Quantum deformed algebras: coherent states and special functions

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Abstract

The Heisenberg algebra is first deformed with the set of parameters \( \{q, l, \lambda\} \) to generate a new family of generalized coherent states. The matrix elements of relevant operators are exactly computed. A proof on sub-Poissonian character of the statistics of the main deformed states is provided. This property is used to determine a generalized metric. A unified method of calculating structure functions from commutation relations of deformed single-mode oscillator algebras is then presented. A natural approach to building coherent states associated to deformed algebras is deduced. Known deformed algebras are given as illustration.

Furthermore, we generalize a class of two-parameter deformed Heisenberg algebras related to meromorphic functions, called \( \mathcal{R}(p,q)\)-deformed algebra. Relevant families of coherent states maps are probed and their corresponding hypergeometric series are computed. The latter generalizes known hypergeometric series and gives a generalization of the binomial theorem. We provide new noncommutative algebras and show that the involved notions of differentiation and integration generalize the usual \( q \)- and \( (p,q) \)-differentiation and integration. A Hopf algebra structure compatible with the \( \mathcal{R}(p,q) \)-algebra is deduced.

Besides, we succeed in giving a new characterization of Rogers-Szegö polynomials, called \( \mathcal{R}(p,q)\)-deformed Rogers-Szegö polynomials. Continuous \( \mathcal{R}(p,q)\)-deformed Hermite polynomials and their recursion relation are also deduced. Novel algebraic relations are provided and discussed.

The whole formalism is performed in a unified way, generalizing known relevant results which are straightforwardly derived as particular cases.

1 Introduction

Much attention has been paid to the study of quantum algebras (or groups) in their both physical and mathematical aspects, motivated by the passage from
classical physical systems to quantum systems. Recall that the classical and quantum mechanics share two basic concepts of states and observables. Indeed, in classical mechanics, states are generally points of a symplectic manifold $M$, and observables are real-valued functions on it. Of course they may be regarded as belonging to the space $\mathcal{F}(M)$ of differentiable complex-valued functions on $M$, but the outcomes of classical measurements are real. In quantum mechanics, states are one-dimensional subspaces of a complex Hilbert space $\mathcal{H}$, and observables are self-adjoint linear operators on $\mathcal{H}$, with real spectrum. The connection between classical and quantum mechanics is well made in the language of observables. In both classical and quantum mechanics, the observables form an associative algebra, which is commutative under the pointwise multiplication in the classical case and noncommutative under the composition of linear operator in the quantum case. Moreover, the time evolution for the classical system is expressed by the equation

$$\frac{df(x(t))}{dt} = \{H_c, f\} (x(t)), \quad f \in \mathcal{F}(M),$$

(1)

where $H_c$ is the classical Hamiltonian and $x(t) \in M$ is the state of the system at time $t$, while for the quantum system the time evolution of an operator $A$ is given by

$$\frac{dA}{dt} = [H_q, A],$$

(2)

where $H_q$ is some operator called quantum Hamiltonian.

Hence, the quantization or the passage from classical to quantum mechanics is somewhat like replacing a Poisson algebra by a Lie algebra.

The illustrative example is given by a particle moving along the real line. In the classical case, the manifold $M$ is the cotangent bundle $T^* \mathbb{R}$, and if $q$ is the coordinate on $\mathbb{R}$ (position) and $p$ the coordinate in the fiber direction (momentum), the Poisson bracket is

$$\{f_1, f_2\} = \frac{\partial f_1}{\partial q} \frac{\partial f_2}{\partial p} - \frac{\partial f_2}{\partial q} \frac{\partial f_1}{\partial p}, \quad f_1, f_2 \in C^\infty(\mathbb{R}).$$

(3)

In particular, the Poisson bracket of coordinate functions is

$$\{q, p\} = 1.$$  

(4)

In quantum case, the Hilbert space $\mathcal{H}$ is the space $L^2(\mathbb{R})$ of square integrable function of $q$, and operators are combinations of the operators $\hat{q}$ and $\hat{p}$ whose actions correspond to the multiplication by $q$ and the derivative $-i\hbar \frac{\partial}{\partial q}$, respectively. So,

$$[\hat{q}, \hat{p}] = i\hbar \mathbf{1}; \quad [\hat{q}, \mathbf{1}] = 0; \quad [\hat{p}, \mathbf{1}] = 0$$

(5)
where \([A, B] = AB - BA\), \(h\) is the Planck’s constant and \(1\) the identity operator. Given physical parameters \(m\) and \(\omega\) carrying physical dimensions, it is possible to present, in an equivalent way, the same algebra in terms of the following combinations:

\[ \hat{q} = \sqrt{\frac{\hbar}{2m\omega}} (b + b^\dagger), \quad \hat{p} = -i\sqrt{(m\hbar\omega/2)(b - b^\dagger)} \]  \(\text{(6)}\)

where \(b\) and \(b^\dagger\) are called \textit{annihilation and creation operators} of the harmonic oscillator obeying

\[ [b, b^\dagger] = 1; \quad [b, 1] = 0; \quad [b^\dagger, 1] = 0. \]  \(\text{(7)}\)

The algebra generated by \(\{\hat{q}, \hat{p}, 1\}\) or \(\{b, b^\dagger, 1\}\) satisfying (5) or (7) is called \textit{Weyl-Heisenberg algebra}.

From (7), one defines the operator \(N := b^\dagger b\), also called \textit{number operator}, with the properties:

\[ [N, b] = -b; \quad [N, b^\dagger] = b^\dagger; \quad [N, 1] = 0. \]  \(\text{(8)}\)

Let \(\mathcal{F}\) be a Fock space and \(\{|n\rangle|n \in \mathbb{N} \cup \{0\}\}\) be its orthonormal basis. The actions of \(b, b^\dagger\) and \(N\) on \(\mathcal{F}\) are given by

\[ b|n\rangle = \sqrt{n}|n - 1\rangle, \quad b^\dagger|n\rangle = \sqrt{n + 1}|n + 1\rangle \quad \text{and} \quad N|n\rangle = n|n\rangle \]  \(\text{(9)}\)

where \(\{|0\rangle\}\) is a normalized vacuum: \(b|0\rangle = 0,\quad \langle 0|0\rangle = 1\).

From (9) the states \(\{|n\rangle\}\) for \(n \geq 1\) are built as follows:

\[ |n\rangle = \frac{1}{\sqrt{n!}} (b^\dagger)^n |0\rangle \]  \(\text{(10)}\)

satisfying the orthogonality and completeness conditions:

\[ \langle n|m \rangle = \delta_{n,m} \quad \text{and} \quad \sum_{n=0}^\infty |n\rangle\langle n| = 1. \]  \(\text{(11)}\)

Thus, the Hilbert \(\mathcal{H}\) space is the Fock space \(\mathcal{F}\) and the operators are elements of the algebra generated by \(\{b, b^\dagger, 1\}\) with the relations (7)-(8). This algebra, also called the \textit{Fock algebra} or \textit{quantum oscillator algebra}, successfully describes physical phenomena (widely spread in nonrelativistic quantum mechanics) but unfortunately fails in solving some important problems (like divergences in field theories, physics under the Planck scale, symmetry breaking, etc.)

Generalization of \textit{canonical commutation relations} (5) or (7) was suggested long before the discovery of quantum groups, by Heisenberg to achieve the regularization for (nonrenormalizable) nonlinear spinor field theory. The issue was
considered as small additions to the canonical commutations relations \[27, 104\]. Snyder, investigating the infrared catastrophe of soft photons in the Compton scattering, raised this issue and built a noncommutative Lorentz invariant space-time where the non-commutativity of space operators is proportional to non-linear combinations of phase space operators \[101\]. Further modifications of the oscillator algebra and their possible physical interpretations as spectrum generating algebras of non standard statistics have been made since the earlier work of Snyder. As matter of citation, let us mention the \(q\)-oscillator algebras by Coon and coworkers \[7, 111\], Kuryshkin \[78\], Jannussis and collaborators \[59, 60, 61, 62\].

With the development of quantum groups, new aspects of \(q\)-oscillators have been identified \[12, 23, 79, 103, 107\] as a tool for providing a boson realization of quantum algebra \(su_q(2)\) using a \(q\)-analogue of the harmonic oscillator and the Jordan-Schwinger mapping, and then generalizations in view of unifying or extending different existing \(q\)-deformed algebras were elaborated \[13, 19, 83, 87\].

Quantum groups made their first appearance in the mathematical physics literature in connection with the quantum inverse scattering method, a technique for studying integrable quantum systems \[33, 34, 88, 106, 108\]. It was shown in Ref. \[77\] that the quantum linear problem of the quantum sine-Gordon equation is associated with the deformation of the Lie algebra \(sl_2\) unlike the classical problem which is associated with \(sl_2\) itself. Later Sklyanin \[99\] showed that deformations of Lie algebraic structures were not bound to this particular equation but they were a part of a more general theory.

In the second half of 1980’s, Drinfel’d realized that the algebraic structure associated to quantum inverse scattering method can be reproduced by suitable algebraic quantization of the Poisson Lie algebras \[29, 30\]. Jimbo obtained independently the same result using the representation theory of the corresponding algebra \[63, 64\]. In fact, they discovered that \textit{quantum groups are dual category of Hopf algebras which are neither commutative nor co-commutative} \[30, 64\]. Notice that most of the well studied concrete examples of quantum groups are deformations of the universal enveloping algebra of the semi-simple Lie algebras \[23, 30, 35, 63, 77, 78, 96, 99\].

Despite all useful properties and applications motivated by various one-parameter deformed algebras, the multi-parameter deformations aroused much interest because of their flexibility when dealing with concrete physical models \[9, 20, 21, 22, 24, 38, 39, 45, 51, 52, 53\].

Coherent states have practically followed the same trend as the quantum algebras. They were invented by Schrödinger in 1926 in the context of the quantum harmonic oscillator. They were defined as minimum-uncertainty states that exhibit classical behavior \[97\].

In 1963, coherent states were simultaneous rediscovered by Glauber \[48, 49\],
Klauder and Sudarshan in quantum optics of coherent light beams emitted by lasers. The following definition summarizes the concept of Schrödinger-Glauber-Klauder-Sudarshan coherent states:

**Definition 1.1** Coherent states (CS) are normalized states $|z\rangle \in \mathcal{H}$ satisfying one of the following three equivalent conditions:

(i) they saturate the Heisenberg inequality

$$\left(\Delta \hat{q}\right)\left(\Delta \hat{p}\right) \geq \frac{\hbar}{2}, \quad (12)$$

where $\left(\Delta X\right)^2 := \langle z|X^2 - \langle X \rangle^2 |z\rangle$ with $\langle X \rangle := \langle z|X|z\rangle$;

(ii) they are eigenvectors of the annihilation operator, with eigenvalue $z \in \mathbb{C}$:

$$b|z\rangle = z|z\rangle; \quad (13)$$

(iii) they are obtained from the ground state $|0\rangle$ of the harmonic oscillator by a unitary action of the Weyl-Heisenberg group:

$$|z\rangle = e^{zb^\dagger - zb}|0\rangle. \quad (14)$$

The important feature of these coherent states resides in the partition (resolution) of identity:

$$\int_{\mathbb{C}} \frac{d^2 z}{\pi} |z\rangle\langle z| = 1, \quad (15)$$

where we have put $[d^2 z] = d(Rez)d(Imz)$ for simplicity.

Since there, CS became very popular objects in mathematics (specially in functional analysis, group theory and representations, geometric quantization, etc.), and in nearly all branches of quantum physics (nuclear, atomic and solid state physics, statistical mechanics, quantum electrodynamics, path integral, quantum field theory, etc.). For more information we refer the reader to the references [4, 72, 89, 112].

The vast field covered by coherent states motivated their generalizations to other families of states deducible from noncanonical operators and satisfying not necessarily all above mentioned properties.

The first class of generalizations, based on the equivalent conditions given in definition 1.1, include:

a) The approach by Barut and Girardello considering coherent states as eigenstates of the annihilation operator. This approach was unsuccessful because of its
drawbacks from both mathematical and physics point of view as detailed in [47, 89].

b) The approach based on the minimum-uncertainty states, i.e. essentially on the original motivation of Schrödinger in his construction of wavepackets which follow the motion of a classical particle while retaining their shapes. This was the basis for building the intelligent coherent states for various dynamical systems [5, 6, 84, 85, 86]. Nevertheless, as has been emphasized by Zhang et al [112], such a generalization has several limitations.

c) The approach related to the unitary representation of the group generated by the creation and annihilation operators. In two papers by Klauder [69, 70] devoted to a set of continuous states, one finds the basic ideas of coherent states construction for arbitrary Lie groups, which have been exploited by Gilmore [46] and Perelomov [89, 90] to formulate a general and complete formalism of building coherent states for Heisenberg groups with various properties similar to those of the harmonic oscillator. The key result of this development was the intimate connection of the coherent states with the dynamical group of a given physical system.

Two other generalizations complete this first class of generalizations: (i) the covariant coherent states introduced in Ref. [4], considered as a generalization of Gilmore-Perelomov formalism in the sense that the CS are built from more general groups (homogeneous spaces), and (ii) the nonlinear coherent states related to nonlinear algebras. Even though nonlinear coherent states have been used to analyze some quantum mechanical systems as the motion of a trapped ion [65, 81], they are not merely mathematical objects. They were defined as right eigenstates of a generalized annihilation operator [80, 81].

The second class of generalizations is essentially based on the overcompleteness property of coherent states. This property was the raison d’être of the mathematically oriented construction of generalized coherent states by Ali et al [43, 44] or of the ones with physical orientations [43, 44, 71]. The construction of generalized CS corresponds to the problem of finding a map from a row set $X$, equipped with a measure $\mu(dx)$, to a (projective) Hilbert space of quantum states $\mathcal{H}$ (a closed subspace of $L^2(X, \mu)$), $x \mapsto |x\rangle$, defining a family of states $\{|x\rangle\}_{x \in X}$ obeying the following two conditions:

1. Normalization: $\langle x|x \rangle = 1$;
2. Resolution of the unit in $\mathcal{H}$:

$$\int_X |x\rangle\langle x|\nu(dx) = 1_{\mathcal{H}},$$

where $\nu(dx)$ is another measure on $X$, usually absolutely continuous with respect to $\mu(dx)$. This means that there exists a positive measurable function $h(x)$ such that $\nu(dx) = h(x)\mu(dx)$ [43].
Numerous publications continue to appear using this property, see for example [28, 53, 54, 92] and references therein. The overcompleteness property is the most important criteria to be satisfied by CS as required by Klauder’s criteria [71].

Let us also mention the generalization performed through the so-called coherent state map, elaborated by Odzijewicz [87] in 1998. It is now known that the coherent state map may be used as a tool for the geometric quantization à la Kostant-Souriau [82, 87]. See the works by Kirillov [68] and Kostant [75] for details on geometric quantization. Further generalization of the latter approach is performed in the framework of this thesis.

More recently, Horzela and Szafraniec [50] have introduced the measure-free approach of building CS, requiring two main objects:

\((1')\) a separable Hilbert space \(\mathcal{H}\) with a fixed orthonormal basis \(\{e_n\}_{n=0}^d\), \(d + 1 = \dim \mathcal{H}\);

\((2')\) a sequence \(\{\phi_n\}_{n=0}^d\) of complex valued functions on \(X\) satisfying

\[
\sum_{n=0}^d |\phi(x)|^2 < \infty \quad x \in X
\]

and

\[
\{\alpha_n\}_{n=0}^d \text{ and } \sum_{n=0}^d \alpha_n \phi_n(x) = 0 \, \text{ for all } x \text{ imply that all } \alpha_n \text{'s are 0.} \quad (18)
\]

The formula

\[
K(x, y) := \sum_{n=0}^d \phi_n(y) \overline{\phi_n(x)}
\]

is regarded as the definition of the reproducing kernel. Therefore, if \(K(x, x) \neq 0, x \in X\), then the prospective CS may be defined at \(x\) by

\[
|x| := \sum_{n=0}^d K(x, x)^{-\frac{1}{2}} \overline{\phi_n(x)} e_n.
\]

Hence,

\[
\langle x|y \rangle = (K(x, x)K(y, y))^{-\frac{1}{2}} K(x, y)
\]

which means that the states \(|x\rangle\) are not normalized. To avoid any further renormalization, the authors of [50] assumed that \(K(x, x) = 1, x \in X\) that leads to the normalized CS

\[
C_x := \sum_{n=0}^d \phi_n(x) e_n.
\]
This work is organized as follows. In Section 2, the Heisenberg algebra is deformed with the set of parameters \{q, l, \lambda\} and the structure function is deduced. The spectrum of the associated deformed oscillator is computed. Then, we built the associated deformed coherent states using the Klauder approach and investigate the quantum statistics and geometry of the deformed coherent states.

Section 3 is devoted to the unification of deformed single-mode oscillator algebras. We give a method of computing the so-called structure function which is the basis of coherent states construction related to a given algebra. We analyse known deformed oscillator algebras and compute corresponding structure functions and give coherent states satisfying the Klauder criteria.

Section 4 generalizes a class of two-parameter deformed Heisenberg algebras related to meromorphic functions. Relevant families of coherent states maps are probed and their corresponding hypergeometric series are computed. Moreover, an \( \mathcal{R}(p,q) \)-binomial theorem, generalizing the \((p,q)\)-binomial theorem given in [57] is deduced. This chapter provides the definitions of the \( \mathcal{R}(p,q) \)-trigonometric, hyperbolic and \((p,q)\)-Bessel functions, including their main relevant properties. The framework developed in this chapter can be considered as a reverse of the previous one.

In Section 5 we build a framework for \( \mathcal{R}(p,q) \)-deformed calculus, which provides a method of the computation for a deformed \( \mathcal{R}(p,q) \)-derivative, generalizing known deformed derivatives of analytic functions defined on a complex disc as particular cases corresponding to conveniently chosen meromorphic functions. We introduce a new algebra generated by four quantities provided some conditions are satisfied and define the \( \mathcal{R}(p,q) \)-differential calculus yielding the \( \mathcal{R}(p,q) \)-integration. Also, a construction of Hopf algebra structure is given in Subsection 5.3, while in Subsection 5.4 we show that some particular cases can be deduced from the constructed general formalism.

Section 6 addresses a new characterization of \( \mathcal{R}(p,q) \)-deformed Rogers-Szegö polynomials by providing their three-term recursion relations and the associated quantum algebra built with corresponding creation and annihilation operators. The whole construction is performed in a unified way, generalizing all known relevant results which are straightforwardly derived as particular cases. Continuous \( \mathcal{R}(p,q) \)-deformed Hermite polynomials and their recursion relations are also deduced. Novel relations are provided and discussed. Finally, there follow the concluding remarks.
2 The \((q; l, \lambda)\)-deformed Heisenberg algebra: coherent states, their statistics and geometry

The Heisenberg algebra is deformed with the set of parameters \(\{ q, l, \lambda \}\) to generate a new family of generalized coherent states respecting the Klauder criteria. In this framework, the matrix elements of relevant operators are exactly computed. Then, a proof on the sub-Poissonian character of the statistics of the main deformed states is provided. This property is used to determine the induced generalized metric.

2.1 \((q; l, \lambda)\)-deformed Heisenberg algebra

Consider now the following \((q; l, \lambda)\)-deformed Heisenberg algebra \[17\] generated by operators \(N, a, a^\dagger\) satisfying

\[
[N, a] = -a, \quad [N, a^\dagger] = a^\dagger,
\]

with the operator products

\[
aa^\dagger - a^\dagger a = l^2q^{-N+\lambda-1},
\]

where \(l\) and \(\lambda\) are real numbers with \(l \neq 0\).

One can readily check that the commutator \([., .]\) of operators is antisymmetric and satisfies the Jacobi identity conferring a Lie algebra structure to the \((q; l, \lambda)\)-deformed Heisenberg algebra. This algebra plays an important role in mathematical sciences in general, and, in particular, in mathematical physics. In a notable work \[66\], similar associative algebra has been investigated by Kalnins et al under the form:

\[
[H, E_+] = E_+ \quad [H, E_-] = -E_- \\
[E_+, E_-] = -q^{-H}\mathcal{E} \quad [\mathcal{E}, E_{\pm}] = 0 = [\mathcal{E}, H],
\]

where \(q\) is a real number such that \(0 < q < 1\). These authors showed that the elements \(\mathcal{C} = qq^{-H}\mathcal{E} + (q - 1)E_+E_-\) and \(\mathcal{E}\) lie in the center of this algebra. It admits a class of irreducible representations for \(\mathcal{C} = l^2I\) and \(\mathcal{E} = l^2 q^{\lambda-1}I\).

The \((q; l, \lambda)\)-deformed Heisenberg algebra \((23)\) is a generalized algebra in the sense that it can generate a series of existing algebras as particular cases. For instance, even the generalization of the Quesne-algebra performed in \[52, 94\] can be deduced from \((23)\) by setting \(l = 1\) and \(\lambda = 0\).

In the sequel, we consider the Fock space of the Bose oscillator constructed as follows. From the vacuum vector \(|0\rangle\) defined by \(a|0\rangle = 0\), the normalized vectors \(|n\rangle\) for \(n \geq 1\), i.e. eigenvectors of the operator \(N\), are obtained as \(|n\rangle = C_n(a^\dagger)^n|0\rangle\), where \(C_n\) stands for some normalization constant to be determined.
Proposition 2.1 The structure function of the \((q; l, \lambda)\)-deformed Heisenberg algebra \((23) - (24)\) is given by
\[
\varphi(n) = l^2 q^\lambda \frac{1 - q^{-n}}{q - 1} = l^2 q^\lambda [n]_q, \quad q > 0,
\]
where \([n]_q = \frac{1 - q^n}{1 - q}\), with \(0 < q < 1\) or \(1 < q\), is the \(q\)-numbers (also known as \(q\)-deformed numbers in Physics literature [42]).

Proof: From the definition (85), \(a^\dagger a = \varphi(N)\) and \(aa^\dagger = \varphi(N + 1)\). Thus, (24) is written as
\[
\varphi(N + 1) - \varphi(N) = l^2 q^{-N+\lambda-1}.
\]
Applying this relation to the vectors \(|n\rangle\), we obtain the recursion relation
\[
\varphi(n + 1) - \varphi(n) = l^2 q^{-n-\lambda-1}, \quad \forall n \in \mathbb{N}
\]
from which we deduce
\[
\varphi(n) = \varphi(0) + l^2 q^{-n} \frac{1 - q^{-n}}{q - 1}.
\]
Since, in particular, \(\varphi(N)|0\rangle = a^\dagger a|0\rangle = 0\) implies \(\varphi(0)|0\rangle = 0\), we have \(\varphi(0) = 0\). Then (26) follows. The structure function is also a strictly increasing function for \(x \in \mathbb{R}\) since
\[
\frac{d\varphi(x)}{dx} = l^2 q^{\lambda-x} \ln \left(\frac{q}{q - 1}\right) > 0, \quad \text{for } q > 0.
\]
Since \(\varphi(0) = 0\), it follows that \(\varphi(x) \geq 0\) for any real \(x > 0\) and in particular \(\varphi(n) \geq 0, \forall n \geq 0\). □

Proposition 2.2 The orthonormalized basis of the Fock space \(\mathcal{F}\) is given by
\[
|n\rangle = \frac{q^{n(n+1)/4}}{\sqrt{([P^2 q^\lambda]^{n})_q!}} (a^\dagger)^n |0\rangle, \quad n = 0, 1, 2, ...
\]
where \([0]_q! := 1\) and \([n]_q! := [n]_q[n-1]_q...[1]_q\).
Moreover, the action of the operators \(a\), \(a^\dagger\), \(N\), \(a^\dagger a\) and \(aa^\dagger\) on the vectors \(|n\rangle\) for \(n \geq 1\) are given by
\[
a|n\rangle = \sqrt{P^2 q^{\lambda-n}[n]_q} |n - 1\rangle, \quad (28)
\]
\[
a^\dagger|n\rangle = \sqrt{P^2 q^{\lambda-n-1}[n+1]_q} |n + 1\rangle, \quad (29)
\]
\[
N|n\rangle = n|n\rangle, \quad (30)
\]
\[
a^\dagger a|n\rangle = l^2 q^{\lambda-n}[n]_q |n\rangle, \quad (31)
\]
\[
aa^\dagger|n\rangle = l^2 q^{\lambda-n-1}[n+1]_q |n\rangle. \quad (32)
\]
Proof: To determine the constant of normalization $C_n$, we set

$$1 =: \langle n|n \rangle = |C_n|^2 \langle 0|a^n(a^\dagger)^n|0 \rangle = |C_n|^2 \varphi(n)\varphi(n-1)\cdots\varphi(1)\langle 0|0 \rangle$$

leading to $C_n = \sqrt{\frac{\varphi(n+1)/4}{(l^2 q^{\lambda})^n|n\rangle^\dagger}}$. Replacing $C_n$ by its value in the definition of $|n\rangle$ given above yields (27). The orthogonality of the vectors $|n\rangle$ is a direct consequence of $a|0\rangle = 0$. The rest of the proof is obtained from (27) using (23), (24) and (26). □

**Theorem 2.3** The operators $(a + a^\dagger)$ and $i(a - a^\dagger)$, defined on the Fock space $\mathcal{F}$, are bounded and, consequently, self-adjoint if $q > 1$. If $q < 1$, they are not self-adjoint.

Proof: The matrix elements of the operator $(a + a^\dagger)$ on the basis $|n\rangle$ are given by

$$\langle m|(a + a^\dagger)|n \rangle = x_n \delta_{m,n-1} + x_{n+1} \delta_{m,n+1}, n, m = 0, 1, 2, \cdots$$  \hspace{1cm} (33)

while the matrix elements of the operator $i(a - a^\dagger)$ are given by

$$\langle m|i(a - a^\dagger)|n \rangle = ix_n \delta_{m,n-1} - ix_{n+1} \delta_{m,n+1}, n, m = 0, 1, 2, \cdots$$  \hspace{1cm} (34)

where $x_n = (l^2 q^{\lambda-n}|n\rangle|q\rangle^{1/2}$. Besides, the operators $(a + a^\dagger)$ and $i(a - a^\dagger)$ can be represented by the two following symmetric Jacobi matrices, respectively:

$$\begin{pmatrix}
0 & x_1 & 0 & 0 & 0 & \cdots \\
x_1 & 0 & x_2 & 0 & 0 & \cdots \\
0 & x_2 & 0 & x_3 & 0 & \cdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots
\end{pmatrix}$$  \hspace{1cm} (35)

and

$$\begin{pmatrix}
0 & -ix_1 & 0 & 0 & 0 & \cdots \\
ix_1 & 0 & -ix_2 & 0 & 0 & \cdots \\
0 & ix_2 & 0 & -ix_3 & 0 & \cdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots
\end{pmatrix}$$  \hspace{1cm} (36)

Two situations deserve investigation:

• Suppose $q > 1$. Then,

$$|x_n| = \left(\frac{l^2 q^{\lambda} - q^n - 1}{q - 1 \frac{q^n}{q^n}}\right)^{1/2} < \left(\frac{l^2 q^{\lambda}}{q - 1}\right)^{1/2}, \forall n \geq 1.$$ 

Therefore, the Jacobi matrices in (35) and (36) are bounded and self-adjoint (Theorem 1.2., Chapter VII in Ref. [11]). Thus, $(a + a^\dagger)$ and $i(a - a^\dagger)$ are bounded
and, consequently, self-adjoint.

- Contrarily, if \( q < 1 \), then

\[
\lim_{n \to \infty} x_n = \lim_{n \to \infty} \left( l^2 q^\lambda \frac{1 - q^{-n}}{q - 1} \right)^{1/2} = \infty. \tag{37}
\]

Considering the series \( \sum_{n=1}^{\infty} 1/x_n \), we obtain

\[
\lim_{n \to \infty} \left( \frac{1}{x_{n+1}} \right) = \lim_{n \to \infty} \left( \frac{1 - q^{-n}}{1 - q^{-n-1}} \right)^{1/2} = q^{1/2} < 1.
\]

This ratio test leads to the conclusion that the series \( \sum_{n=1}^{\infty} 1/x_n \) converges. Moreover, \( 1 - 2q + q^2 = (1 - q)^2 \geq 0 \implies q^{-1} + q \geq 2 \). Hence,

\[
0 \leq \left( \frac{l^2 q^\lambda}{q - 1} \right)^2 \left( 1 - q^{-n}(q + q^{-1}) + q^{-2n} \right) \leq \left( 1 - 2q^{-n} + q^{-2n} \right) \left( \frac{l^2 q^\lambda}{q - 1} \right)^2
\]

\[
\Leftrightarrow 0 \leq \left( \frac{l^2 q^\lambda \left( 1 - q^{-n+1} \right)}{q - 1} \right) \left( \frac{l^2 q^\lambda \left( 1 - q^{-n-1} \right)}{q - 1} \right) \leq \left( \frac{l^2 q^\lambda \left( 1 - q^{-n} \right)}{q - 1} \right)^2
\]

\[
\Leftrightarrow 0 \leq x_{n-1}x_{n+1} \leq x_n^2.
\]

Therefore, the Jacobi matrices in (35) and (36) are not self-adjoint (Theorem 1.5., Chapter VII in Ref. [11]). \( \square \)

**Definition 2.4** The \((q; l, \lambda)\)-deformed position, momentum and Hamiltonian operators denoted by \( X_{l,\lambda,q} \), \( P_{l,\lambda,q} \) and \( H_{l,\lambda,q} \), respectively, are defined as follows:

\[
X_{l,\lambda,q} := \left( \frac{\hbar}{2m\omega} \right)^{1/2} (a + a^\dagger),
\]

\[
P_{l,\lambda,q} := -i \left( \frac{\hbar \hbar \omega}{2} \right)^{1/2} (a - a^\dagger)
\]

\[
H_{l,\lambda,q} := \frac{1}{2m} (P_{l,\lambda,q})^2 + \frac{1}{2} m\omega^2 (X_{l,\lambda,q})^2
\]

\[
= \frac{\hbar \omega}{2} (a^\dagger a + aa^\dagger).
\] \tag{38}

**Proposition 2.5** The following system characterization holds:

- The vectors \(|n\rangle\) are eigenvectors of the \((q; l, \lambda)\)-deformed Hamiltonian with respect to the eigenvalues

\[
E_{l,\lambda,q}(n) = \frac{\hbar \omega}{2} l^2 q^{\lambda - n - 1} (q[n]_q + [n + 1]_q). \tag{39}
\]

\[
\]
The first two relations (33) and (34) in the proof of the previous Theorem 2.3 yield
\[ f \text{ as defined in Definition 2.6} \]
The coherent states associated with the algebra (23)-(24) are defined as

Proof: Indeed, using the result of the Proposition 2.2, we get

\[ (\Delta X_{l,\lambda,q})_n^2 = \frac{m\hbar^2}{2} q^{\lambda-n-1}(q[n]_q + [n+1]_q), \]
\[ (\Delta P_{l,\lambda,q})_n^2 = \frac{\hbar^2}{2m\omega} q^{\lambda-n-1}(q[n]_q + [n+1]_q), \]
where \((\Delta A)^2_n = \langle A^2 \rangle_n - \langle A \rangle_n^2 \) with \( \langle A \rangle_n = \langle n|A|n \rangle \).

The position-momentum uncertainty relation is given by

\[ (\Delta X_{l,\lambda,q})_n (\Delta P_{l,\lambda,q})_n = \frac{\hbar^2}{2} q^{\lambda-n-1}(q[n]_q + [n+1]_q) \]
which is reduced, for the vacuum state, to the expression

\[ (\Delta X_{l,\lambda,q})_0 (\Delta P_{l,\lambda,q})_0 = \frac{\hbar^2}{2} q^{\lambda-1}. \]

Proof: Indeed, using the result of the Proposition 2.2 we get

\[ H_{l,\lambda}|n\rangle = \frac{\hbar\omega}{2}(a^\dagger a + aa^\dagger)) = \frac{\hbar\omega}{2} q^{\lambda-n-1}(q[n]_q + [n+1]_q)|n\rangle. \]
The first two relations (33) and (34) in the proof of the previous Theorem 2.3 yield

\[ \langle n|(a + a^\dagger)|n\rangle = 0 = \langle n|i(a - a^\dagger)|n\rangle \text{ and } \langle n|(a + a^\dagger)^2|n\rangle = x_n^2 + x_{n+1}^2 = \langle n|i^2(a - a^\dagger)^2|n\rangle. \]
Therefore, \( \langle n|X_{l,\lambda}|n\rangle = 0 = \langle n|P_{l,\lambda}|n\rangle, \langle n|X_{l,\lambda}^2|n\rangle = \frac{m\hbar\omega}{2}(x_n^2 + x_{n+1}^2) \) and \( \langle n|P_{l,\lambda}^2|n\rangle = \frac{\hbar^2}{2m\omega}(x_n^2 + x_{n+1}^2). \) The rest of the proof is obtained replacing \( x_n \) and \( x_{n+1} \) by their expressions.

\[ \square \]

2.2 Coherent states \(|z\rangle_{l,\lambda}\)

Definition 2.6 The coherent states associated with the algebra (23)-(24) are defined as

\[ |z\rangle_{l,\lambda} = \mathcal{N}_{l,\lambda}^{-1/2} (|z|^2) \sum_{n=0}^{\infty} \frac{q^{n(n+1)/4} z^n}{\sqrt{(l^2 q^\lambda)^n [n]_q!}} |n\rangle, \quad z \in \mathbb{D}_{l,\lambda}, \]

where

\[ \mathcal{N}_{l,\lambda}(x) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2} x^n}{(l^2 q^\lambda)^n [n]_q!} = \sum_{n=0}^{\infty} \frac{q^{n(n-1)/2}}{(q; q)_n} \left( \frac{1 - q}{l^2 q^\lambda} \right)^n \]

and

\[ \mathbb{D}_{l,\lambda} = \{ z \in \mathbb{C} : |z|^2 < R_{l,\lambda} \}, \text{ with } R_{l,\lambda} = \begin{cases} \infty & \text{if } 0 < q < 1 \\ \frac{p^2 q^\lambda}{q-1} & \text{if } q > 1. \end{cases} \]
$R_{l,\lambda}$ is the convergence radius of the series $N_{l,\lambda}(x)$.

Remark that the $q$-deformed coherent states introduced in [94] are recovered as a particular case corresponding to $l = 1$ and $\lambda = 0$.

We now aim at showing that the coherent states (44) satisfy the Klauder’s criteria [72, 71]. To this end let us first prove the following lemma:

**Lemma 2.7** If $q > 1$, then

$$\frac{N_{l,\lambda}(x)}{N_{l,\lambda}(q^{-1}x)} = \frac{1}{1 - (q - 1)x/(l^2 q^\lambda)}, \quad (47)$$

$$N_{l,\lambda}(x) = \frac{1}{((q - 1)x/(l^2 q^\lambda); q^{-1})_{\infty}}, \quad (48)$$

$$\int_{0}^{R_{l,\lambda}} x^n (N_{l,\lambda}(q^{-1}x))^{-1} d_q^l x = (l^2 q^\lambda)^n q^{-n(n+1)/2} [n]_q!.$$

**Proof:** We use the $(q; l, \lambda)$-derivative defined by

$$\partial_q^{l,\lambda} f(x) = l^2 q^\lambda f(x) - f(q^{-1}x) \quad (50)$$

to obtain

$$N_{l,\lambda}(x) = \partial_q^{l,\lambda} N_{l,\lambda}(x) = l^2 q^\lambda \frac{N_{l,\lambda}(x) - N_{l,\lambda}(q^{-1}x)}{(q - 1)x}$$

which leads to (47) and

$$N_{l,\lambda}(x) = \frac{N_{l,\lambda}(q^{-n}x)}{\prod_{k=0}^{n-1} (1 - (q - 1)q^{-k}x/(l^2 q^\lambda))}, \quad n = 1, 2, \ldots \quad (51)$$

Letting $n$ to $+\infty$ and taking into account the fact that $N_{l,\lambda}(0) = 1$ lead to (48).

Next, we use the $(q; l, \lambda)$-integration given by

$$\int_{0}^{a} f(x) d_q^l x = \frac{q - 1}{l^2 q^\lambda a} \sum_{k=0}^{\infty} q^{-k} f(aq^{-k}) \quad (52)$$

to get

$$\int_{0}^{R_{l,\lambda}} x^n (N_{l,\lambda}(q^{-1}x))^{-1} d_q^l x = \sum_{k=0}^{\infty} q^{-(n+1)k} \frac{(l^2 q^\lambda)^n}{(q - 1)^n} (q^{-k+1}; q^{-1})_{\infty} \frac{q^{-n(k+1)}k}{(q^{-1}; q^{-1})_{\infty}}$$

$$= \frac{(l^2 q^\lambda)^n}{(q - 1)^n} (q^{-1}; q^{-1})_{\infty} \sum_{k=0}^{\infty} q^{-(n+1)k} \frac{q^{-n(k+1)}k}{(q^{-1}; q^{-1})_{k}}$$
\[
(n^2 q^\lambda)^n (q^{-1}; q^{-1})_\infty
\]
\[
= \frac{(q-1)^n (q^{-(n+1)}; q^{-1})_\infty}{(n^2 q^\lambda)^n (q^{-1}; q^{-1})_n (q^\lambda)^n q^{-n(n+1)/2} [n]_q!}.
\]

**Proposition 2.8** The coherent states defined in \[44\] (i) are normalized eigenvectors of the operator \(a\) with eigenvalue \(z\), i.e.
\[
a |z\rangle_{l,\lambda} = z |z\rangle_{l,\lambda}, \quad _{l,\lambda}\langle z|z\rangle_{l,\lambda} = 1;
\]
(ii) are not orthogonal to each other, i.e.
\[
_{l,\lambda}\langle z_1|z_2\rangle_{l,\lambda} \neq 0, \quad \text{when } z_1 \neq z_2;
\]
(iii) are continuous in their labels \(z\);
(iv) resolve the identity, i.e.
\[
1 = \int_{D_{l,\lambda}} d\mu_{l,\lambda}(\tilde{z}, z) |z\rangle_{l,\lambda}\langle z|,
\]
where
\[
d\mu_{l,\lambda}(\tilde{z}, z) = \frac{1 - q}{l^2 q^\lambda \ln q^{-1}} \frac{N_{l,\lambda}(z) d^2 z}{N_{l,\lambda}(\tilde{z}z/q) \pi}, \quad \text{if } 0 < q < 1,
\]
and
\[
d\mu(\tilde{z},z) = \frac{1}{2\pi} \frac{d^2 \chi}{1 - (q - 1)\chi/(l^2 q^\lambda)}, \quad x = |z|^2, \quad \theta = \arg(z),
\]
with \(0 < x < \frac{2q^\lambda}{q-1}\) and \(0 \leq \theta \leq 2\pi\) for \(q > 1\).

