Efficient Convergent Maximum Likelihood Decoding on Tail-Biting Trellises

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Abstract
An algorithm for exact maximum likelihood (ML) decoding on tail-biting trellises is presented, which exhibits very good average case behavior. An approximate variant is proposed, whose simulated performance is observed to be virtually indistinguishable from the exact one at all values of signal to noise ratio, and which effectively performs computations equivalent to at most two rounds on the tail-biting trellis. The approximate algorithm is analyzed, and the conditions under which its output is different from the ML output are deduced. The results of simulations on an AWGN channel for the exact and approximate algorithms on the 16 state tail-biting trellis for the (24,12) Extended Golay Code, and tail-biting trellises for two rate 1/2 convolutional codes with memories of 4 and 6 respectively, are reported. An advantage of our algorithms is that they do not suffer from the effects of limit cycles or the presence of pseudocodewords.

I. INTRODUCTION
Tail-biting trellises are perhaps the simplest instances of decoding graphs with cycles. A tail-biting trellis has a Tanner graph [31] with a single cycle and usually approximate algorithms are used for decoding, as exact algorithms are believed to be too expensive. These approximate algorithms iterate around the trellis until either convergence is reached, or for a preset number of cycles. To the best of our knowledge, no exact decoding algorithms other than the brute force algorithm have been proposed so far for the general case, though there are several approximate algorithms for maximum-likelihood decoding [28], [22], [34], [33], [7], [20] and exact algorithms for bounded distance decoding [4]. The problem of Maximum A-Posteriori Probability (MAP) decoding is not addressed here. We propose an exact recursive algorithm, which exhibits very good average case behavior. The algorithm exploits the fact that a linear tail-biting trellis can be viewed as a coset decomposition

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of the group corresponding to the linear code with respect to a specific subgroup and is based on the \( A^* \) algorithm [23]. We also propose two approximate variants that always converge, and observe their performance on tail-biting trellises for the (24,12) extended Golay code and two convolutional codes of rate 1/2 and memory of 4 and 6 respectively. The performance of the first approximate variant is indistinguishable from that of the exact algorithm in terms of bit error rate for the two convolutional codes, and it is guaranteed to update each node in the tail-biting trellis at most twice i.e it performs a computation equivalent to at most two rounds on the trellis. Section II briefly mentions related work. Section III provides some background. Section IV describes the algorithm, while Section V analyses the algorithm. Section VI describes the approximate algorithm and provides an analysis for its good performance. Section VII reports the results of simulations on an AWGN channel and section VIII concludes the paper.

II. RELATED WORK

Aji et al. [3] have shown that iterative maximum-likelihood (ML) decoding on tail-biting trellises will asymptotically converge to exact maximum likelihood decoding for certain codes. They provide experimental evidence that practically ML decoding is achieved for the (8, 4) Hamming code with five rounds of the tail-biting trellis. The presence of pseudocodewords sometimes results in sub-optimal decoding and it is also possible to have situations where the iterative message passing algorithm does not converge. Several maximum likelihood decoding algorithms on tail-biting trellises have been proposed without a theoretical analysis [22], [33], [34], [30], [28], [20], but with good experimental results. Most of these are sub-optimal algorithms in that they may not produce the exact maximum-likelihood result on termination. Anderson and Hladik [4] have given an algorithm that is optimal for bounded distance decoding. The \( A^* \) algorithm [23] has been used for maximum likelihood soft decision decoding on conventional trellises for block codes [10], [9], [19], [11], [12], [26]. In [10] Han et al. propose the use of the \( A^* \) algorithm for ML decoding of block codes on their conventional trellises and report significant experimental gains in decoding complexity for signal to noise ratios ranging from 5 dB to 8 dB. This algorithm has been analyzed in [14] and shown to be efficient for many practical communication systems. In [11] a modified algorithm is proposed which searches through error patterns instead of codewords and similar gains are reported. Heuristic search algorithms are proposed in [26] which combine previously proposed algorithms and are able to outperform other practical decoders. A tutorial paper on the application of the \( A^* \) algorithm to soft decision decoding appears in [9]. Sorokine and Kschischang [19] propose a metric called the variable bias term that is used in an \( A^* \) algorithm, which has low computational complexity. Aguado and Farrell [1] discuss modified sequential algorithms on conventional trellises for block codes, which offer reduced complexity in comparison with the original stack algorithm [15] for sequential decoding. Han et al. [13] propose a trellis based ML soft-decision decoder for convolutional codes which uses a stack and a metric that ensures ML decoding.

III. BACKGROUND

We first present some background on tail-biting trellises. Tail-biting trellises for convolutional codes were introduced in [30]. Minimal tail-biting trellises for block codes have been discussed in [6], [16], [17].
Definition 3.1: A tail-biting trellis $T = (V, E, F_q)$ of depth $n$ is an edge-labeled directed graph with the property that the set $V$ can be partitioned into $n$ vertex classes

$$V = V_0 \cup V_1 \cup \ldots \cup V_{n-1}$$

such that every edge in $T$ is labeled with a symbol from the alphabet $F_q$, and begins at a vertex of $V_i$ and ends at a vertex of $V_{i+1 \pmod n}$, for some $i \in \{0, 1, \ldots, n-1\}$.

We identify $I$ the set of time indices with $\mathbb{Z}_n$, the residue classes of integers modulo $n$. An interval of indices $[i, j]$ represents the sequence $\{i, i+1, \ldots, j\}$ if $i < j$, and the sequence $\{i, i+1, \ldots, n-1, 0, \ldots, j\}$ if $i > j$. Every cycle in $T$ starting at a vertex of $V_0$ defines a vector $(a_1, a_2, \ldots, a_n) \in F_q^n$ which is an edge-label sequence. We assume that every vertex and every edge in the tail-biting trellis lies on some cycle, that is the tail-biting trellises we are dealing with are reduced [17]. The trellis $T$ represents a block code $C$ over $F_q$ if the set of all edge-label sequences in $T$ is equal to $C$. Let $C(T)$ denote the code represented by a trellis $T$.

A linear tail-biting trellis, for an $(n, k)$ linear block code $C$ over $F_q$ can be constructed as a trellis product [18] of the representation of the individual trellises (called elementary trellises) corresponding to each of the $k$ rows of the generator matrix $G$ for $C$ [17]. Let $T_1$ and $T_2$ be the component trellises. The set of vertices $V_i(T_1 \times T_2)$ of the product trellis $T_1 \times T_2$ at time index $i$, is just the Cartesian product of the vertices of the component trellises. Thus $V_i(T_1 \times T_2) = V_i(T_1) \times V_i(T_2)$. Consider $E_i(T_1) \times E_i(T_2)$, and interpret an element $((v_1, \alpha_1, v'_1), (v_2, \alpha_2, v'_2))$ in this product, where $v_i, v'_i$ are vertices and $\alpha_1, \alpha_2$ edge labels, as the edge $((v_1, v_2), \alpha_1 + \alpha_2, (v'_1, v'_2))$ where $+$ denotes addition in the field $F_q$. If we define the $i^{th}$ section as the set of edges connecting the vertices at time index $i$ to those at time index $i+1$, then the edge count in the $i^{th}$ section is the product of the edge counts in the $i^{th}$ section of the individual trellises.

