Transport of the repulsive Bose–Einstein condensate in a double-well trap: interaction impact and relation to the Josephson effect

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Abstract
Two aspects of the transport of a repulsive Bose–Einstein condensate (BEC) in a double-well trap are inspected: The impact of the interatomic interaction and the analogy with the Josephson effect. The analysis employs a numerical solution of a 3D time-dependent Gross–Pitaevskii equation for a total order parameter covering the whole trap. The population transfer is driven by a time-dependent shift of a barrier separating the left and right wells. The sharp and soft profiles of the barrier velocity are tested. The evolution of the relevant characteristics, involving phase differences and currents, is inspected. It is shown that the repulsive interaction substantially supports the transfer making it possible (i) in a wide velocity interval and (ii) three orders of magnitude faster than in the ideal BEC. The transport can be approximately treated as the dc Josephson effect. The dual origin of the critical barrier velocity (break of the adiabatic following and dc/ac transition) is discussed. Following the calculations, the robustness of the transport (dc) crucially depends on the interaction and barrier velocity profile. Only soft profiles which minimize undesirable dipole oscillations are acceptable.

Keywords: trapped Bose–Einstein condensate, quantum transport, Josephson effect

1. Introduction
Population transfer is a typical problem met in various branches of physics (ultracold gases and condensates [1–9], atomic and molecular physics [10], etc). The problem is easily solvable, if it is linear and accepts an adiabatic evolution, see for example the Landau–Zener scenario [11, 12]. However, if there are significant nonlinear effects and/or we need a rapid but robust transfer, the problem becomes nontrivial, like in the case of the irreversible nonlinear transport (NLT) of Bose–Einstein condensate (BEC) in multi-well traps [13, 14]. The trapped BEC is especially suited for the investigation of nonlinear transport because the BEC’s features, including the interaction-induced nonlinearity, can be precisely controlled and manipulated. Besides, by driving the trap parameters one can simulate various transport protocols.

Despite numerous experimental and theoretical studies (see early [1–6] and recent [7, 15] reviews), some important NLT features are still poorly understood. In particular, it is not well-established in which cases nonlinearity favors the transport and how essential the effect is.

In the present study, we address these general questions for a typical NLT scheme: an external Bose Josephson junction (EBJJ) produced in a double-well trap. Here the left and right BEC fractions are coupled through the barrier separating the trap. The nonlinear effects are caused by the interaction...
between the BEC atoms. The NLT is a population inversion driven by converting the trap from an initial to a final (opposite) asymmetric configurations. Nowadays, such an NTL is a routine experimental operation, which can be produced using various methods: from familiar Rabi oscillations (π pulses) [16] and (quasi)adiabatic population transfer [11, 12, 14] to modern shortcut-to-adiabaticity methods (see review [7] and particular relevant options [17–19]) promising a fast and robust population inversion. The goal of the present study is to use this simple operation for the exploration of: (i) Strong nonlinear effects predicted for this configuration within a simple two-mode model [14], (ii) an analogy between NLT and the dc (direct current) Josephson effect in superconductors [20], as predicted [21–26] and observed [27] in the EBJJ.

For this purpose, the three-dimensional (3D) time-dependent Gross–Pitaevskii equation (GPE) [28] for the total order parameter covering both the left and right parts of the condensate in a double-well trap is numerically solved. The calculations are free from the two-mode approximation (TMA) [29] and other simplifications used in our previous estimations [14]. Furthermore, our study closely follows the conditions and parameters of Heidelberg’s experiments [31, 32], thus providing a realistic picture. The population transfer is determined by a time-dependent barrier shift driving the system between the initial and final asymmetric configurations. This technique allows us to reach two aims simultaneously: (i) To exercise the generalized Landau–Zener/Rosen–Zener transport protocol implemented in our previous study [14] and (ii) to simulate an external current required for the generation of a Josephson dc in an EBJJ [24]. To highlight the nonlinear effects, the dynamics of both an ideal and a repulsive BEC are compared.

In our previous TMA study, a strong support of the transport by a repulsive interaction was found [14]. It was shown that this interaction leads to a wide range (plateau) of process rates, where a complete (quasi)adiabatic transport is realized. In the present study, we test these results within a more realistic model beyond the TMA. The scale of the nonlinear effects is estimated for the particular Heidelberg setup [31, 32]. It is shown that a repulsive BEC can be transferred by three orders of magnitude faster than an ideal condensate. A pollution of NLT by dipole oscillations is estimated and a smooth velocity of magnitude faster than an ideal condensate. A pollution effect of the dipole oscillations is estimated. Furthermore, our study closely follows the conditions and parameters of Heidelberg’s experiments [31, 32], thus providing a realistic picture. The population transfer is determined by a time-dependent barrier shift driving the system between the initial and final asymmetric configurations. This technique allows us to reach two aims simultaneously: (i) To exercise the generalized Landau–Zener/Rosen–Zener transport protocol implemented in our previous study [14] and (ii) to simulate an external current required for the generation of a Josephson dc in an EBJJ [24].

