PERFECT BUT NOT GENERATING DELAUNAY POLYTOPES

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Abstract. In his seminal 1951 paper “Extreme forms” Coxeter [Co51] observed that for \( n \geq 9 \) one can add vectors to the perfect lattice \( A_9 \) so that the resulting perfect lattice, called \( A_3^2 \) by Coxeter, has exactly the same set of minimal vectors. An inhomogeneous analog of the notion of perfect lattice is that of a lattice with a perfect Delaunay polytope: the vertices of a perfect Delaunay polytope are the analogs of minimal vectors in a perfect lattice. We find a new infinite series \( P(n, s) \) for \( s \geq 2 \) and \( n + 1 \geq 4s \) of \( n \)-dimensional perfect Delaunay polytopes. A remarkable property of this series is that for certain values of \( s \) and all \( n \geq 13 \) one can add points to the integer affine span of \( P(n, s) \) in such a way that \( P(n, s) \) remains a perfect Delaunay polytope in the new lattice. Thus, we have constructed an inhomogeneous analog of the remarkable relationship between \( A_9 \) and \( A_3^2 \).

1. Introduction

Given a \( n \)-dimensional lattice \( L \), a polytope \( D \) is called a Delaunay polytope if the set of its vertices is \( S \cap L \) with \( S \) being a sphere containing no lattice points in its interior. If \((v_1, \ldots, v_n)\) is a basis of \( L \) then the Gram matrix \( Q = \langle (v_i, v_j) \rangle_{1 \leq i, j \leq n} \) characterizes \( L \) up to isometry. It has long been observed that for computations it is preferable to work with Gram matrices instead of lattices. Then one defines \( S^n_{>0} \) the cone of positive definite \( n \times n \)-symmetric matrices, identifies the quadratic forms with symmetric matrices and defines \( A[X] = X^tAX \) for a column vector \( X \) and a symmetric matrix \( A \).

Voronoi [Vo08] remarked that if \( D \) is a polytope with coordinates in \( \mathbb{Z}^n \) then the condition that \( D \) is a Delaunay polytope is expressed by linear equalities and inequalities on the coefficients of the Gram matrix.

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That is if one defines

\[ \text{SC}(D) = \left\{ Q \in S^n_{>0} : \exists c \in \mathbb{R}^n, r > 0 \text{ such that } Q[v - c] = r \text{ for } v \in \text{vert } D \right. \]

\[ \text{and } Q[v - c] > r \text{ for } v \in \mathbb{Z}^n - \text{vert } D \left. \right\} \]

then \( \text{SC}(D) \) (called Baranovskii cone in [Sc09]) is a polyhedral cone. The dimension of \( \text{SC}(D) \) is called the rank of \( D \). \( D \) is called perfect if it is of rank 1 (see [Er92] and [DGL93] for more details).

The only perfect Delaunay polytope of dimension \( n \leq 6 \) are the interval \([0, 1]\) and Schlafli polytope \( 2_{21} \), which are Delaunay polytopes of the root lattices \( A_1 \) and \( E_6 \) (see [DD04]). Several infinite series of perfect Delaunay polytopes were built in [Er02], [Du05] and [Gr06]. Some, conjectured to be complete, lists are given in [DER07] for dimension 7 and 8. A polytope \( P \) is called centrally symmetric if there exist a point \( c \), called center, such that for any vertex \( v \in P \) we have \( 2c - v \in P \). In this paper for every \( 4s \leq n + 1 \), we build a Delaunay polytope \( P(n, s) \) such that:

(i) \( P(n, s) \) has dimension \( n \), is centrally symmetric and has \( 2 \binom{n+1}{s} \) vertices.

(ii) \( P(n, s) \) is perfect for \( s \geq 2 \).

Given a Delaunay polytope \( P \) in a lattice \( L \), we denote by \( L(P) \) the set of lattice points that can be expressed as integral sum of vertices of \( P \). \( P \) is generating if \( L(P) \) coincides with \( L \).

All perfect Delaunay polytopes known so far were generating and the main interest of \( P(n, s) \) is that if

\[ 6s < \left\{ \begin{array}{ll} n + 1 & \text{if } n \text{ is odd}, \\ n & \text{if } n \text{ is even}, \end{array} \right. \]

then there exists a lattice \( L' \) such that \( P(n, s) \) is a Delaunay polytope in \( L' \) and \( L(P) \neq L' \).

