Proper temperature of the Schwarzschild AdS black hole revisited

Myungseok Eune\textsuperscript{1,*} and Wontae Kim\textsuperscript{2,†}

\textsuperscript{1}Department of Civil Engineering, Sangmyung University, Cheonan, 31066, Republic of Korea
\textsuperscript{2}Department of Physics, Sogang University, Seoul, 04107, Republic of Korea

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Abstract

The Unruh temperature calculated by using the global embedding of the Schwarzschild AdS spacetime into the Minkowski spacetime was identified with the local proper temperature; however, it became imaginary in a certain region outside the event horizon. So, the temperature was assumed to be zero of non-thermal radiation for that region. In this work, we revisit this issue in an exactly soluble two-dimensional Schwarzschild AdS black hole and present an alternative resolution to this problem in terms of the Tolman’s procedure. However, the process appears to be non-trivial in the sense that the original procedure assuming the traceless energy-momentum tensor should be extended in such a way that it should cover the non-vanishing case of the energy-momentum tensor in the presence of the trace anomaly. Consequently, we show that the proper temperature turns out to be real everywhere outside the event horizon without any imaginary value, in particular, it vanishes at both the horizon and the asymptotic infinity.

Keywords: Hawking temperature, Schwarzschild AdS spacetimes, proper temperature, effective Tolman temperature

\textsuperscript{*} eunems@smu.ac.kr
\textsuperscript{†} wtkim@sogang.ac.kr
I. INTRODUCTION

One of the most outstanding theoretical results in quantum mechanics of black holes would be Hawking radiation [1, 2]. On general grounds, the thermal distribution of Hawking radiation could be characterized by the two black hole temperatures associated with the observers. One is the fiducial temperature measured by a fixed observer who undergoes acceleration, which is usually given as the redshifted Hawking temperature at a finite distance outside the event horizon [3]. The other is the proper temperature measured by a freely falling observer from rest, which is expressed by the Tolman temperature [4, 5].

Surprisingly, the fiducial temperature takes the same form as the proper temperature even though the respective observers belong to different frames. At first glance, one might conclude that the equivalence principle could be violated, particularly at the horizon from the fact that the Tolman temperature could be divergent there. However, it is worth noting that the equivalence principle could be restored just at the horizon as seen from the calculations using the particle detector method [6]. This fact could also be confirmed by showing that the Tolman temperature could vanish effectively at the horizon [7].

As a matter of fact, the Tolman temperature for the freely falling observer should be modified effectively, so that its behavior could be shown to be definitely different from that of the fiducial temperature [7]. Recently, a similar argument for the proper temperature [8] could also be obtained from a different point of view by clarifying the Hawking effect [1] and the Unruh effect [9]. Note that all these arguments are for asymptotically flat black holes, and it appears to be natural to ask how to get the proper temperatures in asymptotically non-flat spacetimes such as the Schwarzschild anti-de Sitter (SAdS) black hole.

Regarding the calculations of the proper temperature in the SAdS black hole, there has been pioneering works employing the global embedding in Minkowski spacetime (GEMS) approach, where an accelerating observer in a higher-dimensional Minkowski spacetime perceives thermal radiation characterized by the Unruh temperature which will be identified with the proper temperature in the original spacetime [10, 11]; however, the proper temperature suffers from an imaginary value. So, it was claimed that the imaginary valued proper temperature would indicate non-thermal radiation [12]. The non-thermal condition to evade the imaginary temperature seems to be somehow ad hoc. Obviously, the temperature could be made real in the near horizon limit when a reduced embedding was used [13].
Now it raises a question: is there any other way to resolve this imaginary value problem for the proper temperature in the SAdS black hole? In fact, there is another way to calculate the proper temperature directly, which is the old-fashioned but clear Tolman procedure [4, 5], which might provide a plausible solution to this question. However, this approach appears to be conceptually non-trivial in the sense that the conventional Tolman temperature derived from the conventional Stefan-Boltzmann law rests upon the traceless condition of the energy-momentum tensor. If one were to study the proper temperature on the background of the asymptotically anti-de Sitter (AdS) spacetimes, the traceless condition for the energy-momentum tensor should be released in order to take into account the non-vanishing trace of the quantized energy-momentum tensor in the presence of the trace anomaly.

