The Korteweg-de Vries Hierarchy and Long Water-Waves

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Abstract

By using the multiple scale method with the simultaneous introduction of multiple times, we study the propagation of long surface-waves in a shallow inviscid fluid. As a consequence of the requirements of scale invariance and absence of secular terms in each order of the perturbative expansion, we show that the Korteweg-de Vries hierarchy equations do appear in the description of such waves. Finally, we show that this procedure of eliminating secularities is closely related to the renormalization technique introduced by Kodama and Taniuti.

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I. INTRODUCTION

In 1967, Gardner, Greene, Kruskal and Miura [1], by making use of the ideas of direct and inverse scattering, showed that the Korteweg-de Vries (KdV) equation could be solved exactly as an initial value problem. Shortly after, Lax [2] generalized these ideas to a general evolution equation of the type

\[
\frac{\partial u}{\partial t} = K[u], \quad (1)
\]

with \( K \) a nonlinear operator that could be written in the form

\[
K[u] = [B, L], \quad (2)
\]

and with \( B \) and \( L \) self-adjoint linear operators depending on \( u \). As a consequence of this formalism, Lax showed the existence of an infinite sequence of integrable partial differential equations of the form

\[
\frac{\partial u}{\partial t} = K_n[u], \quad n = 1, 2, \ldots, \quad (3)
\]

where

\[
K_n = [B_n, L] \quad (4)
\]

with

\[
B_n = D^{2n+1} + \sum_{j=1}^{n} \left( b_j D^{2j+1} + D^{2j+1} b_j \right) \quad (5)
\]

and

\[
L = D^2 + \frac{1}{6} u. \quad (6)
\]

In these equations, \( b_j \) are coefficients that can be determined [2], and \( D = \partial / \partial x \). For \( n = 1 \), the KdV equation is obtained. For \( n \geq 2 \), we have the higher order KdV equations. This infinite sequence of integrable nonlinear partial differential equations form the so called KdV hierarchy [2]. Along with KdV, they are all integrable by the inverse scattering method,
and they have the same integrals of motion as KdV. However, in contrast to KdV, which has been shown to govern the nonlinear long-wave dynamics of general dispersive systems, the physical relevance of the higher order equations of the KdV hierarchy has remained, up to now, obscure.

We will show in this paper that the equations of the KdV hierarchy do appear in the description of physical systems. More specifically, we will first argue that the same amplitude that satisfies the KdV equation, also satisfies the higher order equations of the KdV hierarchy, each one in a different time-scale. As a consequence, we will then show that this property prevents divergent solutions to the physical problem. The infinite sequence of time-scales are defined from the basic time variable $t$ by

$$
\tau_1 = \epsilon^2 t, \tau_3 = \epsilon^4 t, \tau_5 = \epsilon^6 t, \ldots,
$$

where $\epsilon$ is a parameter satisfying $\epsilon \ll 1$. The primary reason for introducing them is that they may be used as a tool to eliminate the secularities that appear in the problem when a perturbative solution for the higher order terms of the amplitude is searched. However, when a function of a single time variable $F(t)$ is extended to a function of several time variables $F(\tau_1, \tau_3, \tau_5, \ldots)$, the problem of knowing the allowed evolution in each one of the different time-scales is posed. Indeed, as we are going to see, this evolution is not arbitrary. It is determined by a scale invariance requirement, which must hold to ensure the ordering of the expansion, and by the compatibility condition

$$
\frac{\partial^2 F}{\partial \tau_3 \partial \tau_{2n+1}} = \frac{\partial^2 F}{\partial \tau_{2n+1} \partial \tau_3}.
$$

If, for a physical system, the evolution of a certain amplitude is governed by the KdV equation in the time $\tau_3$, we will show that the above compatibility condition constrains the evolution on higher order times of that amplitude to be governed by the higher order equations of the KdV hierarchy, leaving only one free parameter at each order which is related to the possibility of redefining the time $\tau_{2n+1}$ by a multiplicative factor $\alpha_{2n+1}$. By choosing specific values for the free parameters, the task of secularity elimination can then be accomplished. As a consequence, our construction of a bona fide perturbative expansion will provide a link between the equations of the KdV hierarchy and the evolution of a physical
quantity. The physical system studied here, that of long surface water waves, is a kind of classical system where our ideas are well illustrated. However, the results obtained are, to a certain extend, model independent, and in this sense the study made here can be considered as representative of a general physical situation.

