Approximate solutions of a generalized functional equation on restricted domains and their asymptotic behaviors

Muaadh Almahalebi · Abdellatif Chahbi

Abstract In this paper, we prove the Hyers-Ulam stability on restricted domains of the following generalized Cauchy and quadratic functional equation

\[ \sum_{k=0}^{m-1} f(x + b_k y) = mf(x) + mf(y), \quad x, y \in E, \]

where \( b_k = \exp\left(\frac{2\pi i k}{m}\right) \) for \( 0 \leq k \leq m - 1 \). These results are applied to study of an asymptotic behavior of this functional equation.

Keywords Cauchy functional equation · quadratic functional equation · Hyers-Ulam stability

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1 Introduction

The concept of stability for functional equations arises when we replace a functional equation by an inequality which acts as a perturbation of the equation. The starting point of studying stability of functional equations seems to be the famous talk of S.M. Ulam of 1940 [11], in which he discussed a number of important unsolved problems. Among those was the question concerning the stability of group homomorphisms:

*Given a group \( G_1 \), a metric group \( G_2 \) with metric \( d(\cdot, \cdot) \) and a positive number \( \varepsilon \), does there exist a \( \delta > 0 \) such that if \( f : G_1 \to G_2 \) satisfies \( d(f(xy), f(x)f(y)) \leq \delta \) for all \( x, y \in G_1 \), then a homomorphism \( \phi : G_1 \to G_2 \) exists with \( d(f(x), \phi(x)) \leq \varepsilon \) for all \( x \in G_1 \)？*

These kinds of questions form the material of the stability theory. The case of approximately additive mappings was solved by D. H. Hyers [4] under the assumption that \( G_1 \) and \( G_2 \) are Banach spaces.
In 1950, T. Aoki [1] provided a generalization of the Hyers’ theorem for additive mappings and in 1978 Th. M. Rassias [10] generalized the Hyers’ theorem for linear mappings by considering an unbounded Cauchy difference.

In 1998, S. M. Jung [6] investigated the Hyers-Ulam stability for additive and quadratic mappings on restricted domains. In 2002, J. M. Rassias [9] improved the bounds and thus the stability results obtained by Jung [6]. Besides, he established the Ulam stability for more general equations of two types on a restricted domain then he applied his recent results to the asymptotic behavior of functional equations of different types.

Throughout this paper, let $E$ be a $\mathbb{C}$-vector space, $F$ be a $\mathbb{C}$-Banach space and $2 \leq m \in \mathbb{N}$. Our aim is to prove the Hyers-Ulam stability on restricted domains of the following generalized Cauchy and quadratic functional equation

$$
\sum_{k=0}^{m-1} f(x + b_k y) = mf(x) + mf(y), \ x, y \in E,
$$

(1.1)

where $b_k = \exp(\frac{2\pi ik}{m})$ for $0 \leq k \leq m - 1$. Indeed, if $m = 1$ in (1.1), we get the Cauchy functional equation

$$
f(x + y) = f(x) + f(y).
$$

When we put $m = 2$ in (1.1), then we obtain the quadratic functional equation

$$
f(x + y) + f(x - y) = 2f(x) + 2f(y).
$$

These results are applied to study of an asymptotic behavior of this functional equation.

2 Notations and preliminary results

In this section, we need to introduce some notions and notations.

**Definition 2.1** [2] A function $A : E \to F$ between two vector spaces $E$ and $F$ is said to be additive provided if $A(x + y) = A(x) + A(y)$ for all $x, y \in E$. In this case, it is easily seen that $A(rx) = rA(x)$ for all $x \in E$ and all $r \in \mathbb{Q}$.

**Definition 2.2** [2] Let $k \in \mathbb{N}$ and $A : E^k \to F$ be a function, then we say that $A$ is $k$-additive provided if it is additive in each variable. In addition, we say that $A$ is symmetric provided if

$$
A(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(k)}) = A(x_1, x_2, \ldots, x_k)
$$

whenever $x_1, x_2, \ldots, x_k \in E$ and $\sigma$ is a permutation of $\{1, 2, \ldots, k\}$. 

