On the discrete spectrum of non-analytic matrix-valued Friedrichs model

I. A. Ikramov, F. Sharipov

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Abstract

We have found the sufficient conditions for the spectrum of matrix-valued Friedrichs model to be finite.

1 Introduction

Consider a self-adjoint operator \( H \) in the Hilbert space \( L^2(T^\nu, \mathbb{C}^n) \) given by the formula

\[
(Hf)(x) = U(x)f(x) + \int_{T^\nu} K(x, y)f(y) \, dy
\]

where \( T^\nu \) is a \( \nu \)-dimensional torus, \( \mathbb{C}^n \) is the \( n \)-dimensional complex Euclidean space, \( L^2(T^\nu, \mathbb{C}^n) \) is the complex Hilbert space of the square-integrable (with respect to the norm) functions on \( T^\nu \) taking values in \( \mathbb{C}^n \), \( U(x) \) and \( K(x, y) \) are continuous functions on \( T^\nu \) and \( T^\nu \times T^\nu \) respectively with values in the space \( M_n(\mathbb{C}) \) of complex \( n \times n \)-matrices, satisfying the following conditions:

\[
U^*(x) = U(x), \quad K^*(x, y) = K(y, x),
\]

where * denotes the adjoint matrix.

Operator of the form (1) for the first time was treated by Friedrichs [1], [2] in the case \( \nu = 1, n = 1, U(x) = x \) as a simple model of the perturbation theory for continuous spectrum. His results were further developed by O. A. Ladyzhenskaya and L. D. Faddeev [3], L. D. Faddeev [4], S. N. Lakaev [5], [6], etc.

In the case when \( U(x) \) and \( K(x, y) \) are matrix-valued analytic functions the spectral properties of the operator \( H \) were studied in [3], [4], [5], [6]. In particular the following theorem was proved:
Theorem 1. [3], [4]. Let $U(x) = (\delta_{ij} u_j(x))$, $K(x, y) = (K_{ij}(x, y))$ where $\delta_{ij}$ is the Kronecker symbol and suppose that the matrix elements $u_j(x)$, $K_{ij}(x, y)$ $(i, j = 1, 2, \ldots, n)$ are real-valued analytic functions on $T^\nu$ and on $T^\nu \times T^\nu$ respectively. If one of the following two conditions is valid then the operator $H$ defined by (1) has a finite number of eigenvalues (counted with multiplicities) outside the essential spectrum.

1. $\nu = 1$ and for every $j = 1, 2, \ldots, n$ all critical points of the functions $u_j(x)$ are isolated;
2. $\nu \geq 1$ and for every $j = 1, 2, \ldots, n$ all critical points of the functions $u_j(x)$ are non-degenerate.

In the present paper we show the finiteness of the discrete spectrum of the operator $H$ for a more wide class of functions $U(x)$ and $K(x, y)$.

2 Main theorem

Denote by $\sigma_{ess}(H)$ the essential spectrum of the operator $H$ and denote by $\Delta(x, z)$ the determinant of the matrix $U(x) - z E$, where $E$ is the unit matrix and $z \in \mathbb{C}$.

Lemma 2. The essential spectrum of $H$ is of the form

$$\sigma_{ess}(H) = \bigcup_{x \in T^\nu} \{ z \in \mathbb{C} : \Delta(x, z) = 0 \} = \bigcup_{i=1}^k [m_i, M_i]$$

with mutually non-intersecting segments $[m_i, M_i]$ $(i = 1, 2, \ldots, k)$.

Lemma 2 follows from the Weyl’s theorem about the essential spectrum and from the minimax principle (cf. [13]).

Denote now by $\Gamma$ the set

$$\Gamma = \{ m_1, m_2, \ldots, m_k, M_1, M_2, \ldots, M_k \}$$

Lemma 3. For any $z \in \Gamma$ the value $A = 0$ is globally extremal value for the continuous real-valued function $\varphi_z(x) = \Delta(x, z)$ on the torus $T^\nu$.

