Shapley-Snow kernels in zero-sum stochastic games

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Abstract

We establish, for the first time, a connection between stochastic games and multi-parameter eigenvalue problems, using the theory developed by Shapley and Snow (1950) in the context of matrix games. This connection provides new results, new proofs, and new tools for studying stochastic games.

Contents

1 Introduction 2

2 Model and main results 3
  2.1 Standard stochastic games 4
  2.2 Main results 5

3 A new approach 6
  3.1 Shapley-Snow kernels for matrix games 6
  3.2 Multi-parameter eigenvalue problems 8

4 Application to discounted stochastic games 12
  4.1 Some preliminaries 12
  4.2 A polynomial equality: first approach 13
  4.3 A polynomial equality: second approach 15

5 Asymptotic behaviour of the values 16
  5.1 Convergence of the values 17
  5.2 Speed of convergence and Puiseux expansion 18
  5.3 Determining the exact limit values 20

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1 Introduction

Two player stochastic games are repeated games in which a state variable follows a Markov chain controlled by the actions of both players, and where the players receive payoffs at every stage. The players maximize a discounted sum of these payoffs, where the discount factor stands for their impatience. The model was introduced in the 1950’s by Shapley [9] in a finite framework, that is, the set of states $K$ is finite, and to each state corresponds a matrix game. In this seminal paper, the author proved that the stochastic game with discount factor $\lambda \in (0, 1]$ has a value for each initial state $k$, and that the vector of values $v_\lambda \in \mathbb{R}^K$ is the unique fixed point of a contracting operator $\Phi(\lambda, \cdot) : \mathbb{R}^K \rightarrow \mathbb{R}^K$, that is $v_\lambda = \Phi(\lambda, v_\lambda)$. In the late 70’s, the convergence of the values as $\lambda$ vanishes was established by Bewley and Kohlberg [3] using the Tarski-Seidenberg elimination theorem from mathematical logic and Puiseux theorem. Three alternative proofs have been provided since then, by Szczekla, Connell, Filar and Vrieze [11], Oliu-Barton [7] and Attia and Oliu-Barton [2]. Besides the convergence, the latter provided a characterisation of the limit, and an alternative and uncoupled characterisation of the discounted values.

The problem of characterizing the limit, which has remained open since the pioneer work of Bewley and Kohlberg, was recently solved by Attia and Oliu-Barton [2], whose approach also provides an alternative proof of the convergence.

But let us go back to matrix games. In the late 20’s, Von Neumann proved the celebrated minmax theorem, stating the existence of the value and of a couple of optimal strategies for any matrix game $G$. The set of optimal strategies was characterized in the 50’s by Shapley and Snow [10] as a polytope; each of its extreme points is given explicitly in terms of some square sub-matrix $\hat{G}$ of $G$, for which the following formula holds:

$$\text{val}(G) = \frac{\det(\hat{G})}{S(\text{co}(\hat{G}))} \quad (1.1)$$

where for any matrix $M$, $S(\text{co}(M))$ denotes the sum of all the entries of the cofactor matrix of $M$, and $S(\text{co}(\hat{G})) \neq 0$. The sub-matrix $\hat{G}$ will be referred here as a Shapley-Snow kernel for $G$.

The theory developed by Shapley and Snow in the context of matrix games was used in the late 90’s by Szczekla, Connell, Filar and Vrieze [11] to study stochastic games with the tools of complex analytic varieties. Their stating point was to express Shapley’s fixed point equation as a system of $n := |K|$ polynomial equalities, and can be described as follows. For each $k \in K$ and $z \in \mathbb{R}^K$, the $k$-th coordinate of Shapley’s operator $\Phi^k(\lambda, z)$ is defined as the value of a matrix game $\hat{G}^k$ whose entries are polynomial in the variables $(\lambda, z^1, \ldots, z^n)$. Suppose that $\hat{G}^1, \ldots, \hat{G}^n$ are Shapley-Snow kernels of $\hat{G}^1, \ldots, \hat{G}^n$ respectively, for some $\lambda$. Then, the following polynomial system holds:

$$P_k(\lambda, z^1, \ldots, z^n) := \det(\hat{G}^k) - z^k S(\text{co}(\hat{G}^k)) = 0, \quad k = 1, \ldots, n \quad (1.2)$$
Each of these polynomial systems defines a complex analytical variety in $\mathbb{C}^{n+1}$, and one can set $W$ to be the union of all such varieties. By Shapley and Snow [10], for each $\lambda$ one has $(\lambda, v^1_\lambda, \ldots, v^n_\lambda) \in W$. The authors prove that $\lambda \mapsto v_\lambda$ is a regular connected component of dimension 1 of $W$, from which they deduce the convergence of the values $v_\lambda$ as $\lambda$ vanishes. The expansion of the maps $\lambda \mapsto v^k_\lambda$ as Puiseux series is also obtained, but not from Tarski-Seidenberg elimination theorem, but as a consequence of the geometry of varieties.

In the present paper, we propose a new approach for studying stochastic games based on the observation that the polynomial systems defined in [11], which are a direct consequence of Shapley and Snow results, can be expressed as a multi-parameter eigenvalue problem, in the terminology of Atkinson [1]. That is, a system with $n$ equalities of the form:

$$\det(M^k_0 + z^1 M^k_1 + \cdots + z^n M^k_n) = 0, \quad k = 1, \ldots, n$$ (1.3)

where $M^k_0, \ldots, M^k_n$ are $n+1$ square matrices of equal size, for each $k$. One of the main tools for solving these systems is to use Kronecker products to transform (1.3) into an uncoupled system:

$$\det(\Delta^k - z^k \Delta^0) = 0, \quad k = 1, \ldots, n$$ (1.4)

where $\Delta^0, \ldots, \Delta^n$ are $n+1$ square auxiliary matrices. The transformation from a coupled system into an uncoupled one auxiliary has already been used by the authors in [2] to provide a characterization of the limit values. Here, we use this approach to obtain new and simple proofs of important known results in the theory of stochastic games, such as the convergence of the discounted values $v^k_\lambda$ as $\lambda$ vanishes, and the expansion of the value function in Puiseux series near 0. Furthermore, we provide some new results such as a new upper bound for the speed of convergence, and for the algebraic order of the values (provided that the data is rational), and an explicit finite set $V^k$ containing the limit value. This set allows us to refine the algorithm provided by the authors in [2] in order to determine the limit value exactly.

Section 2 is devoted to present the model and main results. In Section 3 we present our new approach, which establishes the connection between the theory of Shapley and Snow [10] for matrix games, and the theory of multi-parameter eigenvalue problems developed by Atkinson [1]. Section 4 is devoted to the application of the previously introduced tools to the study of discounted stochastic games, for a fixed discount factor. Finally, Section 5 derives from our approach some consequences on the asymptotic behavior of the discounted values.

## 2 Model and main results

The aim of this section is to present standard stochastic, introduced and studied by Shapley [9], and our main results.
2.1 Standard stochastic games

Stochastic games are described by a 5-tuple $\Gamma = (K, I, J, g, q)$, where $K$ is the set of states, $I$ and $J$ are the action sets of Player 1 and 2 respectively, $g : K \times I \times J \to \mathbb{R}$ is the payoff function and $q : K \times I \times J \to \Delta(K)$ is the transition function, where for each finite set $X$, we denote by $\Delta(X)$ the set of probability distributions over $X$.

We assume throughout the paper that $K$, $I$ and $J$ are finite sets, and $K = \{1, \ldots, n\}$.

2.1.1 The stochastic game $\Gamma^k_\lambda$

The stochastic game $\Gamma$ proceeds by stages as follows: at every stage $m \geq 1$, knowing the current state $k_m$, the players choose simultaneously and independently actions $i_m$ and $j_m$. The triplet $(k_m, i_m, j_m)$ has two effects: it produces a stage payoff $g_m = g(k_m, i_m, j_m)$ and determines the law $q(k_{m+1} | k_m, i_m, j_m)$ of the state at stage $m + 1$. For each $\lambda \in (0, 1)$ we denote by $\Gamma^\lambda$ the $\lambda$-discounted game, that is, the game in which Player 1 maximizes the expectation of $\sum_{m \geq 1} \lambda(1 - \lambda)^{m-1} g_m$, whereas Player 2 minimizes the same amount. For each $k \in K$, we denote by $\Gamma^k_\lambda$ the $\lambda$-discounted game with initial state $k$.