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Definition 2.3 [2] Let $k \in \mathbb{N}$ and $A : E^k \to F$ be symmetric and $k$-additive and let $A_k(x) = A(x, x, \ldots, x)$ for $x \in E$ and note that $A_k(rx) = r^k A_k(x)$ whenever $x \in E$ and $r \in \mathbb{Q}$.

In this way, a function $A_k : E \to F$ which satisfies $A_k(rx) = r^k A_k(x)$ for all $r \in \mathbb{Q}$ and $x \in E$, will be called a rational-homogeneous form of degree $k$ (assuming $A_k \neq 0$).

Definition 2.4 [2] A function $p : E \to F$ is called a generalized polynomial (GP) function of degree $m \in \mathbb{N}$ if there exist $a_0 \in E$ and a rational-homogeneous form $A_k : E \to F$ (for $1 \leq k \leq m$) of degree $k$, such that

$$p(x) = a_0 + \sum_{k=1}^{m} A_k(x)$$

for all $x \in E$.

Definition 2.5 [2] Let $F^E$ denote the vector space, over a field $K$, consisting of all maps from $E$ into $F$. For each $h \in E$ define the linear difference operator $\Delta_h$ on $F^E$ by

$$\Delta_h f(x) = f(x + h) - f(x)$$

for all $f \in F^E$ and all $x \in E$.

Notice that these difference operators commute ($\Delta_{h_1} \Delta_{h_2} = \Delta_{h_2} \Delta_{h_1}$ for all $h_1, h_2 \in E$ ) and if $h \in E$ and $n \in \mathbb{N}$, then $\Delta_n^h$ the $n$-th iterate of $\Delta_h$ satisfies

$$\Delta_n^h f(x) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f(x + kh)$$

for all $f \in F^E$ and all $x, h \in E$, where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.

The following theorem was proved by Mazur and Orlicz [7], [8] then in greater generality by Djoković [3].

Theorem 2.6 Let $n \in \mathbb{N}$ and $f : E \to F$ be a function between two vector spaces $E$ and $F$, then the following assertions are equivalent.

1. $\Delta_n^h f(x) = 0$ for all $x, h \in E$.
2. $\Delta_{h_1} \ldots \Delta_{h_n} f(x) = 0$ for all $x, h_1, \ldots, h_n \in E$.
3. $f$ is a GP function of degree at most $n - 1$. 

3
3 Main results

Lemma 3.1 Suppose that $2 \leq m \in \mathbb{N}$, $b_k \in K$ for $0 \leq k \leq m$, $E$ is a $\mathbb{C}$-vector space, $F$ is a $\mathbb{C}$-Banach space, $\delta \geq 0$ and $f_k : E \to F$ for $0 \leq k \leq m$, be mappings fulfilling

$$\left\| \sum_{k=0}^{m} [f_k(x + b_ky) - f_k(b_ky)] \right\| \leq \delta$$

(3.1)

for all $x, y \in E$ and assume that $b_k - b_j \neq 0$ and $b_0 \neq 0$ whenever $0 \leq j < k \leq m$. Then,

$$\left\| \Delta_{h_{m+1}} \cdots \Delta_{h_1} f_0(x) \right\| \leq 3^m \delta$$

(3.2)

for all $x, h_1, \ldots, h_{m+1} \in E$.

