Two measurements are sufficient for certifying high-dimensional entanglement

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High-dimensional encoding of quantum information provides a promising method of transcending current limitations in quantum communication. One of the central challenges in the pursuit of such an approach is the certification of high-dimensional entanglement. In particular, it is desirable to do so without resorting to inefficient full state tomography. Here, we show how carefully constructed measurements in two or more bases can be used to efficiently certify high-dimensional states and their entanglement under realistic conditions. We considerably improve upon existing criteria and introduce new entanglement dimensionality witnesses which we put to the test for photons entangled in their orbital angular momentum. In our experimental setup, we are able to verify 8-dimensional entanglement for 11-dimensional subspaces, at present the highest amount certified without assumptions on the state itself.

Quantum communication offers advantages such as enhanced security in quantum key distribution (QKD) protocols [1] and increased channel capacities [2] with respect to classical means of communication. All of these improvements, ranging from early proposals [3] to recent exciting developments such as fully device-independent QKD [4, 5], rely on one fundamental phenomenon: quantum entanglement. Currently, the workhorse of most implementations is entanglement between qubits, i.e., between two-dimensional quantum systems (e.g., photon polarization). However, it has long been known that higher-dimensional entanglement can be useful in overcoming the limitations of qubit entanglement [6, 7], offering better key rates [8], higher noise resistance [9, 10] and improved security against different attacks [11].

Attempting to capitalize on this insight, recent experiments have successfully generated and certified high-dimensional entanglement in different degrees of freedom. In particular, the canonical way of generating two-dimensional polarization entanglement in down-conversion processes already offers the potential for exploring entanglement in higher dimensions. This can be achieved by exploiting spatial degrees of freedom [12, 13], orbital angular momentum (OAM) [14, 15], energy-time based encodings [16–19], or combinations thereof in hyper-entangled quantum systems [20, 21]. High-dimensional quantum systems are hence not only of fundamental interest but are also becoming more readily available. In this context, the certification and quantification of entanglement in many dimensions is a crucial challenge since full state tomography for bipartite systems of local dimension \(d\) requires \(d^2(d+1)^2\) single-outcome measurements [22] or \((d+1)^2\) \(d\)-outcome measurements [23]. Unwieldy or inefficient entanglement estimation would hence strongly mitigate possible advantages from high-dimensional encoding. It is therefore of high significance to determine efficient and practical strategies for certifying and quantifying high-dimensional entanglement.

In this work, we quantify entanglement by lower-bounding the Schmidt number of the state of interest. Due to the complexity of realizing measurements in high-dimensional spaces, previous experiments that aimed to certify Schmidt numbers often had to resort to assumptions about the underlying quantum state \(\rho\), including, amongst others, conservation of angular momentum [24], subtraction of accidentals [25], or perfect correlations in a desired basis [26]. Although relying on such assumptions allows for a plausible quantification of entanglement dimensionality, it is not enough for unambiguous certification, which is desirable for secure quantum communication based on high-dimensional entanglement.

Here, we present an efficient and adaptive method that is tailored to better harvest the information about entangled states generated in a given experiment, without the need for any assumptions about \(\rho\). This method is based on certifying the fidelity \(F(\rho, \Phi)\) to a previously identified suitable target state \(|\Phi\rangle\). We show that measurements in only two local bases \(\{\langle mn\rangle\}_{m,n}\) and \(\{\langle ij\rangle\}_{i,j}\) are sufficient to select \(|\Phi\rangle\) and to bound the fidelity from below by a quantity \(F^{(M-1)}(\rho, \Phi)\). Depending on the Schmidt coefficients of the target state, the required measurement settings exhibit a certain degree of biasedness, and become unbiased for maximally entangled target states. For any target state the fidelity bound becomes exact when the setup generates \(|\Phi\rangle\) up to pure dephasing. We demonstrate that this method can be generalized to measurements in multiple local bases \((1 \leq M \leq d)\), and the fidelity bounds become exact in prime dimensions for \(M = d\). Moreover, deriving general decompositions for dephased maximally entangled states further allows us to...
prove that unbiased measurements indeed provide a necessary and sufficient condition for tight Schmidt number bounds for all pure states $\rho = |\Phi \rangle \langle \Phi |$ and for maximally entangled states subject to pure dephasing. Finally, our method can be used also for entanglement quantification by providing lower bounds on the entanglement of formation [27], where our bounds outperform previous methods in terms of their noise robustness and the number of certified e-bits. Before we continue, let us briefly summarize our key results:

| **Fidelity bound:** | $F_{a}(\rho, \Phi) \leq F(\rho, \Phi)$ |
|---|---|
| - | Free of assumptions about the state $\rho$; |
| - | Can be obtained from measurements in only two local bases ($M = 1$); |
| - | Extendible to multiple bases ($2 \leq M \leq d$) for better performance and noise resistance; |
| - | Exact for depolarized pure states for $M = 1$; |
| - | Exact in prime dimensions for $M = d$; |

| **Schmidt number witness:** | $F(\rho, \Phi) \leq B_{k}(\Phi)$ |
|---|---|
| - | Exact for all pure states; |
| - | Exact for depolarized max. entangled states; |

| **Entanglement bound:** | $F(\rho, \Phi^{+}) \rightarrow E_{a}(\rho)$ |
|---|---|
| - | Improvement w.r.t. previous bounds [28]. |

To put these theoretical predictions to the test in realistic circumstances with actual noise, we devise and carry out an experiment based on OAM-entangled photons that allows our approach to prove its mettle. In our experimental implementation, measurements are realized using computer programmable holograms implemented on spatial light modulators (SLMs). Employing the theoretical methods developed here, we are able to certify high target-state fidelities and detect record entanglement dimensions (e.g., dimension 8 for subspaces of dimension $d = 11$, without any assumptions on the state itself). We use our experimental setup to fully explore the performance of our criteria for non-maximally entangled states and different numbers of measurements, showcasing the flexibility of the derived results.

In the following, we first clarify the relevant theoretical concepts. Then we state our main theoretical results and give a step-by-step set of instructions for using the method in any arbitrary quantum experiment. We continue with a discussion of our experimental setup and conclude with a discussion of our theoretical and experimental results.

**Entanglement dimensionality.** Let us consider a typical laboratory situation for preparing a high-dimensional quantum system in a bipartite state $\rho$ that is to be employed for quantum information processing between two parties $A$ and $B$. In order to be useful, this state should be close to some highly entangled target state that is normally required to have a high purity. We therefore choose a pure target state $|\Phi \rangle$ with a desired Schmidt rank $k = k_{\text{max}}$. The Schmidt rank is a measure of the entanglement dimensionality of the state and represents the minimum number of levels one needs to faithfully represent the state and its correlations in any local basis. More quantitatively, $k$ is the number of non-vanishing coefficients $\lambda_{m}$ in the Schmidt decomposition $|\Phi \rangle = \sum_{m=0}^{k-1} \lambda_{m} |mm \rangle$ of $|\Phi \rangle$ w.r.t. the bipartition into $A$ and $B$, $\lambda_{m}$ is real and positive w.l.o.g., and $|mm \rangle = |\phi_{m} \rangle_{A} |\chi_{m} \rangle_{B}$ is a product state of the $m$-th vectors of the respective Schmidt bases. Ideally, the target state’s Schmidt rank is equal (or close) to the (accessible) local dimension, $k_{\text{max}} = d$, where we take the local Hilbert spaces to have the same dimension, $\mathcal{H}_{A} = \mathcal{H}_{B} = d$.

For mixed states $\rho$ the Schmidt rank generalizes to

$$k(\rho) = \inf_{\mathcal{D}(\rho)} \left\{ \max_{\{\psi_{i}\} \in \mathcal{D}(\rho)} \left\{ \text{rank}\left( \text{Tr}_{B} |\psi_{i} \rangle \langle \psi_{i}| \right) \right\} \right\},$$

where the infimum is taken over all pure state decompositions, i.e., $\mathcal{D}(\rho)$ is the set of all sets $\{\{p_{i}, |\psi_{i}\rangle\}\}$, for which $\rho = \sum_{i} p_{i} |\psi_{i}\rangle \langle \psi_{i}|$, $\sum_{i} p_{i} = 1$, and $0 \leq p_{i} \leq 1$. The Schmidt rank hence quantifies the maximal local dimension in which any of the pure state contributions to $\rho$ can be considered to be entangled and we hence call $k(\rho)$ the entanglement dimensionality of $\rho$. Note that this implies $k \leq d$. As an example, any entangled state of two qubits, for which $d = 2$, has an entanglement dimensionality $k = 2$, while separable states have $k = 1$.

