QUASI-ARITHMETICITY OF LATTICES IN $\text{PO}(n,1)$

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Abstract. We show that the non-arithmetic lattices in $\text{PO}(n,1)$ of Belolipetsky and Thomson [BT11], obtained as fundamental groups of closed hyperbolic manifolds with short systole, are quasi-arithmetic, and, by contrast, the well-known non-arithmetic lattices of Gromov and Piatetski-Shapiro are not quasi-arithmetic.

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1. Introduction

1.1. Background, motivation and discussion. The reader will find precise statements of theorems in §1.3 (p.5), following the definitions given in §1.2. Beforehand, we give some background and context.

The study of locally symmetric spaces is often re-framed as the study of discrete subgroups of semisimple Lie groups and in particular of those groups acting with finite co-volume (which are usually called ‘lattices’). A well known source of examples of lattices in semisimple Lie groups are the ‘arithmetic’ lattices; i.e., subgroups of algebraic groups (over $\mathbb{Q}$) that are in some sense defined over $\mathbb{Z}$. That the arithmetic subgroups of semisimple Lie groups have finite co-volume was shown by A. Borel and Harish-Chandra [BHC62], who also obtained co-compactness criteria. (The co-compactness criteria were also obtained by G. Mostow and T. Tamagawa [MT62].) If the real rank of a semisimple Lie group $G$ is at least 2, then by results of G. Margulis it is known that any lattice in $G$ is arithmetic [Mar91, (A), p.298]. For the case of real rank 1 Lie groups, and in particular for $\text{PO}(n,1)$ (the isometry group of real hyperbolic $n$-space), it was not until the work...
of M. Gromov and I. Piatetski-Shapiro [GPS87] that one knew that there are non-arithmetic lattices in \( \text{PO}(n, 1) \), for every \( n \geq 2 \), alongside the previously known arithmetic ones. The examples of Gromov and Piatetski-Shapiro arise as fundamental groups of finite-volume hyperbolic manifolds, constructed by ‘gluing’ together pieces of non-commensurable arithmetic hyperbolic manifolds along isometric totally geodesic boundaries. Before this point, examples of non-arithmetic lattices in \( \text{PO}(n, 1) \) were known for some small \( n \), and these arise as reflection groups whose non-arithmeticity may be deduced from criteria determined by E. Vinberg [Vin67] [VS93, 3.2, p.227].

In his work on the arithmeticity of these reflection groups, Vinberg introduced the class of ‘quasi-arithmetic’ lattices (in a given Lie group) [Vin67, p.437]. Every arithmetic group is quasi-arithmetic, whilst on the other hand Vinberg himself gave examples of lattices that are quasi-arithmetic but not arithmetic (lattices that we will call ‘properly quasi-arithmetic’). However, Vinberg’s examples are only given for dimensions no greater than 4, and it appears that his definition has not since been considered a great deal in the literature, save once, to the author’s knowledge [HLMA92].

It was more recently shown by M. Belolipetsky and S. Thomson [BT11] that one may obtain a class of non-arithmetic lattices in \( \text{PO}(n, 1) \), by a construction of closed hyperbolic manifolds with very short closed geodesics:

**Theorem** (B.–T. [BT11]). Let \( \varepsilon > 0 \) and let \( n \geq 2 \). Then there exists a closed hyperbolic \( n \)-manifold \( M \) such that \( M \) contains a non-contractible closed geodesic of length less than \( \varepsilon \). Moreover, for \( \varepsilon \) small enough (depending on some parameters in the construction of \( M \)), \( M \) is a non-arithmetic hyperbolic manifold.

This construction involves ‘gluing’ in the spirit of Gromov and Piatetski-Shapiro, but in these more recent examples the manifolds being glued together are still commensurable with one another and so one might expect that the ‘non-arithmeticity’ that arises should be ‘weaker’ than that of Gromov and Piatetski-Shapiro. Here, we show that the non-arithmetic examples of Belolipetsky and Thomson are, in fact, properly quasi-arithmetic lattices (§2.2).

In order to establish a distinction between the two above classes of examples of non-arithmetic lattices, we also show here that the lattices of Gromov and Piatetski-Shapiro are not quasi-arithmetic (§3.2).