Proof. Letting $d_{jk} = b_k - b_j$ for all $0 \leq k \leq m$, so we get that $d_{jk} \neq 0$ where $j < k$ and $d_{kk} = 0$. We get

$$(x + b_m h_1) + b_k(y - h_1) = x + b_k y + d_{km} h_1$$

for all $0 \leq k \leq m$ and all $x, y, h_1 \in E$. By using these equalities and (3.1), we find that

$$\left\| \sum_{k=0}^{m} [f_k(x + b_k y + d_{km} h_1) - f_k(x + b_k y) - \sum_{k=0}^{m} [f_k(b_k y + d_{km} h_1) - f_k(b_k y)] \right\|$$

$$\leq \left\| \sum_{k=0}^{m} [f_k(x + b_k y + d_{km} h_1) - f_k(b_k y - h_1)] \right\|$$

$$+ \left\| \sum_{k=0}^{m} [f_k(b_k y + d_{km} h_1) - f_k(b_k y - h_1)] \right\|$$

$$+ \left\| \sum_{k=0}^{m} [f_k(x + b_k y) - f_k(b_k y)] \right\| \leq 3 \delta$$

and, since $d_{mm} = 0$,

$$\left\| \sum_{k=0}^{m-1} \Delta_{d_{km} h_1} f_k(x + b_k y) - \Delta_{d_{km} h_1} f_k(b_k y) \right\| \leq 3 \delta$$

(3.3)

for all $x, y, h_1 \in E$. By repeating the argument that leads from (3.1) to (3.3), we find that

$$\left\| \sum_{k=0}^{m-2} [\Delta_{d_{k,m-1} h_2} \Delta_{d_{km} h_1} f_k(x + b_k y) - \Delta_{d_{k,m-1} h_2} \Delta_{d_{km} h_1} f_k(b_k y)] \right\| \leq 3^2 \delta$$

(3.4)
for all \( x, y, h_1, h_2 \in E \). By applying this reasoning \( m - 2 \) more times, we are inclined to admit that
\[
\left\| \Delta_{d_0,h_m} \cdots \Delta_{d_0,h_1} \left[ f_0(x + b_0 y) - f_0(b_0 y) \right] \right\| \leq 3^m \delta \tag{3.5}
\]
for all \( x, y, h_1, \ldots, h_m \in E \). Consequently,
\[
\left\| \Delta_{b_0 y} \cdots \Delta_{h_1} f_0(x) \right\| \leq 3^m \delta \tag{3.6}
\]
for all \( x, y, h_1, \ldots, h_m \in E \). Since \( d_{0k} \neq 0 \) for \( 1 \leq k \leq m \), and \( b_0 \neq 0 \) the last inequalities assert that
\[
\left\| \Delta_{h_{m+1}} \cdots \Delta_{h_1} f_0(z) \right\| \leq 3^m \delta \tag{3.7}
\]
for all \( z, h_1, \ldots, h_{m+1} \in E \).

Theorem 3.2 Suppose that \( E \) is a \( \mathbb{C} \)-vector space and \( F \) is a \( \mathbb{C} \)-Banach space. If \( f : E \to F \) satisfies
\[
\left\| \sum_{k=0}^{m-1} f(x + b_k y) - m f(x) - m f(y) \right\| \leq \delta \tag{3.8}
\]
for all \( x, y \in E \), where \( 2 \leq m \in \mathbb{N} \), \( b_k = \exp(\frac{2ik\pi}{m}) \) for \( 0 \leq k \leq m-1 \) and \( \delta \geq 0 \). Then there exists a unique GP function \( p : E \to F \) of degree at most \( m \) such that \( p(0) = 0 \) and
\[
\left\| f(x) - f(0) - p(x) \right\| \leq 2 \cdot 3^m \delta \text{ for all } x \in E.
\]

Moreover
\[
\sum_{k=1}^{m} p(x + b_k y) - mp(x) - mp(y) = 0 \text{ for all } x, y \in E.
\]

Proof. Letting \( x = 0 \) in (3.8), we get
\[
\left\| \sum_{k=0}^{m-1} f(b_k y) - m f(0) - m f(y) \right\| \leq \delta \text{ for all } x, y \in E. \tag{3.9}
\]

