Physical sectors of the confluent hypergeometric functions space

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Abstract

A relaxed factorization is used to obtain many of the properties obeyed by the confluent hypergeometric functions. Their implications on the analytical solutions of some interesting physical problems are also studied. It is quite remarkable that, although these properties appear frequently in solving the Schrödinger equation, it has been not clear the role they play in describing the physical systems. The main objective of this communication is precisely to throw some light on the subject.

1 Introduction

There is no doubt on the main role played by special functions in theoretical and mathematical physics. In general, they are used to simplify the original problem by transforming its mathematical description from a rather cumbersome form into a simpler and well known one. The physical scenery is thereby clarified: the solutions of the simplified problem can be easily analyzed and their most relevant qualitative features could be depicted in terms of the involved parameters. As a matter of fact, a considerable amount of basic research has been developed in the study of the differential equations obeyed by special functions, mainly in their connections with the problems appearing in all the branches of the physical theories. Of particular interest are the hypergeometric and confluent hypergeometric functions (h.f. and c.h.f. respectively), in terms of which almost all the solutions of exactly solvable problems in quantum mechanics can be written (e.g. those related with the linear and p-dimensional harmonic oscillator, hydrogen-like, Pöschl-Teller, Wood-Saxon, Hulthén, Morse, Eckart and Scarf potentials among others). The standard mechanism in these cases takes into account appropriate transformations of the involved variables and functions from the Schrödinger into an hypergeometric or confluent hypergeometric equation (h.e. and c.h.e. for the last two respectively) [1]. Remark on the fact that the mathematical properties of the h.f. and c.h.f. lead then to the characterization of some important physical features of the solutions. That is the case, for example, of the quantization of the energy eigenvalues by imposing the Schrödinger solutions of classically confined systems as square integrable functions. On the other hand, an interesting perspective arises by considering
the procedure in the reverse order: one departs from the h.e. or the c.h.e. and the conditioning
transformations leading to the Schrödinger equation are to be determined (see important works
by Natanzon [2] and by Nikiforov and Uvarov [3]). Such a procedure allows the identification of
a wide set of functions that can be understood as physically meaningful potentials.

Considering all this theoretical richness, it seems apparent the presence of relations con-
ecting the stationary Schrödinger solutions of diverse quantum systems. Although they have
been frequently mentioned in the literature since long time ago, there is still lacking an updated
exposition of the topic. Quite recently, the present authors have reported results which could
shed a new light on this subject [4]. In that work a refined factorization was applied to analize
the c.h.e. and its solutions, as well as the implications of their mathematical properties on the
wave-functions of some interesting physical problems. The refined factorization goes deeply into
the possibilities of the conventional factorization method providing with additional significative
information [5].

In this contribution we shall discuss some of the main results published in [4] whereas it
is also reported more on their physical consequences. The Section 2 is devoted to the refined
factorization of the c.h.e. The action of the involved factorization operators on the confluent
hypergeometric parameters underlies the analytical and algebraic properties of the c.h.f space.
Section 3 deals with the mapping from the c.h.e. into diverse Schrödinger equations. The
identification of the involved conventional physical potentials gives the chance to translate the
results obtained in Section 2 from a mathematical into a physical language.

2 Confluent hypergeometric intertwiners

We shall work on second order differential operators of the form

\[ L_{(a,c)} \equiv x \frac{d^2}{dx^2} + (c - x) \frac{d}{dx} - a \] (1)

where the pair \((a, c)\) represents a point on \(\mathbb{R}^2\). Once the values of \(a\) and \(c\) have been given, the
kernel elements of \(L_{(a,c)}\) can be characterized as c.h.f., i.e., \(f(a,c;x)\) is a c.h.f. iff \(L_{(a,c)}f(a,c;x) = 0\), or in other words, \(f(a,c;x) \in \mathcal{K}_{(a,c)}\). Thereby, one can look for a way to connect the kernel
\(\mathcal{K}_{(a,c)}\) with another one \(\mathcal{K}_{(\tilde{a},\tilde{c})}\) and the conditions relating their respective parameters \((a,c)\) and
\((\tilde{a},\tilde{c})\). With this aim let us consider an arbitrary differential operator \(X\), defined by its action
on a kernel element \(f(a,c;x)\) of \(L_{(a,c)}\), as follows

