Gauge Invariance and Noncommutativity*

Corneliu Sochichi

Institutul de Fizică Aplicată AŞ,
str. Academiei, nr. 5, Chişinău, MD2028
MOLDOVA
Bogoliubov Laboratory of Theoretical Physics
Joint Institute for Nuclear Research
141980 Dubna, Moscow Reg.
RUSSIA
email: sochichi@thsun1.jinr.ru

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Abstract

The role of the gauge invariance in noncommutative field theory is discussed. A basic introduction to noncommutative geometry and noncommutative field theory is given. Background invariant formulation of Wilson lines is proposed. Duality symmetries relating various noncommutative gauge models are being discussed.

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1 Introduction

Understanding the structure of the space-time has always been a challenge for the theoretical physics.

Since Einstein the gravity interactions are understood as deformation of the space-time metric. The description in the framework of General Relativity gives satisfactory results for a wide range of scales: from Plank to that Galaxy sizes and more which covers, in fact, all experimentally and astrophysically accessible scales up to date. However, the sizes smaller than the Plank scale remain beyond the consistency limit of General Relativity. Due to a number of problems the General Relativity appeared to be incompatible with the quantum approach thus making such extension impossible.

The lack of reliable data on the space-time structure at very small sizes (or very high energies) provides a good ground for various models, even extremely exotic ones, to be constructed. In this context the geometry including the topology can be accepted to have a form which is quite different from what we are “familiar” with at larger scales. In this case the “familiar” geometry is obtained as an “average” of such exotic structures.

As far as in 50’s of the last century it was proposed to consider the possibility that at small distances coordinates satisfy the Heisenberg-like commutation relation rather than commute “as usually”. Surprisingly, it appeared that such “spaces” possess many features allowing to build over them quantum field theory as it were an ordinary space. Even more surprising is that these models revived recently, however, from different considerations.

An alternative approach to get a quantum description of gravity, or the description of the space-time structure at very small distances is to consider the gravity as an effective model to one which is a more “quantum-friendly”. Such a model was discovered some 30 years ago and it is known as String Theory or its supersymmetric extension(s): Superstring Theory.

This conceptually simple model of a relativistic one-dimensional extended object moving in a $D$-dimensional space-time (target space) appears to have extremely rich and complicate structure. The self-consistence of the quantum model impose tight conditions on the geometry of the target space. Thus, cancellation of the conformal (modular) anomaly requires the space-time dimensionality to be 26 in the bosonic case or 10 in the supersymmetric one.

The mathematical rigidity of the string theory allows the only bosonic string model (which, however, appears to be unstable due to the presence of a tachyonic mode in the spectrum), and a few supersymmetric ones, basically they are, type I, type IIA, type IIB, heterotic string and their modifications. These models differ by the number and relative chirality of supercharges, but they are related with each other by duality relations. Basing on this duality symmetry it was conjectured that all above string models (maybe, except
the bosonic string) are different perturbative regimes of the same larger
theory called \textit{M-theory}, about which we know that its low energy effective
field theory is the eleven dimensional supergravity.

One conceptual problem arising when the superstring models are pro-
posed as fundamental theory is that they are multi dimensional while the
observable world has only four dimensions. This problem was proposed to
be solved by compactification extra dimensions to sizes less than observable
by our today’s tools. Although this mechanism could provide an explana-
tion of the four-dimensionality of the low energy world, but until now there
is no clear mechanism demonstrating why and how this realizes dynamically
in the string theory.

Developing of the string theory in recent years showed that the com-
pactification is not the only way to obtain the four-dimensional world from
the string theory. Nonperturbative analysis of the string dynamics unveiled
a large set of extended objects of various dimension in the nonperturbative
spectrum of the string theory. These objects are called D-branes as it comes
from [membrane]. In particular, existence of D-branes is required by duality
symmetries \cite{3,4} (see also \cite{5,6}.

It was also advocated \cite{7,8,9,10,11} that in the background of constant
stringy NSNS field $B_{\mu\nu}$ the low energy dynamics of the nonperturbative ob-
jects is described by field theory models on noncommutative spaces. Namely,
the quantum algebra of string observables is such that in the limit of large
constant $B_{\mu\nu}$ the subalgebra corresponding to the dynamics of the string
ends decouples and can be described by a field theory on a space whose
coordinates satisfy,

$$[x^{\mu}, x^{\nu}] = i \theta^{\mu\nu},$$

(1.1)

where $\theta^{\mu\nu} \neq 0$ is a constant matrix depending on $B_{\mu\nu}$. In such a way
we “rediscovered” the noncommutative space-time as a consequence of the
string description of gravity.

It appears that in many cases it is possible to develop an analysis of
field theoretical models over noncommutative space very similar to one we
have in ordinary spaces. This analysis sometimes unveils very surprising
properties of such models. Thus, even in classical or tree level these models
are very different from their relatives on commutative spaces. Among such
properties one could enumerate the so called IR/UV mixing, which consists
in dependence of the small distance behaviour on large distance data, non-
commutative solitons and lump configurations, which are localized solutions
of the size of noncommutativity and duality relations which we are consid-
ering in the present paper. Here we discuss the noncommutative theories in
the classical approach. In perturbation theory there are additional problems
which we are not going to discuss here, the relevant reference is \cite{12,13} and
consecutive works.
The plan of the paper is the following. The first part is an introductory one. There we introduce the notion of noncommutative space, including a brief review of the Connes concept of the noncommutative space. As an illustration we consider a lattice example. We also consider the simplest examples of noncommutative spaces such as noncommutative plane $\mathbb{R}^n_\theta$, noncommutative torus $T^n_q$, and noncommutative or fuzzy sphere $S^n_\theta$. The last two noncommutative spaces can be embedded into the first one. We review the notion of Weyl ordering, Weyl symbols and introduce noncommutative calculus. This allows one the introduction of the notion of noncommutative field theory.

The second part is devoted exclusively to the field theory. Here we introduce scalar field. Analyzing it we conclude that it possess a broken gauge invariance which is restored by introduction of the gauge field. In the noncommutative theory the gauge symmetry plays the role of representation independence and is fundamental to the model. (The algebra (1.1) allows a class of unitary equivalent representations, while for $\theta = 0$ one has the only representation.) Respectively the role of gauge fields appears to be extremely important in noncommutative theories. Thus, as we will see different gauge field configurations can be interpreted as different noncommutative spaces. A particular interest present the constant curvature ones which generate flat noncommutative spaces or “vector bundles” over flat noncommutative spaces. This relates, in fact, gauge models on different noncommutative spaces having different noncommutative structure, different local gauge group or even different dimensionality. This equivalence is similar to duality relations in string theory. According to it many different noncommutative gauge models appear to be different perturbative limits of the same model. This model appears to be the infinite dimensional Hilbert space version of the IKKT or BFSS matrix model [14,15] depending if there is a commutative time or not.

As this paper was initially intended as lecture notes, we supply each section with exercises.

Another reviews on noncommutative field theory can be found in [16,17,18,19,20,21,22,23].
2 Noncommutative Space

2.1 The Connes’ Concept of the Noncommutative Space

In this section we give a brief description of Connes idea of noncommutative geometry. For details and mathematical rigor readers are referred to the original Connes book [24] as well as to later review papers e.g. [25]. Since in the remaining part of this paper we will work with simplest examples and will not make use of of noncommutative spaces this section has mainly a “philosophical” character and therefore may be skipped without injury the understanding the rest. However, we decided to put it here, since the Connes formalism gives a natural way to generalize the analysis developed in next sections (but not necessarily an easy one).

Let us start the applying Connes description first to a commutative Riemann or spin manifold.

It is considered that the Connes approach comes from a statement that a topological space can be recovered from the algebra of continuous functions on it. (Which is a topological space too!) In other words, knowing all possible ways a topological space can be continuously mapped into $\mathbb{R}^n$ or $\mathbb{C}$ we know the topological space itself.

This idea seems plausible since the algebra of functions is much “bigger” than the space itself, but let illustrate it on the following discrete example.

Consider a (possibly irregular) lattice $\Gamma$ like in Fig. 1.

The topology on the lattice is given by characteristic of two points as being neighbor or not, which graphically can be represented by drawing a link connecting two neighbor points. Thus, the lattice topology describes how points are arranged with respect to each other. Obviously, this arrangement is invariant against small changes in the positions of points.

Lattice analog of continuous function is a map from $\Gamma$ to $\mathbb{Z}$ which translates neighbor points of $\Gamma$ to neighbor points of $\mathbb{Z}$, which in the natural topology of $\mathbb{Z}$ are $n$ and $n + 1$ for some $n \in \mathbb{Z}$. All such functions for an algebra $\mathfrak{A}$ with respect to multiplication and sum. The induced topology on the algebra of such functions is described as follows. Two maps are neighbor on a point if they map it to mutually neighbor points.

Let us note that all maps that send given point $x \in \Gamma$ to the origin $0\mathbb{Z}$ (vanish on $x$) form an ideal $\mathfrak{A}_x$ of the above algebra. Conversely, to each minimal ideal $\mathfrak{A}_x$ of algebra of functions one can put into correspondence a point labeling it. (Alternatively, one could consider ideals of functions vanishing everywhere except $x$, but in this case these are represented by functions in continuous case.) Thus, counting all minimal ideals of algebra $\mathfrak{A}$ give us the lattice $\Gamma$ as a simple set of points.

The topology of the set of $\Gamma$ is recovered as follows. Two points $x$ and $y$ represented by $\mathfrak{A}_x$ and $\mathfrak{A}_y$ iff any two elements $f \in \mathfrak{A}_x$ and $g \in \mathfrak{A}_y$ are neighbor on $\Gamma$.
Figure 1: An example of irregular lattice. Its topology is given by the neighborhood relations of points. Neighbor points are those connected by links. In order to count all links just once, they are given the orientation denoted by arrows. The lengths of links introduce the metric on the lattice.

The continuous picture one can obtain by assuming continuum limit of the above, e.g. when the lattice points become infinitely close to each other.

The next question which may arise is if it is possible to restore more than that? The answer to this is positive. For example one can recover the metric if an additional structure to the algebra of continuous functions is assumed. Indeed, let us assume that each point has a fixed number \( l \) of neighbors and introduce a set of operators \( i_p \) \( i = 1, \ldots, l \), which substitute the point \( x \) by its \( i \)-th neighbor: \( x \rightarrow x + l_i \) and divides each function on \( l_i \):

\[
[i_p, f](x) = \frac{1}{l_i} (f(x + l_i) - f(x)),
\]

where \( l_i \) is the distance between two neighbors \( x \) and \( x + l_i \).

Having such operators in addition to the algebra of functions \( \mathcal{A} \) we can extract the information about the distance on the lattice. Indeed, equation,

\[
d(x, y) = \sup_{\sum_i |p_i| f \leq 1} |f(x) - f(y)|,
\]

in the case of neighbor points gives the correct answer – \( d(x, x + l_i) = l_i \).

To obtain the continuum limit one has to send \( l_i \rightarrow 0 \) in a way that ensures that lines connecting neighbor points form continuous paths which become coordinate lines. In this case, the commutators \( [i_p, f] \) become derivatives with respect to this coordinate lines. Physically the continuum limit is reached when all \( l_i \) become smaller than the typical physical scale \( l_{ph} \). In the same manner one can encode as many structures of the manifold as one
wishes by adding respective operator to the algebra of functions, e.g. the spin structure can be recovered from the Dirac operator.

