A NOTE ON ARITHMETIC BREUIL-KISIN-FARGUES MODULES

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Abstract. Let $K$ be a discrete valuation field, we combine the construction of Fargues-Fontaine of $G_K$-equivariant modifications of vector bundles over the Fargues-Fontaine curve $X_{FF}$ using weakly admissible filtered $(\varphi, N, G_K)$-modules over $K$, with Scholze and Fargues’ theorems that relate modifications of vector bundles over the Fargues-Fontaine curve with mixed characteristic shtukas and Breuil-Kisin-Fargues modules. We give a characterization of Breuil-Kisin-Fargues modules with semilinear $G_K$-actions that produced in this way and compare those Breuil-Kisin-Fargues modules with Kisin modules.

1. Introduction

1.1. Review of the work of Fargues-Fontaine and Scholze. Fargues and Fontaine in [FF] construct a complete abstract curve $X_{FF}$, the Fargues-Fontaine curve (constructed using the perfectoid field $\mathbb{C}_p^\flat$ and $p$-adic field $\mathbb{Q}_p$). For any $p$-adic field $K$, they show $\mathcal{O}_X = \mathcal{O}_{X_{FF}}$ carries an action of $G_K$, and they define $\mathcal{O}_X$-representations of $G_K$ as vector bundles over $X_{FF}$ that carries a continuous $\mathcal{O}_X$-semilinear action of $G_K$. They can show $\mathcal{O}_X$-representations of $G_K$ are related to $p$-adic representations of $G_K$ in many aspects. For example, Fargues-Fontaine show that $X_{FF}$ is complete in the sense that there is a Harder-Narasimhan theorem holds for coherent $\mathcal{O}_X$-modules over $X_{FF}$, and they prove that the category of $\mathcal{O}_X$-representations of $G_K$ such that the underlying vector bundles over $X_{FF}$ are semistable of pure slope 0 is equivalence to the category of $p$-adic representations of $G_K$ over $\mathbb{Q}_p$. Moreover, they give an explicit construction of slope 0 $\mathcal{O}_X$-representations from weakly admissible filtered $(\varphi, N)$-modules $D$ over $K$. Their construction is that: first using $D$ and the $(\varphi, N)$-structure, they construct an $\mathcal{O}_X$-representation $\mathcal{E}(D, \varphi, N)$ of $G_K$ whose underlying vector bundle is not semistable in general, then using the filtration structure of $D_K$, they constructed a $G_K$-equivariant modification $\mathcal{E}(D, \varphi, N, \text{Fil}^*)$ of $\mathcal{E}(D, \varphi, N)$ along a special closed point called $\infty$ on $X_{FF}$. They can show if $D$ is weakly admissible, then $\mathcal{E}(D, \varphi, N, \text{Fil}^*)$ is pure of slope 0, and the $\mathbb{Q}_p$ representation corresponds to $\mathcal{E}(D, \varphi, N, \text{Fil}^*)$ is nothing but the log-crystalline representation corresponds to the data $(D, \varphi, N, \text{Fil}^*)$. By going through such a construction, they give new proofs of some important theorems in $p$-adic Hodge theory, for instance, they give a lovely proof of the fact that being admissible is the same as being weakly admissible.
The abstract curve $X_{FF}$ also plays a role in Scholze’s work. In his Berkeley lectures on $p$-adic geometry [SW], Scholze defined a mixed characteristic analog of shtukas with legs. To be more precise, he introduced the functor $\text{Spd}(\mathbb{Z}_p)$ which plays a similar role of a proper smooth curve in the equal characteristic story, and for any perfectoid space $S$ in characteristic $p$, he was able to define shtukas over $S$ with legs. If we restrict us to the case that when $S = \text{Spa}(C)$ is just a point, with $C = \mathbb{C}_p^\flat$ an algebraically closed perfectoid field in characteristic $p$, and assume there is only one leg which corresponds to the untilt $C_p$, then he can realize shtukas over $S$ as modifications of vector bundles over $X_{FF}$ along $\infty$. Here $\infty$ is the same closed point on $X_{FF}$ as we mentioned in the work of Fargues-Fontaine. Fargues and Scholze also show that those shtukas can be realized using some commutative algebra data, called the (free) Breuil-Kisin-Fargues modules, which are modules over $A_{\text{inf}} = \text{W}(\mathcal{O}_C)$ with some additional structures.

1.2. Arithmetic Breuil-Kisin-Fargues modules and our main results. If we combine the construction of Fargues and Fontaine of modifications of vector bundles over $X_{FF}$ from log-crystalline representations and the work of Fargues and Scholze that relates modifications of vector bundles over $X_{FF}$ with shtukas and Breuil-Kisin-Fargues modules, one can expect that if starting with a weakly admissible filtered $(\varphi, N)$-module over $K$, one can produce a free Breuil-Kisin-Fargues module (actually only up to isogeny if we do not specify an integral structure of the log-crystalline representation) using the modification constructed by Fargues-Fontaine. Moreover, since the modification is $G_K$-equivariant and all the correspondences of Fargues and Scholze we have mentioned are functorial, we have the Breuil-Kisin-Fargues module produced in this way carries a semilinear $G_K$-action that commutes with all other structures of it. In this paper, we will give a naive generalization of Fargues-Fontaine’s construction of $G_K$-equivariant modifications of vector bundles over $X_{FF}$ when the inputs are weakly admissible filtered $(\varphi, N, G_K)$-modules over $K$, which is also mentioned in the work of Fargues-Fontaine but using descent. Fix a $p$-adic field $K$, and we make the following definition.

**Definition 1.** A Breuil-Kisin-Fargues $G_K$-module is a free Breuil-Kisin-Fargues module with a semilinear action of $G_K$ that commutes with all its other structures. A Breuil-Kisin-Fargues $G_K$-module is called arithmetic if, up to isogeny, it comes from the construction mentioned above using a weakly admissible filtered $(\varphi, N, G_K)$-module.

If $\mathfrak{M}_{\text{inf}}$ is a free Breuil-Kisin-Fargues module, then one defines its étale realization $T(\mathfrak{M}_{\text{inf}})$ as

$$T(\mathfrak{M}_{\text{inf}}) = (\mathfrak{M}_{\text{inf}} \otimes_{A_{\text{inf}}} W(C))^{\varphi = 1}$$

which is a finite free $\mathbb{Z}_p$-module, and recall the following theorem of Fargues and Scholze-Weinstein:
Theorem 1. The functor
\[ \mathfrak{M}^{\inf} \to (T(\mathfrak{M}^{\inf}), \mathfrak{M}^{\inf} \otimes_{A^{\inf}} B^{+}_{dR}) \]
defines an equivalence of the following categories
\[ \{ \text{free Breuil-Kisin-Fargues modules over } A^{\inf} \} \]
and
\[ \{ (T, \Xi) \mid T : \text{finite free } \mathbb{Z}_p\text{-modules, } \Xi : \text{full } B^{+}_{dR}\text{-lattice in } T \otimes_{\mathbb{Z}_p} B_{dR} \} \]

The category of pairs \((T, \Xi)\) is what Fargues called Hodge-Tate modules in [Far]. Note that if \(\mathfrak{M}^{\inf}\) is a Breuil-Kisin-Fargues \(G_K\)-module, then the Hodge-Tate module corresponds to \(\mathfrak{M}^{\inf}\) carries a \(G_K\) action by functoriality. Using the Hodge-Tate module description of Breuil-Kisin-Fargues modules, we give the following easy characterization of arithmetic Breuil-Kisin-Fargues modules.

Proposition 1. A Breuil-Kisin-Fargues \(G_K\)-module \(\mathfrak{M}^{\inf}\) is arithmetic if and only if its \(\acute{e}tale\) realization \(T = T(\mathfrak{M}^{\inf})\) is potentially log-crystalline as a representation of \(G_K\) over \(\mathbb{Z}_p\), and there is a \(G_K\)-equivariant isomorphism of
\[ (T(\mathfrak{M}^{\inf}), \mathfrak{M}^{\inf} \otimes_{A^{\inf}} B^{+}_{dR}) = (T, (T \otimes_{\mathbb{Q}_p} B_{dR})^{G_K} \otimes_{K} B^{+}_{dR}). \]
Moreover, if the isogeny class of \(\mathfrak{M}^{\inf}\) corresponds the \(G_K\)-equivariant modification coming from a weakly admissible filtered \((\varphi, N, G_K)\)-module \(D\) over \(K\), then \(T(\mathfrak{M}^{\inf}) \otimes_{\mathbb{Q}_p}\) is the potentially log-crystalline representation of \(G_K\) corresponds to the weakly admissible filtered \((\varphi, N, G_K)\)-module \(D\).

Remark 1.

1. We want to remind the readers that \(p\)-adic monodromy theorem for de Rham representations, which was first proved in the work of Berger [Ber], tells us that if a \(p\)-adic representation is de Rham, then it is potentially log-crystalline (and the converse is true and actually much easier to prove). We will use the equivalence of being de Rham and potentially log-crystalline through out this paper.

