Towards Baxter equation in supersymmetric Yang-Mills theories

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Abstract

We perform an explicit two-loop calculation of the dilatation operator acting on single trace Wilson operators built from holomorphic scalar fields and an arbitrary number of covariant derivatives in $\mathcal{N} = 2$ and $\mathcal{N} = 4$ supersymmetric Yang-Mills theories. We demonstrate that its eigenspectrum exhibits double degeneracy of opposite parity eigenstates which suggests that the two-loop dilatation operator is integrable. Moreover, the two-loop anomalous dimensions in the two theories differ from each other by an overall normalization factor indicating that the phenomenon is not sensitive to the presence of the conformal symmetry. Relying on these findings, we try to uncover integrable structures behind the two-loop dilatation operator using the method of the Baxter $Q-$operator. We propose a deformed Baxter equation which exactly encodes the spectrum of two-loop anomalous dimensions and argue that it correctly incorporates a peculiar feature of conformal scalar operators – the conformal $SL(2)$ spin of such operators is modified in higher loops by an amount proportional to their anomalous dimension.
1 Introduction

The analysis of one-loop anomalous dimensions of composite high-twist operators in QCD revealed that its dilatation operator possesses nontrivial integrability symmetry. It was observed in Refs. [1, 2, 3] that the one-loop mixing matrix for the so-called maximal-helicity quasipartonic [4] Wilson operators can be mapped in the multi-color limit into the Hamiltonian of the $SL(2;\mathbb{R})$ Heisenberg spin chain and its eigenspectrum can be computed exactly with the help of Bethe Ansatz. The length of the chain is determined by the number of elementary fields in the Wilson operator and the spin operators on its sites are defined by the generators of the collinear subgroup of the conformal group $SO(4, 2)$. Let us emphasize that similar structures were discovered earlier in the Regge limit of QCD [5, 6]. Integrability imposes a very nontrivial analytic structure on the anomalous dimensions, which is reflected, in particular, in pairing of opposite-parity eigenstates in the spectrum.

Integrability observed in QCD anomalous dimensions at one-loop order is a generic phenomenon of four-dimensional Yang-Mills theories and it is ultimately related to the presence of massless spin-one gauge bosons in the particle spectrum. It is also present, as was found in Refs. [7, 8, 9], in maximally supersymmetric Yang-Mills (SYM) theory. Theories with less supersymmetries also inherit integrability although the number of integrable sectors strongly depends on the particle content of the models and is enhanced for theories with more supercharges [10]. In this regard the maximally supersymmetric $\mathcal{N} = 4$ Yang-Mills theory currently occupies a distinguished niche in light of the gauge/string duality [11] which allows one to establish the correspondence between the anomalous dimensions of composite operators in $\mathcal{N} = 4$ theory and energies of string excitations on the $AdS_5 \times S^5$ background [12, 13, 14]. It was recently shown that classical string sigma models with anti-de Sitter space as a factor of the target space possess an infinite set of integrals of motion and therefore are integrable [15, 16, 17]. On gauge theory side, this suggests that the dilatation operator for Wilson operators carrying large quantum numbers should be integrable in the $\mathcal{N} = 4$ theory in the strong coupling regime.

The question arises whether the one-loop integrability of the dilatation operator and integrability of the classical string sigma model is a manifestation of the same universal phenomenon at weak and strong coupling, respectively, and if so then whether the “perturbative” dilatation operator exhibits integrability order-by-order in the coupling constant. Since the range of interaction in the spin chain increases with order in ’t Hooft coupling constant $\lambda = g_{YM}^2 N_c / (8\pi^2)$, – being merely nearest-neighbor at one loop, then stretching to three adjacent neighbors at two loops, etc. – the spectrum of anomalous dimensions should be determined by yet unidentified long-range integrable spin chain. Recent extensive perturbative studies indeed support this conjecture [18, 19, 20, 21, 22, 23, 24, 25, 26].

In this paper, we perform an explicit two-loop calculation of the dilatation operator acting on single-trace Wilson operators built from holomorphic scalar fields and an arbitrary number of covariant derivatives in $\mathcal{N} = 2$ and $\mathcal{N} = 4$ supersymmetric Yang-Mills theories. The motivation behind doing it is two-fold: (i) to establish the anticipated integrability as well as its dependence on the number of supercharges and (ii) to unravel the underlying integrable long-range interaction. In our previous publications [23] we have addressed the same questions in the sector of three-particle gaugino operators with arbitrary number of derivatives in super Yang-Mills theories with $\mathcal{N} = 1, 2, 4$ supercharges. We have computed the spectra of their anomalous dimensions and have found that the energy of the states with zero quasimomentum is double degenerate indicating the existence of higher conserved charges in addition to the quadratic conformal Casimir.
In the present paper we continue the analysis of noncompact sectors closed under renormalization and extend our consideration to single-trace Wilson operators built from $L$ scalar fields $X$ and arbitrary number of covariant derivatives,

$$O_n(0) = \text{tr} \{(iD_+)^{n_1}X(0)(iD_+)^{n_2}X(0)\ldots(iD_+)^{n_L}X(0)\},$$

(1.1)

where $n = (n_1, n_2, \ldots, n_L)$ denotes a set of $L$ nonnegative integers. Here $n^\mu D_\mu = D_+ = \partial_+ - ig[A_+,\cdots]$ is the covariant derivative projected onto the light cone with the help of the light-like vector $n^\mu$. This projection automatically selects the maximal spin component. In order to avoid mixing with operators built from gauginos and gluons, we choose all $X$’s to be the same and to possess the maximal charge with respect to the internal $R$-symmetry group, that is, $X = \phi$ in $\mathcal{N} = 2$ theory and $X = \phi_1 + i\phi_2$, conventionally called $Z$, in $\mathcal{N} = 4$ theory. The scalar operators (1.1) carry the Lorentz spin $N = n_1 + \cdots + n_L$ and the canonical dimension $N + L$ and they can mix under renormalization with operators having the same $N$ and $L$.

The paper is organized as follows. In Section 2 we employ a well-developed QCD technique for perturbative computation of the dilatation operator in the momentum representation. In this representation, the dilatation operator can be realized as an integral operator acting on light-cone momenta of scalar fields. The mixing matrix in the basis of local Wilson operators is simply obtained by forming Mellin moments of its integral kernel and the explicit expressions can be found in Appendix. In Section 3 we discuss the spectrum of anomalous dimensions for scalar operators (1.1) and describe integrable structures behind the two-loop dilatation operator using the method of the Baxter $\bar{Q}$–operator. Section 4 contains concluding remarks.

2 Two-loop noncompact dilatation operator

A concise representation of the entire tower of local Wilson operators is achieved by means of nonlocal operators with elementary fields located at positions $z_i$ on the light-ray, $X(z_in^\mu) \equiv X(z_i)

$$O(z) = \text{tr} \{X(z_1)[z_1, z_2]X(z_2)[z_2, z_3]\ldots X(z_L)[z_L, z_1]\},$$

(2.1)

where $z = (z_1, z_2, \ldots, z_L)$ and the Wilson line $[z_j, z_{j+1}] = ig\int_{z_{j+1}}^{z_j} dz A_+(z)$ is stretched between two fields to make the composite operators gauge invariant. The local Wilson operators (1.1) are deduced from $O(z)$ by means of the Taylor expansion

$$O(z) = \sum_{n_1, n_2, \ldots, n_L \geq 0} \frac{(-iz_1)^{n_1}}{n_1!} \frac{(-iz_2)^{n_2}}{n_2!} \ldots \frac{(-iz_L)^{n_L}}{n_L!} O_n(0).$$

(2.2)

For our purposes yet it will be extremely useful to use a representation of the same operators in the reciprocal momentum space. It is given by the Fourier transform of $O(z)$ with respect to the light-cone coordinates

$$\tilde{O}(u) = \int_{-\infty}^{\infty} \frac{dz_1}{2\pi} e^{iz_1 u_1} \ldots \int_{-\infty}^{\infty} \frac{dz_L}{2\pi} e^{iz_L u_L} O(z),$$

(2.3)

with $u = (u_1, u_2, \ldots, u_L)$ being the vector of the light-cone momenta. These operators obey the renormalization group (Callan-Symanzik) equation

$$\left(\mu \frac{\partial}{\partial \mu} + \beta_N(g) \frac{\partial}{\partial g}\right) \tilde{O}(u) = \int [dv]_L \nabla(u|v) \tilde{O}(v),$$

(2.4)
with the integration measure \( [dv]_L = dv_1 \ldots dv_L \delta(\sum_k v_k - \sum_m u_m) \) and the dilatation operator in the momentum-space representation, \( V(u|v) \), admitting perturbative expansion in 't Hooft coupling \( \lambda = g^2_{YM} N_c/(8\pi^2) \)

\[
V(u|v) = \lambda V^{(0)}(u|v) + \lambda^2 V^{(1)}(u|v) + \mathcal{O}(\lambda^3). \tag{2.5}
\]

A detailed account on the technique used for the perturbative calculation of the kernels \( V^{(0)}(u|v) \) and \( V^{(1)}(u|v) \) can be found in our previous publication [23].

