SPARSE VARIANCE FOR PRIMES IN ARITHMETIC PROGRESSION

ROGER BAKER AND TRISTAN FREIBERG

ABSTRACT. We obtain an analog of the Montgomery–Hooley asymptotic formula for the variance of the number of primes in arithmetic progressions. In the present paper the moduli are restricted to the sequences of integer parts \([F(n)]\), where \(F(t) = t^c\) \((c > 1, c \notin \mathbb{N})\) or \(F(t) = \exp(\log t)^\gamma\) \((1 < \gamma < 3/2)\).

1. Introduction

Let \(F\) be a real differentiable function on \((1, \infty)\) with the property that
\[
F(y) \geq 2, \quad F'(y) \geq 1 \quad (y \geq y_0(F)).
\]

We write
\[
f(y) = \lfloor F(y) \rfloor.
\]

We are concerned with the remainders
\[
E(x; h, \ell) = \sum_{\substack{p \leq x \not\equiv \ell \mod h}} \log p - \frac{x}{\phi(h)}, \quad (\ell, h) = 1,
\]
where \(x\) is large and the moduli \(h\) are restricted to the values \(f(k)\). Here and below, \(p\) denotes a prime number. Let \(V_F(x, y)\) denote the variance
\[
V_F(x, y) = \sum_{y_0(F) < k \leq y} F'(k) \sum_{\substack{\ell = 1 \\ (\ell, f(k)) = 1}} E(x; f(k), \ell)^2.
\]

When \(F(k) = f(k) = k\), the Montgomery–Hooley theorem [3, 6] states that for \(1 \leq y \leq x\),
\[
V_F(x, y) = xy \log y + c_0 xy + O\left(x^{1/2} y^{3/2} + x^2 (\log x)^{-A}\right).
\]

Here and below, \(A\) denotes an arbitrary positive constant; we take \(A > 1\). (Implied constants depend on \(A\) throughout: dependencies on constants such as \(c\) are indicated in context.) The constant \(c_0\) can be given explicitly. This asymptotic formula was generalized by Brüdern and Wooley [2] to the case where \(F = f\) is an integer-valued polynomial of degree \(\geq 2\) with positive leading coefficient. They found that for \(1 \leq F(y) \leq x\),
\[
V_F(x, y) = x f(y) \log f(y) + C(f) x f(y) + O\left(x^{1/2} f(y)^{3/2} + x^2 (\log x)^{-A}\right) \quad (1.1)
\]

Date: March 20, 2018.
2010 Mathematics Subject Classification. Primary 11N13, Secondary 11P55.
Key words and phrases. Variance for primes in arithmetic progressions, Hardy–Littlewood method, exponential sums with integer part functions.
In the present paper, we give two further variants of the Montgomery–Hooley theorem.

**Theorem 1.1.** Let \( F(k) = k^c \), where \( c > 1 \) and \( c \notin \mathbb{N} \). Then (1.1) holds for \( F(y) \leq x \), with \( C(f) \) replaced by a constant \( C \) independent of \( f \).

The constant \( C \) is evaluated in Section 5 (see (5.10)).

**Theorem 1.2.** Let \( F(k) = \exp((\log k)\gamma) \), where \( 1 < \gamma < 3/2 \). Let \( C_1 > 1/(3 - 2\gamma) \).

For \( F(y) \leq x \) we have, with \( C \) as in Theorem 1.1,

\[
V_F(x, y) - V_F(x, \exp((\log \log x)^{C_1})) = xf(y) \log f(y) + Cx f(y) + O(x^{1/2} f(y)^{3/2} + x^2 (\log x)^{-A}).
\]

If we knew more about either Siegel zeros or exponential sums, we would not have to omit small moduli in Theorem 1.2; see [1, Section 6].

**Acknowledgements.** The arguments in Sections 4 and 5 are adapted from [2] with some notable differences. We thank Trevor Wooley for insightful comments about these differences. Thanks are also due from R. B. to the Department of Pure Mathematics, University of Waterloo for hospitality, and to the Simons Foundation for a Collaboration Grant.

**Notation**

As is customary, \( \phi \) denotes Euler’s totient function, \( \mu \) denotes the Möbius function, \( e(\theta) \) abbreviates \( e^{2\pi i \theta} \), and \( \|t\| = \min\{|t|, 1 - |t|\} \), where \( \{t\} = t - [t] \) denotes the fractional part of \( t \). Throughout, we regard the quantities \( c, \gamma \) and \( A \) as fixed and independent of all other quantities: we only assume that \( c > 1 \), \( c \notin \mathbb{N} \), \( 1 < \gamma < 3/2 \), and \( A > 1 \) (arbitrarily large). We regard \( B \) as fixed, but sufficiently large in terms of \( A \) and \( c \) (respectively, \( A \) and \( \gamma \)) in the case \( F(t) = t^c \) (respectively, \( F(t) = \exp((\log t)^\gamma) \)). We write \( C_1, C_2, \ldots \) for ‘large’ positive constants and \( c_1, c_2, \ldots \) for ‘small’ positive constants: each \( C_i \) may depend on \( c, \gamma, A, B, C_1, \ldots, C_{i-1} \) and \( c_1, \ldots, c_{i-1} \) (indicated in context); likewise for each \( c_i \). We view \( x \) as a real parameter tending to infinity, and write \( U \ll V, V \gg U \), or \( U = O(V) \) to denote that \( |U| \leq \kappa |V| \) for all sufficiently large \( x \), where \( \kappa \) is a constant which may depend on \( A \), as well as other fixed quantities (indicated in context). We write ‘\( U = V \)’ for ‘\( U \ll V \) and \( V \gg U \)’.

