Quantum Chaos of Unitary Fermi Gases in Strong Pairing Fluctuation Region

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The growth rate of the out-of-time-ordered correlator in a N-flavor Fermi gas is investigated and the Lyapunov exponent $\lambda_L$ is calculated to the order of $1/N$. We find that the Lyapunov exponent monotonically increases as the the interaction strength increases from the BCS limit to the unitary region. At the unitarity the Lyapunov exponent increases while the temperature drops and it can reach to the order of $\lambda_L \sim T$ around the critical temperature for the $N = 1$ case. The system scrambles faster for stronger pairing fluctuations. At the BCS limit, the Lyapunov exponent behaviors as $\lambda_L \propto e^{\mu/T} a^2 T^2 / N$.

I. INTRODUCTION

Information scrambling is a crucial stage in thermalization of a closed system. During this process the quantum entanglement spreads across all the freedoms of the system and the memory of the initial state is lost, which is taken as a key prerequisite for thermalization. Recently, the studies in gauge gravity duality have inspired some new insights into the quantum chaos\cite{1–8}. It is suggested that black holes are the fastest scramblers in nature\cite{1}. New insights into the quantum chaos\cite{1–8} have been conducted \cite{14–17}. Several experiments on measurement of OTOC (out-of-time-ordered correlator) have been conducted \cite{14–17}. Usually, in stead of the Lyapunov exponent have been studied at both high temperature and the memory of the initial state is lost, which is a key prerequisite for thermalization. Recently, this subject is revived by the discovery of an unexpected bound on the Lyapunov exponent that is extracted from OTOC (out-of-time-ordered correlator). The horizontal direction represents the real time evolution and the vertical direction represents the imaginary time evolution. It contains two real time folds, which are separated by $i\beta/2$.

In condensed matter physics, the systems usually don’t possess conformal symmetry. However, there exist some exceptions. At the critical point the conformal symmetry can emerge for low energy and long distance. Investigations have been done in this regime \cite{19, 23, 24}. In these system there are no quasi-particle excitations and the temperature is the only relevant scale. The Lyapunov exponents are found to obey the relationship of $\lambda_L \sim \kappa T$. The unitary Fermi gas is another example with scaling invariance. With the properties of highly controllable and hyper clean it can be a perfect playground to investigate the information scrambling\cite{22,26} and thermalization in closed quantum systems. At the unitary point, the non-relativistic conformal symmetry emerges and investigations have been taken to discuss its duality to a gravity theory\cite{27, 28}. The behaviors of the Lyapunov exponent have been studied at both high tem-

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\includegraphics[width=0.5\textwidth]{fig1.png}

FIG. 1. The complex time contour for calculating the out-of-time-ordered correlators. The horizontal direction represents the real time evolution and the vertical direction represents the imaginary time evolution. It contains two real time folds, which are seperated by $i\beta/2$.
temperature and low temperature limits \[29\]. However, it is more interesting to investigate the behavior around the critical temperature, where it has been shown more close to a non-Fermi liquid behavior \[22,23\].

In this work, we calculate the Lyapunov exponent of a N-flavor Fermi gas with tunable interaction. The OTOC is evulated by a series of ladder diagrams and the Lyapunov exponent is calculated to the order of 1/\[N\]. As the interaction strength increases from the BCS limit to the unitary regime we find that the Lyapunov exponent monotonically increases while the temperature is fixed. We also investigate the temperature dependence of the Lyapunov exponent at the unitarity. \(\lambda_s\) can increase to \(\lambda_L \sim T\) for \(N = 1\) case when the temperature is close to the critical temperature. Furthermore, we also find that the Lyapunov exponent behaves as \(\lambda_L \propto za_T^2T^2/N\) for high temperature at the BSC limit, where \(a_s \to 0^−\).

II. MODEL

We will start from a system with \(N\) fermion flavors. The Hamiltonian can be cast as

\[
\hat{H} = \int d^3 \mathbf{r} \left\{ \sum_{i\sigma} \hat{\psi}^\dagger_{i\sigma}(\mathbf{r})(-\nabla^2/2m - \mu)\hat{\psi}_{i\sigma}(\mathbf{r}) - \frac{g}{N} \sum_{ij} \hat{\psi}^\dagger_{i\downarrow}(\mathbf{r})\hat{\psi}^\dagger_{j\uparrow}(\mathbf{r})\hat{\psi}_{j\downarrow}(\mathbf{r})\hat{\psi}_{i\uparrow}(\mathbf{r}) \right\},
\]

