as above, the solution \(K_{q+1}(r; t)\) to the initial value problem \((23)\) and \((24)\) is given by

\[
K_{q+1}(r; t) = (-q)^t \left[ q^{-r/2} I_r^{-2/\sqrt{q}}(t) - (q - 1) \sum_{j=1}^{\infty} q^{-(r+2j)/2} I_{r+2j}^{-2/\sqrt{q}}(t) \right]. \tag{25}
\]

**Proof.** Parts (i) and (ii) of Lemma 8 imply that right-hand side of \((25)\) satisfies the initial condition \((24)\) when \(t = 0\). Hence, it is left to prove that the right-hand side of \((25)\) satisfies \((23)\). To begin, let us define

\[
F(r; t) := q^{-r/2} I_r^{-2/\sqrt{q}}(t) - (q - 1) \sum_{j=1}^{\infty} q^{-(r+2j)/2} I_{r+2j}^{-2/\sqrt{q}}(t).
\]

If the above stated assertion is true, then \(F(r; t) = (-q)^{-t} K_{q+1}(r; t)\). It is elementary to verify that the right-hand side of \((25)\) satisfies \((23)\) if and only if \(F(r; t)\) satisfies the equation

\[
\partial_t F(r; t) = \begin{cases} 
\frac{-(q+1)}{q} F(1; t), & \text{if } r = 0 \\
-F(r + 1; t) - \frac{1}{q} F(r - 1; t), & \text{if } r \geq 1.
\end{cases} \tag{26}
\]

We now shall prove that \((26)\) is true. When \(r = 0\) we have that

\[
\partial_t F(0; t) = \partial_t I_0^{-2/\sqrt{q}}(t) - (q - 1) \sum_{j=1}^{\infty} q^{-j} \partial_t I_{2j}^{-2/\sqrt{q}}(t).
\]
Using parts (iii) and (iv) of Lemma 8 we get that

\[
\partial_t F(0; t) = -\frac{2}{\sqrt{q}} I_1^{-2/\sqrt{q}}(t) - (q - 1) \sum_{j=1}^{\infty} q^{-j - \frac{1}{2}} \left( I_{2j+1}^{-2/\sqrt{q}}(t) + I_{2j-1}^{-2/\sqrt{q}}(t) \right)
\]

\[
= I_1^{-2/\sqrt{q}}(t) \left( -\frac{2}{\sqrt{q}} + \frac{q - 1}{q\sqrt{q}} \right) + (q - 1) \sum_{j=1}^{\infty} q^{-(j+1)q} I_{2j+1}^{-2/\sqrt{q}}(t)
\]

\[
= -\frac{q + 1}{q} \left( q^{-1/2} I_1^{-2/\sqrt{q}}(t) - (q - 1) \sum_{j=1}^{\infty} q^{-(2j+1)/2} I_{2j+1}^{-2/\sqrt{q}}(t) \right)
\]

\[
= -\frac{q + 1}{q} F(1; t),
\]

which proves (26) in the case when \( r = 0 \).

Assume now that \( r \geq 1 \). From part (iv) of Lemma 8 we get that

\[
\partial_t F(r; t) = -\frac{2q^{-r/2}}{2\sqrt{q}} \left( I_{r+1}^{-2/\sqrt{q}}(t) + I_{r-1}^{-2/\sqrt{q}}(t) \right)
\]

\[
- (q - 1) \sum_{j=1}^{\infty} q^{-(r+2j)/2} \frac{-2}{2\sqrt{q}} \left( I_{r+2j+1}^{-2/\sqrt{q}}(t) + I_{r+2j-1}^{-2/\sqrt{q}}(t) \right)
\]

\[
= -q^{-(r+1)/2} I_{r+1}^{-2/\sqrt{q}}(t) + (q - 1) \sum_{j=1}^{\infty} q^{-(r+2j)/2} I_{r+2j+1}^{-2/\sqrt{q}}(t)
\]

\[
- \frac{1}{q} q^{-(r-1)/2} I_{r+1}^{-2/\sqrt{q}}(t) + \frac{q - 1}{q} \sum_{j=1}^{\infty} q^{-(r+2j)/2} I_{r+2j-1}^{-2/\sqrt{q}}(t)
\]

\[
= - \left( F(r + 1; t) + \frac{1}{q} F(r - 1; t) \right)
\]

This proves (26) when \( r \geq 1 \), and completes the proof of the proposition.

**Remark 16.** Note that for fixed \( t \in \mathbb{N}_0 \), the series on the right-hand side of formula (26) is actually a finite sum. Indeed, \( I_{r+2j}^{-2/\sqrt{q}}(t) = 0 \) for all \( j \) such that \( r + 2j > t \), so the terms in the series in (26) are non-zero only if \( j \) is such that \( r + 2j \leq t \).

### 3.2 Proof of Theorem 1

Let \( X \) be any \((q + 1)\)-regular graph. Let \( T_{q+1} \) be the universal cover of \( X \) with covering map \( \pi \). Then we can write the heat kernel \( K_X(x_0, x; t) \) on \( X \) by

\[
K_X(x_0, x; t) = \sum_{\tilde{x} \in \pi^{-1}(x)} K_{q+1}(x_0, \tilde{x}; t). \tag{27}
\]

If we view the base point \( x_0 \) as fixed, then we may suppress that part of the notation and simply write \( K_X(x; t) = K_X(x_0, x; t) \). Let \( c_r(x) \) be the number of geodesics in \( X \) of length \( r \) which go from \( x_0 \) to \( x \). Then, as argued in section 3.3 of [CJK14], we can rewrite (27) as

\[
K_X(x; t) = \sum_{r \geq 0} c_r(x) K_{q+1}(r; t).
\]
Note that the series on the right-hand side of the above display is actually a finite sum for any fixed \( t \in \mathbb{N}_0 \) since \( K_{q+1}(r; t) = 0 \) when \( r > t \). As a result, there is no question regarding the convergence of the series. By using the expression (25) for \( K_{q+1}(r; t) \), we immediately obtain (10).