The paper is organized as follows. In section II, we obtain the basic equations describing surface water waves in the multiple space and time formalism. In section III, the first three evolution equations are obtained. In section IV, by using the multiple time formalism, we examine how the symmetry of the time derivatives determines the evolution of the wave amplitude in any time scale. The use of the KdV hierarchy equations to eliminate the secularities in the evolution of the higher order terms of the wave amplitude is discussed in section V. In section VI, we consider the case of a traveling-wave solution to the equations of the KdV hierarchy, and we show that the method of eliminating secularities by using these equations can be related to the results of Kodama and Taniuti [1], where a renormalization technique was introduced to obtain a secular free perturbative expansion. And finally, in section VII, we summarize and discuss the results obtained.

II. THE MULTIPLE SPACE AND TIME FORMALISM FOR WATER WAVES

We consider a two-dimensional inviscid incompressible fluid in a constant gravitational field. The space coordinates are denoted by \((x, z)\) and the corresponding components of the velocity \(\vec{v}\) by \((u, w)\). The gravitational acceleration \(\vec{g}\) is in the negative \(z\) direction. The equations describing such a fluid are:

\[ \vec{\nabla} \cdot \vec{v} = 0, \]  
\[ \frac{D\vec{v}}{Dt} \equiv \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla}) \vec{v} = -\frac{1}{\rho} \vec{\nabla} p + \vec{g}, \]  

where \(\rho\) is the fluid density, \(p\) is the pressure and \(\vec{k}\) is an unit vector in the \(z\) direction. If we assume the fluid to be irrotational,
\[ \nabla \times \vec{v} = 0, \quad (9) \]

and consequently a velocity potential \( \phi \) can be introduced through

\[ \vec{v} = \nabla \phi. \quad (10) \]

Using this definition, Eq. (9) becomes

\[ \nabla^2 \phi = 0, \quad (11) \]

while Eq. (8), after an integration, reads

\[ \frac{p - p_0}{\rho} = -\frac{\partial}{\partial t} \phi - \frac{1}{2} (\nabla \phi)^2 - gz, \quad (12) \]

with \( p_0 \) an integration constant.

We consider the case of a fluid of height \( h \), limited above by a passive gas exerting a constant pressure \( p_0 \) on it, and let the upper surface to be described by

\[ z = \zeta(x, t). \quad (13) \]

The kinematic boundary condition at this surface is then written in the form

\[ \frac{D \zeta}{Dt} \equiv \zeta_t + \phi_x \zeta_x = \phi_z, \quad (14) \]

with the subscripts denoting partial derivatives. However, there is also a dynamic boundary condition obtained from Eq. (12), which reads

\[ \phi_t + \frac{1}{2} \left[ (\phi_x)^2 + (\phi_z)^2 \right] + g\rho = 0, \quad (15) \]

on \( z = \zeta(x, t) \). Finally, the lower boundary is supposed to be a rigid horizontal flat bottom, localized at \( z = -h \). In this case, the corresponding boundary condition implies that the normal velocity of the fluid must vanish:

\[ \phi_z = 0. \quad (16) \]

We now wish to consider the long-wave in shallow-water approximation to the above equations. This may be done, for instance, by using the reductive perturbation method.
of Taniuti [1], which introduces slow space and time variables. The slow space variable is given by:

\[ \xi' = \epsilon^{1/2} x. \]  

(17)

Next, we introduce a very general set of slow time variables [7]:

\[ \tau_1 = \epsilon^{1/2} t, \quad \tau_3 = \epsilon^{3/2} t, \quad \tau_5 = \epsilon^5 t, \ldots. \]  

(18)

In addition, we expand \( \zeta \) and \( \phi \) in a suitable power series in the parameter \( \epsilon \):

\[ \zeta = \epsilon^{1/2} \zeta \equiv \epsilon \left( \zeta_0 + \epsilon \zeta_1 + \epsilon^2 \zeta_2 + \ldots \right) \]  

(19)

\[ \phi = \epsilon^{1/2} \phi \equiv \epsilon^{1/2} \left( \phi_0 + \epsilon \phi_1 + \epsilon^2 \phi_2 + \ldots \right). \]  

(20)

Introducing these expansions and the slow variables into the water wave equations, we obtain:

\[ \epsilon \hat{\phi}_{\zeta^2} \hat{\phi}_z + \hat{\phi}_{zz} = 0, \quad -h < z < \epsilon \zeta \]  

(21)

\[ \hat{\phi}_z = 0, \quad z = -h \]  

(22)

\[ \hat{\phi}_z = \epsilon \hat{\zeta}_{\tau_1} + \epsilon^2 \hat{\zeta}_{\tau_3} + \epsilon^3 \hat{\zeta}_{\tau_5} + \ldots + \epsilon^2 \hat{\phi}_{\zeta^2} \zeta \zeta', \quad z = \epsilon \zeta \]  

(23)

\[ 2g \zeta + 2\hat{\phi}_{\tau_1} + 2\epsilon \hat{\phi}_{\tau_3} + 2\epsilon^2 \hat{\phi}_{\tau_5} + \ldots + \epsilon \hat{\phi}_{\zeta^2} \zeta \zeta' + \hat{\phi}_z \hat{\phi}_z = 0 \,, \quad z = \epsilon \zeta. \]  

(24)

These are the basic equations we are going to use to obtain the evolution equations describing the system.

