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Tighter sum uncertainty relations via metric-adjusted skew information

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Abstract

In this paper, we first provide three general norm inequalities, which are used to give new uncertainty relations of any finite observables and quantum channels via metric-adjusted skew information. The results are applicable to its special cases as Wigner-Yanase-Dyson skew information. In quantifying the uncertainty of channels, we discuss two types of lower bounds and compare the tightness between them, meanwhile, a tight lower bound is given. The uncertainty relations obtained by us are stronger than the existing ones. To illustrate our results, we give several specific examples.

1. Introduction

Uncertainty principle as a quintessential manifestation of quantum mechanics reveals the insights that distinguish quantum theory from classical theory. Heisenberg originally came up with the uncertainty principle [1] in 1927, which enunciates that the position and momentum of a particle cannot be determined simultaneously. Since then, the quantitative characterization of the uncertainty relation has received extensive attention, and many results have been obtained.

There are a host of methods to characterize the uncertainty principle, one of which is variance. This method, which was adopted by Robertson [2] and Schrödinger [3], has been found that there exist lower bounds on the variances product for any two non-commuting observables. Subsequently, with regard to the sum of variance, the stronger uncertainty relations were provided [4]. And Wang et al [5] verified the results in [4] through experiment. Later, some tighter uncertainty relations with respect to variance were given [6–9].

The other well-known method of characterizing uncertainty relation is entropy. Deutsch [10] first proposed a quantitative expression of the uncertainty principle by entropy for any two non-commuting observables, and then Maassen and Uffink [11] optimized the result in 1988. Furthermore, many scholars have put forward diverse uncertainty relations respecting distinct entropies [12–15]. The uncertainty relations of entropy have numerous applications ranging from entanglement witnesses [16, 17], quantum teleportation [18], quantum steering [19], quantum key distribution [20, 21] to quantum metrology [22].

Recently, Luo [23] confirmed that skew information is an alternative approach to characterize the uncertainty relation. In [24], Wigner and Yanase introduced the definition of skew information

\[ I_\rho(M) = \frac{1}{2} \text{Tr} \left[ \rho^{1/2}, M \right]^2 = \frac{1}{2} \| [\rho^{1/2}, M] \|_2^2. \]  

Here \( \rho \) and \( M \) represent the quantum state and the observable, respectively. It can be considered as a measure that quantifies the information content included in the state \( \rho \) regarding the conserved observables. Meanwhile, compared to the usual variance, it is better at times. The skew information, for pure states, is the same as the variance [25], but they differ in mixed states. In the space of quantum states, skew information is convex, on the contrary for variance [25], which is one of the remarkable differences between them. Later, Dyson put forth a
quantitative way which is a generalization of skew information, its specific expression is

\[ I^\alpha_\rho (M) = - \frac{1}{2} \text{Tr} [\rho ^\alpha, M] [\rho ^{1-\alpha}, M], \quad 0 < \alpha < 1, \]  

(2)
termed as Wigner-Yanase-Dyson skew information, and Lieb [26] resolved the convexity of this form on quantum states.

The sum of quantum uncertainty is crucial, because it is an effective tool for detecting quantum entanglement [27–31]. To this end, the sum of quantum Fisher information (QFI) defined by means of symmetric logarithmic derivative probably is superior to Wigner-Yanase skew information [29], as in the quantum Cramer-Rao inequality. In [32], Petz proposed the concept of monotone metric. After that, Hansen [33] defined a class of QFI by using operator monotone metrics called metric-adjusted skew information.

The quantum channel is essential in quantum theory. The uncertainty relation of channels has also been investigated extensively, and a large number of results have been yielded [34, 35]. Recently, some scholars have generalized uncertainty inequalities to metric-adjusted skew information for arbitrary finite quantum channels [36–38].

The study of uncertainty relation is helpful for us to understand the wave-particle duality of quantum physics and to implement quantum precision measurement, and the more accurate the result of uncertainty relation, the more beneficial it is for quantum precision measurement. Consequently, we would like to further study tighter uncertainty relations regarding metric-adjudged skew information.

The overall structure of this paper is as follows. In section 2, we recall the notion of metric-adjusted skew information. In section 3, we present some norm inequalities, and then new uncertainty relations of observables are given regarding metric-adjusted skew information. The distinct types of uncertainty relations of quantum channels with respect to metric-adjusted skew information are discussed in section 4, and we prove that which of the two corresponding lower bounds obtained by the same norm inequality is better. At the same time, these conclusions still hold for its special metrics. We also give several examples and compare the lower bounds obtained by us with the lower bounds in [36–38]. This more intuitively shows that our results are more accurate than the ones in [36–38]. The main conclusions are summarized in section 5.

2. Metric-adjusted skew information

Suppose that \( M_n(C) \) is the set of all complex \( n \times n \) matrices, \( \mathcal{R}_n \) is the set of all positive definite \( n \times n \) matrices with trace 1, where \( n \in \mathbb{N} \). For any \( A, B \in M_n(C) \), \( \rho \in \mathcal{R}_n \), \( K_\rho (\cdot, \cdot): M_n(C) \times M_n(C) \to C \) is termed as symmetric monotone metric [32] which satisfies

(i) \( (A, B) \mapsto K_\rho (A, B) \) is sesquilinear, that is, the function \( K_\rho (A, \cdot) \) is linear and \( K_\rho (\cdot, B) \) conjugate linear.

(ii) \( K_\rho (A, A) \) is nonnegative, \( K_\rho (A, A) = 0 \) if and only if \( A = 0 \).

(iii) \( \rho \mapsto K_\rho (A, A) \) is continuous on \( \mathcal{R}_n \).

(iv) \( K_\rho (T(A), T(A)) \leq K_\rho (A, A) \) for any stochastic mapping \( T \). A linear mapping \( T: M_n(C) \to M_m(C) \) is called stochastic mapping if \( T(\mathcal{R}_n) \subseteq \mathcal{R}_m \) and \( T \) is a completely positive.

(v) \( K_\rho (A, B) = K_\rho (B^T, A^T) \).

The symmetric monotone metric \( K_\rho (A, B) \) can be expressed as

\[ K_\rho (A, B) = \text{Tr}[A^* \epsilon(L_\rho, R_\rho) B], \]  

(3)
where \( L_\rho \) and \( R_\rho \) are respectively left and right multiplication operators, \( \epsilon \) is termed as Morozova-Chentsov function, and its form is

\[ \epsilon(x, y) = \frac{1}{\sqrt{y \cdot (xy)^{-1}}}, \quad x, y > 0, \]  

(4)
where the function \( f \) satisfies the conditions: (a) \( f: R_+ \to R_+ \) is an operator monotone, where \( R_+ \) is the set of all positive real number, namely, if \( A \geq B \), then \( f(A) \geq f(B) \) for arbitrary \( A, B > 0 \); (b) if \( t^{-1} = f(0) \) for every \( t > 0 \). Especially, it has been shown that if \( K_{f(I, I)}(A, B) = 1 \) holds, then the associated normalized function \( f \) requires to admit \( f(1) = 1 \). Here \( I \) is the \( n \)-dimensional identity operator.

In addition, in the space of quantum states, if the Morozova-Chentsov function associated with the symmetric monotone metric \( K_\rho (\cdot, \cdot) \) satisfies
\[ m(c) = \lim_{x \to 0} c(x, 1)^{-1} > 0, \]  
(5)

then \( K_c(\cdot, \cdot) \) is known as regular \([33]\), \( m(c) \) is called metric constant and \( m(c) = f(0) \).

In \([33]\), Hansen introduced the metric-adjusted skew information \( I^a_c(M) \) which is

\[
I^a_c(M) = \frac{m(c)}{2} K^c_c[i\rho, M], i\rho, M\]

\[
= \frac{m(c)}{2} \text{Tr} \{i\rho, M\} c(L_{\rho}, R_{\rho})i\rho, M\},
\]

(6)

where \( c \) satisfies the constraint \([5] \). Due to \( i\rho, M = i(L_{\rho} - R_{\rho})M \), then equation \((6)\) can be rewritten as

\[
I^a_c(M) = \frac{m(c)}{2} \text{Tr} \{MC(L_{\rho}, R_{\rho})M\},
\]

(7)

where \( \hat{c}(x, y) = (x - y)^2 c(x, y), x, y > 0 \).

