3d Quantum Gravity and Effective Non-Commutative Quantum Field Theory

Laurent Freidel† and Etera R. Livine‡
Perimeter Institute, 31 Caroline Street North Waterloo, Ontario, Canada N2L 2Y5 and
Laboratoire de Physique, ENS Lyon, CNRS UMR 5672, 46 Allée d’Italie, 69364 Lyon Cedex 07

ABSTRACT

We show that the effective dynamics of matter fields coupled to 3d quantum gravity is described after integration over the gravitational degrees of freedom by a braided non-commutative quantum field theory symmetric under a κ-deformation of the Poincaré group.

PACS numbers:

One of the most pressing issues in quantum gravity (QG) is the semi-classical regime. In this letter, we address this question in the context of matter coupled to 3d gravity. We show how to recover standard quantum field theory (QFT) amplitudes in the no-gravity limit and how to compute the QG corrections.

Let us consider a matter field φ coupled to gravity,

\[ Z = \int Dg \int D\phi e^{iS_\text{GR}[g] + iS_{\phi}[g]}, \]  

where g is the space-time metric, \( S_{\text{GR}}[g] \) the Einstein gravity action and \( S[\phi, g] \) the action defining the dynamics of φ in the metric g. Our goal is to integrate out the quantum gravity fluctuations and derive an effective action for φ taking into account the quantum gravity correction:

\[ Z = \int D\phi e^{iS_{\text{eff}}[\phi]}. \]

We propose to expand the φ integration into Feynman diagrams, which depend on the “background” metric g and to compute the quantum gravity effects on these Feynman diagram evaluations:

\[ Z = \sum_\Gamma C_\Gamma \int Dg I_\Gamma[g] e^{iS_{\text{eff}}[g]} = \sum_\Gamma C_\Gamma I_\Gamma. \]  

Finally, we re-sum these deformed Feynman diagrams to identify the effective action \( S_{\text{eff}}[\phi] \) taking into account the QG corrections to the matter dynamics. Here, we prove that this program can be explicitly realized for 3d quantum gravity. The resulting effective matter theory is a non-commutative field theory invariant under the κ-deformed Poincaré group. The deformation parameter κ is simply related to the Newton constant for gravitation κ = 4πG. All technical proofs can be found in [1].

In a first order formalism, Riemannian 3d gravity is described in terms of a frame field \( e_\mu^i \)dx\( ^\mu \) and a spin connection \( \omega_\mu^i dx^\mu \), both valued in the Lie algebra \( \mathfrak{so}(3) \). Indices \( i \) and \( \mu \) run from 0 to 2. The action is defined as:

\[ S[c, \omega] = \frac{1}{16\pi G} \int e^i \wedge F_i[w], \]  

where \( F \equiv d\omega + \omega \wedge \omega \) is the curvature tensor of the 1-form \( \omega \). The equations of motion for pure gravity impose that the connection is flat and the torsion vanishes,

\[ F[\omega] = 0, \quad T[\omega, e] = d\omega e = 0. \]  

This is actually a topological field theory. Particles are introduced as topological defects [2]. Spinless particles are source of curvature (the spin introduces torsion):

\[ F^i[\omega] = 4\pi Gp^i|\delta(x). \]

Outside the particle, the space-time remains flat and the particle creates a conical singularity with deficit angle related to the particle’s mass [3]:

\[ \theta = \kappa m. \]  

This deficit angle describes the feedback of the particle on the space-time geometry. Since \( \theta \) is obviously bounded by 2\( \pi \), particles have a maximal allowed mass \( m_P = (2G)^{-1} \). Note that the Planck mass \( m_P \) in 3d does not depend on the Planck constant unlike the Planck length \( l_P = h m_P^{-1} \sim hG \). This feature is specific to 3d QG and does not apply to the 4d theory.

