Exact solutions for a family of spin-boson systems

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Abstract
We obtain the exact solutions for a family of spin-boson systems. This is achieved through application of the representation theory for polynomial deformations of the $su(2)$ Lie algebra. We demonstrate that the family of Hamiltonians includes, as special cases, known physical models which are the two-site Bose–Hubbard model, the Lipkin–Meshkov–Glick model, the molecular asymmetric rigid rotor, the Tavis–Cummings model and a two-mode generalization of the Tavis–Cummings model.

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1. Introduction

The study of polynomial deformations of Lie algebras is an area of research which has found many applications in systems involving non-linear interactions [1–3]. In recent publications [4] we have formulated such methods for the analysis of a class of multi-boson systems. The approach of [4] is to express the Hamiltonian of the systems in terms of the generators of polynomial deformations of the $su(2)$ Lie algebra, through the explicit construction of Fock-space representations. By utilizing a correspondence between the Fock-space representations and differential operator realizations, it was shown that exact solutions are obtained in terms of a system of coupled equations. These equations can be viewed as providing a Bethe ansatz type of solution for the calculation of the energy spectrum and associated eigenstates. The generality of this approach allows for application on a wider level. The work described below is concerned with extending these methods to the study of a family of Hamiltonians which couple multi-boson degrees of freedom to a spin degree of freedom. In this manner we unify the problem of exactly solving spin-boson Hamiltonians to a particular class which contains within it a number of models which are already known in the literature, as we will discuss.

The main result of this paper is the derivation of the exact eigenfunctions and energy eigenvalues of the infinite family of spin-boson systems defined by the Hamiltonian

$$H \equiv \sum_{i=1}^{M} w_i N_i + g' J_0 + g \left( J_r a_1^{k_1} \cdots a_r^{k_r} + J'_r a_1^{k_1} \cdots a_r^{k_r} \right), \quad (1.1)$$
where throughout \( r, s \in \mathbb{Z}_+, M, k_1, \ldots, k_M \in \mathbb{N} \), \( a_i, a_i^\dagger \) and \( N_i = a_i^\dagger a_i \) are bosonic annihilation, creation and number operators, respectively, \( J_{\pm,0} \) are the generators for the \( su(2) \) spin algebra, and \( w_i, w_{ij}, g \) are real coupling constants. The Hamiltonian of the form (1.1) appears in the description of various physical systems of interest in atomic, molecular, nuclear and optical physics. We will explicitly demonstrate that (1.1) includes, as special cases, several known models which are the two-site Bose–Hubbard model [5], the Lipkin–Meshkov–Glick model [6], the molecular asymmetric rigid rotor [7], the many-atom Tavis–Cummings model [8, 9] and a two-mode generalized Tavis–Cummings model [10].

This paper is organized as follows. In section 2 we introduce new higher order polynomial algebras which are dynamical symmetry algebras of Hamiltonian (1.1), which enable an algebraization of the spin-boson systems. We construct finite-dimensional unitary representations of the dynamical symmetry algebras in section 3 and the corresponding single-variable differential operator realizations in section 4. This leads to the higher order differential operator realizations of the Hamiltonian (1.1). In section 5 we establish the quasi-exact solvability of this differential operator [11–13] and solve for the eigenvalue problem via the functional Bethe ansatz method (see, e.g. [14–16]). In section 6 we present explicit results for several special cases, thus providing a unified derivation of exact solutions to the widely studied models mentioned above. We summarize our results in section 7 and discuss further avenues for investigation.

2. Algebraization

In this section we introduce new higher order polynomial deformations of \( sl(2) \) and give an algebraization of the Hamiltonian (1.1). Our approach extends previous studies [1–4] where more restricted classes of systems have been exactly solved using polynomial algebra structures.