**Target state identification.** The task at hand is then to certify that the state $\rho$ generated in the lab indeed provides the desired high-dimensional entanglement. One immediate first approach is to start with local projective measurements in the Schmidt bases $\{|mn\rangle\}_{m,n=0,\ldots,d-1}$, which we will designate as our standard basis. These bases can typically be identified from conserved quantities or the setup design, but depending on the physical setup, the corresponding measurements are realized in different ways. In essence, a good guess for the Schmidt basis provides a good target state. For instance, in an optical setting using OAM (as we employ in the experiment reported on later in this article) these measurements are performed by coincidence post-selection after local projections. That is, SLMs programmed with the phase pattern of a specific state $|mn\rangle$ act as local unitary operations, which are followed by single mode fibers (SMF) as local filters, and the number $N_{mn}$ of coincidences between local photon detectors is counted for each setting corresponding to fixed values of $m$ and $n$. In this way one can obtain the matrix elements

$$\langle mn | \rho | mn \rangle = \frac{N_{mn}}{\sum_{i,j} N_{ij}}.$$

Using $d$ local settings for each side ($d^{2}$ settings globally)
one may thus obtain the values \( \langle mm|\rho|mm \rangle \) and use these to nominate a target state \( |\Phi\rangle = \sum_{m=0}^{d-1} \lambda_m |mm\rangle \) by identifying
\[
\lambda_m = \sqrt{\frac{\langle mm|\rho|mm \rangle}{\sum_n \langle mn|\rho|nn \rangle}}. \tag{3}
\]

Of course this association alone by no means guarantees that the state \( \rho \) really is equivalent to the target state \( |\Phi\rangle \). Although the information about the diagonal elements of \( \rho \) provides an informed guess, it is not enough to infer entanglement properties. In order to access this information, one could in principle perform costly full-state tomography. This requires \( d^4 - 1 \) global measurement settings [22] and hence quickly becomes impractical in high dimensions. Here, we propose a much more efficient alternative method to obtain a lower bound on the Schmidt rank of \( \rho \). To this end, we will employ an adaptive strategy summarized in Fig. 1, where the results of the initial estimates of the \( \lambda_m \) are used to perform measurements in a single, carefully constructed basis in addition to the standard basis. Our approach can hence provide a lower bound for \( k(\rho) \) from as little as \( 2d^2 \) global measurement settings. For instance, for the state in a \( 11 \times 11 \)-dimensional Hilbert space that we have studied experimentally this corresponds to 242 measurement settings, versus the 17,424 settings required for full state tomography, which is a reduction by approximately two orders of magnitude.

**Dimensionality witnesses.** For the certification of the Schmidt rank of \( \rho \) we consider the fidelity \( F(\rho, \Phi) \) to the target state \( |\Phi\rangle \), given by
\[
F(\rho, \Phi) = \text{Tr}(|\Phi\rangle\langle\Phi|\rho) = \sum_{m,n=0}^{d-1} \lambda_m \lambda_n \langle mm|\rho|nn \rangle. \tag{4}
\]

For any state \( \rho \) of Schmidt rank \( k \leq d \) the fidelity of Eq. (4) is bounded by [29, 30]
\[
F(\rho, \Phi) \leq B_k(\Phi) := \sum_{m=0}^{k-1} \lambda_m^2, \tag{5}
\]

where the sum runs over the \( k \) largest Schmidt coefficients, i.e., \( i_m, m \in \{0, \ldots, d-1\} \) such that \( \lambda_m \geq \lambda_m', \forall m \leq m' \). Consequently, any state for which \( F(\rho, \Phi) > B_k(\Phi) \) is incompatible with a Schmidt rank of \( k \) or less, implying an entanglement dimensionality of at least \( k + 1 \). In the following, we will provide efficiently measurable lower bounds on the fidelity \( F(\rho, \Phi) \) that allow certifying minimum Schmidt ranks in this way.

**Fidelity bounds.** The next step is hence to experimentally estimate the value of the fidelity \( F(\rho, \Phi) \). To see how this can be done we split the fidelity in two contributions, i.e., \( F(\rho, \Phi) = F_1(\rho, \Phi) + F_2(\rho, \Phi) \). The first term is
\[
F_1(\rho, \Phi) := \sum_m \lambda_m^2 \langle mm|\rho|mm \rangle \tag{6}
\]
and takes into account diagonal terms of the density matrix w.r.t. the standard basis, while the second term
\[
F_2(\rho, \Phi) := \sum_{m,n=0}^{d-1} \lambda_m \lambda_n \langle mm|\rho|nn \rangle, \tag{7}
\]

collects the corresponding off-diagonal terms. On the one hand, \( F_1(\rho, \Phi) \) can be calculated directly from the already performed measurement in the basis \( \{ |mm\rangle \}_{m,n} \). Exactly determining the term \( F_2(\rho, \Phi) \), on the other hand, would require costly full state tomography.

To avoid such a high overhead, we employ bounds for \( F_2(\rho, \lambda) \) that can be calculated from measurements in one additional basis. Using the previously obtained values \( \{ \lambda_m \}_m \), we define the basis \( \{ |j\rangle \}_{j=0,...,d-1} \) according to
\[
|j\rangle = \frac{1}{\sqrt{\sum_n \lambda_n}} \sum_{m=0}^{d-1} \omega^{jm} \sqrt{\lambda_m} |m\rangle, \tag{8}
\]

where \( \omega = e^{2\pi i/d} \) and \( \{ |m\rangle \}_m \) is the standard basis. Notice that, although the basis vectors \( |j\rangle \) are normalized by construction, they are not necessarily orthogonal and we hence refer to \( \{ |j\rangle \}_j \) as the tilted basis.

As we discuss in detail in Appendix A, the sum over off-diagonal terms \( F_2(\rho, \lambda) \) can be lower-bounded by \( \tilde{F}_2(\rho, \Phi) \leq F_2(\rho, \Phi) \), where
\[
\tilde{F}_2 := \frac{(\sum_m \lambda_m)^2}{d} \sum_{j=0}^{d-1} \langle jj^* | \rho | jj^* \rangle - \sum_{m,n=0}^{d-1} \lambda_m \lambda_n \langle mm|\rho|nn \rangle - \sum_{m,m'\neq n,n' \neq m} \tilde{\gamma}_{mm'nn'} \sqrt{\langle mm'|\rho|nn' \rangle \langle mm|\rho|nn \rangle}. \tag{9}
\]

Here, the second term and third term can be calculated from the original measurements in the standard basis, and the prefactor \( \tilde{\gamma}_{mm'nn'} \) of the third term is given by
\[
\tilde{\gamma}_{mm'nn'} = \begin{cases} 0 & \text{if } (m-m' - n-n') \bmod d \neq 0 \\ \sqrt{\lambda_m \lambda_{m'} \lambda_n \lambda_{n'}} & \text{otherwise}. \end{cases} \tag{10}
\]

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**FIG. 1: Adaptive strategy for certifying entanglement dimensionality**

1. Identify standard basis \( \{ |mn\rangle \} \) and measure coincidences \( \{ N_{mn} \} \) to obtain \( \{ \langle mn|\rho|mn \rangle \} \).
2. Calculate \( \{ \lambda_m \} \) and nominate target state \( |\Phi\rangle \).
3. Construct tilted basis \( \{ |j\rangle \} \) and measure coincidences \( \{ N_{ij} \} \) to obtain \( \{ \langle jj^* | \rho | jj^* \rangle \} \).
4. Evaluate \( \tilde{F}(\rho, \Phi) \) and \( B_{k=1}(\Phi), \ldots, B_{k=d-1}(\Phi) \).

The certified entanglement dimensionality is
\[
d_{\text{ent}} = \max \{ k | \tilde{F}(\rho, \Phi) > B_{k-1}(\Phi) \}.
\]
The first term of $\tilde{F}_k(\rho, \Phi)$ can be obtained via local projective measurements in the tilted basis, i.e., projections of $\rho$ onto the subspace spanned by $|ij\rangle$ for fixed $i$ and $j$, where the asterisk denotes complex conjugation w.r.t. to the standard basis. Due to the general non-orthogonality of the tilted basis, the relation of Eq. (2) between the diagonal matrix elements $(jj^*)|\rho|jj^*)$ and the coincidence counts $N_{ij}$ for the local setting $|ij\rangle$ requires a small modification in terms of an additional normalization factor $c_\lambda := \frac{d^2}{\sum \lambda_n^2} \sum \lambda_m \lambda_n \langle mn | \rho | mn \rangle$ (see Appendix A.II), i.e.,

\[
(jj^*)|\rho|jj^*) = \frac{N_{jj}}{\sum_{i,j} N_{ij}} c_\lambda. \tag{11}
\]

Apart from the inclusion of $c_\lambda$, measurements in the tilted basis are in principle not different from measurements in any orthonormal basis. For instance, in an OAM setup with measurements based on post-selection this is just a matter of appropriately programming the SLMs. The lower bound $\tilde{F}(\rho, \Phi) = F_1(\rho, \Phi) + \tilde{F}_2(\rho, \Phi)$ for the fidelity is hence experimentally easily accessible. In turn, we immediately obtain the dimensionality witness inequality

\[
B_k(\Phi) \geq F(\rho, \Phi) \geq \tilde{F}(\rho, \Phi), \tag{12}
\]

which is satisfied by any state $\rho$ with Schmidt rank $k$ or less. Conversely, the entanglement dimensionality $d_{\text{ent}}$ that is certifiable with our method is the maximal $k$ such that $\tilde{F}(\rho, \Phi) > B_{k-1}(\Phi)$. Crucially, this witness requires only $2d^2$ global measurement settings to be evaluated, and is hence significantly more efficient than full state tomography using $d^4 - 1$ settings. Note that the bounds may in principle be improved by including measurements in more than one tilted basis, leading to a trade-off between increases in the performance of the dimensionality witness and in the number of required measurement settings. As we show in Appendix A.III, the previous fidelity bound $\tilde{F}(\rho, \Phi)$ then becomes a special case, i.e., $\tilde{F} = F^{(M-1)} \leq F^{(M)}$, and the fidelity bound $F^{(M)} \leq F$ becomes tight in prime dimensions for the maximal number of $M = d$ tilted bases.