In this work, we would like to revisit the proper temperature of the SAdS black hole and show how to get the well-defined real-valued proper temperature. In essence, we shall obtain a modified Stefan-Boltzmann law, which is commensurate with the presence of the non-vanishing trace of the energy-momentum tensor. Then, from the modified Stefan-Boltzmann law, we shall derive an effective Tolman temperature and obtain the desired result. In fact, such a modification of the Stefan-Boltzmann law has already been applied to various models: thermodynamics of particle physics in flat spacetime [14], thermodynamics of black hole in curved spacetime [7], and warm inflation models in cosmology [15], so that some puzzling problems have been successfully resolved.

Our calculations will be done in a two-dimensional amenable model in order to solve exactly without losing any essential physics. In Sec. II, the proper temperature will be obtained in the two-dimensional SAdS black hole by using the GEMS [10–12] in the self-contained manner in comparison with our result. As expected, we find that the imaginary temperature is unavoidable in a certain region. Next, in Sec. III, we will calculate the proper temperature from the Tolman’s procedure [4, 5] by releasing the traceless condition for the energy-momentum tensor. We shall show that the proper temperature for the SAdS black hole turns out to be real everywhere outside the horizon without any imaginary value, so that it becomes smooth without any cusp. Summary and discussion will be given in Sec. IV.
II. PROPER TEMPERATURE FROM THE GEMS

We recapitulate how the proper temperature for the two-dimensional SAdS black hole could be derived from the framework of the GEMS employed in Ref. [12]. Let us start with the two-dimensional SAdS black hole described by

\[ ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)}, \]  

(1)

where \( f(r) = 1 - 2M/r + r^2/\ell^2 \). The metric element can be rewritten as

\[ f(r) = \frac{1}{\ell^2} \left( 1 - \frac{r_h}{r} \right) \left( r^2 + rr_h + r_h^2 + \ell^2 \right), \]  

(2)

where \( r_h \) is the horizon of the black hole and the mass is related to the horizon as

\[ M = \frac{r_h}{2\ell^2} (r_h^2 + \ell^2). \]  

(3)

And the surface gravity is also given by

\[ \kappa = \frac{f'(r_h)}{2} = \frac{3r_h^2 + \ell^2}{2\ell^2 r_h}, \]  

(4)

where the prime denotes the derivative with respect to \( r \).

Performing the global embedding of the SAdS spacetime into the higher dimensional Minkowski spacetime, a free-fall observer on the SAdS black hole could be identified with the accelerated observer in the higher dimensional Rindler spacetime [10, 12], so that the Rindler observer could find the Unruh temperature as [9]

\[ T = \frac{a}{2\pi}, \]  

(5)

where \( a \) is the proper acceleration of the observer in the higher dimensional Minkowski spacetime.

The higher dimensional Minkowski spacetime can be obtained by the following transformation [11]

\[ X^0 = \kappa^{-1} \sqrt{f} \sinh \kappa t, \quad X^1 = \kappa^{-1} \sqrt{f} \cosh \kappa t, \quad X^2 = r, \]  

\[ X^3 = \int dr \frac{\ell (r_h^2 + \ell^2)}{3r_h^2 + \ell^2} \frac{r_h (r^2 + rr_h + r_h^2)}{r^3 (r^2 + rr_h + r_h^2 + \ell^2)}, \]  

\[ X^4 = \int dr \frac{1}{3r_h^2 + \ell^2} \sqrt{f} \frac{(9r_h^4 + 10r_h^2 \ell^2 + \ell^4) (r^2 + rr_h + r_h^2)}{r^2 + rr_h + r_h^2 + \ell^2}, \]  

(6)
where the line element is \( ds^2 = \eta_{IJ} dX^I dX^J \) with \( \eta_{IJ} = \text{diag}(-1, 1, 1, 1, -1) \). In this space-time, the square of the proper acceleration is calculated as

\[
a^2 = \eta_{IJ} a^I a^J = \frac{[2 + (3 + c^2)x + (1 + c^2)x^3][-2 + (1 + c^2)(1 + x)x]}{4\ell^2[1 + x + (1 + c^2)x^2]}, \tag{7}
\]

where \( x = r_h/r \) and \( c = \ell/r_h \). Substituting this acceleration (7) into Eq. (5), the Unruh temperature regarded as the proper temperature is obtained as

\[
T = \frac{1}{4\pi\ell} \frac{\sqrt{[2 + (3 + c^2)x + (1 + c^2)x^3][-2 + (1 + c^2)(1 + x)x]}}{\sqrt{1 + x + (1 + c^2)x^2}}. \tag{8}
\]