Equation (11) follows by combining (10) and (15).

### 3.3 Proofs of corollaries 2 and 3

Let us write (1) as

\[
K_X(x_0, x; t + 1) = (A_X - q \text{Id})K_X(x_0, x; t).
\]

Since the degree of \( X \) is \( q + 1 \), the adjacency operator is bounded with the spectrum lying in \([-q - 1, q + 1] \); see, for example, [Lo75] or [Mo82]. Therefore, from the discussion in Chapter 4 of [MW89], we have that the heat kernel \( K_X(x_0, x; t) \) can be expressed in terms of the spectral measure \( \mu_x \) by the formula

\[
K_X(x_0, x; t) = \int_{-q+1}^{q+1} (\lambda - q)^t d\mu_x(\lambda).
\]

Note that, as above, we have suppressed the notation of the point \( x_0 \) in the spectral measure. With this, the first part of Corollary 2 is proved.

When \( X \) is a finite graph of degree \( q + 1 \) with \( M \) vertices, the adjacency operator can be identified with the adjacency matrix \( A_X \) which possesses real eigenvalues \( \lambda_0 = q + 1 > \lambda_1 \geq \ldots \geq \lambda_{M-1} \geq -(q+1) \), counted according to their multiplicities. The associated orthonormal eigenvectors are denoted by \( \psi_j \in \mathbb{R}^M \) for \( j = 0, \ldots, M-1 \). In this case, the spectral expansion of the heat kernel reads as

\[
K_X(x_0, x; t) = \sum_{j=0}^{M-1} (\lambda_j - q)^t \overline{\psi_j(x)} \overline{\psi_j(x_0)},
\]

which proves the second part of Corollary 2.

It remains to prove Corollary 3. We first take \( x = x_0 \) in (12). After multiplying (12) by \((-q)^{-t}\), we then get that

\[
\sum_{j=0}^{M-1} \left(1 - \frac{\lambda_j}{q}\right)^t \overline{\psi_j(x)} \overline{\psi_j(x_0)} = \sum_{m=0}^{t} N_m(x)q^{-m/2}I_m^{-2/\sqrt{q}}(t) + (q - 1)q^{-j}I_{2j}^{-2/\sqrt{q}}(t).
\]

Now, by taking the sum over all vertices \( x \in V_X \) one gets (13) because the eigenvectors \( \psi_j \) are normalized to have \( L^2 \)-norm equal to 1.

### 3.4 Proof of Theorem 4

Let \( h(z) \) be a function which is holomorphic for \(|z| > 1/a\) for some positive real number \( a \). Let us assume that \( a > 3 + 2/q > (2q + 1)/q \). As such, \( h(z) \) can be written as a Taylor series centered at \( z_0 = \infty \), namely

\[
h(z) = \sum_{t=0}^{\infty} g(t)z^{-t}.
\]
In this notation, \( h(z) \) is the one-sided \( \mathbb{Z} \)-transform of function \( g : \mathbb{N}_0 \to \mathbb{C} \). Equivalently, we can say that \( \{g(t)\} \) is the set of Taylor series coefficients of \( h \). Moreover, the convergence of the series which defines \( h \), together with assumption that \( a > 3 + 2/q > (2q + 1)/q \) implies the existence of some \( \epsilon > 0 \) with \( 0 < \epsilon < \frac{1}{m}(a - (2q + 1)/q) \) such that

\[
g(t)((2q + 1)/q + \epsilon)^t \to 0 \quad \text{as} \quad t \to \infty.
\]  

Since, \( aq > 2q + 1 \) and \( -(q + 1) \leq \lambda_j \leq q + 1 \) for every eigenvalue \( \lambda_j \), we have that \( |\frac{q}{q - \lambda_j}| > 1/a \). Therefore,

\[
\sum_{j=0}^{M-1} h \left( \frac{q}{q - \lambda_j} \right) = \sum_{t=0}^{\infty} g(t) \sum_{j=0}^{M-1} \left( 1 - \frac{\lambda_j}{q} \right)^t.
\]  

(30)

The above expression can be obtained from the left-hand side of (13) after multiplying (13) by \( g(t) \) and then summing over all \( t \geq 0 \). Let us now apply the same operations to the right-hand side of (13) and compute the outcome.

As stated, \( (2q + 1)/q \geq 1 + 2/\sqrt{q} \) for \( q \geq 1 \). In (20) we have proved the asymptotic behavior of the discrete \( I \)-Bessel function as \( t \to \infty \) with \( c = -2/\sqrt{q} \). By combining (20) with (29), we conclude that the series \( \sum_{t=0}^{\infty} g(t)I_m^{-2/\sqrt{q}}(t) \) converges absolutely.

Moreover, from Proposition 14 and the power series representation (21), we have that for all \( z \) such that \( 1/a < |z| < (1 + 2/\sqrt{q})^{-1} \) that

\[
f_n^{-2/\sqrt{q}}(z)h(z) = \sum_{t_1=0}^{\infty} I_n^{-2/\sqrt{q}}(t_1)z^{t_1} \sum_{t_2=0}^{\infty} g(t_2)z^{-t_2}.
\]  

(31)

Note that the annulus \( 1/a < |z| < (1 + 2/\sqrt{q})^{-1} \) is not empty because \( a > (2q + 1)/q \geq 1 + 2/\sqrt{q} \). The residue theorem implies that the constant term in (31) can be expressed in terms of the integral of \( f_n^{-2/\sqrt{q}}(z)h(z)z^{-1} \) along any circle inside the annulus \( 1/a < |z| < (1 + 2/\sqrt{q})^{-1} \).

