HORIZON CONDITION HOLDS POINTWISE ON FINITE LATTICE WITH FREE BOUNDARY CONDITIONS

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ABSTRACT

It is proven that the "horizon condition", which was found to characterize the fundamental modular region in continuum theory and the thermodynamic limit of gauge theory on a periodic lattice, holds for every (transverse) configuration on a finite lattice with free boundary conditions.

1. Introduction

In his seminal work, Gribov\textsuperscript{1} pointed out that the non-triviality of the fundamental modular region in Coulomb or Landau gauge has dynamical consequences. [The fundamental modular region is defined explicitly in Eq. (4) below.] Recently\textsuperscript{2}, the fundamental modular region was studied in the infinite-volume or thermodynamic limit of a periodic lattice. It was found that the fundamental modular region is characterized by a "horizon condition" in the sense that the Euclidean probability gets concentrated where the horizon function $H(U)$, a bulk quantity of order $V$, defined in Eq. (41) below, vanishes. More precisely, it was shown that at large volumes the vacuum expectation value of $H(U)$ is of order unity, $\langle H(U) \rangle = O(1)$, (instead of $V$) that its variance is of order $V$, $\langle H^2(U) \rangle = O(V)$, instead of $V^2$. It was also found\textsuperscript{2} that the horizon condition may be implemented by a Boltzmann factor $\exp[-\alpha H(U)]$, where $\alpha$ is a thermodynamic parameter determined by the constraint $\langle H(U) \rangle = 0$. Recently a calculation of glueball masses has been carried out\textsuperscript{3} in which the mass scale is set by $\alpha$. Previous calculations of glueball masses from the fundamental modular region of gauge theory have been done using a mode expansion by Cutkosky and co-workers\textsuperscript{4} and by Van Baal and co-workers\textsuperscript{5}.

The argument whereby the horizon condition is established in the infinite-volume limit of the periodic lattice, relies on two technical hypotheses\textsuperscript{2}. In the present contribution, we shall prove that, remarkably, the horizon condition holds point-wise for every transverse configuration on the finite lattice with free boundary conditions, $H(U) = 0$.

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2. Lattice with Free Boundary Conditions in Minimal Landau Gauge

We consider a finite hypercubic lattice without periodicity conditions, in $D$ Euclidean space-time dimensions. Along each principal axis of the lattice there are $L$ links and $L + 1$ sites. Lattice configurations $U$ are described by link variables $U_{xy} = U_{yx}^\dagger$, and local gauge transformations $g$ by the site variables $g_x$, where $U_{xy}$ and $g_x$ are both elements of $SU(n)$. The gauge-transform $U^g$ of the configuration $U$ by a local gauge transformation $g$, is given explicitly by

$$(U^g)_{xy} = g_x U_{xy} g_y.$$  

(1)

For the purposes of analytic calculations, it is convenient to minimize gauge fluctuations by choosing a gauge which makes the link variables $U_{xy}$, on each link $(xy)$ of the lattice, as close to unity as possible, in an equitable way over the whole lattice. For this purpose, we take as a measure of the deviation of the link variables from unity, the quantity

$I(U) = \sum_{(xy)} (n - 1) \text{Re} \text{tr}(1 - U_{xy}),$  

(2)

where the sum extends over all links $(xy)$ of the lattice. It is positive, $I(U) \geq 0$, and vanishes, $I(U) = 0$, if and only if $U_{xy} = 1$ on every link $(xy)$. [The continuum analog of this expression is the Hilbert norm of the connection $A$, $I[A] = \int d^D x |A(x)|^2$.] The restriction of this quantity to the gauge orbit through an arbitrary configuration $U$ is given by the Morse function

$F_U(g) \equiv I(U^g),$  

(3)

regarded as a function on the local gauge group $G$, for fixed $U$. Gauge fluctuations are minimized by the gauge-fixing which consists in choosing as the representative, on each gauge orbit, that configuration $U$ which yields the absolute minimum of $F_U(g)$ at $g_x = 1$. The set $\Lambda$ of all configurations $U$ such that this function is an absolute minimum on each gauge orbit at $g_x = 1$, constitutes the fundamental modular region,

$\Lambda \equiv \{ U : F_U(1) \leq F_U(g) \text{ for all } g \}.$  

(4)

Degenerate absolute minima occur only on the boundary of the fundamental modular region, and are identified topologically.

