An alternative approach to KP hierarchy
in matrix models

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Abstract

We show that there exists an alternative procedure in order to extract differential hierarchies, such as the KdV hierarchy, from one–matrix models, without taking a continuum limit. To prove this we introduce the Toda lattice and reformulate it in operator form. We then consider the reduction to the systems appropriate for one–matrix model.
1 Introduction

The one Hermitean matrix model \[1\] (\[2\] contains a few of the numer-
ous existing reviews) has attracted a lot of attention recently: it is a
new and powerful tool in the study of two dimensional gravity (coupled
to matter) and it has revealed an extremely interesting mathematical
structure. The latter is synthesized by the appearence of a KdV hier-
archy of which the partition function is a \(\tau\)–function, and by the string
equation which can be cast in the form of Virasoro constraints over
the partition function. It has been shown recently that the KdV hi-
erarchy characterizes both topological gravity and intersection theory
of the moduli space of Riemann surfaces \[3\],\[4\],\[5\]. On the other hand
the Virasoro constraints appear naturally in topological field theory
\[6\],\[7\]. One matrix models are at the core of all these developments,
and although they have been the object of intensive research, still they
may hide some surprises.

In this paper we will eventually deal with the KdV hierarchy en-
iuing from one–matrix models. It is well–known (see \[4\],\[8\] ) that one–
matrix models are characterized by a linear system whose integrability
conditions form a discrete hierarchy (i.e. a hierarchy of differential–
difference equations). This turn out to be a reduced case of the so–
called Toda lattice hierarchy. What in the literature is kno-
wn as

The KdV hierarchy associated to a one–matrix model is obtained as
a continuum limit of the above discrete hierarchy: it is formed by a
hierarchy of purely differential equations, among which we find the
celebrated KdV equation.

To avoid misunderstandings let us insist on the difference be-
tween
the two above–mentioned types of hierarchies. They are typically
represented by the two hierarchies (11) and (24) below. They
have the same form, but in the first case \(Q\) is an (infinite) matrix and
the equations can be interpreted as differential–difference equations;
for this reason we call this hierarchy discrete. In the second \(Q_n\) is
a pseudo–differential operator and the equations involved are purely
differential; for this reason we call this hierarchy differential. Of the
latter type is the KdV hierarchy met in the literature.

In this letter we want to point out that taking a continuum limit is
not necessary in order to get a differential hierarchy. There exists an
alternative procedure by which we can extract a differential hierarchy
from the discrete linear system associated to one–matrix models with-
out reference to any limiting procedure. Said differently, the discrete linear system contains already a differential hierarchy, which can be reduced to the KdV hierarchy, without us being obliged to resort to a limiting procedure – which presumably causes some loss of information. In this hierarchy the first flow parameter $t_1$ plays the role of space coordinate.

The general setting proposed here seems to be closer than the usual one, based on the continuum limit, to the spirit of Kontsevich model.

The paper is organized as follows. We first introduce the Toda lattice in general (section 2), which involves $\infty \times \infty$ matrices, and we show in this general case how to pass from the matrix formulation to the (differential) operator formulation (section 3). Then we consider reductions of the above system to semi–infinite matrices (section 4). Finally (section 5) we consider the reduction to the linear system appropriate for one–matrix models (the Toda chain), and, in particular, by a further reduction we recover the KdV hierarchy.

2 The Toda lattice hierarchy

In this section we introduce the so-called Toda lattice hierarchy. The main reference in this context is the paper of Ueno and Takasaki [9]. However we will present it in a form which is more suitable for our purposes, i.e. mainly by means of the associated linear system.