**Role of the target state.** Moreover, no assumptions about the state $\rho$ need to be made. The initial designation of the target state $|\Phi\rangle$ simply helps to suitably adapt the dimensionality witness to the experimental situation. Although identifying the Schmidt basis from the setup could in principle be seen as an assumption about the underlying state, choosing a basis that is far from the Schmidt basis doesn’t invalidate our certification method. Since the latter is based on lower-bounding the fidelity to the target state such a mis-identification would simply result in a reduced performance by using lower bounds on the fidelity to a state that is far from the actual state. In principle, one could select a fixed target state independently of the particular circumstances in the laboratory. For instance, one may consider a pure state subject to pure dephasing, i.e., $\rho = p |\Phi\rangle \langle \Phi| + \frac{1 - p}{d} \sum m |mm\rangle \langle mm|$, for which it is easy to see that the last term in Eq. (9) vanishes and the fidelity bound $\tilde{F}(\rho, \Phi) \leq F(\rho, \Phi)$ is hence tight. For any pure states or dephased maximally entangled states one can further show that the Schmidt number bound $B_k(\Phi) \geq F(\rho, \Phi)$ is also tight, see Appendix A.IV. Alternatively, one may select the maximally entangled state $|\Phi^+\rangle = |1\rangle |\Phi\rangle - |\Phi\rangle |1\rangle$, i.e., where $\lambda_m = \frac{1}{\sqrt d} \forall m$. In this case we have $B_k(\Phi^+) = \frac{d}{2}$ and the tilted basis becomes an orthonormal basis that is mutually unbiased w.r.t. to the standard basis, i.e., $|\langle m|j\rangle|^2 = \frac{1}{d} \forall m,j$.

This special case is particularly interesting for several reasons. First, it provides a simple theoretical testing ground to evaluate the performance of our method in the presence of noise, as illustrated in Fig. 2. There, we assume $\rho$ to be a mixture of $|\Phi^+\rangle$ with a maximally mixed state, i.e., an isotropic state $\rho_{\text{iso}} = p |\Phi^+\rangle \langle \Phi^+| + \frac{1 - p}{d^2} \mathbb{1}$, where the visibility $p$ satisfies $0 \leq p \leq 1$ and $\mathbb{1}$ is the identity in dimension $d^2$. This allows to identify the visibility thresholds for the certification of the Schmidt ranks of maximally entangled states subject to white noise. Second, the fidelity bounds for the target state $|\Phi^+\rangle$ can be used to construct bounds on the entanglement of formation, as explained in Appendix A.V. Although the selection of $|\Phi^+\rangle$ as a target state may not be optimally suited for a given experimental situation, it thus nonetheless provides an efficient method for the direct certification of the number of e-bits in the system. In Appendix A.VI, we show that this entanglement quantification method outperforms previous approaches [28] in terms of detected e-bits and noise robustness.

**Experimental certification of high-dimensional entanglement.** We now apply our witness for efficiently certifying high-dimensional orbital angular momentum entan-
FIG. 3. Experimental Setup: (a) A 405nm CW laser pumps a ppKTP crystal to generate two photons entangled in their orbital angular momentum (OAM). The pump is removed by a dichroic mirror (DM) and the two photons are separated by a polarizing beam splitter (PBS) and incident on spatial light modulators (SLMs). In combination with single-mode fibers (SMFs), the SLMs act as spatial mode filters. The filtered photons are detected by single-photon avalanche photodiodes (not shown) and time-coincident events are registered by a coincidence counting logic (CC). (b) Examples of computer-generated holograms displayed on the SLMs for measuring the photons in the standard basis (e.g., for azimuthal quantum numbers \(k = 0, 3,\) and \(5\) in subspaces with \(d = 11\) as shown in Fig. 3 (b)).

For local dimensions up to \(d = 11\) (i.e., for azimuthal quantum numbers \(\ell = -5, \ldots, 5\)) we then proceed in the following way. First, we measure the two-photon state in the standard OAM basis \(\{m\}\) to obtain a cross-talk matrix of coincidence counts \(N_{mn}\) [Fig. 4 (a)]. This allows us to calculate the density matrix elements \(\langle mn|\rho|m'n'\rangle\), estimate the \(\lambda_m\), and nominate the target state \(|\Phi\rangle\). We then use the set \(\{\lambda_m\}\) to construct the tilted basis \(\{j\}\), according to Eq. (8) and perform correlation measurements that allow us to calculate \(\langle j^*|\rho|j^\prime\rangle\). From these measurements, we calculate the fidelity to the target state, for which we find high values, e.g., \(F(\rho, \Phi) = 71.4 \pm 2.9\%\) for \(d = 11\) (data for other dimensions is presented in Table I). However, in our setup, the certification thresholds \(B_0\) for the tilted basis are also rather high (e.g., \(B_0 = 0.675\) for \(d = 11\)). We therefore also measure the correlations in the first mutually unbiased basis \(\{j\}\) [Fig. 4 (b)] following the standard MUB construction by Wootters et al. [23], corresponding to \(\lambda_m = 1/\sqrt{d}\) for all \(m\) in Eq. (8). Using these measurements, we calculate the fidelity to the maximally entangled state according to Eqs. (6) and (9), and find \(\tilde{F}(\rho, \Phi^\ast) = 70.1 \pm 2.9\%\) for \(d = 11\), which is significantly above the bound of \(B_f(\Phi^\ast) = \frac{7}{11} \approx 0.636\), but below \(B_0(\Phi^\ast) = \frac{2}{11} \approx 0.272\), and we are hence able to certify 8-dimensional entanglement in this way. This demonstrates that our new witness does indeed work for efficiently certifying high-dimensional entanglement. Moreover, this shows that although the tilted basis measurements can achieve higher fidelities, one pays a price in terms of increased certification thresholds, and thus an increased sensitivity to noise. Our method hence provides the flexibility to exploit this trade-off by being tailored to general states such as the non-maximally entangled states that are designed to manipulate both the phase and amplitude of the incident photons [31].

**TABLE I: Experimental Results**

| \(d\) | \(d_{ent}\) | \(F(\rho, \Phi^\ast)\) | \(F(\rho, \Phi)\) |
|---|---|---|---|
| 3 | 3 | 79.5 \pm 2.0\% | 90.4 \pm 1.0\% |
| 5 | 5 | 82.3 \pm 2.5\% | 90.5 \pm 1.7\% |
| 7 | 6 | 74.1 \pm 3.8\% | 88.1 \pm 2.0\% |
| 11 | 8 | 70.1 \pm 2.9\% | 71.4 \pm 2.9\% |

Fidelities \(\tilde{F}(\rho, \Phi^\ast)\) and \(\tilde{F}(\rho, \Phi)\) to the maximally entangled state and to the target state, obtained via measurements in two MUBs and two \((M = 1)\) tilted bases in dimension \(d\), respectively. The second column lists the entanglement dimensionality \(d_{ent}\) certified using \(\tilde{F}_2(\rho, \Phi^\ast)\).
tangled state produced in our experimental setup or to perform mutually unbiased measurements if this proves to be more useful. This further illustrates the usefulness of MUBs for the detection of entanglement [28, 32–36] and correlations [37].

Conclusion. A remarkable trait of high-dimensional entanglement is that measurements in two bases are enough to certify any entangled pure state for arbitrarily large Hilbert space dimension. We make use of this insight to establish fidelity bounds for states produced under realistic laboratory conditions. Using two (or, if desirable more) local basis choices, these bounds can be employed to certify the Schmidt rank and entanglement of formation in a much more efficient way than is possible via full state tomography or even complete measurements of the fidelity. It is also interesting to note that the two measurement bases required for optimal fidelity certification become unbiased whenever the target state is maximally entangled.