In this paper, we calculate the two-loop kernel \( V(u|v) \) for nonlocal light-cone scalar operators (2.1) in supersymmetric Yang-Mills theories with \( \mathcal{N} = 2 \) and \( \mathcal{N} = 4 \) supercharges. We shall employ the light-cone formalism of Ref. [10] which heavily relies on the light-cone gauge \( A_+(x) = 0 \) and which allows one to treat all supersymmetric theories in a unified fashion. Under this gauge condition, the Wilson lines in (2.1) reduce to unity \([z_k, z_{k+1}] = 1\) and the number of relevant Feynman diagrams decreases significantly. In the light-cone formalism, the Lagrangian of super Yang-Mills theory depends on physical components of elementary fields only since all non-propagating degrees of freedom can be integrated out. This allows one to switch from covariant spinor and vector fields to single-component fermionic and bosonic fields carrying a definite helicity and, then, introduce dimensional regularization inside Feynman integrals by continuing them to \( D = 4 - 2\varepsilon \) dimensions without breaking supersymmetry of the underlying gauge theory. The resulting regularization procedure is equivalent to the dimensional reduction scheme.

### 2.1 Evolution kernel

Let us start with the one-loop kernel \( V^{(0)}(u|v) \). In the multi-color limit, it has a simple nearest neighbor structure

\[
V^{(0)}(u|v) = \sum_{k=1}^L \left\{ V^{(0)}_{k,k+1} - \Gamma^{(0)} \delta(u_k - v_k) \right\} \prod_{j \neq k,k+1} \delta(u_j - v_j), \tag{2.6}
\]

with the periodicity condition \( L + 1 = 1 \) and the total momentum conservation \( \sum_k v_k = \sum_m u_m \) absorbed into the integration measure in (2.4). Here, \( \Gamma^{(0)} = \frac{1}{2}(\mathcal{N} - 4) \) is an additive constant and the two-particle kernel is defined as

\[
V^{(0)}_{k,k+1} = [\Theta(u_k, v_k) f_s(u_k, v_k)]^{(u_k)}_+ + [\Theta(u_{k+1}, v_{k+1}) f_s(u_{k+1}, v_{k+1})]^{(u_{k+1})}_+, \tag{2.7}
\]

where the generalized step-function \( \Theta(u, v) \) specifies possible values of the momentum fractions and a notation was introduced for the decoration factor \( f_s \),

\[
\Theta(u, v) = \theta(u)\theta(u - v) - \theta(-u)\theta(v - u), \quad f_s(u, v) = \frac{1}{v - u}. \tag{2.8}
\]

The plus-distribution in (2.7) regularizes the end-point singularities of \( f_s(u, v) \) as \( u - v \to 0 \) and is conventionally defined as

\[
\left[ \frac{\tau(u)}{v - u} \right]_+^{(u)} = \frac{\tau(u)}{v - u} - \delta(u - v) \int dw \frac{\tau(w)}{v - w}, \tag{2.9}
\]

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The three- and two-particle kernels $\mathcal{V}$ which produce non-vanishing contributions to Figs. 1 and 2, respectively. Finally, $\Gamma^{(1)}$ three-particle step-functions are defined following the conventions of Ref. [23] as permutation of indices under the periodic boundary condition $L$ for $V_{\mathcal{N}}$. We will not dwell on the computation procedure since it was spelled out in details in Ref. [23].

In general, the other kernels $\mathcal{V}_{k+1,k+2}$ can be obtained from this expression through cyclic permutation of indices under the periodic boundary condition $L + k \equiv k$. Here the generalized three-particle step-functions are defined following the conventions of Ref. [23] as

$\Theta^{(1)}_{123} = \Theta(u_1, v_1) [\Theta(u_2, v_1 + v_2 - u_1) - \Theta(v_2 + v_3 - u_3, v_3)]$,

$\Theta^{(4)}_{123} = \Theta(u_2, v_2 + v_3 - u_3) \Theta(v_2 + v_3 - u_3, v_2)$.

For $v_{1,2,3} \geq 0$ and $v_1 + v_2 + v_3 = 1$ they define the regions in $(u_1, u_3) -$plane (with $u_2 = 1 - u_1 - u_3$) which produce non-vanishing contributions to $\mathcal{V}^{(1)}_{123}$:

$\Theta^{(1)}_{123} = \begin{cases} 0 \leq u_1 \leq v_1, \\ v_2 + v_3 \leq u_3 \leq 1 - u_1 \end{cases}$,  

$\Theta^{(4)}_{123} = \begin{cases} v_1 \leq u_1 \leq 1 - u_3, \\ v_3 \leq u_3 \leq 1 \end{cases}$.
Remarkably, the scalar kernel (2.11) can be obtained from the three-particle gaugino kernel (see Eqs. (5.7)-(5.14) in [23]) by substituting the gaugino decorating factor \( f_q(u, v) = u/(v(v - u)) \) with the scalar one \( f_s(u, v) \).

In Eq. (2.11), in the first term the superscript \( (u_3) \) indicates that the plus-distribution is taken only with respect to the variable \( u_3 \) as shown above in Eq. (2.9). The second term in (2.11) involves the double plus-prescription defined as

\[
\left[ \frac{\tau(u, u')}{(v - u)(v' - u')} \right]_{++}^{(uv')} = \frac{\tau(u, u')}{(v - u)(v' - u')} + \delta(u - v)\delta(u' - v') \int dw dw' \frac{\tau(u, w')}{(v - u)(v' - w')}
- \delta(u - v) \int dw \frac{\tau(w, w')}{(v - w)(v' - w')}
- \delta(u' - v') \int dw' \frac{\tau(w, w')}{(v - w)(v' - w')},
\]

with \( \tau(u, u') \) being a test function.

The two-particle kernel \( V_{k,k+1}^{(1)} \) receives contributions from the diagrams presented in Fig. 2 as well as from subtraction terms coming from the single and double plus-distributions. The expression for \( V_{12}^{(1)} \) looks like

\[
V_{12}^{(1)} = \left[ \Theta(u_1, v_1) f_s(u_1, v_1) \left( 4 - \mathcal{N} - \frac{\pi^2}{6} + \frac{1}{4} \ln^2 \frac{u_1}{v_1} - \ln \frac{u_1}{v_1} \ln \left( \frac{1 - u_1}{v_1} \right) \right) \right]_{+}^{(u_1)} + \{ u_1 \leftrightarrow u_2, v_1 \leftrightarrow v_2 \}.
\]  

(2.15)

As before, all other \( V_{k,k+1}^{(1)} \) can be obtained from \( V_{12}^{(1)} \) by cyclically permuting the indices. Finally, one has to add a single-particle contribution due to two-loop renormalization of the scalar fields. The former can be read off from the renormalization constant of the gauginos in the dimensional reduction scheme discussed at length in Ref. [23]

\[
\Gamma^{(1)} = \frac{1}{2}(\mathcal{N} - 4)(2 - \mathcal{N}),
\]

(2.16)

so that \( \Gamma^{(1)} = 0 \) in \( \mathcal{N} = 2 \) and \( \mathcal{N} = 4 \) theories. This result has a simple interpretation (see Eq. (2.21) below).