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**1.2. Definitions and foundational material.** Here we give a definition of quasi-arithmetic lattices and establish some notation for the rest of the article. For a general reference on the theory of arithmetic groups the reader
may wish to consult the books of Margulis [Mar91] and Platonov and Rapinchuk [PR94]; whereas for an introduction to the subject, the book of Morris [WM14] (for example) is directed towards the newcomer.

**Definition 1.1** (Admissible algebraic groups). Let $G$ be a connected semisimple real Lie group without compact factors and with trivial centre. We say that an algebraic group $G$ is admissible for $G$ if

1. $G$ is defined over $\mathbb{Q}$;
2. there exists a surjective homomorphism $\phi : G(\mathbb{R})^0 \to G$; and
3. the homomorphism $\phi$ has compact kernel.

We think of the algebraic group $G$ as a model for the Lie group $G$, whose algebraic structure allows us to easily obtain interesting classes of subgroups as described below. From a geometric point of view, we are interested in these subgroups’ images in $G$.

Recall that a discrete subgroup $\Gamma < G$ is a lattice if the quotient $\Gamma \backslash G$ has finite volume induced by the Haar measure on $G$.

**Definition 1.2** (Quasi-arithmeticity). Suppose that $G$ is as in Definition 1.1 and that $\Gamma < G$ is a lattice. We say that $\Gamma$ is quasi-arithmetic if

1. there exists an admissible algebraic group $G$ for $G$;
2. there exists a finite-index subgroup $\Gamma' \subseteq \Gamma$ such that $\Gamma' \subseteq \phi(G(\mathbb{Q}))$.

We will say that $\Gamma$ is arithmetic if, in addition to being quasi-arithmetic, the following stronger statement holds:

3. $\Gamma$ is commensurable with $\phi(G(\mathbb{Z}))$.

(Note that (3) implies (2).)

1.2.1. **Hyperbolic space.** We adopt the Lorentz model of hyperbolic $n$-space, where we first equip $\mathbb{R}^{n+1}$ with the quadratic form $f_n$, given in the standard basis by

$$f_n(x) = -x_0^2 + x_1^2 + \cdots + x_n^2,$$

and then denote

$$\mathbb{H}^n = \{x \in \mathbb{R}^{n+1} \mid f_n(x) = -1 \text{ and } x_0 > 0\}$$

with metric $d_{\mathbb{H}^n}$ derived from the Lorentzian inner product $(\cdot, \cdot)$ associated to the quadratic form $f_n$ by

$$\cosh(d_{\mathbb{H}^n}(x, y)) = -(x, y).$$

We then have $\text{Isom}(\mathbb{H}^n) \cong \text{PO}(n, 1) = O(\mathbb{R}^{n+1}, f_n)/\{\pm 1\}$ where the equality is by definition and where $O(V, f)$ is the group $\{g \in \text{GL}(V) \mid f(g(x)) = f(x) \ \forall x \in V\}$ [Rat06]. The group of orientation-preserving isometries of $\mathbb{H}^n$ (denoted $\text{Isom}^+(\mathbb{H}^n)$) is isomorphic to $\text{PSO}(n, 1) = \text{SO}(\mathbb{R}^{n+1}, f_n)/\{\pm 1\}$ and has index 2 in $\text{PO}(n, 1)$. The groups $\text{PO}(n, 1)$ and $\text{PSO}(n, 1)$ are real Lie groups. If $f$ is any other form over a number field $K \subset \mathbb{R}$ with signature $(n, 1)$ then we may consider the algebraic group $\text{SO}_f$, and we have a Lie group isomorphism $\text{SO}_f(\mathbb{R})^0 \to \text{PSO}(n, 1)$. We think of $\text{PSO}(n, 1)$ as a
fixed concrete realisation of $\text{Isom}^+(\mathbb{H}^n)$ and $\text{SO}_f(\mathbb{R})$ as an algebraic ‘model’ over $K$.

In what follows we will have $G = \text{Isom}(\mathbb{H}^n) = \text{PO}(n,1)$. Then, the group $\mathcal{G} = G^o$ will satisfy the hypothesis of Definition 1.1. We will not be considering non-orientable lattices here, though we will need to consider some isometries lying in $G \setminus G^o$.