### 2. Some Lemmas

Most of these preliminary results come from [1]. Whether \( F \) be as in Theorem 1.1 or Theorem 1.2, let us write

\[
S_Q = \{[F(n)] : Q < [F(n)] \leq 2Q\}. \tag{2.1}
\]
Lemma 2.1. Let \( F \) be as in Theorem 1.1. For \( Q \in \mathbb{N} \) and \( S_Q \) as in (2.1), we have
\[
\sum_{q \in S_Q} \sum_{\ell=1}^{q} E(x; q, \ell)^2 \ll \frac{|S_Q|x^2}{Q(\log x)^{2A}} \tag{2.2}
\]
provided that \( Q \leq x(\log x)^{-B} \). Here \( B \) is a positive constant depending on \( A \) and \( c \).

Proof. [1, Theorem 1.1]. \qed

Lemma 2.2. Let \( F \) be as in Theorem 1.2. For \( Q \in \mathbb{N} \) and \( S_Q \) as in (2.1), we have (2.2) provided that
\[
\exp \left( (\log \log x)^{C_2} \right) \leq Q \leq x(\log x)^{-B},
\]
where \( C_2 > \gamma/(3 - 2\gamma) \). Here \( B \) is a positive constant depending on \( A \) and \( \gamma \). The implied constant (in (2.2)) depends on \( \gamma, C_2 \) and \( A \).

Proof. [1, Theorem 1.2]. \qed

Lemma 2.3. Let \( 2 \leq N \leq N_1 \leq 2N \).
(i) Let \( F(y) = \exp \left( (\log y)^{c} \right) \). Let \( 0 < \beta < F(N)^{c_2} \) with \( 0 < c_2 < \gamma - 1 \), and
\[
\beta F'(N) \geq \frac{1}{2}, \tag{2.3}
\]
Then
\[
\left| \sum_{N < n \leq N_1} e(\beta F(n)) \right| \leq C_3 N \exp \left( -c_3(\log N)^{3-2\gamma} \right)
\]
where \( C_3, c_3 \) depend on \( \gamma \) and \( c_2 \).
(ii) Let \( F(y) = y^c \). Let \( 0 < \beta < N \) and suppose that (2.3) holds. Then
\[
\left| \sum_{N < n \leq N_1} e(\beta F(n)) \right| \leq C_4 N^{1 - c_4}
\]
where \( C_4, c_4 \) depend on \( c \) and \( c_4 < 1/20 \).

Proof. [1, Lemma 2.2]. \qed

For the remainder of the paper, \( B \) and \( c_4 \) are as in Lemmas 2.1–2.3 and \( C_5, C_6, C_7 \) are constants satisfying
\[
C_5 \geq 8A + 1 + B, \quad C_6 \geq C_5 + 12A + 2, \quad C_7 \geq \frac{8A + 2}{c_4}. \tag{2.4}
\]

Lemma 2.4. Let \( P = (\log x)^{C_5}, R = x(\log x)^{-C_6} \). Let \( \alpha > 0 \) and suppose there is no rational number \( a/q, (a, q) = 1, 1 \leq q \leq P \) satisfying
\[
|\alpha - \frac{a}{q}| \leq \frac{1}{qR}.
\]
Suppose that either
(a) \( F(y) = y^c (c > 1, c \notin \mathbb{N}) \), or
(b) \( F(y) = \exp \left( (\log y)^{\gamma} \right) (1 < \gamma < 3/2) \).
Let \((\log x)^{C_7} \leq K \leq K_1 \leq 2K\) and \(M \geq 1/2\), \(MF(K) \leq 2x\). Suppose further in case (b) that
\[
\log K > (\log \log x)^{C_1}, \quad C_1 > 1/(3 - 2\gamma).
\]
Then
\[
\sum_{M < m \leq 2M} \sum_{K < k \leq K_1 \atop m^f(k) \leq x} e(\alpha m f(k)) \ll \frac{Kx}{F(K)(\log x)^{4A}}.
\]
The implied constant depends on \(c, A\) in case (a) and \(\gamma, C_1, A\) in case (b).

**Proof.** This follows at once from [1, Theorem 2.5].

**Lemma 2.5.** Make the hypotheses of Lemma 2.4 and suppose further that
\[
 x(\log x)^{-B} \leq F(K) \leq x.
\]
Let \(N(K, K_1, q, \ell)\) be the number of solutions to
\[
f(k) \equiv \ell \pmod{q}, \quad K < k \leq K_1.
\]
Then for \(1 < q \leq (\log x)^{C_5}\), we have
\[
N(K, K_1, q, \ell) = \frac{K_1 - K}{q} + O(K(\log x)^{-4A}).
\]
The implied constant depends on \(c, A, B\) in case (a) and \(\gamma, C_1, A, B\) in case (b).