The polytope \( P(7, 2) \) is the Gosset polytope \( 3_{21} \), which is a Delaunay polytope of the root lattice \( E_7 \) and \( P(8, 2) \) is the Delaunay polytope \( D_8^8 \) of [DER07]. The infinite series \( P(n, s) \) were found by looking at \( D_8^8 \) and the lattice \( L' \) was found by an exhaustive search using the computer package [Du08].

2. The lattice \( A_n \)

The lattice \( A_n \) is defined as

\[ A_n = \left\{ x = (x_0, \ldots, x_n) \in \mathbb{Z}^{n+1} : \sum_{i=0}^{n} x_i = 0 \right\} \]
\( \mathbb{A}_n \) is an \( n \)-dimensional lattice, but best seen as embedded into \( \mathbb{R}^{n+1} \) with the standard Euclidean metric \( \sum_{i=0}^{n} x_i^2 \). Define \( (e_i)_{1 \leq i \leq n+1} \) the standard basis of \( \mathbb{R}^{n+1} \).

It is often useful to think of \( \mathbb{A}_n \) as a point lattice. More formally, define

\[
V_{n,s} = \left\{ x = (x_0, \ldots, x_n) \in \mathbb{Z}^{n+1} : \sum_{i=0}^{n} x_i = s \right\}.
\]

Then the difference set \( V_{n,s} - V_{n,s} \) is the lattice \( \mathbb{A}_n \). Let

\[
J(n+1, s) = \text{conv} \left\{ x \in \{0,1\}^{n+1} : \sum_{i=0}^{n} x_i = s \right\}.
\]

It is easily seen that \( J(n+1, s) \) is a lattice polytope in the point lattice \( V_{n,s} \). Since \( V_{n,s} - V_{n,s} = \mathbb{A}_n \), we know that \( \mathbb{A}_n \) contains lattice polytopes isometric to \( J(n+1, s) \).

For \( \alpha_0, \ldots, \alpha_n \in \mathbb{R} \), we define

\[
q_{\alpha_0, \ldots, \alpha_n}(x) = \sum_{i=0}^{n} \alpha_i x_i^2
\]

and denote by \( Q_P \) the cone of all \( q_{\alpha_0, \ldots, \alpha_n} \) with \( \alpha_i > 0 \). Clearly the polytopes \( J(n+1, s) \) are Delaunay polytopes of \( \mathbb{A}_n \) for the scalar product induced by \( q \in Q_P \).

The following theorem is a reformulation of Proposition 8 of [BaGr01].

**Theorem 1.** (i) The lattice \( \mathbb{A}_n \) has \( n \) translation classes of Delaunay polytopes. These classes are represented by polytopes \( J(n+1, s) \) for \( 1 \leq s \leq n \).

(ii) The scalar products on \( \mathbb{A}_n \) having the polytopes \( J(n+1, s) \) as Delaunay polytopes are the ones induced by some \( q \in Q_P \).

According to the terminology of [BaGr01] this means that the non-rigidity degree of \( \mathbb{A}_n \) is \( n + 1 \). Note that the forms \( x_0^2, \ldots, x_n^2 \) remain independent when restricted to \( \mathbb{A}_n \). One classic example is the Delaunay tessellation of \( \mathbb{A}_3 \): It is formed by the regular simplex \( J(4,1) \), its antipodal \( J(4,3) \) and the regular octahedron \( J(4,2) \).

Clearly, the rank of the polytopes \( J(n+1,1) \) and \( J(n+1,n) \) is \( \frac{n(n+1)}{2} \) since those polytopes are \( n \)-dimensional simplices.

**Theorem 2.** Let \( n, s \in \mathbb{N} \) and \( 2 \leq s \leq n - 1 \).

(i) The rank of \( J(n+1, s) \) is \( n + 1 \) and every scalar product on \( \mathbb{A}_n \) having \( J(n+1, s) \) as Delaunay is induced by some \( q \in Q_P \).
(ii) The center $c_{\alpha_0,\ldots,\alpha_n}$ of the empty ellipsoid around $J(n+1, s)$ with respect to the quadratic form $q_{\alpha_0,\ldots,\alpha_n}$ is given by

$$c_{\alpha_0,\ldots,\alpha_n} = \left(\frac{1}{2} + \frac{C}{\alpha_0}, \ldots, \frac{1}{2} + \frac{C}{\alpha_n}\right) \text{ with } C = \frac{s - \frac{n+1}{2}}{\sum_{i=0}^{n} \frac{1}{\alpha_i}}.$$