**Proof:**

• **Non orthogonality and normalizability**
\[
_{l,\lambda}\langle z_1|z_2\rangle_{l,\lambda} = \frac{N_{l,\lambda}(z_1 z_2)}{(N_{l,\lambda}(|z_1|^2) N_{l,\lambda}(|z_2|^2))^{1/2}} \neq 0
\]
implies that the coherent states are not orthogonal.

• **Normalizability**
From the above relation taking \(z_1 = z_2 = z\) we obtain \(_{l,\lambda}\langle z|z\rangle_{l,\lambda} = 1\). Also,
\[
a |z\rangle_{l,\lambda} = \mathcal{N}^{-1/2}(|z|^2) \sum_{n=0}^{\infty} \frac{q^{n(n+1)/4} z^n}{\sqrt{(l^2 q^\lambda)^n [n]_q!}} a |n\rangle
\]
\[ N_{l,\lambda}^{-1/2}(|z|^2) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)/4 + q} n}{\sqrt{(l^2 q^4) n + 1}[n]} |n| \]

\[ z N_{l,\lambda}^{-1/2}(|z|^2) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)/4 + q} n}{\sqrt{(l^2 q^4) n + 1}[n]} |n|. \]

**Continuity in the labels** \( z \)

\[ |||z_1|| - ||z_2||^2 = 2 (1 - R \Re \langle z_1 | z_2 \rangle). \]

So, \( |||z_1|| - ||z_2||^2 \rightarrow 0 \) as \( |z_1 - z_2| \rightarrow 0 \), since \( \Re \langle z_1 | z_2 \rangle \rightarrow 1 \) as \( |z_1 - z_2| \rightarrow 0 \).

**Resolution of the identity**

The computation of the RHS of (55) gives

\[ \int_{D_{l,\lambda}} d\mu_{l,\lambda}(\bar{z}, z) \langle z \rangle_{l,\lambda}(|z|^2) = \sum_{n,m} |n\rangle \langle m| \frac{q^{n(m+1)+m+1}/4}{\sqrt{(l^2 q^4) n + 1}[n]} \langle m| \int_{D_{l,\lambda}} \bar{z}^n z^m d\mu_{l,\lambda}(\bar{z}, z) \frac{N_{l,\lambda}(|z|^2)}{N_{l,\lambda}(|z|^2)}. \]

So, in order to satisfy (55) it is required

\[ \int_{D_{l,\lambda}} \bar{z}^n z^m \frac{d\mu_{l,\lambda}(\bar{z}, z)}{N_{l,\lambda}(|z|^2)} = \delta_{mn} (l^2 q^4)^n q^{m^2(1+\lambda)/2} \frac{[n]_q!}{[m]_q!}, \quad n, m = 0, 1, 2, \ldots \]  

(59)

Upon passing to polar coordinates, \( z = \sqrt{x} e^{i\theta} \), \( d\mu_{l,\lambda}(\bar{z}, z) = d\omega_{l,\lambda}(x) d\theta \) where \( 0 \leq \theta \leq 2\pi \), \( 0 < x < R_{l,\lambda} \) and \( \omega_{l,\lambda} \) is a positive valued function, this is equivalent to the classical Stieltjes power moment problem when \( 0 < q < 1 \) or the Hausdorff power moment problem when \( q > 1 \) [2]:

\[ \int_{0}^{R_{l,\lambda}} x^n \frac{2\pi d\omega_{l,\lambda}(x)}{N_{l,\lambda}(x)} = (l^2 q^4)^n q^{-n(n+1)/2} \frac{[n]_q!}{[n]_q!}, \quad n = 0, 1, 2, \ldots \]

(60)

If \( 0 < q < 1 \), then we have the following Stieltjes power moment problem:

\[ \int_{0}^{+\infty} x^n \frac{2\pi d\omega_{l,\lambda}(x)}{N_{l,\lambda}(x)} = (l^2 q^4)^n q^{-n(n+1)/2} \frac{[n]_q!}{[n]_q!}, \]

(61)
or, equivalently,

\[ \int_{0}^{+\infty} y^n \frac{2\pi d\omega_{l,\lambda}(l^2 q^4 y)}{E_q((1-q)y)} = q^{-n(n+1)/2} \frac{[n]_q!}{[n]_q!}, \]

(62)

where the change of variable \( y = \frac{x}{l^2 q^4} \) has been made. Atakishiyev and Atakishiyeva [8] have proved that

\[ g_q(n) = \int_{0}^{+\infty} \frac{y^{n-1} dy}{E_q((1-q)y)} = \frac{\ln q^{-1}}{1-q} q^{-n(n-1)/2} \frac{[n-1]_q!}{[n-1]_q!}. \]

(63)
Therefore we deduce
\[ d\omega_{l,\lambda}(l^2 q^\lambda y) = \frac{1}{2\pi \ln q^{-1}} \frac{1 - q \ E_q((1 - q)qy)dy}{E_q((1 - q)y)} \]
or
\[ d\omega_{l,\lambda}(x) = \frac{1}{2\pi l^2 q^\lambda \ln q^{-1}} \frac{1 - q \ E_q((1 - q)x/(l^2 q^\lambda))dx}{E_q((1 - q)x/(l^2 q^\lambda))} = \frac{1}{2\pi l^2 q^\lambda \ln q^{-1}} N_{l,\lambda}(x/q). \]

Hence
\[ d\mu_{l,\lambda}(\bar{z},z) = \frac{1 - q}{l^2 q^\lambda \ln q^{-1}} \frac{N_{l,\lambda}(|\bar{z}z|) d^2 z}{N_{l,\lambda}(\bar{z}z/q \pi)}. \]

On the other hand, if \( q > 1 \), then combining (60), (47) and (48) of the Lemma 2.7 we get
\[ d\mu(\bar{z},z) = \frac{1}{2\pi 1 - (q - 1)x/(l^2 q^\lambda)}, \quad x = |z|^2, \quad \theta = \arg(z), \]
\[ 0 < x < \frac{B_{\lambda}^\lambda}{q - 1} \quad \text{and} \quad 0 \leq \theta \leq 2\pi. \]

\subsection*{2.3 Statistics and geometry of coherent states |z⟩_{l,\lambda}}

The conventional boson operators \( b \) and \( b^\dagger \) may be expressed in terms of the deformed operators \( a \) and \( a^\dagger \) as
\[ b = a \sqrt{\frac{N}{\varphi(N)}} \quad \text{and} \quad b^\dagger = \sqrt{\frac{N}{\varphi(N)}} a^\dagger, \quad \varphi(N) \neq \varphi(0) \]
and their actions on the states \(|n⟩\) are given by
\[ b|n⟩ = \sqrt{n} |n - 1⟩, \quad \text{and} \quad b^\dagger|n⟩ = \sqrt{n + 1} |n + 1⟩. \]

Besides,
\[ b^r|n⟩ = \sqrt{\frac{n!}{(n-r)!}} |n - r⟩, \quad 0 \leq r \leq n \]
and
\[ (b^\dagger)^s|n⟩ = \sqrt{\frac{(n+s)!}{n!}} |n + s⟩. \]
2.3.1 Quantum statistics of the coherent states $|z\rangle_{l,\lambda}$

**Proposition 2.9** The expectation value of monomials of boson creation and annihilation operators $b^\dagger$, $b$ in the coherent states $|z\rangle_{l,\lambda}$ are given by

$$
\langle (b^\dagger)^s b^r \rangle = \frac{z^s z^r}{\mathcal{N}_{l,\lambda}(|z|^2)} \sum_{n=0}^{\infty} \sqrt{q^{(n+s)(n+s+1)+(n+r)(n+r+1)/2}(n+r)!(n+s)!} \frac{|z|^{2n}}{n!},
$$

where $s = 0, 1, 2, \ldots$ and $r = 0, 1, 2, \ldots$.

In particular,

$$
\langle (b^\dagger)^r b^r \rangle = \frac{x^r}{\mathcal{N}_{l,\lambda}(x)} \left( \frac{d}{dx} \right)^r \mathcal{N}_{l,\lambda}(x), \quad x = |z|^2, \quad r = 0, 1, 2, \ldots,
$$

and

$$
\langle N \rangle = \frac{\mathcal{N}_{l,\lambda}'(x)}{\mathcal{N}_{l,\lambda}(x)},
$$

where $\mathcal{N}_{l,\lambda}'(x)$ denotes the derivative with respect to $x$.

**Proof:** Indeed, for $s = 0, 1, 2, \ldots$ and $r = 0, 1, 2, \ldots$, we have

$$
\langle (b^\dagger)^s b^r \rangle := \langle l,\lambda | (b^\dagger)^s b^r | z \rangle_{l,\lambda}
= \frac{1}{\mathcal{N}_{l,\lambda}(|z|^2)} \sum_{m=0}^{\infty} \sum_{n=r}^{\infty} \sqrt{q^{m(m+1)+n(n+1)/2}n!(n-r+s)!} \frac{|z|^{2m}}{\mathcal{N}_{l,\lambda}(|z|^2)} \sum_{n=0}^{\infty} \sqrt{q^{(n+s-r)(n+s-r+1)+(n+r)(n+r+1)/2}n!(n-r+s)!} \frac{|z|^{2n}}{\mathcal{N}_{l,\lambda}(|z|^2)}. \quad (71)
$$

In the special case $s = r$, we have

$$
\langle (b^\dagger)^r b^r \rangle
= \frac{x^r}{\mathcal{N}_{l,\lambda}(x)} \sum_{n=0}^{\infty} \frac{q^{(n+r)(n+r+1)/2}(n+r)!}{(q^2\lambda)^{n+r}[n+r]_q!} \frac{|z|^{2n}}{n!}
= \frac{x^r}{\mathcal{N}_{l,\lambda}(x)} \sum_{n=r}^{\infty} \frac{q^{n(n+1)/2}(n)!}{(q^2\lambda)^n[n]_q!} \frac{|z|^{2n}}{n!}
= \frac{x^r}{\mathcal{N}_{l,\lambda}(x)} \left( \frac{d}{dx} \right)^r \mathcal{N}_{l,\lambda}(x), \quad x = |z|^2.
$$
In particular
\[
\langle N \rangle \equiv \langle b^\dagger b \rangle = x \frac{N''_{l,\lambda}(x)}{N_{l,\lambda}(x)}.
\]

The probability of finding \( n \) quanta in the deformed state \(|z\rangle_{l,\lambda}\) is given by
\[
P_{l,\lambda}(n) := |\langle n | z \rangle_{l,\lambda}|^2 = \frac{q^n(n+1)2^{n}}{(l^2q^l)^n[n]_q!}N_{l,\lambda}(x).
\]

(74)

The Mendel parameter measuring the deviation from the Poisson statistics is defined by the quantity
\[
Q_{l,\lambda} := \frac{\langle N^2 \rangle - \langle N \rangle^2 - \langle N \rangle}{\langle N \rangle}.
\]

(75)

Let us evaluate it explicitly. From the expectation value of the operator \( N^2 = (b^\dagger b)^2 + N \) provided by
\[
\langle N^2 \rangle = x^2 \frac{N''_{l,\lambda}(x)}{N_{l,\lambda}(x)} + x \frac{N'_{l,\lambda}(x)}{N_{l,\lambda}(x)},
\]

(76)

we readily deduce
\[
Q_{l,\lambda} = x \left( \frac{N''_{l,\lambda}(x)}{N_{l,\lambda}(x)} - \frac{N'_{l,\lambda}(x)}{N_{l,\lambda}(x)} \right).
\]

(77)

It is then worth noticing that for \( x \ll 1 \),
\[
Q_{l,\lambda} = -\frac{q(1-q)}{l^2q^l(1+q)}x + o(x^2)
\]

(78)

meaning that the \( P_{l,\lambda}(n) \) is a sub-Poissonian distribution \[71\].

2.3.2 Geometry of the states \(|z\rangle_{l,\lambda}\)

The geometry of a quantum state space can be described by the corresponding metric tensor. This real and positive definite metric is defined on the underlying manifold that the quantum states form, or belong to, by calculating the distance function (line element) between two quantum states. So, it is also known as a Fubini-Study metric of the ray space. The knowledge of the quantum metric enables one to calculate quantum mechanical transition probability and uncertainties.
In the case \( q < 1 \), the map from \( z \) to \( |z⟩_{l,λ} \) defines a map from the space \( \mathbb{C} \) of complex numbers onto a continuous subset of unit vectors in Hilbert space and generates in the latter a two-dimensional surface with the following Fubini-Study metric:

\[
dx^2 := ||d|z⟩_{l,λ}|^2 - |l,λ⟨z|d⟩_{l,λ}|^2
\]  

(79)

**Proposition 2.10** The above Fubini-Study metric is reduced to

\[
dx^2 = W_{l,λ}(x)d\bar{z}dz,
\]  

(80)

where \( x = |z|^2 \) and

\[
W_{l,λ}(x) = \left( \frac{N'_{l,λ}(x)}{N_{l,λ}(x)} \right)' = \frac{d}{dx}(N).
\]  

(81)

In polar coordinates, \( z = re^{iθ} \),

\[
dx^2 = W_{l,λ}(r^2)(dr^2 + r^2dθ^2).
\]  

(82)

**Proof:** Computing \( d|z⟩_{l,λ} \) by taking into account the fact that any change of the form \( d|z⟩_{l,λ} = α|z⟩_{l,λ} \), \( α \in \mathbb{C} \), has zero distance, we get

\[
d|z⟩_{l,λ} = N_{l,λ}(|z|^2)^{-1/2} \sum_{n=0}^{∞} \frac{q^n(n+1)/4nz^n}{(l^2q^λ)^n[n]_{q}!} |n⟩ dz.
\]

Then,

\[
||d|z⟩_{l,λ}||^2 = N_{l,λ}(|z|^2)^{-1} \sum_{n=0}^{∞} \frac{q^n(n+1)/2n^2|z|^{2n-1}}{(l^2q^λ)^n[n]_{q}!} d\bar{z}dz
\]

\[
= N_{l,λ}(|z|^2)^{-1} \left( \sum_{n=0}^{∞} \frac{q^n(n+1)/2n|z|^{2n-1}}{(l^2q^λ)^n[n]_{q}!} \right) + |z|^2 \sum_{n=0}^{∞} \frac{q^n(n+1)/2n(n-1)|z|^{2n-2}}{(l^2q^λ)^n[n]_{q}!} d\bar{z}dz
\]

\[
= N_{l,λ}(x)^{-1} \left( N'_{l,λ}(x) + xN''_{l,λ}(x) \right) d\bar{z}dz
\]

\[
= N_{l,λ}(x)^{-1} \left( xN'_{l,λ}(x) \right)' d\bar{z}dz
\]

and

\[
|l,λ⟨z|d⟩_{l,λ}|^2 = \left| N_{l,λ}(|z|^2)^{-1} \sum_{n=0}^{∞} \frac{q^n(n+1)/2n|z|^{2n-1}}{(l^2q^λ)^n[n]_{q}!} \bar{z}dz \right|^2
\]

20
\[ = x\mathcal{N}_{l,\lambda}(x)^{-2} (\mathcal{N}'_{l,\lambda}(x))^2 \, d\bar{z}dz. \]

Therefore,
\[
d\sigma^2 = \left( \mathcal{N}_{l,\lambda}(x)^{-1} (\mathcal{N}'_{l,\lambda}(x) + x\mathcal{N}''_{l,\lambda}(x)) - x\mathcal{N}_{l,\lambda}(x)^{-2} (\mathcal{N}'_{l,\lambda}(x))^2 \right) \, d\bar{z}dz
\]
\[
= \left( \frac{\mathcal{N}'_{l,\lambda}(x)}{\mathcal{N}_{l,\lambda}(x)} \right)' \, d\bar{z}dz = \left( \frac{d}{dx}(\mathcal{N}) \right) \, d\bar{z}dz. \]

\[\square\]

For \( x << 1 \), we have
\[
W_{l,\lambda}(x) = \frac{q}{t^2q^\lambda} \left[ 1 - \frac{2q(1-q)}{t^2q^\lambda(1+q)}x + o(x^2) \right]. \tag{83}\]

3 On generalized oscillator algebras and their associated coherent states

A unified method of calculating structure functions from commutation relations of deformed single-mode oscillator algebras is presented. A natural approach to building coherent states associated to deformed algebras is then deduced [16]. Known deformed algebras are given as illustration and such mathematical properties as the continuity in the label, normalizability and resolution of the identity of the corresponding coherent states are discussed.

3.1 Unified deformed single-mode oscillator algebras

Definition 3.1 We call deformed Heisenberg algebra, an associative algebra generated by the set of operators \( \{ 1, a, a^\dagger, N \} \) satisfying the relations
\[
[N, a^\dagger] = a^\dagger, \quad [N, a] = -a, \tag{84}\]
such that there exists a non-negative analytic function \( f \), called the structure function, defining the operator products \( a^\dagger a \) and \( aa^\dagger \) in the following way:
\[
a^\dagger a := f(N), \quad aa^\dagger := f(N + 1), \tag{85}\]
where \( N \) is a self-adjoint operator, \( a \) and its Hermitian conjugate \( a^\dagger \) denote the deformed annihilation and creation operators, respectively.
Afore-mentioned deformed Heisenberg algebras have a common property characterized by the existence of a self-adjoint number operator $N$, a lowering operator $a$ and its formal adjoint, called raising operator, $a^\dagger$ and differ by the expression of the structure function $f$.

The associated Fock space $\mathcal{F}$ is now spanned by the orthonormalized eigenstates of the number operator $N$ given by:

$$|n\rangle = \frac{1}{\sqrt{f(n)!}}(a^\dagger)^n|0\rangle, \quad n \in \mathbb{N} \cup \{0\},$$  

where

$$f(n)! = f(n)f(n-1)...f(1) \quad \text{with} \quad f(0) = 0. \quad (87)$$

Moreover,

$$a|n\rangle = \sqrt{f(n)}|n-1\rangle, \quad a^\dagger|n\rangle = \sqrt{f(n+1)}|n+1\rangle. \quad (88)$$

We emphasize that the structure function $f$ is a key unifying methods of coherent state construction corresponding to deformed algebras. To this end let us first recall the definition of the canonical coherent states.

**Definition 3.2** The canonical coherent states (CS) are normalized states $|z\rangle \in \mathcal{H}$ satisfying one of the following three equivalent conditions [48, 49, 69, 70, 97, 102]:

(i) they saturate the Heisenberg inequality:

$$\langle \Delta \hat{q} \rangle \langle \Delta \hat{p} \rangle = \frac{\hbar}{2}, \quad (89)$$

where $\langle \Delta A \rangle^2 := \langle z|A^2 - \langle A \rangle^2|z\rangle$ with $\langle A \rangle := \langle z|A|z\rangle$;

(ii) they are eigenvectors of the annihilation operator, with eigenvalue $z \in \mathbb{C}$:

$$b|z\rangle = z|z\rangle; \quad (90)$$

(iii) they are obtained from the ground state $|0\rangle$ of the harmonic oscillator by a unitary action of the Weyl-Heisenberg group:

$$|z\rangle = e^{zb^\dagger - zb}|0\rangle. \quad (91)$$

From (91) and using the famous Baker-Campbell-Hausdorff formula

$$e^{A+B} = e^{\frac{1}{2}[A,B]}e^Ae^B \quad (92)$$
whenever \([A, [A, B]] = [B, [A, B]] = 0\), one obtains
\[
|z\rangle = e^{-|z|^2/2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} |n\rangle = e^{-|z|^2/2} e^{z^\dagger} |0\rangle, \quad z \in \mathbb{C}.
\] (93)

The important feature of these coherent states resides in the partition (resolution) of identity:
\[
\int_{\mathbb{C}} \frac{[d^2 z]}{\pi} |z\rangle \langle z| = 1,
\] (94)
where we have put \([d^2 z] = d(\text{Re}z) d(\text{Im}z)\) for simplicity.

Suppose that \(f(0) = 0, f(n) > 0\) for all \(n \in \mathbb{N}\) and denote \(D_f = \{ z \in \mathbb{C} : |z|^2 < R_f \}\), where \(R_f\) is the radius of convergence of the series (called deformed exponential function):
\[
N_f(x) := \sum_{n=0}^{\infty} \frac{x^n}{[f(n)]!}.
\] (95)

Then, the following holds:

**Proposition 3.3** The states
\[
|z, f\rangle := (N_f(|z|^2))^{-1/2} \sum_{n=0}^{\infty} \frac{z^n}{[f(n)]!} (a^\dagger)^n |0\rangle
\] \[= (N_f(|z|^2))^{-1/2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{[f(n)]!}} |n\rangle, \quad z \in D_f,
\] (96)
are normalized eigenstates of the raising operator \(a\) with eigenvalue \(z\). They are not orthogonal to each other. Moreover, the map \(z \mapsto |z, f\rangle\) from \(D_f \subset \mathbb{C}\) to the Fock space \(\mathcal{F}\) is continuous.

**Proof:** The first assertion is true by definition of states \(|z, f\rangle\) and the action of the raising operator \(a\). To prove the non orthogonality, let \(z_1, z_2 \in D_f\). Then,
\[
\langle z_1, f | z_2, f \rangle = \frac{N_f(\bar{z}_1 z_2)}{(N_f(|z_1|^2)^{1/2}N_f(|z_2|^2)^{1/2})} \neq 0, \quad \text{when} \quad z_1 \neq z_2.
\] (97)

Furthermore,
\[
|||z_1, f\rangle - |z_2, f\rangle||^2 = 2 (1 - \Re \langle z_1, f | z_2, f \rangle) \to 0 \quad \text{as} \quad |z_1 - z_2| \to 0
\] (98)
that means the map \(D_f \ni z \mapsto |z, f\rangle \in \mathcal{F}\) is continuous. \(\square\)

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The family \( \{ |z, f \rangle : z \in D_f \} \) will be called **coherent states** whether there exists a positive measure \( \mu_f \) such that \( \int_{D_f} d\mu_f(\tilde{z}, z)|z, f \rangle \langle z, f| = \sum_{n=0}^{\infty} |n\rangle \langle n| = 1 \), thus forming an overcomplete set of states, or equivalently

\[
\int_{D_f} \bar{z}^n z^m \frac{d\mu_f(\tilde{z}, z)}{N_f(|z|^2)} = \delta_{nm}[f(n)]!, \quad n, m = 0, 1, 2, ... \tag{100}
\]

Passing to polar coordinates, \( z = \sqrt{x}e^{i\theta} \), where \( 0 \leq \theta \leq 2\pi \), \( 0 < x < R_f \), and \( d\mu(\tilde{z}, z) = d\omega(x) d\theta \), the latter equation leads to the following classical Stieltjes (for \( R_f = \infty \)) or Hausdorff (\( R_f < \infty \)) power-moment problem \([2, 98] \):

\[
\int_0^{R_f} x^n \frac{2\pi d\omega_f(x)}{N_f(x)} = [f(n)]!, \quad n = 0, 1, 2, ... \tag{101}
\]

Note immediately that not all deformed algebras lead to coherent states because the moment problem (101) does not always have solution \([2, 98] \). Nevertheless, it is remarkable that the structure function \( f \) plays an important role in the construction of coherent states associated to an algebra. So, the question arises is then how to determine the structure function corresponding to a given algebra.

Many techniques have been proposed in literature \([9, 13, 19, 76, 83] \). A more general answer to this question can be given starting from the Meljanac et al \([83] \) point of view. Indeed, these authors introduced the generalized \( q \)-deformed single-mode oscillator algebra through the identity operator \( 1 \), a self-adjoint number operator \( N \), a lowering operator \( a \) and an operator \( \bar{a} \) which is not necessarily conjugate to \( a \) satisfying

\[
[N, a] = -a, \quad [N, \bar{a}] = \bar{a}, \tag{102}
\]

\[
a\bar{a} - F(N)\bar{a}a = G(N) \tag{103}
\]

where \( F \) and \( G \) are arbitrary complex analytic functions.

Such an algebra furnishes an appropriate approach for the unification of classes of deformed algebras known in the literature.

For the purpose, let us start from the relations (102) to get

\[
[N, a\bar{a}] = 0 = [N, \bar{a}a] \tag{104}
\]

implying the existence of a complex analytic function \( \varphi \) such that

\[
\bar{a}a = \varphi(N) \quad \text{and} \quad a\bar{a} = \varphi(N + 1). \tag{105}
\]
Therefore, Eq. (103) can be rewritten as follows

\[ \varphi(N + 1) - F(N)\varphi(N) = G(N). \]  

(106)

Denote now \( a^\dagger \) the Hermitian conjugate of the operator \( a \). Then,

\[ [N, a^\dagger] = a^\dagger, \quad \text{and} \quad \bar{a} = c(N)a^\dagger, \]  

(107)

where \( c(N) \) is a complex function. For convenience take \( c(N) = e^{i \arg \varphi(N)} \). Therefore, from (105) and the fact that \( a^\dagger a \) and \( aa^\dagger \) are Hermitian operators we necessarily have

\[ a^\dagger a = |\varphi(N)| \quad \text{and} \quad aa^\dagger = |\varphi(N + 1)|. \]  

(108)

We now assume the existence of a ”vacuum state” \( |0\rangle \) such that

\[ N|0\rangle = 0, \quad a|0\rangle = 0 \quad \text{and} \quad \langle 0|0 \rangle = 1, \]  

(109)

and construct the non normalized eigenvectors \( (a^\dagger)^n|0\rangle \) of the operator \( N \). It follows that

\[ \langle 0|(a^\dagger)^m(a^\dagger)^n|0\rangle = \delta_{mn} \prod_{k=1}^{n} |\varphi(k)| =: (|\varphi(n)|!) \delta_{mn}, \quad m, n = 0, 1, 2, \ldots \]  

(110)

and we have the following proposition

**Proposition 3.4** Suppose that the initial condition \( \varphi(0) = 0 \) is satisfied. Then

\[ \varphi(n) = [F(n - 1)]! \sum_{k=0}^{n-1} \frac{G(k)}{[F(k)]!}, \quad n \geq 1, \]  

(111)

where

\[ [F(k)]! = \begin{cases} F(k)F(k-1)\cdots F(1) & \text{if} \quad k \geq 1 \\ 1 & \text{if} \quad k = 0 \end{cases}. \]  

(112)

**Proof:** Applying (106) to the vector \( (a^\dagger)^n|0\rangle \), we obtain

\[ \varphi(n + 1) - F(n)\varphi(n) = G(n), \quad n = 0, 1, 2 \ldots \]  

(113)

Then the result follows. \( \square \)

Notice that for all \( n = 1, 2, \ldots \) each ”excited” state \( (a^\dagger)^n|0\rangle \) is an eigenstate of the operator \( N \) corresponding to the eigenvalue \( n \) with norm

\[ ||(a^\dagger)^n|0\rangle|| = \sqrt{|\varphi(n)|!}. \]  

(114)
Of course, these states are orthogonal, i.e.

\[ \langle 0 | a^m (a^\dagger)^n | 0 \rangle = 0 \text{ for } m \neq n. \]  

(115)

Now, if \( \varphi(n) \neq 0 \) for all \( n \geq 1 \), one normalizes the eigenstates \( (a^\dagger)^n | 0 \rangle \) of \( N \) and gets the vectors \( |n\rangle \in \mathcal{F} \) as

\[ |n\rangle = \frac{(a^\dagger)^n}{\sqrt{||\varphi(n)||!}} | 0 \rangle. \]  

(116)

In the opposite, if \( \varphi(n_0) = 0 \) for some \( n_0 \), then the state \( (a^\dagger)^{n_0} | 0 \rangle \) has zero norm and, consistently, we can put \( |n_0, \varphi\rangle \equiv 0 \). The corresponding Hilbert space is the finite-dimensional space \( \mathbb{C}^{n_0} \).

Besides, the following relations hold:

\[ a |n, \varphi\rangle = \sqrt{||\varphi(n)||} |n - 1, \varphi\rangle, \quad a^\dagger |n, \varphi\rangle = \sqrt{||\varphi(n+1)||} |n + 1, \varphi\rangle, \]  

(117)

\[ a^\dagger a |n, \varphi\rangle = |\varphi(n)||n, \varphi\rangle \quad \text{and} \quad aa^\dagger |n, \varphi\rangle = |\varphi(n+1)||n, \varphi\rangle \]  

(118)

showing that the structure function characterizing a given deformation is defined as follows: \( f(n) = \varphi(n) \) if \( \varphi(n) \geq 0 \), and \( f(n) = |\varphi(n)| \), in general.

Moreover, the conventional boson operators \( b \) and \( b^\dagger \) may be expressed in terms of the deformed operators \( a \) and \( a^\dagger \) as

\[ b = a \sqrt{\frac{N}{f(N)}}, \quad b^\dagger = \sqrt{\frac{N}{f(N)}} a^\dagger. \]  

(119)

Thus the actions of \( b \) and \( b^\dagger \) on the states are as usual

\[ b |n, \varphi\rangle = \sqrt{n} |n - 1, \varphi\rangle, \quad \text{and} \quad b^\dagger |n, \varphi\rangle = \sqrt{n + 1} |n + 1, \varphi\rangle. \]  

(120)

So, for simplicity of notations we set \( |n, \varphi\rangle = |n\rangle \). Note also that the Weyl-Heisenberg algebra oscillator corresponds to \( F(N) = G(N) = 1 \).

In the next section, we analyse known algebras in the light of the above developed formalism.

### 3.2 Application to known deformed algebras

#### 3.3 The Tamm-Dancoff deformed algebra

This algebra appeared in the frame of Tamm-Dancoff method [27, 104], in quantum field theory, and was defined by the commutation relations

\[ [N, a^\dagger] = a^\dagger, \quad [N, a] = -a, \]  

(121)

\[ aa^\dagger - qa^\dagger a = q^N, \]  

(122)
where \( q \) is an arbitrary complex non-zero number. This corresponds to the case \( F(N) = q \) and \( G(N) = q^N \), and yields

\[ \varphi(n) = nq^{n-1} \quad \text{and} \quad f(n) = n|q|^{n-1}. \] (123)

In this case, the exponential function \((95)\) written as

\[ N_q(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!|q|^{n(n-1)/2}} \] (124)

converges on the whole complex plane \( \mathbb{C} \) for \( |q| \geq 1 \).

**Proposition 3.5** The moment problem \((101)\) with \( R_f = +\infty \) has the following solution

\[ d\omega_q(x) = \frac{N_q(x)}{2\pi} \int_0^{\infty} \sqrt{|q|} \exp \left\{ - \left( \sqrt{|q|} \frac{t + \ln^2(x/t)}{2\ln|q|} \right) \right\} dx \] (125)

**Proof:** Setting \( \tilde{W}_q(x)dx = 2\pi d\omega_q(x) \), the moment problem \((101)\) is then written as follows:

\[ \int_0^{\infty} x^n \tilde{W}_q(x)dx = n|q|^{n(n-1)/2}, \quad n = 0, 1, 2, ... \] (126)

The inverse Mellin transforms of \( \tilde{f}_1(s) = |q|^{s^2/2} \) and \( \tilde{f}_2(s) = |q|^{-s/2} \Gamma(s) \) gives \( [93, 100] \)

\[ \mathcal{M}^{-1} \{ \tilde{f}_1(s) \} = \frac{e^{-\frac{\ln^2(x/t)}{2\ln|q|}}}{\sqrt{2\pi \ln |q|}} =: f_1(x) \quad \text{and} \quad \mathcal{M}^{-1} \{ \tilde{f}_2(s) \} = e^{-|q|x} =: f_2(x) \] (127)

respectively. Thus, the solution of the integral equation

\[ \int_0^{\infty} x^s \tilde{W}_q(x)dx = |q|^{s(s-1)/2} \Gamma(s+1), \] (128)

can be obtained using the Mellin integral formula \([91]\)

\[ \mathcal{M} \left\{ \int_0^{\infty} f_1(x/t) f_2(t)dt \right\} = \tilde{f}_1(s) \tilde{f}_2(s+1) \] (129)

which, in particular, gives

\[ \int_0^{\infty} x^{s-1} \left( \int_0^{\infty} \frac{e^{-\frac{\ln^2(x/t)}{2\ln|q|}}}{\sqrt{2\pi \ln |q|}} e^{-|q|x} dt \right) dx = |q|^{-1/2} |q|^{s(s-1)/2} \Gamma(s+1) \]
or equivalently
\[
\int_0^\infty x^s \left( \int_0^\infty \frac{\sqrt{|q|}}{x} \exp \left\{ - \left( \frac{\sqrt{|q|}}{x} t + \frac{\ln^2(x/t)}{2 \ln |q|} \right) \right\} dt \right) dx = |q|^{s(s-1)/2} \Gamma(s+1),
\] (130)

Therefore,
\[
\tilde{W}_q(x) = \int_0^\infty \frac{\sqrt{|q|}}{x} \exp \left\{ - \left( \frac{\sqrt{|q|}}{x} t + \frac{\ln^2(x/t)}{2 \ln |q|} \right) \right\} dt.
\] (131)

Then follows the result. □

Hence, the states defined in \( \mathbb{C} \) by
\[
|z, q \rangle = (N_q(|z|^2))^{-1/2} \sum_{n=0}^{\infty} \frac{z^n}{n! |q|^{n(n-1)/2}} (a^\dagger)^n |0\rangle
\]
\[
= (N_q(|z|^2))^{-1/2} \sum_{n=0}^{\infty} \frac{z^n}{n! |q|^{n(n-1)/2}} |n\rangle
\] (132)

constitute a family of coherent states.

### 3.3.1 The Arick-Coon-Kuryskin deformed algebra (1976)

Arick and Coon first introduced this algebra \[7\] whose generators satisfy the following relations:
\[
[N, a^\dagger] = a^\dagger, \quad [N, a] = -a, \quad aa^\dagger - qa^\dagger a = 1.
\] (133)

The same algebra was examined independently by other authors like Kuryshkin \[78\], Jannussis \[59\], etc., and has gained popularity because of its connection to the developed mathematical \( q \)-analysis theory.

One can check that \( F(N) = q \) and \( G(N) = 1 \) leading to
\[
\varphi(n) = \frac{q^n - 1}{q - 1} =: [n]_q \quad \text{and} \quad f(n) = \varphi(n) \quad \text{for} \quad q \in [-1, 1[ \cup ]1, \infty). \] (135)

which is one of the forms of the so called \( q \)-numbers. This result is also obtained replacing \[136\] by
\[
[a, a^\dagger] = q^N.
\] (136)
Then follows the series
\[ N_q(x) = \sum_{n=0}^{\infty} \frac{x^n}{[n]_q!}. \] (137)

Arick and Coon have considered the case \(0 < q < 1\) for which \(N_q(x) = e_q((1 - q)x)\) where \(e_q(x)\) is one of the famous Jackson \(q\)-exponential functions, which converges for \(|x| < 1\) [32]. In this case the radius of convergence of \(N_q(x)\) is \(R_q = \frac{1}{1-q}\). Note also that,
\[ \partial_q N_q(x) = N_q(x), \] (138)
where the \(q\)-derivative \(\partial_q\) is defined as follows
\[ \partial_q f(x) = \frac{f(x) - f(qx)}{(1-q)x}. \] (139)

The following statements hold:

**Lemma 3.6**
\[ \frac{N'_q(x)}{N_q(qx)} = \frac{1}{1 - (1-q)x}, \quad \frac{N'_q(x)}{(1-q)x; q)_{\infty}} = \frac{1}{((1-q)x; q)_{\infty}}, \] (140)
\[ e_q(x) = \sum_{k=0}^{\infty} \frac{x^n}{(q; q)_n} = N_q(x/(1-q)) = \frac{1}{(x; q)_{\infty}}. \] (141)

where \((a; q)_n = \prod_{k=0}^{n-1} (1-aq^k)\). Moreover,
\[ \int_0^{(1-q)^{-1}} x^n (N_q(qx))^{-1} d_qx = [n]_q!, \quad n = 0, 1, 2, \ldots, \] (142)
where
\[ \int_0^a f(x)d_qx = a(1-q) \sum_{k=0}^{\infty} f(aq^n)q^n \] (143)
defines the Jackson integral [32] of a function \(f\).

**Proof:** Equations (138) and (139) lead to (140)-(141). Using the Jackson integral we obtain
\[ \int_0^{(1-q)^{-1}} x^n (N_q(qx))^{-1} d_qx = \sum_{i=0}^{\infty} \frac{q(n+1)!}{(1-q)^n} \left(\frac{N_q(q^{i+1}/(1-q))}{N_q(q)}\right)^{-1} \]
\[
\sum_{l=0}^{\infty} \frac{q^{(n+1)l}}{(1-q)^n} (q^{l+1}; q)_\infty = \frac{(q; q)_\infty}{(1-q)^n} \sum_{l=0}^{\infty} \frac{q^{(n+1)l}}{(q; q)_l}
\]

Then (142). □

**Proposition 3.7** The solution of the moment problem (101) with the \( R_q = \frac{1}{1-q} \) is given by

\[
d\omega_q(x) = \frac{1}{2\pi} \frac{dq}{1 - (1-q)x}, \quad x = |z|^2
\] (144)

**Proof:** In this case the moment problem (101) becomes

\[
\int_{\frac{1}{1-q}}^{\frac{1}{1-q}} x^n \frac{2\pi d\omega_q(x)}{N_q(x)} = [n]_q^{-1}, \quad n = 0, 1, 2, ...
\] (145)

The comparison of moment problem (145) and (142) with the use of the first equality of previous relations (140) leads to (144). □

Thus, the states

\[
|z\rangle_q = (N_q(|z|^2))^{-1/2} \sum_{n=0}^{\infty} \frac{z^n}{[n]_q!} (a^\dagger)^n |0\rangle
\]

\[
= (N_q(|z|^2))^{-1/2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{[n]_q!}} |n\rangle
\] (146)

define a family of coherent states on \( D_q = \{ z \in \mathbb{C} : |z| < (1-q)^{-1/2} \} \).