**Proof.** We have
\[
N(K, K_1, q, \ell) = \sum_{K < k \leq K_1} \frac{1}{q} \sum_{a=1}^{q} e\left(\frac{a(f(k) - \ell)}{q}\right).
\]
Separating the contribution from \(a = q\),
\[
N(K, K_1, q, \ell) - \frac{K_1 - K}{q} = \frac{1}{q} \sum_{a=1}^{q-1} e\left(-\frac{a\ell}{q}\right) \sum_{K < k \leq K_1} e\left(\frac{af(k)}{q}\right).
\]
(2.6)
The remainder of the proof is a variant of the proof of [1, Theorem 2.5] in the case \(M = 1/2\); we have, for \(1 \leq a < q\),
\[
\sum_{K < k \leq K_1} e\left(\frac{af(k)}{q}\right) = T_1(\alpha) + O(T_2(\alpha)),
\]
(2.7)
where, with \(H = (\log x)^{4A + 1}\),
\[
T_1(\alpha) = \sum_{h \in \mathbb{Z} \atop \left|h+\frac{a}{q}\right| \leq H} c_h\left(\frac{a}{q}\right) \sum_{K < k \leq K_1} e\left(\frac{h + a}{q} F(k)\right)
\]
\[
T_2(\alpha) = \sum_{K < k \leq K_1} \min\left(\log x, \frac{1}{H\|F(k)\|}\right).
\]
Here,
\[
c_h(\beta) := \frac{1 - e(-\beta)}{2\pi i(h + \beta)}.
\]
Just as in the proof of [1, Lemma 2.4] we have

\[ T_2(\alpha) \ll K (\log x)^{-4A}, \quad (2.8) \]

\[ T_1(\alpha) \ll \sum_{|h+a| \leq K} \sum_{K \leq k \leq K_1} e\left(\left( h + \frac{a}{q} \right) F(k) \right). \quad (2.9) \]

Note that \( |h + \frac{a}{q}| < F(K)^{c_2} \), while

\[ \left| h + \frac{a}{q} \right| F'(2K) \geq \frac{1}{2}, \quad (2.10) \]

To see this,

\[ \left| h + \frac{a}{q} \right| \geq (\log x)^{-C_5}, \quad (2.11) \]

while

\[ F'(2K) \gg \frac{F(K)}{K} \gg \begin{cases} F(K)^{1-\frac{c}{3}} & \text{in case (a)} \\ F(K)^{1/2} & \text{in case (b)} \end{cases}, \quad (2.12) \]

and

\[ F(K) \gg x (\log x)^{-B}. \quad (2.13) \]

Combining (2.11)–(2.13) yields (2.10). We now use Lemma 2.3, noting that, in case (b),

\[ \exp (-c_3(\log K)^{3-2\gamma}) \ll (\log x)^{-9A} \]

since \( (\log K)^\gamma > \frac{1}{2} \log x \). This gives

\[ \sum_{K \leq k \leq K_1} e\left(\left( h + \frac{a}{q} \right) F(k) \right) \ll (\log x)^{-9A} \]

and the lemma now follows from (2.7)–(2.9). \( \square \)

**Lemma 2.6.** Let

\[ c_q(h) = \sum_{a=1}^{q} e\left( \frac{ah}{q} \right) \quad \text{and} \quad w_h(q) = \frac{1}{q} \sum_{a=1}^{q} c_q(ha). \]

Then for squarefree \( q \),

\[ w_h(q) = \begin{cases} \phi(q) & \text{when } q \mid h \\ 0 & \text{when } q \nmid h. \end{cases} \]

**Proof.** This is a special case of [2, Lemma 4.2]. \( \square \)

### 3. First stage of proof of Theorems 1.1 and 1.2

This section is similar to material in [1,2,5], but there are enough differences to give the details. Define \( y_1 \) by

\[ F(y_1) = x (\log x)^{-B}. \]

We are concerned with values of \( k \) satisfying

\[ y_1 < k \leq y, \quad \text{where} \quad F(y) \leq x. \quad (3.1) \]
We note that
\[ \frac{f(k)}{k} \leq \frac{F(k)}{k} \ll F'(k) \ll \frac{F(k)}{k} \log x \ll \frac{f(k)}{k} \log x. \] (3.2)

Our objective is to evaluate
\[ V'_F(x, y) := \sum_{y_1 < k \leq y} F'(k) \sum_{\ell=1 \atop (\ell, f(k)) = 1} \frac{f(k)}{\phi(f(k))} E(x; f(k), \ell)^2 \] (3.3)
asymptotically. Let
\[ \theta(x; k, \ell) = \sum_{p \leq x \atop p \equiv \ell \mod k} \log p, \]
\[ \Phi_F(z, y) = \sum_{z < k \leq y} \frac{F'(k)}{\phi(f(k))}. \] (3.4)

Opening the square in (3.3), we find that
\[ V'_F(x, y) = S_1 - 2x S_2 + x^2 \Phi_F(y_1, y), \] (3.5)
where
\[ S_1 = \sum_{y_1 < k \leq y} F'(k) \sum_{\ell=1 \atop (\ell, f(k)) = 1} \theta(x; f(k), \ell)^2, \]
\[ S_2 = \sum_{y_1 < k \leq y} \frac{F'(k)}{\phi(f(k))} \sum_{\ell=1 \atop (\ell, f(k)) = 1} \theta(x; f(k), \ell). \]

