A subexponential upper bound for van der Waerden numbers $W(3, k)$

Tomasz Schoen
Faculty of Mathematics and Computer Science
Adam Mickiewicz University
Uniwersytetu Poznańskiego 4
61-614 Poznań, Poland
schoen@amu.edu.pl

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Abstract

We show an improved upper estimate for van der Waerden number $W(3, k)$: there is an absolute constant $c > 0$ such that if \{1,\ldots,N\} = X \cup Y is a partition such that $X$ does not contain any arithmetic progression of length 3 and $Y$ does not contain any arithmetic progression of length $k$ then

$$N \leq \exp(O(k^{1-c})).$$

Mathematics Subject Classifications: 05D10, 11B25

1 Introduction

Let $k$ and $l$ be positive integers. The van der Waerden number $W(k, l)$ is the smallest positive integer $N$ such that in any partition \{1,\ldots,N\} = X \cup Y there is an arithmetic progression of length $k$ in $X$ or an arithmetic progression of length $l$ in $Y$. The existence of such numbers was established by van der Waerden [22], however the order of magnitude of $W(k, l)$ is unknown for $k, l \geq 3$. Clearly, $W(k, l)$ is related to Szemerédi’s theorem on arithmetic progressions [20] and any effective estimate in this theorem leads to an upper bound on the van der Waerden numbers. Currently the best known bounds in the most important diagonal case are

$$(1 - o(1))\frac{2^{k-1}}{ek} \leq W(k, k) \leq 2^{2\times2^{k+9}}.$$

The upper bound follows from the famous work of Gowers [12] and the lower bound was proved by Szabó [19] using a probabilistic argument. Furthermore, Berlekamp [3] showed
that if $k - 1$ is a prime number then

$$W(k, k) \geq (k - 1)2^{k-1}.$$ 

Another very intriguing instance of the problem is the estimation of the numbers $W(3, k)$, as these are related to Roth’s theorem [15] concerning estimates for sets avoiding three-term arithmetic progressions. Let us denote by $r(N)$ the size of the largest progression-free subset of $\{1, \ldots, N\}$. We know that

$$r(N) \ll \frac{N}{\log N^{1-o(1)}},$$ 

see [4, 5, 17, 18], which implies $W(3, k) \leq \exp(O(k^{1+o(1)}))$.

Green [13] proposed a very clever argument based on arithmetic properties of sumsets to bound $W(3, k)$. Building on this method and applying results from [10] it was shown in [11] that

$$W(3, k) \leq \exp(O(k \log k)).$$ 

The best known lower bound was obtained by Li and Shu [14] (see also [8]), who showed that

$$W(3, k) \gg \left(\frac{k}{\log k}\right)^2.$$ 

The purpose of this paper is to prove a subexponential bound on $W(3, k)$.

**Theorem 1.** There are absolute constants $C, c > 0$ such that for every $k$ we have

$$W(3, k) \leq \exp(Ck^{1-c}).$$ 

Our argument is based on the method of [18], which explores in details the structure of a large spectrum. This method can be partly applied (see Lemma 5) in our approach and it deals only with a progression-free partition class. The second part of the proof exploits the structure of both partition classes and in this case the argument of [18] has to be significantly modified.

Let us remark that during the review process a preprint of Bloom and Sisask [6], which improves an upper bound in Roth’s theorem to $N/(\log N)^{1+c}$ for $c \approx 2^{-2^{2100}}$, has appeared. That result implies directly that $W(3, k) \leq \exp(Ck^{1-c})$ with $c \approx 2^{-2^{2100}}$.

2 Notation

Given functions $f, g : \mathbb{Z}/\mathbb{N}\mathbb{Z} \to \mathbb{C}$, the convolution of $f$ and $g$ is defined by

$$(f * g)(x) = \sum_{t \in \mathbb{Z}/\mathbb{N}\mathbb{Z}} f(t) g(x - t).$$
The Fourier coefficients of a function \( f : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C} \) are defined by
\[
\hat{f}(r) = \sum_{x=0}^{N-1} f(x)e^{2\pi i r x/N},
\]
where \( r \in \mathbb{Z}/N\mathbb{Z} \). The inversion formula states that
\[
f(x) = \frac{1}{N} \sum_{r=0}^{N-1} \hat{f}(r)e^{2\pi i x r/N}.
\]

