ISO-CONTACT EMBEDDINGS OF MANIFOLDS IN CO-DIMENSION 2

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Abstract. The purpose of this article is to study co-dimension 2 iso-contact embeddings of closed contact manifolds. We first show that a closed contact manifold \((M^{2n-1}, \xi_M)\) iso-contact embeds in a contact manifold \((N^{2n+1}, \xi_N)\), provided \(M\) contact embeds in \((N, \xi_N)\) with a trivial normal bundle and the contact structure induced on \(M\) via this embedding is homotopic as an almost-contact structure to \(\xi_M\). We apply this result to first establish that a closed contact 3–manifold having no 2–torsion in its second integral cohomology iso-contact embeds in the standard contact 5–sphere if and only if the first Chern class of the contact structure is zero. Finally, we discuss iso-contact embeddings of closed simply connected contact 5–manifolds.

1. Introduction

The study of embeddings of manifolds in Euclidean spaces has been a very classical and well studied topic which has lead to the development of many important tools in geometric topology. H. Whitney in [Wh] established that every smooth \(n\)-manifold admits an embedding in \(\mathbb{R}^{2n}\). He also demonstrated that \(\mathbb{R}P^2\) does not admit an embedding in \(\mathbb{R}^4\). This showed that the result Whitney obtained is optimal in general. However, M. Hirsch generalized the result for odd dimensional closed orientable manifolds to establish that every \((2n+1)\)-dimensional manifold admits an embedding in \(\mathbb{R}^{4n-1}\). This, in particular, implies that every closed orientable 3–manifold admits an embedding in \(\mathbb{R}^5\).

Another direction along which there is a great development related to embeddings of manifolds can be described as follows. Let \(M\) be a manifold together with a geometric structure \(G\). Is it possible to embed the manifold \(M\) together with the given geometric structure \(G\) inside a Euclidean space such that the standard geometric structure on the Euclidean space induces the given geometric structure \(G\) on the manifold?

J. Nash in [Na] established that every closed Riemannian \(n\)-manifold admits a \(c^1\)-isometric embedding in \(\mathbb{R}^{3n(3n+11)}\)-dimensional flat Euclidean space. M. Gromov in [Gr] formally initiated the study of embeddings of manifolds in Euclidean spaces such that the standard symplectic or contact structure on Euclidean spaces when pull-backed by these diffeomorphisms induces the given symplectic or contact structure on the manifolds under consideration. Gromov in [Gr] using his convex integration technique which generalizes the technique developed by Nash in [Na] essentially provided a complete understanding when the co-dimension of the embedding is bigger than or equal to 4. We will make this more precise a little later after recalling all the necessary terminologies.

The purpose of this article is to study iso-contact embeddings of closed contact manifolds in co-dimension 2, the co-dimension for which the techniques developed by Gromov in [Gr] are generally not sufficient for a complete answer. The study of co-dimension 2 iso-contact embeddings of closed contact manifolds was initiated by J. Etnyre and R. Fukuwara in [EF].

Recall that by a contact structure on a manifold \(M\), we mean a nowhere integrable hyperplane field \(\xi\) on \(M\). The contact structure is said to be co-orientable, provided \(\xi\) is the kernel of a 1–form defined on \(M\). A contact manifold \(M\) with the contact structure \(\xi\) is denoted by the pair \((M, \xi)\). When \(\xi\) is co-oriented and \(\xi\) is the kernel of a 1–form \(\alpha\) defined on \(M\), then we also denote the contact manifold \((M, \xi)\) by the pair...
(M, Ker{α}). In this article, we will always work with co-orientable contact structures defined on orientable manifolds.

Let (M₁, Ker{α₁}) and (M₂, Ker{α₂}) be two contact manifolds. We say that (M₁, Ker{α₁}) admits an iso-contact embedding in (M₂, Ker{α₂}), provided there exists a smooth embedding \( f : M₁ \hookrightarrow M₂ \) such that \( f^∗\alpha₂ = gα₁ \), for some everywhere positive function \( g : M \to \mathbb{R} \).

In case, a manifold \( M₁ \) admits an embedding into a contact manifold \( (M₂, ρ₂) \) such that the restriction of \( ρ₂ \) to \( M₁ \) is a contact structure on \( M₁ \), we say \( M₁ \) admits a contact embedding in \( (M₂, ρ₂) \).

For a contact manifold \( (M₁, Ker{α₁}) \) to admit an iso-contact embedding in the manifold \( (M₂, Ker{α₂}) \), there must exist a smooth embedding \( f : M₁ \hookrightarrow M₂ \) and a monomorphism \( F : TM₁ \to T\partial M₂ \) which covers the map \( f \) and satisfies the property that the bundle \( (F_∗ξ₁, F_∗dα₁) \) is a conformal symplectic sub-bundle of \( (M₂, Ker{α₂}) \). If such a pair \( (f, F) \) exists, then we say that we have a formal iso-contact embedding of \( (M₁, Ker{α₁}) \) in \( (M₂, Ker{α₂}) \). We refer to [EM, Chpt-12] for more on formal iso-contact embeddings.

Let \( (N, ρ) \) be a contact manifold. Following [EM], we say that the problem of iso-contact embeddings of a collection \( A \) of contact manifolds abide by the \( h \)-principle, provided every formal iso-contact embedding of a contact manifold \( (M, ρ_M) \in A \) in \( (N, ρ) \) can be isotoped to an iso-contact embedding of \( (M, ρ_M) \) in \( (N, ρ) \).

Questions related to iso-contact embeddings of closed contact manifolds in arbitrary contact manifold \( (N, ρ) \) abide by the \( h \)-principle provided, either the co-dimension of the embedding is bigger than 4 or the target manifold is overtwisted in the sense of [BEM]. See for example, [EM, Theorem 12.3.1] to understand the case of iso-contact embeddings when the co-dimension of the embedding is bigger than or equal to 4.

Iso-contact embeddings were first studied by Gromov. He established in [Gr] the \( h \)-principle for iso-contact embeddings for the category of open contact manifolds provided the co-dimension of the embeddings is bigger than or equal to 2. As a result, he also obtained the \( h \)-principle for closed contact manifolds, provided the co-dimension of the embeddings is at least 4. Gromov in [Gr] also established the \( h \)-principle for co-dimension 2 immersions. To clearly understand the case of iso-contact embeddings in overtwisted contact manifolds, see discussions related to iso-contact embeddings in [EM] and [EL]. In particular, in [EL] it is shown that every closed contact 3-manifold admits an iso-contact embedding in an overtwisted contact \( S^2 \times S^3 \). We would also like to point out that A. Mori in [Mr] also produced iso-contact embeddings of all contact 3–manifolds in the contact manifold \( (\mathbb{R}^7, Ker\{dz + \sum_{i=1}^{3} x_idx_i\}) \) using open books and D. Martinez-Torres in [Ma] produced an iso-contact embedding of any contact manifold \( M^{2n+1} \) in \( (\mathbb{R}^{4n+3}, Ker\{dz + \sum_{i=1}^{2n+1} x_idx_i\}) \).

In this article, we first establish an \( h \)-principle type result for co-dimension 2 iso-contact embeddings of closed manifolds. In order to state this result, we need the notion of overtwisted contact manifolds due to M. Borman, Y. Eliashberg and E. Murphy discussed in [BEM].

Recall that a contact manifold \( (M, ρ) \) is said to be overtwisted, provided it admits an iso-contact embedding of an overtwisted ball. For a precise definition of an overtwisted ball, refer [BEM]. For the purpose of this article what is important is the following fact established in [BEM].

In every homotopy class of almost contact structures, there exists a unique overtwisted contact structure up to isotopy.

Here, by an almost-contact structure on a manifold \( M \), we mean a hyperplane-field \( ρ \) together with a conformal class of a symplectic structure on it. We would like to remark that a contact structure \( Ker{α} \) can be naturally regarded as an almost-contact structure. This is because \( dα \) restricted to \( Ker{α} \) provides the conformal class of symplectic structure on the hyperplane \( Ker{α} \).

We will always denote the unique overtwisted contact structure in the homotopy class of an almost contact structure \( ρ \) on a manifold \( M \) by \( ρ_{ot} \). Now, we state our main result of this article.

**Theorem 1.** Let \( (M, ρ_M) \) be a closed contact manifold. If \( (M, ρ_M^{ot}) \) admits an iso-contact embedding in a contact manifold \( (N, ρ_N) \) with the trivial normal bundle, then so does \( (M, ρ) \).
Our proof of the Theorem relies on certain flexibility discovered in iso-contact embeddings of contact manifolds in neighborhoods of a special class of closed contact overtwisted manifolds, which are assumed to be embedded in a given contact manifold. See the Proposition for a precise statement.

Next, we discuss some applications of the Theorem. We first discuss co-dimension 2 iso-contact embeddings of contact manifolds in the standard contact spheres. Recall that by the standard contact structure \( \xi_{std} \) on the unit sphere \( S^{2n-1} \subset \mathbb{R}^{2n} \), we mean the kernel of the 1-form \( \sum_{i=1}^{n} x_i dy_i - y_i dx_i \) restricted to \( S^{2n-1} \).

The version of the Theorem which is most useful in applications is stated as the following:

**Corollary 2.** A closed contact manifold \((M^{2n-1}, \xi_M)\) admits an iso-contact embedding in the standard contact sphere \((S^{2n+1}, \xi_{std})\) if and only if \( M \) admits a contact embedding in \((S^{2n+1}, \xi_{std})\) and the induced contact structure on \( M \) by the embedding is homotopic to \( \xi_M \) as an almost-contact structure.

There are many interesting classes of smooth manifolds which admit smooth co-dimension 2 embeddings in the standard spheres. For example, as mentioned earlier, Hirsch in \([Hi]\) showed that every closed smooth 3–manifold admits a smooth embedding in \( S^5 \). There are now many proofs of this result. See for example, \([HLM]\) for what is now known as braided embedding and \([PPS]\) for embeddings using open books.

N. Kasuya in \([Ka]\) first observed that not all contact 3–manifolds admit iso-contact embeddings in the standard contact \( S^5 \). He showed that the necessary condition for the existence of such an embedding is that the first Chern class of the contact structure must be zero. In \([Ka]\), Kasuya also established that every closed contact 3–manifold \((M, \xi)\) admits an iso-contact embedding in some contact \( \mathbb{R}^5 \).

