Biquadratic tensors, biquadratic decompositions, and norms of biquadratic tensors

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Abstract Biquadratic tensors play a central role in many areas of science. Examples include elastic tensor and Eshelby tensor in solid mechanics, and Riemannian curvature tensor in relativity theory. The singular values and spectral norm of a general third order tensor are the square roots of the M-eigenvalues and spectral norm of a biquadratic tensor, respectively. The tensor product operation is closed for biquadratic tensors. All of these motivate us to study biquadratic tensors, biquadratic decomposition, and norms of biquadratic tensors. We show that the spectral norm and nuclear norm for a biquadratic tensor may be computed by using its biquadratic structure. Then, either the number of variables is reduced, or the feasible region can be reduced. We show constructively that for a biquadratic tensor, a biquadratic rank-one decomposition always exists, and show that the biquadratic rank of a biquadratic tensor is preserved under an independent biquadratic Tucker decomposition. We present a lower bound and an upper bound of the nuclear norm of a biquadratic tensor. Finally, we define invertible biquadratic tensors, and present a lower bound for the product of the nuclear norms of an invertible biquadratic tensor and its inverse, and a lower bound for the product of the nuclear norm of an invertible biquadratic tensor, and the spectral norm of its inverse.

Keywords Biquadratic tensor, nuclear norm, tensor product, biquadratic rank-one decomposition, biquadratic Tucker decomposition

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1 Introduction

In this paper, unless otherwise stated, all the discussion will be carried out in the field of real numbers.

Suppose that \( m \) and \( n \) are positive integers. Without loss of generality, we may assume that \( m \leq n \).

As in [7], we use \( \circ \) to denote the operation of tensor outer product. Then for \( x \in \mathbb{R}^m \) and \( y \in \mathbb{R}^n \), \( x \circ y \circ x \circ y \) is a fourth order rank-one tensor in \( \mathbb{R}^{m \times n \times m \times n} \). By the following definition, it is actually a biquadratic rank-one tensor.

**Definition 1** Let \( \mathbb{R}^{m \times n \times m \times n} \) be the space of fourth order tensors of dimension \( m \times n \times m \times n \). Let

\[
A = (a_{i_1 j_1 i_2 j_2}) \in \mathbb{R}^{m \times n \times m \times n}.
\]

The tensor \( A \) is called biquadratic if for all \( i_1, i_2 = 1, \ldots, m \) and \( j_1, j_2 = 1, \ldots, n \), we have

\[
a_{i_1 j_1 i_2 j_2} = a_{i_2 j_1 i_1 j_2} = a_{i_1 j_2 i_2 j_1}.
\]

The tensor \( A \) is called positive semi-definite if for any \( x \in \mathbb{R}^m \) and \( y \in \mathbb{R}^n \),

\[
\langle A, x \circ y \circ x \circ y \rangle = \sum_{i_1, i_2=1}^{m} \sum_{j_1, j_2=1}^{n} a_{i_1 j_1 i_2 j_2} x_{i_1} y_{j_1} x_{i_2} y_{j_2} \geq 0.
\]

The tensor \( A \) is called positive definite if for any \( x \in \mathbb{R}^m \), \( x^\top x = 1 \) and \( y \in \mathbb{R}^n \), \( y^\top y = 1 \),

\[
\langle A, x \circ y \circ x \circ y \rangle = \sum_{i_1, i_2=1}^{m} \sum_{j_1, j_2=1}^{n} a_{i_1 j_1 i_2 j_2} x_{i_1} y_{j_1} x_{i_2} y_{j_2} > 0.
\]

Denote the set of all biquadratic tensors in \( \mathbb{R}^{m \times n \times m \times n} \) by \( \text{BQ}(m, n) \). Then \( \text{BQ}(m, n) \) is a linear space.

Biquadratic tensors play a central role in many areas of science. Examples include the elastic tensor and the Eshelby tensor in solid mechanics, and the Riemannian curvature tensor in relativity theory. The elastic tensor may be the most well-known tensor in solid mechanics and engineering [9]. The Eshelby inclusion problem is one of the hottest topics in modern solid mechanics [18]. Furthermore, the Riemannian curvature tensor is the backbone of Einstein’s general relativity theory [13].

Biquadratic tensors have very special structures. The tensor product of two biquadratic tensors is still a biquadratic tensor. This makes them very special. Biquadratic tensors also have an M-eigenvalue structure. An important problem in solid mechanics is whether strong ellipticity condition holds or not [6,14]. In 2009, M-eigenvalues were introduced for the elastic tensor to characterize the strong ellipticity condition in [10]. An algorithm for computing the largest M-eigenvalue was presented in [15]. The biquadratic optimization problem
was studied in [8]. The M-eigenvalue structure was further extended to the Riemannian curvature tensor [16]. As the big data era arrived, the tensor completion problem came to the stage. It was shown that the nuclear norm of tensors plays an important role in the tensor completion problem [17]. A typical model in the tensor completion problem for higher order models is a general third order tensor [4,17]. The nuclear norm is the dual norm of the spectral norm [3,4,17]. The spectral norm of a tensor is its largest singular value. In [11], it was shown that if we make contraction of a third order tensor with itself on one index, then we get a positive semi-definite biquadratic tensor. A real number is a singular value of that third order tensor if and only if it is the square root of an M-eigenvalue of that positive semi-definite biquadratic tensor. Thus, the spectral norm of that third order tensor is the square root of the spectral norm of that positive semi-definite biquadratic tensor.