We introduce generators

\[
\mathcal{P}_+ = P_+ \prod_{i=1}^M Q(i)_+, \quad \mathcal{P}_- = P_- \prod_{i=1}^M Q(i)_-, \quad \mathcal{P}_0 = P_0 - \frac{\sum_{i=1}^M Q(i)_0}{M+1},
\]

where

\[
P_0 = \frac{J_0}{r}, \quad P_+ = J^+_r, \quad P_- = J^-_r,
\]

\[
Q(i)_+ = \frac{a_i^+_k}{\sqrt{k_i}}, \quad Q(i)_- = \frac{a_i^-_k}{\sqrt{k_i}}, \quad Q(i)_0 = \frac{1}{k_i} \left( a_i^+_k a_i^-_k + 1 \right)
\]

are \( M+1 \) mutually commuting operators. It can be shown that \( \mathcal{P}_{\pm,0} \) satisfy the following commutation relations:

\[
[\mathcal{P}_0, \mathcal{P}_\pm] = \pm \mathcal{P}_\pm,
\]

\[
[\mathcal{P}_+, \mathcal{P}_-] = \psi(2r) (\mathcal{K}, \mathcal{P}_0 - 1, \{\mathcal{C}\}) \prod_{i=1}^M \phi^{(k_i)} (\mathcal{K}, \mathcal{P}_0 - 1, \{\mathcal{L}\})
\]

\[
- \psi(2r) (\mathcal{K}, \mathcal{P}_0, \{\mathcal{C}\}) \prod_{i=1}^M \phi^{(k_i)} (\mathcal{K}, \mathcal{P}_0, \{\mathcal{L}\}),
\]

where \( C \) is the Casimir operator of \( su(2) \),

\[
\mathcal{K} = \frac{M P_0 + \sum_{i=1}^M Q(i)_0}{M+1}, \quad \mathcal{L}_i = Q(i)_0 - Q(i+1)_0, \quad i = 1 \cdots M - 1,
\]

(2.3)
Exact solutions for a family of spin-boson systems

are $M$ central elements of (2.2) and

\[
\psi^{(2r)}(K, P_0, C) = -\prod_{i=1}^{r} \left[ C - (rK + rP_0 + r - i + 1)(rK + rP_0 + r - i) \right],
\]

\[
\phi^{(k)}(K, P_0, L) = -\prod_{i=1}^{k} \left( \frac{K}{M} - (P_0 + 1) - \frac{1}{M} \sum_{\mu=1}^{M-1} \mu L_\mu + \sum_{\mu=i}^{M-1} L_\mu + \frac{ik_i - 1}{k_i^2} \right)
\]

are polynomial functions of degree $2r$ and $k_i$, respectively. Thus (2.2) defines a polynomial algebra of degree $2r + \sum_{i=1}^{M} k_i - 1$.

In terms of the generators of the polynomial algebra (2.2), the Hamiltonian (1.1) can be written as

\[
H = \sum_{i=1}^{M} w_i N_i + g' r^s (P_0 + C)^s + g \left[ \prod_{i=1}^{M} \left( \sqrt{k_i} \right)^{k_i} \right] (P_+ + P_-) \tag{2.4}
\]

with the number operators having the following expression in $P_0, K$ and $L_i$:

\[
N_i = k_i \left( -P_0 + \frac{K}{M} - \frac{1}{M} \sum_{\mu=1}^{M-1} \mu L_\mu - \sum_{\mu=i}^{M-1} L_\mu - \frac{1}{k_i} \right).
\]

It follows that the polynomial algebra (2.2) is the dynamical symmetry algebra of the Hamiltonian (1.1).

3. Unitary irreducible representations

Irreducible representations of the polynomial algebra (2.2) can be constructed in the tensor product space of the representation space of $P_{\pm,0}$ and the Fock spaces of $\{Q_{\pm,0}\}$. As shown in [4], the Fock states for irreducible representations of $\{Q_{\pm,0}\}$ are labelled by quantum numbers $q_i = \frac{1}{k^i}, \frac{k_i+1}{k^i}, \ldots, \frac{(k_i-1)k_i+1}{k^i}$, through

\[
|q_i, m_i\rangle = \frac{a_i^{(k_i(m_i+q_i-k_i^{-2})}}{\sqrt{k_i(m_i+q_i-k_i^{-2})}} |0\rangle, \quad m_i = 0, 1, \ldots. \tag{3.1}
\]

The action of $Q_{\pm,0}^{(i)}$ on these states is

\[
Q_0^{(i)}|q_i, m_i\rangle = (q_i + m_i)|q_i, m_i\rangle,
\]

\[
Q_+^{(i)}|q_i, m_i\rangle = \prod_{j=1}^{k_i} \left( m_i + q_i + \frac{jk_i - 1}{k_i^2} \right)^{1/2} |q_i, m_i + 1\rangle,
\]

\[
Q_-^{(i)}|q_i, m_i\rangle = \prod_{j=1}^{k_i} \left( m_i + q_i - \frac{(j-1)k_i + 1}{k_i^2} \right)^{1/2} |q_i, m_i - 1\rangle.
\]