Let us first introduce some notations. Given a matrix $M$, we will denote by $M^-$ the strictly lower triangular part and by $M^+$ the upper triangular part including the main diagonal. Unless otherwise specified, we will be dealing with $\infty \times \infty$ matrices. As usual $E_{ij}$ will denote the matrix $(E_{ij})_{kl} = \delta_{ik} \delta_{jl}$. We will also use

$$I_{\pm} \equiv \sum_{i=-\infty}^{\infty} E_{i,i\pm 1}, \quad \rho = \sum_{i=-\infty}^{\infty} iE_{ii}$$

Throughout this paper $\lambda$ denotes the spectral parameter, and $\Lambda$ represents an infinite dimensional column vector whose components $\Lambda_n$, $n \in \mathbb{Z}$ are given by

$$\Lambda_n = \lambda^n$$

The vector $\Lambda$ is our elementary starting point. From it, by means of matrix transformations, we obtain other vectors. A useful one is
\[ \eta = \exp\left(\sum_{r=1}^{\infty} t_r \lambda^r\right) \Lambda \] (1)

whose components are

\[ \eta_n = \exp\left(\sum_{r=1}^{\infty} t_r \lambda^r\right) \lambda^n \]

In (1) \( t_r \) are time or flow parameters. On \( \eta \) one can naturally define a (elementary) linear system

\[
\begin{align*}
\lambda \eta &= \partial_t \eta = I_+ \eta \\
\lambda^r \eta &= \partial_{t_r} \eta = \partial^r \eta = I^r_+ \eta \\
\lambda^{-r} \eta &= \partial^{-r} \eta = I^{-r}_- \eta \\
\lambda^{-r} \eta &= \partial^{-r} \eta = I^{-r}_- \eta \\
\frac{\partial}{\partial \lambda} \eta &= P_0 \eta, \quad P_0 = \rho I_- + \sum_{r=1}^{\infty} r t_r I_{-}^{r-1} \\
\end{align*}
\]

(2)

Thoughout the paper \( \partial \) denotes the derivative \( \frac{\partial}{\partial t} \). Moreover \( \partial^{-1} \) denotes formal integration. Since for \( (\infty \times \infty) \) matrices,

\[ [I_+, \rho I_-] = 1 \]

we see that the spectral and flow equations are automatically compatible.

A crucial ingredient in the following construction is the (invertible) “wave matrix” \( W \):

\[ W = 1 + \sum_{i=1}^{\infty} w_i I^i_- = 1 + \sum_{i=1}^{\infty} \sum_{n=-\infty}^{+\infty} w_i(n) E_{n,n-i} \] (3)

So \( w_i = \{w_i(n) | n \in \mathbb{Z}\} \) are infinite diagonal matrices, and \( w_i(n) \) are functions of the time parameters. We impose them to be determined by the equations of motion

\[ \frac{\partial}{\partial t_r} W = Q^r_+ W - W T^r_+ \] (4)

where \( Q \) is the infinite matrix

\[ Q = W I_+ W^{-1} \] (5)
Another important object is the vector \( \Psi \)

\[
\Psi = W \eta
\]  

(6)

In terms of all these objects the dynamical system we have defined can be written as

\[
Q \Psi = \lambda \Psi
\]  

(7)

\[
\frac{\partial}{\partial t_r} \Psi = Q^+_r \Psi
\]  

(8)

\[
\frac{\partial}{\partial \lambda} \Psi = P \Psi
\]  

(9)

where

\[
P = WP_0W^{-1}
\]  

(10)

The compatibility conditions of this linear system form the so-called discrete KP–hierarchy

\[
\frac{\partial}{\partial t_r} Q = [Q^+_r, Q]
\]  

(11)

together with the trivial relation

\[
[Q, P] = 1
\]  

(12)

We should perhaps recall that what we have done so far is at a purely formal level and does not bear yet any relation to matrix models. In particular we insist that eq.(12) does not imply any constraint on the dynamical system.

To end this section let us make the above formulas more explicit and extract a few relations that will be useful in the following. From the equation of motion (4) we consider in particular

\[
\frac{\partial}{\partial t_1} w_i(n) = w_{i+1}(n + 1) - w_{i+1}(n) - w_i(n)(w_1(n + 1) - w_1(n))
\]  

(13)

Let us introduce a piece of terminology by saying that for any matrix its elements in the \( n - th \) row belong to the \( n - th \) sector. Therefore, in regard to eq.(13) we can say that the flow in the \( n - th \) sector depends only on the coordinates of the \( n - th \) and \( n + 1 - th \) sectors.
From (5) we see that

\[ Q = I_+ + \sum_{i=0}^{\infty} a_i I^i_+ \]  