### 3.3.2 The Feinsilver deformed algebra (1987)

The generators of the algebra by Feinsilver algebra \[36\ 37\] verify

\[
[a, a^\dagger] = q^{-2N}, \quad [N, a] = -a, \quad [N, a^\dagger] = a^\dagger,
\] (147)

where \( q \) is non-zero real number. It follows that \( F(N) = 1 \) and \( G(N) = q^{-2N} \) implying

\[
f(n) = \varphi(n) = \frac{1 - q^{-2n}}{1 - q^{-2}} = [n]_{q^{-2}}.
\] (148)

The change of parameters \( \tilde{q} = q^{-2} \), with \( q > 1 \), leads to the previous Arick-Coon-Kuryshkin algebra.
3.3.3 The Biedenharn-Macfarlane oscillator algebra (1989)

The generators of the deformed algebra introduced by Biedenharn [12] and independently by Macfarlane [79], in the context of oscillator realization of the quantum algebra $su_q(2)$, satisfy

$$[N, a^\dagger] = a^\dagger, \quad [N, a] = -a, \quad (149)$$
$$aa^{\dagger} - qa^{\dagger}a = q^{-N} \quad \text{or} \quad a^{\dagger}a - q^{-1}a^{\dagger}a = q^N, \quad q^2 \neq 1. \quad (150)$$

Here $F(N) = q$ and $G(N) = q^{-N}$. Therefore,

$$\varphi(n) = q^{n-1} \sum_{j=0}^{n-1} \frac{q^{-j}}{q^j} = \frac{q^n - q^{-n}}{q - q^{-1}}, \quad q^2 \neq 1 \quad (151)$$

Then, the structure function is given by

$$f(n) = \frac{q^n - q^{-n}}{q - q^{-1}} =: [n]_q^n, \quad q \in \mathbb{R}^*_+ \setminus \{1\}. \quad (152)$$

Hence, considering the series

$$N_q(x) = \sum_{n=0}^{\infty} \frac{x^n}{[n]_q^n}, \quad (153)$$

one notices that its radius of convergence $R_q = +\infty$.

Using the deformed derivative and integration defined by

$$\partial_q^B f(x) := \frac{f(qx) - f(q^{-1}x)}{(q - q^{-1})} \quad (154)$$

and

$$\int_{x_i}^{x_f} f(x) d_q^B x := (q - q^{-1}) \sum_{i=0}^{f} q^{|i|}f(q^{2i}x), \quad x_i = xq^{i}, \quad x_f = xq^{f} \quad (155)$$

respectively, You-quan and Zheng-mao [110] showed that

$$\int_0^{\infty} x^n N_q(-x) d_q^B x = [n]_q^n \quad (156)$$

Therefore, the power-moment problem [101] has a solution given by

$$d\omega_n(x) = \frac{1}{2\pi} N_q(x) N_q(-x) d_q^B x \quad (157)$$
and the states
\[ |z\rangle_B = (\mathcal{N}_B(|z|^2))^{-1/2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{|n|_q^B}} |n\rangle \] (158)
define a family of coherent states with \( z \in \mathbb{C} \).

**Remark 3.8**

- El Baz and Hassouni [32] demonstrated, for \( |q| = 1 \) and using the Fourier transforms, that the power-moment problem (101) has the solution
\[ d\omega_B(x) = \frac{1}{2\pi} \tilde{\mathcal{N}}_B(x) \tilde{W}_B(x) dx \] (159)
where \( \tilde{W}_B(x) \) is the Fourier transform of the series
\[ \tilde{W}_B(y) = \sum_{n=0}^{\infty} \frac{|n|_q^B!}{n!} (iy)^n \] (160)
i.e.
\[ \tilde{W}_B(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ixy} \tilde{W}_B(y) dy. \] (161)
Notice that in this case the function \( \mathcal{N}_B(x) \) is replaced by \( \tilde{\mathcal{N}}_B(x) = \sum_{n=0}^{\infty} \frac{x^n}{|[n]|_q^B!} \).

- Yan [109] has later examined this algebra with relations
\[ a^\dagger a = [N], \quad aa^\dagger = [N + 1], \quad [a, a^\dagger] = [N + 1] - [N]. \] (162)

### 3.3.4 The Calogero-Vasiliev oscillator algebra (1991)

In 1991 Vasiliev [105] introduced a deformed algebra whose generators satisfy
\[ [N, a] = -a, \quad [N, a^\dagger] = a^\dagger, \quad [N, K] = 0, \] (163)
\[ aK = -Ka, \quad a^\dagger K = -Ka^\dagger, \quad K^2 = 1, \] (164)
\[ [a, a^\dagger] = 1 + \nu K, \] (165)
where \( \nu \) is a real such that \( \nu > -\frac{1}{2} \) and \( K = (-)^N \) is the Klein operator interpreted as the generator of the symmetric group \( S_2 \). From (165), we have \( F(N) = 1 \) and \( G(N) = 1 + \nu(-)^N \). Therefore
\[ \varphi(2n) = 2n \quad \text{and} \quad \varphi(2n + 1) = 2(n + \nu) + 1, \quad n = 0, 1, 2, \ldots \] (166)
The exponential function \( N_\nu(x) \) written as
\[
N_\nu(x) = \sum_{n=0}^{\infty} \frac{x^n}{\varphi(n)!}
\]
converges everywhere \( x \). However, the corresponding moment problem \( \int_0^\infty x^n \frac{2\pi d\omega_\nu(x)}{N_\nu(x)} = \varphi(n)!, \quad n = 0, 1, 2, \ldots \)
remains to solve.

### 3.3.5 The \((p, q)\)-Chakrabarti-Jagannathan oscillator algebra (1991)

The two-parameter quantum algebra was first introduced by Chakrabarti and Jagannathan \([24]\) in order to generalize or/and unify the Arick-Coon-Kurskin oscillator algebra \((p = 1)\) and Biedenharn-Macfarlane oscillator algebra \((p = q)\). The generators satisfy
\[
[N, a] = -a, \quad [N, a^\dagger] = a^\dagger, \\
Pa^\dagger - qPa = p^{-N}, \quad \text{or} \quad Pa^\dagger - p^{-1}a^\dagger a = q^N,
\]
where \( p, q \in \mathbb{R}_+^* \).

From the first relation of (170) we deduce \( F(N) = q \) and \( G(N) = p^{-N} \). Therefore,
\[
\varphi(n) = q^{n-1}\sum_{j=0}^{n-1} \frac{p^{-j}}{q^j} = q^{n-1}\frac{1-((pq)^{-1})^n}{1-(pq)^{-1}} = \frac{p^n - q^n}{p^{-1} - q} =: [n]_{p^{-1}, q}.
\]
Notice that the second relation of (170) gives the same result. Hence, it suffices to consider only one of the two relations (170).

The Fock space of the Bose oscillator \( \mathcal{F}_{CJ} \) associated to this deformation is generated by the orthonormalized states
\[
|n\rangle = \frac{(a^\dagger)^n}{\sqrt{[n]_{p^{-1}, q}}}|0\rangle, \quad n = 0, 1, 2, \ldots
\]
where
\[
[n]_{p^{-1}, q}! = \begin{cases} 
1 & \text{if } n = 0 \\
[n]_{p^{-1}, q}[n-1]_{p^{-1}, q}\ldots[1]_{p^{-1}, q} & \text{if } n \geq 1
\end{cases}
\]
3.3.6 The Kalnins-Mukherjee-Miller oscillator algebra (1993)

This $q$-oscillator algebra generated by four elements $H$, $E_+$, $E_-$ and $E$ satisfying
\begin{align}
[H, E_+] &= E_+ \\
[H, E_-] &= -E_- \\
[E_+, E_-] &= -q^{-H}E \\
[E, E_+] &= 0 = [E, H],
\end{align}
where $0 < q < 1$, was introduced by Kalnins et al [66].

The elements $C = qq^{-H}E + (q-1)E_+E_-$ and $E$ lie in the center of this algebra. It admits a class of irreducible representations for $C = l^2 \mathbf{1}$ and $E = l^2q^{\lambda-1}\mathbf{1}$, where $l$ and $\lambda$ are real numbers with $l \neq 0$. Setting $N = H$, $a = E_-$ and $a^\dagger = E_+$, we get
\begin{align}
[N, a] &= -a, \\
[N, a^\dagger] &= a^\dagger, \\
\sum_{k=0}^{n-1} l^2q^{-k+\lambda-1} &= l^2q^{\lambda-\frac{n}{2}} \frac{1-q^{-n}}{q-1} = l^2q^{\lambda-n}[n]_q
\end{align}

Suppose $q < 1$. Then, the series
\begin{align}
\mathcal{N}_{l,\lambda}(x) &= \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}x^n}{(1-q)x_n} = \sum_{n=0}^{\infty} \frac{q^{n(n-1)/2}}{(q; q)_n} \left( \frac{(1-q)x_n}{l^2q^\lambda} \right)^n \\
&= E_q((1-q)x_n/(l^2q^\lambda)).
\end{align}

has a radius of convergence $R_{l,\lambda} = \infty$. While in the case $q > 1$, the factors $\mathcal{N}(n)$ remain positive for every $n \geq 0$, but the series $\mathcal{N}_{l,\lambda}(x)$ converges only for $|x| < \frac{l^2q^\lambda}{q-1} := R_{l,\lambda}$.

We have the following result:

Proposition 3.9 The power-moment problem (177) has a solution given by
\begin{align}
d\omega_{l,\lambda}(x) &= \frac{1}{2\pi l^2q^\lambda \ln q^{-1}} \frac{\mathcal{N}_{l,\lambda}(x)}{\mathcal{N}_{l,\lambda}(x/q)} \, dx, \text{ for } 0 < q < 1, \\
\int_{0}^{l^2q^\lambda} d\omega_{l,\lambda}(x) &= \int_{0}^{l^2q^\lambda} \frac{d\omega_{l,\lambda}(x)}{y}, \text{ for } q > 1.
\end{align}

\textbf{Proof:} See proof of proposition 2.8

Hence, the states $|z\rangle_{l,\lambda} = N_{l,\lambda}^{-1/2}(|z|^2) \sum_{n=0}^{\infty} \frac{q^{n(n+1)/4}z^n}{\sqrt{(l^2q^\lambda)^n}|n\rangle_q}$ form a family of coherent states for $z \in D_{l,\lambda} = \{ z \in \mathbb{C}; |z|^2 \leq R_{l,\lambda} \}$. 34
3.3.7 The Chung-Chung-Nam-Um oscillator algebra (1993)

The generalized \((q, \alpha, \beta)\)-algebra was introduced by Chung et al \cite{26} with the generators satisfying
\[
[N, a] = -a, \quad [N, a^\dagger] = a^\dagger, \tag{183}
\]
\[
aa^\dagger - qa^\dagger a = q^{\alpha N + \beta} \tag{184}
\]
where \(q \in \mathbb{R}^*_+\) and \(\alpha, \beta\) are real parameters. One notices that \(F(N) = q\) and \(G(N) = q^{\alpha N + \beta}\). Therefore,
\[
\varphi(n) = f(n) = F_{\alpha, \beta}(n; q) = \begin{cases} \frac{nq^\gamma(n-1)+\beta}{q^\gamma q^n-q^\gamma} & \text{if } \alpha = 1 \\ \frac{nq^\gamma(n-1)-\gamma}{q^\gamma q^n-q^\gamma} & \text{if } \alpha \neq 1. \end{cases} \tag{185}
\]
This algebra generalizes the algebras introduced by:
- Tamm-Dancoff, with \(\alpha = 1, \beta = 0\);
- Arick-Coon-Kuryshkin, with \(\alpha = \beta = 0\); and
- Biedenharn-Macfarlane, with \(\alpha = -1, \beta = 0\).

3.3.8 The Borzov-Damasky-Yegorov oscillator algebra (1993)

The generalized \(W_{\alpha, \beta}(q)\)-algebra was introduced by Borzov et al \cite{13} in order to unify a large class of known \(q\)-deformed oscillator algebras. The generators satisfy
\[
[N, a] = -a, \quad [N, a^\dagger] = a^\dagger, \tag{186}
\]
\[
aa^\dagger - q^n a^\dagger a = q^{\alpha N + \beta} \tag{187}
\]
where \(q \in \mathbb{R}^*_+\) and \(\alpha, \beta, \gamma\) are real parameters. Here \(F(N) = q^\gamma\) and \(G(N) = q^{\alpha N + \beta}\) leading to
\[
\varphi(n) = f(n) = F_{\alpha, \beta}(n; q) = \begin{cases} \frac{nq^\gamma(n-1)+\beta}{q^\gamma q^n-q^\gamma} & \text{if } \alpha = \gamma \\ \frac{nq^\gamma(n-1)-\gamma}{q^\gamma q^n-q^\gamma} & \text{if } \alpha \neq \gamma. \end{cases} \tag{188}
\]

3.3.9 The Brzeziński - Egusquinza - Macfarlane oscillator algebra (1993)

Brzeziński et al \cite{14} introduced this algebra as \(q\)-deformation à la Biedenharn - Macfarlane of the Calogero-Vasiliev oscillator algebra. It is governed by the relations:
\[
[N, a] = -a, \quad [N, a^\dagger] = a^\dagger, \quad [N, K] = 0, \tag{189}
\]
\[
aK = -Ka, \quad a^\dagger K = -Ka^\dagger, \quad K^2 = 1, \tag{190}
\]
\[
aa^\dagger - qa^\dagger a = q^{-N}(1 + 2\alpha K), \tag{191}
\]
where \( \alpha \in \mathbb{R}^*, q \in \mathbb{R}_+ \) and \( K = (-)^N \) is the Klein operator. Here \( F(N) = q \) and 
\[ G(N) = q^{-N}(1 + 2\alpha K) \]
leading to
\[
\varphi(n) =: f_\alpha(n) = \frac{q^n - q^{-n}}{q - q^{-1}} + 2\alpha \frac{q^n - (-1)^nq^{-n}}{q + q^{-1}}. \tag{192}
\]

### 3.3.10 The Quesne oscillator algebra (2002)

The coherent states introduced by Quesne \[94\] may be associated with the \( q \)-deformed algebra satisfying the relations \[52\]:
\[
[N, a] = -a, \quad [N, a^\dagger] = a^\dagger, \tag{193}
\]
\[ aa^\dagger - a^\dagger a = q^{-N-1} \] or \[ qaa^\dagger - a^\dagger a = 1 \] \tag{194}
where \( 0 < q < 1 \).

The first relation of \((194)\) suggests \( F(N) = 1 \) and \( G(N) = q^{-N-1} \) leading to
\[
\varphi(n) = \frac{1 - q^{-n}}{q - 1} = q^{-n}[n]_q =: [n]_q^Q. \tag{195}
\]

It is a particular case of the \textit{Kalnins-Miller-Mukherjee} algebra developed above with \( l = 1, \lambda = 0 \).

### 3.3.11 The \((q; \alpha, \beta, \gamma; \nu)\)-Burban oscillator algebra (2007)

In 2007, Burban \[19\] introduced the \((q; \alpha, \beta, \gamma; \nu)\)-oscillator algebra whose generators satisfy the relations
\[
[N, a] = -a, \quad [N, a^\dagger] = a^\dagger, \quad [N, K] = 0, \tag{196}
\]
\[ aK = -Ka, \quad a^\dagger K = -K a^\dagger, \quad K^2 = 1, \tag{197}
\]
\[ aa^\dagger - q^\gamma a^\dagger a = q^{\alpha N + \beta(1 + 2\nu K)}, \tag{198}
\]
where \( \nu \in \mathbb{R}^*, \alpha, \beta \in \mathbb{R}, q \in \mathbb{R}_+ \) and \( K = (-)^N \) is the Klein operator. Here \( F(N) = q^\gamma \) and \( G(N) = q^{\alpha N + \beta(1 + 2\nu K)} \) leading to
\[
\varphi(n) = q^{\gamma(n-1)} \sum_{j=0}^{n-1} \frac{q^{\alpha j + \beta(1 + 2\nu(-1)^j)}}{q^j} \tag{199}
\]
\[
\begin{cases}
q^{\gamma(n-1)+\beta(n + \nu(1 - (-1)^n))} & \text{if } \alpha = \gamma \\
q^\beta \left( \frac{q^{-n} - q^n}{q^\gamma - q^{-\gamma}} + 2\nu \frac{q^\gamma - (-1)^n q^n}{q^\gamma + q^{-\gamma}} \right) & \text{if } \alpha \neq \gamma.
\end{cases}
\]

For \( \alpha = \gamma \) and \( 1 + 2\nu > 0 \)
\[
f(n) = F^\gamma_{\alpha,\beta,\nu}(n; q) = q^{\gamma(n-1)+\beta(n + \nu(1 - (-1)^n))}. \tag{200}
\]

For each of the following cases
1. \((q < 1, \alpha < \gamma \text{ and } -1 < 2\nu < \frac{q^\gamma + q^\alpha}{q^\gamma - q^\alpha});\)

2. \((q > 1, \alpha > \gamma \text{ and } -1 < 2\nu < \frac{q^\gamma + q^\alpha}{q^\gamma - q^\alpha});\)

3. \((q < 1, \alpha > \gamma \text{ and } 1 + 2\nu > 0);\)

4. \((q > 1, \alpha < \gamma \text{ and } 1 + 2\nu > 0)\)

we have

\[
f(n) = F_{\alpha, \beta}^{\gamma} (n; q) = q^\beta \left( \frac{q^{7n} - q^{n^2}}{q^\gamma - q^\alpha} + 2\nu \frac{q^{7n} - (-1)^n q^{n^2}}{q^\gamma + q^\alpha} \right). \tag{201}
\]

3.3.12 The \((p, q)\) and \((p, q; \mu, \nu, f)\)-oscillator algebras (2007)

In 2007, our group \cite{52} introduced an algebra generalizing the Quesne oscillator algebra:

\[
\begin{align*}
[N, a] &= -a, \quad [N, a^\dagger] = a^\dagger, \tag{202} \\
p^{-1}aa^\dagger - a^\dagger a &= q^{-N-1}, \quad qaa^\dagger - a^\dagger a = p^{N+1} \tag{203}
\end{align*}
\]

From the first relation \eqref{203}, we get \(F(N) = p\) and \(G(N) = pq^{-N-1}\) and then

\[
\varphi(n) = p^{n-1} \sum_{j=0}^{n-1} \frac{pq^{-j-1}}{p^j} = p^n - q^{-n} \frac{q^n - q^{-n}}{q - p^{-1}} =: [n]^q_p. \tag{204}
\]

The \((p, q; \mu, \nu, f)\)-oscillator algebra is defined through the following commutation relations

\[
\begin{align*}
[N, a] &= -a, \quad [N, a^\dagger] = a^\dagger, \tag{205} \\
p^{-1}aa^\dagger - a^\dagger a &= \left( \frac{q^{\nu}}{p^{\mu-1}} \right)^N f(p, q) \tag{206}
\end{align*}
\]

where \(p, q, \mu, \nu\) are real numbers such that \(0 < pq < 1, p^\mu < q^{\nu-1}, p > 1\) and \(f\) a well behaved real and non-negative function of deformation parameters \(p\) and \(q\), satisfying \(\lim f(p, q) = 1\) as \((p, p) \to (1, 1)\).

Here, \(F(N) = \frac{q^{\nu-1}}{p^\mu}\) and \(G(N) = f(p, q) \frac{q^{\nu-1}}{p^{\mu-1}} \left( \frac{q^{\nu}}{p^{\mu-1}} \right)^N\), so that

\[
\varphi(n) = f(p, q) \left( \frac{q^{\nu}}{p^\mu} \right)^n \frac{p^n - q^{-n}}{q - p^{-1}} =: [n]^\mu_\nu. \tag{207}
\]
The series
\[ N_{p,q,f}^{\mu,\nu}(x) = \sum_{n=0}^{\infty} \frac{x^n}{[n]_{p,q,f}^{\mu,\nu}} \]  
has a radius of convergence \( R = +\infty \). It had also shown in [52] that the moment problem (101) has, for \( \mu = 1 \) and \( \nu = 0 \), the following solution
\[ d\omega_{p,q,f}^{1,0}(x) = \frac{1}{2\pi f(p,q)} \frac{p^{-1} - q}{(pq)^{-1} N_{p,q,f}^{1,0}(x/(pq))} dx. \]  
Finally, the states
\[ |z\rangle_{p,q,f}^{1,0} = \left( N_{p,q,f}^{1,0}(|z|^2) \right)^{-1/2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{[n]_{p,q,f}^{1,0}}} |n\rangle, \ z \in \mathbb{C}, \]  
form a family of coherent states.

### 3.3.13 Unified \((p, q; \alpha, \beta, \nu; \gamma)\)-deformed oscillator algebra (2012)

More recently, Baloiţa et al [9] introduced the unified \((p, q; \alpha, \beta, \nu; \gamma)\)-deformed oscillator algebra whose generators satisfy:
\[ [N, a] = -a, \quad [N, a^\dagger] = a^\dagger, \quad [N, K] = 0, \quad aK = -Ka, \quad a^\dagger K = -Ka^\dagger, \quad K^2 = 1, \quad aa^\dagger - p^\nu a^\dagger a = (1 + 2\gamma K)q^{\alpha N + \beta}, \]  
where, \( \alpha, \beta, \gamma, \nu \in \mathbb{R}, \ p, q \in \mathbb{R}_+ \) and \( K = (-)^N \) is the Klein operator.

Here, \( F(N) = p^\nu \) and \( G(N) = (1 + 2\gamma (-)^N)q^{\alpha N + \beta} \) and
\[ \varphi(n) = \begin{cases} q^\beta \left( \frac{p^{\nu n} - q^\alpha}{p^\nu - q^\alpha} + 2\gamma \frac{p^{\nu n} - (-1)^n q^\alpha}{p^\nu + q^\alpha} \right) & \text{if } p^\nu \neq q^\alpha \\ q^{\beta + \alpha(n-1)} \left( n + 2\gamma \frac{1 - (-)^n}{2} \right) & \text{if } p^\nu = q^\alpha. \end{cases} \]  
For \( p^\nu = q^\alpha \) and \( 1 + 2\gamma > 0 \)
\[ f(n) =: F_{\alpha,\beta;\gamma}^{\nu}(n; p, q) = q^{\beta + \alpha(n-1)} \left( n + 2\gamma \frac{1 - (-)^n}{2} \right). \]  
For each of the following cases:

1. \( p^\nu > q^\alpha \) and \( 1 + 2\gamma > 0 \), and

2. \( p^\nu < q^\alpha \) and \(-1 < 2\gamma < -\frac{p^\nu + q^\alpha}{p^\nu - q^\alpha}\)
we have
\[ f(n) =: F_{\alpha,\beta;\gamma}^{\nu}(n; p, q) = q^\beta \left( \frac{p^{\nu n} - q^\alpha}{p^\nu - q^\alpha} + 2\gamma \frac{p^{\nu n} - (-1)^n q^\alpha}{p^\nu + q^\alpha} \right). \]  

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4 \( R(p,q) \)-deformed quantum algebras: coherent states and special functions

We provide with a generalization of well known \((p,q)\)-deformed Heisenberg algebras, called \( R(p,q) \)-deformed quantum algebras, and study the corresponding \( R(p,q) \)-series. A general formulation of the binomial theorem is given. Special functions are obtained as limit cases. This work well prolongs a previous work by Odzijewicz [87]. Known results in the literature are recovered.

4.1 Theoretical framework

The development displayed in this section is essentially based on the formalism elaborated by Odzijewicz [87] in a nice, mathematically based work published in 1998, but unfortunately hushed up in the recent literature on the topic. In the mentioned work, this author investigated the quantum algebras generated by the coherent state maps of the disc, leading to a generalized analysis which includes standard analysis as well as \( q \)-analysis. He provided with the meromorphic continuation of the generalized basic hypergeometric series and constructed a reproducing measure, when the series is treated as a reproducing kernel. Indeed, much to our very great surprise, most all the remarkable coherent state generalizations, performed from the generalization of exponential function by different authors, can be generated from this more general theory.

Let \( \mathcal{H} \) be an infinite dimensional separable Hilbert space and \( \{ |n\rangle \}_{n=0}^{\infty} \) its canonical basis. Assume that there exists a sequence \( \{ f_n \}_{n=0}^{\infty} \) in \( \mathcal{H} \) such that

\[
f_n = c_n C |n\rangle \quad (217)
\]

where \( C \) and its inverse \( C^{-1} \) are bounded operators on \( \mathcal{H} \), and \( c_n \ (n = 0, 1, 2, \cdots) \) are real positive numbers satisfying the conditions

\[
\sup_{n\in\mathbb{N}} \frac{c_{n-1}}{c_n} < +\infty \quad \text{and} \quad R^{-1} = \lim_{n\to\infty} \sup_n \sqrt{c_n}. \quad (218)
\]

**Definition 4.1** [87] A coherent states map is a complex analytic map

\[
K : \mathbb{D}_R \rightarrow \mathcal{H} \setminus \{ 0 \}
\]

\[
z \mapsto K(z) = \sum_{n=0}^{\infty} f_n z^n \quad (219)
\]
where $\mathbb{D}_R = \{ z \in \mathbb{C} : |z| < R \}$. The states $K(z)$ are called coherent states and the operator $A$ admitting these states as eigenstates with eigenvalues $z \in \mathbb{D}_R$, i.e.

$$AK(z) = zK(z), \quad (220)$$

is said to be the annihilation operator.

The relations (217)-(220) lead to

$$C^{-1}AC|0\rangle = 0 \quad \text{and} \quad C^{-1}AC|n\rangle = \frac{c_n^{-1}}{c_n} |n-1\rangle \quad \forall n \geq 1. \quad (221)$$

Therefore, $\|C^{-1}AC\| < +\infty$ meaning that $A$ is a bounded operator. Its Hermitian conjugate, called the creation operator and denoted by $A^\dagger$, is also bounded. The algebra closure, spanned by the operators $\{A, A^\dagger\}$, in their norm topology, gives the so-called $C^*$-algebra $A_K$.

**Proposition 4.2** Let $\mathcal{O}(\mathbb{D}_R)$ be a set of holomorphic functions defined on the disc $\mathbb{D}_R$.

The map $I_K : \mathcal{H} \rightarrow \mathcal{O}(\mathbb{D}_R)$ such as

$$I_K(v) := <v | K(\cdot) >, \quad v \in \mathcal{H} \quad (222)$$

is an antilinear monomorphism of complex vector spaces. Moreover, considering the topology of simple convergence, $I_K(\mathcal{H})$ is dense on $\mathcal{O}(\mathbb{D}_R)$.

**Proof 4.1** From (219)

$$f_{N+1} = \lim_{z \rightarrow 0} \frac{1}{z^{N+1}} (K(z) - \sum_{k=0}^{N} f_k z^k) \quad \text{and} \quad f_0 = K(0), \quad (223)$$

it results, by induction, that $f_N$ belongs to the closure of the linear space spanned by the elements of the subset $\{K(z), |z| < \epsilon < R\} \subset \mathcal{H}$. Otherwise, the coherent states $K(z)$, $z \in \mathbb{D}_R$, form a linearly subset dense in $\mathcal{H}$. One can easily check that $I_K$ is antilinear, i.e.,

$$I_K(\alpha u + \beta v) = \bar{\alpha} I_K(u) + \bar{\beta} I_K(v), \quad \forall \alpha, \beta \in \mathbb{C} \quad u, v \in \mathcal{H}, \quad (224)$$

and

$$z^n = I_K \left( c_n^{-1}(C^\dagger)^{-1}|n\rangle \right) \quad (225)$$

implying $z^n \in I_K(\mathcal{H})$, $\forall n \geq 0$. }
Therefore, $I_K(\mathcal{H})$ inherits the Hilbert space structure endowed with the scalar product

$$\langle I_K(v_1)|I_K(v_2)\rangle := \langle v_1|v_2 \rangle \quad \forall v_1, v_2 \in \mathcal{H}. \quad (226)$$

Thus, the Hilbert space $\mathcal{H}$ can be identified with the dense subspace $I_K(\mathcal{H})$ of $\mathcal{O}(\mathbb{D}_R)$. Hence, $I_K \circ \mathcal{A}_K \circ I_K^{-1}$ is the analytic representation of the algebra $\mathcal{A}_K$ as shown in the following diagram:

![Diagram]

So, given an operator $X \in \mathcal{A}_K$, we find a unique linear operator $\beta_X$ into $\mathcal{O}(\mathbb{D}_R)$ such that $\beta_X = I_K \circ X \circ I_K^{-1}$. Consequently, the action of the analytic representation $(I_K \circ A^\dagger \circ I_K^{-1})$ of the creation operator $A^\dagger$ on $\varphi \in \mathcal{O}(\mathbb{D}_R)$ yields:

$$(I_K \circ A^\dagger \circ I_K^{-1}) \varphi(z) = z \varphi(z), \quad (227)$$

while that of the analytic representation $\partial := (I_K \circ A \circ I_K^{-1})$ of the annihilation operator, the so-called $K-$derivative, depends on the operator $C$ and parameters $c_n$. On the basis elements $z^n$, it gives

$$\partial z^n = \sum_{k=0}^{\infty} c_k c_n^{-1} \langle n|C^{-1}A^\dagger C|k \rangle z^k. \quad (228)$$

Without loss of generality and as a matter of convenience, we restrict, in the sequel, the analysis to a unity operator $C$, (i.e. $C = I$) leading to

$$\partial z^n = [n] z^{n-1} \quad (229)$$

where

$$[n] = \begin{cases} 0 & \text{if } n = 0 \\ \left(\frac{c_n^{-1}}{c_n}\right)^2 & \text{if } n \geq 1. \end{cases} \quad (230)$$

Therefore,

$$c_n = \frac{c_0}{\sqrt{|n|!}} \quad (231)$$

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with $[0]! = 1$ and $[n]! = [n][(n - 1)]!$, and

$$AA^\dagger |n\rangle = [n + 1]|n\rangle, \quad A^\dagger A |n\rangle = [n]|n\rangle,$$

$$c_n \geq \frac{c_0}{\|A\|^n} \quad \text{for} \quad \|A\| \geq R. \quad (232)$$

The sequence $\{[n]\}_{n \geq 0}$ converges and

$$R = \lim_{n \to \infty} \sqrt{[n]}.$$ \quad (234)

Similarly, the $K$—exponential function is defined by

$$\text{Exp}(\bar{v}, z) := \langle K(v) | K(z) \rangle \quad (235)$$

and satisfies the equation

$$\partial \text{Exp}(\bar{v}, z) = \bar{v} \text{Exp}(\bar{v}, z). \quad (236)$$

Setting $v = 1$, one has

$$\text{Exp}(z) := \text{Exp}(1, z) \quad (237)$$

which satisfies the equation

$$\partial \text{Exp}(z) = \text{Exp}(z). \quad (238)$$

Hence, provided the $K$—derivative ($\partial$) realization, we can determine the analytic representation, $I_K \circ A_K \circ I_K^{-1}$, of the $C^*$— algebra $A_K$. Moreover through the equalities (229) and (231), the coherent states map $K : \mathbb{D}_R \to \mathcal{H} \setminus \{0\}$ generate the coherent states

$$K(z) = \sum_{n=0}^{\infty} c_n z^n |n\rangle = \sum_{n=0}^{\infty} \frac{c_0}{\sqrt{[n]!}} z^n |n\rangle,$$

and the exponential functions (235) and (237) are reduced to

$$\text{Exp}(\bar{v}, z) = \sum_{n=0}^{\infty} \frac{c_0^2}{[n]!} (\bar{v} z)^n \quad (240)$$

and, after normalization (i.e. $c_0 = 1$), to

$$\text{Exp}(z) = \sum_{n=0}^{\infty} \frac{z^2}{[n]!} \quad (241)$$

respectively.

In the next section, we aim at extending the above formalism to a general $(p, q)$—analysis, constructing a $(\mathcal{R}, p, q)$— derivative, where $\mathcal{R}$ is a meromorphic function defined on $\mathbb{C} \times \mathbb{C}$ and can be, in particular cases, a rational function.
4.2 $\mathcal{R}(p, q)$–basic hypergeometric series related to meromorphic functions

The problem we set here consists in defining the derivative which leads to a generalization of $(p, q)$–algebras and $(p, q)$–basic hypergeometric series. Let $p$ and $q$ be two positive real numbers such that $0 < q < p$. Consider a meromorphic function $R$, defined on $\mathbb{C} \times \mathbb{C}$ by

$$R(x, y) = \sum_{k,l=-L}^{\infty} r_{kl} x^k y^l$$

with an eventual isolated singularity at the zero, where $r_{kl}$ are complex numbers, $L \in \mathbb{N} \cup \{0\}$, $\mathcal{R}(p^n, q^n) > 0 \ \forall n \in \mathbb{N}$, and $\mathcal{R}(1, 1) = 0$, and the following linear operators defined on $\mathcal{O}(\mathbb{D}_R)$ \cite{24, 57, 87} by:

$Q : \varphi \mapsto Q\varphi(z) = \varphi(qz)$

$P : \varphi \mapsto P\varphi(z) = \varphi(pz)$

$\partial_{p,q} : \varphi \mapsto \partial_{p,q}\varphi(z) = \frac{\varphi(pz) - \varphi(qz)}{z(p-q)}$.  \hfill (243)

Then, we define the analytic representation of the annihilation operator $A$, called the $\mathcal{R}(p, q)$–derivative, by

$$\partial_{\mathcal{R}(p,q)} := \partial_{p,q} \frac{p-q}{P-Q} \mathcal{R}(P, Q) = \frac{p-q}{pP - qQ} \mathcal{R}(pP, qQ) \partial_{p,q}.$$  \hfill (244)

Using relations \cite{240}, \cite{231}, and \cite{230}, we define the $\mathcal{R}(p, q)$-factors (also called $\mathcal{R}(p, q)$-numbers) by

$$\mathcal{R}!(p^n, q^n) := \begin{cases} 1 & \text{for } n = 0 \\ \mathcal{R}(p, q) \cdots \mathcal{R}(p^n, q^n) & \text{for } n \geq 1, \end{cases}$$ \hfill (245)

and inducing the coefficients $c_n$ and the coherent states map $K$ in the form

$$c_n^2 = \frac{c_0^2}{\mathcal{R}!(p^n, q^n)}$$ \hfill (246)

$$K_{\mathcal{R}(p,q)}(z) = \sum_{n=0}^{\infty} \frac{c_0}{\sqrt{\mathcal{R}!(p^n, q^n)}} z^n |n\rangle.$$ \hfill (247)

Besides, the relations \cite{240} and \cite{241} can be readily generalized to take the form

$$\text{Exp}_{\mathcal{R}(p,q)}(\bar{v}, z) = \sum_{n=0}^{\infty} \frac{c_0^2}{\mathcal{R}!(p^n, q^n)} (\bar{v}z)^n$$ \hfill (248)
and
\[ \text{Exp}_{(p,q)}(z) = \sum_{n=0}^{\infty} \frac{1}{n!} q^n z^n, \] (249)

respectively, with the virtue that \( c_0^2 = \text{Exp}_{(p,q)}(0) = 1. \) Unless we say otherwise \( R \) will represent the radius of convergence of the series. The equations (236) and (238) remain valid and (249) is a solution of the \( R(p,q) \)-difference equation
\[ R(P,Q)\text{Exp}_{(p,q)}(z) = z\text{Exp}_{(p,q)}(z). \] (250)

The following two statements are essential to perform a generalization of the \((p,q)\)-binomial theorem.

**Lemma 4.3** Let
\[ F(z) = \frac{z}{z - R(1,0)}, \]
\[ G(P,Q) = \frac{p(Q - P)R(pP,qQ) + (pP - qQ)R(1,0)}{pQR(pP,qQ)} \] (251)

if \( R(1,0) \neq 0, \) and
\[ F(z) = z, \]
\[ G(P,Q) = \frac{qQ - pP}{pQR(pP,qQ)} \] (252)

if \( R(1,0) = 0. \) Then the exponential function in (249) satisfies
\[ \text{Exp}_{(p,q)}(z) = [1 - F(z)G(P,Q)] \text{Exp}_{(p,q)} \left( q \frac{z}{p} \right). \] (253)

**Proof 4.2** From definitions (243) and (244), we deduce
\[ 1 = Q \frac{pP - qQ}{pQR(pP,qQ)} \frac{Q}{P} \partial_{R(p,q)}. \] (254)

The action of the operator (254) on the exponential function (249) gives
\[ \text{Exp}_{R(p,q)}(z) = \left[ 1 - z \frac{qQ - pP}{pQR(pP,qQ)} \right] \text{Exp}_{R(p,q)} \left( q \frac{z}{p} \right) \]

which corresponds to the case \( R(1,0) = 0. \) Now, if \( R(1,0) \neq 0, \) the identity in (254) can be rewritten
\[ 1 - z \frac{1}{R(1,0)} \partial_{R(p,q)} = \left[ Q \frac{p}{P} - z \frac{1}{R(1,0)} \frac{Q}{P} \partial_{R(p,q)} \right] \]
Finally, let Lemmas 4.3 and 4.4 be satisfied. Then, a generalization of Theorem 4.5 by Proof of theorem 4.1: The result follows from (256), tending \( r \) to \( \infty \).

\[
\text{Exp}_{\mathcal{R}(p,q)}(z) = \left[ 1 - \frac{p(Q - P)\mathcal{R}(pP, qQ) + (pP - qQ)\mathcal{R}(1, 0) Q}{pQ\mathcal{R}(pP, qQ)\mathcal{R}(1, 0)} \right] \mathcal{G}(P, Q) \text{Exp}_{\mathcal{R}(p,q)}\left( \frac{q^n}{p^n}z \right).
\]

\[\square\]

Lemma 4.4 Under assumptions of the Lemma 4.3

\[
\text{Exp}_{\mathcal{R}(p,q)}(z) = \prod_{k=0}^{n-1} \left[ 1 - F\left( \frac{q^k}{p^k}z \right) G(P, Q) \right] \text{Exp}_{\mathcal{R}(p,q)}\left( \frac{q^n}{p^n}z \right).
\]

\[\square\]

Proof 4.3: It is immediate by induction using Lemma 4.3.

Finally, the \( \mathcal{R}(p,q) \)-binomial formula is given through the following statement.

Theorem 4.5 Let Lemmas 4.3 and 4.4 be satisfied. Then, a generalization of the \((p,q)\)-binomial theorem can be expressed as:

\[
\sum_{n=0}^{\infty} \frac{1}{n!\mathcal{R}(p^n, q^n)}(z)^n = \prod_{n=0}^{\infty} \left[ 1 - F\left( \frac{q^n}{p^n}z \right) G(P, Q) \right] \cdot 1.
\]

\[\square\]

Proof of theorem 4.1: The result follows from (256), tending \( n \) to \( +\infty \).

