Clubbed Binomial Approximation for the Lightbulb Process

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Abstract

In the so called lightbulb process, on days \( r = 1, \ldots, n \), out of \( n \) lightbulbs, all initially off, exactly \( r \) bulbs selected uniformly and independent of the past have their status changed from off to on, or vice versa. With \( W_n \) the number of bulbs on at the terminal time \( n \) and \( C_n \) a suitable clubbed binomial distribution,

\[
d_{TV}(W_n, C_n) \leq 2.7314 \sqrt{ne^{-(n+1)/3}} \quad \text{for all } n \geq 1.
\]

The result is shown using Stein’s method.

1 Introduction

The lightbulb process introduced by [3] was motivated by a pharmaceutical study of the effect of dermal patches designed to activate targeted receptors. An active receptor will become inactive, and an inactive one active, if it receives a dose of medicine released from the dermal patch. On each of \( n \) successive days \( r = 1, \ldots, n \) of the study, exactly \( r \) randomly selected receptors will each receive one dose of medicine from the patch, thus changing, or toggling, their status between the active and inactive states. We adopt the more colorful language of [3], where receptors are represented by lightbulbs that are being toggled between their on and off states.

Some fundamental properties of \( W_n \), the number of light bulbs on at the end of day \( n \), were derived in [3]. For instance, Proposition 2 of [3] shows that when \( n(n+1)/2 = 0 \) mod 2, or, equivalently, when \( n \) mod 4 \( \in \{0, 3\} \), the support of \( W_n \) is a subset of even integers up to \( n \), and that otherwise the support of \( W_n \) is a set of odd integers up to \( n \). Further, in [3], the mean and variance of \( W_n \) were computed, and based on numerical computations, an approximation of the distribution of \( W_n \) by the ‘clubbed’ binomial distribution was suggested.

To describe the clubbed binomial, let \( Z_n \) be a binomial \( \text{Bin}(n - 1, 1/2) \) random variable, and for \( i \in \mathbb{Z} \) let \( \pi^*_i = P(Z_n = i) \), that is

\[
\pi^*_i = \begin{cases} \binom{n-1}{i} \left(\frac{1}{2}\right)^{n-1} & \text{for } i = 0, 1, \ldots, n-1, \\ 0 & \text{otherwise.} \end{cases}
\]

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Let \( L_{1,n} \) and \( L_{0,n} \) denote the set of all odd and even numbers in \( \{0, 1, \ldots, n\} \), respectively. Define, for \( m = 0, 1 \),
\[
\pi_i^m = \begin{cases} 
\pi_{i-1}^* + \pi_i^*, & i \in L_{m,n}, \\
0, & i \notin L_{m,n}.
\end{cases}
\]
Summing binomial coefficients using ‘Pascal’s triangle’ yields
\[
\pi_i^m = \begin{cases} 
\binom{n}{i} \left(\frac{1}{2}\right)^{n-1}, & i \in L_{m,n}, \\
0, & i \notin L_{m,n}.
\end{cases}
\]
(1)

We say that the random variable \( C_{m,n} \) has the clubbed binomial distribution if \( P(C_{m,n} = i) = \pi_i^m \) for \( i \in L_{m,n} \). In words, the clubbed binomial distribution is formed by combining two adjacent cells of the binomial.

It was observed in \cite{3} that the clubbed binomial distribution appeared to approximate the lightbulb distribution \( W_n \) exponentially well. Here we make that observation rigorous by supplying an exponentially decaying bound in total variation. First, recall that if \( X \) and \( Y \) are two random variables with distributions supported on \( Z \), then the total variation distance between the (laws of) \( X \) and \( Y \), denoted \( d_{TV}(X, Y) \), is given by
\[
d_{TV}(X, Y) = \sup_{A \subseteq Z} |P(X \in A) - P(Y \in A)|.
\]
(2)

**Theorem 1.1** Let \( W_n \) be the total number of bulbs on at the terminal time in the lightbulb process of size \( n \) and let \( C_n = C_{m,n} \) where \( m = 0 \) for \( n \) mod 4 \( \in \{0, 3\} \) and \( m = 1 \) for \( n \) mod 4 \( \in \{1, 2\} \). Then
\[
d_{TV}(W_n, C_n) \leq 2.7314\sqrt{n}e^{-(n+1)/3}.
\]

In particular, the approximation error is less than 1\% for \( n \geq 21 \) and less than 0.1\% for \( n \geq 28 \).