It is interesting to notice that the three-particle kernel \( V_{123}^{(1)} \) does not depend on the number of supercharges \( \mathcal{N} \). At the same time, examining the \( \mathcal{N} \)-dependence of the two-loop kernel \( V_{12}^{(1)} \) one finds the following remarkable relation between the two-loop dilatation operators in \( \mathcal{N} = 2 \) and \( \mathcal{N} = 4 \) theories,

\[
V_{\mathcal{N}=2}(u|v) = (1 + 2\lambda) V_{\mathcal{N}=4}(u|v).
\]  

(2.17)
The same relation was previously found for the gaugino operators in Ref. [23]. It suggests that the dilatation operators in the two theories share the same properties to two loops at least, and their eigenvalues are, obviously, related via Eq. (2.17).

Being combined together, Eqs. (2.5), (2.10), (2.11), (2.15) and (2.16) provide explicit expressions for the two-loop dilatation operator for the scalar operators (2.1) in $\mathcal{N} = 2$ and $\mathcal{N} = 4$ SYM theories in the multi-color limit. In section 3, we will diagonalize the dilatation operator (2.5) and determine the spectrum of anomalous dimensions of scalar operators. To accomplish this goal however, we will first reconstruct the mixing matrix for the operators (1.1).

2.2 Mixing matrix

It follows from (2.2) and (2.3) that the local Wilson operators (1.1) are related to the moments of the operators in the momentum representation

$$\mathcal{O}_n(0) = \int dv_1 v_1^{n_1} \cdots \int dv_L v_L^{n_L} \tilde{\mathcal{O}}(v). \qquad (2.18)$$

Together with (2.4) this implies that the operators $\mathcal{O}_n(0)$ obey the renormalization group equation

$$\left( \mu \frac{\partial}{\partial \mu} + \beta_{\mathcal{N}}(g) \frac{\partial}{\partial g} \right) \mathcal{O}_n(0) = - \sum_m \Lambda^{m}_n(\lambda) \mathcal{O}_m(0) \qquad (2.19)$$

with $m = (m_1, \ldots, m_L)$ such that $m_i \geq 0$ and $\sum_k n_k = \sum_k m_k$. The mixing matrix is introduced as

$$\int [du]_L v_1^{n_1} u_2^{n_2} \cdots u_L^{n_L} \mathcal{V}(u|v) = - \sum_m \Lambda^{m}_n(\lambda) v_1^{m_1} u_2^{m_2} \cdots u_L^{m_L}. \quad (2.20)$$

It defines the representation of the evolution kernel $\mathcal{V}(u|v)$ in the space spanned by the polynomials $v_1^{n_1} v_2^{n_2} \cdots v_L^{n_L}$. Substituting the evolution kernel in (2.20) by its two-loop expression (2.5), we obtain the corresponding mixing matrix in the multi-color limit. In particular, for $n_1 = \ldots = n_L = 0$ one finds that the terms in (2.5) containing ‘+’ and ‘++’ distributions provide vanishing contribution to the left-hand side of (2.20) leading to

$$\Lambda_0^0(\lambda) = L \left[ \Lambda\Gamma^{(0)} + \lambda^2 \Gamma^{(1)} + \mathcal{O}(\lambda^3) \right] = \frac{L}{2} \left[ \lambda (\mathcal{N} - 4) + \lambda^2 (\mathcal{N} - 4)(2 - \mathcal{N}) + \mathcal{O}(\lambda^3) \right] \quad (2.21)$$

According to (2.19) and (2.20), $\Lambda^0_0(\lambda)$ defines the anomalous dimension of the local operator $\mathcal{O}(0) = \text{tr} [X^L(0)]$ in $\mathcal{N} = 2$ and $\mathcal{N} = 4$ theories. These operators are known to be protected [29] and, therefore, their anomalous dimension is equal, in our notations, to $L\beta_{\mathcal{N}}(g)/2$. This implies that (2.21) is exact to all loops.

The perturbative expansion of the mixing matrix $\Lambda^m_n(\lambda)$ is similar to that of the evolution kernel (2.5) and reads in the multi-color limit

$$\Lambda^m_n(\lambda) = \lambda \Lambda^{(0)m}_n + \lambda^2 \Lambda^{(1)m}_n + \mathcal{O}(\lambda^3) \quad (2.22)$$

with $\lambda = g^2 N_c/(8\pi^2)$. The matrices $\Lambda^{(0)}$ and $\Lambda^{(1)}$ are given by the moments (2.20) of the kernels.
\( \mathcal{V}^{(0)}(u|v) \), Eq. (2.6), and \( \mathcal{V}^{(1)}(u|v) \), Eq. (2.10), respectively,

\[
\Lambda^{(0)}(m) = \sum_{k=1}^{L} \left\{ \lambda^{(0)}(m_{k} m_{k+1}) + \gamma^{(0)}(m_{k}) \delta_{m_{k+1}} \right\} \prod_{j=1, j \neq k, k+1}^{L} \delta_{m_{j}},
\]

\[
\Lambda^{(1)}(m) = \sum_{k=1}^{L} \left\{ \lambda^{(1)}(m_{k} m_{k+1} m_{k+2}) + \gamma^{(1)}(m_{k}) \delta_{m_{k+1}} \delta_{m_{k+2}} \right\} \prod_{j=1, j \neq k, k+1, k+2}^{L} \delta_{m_{j}}.
\] (2.23)

Here \( \delta_{m} \) is the Kronecker symbol and periodic boundary conditions \( L + k = k \) are implied. We also introduced a notation for the moments of the two- and three-particle kernels. The explicit expressions for the corresponding matrices \( \Lambda^{(0)}(m_{1} m_{2}) \), \( \Lambda^{(1)}(m_{1} m_{2}) \) and \( \Lambda^{(1)}(m_{1} m_{2} m_{3}) \) are rather lengthy and can be found in Appendix.

### 3 Eigenspectrum and integrability

To solve the evolution equation (2.19), one examines the eigenproblem for the mixing matrix (2.22)

\[
\sum_{n} \lambda^{m}_{n}(\lambda) \Psi(n) = \gamma(\lambda) \Psi(m),
\] (3.1)

where the sum runs over \( n = (n_{1}, \ldots, n_{L}) \) such that \( n_{i} \geq 0 \) and \( \sum_{k} n_{k} = \sum_{k} m_{k} \). We remind that the \( \beta \)-function vanishes in the \( N = 4 \) SYM theory \( \beta_{N=4}(g) = 0 \), while its exact value in the \( N = 2 \) SYM theory is given by the one-loop expression \( \beta_{N=2}(g) = -2\lambda \). Then, it follows from Eq. (2.19) that in \( N = 4 \) theory, the conformal operators \( \mathcal{O}_{\text{conf}}(0) = \sum_{n} \Psi(n) \mathcal{O}_{n}(0) \) have an autonomous scale dependence with \( \gamma(\lambda) \) being the corresponding anomalous dimensions. In \( N = 2 \) theory the scale dependence of the operators \( \mathcal{O}_{\text{conf}}(0) \) is more involved due to the dependence of the eigenstates \( \Psi(n) \) on the running coupling constant but one still refers to \( \gamma(\lambda) \) as an anomalous dimension.

