COMPARISON THEOREMS IN LORENTZIAN GEOMETRY AND APPLICATIONS TO SPACELIKE HYPERSURFACES

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ABSTRACT. In this paper we prove Hessian and Laplacian comparison theorems for the Lorentzian distance function in a spacetime with sectional (or Ricci) curvature bounded by a certain function by means of a comparison criterion for Riccati equations. Using these results, under suitable conditions, we are able to obtain some estimates on the higher order mean curvatures of spacelike hypersurfaces satisfying an Omori-Yau maximum principle for certain elliptic operators.

1. INTRODUCTION

In general relativity each point of a Lorentzian manifold corresponds to an event. The events that we may experience in the universe are the ones in our chronological future, hence it may be interesting to investigate the geometry of this one. This can be done by means of the analysis of the Lorentzian distance function. Unfortunately this function is not differentiable in any spacetime; precisely, it is not even continuous in general. Nevertheless, in strongly causal spacetimes, the Lorentzian distance function from a point is differentiable at least in a “sufficiently near” chronological future of each point. In this case is possible to analyze the geometry of spacetimes by means of the level sets of the Lorentzian distance function with respect to this point. To do that, the main tools are Hessian and Laplacian comparison theorems for the Lorentzian distance of the spacetime, hence many works have been written in this spirit. For instance, in a recent paper by F. Erkekoglu, E. García-Rio and D. N. Kupeli [5], following the approach of R. E. Greene and H. Wu in [12], the authors obtain Hessian and Laplacian comparison theorems for the Lorentzian distance functions of Lorentzian manifolds comparing their sectional curvatures. Afterwards, in [3], L. J. Alías, A. Hurtado and V. Palmer use these theorems to study the Lorentzian distance function restricted to a spacelike hypersurface \( \Sigma^n \) immersed into a spacetime \( M^{n+1} \). In particular, under suitable conditions, they derive sharp estimates for the mean curvature of spacelike hypersurfaces with bounded image in the ambient spacetime.

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In this paper we obtain Hessian and Laplacian comparison theorems for Lorentzian manifolds with sectional curvature of timelike planes bounded by a function of the Lorentzian distance, improving in this way on classical results, and we give some applications to the study of spacelike hypersurfaces.

The paper is organized as follows. In Sections 2 and 3 we present some basic concepts and terminology involving the Lorentzian distance function from a point and we prove our Hessian and Laplacian comparison theorems. To obtain these theorems we use an ‘analytic’ approach inspired by P. Petersen ([13]) avoiding, in this way, the ‘geometric’ approach used by Greene and Wu. In Section 4 we focus on the study of the Lorentzian distance function restricted to spacelike hypersurfaces. Hence, using the Omori-Yau maximum principle, we derive some estimates on the mean curvature that generalize the ones in [4]. Moreover, using a generalized Omori-Yau maximum principle for certain elliptic operators, we also obtain some estimates for the higher order mean curvatures associated to the immersion. Finally, in Section 5, we restrict ourselves to the case when the ambient space has constant sectional curvature and we prove a Bernstein-type theorem for spacelike hypersurfaces with constant $k$-mean curvature that generalizes Corollary 4.6 in [4].

2. Preliminaries

Let $M^{n+1}$ be an $n+1$-dimensional spacetime, that is, an $n+1$-dimensional time-oriented Lorentzian manifold and let $p, q \in M$. Using the standard terminology and notation in Lorentzian geometry, we say that $q$ is in the chronological future of $p$, written $p \ll q$, if there exists a future-directed timelike curve from $p$ to $q$. Similarly, we say that $q$ is in the causal future of $p$, written $p \leq q$, if there exists a future-directed causal (that is nonspacelike) curve from $p$ to $q$. For a subset $S \subset M$, we define the chronological future of $S$ as

$$I^+(S) = \{q \in M | p \ll q \text{ for some } p \in S\},$$

and the causal future of $S$ as

$$J^+(S) = \{q \in M | p \leq q \text{ for some } p \in S\},$$

where $p \leq q$ means that either $p < q$ or $p = q$. In particular, the chronological and the causal future of a point $p \in M$ are, respectively

$$I^+(p) = \{q \in M | p \ll q\}, \quad J^+(p) = \{q \in M | p \leq q\}.$$

It is well known that $I^+(p)$ is always open, while $J^+(p)$ is neither open nor closed in general. Let $q \in J^+(p)$. Then the Lorentzian distance $d(p, q)$ is defined as the supremum of the Lorentzian lengths of all the future-directed causal curves from $p$ to $q$. If $q \notin J^+(p)$, then $d(p, q) = 0$ by definition. Moreover, $d(p, q) > 0$ if and only if $q \in J^+(p)$. Given a point $p \in M$ one can
define the Lorentzian distance function \( d_p : M \to [0, +\infty) \) with respect to \( p \) by
\[
d_p(q) = d(p, q).
\]
Let
\[
T_{-1}M_p = \{ v \in T_p M | v \text{ is a future-directed timelike unit vector} \}
\]
be the fiber of the unit future observer bundle of \( M^{n+1} \) at \( p \). Define the function
\[
s_p : T_{-1}M_p \to [0, +\infty], \quad s_p(v) = \sup \{ t \geq 0 | d_p(\gamma_v(t)) = t \},
\]
where \( \gamma_v : [0, a) \to M \) is the future timelike geodesic with \( \gamma_v(0) = p, \gamma_v'(0) = v \). The future timelike cutlocus \( \Gamma^+(p) \) of \( p \) in \( T_p M \) is defined as
\[
\Gamma^+(p) = \{ s_p(v)v | v \in T_p M \text{ and } 0 < s_p(v) < +\infty \}
\]
and the future timelike cutlocus \( C_I^+(p) \) of \( p \) in \( M \) is \( C_I^+(p) = \exp_p(\Gamma^+(p)) \) wherever the exponential map \( \exp_p \) at \( p \) is defined on \( \Gamma^+(p) \).

It is well known that the Lorentzian distance function on arbitrary spacetimes may fail in general to be continuous and finite valued. It is known that this is true for globally hyperbolic spacetimes. We recall that a spacetime \( M \) is said to be \textit{globally hyperbolic} if it is strongly causal and it satisfies the condition that \( J^+(p) \cap J^-(q) \) is compact for all \( p, q \in M \). Moreover, a Lorentzian manifold \( M \) is said to be \textit{strongly causal} at a point \( p \in M \) if for any neighborhood \( U \) of \( p \) there exists no timelike curve that passes through \( U \) more than once. In general, in order to guarantee the smoothness of this function we need to restrict it on certain special subsets of \( M \). Let
\[
\tilde{I}^+(p) = \{ tv | v \in T_{-1}M_p \text{ and } 0 < t < s_p(v) \},
\]
and let
\[
\mathcal{I}^+(p) = \exp(\text{int}(\tilde{I}^+(p))) \subset I^+(p).
\]
Since
\[
\exp_p : \text{int}(\tilde{I}^+(p)) \to \mathcal{I}^+(p)
\]
is a diffeomorphism, \( \mathcal{I}^+(p) \) is an open subset of \( M \). In the lemma below we summarize the main properties of the Lorentzian distance function.

\textbf{Lemma 1} \cite{9}, Section 3.1. \textit{Let \( M \) be a spacetime and \( p \in M \).}

\begin{enumerate}
  \item If \( M \) is strongly causal at \( p \), then \( s_p(v) > 0 \forall v \in T_{-1}M_p \) and \( \mathcal{I}^+(p) \neq \emptyset \),
  \item If \( \mathcal{I}^+(p) \neq \emptyset \), then the Lorentzian distance function \( d_p \) is smooth on \( \mathcal{I}^+(p) \) and \( \nabla d_p \) is a past-directed timelike (geodesic) unit vector field on \( \mathcal{I}^+(p) \).
\end{enumerate}
Remark 2. If $M$ is a globally hyperbolic spacetime and $\Gamma^+(p) = \emptyset$, then $I^+(p) = I^+(p)$ and hence the Lorentzian distance function $d_p$ with respect to $p$ is smooth on $I^+(p)$ for each $p \in M$.