\[ Xf(a,c;x) := X_{(a,c)}f(a,c;x) = \xi(\tilde{a},\tilde{c};x) \] (2)

where, according with our earlier convention, \(\xi(\tilde{a},\tilde{c};x)\) is a c.h.f. iff \(\xi(\tilde{a},\tilde{c};x) \in \mathcal{K}_{(\tilde{a},\tilde{c})}\) and the pair \((\tilde{a},\tilde{c})\) depends on \(a\) and \(c\). We shall say that the operator \(X\) is in a free index notation
whereas it is written in the confluent hypergeometric notation \(X_{(a,c)}\) in dependence on the kernel
\(\mathcal{K}_{(a,c)}\) on which it is acting. The benefits of this notation become clear in the composition of
any pair \(X,Y\) of these operators. For instance, if \(Y\) is such that \(Yf(a,c;x) = g(\tilde{a},\tilde{c};x)\), then
the action of \(XY\) on the kernel \(\mathcal{K}_{(a,c)}\) reads

\[ (XY)f(a,c;x) := X_{(a,c)}\left(Y_{(a,c)}f(a,c;x)\right) = X_{(a,c)}g(\tilde{a},\tilde{c};x). \]
On the other hand, the action of \( YX \) on the same kernel gives
\[
(YX)f(a,c;x) := Y_{(\bar{a},\bar{c})} \left( X_{(a,c)}f(a,c;x) \right) = Y_{(\bar{a},\bar{c})}\xi(\bar{a},\bar{c};x).
\]

Taking full advantage of our considerations we look now for a couple of differential operators \( A \) and \( B \) such that
\[
A : \mathcal{K}_{(a,c)} \mapsto \mathcal{K}_{(\bar{a},\bar{c})}, \quad B : \mathcal{K}_{(\bar{a},\bar{c})} \mapsto \mathcal{K}_{(a,c)}.
\]  \hspace{1cm} (3)

The initial parameters \((a,c)\) can also play the role of being the final ones. In that case we use the parameters \((a,\bar{c})\),
\[
A : \mathcal{K}_{(a,\bar{c})} \mapsto \mathcal{K}_{(a,c)}, \quad B : \mathcal{K}_{(a,c)} \mapsto \mathcal{K}_{(a,\bar{c})}.
\]  \hspace{1cm} (4)

It is a matter of substitution to verify that operators \( A \) and \( B \) intertwine the elements of \( \mathcal{K}_{(a,\bar{c})} \) with those of \( \mathcal{K}_{(\bar{a},\bar{c})} \) and vice versa. Table 1 displays the set of basic intertwiners of diverse orders, all of them written in the confluent hypergeometric notation.

| Order | Intertwiner | Expression | \( a \) | \( c \) |
|-------|-------------|------------|------|------|
| zero  | \( Q_{(a,c)} \) | \( x^{-1} \) | \( a-c+1 \) | \( 2-c \) |
| First | \( A^1_{(a,c)} \) | \( \frac{d}{dx} - 1 \) | \( a \) | \( c+1 \) |
| First | \( B^1_{(a,c)} \) | \( x \frac{d}{dx} + c - 1 \) | \( a \) | \( c-1 \) |
| First | \( A^2_{(a,c)} \) | \( x (\frac{d}{dx} - 1) + c - 1 \) | \( a-1 \) | \( c-1 \) |
| First | \( B^2_{(a,c)} \) | \( \frac{d}{dx} \) | \( a+1 \) | \( c+1 \) |
| First | \( A^3_{(a,c)} \) | \( x^c (\frac{d}{dx} - 1) \) | \( a-c \) | \( 1-c \) |
| First | \( B^3_{(a,c)} \) | \( x^c \frac{d}{dx} \) | \( a-c+1 \) | \( 1-c \) |
| First | \( A^4_{(a,c)} \) | \( x^{c-2} [x (\frac{d}{dx} - 1) + c - 1] \) | \( a-c+1 \) | \( 3-c \) |
| First | \( B^4_{(a,c)} \) | \( x^{c-2} (x \frac{d}{dx} + c - 1) \) | \( a-c+2 \) | \( 3-c \) |
| non-finite | \( V_{(a,c)} \) | \( e^x R_x \) | \( c-a \) | \( c \) |
| non-finite | \( W_{(a,c)} \) | \( x^{c-1} e^x R_x \) | \( 1-a \) | \( 2-c \) |