The strength of our method has its origin in the fact that we use prior knowledge about the quantum system under investigation in terms of an educated guess for the Schmidt bases. This is close in spirit to assumptions commonly used in many experiments where preserved quantities in non-linear processes are harnessed to create entanglement (e.g., from transverse momenta in OAM experiments [15] to energy in time-bin setups [16]). In contrast to such assumptions, deviations from the assumed situation do not invalidate the bounds employed in our approach but lead (at most) to suboptimal performance, and an unambiguous certification is still ensured.

As we have discussed, our certification method can also be generalized to operate with more than two bases, enabling an adaptable increase in noise resistance when required. To define and carry out the necessary measurements, as well as the subsequent data analysis, no complex computer optimization or specialized programs are needed. Consequently, the tools developed here are directly applicable to any quantum experiment that allows for well-tuned local measurements.

To demonstrate this practical utility of our method, we have performed an experiment using two photons entangled in their orbital angular momenta. We were able to certify 8-dimensional entanglement in a $11 \times 11$-dimensional Hilbert space, which is the highest number achieved so far without further assumptions on the underlying quantum state. This is achieved using only two local, unbiased measurement settings (11-outcomes each), which are realized by 242 local filters and coincidence counting. Using similar measurements in the tilted bases we are able to achieve target state fidelities of 96.4% in 3 dimensions and 71.4% in 11 dimensions. While our experimental demonstration utilizes single-outcome measurements, recently developed techniques for multi-outcome measurements of OAM [38, 39] would reduce the total number of measurements required to only two. As we have shown, the certification method proposed here is thus surprisingly robust to noise and enables straightforward and assumption-free entanglement characterization in realistic quantum optics setups.

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APPENDIX

In the appendices, we provide detailed proofs and additional calculations illustrating the versatility of the results presented in the main text. A proof of the fidelity bound based on measurements in two local bases is given in Appendix A.I, while Appendix A.II details how measurements in the tilted basis can be performed. In Appendix A.III, these results are generalized to measurements in more than two bases, while Appendix A.IV discusses the tightness of the bounds for certain states. Finally, we discuss some simple bounds for the entanglement of formation in Appendix A.V, before showing the results presented in the main text. A proof of the fidelity bound based on measurements in two local bases is given in Appendix A.I, while Appendix A.II details how measurements in the tilted basis can be performed. In Appendix A.III, these results are generalized to measurements in more than two bases, while Appendix A.IV discusses the tightness of the bounds for certain states. Finally, we discuss some simple bounds for the entanglement of formation in Appendix A.V.

A.1. Fidelity bounds from measurements in two bases

In this appendix, we provide a proof for the fidelity bound

\[ F(\rho, \Phi) \geq \bar{F}(\rho, \Phi), \]  

(A.1)
i.e., the right-hand side of Eq. (12) of the main text, where \( F(\rho, \Phi) = F_1(\rho, \Phi) + F_2(\rho, \Phi) \) and \( \bar{F}(\rho, \Phi) = F_1(\rho, \Phi) + \bar{F}_2(\rho, \Phi) \) each split into two contributions. Since the first of these, given by

\[ F_1(\rho, \Phi) := \sum_m \lambda_m^2 \langle m m | \rho | m m \rangle \]  

(A.2)
is the same for both \( F \) and \( \bar{F} \), we hence want to concentrate on showing that \( F_2 \geq \bar{F}_2 \), where

\[ F_2(\rho, \Phi) := \sum_{m,n} \lambda_m \lambda_n \langle m m | \rho | m n \rangle, \]  

(A.3)

whereas the lower bound to \( F_2(\rho, \Phi) \) is

\[ \bar{F}_2 := \frac{(\sum \lambda_m \lambda_n)}{d} \sum_{j=0}^{d-1} \langle j j^* | \rho | j j^* \rangle - \sum_{m,n=0}^{d-1} \lambda_m \lambda_n \langle m n | \rho | m n \rangle \]  

(A.4)
and the prefactor \( \tilde{\gamma}_{mn,m'n'} \) is given by

\[ \tilde{\gamma}_{mn,m'n'} = \begin{cases} 0 & \text{if } (m - m' - n - n') \mod d \neq 0 \\ \sqrt{\lambda_m \lambda_n \lambda_m' \lambda_n'} & \text{otherwise.} \end{cases} \]  

(A.5)
Here, the quantity \( F_1(\rho, \Phi) \), as well as the second and third terms of \( \bar{F}_2 \) in Eq. (A.4) can be obtained directly from measurements in the standard basis \( \{|m n\}_{m,n} \), whereas the first term of \( \bar{F}_2 \) is constructed from diagonal density matrix elements w.r.t. to the tilted bases with elements

\[ |\tilde{\gamma}_j \rangle = \frac{1}{\sqrt{\sum_n \lambda_n}} \sum_{m=0}^{d-1} \omega^{jm} \sqrt{\lambda_m} |m \rangle, \]  

(A.6)
where \( \omega = e^{2\pi i/d} \).

For the proof, we then focus on the matrix elements obtained from measurements w.r.t. the tilted basis. That is, we define the quantity

\[ \Sigma := \sum_{j=0}^{d-1} \langle \tilde{\gamma}_j^* | \rho | \tilde{\gamma}_j \rangle = \frac{1}{(\sum \lambda_j)^2} \sum_{m',m''} \lambda_m \lambda_n \lambda_{m'} \lambda_{n'} \]  

(A.7)
\[ \times \sum_{j=0}^{d-1} \omega^{j(m-m' - n + n')} \langle m'n' | \rho | m n \rangle, \]  

where the asterisk denotes complex conjugation of the vector components w.r.t. \( \{|m\}_m \). The sums over the standard basis components can then be split into several contributions. When \( m = m' \) and \( n = n' \), the phases all cancel, the sum over the tilted basis elements has \( d \) equal contributions, and we hence have

\[ \Sigma_1 := \frac{1}{d (\sum \lambda_j)^2} \sum_{m,n} \lambda_m \lambda_n \langle m n | \rho | m n \rangle. \]  

(A.8)
When \( m = m' \) but \( n \neq n' \) (or vice versa) one finds terms containing the sum

\[ \sum_{j=0}^{d-1} \omega^{j(n' - n)} = \delta_{nn'}. \]  

(A.9)
Since \( n \neq n' \), these terms vanish. For all remaining contributions to \( \Sigma \) one has \( m = m' \) and \( n \neq n' \). These terms then again split into three sets. First, for \( m = n \) and \( m' = n' \) we recover the desired terms of the form

\[ \Sigma_2 := \frac{1}{(\sum \lambda_j)^2} \sum_{m,n} \lambda_m \lambda_n \langle m n | \rho | n m \rangle, \]  

(A.10)
which also appear in \( F_2(\rho, \Phi) \) in Eq. (A.3). The terms where \( m = n \) but \( m' \neq n' \) (or vice versa) again vanish due to Eq. (A.9). Finally, this leaves the term

\[ \Sigma_3 := \frac{1}{(\sum \lambda_j)^2} \sum_{m,n} \sqrt{\lambda_m \lambda_n \lambda_{m'} \lambda_{n'}} \]  

(A.11)
\[ \times \sum_{j=0}^{d-1} \omega^{j(m-m' - n + n')} \langle m'n' | \rho | m n \rangle, \]  

where we have used the abbreviation \( c_{mnm'n'} := \sum_j \omega^{j(m-m' - n + n')} \). In the last step we have replaced \( c_{mnm'n'} \) by its real part, since for each combination of
values for \(m,n,m',n'\) the sum contains a term where the pairs \((m,n)\) and \((m',n')\) are exchanged. Each term in the sum is hence paired with another term that is its complex conjugate, and the total sum is hence real.

