Privacy-Utility Tradeoff for Hypothesis Testing Over A Noisy Channel

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Abstract—We study a hypothesis testing problem with a privacy constraint over a noisy channel and derive the performance of optimal tests under the Neyman-Pearson criterion. The fundamental limit of interest is the privacy-utility tradeoff (PUT) between the exponent of the type-II error probability and the leakage of the information source subject to a constant constraint on the type-I error probability. We provide an exact characterization of the asymptotic PUT for any non-vanishing type-I error probability. Our result implies that tolerating a larger type-I error probability cannot improve the PUT. Such a result is known as a strong converse or strong impossibility theorem. To prove the strong converse theorem, we apply the recently proposed technique in (Tyagi and Watanabe, 2020) and further demonstrate its generality. The strong converse theorems for several problems, such as hypothesis testing against independence over a noisy channel (Sreekumar and Gündüz, 2020) and hypothesis testing with communication and privacy constraints (Gilani et al., 2020), are established or recovered as special cases of our result.

Index Terms—Strong converse, information leakage, noisy channel, non-asymptotic converse, Euclidean information theory

I. INTRODUCTION

In the binary hypothesis testing problem, given a test sequence $X^n$ and two distributions $P$ and $Q$, one is asked to determine whether the test sequence $X^n$ is generated i.i.d. from $P$ or $Q$. The performance of any test is evaluated by the tradeoff between the type-I and type-II error probabilities. Under the Neyman Pearson setting where the type-I error probability is upper bounded by a constant, the likelihood ratio test [1] is proved optimal. Chernoff-Stein lemma [2] states that the type-II error probability decays exponentially fast with exponent $D(Q||P)$ when the type-I error probability is upper bounded by one half and the length of the test sequence tends to infinity. This result was later refined by Strassen [3] who provided exact second-order asymptotic characterization of the type-II error exponent for any non-vanishing type-I error probability. Strassen’s result implies the asymptotic type-II error exponent remains $D(Q||P)$ regardless of the non-vanishing type-I error probability. Such a result is known as a strong converse theorem, which implies that tolerating a larger type-I error probability cannot increase the asymptotic decay rate of the type-II error probability of an optimal test.

The simple binary hypothesis testing problem was later generalized to various scenarios. Motivated by the application where the source sequence might be only available to a decision maker via rate-limited communication, Ahlswede and Csiszár [4] initiated the study of the hypothesis testing problem with communication constraints. The authors of [4] gave exact asymptotic characterization of the rate-exponent tradeoff subject to a vanishing type-I error probability and proved a strong converse result for the special case of testing against independence. Recently, motivated by the fact the source sequence is transmitted over a noisy channel in certain applications, e.g., in a sensor network [5], Sreekumar and Gündüz [6] further generalized the setting of [4] by adding a noisy channel between the transmitter and the decision maker. However, the authors of [6] derived only a weak converse result which holds for vanishing type-I error probability. For the case of testing against independence, a strong converse result was proved in [7] when the bandwidth expansion ratio $\tau$ (defined as the ratio between the number of channel uses $n$ and the length of the source sequence $k$) is 1 by combining the blowing up lemma [8] and the strong converse technique recently proposed in [9].

Another generalization of the binary hypothesis testing framework takes privacy into consideration. Privacy gains increasing attention from all parties. Releasing collected raw data for statistical inference can potentially leak critical information of individuals (cf. [10, Fig. 1]). Motivated by the privacy concerns in modern data analyses and machine learning, Liao et al. [11] applied a privacy mechanism to the original sequences to remove private parts and then studied the hypothesis testing problem with a privacy constraint. In particular, the authors of [11] derived the privacy-utility tradeoff [10] between the decay rate of the type-II error probability and the leakage of the information sources measured with mutual information. Subsequently, the setting in [11] was generalized to the case under the maximal leakage privacy constraint in [12] and to the case with communication constraints by Gilani et al. [13].

Motivated by i) practical applications where there is a noisy channel between the detector and the transmitter for a hypothesis test and ii) the privacy concerns of statistical data inference problems, we study the privacy-utility tradeoff for a generalized model of [13], [14]. In particular, we consider a hypothesis testing problem over a noisy channel with a privacy constraint as shown in Figure 1. We use mutual information as the privacy metric.
information as the privacy measure, which is consistent with existing literature in terms of measuring privacy [10], [12], [13], [15], [16] or security [17]–[19]. Such a formulation is intuitive since a small value of the mutual information between two random variables implies a low dependence level. The extreme case of vanishing mutual information privacy constraint, a.k.a. the high privacy limit, ensures almost perfect privacy where virtually no information about the raw data is disclosed. Furthermore, a mutual information constraint can also be motivated by the communication rate constraint as in distributed detection [20].

There are many other privacy measures, such as the maximal leakage [12], the distortion function [14], the differential privacy [21] and the maximal α-leakage [22]. Among all these privacy measures, the different privacy is probably the most popular one and finds wide applications in various domains [23]. But a differentially private mechanism might have high leakage under the mutual information privacy measure [24]. Furthermore, the authors of [25] show that the expected information leakage under any privacy measure can be upper bounded by a function of the mutual information privacy constraint. Motivated by the results in [24], [25] and consistent with pioneering works [10], [11], we choose mutual information as the privacy measure in this paper. It is of definite interest to generalize our results to other privacy measures and compare the privacy-utility tradeoffs under different privacy measures as done in [26].

Our main contribution is the exact characterization of the privacy-utility tradeoff (PUT) between the decay rate of type-II error probability and the information leakage at the transmitter subject to a constraint on the type-I error probability. It turns out that the exact PUT is a non-convex optimization problem, which can not be solved efficiently. Under the high privacy limit, we derive an easily computable approximation to the PUT using the Euclidean information theory [27], [28]. Euclidean information theory is based on the local approximation of the KL divergence $D(P∥Q)$ using Taylor expansions when two distributions $P$ and $Q$ are close to each other.

The rest of the paper is organized as follows. In Section II, we formally set up the notation, formulate the hypothesis testing problem with a privacy constraint over a noisy channel and define the privacy-utility tradeoff. Subsequently, we present our characterization of the PUT in Section III. The proofs of our results are given in Section IV. Finally, in Section V, we conclude the paper and discuss future research directions. The proofs of all supporting lemmas are deferred to appendices.
complete with each other and naturally introduce a privacy-utility tradeoff. Our main results provide exact characterization of the PUT in the asymptotic setting.

Formally, a communication protocol is defined as follows.

**Definition 1.** A communication protocol \((f^{n,k}, g^{n,k})\) with \(n\) channel uses for hypothesis testing against independence over a noisy channel consists of

(i) a potentially stochastic encoder \(f^{n,k} : Z^k \rightarrow X^n\)
(ii) a decoder \(g^{n,k} : Y^n \times T^k \rightarrow \{H_1, H_2\}\)

- \(H_1: \) the sequences \(U^k\) and \(V^k\) are correlated, i.e., \((U^k, V^k) \sim P_{U,V}^{k}\)
- \(H_2: \) the sequences \(U^k\) and \(V^k\) are independent, i.e., \((U^k, V^k) \sim P_{U}^{k}P_{V}^{k}\).

