Fractal geometry of higher derivative gravity

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We determine the scaling properties of geometric operators such as lengths, areas, and volumes in models of higher derivative quantum gravity by renormalizing appropriate composite operators. We use these results to deduce the fractal dimensions of such hypersurfaces embedded in a quantum spacetime at very small distances.

It was shown a long time ago by Stelle [1] that the action

\[ S[g] = \int d^4x \sqrt{-g} \left\{ \frac{1}{f_2} \left( \frac{1}{3} R^2 - R_{\mu\nu} R^{\mu\nu} \right) + \frac{1}{6f_2} R^2 \right\} \tag{1} \]

is perturbatively renormalizable in four dimensions. Stelle’s model became even more attractive once it was shown to be asymptotically free in the coupling \( \lambda \equiv (f_2)^2 \). However it was soon realized that the model is non-unitary because of the higher derivative propagator. Nevertheless, solutions to this problem have been proposed early on and invoked a variety of ideas including in particular self-stabilization and the Lee-Wick mechanism [8]. The interest towards higher derivative quantum gravity has resurfaced over the years, and recently has returned thanks to the appearance of two new proposals which are spiritual successors of the aforementioned ideas: a gravity [11, 12] and a perturbatively unitary mechanism based on quantizing some degrees of freedom as fakeons [13, 14].

Alongside the development of higher derivative gravity the idea of non-perturbative renormalizability of standard Einstein gravity has gained momentum and culminated in the asymptotic safety conjecture [15, 16], which has evidence based on non-perturbative renormalization group methods [17, 18]. The status of the relation between asymptotically free Stelle’s gravity and asymptotically safe Einstein’s gravity has been debated by theorists for some time [19, 20], especially because the latter is believed to originate from the continuation of (2 + \( \varepsilon \)-gravity [21]. Explicit results based on mass-dependent regulators suggest that in four dimensions there could be two distinct universality classes [22, 23].

The increasing attention towards Stelle’s gravity and its high energy properties opens the avenue to the discussion of its physical implications in search for possible phenomenological signatures. In fact, model specific implications have already been explored in various contexts [24, 26]. The geometric characterization of the quantum theory of [1], which could be expected to have a fractal nature induced by radiative corrections, is however still lacking. One straightforward tool to explore the geometry of quantum spacetimes is the inclusion of composite operators into the renormalization process which have a geometric meaning [27, 28] and thus can be used to deduce meaningful quantities such as, for example, the fractal dimensions of embedded hypersurfaces of various (bare) dimensionalities. This work is dedicated to the renormalization of some geometric operators which allow to read off such fractal dimensions.

Many quantum gravity scenarios predict that spacetime has a fractal behaviour at very small scale, often implying that the dimension of spacetime is smaller than four. Interestingly this happens both in the asymptotic safety scenario and in the causal dynamical triangulations approach (see [29] for a comprehensive review). It must be emphasized that there are in principle several possible working definitions of the spacetime dimension. Examples include the spectral dimension, the walk dimension and the Hausdorff dimension, and all these definitions could give different estimates of the fractal dimension [30].

Renormalization. We begin by recalling basic facts on the renormalization of Stelle’s gravity to set the stage for our results. In the following we adopt the notation of [11], which we refer to for more details on the couplings’ renormalization. The bare action (1) is the most general power-counting renormalizable action containing the two couplings’ renormalization. The bare action (1) is the

\[ C^2 = C_{\mu\nu\rho\theta} C^{\mu\nu\rho\theta} \]

which is weighted by the coupling \( f_2 \) because it fulfills \( \int C^2 = -2 \int (\frac{1}{3} R^2 - R_{\mu\nu} R^{\mu\nu}) \) by neglecting the contribution of the Euler characteristic. The parametrization is chosen such that the only term manifestly breaking the Weyl symmetry is \( R \) which is weighted by the coupling \( f_0 \). We refer to the conformally invariant limit \( f_0 \to \infty \) as Weyl’s higher derivative gravity [10]. Operators with less derivatives, such as the scalar curvature \( R \) which couples through Newton’s constant or the spacetime volume which couples through the cosmological constant, can in principle be included as relevant deformations of \( S[g] \), but we will stick to (11).