All of these make biquadratic tensors a research interest. In this paper, we study biquadratic tensors, biquadratic decomposition, and norms of biquadratic tensors. In the next section, we show that the spectral norm and nuclear norm for a biquadratic tensor may be computed by using its biquadratic structure. Then, either the number of variables is reduced, or the feasible region can be reduced. In Section 3, we show constructively that for a biquadratic tensor, a biquadratic rank-one decomposition always exists. This gives an upper bound for the biquadratic rank of a biquadratic tensor. In Section 4, we show that the biquadratic rank of a biquadratic tensor is preserved under an independent biquadratic Tucker decomposition. In Section 5, we present a lower bound and an upper bound of the nuclear norm of a biquadratic tensor. In Section 6, we define invertible biquadratic tensors, and present a lower bound for the product of the nuclear norms of an invertible biquadratic tensor and its inverse, and a lower bound for the product of the nuclear norm of an invertible biquadratic tensor, and the spectral norm of its inverse. Some final remarks are made in Section 7.

We use small letters $\lambda, x_i, u_i$, etc., to denote scalars, small bold italic letters $\mathbf{x}, \mathbf{u}, \mathbf{v}$, etc., to denote vectors, capital letters $A, B, C$, etc., to denote matrices, and calligraphic letters $\mathcal{A}, \mathcal{B}, \mathcal{C}$, etc., to denote tensors.

2 Norms and M-eigenvalues of biquadratic tensors

For a vector $\mathbf{u} = (u_1, \ldots, u_m)^\top$, we use $\|\mathbf{u}\|$ to denote its standard Euclidean norm. That is,

$$
\|\mathbf{u}\| := \sqrt{u_1^2 + \cdots + u_m^2}.
$$

For a tensor $\mathcal{A} \in \mathbb{R}^{m \times n \times m \times n}$, its spectral norm is defined as [3–5,17]

$$
\|\mathcal{A}\|_S := \max\{\langle \mathcal{A}, \mathbf{x} \circ \mathbf{y} \circ \mathbf{u} \circ \mathbf{v} \rangle : \|\mathbf{x}\| = \|\mathbf{y}\| = \|\mathbf{u}\| = \|\mathbf{v}\| = 1, \mathbf{x}, \mathbf{u} \in \mathbb{R}^m, \mathbf{y}, \mathbf{v} \in \mathbb{R}^n\}.
$$

We have the following theorem.
Theorem 1  Suppose $\mathcal{A} \in BQ(m,n)$. Then
\[
\|\mathcal{A}\|_S = \max\{\langle \mathcal{A}, x \circ y \circ x \circ y \rangle : \|x\| = \|y\| = 1, x \in \mathbb{R}^m, y \in \mathbb{R}^n\}. \tag{2}
\]

Proof  Suppose that the maximum of (1) is attained at $\bar{x}$, $\bar{y}$, $\bar{u}$, and $\bar{v}$. Then
\[
\|\mathcal{A}\|_S = \max\{\langle \mathcal{A}, x \circ y \circ u \circ v \rangle : \|x\| = \|u\| = 1, x, u \in \mathbb{R}^m\}.
\]

Note that this is a homogeneous quadratic optimization. Then there is a $\hat{x} \in \mathbb{R}^m$ such that
\[
\|\mathcal{A}\|_S = |\langle \mathcal{A}, \hat{x} \circ \hat{y} \circ \hat{x} \circ \hat{v} \rangle|.
\]

Then
\[
\|\mathcal{A}\|_S = \max\{\langle \mathcal{A}, \hat{x} \circ \hat{y} \circ \hat{x} \circ \hat{v} \rangle : \|y\| = \|v\| = 1, y, v \in \mathbb{R}^n\}.
\]

Again, this is a homogeneous quadratic optimization. Then there is a $\hat{y} \in \mathbb{R}^n$ such that
\[
\|\mathcal{A}\|_S = |\langle \mathcal{A}, \hat{x} \circ \hat{y} \circ \hat{x} \circ \hat{y} \rangle|.
\]

This proves (2). \qed

In this way, $\| \cdot \|_S$ also defines a norm in $BQ(m,n)$.

Recall that the nuclear norm of $\mathcal{A} \in \mathbb{R}^{m \times n \times m \times n}$ is defined as
\[
\|\mathcal{A}\|_* = \inf \left\{ \sum_{j=1}^{r} |\lambda_j| : \mathcal{A} = \sum_{j=1}^{r} \lambda_j x^{(j)} \circ y^{(j)} \circ u^{(j)} \circ v^{(j)}, \right. \\
|\|x^{(j)}\|| = |\|y^{(j)}\|| = |\|u^{(j)}\|| = |\|v^{(j)}\|| = 1, \\
x^{(j)}, u^{(j)} \in \mathbb{R}^m, y^{(j)}, v^{(j)} \in \mathbb{R}^n, r \in \mathbb{N} \right\}.
\]

By [3, Corollary 5.4], we have
\[
\|\mathcal{A}\|_* = \min \left\{ \sum_{j=1}^{r} |\lambda_j| : \mathcal{A} = \sum_{j=1}^{r} \lambda_j x^{(j)} \circ y^{(j)} \circ x^{(j)} \circ y^{(j)}, \right. \\
|\|x^{(j)}\|| = |\|y^{(j)}\|| = 1, x^{(j)} \in \mathbb{R}^m, y^{(j)} \in \mathbb{R}^n, r \in \mathbb{N} \right\}.
\]

It can be calculated as [3–5,17]
\[
\|\mathcal{A}\|_* := \max\{|\langle \mathcal{A}, \mathcal{B} \rangle| : \|\mathcal{B}\|_S = 1, \mathcal{B} \in \mathbb{R}^{m \times n \times m \times n}\}. \tag{3}
\]

For a biquadratic tensor, we have the following theorem.

