RANDOM WALK ON DISCRETE POINT PROCESSES

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ABSTRACT. We consider a model for random walks on random environments (RWRE) with random subset of $\mathbb{Z}^d$ as the vertices, and uniform transition probabilities on $2d$ points (two "coordinate nearest points" in each of the $d$ coordinate directions). We prove that the velocity of such random walks is almost surely 0, and give partial characterization of transience and recurrence in the different dimensions. Finally we prove Central Limit Theorem (CLT) for such random walks, under a condition on the distance between coordinate nearest points.

1. Introduction

1.1. Background.

Random walk on random environments is the object of intensive mathematical research for more then 3 decades. It deals with models from condensed matter physics, physical chemistry, and many other fields of research. The common subject of all models is the investigation of movement of particles in an inhomogeneous media. It turns out that the randomness of the media (i.e. the environment) is responsible for some unexpected results, especially in large scale behavior. In the general case, the random walk takes place in a countable graph $(V, E)$, but the most investigated models deals with the graph of the $d$-dimensional integer lattice, (i.e. $V = \mathbb{Z}^d$). For some of the results on those models see [Zei04], [BS02], [Hug96] and [R´ ev05]. The definition of RWRE involves two steps: First the environment is randomly chosen by some given probability, then the random walk, which takes place on this given fixed environment, is a Markov chain with transition probabilities that depend on the environment. We note that the environment is kept fixed and does not evolve during the random walk, and that the random walk, given an environment, is not necessarily reversible. The questions on RWRE come in two major types: quenched, in which the walk is distributed according to a given typical environment, and annealed, in which the distribution of the walk is taken according to an average on the environments. The two main differences between the quenched and the annealed are: First the quenched is Markovian, while the annealed distribution is usually not. Second, in most of the models we assume some kind of translation invariance on the environments and therefore annealed is usually translation invariance while quenched is not. In contrast to most of the models for RWRE on $\mathbb{Z}^d$, this work deals with non nearest neighbor random walks. The subject of non nearest neighbor random walks has not been systematically studied. For results on long range percolation see [Ber02]. For literature on the subject
in the one dimensional case see [BG08], [Bre02], [CS09]. For some results on bounded non nearest neighbors see [Key84]. For some results that are valid in that general case see [Var04] and [CFP09]. For recurrence and transience criteria for random walks on random point processes, with transition probabilities between every two points proportional to their distance, see [CFG08]. Our model also has the property that the random walk is reversible. For some results in this topic see [BBHK08], [BP07], [MP07] and [SS09].

1.2. The Model.

Let \( \mathbb{Z}^d \) be the \( d \)-dimensional lattice of integers. We define \( \Omega = \{0, 1\}^{\mathbb{Z}^d} \) and \( \mathcal{B} \) the Borel \( \sigma \)-algebra (with respect to the product topology) on \( \Omega \). Let \( Q \) be a probability measure on \( \Omega \). We assume the following about \( Q \):

**Assumption 1.1.**

1. \( Q \) is stationary and ergodic with respect to each of \( \{\theta_{e_i}\}_{i=1}^{d} \), where \( e_i \) is the \( i \)th principal axes and for \( x \in \mathbb{Z}^d \) we define \( \theta_x : \Omega \to \Omega \) as the shift in direction \( x \), i.e for every \( y \in \mathbb{Z}^d \) and every \( \omega \in \Omega \) we have \( \theta_x(\omega)(y) = \omega(x+y) \).

2. \( Q(\mathcal{P}(\omega) = \emptyset) < 1 \), where \( \mathcal{P}(\omega) = \{x \in \mathbb{Z}^d : \omega(x) = 1\} \).

We denote by \( E = \{\pm e_i\}_{i=1}^{d} \) the set of \( 2d \) points in \( \mathbb{Z}^d \) with length 1.

Let \( \Omega_0 = \{\omega \in \Omega : \omega(0) = 1\} \), it follows from assumption 1.1 that \( Q(\Omega_0) > 0 \). We can therefore define the probability \( P \) on \( \Omega_0 \) as the conditional probability on \( \Omega_0 \) of \( Q \), i.e.:

\[
P(B) = Q(B|\Omega_0) = \frac{Q(B \cap \Omega_0)}{Q(\Omega_0)} \quad \forall B \in \mathcal{B}.
\]

(1.1)

We denote by \( \mathbb{E}_Q \) and \( \mathbb{E}_P \) the expectation with respect to \( Q \) and \( P \) respectively.

**Claim 1.2.** Given \( \omega \in \Omega \) and \( v \in \mathcal{P}(\omega) \), for every vector \( e \in E \) there exist \( Q \) almost surely infinitely many \( k \in \mathbb{N} \) such that \( v + ke \in \mathcal{P}(\omega) \).

**Proof.** Given \( \omega, v \) and a vector \( e \) as above, since \( \theta_e \) is measure preserving and ergodic with respect to \( Q \), if we define \( \Omega_v = \{\omega \in \Omega : v \in \mathcal{P}(\omega)\} \) then \( \mathbb{1}_{\Omega_v} \in L^1(\Omega, \mathcal{B}, Q) \), and therefore by Birkhoff’s Ergodic Theorem

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \theta_e^k \mathbb{1}_{\Omega_v} = \mathbb{E}_Q(\mathbb{1}_{\Omega_v}) = Q(\Omega_v) = Q(\Omega_0) > 0 \quad Q \text{ a.s.}
\]

Consequently, there \( Q \) almost surely exist infinitely many integers such that \( \theta_e^k \mathbb{1}_{\Omega_v} = 1 \), and therefore infinitely many \( k \in \mathbb{N} \) such that \( v + ke \in \mathcal{P}(\omega) \). \( \square \)

We define for every \( v \in \mathbb{Z}^d \) the set \( N_v(\omega) \) of the \( 2d \) ”coordinate nearest neighbors” in \( \omega \), one for each direction. By Claim 1.2 \( N_v(\omega) \) is \( Q \) almost surely a set of \( 2d \) points in \( \mathbb{Z}^d \).
We can now define a random walk for $P$-almost every $\omega \in \Omega_0$ (on the space $((\mathbb{Z}^d)^N, \mathcal{G}, P_\omega)$, where $\mathcal{G}$ is the $\sigma$-algebra generated by cylinder functions) as the Markov chain taking values in $\mathcal{P}(\omega)$ with initial condition

$$P_\omega(X_0 = 0) = 1,$$

and transition probability

$$P_\omega(X_{n+1} = u | X_n = v) = \begin{cases} 0 & u \notin N_v(\omega) \\ \frac{1}{2d} & u \in N_v(\omega) \end{cases},$$

which will be called the quenched law of the random walk. We denote the corresponding expectation by $E_\omega$.

Finally, since for each $G \in \mathcal{G}$, the map $\omega \mapsto P_\omega(G)$, is $\mathcal{B}$ measurable, we may define the probability measure $P = P \otimes P_\omega$ on $(\Omega_0 \times (\mathbb{Z}^d)^N, \mathcal{B} \times \mathcal{G})$ by

$$P(B \times G) = \int_B P_\omega(G)P(d\omega), \quad \forall B \in \mathcal{B}, \ \forall G \in \mathcal{G}.$$ 

The marginal of $P$ on $(\mathbb{Z}^d)^N$, denoted by $P$, is called the annealed law of the random walk $\{X_n\}_{n=0}^\infty$. We denote by $E$ the expectation with respect to $P$.

We will need one more definition:

**Definition 1.3.** For every $e \in E^d$ we define $f_e : \Omega \to \mathbb{N}^+$ by

$$f_e(\omega) = \min\{k > 0 : \theta^k_e(\omega)(0) = \omega(ke) = 1\}.$$ 

**Figure 1.1.** An example for nearest coordinate points
In order to prove high dimensional Central Limit Theorem we will assume in addition to assumption 1.1 the following:

**Assumption 1.4.**

(3) There exists $\epsilon_0 > 0$ such that for every coordinate direction $e \in \mathcal{E}$, $E_P(f_e^{2+\epsilon_0}) < \infty$.

### 1.3. Main Results.

Our main goal is to characterize these kind of random walks on random environments. The characterization is given by the following theorems:

1. **Law of Large Numbers** - For $P$ almost every $\omega \in \Omega_0$, the limiting velocity of the random walk exists and equals zero. More precisely:

**Theorem 1.5.** Define the event 

$$A = \left\{ \lim_{n \to \infty} \frac{X_n}{n} = 0 \right\}.$$

Then $\mathbb{P}(A) = 1$.

2. **Recurrence Transience Classification** - We give a partial classification of recurrence transience for the random walk on a discrete point process. The precise statements are:

**Proposition 1.6.** The one dimensional random walk on a discrete point process is $\mathbb{P}$-almost surely recurrent.

**Theorem 1.7.** Let $(\Omega, \mathcal{B}, P)$ be a two dimensional discrete point process and assume there exists a constant $C > 0$ such that

$$\sum_{k=N}^{\infty} \frac{k \cdot P(f_{e_i} = k)}{E(f_{e_i})} \leq \frac{C}{N} \quad \forall i \in \{1, 2\} \quad \forall N \in \mathbb{N}. \quad (1.5)$$

which in particular holds, whenever $f_{e_i}$ has a second moment for $i \in \{1, 2\}$. Then the random walk is $\mathbb{P}$ almost surely recurrent.

**Theorem 1.8.** Let $(\Omega, \mathcal{B}, P)$ be a $d$-dimensional discrete point process with $d \geq 3$ then the random walk is $\mathbb{P}$ almost surely transient.

3. **Central Limit Theorems** - We prove that one-dimensional random walks on discrete point processes satisfy a Central Limit Theorem. We also prove that in dimension $d \geq 2$, under the additional assumption, assumption 1.4, the random walks on a discrete point process satisfy a Central Limit Theorem. The precise statements are:

**Theorem 1.9.** Let $d = 1$ and denote $e = 1$ then for $P$ almost every $\omega \in \Omega_0$

$$\lim_{n \to \infty} \frac{X_n}{\sqrt{n}} \overset{D}{=} N(0, E_P^2(f_e)). \quad (1.6)$$
**Theorem 1.10.** Fix $d \geq 2$. Assume the additional assumption, assumption [I.4], then for $P$ almost every $\omega \in \Omega_0$
\[ \lim_{n \to \infty} \frac{X_n}{\sqrt{n}} \overset{d}{=} N(0, D), \]
(1.7)
where $N(0, D)$ is a $d$-dimensional normal distribution with covariance matrix $D$ that depends only on $d$ and the distribution of $P$.

**Structure of the paper.** Sect. 2 collects some facts about the Markov chain on environments and some ergodic results related to it. This section is based on previously known material. In Sect. 3-4 the one dimensional case, i.e, Law of Large Numbers and Central Limit Theorem, are introduced. The Recurrence Transience classification is discussed in Sec. 5. The novel parts of the high dimensional Central Limit proof - asymptotic behavior of the random walk, construction of the corrector and sublinear bounds on the corrector - appear in Sect. 6-9. The actual proof of the high dimensional Central Limit Theorem is carried out in Sect. 10. Finally Sect. 11 contains further discussion, some open questions and conjectures.

2. The Induced shift And The Environment Seen From The Random Walk

The content of this section is a standard textbook material. The form in which it appears here is taken from [BB07]. Even though it was all known before, [BB07] is the best existing source for our purpose.

Let us define the induced shift on $\Omega_0$ as follows. Let $f_e(\omega)$ be as in definition 1.3. By Claim 1.2 we know that $f_e(\omega) < \infty$ $Q$ almost surely Therefore we can define the maps $\sigma_e : \Omega_0 \to \Omega_0$ by
\[ \sigma_e(\omega) = \theta^{f_e(\omega)} e. \]
We call $\sigma_e$ the induced shift.

**Theorem 2.1.** For every $e \in \mathcal{E}$, the induced shift $\sigma_e : \Omega_0 \to \Omega_0$ is $P$-preserving and ergodic with respect to $P$.

Theorem 2.1 will follow from a more general statement. Let $(\Delta, \mathcal{F}, \mu)$ be a probability space, and let $T : \Delta \to \Delta$ be invertible, measure preserving and ergodic with respect to $\mu$. Let $\mathcal{E}$ be of positive measure, and define $n : A \to \mathbb{N} \cup \{\infty\}$ by
\[ n(x) = \min\{k > 0 : T^k(x) \in A\}. \]

The Poincaré recurrence theorem tells us that $n(x) < \infty$ almost surely. Therefore we can define, up to a set of measure zero, the map $S : A \to A$ by
\[ S(x) = T^{n(x)}(x), \quad x \in A. \]

Then we have:
Lemma 2.2. $S$ is measure preserving and ergodic with respect to $\mu(\cdot|A)$. It is also almost surely invertible with respect to the same measure.

Proof. (1) $S$ is measure preserving: For $j \geq 1$, let $A_j = \{x \in A : n(x) = j\}$. Then the $A_j$'s are disjoint and $\mu(A \setminus \cup_{j \geq 1} A_j) = 0$. First we show that

$$i \neq j \Rightarrow S(A_i) \cap S(A_j) = \emptyset.$$  

To do this, we use the fact that $T$ is invertible. Indeed, if $x \in S(A_i) \cap S(A_j)$ for $1 \leq i < j$, then $x = T^i(y) = T^j(z)$ for some $y,z \in A$ with $n(y) = i$, $n(z) = j$. But the fact that $T$ is invertible implies that $y = T^{j-i}(z)$, which means $n(z) \leq j - i < j$, a contradiction. To see that $S$ is measure preserving, we note that the restriction of $S$ to $A_j$ is $T^j$, which is measure preserving. Hence, $S$ is measure preserving on $A_j$ and, since the sets $A_j$ are disjoint, $S$ is measure preserving on the union $\cup_{j \geq 1} A_j$ as well.

(2) $S$ is almost surely invertible: $S^{-1}(\{x\}) \cap \{S \text{ is well defined}\}$ is a one-point set by the fact that $T$ is itself invertible.

(3) $S$ is ergodic: Let $B \in \mathcal{C}$ be such that $B \subset A$ and $0 < \mu(B) < \mu(A)$. Assume that $B$ is $S$-invariant. Then $S^n(x) \notin A \setminus B$ for all $x \in B$ and all $n \geq 1$. This means that for every $x \in B$ and every $k \geq 1$ such that $T^k(x) \in A$, we have $T^k(x) \notin A \setminus B$. It follows that $C = \cup_{k \geq 1} T^k(B)$ is (almost surely) $T$-invariant and $\mu(C) \in (0,1)$, contradicting the ergodicity of $T$.

□

Proof of Theorem (2.1). We know that the shift $\theta_e$ is invertible, measure preserving and ergodic with respect to $Q$. By Lemma (2.2), the induced shift is $P$-preserving, almost surely invertible and ergodic with respect to $P$. □

Under the present circumstances, Theorem 2.1 has one important corollary:

Lemma 2.3. Let $B \in \mathcal{B}$ be a subset of $\Omega_0$ such that for almost every $\omega \in B$

$$P_{\omega}(\theta_{X_1} \omega \in B) = 1.\quad (2.1)$$

Then $B$ is a zero-one event under $P$.

Proof. The Markov property and (2.1) imply that $P_{\omega}(\theta_{X_n} \omega \in B) = 1$ for all $n \geq 1$ and that $P$-almost every $\omega \in B$. We claim that $\sigma_e(\omega) \in B$ for $P$-almost surely $\omega \in B$. Indeed, let $\omega \in B$ be such that $\theta_{X_n} \omega \in B$ for all $n \geq 1$, $P_\omega$-almost surely. note that we have $f_e(\omega) e \in \mathcal{P}(\omega)$. Therefore we have $P_\omega(X_1 = f_e(\omega) e) = \frac{1}{2d} > 0$. This means that $\sigma_e(\omega) = \theta_{X_1}^{\omega}(\omega) \in B$, i.e., $B$ is almost surely $\sigma_e$-invariant. By the ergodicity of the induced shift, $B$ is a zero-one event. □

Our next goal will be to prove that the Markov chain on environments is ergodic. Let $\Xi = \Omega_0^\mathbb{Z}$ and define $\mathcal{H}$ to be the product $\sigma$-algebra on $\Xi$. The space $\Xi$ is a space of two-sided
Hence, in order to prove that \( \mu \) with equalities valid 

\[
\mu((\omega_n, \ldots, \omega_1) \in B) = \int_B P(d\omega_n)\Lambda(\omega_n, d\omega_{n-1}) \ldots \Lambda(\omega_1, d\omega_n),
\]

where \( \Lambda : \Omega_0 \times \mathcal{B} \to [0,1] \) is the Markov kernel defined by

\[
\Lambda(\omega, A) = \frac{1}{2d} \sum_{x \in \mathbb{Z}^d} (\mathbb{1}_{\{x \in N_0(\omega)\}} \mathbb{1}_{\{\theta^x \omega \in A\}}).
\]  

(2.2)

Note that the sum is finite since for almost every \( \omega \in \Omega \) there are exactly \( 2d \) elements in \( N_0(\omega) \). \( \mu \) exists and is unique by Kolmogorov’s Theorem, because \( P \) is preserved by \( \Lambda \), and therefore the finite dimensional measures are consistent. \( \{\theta_X(\omega)\}_{k \geq 0} \) has the same law in \( E(P(\omega(\cdot))) \) as \( (\omega_0, \omega_1, \ldots) \) has in \( \mu \). Let \( \tilde{T} : \Xi \to \Xi \) be the shift defined by \( (\tilde{T}\omega)_n = \omega_{n+1} \). Then \( \tilde{T} \) is measure preserving.