Specifically, for any \( b \) such that \( 1/a < b < q/(3q + 2) \) we have that

\[
\sum_{t=0}^{\infty} I_n^{-2/\sqrt{q}}(t)g(t) = \frac{1}{2\pi i} \int_{c(0,b)} f_n^{-2/\sqrt{q}}(z)h(z) \frac{dz}{z}.
\]  

(32)

In order to complete the proof, it suffices to show that the series

\[
\sum_{m=0}^{\infty} N_m q^{-m/2} \frac{1}{2\pi} \int_{c(0,b)} \left| f_m^{-2/\sqrt{q}}(z)h(z) \frac{dz}{z} \right|
\]  

(33)

is convergent. In order to do so, we estimate function \( f_m^{-2/\sqrt{q}}(z) \) for \( z \) such that \( |z| = b < q/(3q + 2) \). Recall that

\[
f_m^c(z) = \frac{1}{\sqrt{(1 - z)^2 - c^2z^2}} \left( \frac{cz}{(1 - z) + \sqrt{(1 - z)^2 - c^2z^2}} \right)^m.
\]

Trivially, we have that

\[
|(1 - z) + \sqrt{(1 - z)^2 - c^2z^2}| \geq \text{Re}((1 - z) + \sqrt{(1 - z)^2 - c^2z^2}) \geq \text{Re}(1 - z) \geq (1 - b).
\]
Hence
\[ |f_m^{-2/\sqrt{q}}(z)| \ll \left( \frac{2bq^{-1/2}}{1-b} \right)^m \]
where the implied constant is independent of \( m \). The function \( h(z) \) is holomorphic, hence bounded along the circle \(|z| = b\). The number \( N_m \) of geodesic paths is bounded by the number of total paths, i.e. \( N_m \ll (q+1)^m \). Therefore,
\[ N_m q^{-\frac{m}{2}} \frac{1}{2\pi} \int_{c(0,b)} |f_m^{-2/\sqrt{q}}(z)h(z)\frac{dz}{z}| \ll \left( \frac{2b(q+1)}{q(1-b)} \right)^m. \]
The value of \( b \) is chosen so that \( b < q/(3q+2) \), hence \( \frac{2b(q+1)}{q(1-b)} < 1 \) which proves that the series is convergent.

With all this, we can summarize the proof of Theorem 4. First, multiply equation (13) by \( g(t) \) and take the sum over all \( t \geq 0 \). Then, interchange the summation over \( m \) and \( t \) and arrive at the equation
\[ \sum_{j=0}^{M-1} h \left( \frac{q}{q - \lambda_j} \right) = \sum_{m=0}^{\infty} N_m q^{-m/2} \sum_{i=0}^{\infty} g(t)I_m^{-2/\sqrt{q}}(t) + M(q-1) \sum_{j=0}^{\infty} q^{-2j} \sum_{i=0}^{\infty} g(t)I_{2j}^{-2/\sqrt{q}}(t). \]
Finally, by inserting the identity (32), one obtains Theorem 4. In effect, the detailed analysis above justifies the operations described which yeild Theorem 4 from (13).

Remark 17. Corollary 5 follows directly from Theorem 1 and equation (14). The proof of (14) is given in section 4 below. Once the proof of (14) is established, all statements in the introduction are proven.

4 Uniform random walk on a \((q + 1)\)-regular graph

Let \( X \) denote a \((q + 1)\)-regular graph, which is either finite or countably infinite, which has a base point \( x_0 \). A uniform random walk on \( X \), starting at \( x_0 \) is the discrete time random walk at which the particle located at any vertex of \( X \) at time \( t = n \geq 0 \) moves along any of the \((q + 1)\) neighboring edges with the same probability \( 1/(q + 1) \).

This process can be described as a solution to the diffusion equation associated to the uniform random walk Laplacian \( \Delta_{X}^{rw} \) on \( X \), which is defined as
\[ \Delta_{X}^{rw} = \text{Id} - \frac{1}{q + 1} A_X. \]
Namely, the probability that a particle which starts walking at the base point \( x_0 \), after \( t \) steps is located at the point \( x \) equals \( K_{X}^{rw}(x_0, x; t) \), where \( K_{X}^{rw}(x_0, x; t) \) is the discrete time random walk heat kernel. In other words, \( K_{X}^{rw}(x_0, x; t) \) is the solution to the discrete time diffusion equation
\[ \Delta_{X}^{rw} K_{X}^{rw}(x_0, x; t) + \partial_t K_{X}^{rw}(x_0, x; t) = 0 \tag{34} \]
subject to the initial condition
\[ K_{X}^{rw}(x_0, x; 0) = \begin{cases} 1, & \text{if } x = x_0 \\ 0, & \text{otherwise}. \end{cases} \tag{35} \]
Note that (34) can be written as

\[ K_{X}^{\text{rw}}(x_0, x; t+1) = \frac{1}{q+1} A_X K_{X}^{\text{rw}}(x_0, x; t) \]

which nicely describes the dynamics of this process.

Heat kernels \( K_X(x_0, x; t) \) and \( K_{X}^{\text{rw}}(x_0, x; t) \) have the same initial condition, meaning that \( K_{X}^{\text{rw}}(x_0, x; 0) = K_X(x_0, x; 0) \). With this, we can solve for one heat kernel in terms of the other. The result is given in the following lemma.

**Lemma 18.** Let \( X \) be a \((q+1)\)-regular graph, which is either finite or countably infinite, with a base point \( x_0 \). For any \( x \in X \) the discrete time heat kernel \( K_X(x_0, x; t) \) and the discrete time random walk heat kernel \( K_{X}^{\text{rw}}(x_0, x; t) \) are related through the identities

\[
K_{X}(x_0, x; t) = (-q)^t \sum_{k=0}^{t} (-1)^k \binom{t}{k} \left( 1 + \frac{1}{q} \right)^k K_{X}^{\text{rw}}(x_0, x; k). \tag{36}
\]

and the identity (14).

