HAUSDORFF DIMENSION ESTIMATES FOR RESTRICTED FAMILIES OF PROJECTIONS IN $\mathbb{R}^3$

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ABSTRACT. This paper is concerned with restricted families of projections in $\mathbb{R}^3$. Let $K \subset \mathbb{R}^3$ be a Borel set with Hausdorff dimension $\dim K = s > 1$. If $\mathcal{G}$ is a smooth and sufficiently well-curved one-dimensional family of two-dimensional subspaces, the main result states that there exists $\sigma(s) > 1$ such that $\dim \pi_V(K) \geq \sigma(s)$ for almost all $V \in \mathcal{G}$. A similar result is obtained for some specific families of one-dimensional subspaces.

CONTENTS

1. Introduction 1
2. Acknowledgements 5
3. Projections onto planes 5
4. Projections onto lines 10
Appendix A. Proof of the two cones lemma 17
A.1. Overview of the proof 19
A.2. The details 22
Appendix B. Proof of the Three cones lemma 27
References 31

1. INTRODUCTION

This paper continues a line of research motivated by the question: are there Marstrand-Mattila type projection theorems for restricted families of projections? The original result of J. Marstrand [9] and P. Mattila [11] states that if $B \subset \mathbb{R}^d$ is an analytic set with Hausdorff dimension $\dim B \leq m$, then $\dim \pi_V(B) = \dim B$ for almost all $m$-planes $V \in \mathcal{G}(d, m)$. Here $\pi_V : \mathbb{R}^d \to V$ is the orthogonal projection onto $V$, $\dim$ stands for Hausdorff dimension, and an $m$-plane refers to an $m$-dimensional subspace of $\mathbb{R}^d$.

In the ‘restricted projections’ framework, one chooses a smooth submanifold $\mathcal{G} \subset \mathcal{G}(d, m)$ with $\dim \mathcal{G} < \dim \mathcal{G}(d, m)$ and asks whether $\dim \pi_V(B) = \dim B$ for almost all $V \in \mathcal{G}$. To date, several answers are known. First, I mention the
results of E. Järvenpää, M. Järvenpää, T. Keleti, M. Leikas and F. Ledrappier contained in the papers [7] and [6] (the latter of which generalises the theorems in the former). These papers provide a complete answer in the setting where no ‘curvature conditions’ are placed on $\mathcal{G}$. Indeed, [6, Theorem 3.2] gives an almost sure lower bound for $\dim \pi_V(B)$ in terms of $\dim B$ and $\dim \mathcal{G}$. In the typical situation, there exists a number $0 < \sigma < \dim B$, depending on $\dim B$ and $\dim \mathcal{G}$ such that $\dim \pi_V(B) \in [\sigma, \dim B]$ for almost every $V \in \mathcal{G}$. Examples in [6] show that the lower bounds are sharp.

A natural follow-up question, whether $V \mapsto \dim \pi_V(B)$ is almost surely a constant (depending on $B$ and $\mathcal{G}$), was studied by K. Fässler and the author in [3]; positive answers were obtained in some special cases, in particular for the one-dimensional family of planes in $\mathbb{R}^3$ containing the $z$-axis. On the other hand, there are some trivial counterexamples, such as the concatenation of the one-dimensional families of planes in $\mathbb{R}^3$ containing the $z$-axis and the $x$-axis.

There is one notable example of a strict submanifold $\mathcal{G} \subset \mathcal{G}(d, m)$, for which it is known that $\dim \pi_V(B) = \dim B$ for almost all $V \in \mathcal{G}$, and for all analytic sets $B$ with $\dim B \leq m$. This is the isotropic Grassmannian $\mathcal{G} = \mathcal{G}_h(d, m)$, a submanifold of $\mathcal{G}(2d, m)$ with positive codimension. The projection theorem for $\mathcal{G}_h(d, m)$ is due to Z. Balogh, K. Fässler, P. Mattila and J. Tyson [1]; a different proof based on the notion of transversality was given by R. Hovila [5].

As mentioned above, the papers [7] and [6] do not impose any ‘curvature conditions’ on the manifold $\mathcal{G}$. In particular, the framework of these papers allows for two counterexamples, which serve well to motivate the definitions below.

(I) In the first one, all the $m$-planes in $\mathcal{G}$ are contained in a single non-trivial subspace $W \subset \mathbb{R}^d$. Then $\pi_V(W^\perp) = \{0\}$, for all $V \in \mathcal{G}$, which means that there is no non-trivial dimension conservation result for the projection family $(\pi_V)_{V \in \mathcal{G}}$.

(II) In the second – and slightly more subtle – counterexample, the $m$-planes in $\mathcal{G}$ may cover the whole of $\mathbb{R}^d$, but they are co-contained in a single subspace $W \subset \mathbb{R}^d$ with $\dim W \leq m < d$, in the sense that $V^\perp \subset W$ for all $V \in \mathcal{G}$. Then $\pi_V(W) \subset V \cap W$ for all $V \in \mathcal{G}$. (To see this, pick $w \in W$, and write $w = \pi_V(w) + v^\perp$ with $v^\perp \in V^\perp \subset W$. It follows that $\pi_V(w) = w - v^\perp \in V \cap W$.) In particular, $\dim \pi_V(W) < \dim W$ for all $V \in \mathcal{G}$. Since $\dim W \leq m$, this means that $(\pi_V)_{V \in \mathcal{G}}$ does not satisfy the classical Marstrand-Mattila projection theorem (as formulated in the first paragraph). The simplest case of this type of counter example is the family $\mathcal{G}$ of all planes in $\mathbb{R}^3$ containing the $z$-axis.

In three dimensions, at least, these are – essentially – the only counterexamples known to date. Informally speaking, one could conjecture that any (smooth) one-dimensional family of one- or two-planes, no “large part” of which is contained or co-contained in a single non-trivial subspace, should satisfy the Marstrand-Mattila projection theorem.
RESTRICTED FAMILIES OF PROJECTIONS IN $\mathbb{R}^3$

To formulate the hypothesis of the conjecture in precise terms, K. Fässler and the author proposed in [4] the following curvature condition for one-dimensional families of one- and two-planes in $\mathbb{R}^3$.

**Definition 1.1** (Non-degenerate families). Assume that $J \subset \mathbb{R}$ is an open interval, and $\gamma: J \to S^2$ is a $C^3$-curve satisfying
\[
\text{span}\{\gamma(\theta), \dot{\gamma}(\theta), \ddot{\gamma}(\theta)\} = \mathbb{R}^3, \quad \theta \in (0, 1).
\] (1.2)
Write $\ell_\theta := \text{span}(\gamma(\theta)) \in \mathcal{G}(3, 1)$, and $V_\theta := \ell_\theta^\perp \in \mathcal{G}(3, 2)$. Then, the families $\{\ell_\theta\}_{\theta \in J}$ and $\{V_\theta\}_{\theta \in J}$ are referred to as non-degenerate families of lines and planes, respectively. The orthogonal projections onto $\ell_\theta$ and $V_\theta$ are denoted by $\rho_\theta := \pi_{\ell_\theta}$ and $\pi_\theta := \pi_{V_\theta}$, and the families $\{\rho_\theta\}_{\theta \in J}$ and $\{\pi_\theta\}_{\theta \in J}$ are called non-degenerate families of projections.

In the sequel, $\rho_\theta$ and $\pi_\theta$ will always refer to members of non-degenerate families of projections. Classical techniques, dating back as far as Kaufman’s work [8] in 1968, can be used to show that the lower bounds in [7] and [6] (obtained without any curvature assumptions) are no longer sharp for the projections $\rho_\theta$. In [4], we verified the following proposition:

**Proposition 1.3** (Proposition 1.4 in [4]). If $B \subset \mathbb{R}^3$ is an analytic set, then
\[
\dim \rho_\theta(B) \geq \min\{\dim B, \frac{1}{2}\} \quad \text{and} \quad \dim \pi_\theta(B) \geq \min\{\dim B, 1\} \quad \text{for a.e. } \theta \in J,
\]

The lower bound for $\dim \pi_\theta(B)$ holds without the curvature condition (1.2) and was already established in [7]. In contrast, the bounds in [7] and [6] give no information about $\dim \rho_\theta(B)$ in this situation (at least in case $\dim B \leq 1$) – the reason being example (I) above. But even if Proposition 1.3 improves on [7] and [6] under the curvature hypothesis (1.2), there is no longer reason to believe that the bounds $\min\{\dim B, 1/2\}$ and $\min\{\dim B, 1\}$ are sharp. The guess that they are not is the content of the following conjecture, formalising the discussion above:

**Conjecture 1.4.** If $B \subset \mathbb{R}^3$ is an analytic set, then
\[
\dim \rho_\theta(B) = \min\{\dim B, 1\} \quad \text{and} \quad \dim \pi_\theta(B) = \min\{\dim B, 2\} \quad \text{for a.e. } \theta \in J.
\]

The main results in [4] were the verification of the first part of this conjecture for self-similar sets in $\mathbb{R}^3$ without rotations, and a slight improvement over the $\min\{\dim B, 1/2\}$ and $\min\{\dim B, 1\}$ bounds for packing dimension $\dim_p$.

**Theorem 1.5** (Theorem 1.6 in [4]). Assume that $B \subset \mathbb{R}^3$ is an analytic set with $\dim B = s$. If $s > 1/2$, there exists a constant $\sigma_1(s) > 1/2$ such that
\[
\dim_p \rho_\theta(B) \geq \sigma_1(s) \quad \text{for a.e. } \theta \in J.
\]

If $s > 1$, there exists a constant $\sigma_2(s) > 1$ such that
\[
\dim_p \pi_\theta(B) \geq \sigma_2(s) \quad \text{for a.e. } \theta \in J.
\]
The appearance of \( \text{dim}_p \) in the theorem above was unfortunate, but the method of proof simply did not yield the same conclusion for \( \text{dim} \). The first version of the present paper addressed the issue in the special case, where \( \rho_0 \) and \( \pi_\theta \) are obtained from the curve

\[
\gamma(\theta) = \frac{1}{\sqrt{2}}(\cos \theta, \sin \theta, 1), \quad \theta \in (0, 2\pi).
\] (1.6)

In other words, Theorem 1.5 was proven with \( \text{dim}_p \) replaced by \( \text{dim} \), but only for these specific families of projections. The first results for Hausdorff dimension in the general situation were, soon afterwards, obtained by D. and R. Oberlin [12]. Here is their main result:

**Theorem 1.7** (Theorem 1.1 in [12]). Let \( B \subset \mathbb{R}^3 \) be an analytic set with \( \text{dim} B \geq 1 \). Then, for almost every \( \theta \in J \), one has the lower bounds

\[
\text{dim} \pi_\theta(B) \geq \begin{cases} 
(3/4) \text{dim} B, & \text{if } 1 \leq \text{dim} B \leq 2, \\
\min\{\text{dim} B - 1/2, 2\}, & \text{if } 2 \leq \text{dim} B \leq 3.
\end{cases}
\]

The proof of Theorem 1.7 is based on a Fourier restriction estimate. One should note that the technique does not seem to yield improvements over the bound \( \min\{\text{dim} B, 1\} \) bound, when \( 1 \leq \text{dim} B \leq 4/3 \). Such an improvement is the main result of this paper:

**Theorem 1.8.** Let \( B \subset \mathbb{R}^3 \) be an analytic set with \( \text{dim} B = s > 1 \). There exists a constant \( \sigma(s) > 1 \) such that

\[
\text{dim} \pi_\theta(B) \geq \sigma(s) \quad \text{for a.e. } \theta \in J.
\]

For the projections onto lines, the result is analogous but only concerns the specific family arising from the curve (1.6):

**Theorem 1.9.** Assume that \( \rho_0 \) is the orthogonal projection onto the line \( \ell_\theta = \text{span}(\gamma(\theta)) \), where \( \gamma \) is the curve from (1.6). Let \( B \subset \mathbb{R}^3 \) be an analytic set with \( \text{dim} B = s > 1/2 \). There exists a constant \( \tilde{\sigma}(s) > 1/2 \) such that

\[
\text{dim} \rho_0(B) \geq \tilde{\sigma}(s) \quad \text{for a.e. } \theta \in (0, 2\pi).
\]

The manner in which \( \sigma(s) \) and \( \tilde{\sigma}(s) \) are derived would, in principle, allow for their explicit determination, but I will not pursue this below. Speaking off the record, it seems likely that one could obtain \( \sigma(s) = 1 + O((s - 1)^\delta) \) for \( s \) near 1, and \( \tilde{\sigma}(s) = 1/2 + O((s - 1/2)^2) \) for \( s \) close to 1/2.

The introduction is closed with a word on notation. For technical purposes, it is convenient to view \( \rho_0 \) and \( \pi_\theta \) as mappings from \( \mathbb{R}^3 \) to \( \mathbb{R}^2 \), respectively. Throughout the paper I will write \( a \preceq b \), if \( a \leq Cb \) for some constant \( C \geq 1 \).

The two-sided inequality \( a \preceq b \preceq a \), meaning \( a \leq C_1 b \leq C_2 a \), is abbreviated to \( a \sim b \). Should I wish to emphasise that the implicit constants depend on a parameter \( p \), I will write \( a \preceq_p b \) and \( a \sim_p b \). The closed ball in \( \mathbb{R}^d \) with centre \( x \) and radius \( r > 0 \) will be denoted by \( B(x, r) \). For \( A \subset \mathbb{R}^d \) and \( \delta > 0 \), I denote by \( A(\delta) := \{ x \in \mathbb{R}^d : \text{dist}(x, A) \leq \delta \} \) the closed \( \delta \)-neighbourhood of \( A \).
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3. Projections onto Planes

The proofs of Theorems 1.8 and 1.9 have a lot in common, but the former is technically simpler. So, I start there.

