Twistorial and space-time descriptions of massless infinite spin (super)particles and fields

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Abstract

We develop a new twistorial field formulation of a massless infinite spin particle. Unlike our previous approach arXiv:1805.09706, the quantization of such a world-line infinite spin particle model is carried without any gauge fixing. As a result, we construct a twistorial infinite spin field and derive its helicity decomposition. Using the field twistor transform, we construct the space-time infinite (continuous) spin field, which depends on the coordinate four-vector and additional commuting Weyl spinor. The equations of motion for infinite spin fields in the cases of integer and half-integer helicities are derived. We show that the infinite integer-spin field and infinite half-integer-spin field form the $\mathcal{N}=1$ infinite spin supermultiplet. The corresponding supersymmetry transformations are formulated and their on-shell algebra is derived. As a result, we find the field realization of the infinite spin $\mathcal{N}=1$ supersymmetry.

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1 Introduction

Recently, there has been a surge of interest in the description of particles and fields related to infinite (continuous) spin representations of the Poincaré group \([1, 2, 3]\). Although the physical status of such unitary representations is still not very clear, interest in them is caused by an identical spectrum of states of the infinite spin theory \([4]\) and the higher-spin theory \([5, 6, 7]\) (see also the reviews \([8, 9, 10]\)) and by its potential relation to the string theory (see \([11]\) and recent paper \([12]\) and references in it). Various problems related to the quantum-mechanical and field descriptions of such states were considered in a wide range of works devoted to particles and fields of an infinite spin (see, e.g., \([13] – [34]\)).

In this paper, we continue to develop an approach to the description of infinite spin particles and fields, initiated in our previous paper \([30]\), where we constructed a new model of an infinite (continuous) spin particle, which is a generalization of the twistor formulation of standard (with fixed helicity) massless particle \([35, 36, 37]\) to massless infinite spin representations. Making the quantization of the twistor model with gauge fixing and using some type of twistor transform, we reproduced these space-time infinite spin fields, which depend on the position four-vector and obey the Wigner-Bargmann equations \([1, 2, 3]\).

In paper \([30]\), when quantizing the constructed twistor model, we imposed partial gauge fixings for gauge symmetries generated by some first class constraints. This yielded after quantization “limited” fields describing the massless representations of the infinite spin. Now we will carry out the quantization of the twistor model without any gauge fixing and use a different field twistor transform. As a result of such a quantization procedure, we obtain infinite spin fields that have a more transparent decomposition into helicities in the twistor formulation. Besides, we show that our twistorial model reproduces the space-time–spinorial formulation of infinite spin fields with integer helicities proposed in \([31]\). In addition, using the field twistor transform, we can now get the space-time–spinorial formulation of infinite spin fields with half-integer helicities. Moreover, such a formulation allows us to construct a supermultiplet of infinite spins \([13]\) (see also the recent development in \([25]\)) and describe some of its properties. It is worth pointing out that the aspects of infinite spin supersymmetry have almost been unconsidered in the literature earlier.

The plan of this paper is as follows. In Sect. 2, we describe the world-line twistor formulation of the infinite (continuous) spin particle, which we built in \([30]\). We present the twistorial constraints of the model and coordinate twistor transform to space-time formulation. In Sect. 3, by using canonical transformation, we introduce suitable phase variables into the twistor formulation, in which all twistor constraints take a very simple form appropriate for quantization. The twistor field of the infinite spin particle is obtained as a solution of equations obtained from the canonical quantization of the first class constraints. The resulting twistor field has the U(1) charge that corresponds to the method of field description of the infinite spin particle. Integer or half-integer values of this charge correspond to infinite spin particles with integer or half-integer values of helicities. Also, the decomposition of the twistor field in an infinite number of states with fixed helicities is found. In Sect. 4, we construct a field twistor transform, which determines the space-time field of the infinite (continuous) spin particle, according to the twistor field obtained in the previous section. These space-time fields depend, in addition to the four-vector coordinates, also on the components of the additional commuting Weyl spinor. We have found the equations of motion for the fields of infinite spin particles, both in the case of integer and half-integer helicities. In Sect. 5, the \(\mathcal{N}=1\) infinite spin supermultiplet is constructed. On the mass-shell this supermultiplet consists of two complex infinite spin fields with integer and half-integer helicities,
respectively. In Sect. 6, we give some comments on the results obtained. In two Appendices, we give the proof of the equations of motion for the space-time infinite spin fields and present the explicit form of the momentum wave function in the space-time formulation.

2 Twistorial formulation of infinite spin particle

In [30], we constructed the twistorial formulation of the infinite (continuous) spin particle. It is described by the bosonic Weyl spinors $\pi$, $\bar{\pi}$, $\rho$, $\bar{\rho}$ and their canonically conjugated spinors $\omega$, $\bar{\omega}$, $\eta$, $\bar{\eta}$ with the components

$$\pi_\alpha, \quad \bar{\pi}_\dot{\alpha} := (\pi_\alpha)^*, \quad \rho_\alpha, \quad \bar{\rho}_\dot{\alpha} := (\rho_\alpha)^*, \quad \omega^\alpha, \quad \bar{\omega}^\dot{\alpha} := (\omega^\alpha)^*, \quad \eta^{\alpha}, \quad \bar{\eta}^{\dot{\alpha}} := (\eta^{\alpha})^*. \quad (2.1)$$

The nonzero Poisson brackets of these spinors are

$$\{\omega^\alpha, \pi_\beta\} = \{\eta^{\alpha}, \rho_\beta\} = \delta^\alpha_\beta, \quad \{\bar{\omega}^{\dot{\alpha}}, \bar{\pi}_\dot{\beta}\} = \{\bar{\eta}^{\dot{\alpha}}, \bar{\rho}_\dot{\beta}\} = \delta^{\dot{\alpha}}_\dot{\beta}. \quad (2.2)$$

We assume that all spinors are functions of a time parameter $\tau$. Twistorial Lagrangian of the infinite (continuous) spin particle is written in the form [30]:

$$\mathcal{L}_{\text{twistor}} = \pi_\alpha \dot{\omega}^\alpha + \bar{\pi}_\dot{\alpha} \dot{\bar{\omega}}^{\dot{\alpha}} + \rho_\alpha \dot{\eta}^\alpha + \bar{\rho}_\dot{\alpha} \dot{\bar{\eta}}^{\dot{\alpha}} + l \mathcal{M} + k \mathcal{U} + \ell \mathcal{F} + \bar{\ell} \bar{\mathcal{F}}, \quad (2.3)$$

where $l(\tau), k(\tau), \ell(\tau), \bar{\ell}(\tau)$ are the Lagrange multipliers for the constraints

$$\mathcal{M} := \pi^\alpha \rho_\alpha \bar{\rho}^{\dot{\alpha}} - M^2 \approx 0, \quad (2.4)$$
$$\mathcal{F} := \eta^\alpha \pi_\alpha - 1 \approx 0, \quad \bar{\mathcal{F}} := \bar{\pi}_\dot{\alpha} \bar{\eta}^{\dot{\alpha}} - 1 \approx 0, \quad (2.5)$$
$$\mathcal{U} := i (\omega^\alpha \pi_\alpha - \bar{\pi}_\dot{\alpha} \omega^{\dot{\alpha}} + \eta^\alpha \rho_\alpha - \bar{\rho}_\dot{\alpha} \eta^{\dot{\alpha}}) \approx 0, \quad (2.6)$$

and we use standard notation $\dot{x}(\tau) := \partial_\tau x(\tau)$ for any function of $\tau$. The real constant $M$ in (2.4) is represented in the form

$$M^2 = \mu^2 / 2, \quad (2.7)$$

where the dimensionful parameter $\mu$ fixes the irreducible representation of the Poincare group describing massless infinite (continuous) spin particles (see (2.22) below). This constant $\mu$ has nothing to do with the mass parameter in the twistor formulations of massive particles with spin (see, for example, [38] [39] [40] [41] [42] and references therein). One can check that four first class constraints (2.4), (2.5), (2.6) generate abelian Lie algebra with respect to the twistorial Poisson brackets (2.2). The Lie group $\mathcal{H}$ corresponding to this abelian Lie algebra acts in the space of

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1As in [30] we use the following notation. The space-time metric is $\eta_{mn} = \text{diag}(+1, -1, -1, -1)$. The totally antisymmetric tensor $\epsilon_{mnkl}$ has the component $\epsilon_{0123} = -1$. The two-component Weyl spinor indices are raised and lowered by $\epsilon_\alpha^\beta, \epsilon^{\alpha\beta}, \epsilon_{\dot{\alpha}\dot{\beta}}, \epsilon^{\dot{\alpha}\dot{\beta}}$ with the nonvanishing components $\epsilon_{12} = -\epsilon_{21} = \epsilon_{21}^\alpha = -\epsilon_{12} = 1: \psi_\alpha = \epsilon_\alpha^\beta \psi^\beta, \bar{\psi}^{\dot{\alpha}} = \epsilon^{\dot{\alpha}\dot{\beta}} \bar{\psi}_\dot{\beta}, \text{etc.}$ Relativistic $\sigma$-matrices are $(\sigma_m)_{\alpha\dot{\beta}} = (12; \sigma_1, \sigma_2, \sigma_3)_{\alpha\dot{\beta}}$, where $\sigma_1, \sigma_2, \sigma_3$ are the Pauli matrices. The matrices $(\sigma_m)^{\alpha\dot{\beta}} = \epsilon^{\dot{\alpha}\dot{\beta}\gamma}(\sigma_m)_{\gamma\delta}(12; -\sigma_1, -\sigma_2, -\sigma_3)_{\alpha\delta}$ satisfy $\sigma_m^{\alpha\dot{\gamma}} \sigma_m^{\gamma\delta} + \sigma_m^{\alpha\dot{\gamma}} \bar{\sigma}_m^{\dot{\gamma}\delta} = 2 \eta_{mn} \delta^{\alpha\dot{\beta}}$ and $\sigma_m^{\alpha\dot{\gamma}} \bar{\sigma}_m^{\dot{\gamma}\delta} = 2 \delta_{\alpha\dot{\beta}}^m$. The link between the Minkowski four-vectors and spinorial quantities is given by $A_{\alpha\dot{\beta}} = 1 / \sqrt{2} A_m (\sigma_m)^{\alpha\dot{\beta}}, \bar{A}_m = 1 / \sqrt{2} A_{\alpha\dot{\beta}} (\sigma_m)^{\dot{\alpha}\beta}, A_{\alpha\dot{\beta}} = 1 / \sqrt{2} A_m (\bar{\sigma}_m)^{\dot{\alpha}\beta}, A_m = 1 / \sqrt{2} A_{\alpha\dot{\beta}} (\bar{\sigma}_m)^{\dot{\alpha}\beta}$, so that $A^m B_m = A_{\alpha\dot{\beta}} B^{\alpha\dot{\beta}}$. 

spinors (2.1) as follows:

\[
\begin{pmatrix}
\pi_1 \\ \pi_2
\end{pmatrix}
\begin{pmatrix}
\rho_1 \\ \rho_2
\end{pmatrix}
\rightarrow
\begin{pmatrix}
\pi_1 \\ \pi_2
\end{pmatrix}
\begin{pmatrix}
\rho_1 \\ \rho_2
\end{pmatrix}
\left(
\begin{array}{cc}
e^{i\beta} & 0 \\
0 & e^{-i\beta}
\end{array}
\right),
\] (2.8)

\[
\begin{pmatrix}
\eta_1 \\ \eta_2
\end{pmatrix}
\begin{pmatrix}
\omega_1 \\ \omega_2
\end{pmatrix}
\rightarrow
\begin{pmatrix}
\eta_1 \\ \eta_2
\end{pmatrix}
\begin{pmatrix}
\omega_1 \\ \omega_2
\end{pmatrix}
\left(
\begin{array}{cc}
e^{-i\beta} & -\alpha \\
0 & e^{i\beta}
\end{array}
\right) + \left(\bar{\rho}_\alpha \pi^\alpha\right)
\begin{pmatrix}
\pi_1 \\ \pi_2
\end{pmatrix}
\begin{pmatrix}
\gamma \\ 0
\end{pmatrix}
\left(
\begin{array}{cc}
0 & -\gamma
\end{array}
\right),
\] (2.9)

plus complex conjugated transformations,

where \(\beta(\tau), \gamma(\tau) \in \mathbb{R}\) and \(\alpha(\tau) \in \mathbb{C}\backslash 0\) are the parameters of the gauge transformation. Constraints (2.4), (2.5), (2.6) eliminate (if we also add four gauge fixing conditions) eight degrees of freedom from sixteen degrees in (2.1).