The squared proper temperature is positive \( r < r_c \), while it is negative for \( r > r_c \) where the critical radius is given by \( r_c = (r_h^2 + \ell^2 + \sqrt{9r_h^4 + 10r_h^2\ell^2 + \ell^4})/(4r_h) \). In particular, the squared temperature at the horizon becomes

\[
T^2(r_h) = \frac{1}{4\pi^2} \left[ \frac{\ell^2}{2r_h^4} + \frac{3}{3r_h^2 + \ell^2} \right], \tag{9}
\]

which is positive finite. By the way, at the asymptotic infinity, the squared temperature takes the form of

\[
T^2(\infty) \rightarrow -\frac{1}{4\pi^2} \left[ \frac{1}{\ell^2} + \frac{(r_h^2 + \ell^2)^2}{2r_h^4\ell^2} \right], \tag{10}
\]

which is negative. Thus one can find that the proper temperature becomes imaginary for \( r > r_c \). In fact, it was claimed that the imaginary proper temperature would indicate non-thermal radiation [12], and the proper temperature was assumed to be zero for \( r > r_c \). It means that there would appear a cusp at \( r_c \) for the temperature curve. In the next section, we will find another way to resolve this imaginary value problem by directly calculating the proper temperature through the Tolman’s procedure.

### III. PROPER TEMPERATURE FROM THE TOLMAN PROCEDURE

We calculate the temperature measured by a freely falling observer released from rest on the SAdS black hole. Here, we shall release the traceless condition employed in the conventional formulation of the Tolman temperature [4, 5] in order to get the effective Tolman temperature.
Let us start with the proper velocity of a particle obeying the geodesic equation of motion

\[ u^\mu = \left( \frac{\alpha}{f}, -\sqrt{\alpha^2 - f} \right), \tag{11} \]

where \( \alpha \) is an integration constant. The freely falling observer is released at \( r = r_0 \) with the zero velocity, and then the integration constant can be determined by \( \alpha = \sqrt{f(r_0)} \). In the conformal gauge of \( ds^2 = -e^{2\sigma} dx^+ dx^- \) with \( e^\sigma = \sqrt{f(r)} \), the proper velocity (11) is rewritten as

\[ u^\pm = \frac{\sqrt{f(r_0)} \mp \sqrt{f(r_0) - f(r)}}{f(r)}, \tag{12} \]

and the unit normal vector is chosen as \( n^+ = u^+ \) and \( n^- = -u^- \), where they satisfy \( u^\mu u_\mu = -1 \), \( u^\mu n_\mu = 0 \), and \( n^\mu n_\mu = 1 \). We consider a free-fall frame at \( r = r_0 \), and then Eq. (12) reduces to

\[ u^\pm = \frac{1}{\sqrt{f(r_0)}}, \tag{13} \]

where \( r_0 \) will be replaced by \( r \) for a simple notation hereafter.

On the other hand, it has been well-known that Hawking radiation is related to the trace anomaly [16], which means that the traceless condition of the energy-momentum tensor should be released in the thermodynamic black hole system. Explicitly, the trace anomaly for a single scalar field in two dimensions is given by

\[ T^\mu_\mu = \frac{1}{24\pi} R, \tag{14} \]

where the scalar curvature is written as \( R = -f'' \) for the line element (1). From the trace anomaly (14) with the help of the conservation law for the energy-momentum tensor, the energy-momentum tensor is written as [17]

\[ T_{\pm \pm} = \frac{1}{96\pi} \left[ f f'' - \frac{1}{2} (f')^2 + t_\pm \right], \tag{15} \]

\[ T_{+-} = \frac{1}{96\pi} f f''. \tag{16} \]

where \( t_\pm \) reflect the nonlocality of trace anomaly. For the Hartle-Hawking-Israel state [18, 19], \( t_\pm \) are explicitly determined by

\[ t_\pm = \frac{1}{2} f'(r_h)^2, \tag{17} \]
where $t_+ = t_-$ in thermal equilibrium and so the net flux automatically vanishes.