We also observe that if $M$ is a Lorentzian space form, then it is globally hyperbolic and geodesically complete. Moreover, every timelike geodesic realizes the distance between its points. Hence $\Gamma^+(p) = \emptyset$ and we conclude again that the Lorentzian distance function $d_p$ is smooth on $I^+(p)$ for each $p \in M$.

3. Hessian and Laplacian Comparison Theorems

This section is devoted to exhibit estimates for the Hessian and the Laplacian of the Lorentzian distance function in Lorentzian manifolds under conditions on the sectional or Ricci curvature. To prove our theorems we will need the following Sturm comparison result.

Lemma 3. Let $G$ be a continuous function on $[0, +\infty)$ and let $\phi, \psi \in C^1([0, +\infty))$ with $\phi', \psi' \in AC([0, +\infty))$ be solutions of the problems

\[
\begin{cases}
\phi'' - G\phi \leq 0 & \text{a.e. in } (0, +\infty) \\
\phi(0) = 0
\end{cases}
\quad \text{and} \quad
\begin{cases}
\psi'' - G\psi \geq 0 & \text{a.e. in } (0, +\infty) \\
\psi(0) = 0, \quad \psi'(0) > 0
\end{cases}
\]

If $\phi(r) > 0$ for $r \in (0, T)$ and $\psi'(0) \geq \phi'(0)$, then $\psi(r) > 0$ in $(0, T)$ and

$$\frac{\phi'}{\phi} \leq \frac{\psi'}{\psi} \quad \text{and} \quad \psi \geq \phi \quad \text{on } (0, T).$$

For a proof of the lemma see [15]. Using the Sturm comparison result, we obtain a comparison result for solutions of Riccati inequalities with appropriate asymptotic behaviour.

Corollary 4. Let $G$ be a continuous function on $[0, +\infty)$ and let $g_i \in AC((0, T_i))$ be solutions of the Riccati differentials inequalities

\[
g'_1 - \frac{g_2^2}{\alpha} + \alpha G \geq 0, \quad \text{(resp. } \leq 0) \quad g'_2 + \frac{g_1^2}{\alpha} - \alpha G \geq 0, \quad \text{(resp. } \leq 0)
\]

a.e. in $(0, T_i)$, satisfying the asymptotic conditions

$$g_i(t) = \frac{\alpha}{t} + o(t) \quad \text{as} \quad t \to 0^+,$$

for some $\alpha > 0$. Then $T_1 \leq T_2$ (resp. $T_1 \geq T_2$) and $-g_1(t) \leq g_2(t)$ in $(0, T_1)$ (resp. $-g_2(t) \leq g_1(t)$ in $(0, T_2)$).

Proof. Since $\tilde{g}_i = \alpha^{-1} g_i$ satisfies the conditions in the statement with $\alpha = 1$, without loss on generality we may assume that $\alpha = 1$. Notice that $g_i(s) - \frac{1}{s}$
is bounded and integrable in a neighbourhood of \( s = 0 \). Hence the same is true for the function \(-g_1(s) - \frac{1}{s}\). Indeed

\[-(g_1(s) + \frac{1}{s}) < -(g_1(s) - \frac{1}{s}) \leq \left| g_1(s) - \frac{1}{s} \right| \leq C,
\]

for some constant \( C > 0 \). Now let \( \phi_i \in C^1([0, T_i]) \) be the positive functions defined by

\[\phi_1(t) = t \exp \left( - \int_0^t (g_1(s) + \frac{1}{s}) ds \right), \phi_2(t) = t \exp \left( \int_0^t (g_2(s) - \frac{1}{s}) ds \right).\]

Then \( \phi_i(0) = 0, \phi_i' \in AC((0, T_i)), \phi_i'(0) = 1 \) and

\[\phi_1'(t) = -g_1(t)\phi_1(t), \phi_2'(t) = g_2(t)\phi_2(t)\]

Hence

\[\phi_1'' \leq G\phi_1, \phi_2'' \geq G\phi_2 \quad \text{(resp. } \phi_1'' \geq G\phi_1, \phi_2'' \leq G\phi_2).\]

Then, it follows by Lemma 3 that \( T_1 \leq T_2 \) (resp. \( T_1 \geq T_2 \)) and

\[-g_1(t) = \frac{\phi_1'(t)}{\phi_1(t)} \leq \frac{\phi_2'(t)}{\phi_2(t)} = g_2(t) \quad \text{(resp. } -g_2(t) = \frac{\phi_2'(t)}{\phi_2(t)} \leq \frac{\phi_1'(t)}{\phi_1(t)} = g_1(t)).\]

\[\square\]

We are now ready to prove the Hessian and Laplacian comparison theorems. In both cases we will follow the proofs given by S. Pigola, M. Rigoli and A. G. Setti in [15] of the corresponding theorems in the Riemannian setting.

We will denote by \( \nabla \) and \( \Delta \) respectively the Levi-Civita connection and the Laplacian on the spacetime \( M \). Moreover, for a given function \( f \in C^2(M) \), we denote by \( \mathbf{hess} f : TM \rightarrow TM \) the symmetric operator given by

\[\mathbf{hess} f(X) = \nabla_X \nabla f \text{ for every } X \in TM, \text{ and by } \mathbf{Hess} f : TM \times TM \rightarrow C^\infty(M)\]

the metrically equivalent bilinear form given by

\[\mathbf{Hess} f(X, Y) = \langle \mathbf{hess} f(X), Y \rangle.\]

**Theorem 5** (Hessian Comparison Theorem). Let \( M^{n+1} \) be an \( n+1 \)-dimensional spacetime. Assume that there exists a point \( p \in M \) such that \( \mathcal{I}^+(p) \neq \emptyset \) and let \( r(\cdot) = d_p(\cdot) \) be the Lorentzian distance function from \( p \). Given a smooth even function \( G \) on \( \mathbb{R} \), let \( h \) be a solution of the Cauchy problem

\[
\begin{cases}
    h'' - Gh = 0 \\
    h(0) = 0, \ h'(0) = 1
\end{cases}
\]

and let \( I = [0, r_0) \subset [0, +\infty) \) be the maximal interval where \( h \) is positive and \( q \in \mathcal{I}^+(p) \cap B^+(p, r_0), \) where

\[B^+(p, r_0) = \{ q \in \mathcal{I}^+(p) | d_p(q) < r_0 \}\].
If
\[(1)\quad K_M(\Pi) \leq G(r)\]
for all timelike planes $\Pi$, then
\[
\text{Hess}_r(X,X) \geq -\frac{h'(r)}{h}(r)\langle X,X \rangle
\]
for every spacelike $X \in T_qM$ which is orthogonal to $\nabla r$. Analogously, if
\[(2)\quad K_M(\Pi) \geq G(r)\]
for all timelike planes $\Pi$, then
\[
\text{Hess}_r(X,X) \leq -\frac{h'(r)}{h}(r)\langle X,X \rangle
\]
for every spacelike $X \in T_qM$ which is orthogonal to $\nabla r$.

Proof. Let $v \in \exp_p^{-1}(q) \in \text{int}(\bar{I}^+(p))$ and let $\gamma(t) = \exp_p(tv)$, $0 \leq t \leq s_p(v)$, be the radial future directed unit timelike geodesic with $\gamma(0) = p$, $\gamma(s) = q$, $s = r(q)$. Recall that $\gamma'(s) = -\nabla r(q)$ and $\nabla_{\gamma'}\nabla r(q) = 0$. Since $\nabla r$ satisfies the timelike eikonal inequality, $\text{Hess}_r$ is diagonalizable (see [11] Chapter 6 or [10] for more details) and $T_qM$ has an orthonormal basis consisting of eigenvectors of $\text{Hess}_r$. Let us denote by $\lambda_{\max}(q)$ and $\lambda_{\min}(q)$ respectively its greatest and smallest eigenvalues in the orthogonal complement of $\nabla r(q)$.

