Quick Streaming Algorithms for Cardinality-Constrained Maximization of Non-Monotone Submodular Functions in Linear Time

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Abstract

For the problem of maximizing a nonnegative, (not necessarily mono-
tone) submodular function with respect to a cardinality constraint, we
propose deterministic algorithms with linear time complexity; these are
the first algorithms to obtain constant approximation ratio with high prob-
ability in linear time. Our first algorithm is a single-pass streaming algo-
rithm that obtains ratio \( 9.298 + \varepsilon \) and makes only two queries per received
element. Our second algorithm is a multi-pass streaming algorithm that
obtains ratio \( 4 + \varepsilon \). Empirically, the algorithms are validated to use fewer
queries than and to obtain comparable objective values to state-of-the-art
algorithms.

1 Introduction

A nonnegative, set function \( f : 2^U \to \mathbb{R}^+ \), where ground set \( U \) is of size \( n \), is sub-
modular if for all \( S \subseteq T \subseteq U, u \in U \setminus T, f(T \cup \{u\}) - f(T) \leq f(S \cup \{u\}) - f(S) \).
Submodular objective functions arise in many learning objectives, e.g. inter-
preting neural networks [9], nonlinear sparse regression [10]. Some applications
yield submodular functions that are not monotone (a set function is monotone
if \( A \subseteq B \) implies \( f(A) \leq f(B) \)): for example, image summarization with di-
versity [24], MAP Inference for Determinantal Point Processes [14], or revenue
maximization on a social network [17]. In these applications, the task is to
optimize a submodular function \( f \) subject to a variety of constraints. In this
work, we study the problem of submodular maximization with respect to a car-
dinality constraint (SMCC): i.e. given submodular \( f \) and integer \( k \), determine
\( \arg \max_{|S| \leq k} f(S) \). We consider the value query model, in which the function \( f \)
is available to an algorithm as an oracle that returns, in a single operation, the
value \( f(S) \) of any queried set \( S \).

Because of the recent phenomenon of big data, in which data size has ex-
hibited exponential growth [26, 24], there has been substantial effort into the
Table 1: The symbol * indicates that the ratio holds only in expectation; to obtain its ratio with high probability, such an algorithm must be independently repeated. The second-to-last column gives the queries per element per pass (QpEpP) of each streaming algorithm. The algorithm \( \mathcal{A} \) is an offline algorithm for SMCC with ratio \( \alpha \) and time complexity \( T(\mathcal{A}, m) \) on input of size \( m \). The notation \( O_\varepsilon \) indicates that the constants dependent on accuracy parameter \( \varepsilon \) have been suppressed.

| Reference | Ratio | Time | Passes | QpEpP | Memory |
|-----------|-------|------|--------|-------|--------|
| [8]       | 9     | \( O(kn) \) | 1      | \( O(k) \) | \( O(k) \) |
| [12]      | 5.828* | \( O(kn) \) | 1      | \( O(k) \) | \( O(k) \) |
| [15]      | \( 2 + \alpha + \varepsilon \) \( O_\varepsilon(n \log k + T(\mathcal{A}, k)) \) | 1      | \( O_\varepsilon(\log k) \) | \( O_\varepsilon(k) \) |
| [11]      | \( 1 + \alpha + \varepsilon \) \( O_\varepsilon(n \log k + T(\mathcal{A}, k)) \) | 1      | \( O_\varepsilon(\log k) \) | \( O_\varepsilon(k) \) |
| [6]       | \( e^* \) | \( O(n) \) | N/A    | N/A   | \( O(n) \) |
| [18]      | \( 4 + \varepsilon \) \( O_\varepsilon(n \log k) \) | \( O_\varepsilon(\log k) \) | 1      | \( O_\varepsilon(k \log(k)) \) |

This paper 9.298 + \( \varepsilon \) \( O(n) \) | 1 | 2 | \( O_\varepsilon(k \log(k)) \) |
This paper \( 4 + \varepsilon \) \( O_\varepsilon(n) \) | \( O_\varepsilon(1) \) | 1 | \( O_\varepsilon(k \log(k)) \) |

design of algorithms for SMCC with efficient time complexity\(^1\), e.g. Badanidiyuru and Vondrák \cite{2}, Mirzasoleiman et al. \cite{23}, Fahrbach et al. \cite{11}. An algorithm introduced by Buchbinder et al. \cite{6} obtains an expected ratio of \( e \) in \( O(n) \) time, which is close to the best known factor in polynomial time of 2.597 \cite{4}. However, an algorithm with an expected ratio may produce a poor solution with constant probability unless \( O(\log n) \) independent repetitions are performed. Moreover, the derandomization of algorithms for submodular optimization has proven difficult; a method to derandomize some algorithms at the cost of a polynomial increase in time complexity was recently given by Buchbinder and Feldman \cite{5}. No algorithm in prior literature has been shown to obtain a constant approximation ratio with high probability in linear time.

In addition to fast algorithms, much work has been done on streaming algorithms for SMCC, e.g. Badanidiyuru et al. \cite{3}, Chakrabarti and Kale \cite{7}, Mirzasoleiman et al. \cite{25}; a streaming algorithm makes one or more passes through the ground set while using a small amount of memory, i.e. \( O(k \log^\ell(n)) \) space, for some constant \( \ell \). No streaming algorithm has been proposed in prior literature with linear time complexity.

**Contributions** We provide two deterministic algorithms that achieve a constant approximation factor in linear time. These are the 1) first algorithms

\(^1\)In addition to the time complexity of an algorithm, we also discuss the number of oracle queries an algorithm makes or the query complexity; this information is important as the function evaluation may be expensive.
that achieve a constant ratio with high probability for SMCC in linear time; 2) first deterministic algorithms to require a linear number of oracle queries; and 3) the first linear-time streaming algorithms for SMCC, including streaming algorithms that obtain a ratio in expectation only. Our first algorithm, QUICKSTREAM, is a single-pass algorithm that obtains ratio \( \leq 9.298 + \varepsilon \) in \( 2n + 1 \) oracle queries and \( O(n) \) time. Our second algorithm, MULTIPLSELINEAR, is a multi-pass algorithm that obtains a ratio of \( 4 + \varepsilon \) in \( O \left( \frac{n}{\varepsilon \log \left( \frac{1}{\varepsilon} \right)} \right) \) time. Table 1 shows how our algorithms compare theoretically to state-of-the-art algorithms designed for SMCC.