**Proof.** First, we notice that (1) is equivalent to

\[ K_{X}(x_0, x; t+1) = (A_X - q \text{Id}) K_{X}(x_0, x; t). \]

In particular, one has that

\[ K_{X}^{\text{rw}}(x_0, x, t) = \frac{1}{(q+1)^t} A_X^t K_{X}^{\text{rw}}(x_0, x, 0), \quad \text{and} \quad K_X(x_0, x, t) = (A_X - q \text{Id})^t K_X(x_0, x, 0). \]

To prove (36) one writes

\[
K_{X}(x_0, x; t) = (A_X - q \text{Id})^t K_{X}(x_0, x, 0) = (A_X - q \text{Id})^t K_{X}^{\text{rw}}(x_0, x, 0)
\]

\[
= \sum_{k=0}^{t} \binom{t}{k} (-q)^{t-k} (A_X)^k K_{X}^{\text{rw}}(x_0, x, 0)
\]

\[
= (-q)^t \sum_{k=0}^{t} (-1)^k \binom{t}{k} \left( \frac{q+1}{q} \right)^k \frac{1}{(q+1)^k} (A_X)^k K_{X}^{\text{rw}}(x_0, x, 0)
\]

\[
= (-q)^t \sum_{k=0}^{t} (-1)^k \binom{t}{k} \left( 1 + \frac{1}{q} \right)^k K_{X}^{\text{rw}}(x_0, x, k),
\]

as claimed. On the other hand, for \( B_X = A_X - q \text{Id} \), one has that

\[
K_{X}^{\text{rw}}(x_0, x, t) = \frac{1}{(q+1)^t} (B_X + q \text{Id})^t K_{X}^{\text{rw}}(x_0, x, 0)
\]

\[
= \frac{1}{(q+1)^t} \sum_{k=0}^{t} \binom{t}{k} q^{-k} B_X^k K_{X}(x_0, x, 0)
\]

\[
= \frac{1}{(q+1)^t} \sum_{k=0}^{t} \binom{t}{k} q^{-k} K_{X}(x_0, x, k),
\]

which proves (14).
By combining (14) with (10), we have proved Corollary 5.

As a special case of the above computations, one can take $X$ to be the $(q+1)$-regular tree. By combining (25) with (14), we immediately obtain an explicit expression of the random walk heat kernel $K^\text{rw}_{q+1}(r;t)$ associated to a $(q+1)$-regular tree in terms of the discrete $I$-Bessel function. Specifically, for $t \geq r$, one has that

$$K^\text{rw}_{q+1}(r;t) = \frac{q^t}{(q + 1)^t} \sum_{k=r}^{t} \binom{t}{k} (-1)^k \left( q^{-r/2} I_{r^{-2/\sqrt{q}}}(k) - (q - 1) \sum_{j=1}^{\lfloor \frac{k-r}{2} \rfloor} q^{-j} I_{(r+2j)^{-2/\sqrt{q}}}(k) \right),$$

where, as before, $r$ denotes the radial variable on the $(q+1)$-regular tree relative to a fixed base point $x_0$.

**Remark 19.** The random walk heat kernel on homogeneous regular trees was studied in [Ur97], Section 2, where certain properties of $K^\text{rw}_{q+1}(r;t)$ were derived. However, those results were based on recurrence formula satisfied by this kernel and no closed formula for its evaluation was deduced. A more general setting of semi-regular infinite graphs was studied in [Ur03] where the upper and lower bounds for the discrete time random walk heat kernel were deduced. Again, no closed formula for this heat kernel was obtained.

**Remark 20.** By taking $r = 0$ in equation (37) we deduce the return probability after $t$ steps of a random walk on a $(q+1)$-regular tree, meaning the probability that a uniform random walk on the tree starting at the root comes back to the root after $t$ steps. This probability is given by

$$K^\text{rw}_{q+1}(0;t) = \frac{q^t}{(q + 1)^t} \sum_{k=0}^{t} \binom{t}{k} (-1)^k \left( I_{0^{-2/\sqrt{q}}}(k) - (q - 1) \sum_{j=1}^{\lfloor \frac{t}{2} \rfloor} q^{-j} I_{2j^{-2/\sqrt{q}}}(k) \right).$$

## 5 A limiting distribution result

The quantity $K^\text{rw}_X(x_0, x_0; t)$ can be viewed as the return probability of the random walk on $X$ to the starting point after $t$ steps. Using the above formulas, we can determine the limiting behavior of the return probability as the number of vertices of $M$ tends to infinity.

**Proposition 21.** Let $\{X_h\}$ be a sequence of finite $(q+1)$-regular graphs. Assume that $X_h$ has $M_h$ vertices, and that $M_h$ tends to infinity as $h$ tends to infinity. Assume further that the length of the shortest closed geodesic on $X_h$ which passes through $x_0$ tends to infinity as $M_h$ goes to infinity. Then for fixed $t \geq 0$ we have for sufficiently large $h$, the equality

$$K^\text{rw}_{X_h}(x_0, x_0; t) = \frac{q^t}{(q + 1)^t} \sum_{k=0}^{t} \binom{t}{k} (-1)^k \left( I_{0^{-2/\sqrt{q}}}(k) - (q - 1) \sum_{j=1}^{\lfloor \frac{t}{2} \rfloor} q^{-j} I_{2j^{-2/\sqrt{q}}}(k) \right).$$

**Remark 22.** In a slight abuse of notation, we will write the statement in Proposition 21 as saying that for fixed $t$, we have that

$$K^\text{rw}_X(x_0, x_0; t) = K^\text{rw}_{q+1}(0; t).$$

for $h$ sufficiently large.
Proof. Let $X$ be any fixed $(q+1)$-regular graph with $M$ vertices. Then, from (28), we have that the heat kernel $K_X(x_0, x_0; t)$ can be written as

$$K_X(x_0, x_0; t) = (-q)^t \left[ \sum_{m=0}^{t} N_m(x_0) q^{-m/2} I_m^{-2/\sqrt{q}}(t) - (q - 1) \sum_{j=1}^{[\frac{t}{2}]} q^{-j} I^{-2/\sqrt{q}}_{2j}(t) \right],$$

where, as stated above, $N_m(x_0)$ is the number of closed geodesics of length $m$, with distinct direction, beginning at $x_0$. For the sequence $\{X_h\}$ under consideration, and for fixed $t$, it is assumed that for sufficiently large $h$ we have that $N_m(x_0) = 0$ for all $0 < m \leq t$. Recall the convention that $N_0(x_0) = 1$. Therefore, for fixed $t$ and sufficiently large $h$, we have that

$$K_X(x_0, x_0; t) = (-q)^t \left( I_0^{-2/\sqrt{q}}(t) + (q - 1) \sum_{j=1}^{[\frac{t}{2}]} q^{-j} I^{-2/\sqrt{q}}_{2j}(t) \right). \quad (40)$$