Theorem 2  Suppose $\mathcal{A} \in BQ(m,n)$. Then
\[
\|\mathcal{A}\|_* = \max\{|\langle \mathcal{A}, \mathcal{B} \rangle| : \|\mathcal{B}\|_S = 1, \mathcal{B} \in BQ(m,n)\}. \tag{4}
\]
Proof Without loss of generality, assume that $\mathcal{A}$ is nonzero. Suppose that the maximum of (3) is attained at $\mathcal{B} = (b_{ij1_i j2_j}) \in \mathbb{R}^{m \times n \times m \times n}$ with $\|\mathcal{B}\|_S = 1$. Let $\hat{\mathcal{B}} = (\hat{b}_{ij1_i j2_j})$ with

$$\hat{b}_{ij1_i j2_j} = \frac{1}{4}(b_{ij1_i j2_j} + \bar{b}_{i'j1_i' j2_j} + \bar{b}_{i1 j2_i 1 j1} + \bar{b}_{i2 j2_i 1 j1}).$$

Then $\hat{\mathcal{B}} \in \text{BQ}(m, n)$, $\|\hat{\mathcal{B}}\|_S \leq 1$, and

$$\|\mathcal{A}\|_* = |\langle \mathcal{A}, \mathcal{B} \rangle| = |\langle \mathcal{A}, \hat{\mathcal{B}} \rangle|. $$

Since $\mathcal{A}$ is not a zero tensor, $\hat{\mathcal{B}}$ is also not a zero tensor. Then $\|\hat{\mathcal{B}}\|_S \neq 0$. Let

$$\tilde{\mathcal{B}} = \frac{\hat{\mathcal{B}}}{\|\hat{\mathcal{B}}\|_S}. $$

We have $\tilde{\mathcal{B}} \in \text{BQ}(m, n)$, $\|\tilde{\mathcal{B}}\|_S = 1$, and

$$|\langle \mathcal{A}, \tilde{\mathcal{B}} \rangle| \geq |\langle \mathcal{A}, \mathcal{B} \rangle|. $$

Since $\mathcal{B}$ is a maximizer of (3), we have

$$|\langle \mathcal{A}, \tilde{\mathcal{B}} \rangle| = |\langle \mathcal{A}, \mathcal{B} \rangle|. $$

This proves (4). \hfill \square

These two theorems show that we may compute the spectral norm and nuclear norm for a biquadratic tensor by using its biquadratic structure. Then, either the number of variables is reduced, or the feasible region of the maximization problem can be reduced. Furthermore, a biquadratic tensor has its own M-eigenvalue structure which is closely related to its spectral norm.

**Definition 2** Suppose $\mathcal{A} = (a_{ij1_i j2_j}) \in \text{BQ}(m, n)$. A real number $\lambda$ is called an M-eigenvalue of $\mathcal{A}$ if there are vectors $\mathbf{x} = (x_1, \ldots, x_m)^\top \in \mathbb{R}^m$, $\mathbf{y} = (y_1, \ldots, y_n)^\top \in \mathbb{R}^n$ such that the following equations are satisfied:

$$\sum_{i_2=1}^{m} \sum_{j_2=1}^{n} a_{ij1_i j2_j} y_{j1_j} x_{i2_i} y_{j2_j} = \lambda x_{i1}, \quad i_1 = 1, \ldots, m,$$

$$\sum_{i_1=1}^{m} \sum_{j_1=1}^{n} a_{ij1_i j2_j} x_{i1_i} x_{i2_i} y_{j2_j} = \lambda y_{j1}, \quad j_1 = 1, \ldots, n,$$

and

$$\mathbf{x}^\top \mathbf{x} = \mathbf{y}^\top \mathbf{y} = 1.$$ 

Then $\mathbf{x}$ and $\mathbf{y}$ are called the corresponding M-eigenvectors.
Theorem 3 Suppose $\mathcal{A} = (a_{i_1,j_1 i_2 j_2}) \in \text{BQ}(m,n)$. Then its $M$-eigenvalues always exist. The spectral norm of $\mathcal{A}$ is equal to the largest absolute value of its $M$-eigenvalues. Furthermore, $\mathcal{A}$ is positive semi-definite if and only if all of its $M$-eigenvalues are nonnegative; $\mathcal{A}$ is positive definite if and only if all of its $M$-eigenvalues are positive. If $\mathcal{A}$ is positive semi-definite, then its spectral norm is equal to its largest $M$-eigenvalue.

This theorem was proved in [11].

For $m = n = 3$, the elastic tensor in solid mechanics falls in the category of biquadratic tensors, with one additional symmetric properties between indices $i_1$ and $j_1$. Then, the positive definiteness condition of $\mathcal{A}$ corresponds to the strong ellipticity condition in solid mechanics.

3 Biquadratic rank-one decomposition

Let $\mathcal{A} \in \text{BQ}(m,n)$. Then $\mathcal{A}$ has a rank-one decomposition in the form

$$\mathcal{A} = \sum_{k=1}^{r} x^{(k)} \circ y^{(k)} \circ s^{(k)} \circ w^{(k)},$$

where

$$x^{(k)}, s^{(k)} \in \mathbb{R}^m, \quad y^{(k)}, w^{(k)} \in \mathbb{R}^n, \quad k = 1, \ldots, r.$$ 

The smallest $r$ for such a rank-one decomposition is called the rank of $\mathcal{A}$.

