PAPPU'S THEOREM IN GRASSMANNIAN $Gr(3, \mathbb{C}^n)$

S. SAWADA, S. SETTEPANELLA, AND S. YAMAGATA

Abstract. In this paper we study intersections of quadrics, components of the hypersurface in Grassmannian $Gr(3, \mathbb{C}^n)$ introduced in [10]. This lead to an alternative statement and proof of Pappus’s Theorem retrieving Pappus’s and Hesse configurations of lines as special points in complex projective Grassmannian. This new connection is obtained through a third purely combinatorial object, the intersection lattice of Discriminantal arrangement.

1. Introduction

Pappus’s hexagon Theorem, proved by Pappus of Alexandria in the fourth century A.D., began a long development in algebraic geometry.

In its changing expressions one can see reflected the changing concerns of the field, from synthetic geometry to projective plane curves to Riemann surfaces to the modern development of schemes and duality (D. Eisenbud, M. Green and J. Harris [3]).

There are several known proofs of Pappus’s Theorem including its generalizations such as Cayley Bacharach Theorem (see Chapter 1 of [5] for a collection of proofs of Pappus’s Theorem and [3] for proofs and conjectures in higher dimension).

In this paper, by means of recent results in [7] and [10], we connect Pappus’s hexagon configuration to intersections of well defined quadrics in Grassmannian providing a new statement and proof of Pappus’s Theorem as an original result on dependency conditions for defining polynomials of those quadrics. This result enlightens a new connection between special configurations of points (lines) in projective plane and hypersurfaces in projective Grassmannian $Gr(3, \mathbb{C}^n)$. This connection is made through a third combinatorial object, the intersection lattice of the Discriminantal arrangement. Introduced by Manin and Schechtman in 1989, it is an arrangement of hyperplanes generalizing classical braid arrangement (cf. [8], [1] and [2]). Fixed a generic arrangement $A = \{H^0_1, ..., H^0_n\}$ in $\mathbb{C}^k$, Discriminantal arrangement $B(n, k, A), n, k \in \mathbb{N}$ for $k \geq 2$ ( $k = 1$ corresponds to Braid arrangement ), consists of parallel translates $H^t_1, ..., H^t_n, (t_1, ..., t_n) \in \mathbb{C}^k$, of $A$ which fail to form a generic arrangement in $\mathbb{C}^k$. The combinatorics of $B(n, k, A)$ is known in the case of very generic arrangements, i.e. $A$ belongs to an open Zariski set $Z$ in the space of generic arrangements $H^0_i, i = 1, ..., n$ (see [3], [1] and [2]), but still almost unknown for $A \notin Z$. In 2016, Libgober and Settepanella (cf.[7]) gave a sufficient geometric condition for an arrangement $A$ not to be very generic, i.e. $A \notin Z$. In particular in case $k = 3$, their result shows that multiplicity 3 codimension 2 intersections of hyperplanes in $B(n, 3, A)$ appears if and only if collinearity conditions for points at infinity of lines, intersections of certain planes in $A$, are satisfied (Theorem 3.8 in [7]). More recently (see [10]) authors applied this result to show that points in specific degree 2 hypersurface in Grassmannian $Gr(3, \mathbb{C}^n)$ correspond to generic arrangements of $n$ hyperplanes in $\mathbb{C}^3$ with associated discriminantal arrangement having intersections of multiplicity 3 in

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We will use the compactification of \( C \) the coordinate system in and definitions and results from [10]. In section 3 we provide an example of the case of 6 hyperplanes in the space of all arrangements of \( n \) lines in \( (\mathbb{P}^2)^n \). On the other hand in \( Gr(3, \mathbb{C}^n) \) there is open set \( U' \) consisting of 3-spaces intersecting each coordinate hyperplane transversally (i.e. having dimension of intersection equal 2). One has also one set \( \tilde{U} \) in \( Hom(C^3, \mathbb{C}^n) \) consisting of embeddings with image transversal to coordinate hyperplanes and \( \tilde{U}/GL(3) = U' \) and \( \tilde{U}/(\mathbb{C}^*)^n = U \). Hence generic arrangements in \( C^3 \) can be regarded as points in \( Gr(3, \mathbb{C}^n) \). Let \( \{s_1 \leq \cdots \leq s_6\} \subset \{1, \ldots, n\} \) be a set of indices of a generic arrangement \( A = \{H_1^0, \ldots, H_n^0\} \) in \( C^3 \), \( \alpha_i \), the normal vectors of \( H_i^0 \)'s and \( \beta_{ij} = det(\alpha_i, \alpha_j, \alpha_\ell) \). For any permutation \( \sigma \in S_6 \) denote by \( \{\sigma\} = \{(i_1, i_2), (i_3, i_4), (i_5, i_6)\} \), \( l_j = s_{\sigma(j)d} \), and by \( Q_{\sigma} \) the quadric in \( Gr(3, \mathbb{C}^n) \) of equation 
\[
\beta_{i_1i_2i_3}\beta_{i_4i_5i_6} - \beta_{i_1i_3i_4}\beta_{i_5i_6i_2} - \beta_{i_1i_2i_4}\beta_{i_5i_6i_3} = 0
\]
The following theorem, equivalent to the Pappus’s hexagon Theorem, holds.

**Theorem 5.3.** (Pappus’s Theorem) For any disjoint classes \( [\sigma_1] \) and \( [\sigma_2] \), there exists a unique class \( [\sigma_3] \) disjoint from \( [\sigma_1] \) and \( [\sigma_2] \) such that 
\[
Q_{\sigma_1} \cap Q_{\sigma_2} = \bigcap_{i=1}^{3} Q_{\sigma_i}
\]
for any \( \{i_1, i_2\} \subset \{3\} \).

In the rest of the paper, we retrieve the Hesse configuration of lines studying intersections of six quadrics of the form \( Q_{\sigma} \) for opportunely chosen \( [\sigma] \). This lead to a better understanding of differences in combinatorics of Discriminantal arrangement in complex and real case. Indeed it turns out that this difference is connected with existence of the Hesse arrangement (see [9]) in \( \mathbb{P}^2(\mathbb{C}) \), but not in \( \mathbb{P}^2(\mathbb{R}) \).

From above results it seems very likely that a deeper understanding of combinatorics of Discriminantal arrangements arising from non very generic arrangements of hyperplanes in \( C^k \) (i.e. \( \mathcal{A} \not\in \mathcal{Z} \) ), could lead to new connections between higher dimensional special configurations of hyperplanes (points) in projective space and Grassmannian. Viceversa, known results in algebraic geometry could help in understanding combinatorics of Discriminantal arrangements in non very generic case. Moreover we conjecture that regularity in the geometry of Discriminantal arrangement could lead to results on hyperplanes arrangements with high multiplicity intersections, e.g., in case \( k = 3 \), line arrangements in \( \mathbb{P}^2 \) with high number of triple points (see Remark 6.6). This will be object of further studies.

The content of the paper is the following.

In section 2 we recall definition of Discriminantal arrangement from [8], basic notions on Grassmannian, and definitions and results from [10]. In section 3 we provide an example of the case of 6 hyperplanes in \( C^3 \).