Proof of Theorem 1.8. Let $B \subset \mathbb{R}^3$ be an analytic set with $\dim B = s > 1$. Make a counter assumption: there exists a compact set $E \subset J$ with positive length such that

$$\mathcal{H}^\sigma(\pi_\theta(B)) \leq 1 \quad \text{for all } \theta \in E. \quad (3.1)$$

The parameter $\sigma \in (1, s)$ will be fixed during the proof; in the end, it will only depend on $s$ how close $\sigma$ has to be chosen to one. Roughly speaking, the plan is to extract structural information about $B$ based on our counter assumption – and to show that if $\sigma$ is close to one, no $s$-dimensional set can have such structure.

There will be occasions, when it is required or useful to assume that $J$ is "short enough" for various purposes. Since this can always be done without loss of generality – by covering $J$ by short subintervals and proving the theorem individually for those – I will not make further remark about the issue.

The first task is to find small 'bad' scales $\delta > 0$, where the counter assumption (3.1) has a tractable geometric interpretation. This pigeonholing argument is essentially the same as [2, p. 222] by Bourgain.

Lemma 3.2. Let $A \subset \mathbb{R}^d$ be a set with $\mathcal{H}^\sigma(A) \leq 1$. Then, for any $\delta_0 > 0$, there exist collections of balls $G_k, 2^{-k} < \delta_0$, with the properties that (i) the balls in $G_k$ have bounded overlap, (ii) they have diameter $\sim 2^{-k}$, (iii) there are no more than $\lesssim d 2^{k\sigma}$ balls in $G_k$, and

$$A \subset \bigcup_{2^{-k} < \delta_0} \bigcup_{B \in G_k} B. \quad (iv)$$

Proof. By the very definition of $\mathcal{H}^\sigma(A) \leq 1$, one may find collections of balls $G_k^0, 2^{-k} < \delta_0$, satisfying conditions (ii)–(iv). In order to have (i), one first uses the $5r$-covering theorem to extract a disjoint subcollection $G_k^1 \subset G_k^0$ such that

$$\bigcup_{B \in G_k^0} B \subset \bigcup_{B \in G_k^1} 5B.$$ 

Now, the collection $G_k := \{5B : B \in G_k^1\}$ satisfies all the requirements. \qed

Fix $\delta_0 > 0$ and $\theta \in E$. Based on the counter assumption (3.1) and the lemma above, find collections $G_{\theta,k}, 2^{-k} < \delta_0,$ of discs in $\mathbb{R}^2$ such that the properties (i)–(iv) listed in the lemma are satisfied with $A = \pi_\theta(B) \subset \mathbb{R}^2$. Without loss of generality
(by Frostman’s lemma), assume that \( B = \text{spt } \mu \subset B(0, 1) \), where \( \mu \) is a Borel probability measure on \( \mathbb{R}^3 \) satisfying
\[
\mu(B(x, r)) \preceq r^s \quad \text{for } x \in \mathbb{R}^3 \text{ and } r > 0.
\]
Then, Lemma 3.2 (iv) implies that
\[
\sum_{2^{-k} < \delta_0} \mu(\pi^{-1}_\theta(\bigcup G_{k,\theta})) \geq 1,
\]
where \( \bigcup G_{k,\theta} \) stands for the union of the discs in \( G_{k,\theta} \). In particular, there exists \( k \in \mathbb{N} \) with \( 2^{-k} < \delta_0 \) such that
\[
\mu\left( \pi^{-1}_\theta(\bigcup G_{k,\theta}) \right) \gtrsim k^{-2}. \tag{3.3}
\]
Since the conclusion holds for every \( \theta \in E \), one may further pigeonhole \( k \in \mathbb{N} \) so that (3.3) holds for all \( \theta \in E_k \subset E \), where \( |E_k| \gtrsim |E| k^{-2} \). For this \( k \in \mathbb{N} \), write \( \delta := 2^{-k} < \delta_0 \), \( G_\theta := G_{k,\theta} \) and \( E_\delta := E_k \). In the sequel, whenever the text says ‘by taking \( \delta > 0 \) small enough’ or something similar, one should understand it as ‘first choose \( \delta_0 > 0 \) small enough, and then run through the pigeonholing argument above to find \( \delta < \delta_0 \)’.

Given \( \theta \in [0, 2\pi) \) and \( x, y \in \mathbb{R}^3 \), define the relation \( x \sim_\theta y \) by
\[
x \sim_\theta y \iff x, y \in \pi^{-1}_\theta(B) \text{ for some } B \in G_\theta.
\]
So, the condition \( x \sim_\theta y \) means that \( x \) and \( y \) share a common \( \delta \)-tube in \( \mathbb{R}^3 \). We now define the energy \( \mathcal{E} \) by
\[
\mathcal{E} := \int_0^{2\pi} \mu \times \mu(\{(x, y) : x \sim_\theta y\}) d\theta = \iint |\{\theta \in [0, 2\pi) : x \sim_\theta y\}| d\mu x d\mu y.
\]
The next aim is to bound \( \mathcal{E} \) from below; this will be accomplished using the first expression above. Fix \( \theta \in E_\delta \). Then (3.3) holds, so there is a collection of \( \delta \)-tubes \( T_1, \ldots, T_N \) of the form \( T_j = \pi^{-1}_\theta(B_j) \), \( B_j \in G_\theta \), such that the total \( \mu \)-mass of the tubes \( T_j \) is \( \gtrsim (\log 1/\delta)^{-2} \), and \( N \lesssim \delta^{-\sigma} \). For each \( T_j \), one has \( T_j \times T_j \subset \{(x, y) : x \sim_\theta y\} \). Using this fact, the bounded overlap of the product sets \( T_j \times T_j \) and the Cauchy-Schwarz inequality, one obtains the following estimate:
\[
\mu \times \mu(\{(x, y) : x \sim_\theta y\}) \gtrsim \sum_{j=1}^N [\mu(T_j)]^2
\geq \frac{1}{N} \left( \sum_{j=1}^N [\mu(T_j)] \right)^2
\geq \delta^{\sigma} \cdot \mu(\bigcup_{j=1}^N T_j)^2
\geq \delta^{\sigma} \cdot \left( \log\left( \frac{1}{\delta} \right) \right)^{-4}.
\]
Integrating over $\theta \in E_\delta$ and recalling that $|E_\delta| \gtrsim (\log(1/\delta))^{-2}$ yields

$$E \gtrsim \delta^\sigma \cdot \left( \log \left( \frac{1}{\delta} \right) \right)^{-6}.$$  \hfill (3.4)

The next question is: what structural information about $B = \text{spt} \mu$ does (3.4) provide? Write

$$C := \bigcup_{\theta \in J} \ell_\theta(\delta),$$

where $\ell_\theta := \text{span}(\gamma(\theta)) = \pi^{-1}_\theta \{0\}$. Thus, $C$ is the closed $\delta$-neighbourhood of a "conical" surface in $C \subset \mathbb{R}^3$. This intuition is correct, if the $\gamma(J)$ is contained in a small disc in a single hemisphere of $S^2$. This can be assumed without loss of generality, and such an assumption is indeed required a little later. The rest of the proof runs as follows. If $\delta > 0$ is small, one uses (3.4) to find two points $x_1, x_2 \in \mathbb{R}^3$ such that

$$|x_1 - x_2| \geq \delta^\kappa,$$  \hfill (3.5)

and

$$\mu((x_1 + \mathcal{C}) \cap (x_2 + \mathcal{C})) \geq \delta^\kappa.$$  \hfill (3.6)

Here $\kappa > 0$ is a number depending on $s$ and $\sigma$ with the crucial property that it can be chosen arbitrarily close to zero by letting $\sigma \searrow 1$. On the other hand, there is Lemma A.1 below, stating (informally speaking) that if two conical surfaces – such as $C$ – in $\mathbb{R}^3$ are well separated, then the intersection of their $\delta$-neighbourhoods behaves like a one-dimensional object. But $\mu$ is a Frostman measure with index $s > 1$, so such objects cannot have so much mass as (3.6) postulates for small $\kappa$. This will, eventually, show that (3.5) and (3.6) are mutually incompatible and conclude the proof.

The hunt for the points $x_1, x_2 \in \mathbb{R}^3$ begins. First, observe that

$$E = \int \int_{y+C} |\{\theta \in [0,2\pi) : x \sim_\theta y\}| \, d\mu_x \, d\mu_y.$$  \hfill (3.7)

Indeed, if $x \notin y + C$, then the distance of $x$ to any of the lines $y + \ell_\theta, \theta \in [0,2\pi)$, is greater than $\delta$, and consequently $|\pi_\theta(x - y)| > \delta$ for all $\theta \in [0,2\pi)$. In particular, $x \not\sim_\theta y$ for all $\theta \in [0,2\pi)$. To estimate the integral in (3.7) further, the following universal bound is needed:

**Lemma 3.8.** If $x, y \in \mathbb{R}^3$ are distinct points, then

$$|\{\theta \in [0,2\pi) : x \sim_\theta y\}| \lesssim \frac{\delta}{|x-y|}.$$  

**Proof.** Observe that

$$\{\theta \in [0,2\pi) : x \sim_\theta y\} \subset \{\theta \in [0,2\pi) : |\pi_\theta(x-y)| \leq \delta\}.$$

The length of the set on the right hand side can be estimated by studying the smooth function $\theta \mapsto |\pi_\theta(\xi)|^2, \xi \in S^2$. The crucial observation is that this function can have at most second order zeros. The details can be found above [4, (3.9)]. □
Now, in order to estimate the right hand side of (3.7), define
\[ G := \{ y \in \mathbb{R}^3 : \mu(y + C) \geq \delta^r \}, \]
where \( \tau = \kappa/5 > 0 \). Write
\[
E = \int_G \int_{y+C} |\{ \theta \in [0, 2\pi) : x \sim_{\theta} y \}| \, d\mu_x 
\]
\[
+ \int_{\mathbb{R}^3 \setminus G} \int_{y+C} |\{ \theta \in [0, 2\pi) : x \sim_{\theta} y \}| \, d\mu_x 
\]

The terms will be referred to as \( I_G \) and \( I_{\mathbb{R}^3 \setminus G} \). The term \( I_G \) is estimated using the bound from Lemma 3.8, and recalling the uniform bound \( \mu(B(x, r)) \lesssim r^s, s > 1 \).

\[
I_G \lesssim \delta \cdot \int_G \int \frac{1}{|x-y|} \, d\mu_x \, d\mu_y \lesssim \delta \cdot \mu(G). \tag{3.9}
\]

In order to estimate \( I_{\mathbb{R}^3 \setminus G} \), write \( A_j(y) := \{ x \in \mathbb{R}^3 : 2^j \leq |x-y| \leq 2^{j+1} \} \). For every \( j \in \mathbb{Z} \) with \( \delta \leq 2^j \leq 1 \), couple the bound from Lemma 3.8 with the estimate \( \mu((y+C) \cap A_j(y)) \lesssim \min \{ \delta^r, 2^{js} \} \leq \delta^{r(1-1/s)} \cdot 2^j \), valid for \( y \in \mathbb{R}^3 \setminus G \).

\[
I_{\mathbb{R}^3 \setminus G} \lesssim \int_{\mathbb{R}^3 \setminus G} \int_{B(y, \delta)} d\mu_x \, d\mu_y 
\]
\[
+ \int_{\mathbb{R}^3 \setminus G} \sum_{\delta \leq 2^j \leq 1} \int_{(y+C) \cap A_j(y)} |\{ \theta \in [0, 2\pi) : x \sim_{\theta} y \}| \, d\mu_x 
\]
\[
\lesssim \delta^s + \delta \cdot \int_{\mathbb{R}^3 \setminus G} \sum_{\delta \leq 2^j \leq 1} 2^{-j} \cdot \mu((y+C) \cap A_j(y)) \, d\mu_y 
\]
\[
\lesssim \delta^s + \delta^{1+r(1-1/s)} \cdot \log \left( \frac{1}{\delta} \right). 
\]

Comparing the upper bounds for \( I_G \) and \( I_{\mathbb{R}^3 \setminus G} \) with the lower bound (3.4) results in

\[
\delta^\sigma \cdot \left( \log \left( \frac{1}{\delta} \right) \right)^{-6} \lesssim \delta \cdot \mu(G) + \delta^s + \delta^{1+r(1-1/s)} \cdot \log \left( \frac{1}{\delta} \right). 
\]

One of the three terms on the right hand side must dominate the left hand side. The middle term clearly can never do that, since \( \sigma < s \). Neither can the last term, if one chooses \( \sigma < 1 + \tau(1-1/s) \). Then, the only possibility remaining is that \( \mu(G) \gtrsim \delta^{\sigma-1} \cdot \left( \log \left( \frac{1}{\delta} \right) \right)^{-6} \geq \delta^r \).