In the second part of our exploration, the NLT is compared with the Josephson dc and ac effects [20] represented for the BEC by the equations [24]:

\[ I = I_0 \sin \left( \frac{\Delta \mu}{\hbar} \right), \quad \dot{\theta} = \frac{\Delta \mu}{\hbar}, \quad (1) \]

where \( I \) is the supercurrent, \( I_0 \) is its critical value and \( \Delta \mu \) is the difference between the chemical potentials of the wells. As predicted [24] and then experimentally observed [27], the dc can be generated in an EBJJ by an adiabatic movement of the barrier across the trap with a constant velocity, thus simulating the driving current. This shift can drive the trap from an asymmetric to a symmetric configuration [27], or vice versa [20]. The adiabatic evolution assumes that the system change is so slow that the tunneling of the atoms between the wells is sufficient to lock \( \Delta \mu \) to zero. Then we get the Josephson dc driven by the constant phase difference \( \theta \). The critical current \( I_0 \) should be proportional to the critical velocity \( v_{\text{crit}} \) of the barrier shift. Above this velocity, the adiabatic flow breaks down, the non-zero \( \Delta \mu \) develops, and the process becomes ac in character with \( I = I_0 \sin \left( \Delta \mu \theta / \hbar \right) \) [24–26].

It is easy to see that this scenario corresponds to an adiabatic NLT described in the TMA in our previous study [14]. The plateau in transport rates [14] is precisely the region \( I < I_0 \) where the adiabatic evolution takes place. The critical rate [14] marking the break of the adiabatic transport seems to correspond to \( v_{\text{crit}} \) and \( I_0 \) in [24, 27]. The analogy should take place despite the fact that the population transfer in [14] is driven not by the barrier shift but by another technique generalizing the Landau–Zener and Rosen–Zener schemes. Both scenarios have to be physically similar since they satisfy the principal requirements: Weak coupling, inherent phase difference, and adiabatic evolution.

In the present study, we continue the analysis of dc and ac in an EBJJ, but now with the focus on nonlinear effects. As compared to the previous studies [22–26], which were limited to inspections of population imbalance \( z \) and chemical potential difference \( \Delta \mu \), we also scrutinize the evolution of the phase difference \( \theta \), a principle factor in Josephson dynamics. In particular, we provide a detailed analysis of \( \theta \) near \( v_{\text{crit}} \).

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Also, a pollution effect of the dipole oscillations is estimated. It is shown that the constant barrier velocity [24] results in strong oscillations, which massively disrupt the process and complicate the analysis. Thus, a soft velocity profile is proposed to circumvent this problem. It is shown that the repulsive interaction and soft velocity profile make the NLT (and the dc/ac) much more suitable for analysis and experimental observation.

Note that in recent years EBJJs have been widely used in diverse actual areas (shortcuts to adiabaticity and optimal control [7, 15], spin squeezing, entanglement and quantum metrology [33, 34], Josephson dynamics in a spin–orbit BEC [35, etc]. At the same time, investigations of dc/ac regimes in an EBJJ are sparse [36], despite the interesting flaring similarity of dc with adiabatic population transfer scenarios. The present detailed study of ac/dc in a double-well trap aims to partly fill this gap.

This paper is organized as follows. The theory and calculation framework are outlined in section 2. The results are discussed in section 3. The summary is given in section 4.

2. Calculation scheme

2.1. Trap setup and well populations

The calculations are performed using a 3D time-dependent Gross–Pitaevskii equation (GPE) [28]:

\[ i \hbar \frac{\partial \Psi}{\partial t} (r, t) = \left[ -\frac{\hbar^2}{2m} \nabla^2 + V(r, t) + g_0 |\Psi(r, t)|^2 \right] \Psi(r, t) \quad (2) \]

for the total order parameter \( \Psi(r, t) \) describing the BEC in both the left and right wells of the trap. Here \( g_0 = 4\pi \hbar^2 a, m^{-1} \)}
is the interaction parameter, \(a_i\) is the scattering length, and \(m\) is the atomic mass. The trap potential:

\[
V(\mathbf{r}, t) = V_{\text{con}}(\mathbf{r}) + V_{\text{bar}}(x, t) = \frac{m}{2} \left( \omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2 \right) + V_0 \cos^2(\pi(x - x_0(t))/q_0)
\]

(3)

includes the anisotropic harmonic confinement and the barrier in \(x\)-direction, whose position is driven by the control function \(x_0(t)\) [24, 26]; \(V_0\) is the barrier height and \(q_0\) determines the barrier width.

