Symmetric spaces uniformizing Shimura varieties in the Torelli locus

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Abstract
An algebraic subvariety \( Z \) of \( \mathbb{A}_g \) is totally geodesic if it is the image via the natural projection map of some totally geodesic submanifold \( X \) of the Siegel space. We say that \( X \) is the symmetric space uniformizing \( Z \). In this paper we determine which symmetric space uniformizes each of the low genus counterexamples to the Coleman-Oort conjecture obtained studying Galois covers of curves. It is known that the counterexamples obtained via Galois covers of elliptic curves admit two fibrations in totally geodesic subvarieties. The second result of the paper studies the relationship between these fibrations and the uniformizing symmetric space of the examples.

Keywords Modular and Shimura varieties · Families · Moduli (analytic) · Jacobians · Prym varieties

Mathematics Subject Classification 14G35 · 14H15 · 14H40

1 Introduction
Let \( \mathbb{M}_g \) denote the moduli space of smooth complex algebraic curves of genus \( g \), \( \mathbb{A}_g \) the moduli space of principally polarized abelian varieties of dimension \( g \) over \( \mathbb{C} \) and \( j: \mathbb{M}_g \rightarrow \mathbb{A}_g \) the Torelli map. The moduli space \( \mathbb{A}_g = \mathbb{G}/Sp(2g, \mathbb{Z}) \) is a quotient of the Siegel space \( \mathbb{G}_g \), which is an irreducible hermitian symmetric space of the non-compact type. The symmetric metric on \( \mathbb{G}_g \) is \( Sp(2g, \mathbb{R}) \)-invariant. Hence, it induces on \( \mathbb{A}_g \) a locally symmetric metric, called the Siegel metric. Denote by \( \pi: \mathbb{G}_g \rightarrow \mathbb{A}_g \) the natural projection map.
Definition 1.1 An algebraic subvariety $Z \subset \mathbb{A}_g$ is **totally geodesic** if $Z = \pi(X)$ for some (connected) totally geodesic submanifold $X \subset \mathbb{G}_g$. We say that $X$ is the symmetric space uniformizing $Z$.

In [3, 5, 7–11, 13, 17, 18, 22, 23, 25, 26] totally geodesic subvarieties of $\mathbb{A}_g$ have been studied in relation with the Coleman-Oort conjecture, which for large genus predicts the non-existence of positive-dimensional Shimura subvarieties $Z$ of $\mathbb{A}_g$ generically contained in $\bar{j}(\mathcal{M}_g)$, i.e such that $Z \subset \bar{j}(\mathcal{M}_g)$ and $Z \cap \bar{j}(\mathcal{M}_g) \neq \emptyset$. (See [2, 28] and [26] for a thorough survey). **Shimura subvarieties** of $\mathbb{A}_g$ are defined as Hodge loci for the natural variation of Hodge structure on $\mathbb{A}_g$. In our context, we will make use of the following characterization of Shimura subvarieties of $\mathbb{A}_g$ due to Mumford and Moonen (see [24, 27]): an algebraic subvariety of $\mathbb{A}_g$ is Shimura if and only if it is totally geodesic and contains a CM point.

In this paper, we study the known counterexamples to the Coleman-Oort conjecture. Indeed, for low genus there exists examples of (positive-dimensional) Shimura subvarieties $Z$ of $\mathbb{A}_g$ such that $Z \subset \bar{j}(\mathcal{M}_g)$ and $Z \cap \bar{j}(\mathcal{M}_g) \neq \emptyset$. All the examples known so far are in genus $g \leq 7$ and can be divided in three classes:

1. those obtained as families of Galois covers of $\mathbb{P}^1$ (for a complete list see [7]);
2. those obtained as families of Galois covers of elliptic curves (see [8]);
3. those obtained via fibrations constructed on the examples in (2) (see [11]).

There are some overlaps between (1) and (2). Roughly speaking, the examples in (1) and (2) are constructed as follows: one describes families of Galois covers via some combinatorial data (given by the finite group $G$, genera, number of ramification points, and monodromy). A numerical condition ($\dagger$) on the datum ensures that the family’s image in $\mathbb{A}_g$ is a Shimura variety. In [7] and [8] a computer program found all the examples satisfying ($\dagger$) for $g \leq 9$. Moreover, it was proved in [11] that there are no families of Galois covers of curves with genus $\geq 2$ satisfying ($\dagger$). Recently, it has been proved in [4] that, indeed, the one listed in the references above are the only positive-dimensional families of Galois coverings satisfying ($\dagger$) with $2 \leq g \leq 100$. Note that, in general, it is not known whether ($\dagger$) is also necessary for a family to yield a Shimura variety. More generally, very little is known about other counterexamples.

In the following, we determine which symmetric space uniformizes each of the known counterexamples. Since all 1–dimensional irreducible hermitian symmetric spaces of the non-compact type are isomorphic to the Poincaré disc, we will focus on counterexamples of dimension greater than one. More precisely, it is first computed the uniformizing symmetric space for the examples belonging only to (1), next for the six examples in (2) (actually five, since one of them has dimension 1), including the four examples that also appear in (1). Denote by $B_n(\mathbb{C}) = \{ w \in \mathbb{C}^n : \sum_{k=1}^n w_k \overline{w_k} < 1 \}$ the open unit ball in $\mathbb{C}^n$. Our results are the following:

**Theorem 1.2** The uniformizing symmetric spaces for the examples in [7] of dimension $\geq 2$ are the following:
Theorem 1.3 The uniformizing symmetric spaces for the examples in [8] of dimension $\geq 2$ are the following:

| Family | $g$ | $G$ | $\text{dim}_C$ | Uniformizer |
|--------|-----|-----|---------------|-------------|
| (1e)   | 2   | $\mathbb{Z}/2$ | 3 | $\mathbb{E}_2$ |
| (2e)   | 3   | $\mathbb{Z}/3$ | 2 | $B_2(\mathbb{C})$ |
| (3e)   | 3   | $\mathbb{Z}/4$ | 2 | $B_2(\mathbb{C})$ |
| (4e)   | 4   | $\mathbb{Z}/3$ | 3 | $B_3(\mathbb{C})$ |
| (6e)   | 4   | $\mathbb{Z}/6$ | 2 | $B_3(\mathbb{C})$ |
| (26)   | 6   | $\mathbb{Z}/5$ | 2 | $B_2(\mathbb{C})$ |
| (27)   | 3   | $\mathbb{Z}/2 \times \mathbb{Z}/2$ | 3 | $B_1(\mathbb{C}) \times B_1(\mathbb{C}) \times B_1(\mathbb{C})$ |
| (31)   | 3   | $\mathbb{S}_3$ | 2 | $B_1(\mathbb{C}) \times B_1(\mathbb{C})$ |
| (32)   | 3   | $D_4$ | 2 | $B_1(\mathbb{C}) \times B_1(\mathbb{C})$ |

The 6 families of Galois covers of elliptic curves have been further studied in [11], where it is shown that these families admit two fibrations in totally geodesic subvarieties, countably many of which in each family are Shimura. If $f : C \to C'$ is a Galois covering of an elliptic curve $C$ with group $G$, these fibrations are given by the maps

$$P : [C \to C'] \mapsto \text{Prym}(C, C') \in \mathcal{A}_{g-1}$$

$$\phi : [C \to C'] \mapsto [JC'] \in \mathcal{A}_1.$$

In the second part of this paper we discuss the relationship between the fibrations and the uniformizing symmetric space of the examples. More precisely, we prove the following result.

Theorem 1.4 Let $X$ be the uniformizing symmetric space associated to one of the families (1e), (2e), (3e), (4e), (6e). Then

(i) $X$ decomposes as $B_1(\mathbb{C}) \times M$, where $M$ is an hermitian symmetric space of codimension 1.

(ii) The image in $\mathbb{A}_g$ of $M$ is an irreducible component of the fiber of $\phi$. In particular, $M$ uniformizes the irreducible components of the fibers of $\phi$.

(iii) The image in $\mathbb{A}_g$ of $B_1(\mathbb{C})$ is an irreducible component of the fiber of the Prym map.
In particular, the result also shows how to uniformize the examples in (3). Note that family $(5e)$ doesn’t appear in the statement since it is one-dimensional.

## 2 Cartan decomposition

We devote this section to some considerations on Lie algebras and symmetric spaces that will be relevant in the sequel. For the basic definitions and facts, we refer to [19, 21, 30].

We start by recalling that Riemannian symmetric spaces are Riemannian manifolds of the form $G/K$, where $(G, K, \sigma)$ is a symmetric pair. These are defined as follows.

**Definition 2.1** A symmetric pair $(G, K, \sigma)$ is the datum of

1. A connected Lie group $G$;
2. An involutive automorphism $\sigma : G \to G$;
3. A compact subgroup $K \subset G$ such that $G_0^\sigma \subset K \subset G^\sigma$;

where $G^\sigma = \{ g \in G : \sigma(g) = g \}$ denotes the subgroup of $G$ of the elements fixed by $\sigma$, and $G_0^\sigma$ its connected component of the identity. A symmetric pair $(G, K, \sigma)$ is said to be effective (resp. almost effective) if $G$ acts effectively (resp. almost effectively, i.e., with finite stabilizer) on $G/K$.