At any local or absolute minimum, the minimizing function is stationary, which gives the local gauge condition. The second variation is non-negative. To obtain the explicit form of these quantities, we write the local gauge transformation $g_x = \exp(\omega_x)$, where $\omega_x = \omega_x^a t^a$. Here the $t^a$ form an anti-hermitian basis of the defining representation of the Lie algebra of $SU(n)$, $[t^a, t^b] = f^{abc} t^c$, normalized to $\text{tr}(t^a t^b) = -\frac{1}{2} \delta^{ab}$. We have, to second order in $\omega$,

$F_U(g) = F_U(1) - (2n)^{-1} \sum_{(xy)} \text{tr} \{[U_{xy} - U_{yx}^\dagger](\omega_y - \omega_x + \frac{1}{2} [\omega_x, \omega_y])\}.$
\begin{align*}
F_U(g) = F_U(1) + (2n)^{-1} & \sum_{(xy)} [A^c_{xy}(\omega^c_y - \omega^c_x + \frac{1}{2} f^{abc} w^a w^b) \\
& + \frac{1}{2}(\omega^a_y - \omega^a_x) G^{ab}_{xy}(\omega^c_y - \omega^c_x)]. \tag{5}
\end{align*}

Here we have introduced the real link variables
\begin{align*}
A^c_{xy} & \equiv -\text{tr}[(U_{xy} - U_{xy}^\dagger) t^c], \tag{6} \\
G^{ab}_{xy} & \equiv -\frac{1}{2}\text{tr}\{[t^a t^b + t^b t^a][U_{xy} + U_{xy}^\dagger]\}. \tag{7}
\end{align*}

We introduce some elementary geometry of this lattice that will be useful. Let \(\phi_x \equiv \phi(x)\) be a site variable, and let \((xy)\) denote a link of a lattice. Corresponding link variables are defined by
\begin{align*}
(\nabla \phi)_{xy} & \equiv \phi_x - \phi_y \tag{8} \\
(a \phi)_{xy} & \equiv \frac{1}{2}(\phi_x + \phi_y). \tag{9}
\end{align*}

These are defined for all links, including those for which \(x\) or \(y\) may lie on a face, edge or vertex. (For \(\nabla \phi\) we could alternatively use the Cartan notation \(d\phi\).) We may also uniquely designate links by \((xy) = (x, \mu)\), for \(y = x + e_\mu\), where \(e_\mu\) points in the positive \(\mu\)-direction, and we shall also use, as convenient, the notation for link variables
\begin{align*}
\nabla^\mu \phi(x) & \equiv (\nabla \phi)_{yx} \tag{10} \\
a^\mu \phi(x) & \equiv (a \phi)_{xy}. \tag{11}
\end{align*}

(These expressions are undefined for sites \(x\) where \(y = x + e_\mu\) is not a site of the lattice.) Given two link variables \(V_{xy}\) and \(W_{xy}\), we may form the link variable \(P_{xy} = V_{xy} W_{xy}\), etc.

We also introduce the lattice divergence of a link variable \(A_\mu(x)\), which is a site variable \((\nabla \cdot A)_x\), defined by the dual
\begin{align*}
(\nabla \cdot A, \phi) & \equiv -(A, \nabla \phi) = - \sum_{(xy)} A_{xy}(\nabla \phi)_{xy} = \sum_x (\nabla \cdot A)_x \phi_x, \tag{12}
\end{align*}

where the sums extend over all links and sites of the lattice. Similarly the site variable \((a \cdot A)_x\) is defined by
\begin{align*}
(a \cdot A, \phi) & \equiv (A, a \phi) = \sum_{(xy)} A_{xy}(a \phi)_{xy} = \sum_x (a \cdot A)_x \phi_x. \tag{13}
\end{align*}