Following the conditions of the Heidelberg experiment [31, 32] (where Josephson oscillations (JO) and macroscopic quantum self-trapping (MQST) have been observed), we consider a BEC of \(N = 1000\) \(^{87}\)Rb atoms with \(a_i = 5.75\) nm. The trap frequencies are \(\omega_x = 2\pi \times 78\) Hz, \(\omega_y = 2\pi \times 66\) Hz, \(\omega_z = 2\pi \times 90\) Hz, i.e. \(\omega_x + \omega_z = 2\omega_0\). The barrier parameters are \(V_0 = 420 \times h\) Hz and \(q_0 = 5.2\) nm. For the symmetric trap \((x_0(t) = 0)\), the distance between the centers of the left and right wells is \(d = 4.4\) mm. This setup was used earlier in our exploration of JO/MQST in both a weak and strong coupling [39]. It corresponds to the so-called Josephson (classical) regime, when the quantum fluctuations of both the population imbalance and the phase difference are not essential.

The static solutions of GPE are found within the damped gradient method [37], while the time evolution is computed using the time-splitting technique [38]. The total order parameter \(\Psi(\mathbf{r}, t)\) is determined on a 3D cartesian grid. The conservation of the number of atoms, \(\int_{\text{space}} d^3 \mathbf{r} |\Psi(\mathbf{r}, t)|^2 = N\), is directly fulfilled by using an explicit unitary propagator. No time-space factorization of the order parameter is implemented. The conservation of the total energy \(E\) is controlled.

The population of the left (L) and right (R) wells is computed as:

\[
N_j(t) = \int_{-\infty}^{\infty} dr^3 \Psi_j(\mathbf{r}, t) \mathbf{r}^2,
\]

(4)

with \(j = L, R\). \(\Psi_L(\mathbf{r}, t) = \Psi(x \leq x_0(t), y, z, t)\), \(\Psi_R(\mathbf{r}, t) = \Psi(x \geq x_0(t), y, z, t)\) and \(N_L(t) + N_R(t) = N\). The normalized population imbalance is

\[
z(t) = (N_L(t) - N_R(t))/N.
\]

(5)

The NLT to be considered means that initial \((t = 0)\) BEC population \(N_L(0) > N_R(0)\) is inverted during time \(T\) to the final population \(N_L(T) < N_R(T)\) where \(N_L(T) = N_R(0)\) and \(N_R(T) = N_L(0)\). The initial stationary asymmetric BEC state is produced by adjusting the barrier right-shift \((x(0) > 0)\) so as to provide the required initial populations \(N_L(0)\) and \(N_R(0)\). The NLT is achieved by a barrier left shift from \((x(0))\) to \(x(T) = -x(0)\) with a shift velocity of \(v(t)\). Thus the trap asymmetry is changed with the opposite one.

Two velocity time profiles are used: (i) A sharp rectangular one with constant \(v(t) = v_0^L\) at \(0 < t < T\) and \(v(t) = 0\) beyond the transfer time, and (ii) a soft one \(v(t) = v_0^L \cos^2(\pi(x - x_0(t))/q_0)\) with \(v_0^L = v_0(T) = 0\) and \(v_0(T/2) = v_0^L\). For the total barrier shift \(D = 2x(0)\) during the inversion process of duration \(T\), the velocity amplitudes are \(v_0^L = D/T\) and \(v_0^R = 2D/T\). The average velocities \(v_0 = D/T\) are:

\[
v_0^L = v_0^R, \quad v_0^L = v_0^R/2.
\]

(6)

The profile \(v(t)\) was used in previous studies [24, 26]. It changes sharply from 0 to \(v_0^L\) at \(t = 0\) and back at \(t = T\) and, in this sense, is not adiabatic. As shown below, the sharp changes cause undesirable dipole oscillations, which can significantly pollute the population transfer. The second profile \(v(t)\) is softer and thus closer to the adiabatic evolution.

The NLT quality is characterized by its completeness \(P = 1 - z(T)/z(0)\) (the ratio of the final and initial population imbalance) and noise \(n = A_d/N\), where \(A_d\) is the amplitude of the dipole oscillations in the final state, i.e. \(A_d = \max\{N_{L,R}\} - \min\{N_{L,R}\}\) for \(t > T\).

Note that previous studies also used 3D [24] and 1D [26] numerical time-dependent GPE simulations.