2. From the work of [BMS], we know there is a large class of Breuil-Kisin-Fargues \(G_K\)-modules comes from geometry: start with a proper smooth formal scheme \(\mathfrak{X}\) over \(\mathcal{O}_K\), and let \(\mathfrak{X}\) be its base change to \(\mathcal{O}_{\mathbb{C}_p}\). Then there is a \(A^{\inf}\)-cohomology theory attaches to \(\mathfrak{X}\) which is functorial in \(\mathfrak{X}\), so all the \(A^{\inf}\)-cohomology groups \(H^i_{\text{rig}}(\mathfrak{X})\) carry natural semi-linear \(G_K\)-actions that commute with all other structures. If we take the maximal free quotients of the cohomology groups, then they are all arithmetic automatically from the \(\acute{e}tale\)-de Rham comparison theorem. So being arithmetic is the same as to ask an abstract Breuil-Kisin-Fargues \(G_K\)-module to satisfy \(\acute{e}tale\)-de Rham comparison theorem.
(3) The terminology of being arithmetic was first introduced in the work of Howe in [How §4], the above proposition shows our definition are the same. The advantage of our definition is that it enables us to see how arithmetic Breuil-Kisin-Fargues behavior over \(\text{Spa}(A_{\text{inf}})\) instead of only look at the stalks at closed points \(\mathcal{O}_C\) and \(\mathcal{O}_{C_p}\).

As we mentioned in the above remark, if we study the behavior of arithmetic Breuil-Kisin-Fargues modules at the closed point corresponds to \(W(k)\) of \(\text{Spa}(A_{\text{inf}})\), we will have:

**Corollary 1.** Let \(\mathfrak{M}_{\text{inf}}\) be an arithmetic Breuil-Kisin-Fargues module \(\mathfrak{M}_{\text{inf}}\) then

1. \(T(\mathfrak{M}_{\text{inf}})\) is log-crystalline if and only if the inertia subgroup \(I_K\) of \(G_K\) acts trivially on \(\mathfrak{M}_{\text{inf}} = \mathfrak{M}_{\text{inf}} \otimes_{A_{\text{inf}}} W(\overline{k})\).
2. \(T(\mathfrak{M}_{\text{inf}})\) is potentially crystalline if and only if there is a \(\varphi\) and \(G_K\) equivariant isomorphism \(\mathfrak{M}_{\text{inf}} \otimes_{A_{\text{inf}}} B_{\text{cris}}^+ \cong \mathfrak{M}_{\text{inf}} \otimes_{W(k)} B_{\text{cris}}^+\).
3. \(T(\mathfrak{M}_{\text{inf}})\) is crystalline if and only if it satisfies the conditions in (1) and (2).

**Remark 2.** We continue the discussion in Remark (2), in [BMS] Theorem 14.1] one can see the \(A_{\text{inf}}\)-crystalline comparison theorem should give us condition (1) in Corollary (1). And Proposition 13.21. of loc.cit. shows that there is a canonical isomorphism

\[
\widetilde{H}^i_{\text{crys}}(\mathcal{X}_{O_p/p}/A_{\text{cris}})[1/p] \cong \widetilde{H}^i_{\text{crys}}(\mathcal{X}_{\overline{F}/W(\overline{k})}) \otimes_{W(k)} B_{\text{cris}},
\]

which means the condition in Corollary (2) is always satisfied. And we know their étale realizations are crystalline since [BMS] is in the good reduction case. \(A_{\text{inf}}\)-crystalline comparison theorem were extended to the case of semistable reduction by Česnavičius-Koshikawa in [CK] and Zijian Yao in [Yao], and one can show (1) in Corollary is satisfied in the semistable reduction case.

Another natural question is if one starts with a potentially log-crystalline representation \(T\) of \(G_K\) over \(\mathbb{Z}_p\), one can construct a Hodge-Tate module \((T, (T \otimes_{\mathbb{Q}_p} B_{\text{dR}})^{G_K} \otimes B_{\text{dR}}^+)\) which corresponds to an arithmetic Breuil-Kisin-Fargues module \(\mathfrak{M}_{\text{inf}}(T)\) as in Proposition 1. On the other hand, the theory of Kisin in [Kis] shows that if \(L\) is a finite Galois extension of \(K\) such that \(T|_{G_L}\) is log-crystalline, fixing a Kummer tower \(L_{\infty}\) over \(L\), then there is a finite free module \(\mathfrak{M}(T)\) over a subring \(\mathfrak{S}\) of \(A_{\text{inf}}\) with a \(\varphi\)-structure that captures many properties of \(T\). We have the following proposition that enables us to compare \(\mathfrak{M}(T)\) with \(\mathfrak{M}_{\text{inf}}(T)\).
Proposition 2. Let $\mathcal{M}_0^{\text{inf}}(T) = \mathcal{M}(T) \otimes_{\mathcal{O}, \varphi} A_{\text{inf}}$, then $\mathcal{M}_0^{\text{inf}}(T)$ is a Breuil-Kisin-Fargues module with a semilinear action of $G_{L_\infty}$ by construction. There is a unique way to extend the $G_{L_\infty}$-action to $G_K$ such that the étale realization of $\mathcal{M}_0^{\text{inf}}(T)$ is exactly the $G_K$-representation $T$. Moreover, $\mathcal{M}_0^{\text{inf}}(T)$ with such $G_K$ action is isomorphic to $\mathcal{M}^{\text{inf}}(T)$. In particular, $\mathcal{M}_0^{\text{inf}}(T)$ is arithmetic.

The above proposition is related to one of the main results in [GL], and we will give another proof in this paper. We want to emphasize that the above proposition can be proved nothing but by comparing the construction of the $\mathcal{O}$-module by Kisin, and the construction of arithmetic Breuil-Kisin-Fargues module by Fargues-Fontaines. One easy consequence of this proposition is that one observe that $\mathcal{M}^{\text{inf}}(T)$ is defined only using the datum $T$ while the $\mathcal{M}(T)$ is defined with respect to the choice of the Kummer tower $L_\infty$, so $\mathcal{M}_0^{\text{inf}}(T)$ is independent of the choice of the tower $L_\infty$. In [EG], they use the terminology of admitting all descents over $K$ for a Breuil-Kisin-Fargues $G_K$-module, and they can prove that the condition of admitting all descents over $K$ is equivalence to that the étale realization of the Breuil-Kisin-Fargues $G_K$-module is log-crystalline. To compare with their result, we have the following proposition:

Proposition 3. A Breuil-Kisin-Fargues $G_K$-module $\mathcal{M}^{\text{inf}}$ admits all descents over $K$ if and only if it is arithmetic and satisfies the condition (1) in Corollary [GL].

We want to mention that in the proof of [GL Theorem F.11], they use the following criterion of the author about arithmetic Breuil-Kisin-Fargues modules.

Lemma 1. [GL F.13.] A Breuil-Kisin-Fargues $G_K$-module $\mathcal{M}^{\text{inf}}$ is arithmetic if and only if $\mathcal{M}^{\text{inf}} \otimes_{A_{\text{inf}}} B^{+}_{\text{dR}}/(\xi)$ has a $G_K$-stable basis.

Remark 3. (1) The terminology of admitting all descents over a $K$, as been mentioned in [EG], is very likely to be related to the prismatic-$A_{\text{inf}}$ comparison theorem of Bhatt-Scholze [BS], while our condition seems only relates to $A_{\text{inf}}$-cohomology as we mentioned in Remark [GL].

(2) It is natural to ask if one can make sense of “moduli space of arithmetic shtukas”, and compare it with the Emerton-Gee stack defined in [EG].

1.3. Structure of the paper. In section 2 we will first review Scholze’s definition of shtukas in mixed characteristic and how to relate shtukas with Breuil-Kisin-Fargues modules. Then we will give a brief review on how to realize Scholze’s shtukas using Fargues-Fontaine curve. For readers familiar with their theories, they can skip this section. In section 3 we will give an explicit construction of $G_K$-equivariant modifications of vector bundles over $X_{FF}$ using data come from weakly admissible filtered $(\varphi, N, G_K)$-modules following Fargues-Fontaine’s method, and we will give a characterization...
of Breuil-Kisin-Fargues $G_K$-modules come from modifications constructed in this way. In section 4 we will study the relationship between arithmetic Breuil-Kisin-Fargues modules and Kisin modules, and also prove some propositions relate to the work of [EG].

1.4. Notions and conventions. Throughout this paper, $k_0$ will be a perfect field in characteristic $p$ and $W(k_0)$ the ring of Witt vectors over $k_0$. Let $K$ be a finite extension of $W(k_0)[\frac{1}{p}]$. Let $\mathcal{O}_K$ be the ring of integers of $K$, $\varpi$ be any uniformizer and let $k = k_K = \mathcal{O}_K/(\varpi)$ be the residue field. Define $K_0 = W(k)[\frac{1}{p}]$. By a compatible system of $p^n$-th roots of $\varpi$, we mean a sequence of elements $\{\varpi_n\}_{n \geq 0}$ in $\overline{K}$ with $\varpi_0 = \varpi$ and $\varpi_{n+1}^p = \varpi_n$ for all $n$.

Define $\mathbb{C}_p$ as the $p$-adic completion of $\overline{K}$, there is a unique valuation $v$ on $\mathbb{C}_p$ extending the $p$-adic valuation on $K$. Let $\mathcal{O}_{\mathbb{C}_p} = \{x \in \mathbb{C}_p | v(x) \geq 0\}$ and let $\mathfrak{m}_{\mathbb{C}_p} = \{x \in \mathbb{C}_p | v(x) > 0\}$. We will have $\mathcal{O}_{\mathbb{C}_p}/\mathfrak{m}_{\mathbb{C}_p} = \overline{K}$.