Past actions need not be perfectly observed by the players during the game. Rather, each player receives a signal at each stage. Formally, there exist sets $A$ and $B$ and a mapping $\varphi : K \times I \times J \to \Delta(A \times B)$ such that, at every stage $m \geq 1$, a signal $(a_m, b_m) \in A \times B$ is chosen according to $\varphi(k_m, i_m, j_m)$, player 1 is informed only of $a_m$ and player 2 only of $b_m$. A strategy for player 1 is a function from the set of possible past histories $\cup_{m \geq 1} (K \times A)^{m-1} \times K$ to the set of mixed actions $\Delta(I)$, and similarly a strategy for player 2 is a function $\tau : \cup_{m \geq 1} (K \times B)^{m-1} \times K \to \Delta(J)$. Strategies that depend, at each stage, only on the current state, are called stationary strategies. The set of stationary strategies for player 1 (resp. 2) is $\Delta(I)^K$ (resp. $\Delta(J)^K$).

As already observed by Shapley [9], the assumption that the current state is observed implies the existence of optimal stationary strategies. For this reason, we will restrict our attention to stationary strategies throughout the paper. For any couple of stationary strategies $(x, y) \in \Delta(I)^K \times \Delta(J)^K$ and any initial state $k$, we denote by $\mathbb{P}_{x,y}^k$ the unique probability measure on the set of plays $(K \times I \times J)^\mathbb{N}$ induced by a couple $(x, y)$. Similarly, we denote by $\mathbb{E}_{x,y}^k$ the expectation with respect to $\mathbb{P}_{x,y}^k$. Finally, we denote by $\gamma^k(x, y)$ the corresponding expected payoff, i.e.:

$$\gamma^k_\lambda(x, y) := \mathbb{E}_{x,y}^k \left[ \sum_{m \geq 1} \lambda(1 - \lambda)^{m-1} g(k_m, i_m, j_m) \right]$$ (2.1)

Before we present the results of Shapley [9], let us introduce some useful notions.

**Definition 2.1** For each $k \in K$, $\lambda \in (0, 1]$ and $u \in \mathbb{R}^K$, the local game $G^k(\lambda, u)$ is the $I \times J$-matrix game with payoff:

$$\rho^k(i, j) := \lambda g(k, i, j) + (1 - \lambda) \sum_{\ell \in K} q(\ell | k, i, j) u^\ell$$
Definition 2.2 For each \( \lambda \in (0, 1] \) the Shapley operator \( \Phi(\lambda, \cdot) : \mathbb{R}^K \to \mathbb{R}^K \) is defined as follows. For each \( k \in K \) and \( u \in \mathbb{R}^K \):

\[
\Phi^k(\lambda, u) := \text{val}(G^k(\lambda, u))
\]

Definition 2.3 An optimal stationary strategy for Player 1 (resp. 2) in \( \Gamma_\lambda \) is a stationary strategy \( x \in \Delta(I)^K \) (resp. \( y \in \Delta(J)^K \)) that is optimal in \( \Gamma_k^\lambda \), for all \( k \in K \).

For any fixed \( \lambda \in (0, 1] \), the main results in [9] can be stated as follows:

1. For each initial state \( k \) the stochastic game \( \Gamma_k^\lambda \) has a value, which satisfies:

\[
v_k^\lambda = \max_{x \in \Delta(I)^K} \min_{y \in \Delta(J)^K} \gamma_k^\lambda(x, y) = \min_{y \in \Delta(J)^K} \max_{x \in \Delta(I)^K} \gamma_k^\lambda(x, y)
\]

In this respect, the restriction to stationary strategies is without loss.

2. The vector of values \( v_\lambda \in \mathbb{R}^K \) is the unique fixed point of \( \Phi(\lambda, \cdot) \) which is a strict contraction of \( \mathbb{R}^K \) with respect to the \( L^\infty \)-norm. That is, for all \( u_1, u_2 \in \mathbb{R}^K \):

\[
\|\Phi(\lambda, u_1) - \Phi(\lambda, u_2)\|_\infty \leq (1 - \lambda)\|u_1 - u_2\|_\infty
\]

3. Both players have optimal stationary strategies.

### 2.2 Main results

The main contributions of this paper are the following:

(1) We establish, for the first time, a connection between stochastic games and multi-parameter eigenvalue problems, using Shapley and Snow theory. This connection provides new tools for studying stochastic games, and the following results.

The rest of our statements hold for each state \( k = 1, \ldots, n \).

(2) Using our approach, we construct a finite set of non-zero bi-variate polynomials \( E^k \) with the following property: there exists \( \lambda_0 > 0 \) and \( P \in E^k \) such that:

\[
P(\lambda, v_\lambda^k) = 0, \quad \forall \lambda \in (0, \lambda_0)
\]

(3) From (2), we construct a finite set \( V^k \subset \mathbb{R} \) containing any accumulation point of the discounted values \( (v_\lambda^k)_\lambda \). From the finiteness of \( V^k \) we derive an alternative proof of the convergence of the values, as \( \lambda \) vanishes.

(4) From (2) we obtain the expression of \( \lambda \mapsto v_\lambda^k \) as a Puiseux series using Puiseux’s theorem. Though not new, our result shows a direct way to obtain this property without invoking neither Tarski-Seidenberg elimination principle (as in [3]) nor the geometry of complex analytic varieties (as in [11]).

(5) We then deduce a new upper bound for the speed of convergence of the values to the limit, and for the algebraic order of \( v_\lambda^k \) and \( \lim_{\lambda \to 0} v_\lambda^k \) (provided that the data matrix \( D(\lambda) \) takes rational values), namely \( \min(|I|, |J|)^n \). This bound beats the best known upper bound of \( (2 \min(|I|, |J|) + 5)^n \) obtained by Hansen et al. [4]. Our bound is tight in the class of absorbing games. In general, there exists stochastic games where the lower bound \( \min(|I|, |J|)^{n-1} \) holds, according to [4].

(6) We use (3) to refine the algorithm provided by the authors in [2] to compute the limit value exactly.
3 A new approach

3.1 Shapley-Snow kernels for matrix games

Throughout this section, \( G \) will denote a fixed \( I \times J \)-matrix game with value \( v = \text{val}(G) \). The set of optimal strategies for player 1 and 2 will be denoted by \( X^* \subset \Delta(I) \) and \( Y^* \subset \Delta(J) \), respectively. These sets are compact, non-empty polytopes, so that they can be entirely described by their (finitely many) extreme points. A characterization of the extreme points of \( X^* \times Y^* \), called basic solutions of \( G \), was provided Shapley and Snow’s [10]. Before we state their result, we will introduce some notation.

Notation throughout the paper

- For any finite set \( Z \), we denote its cardinal by \(|Z|\).
- We denote by \( 1 \) and \( U \), respectively, a column vector and a matrix of 1’s. The size shall be clear in every context.
- For any square matrix \( M = (M_{i,j})_{i,j} \), we denote by \( \text{co}(M) \) its cofactor matrix, and by \( S(M) \) the sum of the entries of \( M \).
- Mixed strategies will be considered as column vectors, so that the expected payoff induced by a couple of mixed strategies \((x, y) \in \Delta(I) \times \Delta(J)\) is \( x^G y \).
- We denote by \( \hat{G} \) a sub-matrix of \( G \). For any sub-matrix \( \hat{G} \) of \( G \), we denote by \( \hat{x} \) the restriction of \( x \) to the row indices of \( \hat{G} \). Similarly, \( \hat{y} \in \mathbb{R}^J \) is the restriction of \( y \) to the column indices of \( \hat{G} \).

3.1.1 Characterization of basic solutions

We are now ready to state the main result in Shapley and Snow [10], and to introduce Shapley-Snow kernels. The following theorem is a convenient restatement of their results.

**Theorem 3.1 (Shapley and Snow 1950)** Let \( G \) be a \( I \times J \) matrix game with optimal strategies \( X^* \) and \( Y^* \). Then \((x, y) \in X^* \times Y^* \) is a basic solution of \( G \) if and only if there exists a square sub-game \( \hat{G} \) satisfying (1) and (2):

1. \( S(\text{co}(\hat{G})) \neq 0 \)
2. \( \hat{x} = \frac{\text{co}(\hat{G})}{S(\text{co}(\hat{G}))} \mathbf{i} \) and \( \hat{y} = \frac{\text{tr}(\hat{G})}{S(\text{co}(\hat{G}))} \mathbf{i} \)

Moreover, in this case one also has:

3. \( \text{val}(G) = \text{val}(\hat{G}) = \frac{\det(\hat{G})}{S(\text{co}(\hat{G}))} \)
4. \( \hat{x}^G = \text{val}(G) \mathbf{1} \) and \( \hat{G}\hat{y} = \text{val}(G) \mathbf{1} \)

**Definition 3.2** A Shapley-Snow kernel of \( G \) associated to a basic solution \((x, y)\) is a square sub-matrix \( \hat{G} \) satisfying the four conditions of Theorem 3.1.
### 3.1.2 Other properties

The purpose of the remaining of this section is to derive some consequences from Theorem 3.1. The properties obtained here will be useful in establishing a connection between the theory of basic solutions of matrix games, and that of multi-parameter eigenvalue problems in linear algebra.