Notice that the theorem is proved once one shows that

(a) if (1) holds, then
\[
\lambda_{\min}(q) \geq -\frac{h'(r)}{h}(r(q)).
\]

(b) if (2) holds, then
\[
\lambda_{\max}(q) \leq -\frac{h'(r)}{h}(r(q)).
\]

Let us prove claim (a) first. We claim that if (1) holds, then $\lambda_{\min}$ satisfies
\[
(3)\quad \begin{cases}
\frac{d}{dt}(\lambda_{\min} \circ \gamma) - (\lambda_{\min} \circ \gamma)^2 & \geq -G \\
\lambda_{\min} \circ \gamma & = \frac{1}{t} + o(t) \quad \text{as } t \to 0^+
\end{cases}
\]

Namely, by the definition of covariant derivative
\[
(\nabla_X \text{hess}_u)(Y) = \nabla_X(\text{hess}_u(Y)) - \text{hess}_u(\nabla_X Y).
\]
Hence, recalling the definition of the curvature tensor we find
\[
(\nabla_{\nabla_Y \text{hess}_u}(X)) - (\nabla_X \text{hess}_u)(Y) = \overline{R}(X,Y)\nabla u.
\]
Choose \( u = r \), \( X = \nabla r \). For every spacelike unit vector \( Y \in T_qM \), \( Y \) is orthogonal to \( \gamma'(s) \) and we can define a vector field \( Y \) orthogonal to \( \gamma' \) by parallel translation along \( \gamma \). Then

\[
\nabla_{\gamma'(s)}(\text{hess}(Y)) = (\nabla_{\gamma'(s)}\text{hess})(Y) + \text{hess}(\nabla_{\gamma'(s)}Y)
\]

\[
= - (\nabla_{\nabla r}\text{hess})(Y)
\]

\[
= - (\nabla Y\text{hess})(\nabla r) + R(\nabla r, Y)\nabla r
\]

\[
= \text{hess}(\nabla Y\nabla r) + R(\nabla r, Y)\nabla r.
\]

On the other hand, since \( Y \) is parallel

\[
\frac{d}{dt}\langle\text{hess}(Y), Y\rangle\bigg|_s = \langle \nabla_{\gamma'(s)}\text{hess}(Y), Y\rangle.
\]

Hence

\[
\frac{d}{dt}\text{Hess}(\gamma)(Y, Y) - \langle \text{hess}(\gamma)(Y), \text{hess}(\gamma)(Y)\rangle = -K_M\gamma(Y \wedge \gamma')
\]

Notice that

\[
\text{Hess}(X, X) \geq \lambda_{\text{min}}
\]

for every spacelike unit vector field \( X \perp \nabla r \). Let us choose \( Y \) so that at \( s \)

\[
\text{Hess}(\gamma)(Y, Y) = \lambda_{\text{min}}(\gamma(s)).
\]

Then, the function \( \text{Hess } r(\gamma)(Y, Y) - \lambda_{\text{min}} \circ \gamma \) attains its minimum at \( s \). Hence

\[
\frac{d}{dt}\text{Hess}(\gamma)(Y, Y)\bigg|_s = \frac{d}{dt}(\lambda_{\text{min}} \circ \gamma)\bigg|_s
\]

and we have proved that \( \lambda_{\text{min}} \) satisfies the first equation in (3), since \( K(Y \wedge \gamma') \leq G \). The asymptotic behaviour follows from the expression

\[
\text{Hess } r = \frac{1}{r}(\langle , \rangle + dr \otimes dr) + o(1)
\]

that can be proved using normal coordinates around \( p \). Now, if we set \( \phi = \frac{h'}{h} \), we find that \( \phi \) satisfies

\[
\begin{cases}
\phi' + \phi^2 = G & \text{on } (0, r_0) \\
\phi = \frac{1}{t} + o(t) & \text{as } t \to 0^+
\end{cases}
\]

Then, using Corollary 4 with \( g_1 = \lambda_{\text{min}}, g_2 = \phi \) and \( \alpha = 1 \) we conclude that

\[
\lambda_{\text{min}}(q) \geq -\frac{h'}{h}(r(q))
\]

and this concludes the proof of (a).

Finally, for what concerns claim (b), we observe that reasoning as in the proof of claim (a) and choosing \( Y \) so that at \( s \)

\[
\text{Hess}(\gamma)(Y, Y) = \lambda_{\text{max}}(\gamma(s))
\]
we can prove that, if (2) holds, \( \lambda_{\text{max}} \) satisfies
\[
\begin{align*}
\frac{d}{dt}(\lambda_{\text{max}} \circ \gamma) - (\lambda_{\text{max}} \circ \gamma)^2 & \leq -G \\
\lambda_{\text{max}} \circ \gamma = \frac{1}{t} + o(t) & \quad \text{as } t \to 0^+
\end{align*}
\]
In this case, setting again \( \phi = \frac{h'}{h} \), we find that \( \phi \) satisfies
\[
\begin{align*}
\phi' + \phi^2 &= G & \text{on } (0, r_0) \\
\phi &= \frac{1}{t} + o(t) & \text{as } t \to 0^+
\end{align*}
\]
Then, we can conclude again using Corollary 4 with \( g_1 = \lambda_{\text{max}}, g_2 = \phi \) and \( \alpha = 1. \)

\[ \square \]

**Theorem 6** (Laplacian Comparison Theorem). Let \( M^{n+1} \) be an \( n + 1 \)-dimensional spacetime. Assume that there exists a point \( p \in M \) such that \( \mathcal{I}^+(p) \neq \emptyset \) and let \( q \in \mathcal{I}^+(p) \). Let \( r(\cdot) = d_p(\cdot) \) be the Lorentzian distance function from \( p \). Given a smooth even function \( G \) on \( \mathbb{R} \), let \( h \) be a solution of the Cauchy problem
\[
\begin{align*}
{h''} - Gh & \geq 0 \\
h(0) = 0, & \quad h'(0) = 1
\end{align*}
\]
and let \( I = [0, r_0) \subset [0, +\infty) \) be the maximal interval where \( h \) is positive. If
\( (5) \quad \text{Ric}_M(\nabla r, \nabla r) \geq -nG(r), \)
then
\[
\Delta r \geq -n\frac{h'}{h}(r)
\]
holds pointwise on \( \mathcal{I}^+(p) \cap B^+(p, r_0). \)

**Proof.** Let \( v \in \exp^{-1}_p(q) \in \text{int}(\mathcal{I}^+(p)) \) and let \( \gamma(t) = \exp_p(tv), 0 \leq t \leq s_p(v), \) be the radial future directed unit timelike geodesic with \( \gamma(0) = p, \gamma(s) = q, \)
\( s = r(q). \) Recall that \( \gamma'(s) = -\nabla r(q) \) and \( \nabla_{\nabla r} \nabla r(q) = 0. \) Define
\[
\varphi(t) = \Delta r \circ \gamma(t), \quad t \in (0, s].
\]
Then tracing Equation (1)
\[
\varphi(t) = \frac{n}{t} + o(t) \quad \text{as } t \to 0^+.
\]
Recall that given \( f \in C^\infty(M) \) the following Bochner formula holds
\[
\frac{1}{2} \Delta \langle \nabla f, \nabla f \rangle = \|\text{hess}f\|^2 + \text{Ric}_M(\nabla f, \nabla f) + \langle \nabla \Delta f, \nabla f \rangle.
\]
See [11] for more details. Since \( \|\nabla r\|^2 = -1 \), it follows that
\[
0 = \|\text{hess}r\|^2 + \text{Ric}_M(\nabla r, \nabla r) + \langle \nabla \Delta r, \nabla r \rangle.
\]
Since $\|\text{Hess} r\|^2 \geq \frac{(\Delta r)^2}{n}$ and $\text{Ric}_M (\nabla r, \nabla r) \geq -nG(r)$, we have

$$\frac{1}{n} (\Delta r)^2 + \langle \nabla \Delta r, \nabla r \rangle \leq nG(r).$$

Computing along $\gamma$

$$\varphi'(t) = \frac{d}{dt} (\Delta r(\gamma(t))) \bigg|_s = \langle \nabla \Delta r(\gamma(t)), \gamma'(t) \rangle \bigg|_s = -\langle \nabla \Delta r, \nabla r \rangle.$$

Hence the function $\varphi$ satisfies

$$\begin{cases}
\varphi'(t) - \frac{\varphi^2(t)}{n} \geq -nG \\
\varphi(t) = \frac{\varphi'}{t} + o(t) \quad \text{as } t \to 0^+
\end{cases}$$

Set $\phi = n \frac{\varphi'}{\varphi}$. Then $\phi$ satisfies

$$\begin{cases}
\phi'(t) + \frac{\phi^2(t)}{n} \geq nG \quad \text{on } (0, r_0) \\
\phi(t) = \frac{\phi'}{t} + o(t) \quad \text{as } t \to 0^+
\end{cases}$$

Then we conclude again using Corollary 4.