On the other hand, if we have

$$\mathcal{A} = \sum_{k=1}^{r} x^{(k)} \circ y^{(k)} \circ x^{(k)} \circ y^{(k)}, \quad (5)$$

where $x^{(k)} \in \mathbb{R}^m, y^{(k)} \in \mathbb{R}^n$ for $k = 1, \ldots, r$ for some positive integer $r$, then we say that $\mathcal{A}$ has a biquadratic rank-one decomposition. The smallest $r$ for such a biquadratic rank-one decomposition is called the biquadratic rank of $\mathcal{A}$. Denote it by $\text{BR}(\mathcal{A})$. The question is whether such a biquadratic rank-one decomposition always exists. We may follow the approach in [1] to show this by introducing biquadratic polynomials and use them as a tool for the proof. [3, Corollary 5.4] also implies this. Here, we give a constructive proof. This also gives an upper bound of the biquadratic rank of a biquadratic tensor.

Theorem 4 For $\mathcal{A} = (a_{i_1,j_1 i_2 j_2}) \in \text{BQ}(m,n)$, such a biquadratic rank-one decomposition always exists. We also have

$$\text{BR}(\mathcal{A}) \leq \frac{1}{2} mn \min\{m(m+1),n(n+1)\}.$$ 

Proof For $\mathcal{A} = (a_{i_1,j_1 i_2 j_2}) \in \text{BQ}(m,n)$, define a matrix

$$P = (p_{st}) \in \mathbb{R}^{\frac{m(m+1)}{2} \times \frac{n(n+1)}{2}}.$$
by

\[ p_{st} = a_{i_1 j_1 i_2 j_2}, \]

for

\[ s = \frac{i_1(i_1 - 1)}{2} + i_2, \quad t = \frac{j_1(j_1 - 1)}{2} + j_2, \]

with

\[ 1 \leq i_2 \leq i_1 \leq m, \quad 1 \leq j_2 \leq j_1 \leq n. \]

Then \( P \) has a singular value decomposition

\[ P = \sum_{k=1}^{q} \sigma_k u^{(k)} (v^{(k)})^\top, \]

where

\[ u^{(k)} \in \mathbb{R}^{m(m+1)/2}, \quad v^{(k)} \in \mathbb{R}^{n(n+1)/2}, \quad \|u^{(k)}\| = \|v^{(k)}\| = 1, \quad k = 1, \ldots, q, \]

and

\[ q \leq \frac{1}{2} \min\{m(m + 1), n(n + 1)\}. \]

For \( u^{(k)} \in \mathbb{R}^{m(m+1)/2} \), we may fold it to a symmetric matrix \( U^{(k)} \in \mathbb{R}^{m \times m} \). Similarly, for \( v^{(k)} \in \mathbb{R}^{n(n+1)/2} \), we may fold it to a symmetric matrix \( V^{(k)} \in \mathbb{R}^{n \times n} \). Suppose that \( U^{(k)} \) has an eigenvalue decomposition

\[ U^{(k)} = \sum_{l_u=1}^{m} \lambda_{l_u} x^{(k,l_u)} (x^{(k,l_u)})^\top, \]

where

\[ x^{(k,l_u)} \in \mathbb{R}^{m}, \quad \|x^{(k,l_u)}\| = 1, \quad l_u = 1, \ldots, m, \quad k = 1, \ldots, q. \]

Similarly, suppose that \( V^{(k)} \) has an eigenvalue decomposition

\[ V^{(k)} = \sum_{l_v=1}^{n} \mu_{l_v} y^{(k,l_v)} (y^{(k,l_v)})^\top, \]

where

\[ y^{(k,l_v)} \in \mathbb{R}^{n}, \quad \|y^{(k,l_v)}\| = 1, \quad l_v = 1, \ldots, n, \quad k = 1, \ldots, q. \]

Then we have

\[ \mathcal{A} = \sum_{k=1}^{q} \sum_{l_u=1}^{m} \sum_{l_v=1}^{n} \sigma_k \lambda_{l_u} \mu_{l_v} x^{(k,l_u)} \circ y^{(k,l_v)} \circ x^{(k,l_u)} \circ y^{(k,l_v)}. \]

We have the conclusions. \( \square \)
Clearly, the biquadratic rank of a biquadratic tensor is always not less than its rank. In which cases are these two ranks equal? We do not go further discussion on this in this paper.

Let \( \mathcal{A} = (a_{i_1j_1i_2j_2}) \in \text{BQ}(m,n) \). Fix \( j_1, i_2, \) and \( j_2 \). Then we have an \( m \)-vector \( a_{j_1i_2j_2} \). Denote by \( A^{(1)} \) the \( m \times mn^2 \) matrix whose column vectors are such \( m \)-vectors for \( i_2 = 1, \ldots, m \) and \( j_1, j_2 = 1, \ldots, n \). Then \( A^{(1)} \) is the matrix flattening of \( \mathcal{A} \) by the first index. Here, we do not specify the order of such column vectors in \( A^{(1)} \) as this is not related. Denote the rank of \( A^{(1)} \) by \( R_1(\mathcal{A}) \).