Let us start by three Lemmas.

**Lemma 3.3** For any square matrix $M$ one has:

1. $\det(M + zU) = \det(M) + zS(\text{co}(M))$, for all $z \in \mathbb{R}$
2. The map $z \mapsto S(\text{co}(M + zU))$ is constant
3. The maps $z \mapsto \text{co}(M + zU)1$ and $z \mapsto ^t\text{co}(M + zU)1$ are constant

**Proof.**

1. Let $M$ be some square matrix. The function $z \mapsto \det(M + zU)$ is a polynomial in $z$. Subtracting one row to all other rows of $M + zU$, it is clear that its degree is at most 1. From the formulae $\text{tr}(^tMU) = S(M)$ and

\[
\det(M + H) = \det(M) + \text{tr}(^t\text{co}(M)H) + o(||H||), \quad \text{as } H \to 0
\]

which hold for any square matrix $M$, one deduces $\frac{\partial}{\partial z} \det(M + zU)(0) = S(\text{co}(M))$. Hence, $\det(M + zU) = \det(M + zU) - zS(\text{co}(M))$ for any square matrix $M$ and any $z \in \mathbb{R}$.

2. Applying (i) to $M + zU$ and $-z$ yields:

\[
\det(M) = \det((M + zU) - zU) = \det(M + zU) - zS(\text{co}(M + zU))
\]

Comparing with (i), one obtains $S(\text{co}(M)) = S(\text{co}(M + zU))$ for any $M$ and $z$.

3. By the symmetric role of both players, it is enough to prove the first statement. Let $m \in \mathbb{N}^*$ be the size of $M$, and let $M_1, \ldots, M_m$ be its rows. Then, for each $\ell = 1, \ldots, m$ the $\ell$-th component of the vector $\text{co}(M)1$ satisfies:

\[
(\text{co}(M)1)^\ell = \det(M_1, \ldots, M_{\ell-1}, 1, M_\ell, \ldots, M_m) = \det(M_1 + z1, \ldots, M_{\ell-1} + z1, 1, M_\ell, \ldots, M_m + z1) = (\text{co}(M + zU)1)^\ell
\]

thanks to the properties of the determinant (that is, we added $z$ times the $\ell$-th column to the other columns).

The next result is a direct consequence of Lemma 3.3.

**Lemma 3.4** Let $\hat{G}$ be a Shapley-Snow kernel for $G$. Then $\hat{G} + z\hat{U}$ is a Shapley-Snow kernel of the translated game $G + zU$, for any $z \in \mathbb{R}$.

**Proof.** To prove that $\hat{G} + z\hat{U}$ is a Shapley-Snow kernel, it is enough to check properties (1) and (2) of Theorem 3.1. Yet, on the one hand, $S(\text{co}(\hat{G} + zU)) = S(\text{co}(\hat{G})) \neq 0$ by Lemma 3.3 (ii). On the other, by Lemma 3.3 (iii), the following strategies do not depend on $z$:

\[
\hat{x}(z) = \frac{\text{co}(\hat{G} + zU)}{S(\text{co}(G + zU))}1, \quad \hat{y}(z) = \frac{^t\text{co}(\hat{G} + zU)}{S(\text{co}(G + zU))}1
\]
which completes the proof.

Let us now state a result from linear algebra, whose proof can be found in the Appendix.

**Lemma 3.5** Let \( M \) be a square matrix of size \( a \in \mathbb{N}^* \) and rank \( a - 1 \), and let \( x \) and \( y \) be such that \( \text{Ker}(\overline{t}M) = \langle x \rangle \) and \( \text{Ker}(M) = \langle y \rangle \). Then there exists a constant \( \alpha \neq 0 \) such that \( \text{co}(\overline{t}xy) = \alpha^txy \).

The next result, which relies on Theorem 3.1 and Lemmas presented so far, gathers all the properties that will be used in the sequel.

**Proposition 3.6** Let \( G \) be an \( I \times J \)-matrix game with value \( v := \text{val}(G) \), and let \( \hat{G} \) be a Shapley-Snow kernel for \( G \) defining strategies \((\hat{x}, \hat{y})\). Then:

(i) \( S(\text{co}(\hat{G} - v\hat{U})) \neq 0 \)

(ii) \( \det(\hat{G} - v\hat{U}) = \text{val}(\hat{G} - v\hat{U}) = \text{val}(G - vU) = 0 \)

(iii) \( \text{Ker}(\hat{G} - v\hat{U}) = \langle \hat{y} \rangle \) and \( \text{Ker}(\overline{t}(\hat{G} - v\hat{U})) = \langle \hat{x} \rangle \)

(iv) \( \text{co}(\hat{G} - v\hat{U}) = S(\text{co}(\hat{G} - v\hat{U})) \overline{t}\hat{x}\hat{y}. \) In particular, \( \text{co}(\hat{G} - v\hat{U}) \) is non-zero, and has all its non-zero entries of same sign.

**Proof.**

(i) By Lemma 3.4, \( \hat{G} - v\hat{U} \) is a Shapley-Snow kernel for \( G - vU \) so that, in particular, \( S(\text{co}(\hat{G} - v\hat{U})) \neq 0 \).

(ii) For any matrix \( M \) and \( z \in \mathbb{R} \), clearly \( \text{val}(M + zU) = \text{val}(M) + z \). Hence, the formulae of Theorem 3.1 (3) yield:

\[
0 = \text{val}(G - vU) = \text{val}(\hat{G} - v\hat{U}) = \frac{\det(\hat{G} - v\hat{U})}{S(\text{co}(\hat{G} - v\hat{U}))}
\]

(iii) By the symmetric role of both players, it is enough to prove the first statement. The matrix \( \hat{G} - v\hat{U} \) is not invertible and its matrix of cofactors \( \text{co}(\hat{G} - v\hat{U}) \) is non-zero, thanks to (i). Hence, if \( 1 \leq b \leq \min(|I|, |J|) \) denotes its size, one has \( \text{rank}(\hat{G} - v\hat{U}) = b - 1 \) or, equivalently \( \text{dim}(\text{Ker}(\hat{G} - v\hat{U})) = 1 \). Yet by Theorem 3.1 (4), one has \( \hat{G}\hat{y} = v\hat{1} \) and \( \hat{y} \neq 0 \), so that \( (\hat{G} - v\hat{U})\hat{y} = 0 \). Consequently \( \text{Ker}(\hat{G} - v\hat{U}) = \langle \hat{y} \rangle \).

(iv) It follows from Lemma 3.5 as the hypotheses are satisfied thanks to (ii) and (iii). Hence, \( \text{co}(\hat{G} - v\hat{U}) = \alpha^t\hat{x}\hat{y} \) for some \( \alpha \neq 0 \), so that \( S(\text{co}(\hat{G} - v\hat{U})) = S(\alpha^t\hat{x}\hat{y}) = \alpha \) because \( S(\overline{t}\hat{x}\hat{y}) = 1 \).

### 3.2 Multi-parameter eigenvalue problems

Throughout this section, we consider an array \( n \times (n + 1) \) square matrices:

\[
D = \begin{pmatrix}
M_0^1 & M_1^1 & \cdots & M_n^1 \\
\vdots & \vdots & \ddots & \vdots \\
M_0^n & M_1^n & \cdots & M_n^n
\end{pmatrix}
\]

(3.1)

8
such that for each $k$, all matrices of the $k$-th row are of equal size. We briefly present
the so-called multi-parameter eigenvalue problem, terminology introduced by Atkinson.
That is, the problem of finding some vector $z = (z^1, \ldots, z^n) \in \mathbb{R}^n$ satisfying:

$$
\begin{align*}
\text{det}(M_0^1 + z^1 M_1^1 + \cdots + z^n M_n^1) &= 0 \\
\vdots & \\
\text{det}(M_0^n + z^1 M_1^n + \cdots + z^n M_n^n) &= 0
\end{align*}
$$

(3.2)

In order to study such systems, one introduces the following $n + 1$ auxiliary matrices
using the Kronecker product (or tensor product) of matrices.

### 3.2.1 Preliminaries on Kronecker products

Let us start by recalling the definition of the Kronecker product $\otimes$ of two matrices and
the corresponding determinant $\det \otimes$.