It is remarkable the presence of the zero order differential operator \( Q \), which gives rise to a nontrivial intertwining operation: \( L_{(a-c+1,2-c)} Q_{(a,c)} = Q_{(a,c)} L_{(a,c)} \). On the other hand, the non-finite order intertwiners \( V \) and \( W \) underlies some of the very basic relations obeyed by the c.h.f. such as the well known Kummer’s first formula \[3\]. The first order intertwiners \( \{A^i, B^i\}_{i=1}^4 \) deserve as much attention because they factorize the c.h.o. \( L_{(a,c)} \) and \( L_{(\bar{a},\bar{c})} \):
\[
L_{(a,c)} = B^1_{(\bar{a},\bar{c})} A^i_{(a,c)} - q^i_{(a,c)}; \quad L_{(a,c)} = A^i_{(a,c)} B^i_{(a,c)} - q^i_{(a,c)}; \quad i = 1, 2, 3, 4. \] \hspace{1cm} (5)
\[
L_{(\bar{a},\bar{c})} = A^i_{(a,c)} B^i_{(\bar{a},\bar{c})} - q^i_{(a,c)}; \quad L_{(\bar{a},\bar{c})} = B^i_{(a,c)} A^i_{(a,c)} - q^i_{(a,c)}; \quad i = 1, 2, 3, 4. \] \hspace{1cm} (6)

where, by convention, we have implemented the induced action of the operators \( A \) and \( B \) on \( \mathbb{R}^2 \) as follows: \( A^i(a,c) := A^i_{(a,c)}(a,c) = (\bar{a},\bar{c}) \), \( B^i(a,c) := B^i_{(a,c)}(a,c) = (a,\bar{c}) \). The factorization constants \( q^i_{(a,c)} \) are given by
\[
q^1_{(a,c)} = a-c; \quad q^2_{(a,c)} = a-1; \quad q^3_{(a,c)} = a-c; \quad q^4_{(a,c)} = a-1.
\]

Now, in order to determine the explicit action of the free index operators on an arbitrary kernel \( \mathcal{K}_{(a,c)} \), it is enough to consider a basis for \( \mathcal{K}_{(a,c)} \) as follows
\[
_1 F_1(a,c;x), \quad u(a,c;x) \equiv x^{1-c} _1 F_1(a-c+1,2-c;x),
\] \hspace{1cm} (7)
where it is assumed that $c \notin \mathbb{Z}$ (the general case $c \in \mathbb{Z}$ is easily recovered after an straightforward limiting procedure\footnote{This unpleasant situation can be solved by introducing maximal shape-invariant functions preserving their form under the action of all the intertwiners on Table 1\footnote{al-bamid}.}). The action of the intertwiners $Q, V, W$ and $\{A^i, B^i\}_{i=1}^4$ on this basis is displayed on Table 2.