III. THE FIRST EVOLUTION EQUATIONS

Equation (21) for the velocity potential can be solved for the boundary condition (22), independently of equations (23) e (24). At order \( \epsilon^0 \), the solution is
\[ \phi_0 = F, \]  

with \( F = F(\xi'; \tau_1, \tau_3, \tau_5, \ldots) \) an arbitrary function. At order \( \epsilon^1 \), it is given by

\[ \phi_1 = -\left( \frac{z^2}{2} + hz \right) F_{\xi'\xi'} + G, \]  

with \( G = G(\xi'; \tau_1, \tau_3, \tau_5, \ldots) \) another arbitrary function. At order \( \epsilon^2 \), we get

\[ \phi_2 = \frac{-1}{24} \left( z^4 + 4hz^3 - 8h^3z \right) F_{(4\xi')} - \frac{1}{2} \left( z^2 + 2hz \right) G_{\xi'\xi'} + \mathcal{H}, \]  

with \( \mathcal{H} = \mathcal{H}(\xi'; \tau_1, \tau_3, \tau_5, \ldots) \) another arbitrary function. At order \( \epsilon^3 \), the solution is

\[ \phi_3 = \frac{-1}{720} \left( z^6 + 6hz^5 - 40h^3z^3 + 16h^6 \right) F_{(6\xi')} + \frac{1}{24} \left( z^4 + 4hz^3 - 8h^3z \right) G_{(4\xi')} - \frac{1}{2} \left( z^2 + 2hz \right) \mathcal{H}_{\xi'\xi'} + \mathcal{I}, \]  

with \( \mathcal{I} = \mathcal{I}(\xi'; \tau_1, \tau_3, \tau_5, \ldots) \) again an arbitrary function. We could proceed further and calculate \( \phi_4, \phi_5, \ldots \). But, we stop here since this is all we are going to need.

We pass now to eqs. (23) and (24), and solve the first few orders in our perturbative scheme. At order \( \epsilon^0 \), Eq. (23) gives nothing, while Eq. (24) reads

\[ g\zeta_0 + \phi_{0\tau_1} = 0. \]  

Substituting \( \phi_0 \), we get

\[ \zeta_0 = \frac{-1}{g} F_{\tau_1}. \]  

At order \( \epsilon^1 \), Eq. (23) is

\[ F_{\tau_1\tau_1} - gh F_{\xi'\xi'} = 0. \]  

Defining

\[ gh \equiv c^2, \]  

with \( c \) a velocity, its solution can be written in the form

\[ f = F(\xi' - c\tau_1) + G(\xi' + c\tau_1), \]
where the sign $- (+)$ refers to a wave moving to the right (left) with a velocity $c$. For definiteness, let us choose the wave moving to the right, and let us define a new coordinate system by

$$\xi = \xi' - cr_1.$$  \hspace{1cm} (33)

Therefore,

$$F_{r_1} = -cF_{\xi},$$ \hspace{1cm} (34)

and from Eq.(30) we get

$$\zeta_0 = \frac{h}{c}F_{\xi}.$$ \hspace{1cm} (35)

Consequently,

$$\zeta_{0r_1} + c\zeta_{0\xi} = 0,$$ \hspace{1cm} (36)

which is the usual linear wave equation.

Now, using eqs.(17) and (18), we can rewrite Eq.(33) in the form:

$$\xi = \epsilon^\frac{1}{2} (x - ct),$$ \hspace{1cm} (37)

from which we see that $\xi$ is a coordinate system moving with a velocity $c$ of the linear waves, in relation to the laboratory coordinate $x$. Therefore, from now on, all dependent variables will be assumed to depend on $(\xi', r_1)$ through $\xi$ only, which automatically implies that they satisfy Eq.(34).