Finally, let us consider the particular case when \( \mathcal{R} \) is a rational function defined by

\[
\mathcal{R}_s(x, y) = \frac{x^{1+s-r}(x - y)(c_1p^{-1}x - d_1q^{-1}y)\ldots(c_{s-r}p^{-1}x - d_{s-r}q^{-1}y)}{(-y)^{1+s-r}(a_1p^{-1}x - b_1q^{-1}y)\ldots(a_{s-r}p^{-1}x - b_{s-r}q^{-1}y)}
\]

where \( a_1, \ldots, a_{s-r}; b_1, \ldots, b_{s-r}; c_1, \ldots, c_{s-r}; d_1, \ldots, d_{s-r} \) are complex numbers, \( r \) and \( s \) being non negative integers. Then, the \( \mathcal{R}_s(p,q) \)-numbers and factorials, as well as the coefficients \( c_n \) of the corresponding exponential function are readily found to be

\[
[n],_{\mathcal{R}_s(p,q)} = \frac{(p^n - q^n)(c_1p^{n-1} - d_1q^{n-1})\ldots(c_{s-n}\cdot p^{n-1} - d_{s-n}\cdot q^{n-1})}{(-q/p)^{1+r-s}(a_1p^{n-1} - b_1q^{n-1})\ldots(a_{s-r}p^{n-1} - b_{s-r}q^{n-1})},
\]

\[
[n]!,_{\mathcal{R}_s(p,q)} = \frac{(a_1, b_1, \ldots, a_{s-r}, b_{s-r}; (p,q))_n \left[ (-1)^n(q/p)^{\frac{(s-r)}{2}} \right]^{1+r-s}}{(a_1, b_1, \ldots, a_{s-r}, b_{s-r}; (p,q))_n}, \quad n \geq 1,
\]

\[
c_n^2 = \frac{(a_1, b_1, \ldots, a_{s-r}, b_{s-r}; (p,q))_n \left[ (-1)^n(q/p)^{\frac{(s-r)}{2}} \right]^{1+r-s}}{(p,q), (c_1, d_1, \ldots, c_{s-r}, d_{s-r}; (p,q))_n}
\]
where
\[(a_1, b_1), \cdots, (a_r, b_r); (p, q)\]_n = ((a_1, b_1); (p, q))_n \cdots ((a_r, b_r); (p, q))_n, \\
((a_i, b_i); (p, q))_0 = 1 \quad \text{and} \\
((a_i, b_i); (p, q))_n = \prod_{k=0}^{n-1} (a_i p^k - b_i q^k) \quad \text{for} \ n \geq 1. \quad (262)

Therefore,
\[
\text{Exp}_{\mathcal{R},(p,q)}(z) = \sum_{n=0}^{\infty} \left(\frac{(a_1, b_1), \cdots, (a_r, b_r); (p, q)}{((p, q), (c_1, d_1), \cdots, (c_s, d_s); (p, q))} \frac{z^n}{n!} \right) \left[(-1)^n (q/p)^{2n}\right]^{1+s-r}
\]
proving that the \(\mathcal{R}_s(p,q)\)–exponential function corresponds to the twin-basic hypergeometric series,\( \Phi_s \) [57, 24]:
\[
\text{Exp}_{\mathcal{R},(p,q)}(z) = \Phi_s((a_1, b_1), \cdots, (a_r, b_r); (c_1, d_1), \cdots, (c_s, d_s); (p, q); z). \quad (263)
\]

For \(r = s + 1\), the coherent states map \(K_{\mathcal{R},(p,q)}\) and the exponential function \(\text{Exp}_{\mathcal{R},(p,q)}\) are defined on the unit disc \(D_1\) while, for \(r < s + 1\), they are defined on the whole complex plane. The exponential function \(\text{Exp}_{\mathcal{R}(p,q)}\) thus appears as a natural generalization of twin-basic (or \((p,q)\)–) hypergeometric series.

4.3 \(\mathcal{R}(p,q)\)– deformed quantum algebras

In this section, we deal with the study of the \(\mathcal{R}(p,q)\)– deformed quantum algebra. Relations between the annihilation and creation operators, and the operators \(Q\) and \(P\) as well as the algebra generated by the meromorphic function \(\mathcal{R}\) are obtained. Some relevant particular representations recovered in this framework are also investigated.

The use of the relation (232) and the \((\mathcal{R}(p,q)\)-factors engenders
\[
AA^\dagger K(z) = \sum_{n=0}^{\infty} \mathcal{R}(p^{n+1}, q^{n+1}) c_n z^n |n\rangle = \mathcal{R}(pP, qQ) K(z)
\]
giving
\[
AA^\dagger = \mathcal{R}(pP, qQ). \tag{264}
\]
By analogy, one can show that
\[
A^\dagger A = \mathcal{R}(P, Q). \tag{265}
\]

If one passes to analytic representation in which \(Q\), \(P\) and \(\partial_{p,q}\) are given by (243), and \(A^\dagger\) acts as multiplication by \(z\), one obtains the relations:
\[
AQ = qQA, \quad AP = pPA, \tag{266}
\]

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\[ QA^\dagger = qA^\dagger Q, \quad PA^\dagger = pA^\dagger P. \] (267)

It is clear that
\[ QP = PQ. \] (268)

So, the \( \mathcal{R}(p, q) \)-deformed quantum algebra is generated by the operators \( \{ 1, A, A^\dagger, Q, P \} \) that verify the relations (267)-(268).

Let us determine the analytic representation of the number operator \( N \). Taking into account \( N|n\rangle = n|n\rangle \) and (222), one obtains
\[ I_K \left( \frac{1}{c_n} N|n\rangle \right) (z) = I_K \left( \frac{n}{c_n} |n\rangle \right) (z) = nz^n = z \frac{dz^n}{dz} \] (269)
implying
\[ (I_K \circ N \circ I_K^{-1}) (z^n) = z \frac{dz^n}{dz}. \] (270)

Therefore, \( \forall f \in \mathcal{O}(\mathbb{D}_R) \) we infer
\[ (I_K \circ N \circ I_K^{-1}) f(z) = z \frac{d}{dz} f(z) \] (271)
and
\[ p^N f(z) = Pf(z) = f(pz) \quad \text{and} \quad q^N f(z) = Qf(z) = f(qz) \] (272)
for \( f \in \mathcal{O}(\mathbb{D}_R) \).

To sum up, the \( \mathcal{R}(p, q) \)-deformed quantum algebra is then generated by the set of operators \( \{ 1, A, A^\dagger, N \} \) and the commutation relations
\[ [N, A] = -A \quad \text{and} \quad [N, A^\dagger] = A^\dagger \] (273)
and
\[ [A, A^\dagger] = \mathcal{R}(p^{N+1}, q^{N+1}) - \mathcal{R}(p^{N}, q^{N}). \] (274)

**Remarks**

Particulars cases are readily recovered.

(i). **Odziejewicz generalization** \[87\] is recovered for \( 0 < q < p = 1 \).

Consider the meromorphic function \( \mathcal{R} \) defined on \( \mathbb{C} \) by
\[ \mathcal{R}(z) = \sum_{k=-L}^{\infty} r_k z^k \] (275)

which may have an isolated singularity at the zero and such that \( L \in \mathbb{N} \cup \{0\} \), \( \mathcal{R}(q^n) > 0 \) for \( n > 0 \), \( \mathcal{R}(0) > 0 \), and \( \mathcal{R}(1) = 0 \). The following results hold:
• The $\mathcal{R}(q)$-derivative is given by

$$\partial_{\mathcal{R}(q)} = \partial_q \frac{1 - q}{1 - q} \mathcal{R}(Q) = \frac{1 - q}{1 - q} \mathcal{R}(qQ) \partial_q$$  \hspace{1cm} (276)$$

where $Q \varphi(z) = \varphi(qz)$ and $\partial_q = \frac{1 - q}{1 - q} z$.

• The $\mathcal{R}(q)$-factors and $\mathcal{R}(q)$-factorials are given by

$$\mathcal{R}(q^n) \quad \text{i.e.} \quad \partial_{\mathcal{R}} z^n = \mathcal{R}(q^n) z^{n-1} \quad \text{for} \quad n \geq 0,$$

$$\mathcal{R}!(q^n) = \begin{cases} 1 & \text{for} \quad n = 0 \\ \mathcal{R}(q) \cdots \mathcal{R}(q^n) & \text{for} \quad n \geq 1, \end{cases}$$  \hspace{1cm} (277)\hspace{1cm} (278)

respectively.

• The coefficients $c_n$, the coherent states map $K_{\mathcal{R}}$ and the exponential functions $\text{Exp}_{\mathcal{R}}$ are given by

$$c_n^2 = \frac{c_0^2}{\mathcal{R}!(q^n)},$$  \hspace{1cm} (279)$$

$$K_{\mathcal{R}}(z) = \sum_{n=0}^{\infty} \frac{c_0}{\sqrt{\mathcal{R}!(q^n)}} z^n,$$  \hspace{1cm} (280)$$

$$\text{Exp}_{\mathcal{R}}(\bar{v}, z) = \sum_{n=0}^{\infty} \frac{1}{\mathcal{R}!(q^n)} (\bar{v}z)^n,$$  \hspace{1cm} (281)$$

$$\text{Exp}_{\mathcal{R}}(z) = \sum_{n=0}^{\infty} \frac{1}{\mathcal{R}!(q^n)} z^n,$$  \hspace{1cm} (282)$$

respectively.

• The latter exponential function is solution of the $q$–difference equation

$$[\mathcal{R}(Q)\text{Exp}_{\mathcal{R}}](z) = z\text{Exp}_{\mathcal{R}}(z),$$  \hspace{1cm} (283)$$

and satisfies the following generalization of the $q$–binomial theorem

$$\text{Exp}_{\mathcal{R}}(z) = \sum_{n=0}^{\infty} \frac{1}{\mathcal{R}!(q^n)} z^n = \prod_{n=0}^{\infty} [1 - F(zq^n)G(Q)].1$$  \hspace{1cm} (284)$$

where

$$F(z) = \frac{z}{z - \mathcal{R}(0)},$$

$$G(Q) = \frac{(Q - 1)\mathcal{R}(Q) + (1 - qQ)\mathcal{R}(0)}{QR(qQ)},$$  \hspace{1cm} (285)$$

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for \( L = 0 \), and
\[
F(z) = z, \\
G(Q) = \frac{qQ - 1}{QR(qQ)}
\] (286)
for \( L > 0 \).

- The quantum algebra \( A_{K(q)} \) generated by the set of operators
\( \{1, A, A^\dagger, Q\} \) satisfies the relations:
\[
AA^\dagger = R(qQ),
\]
\[
A^\dagger A = R(Q),
\]
from which one obtains
\[
\|A\| = \|A^\dagger\| = \sqrt{\sup_{n \in \mathbb{N}} R(q^n)},
\]
for \( L = 0 \), i.e. \( A \) and \( A^\dagger \) are bounded. Due to (287) and (288), \( A \) and \( A^\dagger \) are unbounded if \( L > 0 \).

Indeed, setting \( R(z) = R(1, z) \), (275) readily follows from (242). The derivative (276) is then deduced from (243). Using (245), one gets (277) and (278). Coefficients (279) and coherent states maps (280) are obtained from (246) and (247), while the exponential functions in (248) and (249) are reduced to (281) and (282), respectively. In the other hand, the \( q \)–difference equation (283) is directly obtained from (250). In the same way and using Theorem 4.5, we get the generalization of the \( q \)–binomial theorem (284). Finally, the relations (287) and (288) between \( A, A^\dagger \), and \( Q \) simply follow from (264) and (265).

(ii). Jagannathan’s generalization [57] The main results summarized as follows are particular cases of Theorem 4.5

- The \((p, q)\)–binomial theorem is:
\[
\Phi_0((a, b); -; (p, q); z) = \frac{(p, bz); (p, q))_\infty}{(p, az); (p, q))_\infty} \quad (290)
\]

- The exponential functions, denoted by \( e_{p,q} \) and \( E_{p,q} \), are
\[
e_{p,q}(z) := \Phi_0((1, 0); -; (p, q); z) = \sum_{n=0}^{\infty} \frac{p^{n(n-1)/2}}{(p, q); (p, q)_n} z^n \quad (291)
\]
\[ E_{p,q}(z) := 1 \Phi_0((0, 1); -(p, q); -z) = \sum_{n=0}^{\infty} \frac{q^{n(n-1)/2}}{((p, q); (p, q))_n} z^n \] 
and
\[ e_{p,q}(z) E_{p,q}(-z) = 1. \]

Indeed, if we consider
\[ 1 \mathcal{R}_0(x, y) = \frac{x - y}{\frac{a}{p} x - \frac{b}{q} y} \]
where \(0 < q < p\) and \(a, b\) are complex numbers, then we get
\[ 1 \mathcal{R}_0(p^n, q^n) = \frac{p^n - q^n}{ap^{n-1} - bq^{n-1}}, \]
\[ 1 \mathcal{R}_0!(p^n, q^n) = \frac{((p, q); (p, q))_n}{((a, b); (p, q))_n}, \]
\[ c_n^2 = \frac{((a, b); (p, q))_n}{((p, q); (p, q))_n}, \]
and
\[ \text{Exp}_{1 \mathcal{R}_0(p,q)} = \sum_{n=0}^{\infty} \frac{(a - b)(ap - bq)\ldots(ap^{n-1} - bq^{n-1})}{(p - q)\ldots(p^n - q^n)} z^n \]
\[ = \sum_{n=0}^{\infty} \frac{((a, b); (p, q))_n}{((p, q); (p, q))_n} z^n, \]
where \(((a, b); (p, q))_n = (a - b)(ap - bq)\ldots(ap^{n-1} - bq^{n-1})\), meaning that
\[ \text{Exp}_{1 \mathcal{R}_0(p,q)} = 1 \Phi_0((a, b); -(p, q); z). \]

In the other hand, since \(1 \mathcal{R}_0(1, 0) = \frac{x}{z}\), we get
\[ F(z) = \frac{az}{az - p} \quad \text{and} \quad G(P, Q) = \frac{a - b}{a}. \]

Then, the application of Theorem 4.5 yields
\[ \text{Exp}_{1 \mathcal{R}_0(p,q)} = \prod_{n=0}^{\infty} \left[ 1 - \frac{z(q/p)^n}{p - az(q/p)^n} (a - b) \right] \]
\[ = \prod_{n=0}^{\infty} \left[ \frac{p - bz(q/p)^n}{p - az(q/p)^n} \right] \]
\[ = \prod_{n=0}^{\infty} \left[ \frac{pp^n - bzq^n}{pp^n - azq^n} \right]. \]
Thus,
\[
\text{Exp}_{\mathcal{R}_{0}(p,q)} = \frac{((p, bz); (p, q))_{\infty}}{((p, az); (p, q))_{\infty}}.
\] (300)

So, (298) and (300) lead to (290). Finally, (291), (292) and (293) are straightforwardly obtained. It is worth noticing that one can also consider the meromorphic functions
\[
i\mathcal{R}_{0}(x, y) = \frac{x - y}{z^p},
\] (301)
\[
i\mathcal{R}_{0}(x, y) = \frac{x - y}{-z^q},
\] (302)

and use Theorem 4.5 to immediately obtain (291), (292) and (293).

### 4.4 \(\mathcal{R}(p,q)-\)coherent states

This section aims at proving that the coherent states derived from the coherent states map (247) satisfy the following conditions [4, 69, 70]:

(i). normalizability (as any vector of Hilbert space)

(ii). continuity in the label \(z\), and

(iii). existence of a resolution of identity with a positive definite weight function, implying that the states form an overcomplete set.

#### 4.4.1 Normalizability

The coherent states defined as
\[
|z\rangle_{\mathcal{R}(p,q)} := \left(\text{Exp}_{\mathcal{R}(p,q)}(|z|^2)\right)^{-1/2} \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} \frac{1}{(p^n, q^n)} z^n |n\rangle.
\] (303)

are normalized so that
\[
\mathcal{R}(p,q) \langle z | z \rangle_{\mathcal{R}(p,q)} = 1.
\] (304)

#### 4.4.2 Continuity in \(z\)

The coherent states \(|z\rangle_{\mathcal{R}(p,q)}\) are continuous in \(z\). Indeed,
\[
\left\| |z\rangle_{\mathcal{R}(p,q)} - |z'\rangle_{\mathcal{R}(p,q)} \right\|^2 = 1 - 2\text{Re} \left(\mathcal{R}(p,q) \langle z | z' \rangle_{\mathcal{R}(p,q)}\right) + 1,
\] (305)
where
\[ R(p,q)\langle z | z' \rangle R(p,q) = \left( \text{Exp}_{R(p,q)}(|z|^2) \text{Exp}_{R(p,q)}(|z'|^2) \right)^{-1/2} \text{Exp}_{R(p,q)}(\bar{z}, z'). \]

So,
\[ |z - z'| \to 0 \implies \| |z\rangle_{R(p,q)} - |z'\rangle_{R(p,q)} \|^2 \to 0. \] (306)

### 4.4.3 Resolution of identity

Assume that there exists a positive measure \( \mu \) on the disc \( \mathbb{D}_R \) for which the resolution of the identity
\[ I = \int_{\mathbb{D}_R} \frac{|K(z)\rangle\langle K(z)|}{\text{Exp}_{R(p,q)}(\bar{z}, z)} d\mu(\bar{z}, z) \] (307)
holds.

**Proposition 4.6** \( I_K(\mathcal{H}) \) is a subspace of the Hilbert space \( L^2(\mathbb{D}_R, \frac{1}{\text{Exp}_{R(p,q)}} d\mu) \).

Moreover, for \( \varphi, \psi \in I_K(\mathcal{H}) \)
\[ \langle \varphi | \psi \rangle = \int_{\mathbb{D}_R} \overline{\varphi(z)} \psi(z) \frac{1}{\text{Exp}_{R(p,q)}(\bar{z}, z)} d\mu(\bar{z}, z), \] (308)
and
\[ \varphi(z) = \int_{\mathbb{D}_R} \varphi(v) \text{Exp}_{R(p,q)}(\bar{v}, z) \frac{1}{\text{Exp}_{R(p,q)}(\bar{v}, v)} d\mu(\bar{v}, v). \] (309)

**Proof 4.4**: Let \( \varphi \in I_K(\mathcal{H}) \). There exists \( \zeta \in \mathcal{H} \) such that \( I_K(\zeta) = \varphi \). So,
\[ \langle \varphi | \varphi \rangle = \int_{\mathbb{D}_R} \overline{\varphi(z)} \varphi(z) \frac{1}{\text{Exp}_{R(p,q)}(\bar{z}, z)} d\mu(\bar{z}, z) \]
\[ = \int_{\mathbb{D}_R} I_K(\zeta)(z) I_K(\zeta)(z) \frac{1}{\text{Exp}_{R(p,q)}(\bar{z}, z)} d\mu(\bar{z}, z) \]
\[ = \int_{\mathbb{D}_R} \langle \zeta | K(z) \rangle \langle K(z) | \zeta \rangle \frac{1}{\text{Exp}_{R(p,q)}(\bar{z}, z)} d\mu(\bar{z}, z) \]
\[ = \langle \zeta | \int_{\mathbb{D}_R} |K(z)| \frac{1}{\text{Exp}_{R(p,q)}(\bar{z}, z)} d\mu(\bar{z}, z) | \zeta \rangle \]
\[ = \langle \zeta | \zeta \rangle = \| \zeta \|^2 < \infty. \]

Therefore, \( \varphi \in L^2(\mathbb{D}_R, \frac{1}{\text{Exp}_{R(p,q)}} d\mu) \).

Thus, since \( I_K(\mathcal{H}) \) is a vector space as subspace of \( O(\mathbb{D}_R) \), we conclude that \( I_K(\mathcal{H}) \) is a subspace of the Hilbert space \( L^2(\mathbb{D}_R, \frac{1}{\text{Exp}_{R(p,q)}} d\mu) \).
Hence, the scalar product (308) is well defined. Moreover, for $\varphi \in I_K(\mathcal{H})$ and $z \in \mathbb{D}_R$, we get
\[
\varphi(z) = \langle I_K \circ I_K^{-1} \circ \varphi \rangle(z) = \langle I_K^{-1} \circ \varphi | K(z) \rangle
\]
\[
= \langle I_K^{-1} \circ \varphi | \int_{\mathbb{D}_R} |K(v)| \langle K(v) | K(z) \rangle d\mu(\bar{v}, v) d\mu(\bar{v}, v) \rangle
\]
\[
= \int_{\mathbb{D}_R} \langle I_K^{-1} \circ \varphi | K(v) \rangle \langle K(v) | K(z) \rangle d\mu(\bar{v}, v)
\]
\[
= \int_{\mathbb{D}_R} \varphi(v) \frac{\Exp_{\mathbb{R}(p,q)}(\bar{v}, z)}{\Exp_{\mathbb{R}(p,q)}(\bar{v}, v)} d\mu(\bar{v}, v).
\]

Proposition 4.7 For $n, m \in \mathbb{N} \cup \{0\}$
\[
\int_{\mathbb{D}_R} \frac{z^n z^m}{\Exp_{\mathbb{R}(p,q)}(\bar{z}, z)} d\mu(\bar{z}, z) = \mathcal{R}!(p^n, q^n) \delta_{m,n}.
\] (310)

Passing to polar coordinate $z = \sqrt{x} e^{i\varphi}$, we get
\[
\int_{0}^{\mathbb{R}^2} x^n \frac{d\nu(x)}{\Exp_{\mathbb{R}(p,q)}(x)} = \frac{\mathcal{R}!(p^n, q^n)}{2\pi}, \text{ for } n = 0, 1, 2, \ldots.
\] (311)

Proof 4.5: For $n \in \mathbb{N} \cup \{0\}$, $z^n \in I_K(\mathcal{H})$, since $z^n = I_K(\sqrt{\mathcal{R}!(p^n, q^n)}|n)(z)$. So,
\[
\langle z^n | z^m \rangle = \int_{\mathbb{D}_R} \frac{z^n z^m}{\Exp_{\mathbb{R}(p,q)}(\bar{z}, z)} d\mu(\bar{z}, z)
\]
\[
= \int_{\mathbb{D}_R} \frac{I_K(\sqrt{\mathcal{R}!(p^n, q^n)}|n)) (z) I_K(\sqrt{\mathcal{R}!(p^n, q^n)}|m)) (z) d\mu(\bar{z}, z)}{\Exp_{\mathbb{R}(p,q)}(\bar{z}, z)}
\]
\[
= \sqrt{\mathcal{R}!(p^n, q^n)} \sqrt{|m|!_{\mathcal{R}(p,q)}} \int_{\mathbb{D}_R} \frac{\langle K(z) | n \rangle \langle K(z) | m \rangle}{\Exp_{\mathbb{R}(p,q)}(\bar{z}, z)} d\mu(\bar{z}, z)
\]
\[
= \sqrt{\mathcal{R}!(p^n, q^n)} \sqrt{|m|!_{\mathcal{R}(p,q)}} \int_{\mathbb{D}_R} \frac{\langle K(z) \rangle \langle K(z) \rangle}{\Exp_{\mathbb{R}(p,q)}(\bar{z}, z)} d\mu(\bar{z}, z) |n\rangle
\]
from which (310) follows. Setting $z = \sqrt{x} e^{i\varphi}$, $0 \leq x \leq \mathbb{R}^2$ and $0 \leq \varphi \leq 2\pi$, we get
\[
d\mu(\bar{z}, z) = d\mu(\sqrt{x} e^{-i\varphi}, \sqrt{x} e^{i\varphi}) = d\nu(x) d\varphi \text{ and } \Exp_{\mathbb{R}(p,q)}(\bar{z}, z) = \Exp_{\mathbb{R}(p,q)}(x).
\]
Then, from (310) and taking $m = n$ we have
\[
\mathcal{R}!(p^n, q^n) = \int_{0}^{2\pi} d\varphi \int_{0}^{\mathbb{R}^2} x^n \frac{d\nu(x)}{\Exp_{\mathbb{R}(p,q)}(x)} = 2\pi \int_{0}^{\mathbb{R}^2} x^n \frac{d\nu(x)}{\Exp_{\mathbb{R}(p,q)}(x)}.
\]
Dividing the left and right sides of the above equalities by \(2\pi\) we obtain (311).

As in [87] the quantities (311) may also be treated as defining properties of the measure \(\nu\). This is the famous moment problem of finding \(\nu\) from the knowledge of the fixed moments \(\mathcal{R}!(p^n, q^n), n = 0, 1, \cdots\).

It may be interesting to formulate the moment problem in a way more adequate for \(K\)-analysis. So, the integration with respect to the measure \(d\nu = \exp \mathcal{R}(p,q)\) is replaced by the \(K\)-integration

\[
\mathcal{I}x^n := \frac{1}{\mathcal{R}(p^{n+1}, q^{n+1})} x^{n+1},
\]

with some unknown analytic weight function \(\sigma\) such that its Taylor expansion

\[
\sigma(x) = \sum_{k=0}^{\infty} a_k x^k
\]

has \(R\) as its convergence radius. The \(K\)-integration is just the right inverse of the \(K\)-differentiation, i.e. \(\mathcal{I} \partial_{\mathcal{R}(p,q)} = \mathbb{I}\). Then, instead of looking for a measure \(d\nu\) which satisfies (311), the problem is to find an analytic function \(\sigma\) which satisfies the moment conditions

\[
\mathcal{I}x^n \sigma(x) \big|_{x=R^2} = \frac{\mathcal{R}!(p^n, q^n)}{2\pi} \quad \text{for} \quad n = 0, 1, 2, \cdots,
\]

i.e.

\[
\sum_{k=0}^{\infty} \frac{R^{2(n+k+1)}}{\mathcal{R}(p^{n+k+1}, q^{n+k+1})} a_k = \frac{\mathcal{R}!(p^n, q^n)}{2\pi} \quad \text{for} \quad n = 0, 1, 2, \cdots.
\]

**Remarks**

Special cases can be recovered:

(i). The \((p,q)\)-algebra of Chakrabarty and Jagannathan [24]

\[
AA^\dagger - qA^\dagger A = q^{-N} \quad AA^\dagger - qA^\dagger A = p^N
\]

\[
[N, A] = -A \quad [N, A^\dagger] = A^\dagger
\]

affords the associated coherent states

\[
|z\rangle = \left[\mathcal{N}_{p,q}(|z|^2)\right]^{-1/2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{[n]_{p,q}}} |n\rangle
\]
where

\[ \mathcal{N}_{p,q}(z) = \sum_{n=0}^{\infty} \frac{z^n}{[n]_{p,q}!}, \]  

(318)

as well as the \((p,q)\)-number and the \((p,q)\)-factorial given by

\[ [n]_{p,q} = \frac{p^{-n} - q^n}{p - 1 - q} \]

(319)

and

\[ [n]_{p,q}! = [1]_{p,q}[2]_{p,q}...[n]_{p,q}, \]

(320)

respectively, taking \( R(x,y) = \frac{1-xy}{(p^{-1}-q)x} \).

Indeed, one obtains relations in (316) through relations (273) and (274), whereas coherent states (317) become particular cases of coherent states (313) with the function \( \mathcal{N} \) in (318) as the analog of the exponential function (249); the \((p,q)\)-number and \((p,q)\)-factorial are deduced from the \( R(p,q) \)-factors and (245), respectively.

(ii). \((p,q)\)-generalization of \(q\)-Quesne algebra [52, 95]:

\[
\begin{align*}
p^{-1}AA^\dagger - A^\dagger A &= q^{N-1} & qAA^\dagger - A^\dagger A &= p^{N+1} \\
[N, A] &= -A & [N, A^\dagger] &= A^\dagger
\end{align*}
\]

(321)

generates the associated coherent states

\[ |z\rangle = \left[ \mathcal{N}_{p,q}(|z|^2) \right]^{-1/2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{[n]_{p,q}!}} |n\rangle \]

(322)

where

\[ \mathcal{N}_{p,q}(z) = \sum_{n=0}^{\infty} \frac{z^n}{[n]_{p,q}^Q!} \]

(323)

while the \((p,q)\)-number is given by

\[ [n]_{p,q}^Q = \frac{p^{-n} - q^n}{p - 1 - q}, \]

(324)

setting \( R(x,y) = \frac{xy^{-1}}{(q-p^{-1})y} \).

(iii). \((p,q; \mu, \nu, f)\)-deformed states of Hounkonnou and Ngompe [52]:

\[ |z\rangle_{p,q,\mu, \nu, f} = \left[ \mathcal{N}_{p,q}(|z|^2) \right]^{-1/2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{[n]_{p,q,\mu, \nu, f}!}} |n\rangle \]

(325)
where
\[ N_{p,q,f}^{\mu,\nu}(z) = \sum_{n=0}^{\infty} \frac{z^n}{[n]_{p,q,f}^{\mu,\nu}}, \] (326)

with the \((p,q)-\)number given by
\[ [n]_{p,q,f}^{\mu,\nu} = f(p,q)^{p^n q^{-n}} \frac{p^n - q^{-n}}{q - q^{-1}}, \] (327)
such that \(0 < pq < 1, \ p^\mu < q^{\nu-1}, \ p > 1, \) and \(f\) a well behaved real and non-negative function of deformation parameters \(p\) and \(q\), satisfying
\[ \lim_{(p,q)\to(1,1)} f(p,q) = 1, \] (328)
becomes a particular case in the generalization provided in this work, setting \(R(x,y) = f(p,q)^{\frac{y^n x^{-n}}{x^n - y^{-n} q - p^{-1}}}, \) \(f\) being meromorphic.

From the above mentioned \((p,q;\mu,\nu,f)-\)deformed states, other deformed states known in the literature can be easily recovered. See [52] for more details.

### 4.5 \(R(p,q)\)-trigonometric and hyperbolic functions

From the expression (249) of the exponential function, we obtain
\[
\begin{align*}
\text{Exp}_{R(p,q)}(iz) &= \sum_{n=0}^{\infty} \frac{(iz)^n}{R!(p^n, q^n)} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{R!(p^{2n}, q^{2n})} + i \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{R!(p^{2n+1}, q^{2n+1})} \tag{329}
\end{align*}
\]
and
\[
\begin{align*}
\text{Exp}_{R(p,q)}(-iz) &= \sum_{n=0}^{\infty} \frac{(-iz)^n}{R!(p^n, q^n)} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{R!(p^{2n}, q^{2n})} - i \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{R!(p^{2n+1}, q^{2n+1})}. \tag{330}
\end{align*}
\]

We then define the \(R(p,q)-\)cosine, sinus, hyperbolic cosine and sinus functions by
\[
\begin{align*}
\cos_{R(p,q)}(z) &= \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{R!(p^{2n}, q^{2n})}, \tag{331}
\end{align*}
\]
\[
\sin_{R(p,q)}(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{R!(p^{2n+1}, q^{2n+1})},
\]

(332)

\[
\cosh_{R(p,q)}(z) = \sum_{n=0}^{\infty} \frac{z^{2n}}{R!(p^{2n}, q^{2n})}
\]

(333)

and

\[
\sinh_{R(p,q)}(z) = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{R!(p^{2n+1}, q^{2n+1})},
\]

(334)

respectively. It is readily checked that

\[
\cos_{R(p,q)}(z) = \frac{1}{2} \left[ \text{Exp}_{R(p,q)}(iz) + \text{Exp}_{R(p,q)}(-iz) \right],
\]

(335)

\[
i \sin_{R(p,q)}(z) = \frac{1}{2} \left[ \text{Exp}_{R(p,q)}(iz) - \text{Exp}_{R(p,q)}(-iz) \right],
\]

(336)

\[
\text{Exp}_{R(p,q)}(iz) = \cos_{R(p,q)}(z) + i \sin_{R(p,q)}(z).
\]

(337)

In particular, the Euler formula is expressed as follows:

\[
\text{Exp}_{R(p,q)}(i\theta) = \cos_{R(p,q)}(\theta) + i \sin_{R(p,q)}(\theta).
\]

(338)

Besides, there follow the relations

\[
\cosh_{R(p,q)}(z) = \frac{1}{2} \left[ \text{Exp}_{R(p,q)}(z) + \text{Exp}_{R(p,q)}(-z) \right],
\]

(339)

\[
\sinh_{R(p,q)}(iz) = \frac{1}{2} \left[ \text{Exp}_{R(p,q)}(z) - \text{Exp}_{R(p,q)}(-z) \right],
\]

(340)

\[
\text{Exp}_{R(p,q)}(z) = \cosh_{R(p,q)}(z) + \sinh_{R(p,q)}(z).
\]

(341)

The derivatives are immediately expressible from their definition:

\[
\partial_{R(p,q)} \sinh_{R(p,q)}(az) = a \cosh_{R(p,q)}(az),
\]

(342)

\[
\partial_{R(p,q)} \cosh_{R(p,q)}(az) = -a \sinh_{R(p,q)}(az),
\]

(343)

\[
\partial_{R(p,q)} \sin_{R(p,q)}(az) = a \cos_{R(p,q)}(az)
\]

and

\[
\partial_{R(p,q)} \cos_{R(p,q)}(az) = -a \sin_{R(p,q)}(az), \quad a \in \mathbb{C}.
\]

(344)

(345)

Therefore, the \(R(p,q)\)-oscillator equation

\[
\partial^2_{R(p,q)} f(z) + \omega^2 f(z) = 0
\]

(346)

can be solved to give the solution

\[
f(z) = C_1 \cos_{R(p,q)}(\omega z) + C_2 \sin_{R(p,q)}(\omega z).
\]

(347)

57
Proposition 4.9: The following relation between $J_s^{(1)}$ and $J_s^{(2)}$ holds:

$$J_s^{(2)}(z|p, q) = \frac{(p^{s+2}, z^2; p, q)_\infty}{(p^{s+2}, 0; p, q)_\infty} J_s^{(1)}(z|p, q).$$  (354)

So, the $(p, q)$–Bessel functions generalize the $q$–Bessel functions [74].
Proof 4.6

\[ \sum_{n=0}^{\infty} \frac{p^{n(\frac{n-1}{2})}(-1)^n(z^2)^{2n}}{((p, q), (p^{s+1}, q^{s+1}); (p, q))_n} = \sum_{n=0}^{\infty} \frac{(-1)^n(z^2)^{2n}}{((\frac{p}{z}), (\frac{z}{p})^{s+1}; (\frac{z}{p}))_n} \]

\[ = 2\phi_1(0, 0; \left(\frac{q}{p}\right)^{s+1}; \left(\frac{q}{p}\right); -\frac{z^2}{4p^{s+2}}) \]

\[ = \lim_{n \to \infty} \frac{1}{\left(\frac{z^2}{4p^{s+2}}; \frac{q}{p}\right)_n} \sum_{n=0}^{\infty} \frac{(-1)^n(z^2q^{s+1})^n}{((\frac{p}{z}), (\frac{z}{p})^{s+1}; (\frac{z}{p}))_n} \]

\[ = (p^{s+2}, 0); (p, q)_\infty \sum_{n=0}^{\infty} \frac{(-1)^nq^{n(n-1)}(\frac{z}{p})^{2n}}{((p, q), (p^{s+1}, q^{s+1}); (p, q))_n} \]

using the following Heine’s transformation \[ [74]\]:

\[ 2\phi_1(a, b; c, q, z) = \frac{1}{(z; q)_\infty} q\phi_1(-c, z; q, cz). \]

Thus

\[ \sum_{n=0}^{\infty} \frac{(-1)^nq^{n(n-1)}(\frac{z}{p})^{2n}}{((p, q), (p^{s+1}, q^{s+1}); (p, q))_n} = \frac{(p^{s+2}, \frac{z^2}{4p^{s+2}}); (p, q)_\infty}{(p^{s+2}, 0); (p, q)_\infty} \sum_{n=0}^{\infty} \frac{p^{n(n-1)}(\frac{z}{p})^{2n}}{((p, q), (p^{s+1}, q^{s+1}); (p, q))_n}. \]

Multiplying both sides of this equality by \( B(s + 1|p, q)(\frac{z}{p})^2 \) leads to (354). \( \square \)

Proposition 4.10 The following three-term recursion relation:

\[ \left( p^{s} P_{\frac{s}{2}} - q^{s} Q_{\frac{s}{2}} \right) (p^{s+1} P_{\frac{s+1}{2}} - q^{s+1} Q_{\frac{s+1}{2}}) + \frac{z^2}{4} J_{s+1}^{(1)}(z|p, q) = \frac{z}{2} (p^{s+1} P_{\frac{s+1}{2}} - q^{s+1} Q_{\frac{s+1}{2}}) J_{s+1}^{(1)}(z|p, q) + J_{s-1}^{(1)}(z|p, q) \]

is satisfied for the \( p, q \)-Bessel function \( J_{s}^{(1)}(z|p, q) \).

Proof 4.7 : We have

\[ \left( p^{s} P_{\frac{s}{2}} - q^{s} Q_{\frac{s}{2}} \right) J_{s}^{(1)}(z|p, q) \]

\[ = \frac{z}{2} B(s + 1|p, q) \sum_{n=0}^{\infty} \frac{p^{n(n-1)}(-1)^n(z^2)^{2n+s}(p^{s+n} - q^{s+n})}{((p, q), (p^{s+1}, q^{s+1}); (p, q))_n} \]

\[ = \frac{z}{2} \left( \frac{z}{2} \right) J_{s-1}^{(1)}(z|p, q) \]

(358)
Adding (358) and (359), we obtain (357). □

5 \( \mathcal{R}(p, q) \)-calculus: differentiation, integration and Hopf algebras

In this section we build a framework for \( \mathcal{R}(p, q) \)-deformed calculus, which provides a method of computation for a deformed \( \mathcal{R}(p, q) \)-derivative, generalizing known deformed derivatives of analytic function defined on a complex disc as particular cases corresponding to conveniently chosen meromorphic functions. Under prescribed conditions, we define the \( \mathcal{R}(p, q) \)-derivative. The main result resides in the proof that \( \mathcal{R}(p, q) \)-algebra is a Hopf algebra. Relevant examples are also given.