2.2. Phases

The phases \(\phi_j(t)\) of the left and right BEC fractions are defined as [39]:

\[
\phi_j(t) = \arg\frac{\Psi_j(t)}{\Psi_j(0)}
\]

(7)

with the averages:

\[
\bar{\phi}_j(t) = \frac{1}{N} \int_{-\infty}^{\infty} d^3 \mathbf{r} |\Psi_j(\mathbf{r}, t)|^2 \bar{\Psi}_j(\mathbf{r}, t) \mathbf{r}^2,
\]

(8)

\[
\chi_j(t) = \frac{1}{N} \int_{-\infty}^{\infty} d^3 \mathbf{r} \text{Re}(\bar{\Psi}_j(\mathbf{r}, t)) |\Psi_j(\mathbf{r}, t)|^2.
\]

(9)

Since the computation of the phase time evolution through \(\arg\) may be cumbersome, we use (7) only for the static case while the time evolution is calculated through the phase increments \(\delta\phi(t + \delta t) = \phi(t) + \delta \phi(t)\) for a small time step \(\delta t\). Namely, we use:

\[
\delta\phi_j(t) = \left[\begin{array}{c}
\bar{\delta}\bar{\phi}_j(t) \\
\bar{\delta}\chi_j(t) + \chi_j(t + \delta t)
\end{array}\right]
\]

(10)

with:

\[
\bar{\delta}\bar{\phi}_j(t) = \bar{\phi}_j(t + \delta t) - \bar{\phi}_j(t), \quad \bar{\delta}\chi_j(t) = \chi_j(t + \delta t) - \chi_j(t).
\]

The phase difference is:

\[
\theta(t) = \phi_R(t) - \phi_L(t).
\]

(11)

2.3. Energy estimations

To discriminate weak and strong couplings between BEC fractions, it is useful to compare the energy of the occupied state with the barrier height \(V_0\). Since the barrier takes place in the \(x\)-direction, only the part of the ground state energy in the same direction is relevant. In the linear case \((g_0 = 0)\), the total ground state energy reads as in the anisotropic harmonic oscillator, \(\mu_0 = \mu_0^L + \mu_0^R\), and its relevant \(x\)-part is:

\[
\mu_0^L = \frac{\hbar}{2} (\omega_x + \omega_z) = \mu_0 - \hbar \omega_x = \alpha \mu_0
\]

(12)

where the relation: \(\omega_x + \omega_z = 2\omega_0\) [31, 32] is used. The numerical GPE estimation gives \(\alpha = \mu_0/\mu_0 \approx 3/4\) [39].
In the nonlinear case \((g_0 \neq 0)\), the estimation of \(\mu_z\) is straightforward for a 1D system, but the demand for a 3D case considered here. So we use the simple ansatz:

\[
\mu_z = \alpha \mu
\]

where \(\mu\) is the total nonlinear ground state energy and \(\alpha = 3/4\) is as in the linear case. This phenomenological relation was shown to be accurate in the investigation of the evolution of JO/MQST dynamics under the transition from a weak to a strong coupling [39]. In this study, it is used only for illustrative aims, namely for comparison with the barrier height \(V_0\), as shown in figure 1.

The energies \(\mu_L\) and \(\mu_R\) can be treated as chemical potentials in the Josephson setup [1–4]. In the rapid evolution of the system, initiated by the barrier shift, the difference between the chemical potentials of the left and right wells, \(\Delta \mu = \mu_L - \mu_R\), can be created [24]. In the NLT, \(\Delta \mu\) can be estimated through \(\theta\), as can be seen in equation (1).

### 2.4. Josephson current

The Josephson current is defined as:

\[
I(t) = -\frac{\dot{z}(t)}{2} = -\frac{\dot{N}_L(t)}{N} = \frac{\dot{N}_R(t)}{N}.
\]

This explicit current may be compared to an approximate one:

\[
\ddot{I}(t) = I_0 \sqrt{1 - \dot{z}(t)^2} \sin \theta(t)
\]

following from the first of the GPE–TMA equations [14, 21, 23, 29]:

\[
\dot{z} = -2K \sqrt{1 - \dot{z}^2} \sin \theta,
\]

\[
\dot{\theta} = \frac{\Delta \mu}{2} + K \frac{\dot{z}}{\sqrt{1 - \dot{z}^2}} \cos \theta + \frac{NU}{2} \dot{z}.
\]

Here \(I_0\) is the EBJJ critical current, \(K\) is the coupling between the BEC fractions through the barrier, and \(U\) is the interaction between the BEC atoms inside the trap wells. In the TMA, we have \(I_0 = 2K\). Equations (16) and (17) are mathematically similar to those for resonantly generated coherent modes [30]. Equations (16) and (17) remind the Josephson equations (1) of what is important regarding our aims.

In our study, we get the population imbalance \(z(t)\), the phase difference \(\theta(t)\), and the currents \(R(t)\) and \(\dot{V}_0\), not from (16) and (17), but from a direct solution of the GPE (2). Then a comparison of the explicit (14) and approximate (15) currents at a reasonable point, say at \(t = T/2\), allows us to estimate the critical current \(I_0\).