A Berry-Esseen bound in the Kolmogorov metric of order \( 1/\sqrt{n} \) for the distance between the standardized value of \( W_n \) and the unit normal was derived in \cite{2}. The lightbulb chain was also studied in \cite{4}, and served there as a basis for the exploration of the more general class of Markov chains of multinomial type. One feature of such chains is their easily obtainable spectral decomposition, which informed the analysis in \cite{2}. In contrast, here we demonstrate the exponential bound in total variation using only simple properties of the lightbulb process.

After formalizing the framework for the lightbulb process in the next section, we prove Theorem 1.1 by Stein’s method. In particular, we develop a Stein operator \( \mathcal{A} \) for the clubbed binomial distribution and obtain bounds on the solution \( f \) of the associated Stein equation. The exponentially small distance between \( W_n \) and the clubbed binomial \( C_n \) can then be seen to be a consequence of the vanishing of the expectation of \( \mathcal{A}f \) except on a set of exponentially small probability.

## 2 The lightbulb process

We now more formally describe the lightbulb process. With \( n \in \mathbb{N} \) fixed we will let \( X = \{X_{rk} : r = 0, 1, \ldots, n, k = 1, \ldots, n\} \) denote a collection of Bernoulli variables. For \( r \geq 1 \) these ‘switch’ or ‘toggle’ variables have the interpretation that
\[
X_{rk} = \begin{cases} 
1 & \text{if the status of bulb } k \text{ is changed at stage } r, \\
0 & \text{otherwise}.
\end{cases}
\]
We take the initial state of the bulbs to be given deterministically by setting the switch variables \( \{X_{0k}, k = 1, \ldots, n\} \) equal to zero, that is, all bulbs begin in the off position. At stage \( r \) for \( r = 1, \ldots, n \), \( r \) of the \( n \) bulbs are chosen uniformly to have their status changed, with different stages mutually independent. Hence, with \( e_1, \ldots, e_n \in \{0, 1\} \), the joint distribution of \( X_{r1}, \ldots, X_{rn} \) is given by

\[
P(X_{r1} = e_1, \cdots, X_{rn} = e_n) = \begin{cases} \binom{n}{r}^{-1} & \text{if } e_1 + \cdots + e_n = r, \\ 0 & \text{otherwise,} \end{cases}
\]

with the collections \( \{X_{r1}, \ldots, X_{rn}\} \) independent for \( r = 1, \ldots, n \).

Clearly, at each stage \( r \) the variables \( (X_{r1}, \cdots, X_{rn}) \) are exchangeable.

For \( r, i = 1, \ldots, n \), the quantity \( (\sum_{s=1}^r X_{si}) \mod 2 \) is the indicator that bulb \( i \) is on at time \( r \) of the lightbulb process, so letting

\[
I_i = \left( \sum_{r=0}^n X_{ri} \right) \mod 2 \quad \text{and} \quad W_n = \sum_{i=1}^n I_i,
\]

the variable \( I_i \) is the indicator that bulb \( i \) is on at the terminal time, and \( W_n \) is the number of bulbs on at the terminal time.

The lightbulb process is a special case of a class of multivariate chains studied in [4], where randomly chosen subsets of \( n \) individual particles evolve according to the same marginal Markov chain. As shown in [4], such chains admit explicit full spectral decompositions, and in particular, the transition matrices for each stage of the lightbulb process can be simultaneously diagonalized by a Hadamard matrix. These properties were applied in [3] for the calculation of the moments needed to compute the mean and variance of \( W_n \) and to develop recursions for the exact distribution, and in [2] for a Berry-Esseen bound of the standardized \( W_n \) to the normal.