Using the prime number theorem and the fact that \( \ll \log x \) primes divide \( f(k) \) \( (k \leq y) \), we rewrite \( S_2 \) as
\[ S_2 = \sum_{y_1 < k \leq y} \frac{F'(k)}{\phi(f(k))} \left( x + O(x(\log x)^{-3A}) \right) \]
\[ = x \Phi_F(y_1, y) + O \left( x(\log x)^{-3A+1} \sum_{y_1 < k \leq y} \frac{f(k)}{k \phi(f(k))} \right) \] in view of (3.2). Since \( \phi(f(k)) \gg f(k)(\log x)^{-1} \), we find that
\[ S_2 = x \Phi_F(y_1, y) + O \left( x(\log x)^{-3A+3} \right). \] (3.6)

We may easily derive the relation
\[ \sum_{\ell=1 \atop (\ell, f(k)) = 1} \theta(x; f(k), \ell)^2 \ll \sum_{p_1 \leq x \atop p_1 \equiv p_2 \mod f(k)} (\log p_1)(\log p_2). \] (3.7)
The conditions \( p_1 \mid f(k), p_1 \equiv p_2 \mod f(k) \) imply that \( p_1 = p_2 \). We may accordingly ignore the constraint \( p_j \not\mid f(k) \) \((j = 1, 2)\) when considering the off-diagonal terms. Consequently,

\[
\sum_{\ell=1}^{f(k)} \theta(x; f(k), \ell)^2 = \sum_{p \leq x} (\log p)^2 + 2 \sum_{p_1 < p_2 \leq x, p_1 \equiv p_2 \mod f(k)} (\log p_1)(\log p_2). \tag{3.8}
\]

We note the bounds

\[
\sum_{k=k_0}^{k_1} F'(k - 1) \leq \sum_{k=k_0}^{k_1} (F(k) - F(k - 1)) \leq \sum_{k=k_0}^{k_1} F'(k) \tag{3.9}
\]

valid for any \( k_0, k_1 \in \mathbb{N}, k_0 \leq k_1 \). It follows that

\[
\sum_{y_1 < k \leq y} F'(k) = F(y) + O\left(x(\log x)^{-2A}\right),
\]

\[
\sum_{y_1 < k \leq y} F'(k) \sum_{p \leq x} (\log p)^2 = F(y) \sum_{p \leq x} (\log p)^2 + O\left(x^2(\log x)^{-A}\right). \tag{3.10}
\]

We let

\[
S_0 := \sum_{y_1 < k \leq y} F'(k) \sum_{p_1 < p_2 \leq x, p_1 \equiv p_2 \mod f(k)} (\log p_1)(\log p_2).
\]

We deduce easily from (3.8), (3.10) that

\[
S_1 = 2S_0 + f(y) \sum_{p \leq x} (\log p)^2 + O\left(x^2(\log x)^{-A}\right).
\]

Combining this with (3.5), (3.6), we have

\[
V_F'(x, y) = 2S_0 - x^2 \Phi_F(y_1, y) + f(y) \sum_{p \leq x} (\log p)^2 + O\left(x^2(\log x)^{-A}\right). \tag{3.11}
\]

Let

\[
T(\alpha) = \sum_{y_1 < k \leq y} F'(k) \sum_{h \leq x/f(k)} e(\alpha h f(k)), \quad U(\alpha) = \sum_{p \leq x} (\log p)e(\alpha p).
\]

It is straightforward to verify that

\[
S_0 = \int_0^1 T(\alpha)|U(\alpha)|^2 \, d\alpha. \tag{3.12}
\]

Let \( P, R \) be as in Lemma 2.4. Define the major arcs \( \mathcal{M} \) to be the union of the pairwise disjoint intervals

\[
\left\{ \alpha : \left|\frac{\alpha - a}{q}\right| \leq \frac{1}{qP} \right\} \quad (1 \leq a \leq q \leq P, (a, q) = 1)
\]

and the minor arcs \( \mathcal{m} \) by

\[
\mathcal{m} = \left[\frac{1}{R}, 1 + \frac{1}{R}\right] \setminus \mathcal{M}.
\]
A splitting-up argument gives

\[ J_m := \int_m^\infty T(\alpha)|U(\alpha)|^2 \, d\alpha \ll (\log x)^2 \int_m^\infty |U(\alpha)|T^*(\alpha) \, d\alpha \]  

(3.13)

where \( T^*(\alpha) \) is the contribution to \( T(\alpha) \) from \( K < k \leq K_1 \) and \( M < h \leq 2M \). Here \( 1/2 \leq M \leq x/f(K) \) while \( y_1 \leq F(K) < y \) and \( K < K_1 \leq 2K \). Moreover,

\[ \int_m^\infty |U(\alpha)|^2 \, d\alpha \leq \int_{1+\frac{1}{M}}^\infty |U(\alpha)|^2 \, d\alpha = \sum_{p \leq x} (\log p)^2 \ll x \log x. \]  

(3.14)

Now,

\[ T^*(\alpha) = \int_K^{K_1} F'(t) \, dS(t) \]

with

\[ S(t) = \sum_{y_1 < k \leq t} \sum \frac{e(ahf(k))}{M < h \leq 2M} \]