We denote by \( 1_A(x) \) the indicator function of set \( A \). Thus using the inversion formula and the fact that \((f * g)(r) = \hat{f}(r)\hat{g}(r)\) one can express the number of three–term arithmetic progressions (including trivial ones) in a set \( A \subseteq \mathbb{Z}/N\mathbb{Z} \) by
\[
\frac{1}{N} \sum_{r=0}^{N-1} \overline{1_A(r)}^2 \overline{1_A(-2r)}.
\]

Parseval’s identity asserts in particular that
\[
\sum_{r=0}^{N-1} |\overline{1_A(r)}|^2 = |A|N.
\]

Let \( \theta \geq 0 \) be a real number. The \( \theta \)–spectrum of \( A \) is defined by
\[
\Delta_\theta(A) = \{ r \in \mathbb{Z}/N\mathbb{Z} : |\overline{1_A(r)}| \geq \theta |A| \}.
\]
If \( A \) is specified then we write \( \Delta_\theta \) instead of \( \Delta_\theta(A) \).

By the span of a finite set \( S \) we mean
\[
\mathrm{Span}(S) = \left\{ \sum_{s \in S} \varepsilon_s s : \varepsilon_s \in \{-1, 0, 1\} \text{ for all } s \in S \right\}
\]
and the dimension of \( A \) is defined by
\[
\dim(A) = \min \left\{ |S| : A \subseteq \mathrm{Span}(S) \right\}.
\]

Chang’s Spectral Lemma provides an upper bound for the dimension of a spectrum.

**Lemma 2.** [9] Let \( A \subseteq \mathbb{Z}/N\mathbb{Z} \) be a set of size \( |A| = \delta N \) and let \( \theta > 0 \). Then
\[
\dim(\Delta_\theta(A)) \ll \theta^{-2} \log(1/\delta).
\]

We are going to use Bohr sets [7] to prove the main result. Let \( \Gamma \subseteq \hat{G} \) and \( \gamma \in (0, \frac{1}{2}] \) then the Bohr set generated by \( \Gamma \) with radius \( \gamma \) is
\[
B(\Gamma, \gamma) = \left\{ x \in \mathbb{Z}/N\mathbb{Z} : \|tx/N\| \leq \gamma \text{ for all } t \in \Gamma \right\},
\]
where \( \|x\| = \min_{y \in \mathbb{Z}} |x - y| \). The rank of \( B \) is the size of \( \Gamma \) and we denote it by \( \text{rk}(B) \).

Given \( \eta > 0 \) and a Bohr set \( B = B(\Gamma, \gamma) \), by \( B_\eta \) we mean the Bohr set \( B(\Gamma, \eta \gamma) \). We will also use the notation \( \beta = \frac{1}{|B|}1_B \). We will use two basic properties of Bohr sets concerning their size and regularity, see [21].
Lemma 3. [7] For every $\gamma \in (0, \frac{1}{2}]$ we have
\[
\gamma|\Gamma|N \leq |B(\Gamma, \gamma)| \leq 8|\Gamma|+1|B(\Gamma, \gamma/2)|.
\]

We call a Bohr set $B(\Gamma, \gamma)$ regular if for every $\eta$, where $|\eta| \leq 1/(100|\Gamma|)$ we have
\[
(1 - 100|\Gamma||\eta|)|B| \leq |B_{1+\eta}| \leq (1 + 100|\Gamma||\eta|)|B|.
\]

Bourgain [7] showed that regular Bohr sets are ubiquitous.

Lemma 4. [7] For every Bohr set $B(\Gamma, \gamma)$, there exists $\gamma'$ such that $\frac{1}{2}\gamma \leq \gamma' \leq \gamma$ and $B(\Gamma, \gamma')$ is regular.

3 Proof of Theorem 1

Our main tool is the next lemma, which can be extracted from [18], by conjugation of results concerning the case of 'small' Fourier coefficients (see Lemmas 7 and 9 of [18]) and the case of 'middle' Fourier coefficients (Lemmas 12 and 13 of [18]). Its proof makes use of the deep result by Bateman and Katz in [1, 2] describing the structure of the large spectrum. The case of 'large' Fourier coefficients is treated similarly as in [18], however using partition properties we will be able to obtain much better estimate.