Let $(G, K, \sigma)$ be a symmetric pair. Denote by $\mathfrak{k}$ and $\mathfrak{p}$, respectively, the $\pm 1$ eigenspaces of the linear involution $d\sigma$. The Lie algebra $\mathfrak{g}$ of the group $G$ decomposes, as a vector space, as $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. This decomposition has the following properties

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}, \quad [\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}, \quad [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}.$$ 

Moreover $\text{Ad}_K(\mathfrak{p}) \subset \mathfrak{p}$, in particular $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$. The decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is the *Cartan decomposition* associated with the symmetric space. Conversely, starting by such a decomposition of a Lie algebra, one can reconstruct a symmetric space.

Secondly, we recall that symmetric spaces divide into symmetric spaces of compact type, non-compact type, and euclidean type, depending on the sign of their sectional curvature. A symmetric space is of euclidean type if and only if $[\mathfrak{p}, \mathfrak{p}] = 0$, while for those of compact and non-compact type $G$ is semisimple. In general, a symmetric space does not necessarily fall into one of these three types. However, a simply connected symmetric space is isometric to the Riemannian product

$$M = \mathbb{R}^n \times M_1 \times M_2,$$

where $M_1$ is a symmetric space of compact type and $M_2$ of non-compact type. In particular, one says that a symmetric space $M$ has no euclidean factors if its universal cover does not contain factors of euclidean type. Symmetric spaces of the three types have special geometric properties, e.g., symmetric spaces of non-compact type are determined up to isomorphism by their Cartan decomposition.

Now we move to the analysis of the Cartan decomposition associated with the symmetric spaces of our interest. Fix a rank $2g$ lattice $\Lambda$ and an alternating form $Q : \Lambda \times \Lambda \to \mathbb{Z}$ of type $(1, \ldots, 1)$. Let $G := \text{Sp}(\Lambda_\mathbb{R}, Q)$ and $\mathcal{G} = \mathcal{G}(\Lambda_\mathbb{R}, Q)$ the Siegel space, which can be defined as:

\[\mathcal{G} \subseteq \mathbb{R}^n \times M_1 \times M_2,\]
\[ \mathcal{S} = \{ J \in GL(\Lambda_R) : J^2 = -I, \text{ } J^*Q = Q, \text{ } Q(x, Jx) > 0, \forall x \neq 0 \}, \quad \text{(2.1)} \]

Fix a finite subgroup \( \Gamma \subset G \) and consider \( \mathcal{S}^\Gamma \). As a consequence of the Cartan fixed point theorem, \( \mathcal{S}^\Gamma \) is nonempty and hence it is a connected complex submanifold of \( \mathcal{S} \) (see e.g., [7, Lemma 3.3]). For \( J \in \mathcal{S} \), denote by \( \mathfrak{g} = \mathfrak{t} \oplus \mathfrak{p} \) the Cartan decomposition associated to \( \mathcal{S} \) and by \( K = Stab_J(G) \). Set \( \mathcal{Z}_\mathfrak{g}(\Gamma) = \{ X \in \mathfrak{g} : \text{Ad}_\mathfrak{t}(X) = X \} \), and define similarly \( \mathcal{Z}_\mathfrak{t}(\Gamma) \) and \( \mathcal{Z}_\mathfrak{p}(\Gamma) \). For \( J \in \mathcal{S}^\Gamma \), \( T_J \mathcal{S}^\Gamma = (T_J \mathcal{S})^\Gamma = \mathcal{Z}_\mathfrak{p}(\Gamma) \). The following is standard, we include the proof for the reader's convenience.

**Lemma 2.2** Let \( J \in \mathcal{S}^\Gamma \). Then \( \mathcal{Z}_\mathfrak{g}(\Gamma) = \mathcal{Z}_\mathfrak{t}(\Gamma) \oplus \mathcal{Z}_\mathfrak{p}(\Gamma) \). In particular, \( \mathcal{Z}_\mathfrak{p}(\Gamma) = \mathcal{Z}_\mathfrak{g}(\Gamma) \cap \mathfrak{p} \).

**Proof** Let \( X \in \mathcal{Z}_\mathfrak{g}(\Gamma) \), and write \( X = u + v \) with \( u \in \mathfrak{t} \) and \( v \in \mathfrak{p} \). Since \( J \in \mathcal{S}^\Gamma \) = \{ \( J \in \mathcal{S} : \gamma J \gamma^{-1} = J \forall \gamma \in \Gamma \} \), we have that \( \Gamma \subset Stab_J = K \). It follows that \( \text{Ad}_\mathfrak{t}(\mathfrak{p}) \subset \mathfrak{p} \) and \( \text{Ad}_\mathfrak{t}(\mathfrak{t}) \subset \mathfrak{t} \). Thus \( X = \text{Ad}_\mathfrak{t}(X) = \text{Ad}_\mathfrak{t}(u) + \text{Ad}_\mathfrak{t}(v) \), with \( \text{Ad}_\mathfrak{t}(u) \in \mathfrak{t} \) and \( \text{Ad}_\mathfrak{t}(v) \in \mathfrak{p} \). We conclude that \( u = \text{Ad}_\mathfrak{t}(u) \) and \( \text{Ad}_\mathfrak{t}(v) = v \). \( \square \)

Observe that, since \( \mathfrak{t} \) is a group of isometries of \( \mathcal{S} \), \( \mathcal{S}^\Gamma \) is a totally geodesic submanifold of \( \mathcal{S} \) (see e.g., [20, Theorem 5.1, p. 59]).

In particular, it is an hermitian symmetric space of the non-compact type. The image of \( \mathcal{S}^\Gamma \) in \( \Lambda_g \) is a Shimura variety (See e.g., [7, Proposition 3.7]).

**Lemma 2.3** Let \( Z \) be a complex totally geodesic submanifold of \( \mathcal{S} \). Then \( Z \) has no euclidean factor.

**Proof** Since \( \mathcal{S} \) is an hermitian symmetric space, the complex structure \( \hat{I} \) on \( \mathcal{S} \) is induced by \( \text{Ad}(z)|_\mathfrak{p} : \mathfrak{p} \rightarrow \mathfrak{p} \) with \( z \in Z(K) \). More precisely, one can show that the matrix representing \( z \) in an appropriate basis is given by \( z = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & I \\ -I & 1 \end{pmatrix} \in Sp(2g, \mathbb{R}) \simeq G \). With this choice it easy is to see that for \( X \in \mathfrak{p}, X \neq 0 \), we have \( [X, \hat{I}X] \neq 0 \). \( \square \)

**Lemma 2.4** Let \((G, K)\) be a symmetric pair with no euclidean factor and Cartan decomposition \( \mathfrak{g} = \mathfrak{t} \oplus \mathfrak{p} \). Suppose that \( G \) acts almost effectively on the coset space \( M = G/K \). Then \( \mathfrak{t} = [\mathfrak{p}, \mathfrak{p}] \).

For a proof see e.g., [19, Theorem 4.1, p. 243]. As a consequence of Lemma 2.3, \( \mathcal{S}^\Gamma \) has no euclidean factor. We can thus apply Lemma 2.4 to an almost effective symmetric pair associated with \( \mathcal{S}^\Gamma \). Recalling also Lemma 2.2, we get the following.

**Corollary 2.5** If \((G', K')\) is an almost effective symmetric pair associated to \( \mathcal{S}^\Gamma \), then its Cartan decomposition \( \mathfrak{g}' = \mathfrak{t}' \oplus \mathfrak{p}' \) is given by
\[
\mathfrak{p}' = \mathcal{Z}_\mathfrak{p}(\Gamma) = \mathcal{Z}_\mathfrak{g}(\Gamma) \cap \mathfrak{p}, \quad \mathfrak{t}' = [\mathcal{Z}_\mathfrak{p}(\Gamma), \mathcal{Z}_\mathfrak{p}(\Gamma)].
\]

Notice that since \( \mathcal{S}^\Gamma \) is of the noncompact type, the Cartan decomposition in Corollary 2.5 determines the symmetric space up to isomorphism.

The following table exhausts the list of irreducible Hermitian symmetric spaces of the non-compact type up to dimension 3. (See [19, Table V, p. 518]).
SU(C → q → g) is a Shimura variety (See e.g., [7, Proposition 3.7]). Moreover, $Z \subset \pi(\mathbb{G})$ (7, Theorem 3.9]). Then, the condition

$$(\dim Z =) \dim(S^2H^0(C, K_C))^G = \dim H^0(C, 2K_C)^G (= \dim \pi(\mathbb{G})),$$  

implies that $Z = \pi(\mathbb{G})$. Thus, under the condition (*), $Z$ provides an example of a special subvariety of $A_g$ contained in the Torelli locus, whose uniformizing symmetric space is $\mathbb{G}$. In particular, it is a Shimura subvariety of $A_g$ generically contained in $j(M_g)$.

If $f : C \to C'$ is one of the coverings in the family above, the action of $G$ on $C$ induces the following action of $G$ on holomorphic 1–forms

$$\rho : G \to GL(H^0(C, K_C)),$$

$$\rho(g)(\omega) = g.\omega := (g^{-1})^*(\omega).$$

Notice that the equivalence class of the representation $\rho$ only depends on the family (not on the point of the family). The homomorphism $\rho$ maps $G$ injectively into $Sp(\Lambda, Q)$, where $\Lambda = H^1(C, \mathbb{Z})$ and $Q$ is the cup product (see e.g., [6, p. 270]). Denote by $\Gamma$ the image of $G$ in $Sp(\Lambda, Q)$ and $\mathbb{G} = \mathbb{G}(\mathbb{A}_\mathbb{F}, Q)$, defined as in (2.1). In the sequel, our technique will be the following: we think $Sp(\Lambda, Q) \subset GL(H^1(C, \mathbb{C}))$. Through the study of the action of $\Gamma$ on $H^1(C, \mathbb{C})$, we fix an appropriate basis of $H^1(C, \mathbb{C})$ so that we can nicely describe first $\Gamma \subset Sp(\Lambda, Q)$, and next the spaces $Z_p(\Gamma)$, and $[Z_p(\Gamma), Z_p(\Gamma)]$. By Corollary 2.5, these spaces give the Lie algebra decomposition associated with the symmetric space $\mathbb{G}$ that uniformizes the family.