In section 4 we define and study Pappus hypersurface. Section 5 contains Pappus’s theorem in \( Gr(3, \mathbb{C}^n) \) and its proof. In the last section we study intersections of higher numbers of quadrics and Hesse configuration.

### 2. Preliminaries

#### 2.1. Discriminantal arrangement

Let \( H_i^0, i = 1, \ldots, n \) be a generic arrangement in \( \mathbb{C}^k \), \( k < n \) i.e. a collection of hyperplanes such that \( \text{codim} \bigcap_{i \in K, K \neq P} H_i^0 = p \). Space of parallel translates \( \mathcal{S}(H_1^0, \ldots, H_n^0) \) (or simply \( \mathcal{S} \) when dependence on \( H_i^0 \) is clear or not essential) is the space of \( n \)-tuples \( H_1, \ldots, H_n \) such that either \( H_i \cap H_j = \emptyset \) or \( H_i = H_j^0 \) for any \( i = 1, \ldots, n \). One can identify \( \mathcal{S} \) with \( n \)-dimensional affine space \( \mathbb{C}^n \) in such a way that \( (H_0^1, \ldots, H_0^n) \) corresponds to the origin. In particular, an ordering of hyperplanes in \( \mathcal{A} \) determines the coordinate system in \( \mathcal{S} \) (see [8]).

We will use the compactification of \( C^k \) viewing it as \( \mathbb{P}^k(\mathbb{C}) \setminus H_{\infty} \) endowed with collection of hyperplanes \( H_i^0 \).
which are projective closures of affine hyperplanes $H^0_i$. Condition of genericity is equivalent to $\bigcup_i H^0_i$ being a normal crossing divisor in $\mathbb{P}^k(\mathbb{C})$.

Given a generic arrangement $\mathcal{A}$ in $\mathbb{C}^k$ formed by hyperplanes $H_i, i = 1, \ldots, n$ the trace at infinity, denoted by $\mathcal{A}_{\infty}$, is the arrangement formed by hyperplanes $H_{\infty,i} = H^0_i \cap H_{\infty}$. The trace $\mathcal{A}_{\infty}$ of an arrangement $\mathcal{A}$ determines the space of parallel translates $\mathbb{S}$ (as a subspace in the space of $n$-tuples of hyperplanes in $\mathbb{P}^k$).

Fixed a generic arrangement $\mathcal{A}$, consider the closed subset of $\mathbb{S}$ formed by those collections which fail to form a generic arrangement. This subset is a union of hyperplanes with each hyperplane $D_L$ corresponding to a subset $L = \{i_1, \ldots, i_{k+1}\} \subset [n] := \{1, \ldots, n\}$ and consisting of $n$-tuples of translates of hyperplanes $H^0_{i_1}, \ldots, H^0_{i_{k+1}}$ in which translates of $H^0_{i_1}, \ldots, H^0_{i_{k+1}}$ fail to form a generic arrangement. The arrangement $\mathcal{B}(n, k, \mathcal{A})$ of hyperplanes $D_L$ is called Discriminantal arrangement and has been introduced by Manin and Schechtman in [8]. Notice that $\mathcal{B}(n, k, \mathcal{A})$ only depends on the trace at infinity $\mathcal{A}_{\infty}$ hence it is sometimes more properly denoted by $\mathcal{B}(n, k, \mathcal{A}_{\infty})$.

2.2. Good 3s-partitions. Given $s \geq 2$ and $n \geq 3s$, a good 3s-partition (see [10]) is a set $\mathcal{T} = \{L_1, L_2, L_3\}$, with $L_i$ subsets of $[n]$ such that $|L_i| = 2s, |L_i \cap L_j| = s$ ($i \neq j$), $L_i \cap L_j \cap L_3 = \emptyset$ (in particular $|\bigcup L_i| = 3s$), i.e. $L_1 = \{i_1, \ldots, i_{2s}\}, L_2 = \{i_1, \ldots, i_s, i_{2s+1}, \ldots, i_{3s}\}, L_3 = \{i_{r+1}, \ldots, i_{3s}\}$.

Let $P_{k+1}(\mathbb{C}^n) = \{L \subset [n] \mid |L| = k + 1\}$ be the set of cardinality $k + 1$ subsets of $[n]$. Following [10] we denote by

$$A(\mathcal{A}_{\infty}) = (a_L)_{L \in P_{k+1}(\mathbb{C}^n)}$$

the matrix having in each row the entries of vectors $a_L$ normal to hyperplanes $D_L$ and by $A_T(\mathcal{A}_{\infty})$ the submatrix of $A(\mathcal{A}_{\infty})$ with rows $a_L, L \in \mathcal{T}, \mathcal{T} \subset P_{k+1}(\mathbb{C}^n)$. In this paper we are mainly interested in matrix $A_T(\mathcal{A}_{\infty})$ in the case of $\mathcal{T}$ good 6-partition.

2.3. Matrices $A(\mathcal{A}_{\infty})$ and $A_T(\mathcal{A}_{\infty})$. Let $a_i = (a_{i1}, \ldots, a_{ik})$ be the normal vectors of hyperplanes $H_i, 1 \leq i \leq n$, in the generic arrangement $\mathcal{A}$ in $\mathbb{C}^k$. Normal here is intended with respect to the usual dot product

$$(a_1, \ldots, a_k) \cdot (v_1, \ldots, v_k) = \sum_i a_i v_i \quad .$$

Then the normal vectors to hyperplanes $D_L, L = \{s_1 < \cdots < s_{k+1}\} \subset [n]$ in $\mathbb{S} \cong \mathbb{C}^n$ are nonzero vectors of the form

$$a_L = \sum_{i=1}^{k+1} (-1)^i \det(a_{s_1}, \ldots, a_{s_i}, \ldots, a_{s_{k+1}}) e_i \quad ,$$

where $\{e_i\}_{1 \leq i \leq n}$ is the standard basis of $\mathbb{C}^n$ (cf. [2]).

Let $P_{k+1}(\mathbb{C}^n) = \{L \subset [n] \mid |L| = k + 1\}$ be the set of cardinality $k + 1$ subsets of $[n]$. Following [10] we denote by

$$A(\mathcal{A}_{\infty}) = (a_L)_{L \in P_{k+1}(\mathbb{C}^n)}$$

the matrix having in each row the entries of vectors $a_L$ normal to hyperplanes $D_L$ and by $A_T(\mathcal{A}_{\infty})$ the submatrix of $A(\mathcal{A}_{\infty})$ with rows $a_L, L \in \mathcal{T}, \mathcal{T} \subset P_{k+1}(\mathbb{C}^n)$. In this paper we are mainly interested in matrix $A_T(\mathcal{A}_{\infty})$ in the case of $\mathcal{T}$ good 6-partition.