At order $\epsilon^1$, Eq.(24) gives

$$2g\zeta_1 - 2cG_{\xi} + 2F_{r_1} + F_{\xi}F_{\xi} = 0,$$ \hspace{1cm} (38)

where we have already used Eq.(34) for $G$. Derivating in relation to $\xi$, and using Eq.(37), we obtain

$$c\zeta_{1\xi} - hG_{\xi\xi} = -\zeta_{0r_1} - \frac{c}{h}\zeta_0\zeta_{0\xi}.$$ \hspace{1cm} (39)
Now we go to the order $\epsilon^2$. From Eq. (23), we have

$$c \zeta_1 + h G_{\xi} = \zeta_0 + 2 \frac{c}{h} \zeta_0 \zeta_\xi + \frac{ch^2}{3} \zeta_0 \xi \xi \xi \xi . \quad (40)$$

Substituting into Eq. (39), we get

$$\zeta_0 + \frac{3c}{2h} \zeta_0 \zeta_\xi + \frac{ch^2}{6} \zeta_0 \xi \xi \xi = 0, \quad (41)$$

which is the KdV equation. It is the first nontrivial equation in the KdV hierarchy. At this point we can see the importance of introducing multiple times: while the linear wave equation (36) describes the evolution of $\zeta_0$ in the time $\tau_1$, the KdV equation describes the evolution of $\zeta_0$ in the time $\tau_3$. In other words, the phenomena described by each one of these equations occur in different time scales. And this will be the case for all the higher order equations of the KdV hierarchy. Now, from Eq. (24) at order $\epsilon^2$, we obtain

$$2g \zeta_2 + 2c^2 \zeta_0 \xi \xi \xi - 2c H_\xi + 2G_{\tau_3} + 2F_\tau_3 + 2F_\xi G_\xi + c^2 \zeta_0 \zeta_\xi = 0. \quad (42)$$

Derivating in relation to $\xi$, and using Eq. (35), it becomes

$$c \zeta_2 - h H \xi \xi + \frac{h}{c} G_{\xi \tau_3} + (\zeta_0 G_\xi + \zeta_0 G_{\xi \xi}) = -\zeta_0 + 2ch \zeta_0 \zeta_0 \xi \xi - ch \zeta_0 \zeta_0 \xi \xi \xi. \quad (43)$$

We pass now to the order $\epsilon^3$. From Eq. (23), we have

$$c \zeta_3 - h H_\xi - (\zeta_0 G_{\xi \xi} + \zeta_0 G_\xi)$$

$$- \frac{c}{h} (\zeta_0 \zeta_1 + \zeta_0 \zeta_\xi) - \frac{h^3}{3} G_{(4\xi)} = \zeta_0 + \frac{2}{15} ch^4 \zeta_0 (5\xi). \quad (44)$$

Equations (43) and (44) can be combined to yield an evolution equation involving $\zeta_1 \tau_3$, $\zeta_0 \tau_3$ and $G_{\xi \tau_3}$:

$$2\zeta_0 \tau_3 + \frac{2}{15} ch^4 \zeta_0 (5\xi) + 2ch \zeta_0 \zeta_0 \xi \xi + ch \zeta_0 \zeta_0 \xi \xi \xi + \zeta_1 \tau_3$$

$$+ \frac{c}{h} (\zeta_0 \zeta_1 + \zeta_0 \zeta_\xi) + \frac{h}{c} G_{\xi \tau_3} + 2 (\zeta_0 G_{\xi \xi} + \zeta_0 G_\xi) + \frac{h^3}{3} G_{(4\xi)} = 0. \quad (45)$$

Now, making use of Eq. (11) to describe $F_{\tau_3}$, Eq. (38) can be put in the form

$$\zeta_1 - \frac{h}{c} G_\xi = \frac{1}{4h} \zeta_0^2 + \frac{h^2}{6} \zeta_0 \xi \xi . \quad (46)$$
We can then use this equation to eliminate $G_\xi$ from Eq.(45). The result is
\[ \zeta_{1\tau_3} + \frac{3c}{2h} (\zeta_0 \zeta_1)_\xi + \frac{ch^2}{6} \zeta_1 \xi \xi \xi = S(\zeta_0), \] (47)
where
\[ S(\zeta_0) = -\zeta_{0\tau_5} - \frac{19}{360} ch^4 \zeta_0 (5\xi) - \frac{5}{12} ch \zeta_0 \zeta_0 \xi \xi \xi - \frac{23}{24} ch \zeta_0 \zeta_0 \xi \zeta_0 \xi + \frac{3c}{8h^2} \zeta_0^2 \zeta_0 \xi. \] (48)

Equation (47), as it stands, cannot be viewed as an evolution equation for $\zeta_1$ in the time $\tau_3$ because the non-homogeneous term involves $\zeta_{0\tau_5}$, and the evolution of $\zeta_0$ in $\tau_5$ is not known up to this point. Moreover, the term proportional to $\zeta_0 (5\xi)$ in $S(\zeta_0)$ is a resonant term, that is, it is a secular producing term to the solution $\zeta_1$ [4]. We will show in the next sections how the evolution of $\zeta_0$ in $\tau_5$ may be used to cancel the secular term. However, let us first state a mathematical result related to the KdV hierarchy, which will be used later.

