FINITE ELEMENT METHOD FOR TWO-DIMENSIONAL LINEAR ADVECTION EQUATIONS BASED ON SPLINE METHOD

KAI QU*, QI DONG, CHANJIE LI AND FEIYU ZHANG
College of Science, Dalian Maritime University
Dalian, 116026, China

Abstract. A new method for some advection equations is derived and analyzed, where the finite element method is constructed by using spline. A proper spline subspace is discussed for satisfying boundary conditions. Meanwhile, in order to get more accuracy solutions, spline method is connected with finite element method. Furthermore, the stability and convergence are discussed rigorously. Two numerical experiments are also presented to verify the theoretical analysis.

1. Introduction. In recent years, there are more researches on developing numerical algorithms for solving linear advection equations

\[
\partial_t u(x, y, t) + \vec{v}(x, y) \cdot \nabla u(x, y, t) = 0, \\
u(x, y, 0) = u_0(x, y),
\]

(1)

(2)

Where \( \vec{v}(x, y) = (v(x, y), w(x, y)) \) is a variable velocity. Advection equations play an important role on applications. In order to distinguish the different physical properties of fluids, advection equations are used to track an interface between the phases of two-phase flow of immiscible fluids. See references [5], [10], [13] and [29] for details. They can used to track the water table for groundwater flows and the fire front in forests[6]. Also, they can used for the description of transport dynamics in the complex systems. We know that almost all of complex systems are controlled by the non-exponential relaxation patterns and the anomalous diffusion [20].

Many researchers constructed semi-implicit finite volume schemes for some advection equations [21]. The important thought is that the values of numerical solution at inflow boundaries of computational cells can be solved by the implicit time discretization. Momani et al. [24] using the Adomian decomposition method to construct a reliable algorithm for the time-space advection equations. In [22], a so-called “extension” velocity in level set methods is advantageous because of the semi-implicit schemes. This approach is also used in solving the linear advection equations which have a time dependent domain given by positions of a fire front [7]. A new representation of semi-implicit scheme is constructed in [8], a novel approach of partial Lax-Wendroff procedure is used in this new scheme.

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* Corresponding author: Kai Qu.
Based on the fundamental solutions to the corresponding Cauchy and source problems for one spatial variable, Povstenko and Kyrylych [26] investigated two different generalizations of the space-time advection equations. Magin et al. calculated advection equations generalized in space and time using Total Shannon spectral entropy in [18], because it can be used as a measure of the information content. One-dimensional space advection equations with variable coefficients on a finite domain is solved by Meerschaert et al. [19]. Liu et al. used variable transformation, Mellin and Laplace transforms, and H-functions to solve time fractional advection equations [15]. Huang and Liu [12] developed new method using Green functions to solve advection equations. By using the homotopy analysis method, Tripathi et al. discussed the approximate analytical solution of fractional order nonlinear diffusion equations [28].

Finite element methods [4], finite difference methods [16], finite volume methods [11], spectral methods [2][32] and homotopy perturbation methods [9] can also solve some advection equations. Mundewadirk et al. developed a numerical method for solving the Abel’s integral equations with using Hermite wavelet approximations [25]. Liu et al. [17] used finite difference methods to proposed an approximation of the Lévy-Feller advection-dispersion process. For taking second-order and fourth-order non-linear hyperbolic equation as examples, a full discrete convergence analysis method for non-linear hyperbolic equation based on finite element analysis is proposed [31]. In [27], Sadia et al. transformed the fractional differential equations into equivalent Volterra integral equations and constructed a new numerical scheme. In [1], a finite element method with multigrid for multi-term time advection diffusion equations was presented. In [30], the stability of the finite difference method is studied with the aid of Von Neumann stability analysis for the fractional Harry Dym equation. A fully implicit finite difference scheme using extended cubic B-splines for advection diffusion equations was obtained in [23]. The structure of Toeplitz or Toeplitz-times-diagonal type and the stability were discussed in [3] and [14].

