Dimension variation of Gouvêa-Mazur type for Drinfeld cuspforms of level $\Gamma_1(t)$

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Notation

- For any field $F$, denote by $\overline{F}$ its algebraic closure
- $p$: rational prime, $q$: $p$-power, $A = \mathbb{F}_q[t]$, $K_\infty = \mathbb{F}_q((1/t))$
- $\mathbb{C}_\infty$: the $(1/t)$-adic completion of $\overline{K_\infty}$
- $\Omega = \mathbb{C}_\infty \setminus K_\infty$: Drinfeld upper half plane
- $\Gamma_1(t) = \left\{ \gamma \in SL_2(A) \mid \gamma \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \text{ mod } t \right\}$
- $\nu_t : \mathbb{F}_q((t)) \rightarrow \mathbb{Z} \cup \{+\infty\}$: additive valuation, $\nu_t(t) = 1$, extended to $\overline{\mathbb{F}_q((t))} \rightarrow \mathbb{Q} \cup \{+\infty\}$
Drinfeld modular form

**Definition**

- A rigid analytic function $f : \Omega \rightarrow \mathbb{C}_\infty$ is called Drinfeld modular form of weight $k$ and level $\Gamma_1(t)$ if

$$f \left( \frac{az + b}{cz + d} \right) = (cz + d)^k f(z) \text{ for any } z \in \Omega, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(t)$$

and satisfies a holomorphy condition at cusps

- $f$ is called Drinfeld cuspform if it vanishes at cusps

- $S_k(\Gamma_1(t))$: $\mathbb{C}_\infty$-vector space of DCFs of weight $k$ and level $\Gamma_1(t)$
Slope

Definition

$U$: endomorphism of $S_k(\Gamma_1(t))$ defined by

$$Uf(z) = \frac{1}{t} \sum_{b \in \mathbb{F}_q} f\left(\frac{z + b}{t}\right)$$

- $S_k(\Gamma_1(t))$ has an $\mathbb{F}_q(t)$-structure preserved by $U$
- So we may think them over $\mathbb{F}_q(t)$, $\mathbb{F}_q((t))$ and $\overline{\mathbb{F}_q((t))}$

Definition

$v_t$ of any eigenvalue of $U$ is called slope, which is in $\mathbb{Q}_{\geq 0} \cup \{+\infty\}$

- $d(k, \alpha) := \text{dimension of generalized eigenspace for } U \sim S_k(\Gamma_1(t)) \text{ with eigenvalues of slope } \alpha$
interlude: $p$-adic slope for elliptic modular forms

- For elliptic cuspform $f$ of weight $k$ and level $\Gamma_0(Np)$, we have analogous

$$U_f(z) = \frac{1}{p} \sum_{b=0,1,\ldots,p-1} f \left( \frac{z + b}{p} \right),$$

slopes using normalized $p$-adic valuation and $d_0(k, \alpha)$

Gouvêa-Mazur conjecture, refuted by Buzzard-Calegari

For any integer $m \geq \alpha$,

$$\begin{cases} k_1, k_2 \geq 2\alpha + 2 \\ k_1 \equiv k_2 \mod p^m(p-1) \end{cases} \implies d_0(k_1, \alpha) = d_0(k_2, \alpha)$$
interlude: $p$-adic family of eigenforms

- $p$-adic weight space $\mathcal{W}$: a rigid analytic space over $\mathbb{Q}_p$ with $\mathcal{W}(\mathbb{C}_p) = \text{Hom}_{\text{cont.}}(\mathbb{Z}_p^\times, \mathbb{C}_p^\times)$
- “weight $k$, level $\Gamma_1(N) \cap \Gamma_0(p)$” $\leftrightarrow$ $(t \mapsto t^k) \in \mathcal{W}(\mathbb{Q}_p)$

Hida, Coleman, Coleman-Mazur, ...

There exist families of elliptic eigenforms of finite slope parametrized by rigid analytic spaces over $\mathcal{W}$

Consequence (Coleman)

For any non-negative rational number $\alpha$, $\exists m(\alpha) \in \mathbb{Z}_{\geq 0}$ such that

$$k_1, k_2 > \alpha + 1$$
$$k_1 \equiv k_2 \mod p^{m(\alpha)}(p - 1) \quad \Rightarrow \quad d_0(k_1, \alpha) = d_0(k_2, \alpha)$$
Why not mimic elliptic case?