When one chooses

\[
\hat{c}^{\text{WY}}(x, y) = \frac{4}{(\sqrt{x} + \sqrt{y})^2}, \quad x, y > 0,
\]

(8)

and

\[
\hat{c}^{\alpha}(x, y) = \frac{1}{\alpha(1 - \alpha)} \frac{(x^\alpha - y^\alpha)(x^{1-\alpha} - y^{1-\alpha})}{(x - y)^2}, \quad 0 < \alpha < 1,
\]

(9)

the associated monotone metrics

\[
K^\text{WY}_c(A, B) = \text{Tr}[A^\dagger c^{\text{WY}}_c(L_{\rho}, R_{\rho})B]
\]

(10)

and

\[
K^\alpha_c(A, B) = \text{Tr}[A^\dagger c^{\alpha}_c(L_{\rho}, R_{\rho})B]
\]

(11)

are known as Wigner-Yanase metric and Wigner-Yanase-Dyson metric, respectively. Therefore, when \( c = c^{\alpha} \), equation \((6)\) turns into equation \((2)\) which is Wigner-Yanase-Dyson skew information \( I^\alpha_c(M) \). When \( \alpha = \frac{1}{2} \), equation \((2)\) reduces to equation \((1)\) which is Wigner-Yanase skew information \( I_c(M) \).

### 3. Uncertainty relations of finite observables

In this section, we first present some norm inequalities. By using these inequalities the new sum uncertainty relations of finite observables are given via metric-adjusted skew information, and the results also hold for its special metrics, such as those mentioned in section 2 above. Then we provide two examples which show that our results are better than existing ones.

For finite \( n \) observables \( M_1, M_2, \ldots, M_n \) \((n > 2)\), Cai \([36]\) showed the uncertainty relation

\[
\sum_{i=1}^{n} I^a_c(M_i) \geq \frac{1}{n} - \frac{1}{2} \left[ \sum_{1 \leq i < j \leq n} I^a_c(M_i + M_j) - \frac{1}{(n - 1)^2} \left( \sum_{1 \leq i < j \leq n} \sqrt{I^a_c(M_i + M_j)} \right)^2 \right].
\]

(12)

Ren et al \([37]\) gave an uncertainty inequality

\[
\sum_{i=1}^{n} I^a_c(M_i) \geq \frac{1}{n} \left[ \sum_{i=1}^{n} M_i \right] + \frac{2}{n^2(n - 1)} \left( \sum_{1 \leq i < j \leq n} \sqrt{I^a_c(M_i - M_j)} \right)^2.
\]

(13)

Recently, Zhang et al \([38]\) provided an uncertainty inequality

\[
\sum_{i=1}^{n} I^a_c(M_i) \geq \max_{z \in [0,1]} \frac{1}{2n - 2} \left( \frac{2}{n(n - 1)} \left( \sum_{1 \leq i < j \leq n} \sqrt{I^a_c(M_i + (-1)^z M_j)} \right)^2 + \sum_{1 \leq i < j \leq n} I^a_c(M_i + (-1)^z M_j) \right).
\]

(14)

The inequalities \((13)\) and \((14)\) hold when \( n \geq 2 \). For simplicity, the lower bounds in \((12), (13), \) and \((14)\) are marked by \( Lb_1, Lb_2, \) and \( Lb_3, \) respectively.

Next we show various inequalities of the norm which are essential for the discussion of main results, so we take the inequalities as a Lemma.
Lemma 1. Suppose that $x_i$ is a complex matrix, we can get

$$\sum_{i=1}^{n} \|x_i\|^2 \geq \frac{1}{mn + (n-2)l} \left[ \frac{2l}{n(n-1)} \left( \sum_{1 \leq i < j \leq n} \|x_i + x_j\|^2 \right) + m \sum_{1 \leq i < j \leq n} \|x_i - x_j\|^2 + (m - l) \left\| \sum_{i=1}^{n} x_i \right\|^2 \right]$$  (15)

and

$$\sum_{i=1}^{n} \|x_i\|^2 \geq \frac{1}{mn + (n-2)l} \left[ l \sum_{1 \leq i < j \leq n} \|x_i + x_j\|^2 + \frac{2m}{n(n-1)} \left( \sum_{1 \leq i < j \leq n} \|x_i - x_j\|^2 \right) + (m - l) \left\| \sum_{i=1}^{n} x_i \right\|^2 \right]$$  (16)

for arbitrary $m, l > 0$, and

$$\sum_{i=1}^{n} \|x_i\|^2 \geq \frac{1}{mn + (n-2)l} \left[ l \sum_{1 \leq i < j \leq n} \|x_i + x_j\|^2 + m \sum_{1 \leq i < j \leq n} \|x_i - x_j\|^2 + \frac{m - l}{n(n-1)} \left( \sum_{1 \leq i < j \leq n} \|x_i + x_j\|^2 \right) \right]$$  (17)

for $l > m > 0$. Specially we have

$$\sum_{i=1}^{n} \|x_i\|^2 \geq \frac{1}{3n - 2} \left[ \frac{2}{n(n-1)} \left( \sum_{1 \leq i < j \leq n} \|x_i + x_j\|^2 \right) + 2 \sum_{1 \leq i < j \leq n} \|x_i - x_j\|^2 + \left\| \sum_{i=1}^{n} x_i \right\|^2 \right], \quad (18)$$

and

$$\sum_{i=1}^{n} \|x_i\|^2 \geq \frac{1}{3n - 4} \left[ 2 \sum_{1 \leq i < j \leq n} \|x_i + x_j\|^2 + \frac{2}{n(n-1)} \left( \sum_{1 \leq i < j \leq n} \|x_i - x_j\|^2 \right) - \left\| \sum_{i=1}^{n} x_i \right\|^2 \right], \quad (19)$$

and

$$\sum_{i=1}^{n} \|x_i\|^2 \geq \frac{1}{3n - 4} \left[ 2 \sum_{1 \leq i < j \leq n} \|x_i + x_j\|^2 + \sum_{1 \leq i < j \leq n} \|x_i - x_j\|^2 - \frac{1}{(n-1)^2} \left( \sum_{1 \leq i < j \leq n} \|x_i + x_j\|^2 \right) \right], \quad (20)$$

where $\|\|$ denotes the operator norm of a matrix.

The proof of lemma 1 is shown in appendix A.

When $n (\geq 2)$ is determined, the larger $m$ and the smaller $l$, the bigger right side of inequalities (15) and (17), the larger $l$ and the smaller $m$, the bigger right side of inequality (16).

We observe some relations between the norm inequalities, which are presented in appendix B.

These inequalities are helpful for us to explore tighter uncertainty relations. Based on the inequalities (15)—(20), we give tighter sum uncertainty relations in the following Theorem.

Theorem 1. For finite $n$ observables $M_1, M_2, \cdots, M_n$ ($n \geq 2$), the sum uncertainty relation with respect to metric-adjusted skew information is

$$\sum_{i=1}^{n} I^r_i(M_i) \geq \max\{P_1, P_2, P_3\}. \quad (21)$$

Specially, we have

$$\sum_{i=1}^{n} I^r_i(M_i) \geq \max\{P_1', P_2', P_3'\}, \quad (22)$$
where

\[
P_1 = \frac{1}{mn + (n - 2)l} \left( \frac{2l}{n(n - 1)} \left( \sum_{1 \leq i < j \leq n} \sqrt{\mathcal{L}_p(M_i + M_j)} \right)^2 + m \sum_{1 \leq i < j \leq n} \mathcal{L}_p(M_i - M_j) + (m - l)I_p \left( \sum_{i=1}^n M_i \right) \right),
\]

\[
P_2 = \frac{1}{mn + (n - 2)l} \left( \frac{2m}{n(n - 1)} \left( \sum_{1 \leq i < j \leq n} \sqrt{\mathcal{L}_p(M_i - M_j)} \right)^2 + l \sum_{1 \leq i < j \leq n} \mathcal{L}_p(M_i + M_j) + (m - l)I_p \left( \sum_{i=1}^n M_i \right) \right),
\]

\[
P_3 = \frac{1}{mn + (n - 2)l} \left( l \sum_{1 \leq i < j \leq n} \mathcal{L}_p(M_i + M_j) + m \sum_{1 \leq i < j \leq n} \mathcal{L}_p(M_i - M_j) + \frac{m - l}{(n - 1)^2} \left( \sum_{1 \leq i < j \leq n} \sqrt{\mathcal{L}_p(M_i + M_j)} \right)^2 \right),
\]