Let $C$ be the tilt of $\mathbb{C}_p$, then by the theory of perfectoid fields, $C$ is algebraically closed of characteristic $p$, and complete with respect to a nonarchimedean norm. Let $\mathcal{O}_C$ be the ring of the integers of $C$, then $\mathcal{O}_C = \mathcal{O}_{\mathbb{C}_p} = \varprojlim \mathcal{O}_{\mathbb{C}_p}/p$. Define $A_{\inf} = W(\mathcal{O}_C)$, there is a Frobenius $\varphi_{A_{\inf}}$ acts on $A_{\inf}$. There is a surjection $\theta : A_{\inf} \rightarrow \mathcal{O}_{\mathbb{C}_p},$ whose kernel is principal and let $\xi$ be a generator of $\text{Ker}(\theta)$. Let $\tilde{\xi} = \varphi(\xi)$ as in [BMS]. There is a $G_K$ action on $A_{\inf}$ via its action on $C = \mathbb{C}_p^\times$, one can show $\theta$ is $G_K$-equivariant.

In this paper, we will use the notion log-crystalline representations instead of semistable representations to make a difference to the semistability of vector bundles over complete regular curves.

A filtered $(\varphi, N)$-module over $K$ is a finite dimensional $K_0$-vector space $D$ equipped with two maps

$$\varphi, N : D \rightarrow D$$

such that

(1) $\varphi$ is semi-linear with respect to the Frobenius $\varphi_{K_0}$.
(2) $N$ is $K_0$-linear.
(3) $N\varphi = p\varphi N$.

And a decreasing, separated and exhaustive filtration on the $K$-vector space $D_K = K \otimes_{K_0} D$.

Let $L$ be a finite Galois extension of $K$ and let $L_0 = W(k_L)\mathbb{Q}$. A filtered $(\varphi, N, \text{Gal}(L/K))$-module over $K$ is a filtered $(\varphi, N)$-module $D'$ over $L$ together with a semilinear action of $\text{Gal}(L/K)$ on the $L_0$ vector space $D'$, such that:

(1) The semilinear action is defined by the action of $\text{Gal}(L/K)$ on $L_0$ via $\text{Gal}(L/K) \rightarrow \text{Gal}(k_L/k) = \text{Gal}(L_0/K_0)$.
(2) The semilinear action of $\text{Gal}(L/K)$ commutes with $\varphi$ and $N$.
(3) Define diagonal action of $\text{Gal}(L/K)$ on $D' \otimes_{L_0} L$, then the filtration on $D' \otimes_{L_0} L$ is stable under this action.
If $L'$ is another finite Galois extension of $K$ containing $L$, then one can show there is a fully faithful embedding of the category of filtered $(\varphi, N, \text{Gal}(L/K))$-modules into the category of filtered $(\varphi, N, \text{Gal}(L'/K))$-modules. One defines the category of filtered $(\varphi, N, G_K)$-modules to be the limit of filtered $(\varphi, N, \text{Gal}(L/K))$-modules over all finite Galois extensions $L$ of $K$.

2. Shtukas in mixed characteristic, Breuil-Kisin-Fargues modules and modifications of vector bundles

2.1. Shtukas in mixed characteristic and Breuil-Kisin-Fargues modules. In this subsection we briefly review Scholze’s definition of shtukas (with one leg) in mixed characteristic and its relation with Breuil-Kisin-Fargues modules following Scholze’s Berkeley notes [SW] and Kedlaya’s AWS notes [Ked1].

Definition 2. [SW, Definition 11.4.1] Let $\text{Pfd}$ be the category of perfectoid spaces of characteristic $p$, for any $S \in \text{Pfd}$, a shtuka with one leg over $S$ is the following data:

- A morphism $x : S \to \text{Spd}(\mathbb{Z}_p)$, (the leg)
- $\mathcal{E}$ a vector bundle over $\text{Spd}(\mathbb{Z}_p) \times S$ together with
  \[ \varphi_\mathcal{E} : \text{Fr}^*_{\mathcal{E}}(\mathcal{E})|_{\text{Spd}(\mathbb{Z}_p) \times S \setminus \Gamma_x} \to \mathcal{E}|_{\text{Spd}(\mathbb{Z}_p) \times S \setminus \Gamma_x} \]

Here $\Gamma_x$ denotes the graph of $x$ and $\to$ means $\varphi_\mathcal{E}$ is an isomorphism over $(\text{Spd}(\mathbb{Z}_p) \times S) \setminus \Gamma_x$ and meromorphic along $\Gamma_x$.

The most revolutionary part in Scholze’s definition is he came up with the object “$\text{Spa}\mathbb{Z}_p$” (as well as the $\text{Spd}(\mathbb{Z}_p)$ we use in the definition) which he used as the replacement of the curve $\mathcal{C}/\mathbb{F}_p$ in the equal characteristic case. Instead of go into the details in the definition, we will unpack concepts in this definition when $S = \text{Spa}(C)$ is just a point, i.e., we assume $C$ is a perfectoid field in characteristic $p$. Then for the first datum in the definition of shtukas, we have:

Lemma 2. [SW, Proposition 11.3.1] For $S = \text{Spa}(C)$ is a perfectoid field in characteristic $p$, the following sets are naturally identified:

- A morphism $x : S \to \text{Spd}(\mathbb{Z}_p)$.
- The set of isomorphism classes of untilts of $S$, or more precisely of pairs $(F, i)$ in which $F$ is a perfectoid field and $i : (S^\#)^{\flat} \to S$ is an isomorphism and isomorphism classes are taken from $F \simeq F'$ that commutes with $i$ and $i'$.
- Sections of $S^\diamond \times \text{Spd}\mathbb{Z}_p \to S^\diamond$

Here $S^\diamond$ denotes the diamond associated with $S$ by identifying $S$ with the functor it represents as a pro-étale sheaf of sets. Again, instead of go into Scholze’s definition of diamonds, we unpack the concept with the following lemma in the case of $S = \text{Spa}(C)$ a perfectoid field.
Lemma 3. [Ked1, Lemma 4.3.6.] For $S = \text{Spa}(C)$, let $\varpi$ be a (nonzero) topological nilpotent element in $C$ and put $A_{\text{inf}} = W(\mathcal{O}_C)$. Let

$$U_S = \{v \in \text{Spa}(A_{\text{inf}}, A_{\text{inf}})|v([\varpi]) \neq 0\},$$

Then for any $Y \in \text{Pfd}$, there is a natural identifacation of

- morphisms of $Y \to S^0 \times \text{Spd}\mathbb{Z}_p$,
- pairs of $(X, f)$ in which $f$ is an isomorphism class of untilts of $Y$ and $f : X \to U_S$ is a morphism of adic spaces.

Remark 4. By tilt equivalence in the relative setting, we have that if $S$ is a perfectoid space (not necessary in $\text{Pfd}$), and $Y \in \text{Pfd}$, then $S^\phi(Y)$ is naturally isomorphic to pairs $(X, f)$ where $X$ is an isomorphism class of untilts of $Y$ and $f : X \to S$ is a morphism of perfectoid spaces. Scholze generalize this notion and define $S^\phi$ for any analytic adic spaces $S$ on which $p$ is topologically nilpotent as be the functor that for $Y \in \text{Pfd}$,

$$S^\phi(Y) = \{(X, f)|X \text{ is an isomorphism class of untilts of } Y \text{ and } f : X \to S\}$$

Using the terminology in the above remark, Lemma[8] says $(S^0 \times \text{Spd}\mathbb{Z}_p)(Y)$ is naturally isomorphic to $U^\phi_S(Y)$, so if we go back to the definition of shtuka, the second data can be taken as a vector bundle over $U_S$, note that over $U_S$ there is a natural Frobenius induced for the Frobenius on $\mathcal{O}_C$.

Now let’s restrict to the case that the leg of the shtuka correspondence to an untilt of $C$ in characteristic 0, then we will have the following lemma:

Lemma 4. [Ked1, Lemma 4.5.14.] For $S = \text{Spa}(C)$ be a perfectoid space in characteristic $p$, and let

$$Y_S = \{v \in \text{Spa}(A_{\text{inf}}, A_{\text{inf}})|v([\varpi]) \neq 0, v(p) \neq 0\}.$$

Then a shtukas over $S$ with leg $x$ correpondence to an untilt of $C$ in characteristic 0, is the same as the following data

$$\mathcal{E}_0 \to \mathcal{E}_1$$

- $\mathcal{E}_0$ is a $\varphi$-equivariant vector bundle over $U_S$,
- $\mathcal{E}_1$ is a $\varphi$-equivariant vector bundle over $Y_S$.
- $\mathcal{E}_0 \to \mathcal{E}_1$ is an isomorphism of $\mathcal{E}_0$ with $\mathcal{E}_1$ over $Y_S \setminus \cup_{n \in \mathbb{Z}} \Gamma_{\varphi^n(x)}$.

Remark 5. Here $\Gamma_x$ is Cartier divisor correspondence to the leg $x$, and the assumption $x$ correspondence to an untilt of $C$ in characteristic 0 is the same as $\Gamma_x$ is inside $Y_S$. Instead of proving the lemma, we just recall that $\mathcal{E}_0$ (resp. $\mathcal{E}_1$) comes from restricting $\mathcal{E}$ in Definition[2] to a “neighborhood” of $V(p)$ (resp. $V([\varpi])$), and use the Frobenius to extend it to $U_S$ (resp. $Y_S$).

