Abstract

We use a version of the Skorokhod integral to give a simple and rigorous formulation of the Wick-ordered (stochastic) heat equation with planar white noise, representing the free energy of an undirected random polymer. The solution for all times is expressed as the $L^1$ limit of a martingale given by the Feyman-Kac formula and defines a randomized shift, or Gaussian multiplicative chaos. The fluctuations far from the centre are shown to be given by the one-dimensional KPZ equation.

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1 Introduction and main results

Although the KPZ universality class should describe random planar geometry, until recently models shown to be universal have all been directed, missing crucial symmetries. In this article we study the planar Wick-ordered heat equation

\[ \partial_t u = \frac{1}{2} \Delta u + u \xi, \quad u(\cdot, 0) = \varsigma \]  

(1)

where \( \xi \) is a planar white noise that does not depend on time (Figure 1). The precise definition uses the mild form (3) of the heat equation, in which the white noise term is given by a Skorokhod integral, a close relative of the Itô integral that does not need a specific time direction. The term “Wick-ordered” refers to the fact that the solutions of such equations are given in terms of Gaussian exponentials with quadratic correction.

The solution \( u(x, t) \) was known to exist and be expressed in terms of a chaos series, but only up to a critical time, see Nualart and Zakai (1989) and Hu (2002), after which
the $L^2$ norm blows up, and the chaos expansion fails to converge. The blow-up time, $t_c$, which coincides with the largest time for which the mutual intersection local time of a pair of independent standard planar Brownian motions has a rate one exponential moment, is known exactly. It is half the optimal constant in the Gagliardo-Nirenberg-Sobolev inequality, see [Chen (2004)] and [Bass and Chen (2004)].

In this article we extend these results in several ways. First, we use an elementary version of the Skorokhod integral to define the solution (1) for all times, including $t > t_c$. We give a construction of $u$ as a randomized shift, or as the free energy of an undirected polymer in a random environment. This holds for all times as an $L^1$ functional of the white noise, and coincides for $t < t_c$ with the chaos series. Let $p$ denote the planar heat kernel,

$$p(x,t) = \frac{1}{2\pi t} \exp\{-|x|^2/2t\}$$

and $(\Xi, \mathcal{F}, P)$ be a probability space containing a Gaussian sequence that we will use to define our planar white noise and Skorokhod integral, planar in this article always referring to $\mathbb{R}^2$. We have the following definition.

**Definition 1.** Let $\varsigma$ be a finite measure on $\mathbb{R}^2$. A measurable function $u : (0,T] \times \mathbb{R} \times \Xi \to [0, \infty)$ is a solution of the planar **Wick-ordered heat equation** (1) with initial condition $\varsigma$ if for almost all $(x,t) \in (0,T] \times \mathbb{R}^2$ the random function of $y$,

$$\mathcal{K}_{x,t}(u)(y) = \int_0^t p(x - y, t - s) u(y, s)ds,$$

is Skorokhod-integrable (see Definition 7) and satisfies

$$u(x,t) = \int p(x - y, t) d\varsigma(y) + \int \mathcal{K}_{x,t}(u) \xi.$$
By linearity, it suffices to understand Dirac $\delta$ initial conditions. The explicit solution is given as follows.

**Theorem 2.** With $\varsigma = \delta_0$, the planar Wick-ordered heat equation has the following unique solution. Let $e_j$ be bounded functions forming an orthonormal basis of $L^2(\mathbb{R}^2)$, let $\xi_j = \langle e_j, \xi \rangle$, let $B$ be a planar Brownian bridge from 0 to $x$ in time $t$ defined on a probability space $(\Omega, \mathcal{G}, Q)$ independent of $\xi$. Then

$$u(x, t) = p(x, t) \lim_{n \to \infty} Z_n, \quad Z_n = E_Q \exp \left\{ \sum_{j=1}^n m_j \xi_j - \frac{1}{2} \sum_{j=1}^n m_j^2 \right\}, \quad m_j = \int_0^t e_j(B(s)) \, ds. \quad (4)$$

$Z_n$ is a uniformly integrable martingale, so it converges almost surely and in $L^1(P)$. For $t < t_c$, it converges in $L^2(P)$. The solution does not depend on the choice of basis $e_j$.

The choice of the Skorokhod integral is motivated by the interpretation of (1) as a polymer free energy. By the Feynman-Kac formula one expects the solution of the stochastic heat equation (1) to have a representation

$$u(x, t) = \lim_{n \to \infty} E_Q \exp \left\{ \int_0^t \xi_{[n]}(B(s)) \, ds - \text{norm}(n, \omega) \right\} p(x, t). \quad (5)$$

where $\xi_{[n]}$ is a mollified version of the noise, and norm is some normalization. Our solution (4) is exactly in this form, with $\xi_{[n]} = \sum_{j=1}^n \xi_j e_j$, a Gaussian noise with $n$ degrees of freedom. We will use this mollification throughout the paper.

We expect the random polymer measure $M_\xi$ to be the a measure on continuous paths $[0, t] \to \mathbb{R}^2$ written as

$$M_\xi = \lim_{n \to \infty} \exp \left\{ \int_0^t \xi_{[n]}(B(s)) \, ds - \text{norm}(n, \omega) \right\} Q, \quad Q : \text{Brownian bridge law.} \quad (6)$$

We might write integral as $\langle \xi_{[n]}, X \rangle$ where $X(A) = \int_0^t 1(B(s) \in A) \, ds$ is the occupation measure of the Brownian bridge up to time $t$. Since $X$ is singular with respect to Lebesgue measure, $\langle \xi_{[n]}, X \rangle$ will not have a limit.

However, we expect that the $L^2$ norm $\langle X_n, X_n \rangle$ of the mollified version $X_n = \sum_{j=1}^n e_j \langle X, e_j \rangle$ converges after subtracting a normalizing constant, to twice the two-dimensional self-intersection local time $\langle X_n, X_n \rangle - E_{Q_{BM}} \langle X_n, X_n \rangle \to 2 \gamma_t$. Here $Q_{BM}$ signifies that the normalization uses Brownian motion from 0 rather than Brownian bridge from 0 to $x$. Such results are shown in Varadhan (1969), Le Gall (1994) for certain mollifications.

If we take expectation of $u(x, t)$ over the randomness of the white noise, we get

$$E_p E_{Q_{BM}} e^{\langle \xi_{[n]}, X \rangle} = E_{Q_{BM}} E_p e^{\langle \xi, X_n \rangle} = E_{Q_{BM}} \frac{1}{2} \langle X_n, X_n \rangle \quad (7)$$


which suggests that we could use \( \text{norm}(n, \omega) = E_{Q_{BM}} \langle X_n, X_n \rangle / 2 \). This breaks the semigroup property slightly, but the problem can be fixed by adding a \( t \)-dependent term to get \( \text{norm}(n, \omega) = E_{Q_{BM}} \langle X_n, X_n \rangle / 2 - \frac{1}{2\pi} t \log t \), see Example \[53\].

This choice corresponds to renormalizing the solution of the equation \( \partial_t u_n = \frac{1}{2} \Delta u_n + u_n \xi_n \) as \( e^{-E_{Q_{BM}} \langle X_n, X_n \rangle / 2 + \frac{1}{4\pi} t \log t} u_n \) to obtain a limit. The result is called PAM in the literature, for parabolic Anderson model, see, for example \[Hairer and Labbé\, (2015)\]. PAM satisfies the usual semigroup property. Note however, that from the above construction the expectation of the polymer measure over the noise \( \xi \) is not the original Brownian bridge \( Q \), but instead the re-weighted bridge measure \( e^{\gamma t + \frac{1}{4\pi} t \log t} Q \). So \( M_\xi \) is a randomized version of the \( e^{\gamma t + \frac{1}{4\pi} t \log t} Q \), instead of \( Q \).

Starting from the physical assumption that the polymer measure should be a randomized version of the original Brownian bridge, one is led to renormalize by the self-intersections themselves instead of just their expectations. This corresponds to renormalizing Gaussian exponentials as

\[
\exp\{W - \frac{1}{2} E_P W^2\}. \tag{8}
\]

Our normalization is essentially that, up to the \( t \)-dependent factor: we use \( \text{norm}(n, \omega) = \langle X_n, X_n \rangle / 2 \) instead of \( E_{Q_{BM}} \langle X_n, X_n \rangle / 2 - \frac{1}{2\pi} t \log t \). This leads to \[4\], and, remarkably, to the Skorokhod interpretation of the planar SHE. The price is that the resulting solution of \[1\] is no longer a semigroup, since the Brownian motions weighted by their renormalized self-intersection local time no longer have the Markov property.

In summary, one has two choices: keep the expected path measure as Brownian bridge, or keep the semigroup property. As in the Itô-Stratonovich dichotomy, the choice depends on which symmetries are more intrinsic to the scientific problem at hand. One advantage of the Skorokhod approach is that it leads to a simple construction of PAM as a re-weighting of the paths in this measure by the exponential of self-intersection local time, see Example \[53\].

The Skorokhod integral also has some pleasant computational properties. To solve the planar Wick-ordered heat equation, and prove Theorem 2, we use the fact that projections of this equation to a finite-dimensional Gaussian space satisfy a version of \[3\], a key property of the Skorokhod integral. These projections are in fact deterministic finite-dimensional linear PDEs, that can be solved explicitly via the Feynman-Kac formula. The solutions are given by \( p(x, t) Z_n(x, t) \) as in Theorem \[2\] and they are real-analytic in the noise coordinates.

To understand the convergence of the projections, we think of the sequence \( m_j \) as a randomized shift. A general theory around this notion was developed by \[Shamov\, (2016)\].

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in studying Gaussian multiplicative chaos. We develop a simple version of this theory, reviewing and establishing existence and convergence theorems along the way.

The Gaussian multiplicative chaos in this setting is the random polymer measure, so we obtain a rigorous version of the polymer measure implied in (5) for free.

**Theorem 3 (GMC representation).** With the notation of Theorem 2, the law of the marginal law of $\xi + m$ is absolutely continuous with respect to the law of $\xi$ and the Radon-Nikodym derivative is given by the partition function $Z$. The (unnormalized) random polymer is the unique $P$-random measure $M_\xi$ on $\Omega$ satisfying

$$E_P E_{M_\xi} F(\xi, \omega) = E_{P \times Q} F(\xi + m(\omega), \omega).$$

for all bounded measurable functions $F : \Xi \times \Omega \to \mathbb{R}$.

In fact, we can write

$$M_\xi = \lim_{n \to \infty} e^{\sum_{j=1}^n m_j(\omega) \xi_j - \frac{1}{2} \sum_{j=1}^n m_j(\omega)^2} Q$$
in $P$-probability with respect to the weak topology of measures on path space.

The key technical input to Theorems 2 and 3 is to show uniform integrability of $Z_n$. For this we use that, even for $t > t_c$, off sets of arbitrarily small probability, the mutual intersection local time of planar Brownian motion has exponential moments.

The solution of the one-dimensional stochastic heat equation with space-time white noise is given by the same simple construction. Consequences of the construction are a new criterion (Proposition 46) for convergence in the class of such models and the apparently new observation (13) that the shift representation holds for the KPZ equation and the continuum random directed polymer (CDRP). Indeed, while the full polymer measures were studied before in a mollified setting (see, for example, Mukherjee, Shamov and Zeitouni (2016), Bröker and Mukherjee (2019)), only the partition function or the endpoint law were considered without mollification. We will use the randomized shift representation of the full polymer measure in the proof of Theorems 4 and 6.

Recall the Kardar-Parisi-Zhang equation,

$$\partial_t h = \frac{1}{2} (\partial_x h)^2 + \frac{1}{2} \partial_x^2 h + \xi,$$

where $x \in \mathbb{R}$, $t \geq 0$ and $\xi$ is white noise in one space and one time dimensions. The solution of this equation is defined through the Hopf-Cole transformation $h(x, t) = \log z(x, t)$ that turns it into the one-dimensional multiplicative stochastic heat equation

$$\partial_t z = \frac{1}{2} \partial_x^2 z + \xi z, \quad z(0, \cdot) = \delta_0.$$
Let
\[ q(x, t) = \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t} \] (12)
be the one-dimensional heat kernel.

**Theorem 4.** The Wick-ordered version, Definition 29, of the one-dimensional stochastic heat equation (11) with space-time noise has the following unique solution:
\[ z(x, t) = q(x, t) \lim_{n \to \infty} E_Q \exp \left\{ \sum_{j=1}^{n} m_j \xi_j - \frac{1}{2} \sum_{j=1}^{n} m_j^2 \right\}, \quad m_j = \int_0^t e_j(b(s), s). \]

Here \( b \) is a one-dimensional Brownian bridge from 0 to \( x \) in time \( t \). The limit exists in \( L^2(P) \) and coincides with the \( \text{Itô} \) solution of (11). The corresponding polymer measure \( M_\xi \) coincides with the continuum directed polymer constructed in [Alberts, Khanin and Quastel (2014)]. It has the shift representation
\[ E_P E_{M_\xi} F(\xi, \omega) = E_{P \times Q} F(\xi + m(\omega), \omega). \] (13)

Next, we show that on appropriate scales, the solution \( u \) of the planar Wick-ordered heat equation (1) has KPZ fluctuations. In particular, we prove the following.

**Theorem 5.** For any \( t > 0 \) and \( a \in \mathbb{R} \) as \( N \to \infty \), with \( \varsigma = \delta_0 \) we have
\[ P(u(((0, N^{3/2}t), Nt) \times Ne^{N^{2t/2} - \sqrt{2\pi}t} \leq a) \to F_{KPZ}(t, a), \] (14)
where \( F_{KPZ}(t, a) = P(z(0, t) \leq a) \) are the KPZ crossover distributions, which have a determinantal representation (see, for example, [Amir, Corwin and Quastel (2011)]).

We also have a process-level version of Theorem 5. To understand the almost sure dependence on instances of white noise in the next theorem, couple them as
\[ \xi_N(\cdot, \cdot) = N\xi(N^{-1/2}, N^{-3/2}, \cdot). \] (15)

The law of \( \xi_N \) does not depend on \( N \), but now these white noises are defined on the same probability space \((\Xi, \mathcal{F}, P)\). We introduce extra parameters to both equations,
\[ \partial_t u = \frac{1}{2}(\nu \partial_x^2 + \partial_y^2) u + \beta u \xi_N, \quad \partial_t z = \frac{1}{2}\nu \partial_x^2 z + \beta z \xi, \] (16)
and write \( u_{\nu, \beta, N}(x_0, y_0; x, y; t) \) to be the solution with initial condition \( \delta_{(x_0, y_0)} \). Similarly, write \( z_{\nu, \beta}(x_0, t_0; x, t) \) for the solution started from \( \delta_{x_0} \) at time \( t_0 \). Note that for \( u \), the noise \( \xi_N \) is used as \( \xi_N(x, y) \) and for \( z \) the noise \( \xi \) means \( \xi(x, t) \).
Theorem 6. Let $\rho, \nu, \beta, s > 0$ and $x, y, t \in \mathbb{R}$. Let $p = (x, t)$ and $q = (y, t + s)$. Let $p_N = (N^{1/2}x, N^{3/2}t)$ and let $q_N = (N^{1/2}y, N^{3/2}(t + s))$. As $N \to \infty$, we have

$$u_{\nu,\beta,N}(p_N; q_N; N\rho s) \times \sqrt{2\pi s} Ne^{N^2 s/(2\rho)} \to z_{\nu,\beta}(p; q) \quad \text{in } L^1(\Xi, \mathcal{F}, P).$$

Moreover, the polymer measure $M_N$ on paths from $p_N$ to $q_N$ in time $N\rho s$ corresponding to $Z_N$ converges to the 1+1-dimensional continuum directed random polymer measure (CDRP, the polymer measure corresponding to $z$) $M$ on paths $b$ between space-time points $p$ and $q$

$$\left((N^{-1/2}B_N(N\rho r)_1, N^{-3/2}B_N(N\rho r)_2), \ r \in [0, s]\right) \text{ under } M_N$$

$$\to \left((b(t + r), t + r), \ r \in [0, s]\right) \text{ under } M, \quad \text{in } P\text{-probability.}$$

Since (16) is convergence in $L^1$, and not just in law, it holds jointly in all seven parameters $\beta, \nu, \rho, x, y, s, t$. For example, with all parameters except $y$ fixed, Theorem 6 implies convergence of finite dimensional distributions to those of the top line of the KPZ line ensemble.

As a result, in the double limit when $t \to \infty$ after the first limit, the planar Wick-ordered heat equation converges to the Airy process, see Quastel and Sarkar (2020), Virág (2020). By linearity, Theorem 6 implies convergence for general initial conditions often considered in KPZ. The KPZ fixed point appears in the double limit.

The Feynman-Kac representation makes the proof of Theorem 6 transparent and explains the scaling as well.

Proof of Theorem 6. Fix an orthonormal basis of $L^2(\mathbb{R}^2)$ consisting of bounded continuous functions $e_j$, and let

$$e_{N,j}(x, y) := \frac{1}{N} e_j(N^{-1/2}x, N^{-3/2}y).$$

This, together with our choice of the noise coupling (15) is compatible with the space-time scaling. It is chosen so that the coordinate representation $\xi_j = \langle e_{N,j}, \xi_N(\cdot) \rangle = \langle e_j, \xi(\cdot) \rangle$ does not depend on $N$.

We will solve for $u$ in the basis $e_{N,j}$. Theorem 2 shows that the solution is basis independent, and, with parameters added in a straightforward way, we can write

$$u(p_N, q_N, N\rho s) = p(q_N - p_N, N\rho s) \lim_{n \to \infty} Z_{N,n},$$

$$Z_{N,n} = E_Q \exp \left\{ \sum_{j=1}^{n} m_{N,j} \xi_j - \frac{1}{2} \sum_{j=1}^{n} m^2_{N,j} \right\}, \quad m_{N,j} = \beta \int_0^{N\rho s} e_{N,j}(B_N(r)) \, dr,$$

where $B_N$ is a two-dimensional Brownian bridge $p_N \to q_N$ in time $N\rho s$ with covariance matrix diag($\nu, 1$). We couple the bridges on a single probability space $Q$ by writing the
limiting spatial and temporal directions separately:

\[ B_N(Nρr) = (N^{1/2}b_{sp}(t + r), N^{1/2}b_{ti}(t + r) + (t + r)N^{3/2}), \quad r \in [0, s], \]

where \( b_{sp}, b_{ti} \) are independent 1-dimensional Brownian bridges run from time \( t \) to \( t + s \) from \( x \) to \( y \) and from \( 0 \) to \( 0 \), and having variances \( νρ \) and \( ρ \) respectively. It is convenient to rewrite

\[ m_{N,j} = β \int_0^s e_j(b_{sp}(t + r), b_{ti}(t + r)/N + t + r) dr \]

as the change-of-variable and scaling factors \( N \) cancel. Then for each \( j \),

\[ \lim_{N \to \infty} m_{N,j} = m_j = β \int_0^s e_j(b_{sp}(t + r), t + r) dr, \quad Q - a.s. \]

and by Theorem 4, with parameters added in a straightforward way, we have

\[ z_{νρ,β}(p, q) = q(q - p) \lim_{n \to \infty} E_Q \exp \left( \sum_{j=1}^n m_jξ_j - \frac{1}{2}m_j^2 \right). \]

This establishes what the limit should be, including matching the parameters. We justify the exchange of limits in \( n \) and \( N \) and complete the proof in Section 7.3.

The key technical input for Theorem 6 is again a mutual intersection local time estimate for a Brownian bridge. This time, we need a uniform bound on exponential moments of a small set for large \( t \) and distant endpoint. Fortunately, the large drift in bridges to far away points makes them intersect less and allows us to establish such a bound.

### 1.1 History

Essentially all earlier work on exact KPZ fluctuations has been on directed models. An exception is the breakthrough work on the boundary fluctuations of lozenge tilings of polygons [Aggarwal and Huang (2021)]. These models, however, are still non-intersecting line ensembles, and the challenge there is to localize previous methods.

In terms of underlying polymer paths, the technical difficulty is to show that self-intersections do not affect the large-scale behaviour. Previously the only tool was to identify a regeneration structure (for example, Ioffe and Velenik (2012)). Besides leading to non-explicit coefficients defined in terms of the regeneration times, such methods failed to identify universal fluctuations except in some trivial (Gaussian) situations.

The present paper treats the undirected two dimensional continuous case, represented by the stochastic heat equation and its underlying polymer. There is a choice of interpretation of the stochastic integral in the Duhamel form of the equation, and here we study
the model resulting from the Skorokhod interpretation. The Skorokhod integral was introduced by Skorohod (1975). Presentations in Nualart (2006), Janson (1997a) and Hairer (2021) are highly recommended. Gaveau and Trauber (1982) identified it as the adjoint of the Malliavin derivative, and it is often defined that way. This makes sense for $L^p$, $p > 1$, as the Malliavin derivative is closable on the duals $L^q$, $1 < q < \infty$. The technical challenge in our work is the necessity to extend to appropriate $L^1$ integrands so that the integration in the Duhamel form of the stochastic heat equation (1) makes sense. This is achieved by the use of martingale theory and an appropriately checkable condition for uniform integrability.

The martingale approximations take the same form as approximations to the Gaussian multiplicative chaos studied by Shamov (2016). In particular, this gives a representation of the solution as the Radon-Nikodym derivative of a shift of the underlying white noise, in our case by the occupation measure of the underlying paths. Such representations were used in the mollified setting by, for example, Mukherjee et al. (2016) and Bröker and Mukherjee (2019).

The technical input for the uniform integrability of the approximating martingales are exponential moments of the mutual intersection local times off a small set. Geman, Horowitz and Rosen (1984), Le Gall (1994), Chen (2004) and Bass and Chen (2004) are recommended references for the mutual intersection local time and finiteness of rate one exponential moments without the cutoff.

## 2 Skorokhod integral

### 2.1 White noise

Let $(R, B, \mu)$ be a measure space. We will always assume $L^2(R, B, \mu)$ is separable, with orthonormal basis $\{e_j\}_{j=1,2,...}$. White noise defined on a probability space $(\Xi, \mathcal{F}, P)$, is a linear isometry $\xi : f \mapsto \langle f, \xi \rangle$ from $L^2(R, B, \mu)$ to $L^2(\Xi, \mathcal{F}, P)$ such that $\langle f, \xi \rangle, f \in L^2(R, B, \mu)$ form a mean zero Gaussian family. The isometry property says that

$$E_P[\langle f, \xi \rangle \langle g, \xi \rangle] = \langle f, g \rangle_{L^2(R, B, \mu)}$$

(18)

for all $f, g \in L^2(R, B, \mu)$. The bracket notation $\langle f, \xi \rangle$ suggests but does not rigorously mean an $L^2(R, \mathcal{R}, \mu)$ inner product, since $\xi$ cannot be realized in $L^2(R, B, \mu)$.