By combining this with formula (14), the proof is complete. \qed

Remark 23. According to [TBK21], the limit as $M \to \infty$ of the first return probability distribution of a random walk on a $(q+1)$-regular randomly chosen graph with $M$ vertices equals the first return probability distribution on the $(q+1)$-regular tree. The above computations show that one has a similar result when considering the (total) return probability distribution for certain sequences of graphs since the return probability distribution (38) on the tree $T_{q+1}$ coincides with the distribution obtained by combining (40) with (14).

6 Relating spectral data to length spectrum data

As with trace formulas in general, one can use Theorem (4) to express spectral data of the Laplacian, or adjacency operator, to topological data, namely the length spectrum. In the case $X$ is finite, such expressions are stated in Corollary 2 of [Mn07], under the additional assumption that $X$ is connected, without multiple edges and loops. In this section, we will use Corollary (3) to show how to express the length spectrum, i.e. the number $N_m$ of distinct closed geodesics of length $m$ in terms of moments of the spectrum of the adjacency matrix $A_X$. Going further, we will discuss in Section 6.4 below how to use Corollary (2) to prove similar results in the case $X$ is an infinite $(q+1)$-regular graph.

6.1 Explicit evaluations of $N_j$ for $j \leq 3$

Let us explicitly compute $N_j$ for all $j \leq 3$ for any finite $(q+1)$-regular graph with $M$ vertices using formula (13). In each case, the evaluations use the precise formula for the discrete $I$-Bessel function, as stated in (18).

Using that $I_0^{-2/\sqrt{q}}(0) = 1$ and $I_2^{-2/\sqrt{q}}(0) = 0$, we get that (13) in the case $t = 0$ yields the formula $N_0 = M$, as is set by convention.

Let us now consider $t = 1$. From (18), we get that $I_0^{-2/\sqrt{q}}(1) = 1$, $I_1^{-2/\sqrt{q}}(1) = -2/(2\sqrt{q})$ and $I_2^{-2/\sqrt{q}}(1) = 0$. Thus, (13) in the case $t = 1$ becomes

$$M - \frac{1}{q} \text{Tr}(A_X) = N_0 + N_1 q^{-1/2} \cdot \frac{1}{\sqrt{q}}.$$
Since $N_0 = M$, we get that $N_1 = \text{Tr}(A_X)$.

The case when $t = 2$ is the first where the second sum in Corollary (3) is non-zero. From [18] we get that $I_0^{-2/\sqrt{q}}(2) = 1 + \frac{2}{q}$. Additionally, [18] can be used to show that

$$I_1^{-2/\sqrt{q}}(2) = \frac{-2}{2\sqrt{q}} = \frac{-2}{\sqrt{q}} \quad \text{and} \quad I_2^{-2/\sqrt{q}}(2) = \left(\frac{-2}{2\sqrt{q}}\right)^2 = \frac{1}{q}.$$ 

Therefore, from [13], we arrive at the expression

$$M - \frac{2}{q} \text{Tr}(A_X) + \frac{1}{q^2} \text{Tr}(A_X^2) = N_0 \left(1 + \frac{2}{q}\right) + N_1 \frac{-2}{q} + N_2 \frac{1}{q^2} - M(q-1) \frac{1}{q}.$$

From the above evaluations of $N_0$ and $N_1$, we conclude that

$$N_2 = \text{Tr}(A_X^2) - M(q+1).$$

The calculations in the case $t = 3$ follow a similar pattern, but are a bit more involved. From [18] we obtain the following evaluations:

$$I_0^{-2/\sqrt{q}}(3) = \sum_{\alpha=0}^{1} \frac{3!}{\alpha!(3-2\alpha)!} \left(\frac{-2}{2\sqrt{q}}\right)^{2\alpha} = 1 + \frac{6}{q},$$

$$I_1^{-2/\sqrt{q}}(3) = \sum_{\alpha=0}^{1} \frac{3!}{\alpha!(3-2\alpha-1)!\alpha!} \left(\frac{-2}{2\sqrt{q}}\right)^{2\alpha+1} = \frac{-3}{\sqrt{q}} - \frac{3}{q\sqrt{q}},$$

$$I_2^{-2/\sqrt{q}}(3) = \left(\frac{-2}{2\sqrt{q}}\right)^2 \cdot 3 = \frac{3}{q}; \quad I_3^{-2/\sqrt{q}}(3) = \left(\frac{-2}{2\sqrt{q}}\right)^3 = \frac{-1}{q^{3/2}}.$$

We now substitute these evaluations into [13] to get, upon expanding the left-hand-side, the expression

$$M - \frac{3}{q} \text{Tr}(A_X) + \frac{3}{q^2} \text{Tr}(A_X^2) - \frac{1}{q^3} \text{Tr}(A_X^3)$$

$$= N_0 \left(1 + \frac{6}{q}\right) - \frac{1}{q} N_1 \left(3 + \frac{3}{q}\right) + N_2 q^{-1} \cdot \frac{3}{q} + N_3 q^{-3/2} \cdot \frac{-1}{q^{3/2}} - M(q-1) q^{-1} \cdot \frac{3}{q}.$$

When solving for $N_3$, one uses the values for $N_0$, $N_1$ and $N_2$ obtained above to get that

$$N_3 = \text{Tr}(A_X^3) - 3q \text{Tr}(A_X).$$

The formulas for $N_j$ for $j \leq 3$ obtained in these calculations agree with the expressions obtained from equation (34) of [Mun07].

### 6.2 Evaluating $N_j$ for general $j$

Going further, one can use [18] to obtain closed-form expressions for each $\tilde{N}_t := q^{-t/2} N_t$ for finite $(q+1)$-regular graph with $M$ vertices. The approach is as follows.