*Notation:* We will denote by $M(n, m, \mathbb{C})$ the space of $n \times m$ complex matrices.

### 3.1 Galois coverings of the line. Cyclic case

We start with the case of a cyclic group $G$. Fix a family of covers, and let $a_i$ be an element of order $m_i$ in $G$ that represents the local monodromy of the covering $\pi : C \to \mathbb{P}^1$ at the
Lemma 3.1 Let \( \mathbb{Z}/m = \langle \zeta \rangle \) be the cyclic group of order \( m \). Consider the \( \mathbb{Z}/m \)-cover \( \pi : C_t \to \mathbb{P}^1 \), with \( t = (t_1, \ldots, t_N) \) branch points in \( \mathbb{P}^1 \) with local monodromy \( a_i \) about \( t_i \). For \( n \in \mathbb{Z}/m \), we write \( H^0(C_t, K_{C_t})(n) := \{ \omega \in H^0(C_t, K_{C_t}) : \zeta^n \omega = \xi \omega \} \). Then

\[
\dim H^0(C_t, K_{C_t})(n) = -1 + \sum_{i=1}^N \frac{na_i}{m},
\]

where \( [a]_m \) denotes the unique representative of \( a \in \mathbb{Z}/m \) in \( \{0, \ldots, m-1\} \).

Let \( \langle , \rangle \) be the Hodge Hermitian product on \( H^0(C, K_C) \), defined by \( \langle \omega, \omega' \rangle := i \int_C \omega \wedge \overline{\omega'} \). Fixing a unitary basis of \( H^0(C, K_C)(0) \) for each \( n \in \mathbb{Z}/m \), one obtains a unitary basis \( \{\omega_1, \ldots, \omega_g\} \) of \( H^0(C, K_C) \) with respect to which the matrix \( A_0 \) that represents the generator \( \rho(\zeta) \) of \( \Gamma \) is diagonal. The multiplicity of each \( m \)-th root of unity on the diagonal is given by Lemma 3.1. Consider now the action \( \tilde{\rho} : G \to GL(H^1(C, \mathbb{C})) \) of \( G \) on the whole \( H^1(C, \mathbb{C}) \) by pullback. Notice that if \( \omega \in H^0(C, K_C) \), then \( \tilde{\rho}(y)(\omega) = \tilde{\rho}(y)(\omega) \). Thus, with respect to the basis \( \{\omega_1, \ldots, \omega_g, \overline{\omega_1}, \ldots, \overline{\omega_g}\} \) of \( H^1(C, \mathbb{C}) \), \( \tilde{\rho}(\zeta) \) is represented by the matrix

\[
A = \begin{pmatrix} A_0 & 0 \\ 0 & A_0 \end{pmatrix}.
\] (3.1)

Lemma 3.2 Let \( \mathfrak{g}' = \mathfrak{p}' \oplus \mathfrak{t}' \) be the Cartan decomposition of the uniformizing symmetric space \( \mathfrak{g}^\Gamma \) associated to a family of cyclic covering of \( \mathbb{P}^1 \). Then, with the notation above, \( \mathfrak{t}' = [\mathfrak{p}', \mathfrak{p}'] \) and

\[
\mathfrak{p}' = \mathfrak{z}_p(A) = \{ \begin{pmatrix} D & 0 \\ 0 & D \end{pmatrix} : D \in M(2, 2, \mathbb{C}) \}, \quad D = D', \quad DA_0 = \overline{A_0D}.
\]

Proof Clearly, \( \mathfrak{g}^\Gamma = \mathfrak{g}^\Lambda = \{ J \in \mathfrak{g} : JA = AJ \} \). For \( J \in \mathfrak{g}^\Gamma \), denote by \( \mathfrak{g} = \mathfrak{t} \oplus \mathfrak{p} \) the Cartan decomposition associated to \( \mathfrak{g} \). Fix the unitary basis \( \{\omega_1, \ldots, \omega_g, \overline{\omega_1}, \ldots, \overline{\omega_g}\} \) of \( H^1(C, \mathbb{C}) \) defined above. Observe that, since \( \langle \omega, \omega' \rangle = iQ(\omega, \omega') \), the matrix representing \( Q \) in this basis is

\[
Q = \begin{pmatrix} 0 & il_g \\ -il_g & 0 \end{pmatrix}.
\]

An element \( U \in \mathfrak{g}(H^1(X, \mathbb{C})) \) belongs to \( \mathfrak{g} \simeq \mathfrak{sp}(2g, \mathbb{R}) \) if and only if it is real, and thus is a block matrix of the form \( \begin{pmatrix} C & \overline{D} \\ D & \overline{C} \end{pmatrix} \), where \( C, D \in M(g, g, \mathbb{C}) \), and satisfies

\[
0 = U'Q + QU = i \begin{pmatrix} D - D' & C' + \overline{C} \\ -C' & D \end{pmatrix}.
\]

Thus \( D = D' \) and \( C \in \mathfrak{u}(g) \). One immediately gets the Cartan decomposition

\[
U = \begin{pmatrix} C & 0 \\ 0 & \overline{C} \end{pmatrix} + \begin{pmatrix} 0 & D \\ D & 0 \end{pmatrix}, \quad \text{with} \quad \begin{pmatrix} C & 0 \\ 0 & \overline{C} \end{pmatrix} \in \mathfrak{t}, \quad \begin{pmatrix} 0 & D \\ D & 0 \end{pmatrix} \in \mathfrak{p}.
\] (3.2)
Now $U \in \mathcal{Z}_g(A)$ if and only if $UA = AU$, i.e.,

$$
\begin{pmatrix}
C & D \\
D & C
\end{pmatrix}
\begin{pmatrix}
A_0 & 0 \\
0 & \overline{A}_0
\end{pmatrix}
= 
\begin{pmatrix}
A_0 & 0 \\
0 & \overline{A}_0
\end{pmatrix}
\begin{pmatrix}
C & D \\
D & C
\end{pmatrix}.
$$

In other words, $U$ must preserve the eigenspaces of $A$. This concludes the study of $\mathcal{S}^I$. Indeed, if $\mathcal{S}^I$ has Cartan decomposition $\mathfrak{g}' = \mathfrak{p}' \oplus \mathfrak{t}'$, it follows from Corollary 2.5, that $\mathfrak{t}' = \{\mathfrak{p}', \mathfrak{p}'\}$ and

$$
\mathfrak{p}' = \mathcal{Z}_p(A) = \mathcal{Z}_g(A) \cap \mathfrak{p} = \left\{ \begin{pmatrix} 0 & \overline{D} \\ D & 0 \end{pmatrix}, \ D = D', \ DA_0 = \overline{A}_0 D \right\}.
$$

With this Lemma, we can determine the uniformizing symmetric space for all the cyclic cases. We will follow the same outline of the argument above. In a few words, the procedure is the following: first, consider the decomposition (3.2) and see which conditions $C$ and $D$ must satisfy to preserve the eigenspaces of the generator $A$. This gives $\mathfrak{p}'$ as in Lemma 3.2. Next calculate $\mathfrak{t}' = \{\mathfrak{p}', \mathfrak{p}'\}$. Finally, identify $\mathcal{S}^I$ in terms of the classification of Hermitian symmetric spaces of the non-compact type (see Section 2.4). Note that $\mathcal{S}^I$ may be a reducible symmetric space. For the reader’s convenience, we will carry out all the explicit calculations for the first family we present. The obtained results are summarized in Theorem 1.2.

Family (8) This is the family of cyclic covers $\pi : C \rightarrow \mathbb{P}^1$ of $\mathbb{P}^1$ with group $G = \mathbb{Z}/4$, ramification data $m = (2^3, 4^2)$, $g = 3$ and dimension 2. In this case $a = (2^3, 1^2)$. Consider the generator $\zeta = e^{2\pi i/4}$ of $G = \langle \zeta \rangle$. Applying Lemma 3.1, we get $\dim H^0(C, K_C)_{(1)} = 1$, $\dim H^0(C, K_C)_{(2)} = 0$, $\dim H^0(C, K_C)_{(3)} = 2$, $\dim H^0(C, K_C)_{(4)} = 0$; that is, the eigenspace of $\zeta^3$ has dimension 2, and the eigenspace of $\zeta$ has dimension 1. Now fix $\{\omega_1, \omega_2\}$ an $\langle \ , \ \rangle$-unitary basis of $H^0(C, K_C)_{(3)}$ and $\{\omega_3\}$ a generator of $H^0(C, K_C)_{(1)}$ with $\langle \omega_1, \omega_2 \rangle = 1$. Since $\omega_3$ and $\omega_1, \omega_2$ are eigenvectors of $\rho(\zeta)$ with respect to different eigenvalues, $\{\omega_1, \omega_2, \omega_3\}$ is a $\langle \ , \ \rangle$-unitary basis of $H^0(C, K_C)$. With respect to this basis $\rho(\zeta)$ is represented by the matrix

$$
A_0 = \begin{pmatrix}
\zeta^3 & 0 & 0 \\
0 & \zeta^3 & 0 \\
0 & 0 & \zeta
\end{pmatrix}.
$$

