The Sherrington-Kirkpatrick spin glass model in the presence of a random field with a joint Gaussian probability density function for the exchange interactions and random fields

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Abstract

The magnetic systems with disorder form an important class of systems, which are under intensive studies, since they reflect real systems. Such a class of systems is the spin glass one, which combines randomness and frustration. The Sherrington-Kirkpatrick Ising spin glass with random couplings in the presence of a random magnetic field is investigated in detail within the framework of the replica method. The two random variables (exchange integral interaction and random magnetic field) are drawn from a joint Gaussian probability density function characterized by a correlation coefficient $\rho$. The thermodynamic properties and phase diagrams are studied with respect to the natural parameters of both random components of the system contained in the probability density. The de Almeida-Thouless line is explored as a function of temperature, $\rho$ and other system parameters. The entropy for zero temperature as well as for non zero temperatures is partly negative or positive, acquiring positive branches as $h_0$ increases.

Key words: Ising model, spin glass, frustration, replica method, random field, Gaussian probability density

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1 Introduction

The critical properties of magnetic systems with quenched disorder has been a topic of growing interest in statistical physics over the last years and still attracts considerable attention in spite of their long history, since it has been established that the introduction of randomness can cause important effects on their thermodynamic behavior in comparison to the pure ones; the type of the phase transition as well as the universality class may change [1,2,3]. In two-dimensions an infinitesimal amount of disorder converts a second-order phase transition (SOPT) into a first-order phase transition (FOPT), whereas in three-dimensions it is converted to an FOPT only when disorder exceeds a threshold. Randomness is encountered in the form of vacancies, variable or diluted bonds, impurities [4,5], random fields [6,7,8,9,10,11,12,13] and spin glasses [13,14,15,16,17,18,19]. In addition, the study of disordered systems is a necessity because homogeneous systems are, in general, an idealization, whereas real materials contain impurities, nonmagnetic atoms or vacancies randomly distributed within the system consisting of magnetic atoms, or variable bonds in magnitude and/or in sign; such systems have attracted wide interest and have been studied intensively theoretically, numerically and experimentally, since it is a matter of great urgency to develop an understanding of the role of these non ideal effects. However, to facilitate their study, these are modeled by relying on pure-system models modified accordingly, e.g. Ising, Potts, Baxter-Wu, etc.

An important manifestation of disorder is the presence of random magnetic fields acting on each spin of the magnetic system under consideration in an otherwise free of defects lattice, whose pure version is modeled according to a current one, e.g. the Ising model; the system now in the presence of such fields is called random field Ising model (RFIM) [6,7,8,9,10,11,12,13]. Associated with this model are the notions of lower critical dimension, tricritical points, scaling laws, crossover phenomena, higher order critical points and random field probability distribution function (PDF). RFIM had been the standard vehicle for studying the effects of quenched randomness on phase diagrams and critical properties of lattice spin systems and had been studied for many years since the seminal work of Imry and Ma [11]. The RFIM Hamiltonian, in the case of constant exchange integral, is

\[ H = -J \sum_{<i,j>} S_i S_j - \sum_i h_i S_i, \quad S_i = \pm 1 \]  

This Hamiltonian describes the competition between the long-range order (expressed by the first summation) and the random ordering fields. We also consider that \( J > 0 \) so that the ground state is ferromagnetic in absence of random fields. The RFIM model, despite its simple definition and apparent simplicity,
together with the richness of the physical properties emerging from its study, has motivated a significant number of investigations; however, these properties have proved to be a source of much controversy, primarily due to the lack of a reliable theoretical foundation. Still, substantial efforts to elucidate the basic problems of the RFIM continue to attract considerable attention, because of its direct relevance to a number of significant physical problems. The presence of random fields requires two averaging procedures, the usual thermal average, denoted by angular brackets $\langle \ldots \rangle$, and disorder average over the random fields denoted by $\langle \ldots \rangle_r$ for the respective PDF, which is usually a version of the bimodal, trimodal or Gaussian distributions. The most frequently used PDF for random fields is either the bimodal or the single Gaussian; the former is,

$$P(h_i) = p\delta(h_i - h_0) + q\delta(h_i + h_0)$$  \hspace{1cm} (2)$$

where $p$ is the fraction of lattice sites having a magnetic field $h_0$, while the rest sites have a field $(-h_0)$ with site probability $q$ such that $p + q = 1$; the usual choice was $p = q = \frac{1}{2}$, symmetric case. The latter PDF is,

$$P(h_i) = \frac{1}{(2\pi\sigma^2)^{1/2}} exp\left[-\frac{h_i^2}{2\sigma^2}\right]$$  \hspace{1cm} (3)$$

with zero mean and standard deviation $\sigma$.

One of the main issues was the experimental realization of random fields. Fishman and Aharony [20] showed that the randomly quenched exchange interactions Ising antiferromagnet in a uniform field $H$ is equivalent to a ferromagnet in a random field with the strength of the random field linearly proportional to the induced magnetization. This identification gave new impetus to the study of the RFIM, the investigation gained further interest and was intensified resulting in a large number of publications (theoretical, numerical, Monte Carlo simulations and experimental) in the last thirty years. Although much effort had been invested towards this direction, the only well-established conclusion drawn was the existence of a phase transition for $d \geq 3$ ($d$ space dimension), that is, the critical lower dimension $d_l$ is 2 after a long controversial discussion [11,21], while many other issues are still unanswered; among them is the order of the phase transition (first or second order), the universality class and the dependence of these points on the form of the random field PDF. Galam, via MFA, has shown that the Ising antiferromagnets in a uniform field with either a general random site exchange or site dilution have the same multicritical space as the random-field Ising model with bimodal PDF [22]. The study of RFIM has also highlighted another feature of the model, that of tricriticality and its dependence on the assumed distribution function of the random fields. A very controversial issue has arisen concerning the effect of the random-field probability distribution function on the equilibrium phase
diagram of the RFIM. The choice of the random field distribution can lead to a continuous ferromagnetic/paramagnetic (FM/PM) boundary as in the single Gaussian PDF, whereas for the bimodal one this boundary is divided into two parts, an SOPT branch for high temperatures and an FOPT branch for low temperatures separated by a tricritical point (TCP) at $kT_c^t/(zJ) = 2/3$ and $h_c^t/(zJ) = (kT_c^t/(zJ)) \times \arg \tanh(1/\sqrt{3}) \simeq 0.439$ [12], where $z$ is the coordination number and $k$ the Boltzmann constant, such that for $T < T_c^f$ and $h > h_c^t$ the transition to the FM phase is of first order for $p = \frac{1}{2}$. However, this behavior is not fully elucidated since in the case of the three-dimensional RFIM, the high temperature series expansions by Gofman et al [23] yielded only continuous transitions for both probability distributions, whereas according to Houghton et al [24] both distributions predicted the existence of a TCP, with $h_c^t = 0.28 \pm 0.01$ and $T_c^t = 0.49 \pm 0.03$ for the bimodal and $\sigma_t^t = 0.36 \pm 0.01$ and $T_c^t = 0.36 \pm 0.04$ for the single Gaussian. In the Monte Carlo studies for $d = 3$, Machta et al [25], using single Gaussian distribution, could not reach a definite conclusion concerning the nature of the transition, since for some realizations of randomness the magnetization histogram was two-peaked (implying an SOPT) whereas for other ones three-peaked, implying an FOPT; Middleton and Fisher [26], using a similar distribution for $T = 0$, suggested an SOPT with a small order parameter exponent $\beta = 0.017(5)$. Malakis and Fytas [27], by applying the critical minimum-energy subspace scheme in conjunction with the WangLandau and broad-histogram methods for cubic lattices, proved that the specific heat and susceptibility are non-self-averaging using the bimodal distribution.