In other words, if the counter assumption is strong enough (\( \sigma \) is close enough to one), the ‘good set’ \( G \) has relatively large \( \mu \) measure. Now, a small technical point: the conical surface \( C \) was "two-sided" to begin with, but later it will be easier to deal with just a one-sided versions of \( C \) and \( \mathcal{C} \). So, define \( C^+ := C \cap \{(x, y, h) : \)}
Let \( h \geq 0 \) and note that \( C \setminus C^+ \subset -C^+ \). Then, the estimate \( \mu(G) \geq \delta^7 \) implies that either \( \mu(G^+) \geq \delta^7/2 \) or \( \mu(G^-) \geq \delta^7/2 \), where
\[
G^+ := \{ y \in \mathbb{R}^3 : \mu(y + C^+) \geq \delta^7/2 \} \quad \text{and} \quad G^- := \mu(\{ y \in \mathbb{R}^3 : \mu(y - C^+) \geq \delta^7/2 \}).
\]
Assume, for instance, that \( \mu(G^-) \geq \delta^7/2 \). This will easily yield the existence of the points \( x_1, x_2 \in \mathbb{R}^3 \). First, one uses H"{o}lder’s inequality to make the following estimate:
\[
A := \int \int \mu((x_1 + C^+) \cap (x_2 + C^+)) \, d\mu x_1 \, d\mu x_2
= \int \int \chi_{x_1+C^+}(y) \chi_{x_2+C^+}(y) \, d\mu y \, d\mu x_1 \, d\mu x_2
= \int \mu(y - C^+)^2 \, d\mu y \geq \left( \int \mu(y - C^+) \, d\mu y \right)^2 \geq \delta^{4\tau}.
\]
Recall that the aim is to find two points \( x_1, x_2 \in \text{spt} \, \mu \subset B(0,1) \) such \( (3.6) \) holds – with \( C \) replaced by \( C^+ \), in fact – and the mutual distance of the points \( x_i \) is at least \( \delta^\kappa = \delta^{5\tau} \). If this cannot be done, then
\[
|x_1 - x_2| \geq \delta^{5\tau} = \delta^\kappa \implies \mu((x_1 + C^+) \cap (x_2 + C^+)) < \delta^{5\tau} = \delta^\kappa
\]
for all \( x_1, x_2 \in \text{spt} \, \mu \). Thus,
\[
A \leq \int \int_{B(x_2,\delta^{5\tau})} d\mu x_1 \, d\mu x_2 + \int_{\{|x_1-x_2| \geq \delta^{5\tau}\}} \delta^{5\tau} \, d\mu x_1 \, d\mu x_2 \lesssim \delta^{5\tau} + \delta^{5\tau}.
\]
Since \( s > 1 \), for small enough \( \delta > 0 \) this violates the lower bound for \( A \) obtained above. The conclusion is that there exist points \( x_1, x_2, x_3 \in B(0,1) \) satisfying \( (3.5) \) and \( (3.6) \) with \( C \) replaced by \( C^+ \). For simplicity of notation, assume that \( x_1 = 0 \).

Now, it is time to state the main geometric lemma. The proof is a bit technical, so it is postponed to Appendix A.

**Lemma 3.10 (Two cones lemma).** The following holds for small enough \( \epsilon > 0 \), and for all short enough intervals \( J \subset \mathbb{R} \) (the precise requirements will be explained in the appendix). There is a constant \( \tau(\epsilon) \in (0,1/2) \) such that \( \tau(\epsilon) \searrow 0 \), as \( \epsilon \to 0 \), and so that the following is true for small enough \( \delta > 0 \) and \( \tau(\epsilon) \leq \tau < 1/2 \). If \( p \in \mathbb{R}^3 \) is a point with \( |p| \geq \delta^\epsilon \), then the intersection
\[
C^+ \cap (C^+ + p) \cap B(0,1)
\]
can be covered by two balls of diameter \( \lesssim \delta^\epsilon \), plus either
\[
(a) \lesssim \delta^{-1/2-2\epsilon-R} \text{ balls of diameter } \lesssim \delta^{1/2-R}, \text{ or } \\
(b) \lesssim \delta^{-\tau/4-R} \text{ balls of diameter } \lesssim \delta^{\tau/4-R},
\]
where \( R \geq 1 \) is an absolute constant.

The correct interpretation is that either option (a) or (b) holds depending on \( p \) – and not that one can choose at will between them. Assuming the lemma, the proof of Theorem 1.8 is completed as follows. Apply the lemma with \( \epsilon = \kappa \) and
p = x_2. Since \(\mu\) is a measure on \(\mathbb{R}^3\) with \(\mu(B) \leq d(B)^s\) for all balls \(B\) and for some \(s > 1\), the lemma shows that

\[
\mu(\mathcal{C}^+ \cap (\mathcal{C}^+ + x_2)) \leq \delta^{\kappa s} + \delta^{-\tau - R\kappa(s+1)} + \delta^{\tau(s+1)/4 - R\kappa(s+1)}.
\]

(3.11)

for small enough \(\delta > 0\), and for any \(\tau > \tau(\kappa)\). It remains to fix the parameters. First choose \(\kappa_0 > 0\) so small that \((s - 1)/2 - 3\tau(\kappa_0) > 2\kappa_0\). Next, pick \(\kappa \in (0, \kappa_0)\) so small that

\[
(s - 1)/2 - 2\tau(\kappa_0) - R\kappa(s+1) \geq (s - 1)/2 - 3\tau(\kappa_0) > 2\kappa_0
\]

and

\[
\tau(\kappa_0)(s - 1)/4 - R\kappa(s+1) \geq \tau(\kappa_0)(s - 1)/8 \geq 2\kappa.
\]

Finally, since the lemma states that (3.11) holds for any \(\tau \in [\tau(\kappa), 1/2]\) under the assumption \(|x_2| \geq \delta^\kappa\), the inequality holds for \(\tau = \tau(\kappa_0) \geq \tau(\kappa)\) in particular. This leads to \(\mu(\mathcal{C}^+ \cap (\mathcal{C}^+ + x_2)) \leq \delta^\eta\) whenever \(|x_2| \geq \delta^\kappa\) where

\[
\eta = \min\{\kappa s, 2\kappa_0, 2\kappa\} > \kappa,
\]

contradicting the choice of \(x_2\) and concluding the proof. \(\square\)

4. PROJECTIONS ONTO LINES

Theorem 1.9 is established in this section. As a quick reminder, it concerns the projections \(\rho_\theta : \mathbb{R}^3 \to \mathbb{R}\) onto the lines \(\ell(\theta) := \text{span}(\gamma(\theta))\), where

\[
\gamma(\theta) = \frac{1}{\sqrt{2}}(\cos \theta, \sin \theta, 1).
\]

The lines \(\ell_\theta\) foliate the (classical) conical surface \(C = \{(x, y, z) : x^2 + y^2 = z^2\}\).

On first sight, it appears that an argument of the kind used in the previous section cannot work for the projections \(\rho_\theta : \mathbb{R}^3 \to \mathbb{R}\). In fact, one can still make a counter assumption – this time that \(\dim \rho_\theta(B) \approx 1/2\) for many \(\theta \in [0, 2\pi]\) – and use it to find two well-separated copies of \(C\) with the property that a large portion of \(B\) is contained in the \(\delta\)-neighbourhoods of both surfaces. This time, however, there is no contradiction: the sets \(B\) one is (mainly) interested in have \(\dim B \leq 1\), so they can be easily contained in \((C + p) \cap (C + q)\) for some \(p \neq q\).

The first new idea is to use three copies of \(C\) instead of two. The difference in the argument is mostly cosmetic, and the upshot is that one finds three well-separated points \(p, q, r \in \mathbb{R}^3\) such that a large part of \(B\) is contained near \(p + C, q + C\) and \(r + C\) each. Now, the intersection \((p + C) \cap (q + C) \cap (r + C)\) is either empty or contained in a line, which – after a lengthy geometric argument given in Appendix B – shows that the intersection of the \(\delta\)-neighbourhoods of \(p + C, q + C\) and \(r + C\) is contained in the neighbourhood of a line on \(p + C\), with quantitative bounds. Still, there is no contradiction, since \(B\) could actually be contained in such a line. It is also worth noting that increasing the number of intersections beyond three gives no new information.

The final new trick is to start the whole proof by asking: if \(B\) is fixed, how many parameters \(\theta \in [0, 2\pi]\) can there be such that a large part of \(B\) is contained near
RESTRICTED FAMILIES OF PROJECTIONS IN $\mathbb{R}^3$

$p + \ell_\theta$ for some $p$? It feels intuitive that there cannot be many such values of $\theta$, and this is not hard to prove either: roughly speaking, the "bad" parameters $\theta$ have measure zero. After this is established, the counter assumption $\dim \rho_\theta(B) \approx 1/2$ must also hold for positively many "good" $\theta$. Finally, one can replace $C$ by

$$C_{\text{good}} := \bigcup_{\theta \text{ is good}} \ell_\theta$$

and run the argument through – with three copies of $C_{\text{good}}$ as outlined above. The conclusion is that a large part of $B$ must be contained near $p + C_{\text{good}}, q + C_{\text{good}}$ and $r + C_{\text{good}}$ each, and then it follows that a large part of $B$ is contained near $p + \ell_\theta$ for some "good" $\theta$. This is a contradiction.

**Proof of Theorem 1.9.** For the "final new trick" described above, one needs to consider the family of projections onto planes $\tilde{\pi}_\theta : \mathbb{R}^3 \to \tilde{V}_\theta$, $\theta \in [0, 2\pi)$, where

$$\tilde{V}_\theta = \text{span}(b_\theta)^\perp,$$

and $b_\theta$ is the line $b_\theta = \text{span}(\gamma(\theta) \times \dot{\gamma}(\theta)) = \text{span}((\cos \theta, \sin \theta, -1))$. As before, it suffices to prove Theorem 1.9 in the case $B = \text{spt} \mu \subset B(0, 1)$, where $\mu$ is a Borel probability measure on $\mathbb{R}^3$ satisfying

$$I_s(\mu) := \iint d\mu x d\mu y \frac{|x - y|^s}{s} < \infty.$$

and the growth condition $\mu(B(x, r)) \lesssim r^s$ for all balls $B(x, r) \subset \mathbb{R}^3$. Moreover, one may assume that $1/2 < s < 1$. Under these hypotheses

$$\int_0^{2\pi} I_s(\tilde{\pi}_{\theta \sharp} \mu) d\theta < \infty, \quad (4.1)$$

where $\tilde{\pi}_{\theta \sharp} \mu$ is the measure on $\tilde{V}_\theta$ defined by $\tilde{\pi}_{\theta \sharp} \mu(A) = \mu(\tilde{\pi}_\theta^{-1}(A))$. The finiteness of the integral in (4.1) follows from the sub-level set estimate

$$|\{\theta : |\tilde{\pi}_\theta(x)| \leq \lambda\}| \lesssim \lambda,$$

valid for all $x \in S^2$ and all sufficiently small $\lambda > 0$. For more details on how to prove (4.1), see [4, §3.1], in particular [4, (3.9)].

From (4.1), one sees that $|\{\theta : I_s(\tilde{\pi}_{\theta \sharp} \mu) \geq C\}| \to 0$ as $C \to \infty$. Combining this fact with a counter assumption to Theorem 1.9, one finds a constant $C > 0$ and a compact positive length set $E \subset [0, 2\pi)$ with the properties that

$$I_s(\tilde{\pi}_{\theta \sharp} \mu) \leq C, \quad \theta \in E, \quad (4.2)$$

and

$$\mathcal{H}^\sigma(\rho_\theta(B)) \leq 1, \quad \theta \in E. \quad (4.3)$$

This time, $\sigma > 1/2$ is a parameter close to $1/2$, to be fixed in the course of the proof. The assumption (4.2) is the "final new trick": it guarantees that tubes perpendicular to the planes $\tilde{V}_\theta$ cannot carry too much $\mu$ mass. This is quantified by the following lemma:
Lemma 4.4. Let $\nu$ be a probability measure on $\mathbb{R}^2$. Then
$$\nu(B) \leq I_s(\nu)^{1/2} d(B)^{s/2}$$
for all $\nu$-measurable sets $B \subset \mathbb{R}^2$.

Proof. Observe that
$$\int_0^\infty \nu \times \nu(\{(x, y) : |x - y|^{-s} \geq \lambda\}) d\lambda = I_s(\nu). \tag{4.5}$$
Now, let $B \subset \mathbb{R}^2$ be a $\nu$-measurable set. Then, as long as $x, y \in B$ and $\lambda \leq d(B)^{-s}$, one has $|x - y|^{-s} \geq d(B)^{-s} \geq \lambda$. This yields the lower bound
$$\int_0^\infty \nu \times \nu(\{(x, y) : |x - y|^{-s} \geq \lambda\}) d\lambda \geq \int_0^{d(B)^{-s}} [\nu(B)]^2 d\lambda = d(B)^{-s} \cdot [\nu(B)]^2.$$
A comparison with (4.5) completes the proof. \hfill \Box

It follows from (4.2) and the lemma, that if $T$ is an $\epsilon$-tube perpendicular to a plane $\tilde{V}_\theta$, $\theta \in E$, then $\mu(T) \lesssim \epsilon^{s/2}$. However, given the counter assumption (4.3), and assuming that $\sigma$ is very close to $1/2$, one can extract such tubes $T$ with mass far greater than $\epsilon^{s/2}$. This contradiction will complete the proof in the end.