Let $\{g_1, g_2, \ldots, g_k\}$ be the rows of a generator matrix $G$ for the linear code $C$. Each vector $g_i$ generates a one-dimensional subcode of $C$, which we denote by $(g_i)$. Therefore $C = (g_1) + (g_2) + \cdots + (g_k)$, and the trellis representing $C$ is given by $T = T_1 \times T_2 \times \cdots \times T_k$, where $T_i$ is the trellis for $(g_i)$, $1 \leq i \leq k$. To specify the component trellises in the trellis product above, we will need to introduce the notions of linear[18] and circular spans [17] and elementary trellises [18], [17]. Given a codeword $c = (c_1, c_2, \ldots, c_n) \in C$, the linear span of $c$, is the smallest interval $[i, j] \in I = \{1, 2, \ldots, n\}, i \leq j$ which contains all the non-zero positions of $c$. A circular span has exactly the same definition with $i > j$. Note that for a given vector, the linear span is unique, but circular spans are not-- they depend on the runs of consecutive zeros chosen for the complement of the span with respect to the index set $I$. For a vector $x = (x_1, \ldots, x_n)$ over the field $F_q$ and a specified span $[i, j]$, there is a unique linear elementary trellis representing $(x)$ [17]. This trellis has $q$ vertices at time indices $i$ to $(j - 1) \mod n$, and a single vertex at other positions. Consequently, $T_i$ in the trellis product mentioned earlier, is the elementary trellis representing $(g_i)$ for some choice of span (either linear or circular). Koetter and Vardy [17] have shown that any linear trellis, conventional or tail-biting can be constructed from a generator matrix whose rows can be partitioned into two sets, those which have linear span, and those taken to have circular span. The trellis for the code is formed as a product of the elementary trellises corresponding to these rows. We will represent such a generator matrix as $G_{KV} = \begin{bmatrix} G_l \\ G_c \end{bmatrix}$, where $G_l$ is the submatrix consisting of rows with linear span, and $G_c$ the submatrix of rows with circular span.
Definition 3.2: For a vector \( v \) of circular span \([i, j]\) in \( G_c \), the interval \([j \mod n, (i - 1) \mod n]\) is called the zero run of the vector.

The path in the trellis corresponding to this vector shares all states at time indices in the zero run with the path corresponding to the all-zero codeword in the product trellis.

For example, consider the codeword 0100011 with circular span \([6, 2]\). This has zero run \([2, 5]\). The elementary trellis corresponding to this vector has state cardinality profile \((2, 2, 1, 1, 1, 1, 2)\). (Recall the time indices are numbered from 0 to \( n - 1 \) where \( n \) is the length of the code).

As an example we display a tail-biting trellis for a binary \((7, 4)\) Hamming code. Though this is not a minimal trellis for the code, it serves to illustrate some of the definitions above. The spans of the rows are shown alongside the rows. All spans \([i, j]\) with \( i \) greater than \( j \) are circular spans.

Example 3.1: Let \( C \) be a \((7, 4)\) Hamming code with a product generator matrix \( G_{KV} \) defined as

\[
G_{KV} = \begin{bmatrix}
1 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 & 0 & 0 & 1
\end{bmatrix}
\]

The product tail-biting trellis for this generator matrix is given in Figure 1.

Fig. 1. A product tail-biting trellis for the \((7, 4)\) binary Hamming code.

Definition 3.3: A subtrellis of a tailbiting trellis consists of a start node at time index zero and all edges and nodes which can be traversed in any cycle of the graph that begins and ends at this start node.

Let \( T_l \) denote the minimum conventional trellis for the code generated by \( G_l \). Clearly \( T_l \) is a subtrellis of the tail-biting trellis. If \( l \) is the number of rows of \( G \) with linear span and \( c \) the number of rows of circular span, the tail-biting trellis constructed using the product construction will have \( q^c \) start states. Each such start state defines a subtrellis whose codewords form a coset of the subcode corresponding to the subtrellis containing the all 0 codeword. The coset structure is well known and has been reported in [29], [24], [8], [27], [30]. Each
vector in the circular span can be considered to be a coset leader. The set of zero runs, of the coset leaders determines the structure of the tail-biting trellis in the following way. If a coset leader has zero run $[i, j]$ then the subtrellis associated with that coset shares all states at time indices in the interval $[i, j]$ with the subtrellis corresponding to the subcode defined by vectors of linear span. Further, we recall, the coset leader shares all states in the interval $[i, j]$ with the states corresponding to the all-zero codeword.
The four subtrellises of the tail-biting trellis of Figure 1 are shown in Figures 2, 3, 4, and 5 along with their associated coset leaders and zero runs.

**Definition 3.4:** If subtrellises $T_1$ and $T_2$ share states from time indices $i$ to $j$ then the interval $[i, j]$ is called the *merging* interval of $T_1$ and $T_2$.

It is easy to see that two subtrellises do not share any states outside their merging interval.

A tail-biting trellis is said to satisfy the *intersection property* if the intersection of all the zero runs of the members of $G_c$ is non-empty. The tail-biting trellis for the Hamming code given in Example 3.1 satisfies the intersection property as the interval $[4, 5]$ is contained in the intersection of all the zero runs of $G_c$.

### IV. Decoding

The decoding algorithm proposed here is different from the sub-optimal algorithms mentioned in Section II that go round and round the tail-biting trellis updating all the nodes of the trellis in every round. It makes one round of the tail-biting trellis and subsequently judiciously uses the information gathered to further update as few nodes as it can before it closes in on the most likely codeword. Our algorithm has two phases. In the first phase a Viterbi algorithm is performed on the tail-biting trellis. This phase performs computations at every node of the tail-biting trellis. In the second phase however, *only one path* is tracked at a time, this being the most likely path. The initial estimate of the most likely path is obtained from the first phase. This path is present in some subtrellis and is followed until the algorithm decides that some other path (perhaps in another subtrellis) looks more promising based on some metric. When such a situation is encountered, computation on this path is suspended and the more promising path is taken up. While this strategy at first glance looks like the stack algorithm [15] for decoding convolutional codes, it differs from it because it has the property that it *always* delivers the optimal path as the metric used satisfies the property required by the $A^*$ algorithm. (We will prove this property formally).