1.2.2. Standard arithmetic lattices. As before, we still have $G = \text{Isom}(\mathbb{H}^n)$ and $G$ is some algebraic group over a number field $K$, then we can form a related algebraic group over $Q$, denoted $\text{Res}_{K/Q}(G)$, such that $G(K) \cong \text{Res}_{K/Q}(G)(Q)$. If $K$ is totally real, of degree $d$ over $Q$, then we have $G(K \otimes Q \mathbb{R}) \cong \text{Res}_{K/Q}(G)(\mathbb{R})$; moreover since $K \otimes Q \mathbb{R} \cong \mathbb{R}^d$ we obtain a direct product

$$\text{Res}_{K/Q}(G)(\mathbb{R}) \cong \prod_{\sigma \in \text{Gal}(K/Q)} G^*(\mathbb{R}).$$

The algebraic group $\text{Res}_{K/Q}(G)$ is called the Weil restriction of $G$, and we call $\text{Res}_{K/Q}$ the restriction of scalars functor [PR94, p.50] [Mar91 I.1.7].

Definition 1.3 (Admissible field-form pair). Fix some $n \geq 2$. Let $K$ be a totally real number field, and let $f$ be a (K-valued) non-degenerate quadratic form on $K^{n+1}$. Let $SO_f$ be the algebraic $K$-group determined by $\{x \in O(K^{n+1}, f) \mid \det(x) = 1\}$. We say that $(K, f)$ is an admissible field-form pair if the group $\text{Res}_{K/Q}(SO_f)$ is admissible for $\mathcal{G}$. Equivalently, by (1), the pair $(K, f)$ is admissible if $SO_f(K \otimes Q \mathbb{R})^0$ is isomorphic to $\mathcal{G}$, modulo a compact kernel.

In more concrete terms, we may fix some basis for $K^{n+1}$, so obtaining a representation of $f$ as a homogeneous quadratic polynomial with coefficients in $K$. We also regard $K$ as embedded in $\mathbb{R}$. Then for $(K, f)$ to be admissible for $\text{PO}(n,1)$ requires that $f$ has signature $(n, 1)$ on $\mathbb{R}^{n+1}$ and that for all elements $\sigma \in \text{Gal}(K/Q) \setminus \{\text{id}\}$ the conjugate forms $f^\sigma$ (obtained by applying $\sigma$ to the coefficients of $f$) are positive definite.

If $(K, f)$ is an admissible pair, and if $O_K$ denotes the ring of integers in $K$, then it is well-known that $SO_f(O_K)$ can be realised as an arithmetic lattice in $\text{PO}(n,1)$ [Mar91 (3.2.7)]. This follows from [1], since $\text{Res}_{K/Q}(SO_f)(\mathbb{Z})$ is a lattice in $\text{Res}_{K/Q}(SO_f)(\mathbb{R})$ and on projecting to the only non-compact factor (which is the only such factor by admissibility of $(K, f)$), we obtain a lattice in $SO_f(\mathbb{R})$, isomorphic to $SO_f(O_K)$. (For a lattice in $\text{PO}(n,1)$, note that we have a homomorphism $SO_f(\mathbb{R})^0 \rightarrow \text{PO}(n,1)$ with finite co-kerne.)

We call $SO_f(O_K)$ the standard arithmetic lattice associated to the pair $(K, f)$. Lattices of this type may also be called arithmetic lattices of the simplest type [VS93 p.217]. By abuse of notation we will also refer to

*One could do away with reference to $K$ as this field is implicit in the definition of $f$. We keep $K$ for emphasis on the field of definition, especially as we will later have occasion to consider extensions of scalars $K \otimes L$. 
PO\(f(O_K)\) as the associated standard arithmetic lattice, when we wish to work with the image of this lattice in PO\(f(\mathbb{R})\).

1.3. Summary and results. In \[2.1\] we will recall the construction of I. Agol, generalised by Belolipetsky and Thomson, of hyperbolic manifolds with short closed geodesics, and then show that their associated lattices are quasi-arithmetic. This leads to the following theorem:

**Theorem 1.4.** For any admissible field-form pair \((K,f)\) there are infinitely many commensurability classes of properly quasi-arithmetic lattices \(\Gamma < \text{PO}(n,1)\) arising from \(\text{SO}(K)\).