**Proposition 2.4.** \( \tilde{T} \) is ergodic with respect to \( \mu \).

**Proof.** Let \( E_\mu \) denote expectation with respect to \( \mu \). Pick \( A \subset \Xi \) that is measurable and \( \tilde{T} \)-invariant. We need to show that \( \mu(A) \in \{0,1\} \).

Let \( f : \Omega_0 \to \mathbb{R} \) be defined as \( f(\omega_0) = E_\mu(1_A|\omega_0) \). First we claim that \( f = 1_A \) almost surely. Indeed, since \( A \) is \( \tilde{T} \)-invariant, there exist \( A_+ \in \sigma(\omega_k : k > 0) \) and \( A_- \in \sigma(\omega_k : k < 0) \) such that \( A \) and \( A_\pm \) differ only by null sets from one another (This follows by approximation of \( A \) by finite-dimensional events and using the \( \tilde{T} \)-invariance of \( A \)). Now, conditional on \( \omega_0 \), the event \( A_+ \) is independent of \( \sigma(\omega_k : k < 0) \) and so Lévy’s Martingale Convergence Theorem gives us

\[
E_\mu(1_A|\omega_0) = E_\mu(1_A_+|\omega_0, \omega_1, \ldots, \omega_n)
\]

\[
= E_\mu(1_{A_-}|\omega_0, \ldots, \omega_n) \xrightarrow{n \to \infty} 1_{A_-} = 1_A,
\]

with equalities valid \( \mu \)-almost surely. Next let \( B \subset \Omega_0 \) be defined by \( B = \{\omega_0 : f(\omega_0) = 1\} \). Clearly \( B \) is \( \mathcal{B} \)-measurable and, since the \( \omega_0 \)-marginal of \( \mu \) is \( P \),

\[
\mu(A) = E_\mu(f) = P(B)
\]

Hence, in order to prove that \( \mu(A) \in \{0,1\} \), we need to show that \( P(B) \in \{0,1\} \). But \( A \) is \( \tilde{T} \)-invariant and so, up to set of measure zero, if \( \omega_0 \in B \) then \( \omega_1 \in B \). This means that \( B \) satisfies the condition of the lemma [2.3] and so \( B \) is a zero-one event. \( \square \)

**Theorem 2.5.** Let \( f \in L^1(\Omega_0, \mathcal{B}, P) \). Then for \( P \)-almost all \( \omega \in \Omega_0 \)

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} f \circ \theta_X(\omega) = E_P(f) \quad P_\omega \text{ almost surely.}
\]
Similarly, if \( f : \Omega \times \Omega \to \mathbb{R} \) is measurable with \( \mathbb{E}_P(E_\omega(f(\omega, \theta X_1, \omega))) < \infty \), then

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} f(\theta X_k \omega, \theta X_{k+1} \omega) = \mathbb{E}_P(E_\omega(f(\omega, \theta X_1, \omega))).
\]

for \( P \)-almost all \( \omega \) and \( P_\omega \)-almost all trajectories of \((X_k)_{k \geq 0}\).

**Proof.** Recall that \( \{\theta X_k(\omega)\}_{k \geq 0} \) has the same law in \( \mathbb{E}_P(P_\omega(\cdot)) \) as \((\omega_0, \omega_1, \ldots)\) has in \( \mu \). Hence, if \( g(\ldots, \omega_{-1}, \omega_0, \omega_1, \ldots) = f(\omega_0) \) then

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} f \circ \theta X_k \overset{D}{=} \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} g \circ \tilde{T}^k.
\]

The latter limit exists by Birkhoff’s Ergodic Theorem (we have already seen that \( \tilde{T} \) is ergodic) and equals \( E_\mu(g) = \mathbb{E}_P(f) \) almost surely. The second part is proved analogously. \( \square \)

### 3. Law of Large Numbers

We turn now to prove Theorem 1.5 - i.e. Law of Large Numbers for random walks on a discrete point process. For completeness we state the theorem again:

**Theorem.** 1.5 Define the event

\[
A = \left\{ \lim_{n \to \infty} \frac{X_n}{n} = 0 \right\}.
\]

Then \( \mathbb{P}(A) = 1 \).

**Proof.** Using linearity, it is enough to prove that for every \( e \in \mathcal{E} \) we have \( \mathbb{P}(A_e) = 1 \), where

\[
A_e = \left\{ \lim_{n \to \infty} \frac{X_n \cdot e}{n} = 0 \right\}.
\]

For every \( e \in \mathcal{E} \) let \( f_e \) be as in Definition 1.3. By (1.2) \( f_e \) is \( P \)-a.s finite. We first prove that \( \mathbb{E}_P(f_e) < \infty \). Assume for contradiction that \( \mathbb{E}_P(f_e) = \infty \), since \( f_e \) is positive then

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} f_e(\sigma^k_\omega) = \infty \quad P \text{ a.s.} \tag{3.1}
\]

Indeed, for every \( M > 0 \) define

\[
f_e^M(\omega) = \begin{cases} f_e(\omega) & f_e(\omega) \leq M \\ M & f_e(\omega) > M \end{cases},
\]

then, since \( f_e^M \) is finite by Birkhoff Ergodic Theorem

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} f_e(\sigma^k_\omega) \geq \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} f_e^M(\sigma^k_\omega) = \mathbb{E}_P(f_e^M), \quad P \text{ a.s.}
\]
Taking now M to infinity we get
\[ \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} f_e(\sigma_e^k(\omega)) \geq \lim_{M \to \infty} \mathbb{E}_P(f_e^M) = \mathbb{E}_P(f_e) = \infty, \quad P \text{ a.s.} \]

Let \( S(k) = \max \{ n \geq 0 : \sum_{m=0}^{n-1} f(\sigma_e^m(\omega)) < k \} \), by (3.1)
\[ \lim_{k \to \infty} \frac{S(k)}{k} = 0 \quad P \text{ a.s.} \]

On the other hand, let \( g : \Omega \to \{0, 1\} \) be defined by
\[ g(\omega) = 1_{\Omega_0}(\omega), \]
then
\[ S(k) = \sum_{j=0}^{k-1} g(\theta_e^j(\omega)), \]
and therefore by Birkhoff’s Ergodic Theorem
\[ Q(\Omega_0) = \lim_{k \to \infty} \frac{1}{k} \sum_{j=0}^{k-1} g(\theta_e^j(\omega)) = \lim_{k \to \infty} \frac{S(k)}{k} = 0, \quad P \text{ a.s.} \]

contradicting assumption [3.2]. It follow that \( \mathbb{E}_P(f_e) < \infty \), and therefore by Birkhoff Ergodic Theorem
\[ \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} f_e(\sigma_e^k(\omega)) = \mathbb{E}_P(f_e) < \infty \quad P \text{ a.s.} \quad (3.2) \]

Notice that
\[ P(f^{-e}(\omega) = k) = P(\sigma_e^{-1}(\omega) = k) = P(f_e(\omega) = k), \quad (3.3) \]
where the last equality is true since \( P \) is stationary. It therefore follows that
\[ \mathbb{E}_P(f_e) = \mathbb{E}_P(f^{-e}) \quad (3.4) \]

For \( e \in \mathcal{E} \) let \( g_e : \Omega \times \Omega \to \mathbb{Z} \) be as follows:
\[ g_e(\omega, \omega') = \begin{cases} f_e(\omega) & \omega' = \sigma_e(\omega) \\ -f^{-e}(\omega) & \omega' = \sigma^{-e}(\omega) \\ 0 & \text{otherwise} \end{cases} \]

Now, \( g_e \) is measurable and using (3.4) we get
\[ \mathbb{E}_P(E_{\omega}(g_e(\omega, \theta^{\chi_1}(\omega)))) = \mathbb{E}_P \left( \frac{1}{2d} f_e(\omega) - \frac{1}{2d} f^{-e}(\omega) \right) = 0. \]
It therefore follows that for every \( e \in \mathcal{E} \), for almost every \( \omega \in \Omega_0 \) and \( P_\omega \) almost every random walk \( \{X_k\}_{k \geq 0} \), we have for \( Z_k = X_k - X_{k-1}, \ k \geq 1 \) that
\[
\lim_{n \to \infty} \frac{X_n \cdot e}{n} = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} Z_k \cdot e,
\]
and from (2.5) this equals to
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} g_e(\theta X_k \omega, \theta X_{k+1} \omega) = \mathbb{E}_P(E_\omega(g_e(\omega, \theta X_1 \omega))) = 0, \ \mathbb{P} \text{ a.s.}
\]
□

4. ONE DIMENSIONAL CENTRAL LIMIT THEOREM

Here we prove Theorem 1.9 - i.e. Central Limit Theorem for one dimensional random walks on discrete point processes. We start by stating the theorem

**Theorem.** \( \text{1.9} \) \( d = 1 \) and denote \( e = 1 \) then for \( P \) almost every \( \omega \in \Omega_0 \)
\[
\lim_{n \to \infty} \frac{X_n}{\sqrt{n}} \overset{D}{=} N(0, \mathbb{E}^2_P(f_e)). \quad (4.1)
\]

**Proof.** We first notice that for \( d = 1 \), a random walk on a discrete point process is almost surely a simple one dimensional random walk with changed distances between points. Secondly the expectation of the distance between points, given by \( \mathbb{E}_P(f_e) \), is finite.

Given an environment \( \omega \in \Omega_0 \) and a random walk \( \{X_k\}_{k \geq 0} \), we define the simple one-dimensional random walk \( \{Y_k\}_{k \geq 0} \) associated with \( \{X_k\}_{k \geq 0} \) as follows: First we define \( Z_k = X_k - X_{k-1} \) for every \( k \geq 1 \), then we define \( W_k = \frac{Z_k}{|Z_k|} \). Finally we define \( Y_0 = 0 \) and for \( k \geq 1 \) we define \( Y_k = \sum_{j=1}^{k} W_k \). Since \( \{Y_k\}_{k \geq 0} \) is a simple one dimensional random walk on \( \mathbb{Z} \), it follows from the Central Limit Theorem that for \( P \) almost every \( \omega \in \Omega_0 \)
\[
\lim_{n \to \infty} \frac{1}{\sqrt{n}} \cdot Y_n \overset{D}{=} N(0, 1). \quad (4.2)
\]

We now turn to define for every \( \omega \in \Omega \) the points of the environment. For every \( n \in \mathbb{Z} \) let \( t_n \) be the \( n^{th} \) place on the grid with a point, i.e. \( t_0 = 0 \),
\[
t_n = \sum_{k=0}^{n-1} f_e(\sigma^k_e \omega) \quad n > 0,
\]
and
\[
t_n = \sum_{k=-n}^{-1} f_e(\sigma^k_e \omega) \quad n < 0.
\]
For every $a > 0$ we have
\[
\lim_{n \to \infty} \frac{1}{\sqrt{n}} t_{\lfloor a \sqrt{n} \rfloor} = a \cdot \lim_{n \to \infty} \frac{1}{a \sqrt{n}} \sum_{k=0}^{\lfloor a \sqrt{n} \rfloor} f_e(\sigma^k \omega) = a \cdot \mathbb{E}_P(f_e),
\]
where the last equality holds since this sequence contain the same elements as the sequence in (3.2) and every element in the original sequence appears only a finite number of times, therefore those sequences have the same partial limits, and the original sequence (the one in (3.2)) converges.

By the same argument for every $a \in \mathbb{R}$ we have that
\[
\lim_{n \to \infty} \frac{1}{\sqrt{n}} t_{\lfloor a \sqrt{n} \rfloor} = a \cdot \mathbb{E}_P(f_e). \tag{4.3}
\]

Using (4.3) and the fact that $\lim_{n \to \infty} \frac{Y_n}{\sqrt{n}}$ exists and finite $\mathbb{P}$ almost surely, we get that
\[
\lim_{n \to \infty} \frac{1}{\sqrt{n}} t_{Y_n} = \lim_{n \to \infty} \frac{1}{\sqrt{n}} t_{\frac{Y_n}{\sqrt{n}}} \leq a \iff \lim_{n \to \infty} \frac{Y_n}{\sqrt{n}} \leq \frac{a}{\mathbb{E}_P(f_e)},
\]
and therefore
\[
\mathbb{P} \left( \lim_{n \to \infty} \frac{1}{\sqrt{n}} t_{Y_n} \leq a \right) = \mathbb{P} \left( \lim_{n \to \infty} \frac{Y_n}{\sqrt{n}} \leq \frac{a}{\mathbb{E}_P(f_e)} \right) = \Phi \left( \frac{a}{\mathbb{E}_P(f_e)} \right),
\]
where $\Phi$ is the standard normal cumulative distribution function. Finally, we notice that
\[
X_n = t_{Y_n},
\]
and therefore we conclude that
\[
\mathbb{P} \left( \lim_{n \to \infty} \frac{X_n}{\sqrt{n}} \leq a \right) = \Phi \left( \frac{a}{\mathbb{E}_P(f_e)} \right),
\]
as required. \qed

5. Transience and Recurrence

Before we continue the discussion on Central Limit Theorem in higher dimensions, we turn to deal with transience and recurrence of random walks on discrete point processes.

5.1. One-dimensional case.

**Proposition 1.6** The one dimensional random walk on a discrete point process is $\mathbb{P}$-almost surely recurrent.

**Proof of Proposition 1.6** Using the notation from the previous section, since $Y_n$ is a one-dimensional simple random walk, it is recurrent $\mathbb{P}$ almost surely. Therefore we have $\# \{ n : Y_n = 0 \} = \infty$ $\mathbb{P}$ almost surely, but since $X_n = t_{Y_n}$ and $t_0 = 0$ we have $\# \{ n : X_n = 0 \} = \infty$ $\mathbb{P}$ almost surely, and therefore the random walk is recurrent. \qed
5.2. Two-dimensional case. The theorem we wish to prove is the following:

**Theorem.** Let \((\Omega, \mathcal{B}, P)\) be a two dimensional discrete point process and assume there exists a constant \(C > 0\) such that

\[
\sum_{k=N}^{\infty} k \cdot P(f_{e_i} = k) \leq \frac{C}{N}, \quad \forall i \in \{1, 2\}, \forall N \in \mathbb{N}, \tag{5.1}
\]

which in particular holds, whenever \(f_{e_i}\) has a second moment for \(i \in \{1, 2\}\). Then the random walk is \(P\) almost surely recurrent.

The proof is based on the connection between random walks, electrical networks and the Nash-William criteria for recurrence of random walks. For a proof of the Nash-William criteria and some background on the subject see [DS84] and [LP04].

We start with the following definition:

**Definition 5.1.** Let \((\tilde{\Omega}, \tilde{\mathcal{B}}, \tilde{P})\) be a probability space. We say that a random variable \(X : \tilde{\Omega} \to [0, \infty)\) has a Cauchy tail if there exist a positive constant \(C\) such that for every \(n \in \mathbb{N}\) we have

\[
\tilde{P}(X \geq n) \leq \frac{C}{n}.
\]

Note that if \(\tilde{E}(X) < \infty\), then \(X\) has a Cauchy tail.

In order to prove theorem 5.2 we will need the following lemmas taken from [Ber02].

**Lemma 5.2 (Ber02 Lemma 4.1).** Let \(\{f_i\}_{i=1}^{\infty}\) be identically distributed positive random variables, on a probability space \((\tilde{\Omega}, \tilde{\mathcal{B}}, \tilde{P})\), that have a Cauchy tail. Then, for every \(\epsilon > 0\), there exist \(K > 0\) and \(N \in \mathbb{N}\) such that for every \(n > N\)

\[
\tilde{P}\left(\frac{1}{n} \sum_{k=0}^{n} f_i > K \log n\right) < \epsilon.
\]

**Proof.** \(f_i\) has a Cauchy tail, so there exists \(C_0\) such that for every \(n \in \mathbb{N}\)

\[
\tilde{P}(f_i > n) < \frac{C_0}{n},
\]

Let \(M > \frac{\epsilon}{2}\) be a large number, and \(N\) large enough that \(C_0 N^{1-M} < \frac{\epsilon}{2}\). Fix \(n > N\), and let \(g_i = \min\{f_i, n^M\}\) for all \(1 \leq i \leq n\). Then,

\[
\tilde{P}\left(\frac{1}{n} \sum_{i=1}^{n} f_i \neq \frac{1}{n} \sum_{i=1}^{n} g_i\right) \leq \sum_{k=1}^{n} \tilde{P}(f_k \neq g_k) = n \cdot \tilde{P}(f_1 \neq g_1).
\]

The last term is equal to

\[
n \cdot \tilde{P}(f_1 > n^M) < \frac{n \cdot C_0}{n^M} < \frac{\epsilon}{2}.
\]
Now, since $E(g_i) \leq C_0 M \log n$, and $g_i$ is positive, by Markov’s inequality, choosing $K = C_0 M^2$ we get

$$\tilde{P}\left(\frac{1}{n} \sum_{i=1}^{n} g_i > K \log n\right) < C_0 M \log n \quad \frac{1}{C_0 M^2 \log n} = \frac{1}{M} < \frac{\epsilon}{2}$$

and so

$$\tilde{P}\left(\frac{1}{n} \sum_{i=1}^{n} f_i > K \log n\right) < \epsilon$$

\[\square\]

**Lemma 5.3** ([Ber02] Lemma 4.2). Let $A_n$ be a sequence of events such that $\tilde{P}(A_n) > 1 - \epsilon$ for all sufficiently large $n$, and let $\{a_n\}_{n=1}^{\infty}$ be a sequence such that

$$\sum_{n=1}^{\infty} a_n = \infty.$$

Then, with probability of at least $1 - \epsilon$

$$\sum_{n=1}^{\infty} \mathbb{1}_{A_n} \cdot a_n = \infty.$$

**Proof.** It is enough to show that there exists $N$ such that for any $M$,

$$P\left(\sum_{n=N}^{\infty} \mathbb{1}_{A_n} \cdot a_n < M\right) \leq \epsilon. \quad (5.2)$$

Define $N$ such that for every $n > N$ we have $P(A_n) > 1 - \epsilon$, and assume that for some $M$ (5.2) is false. Define $B_M$ to be the event

$$B_M = \left\{ \sum_{n=N}^{\infty} \mathbb{1}_{A_n} \cdot a_n < M \right\}.$$

Since $P(B_M) > \epsilon$, we know that there exist $\delta > 0$ such that for every $n$

$$P(A_n|B_M) = \frac{P(A_n \cap B_M)}{P(B_M)} \geq \frac{P(B_M) - \epsilon}{P(B_M)} > \delta > 0.$$

Therefore,

$$E\left[ \sum_{n=N}^{\infty} \mathbb{1}_{A_n} \cdot a_n \bigg| B_M \right] \geq \delta \sum_{n=N}^{\infty} a_n = \infty,$$

which contradicts the definition of $B_M$. \[\square\]
In the proof of Theorem 1.7, we will use the following notation: Given a graph $G = (V, E)$ with $V \subset \mathbb{Z}^d$, for every $e \in E$ define $e^+ \in V$ and $e^- \in V$ to be the end points of $e$, such that if $(e^+ - e^-) \cdot e_i \neq 0$ then $(e^+ - e^-) \cdot e_i > 0$. In addition for every $e \in E$ we write $l(e) = |e^+ - e^-|_1$.

