Classical and quantum behavior of the generic cosmological solution

Giovanni Imponente 1,† and Giovanni Montani 2,‡

Centro Studi e Ricerche “Enrico Fermi”, Compendio Viminale – 00186 Roma, Italy
†Dipartimento di Fisica, Università “La Sapienza”, Piazzale A.Moro 5, 00185 Roma, Italy
ENEAC.R. Frascati (U.T.S. Fusione), Via E. Fermi 45, 00044 Frascati, Roma, Italy
‡ICRA — International Center for Relativistic Astrophysics,
Dipartimento di Fisica (G9), Università “La Sapienza”, Piazzale A.Moro 5, 00185 Roma, Italy

Abstract. In the present paper we generalize the original work of C.W. Misner [1] about the quantum dynamics of the Bianchi type IX geometry near the cosmological singularity. We extend the analysis to the generic inhomogeneous universe by solving the super-momentum constraint and outlining the dynamical decoupling of spatial points. Firstly, we discuss the classical evolution of the model in terms of the Hamilton-Jacobi approach as applied to the super-momentum and super-Hamiltonian constraints; then we quantize it in the approximation of a square potential well after an ADM reduction of the dynamics with respect to the super-momentum constraint only. Such a reduction relies on a suitable form for the generic three-metric tensor which allows the use of its three functions as the new spatial coordinates.

We get a functional representation of the quantum dynamics which is equivalent to the Misner-like one when extended point by point, since the Hilbert space factorizes into $\infty^3$ independent components due to the parametric role that the three-coordinates assume in the asymptotic potential term.

Finally, we discuss the conditions for having a semiclassical behavior of the dynamics and we recognize that this already corresponds to having mean occupation numbers of order $O(10^2)$.

Keywords: Generic Cosmology, Early Universe, Quantum Cosmology

PACS: PACS 04.20.Jb, 83.C, 98.80.Bp-Cq

1. INTRODUCTION

The evolutionary scheme of the universe is well described by the Standard Cosmological Model (SCM) from the light-elements nucleosynthesis ($10^3 - 10^2$ sec.) up to the present stage of the observations; however, the very early phases of the cosmological dynamics (i.e. above $10^{14} - 10^{15}$ GeV) requires more general schemes for being completely understood.

In fact, either the backward instability of the isotropic universe with respect to tensor perturbations [2, 3], either the impossibility to preserve symmetry in quantum gravity, strongly support the idea that the early universe must be represented by a generic inhomogeneous model, as discussed in [4, 5].

As well known, the point-like dynamics of such a generic picture closely resembles the one characterizing the Bianchi type VIII and IX geometries, i.e. the so-called Mixmaster [6]. First investigations about the classical and quantum behavior of the Mixmaster were provided by C.W. Misner in [6] and [1], respectively; his analysis relies on a Hamiltonian formulation of the dynamics and on a suitable approximation of the spatial curvature which here plays the role of a potential term for the generalized coordinates (universe volume and anisotropies).

The present work aims to generalize the main features of the Misner classical and quantum approaches toward the generic inhomogeneous universe, by virtue of the dynamical decoupling characterizing spatial points near the Big Bang.

The classical inhomogeneous dynamics is investigated in Section 2 via a Hamilton-Jacobi approach to the fundamental constraints of the theory; the solution outlines the right number of four physically arbitrary functions of the spatial coordinates in the configuration space as required by the generality.

In Section 3 before proceeding to the quantization, we perform an Arnowitt-Deser-Misner (ADM) reduction of the
super-momentum constraints; in this way, we remove any dynamical role of the spatial gradients from the quantum behavior. Indeed, asymptotically to the singularity, the three coordinates enter the potential term as parameters only and then we can apply the long-wavelength approximation. Such a framework has the physical meaning of dealing with inhomogeneous scales which are super-horizon sized; thus, when below we will speak about spatial points, we will refer to causal regions of the universe which asymptotically are dynamically decoupled. This fact provides a sort of causal structure to our quantization procedure in the sense that no-interference phenomena take place on super-horizon scales.

In Section 4, we give a quantum representation of the dynamics in terms of a functional approach which outlines the factorization of the Hilbert space into $\infty^3$ point-like components. The quantum dynamics relies on an adiabatic approximation ensured by the potential term behavior, which reduces to an infinite well; according to C.W. Misner [6], we model the potential as an infinite square box having the same measure of the original triangular picture. The volume-dependence of the wave functional acquires increasing amplitude and frequency of oscillations as the Big Bang is approached and the occupation number grows, respectively.

