How to correct small quantum errors

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November 10, 2018

Abstract

The theory of quantum error correction is a cornerstone of quantum information processing. It shows that quantum data can be protected against decoherence effects, which otherwise would render many of the new quantum applications practically impossible. In this paper we give a self contained introduction to this theory and to the closely related concept of quantum channel capacities. We show, in particular, that it is possible (using appropriate error correcting schemes) to send a non-vanishing amount of quantum data undisturbed (in a certain asymptotic sense) through a noisy quantum channel $T$, provided the errors produced by $T$ are small enough.

This text is part of a volume entitled: “Coherent evolution in noisy environments” to be published in Lecture notes in physics, Springer Verlag, http://link.springer.de/series/lnpp/, Copyright: Springer Verlag, Berlin, Heidelberg, New York

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1 Introduction

Controlling decoherence is one of the key problems for making quantum information processing and quantum computation work. From the outset, when Peter Shor announced his algorithm [18, 19], many physicists felt that somewhere there would be a price to pay for the miraculous exponential speedup. For example, if the algorithm would require exponentially good adherence to specifications for the quantum circuitry and exponentially low noise levels, it would have been totally useless. Indeed it is far from easy to show that it does not make such requirements.

In this article we look at the simpler, but equally fundamental problem of quantum information transmission or storage. Is it possible to encode the quantum data in such a way that even after some degradation they can be restored nearly perfectly by a suitable decoding operation? Assuming that the degrading decoherence effects are small to begin with, can restoration be made nearly perfect?

For classical information it is very simple to do this, namely by redundant coding. If we want to send one bit through a noisy channel, we can reduce errors by sending it three times and deciding by majority vote which value we take at the output. Clearly, if errors have a small probability $\varepsilon$ for a single channel, they will have order $\varepsilon^2$ for the triple channel, because we go wrong only when two independent errors occur. Unfortunately, such a scheme cannot work in the quantum case because it involves a copying operation, which is forbidden by the No-Cloning Theorem [23]. So we have to look for subtler ways of distributing quantum information among several systems and thereby reducing the probability of errors. Indeed such schemes exist [3, 20] and are the subject of the exciting new field of quantum error correcting codes.

The efficiency of such a scheme is measured by two parameters, namely how many uses of the noisy channel are required, and the error level after correction. The above simple classical scheme can be iterated to get the errors for a single bit down to $\varepsilon^{2^n}$ with $3^n$ parallel uses of the channel. This is a large overhead to correct a single bit. Better procedures work classically by coding several bits at a time, and one can manage to make errors as small as desired with only a finite overhead per bit. The minimal required overhead (or rather its inverse) is, in fact, the central quantity of the coding theory for noisy channels: one defines the capacity of a channel as the number of bit transmissions per use of the channel, in an optimal coding scheme for messages of length $L \to \infty$ with the property that the error probability goes to zero in this limit.

It is not a priori clear that the notion of channel capacity makes sense for quantum information, i.e. that the capacity of a channel which produces only small errors can be corrected with an asymptotically small effort. To this end the paper is organized as follows. We first review the basic notions concerning quantum channels (Section 2), and give an abstract definition of the capacity together with some elementary properties (Section 3). Then we
discuss the theory of error correcting codes (Section 4) and a particular scheme to construct such codes which is based on graph theory (Section 5). In Section 6 and 7 we apply this scheme to channel capacities and finally we draw our conclusions in Section 8.

2 Quantum channels

According to the rules of quantum mechanics, every kind of quantum systems is associated with a Hilbert space $\mathcal{H}$, which for the purpose of this article we can take as finite dimensional. Since even elementary particles require infinite dimensional Hilbert spaces, this means that we are usually only trying to coherently manipulate a small part of the system. The simplest quantum system has a two dimensional Hilbert space $\mathcal{H} = \mathbb{C}^2$, and is called a qubit, for ‘quantum bit’. The observables of the system are given by bounded operators. This space will be denoted by $\mathcal{B}(\mathcal{H})$. The preparations (states) are given by density operators $\rho \in \mathcal{B}^*(\mathcal{H})$, where the latter denotes the space of trace class operators on $\mathcal{H}$. Of course, on finite dimensional Hilbert spaces all linear operators are bounded and trace class. So we use this notation mostly to keep track of the distinction between spaces of observables and spaces of states.

A quantum channel $T$, which transforms input systems described by a Hilbert space $\mathcal{H}_1$ into output systems described by a (possibly different) Hilbert space $\mathcal{H}_2$ is represented mathematically by a completely positive, unital map $T: \mathcal{B}(\mathcal{H}_2) \to \mathcal{B}(\mathcal{H}_1)$. Each $T$ can be written in the form

$$T(A) = \sum_{j=1}^n F_j^* A F_j,$$

where the $F_j$ are (bounded) operators $\mathcal{H}_2 \to \mathcal{H}_1$, called Kraus operators. The equivalence of this form to the condition of complete positivity is a simple consequence of the Stinespring theorem.

The physical interpretation of $T$ is the following. The expectation value of an $A$ measurement ($A \in \mathcal{B}(\mathcal{H}_2)$) at the output side of the channel, on a system which is initially in the state $\rho \in \mathcal{B}_+(\mathcal{H}_1)$ is given in terms of $T$ by $\text{tr}[\rho T(A)]$. Alternatively we can introduce the map $T_\# : \mathcal{B}_+(\mathcal{H}_1) \to \mathcal{B}_+(\mathcal{H}_2)$ which is dual to $T$, i.e. $\text{tr}[T_\#(\rho)A] = \text{tr}[\rho T(A)]$. It is uniquely determined by $T$ (and vice versa) and we can say that $T_\#$ represents the channel in the Schrödinger picture, while $T$ provides the Heisenberg picture representation.

Let us consider now the special case that $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}$. For example $T$ describes the transmission of photons through an optical fiber or the storage in some sort of quantum memory. Ideally we would prefer channels which do not affect the information at all, i.e. $T = \text{Id}$, the identity map on $\mathcal{B}(\mathcal{H})$. We will call this case the ideal channel. In real situations, however, interaction with the environment, i.e. additional, unobservable degrees of freedom, can not be avoided. The general structure of such a noisy channel is given by

$$\rho \mapsto T_\#(\rho) = \text{tr}_K(U(\rho \otimes \rho_0)U^*).$$

where $U : \mathcal{H} \otimes \mathcal{K} \to \mathcal{H} \otimes \mathcal{K}$ is a unitary operator describing the common evolution of the system (Hilbert space $\mathcal{H}$) and the environment (Hilbert space $\mathcal{K}$) and $\rho_0 \in \mathcal{S}(\mathcal{K})$ is the initial state of the environment (cf. Figure 1). Note
that each $T$ can be represented in this way (this is again an easy consequence of the Stinespring theorem), however there are in general many possible choices for such an “ancilla representation”.

3 Channel capacities

As we have already pointed out in the introduction, the capacity of a quantum channel is, roughly speaking, the number of qubits transmitted per channel usage. In this section we will come to a more precise description.

3.1 The cb-norm

As a first step we need a measure for the difference between a noisy channel $T : B(\mathcal{H}) \to B(\mathcal{H})$ and its ideal counterpart. There are several mathematical ways of expressing this, which turn out to be equivalent for our purpose. We find it most convenient to take a certain norm difference, i.e., to consider $\| T - \text{Id} \|_{\text{cb}}$ as a quantitative description of the noise level in $T$, where $\| \cdot \|_{\text{cb}}$ denotes a certain norm, called the norm of complete boundedness (“cb-norm” for short). Its physical meaning is that of the largest difference between probabilities measured in two experimental setups, differing only by the substitution of $T$ by $\text{Id}$. Since this setup may involve further subsystems, and the measurement and preparation may be entangled with the systems under consideration, we have to take into account such additional systems in the definition of the norm. For a general linear operator $T : B(\mathcal{H}_2) \to B(\mathcal{H}_1)$ we set

$$
\| T \|_{\text{cb}} = \sup \left\{ \| (T \otimes \text{Id}_n)(A) \| \mid n \in \mathbb{N}; A \in B(\mathcal{H}_2 \otimes \mathbb{C}^n); \| A \| \leq 1 \right\}. \quad (3)
$$

The cb-norm improves the sometimes annoying property of the usual operator norm that quantities like $\| T \otimes \text{Id}_{\mathbb{C}^d} \|$ may increase with the dimension $d$. On infinite dimensional Hilbert spaces $\| T \|_{\text{cb}}$ can be infinite although the supremum for every fixed $n$ is finite. A particular example for a map with such a behavior is the transposition. A map with finite cb-norm is therefore called completely bounded. In a finite dimensional setup each linear map is completely bounded.
For the transposition $\Theta$ on $\mathbb{C}^d$ we have in particular $\|\Theta\|_{cb} = d$. The cb-norm has some nice features which we will use frequently. This includes its multiplicativity $\|T_1 \otimes T_2\|_{cb} = \|T_1\|_{cb} \|T_2\|_{cb}$ and the fact that $\|T\|_{cb} = 1$ for every channel. For more properties of the cb-norm we refer to [14].