When \(f^{n,k}\) is a stochastic encoder, we use \(P_{f^{n,k}}(x^n|z^k)\) to denote the probability that the output of the encoder is \(x^n\) when the input is \(z^k\). In particular, when \(f^{n,k}\) is deterministic, \(P_{f^{n,k}}(x^n|z^k)\) is simply an indicator function and outputs 1 if and only if \(x^n = f^{n,k}(z^k)\). Given any communication protocol \((f^{n,k}, g^{n,k})\) and any privacy mechanism \(P_{Z^k|U^k}\), their performance is evaluated by the type-I and type-II error probabilities:

\[
\begin{align*}
\beta_1(f^{n,k}, g^{n,k}) &:= \Pr\{g^{n,k}(Y^n, V^k) = H_2|H_1\}, \\
\beta_2(f^{n,k}, g^{n,k}) &:= \Pr\{g^{n,k}(Y^n, V^k) = H_1|H_2\},
\end{align*}
\]

where \(Y^n\) is the output of passing \(X^n = f^{n,k}(Z^k)\) over the noisy memoryless channel \(P_{Y|X}\). Thus, the probability terms in the right-hand side of (1) and (2) depend on the encoding function \(f^{n,k}\) and the privacy mechanism \(P_{Z^k|U^k}\) implicitly via the noisy output \(Y^n\).

**B. Definition of the Privacy-Utility Tradeoff**

We restrict ourselves to memoryless privacy mechanisms, i.e., \(P_{Z^k|U^k} = P_{Z_k|U}\) for some \(P_{Z_k|U} \in \mathcal{P}(Z|U)\). In fact, the adoption of a memoryless privacy mechanism is consistent with a large body of existing literature [11]–[14], [29]. Furthermore, the memoryless privacy scheme enjoys low complexity and is motivated by the case where each respondent can apply the same randomized privacy mechanism before submitting replies to queries. In contrast, if one adopts a non-memoryless privacy mechanism, then as the length \(k\) of the source sequence increases, one needs to design a different privacy mechanism and suffers from higher complexity, especially in the case of large \(k\). Finally, adopting a memoryless privacy mechanism does not trivialize the problem. In fact, our proof, especially the converse proof in Section IV, requires us to judiciously combine the analyses for the utility and the privacy.

Under the Neyman-Pearson formulation, we are interested in the maximal type-II error exponent subject to a constant constraint on the type-I error probability \(\varepsilon \in (0, 1)\), a bandwidth expansion ratio \(\tau \in \mathbb{R}_+\) and a privacy constraint \(L \in \mathbb{R}_+\) for \(n \in \mathbb{N}\) channel uses, i.e.,

\[
E^*(k, \tau, L, \varepsilon) := \sup\{E \in \mathbb{R}_+: \exists (f^{n,k}, g^{n,k}, P_{Z|U}) \text{ s.t. } n \leq k\tau, I(P_U, P_{Z|U}) \leq L, \beta_1(f^{n,k}, g^{n,k}) \leq \varepsilon, \beta_2(f^{n,k}, g^{n,k}) \leq \exp(-kE)\}.
\]

The privacy constraint \(I(P_U, P_{Z|U}) \leq L\) implies that the leakage of information source \(U^k\) from the privatized version \(Z^k\) satisfies that \(I(U^k; Z^k) \leq kL\). Note that the privacy is measured using mutual information [30]. Such a choice of the privacy measure is consistent with most literature studying physical layer security, e.g., [11], [13], [15], [16], [19].

We remark that \(E^*(k, \tau, L, \varepsilon)\) represents a tension between the privacy and the utility. Evidently, the looser the privacy constraint \(L\), the better the utility \(E^*(k, \tau, L, \varepsilon)\). In the extreme case of \(L \geq H(U)\), our setting reduces to the case without a privacy constraint as in [6, Theorem 2] and achieves the best utility. In the other extreme of \(L = 0\), we achieve the perfect privacy while the utility \(E^*(k, \tau, L, \varepsilon) = 0\). This is because to ensure perfect privacy, we generate a private sequence \(Z^k\), which is independent of the source sequence \(U^k\), and therefore, even the full knowledge of \(Z^k\) provides no information about the correlation with side information \(V^k\), let alone noisy observations of \(Z^k\). To better understand the privacy-utility tradeoff for non-extremal values of \(L\), we provide exact characterization of \(E^*(k, \tau, L, \varepsilon)\) in the limit of large \(k\) for any parameters \((\tau, L, \varepsilon) \in \mathbb{R}_+^3 \times (0, 1)\).

**III. MAIN RESULTS**

In this section, we present our main results, which exactly characterize the privacy-utility tradeoff in the limit of large \(k\).

**A. Achievability**

In this subsection, we present our achievability result, which provides a lower bound on \(E^*(k, \tau, L, \varepsilon)\). Several definitions are needed. The capacity [30] of a noisy channel with transition matrix \(P_{Y|X}\)

\[
C(P_{Y|X}) = \max_{P_X \in \mathcal{P}(X)} I(P_X, P_{Y|X}).
\]

Furthermore, let \(W\) be an auxiliary random variable taking values in the alphabet \(W\) and let \(Q\) denote the set of all joint distributions defined on the alphabet \(U \times V \times Z \times W\). Given any \(P_{Z|U} \in \mathcal{P}(Z|U)\), define the following set of distributions

\[
Q(P_{U,V}, P_{Z|U}) := \{Q_{UVZW} \in Q: Q_{UV} = P_{UV}, Q_{Z|U} = P_{Z|U}, V - U - Z - W, |W| \leq |Z| + 1\}.
\]

Given any \(Q_{UVZW}\), let other distributions denoted by \(Q\) be induced distributions. For any \((\tau, L) \in \mathbb{R}_+^2\), define the following optimization problem

\[
f(\tau, L, P_{UV}, P_{Z|U}, P_{Y|X}) := \max_{Q\in\mathcal{Q}(P_{UV}, P_{Z|U})} I(Q_{VQW}|V). \]  

Since \(V - U - Z - W\) forms a Markov chain under any distribution \(Q_{UVZW} \in Q(P_{UV}, P_{Z|U})\), we have

\[
f(\tau, L, P_{UV}, P_{Z|U}, P_{Y|X}) \leq L.
\]

Our achievability result states as follows.

**Theorem 1.** For any \((\tau, L) \in \mathbb{R}_+^2, \varepsilon \in (0, 1),\)

\[
\lim_{k \to \infty} E^*(k, \tau, L, \varepsilon) \geq \max_{P_{Z|U}} f(\tau, L, P_{UV}, P_{Z|U}, P_{Y|X}).
\]
Theorem 1 is a straightforward extension of [6, Theorem 2] and thus its proof is omitted. The proof of Theorem 1 proceeds in three steps. Firstly, we calculate the optimal memoryless privacy scheme $P^*_{Z|U}$. Secondly, we apply the memoryless privacy mechanism $P^*_{Z|U}$ to privatize the original information source $U^k$ and obtain the non-private information counterpart $Z^k$. Finally, we study a hypothesis testing problem against independence over a noisy channel for the new source sequence $Z^k$ and the side information $V^k$ at the decoder. The final step is exactly the same as [6, Theorem 2] when specialized to the case of testing against independence.

**B. Converse and Discussions**

Our main contribution in this paper is the following theorem, which presents a non-asymptotic upper bound on the optimal type-II exponent $E^*(k, \tau, L, \varepsilon)$.