The irreducible representations of $P_{\pm,0}$ can be deduced from the $su(2)$-module $V_j$, $j = 0, \frac{1}{2}, 1, \ldots$ as follows. First, it can be shown that $P_{0,\pm}$ satisfy the relations

\[
[P_0, P_\pm] = \pm P_\pm,
\]

\[
[P_+, P_-] = \psi^{(2r)}(P_0, C) - \psi^{(2r)}(P_0 - 1, C), \tag{3.2}
\]
where
\[ \psi^{(2r)}(P_0, C) = - \prod_{i=1}^{r} (C - (rP_0 + r - i + 1)(rP_0 + r - i)) \]  
(3.3)
is a polynomial in \( P_0 \) and \( C \) of degree \( 2r \). Thus (3.2) is a polynomial algebra of degree \( 2r - 1 \).
The Casimir operator of (3.2) takes fixed value \( \prod_{i=1}^{r}(C - i(i - 1)) \).

It is easily verified that there are \( \min\{r, 2j + 1\} \) lowest weight states,
\[ |j, 0; p\rangle \sim J^r_j|j, 0\rangle, \quad p = 0, 1, \ldots, \min\{r - 1, 2j\}, \]
where \(|j, 0\rangle\) is the lowest weight state of \( su(2) \). This implies that finite-dimensional irreducible representations of (3.2), denoted as \( V_{j,p} \), are labelled by quantum numbers \( j \) and \( p \), \( j = 0, \frac{1}{2}, 1, \ldots \), and \( p = 0, 1, \ldots, \min\{r - 1, 2j\} \). Thus we have the branching rule from \( su(2) \) representation \( V_j \) into \( V_{j,p} \) of (3.2):
\[ V_j = \bigoplus_{p=0}^{\min\{r-1,2j\}} V_{j,p}. \]

General basis vectors in the irreducible representation space \( V_{j,p} \) are given by \( |j, n; p\rangle \sim (P_n)^r|j, 0; p\rangle \). Explicitly,
\[ |j,n; p\rangle = \sqrt{\frac{(2j - p - rn)!}{(p + rn)! (2j)!}} P_p^n |j, 0\rangle. \]  
(3.4)
The action of \( P_{0,\pm} \) on these vectors is given by
\[ P_0|j, n; p\rangle = \left( \frac{p - j}{r} + n \right)|j, n; p\rangle, \]
\[ P_+|j, n; p\rangle = \prod_{i=1}^{r} \sqrt{(p + i + rn)(2j - p - i + 1 - rn)} |j, n + 1; p\rangle, \]
\[ P_-|j, n; p\rangle = \prod_{i=1}^{r} \sqrt{(p - i + 1 + rn)(2j - p + i - rn)} |j, n - 1; p\rangle. \]  
(3.5)
It can also be shown that
\[ P_-|j, 0; p\rangle = 0, \quad P_+|j, 2j - p - \lambda; p\rangle = 0, \]
where \( \lambda \) is a non-negative integer taking specific values \( \lambda = 0, 1, \ldots, \min\{r - 1, 2j\} \) according to \( j \) and \( p \). Moreover, \( \frac{2j - p - \lambda}{r} \) is always a non-negative integer. Therefore \( n = 0, 1, \ldots, \frac{2j - p - \lambda}{r} \), and (3.5) is a finite-dimensional representation of (3.2) with dimension \( \frac{2j - p - \lambda}{r} + 1 \).

