CONVEXITY OF THE RENORMALIZED VOLUME OF HYPERBOLIC 3-MANIFOLDS

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ABSTRACT. The Hessian of the renormalized volume of geometrically finite hyperbolic 3-manifolds without rank-1 cusps, computed at the hyperbolic metric $g_{\text{geod}}$ with totally geodesic boundary of the convex core, is shown to be a strictly positive bilinear form on the tangent space to Teichmüller space. The metric $g_{\text{geod}}$ is known from results of Bonahon and Storm to be an absolute minimum for the volume of the convex core. We deduce the strict convexity of the functional volume of the convex core at its minimum point.

1. INTRODUCTION

The renormalized volume is a functional on the moduli space of hyperbolic 3-manifolds of finite geometry. It has been introduced in this context by Krasnov [9], after initial work by Henningson and Skenderis [8] for more general Poincaré-Einstein manifolds. As 3-dimensional geometrically finite 3-manifolds are closely related to Riemann surfaces, $\text{Vol}_R$ defines in a natural way a Kähler potential for the Weil-Petersson symplectic form on the Teichmüller space. This follows for quasi-fuchsian manifolds by the identity between the renormalized volume and the so-called classical Liouville action functional, a topological quantity known by work of Takhtadzhyan and Zograf [14] to provide a Kähler potential. For geometrically finite hyperbolic 3-manifolds without rank-1 cusps, the Kähler property of the renormalized volume was proved by Colin Guillarmou and the author in [6], by constructing a Chern-Simons theory on the Teichmüller space. The case of cusps of rank 1 is studied in a joint upcoming paper with Guillarmou and Frédéric Rochon.

Here we look at a certain moduli space of complete, infinite-volume hyperbolic metrics $g$ on a fixed 3-manifold $X$. The metrics we consider are geometrically finite quotients $\Gamma \backslash \mathbb{H}^3$ (i.e., they admit a fundamental polyhedron with finitely many faces) and do not have cusps of rank 1, in the sense that every parabolic subgroup of $\Gamma$, if any, must have rank 2. We define the moduli space $\mathcal{M}$ as the quotient of the above set of metrics on $X$ by the group $\text{Diff}^0(X)$ of diffeomorphisms isotopic to the identity. The existence of such metrics on $X$ implies that $X$ is diffeomorphic to the interior of a manifold-with-boundary $K$. Let $2K$ be the smooth manifold obtained by doubling $K$ across $\Sigma$. We make the following assumption throughout the paper:

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There exists on \(2K\) a complete hyperbolic metric of finite volume. It follows from Mostow-Prasad rigidity that up to a diffeomorphism of \(2K\) isotopic to the identity, the boundary \(\Sigma = \partial K\) is totally geodesic for this metric. Since \(2K\) must be aspherical and atoroidal, the connected components of \(\Sigma\) cannot be spheres or tori.

Examples of manifolds where our assumption is not fulfilled are quasi-fuchsian manifolds and Schottky manifolds, since their double is not atoroidal. With the above assumption, a distinguished point \(g_{\text{geod}}\) in \(M\) is obtained from \(K\) by gluing infinite-volume funnels with vanishing Weingarten operator (see Section 2) to each boundary component of \(K\). We call this metric the totally geodesic metric, and note that \(K\) is the convex core of \((X, g_{\text{geod}})\). It was remarked by Thurston, again as a simple consequence of Mostow rigidity, that \(g_{\text{geod}}\) is the unique metric in \(M\) with smooth boundary of the convex core.

By work of Bonahon [3] it is known that the volume of the convex core \(\text{Vol}(C(X, g))\) has a minimum at \(g_{\text{geod}}\) when viewed as a functional on \(M\). When \(X\) is convex co-compact, i.e., without cusps, Storm [11] proved that the minimum point \(g_{\text{geod}}\) is strict. We shall apply here our results on \(\text{Vol}_R\) to deduce the convexity of \(\text{Vol}(C(X, g))\) at this special point in \(M\) for \(X\) geometrically finite without cusps of rank 1, but possibly with cusps of rank 2 as in [3].

It is instructive to compare those results to the situation for quasi-fuchsian manifolds. Combining results of Schlenker [12] and Brock [4], the renormalized volume of quasi-fuchsian manifold is commensurable on Teichmüller space to the volume of the convex core. In particular, it is not proper as a function on Teichmüller space, since it remains bounded under iterations of a Dehn twist. It has been stated without proof by Krasnov and Schlenker [10] that \(\text{Vol}_R\) is non-negative on the quasi-fuchsian space. There is some compelling evidence for this claim: it was proved in [10] that the only critical point in a Bers slice is at the fuchsian locus, and there the Hessian of the renormalized volume equals a multiple of the Weil-Petersson scalar product. However, the lack of properness does not allow one to conclude that \(\text{Vol}_R\) is globally non-negative. In a recent joint paper with Corina Ciobotaru, we proved that \(\text{Vol}_R\) is non-negative on the almost-fuchsian space, an open neighborhood of the fuchsian locus inside the quasi-fuchsian space, and that it vanishes there only at the fuchsian locus.