1) **Preliminaries:** To present our result, for any $(\lambda_1, \lambda_2) \in \mathbb{R}_+^2$, define two constants

$$c(\lambda_1, \lambda_2, \tau) := \log |\mathcal{Y}| + (\lambda_1 + \lambda_2) \log |\mathcal{Z}| + \lambda_1 \tau \log |\mathcal{Y}|,$$

$$\zeta(\lambda_1, \lambda_2, \gamma, \tau) := 3 \left( \frac{2 \lambda_1 + \lambda_2 - \gamma}{\lambda_1 + \lambda_2 - \gamma} \right)^{\lambda_1} \left( \frac{|\mathcal{V}|}{|\mathcal{Y}|} \right)^{\lambda_1}$$

$$+ \lambda_1 \log \left( \frac{|\mathcal{Z}|}{\gamma} \right) + \lambda_2 \log \left( \frac{|\mathcal{V}|}{\gamma} \right) + 3 \lambda_1 \tau \log \left( \frac{|\mathcal{X}|}{\gamma} \right),$$

where $|\mathcal{V}|$ is a finite constant. Given any distributions $(Q_{UVZW}, Q_{XY})$, for any $(\lambda_1, \lambda_2) \in \mathbb{R}_+^2$, define the following linear combination of mutual information terms

$$R^*_{\lambda_1, \lambda_2}(Q_{UVZW}, Q_{XY}) := I(Q_{UVZW}) - \lambda_1 (I(Q_{Z|W}|Z) - \tau I(Q_{X|Y})) - \lambda_2 (I(Q_{XY}) - L).$$

Furthermore, define the following optimization value

$$g^*_{\lambda_1, \lambda_2}(P_{UV}, P_{Z|U}, P_{Y|X}) := \sup_{Q_{UVZW}, Q_{XY} \in C} R^*_{\lambda_1, \lambda_2}(Q_{UVZW}, Q_{XY}),$$

where $C$ denotes the set of all joint distributions defined on the alphabet $X \times Y$. As we shall show, $g^*_{\lambda_1, \lambda_2}(P_{UV}, P_{Z|U}, P_{Y|X})$ is closely related with $f(\tau, L, P_{UV}, P_{Z|U}, P_{Y|X})$ established via the supporting hyperplane, we managed to obtain the desired result in Theorem 2. Our proof applies the strong converse technique by Tyagi and Watanabe [9] to a hypothesis testing problem over a noisy channel with a privacy constraint and thus demonstrates the generality of the technique.

We make several additional remarks. Combining the strong converse result in (17) and Theorem 1, we conclude that given any $(L, \tau) \in \mathbb{R}_+^2$, for any $\varepsilon \in (0, 1)$,

$$\lim_{k \to \infty} E^*(k, \tau, L, \varepsilon) = \max_{P_{Z|U}} f(\tau, L, P_{UV}, P_{Z|U}, P_{Y|X}).$$

Thus, we provide a complete characterization of the asymptotically optimal PUT is independent of the type-I error probability for any given privacy constraint. Therefore, tolerating a larger type-I error probability cannot increase the privacy-utility tradeoff of optimal privacy and communication protocols when the lengths of sequences tend to infinity. Such a result is known as strong converse in information theory (cf. [33–35]), which refines the classical weak converse argument valid only for vanishing type-I error probability.

Furthermore, since several problems are special cases of our formulation, the result in (18) implies strong converse and provides complete asymptotic characterization of fundamental limits for all these special cases, e.g., [4, 6, 13]. In particular, by letting $L \geq H(P_U)$, our setting reduces to the hypothesis testing problem against independence (special
\[ L(P_{Z|U}, \varepsilon) = \begin{cases} \sqrt{V(P_{Z|U})}Q^{-1} \left( \varepsilon - \frac{T(P_{Z|U})}{\theta \sqrt{V(P_{Z|U})}} \right) & \text{if } V(P_{Z|U}) > 0, \\ 0 & \text{otherwise} \end{cases} \]  

(15)

case of [6, Theorem 2]). A strong converse theorem was not established for any \( \tau \neq 1 \) prior to our work. If one considers a memoryless channel and imposes a communication constraint, i.e., \( P_{Y|X} \) is the identity matrix and \( \mathcal{X} = \mathcal{Y} = \{1, \ldots, M\} \) for some \( M \in \mathbb{N} \), our setting then reduces to the setting of hypothesis testing with both communication and privacy constraints considered in [13]. The authors of [13] proved a strong converse result for their setting using the complicated blowing up lemma idea [8]. Our result here provides an alternative yet simpler proof for their setting.

Finally, we compare our converse result with existing works on hypothesis testing over a noisy channel or with a privacy constraint, especially [13] and [7]. The former one corresponds to the special case where the channel is noiseless. By considering a noisy channel in this paper, our analysis is more complicated since we need to account for additional errors due to the noisy nature of the channel. Our results imply the strong converse result in [13, Theorem 2] but not vice versa. In [7], without a privacy constraint by letting \( L \geq H(U) \), the authors proved a strong converse result by combining the techniques in [9] and the blowing up lemma [8]. In contrast, our proof is more transparent and much simpler by getting rid of the blowing up lemma.

### C. Illustration of the PUT via a Numerical Example

Let \( \mathcal{U} = \mathcal{V} = \mathcal{Z} = \{1, 2\} \). Let \( P_U \) be the uniform distribution over \( \mathcal{U} \) and let the transition probability \( P_{Y|U} \) be

\[ P_{Y|U}(v|u) = q1_{v=u} + (1-q)1_{v \neq u}, \]

(20)

for some \( q \in [0, 1] \). Let the channel \( P_{Y|X} \) be a binary symmetric channel with crossover probability 0.2 and let the privacy mechanism \( P_{Z|U} \) be a binary symmetric channel with parameter \( p \), which is later optimized over all choices of \( p \) to obtain the best privacy mechanism. Using [13, Proposition 1], we can obtain the exact formula of \( f(\tau, L, P_{U|V}, P_{Z|U}, P_{Y|X}) \).

In Figure 2, we plot the privacy-utility tradeoff for \( q = 0.8 \) and various values of \( \tau \). Note that \( f(\tau, L, P_{U|V}, P_{Z|U}, P_{Y|X}) \) attains the maximal value for any \( L \geq H(P_U) = \log 2 \) and \( f(\tau, L, P_{U|V}, P_{Z|U}, P_{Y|X}) = 0 \) if \( L = 0 \). For any non-degenerate values of \( L \in (0, H(P_U)) \), we observe a privacy-utility tradeoff.

### D. Approximation to the PUT under the High Privacy Limit

Note that the exact PUT presented in (18) is a non-convex optimization problem, which cannot be solved efficiently. In this subsection, under the high privacy limit, i.e., when \( I(P_U, P_{Z|U}) \) tends to zero, we derive an easily computable approximation to the PUT using Euclidean information theory [27]. Furthermore, as argued in [11], the PUT under the high privacy limit is desirable as we always seek privacy mechanism as strong as possible.