### 3 Stein Operator

In order to apply Stein’s method, we first develop a Stein equation for the clubbed binomial distribution \( C_{m,n} \) and then present bounds on its solution. With \( \pi_m \) given by (1), let \( \pi^m(A) = \sum_{x \in A} \pi^m_x \). Set \( \alpha_x = (n-x)(n-1-x) \) and \( \beta_x = x(x-1) \) for \( x \in \{0, \ldots, n\} \). One may easily directly verify the balance equation

\[
\alpha_{x-2}\pi^m_{x-2} = \beta_x\pi^m_x \quad \text{for } x \in L_{m,n},
\]

which gives the generator of the distribution of \( C_{m,n} \) as

\[
Af(x) = \alpha_x f(x+2) - \beta_x f(x), \quad \text{for } x \in L_{m,n}.
\]

For \( A \subset L_{m,n} \), we consider the Stein equation

\[
Af_A(x) = 1_A(x) - \pi^m(A), \quad x \in L_{m,n}.
\]

For a function \( g \) with domain \( A \) let \( \|g\| \) denote \( \sup_{x \in A} |g(x)| \).

**Lemma 3.1** For \( m \in \{0, 1\} \) and \( A = \{r\} \) with \( r \in L_{m,n} \), the unique solution \( f^m_r(x) \) of (5) on \( L_{m,n} \) satisfying the boundary condition \( f^m_r(m) = 0 \) is given, for \( m < x \leq n, x \in L_{m,n} \), by

\[
f^m_r(x) = \begin{cases} -\frac{\pi^m([0,x-2] \cap L_{m,n})\pi^m_m}{\pi^m([x,n] \cap L_{m,n})\beta_x\pi^m_x} & \text{for } m < x < r + 2 \\ \frac{\beta_x\pi^m_x}{\pi^m([x,n] \cap L_{m,n})\pi^m_m} & \text{for } r + 2 \leq x \leq n. \end{cases}
\]
Furthermore, for all $A \subset L_{m,n}$, $f^m_A(x) = \sum_{r \in A} f^m_r(x)$ is a solution of (5) and satisfies
$$\|f^m_A\| \leq \frac{2.7314}{\sqrt{n(n-1)}} \text{ for } n \geq 1.$$

Lemma 3.1 is proved in Section 4.

Applying Lemma 3.1, we now prove our main result.

**Proof of Theorem 1.1:** Fix $m \in \{0,1\}$ and $A \subset L_{m,n}$, and let $f := f^m_A$ be the solution to (5). Dropping subscripts, let $W_i = \sum_{i=1}^n I_i$, where $I_i$ is the indicator that bulb $i$ is on at the terminal time. For $i, j \in \{1, \ldots, n\}$, now with slight abuse of notation, let $W_i = W - I_i$, and for $i \neq j$ set $W_{ij} = W - I_i - I_j$. Then

$$E(n - W)(n - 1 - W)f(W + 2) = E \sum_{i=1}^n (1 - I_i)(n - 1 - W)f(W_i + 2) = E \sum_{i \neq j} (1 - I_i)(1 - I_j)f(W_{ij} + 2),$$

and similarly,

$$EW(W - 1)f(W) = E \sum_{i=1}^n I_i W_i f(W_i + 1) = E \sum_{i \neq j} I_i I_j f(W_i + 1) = E \sum_{i \neq j} I_i I_j f(W_{ij} + 2).$$

By Proposition 2 of [3], $P(W \in L_{m,n}) = 1$, and hence (5) holds upon replacing $x$ by $W$. Taking expectation and using the expression for the generator in (4), we obtain

$$P(W \in A) - \pi^m(A) = EAf(W) = E \sum_{i \neq j} ((1 - I_i)(1 - I_j) - I_i I_j) f(W_{ij} + 2). \quad (7)$$