Schlenker’s results from [12] have been recently extended to convex co-compact hyperbolic 3-manifolds by Bridgeman and Canary [11]. They obtain quite nice global results bounding the renormalized volume in terms of the convex core.

The main result of this paper describes the local behavior of \(\text{Vol}_R\) near \(g_{\text{geod}}\).

**Theorem 1.** Let \(g_{\text{geod}}\) be a geometrically finite hyperbolic metric on \(X\) without rank 1-cusps and with totally geodesic boundary of the convex core. Then the Hessian of the renormalized volume functional on \(M\) at \(g_{\text{geod}}\) is positive definite.

The proof is done in two steps. First we look at the volume enclosed by minimal surfaces near the boundary of the convex core, proving that it is convex, and then compare it to the “optimal” renormalization with respect to the unique hyperbolic metric in the conformal class at infinity. In the first step we use a boundary-value problem for the linearized Einstein equation in Bianchi gauge at a metric with geodesic boundary. The
Hessian of the volume appears as a Dirichlet-to-Neumann operator, which we prove to be strictly positive by an appropriate Weitzenböck formula. The second step uses the analysis of the uniformizing conformal factor, together with some elementary elliptic theory.

As a consequence, we obtain the convexity of the convex core functional:

**Theorem 2.** Let \((X, g_{\text{geod}})\) be a geometrically finite hyperbolic 3-manifold with totally geodesic boundary of the convex core, and without rank-1 cusps. Then for metrics \(g \in \mathcal{M}\) near \(g_{\text{geod}}\), the Hessian at \(g_{\text{geod}}\) of the functional

\[
\mathcal{M} \rightarrow \mathbb{R}, \quad g \mapsto \text{Vol}(C(X, g))
\]

is positive definite as a bilinear form on \(T_{g_{\text{geod}}} \mathcal{M}\).

By the simultaneous uniformization result of Ahlfors and Bers valid for quasi-fuchsian manifolds, extended by Marden [11] to the geometrically finite case, \(\mathcal{M}\) is identified with the Teichmüller space of \(\Sigma\), keeping in mind that the connected components of \(\Sigma\) have genus at least 2. We identify therefore \(T\mathcal{M}\) with \(T\mathcal{T}_{\Sigma}\). Our strategy in proving Theorem 1 will be to bound from below the Hessian of the renormalized volume in terms of the Weil-Petersson metric on \(T_{\Sigma}\).

**Acknowledgments.** This paper originated from discussions with Colin Guillarmou and Jean-Marc Schlenker about the renormalized volume of hyperbolic 3-manifolds. I owe in particular to Jean-Marc the observation that Theorem 1 has implications about the volume of the convex core.

### 2. Funnels

Let \((X, g)\) be a geometrically finite hyperbolic 3-manifold without rank-1 cusps. Such a manifold can be decomposed in a finite-volume part \(K\) (a smooth manifold-with-boundary with a finite number of cusps of rank 2), and a finite number of funnels. These funnels play an important role in this paper, so we review them below.

A **funnel** is a hyperbolic half-cylinder \((F, g)\), where \(F = [0, \infty) \times \Sigma\), for some compact, possibly disconnected Riemannian surface \((\Sigma, h)\), while

\[
g = dt^2 + h_t, \quad h_t = h \left((\cosh t + A \sinh t)^2, \cdot, \cdot\right).
\]

Here \(A\) is a symmetric field of endomorphisms of \(T\Sigma\), namely the Weingarten operator of the isometric inclusion \(\{0\} \times \Sigma \hookrightarrow F\). The tensors \(h_t\) are Riemannian metrics on \(\Sigma\) whenever \(t\) is such that the eigenvalues of \(A\) are larger than \(-\coth t\). Hence, we must assume that \(A + 1\) is positive definite in order for \(g\) to be well-defined on the whole half-cylinder. For notational simplicity, we allow disconnected funnels.

The necessary and sufficient conditions for \(g\) to be hyperbolic are the hyperbolic version of the Gauss and Codazzi–Mainardi equations:

\[
\text{(2)} \quad \det(A) = \kappa_h + 1,
\]

\[
\text{(3)} \quad (d^* \Gamma)^* A + d\text{Tr}(A) = 0
\]
where $\kappa_h$ is the Gaussian curvature of $h$. Let

$$H := \text{Tr}(A) : \Sigma \to \mathbb{R}$$

be the mean curvature function (without the customary 1/2 factor) of $\{0\} \times \Sigma \hookrightarrow X$ with respect to the direction $\partial_t$ escaping from $K$. Let $A_t, H_t, \kappa_{ht}$ be the Weingarten map, the mean curvature, respectively the Gaussian curvature, of $\{t\} \times \Sigma \hookrightarrow F$. We have

$$A_t = \frac{1}{2} g^{-1} \partial_t g = (\cosh t + A \sinh t)^{-1} (\sinh t + A \cosh t)$$

The Gauss and Codazzi-Mainardi equations continue to hold at every $t$, so

$$\kappa_{ht} = \det(A_t) - 1.$$
Proposition 3. For small deformations $g_s$ of $g$, in the homotopy class of $\Sigma$ there exists a unique family of surfaces $\Sigma_s$ which are minimal for $g_s$.