Since \( F'' \) is monotonic, taking sup norms on \([K, K_1]\) we have

\[ T^*(\alpha) = \left[ F'(t)S(t) \right]_K^{K_1} - \int_{y_1}^y F''(t)S(t) \, dt \ll \|F''\|_\infty \left\| S \right\|_\infty \ll \frac{K}{F(K)(\log x)^{4A}} \|F'\|_\infty \ll x(\log x)^{-3A}, \]

where we have used Lemma 2.4 and (3.2) for the second last and last bounds respectively. Combining this with (3.13), (3.14) we have

\[ J_m \ll x^2(\log x)^{-A}. \]  

(3.15)

We turn to the major arcs, beginning with

\[ J_{\mathfrak{M}} := \int_{\mathfrak{M}} T(\alpha)|U(\alpha)|^2 \, d\alpha = \sum_{q \leq P} \sum_{\substack{b=1 \atop (b,q)=1}}^q \int_{y_1}^{y_2} \left| U \left( \frac{b}{q} + \beta \right) \right|^2 T \left( \frac{b}{q} + \beta \right) \, d\beta. \]

Let

\[ v(\beta) = \sum_{m \leq x} e(m\beta). \]

From Vaughan [7, Lemma 3.1] we see that in the last integral,

\[ \left| U \left( \frac{b}{q} + \beta \right) \right|^2 = \frac{\mu(q)^2}{\phi(q)} |v(\beta)|^2 + O(x^2 \exp \left( -c_5(\log x)^{1/2} \right)). \]

Now

\[ T(\alpha) \ll x \sum_{y_1 < k \leq y} \frac{F'(k)}{f(k)} \ll x(\log x) \sum_{y_1 < k \leq y} \frac{1}{k} \ll x(\log x)^2. \]

Hence

\[ \left| U \left( \frac{b}{q} + \beta \right) \right|^2 T \left( \frac{b}{q} + \beta \right) \]

\[ = \frac{\mu(q)^2}{\phi(q)} |v(\beta)|^2 T \left( \frac{b}{q} + \beta \right) + O(x^3(\log x)^2 \exp \left( -c_5(\log x)^{1/2} \right)), \]
where

This yields

\[ J_{2R} = \sum_{q \in \mathcal{P}} \frac{\mu(q)^2}{\phi(q)^2} H(q) + O\left(x^2(\log x)^{-A}\right) \]  \hspace{1cm} (3.16)

where

By orthogonality,

\[ H(q) = \sum_{b=1}^{q} \sum_{(b,q)=1} \sum_{y_1 < k \leq y} \sum_{h \leq x/f(k)} \sum_{n_1 < x, n_2 < x} \sum_{n_1 - n_2 = h f(k)} e\left(\frac{b h f(k)}{q}\right) \]  \hspace{1cm} (3.17)

Replacing \([x]\) by \(x\) introduces an error in (3.17) of

\[ \ll (\log x)^{C_5} \sum_{y_1 < k \leq y} F'(k) \frac{x}{f(k)} \ll x(\log x)^{C_5 + 2} \]

by (3.2). Combining this with (3.12), (3.15) and (3.16) we reach the expression

\[ S_0 = M_0 + O\left(x^2(\log x)^{-A}\right), \]  \hspace{1cm} (3.18)

where

\[ M_0 = \sum_{q \in \mathcal{P}} \frac{\mu(q)^2}{\phi(q)^2} \sum_{y_1 < k \leq y} F'(k) \sum_{h \leq x/f(k)} c_q(h f(k))(x - h f(k)). \]

4. Proof of Theorems 1.1 and 1.2: second stage

We first show that \(M_0\) can be simplified to the form

\[ M_0 = \sum_{h \leq x/f(y_1)} \frac{h}{\phi(h)} \int_{f(y_1)}^{f(y(h))} (x - h t) \, dt + O\left(x^2(\log x)^{-A}\right) \]  \hspace{1cm} (4.1)

where \(y(h)\) is defined by

\[ F(y(h)) = \min\left(F(y), \frac{x}{h}\right). \]
Sorting the integers \( k \) according to the value of \( f(k) \mod q \),

\[
M_0 = \sum_{q \in \mathcal{P}} \frac{\mu(q)^2}{\phi(q)^2} \sum_{b \leq x/f(y_1)} \sum_{h=1}^q c_q(hb)S_0(y(h), b),
\]

(4.2)

where

\[
S_0(z, b) = \sum_{y_1 < k \leq z \atop f(k) \equiv b \mod q} F'(k)(x - hf(k)).
\]

(4.3)

Let

\[
S_1(z, b) = \sum_{y_1 < k \leq z \atop f(k) \equiv b \mod q} F'(k)(x - hF(k)),
\]

(4.4)

then

\[
S_0(z, b) - S_1(z, b) \ll h \sum_{y_1 < k \leq z} F'(k) \ll hF(y(h)) \ll x
\]

(4.5)

for \( y_1 < z \leq y(h) \), by (3.9). Now let

\[
S_2(z) = \sum_{y_1 < k \leq z} F'(k)(x - hF(k))
\]

and, in the notation of Lemma 2.5,

\[
N_b(t) = N(y_1, t, q, b) - \frac{1}{q} \sum_{y_1 < n \leq t} 1.
\]

We have, for \( z \in (y_1, y(h)] \),

\[
S_1(z, b) - \frac{S_2(z)}{q} = \int_{y_1}^{z} F'(t)(x - hF(t)) dN_b(t)
\]

\[
= \left[ F'(t)(x - hF(t))N_b(t) \right]_{y_1}^{z}
\]

\[
- x \int_{y_1}^{z} N_b(t)F''(t) dt + h \int_{y_1}^{z} N_b(t) \frac{d}{dt}(F(t)F'(t)) dt
\]

\[
= T_1 - T_2 + T_3,
\]

say.