**Definition 3.7**

- The Kronecker product of an $m \times n$ matrix $A$ and a $p \times q$ matrix $B$ is a $mp \times nq$
  matrix defined by blocks as follows:

$$
A \otimes B = \begin{pmatrix}
a_{11}B & \cdots & a_{1n}B \\
\vdots & \ddots & \vdots \\
a_{m1}B & \cdots & a_{mn}B
\end{pmatrix}
$$

- Suppose that $A_1, \ldots, A_{nn}$ are $n^2$ matrices such that for each row $k$, $A_{k1}, \ldots, A_{kn}$
  are of same size. The Kronecker determinant is defined by:

$$
\det \otimes \begin{pmatrix}
A_{11} & \cdots & A_{1n} \\
\vdots & \ddots & \vdots \\
A_{n1} & \cdots & A_{nn}
\end{pmatrix} := \sum_{\sigma \in S(n)} \epsilon(\sigma) A_{1, \sigma(1)} \otimes \cdots \otimes A_{n, \sigma(n)}
$$

where $S(n)$ denotes the set of permutations of the set $\{1, \ldots, n\}$ and $\epsilon(\sigma)$ stands
for the signature of $\sigma$.

We can now define the matrices $\Delta^0, \Delta^1, \ldots, \Delta^n$, which are all square of same size,
namely $\prod_{k=1}^n a^k$, where $a^k \in \mathbb{N}^*$ denotes the common size of the matrices $M_0^k, \ldots, M_n^k$,
for each $k$.

**Definition 3.8** For each $k = 0, \ldots, n$ let $D^k$ denote the $n \times n$ block matrix obtained
by deleting the $k$-th column (of blocks) of the matrix $D$ defined in (3.1). Then:

$$
\Delta^k := (-1)^k \det \otimes D^k
$$

The following canonical bijection between a product of finite sets, and a set of same
cardinal will be useful.

---

1In the original work of Atkinson, the author considers the homogeneous case $z^0 M_0^k + z^1 M_1^k + \cdots + z^n M_n^k$. 

Definition 3.9 Let $S^1, \ldots, S^n$ be $n$ finite sets and set $t_k := \prod_{\ell=k+1}^n |S^\ell|$ for all $k = 0, \ldots, n$. The canonical bijection between $S^1 \times \cdots \times S^n$ and $\{1, \ldots, t_0\}$ is given by:

$$S^1 \times \cdots \times S^n \to \{1, \ldots, t_0\}$$

$$(s_1, \ldots, s_n) \mapsto (s_1 - 1)t_1 + \cdots + (s_n - 1)t_n + 1$$

Let us now state some useful properties of Kronecker products:

(K1) The product $\otimes$ is bilinear and associative, but not commutative.

(K2) The determinant $\det \otimes$ has the same properties as the usual determinant, i.e. it is multilinear and alternating, but only with respect to the columns.

(K3) Let $A_1, \ldots, A_n$ and $B_1, \ldots, B_n$ some matrices such that the products $A_kB_k$ are well defined. Then $(A_1 \otimes \cdots \otimes A_n)(B_1 \otimes \cdots \otimes B_n) = (A_1B_1) \otimes \cdots \otimes (A_nB_n)$.

(K4) Let $A_1, \ldots, A_n$ some matrices over the sets $I_1 \times J_1, \ldots, I_n \times J_n$, respectively. Let $r$ and $(i_1, \ldots, i_n)$, and $s$ and $(j_1, \ldots, j_n)$ be related by the corresponding canonical bijections. Then:

$$(A_1 \otimes \cdots \otimes A_n)^{r,s} = A_1^{i_1 j_1} \cdots A_n^{i_n j_n}$$

(K5) Let $A_{11}, \ldots, A_{nn}$ be an $n \times n$ array of matrices such that $A_{k1}, \ldots, A_{kn}$ are matrices over the same set $I_k \times J_k$, for all $k = 1, \ldots, n$. Let $r$ and $(i_1, \ldots, i_n)$, and $s$ and $(j_1, \ldots, j_n)$ be related by the corresponding canonical bijections. Then the $(r, s)$-th entry of the matrix $\det \otimes (A_{11}, \ldots, A_{nn})$ is:

$$\det \begin{pmatrix} A_{11}^{i_1 j_1} & \cdots & A_{1n}^{i_1 j_1} \\ \vdots & \ddots & \vdots \\ A_{n1}^{i_n j_n} & \cdots & A_{nn}^{i_n j_n} \end{pmatrix}$$

3.2.2 The uncoupled system

Consider now the following uncoupled system:

$$\begin{cases} 
\det(\Delta^1 - z^1\Delta^0) = 0 \\
\vdots \\
\det(\Delta^n - z^n\Delta^0) = 0 
\end{cases} \tag{3.3}$$

Notation. In the sequel, we will use the following notation: $S^M$ denotes the set of solutions to (3.2), and $S^\Delta$ denotes the set of solutions to (3.3).

The importance of the auxiliary matrices $\Delta^0, \ldots, \Delta^n$ becomes clear with the following result, which can be found in Atkinson [11, Chapter 6].

Proposition 3.10 (Atkinson 1972) $S^M \subset S^\Delta$.

\[ \text{Because of the non-commutativity of the direct product, rows and columns do not play the same role. The determinant needs to be developed by columns.} \]
Proof. Fix \( k \) and let \((z^1, \ldots, z^n) \in S^M\). Let \( y^\ell \in \text{Ker}(M^\ell_0 + z^1 M^\ell_1 + \cdots + z^n M^\ell_n)\), \( y^\ell \neq 0 \), for all \( \ell = 1, \ldots, n \), which exist because the matrices are singular. Then \((y^1 \otimes \cdots \otimes y^n) \neq 0\) and:

\[
\Delta^k(y^1 \otimes \cdots \otimes y^n) = (-1)^k \det_\otimes \begin{pmatrix}
M^1_0 y^1 & \cdots & M^1_k y^1 & \cdots & M^n_k y^1 \\
\vdots & & \vdots & & \vdots \\
M^n_0 y^n & \cdots & M^n_k y^n & \cdots & M^n_n y^n
\end{pmatrix}
\]

\[
= (-1)^{k+1} \sum_{\ell=1}^n z^\ell \det_\otimes \begin{pmatrix}
M^1_\ell y^1 & \cdots & M^1_k y^1 & \cdots & M^n_k y^1 \\
\vdots & & \vdots & & \vdots \\
M^n_\ell y^n & \cdots & M^n_k y^n & \cdots & M^n_n y^n
\end{pmatrix}
\]

\[
= (-1)^{k+1} z^k \Delta^0(y^1 \otimes \cdots \otimes y^n)
\]

where the first equality follows from \((K3)\), the second is a consequence of the equalities \(M^\ell_0 y^r = -\sum_{\ell=1}^n z^\ell M^\ell_\ell y^r\) for \( r = 1, \ldots, n\) and \((K2)\), the third follows from the fact that, for all \( \ell \neq k \), the (block) matrix has two equal columns (of blocks) so that its determinant vanishes, and finally the last equality is obtained by taking a cyclic permutation of the columns (of blocks) which has signature \((-1)^{k+1}\). Hence

\[
(\Delta^k - z^k \Delta^0)(y^1 \otimes \cdots \otimes y^n) = 0
\]

so that \(\text{Ker}(\Delta^k - z^k \Delta^0) \neq \emptyset\). As this is true for all \( k \), it follows that \((z^1, \ldots, z^n)\) satisfies the system \((3.3)\).

In the proof of Proposition \(3.10\) we have, in fact, proved the following slightly stronger result.

Corollary 3.11 Let \( z \in S^M\) and let \( y^k \in \text{Ker}(M^k_0 + z^1 M^k_1 + \cdots + z^n M^k_n)\), for all \( k \in K\). Then \( z \in S^\Delta\) and \((y^1 \otimes \cdots \otimes y^n) \in \text{Ker}(\Delta^k - z^k \Delta^0)\) for all \( k \in K\).

Let us end this section with a result that will be useful in the sequel, inspired in a similar lemma from Muic and Plstenjak \([5]\).

Lemma 3.12 Let \( A, B \) be two square matrices of the same size \( m \), consider \( z_0 \) such that \( \det(A + z_0 B) = 0 \). Suppose there exists \( x_0 \in \text{Ker}((A + z_0 B)) \) and \( y_0 \in \text{Ker}(A + y_0 B) \) such that \( ^tx_0 By_0 \neq 0 \). Then \( \text{rank}(A + z_0 B) < \max_{z \in \mathbb{R}} \text{rank}(A + z B) \).