This paper is organized as follows. In Section 2, based on the bivariate spline space, we discuss some spline spaces to satisfy the test spaces and the boundary conditions. Section 3 contains the analysis of the spline method, such as the stability and convergence analysis. In Section 4, two numerical examples are given, also included in Section 4 is the comparison of the spline method and other methods. Some conclusions are given in Section 5.

2. Spline theory. Let Ω be a domain in $\mathbb{R}^2$. $P_k$ denote the collection of all bivariate polynomials which have real coefficients and total degree no more than $k$, i.e.,

$$P_k := \left\{ p = \sum_{i=0}^{k} \sum_{j=0}^{k-i} c_{ij} x^i y^j \middle| c_{ij} \in \mathbb{R} \right\}$$

By using a number of irreducible algebraic curves, the partition $\Delta$ of the domain $\Omega$ is constructed. Then $M$ sub-domains $\delta_1, \cdots, \delta_M$ are obtained and each sub-domains is a cell of $\Delta$. The space of bivariate splines with degree $k$ and smoothness $\mu$ over $\Delta$ is defined by

$$S^\mu_k(\Delta) := \{ s \in C^\mu(\Omega) \mid s|_{\delta_i} \in P_k, i = 1, \cdots, M \}$$
In this paper, we mainly focus on uniform type-2 triangulations, see Fig. 1. Here, \(\Omega\) is a unit square region as follows:

\[
\Omega = (0, 1) \otimes (0, 1)
\]

The type-2 triangulation \(\Delta_{m,n}^{(2)}\) is obtained by the following lines:

\[
\begin{align*}
mx - i &= 0, \quad ny - i = 0, \\
ny - mx - i &= 0, \quad ny + mx - i = 0
\end{align*}
\]

where \(i = \cdots -1, 0, 1, \cdots\).

2.1. Quadratic spline spaces \(S_{1,2}^2(\Delta_{m,n}^{(2)})\). A locally supported spline with its support octagon \(Q\) centered at \((0, 0)\) is shown in Fig. 2. We call it \(S_{1,2}^2(\Delta_{m,n}^{(2)})\). We can determine a bivariate polynomial of degree 2 on \(S_{1,2}^2(\Delta_{m,n}^{(2)})\) uniquely by the values of three midpoints and three vertices of the edges. See Fig. 2, we give some values on triangles. Then we can obtain and the other values by the symmetry of lines \(x = 0\), \(y = 0\), \(x + y = 0\), \(x - y = 0\).

Suppose \(\Phi(x, y)\) is a piecewise polynomial with degree 2 defined in \(\mathbb{R}^2\). Then \(\Phi(x, y) = 0, (x, y) \notin Q\). We all know that \(\Phi(x, y) \in C^1(\mathbb{R}^2)\), and \(\Phi(x, y) > 0, (x, y) \in Q\). Hence, \(\Phi(x, y)\) can be uniquely determined by the symmetry of lines \(x = 0, y = 0, x + y = 0, x - y = 0\), and normalized condition \(\Phi(0, 0) = 1/2\) by using the conformality conditions at vertices.

Denote \(\Phi_{ij}(x, y) : = \Phi(mx - i + 1/2, ny - j + 1/2)\frac{n!}{r!(n-r)!}\)

then collection

\[
A = \{\Phi_{ij}(x, y) : i = 0, \cdots, m + 1, \quad j = 0, \cdots, n + 1\}
\]

is a subspace of \(S_{1,2}^2(\Delta_{m,n}^{(2)})\). \(\text{dim}S_{1,2}^2(\Delta_{m,n}^{(2)}) = (m + 2)(n + 2) - 1\).

2.2. Quartic spline spaces \(S_{4,3,3}^2(\Delta_{m,n}^{(2)})\). Now, we try to construct a locally supported spline \(s\) which satisfies the following three conditions: (i) On the rectangle grid segments, \(s\) is \(C^2\) continuous; (ii) On the diagonal grid segments, \(s\) is \(C^2\) continuous. (iii) \(s\) is a piecewise polynomial of degree 4. We call this supported spline \(S_{4,3}^2(\Delta_{m,n}^{(2)})\). This \(S_{4,3}^2(\Delta_{m,n}^{(2)})\) is more convenient than classic finite element method, because it can obtain the bases immediately. Furthermore, It has been proved that a proper subspace of \(S_{4,3}^2(\Delta_{m,n}^{(2)})\) can be only spaned by the locally supported B-splines.

