Relation between irreversibility and entanglement in classically chaotic quantum kicked rotors

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received 19 March 2016; accepted in final form 1 July 2016
published online 25 July 2016

PACS 05.45.Mt - Quantum chaos; semi-classical methods
PACS 05.45.-a - Nonlinear dynamics and chaos
PACS 03.65.-w - Quantum mechanics

Abstract – The relation between the degree of entanglement and time scale of time-irreversible behavior is investigated for classically chaotic quantum coupled kicked rotors by comparing the entanglement entropy (EE) and the lifetime of correspondence with classical decay of correlation, which was recently introduced. Both increase on average drastically with a strong correlation when the strength of coupling between the kicked rotors exceeds a certain threshold. The EE shows an anomalously large fluctuation resembling a critical fluctuation around the threshold value of coupling strength where the entanglement sharply increases toward full entanglement. In this regime it can be shown that, although the correlation is hidden, EE and the lifetime of individual eigenfunctions also have a positive correlation that can be seen via another measure.

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Introduction. – Classically chaotic quantum systems can be considered as the simplest systems exhibiting apparently time-irreversible behavior [1], and their quantum counterparts can also be the simplest systems realizing quantum irreversibility.

Various examples of phenomena indicating evolution toward irreversibility, such as normal diffusion [2,3], energy dissipation [4], energy spreading [5] and so on, have been presented. Moreover, by coupling with a proper classically chaotic quantum system as a "quantum noise source", we can make a quantum system work as a quantum damper [6].

However, quantum-mechanical properties due to the quantum uncertainty principle prevent a quantum system from becoming irreversible in the same sense as its classical counterpart [2,3]. Indeed unbounded chaotic diffusion of a classical system is inhibited in its quantum counterpart. But if such systems are coupled even at classically negligible coupling strength, the diffusive motion exactly mimicking classical unbounded diffusion is recovered [7]. Such diffusive motion exhibits characteristics of time-irreversibility identical to classical diffusive motion, that is complete decay of correlation and loss of past memory [8–10]. A typical phenomenon showing the quantum recovery of chaotic irreversibility is the Anderson transition in the quantum standard map, which exhibits features of a critical phenomenon occurring due to cooperative effects [11,12]. The drastic change of the coupled system is a reflection of the growth of entanglement among the constituent systems, which can be quantitively measured by the entanglement entropy [13], which is hereafter abbreviated by EE.

We hypothesize that the spontaneous and cooperative recovery of classical chaotic irreversibility is a generic feature of a classically chaotic quantum system even if the system is bounded in a finite phase space region and so cannot exhibit a truly diffusive behavior. To quantitatively characterize the growth of time-irreversibility in quantum systems and their associated quantum states, we proposed a method to measure the lifetime of correspondence with the decay of temporal correlation (hereafter called lifetime for brevity) as an indicator of the time scale on which a quantum system exhibits classical irreversible behavior [14]. The aim of the present paper is to show that the growth of the quantum time-irreversible character
quantified by the lifetime has evident correlation with the development of entanglement measured by the EE, taking a quantum coupled kicked rotor (CKR) as an example. The EE for individual eigenstates fluctuates from state to state anomalously like a critical fluctuation in the threshold regime prior to the full entanglement, and we also show that in such a regime the correlation between the lifetime and the EE exists at the level of individual eigenstate.

Irreversibility in quantum system has been explored directly by the time-reversal experiment [8–10,15], and further it has been extensively investigated by many authors in the context of fidelity [16,17]. The purpose of our method overlaps with that of the fidelity method [18,19]. However, the advantage of our method is that it enables us to observe the features of irreversibility related to the decay of correlation over an extremely long time scale, and it enables us to measure the lifetime of classical irreversibility mentioned above.

Model and method. – Our method is to convert the quantum motion in the bounded phase space of the object system to an extended motion in an infinitely extended homogeneous action space. If the motion of the system is classically chaotic and correlation decays, the motion in the action space is a Brownian motion in the classical limit [14].

We showed that if the system’s classical counterpart is fully chaotic with definitely decaying correlation, the lifetime of mimicking chaotic irreversible behavior can be measured as the time at which an evident deviation from the ideal Brownian-motion–like behavior takes place. The lifetime depends on the number of eigenstates composing the examined state, and the maximal lifetime was proportional to the square of the Hilbert dimension of the system in the fully developed entanglement regime [14].

