DIFFERENTIAL ISOMORPHISM AND EQUIVALENCE OF
ALGEBRAIC VARIETIES

YURI BEREST AND GEORGE WILSON

To Graeme Segal for his sixtieth birthday

1. Introduction

Let $X$ be an irreducible complex affine algebraic variety, and let $\mathcal{D}(X)$ be the ring of (global, linear, algebraic) differential operators on $X$ (we shall review the definition in Section 2). This ring has a natural filtration (by order of operators) in which the elements of order zero are just the ring $\mathcal{O}(X)$ of regular functions on $X$. Thus, if we are given $\mathcal{D}(X)$ together with its filtration, we can at once recover the variety $X$. But now suppose we are given $\mathcal{D}(X)$ just as an abstract noncommutative $\mathbb{C}$-algebra, without filtration; then it is not clear whether or not we can recover $X$. We shall call two varieties $X$ and $Y$ differentially isomorphic if $\mathcal{D}(X)$ and $\mathcal{D}(Y)$ are isomorphic.

The first examples of nonisomorphic varieties with isomorphic rings of differential operators were found by Levasseur, Smith and Stafford (see [LS S] and Section 9 below). These varieties arise in the representation theory of simple Lie algebras; they are still the only examples we know in dimension $> 1$ (if we exclude products of examples in lower dimensions). For curves, on the other hand, there is now a complete classification up to differential isomorphism; the main purpose of this article is to review that case. The result is very strange. It turns out that for curves, $\mathcal{D}(X)$ determines $X$ (up to isomorphism) except in the very special case when $X$ is homeomorphic to the affine line $\mathbb{A}^1$ (we call such a curve a framed curve). There are uncountably many nonisomorphic framed curves (we can insert arbitrarily bad cusps at any finite number of points of $\mathbb{A}^1$). However, the differential isomorphism classes of framed curves are classified by a single non-negative integer $n$. This invariant $n$ seems to us the most interesting character in our story: it appears in many guises, some of which we describe in Section 8.

We can also ask to what extent $X$ is determined by the Morita equivalence class of $\mathcal{D}(X)$: we call two varieties $X$ and $Y$ differentially equivalent if $\mathcal{D}(X)$ and $\mathcal{D}(Y)$ are Morita equivalent (as $\mathbb{C}$-algebras). A complete classification of curves up to differential equivalence is not available; however, it is known that the differential equivalence class of a smooth affine curve $X$ consists of all the curves homeomorphic to $X$. In particular, all framed curves are differentially equivalent to each other: that is one reason why the invariant $n$ which distinguishes them has to be somewhat unusual. In dimension $> 1$, there are already some interesting results about differential equivalence; we include a (very brief) survey in Section 9 where we also mention some generalizations of our questions to non-affine varieties.

At the risk of alienating some readers, we point out that most of the interest in this paper is in singular varieties. For smooth varieties it is a possible conjecture...
that differential equivalence implies isomorphism: indeed, that is true for curves. However, in dimension \( > 1 \) the conjecture would be based on no more than lack of counterexamples.

Our aim in this article has been to provide a readable survey, suitable as an introduction to the subject for beginners; most of the material is already available in the literature. For the convenience of readers who are experts in this area, we point out a few exceptions to that rule: Theorem 8.7 is new, and perhaps Theorem 3.3; also, the formulae (7.1) and (8.3) have not previously appeared explicitly.

2. Generalities on Differential Operators

We first recall the definition of (linear) differential operators, in a form appropriate for applications in algebraic geometry (see [G]). If \( A \) is a (unital associative) commutative algebra over (say) \( \mathbb{C} \), the filtered ring
\[
\mathcal{D}(A) = \bigcup_{r \geq 0} \mathcal{D}^r(A) \subset \text{End}_\mathbb{C}(A)
\]
of differential operators on \( A \) may be defined inductively as follows. First, we set \( \mathcal{D}^0(A) := A \) (here the elements of \( A \) are identified with the corresponding multiplication operators); then, by definition, a linear map \( \theta : A \to A \) belongs to \( \mathcal{D}^r(A) \) if
\[
\theta a - a \theta \in \mathcal{D}^{r-1}(A) \quad \text{for all} \quad a \in A.
\]
The elements of \( \mathcal{D}^r(A) \) are called differential operators of order \( \leq r \) on \( A \). The commutator of two operators of orders \( r \) and \( s \) is an operator of order at most \( r + s - 1 \); it follows that the associated graded algebra
\[
\text{gr} \mathcal{D}(A) := \bigoplus_{r \geq 0} \mathcal{D}^r(A)/\mathcal{D}^{r-1}(A)
\]
is commutative (we set \( \mathcal{D}^{-1}(A) := 0 \).

Slightly more generally, we can define the ring \( \mathcal{D}_A(M) \) of differential operators on any \( A \)-module \( M \): the operators of order zero are the \( A \)-linear maps \( M \to M \), and operators of higher order are defined inductively just as in the special case above (where \( M = A \)).

**Example 2.1.** If \( A = \mathbb{C}[z_1, \ldots, z_m] \), then \( \mathcal{D}(A) = \mathbb{C}[z_i, \partial/\partial z_i] \) is the \( m \)th Weyl algebra (linear differential operators with polynomial coefficients).

**Example 2.2.** Similarly, if \( A = \mathbb{C}(z_1, \ldots, z_m) \), then \( \mathcal{D}(A) = \mathbb{C}(z_i)[\partial/\partial z_i] \) is the algebra of linear differential operators (in \( m \) variables) with rational coefficients.

The definition of \( \mathcal{D}(A) \) makes sense for an arbitrary \( \mathbb{C} \)-algebra \( A \); however, in this paper we shall use it only in the cases when \( A \) is either the coordinate ring \( \mathcal{O}(X) \) of an irreducible affine variety \( X \), or the field \( \mathbb{K} \equiv \mathbb{C}(X) \) of rational functions on such a variety. Let us consider first the latter case. If we choose a transcendence basis \( \{z_1, \ldots, z_m\} \) for \( \mathbb{K} \) over \( \mathbb{C} \) (where \( m = \dim X \)), then there are (unique) \( \mathbb{C} \)-derivations \( \partial_1, \ldots, \partial_m \) of \( \mathbb{K} \) such that \( \partial_i(z_j) = \delta_{ij} \), and each element of \( \mathcal{D}^r(\mathbb{K}) \) has a unique expression in the form
\[
\theta = \sum_{|\alpha| \leq r} f_\alpha \partial^\alpha
\]
Differential Isomorphism

Let $X$ be a smooth (irreducible) affine variety. Then

(i) $\mathcal{D}(X)$ is a simple (left and right) Noetherian ring without zero divisors;
(ii) $\mathcal{D}(X)$ is generated as a $\mathbb{C}$-algebra by finitely many elements of $\mathcal{D}^1(X)$;
(iii) the associated graded algebra $\text{gr} \mathcal{D}(X)$ is canonically isomorphic to $\mathcal{O}(T^*X)$;
(iv) $\mathcal{D}(X)$ has global (that is, homological) dimension equal to $\dim X$.

If $X$ is singular, the situation is less clear. We can still consider the ring $\Delta(X)$ of ($\mathbb{C}$-linear) operators on $\mathcal{O}(X)$ generated by the multiplication operators and the derivations of $\mathcal{O}(X)$; however, in general $\Delta(X)$ is smaller than $\mathcal{D}(X)$. Our main reason to prefer $\mathcal{D}(X)$ to $\Delta(X)$ is the following. Each differential operator on $\mathcal{O}(X)$ has a unique extension to a differential operator (of the same order) on $\mathbb{K}$, so we may view $\mathcal{D}(X)$ as a subalgebra of $\mathcal{D}(\mathbb{K})$. Furthermore, a differential operator on $\mathbb{K}$ which preserves $\mathcal{O}(X)$ is a differential operator on $\mathcal{O}(X)$ (this last statement would in general not be true for $\Delta(X)$). Thus we have:

**Proposition 2.4.** Let $X$ be an affine variety with function field $\mathbb{K}$. Then

$$\mathcal{D}(X) = \{ D \in \mathcal{D}(\mathbb{K}) : \partial \mathcal{O}(X) \subseteq \mathcal{O}(X) \}.$$ 

For the purposes of the present paper we could well take this as the definition of $\mathcal{D}(X)$. It follows from Proposition 2.4 that $\mathcal{D}(X)$ is without zero divisors also for (irreducible) singular varieties $X$.

**Example 2.5.** Let $X$ be the rational curve with coordinate ring $\mathcal{O}(X) := \mathbb{C}[z^2, z^3]$ (thus $X$ has just one simple cusp at the origin). Then $\Delta(X)$ is generated by $\mathcal{O}(X)$ and the derivations $\{ z^r \partial : r \geq 1 \}$ (we set $\partial := \partial/\partial z$). But $\mathcal{D}^2(X)$ contains the operators $\partial^2 - 2z^{-1}\partial$ and $z\partial^2 - \partial$, neither of which belongs to $\Delta(X)$.

To obtain a concrete realization of $\mathcal{D}_A(M)$ similar to that in Proposition 2.4 we need to suppose that $M$ is embedded as an $A$-submodule of some $\mathbb{K}$-vector space; to fix ideas, we formulate the result in the case that will concern us, where $M$ has rank 1.

**Proposition 2.6.** Suppose $M \subset \mathbb{K}$ is a (nonzero) $A$-submodule of $\mathbb{K}$. Then

$$\mathcal{D}_A(M) = \{ D \in \mathcal{D}(\mathbb{K}) : D.M \subseteq M \}.$$ 

**Notes.** 1. To part (iii) of Proposition 2.3 we should add that the commutator on $\mathcal{D}(X)$ induces on $\text{gr} \mathcal{D}(X)$ the canonical Poisson bracket coming from the symplectic structure of $T^*X$; that is, $\mathcal{D}(X)$ is a deformation quantization of $\mathcal{O}(T^*X)$.

2. For singular varieties, the rings $\Delta(X)$ and $\mathcal{D}(X)$ have quite different properties: for example, $\Delta(X)$ is simple if and only if $X$ is smooth (cf. Theorem 3.2 below). It follows that if $X$ is smooth, then $\Delta(X)$ is never isomorphic, or even Morita equivalent, to $\Delta(Y)$ for any singular variety $Y$. Thus the present paper would
probably be very short and dull if we were to work with $\Delta(X)$ rather than with $D(X)$.