Lemma 5. [18] There exists an absolute constant $c > 0$ such that the following holds. Let $A \subseteq \mathbb{Z}/N\mathbb{Z}$, $|A| = \delta N$ be a set such that
\[
\sum_{r: \delta^{1+c}|A| \leq |\hat{A}(r)| \leq \delta^{1/10}|A|} |\hat{A}(r)|^3 \geq \frac{1}{10} \delta^{c/5}|A|^3.
\]
Then there is a regular Bohr set $B$ with $\text{rk}(B) \ll \delta^{-1+c}$ and radius $\Omega(\delta^{1-c})$ such that for some $t$
\[
|(A + t) \cap B| \gg \delta^{1-c}|B|.
\]

Furthermore, we apply Bloom’s iterative lemma, that provides a density increment by a constant factor greater than 1 for progression-free sets and Sanders’ lemma on a containment of long arithmetic progressions in dense subsets of regular Bohr sets.

Lemma 6. [4] There exists an absolute constant $c_1 > 0$ such that the following holds. Let $B \subseteq \mathbb{Z}/N\mathbb{Z}$ be a regular Bohr set of rank $d$. Let $A_1 \subseteq B$ and $A_2 \subseteq B_\varepsilon$, each with relative densities $\alpha_i$. Let $\alpha = \min(c_1, \alpha_1, \alpha_2)$ and assume that $d \leq \exp(c_1(\log^2(1/\alpha)))$. Suppose that $B_\varepsilon$ is also regular and $c_1\alpha/(4d) \leq \varepsilon \leq c_1\alpha/d$. Then either
(i) there is a regular Bohr set $B'$ of rank $\text{rk}(B') \leq d + O(\alpha^{-1}\log(1/\alpha))$ and size
\[
|B'| \geq \exp\left(-O(\log^2(1/\alpha)(d + \alpha^{-1}\log(1/\alpha)))\right)|B|
\]
such that
\[
|(A_1 + t) \cap B'| \gg (1 + c_1)\alpha_1|B'|
\]
for some $t \in \mathbb{Z}/N\mathbb{Z}$;
(ii) or there are \(\Omega(\alpha_1^2\alpha_2|B||B_\varepsilon|)\) three-term arithmetic progressions \(x + y = 2z\) with \(x, y \in A_1, z \in A_2\);

**Lemma 7.** [16] Let \(B(\Gamma, \gamma) \subseteq \mathbb{Z}/N\mathbb{Z}\) be a regular Bohr set of rank \(d\) and let \(\varepsilon\) be a positive number satisfying \(\varepsilon^{-1} \ll \gamma d^{-1}N^{1/d}\). Suppose that \(A \subseteq B\) contains at least a proportion \(1 - \varepsilon\) of \(B(\Gamma, \gamma)\). Then \(A\) contains an arithmetic progression of length at least \(1/(4\varepsilon)\).

Furthermore, to apply Bloom’s result we will need to prove a standard fact on Bohr sets. Let us also remark that it follows from the proof of Bloom’s lemma that we can take \(c_1 < 1/1000\). We will use the following basic property of Bohr sets.

**Lemma 8.** [11] Let \(B\) be a regular Bohr set of rank \(d\) and radius \(\gamma\) and let \(A\) be a set with \(|A| = \alpha|B|\). Suppose that \(\varepsilon < \kappa\alpha/(100d)\) for some \(\kappa \in (0, 1)\). Then

\[
\sum_{x \in B} (1_A * 1_{B_\varepsilon})(x) \geq (1 - \kappa)|B||B_\varepsilon|.
\]

**Lemma 9.** Let \(B\) be a regular Bohr set of rank \(d\) and suppose that \(A \subseteq B\) and \(|A| = \alpha|B|\). Then there is \(\varepsilon\) with \(c_1\alpha/(4d) \leq \varepsilon \leq c_1\alpha/d\) (\(c_1\) is a constant given by Lemma 6), and regular Bohr sets \(B^1\) and \(B^\varepsilon\) of rank \(d\) and size

\[
|B^\varepsilon| \geq \exp(-O(d\log(1/\alpha)\log(d/\alpha)))|B|
\]

such that

\[
|(A + t) \cap B^\varepsilon| \geq \frac{1}{2}\alpha|B^\varepsilon| \quad \text{and} \quad |(A + t) \cap B^1| \geq \frac{1}{2}\alpha|B^1|
\]

for some \(t\).