In the present paper we investigate the relation between the growth of irreversibility and the degree of entanglement in classically chaotic quantum systems. As a typical example, we take the CKRs system composed of two kicked rotors (KR) which are classically fully chaotic. The CKR is represented by

\[ H(\hat{p}, \hat{q}, t) = (\hat{p}_1^2 + \hat{p}_2^2)/2 + \delta_T(t)[V(\hat{q}_1) + V(\hat{q}_2) + \epsilon V_{12}(\hat{q}_1, \hat{q}_2)]. \]  

(1)

The two KRs interact via the interaction \( V_{12}(\hat{q}_1, \hat{q}_2) = \cos(\hat{q}_1 - \hat{q}_2) \) of strength \( \epsilon \), where \( \delta_T(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT) \). We call the above system “S”. The KRs, say KR1 and KR2, are defined in the bounded phase space \((q_i, p_i) \in [0, 2\pi] \times [0, 2\pi]\) and so diffusive motion does not happen. The dimensions of the Hilbert space \( N_1, N_2 \) of the two KRs are equal, \( N_1 = N_2 \), and we take \( h = 2\pi/N \). We take \( T = 1 \) hereafter, although the notation “T” is retained in the mathematical expressions.

Thus the dimension of the Hilbert space for the CKR is \( N = N_1 N_2 = N^2 \). Arnold’s cat map \( V(\hat{q}) = -K\hat{q}^2/2 \) and the bounded standard map \( V(\hat{q}) = K\cos\hat{q} \) in the fully chaotic regime \( K \gg 1 \) are taken as examples of classically chaotic quantum systems. The development of entanglement of eigenstates can be controlled by the system parameter \( \epsilon \).

Our method is to introduce a linear oscillator “L” which is very weakly coupled with the system S and converts S’s quantum motion to a Brownian motion in the homogeneously extended action space of the linear oscillator. L is represented by the angle-action canonical pair operators \( \hat{q} \) and \( \hat{J} = -i\hbar d/d\hat{q} \) with the Hamiltonian \( \hat{J} \) of the frequency \( \omega \) [14]:

\[ \hat{H}_{tot}(t) = \hat{H}(\hat{p}, \hat{q}, t) + \eta \hat{v}(\hat{p}) \hat{g}(\hat{\theta}) + \omega \hat{J}, \]  

(2)

where \( \hat{v}(\hat{p}) \) and \( \hat{g}(\hat{\theta}) \) are Hermitian operators, for which we take here \( \hat{v}(\hat{p}) = i\eta \hat{p} \) and \( \hat{g}(\hat{\theta}) = \cos(\hat{\theta}) \). \( \hat{J} \) has the eigenvalue \( J = jh \) \((j \in \mathbb{Z})\) for the eigenstate \( |\hat{\theta}|J \rangle \propto e^{-i\theta/\hbar} \). Observing the system at the integer multiples of the fundamental period \( T(=1) \), that is \( t = nT + 0 \), where \( n \in \mathbb{Z} \), the one-step evolution of CKR from \( t = rT + 0 \) to \( t + T \) occurs by the unitary operator \( \hat{U} = e^{-i[V(\hat{q}_1) + V(\hat{q}_2) + \epsilon V_{12}(\hat{q}_1, \hat{q}_2)]/2\hbar} e^{-i\epsilon V_{12}(\hat{q}_1, \hat{q}_2)/2\hbar} \), and the equation of the motion for the operators in the Heisenberg picture leads to the formula for the deviation of \( \hat{J} \) from its initial value, namely, \( \hat{J}(\tau) = J(0) + \eta \omega \sum_{s=0}^{\tau-1} \hat{f}_s \), where \( \eta \omega = 2\eta \sin(\omega T/2)/\omega \) and \( \hat{f}_s = \hat{v}(\hat{p})(s) \sin(\omega Ts + \theta + \omega T/2) \). Note that the motion of \( \hat{p}(s) = \hat{U}^{-\ast} \hat{p} \hat{U} \) is not influenced by the linear oscillator L in the limit of \( \eta \rightarrow 0 \), and further the expectation value \( \langle \hat{J}(\tau) \rangle \) vanishes if we take \( |J(0) = 0 \) as the initial state of L. This formula says that, in the classical limit, \( \hat{J} \) exhibits a Brownian motion driven by the chaotic force \( \hat{f}_s \). The mean square displacement (MSD) \( \Delta J^2 = \langle (\hat{J}(\tau) - \langle \hat{J}(\tau) \rangle)^2 \rangle \) is then

\[ \Delta J^2(\tau) = \sum_{s=0}^{\tau-1} D_\omega(s), \]  