Another notable manifestation of randomness is the spin glass (SG) phase exhibited by many systems under certain conditions. These are random magnetic systems in which the interactions between the spins are in conflict to each other, a phenomenon known as frustration, a result of strong frozen-in structural disorder according to which no single spin configuration is favored by all interactions, quenched randomness. Moreover, SG is an emergent phase of matter in random magnetic systems and thus a lot of studies on disordered systems concern that phase, where the magnetic and non-magnetic components, making up the material, are randomly distributed in space; the disorder is present in the Hamiltonian in the form of random couplings between two constituent spins, which vary, in general, in their values and signs according to a PDF $P(J_{ij})$ chosen suitably [13,14,15,16,17,18,19]. The competing interactions are ferromagnetic and antiferromagnetic. Conventional SGs are dilute magnetic alloys such as AuFe or CuMn. The main objective is to understand better, at a theoretical level, what are the microscopic mechanisms leading to such a behavior and how to describe them. Many studies, mainly using the mean-field analysis, have been successful in elucidating various concepts for understanding SGs. One of the current issues in SGs is their nature in finite dimensions below the upper critical dimension. Unfortunately, for finite dimensions, the calculations often rely on numerical simulations, because there
are few ways to analytically study SGs. Long equilibration times for their numerical simulations are needed and average over many realizations of random systems to make error bars small enough. It is thus difficult to gain a conclusive understanding on the nature of them in finite dimensions. Establishing reliable analytical theories of SGs have been one of the most challenging problems for years. Theoretical physicists have developed mathematically heuristic tools based on what is called the “replica trick”. Another successful analysis to elucidate their properties is the use of gauge symmetry, by which one can obtain the exact value of the internal energy, evaluate the upper bound for the specific heat, and obtain some correlation inequalities in a subspace known as the Nishimori line [28]. Since the Nishimori line is also invariant under renormalization group transformations, the intersection of the Nishimori line and the FM/PM transition line must be a fixed point. The so-called Nishimori point corresponds to a new universality class belonging precisely to the family of strong disorder fixed points. Moreover, Kaneyoshi has also applied the effective field theory to the SGs [29].

In the parameter plane spanned by temperature and external magnetic field the high temperature phase is separated from the spin glass one by the so called de Almeida-Thouless line (AT-line); consequently, the determination of the AT-line is a matter of great urgency in the theoretical analysis of the SGs model. The equilibrium properties of mean field SGs are calculated by using the two available different approaches. The first one is the replica method starting with $n$ replicas of the system under consideration. The free energy can be determined by the saddle point method. The main feature of the replica method is that the mathematically problematic limit $n \rightarrow 0$ is usually taken at the end. In this framework, the AT-line is determined by the local stability of the replica symmetric saddle point. In the other method, the cavity one, one spin is added to a system of $N$ spins and a stochastic stability of the thermodynamic limit $N \rightarrow \infty$ is used to derive self-consistent equations for the order parameters. In the latter procedure, the AT-line is obtained by investigating the correlations between two spins which vanish in the thermal limit for a pure state of a mean field system [30,31,32,33,34,35,36,37].

An early attempt in the theory of SGs was put forward by Edwards and Anderson (EA) [30], based on Ising model with the disorder in the exchange integral between nearest neighbors; they managed to demonstrate the existence of the spin glass phase within the mean field theory in conjunction with the replica trick; they also identified two features for a spin glass theory, frustration and disorder. As their main result was the introduction of a new type of ”order” parameter, which describes the long-time correlations $q = \langle < S^z_i >_T >, j \neq i \neq 0$, where $< >_T$ means configuration averaging over the distributions $P(J_{ij})$ for all spin pairs $(ij)$ and $< >_T$ means thermal averaging. A simple mean field approximation leads to $q(T) \neq 0$ below a characteristic temperature $T_f$ and to a sharp second order phase transition at $T_f$. Their model was a generalization
of the Ising model (Ising spin-glass, ISG) but with a non constant exchange integral interaction, namely,

\[ H = - \sum_{<i,j>} J_{ij} S_i S_j \quad , \quad S_i = \pm 1 \]  (4)

where \(<i,j>\) implies summation over nearest neighbors, \(J_{ij}\) is the bilinear exchange interaction between nearest-neighbor pairs, randomly quenched variables, identically and independently distributed according to the single Gaussian probability distribution function

\[ P(J_{ij}) = \left[ (2\pi)^{1/2} J \right]^{-1} \exp \left[ -J_{ij}^2 / (2J^2) \right] \]  (5)

with zero mean value and variance \(J^2\). The disorder is quenched, in that \(J_{ij}\), initially, are chosen randomly but then fixed for all thermodynamic processes. The ISG together with the RFIM constitute two of the most-studied subjects in the area of the disordered magnetic systems. The EA model is far too difficult to be analyzed theoretically in detail, thus Sherrington and Kirkpatrick (SK) [31] introduced in 1975 a simplified version of this model by replacing the pair interaction by a long-range one, in that, each particle interacts with the remaining ones, so that the Hamiltonian (4) is replaced by

\[ H = -\frac{1}{2} \sum_{(ij)} J_{ij} S_i S_j \quad , \quad S_i = \pm 1 \]  (6)

where \((ij)\) implies summation over all pairs of spins; the bilinear spin interactions \(J_{ij}\) are randomly quenched variables, specified by a symmetric matrix \(\{J_{ij}\}\) and are distributed according to the probability distribution function

\[ P(J_{ij}) = \left[ (2\pi)^{1/2} J \right]^{-1} \exp \left[ -(J_{ij} - J_0)^2 / (2J^2) \right] \]  (7)

with \(J\) and \(J_0\) scaled by,

\[ J = \bar{J}/N^{1/2}, \quad J_0 = \bar{J}_0/N \]  (8)

both \(\bar{J}\) and \(\bar{J}_0\) are intensive quantities, so that in the SK model each spin interacts with each other one via weak interaction of the order \(N^{-1/2}\). The presence of \(N\) (the total number of spins in the system) is necessary to ensure that the respective thermodynamic quantities are extensive. This simple model captures the basic ingredients of spin glass physics, namely, quenched randomness and frustration, and was solved “exactly” at the mean field level. The fluctuations around the SK saddle point are described by an \(\left( \frac{n(n-1)}{2} \right) \ast \left( \frac{n(n-1)}{2} \right)\)
matrix and its eigenvalues have been determined in Ref. [36]. The temperature dependence of these eigenvalues shows that the replica symmetric saddle point loses its stability at the phase boundary of the SG phase. The SK model, since its inception, is still alive and attractive presenting several challenging issues; it presents a continuous phase transition in which the spin glass phase has the free energy landscape composed by many almost degenerated thermodynamic states separated by infinitely high barriers. Its formalism has gone far beyond the area of disordered magnetic systems being employed in many other complex systems, like neural networks, optimization problems as well as in stock markets and in wireless network communications [38]. Many experimental observations seem to be in good agreement with the predictions of this model. An important suggestion concerning the behavior of various physical parameters in the SG phase is the so-called Parisi-Toulouse (PaT) hypothesis, or projection hypothesis, for the SK model, according to which the entropy and the EA order parameter \( q \) are field independent \( \left( S(T, h) = S(T), q(T, h) = q(T) \right) \), whereas magnetization is temperature independent \( \left( m(T, h) = m(h) \right) \), where \( h \) is an external magnetic field; Monte Carlo results show strong evidence that entropy is independent of the applied field, [39,40,41,42,43].

In addition to the SK model other infinite-range spin glass models have been proposed, including Blume-Emery-Capel, Potts, spin-\( S \) (\( S > 1/2 \)) and vector spin glass models [14,18,19]. New phases, characterized by different classes of order parameters, have emerged, opening many controversial problems from both theoretical and experimental points of view.

The paper is organized as follows. In the next section, we discuss the limiting procedure between the canonical partition function and the Helmholtz free energy, the replica approach and suggest the joint Gaussian PDF with correlation function \( \rho \). In section 3, we introduce the current model with the random field and calculate the respective free energy functional as well as the magnetization and the Edwards-Anderson parameter. In section 4 we present the numerical results, the phase diagram and other thermodynamic quantities and we close with the conclusions and discussions in section 5.

2 The free energy and replica approach

As in every problem in equilibrium statistical physics the central issue is the calculation of the free energy per particle from the respective one of the \( N \)-particle system \( F(\beta, N) \) in the thermodynamic limit, namely

\[
    f(\beta) = \lim_{N \to \infty} \frac{1}{N} F(\beta, N) \tag{9}
\]

where \( -\beta F(\beta, N) = \ln Z(\beta, N) \), \( Z(\beta, N) \) is the canonical partition function. However, after the introduction of randomness, as in the case of spin glasses, its
influence on the system has to be considered, so that the relation (9) converts into,

\[- \beta f(\beta) = \lim_{N \to \infty} \frac{1}{N} \langle \ln Z(\beta, N) \rangle_r\]  

(10)

where \(\langle \ldots \rangle_r\) represents the thermal average as well as the one with respect to the randomness, thus the calculation of the free energy per particle is transferred to the calculation of \(\langle \ln Z(\beta, N) \rangle_r\).