### 3. Results and discussion

#### 3.1. Confinement, density and chemical potential

Figure 1 shows the trap potential in the \(x\)-direction:

\[
V_x(x, t) = \frac{m}{2} \alpha^2 x^2 + V_0 \cos^2 (\pi (x - x_0(t)) / q_0),
\]

calculated for the initial \(t = 0\), intermediate \(t = T/2\), and final \(t = T\) times of the inversion process. For the same times, the BEC density profile in the \(x\)-direction:

\[
\rho(x, t) = \int_{-\infty}^{+\infty} dy dz |\Psi(x, y, z, t)|^2,
\]

obtained for an adiabatic inversion of a long duration \(T\) is shown. The ideal and repulsive BECs with \(N = 1000\) atoms are considered. In the plots \((a)\) and \((d)\), the initial populations of the left and right wells are \(N_L(0) = 800\) and \(N_R(0) = 200\) with a population imbalance \(z(0) = 0.6\). An adiabatic evolution provides a robust population inversion to its final state with \(N_L(T) = 200, N_R(T) = 800\), and \(z(T) = -0.6\). At the intermediate time \(t = T/2\), the trap and populations are symmetrical. The initial state is stationary by construction. The intermediate and final states, being obtained adiabatically, can be also treated as stationary.

The upper plots in figure 1 show that to get the initial \(z(0) = 0.6\) in an ideal BEC, a small trap asymmetry with \(x_0(0) = 0.0064 \mu m\) is sufficient. The overlap of the left and right parts of the condensate at the center of the trap is very small. The chemical potential \(\mu\) from (12) lies well below the barrier top. The energy difference between the ground and the first excited states in the middle of the transfer (plot \(b\)) is \(\Delta \mu(T/2)/\hbar = 5\) Hz, i.e. much smaller than the well depths and trap frequencies. Altogether, all these factors indicate a weak coupling incidence.

For a repulsive BEC (bottom plots), the initial \(N_L(0) = 800\) and \(N_R(0) = 200\) are obtained at a much larger asymmetry, with \(x_0(0) = 0.5 \mu m\). The energy splitting \(\Delta \mu(T/2)/\hbar\) reaches 36 Hz. The repulsive interaction significantly increases the chemical potential \(\mu\) (13), and thus the coupling between the left and right BEC fractions. In this case, to get the initial stationary population imbalance \(z(0) = 0.6\), one should appreciably weaken the coupling by correspondingly increasing the asymmetry. As compared to an ideal BEC, the repulsive condensate has wider density bumps, which significantly overlap at the center of the trap. The coupling between the left and right BEC fractions is no longer weak, although the NLT considered below is still realized through tunneling.

#### 3.2. Linear and nonlinear dynamics

Some examples of the time evolution of the populations \(N_{L,R}(t)\) in an ideal BEC are given in figure 2. The evolution is driven by the barrier shift with the rectangular \(v_c(t)\) (upper plot) and the soft \(v_s(t)\) (bottom plots) velocity profiles. In both cases, the same average velocities are used. The total barrier shift is \(D = 12.8\) nm. It is seen that, at low (adiabatic) velocities corresponding to a long time \(T = 2\) s (plots \(a\) and \(c\)), we get a robust population inversion. The final state is almost stationary for \(v_s(t)\) and somewhat spoiled by dipole oscillations for \(v_c(t)\). The latter is caused by the sharp change in \(v_c(t)\) at the beginning and end of the process. Instead, the \(v_s(t)\)-transfer is softer and more adiabatic. Following plots (b) and (d), the inversion becomes worse or even breaks down at high velocities.
In figure 3, similar examples are given for the repulsive BEC. At first glance, the non-linear evolution resembles the linear one in figure 2. Like in the linear case, a slow transfer (plots (a) and (c)) results in a robust NLT, while a faster process (plots (b) and (d)) spoils the final state due to dipole oscillations (b), or even breaks the inversion altogether (d). However, nonlinearity essentially changes the rates of this process. Robust NLTs are produced for larger barrier shifts (1 µm instead of 0.013 µm), for much shorter periods (\(T = 250\) ms instead of \(T = 1800\) ms for an ideal BEC), and at much faster velocities (\(\mu m \ s^{-1}\) instead of nm s^{-1}). The velocities are three orders of magnitude higher (!) than in the linear case. The
repulsive interaction greatly favors the population inversion (the transfer parameters become more comfortable for experimental implementation) and the effect is indeed huge. The reason is in the growth of the chemical potential $\mu$, caused by the repulsive interaction. This leads to a dramatic increase in barrier penetrability. The coupling between the BEC fractions becomes strong and the inversion is realized much faster.