Proof of theorem 1.7. For every $\omega \in \Omega$, we define the corresponding network with conductances $G(\omega) = (V(\omega), E(\omega), c(\omega))$ as follows: First let $G''(\omega) = (V''(\omega), E''(\omega), c''(\omega))$ be the network with $V''(\omega) = \mathcal{P}(\omega)$ and $E''(\omega) = \{\{x, y\} \in V'' \times V'': y \in \{x \pm f_{e_1}(\omega)e_1, x \pm f_{e_2}(\omega)e_2\}\}$, i.e. the set of edges from each point to its four “nearest neighbors”. We also define the conductance $c''(\omega)(e) = 1$ for every $e \in E''(\omega)$. We now define $G'(\omega)$ to be the network generated from $G''(\omega)$ by ”cutting” every edge of length $k$ into $k$ edges of length 1, each cut with conductance $k$. Formally we define $V'(\omega) = V^1(\omega) \cup V^2(\omega) \subset \mathbb{Z}^2 \times \{0, 1\}$ where

$$V^i(\omega) = \left\{ (x, i) : \exists e \in E''(\omega) \exists 0 \leq k \leq l(e) \text{ such that } (e^+ - e^-) \cdot e_i \neq 0 \land x = e^- + ke_i \right\},$$

and we define $E'(\omega) = E^1(\omega) \cup E^2(\omega)$ by

$$E^i(\omega) = \left\{ (v, i), (w, i) : \exists e \in E''(\omega) \exists 0 \leq k < l(e) \text{ such that } (e^+ - e^-) \cdot e_i \neq 0 \land v = e^- + ke_i, w = e^- + (k + 1)e_i \right\}.$$

We also define the conductance $c'(\omega)(e)$ of an edge $e \in E'(\omega)$ to be $k$, given that the length of the original edge it was part of was $k$. Finally we define $G(\omega)$ to be the graph generated from $G'(\omega)$ by identifying every $v \in V''(\omega)$ on both levels i.e. we take the graph $G'(\omega)$ modulo the equivalence relations $(v, 1) = (v, 2) \quad \forall v \in V''(\omega)$. We now turn to prove the recurrence using the Nash-Williams Criteria. Let $\Pi_n$ be the set of edges exiting the box $([-n, n] \times [-n, n], [1, 2])$ in the graph $G(\omega)$. Then $\Pi_n$ defines a sequence of pairwise disjoint cutsets in the network $G(\omega)$. Let $e \in \Pi_n$ be such that $(e^+ - e^-) \cdot e_i \neq 0$ then

$$\mathbb{P}(c(e) = k) = \mathbb{P}\left( \text{the original edge that contained } e \text{ is of length } k \right) = \frac{k \cdot \mathbb{P}(f_{e_i} = k)}{\mathbb{E}(f_{e_i})}.$$

Indeed, the probability that the edge $e$ was part of an edge of length $k$ in the original graph, needs to be multiplied by $k$, since it can be in any part of the edge. From assumption (5.1) it follows that $c(e)$ has a Cauchy tail. In $\Pi_n$ there are $2n + 4$ edges in the first level and $2n + 4$ in the second level, all of them with the same distribution (and by (5.1) a Cauchy tail), though they may be dependent. By Lemma 5.2, for every $\epsilon > 0$ there exist $K > 0$ and $N \in \mathbb{N}$ such that for every $n > N$, we have

$$\mathbb{P}\left( \sum_{e \in \Pi_n} C(e) \leq K(4n + 8) \log 4n + 8 \right) > 1 - \epsilon. \quad (5.3)$$
Define $A_n$ to be the event in equation (5.3), and set $a_n = (K(4n + 8) \log(4n + 8))^{-1}$ for $n \geq N$. Now,

$$\sum_{n=1}^{\infty} C_{\Pi_n}^{-1} \geq \sum_{n=N}^{\infty} 1_{A_n} \cdot a_n.$$ 

By the definition of $\{a_n\}$,

$$\sum_{n=N}^{\infty} a_n = \infty.$$ 

On the other hand, $\mathbb{P}(A_n) > 1 - \epsilon$ for all $n$. So by Lemma 5.3,

$$\mathbb{P} \left( \sum_{n=1}^{\infty} C_{\Pi_n}^{-1} = \infty \right) \geq 1 - \epsilon.$$ 

Since $\epsilon$ is arbitrary, we get that $\mathbb{P}$ a.s.

$$\sum_{n=1}^{\infty} C_{\Pi_n}^{-1} = \infty.$$ 

Therefore by the Nash-Williams criteria, the random walk is $\mathbb{P}$ almost surely recurrent on $G(\omega)$.

\[ \Box \]

5.3. Higher dimensions $(d \geq 3)$.

We start by stating the theorem:

**Theorem.** Let $(\Omega, \mathcal{B}, \mathbb{P})$ be a $d$-dimensional discrete point process $d \geq 3$ then the random walk is $\mathbb{P}$ almost surely transient.

The main idea beyond the proof is as follows: first we show that the boundary of every set of volume $n$ in $\mathbb{Z}^d$ is at least a positive constant times $n^{d-1}$, then we will use the known fact that for every graph $G = (V, E)$ with bounded degree, such that for every set of vertices of volume $n$, the boundary is at least a constant times $n^\alpha$, with $\alpha > \frac{1}{2}$, a simple random walk on $G$ is transient.

We start by proving an isoperimetric inequality.

**Lemma 5.4.** Let $A = \{x^i = (x^i_1, x^i_2, \ldots, x^i_d)\}_{i=1}^n$ be a finite subset of $\mathbb{Z}^d$. We define $\Pi^j : \mathbb{Z}^d \to \mathbb{Z}^{d-1}$ to be the projection on all but the $j^{th}$ coordinate, i.e, $\Pi^j(x) = \Pi^j((x_1, x_2, \ldots, x_d)) = (x_1, x_2, \ldots, x_{j-1}, x_{j+1}, \ldots, x_d)$. Define $A_j = \Pi^j(A)$. Then there exists $C > 0$ such that

$$\max_{1 \leq j \leq d} |A_j| \geq C \cdot |A|^{\frac{d-1}{d}},$$

(5.4)

where $|\cdot|$ denotes the cardinality of the set.
Proof. Using translation, we can assume without loss of generality that \(x_i^j > 0\) for every \(1 \leq i \leq n\) and \(1 \leq j \leq d\). For every point \(x\) in the quadrat, where all coordinates are positive, we define the energy of a point \(E(x)\) by

\[
E(x) = x \cdot (1, 1, \ldots, 1) = \sum_{j=1}^{d} x_j. \tag{5.5}
\]

In addition we define the energy of a finite set \(A\) in this quadrat as

\[
E(A) = \sum_{x \in A} E(x). \tag{5.6}
\]

For each point \((x_2, x_3, \ldots, x_d)\) in \(\mathbb{Z}^{d-1}\) with positive entries we define the set \(A_{(x_2, x_3, \ldots, x_d)} = \{x_1 : (x_1, x_2, \ldots, x_d) \in A\}\), which we will call the \((x_2, x_3, \ldots, x_d)\) fiber of \(A\). We now define a new set \(A^1\), with the following property: For each point \((x_2, x_3, \ldots, x_d)\) in \(\mathbb{Z}^{d-1}\) the \((x_2, x_3, \ldots, x_d)\) fiber of \(A\) as the same size as the \((x_2, x_3, \ldots, x_d)\) fiber of \(A^1\), and in addition the \((x_2, x_3, \ldots, x_d)\) fiber of \(A^1\) is the one with least energy (when thought as a set in \(\mathbb{Z}\)). We claim that the following set fulfills this property:

\[
A^1 = \bigcup_{x_2 \in \mathbb{N}} \bigcup_{x_3 \in \mathbb{N}} \ldots \bigcup_{x_d \in \mathbb{N}} \{(a, x_2, x_3, \ldots, x_d) : a \in \mathbb{N} \land 1 \leq a \leq |A_{(x_2, x_3, \ldots, x_d)}|\}. \tag{5.7}
\]

Indeed, the \((x_2, x_3, \ldots, x_d)\) fiber of \(A^1\) is \(\{(a, x_2, x_3, \ldots, x_d) : a \in \mathbb{N} \land 1 \leq a \leq |A_{(x_2, x_3, \ldots, x_d)}|\}\) which has the same size as the \((x_2, x_3, \ldots, x_d)\) fiber of \(A\). In addition, for any fixed \(m \in \mathbb{N}\), the unique set \(B \subset \mathbb{N}\) of size \(m\) and minimal energy is \(B = \{1, 2, \ldots, m\}\). Therefore the set \(A^1\) has the following properties:

1. \(|A^1| = n\).
2. \(|\Pi^j(A^1)| \leq |\Pi^j(A)|\) for every \(1 \leq j \leq d\).
3. \(E(A^1) \leq E(A)\), and equality holds if and only if \(A^1 = A\).

Indeed,

1. This follows from the fact that the size of the fibers don’t change in the process, and that the fibers are disjoint.

\[
|A| = \sum_{x_2 \in \mathbb{N}} \sum_{x_3 \in \mathbb{N}} \ldots \sum_{x_d \in \mathbb{N}} |A_{(x_2, x_3, \ldots, x_d)}| = \sum_{x_2 \in \mathbb{N}} \sum_{x_3 \in \mathbb{N}} \ldots \sum_{x_d \in \mathbb{N}} |A^1_{(x_2, x_3, \ldots, x_d)}| = |A^1|.
\]

2. For \(j = 1\) this is true since

\[(x_2, x_3, \ldots, x_d) \in \Pi^1(A) \iff \exists a \in \mathbb{N} \text{ such that } (a, x_2, x_3, \ldots, x_d) \in A \iff \exists b \in \mathbb{N} \text{ such that } (b, x_2, x_3, \ldots, x_d) \in A^1 \iff (x_2, x_3, \ldots, x_d) \in \Pi^1(A^1),\]
and therefore $|\Pi^1(A^1)| = |\Pi^1(A)|$. For $2 \leq j \leq d$, we assume for contradiction that, $|\Pi^j(A^1)| > |\Pi^j(A)|$. Then there exist $(x_1, x_2, \ldots, x_{j-1}, x_{j+1}, \ldots, x_d) \in \Pi^j(A^1)$ such that

$$|\Pi^j(A^1)(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_d)| > |\Pi^j(A)(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_d)|.$$  

From the definition of $A^1$ there exists $m \in \mathbb{N}$ such that

$$|\Pi^j(A^1)(x_2, \ldots, x_{j-1}, x_{j+1}, \ldots, x_d)| = |A^1_{(x_2, \ldots, x_{j-1}, x_{j+1}, \ldots, x_d)}|,$$

and since for every $k \in \mathbb{N}$

$$|A_{(x_2, \ldots, x_{j-1}, k, x_{j+1}, \ldots, x_d)}| \leq |\Pi^j(A)(x_2, \ldots, x_{j-1}, x_{j+1}, \ldots, x_d)|.$$  

It follows that

$$|A^1_{(x_2, \ldots, x_{j-1}, m, x_{j+1}, \ldots, x_d)}| > |A_{(x_2, \ldots, x_{j-1}, m, x_{j+1}, \ldots, x_d)}|,$$

which contradicts the fact that the size of fibers in $A$ and $A^1$ is the same.

(3) By definition

$$\mathcal{E}(A^1) = \sum_{x_2 \in \mathbb{N}} \sum_{x_3 \in \mathbb{N}} \cdots \sum_{x_d \in \mathbb{N}} \sum_{x_1 \in A^1_{(x_2, \ldots, x_d)}} (x_1 + x_2 + \ldots + x_d)$$

$$= \sum_{x_2 \in \mathbb{N}} \sum_{x_3 \in \mathbb{N}} \cdots \sum_{x_d \in \mathbb{N}} \left[ |A^1_{(x_2, \ldots, x_d)}| (x_2 + x_3 + \ldots + x_d) + \mathcal{E}(A^1_{(x_2, x_3, \ldots, x_d)}) \right]$$

$$= \sum_{x_2 \in \mathbb{N}} \sum_{x_3 \in \mathbb{N}} \cdots \sum_{x_d \in \mathbb{N}} \left[ |A_{(x_2, x_3, \ldots, x_d)}| (x_2 + x_3 + \ldots + x_d) + \mathcal{E}(A_{(x_2, x_3, \ldots, x_d)}) \right]$$

$$\leq \sum_{x_2 \in \mathbb{N}} \sum_{x_3 \in \mathbb{N}} \cdots \sum_{x_d \in \mathbb{N}} \left[ |A_{(x_2, x_3, \ldots, x_d)}| (x_2 + x_3 + \ldots + x_d) + \mathcal{E}(A_{(x_2, x_3, \ldots, x_d)}) \right]$$

$$= \mathcal{E}(A)$$

where the inequality is true since the energy of the $(x_2, x_3, \ldots, x_d)$ fiber of $A^1$ is the one with least energy from all $(x_2, x_3, \ldots, x_d)$ fibers of $A$. In addition equality holds if and only if for every $(x_2, x_3, \ldots, x_d)$ fiber of $A$ we have $\mathcal{E}(A_{(x_2, x_3, \ldots, x_d)}) = \mathcal{E}(A^1_{(x_2, x_3, \ldots, x_d)})$ which is possible if and only if $A_{(x_2, x_3, \ldots, x_d)} = A^1_{(x_2, x_3, \ldots, x_d)}$, since $A^1$ fibers were chosen to be with minimal energy.

Repeating the last procedure for the set $A^i$ with the $i+1^{\text{th}}$ coordinate instead of the first one we obtain the sets $A^2, \ldots, A^d$, with the same number of point, decreasing energy and decreasing size of projections. Let $\tilde{A}_0 \equiv A$, and define by induction $\tilde{A}_{n+1} = \tilde{A}_n^d$ be the set generated from $\tilde{A}_n$ by repeating the last procedure. It follows that sequence of sets $\{\tilde{A}_n\}_{n=0}^\infty$ contains only finite number of sets. Indeed since the energy of a set is a natural number, and the energy can only decrease as $n$ increases, there exist $N$ such that for every $n \geq N$ the energy is constant. Using now property (3) it follows that $\tilde{A}_n = \tilde{A}_{n+1}$ for every $n \geq N$ and therefore there is only finite number of sets in the sequence. Let $\tilde{A}$ be the limiting set
of the sequence. Note that the boundary of $\hat{A}$ is exactly $2 \sum_{i=1}^{d} \Pi^i(\hat{A})$, because otherwise one can decrease the energy. Using the fact that the boundary of every set of size $n$ in $d$ dimensions is at least $C_0 \cdot n^{d-1}$ for some positive constant $C_0$, see [DP96], we get that there exist a positive constant $C$ and at least one $i_0 \in \{1, 2, \ldots, d\}$ such that $\Pi^{i_0}(\hat{A}) \geq C \cdot n^{d-1}$, and therefore $\Pi^{i_0}(A) \geq C_0 n^{d-1}$ for the original set $A$ too, as required. □

We now turn to define the isoperimetric profile of a graph. Let $\{p(x, y)\}_{x, y \in V}$ be transition probabilities for an irreducible Markov chain on a countable state space $V$ (we will think about this Markov chain as the random walk on a weighted graph $G = (V, E, C)$, with $\{x, y\} \in E$ if and only if $p(x, y) > 0$ and for every $\{x, y\} \in E$ we define the conductance $C(x, y) = p(x, y)$. For $S \subset V$, the “boundary size” of $S$ is measured by $|\partial S| = \sum_{s \in S} \sum_{a \in S^c} p(s, a)$. We define $\Phi_S$, the conductance of $S$, by $\Phi_S = \frac{|\partial S|}{|S|}$. Finally, define the isoperimetric profile of the graph $G$, with vertices $V$ and conductances induced from the transition probabilities by:

$$\Phi(u) = \inf \{\Phi_S : S \subset V, |S| \leq u\}. \tag{5.9}$$

We can now state Theorem 1 of [MP05].