Proof. Note that $\pi_1(\Sigma)$ does not necessarily inject into $\pi_1(X)$. For each connected component $\Sigma_j$ of $\Sigma$ cut out the funnel containing it and complete it to a quasi-fuchsian manifold (this is possible since for small enough $s$, the eigenvalues of $A_s$ are close to 0). In that quasi-fuchsian manifold it is known e.g. by Uhlenbeck [15] that there exists a unique minimal surface homotopic to $\Sigma_j$. This surface will live in the original funnel of $X$ for small enough $s$.

For $s$ close to 0 let therefore $\Sigma_s \subset X$ be the unique minimal surface inside $X$ homotopic to $\Sigma$. Choose $\{\Phi_s\}$ a family of diffeomorphisms of $X$ mapping $\Sigma$ onto $\Sigma_s$, with $\Phi_0$ equal to the identity, and furthermore such that the outgoing geodesics on $\Sigma$ are mapped isometrically on the corresponding geodesics normal to $\Sigma_s$ with respect to the metric $g^s$. Hence, by pulling back $g^s$ via $\Phi_s$ we may assume that $[0, \infty) \times \Sigma$ is the underlying space of every funnel in the family $g^s$, of course with different metric $h^s$ and Weingarten map $A^s$.

By composing with an additional family of diffeomorphisms preserving the surface $\Sigma$ and the funnel structure, we can further assume that the first-order variation $\dot{h}$ of the metrics $h^s$ induced by $g^s$ on $\Sigma$ (the dot on top of some tensor denotes $s$-derivative at $s = 0$) is divergence-free:

$$\delta^h \dot{h} = 0.$$ 

Let $\dot{A}$ be the first-order variation of the Weingarten map.

Lemma 4. The tensors $\dot{h}$ and $\dot{A}$ along $\Sigma$ are trace- and divergence-free.

Proof. Since $h^s$ is minimal, we have $\text{Tr}(A^s) = 0$, and hence $\text{Tr}(\dot{A}) = 0$. On one hand, differentiating the Gauss equation implies that the variation of the curvature of $h^s$ is 0:

$$\frac{\partial}{\partial s} \kappa(h^s)|_{s=0} = \text{Tr}(\dot{A}) = 0.$$ 

On the other hand, the variation of $\kappa$ at the hyperbolic metric $h$ is given by the following intrinsic formula (cf. e.g. [2] Theorem 1.174(e)):

$$2\ddot{\kappa} = (\Delta_h + 1)\text{Tr}(\dot{h}) + d^* \delta^h \dot{h}. \tag{4}$$

Since $\delta^h \dot{h} = 0$ and $\dot{\kappa} = 0$, it follows by positivity of the elliptic operator $\Delta_h + 1$ that $\text{Tr}(\dot{h}) = 0$. Differentiating the Codazzi equation $\delta^h A^s = 0$ shows, since $A = 0$, that $\delta^h \dot{A} = 0$.

Lemma 5. The tensor $\dot{g}$ on $X$ is trace- and divergence-free in a neighborhood of $\Sigma$ containing the funnel.

Proof. We have

$$\dot{g} = \cosh^2(t) \dot{h} + 2 \sinh(t) \cosh(t) h(\dot{A}, \cdot).$$

This tensor is clearly trace-free, since $\dot{A}, \dot{h}$ are trace-free.
Let $T := \partial_t$ (we denote by $\nu$ the restriction of $T$ along $\Sigma$). Write $f = \cosh(t)$ so that

$$g = dt^2 + f^2 h, \quad \dot{g} = f^2 \dot{h} + 2f f' h(\dot{A}, \cdot).$$

For every tangential vector fields $U, V$ independent of $t$, i.e., such that $[T, U] = [T, V] = 0$, we have directly from the Koszul formula

$$\nabla_T T = 0, \quad \nabla_T U = \nabla_U T, \quad \nabla_U V = \nabla_U^\Sigma V - ff' h(U, V) T.$$

For a $1$-forms $\alpha$ with $L_T \alpha = 0$ and $\alpha(T) = 0$, we get by duality

$$\nabla_T dt = 0, \quad \nabla_T \alpha = -\frac{f'}{f} \alpha, \quad \nabla_U \alpha = \nabla_U^\Sigma \alpha - ff' \alpha(U) dt.$$

Since $\dot{g}(T, \cdot) = 0$ it follows from the above table that $(\nabla_T \dot{g})(T, \cdot) = 0$. Moreover, if $\{e_1, e_2\}$ is an orthonormal frame for $h$, then

$$-2 \sum_{i=1}^2 (\nabla_{e_i} \dot{g})(e_i, e_j) = -2 \sum_{i=1}^2 (\nabla_{e_i}^\Sigma \dot{g})(e_i, e_j), \quad -2 \sum_{i=1}^2 (\nabla_{e_i} \dot{g})(e_i, T) = ff' \text{Tr}(\dot{g}).$$

Both these terms vanish by Lemma 4.