Finally, in Section 5, we discuss the conditions under which the WKB semiclassical limit is recognized. We show that this corresponds to having occupation numbers sufficiently greater than unity and therefore semiclassical packets of eigenstates would correspond to average mean occupation number of order $O(10^2)$.

2. GENERIC COSMOLOGICAL MODEL

When considering a cosmological solution containing a number of space functions such that a generic inhomogeneous Cauchy problem is satisfied on a non-singular hypersurface, we refer to it as a generic inhomogeneous model [4, 5]. In the Arnowitt-Deser-Misner (ADM) formalism, the corresponding line element reads as

$$ds^2 = N \gamma^{ij} dx_i dx_j + \gamma_{ij} (dx^i + N^i dt) (dx^j + N^j dt);$$

where $N$ is the lapse-function, $N^i$ the shift-vectors and $q_a, \gamma; x^i$ three scalar functions (the index $a$ is summed over $1, 2, 3$); the vectors $l_a^i$ have components which are generic functions of the spatial coordinates only, available for the Cauchy data. The general case, in which $l_a^i$ are time-dependent vectors is addressed in a following Section to simplify the variational principle. It is convenient to introduce also the reciprocal vectors $l_a^i$, such that $l_i^a l_i^b = \delta^a_b$ and $l_i^a l_i^b = \delta^b_j$. The dynamics of this system is summarized by the action

$$S = \int d^3 x dt \sum_{a} p^a \partial_t q_a - H - NH,$$

$$H = \frac{\chi}{2} \sum_{a} \gamma^{ij} \left( \sum_{m, n} p^m p^n \right) p^j \partial_q q^i,$$

$$H_i = 2\chi \alpha \sum_{m} l_i^m l_i^m p^m \chi \partial_j q^i,$$

being $p^a$ the conjugate momenta to the variables $q_a$, $\gamma = \det \gamma$, $\gamma^{ij}$ the tridimensional Ricci scalar and $\chi$ the Einstein constant. By varying the action (3) with respect to the $N$ and $N^i$, we get the super-Hamiltonian and super-momentum constraints $H = 0$ and $H_i = 0$, respectively.

Let us introduce the Misner-like variables $\alpha, \beta; x^i$ via the transformation

$$q_1 = 2\alpha + 2\beta, \quad q_2 = 2\alpha + 2\beta, \quad q_3 = 2\alpha + 4\beta.$$
in terms of this set of configurational coordinates the Hamiltonian constraints rewrite as

\[ H = \chi e^{3\alpha} \left( \frac{\dot{p}_a}{2\hbar} + p_+^2 + p_- \right) V \]  
\[ H_i = \chi \partial_i \left( \frac{p_a}{3} + \frac{p_+}{6} + \frac{p_- \chi^2}{3} p_+ \right) + \frac{1}{6} \partial_j \left( \frac{p_+}{\beta_+} \right) p_+ + \frac{2}{3} \frac{p_- \chi^2}{\beta_+} \frac{p_+}{p_-} + \frac{2}{3} l_i^j l_j^k \frac{\partial S}{\partial p_-} \]  
\[ V = \frac{\chi}{2\chi} (\partial R) \]  

A detailed analysis of the potential term \( V \) leads to

\[ V = \frac{e^{\alpha}}{4\chi} \left[ a_3^i (\chi') e^{8\beta} + a_2^j (\chi') e^{4\beta} + \frac{\chi}{3} \right] + a_2^j (\chi') e^{4\beta} + a_3^i (\chi') e^{8\beta} + W (\chi'; \alpha; \beta \cdot \partial_j \alpha; \partial_j \beta) \circ ] ; \]  

where \( a_i \) refer to the space quantities

\[ a_i (\chi') \]  

and we regard the operations \( \partial \) and \( \text{rot} \) as taken in Euclidean sense.