While \(\Sigma_1\) and \(\Sigma_2\) are accessible via measurements in the standard basis, the off-diagonal matrix elements in \(\Sigma_3\) cannot be obtained from measurements w.r.t. \(\langle mn\rangle_{m,n}\). In order to provide a useful lower bound for \(\Sigma\) we therefore have to provide a bound for \(\Sigma_3\). To this end, we can bound the real part by the modulus, i.e.,

\[
\text{Re}(c_{mmn'n'}(m'n'|\rho|mn)) \leq |c_{mmn'n'}(m'n'|\rho|mn)| = |c_{mmn'n'}(m'n'|\rho|mn)|.
\]

(A.12)

We then use the Cauchy-Schwarz inequality to bound the second factor on the right-hand side of (A.12) by writing \(\rho = \sum_i p_i |\psi_i\rangle\langle \psi_i|\) such that

\[
\langle m'n'|\rho|mn \rangle = |\sum_i \sqrt{p_i} (m'n'|\psi_i) \sqrt{p_i} (\psi_i|mn) |
\]

\[
\leq \sum_i p_i (m'n'|\psi_i) (\psi_i|m'n') \prod_i p_i (mn|\psi_i) (\psi_i|mn)
\]

\[
= \sqrt{(m'n'|\rho|m'n')(mn|\rho|mn)}.
\]

(A.13)

For the first factor \(c_{mmn'n'}\) simply note that the sum \(\sum_j \omega^j(m-m'-n+n')\) vanishes whenever \((m-m'-n+n')\) mod \(d\neq 0\), and otherwise evaluates to the value \(d\). Collecting \(c_{mmn'n'}/d\) with \(\sqrt{\lambda_m \lambda_n \lambda_{n'} \lambda_{n'}}\) into \(\tilde{c}_{mmn'n'}\) as defined in Eq. (A.5), this allows us to bound the quantity \(\Sigma_3\) according to

\[
\Sigma_3 \leq \frac{d}{(\sum \lambda^2)^2} \sum_{m,n,m',n'} \tilde{c}_{mmn'n'} \sqrt{(m'n'|\rho|m'n')(mn|\rho|mn)}.
\]

(A.14)

Collecting the different contributions to \(\Sigma\) we thus have

\[
\Sigma = \Sigma_1 + \Sigma_2 + \Sigma_3 = \sum_{j=0}^{d-1} \langle j^j^*|j^j^* \rangle
\]

\[
\leq \frac{d}{(\sum \lambda^2)^2} \left( \sum_{m,n} \lambda_m \lambda_n \langle mn|\rho|mn \rangle + \sum_{m,n} \lambda_m \lambda_n \langle mn|\rho|mn \rangle 
\right.

\[
+ \sum_{m,n,m',n',n''} \tilde{c}_{mmn'n''} \sqrt{(m'n'|\rho|m'n')(mn|\rho|mn)}
\]

Conversely, this means that the term \(F_2\) can be bounded by

\[
F_2 = \sum_{m,n} \lambda_m \lambda_n \langle mn|\rho|mn \rangle
\]

\[
\geq \frac{(\sum \lambda^2)^2}{d} \sum_{j=0}^{d-1} \langle j^j^*|j^j^* \rangle - \sum_{m,n} \lambda_m \lambda_n \langle mn|\rho|mn \rangle
\]

\[
- \sum_{m,n,m',n',n''} \tilde{c}_{mmn'n''} \sqrt{(m'n'|\rho|m'n')(mn|\rho|mn)}
\]

as claimed for the quantity \(\tilde{F}_2\) in Eq. (A.4). The fidelity \(F(\rho, \tilde{\Phi})\) can hence be bounded by measurements in only two local bases, \(\{m\}_{m}\) and \(\{j\}_{j}\), for each party.

(A.II) Post-selection measurements in tilted bases

Let us now discuss in more detail, how the measurements in the bases \(\{mn\}_{m,n}\) and \(\{ij\}_{i,j}\) can be performed by means of post-selection. As explained in the main text, local filters (e.g., an appropriately programmed SLM) may be employed to allow only systems in particular states to be detected. For a particular setting with fixed \(m\) and \(n\) corresponding to the orthonormal basis \(\{mn\}_{m,n}\) one then counts the coincidences \(N_{mn}\), which give an estimate of the diagonal density matrix elements via

\[
\langle mn|\rho|mn \rangle = \frac{N_{mn}}{\sum \sum_{ij} N_{ij}}.
\]

(A.17)

By construction, one finds \(\sum_{mn} \langle mn|\rho|mn \rangle = 1\), which is sensible, since this expression corresponds to \(Tr(\rho)\) for an orthonormal basis. In other words, \(\sum_{mn} |mn\rangle \langle mn| = 1\) is a resolution of the identity. However, the same cannot be said for the (generally non-orthogonal) basis \(\{ij\}_{i,j}\). That is, we can calculate

\[
\sum_{i,j} \langle ij|j^j^* \rangle = \frac{1}{(\sum \lambda^2)^2} \sum_{m,n,m',n'} \sqrt{\lambda_m \lambda_n \lambda_{m'} \lambda_{n'}} \langle mn'|\rho|mn' \rangle
\]

\[
\times \sum_{i,j} \omega^{j(m-n') \sum_j \omega^{j(n-n')}} = \frac{d}{(\sum \lambda^2)^2} \sum_{m,n} \lambda_m \lambda_n \langle mn|\rho|mn \rangle =: c_\lambda.
\]

(A.18)

If we simply consider the coincidence counts \(N_{ij}\) in the tilted basis, and the quantity analogous to the right-hand side of Eq. (A.17), we would find \(\sum_{ij} \frac{N_{ij}}{N_{ki} N_{kj}} = 1\), by construction. To relate the coincidences to the matrix elements w.r.t. to the tilted basis, we hence have to include the additional normalization factor \(c_\lambda\) of Eq. (A.18), i.e.,

\[
\langle ij|j^j^* \rangle = c_\lambda \frac{N_{ij}}{N_{ij}}.
\]

(A.19)

as stated in Eq. (11) of the main text.
A.III. Improved bounds using multiple bases

Next, we will show how measurements in more than one tilted basis can be included to improve the fidelity bounds. To this end, first note that the choice of tilted basis in Eq. (A.6) is not unique. For instance, all of the statements made so far about the properties of the tilted basis would remain unaffected if additional phase factors independent of $j$ were to be included in the definition of $\hat{\rho}_j$. That is, we have only relied on using identities such as $\sum_j \omega^{j(m-n)} = d\delta_{mn}$. For example, let us consider a family of tilted bases $\{\hat{\rho}_{jk}\}_{j,k}$ parameterized by an integer $k \geq 0$, such that

$$\hat{\rho}_{jk} = \frac{1}{\sqrt{\sum_n \lambda_n}} \sum_{m=0}^{d-1} \omega^{jm+km^2} \sqrt{\lambda_m} |m\rangle. \quad (A.20)$$

For $k = 0$ we hence recover the original tilted basis. If the target state is maximally entangled, $|\Phi\rangle = |\Phi^+\rangle$, we have $\lambda_n = \sqrt{\gamma_j}$, in which case all of the tilted bases become orthonormal. Moreover, in this case one can recognize this construction as that of Ref. [23], i.e., for prime dimensions the choices $k = 0, 1, \ldots, d-1$ provide a maximal set of $d$ mutually unbiased bases (MUBs), $d+1$ if one includes the standard basis $\{|m\rangle\}_m$. For non-prime dimensions, the construction still provides an MUB w.r.t. to the standard basis for every choice of $k$, but the bases for different $k$ are in general not unbiased w.r.t. each other. We will return to these interesting special cases in Appendix A.VI.

Assuming again arbitrary Schmidt coefficients $\lambda_m$ and accordingly chosen tilted bases $\{\hat{\rho}_{jk}\}_{j,k}$, we note that the only contribution of the additional phases $\omega^{km^2}$ appears in the complex coefficient $c_{nmn'}$ of Eq. (A.11), which can then replace by

$$c^{(k)}_{nmn'} := \sum_{j} \omega^{j(m-m'-n+n')} \omega^{k(m^2-m'^2-n^2+n'^2)} (A.21)$$

Clearly, when using any single one of the bases $\{\hat{\rho}_{jk}\}_{j,k}$, the modification of the constant $c_{nmn'}$ becomes irrelevant again due to the modulus in Eq. (A.12), i.e., $|c^{(k)}_{nmn'}| = |c_{nmn'}|$ for all $k$.

However, we may use several of the tilted bases simultaneously to obtain an advantage. Replacing the term $\Sigma = \sum_{j=0}^{d-1} (\hat{\rho}_j \hat{\rho}_j^\dagger)$ by an average over $M$ different tilted bases as defined by Eq. (A.20), i.e.,

$$\Sigma \rightarrow \Sigma^{(M)} = \frac{1}{M} \sum_{k=0}^{M-1} \sum_{j=0}^{d-1} \langle \hat{\rho}_j \hat{\rho}_j^\dagger | \rho | \hat{\rho}_j \hat{\rho}_j^\dagger \rangle, \quad (A.22)$$

one finds that the only affected term in the bound $\tilde{F}_2$ for $F_2$ in Eq. (A.4) is $\Sigma_3$. That is, we may replace the coefficient $\tilde{\gamma}_{nmn'}$ by the modified coefficient

$$\tilde{\gamma}_{nmn'}^{(M)} = \frac{1}{M} \sum_{k=0}^{M-1} \omega^{k(m^2-m'^2-n^2+n'^2)} \quad (A.23)$$

and define the quantity $\tilde{F}_2^{(M)} := \tilde{F}_1 + \tilde{F}_2^{(M)} \leq F_2$, where

$$\tilde{F}_2^{(M)} := \frac{(\sum_m \lambda_m^2)^{1/2} \lambda_0}{d} \left( - \sum_{m,n=0}^{d-1} \tilde{\gamma}_{nmn'}^{(M)} \sqrt{(m'n'| \rho |m'n') (mn|\rho|mn)} \right) \quad (A.24)$$

In the least favourable possible case all phases in the sum over $k$ are aligned and $\tilde{\gamma}_{nmn'}^{(M)} = \tilde{\gamma}_{nmn'}$, but in general $\tilde{\gamma}_{nmn'}^{(M)} \leq \tilde{\gamma}_{nmn'}$. Consequently, the fidelity bounds can only be improved by including measurements in more than one tilted basis.