In example we considered the algebra $\mathcal{A}$ of functions is a commutative algebra. The next step in the generalization is to consider an arbitrary noncommutative algebra instead. For noncommutative algebra an additional input is needed its representation. (For commutative algebra the representation was trivial.)

Thus, we came to the Connes definition of the noncommutative space or the Connes triple, which is

$$\mathcal{M} = (\mathcal{A}, D, \mathcal{H}),$$  \hspace{1cm} (2.3)

where $\mathcal{A}$ is an algebra of bounded operators (irreducibly) represented on the Hilbert space $\mathcal{H}$, which is not necessarily finite-dimensional\footnote{We will consider the cases where it is infinite-dimensional separable.} and $D$ is an operator (not necessarily bounded) acting self-adjointly on $\mathcal{H}$. $\mathcal{A}$ plays the role of the algebra of functions over noncommutative space while $D$ one of a differential operator e.g. for a spin manifold it is Dirac operator.

In the next section we will consider the simplest cases of flat noncommutative spaces, but one should keep in mind that this space is given by the triple (2.3), and therefore changing a component of (2.3) results in change of the noncommutative space.

### 2.2 Simplest Cases of Noncommutative Spaces

As usual happens from the whole variety of mathematical tools only simplest ones are used in most physical applications. In the present development of the noncommutative field theory only noncommutative plane $\mathbb{R}^n_\theta$, noncommutative torus $T^m_\theta$ and noncommutative, or fuzzy sphere are most commonly used. Therefore, we describe this cases in more details.

#### 2.2.1 Noncommutative $\mathbb{R}^n_\theta$

Let us start with noncommutative $\mathbb{R}^n_\theta$. Consider the algebra generated by $x^\mu$, $\mu = 1, \ldots, p$, subject to commutation relations,

$$[x^\mu, x^\nu] = i\theta^{\mu\nu},$$  \hspace{1cm} (2.4)

where $\theta^{\mu\nu}$ are commutative elements (c-numbers) forming a nondegenerate (with respect to indices $\mu$ and $\nu$) matrix. As a consequence we have that $p$ should be always even.

Generators $x^\mu$ are called noncommutative coordinates of $\mathbb{R}^n_\theta$, while $\theta^{\mu\nu}$ is the noncommutativity matrix. In the case of degenerate noncommutativity matrix one can pass to generators $x^i$ and $x^\alpha$, $\alpha = 1, \ldots, p' \leq p$, $i = p' + 1, \ldots, p$, where $x^i$ corresponding to zero modes of $\theta^{\mu\nu}$ form a commutative
subspace of $\mathbb{R}^n_\theta$, while $x^\alpha$ is its completion where the restriction of $\theta^{\mu\nu}$ is nondegenerate.

Commutation relations (2.4) correspond to the standard $p/2$-dimensional Heisenberg algebra, although in a nonstandard parameterizations. From the courses on Quantum Mechanics we are familiar with the form where $x^\mu$ is split into pairs of $p_i$ and $q^i$, $i = 1, \ldots, p/2$, satisfying,

$$[p_i, q^j] = -i\hbar \delta^j_i.$$  \hspace{1cm} (2.5)

In our case the rotations of the coordinates $x^\mu$ allow to bring the algebra (2.4) to the form with “Plank constant” depending on direction,

$$[p_i, q^j] = -i\theta(i) \delta^j_i,$$  \hspace{1cm} (2.6)

we denoted it by $\theta(i)$. Absorption of $\theta(i)$ is possible in principle by scaling $p$ and $q$, but this transformation alters the metric of $\mathbb{R}^n_\theta$.

Thus, we got a familiar thing, Heisenberg algebra, whose irreducible representations, we hope, are well known from standard QM courses. They are all infinite dimensional and unitary equivalent to each other.

The irreducibility in particular mean that any operator commuting with all $x^\mu$ is a $c$-number in the sense that it is proportional to the unity operator. Later, this property will play an important role in our analysis, as it means that any such operator can be expressed as an operator function of $X^\mu$.

Consider now $\mathbb{R}^2_\theta$, or noncommutative plane as the simplest example. It is given by “coordinates” $x^{1,2}$ satisfying the algebra,

$$[x^1, x^2] = i\theta, \quad \theta \neq 0.$$  \hspace{1cm} (2.7)

This is usual Heisenberg algebra of one-dimensional quantum mechanics with $q = x^1$, $p = x^2$ and the Planck constant equal to $\theta$. The representation of algebra (2.7) is realized on e.g. by $L^2$-integrable functions of $q = x^1$, on which $x^1$ acts by multiplication by its eigenvalue and $x^2$ as derivative,

$$x^1 f(x^1) = x^1 f(x^1), \quad f \in L^2;$$  \hspace{1cm} (2.8)

$$x^2 f(x^1) = -i\theta \frac{\partial}{\partial x^1} f(x^1).$$  \hspace{1cm} (2.9)

The analog of noncommutative complex coordinates is given by the oscillator representation. It is given in terms of operators $a$ and $\bar{a}$ defined as follows,

$$a = \sqrt{\frac{1}{2\theta}}(x^1 + ix^2), \quad \bar{a} = \sqrt{\frac{1}{2\theta}}(x^1 - ix^2).$$  \hspace{1cm} (2.10)

These operators satisfy the commutation relation,

$$[a, \bar{a}] = 1,$$  \hspace{1cm} (2.11)
and can be represented in terms of oscillator basis $|n\rangle$ given by the eigenvectors of operator $N = \bar{a}a$, while $a$ and $\bar{a}$ act as lowering and rising operators,

$$\begin{align*}
a|n\rangle &= \sqrt{n} |n-1\rangle, \\
\bar{a}|n\rangle &= \sqrt{n+1} |n+1\rangle.
\end{align*}$$

(2.12)

In general case of $\mathbb{R}^n_\theta$ one can introduce “complex” coordinates as well. The are given by multi-dimensional oscillator operators $a_i$ and $\bar{a}_i$ defined by,

$$\begin{align*}
a_i &= \sqrt{\frac{1}{2\theta(i)}} (q^i + ip^i), \\
\bar{a}_i &= \sqrt{\frac{1}{2\theta(i)}} (q^i - ip^i),
\end{align*}$$

(2.13)

and satisfying the commutation relations,

$$[a_i, \bar{a}_j] = \delta_{ij}. \quad (2.14)$$

The natural basis is given by states $|\vec{n}\rangle$ which are eigenvectors of operators $\vec{N}$ with “components” $N_i = \bar{a}_i a_i$ (no sum over $i$). Vector $\vec{n}$ takes values on an infinite positive $p/2$ dimensional rectangular lattice, $\vec{n} = \sum_i n_i \vec{e}_i$, $n_i \geq 0$. Operators $a_i$ and $\bar{a}_i$ are respectively lowering and rising operators for eigenvalue $n_i$,

$$\begin{align*}
a_i |\vec{n}\rangle &= \sqrt{n_i} |\vec{n} - \vec{e}_i\rangle, \\
\bar{a}_i |\vec{n}\rangle &= \sqrt{n_i + 1} |\vec{n} + \vec{e}_i\rangle.
\end{align*}$$

(2.15)

2.2.2 Noncommutative $\mathbb{T}^n_\theta$

In the commutative case one can pass from the plane to torus by factorizing the plane with respect to a discrete group of the translation group of the plane. In the simplest case this is obtained by the identification

$$x^\mu \sim x^\mu + l^\mu. \quad (2.16)$$

A similar thing can be done also in the noncommutative case. The identification like (2.16) can be done if one considers the noncommutative algebra of functions as being generated by operators,

$$U_\mu = e^{2\pi i \frac{x^\mu}{l^\mu}}, \quad \mu = 1, \ldots, p,$$

(2.17)

which are noncommutative “coordinates” of the torus.

The generators $U_\mu$ satisfy,

$$U_\mu U_\nu = q_{\mu\nu} U_\nu U_\mu, \quad (2.18)$$

where,

$$q_{\mu\nu} = e^{-\frac{(2\pi i)^2 g_{\mu\nu}}{l^\mu l^\nu}} \equiv e^{2\pi i \Xi_{\mu\nu}} \quad \text{(no sum).} \quad (2.19)$$
The above equations were computed using the Campbell–Bjorken–Hausdorff formula,

$$e^A e^B = e^{A+B + \frac{1}{2}[A,B] + ...}$$

which is also a useful tool for our analysis.

In general one can define the noncommutative torus by the algebra (2.18) for some $\Xi^{\mu\nu}$ with no direct reference to the noncommutative plane. From the commutative quantum field theory we are familiar that torus provides an IR regularization of theories on the plane.

In the noncommutative case it turns out that the noncommutative torus provide not only IR but also the UV regularization due to the phenomenon known as IR/UV mixing. Moreover, it appears that in the special case when $\Xi^{\mu\nu}$ is a rational valued matrix, the respective irreducible representation become finite dimensional. This kind of noncommutative lattice theories were also studied (see Refs. [24, 27]).

From the other hand, the noncommutative torus can be always embed- ded into the noncommutative plane of the same dimension. Due to this property in the rest of this paper we will consider mainly the theories on noncommutative planes.

### 2.2.3 Noncommutative or Fuzzy $S^n$

Another natural example of compact noncommutative space is noncommu-tative, or fuzzy sphere. Here we describe first the noncommutative analog of the two dimensional sphere. Noncommutative sphere of arbitrary dimensions is obtained in a similar way.

Two dimensional commutative sphere can be defined in multiple ways, for example like a factor $SO(3)/SO(2)$, or like a submanifold of $\mathbb{R}^3$, defined by condition $x^2 = r^2$. Both these possibilities can be generalized to the noncommutative case. In what follows choose the second one.

Consider noncommutative $\mathbb{R}^3$ with a nonconstant noncommutativity,

$$[x^i, x^j] = i\alpha \epsilon^{ijk} x^k, \quad 1 = 1, 2, 3,$$  

(2.21)

where $\alpha$ is a dimensional parameter. The commutation relations (2.21) are nothing else than ones of $su(2)$ algebra. Irreducible representations of algebra (2.21) are possible when,

$$r^2 = \alpha^2 j(j + 1),$$

(2.22)

where $l$ can be half integer, which restricts the radius of noncommutative sphere to quantized values in terms of $\alpha$. (Or, equivalently, $\alpha$ is restricted in terms of $r$.)

Therefore, the the two dimensional noncommutative sphere of the radius $r^2 = \alpha^2 j(j + 1)$ is given by the algebra of operators acting on the irreducible representation of $su(2)$ of the spin $j$. 

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Again, the data corresponding to the noncommutative sphere can be expressed in terms of those of noncommutative plane. In terms of oscillator operators one can write the \( su(2) \) algebra (2.21) in the Chevalley basis \( x_{\pm}, x_3 \) as follows,

\[
(x_{\pm}/\alpha) = \bar{a} \sqrt{j(j+1)-(N-j)(N-j+1)/N}, \quad (x_3/\alpha) = N-j, \quad (2.23)
\]

where \( N \) is the oscillator number operator. Operators (2.23) generate an irreducible representation on the first \( 2j+1 \) oscillator states.