In \([EF]\), Etnyre and Fukuhara obtained many interesting iso-contact embedding results. One of the most striking result which they established states that every overtwisted contact 3–manifold \((M, \xi_{ot})\) with no 2–torsion in second integral cohomology iso-contact embeds in the standard contact \( S^5 \) if and only if the first Chern class of the overtwisted contact structure \( \xi_{ot} \) is zero.

Applying the Theorem about iso-contact embeddings in spheres and a result about iso-contact embeddings of overtwisted 3–manifolds in \( S^5 \) proved in \([EF]\), we establish the following:

**Theorem 3.** Let \( M \) be a closed orientable 3–manifold. Then, we have the following:

1. In case, \( M \) has no 2–torsion in \( H^2(M, \mathbb{Z}) \), then \( M \) together with any contact structure \( \xi \) on it admits an iso-contact embedding in \((S^5, \xi_{std})\) if and only if the first Chern class \( c_1(\xi) \) is zero.
2. In case, \( M \) has a 2–torsion in \( H^2(M, \mathbb{Z}) \), then there exists a homotopy class \([\xi]\) of plane fields on \( M \) such that \( M \) together with any contact structure homotopic to a plane field belonging to the class \([\xi]\) over a 2–skeleton of \( M \) admits an iso-contact embedding in \((S^5, \xi_{std})\).

Finally, we discuss iso-contact embeddings of simply-connected contact 5–manifolds in \((S^7, \xi_{std})\). In particular, we establish:

**Theorem 4.** Let \((M, \xi)\) be a closed simply connected contact 5–manifold. Then, \((M, \xi)\) admits an iso-contact embedding in \((S^7, \xi_{std})\) if and only if \((M, \xi)\) admits a formal iso-contact embedding in \((S^7, \xi_{std})\).

**Acknowledgment**

The first author is extremely grateful to Yakov Eliashberg for asking various questions regarding iso-contact embeddings which stimulated this work. During the course of the development of this article, he provided us constant encouragement and support. This work would not have been possible without his critical comments and suggestions. We are also thankful to Roger Casals, John Etnyre and Francisco Presas for various helpful comments and suggestions. The first author is also thankful to Simons Foundation for providing support to travel to Stanford, where a part of work of this project was carried out. The first author is thankful to ICTP, Trieste, Italy and Simons Associateship program without which this work would not have been possible.
2. Preliminaries

In this section, we quickly review notions necessary for the article pertaining open books, contact structures and relationship between them.

2.1. Open books.

Let us review few results related to open book decomposition of manifolds. We first recall the following:

**Definition 5** (Open book decomposition). An open book decomposition of a closed oriented manifold $M$ consists of a co-dimension 2 oriented sub-manifold $B$ with a trivial normal bundle in $M$ and a locally trivial fibration $\pi : M \setminus B \to S^1$ such that $\pi^{-1}(\theta)$ is an interior of a co-dimension 1 sub-manifold $N_\theta$ and $\partial N_\theta = B$, for all $\theta \in S^1$. Furthermore, the normal bundle $N(B)$ of the sub-manifold $B$ is trivialized such that $\pi$ restricted to $N(B) \setminus B \to S^1$ is given by the angular co-ordinate in $D^2$-factor.

The sub-manifold $B$ is called the binding and $N_\theta$ is called a page of the open book. We denote the open book decomposition of $M$ by $(M, Ob(B, \pi))$ or sometimes simply by $Ob(B, \pi)$.

Next, we discuss the notion of an abstract open book decomposition. To begin with, let us recall that the mapping class group of a manifold $(\Sigma, \partial \Sigma)$ is the group of isotopy classes of diffeomorphisms of $\Sigma$ which are the identity near the boundary $\partial \Sigma$.

**Definition 6** (Mapping torus). Let $\Sigma$ be a manifold with non-empty boundary $\partial \Sigma$. Let $\phi$ be an element of the mapping class group of $\Sigma$. By the mapping torus $MT((\Sigma, \partial \Sigma), \phi)$, we mean

\[ \Sigma \times [0, 1]/\sim, \]

where $\sim$ is the equivalence relation identifying $(x, 0)$ with $(\phi(x), 1)$.

Observe that by the definition of $MT((\Sigma, \partial \Sigma), \phi)$, there exists a collar of the boundary $\partial MT((\Sigma, \partial \Sigma), \phi)$ in $MT((\Sigma, \partial \Sigma), \phi)$ which can be identified with $(-\epsilon, 0] \times \partial \Sigma \times S^1$ as the diffeomorphism $\phi$ is the identity in a collar $(-\epsilon, 0] \times \partial \Sigma$ of the boundary of $\Sigma$. We will sometimes denote the mapping torus $MT((\Sigma, \partial \Sigma), \phi)$ just by $M(\Sigma, \phi)$. We are now in a position to define an abstract open book decomposition.

**Definition 7** (Abstract open book). Let $\Sigma$ and $\phi$ as in the previous definition. An abstract open book decomposition of $M$ is pair $(\Sigma, \phi)$ such that $M$ is diffeomorphic to

\[ MT(\Sigma, \phi) \cup id \partial \Sigma \times D^2, \]

where $id$ denotes the identity mapping of $\partial \Sigma \times S^1$.

The map $\phi$ is called the monodromy of the open book. We will denote an abstract open book decomposition by $Ob(\Sigma, \phi)$. Note that the mapping class $\phi$ uniquely determines $M = Ob(\Sigma, \phi)$ up to diffeomorphism.

One can easily see that an abstract open book decomposition of $M$ gives an open book decomposition of $M$ up to diffeomorphism and vice versa. Hence, sometimes we will not distinguish between open books and abstract open books. In particular, we will continue to use the notation $Ob(\Sigma, \phi)$ to denote the open book decomposition associated to the abstract open book $Ob(\Sigma, \phi)$.

**Examples 8.**

1. Notice that $S^n$ admits an open book decomposition with pages $D^{n-1}$ and the monodromy the identity map of $D^{n-1}$. We call this open book the trivial open book of $S^n$. For more details regarding open books, refer the lecture notes [Gi, Chpt-4.4.2] and [Et].

2. The manifold $S^3 \times S^2$ admits an open book decomposition with pages disk co-tangent bundle $DT^*S^2$ and monodromy the identity. We call this open book decomposition of $S^3 \times S^2$ the standard open book decomposition of $S^3 \times S^2$.

3. In [Qu], it was shown that every closed orientable 3–manifold admits an open book decomposition. This result was further generalized to all odd dimensional closed orientable manifold of dimension bigger than 5 by Quinn in [Qu].
Given two abstract open books $M^\alpha_1 = \text{Ob}(\Sigma_1, \phi_1)$ and $M^\alpha_2 = \text{Ob}(\Sigma_2, \phi_2)$, if we make the band connected sum of pages of $\text{Ob}(\Sigma_1, \phi_1)$ and $\text{Ob}(\Sigma_1, \phi_2)$, then we get an abstract open book decomposition $\text{Ob}(\Sigma_1 \# \Sigma_2, \phi_1 \# \phi_2)$ of $M_1 \# M_2$. This was first established in [Ga] by D. Gabai for 3–manifolds.

There exists an intimate connection between open books and contact structures on manifolds discovered by E. Giroux in [Gi]. In order to understand this correspondence, we first recall the notion of a contact structure supported by an open book.

2.2. Contact manifolds and supporting open books.

Giroux in [Gi] introduced the notion of a contact structure supported by an open book. We now recall this notion.

**Definition 9** (Open book supporting a contact form). Let $(M, \ker \{\alpha\})$ be a contact manifold. We say that an open book decomposition $\text{Ob}(B, \pi)$ supports a contact form $\alpha$ provided:

1. The binding $B$ is a contact sub-manifold of $M$.
2. $d\alpha$ is a symplectic form on each page of the open book.
3. The boundary orientation on $B$ coming from the orientation of the pages induced by $(d\alpha)^n$ is the same as the orientation given by $\alpha|_B \wedge (d\alpha|_B)^{n-1}$.

We would like to remark that if $\alpha_1$ and $\alpha_2$ are two contact forms on a contact manifold $M$ which are supported by the same open book $\text{Ob}(B, \pi)$, then they are isotopic as contact structures. See, for example, [Ko].

Let $(M, \xi)$ be a contact manifold. We say that $\xi$ is supported by an open book decomposition $\text{Ob}(B, \pi)$ of $M$ provided there exists a contact 1–form $\alpha$ inducing the contact structure $\xi$ on $M$ such that $\alpha$ is supported by $\text{Ob}(B, \pi)$.

Giroux in [Gi] established a one to one correspondence between the open books up to *positive stabilizations* and the supported contact structures up to isotopy for closed orientable 3–manifolds. See the notes [Ko1] for more on this. The purpose of the next subsection is to recall a few notions and results associated to Giroux’s correspondence.

2.3. Contact abstract open book and Giroux’s correspondence.

We begin this subsection by recalling the notion of the *Generalized Dehn twist*. This notion is necessary to understand the notion of positive stabilization. This notion was first introduced in [Se1]. See also [Se2].

**Definition 10** (Generalized Dehn twist). Consider,

$$T^*S^n = \{(x, y) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} | x.y = 0, ||y|| = 1\}.$$ 

Define a diffeomorphism $\tau$ of $T^*S^n$ as follows:

$$\tau(x, y) = \begin{pmatrix} \cos g(y) \\ -|y| \sin g(y) \\ |y|^{-1} \sin g(y) \\ \cos g(y) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

where, $g$ is a function of $y$ which is the identity near 0 and is zero outside a compact set containing 0. The diffeomorphism $\tau$ is called the Generalized Dehn twist while $\tau^{-1}$ is called the negative generalized Dehn twist.

It is relatively easy to check that $\tau$ is a compactly supported symplectomorphism of $T^*S^n$. Furthermore, $\tau$ can be isotoped to a symplectomorphism which is compactly supported in an arbitrary small neighborhood of the zero section of $T^*S^n$. This, in particular, implies that $\tau$ and $\tau^{-1}$ can be regarded as diffeomorphisms of the disk co-tangent bundle $DT^*S^n$. We refer to [MS page-186] and the notes [Ko] for more details. See also [KN] for a nice exposition on how to produce a compactly supported Generalized Dehn twist.