Having the explicit expression for the two-loop mixing matrix (2.22) at our disposal, we can solve the spectral problem (3.1) for arbitrary lengths \( L \) and look for manifestation of symmetries of the dilatation operator in its spectrum. The evolution kernel \( \mathcal{V}(u|v) \) preserves the total momentum \( P = \sum_{n} u_{n} = \sum_{k} v_{k} \) and its eigenvalues \( \gamma(\lambda) \) do not depend on \( P \). This allows one to simplify the analysis by going over to the forward limit, \( P = 0 \), see Ref. [23]. Still, the anomalous dimensions \( \gamma(\lambda) \) depend on the total number of derivatives \( N = \sum_{i=1}^{L} n_{i} \) which is one of the integrals of motion for the Schrödinger-like equation (3.1). Another conserved charge follows from the invariance of the mixing matrix (2.22) under discrete cyclic \( \mathbb{P} \) and mirror transformations \( \mathbb{M} \) defined as

\[
\mathbb{P} \Psi(n_{1}, n_{2}, \ldots, n_{L}) = \Psi(n_{2}, n_{3}, \ldots, n_{1}),
\]

\[
\mathbb{M} \Psi(n_{1}, n_{2}, \ldots, n_{L}) = \Psi(n_{L}, n_{L-1}, \ldots, n_{1}).
\] (3.2)

Since these two operators do not commute with each other, the solutions to (3.1) can be classified according to the eigenvalues of only one of them, say \( \mathbb{P} \)

\[
\mathbb{P} \Psi(n_{1}, n_{2}, \ldots, n_{L}) = e^{i\theta} \Psi(n_{1}, n_{2}, \ldots, n_{L}),
\] (3.3)
where the quasimomentum $\theta$ takes $L$ distinct values, $\theta = \frac{2\pi n}{L}$ with $n = 0, 1, \ldots, L - 1$. Making use of the relation $\mathbb{P} \mathbb{M} \mathbb{P} = \mathbb{M}$, one immediately finds from (3.1) that for $\theta \neq 0$ the eigenstates $\Psi(n)$ and $\mathbb{M} \Psi(n)$ have the same “energy” $\gamma(\lambda)$ and opposite values of the quasimomentum, $\theta$ and $-\theta$, respectively. Thus, the solutions to (3.1) with nonzero quasimomentum are necessarily double degenerate. For the eigenstates with $\theta = 0$ the discrete symmetry alone does not imply any degeneracy. We recall that the eigenstates $\Psi(n)$ determine the form of conformal operators $O_{\text{conf}}(0) = \sum_n \Psi(n) O_n(0)$ with the basis operators $O_n(0)$ defined in (1.1). Since the latter operators are cyclically symmetric with respect to $n$, the eigenstates $\Psi(n)$ should possess the same symmetry, that is, they ought to have zero quasimomentum $\theta = 0$. This leads to an additional selection rule for solutions to Eq. (3.1). We would like to stress that the mixing matrix (2.22) possesses eigenvalues with both quasimomenta $\theta \neq 0$ and $\theta = 0$ but only the latter define eigenvalues of the dilatation operator in gauge theory.

To two-loop accuracy, the anomalous dimensions in the multi-color limit have the perturbative expansion

$$\gamma(\lambda) = \lambda \varepsilon^{(0)} + \lambda^2 \varepsilon^{(1)} + \mathcal{O}(\lambda^3),$$

(3.4)

with $\varepsilon^{(0)}$ and $\varepsilon^{(1)}$ being functions of the length of the operator $L$, the total number of derivatives $N$ and some other quantum numbers, yet to be determined. To find the explicit form of these functions one has to diagonalize the mixing matrix for various $L$ and $N$. In particular, using the expression for the two-loop mixing matrix, Eqs. (2.22) and (2.24), and solving the eigenproblem (3.1) for scalar operators of length $L = 3$ and the total number of derivatives $0 \leq N \leq 20$, we calculated the values of $\varepsilon^{(0)}$ and $\varepsilon^{(1)}$ with $\theta = 0$ in the $\mathcal{N} = 2$ and $\mathcal{N} = 4$ SYM theories. We summarized our results in Fig. 3. We found that in both theories all eigenvalues (3.4) (except of a single lowest eigenvalue for each even $N$) are double degenerate to two loops. We would like to stress that the two-loop evolution kernel in the $\mathcal{N} = 2$ theory contains conformal symmetry breaking terms proportional to the $\beta$–function. In the same fashion as in Ref. [23], the fact that the degeneracy is present in the $\mathcal{N} = 2$ theory suggests that the phenomenon is not directly tied to the conformal symmetry.

Figure 3: The spectrum of anomalous dimensions of $L = 3$ operators (see text) at one loop (a), and two loops for $\mathcal{N} = 2$ (b) and $\mathcal{N} = 4$ (c) super Yang-Mills theories.
3.1 One loop

We recall that for eigenstates with vanishing quasimomentum $\theta = 0$, the discrete symmetry (3.2) is not sufficient to warrant the double degeneracy of the spectrum. To one-loop order, this property immediately follows from hidden integrability of the one-loop dilatation operator [27, 11, 2, 18]. Namely, the one-loop mixing matrix $\Lambda^{(0)}$ can be mapped into a Hamiltonian of the XXX Heisenberg spin chain of length $L$ and spin operators in all sites being generators of (infinite-dimensional) discrete series representation of the $SL(2; \mathbb{R})$ group of spin $j = \frac{1}{2}$. On the gauge theory side, this group emerges as the so-called collinear subgroup of the full (super)conformal group. Its representation is determined by the conformal spin of scalar fields entering the Wilson operators (1.1). It is important that for the one-loop dilatation operator, the conformal spin of scalar field can be substituted by its value at order $\lambda^0$, that is, by its classical value $j = \frac{1}{2}$. As we will argue below the situation becomes more complex starting from two loops.

Thanks to integrability, the spectral problem (3.1) can be solved exactly to one-loop order using the Quantum Inverse Scattering Method [30]. This method allows us to identify the complete set of conserved charges $q^{(0)}_k$, whose eigenvalues parameterize solutions to (3.1). For scalar operators of length $L$, they can be obtained as coefficients in the expansion of the auxiliary transfer matrix in powers of the spectral parameter $u$

$$t^{(0)}_{L}(u) = \text{tr} \{L_L(u)L_{L-1}(u)\ldots L_1(u)\} = u^L \left[ 2 + q^{(0)}_2 u^{-2} + q^{(0)}_3 u^{-3} + \ldots + q^{(0)}_L u^{-L} \right].$$

Here $L_k(u) = \mathbb{1} \cdot u + i\sigma \cdot S_k$ is the standard Lax operator for the XXX Heisenberg spin chain and $\sigma^a$ are Pauli matrices. It is given by a $2 \times 2$ matrix whose entries are spin operators $S^a_k$ in $k$th site. In gauge theory, the latter are generators of the collinear $SL(2)$ subgroup acting on $k$th scalar field in (1.1). The “lowest” integral of motion $q^{(0)}_2$ is related to the total conformal spin of the Wilson operator, $J = N + L/2$,

$$q^{(0)}_2 = -(N + \frac{1}{2}L)(N + \frac{1}{2}L - 1) - \frac{1}{4}L.$$  

(3.6)

The explicit form of higher conserved charges $q^{(0)}_{k \geq 2}$ can be found in [27]. In particular, they possess a definite parity with respect to the discrete transformations (3.2), $[\mathbb{P}, q^{(0)}_n] = [\mathbb{M}, q^{(0)}_{2k+1}] = 0$. This implies that the one-loop “energy” $\varepsilon^{(0)}$ as a function of the conserved charges $q^{(0)}_k$ satisfies the relation

$$\varepsilon^{(0)}(q^{(0)}_3, q^{(0)}_4, \ldots, q^{(0)}_L) = \varepsilon^{(0)}(-q^{(0)}_3, q^{(0)}_4, \ldots, (-1)^L q^{(0)}_L),$$