### 3.2 Achievable rates and capacity

How can we reduce the error level $\|T - \text{Id}\|_{cb}$? As an example, consider a small unitary rotation, i.e., $T(X) = U^*XU$, with $\|T - \text{Id}\|_{cb} \leq 2\|U - \text{Id}\|$ small. Then if we know $U$, it is easy to correct $T$ by the inverse rotation, either before $T$, as an “encoding”, or afterwards, as a “decoding” operation. More generally, we may use both, i.e., we are trying to make the combination $ETD \approx \text{Id}$, by careful choice of the channels $E$ and $D$. Note that in this way we may look at channels $T$, which have different input and output spaces, and hence cannot be compared directly with the ideal channel on any system. For such channels there is no intrinsic way of defining “errors” as deviations from a desired standard. Moreover, we are free to choose the Hilbert space $\mathcal{H}_0$ such that $ETD : \mathcal{B}(\mathcal{H}_0) \rightarrow \mathcal{B}(\mathcal{H}_0)$. For the product $ETD$ to be defined, it is then necessary that $D : \mathcal{B}(\mathcal{H}_0) \rightarrow \mathcal{B}(\mathcal{H}_2)$ and $E : \mathcal{B}(\mathcal{H}_1) \rightarrow \mathcal{B}(\mathcal{H}_0)$. The best error level we can achieve deserves its own notation. We define

$$\Delta(T, M) = \inf_{E,D} \|ETD - \text{Id}\|_{cb},$$

where the infimum is taken over all encodings $E$ and decodings $D$ and $M$ is the dimension of the space $\mathcal{H}_0$. Now for longer messages, e.g., a message of $m$ qubits (so that $M = 2^m$) we need to use the channel more often. In the language of classical information theory, we are using longer code words, say of length $n$. The error for coding $m$ qubits through $n$ uses of the channel $T$ is then $\Delta(T^\otimes n, 2^m)$. Can we make this small while retaining a good rate $m/n$ of bits per channel? Clearly there will be a trade-off between rate and errors, which is the basis of the following Definition. The notation $\lfloor x \rfloor$, read “floor $x$”, denotes the largest integer $\leq x$.

**Definition 3.1** $c \geq 0$ is called an achievable rate for $T$, if

$$\lim_{n \to \infty} \Delta(T^\otimes n, \lfloor 2^m \rfloor) = 0.$$ (5)

The supremum of all achievable rates is called the quantum-capacity of $T$ and is denoted by $Q(T)$.

Because $c = 0$ is always an achievable rate we have $Q(T) \geq 0$. On the other hand, if every $c > 0$ is achievable we write $Q(T) = \infty$.

Often a coding scheme construction does not work for arbitrary integers, but only for specific values of $n$, or the dimension of the coding space. However, this is no serious restriction, as the following Lemma shows.

**Lemma 3.2** Let $(n_{\alpha})_{\alpha \in \mathbb{N}}$ be a strictly increasing sequence of integers such that $\lim_{\alpha} n_{\alpha + 1}/n_{\alpha} = 1$. Suppose $M_{\alpha}$ are integers such that $\lim_{\alpha} \Delta(T^\otimes n_{\alpha}, M_{\alpha}) = 0$. Then any

$$c < \liminf_{\alpha} \frac{\log_2 M_{\alpha}}{n_{\alpha}},$$

(6)
is an admissible rate. Moreover, if the errors decrease exponentially, in the sense that
\[ \Delta(T^\otimes n, M) \leq \mu e^{-\lambda n} \] (\( \mu, \lambda \geq 0 \)), then they decrease exponentially for all \( n \) with rate
\[ \liminf_{n \to \infty} \frac{1}{n} \log \Delta(T^\otimes n, |2^{cn}|) \geq \lambda. \] (7)

Proof. Let us introduce the notation
\[ c_+ = \liminf_{\alpha} (\log_2 M_{\alpha})/n_{\alpha}, \] so \( c < c_+ \). We pick \( \eta > 0 \) such that \( (1 + \eta)c < c_+ \). Then for sufficiently large \( \alpha \geq \alpha_0 \) we have \( (n_{\alpha+1}/n_{\alpha}) \leq (1 + \eta) \), and \( (\log_2 M_{\alpha}/n_{\alpha}) \geq (1 + \eta)c \). Now let \( n \geq n_{\alpha_0} \), and consider the unique index \( \alpha \) such that \( n_{\alpha} \leq n \leq n_{\alpha+1} \). Then \( n \leq (1 + \eta)n_{\alpha} \) and
\[ |2^{cn}| \leq 2^{cn} \leq 2^{c(1+\eta)n_{\alpha}} \leq M_{\alpha}. \] (8)

Clearly, \( \Delta(T^\otimes n, M) \) decreases as \( n \) increases, because good coding becomes easier if we have more parallel channels and increases with \( M \), because if a coding scheme works for an input Hilbert space \( \mathcal{H}_0 \), it also works at least as well for states supported on a lower dimensional subspace. Hence \( \Delta(T^\otimes n, |2^{cn}|) \leq \Delta(T^\otimes n_{\alpha}, M_{\alpha}) \to 0. \) It follows that \( c \) is an admissible rate.

With the exponential bound on \( \Delta \) we find similarly that
\[ \Delta(T^\otimes n, |2^{cn}|) \leq \mu e^{-\lambda n_{\alpha}} \leq \mu e^{-\lambda/(1+\eta)n}, \] so that the liminf in (7) is \( \geq \lambda/(1 + \eta) \). Since \( \eta \) was arbitrary, we get the desired result. \( \square \)

3.3 Elementary properties

To determine \( Q(T) \) in terms of Definition 3.1, is fairly difficult, because optimization problems in spaces of exponentially fast growing dimensions are involved. This renders in particular each direct numerical approach practically impossible. In the classical situation, i.e. if we transfer classical information through a classical channel \( \Phi \), we can define a capacity quantity \( C(\Phi) \) in the same way as above. An explicit calculation of \( C(\Phi) \), however, can be reduced, according to Shannons “noisy channel coding theorem” [17], to an optimization problem over a low dimensional space, which does not involve the limit of infinitely many parallel channels. A similar coding theorem for the quantum case is not yet known – this is the biggest open problem concerning channel capacities.