Recall that both \( \mathcal{U} \) and \( \mathcal{Z} \) are finite alphabets. Without loss of generality, in this subsection, we let \( \mathcal{U} = [|\mathcal{U}|] = \{1, \ldots, |\mathcal{U}|\} \) and let \( \mathcal{Z} = [|\mathcal{Z}|] \). Furthermore, we let \( \mathcal{W} := [|\mathcal{W}|] = [|\mathcal{Z}| + 1] \). Under the perfect privacy, i.e., \( L = 0 \), we conclude that the privacy mechanism is \( P_{Z|U} = Q_Z \) for each \( u \in \mathcal{U} \) where \( Q_Z \in \mathcal{P}(\mathcal{Z}) \) is arbitrary. Furthermore, given any \( P_{W|Z} \), let \( Q_W \) be induced by \( Q_Z \) and \( P_{W|Z} \), i.e.,

\[ Q_w(w) = \sum_z Q_Z(z)P_{W|Z}(w|z). \]

Given any two finite alphabets \( \mathcal{A}, \mathcal{B} \) and any distribution \( P_A \in \mathcal{P}(|\mathcal{A}|) \), let \( \mathcal{J}(\mathcal{A}, \mathcal{B}, P_A) \) be the collection of all \( |\mathcal{A}| \times |\mathcal{B}| \) matrices \( J = \{J(a, b)\}_{a \in \mathcal{A}, b \in \mathcal{B}} \) such that

\[ |J(a, b)| \leq 1, \quad \forall \ (a, b) \in \mathcal{A} \times \mathcal{B}, \]

\[ \sum_{b \in \mathcal{B}} J(a, b) = 0, \quad \forall \ a \in \mathcal{A}, \]

\[ \sum_{a \in \mathcal{A}} P_A(a)J(a, b) = 0, \quad \forall \ b \in \mathcal{B}. \]

Let \( J \in \mathcal{J}(\mathcal{U}, \mathcal{Z}, P_U) \) be an arbitrary. For any \( (v, w) \), define

\[ h(J, \rho) := \rho^2 \frac{1}{2} \sum_{v, u} \frac{P_{U|V}(v)}{Q_W(w)} \left( \sum_{u, z} P_{U|V}(u|v)P_{W|Z}(w|z)J(u, z) \right)^2, \]

(25)

where we use \( J(u, z) \) to denote the \( u \)-th element of \( z \)-th row of the matrix \( J \).
Under the high privacy limit, $L$ can be chosen as \( \frac{1}{2} \rho^2 \) for an arbitrary small $\rho \in (0, 1)$. Using Euclidean information theory [27], [28], we have that when $\rho$ is small,

\[
f \left( \tau, \frac{\rho^2}{2}, P_{UV}, P_{Y|X} \right) \approx \max_{Q_Z, Q_{W|Z}} l(J, \Theta, \rho, Q_Z, Q_W).
\]

In Figure 3, the approximation value in (26) is plotted and compared with the exact value for the binary example considered in Section III-C. We observe that the Euclidean approximation in (26) is quite tight when the privacy constraint $L$ is small.

We then consider the case where the channel $P_{Y|X}$ is extremely noisy so that $C(P_{Y|X})$ is arbitrarily small. Let $Q_W \in P(W)$ be an arbitrary distribution, let $\Theta$ be an arbitrary $|W| \times |Z|$ matrix and define

\[
l(J, \Theta, \rho, Q_Z, Q_W) := \frac{\rho^4}{2} \sum_{u,v,w} P_{V|W}(w) \left( \sum_{u,z} P_{U|V}(u|v)J(u, z)\Theta(z, w) \right)^2,
\]

where we use $\Theta(z, w)$ to denote the $z$-th element of $w$-th row of the matrix $\Theta$.

If we further assume that the channel $P_{Y|X}$ is extremely noisy such that $\tau C(P_{Y|X}) = \frac{\rho^2}{2}$, then

\[
f \left( \tau, \frac{\rho^2}{2}, P_{UV}, P_{Y|X} \right) \approx \max_{Q_Z, Q_{W|Z}, \Theta \in J(Z,W,Q_Z), J \in J(U,Z,P_U)} l(J, \Theta, \rho, Q_Z, Q_W).
\]

The proofs of (26) and (28) are provided in Appendix A.

In Figure 4, the approximation value given in (28) is plotted and compared with the exact value for the binary example considered in Section III-C. We observe that the Euclidean approximation in (28) is very tight when the privacy constraint $L$ is small and the channel is extremely noisy.

IV. PROOF OF THEOREM 2

A. Alternative Characterization of the Optimal PUT

We first provide an alternative characterization of the optimal privacy-utility tradeoff $f(\tau, L, P_{UV}, P_{Z|U}, P_{Y|X})$ in (6) using the supporting hyperplanes [32], [36]. This result is critical to our converse proof.

Recall that $P_{UV}$ is the generating distribution of $(U^k, V^k)$ under hypothesis $H_1$ and $P_{Y|X}$ denotes the memoryless channel between the transmitter and the detector. For any memoryless privacy mechanism $P_{Z|U}$, let $P_U$, $P_Z$, $P_{U|Z}$ and $P_{V|U}$ be distributions induced by $P_{UV}$ and $P_{Z|U}$. Furthermore, recall that $\mathcal{Q}$ denotes the set of all joint distributions defined on the alphabet $\mathcal{U} \times \mathcal{V} \times \mathcal{Z} \times \mathcal{W}$ and that $\mathcal{C}$ denotes the set of all joint distributions defined on the alphabet $\mathcal{X} \times \mathcal{Y}$. Given any $(Q_{UVZW}, Q_{XY}) \in \mathcal{Q} \times \mathcal{C}$, for any $(\lambda_1, \lambda_2, \gamma) \in \mathbb{R}^3_+$, let

\[
\Delta^{\gamma}_{L}(Q_{UVZW}, Q_{XY}, P_{UV}, P_{Z|U}, P_{Y|X}) := \gamma D(Q_Z || P_Z) + \gamma D(Q_{UVZW} || P_{U|Z}P_{V|U}|Q_{Z})
\]

\[
+ \tau \gamma D(Q_{Y|X} || P_{Y|X}|Q_{X}),
\]

\[
R^{\gamma,L}_{\lambda_1, \lambda_2, \gamma}(Q_{UVZW}, Q_{XY}, P_{UV}, P_{Z|U}, P_{Y|X}) := R^{\gamma,L}_{\lambda_1, \lambda_2}(Q_{UVZW}, Q_{XY}) - \Delta^{\gamma}_{L}(Q_{UVZW}, Q_{XY}, P_{UV}, P_{Z|U}, P_{Y|X}),
\]

where $R^{\gamma,L}_{\lambda_1, \lambda_2}(Q_{UVZW}, Q_{XY})$ was defined in (10).

Finally, let

\[
g^{\gamma,L}_{\lambda_1, \lambda_2, \gamma}(P_{UV}, P_{Z|U}, P_{Y|X}) := \sup_{Q_{UVZW} \in \mathcal{Q}, Q_{XY} \in \mathcal{C}} R^{\gamma,L}_{\lambda_1, \lambda_2, \gamma}(Q_{UVZW}, Q_{XY}, P_{UV}, P_{Z|U}, P_{Y|X}).
\]
Recall the definitions of $\zeta(\lambda_1, \lambda_2, \gamma, \tau)$ in (9), $P(\tau, L, P_{UV}, P_{Z|U}, P_{Y|X})$ in (6) and $g_{n,k}^r, L(\cdot)$ in (11). We have the following lemma.

