No minimally coupled scalar black hole hair in Lanczos–Lovelock gravity

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Abstract

We extend here the result of Bekenstein (1995 Phys. Rev. D 51 6608–11, 2000 Cosmology and Gravitation pp 1–85), proving the non-existence of minimally coupled scalar black hole hair in general relativity to the Lanczos–Lovelock gravity in arbitrary dimension with non-negative coupling constants. The only physical requirement on the multiplet of minimally coupled scalar fields is that it fulfills the weak energy condition. We also assume, similarly to Bekenstein, spherical symmetry and asymptotic flatness.

Keywords: Lanczos–Lovelock gravity, black holes, no-hair theorem

1. Introduction

The no-hair theorems in black-hole physics have a long and interesting history [3]. The most important ‘no hair’ results focused on proving the uniqueness of the Kerr–Newman metric in the electro-vacuum. However, another important line of research focused on proving the non-existence of other independent black holes’ quantum numbers beyond mass, angular momentum and charge, quantum numbers associated with the fields in the black-hole exterior. In this context it is certainly interesting to explore if a black hole can be endowed with a minimally coupled scalar hair. This question was investigated some time ago in the context of general relativity (GR) [1, 2] and also a long time ago in a more general context [2, 4]. (For non-minimally coupled scalar hair, see e.g. [5–7].)

Let us now briefly summarize some of the basic results regarding minimally coupled scalar hair. Take a minimally coupled self-interacting scalar field with the action

\[ S = -\frac{1}{2} \int d^Dx \sqrt{-g} \left( \partial_i \psi \partial^i \psi + V(\psi) \right). \]

Bekenstein had shown [2, 4], that if the scalar field potential everywhere fulfills the condition \( V(\psi) \geq 0 \), where the lowercase index stands for the derivative with respect to the field variable
squared, then by using only the wave equation together with some mild assumptions (such as asymptotic flatness and finiteness of the stress-energy tensor invariants at the horizon), one can prove that a stationary non-trivial scalar field cannot form around a black hole. Since this proof uses only the scalar field wave equation it counts as a proof for an arbitrary theory of gravity.

However, the condition on the field’s potential ($V_{\phi} \geq 0$) is quite restrictive. Even the Higgs field potential does not fulfill such a condition. Therefore, Bekenstein partly extended this result [1, 2] to a general minimally coupled scalar field fulfilling the weak energy condition. Moreover, this result can be easily generalized to a multiplet of scalar fields [1]. However, the extension was achieved ‘only’ partly since the extended result uses the spherical symmetry condition and, more importantly, it also uses the equations of the general theory of relativity. This means the second result, valid for a more general scalar field, depends, unlike the previous one, on the theory of gravity.

In this work, we want to extend the second result of Bekenstein [1, 2] to Lanczos–Lovelock gravity theories. (Our paper thus falls in the line of research that occurred in the last two decades and is looking at generalizations of the older no-hair results to the higher-dimensional black holes, e.g. [3, 8–10]. Another interesting line of research explores what happens with black-hole uniqueness if one looks to some other, not necessarily higher-dimensional, generalizations of Einstein gravity. One very powerful result proves that in scalar–tensor gravity theories, black hole uniqueness in a vacuum is the same as in GR [11].) We prove here that the no general-minimally-coupled scalar hair result in spherical symmetric space–time obtained for GR, (with the scalar field fulfilling the weak energy condition), extends (at least) to a significant subclass of Lovelock theories. The proof is contained in the next section of the paper, and at the end of the paper, we discuss our results. We use the signature convention $(-, +, +, +)$.

2. No minimally coupled scalar hair for Lovelock spherically symmetric black holes

The Lanczos–Lovelock gravity [12, 13] is defined by an action of the form:

$$S_L = \int d^D x \sqrt{-g} \sum_{p=0}^{[D/2]} \alpha_p L_p,$$

where the Lagrangians $L_p$ are defined by:

$$L_p = \frac{1}{2^p} \alpha_{a_1 \ldots a_p b_1 \ldots b_p} R^{c_{a_1} d_1 \ldots} R^{c_{a_p} d_p \ldots}.$$

It represents a most natural higher-dimensional extension of the GR theory. (For a recent review see [14].) The Lanczos–Lovelock theory of gravity is for the case of our theorem constrained by the following conditions:

- All the coupling constants $\alpha_p$ in Lanczos–Lovelock theory are non-negative, $\alpha_p \geq 0$. Let us mention that in the case of Gauss–Bonnet gravity (second order Lanczos–Lovelock theory) it is known that $\alpha_2 < 0$ leads to ghosts [15]. Hence, only a positive coupling constant is allowed.
- We assume the theory includes GR, therefore, $\alpha_1 \neq 0$.
- Space–time is asymptotically flat, which is equivalent to the condition $\alpha_0 = 0$, as shown in [16]. (This means we exclude the cosmological constant.)
We also need to mention that, as did Bekenstein in his original proof, we are constraining the space–time by the condition of spherical symmetry. This means, take the $D$ dimensional spherically symmetric static space–time with the metric:

$$
-g^\nu{}^\mu (r) dr^2 + g^\nu{}^\mu (r) r^2 dt_i dx^i, \quad i, j = 2, 3, \ldots, D - 1,
$$

where $g$ is the metric of the $D - 2$ dimensional sphere. The line element is supposed to represent a black hole in the asymptotically flat space–time. Both $f$ and $g$ are everywhere positive in the black-hole exterior and the function $f$ must vanish at the black hole horizon $r_H$. Moreover, by calculating the scalar curvature of such a geometry, it can be easily observed that $f(r_H) = 0$ and $g(r_H) \neq 0$ implies curvature singularity at the horizon. Therefore, $g$ is also required to always vanish at the horizon.

### 2.1. Part of the proof that uses the scalar field equation

Similarly to Bekenstein [1, 4], consider the action of a multiplet of $n = 1, \ldots, N$ scalar fields with generalized dynamics

$$
S = S_L + \int L \left( \psi_1, \ldots, \psi_N, I_1, \ldots, I_N \right) \cdot \sqrt{-g} \cdot d^Dx,
$$

where $S_L$ is (this time) the action of the Lovelock theory. The stress-energy tensor of the scalar field multiplet reads:

$$
T^\nu{}^\mu = 2 \sum_{n=1}^N \frac{\partial L}{\partial I^\nu_n} \psi^\mu_n - L \delta^\nu{}^\mu.
$$

Now, one can easily observe from equation (2) that

$$
T^i_i = T^i_i = -L,
$$

where $T^i_i$ is an arbitrary angular diagonal element of the stress-energy tensor, and

$$
T^r_r = 2g \sum_{n=1}^N \frac{\partial L}{\partial I^r_n} \left( \psi^r_n \right)^2 - L.
$$

The weak energy condition implies $T^r_r \leq 0$. Furthermore, [4], the causality condition together with the weak energy condition imply the following:

$$
-T^i_i + T^r_r \geq 0.
$$

We therefore require these conditions to hold in our proof.

We can further recollect part of Bekenstein’s argumentation [1, 2], which uses purely the wave equation and therefore holds for any theory of gravity. The $r$-component of the wave equation $T^\nu{}^\mu = 0$ can be written as:

$$
\left( T^r_r \right)_r = - \left( \frac{D - 2}{r} + \frac{f_r}{2f} \right) \left( - T^r_r + T^r_r \right).
$$

The argument goes as follows: equation (6) together with equation (5) means that $T^r_r \leq 0$ near the horizon. (The surface gravity of the black-hole horizon is for a non-extremal black hole always positive; therefore, $f_r > 0$ near the horizon. For the extremal black hole $f_r = 0$, but as $f$ is in the black hole exterior positive and at the horizon zero, there must still
exist a neighborhood of the horizon where \( f_r \geq 0 \). Furthermore equation (6) can be expressed as:

\[
\left( \frac{D-2}{\sqrt{f}} \cdot T^\prime_r \right)_r = \left( \frac{D-2}{\sqrt{f}} \right)_r T^\prime_r
\]

and integrating leads to

\[
T^\prime_r = \frac{1}{r^{D-2}} \int_{\eta_0}^\eta \left( \frac{D-2}{\sqrt{f}} \right)_r \cdot T^\prime_r d\eta.
\]

(The integration constant in equation (8) is chosen to avoid the divergence of radial pressure at the horizon.) Equation (8) tells us that, since \( f \) is an increasing function near the horizon, \( T^\prime_r \leq 0 \) near the horizon.

Furthermore, equations (5) and (6) mean that at the asymptotic infinity \( T^\prime_{r, \infty} \leq 0 \). (At the asymptotic infinity, the fact that gravity attracts matter to the gravitating body means \( f_r > 0 \). This statement, which in GR corresponds to the positivity of an ADM mass will be assumed in the proof, but it will be independently shown to hold at the end of the section.) At the same time, \( T^\prime_r \) has to vanish at infinity. Therefore, the radial pressure near the horizon is non-positive and non-increasing and near the point of asymptotic infinity, non-negative and non-increasing.