To outline the relative behavior of the two terms in the potential as the singularity is approached for \( \alpha \) \( \to \infty \), let us consider the quantities

\[ D = e^{3\alpha} \]  
\[ H_1 = \frac{1}{3} \left( 1 + \frac{\beta_+}{3\alpha} \right) \]  
\[ H_2 = \frac{1}{3} \left( 1 + \frac{\beta_+}{3\alpha} \right) \]  
\[ H_3 = \frac{1}{3} \left( 1 + \frac{\beta_+}{3\alpha} \right) \]  
\[ \sum H_i = 1 \]  

Taking into account these definitions, the potential \( V \) rewrites as

\[ V = \sum a_i^j D^{H_i} + W \]  
\[ W \sum H_j + H_k : \]  

Near the cosmological singularity \( \alpha \) \( \to \infty \) and \( D \to 0 \), so that the term \( W \) becomes negligible. Indeed this conclusion is supported by the behavior of the spatial gradients, which does not destroy the feature above outlined (see [7, 8]).

Through the canonical replacements

\[ p_\alpha = \frac{\partial S}{\partial \alpha} \]  
\[ p_\beta = \frac{\partial S}{\partial \beta} \]  

the classical evolution is summarized by the Hamilton–Jacobi system

\[ \frac{\partial S}{\partial \alpha} + \frac{\partial S}{\partial \alpha} + \frac{\partial S}{\partial \beta} + V (\alpha; \beta, \beta) = 0 \]  
\[ \frac{1}{6} \partial_j \left( \frac{\partial S}{\partial \alpha} + \frac{\partial S}{\partial \beta} + \frac{\chi}{3} \frac{\partial S}{\partial \beta} \right) + \partial_j \left( \frac{\partial S}{\partial \beta} + \frac{\chi}{3} \right) + \frac{2}{3} l_i^j l_j^k \frac{\partial S}{\partial \beta} + \frac{\chi}{3} l_i^j l_j^k \frac{\partial S}{\partial \beta} + \psi (\alpha) \frac{\partial S}{\partial \alpha} + \psi (\beta) \frac{\partial S}{\partial \beta} + \phi (\beta) \frac{\partial S}{\partial \beta} = 0 \]
Since sufficiently close to the cosmological singularity the potential term becomes negligible, then the solution of (17a) reads as

\[ S = \frac{q}{k^2 + k^2} \alpha + k_+ \beta_+ + k \beta ; \]  

where \( k = k(\nu') \) are arbitrary functions of the coordinates and the minus sign in front of the square root has been taken because we are considering an expanding Universe.

According to the Jacobi prescription, the functional derivatives of the above action \( \alpha \) with respect to \( k \) have to be set equal to stationary quantities \( c(\nu') \) and therefore we get the following expressions for \( \beta \) in terms of \( \alpha \)

\[ \beta = \pi \alpha ; \]  

where

\[ \pi = \frac{k}{k^2 + k^2} \]  

(20a)

\[ \pi^2 + \pi^2 = 1 ; \]  

(20b)

Substituting the solution (18) with (19) in the Hamilton–Jacobi equation (17b) corresponding to the super-momentum and taking into account the relations (20), the last sum cancels out leaving the equations

\[ 2 \frac{k_+ + k}{k_+^2 + k^2} \partial_i (k_+ + k) + \partial_j k_+ + \partial_{ij} \frac{h}{l_+^2} k_+ + \frac{\hbar}{\pi k} + 2 \frac{\pi}{\pi} = 0 ; \]  

(21)

which are constraints on the spatial functions only. The above mentioned functions \( c(\nu') \) have been set equal to zero because their presence would simply correspond to a rescaling of the vectors \( l^i(\nu') \). Thus, our solution contains ten arbitrary functions of the spatial coordinates, namely the nine components of the vectors \( l^i \) and one of the two functions \( \pi \). These ten free functions have to satisfy the three constraints (21); the choice of the coordinate frame eliminates the arbitrariness of three more degrees of freedom. Therefore our solution is characterized by *four physically* arbitrary functions of the spatial coordinates and, in this sense, it is a generic one.

Above we neglected the role of the potential because it influences the point-Universe evolution only via the bounces producing the establishment of a new free motion (for a detailed discussion about the chaotic properties of the random behavior that the point-universe performs in the potential, see [9, 10, 11]). This effect of the potential is summarized by the reflection law

\[ \sin \theta_f = \sin \theta = \frac{1}{2} \sin (\theta_f + \theta_i) ; \]  

(22)

where \( \theta_i \) and \( \theta_f \) denote the angles of incidence and deflection, respectively, of the point-Universe for a bounce on one of the three equivalent walls of the triangular potential \( V(\alpha; \beta_+ ; \beta) \), see Figure 1.