**Lemma 3.** The following claims hold:

(i) $g_{n,k}^r, L(\cdot)$ is related with $f_{\lambda_1, \lambda_2}(\cdot)$ as follows:

$$f_{\lambda_1, \lambda_2}(\tau, L, \cdot) = \min_{(\lambda_1, \lambda_2) \in \mathbb{R}_+^2} g_{n,k}^r, L(\cdot).$$

(ii) $g_{n,k}^r, L(\cdot)$ is related with $g_{n,k}^r, L(\cdot)$ as follows:

$$g_{n,k}^r, L(\cdot) \geq g_{n,k}^r, L(\cdot),$$

$$g_{n,k}^r, L(\cdot) \leq g_{n,k}^r, L(\cdot) + \zeta(\lambda_1, \lambda_2, \gamma, \tau),$$

where $\cdot$ denotes the triple of (conditional) distributions $P_{UV}, P_{Z|U}, P_{Y|X}$.

The proof of Lemma 3 uses the Lagrange multiplier method in convex optimization [37] and is provided in Appendix B.

**B. Equivalent Expressions for Probability Error**

Fix any $k \in \mathbb{N}$ and consider any $n \leq \tau k$. Given a memoryless privacy mechanism $P_{Z|U}$ and a communication protocol with a potentially stochastic encoder $f_{n,k}$ and a decoder $g_{n,k}$, define the following joint distributions:

$$P_{U|V} = P(v, u, k) = P_{Z|U} = P(z|u) = P_{F, n, k} = P(x^n|z) = P_{P|X} = P(y^n|x^n).$$

$$Q_{U|V} = Q(v, u, k) = Q_{Z|U} = Q(z|u) = Q_{F, n, k} = Q(x^n|z) = Q_{P|X} = Q(y^n|x^n),$$

where $P_{F, n, k}(x^n|z)$ denotes the probability that the output of the encoder is $x^n$ when the input is $z^n$.

Define the acceptance region

$$\mathcal{A} := \{(y^n, v^n) : g_{n,k}(y^n, v^n) = 1\}.$$  

Furthermore, let $P_{Z^n}, P_{Y^n}, P_{U^n|Z^n}$, $P_{Y^n|U^n}$ and $P_{Y^n|V^n|U^n|Z^n}$ be induced by the joint distribution $P_{U|V} = P(z^n, x^n)$ and let $Q_{Y^n|V^n}$ be induced by $Q_{U|V}$ in $Q_{Z^n, x^n}$. Note that the marginal distribution of $(U^n, Z^n)$ is $Q_{U^n}$ and the marginal distribution of $Z^n$ is $P_{Z^n}$ under both distributions $P_{U^n|Z^n, x^n}$ and $Q_{U^n|Z^n, x^n}$. The marginal distribution of $Y^n$ is the same under both joint distributions and denoted as $P_{Y^n}$, i.e.,

$$P_{Y^n}(y^n) := \sum_{u^n, z^n, x^n} P_{U^n}(u^n) P_{Z^n}(z^n|u^n) \times P_{F, n, k}(x^n|z^n) P_{P|X}(y^n|x^n).$$

Then the type-I and type-II error probabilities are equivalently expressed as follows:

$$\beta_1(f_{n,k}, g_{n,k}) = P_{Y^n|V^n}(A^c),$$

$$\beta_2(f_{n,k}, g_{n,k}) = Q_{Y^n|V^n}(A).$$

**C. Construct the Truncated Distribution**

We consider any memoryless privacy mechanism $P_{Z|U}$ and any communication protocol $(f_{n,k}, g_{n,k})$ such that i) the privacy constraint is satisfied with parameter $L$ and ii) the type-I error probability is upper bounded by $\varepsilon \in (0, 1)$, i.e.,

$$I(P_U, P_{Z|U}) \leq L, \quad \beta_1(f_{n,k}, g_{n,k}) \leq \varepsilon.$$  

Define a set concerning the detection probability at the decoder

$$B_1 := \left\{(u^n, z^n, x^n) : P_{Y^n|V^n|U^n|Z^n}(A|u^n, z^n, x^n) \geq 1 - \frac{\varepsilon}{4}\right\}.$$  

Then we have,

$$1 - \varepsilon \leq P_{Y^n|V^n}(A) \leq \sum_{(u^n, v^n, z^n, x^n, y^n) \in A} P_{U^n|V^n} = P_{U^n|V^n} = P_{Y^n|U^n} = P_{Y^n|V^n}(A|u^n, z^n, x^n) \leq \frac{1 - \varepsilon}{4},$$

where (44) follows from the equivalent expression of the type-I error probability in (39) and the constraint on the type-I error probability in (42), and (48) follows from the definition of $B_1$ in (43). Thus,

$$P_{U^n|V^n}(B_1) \geq \frac{3(1 - \varepsilon)}{4}.$$  

Recall the definitions of $i(u; z|P_{U^n})$ in (12) and $L(P_{Z|U}, \varepsilon)$ in (15). Define another set concerning the privacy constraint

$$B_2 := \left\{(u^n, z^n, x^n) : \sum_{i \in [K]} i(u_i; z_i|P_{U^n}) \leq k I(P_U, P_{Z|U}) + \sqrt{k} L(P_{Z|U}, (1 - \varepsilon)/4)\right\}. \quad (50)$$

Applying the Berry-Esseen theorem [38], [39], we have

$$P_{U^n|Z^n,X^n}(B_2^c) \leq \frac{1 - \varepsilon}{4}. \quad (51)$$

Define the intersection of two sets as

$$B := B_1 \cap B_2. \quad (52)$$
The results in (49) and (51) imply that
\[
P_{U^k Z^k X^n}(B) = P_{U^k Z^k X^n}(B_1 \cap B_2) = 1 - P_{U^k Z^k X^n}(B_1^c \cup B_2^c) \leq 1 - P_{U^k Z^k X^n}(B_2^c) = 1 - P_{U^k Z^k X^n}(B_1) + P_{U^k Z^k X^n}(B_1 \cap B_2^c) \geq 1 - P_{U^k Z^k X^n}(B_1) - P_{U^k Z^k X^n}(B_2^c) \geq 1 - \left(1 - \frac{3(1 - \varepsilon)}{4}\right) - 1 - \varepsilon - \frac{1}{4} \geq 1 - \varepsilon.
\]

Consider random variables \((\tilde{U}^k, \tilde{Z}^k, \tilde{X}^n, \tilde{Y}^n)\) with joint distribution \(P_{\tilde{U}^k \tilde{Z}^k \tilde{X}^n \tilde{Y}^n}\) such that
\[
P_{\tilde{U}^k}(u^k) P_{\tilde{Z}^k|\tilde{U}^k}(z^k|u^k) P_{\tilde{X}^n|\tilde{U}^k}(x^n|z^k) 1\{(u^k, z^k, x^n) \in B\} \cdot \frac{P_{\tilde{U}^k \tilde{Z}^k \tilde{X}^n}(\tilde{B})}{P_{\tilde{Y}^n \tilde{V}^k|\tilde{U}^k \tilde{Z}^k \tilde{X}^n}(\tilde{A}|u^k, z^k, x^n)}
\]

Note that the joint distribution in (58) is a truncated distribution of the original one \(P_{U^k Z^k X^n Y^n}\) by considering only \((u^k, z^k, x^n) \in \mathcal{B}\) and \((y^n, v^k) \in \mathcal{A}\). The truncated distribution in (58) allows us to apply change-of-measure and then use the strong converse technique introduced in [9]. As we shall show shortly, under the truncated distribution, the type-I error probability is zero and the truncated distribution in (58) is close to the original distribution in terms of KL divergence.