(3.7)

and, therefore, all eigenstates including those with zero quasimomentum states are double degenerate provided that $q^{(0)}_{2k+1} \neq 0$. By explicit diagonalization of the dilatation operator in $\mathcal{N} = 2$ and $\mathcal{N} = 4$ SYM, evaluated in the previous section, we found that its eigenspectrum (3.4) is double degenerate not only at one loop but also in two-loop order. Together with (3.7) this suggests that the all-loop anomalous dimension (3.4) is a function of the conserved charges $q_k(\lambda)$

$$q_k(\lambda) = q^{(0)}_k + \lambda q^{(1)}_k + O(\lambda^2),$$

(3.8)

such that it verifies the same parity relation (3.7). We shall argue in Sect. 3.2 that this is indeed the case to two loops, at least.
The eigenspectrum of the one-loop anomalous dimension $\varepsilon^{(0)}$ can be easily found using the method of the Baxter $Q$–operator for the $SL(2; \mathbb{R})$ Heisenberg spin chains. The method relies on the existence of a commuting family of operators $Q(u)$ which depend on an arbitrary spectral parameter $u$ and, in addition, commute with the Hamiltonian (i.e., the one-loop dilatation operator). The one-loop anomalous dimensions $\varepsilon^{(0)}$ and the corresponding quasimomenta $\theta$, Eq. (3.3), can be expressed in terms of the eigenvalues of Baxter operator that we shall denote as $Q^{(0)}(u)$

$$
\varepsilon^{(0)} = i \left( \ln Q^{(0)}(i\frac{\tau}{2}) \right)' - i \left( \ln Q^{(0)}(-i\frac{\tau}{2}) \right)',
$$

$$
e^{i\theta} = \frac{Q^{(0)}(i\frac{\tau}{2})}{Q^{(0)}(-i\frac{\tau}{2})}, \quad (3.9)
$$

For the $SL(2; \mathbb{R})$ magnet of spin $j$ and length $L$ the function $Q^{(0)}(u)$ satisfies the second order finite-difference equation \[3.10\], the so-called “$tQ$” or Baxter equation

$$
\Delta^{(0)}_+(u)Q^{(0)}(u + i) + \Delta^{(0)}_-(u)Q^{(0)}(u - i) = t^{(0)}_L(u) Q^{(0)}(u),
$$

where the “dressing factors” $\Delta^{(0)}_{\pm}(u) = (u \pm ij)^L$ depend on the spin $j$ and $t^{(0)}_L(u)$ is a polynomial in $u$ of degree $L$ defined in \[3.5\]. We recall that for the scalar operators \[1.1\], to one-loop order, the spin $j$ is given by the conformal spin of scalar field at $\lambda=0$ order, i.e., $j = \frac{1}{2}$. The Baxter equation \[3.10\] alone does not specify $Q^{(0)}(u)$ uniquely and it has to be supplemented by an additional condition that $Q^{(0)}(u)$ should be a polynomial in $u$ of degree $N \geq 0$ \[27\]. Then, $Q^{(0)}(u)$ can be parameterized (modulo an overall normalization) by its roots

$$
Q^{(0)}(u) = \prod_{k=1}^{N} (u - u_k^{(0)}). \quad (3.11)
$$

It is known that for the $SL(2; \mathbb{R})$ spin chain the roots $u_k^{(0)}$ take real values only \[27\]. In gauge theory, the nonnegative integer $N$ coincides with the total number of derivatives in \[1.1\]. Being combined together, Eqs. \[3.10\] and \[3.11\] uniquely define $Q^{(0)}(u)$ and allow one to calculate the one-loop anomalous dimensions \[3.9\] (see Fig. 3a) and determine the corresponding values of the conserved charges $q_k^{(0)}$ entering \[3.5\].

The method of the Baxter $Q$–operator is equivalent to the Bethe Ansatz. Indeed, substituting $u = u_k^{(0)}$ into both sides of \[3.10\], one finds that $u_k^{(0)}$ satisfy the Bethe-root equations. In the same manner, substitution of \[3.11\] into \[3.9\] leads to the well-known expressions for energy in terms of Bethe roots. In the next section, we will discuss a “deformation” of the one-loop Baxter equation \[3.10\] in order to accommodate the two-loop corrections to the anomalous dimensions that were computed in Sect. 2. The reason why we prefer to deal with the Baxter $Q$–operator is that its eigenvalues have a direct physical meaning which should be preserved to all loops – for real $u$ the function $Q(u)$ determines the wave function of the spin chain in separated (“collective”) variables \[32\ \[33\] (see also discussion in Sect. 4). Defined in this way, $Q(u)$ should oscillate on the real $u$–axis and the number of its nodes should be equal to the excitation number $N$.

### 3.2 Two loops and beyond

The double degeneracy of the spectrum of two-loop anomalous dimension established in Sect. 2 combined with the exact integrability of the one-loop spectrum suggests to generalize the Baxter
equation to higher loops. Going over to two loops, we expect that the Bethe roots will be corrected by perturbative corrections

\[ u_k(\lambda) = u_k^{(0)} + \lambda u_k^{(1)} + \mathcal{O}(\lambda^2) \quad (3.12) \]

and \( u_k \) will verify “modified” Bethe equations. For the scalar operators under consideration, such equations have been conjectured in \[21, 34\]

\[ \left( \frac{x_k^+}{x_k^-} \right)^L = \prod_{j \neq k}^N \frac{x_k^- - x_j^+ 1 - \lambda/(2x_k^+ x_j^-)}{x_k^+ - x_j^- 1 - \lambda/(2x_k^- x_j^+)}; \quad (3.13) \]

where \( x_k^\pm = x(u_k \pm \frac{i}{2}) \) and the deformed spectral parameter \( x = x(u) \) is defined as 35

\[ x(u) = \frac{1}{2} u \left[ 1 + \sqrt{1 - 2\lambda/u^2} \right]. \quad (3.14) \]

Then, the anomalous dimension and the corresponding quasimomentum are determined in terms of \( x_k^\pm \) parameters as follows 21, 34

\[ \gamma(\lambda) = \lambda \sum_{k=1}^N \left( \frac{i}{x_k^+} - \frac{i}{x_k^-} \right), \quad e^{i\theta} = \prod_{k=1}^N \frac{x_k^+}{x_k^-}. \quad (3.15) \]

This relation coincides with (3.9) to one loop and it is believed that it should reproduce the anomalous dimensions of scalar operators (1.1) in the \( \mathcal{N} = 4 \) SYM theory to three loops at least.

As was already mentioned above, the one-loop Bethe equations for the parameters \( u_k^{(0)} \) are equivalent to the Baxter equation (3.10) for the polynomial \( Q(u) \), Eq. (3.11). We assume that the same relation between the Bethe Ansatz and the Baxter equation also holds in higher loops and introduce into consideration a polynomial with roots given by parameters \( u_k \), Eq. (3.12)

\[ Q(u) = \prod_{k=1}^N (u - u_k(\lambda)) = Q^{(0)}(u) + \lambda Q^{(1)}(u) + \lambda^2 Q^{(2)}(u) + \ldots. \quad (3.16) \]

Here the leading term \( Q^{(0)}(u) \) coincides with the solution to the Baxter equation (3.10) and is given by a polynomial of degree \( N \) with real roots \( u_k^{(0)} \). By construction, the subleading terms \( Q^{(n)}(u) \) do not depend on the coupling constant and are given by polynomials of degree \( N - 1 \), that is, \( Q^{(n)}(u) \sim u^{L-1} \) for \( n \geq 1 \).