| Operator | $1 \, F_1(a, c; x)$ | $u(a, c; x)$ |
|---------|-----------------|-------------|
| $Q$     | $u(a + 1 - c, 2 - c; x)$ | $1 \, F_1(a + 1 - c, 2 - c; x)$ |
| $A^1$   | $\left(\frac{a-c}{x}\right) \, 1 \, F_1(a, c + 1; x)$ | $(1 - c) \, u(a, c + 1; x)$ |
| $B^1$   | $(c - 1) \, 1 \, F_1(a, c - 1; x)$ | $\left(\frac{a-c+1}{x}\right) \, u(a, c - 1; x)$ |
| $A^2$   | $(c - 1) \, 1 \, F_1(a + 1, c - 1; x)$ | $\left(\frac{a-c}{x}\right) \, u(a - 1, c - 1; x)$ |
| $B^2$   | $\left(\frac{a-c}{x}\right) \, 1 \, F_1(a + 1, c + 1; x)$ | $(1 - c) \, u(a + 1, c + 1; x)$ |
| $A^3$   | $\left(\frac{a-c}{x}\right) \, u(a - c - 1, 1 - c; x)$ | $(1 - c) \, 1 \, F_1(a - c, 1 - c; x)$ |
| $B^3$   | $\left(\frac{a-c}{x}\right) \, u(a + c + 1, 1 - c; x)$ | $(1 - c) \, 1 \, F_1(a + c, 1 - c; x)$ |
| $A^4$   | $(c - 1) \, u(a - c + 1, 3 - c, x)$ | $\left(\frac{a-c+1}{x}\right) \, 1 \, F_1(a - c + 1, 3 - c; x)$ |
| $B^4$   | $(c - 1) \, u(a - c + 2, 3 - c; x)$ | $\left(\frac{a-c+2}{x}\right) \, 1 \, F_1(a - c + 2, 3 - c; x)$ |
| $V$     | $1 \, F_1(c - a, c; x)$ | $(-1)^{1-c} \, u(c - a, c; x)$ |
| $W$     | $u(1 - a, 2 - c; x)$ | $(-1)^{1-c} \, 1 \, F_1(1 - a, 2 - c; x)$ |

As regards the composition of intertwiners, we first notice that $Q^2 = 1$. Hence, the kernel $\mathcal{K}_{(a, c)}$ is invariant under the twice iterated action of $Q$. The other operators are interrelated by means of $Q$:

$$W = QV, \quad A^2 = QA^1Q, \quad A^3 = QA^1, \quad A^4 = QA^2, \quad B^2 = QB^1Q, \quad B^3 = QB^2, \quad B^4 = QB^3.$$  \hspace{1cm} (8)

These expressions can be now used to compute the commutation rules obeyed by the intertwiners. On the other hand, according with Table 2, the functions $1 \, F_1(a, c; x)$ and $u(a, c; x)$ are such that they interchange roles under the action of $Q$. The same holds for other operators such that $W$ or $A^4$ but adding a multiplicative non trivial constant\footnote{al-bamid}. Thereby, $Q$ becomes an important intertwiner in our approach, it works simply by transforming one element of the basis into the other one and vice versa. Such a behaviour is translated from the c.h.f. space into the space of parameters as a reflection operation. That statement is clear by considering a more convenient parametrization of $\mathbb{R}^2$, defined by the transformation

$$a' = 2a - c, \quad c' = c - 1, \quad \forall a, c \in \mathbb{R}^2.$$  \hspace{1cm} (9)

With this new parametrization the operators $Q$ and $V$ are rewritten in such a way that their induced action on $\mathbb{R}^2$ becomes linear, homogeneous and diagonal:

$$Q(a', c') = (a', -c'), \quad V(a', c') = (-a', c').$$  \hspace{1cm} (10)

The induced action of the first order intertwiners in terms of the new labels is also simpler