Recalling that $X_{rk}$ is the value of the switch variable at time $r$ for bulb $k$, let $A_{ij}$ be the event that the switch variables of the distinct bulbs $i$ and $j$ differ in at least one stage, that is, let

$$A_{ij} = \bigcup_{r=1}^n \{X_{ri} \neq X_{rj}\}. \quad (8)$$

Now using (7) we obtain

$$|P(W \in A) - \pi^m(A)| = \left| E \sum_{i \neq j} ((1 - I_i)(1 - I_j) - I_i I_j) f(W_{ij} + 2) \right| \leq \sum_{i \neq j} E \left| ((1 - I_i)(1 - I_j) - I_i I_j) f(W_{ij} + 2) 1_{A_{ij}} \right| \leq \sum_{i \neq j} E \left| ((1 - I_i)(1 - I_j) - I_i I_j) f(W_{ij} + 2) 1_{A_{ij}} \right|. \quad (9)$$

Note that $I_i, I_j \in \{0, 1\}$ implies

$$(1 - I_i)(1 - I_j) 1_{I_i \neq I_j} = 0 = I_i I_j 1_{I_i \neq I_j},$$

and hence

$$P(W \in A) - \pi^m(A) = E \sum_{i \neq j} ((1 - I_i)(1 - I_j) - I_i I_j) f(W_{ij} + 2) 1_{A_{ij}}.$$
and hence for the first term in (9) we obtain

$$\sum_{i \neq j} ((1 - I_i)(1 - I_j) - I_iI_j) f(W_{ij} + 2) 1_{A_{ij}} = \sum_{i \neq j} ((1 - I_i)(1 - I_j) - I_iI_j) f(W_{ij} + 2) 1_{A_{ij}, I_i = I_j}. \tag{10}$$

For a given pair $i, j$, on the event $A_{ij}$ let $t$ be any index for which $X_{ti} \neq X_{tj}$, and let $X^{ij}$ be the collection of switch variables given by

$$X^{ij}_{rk} = \begin{cases} X_{rk} & r \neq t, \\ X_{tk} & r = t, k \notin \{i, j\}, \\ X_{ti} & r = t, k = j, \\ X_{tj} & r = t, k = i. \end{cases}$$

In other words, in stage $t$, the unequal switch variables $X_{ti}$ and $X_{tj}$ are interchanged, and all other variables are left unchanged. Let $I^{ij}_i$ be the status of bulb $k$ at the terminal time when applying switch variables $X^{ij}$, and similarly set $W^{ij}_{ij} = \sum_{k \notin \{i, j\}} I^{ij}_k$. Note that as the status of both bulbs $i$ and $j$ are toggled upon interchanging their stage $t$ switch variables, and all other variables are unaffected, we obtain

$$I^{ij}_i = 1 - I_i, \quad I^{ij}_j = 1 - I_j \quad \text{and} \quad W^{ij}_{ij} = W_{ij}.$$  

In particular, $I_i = I_j$ if and only if $I^{ij}_i = I^{ij}_j$, and, with $A^{ij}_{ij}$ as in (8) with $X^{ij}_{rk}$ replacing $X_{rk}$, we have additionally that $A^{ij}_{ij} = A_{ij}$. Further, by exchangeability we have $\mathcal{L}(X) = \mathcal{L}(X^{ij})$. Therefore,

$$E(1 - I_i)(1 - I_j)f(W_{ij} + 2) 1_{A_{ij}, I_i = I_j} = E(1 - I^{ij}_i)(1 - I^{ij}_j)f(W^{ij}_{ij} + 2) 1_{A^{ij}_{ij}, I^{ij}_i = I^{ij}_j} = EI_iI_jf(W_{ij} + 2) 1_{A_{ij}, I_i = I_j},$$

showing, by (10), that the first term in (9) is zero. Therefore,

$$|P(W \in A) - \pi^m(A)| \leq \sum_{i \neq j} E ((1 - I_i)(1 - I_j) - I_iI_j) f(W_{ij} + 2) 1_{A^{ij}_{ij}} \leq \|f\| \sum_{i \neq j} P(A^{ij}_{ij}).$$