It turns out that the function \( Q(u) \) defined in (3.16) obeys a second-order finite difference equation very similar to the Baxter equation (3.10)

\[ \Delta_+(x(u + \frac{i}{2}))Q(u + i) + \Delta_-(x(u - \frac{i}{2}))Q(u - i) = t_L (x(u)) Q(u), \quad (3.17) \]

where the dressing factors \( \Delta_\sigma(x) \) (with \( \sigma = \pm \)) satisfy the condition \( \Delta_-(x) = \overline{\Delta_+(x)} \) and read to three-loop accuracy

\[ \Delta_\sigma(x) = x^L \Delta^{(\text{ren})}_\sigma(x), \quad (3.18) \]

with the function \( x = x(u) \) defined in (3.14) and

\[ \Delta^{(\text{ren})}_\sigma(x) = \exp \left( -\frac{\lambda}{x} (\ln Q(\frac{i}{2}\sigma))' - \frac{\lambda^2}{4x^2} \left[ (\ln Q(\frac{i}{2}\sigma))'' + x (\ln Q(\frac{i}{2}\sigma))''' \right] + \mathcal{O}(\lambda^3) \right). \quad (3.19) \]
To lowest order in \( \lambda \) one has \( x(u) = u + \mathcal{O}(\lambda) \) with \( \Delta_+^{(\text{ren})}(x) = 1 + \mathcal{O}(\lambda) \) leading to \( \Delta_\pm(u) = (u \pm \frac{1}{2})^L \). It is straightforward to verify that the roots of \( Q(u) \) verify the modified Bethe equations (3.10) to three-loop accuracy [21]. The apparently unusual feature of (3.17) and (3.18) compared to (3.10) is that the dressing factors \( \Delta_+^\pm(u) \) and \( \Delta_-^\pm(u) \) depend on derivatives of the function \( Q(u) \) evaluated at \( u = \frac{1}{2} \) and \( u = -\frac{1}{2} \), respectively. We shall elucidate the origin of this property in a moment.

The auxiliary transfer matrix \( t_L(u) \) entering the right-hand side of (3.17) is a generating function for the conserved charges (3.8)

\[
t_L(x) = \sqrt{\Delta_+(x)\Delta_-(x)} \left( 2 + \sum_{n \geq 2} q_n(\lambda) x^{-n} \right). \tag{3.20}
\]

Its perturbative expansion starts with (3.3) and includes higher-loop perturbative corrections to the integrals of motion. In distinction with (3.5), the series in the right-hand side of (3.20) does not truncate, thus, reflecting the asymptotic character of the Baxter equation (3.17). Notice the integrals of motion. In distinction with (3.5), the series in the right-hand side of (3.20) does not involve \( \sim x^{-1} \) term. To see this, it suffices to substitute (3.16) and (3.20) into (3.17), expand its both sides at large \( u \) and match the coefficients in front of powers of \( u \). In this manner, one deduces from Eqs. (3.17) that the ‘lowest’ charge \( q_2(\lambda) \) equals

\[
q_2(\lambda) = -(N + \frac{1}{2}L + \frac{1}{2} \gamma(\lambda)) \left( N + \frac{1}{2}L + \frac{1}{2} \gamma(\lambda) - 1 \right) - \frac{1}{4}L, \tag{3.21}
\]

where \( \gamma(\lambda) \) is given by the sum of two functions \( \gamma_\sigma(\lambda) \) (with \( \sigma = \pm \)) parameterizing leading asymptotic behaviour of \( \Delta_\sigma^{(\text{ren})}(x) \) at large \( x \)

\[
\gamma(\lambda) = \gamma_+(\lambda) - \gamma_-(\lambda), \quad \Delta_\sigma^{(\text{ren})}(x) = \exp \left( i \gamma_\sigma(\lambda) x^{-1} + \mathcal{O}(x^{-2}) \right) . \tag{3.22}
\]

They satisfy the relations \( \gamma_+(\lambda) = - (\gamma_-(\lambda))^* \) and \( \gamma_\sigma(\lambda) = \mathcal{O}(\lambda) \) which ensure that \( \gamma(\lambda) \) takes real values and vanishes for \( \lambda \to 0 \). As a result, the charge \( q_2(\lambda) \), Eq. (3.21), also takes real values and approaches its lowest order value (3.5) for \( \lambda = 0 \). The reason we used in (3.22) the same notation as in (3.15) is that \( \gamma(\lambda) \) defines the multi-loop anomalous dimension. Combining together (3.18) and (3.22) we get

\[
\gamma(\lambda) = \left[ \frac{\lambda}{du} + \frac{\lambda^2}{4 du^3} + \frac{\lambda^3}{48 du^5} + \ldots \right] \left( i \ln Q(u) \right) \bigg|_{u=i/2}^{u=-i/2} = \lambda \varepsilon^{(0)} + \lambda^2 \varepsilon^{(1)} + \lambda^3 \varepsilon^{(2)} + \ldots . \tag{3.23}
\]

To leading order in \( \lambda \), it coincides with (3.9) and matches (3.13) up to three loops. One can apply Eqs. (3.17) and (3.23) to determine the spectrum of anomalous dimensions of scalar operators in \( N = 4 \) SYM theory, Eq. (1.1), of arbitrary length \( L \geq 2 \) and total number of derivatives \( N \geq 0 \).

Solving the Baxter equation (3.17), we can determine the function \( Q(u) \) up to three-loop order, Eq. (3.16), as well as the conserved charges \( q_k(\lambda) \), Eq. (3.8). In particular, \( q_k(\lambda) \)'s take real values and \( Q(u) \) is a real function of \( u \). The fact that the leading function \( Q^{(0)}(u) \) has only real roots ensures that the three-loop Bethe roots \( u_k(\lambda) \) are also real. The corresponding quasimomentum is given by the leading order relation (3.9) involving the \( Q^{(0)} \)–function and it is protected from perturbative correction in \( \lambda \).

To elucidate the physical meaning of the factor \( \Delta_\sigma^{(\text{ren})}(x) \), Eqs. (3.18) and (3.19), it is instructive to compare the \( SL(2) \) Baxter equation (3.17) with a similar equation describing the
multi-loop anomalous dimension of Wilson operators in the $SU(2)$ sector in $\mathcal{N} = 4$ SYM theory \cite{[35]}, i.e., single-trace operators of canonical dimension $L$ built from $N$ holomorphic scalars $X = \phi_3 + i\phi_4$ and $L - N$ fields $Z = \phi_1 + i\phi_2$. In that case, the $SU(2)$ Baxter equation takes the same form as (3.17) with the only difference that the “dressing” factors are merely given by $\Delta_{\pm}^{SU(2)}(u) = (x(u \mp x))^{L}$ and do not involve additional factors similar to $\Delta_{\pm}^{(ren)}(x)$. A natural question arises what is the reason for such difference? We recall that the anomalous dimensions (3.4) are eigenvalues of the dilatation operator which, in its turn, is one of the generators of the (super)conformal group of the underlying gauge theory. In the $SU(2)$ sector, the mixing occurs between Wilson operators carrying the same canonical dimension $L$ and the isotopic charge $N$. Then, in the multi-color limit, the dilatation operator in this sector can be mapped into the spin chain with the spin operators being the generators of the $SU(2)$ subgroup of the full $R$-symmetry group. This should be compared with the $SL(2)$ sector in which case the dilatation operator is mapped into the spin chain in such a way that the spin operators are generators of the collinear $SL(2)$ subgroup and the dilatation operator is one of these generators!

To lowest order in $\lambda$ the recursion works as follows. The one-loop dilatation operator is identified as a Hamiltonian of the $SL(2)$ spin chain with the spin operators being the generators of the collinear subgroup in gauge theory to zero-loop order\footnote{This explains why the one-loop dilatation operator inherits the conformal symmetry of the classical Lagrangian. Conformal anomaly affects the anomalous dimensions starting from two loops only.}. To this order, the generators of the collinear subgroup depend on the classical value of the conformal spin of the scalar field $j = \frac{1}{2}$ and, as a consequence, the one-loop dilatation operator coincides with the $SL(2)$ spin chain of spin $j = \frac{1}{2}$. Going over to higher orders one expects that the dilatation operator to $n^{th}$ loop is given by the $SL(2)$ spin chain with spins corrected by perturbative corrections to the dilatation operator to $(n - 1)^{st}$-loop accuracy. This property finds its manifestation in the structural form of the Baxter equation (3.17).