Next, we should construct a proper subspace of \(S_{4,3}^2(\Delta_{m,n}^{(2)})\) for solving the bound conditions. It means that all the splines in this subspace must satisfy homogenous boundary conditions on type-2 triangulations. In order to achieve this subspace, we use the linear combination of \(B(x, y)\) in \(S_{4,3}^2(\Delta_{m,n}^{(2)})\) and their translations.

Let

\[
B_{i,j}(x, y) = B(mx - i, ny - j)
\]

Define the basis functions \(\tilde{B}_{i,j}(x, y)\) as follows:
\[
\begin{align*}
\tilde{B}_{1,1}(x, y) &= B_{1,1}(x, y) - B_{-1,1}(x, y) - B_{1,-1}(x, y) + B_{-1,-1}(x, y) \\
\tilde{B}_{m-1,1}(x, y) &= B_{m-1,1}(x, y) - B_{m+1,1}(x, y) - B_{m-1,-1}(x, y) + B_{m+1,-1}(x, y) \\
\tilde{B}_{1,n-1}(x, y) &= B_{1,n-1}(x, y) - B_{-1,n-1}(x, y) - B_{1,n+1}(x, y) + B_{-1,n+1}(x, y) \\
\tilde{B}_{m-1,n-1}(x, y) &= B_{m-1,n-1}(x, y) - B_{m+1,n-1}(x, y) - B_{m-1,n+1}(x, y) + B_{m+1,n+1}(x, y)
\end{align*}
\]

(3)

\[
\begin{align*}
\tilde{B}_{i,1}(x, y) &= B_{i,1}(x, y) - B_{i,-1}(x, y), i = 2, 3, \ldots, m - 2 \\
\tilde{B}_{i,m-1}(x, y) &= B_{i,m-1}(x, y) - B_{i,m+1}(x, y), i = 2, 3, \ldots, m - 2 \\
\tilde{B}_{i,1,j}(x, y) &= B_{i,1,j}(x, y) - B_{i,-1,j}(x, y), j = 2, 3, \ldots, m - 2 \\
\tilde{B}_{i,n-1,j}(x, y) &= B_{i,n-1,j}(x, y) - B_{i,n+1,j}(x, y), j = 2, 3, \ldots, m - 2 \\
\tilde{B}_{i,j,1}(x, y) &= B_{i,j,1}(x, y), i = 2, 3, \ldots, m - 2, j = 2, 3, \ldots, n - 2
\end{align*}
\]

(4)

We call Eq. (3)-(5) corner, side and interior B-spline bases, respectively. Their supports are shown in Fig. 3. Note that, all the bases are \( C^0 \) across the double marked mesh segments and \( C^1 \) across the single marked mesh lines.

All \( B_{i,j}(x, y) : 1 \leq i \leq m - 1, 1 \leq j \leq n - 1 \) can only span the proper subspace of \( S_{4}^{2,3}(\Delta_{m,n}^{2}) \) and we call them \( S_{4}^{2,3,0}(\Delta_{m,n}^{2}) \). They satisfy the homogenous boundary conditions on type-2 triangulations.

3. Analysis on advection equations. Consider the classic advection equations Eq.(1)-(2). In this paper, we construct a finite spline method by using bivariate spline to get the approximate values \( U_{ij}^n \approx u_{ij}^n \), where \( u_{ij}^n = u(x_i, y_j, t_n) \). We change them to the discrete form:

\[
u_{ij}^{n+1} + \sum_{k=-2}^{2} \left( \alpha_{ijk}^x u_{i+k,j}^{n+1} + \alpha_{ijk}^y u_{i,j+k}^{n+1} \right) = u_{ij}^n + \sum_{k=-2}^{2} \left( \beta_{ijk}^x u_{i+k,j}^{n} + \beta_{ijk}^y u_{i,j+k}^{n} \right)
\]