3. Nakai (cf. [Na]) has conjectured that $D(X) = \Delta(X)$ if and only if $X$ is smooth. The conjecture has been proved for curves (see [MV]) and, more generally, for varieties with smooth normalization (see [T]). In [Be] and [R] it is shown that Nakai’s conjecture would imply the well known Zariski-Lipman conjecture: if the module of derivations of $O(X)$ is projective, then $X$ is smooth.

4. If $X$ is singular, then in general $D(X)$ may have quite bad properties. In [BGG] it is shown that if $X$ is the cone in $A^3$ with equation $x^3 + y^3 + z^3 = 0$, then $D(X)$ is not a finitely generated algebra, nor left or right Noetherian. In this example $X$ is a normal variety, and has only one singular point (at the origin). In [SS], Section 7, it is shown that if $X$ is a variety of dimension $\geq 2$ with smooth normalization and isolated singularities, then $D(X)$ is right Noetherian but not left Noetherian.

5. In the situation of Proposition 2.6, it may happen that the ring $B := D^0_A(M)$ is larger than $A$. In that case the ring $D_A(M) \subset D(\mathbb{K})$ would not change if we replaced $A$ by $B$; thus there is no loss of generality if we restrict attention to modules $M$ for which $B = A$. We call such $A$-modules maximal.

6. Of course, all the statements in this section (and, indeed, in most of the other sections) would remain true if we replaced $\mathbb{C}$ by any algebraically closed field of characteristic zero. If we work over a field of positive characteristic, the above definition of differential operators is still generally accepted to be the correct one, but some of the properties of the rings $D(X)$ are very different: for example, $D(X)$ is not Noetherian, or finitely generated, or without zero divisors (see, for example, [Sm]). In particular, in positive characteristic $D(A^1)$ is not at all like the Weyl algebra.

7. A convenient reference for this section is the last chapter of the book [MR], where one can find proofs of all the facts we have stated (except for Proposition 2.6 whose proof is similar to that of Proposition 2.4).

3. Differential equivalence of curves

From now on until Section 9, $X$ will be an affine curve, probably singular. In this case the problems mentioned in Section 2 Note 4 do not occur.

**Proposition 3.1.** Let $X$ be an (irreducible) affine curve. Then $D(X)$ is a (left and right) Noetherian ring, and is finitely generated as a $\mathbb{C}$-algebra.

However, the associated graded ring $\text{gr} D(X)$ is in general not a Noetherian ring (and hence not a finitely generated algebra either). The following theorem of Smith and Stafford shows that for our present purposes there is a very stark division of curve singularities into “good” and “bad”.

**Theorem 3.2.** Let $X$ be an affine curve, and let $\tilde{X}$ be its normalization. Then the following are equivalent.

1. The normalization map $\pi : \tilde{X} \rightarrow X$ is bijective.
2. The algebras $D(\tilde{X})$ and $D(X)$ are Morita equivalent.
3. The ring $D(X)$ has global dimension 1 (that is, the same as $D(\tilde{X})$).
4. The ring $D(X)$ is simple.
5. The algebra $\text{gr} D(X)$ is finitely generated.
The ring \( \text{gr} \, \mathcal{D}(X) \) is Noetherian.

Perhaps the most striking thing about Theorem 3.2 is that the “good” singularities (from our present point of view) are the cusps (as opposed to double points, or higher order multiple points). If \( X \) has even one double point, the ring \( \mathcal{D}(X) \) is somewhat wild; whereas if \( X \) has only cusp singularities, no matter how “bad”, then \( \mathcal{D}(X) \) is barely distinguishable from the ring of differential operators on the smooth curve \( \tilde{X} \).

Theorem 3.2 does not address the question of when two smooth affine curves are differentially equivalent. However, the answer to that is very simple.

**Theorem 3.3.** Let \( X \) and \( Y \) be smooth affine curves. Then \( \mathcal{D}(X) \) and \( \mathcal{D}(Y) \) are Morita equivalent (if and) only if \( X \) and \( Y \) are isomorphic.

Theorems 3.2 and 3.3 together determine completely the differential equivalence class of a smooth curve \( X \): it consists of all curves obtained from \( X \) by pinching a finite number of points to (arbitrarily bad) cusps.

**Notes.**

1. Apparently, not much is known about the differential equivalence class of a curve with multiple points. From Theorem 3.2 one might guess that if \( \pi : Y \to X \) is regular surjective of degree one, then \( X \) and \( Y \) are differentially equivalent if and only if \( \pi \) is bijective. However, in [SS] (5.8) there is a counterexample to the “if” part of this statement. The paper [CH2] contains some curious results about the Morita equivalence class of \( \mathcal{D}(A) \) when \( A \) is the local ring at a multiple point of a curve.

2. Another natural question that is not addressed by Theorem 3.2 is: what is the global dimension of \( \mathcal{D}(X) \) if \( X \) has multiple points? In [SS] it is proved that if the singularities are all ordinary multiple points, then the answer is 2; but for more complicated singularities it seems nothing is known.

3. We have not found Theorem 3.3 stated explicitly in the literature, but it is an easy consequence of the results of [CH1] and [M-L]: we will sketch a proof in Section 6, Note 5.

4. Proposition 3.1 is proved in [SS] and (also in the case of a reducible (but reduced) curve) in [M].

5. We refer to [SS] for the proofs of the various assertions in Theorem 3.2. Here we mention only that a key role is played by the space

\[
P \equiv \mathcal{D}(\tilde{X}, X) := \{ D \in \mathcal{D}(\mathbb{K}) : \text{D} \cdot \mathcal{O}(\tilde{X}) \subseteq \mathcal{O}(X) \}.
\]

Clearly, \( P \) is a right ideal in \( \mathcal{D}(\tilde{X}) \) and a left ideal in \( \mathcal{D}(X) \); the Morita equivalence in Theorem 3.2 is defined by tensoring with the bimodule \( P \). Another notable property of \( P \) is the following: each of the statements in Theorem 3.2 is equivalent to the condition

\[
P \cdot \mathcal{O}(\tilde{X}) = \mathcal{O}(X).
\]

The formulae (3.1) and (3.2) provide the starting point for the theory of Cannings and Holland which we explain in Section 6; there \( P \) is replaced by an arbitrary right ideal in \( \mathcal{D}(\tilde{X}) \).
4. Differential Isomorphism of Curves

We now turn to our main question, concerning differential isomorphism. We begin by sketching the history of this subject.

To our knowledge, the papers \([St] , [Sm] \) are the first that explicitly pose the question: does \(D(X) \simeq D(Y) \) imply \(X \simeq Y \) ? In \([St]\), Stafford proved that this is true if \(X\) is the affine line \(A^1\) (in which case \(D(X)\) is the Weyl algebra), and also if \(X\) is the plane curve with equation \(y^2 = x^3\), that is, the rational curve obtained from \(A^1\) by introducing a simple cusp at the origin. The first general result in the subject is due to L. Makar-Limanov (see \([M-L]\)). His idea was as follows. Recall that if we take the commutator \((\text{ad}\, f)L := fL - LF\) of a function \(f \in O(X)\) with an operator \(L \in D(X)\) of order \(n\), then we get an operator of order at most \(n - 1\) (indeed, this is essentially the definition of \(D(X)\), see Section 2 above). It follows that \((\text{ad}\, f)^{n+1}L = 0\), so that \(f\) is a (locally) \(\text{ad}\)-nilpotent element of \(D(X)\).

If it happens (as seems likely) that the set \(N(X)\) of all \(\text{ad}\)-nilpotent elements of \(D(X)\) coincides with \(O(X)\), then we have a purely ring-theoretical description of \(O(X) \subset D(X)\), namely, it is the unique maximal abelian \(\text{ad}\)-nilpotent subalgebra (for short: \(\text{mad subalgebra}\) of \(D(X)\)). So in this way \(D(X)\) determines \(X\). Makar-Limanov’s main remark was the following.

**Lemma 4.1.** Let \(K\) be the function field of a curve, and let \(D \in D(K)\) have positive order. Let \(N \subset K\) be the set of elements of \(K\) on which \(D\) acts \(\text{ad}\)-nilpotently. Then there is an element \(q\) in some finite extension field of \(K\) such that \(N \subseteq \mathbb{C}[q]\).

If now \(X\) is a curve such that \(N(X) \neq O(X)\), that is, such that \(N(X)\) contains an operator of positive order, then it follows from Lemma 4.1 that \(O(X) \subseteq \mathbb{C}[q]\) for suitable \(q\). Equivalently:

**Theorem 4.2.** If \(N(X) \neq O(X)\), then the normalization \(\tilde{X}\) of \(X\) is isomorphic to \(A^1\).

In his thesis (see \([P1]\)), P. Perkins refined this result.

**Theorem 4.3.** Let \(X\) be an affine curve. Then \(N(X) \neq O(X)\) if and only if

(i) \(\tilde{X}\) is isomorphic to \(A^1\); and

(ii) the normalization map \(\pi : \tilde{X} \rightarrow X\) is bijective.

In other words, the differential isomorphism class of a curve \(X\) consists just of (the class of) \(X\) itself, except, possibly, when \(X\) has the properties (i) and (ii) above.

For short, we shall call a curve with these two properties a **framed curve**. More precisely, by a framed curve we shall mean a curve \(X\) together with a regular bijective map \(\pi : A^1 \rightarrow X\): the choice of “framing” (that is, of the isomorphism \(\tilde{X} \simeq A^1\)) is fairly harmless, because any two choices differ only by an automorphism \(z \mapsto az + b\) of \(A^1\). The two curves considered by Stafford are certainly framed curves: Stafford’s results do not contradict those of Perkins, because although the rings \(D(X)\) in these examples have many \(\text{ad}\)-nilpotent elements not in \(O(X)\), their mad subalgebras are all isomorphic, so we can still extract \(O(X)\) (up to isomorphism) from \(D(X)\). For a while it might have seemed likely that the situation is similar for any framed curve; but counterexamples were found by Letzter \([L]\) and
The following example of Letzter is perhaps the simplest and most striking. Let \( X \) and \( Y \) be the curves with coordinate rings
\[
\mathcal{O}(X) = \mathbb{C} + z^4 \mathbb{C}[z]; \quad \mathcal{O}(Y) = \mathbb{C}[z^2, z^5].
\]
Each of \( X \) and \( Y \) is obtained from \( \mathbb{A}^1 \) by introducing a single cusp at the origin; \( X \) and \( Y \) are clearly not isomorphic. Indeed, we have \( \mathcal{O}(X) \subset \mathcal{O}(Y) \), so the singularity of \( X \) is strictly “worse” than that of \( Y \). Nevertheless, Letzter proved that \( X \) and \( Y \) are differentially isomorphic. This example, and others in [P2], [L], shows that the problem of classifying framed curves up to differential isomorphism is nontrivial.