**Theorem 5.5 [MP05] Theorem 1.** Let $G = (V, E)$ be a graph with countable vertices and bounded degree. Suppose that $0 < \gamma \leq \frac{1}{2}$ and $p(x, x) \geq \gamma$ for all $x \in V$. If

$$n \geq 1 + \frac{(1 - \gamma)^2}{\gamma^2} \int_4^{4/\epsilon} \frac{4du}{u\Phi^2(u)}, \tag{5.10}$$

then

$$|p^n(x, y)| \leq \epsilon. \tag{5.11}$$

Next we will prove the following claim:

**Claim 5.6.** Let $p^n_\omega(x, y)$ be the probability that the random walk moves from $x$ to $y$ in $n$ steps in the environment $\omega$. Then there exist positive constants $K_1, K_2$ depending only on $d$, and a natural number $N$ such that for every $n > N$ and every $x, y \in \mathcal{P}(\omega)$

$$p^n_\omega(x, y) \leq \frac{K_2}{(n - K_1)^{d/2}}, \quad P \text{ a.s.} \tag{5.12}$$

**Proof.** We start by dealing with even steps of the Markov chain, and at the end extend the argument to the odd ones. Since $p^2(x, x) = \frac{1}{2d}$, we can use Theorem 5.5 with $\gamma = \frac{1}{2d}$. Let $\omega \in \Omega_0$ and $S \subset \mathcal{P}(\omega)$ such that $|S| = n$. By Lemma 5.4 there exists a positive constant $C$, such that at least one of the projections $\{\Pi^i(S)\}_{i=1}^{d}$ satisfy $\Pi^i(S) \geq C \cdot n^{d-1}$. We will assume without loss of generality that this holds for $i = 1$. We now look at the set

$$\tilde{S} = \{(x_1, x_2, \ldots, x_d) : (x_2, \ldots, x_d) \in \Pi^1(S), x_1 = \max\{a : (a, x_2, x_3, \ldots, x_d) \in S\}\}.
\tag{5.13}$$
We note that $|\tilde{S}| = |\Pi^1(s)| \geq C_n^{(d-1)/d}$. In addition since $|\partial S|$ equals in our case to $\frac{1}{2d}$ times the number of edges $e \in E$ with one end point in $S$ and the other in $S^c$, then $|\partial S| \geq c_n^{(d-1)/d}$. This is true since every element in $\tilde{S}$ contributes at least one edge to the boundary. Using these two properties it follows that there exists a positive constant $C_0$ such that
\[
\Phi(u) \geq C_0 \frac{1}{u^{1/d}},
\] (5.14)
and therefore
\[
1 + (2d - 1)^2 \int_4^{4/\epsilon} \frac{4du}{u \Phi^2(u)} \leq 1 + (2d - 1)^2 \int_4^{4/\epsilon} \frac{4u^{2/3} - 1du}{C_0^2}
\leq \left[ 1 - \frac{2d(2d - 1)^2}{c_0^2} \frac{4^{2/3}}{4^{2/3}} + \frac{2d(2d - 1)^2}{c_0^2} \frac{4^{2/3}}{4^{2/3}} \epsilon^{-\frac{2}{d}} \right].
\]
Notice that $1 - \frac{2d(2d - 1)^2}{c_0^2} \frac{4^{2/3}}{4^{2/3}}$ is negative for all but a finite number of dimensions, and therefore we can find a natural number $K_1(d)$ such that the last term in (5.15) is less than or equal to
\[
n(\epsilon) \equiv [K_1 + K_2 \epsilon^{-\frac{2}{d}}],
\] (5.15)
where $K_2 = K_2(d) = \frac{2d(2d - 1)^2}{c_0^2} \frac{4^{2/3}}{4^{2/3}}$. It therefore follows that
\[
\epsilon \leq \left( \frac{n(\epsilon) - K_1 - 1}{K_2} \right)^{-\frac{d}{2}}.
\] (5.16)

Let $K_2 = (\tilde{K}_2)^{-\frac{d}{2}}$, since the condition in Theorem 5.5 is fulfilled, for $P$ almost every environment $\omega$, for every $n > N$ and every $x, y \in \mathcal{P}(\omega)$
\[
p^{2n}_{\omega}(x, y) \leq \frac{K_2}{(2n - K_1 - 1)^{\frac{d}{2}}},
\] (5.17)

Moving to deal with transition probabilities for odd times, if $n > N + 1$ we have for $P$ almost every environment $\omega$
\[
p^{2n+1}_{\omega}(x, y) = \sum_{z \in \mathcal{P}(\omega)} p_{\omega}(x, z) p^{2n}_{\omega}(z, y)
\leq \sum_{z \in \mathcal{P}(\omega)} p_{\omega}(x, z) \frac{K_2}{(2n - K_1 - 1)^{\frac{d}{2}}}
= \frac{K_2}{(2n + 1 - K_1 - 2)^{\frac{d}{2}}}.
\]
Taking $K_1 = \tilde{K}_1 + 2$ we get the desired inequality both for even times and odd ones. □

We are now ready to prove Theorem 1.8.
Proof of Theorem 1.8. Since our graph is connected, it is enough to show that
\[ \sum_{n=0}^{\infty} p^n(0, 0) < \infty. \] (5.18)
Using claim 5.6, we get that for \( P \) almost every environment \( \omega \in \Omega_0 \)
\[ \sum_{n=0}^{\infty} p^n(0, 0) \leq \sum_{n=0}^{N-1} p^n(0, 0) + \sum_{n=N}^{\infty} \frac{2K_2}{(2n - K_1)^2} < \infty. \] (5.19)
\[ \square \]

6. ASYMPTOTIC BEHAVIOR OF THE RANDOM WALK

In this section we prove asymptotic behavior of \( \mathbb{E}(\|X_n\|) \). This will be used in section 10 to prove the high dimensional Central Limit Theorem. Therefore we assume here the additional assumption, assumption 1.4. The estimation follows closely [Bar04] with the following changes:

- The minor change is that we work in discrete time setting and not in continuous time.

- The major change is that the average variance of the distance at the \( n^{th} \) step of the random walk is not bounded by 1 as in the percolation case. Nevertheless we can show that if we assume in addition assumption 1.4, it is still bounded.

Other than that problem, in which we deal in part (3) of Theorem 6.1, the rest of the proof doesn’t contain new ideas and follows [Bar04].

**Theorem 6.1.** Assuming assumption 1.4, there exists a random variable \( c : \Omega_0 \to [0, \infty) \) which is finite almost surely such that for \( P \) almost every \( \omega \in \Omega_0 \)
\[ \mathbb{E}_\omega(\|X_n\|) \leq c\sqrt{n} \quad \forall n \in \mathbb{N}. \] (6.1)

We begin with a few definitions

**Definition 6.2.** Fix \( \omega \in \Omega_0 \). For \( n \in \mathbb{N} \) we denote \( p^n(x, y) = P_\omega(X_n = y|X_0 = x) \) and introduce the following functions, with the understanding that \( 0 \cdot \log(0) = 0 \):

1. \( g_n : \mathcal{P}(\omega) \to \mathbb{R} \), given by
   \[ g_n(x) = \frac{1}{2} \left( p^n(0, x) + p^{n-1}(0, x) \right). \] (6.2)

2. We define \( M : \mathbb{N} \to \mathbb{R}^+ \) by \( M(0) = 0 \) and for \( n > 0 \) by:
   \[ M(n) = \frac{1}{2} \mathbb{E}_\omega(\|X_n\| + \|X_{n-1}\|) := \sum_{y \in \mathcal{P}(\omega)} \|y\|g_n(y). \] (6.3)
(3) We define $Q: \mathbb{N} \to \mathbb{R}^+$ by $Q(0) = 0$ and for $n > 0$ by:

$$Q(n) = -\sum_{y \in \mathcal{P}(\omega)} g_n(y) \log(g_n(y)), \quad (6.4)$$

i.e. $Q$ is the entropy of $g_n$.

In order to prove Theorem 6.1, we will prove some inequalities introduced in the following proposition:

**Proposition 6.3.** There exists $N = N(\omega) \in \mathbb{N}$ and constants $c_1, c_2, c_3, K_1 < \infty$ such that for every $n > N$ we have

1. $$Q(n) \geq c_1 + \frac{d}{2} \log(n - K_1), \quad (6.5)$$
2. $$M(n) \geq c_2 \cdot e^{\frac{Q(n)}{2}}, \quad (6.6)$$
3. $$\sum_{x \in \mathcal{P}(\omega)} \sum_{y \in \mathcal{P}(\omega)} 1_{\{y \in N_x(\omega)\}}(g_n(x) + g_n(y))\|x - y\|^2 < \infty, \quad (6.7)$$
4. $$(M(n + 1) - M(n))^2 \leq c_3 (Q(n + 1) - Q(n)). \quad (6.8)$$

We note that we don’t have any estimation on the tail of $N(\omega)$.

**Proof.**

(1) From the definition of $Q(n)$ we have that

$$Q(n) \geq \inf_{y \in \mathcal{P}(\omega)} (-\log(g_n(y))) = -\sup_{y \in \mathcal{P}(\omega)} (\log(g_n(y))).$$

Using now Claim 5.6 for sufficiently large $n$ we have $\forall y \in \mathcal{P}(\omega)$ that $g_n(y) \leq \frac{K_2}{(n - K_1)^{\frac{d}{2}}}$ and therefore

$$Q(n) \geq -\log \left( \frac{K_2}{(n - K_1)^{\frac{d}{2}}} \right) = -\log(K_2) + \frac{d}{2} \log(n - K_1). \quad (6.9)$$

Taking $c_1 = -\log(2K_2)$ we get the desired inequality.

(2) Let $D_n = B_{2^n}(0) \setminus B_{2^{n-1}}(0)$ for $n > 0$ and $D(0) = \{0\}$, where $B_n(0) = \{x \in \mathbb{Z}^d : \|x\| \leq n\}$. Then for $0 \leq a \leq 2$ we have:

$$\sum_{y \in \mathcal{P}(\omega)} e^{-a\|y\|} \leq \sum_{n=0}^{\infty} \sum_{y \in D_n} e^{-a \cdot 2^n} \leq \sum_{n=0}^{\infty} e^{-a \cdot 2^n} \cdot c_{2,1} \cdot 2^{nd} \leq c_{2,2} \cdot a^{-d}, \quad (6.10)$$
where \( c_{2.2} = c_{2.2}(d) \) depends on \( d \). Indeed, the first inequality is true since \( a \leq 2 \), the second inequality follows from the fact that the set of points in \( \mathcal{P}(\omega) \) with distance greater than \( 2^{n-1} \) and less than \( 2^n \) is bounded by the number of points in \( \mathbb{Z}^d \) with those properties, which is less than a constant times \( 2^{nd} \). The proof of the last inequality follows by separating the series into two parts, up to some \( n_0 \) and starting from \( n_0 \), and then bounding the second one by a geometric series. The proof of it can be found in the Appendix.

Since for every \( u > 0 \) and every \( \lambda \in \mathbb{R} \) we have \( u (\log(u) + \lambda) \geq -e^{-1-\lambda} \), by taking \( \lambda = a\|y\| + b \) with \( a \leq 2 \) and \( u = g_n(y) \) we get

\[-Q(n) + aM(n) + b = \sum_{y \in \mathcal{P}(\omega)} g_n(y) (\log(g_n(y)) + a\|y\| + b) \geq -\sum_{y \in \mathcal{P}(\omega)} e^{-1-a\|y\|-b} = -e^{-1-b} \sum_{y \in \mathcal{P}(\omega)} e^{-a\|y\|}.
\]

Note that we actually used the last inequality only for those \( y \in \mathcal{P}(\omega) \) such that \( g_n(y) > 0 \), and for \( y \in \mathcal{P}(\omega) \) such that \( g_n(y) = 0 \) we used the fact that \( 0 \geq -e^{-1-a\|y\|-b} \). Combining (6.11) and (6.10) we get that

\[-Q(n) + aM(n) + b \geq -e^{-1-b}c_{2.2}a^{-d}.
\]

But for sufficiently large \( n \) we have

\[M(n) = 0 \cdot g_n(0) + \sum_{y \in \mathcal{P}(\omega), y \neq 0} d(0, y) g_n(y) \geq \sum_{y \in \mathcal{P}(\omega), y \neq 0} g_n(y) = 1 - g_n(0) \geq \frac{1}{2}.
\]

Taking now \( a = \frac{1}{M(n)} \) and \( b = d \cdot \log(M(n)) \), by (6.12) (and since by (6.13) we have \( a \leq 2 \)) it follows that

\[-Q(n) + 1 + d \cdot \log(M(n)) \geq -e^{-1}c_{2.2} = -c_{2.3}.
\]

Note that \( c_{2.3} = c_{2.3}(d) \) also depend on \( d \). Rearranging the last inequality we get that there exists a constant \( c_2 = c_2(d) \) such that

\[M(n) \geq c_2 \cdot e^{\frac{Q(n)}{d}}.
\]
We start by rearranging the sum as
\[
\sum_{x,y \in P(\omega)} 1_{\{y \in N_n(\omega)\}} (g_n(x) + g_n(y))\|x - y\|^2 = 2 \sum_{x \in P(\omega)} g_n(x) \sum_{y \in N_n(\omega)} \|x - y\|^2
\]
\[= 2 \sum_{e \in \{\pm e_i\}_{i=1}^d} \sum_{x \in P(\omega)} g_n(x) f_e^2(\theta^x \omega)
\]
\[= 2 \sum_{e \in \{\pm e_i\}_{i=1}^d} \left( E_\omega(f_e^2 \circ \theta^{X_n}) + E_\omega(f_e^2 \circ \theta^{X_{n-1}}) \right).
\]

In order to show that this sum is finite, we will use a theorem taken from [NS94]. Before we can state the theorem we need the following definitions:

Given a countable group \(\Gamma\) we define \(l^1(\Gamma) = \{\mu = \sum_{\gamma \in \Gamma} \mu(\gamma)\gamma : \sum_{\gamma \in \Gamma} |\mu(\gamma)| < \infty\}\). Let \((X, B, m)\) be a standard Lebesgue probability space, and assume \(\Gamma\) acts on \(X\) by measurable automorphisms preserving the probability measure \(m\). This action induces a representation of \(\Gamma\) by isometries on the \(L^p(X)\) spaces, \(1 \leq p \leq \infty\), and this representation can be extended to \(l^1(\Gamma)\) by \((\mu f)(x) = \sum_{\gamma \in \Gamma} \mu(\gamma) f(\gamma^{-1} x)\).

Let \(B_1 = \{A \in B : m(\gamma A \triangle A) = 0 \ \forall \gamma \in \Gamma\}\) denote the sub \(\sigma\)-algebra of invariant sets, and denote by \(E_1\) the conditional expectation with respect to \(B_1\). We call a sequence \(\nu_n \in l^1(\Gamma)\) a pointwise ergodic sequence in \(L^p\) if, for any action of \(\Gamma\) on a Lebesgue space \(X\) which preserves a probability measure and for every \(f \in L^p(X)\), \(\nu_n f(x) \to E_1 f(x)\) for almost all \(x \in X\), and in the norm of \(L^p(X)\). If \(\Gamma\) is finitely generated, let \(S\) be a finite generating symmetric set. \(S\) induces a length function on \(\Gamma\), given by \(|\gamma| = |\gamma|_S = \min\{n : \gamma = s_1 s_2 \ldots s_n, s_i \in S\}\), and \(|e| = 0\). We can therefore define the following sequences:

**Definition 6.4.**

(i.) \(\tau_n = (\#S_n)^{-1} \sum_{w \in S_n} w\), where \(S_n = \{w : |w| = n\}\).

(ii.) \(\tau'_n = \frac{1}{2} (\tau_n + \tau_{n+1})\).

(iii.) \(\mu_n = \frac{1}{n+1} \sum_{k=0}^n \tau_k\).

(iv.) \(\beta_n = (\#B_n)^{-1} \sum_{w \in B_n} w\), where \(B_n = \{w : |w| \leq n\}\).

We can now state the theorem:

**Theorem 6.5** (Nevo, Stein 94). Consider the free group \(F_r\), \(r \geq 2\). Then:

1. The sequence \(\mu_n\) is a pointwise ergodic sequence in \(L^p\), for all \(1 \leq p < \infty\).

2. The sequence \(\tau'_n\) is a pointwise ergodic sequence in \(L^p\), for \(1 < p < \infty\).

3. \(\tau_{2n}\) converges to an operator of conditional expectation with respect to an \(F_r\)-invariant sub \(\sigma\)-algebra. \(\beta_{2n}\) converges to the operator \(E_1 + ((r - 1)/r)E\), where \(E\) is a projection disjoint from \(E_1\). Given \(f \in L^p(X)\), \(1 < p < \infty\), the convergence is pointwise almost everywhere, and in the \(L^p\) norm.
We actually only need the second part of Theorem 6.5. Taking $S = \{\sigma_{x_i}\}_{i=1}^{d}$, we get that

$$\sum_{e \in \{\pm e_i\}_{i=1}^{d}} (E_{\omega}(f^2_e \circ f^{X_n}) + E_{\omega}(f^2_e \circ f^{X_{n-1}})) \leq 4 \sum_{e \in \{\pm e_i\}_{i=1}^{d}} \tau'_n(f^2_e).$$

Using the additional assumption, we get that there exists $1 < p < \infty$ such that for every coordinate direction $e$, $f^2_e \in L^p(\Omega_0)$. Therefore by Theorem 6.5

$$\lim_{n \to \infty} 4 \sum_{e \in \{\pm e_i\}_{i=1}^{d}} \tau'_n(f^2_e) = E_1 \left( 4 \sum_{e \in \{\pm e_i\}_{i=1}^{d}} f^2_e \right),$$

exists. In addition, since $P$ is ergodic with respect to $\sigma_e$ for every coordinate direction $e$, there exists a constant $C$ such that $4 \sum_{e \in \{\pm e_i\}_{i=1}^{d}} E_1(f^2_e) = C$ $P$-almost surely. Consequently, the original sequence converges to $C$ $P$-almost surely, and therefore in particular it is $P$-almost surely bounded.