The lattice divergence is defined for all sites \(x\), and represents the sum of all link variables leaving the site \(x\). (It is not formed simply of differences of link variables when \(x\) is a boundary point of the lattice.)
With the help of these definitions, we rewrite Eq. (5) in the form

\[ F_U(g) = F_U(1) + (2n)^{-1}[-(A, \nabla \omega) + (\nabla \omega, D(U)\omega)]. \]  

(14)

Here \([D(U)\omega]_{yx}\) is a well-defined link variable which we call the lattice gauge-covariant derivative of the site variable \(\omega_x\),

\[ D_{\mu}^a \omega^c(x) = G_{\mu}^{ac}(x) \nabla_\mu \omega^c(x) + f^{abc} A_b^\mu(x) a_\mu \omega^c(x). \]  

(15)

It is the infinitesimal change in \(A_\mu(x)\) under an infinitesimal gauge transformation \(\omega_x\). We also define the lattice Faddeev-Popov matrix \(M(U)\) by

\[ (\omega, M(U)\phi) = (\nabla \omega, D(U)\phi). \]  

(16)

At a minimum, the minimizing function \(F_U(g)\) is stationary, namely \((A, \nabla \omega) = -(\nabla \cdot A, \omega) = 0\). This holds for all \(\omega\), so the gauge condition is expressed by the transversality of \(A\),

\[ \nabla \cdot A = 0. \]  

(17)

Because of transversality, the gauge just defined falls into the class of lattice Landau gauges, and we call it the “minimal Landau gauge”. At a stationary point of the minimizing function, the matrix of second derivatives is symmetric, so when \(A\) is transverse, the Faddeev-Popov matrix is symmetric,

\[ M \equiv -\nabla \cdot D(U) = -D(U) \cdot \nabla \quad \text{(for} \, \nabla \cdot A = 0). \]  

(18)

This matrix is non-negative at a minimum, and the two conditions together define the Gribov region \(\Omega\),

\[ \Omega \equiv \{U : \nabla \cdot A(U) = 0 \text{ and } M(U) \geq 0\}. \]  

(19)

Because the set \(\Lambda\) of absolute minima is contained in the set of relative minima, we have the inclusion

\[ \Lambda \subseteq \Omega, \]  

(20)

where \(\Lambda\) is the fundamental modular region.

3. Vanishing of the Horizon Function on the Finite Lattice with Free Boundary Conditions

Because there are \(L + 1\) sites, but only \(L\) links on each row of the lattice with free boundary conditions, the transversality condition is more restrictive for free boundary conditions than on a periodic lattice, so various additional constraints hold, as we shall now show.
Choose a particular direction, $\mu = 0$, on the hypercubic lattice, and call the corresponding (Euclidean) coordinate $t$ and the other coordinates $x$. Consider the sum over the hyperplane labelled by $t$, for a transverse configuration, $\nabla \cdot A = 0$,

$$\sum_x (\nabla \cdot A)(t, x) = 0. \quad (21)$$

Contributions to this sum from links that lie within the hyperplane vanish, because all such links are connected to 2 sites within the hyperplane and these 2 contributions cancel. There remain the contributions from perpendicular links. For all $t$ that label interior hyperplanes $1 \leq t < L - 1$, they give

$$Q_0(t) - Q_0(t - 1) = 0,$$

whereas on the boundary hyperplanes they give

$$Q_0(L - 1) = Q_0(0) = 0. \quad (23)$$

where

$$Q_0(t) \equiv \sum_x A_0(t, x). \quad (24)$$

We thus have proved the lemma:

Let $A_\mu(x)$ be a transverse configuration, $\nabla \cdot A(x) = 0$, on a lattice with free boundary conditions. Then the "charges" vanish,

$$Q_0(t) \equiv \sum_x A_0(t, x) = 0 \quad (t = 0, 1, \ldots L - 1) . \quad (25)$$

[For a lattice with periodic boundary conditions, only the weaker condition $Q_0(t) = \text{const.}$ holds.] If we sum the last equation over $t$, for generic $\mu$, we obtain:

Corollary. Let $A_\mu(x)$ be a transverse configuration $\nabla \cdot A(x) = 0$ on a lattice with free boundary conditions. Then the zero-momentum component of $A$ vanishes,

$$\sum_x A_\mu(x) = 0 . \quad (26)$$

Note that for the periodic lattice, transversality imposes no condition whatsoever on the constant component of $A_\mu(x)$. On the other hand, for the periodic lattice, the core of the fundamental modular region was found$^2$ to satisfy Eq. (26).