(A proof is given on p.8.)

In \[3\] we will examine the construction of Gromov and Piatetski-Shapiro and show that the lattices so obtained are not quasi-arithmetic:

**Theorem 1.5.** Let \(\Gamma\) be a Gromov–Piatetski-Shapiro lattice. Then \(\Gamma\) is not quasi-arithmetic.

The definition and construction of a ‘Gromov–Piatetski-Shapiro lattice’ is given in \[3.1\].

Finally, \[4\] includes some further comments as well as open questions and speculation regarding future work.

2. Manifolds with short systole

If \(M\) is a closed Riemannian manifold, then the systole of \(M\) is by definition the (length of) the shortest closed non-trivial curve in \(M\). One may show that the systole of such an \(M\) is always positive [Kat07].

2.1. Constructing hyperbolic manifolds with short systole. Let \(n \geq 2\) and \(\varepsilon > 0\). We will describe how one can construct a a hyperbolic \(n\)-manifold with systole at most \(\varepsilon\), as per the construction of Belolipetsky and Thomson [BT11]. This will allow us to establish some notation. The method is a generalisation of a construction by I. Agol in dimension 4 [Ago06]. (It was also pointed out by N. Bergeron, F. Haglund and D. Wise that some of their own methods allowed Agol’s 4-dimensional construction to be generalised to every dimension using subgroup separability arguments [BHW11, Remark on p.17].)

Fix some admissible pair \((K,f)\) with \(K \neq \mathbb{Q}\), and let \(\Lambda\) be a torsion-free subgroup of the associated standard arithmetic lattice in PO\((n,1)\), so that \(\Lambda \backslash \mathbb{H}^n\) is a compact hyperbolic \(n\)-manifold \(N\). So, for some Galois embedding \(\sigma: K \rightarrow \mathbb{R}\), the form \(f^\sigma\) on \(K^{n+1} \otimes_{\sigma(K)} \mathbb{R}\) has signature \((n,1)\) and we can isometrically identify \(\mathbb{H}^n\) with the more convenient model

\[
\mathbb{H}^n_f = \{ x \in K^{n+1} \otimes_{\sigma(K)} \mathbb{R} \mid f(x) < 0 \} / \sim,
\]

where \(x \sim \lambda x\) for all \(x \in K^{n+1} \otimes_{\sigma(K)} \mathbb{R}\) and all \(\lambda \in \mathbb{R} \setminus 0\). In this model, we have \(\text{Isom}(\mathbb{H}^n_f) \cong \text{PO}(K \otimes_{\sigma(K)} \mathbb{R})\). We will continue to refer to \(\mathbb{H}^n\) rather than \(\mathbb{H}^n_f\). Let \(H_0\) and \(H_1\) be two disjoint ‘\(K\)-rational’ hyperplanes in \(\mathbb{H}^n\);
that is, choosing two vectors \(v_0\) and \(v_1\) in \(K^{n+1}\) with \(f(v_i) > 0\) for \(i = 0, 1\), let \(H_i = \langle v_i \rangle \cap \mathbb{H}^n\), and suppose that \(H_0 \cap H_1 = \emptyset\). If one is interested in obtaining a manifold with short systole then one chooses the \(v_i\) so that the \(H_i\) are at most hyperbolic distance \(\varepsilon/2\) apart. The projections \(\pi(H_i) \subseteq N\) are immersed totally geodesic hypersurfaces in \(N\), but they might not be embedded and they might intersect each other. However, by replacing \(\Lambda\) by a suitable finite-index (congruence) subgroup this can be avoided, so that \(H_i\) \((i = 1, 2\) respectively) projects to a totally geodesic embedded disjoint hypersurface \(N_i\) (respectively), and so that \(N_0 \cap N_1 = \emptyset\) [BTT11, Lem. 3.1].