To obtain our single-pass algorithm, we generalize the linear-time streaming method of Kuhnle [19] to non-monotone objectives. To generalize, we follow a strategy of maintaining two disjoint candidate sets that compete for incoming elements. Variations of this general strategy have been employed for non-monotone objectives in many recent algorithms, e.g. Kuhnle [18], Alaluf et al. [1], Feldman et al. [13], Haba et al. [14], Han et al. [16]. At a high level, one novel component of our work is that our disjoint candidate sets are infeasible and may lose elements, a complicating factor that requires careful analysis and to the best of our knowledge is novel to our algorithm. To obtain our multi-pass algorithm, we use the constant factor from our single-pass algorithm to modify and speed up the greedy-based \((4 + \varepsilon)\)-approximation of Kuhnle [18] that requires \( O_e(n \log k) \) time.

Finally, an empirical validation shows improvement in query complexity and solution value of both our single-pass algorithm and our multi-pass algorithm over the current state-of-the-art algorithms on two applications of SMCC.

1.1 Related Work

The field of submodular optimization is too broad to provide a comprehensive survey. In this section, we focus on the most relevant works to ours.

Kuhnle [19] recently introduced a deterministic, single-pass streaming algorithm for SMCC with monotone objectives that obtains a ratio of 4 in \( O(n) \) time. Our single-pass algorithm may be interpreted as an extension of this approach to non-monotone submodular functions, which requires significant changes to the algorithm and analysis: 1) To control the non-monotonicity, we introduce two disjoint sets \( A \) and \( B \) that compete for incoming elements. 2) Part of our analysis (Lemma 2) includes bounding the loss from set \( A \) (resp. \( B \)), in terms of gains to \( B \) (resp. \( A \)). This bound is unnecessary in Kuhnle [19] and is complicated by the periodic deletions from \( A.B \). 3) The condition on the marginal gain to add an element is relaxed through a parameter \( b \), which we use to optimize the ratio (at \( b = 1.49 \)).

Alaluf et al. [1] introduced a deterministic, single-pass streaming algorithm that obtains the state-of-the-art ratio \( 1 + \alpha + \varepsilon \), where \( \alpha \) is the ratio of an offline post-processing algorithm \( A \) for SMCC with time complexity \( T(A,m) \) on an input of size \( m \). The algorithm of Alaluf et al. [1] uses \( \lceil 1/\varepsilon \rceil \) disjoint, candidate solutions, all of which are feasible solutions. The empirical solution quality of our algorithm may be improved by using similar post-processing, as we evaluate
in Section 4. The time complexity of their algorithm is \( O((\log(k/\varepsilon)/\varepsilon) \cdot (n/\varepsilon + T(A, k/\varepsilon))) \) and requires \( O((\log(k/\varepsilon)/\varepsilon^2)) \) queries per incoming element, whereas our algorithm requires \( O(n) \) time, two queries per incoming element, and one additional query after the stream terminates.

Haba et al. [15] recently provided a general framework to convert a streaming algorithm for monotone, submodular functions to a streaming algorithm for general submodular functions, as long as the original algorithm satisfies certain conditions. The results in Table 1 are an application of this framework with the streaming algorithm of Badanidiyuru et al. [3], which is a 2-approximation if the objective is monotone and submodular. The method of Haba et al. [15] requires an offline algorithm \( A \) for SMCC with ratio \( \alpha \). Unfortunately, this framework cannot be directly applied to the algorithm of Kuhnle [19] to obtain a linear-time streaming algorithm for SMCC because \( O_{\varepsilon}(\log k) \) guesses for \( \text{OPT} \) are required in the conversion, which would increase the time complexity by a log factor. Further discussion is provided in Appendix A.

Feldman et al. [12] provided a randomized, single-pass algorithm that achieves ratio 5.828 in expectation. This algorithm requires time \( O(kn) \) and makes \( O(k) \) oracle queries per element received. Further, the algorithm would have to be repeated to ensure the ratio holds with high probability. Our single-pass algorithm is deterministic and requires \( O(n) \) time.

Kuhnle [18] presented a deterministic algorithm that achieves ratio \( 4 + \varepsilon \) in \( O(n \log k) \) time; this is the fastest deterministic algorithm in previous literature. Feldman et al. [13] and Han et al. [16] independently extended this algorithm to handle more general constraints with the same time complexity. Our multipass algorithm is based upon these algorithms; our algorithm uses the constant factor from our single-pass algorithm to speed up the basic approach and obtain ratio \( 4 + \varepsilon \) in \( O(n) \) time.

**Preliminaries** An alternative characterization of submodularity is the following: \( f \) is submodular iff. \( \forall A, B \subseteq \mathcal{U}, f(A) + f(B) \geq f(A \cup B) + f(A \cap B) \). We use the following notation of the marginal gain of adding \( T \subseteq \mathcal{U} \) to set \( S \subseteq \mathcal{U} \): \( \delta_T(S) = f(S \cup T) - f(S) \). For element \( x \in \mathcal{U} \), \( \delta_x(S) = \delta_{\{x\}}(S) \). Throughout the paper, \( f \) is a set function defined on subsets of \( \mathcal{U} \), and \( |\mathcal{U}| = n \). Technically, all algorithms described in this work as streaming algorithms are semi-streaming algorithms: since the solution size \( k \) may be \( \Omega(n) \), it may take linear space to even store a solution to the problem. Omitted proofs may be found in the Appendices, all of which are located in the Supplementary Material.

## 2 Single-Pass Streaming Algorithm for SMCC

In this section, a linear-time, constant-factor algorithm is described. This algorithm (**QuickStream**, Alg. 1) is a deterministic streaming algorithm that makes one pass through the ground set and two queries to \( f \) per element received.
Algorithm 1 A single-pass algorithm for SMCC.

1: procedure QuickStream\((f, k, \varepsilon, b)\)
2: \textbf{Input:} oracle \(f\), cardinality constraint \(k\), \(\varepsilon > 0\), \(b > 0\)
3: \(\beta \leftarrow \frac{1}{(1 - (1 + b/k)^{-k})}\)
4: \(\ell \leftarrow \lceil \log((6\beta)/\varepsilon + 1) \rceil + 3\)
5: \(A \leftarrow \emptyset, B \leftarrow \emptyset\)
6: \textbf{for} element \(e\) received \textbf{do}
7: \(S \leftarrow \arg \max \{\delta_e(A), \delta_e(B)\}\)
8: \textbf{if} \(\delta_e(S) \geq bf(S)/k\) \textbf{then}
9: \(S \leftarrow S \cup \{e\}\)
10: \textbf{if} \(|S| > 2\ell(k/b + 1) \log_2(k)\) \textbf{then}
11: \(S \leftarrow \{\ell(k/b + 1) \log_2(k)\} \text{ elements most recently added to } S\}
12: \(A' \leftarrow \{k \text{ elements most recently added to } A\}\)
13: \(B' \leftarrow \{k \text{ elements most recently added to } B\}\)
14: \textbf{return} \(S' \leftarrow \arg \max \{f(A'), f(B')\}\)