The spin foam quantization of 3d gravity is given by the Ponzano-Regge model [4], which was the first ever written QG model. It is a discretization of the continuum path integral, \( Z = \int Dc D\omega e^{iS[\epsilon, \omega]} \). Since the theory is topological, the discretization actually provides an exact quantization. Considering a triangulation \( \Delta \) of a 3d manifold \( M \) and a graph \( \Gamma \subset \Delta \), we insert particles with deficit angles \( \theta_e \) for all edges \( e \in \Gamma \) of the graph. The partition function is defined as the product of weights associated to the edges and to the tetrahedra:

\[ I_\Delta[\Gamma] = \sum_{\{j_e\}_{e \in \Gamma}} \prod_{j_e \in \mathbb{N}} J_{j_e} \prod_{j_e \in \mathbb{N}} K_{\theta_e(j_e)} \prod_j \left\{ \begin{array}{c} j_{e_1} \ j_{e_2} \ j_{e_3} \ j_{e_4} \end{array} \right\}, \]  

where we sum over all assignments of \( \text{SO}(3) \) representation \( j_e \in \mathbb{N} \) to the edges of \( \Delta \), \( d_j = (2j + 1) \) is the dimension of the \( j \)-representation and we associate a \{6j\}
symbol to each tetrahedron. \( h_\theta = \exp(i\theta \sigma_3) \) is in the U(1) subgroup and we define the weight:

\[
K_\theta(j) = \frac{i}{2\kappa^2} e^{-i\theta j_3(\theta - i\epsilon)}, \quad \text{Re } K_\theta = \frac{\cos \theta}{2\kappa^2 \sin \theta} \chi_j(\theta)
\]

where \( \epsilon > 0 \) is a regulator and \( \chi_j(\theta) \) the trace of \( h_\theta \) in the \( j \)-representation. \( K_\theta \) defines the insertion of a Feynman propagator while \( \text{Re } K_\theta \) gives a Hadamard propagator and leads back to the same partition function as in [5].

The partition function has a dual formulation in terms of \( \text{SO}(3) \) group elements attached to the faces \( f \in \Delta \):

\[
I_\Delta[\Gamma, \theta_e] = \int \prod_f dg_f \prod_{\gamma \in G} \tilde{K}_\theta(g_e) \prod \delta(g_e), \quad (7)
\]

\[
\tilde{K}_\theta(g_e) = \frac{i}{\kappa^2 (\sin^2 \phi - \sin^2 \theta_e + i\epsilon)} = \sum_j K_\theta(j) \chi_j(\phi),
\]

where \( g_e \) is the oriented product \( \prod_{\gamma \in G} g_f \) and the function \( K_\theta(g) \) is invariant under conjugation. Using the real part of \( K_\theta \) leads to replacing \( \tilde{K}_\theta \) by the distribution \( \delta_\theta(g) \) which fixes the rotation angle of \( g \) to \( \theta \),

\[
\int \text{SO}(3) \text{ dg}_f(g) \delta_\theta(g) = \int \text{SO}(3)/U(1) \text{ dx}_f(\theta h_o x^{-1}).
\]

\( I_\Delta[\Gamma] \) is independent of the triangulation \( \Delta \) and only depends on the topology of \( (M, \Gamma) \). It is finite after suitable gauge fixing of the diffeomorphism symmetry [6], which removes redundancies in the product of \( \delta \)-functions. Then for a trivial topology \( M = [0,1] \times \Sigma_2 \), \( I_\Delta \) is the projector onto the physical states, that is the space the flat connections on \( \Sigma_2 \) [7]. Moreover, this quantization scheme has been shown to be equivalent to the Chern-Simons quantization [8]. Finally, the large \( j \) asymptotics of the \( \{6j\} \) symbols are related to the discrete Regge action for \( 3d \) gravity [9].

We have defined a purely algebraic quantum gravity amplitude \( I_\Delta[\Gamma] \). The Newton constant \( G \) only appears as a unit to translate the algebraic quantities \( j, \theta \) into the physical length \( l = j|P| = jhG \) and the physical mass \( m = \theta/\kappa = \theta/4\pi G \).

The essential point is that the QG amplitudes \( I_\Delta[\Gamma] \) are the Feynman diagram evaluations of a non-commutative field theory. Let us first consider a trivial topology \( M \sim S^3 \) with \( \Gamma \) planar. In this case, we can get rid of the triangulation dependence and rewrite \( I_\Gamma \equiv I_\Delta[\Gamma] \) as [1]:

\[
I_\Gamma = \int \prod_{\gamma \in G} d^3 X_\gamma \frac{1}{8 \pi \kappa^3} \int \prod_{\gamma \in G} dg_\gamma \tilde{K}_\theta(g_e) \prod \text{ e}^{\frac{i\pi}{8} \text{tr}(X_e g_e)}.
\]

