A new computation of pairing probabilities in several multiple-curve models

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Abstract

We give a new, short computation of pairing probabilities for multiple chordal interfaces in the critical Ising model, the harmonic explorer, and for multiple level lines of the Gaussian free field. The core of the argument are the convexity and uniqueness properties of local multiple \(\text{SLE}(\kappa)\) measures, valid for all \(\kappa > 0\) and thus in principle applicable for any underlying random curve model, once suitably connected to local multiple SLEs.

1 Introduction

Schramm–Loewner evolution (SLE) type curves are conformally invariant random curves \cite{Sch00, RS05} that are known or conjectured to describe (scaling limits of) random interfaces in many critical planar models (e.g. \cite{Smi01, LSW04, SS05, SS09, CDCHKS14}). The most common SLE variants are defined in the upper half-plane \(\mathbb{H}\) via the Loewner equation

\[
\frac{\partial t g_t(z)}{g_t(z) - W_t} = \frac{2}{g_t(z) - W_t},
\]

whose solution for a given starting point \(g_0(z) = z\) in \(\mathbb{H}\) is only defined up to the (possibly infinite) explosion time when \(g_t(z)\) and \(W_t\) collide. The sets \(K_t\) where the solution is not defined up to time \(t\) are those carved out by the initial segment of the SLE curve, while \(g_t\) is a conformal map \(H \vb K_t \to \mathbb{H}\). For instance, the chordal SLE(\(\kappa\)) from 0 to \(\infty\) in \(\mathbb{H}\) is obtained by taking \(W_t = \sqrt{\kappa} \beta_t\), where \(\beta_t\) is a standard Brownian motion and \(\kappa > 0\) is the parameter of the model; see \cite{Law05} for an introduction.

The most central model in this note is the local multiple SLE(\(\kappa\)) \cite{BBK05, Dub07} which, e.g., with \(\kappa = 3\) describes the scaling-limit interfaces in the Ising model with alternating boundary conditions (see Figure 1), when mapped conformally to \(\mathbb{H}\). The definition takes as an input a partition function, which is defined as a map \(Z: \mathcal{X} \to \mathbb{R}_{>0}\), where \(\mathcal{X} = \{(x_1, \ldots, x_{2N}) \in \mathbb{R}^{2N} : x_1 < \ldots < x_{2N}\}\), satisfying to the conformal covariance condition (denoting \(\frac{6-\kappa}{2\kappa} = h\))

\[
Z(x_1, \ldots, x_{2N}) = \left( \prod_{i=1}^{2N} \phi'(x_i)^h \right) Z(\phi(x_1), \ldots, \phi(x_{2N}))
\]

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\footnote{We will allow starting points in \(\mathbb{H}\) without explicit mention whenever it is more beneficial.
Figure 1: Simulations of the critical Ising model with alternating boundary conditions on $100 \times 100$ (left) and $400 \times 400$ (right) square grid graphs. The model with $2N$ boundary condition alternation points naturally gives rise to $N$ chordal interfaces (here $N = 4$), pairing up the alternation points in some random manner. Labelling the alternation points in these two examples by the points of compass in the natural manner, the pairings of the alternation points by the interfaces are here \{\{sw, s\}, \{se, e\}, \{ne, n\}, \{nw, w\}\} (left) and \{\{sw, s\}, \{se, w\}, \{e, ne\}, \{n, nw\}\} (right).

Figure 2: A schematic illustration of the setup where local multiple SLEs are studied in this note: $2N$ fixed boundary points $V_{0}^{1} < \ldots < V_{0}^{2N}$, a fixed index $1 \leq j \leq 2N$, and a fixed neighbourhood of $V_{0}^{j}$ in $\mathbb{H}$ not containing any other marked boundary points. The Loewner growth process (1)–(3) starting from $V_{0}^{j}$ is then considered up to the stopping time when $K_t$ first hits the boundary (in $\mathbb{H}$) of this localization neighbourhood.

for all conformal (Möbius) maps $\phi$ from $\mathbb{H}$ to $\mathbb{H}$ with $\phi(x_1) < \ldots < \phi(x_{2N})$, and the partial differential equations (PDEs)

$$\left[\frac{\kappa \partial^2}{2} + \frac{2N}{\sum_{i=1}^{N} (x_i - x_j)^2}\right] Z(x_1, \ldots, x_{2N}) = 0, \quad \text{for all } 1 \leq j \leq 2N. \quad (2)$$

Then, in $\mathbb{H}$, with the $2N$ marked boundary points $V_{0}^{1} < \ldots < V_{0}^{2N}$, the growth of the curve from $V_{0}^{j}$ is described by the Loewner equation (1) where $W_t$ is determined by $W_0 = V_{0}^{j}$ and $2N$ coupled SDEs: $V_{t}^{i} = g_{i}(V_{0}^{j})$ for $i \neq j$ and

$$dW_{t} = \sqrt{\kappa} \beta_{t} + \kappa \frac{(\partial_j Z)(V_{t}^{1}, \ldots, V_{t}^{j-1}, W_t, V_{t}^{j+1}, \ldots, V_{t}^{2N})}{Z(V_{t}^{1}, \ldots, V_{t}^{j-1}, W_t, V_{t}^{j+1}, \ldots, V_{t}^{2N})} dt. \quad (3)$$

To avoid treating the boundary behaviour of the PDE solutions $Z$, the process (3) is usually only studied up to the stopping time when the growing sets $K_t$ hit the boundary of a given localization neighbourhood of $V_{0}^{j}$; see Figure 2. Fixing any such geometry,

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2In contrast to some texts on the subject, we define the local multiple SLE as a measure in a given geometry (rather than a collection of measures indexed by geometries), and study a single process of growing sets $K_t$ in a single localization neighbourhood (rather than so-called iterated growth processes). These choices of convention that play more than a technical role; see, e.g., the discussion before Lemma 3.1.
the laws of two multiple SLE (stopped) driving functions $W_t$ coincide if and only if the partition functions are constant multiples of each other (Lemma 3.1). Note also the set of partition functions is a convex cone; a corresponding convex-space property readily follows for the local multiple SLE measures (Equations (9)–(10)). The emergence of a convex space, rather than a unique measure, is intuitively explained as non-crossing interfaces in an underlying lattice model can pair up the $2N$ marked boundary points into a Catalan number

$$\frac{1}{N+1} \binom{2N}{N}$$

of different planar pair partitions, or pairings, for short (Figure 1). Each pairing-conditional measure (as well as their convex combinations) should then converge to a local multiple SLE.

The main contribution of this note is a new, short computation solving the probabilities of the different pairings in several underlying random curve models. The core lemma is the following.