IV. THE SYMMETRY OF TIME DERIVATIVES AND THE KORTEWEG–DE VRIES HIERARCHY

As is widely known [8], the KdV equation
\[ \zeta_t = 6\zeta \zeta_x - \zeta_{xxx}, \] (49)
is a Hamiltonian system, and can be written in the form
\[ \zeta_t = \frac{\partial}{\partial x} \left( \frac{\delta H}{\delta \zeta} \right), \] (50)
with the Hamiltonian $H$ given by
\[ H = \int_{-\infty}^{+\infty} \left[ \frac{(\zeta_x)^2}{2} + \zeta^3 \right] dx. \] (51)

On the other hand, as is also well known, the KdV equation has an infinite sequence of conservation laws [4,10], with the integrals of motion given by
\[ I_n[\zeta] = \int_{-\infty}^{+\infty} T_n dx, \quad n = 0, 1, 2, 3, \ldots, \] (52)
where $T_n$, the conserved density, is a polynomial in $\zeta, \zeta_x, \zeta_{xx}, \text{etc.}$ Their explicit form can be obtained through a perturbative scheme developed by Miura, Gardner and Kruskal [10], which is based on expansions on a small parameter $\epsilon$. In this method, each integral of motion $I_n$ appears in a different order of the perturbation parameter $\epsilon$. Now, the infinite sequence of equations of the KdV hierarchy can be expressed in terms of these integrals of motion according to [11]

$$
\zeta_t = \frac{\partial}{\partial x} \left( \frac{\delta I_n}{\delta \zeta} \right).
$$

(53)

For $n = 1$, the integral $I_1$ coincides with the Hamiltonian given by Eq.(51), and the KdV equation is obtained, which is the first nontrivial member of the KdV hierarchy. From the explicit form of $I_n$ [10], it is easy to show that each one of Eqs.(53) is invariant, up to a Galilean term, under the transformations

$$
\xi = \epsilon^{\frac{1}{2}}(x - ct)
$$

(54)

$$
\tau_{2n+1} = \epsilon^{n+\frac{1}{2}} t ; \quad n = 1, 2, 3, \ldots
$$

(55)

$$
\zeta = \epsilon \zeta_0.
$$

(56)

In these new coordinates, called slow variables, Eq.(53) is written in the form

$$
\zeta_{0\tau_{2n+1}} = \frac{\partial}{\partial \xi} \left( \frac{\delta I_n}{\delta \zeta_0} \right).
$$

(57)

Notice that an infinite sequence of slow time variables $\tau_3, \tau_5, \tau_7, \ldots$ was introduced in the above transformations. As a consequence, any dependent variable $F(x; t)$ will be a function of these multiple times:

$$
F(x; t) = F(x; \tau_1, \tau_3, \tau_5, \ldots).
$$

(58)

Therefore, the time derivative of $F(x; t)$ turns out to be

$$
\frac{\partial F}{\partial t} = \left( \epsilon^\frac{1}{2} \frac{\partial}{\partial \tau_3} + \epsilon^\frac{3}{2} \frac{\partial}{\partial \tau_5} + \epsilon^5 \frac{\partial}{\partial \tau_7} + \ldots \right) F.
$$

(59)
Let us now return to our physical system. As we have already seen, the evolution of \(\zeta_0\) in the time \(\tau_3\) was found to be the KdV equation

\[
\zeta_{0\tau_3} = \alpha_1 \zeta_0 \xi \xi + \beta_1 \zeta_0 \zeta_0 \xi ,
\]  

(60)
with

\[
\alpha_1 = \frac{ch^2}{6} , \quad \beta_1 = \frac{3c}{2h} .
\]  

(61)

Let us now examine how the evolution of \(\zeta_0\) in times \(\tau_5, \tau_7, \ldots\) can be obtained. The first crucial point of this analysis is to note that, for the evolution of \(\zeta_0\) in the time \(\tau_3\), the two terms in the r.h.s. of Eq.(60) exhaust all possible terms such that, when transforming back from the slow to the laboratory variables, the resulting equation does not depend on \(\epsilon\).