We now cut along the two hyperplanes \(N_i\) to obtain a manifold with boundary. Keeping the connected component \(M'\) that contains the common perpendicular geodesic segment \(c\) between the two \(H_i\), we double along the boundary \(B\) of \(M'\) to obtain a manifold \(M = DM'\) [Lee13, p.226]. Then \(M\) contains the closed geodesic \(Dc\) of length at most \(\varepsilon\). We will suppose that \(B\) has \(\ell\) connected components. Depending on whether or not the cutting separates the manifold \(N\), we have \(2 \leq \ell \leq 4\).

Thus \(M\) is a compact hyperbolic \(n\)-manifold that can be written as \(\Gamma \backslash \mathbb{H}^n\) for some lattice \(\Gamma \in \text{PO}_f(K \otimes \sigma(K))\). If \(\varepsilon\) is small enough then the manifold \(M\) (having systole at most \(\varepsilon\)) is non-arithmetic [BTT11, §5.1].

### 2.2. Quasi-arithmeticity of short systole manifolds

Let \(M\) be as in §2.1 so that \(\Gamma\) is a non-arithmetic lattice with \(M = \Gamma \backslash \mathbb{H}^n\). We still suppose \((K, f)\) to be an admissible pair for \(\text{PO}(n, 1)\). We prove the following:

**Proposition 2.1.** The group \(\Gamma\) can be generated by elements in \(\text{PO}_f(K)\) and so is quasi-arithmetic.

**Proof.** By construction, the manifold \(M\) is a union \(M_1 \cup_B M_2\) (where \(M_i\) is isometric to \(M'\)), so it has a symmetry \(\tau\) that interchanges its two parts \(M_1\) and \(M_2\). By this decomposition we see that the lattice \(\Gamma \cong \pi_1(M)\) splits as the fundamental group of a graph of groups \(\mathcal{G}\), where the graph \(\mathcal{G}\) is the graph with two nodes and edges between the two nodes corresponding to each boundary component along which the double is taken.

That is, the edge groups are the fundamental groups of the boundary components and the vertex groups are the (isomorphic) groups \(\Gamma_1\) and \(\Gamma_2\). (See Fig. [1])

So, \(\Gamma\) is generated by the elements of the two fundamental groups \(\Gamma_i = \pi_1(M_i)\) \((i = 1, 2)\), as well as some extra elements corresponding to the new homotopy classes that appear as a result of the gluing. (See Serre [Ser80] for the properties of graphs of groups.)

In what follows we describe how to concretely realise \(\Gamma\) as a subgroup of \(\text{PO}_f(K)\).

**Claim.** We may assume that \(\Gamma_1\) can be identified with its image in \(\Lambda\), on identifying \(M_1\) with \(M' \subseteq N\).

**Proof of claim.** *A priori* it is not in general possible to make this latter identification; in particular if the hypersurfaces \(N_i\) do not each separate \(N\).
Figure 1. The graph of groups $G$ whose fundamental group is equal to that of the fundamental group of $M$. There are between 2 and 4 edges, one or two for $N_0$ and one or two for $N_1$.

But, there exists, nevertheless, a double cover $\pi: N' \to N$ such that each $\pi^{-1}(N_i)$ does separate $N$ \cite[2.8C]{GPS87}. Then, the piece obtained by cutting along both $\pi^{-1}(N_i)$ is isometric to the original $M'$, and $\pi_1(M') \hookrightarrow \pi_1(N')$. Since $N' \to N$ is a finite cover, the discrete subgroup $\pi_1(N') < \pi_1(N)$ is still arithmetic. □

Now, write $I_0$ for the reflection in the hyperplane $H_0$: we have $\Gamma_2 = I_0^{-1}\Gamma_1 I_0$.

The group $\Gamma_i$ is a discrete convex co-compact group acting on $\mathbb{H}^n$ (for both $i = 1$ and $i = 2$). Choosing some basepoint $x_0 \in H_0$ we may construct Dirichlet fundamental domains $F$, $F_1$ and $F_2$ (about $x_0$) for $\Gamma$, $\Gamma_1$ and $\Gamma_2$ respectively. By the doubling construction it is evident that $F$ will be invariant under the reflection $I_0$. Thus we can view $F$ as a union $F_1 \cup F_2$ of the two pieces exchanged by $I_0$, and each $F_i$ satisfies the inclusion $F_i \subseteq F_i$. We also have the inclusion $F \subseteq F_i$ ($i = 1, 2$) since $\Gamma_i < \Gamma$. The intersection $F_1 \cap F_2$ is a Dirichlet fundamental domain (at $x_0$) for the group $(\Gamma_1, \Gamma_2)$ and so we have $F \subseteq F_1 \cap F_2$ since $(\Gamma_1, \Gamma_2) \leq \Gamma$.