2.1 Description of Algorithm

As input, the algorithm receives the value oracle for \(f\), the cardinality constraint \(k\), an accuracy parameter \(\varepsilon > 0\), and a threshold parameter \(b > 1\). Two disjoint sets \(A\) and \(B\) are maintained throughout the execution of the algorithm. Elements of the ground set are processed one-by-one in an arbitrary order by the \textbf{for} loop on Line 6. Upon receipt of element \(e\), if there exists \(C \in \{A, B\}\) such that \(\delta_e(C) > bf(C)/k\), \(e\) is added to the set to which it has a higher marginal gain; otherwise, the element \(e\) is discarded. The maximum size of \(C \in \{A, B\}\) is controlled by checking if the size of \(C\) exceeds an upper bound. If it does, the size of \(C\) is reduced by a factor of 2 by keeping only the latter elements added to \(C\). Finally, at the termination of the stream, let \(A'\) and \(B'\) be the last \(k\) elements added to \(A\) and \(B\), respectively. Of these two sets, the one with the larger \(f\) value is returned.

2.2 Theoretical Guarantees

In this section, we prove the following theorem concerning the performance of QuickStream (Alg. 1).

**Theorem 1.** Let \(\varepsilon, b \geq 0\), and let \((f, k)\) be an instance of SMCC. The solution \(S'\) returned by QuickStream\((f, k, \varepsilon, b)\) satisfies

\[
\text{OPT} \leq \left( \frac{(2b + 4)}{1 - (1 + b/k)^{-k}} + \varepsilon \right) f(S').
\]

Further, QuickStream makes \(2n + 2\) queries to the value oracle for \(f\), has time complexity \(O(n)\), memory complexity \(O(k \log(k) \log(1/\varepsilon))\), and makes one pass over the ground set.
The ratio decreases (improves) monotonically with increasing \( k \). It is trivial to obtain a \((1/9)\)-approximation for \( k \leq 9 \) by remembering the best singleton, so plugging \( k = 10 \) and optimizing over \( b \) into the ratio Theorem 1 yields worst-case ratio of \( \approx 9.298 + \varepsilon \) at \( b \approx 1.49 \). In the limit as \( k \to \infty \), the ratio converges to \( (2b + 4)e^b) / (e^b - 1) + \varepsilon \), which is optimized at \( b \approx 1.505 \) to yield ratio \( \leq 9.02 + \varepsilon \).

**Overview of Proof** The proof can be summarized as follows. If no deletions occurred on Line 14 then \( f(O \cup C) - f(C) \leq O(f(A) + f(B)) \), for \( C \in \{A,B\} \). Once this is established (Lemmas 2 and 3), a bound (Inequality 16) on ratio of \( \geq \frac{f(O \cup A) + f(O \cup B)}{f(O)} \), which is a consequence of the submodularity of \( f \) and the fact that \( A \cap B = \emptyset \).

Moreover, a large fraction of the value of \( f(A) \) (resp. \( f(B) \)) is concentrated in the last \( k \) elements added to \( A \) (resp. \( B \)), as shown in Lemma 4. Finally, Lemma 1 shows that only a small amount of value is lost due to deletions, although this error creates considerable complications in the proof of Lemma 2.

**Proof of Theorem 1** The time, query, and memory complexities of QuickStream follow directly from inspection of the pseudocode. The rest of the proof is devoted to proving the approximation ratio.

The first lemma (Lemma 1) establishes basic facts about the growth of the value in the sets \( A \) and \( B \) as elements are received. Lemma 1 considers a general sequence of elements that satisfy the same conditions on addition and deletion as elements of \( A \) or \( B \), respectively. The proof is deferred to Appendix B and depends on a condition to add elements and uses submodularity of \( f \) to bound the loss in value due to periodic deletions.

**Lemma 1.** Let \( (c_0, \ldots, c_{m-1}) \) be a sequence of elements, and \( (C_0, \ldots, C_m) \) a sequence of sets, such that \( C_0 = \emptyset \), and \( C_i^+ = C_i \cup \{c_i\} \) satisfies \( f(C_i^+) \geq (1 + b/k)f(C_i) \), and \( C_{i+1} = C_i^+ \setminus C_j \) where \( j = \ell(k/b + 1) \log_2(k) \), in which case \( C_{i+1} = C_i^+ \setminus C_j \), where \( j = \ell(k/b + 1) \log_2(k) \). Then

1. \( f(C_{i+1}) \geq f(C_i) \), for any \( i \in \{0, \ldots, m - 1\} \).

2. Let \( C^* = \{c_0, \ldots, c_{m-1}\} \). Then \( f(C^*) \leq \left(1 + \frac{1}{k^2 - 1}\right) f(C_m) \)

**Notation.** Next, we define notation used throughout the proof. Let \( A_i, B_i \) denote the respective values of variables \( A, B \) at the beginning of the \( i \)-th iteration of for loop; let \( A_{n+1}, B_{n+1} \) denote their respective final values. Also, let \( A^* = \bigcup_{1 \leq i \leq n+1} A_i \); analogously, define \( B^* \). Let \( e_i \) denote the element received at the beginning of iteration \( i \). We refer to line numbers of the pseudocode Alg. 1 Notice that after deletion of duplicate entries, the sequences \( (A_n), (B_n) \) satisfy the hypotheses of Lemma 1 with the sequence of elements in \( A^*, B^* \), respectively. Since many of the following lemmata are symmetric with respect to \( A \) and \( B \), we state them generically, with variables \( C, D \) standing in for one
of $A, B$, respectively. The notations $C_i, C^*, D_i, D^*$ are defined analogously to $A_i, A^*$ defined above. Finally, if $D \in \{A, B\}$, define $\Delta D_i = f(D_{i+1}) - f(D_i)$. Observe that $\sum_{i=0}^n \Delta D_i = f(D_{n+1}) - f(0)$.

Lemma 2. Let $C, D \in \{A, B\}$, such that $C \neq D$. Let $o \in O \cap D^*$. Let $i(o)$ denote the iteration in which $o$ was processed. Then

$$\delta_o \left( C_{i(o)} \right) \leq \frac{\Delta D_{i(o)} + \gamma f(D_{n+1})}{1 + \gamma}.$$  

Proof. Since $o \in O \cap D^*$, we know that $o$ is added to the set $D$ during iteration $i(o)$; therefore, by the comparison on Line 7 of Alg. 1 it holds that

$$\delta_o \left( C_{i(o)} \right) \leq \delta_o \left( D_{i(o)} \right). \quad (1)$$  

If no deletion from $D$ occurs during iteration $i(o)$, the lemma follows from the fact that $\Delta D_{i(o)} = \delta_o \left( D_{i(o)} \right)$.