In fact, when the dimension $d$ is a (non-even) prime, we have $\tilde{F}_2^{(M)} \geq \tilde{F}_2$ for $M \geq M$, and for $M = d$ the prefactor $\tilde{\gamma}_{nmn'}^{(M = d)}$ vanishes exactly and the fidelity bound becomes tight, i.e., $F = \tilde{F}_2^{(M = d)}$. In order to show this, we need to examine the sum in Eq. (A.23). At first it is important to realize that since the value of $\tilde{\gamma}_{nmn'}$ does not depend on $k$, only cases for which $(m-m'-n+n') \mod d = 0$ need to be examined, otherwise $\tilde{\gamma}_{nmn'} = 0$ leads to $\tilde{\gamma}_{nmn'} = 0$. Let us therefore prove the following claim. For parameter choices fulfilling the conditions

$$\begin{align*}
m \neq m', n \neq n', m \neq n, \\
m - m' - n + n' \mod d = 0 \\
m + n' = m' + n \mod d \\
m'^2 + n'^2 \mod d  \\
\end{align*} \quad (A.25)$$

it holds that $(m^2 - m'^2 - n^2 + n'^2) \neq 0$. We will prove this claim by contradiction. In order to do so, suppose that both of the following equalities hold

$$\begin{align*}
m + n' &= m' + n \mod d \\
m^2 + n'^2 &= (m')^2 + n^2 \mod d. \\
\end{align*} \quad (A.26) \quad (A.27)$$

Without loss of generality suppose $m > n$, which also implies $m' > n'$, and let us define $c := m - n = m' - n'$, which allows us to rewrite Eq. (A.27) as

$$\begin{align*}
m^2 + n'^2 &= (n' + c)^2 + (m - c)^2 \mod d \\
m^2 + n'^2 &= (n')^2 + 2cn' + c^2 + m^2 - 2cm + c^2 \mod d \\
0 &= 2c^2 + 2cn' - 2cm \mod d \\
0 &= 2c(c + n' - m) \mod d \\
0 &= 2c(m' - m) \mod d. \\
\end{align*} \quad (A.28)$$

The last equality holds, if and only if $2c(m' - m)$ is a multiple of $d$. Since $d$ is an odd prime, the only possibility is that either $c$ or $(m' - m)$ are multiples of $d$. Clearly, since $c = m - n$, $m > n$ and $m, n \in \{0, \ldots, d-1\}$, $0 < c < d$, and $c$ is therefore not a multiple of $d$. Similarly, since $m \neq m'$ and $m, m' \in \{0, \ldots, d-1\}$, $-d < (m' - m) < d$, therefore $(m' - m)$ is not a multiple of $d$. We hence arrive at a contradiction with Eq. (A.28) and conclude that under the conditions of (A.25) we have $(m^2 - m'^2 - n^2 + n'^2) \neq 0.$
Therefore, when working with \( M \) different tilted bases, \( \sum_{k=0}^{M-1} \omega^{k(m^2-m^2-n^2+n'^2)} \) is a sum of \( M \) different\(^1\) powers of \( \omega \). We subsequently have to show that the absolute value of this sum can be bounded to be strictly lower than \( M \). Moreover, the bound improves with increasing \( M \), and whenever \( M = d \), the sum in Eq. (A.23) [and hence also the sum in the last line of Eq. (A.24)] vanishes. Before we turn to the more general statement for arbitrary \( M \), let us briefly focus on the case \( M = d \), where it can be easily seen that for non-zero \((m^2-m'^2-n^2+n'^2)\)

\[
\sum_{k=0}^{d-1} \omega^{k(m^2-m'^2-n^2+n'^2)} = 0.
\]

For general values \( M < d \) let us now analytically bound \( |\sum_{k=0}^{M-1} \omega^k| \), where \( c \) is a non-zero constant. Naturally, the exact value of this sum depends on the particular value of \( c \), but here we give a general bound. To this end, we first argue that the worst case (the highest possible sum) corresponds to the situation, where \( kc \) ranges over subsequent powers of \( \omega \) (i.e. \( c = 1 \)). This can be seen from the fact that powers of \( \omega \) can be represented in the complex plane as vectors lying on the unit circle with the centre at the origin. The absolute value of the sum of several different powers of \( \omega \) can therefore be seen as the size of the sum of their corresponding vectors. Recall that for odd-prime dimension \( d \), the exponent \( kc \) ranges over \( M \) different numbers between 0 and \( d - 1 \). Now it is not hard to see that by fixing the number of vectors \( M \), the worst case sum (i.e., the largest absolute value) corresponds to the sum of the \( M \) vectors next to each other on the complex plane, which in turn corresponds to the subsequent powers of \( \omega \). With this knowledge, we have to bound one particular worst case sum, given by

\[
\sum_{k=0}^{M-1} \omega^k = \sum_{k=0}^{M-1} e^{\frac{2\pi i k}{M}}.
\]

(A.29)

Using a variant of the Dirichlet kernel \( [40] \), i.e.,

\[
\sum_{k=0}^{M-1} e^{iMx} = e^{iM(x/2)} \frac{\sin(Mx/2)}{\sin(x/2)}
\]

(A.30)

with \( x = \frac{2\pi}{d} \), we have

\[
\sum_{k=0}^{M-1} \omega^k = e^{i(M-1)x} \frac{\sin(Mx/2)}{\sin(x/2)}.
\]

(A.31)

Taking the absolute value reveals that for any choice of non-zero constant \( c \) we have

\[
\left| \sum_{k=0}^{M-1} \omega^{kc} \right| \leq \frac{|\sin(Mx/2)|}{|\sin(x/2)|}.
\]

(A.32)

\(^1\) The difference of the powers results from the fact that in the mod prime multiplicative group, every non-zero element is a generator of the whole group. This means that since \((m^2-m'^2-n^2+n'^2)\) is non-zero, iterating over different values of \( k \) results in different values of the whole exponent.

FIG. A.1. **Improved dimensionality witness for isotropic state:** The curves show \( d\tilde{F}^{(M)}(\rho_{\text{iso}}, \Phi^s) \) for isotropic states of local dimension \( d = 7 \) as functions of the visibility \( p \) for \( M = 1 \) (blue) to \( M = 7 \) (green) in steps of 1. The intersections with the horizontal lines at values \( d\tilde{B}_k(\Phi^s) = k \in \{1, ..., 7\} \) indicate that visibilities beyond certify an entanglement dimensionality of at least \( d_{\text{ent}} = k + 1 \).

After plugging this lower bound into Eq. (A.23), all \( (\text{non-zero}) \) prefactors \( \gamma_{m'm'n'n'}^{(M)} \) become decreasing functions of \( M \), on the interval \( 1 \leq M \leq d \), which concludes the proof that \( F^{(M')} \geq \tilde{F}^{(M)} \) for \( M' \geq M \) in odd prime dimensions.

For general dimension \( d \), however, it is not the case that \( F^{(M')} \geq \tilde{F}^{(M)} \) for \( M' \geq M \), except for the case when \( M = 1 \) (for any dimension). In addition we note that even though one may select any \( M \) values of \( k \), we have numerically verified up to \( d = 14 \) that the sequential choice \( k \in \{0, \ldots, M-1\} \) that we have used is optimal.

An illustration of the improvement obtained by including multiple tilted bases is given in Fig. A.1 for an isotropic state \( \rho_{\text{iso}} = |\Phi^s\rangle\langle\Phi^s| + \frac{1-p}{d} 1 \) in dimension \( d = 7 \). Such a state, highlights the influence of white noise on the certification method, since the isotropic state is a mixture of a maximally entangled and a maximally mixed state. We have hence shown that an improvement of the bounds by using more than two local bases is possible in principle. In Appendix A.VI we will further illustrate this improvement for quantifying entanglement.

**A.IV. Tightness of Schmidt number bounds**

In this appendix, we show that the Schmidt number witness \( F(\rho, \Phi) > B_{k-1}(\Phi) \) for any pure state \( \rho = |\Phi \rangle\langle \Phi| \) and for any dephased maximally entangled state \( \rho_{\text{deph}}(p) = p |\Phi^+ \rangle\langle \Phi^+| + \frac{1-p}{d} 1 \). The sum \( \sum_{m|m}(mm|mm|) \) is not only a sufficient, but also a necessary condition for \( |\Phi \rangle \) or \( \rho_{\text{deph}} \) to have a Schmidt rank larger or equal than \( k \). For the state \( |\Phi \rangle \) this is obvious. Since the coefficients
\(\lambda_m\) are determined by measurements in Schmidt basis of \(\rho = |\Phi\rangle \langle \Phi|\), the fidelity bound is tight, and we have \(F(\rho, \Phi) = F(\rho, \Phi) = 1\) and \(B_k(\Phi)\) is equal to 1 if and only if \(k = d\).