The white noise can be constructed in the following elementary way, which we use repeatedly. Define

$$\xi_j = \langle e_j, \xi \rangle.$$

(19)
From the above definition, these are independent Gaussians with mean 0 and variance 1 and for \( f \in L^2(R, \mathcal{B}, \mu) \) with \( f = \sum_{j=1}^{\infty} f_j e_j \) with \( f_j = \langle f, e_j \rangle \), we can realize
\[
\langle f, \xi \rangle = \sum_{j=1}^{\infty} f_j \xi_j
\]
which converges in \( L^2(\Xi, \mathcal{F}, P) \).

We will often think of functions, measures and noises in terms of their representations \( f_j = \langle f, e_j \rangle \), \( \mu_j = \langle \mu, e_j \rangle \) and \( \xi_j = \langle \xi, e_j \rangle \) in the orthonormal basis, and sometimes refer to these as coordinate representation.

Therefore in this paper we will take \((\Xi, \mathcal{F}, P)\) be a probability space given by an independent standard Gaussian sequence: \( \Xi = \mathbb{R}^N \), \( \mathcal{F} \) is the Borel \( \sigma \)-algebra on product space, and \( P \) is independent standard Gaussian product measure. It comes equipped with a natural filtration
\[
\mathcal{F}_n = \sigma(\xi_1, \ldots, \xi_n).
\]

We will mostly be considering planar white noise, which is the case \( R = \mathbb{R}^2 \), \( \mathcal{B} \) = Borel, \( \mu = \text{Lebesgue measure} \).

### 2.2 Definition of the integral

The Skorokhod integral is a notion of integral against white noise that does not need a time ordering. So it is particularly suited to problems involving higher dimensional time-independent white noise. The definition is elegant, and the concept is powerful, but it is hard to quickly see the motivation behind it. We first give a definition, and then provide motivation.

**Definition 7.** Let \( G \) be a \( \mathbb{R}^D \)-valued random variable, where \( D = \{1, 2, \ldots\} \) is finite or infinite. Let \( \xi_n, n \in D \) be independent standard Gaussian random variables. Assume that \( E|G_i| < \infty \) for each \( i \in \mathbb{N} \). We say that \( G \) is **Skorokhod integrable** if there exists
\[
S = \int G \xi \quad \text{in } L^1
\]
called the **Skorokhod integral** of \( G \) such that for every \( n \in D \) and every bounded differentiable \( F : \mathbb{R}^n \to \mathbb{R} \) with bounded gradient \( \nabla F \) we have
\[
ESF(\xi_1, \ldots, \xi_n) = EG \cdot \nabla F(\xi_1, \ldots, \xi_n).
\]

**Remark 8.** Let \( R \) be a measure space, \( e_i \) an orthonormal basis of \( L^2(R) \) consisting of bounded functions, and \( G \) be a random variable on \((\Xi, \mathcal{F}, P)\) taking values in finite total
absolute mass, signed measure on $\mathbb{R}$. Define $G_i = \langle G, e_i \rangle$. Then $G_1, G_2, \ldots$ is an $\mathbb{R}^N$-valued random variable, and if $\int G \xi$ exists in the sense of Definition 7 we will say it is the Skorokhod integral. This is somewhat closer to the standard usage. What we are doing is thinking of the Skorokhod integral as a renormalized extension of an $L^2$ inner product and working directly in the coordinate representation.

2.3 Projections

For infinite dimensional examples, it helps to define versions of the Skorokhod integral where only finitely many variables are integrated over. A natural projection property then determines the integral, see Proposition 11.

Definition 9 (Projections). Let $\mathcal{P}_n G$ denote the vector $G$ with all coordinates beyond $n$ set to zero and

$$\int_{[n]} G \xi := \int \mathcal{P}_n G \xi.$$

(23)

The following proposition can be thought of as a constructive definition of the Skorokhod integral in nice cases.

Proposition 10. 1. If $G$ is Skorokhod integrable, then its integral is unique a.s.

2. Let $G_j, \xi_j$ be the $j$th coordinate of $G, \xi$, respectively. Assume that for all $j$, and almost all $\xi_1, \ldots, \xi_{j-1}, \xi_{j+1}, \ldots$, the function $G_j$ is continuous and a.e. differentiable in $\xi_j$. If $E|G_i| < \infty$ and with

$$S_n = \sum_{j=1}^{n} G_j \xi_j - \partial \xi_j G_j, \quad E|S_n| < \infty$$

(24)

then

$$\int_{[n]} G \xi = S_n.$$

If $S_n$ converges in $L^1$ to a limit $S$ as $n \to \infty$, then $G$ is Skorokhod integrable and

$$\int G \xi = S.$$

Proof. If $S_1, S_2$ are two Skorokhod integrals, then by the definition $E[(S_1 - S_2)F] = 0$ for all bounded differentiable $F$ with bounded derivatives. Such functions $F : \mathbb{R}^n \to \mathbb{R}$ are dense in $L^\infty(\mathbb{R}^n)$. Hence $E[S_1 - S_2 \mid \mathcal{F}_n] = 0$ for each $n \in \mathbb{N}$, and so $S_1 - S_2 = 0$ a.s. For the second claim, let $F$ be as in the definition. We condition on the $\sigma$-field $\mathcal{J}_j$ generated by
Conditionally on this $\sigma$-field, $G_j$ is a continuous, almost everywhere differentiable function of $\xi_j$. By Gaussian integration by parts, we have

$$E[FG_j\xi_j|\mathcal{J}_j] = E[\partial_{\xi_j}(FG_j)|\mathcal{J}_j]$$

taking expectations, we get

$$E[FG_j\xi_j - F\partial_{\xi_j}G_j] = E[\partial_{\xi_j}(FG_j) - F\partial_{\xi_j}G_j] = E[(\partial_{\xi_j}F)G_j]$$

Note that $\partial_{\xi_j}F = 0$ for $j > n$. Summing over $j \leq m$ this gives that for $m \geq n$

$$E[S_mF] = E[G \cdot \nabla F].$$

Since $S$ converges in $L^1$ and $F$ is bounded, we get

$$E[SF] = \lim_{m \to \infty} E[S_mF] = E[G \cdot \nabla F]$$

as required.

The Skorokhod integral is linear, and has $ES = 0$, as we can see by taking $F = 1$ in the definition. It also behaves nicely under conditional expectations.

**Proposition 11.** If $G$ is Skorokhod integrable, then $E[\mathcal{P}_nG|\mathcal{F}_n]$ is Skorokhod integrable, and

$$E\left[\int G \xi | \mathcal{F}_n\right] = \int E[\mathcal{P}_nG | \mathcal{F}_n] \xi = \int_{[n]} E[G | \mathcal{F}_n] \xi$$

For $n = 0$ we get $E \int G \xi = 0$.

**Proof.** We need to check that for all test functions $F$ as in the definition, we have

$$E\left[F E[S|\mathcal{F}_n]\right] = E\left[\nabla F \cdot E[\mathcal{P}_nG | \mathcal{F}_n]\right]$$

By properties of the conditional expectation, the definition of $S$, we have

$$E\left[F E[S|\mathcal{F}_n]\right] = E\left[E[F|\mathcal{F}_n] S\right] = E\left[\nabla E[F|\mathcal{F}_n] \cdot G\right].$$

Now

$$\partial_j E[F|\mathcal{F}_n] = \begin{cases} E[\partial_j F|\mathcal{F}_n], & j \leq n \\ 0, & j > n \end{cases}$$

so we have

$$E\left[\nabla E[F|\mathcal{F}_n] \cdot G\right] = \sum_{j=1}^n E\left[E[\partial_j F|\mathcal{F}_n]G_j\right] = \sum_{j=1}^n E\left[\partial_j F E[G_j|\mathcal{F}_n]\right] = E\left[\nabla F \cdot E[\mathcal{P}_nG | \mathcal{F}_n]\right]$$

as required, showing the first equality. The last inequality follows by definition of the finite-dimensional version of the Skorokhod integral.
Corollary 12. Let $G$ be a $\mathbb{R}^N$-valued random variable such that $E|G_i| < \infty$ for each $i \in \mathbb{N}$. Let $H_n = E[\mathcal{P}_n G | \mathcal{F}_n]$.

The following are equivalent.

1. For every $n$, $S_n = \int H_n \xi$ exists and the $S_n$ are uniformly integrable.

2. $G$ is Skorokhod integrable.

If these conditions hold, then $S_n$ is a martingale with limit $\int G \xi$ almost surely and in $L^1$.

Proof. If 2 holds, by Proposition 11, $H_n$ is Skorokhod integrable and

$$S_n = E[\int G \xi | \mathcal{F}_n].$$

Since these are conditional expectations of a fixed integrable random variable, we see that $(S_n)$ are uniformly integrable, so 1 and the final conclusion hold.

If 1 holds, then the martingale property of $S_n$ follows from Proposition 11. Then $S_n$ is a Doob martingale and converges to some limit $S$ almost surely and in $L_1$.

$$E[FS] = \lim E[FS_n] = \lim E[\nabla F \cdot H_n] = E[\nabla F \cdot G].$$

In the last two equalities, we used that by definition, the coordinate $(H_n)_j$ for $n \geq j$ is a uniformly integrable martingale with limit $G_j$. Thus $G$ is Skorokhod integrable with integral $S$. \hfill \Box

We call a subspace $S \subset \mathbb{R}^N$ finitary if there is some $n$ so that all coordinates beyond $n$ of all vectors in $S$ are zero. The projection to a finitary subspace is well-defined: simply restrict vector to the first $n$ coordinates and then apply Euclidean projection there. Orthogonal invariance implies the following corollary to Proposition 11.

Corollary 13. Let $\mathcal{P}$ be projection to a finitary subspace of $\mathbb{R}^N$. Let $S$ be $\sigma$-field generated by the random variable $\mathcal{P} \xi$. If $G$ is Skorokhod integrable, then $E[\mathcal{P}G | S]$ is Skorokhod integrable, and

$$E\left[\int G \xi \bigg| S \right] = \int E[\mathcal{P} G | S] \xi.$$

2.4  Examples of Skorokhod integrals

We first consider some one-dimensional problems, as these are as simple as possible.
**Example 14.** Consider the case when $\Xi = \mathbb{R}$ houses a one-dimensional Gaussian $\xi$. Let’s solve the recursion

$$X_0 = 1, \quad X_k = \int X_{k-1} \xi, \quad k \geq 1.$$  

Then the solution is $X_n = H_n(\xi)$, where $H_n$ are the Hermite polynomials. This can be seen by the recursion $H_{k+1}(x) = xH_k(x) - H_k'(x)$ and the formula (24). See Example 20 for further connection between the Skorokhod integral and Hermite polynomials.

The following example is good to keep in mind when trying to construct discrete-time Wick-ordered polymers.

**Example 15.** Consider a one-dimensional Gaussian $\xi$, and the slightly more complex linear recursion

$$X_0 = 1, \quad X_k = X_{k-1} + \beta \int X_{k-1} \xi, \quad k \geq 1.$$  

Write the solution of the form $X_k = \beta^k p_k(\xi)$. Then $p_k$ satisfies

$$p_k = \beta^{-1} p_{k-1} + p_{k-1} \xi + p'_{k-1}, \quad p_0 = 1.$$  

This is just the Hermite recursion with $x = \beta^{-1} + \xi$, so the solution is $X_k = \beta^k H_k(\beta^{-1} + \xi)$, where $H_k$ are the Hermite polynomials.

Can basic functions be non-integrable?

**Example 16.** Still in one-dimension, let $G = 1(\xi \geq 0)$. The Skorokhod integral tries to be $\xi^+ + \delta_0(\xi)$, but of course this is not a random variable, as $\delta_0$ is not an honest function. It’s a simple exercise to show that $\int G \xi$ does not exist. Perhaps it could be defined by extending the probability space, but we do not pursue this direction.

Next, we study the continuous version of Example 14.

**Example 17** (A simple Skorokhod stochastic differential equation). We still work on the one-dimensional Gaussian space, and consider the “stochastic” differential equation $du = \int u \xi$ written in an integral form

$$u(t) = \int_0^t \left( \int_0^s u(s) \xi \right) ds, \quad u(0) = 1. \quad (25)$$  

This is not a usual SDE, as the entire randomness is based on a single Gaussian variable $\xi$! To solve this, we introduce the test function $F = e^{i\lambda \xi}$ satisfying $\nabla F = i\lambda F$, and let $\hat{u}(t) = E[u(t)F]$. We multiply (25) by $F$, take expectations and use the definition of the Skorokhod integral to get

$$\hat{u} = i\lambda \int_0^t \hat{u}, \quad \hat{u}(0) = e^{-\lambda^2/2}.$$  

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this is just the mild version of the equation $\hat{u}' = i\lambda \hat{u}$ solved uniquely by $\hat{u} = e^{i\lambda t - \lambda^2/2}$. This holds for all $\lambda$, so inverting the Fourier transform, we get $u(t) = e^{t\xi - t^2/2}$.

Remarkably, this method will also solve the Wick-ordered planar stochastic heat equation \[1\].

The planar Wick-ordered heat equation will be a continuum version of the following simple example.

**Example 18.** (Continuous time, finite state-space Wick-ordered polymer) Next, consider the evolution of a finite state-space $S$ continuous time Markov chain, think continuous time symmetric random walk on the $d$-cycle. The generator $K$ is a $d \times d$ matrix, and the transition probability vector $p$ from a given site $x$ satisfies the forward equation

\[
\partial_t p(y, t) = (p(\cdot, t)K)(y, t) \quad p(\cdot, 0) = 1_x.
\]

We add a fixed potential $V : S \to \mathbb{R}$, and then the partition function for a non-random "polymer" should look like

\[
\partial_t u(y, t) = (u(\cdot, t)K)(y) + u(y, t)V(y) \quad u(\cdot, 0) = 1_x.
\]

The Skorokhod version of this equation is written as

\[
\partial_t u(y, t) = (u(\cdot, t)K)(y) + \int u(y, t)1_y \xi \quad u(\cdot, 0) = 1_x.
\]

here $\xi$ is a standard $d$-dimensional Gaussian vector defined on $\Xi = \mathbb{R}^d$. The vector $1_y$ is there so that the noise corresponds to a diagonal potential.

Multiplying by $F = e^{i\sum_{y \in S} \xi(y)\lambda(y)}$. setting $\hat{u} = E[uF]$ and taking expectations we get

\[
\partial_t \hat{p}(x, t) = \hat{p}(\cdot, t)K(y) + \hat{p}(y, t)i\lambda(y), \quad \hat{p}(\cdot, 0) = 1_x e^{-\sum_{i} \lambda_i^2/2}.
\]

At this point, we leave it to the reader to use the discrete version of the Feynman-Kac formula and then Fourier inversion to conclude that

\[
p(y, t) = E_Q \left[ \exp \left\{ \sum_{z} X(z)\xi_z - X(z)^2/2 \right\} \mathbf{1}(X_t = y) \right].
\]

The expectation is under a measure $Q$ independent of the Gaussian space $\Xi$. Under the measure $Q$, the variable $X_t$ is the position of the original Markov chain at time $t$ when started from $x$, and $X(z) = \int_0^t \mathbf{1}(X_s = z)ds$ is the time spent at site $z$.

The reader will recognise the formula above, without the quadratic terms in the exponential, as the definition of the random polymer with Gaussian weights. The quadratic correction makes $Eu(y, t) = p(y, t)$. 

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2.5 Classical Skorokhod integral

To provide some motivation for the Skorokhod integral (our definition, as well as the classical one), let’s look at the finite dimensional case. A natural requirement is that for deterministic functions (in this case, vectors \( G \) in \( \mathbb{R}^n \)) the Skorokhod integral should be the inner product,

\[
\int \xi \xi = G_1 \xi_1 + \ldots + G_n \xi_n. \tag{26}
\]

Now let’s say we are in \( n = 1 \), and ask what the integral of a polynomial \( G \) in \( \xi \) should be. One option might be \( G \xi \), but it does not have mean zero, for example when \( G = \xi \).

If we want it to have zero expectation, we need to introduce a correction,

\[
\int G \xi = G \xi + \text{correction}. \tag{27}
\]

It is reasonable that the correction would be a polynomial in \( \xi \) of lower degree. Moreover, to be reminiscent of the Ito integral, we may want iterated integrals of 1 to be orthogonal. For polynomials \( G \), these two requirements define a unique integral

\[
\int G \xi = G \xi + \partial \xi, \quad n = 1,
\]

and there is a unique continuous extension for general integrands.

To treat the general finite dimensional case, it helps to first recall a very general construction.

Let \( H = L^2(R) \). The \( n \)-dimensional case corresponds to \( R = \{1, \ldots, n\} \) with counting measure. For each \( k \geq 1 \), define recursively \( \mathcal{H}_k \) as the set of symmetric polynomials in \( \xi_1, \xi_2, \ldots \) of degree \( k \), orthogonal to \( \mathcal{H}_{k-1} \), under the inner product \( E[FG] \). Then (see, for example, Janson (1997b))

\[
\bigoplus_{k=0}^\infty \mathcal{H}_k = L^2(\Xi, \mathcal{F}, P). \tag{28}
\]

Let \( \pi_k \) denote the orthogonal projection to \( \mathcal{H}_k \).

\( \mathcal{H}_0 \) consists of non-random functions on \( R \), and can be identified with \( H \). For \( k \geq 1 \), one needs some symmetrization (after all, the polynomials \( \xi_1 \xi_2 \) and \( \xi_2 \xi_1 \) mean the same thing.) Let \( H^\otimes_s k = L^2(R^k, \mathcal{R}_s^k, \frac{1}{k!} \mu^k) \), where \( \mathcal{R}_s^k \) is the \( \sigma \)-algebra of symmetric \( \mathcal{R}_s^k \)-measurable functions on \( R^k \). We now explain how to represent \( \mathcal{H}_k \) by \( H^\otimes_s k \). The symmetric Fock space \( \bigoplus_{k=0}^\infty H^\otimes_s k \) is thus identified with \( \bigoplus_{k=0}^\infty \mathcal{H}_k \), and therefore by (28), with the Hilbert space \( L^2(\Xi, \mathcal{F}, P) \) of all functions of \( \xi_1, \xi_2, \ldots \) with finite second moment. Given \( G \in H^\otimes_s k \), call

\[
\text{Sym}(G) = \sum_{\sigma \in S_k} G \circ \sigma
\]

where \( S_k \) is the symmetric group. In particular, for \( g_1, \ldots, g_k \in H \), \( g_1 \otimes \cdots \otimes g_k \) is the element of \( L^2(R^k) \) which takes \( (x_1, \ldots, x_k) \) to \( \prod_{i=1}^k g(x_i) \) and \( \text{Sym}(g_1 \otimes \cdots \otimes g_k) \) is the
function that takes \((x_1, \ldots, x_k)\) to \(\sum_{\sigma \in S_k} \prod_{i=1}^{k} g_i(x_{\sigma_i}).\) Linear combinations of such are dense in \(H^{\otimes k}\).

For \(g \in H\) define \(I_1(g) = \langle g, \xi \rangle\), as in the definition of the white noise. For \(k \geq 2\), and \(g_1, \ldots, g_k \in H\) let

\[
I_k(\text{Sym}(g_1 \otimes \cdots \otimes g_k)) = \pi_k(I_1(g_1) \cdots I_1(g_k)).
\]

(29)

It can then be extended to \(H^{\otimes k}\) by linearity and density. For \(G \in H^{\otimes k}\), \(\int G \xi = I_k(G)\) is its multiple stochastic integral.

**Definition 19.** Let \(G : R \to L^2(\Xi, \mathcal{F}, P)\). It follows from (28) that \(G(x) = \sum_{k=1}^{\infty} \pi_k(G(x))\) in \(L^2\). Since for each \(x\), \(\pi_k(G(x))\) is a function of \(k\) variables, we can think of \(\pi_k(G)\) as function of \(k + 1\) variables and set

\[
\int \pi_k(G) \xi = I_{k+1}(\text{Sym}(\pi_k(G))).
\]

(30)

If \(\int G \xi := \sum_{k=0}^{\infty} \int \pi_k(G) \xi\) converges in \(L^2(\Xi, \mathcal{F}, P)\), we call it the classical Skorokhod integral of \(G\) and call \(G\) classically Skorokhod integrable. Note that we use the notation \(\int G \xi\) to distinguish it from our previous definition of the Skorokhod integral \(\int G \xi\).

**Example 20.** Let \(R = \{1, \ldots, n\}\) with counting measure. Then \(H = L^2(M, \mathcal{M}, \mu)\) is \(\mathbb{R}^n\) with the standard inner product. \(H^{\otimes k}\) is the set of real functions defined on \(\{1, \ldots, n\}^k\), or, equivalently the positions in \(\{1, \ldots, n\}\) of \(k\) distinguishable particles. \(H^{\otimes k}\) then corresponds to the set of real functions defined on the occupation measures of \(k\) indistinguishable particles on \(\{1, \ldots, n\}\) represented by \(\eta = (\eta_1, \ldots, \eta_n) \in \mathbb{N}^n\) with \(|\eta| := \sum_i \eta_i = k\), and inner product \(\sum_{|\eta| = k} G(\eta) F(\eta)\), for \(G, F \in H^{\otimes k}\). Let \(H_k, k = 0, 1, \ldots\) be the Hermite polynomials. They can be defined through their generating function

\[
e^{\lambda x - \frac{1}{2} \lambda^2} = \sum_{k=0}^{\infty} H_k(x) \frac{\lambda^k}{k!}.
\]

(31)

One checks directly that \(I_k : H^{\otimes k} \to H_k\) is given in this representation by

\[
I_k(G) = \sum_{|\eta| = k} G(\eta) H_{\eta_1}(\xi_1) \cdots H_{\eta_n}(\xi_n),
\]

(32)

so the classical Skorokhod integral can just be written, by linearity, as a sum over \(|\eta| = k\) of

\[
\int G(x, \eta) H_{\eta_1}(\xi_1) \cdots H_{\eta_n}(\xi_n) \xi(x) = \sum_{i=1}^{n} \text{Sym}(G)(\eta + 1) H_{\eta_1}(\xi_1) \cdots H_{\eta_{i+1}}(\xi_i) \cdots H_{\eta_n}(\xi_n).
\]

(33)
Using the Hermite polynomial identity
\[ H_{k+1}(x) = xH_k(x) - \partial_x H_k(x), \]
we obtain that for every \( G \in H^{\otimes,k} \),
\[ \int G \xi = \sum_{i=1}^n G_i \xi_i - \partial \xi_i G_i = \int G \xi \]
where in the last equality we have used (24).

We can now deduce the following lemma.