5.1 \( \mathcal{R}(p, q) \)-factors and their associated quantum algebras

In the previous chapter (see also [15, 18, 51]) we have built the \( \mathcal{R}(p, q) \)-factors which are a generalization of Heine \( q \)-factors (also called Heine \( q \)-number in physics literature)

\[
[n]_q = \frac{1 - q^n}{1 - q}, \quad n = 0, 1, 2, \ldots
\]  

and Jagannathan-Srinivasa \( (p, q) \)-factors [57]

\[
[n]_{p,q} = \frac{p^n - q^n}{p - q}, \quad n = 0, 1, 2, \ldots
\]  

as follows. Let \( p \) and \( q \) be two positive real numbers such that \( 0 < q < p \leq 1 \). Consider, as in the previous chapter, a meromorphic function \( \mathcal{R} \), defined on \( \mathbb{C} \times \mathbb{C} \) by

\[
\mathcal{R}(x, y) = \sum_{k.l=-L}^{\infty} r_{kl} x^k y^l
\]  

60
with an eventual isolated singularity at the zero, where \( r_{kl} \) are complex numbers, \( L \in \mathbb{N} \cup \{0\} \), \( \mathcal{R}(p^n, q^n) > 0 \ \forall n \in \mathbb{N} \), and \( \mathcal{R}(1, 1) = 0 \) by definition. Then, the \( \mathcal{R}(p, q) \)-factors denoted by \( \mathcal{R}(p^n, q^n) \), \( n = 0, 1, 2, \cdots \) are used to deduce the \( \mathcal{R}(p, q) \)-factorial

\[
\mathcal{R}!(p^n, q^n) = \begin{cases} 
1 & \text{for } n = 0 \\
\mathcal{R}(p, q) \cdots \mathcal{R}(p^n, q^n) & \text{for } n \geq 1,
\end{cases}
\]

(363)

for \( \mathcal{R}(p, q) \)-binomial coefficient

\[
\binom{m}{n}_{\mathcal{R}(p, q)} = \frac{\mathcal{R}!(p^m, q^m)}{\mathcal{R}!(p^n, q^n) \mathcal{R}!(p^{m-n}, q^{m-n})}, \quad m, n = 0, 1, 2, \cdots, \quad m \geq n,
\]

(364)

and the \( \mathcal{R}(p, q) \)-exponential function

\[
\text{Exp}_{\mathcal{R}(p, q)}(z) = \sum_{n=0}^{\infty} \frac{1}{\mathcal{R}!(p^n, q^n)} z^n.
\]

(365)

Denote by \( \mathbb{D}_R = \{ z \in \mathbb{C} : |z| < R \} \) a complex disc and by \( \mathcal{O}(\mathbb{D}_R) \) the set of holomorphic functions defined on \( \mathbb{D}_R \), where \( R \) is the radius of convergence of the series (365).

We then define the following linear operators on \( \mathcal{O}(\mathbb{D}_R) \) by (see [15, 51] and references therein):

\[
Q : \varphi \mapsto Q\varphi(z) = \varphi(qz), \\
P : \varphi \mapsto P\varphi(z) = \varphi(pz), \\
\partial_{p,q} : \varphi \mapsto \partial_{p,q}\varphi(z) = \frac{\varphi(sz) - \varphi(qz)}{z(p-q)},
\]

(366)

\( \varphi \in \mathcal{O}(\mathbb{D}_R) \), \( 0 < q < p \leq 1 \), and the \( \mathcal{R}(p, q) \)-derivative by

\[
\partial_{\mathcal{R}(p, q)} := \partial_{p,q} \frac{p-q}{pQ} \mathcal{R}(P, Q) = \frac{p-q}{pP - qQ} \mathcal{R}(pP, qQ) \partial_{p,q}.
\]

(367)

Note that the \( \mathcal{R}(p, q) \)-exponential function is invariant under the action of the \( \mathcal{R}(p, q) \)-derivative since

\[
\partial_{\mathcal{R}(p, q)} z^n = \begin{cases} 
0 & \text{for } n = 0 \\
\mathcal{R}(p^n, q^n) z^{n-1} & \text{for } n \geq 1.
\end{cases}
\]

(368)

In [51], we also studied the \( \mathcal{R}(p, q) \)-deformed quantum algebra generated by the set of operators \( \{1, A, A^\dagger, N\} \) and the commutation relations

\[
[N, A] = -A \quad \text{and} \quad [N, A^\dagger] = A^\dagger
\]

(369)
with
\[ AA^\dagger = \mathcal{R}(p^{N+1}, q^{N+1}), \quad \text{and} \quad A^\dagger A = \mathcal{R}(p^N, q^N). \tag{370} \]
This algebra is defined on \( \mathcal{O}(\mathbb{D}_R) \) as:
\[ A^\dagger := z, \quad A := \partial_{\mathcal{R}(p,q)}, \quad N := z\partial_z, \tag{371} \]
where \( \partial_z := \frac{\partial}{\partial z} \) is the usual derivative on \( \mathbb{C} \). Therefore, the following holds:

**Proposition 5.1**
\[ P = p^z\partial_z, \quad Q = q^z\partial_z \tag{372} \]
and the algebra \( \mathcal{A}_{\mathcal{R}(p,q)} \) generated by \( \{1, z, z\partial_z, \partial_{\mathcal{R}(p,q)}\} \) satisfies the relations:
\[ z \partial_{\mathcal{R}(p,q)} = \mathcal{R}(P, Q), \quad \partial_{\mathcal{R}(p,q)} z = \mathcal{R}(pP, qQ), \]
\[ [z\partial_z, z] = z, \quad [z\partial_z, \partial_{\mathcal{R}(p,q)}] = -\partial_{\mathcal{R}(p,q)}. \tag{373} \]

**Proposition 5.2** If there exist two functions \( \Psi_1 \) and \( \Psi_2 : \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C} \) such that
\[ \Psi_i(p, q) > 0 \quad \text{for} \quad i = 1, 2 \tag{374} \]
\[ \left[ \begin{array}{c} n+1 \\ k \end{array} \right]_{\mathcal{R}(p,q)} = \Psi_1^k(p, q) \left[ \begin{array}{c} n \\ k \end{array} \right]_{\mathcal{R}(p,q)} + \Psi_2^{n+1-k}(p, q) \left[ \begin{array}{c} n \\ k-1 \end{array} \right]_{\mathcal{R}(p,q)} \tag{375} \]
\[ ba = \Psi_1(p, q)ab, \quad xy = \Psi_2(p, q)yx, \quad \text{and} \quad [i, j] = 0 \quad \text{for} \quad i \in \{a, b\}, j \in \{x, y\} \tag{376} \]
for quantities \( a, b, x, y \), then
\[ (ax + by)^n = \sum_{k=0}^{n} \left[ \begin{array}{c} n \\ k \end{array} \right]_{\mathcal{R}(p,q)} a^{n-k}b^ky^kx^{n-k}. \tag{377} \]

**Proof:** By induction over \( n \). Indeed, the equality (377) holds for \( n = 1 \) since
\[ (ax + by)^1 = ax + by = \left[ \begin{array}{c} 1 \\ 0 \end{array} \right]_{\mathcal{R}(p,q)} a^{0}b^0y^0x^0 + \left[ \begin{array}{c} 1 \\ 1 \end{array} \right]_{\mathcal{R}(p,q)} a^{0}b^1y^1x^0 = \sum_{k=0}^{1} \left[ \begin{array}{c} 1 \\ k \end{array} \right]_{\mathcal{R}(p,q)} a^{1-k}b^ky^kx^{1-k}. \]
Suppose that the equality (377) holds for \( n \leq m \), this means in particular for \( n = m \),
\[ (ax + by)^m = \sum_{k=0}^{m} \left[ \begin{array}{c} m \\ k \end{array} \right]_{\mathcal{R}(p,q)} a^{m-k}b^ky^kx^{m-k}. \tag{378} \]
and let us prove that it remains valid for \( n = m + 1 \). Indeed,

\[
(ax + by)^{m+1} = (ax + by)^m(ax + by)
\]

\[
= \sum_{k=0}^{m} \binom{m}{k} a^{m-k}b^k y^k x^{m-k} (ax + by)
\]

\[
= \sum_{k=0}^{m} \binom{m}{k} a^{m-k}b^k y^k x^{m-k} a x + \sum_{k=0}^{m} \binom{m}{k} a^{m-k}b^k y^k x^{m-k} by
\]

\[
= \sum_{k=0}^{m} \binom{m}{k} R(p,q) a^{m-k}b^k y^k x^{m+1-k}
\]

\[
+ \sum_{k=0}^{m} \binom{m}{k} R(p,q) \Psi^k_1(p,q) a^{m+1-k}b^k y^k x^{m+1-k}
\]

\[
= a^{m+1} x^{m+1} + \sum_{k=1}^{m} \binom{m}{k} R(p,q) \Psi^k_1(p,q) a^{m+1-k}b^k y^k x^{m+1-k}
\]

\[
+ \sum_{k=1}^{m} \binom{m}{k} R(p,q) \Psi^m_2(p,q) a^{m+1-k}b^k y^k x^{m+1-k}
\]

\[
= a^{m+1} x^{m+1} + b^{m+1} y^{m+1} + \sum_{k=1}^{m} \binom{m}{k} R(p,q) \Psi^k_1(p,q) a^{m+1-k}b^k y^k x^{m+1-k}
\]

\[
+ \sum_{k=1}^{m} \binom{m}{k} R(p,q) \Psi^m_2(p,q) a^{m+1-k}b^k y^k x^{m+1-k}
\]

\[
= a^{m+1} x^{m+1} + b^{m+1} y^{m+1} + \sum_{k=1}^{m} \binom{m}{k} R(p,q) a^{m+1-k}b^k y^k x^{m+1-k}.
\]

\[\square\]

5.2 \( R(p,q) \) —differential and integration calculi

5.2.1 Differential calculus

We define a linear operator \( d_{R(p,q)} \) on \( A_{R(p,q)} \) by

\[
d_{R(p,q)} = (dz) \partial_{R(p,q)}.
\]

(379)

It follows that

\[
d_{R(p,q)} 1 = 0, \quad d_{R(p,q)} z = (dz) R(p,q), \quad d_{R(p,q)} \partial_{R(p,q)} = (dz) \partial_{R(p,q)}^2
\]

63
\[ d_{R(p,q)}(z \partial_z) = (dz)(z \partial_z + 1)\partial_{R(p,q)} \quad \text{and} \quad d^2_{R(p,q)} = 0. \] (380)

Hence, the set of zero-forms \( \Omega^0(\mathcal{A}_{R(p,q)}) \) is naturally \( \mathcal{A}_{R(p,q)} \), while a one form \( \omega \), element of \( \Omega^1(\mathcal{A}_{R(p,q)}) \), is given by

\[ \omega = (dz)\omega_0(z, z \partial_z, \partial_{R(p,q)}), \] (381)

where \( \omega_0(z, z \partial_z, \partial_{R(p,q)}) = \sum_{i,j,k=0}^{\infty} \alpha_{ijk}(z)^i(z \partial_z)^j(\partial_{R(p,q)})^k \) with \( \alpha_{ijk} \) belonging to \( \mathbb{C} \). Therefore, \( d\omega = 0 \) for \( \omega \in \Omega^1(\mathcal{A}_{R(p,q)}) \).

**Proposition 5.3** For a nonnegative integer \( n \), the following equalities hold:

\[ d_{R(p,q)}(z^n) = (dz)R(p^n, q^n)z^{n-1}, \]
\[ d_{R(p,q)}((z \partial_z)^n) = (dz)(z \partial_z + 1)^n \partial_{R(p,q)}, \] (382)
\[ d_{R(p,q)}(\partial_{R(p,q)}^n) = (dz)\partial_{R(p,q)}^{n+1}. \]

Moreover if \( f \in \mathcal{O}(\mathbb{D}_R) \) then

\[ d_{R(p,q)}f(z) = (dz)\partial_{R(p,q)}f(z). \] (383)

**Proof:** The equalities in (382) follow from the definition of the \( R(p, q) \) - derivative (367), the commutation relations (373) and the definition of the differential (379). Then, (383) follows by definition (367). \( \square \)

**Proposition 5.4** The differential \( d_{R(p,q)} \) obeys the two following equivalent Leibniz rules

\[ d_{R(p,q)}(fg) = (dz)\frac{p-q}{pP - qQ}R(pP, qQ)\left\{ \partial_{p,q}(f)(Pg) + (Qf)(\partial_{p,q}(g)) \right\}, \] (384)
\[ d_{R(p,q)}(fg) = (dz)\frac{p-q}{pP - qQ}R(pP, qQ)\left\{ (\partial_{p,q}(f))(Qg) + (Pf)(\partial_{p,q}(g)) \right\}. \] (385)

for \( f, g \in \mathcal{O}(\mathbb{D}_R) \).

**Proof:** This follows from

\[ \partial_{p,q}(fg) = (\partial_{p,q}(f))(Qg) + (Pf)(\partial_{p,q}(f)) = (\partial_{p,q}(f))(Pg) + (Qf)(\partial_{p,q}(g)). \] \( \square \)
5.2.2 \( \mathcal{R}(p, q) \)-integration

We define the operator \( \mathcal{I}_{\mathcal{R}(p, q)} \) over \( \mathcal{O}(\mathbb{D}_R) \) as the inverse image of the \( \mathcal{R}(p, q) \)-derivative. For elements \( z^n \) of the basis of \( \mathcal{O}(\mathbb{D}_R) \), \( \mathcal{I}_{\mathcal{R}(p, q)} \) acts as follows:

\[
\mathcal{I}_{\mathcal{R}(p, q)} z^n := (\partial_{\mathcal{R}(p, q)})^{-1} z^n = \frac{1}{\mathcal{R}(p^{n+1}, q^{n+1})} z^{n+1} + c,
\]

where \( n \geq 0 \) and \( c \) is an integration constant.

Hence, if \( f \in \mathcal{O}(\mathbb{D}_R) \) then

\[
\mathcal{I}_{\mathcal{R}(p, q)} \partial_{\mathcal{R}(p, q)} f(z) = f(z) + c \quad \text{and} \quad \partial_{\mathcal{R}(p, q)} \mathcal{I}_{\mathcal{R}(p, q)} f(z) = f(z) + c',
\]

where \( c \) and \( c' \) are integration constants.

Provided that \( \mathcal{R}(P, Q) \) is invertible, one can define the \( \mathcal{R}(p, q) \)-integration by the following formula

\[
\mathcal{I}_{\mathcal{R}(p, q)} = \mathcal{R}^{-1}(P, Q) \ z,
\]

with \( c = c' = 0 \).

One can also derive the definite integrals:

\[
\int_{\alpha}^{\beta} f(z) d_{\mathcal{R}(p, q)} z = \mathcal{I}_{\mathcal{R}(p, q)}(\beta) - \mathcal{I}_{\mathcal{R}(p, q)}(\alpha), \quad \alpha, \beta \in \mathbb{D}_R; \quad (392)
\]

\[
\int_{\alpha}^{+\infty} f(z) d_{\mathcal{R}(p, q)} z = \lim_{n \to \infty} \int_{\alpha}^{p^n/q^n} f(z) d_{\mathcal{R}(p, q)} z; \quad (390)
\]

\[
\int_{-\infty}^{+\infty} f(z) d_{\mathcal{R}(p, q)} z = \lim_{n \to \infty} \int_{-p^n/q^n}^{p^n/q^n} f(z) d_{\mathcal{R}(p, q)} z. \quad (391)
\]

Moreover, the Eqs. (391) and (392) lead to the following formulae:

\[
\mathcal{I}_{\mathcal{R}(p, q)} \partial_{\mathcal{R}(p, q)} (f(z) g(z)) = f(z) g(z) + c
\]

\[
= \mathcal{I}_{\mathcal{R}(p, q)} \left\{ \frac{p - q}{pP - qQ} \mathcal{R}(pP, qQ) (\partial_{p,q}(f))(Pg) \right\} + \mathcal{I}_{\mathcal{R}(p, q)} \frac{p - q}{pP - qQ} \mathcal{R}(pP, qQ) \{ (Qf)(\partial_{p,q}(g)) \}
\]

and

\[
\mathcal{I}_{\mathcal{R}(p, q)} \partial_{\mathcal{R}(p, q)} (f(z) g(z)) = f(z) g(z) + c
\]

\[
= \mathcal{I}_{\mathcal{R}(p, q)} \left\{ \frac{p - q}{pP - qQ} \mathcal{R}(pP, qQ) (\partial_{p,q}(f))(Qg) \right\} + \mathcal{I}_{\mathcal{R}(p, q)} \frac{p - q}{pP - qQ} \mathcal{R}(pP, qQ) \{ (Pf)(\partial_{p,q}(g)) \},
\]

respectively. These relation can be viewed as formulae of integration by parts.
5.3 $\mathcal{R}(p, q)$-Hopf algebra

The aim of this section is to establish conditions for which the $\mathcal{R}(p, q)$-algebra carries a Hopf algebra structure. This is summarized in a theorem given below. Remind first that the algebra $\mathcal{A}_{\mathcal{R}(p,q)}$ will be a Hopf algebra if it admits operations of homomorphisms of a co-product $\Delta$, a counit $\epsilon$ and an anti-homomorphism of an antipode $S$: [11]

$\Delta : \mathcal{A}_{\mathcal{R}(p,q)} \rightarrow \mathcal{A}_{\mathcal{R}(p,q)} \otimes \mathcal{A}_{\mathcal{R}(p,q)}$,  
$\Delta(\Omega_1 \Omega_2) = \Delta(\Omega_1) \Delta(\Omega_2)$;

$\epsilon : \mathcal{A}_{\mathcal{R}(p,q)} \rightarrow \mathcal{C}$,  
$\epsilon(\Omega_1 \Omega_2) = \epsilon(\Omega_1) \epsilon(\Omega_2)$;

$S : \mathcal{A}_{\mathcal{R}(p,q)} \rightarrow \mathcal{A}_{\mathcal{R}(p,q)}$,  
$S(\Omega_1 \Omega_2) = S(\Omega_2)S(\Omega_1)$

satisfying

$$ (\text{id} \otimes \Delta) \Delta(\Omega) = (\Delta \otimes \text{id}) \Delta(\Omega), $$

$$ (\text{id} \otimes \epsilon) \Delta(\Omega) = \Omega = (\epsilon \otimes \text{id}) \Delta(\Omega), $$

$$ m((\text{id} \otimes S) \Delta(\Omega)) = m((S \otimes \text{id}) \Delta(\Omega)) = \epsilon(\Omega) 1, $$

for all $\Omega, \Omega_1, \Omega_2 \in \mathcal{A}_{\mathcal{R}(p,q)}$. To prove this it is sufficient to show that these relations are satisfied by the generators governing the considered algebra. See [25] and references therein.

Let the Leibniz rule be written as

$$ (\partial_{\mathcal{R}(p,q)} fg)(z) = (\partial_{\mathcal{R},p,f(\partial)} 1)(\Psi(p\partial_z, q\partial_z)g(z)) $$

$$ + (\tilde{\Psi}(p\partial_z, q\partial_z)f(z))\partial_{\mathcal{R}(p,q)}g(z), $$

for $f, g \in \mathcal{O}(\mathbb{D}_R)$, where $\Psi(\ldots)$ and $\tilde{\Psi}(\ldots)$ are meromorphic functions. Let the coproduct $\Delta$, the counit $\epsilon$, and the antipode $S$ be defined as follows:

$$ \Delta(A) = \alpha A \otimes \Psi(p^{\alpha_1 N}, q^{\alpha_2 N}) + \tilde{\alpha} \tilde{\Psi}(p^{\tilde{\alpha}_1 N}, q^{\tilde{\alpha}_2 N}) \otimes A, $$

$$ \Delta(A^\dagger) = \beta A^\dagger \otimes \Psi(p^{\beta_1 N}, q^{\beta_2 N}) + \tilde{\beta} \tilde{\Psi}(p^{\tilde{\beta}_1 N}, q^{\tilde{\beta}_2 N}) \otimes A^\dagger, $$

$$ \Delta(N) = N \otimes 1 + 1 \otimes N + \tau 1 \otimes 1, $$

$$ \Delta(1) = 1 \otimes 1, $$

$$ \epsilon(A) = 0, \quad \epsilon(A^\dagger) = 0, \quad \epsilon(N) = -\tau, \quad \epsilon(1) = 1 $$

$$ S(A) = -s_1 A, \quad S(A^\dagger) = -\tilde{s}_1 A^\dagger, \quad S(N) = -N - 2\tau 1, \quad S(1) = 1, $$

where $\alpha_i, \tilde{\alpha}_i, \beta_i, \tilde{\beta}_i (i = 1, 2)$ and $s_1, \tilde{s}_1, \alpha, \tilde{\alpha}, \beta, \tilde{\beta}$ and $\tau$ are real constants such that the following equations hold:

$$ \alpha \Psi(p^{-\tau \alpha_1}, q^{-\tau \alpha_2}) = 1, \quad \tilde{\alpha} \tilde{\Psi}(p^{-\tau \tilde{\alpha}_1}, q^{-\tau \tilde{\alpha}_2}) = 1, $$

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Theorem 5.5  The more general deformed algebra generated by \{1, N, A, A^\dagger\} satisfying:

\[ [N, A] = -A, \quad [N, A^\dagger] = A^\dagger, \quad [A, A^\dagger] = AA^\dagger + \gamma A^\dagger A, \tag{416} \]

where \( \gamma \) is a real constant such that

\[
\tilde{\Psi}(p^{\tilde{\alpha}_1 N}, q^{\tilde{\beta}_2 N}) \otimes \Psi(p^{\alpha_1 N}, q^{\alpha_2 N}) = \gamma \tilde{\Psi}(p^{\tilde{\alpha}_1(N-1)}, q^{\tilde{\beta}_2(N-1)}) \otimes \Psi(p^{\alpha_1(N-1)}, q^{\alpha_2(N-1)}),
\]

\[
\tilde{\Psi}(p^{\tilde{\alpha}_1(N+1)}, q^{\tilde{\beta}_2(N+1)}) \otimes \Psi(p^{\alpha_1(N+1)}, q^{\alpha_2(N+1)}) = \gamma \tilde{\Psi}(p^{\tilde{\alpha}_1 N}, q^{\tilde{\beta}_2 N}) \otimes \Psi(p^{\alpha_1 N}, q^{\alpha_2 N}),
\]

is a Hopf algebra.

Proof: Notice first that

\[
A^\theta A^\lambda = \theta^{\lambda(N+1)} A, \quad \theta^\lambda A = A^\theta \lambda(N+1),
\]

\[
A^\dagger^\lambda = A^\dagger \lambda(N+1), \quad \theta^\lambda A^\dagger = A^\dagger \theta^\lambda(N+1),
\]

where \( \theta = p, q \), so that, for \( \lambda = \alpha, \beta \) and \( \tilde{\lambda} = \tilde{\alpha}, \tilde{\beta} \).

\[
A \Psi(p^{\lambda_1 N}, q^{\lambda_2 N}) = \Psi(p^{\lambda_1(N+1)}, q^{\lambda_2(N+1)}),
\]

\[
A^\dagger \Psi(p^{\lambda_1 N}, q^{\lambda_2 N}) = \tilde{\Psi}(p^{\tilde{\lambda}_1(N+1)}, q^{\tilde{\lambda}_2(N+1)}),
\]

\[
A^\dagger^\dagger \Psi(p^{\lambda_1 N}, q^{\lambda_2 N}) = \Psi(p^{\lambda_1(N+1)}, q^{\lambda_2(N+1)}),
\]

\[
A^\dagger \tilde{\Psi}(p^{\tilde{\lambda}_1 N}, q^{\tilde{\lambda}_2 N}) = \tilde{\Psi}(p^{\tilde{\lambda}_1(N+1)}, q^{\tilde{\lambda}_2(N+1)}).
\]

Then the following main statement is true:

\[
\alpha \beta = 1, \quad \tilde{\alpha} \tilde{\beta} = 1, \quad \Delta(\Psi(p^{\alpha_1 N}, q^{\alpha_2 N})) = \alpha \Psi(p^{\alpha_1 N}, q^{\alpha_2 N}) \otimes \Psi(p^{\alpha_1 N}, q^{\alpha_2 N}),
\]

\[
\Delta(\tilde{\Psi}(p^{\tilde{\alpha}_1 N}, q^{\tilde{\beta}_2 N})) = \tilde{\alpha} \tilde{\Psi}(p^{\tilde{\alpha}_1 N}, q^{\tilde{\beta}_2 N}) \otimes \tilde{\Psi}(p^{\tilde{\alpha}_1 N}, q^{\tilde{\beta}_2 N}),
\]
Let us now prove that the above definitions of coproduct, counit and antipode satisfy the properties (395)-(397) for Ω ∈ \{A, \tilde{A}, N, 1\}. Indeed, 

\* For Ω = N and using (401) and (402), we have

\[(id \otimes \Delta)(N) = (id \otimes \Delta)(N \otimes 1 + 1 \otimes N + \tau 1 \otimes 1)\]
\[= N \otimes \Delta(1) + 1 \otimes \Delta(N) + \tau 1 \otimes \Delta(1)\]
\[= N \otimes 1 \otimes 1 + 1 \otimes N \otimes 1 + 1 \otimes 1 \otimes N + 2\tau 1 \otimes 1 \otimes 1\]
\[= \Delta(N) \otimes 1 + \Delta(1) \otimes N + \tau \Delta(1) \otimes 1\]
\[= (\Delta \otimes id)(N \otimes 1 + 1 \otimes N + \tau 1 \otimes 1)\]
\[= (\Delta \otimes id)\Delta(N).\]

So, (395) is satisfied. Also,

\[(id \otimes \epsilon)(N) = (id \otimes \epsilon)(N \otimes 1 + 1 \otimes N + \tau 1 \otimes 1)\]
\[= N \otimes \epsilon(1) + 1 \otimes \epsilon(N) + \tau 1 \otimes \epsilon(1)\]
\[= N \otimes 1 - \tau 1 \otimes 1 + \tau 1 \otimes 1 = N\]

and

\[(\epsilon \otimes id)(N) = (\epsilon \otimes id)(N \otimes 1 + 1 \otimes N + \tau 1 \otimes 1)\]
\[= \epsilon(N) \otimes 1 + \epsilon(1) \otimes N + \tau \epsilon(1) \otimes 1\]
\[= -\tau \otimes 1 + 1 \otimes N \otimes \tau \otimes 1 = N,\]

where we use (401) and the fact that \(\epsilon(N) = -\tau\). Hence, \(N\) satisfies (396). Next,

\[m((id \otimes S)\Delta(N)) = m((id \otimes S)(N \otimes 1 + 1 \otimes N + \tau 1 \otimes 1))\]
\[= m(N \otimes S(1) + 1 \otimes S(N) + \tau 1 \otimes S(1))\]
\[= m(N \otimes 1 - 1 \otimes N - 2\tau 1 \otimes 1 + \tau 1 \otimes 1)\]
\[= -\tau m(1 \otimes 1) = -\tau 1 = \epsilon(N)1,\]

and similarly

\[m((S \otimes id)\Delta(N)) = -\tau m(1 \otimes 1) = -\tau 1 = \epsilon(N)1,\]

where we use (401) and the fact that \(S(N) = -N - 2\tau 1\). Therefore \(N\) satisfies (397).

\* For Ω = A, we have

\[(id \otimes \Delta)(A) = (id \otimes \Delta)(\alpha A \otimes \Psi(p^{\alpha 1N}, q^{\alpha 2N}) + \tilde{\alpha} \tilde{\Psi}(p^{\tilde{\alpha} 1N}, q^{\tilde{\alpha} 2N}) \otimes A)\]
\[= \alpha A \otimes \Delta(\Psi(p^{\alpha 1N}, q^{\alpha 2N})) + \tilde{\alpha} \tilde{\Psi}(p^{\tilde{\alpha} 1N}, q^{\tilde{\alpha} 2N}) \otimes \Delta(A)\]
\[= \alpha A \otimes \Delta(\Psi(p^{\alpha 1N}, q^{\alpha 2N})).\]
where we use (399) and the fact that (408) and (409). Hence, (395) holds.

The property (396) also holds since

\[ A(\epsilon \otimes \alpha_1) = A = m \]

where the use of (399), (403) and (405) has been made.

A satisfies also property (397) since

\[ m((S \otimes id)\Delta(A)) = m((S \otimes id)(A \otimes (p_{01}^1, q_{02}^1)) + \alpha \tilde{\Psi}(p_{01}^2, q_{02}^2)) \]

where we use (399), (404) and the fact that \( s_1, \alpha, \tilde{\alpha}, \alpha_i \) and \( \tilde{\alpha}_i \) (\( i = 1, 2 \)) satisfy equations (412) and (413).
For $\Omega = A^\dagger$, one can perform the same computations and use (410), (413), (403), (404), (410), (411), (414) and (415) to prove that (395)-(397) also hold. Computing $\Delta(A)\Delta(A^\dagger)$ and $\Delta(A^\dagger)\Delta(A)$ we obtain:

$$
\Delta(A)\Delta(A^\dagger) = (\alpha A \otimes \Psi(p^{\alpha_1N}, q^{\alpha_2N}) + \tilde{\alpha} \tilde{\Psi}(p^{\tilde{\alpha}_1N}, q^{\tilde{\alpha}_2N}) \otimes A)
$$

$$
\times (\beta A^\dagger \otimes \Psi(p^{\beta_1N}, q^{\beta_2N}) + \tilde{\beta} \tilde{\Psi}(p^{\tilde{\beta}_1N}, q^{\tilde{\beta}_2N}) \otimes A^\dagger)
$$

$$
\alpha\beta AA^\dagger \otimes \Psi(p^{\alpha_1N}, q^{\alpha_2N}) \Psi(p^{\beta_1N}, q^{\beta_2N})
$$

$$
\times (A A^\dagger + \tilde{\alpha} \tilde{\Psi}(p^{\tilde{\alpha}_1N}, q^{\tilde{\alpha}_2N}) \Psi(p^{\alpha_1N}, q^{\alpha_2N}) A^\dagger
$$

$$
+ \tilde{\alpha} \tilde{\Psi}(p^{\alpha_1N}, q^{\alpha_2N}) A^\dagger \otimes A \Psi(p^{\beta_1N}, q^{\beta_2N}) A
$$

$$
+ \tilde{\beta} \tilde{\Psi}(p^{\beta_1N}, q^{\beta_2N}) A \otimes A^\dagger \Psi(p^{\alpha_1N}, q^{\alpha_2N})
$$

$$
+ \tilde{\alpha} \tilde{\Psi}(p^{\alpha_1N}, q^{\alpha_2N}) \tilde{\Psi}(p^{\beta_1N}, q^{\beta_2N}) \otimes AA^\dagger
$$

and

$$
\Delta(A^\dagger)\Delta(A) = (\beta A^\dagger \otimes \Psi(p^{\beta_1N}, q^{\beta_2N}) + \tilde{\beta} \tilde{\Psi}(p^{\tilde{\beta}_1N}, q^{\tilde{\beta}_2N}) \otimes A^\dagger)
$$

$$
\times (\alpha A \otimes \Psi(p^{\alpha_1N}, q^{\alpha_2N}) + \tilde{\alpha} \tilde{\Psi}(p^{\tilde{\alpha}_1N}, q^{\tilde{\alpha}_2N}) \otimes A)
$$

$$
\alpha\beta A A^\dagger \otimes \Psi(p^{\alpha_1N}, q^{\alpha_2N}) \Psi(p^{\beta_1N}, q^{\beta_2N})
$$

$$
\times (A^\dagger A + \alpha \beta A A^\dagger \Psi(p^{\alpha_1N}, q^{\alpha_2N}) \otimes (A A^\dagger - \gamma A^\dagger A)
$$

$$
+ \alpha \beta A A^\dagger \Psi(p^{\alpha_1N}, q^{\alpha_2N}) \otimes (A^\dagger A - \gamma A^\dagger A)
$$

$$
+ \alpha \beta A A^\dagger \Psi(p^{\alpha_1N}, q^{\alpha_2N}) \otimes (A^\dagger A - \gamma A^\dagger A)
$$

$$
- \gamma \Psi(p^{\beta_1N}, q^{\beta_2N}) A \otimes A^\dagger \Psi(p^{\alpha_1N}, q^{\alpha_2N}) A^\dagger
$$

$$
+ \alpha \beta A A^\dagger \Psi(p^{\alpha_1N}, q^{\alpha_2N}) \otimes (A^\dagger A - \gamma A^\dagger A)
$$

respectively. Thus,

$$
\Delta(A)\Delta(A^\dagger) = \gamma \Delta(A^\dagger)\Delta(A)
$$

$$
= (\alpha A A^\dagger - \gamma A^\dagger A) \otimes \Psi(p^{\alpha_1N}, q^{\alpha_2N}) \Psi(p^{\beta_1N}, q^{\beta_2N})
$$

$$
+ \alpha \beta A A^\dagger \Psi(p^{\alpha_1N}, q^{\alpha_2N}) \Psi(p^{\beta_1N}, q^{\beta_2N}) \otimes (A A^\dagger - \gamma A^\dagger A)
$$

$$
+ \alpha \beta A A^\dagger \Psi(p^{\alpha_1N}, q^{\alpha_2N}) \Psi(p^{\beta_1N}, q^{\beta_2N}) A^\dagger
$$

$$
+ \alpha \beta A A^\dagger \Psi(p^{\alpha_1N}, q^{\alpha_2N}) \Psi(p^{\beta_1N}, q^{\beta_2N}) A^\dagger
$$

$$
- \gamma \Psi(p^{\beta_1N}, q^{\beta_2N}) A \otimes A^\dagger \Psi(p^{\alpha_1N}, q^{\alpha_2N}) A^\dagger
$$

$$
+ \alpha \beta A A^\dagger \Psi(p^{\alpha_1N}, q^{\alpha_2N}) \otimes (A^\dagger A - \gamma A^\dagger A)
$$

Therefore, $[A, A^\dagger]_\gamma = AA^\dagger - \gamma A A^\dagger$ implies

$$
\Delta([A, A^\dagger]_\gamma) = [A, A^\dagger]_\gamma \otimes \Psi(p^{\alpha_1N}, q^{\alpha_2N}) \Psi(p^{\beta_1N}, q^{\beta_2N})
$$

$$
+ \Psi(p^{\alpha_1N}, q^{\alpha_2N}) \Psi(p^{\beta_1N}, q^{\beta_2N}) \otimes [A, A^\dagger]_\gamma
$$

provided that

$$
A \tilde{\Psi}(p^{\beta_1N}, q^{\beta_2N}) \otimes \Psi(p^{\alpha_1N}, q^{\alpha_2N}) A^\dagger = \gamma A \tilde{\Psi}(p^{\beta_1N}, q^{\beta_2N}) A \otimes A^\dagger \Psi(p^{\alpha_1N}, q^{\alpha_2N})
$$

$$
\tilde{\Psi}(p^{\beta_1N}, q^{\beta_2N}) A^\dagger \otimes A \Psi(p^{\beta_1N}, q^{\beta_2N}) = \gamma A^\dagger \tilde{\Psi}(p^{\beta_1N}, q^{\beta_2N}) \otimes \Psi(p^{\beta_1N}, q^{\beta_2N}) A,
$$

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\[ \alpha \beta = 1, \quad \text{and} \quad \tilde{\alpha} \tilde{\beta} = \]

or

\[
\tilde{\Psi}(p^{\tilde{\alpha}_1N}, q^{\tilde{\alpha}_2N}) \otimes \Psi(p^{\alpha_1N}, q^{\alpha_2N}) = \gamma \tilde{\Psi}(p^{\tilde{\beta}_1(N-1)}, q^{\tilde{\beta}_2(N-1)}) \otimes \Psi(p^{\alpha_1(N-1)}, q^{\alpha_2(N-1)}),
\]

\[
\tilde{\Psi}(p^{\tilde{\alpha}_1(N+1)}, q^{\tilde{\alpha}_2(N+1)}) \otimes \Psi(p^{\beta_1(N+1)}, q^{\beta_2(N+1)}) = \gamma \tilde{\Psi}(p^{\tilde{\beta}_1N}, q^{\tilde{\beta}_2N}) \otimes \Psi(p^{\beta_1N}, q^{\beta_2N}),
\]

\[ \alpha \beta = 1, \quad \text{and} \quad \tilde{\alpha} \tilde{\beta} = 1, \]

which are (418), (419) and (406), respectively. \(\square\)

5.4 Relevant particular cases

Let us now apply the above general formalism to particular deformed algebras, well spread in the literature.

5.4.1 Jagannathan-Srinivasa deformation

A. Taking \( R(x, y) = \frac{x-y}{p-q} \), we obtain the Jagannathan-Srinivasa \((p, q)\)-factors and \((p, q)\)-factorials

\[ [n]_{p,q} = \frac{p^n - q^n}{p - q}, \]

and

\[ [n]!_{p,q} = \begin{cases} 1 & \text{for } n = 0 \\ \frac{((p,q);(p,q))_n}{(p-q)^n} & \text{for } n \geq 1, \end{cases} \tag{422} \]

respectively.

Referring the readers to [57] for details on \((p, q)\)-calculus, let us restrict the present description to some new relevant properties.

Proposition 5.6 If \( n \) and \( m \) are nonnegative integers, then

\[ [n]_{p,q} = \sum_{k=0}^{n-1} p^{n-1-k} q^k, \]

\[ [n + m]_{p,q} = q^n [n]_{p,q} + p^n [m]_{p,q} = p^n [n]_{p,q} + q^n [m]_{p,q}, \]

\[ [-m]_{p,q} = -q^{-m} p^{-m} [m]_{p,q}, \]

\[ [n - m]_{p,q} = q^{-m} [n]_{p,q} - p^{-m} q^{-m} [m]_{p,q} = p^{-m} [n]_{p,q} - q^{-m} p^{-m} [m]_{p,q}, \]

\[ [n]_{p,q} = [2]_{p,q} [n-1]_{p,q} - pq [n-2]_{p,q}, \]
Proposition 5.7 The \((p, q)\)-binomial coefficients

\[
\binom{n}{k}_{p,q} = \frac{((p, q); (p, q))_n}{((p, q); (p, q))_k((p, q); (p, q))_{n-k}}, \quad 0 \leq k \leq n; \quad n \in \mathbb{N}, \tag{424}
\]

where \(((p, q); (p, q))_m = (p - q)(p^2 - q^2) \cdots (p^m - q^m), m \in \mathbb{N}\) satisfy the following identities

\[
\begin{align*}
\binom{n}{k}_{p,q} &= \binom{n}{n-k}_{p,q}, \\
\binom{n+1}{k}_{p,q} &= p^k \binom{n}{k}_{p,q} + q^{n+1-k} \binom{n}{k-1}_{p,q}, \\
\binom{n+1}{k}_{p,q} &= p^k \binom{n}{k}_{p,q} + p^{n+1-k} \binom{n}{k-1}_{p,q} - (p^n - q^n) \binom{n-1}{k-1}_{p,q}.
\end{align*}
\]

with

\[
\binom{n}{k}_{q/p} = \frac{(q/p; q/p)_n}{(q/p; q/p)_k(q/p; q/p)_{n-k}},
\]

where \((q/p; q/p)_n = (1 - q/p)(1 - q^2/p^2) \cdots (1 - q^n/p^n)\) and the \((p, q)\)-shifted factorial

\[
((a, b); (p, q))_n = (a - b)(ap - bq) \cdots (ap^{n-1} - bq^{n-1}) = \sum_{k=0}^{n} \binom{n}{k}_{p,q} (-1)^k p^{(n-k)(n-k-1)/2} q^{(k-1)/2} a^{n-k} k!.
\]

Proposition 5.8 If the quantities \(x, y, a\) and \(b\) are such that \(xy = qyx, \ ba = pab, [i, j] = 0\) for \(i \in \{a, b\}\) and \(j \in \{x, y\}\), and, moreover, \(p\) and \(q\) commute with each element of the set \([a, b, x, y]\), then

\[
(ax + by)^n = \sum_{k=0}^{n} \binom{n}{k}_{p,q} a^{n-k} b^k y^k x^{n-k}. \tag{428}
\]

The latter result is a generalization of noncommutative form of the \(q\)-binomial theorem \([41]\), which can be obtained setting \(a, b\) and \(p\) equal to 1, i.e.

\[
(x + y)^n = \sum_{k=0}^{n} \binom{n}{k}_q y^k x^{n-k}. \tag{429}
\]
where
\[ \left[ \begin{array}{c} n \\ k \end{array} \right]_q = (q; q)_n/(q; q)_k(q; q)_{n-k}, \]
with \((q; q)_n = (1 - q)(1 - q^2) \cdots (1 - q^n)\).