**Proof.** Let \(\varepsilon_1\) and \(\varepsilon_2\) be any numbers satisfying \(c_1\alpha/(4d) \leq \varepsilon_1, \varepsilon_2 \leq c_1\alpha/d\) and such that the Bohr sets \(B^1 = B_{\varepsilon_1}\) and \(B^2 = B_{\varepsilon_1\varepsilon_2}\) are regular. Put \(\beta_1 = \frac{1}{|B^1|}1_{B^1}\) and \(\beta_2 = \frac{1}{|B^2|}1_{B^2}\) then by Lemma 8 we have

\[
\sum_{x} \left( (\beta_1 * 1_A)(x) + (\beta_2 * 1_A)(x) \right) \geq \frac{9}{5}\alpha|B|,
\]

hence for some \(x\) we have

\[
(\beta_1 * 1_A)(x) + (\beta_2 * 1_A)(x) \geq \frac{9}{5}\alpha.
\]

If

\[
(\beta_1 * 1_A)(x) \geq \frac{1}{2}\alpha \quad \text{and} \quad (\beta_2 * 1_A)(x) \geq \frac{1}{2}\alpha \quad (3)
\]

then we can put \(B^\varepsilon = B^1\). Otherwise, for some \(i \in \{1, 2\}\) we have \((\beta_i * 1_A)(x) \geq (6/5)\alpha\) and we put \(B = B^i\) and apply the same procedure again. Clearly, we can iterate this procedure \(O(\log(1/\alpha))\) times, as the density of a set can not exceed 1. Thus after \(O(\log(1/\alpha))\) iterative steps we obtain a pair of Bohr sets \(B^\varepsilon, B^\varepsilon'\) satisfying (3). From Lemma 3 it follows that

\[
|B^\varepsilon| \geq \exp(-O(d\log(1/\alpha)\log(d/\alpha)))|B|,
\]

which concludes the proof.\[\square\]
Proof of Theorem 1. Put \( M = W(3, k) - 1 \) and let \( \{1, \ldots, M\} = X \cup Y \) be a partition such that \( X \) and \( Y \) avoid 3 and \( k \)-term arithmetic progressions respectively. Clearly, we may assume that \( M \geq 100k \) hence

\[
|Y| \leq M - \lfloor M/k \rfloor \leq M - M/(2k),
\]
as no block of \( k \) consecutive numbers is contained in \( Y \) and therefore \( |X| \geq M/(2k) \). Let \( N \) be any prime number satisfying \( 2M < N \leq 4M \). We embed \( \{1, \ldots, M\} = X \cup Y \) in \( \mathbb{Z}/N\mathbb{Z} \) in a natural way and observe that \( \{1, \ldots, M\} \) considered as a subset of \( \mathbb{Z}/N\mathbb{Z} \) is 2-Freiman isomorphic to \( \{1, \ldots, M\} \) considered as a subset of \( \mathbb{Z} \) and therefore any arithmetic progression in \( \{1, \ldots, M\} \subseteq \mathbb{Z}/N\mathbb{Z} \) is a genuine progression in \( \mathbb{Z} \). Put \( |X| = \delta N \) and note that we can assume that \( \delta \gg (\log N)^{-1.1} \). First let us assume that

\[
\sum_{r: \delta^{1+c} |X| \leq |\hat{1}_X(r)| \leq \delta^{1/10} |X|} |\hat{1}_X(r)|^3 \gg \delta^{c/5} |X|^3, \quad (4)
\]

where \( c > 0 \) is the fixed absolute constant from Lemma 5. Then by Lemma 5 there is \( t \in \mathbb{Z}/N\mathbb{Z} \) and a regular Bohr set \( B^0 \) with \( \text{rk}(B^0) = d \ll \delta^{-1+c} \) and radius \( \Omega(\delta^{1-c}) \) such that

\[
|(X + t) \cap B^0| \gg \delta^{1-c} |B|
\]
for some absolute constant \( c > 0 \). Writing \( X_0 = (X + t) \cap B_0 \) we have

\[
|X_0 \cap B^0| \gg \alpha |B^0|,
\]
where

\[
\alpha \gg \delta^{1-c},
\]
and by Lemma 3

\[
|B^0| \gg \exp\left( - O(\delta^{-1+c} \log(1/\delta)) \right) N.
\]

