**Mean-square dissipativity of numerical methods for a class of resource-competition models with fractional Brownian motion**

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**ABSTRACT**

The concept of dissipativity in dynamical systems generalizes the idea of a Lyapunov stability. In this paper the dissipativity is used to designate that the system possesses a bounded absorbing set. Specifically, this paper studies mean-square dissipativity of two numerical methods for a class of resource-competition models with fractional Brownian motion (fBm). Some conditions under which the underlying systems are mean-square dissipative are established. It is shown that the mean-square dissipativity is preserved by the split-step \(\theta\)-method (SS(\(\theta\))) under some constraints, while the split-step backward Euler method (SSBE) could inherit mean-square dissipativity without any restriction on stepsize. The results of this study indicate that the split-step backward Euler method (SSBE) outperforms the split-step \(\theta\)-method (SS(\(\theta\))) in terms of mean-square dissipativity. Finally, an example is given to illustrate the effectiveness of the results.

**1. Introduction**

Resource competition is referred to the interaction between individual biological species that consume the same environmental resources. In mathematical description of resource competition, resources availability is always denoted by using a scalar variable. For example, in MacArthur & Levins (1964), the authors used a scalar variable to represent availability of proteins. The resource competition theory is first applied to microorganisms and then to phytoplankton in laboratory experiments. In Huisman & Weissing (1999), the authors discussed the use of resource competition models for phytoplankton species. It is necessary to study resources competition models. The dynamics of the species depend on the availabilities of the resources. Simultaneously, the resource availability depends on the rates of resource supply and the quantity of resources consumed by the species. These relations can be described by the following model (Huisman & Weissing, 1999):

\[
\frac{dN_i}{dt} = N_i(\mu_i(R_1, R_2, \ldots, R_k) - m_i), \quad i = 1, 2, \ldots, n \quad (1)
\]

\[
\frac{dR_j}{dt} = D(S_j - R_j) - \sum_{i=1}^{n} c_{ij}\mu_i(R_1, R_2, \ldots, R_k)N_i, \quad j = 1, 2, \ldots, k, \quad (2)
\]

where \(N_i\) denote the population size of species \(i\), and \(R_j\) denote the availability of resource \(j\). \(\mu_i(R_1, R_2, \ldots, R_k) = \min(\tau_i R_1/(K_{1i} + R_1), \ldots, \tau_i R_k/(K_{ki} + R_k))\) is the specific growth rate of species \(i\) as a function of the resource availabilities, and are determined by the resource that is most limiting on the basis of Liebig’s law of the minimum (Danger et al., 2008); \(r_i\) is the maximum specific growth rate of species \(i\); \(K_j\) is the half-saturation constant for resource \(j\) of species \(i\); \(m_i\) is the specific mortality rate of species \(i\); \(D\) is the system's turnover rate; \(S_j\) is the supply concentration of resource \(j\); and \(c_{ij}\) is the content of resource \(j\) in species \(i\). The theory of resource competition has been extensively experimentally verified in microbes, microalgae, higher plants and plankton, which is of great significance to the engineering management of marine fisheries.

On the other hand, in the natural world, the species are often subject to effects of external environment and random noise. In order to better describe the ecological model, many authors introduce white noise to the population systems to disclosure richer dynamics, see Mao et al. (2003), Mandal & Bnerjee (2012), Du & Sam (2006), Liu & Wang (2012), Li & Jin (2012) and Ren et al. (2014). Meanwhile, the external environment impact on marine life is obvious, Such as light, temperature, nutrient salts, and so on. Moreover, the description of white noise, we choose the fractional Brownian motion. Compared with the Brown movement, many important properties of the
fractional Brown motion make it a natural choice for white noise, see Dung (2012), Ma et al. (2012), Xiao et al. (2012) and Higham & Kloeden (2005). In these papers, most of them study the stability of equilibrium points. However, from a biological point of view, it is not always the case that all the trajectories of such systems will approach an equilibrium point. In some circumstance, we do not know whether such system is asymptotically stable at its equilibrium. So it is unreasonable to expect for every model that their orbits will approach a single steady state. Therefore, it is necessary for us to study the dissipation of resource competition. Not to mention, many interesting problems in physics and engineering are modelled by dissipative differential equations. They are characterized by having a bounded positive invariant absorbing set which the trajectory starting from any bounded set enter at certain moment and stay there forever. There are application in diverse areas such as stability theory, chaos and robust control. Therefore, it is a very interesting topic to investigate the dissipativity of dynamical systems. For example, the epidemic model of acquired immune deficiency syndrome (AIDS), Early stage, we may be able to control AIDS, but after a period of time, it probably would break out again. In this case, what we only know is the trajectories of epidemic model will always enter into a bounded region at some finite time and stay there from then on.