We shall now prove that the horizon condition is satisfied point-wise for a finite lattice with free boundary conditions. Consider the integral

$$I \equiv \int d\phi d\phi^* \exp[(\phi^*_i, M(U)\phi_i)],$$

$$\quad = \int d\phi d\phi^* \exp[-(\nabla_\lambda \phi_i^{*\alpha}, D_\lambda^{ac}(U)\phi_i^c)], \quad (27)$$

5
where the $\phi^a_i(x)$ variables are integrated over the real axis, and the $\phi^{*a}_i(x)$ are independent variables that are integrated over the imaginary axis. The lattice gauge-covariant derivative $D^{ac}_i(U)$ acts on the upper index, of $\phi^c_i(x)$, and $i = 1, \ldots f$ is a dummy index. For transverse $A = A(U)$, as we assume, the symmetric matrix $M(U) \equiv - \nabla \cdot D(U) = - D(U) \cdot \nabla$ has a null-space, $H_0$, consisting of constant functions, so the integral (27) would diverge if $\phi(x)$ and $\phi^{*}(x)$ were independent integration variables for every $x$. The integral is made finite by the constraint at a fixed point $y$ of the lattice

$$\phi^c_i(y) = \phi^{*c}_i(y) = 0,$$

(28)

and the definition

$$d\phi d\phi^{*} \equiv \prod_{x \neq y, i, a} d\phi^a_i(x) d\phi^{*a}_i(x).$$

(29)

With this specification, the integral has the value

$$I = \int d\phi \prod_{i=1}^f \delta(M\phi_i) = \det^{-f} M_y,$$

(30)

where $M_y$ is the matrix obtained from $M$ by deleting the rows and columns that bear the label $y$.

Because $M(U)$ has a null space, $H_0$, consisting of constant functions $\nabla \omega = 0$, an alternative expression for $I$ is

$$I = d\phi d\phi^{*} \exp[(\phi^{*}_{i\perp}, M(U)\phi_{i\perp})],$$

(31)

where $\phi_{\perp}$ and $\phi^{*}_{\perp}$ are the projections of $\phi$ onto the orthogonal subspace $H_{\perp}$,

$$\phi_{\perp}(x) = \phi(x) - S^{-1} \sum_x \phi(x),$$

(32)

and $S$ is the total number of sites in the lattice. The variables $\phi(x)$ and $\phi^{*}(x)$ for $x \neq y$ constitute a complete set of (linear) coordinates on $H_{\perp}$, and the change of basis (32) is configuration independent. We conclude that

$$I = \det^{-f} M_y = \det^{-f} M_{\perp},$$

(33)

which shows that $I$ is independent of $y$. (The equation $\det M_y = \det M_{\perp}$ holds for any symmetric matrix $M$ with the property that the sum of each row and each column vanish.)

Now let the dummy index $i$ represent the pair $i = (\mu, a)$, so $\phi^c_{\mu,a}(x)$ is a site variable with a preferred direction $\mu$, and make the shift on $\phi$ and $\phi^{*}$ given by

$$\phi^c_{\mu,a}(x) = \phi^c_{\mu,a}(x) + x_{\mu} \delta^c_a$$

$$\phi^{*c}_{\mu,a}(x) = \phi^{*c}_{\mu,a}(x) + x_{\mu} \delta^c_a$$

(34)
which are the lattice analogs of shifts previously introduced in continuum theory. This shift makes no sense on a (finite) periodic lattice, but it is well defined on a finite lattice with free boundary conditions. (Note that the constraint at the lattice point \( y \) for the shifted fields differs from (28), unless it is imposed at the “origin” \( y = 0 \).) For \( \nabla \cdot A = 0 \), one finds after a simple calculation

\[
I \equiv \int d\phi d\phi^* \exp[(\nabla \lambda \phi_{\mu,a}^b, D_{\lambda}^{bc}(U)\phi_{\mu,a}^c) + (D_{\lambda}^{bc}(U)\phi_{\lambda,a}^c) + \sum_{x,\mu} G_{\mu}^{aa}(x)].
\]