\[
P_4 = \frac{1}{3n - 2} \left( \frac{2}{n(n - 1)} \left( \sum_{1 \leq i < j \leq n} \sqrt{\mathcal{L}_p(M_i + M_j)} \right)^2 + 2 \sum_{1 \leq i < j \leq n} \mathcal{L}_p(M_i - M_j) + \nI_p \left( \sum_{i=1}^n M_i \right) \right),
\]

\[
P_5 = \frac{1}{3n - 4} \left( \frac{2}{n(n - 1)} \left( \sum_{1 \leq i < j \leq n} \sqrt{\mathcal{L}_p(M_i - M_j)} \right)^2 + 2 \sum_{1 \leq i < j \leq n} \mathcal{L}_p(M_i + M_j) - I_p \left( \sum_{i=1}^n M_i \right) \right),
\]

\[
P_6 = \frac{1}{3n - 4} \left( 2 \sum_{1 \leq i < j \leq n} \mathcal{L}_p(M_i + M_j) + \sum_{1 \leq i < j \leq n} \mathcal{L}_p(M_i - M_j) - \frac{1}{(n - 1)^2} \left( \sum_{1 \leq i < j \leq n} \sqrt{\mathcal{L}_p(M_i + M_j)} \right)^2 \right),
\]

the parameters \( l, m \) of \( P_1, P_2, P_3 \) satisfy \( m \geq l > 0, l \geq m > 0, l > m > 0 \), respectively.

**Proof.** Because the symmetric monotone metrics \( K_p (\cdot, \cdot) \) satisfy the norm property, according to inequalities (15), (16), and (17), one has

\[
\sum_{i=1}^n K_p^l (g[i\rho, M_i], i\rho, M_i) \geq \frac{1}{mn + (n - 2)l} \left( \frac{2l}{n(n - 1)} \left( \sum_{1 \leq i < j \leq n} \sqrt{K_p^l (g[i\rho, M_i + M_j], i\rho, M_i + M_j)} \right)^2 + m \sum_{1 \leq i < j \leq n} K_p^l (i\rho, M_i - M_j) \right)
\]

\[
+ (m - l)K_p^l \left( i\rho, \sum_{i=1}^n M_i \right), \text{ for } m \geq l > 0,
\]

\[
\sum_{i=1}^n K_p^l (g[i\rho, M_i], i\rho, M_i) \geq \frac{1}{mn + (n - 2)l} \left( \frac{2m}{n(n - 1)} \left( \sum_{1 \leq i < j \leq n} \sqrt{K_p^l (g[i\rho, M_i - M_j], i\rho, M_i - M_j)} \right)^2 + l \sum_{1 \leq i < j \leq n} K_p^l (i\rho, M_i + M_j) \right)
\]

\[
+ (m - l)K_p^l \left( i\rho, \sum_{i=1}^n M_i \right), \text{ for } l \geq m > 0,
\]

\[
\sum_{i=1}^n K_p^l (i\rho, M_i), i\rho, M_i) \geq \frac{1}{mn + (n - 2)l} \left( l \sum_{1 \leq i < j \leq n} K_p^l (i\rho, M_i + M_j), i\rho, M_i + M_j) \right)
\]

\[
+ m \sum_{1 \leq i < j \leq n} K_p^l (i\rho, M_i - M_j), i\rho, M_i - M_j) \right)
\]

\[
+ \frac{m - l}{(n - 1)^2} \left( \sum_{1 \leq i < j \leq n} \sqrt{K_p^l (i\rho, M_i + M_j), i\rho, M_i + M_j)} \right)^2 \right), \text{ for } l > m > 0.
\]

Multiply both sides of inequalities (23a), (23b), and (23c) by a constant \( \frac{f(0)}{2} \), we can derive respectively

\[
\sum_{i=1}^n f_p^l (M_i) \geq P_1, \text{ for } m \geq l > 0,
\]

\[
\sum_{i=1}^n f_p^l (M_i) \geq P_2, \text{ for } l \geq m > 0,
\]
\[
\sum_{i=1}^{n} I_{p}^{a}(M_{i}) \geq P_{3}, \quad \text{for } l > m > 0.
\] (26)

If we take \( m = 2, l = 1 \) for the inequality (24), and \( m = 1, l = 2 \) for the inequalities (25) and (26), then one gets
\[
\sum_{i=1}^{n} I_{p}^{a}(M_{i}) \geq P_{1},
\] (27)
\[
\sum_{i=1}^{n} I_{p}^{a}(M_{i}) \geq P_{2},
\] (28)
\[
\sum_{i=1}^{n} I_{p}^{a}(M_{i}) \geq P_{4},
\] (29)
respectively. For convenience, the lower bound of formula (21) is marked by \( Lb \), that is, \( Lb = \max\{P_{1}, P_{2}, P_{4}\} \).

By virtue of the relations between the norm inequalities presented in (B1), (B2), and (B3), we can derive that our lower bound \( Lb \) is more accurate than the lower bounds in [36–38]. A detailed illustration is shown in appendix C.1.

It is acknowledged that different results can be obtained by taking different Morozova-Chentsov functions for metric-adjusted skew information. Herein, we first consider the Morozova-Chentsov function with the form of equation (9). Meanwhile, the following results are obtained.

**Corollary 1.** For finite \( n \) observables \( M_{1}, M_{2}, \cdots, M_{n} (n \geq 2) \), the sum uncertainty relations with respect to Wigner-Yanase-Dyson skew information are that for \( m \geq 1 \geq 0 \) we can obtain
\[
\sum_{i=1}^{n} I_{p}^{a}(M_{i}) \geq \frac{1}{mn + (n - 2)l} \left\{ \frac{2l}{n(n - 1)} \left( \sum_{1 \leq i < j \leq n} \sqrt{I_{p}^{a}(M_{i} + M_{j})} \right)^{2} \right. \\
\left. \quad + m \sum_{1 \leq i < j \leq n} I_{p}^{a}(M_{i} - M_{j}) + (m - l)I_{p}^{a}\left( \sum_{i=1}^{n} M_{i} \right) \right\},
\] (30)

and for \( l \geq m > 0 \) one derives
\[
\sum_{i=1}^{n} I_{p}^{a}(M_{i}) \geq \frac{1}{mn + (n - 2)l} \left\{ \frac{2m}{n(n - 1)} \left( \sum_{1 \leq i < j \leq n} \sqrt{I_{p}^{a}(M_{i} - M_{j})} \right)^{2} \right. \\
\left. \quad + l \sum_{1 \leq i < j \leq n} I_{p}^{a}(M_{i} + M_{j}) + (m - l)I_{p}^{a}\left( \sum_{i=1}^{n} M_{i} \right) \right\},
\] (31)

and for \( l > m > 0 \) one reads
\[
\sum_{i=1}^{n} I_{p}^{a}(M_{i}) \geq \frac{1}{mn + (n - 2)l} \left\{ l \sum_{1 \leq i < j \leq n} I_{p}^{a}(M_{i} + M_{j}) + m \sum_{1 \leq i < j \leq n} I_{p}^{a}(M_{i} - M_{j}) \right. \\
\left. \quad + \frac{m - l}{(n - 1)^2} \left( \sum_{1 \leq i < j \leq n} \sqrt{I_{p}^{a}(M_{i} + M_{j})} \right)^{2} \right\}.
\] (32)

Thus we have \( \sum_{i=1}^{n} I_{p}^{a}(M_{i}) \geq \max\{\text{ineq30, ineq31, ineq32}\} \), where \( \text{ineq30, ineq31, and ineq32} \) represent the lower bounds of inequalities (30), (31), and (32), respectively.