The domains $F_i$ may be decomposed into two parts separated by $H_0$. By $\tilde{F}_i$ we mean the part containing the ends corresponding to the boundary components whose lifts to $\mathbb{H}^n$ are $H_1$ and $\gamma_j(H_j)$ ($j = 0, 1$) if either of the latter exist (cf. \cite[2.1]{GPS87}). If $H_0$ intersects a bounding hyperplane of $F_i$ then it must do so orthogonally and hence we have the inclusion $I_0(\tilde{F}_i) \subseteq F_i \cap \tilde{F}_i$.

Let the set $J$ be the index set $\{1, \ldots, \ell - 1\}$, and if $2 \in J$ or $3 \in J$ then denote the two hyperplanes $H_2 = \gamma_0(H_0)$ and $H_3 = \gamma_1(H_1)$. Write $I_j$ for the reflection in $H_j$ (whenever $H_j$ exists), and write $H_j^-$ for the half-space bounded by $H_j$ and containing $H_0$. The set $F$ is the intersection

$$F = (F_1 \cap F_2) \cap \bigcap_{j \in J} H_j^- \cap \bigcap_{j \in J} I_0(H_j^-),$$

and the group $\Gamma$ may be given by the generators

$$\Gamma = \left\langle \Gamma_1, \; I_0^{-1}\Gamma_1 I_0, \; I_j^{-1}I_0 \; (j \in J) \right\rangle.$$  \hfill (2)
Geometrically, the conjugate copy of $\Gamma_1$ by $I_0$ corresponds to matching up two copies of the fundamental domain $F_1$ at $H_0$, and the elements $I^{-1}_jI_0$ correspond to the gluing isometries for the remaining sides of the fundamental domain $F$ corresponding to the $H_1$ and $\gamma_j(H_j)$; equivalently the ends of $F_1$ and $F_2$ away from $H_0$.

We finally note that since the $H_j$ are $K$-rational hyperplanes, their corresponding reflections $I_j$ do indeed lie in $\text{PO}_f(K)$. □

Proof of Theorem 1.4 (cf. p.5). We have already seen (cf. Prop. 2.1) that any lattice constructed as in §2.1 is quasi-arithmetic. That there are infinitely many commensurability classes of such lattices has already been demonstrated in the literature [BT11, §5.2], and follows from that fact that when these lattices are non-arithmetic, their commensurator is also a lattice in $\text{PO}_f(\mathbb{R})$ [Mar91, (B), p.298]. As the systole length in the construction of §2.1 decreases, one obtains a sequence of non-arithmetic lattices $\Gamma_m$, and if these were commensurable then Margulis’ theorem would imply that we have some maximal lattice $\Gamma$ with every $\Gamma_m \subset \Gamma$. But since the $\Gamma_m$ have decreasing systole lengths, this is not possible. □

3. Gromov–Piatetski-Shapiro manifolds

Gromov and Piatetski-Shapiro’s construction is well-known and so we do not revisit it in complete detail, but we present below enough of their construction to examine the quasi-arithmeticity properties of the resulting lattices. The reader interested in the details of the construction should find the article of Gromov and Piatetski-Shapiro [GPS87] accessible even to the non-expert.