Next, we discuss the notion of a *contact abstract open book*. We refer to [Ko Section–2] for a more detailed description of this.

Let $(\Sigma, d\lambda)$ be a Weinstein manifold and $\phi$ an exact symplectomorphism of $\Sigma$ which is the identity near the boundary of $\Sigma$. Giroux generalized the construction of W. Thurston and H. Winkelnkemper given in [TW] to produce a contact form on the manifold with open book $\text{Ob}(\Sigma, \phi)$ such that the contact form is
supported by the open book $\mathcal{A}ob\langle \Sigma, \phi \rangle$ in the sense explained in Subsection 2.2. We will generally denote this contact form by $\alpha_{(\Sigma, \phi)}$. See lecture notes by O. van Koert [Ko] for the details of this construction. See also [CM]. We call this contact manifold a contact abstract open book.

In this article, unless stated otherwise, whenever we talk of a contact structure $\xi$, supported by an abstract open book $\mathcal{A}ob\langle \Sigma, \phi \rangle$, we will always mean that $\Sigma$ is a Weinstein manifold and $\phi$ an exact symplectomorphism of $\Sigma$ which when restricted to a collar of its boundary is the identity and the contact structure $\xi$ is contactomorphic to $\text{Ker}\{\alpha_{(\Sigma, \phi)}\}$ described earlier.

**Examples 11.**

1. Let $D^{2n}$ denote the unit $2n$-disk in $\mathbb{R}^{2n}$. Let $\lambda_{std}$ denote the canonical 1-form on $D^{2n}$ given by $\sum_{i=1}^{n} x_i dy_i - y_i dx_i$ which induces the standard symplectic structure on $D^{2n}$. The standard contact sphere $(S^{2n+1}, \xi_{std})$ is contactomorphic to the open book $(\mathcal{A}ob\langle D^{2n}, \text{id} \rangle, \text{Ker}\{\alpha_{(D^{2n}, \text{id})}\})$.

2. Consider $DT^*S^n$, the unit disk bundle associated to the co-tangent bundle of $S^n$ and the Generalized Dehn twist $\tau$ on $DT^*S^n$. It is well known that the contact abstract open book $\mathcal{A}ob\langle T^*S^n, \tau \rangle$ is contactomorphic to the standard contact sphere $(S^{2n+1}, \xi_{std})$.

3. The contact abstract open book $\mathcal{A}ob\langle DT^*S^n, \tau^{-1} \rangle$ induces an overtwisted contact structure on $S^{2n+1}$. This is clearly discussed for 3-manifolds in [KN1]. In general, this follows from [CMP]. We will denote this overtwisted contact structure by $\xi_{stat}$. We will denote a contact 1-form inducing the contact structure $\xi_{stat}$ by $\alpha_{stat}$.

We now define the notion of a generalized contact abstract open book:

**Definition 12** (Generalized contact abstract open book). Let $((W, \partial W), d\lambda)$ be a Weinstein cobordism with a connected convex boundary $M$. Let $\phi$ be a symplectomorphism of $(W, d\lambda)$ which is the identity in a small collar of the boundary of $W$. Consider the quotient manifold $N$ defined as:

$$N = \mathcal{M}T(W, \phi) \cup_{\text{id}} M \times D^2,$$

notice that $N$ admits a contact structure analogous to the one discussed earlier for the contact abstract open book. We call $N$ a generalized contact abstract open book with the binding $M$, page $W$ and the monodromy $\phi$.

By a slight abuse of notation, we will use the same notation $\mathcal{A}ob\langle W, \phi \rangle$ for the generalized contact abstract open book as well. By a Weinstein manifold or Weinstein domain, we will always mean a Weinstein cobordism with an empty concave boundary and a connected convex boundary. Note that whenever the Weinstein cobordism associated to a generalized contact abstract open book is a Weinstein manifold, we get usual contact abstract open book.

**Definition 13** (Generalized contact abstract connected sum).

Let $(\mathcal{A}ob\langle W_1, \phi_1 \rangle, \alpha_{(W_1, \phi_1)})$ and $(\mathcal{A}ob\langle W_2, \phi_2 \rangle, \alpha_{(W_2, \phi_2)})$ be two generalized contact abstract open books. Observe that we can perform the band connected sum $W_1 \#_b W_2$ of $W_1$ and $W_2$ along their connected convex boundaries to produce a new Weinstein cobordism $W_1 \#_b W_2$ with connected convex boundary $\partial W_1 \#_b \partial W_2$. Let $\mathcal{A}ob\langle W_1, \phi_1 \rangle \#_b \mathcal{A}ob\langle W_2, \phi_2 \rangle$ be the generalized abstract open book obtained by performing the band connected sum of their pages along convex boundaries.

Since the pages of the generalized abstract open book are Weinstein manifold $W_1 \#_b W_2$ with connected convex boundary $\partial W_1 \#_b \partial W_2$, it is clear that this generalized abstract open book carries a natural contact structure supported by the generalized open book having pages $W_1 \#_b W_2$ and the monodromy $\phi_1 \#_b \phi_2$. This contact structure will be denoted by $\text{Ker}\{\alpha_{(W_1 \#_b W_2, \phi_1 \#_b \phi_2)}\}$. We call this contact manifold the generalized contact abstract connected sum.

**Remark 14.**

1. Observe that the binding of a generalized contact abstract band connected sum is the connected sum of the bindings of the generalized contact abstract open books.
(2) When $W_1$ and $W_2$ are Weinstein manifolds, then the generalized contact abstract connected sum $\text{Aob}(W_1, \phi_1) \#_b \text{Aob}(W_2, \phi_2)$ is the contact connected sum of $\text{Aob}(W_1, \phi_1)$ and $\text{Aob}(W_2, \phi_2)$. The contact structure $\text{Ker}\{\alpha(W_1, \#_b W_2, \phi_1 \#_b \phi_2, \alpha)\}$ is supported by the open book with pages $W_1 \#_b W_2$ and the monodromy $\phi_1 \#_b \phi_2$.

(3) We will sometime use the notation $\text{Aob}(W_1, \#_b W_2, \phi_1 \#_b \phi_2)$ to denote the generalized abstract connected sum $\text{Aob}(W_1, \phi_1) \#_b \text{Aob}(W_2, \phi_2)$. This notation will be used to emphasize the abstract open book decomposition of $\text{Aob}(W_1, \phi_1) \#_b \text{Aob}(W_2, \phi_2)$.

2.4. Iso-contact open book embeddings. In this sub-section, we discuss the notion of iso-contact open book embeddings. For more on open book embeddings, refer [EL] and [PPS].

Definition 15. Let $M = \text{Aob}(\Sigma, \phi)$ and $N = \text{Aob}(W, \Psi)$ be two generalized contact abstract open books. Let $F : M \to N$ be a proper iso-contact embedding of $M$ in $N$. We say that this embedding is a contact abstract open book embedding, provided the following diagram commutes:

$$
\begin{array}{ccc}
\text{MT}(\Sigma, \phi) & \xrightarrow{F} & \text{MT}(W, \Psi) \\
\downarrow{\pi_1} & & \downarrow{\pi_2} \\
S^1 & & S^1 \\
\end{array}
$$

Here, $\pi_1 : \text{MT}(\Sigma, \phi) \to S^1$ and $\pi_2 : \text{MT}(W, \Psi) \to S^1$ are the natural projections associated to the mapping tori.

We end this section by establishing a proposition. The proposition, in particular, establishes that if $(M_1, \xi_{M_1})$ iso-contact embeds in $(N_1, \text{Ker}\{\alpha_1\})$ and $(M_2, \xi_{M_2})$ iso-contact embeds in $(N_2, \text{Ker}\{\alpha_2\})$, then the contact connected sum $(M_1 \#_b M_2, \xi_{M_1} \#_b \xi_{M_2})$ iso-contact embeds in the contact connected sum $(N_1 \#_b N_2, \text{Ker}\{\alpha_1 \#_b \alpha_2\}) = (N_1 \#_b N_2, \text{Ker}\{\alpha_1 \# \alpha_2\})$. This was already proved by J.Etnyre and R. Fukuwara in [EF].

Proposition 16. If a contact abstract open book $(\text{Aob}(\Sigma_1, \phi_1), \eta_1)$ iso-contact open book embeds in a generalized contact abstract open book $(\text{Aob}(W_1, \Psi_1), \xi_1)$ for $i = 1, 2$, then the contact abstract connected sum $(\text{Aob}(\Sigma_1, \phi_1) \#_b \text{Aob}(\Sigma_2, \phi_2), \eta_1 \#_b \eta_2)$ iso-contact open book embeds in the generalized contact abstract connected sum $(\text{Aob}(W_1, \Psi_1) \#_b \text{Aob}(W_2, \Psi_2), \xi_1 \#_b \xi_2)$.

Furthermore, if $\Sigma_i$ is contained in an arbitrary small collar of the convex boundary $M_i$ of $\partial W_i$, then we can ensure that the page $\Sigma_1 \#_b \Sigma_2$ of $\text{Aob}(\Sigma_1 \#_b \Sigma_2, \phi_1 \#_b \phi_2)$ is contained in an arbitrary small collar of the convex boundary of the page of $\text{Aob}(W_1 \#_b W_2, \Psi_1 \#_b \Psi_2)$.

Proof. First of all notice that since the band connected sum of $W_1$ with $W_2$ can be regarded as adding a 1-handle to $W_1 \cup W_2$, we can perform the band connected sum of $W_1$ with $W_2$ along their convex boundaries in such a way that the band connected sum of $\Sigma_1$ with $\Sigma_2$ properly symplectically embeds in $W_1 \#_b W_2$. To achieve this, notice that in order to perform the band connected sum, we need to fix a small Darboux ball $U_1$ around a point $p_1$ in $\partial W_1$ and a small Darboux ball $U_2$ around a point $p_2$ in $\partial W_2$. We fix these balls in such a way that they restrict to Darboux balls $\tilde{U}_i$ containing the point $p_i$ in $\partial \Sigma_i$ for each $i = 1, 2$. Now, if we perform the band connected sum of $W_1$ with $W_2$, we get an induced band connected sum of $\Sigma_1$ with $\Sigma_2$ which is contained in $W_1 \#_b W_2$.