**Lemma 21.** Let \( R = \{1, \ldots, n\} \) and let \( G \) be a function of the symmetric Fock space \( \sum_{k=0}^\infty H^{\otimes,k} \). Here \( H^{\otimes,k} \) is the set of real functions defined on \( \{1, \ldots, n\}^k \), such that \( \sum_{k=0}^\infty \sum_{|\eta|=k} E[\pi_k(G(\eta))] < \infty \). Then, if \( G \) has a classical Skorokhod integral, it is Skorokhod integrable. Furthermore, in this case, these integrals are equal.

**Proof.** From the previous example, we know that the classical Skorokhod integral and the Skorokhod integral coincide for each \( k \) in the space \( H^{\otimes,k} \). It follows that for each \( K \), if \( G_K = \sum_{k=0}^K \pi_k(G) \), then \( S_K := \int G_K \xi = \int G_K \xi \). Therefore, for all \( F : \mathbb{R}^n \to \mathbb{R} \) we have that \( E[S_K F] = E[G_K \cdot \nabla F] \). Taking the limit \( K \to \infty \) in the above inequality, we conclude that \( S = \sum_{k=0}^\infty \int \pi_k(G) \xi \) satisfies
\[ E[S F] = E[G \cdot \nabla F], \]
so that \( G \) is Skorokhod integrable and its Skorokhod integral is equal to its classical Skorokhod integral. \( \square \)

**Lemma 22.** Let \( G : R \to L^2(\Xi, \mathcal{F}, P) \) be a random function such that \( \int_{\mathbb{R}} E[G(x)^2] d\mu < \infty \). Then, if \( G \) has a classical Skorokhod integral, it is Skorokhod integrable. Furthermore, in this case, these integrals are equal.

**Proof.** Lemma 21 says that the statement holds in the finite dimensional case. For the general case, we use the fact that integrands \( G \) with
\[ \sum_{k=0}^\infty (k+1) \int_{\mathbb{R}} E[\pi_k(G)^2] d\mu < \infty \]  (34)
form the domain of the classical Skorokhod integral (see, for example, Theorem 7.39 of Janson (1997b)). Take such a \( G \) and let \( G_n \) be the truncation of its coordinate representation.

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Let \( S_n \) be its classical Skorokhod integral. We have shown in Example \ref{example:20} that this coincides with our Skorokhod integral, so that for all \( F : \mathbb{R}^n \to \mathbb{R} \) which are bounded with bounded gradient,

\[
E[S_n F] = E[G_n \cdot \nabla F].
\]  (35)

If \( S \) is the classical Skorokhod integral of \( G \) then

\[
E|S_n - S|^2 = \sum_{k=0}^{\infty} E \left[ \left| \int \pi_k(G_n) - \pi_k(G) \xi \right|^2 \right].
\]  (36)

From the definition \ref{definition:30} of the classical Skorokhod integral as a simple raising operator on the \( \mathcal{H}_k \)-s, we have for any \( G \),

\[
\sum_{k=0}^{\infty} E \left[ \left| \int \pi_k(G) \xi \right|^2 \right] = \sum_{k=0}^{\infty} (k + 1) \int_{\mathbb{R}} E[\pi_k(G)^2] d\mu.
\]

Since \( G_n \to G \) in this space from \ref{equation:34}, \( S_n \) converges to \( S \) in \( L^2(\Xi, \mathcal{F}, P) \). Now we can take limits on both sides of \ref{equation:35} to conclude that \( E[S F] = E[G \cdot \nabla F] \).

Remark 23 (Itô integral in 1 dimension). Let \( \xi \) be a \( d \)-dimensional white noise on \( \mathbb{R} \) and \( B(t) = \xi(1_{[0,t]}), d \)-dimensional Brownian motion on \( (\Xi, \mathcal{F}, \mathbb{P}) \). Let \( \mathcal{F}_t = \sigma(B(s), s \in [0, t]) \).

The space of \( d \)-dimensional, square integrable adapted processes \( G(t) \) is the closure in \( L^2(\Xi, \mathcal{F}, \mathbb{P}; \mathbb{R}^d) \) of those which can be written as \( G(t) = \sum_{i=0}^{n-1} g_i 1_{(t_i, t_{i+1})}(t) \) where \( g_i \) are bounded and measurable with respect to \( \mathcal{F}_{[0,t_i]} \).

For any such \( G(t) \), \( \int G \xi \) coincides with the Itô integral \( \int G(t) dB(t) \).

Remark 24 (Itô integral in 1+1 dimensions). Let \( \xi \) be white noise on \( \mathbb{R}_+ \times \mathbb{R} \) and \( B(x, t) = \xi(1_{[0,t]} \times [0,x]), x \geq 0 \) and \( B(x, t) = -\xi(1_{[0,t]} \times [x,0]), x < 0 \) be the Brownian sheet.

Let \( \mathcal{F}_{[0,t]} = \sigma(B(s, x), s \in [0, t], x \in \mathbb{R}) \). The space of square integrable, time-adapted processes \( G(x, t) \) is the closure in \( L^2(\Xi, \mathcal{F}, \mathbb{P}) \) of those which can be written as

\[
G(x, t) = \sum_{i,j=0}^{n-1} g_{i,j} 1_{(t_i, t_{i+1}) \times (x_i, x_{i+1})}(x, t),
\]

where \( g_{i,j} \) are bounded and measurable with respect to \( \mathcal{F}_{[0,t_i]} \).

For such \( G(x, t) \), \( \int G \xi \) coincides with the Itô integral \( \int G(x, t) B(dt, dx) \).
Crucially for us, the second part of the theorem tells us that the standard $1 + 1$ dimensional multiplicative stochastic heat equation coincides with its Skorokhod version. This can also be seen through the chaos representation.

**Example 25.** We try to solve the integral equation with initial data $\mu = \delta_0$ in the Skorokhod sense

$$Z(x, t) = p(x, t) + \int_0^t \int_0^t p(x - y, t - s) Z(y, s) ds \xi(y) dy. \quad (37)$$

We look for a solution $Z(x, t) \in L^2(P)$. As above we have $Z(x, t) = \sum_{k=0}^{\infty} I_k(g_k)$ for some $g_k(\cdot; t, x)$ in symmetric $L^2((\mathbb{R}^2)^k)$. Using (30) this means that $I_0(g_0) = p(x, t)$ and

$$I_{k+1}(g_{k+1}(x, t)) = \int_0^t \int_0^t p(x - y, t - s) I_k(g_k(y, s)) ds \xi(y) dy \quad (38)$$

Here one should think of $g_k \in H^{\otimes k}$ as depending on the extra parameters $t, x$. It is not hard to see that the solution is

$$g_k(x_1, \ldots, x_k; t, x) = E_Q[m(x_1) \cdots m(x_k)] \quad (39)$$

where $m$ is the occupation measure of the Brownian bridge from $0$ at time $0$ to $x$ at time $t$. Of course, this can also be written explicitly as

$$\int_{0<s_1<\ldots<s_k<t} \prod_{j=0}^{k} p(s_{j+1} - s_j, x_{j+1} - x_j) \quad (40)$$

where $p(x, t)$ is the planar heat kernel (2) and $y_{k+1} = x, y_0 = 0$.

By comparison, in PAM, the $p$’s in (40) are weighted by the exponential of the renormalized self-intersection local time (see [Gu and Huang (2018)]).

In $1 + 1$ dimensions, the analogue of (39) has the occupation measure of $(b(s), s)$ where $b$ is the one-dimensional bridge. The analogue of (40) can be found in [Amir et al. (2011)]. One checks directly that the $L^2$ norm of $Z$, which is the sum over $k$ of the $L^2$ norms of the $g$’s, is finite in $1 + 1$ dimensions, for all $t$, and only finite up to $t_c < \infty$ in the planar case.

### 3 Solving Wick-ordered heat equations

#### 3.1 Projections of the equation and their solution

Perhaps the most useful property on the Skorokhod integral is that the conditional expectation of solutions of (1) with respect to $\mathcal{F}_n$ (defined in (21)) are themselves solutions of a stochastic heat equation, but now with respect to finite dimensional noise. Moreover, we can solve such equations in an explicit way.
Proposition 26. Let \( u \) be a solution of the Wick-ordered heat equation, Definition \( 1 \). Then \( u_n(x, t) := E[u(x, t)|\mathcal{F}_n] \) solves the following problem: For almost all \( x, t \) the random variable \( K_{x,t}(u_n) \) is \( \mathcal{F}_n \)-measurable, Skorokhod integrable, and
\[
u_n(x, t) = \int p(x - y, t) d\xi(y) + \int [K_{x,t}(u_n)] \xi. \tag{41}
\]

Proof. This follows by taking conditional expectation of (3). Only the last term on the right needs explanation. By Proposition \( 11 \)
\[
E[\int [K_{x,t}(u)] \xi|\mathcal{F}_n] = \int [K_{x,t}(u)] \xi.
\]

Since \( K_{x,t}u \) is Skorokhod-integrable, it must have finite expectation. Since \( u \geq 0 \), by Fubini, \( E[K_{x,t}u|\mathcal{F}_n] = K_{x,t}u_n \) as required. \( \square \)

Note that \( u_n \) is defined on an \( n \)-dimensional Gaussian space. Thus the equation (41) is much simpler than a true stochastic PDE. In fact, it simply a weak version of the deterministic linear PDE for \( u(x, t, \xi_1 \ldots \xi_n) \) given by
\[
\partial_t u = \left( \frac{1}{2}(\partial_x^2 + \partial_y^2) + \sum_{j=1}^n e_j(\xi_j - \partial_{\xi_j}) \right) u
\]
and the solution is a real analytic function of \( \xi_1, \ldots, \xi_n \). The next proposition connects the Wick-ordered heat equation to polymers.

Proposition 27 (Existence, uniqueness and explicit solution for finite \( n \)). The solution \( u_n \) to (41) exists and has the following representation. For almost all \( t, x, \)
\[
u_n(x, t) = \int Z_n(y, 0; x, t)p(x - y, t) d\xi(y)
\]
where, for \( Q = Q_{x,t,y} \) the law of a planar Brownian bridge \( B(\cdot) \) from \( y \) at time \( 0 \) to \( x \) at time \( t \)
\[
Z_n(0, y; t, x) = E_Q \exp \left( \sum_{j=1}^n m_j \xi_j - \frac{1}{2}m_j^2 \right), \quad m_j = \int_0^t e_j(B(s)) ds. \tag{42}
\]
Proof. We solve the equation by introducing the test functions
\[
F(\xi) = \exp \left( i \sum_{j=1}^n \lambda_j \xi_j \right), \quad \lambda_i \in \mathbb{R}
\]
as these have nice partial derivatives. If \( G \) is an \( \mathcal{F}_n \)-measurable random variable taking values in \( L^1(\mathbb{R}^2) \), then by definition of the finite-dimensional Skorokhod integral,
\[
E_P \left[ F \int [n] G \xi \right] = E_P \left[ \nabla F \cdot \mathcal{P} G \right] = i \sum_{j=1}^n \lambda_j E[G_j F].
\]
Let us apply this to the Skorokhod integral in our equation, that is, for the case \( G = \mathcal{P}_{x,t} u_n \). The summands on the right become

\[
E \left[ G_j F \right] = E \left[ \left( \mathcal{K}_{x,t} u_n \cdot e_j \right) F \right] = E \left[ F \int \mathcal{K}_{x,t} u_n(y) e_j(y) dy \right]
\]  

(43)

We expand, and apply Fubini to get

\[
\int_0^t \int \text{p}(x - y, t - s) e_j(y) E \left[ u_n(y, s) F \right] dy ds
\]  

(44)

After summing, we get

\[
E \left[ F \int \mathcal{K}_{x,t} u_n \xi \right] = E \left[ F \int_0^t \int \text{p}(x - y, t - s) V(y) E \left[ u_n(y, s) F \right] dy ds \right]
\]

where

\[
V(y) = i \sum_{j=1}^n \lambda_j e_j(y).
\]

Thus if we multiply (41) by \( F \) and take expectations, we see that

\[
\hat{u}(y, s) := E \left[ u_n(y, s) F \right]
\]

solves the Duhamel form of the ordinary heat equation

\[
\partial_t \hat{u}(x, t) = \left( EF \right) \int \text{p}(x - y, t) d\zeta(y) + \int \mathcal{K}_{x,t} (V \hat{u})(y) dy
\]

(45)

where \( EF = e^{-\frac{1}{2} \sum_{j=1}^n \lambda_j^2} \) is a constant. The standard form of this equation is

\[
\partial_t \hat{u} = \frac{1}{2} \left( \partial^2_{x_1} + \partial^2_{x_2} \right) \hat{u} + V \hat{u}.
\]

The solution is represented by the Feynman-Kac formula,

\[
\hat{u}(x, t) = \int E_{Q_{x,t,y}} \exp \left( i \sum_{j=1}^n \lambda_j \int_0^t e_j(B(s)) ds - \frac{1}{2} \lambda_j^2 \right) p(x - y, t) d\zeta(y)
\]

(46)

where \( B \) is a Brownian bridge in \( \mathbb{R}^2 \) starting at \( y \) and ending at \( x \) at time \( t \), and \( Q_{x,t,y} \) is its law. The representation (46) can be obtained directly from (45) by repeated iteration.

Since the solution is linear in the initial data \( \zeta \), it suffices to consider \( \zeta = \delta_y \), and write any other data as a superposition. In this case it is convenient to rewrite

\[
u_n(x, t) = Z_n(y, 0; x, t) p(x - y, t).
\]

(47)
Now fix \((x, t)\), and note that \(\hat{u}\) as a function of the \(\lambda_i\) is the Fourier transform of the function \((\xi_1, \ldots, \xi_n) \mapsto u_n(x, t, \xi_1, \ldots, \xi_n)\varphi(\xi_1) \cdots \varphi(\xi_n)\) where \(\varphi\) is the standard Gaussian density.

Inverting the Fourier transform, we conclude that for almost all \(t \geq 0, x \in \mathbb{R}^2\), with \(m_j\) as in (42) we have

\[
Z_n(0, y; x, t) = E_{Q_{x, t, y}} \exp \left( \sum_{j=1}^{n} m_j \xi_j - \frac{1}{2} \sum_{j=1}^{n} m_j^2 \right).
\]

(48)

\[\square\]

**Remarks.**

1. \(Z_n\) is the Radon-Nikodym derivative of the distribution of \((\xi_1 + m_1, \ldots, \xi_n + m_n)\) under the product measure of the laws of the white noise \(P\) and the Brownian bridge \(Q\), with respect to that of the white noise \(P\).

2. \(Z_n\) is a non-negative martingale with respect to the filtration \(\mathcal{F}_n = \sigma(\xi_1, \ldots, \xi_n)\) and hence converges almost surely to a random variable \(Z\).

3. If \(E_P[Z] = 1\) we have a candidate for the solution of the Wick-ordered heat equation (1).

4. If \(E_P[Z] = 1\), \(Z\) has the interpretation as the Radon-Nikodym derivative of the distribution \(\hat{P}\) of the shifted white noise \(\xi + X\) with respect to the distribution \(P\) of \(\xi\), where \(X\) is the occupation measure of the Brownian bridge up to time \(t\). The expected free energy is equal to the relative entropy of \(\hat{P}\) with respect to \(P\),

\[
E_{\hat{P}} \log Z = H(\hat{P} | P).
\]

5. For a.e. realization of the Brownian bridge, \(X\) is not in the Cameron-Martin space; if it were, we should have \(\int (\frac{du}{dx})^2 dx = \sum_{j=1}^{\infty} m_i^2 = 2 \int_{0 \leq s_1 < s_2 \leq t} \delta_0(B(s_2) - B(s_1)) ds_1 ds_2\), the 2-dimensional self-intersection local time. This sum needs a diverging shift renormalization to exist. \(Z \in L^1(P)\) because of the expectation over \(Q\). We call \(Z\) a randomized shift, see, for example, Shamov (2016). We will formally introduce randomized shifts in Section 4.

6. The solution of (11) does not satisfy the semigroup property,

\[
u(x, t + s) \neq \int u(y, s)u(x - y, t)dy.
\]

On the other hand, \(\hat{u}(x, t; \vec{\lambda})\) does satisfy the semigroup property.
3.2 Solving the planar Wick-ordered heat equation

Proposition 26 shows that for almost all \( x, t \) the solution \( u_n(x, t) \) is a martingale in \( n \).

**Proposition 28 (Solving the Wick-ordered heat equation).** Suppose that for almost all \( (x, t) \) the martingale \( u_n(x, t) \) converges in \( L^1 \). Then the limit \( u(x, t) \) is a solution of the Wick-ordered heat equation (1). The solution is unique \( x, t, \xi \)-almost everywhere.

Almost everywhere uniqueness here means that any two solutions agree \( x, t, \xi \) a.e. Note that there is considerable work involved in showing \( L^1 \) convergence (the assumption of Proposition 28). This will be addressed in the later sections of the article.

**Proof.** First we have to show that for almost all \( x, t \) if we write

\[
 u(x, t) = \lim_{n \to \infty} u_n(x, t)
\]

then we also have

\[
 \int K_{x,t}u \xi \bigg|_{[n]} = \lim_{n \to \infty} \int K_{x,t}u_n \xi.
\] (49)

The finite-dimensional version of the Wick-ordered heat equation (41) says

\[
 \int K_{x,t}u_n \xi = u_n(x, t) - \int p(x - y, t) d\xi(y)
\] (50)

since \( u_n(x, t) \) is uniformly integrable, so is \( \int K_{x,t}u_n \). Thus by Corollary 12, (49) holds almost surely and in \( L^1 \), and so \( u \) satisfies the Wick-ordered heat equation (3).

The solution is unique, since it is determined by its conditional expectations, which are unique by Propositions 26 and 27.

3.3 The SHE in one dimension with space-time noise

In this section, we outline how the \( 1 + 1 \) dimensional stochastic heat equation, usually defined through a chaos expansion, can be rigorously defined and solved using the Skorokhod integral.

The method we describe works for a general stochastic PDE of the form \( \partial_t v = Lv + \xi v \), where \( L \) is a generator for a stochastic process. Algebraically, such equations are equivalent to Example 18, although in many cases there are technical challenges or even analytic obstacles to a solution.

There is no significant algebraic difference between the pure space and space-time version of the processes: the time coordinate can just be added as an extra spatial coordinate.
Definition 29. Let $\varsigma$ be a finite measure on $\mathbb{R}$. A measurable function $z : (0, T] \times \mathbb{R} \times \Xi \to [0, \infty)$ is a solution of the one-dimensional stochastic heat equation with space-time white noise and initial condition $\varsigma$ if for almost all $(x, t) \in (0, T] \times \mathbb{R}$ the random function of $(y, s)$

$$q(x - y, t - s)z(y, s)$$

(51)

where $q(x, t)$ is the one-dimensional heat kernel (12) is Skorokhod-integrable, and satisfies

$$z(x, t) = \int q(x - y, t)d\varsigma(y) + \int \int q(t - \cdot, x - \cdot)z\xi.$$  

(52)

Although we are working in $[0, \infty) \times \mathbb{R}$ we can just think of it as a subset of $\mathbb{R}^2$ and use the same orthonormal basis $e_j$ of $L^2(\mathbb{R}^2)$ of bounded functions.

Proposition 30. Let $z$ be a solution of the one-dimensional stochastic heat equation, Definition 29. Then $z_n(x, t) := E[z(x, t)|\mathcal{F}_n]$ solves the following problem: For almost all $x \in \mathbb{R}$ and $t > 0$,

$$z_n(x, t) = \int q(x - y, t)d\varsigma(y) + \int \int z_n(y, s)q(x - y, t - s)\xi.$$  

(53)

Proof. As in the proof of Proposition 26, it is enough to show that

$$E[\int z(y, s)q(x - y, t - s)\xi|\mathcal{F}_n] = \int \int z_n(y, s)q(x - y, t - s)\xi.$$  

This follows as before by Fubini and the fact that $z$ is Skorokhod-integrable.

In this case $z_n$ satisfies the equation

$$\partial_t z_n = \frac{1}{2} \partial^2_x z_n + \sum_{j=1}^n e_j (\xi_j - \partial_{\xi_j}) z_n

$$

with $e_j = e_j(x, t)$.

Proposition 31. (Existence, uniqueness and explicit solution for finite $n$) The solution $z_n$ of equation (53) exists and is unique in the following sense. For almost all $x, t$ we have

$$z_n(x, t) = \int Z_n(y, 0; x, t)q(x - y, t)d\varsigma(y)$$

where $Q$ is the law of one-dimensional Brownian bridge $B$ from $y$ at time $0$ to $x$ at time $t$ and

$$Z_n(0, y; t, x) = E_Q \exp \left( \sum_{j=1}^n m_j \xi_j - \frac{1}{2} m_j^2 \right), \quad m_j = \int_0^t e_j(B(s), s)ds.$$  

(54)
Proof. The proof is really the same as Proposition 27. We introduce test functions

\[ F(\xi) = \exp \left( i \sum_{j=1}^{\lambda_j \xi_j} \right), \quad \lambda_j \in \mathbb{R}. \]

For \( G \) a \( \mathcal{F}_n \)-measurable random variable with values in \( L^1([0, \infty) \times \mathbb{R}) \), we have

\[ E \left[ F \int_{[n]} G \xi \right] = E[\nabla F \cdot \mathcal{P}_n G] = i \sum_{j=1}^{n} \lambda_j E[G_j F]. \]

We apply this to the case \( G(y, s) = z_n(y, s)q(x - y, t - s) \) so that

\[ E[G_j F] = E[F \int z_n(y, s)q(x - y, t - s)e_j(x, s)dxds] \]
\[ = \int q(x - y, t - s)e_j(x, s)E[z_n(y, s)F]dyds. \]

After summing we get

\[ E[F \int z_n(y, s)q(x - y, t - s)\xi] = \int q(x - y, t - s)V(y, s)E[z_n(y, s)F]dyds, \]

where \( V(y, s) = i \sum_{j=1}^{n} \lambda_j e_j(y, s) \), and it follows that \( \hat{z}(y, s) = E[z_n(y, s)F] \) solves

\[ \partial_t \hat{z}(x, t) = (EF) \int q(x - y, t - s)d\zeta(y) + \int q(x - y, t - s)z(y, s)\hat{z}(y, s)dyds. \]

As before, we invert the Fourier transform to get (54). \( \square \)

To take the limit as \( n \to \infty \), we will need the moment bounds established in the next section. The last step of the solution is given in Section 7.2.

4 Randomized shifts

In this section, we consider the solution of the Wick-ordered heat equation from another angle, that of randomized shifts, or, using different terminology, Gaussian multiplicative chaos. The definition we use is closely related to Shamov (2016). We will first define the notion and give motivation later.