(14)

The \( a_i \)'s are new coordinates of the system, which can be uniquely expressed in terms of the \( w_i \)'s. For example

\[ a_0(n) = w_1(n) - w_1(n + 1) \]
\[ a_1(n) = w_1(n)(w_1(n + 1) - w_1(n)) + w_2(n) - w_2(n + 1) \]
\[ a_2(n) = w_2(n)(w_1(n + 1) - w_1(n)) + w_1(n - 1)(w_2(n + 1) - w_2(n)) + w_1(n)w_1(n - 1)(w_1(n + 1) - w_1(n)) + w_3(n) - w_3(n + 1) \]

Another useful representation is obtained by inverting eq. (5)

\[ I_+ = W^{-1} Q W = Q + \sum_{i=0}^{\infty} q_i Q^{-i} \]

the \( q_i \)'s are another set of diagonal matrices, which can be expressed in terms of \( a_i \)'s or \( w_i \)'s. It is worth noting that

\[ I_- = (I_+)^{-1} = Q^{-1} + \ldots \]

which results in the following equality

\[ Q^r_+ \equiv Q^r - Q^r_- = Q^r + \sum_{i=1}^{\infty} q_{r,i} Q^{-i} \quad \forall r \geq 1 \]  

(15)

Finally, from (10), we have

\[ P = \sum_{r=1}^{\infty} r t_r Q^{r-1} + \sum_{i=0}^{\infty} v_i Q^{-i-1} \]  

(16)

Once again, \( v_i \)'s are diagonal matrices, and \( v_0 = \rho \).

We will see later on that the string equation of matrix models can be obtained by imposing a constraint on the coordinates \( v_i \).

---

1 These equations show that the two sets of the variables \( a_i \)'s and \( w_i \)'s can be obtained from each other. However, strictly speaking, this one-to-one correspondence is only due to the fact that we have chosen a special form of \( W \)-matrix (4). Generally, for a given matrix \( Q, W \) is not uniquely determined.

2 Since \( W \rho L_+ W^{-1} = \rho Q^{-1} + [W, \rho] L_+ W^{-1} \), the commutator is a strictly lower triangular matrix, so the second part at most contributes to the term \( Q^{-2} \), which ensures that \( v_0 = \rho \).
3 From the discrete hierarchy to the differential hierarchy

In the previous section we introduced the usual Toda lattice. The discrete KP hierarchy we obtained is known as the Toda lattice hierarchy. It consists of an infinite set of differential-difference equations. In this section we show that passing from the matrix formalism of the previous section to a related (pseudo–differential) operator formalism, we can obtain a new hierarchy which consists merely of differential equations.

The operator formalism alluded to before is introduced as follows. We recall that equation (1) implies

$$\eta_n = \partial^{n-m} \eta_m, \quad \forall n, m : \text{integers}$$

This leads to

$$\Psi_n = (W \eta)_n = \sum_{i=-\infty}^{n} W_{ni} \eta_i = \sum_{i=-\infty}^{n} W_{ni} \partial^{i-n} \eta_n = \bar{W}_n \eta_n \quad (17)$$

where we have defined

$$\bar{W}_n = 1 + \sum_{i<n} W_{ni} \partial^{i-n} = 1 + \sum_{i=1}^{\infty} w_i(n) \partial^{-i} \quad (18)$$

This tells us that the “wave” matrix $W$ can be considered as an infinite column vector, whose components are differential operators. The operator $\bar{W}_n$ can be inverted

$$\eta_n = \bar{W}_n^{-1} \Psi_n$$

In this formalism the spectral equation (7) becomes

$$\lambda \Psi_n = \lambda \bar{W}_n \eta_n = \bar{W}_n \partial \eta_n = \bar{W}_n \partial \bar{W}_n^{-1} \Psi_n = \bar{Q}_n \Psi_n$$

Here we have introduced an infinite set of KP–type differential operators

$$\bar{Q}_n = \bar{W}_n \partial \bar{W}_n^{-1} = \partial + \sum_{i=1}^{\infty} u_i(n) \partial^{-i} \quad \forall n \text{ integer} \quad (19)$$