By Lemma 9 there is \( \varepsilon \gg \alpha^2 \) and regular Bohr sets \( B' \) and \( B'_c \) of rank \( d \) such that

\[
|(A + t) \cap B'| \geq \frac{1}{2} \alpha |B'| \quad \text{and} \quad |(A + t) \cap B'_c | \geq \frac{1}{2} \alpha |B'_c|,
\]
and

\[
|B'| \geq \exp\left( - O(\delta^{-1+c} \log^2 (1/\delta)) \right) |B^0| \geq \exp\left( - O(\delta^{-1+c} \log^2 (1/\delta)) \right) N.
\]

Next, we iteratively apply Lemma 6 and Lemma 9. Since after each step the density increases by factor \( 1 + c_1 \) it follows that after \( l \ll \log(1/\alpha) \) steps case (ii) of Lemma 6 holds. Let \( B^i \) be Bohr sets obtained in the iterative procedure. Note that \( B^i \) has rank \( \text{rk}(B^i) = d_i \ll \alpha^{-1} \log^2 (1/\alpha) \) for every \( i \leq l \) and

\[
|B^{i+1}| \geq \exp\left( - O(\log^2 (1/\alpha) (d_i + \alpha^{-1} \log(1/\alpha))) \right) |B_i|
\]
for every \( i < l \). Therefore, there are

\[
\Omega(\alpha^3 |B^t||B'_c|)
\]
three-term arithmetic progressions in $X$, where $\varepsilon \geq c_1 \alpha/(4\operatorname{rk}(B')) \gg \alpha^2 \log^2(1/\alpha)$. By Lemma 6 and Lemma 3 we have

$$|B| \geq \exp \left( -O(\alpha^{-1} \log^4(1/\alpha)) \right) N \geq \exp \left( -O(\delta^{-1+c} \log^4(1/\delta)) \right) N,$$

and

$$|B| \geq \exp \left( -O(\alpha^{-1} \log^3(1/\alpha)) \right) \exp \left( -O(\alpha^{-1} \log^4(1/\alpha)) \right) N \geq \exp \left( -O(\delta^{-1+c} \log^4(1/\delta)) \right) N.$$

Thus, $X$ contains

$$\Omega(\delta^{-3c} \exp \left( -O(\delta^{-1+c} \log^4(1/\delta)) \right) N^2)$$

arithmetic progressions of length three. Since there are only $|X|$ trivial progressions in $X$ it follows that

$$|X| \gg \delta^{-3c} \exp \left( -O(\delta^{-1+c} \log^4(1/\delta)) \right) N^2,$$

so

$$W(3, k) \ll N \ll \exp \left( O(\delta^{-1+c} \log^4(1/\delta)) \right) \exp \left( O(\delta^{-1+c} \log^4 k) \right).$$

Next let us assume that (4) does not hold. Let us define $\Delta' = \Delta_{\delta_1/10} \cup 2^{-1} \cdot \Delta_{\delta_1/10}$ and observe that $r \notin \Delta'$ is equivalent to $r \notin \Delta_{\delta_1/10}$ and $2r \notin \Delta_{\delta_1/10}$. By Chang’s lemma

$$\dim(\Delta') \leq 2\dim(\Delta_{\delta_1/10}) \ll \delta^{-1/5} \log(1/\delta),$$

and let $\Lambda$ be any set such that $1 \in \Lambda$, $|\Lambda| \ll \delta^{-1/5} \log(1/\delta)$ and $\Delta' \subseteq \operatorname{Span}(\Lambda)$. Let $B = B(\Lambda, \gamma)$ be a regular Bohr set with radius $\delta^3 \ll \gamma \ll \delta^3$. Since $1 \in \Lambda$ it follows that for every $b \in B$ we have

$$\|b/N\| \leq \gamma N \leq 4\gamma M.$$

Recall that $\beta = \frac{1}{|B|}1_B$ then for every $r \in \Delta'$ we have

$$|\hat{\beta}(r) - 1| \leq \frac{1}{|B|} \sum_{b \in B} |e^{-2\pi irb/N} - 1| \leq \frac{2\pi}{|B|} \sum_{b \in B} \sum_{\lambda \in \Lambda} \|rb/N\| \leq 2\pi \delta^2, \quad (5)$$

and similarly $|\hat{\beta}(2r) - 1| \leq 4\pi \delta^2$. For $t \in \mathbb{Z}/N\mathbb{Z}$ put

$$f(t) = \beta * 1_X(t)$$

and note that if for some $t \in [4\gamma M, (1 - 4\gamma)M]$ we have $f(t) = \frac{1}{|B|} |X \cap (B + t)| \leq \delta^{1+c'}$, where $c' = c/20$, then since $B + t \subseteq [1, M]$ it follows that