We may define \( R_2(\mathcal{A}), R_3(\mathcal{A}), \) and \( R_4(\mathcal{A}) \), respectively. They are the Tucker ranks of \( \mathcal{A} \) [5,7]. Then we have

\[
R_1(\mathcal{A}) = R_3(\mathcal{A}), \quad R_2(\mathcal{A}) = R_4(\mathcal{A}).
\]

Hence, only \( R_1(\mathcal{A}) \) and \( R_2(\mathcal{A}) \) are independent. We also have

\[
R_1(\mathcal{A}) \leq m, \quad R_2(\mathcal{A}) \leq n.
\]

Suppose that \( \mathcal{A} \) has a biquadratic rank-one decomposition as (5). Denote \( X \) as an \( m \times r \) matrix, whose column vectors are \( x^{(1)}, \ldots, x^{(r)} \), and \( Y \) as an \( n \times r \) matrix, whose column vectors are \( y^{(1)}, \ldots, y^{(r)} \). Then, as in [7], we may denote (5) as

\[
\mathcal{A} = [[X,Y]]_{\text{BQ}}.
\]  

(6)

Let \( \mathcal{A} = (a_{i_1j_1i_2j_2}) \in \text{BQ}(m,n) \). Suppose that \( \mathcal{A} \) has a biquadratic rank-one decomposition (6). Denote the ranks of \( X \) and \( Y \) by \( R(X) \) and \( R(Y) \), respectively. Then we have

\[
R(X) = R_1(\mathcal{A}), \quad R(Y) = R_2(\mathcal{A}).
\]

4 Biquadratic Tucker decomposition

We may also extend Tucker decomposition [2,5,7] to biquadratic Tucker decomposition. Denote \( \times_k \) as the mode-\( k \) (matrix) product [2,5,7].

**Definition 3** Let \( \mathcal{A} \in \text{BQ}(m,n) \). Suppose that there are \( \mathcal{B} \in \text{BQ}(d_1,d_2), \ P \in \mathbb{R}^{m \times d_1} \) and \( Q \in \mathbb{R}^{n \times d_2} \) such that

\[
\mathcal{A} = \mathcal{B} \times_1 P \times_2 Q \times_3 P \times_4 Q := [[\mathcal{B};P,Q]]_{\text{BQ}}.
\]  

(7)

Then (7) is called a biquadratic Tucker decomposition of \( \mathcal{A} \). The tensor \( \mathcal{B} \) is called a biquadratic Tucker core of \( \mathcal{A} \). The matrices \( P \) and \( Q \) are called the factor matrices of this decomposition. A biquadratic Tucker decomposition is said to be independent if \( P \) and \( Q \) have full column rank. A biquadratic Tucker decomposition is said to be orthonormal if \( P \) and \( Q \) have orthonormal columns.

Note that if the biquadratic Tucker decomposition (7) is independent, then \( d_1 \leq m \) and \( d_2 \leq n \).
De Lathauwer et al. [2] proposed an algorithm to compute Tucker decomposition (HOSVD) for a given tensor. If we apply their algorithm to a biquadratic tensor, since the first and the third matrix flattenings are the same, the second and the fourth flattenings are the same, we obtain an orthonormal biquadratic Tucker decomposition.

A biquadratic Tucker decomposition is a Tucker decomposition [5,7]. Thus, a biquadratic Tucker core has the properties of a Tucker core. For example, the rank of a biquadratic Tucker core $\mathcal{B}$ is the same as the rank of $\mathcal{A}$, if the biquadratic Tucker decomposition is independent [5]. Similarly, the Tucker ranks will also be preserved by an independent biquadratic Tucker decomposition. The problem is whether some biquadratic properties, such as the biquadratic rank, and M-eigenvalues will be preserved or not.

We now prove the following theorem.

**Theorem 5** Suppose that $\mathcal{A} \in \text{BQ}(m,n)$ has a biquadratic Tucker decomposition (7) and it is independent. Then

$$\text{BR}(\mathcal{A}) = \text{BR}(\mathcal{B}).$$

We first prove a lemma.

**Lemma 1** If the biquadratic Tucker decomposition (7) is independent, then there are $\hat{P} \in \mathbb{R}^{d_1 \times m}$ and $\hat{Q} \in \mathbb{R}^{d_2 \times n}$ such that

$$\mathcal{B} = \mathcal{A} \times_1 \hat{P} \times_2 \hat{Q} \times_3 \hat{P} \times_4 \hat{Q} := [[\mathcal{A}; \hat{P}, \hat{Q}]]_{\text{BQ}}. \quad (8)$$

**Proof** Let

$$\hat{P} = (P^\top P)^{-1} P^\top, \quad \hat{Q} = (Q^\top Q)^{-1} Q^\top.$$

Then the conclusion follows. $\square$

**Proof of Theorem 5** Suppose that $\mathcal{B}$ has a biquadratic rank-one decomposition

$$\mathcal{B} = \sum_{k=1}^{r} \hat{x}^{(k)} \circ \hat{y}^{(k)} \circ \hat{x}^{(k)} \circ \hat{y}^{(k)}, \quad (9)$$

where

$$\hat{x}^{(k)} \in \mathbb{R}^{d_1}, \quad \hat{y}^{(k)} \in \mathbb{R}^{d_2}, \quad k = 1, \ldots, r.$$

Then $\mathcal{A}$ has a biquadratic rank-one decomposition (5) with

$$x^{(k)} = P \hat{x}^{(k)}, \quad y^{(k)} = Q \hat{y}^{(k)}, \quad k = 1, \ldots, r.$$

This shows that

$$\text{BR}(\mathcal{A}) \leq \text{BR}(\mathcal{B}).$$

Since the biquadratic Tucker decomposition (7) is independent, by Lemma 1, we have (8). Thus, if $\mathcal{A}$ has a biquadratic rank-one decomposition (5), then $\mathcal{B}$ has a biquadratic rank-one decomposition (9), with

$$\hat{x}^{(k)} = \hat{P} x^{(k)}, \quad \hat{y}^{(k)} = \hat{Q} y^{(k)}, \quad k = 1, \ldots, r.$$
This shows that
\[ \text{BR}(\mathcal{A}) \geq \text{BR}(\mathcal{B}). \]

Hence, we finish the proof. \( \square \)

Jiang et al. [5] proved the following theorem.