Proof. Let \( r := \text{rank}(A + z_0 B) \) and suppose that \( r = \max_{z \in \mathbb{R}} \text{rank}(A + z B) \). As the rank of a matrix is the maximal size of its invertible square sub-matrices, \( A + z_0 B \) admits some invertible \( r \times r \) sub-matrix. By continuity, there exists \( \varepsilon > 0 \) such that this sub-matrix is invertible in the interval \((z_0 - \varepsilon, z_0 + \varepsilon)\), so that \( \text{rank}(A + z B) \geq r \) in this interval. As we have supposed that \( r \) is the maximal rank, the converse inequality
also holds, so that \( \text{rank}(A + zB) = r \) on \((z_0 - \varepsilon, z_0 + \varepsilon)\). Using Cramer’s rule, one deduces the existence of \( r(m - r) + 1 \) polynomials \( P^1_r, \ldots, P^m_r, P^r r + 1, \ldots, P^r r \) and \( P \) such that, up to a permutation of the rows, the kernel of \( A + zB \) is given by the following set of vectors:

\[
\begin{pmatrix}
\frac{P^1_r(z)}{P(z)} t_{r+1} + \cdots + \frac{P^1_r(z)}{P(z)} t_m \\
\vdots \\
\frac{P^r r(z)}{P(z)} t_{r+1} + \cdots + \frac{P^r r(z)}{P(z)} t_m \\
\vdots \\
t_{r+1} \\
t_m
\end{pmatrix}, \quad t_{r+1}, \ldots, t_m \in \mathbb{R}
\]

where \( t_{r+1}, \ldots, t_m \) are the free variables. \(^3\) Multiplying by \( P(z) \), one deduces the existence of a vector \( y(z) \in \mathbb{R}^m \) with polynomial entries satisfying \((A + \lambda B)y(z) = 0\) on \((z_0 - \varepsilon, z_0 + \varepsilon)\) and \( y(z_0) = y_0 \). Differentiating the first equality, one obtains

\[
By(z) + (A + zB)y'(z) = 0
\]

Multiplication by \( ^t x_0 \) at \( z_0 \) yields then:

\[
^t x_0 By_0 + ^t x_0 (A + z_0 B)y'(z_0) = 0
\]

where \( ^t x_0 (A + z_0 B) = 0 \) by the choice of \( x_0 \). Hence \( ^t x_0 By_0 = 0 \), a contradiction. \( \blacksquare \)

4 Application to discounted stochastic games

In this section we are going to apply Shapley and Snow’s theory and tools from multi-parameter eigenvalue problems to derive the following result for discounted stochastic games: for each initial state \( k \in K \) and each discount factor \( \lambda \in (0, 1] \), there exists a bivariate polynomial \( P^k \) such that \( P^k(\lambda, v^k) = 0 \). As we will see in Section 5 this result has important implications on the asymptotic behavior of the value function.

4.1 Some preliminaries

Let \((K, I, J, g, q)\) be a stochastic games with state space \( K = \{1, \ldots, n\} \). First of all, consider the following array of \( n \times (n + 1) \) matrices constructed directly from the data of the game:

\[
\begin{pmatrix}
-\lambda G_1 & U - (1 - \lambda)Q_{1,1} & -(1 - \lambda)Q_{1,2} & \cdots & -(1 - \lambda)Q_{1,n} \\
-\lambda G_2 & -(1 - \lambda)Q_{2,1} & U - (1 - \lambda)Q_{2,2} & \cdots & -(1 - \lambda)Q_{2,n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-\lambda G_n & -(1 - \lambda)Q_{n,1} & -(1 - \lambda)Q_{n,2} & \cdots & U - (1 - \lambda)Q_{n,n,n}
\end{pmatrix}
\]  

\( \text{(4.1)} \)

\(^3\) More precisely, \( P(z) \) is the determinant of the invertible \( r \times r \) sub-matrix of \( A + zB \), and \( P_t^r(z) \) is the determinant of \( r \times r \) sub-matrix obtained after replacing its \( s \)-th column with the \( t \)-th, for \( s = 1, \ldots, r \) and \( t = r + 1, \ldots, m \).
where for each $k, \ell \in K$ we denote by $Q_{k,\ell}$ and $G_k$ the matrices of size $|I| \times |J|:
Q_{k,\ell} := (q(\ell|k, i, j))_{i,j} \quad \text{and} \quad G_k := (g(k, i, j))_{i,j}
and where $U$ stands for a matrix of $1$’s of size $|I| \times |J|$. The theory of multi-parameter eigenvalue problems can be applied since the matrices are of equal size over each row, as in (3.1). It will be convenient to rename these matrices as follows:
\[
M_{k,\ell} := \begin{cases} 
-(1 - \lambda)Q_{k,\ell} & \text{for } 1 \leq k \neq \ell \leq n \\
U - (1 - \lambda)Q_{k,k} & \text{for } k = 1, \ldots, n \\
-\lambda G_k & \text{for } \ell = 0, k = 1, \ldots, n 
\end{cases}
\]
(4.2)
Define the matrices $\Delta_0, \ldots, \Delta^n$ accordingly, as in Definition 3.8.

Remark 4.1 We have omitted the dependence on $\lambda$ of the entries of $M_{k,\ell}$ and of $\Delta_k$ to simplify the notation. Note, however, that the former depend at most linearly on $\lambda$, while the latter depend polynomially on $\lambda$ with degree at most $n$.

The following result, borrowed from Attia and Oliu Barton [2, Lemma 3.3], is a consequence of the fact that $M_{k,\ell} + \sum_{\ell \neq k} M_{k,\ell} \geq \lambda U$ for all $k = 1, \ldots, n$.

Lemma 4.2 One has $\Delta^0 \geq \lambda^n U$.

The following definition will be useful in the sequel.

Definition 4.3 For each $k$, let $E^k$ be the set of the non-zero bi-variate polynomials
\[
\hat{P}(\lambda, z) := \det(\hat{\Delta}^k - z\hat{\Delta}^0)
\]
obtained by ranging over all the square sub-matrices $\hat{\Delta}^k - z\hat{\Delta}^0$ of $\Delta^k - z\Delta^0$.

Obviously, the sets $E^k$ are finite, as a matrix has finitely many sub-matrices.

4.2 A polynomial equality: first approach

By Shapley and Snow, each local normalized game:
\[
G^k(\lambda, v_\lambda) - v^k_\lambda U = -(M^k_0 + v^1_\lambda M^k_1 + \cdots + v^n_\lambda M^k_n)
\]
admits a Shapley-Snow kernel, which is some square sub-matrix, obtained as the sum of the corresponding square sub-matrices of $M^k_t$, that we denote by $\hat{M}^k_t$. Again, the matrices $\hat{M}^k_0, \ldots, \hat{M}^k_n$ define an array of $n(n + 1)$ matrices with equal sizes over each row, as in (3.1), so that one can define $\hat{\Delta}^0, \hat{\Delta}^1, \ldots, \hat{\Delta}^n$ accordingly (see Definition 3.8).

Remark 4.4 It is worth mentioning that $\hat{\Delta}^k - z^k \hat{\Delta}^0$ is not necessarily a Shapley-Snow kernel of the matrix $\Delta^k - z^k \Delta^0$. Nevertheless, a sufficient condition for this to happen is that $S(\mathrm{co}(\hat{\Delta}^k - z^k \hat{\Delta}^0)) \neq 0$. 

13
Consider the corresponding determinantal systems:

\[
\begin{align*}
\det(M_0^1 + z^1 M_1^1 + \cdots + z^n M_n^1) &= 0 \\
\vdots \\
\det(M_0^n + z^1 M_1^n + \cdots + z^n M_n^n) &= 0
\end{align*}
\]

and

\[
\begin{align*}
\det(\Delta^1 - z^1 \Delta^0) &= 0 \\
\vdots \\
\det(\Delta^n - z^n \Delta^0) &= 0
\end{align*}
\]

Denote their sets of solutions, respectively, as \( S^M \) and \( S^\Delta \). Consider also the set of solutions \( S^\mathcal{G} \) and \( S^\dot{\mathcal{G}} \) to the following value systems, inspired from Shapley [9] and Shapley and Snow [10], respectively:

\[
\begin{align*}
\text{val}(\mathcal{G}^1(\lambda, z) - v^1 \lambda U) &= 0 \\
\vdots \\
\text{val}(\mathcal{G}^n(\lambda, z) - v^n \lambda U) &= 0
\end{align*}
\]

\[
\begin{align*}
\text{val}(\dot{\mathcal{G}}^1(\lambda, z) - z^1 \dot{U}) &= 0 \\
\vdots \\
\text{val}(\dot{\mathcal{G}}^n(\lambda, z) - z^n \dot{U}) &= 0
\end{align*}
\]

The relations between these 4 systems is summarized in the next statement.