As above, consider now the action $\tilde{\rho} : G \rightarrow GL(H^1(C, \mathbb{C}))$ of $G$ on the whole $H^1(C, \mathbb{C})$. With respect to the basis $\beta = \{\omega_1, \omega_2, \omega_3, \omega_1, \omega_2, \omega_3\}$ of $H^1(C, \mathbb{C})$, $\tilde{\rho}(\zeta)$ is represented by the matrix

$$
A = \begin{pmatrix}
A_0 & 0 \\
0 & \overline{A}_0
\end{pmatrix}.
$$

Let $I^-$ denote the image of $G$ inside $Sp(H^1(C, \mathbb{R}), Q) \subset GL(H^1(C, \mathbb{C}))$. $I^-$ is generated by the matrix $A$. Thus the symmetric space uniformizing the image in $A_g$ of the family is $\mathcal{S}^A = \{J \in \mathcal{S}^I : JA = AJ\}$. For $J \in \mathcal{S}^I$, denote by $\mathfrak{q} = \mathfrak{sp}(6, \mathbb{R})$, and $\mathfrak{q} = \mathfrak{f} \oplus \mathfrak{p}$ the Cartan decomposition associated to $\mathfrak{g}$. Recall that $\mathfrak{p} = \mathfrak{g} \cap \{X \in \mathfrak{g} : X = X^t\}$ and $\mathfrak{f} = \mathfrak{u}(3)$. An element $U \in \mathfrak{gl}(H^1(C, \mathbb{C}))$ that belongs to $\mathfrak{g}$ is of the form (3.2):
\[ U = \begin{pmatrix} C & 0 \\ 0 & C \end{pmatrix} + \begin{pmatrix} 0 & \bar{D} \\ D & 0 \end{pmatrix}, \quad \text{with} \ \begin{pmatrix} C & 0 \\ 0 & C \end{pmatrix} \in \mathfrak{f}, \quad \begin{pmatrix} 0 & \bar{D} \\ D & 0 \end{pmatrix} \in \mathfrak{p}. \]

Now \( U \in \mathcal{Z}_g(A) \) if and only if \( UA = AU \), i.e.,

\[ \begin{pmatrix} C & 0 \\ 0 & C \end{pmatrix} \begin{pmatrix} A_d & 0 \\ 0 & A_d \end{pmatrix} = \begin{pmatrix} A_d & 0 \\ 0 & A_d \end{pmatrix} \begin{pmatrix} C & 0 \\ 0 & C \end{pmatrix}. \]

Thus, in this case, \( U \in \mathcal{Z}_g(A) \) satisfies

\[ C = \begin{pmatrix} E & 0 \\ 0 & i\lambda \end{pmatrix}, \quad E \in \mathfrak{u}(2), \quad \lambda \in \mathbb{R}, \quad D = \begin{pmatrix} d & 0 \\ d' & 0 \end{pmatrix}, \quad d \in M(2, 1, \mathbb{C}). \]

Denoted by \( \mathfrak{g}' = \mathfrak{p}' \oplus \mathfrak{t}' \) the Cartan decomposition of \( \mathfrak{g}^\mathbb{C} \), it follows from Lemma 3.2 that

\[ \mathfrak{p}' = \mathfrak{x}_p(A) = \left\{ \begin{pmatrix} \bar{D} & 0 \\ D & 0 \end{pmatrix}, D = \begin{pmatrix} d & 0 \\ d' & 0 \end{pmatrix}, d \in M(2, 1, \mathbb{C}) \right\}. \]

Moreover, \( \mathfrak{t}' = [\mathfrak{p}', \mathfrak{p}'] \). We now compute \( \mathfrak{t}' \). Let \( X, Y \in \mathfrak{p}' \), \( X = \begin{pmatrix} 0 & \bar{D} \\ D & 0 \end{pmatrix} \) and \( Y = \begin{pmatrix} 0 & E \\ E & 0 \end{pmatrix} \). Then

\[ [X, Y] = XY - YX = D = \begin{pmatrix} \bar{E}D - \bar{D}E & 0 \\ 0 & \bar{E}D - \bar{D}E \end{pmatrix}. \]

If \( D = \begin{pmatrix} d & 0 \\ d' & 0 \end{pmatrix} \), with \( d \in M(2, 1, \mathbb{C}) \) and \( E = \begin{pmatrix} e & 0 \\ 0 & e' \end{pmatrix} \), with \( e \in M(2, 1, \mathbb{C}) \), we have

\[ \bar{E}D - \bar{D}E = \begin{pmatrix} \bar{e}d - \bar{d}e' & 0 \\ 0 & \bar{e}d' - \bar{d}e \end{pmatrix}. \]

Set \( F = \bar{e}d - \bar{d}e' \) and notice that \( \bar{d}e - \bar{e}d' = \text{tr}(F) = 2i\text{Im}(\langle e, d \rangle) \), where \( \langle \cdot, \cdot \rangle \) denotes the standard hermitian product in \( \mathbb{C}^2 \). We conclude

\[ \mathfrak{t}' = [\mathfrak{p}', \mathfrak{p}'] \subset \left\{ \begin{pmatrix} C & 0 \\ 0 & C \end{pmatrix}, C = \begin{pmatrix} F & 0 \\ 0 & \text{tr}(F) \end{pmatrix}, F \in \mathfrak{u}(2) \right\}. \]

The choices \( \{ d = (-i/2, 0)' , \quad e = (1, 0)' \}, \quad \{ d = (0, -i/2)' , \quad e = (0, 1)' \}, \quad \{ d = (0, 1)' , \quad e = (-1, 0)' \}, \quad \{ d = (0, 1)' , \quad e = (i, 0)' \} \) for \( D \) and \( E \), give as \( F \) the following matrices:

\[ \begin{pmatrix} i & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & i \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \]

which constitute a basis for \( \mathfrak{u}(2) \). Thus the equality sign holds:

\[ \mathfrak{t}' = [\mathfrak{p}', \mathfrak{p}'] = \left\{ \begin{pmatrix} C & 0 \\ 0 & C \end{pmatrix}, C = \begin{pmatrix} F & 0 \\ 0 & \text{tr}(F) \end{pmatrix}, F \in \mathfrak{u}(2) \right\}. \]

Consider now the adjoint representation \( \text{ad} : \mathfrak{t}' \to \mathfrak{g}(\mathfrak{p}') \) of \( \mathfrak{t}' \) on \( \mathfrak{p}' \). If \( F \in \mathfrak{t}' = \mathfrak{u}(2) \), and \( d \in \mathbb{C}^2 = \mathfrak{p}' \), then \( \text{ad}_F(d) = \bar{E}d - \text{tr}(E)d. \) One easily checks it is an irreducible representation and this proves that \( \mathfrak{g}^\mathbb{C} \) is an irreducible hermitian symmetric space of complex dimension 2. We conclude that \( \mathfrak{g}^\mathbb{C} \) is of type \( A \text{ III} (p = 1, q = 2) \).
By the same procedure one can treat all the other cyclic cases. We list here the results of all the computations.

**Family (10)** $A_0 = \text{diag}(\zeta^2, \zeta^2, \zeta^2, \zeta)$, with $\zeta = e^{2\pi i/3}$.

$p' = \mathfrak{z}_p(A) = \{ \begin{pmatrix} 0 & D \\ D & 0 \end{pmatrix}, D = \begin{pmatrix} 0 & d \\ d' & 0 \end{pmatrix}, d \in M(3,1,\mathbb{C}) \}.$

Let $X, Y \in p'$, $X = \begin{pmatrix} 0 & \bar{D} \\ D & 0 \end{pmatrix}$ and $Y = \begin{pmatrix} 0 & \bar{E} \\ E & 0 \end{pmatrix}$. \then

\[ [X, Y] = XY - YX = \begin{pmatrix} \bar{DE} - \bar{ED} & 0 \\ 0 & \bar{DE} - E\bar{D} \end{pmatrix}. \]

If $D = \begin{pmatrix} 0 & d \\ d' & 0 \end{pmatrix}$, with $d \in M(3,1,\mathbb{C})$ and $E = \begin{pmatrix} 0 & e \\ e' & 0 \end{pmatrix}$, with $e \in M(3,1,\mathbb{C})$, we have

\[ \bar{DE} - E\bar{D} = \begin{pmatrix} \bar{d}e' - \bar{e}d' & 0 \\ 0 & \bar{d}e' - \bar{e}d' \end{pmatrix}. \]

Set $F = \bar{d}e' - \bar{e}d'$ and notice that $\bar{d}e' - \bar{e}d = \text{tr}(F) = 2i\text{Im}(e,d)$, where $(\ , \ )$ denotes the standard hermitian product in $\mathbb{C}^3$. Thus, similarly to the previous family, we get

\[ t' = [p', p'] = \{ \begin{pmatrix} C & 0 \\ 0 & \bar{C} \end{pmatrix}, C = \begin{pmatrix} F & 0 \\ 0 & \text{tr}(F) \end{pmatrix}, F \in \mathfrak{u}(3) \}. \]

In particular $\dim t' = 9$. Looking at the adjoint representation $\text{ad} : t' \to \mathfrak{gl}(p')$, one checks that $\mathfrak{g}^A$ is irreducible. Thus it is an irreducible hermitian symmetric space of dimension 3. Looking at $\dim t'$ and comparing it with the possibilities in Section 2.4, we conclude that $\mathfrak{g}^\Gamma$ is of type A III $(p = 1, q = 3)$.