Let \(P_{\tilde{Y}^n \tilde{V}^k}\) and \(P_{\tilde{Y}^n \tilde{Z}^k}\) be induced by \(P_{\tilde{U}^k \tilde{Z}^k \tilde{X}^n \tilde{Y}^n}\). From (58), we have
\[
P_{\tilde{Y}^n \tilde{V}^k}(\mathcal{A}) = 1.
\]

Note that the constructed distribution \(P_{\tilde{U}^k \tilde{Z}^k \tilde{X}^n \tilde{Y}^n}\) is in fact close to the distribution \(P_{\tilde{U}^k \tilde{Z}^k \tilde{X}^n \tilde{Y}^n}\) in terms of KL divergence, i.e.,
\[
D(P_{\tilde{U}^k \tilde{Z}^k \tilde{X}^n \tilde{Y}^n}) = \sum_{y^n, v^k} P_{\tilde{Y}^n \tilde{V}^k}(y^n, v^k) \log \frac{P_{\tilde{Y}^n \tilde{V}^k}(y^n, v^k)}{P_{\tilde{Y}^n \tilde{V}^k}(y^n, v^k)} \leq 2 \log(1 - \varepsilon) + 3 \log 2,
\]

where (62) follows from the definition of \(\mathcal{B}\) in (43) and the result in (49).

**D. Multileter Bound for PUT**

Let \(P_{\tilde{U}^k}, P_{\tilde{Z}^k}\) and \(P_{\tilde{U}^k \tilde{Z}^k}\) be induced by \(P_{\tilde{U}^k \tilde{Z}^k \tilde{X}^n \tilde{Y}^n}\). It follows that
\[
I(\tilde{U}^k; \tilde{Z}^k) = \mathbb{E}_{P_{\tilde{U}^k \tilde{Z}^k}} \left[ \log \frac{P_{\tilde{U}^k \tilde{Z}^k}(U^k, Z^k)}{P_{\tilde{U}^k}(U^k) P_{\tilde{Z}^k}(Z^k)} \right]
\]

where (71) and (73) follow from the fact that \(P_{\tilde{Y}^n \tilde{V}^k}(\mathcal{A}) = 1\) in (59), (72) follows from the log-sum inequality, (75) follows from the definition of \(Q_{Y^k V^k}\) (cf. (36)), and (77) follows since from (43), (49) and (58),

\[
P_{\tilde{Y}^n}(y^n) = \sum_{u^k, z^k, x^n} P_{\tilde{U}^k \tilde{Z}^k \tilde{X}^n \tilde{Y}^n}(u^k, z^k, x^n, y^n)
\]

and (78) follows similarly to (62) and (69) follows from the privacy constraint in (41).
and (89) follows from (62).

Recall the joint distribution of $(\tilde{U}^k, \tilde{Z}^k, \tilde{V}^k, \tilde{X}^n, \tilde{Y}^n)$ in (58). We have

\[ I(\tilde{Z}^k, \tilde{V}^k; \tilde{Y}^n) - I(\tilde{X}^n; \tilde{Y}^n) \leq I(\tilde{Z}^k, \tilde{V}^k; \tilde{X}^n) - I(\tilde{X}^n; \tilde{Y}^n) \]

where (86) follows from the Markov chain \( Y^n - X^n - (\tilde{Z}^k, V^k) \) under the joint distribution \( P_{\tilde{Z}^k V^k X^n Y^n} \) (cf. (35)) and (89) follows from (62).

Combining (62), (69), (78) and (89), for any \((\lambda_1, \lambda_2) \in \mathbb{R}^2_+\), we have

\[ -\log \beta_2(f^{n,k}, g^{n,k}) \leq I(\tilde{Y}^n; \tilde{V}^k) - \lambda_1(I(\tilde{Z}^k, \tilde{V}^k; \tilde{Y}^n) - I(\tilde{X}^n; \tilde{Y}^n)) - \lambda_2(I(\tilde{U}^k; \tilde{Z}^k) - kL) - (2\lambda_1 + \lambda_2 + 2\gamma) \log(1 - \varepsilon) + (3\lambda_1 + 3\lambda_2 + 3\gamma) \log 2 \]

\[ + \gamma D(P_{\tilde{U}^k \tilde{Z}^k \tilde{V}^k \tilde{X}^n \tilde{Y}^n} \| P_{\tilde{U}^k \tilde{Z}^k \tilde{V}^k \tilde{X}^n \tilde{Y}^n}) + \lambda_2 \sqrt{kL}(P_{\tilde{U}^k \tilde{V}^k}, (1 - \varepsilon)/4). \]

**V. Conclusion**

We derived the privacy-utility tradeoff for a hypothesis testing problem against independence over a noisy channel. In particular, we provided exact asymptotic characterization of the type-II error exponent subject to a mutual information privacy constraint on the information source and a constant constraint on the type-I error probability. Our results imply that the asymptotic privacy-utility tradeoff cannot be increased
by tolerating a larger type-I error probability, which is known as a strong converse theorem. The strong converse theorems for several other important problems, including [4], [13], [40], are either established or recovered from our results.

To better understand the privacy-utility tradeoff, one could develop novel techniques to obtain second-order asymptotic result [41, Chapter 2] for the problem, which reveals the non-asymptotic fundamental limit. Such a result is more intuitive for practical situations where both the observation and communication are limited (i.e., $n$ and $k$ are both finite). It is also interesting to generalize our proof ideas to derive or strengthen the privacy-utility tradeoff for other hypothesis testing or communication problems, e.g., [10], [14]. Furthermore, one can study the privacy-utility tradeoff for the Bayesian setting [42] of the present problem where the utility is the decay rate of the average of type-I and type-II error probabilities. Finally, one can also generalize our results to other privacy measures, such as the differential privacy [21], the Rényi divergence [15], [43], the maximal leakage [12], [44] or the maximal $\alpha$-leakage [22].

APPENDIX

A. Proof of (26) and (28)

Recall the definition of the set $J(A, B, P_A)$. Under the high privacy limit where $L = \frac{\rho^2}{2}$ for arbitrary small $\rho$, the privacy mechanism $P_{Z|U}$ can be written as\(^3\)

$$P_{Z|U}(z|u) = Q_Z(z) + \rho J(u, z),$$

(102)

where $J(u, z)$ is the $z$-th element of $u$-th row of a matrix $J \in J(U, Z, P_U)$.