2.4. Grassmannian $Gr(k, \mathbb{C}^n)$. Let $Gr(k, \mathbb{C}^n)$ be the Grassmannian of $k$-dimensional subspaces of $\mathbb{C}^n$ and

$$\gamma : Gr(k, \mathbb{C}^n) \rightarrow \mathbb{P}(\bigwedge^k \mathbb{C}^n) \quad ,$$

$$<v_1, \ldots, v_k> \mapsto [v_1 \wedge \cdots \wedge v_k] \quad ,$$

the Plücker embedding. Then $[x] \in \mathbb{P}(\bigwedge^k \mathbb{C}^n)$ is in $\gamma(Gr(k, \mathbb{C}^n))$ if and only if the map

$$\varphi_x : \mathbb{C}^n \rightarrow \bigwedge^{k+1} \mathbb{C}^n \quad ,$$

$$v \mapsto x \wedge v \quad ,$$

is injective.
has kernel of dimension $k$, i.e. ker $\varphi_x = \langle v_1, \ldots, v_k \rangle$. If $e_1, \ldots, e_n$ is a basis of $\mathbb{C}^n$ then $e_I = e_{i_1} \wedge \ldots \wedge e_{i_k}$, $I = \{i_1, \ldots, i_k\} \subset [n], i_1 < \ldots < i_k$, is a basis for $\wedge^k \mathbb{C}^n$ and $x \in \wedge^k \mathbb{C}^n$ can be written uniquely as

$$x = \sum_{I \subseteq [n]} \beta_I e_I = \sum_{1 \leq i_1 < \ldots < i_k \leq n} \beta_{i_1 \ldots i_k} (e_{i_1} \wedge \ldots \wedge e_{i_k})$$

where homogeneous coordinates $\beta_I$ are the Plücker coordinates on $\mathbb{P}^{\wedge^k \mathbb{C}^n} \cong \mathbb{P}^{s-1}$. With this choice of basis for $\mathbb{C}^n$ matrix $M_x$ associated to the ordered basis $e_1, \ldots, e_n$ of $\mathbb{C}^n$. Plücker relations, i.e. conditions for dim(ker $\varphi_x$) = $k$, are vanishing conditions of all $(n-k+1) \times (n-k+1)$ minors of $M_x$. It is well known (see for instance [6]) that Plücker relations are degree 2 relations and they can also be written as

$$\sum_{l=0}^{k} (-1)^l \beta_{p_1 \ldots p_l q_1 \ldots q_l} = 0$$

for any 2k-tuple $(p_1, \ldots, p_{k-1}, q_0, \ldots, q_k)$.

**Remark 2.1.** Notice that vectors $\alpha_L$ in equation (1) normal to hyperplanes $D_L$ correspond to rows indexed by $L$ in the Plücker matrix $M_x$, that is

$$A(\mathcal{A}_x) = M_x \ ,$$

up to permutation of rows. Notice that, in particular, $\det(\alpha_{s_1}, \ldots, \alpha_{s_4}, \alpha_{s_{k+1}})$ is the Plücker coordinate $\beta_I$, $I = \{s_1, s_2, \ldots, s_{k+1}\} \setminus \{s_i\}$.

### 2.5. Relation between intersections of lines in $\mathcal{A}_x$ and quadrics in $Gr(3, \mathbb{C}^n)$

Let $\mathcal{A} = \{H_{I_1}^1, \ldots, H_{I_n}^n\}$ be a generic arrangement in $\mathbb{C}^3$. If there exist $L_1, L_2, L_3 \subset [n]$ subsets of indices of cardinality 4, such that codimension of $D_{L_1} \cap D_{L_2} \cap D_{L_3}$ is 2 then $\mathcal{A}$ is non very generic arrangement (see (2)). Let $T = \{L_1, L_2, L_3\}$ be a good 6-partition of indices $\{s_1, \ldots, s_6\} \subset [n]$. In [7], authors proved that the codimension of $D_{L_1} \cap D_{L_2} \cap D_{L_3}$ is 2 if and only if points $\cap_{I \in T} H_{\alpha_{s_i}}$, $\cap_{I \in T} H_{\alpha_{s_j}}$, and $\cap_{I \in T} H_{\alpha_{s_k}}$ are collinear in $H_\infty$ (Lemma 3.1 [7]).

Since $\alpha_L$ is vector normal to $D_L$, the codimension of $D_{L_1} \cap D_{L_2} \cap D_{L_3}$ is 2 if and only if rank $A_T(\mathcal{A}_x) = 2$, i.e. all $3 \times 3$ minors of $A_T(\mathcal{A}_x)$ vanish. In [10] authors proved the following Lemma.

**Lemma 2.2.** (Lemma5.3 [10]) Let $\mathcal{A}$ be an arrangement of $n$ hyperplanes in $\mathbb{C}^3$ and $\sigma, T = \{i_1, i_2, i_3, i_4\}$, $\{i_1, i_2, i_3, i_6\}, \{i_3, i_4, i_5, i_6\}$ a good 6-partition of indices $s_1 < \ldots < s_6 \in [n]$ such that $i_j = s_{\sigma(i_j)}$, $\sigma$ permutation in $S_6$. Then rank $A_{\sigma, T}(\mathcal{A}_x) = 2$ if and only if $\mathcal{A}$ is a point in the quadric of Grassmannian $Gr(3, \mathbb{C}^n)$ of equation

$$\beta_{i_1 i_2 i_3} \beta_{i_2 i_3 i_6} - \beta_{i_1 i_2 i_6} \beta_{i_2 i_3 i_6} = 0 \ .$$

As consequence of above results, we obtain correspondence between points $x = \sum_{I \subseteq [n]} \beta_I e_I, \beta_I \neq 0$, in quadric of equation (3) and generic arrangements of $n$ hyperplanes $\mathcal{A}$ in $\mathbb{C}^3$ such that $H_{\alpha_{s_1}}, H_{\alpha_{s_2}}, H_{\alpha_{s_3}}, H_{\alpha_{s_4}}$ and $H_{\alpha_{s_5}}, H_{\alpha_{s_6}}$ are collinear in $H_\infty$. Notice that condition $\beta_I \neq 0$ is direct consequence of $\mathcal{A}$ being generic arrangement.
In classical projective geometry the following theorem is known as Pappus’s theorem or Pappus’s hexagon theorem.

**Theorem 3.1 (Pappus).** On a projective plane, consider two lines $l_1$ and $l_2$, and a couple of triple points $A, B, C$ and $A', B', C'$ which are on $l_1$ and $l_2$ respectively. Let $X, Y, Z$ be points of $AB' \cap A'B, AC' \cap A'C$ and $BC' \cap B'C$ respectively. Then there exists a line $l_3$ passing through the three points $X, Y, Z$ (see Figure 7).

This theorem was originally stated by Pappus of Alexandria around 290-350 A.D.

In this section, we restate this classical theorem in terms of quadrics in Grassmannian. Indeed the six lines $AB', A'B, BC', B'C, AC', A'C \in \mathbb{P}^2(\mathbb{C})$ correspond to lines in the trace at infinity $\mathcal{A}_\infty$ of a generic arrangement $\mathcal{A}$ in $\mathbb{C}^3$ and lines $l_1, l_2$ and $l_3$ correspond to collinearity conditions for intersection points of lines in $\mathcal{A}_\infty$.