**Family (2)** Here $A_0 = -I_2$. Thus $\mathfrak{g}^\Gamma = \mathfrak{g}^A = \mathfrak{g} = \mathfrak{g}_2$. In fact here $Z = \overline{M_2} = A_2$.

**Family (6)** $A_0 = \text{diag}(\zeta^2, \zeta^2, \zeta)$, with $\zeta = e^{2\pi i/3}$.

$p' = \{ \begin{pmatrix} 0 & D \\ D & 0 \end{pmatrix}, D = \begin{pmatrix} 0 & d \\ d' & 0 \end{pmatrix}, d \in M(2,1,\mathbb{C}) \}.$

$t' = [p', p'] = \{ \begin{pmatrix} C & 0 \\ 0 & \bar{C} \end{pmatrix}, C = \begin{pmatrix} F & 0 \\ 0 & \text{tr}(F) \end{pmatrix}, F \in \mathfrak{u}(2) \}.$

$\mathfrak{g}^\Gamma$ is of type A III $(p = 1, q = 2)$.

**Family (14)** $A_0 = \text{diag}(\zeta^5, \zeta^5, \zeta^2, \zeta)$, with $\zeta = e^{2\pi i/6}$.

$p' = \{ \begin{pmatrix} 0 & D \\ D & 0 \end{pmatrix}, D = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, D \in M(4,\mathbb{C}), d \in M(2,1,\mathbb{C}) \}.$

$t' = [p', p'] = \{ \begin{pmatrix} C & 0 \\ 0 & \bar{C} \end{pmatrix}, C = \begin{pmatrix} F & 0 \\ 0 & 0 \end{pmatrix}, 0 & \text{tr}(F) \end{pmatrix}, F \in \mathfrak{u}(2) \}.$

$\mathfrak{g}^\Gamma$ is of type A III $(p = 1, q = 2)$.

**Family (16)** $A_0 = \text{diag}(\zeta^4, \zeta^4, \zeta^3, \zeta^2)$, with $\zeta = e^{2\pi i/5}$.

$p' = \{ \begin{pmatrix} 0 & D \\ D & 0 \end{pmatrix}, D = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, D \in M(6,\mathbb{C}), d \in M(2,1,\mathbb{C}) \}.$
\[ t' = [p', p'] = \left\{ \begin{pmatrix} C & 0 \\ 0 & C \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & F & 0 \\ 0 & 0 & \text{tr}(F) \end{pmatrix}, F \in \mathfrak{u}(2) \right\}. \]

\( \mathcal{S}^\Gamma \) is of type A III \((p = 1, q = 2)\).

### 3.2 3.4 Galois coverings of the line. Non-cyclic case

Now we turn to the non-cyclic cases. In this section, we will deal with just one case, that is family (27). Indeed, we postpone the analysis of the uniformizing symmetric space \( \mathcal{S}^\Gamma \) of the other families of non-cyclic coverings of the line, namely of families (26, 31, 32), to the next section, where we present the study of these varieties as families of Galois covers of elliptic curves. In the case the group \( G \) is not cyclic, we cannot use Lemma 3.1 to study the action \( \rho \) of \( G \) on \( H^0(C, K_C) \). Instead, we will use the general Chevalley-Weil formula to get the multiplicity of a given irreducible representation of \( G \) in \( H^0(C, K_C) \) (see e. g. \([14, \text{Theorem 1.3.3}]\)). Another difference is that, of course, we will deal with more than one generator of \( \Gamma \).

**Family (27)** This is the family of covers of \( \mathbb{P}^1 \) with group \( G = \mathbb{Z}/2 \times \mathbb{Z}/2 \), ramification data \( m = (2^6), g = 3 \) and dimension 3. Label the characters of \( \mathbb{Z}/2 \times \mathbb{Z}/2 \) as follows:

\[
(0, 0) \quad (0, 1) \quad (1, 0) \quad (1, 1)
\]

| \( x_1 \) | \( 1 \) | \( 1 \) | \( 1 \) | \( 1 \) |
| \( x_2 \) | \( 1 \) | \( -1 \) | \( 1 \) | \( -1 \) |
| \( x_3 \) | \( 1 \) | \( 1 \) | \( -1 \) | \( -1 \) |
| \( x_4 \) | \( 1 \) | \( -1 \) | \( -1 \) | \( 1 \) |

From the Chevalley-Weil formula follows that the character \( \chi_\rho \) of the action \( \rho \) of \( G \) on \( H^0(C, K_C) \) is given by \( \chi_\rho = \chi_2 + \chi_3 + \chi_4 \). Thus there exists \( V_i \subset H^0(C, K_C), i = 2, 3, 4 \), \( \dim V_i = 1 \), such that \( H^0(C, K_C) = V_2 \oplus V_3 \oplus V_4 \) and

\[
\rho : G \to GL(V_2 \oplus V_3 \oplus V_4), \quad \rho(g) = \begin{pmatrix} \rho_2(g) & 0 & 0 \\ 0 & \rho_3(g) & 0 \\ 0 & 0 & \rho_4(g) \end{pmatrix},
\]

where \( \rho_i \) denotes the irreducible representation of \( G \) with character \( \chi_i \). The choice of norm one vectors that span \( V_i, i = 2, 3, 4 \), gives a unitary basis \( \{\omega_1, \omega_2, \omega_3\} \) of \( H^0(C, K_C) \), with respect to which

\[
\rho(0, 1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} =: A_0, \quad \rho(1, 0) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} =: B_0.
\]

Consider now the action \( \tilde{\rho} : G \to GL(H^1(C, \mathbb{C})) \) of \( G \) on \( H^1(C, \mathbb{C}) \), and let \( A, B \) denote the matrices that represent \( \tilde{\rho}(1, 0) \) and \( \tilde{\rho}(0, 1) \) with respect to the basis of \( H^1(C, \mathbb{C}) \) induced by \( \{\omega_1, \omega_2\} \):
The matrices $A$ and $B$ are the generators of the injective image $\Gamma$ of $H$ in $Sp(H^1(C, \mathbb{Z}), Q)$. Thus $\mathfrak{S}^\Gamma = \mathfrak{S}^{A,B} = \{ J \in \mathfrak{S} : JA = AJ, JB = BJ \}$. With the notation (3.2), $U / u_{1D524} u_{1D529} \rightarrow , t$ be one of 6 families of Galois covers of elliptic curve found in [8].

Thus, from Lemma 2.2, the Cartan decomposition associated to $\mathfrak{S}^\Gamma$ is $\mathfrak{g}' = \mathfrak{p}' \oplus \mathfrak{k}'$, with $\mathfrak{p}' = \mathfrak{z}_8(A) = \left\{ \left( \begin{array}{ll} 0 & D \\ D & 0 \end{array} \right) : D = \text{diag} (d_1, d_2, d_3), d_i \in \mathbb{C} \right\}$, and $\mathfrak{k}' = [\mathfrak{p}', \mathfrak{p}']$. To calculate $\mathfrak{k}'$, consider $X, Y \in \mathfrak{p}'$. If $X = \left( \begin{array}{ll} 0 & D \\ D & 0 \end{array} \right)$ and $Y = \left( \begin{array}{ll} 0 & E \\ E & 0 \end{array} \right)$, with $D = \left( \begin{array}{ll} d_1 & 0 \\ 0 & d_2 \end{array} \right)$, and $E = \left( \begin{array}{ll} e_1 & 0 \\ 0 & e_2 \end{array} \right)$, we have $[X, Y] = \left( \begin{array}{ll} D(\overline{E} - E\overline{D}) & 0 \\ 0 & D\overline{E} - E\overline{D} \end{array} \right)$, where $D\overline{E} - E\overline{D} = \text{diag} (d_1, d_2, d_3)$. Thus $\mathfrak{k}' = [\mathfrak{p}', \mathfrak{p}'] = \mathfrak{z}_8(A) \cap \mathfrak{k} = \left\{ \left( \begin{array}{ll} C & 0 \\ 0 & C \end{array} \right) : C = \text{diag} (i\lambda_1, i\lambda_2, i\lambda_3), \lambda_i \in \mathbb{R} \right\}$. In this case, the adjoint representation $ad : \mathfrak{k}' \rightarrow \mathfrak{gl}(\mathfrak{p}')$ is not irreducible. Indeed, let $C = \text{diag} (i\lambda_1, i\lambda_2) \in \mathfrak{k}'$ and $D = \text{diag} (d_1, d_2) \in \mathfrak{p}'$. We have $C(\overline{D} - DC) = \left( \begin{array}{ll} 0 & C\overline{D} - DC \\ C\overline{D} - DC & 0 \end{array} \right)$. Thus $ad_C(D) = \overline{C}D - DC = \text{diag} (-2i\lambda_1 d_1, -2i\lambda_2 d_2, -2i\lambda_3 d_3)$ and hence $W_1 = \text{span} (\text{diag} (1, 0, 0)), W_2 = \text{span} (\text{diag} (0, 1, 0)), W_3 = \text{span} ((0, 0, 1))$ are invariant subspaces of $\mathfrak{p}'$. Hence $\mathfrak{S}^\Gamma$ is of type $A \, \text{III}(1,1) \times A \, \text{III}(1,1) \times A \, \text{III}(1,1)$.