Definition 1. Let $\varphi(x)$ be a continuous real-valued function on the torus $T^\nu$. Extremal point $x^0 \in T^\nu$ of the function $\varphi(x)$ is called a point of finite multiplicity if there exist such numbers $m > 0$, $c > 0$ and such neighbourhood $V(x^0)$ of the point $x^0$ that for any $x \in V(x^0)$ the following inequality holds

$$|\varphi(x) - \varphi(x^0)| \geq c|x - x^0|^m, \quad \text{where} \quad |x - x^0|^2 = \sum_{i=1}^n (x_i - x_i^0)^2.$$

(3)
Else the extremal point \( x^0 \) is called a point of infinite multiplicity. The exact lower bound of the set of numbers \( m > 0 \) satisfying the condition (3) is called multiplicity of the extremal point \( x^0 \) and is denoted by \( m(x^0) \). Multiplicity of an extremal value \( A \) of the function \( \varphi(x) \) is the sum of multiplicities of all extremal points from the inverse image \( \varphi^{-1}(A) \) of \( A \).

Denote by \( C^{\alpha+0}(T^\nu \times T^\nu, M_n(C)) \) the space of the matrix-valued functions \( K(x, y) \) on \( T^\nu \times T^\nu \) such that for any multiindex \( \beta \) with \( |\beta| \leq |\alpha| \) the derivative \( K^{(\beta)}(x, y) \) satisfies the Hölder condition with index \( \{\alpha\} + 0 \) where \( \{\alpha\} \) is the entire part of \( \alpha \) and \( \{\alpha\} = \alpha - [\alpha] \).

**Theorem 4.** Let \( 0 < \mu < \infty \). Suppose that for any \( z \in \Gamma \) number \( A = 0 \) is the extremal value of the function \( \varphi_z(x) = \Delta(x, z) \) of the multiplicity \( \leq \mu \). Let the function \( K(x, y) \) belong to the class \( C^{2\mu-\nu/2+0}(T^\nu \times T^\nu, M_n(C)) \). Then the operator \( H \) has only a finite number of eigenvalues (counted with multiplicities) outside the essential spectrum.

**Proof.** of this theorem consists of the following lemmas.

**Lemma 5.** Let \( z_0 \in \Gamma \). If the matrix-valued function \( \Delta^{-1}(x, z_0) K(x, y) \) is square-integrable with respect to the norm on \( T^\nu \times T^\nu \), then there exists a positive number \( \varepsilon = \varepsilon(z_0) \) such that the operator \( H \) defined by (3) has only a finite number of eigenvalues (counted with multiplicities) in the set \( (z_0 - \varepsilon, z_0 + \varepsilon) \setminus \sigma_{\text{ess}}(H) \).

**Proof.** By the Fredholm theorem it is sufficient to show that \( z_0 \) is not a limit point of the discrete spectrum of the operator \( H \). Suppose that it is not so, i.e. there exists a sequence \( \{z_n\} \) of eigenvalues \( z_n \notin \sigma_{\text{ess}}(H) \), converging to \( z_0 \) and let \( f_n \) be a normed eigenfunction of the operator \( H \) corresponding to the eigenvalue \( z_n \), i.e. a solution of the equation

\[
(U(x) - z_n E)f_n(x) + \int_{T^\nu} K(x, y) f_n(y) \, dy = 0.
\]

Consider a sequence of operators

\[
(\hat{K}(z_n)f_n)(x) = \int_{T^\nu} (U(x) - z_n E)^{-1} K(x, y) f(y) \, dy, \quad n = 1, 2, \ldots
\]

By supposition \( \hat{K}(z_n) \) is a compact operator and

\[
\lim_{n \to \infty} \hat{K}(z_n) = \hat{K}(z_0)
\]

in the uniform operator topology, hence the operator \( \hat{K}(z_0) \) is also compact. Put \( F = \{f_n : n = 1, 2, \ldots\} \). As the set \( \hat{K}(z_0)F \) is precompact and

\[
f_n(x) = -\int_{T^\nu} (U(x) - z_n E)^{-1} K(x, y) f_n(y) \, dy = - (\hat{K}(z_n)f_n)(x), \quad n = 1, 2, \ldots
\]
so by (5) the set $F$ is also precompact. It contradicts the orthonormality of the sequence \{$f_n$\}, and the lemma is proved.

**Lemma 6.** Let $B$ be a bounded self-adjoint operator in a Hilbert space $\mathcal{H}$. If the essential spectrum of $B$ consists of union of finite number of segments and if outside the essential spectrum $B$ has a finite number of eigenvalues, then for any finite-dimensional operator $K$ in $\mathcal{H}$ the operator $B + K$ has finite number of eigenvalues (counted with multiplicities) outside the essential spectrum of $B$.