The space-time (Wigner-Bargmann) formulation of the infinite (continuous) spin particle is given in the phase space with two four-vectors \(x_m \leftrightarrow x_{a\dot{a}}, y_m \leftrightarrow y_{a\dot{a}}\) and their conjugated momenta \(p_m \leftrightarrow p_{a\dot{a}}, q_m \leftrightarrow q_{a\dot{a}}\) with the Poisson brackets

\[
\{x^{\dot{\alpha}}, p_{\dot{\beta}}\} = \delta^{\dot{\alpha}}_{\dot{\beta}}, \quad \{y^{\dot{\alpha}}, q_{\dot{\beta}}\} = \delta^{\dot{\alpha}}_{\dot{\beta}}.
\] (2.10)

The link of twistorial and space-time formulations is carried out by the generalization of the Cartan-Penrose relations \[35, 36, 37\]

\[
p_{a\dot{\beta}} = \pi_a \bar{\pi}_{\dot{\beta}},
\] (2.11)

\[
q_{a\dot{\beta}} = \pi_a \bar{\rho}_{\dot{\beta}} + \rho_a \bar{\pi}_{\dot{\beta}},
\] (2.12)

and by the following incidence relations:

\[
\omega^\alpha = \bar{\pi}_a x^{a\alpha} + \bar{\rho}_a y^{a\alpha}, \quad \bar{\omega}^{\dot{\alpha}} = x^{a\dot{\alpha}} \pi_a + y^{a\dot{\alpha}} \rho_a,
\] (2.13)

\[
\eta^\alpha = \bar{\pi}_a \dot{y}^{a\alpha}, \quad \bar{\eta}^{\dot{\alpha}} = \dot{y}^{a\dot{\alpha}} \pi_a.
\] (2.14)

The space-time formulation of the Lagrangian for the infinite (continuous) spin particle in the first-order formalism is

\[
\mathcal{L}_{\text{sp.-time}}(\tau) = p_m \dot{x}^m + q_m \dot{y}^m + e T + e_1 T_1 + e_2 T_2 + e_3 T_3.
\] (2.15)

Here the functions \(e(\tau), e_1(\tau), e_2(\tau), e_3(\tau)\) are the Lagrange multipliers for constraints \[2, 3\]:

\[
T := p_m p^m \approx 0,
\] (2.16)

\[
T_1 := p_m q^m \approx 0,
\] (2.17)

\[
T_2 := q_m q^m + \mu^2 \approx 0,
\] (2.18)

\[
T_3 := p_m y^m - 1 \approx 0.
\] (2.19)

Note that pairs of spinors \(\pi_\alpha, \bar{\omega}^{\dot{\alpha}}\) and \(\rho_\alpha, \bar{\eta}^{\dot{\alpha}}\) form two Penrose twistors

\[
Z_A := \left(\pi_\alpha, \bar{\omega}^{\dot{\alpha}}\right), \quad Y_A := \left(\rho_\alpha, \bar{\eta}^{\dot{\alpha}}\right).
\] (2.20)

Conjugated spinors \(\bar{\pi}_{\dot{\alpha}}, \omega^\alpha\) and \(\bar{\rho}_{\dot{\alpha}}, \eta^\alpha\) constitute the dual twistors

\[
Z^A := \left(\omega^\alpha, -\bar{\pi}_{\dot{\alpha}}\right), \quad Y^A := \left(\eta^\alpha, -\bar{\rho}_{\dot{\alpha}}\right).
\] (2.21)
So the description of infinite spin particles uses with necessity a couple of twistors as opposed to the one-twistor description of the massless particle with fixed helicity.

On the shell of constraints \((2.4)-(2.6)\), \((2.7)\) (or on the shell of constraints \((2.16)-(2.19)\)) we have got the conditions for the Casimir operators

\[
p_m p^m \approx 0, \quad W_m W^m \approx -\mu^2
\]

where

\[
W_m = \frac{1}{2} \varepsilon_{mnpq} p^n M^{pq},
\]

are the components of the Pauli-Lubański pseudovector and \(p_m\) and \(M_{kl} := (x_k p_l - x_l p_k) + (y_k q_l - y_l q_k)\) are the Poincaré algebra generators. The models with Lagrangians \((2.3)\) and \((2.15)\) are equivalent at the classical level (see [30]) and describe massless particles with the infinite (continuous) spin.

Following [35], [36], [37] we choose the norms of twistors \((2.20), (2.21)\) as

\[
\bar{Z}^A Z_A = \omega^a \pi_\alpha - \bar{\pi}_\dot{\alpha} \bar{\omega}^{\dot{\alpha}}, \quad \bar{Y}^A Y_A = \eta^\rho \rho_\alpha - \bar{\rho}_\dot{\alpha} \bar{\eta}^{\dot{\alpha}},
\]

and write the constraint \((2.6)\) in concise form

\[
\mathcal{U} = i (\bar{Z}^A Z_A + \bar{Y}^A Y_A) \approx 0.
\]

The norm \(\bar{Z}^A Z_A\) of the twistor \(Z\) commutes with constraints \((2.4), (2.5), (2.6)\) and therefore is independent of \(\tau\). For a massless particle with fixed helicity the norm \(\bar{Z}^A Z_A\) defines the helicity operator

\[
\Lambda = i \frac{1}{2} \bar{Z}^A Z_A.
\]

So in the considered model of the infinite (continuous) spin particle, in view of the constraint \((2.25)\), the particle helicity is not fixed since it is proportional to \(-\bar{Y}^A Y_A\).

Let us make some comments on the role of the constraints \((2.5)\). First of all, we recall the statement from [30] that the matrices

\[
|| A^a_b || := \frac{1}{\sqrt{\pi \rho}} \begin{pmatrix} \pi_1 & \rho_1 \\ \pi_2 & \rho_2 \end{pmatrix},
\]

where \(\pi \rho := \pi^a \rho_\alpha\) can be considered as elements of \(SL(2, \mathbb{C})\). Moreover, the Lorentz transformation

\[
A \hat{k} A^+ = \hat{p},
\]

with \(A \in SL(2, \mathbb{C})\) given in \((2.27)\) converts test massless momentum

\[
\hat{k} := || k_{\dot{a}b} || := \begin{pmatrix} \sqrt{2k} & 0 \\ 0 & 0 \end{pmatrix}, \quad k^m = (k; 0, 0, k)
\]

to momentum \((2.11)\)

\[
\hat{p} := || p_{\alpha \beta} || := \begin{pmatrix} \pi_1 \bar{\pi}_1 & \pi_1 \bar{\pi}_2 \\ \pi_2 \bar{\pi}_1 & \pi_2 \bar{\pi}_2 \end{pmatrix},
\]
where we assume $|\pi \rho| = \sqrt{2}k$. Inverse Lorentz transformations that translate $\hat{p}$ into the basis of test momentum $\hat{k}$ are given by the matrix

$$A^{-1} := \| (A^{-1})_{a}^{\beta} \| := \frac{1}{\sqrt{\pi \rho}} \begin{pmatrix} -\rho^{1} - \rho^{2} & \pi^{1} \\ \pi^{2} & \pi^{2} \end{pmatrix}.$$ \hspace{1cm} (2.31)

In the twistor realization the Pauli-Lubanski pseudovector with components (2.23) has the form

$$W_{a b} = \Lambda \cdot p_{a b} - i \left[ (\bar{\pi}_{\gamma} \bar{\eta}^{\gamma}) \pi_{a} \rho_{\beta} - (\pi_{\gamma} \eta^{\gamma}) \rho_{a} \bar{\pi}_{\beta} \right] + i \left[ (\bar{\pi}_{\gamma} \bar{\rho}^{\gamma}) \pi_{a} \bar{\eta}_{\beta} - (\pi^{\gamma} \rho_{\gamma}) \eta_{a} \bar{\pi}_{\beta} \right],$$ \hspace{1cm} (2.32)

where $p_{a b}$ is defined in (2.11). Therefore, this pseudovector in the basis of test momentum is defined by the relation

$$\bar{W}_{a b} = (A^{-1})_{a}^{a} W_{a b} (A^{-1})_{b}^{b},$$ \hspace{1cm} (2.33)

and has the components

$$\bar{W}_{11} = |\pi \rho| V, \quad \bar{W}_{22} = 0, \quad \bar{W}_{12} = -2i|\pi \rho| (\bar{\pi}_{a} \bar{\rho}^{a}), \quad \bar{W}_{21} = 2i|\pi \rho| (\eta^{a} \pi_{a}),$$ \hspace{1cm} (2.34)

where (cf. (2.6))

$$V := \frac{i}{2} \left( \omega^{a} \pi_{a} - \bar{\pi}_{a} \omega^{a} - \eta^{a} \rho_{a} + \bar{\rho}_{a} \bar{\eta}^{a} \right) = \Lambda - \frac{i}{2} \left( \eta^{a} \rho_{a} - \bar{\rho}_{a} \bar{\eta}^{a} \right),$$ \hspace{1cm} (2.35)

Using the standard analysis of the massless representations of the Poincaré group, we find from expressions (2.34) that the quantities

$$V, \quad E = E_{1} + iE_{2} := \sqrt{2}M \left( \bar{\pi}_{a} \bar{\rho}^{a} \right), \quad \bar{E} = E_{1} - iE_{2} := \sqrt{2}M \left( \eta^{a} \pi_{a} \right)$$ \hspace{1cm} (2.36)

are the generators of the small subgroup $E(2)$. The generators $E$ and $\bar{E}$ yield translations in $\mathbb{C}$ which correspond to the parameter $\alpha$ in (2.8), (2.9). Then, constraints (2.3) fix the Casimir operator of the small subgroup,

$$E \bar{E} = (E_{1})^{2} + (E_{2})^{2} = 2M^{2} = \mu^{2},$$ \hspace{1cm} (2.37)

and therefore fix the Casimir operator $W^{2}$ of the Poincaré group.

## 3 Alternative quantization of twistorial model and infinite spin twistor field

The twistorial model with Lagrangian (2.3) possesses the gauge symmetry under transformations (2.9) generated by the first class constraints (2.4), (2.5) and (2.6). In [30], we performed the quantization of the model (2.3) after partial fixing the gauge as $s = 1 = \bar{s}$ (the definition of $s, \bar{s}$ is given below). In this paper, we quantize the model on the base of a more general procedure without any gauge fixing.