As before, let \((\Xi, \mathcal{F}, P)\) be a probability space given by an independent standard Gaussian sequence: \( \Xi = \mathbb{R}^N \), and \( \mathcal{F} \) is the Borel \( \sigma \)-algebra on product space, and \( P \) is independent standard Gaussian measure.

Let \( m = (m_1, m_2, \ldots) \) be a sequence of real valued random variables on another probability space \((\Omega, \mathcal{G}, Q)\).
**Definition 32.** The random variable $Z$ defined on $\Xi$ is called the **partition function** of the randomized shift $m$ if for all bounded measurable functions $F : \Xi \to \mathbb{R}$, we have

$$E_P[ZF] = E_{P\times Q}F(\xi + m).$$  

(55)

Note that $Z$ depends only on the law of $m$, not $m$ as a random variable; nonetheless for simplicity we will sometimes indicate the dependence by writing $Z = Z(m)$.

The definition is much more concrete when we consider $P_n m$, that is, $m$ with all coordinates beyond the $n$th set to zero. In this case, one checks directly that

$$Z_n(m) := Z(P_n m) = E_Q \exp \sum_{j=1}^n m_j \xi_j - m_j^2 / 2$$

(56)

satisfies the definition. Note that the right hand side is well-defined and finite for arbitrary $m$, moreover, it is a non-negative $F_n$-martingale and therefore converges almost surely. We will need to know when the convergence is in $L^1$. This requires uniform integrability.

We have the following representation.

**Proposition 33.** For any bounded $F \in F$, and more generally: for any measurable $F$ for which the left hand side is well-defined, for example any $F \geq 0$, with $P_n$ the projection to the first $n$ coordinates, defined in the preamble to (23),

$$E_P[FZ_n] = E_{Q\times P} F(\xi + P_n m).$$

(57)

In particular,

(a) $E_P Z_n = 1$;

(b) $E_P[Z_n^2] = E_{Q\otimes^2} \exp \sum_{i=1}^n m_i m_i'$. The second expectation is over the product of $Q$ with itself and $m, m'$ are two independent copies. Note that this may be infinite, even for finite $n$.

(c)

$$E_P[Z_n^k] = E_{Q\otimes^k} \exp \left\{ \sum_{j, \ell=1}^k \sum_{j \neq \ell}^n m_{ij}^{(j)} m_{i\ell}^{(\ell)} \right\}$$

where $m_{i1}^{(1)}, \ldots, m_{ik}^{(k)}$ are independent copies with the same caveat as in (b).

(d) For any $A \in \mathcal{G}$, $E|Z_n(m) - Z_n(1_A m)| \leq 2Q(A^c)$.

\footnote{Shamov calls this the total mass of the GMC}
Proof. 1.

\[ E_P[Z_n \mid \mathcal{F}_{n-1}] = E_Q \left[ \exp \left\{ \sum_{i=1}^{n-1} m_i \xi_i - \frac{1}{2} \sum_{i=1}^{n} m_i^2 \right\} E_P[\exp \{m_n \xi_n - \frac{1}{2} m_n^2\}] \right] = Z_{n-1}. \]

(57) is just the definition of the Gaussian measure together with Fubini’s theorem, which applies because the integrand is non-negative. To prove (a-b-c) by induction take \( F = Z_{n-1} \) in (57) and expand the exponential. For \( \delta \), note that if \( m_c \) is the conditional law of \( m \) given \( A^c \), then

\[ |Z_n(m) - Z_n(\mathbb{1}_A m)| = Q(A^c)|Z_n(m_c) - 1|. \]

We already have examples from the previous section.

Example 34. In Example 18 of a continuous time, finite state-space Wick-ordered polymer we found the following. For the polymer ending at time \( t \) and position \( y \) the partition function can be written as

\[ p(y,t) = E_Q \left[ \exp \left\{ \sum_z X(z) \xi_z - X(z)^2 / 2 \right\} \mathbb{1}(X_t = y) \right]. \]

The expectation is under a measure \( Q \) independent of the Gaussian space \( \Xi \). Under the measure \( Q \), the variable \( X_t \) is the position of the original Markov chain at time \( t \) when started from \( x \), and \( X(z) = \int_0^t \mathbb{1}(X_s = z) ds \) is the time spent at site \( z \).

This is exactly in the form of Definition 32 with \( m_j = X(s_j) \) for \( j \leq |S| \) and \( m_j = 0 \) otherwise.

Example 35 (Bayesian statistics). Consider trying to guess the mean of Gaussian random variable with variance 1. The mean, unknown to us, is zero. Our prior distribution on \( \Omega = \mathbb{R} \) is denoted \( Q \). Let \( n \) be fixed and let

\[ m_1(\omega) = \ldots = m_n(\omega) = \omega. \]

If we repeatedly observe \( \xi_1, \ldots, \xi_n \), the likelihood is given by \( Z_n \). A good exercise is to check that the partition function \( Z = \lim_n Z_n \) exists if and only if \( Q\{0\} > 0 \).

After \( n \) samples, the posterior distribution \( M_\xi \) will have the interpretation as the polymer measure or Gaussian multiplicative chaos of \( Z_n \) in this toy setting, but we will leave this to Section 4.6 Example 52.

4.1 Convergence in \( L^1 \)

Since \( Z_n \) given in (56) is a non-negative martingale, we have \( Z_n \to Z \) \( P \)-almost surely. In order to extend (57) to \( Z \), it is important that mass not be lost in the limit.
Lemma 36. Suppose that \( Z_n \to Z \) in \( L^1(\Xi,\mathcal{F},P) \) (or, equivalently, that \( Z_n \) is uniformly integrable). Then the marginal law
\[
\hat{P}(A) := (P \times Q)(\xi + m \in A)
\]
of \( \xi + m \) on \((\Xi,\mathcal{F})\) is absolutely continuous with respect to \( P \). Also, \( Z := \lim_{n \to \infty} Z_n \) in \( L^1(\Xi,\mathcal{F},P) \) and \( P\)-almost surely is the Radon-Nikodym derivative
\[
Z = \frac{d\hat{P}}{dP}.
\]

Proof. For \( A \in \mathcal{F} \) define a probability measure \( \tilde{P}(A) = E_P[Z1_A] \). We want to show that \( \tilde{P} = \hat{P} \). For \( A \in \mathcal{F}_n \), \( E_P[Z1_A] = E_P[Z_n1_A] \) by the martingale property, and
\[
E_P[Z_n1_A] = (P \times Q)(\xi + m \in A)
\]
by the previous proposition. So \( \tilde{P} = \hat{P} \) on \( S = \bigcup_{n=1}^{\infty} \mathcal{F}_n \). Since \( S \) forms a \( \pi \)-system, by the \( \pi - \lambda \) theorem \( \tilde{P} = \hat{P} \) on \( \sigma(S) = \mathcal{F} \). \( \square \)

4.2 Properties of randomized shifts

Lemma 37 (0-1 law). If each \( |m_i| \leq b_i \) for some deterministic \( b_i < \infty \), then \( P(Z = 0) \in \{0,1\} \).

A slightly different 0-1 law for GMC is shown in [Mukherjee et al. (2016)]

Proof. Fix \( k \geq 1 \). We write \( Z = \lim_{n \to \infty} E_Q[XY_n] \) where
\[
X = \exp \sum_{i=1}^{k} \xi_im_i - m_i^2/2, \quad Y_n = \exp \sum_{i=k+1}^{n} \xi_im_i - m_i^2/2
\]
Note that for each fixed \( \xi \), \( X \) is bounded from above, and the assumption implies that \( X \) is also bounded from below by a positive number. So
\[
\lim_{n \to \infty} E_Q[XY_n] = 0 \iff \lim_{n \to \infty} E_Q[Y_n] = 0.
\]
Since the second event is in \( \sigma(\xi_{k+1},\xi_{k+2},\ldots) \), so is the first, and since this holds for all \( k \), \( \{Z = 0\} \) is a tail event for the \( \xi \). Kolmogorov’s 0-1 law implies the claim. \( \square \)

Let \( W_n = \exp\{-\sum_{i=1}^{n} m_i^2/2\} \), and let \( w_n = E_QW_n \). Let \( \tilde{Q}_n \) be the distribution of \( m_1,\ldots,m_n \), biased by \( W_n \), that is, for any test function \( F \)
\[
w_nE_{\tilde{Q}_n}[F] = E_Q[W_nF].
\]
Clearly, the distributions \( \tilde{Q}_n \) determine the law of \( m \). The following proposition tells us that \( Z \) determines \( m \).
Proposition 38. Under $\tilde{Q}_n$ the $m_i$ have all exponential moments. If $E_PZ = 1$, then the Laplace transforms satisfy

$$w_n E_{\tilde{Q}_n}[\exp \sum_{i=1}^{n} \xi_i m_i] = E_P[Z | \xi_1, \ldots, \xi_n].$$

Note that the left hand side is analytic in $\xi_1, \ldots, \xi_n$, so even though the right is only defined almost everywhere, it has a continuous version (which is, of course, unique).

Proof. For the first claim, let $1 \leq k \leq n$. We have for all $t \in \mathbb{R}$,

$$w_n E_{\tilde{Q}_n}\exp\{tm_k\} = E_Q\exp\left\{tm_k - \sum_{i=1}^{n} m_i^2/2 \right\} < \infty.$$

since the expression in the expectation is bounded.

For the second, note that by definition (60) the left hand side equals

$$Z_n = E_Q[\exp \sum_{i=1}^{n} \xi_i m_i - m_i^2/2]$$

our usual martingale approximation of $Z$. So we need to prove $Z_n = E_P[Z | \xi_1, \ldots, \xi_n]$. This is equivalent to uniform integrability, or $E_PZ = 1$. \hfill \square

4.3 Existence of the randomized shift

As usual, the trouble is that none of the equivalent conditions of Lemma 36 are practical to check. Not surprisingly, there is a simple $L^2$ condition, and a restricted variant.

Define the intersection exponential as

$$\epsilon(m) = \lim_{n \to \infty} E_P[Z_n^2] = \lim_{n \to \infty} E_{Q \otimes 2}[\exp \sum_{i=1}^{n} m_i m_i']. \quad (61)$$

Here $m$ and $m'$ are independent copies, and $Q \otimes 2$ denotes the product measure. Since $E_P[Z_n^2]$ is a martingale variance, (61) is a limit of an increasing sequence. Hence $\epsilon(m)$ is always defined, but it may be infinite.

Proposition 39. 1. If $\epsilon(m)$ is finite, then the randomized shift exists,

$$Z = \lim_{n \to \infty} Z_n \quad P\text{-a.s. and in } L^2(\Xi, \mathcal{F}, P),$$

and $E_P[Z^2] = \epsilon(m)$.

2. If for every $p < 1$ there is a $\mathcal{G}$-measurable set $A$ with $QA > p$ so that $\epsilon(1_A m) < \infty$, then $Z$ exists and

$$Z = \lim_{n \to \infty} Z_n \quad P\text{-a.s. and in } L^1(\Xi, \mathcal{F}, P).$$
Proof of Proposition 39. For 1, \( Z_n \) is an \( L^2(P) \)-bounded martingale, so it converges in \( L^2(P) \) and almost surely. Lemma 36 identifies the limit as \( Z \).

For 2, write briefly \( Z(m) \) for the partition function corresponding to \( m \). Then by Proposition 56(d),

\[
E_P|Z_n(1_A m) - Z_n(m)| \leq 2Q(A^c) < 2(1 - p).
\]

By the first part, \( Z_n(m1_A) \) converges in \( L^2(P) \), and so it is a Cauchy sequence in \( L^1(P) \). By the above bound \( Z_n(m) \) is also a Cauchy sequence in \( L^1 \). The rest follows from Lemma 36. \( \square \)

Remark 40 (Is there a rabbit?). A rabbit possibly ran through a field and left its tracks. Can the fox tell if the rabbit has been there?

In our setting, the field is planar white noise, and the tracks of the rabbit are the occupation measure of Brownian motion.

The information of the rabbitless field is \( \xi \), and with the rabbit’s tracks, \( \xi + m \). The fact that \( \xi + m \) is absolutely continuous with respect to \( \xi \) means that the rabbit is lucky: if it passed through the field, the fox still cannot tell this with probability one.

The fox’s observation \( Y \) is a sample either from the law of \( \xi + m \) or from the law of \( \xi \). The fox may use the likelihood ratio test to decide which one. In this test, if the two laws are absolutely continuous with respect to a common measure, the choice depends on whether the density ratio passes a certain threshold \( a \). In fact, the law of \( \xi \) can be the common measure, and then the likelihood ratio is \( Z(Y) \) (here we use the definition of the random variable \( Z \) as a function \( \Xi \to \mathbb{R} \)).

So the fox decides to keep pursuing when \( Z(Y) > a \) for some \( a \) depending on how hungry it is. The Neyman-Pearson Lemma says that this is the most powerful among tests of the same level.

Now consider a discrete model of a simple random walk as in Example 18. One may add a parameter \( \beta \) in front of white noise, representing the density of the brush. The regime will be decided based on the rabbit problem. If the fox has asymptotically no chance, then \( Z \) converges to 1 and we expect an additive heat equation (and a Gaussian regime). On the boundary, we expect a multiplicative heat equation (crossover regime), and the asymptotic KPZ fluctuations – the Tracy-Widom fluctuations – probe the regime of the lucky fox.

See Berger and Peres (2013) for a related problem.

Example 41 (The Gaussian multiplicative chaos on the circle). Heuristically, for \( \gamma \in [0, 2) \)
the Gaussian multiplicative chaos on the circle is a random finite measure given by

\[ M(A) = \int_A e^{\gamma \zeta(\omega) - \frac{\gamma^2}{2} E[\zeta^2(\omega)]} d\omega, \quad A \subset [0, 2\pi), \] (heuristic) \hspace{1cm} (62)

where \( \zeta \) is Gaussian free field, a generalized Gaussian process with covariance

\[ K(\omega, \omega') = 2 \log \frac{1}{|e^{i\omega} - e^{i\omega'}|}. \]

There are many ways to make this rigorous. It fits in the present setting as follows. Let \( \Omega = [0, 2\pi] \) and let \( Q \) be uniform measure. Let \( \gamma \geq 0 \), and define the random variables

\[ m_{2k-1}(\omega) = \frac{\cos(k\omega)}{\sqrt{k/2}}, \quad m_{2k}(z) = \frac{\sin(k\omega)}{\sqrt{k/2}}, \quad k = 1, 2, \ldots \]

The random variable \( Z(\gamma m) \) is the total mass of the Gaussian multiplicative chaos (GMC) on the circle with coupling constant \( \gamma \). The random measure \( M_\xi \) on \([0, 2\pi]\) is the GMC itself. A quick computation gives

\[ \sum_{m=1}^{\infty} m_i(\omega)m_i'(\omega') = K(\omega, \omega'), \]

which blows up at the diagonal, \( \sum m_i^2 = \infty \), so the partition function has to be defined as a limit. The intersection exponential is given by

\[ e(\gamma m) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{|e^{is} - 1|^{2\gamma^2}} ds = \begin{cases} \frac{2\Gamma(-2\gamma^2)}{\Gamma(1-\gamma^2)^2}\Gamma(-\gamma^2), & \gamma < 1/\sqrt{2}, \\ \infty, & \gamma \geq 1/\sqrt{2}. \end{cases} \]

The Gaussian sequence \( \xi_0, \xi_1, \xi_2, \ldots \) can be identified with white noise on the circle \( \sum e_i \xi_i \) using the basis elements \( e_0 = 1, \ e_k = \sqrt{k}m_k, \ k \geq 1 \). The analogue of the path measure in this case has state space given by the circle and is the random function

\[ X = \sum_{k=1}^{\infty} m_k(\omega)e_k(x) = 2\Re \text{Li}_{1/2}(e^{i(x-\omega)}) \]

which has a square-root singularity, and is not in \( L^2[0, 2\pi] \) for any \( \omega \). The rabbit problem, Remark 40 in this setting says that the marginal law of \( \xi + \gamma X \) is absolutely continuous with respect to white noise \( \xi \) exactly when \( \gamma < 2 \). An alternative formulation is in terms of the absolute continuity of \( \zeta + \gamma K(\omega, \cdot) \) with respect to the GFF \( \zeta \).
4.4 Properties of $\sum m_j m'_j$.

In preparation for our main convergence theorem, we need to establish properties of the sequence $\alpha_n(m, m') = \sum_{j=1}^{n} m_j m'_j$, or, more generally a sequence

$$\alpha_n(m, w') = \sum_{j=1}^{n} m_j w'_j$$

(63)

The prime notation here means the following. Consider the product of two copies of $(\Omega, \mathcal{G}, Q)$. For a random variable $U$ defined on $\Omega$ and $(\omega, \omega') \in \Omega^2$ we write $U = U(\omega)$ to define $U$ on $\Omega^2$, and write $U' = U(\omega')$ to define an independent copy.

It helps to think of the sequence $m_j$ as a random signed measure on $\mathbb{N}$, defined by $m(\{j\}) = m_j$. Let $m^\otimes k$ denote the random product measure on $\mathbb{N}^k$, and let $\rho_{k,m} = E m^\otimes k$.

We first establish some facts about $\alpha_n$ and its moments.

**Lemma 42.** Assume $E|m_j|^k, E|w_j|^k < \infty$ for all $j$. Then

1. $E\alpha_n(m, w')^k = \langle 1_{\{1, \ldots, n\}^k} \rho_{k,m}, \rho_{k,w} \rangle^\otimes k$

2. For $k \in 2\mathbb{N}$, $n \geq \ell$ and $\| \cdot \|_k = \| L^k(Q^\otimes 2) \|$ we have

$$\|\alpha_n - \alpha_\ell\|^{2k}_k \leq \left( \|\alpha_n(m, m')\|^{2k}_k - \|\alpha_\ell(m, m')\|^{2k}_k \right) \left( \|\alpha_n(w, w')\|^{2k}_k - \|\alpha_\ell(w, w')\|^{2k}_k \right).$$

**Proof.** 1 follows by expanding the two sums. To prove 2, consider the signed measure $\nu = \nu_{m,w}$ on $\mathbb{N}^k$ with $\nu(x) = \rho_{k,m}(x) \rho_{k,m}(x)$. When $m = w$, we have $\nu \geq 0$ and

$$E\alpha_n^k = \nu(\{1, \ldots, n\}^k) \geq \nu(\{1, \ldots, \ell\}^k) + \nu(\{\ell+1, \ldots, n\}^k) = E\alpha_\ell^k + E(\alpha_n - \alpha_\ell)^k$$

(64)

since the last two sets are disjoint subsets of the first one. The inequality holds for all $k \in \mathbb{N}$, but without absolute values (64) describes $L^k$ norms only for even $k$.

When $X$ and $Y$ may be different, by Cauchy-Schwarz, with $A = \{\ell+1, \ldots, n\}^k$ we have

$$(E(\alpha_n - \alpha_\ell)^k)^2 = (\nu_{X,Y}(A))^2 \leq \nu_{X,X}(A) \nu_{Y,Y}(A)$$

and the claim follows as before. \qed

**Proposition 43.** Let $k_0 \in 2\mathbb{N}$, and assume that $\rho_{k,m}, \rho_{k,w} \in \ell^2(\mathbb{N}^k)$. Then for integers $1 \leq k \leq k_0$,

1. $\alpha_n(m, w')$ has a limit $\alpha(m, w')$ in $L^k(Q^\otimes 2)$.

2. $E\alpha(m, w')^k = \langle \rho_{k,m}, \rho_{k,w} \rangle$.

3. $E[\alpha(m, w')^k]^2 \leq E[\alpha(m, m')^k]E[\alpha(w, w')^k]$
Proof. By Lemma 42 and the assumption, \( \alpha_n \) is Cauchy in \( L^{k_0} \), so also in \( L^k \). Taking limits of 1 of the previous lemma gives 2. 3 follows from the \( \ell^2 \) representation in 2. \( \square \)

The following lemma will be useful to understand the intersection exponential.

**Lemma 44.** If \( \rho_{k,m} \in \ell^2(\mathbb{N}^k) \) for all \( k \), then \( \alpha_n = \alpha_n(m, m') \) has a limit \( \alpha \) in \( L^k \) for every \( k \). If \( E \cosh(\alpha) < \infty \), then

\[
e(m) := \lim_{n \to \infty} E e^{\alpha_n} = E e^{\alpha} < \infty.
\]

**Proof.** The first claim is a consequence of Proposition 43. Using Lemma 42 as well, we get

\[
E \alpha_n^{2k} = \|1_{\{1,\ldots,n\}^k} \cdot \rho_{k,m} \|_{\ell^2(\mathbb{N}^k)}^2 \leq \|\rho_{k,m} \|_{\ell^2(\mathbb{N}^k)}^2 = E \alpha^{2k}.
\]

Summing with coefficients, using positivity (the reason for using \( \cosh \) instead of \( \exp \)) Fubini twice, and dominated convergence, we get

\[
E \cosh \alpha_n = E \sum_{k=0}^{\infty} \alpha_n^{2k} (2k)! = \sum_{k=0}^{\infty} E \alpha^{2k} (2k)! \to \sum_{k=0}^{\infty} E \alpha^{2k} (2k)! = E \cosh \alpha < \infty,
\]

so \( \cosh(\alpha_n) \to \cosh(\alpha) \) in \( L^1 \). But \( f(x) = e^x / \cosh(x) \) is a bounded continuous function, so \( f(\alpha_n) \cosh(\alpha_n) \to f(\alpha) \cosh(\alpha) \) in \( L^1 \). Thus \( E e^{\alpha_n} \to E e^{\alpha} \). \( \square \)

Next, we consider a setting where sequences \( m_\ell \) converge to a limiting sequence \( m \).

**Lemma 45 (Convergence of \( \alpha \)).** Assume that

(i) \( \lim_{\ell \to \infty} m_\ell,j = m_j \) in probability for each \( j \),

(ii) for each \( \ell \), as \( n \to \infty \),

\[
S_{\ell,n} = \sum_{j=1}^{n} m_{\ell,j} m'_{\ell,j} \to \alpha_\ell, \quad S_n = \sum_{j=1}^{n} m_j m'_j \to \alpha \quad \text{in} \quad L^2(Q^{\otimes 2}),
\]

(iii) \( \lim_{\ell \to \infty} E \alpha_\ell^2 = E \alpha^2 \).

Then as \( \ell \to \infty \), \( \alpha_\ell \to \alpha \) in probability.