Nevertheless, there are some special cases in which the capacity can be computed explicitly. The most relevant example is the ideal channel \( \text{Id} = \text{Id}_{\mathcal{B}(\mathcal{H})} \). If \( d^n \geq M \) we can embed \( \mathbb{C}^M \) into \( (\mathbb{C}^d)^{\otimes n} \), hence \( \Delta(\text{Id}^\otimes n, M) = 0 \) and we see that the rate \( \log_2(d) \) can be achieved. Intuitively we expect that this is the best what can be done, because it is impossible to embed a high- into a low-dimensional space. This intuition is in fact correct, i.e. we have \( Q(\text{Id}) = \log_2(d) \) for the ideal channel. A precise proof of this statement is, however, not so easy as it looks like and we skip the details here. Maybe the most easy approach is to use the quantity \( \log_2(\|\Theta T\|_{\text{ch}}) \) (where \( \Theta \) denotes the transposition), which is an upper bound on \( Q(T) \) (cf. [1] or [22]). The same idea can be used to show that the quantum capacity of a classical channel, or more generally a channel \( T \) which uses classical information at an intermediate step, is zero. This is a reformulation of the “no classical teleportation theorem” (cf. again [22]).
Another useful relation concerns the concatenation of two general channels $T_1$ and $T_2$: We transmit quantum information first through $T_1$ and then through $T_2$. It is reasonable to assume that the capacity of the composition $T_2 T_1$ cannot be bigger than the capacity of the channel with the smallest bandwidth. This conjecture is indeed true and known as the “Bottleneck inequality”:

$$Q(T_2 T_1) \leq \min\{Q(T_1), Q(T_2)\}. \quad (10)$$

Alternatively we can use the two channels in parallel, i.e. we consider the tensor product $T_1 \otimes T_2$. In this case the capacity of the resulting channel is at least as big as the sum of $Q(T_1)$ and $Q(T_2)$, i.e. $Q$ is superadditive:

$$Q(T_1 \otimes T_2) \geq Q(T_1) + Q(T_2) \quad (11)$$

(cf. [9] for a proof of both statements). To decide whether $Q$ is even additive, i.e. whether equality holds in (11), is another big open question about channel capacities.

4 Quantum error correction

The definition of capacity requires that we correct errors in a collection of $n$ parallel channels $T^\otimes n$. Here the tensor product means that successive uses of the channel are independent. For example, the physical system used as a carrier is freshly prepared every time we use the channel. This independence is important for error correcting schemes, because it prevents errors happening on different channels to “conspire”.

Suggestive as it may be, quantum mechanics cautions us to be very careful with this sort of language: just as we cannot assign trajectories to quantum systems, it is problematic to speak about errors ‘happening’ in one channel, in a situation where we must expect different classical pictures to ‘occur’ in quantum mechanical superposition. This is to be kept in mind, when we now describe the theory of quantum error correcting codes in the sense of Knill and Laflamme [10], which is very much based on a classification of errors according the place where they occur. For example, the coding/decoding pair $E, D$ will typically have the property that $E(T_1 \otimes T_2 \otimes \cdots \otimes T_n)D = \text{Id}$, whenever the number of positions at which $T_i \neq \text{Id}$, i.e., the number of errors, is small (cf. Figure 2).

In our presentation of the Knill-Laflamme Theory, we start from the error corrector’s dream, namely the situation in which all the errors happen in another part of the system, where we do not keep any of the precious quantum information. This will help us to characterize the structure of the kind of errors which such a scheme may tolerate, or ‘correct’. Of course, the dream is just a dream for the situation we are interested in: several parallel channels, each of which may be affected by errors. But the splitting of the system into subsystems, mathematically the decomposition of the Hilbert space of the total system into a tensor product is something we may change by a suitable unitary transformation. This is then precisely the role of the encoding and decoding operations. The Knill-Laflamme theory is precisely the description of the situation where such a unitary, and hence a coding/decoding scheme exists. Constructing such schemes, however, is another matter, to which we will turn in the next section.
4.1 An error corrector’s dream

So consider a system split into $\mathcal{H} = \mathcal{H}_g \otimes \mathcal{H}_b$, where the indices $g$ and $b$ stand for ‘good’ and ‘bad’. We prepare the system in a state $\rho \otimes \ket{\Omega}\bra{\Omega}$, where $\rho$ is the quantum state we want to protect. Now come the errors in the form of a completely positive map $T(A) = \sum_i F_i^* A F_i$. Then according to the error corrector’s dream, we would just have to discard the bad system, and get the same state $\rho$ as before.

The hardest demands for realizing this come from pure states $\rho = \ket{\phi}\bra{\phi}$, because the only way that the restriction to the good system can again be $\ket{\phi}\bra{\phi}$ is that the state after errors factorizes, i.e.

$$T_* (\ket{\phi \otimes \Omega}\bra{\phi \otimes \Omega}) = \sum_i |F_i(\phi \otimes \Omega)| \langle F_i(\phi \otimes \Omega) | = |\phi\rangle \langle \phi | \otimes \sigma . \quad (12)$$

This requires that

$$F_i(\phi \otimes \Omega) = \phi \otimes \Phi_i , \quad (13)$$

where $\Phi_i \in \mathcal{H}_b$ is some vector, which must be independent of $\phi$ if such an equation is to hold for all $\phi \in \mathcal{H}_g$. Conversely, condition (13) implies (12) for every pure state $|\phi\rangle \langle \phi|$ and, by convex combination, for every state $\rho$.

Two remarks are in order. Firstly, we have not required that $F_i = \mathbb{I} \otimes F'_i$. This would be equivalent to demanding that this scheme works with every $\Omega$, or indeed with every (possibly mixed) initial state of the bad system. This would be much too strong for a useful theory of codes. So later on we must insist on a proper initialization of the bad subsystem by a suitable encoding. Secondly, if we have the condition (13) for the Kraus operators of some channel $T$, then it also holds for all channels whose Kraus operators can be written as linear combinations of the $F_i$. In other words, the “set of correctible errors” is naturally identified with the vector space of operators $F$ such that there is a vector $\Phi \in \mathcal{H}_b$ with $F(\phi \otimes \Omega) = \phi \otimes \Phi$ for all $\phi \in \mathcal{H}_g$. This space will be called the maximal error space of the coding scheme, and will be denoted by $E_{\text{max}}$. Usually, a code is designed for a given error space $E$. Then the statement that these given errors are corrected simply becomes $E \subset E_{\text{max}}$. The key observation,
however, is that the space of errors is a vector space in a natural way, i.e., if we
can correct two types of errors, then we can also correct their superposition.

4.2 Realizing the dream by unitary transformation

Let us now consider the situation in which we want to send states of a small
system with Hilbert space $\mathcal{H}_1$ through a channel $T : \mathcal{B}(\mathcal{H}_2) \rightarrow \mathcal{B}(\mathcal{H}_2)$. The
Kraus operators of $T$ lie in an error space $\mathcal{E} \subset \mathcal{B}(\mathcal{H}_2)$, which we assume to be
given. No more assumptions will be made about $T$. Our task is now to devise
coding $E$ and decoding $D$ so that $ETD$ is the identity on $\mathcal{B}(\mathcal{H}_1)$.

The idea is to realize the error corrector’s dream by suitable encoding. The
‘good’ space in that scenario is, of course, the space $\mathcal{H}_1$. We are looking for a
way to write $\mathcal{H}_2 \cong \mathcal{H}_1 \otimes \mathcal{H}_b$. Actually, an isomorphism may be asking too much,
and we look for an isometry $U : \mathcal{H}_1 \otimes \mathcal{H}_b \rightarrow \mathcal{H}_2$. The encoding, written best in the Schrödinger picture, is tensoring with an initial state $\Omega$ as befo re, but now
with an additional twist by $U$:

$$E_*(\rho) = U(\rho \otimes |\Omega\rangle\langle\Omega|)U^*.$$  \hfill (14)

The decoding operation $D$ is again taking the partial trace over the bad space
$\mathcal{H}_b$, after reversing of $U$. Since $U$ is only an isometry and not necessarily unitary
we need an additional term to make $D$ unit preserving. The whole operation is
is best written in the Heisenberg picture:

$$D(X) = U(X \otimes \mathbb{1})U^* + \text{tr}(\rho_0X)(\mathbb{1} -UU^*),$$ \hfill (15)

where $\rho_0$ is an arbitrary density operator. These transformations are successful,
if the error space (transformed by $U$) behaves as before, i.e., if for all $F \in \mathcal{E}$
there are vectors $\Phi(F) \in \mathcal{H}_b$ such that, for all $\phi \in \mathcal{H}_1$

$$FU(\phi \otimes \Omega) = U(\phi \otimes \Phi(F))$$  \hfill (16)

holds. This equation describes precisely the elements $F \in \mathcal{E}_{\text{max}}$ of the maximal
error space.