This problem was solved completely in the thesis [K] of K. Kouakou. The simplest way to state his result is as follows. For each \( n \geq 0 \), let \( X_n \) denote the curve
\begin{equation}
\mathcal{O}(X_n) := \mathbb{C} + z^{n+1} \mathbb{C}[z].
\end{equation}
(Thus the curves considered by Stafford are \( X_0 \equiv \mathbb{A}^1 \) and \( X_1 \), while the curve \( X \) in Letzter’s example above is \( X_3 \)).

**Theorem 4.4 (Kouakou).** Every framed curve \( X \) is differentially isomorphic to one of the above curves \( X_n \).

On the other hand, Letzter and Makar-Limanov (see [LM]) have proved the following.

**Theorem 4.5.** No two of the curves \( X_n \) are differentially isomorphic to each other.

It follows that each framed curve \( X \) is differentially isomorphic to exactly one of the special curves \( X_n \); we shall call this number \( n \) the differential genus of \( X \), and denote it by \( d(X) \).

**Notes.**
1. Of course, this is very unsatisfactory as a definition of the differential genus, because it does not make sense until after we have proved the two nontrivial Theorems 4.4 and 4.5. In Section 8 we discuss several more illuminating ways to define \( d(X) \). We use the term “genus” because \( d(X) \) is in some ways reminiscent of the arithmetic genus of a curve: it turns out that it is a sum of local contributions from each singular point, so it simply counts the cusps of our framed curve with appropriate weights. In Section 8 we shall explain how to calculate these weights: here we just mention that the weight of a simple (that is, of type \( y^2 = x^3 \)) cusp is equal to 1, so if all the cusps of \( X \) are simple, then \( d(X) \) is just the number of cusps.
2. Recall from Theorem 3.2 that the algebras \( \mathcal{D}(X) \) (for \( X \) a framed curve) are all Morita equivalent to each other: thus the invariant \( d(X) \) that distinguishes them must be fairly subtle.
3. Makar-Limanov’s Lemma [44] (in a slightly disguised form) plays a basic role also in the theory of bispectral differential equations (compare the proof in [M-L] with similar arguments in [DG] or [W1]).
4. There is no convenient reference where the reader can find a complete proof of Kouakou’s theorem: Kouakou’s thesis has never been published, and the (different) proof in [BW1] is mostly omitted. The proof that we shall explain in the next three sections amplifies the sketch given in [W3]: it is not the most elementary possible, but it seems to us the most natural available at present.
5. The Adelic Grassmannian

It is actually easier to prove a more general theorem than Theorem 4.4, as follows. Let $X$ be a framed curve, and let $L$ be any rank 1 torsion-free coherent sheaf over $X$; it corresponds to a rank 1 torsion-free $\mathcal{O}(X)$-module $M$. Then we have the ring $D_L(X) \equiv D_{\mathcal{O}(X)}(M)$ of differential operators on (global) sections of $L$. If $L = \mathcal{O}_X$ is the sheaf of regular functions on $X$, then $D_L(X)$ is just the ring $D(X)$ discussed previously. Generalizing Theorem 4.4, we have the following.

Theorem 5.1. Every algebra $D_L(X)$ is isomorphic to one of the algebras $D(X^n)$. Of course, Theorem 4.5 shows that the integer $n$ in this assertion is unique: we call it the differential genus of the pair $(X, L)$ and denote it by $d(L)$. The reason Theorem 5.1 is easier to prove than Theorem 4.4 is that the space of pairs $(X, L)$ has a large group of symmetries that preserves the isomorphism class of the algebra $D_L(X)$ (but does not preserve the subset of pairs of the form $(X, \mathcal{O}_X)$). In fact the isomorphism classes of these pairs form the adelic Grassmannian $Gr_{\text{ad}}$, a well-studied space that occurs in at least two other contexts, namely, in the theory of the Kadomtsev-Petviashvili hierarchy (cf. [Kr]) and in the problem of classifying bispectral differential operators (see [DG, W1]). The adelic Grassmannian is a subspace of the much larger Grassmannian $Gr$ studied in [SW]. We recall the definition of $Gr_{\text{ad}}$. For each $\lambda \in \mathbb{C}$, we choose a $\lambda$-primary subspace of $\mathbb{C}[[z]]$, that is, a linear subspace $V_\lambda$ such that

$$(z - \lambda)^N \mathbb{C}[z] \subseteq V_\lambda$$

for some $N$. We suppose that $V_\lambda = \mathbb{C}[z]$ for all but finitely many $\lambda$. Let $V = \bigcap \lambda V_\lambda$ (such a space $V$ is called primary decomposable) and, finally, let

$$W = \prod \lambda (z - \lambda)^{-k_\lambda} V \subset \mathbb{C}(z),$$

where $k_\lambda$ is the codimension of $V_\lambda$ in $\mathbb{C}[z]$. By definition, $Gr_{\text{ad}}$ consists of all $W \subset \mathbb{C}(z)$ obtained in this way. The correspondence between points of $Gr_{\text{ad}}$ and pairs $(X, L)$ is a special case of the construction explained in [SW]. Given $W$, we obtain $(X, L)$ by setting

$$\mathcal{O}(X) := \{ f \in \mathbb{C}[z] : fW \subseteq W \};$$

and $W$ is then the rank 1 $\mathcal{O}(X)$-module corresponding to $\mathcal{L}$. Conversely, given $(X, L)$, we let $W$ be the space of global sections of $\mathcal{L}$, regarded as a subspace of $\mathbb{C}(z)$ by means of a certain distinguished rational trivialization of $\mathcal{L}$ (implicitly described above).

Proposition 5.2. This construction defines a bijection between $Gr_{\text{ad}}$ and the set of isomorphism classes of pairs $(X, L)$, where $X$ is a framed curve and $L$ is a maximal rank 1 torsion-free sheaf over $X$.

"Maximal" here means that the $\mathcal{O}(X)$-module corresponding to $\mathcal{L}$ is maximal in the sense of Note 5, Section 2.

Example 5.3. If $X_n$ is the curve defined by (4.1), then $\mathcal{O}(X_n)$ is 0-primary, and the corresponding point of $Gr_{\text{ad}}$ is $W_n = z^{-n}\mathcal{O}(X_n)$. More generally, let $\Lambda \subset \mathbb{N}$ be any (additive) semigroup obtained from $\mathbb{N}$ by deleting a finite number.

It is perhaps the most interesting Grassmannian not mentioned explicitly in [SW].
of positive integers, and let \( \mathcal{O}(X) \) be the subring of \( \mathbb{C}[z] \) spanned by \( \{z^i : i \in \Lambda\} \). Such a curve \( X \) is called a monomial curve; the corresponding point of \( \text{Gr}^{\text{ad}} \) is \( z^{-m}\mathcal{O}(X) \), where \( m \) is the number of elements of \( \mathbb{N} \setminus \Lambda \).

**Example 5.4.** If \( X \) has simple cusps at the (distinct) points \( \lambda_1, \ldots, \lambda_r \in \mathbb{C} \), then \( \mathcal{O}(X) \) consists of all polynomials whose first derivatives vanish at these points, and the corresponding point of \( \text{Gr}^{\text{ad}} \) is

\[
W = \prod_{i=1}^{r}(z - \lambda_i)^{-1} \mathcal{O}(X) .
\]

More generally, if in addition we choose \( \alpha_1, \ldots, \alpha_r \in \mathbb{C} \), then

\[
V = \{ f \in \mathbb{C}[z] : f'(\lambda_i) = \alpha_i f(\lambda_i) \text{ for } 1 \leq i \leq r \}
\]

is primary decomposable, and the corresponding point of \( \text{Gr}^{\text{ad}} \) is

\[
W = \prod_{i=1}^{r}(z - \lambda_i)^{-1} V .
\]

In the pairs \((X, L)\) here, the curve \( X \) is the same as before, and as we vary the parameters \( \alpha_i \) we get the various line bundles \( L \) over \( X \).

The rings \( D_{\mathcal{L}}(X) \) that interest us are easy to describe in terms of \( \text{Gr}^{\text{ad}} \). If \( W \in \text{Gr}^{\text{ad}} \), we define the ring of differential operators on \( W \) by

\[
D(W) := \{ D \in \mathbb{C}(z)[\partial] : D.W \subseteq W \}
\]

(as in Section 2, the dot denotes the natural action of differential operators on functions). Proposition 2.6 shows:

**Proposition 5.5.** Let \( W \in \text{Gr}^{\text{ad}} \) correspond to the pair \((X, L)\) as in Proposition 5.2. Then there is a natural identification

\[
D(W) \simeq D_{\mathcal{L}}(X) .
\]

It remains to discuss the symmetries of \( \text{Gr}^{\text{ad}} \). Some of them are fairly obvious. First, we have the commutative group \( \Gamma \) of the KP flows: it corresponds to the action \((X, L) \mapsto (X, L \otimes L)\) of the Jacobian (that is, the group of line bundles \( L \) over \( X \)) on the space of pairs \((X, L)\). If \( W^{\text{an}} \supset W \) is the space of analytic sections of \( L \), then \( \Gamma \) is the group of maps of the form \( W^{\text{an}} \mapsto e^{p(z)} W^{\text{an}} \), where \( p \) is a polynomial. Another fairly evident symmetry is the adjoint involution \( c \) defined by

\[
c(W) = \{ f \in \mathbb{C}(z) : \text{res}_\infty f(z)g(z)dz = 0 \text{ for all } g \in W \} .
\]

Like the KP flows, \( c \) is just the restriction to \( \text{Gr}^{\text{ad}} \) of a symmetry of the Grassmannian \( \text{Gr} \) of \( \mathbb{C}W \). A more elusive symmetry of \( \text{Gr}^{\text{ad}} \) is the bispectral involution \( b \) introduced in [W1]; it does not make sense on \( \text{Gr} \), and does not have a simple description in terms of the pairs \((X, L)\). It can be characterized by the formula

\[
\psi_b W(x, z) = \psi W(z, x) ,
\]

where \( \psi \) is the stationary Baker function of \( W \) (see, for example, [SW]). Let \( \varphi = bc \), and let \( G \) be the group of symmetries of \( \text{Gr}^{\text{ad}} \) generated by \( \Gamma \) and \( \varphi \). In view of Proposition 5.5, Theorems 4.4, 4.5 and 5.1 are all consequences of
Theorem 5.6. (i) Let $V, W \in \text{Gr}^{\text{ad}}$. Then $\mathcal{D}(V)$ and $\mathcal{D}(W)$ are isomorphic if and only if $V$ and $W$ belong to the same $G$-orbit in $\text{Gr}^{\text{ad}}$.