**Lemma 1.1** Fix launching points $V_0^1 < \ldots < V_0^{2N}$, an index $1 \leq j \leq 2N$, and a localization neighbourhood of $V_j^0$. Let $P$ be a convex combination of finitely many laws $P_\alpha$ of some multiple SLE driving functions with respective partition functions $Z_\alpha$:

$$P = \sum_{\alpha} p_\alpha P_\alpha, \quad \text{where } p_\alpha \geq 0 \text{ and } \sum_{\alpha} p_\alpha = 1.$$

Then, also $P$ is a local multiple SLE driving function, and the corresponding, up-to-constant unique partition function $Z$ is given by

$$Z(x_1, \ldots, x_{2N}) = Z(V_0^1, \ldots, V_0^{2N}) \sum_{\alpha} \frac{p_\alpha}{Z_\alpha(V_0^1, \ldots, V_0^{2N})} Z_\alpha(x_1, \ldots, x_{2N}).$$

Consequently, if some random chordal (scaling limit) curves $\gamma_1, \ldots, \gamma_N$ in $\mathbb{H}$ have known descriptions as local multiple SLEs both as such ($P$ and $Z$) and conditionally on any pairing $\alpha$ ($P_\alpha$ and $Z_\alpha$), then the pairing probabilities $p_\alpha$ can be solved from the linear coefficients of $Z_\alpha$ in $Z$, provided that the $Z_\alpha$’s are linearly independent. We apply this method to compute the pairing probabilities in the Ising model, the multiple harmonic explorer, and the level lines of the Gaussian free field. While this method poses no restriction on the value of $\kappa$, in all three examples, the identification of the $\alpha$-conditional limit is deduced from the theory of the so-called global multiple SLEs [PW19, BPW21], an argument specific for $\kappa \leq 4$. Whenever solvable by this method, the pairing probabilities $p_\alpha$ become ratios of multiple SLE partition functions. This has interesting interpretations in terms of Conformal field theory (e.g., [Pel19]) which we however suppress in this short note.

The pairing probabilities in the Ising and free field models were earlier solved in [PW19] and [PW18], respectively. Those proofs rely on conditional probability martingales which, unlike the convex combination argument here, have direct and well-known analogues for single SLEs. Analyzing the behaviour of these martingales up to the termination time of the curves (in order to relate their termination value to the correct pairing event) however required in [PW19, PW18] a fine analysis of both the SLE process and the martingale, which were based on technical, model-specific arguments. We thus hope that the alternative, general proof in this note clarifies the picture. The present proof may also be interesting in the context of other SLE variants with partition functions (e.g., [Kar20]). Other solutions of pairing probabilities of multiple SLE type curves can be found at least in [Smi01, Dub06, KW11, KKP20, FPW22].

3Note that this equation takes a particularly appealing form if all partition functions are normalized to attain the value 1 at $(V_0^1, \ldots, V_0^{2N})$.3
2 Solving the pairings probabilities

2.1 The Ising model

Let us start by explaining in detail how results about discrete models are inferred from the SLE theory above. We choose to explicate the details for the Ising model.

Let \( \Omega^\delta \) be simply-connected discrete domains on \( \delta \mathbb{Z}^2 \) with marked boundary points \( p^\delta_1, \ldots, p^\delta_{2N} \). Let \( \mathbb{P}^\delta \) denote the critical Ising model on \( (\Omega^\delta; p^\delta_1, \ldots, p^\delta_{2N}) \) with alternating + and – boundary conditions; see Figure 1 for an illustration and Appendix A.1 for the precise definition. We say that the discrete domains with marked boundary points \( (\Omega^\delta; p^\delta_1, \ldots, p^\delta_{2N}) \) on \( \delta \mathbb{Z}^2 \) converge in the Carathéodory sense if the conformal maps \( \varphi^\delta : \mathbb{H} \to \Omega^\delta \) (normalized in some fixed manner, e.g., at the point \( i \in \mathbb{H} \)) converge to some \( \varphi \) uniformly over compact subsets of \( \mathbb{H} \) and also the inverse images of the boundary points converge: \( (\varphi^\delta)^{-1}(p^\delta_i) \to V^i_0 \) for \( 1 \leq i \leq 2N \), for some \( V^1_0, \ldots, V^{2N}_0 \in \mathbb{R} \). We also assume that \( \varphi(\mathbb{H}) \) is bounded and that the limit points \( V^1_1, \ldots, V^{2N}_0 \) are distinct and labelled so that \( V^1_1 < \ldots < V^{2N}_0 \).

**Theorem 2.1** Let \( (\Omega^\delta; p^\delta_1, \ldots, p^\delta_{2N}) \) on \( \delta \mathbb{Z}^2 \) converge in the Carathéodory sense as \( \delta \downarrow 0 \); then, we have

\[
\mathbb{P}^\delta[\text{pairing } \alpha] \to \frac{Z_\alpha(V^1_0, \ldots, V^{2N}_0)}{Z(V^1_0, \ldots, V^{2N}_0)} \quad \text{as } \delta \downarrow 0,
\]

where \( Z_\alpha \) and \( Z \) are as defined in Equation (6) and right below it.

**Proof** Let \( \gamma^\delta_1, \ldots, \gamma^\delta_N \) denote the chordal curves in \( \mathbb{H} \) obtained by mapping the Ising interfaces on \( (\Omega^\delta; p^\delta_1, \ldots, p^\delta_{2N}) \) by \( (\varphi^\delta)^{-1} \). The Ising model conditional on a given pairing \( \alpha \) is denoted below by \( \mathbb{P}^\delta[\cdot | \text{pairing } \alpha] = \mathbb{P}^\delta[\cdot] \).

Given a sequence of Carathéodory converging discrete domains \( (\Omega^\delta; p^\delta_1, \ldots, p^\delta_{2N}) \), it is fairly standard (for details see input (i) below the proof) that a subsequence of \( \delta \)'s can be extracted so that \( (\gamma^\delta_1, \ldots, \gamma^\delta_N) \) converge weakly in the space of \( N \) unparametrized curves under \( \mathbb{P}^\delta \) as well as \( \mathbb{P}^\delta_\alpha \) for all \( \alpha \); denote the limiting laws by \( \mathbb{P} \) and \( \mathbb{P}_\alpha \), respectively. It clearly suffices to prove the claimed limit [4] along such a subsequence, so we fix one for the rest of the proof.