Now,

\[
N_b(t) \ll y(h)(\log x)^{-3A}
\]

by Lemma 2.5 and a splitting-up argument. Hence

\[
T_1 \ll xF'(y(h))y(h)(\log x)^{-3A} \ll \frac{x^2}{h}(\log x)^{-2A},
\]

(4.6)

\[
T_2 \ll xF'(y(h))y(h)(\log x)^{-3A} \ll \frac{x^2}{h}(\log x)^{-2A},
\]

(4.7)

\[
T_3 \ll hy(h)(\log x)^{-3A}F(y(h))F'(y(h))
\]

\[
\ll xy(h)(\log x)^{-3A}F'(y(h)) \ll \frac{x^2}{h}(\log x)^{-2A}.
\]

(4.8)
Next we must estimate the difference
\[ D(z) := S_2(z) - \int_{y_1}^{z} F'(u)(x - hF(u)) \, du \]
for \( z \in (y_1, y(h)] \). By Euler’s formula,
\[
|D(z)|
\leq \int_{y_1}^{z} \left| \frac{d}{du} (F'(u)(x - hF(u))) \right| \, du + \left| F'(y_1)(x - hF(y_1)) \right| + \left| F'(z)(x - hF(z)) \right|
= U_1 + U_2 + U_3,
\]
say. We have
\[
U_1 \ll x \left| \int_{y_1}^{z} \frac{d}{du} (F'(u)) \, du \right| + h \left| \int_{y_1}^{z} \frac{d}{du} (F'(u)F(u)) \, du \right|
\ll xF'(y(h)) + hF'(y(h))F(y(h))
\ll \frac{x F(y(h)) \log x}{y_1}
\ll \frac{x^2 (\log x)^{-2A}}{h},
\]
(4.9)
\[
U_2 + U_3 \ll xF'(y(h)) \ll \frac{x^2 (\log x)^{-2A}}{h}.
\]
(4.10)
Assembling (4.2)–(4.10), we obtain
\[
M_0 = \sum_{q \in \mathcal{P}} \frac{\mu(q)^2}{\phi(q)^2} \sum_{h \in x/f(y_1)} \sum_{b=1}^{q} c_q(hb) \int_{y_1}^{y(h)} F'(u)(x - hF(u)) \, du
+ O\left( \frac{x^2}{(\log x)^{2A}} \sum_{q \in \mathcal{P}} \frac{\mu(q)^2}{\phi(q)} \sum_{h \leq 2(\log x)^h} \frac{1}{h} \right).
\]
(4.11)
The error here is \( O(x^2 (\log x)^{-A}) \). Using a substitution in the integral, and applying Lemma 2.6, the main term in (4.11) is
\[
\sum_{q \in \mathcal{P}} \frac{\mu(q)^2}{\phi(q)^2} \sum_{h \in x/f(y_1)} \sum_{q|h} \phi(q) \int_{F(y_1)}^{F(y(h))} (x - ht) \, dt.
\]
Since \( P > (\log x)^B > h \) in this sum, by (2.4), we may rewrite the main term in the form
\[
\sum_{h \leq x/f(y_1)} \left\{ \sum_{q|h} \frac{\mu(q)^2}{\phi(q)} \right\} \int_{F(y_1)}^{F(y(h))} (x - ht) \, dt
= \sum_{h \leq x/f(y_1)} \frac{h}{\phi(h)} \int_{F(y_1)}^{F(y(h))} (x - ht) \, dt.
\]
Changing the limits of integration by an amount \( O(1) \) incurs a further error
\[
\ll x \sum_{h \leq 2(\log x)^h} \frac{h}{\phi(h)} \ll x^2 (\log x)^{-A},
\]
and this yields (4.1).
We simplify (4.1) further using the formula
\[ \frac{d}{dt}(t(x - \frac{1}{2}ht)) = x - ht. \]

For ease of comparison with [2, Section 4], we write \( W(h) = \frac{h}{\phi(h)} \). The main term in (4.1) is
\[
\sum_{h \leq x/f(y)} W(h)\left[ t\left(x - \frac{1}{2}ht\right)\right]^{f(y)} + \sum_{x/f(y) < h \leq x/f(y_1)} W(h)\left[ t\left(x - \frac{1}{2}ht\right)\right]^{x/h}.
\]

This can be rewritten as
\[
\frac{f(y_1)^2}{2} \sum_{h \leq x/f(y_1)} \left\{ \frac{W(h)}{h}\left(\frac{x}{f(y_1)}\right)^2 - 2\frac{W(h)}{f(y_1)} + W(h)f\right\}
- \frac{f(y)^2}{2} \sum_{h \leq x/f(y)} \left\{ \frac{W(h)}{h}\left(\frac{x}{f(y)}\right)^2 - 2\frac{W(h)}{f(y)} + W(h)f\right\}.
\]