In the very same way, the evolution equation of \(\zeta_0\) in the time \(\tau_5\) must only involve terms presenting the same property as the terms in the KdV equation. If this were not so, the ordering of the perturbative series would not be ensured. In other words, this evolution equation must be formally the same in the slow \((\xi, \tau_5, \zeta_0)\) as well as in the laboratory coordinates \((x, t, \zeta)\), establishing thus a scale invariance requirement. A simple analysis shows that the only possible terms are:

\[
\zeta_{0\tau_5} = \alpha_2 \zeta_0 (5\xi) + \beta_2 \zeta_0 \zeta_0 \xi \xi + (\beta_2 + \gamma_2) \zeta_0 \zeta_0 \xi \xi + \delta_2 \zeta_0^2 \zeta_0 \xi ,
\]  

(62)

where \(\alpha_2, \beta_2, \gamma_2, \delta_2\) are constants. Now comes the second crucial point: the coefficients \(\alpha_2, \beta_2, \gamma_2, \delta_2\) are not completely arbitrary if \(\zeta_0\) is to satisfy also the KdV equation in the time \(\tau_3\). Instead, they are constrained by the relations arising when the natural (in the multiple time formalism) compatibility condition

\[
(\zeta_{0\tau_3})_{\tau_5} = (\zeta_{0\tau_5})_{\tau_3}
\]  

(63)
is imposed. This condition only makes sense in the multiple time formalism, since otherwise it would be redundant and would lead to a trivial identity. A tedious but straightforward calculation shows that the relations arising from this condition are:

12
\[
\frac{\beta_2}{\alpha_2} = \frac{5}{3} \frac{\beta_1}{\alpha_1}, \quad \frac{\gamma_2}{\alpha_2} = \frac{5}{3} \frac{\beta_1}{\alpha_1}, \quad \frac{\delta_2}{\alpha_2} = \frac{5}{6} \left(\frac{\beta_1}{\alpha_1}\right)^2.
\]

Substituting into Eq.(62) leads to

\[
\zeta_{0\tau_5} = \alpha_2 \left[ \zeta_{0(5\xi)} + \frac{5}{3} \left(\frac{\beta_1}{\alpha_1}\right) \zeta_0 \zeta_0 \xi \xi + \frac{10}{3} \left(\frac{\beta_1}{\alpha_1}\right) \zeta_0 \xi \xi + \frac{5}{6} \left(\frac{\beta_1}{\alpha_1}\right)^2 \zeta_0^2 \zeta_0 \right],
\]

which is the 5\textsuperscript{th} order equation of the KdV hierarchy. Different choices of the free parameter \(\alpha_2\) correspond to different normalizations of the time \(\tau_5\). In particular, for \(\alpha_2 = 6\) it acquires the canonical form \cite{12}

\[
\zeta_{0\tau_5} = 6\zeta_{0(5\xi)} + 10\alpha \zeta_0 \zeta_0 \xi \xi + 20\alpha \zeta_0 \zeta_0 \xi \xi + 5\alpha^2 \zeta_0^2 \zeta_0 \xi,
\]

with \(\alpha = \beta_1/\alpha_1\).

This procedure can be extended to any higher order. In other words, if \(\zeta_0\) satisfies the KdV equation in the time \(\tau_3\), by making use of the scale invariance described above, we can first select all possible terms appearing in the evolution of \(\zeta_0\) in the time \(\tau_{2n+1}\). Then, by imposing the general time symmetry condition

\[
\frac{\partial^2 \zeta_0}{\partial \tau_3 \partial \tau_{2n+1}} = \frac{\partial^2 \zeta_0}{\partial \tau_{2n+1} \partial \tau_3},
\]

the parameters \(\alpha_n, \beta_n, \gamma_n, \ldots\) appearing in that equation can be uniquely determined in terms of \(\alpha_1\) and \(\beta_1\), and the resulting equation is found to be the \((2n+1)\textsuperscript{th}\) order equation of the KdV hierarchy. As before, \(\alpha_n\) is left as a free parameter responsible for the arbitrariness in the scale of the time \(\tau_{2n+1}\). One gets easily convinced of the validity of this result by performing the calculation explicitly for the first few equations, but we do not intend here to give a general proof valid for any order.

Before proceeding further, let us remark that one could \textit{a priori} envisage, in place of Eq.(62) a new one including also a dependence on \(\zeta_1\), and still keep the agreement with the scale invariance requirement. However, when introducing this back into the physical equation (60), the compatibility condition (63) makes all the supplementary terms to vanish.
V. HIGHER ORDER EVOLUTION EQUATIONS: THE ELIMINATION OF THE SECULAR PRODUCING TERMS