The search for these ‘bad’ tubes $T$ begins much like the search for the translated cones $x + C$, as seen in the proof of Theorem 1.8. The first step is to fix $\delta_0 > 0$ and find a ‘bad’ scale $\delta < \delta_0$ as before. This process is repeated practically verbatim, so I only state the conclusion. There exists a scale $\delta < \delta_0$, a set $E_\delta \subset E$, and collections of intervals $G_\theta, \theta \in E_\delta$, such that

(i) for every $\theta \in E_\delta$, the collection $G_\theta$ consists of $\lesssim \delta^{-\sigma}$ intervals with length $\sim \delta$ and bounded overlap,

(ii) $E_\delta$ is compact, and

$$|E_\delta| \gtrsim \left( \log \left( \frac{1}{\delta} \right) \right)^{-2},$$

(iii)

$$\mu \left( \rho_\theta^{-1}(\cup G_\theta) \right) \gtrsim \left( \log \left( \frac{1}{\delta} \right) \right)^{-2} \text{ for } \theta \in E_\delta.$$

The relation $x \sim_\theta y$, for $x, y \in \mathbb{R}^3$, is defined analogously with the earlier notion:

$$x \sim_\theta y \iff x, y \in \rho_\theta^{-1}(I) \text{ for some } I \in G_\theta.$$

One also defines the energy $E$ almost as before by
$$\mathcal{E} := \int_{E_\delta} \mu \times \mu(\{(x, y) : x \sim_\theta y\}) d\theta = \iint \left| \{ \theta \in E_\delta : x \sim_\theta y \} \right| d\mu_x d\mu_y.$$
The only difference with the earlier notion is that the domain of the \( \theta \)-integration is restricted to \( E_\delta \). Following the argument leading to (3.4), one obtains the familiar lower bound
\[
E \gtrsim \delta^\tau \cdot \left( \log \left( \frac{1}{\delta} \right) \right)^{-6}.\tag{4.6}
\]
In order to estimate \( E \) from above, I record the following universal bound:

**Lemma 4.7.** If \( x, y \in \mathbb{R}^3 \) are distinct points, then
\[
|\{ \theta \in [0, 2\pi) : x \sim_\theta y \}| \lesssim \left( \frac{\delta}{|x - y|} \right)^{1/2}
\]

**Proof.** Observe that
\[
\{ \theta \in [0, 2\pi) : x \sim_\theta y \} \subset \{ \theta \in [0, 2\pi) : |\rho_\theta(x - y)| \leq \delta \}.
\]
The length of the set on the right hand side can be estimated by studying the function \( \theta \mapsto \rho_\theta(\xi), \xi \in S^2 \). The key observation is that this function can have at most second order zeros. The details can be found above \([4, (3.6)]\). \( \square \)

Next, the proof deviates a little further from the one of Theorem 1.8. One defines the cone
\[
C^E := \bigcup_{\theta \in E_\delta} b_\theta,
\]
where \( b_\theta = \text{span}(\gamma(\theta) \times \dot{\gamma}(\theta)) = \text{span}(\cos \theta, \sin \theta, -1) \), as before. If a difference \( x - y \) stays far from \( C^E \), the universal bound in Lemma 4.7 can be improved as follows.

**Lemma 4.8.** Let \( 0 \leq \tau < 1 \), and assume that \( y - x \notin C^E(\delta^\tau) \). Then
\[
|\{ \theta \in E_\delta : x \sim_\theta y \}| \lesssim \delta^{1 - \tau}.
\]

**Proof.** By definition of \( C^E(\delta^\tau) \), one has \( d(y - x, b_\theta) > \delta^\tau \) for all \( \theta \in E_\delta \). Since \( b_\theta = \ker \tilde{\pi}_\theta \), this implies that \( |\tilde{\pi}_\theta(y - x)| > \delta^\tau \) for \( \theta \in E_\delta \). Rewriting the inequality,
\[
\left[ \left( \frac{(x - y) \cdot \gamma(\theta)}{|\gamma(\theta)|} \right)^2 + \left( \frac{(x - y) \cdot \dot{\gamma}(\theta)}{|\gamma(\theta)|} \right)^2 \right]^{1/2} = |\tilde{\pi}_\theta(x - y)| > \delta^\tau.
\]
Since \( |\gamma(\theta)| \) and \( |\dot{\gamma}(\theta)| \) are both bounded from below on \([0, 2\pi)\), one may infer that, for some suitable constant \( c > 0 \),
\[
\{ \theta \in E_\delta : x \sim_\theta y \} \subset \{ \theta : |(x - y) \cdot \gamma(\theta)| > c\delta^\tau \} \cup \{ \theta : |(x - y) \cdot \dot{\gamma}(\theta)| > c\delta^\tau \}.
\]
On the other hand, the condition \( x \sim_\theta y \) always implies that \( |(x - y) \cdot \gamma(\theta)| \leq \delta \), so, if \( \delta > 0 \) is small,
\[
\{ \theta \in E_\delta : x \sim_\theta y \} \subset \{ \theta \in [0, 2\pi) : |(x - y) \cdot \gamma(\theta)| \leq \delta \} \cup \{ \theta : |(x - y) \cdot \dot{\gamma}(\theta)| > c\delta^\tau \}.
\]
As long as \( x \neq y \), the mapping \( \theta \mapsto (x - y) \cdot \gamma(\theta) = \rho_\theta(x - y) \) has at most two zeroes on \([0, 2\pi)\), and the set \( \{ \theta : |\rho_\theta(x - y)| \leq \delta \} \) is contained in the union of certain intervals around these zeroes. The upper bound on \( |(x - y) \cdot \gamma(\theta)| \) and the
lower bound on $| (x - y) \cdot \gamma(\theta) |$ show that these individual intervals have length $\lesssim \delta^{1 - \tau}$, and the proof of the lemma is complete. □

The next goal is to find three points $x_1, x_2, x_3 \in B(0, 1)$ such that $| x_i - x_j | \geq \delta^{13\kappa}$ for $1 \leq i < j \leq 3$ and

$$\mu([x_1 + C^E(\delta^\tau)] \cap [x_2 + C^E(\delta^\tau)] \cap [x_3 + C^E(\delta^\tau)]) \geq \delta^{13\kappa}. \quad (4.9)$$

As long as one is not interested in optimising the constants in Theorem 1.9, the number $\tau$ can be chosen freely on the open interval $(0, 1/2)$; the value of $\kappa > 0$ will be fixed later, and it will have to be small relative to $\tau$. To reach (4.9), one – almost as before – defines the set $G$ by

$$G := \{ y \in \mathbb{R}^3 : \mu( y + C^E(\delta^\tau)) \geq \delta^\kappa \}. \quad (8.9)$$

Write $E = I_G + I_{\mathbb{R}^3 \setminus G}$, where

$$I_G = \int_G \int | \{ \theta \in E_\delta : x \sim_\theta y \} | \, d\mu_x \, d\mu_y$$

and

$$I_{\mathbb{R}^3 \setminus G} = \int_{\mathbb{R}^3 \setminus G} \int | \{ \theta \in E_\delta : x \sim_\theta y \} | \, d\mu_x \, d\mu_y.$$

The part $I_G$ is estimated using the universal bound from Lemma 4.7:

$$I_G \lesssim \delta^{1/2} \cdot \int_G \frac{1}{|x - y|^{1/2}} \, d\mu_x \, d\mu_y \lesssim_s \delta^{1/2} \cdot \mu(G).$$

In the latter inequality one needs the growth condition $\mu(B(x, r)) \lesssim r^s$ with some $s > 1/2$. To find an upper bound for $I_{\mathbb{R}^3 \setminus G}$, another splitting of the integration is required:

$$I_{\mathbb{R}^3 \setminus G} = \int_{\mathbb{R}^3 \setminus G} \int_{y + C^E(\delta^\tau)} \ldots \, d\mu_x \, d\mu_y + \int_{\mathbb{R}^3 \setminus G} \int_{y + C^E(\delta^\tau)} \ldots \, d\mu_x \, d\mu_y.$$

These terms will be called $I_{\mathbb{R}^3 \setminus G}^1$ and $I_{\mathbb{R}^3 \setminus G}^2$. As regards $I_{\mathbb{R}^3 \setminus G}^2$, the definition of $y \in \mathbb{R}^3 \setminus G$ means that $\mu(y + C^E(\delta^\tau)) < \delta^\kappa$. Let

$$A_j(y) := \{ x \in \mathbb{R}^3 : 2^j \leq |x - y| \leq 2^{j+1} \}. \quad (4.10)$$

Combining the universal bound from Lemma 4.7 with the inequality

$$\mu([y + C^E(\delta^\tau)] \cap A_j(y)) \lesssim \min\{\delta^\kappa, 2^{js}\} \leq \delta^{s(1-1/2s)} \cdot 2^{j/2}, \quad y \in \mathbb{R}^3 \setminus G,$$
RESTRICTED FAMILIES OF PROJECTIONS IN $\mathbb{R}^3$

gives

$$I_2^{E \setminus G} \lesssim \int_{\mathbb{R}^3 \setminus G} \int_{B(y, \delta)} d\mu x \, d\mu y$$

$$+ \int_{\mathbb{R}^3 \setminus G} \sum_{\delta \leq 2' \leq 1} \int_{(y + \mathcal{C}E(\delta^\tau) \cap A_j(y))} |\{\theta \in E_\delta : x \sim_\theta y\}| \, d\mu x \, d\mu y$$

$$\lesssim \delta^s + \delta^{1/2} \cdot \int_{\mathbb{R}^3 \setminus G} \sum_{\delta \leq 2' \leq 1} 2^{-j/2} \cdot \mu([y + \mathcal{C}E(\delta^\tau)] \cap A_j(y)) \, d\mu y$$

$$\lesssim \delta^s + \delta^{1/2 + \kappa(1-1/2s)} \cdot \log \left(\frac{1}{\delta}\right).$$

In estimating $I_2^{E \setminus G'}$, one only needs to know that $y - x \notin \mathcal{C}E(\delta^\tau)$ in the inner integration. This enables the use of Lemma 4.8:

$$I_2^{E \setminus G} \lesssim \int_{\mathbb{R}^3} \int_{\mathbb{R}^3 \setminus (y + \mathcal{C}E(\delta^\tau))} \delta^{1-\tau} \, d\mu x \, d\mu y \leq \delta^{1-\tau}.$$

Collecting the three-part upper estimate for $E$ and comparing it with the lower bound (4.6) yields

$$\delta^s \cdot \left(\log \left(\frac{1}{\delta}\right)\right)^{-6} \lesssim E \lesssim \delta^{1/2} \cdot \mu(G) + \delta^s + \delta^{1/2 + \kappa(1-1/2s)} \cdot \log \left(\frac{1}{\delta}\right) + \delta^{1-\tau}.$$ 

Now, as long as $0 < \kappa, \tau < 1/2$ are fixed parameters, assuming that $\sigma$ is close enough to $1/2$ – as one may – shows that the sum of the three last terms on the right hand side cannot dominate the left hand side for small $\delta$. Thus, one obtains

$$\mu(G) \gtrsim \delta^{\sigma - 1/2} \cdot \left(\log \left(\frac{1}{\delta}\right)\right)^{-6} \geq \delta^\sigma,$$ 

(4.10)

where the second inequality is, once again, reached simply by taking $\delta > 0$ small and $\sigma$ close to $1/2$. Next, an application of Hölder’s inequality similar to the one seen in the proof of Theorem 1.8 gives

$$A := \iint \mu([x + \mathcal{C}E(\delta^\tau)] \cap [y + \mathcal{C}E(\delta^\tau)] \cap [z + \mathcal{C}E(\delta^\tau)]) \, d\mu x \, d\mu y \, d\mu z \gtrsim \delta^{6\kappa}.$$

Recall that the aim is to find a triple $x_1, x_2, x_3 \in \text{spt } \mu \subset B(0, 1)$ such (4.9) holds and the mutual distance of the points $x_i$ is at least $\delta^{13\kappa}$. If this cannot be done, then the condition

$$\min\{|x_i - x_j| : 1 \leq i < j \leq 3\} \geq \delta^{13\kappa}$$

implies that

$$\mu([x_1 + \mathcal{C}E(\delta^\tau)] \cap [x_2 + \mathcal{C}E(\delta^\tau)] \cap [x_3 + \mathcal{C}E(\delta^\tau)]) < \delta^{13\kappa}.$$
for all \( x_1, x_2, x_3 \in \text{spt } \mu \). Thus, one finds that
\[
A \leq \sum_{1 \leq i_1 \leq i_2 \leq i_3 \leq 3} \int \int \int_{B(x_{i_1}, \delta^{13\kappa}) \cup B(x_{i_2}, \delta^{13\kappa})} d\mu x_{i_3} d\mu x_{i_1} d\mu x_{i_2} \\
+ \int \int \int_{\{ \min \{|x_i - x_j|: 1 \leq i < j \leq 3 \} \geq \delta^{13\kappa} \}} \delta^{13\kappa} d\mu x_1 d\mu x_2 d\mu x_3 \lesssim \delta^{13\kappa} + \delta^{13\kappa}.
\]

Since \( s > 1/2 \), for small enough \( \delta > 0 \) this violates the lower for \( A \) obtained above. Thus, there must exist points \( x_1, x_2, x_3 \in B(0, 1) \) such that \( |x_i - x_j| \geq \delta^{13\kappa} \) and (4.9) holds. Without loss of generality, assume that \( x_1 = 0 \).

Now, it is again time to introduce the relevant geometric lemma:

**Lemma 4.11** (Three cones lemma). There is an absolute constant \( c > 0 \) such that the following holds for small enough \( \delta > 0 \). Let \( C = \{(x, y, z) : x^2 + y^2 = z^2\} \), and let \( p, q \in B(0, 1) \) be points satisfying
\[
\min \{|p|, |q|, |p - q|\} \geq \delta^c.
\]
Write
\[
C_0 := C(\delta), \quad C_p := p + C_0, \quad C_q := q + C_0.
\]
Then the intersection
\[
(C_0 \cap C_p \cap C_q) \cap B(0, 1)
\]
is contained in the \( \delta^c \)-neighbourhood of at most two of the lines on \( C \).