We now construct irreducible representation of (2.2) in the tensor space \( V_{j,p} \otimes \mathcal{H}_{q_1}^{(1)} \cdots \otimes \mathcal{H}_{q_M}^{(M)} \), where \( V_{j,p} \) is the representation space of \( P_{\pm,0} \) and \( H_{q}^{(i)} \) is the Fock space of \( Q_{\pm,0}^{(i)} \). From (2.3) we have
\[ Q_0^{(i)} = Q_0^{(M)} + \sum_{\mu=1}^{M-1} \mathcal{E}_\mu, \quad (M + 1)K = MP_0 + MQ_0^{(M)} + \sum_{\mu=1}^{M-1} \mu \mathcal{E}_\mu. \]

This implies that for any irreducible representation of (2.2) defined by basis states \(|j, n; p\rangle \otimes \prod_{i=1}^{M} |q_i, m_i\rangle\),
\[ m_i = m_M + q_M - q_i + \sum_{\mu=1}^{M-1} l_\mu, \quad i = 1, \ldots, M - 1, \]
\[ n + m_M = \frac{M + 1}{M} - q_M - \frac{1}{M} \sum_{\mu=1}^{M-1} \mu l_\mu = \frac{p - j}{r}, \]
where $\kappa$ and $l_\mu$ denote the eigenvalues of central elements $K$ and $L_\mu$, respectively. It follows that $q_i \leq q_M + \sum_{\mu=i}^{M-1} l_\mu$ and
\[
q_i \leq q_M + \sum_{\mu=i}^{M-1} l_\mu - p - j - q_i.
\]
Clearly $A_i$ always take non-negative integer values, i.e. $A_i = 0, 1, \ldots$. Thus, the irreducible representation of (2.2) has basis states
\[
|j, n, p, \{q\}, \{l\}, \kappa\rangle \equiv |j, n, p, \{q\}, \{l\}, \kappa\rangle \otimes \prod_{i=1}^{M} |q_i, m_i\rangle
\]
where
\[
J^\mu_{p+rn}(j, 0) = \frac{\prod_{i=1}^{M} a_i^{\mu_k (A_i + q_i - k_i^2 - n)}}{\sqrt{[k_i (A_i + q_i - k_i^2 - n)]!}}.
\]

This gives an $N + 1$ dimensional representation of the polynomial algebra (2.2), where
\[
N = \begin{cases} 
\min \left\{ A_M, \frac{2j - p - \lambda}{r} \right\} & \text{for } M > 0, \\
2j - p - \lambda & \text{for } M = 0.
\end{cases}
\]

4. Differential operator realization

The finite-dimensional irreducible representations in the proceeding section can be realized by differential operators acting on $N + 1$-dimensional space of monomials with basis
\[ \{1, z, z^2, \ldots, z^N\} \text{ by mapping the basis vectors (3.6) into monomials in } z: \]

\[ |j, n, p, \{q\}, \{l\}, \kappa \rangle \rightarrow \frac{z^n}{\sqrt{(p + rn)! (2j - p - rn)! \prod_{i=1}^M k_i (A_i + q_i - 1_2 - n)!}} \]

The corresponding single-variable differential operator realization of (2.1) in the monomial space takes the following form

\[ P_0 = z \frac{d}{dz} - \kappa + \frac{p - j}{r}, \]

\[ P_+ = z \prod_{i=1}^r \left( 2j - p - i + 1 - rz \frac{d}{dz} \right) \times \prod_{i=1}^M \prod_{i=1}^{k_i} \sqrt{k_i} \left( A_i + q_i - \frac{(v - 1)k_i + 1}{k_i^2} - \frac{z}{d} \right), \]

\[ P_- = \prod_{\mu=1}^M \sqrt{k_\mu} \prod_{i=1}^r \left( rz \frac{d}{dz} + p - i + 1 \right). \]

Note that \( P_- \) contains no singularities as \( \prod_{i=1}^r (p - i + 1) = 0 \) for all allowed \( p \) values.

We can thus equivalently represent Hamiltonian (2.4) as the single-variable differential operator of order \( M \equiv \max \{r + \sum_{i=1}^M k_i, s\}, \)

\[ H = \sum_{i=1}^M w_i N_i + g^r \left( rz \frac{d}{dz} - j + p \right)^s + g^{-1} \prod_{i=1}^r \left( rz \frac{d}{dz} + p - i + 1 \right) + g \prod_{i=1}^r \left( 2j - p - i + 1 - rz \frac{d}{dz} \right) \prod_{i=1}^M \prod_{i=1}^{k_i} \left( A_i + q_i - \frac{(v - 1)k_i + 1}{k_i^2} - \frac{z}{d} \right) \]

with

\[ N_i = k_i \left( -z \frac{d}{dz} + \frac{M + 1}{M} - \kappa + \frac{p - j}{r} + \sum_{\mu=i}^{M-1} \frac{l_\mu}{M} - \frac{1}{M} \sum_{\mu=1}^{M-1} \mu l_\mu \right) - \frac{1}{k_i}. \]