When \( \alpha = \frac{1}{2} \), the inequalities (30), (31), and (32) can be further reduced to the uncertainty inequalities with respect to Wigner-Yanase skew information, as shown below. For \( m \geq l > 0 \) we get
\[
\sum_{i=1}^{n} I_{p}(M_{i}) \geq \frac{1}{mn + (n - 2)l} \left\{ \frac{2l}{n(n - 1)} \left( \sum_{1 \leq i < j \leq n} \sqrt{I_{p}(M_{i} + M_{j})} \right)^{2} \right. \\
\left. \quad + m \sum_{1 \leq i < j \leq n} I_{p}(M_{i} - M_{j}) + (m - l)I_{p}\left( \sum_{i=1}^{n} M_{i} \right) \right\},
\] (33)

and for \( l \geq m > 0 \) one obtains
\[
\sum_{i=1}^{n} I_{\phi}(M_i) \geq \frac{1}{mn + (n-2)l} \left\{ \frac{2m}{n(n-1)} \left( \sum_{1 \leq i < j \leq n} \sqrt{I_{\phi}(M_i - M_j)} \right)^2 + l \sum_{1 \leq i < j \leq n} I_{\phi}(M_i + M_j) + (m - l)I_{\phi} \left( \sum_{i=1}^{n} M_i \right) \right\},
\]
and for \( l > m > 0 \) one reads
\[
\sum_{i=1}^{n} I_{\phi}(M_i) \geq \frac{1}{mn + (n-2)l} \left\{ l \sum_{1 \leq i < j \leq n} I_{\phi}(M_i + M_j) + m \sum_{1 \leq i < j \leq n} I_{\phi}(M_i - M_j) + \frac{m-l}{(n-1)^2} \left( \sum_{1 \leq i < j \leq n} \sqrt{I_{\phi}(M_i + M_j)} \right)^2 \right\}.
\]
So we have \( \sum_{i=1}^{n} I_{\phi}(M_i) \geq \max \{ \text{ineq33, ineq34, ineq35} \} \), where ineq33, ineq34, and ineq35 represent the lower bounds of inequalities (33), (34), and (35), respectively.

It is highly natural to get that the lower bound \( \max \{ \text{ineq33, ineq34, ineq35} \} \) is superior to the lower bounds in [9, 34, 39]. This is because when we take \( \epsilon = e^{\Omega} \), a special case of metric-adjusted skew information, the relations in appendix C.1 also hold.

Next we present two examples in term of Wigner-Yanase-Dyson skew information to illustrate the superiority of our result. In the examples below we consider a special case, where we take \( m = 2, l = 1 \) for inequality (24), and \( m = 1, l = 2 \) for inequalities (25) and (26).

**Example 1.** Assume a qubit state \( \rho = \frac{1 + \rho_{s}}{2} \) with \( \rho = (\frac{\sqrt{3}}{2} \cos \theta, \frac{\sqrt{3}}{2} \sin \theta, 0) \), and regard Pauli operators \( \sigma_x, \sigma_y, \sigma_z \) as observables. The first three figures of figure 1 show the comparison of lower bounds for any \( \alpha \). The (a) depicts the lower bounds \( Lb \) and \( Lb_2 \). The difference value between the lower bound \( Lb \) and \( Lb_2 \) is illustrated in (b), and \( Lb - Lb_2 \) is nonnegative, that is, \( Lb \geq Lb_2 \). Similarly, the lower bounds \( Lb \) and \( Lb_3 \) are compared in (c), and \( Lb \geq Lb_3 \). Evidently, the lower bound \( Lb \) is larger than \( Lb_1, Lb_2, Lb_3 \). Considering a special case, we take \( \alpha = \frac{1}{\sqrt{2}} \). In figure 1(d), we only show the lower bounds \( Lb, Lb_3 \). And one can find that the lower bound obtained by us is closer to the sum \( I_{\rho}^{1/3}(\sigma_x) + I_{\rho}^{1/3}(\sigma_y) + I_{\rho}^{1/3}(\sigma_z) \).

**Example 2.** For a Gisin state \( \rho = \lambda |\varphi(\theta)\rangle \langle \varphi(\theta)| + (1 - \lambda) \sigma \) with \( |\varphi(\theta)\rangle = \sin \theta |01\rangle - \cos \theta |10\rangle \), \( \sigma = \frac{1}{2} (|00\rangle \langle 00| + |11\rangle \langle 11|) \), \( 0 \leq \lambda \leq 1 \), and \( 0 \leq \theta \leq 2\pi \). The operators \( I \otimes \sigma_x, I \otimes \sigma_y, I \otimes \sigma_z \) are viewed as observables. Herein, we take \( \alpha = \frac{1}{\sqrt{2}} \). The figure 2(a) depicts the lower bounds \( Lb \) and \( Lb_3 \). The difference values of lower bounds are shown in figure 2(b) which depicts \( Lb - Lb_3 \) and \( Lb - Lb_2 \), and they are nonnegative. Therefore, the lower bound \( Lb \) obtained by us is the most accurate.

**4. Uncertainty relations of finite quantum channels**

In this section, the different types of uncertainty relations associated with any finite quantum channels are presented in terms of metric-adjusted skew information, and the conclusions also hold for its special metrics, such as those mentioned above in section 2. In addition, we prove which of the two corresponding forms yields a better lower bound, and then an optimal lower bound is given. We also provide two examples for the sake of illustrating our results.

Given a quantum state \( \rho \) and a quantum channel \( \Phi \) represented by Kraus operators \( \Phi(\rho) = \sum_{j} K_{j} \rho K_{j}^{\dagger} \). In [36], Cai gave an uncertainty quantification associated with channel \( \Phi \) with regard to metric-adjusted skew information
\[
I_{\rho}(\Phi) = \sum_{j} I_{\rho}(K_{j}),
\]
where
\[
I_{\rho}(K_{j}) = \frac{m(c)}{2} K_{j}^{\dagger} [i |\rho, K_{j}\rangle, i |\rho, K_{j}\rangle] K_{j} + \frac{m(c)}{2} \operatorname{Tr} [i |\rho, K_{j}\rangle c (L_{\rho}, R_{\rho}) i |\rho, K_{j}\rangle].
\]
Analogously, $I^\alpha_\rho(\Phi)$ reduces to Wigner-Yanase-Dyson skew information $I^\alpha_\rho(\Phi)$ when $c = e^{i\alpha}$, the specific form is

$$I^\alpha_\rho(\Phi) = \sum_j I^\alpha_\rho(K_j) = -\frac{1}{2}\sum_j \text{Tr}[\rho^\alpha, K_j][\rho^{1-\alpha}, K_j].$$

(38)

When $\alpha = \frac{1}{2}$, $I^\alpha_\rho(\Phi)$ can turn into the form

$$I_\rho(\Phi) = \sum_j I_\rho(K_j) = -\frac{1}{2}\sum_j \text{Tr}[\rho^{1/2}, K_j][\rho^{1/2}, K_j]$$

(39)

which is Wigner-Yanase skew information associated with channel.
For arbitrary $N$ quantum channels $\Phi_1, \Phi_2, \ldots, \Phi_N$ ($N \geq 2$), and each channel $\Phi_t$ is represented by Kraus operators, i.e., $\Phi_t(\rho) = \sum_{j=1}^{n_t} K^t_j \rho K^t_j \dagger$, $t = 1, 2, \ldots, N$. In [37], Ren et al gave two sum uncertainty quantifications associated with channels,

$$\sum_{t=1}^{N} I^t_{\rho}(\Phi_t) \geq \max_{\pi_t, \eta_t \in S^k} \frac{1}{N-2} \left\{ \sum_{1 \leq \ell < s \leq N} I^t_{\rho}(K^t_{\pi(\ell)} + K^t_{\pi(s)}) \right\},$$

and

$$\sum_{t=1}^{N} I^t_{\rho}(\Phi_t) \geq \max_{\pi_t, \eta_t \in S^k} \left\{ \frac{1}{N} \sum_{j=1}^{n_t} I^t_{\rho} \left( \sum_{j=1}^{n_t} K^t_{\pi(j)} \right) + \frac{2}{N^2(N-1)} \left\{ \sum_{j=1}^{n_t} \sum_{1 \leq \ell < s \leq N} \sqrt{I^t_{\rho}(K^t_{\pi(\ell)} - K^t_{\pi(s)})} \} \right\}.\quad (41)$$

The formula (40) can be used when $N > 2$, while the formula (41) can be used when $N \geq 2$. For simplicity, the lower bounds in (40) and (41) are marked by $LB_1$ and $LB_2$, respectively.

Next, we will present the sum uncertainty relations of arbitrary finite $N$ quantum channels with regard to metric-adjusted skew information.