(4)

$$M(n+1) - M(n) = \sum_{y \in P(\omega)} (g_{n+1}(y) - g_n(y))\|y\|.$$

Using the discrete Gauss Green formula, this term equals to

$$-\frac{1}{4d} \sum_{x,y \in P(\omega)} \mathbb{1}_{\{y \in N_x(\omega)\}} (\|y\| - \|x\|)(g_n(y) - g_n(x)).$$

(6.13)

Indeed, rearranging the sums we get that $\sum_{y \in P(\omega)} (g_{n+1}(y) - g_n(y))\|y\|$ equals to

$$-\frac{1}{4d} \left[ 2d \sum_{y \in P(\omega)} \|y\|g_n(y) + 2d \sum_{x \in P(\omega)} \|x\|g_n(x) - 2d \sum_{y \in P(\omega)} \|y\|g_{n+1}(y) - 2d \sum_{x \in P(\omega)} \|x\|g_{n+1}(x) \right].$$

Since all sums are finite and for every point in $x \in P(\omega)$ we have $|N_x(\omega)| = 2d < \infty$ we get that the last term is equal to

$$-\frac{1}{4d} \left[ \sum_{y \in P(\omega)} \|y\|g_n(y) \sum_{x \in P(\omega)} \mathbb{1}_{y \in N_x(\omega)} + \sum_{x \in P(\omega)} \|x\|g_n(x) \sum_{y \in P(\omega)} \mathbb{1}_{y \in N_x(\omega)} - \sum_{y \in P(\omega)} \|y\| \sum_{x \in P(\omega)} \mathbb{1}_{y \in N_x(\omega)} g_n(x) - \sum_{x \in P(\omega)} \|x\| \sum_{y \in P(\omega)} \mathbb{1}_{y \in N_x(\omega)} g_n(y) \right].$$
But again all sums are finite and therefore we can change the order of summation getting the following presentation

\[ -\frac{1}{4d} \sum_{x,y \in \mathcal{P}(\omega)} \mathbb{1}_{y \in N_x(\omega)} \|y\| g_n(y) - \mathbb{1}_{y \in N_x(\omega)} \|x\| g_n(x) + \mathbb{1}_{y \in N_x(\omega)} \|x\| g_n(x) \]

\[ = -\frac{1}{4d} \sum_{x,y \in \mathcal{P}(\omega)} \mathbb{1}_{\{y \in N_x(\omega)\}} (\|y\| - \|x\|) (g_n(y) - g_n(x)). \]

Using (6.13) and the triangle inequality we get that \( M(n+1) - M(n) \) is less or equal than

\[ \frac{1}{4d} \sum_{x,y \in \mathcal{P}(\omega)} \mathbb{1}_{\{y \in N_x(\omega)\}} \|x - y\| |g_n(y) - g_n(x)|. \]

Therefore by the Cauchy Schwartz inequality

\[ M(n+1) - M(n) \leq \frac{1}{4d} \left( \sum_{x,y \in \mathcal{P}(\omega)} \mathbb{1}_{\{y \in N_x(\omega)\}} (g_n(x) + g_n(y)) \|x - y\|^2 \right)^{\frac{1}{2}} \cdot \left( \sum_{x,y \in \mathcal{P}(\omega)} \mathbb{1}_{\{y \in N_x(\omega)\}} \left( g_n(y) - g_n(x) \right)^2 \right)^{\frac{1}{2}}. \]

The first sum here is exactly the same sum from (6.7) and therefore is finite, so there exists a positive constant \( c_{3,1} = c_{3,1}(d) \) such that \( M(n+1) - M(n) \) is less or equal to

\[ c_{3,1} \left( \sum_{x,y \in \mathcal{P}(\omega)} \mathbb{1}_{\{y \in N_x(\omega)\}} \left( g_n(y) - g_n(x) \right)^2 \right)^{\frac{1}{2}}. \]

Using the fact that for every \( u, v > 0 \)

\[ \frac{(u - v)^2}{u + v} \leq (u - v) (\log(u) - \log(v)). \]

We get that \( M(n+1) - M(n) \) is less or equal than

\[ c_{3,1} \left( \sum_{x,y \in \mathcal{P}(\omega)} \mathbb{1}_{\{y \in N_x(\omega)\}} \left( g_n(y) - g_n(x) \right)^2 \right)^{\frac{1}{2}} \left( \log(g_n(y)) - \log(g_n(x)) \right). \]

Using the discrete Gauss Green formula in the other direction, the last term equals to

\[ \sqrt{4dc_{3,1}} \left( - \sum_{y \in \mathcal{P}(\omega)} \log(g_n(y)) + 1 \right) \left( g_{n+1}(y) - g_n(y) \right)^{\frac{1}{2}}. \]
Since \(1 - x + \log(x) \leq 0\) for all \(x > 0\) we get that the last term is less or equal to
\[
\sqrt{4d_{3.1}} \left( - \sum_{y \in P(\omega)} \left( g_{n+1}(y) - g_n(y) \right) \log(g_n(y)) + g_{n+1}(y) \log(g_{n+1}(y)) \right)^{\frac{1}{2}}.
\]
But this is exactly
\[
\sqrt{4d_{3.1}} \left( Q(n + 1) - Q(n) \right)^{\frac{1}{2}}.
\]
By taking \(c_3 = (\sqrt{4d_{3.1}})^2\) gives the desired inequality.

\[\square\]

**Proof of Theorem 6.1.** Let \(R(n) : \mathbb{N} \to \mathbb{R}\) be defined by
\[
R(n) = \frac{1}{d} \left( Q(n) - c_1 - \frac{d}{2} \log(n - K_1) \right),
\]
for \(n > [K_1] + 1\) and \(R(n) = 0\) for \(n \leq [K_1] + 1\). By (6.6) for sufficiently large \(n\) we have
\[
M(n) \geq c_2 \cdot e^{\frac{Q(n)}{d}} = c_2 \cdot e^{R(n) + \frac{c_1}{d} + \frac{1}{2} \log(n - K_1)} = c_{4.1} e^{R(n)} \sqrt{n - K_1}.
\] (6.15)

On the other hand, let \(N \in \mathbb{N}\) be such that for all \(n > N\) inequalities (6.5-6.8) hold, then for every \(n > N\) we have (set \(c_{4.3} = \sqrt{c_3}\))
\[
M(n) \leq \sum_{k=1}^{N} M(k) - M(k - 1) + \sum_{k=N+1}^{n} M(k) - M(k - 1)
\leq c_{4.2} + c_{4.3} \cdot \sum_{k=N+1}^{n} \left( Q(k) - Q(k - 1) \right)^{\frac{1}{2}}
= c_{4.2} + c_{4.3} \sqrt{d} \sum_{k=N+1}^{n} \left( R(k) - R(k - 1) + \frac{1}{2} \log \left( \frac{k - K_1}{k - 1 - K_1} \right) \right)^{\frac{1}{2}}.
\]
Using the inequality \((a + b)^{\frac{1}{2}} \leq b^{\frac{1}{2}} + \frac{a}{(2b)^{\frac{1}{2}}}\), we find that this is less than or equal to
\[
c_{4.2} + c_{4.3} \sum_{k=N+1}^{n} \left[ \frac{1}{\sqrt{2}} \log \frac{k - K_1}{k - 1 - K_1} + \frac{R(k) - R(k - 1)}{\log \frac{k - K_1}{k - 1 - K_1}} \right].
\]
which can be written (using discrete integration by parts) as

\[
c_{4.2} + c_{4.3} \sum_{k=N+1}^{n} \frac{1}{\sqrt{2}} \log^{\frac{1}{2}} \left( \frac{k - K_1}{k - 1 - K_1} \right) + c_{4.3} \sum_{k=N+1}^{n} \left[ \frac{R(k)}{\log^{\frac{1}{2}} \left( \frac{k+1-K_1}{k-K_1} \right)} - \frac{R(k-1)}{\log^{\frac{1}{2}} \left( \frac{k-K_1}{k-1-K_1} \right)} \right]
\]

\[
- c_{4.3} \sum_{n=N+1}^{n} R(k) \left[ \frac{1}{\log^{\frac{1}{2}} \left( \frac{k+1-K_1}{k-K_1} \right)} - \frac{1}{\log^{\frac{1}{2}} \left( \frac{k-K_1}{k-1-K_1} \right)} \right].
\]

Since (6.5) holds \( R(k) \) is non-negative and therefore the last sum is positive. Consequently we get

\[
M(n) \leq c_{4.2} + c_{4.3} \sum_{k=N+1}^{n} \frac{1}{\sqrt{2}} \log^{\frac{1}{2}} \left( \frac{k - K_1}{k - 1 - K_1} \right) + c_{4.3} \sum_{k=N+1}^{n} \left[ \frac{R(k)}{\log^{\frac{1}{2}} \left( \frac{k+1-K_1}{k-K_1} \right)} - \frac{R(k-1)}{\log^{\frac{1}{2}} \left( \frac{k-K_1}{k-1-K_1} \right)} \right].
\]

Using the fact that

\[
\log \left( \frac{k - K_1}{k - 1 - K_1} \right) = \log \left( 1 + \frac{1}{k - 1 - K_1} \right) < \frac{1}{k - 1 - K_1}.
\]

The first sum in (6.16) is less than

\[
\sum_{k=N+1}^{n} \frac{1}{(k - 1 - K_1)^{\frac{1}{2}}} \leq c_{4.4} \sqrt{n - K_1}.
\]

Therefore we find that

\[
M(n) \leq c_{4.2} + c_{4.3}c_{4.4} \sqrt{n - K_1} + c_{4.3} \sum_{k=N+1}^{n} \left[ \frac{R(k)}{\log^{\frac{1}{2}} \left( \frac{n+1-K_1}{n-K_1} \right)} - \frac{R(N+1)}{\log^{\frac{1}{2}} \left( \frac{N+1-K_1}{N-K_1} \right)} \right] \leq c_{4.2} + c_{4.3}c_{4.4} \sqrt{n - K_1} + c_{4.3} \frac{R(n)}{\log^{\frac{1}{2}} \left( \frac{n+1-K_1}{n-K_1} \right)}
\]

\[
\leq c_{4.2} + c_{4.3}c_{4.4} \sqrt{n - K_1} + c_{4.3} \cdot c_{4.5} R(n) \sqrt{n - K_1}.
\]

We can thus find a constant \( c_{4.6} \) such that for all sufficiently large \( n \)

\[
M(n) \leq c_{4.6} [1 + R(n)] \sqrt{n - K_1}.
\]
So by (6.15) and (6.17) we have that for sufficiently large $n$
\[ c_{4.1}e^{R(n)n - K_1} \leq M(n) \leq c_{4.6}[1 + R(n)] \sqrt{n - K_1}. \]
It follows that $R(n)$ must be a bounded function, and therefore we can find constants $c_{4.7}, c_{4.8}$ such that for sufficiently large $n$
\[ c_{4.7} \sqrt{n - K_1} \leq M(n) \leq c_{4.8} \sqrt{n - K_1}. \]
Consequently, since $M_n = \frac{1}{2}[E_\omega(\|X_n\|) + E_\omega(\|X_{n-1}\|)]$, it follows that there exists a constant $c > 0$ such that for $P$ almost every $\omega \in \Omega$
\[ E_\omega(\|X_n\|) \leq c \sqrt{n} \quad \forall n \in \mathbb{N}. \]

7. Corrector - Construction and harmonicity

In this section, we adapt the construction presented in [BB07] (which in turn adapts the construction of Kipnis and Varadhan [KV86]) into our analysis.

We start with the following observation concerning the Markov chain "on environments".

**Lemma 7.1.** For every bounded measurable function $f : \Omega_0 \to \mathbb{R}$ and every $x \in N_0(\omega)$ we have
\[ \mathbb{E}_P[(f \circ \theta_{x}) \mathbbm{1}_{\{x \in N_0(\omega)\}}] = \mathbb{E}_P[f \mathbbm{1}_{\{-x \in N_0(\omega)\}}]. \]

As a consequence, $P$ is reversible and, in particular, stationary for the Markov kernel $\Lambda$ defined in (2.2).

**Proof.** We will first prove (7.1). Up to the factor $P(\Omega_0)$, we need to show that
\[ \mathbb{E}_Q[f \circ \theta_{x} \mathbbm{1}_{\{x \in N_0(\omega)\}}] = \mathbb{E}_Q[f \mathbbm{1}_{\{-x \in N_0(\omega)\}}]. \]
This will follow from the fact that $\mathbbm{1}_{\{x \in N_0(\omega)\}} = (\mathbbm{1}_{\{-x \in N_0(\omega)\}} \mathbb{1}_{\Omega_0}) \circ \theta_{x}$. This observation implies that
\[ f \circ \theta_{x} \mathbbm{1}_{\{x \in N_0(\omega)\}} = (f \mathbbm{1}_{\Omega_0} \mathbbm{1}_{\{-x \in N_0(\omega)\}}) \circ \theta_{x}, \]
and (7.2) follows from (7.3) by the shift invariance of $Q$. From (7.1) we deduce that for any bounded measurable functions $f, g : \Omega \to \mathbb{R}$,
\[ \mathbb{E}_P[f \cdot (\Lambda g)] = \mathbb{E}_P[g \cdot (\Lambda f)], \]
where $\Lambda f : \Omega_0 \to \mathbb{R}$ is the function
\[ (\Lambda f)(\omega) = \frac{1}{2d} \sum_{x \in \mathbb{Z}^d} \left( \mathbbm{1}_{\{x \in N_0(\omega)\}} f(\theta_{x}\omega) \right). \]
Indeed
\[ \mathbb{E}_P[f \cdot (\Lambda g)] = \frac{1}{2d} \sum_{x \in \mathbb{Z}^d} \mathbb{E}_P[f \cdot g \circ \theta_{x} \mathbbm{1}_{\{x \in N_0(\omega)\}}]. \]
Applying (7.1) we get
\[ E_P[f \cdot (\Lambda g)] = \frac{1}{2d} \sum_{x \in \mathbb{Z}^d} E_P[f \circ \theta_{-x} \mathbbm{1}_{\{-x \in N_0(\omega)\}} \cdot g] = E_P[(\Lambda f) \cdot g], \]
where we replaced the sign in the sum in order to cancel the negative sign inside the sum. But (7.4) is the definition of reversibility. Setting \( f = 1 \) and noting that \( \Lambda f = 1 \), we get that for every bounded measurable function \( g : \Omega \to \mathbb{R} \)
\[ E_P[\Lambda g] = E_P[g], \]
and therefore \( P \) is stationary with respect to the Markov kernel \( \Lambda \).
\( \square \)

### 7.1. The Kipnis-Varadhan Construction.

Next we will adapt the construction of Kipnis and Varadhan [KV86] cited from [BB07] to the present analysis. Let \( L^2 = L^2(\Omega_0, \mathcal{B}, P) \) be the space of all Borel-measurable square integrable functions on \( \Omega_0 \). We will use the notation \( L^2 \) both for \( \mathbb{R} \)-valued functions as well as for \( \mathbb{R}^d \)-valued functions. We equip \( L^2 \) with the inner product \( (f, g) = E_P[f g] \), when for vector valued functions on \( \Omega \) we interpret "\( fg \)" as the scalar product of \( f \) and \( g \). Let \( \Lambda \) be the operator defined by (7.5), and we expand the definition to vector valued functions by letting \( \Lambda \) act like a scalar, i.e., independently for each component. From (7.4) we get that
\[ (f, \Lambda g) = (\Lambda f, g), \quad (7.6) \]
and so \( \Lambda \) is symmetric. In addition, for every \( f \in L^2 \) we have
\[ |(f, \Lambda f)| \leq \frac{1}{2d} \sum_{x \in \mathbb{Z}^d} |(f, \mathbbm{1}_{\{x \in N_0(\omega)\}} f \circ \theta_{-x})| = \frac{1}{2d} \sum_{x \in \mathbb{Z}^d} |(f \mathbbm{1}_{\{x \in N_0(\omega)\}}, \mathbbm{1}_{\{x \in N_0(\omega)\}} f \circ \theta_{-x})|. \]
Using the Cauchy-Schwartz inequality this is less than or equal to
\[ \frac{1}{2d} \sum_{x \in \mathbb{Z}^d} (f \mathbbm{1}_{\{x \in N_0(\omega)\}}) \cdot (1, \mathbbm{1}_{\{x \in N_0(\omega)\}})^{1/2}. \]
which equals
\[ \frac{1}{2d} \sum_{x \in \mathbb{Z}^d} (f, \mathbbm{1}_{\{x \in N_0(\omega)\}})^{1/2} \cdot (1, \mathbbm{1}_{\{x \in N_0(\omega)\}})^{1/2}. \]
Using (7.1) we find that this this equals
\[ \frac{1}{2d} \sum_{x \in \mathbb{Z}^d} (f, \mathbbm{1}_{\{x \in N_0(\omega)\}})^{1/2} \cdot (f, \mathbbm{1}_{\{-x \in N_0(\omega)\}})^{1/2} = \frac{1}{2d} \sum_{x \in \mathbb{Z}^d} (f, \mathbbm{1}_{\{x \in N_0(\omega)\}}) = (f, f), \]
and so \( \|\Lambda\|_{L^2} \leq 1 \). In particular, \( \Lambda \) is self adjoint and \( sp(\Lambda) \subseteq [-1, 1] \).