(ii) Each orbit contains exactly one of the points $W_n$ from Example 5.3.

Although it is possible to formulate a proof of Theorem 5.6 within our present context, the proof will appear more natural if we use two alternative descriptions of $\text{Gr}^{\text{ad}}$: we explain these in the next sections. First, in Section 6 we shall see that $\text{Gr}^{\text{ad}}$ can be identified with the space of ideals in the Weyl algebra $\mathcal{D}(A^1)$: the ring $\mathcal{D}(W)$ then becomes the endomorphism ring of the corresponding ideal, and $G$ becomes the automorphism group of the Weyl algebra. Part (i) of Theorem 5.6 then turns into a theorem of Stafford (see [St]). In Section 7 we explain how $\text{Gr}^{\text{ad}}$ decomposes into the union of certain finite-dimensional varieties $C_n$ that have a simple explicit description in terms of matrices; part (ii) of Theorem 5.6 then follows from the more precise assertion that these spaces $C_n$ are exactly the $G$-orbits. Since the action of $G$ also has a simple description in terms of matrices, part (ii) of the Theorem becomes a problem in linear algebra.

Notes. 1. The fact that the action of $\Gamma \subset G$ preserves the isomorphism class of $\mathcal{D}(W)$ is almost trivial. Indeed, if $g \in \Gamma$ is given (as above) by multiplication by $e^{p(z)}$, then $\mathcal{D}(gW) = e^{p(z)}\mathcal{D}(W)e^{-p(z)}$. It follows that $\mathcal{D}(gW)$ is even isomorphic to $\mathcal{D}(W)$ as a filtered algebra. Thus the (filtered) isomorphism class of $\mathcal{D}(L_X)$ depends only on the orbit of the Jacobian of $X$ in the space of rank 1 torsion-free sheaves; for example, if $\mathcal{L}$ is locally free, then $\mathcal{D}(\mathcal{L})$ is isomorphic to $\mathcal{D}(X)$. A direct proof that $\varphi$ preserves the isomorphism class of $\mathcal{D}(W)$ is also not too difficult: it follows from the facts that $\mathcal{D}(bW)$ and $\mathcal{D}(cW)$ are anti-isomorphic to $\mathcal{D}(W)$ (cf. [BW2], Sections 7 and 8). We regard the main assertions in Theorem 5.6 to be part (ii) and the “only if” statement in part (i).

2. The spaces $W_n$ are fixed by $b$, so $b$ induces an involutory anti-automorphism on each of the rings $\mathcal{D}(X_n)$. Thus Theorem 5.1 shows that the distinction between isomorphism and anti-isomorphism in the preceding note was immaterial.

3. If $\mathcal{L}$ is not locally free, then in general $\mathcal{D}(\mathcal{L})$ is not isomorphic to $\mathcal{D}(X)$ (see Example 8.4 below).

4. Details of the proof of Proposition 5.2 can be found in [W1]; see also [CH4], 1.4 and [E], p. 945.

6. The Cannings-Holland correspondence

In this section we explain a different realization of $\text{Gr}^{\text{ad}}$ (due to Cannings and Holland) as the space of ideals in the Weyl algebra. Let $A := \mathbb{C}[z, \partial]$ from now on denote the (first) Weyl algebra, and let $\mathcal{I}$ be the set of nonzero right ideals of $A$. Let $\mathcal{S}$ be the set of all linear subspaces of $\mathbb{C}[z]$. If $V, W \in \mathcal{S}$, (or, later, also if $V$ and $W$ are subspaces of $\mathbb{C}(z)$) we set

$$\mathcal{D}(V, W) := \{D \in \mathbb{C}(z)[\partial] : D.V \subseteq W\}.$$  

We define maps $\alpha : \mathcal{S} \to \mathcal{I}$ and $\gamma : \mathcal{I} \to \mathcal{S}$ as follows. If $V \in \mathcal{S}$, we set

$$\alpha(V) := \mathcal{D}(\mathbb{C}[z], V);$$

and if $I \in \mathcal{I}$, we set

$$\gamma(I) := \{D.\mathbb{C}[z] : D \in I\}.$$
Theorem 6.1. (i) We have $\alpha \gamma(I) = I$ if and only if $I \cap \mathbb{C}[z] \neq \{0\}$.
(ii) We have $\gamma \alpha(V) = V$ if and only if $V$ is primary decomposable.
(iii) The maps $\alpha$ and $\gamma$ define inverse bijections between the set of primary decomposable subspaces of $\mathbb{C}[z]$ and the set of right ideals of $A$ that intersect $\mathbb{C}[z]$ nontrivially.
(iv) If $V$ and $W$ are primary decomposable and $I := \alpha(V)$, $J := \alpha(W)$ are the corresponding (fractional) ideals, then

$$\mathcal{D}(V, W) = \{D \in \mathbb{C}(z)[d] : DI \subseteq J\} \simeq \text{Hom}_A(I, J).$$

Example 6.2. Let $I_n$ be the right ideal

$$I_n := z^{n+1}A + \sum_{r=1}^{n}(z^r - r)A.$$

The second generator kills $z, z^2, \ldots, z^n$, so we find that $\gamma(I_n) = O(X_n)$.

The assertions (iii) and (iv) in Theorem 6.1 follow at once from (i) and (ii). Now, not every right ideal of $A$ intersects $\mathbb{C}[z]$ nontrivially; but every ideal is isomorphic (as right $A$-module) to one with this property (see [St], Lemma 4.2). Furthermore, two such ideals $I, J$ are isomorphic if and only if $pI = qJ$ for some polynomials $p(z), q(z)$. On the other hand, two primary decomposable subspaces $V, W$ determine the same point of $\text{Gr}^{ad}$ if and only if $pV = qW$ for some polynomials $p(z), q(z)$; and the bijections $\alpha$ and $\gamma$ are clearly compatible with multiplication by polynomials. Let $\mathcal{R}$ denote the set of isomorphism classes of nonzero right ideals of $A$ (equivalently, of finitely generated torsion-free rank 1 right $A$-modules). Combining the remarks above with Theorem 6.1, we get the following.

Theorem 6.3. (i) The maps defined by the formulae (6.2) and (6.3) define inverse bijections

$$\alpha : \text{Gr}^{ad} \to \mathcal{R} \quad \text{and} \quad \gamma : \mathcal{R} \to \text{Gr}^{ad}.$$

(ii) For $V, W \in \text{Gr}^{ad}$, there is a natural identification

$$\mathcal{D}(V, W) \simeq \text{Hom}_A(\alpha(V), \alpha(W)).$$

As a special case of (ii), we see that if $W \in \text{Gr}^{ad}$ and $I := \alpha(W)$ is the corresponding ideal in $A$, then the algebra $\mathcal{D}(W) \equiv \mathcal{D}(W, W)$ is identified with $\text{End}_A(I)$. On the other hand, if $W$ corresponds to the pair $(X, \mathcal{L})$, then according to Proposition 6.5, $\mathcal{D}(W)$ is just the algebra $\mathcal{D}_\mathcal{L}(X)$ that interests us. In this way Theorem 6.3 translates any question about the algebras $\mathcal{D}_\mathcal{L}(X)$ into a question about ideals in the Weyl algebra. It remains to give the translation into these terms of the group $G$ of symmetries of $\text{Gr}^{ad}$. Note that if $\sigma$ is an automorphism of $A$ and $I$ is a finitely generated torsion-free rank 1 $A$-module, then $\sigma_*(I)$ is a module of the same type: thus the automorphism group $\text{Aut}(A)$ acts naturally on $\mathcal{R}$.

Theorem 6.4. Under the bijection $\alpha$, the action of the group $\Gamma$ of KP flows corresponds to the action on $\mathcal{R}$ induced by the automorphisms $D \mapsto e^{p(z)} D e^{-p(z)}$ of $A$; while the map $\varphi$ corresponds to the map on $\mathcal{R}$ induced by the formal Fourier transform $z \mapsto \partial, \partial \mapsto -z$ of $A$.

Now, if $\sigma$ is an automorphism of (any algebra) $A$, and $M$ is any $A$-module, then it is trivial that $\text{End}_A(M) \simeq \text{End}_A(\sigma_*(M))$. Thus Theorem 6.4 makes the “if” part of Theorem 5.3(i) transparent.
Notes. 1. According to Dixmier (see [D]), the automorphisms mentioned in Theorem 6.4 generate the full automorphism group of $A$; thus we may identify our symmetry group $G$ with $\text{Aut}(A)$.

2. There are two routes available to prove the “only if” part of Theorem 5.6(i). If we use Dixmier’s theorem, we can simply note that it translates into a known theorem of Stafford (see [St]): if $I$ and $J$ are two ideal classes of $A$, then their endomorphism rings are isomorphic (if and only if) if $I$ and $J$ belong to the same orbit of $\text{Aut}(A)$ in $R$. Alternatively, after we have classified the orbits, this fact will follow from Theorem 4.5 (whose proof in [LM] does not use Stafford’s theorem, nor Dixmier’s).