For the rest of the proof, suppose that a deletion from $D$ does occur during iteration $i(o)$. For convenience, denote by $D^-$ the value of $D$ after the deletion from $D_{i(o)}$. By Inequality 17 in the proof of Lemma 1 it holds that

$$f(D_{i(o)}) \leq (1 + \gamma) f(D^-) \quad (2)$$

Hence,

$$\Delta D_{i(o)} + \gamma f(D_{n+1}) = f(D^- + o) - f(D_{i(o)}) + \gamma f(D_{n+1}) \geq (1 + \gamma) f(D^- + o) - f(D_{i(o)}) \quad (3)$$

$$\geq (1 + \gamma) f(D^- - o) - (1 + \gamma) f(D^-) \quad (4)$$

$$= (1 + \gamma) \delta_o \left( D^- \right) \quad (5)$$

$$\geq (1 + \gamma) \delta_o \left( C_{i(o)} \right) \quad (6)$$

where Inequality 3 follows from Lemma 1, Inequality 4 follows from Inequality 2, Inequality 5 follows from submodularity of $f$, and Inequality 6 follows from Inequality 1.  

Lemma 3. Let $C, D \in \{A, B\}$, such that $C \neq D$. Then

$$f(C^* \cup O) - f(C^*) \leq b f(C_{n+1}) + (1 + k\gamma) f(D_{n+1})$$


Proof.  

\[ f (C^* \cup O) - f (C^*) \leq \sum_{o \in O \setminus C^*} \delta_o (C^*) \]  \hspace{1cm} (7) 

\[ \leq \sum_{o \in O \setminus C^*} \delta_o (C_{i(o)}) \]  \hspace{1cm} (8) 

\[ \leq \sum_{o \in O \setminus C^*} \frac{bf (C_{i(o)})}{k} + \frac{\Delta D_{i(o)} + \gamma f (D_{n+1})}{1 + \gamma} \]  \hspace{1cm} (9) 

\[ \leq bf (C_{n+1}) + \frac{1 + k\gamma}{1 + \gamma} f (D_{n+1}) \]  \hspace{1cm} (10) 

\[ \leq bf (C_{n+1}) + (1 + k\gamma) f (D_{n+1}) , \] 

where Inequalities \[7\] \[8\] follow from submodularity of \( f \). Inequality \[9\] holds by the following argument: let \( o \in O \setminus C^* \). If \( o \not\in D^* \), then it holds that \( \delta_o (C_{i(o)}) < bf (C_{i(o)})/k \) by Line \[7\]. Otherwise, if \( o \in D^* \), Lemma \[2\] yields \( \delta_o (C_{i(o)}) \leq \frac{\Delta D_{i(o)} + \gamma f (D_{n+1})}{1 + \gamma} \). Inequality \[10\] follows from the fact that \( |O \setminus C^*| \leq k \), Lemma \[4\] and the fact that \( f (D_{n+1}) - f (\emptyset) = \sum_{i=0}^n \Delta D_i \), where each \( \Delta D_i \geq 0 \). \( \square \)

Lemma 4. Let \( C \in \{A, B\} \), and let \( C' \subseteq C_{n+1} \) be the set of \( \min \{ |C_{n+1}|, k \} \) elements most recently added to \( C_{n+1} \). Then \( f (C_{n+1}) \leq \left( \frac{1}{1 + (1/b)k} \right) f (C') \).

Proof. For simplicity of notation, let \( C = C_{n+1} \). If \( |C| \leq k \), the result follows since \( C' = C \). So suppose \( |C| > k \), and let \( C' = \{c_1, \ldots, c_k\} \) be ordered by the iteration in which each element was added to \( C \). Also, let \( C_i' = \{c_1, c_2, \ldots, c_i\} \), for \( 1 \leq i \leq k \), and let \( C_0' = \emptyset \). By Line \[8\] it holds that \( f ((C \setminus C') \cup C_i') \geq (1 + b/k) f ((C \setminus C') \cup C_{i-1}') \), for each \( 1 \leq i \leq k \); thus, \n
\[ f (C) \geq (1 + b/k) k f (C \setminus C') . \]  \hspace{1cm} (11) 

From Inequality \[11\] and the submodularity, nonnegativity of \( f \), we have 

\[ f (C') \geq f (C) - f (C \setminus C') \geq ((1 + b/k)^k - 1) f (C \setminus C') . \] 

Hence 

\[ f (C) \leq f (C') + f (C \setminus C') \leq \left( 1 + \frac{1}{(1 + b/k)^k - 1} \right) f (C') . \] \( \square \)

By application of Lemma \[3\] with \( A = C \) and then again with \( B = C \), we obtain 

\[ \delta_O (A^*) \leq bf (A_{n+1}) + (1 + k\gamma) f (B_{n+1}) , \quad \text{and} \]  \hspace{1cm} (12) 

\[ \delta_O (B^*) \leq bf (B_{n+1}) + (1 + k\gamma) f (A_{n+1}) . \]  \hspace{1cm} (13)
Next, we have that
\[
    f(O) \leq f(A^* \cup O) + f(B^* \cup O) \\
    \leq f(A^*) + f(B^*) + (b + 1 + k\gamma)(f(A_{n+1}) + f(B_{n+1})),
\]
where Inequality 14 follows from the fact that \( A^* \cap B^* = \emptyset \) and submodularity and nonnegativity of \( f \). Inequality 15 follows from the summation of Inequalities 12 and 13. By application of Property 2 of Lemma 1, we have from Inequality 15
\[
    f(O) \leq (b + 2 + (k + 2)\gamma)(f(A_{n+1}) + f(B_{n+1})) \\
    \leq (2b + 4 + 2(k + 2)\gamma)f(C_{n+1}),
\]
where \( C_{n+1} = \arg \max \{ f(A_{n+1}), f(B_{n+1}) \} \). Observe that the choice of \( \ell \) on Line 4 ensures that \( 2(k + 2)\gamma < \varepsilon (1 - (1 + b/k)^{-k}) \), by Lemma 8. Therefore, by application of Lemma 4, we have from Inequality 16
\[
    f(O) \leq \left( \frac{2b + 4}{1 - (1 + b/k)^{-k}} + \varepsilon \right) f(C'). \quad \Box
\]

### 2.3 Post-Processing: QS++ and Parameter \( c \)

In this section, we briefly describe a modification to QuickStream that improves its empirical performance. Instead of choosing, on Line 14, the best of \( A' \) and \( B' \) as the solution; introduce a third candidate solution as follows: use an offline algorithm for SMCC in a post-processing procedure on the restricted universe \( A \cup B \) to select a set of size at most \( k \) to return. This method can only improve the objective value of the returned solution and therefore does not compromise the theoretical analysis of the preceding section. The empirical solution value can be further improved by lowering the parameter \( b \) as this increases the size of \( A \cup B \), potentially improving the quality of the solution found by the selected post-processing algorithm.