**Proof.** Let \( \varphi(x) = |x| \wedge 1 \), so that \( E \varphi(X) \) metrizes convergence in probability. Note that \( E \varphi(X) \leq E |X| \leq (EX^2)^{1/2} \). Let \( q_{\ell,n} = ES_{\ell,n}^2 \) and \( q_n = ES_n^2 \). By the \( L^2 \) moment formulas of Lemma 42 and Proposition 43, we have

\[
E(\alpha - S_n)^2 \leq E \alpha^2 - q_n, \quad E(\alpha_\ell - S_{\ell,n})^2 \leq E \alpha_\ell^2 - q_{\ell,n}.
\]
Given $\varepsilon > 0$, let $n$ be so that
\[ E\alpha^2 - q_n < \varepsilon. \]
Let $\ell_0$ be so that for all $\ell \geq \ell_0$
\[ q_n < q_{n,\ell} + \varepsilon, \quad E\alpha^2_{\ell} < E\alpha^2 + \varepsilon, \quad E\varphi(S_{n,\ell} - S_n) < \varepsilon. \]
In the first inequality we used Fatou’s Lemma. Then
\[ E\varphi(\alpha_{\ell} - \alpha) \leq (E(\alpha_{\ell} - S_{\ell,n})^2)^{1/2} + E\varphi(S_{\ell,n} - S_n) + (E(\alpha - S_n)^2)^{1/2} \]
\[ \leq (E\alpha^2_{\ell} - q_{\ell,n})^{1/2} + \varepsilon + (E\alpha^2 - q_n)^{1/2} < \sqrt{3\varepsilon} + \varepsilon + \sqrt{\varepsilon}. \]

4.5 Convergence of randomized shifts partition functions

One of the advantages of the randomized shifts approach is that it is very effective for proving convergence of models. We will have a sequence of random sequences $m_{\ell}$ converging to a random sequence $m$ in probability. Since $Z(m_{\ell})$ only depends on the law of $m_{\ell}$, we can couple the $m_{\ell}$ in any way we like to get to the present setting.

**Proposition 46.** Let $m_{\ell} = (m_{\ell,1}, m_{\ell,2}, \ldots)$, $\ell = 1, 2, \ldots$ be a random sequence. Suppose that $m_{\ell,j} \to m_j$ in probability for each $j$. Let $m = (m_1, m_2, \ldots)$.

1. Assume that $\mathcal{C}(m_{\ell}) \to \mathcal{C}(m) < \infty$. Then $Z(m_{\ell}) \to Z(m)$ in $L^2(\Omega, \mathcal{F}, P)$.

2. Assume that for all $k \geq 1$ and each $\ell$, as $n \to \infty$,
\[ S_{\ell,n} = \sum_{j=1}^{n} m_{\ell,j}m'_{\ell,j} \to \alpha_{\ell}, \quad S_n = \sum_{j=1}^{n} m_jm'_j \to \alpha \quad \text{in } L^k(Q^{\otimes 2}). \]
Suppose that $E\alpha^2_{\ell} \to E\alpha^2$ and that for some $\gamma > 1$ the following holds. For each $p < 1$, there exists $G$-measurable sets $A_{\ell}$ with $QA_{\ell} > p$ and
\[ \limsup_{\ell \to \infty} E\cosh(\gamma 1_{A_{\ell}} \alpha_{\ell}) < \infty. \] (65)
Then $Z(m_{\ell}) \to Z(m)$ in $L^1(\Omega, \mathcal{F}, P)$.

Here, for clarity, we use $Z(m)$ to denote the dependence on $m$ even though it really only depends on the law of $m$.

**Proof.** For 1, we can bound $\frac{1}{3} E_P[(Z(m_{\ell}) - Z(m))^2]$ by
\[ E_P[(Z(m_{\ell}) - Z_n(m_{\ell}))^2] + E_P[(Z_n(m_{\ell}) - Z_n(m))^2] + E_P[(Z_n(m) - Z(m))^2] \] (66)
Let
\[ q_{n,\ell} = E_P[(Z_{n+1}(\mathbf{m}_\ell) - Z_n(\mathbf{m}_\ell))^2], \quad q_n = E_P[(Z_{n+1}(\mathbf{m}) - Z_n(\mathbf{m}))^2]. \]
The weak convergence of \( \mathbf{m}_\ell \) to \( \mathbf{m} \) and the fact that \( e^{\lambda x - \frac{1}{2} x^2} \) is bounded and continuous in \( x \), means that for each \( n \),
\[ Z_n(\mathbf{m}_\ell) \to Z_n(\mathbf{m}) \] \tag{67}
\( P \)-almost surely. Let \( \| \cdot \|_1 \) denote the sequence \( \ell^1 \)-norm. By the martingale property,
\[ \epsilon(\mathbf{m}_\ell) = \| q_{\ell} \|_1, \quad \epsilon(\mathbf{m}) = \| q \|_1 \]
Let \( \hat{q}_n = \lim \inf_{\ell \to \infty} q_{n,\ell} \). By Fatou’s lemma applied to sequences, \( \lim_{\ell \to \infty} \| q_{\ell,\ell} \|_1 \geq \| \hat{q} \|_1 \). By \( 67 \) and Fatou’s lemma applied in the probability space \( \Xi \) to \( |Z_{n+1}(\mathbf{m}_\ell) - Z_n(\mathbf{m}_\ell)|^2 \), we have \( q_n \leq \hat{q}_n \). After summing over \( n \) we get
\[ \epsilon(\mathbf{m}) = \| q \|_1 \leq \| \hat{q} \|_1 = \lim_{\ell \to \infty} \epsilon(\mathbf{m}_\ell) = \epsilon(\mathbf{m}), \quad 0 \leq q_n \leq \hat{q}_n, \]
where the last equality is our assumption. This implies \( \hat{q} = q \). It follows that \( q_{n,\ell} \to q_n \) coordinate-wise and also in \( \ell^1 \).

From the martingale property, the first and third terms of \( 66 \) are given by \( \sum_{i=n}^{\infty} q_{i,\ell} \) and \( \sum_{i=n}^{\infty} q_i \). The convergence \( \| q_{\ell,\ell} - q_1 \|_1 \to 0 \) implies that for large enough \( n_0 \) the first term (and the easier third term) in \( 66 \) can be made uniformly small for all \( \ell \geq 1, n \geq n_0 \). Since the \( L^2(P) \)-norm of the left hand side converges to that of the right in \( 67 \), the claimed convergence also holds in \( L^2(P) \). Thus the middle term in \( 66 \) vanishes as \( \ell \to \infty \). This proves Case 1.

For 2, by diagonalization, we can find sets \( A_\ell \) so that \( QA_\ell \to 1 \), and \( 65 \) still holds. Let \( \bar{\mathbf{m}}_\ell = 1_{A_\ell} \mathbf{m}_\ell \) and use bar to denote all truncated quantities with \( \bar{\mathbf{m}}_\ell \) replacing \( \mathbf{m}_\ell \).

By Lemma \ref{lem:45}, we have \( \alpha_\ell \to \alpha \) in probability, which implies \( \bar{\alpha}_\ell \to \alpha \) and \( \cosh \bar{\alpha}_\ell \to \cosh \alpha \) in probability. By our assumption, \( E \cosh(\gamma \bar{\alpha}_\ell) \) is bounded for some \( \gamma > 1 \), and therefore \( \cosh \bar{\alpha}_\ell \) is uniformly integrable, so \( E \cosh \alpha_\ell \to E \cosh \alpha < \infty \). Similarly, \( E e^{\alpha_\ell} \to E e^{\alpha} < \infty \).

By dominated convergence, we have \( \bar{\alpha}_\ell = \lim_{n \to \infty} \tilde{S}_{\ell,n} \) in every \( L^k(Q^{\otimes 2}) \), so by Lemma \ref{lem:44}, we have \( \epsilon(\bar{\mathbf{m}}_\ell) = E e^{\alpha_\ell} \), and similarly, \( \epsilon(\mathbf{m}) = E e^{\alpha} \). Thus
\[ \epsilon(\bar{\mathbf{m}}_\ell) \to \epsilon(\mathbf{m}) < \infty. \]
Now Case 1 applies, and we get \( L^2(P) \) convergence, and hence also \( L^1(P) \) convergence
\[ E_P|Z(1_{A_\ell} \mathbf{m}_\ell) - Z(\mathbf{m})| \to 0. \]
Finally, by taking limits of Proposition 56(d),
\[ E_P |Z(m_\ell) - Z(1_A, m_\ell)| \leq 2Q(\ell) \to 0 \]
and the claim follows by the triangle inequality. \(\square\)

**Remark 47.** \(Z(m_\ell)\) does not depend on our choice of basis \(e_j\). This is because it is arbitrarily close in \(L^1(\Xi, \mathcal{F}, P)\) to \(Z(1_A m)\). The latter is basis-independent since it has \(L^2(\Xi, \mathcal{F}, P)\) as its chaos series is as in Example 25 with (39) replaced by
\[ g_{k,A}(x_1, \ldots, x_k; t, x) = E_{Q,1_A}[m(x_1) \cdots m(x_k)]. \]

The results in this section, as well as the next, parallel those of Theorem 25 of Shamov (2016). In our setting, it reads as follows:

**Proposition 48** (Shamov). Let \(m = (m_1, m_2, \ldots)\), \(m_\ell = (m_{\ell,1}, m_{\ell,2}, \ldots), \ell = 1, 2, \ldots\) be random sequences defined on the same probability space. Suppose that

1. \(Z(m_\ell)\) is uniformly integrable
2. For all \(v \in \ell^2\), we have \(\langle m_\ell, v \rangle \to \langle m, v \rangle\) in \(Q\)-probability.
3. \(\alpha(m_\ell, m'_\ell) \to \alpha(m, m')\) in \(Q^\otimes 2\)-probability for some \(\alpha(m, m')\).

Then \(Z(m_\ell) \to Z(m)\) in \(L^1(\Xi, \mathcal{F}, P)\) and the random measures satisfy \(M_\ell \to M\) in \(L^1\) in the following sense. For any test function \(F \in L^1(\Omega, \mathcal{G}, Q)\), we have
\[ E_{M_\ell} F \to E_M F \quad \text{in } L^1(\Xi, \mathcal{F}, P). \]

Let us comment on the main differences. In Proposition 48, uniform integrability is assumed, while we are instead giving explicit conditions which imply the uniform integrability, conditions we verify in later sections. Also, Assumption 2 of Proposition 48 seems harder to verify than convergence in probability of the entries of \(m\).

### 4.6 The polymer measure

Up to this point, our main focus was random heat flow in one and two dimensions. Random paths came up as a tool for solving heat equations through the Feynman-Kac formula. In fact, one motivation for heat equations, random or deterministic, is that they describe the partition functions of polymers.

As an example, assign an i.i.d. mean one positive random variables to lattice site in spacetime \(\mathbb{N} \times \mathbb{Z}\), and let the weight of a simple random walk path be given by the product
of the weight of sites its graph visits. When the law of each weight is $e^{\beta G - \beta^2/2}$ for a standard Gaussian $G$ and $\beta \in \mathbb{R}$, this construction can be described through a randomized shift.

The remarkable advantage of the random shift formalism is that it works effortlessly in technically difficult settings like Brownian paths weighted by white noise. The price we have to pay is a bit of abstraction.

In short, the polymer measure is given by the Gaussian multiplicative chaos on path space. Recall our setting: a probability space $(\Omega, \mathcal{G}, Q)$ (the path space in the example above) on which there is a random sequence $m$. A Gaussian i.i.d. sequence $\xi$ defined on $(\Xi, \mathcal{F}, P)$. In the special case when the $m_i$ are bounded random variables and $\sum_{i=1}^{\infty} m_i^2 < \infty$, then

$$\sum_{i=1}^{\infty} m_i(\omega) - (\sum_{i=1}^{\infty} m_i(\omega))^2/2$$

exists as a limit of partial sums $Q$-almost surely, we can define the new measure

$$M_{\xi} = e^{\sum_{i=1}^{\infty} m_i(\omega)\xi_i - (\sum_{i=1}^{\infty} m_i(\omega))^2/2} Q$$

(68)

on $\Omega$. The strength of the GMC theory is in the fact that such measures behave much better under limiting operations than the weight functions (68).

The direct description (69) fails since in many cases the sum is not defined and $M_{\xi}$ is in fact singular with respect to $Q$. But it is an exercise to check that the formula (70) holds in the special case above. It turns out that it captures enough information in it to be used as a definition.

**Definition 49.** The polymer measure or Gaussian multiplicative chaos of the randomized shift $m$ is a random measure $M$ on $\Omega$ parametrized by $\xi$. It is defined by the property that for all bounded measurable functions $F$ on $\Xi \times \Omega$

$$E_P E_{M_{\xi}} F(\xi, \omega) = E_P \times Q F(\xi + m(\omega), \omega).$$

(70)

Here $\xi, m(\omega)$ are real sequences and $\xi + m(\omega)$ is just their termwise sum. This formula, simple as it looks, contains the definition of the Wick-ordered polymers. In that case, $\Omega$ is the space of paths, and $m_i(\omega)$ is the integral of the basis element $e_i$ with respect to the occupation measure of $\omega$.

Note that $Z(\xi) = M_{\xi}(\Omega)$ is the partition function, or total mass, of the random measure $M$, and that the $P$-expectation of $M$ is exactly $Q$. The normalized version

$$M_{\xi}(\cdot)/M_{\xi}(\Omega),$$
is a random probability measure.

For the definition to make sense, we need the following, one of the main results of Shamov (2016).

**Theorem 50** (Shamov (2016), Theorem 17 and Corollary 18). *If the marginal law of \( \xi + m(\omega) \) is absolutely continuous with respect to \( \xi \), then the polymer measure/Gaussian multiplicative chaos exists and is unique.*

**Example 51** (Discrete lattice polymer in 1+1 dimension). Let \( n(i) \) be an enumeration of the even lattice points in \( \mathbb{N} \times \mathbb{Z} \), and consider discrete time simple random walk \( X \) with \( X_0 = 0 \) run until time \( t_0 \).

For \( n(i) = (x,t) \) let \( m_i = \begin{cases} 1(X_t = x) & \text{if } t \leq t_0 \\ 0 & \text{otherwise,} \end{cases} \)

in other words, \( m \) encodes the occupation measure of the graph of \( X \). In the polymer measure, each path \( \omega \) has weight weight \( \exp \sum_i \xi_i m_i - m_i^2/2 \). So for any test function \( F \), we have

\[
E_{M_\xi} F(\xi, \omega) = 2^{-n} \sum_\omega \left( \exp \sum_i \xi_i m_i(\omega) - \frac{m_i(\omega)^2}{2} \right) F(\xi, \omega).
\]

using the Radon-Nikodym derivative for the shifted mean Gaussian, \( E_P E_{M_\xi} F(\xi, \omega) \) can be written as

\[
2^{-n} \sum_\omega E_P(\exp \sum_i \xi_i m_i(\omega) - \frac{m_i(\omega)^2}{2}) F(\xi, \omega) = 2^{-n} \sum_\omega E_P(\xi + m, \omega) = E_{P \times Q} F(\xi + m, \omega)
\]

so it satisfies (70).

**Example 52** (Bayesian statistics, continued). Continuing Example 35 recall that \( m_i(\omega) = \omega \). After \( n \) observations, the Gaussian multiplicative chaos is given by the unnormalized posterior measure \( \exp \{ -n\omega^2/2 + \omega \sum_{i=1}^n \xi_i \} Q \). The normalized posterior distribution is \( \exp \{ -n\omega^2/2 + \omega \sum_{i=1}^n \xi_i \} Q/Z_n \).

This example holds in full generality: one can think of any Gaussian multiplicative chaos as the posterior distribution of \( Q \) after observing the (usually different) statistics \( m_1, m_2 \ldots \).

**Example 53** (Definition of PAM). The planar parabolic Anderson model, PAM, is closely related to the planar Wick-ordered polymer, whose partition function is

\[
Z = \lim_{n \to \infty} Z_n, \quad Z_n = E_Q \left\{ \exp \sum_{j=1}^n m_j \xi_j - m_j^2/2 \right\}
\]  

(71)
where $m_j = \int_0^t e_j(B(s)) \, ds$ and $e_j$ is an orthonormal basis of $L^2(\mathbb{R}^2)$ consisting of bounded functions. For this example, we take $Q$ to be standard Brownian motion measure on some interval $[0, t]$.

Our definition corresponds to smoothing out the noise by replacing it the finite dimensional Gaussian process with $\sum_{j=1}^n \xi_j e_j$, using it to define a Wick-ordered polymer, and then taking a limit.

PAM should correspond to the partition function

$$Z_{PAM} = \lim_{n \to \infty} E_Q \left\{ \exp \sum_{j=1}^n m_j \xi_j - c_{n,t} \right\} \quad \text{(heuristic)}$$

for some constants $c_{n,j}$. This means we are solving the deterministic heat equation with respect to the smoothed noise $\sum_{j=1}^n \xi_j e_j$, and renormalizing by $e^{-c_{n,t}}$. Proving that this limit exists can be cumbersome, as, unlike in the Wick-ordered case, there is no natural martingale to consider. A precise definition can be made in a different way. While we don’t do it here, it can be shown that

$$\gamma(B) = \lim_{n \to \infty} \gamma_n(B) - E_Q \gamma_n(B), \quad \gamma_n(B) = \frac{1}{2} \sum_{j=1}^n m_j^2 \quad \text{(heuristic)} \quad (72)$$

exists. The random variable $\gamma(B)$ is called the self-intersection local time. Heuristically, up to centering, $2\gamma$ wants to be the squared $L^2(\mathbb{R}^2)$ norm of the occupation measure of $B$. Indeed, $2\gamma_n$ is the squared $L^2(\mathbb{R}^2)$-norm of the occupation measure projected onto the span of $e_1, \ldots, e_n$. Instead of proving (72), we use the definition given in [Le Gall (1994)] through mutual intersection local times:

$$\gamma(B) = \sum_{n=1}^{\infty} \sum_{k=1}^{2^n-1} A_{n,k} - E_Q A_{n,k}, \quad A_{n,k} = \alpha(B_{[\frac{2k-1}{2^n}, \frac{2k}{2^n}]}, B_{[\frac{2k}{2^n}, \frac{2k+1}{2^n}]}).$$

The arguments of $\alpha$ in $A_{n,k}$ are conditionally independent given $B(\frac{2k-1}{2^n})$, and so each term is defined as in Section 5.

Considering the normalization (71) one could first try to take $c_{n,t}$ to equal $E_Q \gamma_n$. But there is a problem: For the PAM to inherit the semigroup property from the polymer given by Brownian paths weighted by $\exp \sum_{j=1}^n m_j \xi_j$, for $0 < s < t$ the normalization should be asymptotically additive:

$$\lim_{n \to \infty} c_{n,t} - c_{n,s} - c_{n,t-s} = 0. \quad (73)$$

For $0 < s < t$ let $B_1$ and $B_2$ denote the restrictions of $B$ to $[0, s]$ and $[s, t]$, respectively. Then by bilinearity, we have

$$E_Q \gamma_n(B) - E_Q \gamma_n(B_1) - E_Q \gamma_n(B_2) = E_Q \alpha_n(B_1, B_2)$$
which converges to
\[ E_\alpha(B_1, B_2) = \int_{\mathbb{R}^2} \int_0^s \int_0^{t-s} p(x, t_1) p(x, t_2) \, dt_2 \, dt_1 \, dx \]
\[ = \frac{t \log t - s \log s - (t - s) \log(t - s)}{2\pi}. \]
This suggests the time-dependent correction
\[ c_{n,t} = E_\gamma_n(B) - \frac{1}{2\pi} t \log t \]
and this is indeed asymptotically additive (73).

To get a quick definition of PAM, we use $\gamma$ to re-adjust the weight of the Wick-ordered polymer paths:
\[ M_{PAM, \xi} = e^{\gamma(\omega) + \frac{1}{2\pi} t \log t} M_{\xi}, \quad Z_{PAM} = E_{M_{PAM, \xi}}[\Omega] = E_{M_{\xi}} e^{\gamma + \frac{1}{2\pi} t \log t}. \]
By the definition of $M_{\xi}$, we have
\[ E_P Z_{PAM} = E_P E_{M_{\xi}} e^{\gamma + \frac{1}{2\pi} t \log t} = e^{\frac{1}{2\pi} t \log t} E_Q e^{-\gamma}. \]
Le Gall (1994) shows that the right hand side is finite for small $t$, but is infinite for large $t$. It can be shown that in the latter case, $Z_{PAM}$ is still finite $P$-almost surely, but $E_P Z_{PAM} = \infty$.

The following proposition will be used to show that the planar Wick-ordered polymer converges to the continuum random directed polymer.

**Proposition 54.** Assume that condition 1 or 2 of Proposition 46 holds. Then the polymer measures converge in the following sense. For any bounded test function $F : \Omega \to \mathbb{R}$
\[ E_{M_\ell} F \to E_M F \quad \text{in } L^1(P). \] (74)
By (74), $M_\ell \to M$ in $P$-probability with respect to weak topology of measures. The normalized measures $M_\ell / Z_\ell \to M / Z$ as well, as long as $P(Z > 0) = 1$. This holds in our cases of interest by Lemma 37.

**Proof.** By linearity, we may assume $E_Q F = 1$, $F \geq 0$. By bounded convergence, the assumptions of the Proposition 46 hold for the sequences $m_\ell$ and $m$ under the modified probability measure $F_Q$. Let $Z_F(\cdot)$ denote the corresponding partition functions. Then
\[ E_{M_\ell} F = Z_F(m_\ell) \to Z_F(m) = E_M F \quad \text{in } L^1(P). \]
5 Intersection local time

5.1 Intersection local time for random measures

Let $X, Y$ be random variables defined on a probability space $(\Omega, G, Q)$ taking values on the space of finite measures on a measure space $(R, B, \mu)$. In general, we will write $\rho_X$ for the density of the deterministic measure $EX$ with respect to $\mu$ if it exists.

Consider the product of two copies of $(\Omega, G, Q)$. For a random variable $U$ defined on $\Omega$ and $(\omega, \omega') \in \Omega^2$ we write $U = U(\omega)$ to define $U$ on $\Omega^2$, and write $U' = U(\omega')$ to define an independent copy.

We would like to define the mutual intersection local time $\alpha = \alpha(X, Y')$ as the random inner product
\[ \alpha = \int \frac{dX}{d\mu} \frac{dY'}{d\mu} d\mu, \quad \text{(heuristic)} \]
a random variable on $\Omega^2$, even when $X, Y'$ do not have densities. When they do, for any nonnegative bounded random variables $F, G$ on $\Omega$ we have
\[ E_Q[FG'\alpha] = \int E_Q[F\frac{dX}{d\mu}G'\frac{dY'}{d\mu}]d\mu = \langle \rho_F X, \rho_G Y \rangle. \]
This gives rise to a definition.