3. To get an idea of the depth of Stafford’s theorem, let us give a proof (following [CH3]) of a crucial special case: if $I$ is an ideal of $A$ whose endomorphism ring is isomorphic to $\text{End}_A(A) = A$, then $I \simeq A$. Let $(X, L)$ be the pair corresponding to $I$; then $\mathcal{D}_L(X)$ is isomorphic to $A$, hence $\mathcal{O}(X)$ is isomorphic to a mad subalgebra of $A$. Another (nontrivial) theorem of Dixmier (see [D]) says that all the mad subalgebras of $A$ are isomorphic to $\mathbb{C}[z]$; hence $X \simeq \mathbb{A}^1$ and $L$ is the trivial line bundle (because this is the only rank 1 torsion-free sheaf over $\mathbb{A}^1$). According to Theorem 6.3, it follows that $I \simeq A$. The general case of Stafford’s theorem is a relatively formal consequence of this special case (see [St], Corollary 3.2).

4. If we introduce the category $\mathfrak{P}$ with objects the primary decomposable subspaces of $\mathbb{C}[z]$ and morphisms $\mathcal{D}(V, W)$, then we could summarize Theorem 6.1 by saying that we have an equivalence of categories between $\mathfrak{P}$ and the category of ideals in $A$ (regarded as a full subcategory of the category of right $A$-modules).

5. Theorems 6.1 and 6.3 remain true (mutatis mutandis) if we replace the Weyl algebra by the ring of differential operators on any smooth affine curve (see [CH1]). Using this fact, we can sketch a proof of Theorem 6.3. Suppose that $X$ and $Y$ are smooth affine curves such that $\mathcal{D}_L(X)$ is isomorphic to $\mathcal{O}(X)$ for some ideal in $A$, and hence to $\mathcal{D}(V)$ for some primary decomposable subspace $V$ of $\mathcal{O}(X)$. This in turn is isomorphic to some ring $\mathcal{D}_L(X')$, where $X'$ is a curve with bijective normalization $X \rightarrow X'$. If $Y$ is isomorphic to $\mathbb{A}^1$, then Theorem 4.2 shows that $\mathcal{D}(Y)$ is Morita equivalent to $\mathcal{D}(V)$, and hence to $\mathcal{D}(Y)$ for some primary decomposable subspace $V$ of $\mathcal{O}(X)$. This in turn is isomorphic to some ring $\mathcal{D}_L(X')$, where $X'$ is a curve with bijective normalization $X \rightarrow X'$. If $Y$ is not isomorphic to $\mathbb{A}^1$, then $\mathcal{D}(Y)$ has only one mad subalgebra. The same is therefore true of $\mathcal{D}_L(X')$; extracting these mad subalgebras gives $\mathcal{O}(Y) \simeq \mathcal{O}(X')$, hence $Y \simeq X'$. Since $Y$ is smooth, this implies $X \simeq X'$, hence $Y \simeq X$. Finally, if $Y$ is isomorphic to $\mathbb{A}^1$, then $\mathcal{D}(Y)$, and hence also $\mathcal{D}_L(X')$, has more than one mad subalgebra, so Lemma 4.1 implies that $X \simeq \mathbb{A}^1$.

6. Theorem 6.1 is proved in [CH1]; Theorem 6.3 is proved in [BW2].

7. A different view of the construction of Cannings and Holland, and some further generalizations, can be found in [BCK2].

7. The Calogero-Moser spaces

Our third realization of $\text{Gr}^{\text{ad}}$ involves the Calogero-Moser spaces $\mathcal{C}_n$. For each $n \geq 0$, let $\tilde{\mathcal{C}}_n$ be the space of pairs $(X, Y)$ of complex $n \times n$ matrices such that

$$[X, Y] + I \text{ has rank 1,}$$
and let $\mathcal{C}_n := \tilde{\mathcal{C}}_n / \text{GL}(n, \mathbb{C})$, where the action of $g \in \text{GL}(n, \mathbb{C})$ is by simultaneous conjugation: $(X, Y) \mapsto (gXg^{-1}, gYg^{-1})$. One can show that $\mathcal{C}_n$ is an smooth irreducible affine variety of dimension $2n$ ($\mathcal{C}_0$ is supposed to be a point).

**Theorem 7.1.** There is a natural bijection 

$$\beta : \mathcal{C} := \bigsqcup_{n \geq 0} \mathcal{C}_n \to \text{Gr}^{\text{ad}}$$

such that

(i) the action of $\Gamma$ on $\text{Gr}^{\text{ad}}$ corresponds to the maps $(X, Y) \mapsto (X + p'(Y), Y)$ on $\mathcal{C}_n$;

(ii) the action of $\varphi$ on $\text{Gr}^{\text{ad}}$ corresponds to the map $(X, Y) \mapsto (-Y, X)$ on $\mathcal{C}_n$;

(iii) the action of the group $G$ on each $\mathcal{C}_n$ is transitive.

It follows from part (iii) of this Theorem that the spaces $\beta(\mathcal{C}_n)$ are the orbits of $G$ in $\text{Gr}^{\text{ad}}$. To complete the proof of Theorem 5.6 we have only to check that $\beta^{-1}(W_n)$ belongs to $\mathcal{C}_n$: that is done in Example 8.2 below.

The decomposition of $\text{Gr}^{\text{ad}}$ in Theorem 7.1 was originally obtained using ideas from the theory of integrable systems (see [W2]). Here we sketch a different method. In view of Theorem 6.3, it is enough to see why the space $\mathcal{R}$ of ideals in the Weyl algebra should decompose into the finite-dimensional spaces $\mathcal{C}_n$. That can be understood by analogy with the corresponding commutative problem, namely, to describe the space $\mathcal{R}_0$ of isomorphism classes of ideals in $A_0 := \mathbb{C}[x, y]$. This problem is easy, because each ideal class in $A_0$ has a unique representative of finite codimension; hence $\mathcal{R}_0$ decomposes into the disjoint union of the point Hilbert schemes $\text{Hilb}_n(\mathbb{A}^2)$ (that is, the spaces of ideals of codimension $n$) for $n \geq 0$. It is elementary that $\text{Hilb}_n(\mathbb{A}^2)$ can be identified with the space of pairs $(X, Y)$ of commuting $n \times n$ matrices possessing a cyclic vector (see [N], 1.2); thus $\text{Hilb}_n(\mathbb{A}^2)$ is the commutative analogue of the Calogero-Moser space $\mathcal{C}_n$. Because the Weyl algebra has no nontrivial ideals of finite codimension, it is not immediately clear how to adapt this discussion to the noncommutative case; however, there is a less elementary point of view which generalizes more easily. We may regard an ideal of $A_0$ as a rank 1 torsion-free sheaf over $\mathbb{A}^2$; it has a unique extension to a torsion-free sheaf over the projective plane $\mathbb{P}^2$ trivial over the line at infinity. The classification of ideals by pairs of matrices can then be regarded as (trivial) special case of Barth’s classification of framed bundles (of any rank) over $\mathbb{P}^2$ (see [N], Ch. 2). In a similar way, an ideal of the Weyl algebra determines a rank 1 torsion-free sheaf over a suitably defined quantum projective plane $\mathbb{P}^2_q$; these can then be classified much as in the commutative case.

**Notes.** 1. Let us try to give something of the flavour of the noncommutative projective geometry needed to carry out the plan sketched above (see, for example [A], [AZ] for more details). Let $X \subseteq \mathbb{P}^N$ be a projective variety, and let $A = \oplus_{k \geq 0} A_k$ be its (graded) homogeneous coordinate ring. To any quasicoherent sheaf $\mathcal{M}$ over $X$ we can assign the graded $A$-module

$$M := \bigoplus_{k \in \mathbb{Z}} H^0(X, \mathcal{M}(k)).$$

A theorem of Serre (see [S]) states that this defines an equivalence between the category of quasicoherent sheaves over $X$ and a certain quotient of the category of graded $A$-modules (we have to divide out by the so-called torsion modules, in
which each element is killed by some $A_k$). Thus many results about projective varieties can be formulated in a purely algebraic way, in terms of graded $A$-modules; in this form the theory makes sense also for a noncommutative graded ring $A$. The coordinate ring of the space $\mathbb{P}^2_{\mathbb{Q}}$ referred to above is the ring of noncommutative polynomials in three variables $x, y, z$ of degree 1, where $z$ commutes with everything, but $[x, y] = z^2$. It turns out that the homological properties of this ring are similar to those of the commutative graded ring $\mathbb{C}[x, y, z]$; in particular, the classification of bundles (of any rank) over $\mathbb{P}^2_{\mathbb{Q}}$ is similar to that of bundles over $\mathbb{P}^2_{\mathbb{Q}}$ (see [KKO]).

2. The idea of using $\mathbb{P}^2_{\mathbb{Q}}$ to classify the ideals in the Weyl algebra is due to L. Le Bruyn (see [LeB]). However, Le Bruyn’s chosen extension of an ideal in $A$ to a sheaf over $\mathbb{P}^2_{\mathbb{Q}}$ was in general not trivial over the line at infinity, so he did not obtain the decomposition of $R$ into the Calogero-Moser spaces. That was done in [BW3] and (in a different way) in [BGK].

3. The connection between the spaces $\text{Hilb}_n(\mathbb{A}^2)$ and $C_n$ is actually much closer than we have indicated: $\text{Hilb}_n(\mathbb{A}^2)$ is a hyperkähler variety, and $C_n$ is obtained by deforming the complex structure of $\text{Hilb}_n(\mathbb{A}^2)$ within the hyperkähler family. See [N], Ch. 3, especially 3.45.

4. The assertions (i) and (ii) in Theorem 7.1 are proved in [BW2] (using the original construction of $\beta$), and in [BW3] (using the construction sketched above). The fact that the two constructions agree is also proved in [BW3].

5. Parts (i) and (ii) of Theorem 7.1 reduce the proof of part (iii) (transitivity of the $G$-action) to an exercise in linear algebra. Unfortunately, the exercise seems to be quite difficult, and the published solution in [BW2] strays outside elementary linear algebra at one point (see Lemma 10.3 in [BW2]). P. Etingof has kindly pointed out to us that transitivity also follows easily from the fact that the functions $(X, Y) \mapsto \text{tr}(X^k)$ and $(X, Y) \mapsto \text{tr}(Y^k)$ generate $O(C_n)$ as a Poisson algebra (see [EG], 11.33).