As $A^{ij}_{ij}$ is the event that the switch variables of $i$ and $j$ are equal in every stage, recalling that these variables are independent over stages we obtain

$$P(A^{ij}_{ij}) = \prod_{r=1}^{n} \frac{r(r - 1) + (n - r)(n - 1 - r)}{n(n - 1)} = \prod_{r=1}^{n} \left( 1 - \frac{2(nr - r^2)}{n(n - 1)} \right) \leq e^{-\frac{2}{n(n-1)} \sum_{r=1}^{n}(nr-r^2)} = e^{-(n+1)/3}.$$

Hence, by Lemma 3.1

$$|P(W \in A) - \pi^m(A)| \leq \frac{2.7314}{\sqrt{n(n-1)}} n(n-1) e^{-(n+1)/3} = 2.7314 \sqrt{n} e^{-(n+1)/3}.$$

Taking supremum over $A$ and applying definition (2) completes the proof. \qed
4 Bounds on the Stein equation

In this section we present the proof of Lemma 3.1.

Proof: Let \( m \in \{0, 1\} \) be fixed. First, the equalities \( f(m) = 0 \) and

\[
f(x + 2) = \frac{1_A(x) - \pi^m(A) + \beta_x f(x)}{\alpha_x}
\]

specify \( f(x) \) on \( L_{m,n} \) uniquely, hence the solution to \( \alpha \) satisfying the given boundary condition is unique.

Next, with \( r \in L_{m,n} \), we verify that \( f^m_r(x) \) given by \( \beta \) solves \( \gamma \) with \( A = \{r\} \); that \( f^m_r(m) = 0 \) is given. For \( m < x < r, x \in L_{m,n} \), applying the balance equation \( \delta \) to obtain the second equality, we have

\[
\alpha_x f^m_r(x + 2) - \beta_x f^m_r(x) = \alpha_x \left( \frac{\pi^m([0, x] \cap L_{m,n})\pi^m_r}{\beta_x + 2\pi^m_{x+2}} \right) - \beta_x \left( \frac{\pi^m([0, x - 2] \cap L_{m,n})\pi^m_r}{\beta_x \pi^m_x} \right)
\]

\[
= \alpha_x \left( \frac{\pi^m([0, x] \cap L_{m,n})\pi^m_r}{\alpha_x \pi^m_x} \right) - \beta_x \left( \frac{\pi^m([0, x - 2] \cap L_{m,n})\pi^m_r}{\beta_x \pi^m_x} \right)
\]

\[
= -\pi^m_r.
\]

If \( x = r \)

\[
\alpha_x f^m_r(x + 2) - \beta_x f^m_r(x) = \alpha_r \left( \frac{\pi^m([r, x] \cap L_{m,n})\pi^m_r}{\beta_x + 2\pi^m_{x+2}} \right) - \beta_r \left( \frac{\pi^m([0, r - 2] \cap L_{m,n})\pi^m_r}{\beta_x \pi^m_x} \right)
\]

\[
= \alpha_r \left( \frac{\pi^m([r, x] \cap L_{m,n})\pi^m_r}{\alpha_r \pi^m_x} \right) - \beta_r \left( \frac{\pi^m([0, r - 2] \cap L_{m,n})\pi^m_r}{\beta_x \pi^m_x} \right)
\]

\[
= \pi^m([r, x] \cap L_{m,n}) + \pi^m([0, r - 2] \cap L_{m,n}) = 1 - \pi^m_r.
\]

If \( x > r \)

\[
\alpha_x f^m_r(x + 2) - \beta_x f^m_r(x) = \alpha_x \left( \frac{\pi^m([x, r] \cap L_{m,n})\pi^m_r}{\beta_x + 2\pi^m_{x+2}} \right) - \beta_x \left( \frac{\pi^m([x, r] \cap L_{m,n})\pi^m_r}{\beta_x \pi^m_x} \right)
\]

\[
= \alpha_x \left( \frac{\pi^m([x, r] \cap L_{m,n})\pi^m_r}{\alpha_x \pi^m_x} \right) - \beta_x \left( \frac{\pi^m([x, r] \cap L_{m,n})\pi^m_r}{\beta_x \pi^m_x} \right)
\]

\[
= -\pi^m_r.
\]

Hence \( f^m_r(x) \) solves \( \gamma \).