However, most of the stochastic resource competition model do not have explicit solutions. Thus, numerical approximation schemes (Rathinasamy, 2012; Wang & Gan, 2010; Ma et al., 2014, 2015; Huang, 2014) are very valuable tools for exploring its properties. Especially, the literature (Ma et al., 2015) researched mean-square dissipativity of numerical methods for a stochastic resource competition model with fBm, which is more reasonable. The work is still not discussed in the extant literature. And this work differs from existing results in that (a) the stochastic resource competition model with fBm is first proposed, (b) SS, SSBE numerical methods are considered, and (c) mean-square dissipativity is involved.

This paper uses fractional Brown motion (fBm) to describe extrinsic noise, which is a novelty of this paper. The fBm has the property of long-term memory compared with the ordinary Brownian motion. Experimental data of chromatin movement in yeast cells have affirmed that fBm plays a critical role in nuclear dynamics, and have important biological significant compared to standard Brownian motion. Therefore, it is important to research the stochastic calculus concerning fBm and related issues (see Boufoussi & Hajji, 2012; Mishura, 2008).

The rest of this paper is organized as follows. In Section 2, we introduce some conditions, definitions and lemmas, which are useful for the following main result. In Section 3, some criteria for mean-square dissipative of (3) are shown. The mean-square dissipativity of several numerical methods for (3) are established in Section 4. In Section 5 we give an example to expound the main results.

2. Preliminaries

If $A$ is a matrix, its transpose is denoted by $A^T$. Let $R^n$ be a Euclidean space with the inner product $\langle x, y \rangle = y^T x$ and corresponding $|x| = \sqrt{\langle x, x \rangle}$, $x, y \in R^n$. $R^{n \times m}$ denotes the set of $n \times m$ real matrices. The norm of the matrix $A$ is defined as $||A|| = \sup \{|Ax| : |x| = 1\} = \sqrt{\lambda_{\text{max}}(A^T A)}$, $A \in R^{n \times n}$, where $\lambda_{\text{max}}(A^T A)$ denotes the maximum eigenvalue of $A^T A$. $\lambda_{\text{max}}(A)$ denotes the maximum eigenvalue of $A$.

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space with a filtrations $\{\mathcal{F}_t\}_{t \geq 0}$ satisfying the usual conditions (i.e. it is increasing and right continuous while $\mathcal{F}_0$ contains all $P$-null sets). The standard fBm $B^H(t)$ has the following properties (Mishura, 2008):

(i) $B^H(0) = 0$ and $E[B^H(t)] = 0$ for all $t \geq 0$.

(ii) $B^H(t)$ is a continuous and centred Gaussian process with covariance function

$$E[B^H(t)B^H(s)] = \frac{1}{2}(|t|^{2H} + |s|^{2H} + |t-s|^{2H}), \quad t, s \in R.$$
(ii) $B^H(t)$ has homogeneous increments, i.e. $B^H(t+s) - B^H(s)$ has the same law of $B^H(t)$ for $s, t \geq 0$.
(iv) $B^H(t)$ has continuous trajectories.

For $H = \frac{1}{2}$, the fBm is then a standard Brownian motion.

For convenience, the equation (3) is written as a matrix form
\[
dN = (\mu - m)N dt + f(t, N) dt + g(t, N) dB^H(t),
\]
where
\[
N = (N_1, N_2, \ldots, N_n)^T, \quad m = \text{diag}(m_1, m_2, \ldots, m_n),
\]
\[
\mu = \text{diag}(\mu_1(R_1, R_2, \ldots, R_k), \mu_2(R_1, R_2, \ldots, R_k), \ldots, \mu_n(R_1, R_2, \ldots, R_k)),
\]
\[
f(t, N) = (f_1(t, N_1), f_2(t, N_2), \ldots, f_n(t, N_n))^T,
\]
\[
g(t, N) = (g_1(t, N_1), g_2(t, N_2), \ldots, g_n(t, N_n)),
\]
\[
B^H(t) = (B^H_1(t), B^H_2(t), \ldots, B^H_n(t))^T.
\]