The connection probabilities at least converge to some numbers along this subsequence:

\[
\mathbb{P}^\delta[\text{pairing } \alpha] \to \mathbb{P}[\text{pairing } \alpha] =: p_\alpha \in [0, 1].
\]

Hence we also have the convex combination formula

\[
P = \sum_\alpha p_\alpha P_\alpha
\]

for the limiting measures. By input (ii), under \( P \) (resp. \( P_\alpha \)), the driving function of the initial segment of the curve starting from a fixed boundary point in a fixed localization neighbourhood has the local multiple SLE law \( P \) (resp. \( P_\alpha \)) with a known partition function \( Z \) (resp. \( Z_\alpha \)). From (5) and Lemma 1.1, we infer that

\[
Z(x_1, \ldots, x_{2N}) = Z(V^1_0, \ldots, V^{2N}_0) \sum_\alpha \frac{p_\alpha}{Z_\alpha(V^1_0, \ldots, V^{2N}_0)} Z_\alpha(x_1, \ldots, x_{2N}).
\]
Finally, by input (iii), \( Z = \sum \alpha Z_\alpha \) and the functions \( Z_\alpha \) are readily seen to be linearly independent\(^4\) so we deduce

\[
p_\alpha = \frac{Z_\alpha(V_1^0, \ldots, V_{2N}^0)}{Z(V_1^0, \ldots, V_{2N}^0)}, \quad \text{for all } \alpha.
\]

This concludes the proof. \( \square \)

**Inputs of the proof** Let us yet explicate the necessary logical inputs used above

i) For any sequence \( \delta \to 0 \), there exists a subsequence along which the curves \( \gamma_1^\delta, \ldots, \gamma_N^\delta \) under \( \mathbb{P}^{O^\delta} \) (resp. under \( \mathbb{P}_\alpha^{O^\delta} \)) converge weakly to some limiting random curves \( \gamma_1, \ldots, \gamma_N \) (in the metric of unparametrized curves).

Input (i) follows from the weak convergence criterion \([KS17, \text{Theorem 1.5}]\) which is verified for the Ising model in \([CDCH13, \text{Corollary 1.7}]\) (see also \([Kar19, \text{Theorem 4.1}]\)).

ii) Any initial segment of the limit curves \( \gamma_1, \ldots, \gamma_N \) of \( \mathbb{P}^{O^\delta} \) and \( \mathbb{P}_\alpha^{O^\delta} \), respectively, from a given starting point in a given localization neighbourhood, has a Loewner description as a local multiple SLE(3) curve with the respective partition functions \( Z : X \to \mathbb{R}_{>0} \) given by the Pfaffian formula

\[
Z(x_1, \ldots, x_{2N}) = \text{Pf}\left( \frac{1}{x_j - x_i} \right)_{i,j=1}^{2N}
\]

(diagonal elements of the matrix above should be interpreted as zeros), and \( Z_\alpha \) as defined in \([PW19, \text{Equation (3.7)}]\) with \( \kappa = 3 \).

This local limit identification for \( \mathbb{P}^{O^\delta} \) is proven in \([Izy17, \text{Theorem 1.1}]\), using input (i) to deduce the existence of subsequential limits and a martingale observable to identify them. For the conditional measures \( \mathbb{P}_\alpha^{O^\delta} \), in turn, \( \gamma_1, \ldots, \gamma_N \) are identified as global multiple SLEs with pairing \( \alpha \) (which essentially boils down to the uniqueness of the latter) in \([BPW21]\). The local Loewner description of the latter was found in \([PW19, \text{Theorem 1.3}]\), with the relation \( \sum \alpha Z_\alpha = Z \), which we call input (iii), given in \([PW19, \text{Lemma 4.13}]\).

**2.2 The multiple harmonic explorer**

We now briefly overview an analogous result for the multiple harmonic explorer introduced in \([Kar19]\). We use the same notations as for the Ising model in applying parts to highlight the exact similarity of the proofs.

Let \( H \) denote the honeycomb lattice, let \( \Omega \) consist of the faces on or inside a simple loop path on the faces of \( H \) (i.e., on the dual graph \( H^* \)), and colour the faces on this loop into \( N \) white and \( N \) black segments, changing colours at \( 2N \) boundary points \( p_1, \ldots, p_{2N} \). The harmonic explorer explores an interface between black and white faces, starting from the edge emanating from \( p_1 \) into \( \Omega \): if the face right in front of this edge is already coloured, the next edge of the interface is evident; otherwise, launch a dual random walk from this face, colour the face according to the first coloured face hit by the walk, and then deduce the next edge. Iteratively, one then adds new edges and new coloured faces, e.g., in a circular order to the boundary points, until \( N \) entire chordal interfaces have been revealed. We denote by \( \mathbb{P}^{\Omega \delta} \) probability measure of the harmonic explorers on \( (\Omega; p_1, \ldots, p_{2N}) \).

\(^4\)This follows, e.g., directly from the defining property \([PW19, \text{Eq. (ASY)}]\), by induction over \( N \).
Theorem 2.2 Let $(\Omega^\delta;p_1^\delta,\ldots,p_N^\delta)$ be discrete domains on the scaled lattice $\delta H$, as described above, and converging in the Carathéodory sense as $\delta \downarrow 0$; then, we have

$$\mathbb{P}^{\Omega^\delta}[\text{pairing } \alpha] \rightarrow \frac{Z_\alpha(V_0^1,\ldots,V_0^{2N})}{Z(V_0^1,\ldots,V_0^{2N})} \text{ as } \delta \downarrow 0,$$

where $Z_\alpha$ and $Z$ are as defined in Equation (7) and right below it.

The proof is word-by-word identical to the Ising model. Input (i), as well as input (ii) for the unconditional measures $\mathbb{P}^{\Omega^\delta}$ are proven in [Kar19, Theorem 6.10]; the SLE parameter is $\kappa = 4$ and partition function is

$$Z(x_1,\ldots,x_{2N}) = \prod_{1 \leq i < j \leq 2N} (x_j - x_i)^{(1-\lambda)/2}. \quad (7)$$

Input (ii) for the conditional measures $\mathbb{P}^{\Omega^\delta}_\alpha$ is [Kar19 Theorem 5.10] (whose proof relies on the global multiple SLEs similarly to the Ising model case), and the partition functions $Z_\alpha$ are as defined in [PW19, Equation (3.7)] with $\kappa = 4$. Finally, the relation

$$Z = \sum_\alpha Z_\alpha, \quad (8)$$

i.e., input (iii), is proven in [PW19, Lemma 4.14].

2.3 The level lines of the Gaussian free field

We yet explicate the analogous proof for multiple level lines of the Gaussian free field (GFF). Here the underlying model is continuous per se and no limit argument is needed.

Let $\mathbb{P}$ be the GFF measure (see Appendix A.2) in $\mathbb{H}$ with the following alternating boundary conditions: given $V_0^1 < \ldots < V_0^{2N}$, the boundary condition at $x \in \mathbb{R}$ is set to be $\lambda$ if the number of marked boundary points strictly left of $x$ is even, and $-\lambda$ if odd; here we denote $\lambda = \sqrt{\pi/8}$, following the normalization convention of [WP20]. While the GFF cannot be represented as a continuous function, and it therefore has no level lines in the usual sense, level lines do exist in the sense of a suitable coupling (see Proposition A.1 in the appendix). In particular, these level lines are disjoint chordal curves between the boundary points $V_0^1,\ldots,V_0^{2N}$, hence forming some pairing.