**Theorem 2.** For arbitrary $N$ quantum channels $\Phi_1, \Phi_2, \ldots, \Phi_N$ ($N \geq 2$), and each channel $\Phi_t$ is represented by Kraus operators, $\Phi_t(\rho) = \sum_{j=1}^{n_t} K^t_j \rho K^t_j \dagger$, $t = 1, 2, \ldots, N$, one reads

$$\sum_{t=1}^{N} I^t_{\rho}(\Phi_t) \geq \max \{ Q_1, Q_2, Q_3 \}.\quad (42)$$

Specially, we have

$$\sum_{t=1}^{N} I^t_{\rho}(\Phi_t) \geq \max \{ Q_1, Q_2, Q_3 \}.\quad (43)$$
where

\[
Q_1 = \max_{\pi, \pi' \in S_n} \frac{1}{mN + (N - 2)l} \left\{ \frac{2l}{N(N - 1)} \left( \sum_{j=1}^{n} \sum_{1 \leq i < j \leq N} \left( \sqrt{I_p^l(K_{\pi(j)}^l) + K_{\pi(j)}^l} \right)^2 \right) \right\} + m \sum_{1 \leq i < j \leq N} I_p^l(K_{\pi(j)}^l - K_{\pi(j)}^l) + (m - l) \sum_{j=1}^{n} I_p^l \left( \sum_{i=1}^{N} K_{\pi(i)}^l \right),
\]

\[
Q_2 = \max_{\pi, \pi' \in S_n} \frac{1}{mN + (N - 2)l} \left\{ \frac{2m}{N(N - 1)} \left( \sum_{j=1}^{n} \sum_{1 \leq i < j \leq N} \left( \sqrt{I_p^l(K_{\pi(j)}^l) - K_{\pi(j)}^l} \right)^2 \right) \right\} + l \sum_{1 \leq i < j \leq N} I_p^l(K_{\pi(j)}^l + K_{\pi(j)}^l) + (m - l) \sum_{j=1}^{n} I_p^l \left( \sum_{i=1}^{N} K_{\pi(i)}^l \right),
\]

\[
Q_3 = \max_{\pi, \pi' \in S_n} \frac{1}{mN + (N - 2)l} \left\{ \frac{1}{3N - 2} \left( \sum_{j=1}^{n} \sum_{1 \leq i < j \leq N} \left( \sqrt{I_p^l(K_{\pi(j)}^l) + K_{\pi(j)}^l} \right)^2 \right) \right\} + \frac{2}{N(N - 1)} \sum_{1 \leq i < j \leq N} I_p^l(K_{\pi(j)}^l - K_{\pi(j)}^l) + \sum_{j=1}^{n} I_p^l \left( \sum_{i=1}^{N} K_{\pi(i)}^l \right),
\]

\[
Q_4 = \max_{\pi, \pi' \in S_n} \frac{1}{mN + (N - 2)l} \left\{ \frac{2}{3N - 4} \left( \sum_{j=1}^{n} \sum_{1 \leq i < j \leq N} \left( \sqrt{I_p^l(K_{\pi(j)}^l) - K_{\pi(j)}^l} \right)^2 \right) \right\} + \frac{2}{N(N - 1)} \sum_{1 \leq i < j \leq N} I_p^l(K_{\pi(j)}^l + K_{\pi(j)}^l) + \sum_{j=1}^{n} I_p^l \left( \sum_{i=1}^{N} K_{\pi(i)}^l \right) - \frac{1}{(N - 1)^2} \left( \sum_{j=1}^{n} \sum_{1 \leq i < j \leq N} \left( \sqrt{I_p^l(K_{\pi(j)}^l) + K_{\pi(j)}^l} \right)^2 \right),
\]

\[
\pi, \pi' \in S_n \text{ are } n\text{-element permutations, and the range of the parameters } l, m \text{ for } Q_1, Q_2, \text{ and } Q_3 \text{ is } m \geq l > 0, l \geq m > 0, \text{ and } l > m > 0, \text{ respectively.}
\]

**Proof.** According to the norm inequalities (15), (16), and (17) of lemma 1, we can get

\[
\sum_{i=1}^{N} I_p^l(K_{\pi(j)}^l) \geq \frac{1}{mN + (N - 2)l} \left( \frac{2l}{N(N - 1)} \left( \sum_{1 \leq i < j \leq N} \left( \sqrt{I_p^l(K_{\pi(j)}^l) + K_{\pi(j)}^l} \right)^2 \right) \right) + m \sum_{1 \leq i < j \leq N} I_p^l(K_{\pi(j)}^l - K_{\pi(j)}^l) + (m - l) \sum_{j=1}^{n} I_p^l \left( \sum_{i=1}^{N} K_{\pi(i)}^l \right) \text{ for } m \geq l > 0, \tag{44a}
\]

\[
\sum_{i=1}^{N} I_p^l(K_{\pi(j)}^l) \geq \frac{1}{mN + (N - 2)l} \left( \frac{2m}{N(N - 1)} \left( \sum_{1 \leq i < j \leq N} \left( \sqrt{I_p^l(K_{\pi(j)}^l) - K_{\pi(j)}^l} \right)^2 \right) \right) + l \sum_{1 \leq i < j \leq N} I_p^l(K_{\pi(j)}^l + K_{\pi(j)}^l) + (m - l) \sum_{j=1}^{n} I_p^l \left( \sum_{i=1}^{N} K_{\pi(i)}^l \right) \text{ for } l \geq m > 0, \tag{44b}
\]
Both sides of these formulas sum over \( j \), we have
\[
\sum_{t=1}^{N} f_p^j(K_{\tau(j)}) \geq Q_1 \text{ for } m \geq l > 0,
\]
\[
\sum_{t=1}^{N} f_p^j(\Phi_t) \geq Q_2 \text{ for } l \geq m > 0,
\]
\[
\sum_{t=1}^{N} f_p^j(\Phi_t) \geq Q_3 \text{ for } l > m > 0.
\]
Specially, if we take \( m = 2, l = 1 \) for the inequality (45), and \( m = 1, l = 2 \) for the inequalities (46) and (47), one has
\[
\sum_{t=1}^{N} f_p^j(\Phi_t) \geq Q_1,
\]
\[
\sum_{t=1}^{N} f_p^j(\Phi_t) \geq Q_2,
\]
\[
\sum_{t=1}^{N} f_p^j(\Phi_t) \geq Q_3,
\]
respectively. Here the formulas of \( Q_1, Q_2, Q_3, \overline{Q}_1, \overline{Q}_2, \overline{Q}_3 \) are presented in theorem 2.

By means of the norm inequality (B2), for \( l \geq m > 0 \) we can derive
\[
\sum_{t=1}^{N} f_p^j(K_{\tau(j)}) \geq \frac{1}{mn + (N - 2)l} \left\{ \frac{2m}{N(N - 1)} \left( \sum_{1 \leq j < k \leq N} \sqrt{f_p^j(K_{\tau(j)})^2 - f_p^j(K_{\tau(k)})^2} \right)^2 \right\}
\]
\[
+ \frac{l}{N} \left( \sum_{1 \leq j < k \leq N} f_p^j(K_{\tau(j)})^2 - f_p^j(K_{\tau(k)})^2 \right)\left( \sum_{1 \leq j < k \leq N} \sqrt{f_p^j(K_{\tau(j)})^2 - f_p^j(K_{\tau(k)})^2} \right)^2,
\]
which leads to the result obtained by us being more accurate than the lower bound of inequality (41). In the same way, we can also demonstrate that \( Q_3 \) is greater than \( \overline{Q}_3 \) based on inequality (B3).

The above results can be appropriate for its special measures, such as the Wigner–Yanase–Dyson skew information, thus the conclusions can be drawn as follows.