To check that we really have $ETD = \text{Id}$ for any channel $T(A) = \sum_i F_i^*AF_i$
with $F_i \in \mathcal{E}_{\text{max}}$, it suffices to consider pure input states $|\phi\rangle\langle\phi|$, and the measure-
ment of an arbitrary observable $X$ at the output:

$$\text{tr}[|\phi\rangle\langle\phi|ETD(X)] = \sum_i \text{tr}[U|\phi\rangle\langle\phi|\otimes \Omega|U^*F_iU(X \otimes \mathbb{1})U^*F_i]$$
$$= \sum_i \text{tr}[|\phi\rangle\langle\phi|\otimes \Phi(F_i)]|X \otimes \mathbb{1}|$$
$$= \langle\phi, X\phi\rangle\sum_i \|\Phi(F_i)\|^2 = \langle\phi, X\phi\rangle.$$ \hfill (17)

In the last equation we have used that $\sum_i \|\Phi(F_i)\|^2 = 1$, since $E, T,$ and $D$ each
map $\mathbb{1}$ to $\mathbb{1}$.

4.3 The Knill-Laflamme condition

The encoding $E$ defined in Equation (14) is of the form $E_*(\rho) = V\rho V^*$ with the
encoding isometry $V : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ given by

$$V\phi = U(\phi \otimes \Omega).$$ \hfill (18)
If we just know this isometry and the error space we can reconstruct the whole structure, including the decomposition \( \mathcal{H}_2 = \mathcal{H}_1 \otimes \mathcal{H}_b \oplus (\mathbb{I} - UU^* )\mathcal{H}_2 \), and hence the decoding operation \( D \). A necessary condition for this, first established by Knill and Laflamme \cite{10}, is that, for arbitrary \( \phi_1, \phi_2 \in \mathcal{H}_1 \) and error operators \( F_1, F_2 \in \mathcal{E} \):

\[
\langle V \phi_1, F_1^* F_2 V \phi_2 \rangle = \langle \phi_1, \phi_2 \rangle \omega(F_1^* F_2)
\]  

holds with some numbers \( \omega(F_1^* F_2) \) independent of \( \phi_1, \phi_2 \). Indeed, from (16) we immediately get this equation with \( \omega(F_1^* F_2) = \langle \Phi(F_1), \Phi(F_2) \rangle \). Conversely, if the Knill-Laflamme condition (19) holds, the numbers \( \omega(F_1^* F_2) \) serve as a (possibly degenerate) scalar product on \( \mathcal{E} \), which upon completion becomes the ‘bad space’ \( \mathcal{H}_b \), such that \( F \in \mathcal{E} \) is identified with a Hilbert space vector \( \Phi(F) \).

The operator \( U : \phi \otimes \Phi(F) = FV \phi \) is then an isometry, as used at the beginning of this section. To conclude, the Knill-Laflamme condition is necessary and sufficient for the existence of a decoding operation. Its main virtue is that we can use it without having to construct the decoding explicitly.

### 4.4 Example: Localized errors

Let us come back to the problem we are addressing in this paper. In that case the space \( \mathcal{H}_2 \) is the \( n \)-fold tensor product of the system \( \mathcal{H} \) on which the noisy channels under consideration act. We say that a coding isometry \( V : \mathcal{H}_1 \rightarrow \mathcal{H}^\otimes n \) corrects \( f \) errors, if it satisfies the Knill-Laflamme condition (19) for the error space \( \mathcal{E}_f \) spanned linearly by all operators of the kind \( X_1 \otimes X_2 \otimes \cdots \otimes X_n \), where at most \( f \) places we have a tensor factor \( X_i \neq \mathbb{I} \).

When \( F_1 \) and \( F_2 \) are both supported on at most \( f \) sites, the product \( F_1^* F_2 \), which appears in the Knill-Laflamme condition involves \( 2f \) sites. Therefore we can paraphrase the condition by saying that

\[
\langle V \phi_1, X V \phi_2 \rangle = \langle \phi_1, \phi_2 \rangle \omega(X)
\]  

for \( X \in \mathcal{E}_{2f} \). From Kraus operators in \( \mathcal{E}_f \) we can build arbitrary channels of the kind \( T = T_1 \otimes T_2 \otimes \cdots \otimes T_n \), where at most \( f \) of the tensor factors \( T_i \) are channels different from \( \mathbb{I} \). We will use this in the form that \( E(R_1 \otimes R_2 \otimes \cdots \otimes R_n)D = 0 \), whenever at most \( f \) tensor factors are \( R_i \neq \mathbb{I} \), and at least one of them is a difference of two channels.

There are several ways to construct error correcting codes of this type (see e.g. \cite{12, 13}). Most appropriate for our purposes is the scheme proposed in \cite{16}, which is quite easy to describe and admits a simple way to check the error correction condition. This will be the subject of the next section.

### 5 Graph Codes

The general scheme of graph codes works not just for qubits, but for any dimension \( d \) of one site spaces. The code will have some number \( m \) of input systems, which we label by a set \( X \), and, similarly \( n \) output systems, labeled by a set \( Y \). The Hilbert space of the system with label \( x \in X \cup Y \) will be denoted by \( \mathcal{H}_x \), although all these are isomorphic to \( \mathbb{C}^d \), and are equipped with a special basis \( | j_x \rangle \), where \( j_x \in \mathbb{Z}_d \) is an integer taken modulo \( d \). As a convenient shorthand,
we write \( j_X \) for a tuple of \( j_x \in \mathbb{Z}_d \), specified for every \( x \in X \). Thus the \( |j_X \rangle \) form a basis of the input space \( \mathcal{H}_X = \otimes_{x \in X} \mathcal{H}_x \) of the code. An operator \( F \), say, on the output space will be called localized on a subset \( Z \subset Y \) of systems, if it is some operator on \( \otimes_{y \in Z} \mathcal{H}_y \), tensored with the identity operators of the remaining sites.

The main ingredient of the code construction is now an undirected graph with vertices \( X \cup Y \). The links of the graph are given by the adjacency matrix, which we will denote by \( \Gamma \). When we have \( |X| = m \) input vertices and \( |Y| = n \) output vertices, this is an \((n+m) \times (n+m)\) matrix with \( \Gamma_{xy} = 1 \) if node \( x \) and \( y \) are linked and \( \Gamma_{xy} = 0 \) otherwise. We do allow multiple edges, so the entries of \( \Gamma \) will in general be integers, which can also be taken modulo \( d \). It is convenient to exclude self-linked vertices, so we always take \( \Gamma_{xx} = 0 \).

The graph determines an operator \( V = V_\Gamma : \mathcal{H}_X \to \mathcal{H}_Y \) by the formula

\[
\langle j_Y | V_\Gamma | j_X \rangle = d^{-n/2} \exp \left( \frac{i\pi}{d} j_{X \cup Y} \cdot \Gamma \cdot j_{X \cup Y} \right),
\]

where the exponent contains the matrix element of \( \Gamma \)

\[
j_{X \cup Y} \cdot \Gamma \cdot j_{X \cup Y} = \sum_{x,y \in X \cup Y} j_x \Gamma_{xy} j_y.
\]

Because \( \Gamma \) is symmetric, every term in this sum appears twice, hence adding a multiple of \( d \) to any \( j_x \) or \( \Gamma_{xy} \) will change the exponent in (21) by a multiple of \( 2\pi \), and thus will not change \( V_\Gamma \).

The error correcting properties of \( V_\Gamma \) are summarized in the following result [16]. It is just the Knill-Laflamme condition with a special expression for the form \( \omega \), for error operators such that \( F_1^* F_2 \) is localized on a set \( Z \).