The eventual aim is to calculate the various observables, but to achieve this we calculate initially the Helmholtz free energy \(F\); the calculation of the partition function is a very hard task, resulting the need for a new procedure to make the calculation feasible; as far as the function needed is the logarithm of the partition function and not the partition function itself, the following formula can be applied,

\[\ln Z = \lim_{n \to 0} \frac{Z^n - 1}{n}\]  

(11)

implying that we have considered \(n\) replicas of the initial system, which are not interacting, and \(Z^n = \prod_{\alpha=1}^{n} Z_{\alpha}\), where \(\alpha\) is the replica identifier and instead of averaging \(\ln Z\) we average \(Z^n\). Considering this expression for \(\langle \ln Z \rangle\), the relation (10) converts into

\[- \beta f(\beta) = \lim_{n \to 0} \lim_{N \to \infty} \frac{1}{Nn} \left( \langle Z^n \rangle_r - 1 \right)\]  

(12)

the order of the limits is irrelevant [32], although the thermodynamic one precedes that of \(n \to 0\) in order to apply the steepest descent method; the replicated partition function for integer values of \(n\) assumes the form

\[Z^n(\beta) = \sum_{\{S_i^{\alpha}=\pm 1\}} \exp \left[ - \beta \sum_{\alpha=1}^{n} H(\{S_i^{\alpha}\}) \right]\]  

(13)

Sherrington and Kirkpatrick managed to derive an expression for the respective free energy, by calculating initially the free energy functional with respect to the two parameters \(m_\alpha\) and \(q_{\alpha\gamma}\) dependent on replicas, introduced through the Hubbard-Stratonovich transformation, with \(\alpha\) and \(\gamma\) characterizing the replicas and \(\alpha \neq \gamma\), among other parameters; by considering the replica symmetry hypothesis (RS), that is \(m_\alpha = m\) and \(q_{\alpha\gamma} = q\) for every \(\alpha\) and \(\gamma\), and using the analytic continuation \(n \to 0\) succeeded in calculating the system...
free energy, namely
\[
\beta F = N \left\{ -\frac{\tilde{J}^2 \beta^2}{4} (1 - q)^2 + \tilde{J}_0 m^2 \beta - \frac{1}{(2\pi)^{1/2}} \int dz e^{-z^2/2} \ln \left[ 2 \cosh(\tilde{H}(z)) \right] \right\} \tag{14}
\]

where \( \tilde{H}(z) = \beta \tilde{J} q^{1/2} z + \beta \tilde{J}_0 m \), \( m = \langle < S_i >_T \rangle \rangle \) and \( q = \langle < S_i >^2_T \rangle \rangle \), in conjunction with the exponential identity (Hubbard-Stratonovich transformation)
\[
e^{\frac{\lambda x^2}{2}} = \left( \frac{\lambda}{2\pi} \right)^{1/2} \int_{-\infty}^{\infty} e^{-\frac{\lambda x^2}{2} + \lambda \sigma x} dx \tag{15}
\]

The SK model was solved by means of the replica method and, consequently, it was originally thought that it was exactly solvable, although Sherrington and Kirkpatrick were aware that it suffers from a serious drawback, in that, it possesses a negative entropy at zero temperature, specifically \( S(T = 0 \text{ K}) = -Nk/2\pi \). SK model was the subject of a large number of publications and cannot be considered as a trivial one in the mean field sense. Though rather unrealistic, it seems to describe some spin glass properties correctly. The SK model for Ising spins with quenched random bonds is the simplest representative of a class of long-ranged models all successfully describing the interesting phenomena of spin glasses. In addition to this success in physical questions, the research on these models has been fruitful and stimulating in optimization problems, in understanding the neural networks and communications. The zero-temperature entropy anomaly \( (s(T = 0 < 0)) \), appearing in the SK model, has been shown to be associated with the hypothesis of the replica symmetry of the two order parameters, \( m \) and \( q \). The correct low-temperature solution, resulting by breaking the replica symmetry of SK, was proposed by Parisi [34], and consists of a continuous order parameter function (an infinite number of order parameters) associated with many low-energy states, a procedure which is usually called the replica-symmetry breaking (RSB). Parisi was the first who found a satisfactory solution of the SK model in the SG regime. Curiously, the simplest one-step RSB (1S-RSB) procedure improves, in part, this anomaly, in the sense that the zero temperature entropy per particle becomes less negative, from \( s(0)/(Nk_B) \approx -0.16 \) within the RS it rises to \( s(0)/(Nk_B) \approx -0.01 \) within the 1S-RSB, a significant improvement. The complementary approach by Thouless, Anderson and Palmer (TAP), for investigating the spin glass model, does not perform the bond average, and permits a treatment of problems depending on specific configurations [35]. For other questions which are expected to be independent of the special configuration, such as all macroscopic physical quantities, self averaging occurs. This is due to the fact that the random interaction matrices have well-known asymptotic properties in the thermodynamic limit. The situation is in prin-
A principle similar to the central limit theorem in probability theory, where large numbers of random variables also permit the calculation of macroscopic quantities which hold for nearly every realization of the random variables. Thus the investigation of one or some representative systems is sufficient and the bond average is not needed. The TAP equations have been well established for more than two decades and several alternative derivations are known. Nevertheless the TAP approach is still a field of current interest. This is due to the importance of the approach to numerous interesting problems. Moreover it is suspected that not all aspects of this approach have yet been worked out. Furthermore, a transition in the presence of an external magnetic field, known as the Almeida-Thouless (AT) line [36,37], is found in the solution of the SK model: such a line separates a low-temperature region, characterized by RSB, from a high-temperature one, where a simple one-parameter solution, RS solution, is stable. Numerical simulations are very hard to carry out for short-range ISGs on a cubic lattice, due to large thermalization times; as a consequence, no conclusive results in three-dimensional systems are available. However, in four dimensions the critical temperature is much higher, making thermalization easier; in this case, many works claim to have observed some mean-field features.

Reentrant spin-glass (RSG) transition is a well-known phenomenon of spin glasses. The RSG transition is found near the phase boundary between the SG phase and the FM phase. As the temperature decreases from a higher temperature, magnetization once increases and then disappears at a lower temperature. Finally, the SG phase is realized. The phenomenon was first considered as a phase transition between an FM phase and a SG phase. However, neutron diffraction studies have revealed that the SG phase is characterized by FM clusters. Now the RSG transition is believed to be a reentry from a FM phase to a frozen state with FM clusters. The mechanism responsible for this reentrant transition has not yet been resolved. Two ideas have been proposed for describing the RSG transition: (i) an infinite-range Ising bond model, and (ii) a phenomenological random field concept. The essential point of that conception is that the system is decomposed into an FM part and a part with frustrated spins (SG part). At low temperatures, the spins of the SG part yield random effective fields to the spins of the FM part. Nevertheless, no theoretical evidence has yet been presented for this idea in a microscopic point of view. In the last two decades, computer simulations have been performed extensively to solve the RSG transition in various models such as short-range bond models [18,10,11], short-range site models [12,20], and a Ruderman-Kittel-Kasuya-Yoshida model [21,22].

Although the aforementioned types of randomness in magnetic systems consist a significant branch of statistical physics, very few investigations have considered them together [44], but even then the considered probability density function for the random bonds and random fields were considered as distinct
and their joint probability density function was simply their product so that the one type of randomness does not influence the other one directly and in fact these are two independent random variables. There are systems described by ISG in the presence of random fields, such as proton and deuteron glasses, mixtures of hydrogen-bonded ferroelectric and antiferroelectrics [45,46]. The diluted antiferromagnets, such as $Fe_xZn_{1-x}F_2$, under the influence of a uniform magnetic field form the realization of an RFIM [47]. For $x \leq 0.24$ it becomes an ISG, whereas for $x \geq 0.40$ it is an RFIM. For $0.24 \leq x \leq 0.40$ both behaviors appear, RFIM or ISG for small or large magnetic fields, respectively. Also, the compound $CdCr_{1.7}Ir_{0.3}S_4$ in a magnetic field exhibits all the characteristic features of SGs [48] as well as $LiHo_xEr_{1-x}F_4$ for various values of $x$ [49].