Indeed, the Baxter equation (3.17) involves the dressing factors $\Delta_{\pm}^{(ren)}(x)$, Eq. (3.19), which depend on the $Q-$function that satisfies the Baxter equation itself. Replacing $Q(u)$ in (3.17) by its perturbative expansion (3.16) and expanding $\Delta^{(ren)}(x)$ in powers of $\lambda$ it is easy to see that $\Delta_{\pm}^{(ren)}(x)$ induces corrections to the Baxter equation for $Q^{(n)}(u)$ involving lowest-order functions $Q^{(k)}(u)$ with $0 \leq k < n$. In particular, for $n = 1$ the one-loop corrections to the conserved charge $q_{2}(\lambda)$ depend on the $Q^{(0)}$-function, Eq. (3.21). We remind that the charge $q_{2}^{(0)}$, Eq. (3.6), is related to the total $SL(2)$ conformal spin $J = N + \frac{1}{2}L$ of the scalar operator (1.1). Substituting $q_{2}^{(0)}$ in (3.21) by its explicit expression (3.6), one notices that a part of $\lambda$ correction to $q_{2}(\lambda)$ proportional to $\epsilon^{(0)}$ can be absorbed into the lowest order term as follows

$$q_{2}(\lambda) = \left[-(N + \frac{1}{2}L + \frac{1}{2}\lambda\epsilon^{(0)})(N + \frac{1}{2}L + \frac{1}{2}\lambda\epsilon^{(0)} - 1) - \frac{1}{4}L\right] + \mathcal{O}(\lambda^{2}).$$

We recall that $\epsilon^{(0)}$ defines the one-loop correction to the anomalous dimension of scalar operators, Eq. (3.4). Going over to higher orders in $\lambda$, one finds from (3.21) that $\lambda\epsilon^{(0)}$ get replaced in (3.21) by the multi-loop anomalous dimension (3.23). Then, one deduces from (3.24) that the factor $\Delta_{\pm}^{(ren)}(x)$ renormalizes the “bare” conformal spin $J$ of the scalar operator by an amount proportional to its anomalous dimension

$$J = N + \frac{1}{2}L \rightarrow J_{\text{ren}} = N + \frac{1}{2}L + \frac{1}{2}\gamma(\lambda).$$

This result can be interpreted as follows. For a conformal operator $\mathcal{O}_{\text{conf}}(0) = \sum_{n} \Psi(n)\mathcal{O}_{n}(0)$, its conformal $SL(2)$ spin $J = \frac{1}{2}(d + s)$ depends on its scaling dimension, $d$, and projection of
its Lorentz spin on the light-cone, s. To order $\lambda^0$ one has $d = N + L$ and $s = N$ so that $J = N + \frac{1}{2}L$. To higher orders in $\lambda$, the spin $s$ is protected from perturbative corrections while the scaling dimension $d$ receives the anomalous contribution $\gamma(\lambda)$ leading to $J_{\text{ren}} = N + \frac{1}{2}L + \frac{1}{2}\gamma(\lambda)$. One can arrive at the same conclusion from consideration of the conformal Ward identities as explained in details in Refs. [36, 23]. In this formalism the additive correction to the conformal spin proportional to the anomalous dimension comes from the renormalization of the composite operator given by the product of the Wilson operators and the trace anomaly of the energy-momentum tensor in regularized gauge theory [36].

Let us compare (3.23) with the results of explicit diagonalization of the two-loop mixing matrix in the $\mathcal{N} = 4$ SYM, Eqs. (3.1) and (2.22). Similar to the Baxter equation, the mixing matrix (3.1) has eigenvalues with zero and nonzero quasimomentum. Although the anomalous dimensions of single trace operators correspond only to the former, we can perform the comparison for all eigenvalues. In this way, we verified that the two eigenspectra coincide for $L \leq 5$ and $N \leq 20$, to two loops at least. Thus, the relation (3.23) provides the exact solution to the spectral problem (3.1) for two-loop mixing matrix in $\mathcal{N} = 4$ SYM. Making use of the relation (2.17), the correspondence can be further extended to $\mathcal{N} = 2$ theory.

We also checked that the “odd” conserved charges $q_{2k+1}$, corresponding to the paired eigenvalues have opposite signs in agreement with the lowest order expectations, Eq. (3.7), e.g., for the state with $[L = 3, N = 5]$ given in Table 1 one finds for the transfer matrix (3.20)

$$t_{L=3}^\pm(u) = 2u^3 \pm u^2 \left( -\frac{1}{12} \lambda + \frac{4933}{38016} \lambda^2 \right)^{\square \text{1155}}$$

$$+ u \left( -\frac{73}{2} - \frac{111}{2} \lambda + \frac{7595}{96} \lambda^2 \right) \pm \left( \frac{1}{2} + \frac{3794}{7504} \lambda - \frac{4894295}{5018112} \lambda^2 \right)^{\square \text{1155}}. \quad (3.26)$$

For eigenvalues with zero quasimomentum, we summarized our results in Table 1. At two loops, they agree with diagrammatic calculations of the $[L = 3, N = 2]$ anomalous dimension of Refs. [22, 20], which is related to the BMN counterpart of the Konishi current [24] due to multiplet splitting, and the $[L = 3, N = 3]$ result of [21, 22, 26] as well as with eigenspectra of Ref. [25] based on algebraic construction of the dilatation operator.

## 4 Discussion and conclusions

Recently, extensive multiloop calculations in various sectors of SYM theories pointed to persistence of integrability beyond leading perturbative order [18, 19, 20, 21, 22, 23, 24, 25, 26]. In this paper, we continued our study of integrability properties of the two-loop dilatation operator in (supersymmetric) Yang-Mills theories initiated in Ref. [23]. As a case of study, we have chosen the sector of single-trace operators built from holomorphic scalar fields in the $\mathcal{N} = 2$ and $\mathcal{N} = 4$ SYM theories and containing an arbitrary number of covariant derivatives projected onto the light-cone. To one-loop order, in both theories, the dilatation operator in this sector can be mapped in the multi-color limit into a Hamiltonian of the $SL(2; \mathbb{R})$ Heisenberg spin chain and its eigenspectrum can be found by means of the Bethe Ansatz. Our goal was to understand whether integrability survives in high loops and if so then what are the novel features of the underlying spin chain. To this end, we performed an explicit two-loop calculation of the dilatation operator and found that the spectrum of two-loop anomalous dimensions has the same degeneracy properties as to one loop level. We also demonstrated that, in agreement with our previous findings

\footnote{The twist three, spin three anomalous dimension was computed earlier in Refs. [21, 22, 26].}
| \([L, N]\) | \(\varepsilon^{(0)}\) | \(\varepsilon^{(1)}\) | \(\varepsilon^{(2)}\) |
|---|---|---|
| [3, 2] | 4 | -6 | 17 |
| [3, 5] | \(\frac{35}{4}\) | \(-\frac{18865}{1152}\) | \(\frac{1068515}{18432}\) |
| [3, 8] | \(\frac{25}{3}\) \(\frac{5087}{420}\) | \(-\frac{455}{32}\) | \(\frac{1140715}{248832}\) \(\frac{2330723533437143}{26138246400000}\) |
| [4, 2] | \(5 \pm \sqrt{5}\) | \(-\frac{1}{2}(17 \pm 5\sqrt{5})\) | \(\frac{9}{20}(65 \pm 23\sqrt{5})\) |
| [4, 5] | \(\frac{35}{4} \pm \frac{\sqrt{385}}{12}\) | \(-\frac{28139}{1728} \pm \frac{9101\sqrt{385}}{44352}\) | \(\frac{799837}{13824} \pm \frac{3060649313\sqrt{385}}{3688312420}\) |
| [4, 8] | 6.4113 | -8.4697 | 22.4035 |
| | 9.1601 | -14.7918 | 47.6639 |
| | 9.9596 | -18.2198 | 62.7707 |
| | 9.8710 | -18.6154 | 68.0070 |
| | 12.4010 | -24.3757 | 88.7702 |
| | 12.9479 | -25.7258 | 93.3842 |
| | 12.9650 | -25.2831 | 90.0613 |
| | 14.9761 | -30.4673 | 111.0666 |
| | 16.4651 | -33.8137 | 123.7385 |
| [5, 2] | 2 | \(-\frac{3}{2}\) | \(\frac{37}{16}\) |
| | 6 | \(-\frac{21}{2}\) | \(\frac{555}{16}\) |

Table 1: Eigenvalues of the one-, two- and three-loop dilatation operator in \(\mathcal{N} = 4\) SYM.
the two-loop anomalous dimensions in $\mathcal{N} = 2$ and $\mathcal{N} = 4$ theories differ from each other by an overall normalization factor indicating that the phenomenon is not sensitive to the conformal symmetry. These results lay a firm ground to the belief that the dilatation operator in the two theories is integrable beyond one loop.