**Definition 55.** Assume that $\rho_X$ and $\rho_Y$ exists. If there exists $\alpha \in L^1(\Omega^2, \sigma(X, Y'), Q^{\otimes 2})$ so that for all bounded measurable $F, G \geq 0$ on $\Omega$ we have
\[ E_Q[FG'\alpha] = \langle \rho_F X, \rho_G Y \rangle < \infty, \]
then we call $\alpha = \alpha(X, Y')$ the **mutual intersection local time** of $X, Y'$.

**Proposition 56.** 1. If the intersection local time exists, then it is nonnegative and unique.

2. If $X \leq \bar{X}, Y \leq \bar{Y}$ and $\bar{\alpha} = \alpha(\bar{X}, \bar{Y}')$ exists, then $\alpha = \alpha(X, Y')$ exists and $\alpha \leq \bar{\alpha}.

**Proof.** Let $\alpha, \hat{\alpha}$ be two versions of the intersection local time. Consider the subset $S \subset \sigma(G^2)$ on which both random variables have the same expectation, and this integral is nonnegative. Then $S$ is a $\pi$-system containing a $\lambda$ system $G^2$ given by the product sets, since
\[ E[1_{A \times B} \alpha] = E[1_{A \times B} \hat{\alpha}] = \int \rho_{1A}X \rho_{1B}Y d\mu \geq 0. \]
Thus by the $\pi - \lambda$ theorem, $\sigma(G^2) = S$. Then $\{\alpha < 0\}, \{\alpha < \hat{\alpha}\} \in S$, so
\[ E[1_{\alpha < 0} \alpha] \geq 0, \quad E[1_{\alpha < \hat{\alpha}} \alpha] = E[1_{\alpha < \hat{\alpha}} \hat{\alpha}] \]
so $E\alpha^- = 0$, and since $\alpha \in L^1$, we get $E(\hat{\alpha} - \alpha)^+ = 0$. Similarly $E(\hat{\alpha} - \alpha)^- = 0$, and so $\hat{\alpha} = \alpha \geq 0$ a.s.

For 2, note that by the Caratheodory extension theorem $A(A \times B) := \int \rho_{A X} \rho_{1B} Y d\mu$ extends to a measure $A$ on $\Omega^2$. Then $\hat{\alpha} d\mu^{\otimes 2}$ dominates $A$ on product sets. So $A$ has a density in $L^1$, which fits the definition of $\alpha$.

Definition 55 does not give a construction of intersection local time, nor an easily verifiable condition for when it should exist.

Recall the notation $X^{\otimes 2} = X \otimes X$ is the $Q$-random product measure on $R \times R$; it is the product of the same random measure, not independent copies. As a special case of Proposition 60 we show the following.

**Corollary 57.** If the deterministic measure $E_Q[X^{\otimes 2}]$ has a square integrable density with respect to $\mu^{\otimes 2}$, then the mutual intersection local time $\alpha(X, X')$ exists.

Random measures defined on $Q$ form a cone in a real vector space through $(bX + Y)(A) = bX(A) + Y(A)$ for $b \geq 0$. When they are occupation measures of random functions, this has nothing to do with adding function values $bX(t) + Y(t)$ in $\mathbb{R}^d$. The following are immediate from the definition.

**Lemma 58.** If $d \geq 0$ is a constant and $\alpha(X, Y')$, $\alpha(X, Z')$ exist, then the following quantities exist and satisfy

\[ \alpha(cX, Y') = c\alpha(X, Y'), \quad \alpha(X, Y' + Z') = \alpha(X, Y') + \alpha(X, Z') \]

and $\alpha(X, Y')(\omega_1, \omega_2) = \alpha(Y, X')(\omega_2, \omega_1)$. For $Q$-random bounded scalars $C, D \geq 0$, the following random variable on $\Omega^2$ exists and satisfies

\[ \alpha(CX, D'Y') = CD'\alpha(X, Y'). \] (75)

Corollary 57 is based on a coordinate representation of the random measure $X$. Consider an orthonormal basis $e_i$ of $L^2(R, \mathcal{B}, \mu)$ consisting of bounded functions. Then with

\[ m_j = m_{X, j} = \int e_j dX, \quad X_n = \sum_{j=1}^n m_j e_j, \]

we have a signed approximation $X_n d\mu$ of the random measure $X$. Similarly, let $w_j$ be the coordinates of $Y$, and let

\[ \alpha_n(X, Y') = \int X_n Y'_n d\mu = \sum_{j=1}^n m_j w'_j. \] (76)
We are now in the setting of Section 4.4. Indeed, \( \alpha_n(X, Y') \) as defined here, equals \( \alpha_n(m', w') \) of (63). So in order for \( \alpha_n \) to have a limit in \( L^k(Q^\otimes 2) \), \( k \in 2\mathbb{N} \), we just need \( \rho_{k,m}, \rho_{k,w} \in L^2(R^k) \). But

\[
\rho_{k,m}(i_1, \ldots, i_k) = E_Q[m_{i_1} \cdots m_{i_k}],
\]

the coefficient of the basis vector \( e_{i_1} \otimes \cdots \otimes e_{i_k} \) in the orthonormal basis representation of \( \rho_{k,X} \), the density of \( E_Q X^\otimes k \). Thus we have

\[
\langle \rho_{k,m}, \rho_{k,w} \rangle_{\mathcal{N}^k} = \langle \rho_{k,X}, \rho_{k,Y} \rangle_{\mu^\otimes k}.
\] (77)

We will use the results of Section 4.4 to show that \( \alpha_n \) converges, and we will use the following general lemma to identify the limit.

**Lemma 59.** If \( \rho_X, \rho_Y \in L^2(R) \) and \( \alpha_n(X, Y') \) is precompact in \( L^1(Q^\otimes 2) \), then the mutual intersection local time \( \alpha(X, Y') \) exists and \( \alpha_n(X, Y') \to \alpha(X, Y') \) in \( L^1(Q^\otimes 2) \).

**Proof.** Let \( \alpha_\infty \) be a limit point of \( \alpha_n \) along a subsequence \( \mathcal{N} \). Let \( F, G \geq 0 \) be bounded random variables on \( \Omega \). By a direct computation, we have

\[
E_{Q^\otimes 2}[FG'\alpha_n] = \sum_{i=1}^n \langle \rho_{FX}, e_i \rangle \langle \rho_{GY}, e_i \rangle.
\]

Since \( \rho_{FX}, \rho_{GY} \in L^2(R) \), and \( FG'\alpha_n \to FG'\alpha_\infty \) in \( L^1 \) along \( \mathcal{N} \), taking limits we conclude

\[
E_{Q^\otimes 2}[FG'\alpha_\infty] = \langle \rho_{FX}, \rho_{GY} \rangle_{\mu}.
\]

**Proposition 60.** Let \( k_0 \in 2\mathbb{N} \). If \( \rho_{k,m}, \rho_{k,w} \in L^2(\mu^\otimes k) \), then for all integers \( 1 \leq k \leq k_0 \),

1. The intersection local time \( \alpha(X, Y') \) exists and \( \alpha_n \to \alpha(X, Y') \) in \( L^k(Q^\otimes 2) \).
2. \( \|\alpha\|_{L^k(Q^\otimes 2)} = \langle \rho_{k,X}, \rho_{k,Y} \rangle_{\mu^\otimes k} \).
3. \( E[\alpha(X, Y')^k] \leq E[\alpha(X, X')^k] E[\alpha(Y, Y')^k] \).

**Proof.** This is a direct consequence of (77), Proposition 43 and Lemma 59.

In classical settings, with \( R = \mathbb{R}^d \) and Lebesgue measure \( \mu \), intersection local time is often defined as the value of the continuous density of the convolution \( X * \tilde{Y}' \) where \( \tilde{Y} = Y(-\cdot) \). The two notions are essentially the same, as the next lemma shows.

**Lemma 61.** Let \( X, Y \) be random measures on the torus \( R = [-\ell, \ell]^d \) or on \( R = \mathbb{R}^d \). If \( X * \tilde{Y}' \) has continuous density \( \alpha_*(X, Y') \) at zero a.s. and \( \rho_{2,X}, \rho_{2,Y} \in L^2 \) then \( \alpha(X, Y') \) exists and \( Q^\otimes 2 \)-a.s. equals \( \alpha_*(X, Y') \).
Proof. For the torus, we may assume \( \ell = \pi \). Let \( e_j \) be the Fourier modes ordered by frequency. For \( n = n_j = (2j + 1)^d \), the projection to the span of \( e_1, \ldots, e_n \) commutes with translations, so it is a convolution by a function \( \varphi_n \). More explicitly,

\[
\varphi_n = D_n^{\otimes d}, \quad \text{and} \quad D_n(x) = \frac{1}{2\pi} \sum_{k=-j}^{j} e^{ikx} = \frac{\sin((k + 1/2)x)}{2\pi \sin(x/2)}
\]

is the Dirichlet kernel. Hence \( X_n \mu = X \ast \varphi_n \), and

\[
\alpha_n(X, Y') = \int (X \ast \varphi_n)(Y' \ast \varphi_n) = (X \ast \varphi_n \ast \tilde{\varphi}_n \ast \tilde{Y'})(0) = (\varphi_n \ast X \ast \tilde{Y'})(0)
\]

since \( \varphi_n \) is symmetric and a projection, so \( \varphi_n \ast \tilde{\varphi}_n = \varphi_n \). Since \( \varphi_n \) approximates \( \delta_0 \), we see that \( \alpha_{n_i} \to \alpha_s \), which then equals \( \alpha \) by Proposition 60.

For the case of \( \mathbb{R}^d \), apply the torus argument on \([-2\ell, 2\ell]^d\) to \( X \mid_{[-\ell, \ell]^d} \), \( Y \mid_{[-\ell, \ell]^d} \) and use dominated convergence to get the result.

5.2 Moment bounds

The solution and convergence of the planar Wick-ordered heat equation requires estimates of the exponential moments of the mutual intersection local time of planar Brownian motion.

Proposition 60 shows that for random measures \( X \), the mutual intersection local time \( \alpha(X, X') \) exists if for \( k = 2 \) the measure \( \mathbb{E}_Q[X^{\otimes k}] \) has a density \( \rho_k \) in \( L^2 \). When \( \alpha(X, X') \) exists, its \( k \)th moment is given by \( \| \rho_k \|^2 \).

The following bound on the moments holds for any continuous time process with independent stationary increments (Lévy process) \( B \) with joint densities with respect to some fixed measure. In particular, it holds for Brownian motion with drift on \( R = \mathbb{R}^d \). For Brownian motion without drift, related bounds can be found in Le Gall (1994). But the method there does not readily generalize to Brownian motion with drift. The statement involves a parameter \( r \), which will be chosen carefully later.

**Proposition 62.** Let \( k \) be a positive integer, \( r > 0 \) and let \( X(A) = \int_0^1 1_A(B(t)) \, dt \) be the occupation measure of a stochastic process \( B \) in \( R = \mathbb{R}^d \) up to time 1. For a time vector \( s = (s_1, \ldots, s_k) \) let \( p(s, x) \) be the density of the random vector \( (B(s_1), \ldots, B(s_k)) \) at \( x = (x_1, \ldots, x_k) \); assume this density exists for almost all vectors \( s \). Then the random \( k \)-fold product \( X^{\otimes k} \) has density

\[
\rho_k(x) = \int_{s \in [0,1]^k} p(x, s)
\]
on $R^k$. Assume further that $B$ is a Levy process. Let $\tau$ be an independent exponential rate $r$ random variable, and let $\varphi_r(\cdot)$ be the density of $B(\tau)$ on $R$. Then
\[
\|\rho_k\|^2 \leq k!^2 e^{2r} \|\varphi_r/r\|^{2k}.
\] (79)

**Proof.** For any measurable set $A \subset R^k$ we have
\[
E[X^{\otimes k}] (A) = E \int_{s \in [0,t]^k} 1((B(s_1), \ldots, B(s_k)) \in A) = \int_{s \in [0,t]^k} P((B(s_1), \ldots, B(s_k) \in A))
\]
Expressing the probability as an integral and using Fubini we get
\[
E[X^{\otimes k}] (A) = \int_{x \in A} \int_{s \in [0,t]^k} p(x, s),
\]
which implies the density formula (78). Let $[0, 1]^k$ denote the subset of vectors in $s \in [0, 1]^k$ with $s_1 < \cdots < s_k.$
\[
\|\rho_k\|^2 = \int_{R^k} \int_{s, t \in [0,1]^k} p(x, s)p(x, t) = \sum_{\sigma, \eta} \int_{R^k} \int_{s \in [0,1]^k} p(x, \sigma s) \int_{t \in [0,1]^k} p(x, \eta t)
\]
where the sum is over all permutations $\sigma$ and $\eta.$ By Cauchy-Schwarz applied to the $x$-integral, the summand is at most
\[
\int_{R^k} \left( \int_{s \in [0,1]^k} p(x, \sigma s) \right)^2
\] (80)
since $p(\sigma s, x) = p(s, \sigma^{-1} x),$ changing variables $x \mapsto \sigma x$ shows that we can drop the $\sigma$ from (80). We now bound each factor the same way. Rewrite (80) as
\[
\int_{R^k} \int_{s, t \in [0,1]^k} p(x, s)p(x, t).
\] (81)
Let
\[
a(s, t) = \int_{R^k} \int_{(s_1, \ldots, s_{k-1}) \in [0, t]^{(k-1)}} p(x, s)p(x, t),
\]
i.e. the integral on the right hand side of (81) without integrating over the last two variables $s, t.$ Since $1 \leq e^{2r} e^{-rs-rt}$ for $s, t \in [0, 1],$
\[
\int_{R^k} \int_{s, t \in [0,1]^k} p(x, s)p(x, t) = \int_0^1 \int_0^1 a(s, t) ds dt \leq e^{2r} \int_0^\infty \int_0^\infty e^{-rs-rt} a(s, t) ds dt.
\]
Change variables to the increments $u, v$ of $s, t$ and the increments $y$ of $x$ to get
\[
\int_0^\infty \int_0^\infty r^{2k} e^{-rs-rt} a(s, t) ds dt = \int_{R^k} \int_{u, v \in [0,\infty)^k} r^{2k} e^{-r \sum_{i=1}^k u_i + v_i} \prod_{i=1}^k q(y_i, u_i)q(y_i, v_i)
\]
\[
= \left( \int_{R} \varphi_r(x)^2 \right)^k
\]
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where we used the independent increment property of \( B \), and \( q(x,u) \) is the density of \( B(u) - B(0) \) at \( x \) and that \( re^{-ru} \) is the density of the exponential random variable \( \tau \). \( \square \)

Next, we apply this to planar Brownian motion, with and without drift. We need the following fact.

**Lemma 63.** Let \( B \) be planar Brownian motion with covariance \( I \) and drift \((0, N)\), and let \( \varphi_{\mu,r} \) be the density of \( B \) at independent rate \( r \) exponential random time. Then we have

\[
\| \varphi_{N,r} \|^2 = \begin{cases} \frac{r^2}{2\pi}, & N = 0; \\ \frac{r^2}{2\pi} \wedge \frac{3r^2}{\sqrt{8N}}, & N \geq 0 \end{cases}
\]

(82)

**Proof.** Let \( \tau \) be the exponential variable, and let prime denote independent copies. Then \( \| \varphi_{N,r} \|^2 \) is the density of \( B_\tau - B'_\tau \) at zero. Conditionally on \( \tau \) and \( \tau' \), \( B_\tau - B'_\tau \) is a Gaussian with mean \( (\tau - \tau')N \) and variance \( \tau + \tau' \). Therefore

\[
\| \varphi_{N,r} \|^2 = \int_0^\infty \int_0^\infty \frac{r^2 e^{-r(t+t')}}{2\pi(t+t')} \exp \left\{ -\frac{(t-t')^2}{2(t+t')} \right\} dt \, dt'.
\]

Evaluate at 0 to get \( \frac{1}{2\pi} \) and note that it is decreasing in \( N \) to obtain the first part of (82). To get the second statement, change variables \( u = t + t', v = N(t - t') \). The Jacobian is \( 2N \), and we get

\[
\frac{r^2}{4\pi N} \int_0^\infty \frac{e^{-ru}}{u} \int_{-N^u}^{N^u} e^{-\frac{v^2}{2u}} \, dv \, du \leq \frac{r^2}{4\pi N} \int_0^\infty \frac{e^{-ru}}{u} \int_{-\infty}^{\infty} e^{-\frac{v^2}{2u}} \, dv \, du = \frac{r^2}{\sqrt{8rN}}.
\]

\( \square \)

### 5.3 Brownian intersection local time

Next we consider the special case of \( R = \mathbb{R}^2 \), \( \mathcal{B} \) the Borel sets, \( \mu = \text{Lebesgue measure} \).

**Proposition 64.** Let \( B \) be planar Brownian motion with drift \( \mu \) run until time \( t \). Then for every \( k \geq 1 \), \( \alpha(B, B') \) exists as a limit in \( L^k \), and we have

\[
E[\alpha(B, B')^k] \leq ek!\sqrt{k(te/(2\pi))}^k.
\]

(83)

Note that in the driftless case [Le Gall (1994)] shows \( a_1^k k! \leq E\alpha(B, B')^k \leq a_2^k k! \), so (83) is not far from the truth.

**Proof.** By scaling, it suffices to show this for \( t = 1 \). By Proposition 62 and Lemma 63

\[
\| \rho_k \|^2 \leq k!^2 e^{2r} \| \varphi_r \|^2 \leq \frac{k!^2 e^{2r}}{(2\pi r)^k}.
\]

(84)

Since this is finite for all \( k \), Proposition 60 implies that the intersection local time \( \alpha \) exists and \( \| \rho_k \|^2 = E\alpha^k \). Setting \( r = k \) and using the inequality version of Stirling’s formula \( k! \leq ek^{1/2+k}e^{-k} \) for \( k \geq 1 \) gives (83). \( \square \)
Proposition 65 (From BM to BB). Let $R$ be a Brownian bridge with covariance $\Sigma$ from $p$ to $p + \mu s$ in $\mathbb{R}^d$ in time $s$. Let $a \in (0, 1)$ and let $F(R_{[0,a s]})$ be a nonnegative test function depending on the initial part of $R$. Let $B$ be a Brownian motion with drift $\mu$ started at $p$. Then

$$EF(R_{[0,a s]}) \leq (1 - a)^{-d/2}EF(B_{[0,a s]}).$$

By time reversal, a similar claim holds for the last $a$ portion of $R$.

Proof. The conditional distributions $R_{[0,a s]}$ given $R(as)$, and $B_{[0,a s]}$ given $B(as)$ are the same: both are Brownian bridges with the given endpoints. So it suffices to bound the Radon-Nikodym derivative of $R(as)$ with respect to $B(as)$. They are Gaussians with the same mean and variance $as\Sigma$ and $as(1 - a)\Sigma$, respectively. The maximal derivative does not depend on a common scale or shift, and it is $(1 - a)^{-d/2}$ when $s = 1, \Sigma = I$. 

We are now ready to bound intersection local time for Brownian bridges.

Proposition 66. Consider planar Brownian bridge $R$ from 0 at time 0 to $\mu t$ at time $t$. Then for all $k \in \mathbb{N}$, the intersection local time $\alpha(R, R')$ exists as a limit in $L^k$ and

$$E\alpha^k \leq 4ek!\sqrt{k(te/\pi)^k}.$$ 

Proof. Let $R_1, R_2$ denote the bridge restricted to $[0,t/2]$ and $[t/2,t]$, respectively. Let $Q$ denote the law of $R_1$. Then $Q_{\otimes 2}$ is absolutely continuous with respect to $Q_{BM}^{\otimes 2}$ of Brownian motion with drift $\mu$ run until time $t$ with derivative bounded by 4, see Proposition 65. In particular, since $\alpha_n$ converges in $L^k(Q_{BM}^{\otimes 2})$, it also converges in $L^k(Q_{\otimes 2}^{\otimes 2})$, and with $\alpha = \alpha(R_1, R_1')$

$$E_{Q_{\otimes 2}}\alpha^k \leq 4E_{Q_{BM}^{\otimes 2}}\alpha^k \leq 4ek!\sqrt{k(te/(4\pi))}^k.$$ 

The same bound applies to $R_2$, and Lemma 58 implies that $\alpha(R, R') = \sum_{i,j=1}^2 \alpha(R_i, R'_j)$ exists. Expanding the $k$-th power, and using Proposition 60, Hölder’s inequality and symmetry, we see that $E\alpha(R, R')^k \leq 4^kE\alpha(R_1, R_1')^k$, as claimed.

Remark 67. The mutual intersection local time is classically defined as the value at 0 of the continuous density of the random measure

$$A \mapsto \int_0^t \int_0^t 1_A(B(s) - B'(s'))dsds', \quad A \in \mathcal{B}(\mathbb{R}^2),$$

see Geman, Horowitz and Rosen [1984]. By Lemma 61 this coincides with our definition of $\alpha$ as a random variable on $Q_{\otimes 2}^{\otimes 2}$. 

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Finally, we consider the case where $X$ is the occupation measure up to time $t$ of $(b(s), s)$ where $b(s)$ is a one-dimensional Brownian bridge. The computations are the same and we can identify the limiting $\alpha$ again by its moments. Let $q(x, t)$ be the density at $x$ of the centered Gaussian with variance $t$.

**Proposition 68.** Let $X$ be the occupation measure of $(b(s), s)$ where $b(s)$ is a one-dimensional Brownian bridge from $y_0 = (0, 0)$ to $y_* = (x, t)$. Then for every $k \in \mathbb{N},$

1. The measure $E_Q[X^\otimes k]$ is supported on $(\mathbb{R} \times [0, t])^k$ and has density

$$\rho_{k, X}(y) = \frac{1}{q(y_*)} \prod_{i=1}^{k+1} q(y(i) - y(i-1)),$$

where $y_{k+1} = y_*$ and $y(i)$ are the order statistics of $y$ ordered increasingly by the second (time) coordinate.

2. With $s_0 = 0, s_{k+1} = t$ we have

$$\|\rho_{k, X}\|^2_{L^2((\mathbb{R}^2)^k)} = \frac{2^{-k/2} k!}{q(0, t)} \int_{0 < s_1 < \cdots < s_k < t} \prod_{i=1}^{k+1} q(0, s_i - s_{i-1})$$

$$= \frac{\sqrt{k!}}{(4\pi)^{k/2}} \int_{0 < s_1 < \cdots < s_k < t} \prod_{i=1}^{k+1} \frac{1}{\sqrt{s_i - s_{i-1}}} = \frac{\sqrt{\pi} t^{k/2} k!}{2^k \Gamma(\frac{1}{2}(k + 1))} \leq c \left(\frac{tk/(2e)}{\sqrt{k}}\right)^{k/2}.$$

3. The intersection local time $\alpha$ exists, $\alpha = \lim \alpha_n$ in $L^k(Q^\otimes 2)$ and $E_{Q^\otimes 2}[\alpha^k] = \|\rho_k\|^2_{L^2((\mathbb{R}^2)^k)}$.