In the current investigation this restriction concerning the discreteness of the PDFs is lifted by considering a pure joint Gaussian probability density function for $J_{ij}$ and $h_i$ as

$$P(J_{ij}, h_i) = \frac{N^{1/2}}{2\pi\Delta J(1 - \rho^2)^{1/2}} \exp \left\{ -\frac{1}{2(1 - \rho^2)} \left[ N \frac{(J_{ij} - J_0/N)^2}{J^2} - 2\rho N^{1/2} \frac{(J_{ij} - J_0/N)(h_i - h_0)}{\Delta J} + \frac{(h_i - h_0)^2}{\Delta^2} \right] \right\}$$

(16)

where $\rho$ is the correlation function (or, simply, correlation) of the two random variables $J_{ij}, h_i$ with $\rho = \text{Cov}(J_{ij}, h_i)/(J\Delta)$, $\text{Cov}(J_{ij}, h_i)$ their covariance; $h_0, \Delta^2$ are the mean value of the random field and its variance, respectively. This joint PDF shall be used for the study of the SK spin glass model in the presence of a random field.

3 The Sherrington-Kirkpatrick spin glass model in the presence of a random field - Replica formalism

The Sherrington-Kirkpatrick infinite-range model of spin glasses for Ising spins $S_i = \pm 1, i = 1, 2, ..., N$, with random pair interactions (specified by a symmetric matrix $\{J_{ij}\}$) in the presence of a random field $\{h_i\}$ is described by the Hamiltonian

$$H = -\frac{1}{2} \sum_{(i,j)} J_{ij}S_iS_j - \sum_{i=1}^{N} h_iS_i$$

(17)

where the first sum runs over all pairs of spins and indicated by $(i, j)$. The exchange interactions $\{J_{ij}\}$ and random fields $\{h_i\}$ are quenched random variables drawn from the joint Gaussian PDF (16). The analysis of this model shall be relied on the MFA as in Ref. [31]; although this direction of study, in the most cases, for ordinary systems (i.e., ferromagnetic, etc.) is very simple and practically trivial, this does not happen to be for the SG case; for the latter
case this study is highly no trivial even for the simplest SG, the ISG, thus motivating intensive research activity in this direction, deriving a plethora of publications and new ideas.

Considering, now, a realization of the bonds and random fields \( \{J_{ij}\}, \{h_i\} \), the respective free energy \( F(\{J_{ij}\}, \{h_i\}) \) results as an average over both disorders, in addition to the thermal one,

\[
\left\langle F(\{J_{ij}\}, \{h_i\}) \right\rangle_{J,h} = \int \prod_{(ij)} P(J_{ij}, h_i) F(\{J_{ij}\}, \{h_i\}) dJ_{ij} dh_i \tag{18}
\]

so that the free energy per particle \( f \) assumes the form with respect to the partition function,

\[
-\beta f(\beta) = \lim_{n \to 0} \lim_{N \to \infty} \frac{1}{Nn} \left( \left\langle Z^n \right\rangle_{J,h} - 1 \right) \tag{19}
\]

Within the framework of the replica method, the average partition function \( \left\langle Z^n \right\rangle_{J,h} \) over both disorders for integer \( n \) is calculated to be

\[
\left\langle Z^n \right\rangle_{J,h} = \sum_{\{S_{\alpha} = \pm 1\}} \exp \left\{ \frac{(\beta \Delta)^2}{2} \sum_{i=1}^{N} \left( \sum_{\alpha=1}^{n} S_{i}^\alpha \right)^2 + \frac{\beta J^2}{2N} \sum_{(i,j)} \left( \sum_{\alpha=1}^{n} S_{i}^\alpha S_{j}^\alpha \right)^2 + \frac{\beta^2 \rho J \Delta}{\sqrt{N}} \sum_{i=1}^{N} \left( \sum_{\alpha=1}^{n} S_{i}^\alpha \right) \sum_{(k,l)} \left( \sum_{\alpha=1}^{n} S_{k}^\alpha S_{l}^\alpha \right) + \beta h_0 \sum_{i=1}^{N} \left( \sum_{\alpha=1}^{n} S_{i}^\alpha \right) + \frac{\beta J_0}{N} \sum_{(i,j)} \left( \sum_{\alpha=1}^{n} S_{i}^\alpha S_{j}^\alpha \right) \right\} \tag{20}
\]

where \( \alpha \) characterizes the replica and \( i \) the lattice site. The above expression can be linearized in the spins by introducing the replica matrix \( \{q_{\alpha \beta}\} \) as well as the auxiliary quantity \( \{m_\alpha\} \), then by reordering and dropping out terms that disappear in the thermodynamic limit \( (N \to \infty) \) by using the so-called Hubbard-Stratonovich transformation (15) (exponential transformation) one finds

\[
\left\langle Z^n \right\rangle_{J,h} = \exp \left\{ \frac{\beta^2 \Delta^2 n N}{2} + \frac{\beta^2 J^2 n N}{4} \right\} \int_{-\infty}^{\infty} \left( \prod_{\alpha \gamma} \left( \frac{\beta J_0 N}{2\pi} \right)^{1/2} d\alpha_\gamma \right) \left( \prod_{\alpha} \left( \frac{\beta J_0 N}{2\pi} \right)^{1/2} dm_\alpha \right)
\]

\[
\left( \prod_{\kappa} \left( \frac{\nu N}{2\pi} \right)^{1/2} dm_\kappa \right) \left( \prod_{\delta} \left( \frac{\nu N}{2\pi} \right)^{1/2} dm_\delta \right) \left( \prod_{\lambda} \left( \frac{\nu N}{\pi} \right)^{1/2} dm_\lambda \right) \left( \prod_{\varepsilon} \left( \frac{\nu N}{2\pi} \right)^{1/2} dm_\varepsilon \right) \left( \prod_{\eta} \left( \frac{i \nu N}{\pi} \right)^{1/2} dm_\eta \right)
\]
\[
\exp\left\{ -N \left[ \frac{(\beta J)^2}{4} \sum_{(\alpha\gamma)} q_{\alpha \gamma}^2 + \frac{\beta J_0}{2} \sum_{\alpha=1}^{n} m_{\alpha}^2 + \frac{\nu}{2} \sum_{\kappa=1}^{n} m_{\kappa}^2 + \nu \sum_{\delta=1}^{n} m_{\delta} \sum_{\lambda=1}^{n} m_{\lambda}^2 + \right.ight.
\]
\[
\left. -i\nu \sum_{\varepsilon=1}^{n} m_{\varepsilon} \sum_{\eta=1}^{n} m_{\eta}^2 \right] \right\} \text{Tr}_{\{S^{\alpha}\}} \exp \left[ NZ_n \left( \{q_{\alpha \gamma}\}, \{m_\alpha\} \right) \right]
\] (21)

where \(\nu = \beta^2 \rho \Delta J/2\) and

\[
Z_n \left( \{q_{\alpha \gamma}\}, \{m_\alpha\} \right) = \frac{(\beta \Delta)^2}{2} \sum_{(\alpha,\gamma)} S^{\alpha} S^{\gamma} + \beta h_0 \sum_{\alpha=1}^{n} S^{\alpha} + \frac{(\beta J)^2}{2} \sum_{(\alpha\gamma)} q_{\alpha \gamma} S^{\alpha} S^{\gamma} + 
\]
\[
\beta J_0 \sum_{\alpha=1}^{n} m_{\alpha} S^{\alpha} + \nu \sum_{\kappa=1}^{n} m_{\kappa} S^{\kappa} - i\nu \sum_{\varepsilon=1}^{n} m_{\varepsilon} S^{\varepsilon} \] (22)