If we further assume $S = \text{Spa}(C)$ with $C$ an algebraically closed non-archimedean field in characteristic $p$, then we have the following lemma:

Lemma 5. [Ked1, Lemma 4.5.17.] Let $S = \text{Spa}(C)$ with $C$ an algebraically closed non-archimedean field in characteristic $p$, and let

$$Y^+_S = \{v \in \text{Spa}(A_{\text{inf}}, A_{\text{inf}})|v([\varpi]) \neq 0, v(p) \neq 0\}.$$
Then any $\varphi$-equivariant vector bundle over $Y_S$ extends uniquely to a $\varphi$-equivariant vector bundle over $Y_S^\circ$.

Let $x_k$ be the unique closed point of $\text{Spa}(A_{\text{inf}})$, then combine all the lemmas, we will have the following theorem of Fargues and Scholze-Weinstein:

**Theorem 2.** [SW, Theorem 14.1.1][Ked1, Theorem 4.5.18] Let $S = \text{Spa}(C)$ with $C$ an algebraically closed non-archimedean field in characteristic $p$, the following categories are canonically equivalent:

1. a shtuka with one leg $x$ over $S$ such that $x$ corresponds to an untilt in characteristic 0,
2. a vector bundle $E$ over $\text{Spa}(A_{\text{inf}})\{x_k\}$ together with an isomorphism $\varphi_E : \varphi^*(E) \cong E|_{\text{Spa}(A_{\text{inf}})\{x_k\}}(x)$,
3. a finite free module $M$ over $A_{\text{inf}}$ together with $\varphi_M : (\varphi^*M)[\frac{1}{z}] \cong M[\frac{1}{z}]$, where $z$ generate the primitive ideal in $A_{\text{inf}}$ correspondence to the untilt $x$.

**Proof.** (sketch) For the equivalence between (1) and (2), by Lemma 4, we have a shtukas over $S$ with leg $x$ correspondence to an untilt of $C$, then we construct a vector bundle over $\text{Spa}(A_{\text{inf}})\{x_k\}$ by gluing $E_0$ with $E_1^+$ over a rational subdomain “between $x$ and $\varphi(x)$”, where $E_1^+$ is the unique $\varphi$-equivariant vector bundle over $Y_S^\circ$ extending $E_1$ under Lemma 5. The “gluing” process makes sense because of the fact the presheaf over $\text{Spa}(A_{\text{inf}})\{x_k\}$ defined by rational subspaces and their Tate algebras is actually a sheaf.

For the equivalence between (2) and (3), one refers to the following theorem of Kedlaya. \hfill $\square$

**Theorem 3.** [Ked2, Theorem 3.6] There is an equivalence of categories between:

1. Finite free modules $M$ over $A_{\text{inf}}$,
2. Vector bundle $E$ over $\text{Spa}(A_{\text{inf}})\{x_k\}$

### 2.2. Fargues-Fontaine curve and Breuil-Kisin-Fargues modules.

In this subsection, we will review Fargues-Fontaine’s construction of the $p$-adic fundamental curve, and its relation to Scholze’s definition of Shtukas in mixed characteristic.

Keep all the notions as in [14]. Let $S = \text{Spa}(C)$ with $C = \mathbb{C}_p$, we also fix the leg $x = \varphi^{-1}(x_{C_p})$ of shtukas over $S$ to make it corresponds to the untilt $\theta \circ \varphi^{-1} : A_{\text{inf}} \to \mathcal{O}_{C_p}$. Recall we have a Frobenius $\varphi$ acts on the space $Y_S$, we define:

**Definition 3.** Let $B = H^0(Y_S, \mathcal{O}_{Y_S})$ and the schematic Fargues-Fontaine curve is the scheme:

$$X_{FF} = \text{Proj} \oplus_{n \geq 0} B[\varphi^n].$$

We also define the adic Fargues-Fontaine curve to be quotient:

$$X_{FF} = Y_S/\varphi^Z.$$
We have θ induces a map $B \to \mathbb{C}_p$, and this defines a closed point $\infty$ on $X_{FF}$.

**Theorem 4. [FF] Fargues-Fontaine**

1. $X_{FF}$ is a regular noetherian scheme of Krull dimension 1, or an abstract regular curve in the sense of Fargues and Fontaine.
2. The set of closed points of $X_{FF}$ is identified with the set of characteristic 0 untilts of $C$ modulo Frobenius equivalence. Under this identification, $\infty$ sends to the untilt $\mathbb{C}_p$ of $C$. The stalk of $X_{FF}$ at $\infty$ is isomorphic to $B^+_{dR}$.
3. $X_e = X_{FF} \setminus \{\infty\}$ is an affine scheme $\text{Spec}B_e$ with $B_e = B^{p^\infty}$ being a principal ideal domain.
4. Vector bundles $E$ over $X_{FF}$ are equivalence to $B$-pairs $(M_e, M^{+}_{dR}, \iota)$ in the sense of Berger. Here $M_e = \Gamma(X_e, E)$ are finite projective modules over $B_e$ and $M^{+}_{dR}$ are the completion of $E$ at $\infty$ which are finite free over $B^+_{dR}$, $\iota$ is an isomorphism of $M_e$ and $M^{+}_{dR}$ over $B_{dR}$.
5. The abstract curve $X_{FF}$ is also complete in the sense that $\deg(f) := \sum_{x \in |X_{FF}|} v_x(f)$ is 0 for all rational functions on $X_{FF}$.

Fargues-Fontaine shows that there is a Dieudonné-Manin classification for vector bundles over $X_{FF}$.

**Theorem 5. [FF]** Let $(D, \varphi)$ be an isocrystal over $k$, then $(D, \varphi)$ defines a vector bundle $E(D, \varphi)$ over $X_{FF}$ which associated with the graded module

$$\bigoplus_{n \geq 0} (D \otimes K_0 B)^{p^n}.$$ 

Moreover, every vector bundle over $X_{FF}$ is isomorphic to $E(D, \varphi)$ for some $(D, \varphi)$.

Let $E$ be a vector bundle over $X_{FF}$, assume $E \cong E(D, \varphi)$ under the above theorem, and if $\{-\lambda_i\}$ are the slopes of $(D, \varphi)$ in the Dieudonné-Manin classification theorem, then $\lambda_i$ are called the slopes of $E$. Moreover $E$ is called semistable of slope $\lambda$ if and only if $E$ corresponds a semisimple isocrystal of slope $-\lambda$. We define $\mathcal{O}(n) = E(K_0, p^{-n})$, one can show $\mathcal{O}(1)$ is a generator of the Picard group of $X_{FF}$ (which is isomorphic to $\mathbb{Z}$). A simple corollary of Dieudonné-Manin classification is:

**Corollary 2.** The category of finite-dimensional $\mathbb{Q}_p$-vector spaces is equivalent to the category of vector bundles over $X_{FF}$ that are semistable of slope 0 under the functor

$$V \to V \otimes_{\mathbb{Q}_p} \mathcal{O}_X.$$

The inverse of this functor is given by:

$$E \to H^0(X_{FF}, E).$$

Note there is a morphism of locally ringed spaces from $X_{FF} \to X_{FF}$, and pullback along this morphism induces a functor from the category of vector bundles over $X_{FF}$ to vector bundles over $X_{FF}$, we have the following GAGA theorem for the Fargues-Fontaine curve.
Theorem 6. Vector bundles over $X_{FF}$ and vector bundles over $X_{FF}$ are equivalent under the above functor.

We have, by the definition of $X_{FF}$, vector bundles over $X_{FF}$ is the same as $\varphi$-equivariant vector bundles over $Y_S$. So by the theorem of GAGA one can make sense of slopes of $\varphi$-equivariant vector bundles over $Y_S$. We have the following theorem of Kedlaya:

Theorem 7. [KL, Theorem 8.7.7] A $\varphi$-equivariant vector bundle $F$ over $Y_S$ is semistable of slope 0 if and only if it can be extended to a $\varphi$-equivariant vector bundle over $U_S$. The set of such extensions is the same as the set of $\mathbb{Z}_p$-lattices inside the $\mathbb{Q}_p$-vector space $H^0(X_{FF}, \mathcal{E})$, where $\mathcal{E}$ is the vector bundle over $X_{FF}$ corresponds to $F$ under GAGA.

Remark 6. One can also rewrite the above theorem in terms of (étale) $\varphi$-modules over $B$ and $B^+$, where $B^+ = H^0(Y_S^+, \mathcal{E})$, one can show that (when $C$ is algebraically closed) the category of vector bundles over $X_{FF}$ is equivalence to all the following categories [FF, Section 11.4]:

- $\varphi$-modules over $B$.
- $\varphi$-modules over $B^+$.
- $\varphi$-modules over $B^+_{\text{cris}}$.
- Vector bundles over $X_{FF}$.