where \(\hat{\psi}_{i\sigma}(\hat{\psi}^\dagger_{i\sigma})\) is the annihilation(creation) operator of the fermion field with flavor \(i\) and spin \(\sigma\). Parameter \(g\) is the bare interaction strength between the fermions. Here we assume the interaction strengths for different flavors are the same, and it can be related to a s-wave scattering length \(a_s\) by the following renormalization relation

\[
\frac{1}{g} = \frac{m}{4\pi a_s} + \int \frac{d^3 k}{(2\pi)^3} \frac{1}{2E_k},
\]

where \(E_k = k^2/2m\), and \(m\) is the mass of the fermions. By introducing an auxiliary bosonic field \(\varphi\) the four-fermion interaction term can be decoupled through the Hubbard-stratonovich transformation. Then in the imaginary time path integral formulism the partition function can be written as \(\mathcal{Z} = \int \mathcal{D}[\psi_{i\sigma}, \psi^\dagger_{i\sigma}, \varphi, \bar{\varphi}] e^{-S[\psi_{i\sigma}, \psi^\dagger_{i\sigma}, \varphi, \bar{\varphi}]}\), where the action \(S\) is

\[
S[\psi_{i\sigma}, \psi^\dagger_{i\sigma}, \varphi, \bar{\varphi}] = \int d\tau d^3 \mathbf{r} \left\{ \sum_{i\sigma} \bar{\psi}^\dagger_{i\sigma}(\mathbf{r}, \tau) \left( \frac{-\nabla^2}{2m} - \mu \right) \psi_{i\sigma}(\mathbf{r}, \tau) - \sum_{i} \bar{\varphi} \psi^\dagger_{i\uparrow} \psi^\dagger_{i\downarrow} \psi_{i\downarrow} + \frac{N\varphi\bar{\varphi}}{g} \right\}.
\]

In this work we set \(\hbar = 1\). The imaginary time Greens’ functions of fermion and boson are defined as \(\delta_{ij} \delta_{\sigma\sigma'} G(\tau, \mathbf{r}) = \langle \psi_{i\sigma}^\dagger(\mathbf{r}, \tau) \psi_{j\sigma'}(0, 0) \rangle\) and \(G(\tau, \mathbf{r}) = \langle \bar{\varphi}(\tau, \mathbf{r}) \varphi(0, 0) \rangle\), respectively. In the momentum space the free propagators can be simply expressed as

\[
G^{(0)}(i\omega_n^b, \mathbf{k}) = \frac{1}{i\omega_n - \epsilon_k + \mu},
\]

where \(\omega_n^b = (2n + 1)\pi/\beta\) and \(\omega_n = 2n\pi/\beta\) are the Matsubara frequencies for fermions and bosons, respectively, and \(\beta = 1/k_B T\). In order to calculate the Lyapunov exponent up to the order of \([1/N]\) we will involve the dressed propagators of fields \(\psi\) and \(\varphi\) as shown in Fig. 2. The dressed propagator of \(\varphi\) is a resummation of bubble diagrams. Then, it’s written as

\[
G(i\omega_n^b, \mathbf{k}) = \frac{g/N}{1 - g\Pi(i\omega_n^b, \mathbf{k})},
\]

where \(\Pi(i\omega_n^b, \mathbf{k})\) is the one-loop bubble

\[
\Pi(i\omega_n^b, \mathbf{q}) = \int \frac{d^3 \mathbf{k}}{(2\pi)^3} \frac{1}{(1 - n_F(\epsilon_k - \mu) - n_F(\epsilon_{q-k} - \mu)) - i\omega_n^b + \epsilon_k + \epsilon_{q-k} - 2\mu},
\]

\(n_F(\epsilon_k - \mu) = 1/\exp(\beta(\epsilon_k - \mu) + 1)\) is the Fermi-Dirac distribution function. The dressed propagator of field \(\psi_i\) is

\[
G(i\omega_n^f, \mathbf{k}) = \frac{1}{-i\omega_n^f + \epsilon_k - \mu - \Sigma(i\omega_n^f, \mathbf{k})},
\]

where the self-energy of fermions \(\Sigma(i\omega_n^f, \bar{\mathbf{k}})\) is expressed as

\[
\Sigma(i\omega_n^f, \bar{\mathbf{k}}) = \frac{1}{\gamma} \sum_{\omega_m} \int \frac{d^3 \mathbf{q}}{(2\pi)^3} \frac{G(i\omega_m^b, \mathbf{q})}{-i\omega_n^b + i\omega_m^b + \epsilon_{q-k} - \mu},
\]