Obviously, for equation (4), we have $R_j \in [0, S_j]$. Therefore, the main research object of this paper is equation (5). In order to research the mean-square dissipativity of equation (5), we give the following conditions:

(a1) There exist constants $\alpha \geq 0$, $\beta < 0$ such that for all $N \in \mathbb{R}^n$,
\[
|f(t, N)|^2 \leq \alpha + \beta|N|^2. \tag{6}
\]
(a2) There exist constants $\gamma \geq 0$ such that for all $N \in \mathbb{R}^n$,
\[
|g(t, N)|^2 \leq \gamma|N|^2. \tag{7}
\]

Remark 2.1: For the conditions (a1), (a2), this is indispensable because the external environment influence function $f(t, N(t))$ and random noise intensity function $g(t, N(t))$ must be controlled in a small range. At the same time, in the real world, the external environment and random noise is reasonable with regard to the influence of the population.

Definition 2.2: Equation (5) is said to be mean-square dissipative if there exists a bounded set $B \subseteq \mathbb{R}$, such that for any given bounded set $D \subseteq \mathbb{R}$, there is a time $t^* = t^*(D)$, such that for any given initial value contained in $D$, the mathematical expectation of corresponding solution $E|N(t)|^2$ is contained in $B$ for all $t \geq t^*$. Here $B$ is referred to as a mean-square absorbing set of equation (5).

Lemma 2.3: For any $a, b \in \mathbb{R}^n$ and any positive constant $\sigma > 0$, the inequality
\[
2a^Tb \leq \sigma a^TAa + \sigma^{-1}b^TA^{-1}b \tag{8}
\]
holds, in which $A$ is any matrix with $A > 0$.

Lemma 2.4: Let $f, g : [0, T] \rightarrow \mathbb{R}$ be deterministic continuous functions. If
\[
dN(t) = f(t) dt + g(t) dB^H(t), \quad t \in [0, T],
\]
then
\[
\frac{dF(t, N(t))}{dt} = \frac{\partial F(t, N(t))}{\partial t} + \frac{\partial F(t, N(t))}{\partial N} + \sum_{i,j} \frac{\partial^2 F(t, N(t))}{\partial^2 N} g_i(t, N(t)) g_j(t, N(t))
\]
where $F \in C^{1,2}([0, T] \times \mathbb{R})$.

Remark 2.5: If taking $H = \frac{1}{2}$ in Lemma 2.4, then the formula (9) is the classical Itô’s formula (Mao, 2007).

Remark 2.6: The model (4) and (5) is new and more general than those discussed in the previous literature. For example, if we do not consider external factors and random disturbance, then the model (4) and (5) is the same as the resource competition model in Huisman & Weissing (1999).

3. Mean-square dissipativity of stochastic resource competition model with fBm

In this section, we shall give the mean-square dissipativity of equation (5).

Theorem 3.1: Suppose that equation (5) satisfies (6), (7), let $\mu_1 = \text{diag}(r_1, r_2, \ldots, r_n)$, and $I := 2\lambda_{\max}(\mu_1 - m) + \sigma \beta + \sigma^{-1} + 2\gamma H T^{2H-1} < 0$. Then

(1) for any given $\epsilon > 0$, there exists a positive number $t^*$ such that
\[
E|N(t)|^2 \leq \frac{\sigma \alpha}{T} + \epsilon, \quad t \geq t^*.
\]
(2) for any $\epsilon > 0$, equation (5) is mean-square dissipative with a mean-square absorbing set $B = (0, -\sigma \alpha / I + \epsilon)$.