**Corollary 2.** For arbitrary \( N \) quantum channels \( \Phi_1, \Phi_2, \ldots, \Phi_N \) \((N \geq 2)\), and each channel \( \Phi_t \) is represented by Kraus operators, \( \Phi_t(\rho) = \sum_{i=1}^{d} K_{t}^{i} \rho K_{t}^{i} \), \( t = 1, 2, \ldots, N \), for \( m \geq l \geq 0 \) one has
\[
\sum_{t=1}^{N} f_p^j(\Phi_t) \geq \frac{1}{mn + (N - 2)l} \left\{ \frac{2m}{N(N - 1)} \left( \sum_{1 \leq j < k \leq N} \sqrt{f_p^j(K_{\tau(j)})^2 - f_p^j(K_{\tau(k)})^2} \right)^2 \right\}
\]
\[
+ m \sum_{1 \leq j < k \leq N} f_p^j(K_{\tau(j)})^2 - f_p^j(K_{\tau(k)})^2 \left( \sum_{1 \leq j < k \leq N} \sqrt{f_p^j(K_{\tau(j)})^2 - f_p^j(K_{\tau(k)})^2} \right)^2,
\]
and for \( l \geq m \geq 0 \) we have
\[
\sum_{t=1}^{N} f_p^j(\Phi_t) \geq \frac{1}{mn + (N - 2)l} \left\{ \frac{2m}{N(N - 1)} \left( \sum_{1 \leq j < k \leq N} \sqrt{f_p^j(K_{\tau(j)})^2 - f_p^j(K_{\tau(k)})^2} \right)^2 \right\}
\]
\[
+ l \sum_{1 \leq j < k \leq N} f_p^j(K_{\tau(j)})^2 - f_p^j(K_{\tau(k)})^2 \left( \sum_{1 \leq j < k \leq N} \sqrt{f_p^j(K_{\tau(j)})^2 - f_p^j(K_{\tau(k)})^2} \right)^2,
\]
and for \( l > m > 0 \) one obtains

\[
\sum_{t=1}^{N} I_{\rho}(\Phi_t) \geq \max_{\pi,\pi' \in \mathcal{S}_n} \frac{1}{mN + (N - 2)l} \left\{ l \sum_{1 \leq t < s \leq N} \sum_{j=1}^{n} \sqrt{\frac{I_{\rho}(K_{\pi(j)}^i) + K_{\pi(j)}^s)}^{2}} \right\} + m \sum_{1 \leq t < s \leq N} \sum_{j=1}^{n} \left( I_{\rho}(K_{\pi(j)}^i) + K_{\pi(j)}^s) \right) ^{2},
\]

(54)

Therefore, \( \sum_{t=1}^{N} I_{\rho}(\Phi_t) \geq \max \{ \text{ineq52}, \text{ineq53}, \text{ineq54} \} \), where \( \text{ineq52} \), \( \text{ineq53} \), and \( \text{ineq54} \) represent the lower bounds of inequalities (52), (53), and (54), respectively.

When \( \alpha = \frac{1}{2} \), the three uncertainty inequalities of corollary 2 can be further simplified to Wigner-Yanase skew information. Based on the relations between our results and the previous ones in term of metric-adjusted skew information, when \( c = e^\omega \), it is natural to conclude that the lower bound of inequality (53) is superior to the lower bound of [34], theorem 3, and the lower bound of inequality (54) is more precise than the lower bound of [34], theorem 2.

The uncertainty quantification of channel \( \Phi \) with regard to metric-adjusted skew information can also be expressed in the form

\[
I_{\rho}(\Phi) = \frac{m(c)}{2} \text{Tr} \left\{ \sum_{j=1}^{n} i[\rho, K_{j}] \text{e}(L_{\rho}, R_{\rho}) i[\rho, K_{j}] \right\}
\]

\[
= \frac{m(c)}{2} \text{Tr}(\alpha^* \Lambda_n \otimes c(L_{\rho}, R_{\rho}) \alpha).
\]

(55)

Here \( \alpha^* = (i[\rho, K_{j}], \ldots, i[\rho, K_{j}]) \). Therefore, on the basis of the inequalities (15), (16), and (17), for \( m \geq l > 0 \) we have uncertainty relation

\[
\sum_{t=1}^{N} I_{\rho}(\Phi_t) \geq \max_{\pi,\pi' \in \mathcal{S}_n} \frac{1}{mN + (N - 2)l} \left\{ l \sum_{1 \leq t < s \leq N} \sum_{j=1}^{n} \sqrt{I_{\rho}(K_{\pi(j)}^i) + K_{\pi(j)}^s) \right\} + m \sum_{1 \leq t < s \leq N} \sum_{j=1}^{n} \left( I_{\rho}(K_{\pi(j)}^i) + K_{\pi(j)}^s) \right) ^{2},
\]

(56)

and for \( l > m > 0 \) one reads

\[
\sum_{t=1}^{N} I_{\rho}(\Phi_t) \geq \max_{\pi,\pi' \in \mathcal{S}_n} \frac{1}{mN + (N - 2)l} \left\{ l \sum_{1 \leq t < s \leq N} \sum_{j=1}^{n} \sqrt{I_{\rho}(K_{\pi(j)}^i) + K_{\pi(j)}^s) \right\} + m \sum_{1 \leq t < s \leq N} \sum_{j=1}^{n} \left( I_{\rho}(K_{\pi(j)}^i) + K_{\pi(j)}^s) \right) ^{2},
\]

(57)

and for \( l > m > 0 \) one derives

\[
\sum_{t=1}^{N} I_{\rho}(\Phi_t) \geq \max_{\pi,\pi' \in \mathcal{S}_n} \frac{1}{mN + (N - 2)l} \left\{ l \sum_{1 \leq t < s \leq N} \sum_{j=1}^{n} \sqrt{I_{\rho}(K_{\pi(j)}^i) + K_{\pi(j)}^s) \right\} + m \sum_{1 \leq t < s \leq N} \sum_{j=1}^{n} \left( I_{\rho}(K_{\pi(j)}^i) + K_{\pi(j)}^s) \right) ^{2},
\]

(58)

For simplicity, the lower bounds in (56), (57), and (58) are respectively marked by \( Z_{1}, Z_{2}, \) and \( Z_{3} \), let \( LB = \max \{ Z_{1}, Z_{2}, Z_{3} \} \), then \( \sum_{t=1}^{N} I_{\rho}(\Phi_t) \geq LB \).

In [38], Zhang et al provided three lower bounds \( LB1, LB2, \) and \( LB3 \), and the uncertainty relation \( \sum_{t=1}^{N} I_{\rho}(\Phi_t) \geq \max \{ LB1, LB2, LB3 \} \) (see reference [38] in detail).

According to the relations between the norm inequalities given by (B1), (B2), and (B3), it is not hard to show that the result \( LB \) derived by us is larger than the lower bound \( \max \{ LB1, LB2, LB3 \} \) in [38]. Detailed proof is provided in appendix C.2.

The above results (56), (57), and (58) are also satisfied for special cases of metric-adjusted skew information. Note that the lower bounds of inequalities (45) and (56), (46) and (57), (47) and (58) are not equal in general. That is to say, the lower bounds obtained by the two distinct expressions of the sum uncertainty relation
associated with channels are generally different. We compare these lower bounds in appendix D, and find the lower bounds in (56), (57), and (47), respectively, are greater than the lower bounds in (45), (46), and (58).

Therefore, max \{Z_1, Z_2, Q_3\} is more precise than the lower bounds \(\max\{Q_1, Q_2, Q_3\}\) and \(\max\{Z_1, Z_2, Z_3\}\). Then, we can get

\[
\sum_{\ell=1}^{N} I_{\ell}(\Phi_{2}) \geq \max\{Z_1, Z_2, Q_3\}. \tag{59}
\]

For simplicity, the right side of inequality (59) is marked by \(\mathbf{TB}\).

To illustrate the tightness of our results, we compare the results obtained by us with existing results. The following we will show two examples based on Wigner-Yanase-Dyson skew information where we take \(m = 2, l = 1\) for inequality (56), and \(m = 1, l = 2\) for inequalities (47) and (57). One is that each channel has the same number of Kraus operators, and the other is that each channel has a different number of Kraus operators.