Thus, for each $z \in Z$, the induced marginal distribution $P_Z$ of $P_U$ and $P_{Z|U}$ satisfies

$$P_Z(z) = Q_Z(z).$$

(103)

Using Euclidean information theory [27], [28], we have that

$$I(P_U, P_{Z|U})$$

$$= \sum_u P_U(u)D(P_{Z|U=\cdot|u} \| P_Z)$$

$$\approx \frac{1}{2} \sum_u P_U(u) \sum_z \frac{(P_{Z|U}(z|u) - P_Z(z))^2}{P_Z(z)}$$

$$\approx \frac{\rho^2}{2} \sum_u P_U(u) \sum_z J(u, z)^2 \frac{Q_Z(z)}{P_Z(z)}.$$

(106)

Recall the definition of $Q_W$ in (21). The induced distributions $P_W$ and $P_{W|V}$ of $P_Z$ and $P_{W|Z}$ satisfy that for any $(v, w) \in V \times W$,

$$P_W(w) = Q_W(w),$$

(107)

and

$$P_{W|V}(w|v) = Q_W(w) + J(u, z) \sum_{u, z} P^2_{U|V}(u|v) P_W(z|u) J(u, z).$$

(108)

Similar to (106), using the definition of $h(J, \rho)$ in (25), we have

$$I(P_V, P_{W|V})$$

$$= \sum_v P_V(v)D(P_{W|V=\cdot|v} \| P_W)$$

$$\approx \frac{1}{2} \sum_v P_V(v) \sum_w \frac{(P_{W|V}(w|v) - Q_W(w))^2}{Q_W(w)}$$

$$\approx h(J, \rho).$$

(111)

The justification of (26) is completed by combining these approximations.

If we further assume that $\tau C(P_{V|X}) = \frac{\rho^2}{2}$, then the conditional probability $P_{W|Z}$ should satisfy

$$P_{W|Z}(w|z) = Q_W(w) + \rho \Theta(z, w)$$

(112)

for any $Q_w \in \mathcal{Q}(W)$, where $\Theta \in J(Z, W, Q_Z)$. Then we have that induced marginal distribution $P_W$ of $P_Z$ and $P_{W|Z}$ satisfies that for any $w \in W$,

$$P_W(w) = Q_W(w).$$

(113)

Similar to (106), we have

$$I(P_Z, P_{W|Z})$$

$$= \sum_z P_Z(z) \log \frac{P_{W|Z}(w|z)}{P_W(w)}$$

$$\approx \frac{\rho^2}{2} \sum_z P_Z(z) \sum_w \frac{(\Theta(z, w))^2}{Q_W(w)}$$

(114)

$$= \frac{\rho^2}{2} \sum_z Q_Z(z) \frac{(\Theta(z, w))^2}{Q_W(w)},$$

(115)

where (116) follows from (103). The induced distributions $P_{W|V}$ and $P_W$ satisfy

$$P_{W|V}(w|v) = Q_W(w) + \rho^2 \sum_{u, z} P^2_{U|V}(u|v) J(u, z) \Theta(z, w).$$

(117)

Similar to (106), we have

$$I(P_V, P_{W|V})$$

$$\approx \frac{\rho^1}{2} \sum_{v, w} P_V(v) \left( \sum_{u, z} P^2_{U|V}(u|v) J(u, z) \Theta(z, w) \right)^2.$$  

(118)

The justification of (28) is completed by combining above approximations for mutual information terms.

B. Proof of Lemma 3

1) Proof of Claim 1: From the definition of $g_{\lambda_1, \lambda_2}(\cdot)$ in (11), we have

$$g_{\lambda_1, \lambda_2}(P_{UV}, P_{Z|U}, P_{Y|X})$$

$$= \sup_{Q_{UVZW} \in \mathcal{Q}(P_{UV}, P_{Z|U})} R_{\lambda_1, \lambda_2}^{\tau_{UV}}(Q_{UVZW}, Q_{XY})$$

(119)

and

$$\lambda_1 \tau I(Q_X, Q_{Y|X})).$$

(119)
where (120) follows from the definition of $C(P_{Y|X})$ in (4).

The proof is completed by taking $n \to \infty$.

2) Proof of Claim 2: The definition of $g_{\lambda_1,\lambda_2}^{\tau,L}(\cdot)$ in (11) implies

$$g_{\lambda_1,\lambda_2}^{\tau,L}(P_{U,V}, P_{Z|U}, P_{Y|X}) = \sup_{Q_{U,V,Z} \in \mathcal{Q}(P_{U,V}, P_{Z|U})} R_{\lambda_1,\lambda_2}^{\tau,L}(Q_{U,V,Z}, Q_{XY})$$

$$= \sup_{Q_{U,V,Z} \in \mathcal{Q}(P_{U,V}, P_{Z|U})} \left( R_{\lambda_1,\lambda_2}^{\tau,L}(Q_{U,V,Z}, Q_{XY}) - \Delta_{\tau,L}(Q_{U,V,Z}, Q_{XY}, P_{U,V}, P_{Z|U}, P_{Y|X}) \right)$$

$$\leq \sup_{Q_{X,Y,Z} \in \mathcal{Q}} \left( R_{\lambda_1,\lambda_2}^{\tau,L}(Q_{U,V,Z}, Q_{XY}) - \Delta_{\tau,L}(Q_{U,V,Z}, Q_{XY}, P_{U,V}, P_{Z|U}, P_{Y|X}) \right)$$

$$= g_{\lambda_1,\lambda_2}^{\tau,L}(P_{U,V}, P_{Z|U}, P_{Y|X}),$$

where (134) follows since $\Delta_{\tau,L}(Q_{U,V,Z}, Q_{XY}, P_{U,V}, P_{Z|U}, P_{Y|X}) = 0$ (cf. (29)) for $Q_{U,V,Z} \in \mathcal{Q}(P_{U,V}, P_{Z|U})$ and $Q_{XY} \in \mathcal{C}: Q_{Y|X} = P_{Y|X}$, (135) follows since $Q(P_{U,V}, P_{Z|U}) \subset \mathcal{Q}$ and (136) follows from the definition of $g_{\lambda_1,\lambda_2}^{\tau,L}(\cdot)$ in (31).

For any $(\lambda_1, \lambda_2, \gamma) \in \mathbb{R}_+^2$, let $(Q_{U,V,Z}^{\lambda_1,\lambda_2,\gamma}, Q_{XY}^{\lambda_1,\lambda_2,\gamma})$ be an optimizer of $g_{\lambda_1,\lambda_2}^{\tau,L}(P_{U,V}, P_{Z|U}, P_{Y|X})$ and let $Q_{XY}^{\lambda_1,\lambda_2,\gamma}$ be a distribution induced by either $Q_{U,V,Z}^{\lambda_1,\lambda_2,\gamma}$ or $Q_{XY}^{\lambda_1,\lambda_2,\gamma}$. From the support lemma [45, Appendix C], we obtain that the cardinality of $W$ can be upper bounded as a function of $|U|$, $|V|$ and $|Z|$, which is finite. Furthermore, let $P_{U,V,Z}^{\lambda_1,\lambda_2,\gamma}$ and $P_{XY}^{\lambda_1,\lambda_2,\gamma}$ be defined as follows:

$$P_{U,V,Z}^{\lambda_1,\lambda_2,\gamma} = P_{U,V} P_{Z|U} Q_{W|Z}^{\lambda_1,\lambda_2,\gamma},$$

$$P_{XY}^{\lambda_1,\lambda_2,\gamma} = Q_{X|Y}^{\lambda_1,\lambda_2,\gamma} P_{Y|X}.$$
From the definitions of $Q_{VUZW}^{\lambda_1,\lambda_2,\gamma}$ and $(P_{VUZW}^{\lambda_1,\lambda_2,\gamma}, P_{XY}^{\lambda_1,\lambda_2,\gamma})$, we have

$$D(Q_{VUZW}^{\lambda_1,\lambda_2,\gamma} \| P_{VUZW}^{\lambda_1,\lambda_2,\gamma}) = D(Q_{X}^{\lambda_1,\lambda_2,\gamma} \| P_{X})$$
$$+ D(Q_{VUZW}^{\lambda_1,\lambda_2,\gamma} \| P_{UZ}P_Y P_U | Q_{VU}^{\lambda_1,\lambda_2,\gamma})$$