Proof: Applying fBm-Itô formula to $|N(t)|^2$, we have
\[
|N(t)|^2 = |N(0)|^2 + 2 \int_0^t \langle N(s), (\mu - m)N(s) \rangle \, ds + 2 \int_0^t \langle N(s), f(s, N(s)) \rangle \, ds + 2H \int_0^t s^{2H-1} |g(s, N(s))|^2 \, ds + 2 \int_0^t \langle N(s), g(s, N(s)) \rangle \, dB^H(s). \tag{10}
\]
Taking mathematical expectation on both sides of (10), we know
\[
E[N(t)]^2 = E[N(0)]^2 + 2 \int_0^t E[N(s), (\mu - m)N(s)] \, ds \\
+ 2 \int_0^t E[N(s), f(s, N(s))] \, ds \\
+ 2H \int_0^t s^{2H-1} E[g(s, N(s))]^2 \, ds.
\] (11)

Now, using (6) and Lemma 2.3 lead to
\[
\langle N(s), (\mu - m)N(s) \rangle = ((\mu - m)N(s))^T N(t) \\
\leq \sigma f(t, N(s))^T f(t, N(s)) + \sigma^{-1} N(t)^T N(t) \\
\leq \sigma (\alpha + \beta |N(t)|^2) + \sigma^{-1} |N(t)|^2 \\
= \sigma \alpha + (\sigma \beta + \sigma^{-1}) |N(t)|^2.
\] (12)

Therefore, using (7), (12), (13) and Cauchy–Schwarz inequality, it can be derived that, for any \( t \in [0, T] \)
\[
E[N(t)]^2 \leq E[N(0)]^2 + \int_0^t \sigma \alpha \, ds \\
+ (2\lambda_{\text{max}}(\mu_1 - m) + \sigma \beta + \sigma^{-1} + 2\gamma HT^{2H-1}) \\
\times \int_0^t E[N(t)]^2 \, ds \\
= E[N(0)]^2 + \int_0^t \sigma \alpha \, ds + l \int_0^t E[N(t)]^2 \, ds.
\] (14)

By the generalized Bellman–Gronwall-type inequality (see Dung, 2012; Ma et al., 2012)
\[
E[N(t)]^2 \leq \left( E[N(0)]^2 + \int_0^t \sigma \alpha \exp \left( -\int_0^t l \, du \right) \exp \left( \int_0^t l \, du \right) \right) e^{lt}. \tag{15}
\]

Let \( Q = \sup_{x \in D} |N(t)|^2 \), for any given bounded set \( D \). If \( E[N(0)]^2 + \sigma \alpha / l \geq 0 \), for any given initial value \( N(0) \in D \), that \( B = (0, -\sigma \alpha / l + \epsilon) \) is a mean-square absorbing set of equation (5) may be seen from (15) by choosing \( t^* \) such that \( Q + \sigma \alpha / l \leq \epsilon, \epsilon > 0 \). If \( E[N(0)]^2 + \sigma \alpha / l < 0 \), it is easy to see that \( E[N(t)]^2 \leq -\sigma \alpha / l + \epsilon \), for all \( 0 \leq t < T \). This completes the proof.

4. Main results

In the first place, base on Theorem 3.1, we investigate the mean-square dissipativity of split-step \( \theta \)-method (SS\( \theta \)) for equation (5). We introduce SS\( \theta \) method:
\[
y_n^* = y_n + (1 - \theta)((\mu - m)y_n + f(t_n, y_n)) \Delta \\
+ \theta((\mu - m)y_n^* + f(t_n, y_n^*)) \Delta, \tag{16}
\]
\[
y_{n+1} = y_n^* + g(t_n, y_n^*) \Delta B_n^H. \tag{17}
\]

With initial value \( y_0 = N(0), \Delta > 0 \) is the constant stepsize and \( y_n \) is the approximation to \( N(t_n) \) with \( t_n = n \Delta \). \( \Delta B_n^H = B_n(t_{n+1}) - B_n(t_n) \) represent the increments of Brownian motion. If \( \theta = 1 \), the SS\( \theta \) method becomes the SSBE method.

**Theorem 4.1:** Consider the numerical solution generated by the SS\( \theta \) method, equation (5) satisfies (6), (7) with \( l := 2\lambda_{\text{max}}(\mu_1 - m) + \sigma \beta + \sigma^{-1} + 2\gamma HT^{2H-1} \), \( 0, 0 < \theta \leq 1 \) and
\[
(a) \quad \theta H^2 \gamma T^{2H-1} > (1 - \theta) \sigma^{-1}, \\
(b) \quad \|\mu_1 - m\|^2 + \beta > 0, \\
(c) \quad \sigma \|\mu_1 - m\|^2 + \sigma \beta < 2\gamma HT^{2H-1} + 2\sigma^{-1} < 0.
\]

Then, for any given \( \epsilon > 0 \), there exists an \( n_1 \) such that
\[
E[y_{n_1}^2] \leq \frac{\sigma \alpha (1 + 2\gamma HT^{2H-1} \Delta)}{l \theta + (\sigma \|\mu_1 - m\|^2 + \sigma \beta - 2\gamma HT^{2H-1} + 2\sigma^{-1})(1 - \theta) \Delta + \epsilon},
\]
where \( n \geq n_1, \Delta < \Delta_0 \).