The corresponding retarded Green’s functions are defined as usual as \(\delta_{ij} \delta_{\sigma\sigma'} G_R(t, \mathbf{r}) = \langle \psi_{i\sigma}(\mathbf{r}, t) \psi_{j\sigma'}(0, 0) \rangle\) and \(G_R(t, \bar{\mathbf{r}}) = \langle \varphi(\bar{\mathbf{r}}, t) \varphi(0, 0) \rangle\), where \(t(\bar{t})\) is the heaviside step function. In momentum space the forms of the retarded Green’s functions can be obtained by the analytic continuation of the Eq. 4 and 7 as \(G_R(\omega, \mathbf{k}) = G(i\omega_n^b \rightarrow \omega + i0^+, \mathbf{k})\) and \(G_R(\omega, \mathbf{k}) = G(i\omega_n^f \rightarrow \omega + i0^+, \mathbf{k})\). Then \(G_R(\omega, \mathbf{k})\) is written as

\[
G_R(\omega, \mathbf{k}) = \frac{1}{-\omega - i0^+ + \epsilon_k - \mu - \Sigma(\omega + i0^+, \mathbf{k})}.
\]

Hence, in the dressed retarded Green’s function the pole is modified by the self-energy. Working to the first order in \(\Sigma\) the pole can be approximately calculated as \(\omega^* = \epsilon_k - \mu - \text{Re}\Sigma(\epsilon_k - \mu + i0^+, \mathbf{k}) + i\Gamma(\mathbf{k})\), and the quantum scattering rate \(\Gamma(k)\) is defined as \(\Gamma(k) = -\text{Im}\Sigma(\epsilon_k - \mu + i0^+, \mathbf{k})\).

In order to evaluate the OTOC we need to define the symmetrized Wightman function as

\[
\delta_{ij} \delta_{\sigma\sigma'} G_W(t, \mathbf{r}) = \text{Tr}\left\{ \sqrt{\rho}\psi_{i\sigma}(t, \mathbf{r}) \sqrt{\rho}\psi_{j\sigma}^\dagger(0, 0) \right\},
\]
\[ \mathcal{G}_W(t, r) = \text{Tr} \{ \sqrt{p} \varphi(t, r) \sqrt{p} \varphi(0, 0) \}. \]  
\hspace{1cm} (10)

In the momentum space they can be written in terms of the spectral functions of fields \( \psi_{\sigma} \) and \( \varphi \) as

\[ G_W(\omega, k) = \frac{A_F(\omega, k)}{2 \cosh(\omega \beta / 2)}, \]
\[ \mathcal{G}_W(\omega, k) = \frac{A_B(\omega, k)}{2 \sinh(\omega \beta / 2)}, \]  
\hspace{1cm} (11)

The spectral functions can be calculated as the imaginary parts of the retarded Green’s functions, \( A_F(\omega, k) = -2 \text{Im} G_R(\omega, k) \) and \( A_B(\omega, k) = -2 \text{Im} \mathcal{G}_R(\omega, k) \).

### III. THE LYAPUNOV EXPONENT

In order to calculate the Lyapunov exponent it’s convenient to evaluate the “regulated” squared anti-commutator defined as \([19, 24]\)

\[ \mathcal{C}_1(t) = \frac{\theta(t)}{N^2} \sum_{i,j} \int d^3r \text{Tr} \left[ \sqrt{p} \psi_{i\uparrow}(t, r) \psi_{j\uparrow}^\dagger(0, 0) \right] \times \sqrt{p} \left( \psi_{i\downarrow}(t, r) \psi_{j\downarrow}^\dagger(0, 0) \right)^\dagger. \]  
\hspace{1cm} (12)

The factor \( 1/N^2 \) is to normalized the summation of indices \( i, j \). Since the system is symmetric about exchanging spin indice, without losing any generality we investigate the “regulated” squared anti-commutator of field \( \psi_{i\uparrow} \) as above. For the calculation up to the order of \( 1/N \) the squared anti-commutator \( \mathcal{C}_1 \) will couple to another squared anti-commutator \( \mathcal{C}_2 \) as demonstrated in Fig. 2 (c). The squared anti-commutator \( \mathcal{C}_2 \) is written as the following

\[ \mathcal{C}_2(t) = \frac{\theta(t)}{N^2} \sum_{i,j} \int d^3r \text{Tr} \left[ \sqrt{p} \psi_{i\uparrow}(t, r) \psi_{j\uparrow}^\dagger(0, 0) \right] \times \sqrt{p} \left( \psi_{i\downarrow}(t, r) \psi_{j\downarrow}^\dagger(0, 0) \right)^\dagger. \]  
\hspace{1cm} (13)

At the moment of \( t = 0 \) the above anti-commutators vanish because of \( r \neq 0 \). However, in chaotic system the time evolution of the operators may involve increasing degree of freedoms. As a result the fields become nonlocal at later time. It is conjectured that the squared anti-commutators will have an exponential growth \( \mathcal{C}_L(t) \sim e^{\lambda_L t} \) at short time. Analogously to the approach in ref. \[18\], in order to compute the \( \lambda_L \) to the leading order in \( 1/N \) we only keep the fastest-growing diagrams, which is a set of ladder diagrams as shown in Fig 2 (c). The “rungs” of the ladder correspond to the retarded Green’s functions. They are defined on the two real time folds. The two rails are separated by an imaginary time difference \( i\beta/2 \) and they are connected by “rungs”. The “rungs” correspond to the Wightman Green’s functions.