To renormalize a path-integral constructed with the action (1) we adopt the background field method and split the metric in a background and a fluctuation part, \( g_{\mu\nu} \to g_{\mu\nu} + h_{\mu\nu} \). This split is used to fix the gauge, while the background metric is chosen to be flat, \( g_{\mu\nu} = \eta_{\mu\nu} \), from now on, which is enough to determine the counter-
ers. We employ the following gauge fixing action

$$S_{\text{GF}}[\bar{h}] = -\frac{1}{2\xi_0} \int d^4x \chi_\mu \partial^2 \chi^\mu$$

(2)

with $\chi_\mu = \partial^\nu (h_{\mu\nu} - c_0 \eta_{\mu\nu} h_{\alpha}^\alpha)$ and the two gauge fixing parameters $\xi_0$ and $c_0$.

By adopting dimensional regularization and using minimal subtraction one finds the beta functions

$$\beta_{f_2} = -\frac{1}{(4\pi)^2} \frac{133}{20} f_2^3,$$

$$\beta_{f_0} = +\frac{1}{(4\pi)^2} \frac{1}{2} \left( \frac{10 f_2^4}{f_0^2} + 5 f_2^2 + 5 \right) f_0^3,$$

(3)

where $\beta_{f_2} \equiv \frac{d}{d\log \mu} f_2$, with $\mu$ being the reference scale at which the renormalized couplings are defined. Both beta functions admit Gaussian fixed points, while the ratio $\omega = \frac{f_2}{f_0}$ has a beta function $\beta_\omega$ with two non-trivial zeroes. Thus by setting $\omega$ to either fixed point one can obtain a perturbative series controlled solely by $f_2$. The conformal limit $f_0 \rightarrow \infty$ is discontinuous because Weyl invariance must be gauge fixed through, e.g., $h_{\mu\nu} = 0$ and the number of propagating degrees of freedom changes. The renormalization group flow in this case becomes [10].

$$\beta_{f_2} = -\frac{1}{(4\pi)^2} \frac{199}{30} f_2^3.$$  

(4)

Here, we explicitly assume that the beta function in the conformal limit is gauge independent as it is the case for the beta functions of Stelle’s gravity [31].

Scaling dimensions. We now introduce the scaling dimension of an embedded hypersurface and discuss its physical meaning. The scaling dimension is sometimes used to guess the Hausdorff dimension but may differ from it [32]. Let us consider the volume of an $n$-dimensional surface $\sigma_n$ and denote it by $V_{\sigma_n}$. Let us also assume that the volume is characterized by some length $L$. Classically, we expect that $V_{\sigma_n} \sim L^n$. In the quantum regime, however, the gravitational fluctuations might change the classical scaling by modifying the scaling exponent via an anomalous dimension $\gamma_n$, i.e. $V_{\sigma_n} \sim L^{n-\gamma_n}$. In this case, we say that the $n$-dimensional surface $\sigma_n$ has scaling dimension $n - \gamma_n$. One can construct further definitions of scaling dimensions from the building blocks $\sigma_n$. For instance, one may measure the scaling dimension not in terms of the characteristic (coordinate) length $L$, but rather in terms of the length of a given curve $\sigma_1$, whose total length we denote by $V_\sigma$. Combining the scaling behaviours $V_{\sigma_n} \sim L^{n-\gamma_n}$ and $V_{\sigma_1} \sim L^{1-\gamma_1}$, one obtains that $V_{\sigma_n} \sim \bar{V}_{\sigma_1}$, which defines a new scaling exponent for the volume of $\sigma_n$.

Anomalous dimensions. Let us introduce the volume of $\sigma_n$ on the field theoretical side now. The induced metric on $\sigma_n$ is given by the pullback of the spacetime metric onto the surface, which we parametrize by $x^\mu(u)$ via the coordinates $u^a$ with $a = 1, \ldots, n$. The pulled-back induced metric is given by

$$g_{ab}(u) = g_{\mu\nu}(x(u)) \frac{\partial x^\mu}{\partial u^a} \frac{\partial x^\nu}{\partial u^b},$$

(5)

and the volume of the submanifold $\sigma_n$ then is written as

$$V_{\sigma_n} = \int_{\sigma_n} \sqrt{\det g} = \int_D d^n u \sqrt{\det g_{ab}(u)}.$$

(6)

It is easy to see that equation [9] for the case $n = 1$ reproduces the length of a given curve $x^\mu(u)$,

$$V_{\sigma_1} = \int du \sqrt{\det g_{\mu\nu}(x(u)) \dot{x}^\mu(u) \dot{x}^\nu(u)}.$$

The induced volume element $g_{\sigma_n} \equiv \det g_{ab}$ is not present in the bare action, but it can be renormalized as a composite operator. We denote its anomalous dimension by $\gamma_{\sigma_n}$, which at one loop is linear in $f_2^2$ and $f_0^2$.