5. Exact solutions

We will now solve for the Hamiltonian equation

\[ H \psi(z) = E \psi(z) \]

for the differential operator realizations by using the functional Bethe ansatz method [14–16], where \( \psi(z) \) is the eigenfunction and \( E \) is the corresponding eigenvalue. It is straightforward to verify

\[ H z^n = z^{n+1} g \prod_{i=1}^r \left( 2j - p - i + 1 - rz \right) \prod_{i=1}^M \prod_{i=1}^{k_i} \left( A_i + q_i - n - \frac{(v - 1)k_i + 1}{k_i^2} \right) + \text{lower order terms}, \quad n \in \mathbb{Z}_+. \]

This means that the differential operator (4.2) is not exactly solvable. However, it is quasi-exactly solvable, since when \( n = N \) the first term (\( \sim z^{n+1} \)) on the rhs of (5.2) is vanishing. That is \( H \) preserves an invariant polynomial subspace of degree \( N \),

\[ H \mathcal{V} \subseteq \mathcal{V}, \quad \mathcal{V} = \text{span}\{1, z, \ldots, z^N\} \]
Thus up to an overall factor, the eigenfunctions of (4.2) have the form

$$\psi(z) = \prod_{i=1}^{N} (z - \alpha_i), \quad (5.4)$$

where \{\alpha_i | i = 1, 2, \ldots, N\} are roots of the polynomial which will be specified by the associated Bethe ansatz equations (5.7) below. We can rewrite the Hamiltonian (4.2) as

$$H = \sum_{i=1}^{M} P_i(z) \left( \frac{d}{dz} \right)^i + P_0(z), \quad (5.5)$$

where \(P_0(z)\) and \(P_i(z)\) are polynomials in \(z\) determined from the expansion of the products in (4.2).

Dividing the Hamiltonian equation \(H\psi = E\psi\) by \(\psi\) gives us

$$E = \frac{H\psi}{\psi} = \sum_{i=1}^{M} P_i(z) \frac{i!}{\prod_{n_1 < n_2 < \ldots < n_i} (z - \alpha_{n_1}) \cdots (z - \alpha_{n_i})} + P_0(z). \quad (5.6)$$

The lhs of (5.6) is a constant, while the rhs is a meromorphic function in \(z\) with at most simple poles. To be equal, we need to eliminate all singularities on the rhs of (5.6). We may achieve this by demanding that the residues of the simple poles, \(z = \alpha_i, i = 1, 2, \ldots, N\) should all vanish. This leads to the Bethe ansatz equations for the roots \{\alpha_i\}:

$$\sum_{i=2}^{M} \sum_{\mu=1}^{N} \frac{P_i(\alpha_{\mu}) i!}{(\alpha_{\mu} - \alpha_{n_1}) \cdots (\alpha_{\mu} - \alpha_{n_{\mu-1}})} + P_1(\alpha_{\mu}) = 0, \quad \mu = 1, 2, \ldots, N. \quad (5.7)$$

The wavefunction \(\psi(z)\) (5.4) becomes the eigenfunction of \(H\) (4.2) in the space \(\mathcal{V}\) provided that the roots \{\alpha_i\} of the polynomial \(\psi(z)\) (5.4) are the solutions of (5.7).

Let us remark that the Bethe ansatz equation (5.7) is the necessary and sufficient condition for the rhs of (5.6) to be independent of \(z\). This is because when (5.7) is satisfied the rhs of (5.6) is analytic everywhere in the complex plane (including points at infinity) and thus must be a constant by Liouville’s theorem.