Fix $t \geq 1$ and let $V_t$ denote the $(t+1) \times (t+1)$ matrix whose $(j, k)$ entry is $v_{j,k} = I_{k-1}^{-2/\sqrt{q}}(j-1)$ for $j, k = 1, \ldots, t + 1$. Observe that $v_{j,k} = 0$ for $k > j$; see Lemma 4(i). Hence, $V$ is a lower triangular matrix. Furthermore, from [18] we get that

$$v_{k,k} = I_{k-1}^{-2/\sqrt{q}}(k-1) = (-1)^{k-1} q^{-(k-1)/2} \quad \text{for} \quad k = 1, \ldots, t + 1.$$
Let $\mathbb{T}$ be the column vector of length $t + 1$ whose $k$-th entry is $\text{Tr} \left( \left( \frac{A x}{\sqrt{q}} \right)^{k-1} \right)$, where the first entry is $\text{Tr}(\text{Id}) = M$. Let $\tilde{N}$ be the column vector of length $t + 1$ whose $k$-th entry is $N_{k-1}$. Finally, let $E = (e_k)_{k=1}^{t+1}$ be the column vector of length $t + 1$ whose $k$-th entry is $M(q - 1)^{-1} I_2^{-2/\sqrt{q}}(k - 1)$. With all this, the set of equations (13) can be written as

$$B \mathbb{T} = V \tilde{N} - E,$$

where $B = (b_{jk})_{(t+1) \times (t+1)}$ is the matrix with entries $b_{jk} = (\frac{k-1}{j-1}) \left( \frac{-2}{\sqrt{q}} \right)^{j-1}$, for $k \geq j$ and $b_{jk} = 0$ for $k < j$. By convention we set $b_{11} = \left( \frac{0}{0} \right) = 1$.

The matrix $V$ is invertible, hence the solution to the set of equations (13) for any fixed $t \geq 1$ is

$$\tilde{N} = V^{-1} (B \mathbb{T} + E). \quad (41)$$

Let $V^{-1} = (\tilde{v}_{jk})_{(t+1) \times (t+1)}$. Let us show how to compute $V^{-1}$ in terms of powers of discrete $I-$Bessel functions. Let $D$ be the $(t + 1) \times (t + 1)$ diagonal matrix whose $(k,k)$ entry is $d_{k,k} = (-1)^{k-1}(q)^{-k(k-1)/2}$. Then $D^{-1}$ is the diagonal matrix whose $k$-th diagonal element is $(-1)^{k-1}(q)^{k(k-1)/2}$. Let us write $V = D + \tilde{V}$, so $\tilde{V}$ is a strictly lower diagonal matrix.

Since $\tilde{V}$ is strictly lower triangular and $D^{-1}$ is diagonal, then $D^{-1}\tilde{V}$ is also strictly lower triangular, so then $(D^{-1}\tilde{V})^{t+1}$ is the zero matrix. Therefore,

$$V^{-1} = (I + D^{-1}\tilde{V})^{-1}D^{-1} = \left( \sum_{h=0}^{t} \left( -D^{-1}\tilde{V} \right)^{h} \right)D^{-1}, \quad (42)$$

which shows that entries $\tilde{v}_{jk}$ of $V^{-1}$ can be written in terms of sums of products of discrete $I$-Bessel functions. Therefore, formula (11) combined with (12) provides an efficient algorithm for evaluation of $\tilde{N}_j$ for general $j$ in terms of values of discrete $I-$Bessel functions.

### 6.3 Chebyshev polynomials and discrete $I$-Bessel functions

In this section we will compare our results with equation (34) of [Mn07], hence we assume that $X$ is a finite connected $(q + 1)-$regular graph with $M$ vertices, without multiple edges and loops.

For any positive integer $\ell$, Equation (34) of [Mn07] can be written, in our notation, as

$$\tilde{N}_\ell = 2\text{Tr} \left( T_\ell \left( \frac{A x}{2\sqrt{q}} \right) \right) + \frac{1 + (-1)^{\ell}}{2}(q - 1)q^{-\ell/2}M, \quad (43)$$

where $T_\ell(x)$ denotes the $\ell$-th Chebyshev polynomial of the first kind. In other words, results from [Mn07] evaluates the left-hand side of (11) in terms of the Chebyshev polynomial; this evaluation can be viewed as a linear form in variables $x_j(X,q) := \text{Tr} \left( \left( \frac{A x}{\sqrt{q}} \right)^{j} \right); j = 0, 1, \ldots \ell$

plus the term $\frac{1 + (-1)^{\ell}}{2}(q - 1)q^{-\ell/2}M$, which is zero if $\ell$ is odd. (Recall that we adopt the convention that $x_0(X,q) := \text{Tr} \left( \left( \frac{A x}{\sqrt{q}} \right)^{0} \right) = M$.)

In other words, we can write equation (13) as

$$\tilde{N}_\ell = 2 \sum_{j=0}^{\ell} t_\ell(j) x_j(X,q) + \frac{1 + (-1)^{\ell}}{2}(q - 1)q^{-\ell/2}M, \quad (44)$$
where $t_\ell(j)$ is the coefficient multiplying $y^j$ in the expansion of the Chebyshev polynomial $T_\ell(y)$.

Returning to our calculations, note that the $k$-th entry on the right-hand side of (41) equals a linear form in the variable $x_j(X,q)$, $j = 0, \ldots, k - 1$, namely

$$P_{k-1}(x \chi(q)) = \sum_{j=0}^{k-1} a_{k-1,j}x_j(X,q),$$

where $a_{k-1,j} = \sum_{m=1}^k \tilde{v}_{km} b_{m(j+1)}$, for $j = 1, \ldots, k - 1$ and $a_{k-1,0} = \sum_{m=1}^k \tilde{v}_{km} (b_{m1} + e_{k-1})$.