The variables $u_i$’s can also be understood as a set of coordinates of the system. Inverting this relation we obtain

$$
\partial = \hat{W}_n^{-1} \hat{Q}_n \hat{W}_n = \hat{Q}_n + \sum_{i=0}^{\infty} q_i(n) \hat{Q}_n^{-i} \quad \forall n \text{ integer} \quad (20)
$$

It is easy to see that this mapping from matrices to operators maps the upper triangular part of a given matrix into the differential part of the operator, and the lower triangular part of the matrix to the formal integration part of the operator. In particular we have

$$(Q^r_{+} \Psi)_n \longrightarrow (\hat{Q}_n)^r_{+} \Psi_n$$

For an operator, the subscript “+” selects, as usual, the non–negative powers of the derivative $\partial$. Going on with the transcription of the Toda lattice linear system in the operator formalism, we can now rewrite the flow equations [5]

$$\frac{\partial}{\partial t_r} \Psi_n = \left(\frac{\partial}{\partial t_r} \hat{W}_n\right) \eta_n + \hat{W}_n \frac{\partial}{\partial t_r} \eta_n$$

$$= \left(\hat{W}_n^{-1} \hat{Q}_n \hat{W}_n\right) \psi_n$$

$$= (\hat{Q}_n^r)_{+} \psi_n \quad \forall n \text{ integer} \quad (22)$$

Finally we can rewrite the linear system as follows

$$\hat{Q}_n \Psi_n = \lambda \Psi_n \quad (21)$$

$$\frac{\partial}{\partial t_r} \Psi_n = (\hat{Q}_n^r)_{+} \Psi_n, \quad \forall n \text{ integer} \quad (22)$$

$$\frac{\partial}{\partial \lambda} \Psi_n = (\sum_{r=1}^{\infty} r t_r \hat{Q}_n^{r-1} + \sum_{i=0}^{\infty} v_i(n) \hat{Q}_n^{-i-1}) \Psi_n \quad (23)$$

Their compatibility conditions are

$$\frac{\partial}{\partial t_r} \hat{Q}_n = [(\hat{Q}_n)^r_{+}, \hat{Q}_n] \quad (24)$$

or in other coordinates

$$\frac{\partial}{\partial t_r} \hat{W}_n = (\hat{Q}_n)^r_{+} \hat{W}_n - \hat{W}_n \partial^r \quad (25)$$
The last two equations, like the previous ones, hold for any integer \( n \).
Eq.\((24)\) or \((25)\) specifies the \textit{differential} hierarchy we promised in the introduction. It consists of an infinite set of differential equations: in the LHS we have the first order derivatives with respect to the flow parameters, in the RHS we have polynomials in the coordinates and derivatives of coordinates with respect to \( t_1 \).

It is a bit inappropriate to speak of one hierarchy: we have in fact an infinite number of hierarchies, one for each integer \( n \). However these hierarchies are not independent as they are related by the \( t_1 \) flow; we will see later on that in the particular case of one-matrix model all these hierarchies are isomorphic.

Let us remark that each equation in \((25)\) involves only coordinates of the same sector. This is not true for \((24)\); however by using the \( t_1 \) flow one can express each equation only in terms of coordinates of a single sector.

Finally let us comment on the relation between the discrete Toda hierarchy and the hierarchies \((24)\). We have seen that from the discrete lattice (with its hierarchy) through the operator formalism we can unambiguously arrive at a new differential hierarchy. From a practical point of view this corresponds to expressing the difference operations on the lattice in terms of \( t_1 \) derivatives of the coordinates in one sector. Conversely from the latter hierarchy we can reconstruct the discrete one. Let us sketch the argument. One starts from the linear system specified by

\[
\begin{align*}
L(x, t) &= \partial + u_1(x, t)\partial^{-1} + u_2(x, t)\partial^{-2} + \ldots, \quad \partial = \frac{d}{dx} \\
L(x, t)\Psi(x, t, \lambda) &= \lambda\Psi(x, t, \lambda) \\
\frac{\partial}{\partial t_r}\Psi(x, t, \lambda) &= L_+^r\Psi(x, t, \lambda)
\end{align*}
\]