**Proof:** From (16), we have
\[
|y_{n+1}^* - (1 - \theta)((\mu - m)y_n + f(t_n, y_n)) \Delta - \theta((\mu - m)y_n^* + f(t_n, y_n^*)) \Delta|^2 = |y_n^2|^2. \tag{18}
\]

Therefore, by (18), we get
\[
|y_{n+1}^2|^2 \leq 2\Delta (1 - \theta)|y_n^*, (\mu - m)y_n) \\
+ 2\Delta (1 - \theta)|y_n^*, f(y_n, y_n)) + 2\Delta \theta |y_n^*, (\mu - m)y_n^*) \\
+ 2\Delta \theta |y_n^*, f(y_n, y_n^*)) + |y_n|^2. \tag{19}
\]
By Lemma 2.3, we have
\[
2(y^*_n, (\mu - m)y_n) = 2((\mu - m)y_n)^T y_n^*
\]
\[
\leq \sigma ((\mu_1 - m)y_n)^T (\mu_1 - m)y_n + \sigma^{-1} (y_n^*)^T y_n^*
\]
\[
= \sigma y_n^T (\mu_1 - m) y_n + \sigma^{-1} (y_n^*)^T y_n^*
\]
\[
\leq \sigma \lambda_{\max}(\mu_1 - m) |y_n|^2 + \sigma^{-1} |y_n^*|^2
\]
\[
= \|\mu_1 - m\|^2 |y_n|^2 + \sigma^{-1} |y_n^*|^2, \tag{20}
\]
and
\[
2(y_n^*, f(y_n, y_n)) = 2(f(y_n, y_n))^T y_n^*
\]
\[
\leq \sigma (f(y_n, y_n))^T f(y_n, y_n) + \sigma^{-1} (y_n^*)^T y_n^*
\]
\[
\leq \sigma (\alpha + \beta |y_n|^2) + \sigma^{-1} |y_n^*|^2
\]
\[
= \sigma (\alpha + \beta |y_n|^2) + \sigma^{-1} |y_n^*|^2. \tag{21}
\]

Similar (12) and (13), we can show that
\[
(y_n^*, (\mu - m)y_n^*) \leq \lambda_{\max}(\mu_1 - m) |y_n|^2, \tag{22}
\]
\[
(y_n^*, f(y_n, y_n)) \leq \sigma \alpha + (\beta + \sigma^{-1}) |y_n^*|^2. \tag{23}
\]

Substituting (20)–(23) into (19), one obtain
\[
|y_n|^2 \leq \Delta (1 - \theta)[\sigma \|\mu_1 - m\|^2 |y_n|^2 + \sigma^{-1} |y_n^*|^2]
\]
\[
+ \Delta (1 - \theta)[\sigma \alpha + \beta |y_n|^2 + \sigma^{-1} |y_n^*|^2]
\]
\[
+ 2 \Delta \theta \lambda_{\max}(\mu_1 - m) |y_n|^2
\]
\[
+ \Delta \theta[\sigma \alpha + (\beta + \sigma^{-1}) |y_n^*|^2] + |y_n|^2
\]
\[
= [1 + \sigma (1 - \theta)] \|\mu_1 - m\|^2 \Delta
\]
\[
+ \sigma \beta (1 - \theta) |y_n|^2 + \sigma \alpha \Delta
\]
\[
+ [2 \sigma^{-1} (1 - \theta) \Delta + (2 \lambda_{\max}(\mu_1 - m)
\]
\[
+ \sigma \beta + \sigma^{-1} \theta \Delta)] |y_n^*|^2
\]
\[
= [1 + \sigma (1 - \theta)] \|\mu_1 - m\|^2 \Delta
\]
\[
+ \sigma \beta (1 - \theta) |y_n|^2 + \sigma \alpha \Delta
\]
\[
+ [2 \sigma^{-1} (1 - \theta) \Delta - 2 \gamma H \Delta + \theta \Delta |y_n|^2. \tag{24}
\]