Let us now return to the evolution equation for $\zeta_1$ in the time $\tau_3$, which is given by Eqs.(47) and (48). As we have seen, there were two problems related to it: the evolution of $\zeta_0$ in $\tau_5$, and the secular producing term. The last section gave the clue to the first problem. As we have seen, the evolution of $\zeta_0$ in $\tau_5$ can not be chosen arbitrarily, but it is restricted to the KdV hierarchy equations. We show now how the secular producing term $\zeta_0(5\xi)$ can be eliminated. A comparison between Eq.(48) for $S(\zeta_0)$ and the 5th order KdV equation (65), immediately shows that the choice

$$\alpha_2 = -\frac{19}{360}c^4$$

(68)

has the property of eliminating the terms

$$\pm \zeta_0\tau_5 - \frac{19}{360}c^4 \zeta_0(5\xi)$$

(69)

from the non–homogeneous term $S(\zeta_0)$. Consequently, $S(\zeta_0)$ acquires the form

$$S(\zeta_0) = \frac{9}{24}c^4 \zeta_0 \zeta_0 \zeta_0 + \frac{5}{8}c^4 \zeta_0 \zeta_0 \zeta_0 + \frac{63}{16}c^4 \zeta_0^2 \zeta_0 \zeta_0$$

(70)

and we see that it does not present a secular producing term anymore.

It should be noted that the resonant term is the linear term of $S(\zeta_0)$. This property holds in any higher order of the perturbative scheme [4]. Therefore, the elimination of the secular producing term by choosing an appropriate value for the free parameter $\alpha_n$ in the higher–order equation of the KdV hierarchy, also remains possible in any higher order.

VI. TRAVELING WAVE SOLUTIONS: THE RENORMALIZATION OF THE SOLITON VELOCITY

We have already assumed, in the context of the multiple time formalism, that

$$\zeta_0 = \zeta_0(\xi; \tau_3, \tau_5, \tau_7, \ldots)$$

(71)
where

\[ \xi = \xi' - c\tau_1, \quad (72) \]

so that \( \zeta_0 \) automatically satisfies the linear wave equation

\[ \zeta_{0\tau_1} + c\zeta_0\xi = 0. \quad (73) \]

Our concern now will be the solutions to the higher order equations of the KdV hierarchy. Hereafter, for simplicity, we will assume that \( c \) and \( h \) can be set equal to unity, so that the KdV equation reads

\[ \zeta_{0\tau_3} = \alpha_1 \zeta_0\xi\xi + \beta_1 \zeta_0\xi, \quad (74) \]

with

\[ \alpha_1 = \frac{1}{6}, \quad \beta_1 = \frac{3}{2}. \quad (75) \]

We now look for a traveling wave solution \( \zeta_0 \) that satisfies this equation in the time \( \tau_3 \), but that satisfies also the higher order equations of the KdV hierarchy in the times \( \tau_5, \tau_7, \ldots \). This multi–solution can be written in the form

\[ \zeta_0 = \frac{k^2}{3}\text{sech}^2 (k \Lambda) \quad (76) \]

with the argument \( \Lambda \) given by

\[ \Lambda = \xi - \frac{k^2}{A_1}\tau_3 - \frac{k^4}{A_2}\tau_5 - \frac{k^6}{A_3}\tau_7 - \ldots, \quad (77) \]

and with \( A_1, A_2, A_3, \ldots \) parameters depending respectively on the constants \( \alpha_1, \alpha_2, \alpha_3, \ldots \). This argument is to be interpreted in the following way: when the evolution equation under consideration is concerned to the time \( \tau_{2n+1} \), the relevant argument is

\[ \Lambda = \xi - \sum_{i=1}^{n} \frac{k^{2i}}{A_i}\tau_{2i+1} - \theta_n, \]

with \( \theta_n \) a phase involving all the remaining terms of the sum. Then, for the KdV equation (74), \( A_1 = (1/\alpha_1) = 6 \), and the solution is
\[
\zeta_0 = \frac{k^2}{3} \operatorname{sech}^2 \left[ k \left( \xi - \frac{k^2}{6} \tau_3 - \theta_3(k; \tau_5, \ldots) \right) \right],
\] (78)

where \( \theta_3 \) is a phase. On the other hand, the travelling wave solution satisfying simultaneously the KdV equation (74) in the time \( \tau_3 \), and the 5th order KdV equation (65) in the time \( \tau_5 \), is given by

\[
\zeta_0 = \frac{k^2}{3} \operatorname{sech}^2 \left[ k \left( \xi - \frac{k^2}{6} \tau_3 - \frac{k^4}{A_2} \tau_5 - \theta_5(k; \tau_7, \ldots) \right) \right],
\] (79)

and so on, to any higher order.