To obtain the corresponding eigenvalue \(E\), we consider the leading order expansion of \(\psi(z)\),

$$\psi(z) = z^N - z^{N-1} \sum_{i=1}^{N} \alpha_i + \cdots.$$ 

It can be directly shown that the \(\mathcal{P}_{\pm,0}\psi(z)\) have the expansions

$$\mathcal{P}_+ \psi = -z^N g \left( \prod_{j=1}^{r} (2j - p - i + 1 - r(N - 1)) \right) \times \prod_{i=1}^{M} \prod_{v=1}^{k_i} \sqrt{k_i} \left( A_i + q_i - N + 1 - \frac{(v-1)k_i + 1}{k_i^2} \right) \sum_{i=1}^{N} \alpha_i + \cdots,$$

$$\mathcal{P}_- \psi \sim z^{N-1} + \cdots,$$

$$\mathcal{P}_0 \psi = z^N \left( N + \frac{P - j}{r} - \kappa \right) + \cdots.$$
Substituting these expressions into the Hamiltonian equation (5.1) and equating the \( z^N \) terms, we arrive at
\[
E = \sum_{i=1}^{M} w_i \left( k_i \left( \frac{M+1}{M} \kappa \frac{p-j}{r} N - \frac{1}{M} \sum_{\mu=1}^{M-1} l_{\mu} + \sum_{\mu=1}^{M-1} \frac{l_{\mu}}{k_i} \right) \right.
\]
\[
+ g' (r, N - j + p)^i - g \left( \prod_{i=1}^{N} (2j - p - i + 1 - r (N - 1)) \right)
\]
\[
\times \prod_{i=1}^{M} \prod_{\nu=1}^{k_i} \left( A_i + q_i - N + 1 - \frac{(v - 1)k_i}{k_i^2} \right) \right) \sum_{i=1}^{N} \alpha_i,
\]
\[
(5.8)
\]
where \( \{\alpha_i\} \) satisfy the Bethe ansatz equations (5.7). This gives the eigenvalue of the Hamiltonian (1.1) with the corresponding eigenfunction \( \psi(z) \) (5.4).

6. Explicit examples

In this section we give explicit results on the Bethe ansatz equations and energy eigenvalues of the Hamiltonian (1.1) for special cases which correspond to some established models frequently studied in the field of atomic and molecular physics, condensed matter, nuclear physics and quantum optics.

6.1. Two-site Bose–Hubbard model

This model corresponds to the special case with \( M = 0, r = 1, s = 2 \) and its Hamiltonian takes the simple form
\[
H = g' J_0^2 + g (J_+ + J_-).
\]
\[
(6.1)
\]
This model has been widely employed in the context of Josephson-coupled Bose–Einstein condensates via the realization of \( J_{\pm,0} \) in terms of two bosons, \( J_+ = b_1^\dagger b_2, J_- = b_1 b_2^\dagger, J_0 = \frac{1}{2} (b_1^\dagger b_1 - b_2^\dagger b_2) \) (see, e.g. [5] and references therein). Exact solutions of the model in terms of algebraic Bethe ansatz methods were first studied in [17]. From the general results in the preceding section, in this case we have \( \kappa = 0, p = 0, \) and \( N = 2j \). Thus (6.1) takes the form
\[
H = P_2(z) \frac{d^2}{dz^2} + P_1(z) \frac{d}{dz} + P_0(z),
\]
where
\[
P_2(z) = g' z^2,
\]
\[
P_1(z) = g' z(1 - 2j) + g(1 + z^2),
\]
\[
P_0(z) = g' j^2 - 2jzg.
\]
The Bethe ansatz equations are given by
\[
\sum_{i \neq \mu}^{2j} \frac{2}{\alpha_i - \alpha_\mu} = -\frac{\alpha_\mu g' (1 - 2j) + g(1 + \alpha_\mu^2)}{g' \alpha_\mu^2}, \quad \mu = 1, 2, \ldots, 2j
\]
and the energy eigenvalues are
\[
E = g' j^2 - g \sum_{i=1}^{2j} \alpha_i.
\]
This exact solution is equivalent to a case described in [18].
6.2. Lipkin–Meshkov–Glick model