**Exercise 1** Show this.

The \( n \)-dimensional noncommutative sphere is given by \( SO(n+1)/SO(n) \). Therefore, one can define it through irreducible representations of \( SO(n+1) \) which are singlet in the factor \( SO(n) \).

**Exercise 2** How many such representations are there?

### 2.3 Weyl Ordering

In the remaining part of this section we discuss noncommutative planes.

So far, we defined the noncommutative space by means of its noncommutative algebra of functions and its representation. When working with functions in commutative space we prefer to operate with explicit \( x \)-dependent form rather then with abstract algebra element. As it turns the same possibility exists also in noncommutative geometry. It is given by Weyl ordering and respective objects are Weyl symbols. Roughly speaking the Weyl symbol of an operator \( \hat{O}(x) \) with respect to the set of basic operators \( x^\mu \) is the function \( O(x) \) symmetrized with respect to all \( x^\mu \). Since in this forms functions do not depend on the commutation relations \( x^\mu \) have one can substitute the operators by usual functions by the following rule,

\[
x^\mu \to x^\mu, \quad (2.24a)
\]

\[
x^{\mu}x^{\nu} = \frac{1}{2}(x^{\mu}x^{\nu} + x^{\nu}x^{\mu}) + \frac{i}{2}\theta^{\mu\nu} \to x^{\mu}x^{\nu} + \frac{i}{2}\theta^{\mu\nu}, \quad (2.24b)
\]

\[
\ldots
\]

Here \( x^\mu \) are treated as ordinary commutative coordinates.

The transformation (2.24) defines the Weyl symbol for arbitrary polynomial in \( x^\mu \) but our wish is to work with \( L^2 \) functions which are non-polynomial. For non-polynomial functions one could in introduce the Weyl ordered symbol term by term using the operator Taylor expansion,

\[
\hat{O}(x) = O^{(0)} + \frac{1}{1!}O^{(1)}_{\mu}x^{\mu} + \frac{1}{2!}O^{(2)}_{\mu\nu}x^{\mu}x^{\nu} + \cdots. \quad (2.25)
\]

Although, application of the above procedure is possible in principle for \( L^2 \) functions there is a more elegant way to find the Weyl symbol which we are going to consider in the next subsection.
2.4 Noncommutative Calculus and the Star Product

Let us introduce operators $p_\mu$ which are linear combinations of $x^\mu$,

$$p_\mu = -\theta^{-1}_{\mu \nu} x^\nu. \quad (2.26)$$

They have the following commutation relations,

$$[p_\mu, p_\nu] = -i\theta^{-1}_{\mu \nu}. \quad (2.27)$$

Consider adjoint operator,

$$P_\mu = [p_\mu, \cdot]. \quad (2.28)$$

This is operator acting on operators, i.e. it applies to elements of $\mathfrak{A}$: $P_\mu : \mathfrak{A} \to \mathfrak{A}$. It is not difficult to check that $P_\mu$ are,

i. commutative,

$$P_\mu P_\nu - P_\nu P_\mu = 0; \quad (2.29)$$

ii. self-adjoint with respect to the scalar product defined on the algebra $\mathfrak{A}$ of operators,

$$\langle a, P_\mu b \rangle = \langle P_\mu a, b \rangle. \quad (2.30)$$

The scalar product is defined as,

$$\langle a, b \rangle = \text{tr} a^\dagger b. \quad (2.31)$$

iii. They form a complete set of operators acting on $\mathfrak{A}$, in the sense that,

$$\forall F : P_\mu F = 0, (\mu = 1, \ldots, p) \Rightarrow F = c\mathbb{1}, \quad (2.32)$$

where $\mathbb{1}$ is the unity operator. This property follows from the irreducibility of generators $p_\mu$ (equivalently, $x^\mu$).

Due to properties i.-iii. operators $P_\mu$ are diagonalizable and having real eigenvalues. In particular, from property iii. it follows that the dimension of common eigenspace of all $P_\mu$ is one.

From the Leibnitz rule operators $P_\mu$ satisfy,

$$P_\mu (f \cdot g) = P_\mu f \cdot g + f \cdot P_\mu g, \quad (2.33)$$

it follows that eigenfunctions $E_k$ of $P_\mu$ satisfy,

$$P_\mu E_k = k_\mu E_k \quad (2.34)$$

$$E_k \cdot E_{k'} \sim E_{k+k'} \quad (2.35)$$
From (2.35) it is not difficult to guess that eigenfunctions $E_k$ should have the form,
\[ E_k = c_k e^{i k \cdot x}. \] (2.36)

Let us note that “eigenvectors” are elements of $\mathfrak{A}$, i.e. operators acting on the Hilbert space $\mathcal{H}$.

As it can be seen, the spectrum of $P_{\mu}'s$ is continuous and, therefore, eigenvectors have infinite norm with respect to the scalar product (2.31). However, one can fix the quotients $c_k$ from the requirement,
\[ \text{tr} E_{k'}^\dagger \cdot E_k = \delta(k' - k). \] (2.37)

Let us compute the trace and find respective quotients. To do this consider the basis where the set of operators $x^\mu$ splits in pairs $p_i, q_i$ satisfying the standard commutation relations (2.5).

As we know from courses of Quantum Mechanics the trace of the operator
\[ e^{-i k' \cdot x} \cdot e^{i k \cdot x} = e^{i(k-k') \cdot x} \cdot e^{i k' \cdot x}, \] (2.38)
can be computed in $q$-representation as,
\[ \text{tr} e^{i(k-k') \cdot x} \cdot e^{i k' \cdot x} = \int dq \langle q | e^{-i(l'_i-l_i)q^i + (z'_i-z_i)p_i} | q \rangle = 1/|c_k|^2 \delta(k' - k), \] (2.39)
where $|q\rangle$ is the basis of eigenvectors of $q^i$,
\[ q^i |q\rangle = q^i |q\rangle, \quad \langle q' | q \rangle = \delta(q' - q), \] (2.40)
and $l_i, z^i (l_i, z^i)$ are components of $k_{\mu} (k'_\mu)$ in the in the parameterizations: $x^\mu \rightarrow p_i, q^i$. Explicit computation gives,
\[ 1/|c_k|^2 = \frac{(2\pi)^{\frac{p}{2}}}{\sqrt{\det \theta}}. \] (2.41)

Now, we have the basis of eigenoperators $E_k$ and can write any operator $F$ in terms of this basis,
\[ F = \int dk \tilde{F}(k) e^{i k \cdot x}, \] (2.42)
where the “coordinate” $\tilde{F}(k)$ is given by,
\[ \tilde{F}(k) = \frac{\sqrt{\det \theta}}{(2\pi)^{\frac{p}{2}}} \text{tr}(e^{-i k \cdot x} \cdot F). \] (2.43)
Function $\tilde{F}(k)$ can be interpreted as the Fourier transform of an $L^2$ function $F(x)$,

$$F(x) = \int dk \tilde{F}(k)e^{ik\cdot x} = \sqrt{\det \theta} \int \frac{dk}{(2\pi)^{p/2}} e^{ikx} \text{tr} e^{-ik\cdot x} \mathbf{F}. \quad (2.44)$$

Conversely, to any $L^2$ function $F(x)$ from one can put into correspondence an $L^2$ operator $\mathbf{F}$ by inverse formula,

$$\mathbf{F} = \int dx \frac{1}{(2\pi)^{p/2}} \int \frac{dk}{(2\pi)^{p/2}} F(x)e^{ik\cdot (x-x)}. \quad (2.45)$$

Equations (2.44) and (2.45) providing a one-to-one correspondence between $L^2$ functions and operators with finite trace,

$$\text{tr} \mathbf{F}^\dagger \cdot \mathbf{F} < \infty, \quad (2.46)$$

give in fact formula for the Weyl symbols. By introducing distributions over this space of operators one can extend the above map to operators with unbounded trace.

**Exercise 3** Check that (2.44) and (2.45) lead in terms of distributions to (2.24).

Let us note, that the map (2.44) and (2.45) can be rewritten in the following form,

$$F(x) = (2\pi)^{p/2} \sqrt{\det \theta} \text{tr} \hat{\delta}(x-x) \mathbf{F}, \quad \mathbf{F} = \int d^p x \hat{\delta}(x-x) F(x), \quad (2.47)$$

where we introduced the operator,

$$\hat{\delta}(x-x) = \int \frac{d^p k}{(2\pi)^p} e^{i k\cdot (x-x)}. \quad (2.48)$$

This operator satisfy the following properties,

$$\int d^p x \hat{\delta}(x-x) = \mathbb{I}, \quad (2.49a)$$
$$\langle 2\pi \rangle^{p/2} \sqrt{\det \theta} \text{tr} \hat{\delta}(x-x) = 1, \quad (2.49b)$$
$$\langle 2\pi \rangle^{p/2} \sqrt{\det \theta} \text{tr} \hat{\delta}(x-x) \hat{\delta}(x-y) = \delta(x-y), \quad (2.49c)$$

where in the r.h.s. of last equation is ordinary delta function. Also, operators $\hat{\delta}(x-x)$ form a complete set of operators if regarded as a family depending on the parameter $x$,

$$[\hat{\delta}(x-x), \mathbf{F}] \equiv 0 \Rightarrow F \propto \mathbb{I}. \quad (2.49d)$$
The commutation relations of $x^\mu$ also imply that $\hat{\delta}(x - x)$ should satisfy,

$$[x^\mu, \hat{\delta}(x - x)] = i\theta^{\mu\nu}\partial_\nu \hat{\delta}(x - x). \quad (2.49e)$$

In fact one can define noncommutative plane starting from operators $D(x)$ satisfying (2.49), with $x^\mu$ defined by,

$$x^\mu = \int d^p x x^\mu \hat{\delta}(x - x). \quad (2.50)$$

In this case (2.49e) provides that $x^\mu$ satisfy the Heisenberg algebra (2.4), while the property (2.49d) provides that they form a complete set of operators. Relaxing these properties allows one to introduce a more general noncommutative spaces.

Let us the operator $\hat{\delta}(x - x)$ in the simplest case of two-dimensional noncommutative plane. The most convenient is to find its matrix elements $D_{mn}(x)$ in the oscillator basis (2.12),

$$D_{mn}(x) = \langle m | \hat{\delta}^{(2)}(a - z) | n \rangle = \text{tr} \hat{\delta}^{(2)}(a - z) P_{nm}, \quad (2.51)$$

where

$$P_{nm} = | n \rangle \langle m |.$$

As one can see, up to a Hermitian transposition the matrix elements of $\hat{\delta}(x - x)$ correspond to Weyl symbols of operators like $| m \rangle \langle n |$, or so called Wigner functions. The computation of (2.51) gives,

$$D_{mn}^\theta(z, \bar{z}) = (-1)^n \left( \frac{2}{\sqrt{\theta}} \right)^{m-n+1} \sqrt{\frac{n!}{m!}} e^{-z\bar{z}/\theta} \left( \frac{z^m}{\bar{z}^n} \right) L_n^{m-n}(2z\bar{z}/\theta), \quad (2.52)$$

where $L_n^{m-n}(x)$ are Laguerre polynomials,

$$L_n^\alpha(x) = \frac{x^{-\alpha} e^x}{n!} \left( \frac{d}{dx} \right)^n (e^{-x} x^{\alpha+n}). \quad (2.53)$$

It is worthwhile to note that in spite of singular looking definition the symbol of the delta operator is a smooth function rapidly vanishing at infinity. The smoothness comes from the fact that the operator elements are written in an $L^2$ basis. In a non-$L^2$ basis, e.g. in the basis of $x_1$ eigenfunctions $D^\theta$ would have more singular form.