By (3.8) and (3.9), we have
\[
\left\| \sum_{k=0}^{m-1} g(x + b_k y) - mg(x) - \sum_{k=0}^{m-1} g(b_k y) \right\| \leq 2\delta, \ x, y \in E, \tag{3.10}
\]
where \( g = f - f(0) \). If we put \( f_k = g \) for each \( 0 \leq k \leq m - 1 \), \( f_m = -mg \) and \( b_m = 0 \) in (3.10), then we get
\[
\left\| \sum_{k=0}^{m} [f_k(x + b_k y) - f_k(b_k y)] \right\| \leq 2\delta. \tag{3.11}
\]
Thus, by Lemma 3.1

\[ \|\Delta_{h_{m+1}} \cdots \Delta h_1 g(z)\| \leq 2 \cdot 3^m \delta \]

for all \( z, h_1, \ldots, h_{m+1} \in E \).

Hence, according to [Theorem II [5]], there exists a GP function \( p : E \rightarrow F \),

of degree at most \( m \) such that

\[ \|g(x) - p(x)\| \leq 2 \cdot 3^m \delta \] (3.12)

and

\[ p(x) = g(0) + \sum_{j=1}^{m} A_j(x), \] (3.13)

where \( A_j : E^j \rightarrow F \) are symmetric, \( j \)-additive mappings.

By (3.12), we obtain

\[ \left\| \sum_{k=0}^{m-1} (p(x + b_k y)) - mp(x) - mp(y) \right\| \leq \left\| \sum_{k=0}^{m-1} [p(x + b_k y) - g(x + b_k y)] \right\| 

+ \left\| mg(x) - mp(x) \right\| + \left\| mg(y) - mp(y) \right\| , \]

so

\[ \left\| \sum_{k=0}^{m-1} (p(x + b_k y)) - mp(x) - mp(y) \right\| \leq 6m \cdot 3^m \delta \] (3.14)

for all \( x, y \in E \).

In right of (3.13), the inequality (3.14) says that

\[ \left\| \sum_{j=1}^{m} \left( \sum_{k=0}^{m-1} (A_j(x + b_k y)) - mA_j(x) - mA_j(y) \right) \right\| \leq 6m \cdot 3^m \delta \] (3.15)

for all \( x, y \in E \). Replacing \( x \) by \( rx \) and \( y \) by \( ry \) \((r \in \mathbb{Q})\) in (3.15), we conclude

\[ \left\| \sum_{j=1}^{m} \left( \sum_{k=0}^{m-1} (A_j(x + b_k y)) - mA_j(x) - mA_j(y) \right) \right\| \leq 6m \cdot 3^m \delta \] (3.16)

for all \( x, y \in E \) and all \( r \in \mathbb{Q} \). By continuity, (3.16) holds for all real \( r \), and all \( x, y \in E \). Now suppose that \( \phi : F \rightarrow \mathbb{R} \) is a continuous linear functional. Then

\[ \left\| \sum_{j=1}^{m} r^j \phi \left( \sum_{k=0}^{m-1} (A_j(x + b_k y)) - mA_j(x) - mA_j(y) \right) \right\| \leq \|\phi\| 6m \cdot 3^m \delta \]

(3.17)
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for all \( x, y \in E \) and all \( r \in \mathbb{R} \). Since a real polynomial function is bounded if and only if it is constant, from the last inequality we surmise that

\[
\phi \left( \sum_{k=0}^{m-1} \left( A_j(x + b_k y) - mA_j(x) - mA_j(y) \right) \right) = 0 \tag{3.18}
\]

for all \( x, y \in E \) where \( 1 \leq j \leq m - 1 \). Since this is so for every continuous linear functional \( \phi : F \to \mathbb{R} \), by the Hahn-Banach theorem we get that

\[
\sum_{k=0}^{m-1} \left( A_j(x + b_k y) - mA_j(x) - mA_j(y) \right) = 0, \tag{3.19}
\]

for all \( x, y \in E \) and \( 1 \leq j \leq m \). Then, each \( p \) is a \((GP)\) function of degree at most \( m \) and from (3.19) we find that

\[
\sum_{k=0}^{m-1} p(x + b_k y) - mp(x) - mp(y) = 0, \tag{3.20}
\]

for all \( x, y \in E \). Finally, let \( p' \) be another generalized polynomial \((GP)\) of degree at most \( m \) such that \( p(0) = 0 \), \( p' \) is a solution of (1.1) and

\[
\|f(x) - p'(x)\| \leq 2 \cdot 3^m \delta, \quad x \in E.
\]