Combine Lemma 6, Theorem 6 and Theorem 7 we have:

Theorem 8. Let $S = \text{Spa}(C)$ with $C = \mathbb{C}_p$, then a shtuka with one leg $\varphi^{-1}(x_{\mathbb{C}_p})$ over $S$ which corresponds to the untilt $\theta \circ \varphi^{-1} : A_{\text{inf}} \rightarrow \mathcal{O}_{C_p}$ is the same as a quadruple $(F_0, F_1, \beta, T)$ where

- $F_0$ is a vector bundle over $X_{FF}$ that is semistable of slope 0,
- $F_1$ is a vector bundle over $X_{FF}$,
- $\beta$ is an isomorphism of $F_0$ and $F_1$ over $X_{FF} \backslash \{\infty\}$,
- $T$ is a $\mathbb{Z}_p$-lattice of the $\mathbb{Q}_p$ vector space $H^0(X_{FF}, \mathcal{E})$.

Using part (4) of Theorem 4, we have the first three data in the above theorem is the same as a $\mathbb{Q}_p$ vector space $V$ together with a $B_{\text{dR}}^+$ lattice inside $V \otimes B_{\text{dR}}$. Note also we have $V \otimes \mathbb{Q}_p B_{\text{dR}} = T \otimes \mathbb{Z}_p B_{\text{dR}}$ for any $\mathbb{Z}_p$-lattice $T$ inside $V$. So we have:

Corollary 3. Let $S = \text{Spa}(C)$ and $x$ as above, then a shtuka with one leg $x$ over $S$ is the same as a pair $(T, \Xi)$ where $T$ is a $\mathbb{Z}_p$-lattice and $\Xi$ is a $B_{\text{dR}}^+$ lattice inside $T \otimes B_{\text{dR}}$.

Definition 4. The pair $(T, \Xi)$ as above is called a Hodge-Tate module. And we define a free Breuil-Kisin-Fargues module as a finite free module $\mathcal{M}_{\text{inf}}$ over $A_{\text{inf}}$ with an isomorphism

$$\varphi_{\mathcal{M}_{\text{inf}}} : \mathcal{M}_{\text{inf}} \otimes_{A_{\text{inf}}} \varphi A_{\text{inf}} \left[\frac{1}{\xi}\right] \simeq \mathcal{M}_{\text{inf}} \left[\frac{1}{\xi}\right]$$

Where $\tilde{\xi} = \varphi(\xi)$ as we defined in [4]
Corollary 4. (Fargues, Scholze-Weinstein) Let $S = \text{Spa}(C)$ and $\varphi^{-1}(x_{C_p})$ as above, then the following categories are equivalence:

- Shtukas with one leg $\varphi^{-1}(x_{C_p})$ over $S$.
- Hodge-Tate modules.
- Free Breuil-Kisin-Fargues modules.

3. $p$-adic representations and vector bundles on the curve

Now let us briefly recall how Fargues-Fontaine construct $G_K$-equivariant modifications of vector bundles over $X_{FF}$ from potentially log-crystalline representations of $G_K$ in \cite{FF §10.3.2}.

Keep the notions as in 1.4, let $D'$ be a filtered $(\varphi, N, \text{Gal}(L/K))$-modules, Fargues-Fontaine first define the $O_X$-representation $E(D', \varphi, G_K)$ of $G_K$ whose underlying vector bundle is $E(D', \varphi)$ (as we defined in Theorem 5) and the semilinear $G_K$-action coming from the diagonal action of $G_K$ on $D' \otimes_{F_0} B$. Note that this construction is functorial, so the relation $N \varphi = p \varphi N$ tells that $N$ defines a $G_K$-equivariant map

$$E(D', \varphi, G_K) \rightarrow E(D', p \varphi, G_K) = E(D', \varphi, G_K) \otimes O(-1).$$

Let $\varpi \in C$ be any element such that $v(\varpi) = 1$, and for any $\sigma \in G_K$ define $\log_{\varpi, \sigma} = \sigma(\log(\varpi)) - \log(\varpi)$. One can show $(\sigma \mapsto \log_{\varpi, \sigma})$ defines an element in $Z^1(G_K, B^{\varphi=p}) = Z^1(G_K, H^0(O(1)))$. So we know the composition:

$$E(D', \varphi) \xrightarrow{N} E(D', \varphi) \otimes O(-1) \xrightarrow{\text{Id} \otimes \log_{\varpi, \sigma}} E(D', \varphi) \otimes O = E(D', \varphi),$$

defines an element in $Z^1(G_K, \text{End}(E(D', \varphi)))$ whose image actually lies in the nilpotent elements of $\text{End}(E(D', \varphi))$. So we can define

$$\alpha = (\alpha_\sigma)_\sigma = (\exp(-\text{Id} \otimes \log_{\varpi, \sigma} \circ N))_\sigma \in Z^1(G_K, \text{Aut}(E(D', \varphi))),$$

Fargues-Fontaine define the $G_K$-equivariant vector bundle associated with a $(\varphi, N, G_K)$-module $D'$ to be the vector bundle:

$$E(D', \varphi, N, G_K) = E(D', \varphi, G_K) \wedge \alpha,$$

i.e., $E(D', \varphi, N, G_K)$ is isomorphic to $E(D', \varphi)$ as vector bundle, and the $G_K$ action on $E(D', \varphi, N, G_K)$ is given by twisting the $G_K$-action of $E(D', \varphi, G_K)$ with the 1-cocycle $\alpha$.

Lemma 6. We have

1. $\alpha$ becomes trivial when completes at $\infty$.
2. If the data $(D', \varphi, N, G_K)$ comes from a potentially log-crystalline representation $V$ of $G_K$, then the completion of $E(D', \varphi, N, G_K)$ at $\infty$ together with its $G_K$-action is isomorphic to $D_{\text{dr}}(V) \otimes_K B^{\text{dr}_\infty}$.
3. If we rewrite the above construction in terms of $\varphi$-modules over $B^+$ as in Remark 4 then $\alpha$ becomes trivial after the base change $B^+ \rightarrow W(k)[\frac{1}{p}]$. 
Proof. (1) is [FF, Proposition 10.3.18, Remark 10.3.19]. For (3), if we rewrite the above construction in terms of \( \varphi \)-modules over \( B^+ \), then \( \mathcal{E}(D', \varphi) \) corresponds to the \( \varphi \)-module \( D' \otimes_{L_0} B^+ \) and the \( \varphi \)-equivariant map
\[
N \otimes \log_{\varphi, \sigma} : D' \otimes_{L_0} B^+ \to D' \otimes_{L_0} B^+
\]
corresponds to the map
\[
\mathcal{E}(D', \varphi) \xrightarrow{\alpha} \mathcal{E}(D', \varphi) \otimes \mathcal{O} \xrightarrow{-1} \mathcal{E}(D', \varphi) \otimes \mathcal{O} = \mathcal{E}(D', \varphi),
\]
In order to show \( \alpha \) becomes trivial after the base change \( B^+ \to W(\mathbb{F}_p)[1/p] \), one just need to show that \( \log_{\varphi, \sigma} \in B^+ \) maps to 0 under \( B^+ \to W(\mathbb{F}_p)[1/p] \), while this can be seen from the fact \( \log_{\varphi, \sigma} \in (B^+)^{\varphi = p} \), but \( W(\mathbb{F}_p)[1/p]^{\varphi = p} = 0 \).

For (2), if \( D' \) is a weakly admissible filtered \( (\varphi, N, G_K) \)-module and we are assuming \( D' = (V \otimes B_{\mathrm{st}})^{G_L} \) with \( V \) a potentially log-crystalline representation that becomes log-crystalline over the finite Galois extension \( L \) over \( K \). Since \( \alpha \) becomes trivial at the stalk \( \infty \), one has
\[
\mathcal{E}(D', \varphi, N, G_K) = \mathcal{E}(D', \varphi, G_K) = (V \otimes B_{\mathrm{st}})^{G_L} \otimes_{L_0} B^+_{\mathrm{dR}}
\]
with the diagonal action of \( G_K \). We have \( V|_{G_L} \) is log-crystalline representation of \( G_L \), so it is de Rham and satisfies:
\[
(V \otimes B_{\mathrm{st}})^{G_L} \otimes_{L_0} L = (V \otimes B_{\mathrm{dR}})^{G_L}
\]
And since \( V \) is a de Rham representation of \( G_K \). We have:
\[
(V \otimes B_{\mathrm{dR}})^{G_L} = (V \otimes B_{\mathrm{dR}})^{G_K} \otimes_K L.
\]
Tensoring everything with \( B^+_{\mathrm{dR}} \), we get what we want to prove. \( \square \)

From now on, we will always assume the data \( (D', \varphi, N, G_K) \) comes from a potentially log-crystalline representation \( V \) that becomes log-crystalline over a finite Galois extension \( L \) over \( K \). Using the \( G_K \)-equivariant filtration on \( D_L \), Fargues-Fontaine construct a \( G_K \)-equivariant modification \( \mathcal{E}(D', \varphi, N, \Fil^\bullet, G_K) \) of \( \mathcal{E}(D', \varphi, N, G_K) \) by letting:
\[
\mathcal{E}(D', \varphi, N, \Fil^\bullet, G_K)|_{X_{FF}\setminus \infty} = \mathcal{E}(D', \varphi, N, G_K)|_{X_{FF}\setminus \infty}
\]
and
\[
\mathcal{E}(D', \varphi, N, \Fil^\bullet, G_K) = \Fil^0(\mathcal{E}(D', \varphi, N, G_K)) = \Fil^0(D_L \otimes \Fil^\bullet(B_{\mathrm{dR}}))
\]
where the filtration on \( D_L \otimes \Fil^\bullet(B_{\mathrm{dR}}) \) is given by
\[
\Fil^k(D_L \otimes \Fil^\bullet(B_{\mathrm{dR}})) = \sum_{i+j=k} \Fil^i(D_L) \otimes \Fil^j(B_{\mathrm{dR}}).
\]

**Proposition 4.** If the filtered \( (\varphi, N, G_K) \)-module \( D' \) is weakly admissible, then \( \mathcal{E}(D', \varphi, N, \Fil^\bullet, G_K) \) is semistable of slope 0. Moreover, there is a \( G_K \)-equivariant isomorphism
\[
V = H^0(X_{FF}, \mathcal{E}(D', \varphi, N, \Fil^\bullet, G_K)),
\]
where $V$ is the potentially log-crystalline representation corresponds to the data $(D', \varphi, N, \text{Fil}^\bullet, G_K)$.