4. Applications to spacelike hypersurfaces

Let $\psi : \Sigma^n \to M^{n+1}$ be a spacelike hypersurface isometrically immersed into the spacetime $M$. Since $M$ is time-orientable, there exists a unique future-directed timelike unit normal field $\nu$ globally defined on $\Sigma$. We will refer to that normal field $\nu$ as the future-pointing Gauss map of the hypersurface. We let $A : T\Sigma \to T\Sigma$ denote the second fundamental form of the immersion. Its eigenvalues $k_1, ..., k_n$ are the principal curvatures of the hypersurface. Their elementary symmetric functions

$$S_k = \sum_{i_1 < ... < i_k} k_{i_1} \cdots k_{i_k}, \quad k = 1, ..., n,$$

$$S_0 = 1,$$

define the $k$-mean curvatures of the immersion via the formula

$$\binom{n}{k} H_k = (-1)^k S_k.$$

Thus $H_1 = -1/n \text{Tr}(A) = H$ is the mean curvature of $\Sigma$ and $n(n-1)H_2 = \overline{S} - S + 2\text{Ric}(\nu, \nu)$, where $S$ and $\overline{S}$ are, respectively, the scalar curvature of $\Sigma$ and $M^{n+1}$ and $\text{Ric}$ is the Ricci tensor of $M^{n+1}$. Even more, when $k$ is even, it follows from the Gauss equation that $H_k$ is a geometric quantity which is related to the intrinsic curvature of $\Sigma^n$.
The classical Newton transformations associated to the immersion are defined inductively by

\[ P_0 = I, \quad P_k = \binom{n}{k} H_k I + AP_{k-1}, \]

for every \( k = 1, \ldots, n \).

**Proposition 7.** The following formulas hold:

1. \( \text{Tr}(P_k) = c_k H_k \),
2. \( \text{Tr}(AP_k) = -c_k H_{k+1} \),
3. \( \text{Tr}(A^2 P_k) = \binom{n}{k+1} (nH_1 H_{k+1} - (n - k - 1)H_{k+2}) \),

where \( c_k = (n - k) \binom{n}{k} = (k + 1) \binom{n}{k+1} \).

We refer the reader to [3] for the proof of the last proposition and for further details on the Newton transformations (see also [16] and [17] for others details on the Newton transformations in the Riemannian setting).

Let \( \nabla \) be the Levi-Civita connection of \( \Sigma \). We define the second order linear differential operator \( L_k : C^\infty(\Sigma) \to C^\infty(\Sigma) \) associated to \( P_k \) by

\[ L_k f = \text{Tr}(P_k \circ \text{hess} f). \]

It follows by the definition that the operator \( L_k \) is elliptic if and only if \( P_k \) is positive definite. Let us state two useful lemmas in which geometric conditions are given in order to guarantee the ellipticity of \( L_k \) when \( k \geq 1 \) (Recall that \( L_0 = \Delta \) is always elliptic).

**Lemma 8.** Let \( \Sigma \) be a spacelike hypersurface immersed into a spacetime. If \( H_2 > 0 \) on \( \Sigma \), then \( L_1 \) is an elliptic operator (for an appropriate choice of the Gauss map \( \nu \)).

For a proof of Lemma 8 see Lemma 3.10 in [8]. The next Lemma is a consequence of Proposition 3.2 in [7].

**Lemma 9.** Let \( \Sigma^n \) be a spacelike hypersurface immersed into a \( n + 1 \)-dimensional spacetime. If there exists an elliptic point of \( \Sigma \), with respect to an appropriate choice of the Gauss map \( \nu \), and \( H_k > 0 \) on \( \Sigma \), \( 3 \leq k \leq n \), then for all \( 1 \leq j \leq k - 1 \) the operator \( L_j \) is elliptic.

We recall here that given a spacelike hypersurface \( \Sigma \), a point \( p \in \Sigma \) is said to be elliptic if the second fundamental form of the immersion is negative definite at \( p \).

Now consider \( \psi : \Sigma^n \to M^{n+1} \) and assume that there exists a point \( p \in M \) such that \( \mathcal{I}^+(p) \neq \emptyset \) and that \( \psi(\Sigma) \subset \mathcal{I}^+(p) \). Let \( r(\cdot) = d_p(\cdot) \) be the Lorentzian distance function from \( p \) and let \( u = r \circ \psi : \Sigma \to (0, +\infty) \) be the
function \( r \) along the hypersurface, which is a smooth function on \( \Sigma \). Let us calculate the Hessian of \( u \) on \( \Sigma \). Notice that

\[
\nabla_r = \nabla u - \langle \nabla_r, \nu \rangle \nu.
\]

Hence, since \( \|\nabla_r\| = -1 \) and \( \langle \nabla_r, \nu \rangle > 0 \), we have

\[
\langle \nabla_r, \nu \rangle = \sqrt{1 + \|\nabla u\|^2} \geq 1.
\]

Hence

\[
\nabla_r = \nabla u - \nu \sqrt{1 + \|\nabla u\|^2}
\]

Moreover

\[
\nabla_X \nabla_r = \nabla_X \nabla u + \sqrt{1 + \|\nabla u\|^2}AX + \langle AX, \nabla u \rangle \nu - X(\sqrt{1 + \|\nabla u\|^2})\nu
\]

for every spacelike \( X \in T\Sigma \). Thus

\[
\text{Hess } u(X, P_k X) = \text{Hess } r(X, P_k X) - \sqrt{1 + \|\nabla u\|^2} \langle P_k AX, X \rangle
\]

On the other hand, we have the following decompositions

\[
X = X^* - \langle X, \nabla u \rangle \nabla r
\]

\[
P_k X = (P_k X)^* - \langle X, P_k \nabla u \rangle \nabla r;
\]

where \( X^* \), \( (P_k X)^* \) are respectively the components of \( X \), \( P_k X \) orthogonal to \( \nabla r \). Then

\[
\langle X^*, (P_k X)^* \rangle = \langle X, P_k X \rangle + \langle X, P_k \nabla u \rangle \langle X, \nabla u \rangle
\]

and, taking into account that

\[
\nabla_{\nabla r} \nabla r = 0
\]

we find

\[
\text{Hess } r(X, P_k X) = \text{Hess } (X^*, (P_k X)^*).
\]

Hence, if we assume that \( K_M(\Pi) \leq G(r) \) for all timelike planes \( \Pi \), then

\[
\text{Hess } r(X, P_k X) \geq -\frac{h'}{h}(u) \langle X^*, (P_k X)^* \rangle
\]

\[
= -\frac{h'}{h}(u) \langle X, P_k X \rangle + \langle X, \nabla u \rangle \langle X, P_k \nabla u \rangle,
\]

where \( h \) is a solution of the problem

\[
\begin{cases}
  h'' - G h = 0 \\
  h(0) = 0, \ h'(0) = 1
\end{cases}
\]
Therefore

\[
\text{Hess } u(X, P_k X) \geq - \frac{h'}{h}(u) \langle X, P_k X \rangle + \langle X, \nabla u \rangle \langle X, P_k \nabla u \rangle - \sqrt{1 + \|\nabla u\|^2} \langle P_k AX, X \rangle.
\]