Theorem 2.3 The pairing probabilities of the GFF level lines are given by

$$\mathbb{P}[\text{pairing } \alpha] = \frac{Z_\alpha(V_0^1,\ldots,V_0^{2N})}{Z(V_0^1,\ldots,V_0^{2N})},$$

where $Z_\alpha$ and $Z$ are as defined in Equation (7) and right below it.

Proof From Proposition A.1(iv) and Corollary A.2 in the Appendix, we extract that

- the driving function of the initial segment starting from a given boundary point $V_0^j$ in a given localization neighbourhood has the law $\mathbb{P}$ of local multiple SLE(4) with the partition function $Z$ in Equation (7), and
- given the occurrence of a pairing $\alpha$, appearing with a positive probability $p_\alpha > 0$, the conditional law of the GFF level lines is the $\alpha$ global multiple SLE(4).
In particular, the driving function of the initial segment of the latter has by [PW19, Theorem 1.3] the law \( P_\alpha \) the local multiple SLE(4) with the partition function \( Z_\alpha \) defined right below Equation (7). The law of the driving function can thus be written in two ways:

\[
P = \sum_\alpha p_\alpha P_\alpha,
\]

where we may as well include the pairings with \( p_\alpha = 0 \) to the right-hand side. The rest of the proof is identical to the previous cases.

3 The SLE theory lemmas

Proof of Lemma 1.1 We begin by explicating the well-known convex space property of local multiple SLE measures. First, any convex combination of partition functions

\[
\tilde{Z}(x_1, \ldots, x_{2N}) = \sum_\alpha c_\alpha Z_\alpha(x_1, \ldots, x_{2N}), \quad (9)
\]

where \( c_\alpha \geq 0 \) with \( \sum_\alpha c_\alpha = 1 \) are constants in \( x_1, \ldots, x_{2N} \), is clearly also a partition function. By [KP16, Theorem A.4(c)] (whose proof is a short computation with the Girsanov martingales (12)), the law \( \tilde{P} \) of the driving function corresponding to the partition function \( \tilde{Z} \), is a convex combination of those corresponding to \( Z_\alpha \):

\[
\tilde{P} = \sum_\alpha \frac{c_\alpha Z_\alpha(V_0^1, \ldots, V_0^{2N})}{\tilde{Z}(V_0^1, \ldots, V_0^{2N})} P_\alpha. \quad (10)
\]

Choose now

\[
c_\alpha = C \frac{p_\alpha}{Z_\alpha(V_0^1, \ldots, V_0^{2N})}, \quad (11)
\]

with a normalizing constant \( C \) matched so that \( \sum_\alpha c_\alpha = 1 \); the corresponding convex combination measure thus becomes

\[
\tilde{P} = \sum_\alpha C \frac{p_\alpha}{\tilde{Z}(V_0^1, \ldots, V_0^{2N})} P_\alpha = \sum_\alpha p_\alpha P_\alpha,
\]

where we observed that \( C = \tilde{Z}(V_0^1, \ldots, V_0^{2N}) \). The convex combination \( P = \sum_\alpha p_\alpha P_\alpha \) thus coincides with a multiple SLE law \( \tilde{P} \) with the partition function (9)–(11). Since, by Lemma 3.1 below, the law of a multiple SLE determines its driving function up to constant multiplication, this identifies the partition function of \( P \) as

\[
Z(x_1, \ldots, x_{2N}) = C' \sum_\alpha \frac{p_\alpha}{Z_\alpha(V_0^1, \ldots, V_0^{2N})} Z_\alpha(x_1, \ldots, x_{2N}) \quad \text{for all } x_1 < \ldots < x_{2N}.
\]

Specializing to \( (x_1, \ldots, x_{2N}) = (V_0^1, \ldots, V_0^{2N}) \) reveals that \( C' = Z(V_0^1, \ldots, V_0^{2N}) \).

The following uniqueness lemma was actually the main new contribution in Lemma 1.1, and crucial in our applications. We highlight again that we study local multiple SLEs in a fixed geometry, not as collections of measures indexed by geometries — in the latter case the analogous lemma is immediate [KP16, Theorem A.4(a)].

Lemma 3.1 Fix launching points \( V_0^1 < \ldots < V_0^{2N} \), an index \( 1 \leq j \leq 2N \), and a localization neighbourhood of \( V_0^j \). Let \( Z_1 \) and \( Z_2 \) be two local multiple SLE partition functions and \( P_1 \) and \( P_2 \) the corresponding laws of the driving function. Then, \( P_1 \) and \( P_2 \) are equal if and only if \( Z_1 \) and \( Z_2 \) are constant multiples of each other.
Proof  The implication “if” is obvious, so we concentrate on “only if”. Let us assume for a lighter notation that $j = 1$ and that all the processes are stopped at the exit time of the localization neighbourhood, but omit denoting for this stopping. Let $(\mathcal{F}_t)_{t \geq 0}$ be the natural right-continuous filtration of the driving function $W_t$ (hence stopped).

First, denoting $\rho = \frac{\tilde{Z}_1}{Z_1}$, it is fairly standard that

$$M_t := \rho(W_t, V^2_t, \ldots, V^{2N}_t)$$

is the measure-changing martingale from $P_1$ to $P_2$, i.e.,

$$\frac{dP_2}{dP_1}|_{\mathcal{F}_t} = M_t/M_0.$$  \hfill (12)

Indeed, the PDEs [2] for the two partition functions imply by direct computation (see [Duf61, Theorem 1] for a clever way) that $\rho$ satisfies

$$\left[ \frac{k}{2} \partial^2_i + \kappa \frac{\partial_j Z_1(x_1, \ldots, x_{2N})}{Z_1(x_1, \ldots, x_{2N})} + \sum_{i=1, i \neq j}^{2N} \frac{2}{x_i - x_j} \partial_i \right] \rho(x_1, \ldots, x_{2N}) = 0, \quad \forall 1 \leq j \leq 2N, \quad (13)$$

which with $j = 1$ shows that the Itô drift term of $M_t$ vanishes under $P_1$. Next, $M_t$ can be shown bounded up to the stopping time (hence a genuine martingale) by the translation invariance of $\rho$ and a standard harmonic measure argument (e.g., [Kar20, Proof of Lemma B.1]). Equation (13) then follows from Girsanov’s theorem. In particular, the two measures are equal if and only if $M_t$ is $P_1$-almost surely a constant process.