The above computations can be generalized to $p$-dimensions. Written in the complex coordinates $z_i, \bar{z}_i$ corresponding to oscillator operators (2.13), which diagonalize the noncommutativity matrix this looks as follows,

$$D_{\tilde{m}\tilde{n}} = D_{m_1n_1}(z_1, \bar{z}_1)D_{m_2n_2}(z_2, \bar{z}_2) \ldots D_{m_{p/2}n_{p/2}}(z_{p/2}, \bar{z}_{p/2}), \quad (2.54)$$

where,

$$[z_i, \bar{z}_i] = \theta_{(i)}, \quad i = 1, \ldots, p/2. \quad (2.55)$$

Having the above map one can establish following relations between operators and their Weyl symbols.

\[\text{For the details of computation see e.g. [17].}\]
1. It is not difficult to derive that,
\[ (2\pi)^{p/2} \sqrt{\det \theta} \text{tr} F = \int dx \, F(x). \quad (2.56) \]

2. The (noncommutative) product of operators is mapped into the *star* or *Moyal* product of functions,
\[ F \cdot G \rightarrow F \ast G(x), \quad (2.57) \]
where \( F \ast G(x) \) is defined as,
\[ F \ast G(x) = e^{-\frac{i}{2} \theta_{\mu \nu} \partial_\mu F(x) G(x')} \bigg|_{x' = x}. \quad (2.58) \]

In terms of operator \( \hat{\delta}(x - x) \), this product can be written as follows,
\[ F \ast G(x) = \int d^p y d^p z K(x; y, z) F(y) G(z), \quad (2.59) \]
where,
\[ K(x; y, z) = (2\pi)^{p/2} \sqrt{\det \theta} \hat{\delta}(x - x) \hat{\delta}(x - y) \hat{\delta}(x - z) = e^{\frac{i}{2} \theta_{\mu \nu} \partial_\mu F - F \ast p_\mu}(x). \quad (2.60) \]

\( \partial_\mu \) and \( \partial_\mu^* \) are, respectively, \( \partial/\partial y^\mu \) and \( \partial/\partial z^\mu \), and in the last line one has ordinary delta functions.

From the other hand the ordinary product of functions was not found to have any reasonable meaning in this context.

3. One property of the star product is that in the integrand one can drop it once because of,
\[ \int d^p x \, F \ast G(x) = \int d^p x \, F(x) G(x), \quad (2.61) \]
where in the r.h.s the ordinary product is assumed.

4. Interesting feature of this representation is that partial derivatives of Weyl symbols correspond to commutators of respective operators with \( ip_\mu \),
\[ [ip_\mu, F] \rightarrow i(p_\mu \ast F - F \ast p_\mu)(x) = \frac{\partial F(x)}{\partial x^\mu}, \quad (2.62) \]
where \( p_\mu \) is linear function of \( x^\mu \): \( p_\mu = -\theta^{-1}_{\mu \nu} x^\nu \).
This is an important feature of the star algebra of functions distinguishing it from the ordinary product algebra. In the last one can not represent the derivative as an *internal automorphism* while in the star algebra it is possible due to its nonlocal character. This property is of great importance in the field theory since, as it will appear later, it is the source of duality relations in noncommutative gauge models which we turn to in the next section.

**Exercise 4** Derive equations (2.56) – (2.62).
3 Noncommutative Field Theory

As we mentioned in the Introduction the setup of noncommutative geometry allows one to introduce and successfully develop the notion of Noncommutative Field Theory. As we can anticipate from what we learned in the previous section, one can introduce the fields in the noncommutative space the same way as in ordinary spaces except that the ordinary products should be replaced with star products.

Here we will consider the classical aspects of Noncommutative Field Theory. There are, however, many interesting things found in the Noncommutative Perturbative QFT, which we are not going to discuss here but refer the Reader to the literature [12, 13].

Another aspect is related to whether the time is commutative or not. When time can be chosen commutative one can in principle define a canonically quantized theory. In the case of noncommutative time one can not speak even on that. However, one can work with path integrals in Euclidean time.

In classical analysis these subtleties are not important since there one can easily pass from one case to another.

3.1 Noncommutative Scalar Field

The simplest field theory model is one of the real scalar field. It is given by the classical action,

\[ S_{\text{comm}} = \int d^p x \left( \frac{1}{2} \partial_\mu \phi \partial_\mu \phi + V(\phi) \right), \tag{3.1} \]

where \( V(\phi) \) is a polynomial potential,

\[ V(\phi) = \frac{1}{2} m^2 \phi^2 + \frac{g(3)}{3!} \phi^3 + \ldots \tag{3.2} \]

If \( m^2 > 0 \) than this defines a massive self-interacting scalar model, while \( m^2 = -\mu^2 < 0 \) corresponds to Higgs models.

The generalization of above to the noncommutative case is straightforward. The only difference which arise is the substitution of ordinary products in (3.1) by star products,

\[ S_{\text{nc}} = \int d^p x \left( \frac{1}{2} \partial_\mu \phi \star \partial_\mu \phi - V_s(\phi) \right), \tag{3.3} \]

where \( V_s(\phi) \) is the noncommutative interaction potential,

\[ V_s(\phi) = \frac{1}{2} m^2 \phi \star \phi + \frac{g(3)}{3!} \phi \star \phi \star + \ldots \tag{3.4} \]
Using the property [2.61], one can drop the star products from the quadratic free parts of the noncommutative action. Thus, the noncommutativity arises only at the interaction level.

Let us try to rewrite the noncommutative action in the operator form using the Weyl map (2.44) and (2.45). Under this map one has to substitute the real field \( \phi(x) \) by a Hermitian operator \( \hat{\phi} \)

\[
\partial_\mu \phi \rightarrow i[p_\mu, \hat{\phi}].
\]

As a result one has the action in the operator form,

\[
S_{nc} = (2\pi)^{p/2} \sqrt{\det \theta} \text{tr} \left( -\frac{1}{2} [p_\mu, \hat{\phi}]^2 + V(\hat{\phi}) \right),
\]

where the potential \( \hat{V}(\hat{\phi}) \) is inherited from the star potential (3.4),

\[
V(\hat{\phi}) = \frac{1}{2} m^2 \hat{\phi} \cdot \hat{\phi} + \frac{g(3)}{3!} \hat{\phi} \cdot \hat{\phi} \cdot \hat{\phi} + \ldots.
\]

As one can see the all dependence of the model on the data of noncommutative space are now stored in the factor \( \sqrt{\det \theta} \) in front of the action and in operators \( p_\mu \). By rescaling,

\[
\hat{\phi} \rightarrow (2\pi)^{p/4}(\det \theta)^{-\frac{1}{4}} \hat{\phi},
\]

the \( \theta \)-dependence reappears only in the interaction part of the action.

As we see the operator form of the action is more invariant, therefore it should be more fundamental. Later we will see that it is indeed so.

3.2 Gauge Invariance and Gauge Fields

In ordinary field theories real singlet scalar field possesses no special symmetries. The things are different, however, in the noncommutative theory.

As we have mentioned in subsection 2.2.1 the noncommutative algebra (2.4) allows a class of unitary equivalent representations rather one single representation and it would be dubitable why the model shall depend on the particular representation. The change of representation is equivalent to unitary transformation of all operators of the theory,

\[
F \rightarrow U^{-1} \cdot F \cdot U,
\]

for some unitary operator \( U \),

\[
U^\dagger \cdot U = 1.
\]

In terms of Weyl symbols this means that the scalar field \( \phi(x) \) corresponding to the operator \( \hat{\phi} \) undergoes the transformation,

\[
F(x) \rightarrow \bar{U} * F * U(x),
\]
where \( U(x) \) is the respective star unitary function,

\[
\bar{U} \star U = 1,
\]

the bar denotes complex conjugation.

The equation (3.11) indicates that \( \phi(x) \) transforms in adjoint representation of noncommutative U(1) gauge group. If we try to find global gauge transformations with \( U = e^{i\alpha} = \text{constant} \) we find that they act trivially on \( \phi(x) \). From the other hand global gauge transformation correspond in the operator language to constant phase transformation of the Hilbert space vectors,

\[
|\psi\rangle \rightarrow e^{i\alpha} |\psi\rangle.
\]

Therefore the scalar field action (3.3) and (3.6) is obviously invariant with respect to these global transformations, however it fails to be invariant with respect to local transformations (3.11) due to noninvariance of the kinetic term.

In ordinary theory one can extend the global gauge invariant model to be local invariant by gauging the kinetic term. This is obtained by the substitution of all partial derivatives of fields with covariant derivatives built of gauge fields. The same gauging procedure can be applied in noncommutative case [28]. It consists in the substitution of the action (3.3) by the gauged one and addition to it of the pure gauge field part \( S_g \),

\[
S_{\text{nc,gauged}} = \int d^p x \left( \frac{1}{2} \nabla_\mu \phi \star \nabla_\mu \phi - \mathcal{V}_s(\phi) \right) + S_g.
\]

The covariant derivatives,

\[
\nabla_\mu \phi = \partial_\mu \phi + i[A_\mu, \phi](x) \equiv \partial_\mu \phi(x) + i(A_\mu \star \phi - \phi \star A_\mu)(x)
\]

imply that the gauge field transforms under the action local gauge transformation as follows,

\[
A_\mu(x) \rightarrow U^{-1} \star A_\mu \star U(x) - iU^{-1} \star \partial_\mu U(x).
\]

One can see, that due to the noncommutativity we got a nonabelian group of local U(1) transformations for the gauge field as well. Therefore the gauge part of the action should be constructed respecting the gauge transformations of the gauge field. In this case it takes the form,

\[
S_g = -\frac{1}{4g^2} \int d^p x F_{\mu\nu} \star F^{\mu\nu},
\]

where,

\[
F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu].
\]
3.3 Background Independence

Let us now rewrite the gauged action in the operator form. In order to do this let us observe that covariant derivatives of the scalar fields can be represented as,

$$\nabla_\mu \phi = i[(p_\mu + A_\mu), \phi] \rightarrow i[X_\mu, \hat{\phi}],$$

where $X_\mu$ is operator corresponding to the function $X_\mu(x) = p_\mu + A_\mu$. At the same time the gauge field strength can be rendered as,

$$F_{\mu \nu} - \theta_{\mu \nu}^{-1} = [(p_\mu + A_\mu), (p_\nu + A_\nu)] \rightarrow [X_\mu, X_\nu].$$