The results of this Section show how the generic cosmological solution toward the Big-Bang is isomorphic, point by point in space, to the one of the Bianchi types VIII and IX models [12] because the spatial coordinates are involved in the problem only as parameters.

### 3. ADM REDUCTION OF THE DYNAMICAL PROBLEM

In the last Section we have discussed a generic cosmological model in which the vectors \( l^i \) appearing in (23) were functions of the coordinates \( x^i \) only. It is interesting to discuss a more generic framework, that is to allow the more general dependence \( l^i = l^i(\nu; x^i) \).

To this purpose, let us rewrite such vectors in the form (13)

\[ l^i = O^i_0 \partial^j y^b ; \]  

(23)

where \( O^i_0 = O^i_0(\nu; x^i) \) is a \( SO(3) \) matrix and \( y^b = y^b(\nu; x^i) \) are three scalar functions. Considering the metric tensor (4) rewritten as

\[ \gamma_{ij} = \delta_{\nu b} O^b_0 O^d_0 \partial^j y^b \partial^j y^c ; \]  

(24)
FIGURE 1. Equipotential lines showing the potential domain $\Gamma_H$, dynamically closed in the corners: this visualization is in the $\beta_+, \beta_-$ plane of the Misner variables. Asymptotically, the walls flatten to a triangular well, as in Equation (30).

with the indexes $(a;b;c;d;j;i;j)$ running as $1;2;3$, the action (3) becomes

$$S = \sum_{(3)} \int d^3x dt \ p_a \partial_t q^d + \Pi_d \partial_t y^d \ NH \ \hat{N} H_i;$$

(25)

where, apart some details of the potential coordinate dependence, $H$ has the same expression as (4), while the super-momentum explicitly reads

$$H_i = \Pi_c \partial_i y^c + p_a \partial_i q^a + 2p_a \ O \ {^1}_a \partial^c \ O^b_c;$$

(26)

$\Pi_d$ being the conjugate momenta of the variables $y^d$.

Summarizing, we have ten independent functions characterizing the dynamical system: the 3 degrees of freedom of the coordinate choice $y^a$, 3 scale factors $q^a$, 3 components of the shift vector $N^i$ and the lapse function $N$.

The variation of the action (25) with respect to $N$ and $N_i$ leads to the usual constraints $H = 0$ and $H_i = 0$, respectively.

Choosing $y^d$ as spatial coordinates, i.e. $y^d = y^d (\eta;x)$ and $\eta = t$, we can diagonalize and solve the constraint (26) as follows

$$\Pi_b = \frac{\partial q^b}{\partial y^b} \ 2p_a \ O \ {^1}_a \partial^c \ O^b_c;$$

(27)

Given $\mathcal{J}$ as the Jacobian of the change of coordinates, the relations

$$q^a (\eta;x) \nabla q^b (\eta;y)$$

$$p_a (\eta;x) \nabla p^b (\eta;y) = \frac{p_a (\eta;y)}{\mathcal{J}}$$

$$\frac{\partial}{\partial t} \nabla \frac{\partial y^b}{\partial \eta} + \frac{\partial}{\partial \eta}$$

(28)

allow to write the (reduced) action as

$$S_{RED} = \sum_{(3)} \int d^3\eta d^3y \ p_a \partial_\eta q^a + 2p_a \ O \ {^1}_a \partial_\eta O^b_c \ NH;$$

(29)

When approaching the cosmological singularity, the potential term (14a) can be expressed as

$$V = \sum_a \Theta (H_a);$$

(30)
being
\[ \Theta(x) = \begin{cases} +\infty & \text{if } x > 0; \\ 0 & \text{if } x < 0; \end{cases} \] (31)

The dynamics of the universe is decoupled for each space point and the point-universe moves in the domain \( \Gamma_H \) as outlined by the triangular equipotential lines of Figure 1, which is dynamically closed also in the corners.

The vanishing behavior of the potential inside the domain of definition implies that, near the singularity, \( \partial p_a \partial \eta = \partial \eta \partial q^a = 0 \). Therefore the term \( p_a O^1 \partial \eta O^a \) in (29) reads as an exact time differential term, and unaffects the asymptotic dynamics whose variational principle takes the form (13)
\[
S_{RED}^0 = \int_{\Sigma^0} \frac{d\eta d^3 y}{\mathcal{R} d^{3} \eta d^{3} q^{a}} (p_a \partial \eta q^a) \; \text{NH}; \quad (32)
\]
and the dynamics is described by \( H \) only.