**Example 3.** Assume a mixed state \(\rho = \frac{I+\theta\beta}{2}\) with \(\theta = \frac{\sqrt{2}}{2} \cos \theta, \frac{\sqrt{2}}{2} \sin \theta, 0 \leq \theta \leq \pi\), and three channels \(\Lambda(\rho) = \sum_{\ell=1}^{3} E_{\ell} \rho E_{\ell}^{\dagger}\) with \(E_{1} = \sqrt{1-\gamma}(|0\rangle \langle 0| + |1\rangle \langle 1|), E_{2} = \sqrt{\gamma}(|0\rangle \langle 1| + |1\rangle \langle 0|), \varepsilon(\rho) = \sum_{\ell=1}^{2} E_{\ell} \rho E_{\ell}^{\dagger}\) with \(F_{1} = \sqrt{1-\gamma}(|0\rangle \langle 0| + |1\rangle \langle 1|), F_{2} = \sqrt{\gamma}(|0\rangle \langle 1| + |1\rangle \langle 0|), \varepsilon(\rho) = \sum_{\ell=1}^{2} K_{\ell} \rho E_{\ell}^{\dagger}\) with \(K_{1} = |0\rangle \langle 0| + \sqrt{1-\gamma}|1\rangle \langle 1|, K_{2} = \sqrt{\gamma}|1\rangle \langle 1|\), are called bit-flipping channel \(\Lambda\), phase-flipping channel \(\varepsilon\), and amplitude damping channel \(\phi\), respectively, where \(0 \leq \gamma \leq 1\). Then according to (40) and (41), one has \(I_{\ell}^{p}(\Lambda) + I_{\ell}^{q}(\varepsilon) + I_{\ell}^{r}(\phi) \geq \max\{A_{1}, A_{2}, A_{3}, A_{4}\}\) and \(I_{\ell}^{p}(\Lambda) + I_{\ell}^{q}(\varepsilon) + I_{\ell}^{r}(\phi) \geq \max\{B_{1}, B_{2}, B_{3}, B_{4}\}\), where \(A_{j}, B_{j}\) \((j = 1, 2, 3, 4)\) are the lower bounds corresponding to \(\{\eta_{1} = (1), \eta_{2} = (1), \eta_{3} = (1), \eta_{4} = (1)\}\), \(\{\eta_{1} = (1), \eta_{2} = (12), \eta_{3} = (12), \eta_{4} = (12)\}\), \(\{\eta_{1} = (1), \eta_{2} = (12), \eta_{3} = (12), \eta_{4} = (12)\}\) and \(\{\eta_{1} = (1), \eta_{2} = (12), \eta_{3} = (12), \eta_{4} = (12)\}\). Analogously, one can get \(I_{\ell}^{p}(\Lambda) + I_{\ell}^{q}(\varepsilon) + I_{\ell}^{r}(\phi) \geq \max\{C_{1}, C_{2}, C_{3}, C_{4}\}\), \(\max\{D_{1}, D_{2}, D_{3}, D_{4}\}\), \(\max\{N_{1}, N_{2}, N_{3}, N_{4}\}\) by inequalities (56), (57), and (47), respectively. Here the lower bounds \(C_{j}, D_{j}, N_{j}\) are similar to \(A_{j}, B_{j}\), where \(j = 1, 2, 3, 4\). Here \(\eta_{1}, \eta_{2}, \eta_{3}, \eta_{4}\) need to take all of the binary permutations, but the lower bounds in the case \(\{\eta_{1} = (1), \eta_{2} = (1), \eta_{3} = (1)\}\) and the case \(\{\eta_{1} = (12), \eta_{2} = (12), \eta_{3} = (12)\}\) are same, similarly the lower bounds in the cases \(\{\eta_{1} = (1), \eta_{2} = (12), \eta_{3} = (12)\}\) and \(\{\eta_{1} = (12), \eta_{2} = (1), \eta_{3} = (1)\}\), \(\{\eta_{1} = (1), \eta_{2} = (12), \eta_{3} = (1), \eta_{4} = (1)\}\) and \(\{\eta_{1} = (12), \eta_{2} = (1), \eta_{3} = (1)\}\) are same, so we only need to consider four cases. When \(\alpha = \frac{1}{2}\) and \(\gamma = 0.7\), apparently, the lower bound \(\mathbf{TB}\) we had is always greater than the lower bounds \(\mathbf{TB}_{1}\) and \(\mathbf{TB}_{2}\), and our result \(\mathbf{TB}\) is highly close to \(I_{\ell}^{p/3}(\Lambda) + I_{\ell}^{p/3}(\varepsilon) + I_{\ell}^{p/3}(\phi)\), which is illustrated in figure 3(a). The figure 3(b) shows that the lower bound \(\mathbf{TB}\) is greater than the lower bound max \(\{LB_{1}, LB_{2}, LB_{3}\}\) in [38].

**Example 4.** Assume that the chosen quantum state is the same as in Example 3, we consider three channels here which are bit-flipping channel \(\Lambda\), phase-flipping channel \(\varepsilon\), and one unitary channel \(U\), respectively, where \(\Lambda(\rho) = \sum_{\ell=1}^{3} E_{\ell} \rho E_{\ell}^{\dagger}\) with \(E_{1} = \sqrt{1-\gamma}(|0\rangle \langle 0| + |1\rangle \langle 1|), E_{2} = \sqrt{\gamma}(|0\rangle \langle 1| + |1\rangle \langle 0|), \varepsilon(\rho) = \sum_{\ell=1}^{2} E_{\ell} \rho E_{\ell}^{\dagger}\) with \(F_{1} = \sqrt{1-\gamma}(|0\rangle \langle 0| + |1\rangle \langle 1|), F_{2} = \sqrt{\gamma}(|0\rangle \langle 1| + |1\rangle \langle 0|), 0 \leq \gamma \leq 1\), and \(U = \cos \frac{\sqrt{2}}{2}|0\rangle \langle 0| + \sin \frac{\sqrt{2}}{2}|1\rangle \langle 1| + \cos \frac{\sqrt{2}}{2}|0\rangle \langle 1| + \sin \frac{\sqrt{2}}{2}|1\rangle \langle 0|\). Since each channel has a different number of Kraus operators, we adopt the method of supplementing \(0\) proposed by Ren et al in [37]. Then we use the same procedure as in Example 3. When \(\alpha = \frac{1}{2}\) and \(\gamma = 0.7\), one can see that \(\mathbf{TB}\) is stronger than \(\mathbf{TB}_{1}\) and \(\mathbf{TB}_{2}\), which is illustrated in figure 4(a). Compared the lower bound \(\mathbf{TB}\) with the lower bound max \(\{LB_{1}, LB_{2}, LB_{3}\}\) in [38], as shown in figure 4(b), the result \(\mathbf{TB}\) is larger.

5. Conclusion

To sum up, we have obtained the new sum uncertainty relations with regard to metric-adjusted skew information of any finite observables and quantum channels by means of the norm inequalities we constructed, and proved our results are stronger than some results in [36–38]. The results also definitely hold for its special cases, and we have shown that our results are stronger than some results in [9, 34, 39] with respect to Wigner–Yanase skew information. For the two different uncertainty relations of channels, when utilizing the norm inequality (17), the lower bound derived directly by first form is better; when using the norm inequalities (15) and (16), the results yielded by second form are superior. Using this result we gave an optimal bound. Meanwhile, several specific examples were given to illustrate more clearly that the conclusions we have drawn are superior to the lower bounds in [36–38]. We think by using the general form of lemma 1, one can obtain much better result. It is hoped that our results can provide some reference for further research on sum uncertainty relations.
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Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

Appendix A. The proof of lemma 1

By using the equations

\[ \sum_{1 \leq i < j \leq n} \|x_i + x_j\|^2 = \left(\sum_{i=1}^{n} x_i\right)^2 + (n-2)\sum_{i=1}^{n} \|x_i\|^2, \]  

(A1)

and

\[ \sum_{1 \leq i < j \leq n} \|x_i - x_j\|^2 = n\sum_{i=1}^{n} \|x_i\|^2 - \left(\sum_{i=1}^{n} x_i\right)^2, \]  

(A2)

we can derive that

\[ \sum_{i=1}^{n} \|x_i\|^2 = \frac{1}{mn + (n-2)l} \left[ l \sum_{1 \leq i < j \leq n} \|x_i + x_j\|^2 + m \sum_{1 \leq i < j \leq n} \|x_i - x_j\|^2 + (m-l)\left(\sum_{i=1}^{n} x_i\right)^2 \right] \]  

(A3)

for arbitrary \(m, l \neq 0\) holds.