Tracing

\[
L_k u \geq - \frac{h'}{h}(u) (c_k H_k + \langle \nabla u, P_k \nabla u \rangle) + \sqrt{1 + \|\nabla u\|^2} c_k H_{k+1}.
\]

Summarizing, we have proved the following

**Proposition 10.** Let \( M^{n+1} \) be an \( n+1 \)-dimensional spacetime. Assume that there exists a point \( p \in M \) such that \( I^+(p) \neq \emptyset \) and let \( r(\cdot) = d_p(\cdot) \) be the Lorentzian distance function from \( p \). Given a smooth even function \( G \) on \( \mathbb{R} \), let \( h \) be a solution of the Cauchy problem

\[
\begin{cases}
  h'' - G h = 0 \\
  h(0) = 0, \quad h'(0) = 1
\end{cases}
\]

and let \( I = [0, r_0) \subset [0, +\infty) \) be the maximal interval where \( h \) is positive. Let \( \psi : \Sigma^n \to M^{n+1} \) be a spacelike hypersurface such that \( \psi(\Sigma^n) \subset I^+(p) \cap B^+(p, r_0) \). If

\[
K_M(\Pi) \leq G(r)
\]

for all timelike planes \( \Pi \), then

\[
L_k u \geq - \frac{h'}{h}(u) (c_k H_k + \langle \nabla u, P_k \nabla u \rangle) + \sqrt{1 + \|\nabla u\|^2} c_k H_{k+1}.
\]

On the other hand, if we assume that \( K_M(\Pi) \geq G(r) \) for all timelike planes in \( M \), the same computations yield the following

**Proposition 11.** Let \( M^{n+1} \) be an \( n+1 \)-dimensional spacetime. Assume that there exists a point \( p \in M \) such that \( I^+(p) \neq \emptyset \) and let \( r(\cdot) = d_p(\cdot) \) be the Lorentzian distance function from \( p \). Given a smooth even function \( G \) on \( \mathbb{R} \), let \( h \) be a solution of the Cauchy problem

\[
\begin{cases}
  h'' - G h = 0 \\
  h(0) = 0, \quad h'(0) = 1
\end{cases}
\]

and let \( I = [0, r_0) \subset [0, +\infty) \) be the maximal interval where \( h \) is positive. Let \( \psi : \Sigma^n \to M^{n+1} \) be a spacelike hypersurface such that \( \psi(\Sigma^n) \subset I^+(p) \cap B^+(p, r_0) \). If

\[
K_M(\Pi) \geq G(r)
\]

for all timelike planes \( \Pi \), then

\[
L_k u \leq - \frac{h'}{h}(u) (c_k H_k + \langle \nabla u, P_k \nabla u \rangle) + \sqrt{1 + \|\nabla u\|^2} c_k H_{k+1}.
\]
In the following, under suitable bounds on the sectional curvature of the ambient spacetime, we will find some lower and upper bounds for the mean curvature and the higher order mean curvatures associated to the immersion. 

In order to do it we will use the Omori-Yau maximum principle for the Laplacian and for more general elliptic operators (for more details and others applications of this technique see [5], [6]). Namely, if \( L = \text{Tr}(P \circ \text{hess}) \), where \( P \) is a symmetric operator with trace bounded above, using the terminology introduced by S. Pigola, M. Rigoli and A. G. Setti in [14], we say that the Omori-Yau maximum principle holds on \( \Sigma \) for \( L \) if for any smooth function \( u \in C^\infty(\Sigma) \) with \( u^* = \sup_{\Sigma} u < +\infty \) there exists a sequence of points \( \{p_i\}_{i \in \mathbb{N}} \subset \Sigma \) such that

\[
(10) \quad (i) \ u(p_i) > u^* - \frac{1}{i}, \quad (ii) \ |\nabla u(p_i)| < \frac{1}{i}, \quad (iii) \ Lu(p_i) < -\frac{1}{i}.
\]

Equivalently if \( u_* = \inf_{\Sigma} u > -\infty \), we can find a sequence \( \{q_i\}_{i \in \mathbb{N}} \subset \Sigma \) such that

\[
(11) \quad (i) \ u(q_i) > u_* - \frac{1}{i}, \quad (ii) \ |\nabla u(q_i)| < \frac{1}{i}, \quad (iii) \ Lu(q_i) > -\frac{1}{i}.
\]

Clearly the Laplacian belong to this class of operators. In this case, S. Pigola, M. Rigoli and A. G. Setti showed in [14] that a condition of the form

\[
(12) \quad \text{Ric}(\nabla \rho, \nabla \rho) \geq -C^2 G(\rho),
\]

where \( \rho \) is the distance function on \( \Sigma \) to a fixed point and \( G : [0, +\infty) \to \mathbb{R} \) is a smooth function satisfying

\[
(13) \quad (i) \ G(0) > 0, \quad (ii) \ G'(t) \geq 0 \quad \text{on} \ [0, +\infty), \quad (iii) \ G(t)^{-\frac{1}{2}} \in L^1(+\infty), \quad (iv) \ \limsup_{t \to +\infty} \frac{tG(\sqrt{t})}{G(t)} < +\infty.
\]

is sufficient to guarantee the validity of the Omori-Yau maximum principle for the Laplacian on \( \Sigma \). Analogously, in [5], L. J. Alias, M. Rigoli and the author showed that the condition

\[
(14) \quad K(\nabla \rho, X) \geq -G(\rho),
\]

where \( X \) is any vector field tangent to \( \Sigma \) and \( G \) satisfies (13), is sufficient to guarantee the validity of the Omori-Yau maximum principle on \( \Sigma \) for operators \( L \) with the properties described above.

Applying the Omori-Yau maximum principle we find the following estimates for the mean curvature. The proof of the following theorems is essentially the same as that of Theorems 4.1 and 4.2 in [4].

**Theorem 12.** Let \( M^{n+1} \) be an \( n + 1 \)-dimensional spacetime. Assume that there exists a point \( p \in M \) such that \( I^+(p) \neq \emptyset \) and let \( r(\cdot) = d_p(\cdot) \) be the
Lorentzian distance function from $p$. Given a smooth even function $G$ on $\mathbb{R}$, let $h$ be a solution of the Cauchy problem
\[
\begin{cases}
h'' - Gh = 0 \\
h(0) = 0, \quad h'(0) = 1
\end{cases}
\]
and let $I = [0, r_0) \subset [0, +\infty)$ be the maximal interval where $h$ is positive.
Let $\psi : \Sigma^n \to M^{n+1}$ be a spacelike hypersurface such that $\psi(\Sigma^n) \subset I^+(p) \cap B^+(p, \delta)$ with $\delta \leq r_0$. If
\[
(15) \quad \text{Ric}_M(\nabla r, \nabla r) \geq -nG(r),
\]
and the Omori-Yau maximum principle holds on $\Sigma$, then
\[
\inf_\Sigma H_1 \leq \frac{h'}{h} \left( \sup_\Sigma u \right),
\]
where $u$ denotes the Lorentzian distance $d_p$ along the hypersurface.

On the other hand, if we assume that the sectional curvature of timelike planes is bounded from below we obtain

**Theorem 13.** Let $M^{n+1}$ be an $n+1$-dimensional spacetime. Assume that there exists a point $p \in M$ such that $I^+(p) \neq \emptyset$ and let $r(\cdot) = d_p(\cdot)$ be the Lorentzian distance function from $p$. Given a smooth even function $G$ on $\mathbb{R}$, let $h$ be a solution of the Cauchy problem
\[
\begin{cases}
h'' - Gh = 0 \\
h(0) = 0, \quad h'(0) = 1
\end{cases}
\]
and let $I = [0, r_0) \subset [0, +\infty)$ be the maximal interval where $h$ is positive
and let $\psi : \Sigma^n \to M^{n+1}$ be a spacelike hypersurface such that $\psi(\Sigma^n) \subset I^+(p) \cap B^+(p, r_0)$. If
\[
(16) \quad K_M(\Pi) \geq G(r)
\]
for all timelike planes $\Pi$ and if the Omori-Yau maximum principle holds on $\Sigma$, then
\[
\sup_\Sigma H_1 \geq \frac{h'}{h} \left( \inf_\Sigma u \right),
\]
where $u$ denotes the Lorentzian distance $d_p$ along the hypersurface $\Sigma$.