The respective integrals in (21) were evaluated by the steepest descent method, since the exponential argument is proportional to \(N\). In expressions (21) and (22), the site indices \((i\) or \(j\)) are absent on the spins since all of the sites are equivalent, thus the resulting \(N\) can be extracted out simplifying the calculation of the thermodynamic limit \((N \to \infty)\) and \((\alpha \gamma)\) indicates a distinct pair of replicas with \(\alpha \neq \gamma\). The trace in (21) is over the \(n\) replicas at a single spin site. The two limits, the thermodynamic one \((N \to \infty)\) and the analytic continuation \((n \to 0)\), can be interchanged [32]; in order to calculate the free energy per particle from the expressions (19), (20), (21) and (22), the two limits are performed consecutively; first we consider the thermodynamic limit yielding

\[
-\beta f(\beta) = \left( \frac{\beta J}{2} \right)^2 + \frac{(\beta \Delta)^2}{2} - \lim_{n \to 0} \frac{1}{n} \ln \text{Tr} \exp \left\{ \frac{(\beta \Delta)^2}{2} \sum_{(\alpha\gamma)} S^{\alpha} S^{\gamma} + \beta h_0 \sum_{\alpha} S^{\alpha} + \frac{(\beta J)^2}{2} \sum_{(\alpha\gamma)} S^{\alpha} S^{\gamma} q_{\alpha \gamma} + 
\]
\[
(\beta J_0 + \nu) \sum_{\alpha} m_{\alpha} S^{\alpha} - i\nu \sum_{\alpha} m_{\alpha} S^{\alpha} \right\} \] (23)

In order to perform the analytic continuation \(n \to 0\), we invoke the RS hypothesis, namely, \(q_{\alpha \gamma} = q\) and \(m_{\alpha} = m\) for every \(\alpha\) and \(\gamma\), then taking the limit \(n \to 0\) the free energy in (23) assumes the form, in conjunction with the Hubbard-Stratonovich transformation (15)
\[
\beta f(\beta) = -\left(\frac{\beta J}{2}\right)^2 (1-q)^2 + \frac{2\beta J_0 + \beta^2 \rho \Delta J}{4} m^2 - \\
\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dz \, e^{-z^2/2} \ln \left\{ 2 \left[ \cosh(A(z)) \cos(B) + i \sinh(A(z)) \sin(B) \right] \right\} \quad (24)
\]

where \( A(z) = \beta \left[ h_0 + m \left( J_0 + \frac{1}{2} \beta \rho \Delta J \right) + z(qJ^2 + \Delta^2)^{1/2} \right] \) and \( B = -\beta^2 \rho \Delta J m / 2 \).

The expression for \( A(z) \) and \( B \) contain the interdependence of both disorders.

Eq. (24) converts into the one found by Sherrington and Kirkpatrick [31] in the absence of random fields, i.e., for \( h_0 = 0, \Delta = 0 \). The free energy for the system resulting from (24) as a function of the temperature and the PDF's parameters is

\[
f(\beta) = -\frac{\beta J^2}{4} (1-q)^2 + \frac{2J_0 + \beta \rho \Delta J}{4} m^2 - \\
\frac{kT}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} dz \, e^{-z^2/2} \ln \left[ 4 \left( \cosh^2 A - \sin^2 B \right) \right] \quad (25)
\]

The former expression constitutes the principal quantity of the current investigation. The logarithmic function in (25) is not singular, since the functions \( \cosh(A(z)) \) and \( \sin(B) \) do not intersect each other, so the integrand is well-defined. The quantities \( q, m \) are given self-consistently by the extremum conditions (saddle-point conditions), \( \frac{\partial (\beta f)}{\partial q} = 0 \) and \( \frac{\partial (\beta f)}{\partial m} = 0 \) from (25), so that the two principal quantities \( m \) and \( q \) satisfy the simultaneous equations

\[
m = \frac{1}{2J_0 + \beta \rho \Delta J} \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-z^2/2} \frac{2J_0 + \beta \rho \Delta J \sinh(2A(z)) + \beta \rho \Delta J \sin(2B)}{\cosh^2(A(z)) - \sin^2 B} \, dz
\]
\[
q = \frac{1}{4\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-z^2/2} \frac{\sinh^2(2A(z)) - \sin^2(2B)}{\left( \cosh^2(A(z)) - \sin^2 B \right)^2} \, dz \quad (26)
\]

which become identical to those found by Sherrington and Kirkpatrick [31] in case the random fields are absent. Both denominators in (26) do not vanish for the same reason as in a previous paragraph.

Initially, we shall focus our attention on the so called Parisi-Toulouse (PaT) hypothesis to explore what occurs in the low temperature SG phase [14,39,40,41,42,43]; consequently, the integrand in (25) is expanded for small values of the arguments (small \( h_0 \)), [14,39,40,41,42,43], namely
\[- \beta f = \frac{(\beta J)^2}{4} (1 - q)^2 - \beta J_0 + \beta \rho \Delta J \frac{m^2 + \ln 2 + 1}{4} \]

\[
\frac{1}{2} \left[ 2 \ln \cos B + \frac{\alpha_1^2 + \alpha_2^2}{\cos^2 B} - \frac{2 \sin^2 B + 1}{6 \cos^4 B} \left( \alpha_1^4 + 3 \alpha_2^4 + 6 \alpha_1^2 \alpha_2^2 \right) \right]
\]

(27)

where \( \alpha_1 = \beta (h_0 + (2J_0 + \beta \rho J) m/2) \), \( \alpha_2 = \beta (\Delta^2 + q J^2)^{1/2} \). Considering the extremum \( \frac{\partial (-\beta f)}{\partial q} = 0 \) we get

\[
q = \frac{(1 + 2 \sin^2 B)(\alpha_1^2 + \beta^2 \Delta^2) - \sin^2(2B)/4}{\cos^4 B - (1 + 2 \sin^2 B) \beta^2 J^2}
\]

(28)

the respective equation for the magnetization \( m \), resulting from the extremum condition \( \frac{\partial (-\beta f)}{\partial m} = 0 \), is

\[
(\beta J_0 + w)m = -w \tan(v/2) + \frac{\alpha_1 (\beta J_0 + w)}{\cos^2(v/2)} + \frac{w \sin(v)}{2} \frac{\alpha_1^2 + \alpha_2^2}{\cos^4(v/2)}
\]

\[- \frac{w \sin(v)}{6} \frac{2 + \sin^2 B}{\cos^6(v/2)} \left( \alpha_1^4 + 3 \alpha_2^4 + 6 \alpha_1^2 \alpha_2^2 \right)
\]

\[- \frac{\beta J_0 + w}{3} \frac{1 + 2 \sin^2 B}{\cos^4(v/2)} \left( \alpha_1^3 + 3 \alpha_1 \alpha_2^2 \right)
\]

(29)

where \( w = \beta^2 \rho \Delta/2 \) and \( v = \beta^2 \rho \Delta m \). However, since the explicit formulae in (28) and (29) seem to be too lengthy, we resort to consider the special case with \( \rho = 0 \) (constraint); in this case Eqs. (28) and (29) become

\[
q = \beta^2 \frac{(h_0 + m J_0)^2 + \Delta^2}{1 - \beta^2 J^2}
\]

(30)

\[
m = \beta (h_0 + m J_0) \left\{ 1 - \frac{\beta^2}{3} \left[ (h_0 + m J_0)^2 + 3 \left( \Delta^2 + q J^2 \right) \right] \right\}
\]

(31)

Introducing Eq. (30) into (31), yields

\[
m = \beta (h_0 + m J_0) \left\{ 1 - \frac{\beta^2}{3} \left[ (h_0 + m J_0)^2 + 3 \left( \Delta^2 + \beta^2 J^2 (h_0 + m J_0)^2 + \Delta^2 \right) \right] \right\}
\]

(32)

or

\[
m = \beta (h_0 + m J_0) \left\{ 1 - \frac{\beta^2}{3} \frac{(h_0 + m J_0)^2 + 3 \Delta^2 + 2 \beta^2 J^2 (h_0 + m J_0)^2}{1 - \beta^2 J^2} \right\}
\]