6. In [BW3], Section 5 we have given an elementary construction of the map $R \to C$, in a similar spirit to the elementary treatment of the commutative case. It turns out that the inverse map $C \to R$ can also be written down explicitly, as follows. Let $(X, Y) \in C_n$, and choose column and row vectors $v, w$ such that $[X, Y] + I = vw$. Define

$$
\kappa := 1 - w(Y - zI)^{-1}(X - \partial I)^{-1}v
$$

(thus $\kappa$ belongs to the quotient field of the Weyl algebra $A$). Then the (fractional) right ideal

$$
\text{det}(Y - zI) A + \kappa \text{det}(X - \partial I) A \subset \mathbb{C}(z)[\partial]
$$

represents the class in $R$ corresponding to $(X, Y)$. Using these formulae, it is possible to give a completely elementary proof that $R$ decomposes into the spaces $C_n$. More details will appear elsewhere.

8. The invariant $n$

Theorem 7.1 assigns to each $W \in \text{Gr}^{ad}$ a non-negative integer $n$, namely, the index of the “stratum” $C_n$ containing $\beta^{-1}(W)$. Using Proposition 5.2 and

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2We get this formula by combining Remark 5.4 in [BW3] with formula (3.5) in [W2].
Theorem 6.3, we may equally well regard \( n \) as an invariant of a pair \((X, \mathcal{L})\),
or of an ideal (class) in the Weyl algebra \( A \). In this section we discuss various
descriptions of this invariant. The first two begin with an ideal class in \( A \).

\( n \) as a Chern class. We return to the quantum projective plane \( \mathbb{P}^2_q \) explained at
the end of Section 7. Let \( M \) be an ideal class of \( A \), and let \( \mathcal{M} \) denote its unique
extension to a sheaf over \( \mathbb{P}^2_q \) trivial over the line at infinity. Then we claim that
\[
(8.1) \quad n = \dim_C H^1(\mathbb{P}^2_q, \mathcal{M}(-1)).
\]
To see that, we need to give more details of the construction of the map \( R \to C \).
Recall that the homogeneous coordinate ring of \( \mathbb{P}^2_q \) has three generators \( x, y, z \).
It turns out that multiplication by \( z \) induces an isomorphism
\[
H^1(\mathbb{P}^2_q, \mathcal{M}(-2)) \to H^1(\mathbb{P}^2_q, \mathcal{M}(-1) := V).
\]
If we use this isomorphism to identify these spaces, then multiplication by \( x \) and
\( y \) gives us a pair \((X, Y)\) of endomorphisms of \( V \): this is the point of \( C \) associated
with \( M \). Obviously, the size of the matrices \((X, Y)\) is given by \((8.1)\).

Note. By analogy with the commutative case (see [N], Ch. 2), we would like to
interpret \( n \) as the second Chern class \( c_2(M) \). However, at the time of writing,
Chern classes have not yet been discussed in noncommutative projective geometry.

\( n \) as a codimension. Again, let \( M \) be an ideal of \( A \). By [St], Lemma 4.2, we
may suppose that \( M \) intersects \( C[z] \subset A \) nontrivially; let \( I \) be the ideal in \( C[z] \)
generated by the leading coefficients of the operators in \( M \), and let \( p(z) \) be a
generator of \( I \). Then \( p^{-1}M \subset C(z)[\partial] \) is a fractional ideal representing the class
of \( M \). Define a map \( D \to D_+ \) from \( C(z)[\partial] \) to \( A \) by
\[
\left( \sum_i f_i \partial^i \right)_+ = \sum_i (f_i)_+ \partial^i;
\]
here \( f_+ \) denotes the polynomial part of a rational function \( f \) (that is, the polyno-
mial such that \( f - f_+ \) vanishes at infinity). Then we claim that
\( n \) is the codimension of \((p^{-1}M)_+ \) in \( A \).

A proof can be found in [BW3], Section 6, where it is shown that the quotient space
\( A/(p^{-1}M)_+ \) can be identified with the (Čech) cohomology group on the right of
\([\text{S}1]\).

Note. The special representative for an ideal class that we used in this subsection
is the same one as is given by the formula \([\text{S}1]\). It is the unique representative of
the form \( D(C[z], W) \) with \( W \in \text{Gr}^{\text{ad}} \) (cf. Theorem [03]).

The differential genus of a framed curve. The following characterization of
\( n \) was one of the main results of [W2].

**Theorem 8.1.** Let \( W \in \text{Gr}^{\text{ad}} \). Then the integer \( n \) that we have associated to \( W \)
is equal to the dimension of the open cell in \( \text{Gr}^{\text{ad}} \) containing \( W \).

This theorem leads easily to a simple formula for calculating \( n \) in concrete
examples (cf. [PS], 7.4). Recall from Section [3] that \( W \) is constructed from a
family of \( \lambda \)-primary subspaces \( V_\lambda \subseteq C[z] \) (one for each \( \lambda \in C \), and almost all of
them equal to \( C[z] \)). In terms of these \( V_\lambda \), we can calculate \( n \) as follows. First,
we have \( n = \sum \lambda n_\lambda \), where \( n_\lambda \) depends only on \( V_\lambda \) (and is zero if \( V_\lambda = \mathbb{C}[z] \)). To find \( n_\lambda \), let

\[
(8.2) \quad r_0 < r_1 < r_2 < \ldots
\]

be the numbers \( r \) such that \( V_\lambda \) contains a polynomial that vanishes exactly to order \( r \) at \( \lambda \). For large \( i \) we have \( r_i = g + i \), where \( g \) is the number of “gaps” (non-negative integers that do not occur) in the sequence \( (8.2) \). Then we have

\[
(8.3) \quad n_\lambda = \sum_{i \geq 0} (g + i - r_i) .
\]

**Example 8.2.** For the 0-primary space \( V := \mathcal{O}(X_n) \) defined by \((4.1)\), the sequence \((8.2)\) is

\[
0 < n + 1 < n + 2 < \ldots ,
\]

whence \( g = n \), and the right hand side of \((8.3)\) is equal to \( n \).

This calculation completes the proof of Theorem 5.6(ii), and shows that we can identify the number \( n \) associated with a pair \((X, L)\) with the differential genus \( d_L(X) \) introduced in Section 5.

**Example 8.3.** If \( Y_r \) is the curve with coordinate ring \( \mathcal{O}(Y_r) := \mathbb{C}[z^2, z^{2r+1}] \), then again \( \mathcal{O}(Y_r) \) is 0-primary, and the sequence \((8.2)\) is

\[
0 < 2 < 4 < \ldots < 2r < 2r + 1 < \ldots .
\]

Hence \( g = r \), and \( d(Y_r) = r + (r - 1) + \ldots + 2 + 1 = r(r+1)/2 \).

In particular, \( d(Y_2) = 3 \) so \( Y_2 \) is differentially isomorphic to \( X_3 \), in agreement with G. Letzter (see [L]).

**Example 8.4.** Here is the simplest example to show that in general \( d_L(X) \) depends on \( L \), not just on \( X \). Let \( V \) be the 0-primary space spanned by \( \{z^i : i \neq 2, 3\} \). Then the the sequence \((8.2)\) is

\[
0 < 1 < 4 < 5 < \ldots ,
\]

whence \( n = 4 \). Clearly, \( V \) is a maximal module over the ring \( \mathcal{O}(X_3) \), and thus corresponds to a maximal torsion-free (but not locally free) sheaf \( L \) over \( X_3 \). For this sheaf \( L \) we therefore have \( d_L(X_3) = 4 \), and the ring \( \mathcal{D}_L(X_3) \) is isomorphic to \( \mathcal{D}(X_4) \).

**The Letzter-Makar-Limanov invariant.** Next, we describe the invariant originally used in [LM] to distinguish the rings \( \mathcal{D}(X_n) \). We return temporarily to the case of any affine curve \( X \), with normalization \( \hat{X} \) and function field \( K \); as usual (see Proposition 2.4), we view \( \mathcal{D}(X) \) and \( \mathcal{D}(\hat{X}) \) as subalgebras of \( \mathcal{D}(K) \). In general, \( \mathcal{D}(X) \) is not contained in \( \mathcal{D}(\hat{X}) \); however, the associated graded algebra \( \text{gr} \mathcal{D}(X) \) is always contained in \( \text{gr} \mathcal{D}(\hat{X}) \) (see [SS], 3.11). In the case that most concerns us when \( \hat{X} = \mathbb{A}^1 \), this simply means that the leading coefficient of each operator in \( \mathcal{D}(X) \) is a polynomial (although the other coefficients may be rational functions, as we saw in Example 2.5). Continuing Theorem 6.2, we have

**Theorem 8.5.** Each of the conditions in Theorem 6.2 is equivalent to:

\[
\text{gr} \mathcal{D}(X) \text{ has finite codimension in } \text{gr} \mathcal{D}(\hat{X}) .
\]
In our case, when \( \tilde{X} = \mathbb{A}^1 \) and \( X \) is a framed curve, \( \text{gr}D(X) \) is a subalgebra of finite codimension in \( \mathbb{C}[z, \zeta] \); we call its codimension the Letzter-Makar-Limanov invariant of \( X \), and denote it by \( LM(X) \). The definition of \( LM(X) \) uses the standard filtration on \( D(X) \); nevertheless, in \([LM]\) it is proved that it depends only on the isomorphism class of the algebra \( D(X) \); that is, if \( X \) and \( Y \) are differentially isomorphic framed curves, then \( LM(X) = LM(Y) \). On the other hand, it is not hard to calculate that \( LM(X_n) = 2n \) (see \([LM]\), Section 5). Combined with Theorem 5.6 that gives

**Theorem 8.6.** Let \( X \) be any framed curve. Then \( 2d(X) = LM(X) \).

**Notes.** 1. Theorem 8.5 is proved (though not explicitly stated) in \([SS]\), 3.12.
2. In \([LM]\) the rings \( D_L(X) \) (for \( L \neq O_X \)) are not considered; however, it is not hard to extend the discussion to include that case. Thus we can define the invariant \( LM(D(W)) \) for any \( W \in \text{Gr}^{ad} \), and Theorem 5.6 shows that it is equal to \( 2n \).
3. It is possible to prove directly (that is, without using Theorem 5.6) that \( LM(D) \) is twice the number \( n \) defined by (8.1). The interested reader may see \([B]\).

All our descriptions of \( n \) so far have been specific to our particular situation. It is natural to ask whether \( n \) is a special case of some general invariant of rings that is able to distinguish between different Morita equivalent domains. Our last two subsections are attempts in that direction.