(3)

\[ D_\omega(\tau) = D \sum_{s=-\tau}^{\tau} C_{\tau-s}(s) \cos(\omega Ts), \]  

(4)

where \( \langle \ldots \rangle \) means the expectation value with respect to the initial wave packet \( |\Psi_0\rangle \otimes |J = 0\rangle \), where \( |\Psi_0\rangle \) is the initial state of the system S, and \( D = 2\eta^2 \sin^2(\omega T/2)/\omega^2 \), \( C_{\tau-s}(s) = \langle (\hat{\Psi}_0|\hat{v}_\tau \hat{v}_{\tau-s}|\Psi_0) + \text{c.c.} \rangle/2 \) \((s \geq 0) \) and \( C_{\tau-s} = C_{\tau-s}(s) \), where \( \hat{v}_\tau = \hat{U}^{-\ast} \hat{v} \hat{U}^\tau \), is the autocorrelation function.

Since the classical CKR system is an ideally chaotic system, it exhibits persistent diffusion, the autocorrelation function decays exponentially with the Markovian property, and \( D_\omega(\tau) \) converges to a constant value \( D_\omega^{\text{class}} \) in the limit \( \tau \rightarrow \infty \), which are manifestations of continuous loss of memory in classical chaos. However, in quantum systems, convergence of \( D_\omega(\tau) \) to a finite value occurs only for a finite \( \tau \). Indeed the quantum \( D_\omega(\tau) \) is explicitly represented by

\[ D_\omega(\tau) \sim D \sum_{s=-\tau}^{\tau} \sum_{m=1}^{M} |C_m|^2 \sum_{n=1}^{N} |\langle m|\hat{v}|n \rangle|^2 \times \cos((\gamma_m - \gamma_n)s) \cos(\omega Ts) \]  

(4)
using quasi-eigenstate \( |m\rangle \) and eigenangle \( \gamma_m \) of the evolution operator \( \hat{U} \) of \( S \), i.e., \( \hat{U}|m\rangle = e^{-\gamma_m}|m\rangle \), and the initial state \( |\Psi_0\rangle = \sum_{m=1}^{M} C_m|m\rangle \) (see footnote 1). Thus the quantum \( D_{\omega} (\tau) \) is a sum of periodic oscillations and therefore vanishes if it is averaged on a long enough time scale, which means that the stationary diffusion is suppressed beyond a certain time scale \( \tau_L \). We regard this time scale to be the lifetime of correspondence with classical diffusion, and, equivalently, the lifetime of correspondence with the classical decay of correlation. The lifetime \( \tau_L \) indicates the maximal time scale on which the quantum time-irreversibility is observable.

Equations (3) and (4) are not convenient for numerical computation over an extremely long time scale. Practically, with the method of wave packet propagation starting from the initial state, we compute \( D_{\omega} (\tau) \) from the data of MSD [14]. With these data we decide \( \tau_L \) as the time at which there occurs a significant deviation from the linear growth of MSD, which is the most definite feature of chaotic irreversibility.

\begin{enumerate}
    \item First we decide the diffusion exponent \( \alpha(\tau) \) such that \( \Delta J^2(s) \propto s^{\alpha(\tau)} \) defined for \( s \) in an appropriate interval around \( \tau \) with the same width in logarithmic scale.
    \item Next, we decide \( \tau_L \) as the time at which \( \alpha \) deviates from 1, specifically the first time step that \( |\alpha(\tau_L) - 1| > r \), where we choose \( r = 0.5 \) in practice.
    \item The value of \( \tau_L \) is very sensitive to the choice of initial condition and also to the parameter \( \epsilon \) in the regime of transition to full entanglement.
\end{enumerate}

According to the procedure of [14], we can eliminate fluctuations of the lifetime due to such sensitivity: we add classically negligible small perturbation such as \( \zeta_R \cos(\hat{q}_i - q_{iR}) \) of order \( \mathcal{O}(\hbar) \), to the potential \( V(\hat{q}_i) \) and take the average of \( \tau_L \) over the ensemble of the potential parameter \( q_{iR} \) (\( i = 1, 2 \)). We refer to the result of this procedure hereafter as the average lifetime \( \langle \tau_L \rangle \).