Observe that we have not yet achieved the second property. In order to achieve this, we first observe that $\Sigma_1 \#_b \Sigma_2 \subset W_1 \#_b W_2$ can be made disjoint from the core of the 1-handle $B$ associated to $W_1 \#_b W_2$ by a sufficiently small $C^\infty$ perturbation whose support is contained in a small tubular neighborhood of $B \cap \Sigma_1 \#_b \Sigma_2 \subset W_1 \#_b W_2$. See Figure 11 for a pictorial description.

Let $\epsilon_1$ be such that $\Sigma_1$ is contained in the symplectic collar $([0, \epsilon_1] \times \partial W_1, d(e^t \alpha_1))$ of $\partial W_1$, where $e^t \alpha_1$ is the Liouville 1–form on the symplectic collar of the convex boundary $\partial W_1$.

Let $\epsilon_2$ be such that $\Sigma_2$ is contained in the symplectic collar $([0, \epsilon_2] \times \partial W_2, d(e^t \alpha_2))$ of $\partial W_1$, where $e^t \alpha_2$ is the Liouville 1–form on the symplectic collar of the convex boundary $\partial W_2$. iso-contact embeddings of manifolds in co-dimension 2
Let us denote by $B = \mathbb{D}^{2n-1}(\delta) \times \mathbb{D}(1)$ the band of length 1 and radius $\delta$ used in the band connected sum $W_1 \#_b W_2$. Clearly, by the construction $\tilde{B} = \mathbb{D}^{2n-3}(\delta) \times \mathbb{D}(1)$ is the band associated to the induced band connected sum $\Sigma_1 \#_b \Sigma_2$.

Let $A_\delta$ denote the annulus $[\delta \mathbb{D}(1), \delta] \times S^{2n-2} \times \mathbb{D}^1$. Notice that the part of the boundary of the band $B$ corresponding to $\mathbb{D}^{2n-1} \times \partial \mathbb{D}(1)$ can be assumed to have the symplectic collar $A_\delta$. Hence, if the $C^\infty$–perturbation that we perform in order to make $\Sigma_1 \#_b \Sigma_2$ disjoint from the core of 1–handle is done such that perturbed $\Sigma_1 \#_b \Sigma_2$ is contained in $A_\delta$ and the support of the perturbation is contained in the complement of the annulus $[\delta \mathbb{D}(1), \delta] \times S^{2n-1} \times \mathbb{D}^1$, then the perturbed band $\tilde{B}$ associated to $\Sigma_1 \#_b \Sigma_2$ is contained in $A_\delta$ and its intersection with the boundary of the annulus $A_\delta$ is the same as the intersection of unperturbed $\tilde{B}$. Observe that such a perturbation is always possible.

Next, choose $\delta$ such that $\delta < \min \{\epsilon_1, \epsilon_2\}$. Observe that for this choice of $\delta$ the perturbed $\Sigma_1 \#_b \Sigma_2$ lies in a small symplectic neighborhood of $\partial W_1 \partial W_2$ as claimed. See Figure 2.

Finally, observe that since the symplectomorphisms $\Psi_1$ and $\Psi_2$ are the identity in suitable collars of the boundaries of $W_1$ and $W_2$ respectively, the symplectomorphism $\Psi_1 \Psi_2$ naturally induces the symplectomorphism $\phi_1 \phi_2$ on the symplectically embedded $\Sigma_1 \#_b \Sigma_2 \subset W_1 \#_b W_2$ that we just described.

This establishes the proposition. 

\[ \delta \]

\textbf{Figure 1.} The figure depicts a small Darboux neighborhood of the attaching sphere $p_i$ contained in $\partial \Sigma_i \subset \partial W_i$ used in performing the band connected sum $W_1 \#_b W_2$. The picture of on the left depicts the embedding of the Darboux ball of $\Sigma_i$ contained the unperturbed $\Sigma_1 \#_b \Sigma_2$, while the the picture on the right depicts the neighborhood after a sufficiently small perturbation.

3. Proof of the Theorem 1

The purpose of this section is to establish the Theorem 1. There are three steps in establishing the Theorem 1. We first mention the first two steps in the form of the Proposition 17 and the Proposition 18. We give proofs of these propositions in Section 4.

In order to state the Proposition 17 we need to introduce the following notation. Let $(M^{2n+1}, \xi)$ be a contact manifold. The contact structure obtained by the contact connected sum of $(M, \xi)$ with the standard overtwisted sphere $(S^{2n+1}, \xi_{stot})$ will be denoted by $\xi_{stot}$. Notice that if $\xi$ is supported by an open book decomposition $\mathcal{A}ob(\Sigma, \phi)$, then $\xi_{stot}$ is supported by the open book $\mathcal{A}ob(\Sigma \#_b DT^*S^n, \phi \# \tau^{-1})$. This follows from [CMP]. See, the third example in Examples 17.

\textbf{Proposition 17.} Let $M^{2n-1}$ be a closed smooth manifold. Let $\xi$ be a contact structure on $M$. Suppose that $(M, \xi_{stot})$ admits an iso-contact embedding in a contact manifold $(N^{2n+1}, \xi_N)$ with the trivial normal bundle, then $(M, \xi)$ also admits an iso-contact embedding in $(N, \xi_N)$.

This is the key step and its proof is divided in to several smaller steps. As mentioned earlier, we will establish each step in Section 4.

The second step is to establish the following:
Proposition 18. There exists an iso-contact embedding of \((S^{2n-1}, \xi_{\text{std}})\) in the standard contact sphere \((S^{2n+1}, \xi_{\text{std}})\).

Finally, the third step is to establish the following:

Proposition 19. Let \((M^{2n-1}, \xi_1)\) be a contact manifold. If \((M, \xi_1)\) admits an iso-contact embedding in a contact manifold \((N, \xi_2)\), then there exists an iso-contact embedding of \((M, \xi_1^{\text{stot}})\) in the contact manifold \((N, \xi_2)\).

Proof. Observe that it follows from the Proposition 16 that if \((M_1, \xi_1)\) iso-contact embeds in \((N_1, \eta_1)\) and \((M_2, \xi_2)\) iso-contact embeds \((N_2, \eta_2)\), then \((M_1 \# M_2, \xi_1 \# \xi_2)\) iso-contact embeds in \((N_1 \# N_2, \eta_1 \# \eta_2)\).

We know from the Proposition 18 that there exists an iso-contact embedding \((S^{2n+1}, \xi_{\text{std}})\) in the standard disk \((S^{2n+1}, \xi_{\text{std}})\). This implies that \((M \# S^{2n-1}, \xi_1 \# \xi_{\text{stot}})\) admits an iso-contact embedding in \((N \# S^{2n-1}, \xi_2 \# \xi_{\text{stot}})\).

Next, note that the contact manifold \((N \# S^{2n-1}, \xi_2 \# \xi_{\text{stot}})\) is contactomorphic to \((N, \xi_2)\). Hence, we have an iso-contact embedding of \((M, \xi_1^{\text{stot}}) = (M \# S^{2n-1}, \xi_1 \# \xi_{\text{stot}})\) in the contact manifold \((N, \xi_2)\).

Let us now discuss how these three steps imply the Theorem 1.

Proof of the Theorem 1.

When the co-dimension of the embedding of \(M\) in \(N\) is bigger than or equal to 4, the theorem was already established by Gromov in [18]. Hence, from now on, we assume that the co-dimension of \(M\) in \(N\) is 2.

First of all, since an overtwisted contact structure is unique in its homotopy class of almost contact structures, we get that the contact structures \((M, \xi_1^{\text{stot}})\) and \((M, \xi_M^{\text{stot}})\) are contactomorphic. Hence, it follows from the Proposition 10 that there is an iso-contact embedding of \((M, \xi_1^{\text{stot}})\) in \((N, \xi_N)\) with the trivial normal bundle. It follows from the proposition 17 that there is an iso-contact embedding of \((M, \xi_M)\) in \((N, \xi_N)\) as required.

Let us now discuss how the Corollary 2 follows from the Theorem 1.

Proof of the Corollary 2.

Since the Euler class of the normal bundle of any embedded closed orientable manifold \(M\) in \(S^k\) has to be zero, we get that the manifold \(M^{2n-1}\) admits an embedding in \(S^{2n+1}\) with the trivial normal bundle.

Next, assume that \(M\) admits an embedding in \((S^{2n+1}, \xi_{\text{std}})\) such that the induced contact structure \(\xi\) is homotopic to \(\xi_M\). The Proposition 18 implies that there exits an iso-contact embedding of \((S^{2n-1}, \xi_{\text{stot}})\) in \((S^{2n+1}, \xi_{\text{std}})\). Hence, it follows from the Proposition 16 that there exists an iso-contact embedding.
of \((M, \xi \# \xi_{\text{tot}})\) in \((S^{2n+1}, \xi_{\text{tot}})\). Now, since the overtwisted contact structures \(\xi \# \xi_{\text{tot}}\) and the \(\xi_M \# \xi_{\text{tot}}\) are homotopic as almost contact structures, by the uniqueness of an overtwisted contact structure in a given homotopy class of almost contact structures, we get that there is an iso-contact embedding of \((M, \xi_M \# \xi_{\text{tot}}) = (M, \xi_{\text{tot}}^M)\) in the standard contact sphere.

The Proposition 17 now implies that there is an iso-contact embedding of \((M, \xi_M)\) in the standard contact sphere as claimed. This completes our argument.

\[\square\]

The next section is devoted to establish the proof of the Proposition 17. Using the techniques developed to establish the Proposition 17, we will also establish the Proposition 18.

4. Proofs of Proposition 17 and Proposition 18

In order to prove the Proposition 17, we would like to think of \(M \times \mathbb{D}_2\) as an abstract open book. This abstract open book is a special case of a generalized abstract open book. Since we will need it time and again, we introduce a special terminology for it.