More general information on NLTs is shown in figures 4 and 5, where the completeness $P$ and noise $n$ of the inversion are given for a wide range of velocity amplitudes. In figure 4, the sharp velocity profile $v_c(t)$ is used. Following plots (a) and (c) for an ideal BEC, a complete inversion ($P = 1$) takes place only at a low velocity $v_0 < 0.04 \mu m s^{-1}$. This inversion is somewhat spoiled by a noise $n = 0.02 - 0.04$, which weakens it with decreasing the velocity. For $v_0 > 0.04 \mu m s^{-1}$, we see the gradual destruction of the inversion, accompanied by an enhanced noise. For even higher velocities, the inversion breaks down ($P \to 0$) and the final state is characterized by strong Rabi oscillations ($n \to 0.4$). These oscillations are caused by the instant change in velocity from zero to $v_0$ at $t = 0$ and back to $t = T$.

As shown in figures 4(b) and (d), the inclusion of a repulsive interaction dramatically changes the results. We see a wide plateau, $0 < v_0 \leq 19 \mu m s^{-1}$ (with the critical velocity...
v_{\text{crit}} \approx 19 \mu m \ s^{-1}, \text{ where the inversion is about } (P = 1) \text{ complete. As mentioned previously, a repulsive interaction allows us to get an inversion three orders of magnitude faster than for an ideal BEC. These findings are in accordance with our previous results for an NLT, obtained within the simplified TMA model [14].}

Note that for an ideal and repulsive BEC the inversion breaks down in different ways. While in the linear case the transfer completeness \( P \) tends to zero, in the repulsive BEC it becomes negative, \( P = -0.7 \). The latter means that \( z(0) \) and \( z(T) \) have the same sign, i.e. the process results only in a partial population transfer, keeping the initial inequality \( N_L > N_R \) at \( t = T \).

In figure 5, a similar analysis of the softer velocity profile \( v_c(t) \) is shown. Note that, as compared to figures 2 and 3, here we do not use the average \( \bar{v}_c \), but the maximal velocity \( v_c = 2\bar{v}_c \). The results are generally similar to those shown in figure 4. However, in the repulsive BEC (figure 5(d)), the process below the critical velocity is much less noisy than in the previous \( v_c(t) \) case. So, as might be expected, the sofer (more adiabatic) profile \( v_c(t) \) leads to a more robust inversion than the sharp profile \( v_c(t) \).

In the repulsive BEC, the critical velocities for both profiles, \( v_{\text{crit}} \approx 19 \mu m \ s^{-1} \) and \( v_{\text{crit}} \approx 22 \mu m \ s^{-1} \), are rather similar. Note that these upper limits concern maximal (not average for \( v_c(t) \)) velocities. The physical significance of critical velocity is simple: Destruction of the adiabatic following [14]. Namely, if the system is transformed slowly, then the tunneling suffices to arrange the BEC distribution in accordance with the transformation. Thus, we gain an adiabatic NLT. However, at critical velocity, the transformation becomes too fast and the efficient adiabatic transfer (transport) breaks down. This argument is partly confirmed by the fact that \( v_{\text{crit}} < v_{\text{crit}} \), i.e. a softer velocity profile leads to a higher critical velocity. More insight into the nature of \( v_{\text{crit}} \) can be reached by treating an NLT in terms of Josephson direct and alternating currents [24]; see the next subsection. Then \( v_{\text{crit}} \) is associated with the critical current manifesting the dc \( \rightarrow \) ac transition. However, the dc also assumes an adiabatic following and so does not contradict the adiabatic arguments of [14].

### 3.3. Analogy with the Josephson effects

Figure 6 shows the evolution of the phase difference \( \theta(t) \) and Josephson currents (right) in an ideal BEC for the cases (a) and (c) of figure 2. The exact \( \hat{I}(t) \) (solid bold curve) and approximate \( \hat{I}(t) \) (dotted curve) currents are shown. In (b) and (d), the critical current \( I_0 = 4.6 \) Hz is used to scale \( \hat{I}(t) \). For \( v_{\text{crit}}(c) \), the chemical potential difference is indicated. The barrier shift duration \( T = 2 \) s is marked as vertical dotted lines. For the reference, the zero line is given in (b) and (d).

![Figure 6](image)

**Figure 6.** Evolution of the phase difference \( \theta(t) \) (left) and Josephson currents (right) in an ideal BEC for the cases (a) and (c) of figure 2. The exact \( \hat{I}(t) \) (solid bold curve) and approximate \( \hat{I}(t) \) (dotted curve) currents are shown. In (b) and (d), the critical current \( I_0 = 4.6 \) Hz is used to scale \( \hat{I}(t) \). For \( v_{\text{crit}}(c) \), the chemical potential difference is indicated. The barrier shift duration \( T = 2 \) s is marked as vertical dotted lines. For the reference, the zero line is given in (b) and (d).