For purposes of decoding we use the unrolled version of the trellis with start states $s_0, s_1, \ldots, s_l$ and final states $f_0, f_1, \ldots, f_l$ where $l$ is the number of subtrellises. An $(s_i, f_j)$ path is a path from start vertex $s_i$ to final vertex $f_j$, and is consequently a codeword path in trellis $T_i$, whereas an $(s_i, f_j)$ path for $i \neq j$ is a non-codeword path as it starts in subtrellis $T_i$ and ends in subtrellis $T_j$. For purposes of our discussion we term the label sequence along such a path as a *semicodeword*.

Maximum-likelihood decoding for a tail-biting trellis is equivalent to finding the codeword closest to the received sequence measured in terms of a soft decision metric. Assume that the channel is modeled as an additive white noise Gaussian(AWGN) channel and that antipodal signaling is used for communication. A binary code digit 0 is mapped into $\sqrt{E_s}$ and a 1 is mapped into $-\sqrt{E_s}$ where $E_s$ is the signal energy per bit entering the channel. For a discrete additive white Gaussian noise(AWGN) channel we have

$$r_t = x_t + n_t$$

where $r_t$ is the received signal at time $t$, $x_t$ is the transmitted signal and $n_t$ is the value of a white Gaussian noise random variable with variance $N_0/2$ where $N_0$ is the noise spectral density. Without loss of generality we can assume that $E_s = 1$. The signal-to-noise ratio or SNR is the quantity $E_s/N_0$. The decoder uses the
received vector $r$ to determine which codeword was transmitted. It forms an estimate $\hat{x}$ of the codeword $x$ that was transmitted. A decoding error occurs if $x \neq \hat{x}$. The maximum likelihood decoding rule is to decode the received sequence $r$ to codeword $x_m$ whenever $p(r/x_m) \geq p(r/x_l)$ for all $l \neq m$, where $p(r/x_m)$ is the conditional probability of $r$ given $x_m$. Let $S(x)$ be the signal vector corresponding to the codeword $x$. If $d_E(S(x_m), r)$ is the Euclidean distance between $S(x_m)$ and $r$, then the maximum likelihood decoding rule for decoding binary linear block codes transmitted over the AWGN channel using antipodal signaling is to decode $r$ into codeword $x_m$ whenever $d_E(S(x_m), r) \leq d_E(S(x_l), r)$ for all $l \neq m$.

The decoding algorithm is thus cast as a shortest path problem in which each path is associated with a metric, and the problem is to find a codeword path with minimum metric. The $A^*$ algorithm is used to cut down the search space. It does so by using a node metric which is the sum of the length of the shortest path from the source to a node and an underestimate of the length of the shortest path from the node to the goal node to guide the search. As mentioned earlier, only one path is explored at a time and the algorithm derives its advantage from the fact that if the estimates used are close to the actual values then the search space that yields the optimal path is greatly reduced. We give the algorithm below. The algorithm maintains two sets of vertices, $S$ and $\bar{S}$. The set $S$ is the set of closed nodes and represents nodes to which the shortest paths have been finalized. At any iteration, the set $\bar{S}$ is the set of candidate nodes the best of which will be closed in the succeeding iteration. These are called the open or visited nodes. An operation of expanding a node consists of the following three steps:

1. Getting all the immediate successors of the node.
2. Checking for each immediate successor if this successor has been visited before.
3. If the successor has been visited then updating the minimum cost path to the successor by taking the minimum of the cost of the previous path and the cost of this one. All the expanded nodes are put into the closed set and the visited nodes are put into the open set. When the goal node is reached an optimal path has been found.

The following is a formal description of the algorithm. Line 1 performs the initialization of the sets and the costs and paths. Line 3 selects the vertex to be expanded. Line 4 puts the selected vertex into the closed set and deletes it from the open set. Line 5 detects if the algorithm has completed; lines 6 through 9 perform an expansion of a node. They update the cost of an immediate successor as well as the best path to that successor and mark the successor as visited by putting it into the open set.

**Algorithm $A^*$**

**Input** : A trellis $T = (V, E, l)$ where $V$ is the set of vertices, $E$ is the set of edges and $l(u, v) \geq 0$ for edge $(u, v)$ in $E$, a source vertex $s$ and a destination vertex $f$, and an estimate $e(u, f)$ for the shortest path from $u$ to $f$ for each vertex $u \in V$.

**Output** : The shortest path from $s$ to $f$.

```plaintext
/* cost(u) is the cost of the current shortest path from s to u and P(u) is a current shortest path from s to u */
begin
```
1. $S \leftarrow \emptyset, \bar{S} \leftarrow \{s\}, \text{cost}(s) \leftarrow 0, P(u) \leftarrow (\), \forall u \in V, \text{cost}(u) = +\infty, \forall u \neq s$;

2. repeat
   3. Let $u$ be the vertex in $\bar{S}$ with minimum value of $\text{cost}(u) + e(u, f)$.
   4. $S \leftarrow S \cup \{u\}; \bar{S} \leftarrow \bar{S} \setminus \{u\}$;
   5. if $u = f$ then return $P(f)$;
   6. for each $(u, v) \in E$ do
      7. if $v \notin S$ then
         8. begin
            9. $\text{cost}(v) \leftarrow \min(\text{cost}(u) + l(u, v), \text{previous}(\text{cost}(v)))$;
            10. if $\text{cost}(v) \neq \text{previous}(\text{cost}(v))$ then append $(u, v)$ to $P(u)$ to give $P(v)$;
            11. ($\bar{S}$) $\leftarrow (\bar{S}) \cup \{v\}$;
         12. end
   13. forever
end

The $A^*$ algorithm is guaranteed to output the shortest path if the following two conditions hold: Let $L_T(u, f)$ be the shortest path length from $u$ to $f$ in $T$. Let $e(u, f)$ be any lower bound such that $e(u, f) \leq L_T(u, f)$, and such that $e(u, f)$ satisfies the following inequality, i.e, for $u$ a predecessor of $v$, $l(u, v) + e(v, f) \geq e(u, f)$. If both the above conditions are satisfied, then the algorithm $A^*$, on termination, is guaranteed to output a shortest path from $s$ to $f$.

The algorithm proposed here is a variant of the $A^*$ algorithm, which at any given instant, is executing an $A^*$ algorithm on exactly one of the subtrellises, with perhaps suspended executions of the algorithm on a set of other subtrellises. The subtrellis on which the algorithm is currently executing, appears the best in its potential to deliver the minimal cost path. Since the algorithm is not straightforward, we first give an informal explanation of how it works. The algorithm has two phases. The first phase performs a Viterbi algorithm on the tail-biting trellis and examines surviving paths, called survivors here, at all states of the tail-biting trellis. The first phase is described below. Let $*e$ denote the initial vertex of edge $e$. Let $e*$ denote the vertex entered via edge $e$.