Since \( l < 0 \) and (a), then \( 1 - [2 \sigma^{-1} (1 - \theta) \Delta - 2 \gamma H \Delta + \theta \Delta] > 0 \), we can drive that
\[
|y_n|^2 \leq \frac{\sigma \alpha \Delta}{1 - [2 \sigma^{-1} (1 - \theta) \Delta - 2 \gamma H \Delta + \theta \Delta]}
\]
\[
+ \frac{1 + \sigma (1 - \theta)}{1 - [2 \sigma^{-1} (1 - \theta) \Delta - 2 \gamma H \Delta + \theta \Delta]}
\]
\[
|y_n|^2. \tag{25}
\]

It follows from (17) that
\[
|y_{n+1}|^2 = |y_n|^2 + |g(t_n, y_n^*)|^2 |\Delta B^H_n|^2 + 2 y_n^* g(t_n, y_n^*) |\Delta B^H_n|,
\]
\[
\text{By } E(\Delta B^H_n) = 0, E(\Delta B^H_n)^2 = \Delta^2 H, \text{ we find}
\]
\[
E|y_{n+1}|^2 = E|y_n|^2 + \Delta^2 H E|g(t_n, y_n^*)|^2.
\]

Applying (7) and substituting (25) into (27), we obtain
\[
E|y_{n+1}|^2 \leq E|y_n|^2 + \gamma \Delta^2 H E|y_n|^2
\]
\[
= (1 + \gamma \Delta^2 H) E|y_n|^2
\]
\[
\leq (1 + 2 \gamma H \Delta^2 - 1) E|y_n|^2
\]
\[
\leq C_1 E|y_n|^2 + D_1, \tag{28}
\]

where
\[
C_1 = \frac{(1 + \sigma (1 - \theta)) \|\mu_1 - m\|^2 \Delta + \sigma \beta (1 - \theta) \Delta}{(1 + 2 \gamma H \Delta^2 - 1)},
\]
\[
D_1 = \frac{\sigma \alpha \Delta (1 + 2 \gamma H \Delta^2 - 1)}{1 - [2 \sigma^{-1} (1 - \theta) \Delta - 2 \gamma H \Delta^2 + \theta \Delta]}.
\]

By recursive calculation, we have
\[
E|y_n|^2 \leq \frac{D_1}{1 - C_1} - C_0^2 + E|N(0)|^2 C_1^2.
\]

It is straightforward to check that \( 0 < C_1 < 1 \) when \( 0 < \theta < 1 \) and
\[
\Delta \leq \frac{\sigma \alpha (1 + 2 \gamma H \Delta^2 - 1)}{l \theta + (\sigma \|\mu_1 - m\|^2 + \sigma - 2 \gamma H \Delta^2 - 1 + 2 \sigma^{-1}) (1 - \theta)}. \tag{26}
\]

If \( \theta = 1 \), we know \( 0 < C_1 < 1 \) for all \( \Delta > 0 \).

Then we have
\[
\limsup_{n \to \infty} E|y_n|^2 \leq \frac{D_1}{1 - C_1}
\]
\[
= \frac{\sigma \alpha (1 + 2 \gamma H \Delta^2 - 1)}{l \theta + (\sigma \|\mu_1 - m\|^2 + \sigma - 2 \gamma H \Delta^2 - 1 + 2 \sigma^{-1}) (1 - \theta) + 2 \gamma H \Delta^2 - 1 + \theta \Delta}
\]
\[
\Delta \leq \frac{\sigma \alpha (1 + 2 \gamma H \Delta^2 - 1)}{l \theta + (\sigma \|\mu_1 - m\|^2 + \sigma - 2 \gamma H \Delta^2 - 1 + 2 \sigma^{-1}) (1 - \theta) + 2 \gamma H \Delta^2 - 1 + \theta \Delta}.
\]