**Parameter \( c \)** To speed up any algorithm \( A \) by a constant factor (at a cost of \( 1/c \) in the approximation ratio), the following reduction may be employed. Suppose the input instance is \( f : 2^U \to \mathbb{R}^+ \), \( k \). Define \( U \) by arbitrarily grouping elements of \( U \) together in blocks of size \( c \) (except for one block that may have less than \( c \) elements). Define the objective function \( \hat{f} \) on a set of blocks \( B \) to be \( \hat{f}(B) = f(\bigcup B) \); run the algorithm \( A \) on input \((\hat{f}, k)\) to obtain a set \( S \) of at most \( k \) blocks, and let \( S' \subseteq U \) be the corresponding set of at most \( c k \) elements. Then, arbitrarily partition \( S' \) into \( c \) subsets of size \( k \) and take the subset with the highest \( f \) value as a solution to the original instance \((f, k)\).

For most algorithms, this reduction is not useful since it results in a large loss in solution quality. However, with QS++, if the reduction with \( c > 1 \) is applied to the initial run of QuickStream only, and the post-processing is applied with \( c = 1 \), this reduction can result in a large speedup while still retaining a good quality of solution.
Algorithm 2 A multi-pass algorithm for SMCC.

1: procedure MultiPassLinear($f, k, \varepsilon, \Gamma, \alpha$)
2: Input: oracle $f$, cardinality constraint $k$, $\varepsilon > 0$, parameters $\Gamma, \alpha$, such that $\Gamma \leq \text{OPT} \leq \Gamma/\alpha$.
3: $\tau \leftarrow \Gamma/(4k\alpha)$ \texttt{▷ Choice satisfies $\tau \geq \text{OPT}/(4k)$.}
4: while $\tau \geq \varepsilon \Gamma/(16k)$ do
5: for $u \in N$ do
6: $S \leftarrow \arg\max\{\delta_u(X) : X \in \{A, B\} \text{ and } |X| < k\}$ \texttt{▷ If arg max is empty, break from loop.}
7: if $\delta_u(S) \geq \tau$ then
8: $S \leftarrow S \cup \{u\}$
9: $\tau \leftarrow \tau(1 - \varepsilon)$
10: return $S \leftarrow \arg\max\{f(A), f(B)\}$

3 Multi-Pass Streaming Algorithm for SMCC

In this section, we describe a multi-pass streaming algorithm for SMCC that obtains ratio $4 + \varepsilon$ in linear time.

3.1 Description of Algorithm

The algorithm MultiPassLinear (Alg. 2) takes as input a value $\Gamma \in \mathbb{R}^+$, such that $\Gamma \leq \text{OPT}$; and parameter $\alpha \in \mathbb{R}^+$, such that $\text{OPT} \leq \Gamma/\alpha$. Two initially empty, disjoint sets $A, B$ are maintained throughout the algorithm. The algorithm takes multiple passes through the ground set with descending thresholds $\tau$ in a greedy approach, where the stepsize is determined by accuracy parameter $\varepsilon > 0$, and $\tau$ is initially set to $\Gamma/(4k\alpha)$. An element is added to the set $S$ in $\{A, B\}$ to which it has the higher marginal gain, as long as the gain is at least $\tau$, and the set $S$ satisfies $|S| < k$. Finally, the set in $\{A, B\}$ with highest $f$ value is returned.

The initial value for $\tau$ of $\Gamma/(4k\alpha) \geq \text{OPT}/(4k)$ allows MultiPassLinear to achieve its ratio in $O(\log(1/(\alpha\varepsilon))/\varepsilon)$ passes. The linear-time $(4+\varepsilon)$-approximation algorithm is obtained by using QuickStream to obtain the input parameters $\Gamma, \alpha$ for MultiPassLinear.

3.2 Theoretical Guarantees

In this section, we prove the following theorem concerning the performance of MultiPassLinear (Alg. 2).

Theorem 2. Let $0 \leq \varepsilon \leq 1/2$, and let $(f, k)$ be an instance of SMCC, with optimal solution value $\text{OPT}$. Suppose $\Gamma, \alpha \in \mathbb{R}$ satisfy $\Gamma \leq \text{OPT} \leq \Gamma/\alpha$. The solution $S$ returned by MultiPassLinear$(f, k, \varepsilon, \Gamma, \alpha)$ satisfies $\text{OPT} \leq (4 + 6\varepsilon)f(S)$. Further, MultiPassLinear has time and query complexity $O(\frac{\varepsilon}{\alpha} \log \left( \frac{1}{\alpha\varepsilon} \right))$,
memory complexity $O(k)$, and makes $O(\log(1/(\alpha \varepsilon))/\varepsilon)$ passes over the ground set.

Proof. To establish the approximation ratio, consider first the case in which $C \in \{A, B\}$ satisfies $|C| = k$ after the first iteration of the while loop. Let $C = \{c_1, \ldots, c_k\}$ be ordered by the order in which elements were added to $C$ on Line 7, let $C_i = \{c_1, \ldots, c_i\}$, $C_0 = \emptyset$, and let $\Delta C_i = f(C_i) - f(C_{i-1})$. Then $f(C) = \sum_{i=1}^{k} \Delta C_i \geq \Gamma/(4\alpha) \geq \text{OPT}/4$, and the ratio is proven.

Therefore, for the rest of the proof, suppose $|A| < k$ and $|B| < k$ immediately after the execution of the first iteration of the while loop. First, let $C, D \in \{A, B\}$, such that $C \neq D$ have their values at the termination of the algorithm. For the definition of $D'$ and the proofs of the next two lemmata, see Appendix C. These lemmata together establish an upper bound on $\delta_O(C)$ in terms of the gains of elements added to $C$ and $D$.

Lemma 5.

$$\sum_{o \in O \setminus (C \cup D')} \delta_o(C) \leq (1 + 2\varepsilon) \sum_{i : c_i \notin O} \Delta C_i + \varepsilon \text{OPT}/16.$$ 

Lemma 6.

$$\delta_O(C) \leq \sum_{i : d_i \in O} \Delta D_i + (1 + 2\varepsilon) \sum_{i : c_i \notin O} \Delta C_i + \varepsilon \text{OPT}/16.$$ 

Applying Lemma 6 with $C = A$ and separately with $C = B$ and summing the resulting inequalities yields

$$\delta_O(A) + \delta_O(B) \leq (1 + 2\varepsilon) \left[ \sum_{i=1}^{k} \Delta B_i + \sum_{i=1}^{k} \Delta A_i \right] + \varepsilon \text{OPT}/8$$

$$= (1 + 2\varepsilon) [f(A) + f(B)] + \varepsilon \text{OPT}/8.$$ 

Thus,

$$f(O) \leq f(O \cup A) + f(O \cup B)$$

$$\leq (2 + 2\varepsilon)(f(A) + f(B)) + \varepsilon \text{OPT}/8,$$

from which the result follows. \qed

**4 Empirical Evaluation**

In this section, we validate our proposed algorithms in the context of single-pass streaming algorithms and other algorithms for SMCC that require nearly linear time. The source code and scripts to reproduce all plots are given in the Supplementary Material.