Then, from (3.20) and (3.12 ) we get that \( p - p' \) is generalized polynomial \((GP)\) of degree at most \( m \) such that \( \|p(x) - p'(x)\| \leq 4 \cdot 3^m \delta \), \( x \in E \). Thus, necessarily \( p = p' + p(0) - p'(0) \). Since \( p(0) = p'(0) = 0 \) we get that \( p = p' \) which ends the proof.

\[\Box\]

**Theorem 3.3** Suppose that \( E \) is a \( \mathbb{C} \)-vector space and \( F \) is a \( \mathbb{C} \)-Banach space. Let \( d > 0 \) and \( \delta \geq 0 \) be fixed. If a mapping \( f : E \to F \) satisfies the generalized quadratic inequality (3.8) for all \( x, y \in E \), with \( \|x\| + \|y\| \geq d \), then there exists a unique \( GP \) function \( p : E \to F \) such that \( p(0) = 0 \) and

\[
\|f(x) - f(0) - p(x)\| \leq 2 \cdot 3^m+1 \delta \text{ for all } x \in E.
\]

Moreover

\[
\sum_{k=0}^{m-1} p(x + b_k y) - mp(x) - mp(y) = 0 \text{ for all } x, y \in E.
\]

**Proof.** Assume that \( \|x\| + \|y\| < d \) and let \( t \in E \) such that \( \|t\| = d \) where \( x = 0 \) and

\[
t = \left(1 + \frac{d}{\|x\|}\right) x.
\]
where \( x \neq 0 \). We note that
\[
\|t\| = \|x\| + d \geq d,\\
\|x\| + \|t\| \geq d
\]
and
\[
\|t\| + \|x + b_ky\| \geq d.
\]
We have \( b_j \) for all \( 0 \leq j \leq m - 1 \) is a root of unity, if we put \( b_j = e^{ij\theta_j} \), we obtain
\[
\|x + b_jt\|^2 = \left( 2 \left( 1 + \frac{d}{\|x\|} \right) \left( 1 + \cos \theta_j \right) + \frac{d^2}{\|x\|^2} \right) \|x\|^2 \geq d^2
\]
Thus, from (3.6) and the new functional identity we obtain that
\[
\begin{align*}
\sum_{k=0}^{m-1} f(x + b_ky) - m f(y) - m f(x) \\
\sum_{n=0}^{m-1} \left( - m f(x + b_n t) - m f(y) + \sum_{k=0}^{m-1} f(x + b_n t + b_ky) \right)
\end{align*}
\]
\[
+m \sum_{n=0}^{m-1} f(x + b_n t) - m f(x) - m f(t)
\]
then
\[
m \left\| \sum_{k=0}^{m-1} f(x + b_ky) - m f(x) - m f(y) \right\| \leq 3m\delta.
\]
Therefore,
\[
\left\| \sum_{k=0}^{m-1} f(x + b_ky) - m f(x) - m f(y) \right\| \leq 3\delta.
\]
for all \( x, y \in E \). Finally, we get the desired result by applying Theorem 3.2. \( \square \)

**Corollary 3.4** Suppose that \( E \) is a \( \mathbb{C} \)-vector space and \( F \) is a \( \mathbb{C} \)-Banach space. A mapping \( f : E \to F \) such that \( f(0) = 0 \) satisfies the functional equation (1.1) if and only if the asymptotic condition
\[
\left\| \sum_{k=0}^{m-1} f(x + b_ky) - m f(x) - m f(y) \right\| \to 0, \text{ as } \|x\| + \|y\| \to \infty.
\]
Proof. By the asymptotic condition (3.21), there exists a sequence $\delta_n$ monotonically decreasing to 0 such that

$$\left\| \sum_{k=0}^{m-1} f(x + b_k y) - mf(x) - mf(y) \right\| \leq \delta_n$$ (3.22)

for all $x, y \in E$ with $\|x\| + \|y\| \geq n$. Hence, it follows from Theorem 3.3 that there exists a unique generalized polynomial $p_n : E \to F$ such that $p_n(0) = 0$ and

$$\|f(x) - p_n(x)\| \leq 2 \cdot 3^{m+1} \delta_n \text{ for all } x \in E.$$ (3.23)