5. VARIATION OF THE EINSTEIN EQUATION

In dimension 3, a hyperbolic metric means an Einstein metric with constant $-2$:

$$\text{Ric} = -2g.$$ (5)

The first-order variation of this equation along a path of metrics reads ([2, Theorem 1.174.d]):

$$-\delta^* (\delta + \frac{1}{2} d \text{Tr}) \dot{g} + \frac{1}{2} \left[ \nabla^* \nabla \dot{g} + \text{Ric} \circ \dot{g} + \dot{g} \circ \text{Ric} - 2 \hat{R} \dot{g} \right] = -2 \dot{g}$$ (6)

where the action of the curvature tensor $R$ on a symmetric 2-tensor $h$ is defined as

$$(\hat{R}h)_{ij} = \sum_{j,k=1}^3 h_{jk} \langle R_{ij} v_k, v_q \rangle.$$

Using (5), equation (6) is equivalent to

$$-\delta^* (2\delta + d \text{Tr}) \dot{g} + \nabla^* \nabla \dot{g} - 2 \hat{R} \dot{g} = 0.$$ A simple computation shows that for $g$ hyperbolic,

$$\hat{R}h = h - \text{Tr}(h) g$$ (7)

for every symmetric 2-tensor $h$. 

5.1. Weitzenböck formula for symmetric tensors. The following Weitzenböck formulae hold for the rough Laplacian $\nabla^*\nabla$, the twisted Hodge Laplacian $d^*d^* + d^*d^*$ and the Laplacian on functions over a hyperbolic 3-manifold: if $q_0$ is a traceless symmetric 2-tensor and $a$ is a smooth function, then
\[
\nabla^*\nabla q_0 = (d^*d^* + d^*d^* + 3)q_0,
\]
\[
\nabla^*\nabla (ag) = \Delta(a)g.
\]
Moreover, by (7),
\[
\check{R}(ag) = -2ag, \quad \check{R}q_0 = q_0.
\]

5.2. The Laplace equation on 1-forms. Let $\Delta = \nabla^*\nabla$ be the rough Laplacian acting on 1-forms (equivalently, on vector fields) on the compact manifold with boundary $K$. Clearly, $\Delta$ maps $C^\infty(K, TK)$ to itself. Recall that $\nu$ is the unit outgoing vector field orthogonal to the boundary $\Sigma$ of $K$, and $L_\nu$ denotes the Lie derivative.

**Proposition 6.** The restriction
\[
\Delta + 2 : \{V \in C^\infty(K, TK); V|_\Sigma \in T\Sigma, L_\nu V \perp \Sigma\} \rightarrow C^\infty(K, TK)
\]
is an isomorphism.

**Proof.** Let $\mathcal{D}$ denote the initial domain $\{V \in C^\infty(K, TK); V|_\Sigma \in T\Sigma, L_\nu V \perp \Sigma\}$. Then by integration by parts using that $\Sigma$ is totally geodesic, we have for all $V, V' \in \mathcal{D}$:
\[
\langle \nabla^*\nabla V, V' \rangle = \langle \nabla V, \nabla V' \rangle = \langle V, \nabla^*\nabla V' \rangle.
\]
This implies that $\nabla^*\nabla$ is symmetric and non-negative on $\mathcal{D}$. Its self-adjoint Friedrichs extension
\[
\Delta_\mathcal{D} : \mathcal{D} \rightarrow L^2
\]
is therefore also non-negative, so $\nabla^*\nabla + 2 : \mathcal{D} \rightarrow L^2$ is invertible. By elliptic regularity, the preimage of $C^\infty(K, TK)$ by this operator must lie in $\mathcal{D} \cap C^\infty(K, TK) = \mathcal{D}$. \qed

6. The Hessian of the volume of compact hyperbolic 3-manifolds with geodesic boundary

**Theorem 7.** Let $(K, g)$ be a compact hyperbolic 3-manifold with totally geodesic boundary, and $\{g^s\}_{s \in \mathbb{R}}$ a smooth family of hyperbolic metrics on $K$ with minimal boundary. Then the Hessian of the volume functional of $K$ at $g$ is positive.

**Proof.** For small $s$, the principal curvatures along $K$ are smaller than 1, so equation (1) defines a funnel, extending $g^s$ to a complete hyperbolic metric on $X = K \cup F$, unique up to isometry. By hypothesis, $\Sigma$, the possibly disconnected boundary of $K$, is minimal for each of the metrics $g^s$. Moreover, the outgoing normal geodesics to $\Sigma$ with respect to $g$ are also parametrized geodesics for $g^s$. By composing with a family of diffeomorphisms of $X$ preserving $\Sigma$, we can assume that $\dot{h}$, the first-order variation of the metrics $g^s$ restricted to $\Sigma$, is divergence-free. It follows that we can apply the results of Section 4.
in particular \( \dot{h} \) and \( A \) are divergence-free, trace-free along \( \Sigma \), while \( \dot{g} \) is divergence-free, trace free on the funnel.