The previous estimates can be extended to the higher order mean curva-
tures in the following way. To find the estimates we will use the Omori-Yau maximum principle for elliptic operators of the form $L = \text{Tr}(P \circ \text{hess})$, where $P$ is a symmetric operator with trace bounded above. For simplicity, we will refer to that as the generalized Omori-Yau maximum principle.
Theorem 14. Let \( M^{n+1} \) be an \( n+1 \)-dimensional spacetime. Assume that there exists a point \( p \in M \) such that \( \mathcal{I}^+(p) \neq \emptyset \) and let \( r(\cdot) = d_p(\cdot) \) be the Lorentzian distance function from \( p \). Given a smooth even function \( G \) on \( \mathbb{R} \), let \( h \) be a solution of the Cauchy problem
\[
\begin{cases}
 h'' - Gh = 0 \\
 h(0) = 0, \ h'(0) = 1
\end{cases}
\]
and let \( I = [0, r_0) \subset [0, +\infty) \) be the maximal interval where \( h \) is positive. Let \( \psi : \Sigma^n \rightarrow M^{n+1} \) be a spacelike hypersurface such that \( \psi(\Sigma^n) \subset \mathcal{I}^+(p) \cap B^+(p, \delta) \), with \( \delta \leq r_0 \). Assume that \( H_2 > 0 \) and that \( \sup_\Sigma H_1 < +\infty \). If
\[\tag{17} K_M(\Pi) \leq G(r)\]
for all timelike planes \( \Pi \) and if Omori-Yau maximum principle holds on \( \Sigma \), then
\[
\inf_\Sigma \frac{H_2}{h} \leq \left| \frac{h'}{h} \left( \sup_\Sigma u \right) \right|,
\]
where \( u \) denotes the Lorentzian distance \( d_p \) along the hypersurface.

Proof. Consider the operator
\[
\mathcal{L} = L_1 + (n - 1) \frac{1}{\sqrt{1 + \|\nabla u\|^2}} \left( \left| \frac{h'}{h} (u) \right| \right) \Delta = \text{Tr}(P \circ \text{hess}),
\]
where
\[
P = P_1 + (n - 1) \frac{1}{\sqrt{1 + \|\nabla u\|^2}} \left( \left| \frac{h'}{h} (u) \right| \right) I.
\]
Notice that, since \( H_2 > 0 \), the operator \( L_1 \) is elliptic and so is \( \mathcal{L} \). Since \( 0 < u < \sup_\Sigma u < \delta \), \( h'/h(u) \) is bounded. Furthermore, \( \sup_\Sigma H_1 < +\infty \) and \( 1/\sqrt{1 + \|\nabla u\|^2} \leq 1 \), hence we can apply the Omori-Yau maximum principle for the operator \( \mathcal{L} \). We can then find a sequence \( \{p_i\}_{i \in \mathbb{N}} \subset \Sigma \) such that
\[
(i) \ u(p_i) > u^* - \frac{1}{i}, \ (ii) \ \|\nabla u(p_i)\| < \frac{1}{i}, \ (iii) \ \mathcal{L}u(p_i) < \frac{1}{i}.
\]
A simple computation using Proposition 10 shows that
\[ \mathcal{L}u \geq -(n-1) \frac{1}{\sqrt{1 + \|\nabla u\|^2}} \left( \frac{h'(u)}{h} \right)^2 (n + \|\nabla u\|^2) - \left( \left| \frac{h'}{h}(u) \right| \right) (P_1 \nabla u, \nabla u) + n(n-1) \sqrt{1 + \|\nabla u\|^2} H_2 \]
\[ \geq -(n-1) \frac{1}{\sqrt{1 + \|\nabla u\|^2}} \left( \frac{h'(u)}{h} \right)^2 (n + \|\nabla u\|^2) - \left( \left| \frac{h'}{h}(u) \right| \right) (P_1 \nabla u, \nabla u) + n(n-1) \sqrt{1 + \|\nabla u\|^2} \inf \Sigma H_2. \]
Hence
\[ \frac{1}{i} > \mathcal{L}(p_i) \geq -(n-1) \frac{1}{\sqrt{1 + \|\nabla u(p_i)\|^2}} \left( \frac{h'(u(p_i))}{h} \right)^2 (n + \|\nabla u(p_i)\|^2) - \left( \left| \frac{h'}{h}(u(p_i)) \right| \right) (P_1 \nabla u(p_i), \nabla u(p_i)) + n(n-1) \sqrt{1 + \|\nabla u(p_i)\|^2} \inf \Sigma H_2. \]
Taking the limit for \( i \to +\infty \) we find
\[ 0 \geq -n(n-1) \left( \frac{h'(u)}{h} (\sup_{\Sigma} u) \right)^2 + n(n-1) \inf \Sigma H_2. \]
and the conclusion follows. \( \square \)

**Theorem 15.** Let \( M^{n+1} \) be an \( n+1 \)-dimensional spacetime, \( n \geq 3 \). Assume that there exists a point \( p \in M \) such that \( \mathcal{I}^+(p) \neq \emptyset \) and let \( r(\cdot) = d_p(\cdot) \) be the Lorentzian distance function from \( p \). Given a smooth even function \( G \) on \( \mathbb{R} \), let \( h \) be a solution of the Cauchy problem
\[
\begin{cases}
  h'' - Gh = 0 \\
  h(0) = 0, \quad h'(0) = 1
\end{cases}
\]
and let \( I = [0, r_0) \subset [0, +\infty) \) be the maximal interval where \( h \) is positive. Let \( \psi : \Sigma^n \to M^{n+1} \) be a spacelike hypersurface such that \( \psi(\Sigma^n) \subset \mathcal{I}^+(p) \cap B^+(p, \delta) \), with \( \delta \leq r_0 \). Assume that there exists an elliptic point \( p_0 \in \Sigma \), that \( H_k > 0 \), \( 3 \leq k \leq n \), and that \( \sup_{\Sigma} H_1 < +\infty \). If
\[ K_M(\Pi) \leq G(r) \]
for all timelike planes \( \Pi \) and if the generalized Omori-Yau maximum principle holds on \( \Sigma \), then
\[ \inf_{\Sigma} h_k \geq \frac{1}{h} \left( \sup_{\Sigma} u \right) \leq \frac{1}{h} \left( \sup_{\Sigma} u \right), \]
where \( u \) denotes the Lorentzian distance \( d_p \) along the hypersurface.

**Proof.** Consider the operator

\[
L = \sum_{j=0}^{k-1} (1 + \|\nabla u\|^2)^{-\frac{k-j-1}{2}} \left( \left| \frac{h'}{h} (u) \right| \right)^{k-j-1} \frac{c_{k-1}}{c_j} \langle P_j \nabla u, \nabla u \rangle
\]

Notice that, since there exists an elliptic point \( p_0 \in \Sigma \) and \( H_k > 0 \), \( 3 \leq k \leq n \), the operators \( L_j \) are elliptic for all \( 1 \leq j \leq k - 1 \). Since \( 0 < u < \sup_{\Sigma} u < \delta \), \( 1/\sqrt{1 + \|\nabla u\|^2} \leq 1 \) and \( \sup_{\Sigma} H_1 < +\infty \), we can apply the Omori-Yau maximum principle for the operator \( L \). Hence, we can find a sequence \( \{p_i\}_{i \in \mathbb{N}} \subset \Sigma \) such that

1. \( u(p_i) > u^* - \frac{1}{i} \),
2. \( \|\nabla u(p_i)\| < \frac{1}{i} \),
3. \( L u(p_i) < \frac{1}{i} \).