Consider a generic arrangement $\mathcal{A} = \{H_1, \ldots, H_6\}$ of 6 hyperplanes in $\mathbb{C}^3$, $\mathcal{A}_\infty$ its trace at infinity and $T = \{l_1, l_2, l_3\}$ the good 6-partition defined by $L_1 = \{1, 2, 3, 4\}$, $L_2 = \{1, 2, 5, 6\}$, $L_3 = \{3, 4, 5, 6\}$. By Lemma 2.2 we get that the triple points $\bigcap_{i \in L_1 \cap L_2} \bar{H}_i \cap H_\infty$, $\bigcap_{i \in L_2 \cap L_3} \bar{H}_i \cap H_\infty$, $\bigcap_{i \in L_3 \cap L_4} \bar{H}_i \cap H_\infty$ are collinear if and only if $\mathcal{A}$ is a point of the quadric

$$Q_1 : \beta_{134}\beta_{256} - \beta_{234}\beta_{156} = 0$$

in $\text{Gr}(3, \mathbb{C}^6)$.

Analogously if $T'' = \{L'_1, L'_2, L'_3\}$, $L'_1 = \{4, 6, 2, 5\}$, $L'_2 = \{4, 6, 1, 3\}$, $L'_3 = \{2, 5, 1, 3\}$ and $T''' = \{L''_1, L''_2, L''_3\}$, $L''_1 = \{2, 4, 1, 6\}$, $L''_2 = \{2, 4, 3, 5\}$, $L''_3 = \{1, 6, 3, 5\}$ are different good 6-partitions then triple points $\bigcap_{i \in L'_1 \cap L'_2} \bar{H}_i \cap H_\infty$, $\bigcap_{i \in L'_1 \cap L'_3} \bar{H}_i \cap H_\infty$, $\bigcap_{i \in L'_2 \cap L'_3} \bar{H}_i \cap H_\infty$, $\bigcap_{i \in L''_1 \cap L''_2} \bar{H}_i \cap H_\infty$, $\bigcap_{i \in L''_1 \cap L''_3} \bar{H}_i \cap H_\infty$, $\bigcap_{i \in L''_2 \cap L''_3} \bar{H}_i \cap H_\infty$ are collinear if and only if $\mathcal{A}$ is, respectively, a point of quadrics

$$Q_2 : \beta_{423}\beta_{613} - \beta_{623}\beta_{413} = 0$$

and

$$Q_3 : \beta_{214}\beta_{435} - \beta_{416}\beta_{235} = 0$$

With above remarks and notations we can restate Pappus’s theorem as follows (see Figure 7).
Theorem 3.2. (Pappus’s theorem) Let $\mathcal{A} = \{H_1, \ldots, H_n\}$ be a generic arrangement of hyperplanes in $\mathbb{C}^3$. If $\mathcal{A}$ is a point of two of three quadrics $Q_1, Q_2$ and $Q_3$ in Grassmannian $Gr(3, \mathbb{C}^6)$, then $\mathcal{A}$ is also a point of the third. In other words
\[
Q_i \cap Q_j = \bigcap_{l=1}^{3} Q_l, \quad \{i_1, i_2\} \subset [3].
\]

We develop this argument in the following sections.

4. Pappus Variety

In this section, we consider a generic arrangement $\{H_1, \ldots, H_n\}$ in $\mathbb{C}^3$ ($n \geq 6$). Let’s introduce basic notations that we will use in the rest of the paper.

Notation. Let $\{s_1, \ldots, s_6\}$ be a subset of indices $\{1, \ldots, n\}$ and $T = \{L_1, L_2, L_3\}$ be the good 6-partition given by $L_1 = \{s_1, s_2, s_3, s_4\}$, $L_2 = \{s_1, s_2, s_5, s_6\}$ and $L_3 = \{s_3, s_4, s_5, s_6\}$. Then for any permutation $\sigma \in S_6$ we denote by $\sigma T = \{\sigma L_1, \sigma L_2, \sigma L_3\}$ the good 6-partition given by subsets $\sigma L_1 = \{i_1, i_2, i_3, i_4\}$, $\sigma L_2 = \{i_5, i_2, i_5, i_6\}$, $\sigma L_3 = \{i_5, i_4, i_5, i_6\}$ with $i_j = s_{\sigma(j)}$. Accordingly, we denote by $Q_\sigma$ the quadric in $Gr(3, \mathbb{C}^n)$ of equation
\[
Q_\sigma : \beta_{i_1 j_1 k_1} \beta_{i_2 j_2 k_2} - \beta_{i_2 j_1 k_1} \beta_{i_1 j_2 k_2} = 0.
\]

The following lemma holds.

Lemma 4.1. Let $\sigma, \sigma' \in S_6$ be distinct permutations, then $Q_\sigma = Q_{\sigma'}$ if and only if there exists $\tau \in S_3$ such that $\sigma L_i \cap \sigma' L_j = \sigma' L_{\tau(i)} \cap \sigma L_{\tau(j)}$ ($1 \leq i < j \leq 3$).

Proof. By definition of good 6-partition we have that
\[
L_1 = (L_1 \cap L_2) \cup (L_1 \cap L_3),
L_2 = (L_2 \cap L_1) \cup (L_2 \cap L_3),
L_3 = (L_3 \cap L_1) \cup (L_3 \cap L_2).
\]

Then there exists $\tau \in S_3$ such that $\sigma$ and $\sigma'$ satisfy $\sigma L_i \cap \sigma' L_j = \sigma' L_{\tau(i)} \cap \sigma L_{\tau(j)}$ ($1 \leq i < j \leq 3$) if and only if $\sigma L_i = \sigma' L_{\tau(l)}$ for $l = 1, 2, 3$, that is $A_{\sigma' T}(\mathcal{A}_\infty)$ is obtained by permuting rows of $A_{\sigma' T}(\mathcal{A}_\infty)$. It follows that $\text{rank} A_{\sigma' T}(\mathcal{A}_\infty) = 2$ if and only if $\text{rank} A_{\sigma T}(\mathcal{A}_\infty) = 2$ and hence by Lemma 2.2 this is equivalent...
to $Q_\sigma \cap N_{s_1, \ldots, s_6} = Q_\sigma \cap N_{s_1, \ldots, s_6}$, where $N_{s_1, \ldots, s_6} = \{x = \sum_{l \in [n]} \beta_l e_l : \beta_l \neq 0 \text{ for any } l \in \{s_1, \ldots, s_6\}\}$. Since $N_{s_1, \ldots, s_6}$ is dense open set in $\gamma(Gr(3, \mathbb{C}^n))$, $Q_\sigma \cap N_{s_1, \ldots, s_6} = Q_\sigma \cap N_{s_1, \ldots, s_6}$ if and only if $Q_\sigma = Q_\sigma'$. Viceversa if $Q_\sigma \cap N_{s_1, \ldots, s_6} = Q_\sigma \cap N_{s_1, \ldots, s_6}$, then any generic arrangement $\mathcal{A}$ corresponding to a point in $Q_\sigma \cap N_{s_1, \ldots, s_6}$ corresponds to a point in $Q_\sigma \cap N_{s_1, \ldots, s_6}$, that is rank $A_{\sigma, T}(\mathcal{A}_w) = 2$ if and only if rank $A_{\sigma', T}(\mathcal{A}_w) = 2$. It follows that $A_{\sigma, T}(\mathcal{A}_w)$ and $A_{\sigma', T}(\mathcal{A}_w)$ are submatrices of $A(\mathcal{A}_w)$ defined by the same three rows, i.e. $\sigma L_l = \sigma' L_{l(i)}$ for $l = 1, 2, 3$.