**Proposition 5.1** Let \( \Gamma \) be a graph, i.e., a symmetric matrix with entries \( \Gamma_{xy} \in \mathbb{Z}_d \), for \( x, y \in (X \cup Y) \). Consider a subset \( Z \subset Y \), and suppose that the \((Y \setminus Z) \times (X \cup Z)\)-submatrix of \( \Gamma \) is non-singular, i.e.,

\[
\forall_{y \in Y \setminus Z} \sum_{x \in X \cup Z} \Gamma_{yx} h_x \equiv 0 \quad \text{implies} \quad \forall_{x \in X \cup Z} h_x \equiv 0
\]

where congruences are \( \mod d \). Then, for every operator \( F \in \mathcal{B}(\mathcal{H}_Y) \) localized on \( Z \), we have

\[
V_\Gamma^* F V_\Gamma = d^{-n} \text{tr}(F) \mathbb{1}_X
\]

**Proof.** It will be helpful to use the notation for collections of variables, already present in (22) more systematically: for any subset \( W \subset X \cup Y \) we write \( j_W \) for the collection of variables \( j_y \) with \( y \in W \). The Kronecker-Delta \( \delta(j_W) \) is defined to be zero if for any \( y \in W \) \( j_y \neq 0 \), and one otherwise. By \( j_W \cdot \Gamma_{WW'} \cdot k_{W'} \) we mean the suitably restricted sum, i.e., \( \sum_{x \in W, y \in W'} j_x \Gamma_{xy} k_y \). The important sets

**Figure 3**: Two graph codes.
to which we apply this notation are $X' = (X \cup Z)$ and $Y' = Y \setminus Z$. In particular, 
the condition on $\Gamma$ can be written as $\Gamma_{Y'X'} j_{X'} = 0 \implies j_{X'} = 0$.

Consider now the matrix element

$$
\langle j_X | V^*_T F V_T | k_X \rangle = \sum_{j_Y, k_Y} \langle j_X | V^*_T | j_Y \rangle \langle j_Y | F | k_Y \rangle \langle k_Y | V_T | k_X \rangle
$$

$$
= d^{-n} \sum_{j_Y, k_Y} e^{\frac{i\pi}{2} \left(k_{X,Y} - j_{X,Y} \cdot \Gamma_{Y,X} \cdot j_{X,Y}\right)} \langle j_Y | F | k_Y \rangle
$$

Since $F$ is localized on $Z$, the matrix element contains a factor $\delta_{j_y,k_y}$ for every 
y \in Y \setminus Z = Y'$, so we can write $\langle j_Y | F | k_Y \rangle = \langle j_Z | F | k_Z \rangle \delta(j_Y' - k_Y')$. Therefore 
we can compute the sum (25) in stages:

$$
\langle j_X | V^*_T F V_T | k_X \rangle = \sum_{j_Z, k_Z} \langle j_Z | F | k_Z \rangle S(j_{X'}, k_{X'})
$$

where $S(j_{X'}, k_{X'})$ is the sum over the $Y'$-variables, which, of course, still depends 
on the input variables $j_X, k_X$ and the variables $j_Z, k_Z$ at the error positions:

$$
S(j_{X'}, k_{X'}) = d^{-n} \sum_{j_{Y'}, k_{Y'}} \delta(j_{Y'} - k_{Y'}) e^{\frac{i\pi}{2} \left(k_{X,Y} - j_{X,Y} \cdot \Gamma_{Y,X} \cdot j_{X,Y}\right)}
$$

The sums in the exponent can each be split into four parts according to the 
decomposition $X'$ vs. $Y'$. The terms involving $\Gamma_{Y'Y'}$ cancel because $k_{Y'} = j_{Y'}$. 
The terms involving $\Gamma_{X'Y'}$ and $\Gamma_{Y'X'}$ are equal because $\Gamma$ is symmetric, and 
together give $2j_{Y'} \cdot \Gamma_{Y'X'} \cdot (k_{X'} - j_{X'})$. The $\Gamma_{X'X'}$ remain unchanged, but only 
give a phase factor independent of the summation variables. Hence

$$
S(j_{X'}, k_{X'}) = d^{-n} e^{\frac{i\pi}{2} \left(k_{X'} - j_{X'} \cdot \Gamma_{X'} \cdot j_{X'}\right)} \sum_{j_{Y'}} e^{\frac{i\pi}{2} \left(k_{X',Y'} - j_{X',Y'} \cdot \Gamma_{Y',X'} \cdot j_{X',Y'}\right)}
$$

$$
= d^{-n} e^{\frac{i\pi}{2} \left(k_{X'} - j_{X'} \cdot \Gamma_{X'} \cdot j_{X'}\right)} \langle Y' | \langle \Gamma_{Y',X'} \cdot (k_{X'} - j_{X'}) \rangle
$$

$$
= d^{-n+|Y'|} e^{\frac{i\pi}{2} \left(k_{X'} - j_{X'} \cdot \Gamma_{X'} \cdot j_{X'}\right)} \delta(k_{X'} - j_{X'})
$$

Here we used at the first equation that the sum is a product of geometric series 
as they appear in discrete Fourier transforms. At the second equality the main condition of the Proposition enters: if $\sum_{z \in X} \Gamma_{yz} (k_z - j_z)$ vanishes for all $y \in Y'$ 
as required by the delta-function then (and only then) the vector $k_{X'} - j_{X'}$ must 
vanish. But then the two terms in the exponent of the phase factor also cancel.

Inserting this result into (26), and using that $\delta(h_{X'}) = \delta(h_X) \delta(h_Z)$, we find

$$
\langle j_X | V^*_T F V_T | k_X \rangle = \delta(j_X - k_X) d^{-n+|Y'|} \sum_{j_Z} \langle j_Z | F | j_Z \rangle
$$

$$
= \delta(j_X - k_X) d^{-n} \sum_{j_Y} \langle j_Y | F | j_Y \rangle
$$

Here the error operator is considered in the first line as an operator on $\mathcal{H}_Z$, and 
as an operator on $\mathcal{H}_{Y'}$ in the second line, by tensoring it with $\mathbb{1}_{Y'}$. This cancels 
the dimension factor $d^{|Y'|}$.
All that is left to get an error correcting code is to ensure that the conditions of this Proposition are satisfied sufficiently often. This is evident from combining the above Proposition with the example at the end of Section 4.3.

**Corollary 5.2** Let $\Gamma$ be a graph as in the previous Proposition, and suppose that the $(Y \setminus Z) \times (X \cup Z)$-submatrix of $\Gamma$ is non-singular for all $Z \subset Y$ with up to $2f$ elements. Then the code associated to $\Gamma$ corrects $f$ errors.

Two particular examples (which are equivalent!) are given in Figure 3. In both cases we have $N = 1$, $M = 5$ and $K = 1$ i.e. one input node, which can be chosen arbitrarily, five output nodes and the corresponding codes correct one error.

### 6 Discrete to continuous error model

The discrete error correction scheme described in the last section is not really designed to correct small errors: it corrects rare errors in multiple applications of the channel. A typical example of a small (but not rare) error is a small unitary rotation, $T(X) = U^* X U$. Then $\|T - \text{Id}\|_{cb}$ can be small, but since the same small error happens to each of the parallel channels in $T^{\otimes n}$, the error syndromes of discrete error correction at first sight do not seem to be appropriate at all. Nevertheless, the discrete theory can be applied, and this is the content of the following Proposition. It is the appropriate formulation of “reducing the order of errors from $\varepsilon$ to $\varepsilon^{f+1}$”.

**Proposition 6.1** Let $T : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ be a channel, and let $E, D$ be encoding and decoding channels for coding $m$ systems into $n$ systems. Suppose that this coding scheme corrects $f$ errors, and that

$$\|T - \text{Id}\|_{cb} \leq (f + 1)/(n - f - 1). \quad (29)$$

Then

$$\|ET^{\otimes n}D - \text{Id}\|_{cb} \leq \|T - \text{Id}\|_{cb}^{f+1} 2^n H_2((f+1)/n), \quad (30)$$

where $H_2(r) = -r \log_2 r - (1-r) \log_2 (1-r)$ denotes the Shannon entropy of the probability distribution $(r, 1-r)$.