The corresponding semi-implicit k-scheme is:

\[
u_{ij}^{n+1} + \tau V_{ij} \left( \partial_x u_{ij}^{n+1} - 0.5 \partial_x K u_{ij}^{n+1} \right) + \tau W_{ij} \left( \partial_y u_{ij}^{n+1} - 0.5 \partial_y K u_{ij}^{n+1} \right) = u_{ij}^n - 0.5 \tau \left( V_{ij} \partial_x^2 u_{ij}^n + W_{ij} \partial_y^2 u_{ij}^n \right)
\]

(6)

The standard semi-discrete methods for Eq.(6) are defined as follows: for any test function \( v \in V_h \), \( u_{ij}^n \) is given by:

\[
u_{ij}^{n+1} + \tau V_{ij} \left( \partial_x u_{ij}^{n+1} - 0.5 \partial_x K u_{ij}^{n+1} \right) + \tau W_{ij} \left( \partial_y u_{ij}^{n+1} - 0.5 \partial_y K u_{ij}^{n+1} \right) + | C_{ij} D_{ij} | / 6 \left( u_{ij}^{n+1} + u_{i+1,j+1}^{n+1} - u_{i+1,j}^{n+1} - u_{i,j+1}^{n+1} \right) = u_{ij}^n - 0.5 \tau \left( V_{ij} \partial_x^2 u_{ij}^n + W_{ij} \partial_y^2 u_{ij}^n \right) + | C_{ij} D_{ij} | / 12 \left( 2u_{ij}^{n+1} + u_{i+1,j}^{n+1} + u_{i,j+1}^{n+1} - u_{i+1,j+1}^{n+1} - u_{i,j+1}^{n+1} - u_{i+1,j}^{n+1} \right)
\]

and

\[
u_{ij}^{n+1} + \tau V_{ij} \left( \partial_x u_{ij}^{n+1} - 0.5 \partial_x K u_{ij}^{n+1} \right) + \tau W_{ij} \left( \partial_y u_{ij}^{n+1} - 0.5 \partial_y K u_{ij}^{n+1} \right) + | C_{ij} D_{ij} | / 6 \left( u_{ij}^{n+1} + u_{i+1,j+1}^{n+1} - u_{i+1,j}^{n+1} - u_{i,j+1}^{n+1} \right) = u_{ij}^n - 0.5 \tau \left( V_{ij} \partial_x^2 u_{ij}^n + W_{ij} \partial_y^2 u_{ij}^n \right) - | C_{ij} D_{ij} | / 12 \left( 2u_{ij}^{n+1} + u_{i+1,j+1}^{n+1} + u_{i+1,j+1}^{n+1} - u_{i+1,j}^{n+1} - u_{i,j+1}^{n+1} - u_{i,j}^{n+1} \right)
\]
Since $S_{4}^{2,3,0}(\Delta_{m,n}^{(2)})$ can be embedded into $H_{0}^{1}(\Omega)$, it can be selected as the testing function space. The finite element method is to find a solution $v \in S_{4}^{2,3,0}(\Delta_{m,n}^{(2)})$ such that