Next, to consider the solution of \( \beta \) more generally for \( A \subset L_{m,n} \) and \( x \in L_{m,n} \), letting

\[
U_{m,x} = [0, x - 2] \cap L_{m,n} \quad \text{and} \quad U_{m,x}^c = L_{m,n} \setminus U_{m,x},
\]

we may write \( \beta \) more compactly as

\[
f^m_r(x) = \frac{1}{\beta_x \pi^m_x} \left( \pi^m(U_{m,x}^c)\pi^m_r([r] \cap U_{m,x}) - \pi^m(U_{m,x})\pi^m_r([r] \cap U_{m,x}) \right).
\]
By linearity, the solution of (5) for \( A \subset L_{m,n} \) is given by \( f^m_A(m) = 0 \), and for \( x > m, x \in L_{m,n} \), by

\[
f^m_A(x) = \frac{1}{\beta_x \pi_x^m} \left( \pi^m(U^c_{m,x}) \pi^m(A \cap U_{m,x}) - \pi^m(U_{m,x}) \pi^m(A \cap U^c_{m,x}) \right)
\]

(cf [1], p. 7), and so, for all \( x \in L_{m,n} \),

\[
- \frac{1}{\beta_x \pi_x^m} \pi^m(U_{m,x}) \pi^m(U^c_{m,x}) \leq f^m_A(x) \leq \frac{1}{\beta_x \pi_x^m} \pi^m(U^c_{m,x}) \pi^m(U_{m,x}),
\]

or that

\[
|f^m_A(x)| \leq \frac{1}{\beta_x \pi_x^m} \pi^m(U_{m,x}) \pi^m(U^c_{m,x}). \tag{11}
\]

Since \( f^m_A(m) = 0 \) and the upper bound of Lemma 3.1 reduces to \( \infty \) if \( 0 \leq n \leq 1 \), we only need to bound \( f^m_A(x) \) for \( n \geq 2 \) and \( x \geq 2 \). Direct computation using (11) gives \( |f^m_A(2)| \leq 1/4 \) for \( n = 2 \), \( |f^m_A(2)| \leq 1/8 \) and \( |f^m_A(3)| \leq 1/8 \) for \( n = 3 \), \( |f^m_A(2)| = |f^m_A(4)| \leq 7/96 \) and \( |f^m_A(3)| \leq 1/12 \) for \( n = 4 \). Therefore, it remains to prove Lemma 3.1 for \( n \geq 5 \).

Noting that for \( x \geq \frac{n}{2} + 1 \) we have \( \beta_x \geq \left( \frac{n}{2} + 1 \right) \frac{1}{2} \), and for \( x < \frac{n}{2} + 1 \) that \( \alpha_{x-2} = (n - x + 2)(n - x + 1) > \left( \frac{n}{2} + 1 \right) \frac{n}{2} \), using (3), we obtain from (11) that

\[
|f^m_A(x)| \leq \begin{cases} 
\frac{\pi^m(U_{m,x}) \pi^m(U^c_{m,x})}{\beta_x \pi_x^m} & \text{if } x \geq \frac{n}{2} + 1, \\
\frac{\pi^m(U_{m,x}) \pi^m(U^c_{m,x})}{\alpha_{x-2} \pi_{x-2}^m} & \text{if } x < \frac{n}{2} + 1.
\end{cases}
\tag{12}
\]

Clearly, for \( i \geq x \),

\[
\frac{\pi^m_i}{\pi^m_x} = \frac{n}{i} = \left\{ \begin{array}{ll}
\frac{1}{(n-1)\cdots(n-i+1)} & \text{if } i = x \\
1 & \text{if } i \geq x + 2.
\end{array} \right.
\]