4. For all $\gamma > 0$ we have $E \cosh(\gamma \alpha) < \infty$.

**Proof.** Fubini gives 1. To get 2, note that for $s$ fixed, the $x$ integral gives the squared $L^2$-norm of the density of $(b(s_1), \ldots, b(s_k))$. The same computation also gives the density of the self-convolution at zero, which is exactly $2^{-k/2}$ times the original density at 0 for any Gaussian vector. Up to scaling, the integral computes the normalizing constant of a Dirichlet($1/2, \ldots, 1/2$) distribution. Proposition [60] implies 3. Writing the Taylor series of $\cosh$, using Fubini and the bounds 2 gives 4. \qed

**Remark 69.** As in Remark [67] by Lemma [61], $\alpha$ coincides with the classical definition, the value at 0 of the continuous density of the random measure

$$A \mapsto \int_0^t 1_A(b(s) - b'(s))ds, \quad A \in \mathcal{B}(\mathbb{R}),$$

(86)

Let $L$ be a $d \times d$ matrix with $\det L \neq 0$. Then $X \circ L(A)$ and $X \circ L^{-1}(A)$ define the pullback and pushforward of the measure $X$ by the linear transformation $L$.

The definition implies that $\alpha(X, X')$ scales like $\|\rho_X\|^2_2$. More precisely, we have the following.
Lemma 70 (Scaling \( \alpha \)). Let \( X, Y \) be a random measures on \( \mathbb{R}^d \), so that \( \alpha(X, Y') \) The mean densities satisfy \( \langle \rho_{X \circ L}, \rho_{Y \circ L} \rangle_\mu = |\det L| \langle \rho_X, \rho_Y \rangle_\mu \) and

\[
\alpha(X \circ L, Y' \circ L) = |\det L| \alpha(X, Y').
\]

Now let \( t > 0 \), \( (B(s), s \in [0, 1]) \) be a process on \( \mathbb{R}^d \) and let \( \hat{B}(s) = LB(s/t) \) on \([0, t] \). Then the occupation measures satisfy

\[
\hat{X}(A) = |\hat{B}^{-1}(A)| = t|B^{-1}L^{-1}(A)| = tX \circ L^{-1}(A), \quad \alpha(\hat{X}, \hat{X}') = \frac{t}{|\det L|} \alpha(X, X').
\]

Proposition 71. Let \( B^\nu \), be a planar Brownian bridge on the time interval \([0, 1]\) from \((0, 0)\) to \((y, 1)\) with covariance \( \text{diag}(1, \nu^2) \). Then for every \( y \),

\[
\phi(\nu, y) := E\alpha(B^\nu, y, (B^\nu)'Y
\]

is continuous at \( \nu = 0 \). Furthermore,

\[
\phi(0, y) = E[\alpha(b^\nu, (b^\nu)'Y
\]

where \( b^\nu \) is a Brownian bridge from \( 0 \) to \( y \) in time \( 1 \). Moreover, \( \phi(0, y) = \phi(0, 0) \).

Proof. By Propositions 60, 62, and 66 for \( \nu > 0 \) we have the moment formula \( \phi(\nu, y) = ||\rho_{2,B^\nu,Y}||_2^2 \) with \( \rho_2 \) as in (78). For \( \nu > 0 \), it can be written using Fourier transform as

\[
\phi(\nu, y) = ||\rho_{2,B^\nu,Y}||_2^2 = \int_{\mathbb{R}^4} \prod_{j,k=1}^2 \frac{dz_j^k}{2\pi} \int_{[0,1]^4} \prod_{j,k=1}^2 dt_j^k \exp\{\Psi(z, t)\},
\]

where

\[
\Psi = -\frac{\nu^2}{2} \text{Var}(\sum_{j=1}^2 z_j^2(b(t_j^1) - b'(t_j^2))) \frac{1}{2} \text{Var}(\sum_{j=1}^2 z_j^2(b(t_j^1) - b'(t_j^2))) + i \sum_{j=1}^2 (z_j^1 + yz_j^1)(t_j^1 - t_j^2).
\]

(see Geman and Horowitz [1980, p. 43].) Dropping the imaginary part of \( \Psi \) just corresponds to the same problem with the bridges going from \((0, 0)\) to \((0, 0)\). Since for such bridges \( E\alpha^2 < \infty \) by Proposition 66, we see that (89) is absolutely integrable for \( \nu > 0 \). So we can use Fubini and perform the \( z_j^2 \) integrations. We have

\[
\text{Var}(\sum_{j=1}^2 z_j^2(b(t_j^1) - b'(t_j^2))) = z_2^T C z_2, \quad C = \sum_{j=1}^2 \left( t_j^1(1 - t_j^1) \begin{pmatrix} t_j^1(1 - t_j^1) & t_j^2(1 - t_j^2) \\ t_j^2(1 - t_j^2) & t_j^1(1 - t_j^1) \end{pmatrix} \right),
\]

so the \( z_j^2 \) integrations give

\[
\int_{\mathbb{R}^2} \prod_{j=1}^2 \frac{dz_j^2}{2\pi} \int_{[0,1]^4} \prod_{j=1}^2 dt_j^2 \exp\left( -\frac{\nu^2}{2} \text{Var}(\sum_{j=1}^2 z_j^2(b(t_j^1) - b'(t_j^2))) + iy \sum_{j=1}^2 z_j^1(t_j^1 - t_j^2) \right)
\]
Let $\nu \to 0$ to get
\[
\int_{\mathbb{R}^2} \prod_{j=1}^2 \frac{dz_j^i}{2\pi} \int_{[0,1]^2} \prod_{j=1}^2 dt_j \exp\{-\frac{1}{2} \text{Var}\left(\sum_{j=1}^2 z_j^i (b(t_j) - b'(t_j))\right)\} = \phi(0, y),
\]
which we identify as the Fourier transform expression for $\|\rho_{2,b}\|_2^2$ given in Proposition 68. The last claim follows from Lemma 70.

6 Exponential moments off a small set for planar Brownian motion

In this section we use the moment bounds of Section 5.2 to bound the exponential moments of intersection local times off some small exceptional sets. We need this in two different settings. The first is for fixed drift as the size of the exceptional set tends to zero, and the second is a uniform bound as the drift increases. The first is needed to define the solution of the Wick-ordered heat equation for large times, the second is needed for the convergence to the KPZ equation.

6.1 Exponential moments for a short time

We start with establishing exponential moment bounds for short times.

**Lemma 72.** Let $B_1, B'_2$ be independent planar Brownian motions on time interval $[0, t]$ with variance 1 with possibly different drifts and started at possibly different locations. Then
\[
E e^{\gamma \alpha(B_1, B'_2)} \leq 1 + \frac{2\pi \gamma t}{(2\pi/e - \gamma t)^2}, \quad \gamma t \in [0, 2\pi/e).
\]

For independent Brownian bridges $R_1, R'_2$ with variance 1, run until time $t$ but an arbitrary start and endpoint,
\[
E e^{\gamma \alpha(R_1, R'_2)} \leq 1 + \frac{4\pi \gamma t}{(\pi/e - \gamma t)^2}, \quad \gamma t \in [0, \pi/e), \quad E e^{\gamma \alpha(R_1, R'_2)} \leq e^{10\gamma}, \quad \gamma t \in [0, 1].
\]

Moreover, there is a constant $c$ such that if the Brownian motions have a shared drift $(0, N)$, $N \geq 0$ then
\[
E[\alpha(B_1, B'_2)^2] \leq \frac{c t}{1 + N^2 t}, \quad E e^{\gamma \alpha(B_1, B'_2)} \leq 1 + \frac{c \gamma t}{1 + N \sqrt{t}}, \quad \gamma t \in [0, \pi/4].
\]
Proof. By scaling, Lemma 70, we may assume $t = 1$. With prime denoting independent copies, by Proposition 60, we have

$$E_\alpha(B_1, B_2')^k \leq (E_\alpha(B_1, B_1')^k E_\alpha(B_2, B_2')^k)^{1/2} \leq (E_\alpha(B_1, B_1')^k)^{1/2} E_\alpha(B_2, B_2')^k \leq ek! \sqrt{k} (e/(2\pi))^k.$$  

Using Proposition 64 and $\sqrt{k} \leq k$, we have

$$E e^{\gamma \alpha(B_1, B_2')} = \sum_{k=0}^\infty \frac{\gamma^k}{k!} E_\alpha(B_1, B_2')^k \leq 1 + e \sum_{k=0}^\infty (e\gamma/(2\pi))^k k = 1 + \frac{2\pi \gamma}{(2\pi/e - \gamma)^2}.$$  

The bridge bound works the same way, just with the moment bounds of Proposition 66; the exponential bounds the first bound. For the second bound, By Proposition 62 with $k = 2$, $r = 1$ and Lemma 63

$$E_\alpha(B_1, B_1')^2 \leq \frac{c}{1+N^2}. \quad (90)$$  

For the third bound, for any $x \geq 0$ we have $e^x - 1 \leq xe^x$. By Cauchy-Schwarz, for any nonnegative random variable $X$, we have $(Ee^X - 1)^2 \leq EX^2 Ee^{2X}$. So by what we have shown already $E e^{\gamma \alpha} - 1 \leq (E[(\gamma \alpha)^2] Ee^{2\gamma \alpha})^{1/2} \leq c\gamma/(1+N).$ \hfill \qedsymbol

### 6.2 Fixed drift

The goal of this section is to show Corollary 75: on sets of probability arbitrarily close to one, large exponential moments of the intersection local time exist. This is used in the solution of the planar Wick-ordered heat equation for large times.

We will control paths through two parameters. Fix $\kappa \in (0, 1/2)$, and let $\text{box}(\epsilon)$ be the set of $\epsilon \times \epsilon$ closed boxes with corners in $\epsilon\mathbb{Z}^2$.

$$\|B\|_\kappa = \sup_{0 \leq s < t \leq 1} \frac{|B(t) - B(s)|}{(t-s)^\kappa}, \quad M(B, \epsilon, \delta) = \max_{K \in \text{box}(\epsilon)} \sum_{i=0}^{\lfloor 1/\delta \rfloor} 1(B(\delta i) \in K). \quad (91)$$  

The first is the Hölder norm of the paths, the second measures the regularity of the occupation measure at a grid of times.

**Proposition 73.** Let $B, B'$ be independent copies of planar Brownian motion or Brownian bridge on the time interval $[0, 1]$, variance $I$ and arbitrary drift or endpoints. Let $\epsilon, \delta, \gamma > 0$. Let $A$ be the event that

$$\|B\|_\kappa < \epsilon/(5\delta^\kappa) \quad \text{and that} \quad M(B, \epsilon, \delta) \leq 1/(6\sqrt{\gamma \delta}).$$  

Then

$$E \exp \left\{ 1_{A \times A} \gamma \alpha(B, B') \right\} \leq E \exp \left\{ c\gamma \delta \sum_{i,i'} 1(|B(\delta i) - B(\delta i')| < 4\epsilon) \right\} \leq \exp \{c\gamma \delta^{-1}\}.$$  

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Proof. Let \( S = \{(0,0), (0,1), (1,0), (1,1)\} \), let \( \text{box}_{(0,0)}(\varepsilon) \) be the set of boxes with lower left corner in \( 2\varepsilon \mathbb{Z}^2 \), and for \( \sigma \in S \) let \( \text{box}_\sigma(\varepsilon) \) be the set of translates of boxes in \( \text{box}_{(0,0)}(\varepsilon) \) by \( \sigma \varepsilon \). This partitions \( \text{box}(\varepsilon) \) into 4 subsets.

Set \( a_{i,i'} = \alpha(B[i,i+1]\delta, B'[i',i'+1]\delta) \), and consider the versions

\[
a_{i,i'} = a_{i,i'}^* \mathbb{1}\{ |B(s) - B(i\delta)| \leq \varepsilon/5, s \in [i, i+1]\delta \} \\
\times \mathbb{1}\{ |B'(s) - B'(i'\delta)| \leq \varepsilon/5, s \in [i', i'+1]\delta \}
\]

For \( K \in \text{box}(\varepsilon) \) define the asymmetric quantity

\[
Y_K = \sum_{i,i'} \mathbb{1}(B(i\delta) \in K) a_{i,i'}
\]

Then on \( A \times A \) we have

\[
\alpha(B, B') = \sum_{\sigma \in S} \sum_{K \in \text{box}_\sigma(\varepsilon)} Y_K.
\]

and so by Cauchy-Schwarz

\[
E \exp\{1_{A \times A} \gamma \alpha(B, B')\} \leq \prod_{\sigma \in S} \left( E \exp\left\{ 4\gamma \sum_{K \in \text{box}_\sigma(\varepsilon)} Y_K \right\} \right)^{1/4}.
\]

Now condition on the sigma field \( \mathcal{G}_\delta \) given by values of \( B, B' \) at times in \( \delta \mathbb{Z} \).

If \( B(\delta i) \in K \) and \( B(\delta j) \in K' \) for \( K \neq K' \in \text{box}_\sigma(\varepsilon) \) (with the same \( \sigma \)!) then by the modulus of continuity, on the event \( A \times A \) the segments of \( B \) on \( \delta[i, i+1] \) and \( \delta[j, j+1] \) cannot intersect the same segment of \( B' \).

Thus, conditionally on \( \mathcal{G}_\delta \), the collections of mutual intersection local times \( A_K = \{ a_{i,i'} : i, i' \in \mathbb{Z}, B(i) \in K \} \) are independent as \( K \) changes. Thus

\[
E\left[ \exp\left\{ 4\gamma \sum_{K \in \text{box}_\sigma(\varepsilon)} Y_K \right\} | \mathcal{G}_\delta \right] = \prod_{K \in \text{box}_\sigma(\varepsilon)} E\left[ \exp\{ 4\gamma Y_K \} | \mathcal{G}_\delta \right]
\]

Now let \( L_K \) be the number of \( i \) so that \( B(\delta i) \in K \) and let \( L_K^* \) be the number of \( i' \) that \( B'(\delta i') \) is in \( K^* \), the union of 9 boxes in \( \text{box}_\varepsilon \) that intersect \( K \). Then \( L_K L_K^* \) is an upper bound on the number of summands in \( Y_K \). Using Hölder’s inequality again,

\[
E\left[ \exp\left\{ 4\gamma Y_K \right\} | \mathcal{G}_\delta \right] \leq \prod_{i,j:B(\delta i) \in K} E\left[ \exp\left\{ 4L_K L_K^* \gamma \alpha_{ij} \right\} | \mathcal{G}_\delta \right]^{1/L_K L_K^*}
\]

On the set \( A \times A \) we have \( 4L_K L_K^* \gamma^2 < 1 \). By Lemma 72, the conditional expectation is bounded by

\[
E\left[ \exp\left\{ 4L_K L_K^* \gamma \alpha_{ij} \right\} | \mathcal{G}_\delta \right] \leq \exp(c L_K L_K^* \gamma \delta).
\]
Putting everything together, we get
\[ E \exp \{ 1_{A \times A} \gamma \alpha (B, B') \} \leq E \exp \left\{ c \gamma \delta \sum_{K \in \text{box}(\varepsilon)} L_k L^*_k \right\} \]

Where we have
\[ \sum_{K \in \text{box}(\varepsilon)} L_k L^*_k \leq \sum_{i,i'} 1(|B(\delta i) - B'(\delta i')| < 4\varepsilon). \]

The latter can be thought of as the mutual intersection local time of a pair of Gaussian random walks. \( \square \)

It remains to bound \( M(B, \delta, \varepsilon) \). We use a standard argument.

**Lemma 74.** There is \( c, c' > 0 \) so that the following holds. For planar Brownian motion with arbitrary drift and covariance \( I \) on the interval \([0, 1]\) and for all \( 0 < \delta \leq \varepsilon^2 \leq 1/e^2 \) and \( a \geq 1 \) we have
\[ P(M(B, \delta, \varepsilon) > a \delta^{-1} \varepsilon^2 (\log \varepsilon)^2) \leq c' \delta^{-1} \varepsilon^a. \]

The same result holds for any planar Brownian bridge with covariance \( I \).

**Proof.** For planar Brownian \( B \) motion with identity covariance arbitrary starting point and drift, the probability that it will not visit a fixed disk \( D \) of radius \( \varepsilon \) between times \( \varepsilon^2 \) and 1 is bounded below by \( c/|\log \varepsilon| \), uniformly over the starting point. Let
\[ L_j = \sum_{i=0}^j 1(B(\delta i) \in D), \quad L_* = L_{\lfloor 1/\delta \rfloor}. \]

Let \( \ell = \lceil \varepsilon^2/\delta \rceil \) and \( J_k \) be the smallest \( j \) so that \( L_j = \ell k \). By the strong Markov property of \( B \) applied at time \( \delta J_k \), we have
\[ P(L_* \geq \ell (k + 1)|L_* \geq \ell k) \leq c/|\log \varepsilon|. \]

This inequality implies that \( L_* \) is dominated by \( \ell \) times a geometric random variable \( G \) with success probability \( c/|\log \varepsilon| \).

By this and the strong Markov property, applied at time \( i \delta \), the number \( M_i \) of total returns up to time 1 to a box containing \( B(i \delta) \) is dominated by \( \ell G \).

Hence by the union bound for all \( x \geq 0 \),
\[ P(\max_i M_i > x) \leq [\delta^{-1}] P(\ell G > x) = [\delta^{-1}](1 - c/|\log \varepsilon|)^{|x/\ell|}. \]

Setting \( x = a \varepsilon^2 \delta^{-1} (\log \varepsilon)^2 \), the tail bound for the geometric random variable completes the proof for Brownian motion.

For Brownian bridge \( R \) we prove the result for the time intervals \([0, 1/2]\) and \([1/2, 1]\) separately. There, it follows from the Brownian motion case and absolute continuity, Proposition 65. \( \square \)
Corollary 75. Let $R$ be planar Brownian bridge run until an arbitrary time with arbitrary endpoints, and let $R'$ be an independent copy. For every $\gamma \geq 0$, $p < 1$, there is an event $A$ so that

$$PA > p, \quad \text{and} \quad E \exp \{1_{A \times A} \gamma \alpha(R, R')\} < \infty.$$  

Proof. By scaling, it suffices to show this for bridges run until time 1. Fix $1/4 < \kappa' < \kappa < 1/2$. As $\delta \to 0$, set $\varepsilon = \delta^{\kappa'}$. Then by Lemma 74 and Levy’s modulus of continuity,

$$P(M \leq a\delta^{2\kappa'-1}(\log \delta)^2) \to 1, \quad \text{and} \quad P(||B||_\kappa \leq \varepsilon/(5\delta^{\kappa})) \to 1,$$

and for the intersection $A_\delta$ of these events $PA_\delta \to 1$. Choosing $\delta$ small enough so that $PA_\delta > p$ and $\gamma < c\delta^{1-4\kappa'}/(\log \delta)^4$, the conditions of Proposition 73 are satisfied and therefore we indeed have

$$E \exp \{1_{A_\delta \times A_\delta} \gamma \alpha(R, R')\} < \infty. \quad \square$$

6.3 Uniform exponential moments for large drift

We need a bound on the exponential moments which improves with the distance of the starting points. First we show such a bound for short times.

Lemma 76. Let $B, B'$ be independent planar Brownian motions started at $(x, y), (x', y') \in \mathbb{R}^2$ with identity covariance matrix and a shared drift $(0, N), N \geq 0$ and run until time $t$. Let $0 < \gamma, t$ and $\gamma t \leq \pi/4$. Then, there is a constant $c$ such that

$$E e^{\gamma \alpha(B, B')} \leq 1 + \frac{c\gamma t}{1 + N\sqrt{t}} e^{-{|y-y'|^2 \over 8t}} \leq \exp \left\{ \frac{c\gamma t}{1 + N\sqrt{t}} e^{-{|y-y'|^2 \over 8t}} \right\}. \quad (92)$$

and

$$E [\alpha(B, B')^2] \leq \frac{ct^2}{1 + N^2 t} e^{-{|y-y'|^2 \over 8t}}. \quad (93)$$

Proof. By symmetry, we may assume $(x', y') = (0, 0)$ and $y \geq 0$. By Lemma 70, we may assume $t = 1$. Let $\tau, \tau'$ be the first times the second coordinate of $B$, respectively $B'$, equals $y/2$. By symmetry, $\tau'$ is an independent copy of $\tau$ and

$$P(\tau \leq 1) = P(\max_{t \in [0, 1]} B_2(t) \geq y/2) = P(|B_2(1)| \geq y/2) \leq e^{-y^2/8} \quad (94)$$

Let $A_0 = [0, \tau)$, and $A_1 = [\tau, 1]$. Define $A'_i$ analogously. By bilinearity (Lemma 58),

$$\alpha(B, B') = \sum_{i,j \in \{0, 1\}} \alpha(B_{A_i}, B'_{A'_j}) \leq \alpha(B_{A_1}, B') + \alpha(B, B'_{A'}),$$

since $\alpha(B_{A_0}, B'_{A'_0}) = 0$, and $\alpha$ is nonnegative. Cauchy-Schwarz gives

$$E e^{\gamma \alpha(B, B')} \leq E [e^{2\gamma \alpha(B_{A_1}, B')}]^{1/2} E [e^{2\gamma \alpha(B, B'_{A'})}]^{1/2} = E e^{2\gamma \alpha(B_{A_1}, B')}.$$
since the two terms are equal, by symmetry. By conditioning on the \( \sigma \)-field at time \( \tau \) and using Lemma \[72\]

\[
E[e^{2\gamma \alpha(B_{A_1},B')}] - 1 \leq P(\tau \leq 1) \frac{c\gamma}{1 + N}.
\]

By the same argument,

\[
E\alpha^2(B_{A_1},B') \leq P(\tau \leq 1) \frac{c}{1 + N^2}.
\]

Together with \[94\] these give \[92\] and \[93\]. \( \square \)

The following lemma will be needed in the proof of Proposition \[78\].