This means, considering a non-trivial field configuration, there must be an interval where the radial pressure is increasing, which means at the interval it holds that \( T^\prime_{r, r} > 0 \). From equation (6) we see that this is possible only if

\[
f_r < -2f \cdot \frac{D-2}{r} < 0.
\]

Furthermore, since at the asymptotic infinity \( T^\prime_r \geq 0 \) holds, there must exist a point \( r_b \) at which \( T^\prime_r (\eta_b) = 0 \) and in the same time on one of the intervals \( (\eta_b, r_c) \), or \( (\eta_c, \eta_b) \) \( T^\prime_r \) is increasing. However, this fact and equation (9) imply that there must exist an interval on which simultaneously \( T^\prime_r \geq 0 \) and \( f_r < 0 \).

This is all one can obtain purely by using the wave equation; in order to push the argument further, one has to look at a particular theory of gravity. Let us therefore demonstrate by using the Lanczos–Lovelock equations (and therefore generalize Bekenstein’s argument to the Lanczos–Lovelock theory), that the conditions \( f_r < 0 \) and \( T^\prime_r \geq 0 \) cannot be simultaneously fulfilled for non-negative coupling constants.

### 2.2. Part of the proof that uses the gravity equations

Let us write the radial Lanczos–Lovelock equation \( G' = 8\pi G' \) as:

\[
\beta_1 \cdot \frac{\dot{g}}{f} f_r - \beta_2 = 8\pi G' T^\prime_r,
\]
where

$$\beta_1 = \sum_{p=1}^{[D/2]} a_p \left\{ p(D - 2) \cdot \frac{1}{2r} \left( \frac{1 - g}{r^2} \right)^{p-1} \right\},$$

(11)

and

$$\beta_2 = \sum_{p=1}^{[D/2]} a_p \left\{ \frac{(D - 2)(D - 2p - 1)}{2} \left( \frac{1 - g}{r^2} \right)^p \right\}.$$  

(12)

Now from equation (10), one can easily see that, for non-negative coupling constants, if

$$(1 - g) > 0$$

holds everywhere, then $T' \geq 0$ implies $f_r > 0$. This is because $f_r$ can be expressed using equation (10) as

$$f_r = \frac{1}{g} \cdot \frac{8\pi GT'_r + \beta_2}{\beta_1},$$

where $\beta_1, \beta_2$ are positive if $(1 - g)$ is positive.

This means all that remains is to show that $(1 - g)$ is necessarily positive everywhere in the black-hole exterior. To show this, we will use the definition of Misner–Sharp mass together with the equation for the Misner–Sharp mass radial derivative. (Both of the equations can be found in [16].)

The Misner–Sharp mass reads [16] as:

$$M(r) = \frac{(D - 2) V_{D-2}}{16\pi G} \sum_{p=1}^{[D/2]} a_p \frac{(D - 3)!}{(D - 2p)!(D - 2)!} r^{D-1-2p}(1 - g)^p,$$

(13)

and the radial derivative of the Misner–Sharp mass follows the equation [16]:

$$M_r = -V_{D-2}T'_r \cdot r^{D-2}.$$  

(14)

($V_{D-2}$ is the volume of the $(D - 2)$-dimensional sphere). Equation (14) means that Misner–Sharp mass is everywhere a non-decreasing function of $r$. Therefore, if $M$ was negative at infinity (which violates the mass positivity theorem), it also has to be negative at the horizon. In case $M$ is non-negative at infinity, it would have to become non-positive at the horizon, unless $(1 - g) > 0$ everywhere in the black-hole exterior. The reason for the last statement is the following: near the asymptotic infinity $(1 - g)$ is infinitesimally small, hence the dominant term in equation (13) is $p = 1$ (the general relativistic term). This is because the term contains the lowest power in $(1 - g)$ and, simultaneously, the highest power in $r$ (which diverges at infinity).

Since the power of $(1 - g)$ is one, the positivity of mass means that $(1 - g) > 0$ near infinity. However, if $(1 - g)$ turned somewhere in the black-hole exterior negative, as can be seen from equation (13), $M$ would become zero at the same point at which $(1 - g)$ becomes zero. However, this implies that $M$ would become non-positive at the horizon.