Let \( V : \Omega_0 \to \mathbb{R}^d \) be the local drift at the origin i.e.,
\[ V(\omega) = \frac{1}{2d} \sum_{x \in \mathbb{Z}^d} x \mathbbm{1}_{\{x \in N_0(\omega)\}}. \quad (7.7) \]
If the second moment of $f_e$ exists for every $e \in \mathcal{E}$, then $V \in L^2$. Indeed

$$(V, V) = \sum_{e \in \mathcal{E}} (V \cdot e, V \cdot e),$$

and

$$(V \cdot e, V \cdot e) = \frac{1}{2d} \mathbb{E}_P[(V \cdot e)^2] = \frac{1}{2d} \mathbb{E}_P[(f_e)^2 + (f_{-e})^2],$$

which is finite if the second moments exist. For each $\epsilon > 0$, let $\psi_\epsilon : \Omega_0 \rightarrow \mathbb{R}^d$ be the solution of

$$(1 + \epsilon - \Lambda)\psi_\epsilon = V. \quad (7.8)$$

This is well defined since $\text{sp}(\Lambda) \subset [-1, 1]$, so for every $\epsilon > 0$ we get $\text{sp}(1 + \epsilon + \Lambda) \subset [\epsilon, 2 + \epsilon]$. In addition we get that $\psi_\epsilon \in L^2$ for all $\epsilon > 0$. The following theorem is the main result concerning the corrector:

**Theorem 7.2.** There is a function $\chi : \mathbb{Z}^d \times \Omega_0 \rightarrow \mathbb{R}^d$ such that for every $x \in \mathbb{Z}^d$,

$$\lim_{\epsilon \rightarrow 0} 1_{\{x \in \mathcal{P}(\omega)\}}(\psi_\epsilon \circ \theta_x - \psi_\epsilon) = \chi(x, \cdot), \quad \text{in } L^2. \quad (7.9)$$

Moreover, the following properties hold:

1. (Shift invariance) For $P$-almost every $\omega \in \Omega_0$

   $$\chi(x, \omega) - \chi(y, \omega) = \chi(x - y, \theta_y(\omega)), \quad (7.10)$$

   for all $x, y \in \mathcal{P}(\omega)$.

2. (Harmonicity) For $P$-almost every $\omega \in \Omega_0$, the function

   $$x \mapsto \chi(x, \omega) + x, \quad (7.11)$$

   is harmonic with respect to the transition probability given in (1.3).

3. (Square integrability) There exists a constant $C < \infty$ such that

   $$\|\chi(x + y, \cdot) - \chi(x, \cdot)\|_{L^2} \leq C, \quad (7.12)$$

   for all $x, y \in \mathbb{Z}^d$.

The rest of this section deals with proving Theorem 7.2. The proof is based on spectral calculus and closely follows the corresponding arguments from [BB07] and [KV86].

### 7.2. Spectral calculation.

Let $\mu_{\Lambda, V} = \mu_V$ denote the spectral measure of $\Lambda : L^2 \rightarrow L^2$ associated with the function $V$. i.e, for every bounded, continuous function $\Phi : [-1, 1] \rightarrow \mathbb{R}$, we have

$$(V, \Phi(\Lambda)V) = \int_{-1}^{1} \Phi(\lambda)\mu_V(d\lambda). \quad (7.13)$$

Since $\Lambda$ acts as a scalar, $\mu_V$ is the sum of the "usual" spectral measures for the Cartesian components of $V$. In the integral, we used the fact that $\text{sp}(\Lambda) \subset [-1, 1]$, and therefore the
measure $\mu_V$ is supported entirely on $[-1, 1]$. The first observation, made already by Kipnis and Varadhan, is stated as follows:

**Lemma 7.3.** Assume that the second moments of $\{f_{\pm e_i}\}_{i=1}^d$ are finite, then

$$\int_{-1}^1 \frac{1}{1 - \lambda} \mu_V(d\lambda) < \infty. \quad (7.14)$$

**Proof.** The proof follows the proof of Lemma 2.3 in [BB07]. Let $f \in L^2$ be a bounded real-valued function. Using (7.1) we get

$$\sum_{x \in \mathbb{Z}^d} x \mathbb{E}_P[f \mathbb{1}_{\{x \in N_0(\omega)\}}] = \frac{1}{2} \sum_{x \in \mathbb{Z}^d} x \mathbb{E}_P[(f - f \circ \theta_x) \mathbb{1}_{\{x \in N_0(\omega)\}}]. \quad (7.15)$$

Hence, for every $a \in \mathbb{Z}^d$ we get

$$(f, a \cdot V) = \frac{1}{2d} \sum_{x \in \mathbb{Z}^d} x \cdot a \mathbb{E}_P[f \mathbb{1}_{\{x \in N_0(\omega)\}}]$$

$$= \frac{1}{2} \left( \frac{1}{2d} \sum_{x \in \mathbb{Z}^d} (x \cdot a)^2 P(x \in N_0(\omega)) \right)^{1/2}$$

$$\leq \frac{1}{2} \left( \frac{1}{2d} \sum_{x \in \mathbb{Z}^d} \mathbb{E}_P[(f - f \circ \theta_x)^2 \mathbb{1}_{\{x \in N_0(\omega)\}}] \right)^{1/2},$$

where we used (7.15) in the second equality, and the Cauchy-Schwarz inequality for the inequality. Using the assumption that the second moments exist for every $e \in \mathcal{E}$, the first term on the right hand side is less than a finite constant times $|a|$. On the other hand, the second term, using (7.1), can be written as follows:

$$\frac{1}{2d} \sum_{x \in \mathbb{Z}^d} \mathbb{E}_P((f - f \circ \theta_x)^2 \mathbb{1}_{\{x \in N_0(\omega)\}})$$

$$= 2 \frac{1}{2d} \sum_{x \in \mathbb{Z}^d} \mathbb{E}_P(f - f \circ \theta_x) \mathbb{1}_{\{x \in N_0(\omega)\}}$$

$$= 2(f, (1 - \Lambda)f).$$

From the assumption that the second moments exist, there exists a constant $C_0 < \infty$ such that for all bounded $f \in L^2$,

$$|(f, a \cdot V)| \leq C_0 |a| (f, (1 - \Lambda)f)^{1/2}. \quad (7.16)$$

Applying (7.16) for $f$ of the form $f = a \cdot \psi(\Lambda)V$, where $a \in \mathbb{R}^d$, and $\Psi : [-1, 1] \to \mathbb{R}$ is a bounded continuous function, summing over coordinate vectors in $\mathbb{R}^d$ and invoking (7.14),
we get that
\[
\left| \int_{-1}^{1} \psi(\lambda) \mu_V(d\lambda) \right| = \left| \sum_{i=1}^{d} (V \cdot e_i, \psi(\Lambda)V \cdot e_i) \right|
\]
\[
\leq \sum_{i=1}^{d} |(V \cdot e_i, \psi(\Lambda)V \cdot e_i)|
\]
\[
\leq C_0 \sum_{i=1}^{d} (V \cdot e_i, \psi(\Lambda)^2(1 - \Lambda)V \cdot e_i)^{1/2}
\]
\[
\leq C_0 \sqrt{d} \left( \sum_{i=1}^{d} (V \cdot e_i, \psi(\Lambda)^2(1 - \Lambda)V \cdot e_i) \right)^{1/2}
\]
\[
= C_0 \sqrt{d} \left( \int_{-1}^{1} \psi(\lambda)^2(1 - \lambda) \mu_V(d\lambda) \right)^{1/2}.
\]
Substituting \( \psi(\lambda) = \min \left\{ \frac{1}{\epsilon}, \frac{1}{1 - \lambda} \right\} \) for \( \psi \) and noting that \( (1 - \lambda)\psi(\lambda) \leq 1 \), we get
\[
\int_{-1}^{1} \psi(\lambda) \mu_V(d\lambda) \leq C_0 \sqrt{d} \left( \int_{-1}^{1} \psi(\lambda) \mu_V(d\lambda) \right)^{1/2},
\]
and therefore
\[
\int_{-1}^{1} \psi(\lambda) \mu_V(d\lambda) \leq d \cdot C_0^2.
\]
Now, the Monotone Convergence Theorem implies that
\[
\int_{-1}^{1} \frac{1}{1 - \lambda} \mu_V(d\lambda) = \lim_{\epsilon \searrow 0} \int_{-1}^{1} \psi(\epsilon) \mu_V(d\lambda) = \sup_{\epsilon > 0} \int_{-1}^{1} \psi(\epsilon) \mu_V(d\lambda) \leq d \cdot C_0^2 < \infty,
\]
proving the desired claim. \( \square \)

We now turn to prove the following lemma, also taken from [BB07]:

**Lemma 7.4.** Let \( \psi_\epsilon \) be defined as in (7.8), i.e., the solution of \((1 + \epsilon - \Lambda)\psi_\epsilon = V\). Then
\[
\lim_{\epsilon \searrow 0} \epsilon \| \psi_\epsilon \|^2_2 = 0.
\] (7.20)

In addition, for every \( x \in \mathbb{Z}^d \) let
\[
G_x^{(\epsilon)}(\omega) = 1_{\Omega_0}(\omega) \cdot 1_{\{x \in N_0(\omega)\}}(\omega) \cdot (\psi_\epsilon \circ \theta_x(\omega) - \psi_\epsilon(\omega)).
\] (7.21)

Then for all \( x, y \in \mathbb{Z}^d \),
\[
\lim_{\epsilon_1, \epsilon_2 \searrow 0} \| G_x^{(\epsilon_1)} \circ \theta_y - G_x^{(\epsilon_2)} \circ \theta_y \|_2 = 0.
\] (7.22)
Proof. The proof follows the proof in [BB07]. From the definition of \( \psi \), we have,

\[
\epsilon \| \psi \epsilon \|_2^2 = \int_{-1}^{1} \frac{\epsilon}{(1 + \epsilon - \lambda)^2} \mu_V(d\lambda). \tag{7.23}
\]

The integrand is dominated by \( \frac{1}{1 - \lambda} \) and in addition tends to zero as \( \epsilon \searrow 0 \) in the support of \( \mu_V \). Then (7.20) follows by the Dominated Convergence Theorem. The second part of the claim is proved similarly: First we get rid of the \( y \)-dependence by noting the following. Due to the fact that \( G^\epsilon_x \circ \theta^y \neq 0 \) ensure that \( y \in P(\omega) \), and since \( P \) is invariant under translation of the form \( \theta^z \) \( z \in P(\omega) \) we get that:

\[
\| G^{(\epsilon_1)}_x \circ \theta_y - G^{(\epsilon_2)}_x \circ \theta_y \|_2 = \| G^{(\epsilon_1)}_x - G^{(\epsilon_2)}_x \|_2. \tag{7.24}
\]

Therefore, averaging the square of (7.24) over \( x \in N_0(\omega) \) we find that

\[
\frac{1}{2d} \sum_{x \in N_0(\omega)} \| G^{(\epsilon_1)}_x \circ \theta_y - G^{(\epsilon_2)}_x \circ \theta_y \|_2^2 = \frac{1}{2d} \sum_{x \in N_0(\omega)} \| G^{(\epsilon_1)}_x - G^{(\epsilon_2)}_x \|_2^2
\]

\[
= \frac{1}{2d} \sum_{x \in N_0(\omega)} \mathbb{E}_P \left[ (G^{(\epsilon_1)}_x - G^{(\epsilon_2)}_x)^2 \right]
\]

\[
= \frac{1}{2d} \sum_{x \in \mathbb{Z}^d} \mathbb{E}_P \left[ \mathbb{1}_{\Omega_0} \mathbb{1}_{\{x \in N_0(\omega)\}} (\Psi \circ \theta_x - \Psi)^2 \right],
\]

where \( \Psi = \psi_{\epsilon_1} - \psi_{\epsilon_2} \). Expanding the last expression we see that it equals to:

\[
\frac{1}{2d} \sum_{x \in \mathbb{Z}^d} \mathbb{E}_P \left[ \mathbb{1}_{\Omega_0} \mathbb{1}_{\{x \in N_0(\omega)\}} (\Psi^2 \circ \theta_x + \Psi^2 - 2\Psi \cdot \Psi \circ \theta_x) \right]. \tag{7.25}
\]

Since \( P \) is stationary under translation \( \theta_x \) when \( x \in N_0(\omega) \), we get that it can be written as

\[
2(\Psi, \Psi) - 2 \left( \Psi, \frac{1}{2d} \sum_{x \in \mathbb{Z}^d} \mathbb{E}_P \left( \mathbb{1}_{\Omega_0} \mathbb{1}_{\{x \in N_0(\omega)\}} \Psi \circ \theta_x \right) \right) = 2(\Psi, (1 - \Lambda)\Psi). \tag{7.26}
\]

Finally we evaluate \( (\Psi, (1 - \Lambda)\Psi) \):

\[
(\psi_{\epsilon_1} - \psi_{\epsilon_2}, (1 - \Lambda)(\psi_{\epsilon_1} - \psi_{\epsilon_2})) = \int_{-1}^{1} \left( \frac{1}{(1 + \epsilon_1 - \lambda)^2} - \frac{1}{(1 + \epsilon_2 - \lambda)^2} \right) (1 - \lambda) \mu_V(d\lambda)
\]

\[
= \int_{-1}^{1} \left( \frac{(\epsilon_1 - \epsilon_2)^2(1 - \lambda)}{(1 + \epsilon_1 - \lambda)^2(1 + \epsilon_2 - \lambda)^2} \right) \mu_V(d\lambda).
\]

The integrand here is again bounded by \( \frac{1}{1 - \lambda} \) for all \( \epsilon_1, \epsilon_2 > 0 \), and it tends to zero as \( \epsilon_1, \epsilon_2 \searrow 0 \). The claim now follows by the Dominated Convergence Theorem.

\[\square\]

Now we are finally ready to prove Theorem 7.2.
Proof of Theorem 7.2. Again we closely follow the proof of Theorem 2.2 in \[BB07\]. Let \(G^\epsilon_x \circ \theta_y\) be as in (7.21). Using (7.22) we know that \(G^\epsilon_x \circ \theta_y\) converges in \(L^2\) as \(\epsilon \searrow 0\). We denote the limit by \(G_{y,y+x} = \lim_{\epsilon \searrow 0} G^\epsilon_x \circ \theta_y\). Since \(G^\epsilon_x \circ \theta_y\) is a gradient field on \(P(\omega)\), we have \(G_{y,y+x}(\omega) + G_{y+y,x}(\omega) = 0\) and, more generally, \(\sum_{k=0}^{n-1} G_{x_k,x_{k+1}} = 0\) whenever \((x_0, x_1, \ldots, x_n)\) is a closed loop on \(P(\omega)\). Thus we may define

\[
\chi(x, \omega) := \sum_{k=0}^{n-1} G_{x_k,x_{k+1}}(\omega),
\]

(7.27)

where \((x_0, x_1, \ldots, x_n)\) is a “nearest neighbor” (in the sense of \(x_i \in N_{x_{i-1}}(\omega)\)) path on \(P(\omega)\) connecting \(x_0 = 0\) to \(x_n = x\). By the above “loop” conditions, the definition is independent of this path for almost every \(\omega \in \Omega_0 \cap \{ \omega : x \in P(\omega) \}\). The shift invariance (7.10) will now follow from the definition of \(\chi\) and the fact that \(G_{x,x+y} = G_{0,y} \circ \theta_x\). In light of the shift invariance, to prove the harmonicity of \(x \mapsto x + \chi(x, \omega)\) it is sufficient to show that, almost surely,

\[
\frac{1}{2d} \sum_{x \in N_0(\omega)} [x + \chi(x, \cdot)] = \chi(0, \cdot),
\]

(7.28)

which can be written as:

\[
\frac{1}{2d} \sum_{x \in N_0(\omega)} [\chi(0, \cdot) - \chi(x, \cdot)] = V(\omega).
\]

(7.29)

By the definition of \(\chi\) we have for \(x \in N_0(\omega)\) that \(\chi(x, \cdot) - \chi(0, \cdot) = G_{0,x}\), therefore the left hand side is the \(\epsilon \searrow 0\) limit of

\[
- \frac{1}{2d} \sum_{x \in \mathbb{Z}^d} G^\epsilon_x = \frac{1}{2d} \sum_{x \in \mathbb{Z}^d} \mathbb{1}_{N_0(\omega)}(\psi_\epsilon - \psi_\epsilon \circ \theta_x) = (1 - \Lambda) \psi_\epsilon.
\]

(7.30)

Using the definition of \(\psi_\epsilon\) (7.8), we get that \((1 - \Lambda) \psi_\epsilon = V - \epsilon \psi_\epsilon\). From here, using (7.20), we get that the \(\epsilon \searrow 0\) limit is indeed \(V\) in \(L^2\).