Assuming that \( 13\kappa/\tau < c \) and applying the three cones lemma with \( p = x_2, q = x_2 \), and with \( \delta^c \) in place of \( \delta \), one finds that the intersection
\[
(x_1 + C^E(\delta^c)) \cap (x_2 + C^E(\delta^c)) \cap (x_3 + C^E(\delta^c)) \cap B(0, 1)
\]
is contained in the \( \delta^c \)-neighbourhood of at most two lines on \( C \). Let \( L_1, L_2 \subset C \) be these lines. It follows from (4.9) that either \( \mu(C^E(\delta^c) \cap L_1(\delta^c)) \gtrsim \delta^{13\kappa} \) or \( \mu(C^E(\delta^c) \cap L_2(\delta^c)) \gtrsim \delta^{13\kappa} \), assume that the former options holds. Then also
\[
\mu(C^E(\delta^c) \cap L_1(\delta^c)) \gtrsim \delta^{13\kappa}, \quad (4.12)
\]
by monotonicity. There are two options: either \( L_1 \) forms a large angle with all the lines on \( b_0 \subset C^E \), or \( L_1 \) forms a small angle with a certain line on \( C^E \). More precisely, assume first that the angle between \( L_1 \) and each line \( b_0 \subset C^E, \theta \in E, \) is at least \( \delta^{c\tau/2} \). Then, since \( L_1 \) intersects all the lines on \( C^E \) at the origin, simple geometry (as in [13, (4)]) shows that
\[
C^E(\delta^{c\tau}) \cap L_1(\delta^{c\tau}) \subset B(0, \delta^{c\tau/3})
\]
for \( \delta > 0 \) small enough. However, this would imply that \( \mu(C^E(\delta^{c\tau}) \cap L_1(\delta^{c\tau})) \lesssim \delta^{c\tau/3} \), which, using (4.12), can be ruled out by choosing \( \kappa > 0 \) small enough to begin with. The conclusion is that there exists a line \( L = b_0 \subset C^E \) such that the angle between \( L_1 \) and \( L \) is smaller than \( \delta^{c\tau/2} \). It follows that \( L_1(\delta^{c\tau}) \cap B(0, 1) \subset L(\delta^{c\tau/3}) \) for small enough \( \delta > 0 \), and so (4.12) yields
\[
\mu(L(\delta^{c\tau/3})) \gtrsim \delta^{13\kappa}.
\]
To complete the proof of the theorem, apply Lemma 4.4 to the projected measure \( \tilde{\pi}_\theta \mu \), where \( L = b_\theta \). Since \( \theta \in E \), one has (4.2), and then Lemma 4.4 yields an upper bound for the \( \mu \) mass of the pre-images of discs on \( V_\theta \). The neighbourhood \( L(\delta^{c_\tau/3}) \) is such a pre-image, so

\[
\mu(L(\delta^{c_\tau/3})) \lesssim (\delta^{c_\tau/3})^{s/2} \sim \delta^{c_\tau s/6}.
\]

Choosing \( \kappa < c_\tau s/78 \), this contradicts the lower bound from (4.12) and completes the proof of Theorem 1.9. \( \square \)

**Appendix A. Proof of the Two Cones Lemma**

Recall the statement:

**Lemma A.1 (Two cones lemma).** The following holds for small enough \( \epsilon > 0 \), and for all short enough intervals \( J \subset \mathbb{R} \) (see Remark A.2 below). There is a constant \( \tau(\epsilon) \in (0, 1/2) \) such that \( \tau(\epsilon) \searrow 0 \) as \( \epsilon \to 0 \), and so that the following is true for small enough \( \delta > 0 \) and all \( \tau(\epsilon) \leq \tau < 1/2 \). Let

\[
C := \bigcup_{\theta \in J} \ell_\theta(\delta),
\]

where \( \ell_\theta \) is the half-line \( \ell_\theta = \{ r\gamma(\theta) : r \geq 0 \} \), and assume that \( p \in \mathbb{R}^3 \) is a point with \( |p| \geq \delta' \). Then the intersection

\[
C \cap (C + p) \cap B(0, 1)
\]

can be covered by two balls of diameter \( \lesssim \delta' \), plus either

(a) \( \lesssim \delta^{-1/2-2\tau} \) balls of diameter \( \lesssim \delta^{1/2} \), or

(b) \( \lesssim \delta^{-\tau/4} \) balls of diameter \( \lesssim \delta^{\tau/4} \).

**Remark A.2.** The correct interpretation of the lemma is that one of the options (a) or (b) always holds, depending on \( p \), and not that one can choose freely between them. The notation \( A \lesssim B \) means that \( A \leq R\delta^{-\text{Re}B} \) for some absolute constant \( R \geq 1 \), where \( \epsilon > 0 \) is the constant from the lemma. Writing \( A \gtrsim B \) means that \( B \lesssim A \) (that is, \( A \geq (1/R)\delta^{\text{Re}B} \)). It will be made apparent after (A.17) below, how small is a "small enough \( \epsilon > 0 \)", but the meaning of "short enough \( J \)" will be explained right now. First of all, a precise formulation of the phrase would read as follows: one can pick any point \( \theta_0 \) on the interval where \( \gamma \) was originally defined, and then restrict this interval to a neighbourhood \( J \) of \( \theta_0 \), so that the lemma holds for \( J \), and the length requirements for \( J \) depend only on \( \gamma \) and \( \theta_0 \).

Let \( C \) be the surface

\[
C := \bigcup_{\theta \in J} \ell_\theta.
\]

For convenience, assume that \( J \) is closed, and \( \gamma'' \) is defined and continuous at all points of \( J \). It is desirable to be able to parametrise \( C \) as

\[
C = \left\{ \left( t, hf\left( \frac{t}{h} \right), h \right) : h \geq 0, t \in hI \right\}, \tag{A.3}
\]
where $I \subset \mathbb{R}$ is a compact interval, and $f : I \to \mathbb{R}$ is a smooth Lipschitz function satisfying 

$$f'' \geq \eta > 0.$$ 

This can be done, if $J$ is "short enough". To understand the restrictions, assume that $\theta_0 \in J$, and – without loss of generality – $\gamma(\theta_0) = (0, 0, 1) \in S^2$. Then, the tangent plane of $S^2$ at $\gamma(\theta_0)$ is $H = \{(x, y, 1) : x, y \in \mathbb{R}\}$, and, if $J$ is so short that $\gamma(J)$ lies in the well inside the upper hemisphere of $S^2$, one can define a path $\lambda : J \to H$ by

$$\lambda(\theta) := \left(\frac{\gamma_1(\theta)}{\gamma_3(\theta)}, \frac{\gamma_2(\theta)}{\gamma_3(\theta)}, 1\right).$$

Then, it is clear that

$$C = \bigcup_{\theta \in J} \text{span}(\lambda(\theta)),$$

where $\text{span}(\lambda(\theta))$ refers to the half-line spanned by $\lambda(\theta)$. Moreover, since $\gamma = \gamma_3 \lambda$, one has the following relations for the derivatives:

$$\gamma_3' \lambda + \lambda' \gamma_3 = \gamma' \quad \text{and} \quad \gamma_3'' \lambda + 2 \gamma_3' \lambda' + \gamma_3' \lambda'' = \gamma''.$$ 

This leads to

$$\gamma'' \cdot (\gamma \times \gamma') = (\gamma_3^2 \lambda + 2 \gamma_3' \lambda' + \gamma_3' \lambda'') \cdot (\gamma_3 \lambda \times [\gamma_3' \lambda + \lambda' \gamma_3]) = \gamma_3^3 \lambda'' \cdot (\lambda \times \lambda').$$

Since $\text{span}\{\gamma, \gamma', \gamma''\} = \mathbb{R}^3$ implies that $\gamma'' \cdot (\gamma \times \gamma') \neq 0$, the relation above shows that $\lambda''(\theta) \neq 0$ for $\theta \in J$. Moreover, $|\lambda''(\theta)| \geq \tilde{\eta} > 0$ for $\theta \in J$ by compactness. Using this, a routine argument shows that $\lambda(J)$ can be parametrised as

$$\lambda(J) = \{(t, f(t), 1) : t \in I\}, \quad (A.4)$$

where $f : I \to \mathbb{R}$ is a Lipschitz function with $|f''(t)| \geq \eta > 0$ for $t \in I$, and $I \subset \mathbb{R}$ is a compact interval (this may involve a rotation of coordinates by 90 degrees and making $J$ a little shorter around $\theta_0$, if one is so unlucky that $\lambda'(\theta_0)$ is parallel to the $y$-axis). Then, without loss of generality, one may assume that $f''$ is positive on $I$, and $I = [0, 1]$. Finally, (A.4) implies (A.3), since

$$C \cap \{(x, y, h) : x, y \in \mathbb{R}\} = h\lambda(J) = \left\{\left(t, hf\left(\frac{t}{h}\right), h\right) : 0 \leq t \leq h\right\}, \quad h \geq 0.$$ 

So, the parametrisation (A.3) is possible, once $J$ is "short enough". This hypothesis will be also be needed in the proof below – mainly in the form that $\gamma(J)$ is contained in the upper hemisphere – but I will make no further mention about it.

Now that the assumptions and notations have been clarified, I start the preparations for the actual proof. Assuming that $C$ is parametrised as in (A.3), with $I = [0, 1]$, one has

$$C + p = \left\{\left(t, (h + a)f\left(\frac{t + b}{h + a}\right) + c, h\right) : t \in \mathbb{R}, h \geq 0, 0 \leq t + b \leq h + a\right\},$$
where $a, b, c \in \mathbb{R}$ are constants depending on $p$. To see this, simply note that $a$ has the effect of translating $C$ in the direction of the $z$-axis, while $b$ and $c$ have the effects of translating $C$ in the directions of the $x$- and $y$-axes, respectively. So, all possible translations $C + p$ can be obtained by varying $a, b$ and $c$. Moreover, the assumption that $\delta^\epsilon \leq |p| \leq 1$ has the effect that

$$\delta^{\kappa(\epsilon)} \leq \max\{|a|, |b|, |c|\} \lesssim 1,$$

where $\kappa(\epsilon) \to 0$ as $\epsilon \to 0$.

Before getting anywhere, one also needs to declare that

$$\min\{h, h+a\} \geq \delta^\epsilon$$

for all the heights $h$, which one encounters below. Indeed, the "two balls of diameter $\lesssim \delta^\epsilon$" appearing in the statement of the lemma are used to cover the sets

$$C \cap \mathbb{R}^2 \times [-\delta^\epsilon, \delta^\epsilon] \quad \text{and} \quad (C + p) \cap \mathbb{R}^2 \times [-a - \delta^\epsilon, -a + \delta^\epsilon].$$

After this, all the points $(t, y, h) \in C \cap (C + p)$ where (A.6) fails have already been covered, and one can assume (A.6) in the sequel. The upshot is that the functions

$$t \mapsto hf\left(\frac{t}{h}\right) \quad \text{and} \quad t \mapsto (h+a)f\left(\frac{t+b}{h+a}\right)$$

and their difference are $L$-Lipschitz with $L \lesssim 1$ under the assumption (A.6).

Next, write

$$H(h, r) := \mathbb{R}^2 \times [h-r, h+r]$$

for the horizontal slab of width $2r$, with vertical centre at $h$. For $r = 0$, this is abbreviated to $H(h) := H(h, 0)$.

A.1. Overview of the proof. I will now explain the structure of the proof at a semi-technical level, introducing notation as I go. There are two main steps. The first is to restrict the intersection $C \cap (C + p) \cap B(0, 1)$ to some fixed height $h$ satisfying (A.6), and to study a single slice of the form

$$H(h) \cap C \cap (C + p), \quad h \in [-1, 1].$$

The main analytic tool in this task is the function

$$d_h(t) = hf\left(\frac{t}{h}\right) - \left[(h+a)f\left(\frac{t+b}{h+a}\right) + c\right],$$

defined for

$$t \in I_h := [\max\{0, -b\}, \min\{h, h+a-b\}].$$

So, $I_h$ is simply the intersection of the domains of definition of the functions in (A.7). See Figure 1 for the graphical interpretation and recall that $d_h$ is Lipschitz with constant $\lesssim 1$ under the hypothesis (A.6). Now, in the first step of the proof, one is trying to establish that the set $H(h) \cap C \cap (C + p)$ (interpreted as a subset of $\mathbb{R}^2$) can be covered by at most two small discs. To do this, one observes that

$$H(h) \cap C \cap (C + p) \subset \Gamma_1(\lesssim \delta) \cap \Gamma_2(\lesssim \delta),$$
where $\Gamma_1(\lesssim \delta)$ and $\Gamma_2(\lesssim \delta)$ stand for the $A$-neighbourhoods of the graphs of the functions

$$g_1(t) = hf\left(\frac{t}{h}\right) \quad \text{and} \quad g_2(t) = (h + a)f\left(\frac{t + b}{h + a}\right).$$

for some $A \lesssim \delta$. With this notation, $d_h = g_1 - g_2$, and the proof will proceed by establishing that

$$\{t \in I_h : |d_h(t)| \lesssim \delta\}$$

can be covered by two small intervals with midpoints located either at the zeros of $d_h$, or at the endpoints of the interval $I_h$. To obtain from this information the desired disc-cover for $\Gamma_1(\lesssim \delta) \cap \Gamma_2(\lesssim \delta)$, one uses the following simple fact:

**Fact A.8.** Assume that $g_i : I_i \to \mathbb{R}, i \in \{1, 2\}$ are two $L$-Lipschitz functions defined on the intervals $I_1, I_2 \subset \mathbb{R}$, where $I_1 \cap I_2 \neq \emptyset$ and $L \geq 1$. Let $\Gamma_i \subset \mathbb{R}^2$ be the graph of $g_i$,

$$\Gamma_i = \{(t, g_i(t)) : t \in I_i\}.$$

Then $\Gamma_1(\delta) \cap \Gamma_2(\delta)$ is contained in the $6L\delta$-neighbourhood of the set

$$g_1(\{t \in I_1 \cap I_2 : |(g_1 - g_2)(t)| \leq 6L\delta\}).$$

**Proof.** Repeated application of the triangle inequality. □

In the present application, $I_1 \cap I_2 = I_h$, so – to apply the fact – one should make sure that $I_h \neq \emptyset$. In general, the set $H(\neq \emptyset) := \{h \in [-1, 1] : I_h \neq \emptyset\}$ is a closed subinterval of $[-1, 1]$, by inspecting the definition of $I_h$. If it is a **strict** subinterval, one should restrict all further attention to $H(\neq \emptyset)$. To avoid introducing any further notation, however, I will assume that $H(\neq \emptyset) = [-1, 1]$.