Proof. For $i = 1, \ldots, n$ define $v_i = e_i - e_0$. The norm of a vector $x = \sum_{i=0}^{n} x_i e_i \in A_n$ with respect to $q_{\alpha_0,\ldots,\alpha_n}$ is

$$q_{\alpha_0,\ldots,\alpha_n}(x) = q_{\alpha_0,\ldots,\alpha_n}(-\sum_{i=1}^{n} x_i e_0 + \sum_{i=1}^{n} x_i e_i) = \alpha_0(\sum_{i=1}^{n} x_i)^2 + \sum_{i=1}^{n} \alpha_i x_i^2 = X^t A_{\alpha_0,\ldots,\alpha_n} X$$

where $X = (x_1, \ldots, x_n)^t$, and

$$A_{\alpha_0,\ldots,\alpha_n} = \begin{pmatrix} \alpha_0 + \alpha_1 & \alpha_0 & \cdots & \alpha_0 \\ \alpha_0 & \alpha_0 + \alpha_2 & \cdots & \vdots \\ \vdots & \vdots & \ddots & \alpha_0 \\ \alpha_0 & \cdots & \alpha_0 & \alpha_0 + \alpha_n \end{pmatrix}.$$

Expressed in terms of the basis $(v_i)_{1 \leq i \leq n}$ the polytope $J(n+1, s)$ is written as

$$J'(n+1, s) = \text{conv} \left\{ (x_1, \ldots, x_n) \in \{0,1\}^n \text{ with } s - \sum_{i=1}^{n} x_i \in \{0,1\} \right\}.$$

Theorem 1 (ii) then implies that if $\alpha_i > 0$, then $A_{\alpha_0,\ldots,\alpha_n} \in \text{SC}(J'(n+1, s))$. Let us now take $A = (a_{i,j})_{1 \leq i, j \leq n} \in \text{SC}(J'(n+1, s))$.

Select a three element subset $S = \{s_1, s_2, s_3\}$ of $\{1, \ldots, n\}$ and a vector $v \in \{0,1\}^n$. Consider the polytope

$$J_{S,v} = \text{conv}\{w \in \text{vert} J'(n+1, s) : w_i = v_i \text{ for } i \notin S\}.$$

If one chooses $v$ such that $\sum_{i \in S} v_i = s - 2$, then $J_{S,v}$ is affinely equivalent to the octahedron $J(4,2)$. The quadratic form $q(x) = X^t A X$ induces a quadratic form $q_S$ on the affine space spanned by $J_{S,v}$ with $q_S(Y) = Y^t A_S Y$, $Y = (x_{s_1}, x_{s_2}, x_{s_3})^t$ and $A_S = (a_{i,j})_{i,j \in S}$.

The rank of the octahedron $J(4,2)$ is equal to 4 as proved on page 232 of [DeLa97]. The quadratic form $A_{\alpha_0,\alpha_1,\alpha_2,\alpha_3}$ with $\alpha_i > 0$ has 4 independent coefficients and belongs to $\text{SC}(J_{S,v})$ thus we get $A_S = A_{\alpha_0,\ldots,\alpha_3}$ for some coefficients $\alpha_i$. This implies $a_{i,j} = C$ for $i \neq j \in S$ with the constant $C_S$ a priori depending on $S$. If one interprets the value $a_{i,j}$ as colors of an edge between vertices $i$ and $j$ then we get that all triangles of the complete graph on $n$ vertices are monochromatic. This is possible only if there is only one edge color. So, $a_{i,j} = C$ for $i \neq j$ and one can write $A = A_{\alpha_0,\ldots,\alpha_n}$ with $\alpha_i \in \mathbb{R}$. 

Let us find the circumcenter of the empty sphere around $J(n+1, s)$. The point $h_{n+1} = \left( \left( \frac{1}{2} \right)^{n+1} \right)$ is at equal distance from all points of $J(n+1, s)$. However, it does not belong to $V_{n, s}$. To find the circumcenter $c$ of $J(n+1, s)$, we take the orthogonal projection of $h_{n+1}$ on the hyperplane \( \sum_{i=0}^{n} x_i = s \) for the quadratic form $q_{\alpha_0, \ldots, \alpha_n}$. Easy computations give (ii).

Let us prove $\alpha_i > 0$. It is well known that the facets of $J(n+1, s)$ are determined by the inequalities $x_i \geq 0$ and $x_i \leq 1$. It is also easy to see that the Delaunay polytopes adjacent to the facets $x_0 \geq 0$ and $x_0 \leq 1$ are

\[
J_0^- = \{ x \in \{-1, 0\} \times \{0, 1\}^n : \sum_{i=0}^{n} x_i = s \},
\]

and \( J_0^+ = \{ x \in \{1, 2\} \times \{0, 1\}^n : \sum_{i=0}^{n} x_i = s \} \).