The integral is over one copy of \( \mathfrak{so}(3) \sim \mathbb{R}^3 \) for each vertex \( X_e \equiv X_\Gamma \sigma_3 \) and one copy of \( \text{SO}(3) \) for each edge. We define at each vertex \( v \), the ordered product of the edge group elements meeting at \( v \)

\[
G_v = \prod_{e \subset v} g_e^{\epsilon_v(e)},
\]

c_v(e) = \pm 1 \text{ depending on whether the edge is incoming or outgoing at } v \text{. The kernel } \tilde{K}_\theta \text{ defines the Feynman propagator and is given by}

\[
\tilde{K}_\theta(g) = \int \mathbb{R}^+ dTe^{-\frac{i}{8} \text{tr}(\epsilon_2 g \gamma_5 - (\sin \theta \gamma_5)^2)},
\]

with \( 2i\tilde{K}(g) \equiv \text{tr}(g\gamma_5) \) the projection of \( g \) on Pauli matrices. Changing the integration range from \( \mathbb{R}^+ \) to \( \mathbb{R} \), we would obtain the Hadamard function \( \delta_\theta \) instead of \( \tilde{K}_\theta \). To further simplify this expression, we introduce a noncommutative \( \star \)-product on \( \mathbb{R}^3 \) such that

\[
e^{-\frac{i}{8} \text{tr}(X_\Gamma)} \star \text{e}^{-\frac{i}{8} \text{tr}(X_\Gamma)} = \text{e}^{-\frac{i}{8} \text{tr}(X_\Gamma)}
\]

Using the parametrization of \( \text{SO}(3) \) group elements,

\[
g = (P_4 + \kappa P^3 \sigma_1), \quad P_4^2 + \kappa^2 P^3 P_1 = 1, \quad P_4 \geq 0,
\]

the \( \star \)-product deforms the composition of plane waves,

\[
e^{-i\bar{P}_1 \cdot \bar{P}_2 \cdot X} \star e^{-i\bar{P}_1 \cdot \bar{P}_2 \cdot X} = e^{-i\bar{P}_1 \cdot \bar{P}_2 \cdot X} e^{-i\bar{P}_1 \cdot \bar{P}_2 \cdot X},
\]

\[
\bar{P}_1 \star \bar{P}_2 = \sqrt{1 - \kappa^2|\bar{P}_1|^2} \bar{P}_1 + \sqrt{1 - \kappa^2|\bar{P}_2|^2} \bar{P}_2 \quad - \kappa \bar{P}_1 \times \bar{P}_2,
\]

with \( \times \) the 3d vector cross product. To define the \( \star \)-product on all functions, we introduce a new group Fourier transform \( F : C(\text{SO}(3)) \rightarrow C_\kappa(\mathbb{R}^3) \) mapping functions on the group \( \text{SO}(3) \) to functions on \( \mathbb{R}^3 \) with momenta bounded by \( 1/\kappa \):

\[
\phi_\kappa(X) = \int d\gamma \hat{\phi}(\gamma) e^{\frac{i}{8} \text{tr}(X \gamma_5)}.
\]

The inverse group Fourier transform is explicitly written

\[
\hat{\phi}(g) = \int_{\mathbb{R}^3} d^3X \frac{8 \pi \kappa^3}{8 \pi \kappa^3} \phi_\kappa(X) \star e^{\frac{i}{8} \text{tr}(X g^{-1})}
\]

\[
= \int_{\mathbb{R}^3} d^3X \phi_\kappa(X) \sqrt{1 - \kappa^2 P^2} \text{e}^{\frac{i}{8} \text{tr}(X g^{-1})}.
\]

Under this Fourier transform, the \( \star \)-product is dual to the group convolution product. Finally \( F \) is an isometry between \( L^2(\text{SO}(3)) \) and \( C_\kappa(\mathbb{R}^3) \) equipped with the norm

\[
||\phi_\kappa||_{C_\kappa}^2 = \int \frac{dX}{8 \pi \kappa^3} \phi_\kappa(X) \phi_\kappa(X).
\]