$$\begin{align*}
A^1(a', c') &= (a' - 1, c' + 1), \\
B^1(a', c') &= (a' + 1, c' - 1), \\
A^2(a', c') &= (a' - 1, c' - 1), \\
B^2(a', c') &= (a' + 1, c' + 1).
\end{align*}$$  \hspace{1cm} (11)
From (10) and Table 1 we see that \( V \) and \( W \) are also reflection operators in the parameter domain. A different situation arises for the other intertwiners. For instance, according to (11), the action of \( A^1 \) and \( B^1 \) on the point \((a',c') \in \mathbb{R}^2\) produces the displacements of \( a' \) and \( c' \). Therefore, by iterating the action of \( A^1 \) on \( \mathbb{R}^2 \) we obtain \((A^1)^n (a',c') = (a' - n, c' + n)\), while for \( B^1 \) it reads \((B^1)^m (a',c') = (a' + m, c' - m)\). All these points are indeed in a straight line and form a linear discrete lattice on the space of parameters: \{\((a' + s, c' - s), s \in \mathbb{Z}\)\}. For each one of these points we can associate a kernel \( \mathcal{K}_{(a' + s, c' - s)} \) and, in this way, the action of \( A^1 \) or \( B^1 \) on a c.h.f. \( f(a',c';x) \) can be understood as the mapping from \( \mathcal{K}_{(a',c')} \) into \( \mathcal{K}_{(a' + 1, c' + 1)} \) or \( \mathcal{K}_{(a' + 1, c' - 1)} \) respectively. The iteration process is now clear and the complete set of related c.h.f. \( \{f(a' + s, c' - s; x)\}_{s=-\infty}^{\infty} \) is nothing but an invariant subspace under the action of \( A^1 \) and \( B^1 \). A similar situation occurs for the other first order intertwiners. In general, when the kernel of an operator \( X^k \in \{A^k, B^k\}_{k=1}^4 \) is in a kernel \( \mathcal{K}_{(a',c')} \), the involved invariant subspace is bounded. We shall call an annihilation line of the operator \( X^k \), denoted by \( a[X^k] \), to the set of points \((a',c') \in \mathbb{R}^2\) such that \( f(a',c';x) \in \mathcal{K}_{(a',c')} \) iff \( X^k_{(a',c')}(a',c';x) = 0 \). From (5) and (7) it is easy to see that the factorization lines for the operators \( A^k \) and \( B^k \) are given by \( q^k_{a',c'} = 0 \) and \( p^k_{a',c'} = 0 \) respectively. After a simple calculation and using (8) one gets the following relationships

\[
\begin{align*}
al[A^3] &= al[A^1], & al[A^4] &= al[A^2], & al[B^3] &= al[B^2], & al[B^4] &= al[B^1],
\end{align*}
\]

which give the chance to construct a common set of functions \( \mathcal{F} \equiv \{f(a_n, c_m; x)\} \) representing an invariant subspace under the action of \( \{A^k, B^k\}_{k=1}^2 \). In this case the points \( \{(a_n, c_m)\} \) constitute a two dimensional discrete lattice on \( \mathbb{R}^2 \). The most interesting situation arises when there exists a critical point on the lattice because, by definition, this point is shared by two different annihilation lines. A situation giving rise to an invariant sector of \( \mathcal{F} \), bounded by that lines. In fact, there are four critical points on our \((a',c')\) plane\(\footnote{The \( a' \) values run on the horizontal axis in a right hand frame of reference.} \): \((-1,0), (0,1), (0,-1), \) and \((1,0)\). We can then define the following sectors:

**Left Invariant Sector (L.I.S.).** The point \((a',c') = (-1,0)\) is at the intersection of \( al[B^1] = \{(a',c') | c' = a' + 1\} \) and \( al[B^2] = \{(a',c') | c' = -a' - 1\} \). A left sector of the lattice is then bounded by these annihilation lines and it can be filled by successively applying higher powers of \( A^1 \) and \( A^2 \) on \((-1,0)\). There is a left-corner c.h.f. \( f(-1,0;x) = 1 \) connected with this point while the related functions for the other points on the sector are of the form

\[
\begin{align*}
f(-1 - n, n; x) &= (-1)^n, & c' &\geq 0; & f(-1 - m, -m; x) &= (-x)^m, & c &\leq 0.
\end{align*}
\]

In this way, we can characterize a support space generated by the basis functions \( \{f(-1 - m - n, m, n; x), m, n \in \mathbb{Z}^+\} \), called left invariant sector and denoted \( \mathcal{H}_L \), for which there is a (highest weight) irreducible representation of the algebra \( \{A^1, B^1 = -(A^1)^+, A^2, B^2 = -(A^2)^+, Q\} \) (for details see [4]).