As we see covariant derivatives and $F_{\mu \nu} - \theta_{\mu \nu}^{-1}$ can be represented by operators independent of the generator basis $x_\mu$. In fact, as $\theta_{\mu \nu}$ is constant the gauge field action is indistinguishable in what concerns equations of motion from one where all $F_{\mu \nu}$ are substituted by $F_{\mu \nu} - \theta_{\mu \nu}^{-1}$,

$$S'_{g} = -\frac{1}{4g^2} \int d^p x (F_{\mu \nu} - \theta_{\mu \nu}^{-1})^2 = -(2\pi)^{p/2} \sqrt{\det \theta} \frac{1}{4g^2} \text{tr}[X_\mu, X_\nu]^2.$$  

Combining all together we can write down the action of noncommutative gauge model of the scalar field in the operator form,

$$S = -\text{tr} \left( \frac{1}{4g'^2} [X_\mu, X_\nu]^2 + \frac{1}{2} [X_\mu, \hat{\phi}]^2 + \tilde{V}(\hat{\phi}) \right),$$

were we introduced modified couplings,

$$\tilde{V}(\hat{\phi}) = \frac{1}{2} m^2 \hat{\phi} \cdot \hat{\phi} + \frac{g'}{3!} \hat{\phi} \cdot \hat{\phi} \cdot \hat{\phi} + \ldots,$$

$$g' = \frac{g}{(2\pi)^{\frac{p}{2}} (\det \theta)^{\frac{1}{2}}},$$

$$g'_{(n)} = (2\pi)^{\frac{p(2-n)}{4}} (\det \theta)^{\frac{2-n}{4}} g_{(n)}.$$

There is no difficulty to generalize above to the case of a multiplet of scalar fields. Consider now the particular case when there is a scalar multiplet $\phi_a(x)$ $a = 1, \ldots, n$, with the potential

$$V_H(\phi) = -\frac{g^2}{4} [\phi_a, \phi_b]^2.$$  

This potential has a valley of nontrivial vacua and it plays an important role in the dynamics of branes. (The fields $\phi_a(x)$ are believed to describe the transversal degrees of freedom of branes, while gauge fields are responsible for the longitudinal ones.) This gauge model has a fairly simple operator form,

$$S_H = -\frac{1}{4g^2} \text{tr}[X_M, X_N]^2.$$
where,
\[ X_M = \begin{cases} 
X_\mu, & M = \mu, \\
\phi_a, & M = a; 
\end{cases} \quad M = 1, \ldots, p + n. \quad (3.28) \]

**Exercise 5** Prove (3.27) and (3.28).

As one can see the action in the form (3.22) do not depend explicitly on generators \( p_\mu \) or \( x^\mu \). The only input required is the Hilbert space of representation generated by them. Thus, if a different algebra can be represented on the same Hilbert space this equation could equally apply to it. Algebras which can have exact representations on the same Hilbert space are called *Morita equivalent*. In fact all Heisenberg algebras (2.4) are Morita equivalent because all infinite-dimensional separable Hilbert spaces are isomorphic. The isomorphism follows from the existence of countable basis in each separable Hilbert space.

In section 4 we will explore this fact to show duality relations arising between different gauge models. Before turning to it let us consider the topic of Wilson lines in noncommutative gauge models.

**Note:** Background invariance defined by us here is different from one discussed e.g. in [29]. The difference consists in the interpretation of \( X_\mu \) as derivatives rather than noncommutative coordinates. Although, for \( [X, X] = \text{constant background} \) this reduces simply to redefinition \( X_\mu \rightarrow \theta^{\mu\nu}X_\nu \) or, equivalently respective redefinition of the metric \( g_{\mu\nu} \) which contracts \( X_\mu \), for \( [X, X] \neq \text{constant} \) these two possible choices are rather dual then equivalent. The noncommutative space can be naturally defined in terms of derivatives or vector fields on it.

### 3.4 Wilson lines

Wilson lines and Wilson loops in context of noncommutative gauge theory or matrix models where considered in Refs. [30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40].

In commutative gauge theories Wilson loops or closed Wilson lines play an important role of local gauge invariant observables. The generic Wilson line in a nonabelian ordinary (i.e. on a commutative space) Yang–Mills theory is defined as path ordered exponent,
\[ U[C](x, y) = P \exp \int_C A_\mu(x)dx^\mu, \quad (3.29) \]
where \( C \) is a line connecting points \( x \) and \( y \), e.g. parameterized as \( x^\mu(t): x^\mu(0) = x^\mu, \ x^\mu(1) = y^\mu \). The \( P \)-symbol in front of the exponent means that the exponential in r.h.s. of eq. (3.29) is computed by multiplying factors with bigger parameter \( t \) to the right of those with smaller \( t \).
Under an ordinary nonabelian gauge transformation \( A \to g^{-1} Ag + g^{-1} dg \) the Wilson line transforms in the bi-local manner,

\[
U[C](x, y) \to g^{-1}(x)U[C](x, y)g(y).
\] (3.30)

Thus, if one takes \( x = y \) i.e. \( C \) to represent a closed loop then one can make out of \( U[C] \) a gauge invariant quantity,

\[
W[C](x) = \text{tr} U[C](x, x),
\] (3.31)

where the trace is performed over the gauge group.

The Wilson loop observables (3.31) play an important role in analyzing the phase structure of the ordinary four dimensional nonabelian Yang–Mills model as the order parameter. From the other hand locally time-like (in Minkowski space) Wilson loop is the interaction potential between a charged particle-antiparticle pair e.g. interacting quark and antiquark. The behavior of the expectation value of \( W[C] \) describes the character of interaction of opposite charges in the theory.

Thus in particular if the Wilson loop expectation value for large loops has the behavior \( \langle W[C] \rangle \sim \exp(-A(C)) \), where \( A(C) \) is the minimal area spanned by the loop \( C \), then it is not difficult to see that the \( q\bar{q} \) interaction potential increases linearly which means confinement. This is called area-law behaviour of Wilson loop expectation value. It is realized in the strong coupling regime of commutative Yang–Mills model.

Another interesting regime is described by the perimeter law behavior i.e. \( \langle W[C] \rangle \sim \exp(-L(C)) \), where \( L(C) \) is the perimeter of the loop. That is realized in the weak coupled limit. The perimeter low indicates that the interaction is Coulomb-like.

For Abelian theories one of course does not need to write path ordering and trace to define the Wilson loop invariant. The particular meaning of the Wilson loop invariant here is the electro-magnetic flux through \( C \). Therefore, (3.31) can be interpreted as definition of the analog of the field strength flux in the nonabelian theory.

Let us turn to the noncommutative Yang–Mills model. Formally, one can apply the rule given by commutative formula (3.30) for the Weyl symbols to get,

\[
U_*[C](x, y) = P \exp_* \int dx^\mu A_\mu(x),
\] (3.32)

where \( \exp_* \) means that the exponential is computed using the star product (2.58),

\[
\exp_* f(x) = 1 + \frac{1}{1!} f(x) + \frac{1}{2!} f * f(x) + \ldots
\] (3.33)
Due to noncommutativity of the star product we had to impose path ordering even for U(1) gauge group. The quantity (3.32) transform “bi-locally” under star gauge transformations analogously to (3.30). Therefore to make a gauge invariant object out of closed loops we have to trace them out. Recall, however that the tracing in noncommutative theory is equivalent to integration. Therefore, invariant Wilson loop invariants in noncommutative gauge theories look like,

$$W[C] = \text{tr} U[C] \propto \int dx U[C](x, x).$$

(3.34)

As we see not much of $x$-dependence remains in the gauge invariant objects constructed out of the Wilson loops. Indeed, we could not expect to have a set of gauge invariant local functions since any local function must obey adjoint transformation rule under the gauge group action!

Another problem is that the above definition of the Wilson line runs in terms of functions rather then in the terms of operators which we are inclined to attribute more fundamental role.

From the other hand, in noncommutative case one can multiply an open Wilson line (3.32) by the function $e^{-ip \cdot (\Delta x)}$, where $\Delta x^\mu = y^\mu - x^\mu$ as shown in the Fig.2. Since $e^{-ip \cdot (\Delta x)}$ is the translation operator from $y$ to $x$ the modified Wilson line will transform under the gauge group action as a “local” noncommutative function of $x$. Therefore, integrating it out will produce a gauge invariant quantity,

$$W'[C] = \int d^p x \left( Pe^{i \int_C A_\mu(x) dx^\mu} e^{-ip \cdot \Delta x^\mu} \right).$$

(3.35)
Definition of Wilson lines (3.32) strongly depends on the background and therefore there is no indication that Wilson lines should be background independent. However, it surprisingly appears that Wilson line with the shifted end (3.35) is background invariant. (When it is multiplied with the $\theta$-dependent factor, $(2\pi)^{p/2}\sqrt{\det \theta}$.)

To show this, consider first a straight Wilson line $C_{\Delta x}$. In this case, the Wilson line with shifted end acquire the following simple operator form,

$$W'[C_{\Delta x}] = (2\pi)^{p/2}\sqrt{\det \theta} \text{tr} e^{i\Delta x^\mu X_\mu}. \quad (3.36)$$

The equivalence of (3.36) and (3.35) for straight lines can be proved in the following way. Let us divide the line $C_{\Delta x}$ in a large number $N$ of equal pieces. Then, one can split the exponential factor in (3.36) in a product of $N$ factors,

$$W[C_{\Delta x}] \sim \text{tr} e^{i\Delta x^\mu X_\mu} = \text{tr} \prod_n e^{i\Delta_{(n)}^\mu X_\mu} = \text{tr} \prod_n e^{i\Delta^\mu(p_\mu + A_\mu)} = \text{tr}(1 + i\Delta^\mu A_\mu^{(1)}) \cdots (1 + i\Delta^\mu A_\mu^{(n)}) \cdots (1 + i\Delta^\mu A_\mu^{(N)}) \times (1 + i\Delta^\mu p_\mu)^N + O(N^{-1}), \quad (3.37)$$

where each $\Delta_{(n)}^\mu = \Delta x^\mu / N$, and

$$A_\mu^{(n)} = (1 + i\Delta^\mu p_\mu)^n A_\mu(1 - i\Delta^\mu p_\mu)^n. \quad (3.38)$$

Rewriting (3.37) and (3.38) in the Weyl form and taking the limit $N \to \infty$, one gets precisely equation (3.33) for the straight line. In particular, operator $A_\mu^{(N)}$ defined in Eq. (3.38) maps to $A_\mu(x(t))$, where $t = \lim_{N \to \infty} n/N$. Equation (3.36) is not yet background invariant, but can be easily made so. The background invariance of $W'[C_{\Delta x}]$ is affected by the $\theta$-dependent factor in the r.h.s. of Eq. (3.36) and by the the explicit dependence of the $X_\mu$ field only. Therefore, the generalization to background invariant form looks as,

$$W[C_u] = \text{tr} e^{iu_i X_i}, \quad i = 1, \ldots, D. \quad (3.39)$$

Now, let us generalize the background invariant formula (3.39) to arbitrary smooth Wilson lines, not necessarily straight ones. Any smooth Wilson line can be imagined as consisting of infinitesimal straight lines. The observable corresponding to entire line corresponds to the product of straight factors,

$$W[C] = \text{tr} e^{iu_i X_i} e^{iu_1 X_1} \cdots e^{iu_N X_N}, \quad C = C_1^{\text{straight}} C_2^{\text{straight}} \cdots C_N^{\text{straight}}, \quad (3.40)$$
whose smooth limit is the path-ordered integral,

\[ W[C] = \text{tr} \, P_u e^{\int_C du^i X_i}, \quad (3.41) \]

where "\( P_u \)" denotes ordering with respect to parameter \( u \), while \( C \) denotes an arbitrary line in \( \mathbb{R}^D \).