**Proof.** Into $ET^{\otimes n}D$, we insert the decomposition $T = \text{Id} + (T - \text{Id})$ and expand the product. This gives $2^n$ terms, containing tensor products with some number, say $k$, of tensor factors $(T - \text{Id})$ and tensor factors $\text{Id}$ on the remaining $(n-k)$ sites. Now when $k \leq f$, the error correction property makes the term zero. Terms with $k > f$ we estimate by $\|T - \text{Id}\|_{cb}^k$. Collecting terms we get

$$\|ET^{\otimes n}D - \text{Id}\|_{cb} \leq \sum_{k=f+1}^{n} \binom{n}{k} \|T - \text{Id}\|_{cb}^k. \quad (31)$$

The rest then follows from the next Lemma (with $r = (f+1)/n$). It treats the exponential growth in $n$ for truncated binomial sums.
Lemma 6.2 Let $0 \leq r \leq 1$ and $a > 0$ such that $a \leq r/(1 - r)$. Then, for all integers $n$:

$$\frac{1}{n} \log \left( \sum_{k=rn}^{n} \binom{n}{k} a^k \right) \leq \log(a^r) + H_2(r). \quad (32)$$

Proof. For $\lambda > 0$ we can estimate the step function by an exponential, and get

$$\sum_{k=0}^{n-rn} \binom{n}{k} a^k \leq n \sum_{k=0}^{rn} \binom{n}{k} a^k e^{\lambda(k-rn)} = e^{\lambda rn} (1 + ae^{\lambda})^n = M(\lambda)^n \quad (33)$$

with $M(\lambda) = e^{-\lambda r} (1 + ae^{\lambda})$. The minimum over all real $\lambda$ is attained at $ae^{\lambda_{\text{min}}} = r/(1 - r)$. We get $\lambda_{\text{min}} \geq 0$ precisely when the conditions of the Lemma are satisfied, in which case the bound is computed by evaluating $M(\lambda)$. ☐ ☐

Suppose now that we find a family of coding schemes with $n, m \to \infty$ with fixed rate $r \approx (m/n)$ of inputs per output, and a certain fraction $f/n \approx \varepsilon$ of errors being corrected. Then we can apply the Proposition and find that the errors can be estimated above by

$$\Delta(T^\otimes n, d^m) \leq \left(2^{H_2(\varepsilon)} \|T - \text{Id}\|_{\text{cb}}^{\varepsilon} \right)^n, \quad (34)$$

where $d$ is the Hilbert space dimension of each input system. This goes to zero, and even exponentially to zero, as soon as the expression in parentheses is $< 1$. This will be the case whenever $\|T - \text{Id}\|_{\text{cb}}$ is small enough, or, more precisely,

$$\|T - \text{Id}\|_{\text{cb}} \leq 2^{-H_2(\varepsilon)/\varepsilon}. \quad (35)$$

Note in addition that we have for all $n \in \mathbb{N}$

$$2^{H_2(\varepsilon)/\varepsilon} < \frac{\varepsilon - \frac{1}{n}}{1 - \varepsilon + \frac{1}{n}}. \quad (36)$$

Hence the bound from Equation (34) is implied by (35).

The function appearing on the right hand side of (35) looks rather complicated, so we will often replace it by a simpler one, namely

$$\frac{\varepsilon}{e} \leq 2^{-H_2(\varepsilon)/\varepsilon}, \quad (37)$$

where $e$ is the base of natural logarithms; cf. Figure 4. The proof of this inequality is left to the reader as exercise in logarithms. The bound is very good (exact to first order) in the range of small $\varepsilon$, in which we are most interested anyhow. In any case, from $\|T - \text{Id}\|_{\text{cb}} \leq \varepsilon/e$ we can draw the same conclusion as from (35): exponentially decreasing errors, provided we can actually find code families correcting a fraction $\varepsilon$ of errors. This will be the aim of the next section.
7 Coding by random graphs

Our aim in this section is to apply the theory of graph codes to construct a family of codes with positive rate. It is not so easy to construct such families explicitly. However, if we are only interested in existence, and do not attempt to get the best possible rates, we can use a simple argument, which shows not only the existence of codes correcting a certain fraction of errors, but even that “typical graph codes” for sufficiently large numbers of inputs and outputs have this property. Here “typical” is in the sense of the probability distribution, defined by simply setting the edges of the graph independently, and each according to the uniform distribution of the possible values of the adjacency matrix. For the random method to work we need the dimension of the underlying one site Hilbert space to be a prime number. This curious condition is most likely an artefact of our method, and will be removed later on.

We have seen that a graph code corrects many errors if certain submatrices of the adjacency matrix have maximal rank. Therefore we need the following Lemma.

**Lemma 7.1** Let $d$ be a prime, $M < N$ integers and let $X$ be an $N \times M$-matrix with independent and uniformly distributed entries in $\mathbb{Z}_d$. Then $X$ is singular over the field $\mathbb{Z}_d$ with probability at most $d^{-(N-M)}$.

**Proof.** The sum of independent uniformly distributed random variables in $\mathbb{Z}_d$ is again uniformly distributed. Moreover, since $d$ is prime, this distribution is invariant under multiplication by non-zero factors. Hence if $x_j \in \mathbb{Z}_d$ ($j = 1, \ldots, N$) are independent and uniformly distributed, and $\phi_j \in \mathbb{Z}_d$ are non-random constants, not of all of which are zero, $\sum_{j=1}^N x_j \phi_j$ is uniformly
distributed. Hence, for a fixed vector \( \phi \in \mathbb{Z}^d \), the \( N \) components \( (X\phi)_k = \sum_{j=1}^M X_{kj}\phi_j \) are independent uniformly distributed random variables. Hence the probability for \( X\phi = 0 \) for some fixed \( \phi \neq 0 \) is \( d^{-N} \). Since there are \( d^M - 1 \) vectors \( \phi \) to be tested, the probability for some \( \phi \) to yield \( X\phi = 0 \) is at most \( d^{M-N} \).

Proposition 7.2 Let \( d \) be a prime, and let \( \Gamma \) be a symmetric \((n+m) \times (n+m)\)-matrix with entries in \( \mathbb{Z}^d \), chosen at random such that \( \Gamma_{kk} = 0 \) and that the \( \Gamma_{kj} \) with \( k > j \) are independent and uniformly distributed. Let \( P \) be the probability for the corresponding graph code not to correct \( f \) errors (with \( 2f < n \)). Then

\[
\frac{1}{n} \log P \leq \left( \frac{m}{n} + \frac{4f}{n} - 1 \right) \log d + H_2 \left( \frac{2f}{n} \right).
\]

(38)

Proof. Each error configuration is a \( 2f \)-element subset of the \( n \) output nodes. According to Proposition ... we have to decide, whether the corresponding \((n-2f) \times (m+2f)\)-submatrix of \( \Gamma \), connecting input and error positions with the remaining output positions, is singular or not. Since this submatrix contains no pairs \( \Gamma_{ij}, \Gamma_{ji} \), its entries are independent and satisfy the conditions of the previous Lemma. Hence the probability that a particular configuration of \( e \) errors goes uncorrected is at most \( d^{(m+2f)-(n-2f)} \). Since there are \( \left( \begin{array}{c} n \\ 2f \end{array} \right) \) possible error configurations among the outputs, we can estimate the probability of any \( 2f \) site error configuration to be undetected as less than \( \left( \begin{array}{c} n \\ 2f \end{array} \right) d^{m-n+4f} \). Using Lemma 6.2 we can estimate the binomial as \( \log \left( \begin{array}{c} n \\ 2f \end{array} \right) \leq nH_2(2f/n) \), which leads to the bound stated.

In particular, if the right hand side of the inequality in (38) is negative, we get \( P < 1 \), so that there must be at least one matrix \( \Gamma \) correcting \( f \) errors. The crucial point is that this observation does not depend on \( n \), but only on the rate-like parameters \( m/n \) and \( f/n \). Let us make this behaviour a Definition:

Definition 7.3 Let \( d \) be an integer. Then we say a pair \((\mu, \varepsilon)\) consisting of a coding rate \( \mu \) and an error rate \( \varepsilon \) is achievable, if for every \( n \) we can find an encoding \( E \) of \([\mu n]\) \( d \)-level systems into \( n d \)-level systems correcting \([\varepsilon n]\) errors.