$$
Lu = u_{j}^{n+1} - u_{j}^{n} 
$$

$$
= -u^{\alpha} c_{0}^{\alpha} \left[ \mu_{1} \left( \sum_{k=0}^{j+1} \omega_{k}^{(\beta_{1})} u_{j-k+1}^{n+1} + \sum_{k=0}^{M-j+1} \omega_{k}^{(\beta_{1})} u_{j+k-1}^{n+1} \right) \right. \\
- \mu_{2} \left( \sum_{k=0}^{j+1} \omega_{k}^{(\beta_{2})} u_{j-k+1}^{n+1} + \sum_{k=0}^{M-j+1} \omega_{k}^{(\beta_{2})} u_{j+k-1}^{n+1} \right) - P_{2} \right] \\
- u^{\alpha} (a_{n}^{0} - a_{n-1}^{0}) \left[ \mu_{1} \left( \sum_{k=0}^{j+1} \omega_{k}^{(\beta_{1})} u_{j-k+1}^{n+1} + \sum_{k=0}^{M-j+1} \omega_{k}^{(\beta_{1})} u_{j+k-1}^{n+1} \right) \right. \\
+ \mu_{2} \left( \sum_{k=0}^{j+1} \omega_{k}^{(\beta_{2})} u_{j-k+1}^{n+1} + \sum_{k=0}^{M-j+1} \omega_{k}^{(\beta_{2})} u_{j+k-1}^{n+1} \right) - P_{2} \right] \\
+ I^{\alpha}(\xi_{j}, \tau_{n+1} - \delta_{n}) - I^{\alpha}(\xi_{j}, \tau_{n}) + P_{3} \\
= -u^{\alpha} \left[ c_{0}^{\alpha} \left( \sum_{k=0}^{j+1} \omega_{k}^{(\beta_{1})} u_{j-k+1}^{n+1} + \sum_{k=0}^{M-j+1} \omega_{k}^{(\beta_{1})} u_{j+k-1}^{n+1} \right) \right. \\
+ (a_{n}^{0} - a_{n-1}^{0}) \left( \sum_{k=0}^{j+1} \omega_{k}^{(\beta_{1})} u_{j-k+1}^{n+1} + \sum_{k=0}^{M-j+1} \omega_{k}^{(\beta_{1})} u_{j+k-1}^{n+1} \right) \\
+ \sum_{l=0}^{n-1} (d_{l}^{\alpha} - d_{l+1}^{\alpha}) \left( \sum_{k=0}^{j+1} \omega_{k}^{(\beta_{1})} u_{j-k+1}^{n+1} + \sum_{k=0}^{M-j+1} \omega_{k}^{(\beta_{1})} u_{j+k-1}^{n+1} \right) \left] \right. \\
- \mu_{2} \left[ c_{0}^{\alpha} \left( \sum_{k=0}^{j+1} \omega_{k}^{(\beta_{2})} u_{j-k+1}^{n+1} + \sum_{k=0}^{M-j+1} \omega_{k}^{(\beta_{2})} u_{j+k-1}^{n+1} \right) \right. \\
+ (a_{n}^{0} - a_{n-1}^{0}) \left( \sum_{k=0}^{j+1} \omega_{k}^{(\beta_{2})} u_{j-k+1}^{n+1} + \sum_{k=0}^{M-j+1} \omega_{k}^{(\beta_{2})} u_{j+k-1}^{n+1} \right) \\
+ \sum_{l=0}^{n-1} (d_{l}^{\alpha} - d_{l+1}^{\alpha}) \left( \sum_{k=0}^{j+1} \omega_{k}^{(\beta_{2})} u_{j-k+1}^{n+1} + \sum_{k=0}^{M-j+1} \omega_{k}^{(\beta_{2})} u_{j+k-1}^{n+1} \right) \left] \right. \\
+ \phi(\xi_{j}, \tau_{n+1}) - \phi(\xi_{j}, \tau_{n}) + P_{3} \\
\text{where } P_{2} \text{ depends on } h \text{ and } P_{3} \text{ depends on } h \text{ and}

$$
v^{\alpha} = \frac{\tau^{\alpha}}{\Gamma(\alpha + 1)} \phi(\xi_{j}, \tau_{n}) = \frac{1}{\Gamma(\alpha)} \int_{0}^{\tau_{n}} (\tau_{n} - \sigma)^{\alpha-1} \phi(\xi_{j}, \sigma) d\sigma
$$

$$
P_{3} = P_{1} + \left[ c_{0}^{\alpha} (u_{j-k+1}^{n+1} + u_{j+k-1}^{n+1}) + (a_{n}^{0} - a_{n-1}^{0}) (u_{j-k+1}^{0} + u_{j+k-1}^{0}) \right]
$$
here $P_3$ depends on $h$. Since

$$
\|Lu(x)\|^2_{H^1(\Omega)} = ((Lu)(x), (Lu)(x))_{H^1(\Omega)} = [(Lu)(a)]^2 + \int_a^b [(Lu)'(x)]^2 \, dx \\
= \left[\left(\sum_{i=0}^n \varphi_i(x)u^{(i)}(x)\right)\right]^2 \, dx
$$

we have

$$
\int_a^b [(Lu)'(x)]^2 \, dx \geq \int_a^b \left[ u^{(n)}(x) + \sum_{i=0}^{n-1} \left( \varphi_i'(x)u^{(i)}(x) + \varphi_i(x)u^{(i+1)}(x) \right) \right]^2 \, dx