where \( t \) represent the collection of flow parameters \( t_1, t_2, \ldots \), and the associated KP hierarchy. From the last equation we see that it is consistent to identify \( x \) with \( t_1 \). Next we can define the operator

\[
W = 1 + w_1\partial^{-1} + w_2\partial^{-2} + \ldots
\]

such that

\[
L = W\partial W^{-1}
\]
Now we call the $w_i$ in eq.(26) as $w_i(n)$, and we use the $t_1$ flow to define the $w_i(n+1)$ as in eq.(13). In this way we can reconstruct the discrete linear system and the discrete KP hierarchy.

4 Reduction: Semi–infinite matrices

Let us study now the problem of reducing the general system defined in the two previous sections to a simpler one. In the next section we will consider a further reduction, i.e. to the Toda chain which is relevant for one-matrix models. We notice, first of all, that for the linear systems involved in matrix models $\Psi$ does not contain negative powers of $\lambda$. Therefore the Jacobi matrix $Q$ must be semi–infinite,

$$w_i(n) = 0, \quad n < i$$

This is the reduction we will study in this section.

In this reduced system for any positive integer $n$ we have an invertible operator with a finite number of terms

$$\hat{W}_n = 1 + \sum_{i=1}^{n} w_i(n) \partial^{-i}$$

The KP–type operator is

$$\hat{Q}_n = \hat{W}_n \partial \hat{W}_n^{-1}$$

which still contains infinite many terms.

We remark here that we are still formally using $\infty \times \infty$ matrices in order to be able to fully exploit the formalism introduced in the previous sections. However three quadrants of these matrices become irrelevant.

So far we have been using mostly $w_i$ coordinates, but henceforth it will be more convenient to shift to $a_i$ coordinates. We recall that they are defined in the following way through the spectral equation

$$\lambda \Psi_n = \Psi_{n+1} + \sum_{i=0}^{n} a_i(n) \Psi_{n-i} \quad (27)$$

Next we want to express the RHS of this equation in terms of $\Psi_n$ only (this is an example of the procedure outlined in the last section). To this end we use the first flow equation

$$\partial \Psi_n = (Q + \Psi)_n = \Psi_{n+1} + a_0(n) \Psi_n$$
or equivalently
\[ \Psi_{n+1} = (\partial - a_0(n))\Psi_n \] (28)

Inverting this relation, we get
\[ \Psi_n = \hat{B}_n \Psi_{n+1}, \quad \hat{B}_n = \partial^{-1} \sum_{l=0}^{\infty} (a_0(n)\partial^{-1})^l \] (29)

Using this relation repeatedly we can express any \( \Psi_i (i < n) \) in terms of \( \Psi_n \), i.e.
\[ \Psi_{n-r} = \hat{B}_{n-r} \hat{B}_{n-r+1} \ldots \hat{B}_{n-1} \Psi_n \]

Therefore the spectral equation (27) can be rewritten as
\[ \hat{Q}_n \Psi_n = \left( \partial + \sum_{i=1}^{n} a_i(n) \hat{B}_{n-i} \hat{B}_{n-i+1} \ldots \hat{B}_{n-1} \right) \Psi_n \] (30)

and the \( n \)-th KP-type operator becomes
\[ \hat{Q}_n = \partial + \sum_{i=1}^{n} a_i(n) \hat{B}_{n-i} \hat{B}_{n-i+1} \ldots \hat{B}_{n-1} \] (31)

From the first KP-equation in matrix form
\[ \frac{\partial}{\partial t_1} Q = [Q_+, Q] \]
written in terms of coordinates
\[ \frac{\partial}{\partial t_1} a_i(n) = a_i(n)' = a_{i+1}(n+1) - a_{i+1}(n) + a_i(n)(a_0(n) - a_0(n-i)) \]