**Definition 4.2.** For any 6 fixed indices $T = \{s_1, \ldots, s_6\} \subset [n]$ the Pappus Variety is the hypersurface in $Gr(3, \mathbb{C}^n)$ given by

$$\mathcal{P}_T = \bigcup_{\sigma \in S_6} \mathcal{Q}_\sigma \ .$$

For $\sigma, \sigma' \in S_6$ we define the equivalence relation $\sigma \sim \sigma'$ corresponding to $Q_\sigma = Q_\sigma'$ as following:

$$\sigma \sim \sigma' \iff \exists \tau \in S_3 \text{ such that } \sigma L_l \cap \sigma L_{l'} = \sigma' L_{l(i)} \cap \sigma' L_{l'(i)}(1 \leq i \leq 3) \ .$$

We denote by $[\sigma]$ the equivalence class containing $\sigma \sim$ and by $Q_\sigma$ the corresponding quadric (notice that $\sigma$ in notation $Q_\sigma$ can be any representative of $[\sigma]$). By Lemma 4.1 $[\sigma]$ only depends on couples $L_l \cap L_{l'}$ hence for each class $[\sigma]$ we can choose a representative $\tilde{\sigma} \in S_3 = \{(j_1, j_2, j_3, j_4), (j_1, j_2, j_4, j_5), (j_3, j_4, j_5, j_6)\}$ such that $j_1 < j_2, j_3 < j_4, j_5 < j_6$ and $j_1 < j_2 < j_3$ and we can equivalently define

$$[\sigma] = \{(j_1, j_2), (j_3, j_4), (j_5, j_6)\} \ .$$

Since the number of choices of $[\sigma]$ is $\frac{3!}{2!} = 15$, Pappus variety is composed by 15 quadrics. Finally remark that $[\sigma] = \{(j_1, j_2), (j_3, j_4), (j_5, j_6)\}$ and $[\sigma'] = \{(j'_1, j'_2), (j'_3, j'_4), (j'_5, j'_6)\}$ are disjoint, i.e. $[\sigma] \cap [\sigma'] = \emptyset$, if and only if $\{j_{2l-1}, j_2\} \neq \{j'_{2l-1}, j'_{2l}\}$ for any $1 \leq l, l' \leq 3$.

**Definition 4.3.** (Pappus configuration) Let $[\sigma_1], [\sigma_2]$ and $[\sigma_3]$ be disjoint classes, a Pappus configuration is a set $\{Q_{\sigma_1}, Q_{\sigma_2}, Q_{\sigma_3}\}$ of quadrics in $Gr(3, \mathbb{C}^n)$ such that

$$Q_{\sigma_1} \cap Q_{\sigma_2} = \bigcap_{i=1}^{3} Q_{\sigma_i} \ .$$

for any $\{i_1, i_2\} \subset [3]$. Quadrics $Q_{\sigma_1}, Q_{\sigma_2}, Q_{\sigma_3}$ are said to be in Pappus configuration if $\{Q_{\sigma_1}, Q_{\sigma_2}, Q_{\sigma_3}\}$ is a Pappus configuration.

**Remark 4.4.** Fixed a class of good 6-partition $[\sigma] = \{(j_1, j_2), (j_3, j_4), (j_5, j_6)\}$ we shall count the number of disjoint classes. First let’s count the number of classes $[\sigma'] = \{(j'_1, j'_2), (j'_3, j'_4), (j'_5, j'_6)\}$ not disjoint and distinct from $[\sigma]$. Since $[\sigma]$ and $[\sigma']$ are distinct, only one couple $\{j'_i, j'_{i+1}\}$ is contained in $[\sigma]$. Without loss of generality we can assume $\{j_i, j_{i+1}\} = \{j'_i, j'_2\}$ (is either 1, 3 or 5) then pairs $\{j'_3, j'_4\}$ and $\{j'_5, j'_6\}$ are not in the same set, i.e. we have two possibilities:

$$\{j'_3, j'_4\} \in [\sigma] \ ,$$

$$\{j'_5, j'_6\} \in [\sigma] \ .$$

Hence there are $2 \cdot 3 + 1 = 7$ not disjoint classes from $[\sigma]$ and, since the number of all classes is 15, we get that any fixed $[\sigma]$ admits exactly $15 - 7 = 8$ disjoint classes.
5. Pappus’s Theorem

In this section we restate Pappus’s Theorem for quadrics in $Gr(3, \mathbb{C}^n)$ by using notation introduced in previous section. For a fixed class $[\sigma] = \{[j_1, j_2], [j_3, j_4], [j_5, j_6]\}$ let’s denote by $G_{[\sigma]}$ the free group generated by permutations of elements in each subset of $[\sigma]$, that is

\[ G_{[\sigma]} = \langle (j_{2l-1}, j_{2l}) \in S_6 | l = 1, 2, 3 \rangle \]

and, for any class, $[\sigma']$ let’s define the set

\[ \text{orbit}_{G_{[\sigma]}}([\sigma']) = \{ \tau[\sigma'] | \tau \in G_{[\sigma]} \} \]

where $\tau$ acts naturally as permutation of entries of each set in $[\sigma']$.

**Remark 5.1.** The action of $G_{[\sigma]}$ on class $[\sigma']$ disjoint from $[\sigma]$ is faithful. Indeed let $\tau, \tau' \in G_{[\sigma]}$ be such that $\tau[\sigma'] = \tau'[\sigma']$ then $\tau^{-1}[\sigma'] = [\sigma']$, i.e. $\tau^{-1}\tau' \in G_{[\sigma]}$. Thus we get $\tau^{-1}\tau' \in G_{[\sigma]} \cap G_{[\sigma']}$. Since $[\sigma]$ and $[\sigma']$ are disjoint, $G_{[\sigma]} \cap G_{[\sigma']} = \{e\}$, i.e., $\tau = \tau'$. Remark that $|\text{orbit}_{G_{[\sigma]}}([\sigma'])| = |G_{[\sigma]}| = 8$ and $\tau[\sigma] = [\sigma]$ for any $\tau \in G_{[\sigma]}$.