3.3 3.5 Galois coverings of elliptic curves

Let $f_t : C_t \rightarrow C'_t, t \in B$ be one of 6 families of Galois covers of elliptic curve found in [8]. Associated with this family there are two maps: the first is the generalized Prym map $P : B \rightarrow A^8_{k-1}, [C_t \rightarrow C'_t] \mapsto \text{Prym}(C_t, C'_t)$

and the second is the map $\varphi : B \rightarrow A_1, [C_t \rightarrow C'_t] \mapsto [JC'_t]$. 
It is proved in [11] that the connected components of the fibers of both maps have a totally geodesic image in $A_g$. Therefore, in an orbifold sense, the image of the family in $A_g$ admitts two different fibrations in totally geodesic subvarieties. Moreover, it is proved there that countably many of these totally geodesic fibers are Shimura. Linked to the study of these maps is the decomposition, up to isogeny, of the Jacobian $JC$ of $C$, as $JC \sim JC' \times Prym(C, C')$. The aim of this section is to study this decomposition at the level of the Siegel space, relating it to the study of the uniformizing symmetric space of the examples. Consider the natural projection map $\pi : \mathfrak{G} \to A_g$ and let

$$B \to M_g \to A_g$$

be, respectively, the natural map associated to the family, and the Torelli map. Recall that $h$ is generically finite. With this notation Theorem 1.4 reads as follows.

**Theorem 3.3** Let $\mathfrak{G}^\Gamma$ be the uniformizing symmetric space associated to one of the families (1e), (2e) (3e), (4e), (6e). Then

(i) $\mathfrak{G}^\Gamma$ decomposes as $B_1(\mathbb{C}) \times M$, where $M$ is an hermitian symmetric space of codimension 1.

(ii) $\pi(M) = (j \circ h)(F)$, where $F$ is an irreducible component of the fiber of $\varphi$. In particular, $M$ uniformizes $(j \circ h)(F)$.

(iii) $\pi(B_1(\mathbb{C})) = (j \circ h)(F)$, where $F$ is an irreducible component of the fiber of the Prym map.

**Remark 3.4** It is known that the fibers of the Prym map are not irreducible for the family (1e) and irreducible for the family (2e) (see [12] and references therein). In particular, the statement of Theorem 3.3 can be made more precise as follows: the fibers of the Prym map of the family (2e) are irreducible and $\pi(B_1(\mathbb{C})) = (j \circ h)(F)$, where $F$ is the fiber of the Prym map. The fibers of the Prym map of family (1e) are not irreducible and $\pi(B_1(\mathbb{C})) = (j \circ h)(F)$, where $F$ is an irreducible component of the non-irreducible fiber of $P$.

Consider as usual the action of $G$ on the space of holomorphic one–forms and let $H^0(C, K_C) = \bigoplus_\chi V_\chi$ be the associated decomposition in isotypic components. The isotypic component $V_\chi$ is the direct sum of all the copies of the irreducible representation associated with $\chi$ that appear in $H^0(C, K_C)$. If $\chi_0$ is the character of the trivial representation, then $V_{\chi_0} = H^0(C, K_C)^G$. Denote by $V_-$ the direct sum of the other isotypic components. We get $H^0(C, K_C) = H^0(C, K_C)^G \oplus V_-$. We point out that $H^0(C, K_C)^G$ and $V_-$ are the subspaces of $H^0(C, K_C)$ corresponding, respectively, to the construction of $JC'$ and of $Prym(C, C')$. In fact, since the pullback map $f^* : H^0(C', K_{C'}) \to H^0(C, K_C)$ is injective, $H^0(C', K_{C'}) \cong f^*(H^0(C', K_{C'})) = H^0(C, K_C)^G$. Note that this implies $\dim H^0(C, K_C)^G = 1$. Moreover recall that, for a certain sublattice $\Lambda \subset H_1(C, \mathbb{Z})$, we have $Prym(C, C') = (V_-)^* / \Lambda$.

For $J \in \mathfrak{G}^\Gamma$ denote as usual by $g = \mathfrak{sp}(2g, \mathbb{R})$, and $g = \mathfrak{f} \oplus \mathfrak{p}$ the Cartan decomposition associated to $\mathfrak{G}$, we are again in the situation of Lemma 2.2. As usual, we are interested in $T_J \mathfrak{G}^\Gamma \subset T_J \mathfrak{G} = \mathfrak{p}$. The proof of Theorem 3.3 is based on the following two Propositions.
Proposition 3.5 Let $\mathfrak{g}' = \mathfrak{p}' \oplus \mathfrak{f}'$ be the Cartan decomposition of the uniformizing symmetric space $\mathfrak{S}^\Gamma$ of one of the families $(1e), (2e), (3e), (4e), (6e)$. Then

$$\mathfrak{p}' = W_1 \oplus W_2, \quad W_1 = S^2(H^0(K_C)^G)^*, \quad W_2 \subset S^2(V_-)^*$$

where $W_i$ is $\text{ad}_{\mathfrak{g}'}$-invariant. In other words, $\mathfrak{S}^\Gamma = B_1(\mathbb{C}) \times M$, where $M$ is a symmetric space with $T_JM = W_2$ and $T_JB_1(\mathbb{C}) = W_1$.

**Proof** From the decomposition $H^0(K_C) = H^0(K_C)^G \oplus V_-$, we get

$$(S^2H^0(K_C))^G = S^2H^0(K_C)^G \oplus (S^2V_-)^G.$$ Recalling that $\mathfrak{p} \simeq S^2H^0(K_C)^*$, we conclude

$$\mathfrak{p}' = W_1 \oplus W_2, \quad W_1 = S^2(H^0(K_C)^G)^*, \quad W_2 = (S^2(V_-)^*)^G \subset S^2(V_-)^*.$$ (3.3)

We want now to show that the subspaces $W_i$ are $\text{ad}_{\mathfrak{g}'}$-invariant and hence that we get a decomposition of $\mathfrak{S}^\Gamma$ as symmetric space. Choose a norm one vector that spans $H^0(K_C)^G$, choose a unitary basis of $V_-$, and consider the induced basis of $H^1(C, \mathbb{C})$. With the choice of this basis, (3.3) implies that, in terms of the usual matrix notation 3.2, an element $D \in \mathfrak{g}' = \mathfrak{Z}_p(\Gamma)$ is a block matrix of the form

$$D = \begin{pmatrix} d_{11} & 0 \\ 0 & F \end{pmatrix}.$$ Since $\mathfrak{f}' = [\mathfrak{p}', \mathfrak{p}']$, the elements in $\mathfrak{f}'$ also inherit this property. Hence

$$W_1 = \left\{ \begin{pmatrix} d_{11} & 0 \\ 0 & 0 \end{pmatrix}, \quad d_{11} \in \mathbb{C} \right\} \quad \text{and} \quad W_2 = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & F \end{pmatrix} \in \mathfrak{g}' \right\}$$

are $\text{ad}_{\mathfrak{g}'}$-invariant subspaces of $\mathfrak{p}'$. That is, the adjoint representation is not irreducible and $\mathfrak{S}^\Gamma = B_1(\mathbb{C}) \times M$, where $M$ is a symmetric space with tangent space $W_2$ and $T_JB_1(\mathbb{C}) = W_1$. \hfill $\square$

The following Lemma is probably well-known but, since we were not able to locate it in the literature, we recall it with its proof.

**Lemma 3.6** Let $M_1, M_2 \subset \mathfrak{S}$ be closed connected complex submanifolds such that $\pi(M_1) \subset \pi(M_2)$. Then there exists $a \in Sp(2g, \mathbb{L})$ such that $a(M_1) \subset M_2$.

**Proof** For $a \in Sp(2g, \mathbb{L})$, let $f_a : \mathfrak{S} \rightarrow \mathfrak{S}$, $f_a(J) = aJ = aJa^{-1}$. This is the usual action of the symplectic group on the Siegel space in the model of (2.1). By assumption

$$M_1 \subset \bigcup_{a \in Sp(2g, \mathbb{L})} f_a^{-1}(M_2).$$

Indeed, for $J \in M_1$ there exists $a \in Sp(2g, \mathbb{L})$ such that $f_a(J_1) = J_2$. Thus

$$M_1 = \bigcup_{a \in Sp(2g, \mathbb{L})} M_1 \cap f_a^{-1}(M_2).$$
Since $M_1 \cap f_a^{-1}(M_2)$ is closed in $M_1$ and $a$ vary in a countable set, Baire theorem [1, p. 57] implies that there exists $a$ and an open subset $U \subset M_1$, such that $U \subset f_a^{-1}(M_2)$, i.e. $f_a(U) \subset M_2$. Therefore, $f_a^{-1}(M_2) \cap M_1$ is an analytic subset of $M_1$, which contains an open subset $U \subset M_1$. By the Identity Lemma [16, p. 167], this implies that $f_a^{-1}(M_2) \cap M_1 = M_1$ and hence that $f_a(M_1) \subset M_2$.

**Proposition 3.7** Let $\mathcal{O}^1 = B_1(\mathbb{C}) \times M$ be the uniformizing symmetric space of one of the families (1e), (2e) (3e), (4e), (6e). Then

(i) $\pi(M) = (\text{jcoh}(F))$, where $F$ is an irreducible component of the fiber of $\varphi$.

(ii) $\pi(B_1(\mathbb{C})) = (\text{jcoh}(F))$, where $F$ is an irreducible component of the fiber of the Prym map.