Introducing the function
\[
\Theta(H) = \sum_{h \leq H} \frac{W(h)}{h}(H - h)^2,
\]
the main term in (4.1) is
\[
\frac{1}{2}\left\{ f(y_1)^2\Theta\left(\frac{x}{f(y_1)}\right) - f(y)^2\Theta\left(\frac{x}{f(y)}\right) \right\}.
\]

Combining this with (3.18), we have
\[
2S_0 = f(y_1)^2\Theta\left(\frac{x}{f(y_1)}\right) - f(y)^2\Theta\left(\frac{x}{f(y)}\right) + O(x^2(\log x)^{-A}).
\]

We have
\[
\Theta(H) = bH^2 \log H + 2\Gamma_0 H^2 + H \log H + 2\Gamma_1 H + O(H^{1/2})
\]
from [2, Section 5] in the case \( W(h) = \frac{h}{\phi(h)} \); here
\[
b = \sum_{n=1}^{\infty} \frac{\mu(n)^2}{n\phi(n)}.
\]

The constants \( \Gamma_0, \Gamma_1 \) can be calculated explicitly; see [2, p. 13]. Hence
\[
2S_0 = bx^2 \log \left(\frac{f(y)}{f(y_1)}\right) - xf(y) \log \left(\frac{x}{f(y)}\right) - 2\Gamma_1 xf(y)
+ O(x^{1/2}f(y)^{3/2} + x^2(\log x)^{-A}).
\]

5. Completion of the proof of Theorems 1.1 and 1.2

We begin by using the identity
\[
\frac{q}{\phi(q)} = \sum_{r|q} \frac{\mu(r)^2}{\phi(r)}.
\]
to evaluate \( \Phi_F(y_1, y) \) asymptotically. For \( y_1 < z \leq y \),
\[
\sum_{y_1 < k \leq z} \frac{f(k)}{\phi(f(k))} = \sum_{r \leq f(z)} \frac{\mu(r)^2}{\phi(r)} \sum_{\substack{y_1 < k \leq z \atop f(k) \equiv 0 \mod r}} 1
= \sum_{r \leq f(z)} \frac{\mu(r)^2}{\phi(r)} \left( \frac{z - y_1}{r} + O\left( (\log x)^{-4A} \right) \right)
= b(z - y_1) + O\left( (\log x)^{-3A} \right),
\]
where we have used Lemma 2.5 for the second last equality, and where \( b \) is the constant in (4.12). Replacing \( \frac{f(k)}{\phi(f(k))} \) by \( \frac{F(k)}{\phi(f(k))} \) in (5.1) introduces an error that is
\[
\ll \sum_{y_1 < k \leq z} \frac{1}{\phi(f(k))} \ll (\log x) \sum_{y_1 < k \leq z} \frac{1}{f(k)} \ll (\log x) \sum_{y_1 < k \leq z} \frac{(\log x)^B}{x} \ll \frac{(\log x)^{B+1}}{x}.
\]
Hence
\[
N(z) := \sum_{y_1 < k \leq z} \frac{F(k)}{\phi(f(k))} - b(z - y_1) \ll (\log x)^{-3A}.
\]

Now
\[
\Phi_F(y_1, y) - b \int_{y_1}^{y} \frac{F'(t)}{F(t)} \, dt = \int_{y_1}^{y} \frac{F'(t)}{F(t)} \, dN(t)
= \left[ \frac{F'(t)}{F(t)} N(t) \right]_{y_1}^{y} - \int_{y_1}^{y} N(t) \frac{d}{dt} \left( \frac{F'(t)}{F(t)} \right) \, dt.
\]
Since \( F'/F \) is monotonic, we deduce from (5.2) that, for some \( w \in [y_1, y] \),
\[
\Phi_F(x, y) = b \int_{y_1}^{y} \frac{F'(t)}{F(t)} + O\left( \frac{F'(w)}{F(w)} (\log x)^{-3A} \right) = b \log \left( \frac{F(y)}{F(y_1)} \right) + O\left( (\log x)^{-A} \right)
\]
(recalling 3.2). Noting that
\[
\log \frac{F(y)}{F(y_1)} - \log \frac{f(y)}{f(y_1)} = \log \left( 1 + \frac{F(y) - f(y)}{f(y)} \right) - \log \left( 1 + \frac{F(y_1) - f(y_1)}{f(y_1)} \right)
= O\left( \frac{1}{f(y_1)} \right) = O\left( (\log x)^{-A} \right)
\]
we have the more convenient expression
\[
\Phi_F(y_1, y) = b \log \left( \frac{f(y)}{f(y_1)} \right) + O\left( (\log x)^{-A} \right).
\]

We now substitute (4.13) and (5.3) into the expression for \( V'_F(x, y) \) obtained in (3.11). This gives
\[
V'_F(x, y) = x f(y) \log f(y) + f(y) \left( \sum_{p \leq x} (\log p)^2 - x \log x \right) - 2 \Gamma_{-1} x f(y)
+ O\left( x^{1/2} f(y)^{3/2} + x^2 (\log x)^{-A} \right).
\]
By the prime number theorem and partial summation,
\[
V'_F(x, y) = x f(y) \log f(y) + C x f(y) + O\left( x^{1/2} f(y)^{3/2} + x^2 (\log x)^{-A} \right)
\]
with \( C = -(2\Gamma_{-1} + 1) \).