We performed two sets of experiments; the first set evaluates single-pass algorithms, while the second set evaluates algorithms with low overall time complexity.
Figure 1: Evaluation of single-pass streaming algorithms (Set 1) on ca-AstroPh ($n = 18772$), in terms of objective value normalized by the standard greedy value, total number of queries normalized by $n$, and the maximum memory used by each algorithm normalized by $k$. The legend shown in (a) applies to all subfigures.

Figure 2: Evaluation of algorithms (Set 2) on ca-AstroPh and web-Google ($n = 875713$), in terms of objective value normalized by the standard greedy value and total number of queries normalized by $n$. The legend shown in (a) applies to all subfigures.
Set 1: Single-Pass Algorithms  In this set of experiments, we compared our single-pass algorithm with the following algorithms:

- Algorithm 2 (FKK) of Feldman, Karbasi, and Kazemi [12]: this algorithm achieves ratio 5.828 in expectation and has $O(kn)$ time complexity.

- Algorithm 1 (AEFNS) of Alaluf, Ene, Feldman, Nguyen, and Suh [1]; this algorithm achieves ratio $1 + \alpha$ and has time complexity $O(n \log k) + T(A)$, where $\alpha, T(A)$ are the approximation ratio and time complexity, respectively, of post-processing algorithm $A$. The selection of $A$ is discussed below.

- Our Algorithm 1 (QS) with parameters $c = 1$ and $b = 1.49$, as described in Section 2.

- Our Algorithm 1 (QS++) with various blocksize parameters $c$ and with parameter $b = 0.7c$, as described in Section 2.3. The post-processing algorithm is described below.

AEFNS and QS++ used the same post-processing algorithm MultiPassLinear with input parameters as follows: QS++ used its solution value (before post-processing) and its approximation ratio for $\Gamma$ and $\alpha$, respectively. AEFNS does not have a ratio before post-processing, so the maximum singleton value and $k$ were used for $\Gamma$ and $\alpha$, respectively. Both algorithms used their respective value for accuracy parameter $\varepsilon$ for the same parameter in MultiPassLinear.

Set 2: Algorithms with Low Time Complexity  In this set, we compared our algorithms with state-of-the-art algorithms with lowest time complexity in prior literature:

- Algorithm 4 (BFS) of Buchbinder, Feldman, and Schwartz [6], the fastest randomized algorithm, which achieves expected ratio $\varepsilon$ with $O(n)$ time complexity.

- Algorithm 2 (K19) of Kuhnle [18], the fastest deterministic algorithm in prior literature, which achieves ratio $4 + \varepsilon$ with $O(n \log k)$ time complexity.

- Algorithm 2 (K21) of Kuhnle [20], the fastest algorithm with nearly optimal adaptivity, which achieves expected ratio $0.193 - \varepsilon$ in $O(n \log k)$ time.

- Our Algorithm 2 (MPL), with an initial run of QS to determine input parameters $\Gamma$ and $\alpha$.

We also evaluated the performance of QS++ in this context. We remark that MPL and QS++ are similar in that each run QS followed by MultiPassLinear; the difference is that MPL runs MultiPassLinear on the entire ground set (which results in the improved approximation ratio of 4), whereas QS++ runs MultiPassLinear only on its restricted ground set $A \cup B$ (which does not improve the approximation ratio of QS).
All algorithms used lazy evaluations whenever possible as follows. Suppose $\delta_x(S)$ has already been computed, and the algorithm needs to check if $\delta_x(T) \geq \tau$, for some $\tau \in \mathbb{R}$ and $T \supseteq S$. Then if $\delta_x(S) < \tau$, this evaluation may be safely skipped due to the submodularity of $f$. Strictly speaking, this means that MultiPassLinear is not run as a streaming algorithm in our evaluation. However, using this optimization during post-processing does not compromise the streaming nature of the single-pass algorithms. Unless otherwise specified, the accuracy parameter $\varepsilon$ of each algorithm is set to $0.2$. For QS++, no attempt was made to optimize over the parameters $c$ and $b$, although the choice of $b < c$ is important to obtain a large enough set of elements to improve the empirical objective value through post-processing.

Applications and Datasets The algorithms were evaluated on two applications: cardinality constrained maximum cut and revenue maximization on social networks objectives. The maximum cut objective is defined as follows: given graph $G = (V, E)$ with edge weight function $w$, the maximum cut objective is defined by $f(S) = \sum_{u \in S} \sum_{v \in V \setminus S} w(u, v)$. A variety of network topologies from the Stanford Large Network Dataset Collection \cite{dataset} were used. For more details on the applications and datasets, see Appendix D.

Results: Set 1 Results for the objective value of maximum cut, total queries to $f$, and memory are shown for the single-pass algorithms in Fig. 1 as the cardinality constraint $k$ and accuracy parameter $\varepsilon$ vary. Results on other datasets and revenue maximization were analogous and are given in Appendix D.

In summary, QS++ with $c = 1$ exhibited the best objective value of the streaming algorithms and sacrificed less than 1% of the greedy value on all instances (Fig. 1(a)). Further, QS++ with $c = 4$ achieved better than 0.8 of the greedy value while using less than $n$ queries; and the variants of QS used more than an order of magnitude fewer queries than the other streaming algorithms (Fig. 1(b)). In addition, QS++ exhibited better robustness to the accuracy parameter $\varepsilon$ than AEFNS (Fig. 1(d)), although the query complexity of AEFNS improved drastically with larger $\varepsilon$ (Fig. 1(e)).

Results: Set 2 Results for the objective value of maximum cut and total queries to $f$ are shown for the algorithms in Fig. 2 as the cardinality constraint $k$ varies. Results on other datasets and revenue maximization were analogous and are given in Appendix D.

In summary, all of K19, K21, MultiPassLinear, and QS++ ($c = 1$) returned nearly the greedy objective value, while BFS and QS++ ($c = 4$) returned lower objective values (Figs. 2(a), 2(b)). In terms of total queries, our algorithms had the lowest, closely followed by K19. K21 and BFS required substantially more queries (Fig. 2(c)).