The lemma 6 can be easily proved using the Weyl theorem and the Fredholm determinant.

**Proof** of the theorem 4. If the conditions of the theorem 4 are satisfied then for any $z \in \Gamma$ the function $K(x, y)$ can be represented in the form

$$K(x, y) = K_1(x, y) + K_2(x, y)$$

so that the following conditions are valid:

$$\|\Delta^{-1}(x, z)K_1(x, y)\| \in L^2(T^\nu \times T^\nu), \ K_1^*(x, y) = K_1(y, x), \ K_2^*(x, y) = K_2(y, x)$$

and the integral operator with the kernel $K_2(x, y)$ in the space $L^2(T^\nu, C^n)$ is finite-dimensional.

Now using lemmas 2,3,5 and 6 begin proving the theorem 4.

3 Applications

1. Suppose that the matrix-valued function $U(x)$ is analytic. Then for any $z \in \mathbb{R}$ the function $\varphi_z(x) = \Delta(x, z)$ is real-analytic on $T^\nu$. It follows from the Lojasevitch inequality (cf. [11]) that the isolated extremal point of a real-analytic function on the torus $T^\nu$ is an extremal point of finite multiplicity (cf. definition 1). Henceforth by the theorem 4 we obtain the following theorem generalizing the theorem 1.

**Theorem 7.** Let the matrix-valued function $U(x)$ be analytic on $T^\nu$ and let for any $z \in \Gamma$ the set $\varphi_z^{-1}(0)$ is finite (where $\varphi_z(x) = \Delta(x, z)$). Then there exists a positive number $s > 0$ such that for any matrix-valued function $K(x, y)$ from the class $C^{s+0}(T^\nu \times T^\nu, M_n(C))$ the operator $H$ has finite number of eigenvalues (counted with multiplicities) outside the essential spectrum.

The following example shows the necessity of the condition of the theorem 7.
Example. Consider in the space $L^2(T^2)$ (where $T^2 = [0, 2\pi]^2$) operator of the form (1) with $n = 1$, $\nu = 2$, $U(x) = \cos x_1$,

$$K(x, y) = \sum_{k \geq 1} c_k \cos kx_2 \cos ky_2, \quad x = (x_1, x_2), \quad y = (y_1, y_2),$$

where

$$c_k^{-1} = \int_{T^1} (\cos x_1 + 1 + e^{-k})^{-1}dx_1.$$

It is clear that these functions $U(x)$ and $K(x, y)$ are analytic on $T^2$ and on $T^2 \times T^2$ respectively and the extremal points of the function $U(x)$ are not isolated. It can be easily checked that the essential spectrum of the operator $H$ coinsides with the segment $[-1, 1]$ and the numbers $\lambda_n = -1 - e^{-n}$, $n = 1, 2, \ldots$ are eigenvalues of the operator $H$ lying outside its essential spectrum.

2. Let $\varphi \in C^s(T^\nu)$ with natural $s$. Denote by $J_\alpha^s \varphi$, $\alpha \in T^\nu$ a $s$-jet of the function $\varphi$ at the point $\alpha$ (cf. [3]). The following theorem is valid.

**Theorem 8.** Let $\varphi(x)$ be a real-valued function from the class $C^{\mu+3}(T^\nu)$ where $\mu$ is a natural number and let $\alpha \in T^\nu$ be an extremal point of the function $\varphi$. If there exists a smooth function $\psi(x)$ on $T^\nu$ for which the point $\alpha$ is a critical point of multiplicity $n \leq \mu$ (cf. [3]) and if $J_\alpha^{\mu+1} \varphi = J_\alpha^{\mu+1} \psi$, then $m(\alpha) \leq n + 1$.

**Proof.** As the statement of the theorem has a local character so it is sufficient to prove it in the neighbourhood of zero in $\mathbb{R}^\nu$. Without any loss of generality we can suppose that $\alpha = 0$ is the point of minimum of the function $\varphi$ and $\varphi(0) = 0$. As it is shown in [12] the number $n$ is odd.