\( t \sim T/2 \). During the first half of the evolution \( (t < T/2) \), the average chemical potential difference is \( \Delta \mu / \hbar = \hat{\theta} \sim 0.043 \) Hz, i.e. very small. The dc assumes a constant phase difference \( \theta \) and, therefore, zero chemical potential difference \( \Delta \mu \). The present process demonstrates a small \( \Delta \mu \) and so can be approximately treated as a quasi-adiabatic dc. A true dc only occurs for a short time in the middle of the evolution \( (t < T/2) \).

Further insight into the process can be brought by a direct inspection of the Josephson currents. In figure 6(d), the exact current \( I \) is obtained through \( \hat{z} \) and the approximate current \( \hat{I} \) is determined through \( \hat{\theta} \) (see equations (14) and (15)). For the calculation of \( \hat{I} \), the critical current \( I_0 = 4.6 \) Hz, which is obtained from the condition \( \hat{I}(t) = \hat{I}(t) \) at \( t = T/2 \), is used (note that maximal \( I < I_0 \)). Plot (d) shows that, for \( t < T/2 \), both \( I \hat{ } \) and \( \hat{I} \) are similar and closely follow the evolution of \( \theta \). Since \( \hat{I}(t) \propto \sin \hat{\theta} \), we indeed have here a Josephson-like phase-driven process.

For \( t > T/2 \), the behavior of \( I \) and \( \hat{I} \) is different. \( I \) tends to be zero (in accordance with figure 2(c)), while the approximate current \( \hat{I} \) approaches a finite value (in accordance with the behavior of \( \theta \) in figure 6(c)). The difference is obviously caused by the final phase difference of \( \theta \).

In figure 6(a) and (b), the same characteristics are presented for the constant velocity. Despite the fact that the average velocities of the two profiles are the same, \( v_{\text{crit}} = v_{\text{crit}} = 6.4 \) nm s\(^{-1} \), the
evolution in (a, b) is significantly disrupted by dipole oscillations, which once more shows the importance of using soft velocity profiles. In general, up to the dipole oscillations, the behavior of $\theta$ and the currents in (a, b) are similar to those in (c, d). At the same time, plot (b) provides additional information: It shows that the Josephson current $I$ is not constant, even for the constant velocity profile. So, in contrast to statement [24], the Josephson current is not necessarily proportional to the barrier velocity.

In figures 7 and 8, the evolution of the relevant characteristics for the repulsive BEC is presented. Since the velocity profile $v(t)$ leads to dipole oscillations, which complicate the analysis, we will further inspect only the soft profile $v(t)$. As shown in figure 7, the slow ($v_a = 1 \mu m s^{-1}$), medium ($v_a = 4 \mu m s^{-1}$) and fast ($v_a = 8 \mu m s^{-1}$) processes are considered. In all the cases, a successful NLT takes place (left plots). The behavior of $\theta$ and the currents in the repulsive BEC are qualitatively similar to those for an ideal BEC. The main difference is in the significant enhancement of the process rates. In particular, as mentioned above, the average barrier velocities become 3 orders of magnitude higher than for an ideal BEC. The final phase difference $\theta_f$ remains constant with increasing $v_a$. Its relative impact, being decisive at a low velocity, becomes less important at high velocities. It seems that just $\theta_f$ leads to some variance in $I_0$. For a large $v_a = 8 \mu m s^{-1}$, we still have $\theta \approx \pi/2$ and $I < I_0$. The chemical potential difference still remains modest, $\Delta \mu/h \sim 4.0 Hz$. Thus in general agreement with the prediction [24], this NLT can be approximately treated as a quasi-adiabatic, phase-driven dc-like process.

In figure 8, the NLT near $v_{crit}$ is considered (for the soft velocity profile this average critical velocity is twice as low as the maximal critical velocity as shown in figures 4 and 5). It is seen that at the interval $v < v_{crit}$, there is a pronounced transition to the ac-like regime. For $v \approx 12 \mu m s^{-1}$, $\theta$ acquires a linear time dependence when the current starts to oscillate at frequency $\omega = \Delta \mu/h$. The value of $\Delta \mu$ becomes much larger than for $v < v_{crit}$. The approximate current $I$ converges with the supercurrent $I$, while the latter approaches the critical current $I_0$. Altogether, all these factors clearly indicate the ac nature of the final state. The high-frequency ac is modulated by low-frequency dipole oscillations. The ac looks like an MQST [21, 23] (running phase, non-zero average population imbalance $\langle z \rangle$) near the critical point ($v_{crit} = 12 \mu m s^{-1}$), but deviates from the MQST ($\langle z \rangle \to 0$) at higher velocities.