The Callan-Symanzik equation for $\langle g_{\sigma_n}(u) \rangle$ reads

$$\left( \mu \partial_\mu + \beta_{f_2} \frac{\partial}{\partial f_2} + \beta_{f_0} \frac{\partial}{\partial f_0} + \gamma_{\sigma_n} \right) \langle g_{\sigma_n}(u) \rangle = 0.$$  

(7)

In the deep ultraviolet, i.e. for $f_2, f_0 \rightarrow 0$, we can neglect the beta functions, which are cubic in the couplings, and approximate equation [9] as

$$\left( \mu \partial_\mu + \gamma_{\sigma_n} \right) \langle g_{\sigma_n}(u) \rangle \approx 0.$$  

(8)

Next, the metric is dimensionless so that dimensional analysis implies

$$\left( \mu \partial_\mu - u \partial_u \right) \langle g_{\sigma_n}(u) \rangle = 0,$$

(9)

in which we assume that the energy scale of interest is much bigger than all other dimensionful quantities other than the coordinates (e.g. any mass). Combining equations [9] and [8] together, one obtains

$$\langle g_{\sigma_n}(u) \rangle \sim u^{-\gamma_{\sigma_n}}.$$

Thus, one can estimate the scaling behaviour of $V_{\sigma_n}$ via

$$\langle V_{\sigma_n} \rangle = \int_D d^n u \langle g_{\sigma_n}(u) \rangle \sim L^{n-\gamma_{\sigma_n}},$$

where $L$ is the characteristic length of the domain of integration. This proves that at very high energies, or alternatively at very small scales, the exponents coincide.

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1 For instance, the volume is specified by an $n$-dimensional ball of radius $L$ in the coordinate space.

2 We assume here that the expectation value of the composite operator can be expanded perturbatively and that it is non-zero even at zero coupling, which is to be expected since gravity is naturally in the broken phase, i.e. $\langle g_{\mu\nu} \rangle \approx \eta_{\mu\nu} \neq 0$. 
\(\gamma_n = \gamma_{\sigma_n} \), and we can determine the scaling properties by field theoretic methods.

In the case of Stelle's gravity, the ultraviolet fixed-point is Gaussian, thus in the infinite energy limit the anomalous dimensions are zero or, more precisely, radiative corrections to the scaling behavior are only logarithmic. However, in a regime in which the coupling is sufficiently small, we are allowed to neglect the beta functions in (7) and we encounter approximate scale invariance in which a fractal like behaviour of geometrical volumes is present.\(^3\) This is the scaling regime located above any physical mass (including the Planck mass) and below the possibly infinite energy range of validity of (11), which displays scale invariant fractal spacetime properties.

**Geometric composite operators.** To explicitly derive the scaling exponent \(\gamma_{\sigma_n}\) we couple the composite operator of interest to a local source \(\zeta(u)\) in (11). At one loop a new divergence associated to the composite operator \(\zeta\) of interest to a local source \(\zeta\) denotes the gauge-fixed graviton propagator as in \([11]\),

\[
\mathcal{G} = i \frac{1}{k^4} \left[ -2f_2^2 P^{(2)} + f_0 \left( P^{(0)} + \frac{\sqrt{3}c_g T^{(0)}}{2 - c_g} + \frac{3c_g^3 P^{(0\omega)}}{(2 - c_g)^2} \right) \right] \mu \nu \rho \sigma ,
\]

in which we keep all terms up to \(O(f_2^2)\) and \(O(f_0^2)\). Here, \(\mathcal{G}\) denotes the gauge-fixed graviton propagator as in \([11]\),

\[
\Gamma[g, \zeta] = \mathcal{S}[g, \zeta] + \frac{1}{2} \text{Tr} \log (\delta^2 S[g, \zeta]) .
\]