(33)
In the absence of an external magnetic field and for $J_0 = 0$ the transition temperature to the spin glass phase is $T_f = J$ (see [14]), introducing this into (33) yields

$$m = \frac{h_0 + m J_0}{T} \left\{ 1 - \frac{1}{3 T^2} \left( \frac{(h_0 + m J_0)^2 (T^2 + 2 T_f^2) + 3 \Delta^2 T^2}{T^2 - T_f^2} \right) \right\} \quad T > T_f \quad (34)$$

implying that magnetization $m$ is a nonanalytic function of $h_0$ at $T = T_f$. The respective equation for $q$ is

$$q = \frac{(h_0 + m J_0)^2 + \Delta^2}{T^2 - T_f^2} \quad T > T_f \quad (35)$$

In addition, for $T = T_f$, we find by expanding both Eqs. (26)

$$m = \frac{h_0}{T_f} \left\{ 1 + \frac{2 h_0^2}{3 T_f^2} - \frac{h_0}{T_f \sqrt{2}} \left[ 1 + \frac{2 h_0^2}{3 T_f^2} + \frac{\Delta^2}{2 h_0^2} \right] \right\}$$

$$q = \frac{h_0}{T_f \sqrt{2}} \left[ 1 + \frac{2 h_0^2}{3 T_f^2} + \frac{\Delta^2}{2 h_0^2} \right] - \frac{h_0^2 + \Delta^2}{T_f^2} \quad T = T_f \quad (36)$$

Eqs. (34), (35), (36) constitute a generalization of the so called Parisi-Toulouse (PaT) hypothesis (or projection hypothesis) in the presence of an external random magnetic field and for $\Delta = 0$ both equations convert into the respective ones in Refs. [14,39,40,41,42]. However, for $T < T_f$ the Parisi-Toulouse hypothesis, [39,40,41,42], guesses that

$$\frac{m}{h_0} = 1 - \left( \frac{3}{4} \right)^{2/3} h_0^{4/3} + \frac{7}{6} h_0^2 \quad T < T_f \quad (37)$$

4 Numerical results. Phase diagrams

In order to simplify the calculations to follow we use as measure the standard deviation $J$ of the exchange integral in Eq. (16) by setting from now on $J = 1$.

Taking the derivative with respect to $m$ of the first relation in (26) and softening to zero the magnetization $m$, we find

$$\frac{2 J_0 T^2 + \rho \Delta T}{2 J_0 (J_0 T + \rho \Delta)} = \frac{1}{\sqrt{2 \pi}} \int_{-\infty}^{\infty} \frac{e^{-z^2/4}}{\cosh^2 H(z)} dz \quad (38)$$

where $H(z) = \beta \left[ h_0 + z (q + \Delta^2)^{1/2} \right]$, from which the value for $J_0$ is calculated, since the temperature $T$ is known a priori,
\[
J_0 = \frac{1}{2} \left[ \frac{T}{1-q} - \frac{\rho \Delta}{T} + \sqrt{\left( \frac{T}{1-q} \right)^2 + \left( \frac{\rho \Delta}{T} \right)^2} \right]
\]

\[
= \frac{1}{2} \left[ J_0^{(SK)} - \frac{\rho \Delta}{T} + \sqrt{\left( J_0^{(SK)} \right)^2 + \left( \frac{\rho \Delta}{T} \right)^2} \right]
\]

(39)

the positive root is chosen, since the negative one leads to a negative \(J_0\) and \(J_0 > J_0^{(SK)}\). For \(h_0 = \Delta = \rho = 0\), the Sherrington-Kirkpatrick result is recovered, \(J_0^{(SK)} = \frac{T}{1-q}\). From this equation one can calculate analytically its high as well as low temperature behavior, including that for zero temperature \((T = 0^\circ K)\).

![Graphs](image)

**Fig. 1.** The average value \(J_0\) for \(T = 0K\) \((J_{00})\) as a function of \(\Delta\) for specific \(h_0\)-values, (a) \(h_0 = 1.1\) and (b) \(h_0 = 2.0\). \(J_{00}\) possesses a minimum value only when \(h_0 \geq 1.0\), since \(\Delta = \sqrt{h_0^2 - 1}\)

For the zero temperature, one finds

\[
J_0(T = 0) \equiv J_{00} = \sqrt{\frac{\pi}{2}} \sqrt{1 + \Delta^2} \exp \left[ \frac{h_0^2}{2(1 + \Delta^2)} \right]
\]

(40)

which, considered as a function of \(\Delta\), for a specific \(h_0\)-value, possesses a minimum value at \(\Delta = \sqrt{h_0^2 - 1}\); this minimum is present only for \(h_0 \geq 1\) and its plots appear in Fig. 1. The low temperature region one has

\[
\frac{1}{J_0} = \frac{1}{J_{00}} + \frac{2T}{\pi(1 + \Delta^2)^2}
\]

(41)

In order to facilitate the evaluation of the phase diagram (the temperature \(T\) with respect to \(J_0\)) initially we focus on the boundary between the PM \((m = 0, q = 0)\) and SG \((m = 0, q \neq 0)\) phases, which is achieved by expanding the free energy (25) in powers of \(q\) under the constraint \(m = 0\) \([14,50,51,52]\); in this expansion the coefficient of \(q^2\)-term is set equal to zero in order to determine
the respective transition temperature $T_f$ between the aforementioned phases, yielding

$$ T_f = \left\{ 1 \pm \left[ \frac{1 - 16(\Delta^2 + h_0^2)}{2} \right]^{1/2} \right\}^{1/2} $$

being $J_0$-independent, $\frac{dT_f}{dJ_0} = 0$, thus representing a straight line in $(J_0 - T)$-plane. However, as it is clear from (42) the transition temperature $T_f$ possesses two branches, the plus-one and the minus-one; both temperatures lead to a spin glass phase, but in case of absence of an external magnetic field ($h_0 = 0$ and $\Delta = 0$) the resulting temperature from the plus-branch is $T_f^+ = 1$ (in units of $J$) whereas from the minus-branch is $T_f^- = 0$, that is, both specify the two limits for the existence of the SG phase; the former one leads to the existence of a non zero temperature spin glass phase and not to a trivial temperature as the latter yields. However, the functional form of Eq. (42) imposes a severe constraint on the specific values $\Delta$ and $h_0$ can assume for the existence of a spin glass phase, since they have to satisfy the relation $\Delta^2 + h_0^2 \leq 0.0625$ so that the interior square root is meaningful. Due to this constraint, the temperature $T_f^+$ possesses a minimum value, which is $T_f^+ = \sqrt{0.5}$, which is the maximum one for the $T_f^-$, whose minimum value is zero, $T_f^- = 0.0$.

---

Fig. 2. (Color online) (a) The phase diagram in the absence of a random field (i) and in presence of a random field (ii) $\Delta = 0.1, h_0 = 0.0, \rho = 0.1$, (iii) $\Delta = 0.1, h_0 = 0.1, \rho = 0.1$, (iv) $\Delta = 0.1, h_0 = 0.2, \rho = 0.5$. (b) The phase diagram for fixed $\Delta = 0.1, h_0 = 0.2$ and (i) $\rho = 0.0$, (ii) $\rho = 0.5$ and (iii) $\rho = 1.0$; as the correlation $\rho$ increases the extent of the spin glass phase is reduced and simultaneously that for ferromagnetism increases; also, the straight lines defining the boundary of the SG phase are the same for the three cases, since these depend only on $h_0$ and $\Delta$. In panel (c), $(\Delta = 0.1, h_0 = 0.0, \rho = 0.1)$ the upper and lower transition temperature lines ($T_f^+, T_f^-$) are shown for the PM-SG transition; the upper one (corresponding to $T_f^+$) intersects the PM-FM line thus defining an SG region inside the FM-phase whereas the lower line (corresponding to $T_f^-$) does not; the same behavior appears for other values of $h_0$ and $\Delta$.

Considering also the softening to zero of the magnetization $m$, the spin glass
Fig. 3. (Color online) The phase diagram temperature $T$ with respect to $\rho$ for $J_0 = 1.0$ and (a) $h_0 = 0.0$, (b) $h_0 = 0.1$. The plots are labelled by $\Delta$, with (i) $\Delta = 0.1$, (ii) $\Delta = 0.3$, (iii) $\Delta = 0.5$. Because of the limited values for $\rho$ the phase diagram is also restricted; however, the reentrance is evident in both panels.