Now, summing the PDEs (14) over all $j$ yields an elliptic PDE that is hence also satisfied by $\rho$. The strong maximum principle of elliptic PDEs [Duf61, Theorem 2] then states that $\rho$ is either an everywhere constant function, or it is nowhere locally constant. In the former case the proof is finished, so the rest of the proof consists of assuming the latter and showing $M_t$ then cannot be almost surely a constant process.

Let $P$ be a third measure on driving functions, under which $W$ has the law of $V^1_0 + \sqrt{\kappa} \beta_t$, but with the same stopping as throughout. Yet another Girsanov transform is

$$\frac{dP_1}{dP}|_{\mathcal{F}_t} = Z_1(W_t, V^2_t, \ldots, V^{2N}_t) \prod_{i=2}^{2n} (g_i'(V^i_0))^h,$$

where $g_i'$ denotes the derivative of the Loewner mapping-out function. It thus suffices to show that $\rho(W_t, V^2_t, \ldots, V^{2N}_t)$ is not a constant process under $P$. Note that typical SDE results are derived for Lipschitz SDEs, while $dV^j_t = \frac{2dW_t}{V^j_t - W_t}$ is not Lipschitz if small $|V^j_t - W_t|$ are allowed. However, up to the stopping time, the above-mentioned harmonic measure argument lower-bounds $|V^j_t - W_t|$. Thus, rather than $P$, we will actually perform the SDE analysis on the non-stopped process

$$\begin{cases}
\frac{d\tilde{V}^1_t}{dt} = \sqrt{\kappa} d\beta_t \\
\frac{d\tilde{V}^j_t}{dt} = \theta(\tilde{V}^j_t - \tilde{V}^1_t) \frac{2}{V^j_t - V^1_t} dt, \quad 2 \leq j \leq 2N,
\end{cases} \quad (15)$$

with $(\tilde{V}^1_0, \ldots, \tilde{V}^{2N}_0) = (W_0, V^2_0, \ldots, V^{2N}_0)$, where $\theta$ is a smooth cutoff function being one in $\mathbb{R} \setminus (-\epsilon, \epsilon)$ and zero in $[-\epsilon/2, \epsilon/2]$, with a small enough $\epsilon > 0$ so that up to the stopping time, the process $(\tilde{V}^1_t, \ldots, \tilde{V}^{2N}_t)$ is the same as $(W_t, V^2_t, \ldots, V^{2N}_t)$ under $P$ (here and below, we sample the two from the same Brownian motion).
Lemma 3.2 For all fixed \( t > 0 \), the law of \((\tilde{V}_t^1, \ldots, \tilde{V}_t^{2N})\), as defined above, is absolutely continuous with respect to the Lebesgue measure on \( \mathbb{R}^{2N} \).

Taking Lemma 3.2 as given for a moment, the proof is readily finished: fix \( T \) small enough so that the event \( E_T \) that the exit time has not occurred by \( T \) has a positive probability under \( P \). On the event \( E_T \), we have \([W_T, V_T^1, \ldots, V_T^{2N}] = (\tilde{V}_T^1, \ldots, \tilde{V}_T^{2N})\). On the other hand, these \( \tilde{V}_T \)'s have Lebesgue measure zero, so by Lemma 3.2, we have \( P(\tilde{V}_T^1, \ldots, \tilde{V}_T^{2N}) \neq \rho(V_0^1, \ldots, V_0^{2N}) \) with probability one. Hence, \( P[\rho(W_T, V_T^1, \ldots, V_T^{2N}) \neq \rho(V_0^1, \ldots, V_0^{2N})] \geq P[E_T] > 0 \). \( \square \)

Proof of Lemma 3.2 The proof is based on the Hörmander criterion in [Nua06, Theorem 2.3.1]. To coincide with the notations there, denote \( \beta_t = \beta^1_t \) and re-write (15) as

\[
d\tilde{V}_t = A_1(\tilde{V}_t)d\beta^1_t + B(\tilde{V}_t)dt,
\]

where \( A_1 : \mathbb{R}^{2N} \to \mathbb{R}^{2N} \) is here given by \( A_1 = (\sqrt{K}, 0, \ldots, 0) \) and \( B : \mathbb{R}^{2N} \to \mathbb{R}^{2N} \) by \( B(x_1, \ldots, x_{2N}) = (0, \theta(x_2 - x_1), \ldots, \theta(x_{2N} - x_1)) \). Lemma 3.2 can then be deduced by checking that the vector fields (operators)

\[
A_1: \ [A_i, A_j] \text{ with } 0 \leq i, j \leq 1; \quad [A_i, [A_j, A_k]] \text{ with } 0 \leq i, j, k \leq 1; \quad \ldots
\]

at the launching point \((x_1, \ldots, x_{2N}) = (V_0^1, \ldots, V_0^{2N})\) span a 2N-dimensional vector space; here the operator commutator is the usual one and the operators are in our case

\[
A_1 = \sum_{i=1}^{2N} A^i_1(x_1, \ldots, x_{2N}) \frac{\partial}{\partial x_i} = \sqrt{K} \frac{\partial}{\partial x_1} \quad \text{and}
\]

\[
A_0 = \sum_{i=1}^{2N} B(x_1, \ldots, x_{2N}) + \frac{1}{2} \sum_{j=1}^{2N} A^j_1(x_1, \ldots, x_{2N}) \left( \frac{\partial}{\partial x_j} A^j_1(x_1, \ldots, x_{2N}) \right) \frac{\partial}{\partial x_i}
\]

\[
= \sum_{i=2}^{2N} \theta(x_i - x_1) \frac{2}{x_i - x_1} \frac{\partial}{\partial x_i}.
\]

We now verify this criterion\(^5\) Define the vector fields \( G_1 = [A_1, A_0] \) and \( G_k = [A_1, G_{k-1}] \) for \( k \geq 2 \). By direct computation, we have

\[
G_k(V_0^1, \ldots, V_0^{2N}) = C_k \sum_{i=2}^{2N} \frac{2}{(V_0^i - V_0^{i+1})^{k+1}} \frac{\partial}{\partial x_i},
\]

where \( C_k > 0 \) are constant factors that are irrelevant in what follows. In matrix algebra:

\[
\begin{pmatrix}
G_1(V_0^1, \ldots, V_0^{2N}) \\
\vdots \\
G_{2N-1}(V_0^1, \ldots, V_0^{2N})
\end{pmatrix} = \Lambda M \begin{pmatrix}
\frac{\partial}{\partial x_2} \\
\vdots \\
\frac{\partial}{\partial x_{2N}}
\end{pmatrix},
\]

where \( \Lambda \) is a diagonal matrix, \( \Lambda_{k,k} = C_k \), and \( M \) is a Vandermonde (type) matrix \( M_{k,i} = \frac{2}{(V_0^i - V_0^{i+1})^{k+1}} \). Hence, one can invert these two matrices and express the operators \( \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_{2N}} \) as linear combinations of \( G_1, \ldots, G_{2N-1} \). Since in addition \( A_1 \propto \frac{\partial}{\partial x_1} \), the Hörmander criterion is satisfied. \( \square \)