Consider the following boundary-value problem:

\[
\begin{aligned}
(\nabla^* \nabla + 2)V &= -(\delta + \frac{1}{2}d\text{Tr})\dot{g}, \\
V &\in C^\infty(K, TK), \\
V|_{\Sigma} &\in T\Sigma, \\
L_\nu V &\perp \Sigma.
\end{aligned}
\] (8)

By Proposition [6] there exists a unique solution \( V \) to (8). Set

\[ q := \dot{g} + L_V g. \]

If \( \{\phi_s\} \) is the 1-parameter group of diffeomorphisms of \( K \) integrating \( V \) (well-defined since \( V \) is tangent to \( \partial K \)), then \( q \) is the tangent vector field to the 1-parameter family of metrics \( G^s := \phi^*_s g^s \).

Remark that

- \( G^s \) is hyperbolic;
- \( \text{Vol}(K, g^s) = \text{Vol}(K, G^s) \);
- \( \Sigma \) is minimal in \( (K, G^s) \) for every \( s \);
- \( \Sigma \) is totally geodesic at \( s = 0 \);
- \( \nu^s := \phi^*_s \nu \) is the unit normal vector field to \( \Sigma \) with respect to \( G^s \).

The last property holds because \( \nu \) is the unit normal vector field to \( \Sigma \) with respect to \( g^s \) for every \( s \).

By the Schl"afli formula of Rivin-Schlenker (see [7], Lemma 5.1), we have

\[
\partial_s \text{Vol}(K, g^s) = \frac{1}{2} \int_{\Sigma} (\text{Tr}(\dot{A}^s) + \frac{1}{2} \text{Tr}((\dot{h}^s)^{-1} \dot{h}^s A^s))d\text{vol}_h^s = \frac{1}{8} \langle \dot{h}^s, L_\nu g^s \rangle_{L^2(\Sigma, g^s)}.
\]

The same formula for the family of metrics \( G^s = \phi^*_s g^s \) gives

\[
\partial_s \text{Vol}(K, \phi^*_s g^s) = \frac{1}{8} \langle \partial_s G^s, L_{\phi^*_s \nu} G^s \rangle_{L^2(\Sigma, G^s)}.
\]

The term \( L_{\phi^*_s \nu} G^s \), i.e., the second fundamental form of \( \Sigma \) with respect to \( G^s \), vanishes at \( s = 0 \). One more derivative at \( s = 0 \) shows therefore

\[
\partial^2_s \text{Vol}(K, G^s)_{s=0} = \frac{1}{8} \langle q, L_\nu q - L_{[V, \nu]} g \rangle_g.
\]

Since \( \text{Vol}(K, g^s) = \text{Vol}(K, G^s) \), the above formula computes the second variation \( \ddot{\text{Vol}} \) of \( \text{Vol}(K, g^s) \).

**Theorem 8.** The inner product \( \langle q, L_\nu q \rangle_{L^2(\Sigma, g)} \) is non-negative. Explicitly,

\[
\langle q, L_\nu q \rangle_{L^2(\Sigma, g)} \geq \|q\|^2.
\]

**Proof.** The tensor \( q \) is a solution to the linearized Einstein equation because the family \( G^s \) consists of hyperbolic metrics. Use now the following identities on vector fields:

\[
(2\delta + d\text{Tr})\delta^* = \nabla^* \nabla + 2, \quad \quad 2\delta^* V = L_V g.
\]
From (8), it follows that $q = \dot{g} + L_V g$ is in Bianchi gauge, i.e.,
$$(\delta + \frac{1}{2} dTr) g = 0.$$ Equation (9) implies that $q$ is a solution of the elliptic equation
$$(\nabla^* \nabla - 2 \check{R}) q = 0.$$ Decompose $q$ in its trace-free component $q_0$ and its pure trace component $ag$ for some $a \in C^\infty (K)$. Using the Weitzenböck formulae from section [5.1], (11) is equivalent to
$$\begin{cases} (d^* d + d d^* + 1) q_0 = 0, \\
(\Delta + 4) a = 0. \end{cases}$$ Because of these identities, integration by parts on $K$ gives
$$\begin{align}
\int_{\Sigma} \langle q_0, L_\nu q_0 \rangle dvol_h &= \int_K (|d^* q_0|^2 + |d d^* q_0|^2 + |q_0|^2) dvol_g, \\
\int_{\Sigma} \langle a, L_\nu a \rangle dvol_h &= \int_K (|da|^2 + 4a^2) dvol_g.
\end{align}$$ Since $L_\nu g = 0$ (equivalent to $(\Sigma, h) \hookrightarrow (K, g)$ being totally geodesic), (14) is the same as
$$\int_{\Sigma} \langle ag, L_\nu (ag) \rangle dvol_h = \int_K (|da|^2 + 4a^2) dvol_g.$$ Since $\text{Tr}(g^{-1} q_0) = 0$ by definition and $L_\nu g = 0$, it follows by applying $L_\nu$ that
$$\text{Tr}(g^{-1} L_\nu q_0) = 0.$$ Hence $L_\nu q_0$ is trace-free, $L_\nu (ag)$ is a multiple of $g$, and so (13) and (15) give
$$\int_{\Sigma} \langle q, L_\nu q \rangle dvol_h \geq \| q_0 \|^2 + 4 \| a \|^2 \geq \| q \|^2.$$ Returning to (10), we would like to analyze the remaining term. In fact we prove below that it vanishes pointwise on $\Sigma$, thereby ending the proof of Theorem 7.