**Theorem 6** [5, Theorem 7] Suppose that \( \mathcal{A} \in \text{BQ}(m,n) \) has a biquadratic Tucker decomposition (7) and it is orthonormal. Then an M-eigenvalue of \( \mathcal{B} \) is an M-eigenvalue of \( \mathcal{A} \), and a nonzero M-eigenvalue of \( \mathcal{A} \) is also an M-eigenvalue of \( \mathcal{B} \).

By Theorem 3, this shows that the spectral norm is preserved under an orthonormal biquadratic Tucker decomposition.

Thus, biquadratic Tucker decomposition has better properties. It only involves two factor matrices \( P \) and \( Q \). This makes it much simple.

## 5 Lower and upper bounds of nuclear norm of a biquadratic tensor

Let \( \mathcal{A} = (a_{i_1j_1i_2j_2}) \in \mathbb{R}^{m \times n \times m \times n} \). If we regard \( i_1j_1 \) as an index from 1 to \( mn \), and regard \( i_2j_2 \) as another index from 1 to \( mn \), then we have a matrix flattening \( M = M(\mathcal{A}) \in \mathbb{R}^{mn \times mn} \). Then there is a one-to-one relation between \( \mathcal{A} \in \mathbb{R}^{mn \times mn} \) and \( M \in \mathbb{R}^{mn \times mn} \). Hence, we may also write \( \mathcal{A} = M(\mathcal{A}) \) for \( \mathcal{A} \in \mathbb{R}^{mn \times mn} \). If \( M \in \mathbb{R}^{mn \times mn} \) is diagonal, then we also say that \( \mathcal{A} = M(\mathcal{A}) \) is diagonal. In particular, if \( M \) is the identity matrix \( I_{mn} \in \mathbb{R}^{mn \times mn} \), then we denote \( \mathcal{A}(I_{mn}) \) as \( \mathcal{I}_{m,n} \) and call it the identity tensor in \( \mathbb{R}^{m \times n \times m \times n} \).

In the other words, for a fourth order tensor \( \mathcal{A} = (a_{i_1j_1i_2j_2}) \in \mathbb{R}^{m \times n \times m \times n} \), an entry \( a_{i_1j_1i_2j_2} \) is called a diagonal entry if \( i_1 = i_2 \) and \( j_1 = j_2 \). Otherwise, it is called an off-diagonal entry. Then a diagonal tensor in \( \mathbb{R}^{m \times n \times m \times n} \) is a biquadratic tensor in \( \text{BQ}(m,n) \) such that all of its off-diagonal entries are 0, while the identity tensor \( \mathcal{I}_{m,n} \in \mathbb{R}^{m \times n \times m \times n} \) is the diagonal biquadratic tensor in \( \text{BQ}(m,n) \) such that all of its diagonal entries are 1.

Denote the Frobenius norm of a fourth order tensor \( \mathcal{A} \in \mathbb{R}^{m \times n \times m \times n} \) by \( \|\mathcal{A}\|_2 \), and the Frobenius norm of \( M \in \mathbb{R}^{mn \times mn} \) by \( \|M\|_2 \). For \( \mathcal{A}, \mathcal{B} \in \mathbb{R}^{m \times n \times m \times n} \), we use \( \langle M(\mathcal{A}), M(\mathcal{B}) \rangle \) to denote the inner product of matrices \( M(\mathcal{A}) \) and \( M(\mathcal{B}) \). Then for \( \mathcal{A}, \mathcal{B} \in \mathbb{R}^{m \times n \times m \times n} \), we have

\[
\langle \mathcal{A}, \mathcal{B} \rangle = \langle M(\mathcal{A}), M(\mathcal{B}) \rangle,
\]

and
\[
\|\mathcal{A}\|_2 = \|M(\mathcal{A})\|_2.
\]

We first prove a proposition.

**Proposition 1** Let \( \mathcal{A} = (a_{i_1j_1i_2j_2}) \in \mathbb{R}^{m \times n \times m \times n} \). Then
\[
\|M(\mathcal{A})\|_S \geq \|\mathcal{A}\|_S.
\]
Proof Suppose \( x \in \mathbb{R}^m \) and \( y \in \mathbb{R}^n \). Let \( x \otimes y \) be the Kronecker product of \( x \) and \( y \). Then \( x \otimes y \in \mathbb{R}^{mn} \). If \( \|x\| = \|y\| = 1 \), then \( \|x \otimes y\| = 1 \).