Next, one can write down the energy density and pressure for the freely falling observer as follows

$$\rho = T_{\mu\nu}u^\mu u^\nu = \frac{1}{96\pi f} [4ff'' - (f')^2 + t_+ + t_-], \quad (18)$$

and

$$p = T_{\mu\nu}n^\mu n^\nu = \frac{1}{96\pi f} [-(f')^2 + t_+ + t_-], \quad (19)$$

respectively. Then the explicit form of the proper energy density and pressure are

$$\rho = \frac{1}{96\pi f} \left[8 \left(1 - \frac{2M}{r} + \frac{r^2}{\ell^2}\right) \left(-\frac{2M}{r^3} + \frac{1}{\ell^2}\right) - \left(\frac{2M}{r^2} + \frac{2r}{\ell^2}\right)^2 + \left(\frac{2M}{r^2} + \frac{2r_h}{\ell^2}\right)^2\right], \quad (20)$$

and

$$p = \frac{1}{96\pi f} \left[-\left(\frac{2M}{r^2} + \frac{2r}{\ell^2}\right)^2 + \left(\frac{2M}{r_h^2} + \frac{2r_h}{\ell^2}\right)^2\right]. \quad (21)$$

Note that the proper energy density is negative finite at the horizon such as $\rho(r_h) \to -(1/12\pi r_h^2)$ and it is positive finite at the asymptotic infinity, $\rho(\infty) \to 1/(24\pi \ell^2)$. So there appears a special point to divide the region into the negative energy density and the positive energy density. This kind of feature appears even in asymptotically flat black holes [7, 20]. The attendant problem is how to relate the positive and negative energy density to the corresponding temperatures consistently. For this purpose, we have to extend the conventional Stefan-Boltzmann law which is only valid for the positive energy density.

We are now in a position to explain how to get the proper temperature by using the modified Stefan-Boltzmann law to relate the proper energy density to the proper temperature. Let us start with the first law of thermodynamics written as [4, 5]

$$dU = TdS - pdV, \quad (22)$$

where $U$, $S$, $V$, $T$, and $p$ are the internal energy, entropy, volume, temperature, and pressure of a system, respectively. At a fixed temperature, Eq. (22) can be rewritten as

$$\left(\frac{\partial U}{\partial V}\right)_T = T \left(\frac{\partial S}{\partial V}\right)_T - p, \quad (23)$$
where \((\partial U/\partial V)_T\) is just the energy density \(\rho\). Using the Maxwell relation

\[
\left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{\partial p}{\partial T}\right)_V,
\]

one can see that Eq. (23) becomes

\[
\rho = T \left(\frac{\partial p}{\partial T}\right)_V - p.
\]

In addition to this, we note that the trace of the energy-momentum tensor for a perfect fluid is generically non-vanishing, which is given as

\[
T^{\mu \mu} = -\rho + p.
\]

From Eqs. (25) and (26), we can eliminate the pressure term and obtain the first order differential equation as

\[
T \left(\frac{\partial \rho}{\partial T}\right)_V - 2\rho = T^{\mu \mu},
\]

where we used the fact that the trace anomaly is independent of temperature \([21]\). Then, the energy density and pressure are easily solved as

\[
\rho = \gamma T^2 - \frac{1}{2} T^{\mu \mu},
\]

\[
p = \gamma T^2 + \frac{1}{2} T^{\mu \mu},
\]

respectively, where \(\gamma\) is an integration constant determined as \(\gamma = \pi/6\) for a scalar field. Note that the modified Stefan-Boltzmann law (28) and (29) naturally reduce to the conventional Stefan-Boltzmann law \([4, 5]\).