Using this \( \star \)-product, the amplitude (8) reads

\[
I_\Gamma = \int \prod_{\gamma \in G} d^3 X_\gamma \frac{1}{8 \pi \kappa^3} \prod_{\gamma \in G} dg_\gamma \tilde{K}_\theta(g_e) \prod \left( e^{\frac{i}{8} \text{tr}(X_\gamma g_e)} \right)
\]

\[
= \sum_{\Gamma_{\text{trivalent}}} \frac{\lambda|\gamma|}{S_\Gamma} \frac{1}{S_\Gamma} I_\Gamma.
\]
where \( \lambda \) is a coupling constant. \(|\nu_T|\) is the number of vertices of \( \Gamma \) and \( S_T \) is the symmetry factor of the graph. Remarkably, this sum can be obtained from the perturbative expansion of a non-commutative field theory given explicitly by:

\[
S = \int \frac{d^3x}{8\pi \kappa^3} \left[ \frac{1}{2} \left( \partial_t \phi \ast \partial_t \phi \right) - \frac{1}{2} \sin^2 \frac{m \kappa}{\sqrt{2}} \left( \phi \ast \phi \right) + \frac{1}{3!} \left( \phi \ast \phi \ast \phi \right) \right]
\]

where \( \phi \) is in \( C_c(\mathbb{R}^3) \). Its momentum has support in the ball of radius \( \kappa^{-1} \). We can write this action in momentum space

\[
S(\phi) = \frac{1}{2} \int dg \left( P^2(g) - \sin^2 \frac{m \kappa}{\kappa} \right) \tilde{\phi}(g) \tilde{\phi}(g^{-1}) + \frac{1}{3!} \int dg_1dg_2dg_3 \delta(g_1g_2g_3) \tilde{\phi}(g_1) \tilde{\phi}(g_2) \tilde{\phi}(g_3).
\]

This is our effective field theory describing the dynamics of the matter field after integrating out the gravitational sector. This non-commutative field theory action is symmetric under a \( \kappa \)-deformed action of the Poincaré group. Calling \( \Lambda \) the generators of Lorentz transformations and \( T_a \) the generators of translations, the action of these generators on one-particle states is undeformed:

\[
\Lambda \cdot \tilde{\phi}(g) = \tilde{\phi}(Ag \Lambda^{-1}) = \tilde{\phi}(\Lambda \cdot P(g)),
\]

\[
T_a \cdot \tilde{\phi}(g) = e^{iP(g) \cdot a} \tilde{\phi}(g).
\]

The non-commutative deformation of the Poincaré group appears at the level of multi-particle states and only the action of the translations is deformed:

\[
\Lambda \cdot \tilde{\phi}(P_1) \tilde{\phi}(P_2) = \tilde{\phi}(\Lambda \cdot P_1) \tilde{\phi}(\Lambda \cdot P_2),
\]

\[
T_a \cdot \tilde{\phi}(P_1) \tilde{\phi}(P_2) = e^{iP_1 \cdot a} \tilde{\phi}(P_1) \tilde{\phi}(P_2).
\]

It is straightforward to derive the Feynman rules from the action (21) (see fig.1). The effective Feynman propagator is the group Fourier transform of \( \tilde{K}_\phi(g) \),

\[
K_m(X) = i \int dg e^{i \frac{1}{2} \text{tr}(Xg) - \frac{1}{2} \sin \frac{m \kappa}{\kappa} \frac{X}{P^2}}.
\]

The effect of quantum gravity is two-fold. First the mass gets renormalized \( m \to \sin \frac{m \kappa}{\kappa} \). Then the momentum space is no longer the flat space but the homogeneously curved space \( S^3 \sim \text{SO}(3) \). This reflects that the momentum is bounded \( |P| < 1/\kappa \).

At the interaction vertex the momentum addition becomes non-linear with a conservation rule \( P_1 \oplus P_2 \oplus P_3 = 0 \) which implies a non-conservation of momentum \( P_1 + P_2 + P_3 \neq 0 \). Intuitively, part of the energy involved in a collision process is absorbed by the gravitational field: gravitational effects can not be ignored at high energy. This effect, which is stronger at high momenta and for non-collinear momenta, prevents the total momenta from being larger than the Planck energy.

A last subtlety of the Feynman rules is the evaluation of non-planar diagrams. A careful analysis of \( I_T \) shows that we have a non-trivial braiding for each crossing of two edges, we associate a weight \( \delta(g_1g_2^{-1})(g_2^{-1}) \delta(g_2g_2^{-1}) \) (see fig.1). This reflects a non-trivial statistics where the Fourier modes of the fields obey the exchange relation:

\[
\tilde{\phi}(g_1)\tilde{\phi}(g_2) = \tilde{\phi}(g_2)\tilde{\phi}(g_2^{-1}g_1g_2),
\]

which is naturally determined by our choice of star product. Indeed, let us look at the product of two identical fields:

\[
\phi \ast \phi(X) = \int dg_1dg_2 e^{i\frac{1}{3!} \text{tr}(Xg_1g_2)} \tilde{\phi}(g_1) \tilde{\phi}(g_2),
\]