So, once it has been shown that $\{t \in I_h : |d_h(t)| \lesssim \delta\}$ can be covered by two intervals $I_1^h, I_2^h$ of length $\leq \delta^\beta$, for some $\beta > 0$, Fact A.8 shows that the intersection

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**Figure 1.** The function $d_h$ measures the difference between the heights of the graphs on the interval $I_h$, where both graphs are well-defined.
RESTRICTED FAMILIES OF PROJECTIONS IN $\mathbb{R}^3$

$\Gamma_1(\lesssim \delta) \cap \Gamma_2(\lesssim \delta) \supset H(h) \cap C \cap (C + p)$ can be covered by two discs of diameter $\lesssim \delta^3$. More precisely, the centres of the discs can be chosen to be of the form

$$(t, g_1(t), h) = \left(t, h f\left(\frac{t}{h}\right), h\right),$$

where either

$$d_h(t) = 0 \quad \text{or} \quad t \in \partial I_h,$$

as long as this holds for the midpoints $t$ of the intervals $I^h_1$ and $I^h_2$.

The second main step of the proof is "gluing together" the slices $H(h) \cap C \cap (C + p)$ for various $h \in [-1, 1]$. As there is only an "abstract" statement that each $h$-slice can be covered by two small discs, there remains a risk of the – admittedly unbelievable – situation that the centres of the discs vary so much for different $h$ that the union of the slices can no longer be covered by a small number of small balls. Morally, the solution is to parametrise the centres of the discs by a Lipschitz function with Lipschitz constant $\lesssim 1$. Since the centres were connected with the zeros of $d_h$, this sounds like a job for the implicit function theorem (IFT).

A straightforward application of the IFT runs soon into trouble, and it is instructive to see why. I will explain the "argument". One first defines a function $d: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by $d(t, h) = d_h(t)$. Then, as remarked above, the midpoints of the at most two intervals covering $\{t : |d_h(t)| \lesssim \delta\}$ are situated at the zeros of the function $d_h(t)$ (or at the endpoints of $I_h$, but ignore this possibility for now). So, one can start off at some $(t_0, h_0)$ such that $d(t_0, h_0) = 0$ and try to apply the IFT: if everything works out, the theorem pops out a smooth function $\psi$ of the variable $h$ such that $d(\psi(h), h) = 0$ for $h$ close enough to $h_0$. Then, because $d_h$ can have at most two zeros on $I_h$ (easy), and since $d_h(\psi(h)) = 0$, it has to be the case that $\psi(h)$ is among the midpoints of the "abstractly" chosen two intervals covering $\{t : |d_h(t)| \leq \delta\}$. So, this strategy might conceivably produce a smooth parametrisation for the midpoints. As a corollary, one would obtain a smooth parametrisation for the centres of the discs, since – as discussed above – these can be taken to be of the form $(\psi(h), h f(\psi(h)/h), h)$. At this point, the proof would practically be finished.

There are two issues. First, the IFT gives no indication of the size of the interval around $h_0$ such that $g(h)$ is well-defined. However, one essentially needs a global parametrisation here. Second, the principal hypothesis of the IFT in this situation is that $d'_{h_0}(t_0) \neq 0$, and this can easily fail, if the parameters $a, b, c$ are chosen suitably. Such an event is depicted in Figure 2. The second issue would kill the approach, were it not the case that the situation of Figure 2 can be excluded a priori. In fact, if $d'_{h_0}(t_0) \sim 0 \sim d''_{h_0}(t_0)$ for some $(t_0, h_0)$, one can extract an algebraic relation between the parameters $a, b, c$ and use it to finish off the whole proof in an ad hoc manner. This leads to alternative (b) in the lemma. After the bad case has been excluded, one can prove a global "poor man’s version" of the implicit function theorem by hand, and conclude the proof along the lines discussed above.
A.2. The details. According to the proof outline above, the first task is to dispose of the situation, where $d_{h_0}(t_0) \sim 0 \sim d_{h_0}'(t_0)$ for some $(t_0, h_0)$. This is the content of the following proposition:

**Proposition A.9.** Let $4 \kappa < \tau < 1/2$ where $\kappa = \kappa(\epsilon) > 0$ is the constant from (A.5). Assume that there exists a height $h_0$ satisfying (A.6), and a point $t_0 \in I_{h_0}$ such that

$$|d_{h_0}(t_0)| \leq \delta^\tau \quad \text{and} \quad |d_{h_0}'(t_0)| \leq \delta^\tau.$$  

Then, the intersection $C \cap (C + p) \cap B(0, 1)$ can be covered by two balls of diameter $\lesssim \delta^\tau$, plus $\lesssim \delta^{-\tau/4}$ balls of diameter $\lesssim \delta^\tau/4$.

**Proof.** The derivative of $d_h$ has the relatively simple expression

$$d_h'(t) = f'(\frac{t}{h}) - f'(\frac{t + b}{h + a}).$$ \hspace{1cm} (A.10)

Now, recall that $f'' \geq \eta > 0$. In particular, if $|d_{h_0}'(t_0)| \leq \delta^\tau$, it follows that

$$\left|\frac{t_0}{h_0} - \frac{t_0 + b}{h_0 + a}\right| \lesssim \left|f'(\frac{t_0}{h_0}) - f'(\frac{t_0 + b}{h_0 + a})\right| = |d_{h_0}'(t_0)| \leq \delta^\tau.$$ \hspace{1cm} (A.11)

Consequently,

$$\left|\frac{t_0}{h_0} - \frac{b}{a}\right| \lesssim \frac{|h_0 + a|}{|a|} \delta^\tau \lesssim \frac{\delta^\tau}{|a|}.$$ \hspace{1cm} (A.12)

Next, using the fact that $f$ is Lipschitz, one deduces from (A.11) that

$$\left|h_0 f\left(\frac{t_0}{h_0}\right) - h_0 f\left(\frac{t_0 + b}{h_0 + a}\right)\right| \lesssim \delta^\tau,$$

so that (by the triangle inequality and the definition of $d_{h_0}$)

$$\left|a f\left(\frac{t_0 + b}{h_0 + a}\right) + c\right| \lesssim \delta^\tau + |d_{h_0}(t_0)| \lesssim \delta^\tau.$$ \hspace{1cm} (A.13)
Now, if \( b/a \in [0, 1] \) (so that \( f(b/a) \) is well-defined), one can argue as follows (I will come back to this simplifying assumption later). Combining (A.11)–(A.13),

\[
\left| a f \left( \frac{b}{a} \right) + c \right| \lesssim \frac{\delta^r}{|a|}.
\] (A.14)

This is not very useful, if \(|a|\) is small, say \(|a| \leq \delta^{r/4}\). However, since \( \tau/4 > \kappa \), the condition that \(|a| \leq \delta^{r/4}\) forces \(|c| \geq \delta^r\) or \(|b| \geq \delta^r\) by (A.5). But if \(|a| \leq \delta^{r/4}\), then also \(|c| \lesssim \delta^{r/4}\) by (A.13), so it has to be the case that \(|b| \geq \delta^r\). In this case

\[
\left| t_0 - t_0 + \frac{b}{h} \right| = \frac{|bh_0 - at_0|}{h_0(h_0 + a)} \geq \frac{|b|}{h_0 + a} - \frac{|a||t_0|}{h_0(h_0 + a)}.
\]

Since \( t_0/h_0 \leq 1 - \delta \leq 1 - t_0 \in I_{h_0} \) – the last expression is further bounded from below by \(|b| - |a|/(h_0 + a) \gtrsim \delta^s\), which is a contradiction in light of (A.11). The conclusion is that \(|a| \gtrsim \delta^{r/4}\) under the hypotheses of the lemma. Then, (A.14) gives

\[
|d_a(\frac{bh}{a})| = |a f \left( \frac{b}{a} \right) + c| \leq C \delta^{3r/4}
\] (A.15)

for every \( h \in [-1, 1] \), and not just \( h = h_0 \) (the first equality in (A.15) being simply the definition of \( d_h \)). This will have the consequence that the set \( \{ t : |d_h(t)| \leq \delta \} \) is contained in a single (short) interval around \( t(h) = bh/a \). To see why, one has to show that \(|d_h(t)|\) is large, when \(|t - bh/a|\) is large. Assume, for example, that \( t > bh/a \). Now, since the only zero of \( d_h'(t) \) is at \( t = bh/a \), the function \( d_h'' \) has constant sign on the interval \([bh/a, t]\). This sign could be determined from \( a \) and \( b \), but it does not affect the computations; I will simply assume that it is positive. So, using \( f'' \geq \eta \) again,

\[
|d_h(t)| \gtrsim |d_h(t) - d_{\frac{bh}{a}}(\frac{bh}{a})| - C \delta^{3r/4} = \int_{bh/a}^t d_h'(r) dr - C \delta^{3r/4}
\]

\[
\gtrsim \int_{bh/a}^t \left[ \frac{r - \frac{r + b}{h + a}}{h} \right] dr - C \delta^{3r/4} = \int_{bh/a}^t \frac{ra - \frac{hb}{h + a}}{h(h + a)} dr - C \delta^{3r/4}
\]

\[
= \int_{bh/a}^t \frac{(r - \frac{bh}{a})a}{h(h + a)} dr - C \delta^{3r/4} \geq \tau^4(t - \frac{bh}{a})^2 - C \delta^{3r/4}.
\]

This is far larger than \( \delta^{3r/4} \), as soon as \( \tau^4(t - \frac{bh}{a})^2 \geq 2C \delta^{3r/4} \), which happens as soon as \( (t - \frac{bh}{a}) \geq \sqrt{2C} \delta^{r/4} \). So, since \(|d_h(t)| \lesssim \delta\) implies that \(|d_h(t)| \leq \delta^{3r/4}\), this gives

\[
\{ t \in I_h : |d_h(t)| \lesssim \delta \} \subset [bh/a - c\delta^{r/4}, bh/a + c\delta^{r/4}]
\] (A.16)

for some large enough constant \( c > 0 \). So, the sets \( \{ t \in I_h : |d_h| \lesssim \delta \} \), \( \min\{h, h + a\} \geq \delta^s \), can be covered by a single short interval each, the midpoint of which depends smoothly on \( t \). The rest of the argument follows the outline described.
earlier. Here are the details once more: using Fact A.8, the inclusion (A.16) yields a covering of \( H(h) \cap C \cap (C + p) \) by a single disc of diameter \( \lesssim \delta^{r/4} \), centred at
\[
\text{centre}(h) := \left( \frac{hb}{a}, hf \left( \frac{b}{a} \right), h \right).
\]
Since \( h \mapsto \text{centre}(h) \) is Lipschitz with bounded constants (recalling that \( 0 \leq b/a \leq 1 \)), this means that \( C \cap (C + p) \cap B(0, 1) \) can be covered by \( \lesssim \delta^{-r/4} \) balls of diameter \( \lesssim \delta^{r/4} \), and the proof of the proposition is complete.

If \( b/a \not\in [0, 1] \), the details are similar but messier. The extra assumption was not used before (A.14), so (A.11)–(A.13) hold. Also, \(|a| \geq \delta^{r/4}\). Note that
\[
\left| \frac{t_0 + b}{h_0 + a} - \frac{b}{a} \right| \lesssim \delta^{3r/4}
\]
by (A.11) and (A.12). By definition of \( I_{h_{\omega r}} \), one has \((t_0 + b)/(h_0 + a) \in [0, 1]\), so the fact that \( b/a \not\in [0, 1] \) implies that either 0 or 1 has to lie between \((t_0 + b)/(h_0 + a)\) and \( b/a \), at distance \( \lesssim \delta^{3r/4} \) from both numbers. Assume, for instance, that 1 has this property, so that \( b/a > 1 \). Now \( t(h) := h + a - b \) will play the role of the special point \( hb/a \) above (the reason being that \((t(h) + b)/(h + a) = 1\); if 0 was picked instead of 1, the choice \( t(h) = -b \) would be correct). The claim is that \( \{ t \in I_h : |d_h(t)| \lesssim \delta \} \) is contained in a single short interval centred at \( t(h) \). First, note that
\[
d_h(t(h)) = hf \left( \frac{h + a - b}{h} \right) - [(h + a)f(1) + c]
\]
is well-defined for all \( h \in [-1, 1] \), since \( h + a - b \geq 0 \) and \( a - b \leq 0 \): the first condition is necessary for \( I_{h_{\omega r}} \neq \emptyset \) (an assumption I made at the beginning), and the second condition is equivalent to \( b/a > 1 \).