**Definition 20** (\(\varepsilon\)-partial open book). Let \(W\) be the product Weinstein cobordism \([(a, \varepsilon) \times M, d(e^{-r} \alpha)]\), where \(\alpha\) be a contact form on \(M\) and \(\{a\} \times M\) is the convex boundary. Consider the mapping torus \(M \times \mathbb{D}_2 = S^1 \times ([a, \varepsilon] \times M)\) with the contact form \(\varepsilon d\theta + e^{-r} \alpha\). Consider the contact abstract open book \((\text{Aob}(W, id), \omega(W, id)) = (M \times \mathbb{D}_2, \text{Ker}\{h_1(r)\alpha + h_2(r) d\theta\})\) constructed using the pair of functions \(h_1\) and \(h_2\) as depicted in Figure 3 and satisfying the following properties:

1. \(h_1(r) > 0\), decreasing and \(h_1'(0) = 0\) and \(h_1'(r) < 0\) for every \(r \in (0, b)\).
2. \(h_1(r) = e^{-r}\) near \(\varepsilon\).
3. \(h_2(r) = r^2\) near \(r = 0\) and \(h_2\) is non-decreasing and \(h_2(r)\) is the constant \(\varepsilon\) near \(\varepsilon\).
4. \(h_2'(r)h_1(r) - h_1'(r)h_2(r)\) is always positive.

This contact open book is called an \(\varepsilon\)-partial open book associated to \((M, \text{Ker}\{\alpha\})\).

![Figure 3. The figure depicts the graphs of the functions \(h_1\) and \(h_2\). These functions are also used in the construction of contact abstract open book in [Ko].](image)

**Remark 21.**

1. The form \(h_1(r)\alpha + h_2(r)d\theta\) on the \(\varepsilon\)-partial open book is supported by the \(\varepsilon\)-partial open book.
2. The region of an \(\varepsilon\)-partial open book where the form is given by \(\varepsilon d\theta + e^{-r} \alpha\) will be referred as the standard region associated to the partial open book.
3. By changing the co-ordinates \(s = -r\) we will denote the standard region by \(S^1 \times [-\varepsilon, a] \times M\) for some \(a \in (-\varepsilon, 0)\). The contact form in the standard region will be described by the formula \(\varepsilon d\theta + e^s \alpha\).

In the next sub-section, we establish the Proposition 17.
4.1. Proof of Proposition [17]

The proof of the Proposition [17] can be divided into three steps.

The first step, which readily follows from neighborhoods of contact sub-manifolds discussed [Ge] Theorem:2.5.15] is stated as the following:

**Lemma 22.** [Ge] Theorem:2.5.15 Let \((N, \xi_N)\) be a contact manifold. Let \((M, \xi_M)\) be a contact sub-manifold of \((N, \xi_N)\) with the trivial normal bundle. If \(\text{Ker}\{\alpha\}\) is contactomorphic to \(\xi_M\) on \(M\), then there exists an \(\varepsilon_0\)-positive such that there is an iso-contact embedding of an \(\varepsilon\)-partial open book associated to \((M, \text{Ker}\{\alpha\})\) in \((N, \xi_N)\) for every \(\varepsilon\) smaller than \(\varepsilon_0\).

We now state the next two steps in the form of lemmas whose proofs we will provide in the subsequent sub-sections.

The second step is to establish the following:

**Lemma 23.** Let \((M, \text{Ker}\{\alpha\}) = (\text{Aob}(\Sigma, \phi), \text{Ker}\{\alpha(\Sigma, \phi)\})\). Let \(\varepsilon_0 > 0\) be given. There exists a contact abstract open book embedding \(F\) of \((M, \text{Ker}\{\alpha\})\) in \(\varepsilon_0\)-partial open book \(M \times \mathbb{D}^2\).

In particular, this implies the following:

1. \(F\) is constructed such that if \(\mathbb{S}^1 \times [-\varepsilon_0, a] \times M\) is the standard region associated to the \(\varepsilon_0\)-partial open book for some \(0 < -a < \varepsilon_0\), then the following diagram commutes:

\[
\begin{array}{c}
MT(\Sigma, \phi) \xrightarrow{F} \mathbb{S}^1 \times [-\varepsilon_0, a] \times M \\
\pi_1 \downarrow \quad \downarrow \pi \\
\mathbb{S}^1
\end{array}
\]

2. The pull-back under \(F\) of the 1-form \(\varepsilon_0 d\theta + e^*\alpha\) induces the contact structure \(\text{Ker}\{\alpha\}\) restricted to \(MT(\Sigma, \phi)\).

The third step is to establish the following:

**Lemma 24.** For every \(\varepsilon > 0\), there exists an contact abstract open book embedding of the standard contact sphere \((\mathbb{S}^{2n-1}, \xi_{\text{std}})\) in the \(\varepsilon\)-partial open book associated to \((\mathbb{S}^{2n-1}, \text{Ker}\{\alpha_{\text{std}}\})\), where the standard contact sphere is regarded as an abstract open book with pages the standard symplectic \((2n - 2)\)-disc and monodromy the identity.

In particular, this implies the following:

1. \(F\) is constructed such that if \(\mathbb{S}^1 \times [-\varepsilon_0, a] \times \mathbb{S}^{2n-1}\) for some \(0 < -a < \varepsilon_0\), is the standard region associated to the \(\varepsilon\)-partial open book, then the following diagram commutes:

\[
\begin{array}{c}
MT(\mathbb{D}^{2n-2}, \text{id}) \xrightarrow{F} \mathbb{S}^1 \times [-\varepsilon_0, a] \times \mathbb{S}^{2n-1} \\
\pi_1 \downarrow \quad \downarrow \pi \\
\mathbb{S}^1
\end{array}
\]

2. The pull-back under \(F\) of the 1-form \(\varepsilon d\theta + e^*\alpha_{\text{std}}\) induces the contact structure \(\alpha_{\text{std}}\) restricted to \(MT(\mathbb{D}^{2n-2}, \text{id})\).

Now that we have clearly stated all three steps needed for the proof of the Proposition [17] in the form of the Lemmas [22] 23 and 24 we give the proof of the Proposition [17] assuming the Lemmas [23] and [24]

**Proof of the Proposition [17]**

We can assume that \((M, \xi)\) is an abstract open book \((\text{Aob}(\Sigma, \phi), \text{Ker}\{\alpha(\Sigma, \phi)\})\).

We first notice that the Lemma [23] implies that given an \(\varepsilon > 0\), it is sufficient to iso-contact embed \((M, \xi)\) in the \(\varepsilon\)-partial open book associated to the contact manifold \((M \# \mathbb{S}^{2n-1}, \xi \# \xi_{\text{std}}) = (M, \xi_{\text{std}})\).
Let $\Sigma_{\theta} = \pi_1^{-1}(\theta)$, where $\pi_1 : \mathcal{MT}(\Sigma, \phi) \to S^1$ is the fibration associated to $\mathcal{Ob}(\Sigma, \phi)$. Now, consider the standard region $S^1 \times [-\varepsilon, -\varepsilon_0 + \delta] \times M$ of the $\varepsilon$-partial open book associated to $(M \# S^{2n-1}, \text{Ker} \{\beta = \alpha(\Sigma, \phi) \# \alpha_{\text{std}}\})$, where the contact form is given by $\varepsilon d\theta + e^{-r}\beta$.

Observe that if there exists a proper symplectic embedding of the band connected sum $\Sigma_{\theta} \#_b D^{2n-2}$ in $\{\theta\} \times (-\varepsilon_0, -\varepsilon_0 + \delta] \times (M \# S^{2n-1})$ for any $\varepsilon_0$, then there exists an iso-contact embedding of the contact manifold $\mathcal{Ob}(\Sigma \#_b D^{2n-2}, \text{Ker} \{\alpha(\Sigma \#_b D^{2n-2}, \phi \# \text{id})\})$ in the $\varepsilon$-partial open book associated to $(M \# S^{2n-1}, \xi \# \xi_{\text{std}})$.

By the Lemma 23 there exists a contact abstract open book embedding of the contact manifold $(M, \text{Ker} \{\alpha\}) = (\mathcal{Ob}(\Sigma, \phi), \text{Ker} \{\alpha(\Sigma, \phi)\})$ in the $\varepsilon$-partial open book associated to $(M, \text{Ker} \{\alpha\})$. Moreover, the mapping tours $\mathcal{MT}(\Sigma, \phi)$ is properly embedded close to the convex boundary times $S^1$ in the standard region of the $\varepsilon$-partial open book associated to $(M, \text{Ker} \{\alpha\})$. Furthermore, the pull-back of the form $\varepsilon d\theta + e^s \alpha$ induces the contact structure $\text{Ker} \{\alpha\}$ restricted to $\mathcal{MT}(\Sigma, \phi)$.

Also, by the Lemma 24 there exists an iso-contact abstract open book embedding of the standard sphere $(S^{2n-1}, \text{Ker} \{\alpha_{\text{std}}\}) = (\mathcal{Ob}(D^{2n-2}, \text{id}), \text{Ker} \{\alpha(D^{2n-2}, \text{id})\})$ in the $\varepsilon$-partial open book associated to $(S^{2n-1}, \text{Ker} \{\alpha_{\text{std}}\})$. Moreover, the mapping tours $\mathcal{MT}(D^{2n-2}, \text{id})$ is properly embedded close to the convex boundary times $S^1$ in the standard region of the $\varepsilon$-partial open book associated to $(S^{2n-1}, \text{Ker} \{\alpha_{\text{std}}\})$. Also notice that the pull-back of the form $\varepsilon d\theta + e^s \alpha$ induces the contact structure $\text{Ker} \{\alpha_{\text{std}}\}$ restricted to $\mathcal{MT}(D^{2n-2}, \text{id})$.

Hence, by the Proposition 10 there exists an iso-contact abstract open book embedding of $(\mathcal{Ob}(\Sigma \#_b D^{2n-2}, \phi \# \text{id}), \text{Ker} \{\alpha(\#_b D^{2n-2}, \phi \# \text{id})\})$ in the $\varepsilon$-partial open book associated to $(M \# S^{2n-1}, \xi \# \xi_{\text{std}})$.

Since the contact abstract open book $\mathcal{Ob}(\Sigma \#_b D^{2n-2}, \text{Ker} \{\alpha(\Sigma \#_b D^{2n-2}, \phi \# \text{id})\})$ is contactomorphic to $(M, \xi)$ and $(M \# S^{2n-1}, \xi \# \xi_{\text{std}})$ by definition is $(M, \xi_{\text{std}})$, the proposition follows.