Once we have found the solutions to any member of the KdV hierarchy, let us return to the equation for \( \zeta_1 \) in the time \( \tau_3 \), which is now \((c = h = 1)\) written in the form

\[
\zeta_1 \tau_3 + \frac{3}{2} (\zeta_0 \zeta_1)_\xi + \frac{1}{6} \zeta_1 \xi \xi = S(\zeta_0),
\] (80)

with

\[
S(\zeta_0) = \frac{9}{24} \zeta_0 \zeta_0 \xi \xi + \frac{5}{8} \zeta_0 \zeta_0 \zeta_0 \xi + \frac{63}{16} \zeta_0^2 \zeta_0 \xi .
\] (81)

It is important to remember that the linear term of \( S(\zeta_0) \), which is the secular producing term in the equation for \( \zeta_1 \), was eliminated through the use of the 5th order KdV equation (65), with \( \alpha_2 \) given by Eq. (68). Therefore, the \( \zeta_0 \) appearing in Eqs. (80–81) is that given by Eq. (79), since it must be a solution of KdV as well as of the KdV hierarchy 5th order equation (65). The secular–free solution \( \zeta_1 \) can then be found by solving the linear equation (80–81). This solution can be found in Ref. [4].

Finally, let us take the solutions \( \zeta_0 \), and let us make the transformation back from the slow \((\xi, \tau, \zeta_0)\) to the laboratory coordinates \((x, t, \zeta)\). For the case of solution (78) to the KdV equation, we get

\[
\zeta = \frac{k^2}{3} \epsilon \operatorname{sech}^2 \left[ k \epsilon^{1/2} (x - V_3 t) \right],
\] (82)

where

\[
V_3 = c + \epsilon \frac{k^2}{6}
\] (83)
is the solitary–wave velocity in the laboratory coordinates. For the case of solution (73) to the 5th order KdV equation, we get

\[ \zeta = \frac{k^2}{3} \epsilon \text{sech}^2 \left[ k \epsilon^{1/2} (x - V_5 t) \right], \]  

where now

\[ V_5 = V_3 + \epsilon^2 \frac{k^4}{A_2}. \]  

Since \( A_2 \) depends on the parameter \( \alpha_2 \), to choose \( \alpha_2 \) means to define the velocity renormalization itself. In this sense we can say that there is a unique velocity renormalization leading to a secular–free perturbation scheme for \( \zeta_1 \). From these properties, we can see now that this method is equivalent to the renormalization technique developed by Kodama and Taniuti [4], in which the secular–free higher order effects were also given by the renormalization of the KdV soliton velocity. Moreover, in the same way as in the method of Kodama and Taniuti, if higher order scales are introduced, it is possible to continue the secular–free perturbation to higher orders by using the higher order equations of the KdV hierarchy. And in general, depending on the order considered in the perturbative scheme, the renormalized solitary–wave velocity in the laboratory coordinates is given by

\[ V_{2n+1} = c + \sum_{i=1}^{n} \epsilon^i \frac{k^{2i}}{A_i}. \]  

The analysis for multi–soliton solutions proceeds in a similar fashion as well.

**VII. FINAL REMARKS**

By using the reductive perturbation method of Taniuti [5], with the introduction of an infinite sequence of slow time variables \( \tau_1, \tau_3, \tau_5, \ldots \), we studied the propagation of long surface waves in a shallow inviscid fluid. The three main ingredients of our analysis were: (i) the scale–invariance argument, which restricts the possible forms of the evolution equations, and which is necessary for the coherence of the perturbative expansion; (ii) the
compatibility condition (63), which appears when extending a function of a single time-variable to a multiple time-variable; (iii) the secularity elimination procedure, without which the perturbative expansion would be meaningless. By using (i) and (ii) we have shown that, if the amplitude $\zeta_0$ satisfies the KdV equation, it also satisfies the higher-order equations of the KdV hierarchy. Then, by using the $5^{th}$ order equation of the hierarchy with a properly chosen $\alpha_2$, we have shown that the secular producing terms of the equation for $\zeta_1$ in the time $\tau_3$ could be eliminated, a result which can be extended up to any higher order. As the secularity elimination is mandatory for a physical theory, being essentially a finiteness requirement, the results obtained in this paper allowed us to give a physical meaning to the KdV hierarchy equations. Thereafter, by considering a solitary wave solution, we have shown that the elimination of secularities through the use of higher order KdV equations corresponds, in the laboratory coordinates, to a renormalization of the soliton velocities, as obtained previously by Kodama and Taniuti [4].

The study of long-waves in shallow water is representative of a wide class, that of the weak nonlinear, dispersive systems, where the KdV equation has a kind of universal character. In this sense, we can say that the results of the present paper do not depend on the specific physical system under consideration, or, which is the same, on the specific form of the basic equations, except for the values of the coefficients in the perturbative expansion. It is, therefore, legitimate to conjecture that they might be extended to the above mentioned larger class of systems.

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