For dephased maximally entangled states we proceed by showing that there exists a Schmidt–rank \(k\) state \(\rho_{\text{deph}}(p = p_k)\) such that \(F(\rho_{\text{deph}}(p_k), \Phi) = B_k(\Phi)\) for every \(k\). To this end, first note that \(\rho_{\text{deph}}\) can be written as

\[
\rho_{\text{deph}} = p |\Phi\rangle \langle \Phi'\rangle + \frac{1-p}{d} \sum_m |mm\rangle \langle mm|,
\]

where \(p\) is the dephasing parameter. The relevant fidelity then evaluates to

\[
F(\rho, \Phi) = F(\rho_{\text{deph}}, \Phi^\ast) = \frac{1+(d-1)p}{d},
\]

and

\[
F(\rho_{\text{deph}}, \Phi^\ast) = B_k \text{ for } p = p_k = \frac{k-1}{d-1}. \quad \text{All we need to do now is to show that \(\rho_{\text{deph}}(p_k)\) has a Schmidt rank no larger than \(k\). To see this, consider the family of maximally entangled states in dimension \(k\), i.e.,
\]

\[
|\Phi^\ast\rangle := \frac{1}{\sqrt{d}} \sum_m |mm\rangle,
\]

where \(\alpha \in \{0, 1, \ldots, d-1\}\) is a cardinality \(|\alpha| = k\). In dimension \(d\), we can find \(\binom{d}{k}\) such states and consider their coherent mixture, i.e.,

\[
\rho_k = \frac{1}{\binom{d}{k}} \sum_{\alpha \in \{0, 1, \ldots, d-1\}, |\alpha| = k} |\Phi^\ast\rangle \langle \Phi^\ast|.
\]

Since each of the \(\Phi^\ast\) has Schmidt rank \(k\), the convex sum \(\rho_k\) cannot have a Schmidt rank larger than \(k\). Since there are \(\binom{d}{k-1}\) terms contributing to every nonzero diagonal matrix element, we have \(\langle mn | \rho | mn \rangle = \frac{1}{d} \delta_{mn}\). Similarly, every nonvanishing off-diagonal matrix element has \(\frac{d}{k-1}\) contributions, and we hence have \(\langle mn | \rho | ij \rangle = \frac{k-1}{d} \delta_{mn} \delta_{ij}\) for \(m \neq i\). It is then easy to see that the fidelity with the maximally entangled state (in dimension \(d\)) is \(F(\rho_k, \Phi^\ast)\) is \(\frac{k}{d}\). More specifically, by comparison with Eq. (A.33) reveals that \(\rho_{\text{deph}} = \rho_k\) for \(p = p_k = \frac{k-1}{d-1}\). Since the Schmidt rank of \(\rho_k\) is smaller or equal than \(k\), we have hence shown that the Schmidt rank of the dephased maximally entangled state \(\rho_{\text{deph}}(p_k)\) with \(F(\rho_{\text{deph}}(p_k), \Phi) = B_k\) is \(k\) or less. Consequently, \(F(\rho_{\text{deph}}, \Phi^\ast) > B_{k-1}\) is a necessary and sufficient condition for \(\rho_{\text{deph}}\) to have Schmidt rank \(k\).

Moreover, since the fidelity bound \(\tilde{F} \leq F\) is tight for \(\rho_{\text{deph}}\) already for \(M = 1\), we can conclude that measurements in two unbiased bases provide the necessary and sufficient condition \(F(\rho_{\text{deph}}, \Phi^\ast) > \tilde{F}_{k-1}\) for Schmidt rank \(k\) for dephased maximally entangled states.

### A.V. Bounds on the entanglement of formation

In this appendix, we discuss a method for bounding the entanglement of formation in bipartite systems of arbitrary dimension. To provide a self-contained approach, let us first give a pedagogical review of the entanglement of formation and useful bounds for it as also discussed in Ref. [28], before we make use of the fidelity bounds established thus far in Appendix A.VI. To begin, recall that the subsystems \(A\) and \(B\) of a pure bipartite state \(|\psi\rangle_{AB}\) are entangled if and only if their reduced states \(\rho_A = \text{Tr}_B(|\psi\rangle \langle \psi|)\) and \(\rho_B = \text{Tr}_A(|\psi\rangle \langle \psi|)\) are mixed. This fact can easily be seen from the Schmidt decomposition, i.e., that any pure state \(|\psi\rangle_{AB} \in \mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B\) may be written as

\[
|\psi\rangle_{AB} = \sum_{m=0}^{k-1} \lambda_m |\phi_m\rangle_A |\chi_m\rangle_B
\]

with respect to the Schmidt bases \(\{ |\phi_m\rangle\}_m\) and \(\{ |\chi_m\rangle\}_m\), and where \(k \equiv \min(\dim(\mathcal{H}_A), \dim(\mathcal{H}_B))\). The entanglement of the state \(|\psi\rangle_{AB}\) may therefore be quantified by the mixedness \(1 - \text{Tr}(\rho_A^2)\) of the reduced states. More specifically, we can define the entropy of entanglement \(E_L\) via the linear entropy \(S_L\) as

\[
E_L(|\psi\rangle) = S_L(\rho_A) = \sqrt{2(1 - \text{Tr}(\rho_A^2)).}
\]

This method for entanglement quantification can be extended to mixed states via a convex-roof construction, i.e.,

\[
E_L(\rho) := \inf_{D(\rho)} \sum_i p_i S_L(\rho_A^{(i)}),
\]

where the infimum is taken over the set of all pure state decompositions of \(\rho\), i.e.,

\[
D(\rho) = \left\{ \{|p_i, \psi_i\}_i \mid \rho = \sum_i p_i |\psi_i\rangle \langle \psi_i|, 0 \leq p_i \leq 1, \sum_i p_i = 1 \right\},
\]

where \(\rho_A^{(i)} = \text{Tr}_B(|\psi_i\rangle \langle \psi_i|)\).

A simple bound on this convex roof of the linear entropy was derived in Refs. [41, 42]. Defining the quantity

\[
I(\rho) = \sqrt{\frac{2}{(d-1)}} \sum_{m,n} \left| \langle mn | \rho | mn \rangle \right| - \sqrt{\langle mn | \rho | mn \rangle \langle mn | \rho | mn \rangle},
\]

for bipartite systems of equal local dimension \(d\), i.e., \(\dim(\mathcal{H}_A) = \dim(\mathcal{H}_B) = d\), with bases \(\{|\phi_n\rangle_A \equiv |n\rangle_A\}\) and \(\{|\chi_n\rangle_B \equiv |n\rangle_B\}\), it was shown in [41, 42] that

\[
I(\rho) \leq E_L(\rho).
\]

Now, we want to see how \(I(\rho)\) can used to bound also the entanglement of formation (EoF) [43], defined as the convex roof extension of the entropy of entanglement when
the von Neumann entropy $S(\rho) = -\text{Tr}(\rho \log \rho)$ is used instead of the linear entropy, i.e.,

$$\mathcal{E}_{\text{OF}}(\rho) := \inf_{\rho^i} \sum_i p_i S(\rho_i^\alpha).$$

(A.43)

To understand this connection, let us briefly expand upon the derivation given in Ref. [28]. First, note that for pure states $|\psi\rangle$ we have

$$I(|\psi\rangle) \leq \mathcal{E}_L(|\psi\rangle) = \sqrt{2(1 - \text{Tr}(\rho_1^2))}.$$  

(A.44)

Therefore, if $I(|\psi\rangle) \geq 0$ we can write

$$\text{Tr}(\rho_1^2) \leq 1 - \frac{1}{2} I^2(|\psi\rangle),$$

which implies that

$$-\log(\text{Tr}(\rho_1^2)) \geq -\log\left(1 - \frac{1}{2} I^2(|\psi\rangle)\right)$$

(A.46)

since $\log x$ is a monotonically increasing function. With the additional negative sign we can recognize the left-hand side as the Rényi 2-entropy, defined as

$$S_\alpha(\rho) := \frac{1}{1-\alpha} \log \text{Tr}(\rho^\alpha)$$

(A.47)

for $\alpha = 2$. For all $\alpha, \beta \in \mathbb{N}$ and for all $\rho$, the Rényi entropies satisfy $S_\alpha(\rho) \geq S_\beta(\rho)$ for $\alpha \leq \beta$. In particular, this means that

$$S_1(\rho) = \lim_{\alpha \to 1} S_\alpha(\rho) \geq S_2(\rho) = -\log(\text{Tr}(\rho^2))$$