**Algorithm First Phase**

**Input:** An unrolled tail-biting trellis with start nodes $s_1, s_2, \ldots s_l$, final nodes $f_1, f_2, \ldots f_l$ for the $l$ subtrellises, and an edge cost $c(e)$ associated with each edge $e$ of the tail-biting trellis.

**Output:** The cost $\text{cost}(v)$ of a least cost path to each node $v$ from any start node.

begin
   for each node $v$ in the tail-biting trellis initialize $\text{cost}(v) = 0$;
   for $i = 1$ to $n$ do
      for each vertex $v$ at time index $i$ do
         $\text{cost}(v) = \min_{e, e* = v}\{\text{cost}(e*) + c(e)\}$
      for $j = 1$ to $l$ do
At the end of the first phase therefore we have a set of survivors at final nodes $f_1, f_2, \ldots, f_l$ some of which may not correspond to codewords. The costs of these paths are taken as initial estimates for the second phase. We first informally describe the second phase below and then describe a recursive version in more detail.

Algorithm Second Phase

Input: The initial metrics $\text{metric}(T_i), i = 1 \ldots l$ computed in the first phase for the $l$ subtrellises and the costs $\text{cost}(v)$ of the survivors at all vertices $v$ of the tail-biting trellis.

Output: The maximum likelihood path.

1. Sort the metrics $\text{metric}(T_i), i = 1 \ldots l$ in increasing order; if the lowest metric is that of a codeword path then output that path as the ML path and return, else go to next step.
2. $\text{low} = \text{cost of lowest codeword survivor if there is one, otherwise, otherwise } \text{low} = \infty.$
3. If any of the metrics $\text{metric}(T_j)$ is greater than $\text{low}$ then discard subtrellis $T_j$ from the set of participants in the second phase.
4. $\text{Residual-trellises} = \text{set of all non-discarded trellises with non-codeword survivors};$
5. Create a set $\bar{S}$ of the initial vertices along with metrics, of all residual trellises, and let the start node $s$ of the $A^*$ algorithm be the start node of the residual trellis with a minimum initial metric;
6. Execute lines 2 to 11 of Algorithm $A^*$ modifying statement in line 11 as if $\text{cost}(v) < \text{low}$ then $(\bar{S}) \leftarrow (\bar{S}) \cup \{v\}$ and statement $u = f$ in line 5 by $u \in \{f_1, f_2, \ldots, f_l\}$
7. If the open set $(\bar{S})$ becomes empty before a final node is reached, then the codeword with cost $\text{low}$ is output as the decoder’s estimate of the transmitted codeword.

The algorithm above is therefore different from the standard $A^*$ algorithm in the following ways:

1) It may switch from one subtrellis to another depending on which subtrellis the node with minimum metric is located in.
2) Each shared node in a subtrellis is regarded as a distinct node for purposes of the algorithm. Thus there will be as many distinct copies of a given node of the tail-biting trellis as there are residual subtrellises sharing that node.
3) Before adding an element to the open set, we check to see that its metric is less than that of the best codeword survivor stored in $\text{low}.$ In the traditional algorithm there is no such check.
4) If the open set $\bar{S}$ becomes empty before a final node is reached then the codeword with cost $\text{low}$ is output.

We need to define the estimate $e(u, f)$ in line 3 of Algorithm $A^*.$ Recall that this has to be an underestimate of the path length from node $u$ to the final node if the ML path is to be output. The estimate we use for node $v$ in subtrellis $T_j$ is the difference between the initial metric for trellis $T_j$ computed in the first phase and the cost of the survivor at node $v$ in the first phase. We will prove later that this is indeed an underestimate and therefore guarantees that the ML path is output on termination. We implement the open set $\bar{S}$ as a heap [2]. This ensures that the minimum element can be retrieved in constant time and that whenever an element is
inserted into the heap, restructuring it in order to preserve the property of constant time access to the minimum element, has complexity logarithmic in the size of the heap.

We now describe the second phase of the algorithm more formally beginning with the notation used.

1. Variable $e(s_i, f_i)$ is the estimate obtained for the shortest path from the start to the final node in subtrellis $T_i$ in the first phase.

2. Variable $e(v, f_i)$ is the estimate for the shortest path from node $v$ to node $f_i$ in subtrellis $T_i$ which is computed when an update occurs at node $v$. This is the difference between the initial estimate at $s_i$ in trellis $T_i$, and the cost of the survivor at node $v$ in the first phase.

3. Variable $h$ is a pointer to a structure representing a node in the trellis; $h.state$ is the state, $h.trellis$ indicates which trellis that state belongs to; $h.metric$ stores the current metric which is the sum of the length of the path from the start node in trellis $h.trellis$ to $h.node$ and the estimate of the path length from $h.node$ to the final node in that trellis.

4. Variable $succ$ is a pointer to the successor of a node; $succ.state$ and $succ.metric$ have meanings that can be deduced from 3 above.

5. Variable $index$ refers to the time index and takes on values from 0 to $n-1$ where $n$ is the length of the code.

6. Variable $trellisnumber$ is a unique number associated with a subtrellis.

7. Function $InsertHeap$ inserts a node into the heap; function $DeleteMin$ extracts the node with minimum value of metric from the heap.

8. Function $IsEmpty$ returns a boolean value which is true if the heap is empty and false otherwise.

9. Variable $node.cost$ represents the actual cost of the path from the start state of a subtrellis that ends at the node $node$. Variable $node.cost1$ represents the cost of the survivor in the first phase at that node.

10. Variable $metric$ is the updated metric at a successor of a node in a trellis using function $Update$, which is called when that node is closed using $Expand$.

11. Variable $P(state)$ is the sequence of nodes representing the winning path at the state $state$.

12. Variable $low$ is the cost of the lowest cost $(s_i, f_i)$ path in the first phase.

13. Variable $flag$ is used to detect whether the winning path is the one identified in the first or second phase. It is initialized to 0. If the heap becomes empty without reaching a final node in the second phase then the lowest cost $(s_i, f_i)$ path is output as flag remains 0. Else the path that first reaches a final node in the second phase is the winning path.