= \int_a^b \left[ u^{(n)}(x) \right]^2 \, dx + \int_a^b \left[ \sum_{i=0}^{n-1} \left( \varphi_i'(x)u^{(i)}(x) + \varphi_i(x)u^{(i+1)}(x) \right) \right]^2 \, dx

+ 2\int_a^b \left[ u^{(n)}(x) \sum_{i=0}^{n-1} \left( \varphi_i'(x)u^{(i)}(x) + \varphi_i(x)u^{(i+1)}(x) \right) \right] \, dx
$$

where

$$
\int_a^b \left[ u^{(n)}(x) \right]^2 \, dx \leq \|u(x)\|^2
$$

and

$$
\int_a^b \left[ u^{(n)}(x) \sum_{i=0}^{n-1} \left( \varphi_i'(x)u^{(i)}(x) + \varphi_i(x)u^{(i+1)}(x) \right) \right] \, dx

\leq \left\{ \int_a^b \left[ u^{(n)}(x) \right]^2 \, dx \right\}^{1/2} \left\{ \int_a^b \left[ \sum_{i=0}^{n-1} \left( \varphi_i'(x)u^{(i)}(x) + \varphi_i(x)u^{(i+1)}(x) \right) \right]^2 \, dx \right\}^{1/2}
$$

Therefore $L$ is a bounded operator. Letting $\eta_1 = \mu_1 v^\alpha \geq 0$, $\eta_2 = \mu_2 v^\alpha \leq 0$, we obtain

$$
v_j^{n+1} - v_j^n = -\eta_1 \left[ c_0^j \left( \sum_{k=0}^{j+1} \omega_k^{(\beta_1)} v_{j-k+1}^{n+1} + \sum_{k=0}^{M-j+1} \omega_k^{(\beta_1)} v_{j-k+1}^{n+1} \right) \right]

+ \left( a_n^\alpha - a_n^{\alpha} \right) \left( \sum_{k=0}^{j+1} \omega_k^{(\beta_1)} v_{j-k+1}^0 + \sum_{k=0}^{M-j+1} \omega_k^{(\beta_1)} v_{j-k+1}^0 \right)

+ \left( a_n^\alpha - a_n^{\alpha} \right) \left( \sum_{k=0}^{j+1} \omega_k^{(\beta_1)} v_{j-k+1}^0 + \sum_{k=0}^{M-j+1} \omega_k^{(\beta_1)} v_{j-k+1}^0 \right)

+ \sum_{\lambda=0}^{n-1} \left( \delta_{\lambda+1}^\alpha - \delta_{\lambda}^\alpha \right) \left( \sum_{k=0}^{j+1} \omega_k^{(\beta_1)} v_{j-k+1}^{n-\lambda} + \sum_{k=0}^{M-j+1} \omega_k^{(\beta_1)} v_{j-k+1}^{n-\lambda} \right)

- \eta_2 \left[ c_0^j \left( \sum_{k=0}^{j+1} \omega_k^{(\beta_2)} v_{j-k+1}^{n+1} + \sum_{k=0}^{M-j+1} \omega_k^{(\beta_2)} v_{j-k+1}^{n+1} \right) \right]
$$

(7)
where \( \tilde{\Phi} \) is an \((M-1) \times (M-1)\) identity matrix, \( B_1 \) and \( B_2 \) are \((M-1) \times (M-1)\) matrices that satisfy