**Proof of Proposition 5.8** A proof has been proposed in [57]. Here we provide another one by induction on \(n\). Indeed, the result is true for \(n = 1\).

Suppose it remains valid for all \(n \leq m\) and prove that this is also true for \(n = m + 1\):

\[
(ax + by)^{m+1} = (ax + by)^m(ax + by)
= \sum_{k=0}^{m} \binom{m}{k}_{p,q} a^{m-k} b^k y^k x^{m-k} (ax + by)
= \sum_{k=0}^{m} \binom{m}{k}_{p,q} p^k a^{m+1-k} b^k y^k x^{m+1-k}
+ \sum_{k=0}^{m} \binom{m}{k}_{p,q} q^{m-k} a^{m-k} b^{k+1} y^{k+1} x^{m-k}
= a^{m+1} x^{m+1} + \sum_{k=1}^{m} \binom{m}{k}_{p,q} p^k a^{m+1-k} b^k y^k x^{m+1-k}
+ \sum_{k=0}^{m} \binom{m}{k}_{p,q} q^{m-k} a^{m-k} b^{k+1} y^{k+1} x^{m-k} + b^{m+1} y^{m+1}
= a^{m+1} x^{m+1} + \sum_{k=1}^{m} \binom{m}{k}_{p,q} p^k a^{m+1-k} b^k y^k x^{m+1-k}
+ \sum_{k=1}^{m} \binom{m}{k}_{p,q} q^{m-k} a^{m-k} b^{k+1} y^{k+1} x^{m-k}
+ b^{m+1} y^{m+1},
\]

where the use of (426) has been made. Hence the result is true for all \(n \in \mathbb{N}\).

The \(\mathcal{R}(p, q)\)-derivative is thus reduced to the \((p, q)\)-derivative [57]

\[
\partial_{p,q} = \frac{1}{(p - q)z} (P - Q),
\]

namely, for \(f \in \mathcal{O}(\mathbb{D}_R)\),

\[
\partial_{p,q} f(z) = \frac{f(pz) - f(qz)}{z(p - q)}.
\]
The associated algebra $A_{p,q}$, generated by $\{1, A, A^\dagger, N\}$, satisfies the relations:

$$A A^\dagger - p A^\dagger A = q^N, \quad A A^\dagger - q A^\dagger A = p^N$$  \hspace{1cm} (432)

$$[N, A^\dagger] = A^\dagger \quad [N, A] = -A,$$  \hspace{1cm} (433)

and its realization on $O(D_R)$, engendered by $\{1, z, z\partial_z, \partial_{p,q}\}$, satisfies the relations

$$z \partial_{p,q} - p \partial_{p,q} z = q z \partial_{p,q} \partial_{p,q} \quad z \partial_{p,q} - q \partial_{p,q} z = p^\partial_z$$  \hspace{1cm} (434)

Therefore, the differential operator $d_{p,q}$ is then given by

$$d_{p,q} = (dz) \frac{1}{(p-q)z}(P - Q)$$  \hspace{1cm} (435)

with the following properties:

$$d_{p,q} 1 = 0, \quad d_{p,q} z = (dz), \quad d_{p,q} \partial_{p,q} = (dz) \partial_{p,q}^2$$  \hspace{1cm} (436)

$$d_{p,q}(z\partial_z) = (dz)(z\partial_z + 1)\partial_{p,q} \quad \text{and} \quad d_{p,q}^2 = 0.$$

The differential of $f \in O(D_R)$ is then

$$d_{p,q} f(z) = (dz) \frac{f(pz) - f(qz)}{(p-q)z}$$  \hspace{1cm} (437)

affording the Leibniz rule

$$d_{p,q}(fg)(z) = (dz) \frac{f(pz) - f(qz)}{(p-q)z} g(qz)$$  \hspace{1cm} (438)

$$+ (dz) f(pz) \frac{g(pz) - g(qz)}{(p-q)z}$$

$$= \{d_{p,q} f(z)\} \cdot g(qz) + f(pz) \cdot d_{p,q} g(z)$$

or, equivalently,

$$d_{p,q}(fg)(z) = (dz) \frac{f(pz) - f(qz)}{(p-q)z} g(pz)$$  \hspace{1cm} (439)

$$+ (dz) f(qz) \frac{g(pz) - g(qz)}{(p-q)z}$$

$$= \{d_{p,q} f(z)\} \cdot g(pz) + f(qz) \cdot d_{p,q} g(z).$$  

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The \((p, q)\)-integration is obtained from (388) as follows:

\[
I_{p,q} f(z) = \frac{p-q}{P-Q} z f(z) = (p-q) \sum_{\nu=0}^{\infty} \frac{Q^{\nu}}{P^{\nu+1}} z f(z) \tag{440}
\]

\[
= (p-q) z \sum_{\nu=0}^{\infty} f(z q^{\nu}/p^{\nu+1}) q^{\nu}/p^{\nu+1}.
\]

Setting \(p = 1\), one recovers the \(q\)-derivative and \(q\)-integral of Jackson [74].

B. Tacking \(R(x,y) = \frac{x-y}{p x - q y}\), where \(a, b \in \mathbb{C}\) with \(a \neq b\), the \(R(p,q)\)-factors are given by

\[
R(p^n, q^n) = \left[ \begin{array}{c} n \\ a, b \end{array} \right]_p q^n = \frac{p^n - q^n}{ap^n - bq^n} - 1, \quad n = 0, 1, \ldots \tag{441}
\]

The \(R(p,q)\)-factorials become

\[
[n]_{p,q}^{a,b} = \left\{ \begin{array}{ll} 1 & \text{for } n = 0 \\ \frac{((p,q)(p,q))_n}{(a,b)(p,q))_n} & \text{for } n \geq 1, \end{array} \right. \tag{442}
\]

The derivative is now given by

\[
\partial R(p,q) = \partial p,q \frac{P - Q}{PQ} \frac{aP - bQ}{ap - bq} = \frac{1}{z} \frac{P - Q}{z P - \frac{b}{q}Q}, \tag{443}
\]

so that for \(f \in \mathcal{O}(D_R)\) we have

\[
\partial R(p,q) f(z) = \frac{1}{z} \frac{P - Q}{z P - \frac{b}{q}Q} f(z) \tag{444}
\]

\[
= \frac{1}{z} (P - Q) \sum_{\nu=0}^{\infty} \frac{(bp/aq)^\nu (Q/P)^\nu}{ap} f(z)
\]

\[
= \frac{z}{P} \sum_{\nu=0}^{\infty} \left[ (Q/P)^\nu - (Q/P)^{\nu+1} \right] f(z)
\]

\[
= \frac{z}{P} \sum_{\nu=0}^{\infty} \left[ f((q/p)^\nu z) - f((q/p)^{\nu+1} z) \right].
\]

Moreover,

\[
I_R(p,q) = \frac{aP - \frac{b}{q}Q}{P - Q} z. \tag{445}
\]

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Applying this to $f \in \mathcal{O}(\mathbb{D}_R)$ we obtain

$$I_{R(p,q)} f(z) = \frac{aP - bQ}{P - Q} z f(z)$$

(446)

$$= \left( \frac{a}{p} P - \frac{b}{q} Q \right) \frac{1}{P} \sum_{\nu=0}^{\infty} (Q/P)^\nu \ z f(z)$$

$$= \left( \frac{a}{p} (Q/P) \right) \sum_{\nu=0}^{\infty} (Q/P)^\nu \ z f(z)$$

$$= \sum_{\nu=0}^{\infty} \left[ \left( \frac{a}{p} (Q/P)^\nu - \frac{b}{q} (Q/P)^\nu + 1 \right) z f(z) \right]$$

$$= (z/p) \sum_{\nu=0}^{\infty} \left( \frac{q}{p} \right)^\nu \left[ a f ((q/q)^\nu z) - b f ((q/p)^{\nu+1}) \right].$$

Let us display the Hopf algebra structure of Jagannathan-Srinivasa algebra according to the theorem 5.5. To this end notice first that the Leibniz rule of the derivative is given by

$$\partial_{p, q}(fg)(z) = \left( \partial_{p, q} f(z) \right) p \partial_z g(z) + \left( \partial_z f(z) \right) \partial_{p, q} g(z)$$

(447)

from which we deduce $\Psi(x, y) = x$ and $\tilde{\Psi}(x, y) = y$. So, $\alpha_2 = 0$, $\tilde{\alpha}_1 = 0$, $\beta_2 = 0$, and $\tilde{\beta}_1 = 0$. Equations (405)-(407) yield

$$\alpha = p^{\alpha_1 \tau}, \quad \tilde{\alpha} = q^{\tilde{\alpha}_2 \tau}, \quad \beta = p^{\beta_1 \tau}, \quad \tilde{\beta} = q^{\tilde{\beta}_2 \tau}$$

(448)

$$\tau (\alpha_1 + \beta_1) = 0 \quad \text{and} \quad \tau (\tilde{\alpha}_2 + \tilde{\beta}_2) = 0,$$

(449)

while equations (412)-(415) are reduced to

$$s_1 p^{\alpha_1 (\tau + 1)} p^{-\alpha_1 N} = q^{\tilde{\alpha}_2 (\tau + 1)} q^{-\tilde{\alpha}_2 N}, \quad p^{-\alpha_1 \tau} p^{-\alpha_1 N} = s_1 q^{\tilde{\alpha}_2 (\tau + 1)} q^{-\tilde{\alpha}_2 N};$$

$$p^{-\beta_1 \tau} p^{-\beta_1 N} = s_1 q^{\tilde{\beta}_2 (\tau + 1)} q^{-\tilde{\beta}_2 N}, \quad s_1 p^{\beta_1 (\tau + 1)} p^{-\beta_1 N} = q^{\tilde{\beta}_2 (\tau + 1)} q^{-\tilde{\beta}_2 N}.$$ 

Therefore,

$$\alpha_1 = \tilde{\alpha}_2 = \beta_1 = \tilde{\beta}_2 = 0, \quad \alpha = \tilde{\alpha} = \beta = \tilde{\beta} = s_1 = \tilde{s}_1 = 1$$

(450)

and equations (408)-(411) are satisfied and (418)-(419) yield $\gamma = 1$. Thus, the coproduct, the counit and the antipode are given by

$$\Delta(A) = A \otimes 1 + 1 \otimes A,$$

(451)
\[ \Delta(A^\dagger) = A^\dagger \otimes 1 + 1 \otimes A^\dagger, \]
\[ \Delta(N) = N \otimes 1 + 1 \otimes N + \tau 1 \otimes 1, \]
\[ \Delta(1) = 1 \otimes 1, \]
\[ \epsilon(A) = 0, \quad \epsilon(A^\dagger) = 0, \quad \epsilon(N) = -\tau, \quad \epsilon(1) = 1 \]
\[ S(A) = -A, \quad S(A^\dagger) = -A^\dagger, \quad S(N) = -N - 2\tau 1, \quad S(1) = 1 \]
respectively, where \( \tau \) is a real number, usually set equal to 0.

### 5.4.2 Chakrabarty and Jagannathan deformation

The algebra of Chakrabarty and Jagannathan \([24]\) is obtained by taking \( R(x, y) = \frac{1-xy}{(p-1-q)x} \). Indeed, the \( R(p, q) \)-factors and \( R(p, q) \)-factorials are reduced to \( (p^{-1}, q) \)-factors and \( (p^{-1}, q) \)-factorials, namely

\[ [n]_{p^{-1},q} = \frac{p^{-n} - q^n}{p^{-1} - q}, \]

and

\[ [n]!_{p^{-1},q} = \begin{cases} 1 & \text{for } n = 0 \\ \frac{1}{(p^{-1}-q)^n} & \text{for } n \geq 1, \end{cases} \]

respectively. The properties of this deformation can be readily recovered from the previous section 5.4.1 by replacing the parameter \( p \) by \( p^{-1} \).

The \( R(p, q) \)-derivative is also reduced to \( (p^{-1}, q) \)-derivative. Indeed,

\[ \partial_{R(p,q)} = \partial_{p,q} \frac{p - q}{P - Q} \frac{1 - PQ}{(p^{-1} - q)P} \]
\[ = \frac{1}{(p^{-1} - q)z} (P^{-1} - Q) =: \partial_{p^{-1},q}. \]

Therefore, for \( f \in \mathcal{O}(D_R) \)

\[ \partial_{p^{-1},q} f(z) = \frac{f(p^{-1}z) - f(qz)}{z(p^{-1} - q)} \]

and the differential of \( f \in \mathcal{O}(D_R) \) is given by

\[ d_{p^{-1},q} f(z) = (dz) \frac{f(p^{-1}z) - f(qz)}{z(p^{-1} - q)}. \]

Computing the Leibniz rule we get

\[ d_{p^{-1},q}(fg)(z) = (dz) \frac{f(p^{-1}z) - f(qz)}{z(p^{-1} - q)} g(qz) + (dz) f(p^{-1}z) \frac{g(p^{-1}z) - g(qz)}{z(p^{-1} - q)} \]
\[ d_{p^{-1}, q} (fg)(z) = (dz) \frac{f(p^{-1}z) - f(qz)}{z(p^{-1} - q)} g(p^{-1}z) + (dz) \frac{g(p^{-1}z) - g(qz)}{z(p^{-1} - q)} f(qz) = \{d_{p^{-1}, q} f(z)\} \cdot g(p^{-1}z) + f(qz) \cdot d_{p^{-1}, q} g(z). \]

or, equivalently,

\[ \{d_{p^{-1}, q} f(z)\} \cdot g(p^{-1}z) + f(qz) \cdot d_{p^{-1}, q} g(z). \]

We obtain from (388) the action of the \((p^{-1}, q)\)-integration on \(f \in \mathcal{O}(\mathbb{D}_R)\) as follows:

\[ I_{p^{-1}, q} f(z) = \frac{p^{-1} - q}{P - 1 - Q} z f(z) = (p^{-1} - q) \sum_{\nu=0}^{\infty} Q^\nu P^{\nu+1} z f(z) \]

\[ = (1 - pq) z \sum_{\nu=0}^{\infty} f(z q^\nu P^{\nu+1})(pq)^\nu. \]

The same Hopf algebra structure as that of Jagannathan-Srinivasan is also obtained for Chakrabarty and Jagannathan deformation.

5.4.3 Generalized \(q\)-Quesne deformation

The generalized Quesne algebra [53, 95] is found by taking \(\mathcal{R}(x, y) = \frac{xy - 1}{(q - p - 1)y}\).

Indeed, the \((p, q)\)-Quesne factors and factorials are given by

\[ [n]_{p, q}^Q = \frac{p^n - q^{-n}}{q - p^{-1}}, \]

and

\[ [n]_{p, q}^! = \begin{cases} 1 & \text{for } n = 0 \\ \frac{(p^{-1} - 1)(p^{-1} - q)}{q^{-1}} & \text{for } n \geq 1, \end{cases} \]

respectively. There follow some relevant new properties:

**Proposition 5.9** If \(n\) and \(m\) are nonnegative integers, then

\[ [-m]_{p, q}^Q = -p^{-m} q^m [m]_{p, q}^Q, \]

\[ [n + m]_{p, q}^Q = q^{-m} [n]_{p, q}^Q + p^n [m]_{p, q}^Q = p^n [n]_{p, q}^Q + q^{-m} [m]_{p, q}^Q, \]

\[ [n - m]_{p, q}^Q = q^m [n]_{p, q}^Q - p^{-n} q^m [m]_{p, q}^Q = p^{-m} [n]_{p, q}^Q + p^{-n} q^m [m]_{p, q}^Q, \]

\[ [n]_{p, q}^Q = \frac{q - p^{-1}}{p - q^{-1}} [2]_{p, q}^Q [n - 1]_{p, q}^Q - pq^{-1} [n - 2]_{p, q}^Q. \]
Proof: We obtain Eqs. (465) and (466) applying the relations
\[ p^m - q^m = -p^{-m}q^m(p^m - q^{-m}) \]
and
\[ p^{n+m} - q^{-n-m} = q^m(p^n - q^n) + p^n(p^m - q^{-m}) = p^m(p^n - q^n) + q^{-n}(p^m - q^{-m}), \]
respectively. Eq. (467) follows combining Eqs. (465) and (466). Note that
\[ [n]_{p,q^{-1}} = \frac{p^n - q^{-n}}{p - q^{-1}} = \frac{q - p^{-1}p^n - q^{-n}}{p - q^{-1}q - p^{-1}} = [n]_{p,q}^Q, \quad n = 1, 2, \ldots \] (469)
which, combined with the following identity
\[ [n]_{p,q^{-1}} = [2]_{p,q^{-1}}[n-1]_{p,q^{-1}} - pq^{-1}[n-2]_{p,q^{-1}}, \]
gives Eq. (468).

Proposition 5.10 The \((p, q)\)-Quesne binomial coefficients
\[
\left[ \begin{array}{c} n \\ k \end{array} \right]_{p,q}^Q = \frac{(p, q^{-1}); (p, q^{-1})_n}{(p, q^{-1}); (p, q^{-1})_k((p, q^{-1}); (p, q^{-1}))_{n-k}}, \quad (470)
\]
where \(0 \leq k \leq n\), \(n \in \mathbb{N}\), satisfy the following properties:
\[
\left[ \begin{array}{c} n \\ k \end{array} \right]_{p,q}^Q = \left[ \begin{array}{c} n \\ n-k \end{array} \right]_{p,q}^Q = p^{k(n-k)} \left[ \begin{array}{c} n \\ k \end{array} \right]_{1/qp}, \quad (471)
\]
\[
\left[ \begin{array}{c} n+1 \\ k \end{array} \right]_{p,q}^Q = p^k \left[ \begin{array}{c} n \\ k \end{array} \right]_{p,q}^Q + q^{-n-1+k} \left[ \begin{array}{c} n \\ k-1 \end{array} \right]_{p,q}^Q, \quad (472)
\]
\[
\left[ \begin{array}{c} n+1 \\ k \end{array} \right]_{p,q}^Q = p^k \left[ \begin{array}{c} n \\ k \end{array} \right]_{p,q}^Q + p^{n+1-k} \left[ \begin{array}{c} n \\ k-1 \end{array} \right]_{p,q}^Q - (p^n - q^{-n}) \left[ \begin{array}{c} n-1 \\ k-1 \end{array} \right]_{p,q}^Q.
\]

Proof: This is direct using the Proposition 5.6 and
\[
\left[ \begin{array}{c} n \\ k \end{array} \right]_{p,q}^Q = \left[ \begin{array}{c} n \\ k \end{array} \right]_{p,q^{-1}}^Q. \quad (473)
\]
Proposition 5.11 If the quantities \( x, y, a \) and \( b \) are such that \( xy = q^{-1}yx \), \( ba = pab \), \([i, j] = 0 \) for \( i \in \{a, b\} \) and \( j \in \{x, y\} \), and, moreover, \( p \) and \( q \) commute with each element of the set \( \{a, b, x, y\} \), then

\[
(ax + by)^n = \sum_{k=0}^{n} \binom{n}{k}^Q a^{n-k} b^k y^k x^{n-k}.
\]  

(474)

Proof: By induction on \( n \). Indeed, the result is true for \( n = 1 \). Suppose it remains valid for \( n \leq m \) and prove that this is also true for \( n = m + 1 \):

\[
(ax + by)^{m+1} = (ax + by)^m (ax + by)
\]

\[
= \sum_{k=0}^{m} \binom{m}{k}^Q a^{m-k} b^k y^k x^{m-k} (ax + by)
\]

\[
= \sum_{k=0}^{m} \binom{m}{k}^Q p^k a^{m+1-k} b^k y^k x^{m+1-k}
\]

\[
+ \sum_{k=0}^{m} \binom{m}{k}^Q q^{-m+k} a^{m-k} b^{k+1} y^{k+1} x^{m-k}
\]

\[
= a^{m+1} x^{m+1} + \sum_{k=1}^{m} \binom{m}{k}^Q p^k a^{m+1-k} b^k y^k x^{m+1-k}
\]

\[
+ q^{-m-1+k} \binom{m}{k-1}^Q a^{m+1-k} b^k y^k x^{m+1-k} + b^{m+1} y^{m+1}
\]

\[
= a^{m+1} x^{m+1} + \sum_{k=1}^{m+1} \binom{m+1}{k}^Q a^{m+1-k} b^k y^k x^{m+1-k}
\]

\[
+ b^{m+1} y^{m+1}
\]

where the use of (472) has been made. Therefore the result is true for all \( n \in \mathbb{N} \). The \((p, q)\)–Quesne derivative is given by

\[
\partial_{p,q}^2 = \partial_{p,q} \frac{p - q}{P - Q (q - p^{-1})Q} = \frac{1}{(q - p^{-1})} (P - Q^{-1}).
\]

(475)
Therefore, for $f \in \mathcal{O}(\mathbb{D}_R)$
\[
\partial^Q_{p,q} f(z) = \frac{f(pz) - f(q^{-1}z)}{z(q - p^{-1})}
\]
(476)
and the differential is given by
\[
d^Q_{p,q} f(z) = (dz) \frac{f(pz) - f(q^{-1}z)}{z(q - p^{-1})}
\]
(477)
leading to the Leibniz rule
\[
d^Q_{p,q} (fg)(z) = (dz) \frac{f(pz) - f(q^{-1}z)}{z(q - p^{-1})} g(q^{-1}z) + (dz) f(pz) \frac{g(pz) - g(q^{-1}z)}{z(q - p^{-1})}
\]
(478)
or, equivalently,
\[
d^Q_{p,q} (fg)(z) = (dz) \frac{f(pz) - f(q^{-1}z)}{z(q - p^{-1})} g(pz) + (dz) f(q^{-1}z) \frac{g(pz) - g(q^{-1}z)}{z(q - p^{-1})}
\]
(479)
The action of the $(p,q)$–Quesne integration on $f \in \mathcal{O}(\mathbb{D}_R)$ is obtained from (388) as follows:
\[
I^Q_{p,q} f(z) = \frac{q - p^{-1}}{P - Q^{-1}} f(z) = (p^{-1} - q) \sum_{\nu=0}^{\infty} P^\nu Q^\nu z f(z)
\]
(480)
(481)
= $(p^{-1} - q)z \sum_{\nu=0}^{\infty} f(zp^\nu q^{\nu+1}) p^\nu q^{\nu+1}$.

The same structure of Hopf algebra as for Jagannathan-Srinivas a is also found for the generalized $q$–Quesne deformation.

5.4.4 $(p, q; \mu, \nu, h)$–deformation

The deformed Hounkonnou-Ngompe [52] algebra is obtained by taking
\[
\mathcal{R}(x, y) = h(p, q) \frac{y^\nu xy - 1}{x^\mu (q - p^{-1})y}
\]
such that $0 < pq < 1$, $\mu < q^{\nu-1}$, $p > 1$, and $h(p, q)$ is a well behaved real and non-negative function of deformation parameters $p$ and $q$ such that $h(p, q) \to 1$ as $(p, q) \to (1, 1)$. Here the $R(p, q)$--factors become $(p, q; \mu, \nu, h)$-factors, namely

$$[n]_{p, q, h}^{\mu, \nu} = h(p, q)\frac{q^{\nu m} p^n - q^{-n}}{p^{\mu m} q - p^{-1}}. \quad (481)$$

**Proposition 5.12** The $(p, q; \mu, \nu, h)$--factors verify the following properties, for $m, n \in \mathbb{N}$:

$$[-m]_{p, q, h}^{\mu, \nu} = -\frac{q^{-2vm+m}}{p^{-2vm+m}}[m]_{p, q, h}^{\mu, \nu}, \quad (482)$$

$$[n + m]_{p, q, h}^{\mu, \nu} = \frac{q^{vm-m}}{p^{\mu m}}[n]_{p, q, h}^{\mu, \nu} + \frac{q^{vm}}{p^{\mu m} - n}[m]_{p, q, h}^{\mu, \nu}, \quad (483)$$

$$[n - m]_{p, q, h}^{\mu, \nu} = \frac{q^{-vm+m}}{p^{-\mu m} + m}[n]_{p, q, h}^{\mu, \nu} - \frac{q^{vm}}{p^{\mu m} - n}[m]_{p, q, h}^{\mu, \nu}, \quad (484)$$

$$[n]_{p, q, h}^{\mu, \nu} = \frac{q - p^{-1}}{p - q^{-1}} \frac{q^{-\nu} 1}{h(p, q)} [2]_{p, q, h}^{\mu, \nu} [n - 1]_{p, q, h}^{\mu, \nu} - q^{2\nu-1}[n - 2]_{p, q, h}^{\mu, \nu}. \quad (485)$$

**Proof:** This is direct using the Proposition 5.9 and the fact that

$$[n]_{p, q, h}^{\mu, \nu} = h(p, q)\frac{q^{vm}}{p^{\mu m} [n]_{p, q, h}^{Q}}. \quad (486)$$

**Proposition 5.13** The $(p, q, \mu, \nu, h)$-- binomial coefficients

$$\left[ \begin{array}{c} n \\ k \end{array} \right]_{p, q, h}^{\mu, \nu} := \frac{[n]_{p, q, h}^{\mu, \nu}}{[k]_{p, q, h}^{\mu, \nu}[n - k]_{p, q, h}^{\mu, \nu}} = \frac{q^{k(n-k)}}{p^{\mu k(n-k)}} \left[ \begin{array}{c} n \\ k \end{array} \right]_{p, q}^{Q}. \quad (487)$$
where \(0 \leq k \leq n\), \(n \in \mathbb{N}\), satisfy the following properties:

\[
\begin{align*}
\left[ \frac{n}{k} \right]_{\mu,\nu}^{p,q,h} &= \left[ \frac{n}{n-k} \right]_{\mu,\nu}^{p,q,h} \\
\left[ \frac{n+1}{k} \right]_{\mu,\nu}^{p,q,h} &= \frac{q^{\nu k}}{p^{(\nu-1)k}} \left[ \frac{n}{k} \right]_{\mu,\nu}^{p,q,h} + \frac{q^{(\nu-1)(n+1-k)}}{p^{\mu(n+1-k)}} \left[ \frac{n}{k-1} \right]_{\mu,\nu}^{p,q,h} \\
\left[ \frac{n+1}{k} \right]_{\mu,\nu}^{p,q,h} &= \frac{q^{\nu k}}{p^{(\nu-1)k}} \left[ \frac{n}{k} \right]_{\mu,\nu}^{p,q,h} + \frac{q^{(\nu-1)(n+1-k)}}{p^{\mu(n+1-k)}} \left[ \frac{n}{k-1} \right]_{\mu,\nu}^{p,q,h} \\
&\quad - \left( p^n - q^{-n} \right) \frac{q^n}{p^n} \left[ \frac{n-1}{k-1} \right]_{\mu,\nu}^{p,q,h}.
\end{align*}
\]

**Proof:** This is direct using the Proposition 5.10 and the fact that

\[
\left[ n \right]_{\mu,\nu}^{p,q,h} = h^n (p, q) \frac{q^{n(n+1)/2}}{p^{n(n+1)/2} [t]^{Q}} \left[ n \right]_{\mu,\nu}^{p,q,h},
\]

where the use of Eq. (486) has been made. \(\Box\)

**Proposition 5.14** If the quantities \(x, y, a\) and \(b\) are such that \(xy = \frac{q^{\nu-1}}{p^\mu} yx\), \(ba = \frac{q^\nu}{p^{\mu-1}} ab\), \([i, j] = 0\) for \(i \in \{a, b\}\) and \(j \in \{x, y\}\), and, moreover, \(p\) and \(q\) commute with each element of the set \(\{a, b, x, y\}\), then

\[
(ax + by)^n = \sum_{k=0}^{n} \left[ \frac{n}{k} \right]_{\mu,\nu}^{p,q,h} a^{n-k} b^k y^k x^{n-k}.
\]

**Proof:** By induction on \(n\). Indeed, the result is true for \(n = 1\). Suppose it remains valid for \(n \leq m\) and prove that this is also true for \(n = m + 1\):

\[
(ax + by)^{m+1} = (ax + by)^m (ax + by)
\]

\[
= \sum_{k=0}^{m} \left[ \frac{m}{k} \right]_{\mu,\nu}^{p,q,h} a^{m-k} b^k y^k x^{m-k} (ax + by)
\]

\[
= \sum_{k=0}^{m} \left[ \frac{m}{k} \right]_{\mu,\nu}^{p,q,h} \frac{q^{\nu k}}{p^{(\nu-1)k}} a^{m+1-k} b^k y^k x^{m+1-k}
\]

\[
+ \sum_{k=0}^{m} \left[ \frac{m}{k} \right]_{\mu,\nu}^{p,q,h} \frac{q^{(\nu-1)(m-k)}}{p^{\mu(m-k)}} a^{m-k} b^{k+1} y^{k+1} x^{m-k}
\]

\[
= a^{m+1} x^{m+1} + \sum_{k=1}^{m} \left[ \frac{m}{k} \right]_{\mu,\nu}^{p,q,h} \frac{q^{\nu k}}{p^{(\nu-1)k}} a^{m+1-k} b^k y^k x^{m+1-k}
\]

\[
+ \sum_{k=0}^{m} \left[ \frac{m}{k} \right]_{\mu,\nu}^{p,q,h} \frac{q^{(\nu-1)(m-k)}}{p^{\mu(m-k)}} a^{m-k} b^{k+1} y^{k+1} x^{m-k}
\]

\[
= a^{m+1} x^{m+1} + \sum_{k=1}^{m} \left[ \frac{m}{k} \right]_{\mu,\nu}^{p,q,h} \frac{q^{\nu k}}{p^{(\nu-1)k}} a^{m+1-k} b^k y^k x^{m+1-k}
\]

\[
+ \sum_{k=0}^{m} \left[ \frac{m}{k} \right]_{\mu,\nu}^{p,q,h} \frac{q^{(\nu-1)(m-k)}}{p^{\mu(m-k)}} a^{m-k} b^{k+1} y^{k+1} x^{m-k}.
\]

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\[ + \sum_{k=0}^{m-1} \binom{m}{k}_{p,q,h} Q^{(\nu-1)(m-k)} \frac{q^k}{p^{(m-k)}} a^{m-k} b^{k+1} y^{k+1} x^{m-k} \]
\[ = a^{m+1} x^{m+1} + b^{m+1} y^{m+1} + \sum_{k=1}^{m} \binom{m}{k}_{p,q,h} Q^{\nu-1} a^{m-k} b^{k} x^{m+1-k} \]
\[ + b^{m+1} y^{m+1} \]
\[ = \sum_{k=0}^{m+1} \binom{m+1}{k}_{p,q,h} a^{m+1-k} b^{k} y^{k} x^{m+1-k}, \]

where the use of (489) has been made. Therefore, the result is true for all \( n \in \mathbb{N} \).

\[ \square \]

The \( R(p,q) \)-derivative is then reduced to the \( (p,q;\mu,\nu,h) \)-derivative, given by

\[ \partial_{R(p,q)} = \partial_{p,q} \frac{p-q}{P-Q} h(p,q) \frac{Q^\nu P Q - 1}{P^\nu (q-p^{-1})Q} \]

(493)

\[ = \frac{h(p,q)}{(q-p^{-1})z} \frac{Q^\nu}{P^\nu} (P - Q^{-1}) \equiv \partial_{p,q,h}^{\mu,\nu}. \]

Therefore the \( (p,q;\mu,\nu,h) \)-derivative and the \( (p,q;\mu,\nu,h) \)-differential of \( f \in \mathcal{O}(\mathbb{D}_R) \) are given by

\[ \partial_{p,q,h}^{\mu,\nu} = h(p,q) \frac{f(zq^\nu/p^{\mu-1}) - f(zq^{\nu-1}/p^\mu)}{z(q-p^{-1})} \]

(494)

and

\[ d_{p,q,h}^{\mu,\nu} f(z) = (dz)h(p,q) \frac{f(zq^\nu/p^{\mu-1}) - f(zq^{\nu-1}/p^\mu)}{z(q-p^{-1})} \]

(495)

respectively, with the Leibniz rule

\[ d_{p,q,h}^{\mu,\nu} (fg)(z) = (dz)h(p,q) \frac{f(zq^\nu/p^{\mu-1}) - f(zq^{\nu-1}/p^\mu)}{z(q-p^{-1})} \cdot g(zq^{\nu-1}/p^\mu) \]

\[ + (dz) f(zq^\nu/p^{\mu-1}) h(p,q) \frac{g(zq^\nu/p^{\mu-1}) - g(zq^{\nu-1}/p^\mu)}{z(q-p^{-1})} \]

\[ = \{d_{p,q,h}^{\mu,\nu} f(z)\} \cdot g(zq^{\nu-1}/p^\mu) + f(zq^\nu/p^{\mu-1}) \cdot d_{p,q,h}^{\mu,\nu} g(z) \]
which is equivalent to
\[
d^\mu,\nu_{p,q,h}(fg)(z) = (dz)h(p,q) \left( f(zq^\nu/p^{\mu-1}) - f(zq^{\nu-1}/p^\mu) \right) g(zq^\nu/p^{\mu-1})/z(q - p^{-1})
\]
\[
+ (dz)f(zq^{\nu-1}/p^\mu)h(p,q) \left( g(zq^\nu/p^{\mu-1}) - g(zq^{\nu-1}/p^\mu) \right) /z(q - p^{-1})
\]
\[
= \{d^\mu,\nu_{p,q,h}(f)(z)\} \cdot g(zq^\nu/p^{\mu-1}) + f(zq^{\nu-1}/p^\mu) \cdot d^\mu,\nu_{p,q,h}g(z).
\]

From (388) we obtain the action of the \((p,q,\mu,\nu,h)\) follows:
\[
\mathcal{I}^\mu,\nu_{p,q,h}f(z) = \frac{q - p^{-1}}{h(p,q) P - Q^{-1}} zf(z) = \frac{p^{-1} - q P^n/Q^{\nu-1}}{h(p,q) 1 - PQ} zf(z)
\]
\[
= \frac{p^{-1} - q P^n}{h(p,q) Q^{\nu-1}} \sum_{j=0}^\infty P^j Q^j z f(z)
\]
\[
= \frac{p^{-1} - q}{h(p,q)} \sum_{j=0}^\infty Q^{j+1} Q^{\nu} f(z)
\]
\[
= z(p^{-1} - q) \frac{P^n}{h(p,q) q^{\nu-1}} \sum_{j=0}^\infty f(zp^{j+\nu}q^{j+1-\nu}).
\]

From the derivative Leibniz rule
\[
\partial^\mu,\nu_{p,q,h}(f(z)g(z)) = \left( \partial^\mu,\nu_{p,q,h}f(z) \right) \frac{q^{\nu-1} z \partial x}{p^{\mu-1} z \partial y} g(z)
\]
\[
+ \left( \frac{q^{\nu-1} z \partial x}{p^{\mu-1} z \partial y} f(z) \right) \partial^\mu,\nu_{p,q,h}(g(z)),
\]
one deduces \(\Psi(x, y) = x^{-(\mu-1)}y^\nu\) and \(\tilde{\Psi}(x, y) = y^{\nu}x^{\mu-1}\). Hence, (405)-(407) yield
\[
\alpha = p^{-\alpha_1 \tau(\mu-1)} q^{\alpha_2 \tau}, \quad \bar{\alpha} = p^{-\bar{\alpha}_1 \tau(\mu-1)} q^{\bar{\alpha}_2 \tau},
\]
\[
\beta = p^{-\beta_1 \tau(\mu-1)} q^{\beta_2 \tau}, \quad \bar{\beta} = p^{-\bar{\beta}_1 \tau(\mu-1)} q^{\bar{\beta}_2 \tau},
\]
\[
(\alpha_1 + \beta_1)\tau = 0, \quad (\alpha_2 + \beta_2)\tau = 0, \quad (\bar{\alpha}_1 + \bar{\beta}_1)\tau = 0, \quad (\bar{\alpha}_2 + \bar{\beta}_2)\tau = 0.
\]

Equations (408)-(411) are of course satisfied and (418)-(419) give
\[
\gamma = p^{[\bar{\beta}_1 \mu + \bar{\alpha}_1 (\mu-1)]} q^{\bar{\beta}_2 (\nu-1) + \alpha_2 \nu} \quad \text{and} \quad \gamma = p^{[-\bar{\alpha}_1 \mu + \bar{\beta}_1 (\mu-1)]} q^{\bar{\alpha}_2 (\nu-1) + \beta_2 \nu}
\]

From equations (412)-(415) we deduce
\[
s_1 = p^{\alpha_1 (\mu-1)} q^{-\alpha_2 \nu}, \quad s_1 = p^{\beta_1 (\mu-1)} q^{-\beta_2 \nu},
\]
\[
\tilde{\alpha}_1 = -\alpha_1 \frac{\mu - 1}{\mu}, \quad \tilde{\alpha}_2 = -\alpha_2 \frac{\nu}{\nu - 1}; \quad \tilde{\beta}_1 = -\beta_1 \frac{\mu - 1}{\mu}, \quad \tilde{\beta}_2 = -\beta_2 \frac{\nu}{\nu - 1}
\]
so that
\[ \tilde{\alpha} = \alpha^{-1}, \quad \tilde{\beta} = \beta^{-1}, \quad \gamma = 1. \]  
(504)

Setting
\[ \kappa_\alpha = \frac{q^{2\nu}}{p^{\alpha_1}(\mu-1)} \quad \text{and} \quad \kappa_\beta = \frac{q^{2\nu}}{p^{\beta_1}(\mu-1)}, \]
with \( \alpha_1, \alpha_2, \beta_1, \beta_1 \in \mathbb{R}, \)
(505)
we get
\[ \Psi(p^{\alpha_1}, q^{\alpha_2}) = \kappa_\alpha, \quad \tilde{\Psi}(p^{\tilde{\alpha}_1}, q^{\tilde{\alpha}_2}) = \kappa_{\tilde{\alpha}}, \]
\[ \Psi(p^{\beta_1}, q^{\beta_2}) = \kappa_\beta, \quad \tilde{\Psi}(p^{\tilde{\beta}_1}, q^{\tilde{\beta}_2}) = \kappa_{\tilde{\beta}}. \]
(506)

The remaining conditions are \((\alpha_1 + \beta_1)\tau = 0\) and \((\alpha_2 + \beta_2)\tau = 0\) which hold if \( \tau = 0 \) or \( \beta_1 = -\alpha_1 \) and \( \beta_2 = -\alpha_2 \).