The Fourier transformation of \( \mathcal{C}_i(t) \) is denoted as \( \mathcal{C}_i(\omega) \) with \( \mathcal{C}_i(t) = \int d\omega e^{-i\omega t} \mathcal{C}_i(\omega) \). To sum up all the ladder series it’s convenient to define functions \( f_i(\nu; \omega, k) \) as

\[ \mathcal{C}_i(\nu) = \frac{1}{N} \int \frac{d\omega d^3k}{(2\pi)^4} f_i(\nu; \omega, k). \]  
\hspace{1cm} (14)

The lowest order of \( f_1(\nu; \omega, k) \) is simply expressed as \( G_R(\omega, k)G_R^*(\omega - \nu, k) \). Summation of all the ladder diagrams yields the Bethe-Salpeter equations

\[ f_1(\nu; \omega, k) = G_R(\omega, k)G_R^*(\omega - \nu, k) \left( 1 + \int \frac{d\omega' d^3k'}{(2\pi)^4} \right) \]
\[ \times \left( K_1(\nu; \omega, k; \omega', k') f_2(\nu; \omega', k') + K_2(\nu; \omega, k; \omega', k') f_1(\nu; \omega', k') \right), \]
\[ f_2(\nu; \omega, k) = G_R(\omega, k)G_R^*(\omega - \nu, k) \int \frac{d\omega d^3k'}{(2\pi)^4} K_1(\nu; \omega, \omega', k') f_1(\nu; \omega', k'), \]  
\hspace{1cm} (15)

where \( K_1 \) and \( K_2 \) are the integral kernels corresponding to the one-rung and two-rung diagrams in Fig 2 (c), respectively. They are written as

\[ K_1(\nu; \omega, k; \omega', k') = G_R(\omega + \omega, k + k'), \]
\[ K_2(\nu; \omega, k; \omega', k') = \frac{N}{(2\pi)^4} \int \frac{d\omega'' d^3k''}{(2\pi)^4} G_R(\omega'', k'') G_R^*(\omega'' - \nu, k'') \times G_W(\omega + \omega'', k + k') G_W(\omega + \omega'', k' + k''). \]  
\hspace{1cm} (16)

For the following calculation we will take several approximations. Firstly, one expects that the \( f_1(\nu; \omega, k) \) to be exponentially growing, while first term of \( f_1 \) in Eq. \[15\] will be decaying. Hence, this term can be safely dropped without affecting the evaluation of the
growth rate. Secondly, the pair of fermionic Green’s functions \( G_R(\omega, \mathbf{k}) | G_R(\omega - \nu, \mathbf{k}) \) in Eq. \((15)\) can be approximated as \( \frac{2\pi i\delta(\omega - \epsilon_k + \mu)}{\nu - 2\pi i\delta(\mathbf{k})} \). Thirdly, because in the above approximation all pairs of the retarded Green’s functions include a on-shell delta function, it’s natural to postulate the on-shell form of \( f_1(\nu; \omega, \mathbf{k}) \) as \( f_1(\nu; \omega, \mathbf{k}) \approx f_1(\nu; \mathbf{k}) \delta(\omega - \epsilon_k + \mu) \) \( [18][19] \). Please refer to the appendix A for the details of the approximation. With all above approximations the Bethe-Salpeter equations of Eq.(15) can be reduced to

\[
(-i\omega + 2T\tilde{\Gamma}(\tilde{k})) f_1(\omega; \tilde{k}) = \frac{T}{N} \int \frac{d\tilde{k}'\tilde{k}'}{k} \left( \tilde{K}_1(\tilde{k}, \tilde{k}') f_2(\omega; \tilde{k}') + \tilde{K}_2(\tilde{k}, \tilde{k}') f_1(\omega; \tilde{k}') \right),
\]

\[
(-i\omega + 2T\tilde{\Gamma}(\tilde{k})) f_2(\omega; \tilde{k}) = \frac{T}{N} \int \frac{d\tilde{k}'\tilde{k}'}{k} \tilde{K}_1(\tilde{k}, \tilde{k}') f_2(\omega; \tilde{k}'),
\]