Then, using that \( f \) is Lipschitz with bounded constants, combined with the fact that both numbers \( b/a \) and \((t_0 + b)/(h_0 + a)\) are very close to one, and (A.13),
\[
|d_h(t(h))| \leq |h| \left| f \left( 1 + \frac{a - b}{h} \right) - f(1) \right| + |af(1) + c| 
\]
\[
\lesssim |a - b| + |af \left( \frac{t_0 + b}{h_0 + a} \right) + c| + \delta^{3r/4} \lesssim \delta^{3r/4}.
\]
This is the analogue of (A.15), and the proof can now be concluded in the same spirit as before; one should note \( d_h^r \) has constant sign on the whole interval \( I_{h_{\omega r}} \), because \( d_h^r \) could only have a zero at \( hb/a \not\in I_h \). I omit the rest of the details. \( \square \)

In the sequel, one is entitled to assume that
\[
|d_h(t)| \leq \delta^r \implies |d_h^r(t)| \geq \delta^r, \tag{A.17}
\]
if \( \tau > 4\kappa \), and \( t \in I_h \). The constant \( \tau(\epsilon) \in (0, 1/2) \) from the statement of the lemma can be taken to be any number larger than \( 4\kappa \) (so that \( \tau \geq \tau(\epsilon) \) implies \( \tau > 4\kappa \)). In particular, one can pick \( \tau(\epsilon) < 1/2 \) (as required by the statement), as soon as \( \kappa(\epsilon) < 1/8 \).
Proof of Lemma A.1. Assume that \( a \leq 0 \); this corresponds to the case that the vertex of the cone \( C + p \) is above the \( xy \)-plane. The case with \( a > 0 \) is treated similarly. Recall that

\[
d_h'(t) = f'(\frac{t}{h}) - f'(\frac{t + b}{h + a}).
\]

Since \( f' \) is a strictly increasing function, a quick computation gives

\[
d_h'(t) \geq 0 \iff \frac{t}{h} \leq \frac{b}{a}.
\]

So, either \( b/a \notin I_h \), and \( d_h \) is strictly monotone on \( I_h \), or then \( b/a \in I_h \), and the picture looks something like Figure 3. In particular, \( d_h \) can have zero, one or two zeros on \( I_h \). Observe that any zero \( z \) of \( d_h \) must satisfy

\[
|z - \frac{hb}{a}| \gtrsim \delta^r,
\]

(A.18) since otherwise

\[
|d_h'(z)| = \left| f'(\frac{z}{h}) - f'(\frac{z + b}{h + a}) \right| \lesssim \left| \frac{z}{h} - \frac{z + b}{h + a} \right| = \frac{a(z - \frac{hb}{a})}{h(h + a)} < \delta^r,
\]

contrary to (A.17). Also, it is good to keep in mind that if there are two zeros \( z_1, z_2 \), then \( \frac{hb}{a} \in I_h \), and \( z_1, z_2 \) have to be located on different sides of \( \frac{hb}{a} \).

Now, define exactly two special points \( s_1(h), s_2(h) \in I_h \)

\[
s_1(h) < s_2(h)
\]

(A.19) as follows.

- If \( d_h \) has a zero \( z \leq \frac{hb}{a} \), then \( s_1(h) = z \). Otherwise \( s_1(h) \) is the left endpoint of \( I_h \).
- If \( d_h \) has a zero \( z \geq \frac{hb}{a} \), then \( s_2(h) = z \). Otherwise \( s_2(h) \) is the right endpoint of \( I_h \).

Next, the plan is to argue that the set \( Z^h_h := \{ t \in I_h : |d_h(t)| \leq \delta^{1/2 + 2r} \} \) is contained in two intervals of length \( \lesssim \delta^{1/2 + r} \), centred at the special points \( s_1(h) \) and \( s_2(h) \). First, consider the part

\[
Z^h_{\leq \frac{hb}{a}} := \{ t \in I_h \cap (-\infty, \frac{hb}{a}] : |d_h(t)| \leq \delta^{1/2 + 2r} \}.
\]
The function $d_h$ is strictly increasing on $I_h \cap (-\infty, hb/a]$, so it can have at most one zero on this interval. If such a zero exists, it is located at $s_1(h)$, and $Z_h^{\ell}\subset hb/a$ is an interval $I$ around $s_1(h)$. Moreover, since $d'_h(t) \geq \delta^\tau$ for all $t \in I$ by (A.17), the length of $I$ is bounded by $\ell(I) \leq 2\delta^{1/2+\tau}$, as desired.

If there is no zero of $d_h$ on $I_h \cap (-\infty, hb/a]$, then the left endpoint of $I_h$ is $s_1(h)$. Moreover, if $Z_h^{\ell}\subset hb/a$ is non-empty, it is an interval $I$ containing $s_1(h)$. Once more, $d'_h(t) \geq \delta^\tau$ for all $t \in J$ by (A.17), and this implies that $\ell(J) \leq 2\delta^{1/2+\tau}$.

A similar argument shows that $\{t \in I_h \cap [hb/a, \infty) : |d_h| \leq \delta^{1/2+2\tau}\}$ is contained in a single interval of length $\lesssim \delta^{1/2+\tau}$ around one of the special points on $I_h \cap [hb/a, \infty)$. Putting the two pieces together, $Z_h$ is indeed contained in two intervals of length $\lesssim \delta^{1/2+\tau}$ centred at the special points. These intervals are denoted by $I_1(h)$ and $I_2(h)$ (not to be confused with $I_h$ and $I_h$).

Finally, it is time to examine how the special points $s_1(h)$ and $s_2(h)$ vary as functions of $h$. The desirable conclusion has the form

$$|h_1 - h_2| \leq \delta^{1/2+2\tau+C\epsilon} \implies |s_2(h_1) - s_2(h_2)| \lesssim \delta^{1/2}, \quad (A.20)$$

where $C$ is a large enough absolute constant; the same statement holds for $s_1(h)$, and the proof is slightly easier. So, assume that $|h_1 - h_2| \leq \delta^{1/2+2\tau+C\epsilon}$. There are essentially two different cases.

First, it is possible that $s_2(h_1)$ is the right endpoint of $I_h$, and $s_2(h_2)$ is the right endpoint of $I_h$. Then $|s_2(h_1) - s_2(h_2)| = |h_1 - h_2| \leq \delta^{1/2+2\tau+C\epsilon}$, which is good.

The second possibility is that at least one of the points $s_2(h_i)$ is a zero of $d_{h_i}$. Assume, for instance, that this is the case for $s_2(h_1)$. Then, if $C$ is large enough, one can find a point $t \in I_h$ such that $|t - s_2(h_1)| \leq \delta^{1/2+\tau}$, and $|d_{h_2}(t)| \leq \delta^{1/2+2\tau}$: if $s_2(h_1) \in I_h$, a direct computation shows that $t = s_2(h_1)$ is a good choice, whereas in general one should pick the point of $I_h$ closest to $s_2(h_1)$. Now, the point $t$ lies in the set $Z_h^{\ell}$, so it is at distance $\lesssim \delta^{1/2+\tau}$ from either one of the special points $s_1(h_i), i \in \{1, 2\}$ by the previous considerations. This implies that

$$|s_2(h_1) - s_1(h_2)| \lesssim \delta^{1/2+\tau}, \quad (A.21)$$

which looks like a little better than required: the surplus $\tau$ will be lost when proving that one can take $i = 2$.

This is yet another case chase. First, assume that $s_1(h_2)$ is a zero of $d_{h_2}$. Then $s_1(h_2) < h_2 b/a$. Also, since $s_2(h_1)$ is a zero of $d_h$, one has $s_2(h_1) > h_1 b/a$, and indeed $s_2(h_1) - h_1 b/a \gtrsim \delta^\tau$ by (A.18). These facts show that

$$s_2(h_1) - s_1(h_2) \geq s_2(h_1) - h_2 b/a \gtrsim \delta^\tau - |h_1 b/a - h_2 b/a| \geq \delta^\tau/2,$$

since $|h_1 - h_2| \leq \delta^{1/2}$ and $b/a \leq s_2(h_1)/h_1 \leq 1$. In particular, (A.21) is out of the question with $i = 1$, for small enough $\tau > 0$.

So, if $i = 1$, it has to be the case that $s_1(h_2)$ is an endpoint of $I_h$ – namely the left one. Now, if $s_1(h_2) < h_2 b/a$, one can reason exactly as above to show that (A.21) is impossible with $i = 1$. So, one only has to consider the case $s_1(h_2) \geq h_2 b/a$. Recall
that $s_i(h_2)$ was at distance $\lesssim \delta^{1/2+\tau}$ from a certain point $t$ with $|d_{h_2}(t)| \leq \delta^{1/2+2\tau}$ (chosen above (A.21)), which implies that
\[ |d_{h_2}(s_i(h_2))| \lesssim \delta^{1/2+\tau}. \tag{A.22} \]

So, if $i = 1$, the left endpoint $s_1(h_2)$ of $I_{h_2}$ has to satisfy (A.22). Then, by (A.17), there exists a zero $z \in I_{h_2}$ of $d_{h_2}$ with $z - s_1(h_2) \lesssim \delta^{1/2}$ unless $I_{h_2}$ is too short for this to happen, namely $\ell(I_{h_2}) \lesssim \delta^{1/2}$. In both cases, the special points of $I_{h_2}$ are necessarily close to each other, $|s_1(h_2) - s_2(h_2)| \lesssim \delta^{1/2}$, which – combined with (A.21) – shows that $|s_2(h_1) - s_2(h_2)| \lesssim \delta^{1/2}$. Thus, the proof of (A.20) is complete.

Now, it is time to pass from intervals to discs to balls. At the risk of over-repeating an argument, I recall it once more: it has been established that
\[ \{ t \in I_h : |d_h(t)| \lesssim \delta \} \subset \mathbb{Z}^h \]
can be covered by two intervals of length $\lesssim \delta^{1/2}$ centred at $s_1(h)$ and $s_2(h)$. It follows that $H(h) \cap C \cap (C + p)$ can be covered by two discs of diameter $\lesssim \delta^{1/2}$ centred at
centre_1(h) = \left( s_1(h), h f \left( \frac{s_1(h)}{h} \right), h \right) \quad \text{and} \quad \centre_2(h) := \left( s_2(h), h f \left( \frac{s_2(h)}{h} \right), h \right).

Next, take any interval $H \subset [-1, 1]$ such that $\ell(H) \leq \delta^{1/2+2\tau+C \epsilon}$, and $h, h + a \geq \delta^{\epsilon}$ for $h \in H$. It follows from (A.20) (which is the "poor man’s implicit function theorem") that
\[ |\centre_1(h_1) - \centre_1(h_2)| \lesssim \delta^{1/2} \]
for all $h_1, h_2 \in H$. Consequently, $C \cap (C + p) \cap \mathbb{R}^2 \times H$ can be covered by two balls of diameter $\lesssim \delta^{1/2}$. Finally, one can split the set
\[ B_+ := B(0, 1) \cap \{ (t, y, h) : h, h + a \geq \delta^{\epsilon} \} \]
into $\lesssim \delta^{-1/2-2\tau}$, regions of the form $B_+ \cap \mathbb{R}^2 \times H$, where $H \subset [-1, 1]$ satisfies the requirements above. It follows that $C \cap (C + p) \cap B_+$ can be covered by $\lesssim \delta^{-1/2-2\tau}$ balls of diameter $\lesssim \delta^{1/2}$, and the proof of Lemma A.1 is complete.

### Appendix B. Proof of the Three Cones Lemma

Recall the statement:

**Lemma B.1** (Three cones lemma). There is an absolute constant $c > 0$ such that the following holds for small enough $\delta > 0$. Let $C \subset \mathbb{R}^3$ be the cone $C = \{ (x, y, z) : x^2 + y^2 = z^2 \}$, and let $p, q \in B(0, 1)$ be points satisfying
\[ \min \{ |p|, |q|, |p - q| \} \geq \delta^c. \]

Write
\[ C_0 := C(\delta), \quad C_p := p + C_0, \quad C_q := q + C_0. \]

Then the intersection
\[ (C_0 \cap C_p \cap C_q) \cap B(0, 1) \]
is contained in the $\delta^c$-neighbourhood of at most two of the lines on $C$. 
It seems likely that the lemma should hold with one line in place of two, but this way the proof is easier. The argument divides into several propositions. I will not write a heuristic overview of them here, because this would essentially be repeating the paragraph "Proof of Lemma B.1" below; in fact, I suggest the reader take a look there before starting with the technicalities.

In order to avoid writing ‘B(0, 1)’ all the time, the agreement is made that the all the sets below will be intersected with B(0, 1). Thus, any claim concerning, say, C0 ∩ Cp should be interpreted as a claim concerning C0 ∩ Cp ∩ B(0, 1) instead. A similar remark concerns the words taking c, δ > 0 small enough: these should be inserted anywhere in the text, where they appear needed but missing.

**Proposition B.2.** Suppose that either p or q, say, p, lies in the $\delta^{1/4}$-neighbourhood of C. Then C0 ∩ Cp (and in particular C0 ∩ Cp ∩ Cq) is contained in the $\delta^c$-neighbourhood of a single line on C.