In a particular background \( X_\mu = p_\mu, \, X_a = \text{constant} \) the line \( C \) maps to a line \( C' \) in the \( p \)-dimensional space. Applying a method similar to one given by eq. (3.37), one can obtain the Weyl form of the generalized Wilson line (3.41) in this background,

\[ W[C'] = \frac{1}{(2\pi)^{\frac{D-p}{2}} \sqrt{\det \theta}} \int d^p x \, e^i (\int_{C'} dx^\mu A_\mu + \int_{C'} dt \, \dot{u}^a(t) \phi_a). \quad (3.42) \]

The line \( C' \) in above equation is given by the projection of \( C \) to the space spanned by \( p_\mu \), i.e. if \( u^i(t), \, 0 \leq t \leq 1, \, i = 1, \ldots, D \) defines line \( C \), then \( C' \) is defined by equation \( x^\mu(t) = u^\mu(t), \, \mu = 1, \ldots, p \). As we see the line \( C \) in the auxiliary space \( \mathbb{R}^D \) corresponds to a \( p \)-dimensional line with a \((D - p)\)-dimensional vector fibre over it.
4 Noncommutative Gauge Dualities

This section follows the original papers \[41, 42\].

As we have discovered above, the noncommutative gauge model with scalar fields can be reformulated in the operator form where the dependence of the noncommutative space enters only through the representation of the noncommutative space algebra. Spanning algebras corresponding to different noncommutative spaces but having isomorphic representation will give us different equivalent noncommutative models.

As we discussed in the section 2, the noncommutative settings are stored in three factors: the operator algebra, its representation, and the (set of) derivative operators. While representation and operator algebras are generally fixed, the choice of derivative operators is at our disposal. In fact we can switch from one noncommutative $\mathbb{R}^D_\theta$ to another or from one gauge group to a different one by choosing derivative with appropriate symmetries. This is possible because particular gauge field backgrounds in some spaces can be interpreted as pure partial derivatives in different noncommutative spaces. This equivalence relation is described by the Seiberg–Witten map. (See Appendix A.)

Here we consider the cases of maps relating models in different dimensions, and ones relating models with different gauge groups.

4.1 Equations of Motion and Constant Curvature Solutions

In this subsection we are interested mainly in the scalar multiplet model with the potential (3.26).

Gauge field equations of motion corresponding to the action with potential (3.26) take the standard form,

\[ \nabla_\mu F_{\mu\nu} = -j_\nu, \]
\[ \nabla^2 \phi_a = 0, \]

where

\[ j_\mu = -i[\phi_a, \nabla_\mu \phi_a], \]

are, respectively, the noncommutative current generated by the scalar fields and the noncommutative Laplace operator respectively. In the operator form equations of motion have the form,

\[ [X_M, [X_M, X_N]] = 0, \quad M, N = (\mu, a), \]  

or, equivalently,

\[ [X_\mu, [X_\mu, X_\nu]] = -j_\nu, \quad j_\nu = [\phi_a, [X_\mu, \phi_a]], \]
\[ [X_\mu, [X_\mu, \phi_a]] = 0. \]
Of course all three forms of equations of motion (4.1), (4.2) and (4.3) are equivalent, the second (4.2) being also the most compact.

An obvious solution to the equations of motion is the constant gauge field strength one,

\[ F^{(0)}_{\mu\nu}(x) = \text{constant}_{\mu\nu}, \quad \phi^{(0)} = \text{constant}. \] (4.4)

The operator form (4.2) of the equations of motion suggest a more general solution,

\[ [X_M^{(0)}, X_N^{(0)}] = -i \Theta_{MN}, \] (4.5)

which in terms of fields \( F_{\mu\nu} \) and \( \phi \) reads,

\[ F^{(0)}_{\mu\nu} = -\Theta_{\mu\nu}, \quad [\phi_a^{(0)}, \phi_b^{(0)}] = -\Theta_{ab}, \] (4.6a)

\[ \nabla_\mu \phi_a^{(0)} = -\Theta_{\mu a}, \] (4.6b)

where \( \Theta_{MN} \) is split in blocks according to (3.28),

\[ \Theta_{MN} = \left( \begin{array}{cc} \Theta_{\mu\nu} & \Theta_{\mu b} \\ \Theta_{a\nu} & \Theta_{ab} \end{array} \right). \] (4.7)

### 4.2 Dimension Changing Solutions

Let us see what is the effect of the solution (4.5). First assume that this solution is given by a complete (in the sense of Quantum Mechanics) set of operators \( X_M \) satisfying,

\[ \forall F : [X_M, F] = 0, \quad M = 1, \ldots, p + n \Rightarrow F \propto I. \] (4.8)

If \( p' \) is the rank of \( \Theta_{MN} \), then it has \( D - p' \) zero modes. Let us divide the indices \( M, N, \ldots \) in primed early roman indices \( a', b' \) etc., which run in the zero space of \( \Theta_{MN} \) and primed Greek \( \mu', \nu' \) etc., running in the orthogonal completion. In other words, in primed indices introduce the basis where \( \Theta_{MN} \) have the following block structure:

\[ \Theta_{MN} = \left( \begin{array}{cc} \Theta_{\mu'\nu'} & 0 \\ 0 & 0 \end{array} \right). \] (4.9)

Applying now the machinery developed by us in the subsection 2.4 we end up with a gauge model having a different field content: \( A_{\mu'\nu'}, \mu' = 1, \ldots, p' \) and \( \phi_{a'}, a' = 1, \ldots, D - p \). Such an operation can be applied to any field participating in the gauge interaction in the adjoint representation to give rise to a field in the new background. (As we established, any real field, in fact, transforms in the adjoint representation of the gauge group.) As a result any field \( F(x) \) corresponding to a background invariant operator \( F \) in
the "old" theory in the "new" theory will be represented by a field $F'(x')$, according to the formula,

$$F'(x') = \sqrt{\frac{\det \theta'}{\det \theta}} \int \frac{d^p k'}{(2\pi)^{p'/2}} e^{ik'x'} \int d^p x e^{-i\theta'_{\mu'\nu'} k'_{\mu'} p'_{\nu'}(x)} F(x), \quad (4.10)$$

where $\theta'_{\mu'\nu'}$ is inverse matrix to $\theta'_{\mu\nu}$ and $p'_{\mu'}$ is the projection of the solution $p_M(x) \equiv (p_\mu + A_\mu^{(0)}(x), \phi_a^{(0)}(x))$ to the nonzero subspace of $\theta_{MN}$ (see eq. (4.9)).

Exercise 6 Derive (4.10) using the analysis of subsection 2.4.

Exercise 7 Find transformation rules for fields $\phi_a$ and for gauge fields in the case of infinitesimal change of the solution, $\delta p_\mu = \epsilon_{\mu\nu} p_\nu$.

Thus, we established an interesting feature of the noncommutative models involved in gauge interaction: the dimensionality of this models can be changed switching out between fields $A_\mu$ and $\phi_a$.

The question one may still ask is whether such gauge field configurations exist and how to construct them.

The existence of different gauge configurations satisfying irreducibility condition (4.8) follows from the isomorphism of separable infinite dimensional Hilbert spaces realizing irreducible representation of Heisenberg algebras generated by $X_M$. The isomorphism of infinite dimensional Hilbert spaces is a result of separability. Let us remind that by definition the separable infinite dimensional Hilbert space is one having a countable basis. The isomorphism between such spaces is realized by the identification of elements with same numbers in countable bases of such spaces. It is clear that this isomorphism is defined up to unitary transformations due to ambiguity in choosing the Hilbert space basis.

Consider now, as an example, the two-dimensional Yang–Mills–Higgs model with at least two scalar fields and let us construct a solution in vicinity of which the theory is effectively described by the four-dimensional model with two scalar fields less. To construct such a field configuration in the operator form, let us fix the isomorphic map $\sigma$ connecting Hilbert spaces of two and four dimensional Heisenberg algebras.

The map $\sigma$ can be constructed as follows. Consider the oscillator basis (2.15) for $p = 4$. and let us enumerate the oscillator states. Denote the unique number assigned to each state $|\vec{n}\rangle$ as $n(\vec{n}) \equiv n(n_1, n_2)$. We have then,

$$\sigma : |\vec{n}\rangle \mapsto |n = n(\vec{n})\rangle, \quad (4.11)$$

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where $|n\rangle$ is the element of oscillator basis of two-dimensional Heisenberg algebra. Since the number assigned to the lattice vector is unique this map is isomorphic.

Any operator $F_{(2)}$ acting on the Hilbert space of one oscillator can be uniquely mapped to an operator $F_{(4)}$ acting on two oscillator Hilbert space, by the rule

$$F_{(2)} \mapsto F_{(4)} = \sigma^{-1} \cdot F_{(2)} \cdot \sigma.$$  \tag{4.12}

Let us remind that such operators $F_{(4,2)}$ correspond via Weyl map to, respectively, four-dimensional and two-dimensional noncommutative functions. Thus, this is also an isomorphism between spaces of these functions.

Since the map is realized in the operator form it preserves the star algebra in the sense that it is a homomorphism of star algebras, i.e.,

$$F_{(2)} \ast_{(2)} G_{(2)} \mapsto F_{(4)} \ast_{(4)} G_{(4)},$$  \tag{4.13}

where $\ast_{(2,4)}$ are respective star products on the two- and four-dimensional noncommutative functions $F_{(p)}(x_{(p)})$, where the subscript $(p)$, denote the dimension (in our case it is $p = 2$ or $p = 4$). Let us note, that the data about the noncommutativity parameter $\theta^{\mu\nu}$ are contained in the definition of oscillator operators $a_i$ and $\bar{a}_i$.

The above construction was performed in the operator formalism. When trying to pass to the Weyl symbols one faces the problem that expressions defining the Weyl symbols are not valid for such operators as $p_{\mu}$ or $p'_{\mu'}$. Indeed, it is not difficult to see, that at least half of operators satisfying the Heisenberg algebra should have divergent trace, and therefore divergent integral in the Fourier transform. However, as we know, in the theory of commutative functions one can extend the definition of the Fourier transform to non square integrable functions if to work in terms of distributions rather than in that of ordinary functions.