Let $p(x)$ be a $(\mu + 1)$-jet of the function $\varphi$ at the point $\alpha = 0$. It follows from the Tujron theorem [12] that the function $p(x)$ has a local minimum at zero. Consider now a one-parameter deformation of the function $p$ of the form $F_\varepsilon(x) = p(x) - \varepsilon x_1^{\mu+1}$ (where $x = (x_1, x_2, \ldots, x_\nu)$, $\varepsilon > 0$. As the multiplicity of the critical point $\alpha = 0$ is equal to $n$, so $x_1^n \in I_{\nabla p}$ (cf. [3]) where $I_{\nabla p}$ is the local gradient ideal of the function $p$ at zero. Henceforth there exist such smooth functions $\{h_k(x)\}$ that in some neighbourhood of zero the following equality is valid:

$$x_1^n = h_1(x) \frac{\partial p}{\partial x_1} + h_2(x) \frac{\partial p}{\partial x_2} + \ldots + h_\nu(x) \frac{\partial p}{\partial x_\nu}. $$

Hence if $\varepsilon$ is small enough we have $I_{\nabla p} = I_{\nabla F_\varepsilon}$. So the map $\nabla F_\varepsilon$ at the point $\alpha = 0$ has a zero of the multiplicity $n$ (3).

As $\alpha = 0$ is the critical point of finite multiplicity for the polynom $p(x)$, so there exists a ball neighbourhood $V \subset \mathbb{C}^\nu$ of zero such that for all $x \in \partial V$ (where $\partial V$ is a boundary of $V$) the inequality $|\nabla p(x)| \geq \delta > 0$.

Take $\varepsilon$ to satisfy the following inequalities:

5
\begin{itemize}
  \item $F_\varepsilon(x) > 0$ for all $x \in \partial V \cap \mathbb{R}^\nu$,
  \item $|\nabla p(x)| > \varepsilon |x_1^n|$ for all $x \in \partial V$.
\end{itemize}

By the multidimensional Rouchet theorem ([11]) the maps $\nabla F_\varepsilon(x)$ and $\nabla p(x)$ have equal number of zeros (counted with multiplicities) in $V$. Therefore both functions $\nabla F_\varepsilon(x)$ and $\nabla p(x)$ have in $V$ a unique zero of multiplicity $n$ at the point $\alpha = 0$, in particular the function $F_\varepsilon(x)$ has no real critical points in $V$.

Therefore for any $x \in \overline{V} \cap \mathbb{R}^\nu$ and small enough positive $\varepsilon$ we have $F_\varepsilon(x) \geq 0$, i.e.

$$p(x) \geq \varepsilon |x_1|^{n+1}.$$

By the same way we can prove that there exists a neighbourhood $W$ of zero and a positive number $\varepsilon$ such that the estimates

$$p(x) \geq \varepsilon |x_k|^{n+1}, \quad (k = 1, 2, \ldots, \nu)$$

hold for all $x \in W$ and it proves the theorem.

**Remark.** Validity of the theorem 8 for so-called extremally non-degenerate polynomials follows from the theorem 1.5 of [12].

From the theorems 4 and 8 we obtain the following

**Theorem 9.** Let $U(x) \in C^{\mu+3}(\mathbb{R}^\nu \times \mathbb{R}^\nu, M_n(\mathbb{C}))$ with some natural $\mu$. If for any $z \in \Gamma$ and for any $\alpha \in \varphi_z^{-1}(0)$ where $\varphi_z(x) = \Delta(x, z)$ there exists a smooth function $\psi_{z, \alpha}(x)$ on $\mathbb{R}^\nu$ for which the point $\alpha$ is critical of multiplicity $n(\alpha)$ and

$$J^\mu_{\alpha + 1} \varphi_z = J^\mu_{\alpha + 1} \psi_{z, \alpha}, \quad \sum_{\alpha} n(\alpha) \leq \mu,$$

then for any function $K(x, y) \in C^{2^{\mu+2-\nu/2+\mu}}(\mathbb{R}^\nu \times \mathbb{R}^\nu, M_n(\mathbb{C}))$ the operator $H$ has finite number of eigenvalues (counted with multiplicities) outside the essential spectrum.

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Ikramov I. A., Sharipov F.
Dept. of Mathematics
Samarkand State University
Samarkand, 703004, Uzbekistan