The additional sum required to complete the proof of Theorems 1.1 and 1.2 is

\[
\sum_{Y < k \leq y_1} F'(k) \sum_{\ell, f(k)}^{f(k)} E(x; f(k), \ell)^2,
\]

where \( Y = y_0(F) \) (Theorem 1.1), \( Y = \exp((\log \log x)^C_1) \) (Theorem 1.2). By a splitting-up argument it suffices to show that when \( Y \leq Q < F(y_1) \), we have

\[
\sum_{Q < [F(k)]} F'(k) \sum_{\ell, f(k)}^{f(k)} E(x; f(k), \ell)^2 \ll x^2(\log x)^{-2A}. \tag{5.4}
\]

This is a straightforward consequence of Lemma 2.1 in the case of Theorem 1.1. In the case of Theorem 1.2, let

\[
F(K) = Q, \quad F(K_1) = 2Q
\]

so that

\[
F'(k) = (\log K)^{-1} K^{-1} (k \in [K, K_1]).
\]

Now the mean value theorem yields

\[
\frac{|S_Q|}{Q} = \frac{K_1 - K}{F(K_1) - F(K)} = \frac{K}{Q(\log K)^{-1}}.
\]

The left-hand side of (5.4) is

\[
\ll \frac{(\log K)^{-1} Q}{K} \sum_{q \in S_Q} \sum_{\ell, q}^q E(x; q, \ell)^2 \ll \frac{Q}{|S_Q|} \sum_{q \in S_Q} \sum_{\ell, q}^q E(x; q, \ell)^2 \ll x^2(\log x)^{-2A}
\]

by Lemma 2.2. This completes the proof of Theorems 1.1 and 1.2.

The constant \( C \) may be evaluated using material from [2, Section 5], with the function \( \rho(p) \) replaced by 1 to yield the desired function \( W(h) = h/\phi(h) \). Let

\[
D(s) = \zeta(s + 1)\zeta(s + 2)E_2(s),
\]

where \( \zeta \) is Riemann’s zeta function and

\[
E_2(s) = \prod_p \left( 1 + \frac{p^{-s}}{p^2(p-1)} - \frac{p^{-2s}}{p^3(p-1)} \right) \quad (\Re(s) > -3/2). \tag{5.5}
\]

Then in the notation used just before (4.12), we have the residue formula

\[
\text{Res} \left( \frac{D(s)H^{s+2}}{s(s + 1)(s + 2)} , -1 \right) = -\zeta(0)H \log H + \Gamma_{-1}H. \tag{5.6}
\]

We use \( E_2(-1) = 1 \) (see [2, p. 301]). We also need

\[
E_2'(-1) = \sum_p \frac{\log p}{p(p-1)}, \tag{5.7}
\]
which can be obtained from (5.5) by logarithmic differentiation. Now we have the Laurent expansions near $-1$:

$$G(s) := \frac{\zeta(s + 1)H^{s+2}E_2(s)}{s(s + 2)} = -\zeta(0)H + G'(-1)(s + 1) + \cdots,$$

$$\frac{\zeta(s + 2)}{s + 1} = \frac{1}{(s + 1)^2}(1 + \gamma_0(s + 1) + \cdots).$$

See Ivić [4, p. 4] for the coefficient $\gamma_0$ (Euler’s constant) in the latter expansion. We are led immediately to

$$\text{Res} \left( \frac{D(s)H^{s+2}}{s(s + 1)(s + 2)}, -1 \right) = -\gamma_0\zeta(0)H + G'(-1). \tag{5.8}$$

A short calculation yields

$$G'(-1) = - \left\{ \zeta'(0) + \zeta(0)E'_2(-1) \right\} H - \zeta(0)H \log H. \tag{5.9}$$

Combining (5.6)–(5.9),

$$\Gamma_{-1} = \zeta(0) \left( -\gamma_0 - \sum_p \frac{\log p}{p(p - 1)} \right) - \zeta'(0)$$

and

$$C' = -(2\Gamma_{-1} + 1) = 2\zeta(0) \left( \gamma_0 + \sum_p \frac{\log p}{p(p - 1)} \right) + 2\zeta'(0) - 1. \tag{5.10}$$

References

[1] Baker, R. C. “Primes in arithmetic progressions to spaced moduli. II” Q. J. Math. 65(2):597–625, 2014.
[2] Brüdern, J. and T. D. Wooley. “Sparse variance for primes in arithmetic progression.” Q. J. Math. 62(2):289–305, 2011.
[3] Hooley, C. “On the Barban–Davenport–Halberstam theorem. I.” J. Reine Angew. Math. 274/275:206–223, 1975.
[4] Ivić, A. The Riemann zeta-function: theory and applications. Dover Publications, Inc., Mineola, NY, 2003
[5] Mikawa, H. and T. P. Peneva. “Primes in arithmetic progressions to spaced moduli.” Arch. Math. (Basel) 84(3):239–248, 2005.
[6] Montgomery, H. L. “Primes in arithmetic progressions.” Mich. J. Math. 17(1):33–39, 1970.
[7] Vaughan, R. C. The Hardy–Littlewood Method. 2nd edn. Cambridge Tracts in Mathematics, 125. Cambridge University Press, Cambridge, 1997.