Function $SecondPhase$

/* Begin with $r$ residual trellises whose metrics have been sorted in increasing order, and with variable $low$ which stores the metric of the best codeword survivor*/

begin

/* First create a heap $H$ with these $r$ metrics; each element of the heap is a record containing the trellis number, the node, the time index, and the metric*/

for $i = 1$ to $r$ do

$InsertHeap(H, i, startVertex(T_i), 0, e(s_i, f_i))$
endfor
flag = 0;
while IsEmpty(H) = false and flag = 0 do
    h := DeleteMin(H)
    S := S ∪ h.node /*Add h.node to the set of closed nodes*/
    Expand(h.trellisNo, h.state, h.timeindex, h.metric) /* Expand h.node*/
endwhile
if flag = 0 then output the codeword with metric low; return
end

function Expand(trellisnumber, state, index, metric)
begin
    if index = n − 1 then flag = 1; output P(state); return
    else
        for each successor succ of state do
            Update(trellisnumber, state, succ.state, succ.metric, index)
            if succ.metric ≤ metric then S := S ∪ {succ.state};
            Expand(trellisnumber, succ.state, index, succ.metric)
            else
                if succ.metric < low then InsertHeap(H, trellisnumber, succ.state, index, succ.metric)
            endif
        endif
        endfor
    end
end

function Update(i, node1, node2, metric, timeindex);
begin
    timeindex := timeindex + 1
    newcost := node1.cost + edgecost(node1, node2)
    if newcost ≤ node2.cost then
        P(node2) := (P(node1), node2) /* update the current shortest path to node2*/
        node2.cost := newcost /* update the cost of the current shortest path to node 2*/
        metric := node2.cost + e(s_i, f_i) − node2.cost1/* update the metric at node2; node2.cost1 is the cost of the survivor in the first phase*/
    endif
end
V. Analysis of the Decoding Algorithm

We first prove that on termination the algorithm always outputs the optimal path

**Lemma 5.1:** Each survivor at a node \( u \) has a cost which is a lower bound on the cost of the least cost path from \( s_j \) to \( u \) in an \((s_j, f_j)\) path passing through \( u \).

**Proof:** Assume that \( u \) is an arbitrary node on an \((s_j, f_j)\) path and that path \( P \) is the survivor at \( u \) in the first phase. There are two cases. Either \( P \) is a path from \( s_j \) to \( u \) or \( P \) is a path from \( s_i \) to \( u \), \( j \neq i \). If the latter is the case, then the cost of \( P \) is less than the cost of the path from \( s_j \) to \( u \); hence the cost of the survivor at \( u \) is a lower bound on the cost of the least cost path from \( s_j \) to \( u \). \( \square \)

**Lemma 5.2:** The quantity \( e(u, f_j) \) defined in the algorithm satisfies the following two properties:

1) \( e(u, f_j) \leq L_{T_j}(u, f_j) \)
2) \( l(u, v) + e(v, f_j) \geq e(u, f_j) \) where \((u, v)\) is an edge.

**Proof:**

1) \( e(u, f_j) = \text{cost}(\text{survivor}(f_j)) - \text{cost}(\text{survivor}(u)) \)

Also \( \text{cost}(\text{survivor}(f_j)) \leq \text{cost}(\text{survivor}(u)) + L_{T_j}(u, f_j) \), from which the result follows.

2) To prove: \( l(u, v) + e(v, f_j) \geq e(u, f_j) \)

\[
\text{LHS} = l(u, v) + e(v, f_j)
\]

\[
= l(u, v) + e(s_j, f_j) - \text{cost}(\text{survivor}(v))
\]

If survivor at \( v \) is survivor at \( u \) concatenated with edge \((u, v)\), then

\[
\text{LHS} = l(u, v) + e(s_j, f_j) - \text{cost}(\text{survivor}(u)) - l(u, v)
\]

\[
= e(u, f_j)
\]

On the other hand if survivor at \( v \) is not a continuation of the survivor at \( u \),

\[
\text{cost}(\text{survivor}(v)) < \text{cost}(\text{survivor}(u)) + l(u, v)
\]

\[
\text{cost}(\text{survivor}(v)) - l(u, v) < \text{cost}(\text{survivor}(u))
\]

or,

\[
e(s_j, f_j) - \text{cost}(\text{survivor}(v)) + l(u, v) > e(s_j, f_j) - \text{cost}(\text{survivor}(u))
\]

or,

\[
e(v, f_j) + l(u, v) > e(u, f_j)
\]

Therefore, \( l(u, v) + e(v, f_j) \geq e(u, f_j) \) \( \square \)

Lemma 5.2 and the fact that all estimates on trellises on which execution is suspended are underestimates, assures us that *if the final node is reached in any subtrellis then this is indeed the shortest path in the tail-biting trellis or in other words the ML codeword.*

We first make a few observations about the algorithm. During any point in the second phase, the algorithm is exploring some path in a candidate subtrellis called the *current* trellis even though it may do so in discontinuous steps. This path is called the *current path* in that subtrellis. The metric which it uses to decide whether to continue on the current path on the current trellis, say \( T_i \), or forsake it in favour of another path either in the
current trellis or on another candidate trellis is initially \( e(s_i, f_i) \). We have the following lemma specifying how the metric changes along the path.

**Lemma 5.3:** During the second phase, if the current path updates a node \( v \) using function \( \text{Update} \), where the survivor in the first phase was not in the current subtrellis then the metric becomes \( e(s_i, f_i) + \Delta(i, v) \) where \( \Delta(i, v) \) is the difference between the cost of the least cost path ending at \( v \) in the current trellis and the survivor at \( v \) during the first pass.

**Proof:** We know that
\[
\text{cost}(s_i, v) = \text{cost}(s_i, u) + \text{edgecost}(u, v)
\]
and
\[
e(v, f_i) = e(s_i, f_i) - \text{cost(survivor}(v))
\]
The metric is just the sum of the two lefthand sides of the previous two equations. Thus if the survivor is the current path then
\[
\text{cost(survivor}(v)) = \text{cost}(s_i, u) + \text{edgecost}(u, v)
\]
and the lemma follows. If the survivor is not the current path then the metric is increased by the difference between the length of the current path up to \( v \) and the survivor at \( v \).

**Definition 5.1:** A critical node on a path in a subtrellis is one at which the metric for a subtrellis reaches its final value (i.e. the actual cost of the path).

**Lemma 5.4:** During the second phase, once a critical node is closed in a subtrellis, the algorithm goes on to reach the final node in that subtrellis without switching trellises, and outputs an ML path.

**Proof:** The critical node was closed because it had the minimum metric. The metric represents the actual cost of the path at a critical node. This is no greater than the metrics of all other visited nodes which are underestimates of the costs of all other paths. Thus once a critical node is closed, the metric does not change along the continuation of this winning path to the final node. Therefore line 6 of function \( \text{Expand} \) is always true at some successor and no trellis switching takes place.

The following properties hold for the metric. Let \( m_i(N) \) denote the metric in subtrellis \( i \) at node \( N \):

**Lemma 5.5:** Let an \((s_k, f_i)\) path be the winner at \( f_i \) in the first phase and let it win over an \((s_i, f_i)\) path at node \( A \). Then \( m_i(A) = m_i(f_i) \) and \( m_i(B) < m_i(f_i) \) for any proper predecessor \( B \) of \( A \).