**Lemma 5.2.** Let $[\sigma]$ and $[\sigma']$ be disjoint classes, then

\[ \text{orbit}_{G_{[\sigma]}}([\sigma']) = \{ [\sigma''] | [\sigma] \cap [\sigma''] = \emptyset \} \]

**Proof.** First we prove that $\text{orbit}_{G_{[\sigma]}}([\sigma']) \subseteq \{ [\sigma''] | [\sigma] \cap [\sigma''] = \emptyset \}$. Let $[\sigma] = \{[j_1, j_2], [j_3, j_4], [j_5, j_6]\}$ and $[\sigma'] = \{[j'_1, j'_2], [j'_3, j'_4], [j'_5, j'_6]\}$ be disjoint, then $|[j_{2l-1}, j_{2l}] \cap [j'_{2m-1}, j'_{2m}]| \leq 1$. Since $\tau \in G_{[\sigma]}$ permutes only $j_{2l-1}$ and $j_{2l}$ then $\tau[\sigma'] \cap [\sigma] = \emptyset$, that is $\tau[\sigma']$ is disjoint from $[\sigma]'$, i.e. $\tau[\sigma'] \in \{[\sigma''] | [\sigma] \cap [\sigma''] = \emptyset\}$. Since $G_{[\sigma]}$ is faithful, $|\text{orbit}_{G_{[\sigma]}}([\sigma'])| = 8$ and, by calculations in Remark 4.4, $|[\sigma'']| |[\sigma] \cap [\sigma''] = \emptyset| = 8$, it follows that $|\text{orbit}_{G_{[\sigma]}}([\sigma'])| = |[\sigma'']| |[\sigma] \cap [\sigma''] = \emptyset| = 8$. \qed

The following theorem holds.

**Theorem 5.3.** (Pappus’s Theorem) For any disjoint classes $[\sigma]$ and $[\sigma']$, there exists a unique class $[\sigma'']$ disjoint from $[\sigma]$ and $[\sigma']$ such that $\{Q_{[\sigma]}, Q_{[\sigma']}, Q_{[\sigma'']}\}$ is a Pappus configuration.

**Proof.** Following example in section 3, for any class $[\omega_1] = \{[j_1, j_2], [j_3, j_4], [j_5, j_6]\}$ let’s consider disjoint classes $[\omega_2] = \{[j_1, j_3], [j_2, j_4], [j_5, j_6]\}$ and $[\omega_3] = \{[j_1, j_6], [j_2, j_3], [j_4, j_5]\}$.

The corresponding quadrics have equations:

\[ Q_{\omega_1} : \beta_{j_1 j_2} \beta_{j_1 j_3} - \beta_{j_1 j_2} \beta_{j_1 j_3} = 0 \]
\[ Q_{\omega_2} : \beta_{j_2 j_3} \beta_{j_2 j_4} - \beta_{j_2 j_3} \beta_{j_2 j_4} = 0 \]
\[ Q_{\omega_3} : \beta_{j_3 j_4} \beta_{j_3 j_5} - \beta_{j_3 j_4} \beta_{j_3 j_5} = 0 \]

By definition of $\beta_{j_ik}$, equations of $Q_{\omega_2}$ and $Q_{\omega_3}$ can equivalently be written as

\[ Q_{\omega_2} : \beta_{j_1 j_2} \beta_{j_1 j_3} + \beta_{j_2 j_3} \beta_{j_2 j_4} = 0 \]
\[ Q_{\omega_3} : \beta_{j_1 j_3} \beta_{j_1 j_4} + \beta_{j_2 j_3} \beta_{j_2 j_4} = 0 \]

If we denote left side of defining equations of $Q_{\omega_3}$ by $P_{\omega_3}$ then

\[ P_{\omega_1} - P_{\omega_2} = P_{\omega_3} \]

that is zeros of any two polynomials $P_{\omega_1}, P_{\omega_2}$ are zeros of $P_{\omega_3}, \{i_1, i_2, i_3\} = \{1, 2, 3\}$. We get

\[ Q_{\omega_1} \cap Q_{\omega_2} = \bigcap_{i=1}^{3} Q_{\omega_i} \]
for any $\{i_1, i_2\} \subset [3]$, i.e. $Q_{o_1}, Q_{o_2}$ and $Q_{o_4}$ are in Pappus configuration. By Remark 4.4 the number of $Q_{o_1}$ is 15 and by Lemma 5.2 each fixed class $[\sigma]$ admits 8 disjoint classes. The number of Pappus configurations is 20.

Proof. By Remark 4.4 the number of $[\sigma]$ is 15 and by Lemma 5.2 each fixed class $[\sigma]$ admits 8 disjoint classes. By Theorem 5.3 if $[\sigma]$ and $[\sigma']$ are fixed, $[\sigma'']$ is uniquely determined, thus the number of the sets $[[\sigma], [\sigma'], [\sigma'']]$ is 15 $\times$ 8/3! = 20.

6. INTERSECTIONS OF QUADRICS

In this section we study intersections of quadrics in $Gr(3, \mathbb{C}^n)$. In particular we are interested in intersections of sets

$$Q^\circ_{\sigma} = Q_{\sigma} \cap \{ x = \sum_{\substack{I \subset [m] \; \text{such that} \; |I| = 3}} \beta_I e_I | \beta_I \neq 0, \; \text{for any} \; I \subset \{s_1, \ldots, s_6\} \}$$

of points in quadrics $Q_{\sigma}$ that correspond to arrangements of lines in $\mathbb{P}^2(\mathbb{C})$ with subarrangement $[H_{s_1}, \ldots, H_{s_6}]$ generic. The following lemma holds.

Lemma 6.1. If $[\sigma_1], [\sigma_2], [\sigma_3]$ are distinct and pairwise not disjoint classes then $Q^\circ_{\sigma_1} \cap Q^\circ_{\sigma_2} \cap Q^\circ_{\sigma_3} = \emptyset$.

Proof. If $[\sigma_1], [\sigma_2], [\sigma_3]$ are not disjoint then either

(1) $[\sigma_1] \cap [\sigma_2] \cap [\sigma_3] = 1$ or
(2) $[\sigma_i] \cap [\sigma_j] = 1$ (1 $\leq$ $i_1$ $<$ $i_2$ $\leq$ 3) and $[\sigma_1] \cap [\sigma_2] \cap [\sigma_3] = \emptyset$.

Assume $[\sigma_1] \cap [\sigma_2] \cap [\sigma_3] = \{i_1, i_2\}$. Let $[\sigma_1] = \{i_1, i_2, \{i_3, i_4, i_5, i_6\}\}$, $[\sigma_2] = \{i_1, i_2, \{i_3, i_5, i_4, i_6\}\}$, and $[\sigma_3] = \{i_1, i_2, \{i_3, i_5, i_4, i_6\}\}$ then we obtain the following quadrics
Any point \(x \in Q_{\sigma_1} \cap Q_{\sigma_3}^o\) belongs to \(Gr(3, \mathbb{C}^6)\), that is \(x\) satisfies Plücker relations in (2). In particular \(x \in P_{l_1} \cap P_{l_2}\) where \(P_{l_1}\) and \(P_{l_2}\) are the quadrics:

\[
\begin{align*}
Q_{\sigma_1} : & \quad \beta_{i_1i_2i_3} \beta_{i_4i_5i_6} - \beta_{i_1i_3i_4} \beta_{i_2i_5i_6} = 0 , \\
Q_{\sigma_2} : & \quad \beta_{i_1i_3i_4} \beta_{i_2i_5i_6} - \beta_{i_1i_2i_4} \beta_{i_3i_5i_6} = 0 , \\
Q_{\sigma_3} : & \quad \beta_{i_1i_2i_4} \beta_{i_3i_4i_5} - \beta_{i_1i_3i_5} \beta_{i_2i_4i_6} = 0 .
\end{align*}
\]