We now proceed to establish the Lemma 23 and the Lemma 24.

### 4.2. Proof of the Lemma 23

The purpose of this sub-section is to establish the Lemma 23.

**Lemma 25.** Let $(N^{2n-1}, \text{Ker} \{\alpha\}) = (\mathcal{Ob}(\Sigma, \phi), \text{Ker} \{\alpha(\Sigma, \phi)\})$ be a contact abstract open book. Let $\varepsilon > 0$ be such that $[0, \varepsilon] \times \partial \Sigma$ is the collar of $\partial \Sigma$ which satisfies the following:

1. The symplectomorphism $\phi$ is the identity when restricted to this collar.
2. The form $\alpha_{\Sigma, \phi}$ restricted to this collar is the form $e^s \lambda$ for a contact form $\lambda$ defined on $\{0\} \times \partial \Sigma$.

Then, there exists a family $f_{(c, t)}$ of embeddings of $\Sigma$ in the symplectic manifold $([a, b] \times N, d(e^s \alpha))$, which satisfies the following properties:

1. The family $f_{(c, t)}$ is smooth in both $c$ and $t$.
2. $f_{(c, t)}(\Sigma)$ is a properly embedded symplectic sub-manifold of $([a, b] \times N, d(e^s \alpha))$ for every $c$ and $t$. 

![Figure 4](image-url)
(3) \( f_{(c,0)}(\Sigma) = f_{(c,0)}(\Sigma) \) and \( f_{(c,1)}^{-1} \circ f_{(c,0)} = \phi \).

(4) \( \partial f_{(c,1)}(x) = \partial f_{(c,2)}(x) \) for all \( x \) and for any pair of reals \( t_1, t_2 \in [0, 1] \) in the collar neighborhood \( [0, \varepsilon] \times \partial \Sigma \).

(5) The embedding is such that the complement of the collar \( f_{(c,1)}(\Sigma) \setminus \{0, \varepsilon\} \times \partial \Sigma \) is contained in \( \{c\} \times N \).

(6) The form \( f_{(c,1)}(d(e^\phi \lambda)) \) restricted to the collar is a part of the symplectization of the contact manifold \( \{\{0\} \times \partial \Sigma, \text{Ker}\phi(e^\phi \lambda_{\{0\} \times \partial \Sigma})\} \). Furthermore, the primitive of the symplectic form when restricted to the convex boundary \( \{\varepsilon\} \times \partial \Sigma \) is the 1-form \( e^{b+c}\lambda \) and in a neighborhood of \( \{0\} \times \partial \Sigma \), the primitive is given by \( e^{\phi}(e^\phi \lambda) \).

Before we discuss the formal proof, let us discuss briefly the idea behind the proof. Given a contact abstract open book \((M, \text{Ker}\{\alpha\})\) with page \( \Sigma \), we know that \( \Sigma \) admits an embedding as a page in \( M \) at level \( \theta \) for any \( \theta \in S^1 \). Call the image of the embedding as \( \Sigma_\theta \). This embedding is symplectic and its boundary is the binding \( B \).

Hence, for any \( c \in [a,b] \) there exists a piece-wise linear embedding of \( \Sigma \) in \( [a,b] \times M \) consisting of \( \Sigma_\theta \cup [c, b] \times B \), where we regard \( \Sigma_\theta \) as embedded in \( \{c\} \times M \). Our main observation is that we can smoothen the corner along the binding \( B \) to produce a symplectic embedding in \( ([a,b] \times M, d(e^\phi \alpha)) \).

**Proof.** For any \( c \in (a, b] \), since \( e^\phi \alpha \) is supported by the open book decomposition \( \text{Aob}(\Sigma, \phi) \). This implies that there exists a family \( \tilde{f}_{(c,t)} \) of embedding of \( \Sigma \) in \( \{c\} \times N \) which satisfies the following properties:

1. \( \tilde{f}_{(c,t)}(e^\phi \alpha) \) is a symplectic form on \( \Sigma \).
2. \( \tilde{f}_{(c,1)}(\Sigma) = \tilde{f}_{(c,0)}(\Sigma) \) and \( \tilde{f}_{(c,t_1)}(x) = \tilde{f}_{(c,t_2)}(x) \) for every \( x \) in a collar of \( \partial \Sigma \) and for any \( t_1, t_2 \in [0, 1] \).
3. \( \tilde{f}_{(c,0)} \circ \tilde{f}_{(c,1)}^{-1} \) is the symplectomorphism \( \phi \) of \( (\Sigma, \tilde{f}_{(c,0)}(d(e^\phi \lambda))) \)

Fix a \( t_0 \in [0, 1] \). Let us denote by \( \Sigma_{t_0} \) the image \( \tilde{f}_{(c,t_0)}(\Sigma) \). Notice that \( \Sigma_{t_0} \) is a symplectic sub-manifold of \( ([a,b] \times N, d(e^\phi \alpha)) \) which is contained in \( \{c\} \times N \). Observe that by the definition of the abstract open book, we get that the form \( e^\phi \alpha \) restricted to the collar \( \tilde{f}_{(c,t_0)}([0, \varepsilon] \times \partial \Sigma) \) of \( \Sigma_{t_0} \) is pulled back via the symplectomorphism \( \tilde{f}_{(c,t_0)} \) to the 1–form \( e^r e^\phi \lambda \), where \( r \in [0, \varepsilon] \), on the collar \( [0, \varepsilon] \times \Sigma \).

We now describe how to re-embed the collar \( [0, \varepsilon] \times \Sigma \) in the symplectic manifold by an embedding \( F \) such that it satisfies the following properties:

- There exists a \( \delta \in (0, \varepsilon) \) such that \( F = \tilde{f}_{(c,t_0)} \) when restricted to \( [0, \delta] \times \partial \Sigma \).
- The pull-back of the form \( e^\phi \alpha \) by \( F \) induces the form \( e^{b+c}\lambda \) on \( \{\varepsilon\} \times \partial \Sigma \).

In order to achieve this, consider a pair of functions \( f : [0, \varepsilon] \rightarrow [a, b] \) and \( g : [0, \varepsilon] \rightarrow [0, \varepsilon] \) such that \( f \) is constant \( c \) near \( 0 \) and increases to \( b \) while \( g \) is the identity near \( 0 \) and is the constant \( \varepsilon \) in a small neighborhood of the point \( \varepsilon \). See Figure 4 for graphs of \( f \) and \( g \). We now define the embedding \( F \) as \( F(t, x) = (f(t), g(t), x) \). See Figure 4 for a pictorial description of the embedding \( F \).

We now define the embedding \( f_{(c,t_0)} \) of \( \Sigma \) using \( F \) and \( \tilde{f}_{(c,t_0)} \) as:

![Figure 5. The figure depicts graphs of functions f and g.](image-url)
\[ f_{(c,t_0)}(x) = \begin{cases} F(x), & \text{for } x \in [0, \varepsilon] \\ \hat{f}_{(c,t_0)}, & \text{otherwise}. \end{cases} \]

Observe that since \( t_0 \) is arbitrary, doing the construction parametrically, we get the family \( f_{(c,t)} \) from the family \( \hat{f}_{(c,t)} \) with the required properties. This completes our argument.

Lemma 26. Let \((N, \Ker\{\alpha\}) = (\AO\{\Sigma, \phi\}, \Ker\{\alpha_{(\Sigma, \phi)}\})\) be a contact abstract open book. Let \( m = 1 \) or \(-1\).
Let \((M, \Ker\{\alpha_m\})\) be the contact abstract open book \((\AO\{\Sigma, \phi^m\}, \Ker\{\alpha_m = \alpha_{(\Sigma, \phi^m)}\})\). Let \( MT(\Sigma, \phi^m) \) be the contact mapping torus associated to the contact abstract open book \( M \).
Consider the manifold \( S^1 \times [a, b] \times N, \Ker\{\alpha_K = Kd\theta + e^s\alpha\} \).
If \( \pi_m : MT(\Sigma, \phi^m) \rightarrow S^1 \) denotes the bundle projection, then there exists a \( K_0 > 0 \) and a contact embedding of \( MT(\Sigma, \phi^m) \) in \( S^1 \times [a, b] \times N, \Ker\{\alpha_K\} \) for all \( K \geq K_0 \), which satisfies the following properties:

1. The following diagram commutes:
   \[
   \begin{array}{ccc}
   MT(\Sigma, \phi^m) & \xrightarrow{F} & S^1 \times [a, b] \times N \\
   \pi_m \downarrow & & \downarrow \pi_2 \\
   S^1 & & .
   \end{array}
   \]

2. For a fixed \( c \in (a, b) \), the fiber \( \pi_m^{-1}(\theta) \) is a symplectic sub-manifold of the symplectic manifold \([c, b] \times N, d(e^s\alpha)\) for every \( \theta \in S^1 \).

3. The contact structure induced on the embedded \( MT(\Sigma, \phi^m) \) by \( \alpha_K \) is contactomorphic to the contact structure \( \Ker\{\alpha_m\} \) restricted to \( MT(\Sigma, \phi^m) \subset N \) by a contactomorphism which is the identity when restricted to \( \partial MT(\Sigma, \phi^m) \).

4. When \( m = 1 \), we can choose \( K_0 \) to be arbitrarily small.

Proof. Our first task is to produce an embedding which satisfies the first two properties stated in the statement of the lemma. In order to achieve this, let us fix an \( m \in \{-1, 0, 1\} \).
Now, consider the family \( f_t \) of embeddings of \( \Sigma \) in the symplectic manifold \([a, b] \times N, d(e^s\alpha)\) defined as follows:
\[
f_t = \begin{cases} f_{(c,t)}, & t \in [0, 1] \text{ and } m = 1 \\ f_{(c,1-t)}, & t \in [0, 1] \text{ and } m = -1. \end{cases}
\]

The family \( f_{(c,t)} \) is the family constructed in the Lemma 25 earlier.
Consider the embedding of \( I \times \Sigma \subset I \times [a, b] \times N \) given by \( F(t, x) = (t, f_t(x)) \).
Regard \( S^1 \times [a, b] \times N \) as the quotient of \([0, 1] \times [a, b] \times N \) where we identify \( \{0\} \times [a, b] \times N \) to \( \{1\} \times [a, b] \times N \).
Notice that due to the third property associated to the family \( f_{(c,t)} \) constructed in the Lemma 25, we get that \( F \) naturally induces an embedding of \( MT(\Sigma, \phi^m) \) in \( S^1 \times [a, b] \times N \) which satisfies the first two properties mentioned in the statement.
Let us denote this induced embedding of \( MT(\Sigma, \phi^m) \) in \( S^1 \times [a, b] \times N \) by \( F_c \).