Under change of variables \((g_1, g_2) \to (g_2, g_2^{-1}g_1g_2)\), the star product reads

\[
\phi \ast \phi(X) = \int dg_1dg_2 e^{i\frac{1}{3!} \text{tr}(Xg_1g_2)} \tilde{\phi}(g_2) \tilde{\phi}(g_2^{-1}g_1g_2).
\]

The identification of the Fourier modes of \( \phi \ast \phi(X) \) leads to the exchange relation (26). This braiding was first proposed in [10] for two particles coupled to 3d QG and then computed in the Ponzano-Regge model in [5]. It is encoded into a braiding matrix

\[
R \cdot \tilde{\phi}(g_1)\tilde{\phi}(g_2) = \tilde{\phi}(g_2)\tilde{\phi}(g_2^{-1}g_1g_2).
\]

This is the \( R \) matrix of the \( \kappa \)-deformation of the Poincaré group [10]. Such field theories with non-trivial braided statistics are simply called braided non-commutative field theories and were first introduced in [11].

Finally, the \( \ast \)-product induces a non-commutativity of space-time and a deformation of phase space:

\[
[X_i, X_j] = i\kappa \epsilon_{ijk}X_k,
\]

\[
[X_i, P_j] = i\sqrt{1-\kappa^2P^2} \delta_{ij} - i\kappa \epsilon_{ijk}P_k.
\]

This non-commutativity reflects the fact that momentum space is curved. Indeed the coordinates \( X \) are realized as right invariant derivations on momentum space and derivations of a curved manifold do not commute. Moreover, this non-commutativity being related to having bounded momenta implies the existence of a minimal
length scale accessible in the theory. Indeed defining the non-commutative \( \delta \text{-function} \ \delta_0 \ast \phi(X) = \phi(0)\delta_0(X) \), we compute
\[
\delta_0(X) = 2\kappa \frac{J_1 \left( \frac{|X|}{\kappa} \right)}{|X|},
\]
with \( J_1 \) the 1st Bessel function. It is clear that \( \delta_0(X) \) is concentrated around \( X = 0 \) but has a non-zero width.

Using this formalism, one can compute the QG effects order by order in \( \kappa \). The 0th order is defined by the non-gravity limit \( \kappa \to 0 \). Starting either from the spin foam amplitude \( I_F \) given by (6) or from the Feynman evaluations (17), one can show that the limit \( \kappa \to 0 \) is exactly given by the Feynman evaluations of the usual commutative QFT:
\[
I^\text{R}_0 [\Gamma, m_c] \equiv \lim_{\kappa \to 0} \kappa^{3|\epsilon_F|} I_0 [\Gamma, \theta_c] 
= \int_{\mathbb{R}^3} \prod_{j \in \Delta} d^3 \vec{p}_j \prod_{e \in \mathcal{I}} \frac{i}{2\pi} \left( p^e - m^e \right) \prod_{e \in \Delta \setminus \mathcal{I}} \delta(p^e),
\]
where \( |\epsilon_F| \) is the number of edges of the graph \( \Gamma \), \( \vec{p}_j \in \mathbb{R}^3 \) are variables attached to the faces of \( \Delta \) and \( p^e = \sum_{j \in \mathcal{E}} \vec{p}_j \). Moreover, since physical lengths and masses are defined in \( \kappa \)-units, \( l = \kappa j \) and \( m = \theta/\kappa, \) taking \( \kappa \to 0 \) corresponds to \( j \to \infty \) and \( \theta \to 0 \). For \( \theta \to 0 \), the group multiplication on \( \text{SO}(3) \) becomes abelian at first order in \( \kappa \). More precisely, we prove in [1] that the non-gravity limit of the Ponzano-Regge model is actually the topological state sum based on the abelian group \( \mathbb{R}^3 \).

This shows that the usual Feynman evaluations of QFT in 3d can be generically written as amplitudes of a topological theory.