The polytopes $J_0^-$, $J_0^+$ are equivalent under translation to $J(n+1, s+1)$ and $J(n+1, s-1)$.

The square distance of $h_{n+1}$ to the vertices of $J(n+1, s)$ is $d = \sum_{i=0}^{n} \frac{\alpha_i}{4}$ and the square distance of $h_{n+1}$ to the vertices of $J_0^-$, $J_0^+$ not in $J(n+1, s)$ is $d' = \alpha_0^2 + \sum_{i=1}^{n} \alpha_i \frac{1}{4}$. The conditions defining $SC(J'(n+1, s))$ imply $d' > d$ hence $\alpha_0 > 0$ and by symmetry $\alpha_i > 0$. So, the conditions for $J(n+1, s)$ to be a Delaunay polytope imply that $A = A_{\alpha_0, \ldots, \alpha_n}$ with $\alpha_i > 0$. But according to Theorem 1, these conditions are sufficient for the stronger condition of preserving all the Delaunay polytopes of $A_n$ so they are clearly sufficient for just $J(n+1, s)$. □

3. The polytopes $P(n, s)$

We denote an $(n+1)$-vector whose first $a$ coordinates are $A$ and the remaining $n+1-a$ coordinates $B$ by $(A^a; B^{n+1-a})$. Similar convention is used for vectors with three distinct coordinates, e.g. $(A^a; B^b; C^{n+1-a-b})$.

**Definition 1.** Take $n, s \in \mathbb{Z}$ with $s \geq 1$ and $4s \leq n + 1$.

(i) Set $v_{n,s} = \left( \left( \frac{1}{4} \right)^{4s} ; 0^{n+1-4s} \right)$. The polytope $P(n, s)$ is defined as

\[
P(n, s) = \text{conv} \{ v, 2v_{n,s} - v \text{ for } v \in \text{vert } J(n+1, s) \}.
\]

(ii) Define $t_{n,s} = \left( \left( \frac{1}{2} \right)^{2s} ; \left( \frac{1}{2} \right)^{2s} ; 0^{n+1-4s} \right)$ and

\[
V_{n,s}^2 = \{ v, t_{n,s} + v \text{ for } v \in V_{n,s} \}.
\]

**Theorem 3.** Take $n, s \in \mathbb{Z}$ with $s \geq 2$ and $4s \leq n + 1$.

(i) $V_{n,s}^2$ is a lattice and $P(n, s)$ affinely generates it.
(ii) The polytope $P(n, s)$ is perfect with the unique, up to positive multiple, positive definite quadratic form being

$$q_{n,s}(x) = 2 \sum_{i=0}^{4s-1} x_i^2 + \sum_{i=4s}^{n} x_i^2.$$ 

The center of the circumscribed ellipsoid is $v_{n,s}$ and the squared radius is $\frac{4s}{2}$.

Proof. We have $2t_{n,s} \in V_{n,s}$ so $V_{n,s}^2$ is a lattice. $P(n, s)$ generates it since $J(n+1, s)$ generates $A_n$. By its definition, $P(n, s)$ is centrally symmetric of center $v_{n,s}$. It is well known and easy to prove that if a Delaunay polytope is centrally symmetric then the center $c'$ of its empty sphere coincides with the center $c$ of the antisymmetry operation $v \mapsto 2c - v$. So, we should have $v_{n,s} = c_{\alpha_0, \ldots, \alpha_n}$ with

$$c_{\alpha_0, \ldots, \alpha_n} = \left( \frac{1}{2} + \frac{C}{\alpha_0}, \ldots, \frac{1}{2} + \frac{C}{\alpha_n} \right).$$

Thus:

- For $0 \leq i \leq 4s - 1$, we have $c_i = \frac{1}{4}$. This implies $\alpha_i = -4C$.
- For $4s \leq i \leq n$, we have $c_i = 0$. This implies $\alpha_i = -2C$.