Discussion QS++ with $c = 1$ returned nearly the greedy value on all instances while using $\approx 2n$ queries on instances with small $k$ and at most 3n
queries, the most efficient of any prior algorithm. Larger values of $c$ obtained further speedup (to less than $n$ queries) at the cost of objective value. \textsc{MultiPassLinear} improved the objective value of $\text{QS}++$ slightly for larger $k$ values and used roughly the same number of queries as $\text{QS}++$ with $c = 1$; however, \textsc{MultiPassLinear} requires multiple passes through the ground set.

5 Concluding Remarks

In this work, we have presented two deterministic, linear-time algorithms for SMCC, which are the first linear-time algorithms for this problem that yield constant approximation ratio with high probability. The first algorithm is a single-pass streaming algorithm with ratio $\approx 9.298 + \varepsilon$; the second uses the output of the first algorithm in a multi-pass streaming framework to obtain ratio $4 + \varepsilon$. A natural question for future work is if the ratio of $4 + \varepsilon$ could be improved in linear time; currently, the best deterministic algorithm has a ratio of $e$ in time $O(k^3n^2)$ \cite{buchbinder2018deterministic}.

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A Additional Discussion of Related Work

Haba et al. [15] recently provided a general framework to convert a streaming algorithm for monotone, submodular functions to a streaming algorithm for general submodular functions, as long as the original algorithm satisfies certain conditions, namely:

Definition 1 ([15]). Consider a data stream algorithm for maximizing a non-negative submodular function $f : 2^N \rightarrow \mathbb{R}^+$ subject to a constraint $(N, \mathcal{I})$. We say that such an algorithm is an $(\alpha, \gamma)$-approximation algorithm, for some $\alpha \geq 1$ and $\gamma \geq 0$, if it returns two sets $S \subseteq A \subseteq N$, such that $S \in \mathcal{I}$, and for all $T \in \mathcal{I}$, we have

$$
\mathbb{E}[f(T \cup A)] \leq \alpha \mathbb{E}[f(S)] + \gamma.
$$

In the above definition, $\mathcal{I}$ is a set independence system; in our work, $\mathcal{I}$ is always the $k$-uniform matroid: $\mathcal{I} = \{S \subseteq N : |S| \leq k\}$. The algorithm of Kuhnle [19] returns a set $A' \in \mathcal{I}$; it does hold that $A' \subseteq A^\ast$, where $A'$ and $A^\ast$ satisfy the above definition with $\alpha = 4$ and $\gamma = 0$, but $A^\ast$ is not stored by the algorithm. Hence, the technique as outlined in Haba et al. [15] and Chekuri et al. [8] must be followed to create a tradeoff between the size of $A^\ast$ and $\gamma$; since $\gamma$ must be set to $\beta \text{OPT}/k$, logarithmically many guesses for OPT must be employed, which increases the runtime of Kuhnle [19] by an $\Omega(\log k)$ factor. Furthermore, the method of Haba et al. [15] requires an offline algorithm for SMCC. Since no linear-time algorithm exists for SMCC, one cannot obtain a linear-time algorithm by application of the framework of Haba et al. [15] to the streaming algorithm of Kuhnle [19].

B Proof of Lemma 1

Claim 1. For any $y \geq 1$, $b > 0$, if $i \geq (k/b + 1) \log y$, then $(1 + b/k)^i \geq y$.

Proof. Follows directly from the inequality $\log x \geq 1 - 1/x$ for $x > 0$. \qed

Proof of Property 1 of Lemma 7 If no deletion is made during iteration $i$ of the for loop, then any change in $f(C)$ is clearly nonnegative. So suppose deletion of set $D$ from $C$ occurs on line 11 of Alg. 1 during this iteration. Observe that $C_{i+1} = (C_i \setminus D) \cup \{e_i\}$, because the deletion is triggered by the addition of $e_i$ to $C_i$. In addition, at some iteration $j < i$ of the for loop, it holds that $C_j = D$. If $f(C_i \setminus C_j) \geq f(C_j)$, the lemma follows by submodularity and the condition to add $e_i$ to $C_i$. Therefore, for the rest of the proof, suppose $f(C_i \setminus C_j) < f(C_j)$.

From the beginning of iteration $j$ to the beginning of iteration $i$, there have been $\ell(k/b + 1) \log_2(k) - 1 \geq (\ell - 1)(k/b + 1) \log_2(k)$ additions and no deletions to $C$, which add to $C$ precisely the elements in $(C_i \setminus C_j)$.

It holds that

$$f(C_i \setminus C_j) \xleftarrow{(a)} f(C_i) - f(C_j) \xrightarrow{(b)} \left(1 + \frac{b}{k}\right)^{(\ell - 1)(k/b + 1) \log_2 k} \cdot f(C_j) - f(C_j) \xrightarrow{(c)} (k^{\ell-1} - 1)f(C_j),$$

18
where inequality (a) follows from submodularity and nonnegativity of \( f \), inequality (b) follows from the fact that each addition from \( C_j \) to \( C_i \) increases the value of \( f(C) \) by a factor of at least \( 1 + b/k \), and inequality (c) follows from Claim 1. Therefore

\[
f(C_i) \leq f(C_i \setminus C_j) + f(C_j) \leq \left(1 + \frac{1}{k^{\ell-1} - 1}\right) f(C_i \setminus C_j).
\]

Next,

\[
f((C_i \setminus C_j) \cup \{e_i\}) - f(C_i \setminus C_j) \overset{(d)}{\geq} f(C_i \cup \{e_i\}) - f(C_i) \overset{(e)}{\geq} b f(C_i) / k \geq f(C_i \setminus C_j) / k,
\]

where inequality (d) follows from submodularity; inequality (e) is by the condition to add \( e_i \) to \( A_i \) on line 8 and inequality (f) holds since \( b \geq 1 \) and \( f(C_i) > f(C_i \setminus C_j) \). Finally, using Inequalities (17) and (18) as indicated below, we have

\[
f(C_{i+1}) = f(C_i \setminus C_j \cup \{e_i\}) \overset{(18)}{\geq} \left(1 + \frac{1}{k}\right) f(C_i \setminus C_j) \overset{(17)}{\geq} \frac{1 + \frac{1}{k}}{1 + \frac{1}{k^{\ell-1} - 1}} \cdot f(C_i) \geq f(C_i),
\]

where the last inequality follows since \( k \geq 2 \) and \( \ell \geq 3 \).

**Proof of Property 2 of Lemma 1** Next, we bound the total value of \( f(A) \) and \( f(B) \) lost from deletions throughout the run of the algorithm.