Suppose \( \tau = 0 \). Then \( \alpha = \tilde{\alpha} = \beta = \tilde{\beta} = 1 \). Then, the coproduct, the counit, the antipode are defined as follows:
\[ \Delta(A) = A \otimes \kappa_\alpha \otimes A, \]
\[ \Delta(A^\dagger) = A^\dagger \otimes \kappa_{\tilde{\alpha}} \otimes A^\dagger, \]
\[ \Delta(N) = N \otimes 1 + 1 \otimes N, \]
\[ \Delta(1) = 1 \otimes 1, \]
\[ \epsilon(A) = 0, \quad \epsilon(A^\dagger) = 0, \quad \epsilon(N) = 0, \quad \epsilon(1) = 1, \]
\[ S(A) = -\kappa_\alpha, \quad S(A^\dagger) = -\kappa_{\tilde{\alpha}}, \quad S(N) = -N, \quad S(1) = 1. \]
(512)

Suppose now \( \tau \neq 0 \) that means \( \beta_1 = -\alpha_1 \) and \( \beta_2 = -\alpha_2 \). So,
\[ \alpha = \kappa_\alpha, \quad \beta = \kappa_{\tilde{\alpha}}, \quad s_1 = \kappa^{-1}, \quad \tilde{s}_1 = \kappa. \]

Thus, the coproduct, the counit, the antipode are defined as follows:
\[ \Delta(A) = A \otimes \kappa_N \otimes A, \]
\[ \Delta(A^\dagger) = A^\dagger \otimes \kappa_{\tilde{N}} \otimes A^\dagger, \]
\[ \Delta(N) = N \otimes 1 + 1 \otimes N + \tau 1 \otimes 1, \]
\[ \Delta(1) = 1 \otimes 1, \]
\[ \epsilon(A) = 0, \quad \epsilon(A^\dagger) = 0, \quad \epsilon(N) = -\tau, \quad \epsilon(1) = 1, \]
\[ S(A) = -\kappa^{-1}A, \quad S(A^\dagger) = -\kappa A^\dagger, \quad S(N) = -N - 2\tau 1, \quad S(1) = 1. \]
(519)
6 \( \mathcal{R}(p, q) \)-deformed Rogers-Szegő polynomials: associated quantum algebras, deformed Hermite polynomials and relevant properties

This section addresses a new characterization of \( \mathcal{R}(p, q) \)-deformed Rogers-Szegő polynomials by providing their three-term recursion relation and the associated quantum algebra built with corresponding creation and annihilation operators. The whole construction is performed in a unified way, generalizing all known relevant results which are straightforwardly derived as particular cases. Continuous \( \mathcal{R}(p, q) \)-deformed Hermite polynomials and their recursion relation are also deduced. Novel relations are provided and discussed.

The present investigation aims at giving a new realization of the previous generalized deformed quantum algebras and an explicit definition of the \( \mathcal{R}(p, q) \)-Rogers-Szegő polynomials, together with their three-term recursion relation and the deformed difference equation giving rise to the creation and annihilation operators.

6.1 \( \mathcal{R}(p, q) \)-Rogers-Szegő polynomials and related quantum algebras

This section aims at providing realizations of \( (\mathcal{R}, p, q) \)-deformed quantum algebras induced by \( (\mathcal{R}, p, q) \)-Rogers-Szegő polynomials. We first define the latter and their three-term recursion relation, and then following the procedure elaborated in [40, 58], we prove that every sequence of these polynomials forms a basis for the corresponding deformed quantum algebra.

Indeed, Galetti in [40], upon recalling the technique of construction of raising and lowering operators which satisfy an algebra akin to the usual harmonic oscillator algebra, by using the three-term recursion relation and the differentiation expression of Hermite polynomials, has shown that a similar procedure can be carried out to construct a \( q \)-deformed harmonic oscillator algebra, with the help of relations controlling the Rogers-Szegő polynomials. Following this author, Jagannathan and Sridhar in [58] adapted the same approach to construct a Bargman-Fock realization of the harmonic oscillator as well as realizations of \( q \)- and \( (p, q) \)-deformed harmonic oscillators based on Rogers-Szegő polynomials.

As matter of clarity, this section is stratified as follows. We first develop the synoptic schemes of known different generalizations and then display the formalism of \( \mathcal{R}(p, q) \)-Rogers-Szegő polynomials.
6.1.1 Hermite polynomials and harmonic oscillator approach

The Hermite polynomials are defined as orthogonal polynomials satisfying the three-term recursion relation

\[ H_{n+1}(z) = 2zH_n(z) - 2nH_{n-1}(z) \]  \hspace{2cm} (520)

and the differentiation relation

\[ \frac{d}{dz}H_n(z) = 2nH_{n-1}(z). \]  \hspace{2cm} (521)

Inserting Eq. (521) in Eq. (520), one gets

\[ H_{n+1}(z) = \left(2z - \frac{d}{dz}\right)H_n(z) \]  \hspace{2cm} (522)

which includes the introduction of a raising operator (see [40] and references therein), defined as

\[ \hat{a}_+ = 2z - \frac{d}{dz} \]  \hspace{2cm} (523)

such that the set of Hermite polynomials can be generated by the application of this operator to the first polynomial \( H_0(z) = 1 \), i.e.,

\[ H_n(z) = \hat{a}_+^n H_0(z). \]  \hspace{2cm} (524)

From Eq. (521), one defines the lowering operator \( \hat{a}_- \) as

\[ \hat{a}_- H_n(z) = \frac{1}{2} \frac{d}{dz} H_n(z) = nH_{n-1}(z). \]  \hspace{2cm} (525)

Furthermore one constructs a number operator in the form

\[ \hat{n} = \hat{a}_+ \hat{a}_-. \]  \hspace{2cm} (526)

One can readily check that these operators satisfy the canonical commutation relations

\[ [\hat{a}_-, \hat{a}_+] = 1, \quad [\hat{n}, \hat{a}_+] = \hat{a}_+ \quad [\hat{n}, \hat{a}_-] = -\hat{a}_-, \quad [\hat{n}, \hat{a}_+] = \hat{a}_+, \]  \hspace{2cm} (527)

although the operators \( \hat{a}_- \) and \( \hat{a}_+ \) are not the usual creation and annihilation operators associated with the quantum mechanics harmonic oscillator. Thus, we see that one can obtain raising, lowering and number operators from the two basic relations satisfied by the Hermite polynomials, i.e., the three-term recursion relation.

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relation and the differentiation relation, respectively, so that they satisfy the well
known commutation relations.

On the other hand, if one considers the usual Hilbert space spanned by the vec-
tors $|n\rangle$, generated from the vacuum $|0\rangle$ by the raising operator $\hat{a}_+$, then together
with the lowering operator $\hat{a}_-$, the following relations hold

$$\hat{a}_- \hat{a}_+ - \hat{a}_+ \hat{a}_- = 1,$$
$$\langle 0|0 \rangle = 1,$$
$$|n\rangle = \hat{a}_+^n |0\rangle,$$
$$\hat{a}_- |0\rangle = 0.$$  \hspace{1cm} (528)

In particular, the next expressions, established using the previous equations, are
in order:

$$\hat{a}_+ |n\rangle = |n + 1\rangle,$$
$$\hat{a}_- |n\rangle = |n - 1\rangle,$$
$$\langle m|n \rangle = n! \delta_{mn}.$$  \hspace{1cm} (529)

Now, on the other hand, examining the procedure given in [58], the authors
considered the sequence of polynomials

$$\psi_n(z) = \sqrt{n!} h_n(z),$$  \hspace{1cm} (530)

where

$$h_n(z) = (1 + z)^n = \sum_{k=0}^{n} \binom{n}{k} z^k,$$  \hspace{1cm} (531)

obeying the relations

$$d\frac{d}{dz} \psi_n(z) = \sqrt{n} \psi_{n-1}(z),$$  \hspace{1cm} (532)
$$\sqrt{n + 1} \psi_{n+1}(z),$$  \hspace{1cm} (533)
$$\sqrt{n + 1} \psi_{n+1}(z),$$  \hspace{1cm} (534)
$$\sqrt{n + 1} \psi_{n+1}(z),$$  \hspace{1cm} (535)

Here equations (532) and (534) are the recursion relation and the differential equa-
tion for polynomials $\psi_n(z)$, respectively. By analogy to the work done by Galleti,
Jagannathan and Sridhar proposed the following relations:

$$\hat{a}_+ = (1 + z), \quad \hat{a}_- = \frac{d}{dz}, \quad \hat{n} = (1 + z) \frac{d}{dz},$$  \hspace{1cm} (536)

for creation (or raising), annihilation (or lowering) and number operators, re-
spectively, and found that the set \{\psi_n(z) \mid n = 0, 1, 2, \cdots\} forms a basis for the
Bargman-Fock realization of the harmonic oscillator [527].
6.1.2 Rogers-Szegő polynomials and $q$-deformed harmonic oscillator

Here in analogous way as Jagannathan and Sridhar [58], we perform a construction of the creation, annihilation and number operators from the three-term recurrence relation and the $q$–difference equation founding the Rogers-Szegő polynomials. This procedure a little differs from that used by Galetti [40] to obtain raising, lowering and number operators.

The Rogers-Szegő polynomials are defined as

$$H_n(z; q) = \sum_{k=0}^{n} \binom{n}{k}_q z^k, \quad n = 0, 1, 2 \cdots$$

(537)

and satisfy a three-term recursion relation

$$H_{n+1}(z; q) = (1 + z)H_n(z; q) - z(1 - q^n)H_{n-1}(z; q)$$

(538)
as well as the $q$-difference equation

$$\partial_q H_n(z; q) = [n]_q H_{n-1}(z; q).$$

(539)

In the limit case $q \to 1$, the Rogers-Szegő polynomial of degree $n$ ($n = 0, 1, 2, \cdots$) well converges to

$$h_n(z) = \sum_{k=0}^{n} \binom{n}{k} z^k$$

as required. Defining

$$\psi_n(z; q) = \frac{1}{\sqrt{[n]_q}} H_n(z) = \frac{1}{\sqrt{[n]_q}} \sum_{k=0}^{n} \binom{n}{k}_q z^k, \quad n = 0, 1, 2 \cdots, \quad (540)$$

one can straightforwardly infer that

$$\partial_q \psi_n(z; q) = \sqrt{[n]_q} \psi_{n-1}(z; q)$$

(541)

with the property that for $n = 0, 1, 2, \cdots$

$$\partial_q^{n+1} \psi_n(z; q) = 0 \quad \text{and} \quad \partial_q^m \psi_n(z; q) \neq 0 \quad \text{for any } m < n + 1. \quad (542)$$

It follows from Eqs. (538) and (540) that the polynomials $\{\psi_n(z; q) \mid n = 0, 1, 2, \cdots\}$ satisfy the following three-term recursion relation

$$\sqrt{[n+1]_q} \psi_{n+1}(z; q) = (1 + z)\psi_n(z; q) - z(1 - q) \sqrt{[n]_q} \psi_{n-1}(z; q)$$

(543)
and the $q$–difference equation
\[
((1 + z) - (1 - q)z \partial_q) \psi_n(z; q) = \sqrt{[n + 1]_q} \psi_{n+1}(z; q)
\]  (544)
obtained from Eq. (541). Hence, it is natural to formally define the number operator $N$ as
\[
N \psi_n(z; q) = n \psi_n(z; q)
\]  (545)
determined for the creation and annihilation operators expressed as
\[
A^\dagger = 1 + z - (1 - q)z \partial_q \quad \text{and} \quad A = \partial_q
\]  (546)
respectively. Indeed, the proofs of the following relations are immediate:
\[
N \psi_n(z; q) = n \psi_n(z; q),
\]  (547)
\[
A^\dagger \psi_n(z; q) = \sqrt{[n + 1]_q} \psi_{n+1}(z; q),
\]  (548)
\[
A \psi_n(z; q) = \sqrt{[n]_q} \psi_{n-1}(z; q),
\]  (549)
\[
A^\dagger A \psi_n(z; q) = [n]_q \psi_n(z; q) = [N]_q \psi_n(z; q),
\]  (550)
\[
AA^\dagger \psi_n(z; q) = [n + 1]_q \psi_n(z; q) = [N + 1]_q \psi_n(z; q).
\]  (551)
Therefore, one concludes that the set of polynomials \{\psi_n(z; q) | n = 0, 1, 2, \cdots\} provides a basis for a realization of the $q$-deformed harmonic oscillator algebra given by
\[
AA^\dagger - qA^\dagger A = 1, \quad [N, A] = -A, \quad [N, A^\dagger] = A^\dagger
\]  (552)

### 6.1.3 $\mathcal{R}(p, q)$–generalized Rogers-Szegö polynomials and quantum algebras

We can now supply the general procedure for constructing the recursion relation for the $\mathcal{R}(p, q)$–Rogers-Szegö polynomials and the related $\mathcal{R}(p, q)$-difference equation that allow to define the creation, annihilation and number operators for a given ($\mathcal{R}, p, q$)–deformed quantum algebra. This is summarized as follows.

**Theorem 6.1** If $\phi_i(x, y)$ ($i = 1, 2, 3$) are functions satisfying:
\[
\phi_i(p, q) \neq 0 \quad \text{for } i = 1, 2, 3,
\]  (553)
\[
\phi_i(P, Q) z^k = \phi_i^k(p, q) z^k \quad \text{for } z \in \mathbb{C}, \ k = 0, 1, 2, \cdots \quad i = 1, 2
\]  (554)
and if, moreover, the following relation between $\mathcal{R}(p, q)$–binomial coefficients holds
\[
\left[ \begin{array}{c}
 n + 1 \\
 k
\end{array} \right]_{\mathcal{R}(p, q)} = \phi_1^k(p, q) \left[ \begin{array}{c}
 n \\
 k
\end{array} \right]_{\mathcal{R}(p, q)} + \phi_2^{n+1-k}(p, q) \left[ \begin{array}{c}
 n \\
 k - 1
\end{array} \right]_{\mathcal{R}(p, q)}
\]
\[-\phi_3(p, q)\mathcal{R}(p^n, q^n) \left[ \begin{array}{c} n-1 \\ k-1 \end{array} \right] \mathcal{R}(p^n, q^n)\] (555)

for \(1 \leq k \leq n\), then the \(\mathcal{R}(p, q)\)-Rogers-Szegő polynomials defined as

\[H_n(z; \mathcal{R}(p, q)) = \sum_{k=0}^{n} \left[ \begin{array}{c} n \\ k \end{array} \right] \mathcal{R}(p^n, q^n) z^k, \quad n = 0, 1, 2 \cdots\] (556)

satisfy the three-term recursion relation

\[H_{n+1}(z; \mathcal{R}(p, q)) = H_n(\phi_1(p, q) z ; \mathcal{R}(p, q)) + z\phi_2^2(p, q) H_n(z\phi_2^{-1}(p, q); \mathcal{R}(p, q)) - z\phi_3(p, q) \mathcal{R}(p^n, q^n) H_{n-1}(z; \mathcal{R}(p, q))\] (557)

and \(\mathcal{R}(p, q)\)-difference equation

\[\partial_{\mathcal{R}(p,q)} H_n(z; \mathcal{R}(p, q)) = \mathcal{R}(p^n, q^n) H_{n-1}(z; \mathcal{R}(p, q)).\] (558)

**Proof:** Multiplying the two sides of the relation (555) by \(z^k\) and adding for \(k = 1\) to \(n\) we get

\[\sum_{k=1}^{n} \left[ \begin{array}{c} n+1 \\ k \end{array} \right] \mathcal{R}(p^n, q^n) z^k = \sum_{k=1}^{n} \phi_1^k(p, q) \left[ \begin{array}{c} n \\ k \end{array} \right] \mathcal{R}(p^n, q^n) z^k + \sum_{k=1}^{n} \phi_2^{n+1-k}(p, q) \left[ \begin{array}{c} n \\ k-1 \end{array} \right] \mathcal{R}(p^n, q^n) z^k - \phi_3(p, q) \mathcal{R}(p^n, q^n) \sum_{k=1}^{n} \left[ \begin{array}{c} n-1 \\ k-1 \end{array} \right] \mathcal{R}(p^n, q^n) z^k.\] (559)

After a short computation and using the condition (555) we get Eq. (557). Then there immediately results the proof of Eq. (558). \(\square\)

Setting

\[\psi_n(z; \mathcal{R}(p, q)) = \frac{1}{\sqrt{\mathcal{R}(p^n, q^n)}} H_n(z; \mathcal{R}(p, q)),\] (560)

and using the equations (557) and (558) yield the three-term recursion relation

\[\left(\phi_1(P, Q) + z\phi_2^2(p, q)\phi_2^{-1}(P, Q) - z\phi_3(p, q)\partial_{\mathcal{R}(p,q)}\right) \psi_n(z; \mathcal{R}(p, q)) = \sqrt{\mathcal{R}(p^{n+1}, q^{n+1})} \psi_{n+1}(z; \mathcal{R}(p, q))\] (561)

and \(\mathcal{R}(p, q)\)-difference equation

\[\partial_{\mathcal{R}(p,q)} \psi_n(z; \mathcal{R}(p, q)) = \sqrt{\mathcal{R}(p^n, q^n)} \psi_{n-1}(z; \mathcal{R}(p, q))\] (562)
for the polynomials $\psi_n(z; R(p,q))$ with the virtue that for $n = 0, 1, 2, \ldots$

$$\partial^{n+1}_{R(p,q)} \psi_n(z; R(p,q)) = 0 \text{ and } \partial^m_{R(p,q)} \psi_n(z; R(p,q)) \neq 0 \quad \text{for } m < n + 1. \quad (563)$$

Now, formally defining the number operator $N$ as

$$N \psi_n(z; R(p,q)) = n \psi_n(z; R(p,q)), \quad (564)$$

and the raising and lowering operators by

$$A^\dagger = (\phi_1(P,Q) + z\phi_2^N(p,q)\phi_2^{-1}(P,Q) - z\phi_3(p,q)\partial_{R(p,q)}) \quad \text{and} \quad A = \partial_{R(p,q)}, \quad (565)$$

respectively, the set of polynomials $\{\psi_n(z; R, p, q) \mid n = 0, 1, 2, \ldots\}$ provides a basis for a realization of $R(p,q)$—deformed quantum algebra $A_{R(p,q)}$ satisfying the commutation relations $\{369\}$. Provided the above formulated theorem, we can now show how the realizations in terms of Rogers-Szegő polynomials can be derived for different known deformations simply by determining the functions $\phi_i$ ($i = 1, 2, 3$) that satisfy the relations $\{553\} - \{555\}.$

### 6.2 Continuous $R(p,q)$—Hermite polynomials

We exploit here the peculiar relation established in the theory of $q$—deformation between Rogers-Szegő polynomials and Hermite polynomials $\{55\}$ and given by

$$H_n(\cos \theta; q) = e^{in \theta} H_n(e^{-2i \theta}; q) = \sum_{k=0}^{n} \binom{n}{k}_q e^{i(n-2k)\theta}, \quad n = 0, 1, 2, \ldots, \quad (566)$$

where $H_n$ and $H_n$ stand for the Hermite and Rogers-Szegő polynomials, respectively. Is also of interest the property that all the $q$—Hermite polynomials can be explicitly recovered from the initial one $\mathbb{H}_0(\cos \theta; q) = 1$, using the three-term recurrence relation

$$\mathbb{H}_{n+1}(\cos \theta; q) = 2 \cos \theta \mathbb{H}_n(\cos \theta; q) - (1 - q^n)\mathbb{H}_{n-1}(\cos \theta; q) \quad (567)$$

with $\mathbb{H}_1(\cos \theta; q) = 0$.

In the same way we define the $R(p,q)$—Hermite polynomials through the $R(p,q)$—Rogers-Szegő polynomials as

$$\mathbb{H}_n(\cos \theta; R(p,q)) = e^{in \theta} H_n(e^{-2i \theta}; R(p,q)), \quad n = 0, 1, 2, \ldots. \quad (568)$$

Then the next statement is true.
Proposition 6.2 Under the hypotheses of the theorem 6.1, the continuous \( R \)-
Hermite polynomials satisfy the following three-term recursion relation

\[
H_{n+1}(\cos \theta; R(p, q)) = e^{i \theta} \frac{n}{2} (p, q) \phi_1(P, Q) H_n(\cos \theta; R(p, q)) + e^{-i \theta} \frac{n}{2} (p, q) \phi_2^{-1}(P, Q) H_n(\cos \theta; R(p, q)) - \phi_3(p, q) R(p^n, q^n) H_{n-1}(\cos \theta; R(p, q)).
\] (6.59)

Proof: Multiplying the two sides of the three-term recursion relation (557) by \( e^{i(n+1)\theta} \), we obtain, for \( z = e^{-2i\theta} \),

\[
e^{i(n+1)\theta} H_{n+1}(e^{-2i\theta}; R(p, q)) = e^{i(n+1)\theta} H_n(\phi_1(p, q) e^{-2i\theta}; R(p, q)) + e^{i(n-1)\theta} \phi_2(p, q) H_n(\phi_2^{-1}(p, q) e^{-2i\theta}; R(p, q)) - e^{i(n-1)\theta} \phi_3(p, q) R(p^n, q^n) H_{n-1}(e^{-2i\theta}; R(p, q)) = e^{i\theta} e^{in\theta} \phi_1(P, Q) H_n(e^{-2i\theta}; R(p, q)) + e^{-i\theta} \phi_2(p, q) e^{in\theta} \phi_2^{-1}(P, Q) H_n(e^{-2i\theta}; R(p, q)) - \phi_3(p, q) R(p^n, q^n) e^{i(n-1)\theta} H_{n-1}(e^{-2i\theta}; R(p, q)).
\]

The required result follows from the use of the equalities

\[
e^{in\theta} \phi_1(P, Q) H_n(e^{-2i\theta}; R(p, q)) = \phi_1(p, q) \phi_1(P, Q) e^{in\theta} H_n(e^{-2i\theta}; R(p, q)), \quad (570)
\]

\[
e^{in\theta} \phi_2^{-1}(P, Q) H_n(e^{-2i\theta}; R(p, q)) = \phi_2^{-1}(P, Q) \phi_2^{-1}(P, Q) e^{in\theta} H_n(e^{-2i\theta}; R(p, q)). \quad (571)
\]

with

\[
\phi_j(P, Q) e^{-2ik\theta} = \phi_j^k(p, q) e^{-2ik\theta}, \quad j = 1, 2, \quad k = 0, 1, 2, \ldots \quad (572)
\]

6.3 Relevant particular cases

The following pertinent cases deserve to be raised, as their derivation from the
previous general theory appeals concrete expressions for the deformed function \( R(p, q) \).

6.3.1 \( R(x, y) = \frac{x-y}{p-q} \)

In this case, the \( R(p, q) \)-factors are simply given by

\[
[n]_{p, q} = R(p^n, q^n) = \frac{p^n - q^n}{p - q}, \quad n = 0, 1, 2, \ldots
\]
with the $\mathcal{R}(p,q)$–factorials defined by

$$[n]_{p,q}! = \begin{cases} 1 & \text{for } n = 0 \\ \prod_{k=1}^{n} \frac{p^k - q^k}{p - q} = \frac{((p,q);(p,q))_n}{(p - q)^n} & \text{for } n \geq 1. \end{cases}$$ (573)

They correspond to the Jagannathan-Srinivasa $(p,q)$-numbers and $(p,q)$-factorials $[57],[58]$.

There result the following relevant properties.

**Proposition 6.3** If $n$ and $m$ are nonnegative integers, then

$$[n + m]_{p,q} = q^m [n]_{p,q} + p^n [m]_{p,q} = p^m [n]_{p,q} + q^n [m]_{p,q},$$

$$[-m]_{p,q} = -q^{-m} p^{-m} [m]_{p,q},$$

$$[n - m]_{p,q} = q^{-m} [n]_{p,q} - q^{n-m} p^{-m} [m]_{p,q} = p^{-m} [n]_{p,q} - q^{-m} p^{-m} [m]_{p,q},$$

$$[n]_{p,q} = [2]_{p,q} [n-1]_{p,q} - pq [n-2]_{p,q}.$$ (574)

**Proposition 6.4** The $(p,q)$–binomial coefficients

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = \frac{[n]_{p,q}!}{[k]_{p,q}! [n-k]_{p,q}!} = \frac{((p,q);(p,q))_n}{((p,q);(p,q))_k ((p,q);(p,q))_{n-k}},$$ (575)

where $0 \leq k \leq n$, $n \in \mathbb{N}$, and $((p,q);(p,q))_n = (p-q)(p^2-q^2) \cdots (p^m-q^m)$, $m \in \mathbb{N}$, satisfy the following identities:

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = \begin{bmatrix} n \\ n-k \end{bmatrix}_{p,q} = p^{k(n-k)} \begin{bmatrix} n \\ k \end{bmatrix}_{q/p},$$ (576)

$$\begin{bmatrix} n+1 \\ k \end{bmatrix}_{p,q} = p^k \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} + q^{n+1-k} \begin{bmatrix} n \\ k-1 \end{bmatrix}_{p,q},$$ (577)

$$\begin{bmatrix} n+1 \\ k \end{bmatrix}_{p,q} = p^k \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} + p^{n+1-k} \begin{bmatrix} n \\ k-1 \end{bmatrix}_{p,q} - (p^n-q^n) \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_{p,q}.$$ (578)

with

$$\begin{bmatrix} n \\ k \end{bmatrix}_{q/p} = \frac{(q/p; q/p)_n}{(q/p; q/p)_k (q/p; q/p)_{n-k}},$$

where $(q/p; q/p)_n = (1 - q/p)(1 - q^2/p^2) \cdots (1 - q^n/p^n)$ and the $(p,q)$–shifted factorial

$$(a,b);(p,q))_n \equiv (a - b)(ap - bq) \cdots (ap^{n-1} - bq^{n-1})$$

$$= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} (-1)^k p^{(n-k)(n-k-1)/2} q^{k(k-1)/2} a^{n-k} b^k.$$ (579)
The algebra \( \mathcal{A}_{p,q} \), generated by \( \{ 1, A, A^\dagger, N \} \), associated with \( (p,q) \)-Jangathan-Srinivasa deformation, satisfies the following commutation relations [57, 58]:

\[
\begin{align*}
A A^\dagger - pA^\dagger A &= q^N, & A A^\dagger - qA^\dagger A &= p^N \\
[N, A^\dagger] &= A^\dagger, & [N, A] &= -A.
\end{align*}
\]

The \( (p,q) \)-Rogers-Szegö polynomials studied in [58] appear as a particular case obtained by choosing \( \phi_1(x,y) = \phi_2(x,y) = \phi(x,y) = x \) and \( \phi_3(x,y) = x - y \). Indeed, \( \phi(p,q) = p \neq 0, \phi_3(p,q) = p - q \neq 0, \phi(P,Q)z^k = \phi^k_1(p,q)z^k \) and Eq. (578) shows that

\[
\binom{n+1}{k}_{p,q} = p^k \binom{n}{k}_{p,q} + np^{n+1-k} \binom{n}{k-1}_{p,q} - (p - q)[n]_{p,q} \binom{n-1}{k-1}_{p,q}.
\]

Hence, the hypotheses of the above theorem are satisfied and, therefore, the \( (p,q) \)-Rogers-Szegö polynomials

\[
H_n(z; p,q) = \sum_{k=0}^{\infty} \binom{n}{k}_{p,q} z^k \quad n = 0, 1, 2, \ldots
\]

satisfy the three-term recursion relation

\[
H_{n+1}(z; p,q) = H_n(pz; p,q) + zp^n H_n(pz^{-1}; p,q) - z(p^n - q^n)H_{n-1}(z; p,q)
\]

and \( (p,q) \)-difference equation

\[
\partial_{p,q} H_n(z; p,q) = [n]_{p,q} H_{n-1}(z; p,q).
\]

Finally, the set of polynomials

\[
\psi_n(z; p,q) = \frac{1}{\sqrt{[n]_{p,q}}} H_n(z; p,q), \quad n = 0, 1, 2, \ldots
\]

forms a basis for a realization of the \( (p,q) \)-deformed harmonic oscillator and quantum algebra \( \mathcal{A}_{p,q} \) satisfying the commutation relations (580) with the number operator \( N \) defined as

\[
N \psi_n(z; p,q) = n \psi_n(z; p,q),
\]

relating the annihilation and creation operators given by

\[
A = \partial_{p,q} \quad \text{and} \quad A^\dagger = P + zp^N P^{-1} - z(p - q) \partial_{p,q}
\]
respectively. Naturally, setting \( p = 1 \) one recovers the results of the subsection \[6.1.2\]

The continuous \((p,q)\)-Hermite polynomials have been already suggested in \[57\] without any further details. In the above achieved generalization, these polynomials are given by

\[
H_n(\cos \theta; p, q) = e^{i\theta n} H_n(e^{-2i\theta}; p, q) = \sum_{k=0}^{n} \binom{n}{k} \frac{e^{i(n-2k)\theta}}{p^k q^{n-k}}, \quad n = 0, 1, 2, \cdots. \tag{587}
\]

Since for the \((p,q)\)-deformation \( \phi_1(x,y) = \phi_2(x,y) = x \) and \( \phi_3(x,y) = x - y \), from the Proposition \[62\] we deduce that the corresponding sequence of continuous \((p,q)\)-polynomials satisfies the three-term recursion relation

\[
H_{n+1}(\cos \theta; p, q) = p^2 \left( e^{i\theta} P + e^{-i\theta} P^{-1} \right) H_n(\cos \theta; p, q) - (p^n - q^n) H_{n-1}(\cos \theta; p, q), \tag{588}
\]

with \( P e^{i\theta} = p^{-1/2} e^{i\theta} \). This relation turns to be the well-known three-term recursion relation \[567\] for continuous \(q\)-Hermite polynomials in the limit \( p \to 1 \). As matter of illustration, let us explicitly compute the first three polynomials using the relation \[588\], with \( H_0(\cos \theta; p, q) = 0 \) and \( H_0(\cos \theta; p, q) = 1 \):

\[
H_1(\cos \theta; p, q) = p^0 \left( e^{i\theta} P + e^{-i\theta} P^{-1} \right) 1 - (p^0 - q^0) 1 = e^{i\theta} + e^{-i\theta} = 2 \cos \theta
= \left[ \begin{array}{c} 1 \end{array} \right]_{p,q} e^{i\theta} + \left[ \begin{array}{c} 1 \end{array} \right]_{p,q} e^{-i\theta}.
\]

\[
H_2(\cos \theta; p, q) = p^1 \left( e^{i\theta} P + e^{-i\theta} P^{-1} \right) \left( e^{i\theta} + e^{-i\theta} \right) - (p - q) 1 = e^{2i\theta} + e^{-2i\theta} + p + q = 2 \cos 2\theta + p + q
= \left[ \begin{array}{c} 0 \end{array} \right]_{p,q} e^{2i\theta} + \left[ \begin{array}{c} 2 \end{array} \right]_{p,q} e^{0i\theta} + \left[ \begin{array}{c} 2 \end{array} \right]_{p,q} e^{-2i\theta}.
\]

\[
H_3(\cos \theta; p, q) = p \left( e^{i\theta} P + e^{-i\theta} P^{-1} \right) \left( e^{2i\theta} + e^{-2i\theta} + p + q \right)
= e^{3i\theta} + e^{-3i\theta} + (p^2 + pq + q^2) \left( e^{i\theta} + e^{-i\theta} \right)
= 2 \cos 3\theta + 2(p^2 + pq + q^2) \cos \theta
= \left[ \begin{array}{c} 3 \end{array} \right]_{p,q} e^{3i\theta} + \left[ \begin{array}{c} 3 \end{array} \right]_{p,q} e^{i\theta} + \left[ \begin{array}{c} 3 \end{array} \right]_{p,q} e^{-i\theta} + \left[ \begin{array}{c} 3 \end{array} \right]_{p,q} e^{-3i\theta}.
\]

6.3.2 \( \mathcal{R}(x, y) = \frac{1-x y}{(p^{-1}-q)x} \)

The \( \mathcal{R}(p,q) \)-factors and \( \mathcal{R}(p,q) \)-factorials are reduced to \((p^{-1},q)\)-numbers and \((p^{-1},q)\)-factorials, namely,

\[
[n]_{p^{-1}, q} = \frac{p^{-n} - q^n}{p^{-1} - q},
\]

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and

$$[n]_{p^{-1}, q} = \begin{cases} 1 & \text{for } n = 0 \\ \frac{(p^{-1}; q; (p^{-1}, q))_{n}}{(p^{-1} - q)^n} & \text{for } n \geq 1, \end{cases}$$  \hfill (589)$$

respectively, which exactly reproduce the $(p, q)$-numbers and $(p, q)$-factorials introduced by Chakrabarty and Jagannathan [24].

The other properties can be recovered similarly to those of section 6.3.1 replacing the parameter $p$ by $p^{-1}$.

The $R_{(p, q)}$-derivative is also reduced to $(p^{-1}, q)$-derivative. Indeed,

$$\frac{\partial}{\partial_R(p, q)} = \frac{\partial_{p,q}}{(p^{-1} - q)(P - Q)(p^{-1} - q)} \equiv \frac{1}{(p^{-1} - q)^z} (P - Q) \equiv \partial_{p^{-1}, q}$$

obtained by a simple replacement of the dilatation operator $P$ by $P^{-1}$.

The algebra $A_{p^{-1}, q}$, generated by $\{1, A, A^\dagger, N\}$, associated with $(p, q)$–Chakrabarty and Jagannathan deformation satisfies the following commutation relations:

$$[A A^\dagger - p^{-1} A^\dagger A = q^N, \quad A A^\dagger - q A^\dagger A = p^{-N}]$$

$$[N, A] = A^\dagger \quad [N, A] = -A.$$ \hfill (591)$$

Hence, the $(p^{-1}, q)$–Rogers-Szegö polynomials

$$H_n(z; p^{-1}, q) = \sum_{k=0}^{n} \left[ \begin{array}{c} n \\ k \end{array} \right]_{p^{-1}, q} z^k \quad n = 0, 1, 2, \cdots$$ \hfill (592)$$

obey the three-term recursion relation

$$H_{n+1}(z; p^{-1}, q) = H_n(p^{-1} z; p^{-1}, q) + z p^{-n} H_n(pz; p^{-1}, q)$$

$$- z(p^{-n} - q^n) H_{n-1}(z; p^{-1}, q)$$ \hfill (593)$$

and $(p^{-1}, q)$–difference equation

$$\partial_{p^{-1}, q} H_n(z; p, q) = [n]_{p^{-1}, q} H_{n-1}(z; p, q).$$ \hfill (594)$$

Finally, the set of polynomials

$$\psi_n(z; p^{-1}, q) = \frac{1}{\sqrt{|n|_{p^{-1}, q}}} H_n(z; p^{-1}, q), \quad n = 0, 1, 2, \cdots$$ \hfill (595)$$
forms a basis for a realization of the \((p^{-1}, q)\)-deformed harmonic oscillator and quantum algebra \(A_{p^{-1},q}\) generating the commutation relations (591) with the number operator \(N\) formally defined as

\[
N\psi_n(z; p^{-1}, q) = n\psi_n(z; p^{-1}, q),
\]

and the annihilation and creation operators given by

\[
A = \partial_{p^{-1},q} \quad \text{and} \quad A^\dagger = P^{-1} + zp^{-N}P - z(p^{-1} - q)\partial_{p^{-1},q},
\]

respectively. Naturally, setting \(p = 1\) permits to recover the results of the subsection 6.1.2.

6.3.3 \(\mathcal{R}(x, y) = \frac{zy^{-1}}{(q-p^{-1})y}\)

In this case, the \(\mathcal{R}(p,q)\)-factors and \(\mathcal{R}(p,q)\)-factorials are reduced to

\[
[n]_{p,q} = \frac{p^n - q^{-n}}{q - p^{-1}},
\]

and

\[
[n]_{p,q}! = \begin{cases} 
1 & \text{for } n = 0 \\
\left(\frac{p^{-1}}{p-q^{-1}}\right)^n & \text{for } n \geq 1,
\end{cases}
\]

introduced in our previous work [53], generalizing the \(q\)-Quesne algebra [95].

Then follow some remarkable properties:

**Proposition 6.5** If \(n\) and \(m\) are nonnegative integers, then

\[
[-m]_{p,q}^{Q} = -p^{-m}q^{m}[n]_{p,q}^{Q},
\]

\[
[n + m]_{p,q}^{Q} = q^{-m}[n]_{p,q}^{Q} + p^{n}[m]_{p,q}^{Q} = p^{m}[n]_{p,q}^{Q} + q^{-n}[m]_{p,q}^{Q},
\]

\[
[n - m]_{p,q}^{Q} = q^{m}[n]_{p,q}^{Q} - p^{-n}q^{m}[n]_{p,q}^{Q} = p^{-m}[n]_{p,q}^{Q} + p^{-m}q^{-m}[m]_{p,q}^{Q},
\]

\[
[n]_{p,q}^{Q} = \frac{q - p^{-1}}{p - q^{-1}}[2]_{p,q}^{Q}[n - 1]_{p,q}^{Q} - pq^{-1}[n - 2]_{p,q}^{Q}.
\]

**Proof:** Eqs. (599) and (600) are immediate by the application of the relations \(p^{-m} - q^{m} = -p^{-m}q^{m}(p^{m} - q^{-m})\) and \(p^{n+m} - q^{-n-m} = q^{-m}(p^{n} - q^{-n}) + p^{n}(p^{m} - q^{-m}) = p^{m}(p^{n} - q^{-n}) + q^{-n}(p^{m} - q^{-m}),\) respectively, while Eq. (601) results from the combination of Eqs. (599) and (600). Finally, the relation

\[
[n]_{p,q}^{-1} = \frac{p^{n} - q^{-n}}{p - q^{-1}} = \frac{q - p^{-1}}{p - q^{-1}}\frac{p^{n} - q^{-n}}{q - p^{-1}} = \frac{q - p^{-1}}{p - q^{-1}}[n]_{p,q}^{Q}, \quad n = 1, 2, \cdots
\]

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cumulatively taken with the identity
\[ [n]_{p,q-1} = [2]_{p,q-1}[n-1]_{p,q-1} - pq^{-1}[n-2]_{p,q-1} \]
gives Eq. (602).