**Pic and Aut.** Let \( D \) momentarily be any domain (associative algebra without zero divisors) over \( \mathbb{C} \). The following idea for obtaining subtle invariants of the isomorphism class of \( D \) is due to Stafford (see \([St]\)). Consider the group\(^3\) \( \text{Pic}(D) \) of all Morita equivalences of \( D \) with itself, that is, of all self-equivalences of the category \( \text{Mod}-D \) of (say right) \( D \)-modules. Each such equivalence is given by tensoring with a suitable \( D \)-bimodule, so we may also think of \( \text{Pic}(D) \) as the group of all invertible \( D \)-bimodules. Each automorphism of \( D \) induces a self-equivalence of \( \text{Mod}-D \), so there is a natural map

\[
\omega : \text{Aut}(D) \to \text{Pic}(D) .
\]

Although the group \( \text{Pic}(D) \) is a Morita invariant of \( D \), the automorphism group and the map \( \omega \) are not.

We return to our case, where \( D \) is one of the algebras \( \text{End}_A(I) \) (or \( D_L(X) \)). In general, the kernel of \( \omega \) consists of the inner automorphisms of \( D \); in our case these are trivial, so \( \omega \) is injective. For the Weyl algebra \( A \), Stafford showed that \( \omega \) is an isomorphism. We thus have a natural inclusion

\[
\text{Aut}(D) \hookrightarrow \text{Pic}(D) \simeq \text{Pic}(A) = \text{Aut}(A)
\]

(the isomorphism from \( \text{Pic}(D) \) to \( \text{Pic}(A) \) is defined by tensoring with the \( D-A \)-bimodule \( I \)). Recalling that the group \( \text{Aut}(A) \) acts transitively on \( C_n \), one can calculate that the isotropy group of the point in \( C_n \) corresponding to \( I \) is exactly this subgroup \( \text{Aut}(D) \). It follows that we have a natural bijection

\[
C_n \simeq \text{Pic}(D)/\text{Aut}(D) ,
\]

\(^3\)More properly, we should write \( \text{Pic}_C(D) \) to indicate that we consider only equivalences that commute with multiplication by scalars. For a similar reason, we should write \( \text{Aut}_C \) too.
so it is tempting to claim that our invariant $n$ is given by

$$2n = \dim_C \text{Pic}(\mathcal{D})/\text{Aut}(\mathcal{D}).$$

The flaw in this is that the structure of algebraic variety on the quotient “space” in (8.5) has been imposed \textit{a posteriori}, and has not been extracted intrinsically from the algebra $\mathcal{D}$.

\textbf{Note.} In view of the above, we may hope that there should be (at least for some algebras $\mathcal{D}$) a natural structure of (infinite-dimensional) algebraic group on $\text{Pic}(\mathcal{D})$ for which $\text{Aut}(\mathcal{D})$ would be a closed subgroup. In our case, we can identify $\text{Pic}(\mathcal{D})$ with $\text{Aut}(\mathcal{A})$, which does indeed have a natural structure of algebraic group; however, for this structure $\text{Aut}(\mathcal{D})$ is not a closed subgroup (see [BW2], Section 11 for more details).

\textbf{Mad subalgebras.} The idea behind our final description of $n$ is very simple, namely: $n$ should measure the “number” of mad subalgebras of $\mathcal{D}(X)$. Let us formulate a precise statement. For each $W \in \text{Gr}^{\text{ad}}$ with invariant $n$, we may choose an isomorphism $\phi: \mathcal{D}(W) \to \mathcal{D}(X^n)$. Since $\mathcal{D}^0(W)$ is a mad subalgebra of $\mathcal{D}(W)$, $B := \phi(\mathcal{D}^0(W))$ is a mad subalgebra of $\mathcal{D}(X^n)$. Furthermore, $\phi$ extends to an isomorphism of quotient fields, in particular, it maps $z \in \mathbb{C}(z)[\partial]$ to some element $u := \phi(z)$ in the quotient field of $B$. Clearly, $\mathbb{C}[u]$ is the integral closure of $B$. According to [LM], the integral closure $\mathcal{B}$ of any mad subalgebra $B$ is isomorphic to $\mathbb{C}[u]$: we shall call a choice of generator for $\mathcal{B}$ a framing of $B$. Thus the above isomorphism $\phi$ gives us a framed mad subalgebra $(B, u)$ of $\mathcal{D}(X^n)$. Any two choices of $\phi$ differ only by an automorphism of $\mathcal{D}(X^n)$, so the class (modulo the action of $\text{Aut}(\mathcal{D}(X^n))$ of the framed mad subalgebra we have obtained depends only on $W$. Moreover (cf. Section 5, Note 1), if we replace $W$ by $gW$, where $g$ belongs to the group $\Gamma$ of KP flows, then conjugation by $g$ defines an isomorphism of $\mathcal{D}(gW)$ with $\mathcal{D}(W)$ which is the identity on $\mathcal{D}^0$, so the isomorphism $\mathcal{D}(gW) \to \mathcal{D}(W) \xrightarrow{\phi} \mathcal{D}(X^n)$

defines the same framed mad subalgebra as $\phi$. It follows that we have constructed a well-defined map

$$C_n/\Gamma \to \{\text{classes of framed mad subalgebras in } \mathcal{D}(X^n)\}.$$ 

\textbf{Theorem 8.7.} The map (8.6) is a set-theoretical bijection.

We will explain the proof elsewhere. Since the (categorical) quotient $C_n/\Gamma$ is $n$-dimensional, we should like to interpret $n$ as the dimension of the “space” of (classes of framed) mad subalgebras of $\mathcal{D}(X^n)$. However, the word “space” here is open to even more serious objections than in the preceding subsection.

\textbf{Notes.} 1. In the definition of a framed mad subalgebra $(B, u)$ we did not assume \textit{a priori} that the curve Spec $B$ was free of multiple points (indeed, that was not proved in [LM]). This momentary inconsistency of terminology is resolved by Theorem 8.7, which asserts (\textit{inter alia}) that every mad subalgebra $B$ arises from the construction described above; in particular, that Spec $B$ is a framed curve as defined earlier.
2. In the case \( n = 0 \), the left hand side of (8.6) is a point, so Theorem 8.7 becomes a well known result of Dixmier: in the Weyl algebra there is only one class of mad subalgebras (see [D]).

9. Higher dimensions

The examples of Levasseur, Smith and Stafford. Let \( \mathfrak{g} \) be a simple complex Lie algebra, and let \( O \) be the closure of the minimal nilpotent orbit in \( \mathfrak{g} \). Let \( \mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+ \) be a triangular decomposition of \( \mathfrak{g} \); then \( O \cap \mathfrak{n}_+ \) breaks up into several irreducible components \( X_i \). In [LSS] it is shown that in some cases the ring \( \mathcal{D}(X_i) \) can be identified with \( U(\mathfrak{g})/J \), where \( J \) is a certain distinguished completely prime primitive ideal of \( U(\mathfrak{g}) \) (the Joseph ideal). The examples of differential isomorphism arise in the case \( \mathfrak{g} = \mathfrak{so}(2n,C) \) (with \( n \geq 5 \)), because in that case there are two nonisomorphic components \( X_1 \) and \( X_2 \) of this kind. They can be described quite explicitly: \( X_1 \) is the quadric cone \( \sum z_i^2 = 0 \) in \( C^{2n-2} \), and \( X_2 \) is the space of skew-symmetric \( n \times n \) matrices of rank \( \leq 2 \). In contrast to what we saw for curves, these spaces \( X_1 \) and \( X_2 \) are quite different topologically.

Morita equivalence. There are several papers that study differential equivalence in dimension \( > 1 \). In view of Theorem 3.2, attention has focused on the question of when a variety \( X \) is differentially equivalent to its normalization \( \tilde{X} \). Of course, in dimension \( > 1 \) the normalization is not necessarily smooth: in [J1] there are examples of differential equivalence in which \( \tilde{X} \) is not smooth (they can be thought of as generalizations of the monomial curves of Example 5.3). Another point that does not arise for curves is that the condition that \( X \) be Cohen-Macaulay plays an important role (we recall that every curve is Cohen-Macaulay). For example, a theorem of Van den Bergh states that if \( \mathcal{D}(X) \) is simple, then \( X \) must be Cohen-Macaulay (see [VD], Theorem 6.2.5). For varieties with smooth normalization, there are good generalizations of at least some parts of Theorem 3.2. For example, piecing together various results scattered through the literature, we can get the following.

Theorem 9.1. Let \( X \) be an (irreducible) affine variety with smooth normalization \( \tilde{X} \). Then the following are equivalent.

1. The normalization map \( \pi : \tilde{X} \rightarrow X \) is bijective and \( X \) is Cohen-Macaulay.
2. The algebras \( \mathcal{D}(\tilde{X}) \) and \( \mathcal{D}(X) \) are Morita equivalent.
3. The ring \( \mathcal{D}(X) \) is simple.

Beautiful examples are provided by the varieties of quasi-invariants of finite reflection groups (see [BEG], [BC]): here \( \tilde{X} \) is the affine space \( \mathbb{A}^m \), so these examples are perhaps the natural higher-dimensional generalizations of our framed curves.

References for the proof of Theorem 9.1. For the implication “(1) \( \Rightarrow \) (2)” in Theorem 9.1 we are relying on the recent preprint [BN] (at least in dimension \( > 2 \); for surfaces it was proved earlier in [HS]). For the rest, the implication “(2) \( \Rightarrow \) (3)” is trivial, and the fact that \( \mathcal{D}(X) \) simple implies \( X \) Cohen-Macaulay is the theorem of Van den Bergh mentioned above. The only remaining assertion in Theorem 9.1 is that if \( \mathcal{D}(X) \) is simple then \( \pi \) is bijective. Suppose \( \mathcal{D}(X) \) is simple. Then by [SS], 3.3, \( \mathcal{D}(X) \) is isomorphic to the endomorphism ring of the right \( \mathcal{D}(\tilde{X}) \)-module \( P \).
defined by (3.1); the dual basis lemma then implies that $P$ is a projective $D(X)$-module. It now follows from [CS], Theorem 3.1 that $D(X)$ is a maximal order, then from [CS], Corollary 3.4 that $\pi$ is bijective.