Next, observe that \( F^*(Kd\theta) \) is a \( 1 \)-form on \( MT(\Sigma, \phi) \) which is transverse to the fiber of the fibration \( \pi_m \).
Since we have already established that the fibers of \( MT(\Sigma, \phi) \) are symplectic sub-manifolds of \([a, b] \times N, d(e^s\alpha)\), it follows that for a sufficiently large \( K_0 \) the pull-back of the form \( Kd\theta + e^s\alpha \) by \( F \) induces a contact structure on \( MT(\Sigma, \phi^m) \).

We now focus on our claim that \( K \) can be chosen to be arbitrary small in case \( m = 1 \).
Observe that in the calculation of the volume form \( F^*(Kd\theta + e^s\alpha) \wedge (dF^*(Kd\theta + e^s\alpha))^{n-1} \) on \( MT(\Sigma, \phi) \), we get two non-zero terms \( F^*(e^s\alpha) \wedge (F^*(d(e^s\alpha)))^{n-1} \) and \( F^*_m(Kd\theta) \wedge (F^*(de^s\alpha))^{n-1} \).

Since the second term is always positive for any \( K \) positive, we get that whenever the first term is non-negative, the pull-back form defines a contact form on \( MT(\Sigma, \phi) \) for an arbitrary \( K \) positive. Notice that when \( m = 1 \) the term \( F^*(e^s\alpha) \wedge (F^*(d(e^s\alpha)))^{n-1} \) is positive and hence, we get what we required.
So far, we have produced a contact embedding of $M$ in $(\mathbb{S}^1 \times [a, b] \times N, d\theta + e^*\alpha)$. Hence, in order to establish the lemma, we need to show that the contact structure induced by the form $F^*(d\theta + e^*\alpha)$ on $MT(\Sigma, \phi^m)$ is contactomorphic to the contact structure $\ker\{\alpha_m\}$ restricted to $MT(\Sigma, \phi^m)$.

In order to see this, we observe that the embedding $F_\epsilon$ is constructed using a family $f_{(c,t)}$ of embeddings of $\Sigma$ in the symplectic manifold $([a, b] \times N, d(e^*\alpha))$ constructed in the Lemma 25. The Lemma 25 implies that the family $f_{(c,t)}$ is smooth in the parameter $c$. Observe that by varying $c \in [a, b]$ and pulling back the form $Kd\theta + e^*\alpha$ via the family $F_\epsilon$ of embeddings, we can produce a 1-parameter family $\alpha_t, t \in (a, b)$ of contact forms on $MT(\Sigma, \phi_m)$ such that the form $\alpha_0$ is the contact form $e^*\alpha$ restricted to $MT(\Sigma, \phi^m)$. It follows from the Gray’s stability theorem [Ge, Theorem 2.2.2] that the induced contact structure via the embedding $F_\epsilon$ is contactomorphic to $\ker\{\alpha\}$ restricted to $MT(\Sigma, \phi^m)$.

This establishes the lemma.

We are now in a position to establish the Lemma 24.

Proof of the Lemma 24

Notice that it follows from the Lemma 26 applied in the case when $m = 1$ that for a given $\varepsilon$, an arbitrary pair of reals $0 < a < b$, there is a proper iso-contact open book embedding of $MT(\Sigma, \phi)$ inside the contact manifold $(\mathbb{S}^1 \times [a, b] \times M, \varepsilon d\theta + e^*\alpha)$.

Observe this clearly implies that the $\varepsilon$-partial open book admits an iso-contact abstract open book embedding $F$ of $(M, \alpha)$ as desired.

4.3. Proof of the Lemma 24

Let us now establish the Lemma 24.

Proof of the Lemma 24

To begin with observe that it follows from the Lemma 26 that there exist a contact abstract open book embedding of the standard contact $(2n-1)$-sphere $(\mathbb{S}^{2n-1}, \ker\{\alpha_{std}\}) = (\mathbb{A}ob(\mathbb{D}^{2n-2}, id), \alpha(\mathbb{D}^{2n-2}, id))$ in any $\varepsilon$-partial open book $\mathbb{S}^{2n-1} \times \mathbb{D}_\varepsilon$ associated to $(\mathbb{S}^{2n-1}, \alpha_{std})$. Call this embedding $F$.

Next, produce a generalized contact abstract open book – say $N$ – by performing the band connected sum to be disjoint from the image $F(\mathbb{S}^{2n-1}) \cap P$.

It now follows from the Proposition 10 that there exists an abstract open book embedding of $(\mathbb{S}^{2n-1}, \alpha_{std})$ in the $\varepsilon$-partial open book associated to $(\mathbb{S}^{2n-1}, \# \mathbb{S}^{2n-1}, \ker\{\alpha \# \alpha_{stat}\})$. This clearly implies the lemma.

The only proposition left to establish that is used in the proof of the Theorem 1 is the Proposition 18. In the next sub-section we establish this.

4.4. Proof of the Proposition 18

Before we establish this proposition, we would like to point out that this proposition for the overtwisted 3-sphere in the standard 5-sphere was already established in the [EF].

Proof of the Proposition 18

Consider the standard contact sphere $(\mathbb{S}^{2n-1}, \xi_{std})$ as the contact abstract open book $\mathbb{A}ob(T^*\mathbb{S}^{n-1}, \tau)$. Observe that for any large $K$, there is an iso-contact embedding of the $K$-partial open book associated to $(\mathbb{S}^{2n-1}, \ker\{\alpha_{std}\})$ in the standard contact sphere $\mathbb{S}^{2n+1}$. This is because the standard $\mathbb{S}^{2n-1}$ appears as the binding of the trivial open book supporting the standard contact form on $\mathbb{S}^{2n+1}$.

Since $T^*\mathbb{S}^{n-1}$ is a page of the standard open book of $\mathbb{S}^{2n-1}$, it follows from the Lemma 26 applied in the case when $\phi$ is $\tau$ and $m = -1$, that there is an iso-contact abstract open book embedding of $(\mathbb{S}^{2n-1}, \xi_{stat})$ in the $K$-partial open book associated to $(\mathbb{S}^{2n-1}, \ker\{\alpha_{std}\})$ for a very large $K$. Hence the Proposition.
In the next couple of sections, we will give applications of the Theorem [\ref{thm:main}].

5. Contact embedding of 3-manifolds in the standard contact $S^5$

The purpose of this section is to show that every contact 3–manifold $(M, \xi)$ contact embeds in $(S^5, \xi_{\text{std}})$, provided the first Chern class of $\xi$ is zero and $M$ has no 2-torsion in $H^2(M, \mathbb{Z})$. The result essentially follows from [EF, Theorem:1.20] and the Proposition [\ref{prop:Chern}]. However, for the sake of completion, we provide a slightly more detailed argument. We begin this section by reviewing a few facts about homotopy classes of plane fields on an orientable 3–manifold.

5.1. Homotopy classes of oriented plane fields on orientable 3-manifolds.

Let $\xi$ be an oriented 2–plane field on a closed oriented 3–manifold $M$. Recall that any two such plane fields are homotopic over the 1–skeleton of a triangulation of $M$. Gompf in [Go] established that when $M$ has no two torsion in $H^2(M, \mathbb{Z})$, the first Chern class $c_1(\xi)$ completely determines homotopy of plane fields over the 2–skeleton. See [Go, Theorem:4.5].

It also follows from [Go, Theorem:4.5] that if $c_1(\xi) = 0$, then homotopy over the 3–skeleton is completely determined by the 3–dimensional invariant $d_3(\xi)$, which is defined as follows:

It was shown in [Go] that it is possible to choose an almost complex manifold $(X, J)$ whose complex tangencies are $(M, \xi)$. Given this one defines $d_3(\xi)$ as:

$$d_3(\xi) = \frac{1}{4} \left( C_1^2(X, J) - 3\sigma(X) - 2(\chi(X) - 1) \right).$$

We would like to point out that this formula is slightly different from the one given in [Go], as we are subtracting 1 from the Euler characteristic of $X$ in the formula. This is just to ensure that the formula for $d_3$ is additive when one considers the connected sums. More precisely,

Let $(M_1, \xi_1)$ be a contact manifold with $c_1(\xi_1) = 0$ and $(M_2, \xi_2)$ be another contact manifold with $c_1(\xi_2) = 0$, then for the contact connected sum $(M_1 \# M_2, \xi_1 \# \xi_2)$, we have

$$d_3(\xi_1 \# \xi_2) = d_3(\xi_1) + d_3(\xi_2).$$

To begin with, we need the following result of Etnyre and Fukuwara established in [EF]. For the sake of completeness, we will provide a short sketch of the proof of this result here.

**Theorem 27** (Etnyre and Fukuwara).

*Let $M$ be a closed 3–manifold. If $M$ is orientable, then there exists an embedding of $M$ in $S^5$ such that the contact structure $\xi_{\text{std}}$ on $S^5$ induces a contact structure on $M$.*

**Proof.** To begin with, we observe that if there exists an embedding $F : M \to S^3 \times D^2$ given by $F(x) = (f_1(x), f_2(x))$ which satisfies the following properties:

1. The map $f_1 : M \to S^3$ is a branch covering,
2. The branch locus $L$ in $S^3$ for the branch cover $f_1 : M \to S^3$ is transversal to the standard contact structure on $S^3$.