**Proof.** This is stated as [FF, §10.5.3, Remark 10.5.8] and the proof also works for potentially log-crystalline representations. Actually, let $V$ be the potentially log-crystalline representation corresponds to $(D', \varphi, N, \text{Fil}^\bullet, G_K)$ and let $E_V = V \otimes_{\mathbb{Q}_p} \mathcal{O}_X$ be corresponded slope 0 $\mathcal{O}_X$-representation of $G_K$. The theorem is equivalence to show

$$E_V = \mathcal{E}(D', \varphi, N, \text{Fil}^\bullet, G_K)$$

and we can prove it by comparing the $B$-pairs of $\mathcal{E}(D', \varphi, N, \text{Fil}^\bullet, G_K)$ and $\mathcal{E}_V$ by (4) of Theorem 4. While we have the $B_e$-part of the $\mathcal{O}_X$-representation $\mathcal{E}(D', \varphi, N, \text{Fil}^\bullet, G_K)$ is the same as the $B_e$-part of the $\mathcal{O}_X$-representation $\mathcal{E}(D', \varphi, N, G_K)$ by construction, and which is equal to

$$(D' \otimes_{B_{st}} L_0)^{\varphi=1, N=0}$$

by Corollary 10.3.17 of *loc.cit.* The $B_{dR}^+$-part of $\mathcal{E}(D', \varphi, N, \text{Fil}^\bullet, G_K)$ is the $B_{dR}^+$-representation

$$\text{Fil}^0(D_L' \otimes_{B_{dR}} B_{dR})$$

by definition.

On the other hand, the $B$-pair correspond to $\mathcal{E}_V$ is

$$(V \otimes_{\mathbb{Q}_p} B_e, V \otimes_{\mathbb{Q}_p} B_{dR}^+)$$.

Since $V$ is potentially log-crystalline, so we have $V \otimes_{\mathbb{Q}_p} B_e$ is potentially log-crystalline as a $B_e$-representation, which means there is a $G_K$-equivariant isomorphism

$$V \otimes_{\mathbb{Q}_p} B_e = \left( (V \otimes_{\mathbb{Q}_p} B_e) \otimes_{B_{st}} B_{st} \right)^{G_L} \otimes_{L_0} B_{st}^{\varphi=1, N=0}$$

by Proposition 10.3.20 of *loc.cit.* And $V \otimes_{\mathbb{Q}_p} B_e \otimes_{B_{st}} B_{st}^{G_L}$ is nothing but $D'$. Since $V$ is de Rham, so the $B_{dR}^+$-representation $V \otimes_{\mathbb{Q}_p} B_{dR}^+$ is generically flat in the sense of Definition 10.4.1 of *loc.cit.*, so Proposition 10.4.4 of *loc.cit.* shows that there is a $G_K$-equivariant isomorphism

$$V \otimes_{\mathbb{Q}_p} B_{dR}^+ = \text{Fil}^0(D_{dR}(V) \otimes_{K} B_{dR})$$.

The proposition follows from the fact $D_{dR}(V) \otimes_K L = D_{L}'$, and the $G_K$-equivariant filtration on $D_{dR}'$ descents to the filtration $D_{dR}(V)$ by the definition of weakly admissibility filtered $(\varphi, N, G_K)$-modules.

**Definition 5.** If the filtered $(\varphi, N, G_K)$-module $D'$ is weakly admissible, and let $V$ be the corresponded potentially log-crystalline representation, then the $G_K$-equivariant modification

$$\mathcal{E}(D', \varphi, N, \text{Fil}^\bullet, G_K) \longrightarrow \mathcal{E}(D', \varphi, N, G_K)$$

together with a $G_K$-stable lattice $T$ inside $V$ defines a shtuka with one leg at $\varphi^{-1}(x_{C_p})$ over $S = \text{Spa}(C)$. Moreover, since the correspondence in
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Theorem 8 is functorial, the shtuka constructed in this way carries a natural $G_K$-action.

A shtuka (resp. Hodge-Tate module or Breuil-Kisin-Fargues module) is called arithmetic if it (resp. the corresponded shtuka with one leg at $\varphi^{-1}(x_{C_p})$ over $\Spa(C)$) carries a $G_K$-action arisen as above.

Proposition 5. A Hodge-Tate module $(T, \Xi)$ together with a semilinear $G_K$-action is arithmetic if and only if $T$ is de Rham as a $G_K$-representation over $\mathbb{Z}_p$ and there is a $G_K$-equivalence isomorphism

$$(T, \Xi) \cong (T, (T \otimes \mathbb{Z}_p B_{dR})^{G_K} \otimes_K B_{dR}^+).$$

A Breuil-Kisin-Fargues module $\mathfrak{M}^{\text{inf}}$ is arithmetic if and only if its $\acute{\text{e}}$tale realization $T = (\mathfrak{M}^{\text{inf}} \otimes W(C))^{\varphi=1}$ is de Rham as a $G_K$-representation over $\mathbb{Z}_p$ and $\mathfrak{M}^{\text{inf}} \otimes B_{dR}^+$ has a $B_{dR}^+$-basis fixed by the $G_K$-action.

Proof. First, $(T, (T \otimes B_{dR})^{G_K} \otimes B_{dR}^+)$ is a Hodge-Tate module since $T$ is potentially log-crystalline so de Rham, we have $(T \otimes B_{dR})^{G_K} = D_{dR}(T \otimes \mathbb{Q}_p)$ is of full rank.

The rest of the proposition comes from Lemma 6 (2) and the correspondence in Corollary 4. $\square$

Corollary 5. Let $\mathfrak{M}^{\text{inf}}$ be an arithmetic Breuil-Kisin-Fargues module $\mathfrak{M}^{\text{inf}}$ then

1. $T(\mathfrak{M}^{\text{inf}})$ is log-crystalline if and only if the inertia subgroup $I_K$ of $G_K$ acts trivially on $\mathfrak{M}^{\text{inf}} \otimes A_{\text{inf}} W(\overline{k})$.
2. $T(\mathfrak{M}^{\text{inf}})$ is potentially crystalline if and only if there is a $\varphi$ and $G_K$ equivariant isomorphism

$$\mathfrak{M}^{\text{inf}} \otimes A_{\text{inf}} B^+ \cong \mathfrak{M}^{\text{inf}} \otimes_{W(\overline{k})} B^+.$$  

3. $T(\mathfrak{M}^{\text{inf}})$ is crystalline if it satisfies the conditions in (1) and (2).

Proof. Keep the notions as in the proof of Lemma 5. For (1), recall we have the fact that $T$ is log-crystalline if and only if $T|_{I_K}$ is log-crystalline, so we can assume $\overline{k}$ is algebraically closed and $L_0 = K_0$. And we have since $\alpha$ becomes trivial after $B^+ \to W(\overline{k})[\frac{1}{p}] = K_0$ from Lemma 5 (3), we have $\overline{\mathfrak{M}^{\text{inf}}}[\frac{1}{p}]$ with its $G_K$-action is nothing but $D'$ with the $G_K$-action coming from the $G_K$-structure of the filtered $(\varphi, N, G_K)$-module $D'$, and it is log-crystalline if and only if the filtered $(\varphi, N, G_K)$-module $D'$ is a filtered $(\varphi, N)$-module $D'$, i.e. $G_K$ acts trivially on $D'$.