where the momenta have been rescaled to be dimensionless as \( \tilde{k} = k/\sqrt{T} \) and \( \tilde{k}' = k'/\sqrt{T} \). Correspondingly we define a dimensionless quantum scattering rate \( \Gamma = \Gamma/T \). Here we have assumed the function \( f(\omega, \mathbf{k}) \) is rotationally invariant and integrated over the angles. Then the function \( f_{\nu}(\omega, \mathbf{k}) \) is reduced to \( f_{\nu}(\omega, \tilde{k}) \) in Eq. \((17)\). The dimensionless functions \( \tilde{K}_1 \) and \( \tilde{K}_2 \) are written as

\[
\tilde{K}_1(\tilde{k}, \tilde{k}') = N \int \frac{d\tilde{p}d\tilde{p}'}{(2\pi)^2} \tilde{G}_R(\tilde{p}, \tilde{p}') \tilde{G}_W(\tilde{p}, \tilde{p}') \Theta(\tilde{k}, \tilde{k}') f_1(\omega; \tilde{k}'),
\]

\[
\tilde{K}_2(\tilde{k}, \tilde{k}') = N^2 \int \frac{d\tilde{k}''d\tilde{k}'''}{(2\pi)^4} \frac{|\tilde{G}_R(\tilde{k}'', \tilde{k''')}|^2}{128\pi^5} \frac{2\tilde{\Gamma}(\tilde{k}, \tilde{k}')}{\cosh(\frac{\tilde{k}''+\tilde{k}'''}{2}) \cosh(\frac{\tilde{k}''-\tilde{k}'''}{2})},
\]

where \( \tilde{k}' = k'/\sqrt{T} \), \( \tilde{\omega}' = \omega'/T \), \( \tilde{\epsilon}_k = \epsilon_k/T \) and \( \tilde{\mu} = \mu/T \) and the bosonic retarded Green’s functions and Wightman function are also rescaled to be dimensionless by \( \tilde{G}_R = \sqrt{T} \tilde{G}_R \) and \( \tilde{G}_W = \sqrt{T} \tilde{G}_W \). The \( \Theta \) function is defined as \( \Theta(k, k', k'') = \Theta(2k'k'' + \epsilon_k' - \epsilon_k'' + \mu + k'^2 - k'^2) \). To more easily solve for the Lyapunov exponent the Bethe-Salpeter equations of Eq. \((17)\) can be written in a simply form

\[
-i\omega \mathbf{F}(\omega; \tilde{k}) = \frac{T}{N} \int d\tilde{k}' \mathbf{S}(\tilde{k}, \tilde{k}') \mathbf{F}(\omega; \tilde{k}'),
\]

where \( \mathbf{F}(\omega; \tilde{k}) = (\tilde{k} f_1(\omega; \tilde{k}), \tilde{k} f_2(\omega; \tilde{k})) \) and the dimensionless integral kernel \( \mathbf{S}(\tilde{k}, \tilde{k}') \) is defined as following

\[
\mathbf{S}(\tilde{k}, \tilde{k}') = \begin{pmatrix}
\tilde{K}_2(\tilde{k}, \tilde{k}') - 2N\tilde{\Gamma}(\tilde{k}') \delta(\tilde{k} - \tilde{k}') & \tilde{K}_1(\tilde{k}, \tilde{k}') \\
\tilde{K}_1(\tilde{k}, \tilde{k}') & -2N\tilde{\Gamma}(\tilde{k}') \delta(\tilde{k} - \tilde{k}')
\end{pmatrix}.
\]

We do not know how to solve Eq. \((19)\) analytically. However, it can be solved numerically by discretizing the momenta \( \tilde{k} \) and \( \tilde{k}' \) in the integral kernel \( \mathbf{S}(\tilde{k}, \tilde{k}') \). Then, the integral becomes the summation over the discrete momenta and Eq. \((19)\) can be written as

\[
-i\omega \mathbf{F}(\omega; \tilde{k}_i) = \frac{T}{N} \sum_{\tilde{k}_j} \mathbf{S}(\tilde{k}_i, \tilde{k}_j) \mathbf{F}(\omega; \tilde{k}_j),
\]

where \( \tilde{k}_i \) is the discrete momentum with a small internal. Obviously \( -i\omega \) is given by the eigenvalues of the kernel \( \mathbf{S}(\tilde{k}_i, \tilde{k}_j) \) multiplied by a factor \( T/N \). The Lyapunov exponent corresponds to the largest eigenvalue. Please refer to the appendix B for the details of the numerical calculation of the Lyapunov exponent.