Then according to the inequality relations

\[ \sum_{1 \leq i < j \leq n} \|x_i \pm x_j\|^2 \geq \frac{2}{n(n-1)} \left(\sum_{1 \leq i < j \leq n} \|x_i \pm x_j\|^2 \right)^2, \]  

(A4)

we can get

\[ \sum_{i=1}^{n} \|x_i\|^2 \geq \frac{1}{mn + (n-2)l} \left[ \frac{2l}{n(n-1)} \left(\sum_{1 \leq i < j \leq n} \|x_i + x_j\|^2 \right) + m \sum_{1 \leq i < j \leq n} \|x_i - x_j\|^2 + (m-l)\left(\sum_{i=1}^{n} x_i\right)^2 \right] \]  

(A5)
and
\[
\sum_{i=1}^{n} \| x_i \|^2 \geq \frac{1}{mn + (n-2)l} \left[ l \sum_{1 \leq i < j \leq n} \| x_i + x_j \|^2 + \frac{2m}{n(n-1)} \left( \sum_{1 \leq i < j \leq n} \| x_i - x_j \| \right)^2 + (m - l) \left( \sum_{i=1}^{n} x_i \right)^2 \right] 
\]
for \( m, l \geq 0 \). Due to \( \| \sum_{i=1}^{n} x_i \|^2 \leq \frac{1}{(n-1)^2} \left( \sum_{1 \leq i < j \leq n} \| x_i + x_j \| \right)^2 \), when \( l > m > 0 \), we have
\[
\sum_{i=1}^{n} \| x_i \|^2 \geq \frac{1}{mn + (n-2)l} \left[ l \sum_{1 \leq i < j \leq n} \| x_i + x_j \|^2 + m \sum_{1 \leq i < j \leq n} \| x_i - x_j \|^2 + (m - l) \left( \sum_{i=1}^{n} x_i \right)^2 \right]. 
\]

For special case \( m = 2, l = 1 \), we obtain inequality (18). In the case \( m = 1, l = 2 \), one gets inequalities (19) and (20).

**Appendix B. The relation of norm inequalities**

Note that for \( m \geq l > 0 \) we have
\[
\sum_{i=1}^{n} \| x_i \|^2 \geq \frac{1}{mn + (n-2)l} \left[ \frac{2l}{n(n-1)} \left( \sum_{1 \leq i < j \leq n} \| x_i - x_j \| \right)^2 + 2 \sum_{1 \leq i < j \leq n} \| x_i + x_j \|^2 \right] 
\]
and the inequality (19) and the inequalities in [34, 35] have the relation

\[
\sum_{i=1}^{n} \|x_i\|^2 \geq \frac{1}{3n-4} \left[ \frac{2}{n(n-1)} \left( \sum_{1 \leq i < j \leq n} \|x_i - x_j\|^2 \right) + 2 \sum_{1 \leq i < j \leq n} \|x_i + x_j\|^2 - \left( \sum_{i=1}^{n} \|x_i\|^2 \right) \right]
\]

\[
\geq \frac{1}{2n-2} \left[ \frac{2}{n(n-1)} \left( \sum_{1 \leq i < j \leq n} \|x_i - x_j\|^2 \right) + \sum_{1 \leq i < j \leq n} \|x_i + x_j\|^2 \right]
\]

\[
\geq \frac{1}{n} \left[ \sum_{i=1}^{n} \|x_i\|^2 \right] + \frac{2}{n^2(n-1)} \left( \sum_{1 \leq i < j \leq n} \|x_i - x_j\|^2 \right), \quad \text{(B5)}
\]

and when \( n > 2 \), the relation between the inequality (20) and the inequality in [40] is

\[
\sum_{i=1}^{n} \|x_i\|^2 \geq \frac{1}{3n-4} \left[ \frac{2}{n(n-1)} \left( \sum_{1 \leq i < j \leq n} \|x_i + x_j\|^2 - \frac{1}{(n-1)^2} \left( \sum_{1 \leq i < j \leq n} \|x_i - x_j\|^2 \right) \right] \right]
\]

\[
> \frac{1}{n-2} \left[ \sum_{1 \leq i < j \leq n} \|x_i + x_j\|^2 - \frac{1}{(n-1)^2} \left( \sum_{1 \leq i < j \leq n} \|x_i - x_j\|^2 \right) \right]. \quad \text{(B6)}
\]

**Appendix C. The proof of the tightness of our results**

**C.1. The tightness of results relative to observables**

According to the relation between the norm inequality presented in (B1), for \( m \geq l > 0 \) one can directly get

\[
\sum_{i=1}^{n} K_{ij}^m (\bar{g}(\rho, M_i), i\bar{g}(\rho, M_j))
\]

\[
> \frac{1}{mn + (n-2)l} \left( \sum_{1 \leq i < j \leq n} K_{ij}^m (\bar{g}(\rho, M_i + M_j), i\bar{g}(\rho, M_i + M_j)) \right)^2
\]

\[
+ \frac{m}{2} \sum_{1 \leq i < j \leq n} K_{ij}^m (\bar{g}(\rho, M_i - M_j), i\bar{g}(\rho, M_i - M_j)) + (m - l) K_{ij}^m \left( i\bar{g}(\rho, x) \right)^2 \left( i\bar{g}(\rho, x) \right)
\]

\[
> \frac{1}{2n-2} \left[ \sum_{1 \leq i < j \leq n} K_{ij}^m (\bar{g}(\rho, M_i + M_j), i\bar{g}(\rho, M_i + M_j)) \right)^2
\]

\[
+ \sum_{1 \leq i < j \leq n} K_{ij}^m (\bar{g}(\rho, M_i - M_j), i\bar{g}(\rho, M_i - M_j)) \right), \quad \text{(C1)}
\]

Multiply by a constant \( \frac{f(l)}{2} \) on the formula (C1), we obtain the lower bound of inequality (24) is tighter than the lower bound of inequality (14) with \( z = 0 \). By the same method, we get the lower bound of inequality (25) is superior to the lower bounds of inequality (14) with \( z = 1 \) and inequality (13) for \( l \geq m > 0 \) based on (B2); the lower bound of inequality (26) is more accurate than the lower bound of inequality (12) for \( l > m > 0 \) by virtue of (B3). Therefore, the lower bound \( L_b = \max \{ P_3, P_2, P_1 \} \) obtained by us is more accurate than the lower bounds of inequalities (12), (13), and (14).

**C.2. The tightness of results relative to channels**

The accuracy of the lower bounds are actually compared by using the inequality relations in appendix B. We note the fact that

\[
\sum_{1 \leq i < j \leq N} \left( \sum_{i=1}^{n} \left( K_{ij}^m (\bar{g}(\rho, M_i + M_j), i\bar{g}(\rho, M_i + M_j)) \right)^2 \right) \right)
\]

\[
> \frac{1}{2n-2} \left[ \sum_{1 \leq i < j \leq n} K_{ij}^m (\bar{g}(\rho, M_i + M_j), i\bar{g}(\rho, M_i + M_j)) \right)^2
\]

\[
+ \sum_{1 \leq i < j \leq n} K_{ij}^m (\bar{g}(\rho, M_i - M_j), i\bar{g}(\rho, M_i - M_j)) \right), \quad \text{(C1)}
\]

Multiply by a constant \( \frac{f(l)}{2} \) on the formula (C1), we obtain the lower bound of inequality (24) is tighter than the lower bound of inequality (14) with \( z = 0 \). By the same method, we get the lower bound of inequality (25) is superior to the lower bounds of inequality (14) with \( z = 1 \) and inequality (13) for \( l \geq m > 0 \) based on (B2); the lower bound of inequality (26) is more accurate than the lower bound of inequality (12) for \( l > m > 0 \) by virtue of (B3). Therefore, the lower bound \( L_b = \max \{ P_3, P_2, P_1 \} \) obtained by us is more accurate than the lower bounds of inequalities (12), (13), and (14).
lower bound $Z_j$ is stronger than the lower bound $LB_1$. To sum up, the result max \{ $Z_1$, $Z_2$, $Z_3$ \} is more accurate than max \{ $LB_1$, $LB_2$, $LB_3$ \}.

Appendix D. The comparison of two kinds of lower bounds

Because $I_\rho^l$ is nonnegative, the key is to prove \( \sum_{1 \leq i < j \leq N} \frac{1}{2} (K^l_{i,j}(\rho)) \geq \sum_{1 \leq i < j \leq N} \frac{1}{2} (K^l_{i,j}(\rho)) \). If we set $I_\rho^l = I_\rho^l (K^l_{i,j}(\rho))$, then

\[
\sum_{1 \leq i < j \leq N} (\sum_{1 \leq i < j \leq N} I_\rho^l (K^l_{i,j}(\rho)) - \sum_{1 \leq i < j \leq N} I_\rho^l (K^l_{i,j}(\rho)))^2 \geq 0.
\]

Because $I_\rho^l$ holds based on the triangle inequality. Due to the arbitrariness of permutation, the above conclusion holds for every permutation.

Therefore, the lower bounds in (56), (57), and (47), respectively, are greater than the lower bounds in (45), (46), and (58).

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