To simplify the quantization of the model, in [30] we made a canonical transformation of spinorial variables (2.1) which is motivated by the gauge transformation (2.8), (2.9). For variables $\pi_{a}, \rho_{a}$ we have

$$\begin{pmatrix} \pi_{1} \\ \pi_{2} \end{pmatrix} = \sqrt{M} \begin{pmatrix} p_{1}^{(z)} \\ p_{2}^{(z)} \end{pmatrix} \begin{pmatrix} 0 & p_{1}^{(z)} \\ p_{2}^{(z)} / p_{1}^{(z)} \end{pmatrix} \begin{pmatrix} 1 & p_{1}^{(t)} \\ 0 & 1 \end{pmatrix},$$ \hspace{1cm} (3.1)
where new variables are
\[ p^{(z)}_\alpha = \pi_\alpha / \sqrt{M}, \quad p^{(s)} = \pi^\alpha \rho_\alpha / M, \quad p^{(t)} = \rho_1 / \pi_1 \]
(3.2)
For the conjugated momentum variables we have
\[ \bar{\pi}_\alpha, \bar{\rho}_\alpha \rightarrow \bar{p}^{(z)}_\alpha = \bar{\pi}_\alpha / \sqrt{M}, \quad \bar{p}^{(s)} = \bar{\rho}_\alpha \bar{\pi}_\alpha / M, \quad \bar{p}^{(t)} = \bar{\rho}_1 / \pi_1. \]
(3.3)
The corresponding new coordinate variables \( z^\alpha, s, t \) are connected with old twistorial variables by means of relations (3.3) which also motivated by gauge transformations (2.9)
\[ \begin{pmatrix} \eta_1 & \omega_1 \\ \eta_2 & \omega_2 \end{pmatrix} = \begin{pmatrix} 0 & z_1 / \sqrt{M} \\ -t / \pi_1 & z_2 / \sqrt{M} \end{pmatrix} \begin{pmatrix} 1 & -p^{(t)} \\ 0 & 1 \end{pmatrix} + \frac{s}{M} \begin{pmatrix} \pi_1 & \rho_1 \\ \pi_2 & \rho_2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \]
(3.4)
where
\[ \omega^\alpha = \frac{1}{\sqrt{M}} z^\alpha - \frac{1}{M} s \rho^\alpha - \frac{\delta^\alpha_1}{\pi_1} t p^{(t)}, \]
\[ \eta^\alpha = \frac{1}{M} s \pi^\alpha + \frac{\delta^\alpha_1}{\pi_1} t. \]
(3.5)
(3.6)
By complex conjugation we obtain from (3.5), (3.6) the relations for conjugated coordinates \( \bar{z}^\alpha, \bar{s}, \bar{t} \). The nonzero Poisson brackets, which are consistent with (2.2), are
\[ \{ \bar{z}^\alpha, p^{(z)}_\beta \} = \delta^\alpha_\beta, \quad \{ \bar{z}^\alpha, \bar{p}^{(z)}_\beta \} = \delta^\alpha_\beta, \]
\[ \{ s, p^{(s)} \} = \{ \bar{s}, \bar{p}^{(s)} \} = 1, \quad \{ t, p^{(t)} \} = \{ \bar{t}, \bar{p}^{(t)} \} = 1. \]
(3.7)
In terms of new variables the constraints (2.4), (2.5), (2.6) take, up to multipliers, the form
\[ \mathcal{M}' := p^{(s)} \bar{p}^{(s)} - 1 \approx 0, \]
(3.8)
\[ \mathcal{F}' := t - 1 \approx 0, \quad \bar{\mathcal{F}}' := \bar{t} - 1 \approx 0, \]
(3.9)
\[ \mathcal{U}' := \frac{i}{2} \left( z^\alpha p^{(z)}_\alpha - \bar{z}^\alpha \bar{p}^{(z)}_\alpha \right) + i \left( s p^{(s)} - \bar{s} \bar{p}^{(s)} \right) \approx 0, \]
(3.10)
and we write twistorial Lagrangian (2.3) as
\[ \mathcal{L}' = p^{(z)}_\alpha \dot{z}^\alpha + \bar{p}^{(z)}_\alpha \dot{\bar{z}}^\alpha + p^{(s)} \dot{s} + \bar{p}^{(s)} \dot{\bar{s}} + p^{(t)} \dot{t} + \bar{p}^{(t)} \dot{\bar{t}} + l' \mathcal{M}' + k' \mathcal{U}' + \ell' \mathcal{F}' + \bar{\ell}' \bar{\mathcal{F}}', \]
(3.11)
where \( l'(\tau), k'(\tau), \ell'(\tau), \bar{\ell}'(\tau) \) are the Lagrange multipliers for the constraints (3.8), (3.9), (3.10).

The important fact is that in terms of the new variables all constraints (3.8), (3.9), (3.10) acquire a simpler form which is suitable for quantization [..] = i {.,.}. We will perform quantization in the momentum representation, where the operators of the dynamical variables are realized as follows (we take \( \hbar = 1 \)):
\[ \dot{z}^\alpha = i \frac{\partial}{\partial p^{(z)}_\alpha}, \quad \dot{\bar{p}}^{(z)}_\alpha = p^{(z)}_\alpha, \quad \dot{\bar{z}}^\alpha = i \frac{\partial}{\partial \bar{p}^{(z)}_\alpha}, \quad \dot{\bar{p}}^{(z)}_\alpha = \bar{p}^{(z)}_\alpha, \]
\[ \dot{s} = i \frac{\partial}{\partial p^{(s)}}, \quad \dot{p}^{(s)} = p^{(s)}, \quad \dot{\bar{s}} = i \frac{\partial}{\partial \bar{p}^{(s)}}, \quad \dot{\bar{p}}^{(s)} = \bar{p}^{(s)}, \]
\[ \dot{t} = i \frac{\partial}{\partial p^{(t)}}, \quad \dot{\bar{p}}^{(t)} = p^{(t)}, \quad \dot{\bar{t}} = i \frac{\partial}{\partial \bar{p}^{(t)}}, \quad \dot{\bar{p}}^{(t)} = \bar{p}^{(t)}. \]
(3.12)
The corresponding wave function has the form
\[ \Psi = \Psi(p^{(z)}_\alpha, \bar{p}^{(z)}_\alpha; p^{(s)}, \bar{p}^{(s)}; p^{(t)}, \bar{p}^{(t)}) . \] (3.13)

The wave function (3.13) describes physical states and obeys the equations
\[ (p^{(s)} \bar{p}^{(s)} - 1) \Psi^{(c)} = 0 , \] (3.14)
\[ \frac{\partial}{\partial p^{(t)}} \Psi^{(c)} = \frac{\partial}{\partial \bar{p}^{(t)}} \Psi^{(c)} = -i \Psi^{(c)} , \] (3.15)
\[ \left[ \frac{1}{2} \left( p^{(z)}_\alpha \frac{\partial}{\partial p^{(z)}_\alpha} - \bar{p}^{(z)}_\alpha \frac{\partial}{\partial \bar{p}^{(z)}_\alpha} \right) + p^{(s)} \frac{\partial}{\partial p^{(s)}} - \bar{p}^{(s)} \frac{\partial}{\partial \bar{p}^{(s)}} \right] \Psi^{(c)} = c \Psi^{(c)} , \] (3.16)

which are quantum counterparts of constraints (3.8)-(3.10) in the representation (3.12). In equation (3.16) we have introduced the constant parameter \( c \) related to the ambiguity of operator ordering (it is an analog of the vacuum energy in the model of quantum oscillator).

Equations (3.14) and (3.15) can be solved explicitly and solution is
\[ \Psi^{(c)} = \delta \left( p^{(s)} \cdot \bar{p}^{(s)} - 1 \right) e^{-i(p^{(t)} + \bar{p}^{(t)})} \tilde{\Psi}^{(c)}(p^{(z)}_\alpha, \bar{p}^{(z)}_\alpha; (p^{(s)}/\bar{p}^{(s)})^{1/2}) , \] (3.17)

where \( \tilde{\Psi}^{(c)}(p^{(z)}_\alpha, \bar{p}^{(z)}_\alpha; (p^{(s)}/\bar{p}^{(s)})^{1/2}) \) is the function of the complex coordinates \( p^{(z)}_\alpha, \bar{p}^{(z)}_\alpha \in \mathbb{C}^2 = \mathbb{C}^2 \setminus (0,0) \) and coordinate \( (p^{(s)}/\bar{p}^{(s)})^{1/2} = e^{i\varphi} \) on the unit circle: \( p^{(s)} \cdot \bar{p}^{(s)} = 1 \), i.e. \( p^{(s)} = \exp(i\varphi) \) and \( p^{(s)}/\bar{p}^{(s)} = \exp(2i\varphi) \). In general, the function \( \tilde{\Psi} \) which depends on \( (p^{(s)}/\bar{p}^{(s)})^{1/2} = e^{i\varphi} \) can be expanded into the Fourier series:
\[ \tilde{\Psi}^{(c)}(p^{(z)}, \bar{p}^{(z)}; (p^{(s)}/\bar{p}^{(s)})^{1/2}) = \sum_{k = -\infty}^{\infty} \left( \frac{p^{(s)}}{\overline{p}^{(s)}} \right)^{-k/2} \tilde{\psi}^{(c+k)}(p^{(z)}, \bar{p}^{(z)}) . \] (3.18)

Therefore, solution (3.17) to the wave function can be written in the form
\[ \Psi^{(c)} = \delta \left( p^{(s)} \cdot \bar{p}^{(s)} - 1 \right) e^{-i(p^{(t)} + \bar{p}^{(t)})} \sum_{k = -\infty}^{\infty} \left( \frac{p^{(s)}}{\overline{p}^{(s)}} \right)^{-k/2} \tilde{\psi}^{(c+k)}(p^{(z)}, \bar{p}^{(z)}) . \] (3.19)

Due to the last constraint (3.16) the coefficients \( \tilde{\psi}^{(c+k)}(p^{(z)}, \bar{p}^{(z)}) \) in expansion (3.19) satisfy the equations
\[ \frac{1}{2} \left( p^{(z)}_\alpha \frac{\partial}{\partial p^{(z)}_\alpha} - \bar{p}^{(z)}_\alpha \frac{\partial}{\partial \bar{p}^{(z)}_\alpha} \right) \tilde{\psi}^{(c+k)}(p^{(z)}, \bar{p}^{(z)}) = (c + k) \tilde{\psi}^{(c+k)}(p^{(z)}, \bar{p}^{(z)}) . \] (3.20)

Expression (3.19) contains the delta-function \( \delta(p^{(s)} \cdot \bar{p}^{(s)} - 1) \). Therefore, in formula (3.19) we can make replacements
\[ \delta \left( p^{(s)} \cdot \bar{p}^{(s)} - 1 \right) \frac{p^{(s)}}{\overline{p}^{(s)}} \rightarrow \delta \left( p^{(s)} \cdot \bar{p}^{(s)} - 1 \right) (p^{(s)})^2 , \]
\[ \delta \left( p^{(s)} \cdot \bar{p}^{(s)} - 1 \right) \frac{\bar{p}^{(s)}}{p^{(s)}} \rightarrow \delta \left( p^{(s)} \cdot \bar{p}^{(s)} - 1 \right) (\bar{p}^{(s)})^2 , \] (3.21)
and then write the wave function \((3.19)\) as
\[
\Psi^{(c)} = \delta \left( (p^{(s)} \cdot \tilde{p}^{(s)} - 1) e^{-i(p^{(t)} + \tilde{p}^{(t)})} \sum_{k=1}^{\infty} (p^{(s)} \cdot \tilde{p}^{(s)})^k \tilde{\psi}^{(c+k)}(p^{(z)}, \tilde{p}^{(z)}) \right)
\]
\[
+ \sum_{k=1}^{\infty} (p^{(s)} \cdot \tilde{p}^{(s)})^k \tilde{\psi}^{(c-k)}(p^{(z)}, \tilde{p}^{(z)}) \right). \quad (3.22)
\]