Therefore, for any given \( \epsilon > 0 \), there exists an \( n_1 \) such that
\[
E|y_n|^2 \leq \frac{-\sigma \alpha (1 + 2 \gamma H \Delta^2 - 1)}{l \theta + (\sigma \|\mu_1 - m\|^2 + \sigma - 2 \gamma H \Delta^2 - 1 + 2 \sigma^{-1}) (1 - \theta) + 2 \gamma H \Delta^2 - 1 + \theta \Delta}
\]
\[
\Delta \leq \frac{-\sigma \alpha (1 + 2 \gamma H \Delta^2 - 1)}{l \theta + (\sigma \|\mu_1 - m\|^2 + \sigma - 2 \gamma H \Delta^2 - 1 + 2 \sigma^{-1}) (1 - \theta) + 2 \gamma H \Delta^2 - 1 + \theta \Delta} + \epsilon,
\]
where \( n \geq n_1, \Delta < \Delta_0 \).

\[
\Delta_0 = \frac{l \theta + (\sigma \|\mu_1 - m\|^2 + \sigma - 2 \gamma H \Delta^2 - 1 + 2 \sigma^{-1}) (1 - \theta)}{2 \gamma H \Delta^2 - 1 + \theta \Delta}.
\]

The proof is completed. \( \blacksquare \)
In the next place, we discuss the mean-square dissipativity of split-step backward Euler (SSBE) method. Given a stepsize $\Delta > 0$, the SSBE method applied to equation (5), we obtain

$$y_n^* = y_n + (\mu - m)y_n^* \Delta + f(t_n, y_n^*),$$  \hspace{1cm} (29)

$$y_{n+1} = y_n^* + g(t_n, y_n^*) \Delta B_n^H,$$  \hspace{1cm} (30)

with $y_n \approx N(t_n)$, $y_0 = N(0)$, $t_n = n\Delta$, $\Delta B_n^H = B^H(t_{n+1}) - B^H(t_n)$.

**Theorem 4.2:** Assume that equation (5) satisfies (6) and (7) with $l := 2\lambda_{\text{max}}(\mu_1 - m) + \sigma \beta + \sigma^{-1} + 2\gamma H T^{2H-1} < 0$. Then for any given $\epsilon > 0$, there exists an $n_2$ such that

$$E|y_n|^2 \leq \frac{\sigma \alpha (1 + 2H T^{2H-1} \Delta)}{l} + \epsilon, \hspace{1cm} n \geq n_2.$$  \hspace{1cm} (31)

**Proof:** From (29), we have

$$|y_n^* - (\mu - m) y_n^* \Delta - f(t_n, y_n^*)\Delta|^2 = |y_n|^2.$$  \hspace{1cm} (32)

Therefore, by (12) and (13), we get

$$|y_n^*|^2 \leq 2\Delta (y_n^*, (\mu_1 - m) y_n^*) + 2\Delta (y_n^*, f(t_n, y_n^*)) + |y_n|^2$$

$$= (2\lambda_{\text{max}}(\mu_1 - m) + \sigma \beta + \sigma^{-1})|y_n|^2 + \sigma \alpha \Delta + |y_n|^2.$$  \hspace{1cm} (33)

Since $l < 0$, then $2\lambda_{\text{max}}(\mu_1 - m) + \sigma \beta + \sigma^{-1} < 0$, we can show that

$$|y_n^*|^2 \leq \frac{\sigma \alpha \Delta}{1 - (2\lambda_{\text{max}}(\mu_1 - m) + \sigma \beta + \sigma^{-1}) \Delta} + 1$$

$$+ \frac{1}{1 - (2\lambda_{\text{max}}(\mu_1 - m) + \sigma \beta + \sigma^{-1}) \Delta} |y_n|^2.$$  \hspace{1cm} (34)
Figure 2. The pictures on the left are mean-square dissipativity of numerical solutions for equation (35), where $H = \frac{1}{2}$. (a) SSBE method, (c) SS$\theta$ method, $\theta = 0.7$. The subgraphs on the right are mean-square transient response of state variables $y_1^n, y_2^n, y_3^n$, (b) SSBE method, (d) SS$\theta$ method, $\theta = 0.7$.