Notice that \(P_{l_1}\) and \(P_{l_2}\) can be obtained from equations in (2) considering the 6-tuples \( (p_1, p_2, q_0, q_1, q_2, q_3) = (i_1, i_2, i_3, i_4, i_5, i_6) \) and \((i_2, i_3, i_1, i_4, i_5, i_6)\) respectively. We get

\[
Q_{\sigma_2} - Q_{\sigma_1} - P_{l_1} + P_{l_2} = \beta_{i_1i_3i_4} \beta_{i_2i_5i_6} - \beta_{i_1i_2i_4} \beta_{i_3i_5i_6} + 2(\beta_{i_1i_3i_4} \beta_{i_2i_5i_6}) = 0 .
\]

Since \(\beta_{i_1i_2i_3} \neq 0\) and \(\beta_{i_4i_5i_6} \neq 0\) then \(\beta_{i_1i_2i_4} \beta_{i_3i_5i_6} \neq 0\) and hence \(\beta_{i_1i_3i_4} \beta_{i_2i_5i_6} - \beta_{i_1i_2i_4} \beta_{i_3i_5i_6} \neq 0\), that is \(x \notin Q_{\sigma_1}^o\).

Assume \([\sigma_1] \cap [\sigma_2] = \{i_1, i_2\}, [\sigma_1] \cap [\sigma_3] = \{i_3, i_4\}\) and \([\sigma_2] \cap [\sigma_3] = \{i_5, i_6\}\) and name \(P_1 = \{i_1, i_2\}, P_2 = \{i_1, i_4\}, P_3 = \{i_5, i_6\}\). To any point \(x \in Q_{\sigma_1}^o \cap Q_{\sigma_3}^o \) corresponds the existence of an arrangement with a generic sub-arrangement indexed by \(\{i_1, \ldots, i_6\}\) which trace at infinity \(\{H_{\infty,i_1}, \ldots, H_{\infty,i_6}\}\) satisfies collinearity conditions as in Figure 3. That is there exist couples \(P_1 \in [\sigma_1], P_2 \in [\sigma_2]\) and \(P_6 \in [\sigma_3]\) that correspond, respectively, to intersection points \(p_4, p_5\) and \(p_6\) of lines in \(\{H_{\infty,i_1}, \ldots, H_{\infty,i_6}\}\) (see Figure 3).

![Figure 3](image-url)

By definition of \(P_1, P_2\) and \(P_3\) we have

\[
P_3 = \{i_5, i_6\} \in (\{i_1, \ldots, i_6\}\setminus P_1) \cap (\{i_1, \ldots, i_6\}\setminus P_2) .
\]

On the other hand, if \(P_4\) is different from \(P_1\) and \(P_2\) in \(Q_{\sigma_1}^o\) then \(P_4 = (\{i_1, \ldots, i_6\}\setminus P_1) \cap (\{i_1, \ldots, i_6\}\setminus P_2)\). Thus we get \(P_3 = P_4\) and, similarly, \(P_5 = P_2\) and \(P_6 = P_1\), that is \(Q_{\sigma_1}^o = Q_{\sigma_2}^o = Q_{\sigma_3}^o\) which contradict hypothesis. \(\square\)
Lemma 6.2. For any three pairwise disjoint classes \([\sigma_1], [\sigma_2], [\sigma_3]\), either \([Q_{\sigma_1}, Q_{\sigma_2}, Q_{\sigma_3}]\) is a Pappus configuration or \(\bigcap_{i=1}^{3} Q_{\sigma_i} = \emptyset\).

Proof. By Pappus’s Theorem, for any two disjoint classes \([\sigma_i], [\sigma_j]\), there exists \([\sigma_{ij}]\) such that \([Q_{\sigma_i}, Q_{\sigma_j}, Q_{\sigma_{ij}}]\) is Pappus configuration. If \([\sigma_{ij}] = [\sigma_k]\) for some \(k \in [3]\), then \([Q_{\sigma_i}, Q_{\sigma_j}, Q_{\sigma_k}]\) is a Pappus configuration. Thus assume all \([\sigma_{ij}] \neq [\sigma_k]\) for any \(k = 1, 2, 3\). Moreover \([\sigma_{12}], [\sigma_{13}], [\sigma_{23}]\) are distinct since if \([\sigma_{13}] = [\sigma_k]\) then \([\sigma_1] = [\sigma_k]\).

If \([\sigma_{12}] \cap [\sigma_{13}] \neq \emptyset, [\sigma_{12}] \cap [\sigma_{23}] \neq \emptyset\) and \([\sigma_{13}] \cap [\sigma_{23}] \neq \emptyset\), then \(\bigcap_{1 \leq i \leq 3} Q_{\sigma_{i23}} = \emptyset\) by Lemma 6.1 and

\[
\bigcap_{i=1}^{3} Q_{\sigma_i} = \left(\bigcap_{i=1}^{3} Q_{\sigma_i}\right) \cap \left(\bigcap_{1 \leq i < j \leq 3} Q_{\sigma_{ij23}}\right) = \emptyset.
\]

Otherwise assume \([\sigma_{12}] \cap [\sigma_{13}] = \emptyset\), we get a new Pappus configuration. Since the number of disjoint classes is finite, iterating the process, we will eventually get 3 classes \([\sigma_{1i}], [\sigma_{ij}], [\sigma_{ji}]\) pairwise not disjoint and \(\bigcap_{i=1}^{3} Q_{\sigma_i} = \emptyset\).

Lemma 6.3. If \([\sigma_1], [\sigma_2], [\sigma_3]\) are distinct classes such that \([\sigma_1] \cap [\sigma_2] \neq \emptyset\) and \([\sigma_1] \cap [\sigma_3] = \emptyset\) for \(i = 1, 2, 3\), then \(\bigcap_{i=1}^{3} Q_{\sigma_i} = \emptyset\).

Proof. Since \([\sigma_1], [\sigma_2], [\sigma_3]\) are disjoint, there exist \([\sigma_4]\) and \([\sigma_5]\) such that \([Q_{\sigma_1}, Q_{\sigma_2}, Q_{\sigma_4}]\) and \([Q_{\sigma_1}, Q_{\sigma_3}, Q_{\sigma_5}]\) are Pappus configurations and

\([\sigma_1] \cap [\sigma_5] \neq \emptyset, [\sigma_2] \cap [\sigma_4] \neq \emptyset, [\sigma_3] \cap [\sigma_4] \neq \emptyset\).

Indeed if one of them is empty, we obtain 3 disjoint classes not in Pappus configuration and by Lemma 6.2 it follows \(\bigcap_{i=1}^{3} Q_{\sigma_i} = \emptyset\). Since \([\sigma_1] \cap [\sigma_2] \neq \emptyset\), we can assume \([i_1, i_2] = [\sigma_1] \cap [\sigma_2]\) and we can set

\([\sigma_1] = \{[i_1, i_2], [i_3, i_4], [i_5, i_6]\}, [\sigma_2] = \{[i_1, i_2], [i'_3, i'_4], [i'_5, i'_6]\}, [\sigma_3] = \{[j_1, j_2], [j_3, j_4], [j_5, j_6]\}\).