Then we can paraphrase the last proposition as saying that all pairs \((\mu, \varepsilon)\) with

\[
(1 - \mu - 4\varepsilon) \log_2 d > H_2(2\varepsilon)
\]

(39)

are achievable. This is all the input we need for the next section, although a better coding scheme, giving larger \( \mu \) or larger \( \varepsilon \) would also improve the rate estimates proved there. Such improvements are indeed possible. E.g. for the qubit case \( (d = 2) \) it is shown in \[2\] that there is always a code which saturates the quantum Gilbert-Varshamov bound \((1 - \mu - \varepsilon \log_2(3)) > H_2(2\varepsilon) \) which is slightly better than our result.

But there are also known limitations, particularly the so-called Hamming bound. This is a simple dimension counting argument, based on the error correctors dream: Assuming that the scalar product \((F, G) \mapsto \omega(F^*G)\) on the error space \( \mathcal{E} \) is non-degenerate, the dimension of the “bad space” is the same as the
dimension of the error space. Hence with the notations of Section 4 we expect\[\dim \mathcal{H}_0 \cdot \dim \mathcal{E} \leq \dim \mathcal{H}_2.\] We now take \(m\) input systems and \(n\) output systems of dimension \(d\) each, so that \(\dim \mathcal{H}_1 = d^m\) and \(\dim \mathcal{H}_2 = d^n\). For the space of errors happening at at most \(f\) places we introduce a basis as follows: at each site we choose a basis of \(\mathcal{B}(\mathcal{H})\) consisting of \(d^2 - 1\) operators plus the identity. Then a basis of \(\mathcal{E}\) is given by all tensor products with basis elements \(\neq \mathbb{I}\) placed at \(j \leq f\) sites. Hence \(\dim \mathcal{E} = \sum_{j \leq f} \binom{n}{j}(d^2 - 1)^j\). For large \(n\) we estimate this as in Lemma 6.1 as \(\log \dim \mathcal{E} \approx (f/n) \log_2(d^2 - 1) + H_2(f/n)\). Hence the Hamming bound becomes

\[
\frac{m}{n} \log_2 d + H_2(\varepsilon) + \frac{f}{n} \log_2(d^2 - 1) \leq \log_2 d
\]  

which (with \(d^2 \gg 1\)) is just (39) with a factor \(1/2\) on all errors.

If we drop the nondegeneracy condition made above it is possible to find codes which break the Hamming bound [4]. In this case, however, we can consider the weaker singleton bound, which has to be respected by those degenerate codes as well. It reads

\[
1 - \frac{m}{n} \geq df/n.
\]

We omit its proof here (see [13] Sect. 12.4 instead). Both bounds are plotted together with the rate achieved by random graph coding in in Figure 5 (for \(d = 2\)).
8 Conclusions

We are now ready to combine our discussion of channel-capacity from Section 3 with the results about error correction we have derived in the previous sections. Please note that most of the result presented here can be found in [7, 12], in some cases with better bounds.

8.1 Correcting small errors

We first look at the problem which motivated our study, namely estimating the capacity of a channel $T \approx \text{Id}$.

**Theorem 8.1** Let $d$ be a prime, and let $T$ be a channel on $d$-level systems. Suppose that for some $0 < \varepsilon < 1/2$,

$$\|\text{Id} - T\|_{cb} < 2^{-H_2(\varepsilon)/\varepsilon}.$$  \hspace{1cm} (42)

Then

$$Q(T) \geq (1 - 4\varepsilon) \log_2(d) - H_2(2\varepsilon)$$  \hspace{1cm} (43)

**Proof.** For every $n$ set $f = \lfloor \varepsilon n \rfloor$, and $m = \lfloor \mu n \rfloor - 1$, where $\mu$ is, up to a $\log_2(d)$ factor, the right hand side of (13), i.e. $\mu = 1 - 4\varepsilon - \log_2(d)^{-1}H_2(2\varepsilon)$. This ensures that the right hand side of (12) is strictly negative, so there must be a code for $d$-level systems, with $m$ inputs and $n$ outputs, and correcting $f$ errors.

To this code we apply Proposition 6.1, and insert the bound on $\|\text{Id} - T\|_{cb}$ into Equation (34). Thus $\Delta(T^{\otimes n}, d^{\lfloor \mu n \rfloor - 1}) \to 0$, even exponentially. This means that any number $< \mu \log_2(d)$ is an achievable rate. In other words, $\mu \log_2(d)$ is a lower bound to the capacity. 

If $\varepsilon > 0$ is small enough the quantity on the right hand side of Equation (13) is strictly positive (cf. the dotted graph in Figure 5). Hence each channel which is sufficiently close to the identity allows (asymptotically) perfect error correction. Beyond that we see immediately that $Q(T)$ is continuous (in the cb-norm) at $T = \text{Id}$: Since $Q(T)$ is smaller than $\log_2(d)$ and $g(\varepsilon)$ is continuous in $\varepsilon$ with $g(0) = \log_2(d)$ we find for each $\delta > 0$ an $\varepsilon > 0$ exists, such that $\log_2(d) - Q(T) < \varepsilon$ for all $T$ with $\|T - \text{Id}\|_{cb} < \varepsilon/\delta$. In other words if $T$ is arbitrarily close to the identity its capacity is arbitrarily close to $\log_2(d)$. In Corollary 8.3 below we will show the significantly stronger statement that $Q$ is a lower semicontinuous function on the set of all channels.

8.2 Estimating capacity from finite coding solutions

A crucial consequence of the ability to correct small errors is that we do not actually have to compute the limit defining the capacity: if we have a pretty good coding scheme for a given channel, i.e., one that gives us $ET^{\otimes n}D \approx \text{Id}_d$, then we know the errors can actually be brought to zero, and the capacity is close to the nominal rate of this scheme, namely $\log_2(d)/n$.

**Theorem 8.2** Let $T$ be a channel, not necessarily between systems of the same dimension. Let $k, p \in \mathbb{N}$ with $p$ a prime number, and suppose there are channels
E and D encoding and decoding a p-level system through k parallel uses of T, with error \( \Delta = \| \text{Id}_p - ET^{\otimes k}D \|_{cb} < \frac{1}{2\varepsilon} \). Then

\[
Q(T) \geq \frac{\log_2(p)}{n}(1 - 4\varepsilon\Delta) - \frac{1}{n}H_2(2\varepsilon\Delta).
\]

(44)

Moreover, \( Q(T) \) is the least upper bound on all expressions of this form.

Proof. We apply Proposition 8.1 to the channel \( \bar{T} = ET^{\otimes n}D \). With the random coding method we thus find a family of coding and decoding channels \( \bar{E} \) and \( \bar{D} \) from \( m' \) into \( n' \) systems, of \( p \) levels each, such that

\[
\| \text{Id} - \bar{E}(ET^{\otimes k}D)^{\otimes n'}\bar{D} \|_{cb} \to 0.
\]

(45)

This can be reinterpreted as an encoding of \( p^{m'} \)-dimensional systems through \( kn' \) uses of the channel \( T \) (rather than \( \bar{T} \)), which corresponds to a rate \( (kn')^{-1} \log_2(p^{m'}) = (\log_2 p/k)(m'/n') \). We now argue exactly as in the proof of the previous proposition, with \( \varepsilon = \varepsilon\Delta \), so that

\[
\| \text{Id}_p - ET^{\otimes k}D \|_{cb} = \varepsilon/e \leq 2^{H_2(\varepsilon)/\varepsilon}
\]

(46)

by equation (47). By random graph coding we can achieve the coding ratio 
\( \mu \approx (m'/n') = 1 - 4\varepsilon - \log_2(p)^{-1}H_2(2\varepsilon) \), and have the errors \( \Delta(T^{\otimes n'}, p^{m'}) \) go to zero exponentially. Since

\[
\Delta(T^{\otimes kn'}, p^{m'}) \leq \Delta(T^{\otimes n'}, p^{m'}) \leq \| \text{Id} - \bar{E}(ET^{\otimes k}D)^{\otimes n'} \|_{cb},
\]

(47)

we can apply Lemma 3.2 to the channel \( T \) (where the sequence \( n_\alpha \) is given by \( n_\alpha = na \)) and find that the rate \( \mu(\log_2 p/k) \) is achievable. This yields the estimate claimed in Equation (44).