**Right Invariant Sector (R.I.S.).** The critical point \((a',c') = (1,0)\) is at the intersection of \( al[A^1] = \{(a',c') | c' = a' - 1\} \) and \( al[A^2] = \{(a',c') | c' = -a' + 1\} \). The right sector bounded by
Therefore, we can construct a (lowest weight) irreducible representation of the algebra \( A^1B^1 \) from which, the right corner function \( g(1,0;x) = e^x \) can be obtained by making \( n = 0 \) or \( m = 0 \).

The critical point \((0,-1)\) is at the intersection of \( a[A^3] \) and \( a[B^3] \) while \((0,1)\) is at the crossing of \( a[A^4] \) and \( a[B^4] \). However, there are no corner c.h.f. associated with that points and one is unable to construct another invariant sector (outside of the above mentioned ones) by means of any of the first order intertwiners \( A^k \) or \( B^k \). Hence, there are only the two doubly bounded sectors already mentioned.

It is now clear that only the left and the right sectors are of our interest. It is remarkable that the intertwiner \( Q \) becomes a self adjoint operator in the support spaces related with each of these sectors. Moreover, by taking a function \((-1)^n\) in the upper part of the left sector we get, after the action of \( Q \), a function \((-x)^n\) in the lower part of that sector and vice versa. The same can be said about the functions living in the right sector. Hence, this operator intertwins the elements of a given sector with elements of the same sector. A very different situation holds for the reflection operator \( V \) which, by acting on a function \((-x)^m\), gives \( e^x x^m \). In other words, it maps the functions living in the upper part of the left sector into the functions of the upper part of the right sector and vice versa. The same situation occurs for the involved lower parts. The left and right invariant sectors are then intertwined by means of the operators \( Q \) and \( V \) whereas they are generated by the iterated action of the first order intertwiners on the left and right corner functions.

At a first sight, it seems to be enough to consider the points on the lattice living in the left and right sectors. However, there is still a subset of points deserving attention. The straight line defined by the constraint \( c' = -1/2 \) results invariant under the action of \( A^3B^3 \). This line cuts \( a[B^3] \) on \((-1/2,-1/2)\) and \( a[A^3] \) on \((1/2,-1/2)\). Hence, the points \((-1/2 - n, -1/2)\) and \((1/2 + n, -1/2)\) will form the lattices for two irreducible representations of \( A^3B^3 \). A similar argument can be established for the line defined by \( c' = 1/2 \) and the operators \( A^4B^4 \).

Therefore, we have constrained the original \((a',c')\)-plane from all the \( \mathbb{R}^2 \) into its more relevant part, composed by two invariant interrelated sectors plus two invariant lines.

### 3 Schrödinger equations

Let us start this section by introducing a free index operator \( M \), written in the confluent hypergeometric notation as follows

\[
M f(a,c;x) := M_{(a,c)} f(a,c;x), \quad M_{(a,c)} := Y_{(a,c)} \circ \varphi^{-1}(a,c;x)
\]  

where

\[
\varphi(a,c;x) = \left[ \frac{e^{x}}{x^{c} y'(x)} \right]^{1/2}
\]
and $Y_{(a,c)}$ is an auxiliary operator changing the independent variable $x$ into $y$

$$Y_{(a,c)} : \begin{cases} x \rightarrow x(y), \\ F(x) \rightarrow \Phi(y) \equiv F(x(y)). \end{cases}$$

The action of the operator $M$ on the kernel elements of $L_{(a,c)}$ produces the mapping from the confluent hypergeometric equation $L_{(a,c)} f(a; c; x) = 0$ to the time independent Schrödinger equation

$$\left[-\frac{d^2}{dy^2} + V(y)\right] \psi_s(y) = E \psi_s(y), \quad (14)$$

where $2m/\hbar^2 = 1$ and the label $s$ stands for the dependence on certain Schrödinger parameters. In other words,

$$f(a; c; x) \xrightarrow{M} \psi_s(y) = \varphi^{-1}(a; c; x(y)) f(a; c; x(y)). \quad (15)$$