Now we use relations \((3.2)\) and \((3.3)\) to restore the dependence of the wave function \((3.19)\) on the twistor variables. As a result, we obtain the twistor wave function in the form (we leave the same notation for functions as in \((3.19)\))
\[
\Psi^{(c)}(\pi, \pi; \rho, \bar{\rho}) = \delta \left( (\pi \rho)(\bar{\rho} \pi) - M^2 \right) e^{-i \left( \frac{\rho_1}{\pi_1} + \bar{\rho}_1 \bar{\pi}_1 \right) \tilde{\Psi}^{(c)}(\pi, \bar{\pi}; \rho, \bar{\rho})}, \quad (3.23)
\]
where \(\tilde{\Psi}^{(c)}(\pi, \bar{\pi}; \rho, \bar{\rho})\) has the form
\[
\tilde{\Psi}^{(c)}(\pi, \bar{\pi}; \rho, \bar{\rho}) = \tilde{\psi}^{(c)}(\pi, \bar{\pi}) + \sum_{k=1}^{\infty} (\bar{\rho} \pi)^k \tilde{\psi}^{(c+k)}(\pi, \bar{\pi}) + \sum_{k=1}^{\infty} (\pi \rho)^k \tilde{\psi}^{(c-k)}(\pi, \bar{\pi}), \quad (3.24)
\]
and the coefficients \(\tilde{\psi}^{(c \pm k)}(\pi, \bar{\pi}) := M^{-|c \pm k|} \tilde{\psi}^{(c \pm k)}(p^{(z)}, \bar{p}^{(z)})\) (which are the functions on the two-dimensional complex plane \(\mathbb{C}^2 \setminus \{0, 0\}\) with the coordinates \(\pi_\alpha\) are subjected to the condition
\[
\frac{1}{2} \left( \pi_\alpha \frac{\partial}{\partial \pi_\alpha} - \bar{\pi}_\alpha \frac{\partial}{\partial \bar{\pi}_\alpha} \right) \tilde{\psi}^{(c \pm k)}(\pi, \bar{\pi}) = (c \pm k) \tilde{\psi}^{(c \pm k)}(\pi, \bar{\pi}). \quad (3.25)
\]

In \((3.23)\) we have used the concise notation: \((\pi \rho) := \pi^\beta \rho_\beta, (\bar{\rho} \bar{\pi}) := \bar{\rho}_\beta \bar{\pi}^\beta\).

Under the shift of variables \(\rho_\alpha\) and \(\bar{\rho}_\dot{\alpha}\), the twistor wave function, obtained in \((3.23)\), is transformed in the following way:
\[
\Psi^{(c)}(\pi_\alpha, \bar{\pi}_\dot{\alpha}; \rho_\alpha + \kappa \pi_\alpha, \bar{\rho}_\dot{\alpha} + \bar{\kappa} \bar{\pi}_\dot{\alpha}) = e^{-i (\kappa + \bar{\kappa})} \Psi^{(c)}(\pi_\alpha, \bar{\pi}_\dot{\alpha}; \rho_\alpha, \bar{\rho}_\dot{\alpha}), \quad \forall \kappa \in \mathbb{C}, \quad (3.26)
\]
and, therefore, the wave function \((3.23)\) satisfies the equations
\[
i \pi_\alpha \frac{\partial}{\partial \rho_\alpha} \Psi^{(c)} = \Psi^{(c)}, \quad i \bar{\pi}_\dot{\alpha} \frac{\partial}{\partial \bar{\rho}_\dot{\alpha}} \Psi^{(c)} = \Psi^{(c)}, \quad (3.27)
\]
which are quantum counterparts of constraints \((2.5)\). Also, the twistor wave function \((3.23)\) obeys the condition
\[
\left( \pi_\alpha \frac{\partial}{\partial \pi_\alpha} - \bar{\pi}_\dot{\alpha} \frac{\partial}{\partial \bar{\pi}_\dot{\alpha}} + \rho_\alpha \frac{\partial}{\partial \rho_\alpha} - \bar{\rho}_\dot{\alpha} \frac{\partial}{\partial \bar{\rho}_\dot{\alpha}} \right) \Psi^{(c)} = 2c \Psi^{(c)}, \quad (3.28)
\]
which is a quantum counterpart of constraint \((2.6)\). Equation \((3.28)\) is equivalent to the homogeneity condition
\[
\Psi^{(c)}(e^{i\gamma} \pi_\alpha, e^{-i\gamma} \bar{\pi}_\dot{\alpha}; e^{i\gamma} \rho_\alpha, e^{-i\gamma} \bar{\rho}_\dot{\alpha}) = e^{2ic\gamma} \Psi^{(c)}(\pi_\alpha, \bar{\pi}_\dot{\alpha}; \rho_\alpha, \bar{\rho}_\dot{\alpha}) \quad (3.29)
\]
of the twistor field $\Psi^{(c)}$ with respect to $U(1)$ transformations with the constant phase parameter $\gamma$, where the constant $c$ plays the role of the $U(1)$ charge. The uniqueness condition of the twistor field requires integer or half-integer values for the constant $c$:

$$2c \in \mathbb{Z}.$$  \hfill (3.30)

Now we check that the twistor wave function $\Psi^{(c)}$ describes the massless particle of the infinite (continuous) spin. To do this, we use the twistor realization of the Poincaré algebra generators

$$P_{\alpha\beta} = \pi_\alpha \bar{\pi}_\beta, \quad M_{\alpha\beta} = \pi_\alpha \frac{\partial}{\partial \pi^\beta} + \rho_\alpha \frac{\partial}{\partial \rho^\beta}, \quad \bar{M}_{\alpha\beta} = \bar{\pi}_\alpha \frac{\partial}{\partial \bar{\pi}^\beta} + \bar{\rho}_\alpha \frac{\partial}{\partial \bar{\rho}^\beta},$$  \hfill (3.31)

in terms of which the Pauli-Lubanski operator

$$\mathcal{W}_{\alpha\gamma} = \bar{M}_{\gamma\beta} P^\beta_\alpha - M_{\alpha\beta} P^\beta_\gamma,$$

takes the form (see [30])

$$\mathcal{W}_{\alpha\gamma} = \pi_\alpha \pi_\gamma \Lambda + \frac{1}{2} \left[ \pi_\alpha \bar{\rho}_\gamma \frac{\partial}{\partial \rho^\beta} - \rho_\alpha \bar{\pi}_\gamma \frac{\partial}{\partial \bar{\pi}^\beta} \right],$$  \hfill (3.32)

where

$$\Lambda = -\frac{1}{2} \left( \pi_\beta \frac{\partial}{\partial \pi^\gamma} - \bar{\pi}_\beta \frac{\partial}{\partial \bar{\pi}^\gamma} \right).$$  \hfill (3.34)

Direct calculations show that

$$\mathcal{W}_{\alpha\gamma} \pi^\beta_\rho \rho_\beta = \mathcal{W}_{\alpha\gamma} \bar{\rho}_\beta \bar{\pi}_\beta = 0, \quad \mathcal{W}_{\alpha\gamma} \left( \frac{\rho_1}{\pi_1} + \frac{\bar{\rho}_1}{\bar{\pi}_1} \right) = \left( \pi \rho \right) \frac{\epsilon_{\alpha} \epsilon_{\phi} \bar{\pi}_{\gamma}}{\pi_1} - \left( \bar{\rho} \bar{\pi} \right) \frac{\epsilon_{\gamma} \epsilon_{\phi} \pi_{\alpha}}{\bar{\pi}_1}.$$  \hfill (3.35)

By using (3.35) one can find the action of the Pauli-Lubanski operator (3.33) on the twistorial wave function (3.23):

$$\mathcal{W}_{\alpha\gamma} \Psi^{(c)} = \delta \left( \left( \pi \rho \right) \left( \bar{\rho} \bar{\pi} \right) - M^2 \right) e^{-i \left( \frac{\rho_1}{\pi_1} + \frac{\bar{\rho}_1}{\bar{\pi}_1} \right)} D_{\alpha\gamma} \tilde{\Psi}^{(c)},$$

where the operator $D_{\alpha\gamma}$, acting on the reduced twistor field $\tilde{\Psi}^{(c)}$, takes the form

$$D_{\alpha\gamma} := \pi_\alpha \pi_\gamma \Lambda + i \left( \frac{\rho_1}{\pi_1} \epsilon_{\alpha} \epsilon_{\phi} \bar{\pi}_{\gamma} \right) e^{-i \left( \frac{\rho_1}{\pi_1} + \frac{\bar{\rho}_1}{\bar{\pi}_1} \right)} \left( \pi \rho \right) \frac{\epsilon_{\gamma} \epsilon_{\phi} \pi_{\alpha}}{\pi_1}.$$  \hfill (3.37)

Acting on both sides of (3.36) by the operator $\mathcal{W}^{\alpha\gamma}$ and taking into account the identity $D^\alpha_{\alpha\gamma} D_{\alpha\gamma} \tilde{\Psi} = -2 M^2 \tilde{\Psi}$, we obtain that

$$\Psi^{\alpha\gamma} \mathcal{W}^{\alpha\gamma} \Psi^{(c)} = -2 M^2 \Psi^{(c)} = -\mu^2 \Psi^{(c)}.$$  \hfill (3.38)

So the twistor field (3.23) indeed describes a massless particle of the infinite spin.

The states with fixed helicities are the eigenvectors of the helicity operator $\Lambda$ which is defined as a projection of the total angular momentum $\vec{J}$ in the direction of motion with the momentum
operator \( \mathcal{P}_m = (\mathcal{P}_0, \vec{P}) \): \( \Lambda = \vec{J} \vec{P}/\mathcal{P}_0 \). This operator can be rewritten in terms of the Pauli-Lubański pseudovector \((3.33)\) in the form

\[
\Lambda = \frac{\mathcal{W}_0}{\mathcal{P}_0} = \frac{\mathcal{W}_{\alpha \gamma} \bar{\sigma}_0^{\alpha \gamma}}{\pi_\beta \bar{\pi}_\beta \bar{\sigma}_0^{\beta \delta}} = \sum_{\gamma=\gamma} \frac{\mathcal{W}_{\alpha \gamma}}{\pi_\beta \bar{\pi}_\beta \bar{\sigma}_0^{\beta \delta}}, \tag{3.39}
\]

As we see from \((3.36)\) the representation \((3.23)\) of the twistor field is unsuitable for helicity expansion of the continuous spin wave function.

For finding helicity expansion of the twistor wave function \((3.23)\) we represent expression \((3.23)\) in the following form:

\[
\Psi^{(c)}(\pi, \bar{\pi}; \rho, \bar{\rho}) = \delta \left( ((\pi \rho)(\bar{\rho} \bar{\pi}) - M^2) \exp \left( -i(\pi \bar{\rho} \bar{\pi} + \rho \bar{\pi} \bar{\rho}) \bar{\sigma}_0^{\gamma \gamma} \right) \right) \Psi^{(c)}(\pi, \bar{\pi}; \rho, \bar{\rho}), \tag{3.40}
\]

where

\[
\Psi^{(c)}(\pi, \bar{\pi}; \rho, \bar{\rho}) = \exp \left( i \left( \frac{\bar{\rho} \bar{\pi}}{\pi_1 \bar{\pi}_1 + (\pi \rho) \bar{\pi}_1} \right) \right) \Psi^{(c)}(\pi, \bar{\pi}; \rho, \bar{\rho}), \tag{3.41}
\]

and the function \( \tilde{\Psi}^{(c)}(\pi, \bar{\pi}; \rho, \bar{\rho}) \) is defined in \((3.24)\). In fact the exponent in \((3.40)\) is expressed in terms of zero components of four-vectors \((2.11)\) and \((2.12)\). So formula \((3.40)\) takes the form

\[
\Psi^{(c)}(\pi, \bar{\pi}; \rho, \bar{\rho}) = \delta \left( ((\pi \rho)(\bar{\rho} \bar{\pi}) - M^2) \right) e^{-i q_0/\mathcal{P}_0} \tilde{\Psi}^{(c)}(\pi, \bar{\pi}; \rho, \bar{\rho}), \tag{3.42}
\]

in which the quantities \( p_0 \) and \( q_0 \) have the generalized Cartan-Penrose representations \((2.11)\) and \((2.12)\).