**Proposition 6.6** The \((p,q)\)-Quesne binomial coefficients
\[
\left[ \begin{array}{c} n \\ k \end{array} \right]_{p,q}^Q = \frac{((p, q^{-1}); (p, q^{-1}))_n}{((p, q^{-1}); (p, q^{-1}))_{n-k}}
\]
where \(0 \leq k \leq n; \ n \in \mathbb{N}\), satisfy the following properties
\[
\left[ \begin{array}{c} n \\ k \end{array} \right]_{p,q}^Q = \left[ \begin{array}{c} n \\ n-k \end{array} \right]_{p,q}^Q = p^{k(n-k)} \left[ \begin{array}{c} n \\ k \end{array} \right]_{1/qp}^Q = p^{k(n-k)} \left[ \begin{array}{c} n \\ n-k \end{array} \right]_{1/qp}^Q,
\]
\[
\left[ \begin{array}{c} n+1 \\ k \end{array} \right]_{p,q}^Q = p^k \left[ \begin{array}{c} n \\ k \end{array} \right]_{p,q}^Q + q^{-n-k} \left[ \begin{array}{c} n \\ k-1 \end{array} \right]_{p,q}^Q,
\]
\[
\left[ \begin{array}{c} n+1 \\ k \end{array} \right]_{p,q}^Q = p^k \left[ \begin{array}{c} n \\ k \end{array} \right]_{p,q}^Q + p^{n-k} \left[ \begin{array}{c} n \\ k-1 \end{array} \right]_{p,q}^Q - (p^n - q^{-n}) \left[ \begin{array}{c} n-1 \\ k-1 \end{array} \right]_{p,q}^Q.
\]

**Proof:** It is straightforward, using the Proposition 6.3 and
\[
\left[ \begin{array}{c} n \\ k \end{array} \right]_{p,q}^Q = \left[ \begin{array}{c} n \\ k \end{array} \right]_{p,q-1}^Q.
\]

Finally, the algebra \(A_{p,q}^Q\), generated by \(\{1, A, A^\dagger, N\}\), associated with \((p,q)\)-Quesne deformation satisfies the following commutation relations:
\[
p^{-1}A A^\dagger - A^\dagger A = q^{-N-1}, \quad qA A^\dagger - A^\dagger A = p^{N+1}
\]
\[
[N, A^\dagger] = A^\dagger, \quad [N, A] = -A.
\]

The \((p,q)\)-Rogers-Szegö polynomials corresponding to the Quesne deformation \([53]\) are deduced from our generalization by choosing \(\phi_1(x, y) = \phi_2(x, y) = \phi(x, y) = x\) and \(\phi_3(x, y) = y - x^{-1}\). Indeed, it is worthy of attention that we get
in this case $\phi(p, q) = p \neq 0$, $\phi_3(p, q) = q - p^{-1} \neq 0$, $\phi(P, Q)z^k = \phi_1^k(p, q)z^k$ and from Eq. (607)

\[
\begin{bmatrix} n+1 \end{bmatrix}_p^Q = p^k \begin{bmatrix} n \end{bmatrix}_p^Q + p^{n+1-k} \begin{bmatrix} n \end{bmatrix}_{k-1}^Q - (q - p^{-1})[n]_{p, q} \begin{bmatrix} n-1 \end{bmatrix}_{k-1}^Q.
\]

Hence, the hypotheses of the theorem are satisfied and, therefore, the $(p, q)$—Rogers - Szegő polynomials

\[
H_n^Q(z; p, q) = \sum_{k=0}^{n} \begin{bmatrix} n \end{bmatrix}_p^Q z^k, \quad n = 0, 1, 2, \ldots
\]

satisfy the three-term recursion relation

\[
H_{n+1}^Q(z; p, q) = H_n^Q(pz; p, q) + zp^n H_n^Q(p^{-1}z; p, q) - z(p^n - q^{-n})H_{n-1}^Q(z; p, q)
\]

and the $(p, q)$—difference equation

\[
\partial_{p, q}^Q H_n^Q(z; p, q) = [n]_p^Q H_{n-1}^Q(z; p, q).
\]

Thus, the set of polynomials

\[
\psi_n^Q(z; p, q) = \frac{1}{\sqrt{[n]_p^Q}} H_n^Q(z; p, q), \quad n = 0, 1, 2, \ldots
\]

forms a basis for a realization of the $(p, q)$— Quesne deformed harmonic oscillator and quantum algebra $A_{p, q}$ engendering the commutation relations [609] with the number operator $N$ formally defined as

\[
N\psi_n^Q(z; p, q) = n\psi_n^Q(z; p, q),
\]

and the annihilation and creation operators given by

\[
A = \partial_{p, q}^Q \quad \text{and} \quad A^\dagger = P + zp^NP^{-1} - z(q - p^{-1})\partial_{p, q},
\]

respectively. Naturally, setting $p = 1$ gives the Rogers-Szegő polynomials associated with the $q$—Quesne deformation [95].

The continuous $(p, q)$—Hermite polynomials corresponding to the $(p, q)$—generalization of Quesne deformation [53] can be defined as follows:

\[
H_n^Q(\cos \theta; p, q) = e^{in\theta} H_n^Q(e^{-2i\theta}; p, q)
\]
\[
\sum_{k=0}^{n} \binom{n}{k}^Q e^{i(n-2k)\theta}, \quad n = 0, 1, 2, \ldots.
\]  

(616)

Since for the \((p, q)\)-generalization of Quesne deformation, \(\phi_1(x, y) = \phi_2(x, y) = x\) and \(\phi_3(x, y) = y - x^{-1}\), from the Proposition 6.2 we deduce that the corresponding sequence of continuous \((p, q)\)-Hermite polynomials satisfies the three-term recurrence relation

\[
H_{n+1}^Q(\cos \theta; p, q) = p^n(e^{i\theta} P + e^{-i\theta} P^{-1})H_n^Q(\cos \theta; p, q) - (p^n - q^{-n})H_{n-1}^Q(\cos \theta; p, q).
\]  

(617)

6.3.4 \(\mathcal{R}(x, y) = h(p, q)y^\nu / x^\mu\left[xy - 1 \atop (q - p^{-1})y\right]\)

Here \(0 < pq < 1\), \(p^\mu < q^\nu - 1\), \(p > 1\), \(h\) is a well behaved real and non-negative function of deformation parameters \(p\) and \(q\) such that \(h(p, q) \to 1\) as \((p, q) \to (1, 1)\).

The \(\mathcal{R}(p, q)\)-factors become \((p, q; \mu, \nu, h)\)-numbers introduced in our previous work and defined by

\[
[n]_{\mu, \nu}^{p, q, h} = h(p, q) \frac{q^n p^n - q^{-n}}{p^{\mu n} q - p^{-1}}.
\]

(618)

Proposition 6.7 The \((p, q; \mu, \nu, h)\)-numbers verify the following properties, for \(m, n \in \mathbb{N}\):

\[
[-m]_{\mu, \nu}^{p, q, h} = \frac{-q^{-2\nu m + m}}{p^{2\mu m + m}} [m]_{\mu, \nu}^{p, q, h},
\]

(619)

\[
[n + m]_{\mu, \nu}^{p, q, h} = \frac{q^{\nu m - m}}{p^{\mu m}} [n]_{\mu, \nu}^{p, q, h} + \frac{q^{\nu m}}{p^{\mu m - n}} [m]_{\mu, \nu}^{p, q, h} = \frac{q^{\nu m}}{p^{\mu m - n}} [n]_{\mu, \nu}^{p, q, h} + \frac{q^{\nu m - n}}{p^{\mu m}} [m]_{\mu, \nu}^{p, q, h},
\]

(620)

\[
[n - m]_{\mu, \nu}^{p, q, h} = \frac{q^{-\nu m + m}}{p^{\mu m}} [n]_{\mu, \nu}^{p, q, h} - \frac{q^{\nu(n-2m)+m}}{p^{\mu(n-2m)-n+m}} [m]_{\mu, \nu}^{p, q, h} = \frac{q^{-\nu m}}{p^{\mu m + m}} [n]_{\mu, \nu}^{p, q, h} - \frac{q^{\nu(n-2m)-n+m}}{p^{\mu(n-2m)+m}} [m]_{\mu, \nu}^{p, q, h},
\]

(621)

\[
[n]_{\mu, \nu}^{p, q, h} = \frac{q - p^{-1}}{p - q^{-1}} \frac{q^{-\nu}}{p^{\mu}} \frac{1}{h(p, q)} \frac{[2]_{\mu, \nu}^{p, q, h}}{[n - 1]_{\mu, \nu}^{p, q, h}} - \frac{q^{2\nu - 1}}{p^{2\nu - 1}} [n - 2]_{\mu, \nu}^{p, q, h}.
\]

(622)
Proof: It is direct using the Proposition 6.5 and the fact that

\[ [n]_{p,q,h}^{\mu,\nu} = h(p,q) \frac{q^{\nu n}}{p^{\mu n}} [n]_{p,q}^{Q} \]  

(623)

Proposition 6.8 The \((p,q,\mu,\nu,h)\)-binomial coefficients

\[ \binom{n}{k}_{p,q,h}^{\mu,\nu} := \frac{[n]_{p,q,h}^{\mu,\nu}}{[k]_{p,q,h}^{\mu,\nu} [n-k]_{p,q,h}^{\mu,\nu}} = \frac{q^{\nu k(n-k)}}{p^{\mu k(n-k)}} \binom{n}{k}_{p,q}^{Q} \]  

(624)

where 0 ≤ k ≤ n; n ∈ N, satisfy the following properties

\[ \binom{n}{k}_{p,q,h}^{\mu,\nu} = \binom{n}{n-k}_{p,q,h}^{\mu,\nu} \]  

(625)

\[ \binom{n+1}{k}_{p,q,h}^{\mu,\nu} = \frac{q^{\nu k}}{p^{\mu(k-1)}} \binom{n}{k}_{p,q,h}^{\mu,\nu} + \frac{p^{\nu (n+1-k)}}{q^{\mu(n+1-k)}} \binom{n}{k-1}_{p,q,h}^{\mu,\nu} \]  

(626)

\[ \binom{n+1}{k}_{p,q,h}^{\mu,\nu} = \frac{q^{\nu k}}{p^{\mu(k-1)}} \binom{n}{k}_{p,q,h}^{\mu,\nu} + \frac{q^{\nu(n+1-k)}}{p^{\mu(n+1-k)}} \binom{n}{k-1}_{p,q,h}^{\mu,\nu} \]  

(627)

Proof: There follow from the Proposition 6.5 and the fact that

\[ [n]_{p,q,h}^{\mu,\nu} = h^n(p,q) \frac{q^{n(n+1)/2}}{p^{\nu(n+1)/2}} [n]_{p,q}^{Q} \]  

(628)

where use of Eq.(623) has been made. □

The algebra \(A_{p,q,h}^{\mu,\nu}\), generated by \(\{1, A, A^\dagger, N\}\), associated with \((p,q,\mu,\nu,h)\)-deformation, satisfies the following commutation relations:

\[ p^{-1} A A^\dagger - \frac{q^\mu}{p^{\mu}} A^\dagger A = h(p,q) \left( \frac{q^{\mu-1}}{p^{\mu}} \right)^{N+1} \]  

\[ qA A^\dagger - \frac{q^\nu}{p^{\mu}} A^\dagger A = h(p,q) \left( \frac{q^{\nu}}{p^{\mu-1}} \right)^{N+1} \]  

\[ [N, A^\dagger] = A^\dagger, \quad [N, A] = -A. \]  

(629)

The \((p,q,\mu,\nu,h)\)-Rogers-Szegö [52] polynomials are deduced from the above general construction by setting \(\phi_1(x, y) = x^{1-\nu} y^{\nu}, \phi_2(x, y) = x^{\nu} y^{\nu-1}\) and \(\phi_3(x, y) =…\)
Indeed, \( \phi_i(p, q) \neq 0 \) for \( i = 1, 2, 3 \); \( \phi_i(P, Q) z^k = \phi_i(p, q)^k z^k \) for \( i = 1, 2 \) and the property (627) furnishes

\[
\begin{bmatrix}
  n + 1 \\
  \hline
  k
\end{bmatrix}_{p,q,h}^\mu,\nu = \frac{q^{-k}}{\mu^{-1}k} \begin{bmatrix}
  n \\
  \hline
  k
\end{bmatrix}_{P,Q}^\mu,\nu + \frac{q^{\nu(n+1-k)}}{\mu^{(1)(n+1-k)}} \begin{bmatrix}
  n \\
  \hline
  k - 1
\end{bmatrix}_{P,Q}^\mu,\nu - \frac{q - p}{h(p, q)} [n]_{p,q,h}^\mu,\nu \begin{bmatrix}
  n - 1 \\
  \hline
  k - 1
\end{bmatrix}_{p,q,h}^\mu,\nu.
\]

Therefore, the \((p, q, \mu, \nu, h)\)-Rogers-Szegö polynomials are defined as follows:

\[
H_n(z; p, q, \mu, \nu, h) = \sum_{k=0}^{n} \begin{bmatrix}
  n \\
  \hline
  k
\end{bmatrix}_{p,q,h}^\mu,\nu z^k, \quad n = 0, 1, 2 \ldots
\] (630)

with the three-term recursion relation

\[
H_{n+1}(z; p, q, \mu, \nu, h) = H_n \left( \frac{q^{\nu}}{\mu^{(1)}} z : p, q, \mu, \nu, h \right) + z \frac{q^{(\nu-1)n}}{\mu^n} H_n \left( \frac{p^{\nu}}{q^{(1)} z : p, q, \mu, \nu, h} \right) - z \frac{q^{\nu}}{\mu^n} (p^n - q^n) H_{n-1}(z; p, q, \mu, \nu, h)
\] (631)

and \((p, q, \mu, \nu, h)\)--difference equation

\[
\partial_{p,q,h}^{\mu,\nu} H_n(z; p, q, \mu, \nu, h) = [n]^{\mu,\nu}_{p,q,h} H_{n-1}(z; p, q, \mu, \nu, h).
\] (632)

Hence, the set of polynomials

\[
\psi_n(z; p, q, \mu, \nu, h) = \frac{1}{\sqrt{[n]^{\mu,\nu}_{p,q,h}}} H_n(z; p, q, \mu, \nu, h), \quad n = 0, 1, 2, \ldots
\] (633)

forms a basis for a realization of the \((p, q, \mu, \nu, h)\)--deformed algebra \( A_{p,q,\mu,\nu,h} \) satisfying the commutation relations (629), with the number operator \( N \) formally defined as

\[
N \psi_n^Q(z; p, q, \mu, \nu, h) = n \psi_n(z; p, q, \mu, \nu, h),
\] (634)

together with the annihilation and the creation operators given by

\[
A = \partial_{p,q,h}^{\mu,\nu} \quad \text{and} \quad A^\dagger = \frac{Q^{\nu}}{P^{(1)}} + z \left( \frac{Q^{\nu-1}}{P^{\mu}} \right)^N P^{\mu} \frac{Q^{\nu}}{Q^{(1)}} - z \frac{(q - p^{-1})}{h(p, q)} \partial_{p,q,h}^{\mu,\nu},
\] (635)

respectively.
The continuous \((p, q, \mu, \nu, h)\)-Hermite polynomials \([52]\) can be now deduced as:

\[
\mathbb{H}_n(\cos \theta; p, q, \mu, \nu, h) = e^{in\theta} \mathbb{H}_n(e^{-2i\theta}; p, q, \mu, \nu, h) = \sum_{k=0}^{n} \binom{n}{k}_{p,q,h} e^{i(n-2k)\theta}, \quad n = 0, 1, 2, \ldots.
\] (636)

Since for the \((p, q, \mu, \nu, h)\)-deformation \(\phi_1(x, y) = x^{1-\mu}y^{\nu}\), \(\phi_2(x, y) = x^{-\mu}y^{\nu-1}\) and \(\phi_3(x, y) = \frac{y-x}{h(p,q)}\), from the Proposition \([6,2]\) the corresponding sequence of continuous \((p, q, \mu, \nu, h)\)-Hermite polynomials satisfies the three-term recursion relation

\[
\mathbb{H}_{n+1}(\cos \theta; p, q, \mu, \nu, h) = \frac{q^{\nu}}{p^{(\mu-1)\frac{1}{2}}} Q^{\nu}_{P^{\nu-1}} \mathbb{H}_n(\cos \theta; p, q, \mu, \nu, h)
+ \frac{q^{(\nu-1)\frac{1}{2}}}{p^{\mu\frac{1}{2}}} Q^{(\nu-1)}_{P^{-\mu}} \mathbb{H}_n(\cos \theta; p, q, \mu, \nu, h)
- (p^n - q^{-n}) \frac{q^{\mu n}}{p^{\mu n}} \mathbb{H}_{n-1}(\cos \theta; p, q, \mu, \nu, h).
\] (637)

7 Concluding remark

We have first deformed the Heisenberg algebra with the set of parameters \(\{q, l, \lambda\}\) to generate a new family of generalized coherent states respecting the Klauder criteria. In this framework, the matrix elements of relevant operators have been exactly computed and investigated from functional analysis point of view. Then, relevant statistical properties have been examined. Besides, a proof on the sub-Poissonian character of the statistics of the main deformed states has been provided. This property has been finally used to determine the induced generalized metric, characterizing the geometry of the considered system.

Next, a unified method of defining structure functions from commutation relations of deformed single-mode oscillator algebras has been presented. A natural approach to building coherent states associated to deformed algebras has been then deduced. Known deformed algebras have been given as illustration and such mathematical properties as continuity in the label, normalizability and resolution of the identity of their corresponding coherent states have been discussed.

Besides, we have generalized a class of two - parameter deformed Heisenberg algebras related to meromorphic functions. There have been probed relevant families of coherent states maps and their corresponding hypergeometric series. The latter constitutes a generalization of known hypergeometric series. Moreover, a \(\mathcal{R}(p,q)\)-binomial theorem, generalizing the \((p,q)\)-binomial theorem given in \([57]\) has been deduced. We have also defined the \(\mathcal{R}(p,q)\)-trigonometric, hyperbolic and \((p,q)\)-Bessel functions, including their main relevant properties.
Then, we have provided a new noncommutative algebra related to the $\mathcal{R}(p,q)$-deformation and shown that the notions of differentiation and integration can be extended to it, thus generalizing well known $q$ or/and $(p,q)$-differential and integration calculi \cite{22,31,74}. Besides, we have performed a general procedure of constructing the Hopf algebra structure compatible with the $\mathcal{R}(p,q)$-algebra. As illustration, relevant examples have been given.

Finally, we have defined and discussed a general formalism for constructing $\mathcal{R}(p,q)$-deformed Rogers-Szeg"{o} polynomials. The displayed approach not only provides novel relations, but also generalizes well known standard and deformed Rogers-Szeg"{o} polynomials. A full characterization of the latter, including the data on the three-term recursion relations and difference equation, has been provided. We have succeeded in elaborating a new realization of $\mathcal{R}(p,q)$—deformed quantum algebra generalizing the construction of $q$—deformed harmonic oscillator creation and annihilation operators performed in \cite{40,58}. The continuous $\mathcal{R}(p,q)$—Hermite polynomials have been also investigated in detail and relevant particular cases and examples have been exhibited.

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References

[1] E. Abe, *Hopf algebras* (Cambridge University Press, Cambridge, 1977).

[2] N.I. Akhiezer, *The Classical Moment Problem and Some Related Questions in Analysis* (Olivier and Boyd, London, 1965).

[3] S. T. Ali, J.-P. Antoine, J.-P. Gazeau, *Coherent States, Wavelets and their Generalizations* (Springer-Verlag, New-York, 1999).

[4] S. T. Ali, J.-P. Antoine and U.A. Mueller, Coherent states and their generalizations: A mathematical overview, Rev. Math. Phys. 7 (1995), 1013-1104.

[5] C. Aragone, E. Chalbaud, S. Salamo, Intelligent spin states, J. Math. Phys. 17 (1976), 1963-1971.
[6] C. Aragone, G. Guerri, S. Salamo, and J.L. Tani, Intelligent spin states, J. Phys. A: Math. Nucl. Gen. 7 (1974), L149-L151.

[7] M. Arik and D.D. Coon, Hilbert spaces of analytic functions and generated coherent states, J. Math. Phys. 17 (1976), 424-427.

[8] N.M. Atakishiyev, M.K. Atakishiyeva, A q-analogue of the Euler gamma integral, Theoret. Math. Phys. 129 (2001), 1325-1334.

[9] E. Balóitcha, M.N. Hounkonnou and E.B. Ngompe Nkouankam, Unified $(p,q;\alpha,\beta,\nu;\gamma)$-deformation: Irreducible representations and induced deformed harmonic oscillator, J. Math. Phys. 53 (2012), 013504-013514.

[10] A. O. Barut and L. Girardello, New coherent states associated with noncompact groups, Commun. Math. Phys. 21 (1971), 41-55.

[11] Ju. M. Berezanski, Expansions in Eigenfunctions of Selfadjoint Operators, (Amer. Math. Soc., Providence, Rhode Island, 1968).

[12] L. C. Biedenharn, The quantum group $SU_q(2)$ and a q-analogue of the boson operators, J. Phys. A 22 (1989), L873-L878.

[13] V.V. Borzov, E.V. Damaskinsky and S.V. Yegorov, Somme remarks on the representations of the generalized deformed algebra, arXiv:q-alg/9509022.

[14] T. Brzeziński, J.L. Egusquinza, A.J. Macfarlane, Generalised Harmonic Oscillator Systems and Their Fock Space Description, Phys. Lett. B 311 (1993), 202-206.

[15] J.D. Bukweli Kyemba and M.N. Hounkonnou, Characterization of $(R, p, q)$-deformed Rogers-Szegö polynomials: associated quantum algebras, deformed Hermite polynomials and relevant properties, J. Phys. A: Math. Theor. 45 (2012), 225204.

[16] J.D. Bukweli Kyemba and M.N. Hounkonnou, On generalized oscillator algebras and their associated coherent states, submitted.

[17] J.D. Bukweli Kyemba and M.N. Hounkonnou, $(q;l, \lambda)$-deformed Heisenberg algebra: coherent states, their statistics and geometry, Afr. Diaspora J. Math. 14 (2012), 38-56.

[18] J.D. Bukweli Kyemba and M.N. Hounkonnou, $R(p,q)$-calculus: differentiation, integration and Hopf algebras, submitted.
[19] I.M. Burban, Unified $(q; \alpha, \beta, \gamma; \nu)$-deformation of one-parametric $q$-deformed oscillator algebras, Phys. Lett. A 42 (2009), 056201, arxiv.org/pdf/0806.0613

[20] I. M. Burban, Two-parameter deformation of oscillator algebra, Phys. Lett. B 319 (1993), 485-489.

[21] I. M. Burban, On $(p, q; \alpha, \beta, \nu)$-deformed oscillator and its generalized quantum Heisenberg-Weyl algebra, Phys. Lett. A 366 (2007), 308-314.

[22] I. M. Burban and A. U. Klimyk, $P, Q$-differentiation, $P, Q$-integration, and $P, Q$- hypergeometric functions related to quantum groups. Integral Transforms and Special Functions. 2(1994), 15-36.

[23] B. L. Cerchiai, R. Hinterding, J. Madore and J. Wess, The geometry of $q$-deformed phase space, Eur. Phys. J. C 8 (1999), 533-546.

[24] R. Chakrabarti and R. Jagannathan, A $(p, q)$-oscillator realisation of two-parameter quantum algebras. J. Phys. A: Math. Gen. 24 (1991), L711-L718.

[25] V. Chari and A. Pressley, A guide to Quantum groups (Cambridge, Cambridge University Press, 1994).

[26] W.-S. Chung, K.-S Chung, S.-T. Nam and C.-I. Um, Generalized deformed algebra, Phys.Lett.A, 183 (1993), 363-370.

[27] S.M. Dancoff, Non-adiabatic meson theory of nuclear forces, Phys. Rev. 78 (1950), 382-385.

[28] M. Daoud, Photon-added coherent states for exactly solvable Hamiltonians, Phys. Letters A 305 (2002), 135-143.

[29] V. G. Drinfel’d, Hopf Algebras and the quantum Yang-Baxter equation, Dokl. Akad. Nauk SSSR, 283 (1985), 1060-1064.

[30] V.G. Drinfel’d, Quantum Group in Proc. Int. Congr. Math., Berkeley (1986), pp. 798-820.

[31] A. Dobrogowska and A. Odzijewicz, Second order $q$–difference equations solvable by factorization method, J. Comput. Appl. Math. 193 (2006), 319-346.

[32] M. El Baz and Y. Hassouni, Deformed exterior algebra, Quons and their coherent states, Int. Jour. Mod. Phys. 18 (2003), preprint: IC/2002/106.
[33] L. D. Faddeev and L. A. Takhtajan, The Quantum Method of the Inverse Problem and the Heisenberg XYZ model, Russ. Math. Surveys 34 (1979), 11-68.

[34] L. D. Faddeev and N. Yu. Reshetikhin, Hamiltonian structures for integrable models field theory, Teor. Mat. Fiz., 56 (1983), 323-343.

[35] L.D. Faddeev, N.Yu. Reshitikhin and L.A. Takhtajan, Quantization of Lie groups and Lie algebras, Leningrad Math J. 1 (1990), 193-225.

[36] Ph. Feinsilver, Commutators, anti-commutators and Eulerian calculus, Rocky Mountain J. Math., 12 (1982), 171-183.

[37] Ph. Feinsilver, Discrete analogues of the Heisenberg-Weyl algebra, Mn. Math., 104 (1987), 89-108.

[38] R. Floreanini, L. Lapointe and L. Vinet, A note on $(p,q)-$oscillators and bibasic hypergeometric functions, J. Phys. A: Math. Gen. 26 (1993), L611-L614.

[39] R. Floreanini, L. Lapointe and L. Vinet, A quantum algebra approach to basic multivariable special functions, J. Phys. A: Math. Gen. 27 (1994), 6781-6797.

[40] D. Galetti, A realization of the q-deformed harmonic oscillator: Rogers-Szegö and Stieltjes-Wigert polynomials, Braz. J. Phys. 33 (2003), 148-157.

[41] G. Gasper and M. Rahman, Basic Hypergeometric Series (Cambridge, Cambridge University Press, 1990).

[42] G. Gasper and M. Rahman, Basic Hypergeometric Series (Cambridge University Press, Cambridge, 2004).

[43] J.-P. Gazeau, Coherent States in Quantum Physics (WILEY-VCH Verlag GmbH & Co. KGaA, Weinheim, 2009).

[44] J.-P. Gazeau and J.R. Klauder, Generalized Coherent States for Arbitrary Quantum Systems (Academic Press, New-York, 2000).

[45] M.I. Gelfand, M. I. Graev and V. S. Retakh, $(r,s)-$Hypergeometric functions of one variable Russian Acad. Sci. Dokl. Math 48 (1994), 591-596.

[46] R. Gilmore, Geometry of symmetrized states Ann. Phys. (NY) 74 (1972), pp 391-463.
[47] R. Gilmore, On properties of coherent states, Rev. Mex. Fis. 23 (1974), 143-187.

[48] R.J. Glauber, Photon correlations, Phys. Rev. Letters 10 (1963), 84-86.

[49] R.J. Glauber, Coherent and incoherent states of the radiation field, Phys. Rev. 131 (1963), 2766-2788.

[50] A. Horzela and F.H. Szafraniec, A measure-free approach to coherent states, J. Phys. A Math. Theor. 45 (1012), 244018.

[51] M.N. Hounkonnou and J.D. Bukweli Kyemma, Generalized $(\mathcal{R}, p, q)$-deformed Heisenberg algebras: coherent states and special functions, J. Math. Phys. 51 (2010), 063518.

[52] M. N. Hounkonnou and E. B. Ngompe Nkouankam, New $(p, q, \mu, \nu, f)$-deformed states, J. Phys. A: Math. Theor. 40 (2007), 12113-12130.

[53] M. N. Hounkonnou and E. B. Ngompe Nkouankam, On $(p, q, \mu, \nu, \phi_1, \phi_2)$ generalized oscillator algebra and related bibasic hypergeometric functions, J. Phys. A: Math. Theor. 40 (2007), 8835-8843.

[54] M. N. Hounkonnou and K. Sodoga, Generalized coherent states for associated hypergeometric-type functions, J. Phys. A: Math. Gen. 38 (2005), 7851-7857.

[55] E. H. Ismail Moudard, Classical and Quantum Orthogonal Polynomials in one Variable. Enc. Math Appl. Vol. 98 (Cambridge, Cambridge University Press, 2005).

[56] F.H. Jackson, The application of basic numbers to Bessel’s and Legendre’s functions II in Proc. London Math. Soc.(2), 3 (1905), pp 192-220.

[57] R. Jagannathan and K. Srinivasa Rao, Two-parameter quantum algebras, twin-basic numbers, and associated generalized hypergeometric series, [arXiv:math/0602613](http://arxiv.org/abs/math/0602613).

[58] R. Jagannathan and R. Sridhar, $(p, q)$-Rogers-Szegő polynomials and the $(p, q)$-oscillator, [arXiv:1005.4309v1 [math.QA]](http://arxiv.org/abs/1005.4309v1).

[59] A.D. Jannussis, L.C. Papaloucas and P.D. Siafarikas, Eigenfunctions and Eigenvalues of the $Q$-Differential Operators, Hadronic J. 3 (1980), 1622-1632.

[60] A.D. Jannussis, G. Bbedimas, D. Sourlas and V. Zisis, Remarks on the $q$-quantization, Lett. Nuovo Cim. 30 (1981), 123-127.
[61] A. Jannussis, G. Bbodimas, D. Sourlas, L.C. Papaloucas and P.D. Siafaricas, $Q$-algebras without interaction, Lett. Nuovo Cim. 37 (1983), 119-123.

[62] A. Jannussis, G. Bbodimas, D. Sourlas, L. Papaloucas, P.D. Siafaricas and K. Vlachos, Some properties of $Q$-analysis and applications to noncanonical mechanics, Hadronic J. 6 (1983), 1653-1686.

[63] M. Jimbo, A $q$-difference analogue of $U(g)$ and Yang-Baxter equation, Lett. Math. Phys. 10 (1985), 63-69.

[64] M. Jimbo, Quantum $R$-matrix for the generalized Toda system, Comm. Math. Phys. 102 (1986), 537-547.

[65] G. Junker and P. Roy, Conditionally exactly solvable problems and non-linear algebras, Phys. Lett. A 232 (1997), 155-161.

[66] G. Kalnins, W. Miller and S. Mukhejee, Models of $q$-algebra representations: matrix elements of the $q$-oscillator algebra, J. Math. Phys. 34 (1993), 5333-5356.

[67] C. Kassel, Quantum groups (Springer-Verlag, New York, 1995).

[68] A.A. Kirillov, Elements of the theory of representation (Springer, Berlin, Heidelberg, New York, 1976).

[69] R. J. Klauder, Continuous-representation theory I. Postulates of continuous representation theory J. Math. Phys. 4 (1963), 1055-1058.

[70] R. J. Klauder, Continuous-representation theory II. Generalized relation between quantum and classical dynamics J. Math. Phys. 4 (1963), 1058-1073.

[71] J.R. Klauder, K.A. Penson and J.-M. Sixderniers, Constructing coherent states through solutions of Stieljes and Hausdorff moment problems, Phys. Rev. A 64 (2001), 013817.

[72] J.R. Klauder and B.S. Skagerstam, Coherent states: Applications in Physics and Mathematical Physics (Singapore: World Scientific, 1985)

[73] A. Klimyk and K. Schmüdgen, Quantum Groups and their Representation (Berlin Heidelberg, Springer-Verlag, Berlin Heidelberg, 1997).

[74] R. Koekoek and R. F. Swarttouw, The Askey-scheme of orthogonal polynomials and its $q$–analogue, (TUDelft Report No. 98-17, 1998).
[75] B. Kostant, *Group Representation in Mathematics and Physics*, ed. by V. Bargamann, Lecture Notes Phys., Vol 6 (Springer, Berlin, Heidelberg, New York, 1970).

[76] K. Kosiński, M. Majewski and P. Mašlanka, Representations of generalized oscillator algebra, arXiv:q-alg/9501012v1

[77] P.P. Kulish and N.Y. Reshetikhin, Quantum linear problem for the sine-Gordon equation and higher representations, J. Sov. Math. 23 (1983), 2435-2445.

[78] V.V. Kuryshkin, Annales de la Fondation Louis de Broglie 5 (1980), p. 111.

[79] A. J. Macfarlane, On \( q \)-analogues of the quantum harmonic oscillator and quantum group \( SU(2)_q \), J. Phys. A 22 (1989), 4581-4588.

[80] V.I. Man’ko, G. Marmo, E.C.G. Sudarshan and F. Zaccaria, \( f \)-oscillators and nonlinear coherent states, Phys. Scr. 55 (1997), 528-541.

[81] R. L. de Matos Filho and W. Vogel, Nonlinear coherent states, Phys. Rev. A 54 (1996), 4560-4563.

[82] V. Maximov and A. Odzijewicz, The \( q \)–deformation of quantum mechanics of one degree of freedom. J. Math. Phys. 36 (1995), 1681-1689.

[83] S. Meljanac, M. Milekovi, and S. Pallua, Unified view of deformed single-mode oscillator algebras, Phys. Lett. B 328 (1994), 55-59.

[84] M.M. Nieto and L.M. Simmons, Jr., Coherent states for general potentials, Phys. Rev. Lett. 41 (1978), 207-210.

[85] M.M. Nieto and L.M. Simmons, Jr., Coherent states for general potentials: I, II and III, Phys. Rev. D 20 (1979), 1321-1350.

[86] M.M. Nieto, L.M. Simmons, Jr., and V.P. Gutschick, Coherent states for general potentials: IV, Phys. Rev. D 20 (1981), 391-402.

[87] A. Odzijewicz, Quatum algebras and \( q \)-special functions related to coherent states maps of the disc. Commun. Math. Phys. 192 (1998), 183-215.

[88] V. Pasquier and H. Saleur, Common structures between finite systems and conformal field theories through quantum groups, Nucl. Phys. B 330, (1990),523-556.

[89] A. M. Perelomov, Coherent states for arbitrary Lie group, Commun. Math. Phys. 26 (1972), 222-236.
[90] A.M. Perelomov, *Generalized Coherent States and their Applications* (Springer-Verlag, Berlin Heidelberg, 1986).

[91] A.D. Polyanin and A. V. Manzhirov, *Handbook of integral equations* (CRC Press, Boca Raton, 1998).

[92] D. Popov, Gazeau-Klauder quasi-coherent states for the Morse oscillator, Phys. Letters A 316 (2003), 369-381.

[93] A.P. Prudnikov, Yu A. Brychkov and O.I. Marichev, *Integrals and Series*, vol 3 (Gordon and Breach, New York 1990).

[94] C. Quesne, New $q$-deformed coherent states with an explicitly known resolution of unity, J. Phys. A: Math. Gen. 35 (2002), 9213-9226.

[95] C. Quesne, K.A. Penson and V.M. Tkachuk, Maths-type $q$-deformed coherent states for $q > 1$, Phys. Lett. A 313 (2003), 29-36. [arXiv:quant-ph/0303120v2].

[96] V. Sahai, $q$–Representations of Lie algebras $\mathcal{G}(a,b)$, J. Indian Math. Soc. 63 (1997), 83-98.

[97] E. Schrödinger, Der stetige Übergang von der Mikro- zur Makromechanik, Naturwissenschaften, 14 (1926), 664-666.

[98] J.A. Shohat and J.D. Tamarkin, *The Problem of Moments*, (APS, New York, 1943).

[99] E. K. Sklyanin, Some Algebraic Structures Connected with the Yang-Baxter Equation. Representations of Quantum Algebras, Funct. Anal. Appl., 17 (1983), 34-48.

[100] I.N. Sneddon, *The Use of Integral Transforms* (McGraw-Hill, New York, 1974).

[101] H. S. Snyder, Quantized space-time, Phys. Rev. 71 (1947), 38-41.

[102] E.C. G. Sudarshan, Equivalence of semiclassical and quantum mechanical descriptions of statistical light beams, Phys. Rev. Letters 10 (1963), 277-279.

[103] C.-P. Sun, H.-C. Fu, The $q$-deformed boson realisation of the quantum group $SU(n)_q$ and its representations, J. Phys. A: Math. Gen. 22 (1989), L983-L986.
[104] I.E. Tamm, The relativistic interaction of elementary particles, J. Phys. UdSSR 9 (1945), 449-465.

[105] M.A. Vasiliev, High spin algebras and quantization on the sphere and hyperboloid, Int. J. Mod. Phys. A 6 (1991), 1115.

[106] H. J. de Vega, Yang-Baxter Algebras, Integrable Theories and Quantum Groups, Int. J. Mod. Phys. A 4, (1989), 2371.

[107] J. Wess, $q$-deformed phase space and its lattice structure, Int. J. Mod. Phys. A 12 (1997), 4997-5006.

[108] E. Witten, Gauge theories, vertex models, and quantum groups, Nucl. Phys. B 330 (1990), 285-346.

[109] H. Yan, $q$-deformed oscillator algebra as a quantum group , J. Phys. A: Math. Gen. 23 (1990), L1155-L1160.

[110] L. You-quan and S. Zheng-mao, A deformation of quantum mechanics, J. Phys. A: Math. Gen. 25 (1992), 6779-6788.

[111] S. Yu, M. Baker and D. D. Coon, First and Second Factorization in a Dual Multiparticle Theory with Nonlinear Trajectories, Phys. Rev. D 5(1972),3108-3121.

[112] W.M. Zhang, D.H. Feng and R.G. Gilmore, Coherent states: theory and some applications, Rev. Mod. Phys. 62 (1990), 867-927.