which just makes the connection between the coordinates \( a_i(n) (i \leq n) \)'s in the \( n \)-th sector and the coordinates \( a_i(j) \)'s in the \( j \)-th sectors, \( j \leq n+1 \). We conjecture that, using these relations, one should be able to write
\[ \hat{Q}_n = \partial + \sum_{l=1}^{n} u_l(n)\partial^{-l} \]
where the functions \( u_l(n) \)'s only depend on the coordinates in the \( n \)-th sector. We will explicitly show this property below in the case of
one-matrix model. The equations of motion (the remaining $t_r(r \geq 2)$–flows), as expected, take the form of the differential KP–hierarchy

$$\frac{\partial}{\partial t_k} \hat{Q}_n = [(\hat{Q}_n)^k_+, \hat{Q}_n]$$

Let us now discuss the integrability of the reduced system. We construct a bi–Hamiltonian structure as follows. Define an inner product in the differential operator space as usual

$$Tr(A) = \int dx A_{-1}(x), \quad A = \ldots + A_{-1}(x)\partial^{-1} + \ldots$$

Using this trace operation, we can write a functional of the $a_j(x)$’s in the following way

$$f_X(\hat{Q}_j) = Tr(\hat{Q}_j X) \equiv \hat{Q}_j(X)$$

where $X$ is a pure differential operator. Then the coadjoint analysis shows us that there exist two compatible Poisson brackets

$$\{f_X, f_Y\}_1(\hat{Q}_n) = \hat{Q}_n([Y, X])$$

$$\{f_X, f_Y\}_2(\hat{Q}_n) = <(X \hat{Q}_n)_+ Y \hat{Q}_n > - <(\hat{Q}_n X)_+ \hat{Q}_n Y >$$

The relevant conserved quantities (in involution) are

$$H_k = \frac{1}{k} Tr(\hat{Q}_n^k), \quad \forall k \geq 1$$

and the compatibility reads

$$\{H_{r+1}, f\}_1 = \{H_r, f\}_2 \quad \text{for any function } f \quad (35)$$

5 Toda chain and one-matrix models

The case relevant to one–matrix models is specified by the conditions

$$a_0(j) = S_j, \quad a_1(j) = R_j, \quad a_i(j) = 0, \quad \forall i \geq 2$$

The first (i.e. $t_1$) flow equation is

$$\frac{\partial}{\partial t_1} S_j = S'_j = R_{j+1} - R_j \quad (36)$$

$$\frac{\partial}{\partial t_1} R_j = R'_j = R_j(S_j - S_{j-1}) \quad (37)$$
From the second equality we have

\[ S_{j-1} = S_j - \frac{R'_j}{R_j} \]

Therefore the j–th KP–type operator is

\[ \hat{Q}_j = \partial + R_j \hat{B}_{j-1} \equiv \partial + \sum_{l=1} u_l(j) \partial^{-l} \quad (38) \]

The first few \( u_l(j) \)'s are

\[
\begin{align*}
   u_1(j) &= R_j, \\
   u_2(j) &= -R'_j + R_j S_j \\
   u_3(j) &= R''_j - 2R'_j S_j - R_j S'_j + R_j S_j^2 \\
   u_4(j) &= -R'''_j + 3R''_j S_j + 3R'_j S'_j - 3R_j S'_j S_j + R_j S_j^2 \\
   u_5(j) &= R''''_j - 4R'''_j S_j - 4R''_j S'_j - 6R'_j S_j^2 + 4R_j S''_j S_j + R_j S'_j^2 \\
   &\quad + 12R'_j S'_j S''_j + 4R_j S_j S''_j + 3R_j S_j^2 \\
   &\quad - 6R_j S_j^2 S'_j - R_j S''_j^2 + R_j S_j^4
\end{align*}
\]

As anticipated in the previous section the \( u_l \)'s depend only on the coordinates of one sector. The consequence of this is that the infinite many KP–hierarchies we discussed about at the end of section 3 are all isomorphic.

Since the following analysis is universal, i.e. is the same for all sectors, we simply omit the subscript “j”, and denote \( R_j(t_1), S_j(t_1) \) by \( R(t_1) \) and \( S(t_1) \), respectively.