**IV. QUANTUM CHAOS AT THE UNITARY POINT**

In this section we study the case of unitary Fermi gases by setting \( N = 1 \). This is not a fully controllable choice.
However, since we only focus on the variations of the Lyapunov exponent with respect to the scattering length \( a_s \) and the temperature, it may generate qualitative correct interpretation as the large \( N \) cases and inspire useful insight. In Fig. 3(a) we plot \( \lambda_L/T \) as a function of \( 1/a_s k_F \) for fixed temperature \( T/T_c = 0.24, 0.33 \) and 0.44. If we compare these temperatures with the critical temperature calculated in the Nozières and Schmitt-Rink (NSR) scheme [34, 35], which is \( T_c = 0.22 T_F \) at \( 1/a_s k_F = 0 \), they can be written as \( T/T_c = 1.1, 1.5, \) and 2.0. One observes that the Lyapunov exponent monotonically increases as \( 1/a_s k_F \) goes from the BCS limit to the unitary regime. For lower temperature the \( \lambda_L \) increases much faster than the higher temperature cases. At the unitary point \( 1/a_s k_F = 0 \) we plot \( \lambda_L/T \) as a function of temperature \( T/T_F \) in Fig. 3(b). As the temperature drops the Lyapunov exponent monotonically increases and approaches the upper bound \( 2 \pi T \). At the temperature of \( T/T_F = 0.24 \), which corresponds to \( T/T_c = 1.1 \) the Lyapunov exponent can reach a value of \( \lambda_L \approx 3.2 T \). Here we would like to point out that we won’t be able to explore the region very close to \( T_c \), where our numerical calculation becomes unstable since the propagator of Eq. 15 diverges at \( T_c \).

V. THE BEHAVIORS IN THE BCS LIMIT

At the BCS limit the scattering length \( a_s \to 0^- \). The retarded Green’s function \( G_R \) of the field \( \varphi \) can be expanded in terms of small \( a_s \) as the following

\[
G_R(\omega, k) = -\frac{1}{a_s} + \int \frac{d^3k}{(2\pi)^3} \frac{1}{2 \omega} = -\frac{1}{a_s} + \int \frac{d^3k}{(2\pi)^3} \frac{1}{2 \omega} - \int \frac{d^3k}{(2\pi)^3} \frac{1}{a_s} \approx \frac{1}{N}.
\]

Notice that the temperature must be far from the critical temperature. Otherwise, according to the Thouless criterion one has

\[
\int \frac{d^3k}{(2\pi)^3} \frac{1}{a_s} \approx 0 \quad \text{when} \quad T \to T_c
\]

\[
\int \frac{d^3k}{(2\pi)^3} \frac{1}{2 \omega} \to 0 \quad \text{when} \quad T \to T_c
\]

It's worth mentioning that the behavior of \( \lambda_L \) at the BCS limit was calculated as \( \lambda_L \propto a_s^2 T^2 \), which is consistent with the Fermi liquid theory.

VI. CONCLUSIONS

We have computed the Lyapunov exponent for a N-flavor Fermion system using 1/N expansion. The variation of the Lyapunov exponent with respect to the scattering length \( a_s \) and the temperature \( T \) has been investigated. When \( T \) is fixed the Lyapunov exponent monotonically increases as \( 1/a_s k_F \) increases from the BCS limit to the unitary regime. When the scattering length is fixed to \( 1/a_s k_F = 0 \) the Lyapunov exponent increases while the temperature drops. Around the critical temperature it can reach to the order of \( \lambda_L \sim T \) for \( N = 1 \) case. Basically, our results indicate that with strong pairing fluctuations the system exhibits strong chaos. Furthermore, the behavior of \( \lambda_L \) at the BCS limit was calculated as \( \lambda_L \propto a_s^2 T^2 / N \), which is consistent with the Fermi liquid theory.

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Appendix A: Approximations for the reduction of Eq. (17)

With the first approximation the first term of \( f_1 \) in Eq. (15) is dropped. Then the Bethe-Salpeter equations in Eq. (15) is reduced to

\[
f_1(\nu; \omega, k) = G_R(\omega, k) G_R^*(\omega - \nu, k) \int \frac{d\nu' d\nu''}{(2\pi)^4} \left( K_1(\nu; \omega, k; \omega', k') f_2(\nu; \omega', k') + K_2(\nu; \omega, k; \omega', k') f_2(\nu; \omega', k') \right),
\]

\[
f_2(\nu; \omega, k) = G_R(\omega, k) G_R^*(\omega - \nu, k)
\]
\[ \int \frac{d\omega' d^3k'}{(2\pi)^4} K_1(\nu; \omega, k; \omega', k') f_1(\nu; \omega', k'), \quad (A1) \]