Therefore, equation (41) combines with (44) to imply that for any $k \geq 0$ we have that

$$2\sum_{j=0}^{k} t_k(j)x_j(X,q) + \frac{1 + (-1)^k}{2} (q - 1)q^{-k/2} M = \sum_{j=0}^{k} a_{k,j}x_j(X,q).$$

The coefficients of the linear form on the right-hand side are explicitly computable in terms of discrete $I$-Bessel functions $I_{\ell - 2/\sqrt{q}}(\ell)$ for $j \leq \ell \leq k - 1$. In other words, the rows of the matrix

$$\left( \sum_{h=0}^t \left( -D^{-1} \hat{V} \right)^h \right) D^{-1} B$$

amount a listing of the coefficients of the Chebyshev polynomial associated to non-constant terms.

We find it intriguing to see that the special values of the discrete $I$-Bessel function come together through the above computations and produce the coefficients of Chebyshev polynomials.

### 6.4 Further counting problems

The analysis of the previous sections yielded an explicit, closed-form expression for $N_m$, the number of closed geodesics of length $m$ on a finite $(q + 1)$-regular graph, in terms of a finite sum involving the length spectrum of the adjacency matrix. In the setting of continuous time heat kernels, as in [CJK14] or [Mn07], the authors obtain expressions which relate a spectral term, often finite, to a geometric term, which is an infinite sum in the continuous time setting. In the discrete time setting we study in this article, the geometric side of the identities we study also is finite. As a result, the formula for $N_m$ follows from our heat kernel formulas and elementary linear algebra.

Let us now briefly describe how one can expand the approach taken above to the case when $X$ is infinite.

One begins with the pre-trace formula as given in Corollary (2). It is not necessary to take $x = x_0$, but certainly that is a possibility. For a fixed integer $t$, the left-hand side of the formulas in Corollary (2) could be taken as written or expanded into a series involving the various moments of the spectral measure. As in Section 6.2 one can solve for $b_m(x)$, ultimately obtaining an expression for $b_t(x)$ in terms of the $m$-th moments of the spectral measure, for all $m \leq t$, and special values of the discrete $I$-Bessel function. Equivalently, the formulas for $b_t(x)$ will involve the shifted moments

$$\int_{-(q+1)}^{(q+1)} (\lambda - q)^m \mu_x(\lambda) \quad \text{for} \quad m \leq t$$
and special values of the discrete $I$-Bessel function. The resulting formulas will be similar to those in (41) and (42). Finally, one can go one step further and get expressions for $c_m(x)$ using Lemma 7 and one further matrix multiplication.

7 Applications to other diffusion processes

In this section, we briefly indicate how to apply our main results to give a closed formula for the solution of a certain zero-sum diffusion process on the $(q + 1)$-regular tree $T_{q+1}$, which can be viewed as a random walk in presence of a heat or cooling source.

Namely, let $\alpha, \beta > 0$ be such that $\alpha + \beta < 1$. Consider the zero-sum diffusion process on the non-negative integers $\mathbb{N}_0$ as described by the difference equation

$$\partial_t K(x; t) = \begin{cases} (\beta - 1)K(0; t) + (1 - \beta)K(1; t), & \text{if } x = 0; \\ (\beta - 1)K(x; t) + (1 - \beta - \alpha)K(x + 1; t) + \alpha K(x - 1; t), & \text{if } x \geq 1, \end{cases}$$

with initial condition as in (24). One can view the conditions at $x = 0$ as that of a reflecting boundary.

Equation (23) can be suitably rescaled to give a solution to (45). Indeed, by reasoning as in the proof of Proposition 15, one can show that the solution $K(x; t)$ for $x, t \in \mathbb{N}_0$ is given by

$$K(x; t) = \beta^t \left[ \left( \frac{1 - \alpha - \beta}{\alpha} \right)^{-x/2} I^{\alpha}_x(t) + \frac{2\alpha + \beta - 1}{\alpha} \sum_{j=1}^{\infty} \left( \frac{1 - \alpha - \beta}{\alpha} \right)^{-(x+2j)/2} I^{\alpha}_{x+2j}(t) \right],$$

where $a = 2\sqrt{\frac{\alpha(1-\alpha-\beta)}{\beta^2}}$. We will leave the proof to the interested reader.

References

[Ah87] Ahumada, G.: Fonctions périodiques et formule des traces de Selberg sur les arbres, C. R. Acad. Sci. Paris Sér. I Math. 305 (1987), no. 16, 709–712.

[Be97] Bednarchak, D.: Heat kernel for regular trees In: Harmonic functions on trees and buildings (New York, 1995), volume 206 of Contemp. Math. pp. 111–112, Amer. Math. Soc. Providence, RI (1997).

[BC18] Bohner, M., Cuchta, T.: The Bessel difference equation, Proc. Am. Math. Soc. 145 (2017), No. 4, 1567–1580.

[BP01] Bohner, M., Peterson, A.: Dynamic equations on time scales. An introduction with applications, Birkhäuser Boston, Inc., Boston, MA, 2001.

[BGP09] Brasseur, C. E., Grady, R. E., Prassidis, S.: Coverings, Laplacians, and heat kernels of directed graphs, Electron. J. Combin. 16 (2009), no. 1, Research Paper 31, 25 pp.

[CJK10] Chinta, G., Jorgenson, J., Karlsson, A.: Zeta functions, heat kernels, and spectral asymptotics on degenerating families of discrete tori, Nagoya Math. J. 198 (2010), 121–172.

[CJK14] Chinta, G., Jorgenson, J., Karlsson, A.: Heat kernels on regular graphs and generalized Ihara zeta function formulas, Monatsh. Math. 178 (2015), No. 2, 171–190.
[CY99] Chung, F., Yau, S.-T.: *Coverings, heat kernels and spanning trees*, Electron. J. Comb. 6 (1999), Research Paper vol. 12, p. 21.

[CGRTV17] Ciaurri, Ó., Gillespie, T. A., Roncal, L., Torrea, J. L., Varona, J. L.: *Harmonic analysis associated with a discrete Laplacian*, J. Anal. Math. 132 (2017), 109–131.