By Theorem 1, we have
\[
\|\mathcal{A}\|_S = \max\{|\langle \mathcal{A}, x \otimes y \circ u \circ v \rangle|: \|x\| = \|y\| = \|u\| = \|v\| = 1, x, u, y, v \in \mathbb{R}^n\} = \max\{|\langle M(\mathcal{A}), (x \otimes y) \circ (u \otimes v) \rangle|: \|x\| = \|y\| = \|u\| = \|v\| = 1, x, u \in \mathbb{R}^m, y, v \in \mathbb{R}^n\} \leq \max\{|\langle M(\mathcal{A}), z \circ w \rangle|: \|z\| = \|w\| = 1, z, w \in \mathbb{R}^{mn}\} = \|M(\mathcal{A})\|_S.
\]

This proves the proposition. \( \square \)

If \( \mathcal{A} = (a_{ij}) \in \text{BQ}(m,n) \), then its matrix flattening \( M(\mathcal{A}) \) is symmetric. We now have the following theorem.

**Theorem 7** Suppose that \( \mathcal{A} = (a_{ij}) \in \text{BQ}(m,n) \) and \( M = M(\mathcal{A}) \) is its symmetric matrix flattening. Then
\[
\|M\|_* \leq \|\mathcal{A}\|_* \leq \min\{m,n\}\|M\|_*.
\] (11)

In particular, if \( \mathcal{A} \) is diagonal, then we have
\[
\|\mathcal{A}\|_* = \|M\|_* = \sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|.
\] (12)

**Proof** We first prove the first inequality of (11). By (3), we have
\[
\|\mathcal{A}\|_* = \max\{|\langle \mathcal{A}, \mathcal{B} \rangle|: \|\mathcal{B}\|_S = 1, \mathcal{B} \in \mathbb{R}^{m \times n \times m \times n}\} = \max\{|\langle \mathcal{A}, \mathcal{B} \rangle|: \|\mathcal{B}\|_S \leq 1, \mathcal{B} \in \mathbb{R}^{m \times n \times m \times n}\} = \max\{|\langle M(\mathcal{A}), M(\mathcal{B}) \rangle|: \|M(\mathcal{B})\|_S \leq 1, \mathcal{B} \in \mathbb{R}^{m \times n \times m \times n}\} \geq \max\{|\langle M, M(\mathcal{B}) \rangle|: \|M(\mathcal{B})\|_S \leq 1, \mathcal{B} \in \mathbb{R}^{m \times n \times m \times n}\} = \max\{|\langle M, B \rangle|: \|B\|_S \leq 1, B \in \mathbb{R}^{mn \times mn}\} = \|M\|_*,
\]
where the third equality is due to (10), the inequality is by Proposition 1, and the last equality is by the definition of the nuclear norm of a matrix.

We now prove the second inequality of (11). Since \( M(\mathcal{A}) \in \mathbb{R}^{mn \times mn} \) is symmetric, we may assume that \( M(\mathcal{A}) \) has an eigenvalue decomposition
\[
M(\mathcal{A}) = \sum_{k=1}^{mn} \lambda_k z^{(k)}(z^{(k)})^\top,
\]
where \( z^{(k)} \in \mathbb{R}^{mn} \) and \( \|z^{(k)}\| = 1 \) for \( k = 1, \ldots, mn \). For each \( k \), \( z^{(k)} \) corresponds to an \( m \times n \) matrix \( M(z^{(k)}) \). Since \( \|z^{(k)}\| = 1 \), we have \( \|M(z^{(k)})\|_2 = 1 \). Then, by [4, Lemma 5.1], we have
\[
\|M(z^{(k)})\|_* \leq \sqrt{\min\{m,n\}}.
\]
On the other hand,
\[
\|\overline{M}(z^{(k)})\|_* = \sum_{l=1}^{\min\{m,n\}} |\sigma_{k,l}|,
\]
where \(\sigma_{k,l}\) for \(l = 1, \ldots, \min\{m,n\}\) are singular values of \(\overline{M}(z^{(k)})\) for \(k = 1, \ldots, mn\). Then \(\overline{M}(z^{(k)})\) has a singular value decomposition
\[
\overline{M}(z^{(k)}) = \sum_{l=1}^{\min\{m,n\}} \sigma_{k,l} x^{(k,l)} (y^{(k,l)})^\top,
\]
where
\[
x^{(k,l)} \in \mathbb{R}^m, \quad \|x^{(k,l)}\| = 1, \quad k = 1, \ldots, mn, \quad l = 1, \ldots, \min\{m,n\}.
\]
\[y^{(k,l)} \in \mathbb{R}^n, \quad \|y^{(k,l)}\| = 1, \quad k = 1, \ldots, mn, \quad l = 1, \ldots, \min\{m,n\}.
\]
This implies
\[
\sum_{l=1}^{\min\{m,n\}} |\sigma_{k,l}| \leq \sqrt{\min\{m, n\}}, \quad k = 1, \ldots, mn.
\]
Then we have
\[
\mathcal{A} = \sum_{k=1}^{mn} \lambda_k \overline{M}(z^{(k)}) \circ \overline{M}(z^{(k)})
\]
\[
= \sum_{k=1}^{mn} \lambda_k \left( \sum_{l=1}^{\min\{m,n\}} \sigma_{k,l} x^{(k,l)} \circ y^{(k,l)} \right) \circ \left( \sum_{l=1}^{\min\{m,n\}} \sigma_{k,l} x^{(k,l)} \circ y^{(k,l)} \right)
\]
\[
= \sum_{k=1}^{mn} \sum_{l,s=1}^{\min\{m,n\}} \sigma_{k,l} \sigma_{k,s} x^{(k,l)} \circ y^{(k,l)} \circ x^{(k,s)} \circ y^{(k,s)}.
\]
Thus,
\[
\|\mathcal{A}\|_* \leq \sum_{k=1}^{mn} \sum_{l,s=1}^{\min\{m,n\}} |\lambda_k \sigma_{k,l} \sigma_{k,s}| \leq \sum_{k=1}^{mn} |\lambda_k| \min\{m,n\} = \min\{m,n\} \|M(\mathcal{A})\|_*.
\]
Finally, assume that \(\mathcal{A}\) is diagonal. Then
\[
\mathcal{A} = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} e^{(i)} \circ \overline{e}^{(j)} \circ e^{(i)} \circ \overline{e}^{(j)},
\]
where \(e^{(i)}\) for \(i = 1, \ldots, m\) are the unit vectors in \(\mathbb{R}^m\), while \(\overline{e}^{(j)}\) for \(j = 1, \ldots, n\) are the unit vectors in \(\mathbb{R}^n\). This implies that
\[
\|\mathcal{A}\|_* \leq \sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}| = \|M(\mathcal{A})\|_*.
\]
Then, by (11), we have (12).