**Proof:** Since the \((s_k, f_i)\) path was the overall winner at \( f_i \) its length will be the metric at the start node of trellis \( T_i \) and by Lemma 5.3 the metric on the path in \( T_i \) will rise by the appropriate amounts \( \Delta_j \) at each node \( j \) where the path was overtaken by a path from some other subtrellis. When it reaches node \( A \), which is a critical node, the metric will reach its final value, namely \( m_i(f_i) \). Since \( B \) is a predecessor of \( A \) and the metric rises at \( A \), \( m_i(B) < m_i(f_i) \).

For each shortest path in a subtrellis \( i \), the nodes where it was overtaken by paths originating at the start nodes of other subtrellises in the first phase, are the nodes where its metric will rise during the second phase. These nodes are called rising points. Thus the node at the final rising point in a subtrellis is the critical node.

**Lemma 5.6:** Let subtrellises \( T_i \) and \( T_j \) share a node \( N \) and between them, let \( T_i \) be the first to close the node in the second phase. Then \( m_i(N) \leq m_j(N) \).
Proof: Since \( T_i \) is the first to close the node it closes either before \( T_j \) was first opened or after. If the former was the case, then \( m_i(N) \leq m_j(s_j) \leq m_j(N) \). If the latter was the case the least current metric of \( T_j \) is greater than the metric \( m_i(N) \) of \( T_i \) from which the result follows as the metric can only increase.

Lemma 5.7: For nodes \( A \) and \( B \) let \((A, B)\) be a path segment in the merging interval of \( T_i \) and \( T_j \) and let \( m_i(A) \leq m_j(A) \). Then \( m_i(B) \leq m_j(B) \).

Proof: Since at \( A \), \( m_i(A) \leq m_j(A) \) and thereafter all updates to the metrics in trellises \( T_i \) and \( T_j \) until node \( B \) is reached will be identical as the survivors at those node in the first phase will be the same for both trellises \( T_i \) and \( T_j \), \( m_i(B) \leq m_j(B) \).

We next show that any path from an arbitrary start node to any final node represents a vector in a vector space. For the sake of simplicity we restrict our arguments to binary codes.

Lemma 5.8: The set of all labels from an arbitrary start node to any final node is a vector space.

Proof: Assume that each of the \( c \) vectors in the submatrix \( G_c \) of the generator matrix is of the form \( v_i = [h_i, 0, t_i] \) where \( v_i \) has circular span \([j, k]\), where \( h_i \) stands for the sequence of symbols from the first, up to and including the \( k \)th symbol and is called the head, and \( t_i \) stands for the sequence of symbols from positions \( j \) to \( n - 1 \) and is called the tail; \( 0 \) represents the run of zero symbols in between the head and the tail, spanning the appropriate number of codeword indices. (This run may be empty if \( j = k + 1 \).) Let \( \{v_1, v_2 \ldots v_c\} \) be the vectors of \( G_c \). Then the matrix \( G_s \) defined as \( G_s = \begin{bmatrix} G_l \\ G_c' \end{bmatrix} \), where \( G_c' \) consists of \( 2c \) rows of the form \([h_i, 0], [0, t_i], 1 \leq i \leq c \), (where the number of zeroes in \( 0 \) makes up a total of \( n \) elements for the row) generates the set of labels of all paths from any start node to any final node. This set has \( 2^{l+2c} \) elements. This can be verified from the product construction. The set of elements of this vector space consists of semicodewords and codewords. Each semicodeword is the label of an \((s_i, f_j)\) path \( i \neq j \).

Example 5.1: The matrix \( G_s \) corresponding to the matrix \( G_{KV} \) for the Hamming (7,4) code of Example 3.1 is displayed below.

\[
G_s = \begin{bmatrix}
1 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

It can be observed that the semicodeword 1100110 formed by adding rows 1 and 3 of \( G_s \) traces a path from start vertex \( s_2 \) to final vertex \( f_1 \) in the tail-biting trellis of Figure 1.

Lemma 5.9: The algorithm will not close any node whose metric exceeds the cost of the ML path.

Proof: The lemma follows from lines 6 and 7 of function Expand and the observation that calling function Expand on a node is equivalent to closing the node. The test ensures that only nodes with metric value less than the current metric are closed. Since the current metric is a lower bound on the cost of the ML path the lemma follows.

We use a result of Tendolkar and Hartmann [32] stated below.
Lemma 5.10: Let $H$ be the parity check matrix of the code and let a codeword $x$ be transmitted as a signal vector $S(x)$. Let the binary quantization of the received vector $r = r_1, r_2, \ldots r_n$ be denoted by $y$. Let $r' = (|r_1|, |r_2|, \ldots |r_n|)$ and $S = yH^T$. Then ML decoding is achieved by decoding a received vector $r$ into the codeword $y + e$ where $e$ is a binary vector that satisfies $s = eH^T$ and has the property that if $e'$ is any other binary vector such that $s = e'H^T$ then $e.r' < e'.r'$ where $.$ is the inner product.

A direct consequence of Lemma 5.10 is the following result.

Lemma 5.11: If the all-zero codeword is the ML codeword for an error pattern $e$ then

$$e.r' < (c + e).r'$$

for any non-zero codeword $c$.

Since the space explored by the algorithm, namely the space of semicodewords and codewords is a vector space, we can analyse the algorithm assuming that the ML codeword is the all 0 codeword.

Lemma 5.12: Assume the all 0 codeword is the ML codeword. Let $e$ be the binary quantization of the received vector. For the error pattern $e$ the second phase of the decoding algorithm will close the start nodes of only those subtrellises whose initial metric corresponds to a semicodeword $C_s$ satisfying

$$(C_s + e).r' < e.r'$$

Proof: We first note that at the start of the second phase the metrics at the start nodes of all residual subtrellises correspond to the costs of vectors in the vector space of codewords and semicodewords, i.e. the vector space defined by the generator matrix $G_s$. From Lemma 5.8 we have $(C_s + e)H_s^T = e.H_s^T$ where $H_s$ is the parity check matrix corresponding to the matrix $G_s$. From Lemma 5.10 maximum likelihood decoding on the set of semicodewords will initially choose $C_s$, a semicodeword, which satisfies the inequality of the Lemma and the algorithm will close the start node of the subtrellis with that initial metric. As the algorithm proceeds with updating metrics it may close start nodes of other subtrellises. However by Lemma 5.9 it will never close the start node of any trellis $T_j$ whose initial metric exceeds that of the ML codeword, which implies that the all-0 codeword is more likely than the semicodeword survivor in $T_j$, thus implying Equation 6.

The properties of the algorithm proved in this section will be used to explain the good performance of the approximate algorithms described in the following section.

VI. AN APPROXIMATE ALGORITHM

Recall that each shared node is treated as a distinct node in the second phase of the algorithm. We now propose an approximate variant of the exact algorithm which closes a shared node at most once in the second phase. We term this algorithm Approx1.