\(^5\)The reader may observe that, in technical terms, this verification is very similar to [PW19, Proposition 2.6]. There is however no direct logical connection: [PW19] studies the spatial PDEs \( \mathbb{P}[\rho(V_T^1, \ldots, V_T^{2N}) \neq \rho(V_0^1, \ldots, V_0^{2N})] \geq P[E_T] > 0 \) via the original Hörmander criterion, while the origins of the stochastic variant used here are in the spatio-temporal (Fokker–Planck) PDEs (see [Nua06] for details).
A Precise definitions of the models

A.1 The Ising model

Equip $\mathbb{Z}^2$ with the usual square grid graph structure, and denote by $(\mathbb{Z}^2)^*$ its planar dual graph. We study the Ising model on simply-connected discrete domains $\Omega$ of the dual graph $(\mathbb{Z}^2)^*$, interpreting $\Omega$ (a bit abusively) when needed as a collection of faces, a dual subgraph, a primal subgraph, or as a planar domain. Formally, we require that the faces of $\Omega$, as well as their complementary faces, are connected in $(\mathbb{Z}^2)^*$; as a subgraph $\Omega \subset (\mathbb{Z}^2)^*$ then consists of these faces and the dual-edges between them; as a subgraph of $\mathbb{Z}^2$, $\Omega$ consists of the vertices and edges on the boundaries of the faces in $\Omega$; finally, the interior of the union of the faces (closed squares) in $\Omega$ is the (simply-connected) planar domain $\Omega$.

Let $p_1, \ldots, p_{2N} \in \mathbb{Z}^2$ be distinct vertices on $\partial \Omega$ (the boundary of the planar domain), ordered counter-clockwise. The Ising model in $\Omega$ with $+$ and $-$ boundary conditions alternating at $p_1, \ldots, p_{2N}$ is a random vector $\sigma$, with components in $\{\pm 1\}$ and indexed by the faces in $\Omega$ and the edges on $\partial \Omega$ where, for such an edge $u$, $\sigma_u$ is deterministic: $\sigma_u = 1$ if $u$ lies on one of the arcs $p_1 p_2, p_3 p_4, \ldots, p_{2N-1} p_{2N}$, and $\sigma_u = -1$ otherwise. Denoting

$$H(\sigma) = - \sum_{u \sim v} \sigma_u \sigma_v,$$

where the sum runs over all face–face or face–edge adjacency pairs $\{u, v\}$ where $\sigma$ is defined, the Ising model is then finally defined via

$$\mathbb{P}_\Omega[\sigma] \propto e^{-\beta H(\sigma)}.$$

Throughout this paper, we will restrict our attention to the critical temperature

$$\beta = \frac{1}{2} \ln(1 + \sqrt{2}).$$

To define the interfaces of the model, colour the faces of $\Omega$ with two colours, according to the values of $\sigma$ on the faces, and the one-fourth faces outside of and neighbouring to $\Omega$ according to $\sigma$ on the corresponding boundary edges. Then start exploring from $p_i$, $i$ odd, the interface along the graph $\Omega \subset \mathbb{Z}^2$ between the two colours, taking the left-most alternative whenever there is an ambiguity. This produces $N$ non-crossing, edge-simple chordal interfaces on the graph $\Omega \subset \mathbb{Z}^2$, pairing the odd $p_i$’s to the even $p_j$’s and thus determining the colour clusters of the boundary segments. Due to the non-crossing property, the curves form a pairing of the boundary points $p_1, \ldots, p_{2N}$.

A.2 The Gaussian free field and its level lines

Let $D \neq \mathbb{C}$ be a simply-connected planar domain with regular enough boundary$^6$ and $G_D : D \times \overline{D} \to \mathbb{R}$ the Green’s function of the negative Laplacian on $D$. Let $\mathcal{M}_D$ be the set of finite signed Borel measures $\mu$ supported on $D$ with

$$\int_{D \times \overline{D}} G_D(x, y) d\mu(x) d\mu(y) < \infty$$

$^6$Precisely, for any $z \in \partial D$, the Brownian motion launched from $z$ a.s. visits $D^c$ at arbitrarily small positive times.

$^7$Explicitly, $G_H(z, w) = \frac{1}{4\pi} ( \log |z - w|^* - \log |z - w|)$, and if $\phi$ is a conformal map $\mathbb{H} \to D$, then $G_D(x, y) = G_H(\phi^{-1}(x), \phi^{-1}(y))$. 

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Figure 3: The boundary conditions in the conditional law of Proposition A.1(ii) extend the original boundary condition in a natural “level line” manner.

(this is satisfied, e.g., if $\mu$ is absolutely continuous with respect to the Lebesgue measure). The Gaussian free field (GFF) $\Gamma_D$ on $D$ with Dirichlet boundary conditions is a centered Gaussian process indexed by $\mathcal{M}_D$ and determined by the covariance structure

$$\text{Cov}(\Gamma_D(\mu), \Gamma_D(\nu)) = \int_{D \times D} G_D(x,y) d\mu(x) d\nu(y).$$

If $H$ is a harmonic function on $D$ with regular enough boundary behaviour, then the GFF $\hat{\Gamma}_D$ with boundary condition $H|_{\partial D}$ is given by

$$\hat{\Gamma}_D(\mu) = \Gamma_D(\mu) + \int_D H(x) d\mu(x).$$

For smooth functions $\psi : D \to \mathbb{R}$, we denote $\hat{\Gamma}_D(\psi) := \hat{\Gamma}_D(\mu)$, where $\mu[A] = \int_A \psi(x) dx$. We refer the reader to the textbook [WP20] for a more thorough introduction and other key features of the GFF such as its conformal property and the definition of local sets.

Below we state the existence and uniqueness of GFF level lines, in the sense of a suitable coupling. The statement is for experts a simple consequence of well-known results but we outline the argument and collect the references for convenience in the end of this appendix. For a convenient statement, we also modify our conventions a tiny bit for the rest of this appendix: First, we will liberally use curve terminology for (random) Loewner growth processes, which we however study in the topology of the Loewner driving functions as throughout this paper. Second, local multiple SLEs are studied up to a generic stopping time that almost surely occurs before any marked point is swallowed by the hulls $K_t$ (the SDEs (3) thus still have a unique strong solution). Third, by an initial segment of a Loewner growth process, we mean below (the hulls of) the process up to such given stopping time.