- \( \mathbb{C}_t \): \( t \)-adic completion of \( \overline{\mathbb{F}_q((t))} \)
- Why don’t we consider an adic space \( \mathcal{W} \) with
  \[
  \mathcal{W}(\mathbb{C}_t) = \text{Hom}_{\text{cont.}}(\mathbb{F}_q[[t]]^\times, \mathbb{C}_t^\times),
  \]
- and try to find \( t \)-adic analytic families of Drinfeld eigenforms over \( \mathcal{W} \)?

For Drinfeld case
parallel construction to elliptic case does not work (so far)
Why $t$-adic analytic family breaks down

Reason 1: scarce analytic characters (Jeong)

Only analytic characters $1 + t\mathbb{F}_q[[t]] \to \mathbb{C}_t^\times$ are $(t \mapsto t^c)$, $c \in \mathbb{Z}_p$

$\Rightarrow$ No $t$-adic analytic interpolation of weights

Reason 2: no known characteristic power series (Buzzard)

$\mathbb{F}_q[[t]]^\times$ top. infinitely generated, $\mathcal{M}$ locally non-Noetherian

$\Rightarrow$ No known definition of characteristic power series

- Want to define it as a limit (of something)
- For convergence we use Noetherian assumption
- Key: “any submodule of complete module is closed” fails if non-Noether
Slope patterns

Nonetheless there seem some patterns for slopes on $S_k(\Gamma_1(t))$

| weight | slopes for $p = q = 2$ |
|--------|-------------------------|
| 2      | $0^1$                   |
| 3      | $0^1, +\infty^1$        |
| 4      | $0^1, 1^1, +\infty^1$   |
| 5      | $0^1, \frac{3^2}{2}, +\infty^1$ |
| 6      | $0^1, 1^1, 2^1, +\infty^2$ |
| 7      | $0^1, 2^1, \frac{5^2}{2}, +\infty^2$ |
| 8      | $0^1, 1^1, 3^3, +\infty^2$ |
| 9      | $0^1, \frac{3^2}{2}, \frac{7^2}{2}, +\infty^3$ |
| 10     | $0^1, 1^1, 2^1, 4^3, +\infty^3$ |
| 11     | $0^1, 2^1, 4^1, \frac{9^4}{2}, +\infty^3$ |
| 12     | $0^1, 1^1, 3^1, 4^1, 5^3, +\infty^4$ |
Main theorem

Theorem (H.)

For any integer $m \geq \alpha$,

$$k_1, k_2 > \alpha + 1$$

$$k_1 \equiv k_2 \mod p^m$$

$$\Rightarrow d(k_1, \alpha) = d(k_2, \alpha)$$

Natural questions

- What if nebentypus and type allowed?
- What about higher tame level and $\wp$-adic case for $\deg(\wp) > 1$?
- Does it reflect existence of families of DMFs whatsoever?
- Are $n$-th smallest slopes periodic?
- Does anyone want to compute slopes for $g(X_1(\wp)) > 0$ case?
Proof: Bandini-Valentino formula and glissandonelessness

Note: genus of $X_1(t)$ is zero and $\dim(S_k(\Gamma_1(t))) = k - 1$

Theorem (Bandini-Valentino)

For a certain basis $c_0^{(k)}, \ldots, c_{k-2}^{(k)}$ of $S_k(\Gamma_1(t))$, we have

$$U(c_j^{(k)}) = (-t)^j \binom{k-2-j}{j} c_j^{(k)} - t^j \sum_{h \in \mathbb{Z}, h \neq 0} \{\ast\} c_j^{(k)}_{j+h(q-1)},$$

where $\ast = \binom{k-2-j-h(q-1)}{j-h(q-1)} + (-1)^{j+1} \binom{k-2-j-h(q-1)}{j}$

So the representing matrix $U^{(k)}$ of $U \acts S_k(\Gamma_1(t))$ with this basis is glissando, namely

entries of $j$-th column (starting with zeroth) are in $t^j \mathbb{F}_p$
Proof: Consequences of B-V formula

B-V formula and glissandness imply:

- \( d(k, 0) = 1 \)
- \( l \)-th smallest elementary divisor of \( U^{(k)} \) is \( \geq l - 1 \)
- we have

\[
U^{(k+p^m)} \equiv \begin{pmatrix}
U^{(k)} & O \\
* & t^{k-1}D
\end{pmatrix} \pmod{t^{p^m}}
\]

with \( D \in M_{p^m, p^{m-k+1}}(\mathbb{F}_q[[t]]) \)
- Let \( V \) be the matrix on RHS, then

\[
\text{Slopes}(V) \cap [0, k - 1) = \text{Slopes}(U^{(k)}) \cap [0, k - 1),
\]

where \( \text{Slopes}(V) \) is the multiset of \( t \)-adic valuations of eigenvalues of \( V \)
Proof: Perturbation

Definition
For any $B \in M_m(\mathbb{F}_q[[t]])$, define reciprocal characteristic poly by

$$P_B(X) := \det(I - BX), \quad a_n(B) := \text{coefficient of } X^n$$

Lemma (cf. Kedlaya’s book on $p$-adic differential equations)
For any integers $m \geq 1$ and $n \geq 2$, we have

$$\nu_t(a_n(U^{(k+p^m)}) - a_n(V)) > m(n - 1)$$

- $V = U^{(k+p^m)} + t^{p^m}W$ with some $W \in M_{k+p^m-1}(\mathbb{F}_q[[t]])$
- $a_n(V) = (-1)^n \sum \text{(principal } (n \times n)-\text{minors of } V)$$
- By Laplace expansion, LHS is controlled by elementary divisors of $U^{(k+p^m)}$, which are bounded below by glissandoneess
Proof: Analyzing Newton polygons

- For $\alpha \in \mathbb{Q}_{\geq 0}$, multiplicity of slope $\alpha$ in $S_k(\Gamma_1(t))$ equals width of slope $\alpha$ segment of NP of $P_{U(k)}(X)$

Want to show

NPs for $k$ and $k + p^m$ agree on slope $\left\{ \begin{array}{ll} \leq k - 1 \\ \leq m \end{array} \right\}$ segments

- 1st segments: agree by $d(k, 0) = 1$
- Suppose NPs agree up to $N$-th such segments
- Let $0 < \alpha \leq \beta$ be the slopes of the $(N + 1)$-st segments, suppose $\alpha < k - 1$ and $\alpha \leq m$
- Let $n$ be the $x$-coordinate of the terminus of the lower $(N + 1)$-st segment
Proof: Analyzing Newton polygons

- When slope $\alpha$ appears in NP for $k$, 

\[ (n, v_t(a_n(U^{(k)}))) \]
Proof: Analyzing Newton polygons

- Picture $\Rightarrow v_t(a_n(U^{(k)})) \leq \alpha(n-1) \leq m(n-1)$
- Lemma $\Rightarrow v_t(a_n(U^{(k)})) < v_t(a_n(U^{(k+p^m)}) - a_n(V))$
- Slopes($V$) $\cap$ [0, $k-1$) = Slopes($U^{(k)}$) $\cap$ [0, $k-1$) implies
  
  $$v_t(a_n(V)) = v_t(a_n(U^{(k)}))$$

- These imply $v_t(a_n(U^{(k)})) = v_t(a_n(U^{(k+p^m)}))$ and thus
  
  $$\alpha = \beta, \quad d(k + p^m, \alpha) \geq d(k, \alpha)$$

- When $\alpha$ appears in NP for $k + p^m$: can be treated similarly
  $\Rightarrow$ get opposite inequality & ($N+1$)-st segments agree
Proof: Analyzing Newton polygons

- Picture $\Rightarrow v_t(a_n(U^{(k)})) \leq \alpha(n - 1) \leq m(n - 1)$
- Lemma $\Rightarrow v_t(a_n(U^{(k)})) < v_t(a_n(U^{(k+p^m)}) - a_n(V))$
- Slopes$(V) \cap [0, k - 1) = $ Slopes$(U^{(k)}) \cap [0, k - 1)$ implies
  $$v_t(a_n(V)) = v_t(a_n(U^{(k)}))$$
- These imply $v_t(a_n(U^{(k)})) = v_t(a_n(U^{(k+p^m)}))$ and thus
  $$\alpha = \beta, \quad d(k + p^m, \alpha) \geq d(k, \alpha)$$
- When $\alpha$ appears in NP for $k + p^m$: can be treated similarly
  $\rightarrow$ get opposite inequality & $(N + 1)$-st segments agree

Thank you for your attention!