**Proof**

(i) For $t \in B$ generic, $dh_t(T,B) = H^0(C_t, T_{C_t})^G$, and the codifferential of $\varphi$ at $t$ is given by the composition

$$S^2(H^0(K_C)^G) \xrightarrow{\text{moi}} H^0(2K_C)^G \xrightarrow{dh^*} (T,B)^*,$$

where $i : S^2H^0(K_C)^G \hookrightarrow S^2H^0(K_C)$ is the inclusion and $m$ is the multiplication map. Since $m^* = df$, we get $d\varphi = i^* \circ d\varrho$ and we conclude that

$$d\varphi d(\text{jcoh})^{-1} = i^* : (S^2H^0(K_C)^*)^* \rightarrow S^2(H^0(K_C)^G)^*,$$

Since $W_2 \subset S^2(V_-)^*$, it follows that $W_2 \subset \ker(d\varphi d(\text{jcoh})^{-1}) = d(\text{jcoh})(\ker d\varphi) = d(\text{jcoh})(T,F)$. So $\pi(M) \subset (\text{jcoh}(F))$. By [11, Theorem 3.11], $(\text{jcoh}(F)) \subset A_g$ is totally geodesic of dimension $\dim \mathcal{O}_F - 1$. Denote by $M'$ its uniformizing symmetric space. Applying Lemma 3.6 to $M$ and $M'$, we obtain that $a.M \subset M'$ for some $a \in Sp(2g, \mathbb{Z})$. Since they have the same dimension, the equality sign holds and we get $\pi(M) = \pi(a.M) = \pi(M') = (\text{jcoh}(F))$.

(ii) For $t \in B$ generic the codifferential of the Prym mat at $t$ is given by the composition

$$S^2(V_-) \xrightarrow{pr} (S^2(V_-))^G \xrightarrow{\text{moi}} H^0(2K_C)^G \xrightarrow{dh^*} (T,B)^*$$

where $pr$ denotes the natural projection and now $i : (S^2V_-)^G \hookrightarrow S^2H^0(K_C)$. Thus $dP = pr^* i^* \circ d\varrho dh$ and we conclude that

$$dP d(\text{jcoh})^{-1} = pr^* i^* : (S^2H^0(K_C)^*)^* \rightarrow S^2(V_-)^*.$$

Since $i^* : S^2H^0(K_C)^* \rightarrow (S^2V_-)^G^*$ is the restriction and $W_1 = S^2H^0(K_C)^G$, we get $W_1 \subset \ker(dP d(\text{jcoh})^{-1}) = d(\text{jcoh})(\ker dP) = d(\text{jcoh})(T,F)$. So $\pi(B_1(\mathbb{C})) \subset (\text{jcoh}(F))$. By [11, Theorem 3.9], $(\text{jcoh}(F)) \subset A_g$ is totally geodesic of dimension 1. Applying again Lemma 3.6, the same argument as above implies $\pi(B_1(\mathbb{C})) = (\text{jcoh}(F))$. 

We present in the following the analysis of the various examples, reporting the isomorphisms with the families of Galois covers of the line. The obtained results are summarized...
in Theorem 1.3. For $G = \mathbb{Z}/m$, we will denote by $\chi_j$ the character of the irreducible representation $\rho_j$ of $G$ defined as

$$\rho_j(1)(\omega) = \zeta^j\omega, \quad \forall \omega \in H^0(C, K_C),$$

where $\zeta = e^{2\pi i / n}$.

**Family (1e) = (26)** The Chevalley-Weil formula gives $\chi_p = \chi_0 + \chi_1$. It follows that $H^0(C, K_C)$ decomposes as $H^0(C, K_C) = H^0(C, K_C)^G \oplus V_\omega$, with $H^0(C, K_C)^G = V_0$ and $V_\omega = V_1$. Both $V_0$ and $V_1$ have dimension 1. Thus the action $G$ on $H^0(C, K_C)$ is given by

$$\rho : \mathbb{Z}/2\mathbb{Z} \to GL(V_0 \oplus V_1), \quad \rho(g) = \begin{pmatrix} \rho_0(g) & 0 \\ 0 & \rho_1(g) \end{pmatrix}.$$ 

We get $A_0 = \text{diag}(1, -1)$.

$$\mathfrak{p}' = \mathfrak{z}_p(A) = \begin{pmatrix} 0 & D \\ D & 0 \end{pmatrix}, \quad D = \text{diag}(d_1, d_2), \quad d_i \in \mathbb{C}$$

$$\mathfrak{g}' = [\mathfrak{p}', \mathfrak{g}'] = \mathfrak{z}_g(A) \cap \mathfrak{k} = \begin{pmatrix} C & 0 \\ 0 & \mathbb{C} \end{pmatrix}, \quad C = \text{diag}(i\lambda_1, i\lambda_2), \quad \lambda_i \in \mathbb{R}.$$ 

$W_1 = \text{span}(\text{diag}(1, 0)) = S^2(H^0(C, K_C)^G)^*$, 

$W_2 = \text{span}(\text{diag}(0, 1)) = S^2(V_\omega)^*$.

$\mathfrak{g}'$ decomposes in the product of two irreducible hermitian symmetric spaces of complex dimension 1.

**Family (2e)** The Chevalley-Weil formula implies $\chi = \chi_0 + 2\chi_1$. Thus there exists 1-dimensional subspaces $V_0, V_1, V_1' \subset H^0(C, K_C)$, such that $H^0(C, K_C) = V_0 \oplus V_1 \oplus V_1'$ and

$$\rho : \mathbb{Z}/2\mathbb{Z} \to GL(V_0 \oplus V_1 \oplus V_1'), \quad \rho(g) = \begin{pmatrix} \rho_0(g) & 0 & 0 \\ 0 & \rho_1(g) & 0 \\ 0 & 0 & \rho_1(g) \end{pmatrix}.$$ 

$A_0 = \text{diag}(1, -1, -1)$.

$$\mathfrak{p}' = \mathfrak{z}_p(A) = \begin{pmatrix} 0 & D \\ D & 0 \end{pmatrix}, \quad D = \begin{pmatrix} d & 0 \\ 0 & F \end{pmatrix}, \quad d \in \mathbb{C}, \quad F = F'$$

$$\mathfrak{g}' = [\mathfrak{p}', \mathfrak{g}'] = \mathfrak{z}_g(A) \cap \mathfrak{k} = \begin{pmatrix} C & 0 \\ 0 & \mathbb{C} \end{pmatrix}, \quad C = \begin{pmatrix} i\lambda & 0 \\ 0 & E \end{pmatrix}, \quad \lambda \in \mathbb{R}, \quad E \in \mathfrak{u}(2).$$

For $C = \begin{pmatrix} i\lambda & 0 \\ 0 & E \end{pmatrix} \in \mathfrak{g}'$ and $D = \begin{pmatrix} d & 0 \\ 0 & F \end{pmatrix} \in \mathfrak{p}'$, we have

$$\text{ad}_C(D) = \overline{CD} - DC = \begin{pmatrix} -2\lambda d & 0 \\ 0 & \overline{EF} - FE \end{pmatrix}.$$ 

$W_1 = \text{span}(\text{diag}(1, 0, 0)) = S^2(H^0(C, K_C)^G)^*$, 

$W_2 = \{ \begin{pmatrix} 0 & 0 \\ 0 & F \end{pmatrix} \in \mathfrak{g}' \}, \quad F = F' = S^2(V_\omega)^*$.

$\mathfrak{g}' \approx B_1(\mathbb{C}) \times M$, where $M$ is a 3-dimensional irreducible symmetric space with Cartan decomposition $\mathfrak{g}'' = \mathfrak{t}'' \oplus \mathfrak{p}''$ with $\mathfrak{t}'' = \mathfrak{u}(2)$ and $\mathfrak{p}'' = \{ F \in \mathfrak{gl}(2, \mathbb{C}), \quad F = F' \}$. We conclude that $M$ is of type CI ($n = 2$) and $\mathfrak{g}' \approx A_{III}(1,1) \times \text{CI}(n = 2)$.

**Family (3e) = (31)** The Chevalley-Weil formula gives $\chi = \chi_0 + \chi_1 + \chi_2$. Thus there exists $V_i \subset H^0(C, K_C)$, $i = 0, 1, 2$, dim $V_i = 1$, such that $H^0(C, K_C) = V_0 \oplus V_1 \oplus V_2$ and
\[ \rho : \mathbb{Z}/3\mathbb{Z} \to GL(V_0 \oplus V_1 \oplus V_2), \quad \rho(1) = \begin{pmatrix} \rho_0(1) & 0 & 0 \\ 0 & \rho_1(1) & 0 \\ 0 & 0 & \rho_2(1) \end{pmatrix}. \]

To get the decomposition \( H^0(C, K_C) = H^0(C, K_C)^G \oplus V_- \), recall that \( V_0 = H^0(C, K_C)^G \), and \( V_- = V_1 \oplus V_2 \). Denoted by \( \zeta = e^{2\pi i/3} \), we have \( A_0 = \text{diag}(1, \zeta, \zeta^2) \).

\[ \mathfrak{p}' = \mathfrak{z}_p(A) = \{ \begin{pmatrix} 0 & D \\ D & 0 \end{pmatrix}, \quad D = \begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_i \end{pmatrix}, \quad d_i \in \mathbb{C} \} \]

\[ \mathfrak{t}' = \mathfrak{z}_g(A) \cap \mathfrak{k} = \{ \begin{pmatrix} C & 0 \\ 0 & C \end{pmatrix}, \quad C = \text{diag}(i\lambda_1, i\lambda_2, i\lambda_3), \quad \lambda_i \in \mathbb{R} \}, \quad \lambda_i \in \mathbb{R} \}. \]

In this case \( \text{ad}_C(D) = \overline{CD} - DC = \begin{pmatrix} -2i\lambda_1d_1 & 0 & 0 \\ 0 & 0 & -2i\lambda_2d_2 \\ 0 & 0 & 0 \end{pmatrix} \).