The new divergences can be renormalized multiplicatively by introducing a suitable counterterm and adding \(\int d^n u \zeta(u) Z_{\sigma_n} \sqrt{g_{\sigma_n}(u)} \) to the action (11). The anomalous dimension is then given by the coefficient of the pole of \(Z_{\sigma_n}\), which is computed by evaluating the one-point function

\[
\frac{\delta \Gamma}{\delta \xi} \bigg|_{\xi = 0} = \frac{1}{2} \text{Tr} \mathcal{G} \cdot (\delta^2 \mathcal{G}) \bigg|_{\xi = 0} ,
\]

where \(\mathcal{G}\) denotes the gauge-fixed graviton propagator as in \([11]\),

\[
\Gamma[g, \zeta] = \mathcal{S}[g, \zeta] + \frac{1}{2} \text{Tr} \log (\delta^2 S[g, \zeta]) .
\]

which is linear in both couplings \(f_0^2\) and \(f_2^2\) and in the gauge fixing parameter \(\xi_g\). The physical interpretation of this result is that all modes propagating in the gauge-fixed propagator contribute to (11), including both, gauge-invariant spin-2 (graviton) and scalar modes as well as the unphysical vector and pseudoscalar ones.

The gauge dependence of (11) is to be expected because embedded hypersurfaces break diffeomorphism invariance and thus, strictly speaking, are not true observables. We could circumvent this problem by constructing a gauge-invariant observable which combines the volume of an hypersurface with an observable amplitude such that the various gauge dependencies cancel each other. This typically results in very non-local observables such as the correlation length at fixed geodesic length. A similar program works nicely in 2d quantum gravity \([33, 34]\) where computations are typically performed in the conformal gauge (to the best of our knowledge there is no study exploiting the explicit gauge dependence cancellation). However, the problem of constructing interesting and meaningful gauge-invariant observables in four dimensional quantum gravity is a long-standing one \([35-40]\), and is beyond the scope of this work.

In order to find a simpler workaround we first notice, cf. \([12]\), that in the physical gauge \(c_g = \xi_g = 0\) only the gauge-invariant, hence physical, modes propagate. The physical gauge is often associated to the unique Vilkovisky-de Witt effective action in which a nontrivial connection in field space ensures that only physical modes are integrated over and that the effective action is gauge independent (11). Therefore, from now on we work in the gauge \(c_g = \xi_g = 0\) and assume that the information obtained in this way is indeed physical.

**Anomalous scaling in Stelle’s gravity.** On the basis of the fact that only physical modes propagate, we argue that the physical gauge limit of (11) gives a reliable estimate of the scaling dimension of an embedded hypersurface. In the limit \(\xi_g = c_g = 0\) the anomalous dimension in terms of the couplings \(f_2\) and \(\omega = \frac{f_2^2}{2f_0^2}\) reads

\[
\gamma_{\sigma_n} = \frac{n}{(4\pi)^2 288} \left\{ 40(2 - 5n) + \frac{1}{\omega}(26 - 11n) \right\} ,
\]

which can be used in (11) and constitutes one of the main results of this letter.

In four dimensions, the value of \(\gamma_{\sigma_n}\) depends on the scale of the RG flow at which the couplings are located. The anomalous dimension vanishes at the fixed point but is non-zero as soon as we move away from it. For sufficiently small values of the couplings, we find approximate scale invariance characterized by an effective fractal dimension of the geometric operators. More precisely, if \(f_2^2 > 0\) it is straightforward to check that the anomalous dimension is positive or negative depending on the value of \(n\). For \(n = 1, 2\), i.e. lengths and areas, the anomalous dimension is positive for \( \frac{1}{2} > \frac{80 - 200n}{26 + 11n} \), while for
\( n = 3, 4 \), i.e. three- and four-volumes, for \( \frac{1}{\varepsilon} < 80 - 200n \). It follows that quantum fluctuations affect hypersurfaces of different dimensions in different manners: a length can effectively decrease its scaling dimension while the opposite happens for a three-volume. Compared to other models of quantum gravity, which often display only dimensional reduction, this behavior is very peculiar to higher derivative quantum gravity.

As already mentioned, in \( d = 4 - \varepsilon \) the theory exhibits two nontrivial ultraviolet fixed points which are solutions of \( -\varepsilon f_2 + 2\beta f_2 = 0 \) and \( \beta_* = 0 \). The scaling of geometric operators at such fixed points is characterized by \( \gamma_{\sigma_n} \). The system has two non-Gaussian solutions because the equation \( \beta_* = 0 \) has two roots \( \omega_{*,1} = -0.0229 \) and \( \omega_{*,2} = -5.4671 \). We label them by

\[
(f_2^2, \omega)_{*,1} = (-11.8732 \varepsilon, -0.0229), \quad (f_2^2, \omega)_{*,2} = (-11.8732 \varepsilon, -5.4671). 
\]

We argue that the more important solution is the first one because it is fully ultraviolet attractive and because the second one was shown to lead to a non-positive ghost inverse propagator [23]. The numerical expressions for \( \gamma_{\sigma_n} \) at both fixed points is given in Table I.