**Proposition 4.5** The following relations hold:

\[ S^\mathcal{G} = S^\dot{\mathcal{G}} \subset S^M \subset S^\Delta \]

**Proof.** By Shapley [9], \( S^\mathcal{G} = \{ v_\lambda \} \). Applying Proposition 3.6 (ii) to the local games \( (\mathcal{G}^\ell(\lambda, v_\lambda))_\ell \) one obtains, for each \( k \):

\[
0 = \text{val}(\mathcal{G}^k(\lambda, v_\lambda) - v^k \lambda U) = \text{val}(\dot{\mathcal{G}}^k(\lambda, v_\lambda) - v^k \lambda U) = \det(\dot{\mathcal{G}}^k(\lambda, v_\lambda) - v^k \lambda U) = -\det(M_0^k + v_1^k M_1^k + \cdots + v_n^k M_n^k)
\]

so that \( \{ v_\lambda \} = S^\mathcal{G} = S^\dot{\mathcal{G}} \subset S^M \). The last inclusion follows from Proposition 3.10. \( \blacksquare \)

**Remark 4.6** At this point, it is unclear whether the sets \( S^M \) and \( S^\Delta \) contain one or more elements. Examples where the former can contains \( \min(|I|, |J|) \) elements, and where the latter is infinite are described in the Appendix.

Let us now show an algebraic property of the discounted values.

**Proposition 4.7** For each \( k = 1, \ldots, n \) one has:

\[ \text{rank}(\Delta^k - v_\lambda^k \Delta^0) < \max_{z \in \mathbb{R}} \text{rank}(\Delta^k - z \Delta^0) \]
Proof. Let $\dot{x}^k$ and $\dot{y}^k$ be some Shapley-Snow strategies for the local game $G^k(\lambda, v_\lambda)$, for all $k$. By Proposition 3.6 (iii), one has

$$\dot{y}^k \in \text{Ker}(\dot{G}^k(\lambda, v_\lambda) - v_\lambda^k \dot{U}), \quad \forall k = 1, \ldots, n$$

Hence, by Corollary 3.11, one has

$$\dot{\Delta}^k - v_\lambda^k \dot{\Delta}^0(\dot{y}^1 \otimes \cdots \otimes \dot{y}^n) = 0$$

or, equivalently, $\dot{y}^1 \otimes \cdots \otimes \dot{y}^n \in \text{Ker}(\dot{\Delta}^k - v_\lambda^k \dot{\Delta}^0)$. By the symmetric roles of the players, one has $\dot{x}^1 \otimes \cdots \otimes \dot{x}^n \in \text{Ker}(\dot{\Delta}^k - v_\lambda^k \dot{\Delta}^0)$ in the same manner. On the other hand, by Lemma 4.2, all the entries of the matrix $\dot{\Delta}^0$ are of same sign and different from 0. Hence

$$t(\dot{x}^1 \otimes \cdots \otimes \dot{x}^n) \dot{\Delta}^0(\dot{y}^1 \otimes \cdots \otimes \dot{y}^n) \neq 0$$

All the hypotheses of Lemma 3.12 are fulfilled with $A = \dot{\Delta}^k$ and $B = -\dot{\Delta}^0$, which gives the desired result.

We can now state the main result of this section.

Corollary 4.8 For each $k \in K$ and $\lambda \in (0, 1]$, there exists a polynomial $P^k \in E^k$ such that $P^k(\lambda, v^k_\lambda) = 0$.

Proof. Let $k$ and $\lambda$ be fixed, and let $z$ be a real variable. Let $r$ denote the rank of $\dot{\Delta}^k - z\dot{\Delta}^0$, that is, the size of the biggest square sub-matrix such that its determinant, which is a polynomial in $z$ and $\lambda$ (see Remark 4.1), is not identically equal to 0. Let $P^k \neq 0$ denote this bi-variate polynomial. The rank drop condition of Proposition 4.7 yields $P^k(\lambda, v^k_\lambda) = 0$. It remains to show that $P^k \in E^k$, which follows from the canonical bijection between $I^n$ and $\{1, \ldots, |I|^n\}$ given in Definition 3.9. Indeed, any sub-matrix of $\dot{\Delta}^k - z\dot{\Delta}^0$, which corresponds to the choice of sub-matrices $M^0_1, \ldots, M^0_n$ of equal size over each row, is a sub-matrix of $\dot{\Delta}^k - z\dot{\Delta}^0$.

The following result is immediate, since the degree of polynomial in Corollary 4.8 in the second variables is bounded by $\min(|I|, |J|)^n$, the size of the biggest square sub-matrix of $W^k(\lambda, z)$.

Corollary 4.9 Suppose that the data matrix $D(\lambda)$ has rational entries. Then $v^k_\lambda$ is an algebraic number of order at most $\min(|I|, |J|)^{n-1}$.

As already stated in Section 2.2, our bound improves all previously known bounds and almost fits the best known lower bound $\min(|I|, |J|)^{n-1}$.

4.3 A polynomial equality: second approach

The approach presented is last section establishes, for the first time, a connection between stochastic games and multi-parameter eigenvalue problems, using Shapley and Snow theory. As we will show below, one can derive from Corollary 4.8 several properties concerning the asymptotic behavior of the values. Before we proceed, we present an alternative proof of this result, based on [2, Theorem 1], presented here:
Theorem 4.10 (Attia and Oliu-Barton 2018) For each \( k \) and \( \lambda \in (0,1] \):
\[
\text{val}(\Delta^k - v^k\Delta^0) = 0
\]

The proof that we are going to present has the advantage of being more direct and easier to handle, but hides the connections with the multi-parameter eigenvalue theory.

An alternative proof of Corollary 4.8

Let \( \hat{\Delta}^k - v^k\hat{\Delta}^0 \) be a Shapley-Snow kernel of \( \Delta^k - v^k\Delta^0 \), and let:
\[
\hat{P}^k(\lambda, z) := \det(\hat{\Delta}^k - z\hat{\Delta}^0), \quad z \in \mathbb{R}
\]

Clearly, \( \hat{P}^k \) belongs to the set \( E^k \), introduced in Definition 4.3. By Proposition 3.6 (ii) and Theorem 4.10 one has:
\[
0 = \text{val}(\Delta^k - v^k\Delta^0) = \text{val}(\hat{\Delta}^k - v^k\hat{\Delta}^0) = \det(\hat{\Delta}^k - v^k\hat{\Delta}^0) \quad (4.6)
\]

Thus, \( \hat{P}^k(\lambda, v^k\lambda) = 0 \). To conclude, it is enough to prove that \( \hat{P} \neq 0 \). An easy way is to compute its partial derivative. Recall that for any square matrix \( M \), one has \( \det(M + H) = \det(M) + \text{tr}(\text{co}(M)H) + o(||H||) \) as \( H \) is tends to 0. Thus:
\[
\frac{\partial \hat{P}^k}{\partial z}(\lambda, v^k\lambda) = \text{tr}(-^t(\text{co}(\hat{\Delta}^k - v^k\hat{\Delta}^0)\hat{\Delta}^0))
\]

As already argued in (4.6) one has \( \text{val}(\hat{\Delta}^k - v^k\hat{\Delta}^0) = 0 \) and \( \hat{\Delta}^k - v^k\hat{\Delta}^0 \) is a Shapley-Snow kernel of \( \Delta^k - v^k\Delta^0 \). Hence, by Proposition 3.6 (iv), the matrix \( \text{co}(\hat{\Delta}^k - v^k\hat{\Delta}^0) \) is not identically zero and has all its entries of same sign. By Lemma 4.2, all entries of \( \hat{\Delta}^0 \) are larger than \( \lambda^n \). Thus,
\[
\frac{\partial \hat{P}^k}{\partial z}(\lambda, v^k\lambda) \neq 0
\]
which proves the result. \( \square \)

Remark 4.11 The two proofs of Corollary 4.8 provide two polynomials \( P_1 \) and \( P_2 \) in the set \( E^k \) such that \( P_1(\lambda, v^k\lambda) = P_2(\lambda, v^k\lambda) = 0 \). The first one is constructed via the rank drop condition given by the choice of a Shapley-Snow kernel at every local game \( G^k(\lambda, v^k\lambda) \), the second is obtained by considering directly a Shapley-Snow kernel of the matrix \( \Delta^k - v^k\Delta^0 \). As already noticed earlier (see Remark 4.4) these two notions may not coincide, so that \( P_1 \) and \( P_2 \) may differ.