\( W_1 = \text{span}(\text{diag}(1, 0, 0)) = S^2(H^0(C, K_C)^G)^* \),

\( W_2 = \text{span}\left( \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \right) \subseteq S^2(V_-)^* \)

\( \mathfrak{B}^\Gamma \) is product of two irreducible hermitian symmetric spaces of complex dimension 1.

**Family (4e)=(32)** The Chevalley-Weil formula gives \( \chi_\rho = \chi_0 + \chi_1 + \chi_3 \). Thus there exists \( V_i \subset H^0(C, K_C), \ i = 0, 1, 3, \ \text{dim} V_i = 1 \), such that \( H^0(C, K_C) = V_0 \oplus V_1 \oplus V_3 \) and

\[ \rho : \mathbb{Z}/4\mathbb{Z} \to GL(V_0 \oplus V_1 \oplus V_3), \quad \rho(1) = \begin{pmatrix} \rho_0(1) & 0 & 0 \\ 0 & \rho_1(1) & 0 \\ 0 & 0 & \rho_3(1) \end{pmatrix}. \]

\( \mathfrak{A}_0 = \text{diag}(1, i, -i) \).

\[ \mathfrak{p}' = \mathfrak{z}_p(A) = \{ \begin{pmatrix} 0 & D \\ D & 0 \end{pmatrix}, \quad D = \begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_i \end{pmatrix}, \quad d_i \in \mathbb{C} \} \]

\[ \mathfrak{t}' = [\mathfrak{p}', \mathfrak{p}'] = \{ \begin{pmatrix} C & 0 \\ 0 & C \end{pmatrix}, \quad C = \text{diag}(i\lambda_1, i\lambda_2, i\lambda_3), \quad \lambda_i \in \mathbb{R} \}, \quad \lambda_i \in \mathbb{R} \}. \]

The adjoint representation is given by

\[ \text{ad}_C(D) = \overline{CD} - DC = \begin{pmatrix} -2i\lambda_1d_1 & 0 & 0 \\ 0 & 0 & -2i\lambda_2d_2 \\ 0 & 0 & 0 \end{pmatrix}. \]

\( W_1 = \text{span}(\text{diag}(1, 0, 0)) = S^2(H^0(C, K_C)^G)^* \),

\( W_2 = \text{span}\left( \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \right) \subseteq S^2(V_-)^* \).
\( \mathcal{S}^\Gamma \) is product of two irreducible hermitian symmetric spaces of complex dimension 1.

**Family (6e)** This family was originally studied in [29]. The Chevalley-Weil formula gives \( \chi_\rho = \chi_0 + \chi_1 + 2 \chi_2 \).

\[ A_0 = \text{diag}(1, \zeta^2, \zeta^2, \zeta), \text{ with } \zeta = e^{2\pi i/3}. \]

\[ \mathfrak{g}' = \{ \begin{pmatrix} 0 & \overline{D} \\ D & 0 \end{pmatrix}, \quad D = \begin{pmatrix} d_1 & 0 & 0 \\ 0 & 0 & d \\ 0 & d^t & 0 \end{pmatrix}, \quad d_1 \in \mathbb{C}, \quad d \in M(2, 1, \mathbb{C}) \}. \]

\[ \mathfrak{t}' = [\mathfrak{g}', \mathfrak{g}'] = \{ \begin{pmatrix} C & 0 \\ 0 & \overline{C} \end{pmatrix}, \quad C = \begin{pmatrix} i \lambda & 0 & 0 \\ 0 & F & 0 \\ 0 & 0 & \text{tr}(F) \end{pmatrix}, \quad F \in \mathfrak{u}(2) \}. \]

If \( C \in \mathfrak{t}' \) and \( D \in \mathfrak{g}' \), then \( \text{ad}_C(D) = \begin{pmatrix} -2i \lambda d_1 & 0 & 0 \\ 0 & 0 & (\overline{E}d - \text{tr}(E)d)^t \\ 0 & (\overline{E}d - \text{tr}(E)d)^t & 0 \end{pmatrix} \).

\( \mathbb{S}^\Gamma \cong B_1(\mathbb{C}) \times B_2(\mathbb{C}) \).

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**References**

1. Bredon, G.E.: Topology and Geometry. Springer-Verlag, New York (1993)
2. Coleman, R.: Torsion points on curves. In: Ihara, Y. (ed.) Galois Representations and Arithmetic Algebraic Geometry, pp. 235–247. North-Holland, Amsterdam (1987)
3. Colombo, E., Frediani, P., Ghigi, A.: On totally geodesic submanifolds in the Jacobian locus. Int. J. Math. 26(1), 1550005 (2015)
4. Conti D., Ghigi A., Pignatelli R., Some evidence for the Coleman-Oort conjecture, Preprint 2021, ArXiv:2102.12349
5. de Jong J. and Zhang S.-W., Generic abelian varieties with real multiplication are not Jacobians, In: Diophantine geometry, U. Zannier (ed.), volume 4 of CRM Series, pp. 165–172. Ed. Norm., Pisa, 2007
6. Farkas, H.M., Kra, I.: Riemann Surfaces. Springer-Verlag, New York-Heidelberg-Berlin (1980)
7. Frediani, P., Ghigi, A., Penegini, M.: Shimura varieties in the Torelli locus via Galois coverings. Int. Math. Res. Not. IMRN 20, 10595–10623 (2015)
8. Frediani, P., Penegini, M., Porru, P.: Shimura varieties in the Torelli locus via Galois coverings of elliptic curves. Geom. Dedicata 181, 177–192 (2016)
9. Frediani P. and Pirola G.P., On the geometry of the second fundamental form of the Torelli map, Preprint 2019. ArXiv:1907.11407
10. Frediani P. and Porru P., On the bielliptic and bihyperelliptic loci, Preprint 2018. ArXiv: 1807.02073, To appear on Michigan Math. J
11. Frediani P., Ghigi A., Spelta I., Infinitely many Shimura varieties in the Jacobian locus for \( g \leq 4 \), Preprint 2020. ArXiv:1910.13245. To appear in Ann. Scuola Norm. Sup. Pisa Cl. Sci
12. Frediani P., Naranjo J.C., Spelta I., The fibres of the ramified Prym map, Preprint 2020. ArXiv: 2007.02068
13. Ghigi A., Pirola G.P. and Torelli S., Totally geodesic subvarieties in the moduli space of curves, Preprint 2019. ArXiv:1902.06098. To appear on Commun. Contemp. Math
14. Gleissner C., Threefolds Isogenous to a Product and Product quotient Threefolds with Canonical Singularities, Master Thesis, Bayreuth, 2016
15. González Díez G. and Harvey W.J., Moduli of Riemann surfaces with symmetry. In Discrete groups and geometry (Birmingham, 1991), volume 173 of London Math. Soc. Lecture Note Ser., pp. 75–93. Cambridge Univ. Press, Cambridge, 1992
16. Grauert, H., Remmert, R.: Coherent Analytic Sheaves. Springer-Verlag, Berlin (1984)
17. Grushevsky, S., Möller, M.: Shimura curves within the locus of hyperelliptic Jacobians in genus 3. Int. Math. Res. Not. IMRN 6, 1603–1639 (2016)
18. Hain R. Locally symmetric families of curves and Jacobians, In: Moduli of curves and abelian varieties, 91–108. Vieweg, Braunschweig, 1999
19. Helgason, S.: Differential Geometry, Lie Groups, and Symmetric Spaces. Academic Press Inc., New York (1978)
20. Kobayashi, S.: Transformation Groups in Differential Geometry. Springer-Verlag, Berlin Heidelberg (1995)
21. Kobayashi, S., Nomizu, K.: Foundations of Differential Geometry, vol. II. Interscience Publishers, New York-London-Sydney (1969)
22. Liu, K., Sun, X., Yang, X., Yau, S.-T.: Curvatures of moduli spaces of curves and applications. Asian J. Math. 21(5), 841–854 (2017)
23. Lu, X., Zuo, K.: The Oort conjecture for on Shimura curves in the Torelli locus of curves. J. Math. Pures Appl. 123(9), 41–77 (2019)
24. Moonen, B.: Linearity properties of Shimura varieties I. J. Algebraic Geom. 7(3), 539–567 (1998)
25. Moonen, B.: Special subvarieties arising from families of cyclic covers of the projective line. Doc. Math. 15, 793–819 (2010)
26. Moonen, B., Oort, F.: The Torelli locus and special subvarieties. In: Farkas, G., Morrison, I. (eds.) Handbook of Moduli, vol. II, pp. 549–94. International Press, Boston (2013)
27. Mumford, D.: A note of Shimura's paper Discontinuous groups and abelian varieties. Math. Ann. 181, 345–351 (1969)
28. Oort F., Canonical liftings and dense sets of CM-points, In: Arithmetic geometry, Sympos. Math. XXXVII, 228–234. Cambridge Univ. Press, Cambridge, 1997
29. Pirola, G.P.: On a conjecture of Xiao. J. Reine Angew. Math. 431, 75–89 (1992)
30. Ziller W., Lie Groups. Representation Theory and Symmetric Spaces, Notes from courses taught at University of Pennsylvania, Fall 2010, and at IMPA, 2012 https://www2.math.upenn.edu/~wziller/meth650/LieGroupsReps.pdf

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