This model is the special case corresponding to $M = 0$, $r = 2$, $s = 1$. The Hamiltonian is given by [6]

$$H = g'J_0 + g \left( J_x^2 + J_y^2 \right)$$

and continues to be studied extensively (see e.g. [19] and references therein). Exact solution via the algebraic Bethe ansatz method is discussed in [20, 21]. Specializing the general results of the preceding section to this case, we have $p = 0$, $1$, $\kappa = 0$ and $N = \frac{2j - p - \lambda}{2}$ with $\lambda = 0$, $1$ so that $N$ is a non-negative integer. The differential operator representation of the Hamiltonian (6.2) is thus

$$H = P_2(z) \frac{d^2}{dz^2} + P_1(z) \frac{d}{dz} + P_0(z)$$

where

$$P_2(z) = 4gz^3 + 4g z, \quad P_1(z) = g(6 + 4p - 8j)z^2 + 2g'z + g(2 + 4p), \quad P_0(z) = gz(2j - p)(2j - p - 1) + g'(p - j).$$

The Bethe ansatz equations are given by

$$\sum_{i \neq \mu}^{2j - p - 1} \frac{2}{\alpha_i - \alpha_\mu} = -\frac{g(3 + 2p - 4)\alpha_\mu^2 + g'\alpha_\mu + g(1 + 2p)}{2g(\alpha_\mu^3 + \alpha_\mu)}, \quad \mu = 1, 2, \ldots, \frac{2j - p - \lambda}{2}$$

and the energy eigenvalues are

$$E = g'(j - \lambda) - g(\lambda + 1)(\lambda + 2) \sum_{i = 1}^{2j - p - 1} \alpha_i.$$
Finally, we consider the case when $M = 2$. Two-mode generalized Tavis–Cummings model

The Bethe ansatz equations read

$$P_2(z) = (a - b)z^3 + 2(2c - a - b)z^2 + (a - b)z,$$

$$P_1(z) = \frac{a - b}{2}(3 + 2p - 4j)z^2 + 2(2c - a - b)(1 + p - j)z + \frac{a - b}{2}(1 + 2p),$$

$$P_0(z) = \frac{a - b}{4}(2j - p)(2j - p - 1)z + \frac{2c - a - b}{2}(p - j)^2 + \frac{a + b}{2}j(j + 1).$$

The Bethe ansatz equations are given by

$$\sum_{i \neq \mu} \frac{2}{\alpha_i - \alpha_\mu} = \frac{(a - b)(3 + 2p - 4j)\alpha_i^2 + 4(2c - a - b)(1 + p - j)\alpha_i + (a - b)(1 + 2p)}{2(a - b)(\alpha_\mu^3 + \alpha_\mu) + 4(2c - a - b)\alpha_\mu^2},$$

$$\mu = 1, 2, \ldots, 2j - p - \lambda.$$

and the energy eigenvalues are

$$E = \frac{2c - a - b}{2}(j - \lambda)^2 + \frac{a + b}{2}j(j + 1) - \frac{a - b}{4}(\lambda + 1)(\lambda + 2) \sum_{i=1}^{2j - p - \lambda} \alpha_i.$$

6.4. Tavis–Cummings model

This model corresponds to the special case when $M = r = s = k_1 = 1$. The Hamiltonian is given by

$$H = w_1N_1 + g'J_0 + g \left(J_+a_1 + J_-a_1^\dagger\right).$$

This is one of the widely studied models in quantum optics and had been exactly solved via the algebraic Bethe ansatz approach [23, 24]. Applying the results in the preceding section gives $q_1 = 1$, $p = 0$ and $N = \min\{2k + j - 1, 2j\}$. The differential operator representation of the Hamiltonian is

$$H = P_2(z) \frac{d^2}{dz^2} + P_1(z) \frac{d}{dz} + P_0(z),$$

where

$$P_2(z) = g'z^3,$$

$$P_1(z) = -g(3j + 2k - 2)z^2 + (g' - w_1)z + g,$$

$$P_0(z) = w_1(2k + j - 1) - g'j + 2gjz(2k + j - 1).$$

The Bethe ansatz equations read

$$\sum_{i \neq \mu} \frac{2}{\alpha_i - \alpha_\mu} = \frac{g(3j + 2k - 2)\alpha_i^2 + (g' - w)\alpha_i - g}{g\alpha_\mu^2},$$

$$\mu = 1, 2, \ldots, N$$

and the energy eigenvalues are

$$E = w_1(2k + j - N - 1) + g'(N - j) - g(2j - N + 1)(2k + j - N) \sum_{i=1}^{N} \alpha_i,$$

where $N = \min\{2k + j - 1, 2j\}.$

6.5. Two-mode generalized Tavis–Cummings model

Finally, we consider the case when $M = 2$, $r = s = k_1 = k_2 = 1$. This gives the Hamiltonian