Expression of \( \tilde{\Psi}^{(c)}(\pi, \bar{\pi}; \rho, \bar{\rho}) \), defined in \((3.41)\), contains the spinors \( \rho_\alpha, \bar{\rho}_{\bar{\alpha}} \) only in the contractions \((\pi \rho), (\bar{\rho} \bar{\pi})\). But, expanding \( \exp \left( i \left( \frac{(\bar{\rho} \bar{\pi})}{\pi_1 \bar{\pi}_1 (\pi \rho) \bar{\pi}_1} \right) \right) \) in \((3.41)\) as the Fourier series and using \( \delta \left( ((\pi \rho)(\bar{\rho} \bar{\pi}) - M^2) \right) \), we can represent \( \tilde{\Psi}^{(c)}(\pi, \bar{\pi}; \rho, \bar{\rho}) \) in the form

\[
\tilde{\Psi}^{(c)}(\pi, \bar{\pi}; \rho, \bar{\rho}) = \psi^{(c)}(\pi, \bar{\pi}) + \sum_{k=1}^{\infty} (\frac{\bar{\rho} \bar{\pi}}{\pi_1 \bar{\pi}_1})^k \psi^{(c+k)}(\pi, \bar{\pi}) + \sum_{k=1}^{\infty} (\pi \rho)^k \psi^{(c-k)}(\pi, \bar{\pi}), \tag{3.43}
\]

similar to \((3.24)\) for \( \Psi^{(c)}(\pi, \bar{\pi}; \rho, \bar{\rho}) \). The fields \( \psi^{(c+k)}(\pi, \bar{\pi}) \) are subjected to the conditions

\[
\frac{1}{2} \left( \pi_\alpha \frac{\partial}{\partial \pi_\alpha} - \bar{\pi}_{\bar{\alpha}} \frac{\partial}{\partial \bar{\pi}_{\bar{\alpha}}} \right) \psi^{(c+k)}(\pi, \bar{\pi}) = (c \pm k) \psi^{(c \pm k)}(\pi, \bar{\pi}), \tag{3.44}
\]

similar to \((3.25)\).

\(^{2}\)Here we take into account the equality

\[
\frac{\rho_1}{\pi_1} + \frac{\bar{\rho}_{\bar{1}}}{\bar{\pi}_{\bar{1}}} = \sum_{\alpha=\alpha} \frac{\pi_\alpha \bar{\rho}_{\bar{\alpha}} + \rho_\alpha \bar{\pi}_{\bar{\alpha}}}{\pi_\beta \bar{\pi}_{\beta}} \tag{3.39}
\]
The representation (3.42) for the twistorial field of the infinite spin particle is convenient to clarify its helicity content. Considering the relation

\[ \mathcal{W}_{\alpha\dot{\alpha}} \frac{q_0}{p_0} = \frac{1}{2} \left[ \pi_{\alpha} \rho_{\dot{\alpha}} - \rho_{\alpha} \bar{\pi}_{\dot{\alpha}} - \frac{\pi_{\alpha} \bar{\pi}_{\dot{\alpha}}}{\pi_{\delta} \bar{\pi}_{\dot{\delta}}} \sum_{\beta=\dot{\beta}} (\pi_{\beta} \bar{\rho}_{\dot{\beta}} - \rho_{\beta} \bar{\pi}_{\dot{\beta}}) \right] \quad (3.45) \]

which holds for the operator (3.33) and using the generalized Cartan-Penrose representations (2.11), (2.12) for \( p_0, q_0 \), we obtain that the action of the Pauli-Lubanski pseudovector on the field (3.47) gives

\[ \mathcal{W}_{\alpha\dot{\alpha}} \psi^{(c)} = \delta (\pi \rho)(\bar{\rho} \bar{\pi}) - M^2) e^{-i q_0/p_0} \left( \hat{D}_{\alpha\dot{\alpha}} \psi^{(c)} + \sum_{k=1}^{\infty} (\bar{\rho} \pi)^k \hat{D}_{\alpha\dot{\alpha}} \psi^{(c+k)} + \sum_{k=1}^{\infty} (\pi \rho)^k \hat{D}_{\alpha\dot{\alpha}} \psi^{(c-k)} \right) , \quad (3.46) \]

where

\[ \hat{D}_{\alpha\dot{\alpha}} = \pi_{\alpha} \bar{\pi}_{\dot{\alpha}} \Lambda - \frac{i}{2} (\pi_{\alpha} \bar{\rho}_{\dot{\alpha}} - \rho_{\alpha} \bar{\pi}_{\dot{\alpha}}) \]

(3.47)

In contrast to quantity (3.37), the operator (3.47) satisfies the property

\[ \sum_{\alpha=\dot{\alpha}} \hat{D}_{\alpha\dot{\alpha}} = \sum_{\alpha=\dot{\alpha}} \pi_{\alpha} \bar{\pi}_{\dot{\alpha}} \Lambda = \mathbb{P}_0 \Lambda . \]

As a result, the helicity operator (3.39) acts on the twistorial field in the following way:

\[ \Lambda \psi^{(c)} = \frac{\mathcal{W}_{\alpha\dot{\alpha}}}{\mathcal{W}_{\alpha\dot{\alpha}}} \psi^{(c)} = \sum_{\alpha=\dot{\alpha}} \frac{\mathcal{W}_{\alpha\dot{\alpha}}}{\pi_{\beta} \bar{\pi}_{\dot{\beta}}} \psi^{(c)} \quad (3.48) \]

\[ = \delta (\pi \rho)(\bar{\rho} \bar{\pi}) - M^2) e^{-i q_0/p_0} \left( \Lambda \psi^{(c)} + \sum_{k=1}^{\infty} (\bar{\rho} \pi)^k \Lambda \psi^{(c+k)} + \sum_{k=1}^{\infty} (\pi \rho)^k \Lambda \psi^{(c-k)} \right) . \]

As we see from this expression, the action of the helicity operator \( \Lambda \) is defined by the action of the operator \( \Lambda \) on the functions \( \psi^{(c+k)}(\pi, \bar{\pi}) \) (where \( k \in \mathbb{Z} \)) which are parameterized by two dimensional complex variable \( \pi_{\alpha} \). Due to the conditions (3.44), the fields \( \psi^{(c+k)}(\pi, \bar{\pi}) \) are the eigenvectors of the helicity operator (3.34):

\[ \Lambda \psi^{(c+k)}(\pi, \bar{\pi}) = -(c + k) \psi^{(c+k)}(\pi, \bar{\pi}) \quad (3.49) \]

Thus, in view of this equation and (3.48), the twistorial wave function (3.42) of the infinite spin particle describes an infinite number of massless states \( \psi^{(c+k)} \) whose helicities are equal to integer (for integer \( c \)) or half-integer (for half-integer \( c \)) values and these helicities run from \(-\infty\) to \(+\infty\).
Note that the helicity content of the twistor field $\Psi^{(c)}$ is the same for all integer values of the U(1) charge $c$. The distinction is only in the shift of the infinite tower of states (with all possible integer helicities) in $k$ by integer difference of the values of $c$. The same situation occurs for half-integer values of $c$, when the tower of states with half-integer helicities is also the same for various $c \in (\mathbb{Z} + 1/2)$. The choice of number $c$, which takes a fixed value, determines only a specific way of describing the same infinite spin representations for all integer (or half-integer) spins. In fact, there are only two independent values for $c$, namely $c = 0$ and $c = -\frac{1}{2}$.

Recall that the twistorial wave function (3.42) is complex and therefore all component fields in the expansion (3.43) are also complex. Then, in the CPT-invariant theory we must consider together with the field $\Psi^{(c)}$ its complex conjugated field $(\Psi^{(c)})^*$ which (due to the condition (3.28)) has the opposite charge

$$(\Psi^{(c)})^* := \bar{\Psi}^{(-c)}.$$ (3.50)

To describe the bosonic infinite spin representation related to all integer helicities, we put

$$c = 0$$ (3.51)

and consider the twistorial field

$$\Psi^{(0)}(\pi, \bar{\pi}; \rho, \bar{\rho}).$$ (3.52)

The complex conjugate field $\bar{\Psi}^{(0)}$ has also zero charge. Similarly, to describe the infinite spin representation related to half-integer helicities we take for $c$ the value

$$c = -\frac{1}{2}.$$ (3.53)

The corresponding wave function

$$\Psi^{(-1/2)}(\pi, \bar{\pi}; \rho, \bar{\rho})$$ (3.54)

contains in its expansion (3.43) all half-integer helicities. The complex conjugate field

$$\bar{\Psi}^{(+1/2)}(\pi, \bar{\pi}; \rho, \bar{\rho})$$ (3.55)

possesses the charge $c = +1/2$.

To conclude this section, we stress that the choice of the integer or half-integer values of the U(1)-charge $c$ can be considered as fixing of different boundary conditions, by analogy with the choice of the Ramond or Neveu-Schwarz sectors in the fermionic string theories (see, e.g., [43]).

4 Twistor transform for infinite spin fields

In [30], performing a quantization of the twistor model with special gauge fixing, we obtained space-time fields which depended on the coordinates $x^m$ of position four-vector and obeyed the Wigner-Bargmann equations [2, 3] following from Lagrangian (2.15). Now we show that our twistorial model reproduces the formulation of the infinite spin field model developed in [31].

The link of our twistor field with the fields introduced in [31] follows from the explicit solutions (3.42) and (3.43) obtained for the twistor wave functions. For further convenience we introduce the dimensionless spinor

$$\xi_\alpha := M^{-1/2} \rho_\alpha, \quad \bar{\xi}_\dot{\alpha} := M^{-1/2} \bar{\rho}_{\dot{\alpha}}.$$ (4.1)

\footnote{The interpretation of the parameter $c$ as U(1)-charge follows from equations (3.28) and (3.29).}
Then, the twistor wave function (3.52) of infinite integer-spin particle (see (3.42) and (3.43)) takes the form
\[
\Psi^{(0)}(\pi, \bar{\pi}; \xi, \bar{\xi}) = \delta \left( (\pi \xi)(\bar{\xi}\bar{\pi}) - M \right) e^{-iq_0/p_0} \Psi^{(0)}(\pi, \bar{\pi}; \xi, \bar{\xi}), \quad (4.2)
\]

\[
\hat{\Psi}^{(0)} = \psi^{(0)}(\pi, \bar{\pi}) + \sum_{k=1}^{\infty} (\bar{\xi}\bar{\pi})^k \psi^{(k)}(\pi, \bar{\pi}) + \sum_{k=1}^{\infty} (\pi \xi)^k \psi^{(-k)}(\pi, \bar{\pi}).
\]

In the expansion of \(\hat{\Psi}^{(0)}\) all components \(\psi^{(k)}(\pi, \bar{\pi})\) (here \(k \in \mathbb{Z}\)) are complex functions (fields), in general. The wave function of the infinite half-integer spin particle (3.54) is
\[
\Psi^{(-\frac{1}{2})}(\pi, \bar{\pi}; \xi, \bar{\xi}) = \delta \left( (\pi \xi)(\bar{\xi}\bar{\pi}) - M \right) e^{-iq_0/p_0} \Psi^{(-\frac{1}{2})}(\pi, \bar{\pi}; \xi, \bar{\xi}), \quad (4.3)
\]
\[
\hat{\Psi}^{(-\frac{1}{2})} = \psi^{(-\frac{1}{2})}(\pi, \bar{\pi}) + \sum_{k=1}^{\infty} (\bar{\xi}\bar{\pi})^k \psi^{(-\frac{1}{2}+k)}(\pi, \bar{\pi}) + \sum_{k=1}^{\infty} (\pi \xi)^k \psi^{(-\frac{1}{2}-k)}(\pi, \bar{\pi}).
\]