In any case, unless $(1 - g) > 0$ everywhere in the black-hole exterior, the Misner–Sharp mass would become non-positive at the horizon. Non-positivity of the Misner-Sharp mass, however, contradicts the existence of the horizon, as the horizon’s radius $r_H$ is given by equation (13) with $g = 0$. This gives:
However, since the coefficients on the right side of equation (15) are all positive, equation (15) will have no positive real solution \( r_H \) for non-positive \( M \). This proves that \( f_s > 0 \) everywhere in the black-hole exterior. As we have shown, this implies that \( f_s < 0 \) and \( T'_i > 0 \) cannot hold in the same time. As shown before, this was the statement we needed in order to complete the proof that there is no minimally coupled spherically symmetric black-hole scalar hair in the case of Lanczos-Lovelock gravity with non-negative coupling constants.

There is one remaining detail that needs to be shown. Throughout the proof, we assumed as obvious the fact that \( f_s > 0 \) at the asymptotic infinity. However, this statement follows from the fact that \( 1 - g \) > 0 everywhere in the black-hole exterior and \( g \rightarrow 1 \) at the asymptotic infinity. (These two statements are independent on the assumption of \( f_s > 0 \) at infinity; hence, the proof is not circular.) This behavior of \( g \) implies \( g, r > 0 \) at the asymptotic infinity. Let us write now the following combination of Lovelock equations: 
\[
-G^r_i + G^r_i = 8\pi G (-T^r_i + T^r_i).
\]
This reads as: 
\[
\left( \frac{g f^r_s}{f} - g_s \right) \beta_i = 8\pi G \left( -T^r_i + T^r_i \right) \geq 0.
\] (16)
In the inequality in equation (16) we used equation (5). But this means, considering the fact that \( 1 - g \) > 0, that 
\[
f_s \geq g_s > 0,
\] at asymptotic infinity.

### 3. Discussion

In this work, we have shown that Bekenstein’s no-hair result [1, 2] for a general multiplet of minimally coupled scalar fields fulfilling the weak-energy condition generalizes to Lanczos–Lovelock theory of gravity with non-negative coupling constants.

Let us further make two remarks: First, the fact that the stress-energy tensor describes a scalar field given by the action (1) was used only in one single step. It was used in the fact that \( T^r_i = T^r_i \), where by \( T^r_i \) we mean the diagonal angular component of the stress–energy tensor. (All the diagonal angular components of the stress–energy tensor are, in the case of spherical symmetry, necessarily equal.) This means the proof given here for scalar fields holds automatically for any other matter field described by a stress–energy tensor that fulfills the condition \( T^r_i = T^r_i \).

Second, equation (10) can be expressed more generally for a space–time containing a maximally symmetric \( D - 2 \) dimensional subspace as 
\[
\sum_{p=1}^{[D/2]} \alpha_p \left\{ \frac{(D - 2) p (D - 2) \cdot g f^r_s}{2 r f} \left( \frac{k - g}{r^2} \right)^{p-1} \right\} = 8\pi G T^r_i.
\] (17)
Here, $k = \{ \pm 1, 0 \}$ is a sectional curvature of the maximally symmetric subspace (sphere, hyperboloid, plane). One can easily see that equation (17) and $T' \neq 0$ directly imply that $f_r > 0$ for two subcases, which are, however, not of any substantial interest. The first subcase is a general space–time with maximally symmetric subspace in a theory with $\alpha_p \geq 0$ where $\alpha_p = 0$ for $p$ odd. In addition, the same implication one obtains for the case of $k = -1, 0$ in the theory with $\alpha_p \leq 0$ where $\alpha_p = 0$ for $p$ even. Therefore, the proof of no scalar hair simply holds also for these cases. (Note that the wave equation part of the argument trivially generalizes from spherically symmetric space–time to more general spacetime with maximally symmetric $D - 2$ dimensional subspace.) However, it is important to note that, in general, relaxing the condition on $\alpha_p \geq 0$, or departing from spherical symmetry (in any direction), has significant consequences for the logic used in our proof, and new ideas have to be employed. (Note also that with spinning black holes the situation is far more complicated: It is well known that vacuum-rotating black holes in higher-dimensional gravity have more rich structure than in four dimensions (4D) [17]. Furthermore, it has been recently shown that even a non-stationary complex massive scalar field in ordinary 4D GR can lead to a stationary, rotating, non-trivial black-hole exterior [18].)

In future work, it would be interesting to look at what happens with non-minimally coupled scalar fields. In addition, in case of curvature coupled field and GR Bekenstein had shown that the no-hair result holds for certain intervals of values of the scalar field coupling parameter [2, 4]. However, outside these intervals, there is a known non-trivial (albeit unstable) solution describing a configuration of a scalar field around the static black hole [2]. It would be interesting to see to what extent these results generalize in the Lovelock theory. It would be also interesting to explore what happens in theories with non-metric connections (metric–affine or Palatini gravity) [19]. All this is left for future research.

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