It is easy to check that the identification of $E$ and $V(y)$ in $[14]$ depends exclusively on the specific analytical realization of $y'(x)$, i.e., on the function we have taken as the new independent variable $y(x)$. Once we know how to connect the confluent hypergeometric equations to a class of Schrödinger equations, we can translate all the studied mathematical properties into physical ones within the framework of the stationary Schrödinger wave-functions. Therefore, by using the first order intertwiners of Table 1 and $M$, we can establish the following commutative diagram

$$\begin{array}{ccc}
    f(a; c; x) & \xrightarrow{M} & \psi_s(y) \\
    \uparrow & & \uparrow \\
    A^i_{(a,c)} & \xrightarrow{B^i_{(a,c)}} & B^i_s \\
    \downarrow & & \downarrow \\
    f(\tilde{a}; \tilde{c}; x) & \xrightarrow{M} & \psi_{\tilde{s}}(y) \\
\end{array} \quad (16)$$

where the operators $A^i_s$ and $B^i_s$ relate Schrödinger wave-functions with different labels:

$$A^i_s \psi_s(y) \equiv M A^i_{(a,c)} M^{-1} \psi_s(y) \propto \psi_{\tilde{s}}(y),$$

$$B^i_s \psi_{\tilde{s}}(y) \equiv M B^i_{(\tilde{a}, \tilde{c})} M^{-1} \psi_{\tilde{s}}(y) \propto \psi_s(y).$$

Now, we proceed to identify the immediate physical potentials $V(y)$ related with the c.h.f. by means of the above mentioned transformations. We shall focus on the main information drawing the readers attention to our previous work $[4]$ for details. In each case $\ell$ stands for the angular quantum number $\ell \in \mathbf{Z}^+$ and $E$ for the involved energy eigenvalues. Moreover, for all the following examples, the c.h.f. $f(a'; c'; x)$ are understood as appropriate linear combinations of the basis elements $[7]$. Notice that it is always possible to choose between two different combinations of the pair $(a', c')$, which are labeled by an $\pm$ subindex in each case. In addition, we have clearly established the part of the $(a_n, c_m)$-lattice where the involved c.h.f. live. The specific physically allowed values of these points can be obtained by asking for the square integrability property of the corresponding Schrödinger functions $\psi_s$.

**N-dimensional harmonic oscillator**

$$V_{osc}^N(y) \equiv y^2 + \frac{(2\ell + N - 1)(2\ell + N - 3)}{4y^2}, \quad N \geq 2, \quad y = x^{1/2}.$$
\[
\begin{aligned}
\ell &= c'_+ + 1 - N/2 \geq 0 \\
E &= -2a'_+ > 0 \\
\text{upper part of the L.I.S.}
\end{aligned} \quad \begin{aligned}
\ell &= 1 - c'_- - N/2 \geq 0 \\
E &= -2a'_- > 0 \\
\text{lower part of the L.I.S.}
\end{aligned}
\]

One-dimensional harmonic oscillator

\[V_{\text{osc}}(y) = y^2, \quad y = x^{1/2}\]

\[
\begin{aligned}
c'_+ &= 1/2 \\
E &= -2a'_+ > 0 \\
\text{upper left invariant line}
\end{aligned} \quad \begin{aligned}
c'_- &= -1/2 \\
E &= -2a'_- > 0 \\
\text{lower left invariant line}
\end{aligned}
\]