The expansion of the complex conjugated wave function (3.55) is
\[
\Psi^{(+\frac{1}{2})}(\pi, \bar{\pi}; \xi, \bar{\xi}) = \delta \left( (\pi \xi)(\bar{\xi}\bar{\pi}) - M \right) e^{iq_0/p_0} \Psi^{(+\frac{1}{2})}(\pi, \bar{\pi}; \xi, \bar{\xi}), \quad (4.4)
\]
\[
\hat{\Psi}^{(+\frac{1}{2})} = \psi^{(\frac{1}{2})}(\pi, \bar{\pi}) + \sum_{k=1}^{\infty} (\bar{\xi}\bar{\pi})^k \psi^{(\frac{1}{2}+k)}(\pi, \bar{\pi}) + \sum_{k=1}^{\infty} (\pi \xi)^k \psi^{(\frac{1}{2}-k)}(\pi, \bar{\pi}),
\]
where the components (fields) \(\bar{\psi}^{(r)}(\pi, \bar{\pi})\) are complex conjugation of the components (fields) \(\psi^{(r)}(\pi, \bar{\pi})\) in the expansion of (4.3): \[
\left( \psi^{(-\frac{1}{2}+k)} \right)^* = \bar{\psi}^{(-\frac{1}{2}-k)}, \quad k \in \mathbb{Z}. \quad (4.5)
\]

Recall that the quantities \(p_0\) and \(q_0\) in (4.2), (4.3), (4.4) can be resolved by means of the generalized Cartan-Penrose representations (2.11) and (2.12), so we have
\[
\frac{q_0}{p_0} = \sqrt{M} \sum_{\alpha=\alpha'} (\pi_\alpha \xi_\alpha + \xi_\alpha \bar{\pi}_\alpha) \quad (4.6)
\]
\[
\sum_{\beta=1}^{\beta=\beta} \pi_\beta \bar{\pi}_\beta.
\]

In view of (3.44) we assign the U(1)-charges \((c \pm k)\) to the the component fields \(\psi^{(c \pm k)}(\pi, \bar{\pi})\) in (4.2), (4.3), (4.4). Note that in (4.2), (4.3), (4.4) we use the same notation for the component fields \(\psi^{(k)}(\pi, \bar{\pi})\) as in (3.43) so as not to clutter the expressions by new notation. Of course, the fields \(\psi^{(c \pm k)}(\pi, \bar{\pi})\) in (4.2), (4.3), (4.4) equal the fields \(\psi^{(c \pm k)}(\pi, \bar{\pi})\) in (3.43) up to multipliers \(M^{k/2}\).

The \(\delta\)-function in (4.2), (4.3), (4.4) has the form
\[
\delta \left( (\pi \xi)(\bar{\xi}\bar{\pi}) - M \right) = \delta \left( \xi^\alpha p_{\alpha \bar{\alpha}} \xi_{\bar{\alpha}} - M \right), \quad (4.7)
\]

if we take into account \(p_{\alpha \bar{\alpha}} = \pi_\alpha \bar{\pi}_{\bar{\alpha}}\). Such \(\delta\)-function is present in the definition of fields in (31). Moreover, the residual part \(e^{-iq_0/p_0} \Psi^{(c)}(\pi, \bar{\pi}; \xi, \bar{\xi})\) of the field (4.2), (4.3), (4.4) is a polynomial in the spinor variables \(\xi_\alpha\), as it is in (31). So we can interpret the wave functions (4.2), (4.3), (4.4) as the fields in the momentum representation, where momentum \(p_m\) is defined by the spinors \(\pi_\alpha\) and \(\bar{\pi}_{\bar{\alpha}}\).
4.1 Integer spins

We take $c = 0$ for the value of the U(1)-charge $c$ of the twistor field $\Psi^{(c)}$ as in (3.51). The field $\Psi^{(0)}$ in this case is given in (4.2). Consider the space-time wave function determined by means of the integral Fourier transformation

$$\Phi(x; \xi, \bar{\xi}) = \int d^4\pi e^{iP_{\alpha\dot{\alpha}}x^{\alpha\dot{\alpha}}} \Psi^{(0)}(\pi, \bar{\pi}; \xi, \bar{\xi}) = \int d^4\pi e^{i\pi_\alpha \bar{\pi}_{\dot{\alpha}} x^{\alpha\dot{\alpha}}} \Psi^{(0)}(\pi, \bar{\pi}; \xi, \bar{\xi}),$$  \hspace{5pt} (4.8)

where we have used for the momentum the representation $p^{\alpha\dot{\alpha}} = \pi^{\alpha} \bar{\pi}^{\dot{\alpha}}$ (see (2.11)). In the integral (4.8) we perform integration over the two-dimensional complex space with the integration measure $d^4\pi := \frac{1}{8} d\pi^{\alpha} \wedge d\bar{\pi}^{\dot{\alpha}} \wedge d\pi^{\dot{\alpha}} \wedge d\bar{\pi}^{\alpha} = \frac{1}{2} d\pi_1 \wedge d\pi_2 \wedge d\bar{\pi}_1 \wedge d\bar{\pi}_2$. This integration is in fact the integration over four-vector $p = (p_0, p_1, p_2, p_3)$ with measure $d^4p \delta(p^2)$ (see Appendix B) and integration over common phase in $\pi^{\alpha}$ which is not presented in $p^{\alpha\dot{\alpha}}$.

The field (4.8) satisfies four equations

$$\partial^{\alpha\dot{\alpha}} \partial_{\alpha\dot{\alpha}} \Phi(x; \xi, \bar{\xi}) = 0,$$ \hspace{5pt} (4.9)

$$(i \xi^{\alpha} \partial_{\alpha\dot{\alpha}} \bar{\xi}^{\dot{\alpha}} + M) \Phi(x; \xi, \bar{\xi}) = 0,$$ \hspace{5pt} (4.10)

$$\left( i \frac{\partial}{\partial \xi^{\alpha}} \partial_{\alpha\dot{\alpha}} \frac{\partial}{\partial \bar{\xi}^{\dot{\alpha}}} - M \right) \Phi(x; \xi, \bar{\xi}) = 0,$$ \hspace{5pt} (4.11)

$$\left( \xi^{\alpha} \frac{\partial}{\partial \xi^{\alpha}} - \bar{\xi}^{\dot{\alpha}} \frac{\partial}{\partial \bar{\xi}^{\dot{\alpha}}} \right) \Phi(x; \xi, \bar{\xi}) = 0.$$ \hspace{5pt} (4.12)

The proof of these equations is given in Appendix A. Equations (4.9), (4.10), (4.11), (4.12) underlie the definition of the infinite spin fields proposed in [31]. The explicit form of the momentum wave function corresponding to the space-time field (4.8) is given in Appendix B.

4.2 Half-integer spins

In this case, for the U(1)-charge $c$ we choose the value $c = -1/2$ (see (3.53)). The corresponding twistor field $\Psi^{(-1/2)}$ is written in (4.3). Then we use the standard prescription of the twistorial definition of space-time fields with nonvanishing helicities. Namely, we have to insert the twistorial spinor $\pi_{\alpha}$ in the integrand in the integral Fourier transformation (4.8). Therefore, in the case $c = -1/2$ the space-time field for describing the half-integer spins has the form

$$\Phi_{\alpha}(x; \xi, \bar{\xi}) = \int d^4\pi e^{iP_{\beta\dot{\beta}}x^{\beta\dot{\beta}}} \pi_{\alpha} \Psi^{(-1/2)}(\pi, \bar{\pi}; \xi, \bar{\xi}) = \int d^4\pi e^{i\pi_{\beta} \bar{\pi}_{\dot{\beta}} x^{\beta\dot{\beta}}} \pi_{\alpha} \Psi^{(-1/2)}(\pi, \bar{\pi}; \xi, \bar{\xi}),$$ \hspace{5pt} (4.13)

which by construction possesses the external spinor index $\alpha$. This field satisfies the Dirac-Pauli-Fierz equation

$$\partial^{\alpha\dot{\alpha}} \Phi_{\alpha}(x; \xi, \bar{\xi}) = 0.$$ \hspace{5pt} (4.14)
and equations (4.10), (4.11), (4.12):

\[
\left( i\xi^\beta \partial_{\bar{\beta}} \xi^{\bar{\beta}} + M \right) \Phi_\alpha(x;\xi,\bar{\xi}) = 0 ,
\]  

(4.15)

\[
\left( i \frac{\partial}{\partial \xi^\beta} \partial_{\bar{\beta}} \frac{\partial}{\partial \xi^{\bar{\beta}}} - M \right) \Phi_\alpha(x;\xi,\bar{\xi}) = 0 ,
\]  

(4.16)

\[
\left( \xi^\beta \frac{\partial}{\partial \xi^\beta} - \bar{\xi}^{\bar{\beta}} \frac{\partial}{\partial \bar{\xi}^{\bar{\beta}}} \right) \Phi_\alpha(x;\xi,\bar{\xi}) = 0 .
\]  

(4.17)

The complex conjugate twistorial field (4.4) with charge \( c = +1/2 \) is used for definition of the spinor field with the dotted Weyl index

\[
\tilde{\Phi}_\alpha(x;\xi,\bar{\xi}) = \int d^4\pi e^{-i\pi^\beta \pi^{\bar{\beta}} x^{\beta \bar{\beta}}} \tilde{\Phi}^{(+1/2)}(\pi, \bar{\pi}; \xi, \bar{\xi}) .
\]  

(4.18)

The field (4.18) satisfies the Dirac-Pauli-Fierz equation

\[
\partial^{\alpha \dot{\alpha}} \tilde{\Phi}_\alpha(x;\xi,\bar{\xi}) = 0
\]  

(4.19)

and the equations

\[
\left( i\xi^\beta \partial_{\bar{\beta}} \xi^{\bar{\beta}} - M \right) \tilde{\Phi}_\alpha(x;\xi,\bar{\xi}) = 0 ,
\]  

(4.20)

\[
\left( i \frac{\partial}{\partial \xi^\beta} \partial_{\bar{\beta}} \frac{\partial}{\partial \xi^{\bar{\beta}}} + M \right) \tilde{\Phi}_\alpha(x;\xi,\bar{\xi}) = 0 ,
\]  

(4.21)

\[
\left( \xi^\beta \frac{\partial}{\partial \xi^\beta} - \bar{\xi}^{\bar{\beta}} \frac{\partial}{\partial \bar{\xi}^{\bar{\beta}}} \right) \tilde{\Phi}_\alpha(x;\xi,\bar{\xi}) = 0 .
\]  

(4.22)

We stress that although the twistorial fields (4.3), (4.4) have nonvanishing external charges \( c = \pm 1/2 \), their integral transformations (4.13), (4.18) have zero \( U(1) \)-charge defined by equations (4.17), (4.22). This fact is crucial for forming infinite spin supermultiplets, as we will see in the next Section.

5 Infinite spin supermultiplet

The results of the previous section allow us to construct an infinite spin supermultiplet. To do this, we consider the fields \( \Phi(x;\xi,\bar{\xi}) \) and \( \Phi_\alpha(x;\xi,\bar{\xi}) \), which are defined in (4.18), (4.13), satisfy equations (4.9)-(4.12), (4.14)-(4.17) and unify them into one multiplet. These fields contain the bosonic \( \psi^{(k)}(\pi, \bar{\pi}) \) and fermionic \( \psi^{(k-1/2)}(\pi, \bar{\pi}) \) component fields \( (k \in \mathbb{Z}) \) with all integer and half-integer spins, respectively. It is natural to expect that individual components of these fields with fixed spins \( n \) and \( n + \frac{1}{2} \) \( (n = 0, 1 \ldots) \) should form the on-shell \( \mathcal{N} = 1 \) higher spin supermultiplet. Therefore, the bosonic (even) \( \Phi(x;\xi,\bar{\xi}) \) and fermionic (odd) \( \Phi_\alpha(x;\xi,\bar{\xi}) \) fields themselves should form the on-shell \( \mathcal{N} = 1 \) infinite spin supermultiplet containing an infinite number of conventional
supermultiplets. The only thing we should do is to introduce the corresponding supertransformations.