The second approximation is performed on the pair propagators \( G_R(\omega, k)G_R^*(\omega - \nu, k) \). In the free fermion case it’s expressed as

\[ G_R(\omega, k)G_R^*(\omega - \nu, k) = \frac{1}{\omega - \epsilon_k + \mu + i0^+} \frac{1}{\omega - \nu - \epsilon_k + \mu + i0^+}. \quad (A2) \]

The integration over \( \omega \) can be evaluated by the method of residue. Then it’s straightforward to yield

\[ G_R(\omega, k)G_R^*(\omega - \nu, k) = \frac{2\pi i \delta(\omega - \epsilon_k + \mu)}{\nu + 2i \Gamma(k)}. \quad (A3) \]

The approximation is taken by replacing the \( 0^+ \) by the scattering rate \( \Gamma(k) \) for the interacting case. Then, the Eq. (A1) can be written as

\[ (-i\nu + 2\Gamma(k)) f_1(\nu; k) = \int \frac{d\omega' d^3k'}{(2\pi)^4} 2\pi \delta(\omega - \epsilon_k + \mu) \]
\[ \left( K_1(\nu; \omega, k; \omega', k') f_2(\nu; k') + K_2(\nu; \omega; k', \omega', k') f_1(\nu; k') \right), \]
\[ (-i\nu + 2\Gamma(k)) f_2(\nu; k) = \frac{2\pi i \delta(\omega - \epsilon_k + \mu)}{\nu + 2i \Gamma(k)} \]
\[ K_1(\nu; \omega, k; \omega', k') f_1(\nu; k'). \quad (A4) \]

As discussed in the main text the third approximation is to postulate the on-shell form \( f_i(\nu; \omega, k) \approx f_i(\nu; k) \delta(\omega - \epsilon_k + \mu) \). Then the Eq. (A1) can be written as

\[ \left( -i\nu + 2\Gamma(k) \right) f_1(\nu; k) = \int \frac{d\omega' d^3k'}{(2\pi)^4} 2\pi \delta(\omega - \epsilon_k + \mu) \]
\[ \left( K_1(\nu; \omega, k; \omega', k') f_2(\nu; k') + K_2(\nu; \omega; k', \omega', k') f_1(\nu; k') \right), \]
\[ \left( -i\nu + 2\Gamma(k) \right) f_2(\nu; k) = \frac{2\pi i \delta(\omega - \epsilon_k + \mu)}{\nu + 2i \Gamma(k)} \]
\[ K_1(\nu; \omega, k; \omega', k') f_1(\nu; k'). \quad (A5) \]

Assuming \( f_i(\nu; k') \) is rotationally invariant and performing the integration by implementing the delta function \( \delta(\omega - \epsilon_k + \mu) \) one obtains the Eq. (17).

**Appendix B: Remarks on Numerical Technique**

To numerically solve for the Lyapunov exponent we first discretize the momenta \( k \) and \( k' \) of the integral kernel \( S(k, k') \) in Eq. (20) into \( N_{size} \) pieces. The cutoffs of momenta \( k \) and \( k' \) are set to \( \Lambda = 15 \). We have also checked the convergence of the results by performing the calculation for larger cutoffs. The kernel \( S(k, k') \) is symmetric for exchanging \( k \) and \( k' \). Then it can be easily diagonalized to obtain the eigenvalues, which are denoted as \( \lambda_i \) here. The Lyapunov exponent is related to the largest eigenvalue as \( \lambda_L N/T = \text{max}(\lambda_i) \). Then the same calculation is performed for different \( N_{size} \) and the corresponding value of \( \lambda_L N/T \) is obtained. As an example we illustrate the case of \( 1/a_s k_F = 0 \) and \( T/T_F = 0.24 \) in Fig. 4. The final value of \( \lambda_L N/T \) is read by the extrapolation to \( 1/N_{size} = 0 \).