5 Asymptotic behaviour of the values

In this section we prove some useful consequences of Corollary 4.8 on the asymptotic behaviour of the discounted values \( (v^k)_\lambda \). In order to state our results with the best possible bounds, we will no longer assume that the action sets are state-independent.
Rather, let $I_k \times J_k$ denote the action set at state $k$, for all $k \in K$, and let the payoff and transition functions be defined over the set $Z := \{(k, i, j) \mid k \in K, (i, j) \in I_k \times J_k\}$. The results in Section 4 can be easily extended to this case, since the data matrix $D(\lambda)$ still satisfies the conditions of Section 3.2, that is, the blocks are of constant size on every row. Furthermore, without loss of generality, we will assume that $|I_k| = |J_k| = 1$ for any absorbing state $k$, i.e. $Q_{k,k} = U$.

Set $a := \prod_{k=1}^{n} \min(|I_k|, |J_k|)$. For each initial state $k = 1, \ldots, n$ one has:

1. There exists a finite set $V^k$ containing all accumulation points of $(v^k_\lambda)_\lambda$.
2. The limit $w^k := \lim_{\lambda \to 0} v^k_\lambda$ exists.
3. Assume that the data of the game (i.e. the payoff and transition functions) are all rational numbers. Then $w^k$ is an algebraic number of degree bounded by $a$.
4. The following bound holds $|v^k_\lambda - w^k| = O(\lambda^{1/a})$. In particular, for absorbing with non-absorbing state 1 one has $a = \min(|I_1|, |J_1|)$ and this bound is tight.
5. The function $\lambda \mapsto v^k_\lambda$ admits an expansion in Puiseux series near 0.

The following remark will be useful.

**Remark 5.1** All polynomials in $E_k$ are defined as the determinant of some square sub-matrix of size less or equal than $a$, with all its entries depending at most linearly on $z$. Hence, the degree in $z$ of all such polynomials is uniformly bounded by $a$.

### 5.1 Convergence of the values

Let us start by defining the set $V^k$ and establishing (1). Note that (1) clearly implies (3), and also (2) thanks to the continuity of the value function.

An initial state $k$ will be fixed throughout this section. Let us start by introducing some useful notions. Let $\varphi$ be the following map from the set of polynomials in $(\lambda, z)$ to the set of polynomials in $z$:

$$
\varphi_P := \begin{cases} 
1 & \text{if } P = 0 \\
P_s & \text{if } P(\lambda, z) = \sum_{r=s}^{m} P_r(z)\lambda^s \text{ and } P_s \neq 0
\end{cases}
$$

where $m := \deg_1(P)$ denotes the degree of $P$ in the variable $\lambda$ and $P_0, \ldots, P_m$ are the unique polynomials in $z$ such that $P(\lambda, z) = \sum_{r=0}^{m} P_r(z)\lambda^s$. Now let $\bar{E}^k$ be the image of $E^k$ by $\varphi$, that is:

$$
\bar{E}^k := \{ \varphi_P \mid P \in E^k \}
$$

Finally, let $V^k \subset \mathbb{R}$ be the following limit set:

$$
V^k := [C^-, C^+] \cap \{v \in \mathbb{R} \mid P(v) = 0 \text{ for some } P \in \bar{E}^k \}
$$

where $C^- := \min_{k,i,j} g(k, i, j)$ and $C^+ := \max_{k,i,j} g(k, i, j)$.

**Proposition 5.2** Every accumulation point of $(v^k_\lambda)_\lambda$ belongs to $V^k$. 

17
**Proof.** Let \((\lambda_n)_n\) be such that \(\lim_{n \to \infty} \lambda_n = 0\) and \(\lim_{n \to \infty} v^k_{\lambda_n} = w\). By Corollary 4.8, for each \(n\) there exists a polynomial \(P_n \in E^k\) such that \(P_n(\lambda_n, v^k_{\lambda_n}) = 0\). The set \(E^k\) being finite, up to extracting a subsequence we can assume that \(P_n\) is constant, say equal to \(P\). Hence, for each \(n \in \mathbb{N}^*\) one has:

\[
P(\lambda_n, v^k_{\lambda_n}) = 0
\]

Then, by the definition of the map \(\varphi\), one has:

\[
0 = P(\lambda_n, v^k_{\lambda_n}) = \lambda_n^s \varphi_P(v^k_{\lambda_n}) + o(\lambda_n^s)
\]

Dividing by \(\lambda_n^s\) and taking \(n\) to infinity, it follows that \(\varphi_P(w) = 0\). Hence, \(w \in V^k\).

**Corollary 5.3** The values \((v^k_\lambda)_\lambda\) converge, and the limit belongs to \(V^k\).

**Proof.** The set of accumulation points of \((v^k_\lambda)_\lambda\) is non-empty, by boundedness, and is included in \(V^k\) by Proposition 5.2 which is a finite set. If \(v < v'\) are two different accumulation points then, by the continuity of the function \(\lambda \mapsto v^k_\lambda\), every \(c \in [v, v']\) is an accumulation point as well. A contradiction.

Together, Proposition 5.2 and Corollary 5.3 prove (1) and (2). Let us finish this section by proving (3).

**Proposition 5.4** If \(g\) and \(q\) are rational, then all elements of \(V^k\) are algebraic of degree bounded by \(M\). In particular, \(w^k := \lim_{\lambda \to 0} v^k_\lambda\).

**Proof.** Let \(z \in V^k\). By definition of \(V^k\), there exists \(P \in E^k\) such that \(\varphi_P(z) = 0\). By definition of \(\varphi_P\) its degree is bounded by the degree in \(z\) of \(P\). The result follows, as \(w^k \in V^k\) by Corollary 5.3.

### 5.2 Speed of convergence and Puiseux expansion

To prove (4) and (5), let us start by recalling the definition of a Puiseux series.

**Definition 5.5** A map \(f : (0, \varepsilon) \to \mathbb{C}\) is a Puiseux series if there exists \(N \in \mathbb{N}^*\), \(m_0 \in \mathbb{Z}\) and a complex sequence \((a_m)_{m \geq m_0}\) such that:

\[
f(\lambda) = \sum_{m \geq m_0} a_m \lambda^{m/N}
\]

Any bounded Puiseux series satisfies \(m_0 \geq 0\) and, in particular, converges as \(\lambda \to 0\). The following result is based on Puiseux’s Theorem [8].

**Theorem 5.6 (Puiseux, 1850)** Let \(P(\lambda, z)\) be a bi-variate polynomial. There exists \(\lambda_0 > 0\) such that the roots of \(P(\lambda, \cdot)\) are Puiseux series in the interval \((0, \lambda_0)\).

**Proposition 5.7** The following assertions hold:

(i) There exists \(\lambda_0 > 0\) and \(P \in E^k\) such that \(P(\lambda, v^k_\lambda) = 0\) for all \(\lambda \in (0, \lambda_0)\)

(ii) There exists \(\lambda_0 > 0\) such that \(\lambda \mapsto v^k_\lambda\) is a Puiseux series on \((0, \lambda_0)\)
(ii) As $\lambda$ vanishes one has the following bound:

$$|v_\lambda^k - w^k| = O(\lambda^{1/a})$$

Proof.

(i) and (ii). By construction, any polynomial $P \in E^k$ is of degree at most $n$ in $\lambda$. By Puiseux’s theorem, for each of these polynomials, there exists $\lambda_1 > 0$ such that the $m \leq n$ roots $z^1(\lambda), \ldots, z^n(\lambda)$ of $P(\lambda, \cdot)$ are Puiseux series on $(0, \lambda_1)$. By finiteness of the set $E^k$, we can suppose that $\lambda_1$ is common to all polynomials. Ranging over the different polynomials of $E^k$, we thus define a finite set of Puiseux series. By Corollary 4.8 $\lambda \mapsto v_\lambda^k$ belongs to this set. The continuity of the map $\lambda \mapsto v_\lambda^k$ implies that, as $\lambda$ changes, $v_\lambda^k$ may change of branch (i.e. of Puiseux series) only at points where two such series intersect. Yet, two different Puiseux series cannot intersect infinitely many times on $(0, \lambda_1)$. Consequently, there exists $0 < \lambda_0 < \lambda_1$ such that, on $(0, \lambda_0)$, any two Puiseux series are either congruent or disjoint. It follows that $v_\lambda^k = z(\lambda)$ on $(0, \lambda_0)$ for one of these series.