3.1. The construction of Gromov and Piatetski-Shapiro. In order to obtain non-arithmetic lattices $\Gamma < \text{PO}(n, 1)$ Gromov and Piatetski-Shapiro first consider two torsion-free co-compact arithmetic lattices $\tilde{\Gamma}_1 < \text{PO}_{f_1}(\mathbb{R})$ and $\tilde{\Gamma}_2 < \text{PO}_{f_2}(\mathbb{R})$ over a field $K$, such that the two quotient manifolds $\tilde{M}_i = \tilde{\Gamma}_i\backslash \mathbb{H}^n$ ($i = 1, 2$) each contain a co-dimension 1 closed submanifold $M_0^{(i)}$ ($i = 1, 2$) with an isometry $\psi: M_0^{(1)} \rightarrow M_0^{(2)}$. They show that if the forms $f_1$ and $f_2$ are not similar over $K$ then the manifolds $\tilde{M}_1$ and $\tilde{M}_2$ are not commensurable: we assume that the forms are indeed not similar. The manifold $M$ obtained by cutting each of the $\tilde{M}_i$ along $M_0^{(i)}$ and gluing the two together via the boundary isometry $\psi$, would be a cover of $M_1$ and $M_2$, if $M$ was arithmetic [GPS87, 0.2]. However in light of the non-commensurability of $M_1$ and $M_2$, the glued manifold $M$ cannot be a common cover of these spaces and so must be non-arithmetic. Thus $\Gamma = \pi_1(M)$ is a non-arithmetic lattice in $\text{PO}(n, 1)$.

Definition 3.1. By a Gromov–Piatetski-Shapiro lattice (GPS lattice for short) is meant a non-arithmetic lattice $\Gamma < \text{PO}(n, 1)$ obtained by the above procedure; that is, by taking two non-similar admissible quadratic forms $f_1$
and $f_2$ over a common field $K$, containing a common subform $f_0$ giving rise to an embedded codimension 1 hypersurface.

3.2. Quasi-arithmeticity and Gromov–Piatetski-Shapiro lattices. In this section we will prove Theorem 1.5 (cf. p. 5).

Let $\Gamma < \PO(n, 1)$ be a gps lattice. So, there is a hyperbolic manifold $M$ with $\Gamma = \pi_1(M)$, such that $M = M_1 \cup M_2$ with $M_1$ and $M_2$ arising as manifolds with boundary from closed arithmetic manifolds $\tilde{M}_i = \Lambda_i \setminus \mathbb{H}^n$. As in §3.1 suppose $\Lambda_i \subseteq \PO_f(O_K)$ (where we identify $\PO_f(O_K)$ with its image in $\PO(n, 1)$ via an isomorphism over some finite extension of $K$). Write $\Gamma_i = \pi_1(M_i)$ for $i = 1, 2$. Then each $\Gamma_i$ is Zariski dense in $\PO(n, 1) = \PO_f(\mathbb{R})$ [GPS87, 0.1].

Now suppose that there is another admissible pair $(K', f')$ for $\PO(n, 1)$ and such that $\Gamma \subset \PO_f(K')$ (where again we identify the corresponding groups of real points by an isomorphism over $\mathbb{Q}$). Then we would have the configuration

$$
\begin{align*}
\begin{array}{ccc}
\Gamma & \xrightarrow{\phi_1} & \PO_f(\mathbb{R}) \\
\xrightarrow{\phi_2} & & \\
\PO_f(K') & \xrightarrow{\phi_2} & \Gamma_2
\end{array}
\end{align*}
$$

(3)

**Proposition 3.2.** Assuming the above configuration, we have $K = K'$ and for each $i = 1, 2$, there is a $K$-isomorphism $\PO_{f_i} \cong \PO_{f_i}$.

**Proof.** In what follows, $\overline{A}^{(F)}$ will denote the Zariski closure of a set $A$ with respect to the Zariski $F$-topology, for $F$ a field. We will abuse notation and write equalities in place of the isomorphisms $\phi_i$ and $\phi'$.

Fix $i = 1$ or $i = 2$. We denote by $L'$ the smallest extension of $\mathbb{Q}$ over which the morphisms $\phi_i$ and $\phi'$ may be simultaneously defined. Now, $\Gamma_i$ is Zariski dense in $\PO_f(K')$; for if not then $\overline{\Gamma_i}^{(K')}$ would be a $K'$-closed subgroup of $\PO_f(\mathbb{R})$ containing $\Gamma_i$. But this is also an $\mathbb{R}$-closed subgroup, which is impossible by Zariski density of $\Gamma_i$ in $\PO_f(\mathbb{R})$. Similarly, $\Gamma_i$ is also Zariski dense in $\PO_f(K)$. We also have $\Gamma_i$ Zariski dense in $\PO_f(L')$, because $\overline{\Gamma_i}^{(\mathbb{R})} \subseteq \overline{\Gamma_i}^{(L')}$. So, $\PO_f(L') = \overline{\Gamma_i}^{(L')} = \PO_f(L')$. But these observations imply that $\overline{\Gamma_i}^{(L')}$ may be defined over $K$ and $K'$. Thus $K = K' = L'$. □