\[
B_1 = \begin{pmatrix}
\theta_1 & \theta_0 & 0 & 0 & \cdots & 0 \\
\theta_2 & \theta_1 & \theta_0 & 0 & \cdots & 0 \\
\theta_3 & \theta_2 & \theta_1 & \theta_0 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\theta_{M-2} & \theta_{M-3} & \theta_{M-4} & \cdots & \theta_1 & \theta_0 \\
\theta_{M-1} & \theta_{M-2} & \theta_{M-3} & \cdots & \theta_2 & \theta_1 \\
\end{pmatrix}
\]
\[
B_2 = \begin{pmatrix}
\sigma_1 & \sigma_0 & 0 & 0 & \cdots & 0 \\
\sigma_2 & \sigma_1 & \sigma_0 & 0 & \cdots & 0 \\
\sigma_3 & \sigma_2 & \sigma_1 & \sigma_0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\sigma_{M-2} & \sigma_{M-3} & \sigma_{M-4} & \cdots & \sigma_1 & \sigma_0 \\
\sigma_{M-1} & \sigma_{M-2} & \sigma_{M-3} & \cdots & \sigma_2 & \sigma_1
\end{pmatrix}
\]

where \( \theta_i = \omega_i^{(\beta_1)} \), \( \sigma_i = \omega_i^{(\beta_2)} \).

By using the B-spline bases on \( S_{4}^{2,3,0} (\Delta_{m,n}^{(2)}) \), we can write

\[
(I + c_0^\alpha A)Y^{n+1} = Y^n - A(a_n^\alpha - a_{n-1}^\alpha)Y^0 - A \sum_{\lambda=0}^{n-1} (\delta_{\lambda+1}^\alpha - \delta_{\lambda}^\alpha)Y^{n-1} + (\tilde{\Phi}_j^{n+1} - \tilde{\Phi}_j^n) \tag{8}
\]

Therefore, the coefficients can be determined by the system of linear equations (8).

4. Numerical experiment. In this section, we solve the numerical solutions of two dimensional advection equation with periodic boundary condition by using the bivariate spline finite element method.

Example 1.

\[
u_t + u_x + u_y = -\frac{u}{\varepsilon}, 0 \leq x \leq 2\pi, t > 0
\]

Take the initial value

\[u(x, y, 0) = \exp(sin(x + y)) - \exp(-1)\]

Numerical results are compared with existing methods in the literature and with the exact solution at the different \( \varepsilon \) and \( t \).

Here, we consider \( \varepsilon = 1 \) and \( \varepsilon = 2 \) respectively. \( m \) and \( n \) are 32 in spline space \( S_{4}^{2,3,0} (\Delta_{m,n}^{(2)}) \). At \( t = 0.1, 0.3, 0.5, 0.7 \), the exact solution \( u(t) \), the approximate solution \( \hat{u}(t) \) by using spline method, and the approximate solution \( \tilde{u}(t) \) by using classical finite elements method (FEM) are shown in Fig.4 - Fig.11.

Table 1 present the comparison of the numerical solutions which are obtained by using spline method and the finite element method (FEM). Also, we give the errors between its exact solutions and numerical solutions at \( t = 0.1, 0.3, 0.5, 0.7 \). Here, we enumerate the 2-norm errors.

**Table 1.** Comparison of numerical and exact solutions of Example 1

| \( \varepsilon \) = 1 | \( t \) = 0.1 | 3.774902e-005 | 6.416033e-005 |
| \( \varepsilon \) = 1 | \( t \) = 0.3 | 2.721077e-005 | 2.347618e-004 |
| \( \varepsilon \) = 1 | \( t \) = 0.5 | 6.327942e-005 | 7.128474e-004 |
| \( \varepsilon \) = 1 | \( t \) = 0.7 | 6.323704e-005 | 2.739811e-004 |
| \( \varepsilon \) = 2 | \( t \) = 0.1 | 3.573924e-004 | 2.159467e-003 |
| \( \varepsilon \) = 2 | \( t \) = 0.3 | 5.898324e-004 | 4.492941e-003 |
| \( \varepsilon \) = 2 | \( t \) = 0.5 | 8.340885e-004 | 6.032486e-003 |
| \( \varepsilon \) = 2 | \( t \) = 0.7 | 7.448253e-004 | 5.265392e-003 |
Example 2.

\[ u_t - 2\pi y \cdot u_x + 2\pi x \cdot u_y = 0, \quad 0 \leq t \leq 1 \]

Take the initial value

\[ u(x, y, 0) = x \cdot y \]

We choose \( m \) and \( n \) are 32 in spline space \( S^{2,3,0}_{4} (\Delta^{(2)}_{m,n}) \). At \( t = 0.1, 0.2, 0.3, 0.4, 0.5 \), the exact solution \( u(t) \), the approximate solution \( \hat{u}(t) \) by using spline method, and the approximate solution \( \tilde{u}(t) \) by using classical finite elements method (FEM) are shown in Fig.12 - 16.