\section*{4. GENERIC QUANTUM COSMOLOGY}

Let us now analyze the quantum dynamics corresponding to a generic cosmological model within a canonical framework, by replacing the canonical variables with the corresponding operators and implementing the Hamiltonian constraints on the state functional describing the system, \( \Psi = \Psi(\alpha; \beta_+; \beta_-) \). Therefore we adopt the representation
\[
p_\alpha \delta \frac{\delta}{\delta \alpha}; \quad p_\beta \delta \frac{\delta}{\delta \beta}; \quad (33)
\]
and implementing \( H \) to an operator the quantum dynamics is described by the Wheeler-deWitt equation
\[
\hat{H} \psi = \chi e^{3\alpha} \hbar^2 \left( \frac{\delta^2}{\delta \alpha^2} + \frac{\delta^2}{\delta \beta_+^2} + \frac{\delta^2}{\delta \beta_-^2} \right) \psi \quad \delta^4 V \psi = 0; \quad (34)
\]
As discussed at the end of the preceding Section, since the asymptotically the potential term remains isomorphic to the Bianchi IX one, point by point in space, we can find a solution in the form
\[
\psi = \sum_{n} \Gamma_n(\alpha) \varphi_n(\alpha; \beta_+; \beta_-); \quad (35)
\]
where the coefficients \( \Gamma_n \)
\[
\Gamma_n = \exp \int d^3 x \ln C_n(\alpha) = \prod_{x^i} C_n(\nu^i)(\alpha) \quad (36)
\]
keep trace of the inhomogeneities of the system and the index \( n \) has to be regarded as a space function \( n(\nu^i) \).

Thus, equation (34) is reduced to the ADM eigenvalue problem
\[
\delta^2 \frac{\delta^2}{\delta \beta_+^2} + \delta^2 \frac{\delta^2}{\delta \beta_-^2} + e^{4\alpha} V \phi = E_n^2(\alpha) \phi; \quad (37)
\]
According to \( \Pi \) we approximate the triangular infinite walls of the potential by a box having the same measure and then we find the eigenvalues \( E_n \)
\[
E_n(\alpha) = \pi \frac{4}{3s^2} \frac{1}{4} \frac{(\hbar n)^2}{\alpha}; \quad (38)
\]
where \( n^2 = n_+^2 + n_-^2 \), being \( n_+ \neq n_+ \), the two independent quantum numbers corresponding to the variables \( \beta_+; \beta_- \), respectively.

Substituting the expression for \( \Gamma_n \) in equation (34) we get the differential equation
\[
\sum_n \partial_\alpha C_n(\varphi_n) + \sum_n C_n(\partial_\alpha \varphi_n) + 2 \sum_n (\varphi_n C_n) (\partial_\alpha \varphi_n) + \sum_n E_n^2 C_n \varphi_n = 0; \quad (39)
\]
FIGURE 2. Behavior of the solution $C_n(\alpha)$ for three different values of the parameter $k_n = 1; 15; 30$. The bigger $k_n$, the higher the frequency of oscillation.

which, in the limit of the Misner adiabatic approximation of neglecting $\partial_\alpha \phi_n$ (i.e. $\phi(\beta, i\beta)$), simplifies to

$$h^2 \frac{d^2 C_n}{d\alpha^2} + \frac{k_n^2}{\alpha^2} C_n = 0$$

(40)

where

$$k_n^2 = \frac{2}{3\pi} \frac{3^{-2}}{j n \frac{3}{\hbar}}.$$ 

(41)

The above equation is solved by $C_n(\alpha)$ in the form

$$C_n(\alpha) = C_1 \frac{\rho}{\alpha} \sin \left( \frac{1}{2} \frac{\rho}{p_n \ln \alpha} \right) + C_2 \frac{\rho}{\alpha} \cos \left( \frac{1}{2} \frac{\rho}{p_n \ln \alpha} \right);$$

(42)

where $\frac{\rho}{p_n} = \frac{\rho}{p_n}$. Figure 2 shows the behavior of $C_n(\alpha)$ for various values of the parameter $k_n$. Such wave function at fixed $x'$ behaves like an oscillating profile whose frequency increases with occupation number $n$ and approaching the Big Bang, while the amplitude depends on the $\alpha$ variable only.