**Proposition 9.** The scalar product $\langle q, L_{[V,T]} g \rangle_L$ is pointwise zero on $\Sigma$.

**Proof.** Let
$$V = v_0 + tu_1 T + t^2 (v_2 + u_2 T) + O(t^3)$$
be the Taylor expansion of $V$ near $t = 0$, using the fixed product decomposition near $\Sigma$. Here $v_0, v_2$ are vector fields on $\Sigma$, while $u_1, u_2$ are functions. Note that the coefficients $u_0$ and $v_1$ vanish (and we omit them from the formula) as a consequence of (8).

From the definition, $q = \dot{g} + L_V g$ where we can also write $L_V g = 2(\nabla V)_{sym} = 2\delta^* V$. We recall that $\dot{g}$ is tangential (it does not contain terms involving $dt$). The correction term equals at $\Sigma$
$$L_V g = L_{v_0} h + u_1 dt \otimes dt + O(t)$$
and so in particular it has no mixed terms of the type \( dt \otimes \Lambda^1 \Sigma \). The vector field \([T, V]\) equals
\[
[T, V] = u_1 T + 2tv_2 + 2tu_2 T + O(t^2).
\]
The tensor \( L_{[T, V]} g \) does not have any tangential component at \( t = 0 \). The mixed terms do not contribute in the scalar product with \( q \) since that last tensor has no such mixed terms. The coefficient of \( dt \otimes dt \) in \( L_{[T, V]} g \) is \( 2u_2 \). But this term vanishes by the lemma below.

\[\square\]

**Lemma 10.** The second-order normal term \( u_2 \) in \( V \) vanishes.

**Proof.** The free term \(-2\delta \dot{g} - d\text{Tr}(\dot{g})\) in the boundary-value problem [3] determining \( V \) vanishes near \( \Sigma \) by Lemma [5]. At \( t = 0 \) the Hodge Laplacian \( \Delta_H = dd^* + d^*d \) on 1-forms takes the form
\[
(\Delta_H V)_{|t=0} = \Delta h v_0 - 2v_2 - 2u_2 \nu.
\]
Since \( g \) is hyperbolic, Bochner’s formula \( \Delta_H = \nabla^* \nabla + \text{Ric} \) gives \((\Delta_H + 4)V = (\Delta + 2)V = 0\), and using that \( V \) is tangent to \( \Sigma \) we deduce \( u_2 = 0 \). \( \square \)

### 7. The Hessian of the Renormalized Volume on the Funnel

We have seen above that \( \text{vol}(K) \) has positive Hessian at \( g \). But since \( \Sigma \) is minimal for \( g^s \), we remark that
\[
\text{Vol}_R(X, g^s; h^s_\infty) = \text{Vol}(K, g^s).
\]
To prove Theorem 1, we must therefore analyze the Hessian of \( \text{Vol}_R - \text{Vol}(K) \). For this, let \( \omega^s \in C^\infty(\Sigma) \) be the conformal factor so that \( e^{2\omega^s} h^s_\infty \) has constant curvature \(-4\). Such a conformal factor is unique, smooth in \( s \), and \( \omega^0 = 0 \).

Following the proof of [5, Theorem 9] one could obtain by similar methods:

**Proposition 11.** Let \( \omega^s \) be the conformal factor uniformizing the metrics \( h^s_\infty \). Then for small \( s \),
\[
\text{Vol}_R(X, g^s) \geq \text{Vol}_R(X, g^s; h^s_\infty),
\]
with equality at \( s = 0 \).

As a consequence, \( \dot{V}_R \geq \frac{d^2}{ds^2} \text{Vol}_R(X, g^s; h^s_\infty) \big|_{s=0} \). However, rather than adapting the results of [5], we prove below that the functional \( \text{Vol}_R - \text{Vol}(K) \) is convex at \( g = g_{\text{geod}} \), with an explicit lower bound.