Up to now, we have worked in the Riemannian context. All the previous constructions and results can be straightforwardly extended to the Lorentzian theory. The Lorentzian version of the Ponzano-Regge model is expressed in terms of the \( \{6j\} \) symbols of the non-compact group \( \text{SO}(2,1) \) [12]. Holonomies around particles are \( \text{SO}(2,1) \) group elements parametrized as \( g = P_4 + iP_2 \sigma^\tau \) with \( P_2^2 + \kappa^2 P_1 P_4 = 1 \) and \( P_4 \geq 0 \), with the metric \( (+--+) \) and the \( \text{su}(1,1) \) Pauli matrices, \( \tau_0 = \sigma_0, \tau_1, 2 = i\sigma_{1,2} \). Massive particles correspond to the \( P_1 P_4 > 0 \) sector. They are described by elliptic group elements, \( P_4 = \cos \theta, |P| = sin \theta \). The deficit angle is given by the mass, \( \theta = \kappa m \). All the mathematical relations of the Riemannian theory are translated to the Lorentzian framework by changing the signature of the metric. The propagator remains given by the formula (10). The momentum space is now \( \text{AdS}^3 \sim \text{SO}(2,1) \).

The addition of momenta is deformed accordingly to the formula (13). We similarly introduce a group Fourier transform \( F : C(\text{SO}(2,1)) \to C_\kappa (\mathbb{R}^3) \) and a \( \ast \)-product dual to the convolution product on \( \text{SO}(2,1) \). Finally we derive the effective non-commutative field theory with the same expression (19) as in the Riemannian case.

To sum up, we have shown that the 3d quantum gravity amplitudes, defined through the Ponzano-Regge spin foam model, are actually the Feynman diagram evaluations of a (braided) non-commutative QFT. This effective field theory describes the dynamics of the matter field after integration of the gravitational degrees of freedom. The theory is invariant under a \( \kappa \)-deformation of the Poincaré algebra, which acts non-trivially on many-particle states. This is an explicit realization of a QFT in the framework of deformed special relativity (see e.g. [13]), which implements from first principles the original idea of Snyder of using a curved momentum space to regularize the Feynman diagrams.

The formalism can naturally take into account a non-zero cosmological constant \( \Lambda \). The model is based on \( \mathcal{U}_q (\text{SU}(2)) \) and its Feynman rules are given in [1].

A natural question concerns the unitarity of our non-commutative quantum field theory since the non-commutativity affects time [15]. We a priori do not expect a unitary theory: since we have integrated out the gravity degrees of freedom, we expect ghosts to appear at the Planck energy \( m_P = 1/\kappa \sim 1/G \).

Finally the present results suggest an extension to 4d. The standard 4d QFT Feynman graphs would be expressed as expectation values of a 4d topological spin foam model (see e.g. [16, 17]). That model would provide the semi-classical limit of QG and be identified as the zeroth order of an expansion in term of the inverse Planck mass \( \kappa \) of the full QG spin foam amplitudes. QG effects would then appear as deformations of the Feynman graph evaluations and QG corrections to the scattering amplitudes could be computed order by order in \( \kappa \).

[1] L Freidel, ER Livine, Class.Quant.Grav. 23 (2006) 2021-2062, [arXiv:hep-th/0502106]
[2] S Deser, R Jackiw, G t Hooft, Annals Phys.152, 220 (1984)
[3] HJ Matschull, M Welling, Class.Quant.Grav. 15 (1998) 2981, [arXiv:gr-qc/9708054]
[4] G Ponzano, T Regge, Spectroscopic and group theoretical methods in physics (Bloch ed.), North Holland (1968)
[5] L Freidel, D Louapre, Class.Quant.Grav. 21 (2004) 5685
[6] L Freidel, D Louapre, Nucl. Phys. B 662, 279 (2003)
[7] H Ooguri, Nucl.Phys. B382 (1992) 276-304
[8] L Freidel, D Louapre, [arXiv:gr-qc/0410114]
[9] J Roberts, Geom. Topol. Monogr. 4 (2002) 245-261
[10] FA Bais, NM Muller, BJ Schroers, Nucl. Phys. B 640, 3 (2002), [arXiv:hep-th/0205021]
[11] R Oeckl, Int. J. Mod. Phys. B 14 (2000) 2461
[12] L Freidel, Nucl.Phys.Proc.Suppl. 88 (2000) 237-240, [arXiv:gr-qc/0102098]; S Davids, [arXiv:gr-qc/0110114]
[13] J Kowalski-Glikman, [arXiv:hep-th/0405273]
[14] H Snyder, Phys.Rev. 71, 38 (1947)
[15] J Gomis, T Mehen, Nucl. Phys. B 591, 265 (2000)
[16] L Freidel, A Starodubtsev, [arXiv:hep-th/0501191]
[17] A Baratin, L Freidel, [arXiv:gr-qc/0604016]