As a next step, we tried to uncover integrable structures behind the two-loop dilatation operator by applying the method of the Baxter $Q-$operator. The reason for this is the following. It is well known that in classical integrable models admitting the Lax representation one can apply the “magic recipe” to perform a canonical transformation to the separated variables and reduce the original multi-dimensional problem to a set of one-dimensional ones. In quantum integrable models, the canonical transformation is replaced by a unitary transformation to the separated coordinates such that the multi-particle wave function is factorized into a product of single-particle ones. For the $SL(2; \mathbb{R})$ Heisenberg spin chain (= one-loop dilatation operator) the representation of the separated coordinates (SoV) has been constructed in [33]. In this representation, the single-particle wave function is given by the eigenvalue of the Baxter operator, $Q^{(0)}(u)$, and the Schrödinger equation in the separated variables coincides with the Baxter equation (3.10) supplemented with (3.11) and (3.9). Going over to higher loops, we assumed that the spin chain describing the eigenspectrum of the multi-loop dilatation operator admits the SoV representation with the single-particle wave function $Q(u)$ corrected by perturbative corrections (3.16). In the $\mathcal{N} = 4$ SYM, this is in agreement with the fact that the dilatation operator for scalar operators with large canonical dimension $L$ and Lorentz spin $N$ can be identified via the gauge/string correspondence with a Hamiltonian of the classical sigma-model on AdS$_3 \times S^1$ background. This sigma-model is known to be completely integrable and it admits both the Lax and SoV representations [38].

The question remains however how to construct the higher-loop Baxter $Q-$operator and what is the analog of the Baxter equation (3.10) for its eigenvalue $Q(u)$. To answer the second part of this question we first verified that eigenvalues of two-loop dilatation operator calculated in the $\mathcal{N} = 4$ theory are in agreement with the modified Bethe Ansatz equations conjectured in [34]. Identifying the Bethe roots as roots of $Q(u)$ we worked out a deformed Baxter equation which exactly encodes the one- and two-loop spectra of anomalous dimension. Then, we demonstrated that the Baxter equation correctly incorporates a peculiar feature of conformal operators – the conformal $SL(2)$ spin of such operators is modified in higher loops by an amount proportional to their anomalous dimension. From the point of view of spin chains this property implies that the underlying integrable model is rather unusual – the Hamiltonian of the spin chain depends on the total $SL(2)$ spin which in its turn is proportional to the Hamiltonian. Still, to identify this spin chain one needs the explicit form of the $Q-$operator. To one-loop order, this operator has been constructed in [39]. Acting on the Wilson operators in the momentum representation $\tilde{O}(u)$, Eq. (2.3), the one-loop Baxter operator can be realized as an integral operator $Q_u^{(0)}(u|v)$ acting on the momentum fraction, in a close analogy with the dilatation operator $V(u|v)$, Eq. (2.4). The one-loop evolution kernel $V^{(0)}(u|v)$, (2.4), arises as a coefficient in the expansion of the kernel $Q_u^{(0)}(u|v)$ in the spectral parameter around $u = \pm \frac{i}{2}$, in agreement with (3.9). In a similar manner, one can translate (3.23) into the relation between the two-loop evolution kernel (2.5) and the two-loop Baxter operator. Simplicity of the two-loop kernel (2.10) gives us a hope that such operator can be constructed explicitly and the problem deserves additional studies.

For an interpretation of the $Q-$operator in string theory see Ref. [37].
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Appendix

To one-loop order, the moments of the two-particle kernel \( \Psi_{12}^{(0)} \), Eq. (2.7), are given by

\[
\Lambda_{n_1 n_2}^{(0)m_1 m_2} = -S_{n_1}^{(1)} \delta_{n_1}^{m_1} \delta_{n_2}^{m_2} + \frac{n_1! n_2!}{m_1! m_2! n_2 - m_2} \theta_{n_2 m_2} \delta_{n_1 + n_2}^{m_1 + m_2} + \left\{ \begin{array}{c} n_1 \leftrightarrow n_2 \\ m_1 \leftrightarrow m_2 \end{array} \right\},
\]

(A.1)

with discrete step-function taking values \( \theta_{nm} = 1 \) for \( n > m \) and vanishing otherwise. Analogously, to two loops, the moments of the two-particle irreducible kernel \( \Psi_{12}^{(2)} \), Eq. (2.13), are given by

\[
\Lambda_{n_1 n_2}^{(1)m_1 m_2} = \left[ S_{n_1}^{(1)} S_{n_1}^{(2)} + S_{n_1}^{(3)} - 2(4 - N) S_{n_1}^{(1)} \right] \delta_{n_1}^{m_1} \delta_{n_2}^{m_2} + \frac{n_1! n_2!}{m_1! m_2! n_2 - m_2} \theta_{n_2 m_2} \delta_{n_1 + n_2}^{m_1 + m_2} \left\{ 2(4 - N) \right\}
\]

- \frac{1}{2} S_{n_1}^{(2)} - \frac{3}{2} S_{n_1}^{(2)} - \frac{1}{2} \left( S_{m_1}^{(1)} - S_{n_1}^{(1)} \right) \left( S_{m_1}^{(1)} + 3 S_{m_1}^{(1)} - 4 S_{m_1}^{(1)} - m_2 - 2 \right)^2 + \left\{ \begin{array}{c} n_1 \leftrightarrow n_2 \\ m_1 \leftrightarrow m_2 \end{array} \right\},
\]

(A.2)

Here the notation was introduced for harmonic sums

\[
S_{n}^{(1)} = \sum_{\ell=1}^{n} \frac{1}{\ell} = \psi(n + 1) - \psi(1), \quad S_{n}^{(2)} = \sum_{\ell=1}^{n} \frac{1}{\ell^2} = -\psi'(n + 1) + \frac{\pi^2}{6}.
\]

(A.3)

with \( \psi(x) = d \ln \Gamma(x)/dx \) being the Euler digamma function.

The moments of the three-particle irreducible kernel \( \Lambda_{n_1 n_2 n_3}^{(1)m_1 m_2 m_3} \) read for \( n_2 = 0 \)

\[
\Lambda_{n_1 n_2 n_3}^{(1)m_1 m_2 m_3} = \frac{n_1! n_3!}{m_1! m_2! m_3!} \left( \frac{1}{n_1 - m_1} + S_{m_1}^{(1)} - S_{n_1 - m_1}^{(1)} \right)
\]

+ \theta_{n_1 m_1} \left( S_{n_1 - m_1}^{(1)} + S_{n_3 - m_3}^{(1)} - S_{m_2}^{(1)} \right) + \left\{ \begin{array}{c} n_1 \leftrightarrow n_3 \\ m_1 \leftrightarrow m_3 \end{array} \right\},
\]

(A.4)

while for \( n_2 > 0 \) they can be expressed in terms of the \( n_2 = 0 \) moments as

\[
\Lambda_{n_1 n_2 n_3}^{(1)m_1 m_2 m_3} = \sum_{j_1=0}^{n_2} \sum_{j_3=0}^{n_2} \sum_{k_1=0}^{n_2} \sum_{k_3=0}^{n_2} \frac{(-1)^j n_2!}{j! j_1! j_3! k_1! k_3! (n_2 - j - k)!} \Lambda_{n_1 + j_1, n_2 + j, n_3 + j_3}^{(1)m_1 - k_1, m_2 - j, m_3 - k},
\]

(A.5)

where we used shorthand notations \( j = j_1 + j_3 \) and \( k = k_1 + k_3 \).

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