Combining Eqs. (14), (18), and (28), we obtain the squared effective Tolman temperature as

\[
T^2 = \frac{1}{96\pi \gamma f} \left[2f f'' - (f')^2 + t_+ + t_-\right].
\]

Plugging the boundary condition (17) into Eq. (30), we get the proper temperature as

\[
T = \frac{1}{4\pi \ell^2} x \sqrt{\left(1 + c^2\right)(1 - x)[3(3 + 2x + x^2) + c^2(1 + 2x + 3x^2)]} / \sqrt{1 + x + (1 + c^2)x^2},
\]

where \(x = r_h/r\) and \(c = \ell/r_h\). The behavior of the temperature (31) is shown in Fig. 1, where the proper temperature described by the solid curve is real everywhere. In particular,
FIG. 1. The two proper temperatures (8) and (31) are plotted by setting $r_h = 1$, $\ell = 2$, and $M = 5/8$, for convenience. The dashed curve describes the behavior of the proper temperature from the GEMS. It has a cusp at $r_c \approx 3.266$, so that $T = 0$ is imposed at $r > r_c$, where the imaginary temperature appears, and $T_1 = (2\pi r_h)^{-1} = (2\pi)^{-1}$ at $r = r_h$. The solid curve is the proper temperature based on the effective Tolman temperature, which is vanishing at both the event horizon $r_h$ and the asymptotic infinity. It is always real and smooth, and reaches a peak of $T_{\text{max}} \approx 0.0948$ at $r_{\text{max}} \approx 1.675$. It vanishes at both the horizon and the asymptotic infinity, while it approaches the maximum value at the critical radius of $r_{\text{max}}$. The smooth behavior of the effective Tolman temperature is in contrast to the behavior of the proper temperature calculated by using the GEMS method [12], where one should require a non-thermal condition by hand such as $T = 0$ for $r > r_c$ at which the imaginary value appears.

On the other hand, if one takes the limit of $\ell \to \infty$ in Eq. (31) for the limit of the Schwarzschild black hole, then one can find the proper temperature of

$$T = \frac{1}{8\pi M} \sqrt{(1-x)(1+2x+3x^2)}$$

(32)

which exactly agrees with the previous result of the proper temperature for the Schwarzschild black hole [7]. So, in this limit, the proper temperature vanishes at the horizon of $x \to 1$ and it reproduces the Hawking temperature at the spatial infinity of $x \to 0$. 

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IV. CONCLUSION AND DISCUSSION

In summary, the proper temperature on the background of the two-dimensional SAdS black hole has been investigated by using the two different methods. First, it was derived from the framework of the GEMS; however, it became imaginary for a certain region such as $r > r_c$ as shown in Fig. 1. So, it was claimed that the imaginary temperature implies non-thermal radiation in that region [12]. In this work, we revisited this issue by calculating the proper temperature straightforwardly from the Stefan-Boltzmann law without resort to any indirect methods. For this purpose, the conventional Stefan-Boltzmann law was extended to the case of the non-vanishing trace of the energy-momentum tensor. In essence, if one were to consider a black hole system with Hawking radiation, then one should take into account non-trivial trace of the energy-momentum tensor in the calculation of the Stefan-Boltzmann law [7]. Consequently, we could find the effective Tolman temperature whose form is different from the conventional Tolman temperature by the anomalous term in Eq. (28). The resulting effective Tolman temperature (31) as the proper temperature is always real and smooth without encountering any imaginary value.

For the Schwarzschild black hole, it was shown that the equivalence principle could be restored only at the horizon [6], and so it would be natural for the proper temperature to vanish there [7, 8]. The present calculation shows that the above feature could also be found even in the SAdS black hole. This fact can be understood by employing the Unruh effect. For the large black hole, the metric (2) is expressed by the Rindler metric in the near horizon limit. The Unruh effect tells us that the temperature is given as $T_U = a/2\pi$ near the horizon, where the acceleration of the fiducial observer is $a = M/(r^2\sqrt{f})$ [9]. It implies that the free-fall observer could find the vanishing Unruh temperature, if the frame is free from the acceleration. In that sense, it seems to be reasonable for the freely falling observer to find the vanishing temperature at the horizon. On the other hand, at the asymptotic infinity, particles could not pass through the AdS boundary due to the infinite potential. In these regards the proper temperature vanishes at the horizon and the asymptotic infinity.

Finally, one might wonder why the proper temperature calculated by using the GEMS method was different from the result derived from the Tolman procedure. In the GEMS method, the lower dimensional black hole geometry is embedded into the higher dimensional Minkowski spacetime classically, prior to the quantization of the theory. However, the
quantized theory formulated in the higher dimensions might be inequivalent to the quantized theory in the lower dimensions despite the classical equivalence. It means that the classical equivalence does not always warrant the quantum-mechanical equivalence. This speculation might deserve further attention.

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