From the above general discussion, we can derive two compatible Poisson brackets, which are

\[
\begin{align*}
   \{ R(x), R(y) \}_1 &= 0 \\
   \{ S(x), S(y) \}_1 &= 0 \\
   \{ R(x), S(y) \}_1 &= -\partial_x \delta(x - y)
\end{align*}
\]

and

\[
\begin{align*}
   \{ R(x), R(y) \}_2 &= -(2R(x) \partial_x + R'(x)) \delta(x - y) \\
   \{ S(x), S(y) \}_2 &= -2\partial_x \delta(x - y) \\
   \{ R(x), S(y) \}_2 &= (\partial_x^2 - S(x) \partial_x) \delta(x - y)
\end{align*}
\]
We are now going to write down the flow equations of this system. To do this one can directly use equation (32). A more concise way is to introduce the two polynomials
\[ F_r(x) = \frac{\delta H_r}{\delta S(x)} \quad G_r(x) = \frac{\delta H_r}{\delta R(x)} \quad \forall r \geq 1 \]
In particular
\[ F_1 = 0, \quad G_1 = 1 \]
The compatibility condition of two Poisson brackets shows the existence of the following recursion relations among these polynomials
\[ F'_{r+1} = SF'_r - F''_r + 2RG'_r + R'G_r \quad (39) \]
\[ G'_{r+1} = 2F'_r + G''_r + (SG'_r)' \quad (40) \]
In terms of these polynomials, the general equations of motion can be written as
\[ \frac{\partial}{\partial t_r} S = G'_{r+1} \]
\[ \frac{\partial}{\partial t_r} R = F'_{r+1} \]
For example, the first two flows have the following explicit forms
\[ \frac{\partial}{\partial t_2} S = S'' + 2SS' + 2R' \]
\[ \frac{\partial}{\partial t_2} R = -R'' + 2(RS)' \quad (41) \]
and
\[ \frac{\partial}{\partial t_3} S = S''' + 3SS'' + 3S'^2 + 6(RS)' + 3S^2S' \]
\[ \frac{\partial}{\partial t_3} R = R''' - 3RR'' + 6RR' + 3(RS^2)' - 3R'S' \quad (42) \]
This is the hierarchy characterizing the Toda chain and one-matrix models.

Hereafter we show that by further restricting the system we can obtain the KdV hierarchy. This is achieved in the following way. Let us set
\[ S = 0 \]
and let us discard the $t_{2r}$ flows. Then from eqs. ([39],[40]) we get
\begin{align*}
G_{2r} & = 0 \\
F'_{2r+2} & = (\partial^3 + 4R\partial + 2R')F_{2r}
\end{align*}
utilizing the initial condition $F_2 = R$. This is the recursion relation for KdV hierarchy. In particular, as can be seen also from ([12]), we have
\begin{equation}
\frac{\partial}{\partial t_3} R = R''' + 6RR'
\end{equation}
The fact that in one–matrix models the conditions $S_j = 0$ and $t_{2r+1} = 0$ are related does not prevent us from drawing the above conclusions. Here we have been considering the reduction of the system ensuing from a one–matrix model under the condition $S = 0$, and this is of course legitimate. Considered in the context of one–matrix model it seems to correspond to a rather singular situation. This is perhaps not entirely surprising since the KdV hierarchy corresponds to a topological point. On the other hand also in the continuum limit approach the reduction to the even potentials presents not fully understood aspects (see, for example ([8])). We think an understanding of this problem will be facilitated by the analysis of the full hierarchy ([11],[12]). We will return to this point elsewhere.

What we said so far in this section is valid for one–matrix models but does not characterize them completely. As is well known we have to impose the string equation. As we already noticed, the equation
\begin{equation}
[Q, P] = 1
\end{equation}
does not imply any restriction on the model as long as $P$ has the general form ([10]). One-matrix models are characterized by the following restriction on the form of $P$
\begin{equation}
P_\perp = 0, \quad \text{i.e.} \quad P = \sum_{r=1}^\infty r_{t_r}Q_{r+1}^{r-1}
\end{equation}
By string equation we mean ([14]) together with the condition ([15]). Starting from it we can set out for the analysis of the critical points.