Table 2 present the comparison of the numerical solutions which are obtained by using spline method and the finite element method (FEM). Also, we give the errors between its exact solutions and numerical solutions at \( t = 0.1, 0.2, 0.3, 0.4, 0.5 \). Here, we enumerate the 2-norm errors.

**Table 2. Comparison of numerical and exact solutions of Example 2**

| t   | Spline method         | Finite element method |
|-----|-----------------------|-----------------------|
| 0.1 | 4.537264e-006         | 5.276482e-005         |
| 0.2 | 4.822438e-006         | 4.653391e-005         |
| 0.3 | 5.645882e-006         | 5.283764e-005         |
| 0.4 | 5.326159e-006         | 7.563822e-005         |
| 0.5 | 7.438625e-006         | 6.435764e-005         |

5. **Conclusions.** A two-dimensional linear advection equation with known initial condition is studied in this article. Based on the spline theory, a finite element method is obtained. Compared with classical finite element, spline method doesn’t need to construct partitions. This advantage can reduce computational complexity. We use finite difference discretization to discretize time derivatives. Meanwhile, Bivariate spline is applied for space derivatives. Since the bivariate spline method proposed in this paper is simple and straightforward to apply, we use it to construct the test space. Then we obtain the numerical results of advection equations. The validity of this method is tested by data experiment. We will try to solve non-linear advection equations and other partial differential equations use spline method in future.

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Figure 1. Uniform type-2 triangulation, m=4, n=4

Figure 2. A locally supported spline

Figure 3. (a) Corner B-spline Basis (b) Side B-spline Basis Interior (c) B-spline Basis
Figure 4. $\varepsilon = 1$, (a)$u(0.1)$, (b)$\hat{u}(0.1)$, (c)$\tilde{u}(0.1)$

Figure 5. $\varepsilon = 1$, (a)$u(0.3)$, (b)$\hat{u}(0.3)$, (c)$\tilde{u}(0.3)$

Figure 6. $\varepsilon = 1$, (a)$u(0.5)$, (b)$\hat{u}(0.5)$, (c)$\tilde{u}(0.5)$

Figure 7. $\varepsilon = 1$, (a)$u(0.7)$, (b)$\hat{u}(0.7)$, (c)$\tilde{u}(0.7)$
Figure 8. $\varepsilon = 2$, (a)$u(0.1)$, (b)$\hat{u}(0.1)$, (c)$\tilde{u}(0.1)$

Figure 9. $\varepsilon = 2$, (a)$u(0.3)$, (b)$\hat{u}(0.3)$, (c)$\tilde{u}(0.3)$

Figure 10. $\varepsilon = 2$, (a)$u(0.5)$, (b)$\hat{u}(0.5)$, (c)$\tilde{u}(0.5)$

Figure 11. $\varepsilon = 2$, (a)$u(0.7)$, (b)$\hat{u}(0.7)$, (c)$\tilde{u}(0.7)$
Figure 12. (a) $u(0.1)$, (b) $\hat{u}(0.1)$, (c) $\tilde{u}(0.1)$

Figure 13. (a) $u(0.2)$, (b) $\hat{u}(0.2)$, (c) $\tilde{u}(0.2)$

Figure 14. (a) $u(0.3)$, (b) $\hat{u}(0.3)$, (c) $\tilde{u}(0.3)$

Figure 15. (a) $u(0.4)$, (b) $\hat{u}(0.4)$, (c) $\tilde{u}(0.4)$
Figure 16. (a) $u(0.5)$, (b) $\hat{u}(0.5)$, (c) $\tilde{u}(0.5)$

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*E-mail address*: quka18@dlmu.edu.cn

*E-mail address*: dongqi2345@163.com

*E-mail address*: 2365999744@qq.com

*E-mail address*: 981271189@qq.com