The following Polyakov-type formula holds for the conformal variation of the renormalized volume (cf. [7]):
\[
\text{Vol}_R(X, g^s; e^{2\omega^s} h^s_\infty) - \text{Vol}_R(X, g^s; h^s_\infty) = -\frac{1}{4} \int_\Sigma \left( |d\omega^s|_{h^s_\infty}^2 + 2\kappa_{h^s_\infty} \omega^s \right) d\text{vol}_{h^s_\infty}.
\]

Let \( \tilde{\kappa} \) be the second variation of \( \kappa_{h^s_\infty} \) at \( s = 0 \):
\[
\tilde{\kappa} = \partial_s^2 \kappa_{h^s_\infty} \big|_{s=0}.
\]
**Lemma 12.** Let \( \omega^s \in C^\infty(\Sigma) \) such that \( \kappa e^{2\omega^s} h^\infty_s = -4 \). Then \( \omega^s = \omega_2 s^2 + O(s^3) \) with
\[
\omega_2 = -\frac{1}{2} (\Delta h^0_\infty + 8)^{-1}\tilde{\kappa}.
\]

**Proof.** The metrics \( h^s \) are given by
\[
h^s_\infty = \frac{1}{4} h^s ((1 + A^s)^2, \cdot)
\]
hence the first-order variation is
\[
\frac{dh^s_\infty}{ds} = \frac{1}{4} (\dot{h} + 2 h (\dot{A}, \cdot)).
\]
By applying the formula (4), we get for the first-order variation of the Gaussian curvature of \( h^s_\infty \):
\[
2 \dot{\kappa}_\infty = (\Delta_\infty + 1) \text{Tr} (\dot{h}_\infty) + d^* \delta \dot{h}_\infty.
\]
Both \( \text{Tr}(\dot{h}_\infty) \) and \( \delta \dot{h}_\infty \) vanish by Lemma 4, thus \( \dot{\kappa}_\infty = 0 \).
By the conformal change rule for the Gaussian curvature,
\[
-4 = \kappa e^{2\omega^s} h^\infty_s = e^{-2\omega^s} (\kappa h^\infty_s + \Delta h^\infty_s \omega^s).
\]
Let \( \omega^s = s \omega_1 + s^2 \omega_2 + O(s^3) \) be the limited Taylor expansion of \( \omega \). Write the expansion in \( s \) up to errors of order \( s^3 \) of the above identity, using \( \kappa h^\infty_s = -4 + O(s^2) \):
\[
-4 = (1 - 2 \omega_1 s + O(s^2)) (-4 + O(s^2) + (\Delta h^0_\infty + O(s))(s \omega_1 + O(s^2)))
\]
giving for the coefficient of \( s \)
\[
\Delta h^0_\infty \omega_1 + 8 \omega_1 = 0.
\]
It follows by positivity of this elliptic operator on the closed surface \( \Sigma \) that \( \omega_1 = 0 \), and returning to (19),
\[
-4 = (1 - 2 \omega_1 s^2 + O(s^3)) \left(-4 + \frac{1}{2} \dot{\kappa} s^2 + O(s^3) + (\Delta h^0_\infty + O(s))(s^2 \omega_2 + O(s^3))\right).
\]
The coefficient of \( s^2 \) must be 0, hence
\[
8 \omega_2 + \frac{1}{2} \dot{\kappa} + \Delta h^0_\infty \omega_2 = 0,
\]
proving the lemma. \( \square \)

Using this lemma, (17) gives for the quadratic term in the right-hand side:
\[
-\frac{1}{4} \int_{\Sigma} \left| d\omega^s \right|_{h^\infty_s}^2 + 2 \kappa h^\infty_s \omega^s \text{dvol}_{h^\infty_s} = -s^2 \int_{\Sigma} (\Delta h^0_\infty + 8)^{-1} \dot{\kappa} \text{dvol}_{h^\infty_s} + O(s^3).
\]
By decomposing \( \dot{\kappa} \) in eigenmodes for \( \Delta h^0_\infty \), we see that in the integral the only surviving such term is the zero-eigenmode, hence
\[
(20) \quad \text{Vol}_R (X, g^s; e^{2\omega^s} h^\infty_s) - \text{Vol}_R (X, g^s; h^\infty_\infty) = -\frac{1}{8} s^2 \int_{\Sigma} \dot{\kappa} \text{dvol}_{h^\infty_s} + O(s^3).
\]

**Lemma 13.** The second variation of \( \kappa h^\infty_s \) equals \( -8 \text{Tr}(\dot{A}^2) \).
Proof. We have (see [5], proof of Lemma 10):
\[
\kappa_{h|_{s=\infty}} = \frac{4\kappa_{h_s}}{2 + \kappa_{h_s}} = 4 - \frac{8}{2 + \kappa_{h_s}}.
\]
Thus, since \(\partial_s \kappa_{h_s}|_{s=0} = 0\), we get \(\partial_s^2 \kappa_{h_s}|_{s=0} = 8\kappa_{h_s}|_{s=0}\).