For (2), again we can restrict our statement to $I_K$, and (2) means the shtuka comes from just modifying the vector bundle $E(D', \varphi, G_K)$ (there is not a twist by $\alpha$), i.e., $N = 0$. $\square$

4. Kisin modules and arithmetic shtukas

Let $K$ and $O_K$ as before, fix a uniformizer $\varpi$ of $O_K$ and $k = O_K/\varpi$ the residue field. We also fix a compatible system $\{\varpi^n\}$ of $p^n$-th roots of $\varpi$,
we define $K_\infty = \bigcup_{n=1}^{\infty} K(\varpi_n)$. The compatible system $\{\varpi_n\}$ also defines an element $([\varpi_n])$ in $\mathcal{O}_C = \varprojlim \mathcal{O}_{C_\varphi}/p$ and so an element $u = ([\varpi_n])$ in $A_{\inf}$. We have $k = \mathcal{O}_K/\varpi$ and we have $k = \varprojlim \mathcal{O}_K/\varpi \sim \varprojlim \mathcal{O}_{C_\varphi}/\varpi = \varprojlim \mathcal{O}_{C_\varphi}/p = \mathcal{O}_C$ by [BMS] Lemma 3.2, so $W(k)$ is a subring of $A_{\inf}$. Define $\mathcal{S}$ as the sub-$W(k)$-algebra of $A_{\inf}$ generated by $u$. One can check $\varphi_{A_{\inf}}(u) = u^p$, so in particular $\mathcal{S}$ in stable under $\varphi_{A_{\inf}}$, let $\varphi_{\mathcal{S}} = \varphi_{A_{\inf}}|\mathcal{S}$. We also have $G_{K_\infty}$ fix $u$ so $G_{K_\infty}$ acts trivially on $\mathcal{S}$. And we have the following commutative diagram:

\[
\begin{array}{ccc}
\mathcal{S} & \xrightarrow{\varphi_{\mathcal{S}}} & \mathcal{O}_K \\
\downarrow & & \downarrow \\
A_{\inf} & \xrightarrow{\varphi} & \mathcal{O}_{C_\varphi}
\end{array}
\]

the vertical arrows are faithful flat ring extensions by [BMS] and moreover, $\varphi|_{\mathcal{S}}$ is surjective and the kernel is generated by $E(u)$, which is an Eisenstein polynomial. All arrows in this diagram are $G_{K_\infty}$-equivalent.

Let $T$ be a log-crystalline representation of $G_K$ over $\mathbb{Z}_p$ with nonnegative Hodge-Tate weights, then Kisin in [Kis] can associate $T$ with a free Kisin module, i.e. a finite free $\mathcal{S}$-module $\mathcal{M}$ together a $\varphi_{\mathcal{S}}$-semilinear endomorphism $\varphi_{\mathcal{M}}$ such that the cokernel of the $\mathcal{S}$-linearization $1 \otimes \varphi_{\mathcal{M}} : \varphi_{\mathcal{S}} \mathcal{M} \to \mathcal{M}$ is killed by a power of $E(u)$. Moreover, if we define $\mathcal{M}_{\inf}(T) = \mathcal{M} \otimes_{\mathcal{S}, \varphi} A_{\inf}$ then one can show $\mathcal{M}_{\inf}(T)$ is a Breuil-Kisin-Fargues module as in Definition 4, it carries a natural $G_{K_\infty}$-semilinear action. We claim that there is an unique way to define a $G_K$-semilinear action on $\mathcal{M}_{\inf}(T)$ commutes with $\varphi_{\mathcal{M}_{\inf}(T)}$ extending the $G_{K_\infty}$-semilinear action such that

$$\left(\mathcal{M}_{\inf}(T) \otimes W(C)\right)^{\varphi = 1} = T.$$

In the case when $T$ is a potentially log-crystalline representation of $G_K$ over $\mathbb{Z}_p$ with nonnegative Hodge-Tate weights, and assume $L/K$ is a finite Galois extension such that $T|_{G_L}$ is log-crystalline. Then as before, $\mathcal{M}_{\inf}(T|_{G_L})$ carries a natural $G_{L_\infty}$-semilinear action for a choices of $L_\infty$. We make the claim:

**Proposition 6.** There is a unique way to extends the $G_{L_\infty}$-semilinear action on $\mathcal{M}_{\inf}(T) := \mathcal{M}_{\inf}(T|_{G_L})$ to an action of $G_K$, such that it commutes with $\varphi_{\mathcal{M}_{\inf}(T)}$ and

$$\left(\mathcal{M}_{\inf}(T) \otimes W(C)\right)^{\varphi = 1} = T.$$

**Proof.** As we mentioned in the introduction, we will prove this proposition by comparing the construction of Kisin of the $\mathcal{S}$-module and Fargues-Fontaine’s construction.

First, we need a brief review of the construction of Kisin module from log-crystalline representation: let $T$ is a potentially log-crystalline representation of $G_K$ over $\mathbb{Z}_p$, $L/K$ be a finite Galois extension such that $T|_{G_L}$ becomes log-crystalline, and let $L_0 = W(k_L)[\frac{1}{p}]$ and define $D' = (T \otimes B_{st})^{G_L}$.
as the filtered \((\varphi, N, G_K)\)-module associated with \(T \otimes \mathbb{Q}_p\). Then we obtain a filtered \((\varphi, N)\)-module \(D'\) over \(L\) or \((D', \varphi, N, \text{Fil}^\bullet)\) by forgetting the \(G_K\)-action. \(D'\) corresponds to the log-crystalline representation \(T \otimes \mathbb{Q}_p|_{G_L}\). Now let \(\mathcal{O}\) be the ring of rigid analytic functions over the open unit disc over \(L_0\) in the variable \(u\). Let \(\mathfrak{S} = W(k_L)[[u]]\), then one has \(\mathfrak{S}[1/p] \subset \mathcal{O}\) and there is a \(\varphi\O\) extending \(\varphi\mathfrak{S}\). Fix \((\varphi, N, \text{Fil}^\bullet)\) any choice of compatible system of \(p^n\)-th roots of a uniformizer \(\varpi_{L,0}\) of \(L\), then one can easily show that the the inclusion \(\mathfrak{S}[1/p] \rightarrow \mathfrak{A}_{\inf}[1/p]\) with \(u \mapsto [(\varpi_{L,n})]\) extends to an inclusion \(\mathcal{O} \rightarrow B^+\). Geometrically, \(\mathcal{O}\) (resp. \(B^+\)) is the locus \(\{p \neq 0\}\) of \(\text{Spa}(\mathfrak{S})\) (resp. \(\text{Spa}(\mathfrak{A}_{\inf})\)), and restrict the covering map \(\text{Spa}(\mathfrak{A}_{\inf}) \rightarrow \text{Spa}(\mathfrak{S})\) to these loci will give \(\mathcal{O} \rightarrow B^+\).

Given \((D', \varphi, N, \text{Fil}^\bullet)\), Kisin constructs a finite free module \(\mathcal{M}(D')\) over \(\mathcal{O}\) together with a \(\varphi\O\)-semilinear action [Kis §1.2].

To be more precise, for every \(n \geq 0\) consider the composition:

\[
\theta_n : \mathcal{O} \longrightarrow B^+ \xrightarrow{\varphi^{-n}} B^+ \xrightarrow{\theta} \mathbb{C}_p
\]

and let \(x_n\) be the closed points on the rigid open unit disc defined by \(\theta_n\). Now define

\[
\log u = \sum_{i=1}^{\infty} (-1)^{i-1} \frac{(u/\varpi_{L,0} - 1)^i}{i}
\]

One can check \(\log u \in B^+_{\text{dR}}\) and let \(\mathcal{O}_{st} = \mathcal{O}[\log u]\), the \(\mathcal{O}\)-algebra generated by \(\log u\) in side \(B^+_{\text{dR}}\). And extend the \(\varphi\) action to \(\log u\) by \(\varphi(\log u) = \nu \log u\) and define a \(\mathcal{O}\)-derivation \(N\) on \(\mathcal{O}_{st}^+\) by letting \(N(\log u) = -\lambda\) for some \(\lambda \in \mathcal{O}\). Kisin defines \(\mathcal{M}(D')\) as a modification of the vector bundle

\[
(\mathcal{O}[\log u] \otimes_{L_0} D)^{N=0}
\]

over \(\mathcal{O}\) along all the stalks at \(x_n\) for \(n \geq 0\). And the modifications are defined using the filtration on \(D_L\). As a result, the stalks of \(\mathcal{M}(D')\) away from \(\{x_n\}\) are isomorphic to those of \((\mathcal{O}[\log u] \otimes_{L_0} D')^{N=0}\). If we base change \(\mathcal{M}(D')\) to \(B^+\), and consider the closed points \(x_{-1}\) corresponds to

\[
B^+ \xrightarrow{\varphi} B^+ \xrightarrow{\theta} \mathbb{C}_p.
\]

We know the completion of \(B^+\) at \(x_{-1}\) is isomorphic to \(B^+_{\text{dR}}\) and the above arguments tell us that:

\[
\mathcal{M}(D') \otimes_{\mathcal{O}, \varphi} B^+_{\text{dR}} = (\mathcal{O}[\log u] \otimes_{L_0} D')^{N=0} \otimes_{\mathcal{O}, \varphi} B^+_{\text{dR}}.
\]

Moreover, Kisin shows there is a natural isomorphism [Kis Proposition 1.2.8.]:

\[
(\mathcal{O}[\log u] \otimes_{L_0} D')^{N=0} = (L_0[\log u] \otimes_{L_0} D')^{N=0} \otimes_{L_0} \mathcal{O} \xrightarrow{\eta \otimes id} D' \otimes_{L_0} \mathcal{O}.
\]