**Proposition A.1 (GFF level lines)** Let $\hat{\Gamma}$ be the GFF in $\mathbb{H}$ with alternating boundary conditions, introduced in Section 2.3. There exists a coupling of $\hat{\Gamma}$ and $N$ Loewner growth processes starting from each odd-index boundary point, such that

i) the Loewner growth processes are generated by disjoint simple chordal curves $\gamma_1, \ldots, \gamma_N$, traversing from odd-index to even-index marked boundary points, and staying inside $\mathbb{H}$ except at the end points; and

ii) any collection of initial segments $\eta_1, \ldots, \eta_k$ and entire curves $\gamma_{j1}, \ldots, \gamma_{j\ell}$ is a local set for $\hat{\Gamma}$, and the corresponding conditional law of $\hat{\Gamma}$ is a GFF in the reduced domain (independent GFFs on each connected component), with the boundary condition depicted in Figure 3.

---

8In this paper, we may require that $H$ is bounded and the limits of $H \circ \phi : \mathbb{H} \to \mathbb{R}$ exist at all but finitely many real points. Generally, GFF theory often requires allowing much more general functions $H$. 

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Furthermore,

iii) a coupling of \( \hat{\Gamma} \) with such Loewner growth processes \( \gamma_1, \ldots, \gamma_N \) is uniquely determined already by requiring (ii) for any initial segment, and in this unique coupling the curves \( \gamma_1, \ldots, \gamma_N \) are determined by \( \hat{\Gamma} \); and

iv) the law of the curves \( \gamma_1, \ldots, \gamma_N \) is the following: the marginal law of a single curve is the local multiple SLE(4) with the partition function \( \hat{\gamma} \) (which indeed terminates at an even boundary point), and given any collection of full curves, the regular conditional law of any remaining curve is the analogous local multiple SLE in the reduced domain, between the remaining boundary points.

We call the curves \( \gamma_1, \ldots, \gamma_N \) the level lines of the GFF \( \hat{\Gamma} \). Before outlining the proof of Proposition A.1 we point out two some simple consequences that are central in this note. First, part (i) above guarantees that the level lines form some planar pairing between the points \( V_0^1, \ldots, V_0^{2N} \). Second, we have the following (also in [PW19, Theorem 1.4]).

**Corollary A.2** For any pairing \( \alpha \) appearing for the GFF level lines with a positive probability\(^9\) the level lines conditional on the pairing \( \alpha \) are a global multiple SLE(4).

**Proof** Recall first that the global multiple SLE(\( \kappa \)), \( \kappa \leq 4 \), in \( (\mathbb{H}; V_0^1, \ldots, V_0^{2N}) \) with a fixed pairing \( \alpha \), is a collection of random disjoint curves forming the pairing \( \alpha \) between the boundary points, and determined via the following regular conditional law property: given any \( N - 1 \) curves, the remaining one is a chordal SLE(\( \kappa \)) between the remaining two boundary points in their connected component of the reduced domain \[BPW21\]. Let us prove this characterizing property in the \( \alpha \)-conditional level lines, for the conditional law of the \( j \)-th curve \( \gamma_j \) given \( \gamma_i, i \neq j \). Sample first \( \gamma_i, i \neq j \), via Proposition A.1(iv). Note that already these \( (N - 1) \) curves reveal the pairing formed by the level lines. Whatever pairing is observed, by Proposition A.1(iv) again, the conditional law of \( \gamma_j \) given \( \gamma_i, i \neq j \) is the (unique) local multiple SLE(4) between remaining marked boundary points in the remaining sub-domain, which is well-known to coincide with the chordal SLE(4).

**Proof of Proposition A.1** Below we denote the harmonic extension of the alternating GFF boundary values with jumps at \( x_1, \ldots, x_{2N} \) by \( H(z; x_1, \ldots, x_{2N}) \), i.e.

\[
H(z; x_1, \ldots, x_{2N}) = \frac{2\lambda}{\pi} \text{Im} \left( \log(z - x_1) - \log(z - x_2) + \ldots - \log(z - x_{2N}) \right) + \lambda.
\]

(16)

For Loewner evolutions with a driving function \( W \) satisfying \( W_0 = V_0^1 \), we denote for short

\[
H_t(z) := H(g_t(z); V_t^1, \ldots, V_t^{j-1}, W_t, V_t^{j+1}, \ldots, V_t^{2N}).
\]

We present the proof by proving (the obvious restrictions of) the statements for more and more general collections of curves, starting from a single initial segment. By [MS16, Theorem 1.1] (special case), any initial segment of the local multiple SLE(4) with the partition function \( \hat{\gamma} \) (i.e., the one-segment marginal from (iv)) indeed satisfies (ii). As for part (i), such a local multiple SLE is absolutely continuous with respect to the chordal SLE from \( V_0^1 \) (with the same stopping), and thus such an initial segment is indeed generated by a curve which is simple and has no boundary visits. Finally, (iii) follows by [MS16].

\(^9\)Note that, by Theorem 2.3 all planar pairings indeed appear with a positive probability.
Theorem 1.2], but for later use, we review the following argument for determining the only possible the law for a Loewner evolution satisfying (ii). For any smooth test function $\psi$, the conditional-law property of part (ii) gives
\[
E[\hat{\Gamma}(\psi) \mid \mathcal{F}_t] = \int H_t(x) \psi(x) dx,
\]
where $\mathcal{F}_t$ is the right-continuous filtration of the stopped process $W$; the above is hence a martingale. We then identify the only possible the law by the lemma below.

**Lemma A.3** Suppose that $W$ is a random continuous function such that $W_0 = V^j_0$ and
\[
\int_{\mathbb{H}} H_t(x) \psi(x) dx
\]
is a martingale for all smooth compactly-supported test functions $\psi$ supported outside of the localization neighbourhood of $V^j_0$. Then $W$ is the local multiple SLE(4) of Equation (7).

**Proof** It readily follows that $H_t(x)$ must be a martingale for all $x$ outside of the localization neighbourhood. It was proven, e.g., in [Kar19, Proposition 6.12] that this identifies $W$ as the desired local multiple SLE driving function. □

We next prove the statements for one full curve. As for (i), it is known that the full curve of increasing local multiple SLE(4) initial segments from (iv) indeed terminates at an even-index boundary point; for us the simplest proof is through Lemma 1.1 and (8), where the partition functions $Z_{\alpha}$ describe global multiple SLE curves with this property [PW19]. Thus, the marginal law of one entire curve $\gamma_i$ is first of all indeed determined by (iv), and secondly satisfies (i). For part (ii), in the case $N = 1$, $V^2_t = \infty$, the interface becomes a chordal SLE(4) and a modern proof can be found in [WP20, Proposition 5.8]. An analogous argument works here, replacing the infinite termination time of the curve by the finite termination time, and the harmonic martingale $\theta_t(z)$ in [WP20] by the imaginary part in (16). Part (iii) is again in [MS16, Theorem 1.2]. This concludes the second case.