(iii) Let $P \in E^k$ be such that $P(\lambda, v_\lambda^k) = 0$ for all $\lambda \in (0, \lambda_0)$, given by (i). By the definition of $\varphi$, one can write:

$$P(\lambda, z) = \lambda^s \varphi_P(z) + O(\lambda^t)$$

for some integers $s \leq n$ and $t \geq s + 1$. From these two relations, one deduces that:

$$\varphi_P(v_\lambda^k) = O(\lambda^{t-s})$$

In particular, one has $\varphi_P(w^k) = 0$. More precisely, as $\varphi_P \neq 0$ by definition there exists an integer $r \leq n$ and a polynomial $Q$ such that

$$\varphi_P(z) = (z - w^k)^r Q(z) \text{ and } Q(w^k) \neq 0$$

It follows that $|v_\lambda^k - w^k|^r = O(\lambda^{t-s})$. Now, recall from Remark 5.1 that the degree of $P(\lambda, z)$ is bounded by $a$, so that the same bound holds for the degree of $\varphi_P$. Hence, $r \leq M$. On the other hand, $t = s + 1$ in the worst case, which gives the result. 

5.2.1 Tightness of the bound

In the class of absorbing games (that is, stochastic games in which all states except one, say state 1, are absorbing), the bound given in Proposition 5.7 reduces to

$$|v_\lambda^1 - w^1| = O(\lambda^{1/a})$$

where $a := \min(|I^1|, |J^1|)$. The following example, due to Kohlberg [3] shows that the bound is tight in this class of games. For any $p \geq 1$, consider the following absorbing game of size $p \times p$:

$$
\begin{pmatrix}
1^* & 0^* & \ldots & 0^* \\
0 & 1^* & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0^* \\
0 & \ldots & 0^* & 1^*
\end{pmatrix}
$$

where $c^*$ indicates a stage payoff of $c$ and a certain transition to an absorbing state with payoff $c$. Its discounted values are known to be equal to $v_\lambda = \frac{1 - \lambda^{1/p}}{1 - \lambda}$, so that $v_\infty = 1$ and $|v_\lambda - v_\infty| = O(\lambda^{1/p})$. 

19
5.3 Determining the exact limit values

For any \( k \in K \), the limit set \( V^k \) defined in section 5.1 can be computed algorithmically without using any game theoretical skills. It is finite and included in the interval \([C^-, C^+]\) by definition and, more importantly, contains the limit value \( w^k \) thanks to Corollary 5.3. This sets allow us to transform the algorithm provided by the authors in [2], intended to compute an approximation \( w^k \), into an exact calculation. Let us start by recalling the characterisation provided therein.

**Theorem 5.8 (Attia and Oliu-Barton 2018)** For each \( k \), the values \((v^k_\lambda)\lambda\) converge to the unique point \( w^k \in \mathbb{R} \) satisfying:

\[
\begin{align*}
    z > w^k & \Rightarrow F^k(z) < 0 \\
    z < w^k & \Rightarrow F^k(z) > 0
\end{align*}
\]

where for each \( z \in \mathbb{R} \), \( F^k(z) := \lim_{\lambda \to 0} \lambda^{-n} \text{val}(\Delta^k - z \Delta^0) \).

The algorithm provided in [2] goes as follows. Set an error \( \varepsilon > 0 \), and initialize \( w = C^- \) and \( \overline{w} = C^+ \), then let \( d := \overline{w} - w \). Then, for each \( k \), proceed as follows.

While \( w < \overline{w} \) and \( d2^{-n} > \varepsilon \), update these variables according to:

\[
\begin{align*}
    w & := \frac{w + \overline{w}}{2} & \text{if } F^k\left(\frac{w + \overline{w}}{2}\right) \geq 0 \\
    \overline{w} & := \frac{w + \overline{w}}{2} & \text{if } F^k\left(\frac{w + \overline{w}}{2}\right) \leq 0
\end{align*}
\]

The algorithm terminates after at most

\[
\left\lceil \frac{\ln(d/\varepsilon)}{\ln 2} \right\rceil
\]

iterations, and the same number of computations of the sign of \( F^k \), and provides a final interval of size at most \( \varepsilon \), denoted by \([\underline{w}_f, \overline{w}_f] \) containing \( w^k \). Combining it with Corollary 5.3 one obtains a finite set containing the exact value of the limit, whose diameter is less than \( \varepsilon \):

\[
w^k \in V^k \cap [\underline{w}_f, \overline{w}_f]
\]

In particular, setting \( \varepsilon := \min\left\{ \frac{1}{2}|v - v'|, v, v' \in V^k, v \neq v' \right\} \) at the beginning of the algorithm leads to the exact limit value, since \([\underline{w}_f, \overline{w}_f] \) contains a unique element of \( V^k \), which is the limit value.

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**Appendix**

**Proof of Lemma 3.5**

Let us start by recalling the statement of Lemma 3.5.
Let $M$ be a square matrix of size $a \in \mathbb{N}^*$ and rank $a - 1$, and let $x$ and $y$ be such that $\text{Ker}(tM) = \langle x \rangle$ and $\text{Ker}(M) = \langle y \rangle$. Then there exists a constant $\alpha \neq 0$ such that $\text{co}(M) = \alpha txy$.

**Proof.** Using the relation $tA \text{co}(A) = \det(A) \text{Id}$, which is valid for any matrix $A$ and $\det(M) = 0$, we get $tM \text{co}(M) = 0$. Moreover since $\text{Ker}(M) = \langle y \rangle$, all the rows of $\text{co}(M)$ are proportional to $y$. Hence, there exists $x'$ such that $\text{co}(M) = t x' y$. But this equality shows that the columns of $\text{co}(M)$ are proportional to $x'$. A symmetric argument shows that the column are proportional to $x$ so that $x$ and $x'$ are proportional.

Let $\alpha \in \mathbb{R}$ be such that $x' = \alpha x$ so that $\text{co}(M) = \alpha txy$. As $M$ is of rank $n - 1$, the matrix $\text{co}(M)$ has a non-zero column, so that $\alpha \neq 0$, which proves the result. 

**Examples illustrating the sets $S^M$ and $S^\Delta$**

**Example 1:** $S^M$ is not a singleton

Consider the following absorbing game, for some $a > b > 0$.

$$
\begin{pmatrix}
  a^* & 0 \\
  0 & b^*
\end{pmatrix}
$$

where $c^*$ indicates that an absorbing state is reached with probability 1, where the payoff is equal to $c$ regardless of players' action. Model this game over the set of states $\{1, 2, 3\}$, where state 1 is non absorbing, and states 2 and 3 are absorbing with payoffs $a$ and $b$, respectively (w.l.o.g. we assume that players have only one action in these states). For any $z \in \mathbb{R}^3$ the normalized local games $G^k(\lambda, z) = z^kU$ are given by:

$$
\begin{pmatrix}
  a\lambda + (1 - \lambda)z^2 - z^1 & -\lambda z^1 \\
  -\lambda z^1 & b\lambda + (1 - \lambda)z^3 - z^1
\end{pmatrix}, \quad (\lambda(a - z^2)), \quad \text{and} \quad (\lambda(b - z^3))
$$

The unique Shapley-Snow kernel of $G^1(\lambda, v)$ is the entire matrix, so that the system (4.3) reduces to the equalities $z^2 = a$, $z^3 = b$ and:

$$
\det \begin{pmatrix}
  a - z^1 & -\lambda z^1 \\
  -\lambda z^1 & b - z^1
\end{pmatrix} = 0
$$

Yet, this system has 2 solutions for any $a > b > 0$ and $\lambda \in (0, 1]$.

**Example 2:** $S^\Delta$ is infinite

Consider a two-state two-actions stochastic game where the states do not communicate. Formally, $K = \{1, 2\}$, $|I| = |J| = 2$ and $g(k | k, i, j) = 1$, for $k = 1, 2$ and $(i, j) \in I \times J$.

Suppose also that $g(1, i, j) = 1$ and $g(2, i, j) = 1_{\{i=j\}}$ for all $(i, j) \in I \times J$, so that the array of matrices is given by:

$$
\begin{pmatrix}
  \lambda U & -\lambda U & 0 \\
  \lambda \text{Id} & 0 & -\lambda U
\end{pmatrix}
$$
The normalized local games \( G^k(\lambda, z) - z^kU \) are given by:

\[
\lambda(1 - z^1)U \quad \text{and} \quad \lambda \begin{pmatrix}
1 - z^2 & -z^2 \\
-z^2 & 1 - z^2
\end{pmatrix}
\]

so that any entry of the first local game is a Shapley-Snow kernel, and the entire game is the unique Shapley-Snow kernel of the second. Thus, the corresponding \( M \)-system has a unique solution \( S^M = \{(1, \frac{1}{2})\} \). On the other hand, the auxiliary matrices are \( \hat{\Delta}^0 = \hat{\Delta}^1 = \lambda^2 U \) and \( \hat{\Delta}^2 = \lambda^2 \text{Id} \), all of size \( 2 \times 2 \), and \( S^{\hat{\Delta}} = \{(z^1, z^2) \mid z^1 \in \mathbb{R}, \, z^2 = \frac{1}{2}\} \).

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