A straightforward computation using Proposition 10 shows that

\[
Lu \geq - \sum_{j=0}^{k-1} (1 + \|\nabla u\|^2)^{-\frac{k-j-1}{2}} \left( \left| \frac{h'}{h} (u) \right| \right)^{k-j-1} \frac{c_{k-1}}{c_j} \langle P_j \nabla u, \nabla u \rangle
- c_{k-1} \frac{1}{(1 + \|\nabla u\|^2)^{(k-1)/2}} \left( \left| \frac{h'}{h} (u) \right| \right)^k + \sqrt{1 + \|\nabla u\|^2} c_{k-1} H_k.
\]

Hence

\[
\frac{1}{i} > Lu(p_i) \geq - c_{k-1} \frac{1}{(1 + \|\nabla u(p_i)\|^2)^{(k-1)/2}} \left( \left| \frac{h'}{h} (u(p_i)) \right| \right)^k
- \sum_{j=1}^{k-1} (1 + \|\nabla u(p_i)\|^2)^{-\frac{k-j-1}{2}} \left( \left| \frac{h'}{h} (u(p_i)) \right| \right)^{k-j-1} \frac{c_{k-1}}{c_j} \langle P_j \nabla u, \nabla u \rangle (p_i)
+ \sqrt{1 + \|\nabla u(p_i)\|^2} c_{k-1} \inf_{\Sigma} H_k.
\]

Taking the limit for \( i \to +\infty \) we find

\[
0 \geq - c_{k-1} \left( \left| \frac{h'}{h} (\sup u) \right| \right)^k + c_{k-1} \inf_{\Sigma} H_k.
\]

□

On the other hand, if we assume that the sectional curvature of timelike planes is bounded from below we find the following estimates

**Theorem 16.** Let \( M^{n+1} \) be an \( n + 1 \)-dimensional spacetime. Assume that there exists a point \( p \in M \) such that \( \mathcal{I}^+(p) \neq \emptyset \) and let \( r(\cdot) = d_p(\cdot) \) be the
Lorentzian distance function from $p$. Given a smooth even function $G$ on $\mathbb{R}$, let $h$ be a solution of the Cauchy problem
\[
\begin{align*}
  h'' - Gh &= 0 \\
  h(0) &= 0, \quad h'(0) = 1
\end{align*}
\]
and let $I = [0, r_0) \subset [0, +\infty)$ be the maximal interval where $h$ is positive.

Let $\psi : \Sigma^n \to M^{n+1}$ be a spacelike hypersurface such that $\psi(\Sigma^n) \subset N^{+}(p) \cap B^{+}(p, r_0)$. Assume that $H_2 > 0$ and that $\sup_{\Sigma} H_1 < +\infty$. If
\[
K_M(\Pi) \geq G(r)
\]
for all timelike planes $\Pi$ and if the generalized Omori-Yau maximum principle holds on $\Sigma$, then
\[
\sup_{\Sigma} H_{\frac{1}{2}} \geq \frac{h'}{h}(\inf_{\Sigma} u),
\]
where $u$ denotes the Lorentzian distance $d_p$ along the hypersurface.

**Theorem 17.** Let $M^{n+1}$ be an $n+1$-dimensional spacetime, $n \geq 3$. Assume that there exists a point $p \in M$ such that $N^{+}(p) \neq \emptyset$ and let $r(\cdot) = d_p(\cdot)$ be the Lorentzian distance function from $p$. Given a smooth even function $G$ on $\mathbb{R}$, let $h$ be a solution of the Cauchy problem
\[
\begin{align*}
  h'' - Gh &= 0 \\
  h(0) &= 0, \quad h'(0) = 1
\end{align*}
\]
and let $I = [0, r_0) \subset [0, +\infty)$ be the maximal interval where $h$ is positive.

Let $\psi : \Sigma^n \to M^{n+1}$ be a spacelike hypersurface such that $\psi(\Sigma^n) \subset N^{+}(p) \cap B^{+}(p, r_0)$. Assume that there exists an elliptic point $p_0 \in \Sigma$, that $H_k > 0$, $3 \leq k \leq n$, and that $\sup_{\Sigma} H_1 < +\infty$. If
\[
K_M(\Pi) \geq G(r)
\]
for all timelike planes $\Pi$ and if the generalized Omori-Yau maximum principle holds on $\Sigma$, then
\[
\sup_{\Sigma} H_{\frac{1}{k}} \geq \frac{h'}{h}(\inf_{\Sigma} u),
\]
where $u$ denotes the Lorentzian distance $d_p$ along the hypersurface.

We will only prove Theorem 17. The proof of Theorem 16 proceed exactly in the same way.

**Proof of Theorem 17.** If $h'/h(\inf_{\Sigma} u) \leq 0$, the result is trivial since
\[
\frac{h'}{h}(\inf_{\Sigma} u) \leq 0 < \sup_{\Sigma} H_{\frac{1}{k}}.
\]
Conversely, assume $h'/h(\inf_{\Sigma} u) > 0$. Since $u \geq u_* := \inf_{\Sigma} u \geq 0$, we want to apply the Omori-Yau maximum principle for a suitable elliptic operator
with trace bounded above. Notice that it must be $\inf_{\Sigma} u > 0$. Indeed, if $\inf_{\Sigma} u = 0$, since $\lim_{s \to 0} h'/h(s) = +\infty$, it follows by the estimate in Theorem 13 that $\sup_{\Sigma} H_1 = +\infty$, which contradicts our assumptions. The operator that we consider is the following

$$L = \sum_{j=0}^{k-1} (1 + \|\nabla u\|^2)^{-\frac{k-j}{2}} \left( \frac{h'}{h} (\inf_{\Sigma} u) \right)^{k-j-1} L_j$$

$$= \text{Tr}(\mathcal{P} \circ \text{hess}),$$

where

$$\mathcal{P} = \sum_{j=0}^{k-1} (1 + \|\nabla u\|^2)^{-\frac{k-j}{2}} \left( \frac{h'}{h} (\inf_{\Sigma} u) \right)^{k-j-1} c_{k-1}^{-1} P_j.$$

Notice that, since there exists an elliptic point $p_0 \in \Sigma$ and $H_k > 0$, $3 \leq k \leq n$, the operators $L_j$ are elliptic for all $1 \leq j \leq k-1$ and so $L$ is elliptic as well. Furthermore, we observe that

$$\text{Tr} \mathcal{P} = \sum_{j=0}^{k-1} (1 + \|\nabla u\|^2)^{-\frac{k-j}{2}} \left( \frac{h'}{h} (\inf_{\Sigma} u) \right)^{k-j-1} c_{k-1}^{-1} H_j.$$

Since $1/\sqrt{1 + \|\nabla u\|^2} \leq 1$, $h'/h(\inf_{\Sigma} u) < +\infty$ and, by the Newton inequalities

$$H_j \leq H_j^1 < +\infty$$

we conclude that $\mathcal{P}$ has trace bounded above and we can apply the Omori-Yau maximum principle for the operator $L$. Hence, we can find a sequence $\{q_i\}_{i \in \mathbb{N}} \subset \Sigma$ such that

$$(i) \ u(q_i) < u_* + \frac{1}{i}, \ (ii) \ \|\nabla u(q_i)\| < \frac{1}{i}, \ (iii) \ L u(q_i) > -\frac{1}{i}.$$ 

A straightforward computation using Proposition 11 shows that

$$Lu \leq -\frac{h'}{h}(u) \sum_{j=0}^{k-1} (1 + \|\nabla u\|^2)^{-\frac{k-j}{2}} \left( \frac{h'}{h} (\inf_{\Sigma} u) \right)^{k-j-1} c_{k-1}^{-1} \langle P_j \nabla u, \nabla u \rangle$$

$$- c_{k-1} \frac{h'}{h}(u) \frac{1}{(1 + \|\nabla u\|^2)^{(k-1)/2}} \left( \frac{h'}{h} (\inf_{\Sigma} u) \right)^{k-1} + \sqrt{1 + \|\nabla u\|^2} c_{k-1} H_k$$

$$+ c_{k-1} \sum_{j=1}^{k-1} (1 + \|\nabla u\|^2)^{-\frac{k-j}{2}} \left( \frac{h'}{h} (\inf_{\Sigma} u) \right)^{k-j-1} \left( \frac{h'}{h} (\inf_{\Sigma} u) - \frac{h'}{h}(u) \right) H_j.$$
Evaluating the previous expression at \( q_i \), using condition \((iii)\) in \([21]\) and taking the limit for \( i \to +\infty \), we find

\[
0 \leq -c_{k-1} \left( \frac{h'}{h}(\inf_{\Sigma} u) \right)^k + c_k \sup_{\Sigma} H_k
\]

and this concludes the proof. \( \square \)