Suppose now that the joint law of $\gamma_1, \ldots, \gamma_N$, defined via the property that each one is a deterministic function of $\hat{\Gamma}$, was proven to coincide with the sampling procedure (iv) above. We claim that this would actually finish the proof in full generality. Indeed, combining with property (i) for a single local multiple SLE curve (proven above), curves from the sampling procedure (iv) would also verify part (i). As for part (ii), the curves $\eta_1, \ldots, \eta_k; \gamma_1, \ldots, \gamma_N$ were known to be deterministic given $\hat{\Gamma}$, and thus also conditionally independent. Their union is thus also a local set by [WP20, Proposition 4.13], and knowing from (i) the disjointness of these curves, the boundary condition of the GFF given the local set of their union is found by [WP20, Proposition 4.23] to satisfy part (ii). For part (iii), we already saw that property (ii) determines the coupling of $\gamma_j$ and $\hat{\Gamma}$, and since $\gamma_j$ is a deterministic function of $\hat{\Gamma}$ in that coupling, also the coupling with $\gamma_1, \ldots, \gamma_N$ is determined by property (ii).

It thus remains to prove the sampling procedure (iv) for joint law of $\gamma_1, \ldots, \gamma_N$. We start this task by establishing (a more explicit description of) the joint law of several initial segments. Suppose thus that we have two initial segments, say $\eta_1, \eta_2$, and that the conditional law of $\hat{\Gamma}$ given $\eta_1$ (resp. $\eta_1, \eta_2$) is the one in Figure 3. Using the conformal invariance of local multiple SLEs and the GFF (neither of which we present formally in this note), as well as the case of a single initial segment above, one sees that taking the conditional law of $\eta_2$ given $\eta_1$ to be the local multiple SLE(4) curve with (7) in the slit domain $\mathbb{H} \setminus \eta_1$ gives these two conditional laws; this is thus a candidate for the joint
law \( \eta_1, \eta_2 \)). The produced marginal law of \( \eta_2 \) through this iterated sampling is indeed consistent with the one-curve marginals determined above and satisfies property (ii); this follows by the commutation property of local multiple SLEs \([\text{Dub07}]\), as well as property (ii) for shorter initial segments of \( \eta_1, \eta_2 \). We have thus established a law for \( \eta_1, \eta_2 \) satisfying (ii); by the uniqueness of such laws, \( \eta_1, \eta_2 \) coincide with initial segments of \( \gamma_1, \gamma_2 \) (defined via \( \hat{\Gamma} \)). The joint law of several initial segments is found similarly to be an iterated local multiple SLE. Note also that any initial segments \( \eta_1, \ldots, \eta_k \) are indeed disjoint due to this sampling description.

Next, note that this joint law of the initial segments obtained above also yields one explicit description of \( \gamma_1, \ldots, \gamma_N \) (or a sub-collection): grow initial segments \( \eta_1, \ldots, \eta_N \) as iterated SLEs as above, each \( \eta_j \) stopped upon reaching a given distance \( \epsilon \) from the boundary arc \((-\infty, V_0^j + 1)\cup[V_0^j + 1, +\infty)\) of the other boundary points. Then continue such growth iterations, taking \( \epsilon \) smaller at every step. From the commutation property, it follows that \( \eta_1, \ldots, \eta_N \) obtained in this manner indeed has the correct law for all \( \epsilon \) (i.e., the same law as if grown to that \( \epsilon \) in one iteration). Since \( \gamma_j \) is the limit of each \( \eta_j \) as \( \epsilon \) shrinks (as observed in the one-curve case), this joint limit procedure determines the joint law of \( \gamma_1, \ldots, \gamma_N \).

Suppose next inductively that the growing initial segments above yield the same law for the curves \( \gamma_1, \ldots, \gamma_k \) as the sampling procedure (iv). The case of \( k = 1 \) curve was handled above. Let \( \eta \) be the initial segment of \( \gamma_{k+1} \), as in the previous paragraph, with a fixed \( \epsilon_0 \); let \( \eta^j \) be \( \eta \) stopped at capacity \( t_j \), if that capacity is ever reached, and sample \( (\gamma_1, \ldots, \gamma_k, \eta) \) as in the previous paragraph (but only growing \( \gamma_{k+1} \) up to \( \epsilon = \epsilon_0 \)). By our inductive assumption and the disjointness of initial segments, these curves are disjoint, and the results in \([\text{WP20}]\) cited above show that \( (\gamma_1, \ldots, \gamma_k, \eta^j) \) is a local set with the correct conditional law (ii) for \( \hat{\Gamma} \). Set \( \mathcal{G}_t \) be the right-continuous filtration of \( \gamma_j, \ldots, \gamma_k, \eta^j \), and let \( D \) be the connected component of \( \mathbb{H} \setminus (\gamma_1, \ldots, \gamma_k) \) containing the starting point of \( \eta \), with \( 2m < 2N \) marked points on its boundary. Let \( H^D \) be the harmonic function on \( D \) with boundary values alternating between \( \pm \lambda \) at these points, and define \( H^D_t \) similarly by slitting \( D \) with \( \eta^j \). The conditional law of part (ii) implies that

\[
\mathbb{E}[\hat{\Gamma}(\psi) \mid \mathcal{G}_t] = \int_D H^D_t(x) \psi(x) \mathrm{d}x, \tag{17}
\]

for all smooth test functions \( \psi \) with a compact support in \( D \). Using \((17)\) and Lemma \(A.3\) one can then show that, almost surely, the conditional law of \( \eta \) given \( \gamma_1, \ldots, \gamma_k \) is the desired local multiple SLE in \( D \), with \( 2m \) boundary points. Finally, this conditional law property can be improved from \( \eta \) to \( \gamma_{k+1} \), using standard stochastic limit theorems. This concludes the induction step and the entire proof.

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\footnote{Note that this definition makes sense: due the chordality of these local multiple SLEs (proven above), \( \eta_\ell \) thus defined does not hit the slit \( \eta_1 \); in particular it can indeed be grown \textit{in the slit domain} up to its defining stopping time that was assumed to occur before the swallowing of a marked boundary point in the \textit{original domain} \( \mathbb{H} \).}

\footnote{Note carefully that this is the point of the proof where we the boundary values \( \pm \lambda \) come into play; it is only then that the level line process from \([\text{MSS16}] \text{Theorem 1.1}\) also is a local multiple SLE with this commutation property.}
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