5. SEMICLASSICAL LIMIT OF THE QUANTUM DYNAMICS

It is now interesting to consider the evolution of the system in the semiclassical limit, that is to investigate the dynamical behavior in the vicinity of the Big Bang within the WKB approximation of the dynamics (for a discussion of the semiclassical limit in the homogeneous case, see [14]). To this purpose, let us search for a solution of the wave equation (40) in the form

$$C(\alpha) = \frac{\rho e^{i\sigma}}{\rho};$$

(43)

which reduces such equation to the system of the real and complex parts as

$$\frac{d\sigma}{d\alpha} = \frac{\rho}{2\alpha} \frac{d\rho}{d\alpha} \ln \alpha + \frac{1}{2} \frac{d^2 \rho}{d\alpha^2} = 0$$

(44a)

$$\hbar \frac{d}{d\alpha} \rho \frac{d\sigma}{d\alpha} = 0;$$

(44b)
respectively. In the semiclassical limit, neglecting the term in square brackets in Equation (44), we obtain the solution

\[
\sigma(\alpha) = k \ln \alpha; \\
\rho(\alpha) = \text{const}; \quad \alpha,
\]

This behavior remains valid as far as the term in $\hbar^2$ is negligible and therefore we get the constraint

\[
k^2 \frac{\hbar^2}{4};
\]

i.e. $n = 1$. As soon as this constraint is fulfilled, the solution remains semiclassical close to the Big Bang and we see that, in the sense of the WKB approximation, this would imply packets of eigenfunctions whose mean quantum number $<n>$ can be of order $\sim (10^2)$ only.

### 6. CONCLUDING REMARKS

By our analysis, we gave an inhomogeneous extension of the classical and quantum Mixmaster dynamics. Following the original point of view of C.W. Misner, we achieved a quantum description of the model by approximating the potential term as an infinite square box.

The main issue of our investigation has shown how the removal of the spatial gradients dynamics is possible in the asymptotic regime. This result comes out from an ADM reduction of the super-momentum constraints and from the parametric role that the three-coordinates assume in the potential term.

On the quantum level, this achievement turned out into the factorized structure of the Hilbert space; from a physical point of view, this implied to deal with independent quantum behaviors within each cosmological horizon.

Finally, we discussed the conditions allowing a WKB semiclassical approach to the dynamics and we fixed as condition to have occupation numbers much greater than unity.

### ACKNOWLEDGMENTS

One of us, G. Imponente, would like to thank the Royal Astronomical Society (RAS), UK, and the Institute of Physics (IoP), UK, for having partially supported this work.

### REFERENCES

1. C.W. Misner, *Phys. Rev.* **186**, 1319-1327 (1969).
2. E.M. Lifshitz and I.M. Khalatnikov, *Adv. Phys.* **12**, 185 (1963).
3. C.P. Ma and E. Bertschinger, *Astrophys.J.* **455**, 7 (1995).
4. V.A. Belinski, I.M. Khalatnikov, and E.M. Lifshitz, *Adv. Phys.* **19**, 525 (1970).
5. V.A. Belinski, I.M. Khalatnikov, and E.M. Lifshitz, *Adv. Phys.* **31**, 639 (1982).
6. C.W. Misner, *Phys. Rev. D* **186**, 1319 (1969).
7. A.A. Kirillov, *Zh. Eksp. Theor. Fiz.*, **103**, 721-729, (1993).
8. G. Montani, *Class. and Quantum Grav.*, **12**, 2505, (1995).
9. *Deterministic Chaos in General Relativity*, edited by D.Hobill, A. Burd, and A. Coley, World Scientific, Singapore, (1994).
10. A. A. Kirillov and G. Montani, *JETP Letters* **66**, 475 (1997).
11. G. Imponente and G. Montani *Phys. Rev. D* **63**, 103501 (2001).
12. J.D. Barrow *Phys. Rep.* **85**, 1 (1982).
13. R. Benini and G. Montani, *Phys. Rev. D* **70**, 103527 (2004).
14. G. Imponente and G. Montani *Int. Journ. Mod. Phys. D*, **12**, 977 (2003).