5. A Bernstein-type Theorem

Recall now the Gauss equation

\[
R(X,Y)Z = (\overline{R}(X,Y)Z)^T - \langle AX, Z \rangle \langle AY, Z \rangle AX,
\]

for all tangent vector field \( X, Y, Z \in T\Sigma \), where \( (\overline{R}(X,Y)Z)^T \) denotes the tangential component of \( \overline{R}(X,Y)Z \). Hence, if \( \{X,Y\} \) is any orthonormal basis of a tangent plane \( \Pi \leq T_q \Sigma, q \in \Sigma \), the sectional curvature of \( \Sigma \) is given by

\[
K(X,Y) = \overline{K}(X,Y) - \langle AX, X \rangle \langle AY, Y \rangle + \langle AX, Y \rangle \geq \overline{K}(X,Y) - \langle AX, X \rangle \langle AY, Y \rangle \geq \overline{K}(X,Y) - n^2 H_1^2,
\]

where the last inequality follows by applying the Cauchy-Schwartz inequality. In particular, if \( M^{n+1} \) is a Lorentzian space form of constant sectional curvature \( c \), then

\[
K(X,Y) \geq c - n^2 H_1^2.
\]

Hence, if \( \sup_{\Sigma} H_1 < +\infty \) the Omori-Yau maximum principle holds on \( \Sigma \) for semi-elliptic operators of the form \( L = \text{Tr}(P \circ \text{Hess}) \), where \( P \) is a symmetric operator with trace bounded above. Applying the curvature estimates found in the previous section we are able to obtain the main result of this section, that extends Corollary 4.6 in \([1]\) to spacelike hypersurfaces of constant higher order mean curvature. Notice that the previous estimates extend the ones given in \([1]\) and \([2]\). Indeed, in this case the function \( h \) has the expression

\[
h(t) = \begin{cases} 
\frac{1}{\sqrt{c}} \sinh(\sqrt{c}t) & \text{if } c > 0 \text{ and } t > 0 \\
t & \text{if } c = 0 \text{ and } t > 0 \\
\frac{1}{\sqrt{-c}} \sin(\sqrt{-c}t) & \text{if } c < 0 \text{ and } 0 < t < \pi / \sqrt{-c}
\end{cases}
\]

Set \( f_c(t) = h'(t)/h(t) \). Then

\[
f_c(t) = \begin{cases} 
\frac{1}{\sqrt{c}} \coth(\sqrt{c}t) & \text{if } c > 0 \text{ and } t > 0 \\
\frac{1}{t} & \text{if } c = 0 \text{ and } t > 0 \\
\frac{1}{\sqrt{-c}} \cot(\sqrt{-c}t) & \text{if } c < 0 \text{ and } 0 < t < \pi / \sqrt{-c}
\end{cases}
\]
It is worth pointing out that $(f_c(t))^k$ is the $k$-mean curvature of the Lorentzian sphere of radius $t$ in the Lorentzian spaceform $M_{c}^{n+1}$ (when $\mathcal{I}^{+}(p) \neq \emptyset$), that is the level set
\[
\Sigma_c(t) = \{q \in \mathcal{I}^{+}(p)\,|\,d_p(q) = t\}.
\]
The following corollaries are straightforward.

**Corollary 18.** Let $M^{n+1}$ be an $n+1$-dimensional spacetime, $n \geq 3$, such that $K_M(\Pi) \leq c$, $c \in \mathbb{R}$, for all timelike planes $\Pi$. Assume that there exists a point $p \in M$ such that $\mathcal{I}^{+}(p) \neq \emptyset$ and let $\psi : \Sigma^n \to M^{n+1}$ be a spacelike hypersurface such that $\psi(\Sigma^n) \subset \mathcal{I}^{+}(p) \cap B^{+}(p, \delta)$ for some $\delta > 0$ (with $\delta \leq \pi/\sqrt{-c}$ if $c < 0$). Assume that either

(i) $k = 2$ and $H_2$ is a positive function

or

(ii) $H_k$ is a positive function, $3 \leq k \leq n$, and there exists an elliptic point $p_0 \in \Sigma$.

Moreover, suppose that $\sup_{\Sigma} H_1 < +\infty$ and that $\inf_{\Sigma} u < \pi/\sqrt{-c}$ if $c < 0$.

If the generalized Omori-Yau maximum principle holds on $\Sigma$, then
\[
\inf_{\Sigma} H_1^{\frac{1}{k}} \leq f_c(\sup_{\Sigma} u),
\]
where $u$ denotes the Lorentzian distance $d_p$ along the hypersurface.

**Corollary 19.** Let $M^{n+1}$ be an $n+1$-dimensional spacetime, $n \geq 3$, such that $K_M(\Pi) \geq c$, $c \in \mathbb{R}$, for all timelike planes $\Pi$. Assume that there exists a point $p \in M$ such that $\mathcal{I}^{+}(p) \neq \emptyset$ and let $\psi : \Sigma^n \to M^{n+1}$ be a spacelike hypersurface such that $\psi(\Sigma^n) \subset \mathcal{I}^{+}(p)$. Assume that either

(i) $k = 2$ and $H_2$ is a positive function

or

(ii) $H_k$ is a positive function, $3 \leq k \leq n$, and there exists an elliptic point $p_0 \in \Sigma$.

Moreover, suppose that $\sup_{\Sigma} H_1 < +\infty$ and that $\inf_{\Sigma} u < \pi/\sqrt{-c}$ if $c < 0$.

If the generalized Omori-Yau maximum principle holds on $\Sigma$, then
\[
\sup_{\Sigma} H_1^{\frac{1}{k}} \geq f_c(\inf_{\Sigma} u),
\]
where $u$ denotes the Lorentzian distance $d_p$ along the hypersurface.

Using the previous estimates we then obtain the following

**Theorem 20.** Let $M_{c}^{n+1}$ be a Lorentzian spaceform of constant sectional curvature $c$, $n \geq 3$, and let $p \in M_{c}^{n+1}$. Let $\Sigma$ be a complete spacelike hypersurface which is contained in $\mathcal{I}^{+}(p)$ such that either

(i) $k = 2$ and $H_2$ is a positive constant

or
H is constant, $3 \leq k \leq n$, and there exists an elliptic point $p_0 \in \Sigma$. Moreover, assume that $\sup_{\Sigma} H_1 < +\infty$. If $\Sigma$ is bounded from above by a level set of the Lorentzian distance function $d_p$ (with $d_p < \pi/\sqrt{-c}$ if $c < 0$), then $\Sigma$ is necessarily a level set of $d_p$.

Proof. Our hypotheses imply that $\Sigma$ is contained in $I^+(p) \cap B^+(p, \delta)$, with $\delta \leq \pi/\sqrt{-c}$ when $c < 0$ and that $\Sigma$ has sectional curvature bounded from below. In particular the generalized Omori-Yau maximum principle holds on $\Sigma$ and we can apply Corollaries 18 and 19 to obtain

$$f_c(\sup_{\Sigma} u) \geq H^1_k \geq f_c(\inf_{\Sigma} u).$$

Hence, since $f_c$ is a decreasing function, $\sup_{\Sigma} u = \inf_{\Sigma} u = f_c^{-1}(H^1_k)$ and $\Sigma$ is necessarily the level set $d_p = f_c^{-1}(H^1_k)$. \qed

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