The $q$-parameter in (26) takes the form

$$q = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-z^2/2} \tanh^2 \left\{ \beta [h_0 + z(q + \Delta^{2/2})^{1/2}] \right\} dz \quad (43)$$

The ($J_0 - T$) phase diagram appears in Fig. 2(a), in the absence and presence of a random field, using equations (38), (39), (41), (42) and (43); these ones express the interdependence of any two quantities (mainly the temperature and another variable) with the remaining ones considered as constants, thus forming the respective phase diagram, in addition to the normal one ($J_0 - T$), Fig. 2(a). In this figure, the straight lines, parallel to the $J_0$-axis and specifying the boundaries of the SG phase with the aforementioned axis in the presence of an external random magnetic field, are obtained from Eq. (42). If both branches of this equation are considered, the plus-branch intersects the PM/FM boundary enclaving a part of the FM phase inside the SG phase thus yielding a mixed phase (see Fig. 2(c)), whereas the minus-branch does not intersect the PM/FM boundary so that the mixed phase disappears, compare Fig. 2(a) with Fig. 2(c). Also an interesting remark is that for fixed values for $\Delta, h_0$ the extent of the ferromagnetic phase increases at the expense of the spin glass one, Fig. 2(b) as the correlation $\rho$ increases; the straight line, parallel to the $J_0$-axis delimiting the SG phase, is common to the three lines because these ones correspond to the same $\Delta$ and $\rho$ values. In Fig. 2(c) we exhibit the phase diagram for fixed $h_0$ and $\Delta$ values in the presence of both PM-SG boundaries (upper and lower). The plus-branch is used if $\sqrt{0.5} < T_f < 1$ and the minus-one if $0 < T_f < \sqrt{0.5}$. The phenomenon of reentrance is also evident in Fig. 2 in the presence of a random magnetic field. Moreover, one can use as independent variable any one from the natural parameters of the system, as $\rho$; the respective "phase diagram" appears in Fig. 3, in which reentrance is
also evident in both panels.

Sherrington and Kirkpatrick in their seminal paper, [31], apart from their significant contribution to the solution of the SG problem, their finding that the zero-temperature entropy \( s(0) \) was negative had aroused a lot of debate for this peculiar behavior until Parisi resolved this inadequacy by invoking the replica symmetry breaking. The RS solution gives a satisfactorily good phase diagram for the true one; however, the RS hypothesis leads to a significant problem at low temperatures, revealed by the negative entropy at \( T = 0^\circ K \).

Using the general form for the free energy in (25) the respective entropy as a function of the temperature and the random field parameters \( \rho, h_0 \) and \( \Delta \), is

\[
s(T) = -\frac{\partial f}{\partial T} = \beta^2 \left[ \frac{(1 - q)^2}{4} + 2q(q + \Delta^2) \right] - \beta m^2 \left( 2J_0 + 3\beta \rho \Delta / 4 \right) + \frac{1}{2(2\pi)^{1/2}} \int_{-\infty}^{\infty} dz \ e^{-z^2/2} \ln(4\zeta(z)) - \frac{h_0 - mJ_0}{2T(2\pi)^{1/2}} \int_{-\infty}^{\infty} dz \ e^{-z^2/2} \ \frac{\sinh(2A(z))}{\zeta(z)} - \frac{q + \Delta^2}{2T^2(2\pi)^{1/2}} \int_{-\infty}^{\infty} dz \ e^{-z^2/2} \ \frac{2\zeta(z) \cosh(2A(z)) - \sinh^2(2B)}{\zeta(z)^2} \tag{44}
\]

where \( \zeta(z) = \cosh^2(A(z)) - \sin^2 B \).

The low-temperature form of the spin glass parameter \( q \) is given by

\[
q \approx 1 - \sqrt{\frac{2}{\pi \sqrt{1 + \Delta}}} \tag{45}
\]

from (45) the resulting entropy per spin at zero temperature is dependent on the random field average value \( h_0 \) as well as on \( \Delta \),

\[
s(0) = -\frac{\gamma}{\pi (1 + \Delta^2)^{1/2}} \left[ 1 - \frac{\gamma}{2 (1 + \Delta^2)^{1/2}} \right] \tag{46}
\]

remaining negative for any value of \( \Delta \) and \( h_0 \)—values smaller than or equal to one, Fig. 4 as in Ref.[31], generalizing their result in the presence of a random magnetic field; it tends to the zero value asymptotically as \( \Delta \rightarrow \infty \) and for \( h_0 = 0, \Delta = 0, \) \( s(0) = -\frac{1}{2\pi^2} = -0.1591549 \), the Sherrington-Kirkpatrick result is recovered [31]. For \( h_0 \geq 1 \) it possesses a minimum value. However, for \( h_0 = 2 \), entropy \( s(0) \) is mainly positive, acquiring negative values for a small interval of \( \Delta \). From Eq. (44), entropy depends on \( T, h_0, \Delta \); its temperature dependence
Fig. 4. (Color online) The entropy per spin at zero temperature \( (T = 0^\circ K) \) as a function of \( \Delta \); panel (a) from bottom to top \( h_0 = 0.0, 0.50, 0.75, 1.00 \), panel (b) \( h_0 = 2.0 \). For small \( h_0 \) values (panel (a)) it always remains negative tending to zero as \( \Delta \to \infty \), whereas in panel (b) (strong \( h_0 \) field) it is mainly positive tending also to zero as \( \Delta \to \infty \).

Fig. 5. (Color online) The entropy per spin as a function of the temperature \( T \) for (a) \( h_0 = 0.25 \) and (b) \( h_0 = 1.00 \), with \( \Delta = 0.25(i), 0.50(ii), 0.75(iii), 1.00(iv) \). For low temperatures it is negative in agreement with the Sherrington-Kirkpatrick model. In panel (a) The smaller the \( \Delta \) value the lower the plot is, whereas in panel (b) the reverse is true. It appears in Fig. 5 representing a parabola-like curve, for specific \( h_0 \) values and labelled by \( \Delta \); for small \( h_0 \) (panel (a)) as \( \Delta \) increases the entropy increases as a function of \( T \), whereas for \( h_0 = 1.0 \) (panel (b)) the reverse is true. In both cases the entropy is initially negative but, finally, becomes positive. If, now, we consider the entropy in (44) as a function of \( \Delta \) for specific \( T \) values and labelled by \( h_0 \), then its behavior appears in Fig. 6(a, b). In this case it seems to
Fig. 6. (Color online) Panels (a) and (b) the entropy per spin as a function of $\Delta$ labelled by the random field strength $h_0 = 0.0(i), 0.5(ii), 1.0(iii)$ for temperatures, (a) $T = 0.5$, (b) $T = 1.0$. Panel (c) the entropy per spin as a function of $h_0$ labelled by $\Delta = 0.0(i), 0.5(ii), 1.0(iii)$ for $T = 1.0$.

tend to zero as $\Delta \to \infty$. For small temperature ($T = 0.5$), panel (a), and small $h_0$—values it varies, nearly, linearly, but for larger $h_0 = 1.0$ it behaves non-monotonically exhibiting a minimum value. For higher temperature ($T = 1.0$), panel (b), it behaves non-monotonically starting from negative values for small $h_0 = 0.0, 0.5$ whereas for higher $h_0 = 1.0$ it starts from a positive value. In Fig. 6(c) the dependence of the entropy $s$ on $h_0$ for $T = 1.0$ and labelled by $\Delta$ appears; the entropy increases monotonically, the higher the $\Delta$—value the less steeper is the respective entropy plot and the three entropy branches intersect each other.

Fig. 7. (Color online) The de Almeida-Thouless line (temperature $T$ versus the random field strength $h_0$), labelled by the same values of the correlation $\rho = 0.0, 0.5, 1.0$; panel (a) corresponds to $J_0 = 0.5, \Delta = 0.5$; panel (b) corresponds to $J_0 = 1.0, \Delta = 1.0$; the order of the plots in each panel is from top to bottom.

An important issue, arisen from Sherrington-Kirkpatrick paper, is the stability of the solution, which is reduced to the requirement the eigenvalues of the Hessian matrix, associated with the exponential functional following the trace
Fig. 8. (Color online) The de Almeida-Thouless line, namely temperature $T$ versus the random field strength standard deviation $\Delta$, labelled by the natural parameters; panel (a): (i) $J_0 = 0.2, h_0 = 0.2, \rho = 0.2$, (ii) $J_0 = 0.5, h_0 = 0.0, \rho = 0.5$, (iii) $J_0 = 0.5, h_0 = 0.5, \rho = 0.0$; panel (b): (i) $J_0 = 0.5, h_0 = 0.5, \rho = 0.5$, (ii) $J_0 = 1.0, h_0 = 0.3, \rho = 0.3$, (iii) $J_0 = 1.0, h_0 = 0.5, \rho = 0.2$, (iv) $J_0 = 1.0, h_0 = 0.5, \rho = 0.5$.