**Non-affine varieties.** In this paper we have considered only affine varieties. However, the problem of differential equivalence has an obvious generalization to arbitrary (for example, projective) varieties $X$. Namely: on $X$ we have the sheaf $D_X$ of differential operators (whose sections over an affine open set $Spec A$ are the ring $D(A)$), and given two varieties $X$ and $Y$, we can ask whether the categories of $O$-quasicoherent sheaves of modules over $D_X$ and $D_Y$ are equivalent. For $X$ affine, the global section functor gives an equivalence between the categories of $D_X$-modules and of $D(X)$-modules, so we recover our original problem. The available evidence (namely [SS] and [BN]) suggests that results about the affine case carry over to this more general situation.

The question of differential isomorphism does not make sense for sheaves; however, we can always consider the ring $D(X)$ of global sections of $D_X$, and ask when $D(X)$ and $D(Y)$ are isomorphic. In general, $D(X)$ may be disappointingly small: for example, if $X$ is a smooth projective curve of genus $> 1$, then we have no global vector fields, so $D(X) = C$. Probably the question is a sensible one only if $X$ is close to being a $D$-affine variety (for which the global section functor still gives an equivalence between $D_X$-modules and $D(X)$-modules). As far as we know, there are not yet any papers on this subject: however, the question of Morita equivalence of rings $D(X)$ has been studied in [HoS] (where $X = \mathbb{P}^1$, this being the only $D$-affine smooth projective curve); and in [J2] (where $X$ is a weighted projective space). We should like to state one of the results of [HoS], since it is very close to our framed curves. Let $X$ be a “framed projective curve”, that is, we have a bijective normalization map $\mathbb{P}^1 \to X$. Then Holland and Stafford show that the rings $D(X)$ (for $X$ singular) are all Morita equivalent to each other, but not to $D(\mathbb{P}^1)$. A key point is that the although $\mathbb{P}^1$ is $D$-affine, the singular curves $X$ are not.

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**References**

[A] M. Artin, *Geometry of quantum planes*, in Azumaya Algebras, Actions and Groups, Contemporary Math. 124, Amer. Math. Soc., Providence, 1992, pp. 1–15.

[AZ] M. Artin and J. Zhang, *Noncommutative projective schemes*, Adv. in Math. 109 (1994), 228–287.

[BGK1] V. Baranovsky, V. Ginzburg and A. Kuznetsov, *Quiver varieties and a noncommutative $\mathbb{P}^2$*, Compositio Math. 134 (2002), 283–318.

[BGK2] V. Baranovsky, V. Ginzburg and A. Kuznetsov, *Wilson’s Grassmannian and a noncommutative quadric*, Internat. Math. Res. Notices 21 (2003), 1155–1197.

[Be] J. Becker, *Higher derivations and integral closure*, Amer. J. Math. 100(3) (1978), 495–521.

[BN] D. Ben-Zvi and T. Nevins, *Cusps and D-modules*, arXiv:math.AG/0212094.

[B] Yu. Berest, *A remark on Letzter-Makar-Limanov invariants*, in Proceedings of ICRA X, Toronto, August 2002 (to appear), arXiv:math.AG/0304154.

[BC] Yu. Berest and O. Chalykh, *Quasi-invariants of complex reflection groups*, in preparation.

[BEG] Yu. Berest, P. Etingof and V. Ginzburg, *Cherednik algebras and differential operators on quasi-invariants*, Duke Math. J. (to appear), arXiv:math.QA/0111005.
[BW1] Yu. Berest and G. Wilson, Classification of rings of differential operators on affine curves, Internat. Math. Res. Notices 2 (1999), 105–109.

[BW2] Yu. Berest and G. Wilson, Automorphisms and ideals of the Weyl algebra, Math. Ann. 318(1) (2000), 127–147.

[BW3] Yu. Berest and G. Wilson, Ideal classes of the Weyl algebra and noncommutative projective geometry (with an Appendix by M. Van den Bergh), Internat. Math. Res. Notices 26 (2002), 1347–1396.

[BGG] J. N. Bernstein, I. M. Gel’fand and S. I. Gel’fand, Differential operators on the cubic cone, Russian Math. Surveys 27 (1972), 169–174.

[CH1] R. C. Cannings and M. P. Holland, Right ideals of rings of differential operators, J. Algebra 167 (1994), 116–141.

[CH2] R. C. Cannings and M. P. Holland, Differential operators, n-branch curve singularities and the n-subspace problem, Trans. Amer. Math. Soc. 347 (1995), 1439–1451.

[CH3] R. C. Cannings and M. P. Holland, Etale covers, bimodules and differential operators, Math. Z. 216(2) (1994), 179–194.

[CH4] R. C. Cannings and M. P. Holland, Limits of compactified Jacobians and D-modules on smooth projective curves, Adv. Math. 135(2) (1998), 287–302.

[CS] M. Chamarie and J. T. Stafford, When rings of differential operators are maximal orders, Math. Proc. Cambridge Philos. Soc. 102(3) (1987), 399–410.

[D] J. Dixmier, Sur les algébres de Weyl, Bull. Soc. Math. France 96 (1968), 209–242.

[DG] J. J. Duistermaat and F. A. Grünbaum, Differential equations in the spectral parameter, Commun. Math. Phys. 103 (1986), 243–348.

[E] B. El boufi, Idéaux à droite réflexifs dans l’algèbre des opérateurs différentiels, Comm. Algebra 24(3) (1996), 939–947.

[EG] P. Etingof and V. Ginzburg, Symplectic reflection algebras, Calogero-Moser space, and deformed Harish-Chandra homomorphism, Invent. Math. 147 (2002), 243–348.

[G] A. Grothendieck, Eléments de Géométrie Algébrique IV, Publ. Math. 32, IHES, Paris, 1967.

[HoS] M. Holland and J. T. Stafford, Differential operators on rational projective curves, J. Algebra 147 (1992), 176–244.

[J1] A. G. Jones, Some Morita equivalences of rings of differential operators, J. Algebra 173(1) (1995), 180–199.

[J2] A. G. Jones, Non-normal D-affine varieties with injective normalization, J. Algebra 173(1) (1995), 200–218.

[KKO] A. Kapustin, A. Kuznetsov and D. Orlov, Noncommutative instantons and twistor transform, Commun. Math. Phys. 220 (2001), 385–432.

[K] R. M. Konakou, Isomorphismes entre algèbres d’opérateurs différentiels sur les courbes algébriques affines, Thèse, Univ. Claude Bernard Lyon-1, 1994.

[Kr] I. M. Krichever, On rational solutions of the Kadomtsev-Petviashvili equation and integrable systems of Calogero type, Funct. Anal. Appl. 12 (1978), 59–61.

[LeB] L. Le Bruyn, Moduli spaces of right ideals of the Weyl algebra, J. Algebra 172 (1995), 32–48.

[L] G. Letzter, Non-isomorphic curves with isomorphic rings of differential operators, J. London Math. Soc. 45(2) (1992), 17–31.

[LMP] G. Letzter and L. Makar-Limanov, Rings of differential operators over rational affine curves, Bull. Soc. Math. France 118 (1990), 193–209.

[LSS] T. Levasseur, S. P. Smith and J. T. Stafford, The minimal nilpotent orbit, the Joseph ideal, and differential operators, J. Algebra, 116 (1988), 480–501.

[M-L] L. Makar-Limanov, Rings of differential operators on algebraic curves, Bull. London Math. Soc. 21 (1989), 538–540.

[MR] J. C. McConnell and J. C. Robson, Noncommutative Noetherian Rings, Graduate Studies in Mathematics 30, American Mathematical Society, Providence, RI, 2001.

[MV] K. Mount and O. E. Villamayor, On a conjecture of Y. Nakai, Osaka J. Math. 10 (1973), 325–327.

[M] J. Muhasky, The differential operator ring of an affine curve, Trans. Amer. Math. Soc. 307(2) (1988), 705–723.

[Na] Y. Nakai, High order derivations I, Osaka J. Math. 7 (1970), 1–27.
[N] H. Nakajima, Lectures on Hilbert schemes of points on surfaces, University Lecture Series, vol. 18, American Mathematical Society, Rhode Island, 1999.

[P1] P. Perkins, Commutative subalgebras of the ring of differential operators on a curve, Pacific J. Math., 139(2) (1989), 279–302.

[P2] P. Perkins, Isomorphisms of rings of differential operators on curves, Bull. London Math. Soc. 23 (1991), 133–140.

[PS] A. Pressley and G. Segal, Loop Groups, Clarendon Press, Oxford, 1986.

[R] C. J. Rego, Remarks on differential operators on algebraic varieties, Osaka J. Math. 14 (1977), 481–486.

[PS] A. Pressley and G. Segal, Loop Groups, Clarendon Press, Oxford, 1986.

[Sm] S. P. Smith, Differential operators on commutative algebras, Lecture Notes in Math. 1197, Springer, Berlin, 1986, 165–177.

[SS] S. P. Smith and J. T. Stafford, Differential operators on an affine curve, Proc. London Math. Soc. (3) 56 (1988), 229–259.

[St] J. T. Stafford, Endomorphisms of right ideals of the Weyl algebra, Trans. Amer. Math. Soc. 299 (1987), 623–639.

[T] W. N. Traves, Nakai’s conjecture for varieties smoothed by normalization, Proc. Amer. Math. Soc. 127 (1999), 2245–2248.

[VdB] M. Van den Bergh, Differential operators on semi-invariants for tori and weighted projective spaces, Lecture Notes in Math. 1478, Springer, Berlin, 1991, 255–272.

[W1] G. Wilson, Bispectral commutative ordinary differential operators, J. Reine Angew. Math. 442 (1993), 177–204.

[W2] G. Wilson, Collisions of Calogero-Moser particles and an adelic Grassmannian (with an Appendix by I. G. Macdonald), Invent. Math. 133 (1998), 1–41.

[W3] G. Wilson, Bispectral symmetry, the Weyl algebra and differential operators on curves, Proceedings of the Steklov Institute of Mathematics 225 (1999), 141–147.

Department of Mathematics, Cornell University, Ithaca, NY 14853, USA
E-mail address: berest@math.cornell.edu

Department of Mathematics, Imperial College London SW7 2AZ, UK
E-mail address: g.wilson@imperial.ac.uk