Similar to the Wess-Zumino supermultiplet (see, e.g., [44, 45]), we define supersymmetry transformations for the fields $\Phi$ and $\Phi_\alpha$ in the form

$$
\delta \Phi = \varepsilon^\alpha \Phi_\alpha, \quad \delta \Phi_\alpha = 2i \bar{\varepsilon}^\beta \partial_{\alpha\beta} \Phi,
$$

(5.1)

where $\varepsilon^\alpha$, $\bar{\varepsilon}^\dot{\beta}$ are the components of the constant odd Weyl spinor. The commutators of these transformations are

$$
(\delta_1 \delta_2 - \delta_2 \delta_1) \Phi = -2ia^{\beta\dot{\beta}} \partial_{\beta\dot{\beta}} \Phi,
$$

$$
(\delta_1 \delta_2 - \delta_2 \delta_1) \Phi_\alpha = -2ia^{\beta\dot{\beta}} \partial_{\beta\dot{\beta}} \Phi_\alpha + 2ia_{\alpha\dot{\beta}} \partial^{\dot{\beta}} \Phi_\beta,
$$

(5.2)

where

$$
a_{\alpha\dot{\beta}} := \varepsilon_{1\alpha} \bar{\varepsilon}^{\dot{2}\dot{\beta}} - \varepsilon_{2\alpha} \bar{\varepsilon}^{\dot{1}\dot{\beta}}.
$$

(5.3)

As we see, the superalgebra (5.2) is closed on-shell on the translations with the generator $P_{\beta\dot{\beta}} = -i\partial_{\beta\dot{\beta}}$ due to the equation of motion (4.14). Moreover, the set of equations (4.9)-(4.12), (4.14)-(4.17) is invariant with respect to transformations (5.2).

Using the integral transformations inverse to (4.8), (4.13), we rewrite (5.1) as supersymmetry transformations for the twistor fields $\Psi^{(0)}(\pi, \bar{\pi}; \xi, \bar{\xi})$, $\Psi^{(-1/2)}(\pi, \bar{\pi}; \xi, \bar{\xi})$ in the momentum representation:

$$
\delta \Psi^{(0)} = \varepsilon^\alpha \pi_\alpha \Psi^{(-1/2)}, \quad \delta \Psi^{(-1/2)} = -2 \bar{\varepsilon}^{\dot{\alpha}} \pi_\dot{\alpha} \Psi^{(0)}.
$$

(5.4)

Using further component expansions (4.2), (4.3), we find supersymmetry transformations for the bosonic $A^{(k)}(\pi, \bar{\pi}) \equiv \psi^{(k)}(\pi, \bar{\pi})$ and fermionic $\psi^{(-1/2+k)}(\pi, \bar{\pi})$ twistorial components at all $k \in \mathbb{Z}$ in the form

$$
\delta A^{(k)} = \varepsilon^\alpha \pi_\alpha \psi^{(-1/2+k)}, \quad \delta \psi^{(-1/2+k)} = -2 \bar{\varepsilon}^{\dot{\alpha}} \pi_\dot{\alpha} A^{(k)}.
$$

(5.5)

According to (3.49), the bosonic field $A^{(k)}$ and fermionic field $\psi^{(-1/2+k)}$ at fixed $k \in \mathbb{Z}$ describe massless states with helicities $(-k)$ and $(\frac{1}{2} - k)$, respectively. Therefore, the infinite-component supermultiplet of the infinite spin stratifies into an infinite number of levels with pairs of the fields $A^{(k)}$, $\psi^{(-1/2+k)}$ at fixed $k \in \mathbb{Z}$. The supersymmetry transforms the bosonic and fermionic fields into each other inside a given level $k$. The boosts of the Poincare group transform the levels with different $k$ and therefore mix the fields with different values of $k$.

6 Summary and outlook

In this paper, we have presented the new twistorial field formulation of the massless infinite spin particle and field. As opposed to the paper [30], we obtained the field description by the canonical quantization of the world-line twistor model without any gauge fixing. We gave the helicity decomposition of twistorial infinite spin fields and constructed the field twistor transform to define the space-time infinite (continuous) spin fields. These space-time fields, bosonic $\Phi(x; \xi, \bar{\xi})$
and fermionic $\Phi_\alpha(x; \xi, \bar{\xi})$, depend on the coordinate four-vector $x^m$ and on the commuting Weyl spinors $\xi^\alpha, \bar{\xi}^\dot{\alpha}$. We found the equations of motion for $\Phi(x; \xi, \bar{\xi})$ and $\Phi_\alpha(x; \xi, \bar{\xi})$. Moreover, we showed that these fields form the $\mathcal{N}=1$ infinite spin supermultiplet.

In subsequent works we will consider the construction of the Lagrangian field theory of continuous spin, both in the bosonic and fermionic cases and also in the supersymmetric case. One of the commonly used methods for this purpose is the BRST quantization method, which was used in the case of continuous spin particles in [19, 28, 29, 31, 32, 33]. In a recent paper [31] the Lagrangian formulation of the infinite integer-spin field was constructed by using the methods developed in [40, 47]. We plan to construct the Lagrangian formulation for the infinite half-integer field, as well as in the supersymmetric case. Another interesting problem is to develop the Lagrange description of the infinite spin supermultiplet in a superfield approach. We note in this regard that in [25] the Lagrangian formulation of the infinite spin supermultiplets in the $d=3$ space-time was constructed by using the frame-like formalism for corresponding bosonic and fermionic fields.

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**Appendix A: Proof of equations (4.9)-(4.12), (4.14)-(4.17) and (4.19)-(4.22)**

In this Appendix we present the proof of the equations of motion for the infinite spin space-time fields (4.8), (4.13), (4.18).

The field (4.8) $\Phi(x; \xi, \bar{\xi})$ satisfies the massless Klein-Gordon equation (4.9) due to resolved form $p_{a\dot{a}} = \pi_\alpha \bar{\pi}_{\dot{\alpha}}$ of the momentum in the integrand.

Due to the presence of the $\delta$-function (4.7) in the integrand, the field (4.8) satisfies equations (4.10).

The proof of the fulfillment of equation (4.11) is the following:

$$i \frac{\partial}{\partial \xi^\alpha} \frac{\partial}{\partial \bar{\xi}^\dot{\alpha}} \Phi = \int d^4 \pi e^{i\pi_\alpha \bar{\pi}_{\dot{\alpha}} x^\alpha} \delta ((\pi \xi)(\bar{\xi} \bar{\pi}) - M) \left[ -\pi_\alpha \frac{\partial}{\partial \xi^\alpha} \bar{\pi}_{\dot{\alpha}} \frac{\partial}{\partial \bar{\xi}^\dot{\alpha}} e^{-i\eta_0/p_0} \right] \hat{\Psi}(0)(\pi, \bar{\pi}; \xi, \bar{\xi}) = M \Phi.$$
The proof of fulfillment of equation (4.12) is the following:

\[
\left(\xi_\alpha \frac{\partial}{\partial \xi_\alpha} - \bar{\xi}_\alpha \frac{\partial}{\partial \bar{\xi}_\alpha}\right) \Phi = \int d^4 \pi \, e^{i \pi_\alpha \bar{\pi}_\alpha x^{\alpha \bar{\alpha}}} \delta \left( (\pi \xi)(\bar{\xi} \bar{\pi}) - M \right) \left[ \left( \xi_\alpha \frac{\partial}{\partial \xi_\alpha} - \bar{\xi}_\alpha \frac{\partial}{\partial \bar{\xi}_\alpha}\right) e^{-i q_0/p_0} \right] \hat{\Psi}^{(0)}(\pi, \bar{\pi}; \xi, \bar{\xi}) +
\]

\[
\int d^4 \pi \, e^{i \pi_\alpha \bar{\pi}_\alpha x^{\alpha \bar{\alpha}}} \delta \left( (\pi \xi)(\bar{\xi} \bar{\pi}) - M \right) e^{-i q_0/p_0} \left( \xi_\alpha \frac{\partial}{\partial \xi_\alpha} - \bar{\xi}_\alpha \frac{\partial}{\partial \bar{\xi}_\alpha}\right) \hat{\Psi}^{(0)}(\pi, \bar{\pi}; \xi, \bar{\xi}) =
\]

\[
\int d^4 \pi \, e^{i \pi_\alpha \bar{\pi}_\alpha x^{\alpha \bar{\alpha}}} \delta \left( (\pi \xi)(\bar{\xi} \bar{\pi}) - M \right) \left[ - \left( \pi_\alpha \frac{\partial}{\partial \pi_\alpha} - \bar{\pi}_\alpha \frac{\partial}{\partial \bar{\pi}_\alpha}\right) e^{-i q_0/p_0} \right] \hat{\Psi}^{(0)}(\pi, \bar{\pi}; \xi, \bar{\xi}) +
\]

\[
\int d^4 \pi \, e^{i \pi_\alpha \bar{\pi}_\alpha x^{\alpha \bar{\alpha}}} \delta \left( (\pi \xi)(\bar{\xi} \bar{\pi}) - M \right) e^{-i q_0/p_0} \left( \xi_\alpha \frac{\partial}{\partial \xi_\alpha} - \bar{\xi}_\alpha \frac{\partial}{\partial \bar{\xi}_\alpha}\right) \hat{\Psi}^{(0)}(\pi, \bar{\pi}; \xi, \bar{\xi}) =
\]

\[
\int d^4 \pi \, e^{i \pi_\alpha \bar{\pi}_\alpha x^{\alpha \bar{\alpha}}} \delta \left( (\pi \xi)(\bar{\xi} \bar{\pi}) - M \right) e^{-i q_0/p_0} \left( \pi_\alpha \frac{\partial}{\partial \pi_\alpha} - \bar{\pi}_\alpha \frac{\partial}{\partial \bar{\pi}_\alpha} + \xi_\alpha \frac{\partial}{\partial \xi_\alpha} - \bar{\xi}_\alpha \frac{\partial}{\partial \bar{\xi}_\alpha} + 1 \right) \hat{\Psi}^{(0)}(\pi, \bar{\pi}; \xi, \bar{\xi}) = 0.
\]