Hence, we can write, for \( i \geq x + 2 \),

\[
\frac{\pi^m_i}{\pi^m_x} = \frac{(n-x)}{x+1} \frac{(n-x-1)}{x+2} \cdots \frac{(n-i+1)}{i} = \prod_{y=0}^{i-x-1} \frac{n-x-y}{x+1+y} \tag{13}
\]

Note that as \( (n-x)/(x+1) \leq 1 \) for \( x \geq n/2 \), the terms in the product (13) are decreasing. In particular,

\[
\frac{\pi^m_i}{\pi^m_x} \leq 1 \quad \text{for } i \geq x, \quad \prod_{0 \leq y \leq \lfloor \sqrt{n} \rfloor} \frac{n-x-y}{x+1+y} \leq 1 \quad \text{provided } x \geq \frac{n}{2}. \tag{14}
\]

For \( n \) even let \( x_s = n/2 \), and for \( n \) odd let \( x_s = (n-1)/2 \) when \( m = 0 \), and \( x_s = (n+1)/2 \) when \( m = 1 \). Then, except for the case where \( m = 0 \) and \( x = (n+1)/2 \), which we deal with separately, we have

\[
\pi^m(U_{m,x}) \pi^m(U^c_{m,x}) = \pi^m(U_{m,2x_s-x+2}) \pi^m(U^c_{m,2x_s-x+2}),
\]

and we may therefore assume \( x \geq x_s + 1 \), and so \( x \geq n/2 + 1 \).
Since for \( y \geq \sqrt{n}/2 \), recalling \( x \geq n/2 + 1 \), we have

\[
\frac{n - x - y}{x + 1 + y} \leq \frac{n - \left(\frac{n+1}{2}\right) - \frac{\sqrt{n}}{2}}{\left(1 + \frac{n}{2}\right) + 1 + \frac{\sqrt{n}}{2}} = \frac{n - \frac{\sqrt{n}}{2} - 1}{\frac{n}{2} + 2 + \frac{\sqrt{n}}{2}} = 1 - \frac{\sqrt{n} + 3}{\frac{n}{2} + 2 + \frac{\sqrt{n}}{2}},
\]

applying (13) and (14) we conclude that

\[
\frac{\pi_i^m}{\pi_x^m} \leq \left(1 - \frac{\sqrt{n} + 3}{\frac{n}{2} + 2 + \frac{\sqrt{n}}{2}}\right)^{i-x-[\frac{\sqrt{n}}{2}]-1} \quad \text{for } i \geq x + [\frac{\sqrt{n}}{2}].
\]

Hence, applying (14) again, here to obtain the second inequality, we have

\[
\frac{1}{\pi_x^m} \pi_i^m (U_{m,x}^c) \leq \frac{1}{(\frac{n}{2} + 1) \frac{n}{2}} \left(\sum_{x \leq i \leq x + [\frac{\sqrt{n}}{2}], i \in L_{m,n}} \frac{\pi_i^m}{\pi_x^m} + \sum_{i \geq x + [\frac{\sqrt{n}}{2}], i \in L_{m,n}} \frac{\pi_i^m}{\pi_x^m}\right) \leq \frac{1}{(\frac{n}{2} + 1) \frac{n}{2}} \left(\frac{\sqrt{n}}{4} + 1\right) + \sum_{j=0}^{\infty} \left(1 - \frac{\sqrt{n} + 3}{\frac{n}{2} + 2 + \frac{\sqrt{n}}{2}}\right)^{2j} \leq \frac{2.7314}{\sqrt{n}(n-1)} \quad \text{for } n \geq 1.
\]

This final inequality is obtained by determining the maximum of the function

\[
g_1(n) := \frac{1}{(\frac{n}{2} + 1) \frac{n}{2}} \left(\frac{\sqrt{n}}{4} + 1 + \frac{1}{1 - \left(1 - \frac{\sqrt{n} + 3}{\frac{n}{2} + 2 + \frac{\sqrt{n}}{2}}\right)^2}\right) \sqrt{n}(n-1)
\]

by noting \( g_1(n) < 1 + \frac{4}{\sqrt{n}} + \frac{(n+4+\sqrt{n})^2}{(n+2)(n+3)\sqrt{n}} < 2.5 \) for \( n \geq 64 \) and \( \max_{1 \leq n \leq 63} g_1(n) = g_1(9) = 2.7313131 \ldots \)