**Anomalous scaling in Weyl-squared gravity.** The computation of \( \gamma_{\sigma_n} \) in the Weyl-invariant case goes along the same lines as in the case of Stelle’s theory with the difference being that the propagator in [10] contains only spin-2 propagating modes. This propagator is given by \( \mathcal{G} \) with \( f_2^0 = \xi_0 = 0 \) which also employs the additional gauge fixing condition \( h_\mu^\nu = 0 \). The explicit result for the anomalous dimension then reads

\[
\gamma_{\sigma_n} = -\frac{1}{(4\pi)^2} \frac{10}{144} (5n^2 - 2n) f_2^2. \tag{15}
\]

In \( d = 4 \) the anomalous dimension is zero at the fixed point but non-zero in its neighbourhood. It is straightforward to see that the sign of the correction depends on the sign of \( f_2^2 \) for hypersurfaces of dimension \( 1 < n \leq 4 \) and not on \( n \) itself. In \( d = 4 - \varepsilon \) there is only one solution of \( -\varepsilon f_2 + 2\beta f_2 = 0 \), with the beta function [14], in this case and we include the estimates for the scaling dimensions in the last line of Tab. I with the label \( w \) for Weyl. The anomalous dimension at the non-trivial fixed point in \( d = 4 - \varepsilon \) implies an effective dimensional reduction in the UV for \( \varepsilon > 0 \).

**Summary and future prospects.** Since by nature gravity is a geometrical theory, we believe that it is natural to investigate the quantum properties of geometrical objects, such as lines, areas, and volumes in quantum gravity. In this letter we have considered the quantum properties of such geometric operators in higher derivative gravity for the first time. More precisely, we have computed the scaling properties of these geometric operators in Stelle’s and Weyl theories in \( d = 4 \) and \( d = 4 - \varepsilon \).

For the most physically relevant case corresponding to \( d = 4 \), we have found that these geometric operators display a peculiar scaling behaviour: at the Gaussian fixed point the scaling is purely classical while moving away from it we have a regime of approximate scale invariance in which the effective dimension is fractal. The nature of this fractal behaviour depends on the couplings and thus on the precise scale at which the operators are observed.

Remarkably, similar geometric operators can be defined also in other approaches to quantum gravity, such as Loop Quantum Gravity, and Causal Dynamical Triangulations [40, 42]. Therefore this work paves the way to a possible comparison among the predictions of all different quantum gravity models, now including higher derivative quantum gravity.

The biggest open issue of the approach presented here is to find a gauge-invariant generalization of our results on the scaling dimensions. As a matter of fact, this search overlaps with the quest for meaningful gauge invariant observables in theories of quantum gravity. Our approach offers a shortcut based on the choice of propagating only the physical, i.e. gauge invariant, degrees of freedom in full analogy with the Vilkovisky-de Witt formalism. However, to which extent our approach is valid should be tested further. In any case, let us emphasize that the approach developed here can also serve to study fully fledged diffeomorphism invariant observables. For instance, one could consider a correlation function at fixed geodesic length between two operators: \( \langle \int_x \int_y O(x)O(y)\delta (\ell g - r) \rangle \) with \( \ell g \) being the geodesic length. Performing a scaling analysis of such correlation functions involves the computation of the scaling dimension of the geodesic length itself, which can be computed in a way similar to the one outlined in this work.

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**TABLE I.** Leading estimates of the anomalous dimension [12] at the fixed points in \( d = 4 - \varepsilon \) spacetime dimensions which correct the scaling of all hypersurfaces of dimension \( n \) lower than four.

| \( n \) | \( \gamma_{\sigma_n} \) |
|---|---|
| 1 | 0.1012ε  |
| 2 | 0.1291ε  |
| 3 | 0.0839ε  |
| 4 | -0.0348ε |
| 1 | 0.0160ε  |
| 2 | 0.0837ε  |
| 3 | 0.2031ε  |
| 4 | 0.3742ε  |
| 1 | 0.0157ε  |
| 2 | 0.0838ε  |
| 3 | 0.2041ε  |
| 4 | 0.3769ε  |

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