Finally, we need to show the square integrability (7.12). We note that, by the construction of the corrector, \n
\[
[\chi(x + y, \cdot) - \chi(x, \cdot)] \mathbb{1}_{\{x \in P(\omega)\}} \mathbb{1}_{\{y \in N_0(\omega)\}} \circ \theta_x = G_{x,x+y}.
\]

(7.31)

But \(G_{x,x+y}\) is the \(L^2\) limit of \(L^2\)-functions \(G^\epsilon_y \circ \theta_x\) whose \(L^2\) norm is bounded by that of \(G^\epsilon_y\). Hence (7.12) follows with \(C = \max_{\{x \in N_0(\omega)\}} \|G_{0,x}\|_2\). \(\square\)

8. Sublinearity along coordinate directions

We are now ready to start treating the main difficulty of the high dimensional Central limit theorem proof: the sublinearity of the corrector. In this section, we treat the sublinearity along the coordinate directions in \(\mathbb{Z}^d\). Fix \(e \in \mathcal{E}\). We define a sequence \(n_k^e(\omega)\) inductively by \(n_0^e(\omega) = f_\epsilon(\omega)\) and \(n_{k+1}^e = n_{k}^e(\sigma_e(\omega))\) where \(\sigma_e\) is the induced translation defined by \(\sigma_e = \theta_{f_\epsilon(\omega)}\). The numbers \(n_k^e\) are well-defined and finite almost surely. Let \(\chi\)
be the corrector defined in Theorem 7.2. The main goal of this section is to prove the following theorem:

**Theorem 8.1.** For \( P \)-almost all \( \omega \in \Omega_0 \)

\[
\lim_{k \to \infty} \frac{\chi(n_k^\varepsilon(\omega)e, \omega)}{k} = 0.
\] (8.1)

The proof of this theorem is based on the following properties of \( \chi(n_k^\varepsilon(\omega)e, \omega) \):

**Proposition 8.2.**

1. \( \mathbb{E}_P \left[ |\chi(n_1^\varepsilon(\omega)e, \cdot)| \right] < \infty. \)
2. \( \mathbb{E}_P \left[ |\chi(n_1^\varepsilon(\omega)e, \cdot)| \right] = 0. \)

**Proof.** Using the definition of the corrector (7.27), it follows that

\[
\chi(n_1^\varepsilon(\omega)e, \omega) = G_{0,n_1^\varepsilon(\omega)e}(\omega).
\] (8.2)

By (7.22), and since \( G_{0,n_1^\varepsilon(\omega)e}(\omega) \) is the \( \varepsilon \searrow 0 \) limit of \( G_{n_1^\varepsilon(\omega)e}(\cdot) \) in \( L^2 \), it follows that \( G_{0,n_1^\varepsilon(\omega)e}(\omega) \in L^2 \). Since \( P \) is a probability measure, it is in particular a finite measure, and therefore for every \( 1 \leq r < 2 \) it is also true that \( G_{0,n_1^\varepsilon(\omega)e}(\omega) \in L^r \). Taking \( r = 1 \) we find:

\[
\mathbb{E}_P \left[ |\chi(n_1^\varepsilon(\omega)e, \cdot)| \right] = \mathbb{E}_P \left[ |G_{0,n_1^\varepsilon(\omega)e}(\omega)| \right] < \infty.
\] (8.3)

In order to prove part (2), we again use the fact that \( G_{0,n_1^\varepsilon(\omega)e}(\omega) \) is the \( \varepsilon \searrow 0 \) limit in \( L^2 \) of \( G_{n_1^\varepsilon(\omega)e} \); and therefore it’s enough to show that for every \( \varepsilon > 0 \)

\[
\mathbb{E}_P \left[ G_{n_1^\varepsilon(\omega)e} \right] = 0.
\] (8.4)

and indeed

\[
\mathbb{E}_P \left[ G_{n_1^\varepsilon(\omega)e} \right] = \mathbb{E}_P \left[ \mathbb{1}_{\Omega_0} \mathbb{1}_{\{n_1^\varepsilon(\omega)e \in N_0(\omega)\}} (\psi_e \circ \sigma_e^{n_1^\varepsilon(\omega)} - \psi_e) \right]
\]

\[
= \mathbb{E}_P \left[ \mathbb{1}_{\Omega_0} \mathbb{1}_{\{n_1^\varepsilon(\omega)e \in N_0(\omega)\}} \psi_e \circ \sigma_e^{n_1^\varepsilon(\omega)} \right] - \mathbb{E}_P \left[ \mathbb{1}_{\Omega_0} \mathbb{1}_{\{n_1^\varepsilon(\omega)e \in N_0(\omega)\}} \psi_e \right]
\]

\[
= \mathbb{E}_P \left[ (\mathbb{1}_{\Omega_0} \mathbb{1}_{\{n_1^\varepsilon(\omega)e \in N_0(\omega)\}}) \psi_e \circ \sigma_e \right] - \mathbb{E}_P \left[ \mathbb{1}_{\Omega_0} \mathbb{1}_{\{n_1^\varepsilon(\omega)e \in N_0(\omega)\}} \psi_e \right],
\]

which equals zero by Theorem 2.1 and the fact that \( \psi_e \) is absolutely integrable since it is in \( L^2 \).

**Proof of Theorem 8.1.** Let \( g : \Omega \to \mathbb{R}^d \) be defined by \( g(\omega) = \chi(n_1^\varepsilon(\omega)e, \omega) \), and let \( \sigma_e \) be the induced shift in direction \( e \). Then

\[
\chi(n_k^\varepsilon(\omega)e, \omega) = \sum_{i=0}^{k-1} g \circ \sigma_e^i(\omega).
\] (8.5)

Using Proposition 8.2 we have that \( g \in L^1 \) and \( \mathbb{E}_P [g] = 0 \). Since Theorem 2.1 ensures \( \sigma_e \) is \( P \)-preserving and ergodic, the claim follows from Birkhoff’s Ergodic Theorem.

\( \square \)
9. Sublinearity everywhere

**Definition 9.1.** Given $K > 0$ and $\epsilon > 0$, we say that a site $x \in \mathbb{Z}^d$ is $K,\epsilon$-good in configuration $\omega \in \Omega$ if $x \in P(\omega)$ and

$$|\chi(y,\omega) - \chi(x,\omega)| < K + \epsilon|x - y|,$$

holds for every $y \in P(\omega)$ of the form $y = le$, where $l \in \mathbb{Z}$ and $e$ is a unit coordinate vector. We will use $G_{K,\epsilon} = G_{K,\epsilon}(\omega)$ to denote the set of $K,\epsilon$-good sites in configuration $\omega$.

**Theorem 9.2.** For every $\epsilon > 0$ and $P$-almost every $\omega \in \Omega_0$

$$\limsup_{n \to \infty} \frac{1}{(2n + 1)^d} \sum_{x \in P(\omega), |x| \leq n} 1_{\{|\chi(x,\omega)| \geq \epsilon n\}} \leq \epsilon.$$

Before stating the proof, we give a short introduction of the basic idea. This proof is a light modification of the proof from [BB07].

Fix the dimension $d$, and for each $\nu = 1, 2, \ldots, d$ let $\Lambda^n_\nu$ be the $\nu$-dimensional box

$$\Lambda^n_\nu = \{k_1 e_1 + \ldots + k_\nu e_\nu : k_i \in \mathbb{Z}, |k_i| \leq n, \forall i = 1, 2, \ldots, \nu\}.$$  (9.3)

We will run an induction over $\nu$-dimensional sections of the $d$-dimensional box $\{x \in \mathbb{Z}^d : |x| \leq n\}$. The induction eventually gives Theorem 9.2 for $\nu = d$ thus proving it. Since it is not advantageous to assume that $0 \in P(\omega)$, we will carry out the proof for differences of the form $\chi(x,\omega) - \chi(y,\omega)$ with $x, y \in P(\omega)$. For each $\omega \in \Omega$, we thus consider the (upper) density

$$Q_\nu(\omega) = \lim_{\epsilon \downarrow 0} \lim_{n \to \infty} \inf_{y \in P(\omega) \cap \Lambda^n_\nu} \frac{1}{|\Lambda^n_\nu|} \sum_{x \in P(\omega) \cap \Lambda^n_\nu} 1_{\{|\chi(x,\omega) - \chi(y,\omega)| \geq \epsilon n\}}.$$  (9.4)

Note that the infimum is taken only over sites in the one-dimensional box $\Lambda^n_1$. Our goal is to show by induction that $Q_\nu = 0$ almost surely for all $\nu = 1, \ldots, d$. The induction step is given by the following lemma:

**Lemma 9.3.** Let $1 \leq \nu < d$. If $Q_\nu = 0$ $P$-almost surely, then also $Q_{\nu+1} = 0$ $P$-almost surely.

Before we start the formal proof, we give the main idea: Suppose that $Q_\nu = 0$ for some $\nu < d$ $P$-almost surely. Pick $\epsilon > 0$. Then for $P$-almost every $\omega$ and all sufficiently large $n$, there exists a set of sites $\Delta \subset \Lambda^n_\nu \cap P(\omega)$ such that

$$|(\Lambda^n_\nu \cap P(\omega)) \setminus \Delta| \leq \epsilon |\Lambda^n_\nu|,$$  (9.5)

and

$$|\chi(x,\omega) - \chi(y,\omega)| \leq \epsilon n \quad \forall x, y \in \Delta.$$  (9.6)

Moreover, for $n$ sufficiently large, $\Delta$ could be picked so that $\Delta \cap \Lambda^n_1 \neq \emptyset$ and, assuming $K \gg 1$ the non-$K,\epsilon$-good sites could be pitched out with little loss of density to achieve even

$$\Delta \subset G_{K,\epsilon}.$$  (9.7)
that $P(0 \in G_{K,n})$ converges to $P(0 \in \Omega)$ as $k \to \infty$.)

As a result of this construction we have
\begin{equation}
|\chi(z, \omega) - \chi(x, \omega)| \leq K + \epsilon n,
\end{equation}
for any $x \in \Delta$ and any $z \in \Lambda_{n}^{\nu+1} \cap \mathcal{P}(\omega)$ of the form $x + je_{\nu+1}$. Thus, if $r, s \in \mathcal{P}(\omega) \cap \Lambda_{n}^{\nu+1}$ are of the form, $r = x + je_{\nu+1}$ and $s = y + ke_{\nu+1}$, then (9.8) implies
\begin{equation}
|\chi(r, \omega) - \chi(s, \omega)| \leq |\chi(r, \omega) - \chi(x, \omega)| + |\chi(x, \omega) - \chi(y, \omega)| + |\chi(y, \omega) - \chi(s, \omega)| \leq 2K + 2\epsilon n + |\chi(x, \omega) - \chi(y, \omega)|.
\end{equation}

Invoking the induction hypothesis (9.6), the right hand side is less than $2K + 3\epsilon n$, implying a bound of the type (9.6) but one dimension higher. Unfortunately, the above is not sufficient to prove (9.6) but all but a vanishing fraction of sites in $\Lambda_{n}^{\nu+1}$. The reason is that the $r'$s and $s'$s for which (9.9) holds, need to be of the form $x + je_{\nu+1}$ for some $x \in \Delta \cap \mathcal{P}(\omega)$. But $\mathcal{P}(\omega)$ will occupy only about a $P(0 \in \mathcal{P}(\omega))$ fraction of all sites in $\Lambda_{n}^{\nu}$, and so this argument does not permit to control more than a fraction of about $P(0 \in \mathcal{P}(\omega))$ of $\Lambda_{n}^{\nu+1} \cap \mathcal{P}(\omega)$.

To fix this problem, we will have to work with a "stack" of translates of $\Lambda_{n}^{\nu}$ simultaneously. Explicitly, consider the collection of $\nu$-boxes
\begin{equation}
\Lambda_{n,j}^{\nu} = \theta_{e_{\nu+1}}^j(\Lambda_{n}^{\nu}) \quad j = 1, 2, \ldots, L.
\end{equation}
Here $L$ is a deterministic number chosen so that, for a given $\delta > 0$, the set
\begin{equation}
\Delta_0 = \{x \in \Lambda_{n}^{\nu} : \exists j \in \{0, 1, \ldots, L - 1\}, x + je_{\nu+1} \in \Lambda_{n,j}^{\nu} \cap \mathcal{P}(\omega)\},
\end{equation}
is so large that for sufficiently large $n$
\begin{equation}
|\Delta_0| \geq (1 - \delta)|\Lambda_{n}^{\nu}|.
\end{equation}
These choices ensure that $(1 - \delta)$-fraction of $\Lambda_{n}^{\nu}$ is now "covered" which, by repeating the above argument, gives us control over $\chi(r, \omega)$ for nearly the same fraction of all sites $r \in \Lambda_{n}^{\nu+1} \cap \mathcal{P}(\omega)$.

Proof of Lemma 9.3. Let $\nu < d$ and suppose that $Q_{\nu} = 0$ $P$-almost surely. Fix $\delta > 0$ with $0 < \delta < \frac{1}{2} P(0 \in \mathcal{P}(\omega))^2$ and let $L$ be as defined above. Choose $\epsilon > 0$ so that
\begin{equation}
L \epsilon + \delta < \frac{1}{2} P(0 \in \mathcal{P}(\omega))^2.
\end{equation}
For a fixed but large $K$, $P$-almost every $\omega$ and $n$ exceeding an $\omega$-dependent quantity, for each $j = 1, 2, \ldots, L$, we can find $\Delta_j \subset \Lambda_{n,j}^{\nu} \cap \mathcal{P}(\omega)$ satisfying the properties (9.5-9.7) - with $\Lambda_{n}^{\nu}$ replaced by $\Lambda_{n,j}^{\nu}$. Given $\Delta_1, \ldots, \Delta_L$, let $\Lambda$ be the set of sites in $\Lambda_{n}^{\nu+1} \cap \mathcal{P}(\omega)$ whose projection onto the linear subspace $H = \{k_1 e_1 + \ldots + k_{\nu} e_{\nu} : k_i \in \mathbb{Z}\}$ belongs to the corresponding projection of $\Delta_1 \cup \ldots \cup \Delta_L$. Note that the $\Delta_j$ could be chosen so that $\Lambda \cap \Lambda_{n}^{1} \neq \emptyset$. By their construction, the projections of the $\Delta_j's$, $j = 1, \ldots, L$ onto $H$ "fail to
cover” at most $L\epsilon|\Lambda_1^n|$ sites in $\Delta_0$, and so at most $(\delta + L\epsilon)|\Lambda_1^n|$ sites in $|\Lambda_1^n|$ are not of the form $x + ie_{\nu+1}$ for some $x \in \bigcup_j \Delta_j$. It follows that

$$|(\Lambda_1^{n+1} \cap \mathcal{P}(\omega)) \setminus \Lambda| \leq (\delta + L\epsilon)|\Lambda_1^{n+1}|,$$  \hfill (9.14)

i.e. $\Lambda$ contains all except at most $(\delta + L\epsilon)$-fraction of all sites in $\Lambda_1^{n+1}$ that we care about.

Next we note that if $K$ is sufficiently large, then for every $1 \leq i < j \leq L$, the set $\mathbb{H}$ contains $\frac{1}{2}P(0 \in \mathcal{P}(\omega))$-fraction of sites such that

$$z_i \overset{\text{def}}{=} x + ie_\nu \in \mathcal{G}_{K,\epsilon}, \quad z_j = x_j e_\nu \in \mathcal{G}_{K,\epsilon}. \hfill (9.15)$$

Since we assumed $\overset{(9.13)}{\text{true}}$, once $n \gg 1$, for each pair $(i,j)$ with $1 \leq i < j \leq L$ such $z_i$ and $z_j$ can be found so that $z_i \in \Delta_i$ and $z_j \in \Delta_j$. But the $\Delta_j$'s were picked to make $\overset{(9.6)}{\text{true}}$ and so using these pairs of sites we now show that

$$|\chi(y,\omega) - \chi(x,\omega)| \leq |\chi(y,\omega) - \chi(z_j,\omega)| + |\chi(z_j,\omega) - \chi(z_i,\omega)| + |\chi(z_i,\omega) - \chi(x,\omega)| \leq en + K + \epsilon L + en = K + \epsilon L + 2en,$$  \hfill (9.16)

for every $x, y \in \Delta_1 \cup \ldots \cup \Delta_L$. From $\overset{(9.6)}{\text{and}} (9.16)$, we now conclude that for all $r, s \in \Lambda$,

$$|\chi(r,\omega) - \chi(s,\omega)| \leq 3K + \epsilon L + 4en < 5en,$$  \hfill (9.17)

assuming that $n$ is so large that $en > 3K + \epsilon L$. If $Q_{\epsilon,\nu}$ denotes the right-hand side of $\overset{(9.4)}{\text{true}}$ before taking $\epsilon \searrow 0$, the bounds $\overset{(9.14)}{\text{and}} (9.17)$ and the fact that $\Lambda \cap \Lambda_1^n \neq \emptyset$ yield

$$Q_{\nu+1.5\epsilon}(\omega) \leq \delta + L\epsilon,$$  \hfill (9.18)

for $P$-almost every $\omega$, But the left-hand side of this inequality increases as $\epsilon \searrow 0$ while the right hand side decreases. Thus, taking $\epsilon \searrow 0$ and $\delta \searrow 0$ proves that $Q_{\nu+1} = 0$ holds $P$-almost surely. \hfill \square

**Proof of Theorem 9.2** The proof is an easy consequence of Lemma 9.3. First, by Theorem \overset{8.1}{\text{we know that}} $Q(\omega) = 0$ for $P$-almost every $\omega$. Invoking appropriate shifts, the same conclusion applies $Q$ almost surely. Using induction on dimension, Lemma 9.3 then tells us that $Q_d(\omega) = 0$ for $P$ almost every $\omega$. Let $\omega \in \Omega_0$. By Theorem \overset{8.1}{\text{for each}} $\epsilon > 0$ there is $n_0 = n_0(\omega)$ with $P(n_0 < \infty) = 1$ such that for all $n \geq n_0(\omega)$, we have $|\chi(x,\omega)| \leq en$ for all $x \in \Lambda_1^n \cap \mathcal{P}(\omega)$. Using this to estimate away the infimum in $\overset{(9.4)}{\text{true}}$, the fact that $Q_d = 0$ now immediately implies $\overset{(9.2)}{\text{true}}$ for all $\epsilon > 0$. \hfill \square

### 10. High dimensional Central Limit Theorem

The theorem we wish to prove in this section is the following:

**Theorem 10.1.** Fix $d \geq 2$. Assume the additional assumption, assumption $\overset{[1.4]}{\text{true}}$ then for $P$ almost every $\omega \in \Omega_0$

$$\lim_{n \rightarrow \infty} \frac{X_n}{\sqrt{n}} \overset{D}{=\text{N}}(0,D), \hfill (10.1)$$

where $N(0,D)$ is a $d$-dimensional multivariate normal distribution with covariance matrix $D$ that depends only on $d$ and the distribution of $P$. $


Proof. Let
\[ M_n^{(\omega)} = X_n + \chi(X_n, \omega), \quad \forall n \geq 0. \] (10.2)
Then \( \{M_n^{(\omega)}\}_{n=0}^\infty \) is an \( L^2 \)-martingale for the filtration \( \{\sigma(X_0, X_1, \ldots, X_n)\}_{n=0}^\infty \). Moreover, conditional on \( X_{k_0} = x \), the increments \( \{M_k^{(\omega)} - M_{k_0}^{(\omega)}\}_{k=0}^\infty \) have the same law as \( \{M_k^{(\theta \omega)}\}_{k=0}^\infty \).