It follows from (30) that

$$|y_{n+1}|^2 = |y_n|^2 + |g(t_n, y_n^*)|^2 |\Delta B_n^H|^2 + 2 \langle y_n^*, g(t_n, y_n^*) \Delta B_n^H \rangle,$$

(34)

Taking mathematical expectation on both sides of (34), substituting (33) into (34), applying (7), we have

$$E|y_{n+1}|^2 \leq C_2 E|y_n|^2 + D_2,$$

where

$$C_2 = \frac{1 + 2H\gamma T^{2H-1} \Delta}{1 - (2\lambda_{\max}(\mu_1 - m) + \sigma \beta + \sigma^{-1}) \Delta},$$

$$D_2 = \frac{\sigma \alpha (1 + 2H\gamma T^{2H-1} \Delta)}{1 - (2\lambda_{\max}(\mu_1 - m) + \sigma \beta + \sigma^{-1}) \Delta}.$$

By recursive method, we can derive that

$$E|y_n|^2 \leq \frac{D_2}{1 - C_2} (1 - C_2^n) + E|N(0)|^2 C_2^n.$$

Since $l < 0$, it is straightforward to check that $0 < C_2 < 1$. Then we have

$$\limsup_{n \to \infty} E|y_n|^2 \leq \frac{D_2}{1 - C_2} \frac{\sigma \alpha (1 + 2H\gamma T^{2H-1} \Delta)}{l}.$$

Therefore, for any given $\epsilon > 0$, there exist an $n_2$ such that

$$E|y_n|^2 \leq -\frac{\sigma \alpha (1 + 2H\gamma T^{2H-1} \Delta)}{l} + \epsilon, \quad n \geq n_2.$$

The proof is completed.

Remark 4.3: It turned out that SSBE method can reproduce mean-square dissipativity without any restriction on stepsize, whereas SS$\theta$ method can preserve mean-square dissipativity within some strict constraints. Furthermore,
when $\theta = 1$, Theorem 4.1 simplified to Theorem 4.2. According to mean-square dissipativity in Theorems 4.1 and 4.2, the numerical solutions of equation (5) are bounded in mean square sense.

5. Numerical experiments

In this section we discuss an example to verify our results. Consider three species competing for three resources with FBM

$$
\begin{align*}
\frac{dN_1}{dN_2} & = \begin{pmatrix} \mu_1 - 0.62 & 0 \\ 0 & \mu_2 - 0.62 \\ 0 & \mu_3 - 0.62 \end{pmatrix} \begin{pmatrix} N_1 \\ N_2 \\ N_3 \end{pmatrix} \, dt \\
+ & \begin{pmatrix} 0.25 \sin N_1 \\ 0.28 \sin N_2 \\ 0.32 \sin N_3 \end{pmatrix} \, d\mathcal{B}^H(t),
\end{align*}
$$

where

$$
K = \begin{pmatrix} 1 & 0.75 & 0.25 \\ 0.25 & 1 & 0.75 \\ 0.75 & 0.25 & 1 \end{pmatrix},
$$

and

$$
\begin{pmatrix}
\frac{dR_1}{dR_2} \\
\frac{dR_2}{dR_3}
\end{pmatrix} = 0.25 \begin{pmatrix} 10 - R_1 \\ 10 - R_2 \end{pmatrix} \, dt - \begin{pmatrix} 0.10 & 0.20 & 0.15 \\ 0.15 & 0.10 & 0.20 \\ 0.20 & 0.15 & 0.10 \end{pmatrix} \begin{pmatrix} N_1 \\ N_2 \\ N_3 \end{pmatrix} \times \begin{pmatrix} \mu_1 \mu_2 N_1 \\ \mu_2 N_2 \\ \mu_3 N_3 \end{pmatrix} \, dt,
$$

$$(N_1(0), N_2(0), N_3(0))^T = (1.91, 1.92, 1.93)^T,$$

$$(R_1(0), R_2(0), R_3(0))^T = (10, 10, 10)^T. \tag{35}$$
obtain square dissipative with a mean-square absorbing set (solutions which are obtained by SSBE, SS depict mean-square transient response of the state variables). No potential conflict of interest was reported by the authors. Numerical methods. Our method is different from the literature.

6. Conclusions

As is well known, the concept of dissipativity play an important role in the field of physics and control system theory and it has extensive applications in such fields. Thus, in this paper, the mean-square dissipativity for a well-known resource competition model with fractional Brownian motion (fBm) is considered, and the mean-square dissipativity of numerical methods for the model is exquisitely researched. By using fBm-Itô formula, inner product properties, some sufficient conditions are obtained for mean-square dissipativity of numerical methods. Our method is different from the literature mentioned above. In future work, dissipativity of this model with external input vector can be discussed.

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