From the Gauss equation (2) and the fact that \(\text{Tr}(A_s) = 0\) we deduce
\[
\text{Tr}((A_s)^2) = -2\kappa_{h_s} - 2.
\]
Since \(A_s = 0\) at \(s = 0\), we get \(\kappa_{h_s}|_{s=0} = -\text{Tr}(\dot{A}^2)\). \qed

8. PROOF OF THEOREM [1]

From (20), Theorem [8] and Lemma [13], we get
\[
\text{Vol}_{R}(X, g^s) \geq \frac{1}{4} \int_{\Sigma} \text{Tr}(\dot{A}^2) d\text{vol}_h + \int_{K} \|q\|^2.
\]

We would like to translate this into an inequality in terms of the Weil-Petersson metric on the Teichmüller space \(T_{\Sigma}\). Consider a smooth slice
\[
g : T_{\Sigma} \rightarrow \mathcal{M}_{-1}(X)
\]
with \(g(h) = g_{\text{good}}\), where \(h\) is the hyperbolic metric on \(\Sigma\) viewed as the totally geodesic boundary of the convex core of \((X, g_{\text{good}})\). We can assume, up to pulling back by a family of diffeomorphisms of \(X\), that the boundary of \(K\) is minimal with respect to \(g^s := g(s)\) for every \(s \in T\), and moreover that the geodesics normal to \(\partial K\) are the same for all \(s\).

Let \(h^s\) be the metric induced on \(\Sigma = \partial K\) by restriction of \(g^s\). The tangent map to this restriction is the linear map
\[
R : T_h T \rightarrow T_h T, \quad \dot{g} \mapsto \dot{h}
\]
of finite-dimensional vector spaces. By composing with the map \(\dot{h} \mapsto V\), where \(V\) is the solution to the boundary problem (8), we obtain a linear map
\[
Q : T_h T \rightarrow T_{g_{\text{good}}} \mathcal{M}_{-1}(K), \quad \dot{g} \mapsto q = \dot{g} + L_V g.
\]

Proposition 14. There exists a constant \(c > 0\), independent of the metrics \(g^s\), such that
\[
\|q\|_{L^2(K)} \geq c\|\dot{h}\|_{L^2(\Sigma)}.
\]

Proof. If \(q = 0\), it follows that \(\dot{h} + L_V \dot{h} = 0\) as a tensor on \(\Sigma\). Since \(\dot{h}\) is divergence-free, the tensors \(\dot{h}\) and \(L_V \dot{h}\) are orthogonal in \(L^2\), so they both must vanish. Hence, if \(Q(\dot{g}) = 0\) we get that \(R(\dot{g}) = 0\). Therefore \(R\) factors through \(Q\), i.e., there exists a linear map
\[
T : \text{range}(Q) \rightarrow T_h T, \quad q \mapsto \dot{h}
\]
so that \(T \circ Q = R\). The map must be bounded since \(T\) is of finite dimension. \qed
Remark 15. On the quasi-fuchsian space, the Hessian of \( \text{Vol}_R \) at any point of the fuchsian locus equals

\[
\text{Vol}_R = \frac{1}{8} \| \dot{h}^+ - \dot{h}^- \|^2,
\]

Thus the Hessian is only positive semi-definite in that case. It becomes however positive definite when we restrict the renormalized volume functional to a Bers slice, i.e., we keep fixed one conformal boundary. In that case it equals \( 1/8 \) of the Weil-Petersson metric, by a result of Krasnov and Schlenker [10].

9. PROOF OF THEOREM 2

Schlenker [12, Theorem 1.1] proved the following inequality between the renormalized volume and the volume of the convex core of quasi-fuchsian manifolds:

\[
\text{Vol}_R(X, g^s) \leq \text{Vol}(C(X, g^s)) - \frac{1}{4} L_m(l)
\]

where \( L_m(l) \geq 0 \) is the length of the measured bending lamination of the convex core. The proof consists in identifying the right-hand side with the renormalized volume of \( X \) starting from the boundary of the convex core. The induced metric at infinity is Thurston’s grafting metric on \( \Sigma \), which is known to be of class \( C^{1,1} \), of non-positive curvature, and bounded below by the Poincaré (i.e., hyperbolic) metric on \( \Sigma \). These properties imply (21).

This inequality carries over with the same proof to convex co-compact manifolds, as was noted in [1]. In fact, Schlenker’s proof remains valid for manifolds with funnels and with rank 2-cusps (i.e., geometrically finite without rank 1-cusps). Hence we may safely use (21) in our setting.

At the initial metric \( g_{\text{geod}} \) the two sets \( K \) and \( C(X, g_{\text{geod}}) \) are the same, in particular they share the same volume. Moreover, \( \text{Vol}_R(X, g_{\text{geod}}) = \text{Vol}(K) \) since the boundary of \( K \) is totally geodesic with respect to \( g_{\text{geod}} \). Since by Theorem 1 the Hessian of \( \text{Vol}_R(X, g) \) is positive definite at \( g_{\text{geod}} \), the same can be said about the Hessian of the functional \( \text{Vol}(C(X, g)) \) at \( g_{\text{geod}} \), using the inequality (21).

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