N-dimensional Coulomb potential

\[V_{\text{Coul}}^N(r) = -\frac{2}{r} + \frac{(2\ell + N - 1)(2\ell + N - 3)}{4y^2}, \quad y \propto x\]

\[
\begin{aligned}
\ell &= \frac{c'_+ + 2 - N}{2} \\
E &= -\left(\frac{2}{a'_+}\right)^2 \\
\text{upper part of the L.I.S.}
\end{aligned} \quad \begin{aligned}
\ell &= \frac{2 - c'_- - N}{2} \\
E &= -\left(\frac{2}{a'_-}\right)^2 \\
\text{lower part of the L.I.S.}
\end{aligned}
\]

Morse potential

\[V_M(y) = \left(\frac{\alpha}{2}\right)^2 \left(e^{2\alpha y} - 2\lambda e^{\alpha y}\right), \quad y = (\ln x)^{1/\alpha}, \quad \alpha > 0, \ \lambda > 0\]

\[
\begin{aligned}
c'_+ &= \frac{2}{\alpha}\sqrt{-E} \geq 1 \\
\lambda &= -a'_+ \\
\text{upper part of the L.I.S.}
\end{aligned} \quad \begin{aligned}
c'_- &= -\frac{2}{\alpha}\sqrt{-E} \leq 1 \\
\lambda &= -a'_- \\
\text{lower part of the L.I.S.}
\end{aligned}
\]

As we can see, the same Schrödinger function \(\psi_s\) can be constructed by means of the c.h.f. parametrized by \((a'_+, c'_+)\) as well as that with parameters \((a'_-, c'_-)\). As \(c'_-\) is always such that \(c'_- = -c'_+\), these functions live in different parts of the same (left) sector or line. Hence, it could be enough to consider either the upper or the lower part of that sector or line to describe the physical solutions of the related quantum problem. In other words, there is a mapping 2:1 from the kernel \(\mathcal{K}_{(a,c)}\) to the Hilbert space \(\mathcal{H}\) of square integrable functions \(\psi_s\). A situation leading to a representation of \(Q\) in \(\mathcal{H}\) as an operator which does not transform the general solutions of the Schrödinger equation but changing only its parametrization. The same is true for the reflection operator \(V\) developing the first Kummer transformation on the c.h.f.
In summary, the action of the operator $M$ on the c.h.f. (living only in the afore mentioned sectors) gives the stationary Schrödinger solutions for the oscillator, Coulomb, and Morse potentials. As we have seen, $M$ connects the c.h.o. with the Schrödinger equation related with these potentials. Hence, only the left and right invariant sectors, besides the invariant lines we are dealing with, are the responsible for bound states in the physical problems here considered (we will not discuss here the states corresponding to the continuum spectrum of these examples.) This is why they have been called the physical sectors of the confluent hypergeometric functions space. On the other hand, the same mapping $M$ allows the connection among the wave-functions of the diverse quantum systems we have presented. As a very last example, let us consider the $N = 2$ case for the oscillator and the Coulomb potentials. In order to fix the notation, we first write the energy of the discrete spectrum as follows:

$$E_O = 2n_O, \quad E_C = \frac{-1}{(n_C + 1/2)^2}, \quad E_M = -\frac{\alpha^2}{4}\nu_M^2$$

where $O$, $C$ and $M$ mean oscillator, Coulomb, and Morse, respectively. Thus, we have

**Oscillator-Morse:**

$$\psi_M^{(\nu_M, \lambda_M)}(y) \propto e^{-\alpha y/4}\psi_O^{(n_O, \ell_O)}(e^{\alpha y}/2), \quad n_O=\lambda_M, \ell_O=\nu_M.$$  \tag{18}

**Oscillator-Coulomb:**

$$\psi_H^{(n_C, \ell_C)}(y) \propto y^{1/4}\psi_O^{(n_O, \ell_O)}(2y^{1/2}/(2n_O+1)^{1/2}), \quad n_O=2n_C+1, \ell_O=2\ell_C.$$  \tag{19}

For other dimensions $N$ of the oscillator or Coulomb potentials, the relations keep on valid, but with other restrictions on the values of $(n, \ell)$. The above formulas \[(18)-(19)\] do not change the number of nodes but they change the local density probability so that there is a mathematical mapping among these three potentials which does not represent a physical equivalence.

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