Thus if $\Gamma$ is a quasi-arithmetic lattice then it must be contained in $\PO_f(K)$ for both $i = 1, 2$, and the $\PO_f$ must be $K$-isomorphic, which is impossible [GPS87, 2.6].

This concludes the proof of Theorem 1.5 □
4. FURTHER QUESTIONS AND COMMENTS

4.1. Counting arithmetic versus non-arithmetic lattices. Recently the work of J. Raimbault [Rai13], followed by that of T. Gelander and A. Levit [GL14], has allowed us to see in a quantitative way that there are 'more' non-arithmetic lattices in Isom(\(\mathbb{H}^n\)) than arithmetic ones. More precisely, Raimbault showed that if \(L_{\text{max}}(v)\) denotes the number of conjugacy classes of maximal co-compact lattices in SO\((n, 1)\) of co-volume at most \(v\), and if \(L_{\text{max}}^a(v)\) denotes the sub-class of arithmetic such lattices, then

\[
\lim_{v \to \infty} \frac{\log L_{\text{max}}^a(v)}{\log L_{\text{max}}(v)} = 0; \tag{4}
\]

whilst Gelander and Levit have shown that the number of commensurability classes of hyperbolic manifolds of bounded volume \(v\) is super-exponential, in contrast with a sub-exponential bound on arithmetic manifolds previously obtained by M. Belolipetsky [Bel07]. The arguments used involve refinements of the Gromov–Piatetski-Shapiro construction in which one may obtain many different commensurability classes of non-arithmetic manifolds, such that the number of classes obtained may be bounded from above and below and compared with the known bounds for both arithmetic lattices and lattices in general. These three authors consider a generalised Gromov–Piatetski-Shapiro construction and so they work with non-quasi-arithmetic lattices by the observations in §3.2.

Therefore it would be interesting to have some quantitative bounds on numbers of commensurability classes or conjugacy classes of properly quasi-arithmetic lattices in PO\((n, 1)\) (with respect to volume) and then compare these with the bounds for, say, non-quasi-arithmetic lattices, or perhaps arithmetic lattices. This is beyond our present scope but would make for interesting further investigation.

4.2. Sub-arithmeticity. In the original article describing gps lattices, the authors define a sub-arithmetic group \(\Gamma < \text{PO}(n, 1)\) to be a discrete group that is Zariski dense and such that for some arithmetic lattice \(\Lambda < \text{PO}(n, 1)\), we have \(|\Gamma : \Lambda \cap \Gamma| < \infty\) (i.e., \(\Gamma\) is virtually contained in an arithmetic lattice). Both constructions described above (§2.1 and §3.1) are by gluing of sub-arithmetic manifolds with boundary, but Gromov and Piatetski-Shapiro asked whether this is necessarily the only way of constructing non-arithmetic lattices in PO\((n, 1)\) [GPS87, 0.4]. This problem is still open, and it is unclear how quasi-arithmeticity might relate to this question. For instance, if

- \(\mathcal{H}_A\) denotes the collection of closed arithmetic hyperbolic manifolds,
- \(\mathcal{H}_Q\) denotes the collection of quasi-arithmetic closed hyperbolic manifolds,
- \(\mathcal{H}_S\) is the collection of closed hyperbolic manifolds obtained from a finite collection of sub-arithmetic pieces by gluing, and
- \(\mathcal{H}_N\) is the collection of non-arithmetic closed hyperbolic manifolds;
then it is not entirely clear how all of the classes $\mathcal{H}_A$, $\mathcal{H}_Q$, $\mathcal{H}_S$, $\mathcal{H}_N$ intersect each other. Similar questions to those raised in \[\square\] could be posed for the class $\mathcal{H}_S$, but until more is known about the structure of non-arithmetic groups in general, answers are likely to be out of reach.

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