[CMS00] Cowling, M., Meda, S., Setti, A.G.: *Estimates for functions of the Laplace operator on homogeneous trees*, Trans. Am. Math. Soc. 352 (2000), 4271-4293.

[CR62] Curtis, C.W., Reiner, I.: *Representation theory of finite groups and associative algebras*, Reprint of the 1962 original. Wiley Classics Library. A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1988.

[CSP17] Cvitković, M., Smith, A.-S., Pande, J.: *Asymptotic expansions of the hypergeometric function with two large parameters—application to the partition function of a lattice gas in a field of traps*, J. Phys. A 50 (2017), no. 26, 265206, 24 pp.

[FSS14] Friesl, M., Slavík, A., Stehlík, P.: *Discrete-space partial dynamic equations on time scales and applications to stochastic processes*, Appl. Math. Lett. 37 (2014), 86–90.

[Gi95] Giacometti, A.: *Exact closed form of the return probability on the Bethe lattice*, J. Phys. A 28 (1995), no. 1, L13.

[GR07] Gradshteyn, I. S., Ryzhik, I. M.: *Table of Integrals, Series and Products*. Elsevier Academic Press, Amsterdam, 2007.

[HNT06] Horton, M.D., Newland, D.B., Terras, A.A.: *The contest between the kernels in the Selberg trace formula for the (q+1)-regular tree*, In: The ubiquitous heat kernel, volume 398 of Contemp. Math. pp. 265—293, Amer. Math. Soc., Providence, RI (2006).

[HS82] Hughes, B., Sahimi, M.: *Random walks on the Bethe lattice*, J. Statist. Phys. 29 (1982), no. 4, 781–794.

[Jo01] Jones, D.S.: *Asymptotics of the hypergeometric function*, Math. Methods Appl. Sci. 24 (2001) 369–389.

[KN06] Karlsson, A., Neuhauser, M.: *Heat kernels, theta identities, and zeta functions on cyclic groups*, in: Grigorchuk R., Mihalik M., Sapir M., Šunič Z. eds. Topological and asymptotic aspects of group theory, volume 394 of Contemp. Math., pp. 177-189, Amer. Math. Soc., Providence, RI (2006).

[LNY21] Lin, Y., Ngai, S.-M., Yau, S.-T.: *Heat kernels on forms defined on a subgraph of a complete graph*, Math. Ann. 380 (2021), 1891–1931.

[Lo75] Lovász, L.: *Spectra of graphs with transitive groups*, Period. Math. Hungar. 6 (1975), no. 2, 191–195.

[Mn07] Mnëv, P.: *Discrete path integral approach to the Selberg trace formula for regular graphs*, Comm. Math. Phys. 274 (2007), no. 1, 233–241.

[Mo82] Mohar, B.: *The spectrum of an infinite graph*, Linear Algebra Appl. 48 (1982), 245–256.

[MW89] Mohar, B., Woess, W.: *A survey on spectra of infinite graphs*, Bull. London Math. Soc. 21 (1989), no. 3, 209–234.
[Se03] Serre, J.-P.: Trees, Springer monographs in mathematics, Springer, Berlin, Translated from the French original by John Stillwell, Corrected 2nd printing of the 1980 English translation (2003).

[SI18] Slavík, A.: Discrete Bessel functions and partial difference equations, J. Difference Equ. Appl. 24 (2018), No. 3, 425–437.

[SS14] Slavík, A., Stehlík P.: Explicit solutions to dynamic diffusion-type equations and their time integrals, Appl. Math. Comput. 234 (2014), 486–505.

[SS15] Slavík, A., Stehlík P.: Dynamic diffusion-type equations on discrete-space domains, J. Math. Anal. Appl. 427 (2015), no. 1, 525–545.

[SRb-A05] Sood V., Redner, S. and ben-Avraham, D.: First-passage properties of the Erdős–Rényi random graph, J. Phys. A: Math. Gen. 38 (2005), 109–123.

[Te03] Temme, N. M.: Large parameter cases of the Gauss hypergeometric function, J. Comput. Appl. Math. 153 (2003), 441–462.

[TW03] Terras, A., Wallace, D.: Selberg’s trace formula on the k-regular tree and applications, Int. J. Math. Math. Sci. 8 (2003), 501–526.

[TBK21] Tishby, I., Biham, O., Katzav, E.: Analytical results for the distribution of first return times of random walks on random regular graphs, J. Phys. A 54 (2021), no. 32, Paper No. 325001, 25 pp.

[Ur97] Urakawa, H.: Heat kernel and Green kernel comparison theorems for infinite graphs, J. Functional Anal. 146 (1997), 206–235.

[Ur03] Urakawa, H.: The Cheeger constant, the heat kernel, and the Green kernel of an infinite graph, Monatsh. Math. 138 (2003), no. 3, 225–237.

[Wa18] Watson, G.N.: Asymptotic expansions of hypergeometric functions, Trans. Cambridge Philos. Soc. 22 (1918), 277–308.

Carlos A. Cadavid
Department of Mathematics
Universidad Eafit
Carrera 49 No 7 Sur-50
Medellín, Colombia
e-mail: ccadavid@eafit.edu.co

Paulina Hoyos
Department of Mathematics
The University of Texas at Austin
C2515 Speedway, PMA 8.100
Austin, TX 78712 U.S.A.
e-mail: paulinah@utexas.edu

Jay Jorgenson
Department of Mathematics
The City College of New York  
Convent Avenue at 138th Street  
New York, NY 10031 U.S.A.  
e-mail: jjorgenson@mindspring.com

Lejla Smajlović  
Department of Mathematics  
University of Sarajevo  
Zmaja od Bosne 35, 71 000 Sarajevo  
Bosnia and Herzegovina  
e-mail: lejlas@pmf.unsa.ba

Juan D. Vélez  
Department of Mathematics  
Universidad Nacional de Colombia  
Carrera 65 Nro. 59A - 110  
Medellín, Colombia  
e-mail: jdvelez@unal.edu.co