Altogether, our analysis confirms that the NLT can be approximately treated as a phase-driven, quasi-adiabatic, dc-like process occurring at $v < v_{crit}$. At higher velocities $v > v_{crit}$, the NLT breaks down and transforms to an ac. Note
that the dc treatment of an NLT should be taken with care. Indeed, our calculations show that, for \( v < v^{\text{crit}} \), the phase difference \( \theta \) is not constant, and the chemical potential difference \( \Delta \mu \) is not zero. Only the small values of \( \dot{\omega} \) and thus \( \Delta \mu \) permits dc treatment.

It should be emphasized that a cornerstone of dc in a weakly coupled phase-driven system is an adiabatic following. Indeed, a dc is adiabatic by definition (as a weak current still unable to produce quasiparticle excitations). Therefore, \( v^{\text{crit}} \) can be treated as a critical point for both dc \( \rightarrow \) ac [24] and (quasi)adiabatic \( \rightarrow \) non-adiabatic [14] transitions. Then, for example, the critical velocity in a quasi-adiabatic Landau-Zener population transfer of a repulsive BEC in a double-well trap [14] can be viewed both as a break of adiabatic following and as a dc \( \rightarrow \) ac transition.

Finally, in the present study, the trap is transformed from an initial asymmetrical form into a final opposite asymmetrical form, passing through the symmetric configuration at the middle of the process (asym \( \rightarrow \) sym \( \rightarrow \) asym transformation). Instead, the previous theoretical [24] and experimental [27] studies used sym \( \rightarrow \) asym and asym \( \rightarrow \) sym transformations, respectively. Despite these differences, the Josephson physics behind the evolution is essentially the same. However, compared to [24, 27], our analysis is more complete in the sense that (i) the nonlinear impact is explored in detail and (ii) the crucial ingredient of the Josephson effects, the phase difference, is numerically inspected.

4. Summary

The linear and nonlinear transport of a BEC in a double-well trap is investigated within the time-dependent three-dimensional Gross–Pitaevskii equation, in close reference to the parameters of the Heidelberg experiments [31, 32]. The calculations are performed for the total order parameter, thus avoiding typical (two-mode, etc) approximations. The population transfer is driven by a time-dependent barrier shift with sharp (rectangular) and soft (\( \sim \cos^2(\omega t) \)) velocity profiles. It is shown that using a soft profile is crucial for avoiding strong dipole oscillations, which significantly disrupt the transport and complicate its theoretical analysis and experimental observation [27].

The calculations confirm our previous findings (obtained in the simplified model [14]), that repulsive interaction between BEC atoms (and the related nonlinearity of this problem) significantly supports an NLT, making it possible within a wide interval of barrier velocities. As compared to an ideal BEC, the process can be three orders of magnitude faster. Besides, the nonlinearity allows to produce, transport between the stationary states of an essentially anisotropic trap. All these factors should facilitate the experimental investigation of NLTs.

The interaction effect is mainly caused by a rise in chemical potential. Hence this effect should depend on the barrier form, being strong for smooth barriers, whose penetrability increases with excitation energy, and suppressed for sharp barriers with a weak energy dependence of the penetrability.
Furthermore, the relation between NLT and the dc Josephson effect is studied in detail. As compared to previous studies [22–26], the evolution of the phase difference $\theta$ (a crucial ingredient of the Josephson effect) is numerically explored. It is shown that, in accordance with [24, 27], the NLT indeed can be approximately treated as a dc. Above the critical barrier velocity $v_{\text{crit}}$, the NLT decays into an ac. Note that the dc treatment of NLT is actually an approximation because in NLT the phase difference $\theta$ is not constant and the chemical potential difference $\Delta \mu$ is not zero, which contradicts the definition of a dc. However, because of the small values of $\theta$ and $\Delta \mu$, the dc treatment is still reasonable.

The behavior of the transport near the critical velocity $v_{\text{crit}}$ is investigated in detail. It is shown that $v_{\text{crit}}$ marks both dc $\rightarrow$ ac [24] and (quasi)adiabatic $\rightarrow$ nonadiabatic [14] transitions. These results emphasize the adiabatic nature of the dc in Bose–Josephson junctions (BJJ). Actually we are dealing here with a general phase-driven adiabatic following of a weakly-bound two-component system. In this sense, a variety of (quasi)adiabatic population transfer protocols (from the familiar Landau–Zener [11, 12] scheme and its generalizations [14] to modern adiabatic prescriptions [17]) in internal and external BJJ can be roughly considered as manifestations of the dc Josephson effect.

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