As we discussed in the section 2, in the noncommutative case one can also introduce the notion of a “noncommutative” linear distribution. Let us rewrite the map (4.10) in terms of distribution-valued operators,

$$D(x) = \hat{\delta}(x - x), \text{ and } D'(x') = \hat{\delta}(x' - x').$$  \tag{4.14}

Then, the map (4.10) can be rewritten as,

$$F'(x') = (2\pi)^{\frac{d'}{2}} \sqrt{\det \theta'} \int d^p x D'(x', x) F(x),$$  \tag{4.15}

where,

$$D(x', x) = \text{tr} D'(x') D(x) = \sum_{\vec{m}, \vec{n}} D'_{\vec{m}'(\vec{m})\vec{n}'(\vec{n})}(x') D_{\vec{n}, \vec{m}}(x).$$  \tag{4.16}
This can be represented as follows,

\[ D(x', x) = \sum_{\vec{m}, \vec{n}'} D'_{\vec{m}' \vec{n}'}(x') V_{\vec{m}'} \vec{n}' D_{\vec{m}, \vec{n}}(x) V_{\vec{m}}^\dagger \vec{n}, \quad (4.17) \]

where the unitary operator \( V \) defined by,

\[ V = \sum_{\vec{n}'} V_{\vec{n}'} \vec{n}' \langle \vec{n} |, \quad (4.18) \]

can be viewed as the map from representation of \( p \)-dimensional Heisenberg algebra to the space of representation of the \( p' \)-dimensional one. The operator \( V \) has the meaning of element of the equivalence bimodule \( \mathcal{H}(p') \otimes \mathcal{H}(p) \) (see Appendix B) realizing Morita equivalence of two algebras. Heisenberg algebras are trivial from the K-theory point of view since any module is an infinite-dimensional separable Hilbert space and it is isomorphic to the space of operators with bounded trace of squares (see (2.46)) which is an infinite-dimensional separable Hilbert space too.

### 4.3 Gauge Group Changing Solutions

Let us return to the solution (4.5), but relax the condition (4.8). Let, now operators \( X_M = \{ p_\mu, 0 \} \) fail to form a complete set of operators, i.e. there are such operators \( F_a \), which commute with all \( p_\mu \),

\[ [p_\mu, F_a] = 0, \quad \mu = 1, \ldots, p, \quad (4.19) \]

but which are not scalar, \( F_a \neq c_a \mathbb{1} \). Let a set of \( F_a, a = 1, \ldots, n \) be chosen in such a way that the total set consisting of both \( p_\mu \) and \( F_a \) is complete, i.e.

\[ \forall F : [p_\mu, F] = [F_a, F] = 0 \Rightarrow F = c \mathbb{1}. \quad (4.20) \]

(Generically, one can always supplement the set of operators \( p_\mu \) by some other operators \( F_a \), such that the total set to be complete.)

As it can be seen, from (4.19) and (4.20) it follows that operators \( F_a \) should form an algebra,

\[ [F_a, F_b] = iG_{ab}(F), \quad (4.21) \]

where \( C_{ab} \) is an operator function of \( F_a \). In particular, it can be linear in \( F_a \), like

\[ C_{ab}(F) = C_{ab} F_c, \quad (4.22) \]

where \( C_{ab} \), are structure constants of a (semi)simple Lie algebra \( G = \text{Lie}(G) \). In the last case operators \( F_a \) should form an irreducible representation of the respective Lie algebra.
Let us show that the model around such a background looks like non-commutative gauge model with the gauge group $U(1) \times G$. To do this one should perform the analysis similar to one of the subsection 2.4.

The difference of the present case from the standard one analyzed in subsection 2.4 is that now the set of operators $p_\mu$ is not complete and so is the set of adjoint operators $P_\mu$. Therefore, the eigenvalues of $P_\mu$ become degenerate.

Since operators $F_a$ commute with $p_\mu$ they also should commute with $P_\mu$. This means that the eigenspace of $P_\mu$ corresponding to a particular momentum $k$ is invariant under action of $F_a$. Therefore, this eigenspace realizes a representation of the algebra (4.21). Moreover, in virtue of the Schur’s lemma this representation is irreducible, it is the adjoint representation of the algebra $u(1) \oplus G$. The factor $u(1)$ comes from the fact that the unity operator is always present in the algebra of operators. It corresponds to operators which are “singlet” in $G$, i.e. commute with all $F_a$. They, are functions of $p_\mu$ and, therefore, are not trivial.

As a result, we have that the Hilbert space $\mathcal{H}$ can be split into a tensor product as follows,

$$\mathcal{H} \simeq \mathcal{H}’ \otimes V_G,$$

(4.23)

where $p_\mu$ are irreducible on $\mathcal{H}’$, while $G$ on $V_G$.

Obviously, $\mathcal{H}’ \simeq \mathcal{H}$, which means that the infinite dimensional separable Hilbert space is isomorphic to itself times a Hermitian space. Let us construct this isomorphic map for a particular case of $G = su(2)$. For this, fix two bases in, respectively, $\mathcal{H}$ and $\mathcal{H} \otimes V(2)$ where $V(2)$ is two-dimensional Hermitian space,

$$|n⟩ \in \mathcal{H}, \quad n = 0, 1, 2, \ldots,$$

(4.24)

$$|n⟩ \otimes e_\alpha \in \mathcal{H} \otimes V(2), \quad n = 0, 1, 2, \ldots, \quad \alpha = 0, 1,$$

(4.25)

$\{e_\alpha, \alpha = 0, 1\}$ is the basis of $V(2)$. The map is obtained by the identification of the subspace of $\mathcal{H}$ corresponding to even values of $n$ to the subspace $\mathcal{H} \otimes e_0$ of $\mathcal{H} \otimes V(2)$, and respectively, the subspace with odd values of $n$ is mapped to $\mathcal{H} \otimes e_1$. On the basis elements it looks as follows,

$$\sigma : |n⟩ \mapsto |[n/2]⟩ \otimes e_\alpha,$$

(4.26)

where $\alpha = n \mod 2$, and $[n/2]$ is the integer part of $n$.

As above, with the map $\sigma$ at hand we can pull back any noncommutative function from the “nonabelian” $U(1) \times G$ Yang–Mills–Higgs theory to the $U(1)$ Yang–Mills–Higgs model,

$$\mathbf{F}_{U(1)} = \sigma^{-1} \cdot \mathbf{F}_{U(n)} \cdot \sigma,$$

(4.27)

and vice versa using $\sigma^{-1}$.

An analysis in terms of distributions analogous to one carried in the previous subsection can be also performed here.
Exercise 8 Perform the analysis of subsection 2.4, and show that the model around the solution satisfying (4.19) indeed looks like noncommutative U(1) × G Yang–Mills–Higgs model.

Exercise 9 Show that algebra (4.21) can be a central extended Lie algebra. What does the model look like in this case? Try other closed algebras.

Exercise 10 Generalize the map (4.26) to $G$ an arbitrary $su(n)$ algebra.

4.4 Solution with $θ = 0$

A particularly interesting case is given by solutions with $F_{μν} = θ_{μν}^{-1}$ and $φ_{a} =$ constant. The solution is highly degenerate in this case and a modification of the analysis is needed. The action vanishes on such solutions, therefore they correspond to absolute minima or vacua of the model. Another property is that such solution also exist in finite dimensional Hilbert spaces which is not the case of solutions with nonzero $θ$.

As one can see, equation (4.10) is singular in this limit and does not apply. Therefore a more detailed analysis is needed.

In this case $p_{μ}' = p_{μ} + A_{μ}^{(0)}$, form a commutative set,

$$[p_{μ}', p_{ν}'] = 0.$$  \hfill (4.28)

Obviously, commuting operators fail to form an irreducible set, unless the algebra is commutative which is not the case. However let us still assume that the still form a complete set of commutative operators, that is any operator commuting with all $p_{μ}'$ can be expressed as a function of $p_{μ}$. For this function to be unique, it is natural to require also all $p_{μ}$ to be functionally independent.

In other words, this means that all $p_{μ}'$ are chosen so that they are diagonalizable and any their common eigenvector $|ξ⟩$ is defined uniquely by its eigenvalues $ξ_{μ}$,

$$p_{μ}' |ξ⟩ = ξ_{μ} |ξ⟩.$$  \hfill (4.29)

By a redefinition $p_{μ}' = P_{μ}(p')$ one can make eigenvalues to be distributed uniformly in their range. Let us denote this set as spec $p$. (We suppress the tilde in the notations.)

In what follows, consider the case when spec $p$ coincides with $\mathbb{R}^p$, i.e. eigenvalues $ξ_{μ}$ are uniformly and continuously distributed in the range from $−∞$ to $+∞$. Then, one can introduce operators $q^{μ}$ defined as follows,

$$q^{μ} \int d^{p}ξ f(ξ) |ξ⟩ = i \int d^{p}ξ \frac{∂f(ξ)}{∂ξ_{μ}} |ξ⟩,$$  \hfill (4.30)

In the case when one or several $p_{μ}'$ are functionally dependent on other operators they can be dropped, coming to a smaller number of dimensions.
where $\psi = \int d^p \xi f(\xi) |\xi\rangle$ is an arbitrary vector of the Hilbert space written in the basis of $p_\mu$-eigenvectors.

It is not difficult to verify that $p_\mu$ together with $q^\mu$ form a $2p$-dimensional Heisenberg algebra (i.e. the Heisenberg algebra of a $p$-dimensional particle),

$$[p_\mu, q^\nu] = -i \delta^\nu_\mu.$$  \hspace{1cm} (4.31)

Let us note, that $p_\mu$ and $q^\mu$ together already form an irreducible set of operators. Therefore, from this point on one can apply the machinery of the section 2. After a computation one has that the action around the background given by the solution (4.28) takes the form,

$$S_{\theta=0} = \int dp \, dx \left( -\frac{1}{4g^2} F_{\mu\nu}(x,p)^2 + \frac{1}{2}(\nabla_\mu \phi_a(x,p))^2 - V_*(\phi(x,p)) \right),$$  \hspace{1cm} (4.32)

where, \hspace{1cm} (4.33a)

$$F_{\mu\nu}(x,p) = \partial_\mu A_\nu(x,p) - \partial_\nu A_\mu(x,p) + [A_\mu, A_\nu](x,p),$$

$$\nabla_\mu \phi_a(x,p) = \partial_\mu \phi_a(x,p) + [A_\mu, \phi_a](x,p),$$

$$[A, B](x,p) \equiv A* B(x,p) - B* A(x,p),$$

$$A* B(x,p) = e^{\frac{i}{2} \left( \frac{\partial^2}{\partial x^\mu \partial p^\nu} - \frac{\partial^2}{\partial x^\mu \partial p^\nu} \right)} A(x,p) B(x',p') \bigg|_{x'=x, p'=p}. \hspace{1cm} (4.33d)$$

The fields $A_\mu(x,p)$ and $\phi(x,p)$ are expressed in terms of old one as follows,

$$\langle p_\mu + A_\mu(x,p) \rangle = \int d^p x_{\text{old}} K(x,p; x_{\text{old}})(A^{(\text{old})}_\mu(x_{\text{old}}) - \theta_{\mu\nu}^{-1} x^\nu_{\text{old}}),$$  \hspace{1cm} (4.34a)

$$\phi_a(x,p) = \int d^p x_{\text{old}} K(x,p; x_{\text{old}}) \phi_a(x_{\text{old}}),$$  \hspace{1cm} (4.34b)

$$K(x,p; x_{\text{old}}) = \text{tr} [D(x,p)D_{\text{old}}(x_{\text{old}})],$$  \hspace{1cm} (4.34c)

$$D(x,p) = \int \frac{dp dp' dz}{(2\pi)^{2p}} e^{i z(p-p') + i k(q-x)}$$ \hspace{1cm} (4.34d)

As we see, the noncommutative Yang–Mills–Higgs model turns to be equivalent to a commutative Yang–mills–Higgs model with infinite dimensional gauge group of diffeomorphisms.