Assume we replace line 5 of function Expand by

```
if succ.state ∉ S then Update(trellisnumber, state, succ.state, succ.metric, index) else continue
```

What this ensures is that each shared node is closed at most once, that is, by at most one subtrellis, in the second phase. Therefore the total number of Viterbi updates in the first phase and expansions in the second phases is at most $2V$ where $V$ is the number of states in the tail-biting trellis. Since a node is closed by at
most one subtrellis, it is conceivable that a shared node that is on the ML path is closed by a subtrellis that does not contain the ML codeword. In such a case the result produced will not be the ML codeword. We now analyse the conditions under which this happens. The symbols are the same as those defined for Lemma 5.12.

The following theorem gives the conditions under which the approximate algorithm produces a non-ML output. Recall that the intersection property requires that the intersection of all the zero runs of vectors in $G'$ be non-empty.

**Theorem 6.1:** If the tail-biting trellis satisfies the intersection property, the approximate algorithm produces a non-ML output for error patterns $e$ satisfying equation 6 whenever $C_e$ is a semicodeword which is formed as a linear combination of rows of $G_e$ that contain at least one non-zero multiple of a vector from $G_i$.

**Proof:** Let us assume that the all-zero codeword is the ML codeword but that it is not the output of the approximate algorithm $\text{Approx}$. Therefore some trellis say $T_i$ must close a node $N$ on the all 0 path (so that $T_0$ never gets to close it, as only one closure is allowed, and therefore cannot output the all 0 path). Clearly node $N$ must be in the merging interval of $T_0$ and $T_i$. Since $T_i$ is a residual trellis (otherwise it would have not participated in the second phase), let the survivor at $f_i$ in the first phase be an $(s_{ik}, f_i)$ path that overtakes the $(s_i, N, f_i)$ path at node $A$, in other words, $A$ is the critical node for trellis $T_i$.

**Case 1.** Suppose node $A$ is a predecessor of node $N$. By Lemma 5.5 $m_i(A) = m_i(f_i)$, and since $A$ is a critical node, by Lemma 5.4 $T_i$ would have gone on to win in the exact algorithm and therefore the all-zero codeword could not have been the ML codeword giving a contradiction.

**Case 2.** Suppose node $A$ is a successor of $N$ within the merging interval of $T_i$ and $T_0$. By Lemma 5.5 $m_i(A) = m_i(f_i)$. Since 0 is the ML codeword $m_i(f_i) > m_0(f_0)$ implying that $m_i(A) > m_0(f_0)$. Since subtrellis $T_i$ closed node $N$, by Lemma 5.6 $m_i(N) \leq m_0(N)$. By the property of the metric $m_0(N) < m_0(f_0)$ implying that $m_i(N) < m_0(f_0)$. Since $A$ is in the merging interval of $T_0$ and $T_i$ by Lemma 5.7 $m_i(A) \leq m_0(A) \leq m_0(f_0)$ giving a contradiction. Therefore we conclude that if subtrellis $T_i$ closes $N$ and $A$ is a successor of $N$, then $A$ cannot be in the merging interval of $T_i$ and $T_0$.

We thus conclude that $A$ is beyond the merging interval of $T_0$ and $T_i$, and hence the $(s_{ik}, A, f_i)$ path does not touch the all-zero path. Since the intersection property is satisfied, any path which is a linear combination of vectors of $G'_c$ alone must have at least one node on the all-zero path. Hence the semicodeword corresponding to the $(s_{ik}, A, f_i)$ path cannot be formed as a linear combination of rows only in $G'_c$ and therefore it is formed as a linear combination of vectors with at least one member of $G_i$.

Theorem 6.1 and Lemmas 5.11 and 5.12 provide an explanation of the experimental observation that decoding differences between the exact and the approximate algorithm are infrequent, so much so, that the bit error rate curves are practically indistinguishable. Lemma 5.12 tells us that in order for a subtrellis to be opened it must contain a semicodeword satisfying equation 6 being the most likely semicodeword among the possible candidates. Theorem 6.1 establishes the condition that if a node on the all-zero path is closed by some trellis $T_i$ other than $T_0$ when the all-zero codeword was transmitted, then the initial metric of $T_i$ must be that of a semicodeword of pretty high weight (because it is a linear combination of vectors which contain at least one vector in $G_i$). Further, the error $e$ which caused the cost of this high weight semicodeword to drop significantly enough to satisfy Equation 6 should not cause the weight of any non-zero codeword to drop by an amount.
enough to violate Equation 5. Since semi-codewords share prefixes and suffixes with codewords, such events may be quite infrequent.

One could get an even better approximation by allowing a node to be closed at most twice. We have experimented with this and observe that the bit error rate for this approximation is indistinguishable from that of the exact algorithm at all values of signal to noise ratio for all the three codes on which we have run the simulations. The significance of this is that the time complexity can be explicitly bounded by the complexity of at most three computations for each node of the tail-biting trellis, one update in the first Viterbi decoding phase and at most two expansions in the second phase.

A. Complexity Analysis

We now estimate the time complexity of the approximate algorithm. The following bound on the complexity of the Viterbi algorithm is well known[21].

Lemma 6.1: The complexity of the first phase of the decoding algorithm is $O(E)$ where $E$ is the number of edges in the tail-biting trellis.

The next lemma is a statement of a well known result on heap data structures[2].

Lemma 6.2: Each insertion into the heap has complexity $O(\log H)$ where $H$ is the number of elements in the heap.

Theorem 6.2: The algorithm $\text{Approx}_1$ has complexity bounded by $O(E \log V)$ where $V$ is the number of states in the tail-biting trellis.

Proof: The number of vertices that are updated is at most $2V$ as each vertex is expanded at most once in the second phase. Each time a vertex is expanded it results in computations on every edge leaving it and at most a constant number of elements being visited and inserted into the heap $\tilde{S}$, (as this number is bounded by the field size assumed to be a constant). The complexity of each insertion phase is $\log H$ where $H$ is the size of the heap. Since this size is proportional to $V$ the complexity of the second phase is $O(E \log V)$. The sorting operation at the end of the first phase has complexity $O(V_0 \log V_0)$ where $V_0$ is the number of states at time index 0. The complexity is dominated by the $O(E \log V)$ term and hence the theorem.

To reduce the overheads, the heap is implemented as $m$ separate heaps if there are $m$ residual trellises, with a separate heap of pointers, each element of which points to the root of a distinct subtrellis heap. The individual heap sizes are small in practice and the algorithm is practically linear in the size of the trellis. In the next section we present results from profiling the program which bear out the claim that the overheads of heap operations are negligible.

An argument similar to that in Theorem 6.2 establishes the complexity of algorithm $\text{Approx}_2$ as $O(E \log V)$. We next look at the space complexity of the algorithm.

Lemma 6.3: The space requirement for algorithm $\text{Approx}_1$ is $O(V_0 \times V)$ bits.