$$|Y \cap (B + t)| \geq (1 - \delta^{1+c'})|B|.$$

Therefore, by Lemma 7 either $\delta^{-1-c} \gg \gamma^{-1} N^{1/d}$ or $Y$ contains an arithmetic progression of length $\frac{1}{4} \delta^{-1-c}$. The former inequality implies that

$$k^{1+c'} \gg \gamma^{-1} N^{1/d} \gg \delta^4 N^O(\delta^{-1/5} \log^{-1}(1/\delta)) \gg k^{-4} N^O(\delta^{-1/5} \log^{-1} k),$$

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so

\[ W(3, k) \ll N \ll \exp\left(O\left(k^{1/5} \log^2 k\right)\right). \]

If the second alternative holds then

\[ \frac{1}{4} \delta^{-1-c'} < k \]

hence by (1)

\[ k^{-1/(1+c')} \ll \delta \ll (\log N)^{-1+o(1)} \]

so

\[ W(3, k) \leq N \ll \exp\left(O\left(k^{1/5}+o(1)\right)\right). \]

Finally we can assume that for every \( t \in [4\gamma M, (1 - 4\gamma)M] \) we have \( f(t) \geq \delta^{1+c'} \). Let \( T(X) \) denote the number of three-term arithmetic progressions in \( X \) and let

\[ T(f) = \sum_{x+y=2z} f(x)f(y)f(z). \]

Then clearly

\[ T(f) \gg \delta^{3+3c'} M^2 \gg \delta^{3+c/6} N^2 \]

and we will show that \( T(X) \) does not differ much from \( T(f) \)

\[
|T(X) - T(f)| = \frac{1}{N} \left| \sum_{r=0}^{N-1} \hat{1}_X(r)^2 \hat{1}_X(-2r) - \sum_{r=0}^{N-1} \hat{f}(r)^2 \hat{f}(-2r) \right|
\leq \frac{1}{N} \left| \sum_{r=0}^{N-1} \hat{1}_X(r)^2 \hat{1}_X(-2r)(1 - \hat{\beta}(r)^2 \hat{\beta}(-2r)) \right|
= S_1 + S_2,
\]

where \( S_1 \) and \( S_2 \) are summations of (7) respectively over \( \Delta' \) and \( \mathbb{Z}/N\mathbb{Z} \setminus \Delta' \). By (5), the negation of (4), Parseval’s formula and Hölder’s inequality we have

\[
S_1 \ll \delta^2 \frac{1}{N} \sum_{r \in \Delta'} |\hat{1}_X(r)|^3 \leq \delta^3 \sum_{r=0}^{N-1} |\hat{1}_X(r)|^2 = \delta^2 |X|^2, 
\]

\[
S_2 \leq \frac{2}{N} \sum_{r \notin \Delta'} |\hat{1}_X(r)^2 \hat{1}_X(-2r)|
\leq \frac{2}{N} \left( \sum_{r \notin \Delta'} |\hat{1}_X(r)|^3 \right)^{2/3} \left( \sum_{r \notin \Delta'} |\hat{1}_X(2r)|^3 \right)^{1/3}
\leq \frac{2}{N} \sum_{r \notin \Delta_{\delta/10}} |\hat{1}_X(r)|^3
\leq \frac{2}{N} \sum_{r \in \Delta_{\delta/10} \setminus \Delta_{\delta/2}} |\hat{1}_X(r)|^3 + \frac{2}{N} \sum_{r \notin \Delta_{\delta/10}} |\hat{1}_X(r)|^3
\ll \delta^2 |X|^2 + \delta^{1+c/5} |X|^2 + \delta^{1+c} |X|^2 \ll \delta^{1+c/5} |X|^2
\]

\[ \leq \delta^{1+c/5} |X|^2 \]

\[ \leq \delta^{1+c/5} |X|^2 \]

\[ \ll \delta^{1+c/5} |X|^2 \]
Thus,
\[ |T(X) - T(f)| \ll \delta^{3+c/5}N^2, \]
so by (6) and the fact that \( X \) avoids non-trivial three-term arithmetic progression we have
\[ |X| = T(X) \gg \delta^{3+c/6}N^2, \]
hence
\[ W(3, k) \leq N \ll \delta^{-2-c/6} \ll k^3 \]
which concludes the proof. \( \square \)

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