This proves the theorem. □

Comparing Theorem 7 with [4, Theorem 5.2], our theorem is somewhat stronger.

Theorem 7 says that the equality in the first inequality of (11) may hold. How about the second inequality of (11)?

Corollary 1

\[ \| I_{m,n} \|_* = mn. \]

6 Norms of tensor products of biquadratic tensors

We may define products of two biquadratic tensors. Let

\[ \mathcal{A} = (a_{i_1 j_1 i_2 j_2}), \quad B = (b_{i_1 j_1 i_2 j_2}) \in BQ(m,n). \]

Then we have

\[ \mathcal{C} = (c_{i_1 j_1 i_2 j_2}) := \mathcal{A} B \in BQ(m,n), \]

defined by

\[ c_{i_1 j_1 i_2 j_2} = \sum_{i_3=1}^{m} \sum_{j_3=1}^{n} a_{i_1 j_1 i_3 j_3} b_{i_3 j_3 i_2 j_2}, \quad i_1, i_2 = 1, \ldots, m, j_1, j_2 = 1, \ldots, n. \]

Then, for any \( \mathcal{A} \in BQ(m,n) \),

\[ \mathcal{A} I_{m,n} = I_{m,n} \mathcal{A} = \mathcal{A}. \]

If \( \mathcal{A}, \mathcal{B} \in BQ(m,n) \) and \( \mathcal{A} B = I_{m,n} \), then we also have \( \mathcal{B} \mathcal{A} = I_{m,n} \) and we denote \( \mathcal{A}^{-1} = \mathcal{B} \).

We have the following proposition.

**Proposition 2** For any \( \mathcal{A}, \mathcal{B} \in BQ(m,n) \), we have

\[ \| \mathcal{A} B \|_* \leq \| \mathcal{A} \|_* \| \mathcal{B} \|_* . \]

This proposition may be proved directly. It may also be regarded as a special case of [12, Theorem 2.1]. Hence, we do not prove it here.

By Proposition 2 and Corollary 1, we have the following proposition.

**Proposition 3** Suppose that \( \mathcal{A} \in BQ(m,n) \) is invertible. Then we have

\[ \| \mathcal{A} \|_* \| \mathcal{A}^{-1} \|_* \geq mn. \]

In general, for \( \mathcal{A}, \mathcal{B} \in BQ(m,n) \), we may not have

\[ \| \mathcal{A} B \|_S \leq \| \mathcal{A} \|_S \| \mathcal{B} \|_S, \]
see [12, Example 4.1]. On the other hand, for any $\mathcal{A}, \mathcal{B} \in BQ(m,n)$, by [12, Theorem 4.2], we have

$$\|\mathcal{A}\mathcal{B}\|_S \leq \|\mathcal{A}\|_* \|\mathcal{B}\|_S.$$ 

By definition, it is easy to see that

$$\|\mathcal{I}_{m,n}\|_S = 1.$$ 

From these, we have the following proposition.

**Proposition 4** Suppose that $\mathcal{A} \in BQ(m,n)$ is invertible. Then we have

$$\|\mathcal{A}\|_* \|\mathcal{A}^{-1}\|_S \geq 1.$$ 

7 Final remarks

Viewing the importance and the special structure properties of biquadratic tensors, we hope that we may explore more at this direction.

Our study can be extended to the field of complex numbers without difficulties.

Our study can also be extended to bisymmetric tensors. A former definition for bisymmetric tensors are as follows.

**Definition 4** Let $p$ be a positive integer. Let $\mathbb{R}^{n_1 \times \cdots \times n_p \times n_1 \times \cdots \times n_p}$ be the space of $2p$th order tensors of dimension $n_1 \times \cdots \times n_p \times n_1 \times \cdots \times n_p$. Let $\mathcal{A} = (a_{i_1 \cdots i_p j_1 \cdots j_p}) \in \mathbb{R}^{n_1 \times \cdots \times n_p \times n_1 \times \cdots \times n_p}$. The tensor $\mathcal{A}$ is called bisymmetric if for all $i_k, j_k = 1, \ldots, n_k$, $k = 1, \ldots, p$, we have

$$a_{i_1 \cdots i_p j_1 \cdots j_p} = a_{i_1 \cdots i_{k-1} j_k i_{k+1} \cdots i_p j_1 \cdots j_{k-1} i_k j_{k+1} \cdots j_p}.$$ 

Then for $p = 1$, we have symmetric matrices, and for $p = 2$, we have biquadratic tensors.

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