**Lemma 7.** Let \( C \in \{A, B\} \). \( f(C^*) \leq \left(1 + \frac{1}{k^{\ell-1} - 1}\right) f(C_{n+1}) \).

**Proof.** Observe that \( C^* \setminus C_{n+1} \) may be written as the union of pairwise disjoint sets, each of which is size \( \ell(k/b + 1) \log_2(k) + 1 \) and was deleted on line 11 of Alg. 1. Suppose there were \( m \) sets deleted from \( C \); write \( C^* \setminus C_{n+1} = \{D^i : 1 \leq i \leq m\} \), where each \( D^i \) is deleted on line 11 ordered such that \( i < j \) implies \( D^i \) was deleted after \( D^j \) (the reverse order in which they were deleted); finally, let \( D^0 = C_{n+1} \).

**Claim 2.** Let \( 0 \leq i \leq m \). Then \( f(D^i) \geq k^i f(D^{i+1}) \).

**Proof.** There are at least \( \ell(k/b + 1) \log k + 1 \) elements added to \( C \) and exactly one deletion event during the period between starting when \( C = D^{i+1} \) until \( C = D^i \). Moreover, each addition except possibly one (corresponding to the deletion event) increases \( f(C) \) by a factor of at least \( 1 + b/k \). Hence, by Lemma 1 and Claim 1, \( f(D^i) \geq k^i f(D^{i+1}) \).
By Claim 2 for any $0 \leq i \leq m$, $f(C_{n+1} = D^0) \geq k^{\ell i} f(D^i)$. Thus,

$$f(C^*) \leq f(C^* \backslash C_{n+1}) + f(C_{n+1}) \leq \sum_{i=0}^{m} f(D^i)$$

(Submodularity, Nonnegativity of $f$)

$$\leq f(C_{n+1}) \sum_{i=0}^{\infty} k^{-\ell i}$$

(Claim 2)

$$= f(C_{n+1}) \left( \frac{1}{1 - k^{-\ell}} \right)$$

(Sum of geometric series)

\[ \Box \]

**B.1 Justification of choice of $\ell$**

**Lemma 8.** Let $\varepsilon > 0$, and let $\beta = \frac{1}{1-(1+b/k)^{-1}}$. Choose $\ell \geq 1 + \log((6\beta)/\varepsilon + 1)$, and let $\gamma = 1/(k^\ell - 1)$. Then

$$2(\kappa + 2)\gamma < \varepsilon\beta^{-1}.$$

**Proof.** First, one may verify that $\ell > \frac{\log((2k+4)\beta/\varepsilon + 1)}{\log k} \implies 2(\kappa + 2)\gamma < \varepsilon\beta^{-1}$. Next, since $k \geq 1$,

$$\frac{1}{\log k} \left( \log \left( \frac{(2k+4)\beta}{\varepsilon} + 1 \right) \right) \leq \frac{1}{\log k} \left( \log \left( \frac{(2 + 4)\beta}{\varepsilon} + 1 \right) k \right)$$

$$= \frac{1}{\log k} \log \left( \frac{6\beta}{\varepsilon} + 1 \right) + \log(k)$$

$$\leq 1 + \log \left( \frac{6\beta}{\varepsilon} + 1 \right).$$

Hence it suffices to take $\ell$ greater than the last expression. \[ \Box \]

**C Proofs for Section 3**

Let $C, D \in \{A, B\}$, such that $C \neq D$ have their values at the termination of the algorithm. If $|C| = k$, let $D'$ have the value of its corresponding variable when the $k$th element is added to $C$; otherwise, if $|C| < k$ let $D' = D$.

**Proof of Lemma 3** Suppose $|C| = k$. Let $\tau'$ be the value of $\tau$ during the iteration of the while loop in which the last element was added to $C$. Let
Figure 3: Results for single-pass algorithms (Set 1) on the revenue maximization application on ca-AstroPh.

\( o \in O \setminus (C \cup D') \). Then, since \( o \) was not added to \( C \) or \( D' \) during the previous iteration of the while loop, \( \delta_o(C) < \tau'/(1 - \varepsilon) \). Further, \( \Delta C_i \geq \tau' \) for all \( i \). Hence,

\[
\sum_{o \in O \setminus (C \cup D')} \delta_o(C) \leq \frac{1}{1 - \varepsilon} \sum_{i : c_i \notin O} \Delta C_i \\
\leq (1 + 2\varepsilon) \sum_{i : c_i \notin O} \Delta C_i.
\]

Next, suppose that \( |C| < k \). In this case, the last threshold \( \tau \) of the while loop ensures that \( \sum_{o \in O \setminus (C \cup D')} \delta_o(C) < \varepsilon \Gamma/16 \leq \varepsilon \text{OPT}/16 \).

**Proof of Lemma 6.** Observe that

\[
\delta_O(C) \leq \sum_{o \in O \cap D'} \delta_o(C) + \sum_{o \notin (C \cup D')} \delta_o(C) \\
\leq \sum_{i,d_i \in O} \Delta D_i + \sum_{o \notin (C \cup D')} \delta_o(C),
\]

where Inequality 19 follows from submodularity and Inequality 20 follows from submodularity and the comparison on Line 9 for each element \( o \in O \cap D' \). From Inequality 20, the lemma follows from application of Lemma 5. \( \square \)

**D Additional Empirical Results**

**D.1 Applications and Datasets**

The cardinality-constrained maximum cut function is defined as follows. Given graph \( G = (V,E) \), and nonnegative edge weight \( w_{ij} \) on each edge \((i,j) \in E \). For \( S \subseteq V \), let

\[
f(S) = \sum_{i \in V \setminus S} \sum_{j \in S} w_{ij}.
\]
In general, this is a non-monotone, submodular function.

The revenue maximization objective is defined as follows. Let graph $G = (V, E)$ represent a social network, with nonnegative edge weight $w_{ij}$ on each edge $(i, j) \in E$. We use the concave graph model introduced by Hartline et al. [17]. In this model, each user $i \in V$ is associated with a non-negative, concave function $f_i : \mathbb{R}^+ \rightarrow \mathbb{R}^+$. The value $v_i(S) = f_i(\sum_{j \in S} w_{ij})$ encodes how likely the user $i$ is to buy a product if the set $S$ has adopted it. Then the total revenue for seeding a set $S$ is

$$f(S) = \sum_{i \in V \setminus S} f_i \left( \sum_{j \in S} w_{ij} \right).$$

This is a non-monotone, submodular function. In our implementation, each edge weight $w_{ij} \in (0, 1)$ is chosen uniformly randomly; further, $f_i(\cdot) = (\cdot)^{\alpha_i}$, where $\alpha_i \in (0, 1)$ is chosen uniformly randomly for each user $i \in V$.

### D.2 Additional Results

Figs. 3 and 4 show results of all algorithms with the revenue maximization objective function. Results were qualitatively similar to the maximum cut application discussed in Section 4.