**Proof.** Since \( X_n \) is bounded, \( \chi(X_n, \omega) \) is bounded and so \( M_n^{(\omega)} \) is square integrable with respect to \( P_\omega \). Since \( x \mapsto x + \chi(x, \omega) \) is harmonic with respect to the transition probabilities of the random walk \( (X_n) \) with law \( P_\omega \) we have
\[ E_\omega[M_{n+1}^{(\omega)}|\sigma(X_n)] = M_n^{(\omega)} \quad \forall n \geq 0, P_\omega \text{-a.s.} \] (10.3)
Since \( M_n^{(\omega)} \) is \( \sigma(X_n) \)-measurable, \( (M_n^{(\omega)}) \) is a martingale. The stated relation between the laws of \( (M_k^{(\omega)} - M_{k_0}^{(\omega)})_{k \geq 0} \) and \( (M_k^{(\theta \omega)})_{k \geq 0} \) is implied by the shift invariance proved in Theorem 7.10 and the fact that \( (M_n^{(\omega)}) \) is a simple random walk on the deformed graph. \( \square \)

**Theorem 10.3** (The Modified random walk CLT). Fix \( d \geq 2 \), and assume in addition, assumption 7.4. For \( \omega \in \Omega_0 \) let \( \{X_n\}_{n=0}^\infty \) be random walk with transition probabilities \( (1.3) \) and let \( \{M_n^{(\omega)}\}_{n=0}^\infty \) be as defined in (10.2). Then for \( P \) almost every \( \omega \in \Omega_0 \) we have
\[ \lim_{n \to \infty} \frac{M_n^{(\omega)}}{\sqrt{n}} \overset{D}{\to} N(0, D), \] (10.4)
where \( N(0, D) \) is a \( d \)-dimensional multivariate normal distribution with covariance matrix \( D \) which depends only on \( d \) and the distribution \( P \), and is given by \( D_{i,j} = E \left[ \text{cov}(M_1^{(\omega)} \cdot e_i, M_1^{(\omega)} \cdot e_j) \right] \).

**Proof.** Let
\[ V_n^{(\omega)}(\epsilon) = \frac{1}{n} \sum_{k=0}^{n-1} E_\omega \left[ D_k^{(\omega)} \mathbb{1}_{\{|\min_{i,j} |(D_k^{(\omega)})_{i,j}| \geq \epsilon \sqrt{n}\}} X_0, X_1, \ldots, X_k \right], \] (10.5)
where \( D_k^{(\omega)} \) is the covariance matrix for \( M_{k+1}^{(\omega)} - M_k^{(\omega)} \). By the Lindeberg-Feller Central Limit Theorem (see for example [Dur96]), it is enough to show that

1. \( \lim_{n \to \infty} V_n^{(\omega)}(0) = D \) in \( P_\omega \)-probability.

2. \( \lim_{n \to \infty} V_n^{(\omega)}(\epsilon) = 0 \) in \( P_\omega \)-probability for all \( \epsilon > 0 \).
Both conditions are implied from Theorem 2.5. Indeed

\[ V_n^{(\omega)}(0) = \frac{1}{n} \sum_{k=0}^{n-1} h_0 \circ \theta_{X_k}(\omega), \]

where

\[ h_K(\omega) = \mathbb{E}\omega[ D_1^{(\omega)} \mathbb{1}_{\{\min_{i,j} |(D_1^{(\omega)})_{i,j}| \geq K\}}]. \quad (10.6) \]

Therefore by Theorem 2.5 we have for \( P \)-almost every \( \omega \in \Omega_0 \)

\[ \lim_{n \to \infty} V_n^{(\omega)}(0) = \mathbb{E}[h_0(\omega)] = D. \quad (10.7) \]

On the other hand, for every \( K \in \mathbb{R} \) and every \( \epsilon > 0 \) we have \( \epsilon \sqrt{n} > K \) for sufficiently large \( n \), and therefore \( f_{\epsilon \sqrt{n}} \leq f_K \). So \( P \)-almost surely

\[ \limsup_{n \to \infty} V_n^{(\omega)}(\epsilon) \leq \mathbb{E}\left[D_1^{(\omega)} \mathbb{1}_{\{\min_{i,j} |(D_1^{(\omega)})_{i,j}| \geq K\}}\right] \quad (10.8) \]

Where in order to apply the Dominated Convergence, we used the fact that \( M^{(\omega)}_1 \in L^2 \)

\[ \square \]

We are now ready to prove the high dimensional Central Limit Theorem

**Proof of Theorem 1.10.** Due to Theorem 10.3 it is enough to prove that for \( P \)-almost every \( \omega \in \Omega_0 \)

\[ \lim_{n \to \infty} \frac{\chi(X_n,\omega)}{\sqrt{n}} = 0 \quad P_\omega \text{ a.s.} \quad (10.9) \]

This will follow if we will show that there exists a constant \( K > 0 \) such that for every \( \epsilon > 0 \) and for \( P \)-almost every \( \omega \in \Omega_0 \)

\[ \lim_{n \to \infty} P_\omega\{|\chi(X_n,\omega)| > \epsilon \sqrt{n} \} < K \epsilon. \quad (10.10) \]

By Theorem 6.1 and the Markov inequality, there exists a random \( c = c(\omega > 0, P \text{ almost surely finite}, \text{ such that that for } P \text{-almost every } \omega \in \Omega_0 \)

\[ P_\omega \left[ \|X_n\| > \frac{1}{\epsilon} \sqrt{n} \right] \leq \frac{\mathbb{E}_\omega(\|X_n\|)}{\sqrt{n}} \leq c \epsilon. \quad (10.11) \]

We therefore get

\[ P_\omega \left( |\chi(X_n,\omega)| > \epsilon \sqrt{n} \right) \leq P_\omega \left( \|X_n\| > \frac{\sqrt{n}}{\epsilon} \right) + P_\omega \left( \chi(X_n,\omega) > \epsilon \sqrt{n}, \|X_n\| \leq \frac{\sqrt{n}}{\epsilon} \right). \]

By (10.11) we find that this is less or equal than

\[ c \epsilon + \sum_{x \in P(\omega)} P_\omega^n(0,x) \mathbb{1}_{\{\chi(x,\omega)| > \epsilon \sqrt{n}, x \in \left[ -\frac{\sqrt{n}}{\epsilon}, \frac{\sqrt{n}}{\epsilon} \right]\}}. \]

Using now Theorem 5.6 for sufficiently \( n \) if follows that

\[ P_\omega \left( |\chi(X_n,\omega)| > \epsilon \sqrt{n} \right) \leq c \epsilon + \frac{K_1}{(n-K_2)^{\frac{d}{2}}} \sum_{x \in P(\omega) \cap \left[ -\frac{\sqrt{n}}{\epsilon}, \frac{\sqrt{n}}{\epsilon} \right]} \mathbb{1}_{\{\chi(x,\omega)| > \epsilon \sqrt{n}\}}. \]
Therefore by Theorem 9.2 we get that there exist constants $c_0, K$ such that
\[
\lim_{n \to \infty} P_\omega(|\chi(X_n, \omega)| > \epsilon \sqrt{n}) \leq c \epsilon + c_0 \epsilon^2 \leq K \epsilon
\]
As required. □

11. SOME CONJECTURES AND QUESTIONS

While we have full classification of transience recurrence of random walks on discrete point processes in dimensions $d = 1$ and $d \geq 3$, we only have a partial classification in dimension 2. We therefore give the following two conjectures:

Conjecture 11.1. There are transient two dimensional random walks on discrete point processes.

Conjecture 11.2. The condition given in Theorem 1.7 for recurrence of 2-dimensional random walk on discrete point process, i.e, the existence of a constant $C > 0$ such that
\[
\sum_{k=N}^{\infty} k \cdot \frac{P(f_{e_i} = k)}{\mathbb{E}(f_{e_i})} \leq C \frac{1}{N} \quad i \in \{1, 2\} \quad N \in \mathbb{N}
\] (11.1)
is not necessary.

In Theorem 1.10 we gave conditions for the random walk on discrete point processes to satisfy a Central Limit Theorem. However, we didn’t give any example for a random walk without a Central Limit Theorem. We therefore give the following conjecture:

Conjecture 11.3. There are random walks on discrete point processes in high dimensions that don’t satisfy a Central Limit Theorem.

In the proof of Theorem 1.10 we used the additional assumption that there exists $\epsilon_0 > 0$ such that for every coordinate direction $e$ $E_P[f_{e}^2 + \epsilon_0] < \infty$. The assumption that the second moments are finite, is fundamental in our proof in order to build the corrector, and seems to be necessary for the CLT to hold. On the other hand, existence of such $\epsilon_0 > 0$ though needed in our proof, was used only in order to bound (6.7). We therefore give the following condition:

Conjecture 11.4. Theorem 1.10 is true even with the weak assumption that only the second moments are finite.

Even if the theorem is true with the weak assumption that only the second moment of the distances between points is finite, we can still ask the following question:

Conjecture 11.5. Is the condition given in Theorem 1.10 also necessary, or can one find examples for random walks on discrete point processes that satisfy a Central Limit Theorem but don’t have all of their second moments finite? We conjecture that such examples exist, but didn’t verified it.
We also have the following conjecture about the Central Limit Theorem:

**Conjecture 11.6.** Under assumptions 1.1 and 1.4, the Central Limit Theorem, 1.10, can be strengthened as follows: Random walk on discrete point process under appropriate scaling converges to Brownian motion.

Our model describes non nearest neighbors random walk on random subset of $\mathbb{Z}^d$ with uniform transition probabilities. We suggest the following generalization of the model:

**Question 11.7.** Fix $\alpha \in \mathbb{R}$. We look on the same model for the environments with transition probabilities as follows: for $\omega \in \Omega_0$

$$P_\omega(X_{n+1} = u | X_n = v) = \begin{cases} 
1/Z(v) \|u - v\|^{\alpha} & u \notin N_v(\omega) \\
0 & u \in N_v(\omega)
\end{cases}, \quad (11.2)$$

where $Z(v)$ is normalization constant (The case $\alpha = 0$ is the uniform distribution case). What can be proved about the extended model?

**APPENDIX**

In this Appendix we prove there exists a constant $c = c(d) > 0$ such that for every $0 < a \leq 2$

$$\sum_{n=0}^{\infty} e^{-a^{2^n} \cdot 2^{nd}} \leq ca^{-d} \quad (11.3)$$

**Proof.** First, we can restrict ourselves to $0 < a < \epsilon$ for any fixed $\epsilon > 0$. This follows from the fact that both expressions are monotonic in $a$. Next we note that:

$$\sum_{n=0}^{\infty} e^{-a^{2^n} \cdot 2^{nd}} \leq 1 + \sum_{n=1}^{\infty} \sum_{k=2^{n-1}d}^{2^nd} e^{-a^{2^n} \cdot 2^{nd}} \leq \frac{1}{1 - 2^{-d}} \sum_{k=0}^{\infty} e^{-ak^{1/d}}$$

$$= \frac{1}{1 - 2^{-d}} \sum_{j=0}^{\infty} e^{-aj} \# \{j < k^{1/d} \leq j + 1\} = \frac{1}{1 - 2^{-d}} \sum_{j=0}^{\infty} e^{-aj} [(j + 1)^d - j^d].$$

Since there exists a constant $c = c(d) > 0$ such that for every $j \geq 0$ we have $(j + 1)^d - j^d \leq cj^{d-1}$ the last term is less than or equal to

$$\frac{c}{1 - 2^{-d}} \sum_{j=0}^{\infty} e^{-aj} j^{d-1}.$$
For $j \geq 0$ denote $\alpha_j = e^{-aj} j^{d-1}$ and define $j_0 = \min \left\{ j \geq 0 : \forall i \geq j \quad \frac{\alpha_{i+1}}{\alpha_i} < e^{-a/2} \right\}$. From the definition of $j_0$ it follows that (11) is less than

\[
\frac{c}{1 - 2^{-d}} \left[ j_0^{-1} + \sum_{j=j_0}^{\infty} a_j + \sum_{j=j_0}^{\infty} a_j e^{-a(j-j_0)/2} \right] \leq \frac{c}{1 - 2^{-d}} \left[ j_0^{-1} + \frac{\alpha_{j_0}}{1 - e^{-a/2}} \right].
\]  

(11.4)

From the definition of $j_0$ one can see that $j_0 = \left\lceil \frac{1}{e^{a/2} - 1} \right\rceil \leq \left\lceil \frac{2d}{a} \right\rceil$, and therefore (11) equals to

\[
\frac{c}{1 - 2^{-d}} \left( 1 + \frac{e^{-a} \left\lceil \frac{2d}{a} \right\rceil}{\frac{\alpha_{j_0}}{2} e^{-a/2}} \right) \left\lceil \frac{2d}{a} \right\rceil^{-1}
\]

which for an appropriate constant $c = c(d) > 0$ is less than $ca^{-d}$, as required. \( \square \)

References

[Bar04] Martin T. Barlow. Random walks on supercritical percolation clusters. *Ann. Probab.*, 32(4):3024–3084, 2004.

[BB07] Noam Berger and Marek Biskup. Quenched invariance principle for simple random walk on percolation clusters. *Probab. Theory Related Fields*, 137(1-2):83–120, 2007.

[BBHK08] N. Berger, M. Biskup, C. E. Hoffman, and G. Kozma. Anomalous heat-kernel decay for random walk among bounded random conductances. *Ann. Inst. Henri Poincaré Probab. Stat.*, 44(2):374–392, 2008.

[Ber02] Noam Berger. Transience, recurrence and critical behavior for long-range percolation. *Comm. Math. Phys.*, 226(3):531–558, 2002.

[BG08] Erwin Bolthausen and Ilya Goldsheid. Lingering random walks in random environment on a strip. *Comm. Math. Phys.*, 278(1):253–288, 2008.

[BP07] Marek Biskup and Timothy M. Prescott. Functional CLT for random walk among bounded random conductances. *Electron. J. Probab.*, 12:no. 49, 1323–1348 (electronic), 2007.

[Bré02] Julien Brémond. On some random walks on $\mathbb{Z}$ in random medium. *Ann. Probab.*, 30(3):1266–1312, 2002.

[BS02] Erwin Bolthausen and Alain-Sol Sznitman. *Ten lectures on random media*, volume 32 of *DMV Seminar*. Birkhäuser Verlag, Basel, 2002.

[CFG08] P. Caputo, A. Faggionato, and A. Gaudilliere. Recurrence and transience for long-range reversible random walks on a random point process, 2008.

[CFP09] P. Caputo, A. Faggionato, and T. Prescott. Invariance principle for mott variable range hopping and other walks on point processes. *Arxiv preprint arXiv:0912.4591*, 2009.

[CS09] Nicholas Crawford and Allan Sly. Heat kernel upper bounds on long range percolation clusters. 2009.

[DP96] Jean-Dominique Deuschel and Agoston Pisztora. Surface order large deviations for high-density percolation. *Probability Theory and Related Fields*, 104:467–482, 1996.

[DS84] Peter G. Doyle and J. Laurie Snell. *Random walks and electric networks*, volume 22 of *Carus Mathematical Monographs*. Mathematical Association of America, Washington, DC, 1984.

[Dur96] Richard Durrett. *Probability: theory and examples*. Duxbury Press, Belmont, CA, second edition, 1996.
Barry D. Hughes. *Random walks and random environments*. Oxford Science Publications. The Clarendon Press Oxford University Press, New York, 1996. Random environments.

Eric S. Key. Recurrence and transience criteria for random walk in a random environment. *Ann. Probab.*, 12(2):529–560, 1984.

C. Kipnis and S. R. S. Varadhan. Central limit theorem for additive functionals of reversible Markov processes and applications to simple exclusions. *Comm. Math. Phys.*, 104(1):1–19, 1986.

R. Lyons and Y Peres. *Probability on Trees and Networks*. Cambridge University Press, in progress. Current version published on the web at http://php.indiana.edu/~rdlyons, 2004.

B. Morris and Yuval Peres. Evolving sets, mixing and heat kernel bounds. *Probab. Theory Related Fields*, 133(2):245–266, 2005.

P. Mathieu and A. Piatnitski. Quenched invariance principles for random walks on percolation clusters. *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.*, 463(2085):2287–2307, 2007.

Amos Nevo and Elias M. Stein. A generalization of Birkhoff’s pointwise ergodic theorem. *Acta Math.*, 173(1):135–154, 1994.

Pál Révész. *Random walk in random and non-random environments*. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, second edition, 2005.

Vladas Sidoravicius and Alain-Sol Sznitman. Percolation for the vacant set of random interlacements. *Comm. Pure Appl. Math.*, 62(6):831–858, 2009.

S. R. S. Varadhan. Random walks in a random environment. *Proc. Indian Acad. Sci. Math. Sci.*, 114(4):309–318, 2004.

Ofer Zeitouni. Random walks in random environment. In *Lectures on probability theory and statistics*, volume 1837 of *Lecture Notes in Math.*, pages 189–312. Springer, Berlin, 2004.