**Development of entanglement and lifetime.**

In the CKR the coupling parameter \( \epsilon \) sensitively controls its statistical properties. If the standard map with unbounded phase space is used for each KR, the classical chaotic diffusion is recovered as \( \epsilon \) is increased beyond a threshold proportional to \( \hbar \). In a previous paper we showed that a similar transition occurs also for bounded finite-dimensional CKR and the lifetime of correspondence with classical diffusive behavior is drastically enhanced [15].

In fig. 1(a) we show some examples of \( \Delta J^2(\tau) \) as a function of \( \tau \), which indicate the tendency that the time scale on which \( \Delta J^2(\tau) \) follows the classical \( \Delta J^2(\tau) \) increases with \( \epsilon \). In fig. 1(b) we depict how the average lifetime increases with the coupling strength \( \epsilon \), demonstrating that the time scale of correspondence with classical diffusive behavior increases with \( \epsilon \).

We are interested in the relation between growth of the irreversibility and development of the entanglement in the...
CKR, especially for the bounded KR. The von-Neumann entropy is a standard tool to measure the degree of entanglement quantitively [13]. We use it to measure the entanglement between the two KR’s. Considering the reduced density operator \( \rho_{m}^{(1)} = \text{Tr}_2 |m\rangle \langle m| \) traced over the KR2 the von-Neumann entropy for the reduced \( \rho_{m}^{(1)} \), namely

\[
S_{m}^{(1)} = -\text{Tr}_1 (\rho_{m}^{(1)}) \log \rho_{m}^{(1)}
\]  

is the EE for KR1. We show in fig. 1(b) the mean EE, \( \sum_{m=1}^{N} S_{m}^{(1)}/N \), as a function of \( \epsilon \), which is averaged over the ensemble of the perturbation potential parameter \( \epsilon \).

It is evident that the average lifetime is strongly correlated with the mean EE. We cannot, however, know what kind of physical process is happening in the system from EE alone. We explore more closely the connection between EE and \( \tau_{L} \). In the transition regime between the unentangled regime and the fully entangled one, a quite interesting phenomenon is observed numerically: the distribution of EE for individual eigenstates spreads over the whole range of values from 0 to the fully entangled value. The mean EE and the standard deviation of EE around the mean value are shown in fig. 2(a), and the distribution function of EE are depicted in fig. 2(b). The spread of the EE distribution indicates a critical-like phenomenon in the transition regime, which means that the degree of entanglement fluctuates anomalously from state to state in the range between the unentangled value and the fully entangled value, although we could not confirm the presence of a critical phenomenon in the rigorous sense (namely, critical slowing-down and critical divergence of some physical quantities). Then we can expect that such an anomalous fluctuation should be apparent in the fluctuation of the lifetime, and the more entangled eigenstate should have longer lifetime. However, the following question arises: for individual eigenstates, does the correlation between lifetime and EE exist? Unfortunately, the accidental fluctuation of lifetime is in general so large that it is very difficult to numerically extract the correlation between the EE and the lifetime for individual eigenstates.

**Correlation between entanglement and lifetime at the level of individual eigenstates.** To supplement the EE, we introduce a more intuitively comprehensible entropy-like quantity. Since \( \hat{v} (\hat{p}_1) \) is free from KR2, the number of eigenstates connected by \( \hat{v} \) increases as the entanglement between KR1 and KR2 is enhanced. We therefore define the *transition rate entropy* (TE) \( S_{m}^{T} \), representing the variety of transitions from the state \( |m\rangle \) due to the perturbation \( \hat{v} \) as

\[
S_{m}^{T} = -\sum_{n} t_{mn} \log t_{mn},
\]

where \( t_{mn} = |\langle m|\hat{v}|n\rangle|^2 / \sum_{n'} |\langle m|\hat{v}|n'\rangle|^2 \) is the normalized transition rate. Its mean value \( \sum_{m=1}^{N} S_{m}^{T}/N \) as a function of \( \epsilon \) has a definite correlation with the mean EE \( \sum_{m=1}^{N} S_{m}^{(1)}/N \) as is displayed in fig. 3(a).

To elucidate the relation between the EE and the lifetime, we can use the TE introduced above as a mediating quantity between them. We can expect that TE is strongly correlated with \( \tau_{L} \) of individual eigenstate. This is based upon the following theoretical considerations. First, we note that the TE \( S_{m}^{T} \) represents the number of eigenstates which are connected with the eigenstate \( |m\rangle \), by the relation \( B_{m} = e^{S_{m}^{T}} \).