Summarizing we get $q = -2Cq_{n,s}$ and thus that $P(n, s)$ is perfect. The proof of the Delaunay property follows from the fact that the coefficient in front of $x_i^2$ are strictly positive for $0 \leq i \leq n$ and property (i) of Theorem 2. \qed

4. The lattice $V_{n,s}^4$

Define the vector $w_{n,s}$ by

$$w_{n,s} = \begin{cases} \left( \left( \frac{1}{4} \right)^2, \left( -\frac{1}{4} \right)^2, \left( \frac{1}{2} \right)^{n+1-4s} \right) - \frac{n+1-4s}{2} e_1 & \text{if } n \text{ is odd}, \\ \left( \left( \frac{1}{4} \right)^2, \left( -\frac{1}{4} \right)^2, 0, \left( \frac{1}{2} \right)^{n-4s} \right) - \frac{n-4s}{2} e_1 & \text{if } n \text{ is even}. \end{cases}$$

Then define

$$V_{n,s}^4 = V_{n,s}^2 \cup w_{n,s} + V_{n,s}^2.$$ 

Clearly $V_{n,s}^4$ is a lattice that contains $V_{n,s}^2$ as an index 2 sublattice. We want to prove that $P(n, s)$ remains a Delaunay polytope in $V_{n,s}^4$ for some values of $n$ and $s$.

Theorem 4. The polytope $P(n, s)$ is a Delaunay polytope of $V_{n,s}^4$ if

$$6s < \begin{cases} n + 1 & \text{if } n \text{ is odd}, \\ n & \text{if } n \text{ is even}. \end{cases}$$
Proof. We need to solve the closest vector problem for the lattice $V_{n,s}^4$ and the point $v_{n,s}$. For $V_{n,s}^2$ this is solved by Theorem 3. Thus we need to find the closest vectors in $w_{n,s} + V_{n,s}^2$ to $v_{n,s}$. This is equivalent to finding the closest vectors in $V_{n,s}$ to $v_{n,s} - w_{n,s}$ and to $v_{n,s} - w_{n,s} - t_{n,s}$. We have if $n$ is odd:

$$v_{n,s} - w_{n,s} = \left(0^{2s}; \left(\frac{1}{2}\right)^{2s}; (-\frac{1}{2})^{n+1-4s}\right) + \frac{n+1-4s}{2} e_1,$$

$$v_{n,s} - w_{n,s} - t_{n,s} = \left(-\left(\frac{1}{2}\right)^{2s}; 1^{2s}; (-\frac{1}{2})^{n+1-4s}\right) + \frac{n+1-4s}{2} e_1,$$

and if $n$ is even:

$$v_{n,s} - w_{n,s} = \left(0^{2s}; \left(\frac{1}{2}\right)^{2s}; 0; (-\frac{1}{2})^{n-4s}\right) + \frac{n-4s}{2} e_1,$$

$$v_{n,s} - w_{n,s} - t_{n,s} = \left(-\left(\frac{1}{2}\right)^{2s}; 1^{2s}; 0; (-\frac{1}{2})^{n-4s}\right) + \frac{n-4s}{2} e_1.$$

All the vectors occurring have coordinates belonging to $\mathbb{Z}$ or $\mathbb{Z} + \frac{1}{2}$. Since the coordinates of elements of $V_{n,s}$ are integral and $q_{n,s}$ has non-zero coefficients only for $x_i^2$ this gives for $v \in V_{n,s}$ the following lower bounds if $n$ is odd:

$$q_{n,s}(v_{n,s} - w_{n,s} - v) \geq 2 \times 2s \times \frac{1}{4} + (n + 1 - 4s) \frac{1}{4} = \frac{n+1}{4},$$

$$q_{n,s}(v_{n,s} - w_{n,s} - t_{n,s} - v) \geq 2 \times 2s \times \frac{1}{4} + (n + 1 - 4s) \frac{1}{4} = \frac{n+1}{4},$$

and if $n$ is even:

$$q_{n,s}(v_{n,s} - w_{n,s} - v) \geq 2 \times 2s \times \frac{1}{4} + (n - 4s) \frac{1}{4} = \frac{n}{4},$$

$$q_{n,s}(v_{n,s} - w_{n,s} - t_{n,s} - v) \geq 2 \times 2s \times \frac{1}{4} + (n - 4s) \frac{1}{4} = \frac{n}{4}.$$

So, if $n$ and $s$ satisfy the condition of the theorem then the closest points in $w_{n,s} + V_{n,s}^2$ are at a square distance greater than $\frac{3s}{2}$. But $\frac{3s}{2}$ is the square radius of the circumscribing sphere thus proving that $P(n,s)$ is a Delaunay polytope in $V_{n,s}^4$.

The above theorem gives example of Delaunay polytopes, which are perfect but not generating, the first example of which is $P(13,2)$.

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