To any point \(x \in \bigcap_{i=1}^{3} Q_{\sigma_i} \neq \emptyset\) corresponds an arrangement \(A\) with generic subarrangement \([H_{i_1}, \ldots, H_{i_6}]\) with trace at infinity \([H_{i_3,j_1}, \ldots, H_{i_6,j_4}]\) intersecting as in Figures 4 and 5 (up to rename). It follows that \([j_4, j_5] \in [\sigma_4]\) and since \([j_3, j_5] = [i_1, i_2] \in [\sigma_1]\) and \([\sigma_1] \cap [\sigma_4] = \emptyset\) (see Figure 4), there are two possibilities:

\([\sigma_4] = \{[j_4, j_5], (j_1, j_3), (j_2, j_3)\}
\]
or
\([\sigma_4] = \{[j_4, j_5], (j_1, j_5), (j_2, j_3)\}\).

Analogously (see Figure 5) class \([\sigma_5]\) is of the form

\([\sigma_5] = \{[j_4, j_5], (j_1, j_3), (j_2, j_3)\}
\]
or
\([\sigma_5] = \{[j_4, j_5], (j_1, j_5), (j_2, j_3)\}.\)
Figure 4. each \( j, j' \) is \( j_1 \) or \( j_2 \).

Figure 5. each \( j, j' \) is \( j_1 \) or \( j_2 \).

Since \([\sigma_1] \cap [\sigma_5] \neq \emptyset\) and \([\sigma_5] \neq \{j_3, j_5\} = \{i_1, i_2\}\), we deduce that \( \{j_4, j_6\} = \{i_3, i_4\} \) or \( \{i_5, i_6\} \), which is not possible by \([\sigma_1] \cap [\sigma_4] = \emptyset\). Hence \( \bigcap_{i=1}^{3} Q^{\sigma_i} = \emptyset \).

Notice that the Hesse arrangement in \( \mathbb{P}^2(\mathbb{C}) \) (see Figure 6) can be regarded as a generic arrangement of 6 lines which intersection points satisfy 6 collinearity conditions.

**Definition 6.4.** (Hesse configuration) Let \([\sigma_i], 1 \leq i \leq 6\) be distinct classes, we call Hesse configuration a set \( \{Q_{\sigma_1}, \ldots, Q_{\sigma_6}\} \) of quadrics in \( Gr(3, \mathbb{C}^n) \) such that there exist disjoint sets \( I, J \subset [6], |I| = |J| = 3 \) such that \( \{Q_{\sigma_i}\}_{i \in I}, \{Q_{\sigma_j}\}_{j \in J} \) are Pappus configurations and \([\sigma_i] \cap [\sigma_j] \neq \emptyset\) for any \( i \in I, j \in J \).
With above notations, the following classification Theorem holds.

**Theorem 6.5.** For any choice of indices \( \{s_1, \ldots, s_6\} \subset [n] \) sets \( Q^\sigma_{r_i} \), \( \sigma \in S_6 \), in Grassmannian \( \text{Gr}(3, \mathbb{C}^n) \) intersect as follows.

1. For any disjoint classes \([\sigma_1]\) and \([\sigma_2]\), there exist \([\sigma_3], \ldots, [\sigma_6]\) such that \( \{Q^\sigma_{r_1}, \ldots, Q^\sigma_{r_6}\} \) is an Hesse configuration for \( I = \{1, 2, 3\} \), \( J = \{4, 5, 6\} \) and
   \[
   \bigcap_{i=1}^2 Q^\sigma_{r_i} \supseteq \bigcap_{i=1}^3 Q^\sigma_{r_i} \supseteq \bigcap_{i=1}^4 Q^\sigma_{r_i} \supseteq \bigcap_{i=1}^6 Q^\sigma_{r_i} \neq \emptyset .
   \]

2. For any not disjoint classes \([\sigma_1]\) and \([\sigma_2]\), there exist \([\sigma_3], \ldots, [\sigma_6]\) such that \( \{Q^\sigma_{r_1}, \ldots, Q^\sigma_{r_6}\} \) is an Hesse configuration for \( I = \{1, 3, 4\} \), \( J = \{2, 5, 6\} \) and
   \[
   \bigcap_{i=1}^2 Q^\sigma_{r_i} \supseteq \bigcap_{i=1}^3 Q^\sigma_{r_i} \supseteq \bigcap_{i=1}^4 Q^\sigma_{r_i} \supseteq \bigcap_{i=1}^6 Q^\sigma_{r_i} \neq \emptyset .
   \]

All other intersections are empty.

**Remark 6.6.** Notice that, since Hesse configuration only exists in complex case, in \( \text{Gr}(3, \mathbb{C}^n) \) we can find 6 quadrics \( \{Q^\sigma_{r_1}, \ldots, Q^\sigma_{r_6}\} \) such that
   \[
   \bigcap_{i=1}^6 Q^\sigma_{r_i} \neq \emptyset ,
   \]
while in \( \text{Gr}(3, \mathbb{R}^n) \),
   \[
   \bigcap_{j \in J \setminus \{6\}, \#J \geq 4} Q^\sigma_{r_j} = \emptyset .
   \]

It follows that in real case, for any choice of indices \( \{s_1, \ldots, s_6\} \subset [n] \), we have at most 4 collinearity conditions (see Figure 7) corresponding to 15 hyperplanes in Discriminantal arrangement with 4 multiplicity 3 intersections in codimension 2 (see Figure 8). While in complex case Hesse configuration (see Figure 6) gives rise to a Discriminantal arrangement containing 15 hyperplanes intersecting in 6 multiplicity 3 spaces in codimension 2.
This remark allows a better understanding of differences in combinatorics of Discriminantal arrangement in real and complex case. Moreover those observations suggest that some special configuration of lines in projective plane intersecting in a big number of triple points could be understood by studying Discriminantal arrangements with maximum number of multiplicity 3 intersections in codimension 2.

**Figure 7.** Generic arrangement $\mathcal{A}$ in $\mathbb{R}^3$ containing 6 lines satisfying 4 collinearity conditions.

**Figure 8.** Codimension 2 intersections of 15 hyperplanes in $\mathcal{B}(n, 3, \mathcal{A}_\infty)$ indexed in $\{s_1, \ldots, s_6\} \subset [n]$ with 4 multiplicity 3 points ▲ corresponding to intersections $\cap_{i=1}^{3} D_{\sigma_i L_i}$, $\cap_{i=1}^{3} D_{\sigma_i' L_i}$, $\cap_{i=1}^{3} D_{\sigma_i'' L_i}$ and $\cap_{i=1}^{3} D_{\sigma_i''' L_i}$, $\sigma, \sigma', \sigma'', \sigma'''$ as in Figure 7.

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Department of Mathematics, Hokkaido University, Japan.
E-mail address: b.lemon329@gmail.com
E-mail address: s.settepanella@math.sci.hokudai.ac.jp
E-mail address: so.yamagata.math@gmail.com