Next, in the previous paper we showed that if we could suppose that the eigenstate \( |m\rangle \) are connected by \( \hat{v} \) with all other eigenstates almost equally, then the lifetime of the superposed state \( |\Psi_{0}\rangle = \sum_{m=1}^{M} C_{m} |m\rangle \) is given by

\[
\tau_{L} \sim \frac{2DCr(0)MN}{\pi^{2}D^{(m)}}
\]

which predicts a seemingly strange feature that the lifetime depends on \( M \) i.e., the number of eigenstates forming the initial state \( |\Psi_{0}\rangle \), where \( Cr(0) = \cdots \)
\[ \sum_{m=1}^{M} |C_m|^2 \sum_{n=1}^{N} |\langle m|\hat{v}|n\rangle|^2 \ [14]. \] Equation (7) explains the numerical results quite well in the fully entangled regime. This result can be extended in the transition regime if we suppose that \( \hat{v} \) is dominantly connected only with \( B = e^{ST} \) eigenstates among the \( N \) eigenstates, and moreover the \( M \) states forming \( |\Psi_0\rangle \) are so chosen as to have almost the same value of the transition rate entropy \( S^T \) (or \( B \)). Then \( N \) in eq. (7) can be replaced by \( B_0 = e^{S^T} \) and the lifetime of such a \( |\Psi_0\rangle \) satisfies \( \tau_L \propto BM = e^{S^T} M \). Thus we may expect that \( S^T = \log B \) is straightforwardly connected with \( \tau_L \). In the short limit of the correlation time, the classical diffusion coefficient is approximated as \( D_\omega^{(cl)} \sim DC(\tau(0)) \), and then eq. (7) becomes

\[ \tau_L \sim \frac{2M}{\pi^2} e^{S^T}, \quad (8) \]
as is the case for the coupled Arnold’s cat maps. To confirm this conjecture numerically we executed the following procedures: A) Sort \( N = N^2 \) eigenfunctions in order of decreasing TE. B) The sorted eigenfunctions are grouped into \( N/M \) sets in descending order which are numbered as \( \ell = 1, 2, \ldots N/M \), where each of the sets contain \( M \) eigenfunctions. C) Superpose \( M \) eigenfunctions with almost the same TE to form the representative state \( |\Psi_0^{(\ell)}\rangle \) of \( \ell \)-th group and measure its \( \tau_L^{(\ell)} \) and compute the mean TE, say \( \tau_L^\ell \). D) Take the ensemble average of \( \tau_L^{(\ell)} \) and \( \tau_L^\ell \) over the potential parameter \( g_{R,R} \) (\( i = 1, 2 \)). We specifically take \( M = N_1 \). We show in fig. 3(b) the plots (\( \langle \tau_L^{(\ell)} \rangle \), \( \langle \tau_L^\ell \rangle \)) for several \( \epsilon \) in the transition regime. The plotted points are connected by lines to guide the eye. It is evident that the lifetime is almost proportional to \( e^{S^T} \).

On the other hand, as shown in fig. 3(a) the mean values of EE and TE are highly correlated. The question is whether or not the correlation exists even at the level of individual eigenstates. In fig. 3(c) we examined the correlation between EE and TE for an individual eigenstate. It is evident that there exists a clear positive correlation between EE and TE in each of the anomalously fluctuating sets with different values of \( \epsilon \). Combining figs. 3(b) and (c), we can claim that, through the transition rate entropy, the existence of a strong positive correlation between the entanglement entropy and the lifetime of irreversibility is evident at the level of individual eigenfunctions.

**Conclusion.** — We showed that, by taking the classically chaotic coupled quantum kicked rotors as examples, the lifetime of correspondence with classical decay of correlation, which indicates the maximal time scale on which the quantum time irreversibility is observable, can be measured even for individual eigenstates. The lifetime is strongly correlated with the degree of entanglement between the kicked rotors. The lifetime varies from eigenstate to eigenstate particularly in the threshold regime at the onset of full entanglement. But by introducing the transition rate entropy, we succeed in showing that the anomalous fluctuation of entropy has a definite positive correlation with the fluctuation of lifetime for the eigenstates.

The lifetime used here plays the role of an order parameter that will be useful for investigating the cooperative nature of the critical phenomena in the transition process through which time irreversible property is self-organized in bounded quantum systems, which has not been examined yet.

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