Then, there exists an $\varepsilon_0 > 0$ such that for all $\varepsilon$ less than $\varepsilon_0$, the embedding $F_\varepsilon : M^3 \to S^3 \times D^2$ given by $F_\varepsilon(x) = (f_1(x), \varepsilon f_2(x))$ is a contact embedding of $M$ in $(S^3 \times D^2, \{\alpha_{\text{std}} + r^2 d\theta\} = 0)$. See [EF] for the computation establishing that the pulled back form $F_\varepsilon^*(\alpha + r^2 d\theta)$, in fact, induces a contact structure on $M$.

Now, the Remark 3 on the page 375 of [HLM] and the fact that transversality is a generic property implies that there exists an embedding of $M$ in $S^3 \times D^2$ satisfying the two properties mentioned above. This clearly implies that in an arbitrarily small neighborhood of contact $(S^3, \xi_{\text{std}})$ inside $(S^5, \xi_{\text{std}})$ admits an embedding of $M$ such that it is a contact embedding. 

We are now in a position to prove the Theorem [\ref{thm:main}]. Recall that the Theorem [\ref{thm:main}] states that a necessary and sufficient condition for an iso-contact embedding of a contact 3–manifold $(M, \xi)$ in $(S^5, \xi_{\text{std}})$ is that $c_1(\xi) = 0$ provided $M$ has no 2–torsion in $H^2(M, \mathbb{Z})$. In case, $M$ has a 2–torsion in $H^2(M, \mathbb{Z})$, the statement claims
that there is a homotopy class \([\xi]\) of plane fields such that \(M\) with every contact structure homotopic to a plane field in the class \([\xi]\) over the 2–skeleton of \(M\) admits an iso-contact embedding in \((S^5, \xi_{\text{std}})\).

**Proof of the Theorem 3.**

We know from \([Ka]\) that \(c_1(\xi) = 0\) is a necessary condition for having an iso-contact embedding of any contact \((M, \xi)\) in \((S^5, \xi_{\text{std}})\). We know from the Theorem 27 that there exist a contact structure \(\eta\) on every 3–manifold \(M\) with \(c_1(\eta) = 0\) such that \((M, \eta)\) admits an iso-contact embedding in \((S^5, \xi_{\text{std}})\).

In case, \(M\) has no 2–torsion in \(H^2(M, \mathbb{Z})\), it follows from the \([Ge]\) Theorem:4.5 that every overtwisted contact structure \(\eta_{\text{ot}}^n\) on \(M\) which is homotopic to \(\eta\) over a 2–skeleton of \(M\) can be obtained by making a contact connected sum of \(M\) with a suitably chosen overtwisted \(S^3\). We already know from \([EF\] Theorem:1:20] that every contact \(S^3\) embeds in \((S^5, \xi_{\text{std}})\). Hence, we conclude that if \((M, \eta)\) iso-contact embeds in \((S^5, \xi_{\text{std}})\), then so does \((M, \eta_{\text{ot}}^n)\) provided \(\eta_{\text{ot}}^n\) is an overtwisted contact structure on \(M\) which is homotopic to \(\eta\) over the 2–skeleton of \(M\). But, this implies that every \((M, \eta_{\text{ot}}^n)\) iso-contact embeds in \((S^5, \xi_{\text{std}})\), provided the first Chern class of \(\eta_{\text{ot}}^n\) is zero. The case of no 2–torsion in \(H^2(M, \mathbb{Z})\) is now a straightforward consequence of the Corollary 12.

In case, \(M\) has a 2–torsion in \(H^2(M, \mathbb{Z})\) – by an argument similar to the one discussed above – it is clear that every overtwisted contact structure \(\xi_{\text{ot}}\) on \(M\) such that \(\xi_{\text{ot}}\) is homotopic to \(\eta\) as an almost contact plane field over a 2–skeleton on \(M\) admits iso-contact embedding in \((S^5, \xi_{\text{std}})\). Again, applying the Theorem 12 we conclude that every contact structure homotopic as a plane field over 2–skeleton to \(\eta\) admits an iso-contact embedding in \((S^5, \xi_{\text{std}})\). This completes our argument.

\[\square\]

### 6. Embeddings of Simply Connected 5–Manifolds in \((S^7, \xi_{\text{std}})\)

We begin this section by observing the following:

**Proposition 28.** Let \(\xi\) be a contact structure on \(S^{2n-1}\). If \(\xi\) is co-orientable and homotopic as an almost-contact structure to the standard contact structure on \(S^{2n-1}\), then \((S^{2n-1}, \xi)\) admits an iso-contact embedding in \((S^5, \xi_{\text{std}})\). In particular, every contact \((S^5, \xi)\) iso-contact embeds in \((S^7, \xi_{\text{std}})\).

**Proof.** The first part of the proposition is an immediate consequence of the Corollary 12. In order to establish the second part, recall that there exists a unique almost-contact class on \(S^5\). This was established in \([Ge]\). But this implies \(\xi_{\text{std}}\) is homotopic as an almost-contact plane field to \(\xi\). Hence the theorem.\[\square\]

Next, we show that any contact structure on \(S^2 \times S^3\) with trivial first Chern class iso-contact embeds in \((S^7, \xi_{\text{std}})\). More precisely, we establish:

**Lemma 29.** Let \(\xi\) be a co-orientable contact structure on \(S^2 \times S^3\). The contact manifold \((S^2 \times S^3, \xi)\) iso-contact embeds in \((S^7, \xi_{\text{std}})\) if and only if the first Chern class \(c_1(\xi)\) of the contact structure is zero.

**Proof.** Recall that in \([Ge]\) it is established that two almost-contact plane fields \(\xi_1\) and \(\xi_2\) are homotopic as almost-contact structures if and only if their first Chern classes coincide.

Next, Kasuya in \([Ka]\) showed that a necessary condition for a contact manifold \((M^{2n+1}, \xi)\) to admit an iso-contact embedding in \((S^{2n+3}, \xi_{\text{std}})\) is that \(c_1(\xi) = 0\).

Hence, from the Corollary 12 we can see that if there exist a contact embedding of \(S^2 \times S^3\) in \((S^7, \xi_{\text{std}})\), then the lemma follows. So, we now show that there is a contact embedding of \(S^2 \times S^3\) in \((S^7, \xi_{\text{std}})\).

Notice that the contact abstract open book \(A\text{ob}(T^\ast S^2, \text{id})\) is contact manifold diffeomorphic to \(S^2 \times S^3\). Clearly, \(A\text{ob}(T^\ast S^2, \text{id})\) iso-contact open book embeds in the contact abstract open book \(A\text{ob}(D^6, \text{id})\) as there is a symplectic embedding of \(DT^\ast S^2\) in \(S^5\). Since contact abstract open book \(A\text{ob}(D^6, \text{id})\) is contactomorphic to \((S^7, \xi_{\text{std}})\), the theorem follows.\[\square\]

It was established by H. Geiges in \([Ge]\) Chapter–8 that a necessary condition to produce a contact structure on any five manifold is that the third integral Steifel-Whitney class \(W_3\) is zero. D. Barden in \([Ba]\) had given a complete classification of simply connected 5–manifolds. Using this classification, it is easy to list all the simply connected prime 5–manifolds with vanishing \(W_3\). We now proceed to describe this list.
First of all, recall that for each \(2 \leq k < \infty\), there exists a unique prime simply connected manifold \(M_k\) characterized by the property that \(H_2(M_k, \mathbb{Z}) = \mathbb{Z}_k \oplus \mathbb{Z}_k\). Next, recall that there exists a unique non-trivial orientable real rank 4 vector-bundle over \(S^2\). By \(S^2 \times S^3\), we denote the unit sphere bundle associated to this vector-bundle.

We are now in a position to state Barden’s theorem that we will need to establish the Theorem 30.

**Theorem 30 (Barden).** Every closed simply connected almost contact 5–manifold can be uniquely decomposed into a connected sum of prime manifolds \(M_k\), \(2 \leq k < \infty\), \(S^2 \times S^3\) and \(S^5 \times S^3\). Furthermore, the decomposition can have at most one copy of \(S^2 \times S^3\).

**Proof of the Theorem 30**

Let \(M\) be a closed simply connected 5–manifold and let \(\xi\) be a contact structure on it which admits a formal iso-contact embedding in \((S^7, \xi_{std})\). In order to establish the Theorem 30 we first show that \(M\) admits a contact embedding in \((S^7, \xi_{std})\) such that the induced contact structure has its first Chern class 0.

Notice that if \(M\) is a simply connected manifold which embeds in \(S^7\), then \(M\) in its connected sum decomposition can not contain \(S^2 \times S^3\) factor. This is because for a 5–manifold to embed in \(S^7\), the normal bundle of the 5-manifold in \(S^7\) has to be trivial. This implies \(S^2 \times S^3\) can not embed in \(S^7\).

Since our hypothesis assumes that \(M\) admits a formal iso-contact embedding in \(S^7\), \(M\) can not have \(S^2 \times S^3\) in its connected sum decomposition into prime manifolds. Next, observe that if \(M = N_1 \# N_2 \# \cdots \# N_l\) and each \(N_i\) contact embeds in \((S^7, \xi_{std})\), then there exist a contact embedding of \(M\) in \((S^7, \xi_{std})\).

We have already shown in the Lemma 29 that \(S^2 \times S^3\) contact embeds in \((S^7, \xi_{std})\). Hence, in order to show that \(M\) contact embeds in \((S^7, \xi_{std})\), we just need to show that each prime manifold \(M_k\) described in the Theorem 30 above must contact embed in \((S^7, \xi_{std})\). It is well known that each \(M_k\) is a Biskorn 5–sphere. Hence, they admit contact embedding in \((S^7, \xi_{std})\). See, for example, \(\cite{Ko}\).

Thus, we have shown that every simply connected 5–manifold which admits a formal iso-contact embedding in \((S^7, \xi_{std})\) admits a contact embedding in \((S^7, \xi_{std})\). Next, recall that if a 5–manifold admits a formal contact embedding in \((S^7, \xi_{std})\), then it was shown in \(\cite{Ka}\) that the first Chern class of the induced contact structure has to be trivial.

Finally, observe that it was established in \(\cite{Ge} \cite{Ge1}\) Clpt–8 that any two contact structures on a closed simply connected 5–manifold having the identical first Chern classes are homotopic as almost-contact structures. It now follows from the Corollary 2 that \((M, \xi)\) admits an iso-contact embedding in \((S^7, \xi_{std})\).

\[\square\]

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