**Lemma 77.** Let \( B \) be standard one-dimensional Brownian motion. For every \( \gamma, \lambda > 0 \), and integer \( k \geq \gamma^2 \), for the sum over integer times,

\[
E \exp \left\{ \frac{\gamma}{\sqrt{k}} \sum_{i=0}^{k-1} e^{-\lambda B^2(i)} \right\} \leq 2 \exp\left\{ 4\pi \gamma^2 e^{\lambda}/\lambda \right\}.
\]

**Proof.** For \( y \in [0, 1] \), we have \( e^{-y} \leq 1 - y/2 \), so for \( a \in [0, 1] \),

\[
E \exp \left\{ -a \int_0^1 e^{-\lambda(B(t)-x)^2} dt \right\} \leq 1 - a \frac{2}{E} \int_0^1 e^{-\lambda(B(t)-x)^2} dt.
\]

By Jensen’s inequality applied to both the expectation and time integral, this is at most

\[
1 - \frac{a}{2} \exp \left\{ -\lambda \int_0^1 E(B(t)-x)^2 dt \right\} = 1 - 2a_1 e^{-\lambda x^2} \leq \exp\left\{ -2a_1 e^{-\lambda x^2} \right\}, \quad a_1 = \frac{ae^{-\lambda/2}}{4}.
\]

This implies that

\[
E[\exp\left\{-a \int_i^{i+1} e^{-\lambda B^2(t)} dt \right\} | \mathcal{F}_i] \leq \exp\left\{ -2a_1 e^{-\lambda B^2(i)} \right\}
\]

and so

\[
M_j = \exp \left\{ 2a_1 \sum_{i=0}^{j-1} e^{-\lambda B^2(i)} - a \int_0^j e^{-\lambda B^2(t)} dt \right\}
\]

is a supermartingale. By Cauchy-Schwarz

\[
E \exp \left\{ a_1 \sum_{i=0}^{k-1} e^{-\lambda B^2(i)} \right\} \leq (EM_k)^{1/2} \left( E \exp \left\{ a \int_0^k e^{-\lambda B^2(t)} dt \right\} \right)^{1/2}
\]

with \( EM_k \leq 1 \). Setting \( a_1 = \gamma/\sqrt{k}, a = 4\gamma \lambda^{1/2}/\sqrt{k}, a' = 4\gamma \sqrt{\pi/(\lambda k)} e^{\lambda/2} \),

\[
E \exp \left\{ a \int_0^k e^{-\lambda B^2(t)} dt \right\} = E \exp \left\{ a' \int \sqrt{\lambda/\pi} e^{-\lambda x^2} \ell(k, x) dx \right\}
\]

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where \( \ell(k, x) = \int_0^k \delta_0(B(t) - x)dt \) is the local time up to time \( k \) at \( x \). Since \( \int \sqrt{\lambda/\pi} e^{-\lambda x^2} dx = 1 \), we have by Jensen’s inequality,

\[
E \exp \left\{ a' \int \sqrt{\lambda/\pi} e^{-\lambda x^2} \ell(k, x) dx \right\} \leq E \int \sqrt{\lambda/\pi} e^{-\lambda x^2} \exp \{a' \ell(k, x)\} dx.
\]

By stochastic domination,

\[
E \exp \{a' \ell(k, x)\} \leq E \exp \{a' \ell(k, 0)\} = E \exp \{4\gamma \sqrt{\pi/\lambda} e^{\lambda/2} |B_1|\} \leq 2 \exp \{8\pi \gamma^2 e^{\lambda}/\lambda\}
\]

since \( \ell(k, 0) \) has the same distribution as \( |B(k)| \) and \( \sqrt{k} |B(1)| \) and using \( E e^{X} \leq E e^{X} + E e^{-X} \).

**Proposition 78.** Let \( B, B' \) be two independent standard planar Brownian motions with drift \((0, N)\) started at some arbitrary \( z, z' \in \mathbb{R}^2 \) on the time interval \([0, 1]\). There is a \( c < \infty \) such that for all \( N, r \geq 1 \), and \( 0 \leq \gamma \leq \frac{\pi}{nr} \)

\[
E \left[ e^{\gamma N\alpha(B, B')} 1_{A_2^r} \right] \leq e^{c\gamma^2}, \quad A_r := \left\{ \sup_{t \in [0, 1]} |B_2(t) - Nt| \leq r/2 \right\}.
\]

**Proof.** Consider overlapping time intervals

\[
I_j = \left[ j \frac{r}{N}, (j + 1) \frac{r}{N} \right), \quad J_j = \left[ (j - \frac{1}{2}) \frac{r}{N}, (j + \frac{1}{2}) \frac{r}{N} \right).
\]

On \( A_r \), the restrictions \( B_{I_j} \) and \( B'_{I_j} \) of the paths to those intervals can intersect only when \( |i - j| \leq 1 \). Even if \( j = i + 1 \), the intersection can only happen at a time in \( J_j \). This implies that on \( A_r \),

\[
\alpha(B, B') \leq \alpha_1 + \alpha_2, \quad \alpha_1 = \sum_{j=0}^{k-1} \alpha(B_{I_j}, B'_{I_j}), \quad \alpha_2 = \sum_{j=1}^{k-1} \alpha(B_{J_j}, B'_{J_j}).
\]

where \( k = \lceil N/r \rceil \). Let \( \gamma' = 2N\gamma \), so that by assumption \( \gamma' r/N \leq \pi/4 \). Let \( Y_j \) denote the first coordinate of \( B_{I_j}/N - B'_{I_j}/N \). By Lemma 72

\[
E \left[ \exp \{\gamma' \alpha(B_{I_j}, B'_{I_j})\} | \mathcal{F}_{I_j/N} \right] \leq \exp \{d e^{-\frac{1}{\pi} Y_j^2}\}, \quad d = \frac{e^{\gamma' r/N}}{1 + N \sqrt{r/N}} \leq \frac{e^{\gamma' r/N}}{\sqrt{N}},
\]

so the process

\[
M_j = \exp \left\{ \sum_{i=0}^{j-1} \gamma' \alpha(B_{I_i}, B'_{I_i}) - d e^{-\frac{1}{\pi} Y_i^2} \right\}
\]

is a supermartingale. By Cauchy-Schwarz,

\[
(EM e^{2\gamma N\alpha_l})^2 \leq EM_k E \exp \sum_{i=0}^{k-1} \frac{c_{\gamma' r}}{\sqrt{N}} e^{-\frac{m}{\sqrt{r}} Y_i^2} \leq 2 \exp(c\gamma^2), \quad (95)
\]

where the last inequality uses \( EM_k \leq 1 \) and Lemma 77. The same bound holds for \( \alpha_2 \), and we conclude by Cauchy-Schwarz.

\( \square \)
An important shortcoming of Proposition 78 is that the range \([0, \pi/(8r)]\) of exponents \(\gamma\) for which it gives a bound gets worse with the truncation parameter \(r\). The remedy is provided by cutting off paths with large Hölder norm.\(^7\)

**Proposition 79.** Let \(B, B'\) be two independent standard planar Brownian motions with drift \((0, N)\) on the time interval \([0, 1]\). Alternatively, let \(B, B'\) be Brownian bridges from \((0, 0)\) to \(N\). Let \(0 < \kappa < 1/2\). Then for all and \(\gamma > 0\) and \(\alpha > 1\), and \(N > c\gamma r^3 + 1\) we have

\[
E\left[ e^{\gamma N \alpha(B,B')} 1_{H^2} \right] \leq 2e^{b\gamma^2}, \quad H = \{ \|B\|_{\kappa} \leq r/2 \}.
\]

**Proof.** We start with the Brownian motion case. Let \(A_{r,t}\) be the event that \(\sup_{s \in [0,t]} |B_2(s) - Ns| \leq r\). Let

\[
\epsilon(t, N, r, \gamma) = \sup E\left[ e^{\gamma N \alpha(B,B')} 1_{A_{r,t}^2} \right],
\]

where the sup is over all starting points for \(B, B'\), which still have a shared drift \((0, N)\).

Consider the time intervals \(I_j = \left[ \frac{j}{m}, \frac{j+1}{m} \right], J_j = \left( j - \frac{1}{2}, \frac{j+1}{2} \right) \cup \left( j + \frac{1}{2}, \frac{j+3}{2} \right)\). When \(1/m > r/N\) and \(H^2\) holds, then \(B_{I_j}\) and \(B'_{I_j}\) have disjoint support unless \(|i - j| \leq 1\). Moreover, if \(j = i + 1\) they can only have intersecting support within the time interval \(J_j\). This implies that

\[
\alpha(B, B') \leq \alpha_1 + \alpha_2, \quad \alpha_1 = \sum_{j=1}^{m-1} \alpha(B_{I_j}, B_{I_j}), \quad \alpha_2 = \sum_{j=1}^{m-1} \alpha(B_{J_j}, B'_{J_j}).
\]

Let \(G_i\) be the event that \(\sup_{s \in I_i} |B(s) - B(t) - (0, N)(s - t)| \leq r/(m\kappa)\), and abbreviate

\[
\epsilon = \epsilon(1/m, N, rm^{-\kappa}, \gamma) = \epsilon(1, N/m^{1/2}, rm^{1/2-\kappa}, \gamma/m^{1/2}),
\]

where the equality is by scaling, Lemma \(^7\). Then \(R_j = \epsilon^{-1} e^{\gamma N \sum_{i=0}^{j-1} 1_{G_i}} e^{\gamma \alpha(B_{I_i}, B'_{I_i})} \) is a super-martingale, and since on \(H^2\) all \(G_i\) occur, we have

\[
E[e^{\gamma N \alpha} 1_{H^2}] \leq \epsilon^m \epsilon R_m \leq \epsilon^m.
\]

Similarly, \(E[e^{\gamma N \alpha} 1_{H^2}] \leq \epsilon^{m-1}\). By Cauchy-Schwarz and Proposition \(^7\) we get

\[
E[e^{\gamma N \alpha(B,B')} 1_{H^2}] \leq \epsilon(1, N/m^{1/2}, rm^{1/2-\kappa}, 2\gamma/m^{1/2}) \leq \epsilon^{4\gamma^2
\]

as long as \(2\gamma rm^{-\kappa} \leq \pi/8\). The motion case follows with \(m = [((16\gamma r/\kappa)^1/\kappa]\).

For the bridge, on \(1_{H^2}\), for \(N\) large enough, \(\alpha(B_{[0,1/3]}, B'_{[2/3,1]}) = \alpha(B_{[2/3,1]}, B'_{[0,1/3]}) = 0\) and then since \(\alpha\) is bilinear and non-negative we have

\[
\alpha(R, R') \leq \alpha(R_{0\to2/3}, R'_{0\to2/3}) + \alpha(R_{1/3\to1}, R'_{1/3\to1}).
\]

The two quantities have the same distribution by symmetry. The first two-thirds of the bridge is absolutely continuous with bounded derivative with respect to the first two-thirds of the motion, Proposition \(^5\). Cauchy-Schwarz concludes the proof. \(\square\)
7 Proofs of the main theorems

In this section we prove Theorems 2, 3, 4 and 6. Theorem 5 is a special case of Theorem 6. The hard work has been done, but the pieces need to be assembled.

7.1 Proof of Theorems 2 and 3

Explicit solution and uniqueness. Definition 1 is what we rigorously mean by a solution of the Wick-ordered heat equation. Proposition 26 shows that the projections $E_P[u(x,t)|F_n]$ satisfy a finitary version of the equation. These projections are martingales.

Proposition 27 shows that the finitary version of the Wick-ordered heat equation has a unique solution $u_n(x,t) = p(x,t)Z_n(x,t)$ with

$$Z_n(x,t) = E_Q \exp \left\{ \sum_{j=1}^{n} m_j \xi_j - \frac{1}{2} \sum_{j=1}^{n} m_j^2 \right\}, \quad m_j = \int_{0}^{t} e_j(B(s)) \, ds.$$  \hspace{1cm} (97)

Proposition 28 shows that if the martingale $Z_n(x,t)$ is uniformly integrable for all $x,t$, then its limit is the unique solution of the Wick-ordered SHE. So we fix $x,t$.

In Section 4, we identify the martingale $Z_n$ as a randomized shift. Proposition 39.2 gives criteria for $Z_n$ to converge in $L^1$ in terms of the intersection exponential $\epsilon(m)$ (see (61)). Here

$$\epsilon(m) = \lim_{n \to \infty} E_{Q\otimes^2} e^{\alpha_n}, \quad \alpha_n = \sum_{i=1}^{n} m_i m_i'.$$

is defined in terms of two independent copies of $m$ in (97). Proposition 66 shows that $\alpha_n$ has a limit $\alpha \geq 0$ in $L^k$ for every $k > 0$, the mutual intersection local time of two independent copies of $B$. Lemma 44 shows that for $\alpha \geq 0$ we have $\epsilon(m) \leq Ee^\alpha$.

By Proposition 39.2 to show martingale convergence, it suffices to show that for every $p < 1$ there is set of paths $A$ with $QA > p$ so that $\epsilon(1_A m) \leq \infty$.

More concretely, for Brownian bridges, we need to show that there is a set of paths $A$ with probability arbitrarily close to 1 so that the mutual intersection local time satisfies $E_{Q\otimes^2} \exp\{1_{A \otimes A} \alpha\} < \infty$.

For the set $A$, we control two properties of the paths: a Hölder norm and the maximal mass of a discretized version of the occupation measure, see (91).

Proposition 73 shows that under such control, the exponential moments of the intersection local time exist, and Lemma 74 bounds the maximal mass. Using these results, Corollary 75 provides the set $A$ of paths with $QA > p$ so that $E_{Q\otimes^2} \exp\{1_{A \otimes A} e^\alpha\} < \infty$. Thus we have found the solution of the Wick-ordered heat equation.
Convergence in $L^2(P)$. For $t < t_c$ we have $Ee^\alpha < \infty$. By Lemma 44, we have $e(m) < \infty$, and Proposition 39 implies that $Z_n \to Z$ in $L^2(P)$.

Basis independence. The solution we have found is the partition function of a randomized shift with $\alpha$ given by the mutual intersection local time. Proposition 56 shows that when the mutual intersection local time exists, it is unique. In particular, it does not depend on the basis $e_j$. The partition function $Z$ is basis-independent by Remark 47. Uniqueness in this type of construction is due to Shamov (2016) (Theorem 17 and Corollary 18), which show that the underlying extended Gaussian covariance kernel uniquely determines the partition function $Z$. For us, the kernel is on path space $\Omega$, and is given by the mutual intersection local time $\alpha$. Theorem 2 follows.

Polymer measure. The randomized shift representation has been established along the way in Proposition 39. It gives rise to a unique polymer measure by Theorem 50. This completes the proof of Theorem 3. Proposition 54 implies the claimed polymer convergence.

Remark 80. This proof extends to the case when the initial condition $\varsigma$ is a general probability measure.

7.2 Proof of Theorem 4

Explicit solution and uniqueness. By Proposition 31, the conditional expectation $z_n = E_Q[z|F_n]$ of the equation has the unique solution given by

$$z_n(x, t) = Z_n(y, 0; x, t)q(x - y, t)$$

$$Z_n = Z_n(0, y; t, x) = E_Q \exp \left( \sum_{j=1}^{n} m_j \xi_j - \frac{1}{2} m_j^2 \right), \quad m_j = \int_0^t e_j(b(s), s)\,ds.$$ 

and $b(s)$ is a one-dimensional Brownian bridge between space-time points $(0, 0)$ to $(x, t)$.

We need to show that this is an $L^2(P)$ martingale, that is, it has a bounded $L^2$-norm. By Proposition 33, we have

$$E_P Z_n^2 = E_Q e^{\alpha_n}, \quad \alpha_n = \sum_{i=1}^{n} m_{ij}^j.$$ 

By Proposition 68, $\alpha_n$ has a limit in every $L^k(Q^2)$. The limit $\alpha(b, b')$ is the mutual intersection local time of two independent copies of $(b(s), s)$ up to time $t$, and $E_{Q^2} \cosh \alpha < \infty$. By Lemma 44, we have

$$E_P[Z_n^2] = E_{Q^2} e^{\alpha_n} \leq E_{Q^2} \cosh \alpha.$$ 

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so $Z_n \to Z$ in $L^2(P)$. With this construction, taking conditional expectations given $\mathcal{F}_n$ as in Proposition 30, we see that for every $n$ the $\mathcal{F}_n$-conditional expectations of the two sides of the equation (52) of Definition 29 agree. So the limit $z$ is indeed a solution. We have thus found the unique solution in an explicit form.

**Agreement with Itô.** The Skorokhod integral formulation in Definition 29 of the one-dimensional Wick-ordered stochastic heat equation coincides with the mild Itô formulation. This is because by Remark 24, the two integrals are the same. Since there is a unique solution, the solution in Alberts et al. (2014) agrees with ours.

**Polymer measure and CDRP.** Associated to the partition function $Z$ there is a random polymer measure $M$ on $\Omega$ defined by the formula (13), see Section 4.6. We claim that it coincides with the continuum directed random polymer constructed in Alberts et al. (2014), Definition 4.1. Their construction is a random continuous function on $[0,1]$, determined by its finite dimensional distributions given the noise $\xi$. Since $\Omega$ is the space of continuous functions, it suffices to show that their finite dimensional distributions agree with ours. This is the content of the next lemma. Let $z(x, s; y, t) = q(y - x, t - s)Z(x, s; y, t)$ denote the solution of the SHE started at time $s$ from the initial condition $\delta_x$.

**Lemma 81.** Let $M$ be the unnormalized polymer measure on paths $b$ from $p$ to $q$, let $k \geq 1$ and set $(x_0, s_0) = p$, $(x_{k+1}, s_{k+1}) = q$. Pick $s_1 < \cdots < s_k$ in $(s_0, s_{k+1})$. Let $\varphi$ be a bounded nonnegative test function on $\mathbb{R}^k$. Then

$$E_M \varphi(b(s_1), \ldots, b(s_k)) = \int \varphi(x_1, \ldots, x_k) \prod_{i=0}^k z(x_i, s_i; x_{i+1}, s_{i+1}) \, dx_1 \cdots dx_k \quad P-a.s.$$  

**Proof.** To keep the notation manageable, let

$$\varphi = \varphi(b(s_1), \ldots, b(s_k)), \quad Z_i = Z(b(s_i), s_i; b(s_{i+1}), s_{i+1}).$$

By expressing the integral as a $Q$-expectation and accounting for the heat equation $q$ factors between $z$ and $Z$, the claim translates to

$$E_M \varphi = E_Q[\varphi Z_0 \cdots Z_k] \quad P-a.s. \quad (98)$$

Let $F_i$ be a bounded nonnegative test function on $(\Xi, \mathcal{F}, P)$ depending only on the noise at times in between $s_i$ and $s_{i+1}$. Let $F = F_0 \cdots F_k$. Let $S \subset \mathcal{G}$ denote the $\sigma$-field generated by $b(s_1), \ldots, b(s_k)$. By definition of the polymer measure, Fubini, and since $\varphi$ is $S$-measurable, we have

$$E_P E_M[\varphi F] = E_P E_Q[\varphi F(\xi + m)] = E_P E_Q E_Q[\varphi F(\xi + m)|S] = E_Q[\varphi E_P E_Q[F(\xi + m)|S]]. \quad (99)$$
Given $S$ and $\xi$ fixed, by the Markov property of $b$ the random variables $F_0(\xi+m), \ldots, F_k(\xi+m)$ are independent over $Q$, so

$$E_Q[F(\xi+m)|S] = \prod_{i=0}^{k} E_Q[F_i(\xi+m)|S].$$

Because the noises between times $s_i$ and $s_{i+1}$ are $P$-independent as $i$ varies, we have

$$E_P E_Q[F(\xi+m)|S] = \prod_{i=0}^{k} E_P E_Q[F_i(\xi+m)|S] = \prod_{i=0}^{k} E_P[Z_i F_i],$$

by the shift representation of the partition functions $Z_i$. Since the factors are independent over $P$, substituting this into (99) we get

$$E_P E_M[\phi F] = E_Q E_P[\phi \prod_{i=0}^{k} Z_i F_i] = E_P E_Q[\phi Z_0 \cdots Z_k F].$$

As this holds for any product test function $F$, the claim (98) follows. 

Next, we prove the main convergence theorem.

### 7.3 Proof of Theorem 6, tightness and polymer limits

We continue the proof started in the introduction.

**Tightness.** We need to justify the interchange of limits in $n$ and $N$. For the rest of the proof, we will assume that all scaling parameters are standard, namely $\nu = \rho = \beta = s = 1$, $x = t = 0$ and only $y$ is free, since the general argument is identical by scaling, see Lemma 70.

A condition for the interchange of limits is established in Proposition 46.2. It is in terms of the self-intersection local times

$$\alpha_N = \lim_{n \to \infty} \sum_{i=1}^{n} m_N m'_N, \quad \alpha = \lim_{n \to \infty} \sum_{i=1}^{n} m_i m'_i,$$

defined on $\Omega^2$, with prime denoting independent copies. See Section 5.1 for more explanation about the $\alpha$s.

One condition is that the limits defining $\alpha_N$ and $\alpha$ should exist in $L^2$; this is established in Propositions 66 and 68. Next, we need $\alpha_N \to \alpha$ in law, which, by Lemma 45, follows from $E\alpha^2_N \to E\alpha^2$. This is established by an explicit computation in Proposition 71. Finally we need a tightness condition: there is $\gamma > 1$ so that given $p < 1$ we have a set of paths $A_n$ for every $n$, so that $QA_n > p$ and

$$\limsup_{N \to \infty} \epsilon(\gamma 1_{A_N} m_N) < \infty.$$
Since $\alpha_N$ are nonnegative, by Lemma 44 it suffices to show $\lim \sup_N E[e^{\gamma\alpha_N} 1_{A \times A}] < \infty$. We take $A$ to be a set of paths with bounded Hölder-$\kappa$ norm (91) for $\kappa \in (0, 1/2)$. The $L^1(P)$ convergence claim then follows from Proposition 70.

**Polymer convergence.** For every $N$, the formula (9) gives a $P$-random measure $M_N$ on $Q$. These measures are modifications of the law of the pair $(b_{sp}, b_{ti})$. Similarly, the limiting measure $M$ modifies the law of this pair, but in a way that only depends on the first process.

Proposition 54 implies that $M_n \to M$ in probability with respect to weak convergence of measures. In other words,

$$(b_{sp}, b_{ti}) \text{ under } M_N \xrightarrow{\text{law}} (b_{sp}, b_{ti}) \text{ under } M.$$ 

By Theorem 4 under $M$, the process $b_{sp}$ is the continuum directed random polymer (CDRP), and $b_{ti}$ is an independent Brownian bridge. This implies the following laws converge in probability

$$\left( (N^{-1/2} B_N(N \rho r)_1, N^{-3/2} B_N(N \rho r)_2), \ r \in [0, s] \right) \text{ under } M_N \xrightarrow{\text{law}} \left( (b_{sp}(t + r), t + r), \ r \in [0, s] \right) \text{ under } M.$$ 

This completes the proof of Theorem 6.

**Remark 82.** The proof also shows that the fluctuations $b_{ti}$ in the limiting time-direction converge to an unweighted Brownian bridge independent of the limiting polymer. Also, since $Z_N \to Z$ and $Z_N, Z > 0$ a.s. by the 0-1 law, Lemma 37, the normalized polymer measures converge as well: $M_N/Z_N \to M/Z$.

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