Fig. 9. (Color online) The magnetization $m$ versus the random field strength standard deviation $\Delta$ corresponding to the temperature values of the de Almeida-Thouless line in Fig. 7, labelled by the natural parameters; panel (a): (i) $J_0 = 1.0, h_0 = 0.5, \rho = 0.2$, (ii) $J_0 = 1.0, h_0 = 0.5, \rho = 0.5$; panel (b): (i) $J_0 = 1.0, h_0 = 1.0, \rho = 1.0$.

in the expression (23), to be positive [36]; if at least one of them becomes negative the symmetric solution is not correct resulting in replica symmetry breaking. For $n \geq 1$, the eigenvalues are real, but when analytic continuation $n \to 0$ is considered this guarantee is lost. However, within the replica symmetric solution one of the eigenvalues of the aforementioned Hessian matrix
Fig. 10. (Color online) The Edwards-Anderson parameter $q^{1/2}$ versus the random field strength standard deviation $\Delta$ corresponding to temperature values of the de Almeida-Thouless line in Fig. 7, labelled by the natural parameters; panel (a): (i) $J_0 = 1.0, h_0 = 0.5, \rho = 0.2$, (ii) $J_0 = 1.0, h_0 = 0.5, \rho = 0.5$; panel (b): (i) $J_0 = 0.2, h_0 = 0.2, \rho = 0.2$, (ii) $J_0 = 0.5, h_0 = 0.0, \rho = 0.5$; panel (c): $J_0 = 0.5, h_0 = 0.5, \rho = 0.0$, (iii) $J_0 = 1.0, h_0 = 0.3, \rho = 0.3$.

is negative below a temperature defined by the functional

$$T^2 = \left( \frac{1}{2\pi} \right)^{1/2} \int_{-\infty}^{\infty} dz \ e^{-z^2/2} \text{sech}^4(\beta \xi(z)) \quad (47)$$

where $\xi(z) = h_0 + J_0 m + \frac{1}{2} \beta \rho \Delta m + z (\Delta^2 + q)^{1/2}$. Eq. (47) defines a phase boundary between the SG phase and PM (FM) phase, called de Almeida-Thouless line (AT line), see Fig. 7. A characteristic feature of Fig. 7(b) is that all the AT-lines are fluctuating, whereas this phenomenon is missing from Fig. 7(a). In the high temperature region the replica symmetry is stable and the existing phase is the PM solution; at lower temperatures the RS solution is not valid characterizing an SG phase. The physical consequence of the AT-line is that at any temperature $T$ below $T_f$ there exist a magnetic field $H_c(T)$ above which the system can be described by the mean-field equations of Sherrington-Kirkpatrick model [31], whereas below that field a specific hypothesis concerning the structure of the SG-phase must be done. The latter assertion constitutes the cornerstone of the Parisi-Touless hypothesis [39,40,41,42,43]. For $J_0 \neq 0$ the AT line, for any value of $\rho$, results as a simultaneous solution of Eqs. (26) and (47); however, for low temperatures this line assumes a simpler form, by considering mainly the influence of the random field as well as the correlation $\rho$ on the behavior of the random exchange integral

$$T = \frac{4}{3} \frac{1}{\sqrt{2\pi}} \exp \left\{ -\frac{1}{2} \frac{(h_0 + J_0 m + \frac{1}{2} \rho \Delta m)^2}{\Delta^2 + q} \right\} \quad (\Delta^2 + q)^{1/2} \quad (48)$$

However, the current system possesses more natural variables, which influence
Fig. 11. (Color online) The AT profiles with respect to the correlation $\rho$ and labeled by $J_0, h_0, \Delta$ with (i)$J_0 = 0.5, h_0 = 0.5, \Delta = 0.1$, (ii)$J_0 = 0.5, h_0 = 0.5, \Delta = 0.5$, (iii)$J_0 = 1.0, h_0 = 1.0, \Delta = 0.1$, (iv)$J_0 = 1.0, h_0 = 1.0, \Delta = 1.0$: (a) temperature; (b) Edwards-Anderson parameter $q^{1/2}$; (c) magnetization.

It in one or another way, so that one can define an "AT line" as the temperature with respect to one of these variables besides the one corresponding to Fig. 7. As such independent variable, the random field standard deviation $\Delta$ is chosen and the respective "AT line" appears in Fig. 8 for specific values of the remaining natural parameters using (47), as well. Also, in Fig. 9 and Fig. 10 appear the respective profiles for the magnetization and the square root of Edwards-Anderson parameter with respect $\Delta$ for the respective temperatures resulting from (47); these graphs display, in some cases, a non monotonic behavior with maxima and minima. In case the correlation $\rho$ is considered as the main control parameter, the remaining natural parameters are fixed to specific values, then the resulting AT profiles with respect to $\rho$ were calculated along the AT line, Eq. (47), and do not display any significant structure as those with respect to $\Delta$, see Fig. 11.

5 Conclusions and discussions

In the current investigation, we have explored the thermodynamic properties of the infinite-range Sherrington-Kirkpatrick Ising spin glass model in the presence of a random field by employing a joint Gaussian probability density function for both random variables (the exchange integral and random field) with correlation $\rho$ by means of the replica trick formulation. Following the traditional route, the free energy was calculated along the lines of the replica symmetry and, on the basis of the saddle point method, the functional forms for the magnetization $m$ as well as the Edwards-Anderson parameter $q$ were calculated, which form a system of simultaneous equations. Initially, making use of the small argument expansion, the generalized forms for the magnetization and Edwards-Anderson parameter were found corresponding to the
respective formulae of the PaT hypothesis. We have also found a generalized form for the SK zero-temperature value for $J(T = 0) = \sqrt{\frac{\pi}{2}}$ by proposing one depending on $h_0$ and $\Delta$; this expression is non-monotonic as a function of $\Delta$ and labelled by $h_0$. The temperature, delimiting the SG phase, has been calculated analytically depending only on $\Delta$ and $h_0$; from its functional expression it can be determined the possibility of the existence or not of the SG phase. The structure of the phase diagram ($T$ with respect to $J_0$) is more rich and as the correlation increases the extent of the SG phase is reduced, thus increasing that of the FM phase; in addition to the former phase diagram, the random system possesses more ones, i.e., the temperature $T$ with respect to $\Delta$ and $h_0$; however, the random system possesses a plethora of natural variables $(T, J_0, h_0, \Delta, \rho)$, which affect the behavior of the system, implying that one can plot the "phase diagram" by using a pair of these variables keeping the remaining ones fixed.

The zero temperature entropy ($s(0) = -\frac{1}{2\pi}$) was generalized with the new expression depending on $h_0$ and $\Delta$; $s(0)$ as a function of $\Delta$ initially is monotonic but it becomes non-monotonic as well as positive, also the entropy as a function of temperature is represented by a parabola-like curve. A similar complex variation presents the de Almeida-Thouless line ($T$ vs $h_0$) so that as the correlation $\rho$ increases the SG region is reduced whereas that of the FM phase increases. We have also calculated the magnetization as well as the square root of the EA parameter profiles with respect to $\Delta$ for the respective de Almeida-Thouless line temperature; in both cases the respective profiles are mainly non-monotonic.

Although the formulation is along the lines of the mean field theory, this description cannot be considered as a trivial one, in that it is very complex in its description requiring complicated mathematics for the description of the low temperature phase - the spin glass phase. The SG phenomena are essentially dynamic critical ones, and more developments are needed to give us a better understanding of the spin glasses. In this case we shall rely on short range models concentrating on Monte Carlo simulations and the Renormalization Group methods. One may assert that the present situation about spin glasses is successful in the sense that a qualitative understanding of the SG phase has been achieved by introducing innovative methods in Statistical Physics as the Replica Symmetry Breaking as well as the Ultrametricity, since the Replica Theory provides a powerful approach to the complex low temperature phase of these systems, which is described in terms of a "Replica Symmetry Broken" solution. However, an important open issue is to understand whether the RSB scenario also holds beyond the mean-field approximation. Still, the glassy behaviour is observed at low temperatures and the phenomenology much resembles the one of some mean-field spin glass models. For this reason, concepts and techniques from spin glasses have been widely applied to investigate glassy behaviour in these systems.
The current investigation shall be extended towards the Replica Symmetry Breaking formulation.

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