This tells us the stalk \((\mathcal{M}(D') \otimes_{\mathcal{O}} B^+) \otimes_{B^+, \varphi} B^+_{\text{dR}}\) is isomorphic to

\[
(D' \otimes_{L_0, \varphi} L_0) \otimes_{L_0} B^+_{\text{dR}} \cong D' \otimes_{L_0} B^+_{\text{dR}}.
\]
Moreover, using the theory of slope of Kedlaya, Kisin was able to prove that the \( \varphi \)-module \( \mathcal{M}(D') \) over \( \mathcal{O} \) descents to a Kisin module \( \mathfrak{M} \) over \( \mathcal{G} \) when \( D' \) is weakly admissible, i.e., \( \mathfrak{M} \otimes \mathcal{O} = \mathcal{M}(D') \). So we have
\[
\mathfrak{M} \otimes_{\mathcal{G}, \varphi} B^+ = \mathcal{M}(D') \otimes_{\mathcal{O}, \varphi} B^+.
\]
In particular, one has:
\[
\mathfrak{M} \otimes_{\mathcal{G}, \varphi} B_{\text{dR}}^+ = \mathcal{M}(D') \otimes_{\mathcal{O}, \varphi} B_{\text{dR}}^+ = D' \otimes_{L_0} B_{\text{dR}}^+.
\]
A theorem of Fontaine says that the ways of descent \( \mathcal{M}(D') \) to \( \mathfrak{M} \) are canonically corresponded with \( G_{L_0} \)-stable \( \mathbb{Z}_p \)-lattices in \( T \otimes \mathbb{Q}_p \), where \( L_0 = \bigcup_{n=1}^{\infty} L(\varpi_{L,n}) \). Then Kisin define \( \mathfrak{M} \) to be the \( \mathcal{G} \)-module descents \( \mathcal{M}(D') \) using the lattice \( T|_{G_{L_0}} \). This is the same as saying that
\[
(\mathfrak{M} \otimes_{\mathcal{G}} W(C))^{\varphi=1} = T|_{G_{L_0}}.
\]
Note that
\[
(\mathfrak{M} \otimes_{\mathcal{G}, \varphi} W(C))^{\varphi=1} = (\mathfrak{M} \otimes_{\mathcal{G}} W(C))^{\varphi=1}
\]
since \( W(C)^{\varphi=1} = \mathbb{Z}_p \). Now if we let \( \mathfrak{M}_{\text{inf}}(T) = \mathfrak{M} \otimes_{\mathcal{G}, \varphi} A_{\text{inf}} \), then the Hodge-Tate module of \( \mathfrak{M}_{\text{inf}}(T) \) is
\[
(T|_{G_{L_0}}, D' \otimes_{L_0} B_{\text{dR}}^+).
\]
To finish the proof, we let \( \mathfrak{M}_c^{\text{inf}} \) be the Breuil-Kisin-Fargues module corresponds to the constant Hodge-Tate module \( (T, T \otimes B_{\text{dR}}^+) \), then \( \mathfrak{M}_c^{\text{inf}} \) is equipped with an unique semilinear \( G_K \)-action coming from the action on \( T \). It is enough to show that there is an injection \( \mathfrak{M}_{\text{inf}}(T) \to \mathfrak{M}_c^{\text{inf}} \) such that \( \mathfrak{M}_{\text{inf}}(T) \) is stable under \( G_K \). From the construction in Corollary 4, it is enough to show the Hodge-Tate module corresponds to \( \mathfrak{M}_{\text{inf}}(T) \) injects to \( (T, T \otimes B_{\text{dR}}^+) \) and stable under \( G_K \) (the functor is left exact). But we have computed the the Hodge-Tate module corresponds to \( \mathfrak{M}_{\text{inf}}(T) \). Using the fact
\[
D' \otimes_{L_0} L = D_{\text{st}}(T \otimes \mathbb{Q}_p|_{G_L}) \otimes_{L_0} L = D_{\text{dR}}(T \otimes \mathbb{Q}_p|_{G_L}),
\]
and
\[
D_{\text{dR}}(T \otimes \mathbb{Q}_p) = D_{\text{dR}}(T \otimes \mathbb{Q}_p|_{G_L})^{G_K} = (D' \otimes_{L_0} L)^{G_K}.
\]
Then the proposition follows from that \( D_{\text{dR}}(T \otimes \mathbb{Q}_p) \otimes B_{\text{dR}}^+ \) injects into \( T \otimes B_{\text{dR}}^+ \). And when we extends the \( G_{L_0} \) action on \( T|_{G_{L_0}} \) to \( G_K \) by \( T \), \( D' \otimes_{L_0} B_{\text{dR}}^+ \) which equals to \( D_{\text{dR}}(T \otimes \mathbb{Q}_p) \otimes_K B_{\text{dR}}^+ \) is automatically stable under \( G_K \) inside \( T \otimes B_{\text{dR}}^+ \).

From the proof of the above proposition, we have

**Corollary 6.** Let \( T \) be a log-crystalline representation of \( G_K \) over \( \mathbb{Z}_p \) with nonnegative Hodge-Tate weights, and let \( \mathfrak{M}_{\text{inf}}(T) \) be the Breuil-Kisin-Fargues module with the semilinear \( G_K \)-action described as in the pervious proposition. Then \( \mathfrak{M}_{\text{inf}}(T) \) is arithmetic. Moreover, \( \mathfrak{M}_{\text{inf}}(T) \) corresponds to the shtuka associate with the \( G_K \)-equivariant modification:
\[
\mathcal{E}(D, \varphi, N, \text{Fil}^\bullet, G_K) \twoheadrightarrow \mathcal{E}(D, \varphi, N, G_K)
\]
together with the $G_K$-stable $\mathbb{Z}_p$-lattice $T$, where $(D, \varphi, N, \Fil^*, G_K)$ is the filtered $(\varphi, N, G_K)$-module corresponds to $T \otimes \mathbb{Q}_p$.

Proof. We have showed that the Hodge-Tate module of $\mathcal{M}^\inf(T)$ together with the $G_K$-action defined in the previous proposition is isomorphic to $(T, D_{\text{dR}}(T \otimes \mathbb{Q}_p) \otimes_K B_{\text{dR}}^+)$. Then use Proposition 5. □

Definition 6. [GL, F.7. Definition] Let $\mathcal{M}^\inf$ be a Breuil-Kisin-Fargues $G_K$-module. Then we say that $\mathcal{M}^\inf$ admits all descents over $K$ if the following conditions hold.

1. For any choice $\varpi$ of uniformizer of $O_K$ and any compatible system $\varpi^b = (\varpi_n)$ of $p^n$-th roots of $\varpi$, there is a Breuil-Kisin module $\mathcal{M}_{\varpi^b}$ defined using $\varpi^b$ such that $\mathcal{M}_{\varpi^b} \otimes_{\Phi, \varpi} A^\inf$ is isomorphic to $\mathcal{M}^\inf$ and $\mathcal{M}_{\varpi^b}$ is fixed by $G_{K, \varpi^b, \infty}$ under the above isomorphism, where $K_{\varpi^b, \infty} = \cup_n K(\varpi_n)$

2. Let $u_{\varpi^b} = [(\varpi_n)]$, then $\mathcal{M}_{\varpi^b} \otimes_{\Phi, \varpi} (\mathcal{S} / u_{\varpi^b} \mathcal{S})$ is independent of the choice of $\varpi$ and $\varpi^b$ as a $W(k)$-submodule of $\mathcal{M}^\inf \otimes_{A^\inf} W(k)$.

3. $\mathcal{M}_{\varpi^b} \otimes_{\Phi, \varpi} (\mathcal{S} / E(u_{\varpi^b}) \mathcal{S})$ is independent of the choice of $\varpi$ and $\varpi^b$ as a $O_K$-submodule of $\mathcal{M}^\inf \otimes_{A^\inf} (A^\inf / \xi A^\inf)$.

Remark 7. From Corollary 6 and if we further assume that the étale realization $T$ of an arithmetic Breuil-Kisin-Fargues module $\mathcal{M}^\inf$ is log-crystalline, we observe

$$\mathcal{M}^\inf = \mathcal{M}^\inf(T) = \mathcal{M} \otimes_{\Phi, \varpi} A^\inf.$$ We have $\mathcal{M}$ depends on the choice of $\{\varpi_n\}$, while the left hand side only depends on $T$ from Corollary 6.

Proposition 7. Let $\mathcal{M}^\inf$ be a Breuil-Kisin-Fargues $G_K$-module. Then $\mathcal{M}^\inf$ admits all descents over $K$ if and only if $\mathcal{M}^\inf$ is arithmetic and satisfies the condition (1) in Corollary 5, i.e., the inertia subgroup $I_K$ of $G_K$ acts trivially on $\mathcal{M}^\inf = \mathcal{M}^\inf \otimes_{A^\inf} W(k)$.

Proof. The if part of the proposition comes from Corollary 5 (1), Corollary 6 and Remark 7.

For the only if part of the proposition, one uses the lemma about Kummer extensions as stated in [GL, F.15. Lemma,]. Then it will imply that the submodules defined in (2) and (3) of Definition 6 are $G_K$-stable.

Then (3) in Definition 6 together with Lemma 1 will imply that $\mathcal{M}^\inf$ is arithmetic. And similarly, (2) in Definition 6 will force $I_K$ acts trivially on $\mathcal{M}^\inf = \mathcal{M}^\inf \otimes_{A^\inf} W(k)$. □

Remark 8. As been mentioned in [GL, F.12. Remark,], it is plausible that (2) and (3) in Definition 6 are actually consequences of (1). And one observes in the proof of Proposition 7, (2) + (3) implies $\mathcal{M}^\inf$ is arithmetic and $T(\mathcal{M}^\inf)$ is log-crystalline, so by Remark 7, $\mathcal{M}^\inf$ satisfies (1).
Corollary 7. (2) + (3) implies (1) in Definition 6.

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