(35)

It is convenient to introduce a field variable \( B_{\lambda,a}^c(x) \), by duality

\[
(B_{\lambda,a}^c, \phi_{\lambda,a}^c) \equiv -(D_{\lambda}^{bc}(U)\phi_{\lambda,a}^c).
\]

(36)

Like \( \phi_{\lambda,a}^c \), the new variable \( B_{\lambda,a}^c \) is a site variable with a preferred direction \( \mu \). We have

\[
B_{\lambda,a}^c(x) = (\nabla \cdot G_{\lambda}^{ac}(x) - f_{abc}(a \cdot A)^b_{\lambda}(x),
\]

(37)

where \( (\nabla \cdot G_{\lambda}^{ac}(x) \) represents the contribution to the lattice divergence \( \nabla \cdot G^{ac}(x) \) associated with the \( \lambda \) axis, and similarly for \( (a \cdot A)^b_{\lambda}(x) \). With this definition,

\[
I \equiv \int d\phi d\phi^* \exp[(\nabla \lambda \phi_{\mu,a}^b, D_{\lambda}^{bc}(U)\phi_{\mu,a}^c - (B_{\lambda,a}^c, \phi_{\lambda,a}^c) - (\phi_{\lambda,a}^c, B_{\lambda,a}^c) + \sum_{x,\mu} G_{\mu}^{aa}(x)].
\]

(38)

We may now effect the \( \phi \) and \( \phi^* \) integrations by making the shifts

\[
\phi_{\lambda,a}^b = \phi_{\lambda,a}^b + (M^{-1})^{bc} B_{\lambda,a}^c,
\]

\[
\phi_{\lambda,a}^* = \phi_{\lambda,a}^* + (M^{-1})^{bc} B_{\lambda,a}^c.
\]

(39)

This expression is well-defined because \( B_{\lambda,a}^c \) is orthogonal to the null-space of \( M \),

\[
\sum_x B_{\lambda,a}^c(x) = 0,
\]

(40)

and we may choose coordinates adapted to these subspaces. To see this, observe that \( \sum_x (\nabla \cdot G_{\lambda}^{ac}(x) \) vanishes because each link is connected to two sites and gives opposite contributions from each. Moreover \( \sum_x (a \cdot A)^b_{\lambda}(x) = \sum_x A^b_{\lambda}(x) \) vanishes by the preceding corollary. This gives \( I = I \exp[-H(U)] \), where \( H(U) \) is given by Eq. (41).

We have proven:

Theorem. Let \( U \) be a transverse configuration \( \nabla \cdot A(U) = 0 \), and let the horizon function \( H(U) \) on a finite lattice with free boundary conditions be defined by

\[
H(U) = (B_{\lambda,a}^b, (M^{-1})^{bc} B_{\lambda,a}^c) - \sum_{x,\mu} G_{\mu}^{aa}(x),
\]

(41)
where $B$ is defined in (37) and $G_{a}^{ab}(x)$ in (7). Then $H(U)$ vanishes

$$H(U) = 0.$$  \hfill (42)

**Remark.** On a periodic lattice, the horizon condition does not hold point-wise. For example, the vacuum configuration $U_\mu(x) = 1$ on a periodic lattice gives $B_{X,\alpha}^c(x) = 0$, so $H(U) = -(n^2 - 1)DV$. For the vacuum configuration on the finite lattice with free boundary conditions, $B_{X,\alpha}^c(x)$ is entirely supported on the boundary. Thus boundary contributions remain important at arbitrary large volume. Long range boundary effects have also been found by Patrascioiu and Seiler\(^7\).

**Discussion.** The result (42) is quite remarkable and unexpected. Recall that as the volume of the periodic lattice approaches infinity, the probability gets concentrated where the horizon condition holds\(^2\) $H(U) = 0$, and where $\sum_x A(x) = 0$. Here we have just proven that these conditions are satisfied for all transverse configurations $U$ on the finite lattice with free boundary conditions.

4. **References**

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