Furthermore,

$$\gamma D(Q_{VUZW}^{\lambda_1,\lambda_2,\gamma} \| P_{VUZW}^{\lambda_1,\lambda_2,\gamma}) + \tau \gamma D(Q_{XY}^{\lambda_1,\lambda_2,\gamma} \| P_{XY}^{\lambda_1,\lambda_2,\gamma})$$

$$= \Delta L(Q_{VUZW}^{\lambda_1,\lambda_2,\gamma}, Q_{XY}^{\lambda_1,\lambda_2,\gamma}, P_{UV}, P_{UZ}, P_{UY}, P_{X})$$

$$\geq R_{\lambda_1,\lambda_2}^{\tau L}(Q_{VUZW}^{\lambda_1,\lambda_2,\gamma}, Q_{XY}^{\lambda_1,\lambda_2,\gamma}) - g_{\lambda_1,\lambda_2}(P_{UV}, P_{UZ}, P_{UX})$$

$$\leq c(\lambda_1, \lambda_2, \lambda_2, \tau),$$

(146)

where (146) follows from the definition of $\Delta L(\cdot)$ in (29), (147) follows since $(Q_{VUZW}^{\lambda_1,\lambda_2,\gamma}, Q_{XY}^{\lambda_1,\lambda_2,\gamma})$ is an optimizer for $g_{\lambda_1,\lambda_2}(P_{UV}, P_{UZ}, P_{UX})$ (cf. (31)), (150) follows from the definition of $c(\lambda_1, \lambda_2, \tau)$ in (8), and (148) follows since

$$g_{\lambda_1,\lambda_2}(P_{UV}, P_{UZ}, P_{UX}) \geq f(\tau, L, P_{UV}, P_{UZ}, P_{UX})$$

where the result in (150) implies that

$$D(Q_{VUZW}^{\lambda_1,\lambda_2,\gamma} \| P_{VUZW}^{\lambda_1,\lambda_2,\gamma}) \leq \frac{c(\lambda_1, \lambda_2, \lambda_2, \tau)}{\gamma},$$

(154)

Using (154), Pinsker’s inequality and data processing inequality for KL divergence, we have

$$||Q_{VUZW}^{\lambda_1,\lambda_2,\gamma} - P_{VUZW}^{\lambda_1,\lambda_2,\gamma}|| \leq 2D(Q_{VUZW}^{\lambda_1,\lambda_2,\gamma} \| P_{VUZW}^{\lambda_1,\lambda_2,\gamma})$$

$$\leq \frac{2c(\lambda_1, \lambda_2, \lambda_2, \tau)}{\gamma}.$$
where (174) follows from the non-negativity of KL divergence and the result in (62).

Furthermore, we have
\[
I(\tilde{Y}^n; \tilde{V}) = \sum_{i \in [k]} I(\tilde{Y}_i^n; \tilde{V}_i|\tilde{Y}_i^{i-1})
\]
\[
\leq \sum_{i \in [k]} I(\tilde{Y}_i^n - 1; \tilde{Y}_i^n; \tilde{V}_i)
\]
\[
\leq \sum_{i \in [k]} I(\tilde{Z}_i^n - 1; \tilde{Y}_i^n - 1, \tilde{V}_i)
\]
\[
= \sum_{i \in [k]} I(W_i; \tilde{V}_i)
\]
\[
kI(W_j; J; \tilde{V}_j),
\]
and
\[
I(\tilde{X}^n; \tilde{Y}^n)
\]
\[
= H(\tilde{Y}^n) - H(\tilde{Y}^n|\tilde{X}^n)
\]
\[
= \sum_{i \in [n]} H(\tilde{Y}_i^n|\tilde{X}_i^n - 1) - H(\tilde{Y}^n|\tilde{X}^n)
\]
\[
\leq \sum_{i \in [n]} H(\tilde{Y}_i^n) - H(\tilde{Y}^n|\tilde{X}^n)
\]
\[
\leq nH(\tilde{Y}_{J_n}) - nH(\tilde{Y}_{J_n}|\tilde{X}_{J_n})
\]
\[
- nD(P_{\tilde{Y}_{J_n}|\tilde{X}_{J_n}}||P_{\tilde{Y}|\tilde{X}}) + D(P_{\tilde{Y}^n|\tilde{X}^n})
\]
\[
\leq nI(\tilde{X}_{J_n}; \tilde{Y}_{J_n}, J_n) + D(P_{\tilde{Z}^n|\tilde{V}^n, X^n, Y^n})
\]
\[
\leq nI(\tilde{X}_{J_n}; \tilde{Y}_{J_n}, J_n) - 2\log(1 - \varepsilon) + 3 \log 2,
\]
where (183) follows from the result in (168) and (185) follows from the result in (89).

Similarly, we have
\[
I(\tilde{Z}^k, \tilde{V}^k; \tilde{Y}^n)
\]
\[
= H(\tilde{Z}^k, \tilde{V}^k) - H(\tilde{Z}^k, \tilde{V}^k|\tilde{Y}^n)
\]
\[
= kH(\tilde{Z}_j, \tilde{V}_j) + kD(P_{\tilde{Z}^k, \tilde{V}^k}||P_{\tilde{Z}^k, \tilde{V}^k})
\]
\[
- \sum_{i \in [k]} H(\tilde{Z}_i, \tilde{V}_i|\tilde{Z}_i^{i-1}, \tilde{V}_i^{i-1}, \tilde{Y}^n)
\]
\[
\geq kH(\tilde{Z}_j, \tilde{V}_j) + 2\log(1 - \varepsilon) + 3 \log 2 - kH(\tilde{Z}_j, \tilde{V}_j|W_j)
\]
\[
\geq kI(\tilde{Z}_j, \tilde{V}_j; W_j, J) + 2\log(1 - \varepsilon) + 3 \log 2
\]
\[
\geq kI(\tilde{Z}_j; W_j, J) + 2\log(1 - \varepsilon) + 3 \log 2,
\]
where (187) follows from (167), (188) follows similarly to (89).

Furthermore, using non-negativity and convexity of KL divergence [30], we have
\[
D(P_{X^n Z^n X^n Y^n}||P_{U^n Z^n X^n Y^n})
\]
\[
= D(P_{Z^n X^n Y^n}||P_{Z^n X^n Y^n}) + D(P_{X^n Z^n X^n Y^n})
\]
\[
+ D(P_{X^n Z^n X^n Y^n}||P_{Z^n X^n Y^n})
\]
\[
+ D(P_{X^n Z^n X^n Y^n}||P_{X^n Y^n}||P_{Z^n X^n Y^n})
\]
\[
\geq kI(\tilde{Z}_j, \tilde{V}_j; W_j, J) + 2\log(1 - \varepsilon) + 3 \log 2
\]
\[
\geq kI(\tilde{Z}_j; W_j, J) + 2\log(1 - \varepsilon) + 3 \log 2,
\]
where (187) follows from (167), (188) follows similarly to (89).

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