To prove the second statement consider the function \( x \to p(x) \) which associates to each real number \( x \geq 2 \) the biggest prime \( p(x) \) with \( p(x) \leq x \). From known bounds on the length of gaps between two consecutive primes \( \approx \) it follows that \( \lim_{x \to \infty} x/p(x) = 1 \) holds, hence we get \( 2^{kc}/p(2^{kc}) \leq 1 + \delta' \) for an arbitrary \( \delta' > 0 \), provided \( n \) is large enough, but this implies

\[
c - \frac{\log_2[p(2^{kc})]}{k} < \frac{\log_2(1 + \delta')}{k}.
\]

(48)

Since we can choose an achievable rate \( c \) arbitrarily close to the capacity \( Q(T) \) this shows that there is for each \( \delta > 0 \) a prime \( p \) and a positive integer \( k \) such that \( |Q(T) - \log_2(p)/k| \leq \delta \). In addition we can find a coding scheme \( E, D \) for \( T^{\otimes k} \) such that Equation (44) holds, i.e. the right hand side of (44) can be arbitrarily close to \( \log_2(p)/k \), and this completes the proof. \( \square \)

This theorem allows us to derive very easily an important continuity property of the quantum capacity. It is well known that each function \( F \) (on a topological space) which is given as the supremum of a set of real-valued, continuous functions is lower semicontinuous, i.e. the set \( F^{-1}(\{x, \infty\}) \) is open for each \( x \in \mathbb{R} \). Since the right hand side of Equation (44) is continuous in \( T \) and since \( Q(T) \) is (according to Proposition 8.2) the supremum over such quantities, we get:

Corollary 8.3 \( T \mapsto Q(T) \) is lower semi-continuous in cb-norm.

\footnote{If \( p_n \) denotes the \( n \)th prime and \( g(p_n) = p_{n+1} - p_n \) is the length of the gap between \( p_n \) and \( p_{n+1} \), it is shown in 8 that \( g(p) \) is bounded by const\( p^{3/8+\varepsilon} \).}
8.3 Error exponents

Another consequence of Theorem 8.2 concerns the rate with which the error $\Delta(T^\otimes n, 2^{|cn|})$ decays in the limit $n \to \infty$. Theorem 8.2 says, roughly speaking, that we can achieve each rate $c < Q(T)$ by combining a coding scheme $E, D$ with subsequent random-graph coding $\tilde{E}, \tilde{D}$. However, the error $\Delta[(ET^\otimes n D)^{\otimes l}, p^k]$ decays according to (44) and Proposition 7.2 exponentially. A more precise analysis of this idea leads to the following (cf. also the work Hamada [7]):

**Proposition 8.4** If $T$ is a channel with quantum capacity $Q(T)$ and $c < Q(T)$, then, for sufficiently large $n$ we have

$$\Delta(T^\otimes n, 2^{|cn|}) \leq e^{-n\lambda(c)},$$

with a positive constant $\lambda(c)$.

**Proof.** We start as in Theorem 8.2 with the channel $\tilde{T} = ET^\otimes k D$ and the quantity $\Delta = \| \text{Id}_p - ET^\otimes k D\|_{cb}$. However instead of assuming that $\Delta = \varepsilon/e$ holds, the full range $c\Delta \leq \varepsilon \leq 1/2$ is allowed for the error rate $\varepsilon$. Using the same arguments as in the proof of Theorem 8.2 we get an achievable rate

$$c(k,p,\varepsilon) = \frac{\log_2(p)}{k} \left( 1 - 4\varepsilon - \frac{H_2(2\varepsilon)}{\log_2(p)} \right)$$

and an exponential bound on the coding error:

$$\Delta(T^{\otimes kn'}, p^{m'}) \leq \| \text{Id} - \tilde{E}(ET^\otimes k D)^{\otimes m'}\|_{cb} \leq \left( 2^{H_2(\varepsilon)} \Delta^\varepsilon \right)^{m'};$$

cf. Equations (34) and (47).

To calculate the exponential rate $\lambda(c)$ with which the coding error vanishes we have to consider the quantity

$$\lambda(c) = \liminf_{n \to \infty} -\frac{1}{n} \ln \Delta(T^\otimes n, 2^n|c|) \geq \lim_{n \to \infty} -\frac{1}{kn} \ln \left( 2^{H_2(\varepsilon)} \Delta^\varepsilon \right)$$

$$\geq -\frac{\varepsilon}{k} \left( \ln(\Delta) + \ln \left( 2^{H_2(\varepsilon)} \Delta^\varepsilon \right) \right) = -\varepsilon \Lambda(\Delta, \varepsilon)/k$$

where we have inserted inequality (51). Now we can apply Lemma 3.2 (with the sequence $n_a = ka$), which shows that $\lambda(c)$ is positive, if the right hand side of (53) is.

What remains to show is that $\lambda(c) > 0$ holds for each $c < Q(T)$. To this end we have to choose $k, p, \Delta$ and $\varepsilon$ such that $c(k,p,\varepsilon) = c$ and $\Lambda(\Delta, \varepsilon) < 0$. Hence consider $\delta > 0$ such that $c + \delta < Q(T)$ is an achievable rate. As in the proof of Theorem 8.2 we can choose $\log_2(p)/k$ such that $\log_2(p)/k > c + \delta$ holds while $\Delta$ is arbitrarily small. Hence there is an $\varepsilon_0 > 0$ such that $c(k,p,\varepsilon) = c$ implies $\varepsilon > \varepsilon_0$. The statement therefore follows from the fact that there is a $\Delta_0 > 0$ with $\Lambda(\Delta, \varepsilon) > 0$ for all $0 < \Delta < \Delta_0$ and $\varepsilon > \varepsilon_0$.

In addition to the statement of Proposition 8.4 we have just derived a lower bound on the error exponent $\lambda(c)$. Since we can not express the error rate $\varepsilon$ as a function of $k, p$ and $c$ we can not specify this bound explicitly. However we can plot it as a parametrized curve (using Equation (8) and (53) with $\varepsilon$ as the parameter) in the $(c, \lambda)$-space. In Figure 3 this is done for $k = 1, p = 2$ and several values of $\Delta$. 

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We can also tolerate finite errors in encoding. Let $Q_\varepsilon(T)$ denote the quantity defined exactly like the capacity, but with the weaker requirement that $\Delta(T^\otimes n, 2^{[cn]}) \leq \varepsilon$ for large $n$. Obviously we have $Q_\varepsilon(T) \geq Q(T)$ for each $\varepsilon > 0$. Regarded as a function of $\varepsilon$ and $T$ this new quantity admits in addition the following continuity property in $\varepsilon$.

**Proposition 8.5** $\lim_{\varepsilon \to 0} Q_\varepsilon(T) = Q(T)$.

**Proof.** By definition we can find for each $\varepsilon', \delta > 0$ a tuple $n, p, E$ and $D$ such that

$$|| \text{Id}_p - ET^\otimes n D||_{cb} = \frac{\varepsilon' + \varepsilon}{\epsilon}$$

and $|Q_\varepsilon(T) - \log_2(p)/n| < \delta$ holds. If $\varepsilon + \varepsilon'$ is small enough, however, we find as in Theorem 8.2 a random graph coding scheme such that

$$Q(T) \geq \frac{\log_2(p)}{n} (1 - 4(\varepsilon + \varepsilon')) - \frac{1}{n} H_2(2(\varepsilon + \varepsilon')) = g(\varepsilon + \varepsilon').$$

Hence the statement follows from continuity of $g$ and the fact that $g(0) = \log_2(p)/n$ holds. \Box

For a classical channel $\Phi$ even more is known about the similar defined quantity $C_\varepsilon(T)$: If $\varepsilon > 0$ is small enough we can not achieve bigger rates by allowing small errors, i.e. $C(T) = C_\varepsilon(T)$. This is called the “strong converse
of Shannon's noisy channel coding theorem" [17]. To check whether a similar statement holds in the quantum case is one of the big open problem of the theory.

Acknowledgements

Funding by the European Union project EQUIP (contract IST-1999-11053) and financial support from the DFG (Bonn) is greatly acknowledged.

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