\hspace{1cm} 4For details we refer the reader to Ref. [43].
5 Discussions and Outlook

We discussed field theory on noncommutative spaces. In the Conne’s approach the noncommutative space is defined by the algebra of continuous functions on it, its representation and a derivative operator defined together with this algebra. The commutative space appears to be a degenerate case of the above. The difference is that the algebra defining the commutative space is Abelian and this implies that there is the only representation of it which is one-dimensional.

This is in contrast to what one has in the noncommutative case. Noncommutativity of the space leads to existence of a class of unitary equivalent representations instead of one single representation. The physics, however, should look the same irrespective to the chosen element of the equivalence class. Thus we come to the notion of gauge invariance which is an intrinsic feature of the noncommutative space. Therefore, the derivative operator defining the noncommutative space can be identified with the gauge field on it.

Thus, different gauge field configurations represent different noncommutative spaces. Among these the most interesting ones are, of course, those for which the gauge fields satisfy equations of motion. The last are flat noncommutative spaces. Flat spaces with unambiguous connection correspond to noncommutative spaces with (flat) coordinates satisfying the Heisenberg algebra. The Hilbert space representing it is infinite dimensional. Dropping out the requirement of unambiguity allows one to have noncommutative analogue of torus. Depending on defining parameters the noncommutative torus can fit into a finite-dimensional subspace of the Hilbert space. The study of the field theory on such spaces remained so far beyond the scope of the present paper, although they seem to be important at least as a regularization suitable for numeric analysis.

Arbitrary “deformations” of noncommutative space including ones changing the metric and topology as defined by the Connes geometry are encoded in gauge fields. In this sense they play the also role of gravity. If it is so, it would be interesting to separate the gravity component of the noncommutative gauge theory.

So far we considered everything in purely classical approach. The quantum theory even in the perturbative approach is known to face some problems with renormalizability due to so called IR/UV mixing \[12,13\]. Taking this into account some of the above results can be generalized to the quantum level \[14\].

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A Seiberg–Witten Map

In this appendix we give a brief review on the Seiberg–Witten map \[7\]. The bibliography on Seiberg–Witten map and its applications is vast \[45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59\].

The Seiberg–Witten map was proposed as a map realizing equivalences in low energies effective models for superstring theory. The effective model in the presence of the constant background field $B_{\mu\nu}$ computed in zero slope limit using two different regularization schemes, namely in Pauli–Villars and split-point regularization respectively, appears to be different since in the first case it is a commutative theory with a background constant field $B_{\mu\nu}$ while in the last one it is a noncommutative model with no background field $B_{\mu\nu}$, but whose noncommutative parameter $\theta^{\mu\nu}$ approaches the value $(B^{-1})^{\mu\nu}$. The consistency requires this effective models to be equivalent. As it was proposed in \[7\], this equivalence can be realized by a map which relates the field configurations in these two cases. Beyond this one can also consider a regularization scheme which is intermediate between these two cases, and therefore the equivalence should extend to arbitrary noncommutativity parameters.

Since this is a map of gauge models, the natural requirement is that gauge equivalent configurations should map to gauge equivalent. Let $A_\mu(x)$ be the commutative gauge field and $A'_\mu[A](x)$ or shortly $A'[A]$ be its image in the noncommutative model, and let $U = e^{i\lambda(x)}$ be an Abelian gauge transformation,

$$A'_\mu = A_\mu + \partial_\mu \lambda. \quad (A.1)$$

Then there exists a noncommutative gauge transformation $U'[A, U]$, which leads to the image of $A'_\mu$,

$$A'[A^U] = (A')^{U'[A, U]}[A] \equiv U'^{-1} \ast A'[A] \ast U' + U'^{-1} \ast dU', \quad (A.2)$$

where $U'[A, U] = e^{i\lambda'[A, U=e^{i\lambda}]}$, and in the last expression we suppressed the functional arguments of $A'$ and $U'$. The equation \[2\] is called Seiberg–Witten equation, and, respectively the map satisfying it is called Seiberg–Witten map.

For a small variation $\delta\theta^{\mu\nu}$ of the noncommutativity parameter Seiberg–Witten equation \[2\] takes the following infinitesimal form,

$$\delta A'_\mu(A^{(1+\lambda)}) - \delta A'_\mu(A) - \partial_\mu \delta \lambda' - i[\delta \lambda', A_\mu]_* - i[\lambda, \delta A'_\mu]_* = -\frac{1}{2} \delta \theta^{\alpha\beta}(\partial_\alpha \lambda \ast \partial_\beta A_\mu + \partial_\beta A_\mu \ast \partial_\alpha \lambda), \quad (A.3)$$

where $\delta A'_\mu(A)$ and $\delta \lambda'$ are infinitesimal maps of the gauge field and gauge
parameter respectively,

\[ A'_\mu = A_\mu + \delta A'_\mu[A], \]  
\[ \lambda' = \lambda + \delta \lambda'[A, \lambda]. \]

(A.4a) \hspace{1cm} (A.4b)

In [7] the following solution to (A.3) was proposed,

\[ \delta A_\mu = -\frac{1}{4} \delta \theta^{\alpha\beta}[A_\alpha * (\partial_\beta A_\mu + F_{\beta\mu}) + (\partial_\beta A_\mu + F_{\beta\mu}) * A_\alpha], \]

(A.5a)

\[ \delta \lambda = \frac{1}{4} \delta \theta^{\alpha\beta}(\partial_\alpha \lambda * A_\beta + A_\beta * \partial_\alpha \lambda). \]

(A.5b)

This solution, however, is by far not unique. For example, one can make a gauge transformation of either commutative field,

\[ A_\mu \rightarrow A'^\mu_\mu, \hspace{1cm} e^{i\lambda} \rightarrow g^{-1}e^{i\lambda}, \]

(A.6a)

or,

\[ A'_\mu \rightarrow (A')'^\mu_\mu, \hspace{1cm} U' \rightarrow (g')^{-1}U'. \]

(A.6b)

the transformed quantities will continue to satisfy (A.3) or (A.2). In particular \( g \) can depend on \( A \) and \( \lambda \), and respectively \( g' \) on \( A' \) and \( \lambda' \), in this case expressions (A.3) change considerably.
B K-theory and Morita Equivalence

This appendix contains a short introduction to K-theory and Morita equivalence. K-theory and Morita equivalence relation to string theory and noncommutative geometry is discussed in the following papers, [60, 61, 62, 63, 64]. (For a review of K-theory see [66].)

Consider a \( C^\ast \)-algebra \( A \), or an associative complex algebra with involution \( * \). We will mainly think about the algebra of complex functions on a noncommutative space. (In this case it is a noncommutative algebra.) Let \( E \) be its left module i.e.,

\[
(a)(m) = am \in E, \quad (a')(a)(m) = a'(am) = a'am, \tag{B.1}
\]

for arbitrary \( m \in E \), and \( a, a' \in A \). Right module is defined in a similar way but with consequent action of elements of \( A \) from the right.

The algebra \( A \) itself and its tensor products \( A \otimes A \otimes \cdots \otimes A \) is a primitive example of both left and right modules, such modules are called free. A module \( E \) for which exists another module \( E' \) such that \( E \oplus E' \) is free is a projective one. (It is clear that \( E' \) is also a projective module.) The set of left or right projective modules form a semigroup with respect to the direct sum operation. This semigroup can be “upgraded” to a group as follows.

Consider pairs of modules \((E, F)\), with the composition rule \((E, F) + (E', F') = (E \oplus E', F' \oplus F)\) and the equivalence relation \((E, F) \sim (E \oplus G, F \oplus G)\), for arbitrary module \( G \). This equivalence classes form a group whose unity is given by \((G, G)\)-pairs and the opposite element to \((E, F)\) is given by \((F, E)\),

\[
(E, F) + (F, E) = (E \oplus F, E \oplus F) \sim (G, G). \tag{B.2}
\]

This trick is similar to one used to extend the set of positive numbers to real ones. The group one gets in such a way is called the \( K(A) \) or if \( A \) is the algebra of functions on some space \( M \) it is denoted as \( K(M) \).

Let us equip our left or right projective module \( E \), with an \( A \)-valued product \( \langle , \rangle_{A} \), satisfying,

\[
\langle m, m' \rangle_{A} = \langle m', m \rangle_{A} \tag{B.3}
\]

\[
\langle am, m' \rangle_{A} = a \langle m, m' \rangle_{A} \tag{B.4}
\]

\[
\langle m, m' \rangle_{A} \text{ is a positive element in } A. \tag{B.5}
\]

In other words, if we assume that the algebra \( A \) is equipped with trace, \( \text{tr} a^* = (\text{tr} a)^* \), then the eqs. (B.2-B.4) mean that \( \text{tr} \langle m, m' \rangle_{A} \) should define a nondegenerate scalar product whose adjoint is compatible with the involution. The interesting most case of the full module \( E \) is when the linear span of the range of \( \langle , \rangle_{A} \) is dense in \( A \).
One can introduce connection $\nabla_\alpha$ on the $\mathfrak{A}$-module $E$ with respect to infinitesimal automorphisms of $\mathfrak{A}$: $a \to a + \delta_\alpha a$, labelled by some element $\alpha$, which satisfies,

$$\nabla_\alpha(am) = a\nabla_\alpha(m) + \delta_\alpha am,$$

and it is linear in $\alpha$. Using this connection one can built the curvature associated to it,

$$F_{\alpha\beta} = [\nabla_\alpha, \nabla_\beta] - \nabla_{[\alpha, \beta]}.$$

$\mathfrak{A}$-linear maps $T : E \to E$ which have an adjoint with respect to the product (B.2-B.4) and commute with the action of $\mathfrak{A}$ on $E$ form the algebra $\text{End}_\mathfrak{A} E$ of endomorphisms of the $\mathfrak{A}$-module $E$.

By definition an algebra $\mathfrak{B}$ is Morita equivalent to $\mathfrak{A}$ if it is isomorphic to $\text{End}_\mathfrak{B} E$ for some complete module $E$.

There exists the following criterium for Morita equivalence of two algebras $\mathfrak{A}$ and $\mathfrak{B}$. A left $\mathfrak{A}$-module $P$ which is also a right $\mathfrak{B}$-module is called $(\mathfrak{A}, \mathfrak{B})$-bimodule. Assume that $P$ as $\mathfrak{A}$- and $\mathfrak{B}$-module is equipped with $\mathfrak{A}$-valued product $\langle \cdot, \cdot \rangle_\mathfrak{A}$, and $\mathfrak{B}$-valued product $\langle \cdot, \cdot \rangle_\mathfrak{B}$, and it is full as both $\mathfrak{A}$- and $\mathfrak{B}$-module. When it exists such a module is called $(\mathfrak{A}, \mathfrak{B})$ equivalence bimodule, in this case algebras $\mathfrak{A}$ and $\mathfrak{B}$ are Morita equivalent. The Morita equivalence allows one to establish relations between various structures of the equivalent algebras and their modules, like endomorphisms, connections, etc.

It is conjectured [60, 61, 63, 64, 65], that Morita equivalent algebras in string theory correspond to physically equivalent systems e.g. related by duality transformations. In noncommutative theory the gauge models on the dual tori are also known to be Morita equivalent [8, 61].
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