To end this section let us discuss the Virasoro constraints for the system corresponding to one–matrix models. Since they were already
derived along similar lines in a previous paper [8], we will simply sketch the derivation here. From the string equation we can derive

\[ d_{-1} R = 0 \]
\[ d_{-1} S + 1 = 0 \]

(46)

where

\[ d_{-1} = \sum_{r=2}^{\infty} rt_r \frac{\partial}{\partial t_{r-1}} \]

Eqs. (46) can be written in the form

\[ \sum_{r=2}^{\infty} rt_r F'_r = 0 \]
\[ \sum_{r=2}^{\infty} rt_r G'_r + 1 = 0 \]

Using the recursion relations (39, 40) we can obtain other similar equations, for example

\[ \sum_{r=1}^{\infty} rt_r F'_r + 2R = 0 \]
\[ \sum_{r=1}^{\infty} rt_r G'_r + S = 0 \]

etc.

Now using the definition

\[ \frac{\partial}{\partial t_r} \ln Z_N(t) = \sum_{i=0}^{N-1} Q^r_{ii} \]

we can recast the above equations in the familiar form of the Virasoro constraints

\[ \left( \sum_{r=2}^{\infty} rt_r \frac{\partial}{\partial t_{r-1}} + N t_1 \right) Z_N = 0 \]
\[ \left( \sum_{r=1}^{\infty} rt_r \frac{\partial}{\partial t_r} + N^2 \right) Z_N = 0 \]
6 Discussion

Starting from the Toda lattice in the traditional form, we have extracted a continuous linear system (without taking a continuum limit). By reducing it we have been able to show that the discrete system corresponding to one–matrix models gives rise naturally to a differential hierarchy and in particular to the KdV hierarchy.

A few differences with the more common limiting treatment in one-matrix models should be stressed:

– the space parameter in our approach is $t_1$ (not $t_0$, which does not show up here);
– the Virasoro constraints we found in the previous section are the same as the Virasoro constraints of the discrete system \[ \text{[8]}; and therefore differ from the Virasoro constraints one finds after taking the continuum limit;
– the complete differential hierarchy corresponding to one–matrix models does not seem to coincide exactly with any of the continuous hierarchies proposed so far \[ \text{[10]}]; but further research is needed on this point.

Acknowledgements One of us (L.B.) would like to thank the Instituto de Fisica Teorica – UNESP for the kind hospitality extended to him during the completion of this paper.

References

[1] E. Brezin and V. Kazakov, Phys.Lett.236B (90) 144;
M. Douglas and S. Shenker, Nucl.Phys.B335 (90) 635;
D. Gross and A. Migdal, Phys.Rev.Lett.64 (90) 127;
T. Banks, M. Douglas, N. Seiberg and S. Shenker, Phys.Lett.B238 (90) 279;
M. Douglas, Phys.Lett. 238B (90) 176.
[2] V.A.Kazakov, in Proc. Cargése workshop in 2d gravity, O.Alvarez et al. eds. O.Alvarez, E.Marinari and P.Windey;
L.Alvarez-Gaumé, Helv.Phys.Acta 64 (1991) 361;
P.Ginsparg, lectures at 1991 Trieste Summer School, Los Alamos preprint
[3] E.Witten, Nucl.Phys. B340 (1990) 281.
[4] E. Witten, Surveys in Diff. Geom. 1 (1991) 243.

[5] M. Kontsevich *Intersection theory on the moduli space of curves and the matrix Airy function*, Max–Planck-Institut preprint MPI/91-47.

[6] R. Dijkgraaf, H. Verlinde, and E. Verlinde Nucl. Phys. B348 (1991) 435.

[7] M. Fukuma, H. Kawai and R. Nakayama, Int. J. Mod. Phys. A6 (1991) 1385.

[8] L. Bonora, M. Martellini and C. S. Xiong, *Integrable Discrete Linear Systems and One-Matrix Random Models*, SISSA 107/91/EP, to appear in Nucl. Phys. B.

[9] K. Ueno and K. Takasaki, Adv. Studies in Pure Math. 4 (1984) 1.

[10] see, for example, T. Hollowood, L. Miramontes, A. Pasquinnucci, C. Nappi *Hermitean vs. Anti-Hermitean 1-matrix Models and their hierarchies* IASSNS–HEP–91/59, PUPT–1280, and references therein.