**Proof.** Assume, without loss of generality, that p lies in the $\delta^{1/4}$-neighbourhood of the line $\text{span}(0, 1, 1) \subset C$. Then $p = (0, r, r) + e$, where $|e| \leq \delta^{1/4}$ and |r| ≥ $\delta^c$. The idea is to study separately all the intersections $C_0 ∩ C_p ∩ H_t$, $t \in \mathbb{R}$, where $H_t$ is the horizontal plane $H_t = \{(x, y, t) : (x, y) \in \mathbb{R}^2\} \subset \mathbb{R}^3$. Fix $t \in \mathbb{R}$ and make the temporary identification $H_t \cong \mathbb{R}^2$ (that is, drop off the third component from all vectors on $H_t$). Then $C_0 ∩ H_t$ and $C_p ∩ H_t$ are contained in the $\delta$-neighbourhoods of the circles

$$S_0 = S((0, 0), |t|) \subset \mathbb{R}^2 \quad \text{and} \quad S_p = S((p_1, p_2), |p_3 - t|) \subset \mathbb{R}^2,$$

respectively. Since

$$S((p_1, p_2), |p_3 - t|) = S((e_1, r + e_2, |r + e_3 - t|),$$

where $|(e_1, e_2, e_3)| \leq \delta^{1/4}$, one may infer that the $\delta$-neighbourhood $S_p(\delta)$ is contained in the $R\delta^{1/4}$-neighbourhood of the circle $S((0, r), |r - t|)$ for some large enough absolute constant $R \geq 1$. Now, the circles $S(0, |t|)$ and $S((0, r), |r - t|)$ are tangent (either internally or externally) at $(0, t)$, so the intersection of their $R\delta^{1/4}$-neighbourhoods is contained in a small disc $D$ centred at $(0, t)$. The diameter of $D$ depends, of course, on the size of $r$, but choosing $c, \delta > 0$ small enough and assuming $|r| \sim |p| \gtrsim \delta^c$ guarantees that $\text{diam}(D) \leq \delta^c$. For more details, see the proof of [13, Lemma 3.1].

Finally, observe that $(0, t, t)$ – the midpoint of $D$ lifted from $\mathbb{R}^2$ to $H_t$ – lies on the line $L = \text{span}(0, 1, 1) \subset C$. Repeating the argument above for every $t \in \mathbb{R}$ shows that $C_0 ∩ C_p$ is contained in the $\delta^c$-neighbourhood of $L$. □

**Proposition B.3.** Let $A$ and $B$ be sets in a metric space $(X, d)$, and let $r, s > 0$. Then

$$A(r) \cap B(s) \subset [A(r + s) \cap B](s).$$

**Proof.** Let $x \in A(r) \cap B(s)$. Choose $a \in A$, $b \in B$ such that $d(x, a) \leq r$, $d(x, b) \leq s$. Then $b \in A(r + s) \cap B$, so that $x \in [A(r + s) \cap B](s)$. □
**Proposition B.4.** There is an absolute constant $R \geq 1$ such that the intersections $C_0 \cap C_p$ and $C_0 \cap C_q$ are contained in the $R\delta^{1-c}$-neighbourhoods of the planes

$$V_p := \left\{ (x, y, z) : \left( (x, y, z) - \frac{(p_1, p_2, p_3)}{2} \right) \cdot (p_1, p_2, -p_3) = 0 \right\}$$

and

$$V_q := \left\{ (x, y, z) : \left( (x, y, z) - \frac{(q_1, q_2, q_3)}{2} \right) \cdot (q_1, q_2, -q_3) = 0 \right\}.$$

**Proof.** By the previous proposition, it suffices to prove the claim for the intersection $C \cap C_p$. Note that

$$C_p = \bigcup_{r \in B(0, \delta)} p + r + C.$$

We will now prove that $C \cap (p + r + C)$ is contained in the $R\delta^{1-c}$-neighbourhood of $V_p$ for every $r \in B(0, \delta)$. Using the equation $C = \{(x, y, z) : x^2 + y^2 = z^2\}$, one can check that $C \cap (p + r + C)$ is contained in the plane

$$\left\{ \left( (x, y, z) - \frac{(p_1, p_2, p_3) + (r_1, r_2, r_3)}{2} \right) \cdot [(p_1, p_2, -p_3) + (r_1, r_2, -r_3)] = 0 \right\}.$$

Now, if $(x, y, z) \in B(0, 1)$ satisfies the equation above, then it follows from $|r| \leq \delta$ and $p \in B(0, 1)$ that

$$\left| \left( (x, y, z) - \frac{(p_1, p_2, p_3)}{2} \right) \cdot (p_1, p_2, -p_3) \right| \leq 3\delta.$$

Choose $(x', y', z') \in V_p$ such that the difference $(x, y, z) - (x', y', z')$ is parallel to $(p_1, p_2, -p_3)$ (so $(x', y', z')$ is the orthogonal projection of $(x, y, z)$ into $V_p$). Then

$$|(x, y, z) - (x', y', z')||p| = |(x, y, z) - (x', y', z')||(p_1, p_2, -p_3)|$$

$$= |[(x, y, z) - (x', y', z')] \cdot (p_1, p_2, -p_3)|$$

$$= \left| \left( (x, y, z) - \frac{(p_1, p_2, p_3)}{2} \right) \cdot (p_1, p_2, -p_3) \right| \leq 3\delta,$$

proving that $(x, y, z)$ lies in the $(3\delta/|p|)$-neighbourhood of $V_p$. Since $|p| \geq \delta^c$ by hypothesis, the claim follows. \qed

For the remainder of the proof, fix $\tau \in (1/2, 1)$.

**Proposition B.5.** Assume that $p, q \notin C(\delta^{1/4})$ and $\text{dist}(p, \text{span}(q)) \leq \delta^\tau$. Then the intersection $V_p(R\delta^{1-c}) \cap V_q(R\delta^{1-c})$ is empty. In particular, the previous lemma implies that

$$C_0 \cap C_p \cap C_q = \emptyset.$$

**Proof.** It suffices to show that the planes $V_p$ and $V_q$ intersected with $B(0, 1)$ are at distance more than $3R\delta^{1-c}$ apart. Let $\xi = q/|q| \in S^2$, and write $p = r\xi + \epsilon$, where
Translating if necessary, one may assume that the line \(|e| \leq \delta^c\) and \(| r - |q| | \geq \delta^{1/4}\) (for the latter inequality one uses the assumption \(|p - q| \geq \delta^c\) with \(c \leq 1/4\)). Then the equation for the plane \(V_p\) becomes

\[
\left\{(x, y, z) : (r \xi_1 + e_1, r \xi_2 + e_2, -r \xi_3 - e_3) = \left(\frac{(r \xi_1 + e_1)^2 + (r \xi_2 + e_2)^2 - (r \xi_3 + e_3)^2}{2}\right) \right\}.
\]

This means that if \((x, y, z) \in V_p\), then

\[
(x, y, z) \cdot (\xi_1, \xi_2, -\xi_3) = r \cdot \frac{\xi_1^2 + \xi_2^2 - \xi_3^2}{2} \pm O(\delta^\tau) = r \cdot \frac{1 - 2 \xi_3^2}{2} \pm O(\delta^\tau).
\]

On the other hand, if \((x', y', z') \in V_q\), then

\[
(x', y', z') \cdot (\xi_1, \xi_2, -\xi_3) = |q| \cdot \frac{1 - 2 \xi_3^2}{2}.
\]

Thus, for \((x, y, z) \in V_p\) and \((x', y', z') \in V_q\), one finds that

\[
||(x, y, z) - (x', y', z')|| \cdot (\xi_1, \xi_2, -\xi_3) \geq |r - |q|| \cdot \frac{1 - 2 \xi_3^2}{2} - O(\delta^\tau).
\]

The assumption \(q \notin C(\delta^{1/4})\) shows that \(\text{dist}(\xi, C) \geq \delta^{1/4}\). Observing that \(C \cap S^2 = \{(t_1, t_2, t_3) : t_3 \in \{-1/\sqrt{2}, 1/\sqrt{2}\}\} \cap S^2\), this (and \(\xi \in S^2\)) implies further that

\[
\text{dist}(\xi, \{-1/\sqrt{2}, 1/\sqrt{2}\}) \geq \delta^{1/4}.
\]

Since the derivative of the mapping \(t \mapsto 1 - 2t^2\) stays bounded away from zero near \(t = \pm 1/\sqrt{2}\), one may infer that \(|(1 - 2 \xi_3^2)/2| \gtrsim \delta^{1/4}\). All in all, for small enough \(\delta > 0\),

\[
||(x, y, z) - (x', y', z')|| \geq ||(x, y, z) - (x', y', z')|| \cdot (\xi_1, \xi_2, -\xi_3) \gtrsim \delta^{1/2}.
\]

Assuming that \(c < 1/2\), the term on the right hand side dominates \(3R\delta^{1-c}\) for small enough \(\delta > 0\). This proves that \(\text{dist}(V_p \cap B(0, 1), V_q \cap B(0, 1)) \geq 3R\delta^{1-c}\), and so the two \(R\delta^{1-c}\)-neighbourhoods cannot intersect inside \(B(0, 1)\).

**Proposition B.6.** Assume that \(\text{dist}(p, \text{span}(q)) \geq \delta^\tau\). Then, for small enough \(c, \delta > 0\), the intersection \(V_p(\delta^{1-c}) \cap V_q(\delta^{1-c})\) is contained in the \(\delta^c\)-neighbourhood of the the line \(V_p \cap V_q\).

**Proof.** Translating if necessary, one may assume that the line \(L = V_p \cap V_q\) passes through the origin. Let \(y \in V_p(\delta^{1-c}) \cap V_q(\delta^{1-c})\). Then \(y = l + x\), where \(l \in L\) and \(x \in L^+ = \text{span}\{\bar{p}, \bar{q}\}\). Here \(\bar{p} = (p_1, p_2, -p_3)/|p|\) and \(\bar{q} = (q_1, q_2, -q_3)/|q|\) are normal to \(V_p\) and \(V_q\), respectively. Then, since \(\{\bar{p}, \bar{q}, (\bar{q} - (\bar{p} \cdot \bar{q})\bar{p})/|\bar{q} - (\bar{p} \cdot \bar{q})\bar{p}|\}\) is an orthonormal basis for \(\text{span}\{\bar{p}, \bar{q}\}\), one sees that

\[
|q| \leq |\bar{q} - (\bar{p} \cdot \bar{q})\bar{p}| \leq |\bar{q}| + |\bar{p} \cdot \bar{q}|.
\]

Here \(|x \cdot \bar{p}|, |x \cdot \bar{q}| \leq R\delta^{1-c}\), since, for instance, \(|x \cdot \bar{p}| = \text{dist}(y, V_p) \leq R\delta^{1-c}\). On the other hand \(|\bar{q} - (\bar{p} \cdot \bar{q})\bar{p}| \geq \delta^\tau\) by assumption, so one obtains \(\text{dist}(y, L) = |x| \lesssim \delta^{1-c-\tau}\). Hence, the claim is true as long as \(c < 1 - c - \tau\). \(\square\)
**Proposition B.7.** Let $L$ be an arbitrary line in $\mathbb{R}^3$. Then the intersection $L(\delta^c) \cap C_0$ is contained in the $\delta^{2/5}$-neighbourhood of at most two lines on $C$.

**Proof.** Let $L$ be the line $L = \{r\xi + p : r \in \mathbb{R}\}$, where $\xi \in S^2$ and $p \in \mathbb{R}^3$. Assume first that $\xi$ forms a small angle with one of the lines on $C$, say $d(\xi, C) \leq \delta^{4/5}$. Then, if $q \in L(\delta^c)$, one may conclude that $L(\delta^c) \subset q + C(\delta^{4/5})$ for small enough $\delta > 0$. Thus, assuming that $L(\delta^c)$ intersects $C_0$ at even one point, say $q \in C_0$, then certainly $L(\delta^c) \cap C_0 \subset (q + C(\delta^{4/5})) \cap C(\delta^{4/5})$. But now Proposition B.2 is applicable and shows that $C(\delta^{4/5}) \cap (q + C(\delta^{4/5}))$ is contained in the $\delta^{2/5}$-neighbourhood of a single line on $C$.

Next, assume that $d(\xi, C) \geq \delta^{4/4}$. By Proposition B.3, it suffices to prove that $L(\delta^c) \cap C$ is contained in the union of two small balls centred at points on $C$. The neighbourhood $L(\delta^c)$ is the union of the lines $L_q := \{r\xi + q : r \in \mathbb{R}\}$, where $q \in p + B(0, \delta^c)$. We may explicitly find the (at most) two points on $L_q \cap C$, since such points must satisfy

$$(r\xi_1 + q_1)^2 + (r\xi_2 + q_2)^2 - (r\xi_3 + q_3)^2 = 0,$$

amounting to

$$r = \frac{-2(\xi_1 q_1 + \xi_2 q_2 - \xi_3 q_3) \pm \sqrt{4(\xi_1 q_1 + \xi_2 q_2 - \xi_3 q_3)^2 - 4(\xi_1^2 + \xi_2^2 - \xi_3^2)(q_1^2 + q_2^2 - q_3^2)}}{2(\xi_1^2 + \xi_2^2 - \xi_3^2)}.$$  

The denominator is $\gtrsim \delta^{4/4}$, by the assumption $d(\xi, C) \geq \delta^{4/4}$. The numerator, on the other hand is $1/2$-Hölder continuous with respect to moving the point $q = (q_1, q_2, q_3)$ around. So, when $q$ ranges in $p + B(0, \delta^c)$, the solutions $r = r(q)$ can vary only inside intervals of length $\lesssim \delta^{-c/4} \cdot \delta^{c/4} = \delta^{c/4}$. This implies that the intersection $L(\delta^c) \cap C$ is contained in two balls of radius $\lesssim \delta^{c/4}$. \hfill \Box

**Proof of Lemma B.1.** The lemma follows by combining the propositions. If either $p$ or $q$ lies very close to the surface $C$, one is instantly done by Proposition B.2. If both points lie far from $C$, then Proposition B.5 implies that either $C_0 \cap C_p \cap C_q$ is empty, or $p$ does not lie close to the line spanned by $q$. In the latter case, the intersection $C_0 \cap C_p \cap C_q$ is contained in the small neighbourhood of a single line in $\mathbb{R}^3$, according to Proposition B.6. Finally, by Proposition B.7, the intersection of any such neighbourhood with $C_0$ is contained in the neighbourhood of at most two lines on $C$, as claimed. \hfill \Box

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