**Appendix C: Behaviors at BCS limit**

At the BCS limit one has \( a_s^{-1} \to -\infty \). Then the asymptotic behaviors of various propagators and the scattering rate \( \Gamma(k) \) are demonstrated as the following. The full propagator of field \( \varphi \) is

\[ G_R(\omega, k) = \frac{1}{1 - \Pi(\omega, k)} = \frac{1}{1 - 2\pi i \Gamma(k)} \]

where

\[ Re = -\frac{m}{4\pi a_s} + \frac{1}{(2\pi)^3} \frac{1}{2\epsilon_k} \int \frac{d^3k}{(2\pi)^3} \left[ \frac{1}{\omega + \epsilon_k + \epsilon_{q-k} - 2\mu} \right] \]
\[ Im = -\pi \int \frac{d^3k}{(2\pi)^3} \left[ \frac{1}{\omega + \epsilon_{q-k} - 2\mu} \right] \]
\[ \delta(-\omega + \epsilon_{q-k} - 2\mu). \quad (C2) \]

After we rescale all the momenta and frequency by \( k \to k/\sqrt{T}, q \to q/\sqrt{T} \) and \( \omega \to \omega/T \) it’s straightforward to get the following asymptotic behaviors for large \( a_s^{-1} \)

\[ Re \propto a_s^{-1}, \]
\[ Im \propto \sqrt{T}. \quad (C3) \]

Notice that the temperature here must be far from the superfluid critical temperature, otherwise \( Re \to 0 \). Then for large \( a_s^{-1} \) the propagator \( G_R(\omega, k) \) behaves as

\[ G_R(\omega, k) \propto a_s/N. \quad (C4) \]

The imaginary part of \( G_R(\omega, k) \) is

\[ \text{Im} G_R(\omega, k) = -\frac{1}{N} \frac{Im}{Re^2 + Im^2} \propto a_s^2 \sqrt{T}/N. \quad (C5) \]

The Wightman function of field \( \varphi \) behaves as

\[ G_W(\omega_k - 2\mu, k) = \frac{A_B(\omega_k - 2\mu, k)}{2\sinh((\omega_k - 2\mu)\beta/2)} \]
= - \text{Im} G_R(\omega_k - 2\mu, k) / \sinh((\omega_k - 2\mu)/2)) \sim \Im \Sigma(\omega_k + i0^+, k).

(C6)

The self-energy of fermions is

$$\Sigma(\omega^*_n, k) = \frac{1}{\beta} \sum_{\omega^*_m} \int \frac{d^3q}{(2\pi)^3} \frac{G(\omega^*_m, q)}{-i \omega^*_m + \omega^*_n + \epsilon_{q-k} - \mu},$$

(C7)

where the summation over $\omega^*_m$ is equivalent to a contour integration as the following

$$\Sigma(\omega^*_n, k) = \int \frac{d^3q}{(2\pi)^3} \left( \int \frac{dz}{2\pi i} n_B(z)(G_R(z, q) - G_A(z, q)) \right.$$

$$\left. - G(\omega^*_n + \epsilon_{q-k} - \mu, q)n_F(\epsilon_{q-k} - \mu) \right).$$

(C8)

where $G_A$ is the advanced Green’s function for field $\varphi$. After we take a analytical continuation the imaginary part of the self-energy can be calculated as

$$\Im \Sigma(\omega + i0^+, k) = - \int \frac{d^3q}{(2\pi)^3} \left( n_F(\epsilon_{q-k} - \mu)A_B(\omega + \epsilon_{q-k} - \mu) \right.$$  

$$+ \int dz A_B(z) \delta(-z + \omega + \epsilon_{q-k} - \mu)n_B(z) \right).$$

(C9)

The quantum scattering rate is defined as $\Gamma(k) = -\Im \Sigma(\epsilon_k - \mu + i0^+, k)$, then it can be written as

$$\Gamma(k) = \int \frac{d^3q}{(2\pi)^3} G_W(\epsilon_k + \epsilon_{q-k} - 2\mu, q) \cosh((\epsilon_k - \mu)/2) / \cosh((\epsilon_{q-k} - \mu)/2).$$

(C10)

As we have derived in Eq. (C6) the asymptotic behavior of the Wightman function is $G_W(\epsilon_k + \epsilon_{q-k} - 2\mu, q) \sim \Im \Sigma(\omega_k + i0^+)$, then the asymptotic behavior of the quantum scattering rate for large $a_s^{-1}$ is as the following

$$\Gamma(k) \propto a_s^2 T^2 / N.$$  

(C11)

With all above asymptotic forms of $G_R(\omega, k)$, $G_W(\omega, k)$ and $\Gamma(k)$ straightforward calculation yields

$$\tilde{K}_1(\tilde{k}, \tilde{k}') \propto a_s^2 T,$$

$$\tilde{K}_2(\tilde{k}, \tilde{k}') \propto a_s^2 T,$$

(C12)

and hence

$$S(\tilde{k}, \tilde{k}') \propto a_s^2 T.$$ 

(C13)

Then the asymptotic behavior of Lyapunov exponent $\lambda_L$ for large $a_s^{-1}$ is

$$\lambda_L \propto T(a_s^2 T)/N = a_s^2 T^2 / N.$$ 

(C14)

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