$$H = w_1N_1 + w_2N_2 + g'J_0 + g \left(J_-a_1^\dagger a_2 + J_+a_1a_2\right),$$
which belongs to the class of $su(1, 1)$ generalized Tavis–Cummings model discussed in [10]. Applying the results in the preceding section gives $q_1 = q_2 = 1$, $p = 0$ and $\mathcal{N} = \min\{\frac{3j}{2} - 1 + j, 2j\}$. The differential operator representation of the Hamiltonian thus reads

$$H = P_3(z) \frac{d^3}{dz^3} + P_2(z) \frac{d^2}{dz^2} + P_1(z) \frac{d}{dz} + P_0(z),$$

where

$$P_3(z) = -g z^4,$$
$$P_2(z) = g(3\kappa + 4j - 5)z^3,$$
$$P_1(z) = Az^2 + (g' - w_1 - w_2)z + g,$$
$$P_0(z) = zB + F,$$

with

$$A = g \left( -9j\kappa + 10j + 6\kappa + \frac{l_1^2}{4} - 5j^2 - 4 - \frac{9}{4}\kappa^2 \right),$$
$$B = g \left( \frac{9j\kappa}{2} + 6j^2\kappa - 6\kappa + 2j - \frac{jl_1^2}{2} + 2j^3 - 4j^2 \right),$$
$$F = (w_1 + w_2) \left( \frac{3\kappa}{2} - 1 + j \right) + \frac{l_1}{2}(w_1 - w_2) - g'j.$$

The Bethe ansatz equations assume the form

$$\sum_{\mu < \nu \neq \beta}^N \frac{6g\alpha^2_\beta}{(\alpha_\beta - \alpha_\mu)(\alpha_\beta - \alpha_\nu)} - \sum_{i \neq \beta}^N \frac{2g(3\kappa + 4j - 5)\alpha^2_i}{\alpha_\beta - \alpha_i} = A\alpha^2_\beta + (g' - w_1 - w_2)\alpha_\beta + g,$$

$$\beta = 1, 2, \ldots, \mathcal{N},$$

and the energy eigenvalues are

$$E = (w_1 + w_2) \left( \frac{3\kappa}{2} - 1 + j - \mathcal{N} \right) + \frac{l_1}{2}(w_1 - w_2) + g'N - j$$
$$- g(2j - \mathcal{N} + 1) \left[ \left( \frac{3\kappa}{2} + j - \mathcal{N} \right)^2 - \frac{l_1^2}{4} \right] \sum_{i=1}^N \alpha_i,$$

where $\mathcal{N} = \min\{\frac{3j}{2} - 1 + j, 2j\}.$

7. Discussions

We have derived the exact solutions of a family of Hamiltonians with the following general form

$$H = F(Q_0) + g(Q_+ + Q_-)$$

(7.1)

whereby $Q_{\pm,0}$ are particular polynomial deformations of the $sl(2)$ Lie algebra and $F(Q_0)$ is a polynomial function of $Q_0$ with real coefficients. We have seen that via the differential operator realization of these algebras, the block diagonal sectors of the Hamiltonians can be realized as higher order quasi-exactly solvable differential operators. The eigenvalues of the Hamiltonians in these sectors have been obtained via the functional Bethe ansatz approach.

Specific cases of the general Hamiltonian have previously been solved via the algebraic Bethe ansatz approach [17, 20–24] as mentioned earlier. Comparing both methods, it appears
that the functional Bethe ansatz approach has some advantage over the algebraic Bethe ansatz by requiring less algebraic machinery. This advantage manifests itself in the fact that we have been able to give a unified solution for (1.1) through (5.7) and (5.8). Such a unified solution presently appears beyond the limits of algebraic Bethe ansatz approaches which treat the models on a case-by-case basis. It would therefore be interesting to see whether other classes of exactly solvable models can be easily handled by the functional Bethe ansatz approach.

One avenue for further work would be to generalize the functional Bethe ansatz approach to solve for other classes of Hamiltonians, such as $q$-deformed versions of the models discussed above. It would also be worthwhile to explore the role of polynomial algebra structures in connections between higher order ODEs and integrable models, i.e. the ODE/IM correspondence [25].

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