Proof: The algorithm requires $O(V)$ space to store the estimates at each state in the first phase. The additional space required to store the heap is also $O(V)$ as each expanded node can put at most all its successors on the heap. The bit vectors that store trellis membership are of size $V_0$ where $V_0$ is the number of start nodes of
the tail-biting trellis. The space requirements for the bit vectors is therefore $V_0 \times V$ bits. The space requirements for storing the current cost at each node is $O(V)$. This follows from the fact that each shared node is closed at most once. This means that at most one copy of a shared node updates its successors. This in turn means that each successor has at most one update along each of its incoming edges. Since the number of incoming edges is a constant which is at most the size of the field, a constant number of costs are associated with each node in the tail-biting trellis from which the result follows.

VII. Simulations

We have coded the exact and approximate algorithms and show the results of simulations on minimal tail-biting trellises for the 16 state tail-biting trellis [6] for the extended (24,12) Golay code on an AWGN channel with antipodal signaling, and tail-biting trellises for two rate 1/2 convolutional codes with memory 6, circle size 48 (which is the same as the (554,744) convolutional code experimented with in [5], and memory 4, circle size 20 (which is the same as the (72,62) convolutional code used in [4]) respectively. We show the variation of both, the average as well as the maximum number of node computations (counting Viterbi updates in the first phase and expansions in the second phase) with the signal to noise ratio for our exact algorithm, and compare this with the number of Viterbi updates needed for the brute force approach. Note that this number is indicative of the time complexity of the algorithm. The results are encouraging and are displayed in Tables I, II and III respectively for the Golay code and the two convolutional codes. On the average, the number of updates to get the exact ML result requires fewer than two computations at each node of the tail-biting trellis at all values of signal to noise ratio, one in the first pass and one in the second. The maximum number of node computations for the algorithm $\text{Approx}1$ is obviously bounded by twice the number of nodes in the tail-biting trellis. We also display the bit error-rate performance of the approximate algorithms closing nodes at most once for the first approximation $\text{Approx}1$, and at most twice for the second approximation, $\text{Approx}2$ in Figures 6, 7, 8 and find that there is virtually no difference in the bit error rates for the second approximation and the exact ML algorithm. Thus we get virtually ML performance for an explicit linearly bounded update complexity at all values of signal to noise ratio.

VIII. Discussion and Conclusions

We have proposed an exact algorithm for ML decoding on tail-biting trellises and also experimented on two approximate variants. The average time complexity of the exact algorithm is seen to be quite low. The approximate variants perform as well as the exact one in terms of the bit error rate at an explicitly bounded update complexity equivalent to two, or sometimes three rounds on the tail-biting trellis. The algorithm does not suffer from the effects of limit cycles or pseudocodewords which current iterative algorithms are subject to. Profiling measurements carried out on the program are displayed in Table VIII. The execution time was averaged over 10,000 runs of the decoder. The percentage of execution time taken up by each of the five major operations in the decoding process, namely, the initializations of all the arrays, the first pass, the sorting operation at the end of the first pass, the second pass, and the heap operations is displayed. It can be observed that heap operations incur an overhead of only 11% of the program running time at 0 dB and are negligible for higher values of signal to noise ratios.
The results of simulations on the extended (24,12) Golay code, a rate 1/2, memory 6 convolutional code with a circle size of 48 (which is the same as the (554,744) convolutional code used for experiments in [5]) and a rate 1/2 memory 4 convolutional code with a circle size of 20 (which is the same as the (72,62) rate 1/2 convolutional used for experimentation in [4]) have been reported. It is seen that the second approximate
Fig. 7. Bit Error Rates for the Exact and Approximate Algorithms for the rate 1/2 (133,171) Convolutional Code with circle length 48

Fig. 8. Bit Error Rates for the Exact and Approximate Algorithms for the rate 1/2 (35,31) Convolutional Code with circle length 20
| SNR | Maximum Heap Size | Maximum Node Computations | Average Node Computations |
|-----|-------------------|---------------------------|---------------------------|
| 0.0 | 13064             | 22311                     | 4414.1                    |
| 0.5 | 15698             | 24958                     | 4051.4                    |
| 1.0 | 13161             | 20369                     | 3738.5                    |
| 1.5 | 12926             | 18981                     | 3487.9                    |
| 2.0 | 9948              | 16162                     | 3330.0                    |
| 2.5 | 7492              | 11700                     | 3233.5                    |
| 3.0 | 5743              | 11175                     | 3175.0                    |
| 3.5 | 3354              | 7163                      | 3138.2                    |
| 4.0 | 2781              | 6447                      | 3115.0                    |
| 4.5 | 1526              | 5104                      | 3099.5                    |
| 5.0 | 1059              | 4693                      | 3088.2                    |

**TABLE II**

Runtime statistics for the Exact algorithm for the rate 1/2 [133, 171] convolutional code with circle length 48.

A brute force algorithm would typically perform 159552 updates. The tail-biting trellis has 3072 states.

| SNR | Maximum Heap Size | Maximum Node Computations | Average Node Computations |
|-----|-------------------|---------------------------|---------------------------|
| 0.0 | 701               | 1437                      | 426.9                     |
| 0.5 | 784               | 1447                      | 405.4                     |
| 1.0 | 824               | 1554                      | 384.9                     |
| 1.5 | 749               | 1426                      | 367.6                     |
| 2.0 | 623               | 1214                      | 353.5                     |
| 2.5 | 563               | 1179                      | 342.7                     |
| 3.0 | 578               | 1162                      | 334.6                     |
| 3.5 | 503               | 984                       | 329.5                     |
| 4.0 | 412               | 918                       | 326.2                     |
| 4.5 | 292               | 718                       | 323.7                     |
| 5.0 | 241               | 660                       | 322.3                     |

**TABLE III**

Runtime statistics for the Exact algorithm for the rate 1/2 [35, 31] convolutional code with circle length 20. A brute force algorithm would typically perform 4368 updates. The tail-biting trellis has 320 states.

The variant has a bit error rate which is indistinguishable from that of the exact algorithm for all values of signal to noise ratio.

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## Table IV

| SNR | Percentage of Initializations | Phase 1 | Sorting | Phase 2 | Heap operations |
|-----|------------------------------|---------|---------|---------|-----------------|
| 0.0 | 14.75%                       | 34.09%  | 5.88%   | 31.38%  | 11.12%          |
| 1.0 | 8.34%                        | 46.95%  | 8.34%   | 24.94%  | 6.61%           |
| 2.0 | 6.07%                        | 58.84%  | 10.20%  | 17.62%  | 2.74%           |
| 3.0 | 3.21%                        | 69.02%  | 10.79%  | 12.68%  | 0.92%           |
| 4.0 | 1.13%                        | 74.23%  | 11.49%  | 9.62%   | 0.23%           |
| 5.0 | 0.52%                        | 75.70%  | 11.59%  | 8.425%  | 0.09%           |

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