Lastly we handle the situation where \( n \) is odd, \( m = 0 \) and \( x = (n+1)/2 =: x_0 \in L_{0,n} \), in which case \( n = 3 \mod 4 \). In place of (13), we have, for \( y \geq \sqrt{n}/2 \),

\[
\frac{n - x_0 - y}{x_0 + 1 + y} \leq \frac{n - \left(\frac{n+1}{2}\right) - \frac{\sqrt{n}}{2}}{\left(\frac{n+1}{2}\right) + 1 + \frac{\sqrt{n}}{2}} = 1 - \frac{4 + 2\sqrt{n}}{n + 3 + \sqrt{n}}.
\]

Since (14) is valid for all \( x \geq n/2 \), in view of (13) we obtain the bound

\[
\frac{\pi_i^m}{\pi_{x_0}^m} \leq \left(1 - \frac{4 + 2\sqrt{n}}{n + 3 + \sqrt{n}}\right)^{i-x_0-[\frac{\sqrt{n}}{2}]-1} \quad \text{for } i \geq x_0 + [\frac{\sqrt{n}}{2}].
\]
Using (14) again for the first inequality we have
\[
\frac{1}{2 \left( \frac{n}{2} + 1 \right) \frac{n}{2}} \frac{\pi^m(U_{m,x_0})}{\pi^m_{x_0}} \frac{\pi^m(U^c_{m,x_0})}{\pi^m_{x_0}}
\]
\[
= \frac{1}{2 \left( \frac{n}{2} + 1 \right) \frac{n}{2}} \left( \sum_{x_0 \leq i \leq x_0 + \left\lfloor \frac{\sqrt{n}}{2} \right\rfloor, i \in L_{m,n}} \frac{\pi^m_{i}}{\pi^m_{x_0}} + \sum_{i \geq x_0 + \left\lfloor \frac{\sqrt{n}}{2} \right\rfloor + 1, i \in L_{m,n}} \frac{\pi^m_{i}}{\pi^m_{x_0}} \right)
\]
\[
\leq \frac{1}{(\frac{n}{2} + 1) n} \left( (\frac{\sqrt{n}}{4} + 1) + \sum_{j=0}^{\infty} \left( 1 - \frac{4 + 2\sqrt{n}}{n + 3 + \sqrt{n}} \right)^j \right)
\]
\[
= \frac{1}{(\frac{n}{2} + 1) n} \left( \frac{\sqrt{n}}{4} + 1 + \frac{n + 3 + \sqrt{n}}{4 + 2\sqrt{n}} \right)
\]
\[
\leq \frac{1.638496535}{\sqrt{n(n-1)}} \text{ for } n \geq 1,
\]
where the last inequality is from bounding the function
\[
g_2(n) := \frac{1}{(\frac{n}{2} + 1) n} \left( \frac{\sqrt{n}}{4} + 1 + \frac{n + 3 + \sqrt{n}}{4 + 2\sqrt{n}} \right) (n-1)\sqrt{n},
\]
with \( g_2(n) \leq \frac{1}{2} + \frac{2}{\sqrt{n}} + \frac{n + 3 + \sqrt{n}}{n + 2\sqrt{n}} \leq 1.6 \) for \( n \geq 400 \) and \( \max_{1 \leq n \leq 399} g_2(n) = g_2(23) = 1.638496535. \)

The result now follows from combining the estimates (16), (17) and (12).\[\square\]

We remark that a direct argument using Stirling’s formula for the case \( x = \left\lfloor n/2 \right\rfloor \) shows that the best order that can be achieved for the estimate of \( f_{\frac{n}{2}}^m \) is \( O(n^{-3/2}) \).

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