The spinor field \( \Phi_\alpha(x; \xi, \bar{\xi}) \) defined by integral transform (4.13) satisfies the massless Dirac equation (4.14) due to \( p_{0\alpha} = \pi_{0\alpha} \bar{\pi}_{0\bar{\alpha}} \) in integrand and the identity \( \pi^\alpha \bar{\pi}_\alpha \equiv 0 \) for the even Weyl spinor. The proof of fulfillment of equations (4.15), (4.16) for the spinor field \( \Phi_\alpha(x; \xi, \bar{\xi}) \) is completely analogous to the scalar field \( \Phi(x; \xi, \bar{\xi}) \). The proof of fulfillment of equation (4.17) is some generalization of

\[
\left(\xi_\alpha \frac{\partial}{\partial \xi_\alpha} - \bar{\xi}_\alpha \frac{\partial}{\partial \bar{\xi}_\alpha}\right) \Phi_\beta = \int d^4 \pi \, e^{i \pi_\alpha \bar{\pi}_\alpha x^{\alpha \bar{\alpha}}} \delta \left( (\pi \xi)(\bar{\xi} \bar{\pi}) - M \right) \left[ \left( \xi_\alpha \frac{\partial}{\partial \xi_\alpha} - \bar{\xi}_\alpha \frac{\partial}{\partial \bar{\xi}_\alpha}\right) e^{-i q_0/p_0} \right] \hat{\Psi}^{(-1/2)}(\pi, \bar{\pi}; \xi, \bar{\xi}) +
\]

\[
\int d^4 \pi \, e^{i \pi_\alpha \bar{\pi}_\alpha x^{\alpha \bar{\alpha}}} \delta \left( (\pi \xi)(\bar{\xi} \bar{\pi}) - M \right) e^{-i q_0/p_0} \left( \xi_\alpha \frac{\partial}{\partial \xi_\alpha} - \bar{\xi}_\alpha \frac{\partial}{\partial \bar{\xi}_\alpha}\right) \hat{\Psi}^{(-1/2)}(\pi, \bar{\pi}; \xi, \bar{\xi}) =
\]

\[
\int d^4 \pi \, e^{i \pi_\alpha \bar{\pi}_\alpha x^{\alpha \bar{\alpha}}} \delta \left( (\pi \xi)(\bar{\xi} \bar{\pi}) - M \right) \left[ - \left( \pi_\alpha \frac{\partial}{\partial \pi_\alpha} - \bar{\pi}_\alpha \frac{\partial}{\partial \bar{\pi}_\alpha}\right) e^{-i q_0/p_0} \right] \hat{\Psi}^{(-1/2)}(\pi, \bar{\pi}; \xi, \bar{\xi}) +
\]

\[
\int d^4 \pi \, e^{i \pi_\alpha \bar{\pi}_\alpha x^{\alpha \bar{\alpha}}} \delta \left( (\pi \xi)(\bar{\xi} \bar{\pi}) - M \right) e^{-i q_0/p_0} \left( \xi_\alpha \frac{\partial}{\partial \xi_\alpha} - \bar{\xi}_\alpha \frac{\partial}{\partial \bar{\xi}_\alpha}\right) \hat{\Psi}^{(-1/2)}(\pi, \bar{\pi}; \xi, \bar{\xi}) =
\]

\[
\int d^4 \pi \, e^{i \pi_\alpha \bar{\pi}_\alpha x^{\alpha \bar{\alpha}}} \delta \left( (\pi \xi)(\bar{\xi} \bar{\pi}) - M \right) e^{-i q_0/p_0} \left( \pi_\alpha \frac{\partial}{\partial \pi_\alpha} - \bar{\pi}_\alpha \frac{\partial}{\partial \bar{\pi}_\alpha} + \xi_\alpha \frac{\partial}{\partial \xi_\alpha} - \bar{\xi}_\alpha \frac{\partial}{\partial \bar{\xi}_\alpha} + 1 \right) \hat{\Psi}^{(-1/2)}(\pi, \bar{\pi}; \xi, \bar{\xi}) = 0.
\]

Since \( \tilde{\Phi}_\alpha(x; \xi, \bar{\xi}) \) is complex conjugation of the field \( \Phi_\alpha(x; \xi, \bar{\xi}) \), therefore the fulfillment of equations (4.19), (4.20), (4.21), (4.22) is the consequence of equations (4.14), (4.15), (4.16), (4.17).

**Appendix B: Momentum wave function**

In this Appendix we present the exact form of the momentum wave function.
For definiteness we consider the case $c = 0$.

In integrand (4.8) the space-time momentum is represented as a product of spinor components (2.11):

$$ p_\alpha \dot{\pi}_\alpha = \pi_\alpha (p) e^{i\varphi}, \quad \bar{\pi}_{\dot{\alpha}} = \bar{\pi}_{\dot{\alpha}} (p) e^{-i\varphi}, $$

(B.1)

where

$$ \pi_\alpha (p) : \pi_1 = |\pi_1| e^{i\varphi}, \quad \pi_2 = |\pi_2| e^{-i\varphi}, \quad \bar{\pi}_{\dot{\alpha}} (p) : \bar{\pi}_{\dot{1}} = |\pi_{\dot{1}}| e^{-i\varphi}, \quad \bar{\pi}_{\dot{2}} = |\pi_{\dot{2}}| e^{i\varphi}, $$

(B.2)

are the functions of the light-like momentum components:

$$ |\pi_1| = \left( \frac{|p^0 + p^3|}{\sqrt{2}} \right)^{1/2}, \quad |\pi_2| = \left( \frac{|p^0 - p^3|}{\sqrt{2}} \right)^{1/2}, \quad e^{i\varphi} = \left( \frac{p^1 - ip^2}{p^1 + ip^2} \right)^{1/4}. $$

(B.3)

The phase $\varphi$ presented in (B.1) does not give a contribution to the definition of the light-like momentum $p_{\alpha \dot{\alpha}} = \pi_\alpha \bar{\pi}_{\dot{\alpha}}$. As result, we have

$$ d^4 \pi = \frac{1}{8} d\pi^\alpha \wedge d\pi_\alpha \wedge d\bar{\pi}_{\dot{\alpha}} \wedge d\bar{\pi}_{\dot{\alpha}} = d^4 p \delta (p^2) d\varphi. $$

(B.4)

In terms of the variables (B.1) the fraction (4.6) takes the value

$$ \frac{q_0}{p_0} = a(p; \xi, \bar{\xi}) \cos \varphi + b(p; \xi, \bar{\xi}) \sin \varphi, $$

(B.5)

where

$$ a(p; \xi, \bar{\xi}) = \frac{\sqrt{M} \sum_{\alpha = \dot{\alpha}} (\pi_\alpha \bar{\xi}_{\dot{\alpha}} + \xi_{\alpha} \bar{\pi}_{\dot{\alpha}})}{\sum_{\beta = \dot{\beta}} \pi_\beta \bar{\pi}_{\dot{\beta}}}, \quad b(p; \xi, \bar{\xi}) = \frac{i\sqrt{M} \sum_{\alpha = \dot{\alpha}} (\pi_\alpha \bar{\xi}_{\dot{\alpha}} - \xi_{\alpha} \bar{\pi}_{\dot{\alpha}})}{\sum_{\beta = \dot{\beta}} \pi_\beta \bar{\pi}_{\dot{\beta}}}. $$

(B.6)

In addition, equations (3.44) for the component fields $\psi^{(k)} (\pi, \bar{\pi})$ of the wave function (4.2) take the form (at $c = 0$):

$$ \left( \pi_\alpha \frac{\partial}{\partial \pi_\alpha} - \bar{\pi}_{\dot{\alpha}} \frac{\partial}{\partial \bar{\pi}_{\dot{\alpha}}} \right) \psi^{(k)} (\pi, \bar{\pi}) = -i \frac{\partial}{\partial \varphi} \psi^{(k)} (\pi, \bar{\pi}) = 2k \psi^{(k)} (\pi, \bar{\pi}). $$

(B.7)

General dependence of the component fields $\psi^{(k)} (\pi, \bar{\pi})$ on the phase $\varphi$ is defined by the Fourier series. Then, general solutions of equations (B.7) are

$$ \psi^{(k)} (\pi, \bar{\pi}) = e^{2ik\varphi} \psi^{(k)} (\pi, \bar{\pi}) = e^{\pm i k \varphi} \psi^{(k)} (p), $$

(B.8)

where the functions $\psi^{(k)} (p)$ do not depend on $\varphi$.

Inserting (B.1), (B.4), (B.5), (B.6), (B.8) in (4.2) and (4.8) we obtain that the space-time field of the infinite spin particle takes the form

$$ \Phi (x; \xi, \bar{\xi}) = \int d^4 p e^{ipx} \Psi (p, \xi, \bar{\xi}), $$

(B.9)
where
\[
\Psi(p, \xi, \bar{\xi}) = \delta(p^2) \delta(\xi^{\alpha} p_{\alpha\bar{\alpha}} \bar{\xi}^{\bar{\alpha}} - M) \left[ \psi^{(0)}(p) \int_0^{2\pi} d\varphi e^{-i (a \cos \varphi + b \sin \varphi)} \right] 
+ \sum_{k=1}^{\infty} \psi^{(k)}(p) (\bar{\xi}^{\bar{\alpha}} \pi^k)^k \int_0^{2\pi} d\varphi e^{i (k \varphi - a \cos \varphi - b \sin \varphi)} 
+ \sum_{k=1}^{\infty} \psi^{(-k)}(p) (\pi^k \xi^{\alpha})^k \int_0^{2\pi} d\varphi e^{-i (k \varphi + a \cos \varphi + b \sin \varphi)} \right].
\]

We will use the integral (see formulae 337 (9a,b) in [48] and formulae 3.937 (1,2) in [49])
\[
\int_0^{2\pi} e^{i (\pm n \varphi - a \cos \varphi - b \sin \varphi)} d\varphi = 2\pi \left( \frac{a \pm ib}{a \mp ib} \right)^{n/2} I_n(i\sqrt{a^2 + b^2}),
\]
where \(n = 0, 1, 2, \ldots\) and \(I_n\) are the modified Bessel functions\(^4\).

Expressions (B.6) lead to the following equalities:
\[
a + ib = \frac{2\sqrt{M} \sum_{\alpha = \bar{\alpha}} \xi_{\alpha} \bar{\pi}_{\bar{\alpha}}}{\sum_{\beta = \bar{\beta}} \pi_{\beta} \bar{\pi}_{\bar{\beta}}}, \quad a - ib = \frac{2\sqrt{M} \sum_{\alpha = \bar{\alpha}} \pi_{\alpha} \bar{\xi}_{\bar{\alpha}}}{\sum_{\beta = \bar{\beta}} \pi_{\beta} \bar{\pi}_{\bar{\beta}}}. \quad (B.13)
\]

Therefore, there are equalities
\[
\frac{a + ib}{a - ib} = \frac{\sum_{\alpha = \bar{\alpha}} \xi_{\alpha} \bar{\pi}_{\bar{\alpha}}}{\sum_{\beta = \bar{\beta}} \pi_{\beta} \bar{\xi}_{\bar{\beta}}}, \quad (B.14)
\]
\[
y^2 := a^2 + b^2 = \frac{4M \sum_{\alpha = \bar{\alpha}} \xi_{\alpha} \bar{\pi}_{\bar{\alpha}} \sum_{\beta = \bar{\beta}} \pi_{\beta} \bar{\xi}_{\bar{\beta}}}{\left( \sum_{\gamma = \bar{\gamma}} \pi_{\gamma} \bar{\pi}_{\bar{\gamma}} \right)^2}. \quad (B.15)
\]

\(^4\) Modified Bessel functions (or Bessel functions of imaginary argument) \(I_n(z) = I_{-n}(z)\) are expressed in terms of the Bessel functions of the first kind \(J_n(z)\) by the expression
\[
I_n(z) = i^{-n} J_n(iz) \quad (B.12)
\]
in the cases when \(n\) is integer.
Using (B.11)-(B.15), we obtain that the momentum wave function (B.10) takes the form

\[
\Psi(p, \xi, \bar{\xi}) = 2\pi \delta(p^2) \delta(\xi^\alpha p_{\alpha\bar{\alpha}}\bar{\xi}^{\bar{\alpha}} - M) \left[ \psi^{(0)}(p) I_0(iy) \right.
\]

\[
+ \sum_{k=1}^{\infty} \psi^{(k)}(p) \left( \frac{(\xi^\pi \pi^\alpha_\alpha)^2}{\sum_{\beta=\beta}^\pi \xi_\beta^\beta \pi_\beta^\beta} \right)^{k/2} I_k(iy)
\]

\[
+ \sum_{k=1}^{\infty} \psi^{(-k)}(p) \left( \frac{(\pi^\xi)^2}{\sum_{\alpha=\alpha}^\xi \xi_\alpha^\alpha \pi_\alpha^\alpha} \right)^{k/2} I_k(iy) \right],
\]

where \(\psi^{(k)}(p)\) at fixed \(k \in \mathbb{Z}\) describes the massless state of helicity \(-k\).

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