(A.48)

and consequently one has

$$S_1(\rho_1) \geq -\log\left(1 - \frac{1}{2} I^2(|\psi\rangle)\right).$$

(A.49)

For pure states, the (von Neumann) entropy of the subsystem is equal to the EoF and we have hence obtained the desired bound. To see that the bound also holds for mixed states, simply note that $-\log(1 - x^2/2)$ is a convex function. Similarly, the function $I(\rho)$ is convex, since

$$I_1 := \sum_{m,n} |\langle mn|\rho|mn\rangle|$$

(A.50)

is convex, while

$$I_2 := \sum_{m,n} \sqrt{\langle mn|\rho|mn\rangle \langle mn|\rho|mn\rangle}$$

(A.51)

is concave, i.e., by Jensen’s inequality [44]

$$I_1(\sum_i p_i \rho_i) \leq \sum_i p_i I_1(\rho_i),$$

(A.52)

$$I_2(\sum_i p_i \rho_i) \geq \sum_i p_i I_2(\rho_i)$$

(A.53)

for $0 \leq p_i \leq 1$ and $\sum_i p_i = 1$. This allows us to conclude that for all states $\rho$, for which $I(\rho) \geq 0$ one has

$$\mathcal{E}_{\text{OF}}(\rho) \geq -\log\left(1 - \frac{1}{2} I^2(\rho)\right).$$

(A.54)

Here, it is first useful to note here that the value of $I(\rho)$ (in particular, whether or not $I$ is non-negative) for a given state of course depends on the bases $\{|m\rangle\}_m$ and $\{|n\rangle\}_n$ that are chosen. For instance, if both bases are chosen to be the same single-qubit bases and the quantum state in question is the singlet state $|\psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$, where $|0\rangle$ and $|1\rangle$ are assumed to be the eigenstates of the third Pauli matrix $Z = \text{diag}\{1, -1\}$, then $I(|\psi^-\rangle) = -1$. In other words, the bases $\{|m\rangle\}_m$ and $\{|n\rangle\}_n$ should be chosen with a specific family of states in mind. For pure states, it is most useful to choose the Schmidt bases of the two subsystems.

Second, observe that, on the one hand, the term $I_2$ contains only diagonal matrix elements and can hence be practically easily estimated using measurements in one pair of local bases only. That is, counting the coincidences $N_{mm}$ in the basis setting $|m\rangle\langle m|$ we can reconstruct the desired matrix elements as $\langle mn|\rho|mn\rangle = N_{mn}/(\sum_{i,j} N_{ij})$. On the other hand, to estimate the off-diagonal matrix elements of the term $I_1$ precisely, one would be required to reconstruct the entire density matrix by way of state tomography. However, this costly procedure can be avoided by supplementing the measurements in the basis $\{|mn\rangle\}_{m,n}$ by measurements in one (or more) MUBs w.r.t. $\{|mn\rangle\}_{m,n}$ to provide a lower bound on $I_2(\rho)$.

A.VI. Entanglement quantification using mutually unbiased bases

Having established the usefulness of the quantity $I(\rho)$ for bounding the entanglement of formation, let us now relate it to the fidelity bounds we have discussed before. Inspection of the fidelity to the maximally entangled state, i.e.,

$$F(\rho, \Phi^+) = \frac{1}{d} \sum_m \langle mn|\rho|mn\rangle + \frac{1}{d} \sum_m \langle mn|\rho|nn\rangle,$$

(A.55)

immediately lets us obtain the bound

$$\sum_{m,n} |\langle mn|\rho|nn\rangle| \geq \sum_{m,n} \langle mn|\rho|mn\rangle$$

(A.56)

which

$$= d F(\rho, \Phi^+) - \sum_m \langle mn|\rho|mm\rangle.$$
Entanglement bounds for isotropic state: (a) The dashed and solid curves show the lower bounds for $\mathcal{E}_{oF}$ obtained for $M = 1$ and $\rho_{iso}(p)$ using the bounds from Ref. \[28\] (dashed) and using the bound presented here in (A.57) (solid curves), respectively, for dimensions $d = 3$ (blue) to $d = 10$ (green) in steps of 1 and in units of $\log d$. It can be seen that the newly improved bounds can certify higher entanglement for given visibilities $p$. (b) The bound of Ref. \[28\] (orange, dashed) is compared with the bound of (A.57) (solid curves) for fixed dimension $d = 7$ and varying numbers of bases, $M = 1$ (blue) to $M = 7$ (green) in steps of 1.

Since $F(\rho, \Phi^*) \geq \tilde{F}^{(M)}$, this, in turn, implies that

$$I(\rho) \geq \sqrt{\frac{2}{d(d-1)}} \left( d \tilde{F}^{(M)}(\rho, \Phi^*) - \sum_{m} \langle mn | \rho | mn \rangle \right)$$

$$- \sum_{m \neq n} \sqrt{\langle mn | \rho | mn \rangle \langle nm | \rho | nm \rangle}$$

(A.57)

$$\geq \sqrt{\frac{2}{d(d-1)}} \left( d \tilde{F}^{(M)} - 1 - \sum_{m \neq n} \sqrt{\langle mn | \rho | mn \rangle \langle nm | \rho | nm \rangle} \right)$$

$$- \sum_{m \neq n', n \neq n'} \sqrt{\langle m'n' | \rho | m'n' \rangle \langle mn | \rho | mn \rangle},$$

where we have inserted the fidelity bound $\tilde{F}^{(M)}$ for multiple MUBs derived in Appendix A.III. The measurements performed to lower-bound the entanglement dimensionality of $\rho$ may hence directly be used to also obtain a lower bound on the entanglement of formation.

We further note that the bound of (A.57) can also be considered to be a generalization of the bounds discussed in Ref. \[28\], where a similar, but strictly weaker bound for $I(\rho)$ is provided, corresponding to setting $M = 1$ and $\tilde{F}^{(M)} = 1$. To provide direct comparisons of our bounds with the methods of Ref. \[28\], we again turn to the example of the isotropic state $\rho_{iso} = p|\Phi^*\Phi^*| + \frac{1-p}{d} I$, where $0 \leq p \leq 1$, $|\Phi^*| = \frac{1}{\sqrt{\dim}} \sum_{m} |mn\rangle$, and $I$ is the identity in dimension $d^2$. A comparison of the performance of these bounds for entanglement quantification for the assumed state $\rho_{iso}$ is shown in Fig. A.2.

The isotropic state also provides an ideal theoretical testing ground for the noise robustness of these bounds, since it correspond to mixing a maximally entangled state with white noise and hence allows to characterize the robustness of the entanglement bounds against decoherence. To this end, we compare the critical visibilities $p_{\text{crit}}$, that is, the parameters appearing in $\rho_{iso}(p)$ for which the different methods stop detecting entanglement. Ideally, this could be the case for the value $p_{\text{crit}} = \frac{1}{2^{1/4}}$, below which the isotropic state is separable \[9\]. For the bound of Ref. \[28\] we find $p_{\text{crit}}^{BW} = \frac{d^2 - 3d + 4}{(d^2 - 2d + 4)},$ whereas our bound from (A.57) provides $p_{\text{crit}}^{(M)} = \frac{f(M)}{d(d-1) + f(M)}$, where $f(M) = \sum_{m \neq n', n \neq n'} \tilde{F}^{(M)}$. As illustrated in Fig. A.3, the improved bounds presented here significantly improve on the noise resistance of the bounds.

FIG. A.2. Entanglement bounds for isotropic state: (a) The dashed and solid curves show the lower bounds for $\mathcal{E}_{oF}$ obtained for $M = 1$ and $\rho_{iso}(p)$ using the bounds from Ref. \[28\] (dashed) and using the bound presented here in (A.57) (solid curves), respectively, for dimensions $d = 3$ (blue) to $d = 10$ (green) in steps of 1 and in units of $\log d$. It can be seen that the newly improved bounds can certify higher entanglement for given visibilities $p$. (b) The bound of Ref. \[28\] (orange, dashed) is compared with the bound of (A.57) (solid curves) for fixed dimension $d = 7$ and varying numbers of bases, $M = 1$ (blue) to $M = 7$ (green) in steps of 1.

FIG. A.3. Critical visibilities: The curves show the parameters $p$ for which the entanglement of the isotropic states in $d \times d$ dimensions become undetectable using the bound of Ref. \[28\] (upper orange curve) and the bound of (A.57) for $M = 1, 2, 3$ (blue solid, dashed, dotted curves), respectively. The bottom purple curves indicate the value below which $\rho_{iso}$ is separable. The irregular behaviour of the curves for $M > 1$ originates from the fact that the bases we use are all unbiased w.r.t. each other only in prime dimensions (green dots).