Since $\delta_n$ is a monotonically decreasing sequence, the generalized polynomial $p_n$ satisfies (1.1) for all $n \geq N$. The uniqueness of $p_n$ implies for all $n \geq N$, $p_N = p_n$. Hence, by letting $n \to \infty$, we conclude that $f$ is generalized polynomial at most $m$, and it satisfies the functional equation (1.1).

\[\square\]

If we put $m = 1$ then $m = 2$ in Theorem 3.2, we find the following particular cases.

**Corollary 3.5** Suppose that $E$ is a vector space, $F$ is a real (or complex) Banach space and $\delta \geq 0$. If $f : E \to F$ satisfies

$$\|f(x + y) - f(x) - f(y)\| \leq \delta$$

for all $x, y \in E$ such that $\|x\| + \|y\| \geq d$. Then there exists an additive function $A : E \to F$ such that

$$\|f(x) - f(0) - A(x)\| \leq 6\delta \text{ for all } x \in E.$$ (3.24)

**Corollary 3.6** Let $d > 0$ and $\delta \geq 0$ be fixed. If a mapping $f : E \to F$ satisfies

$$\|f(x + y) + f(x - y) - 2f(x) - 2f(y)\| \leq \delta$$

for all $x, y \in E$ such that $\|x\| + \|y\| \geq d$. Then there exists a quadratic function $Q : E \to F$ such that

$$\|f(x) - f(0) - Q(x)\| \leq 18\delta \text{ for all } x \in E.$$ (3.25)

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References

1. Aoki, T. – On the stability of the linear transformation in Banach spaces, J. Math. Soc. Japan, 2 (1950), 64-66.
2. Baker, J. A. – A general functional equation and its stability, Proc. Amer. Math. Soc., 133 (2005), no. 6, 1657-1664.
3. Đoković, D. Ž. – A representation theorem for \((X_1-1)(X_2-1)\cdots(X_n-1)\) and its applications, Ann. Polon. Math., 22 (1969/1970), 189-198.
4. Hyers, D. H. – On the stability of the linear functional equation, Proc. Nat. Acad. Sci. U. S. A., 27 (1941), 222-224.
5. Hyers, D. H. – Transformations with bounded \(m\)-th differences, Pacific J. Math., 11 (1961), 591-602.
6. Jung, S. M. – On the Hyers-Ulam stability of the functional equations that have the quadratic property, J. Math. Anal. Appl., 222 (1998), no. 1, 126-137.
7. Mazur, S. ; Orlicz, W. – Grundlegende Eigenschaften der Polynomischen Operationen, Erste Mitteilung, Studia Mathematica, 5 (1934), no. 1, 50-68.
8. Mazur, S. ; Orlicz, W. – Grundlegende Eigenschaften der Polynomischen Operationen, Zweite Mitteilung, Studia Mathematica, 5 (1934), no. 1, 179-189.
9. Rassias, J. M. – On the Ulam stability of mixed type mappings on restricted domains, J. Math. Anal. Appl., 276 (2002), no. 2, 747-762.
10. Rassias, Th. M. – On the stability of the linear mapping in Banach spaces, Proc. Amer. Math. Soc., 72 (1978), no. 2, 297-300.
11. Ulam, S. M. – A collection of mathematical problems, Interscience Tracts in Pure and Applied Mathematics, no. 8 Interscience Publishers, New York-London 1960, xiii+150 pp.

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