Decomposition formula for jump diffusion models

Raúl Merino¹,³, Jan Pospíšil², Tomáš Sobotka², and Josep Vives¹

¹Facultat de Matemàtiques i Informàtica, Universitat de Barcelona,
Gran Via 585, 08007 Barcelona, Spain,
²NTIS - New Technologies for the Information Society, Faculty of Applied Sciences,
University of West Bohemia, Univerzitní 8, 301 00 Plzeň, Czech Republic,
³VidaCaixa S.A., Investment Risk Management Department,
C/Juan Gris, 2-8, 08014 Barcelona, Spain.

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Abstract

In this paper we derive a generic decomposition of the option pricing formula for models with finite activity jumps in the underlying asset price process (SVJ models). This is an extension of the well-known result by Alós (2012) for Heston (1993) SV model. Moreover, explicit approximation formulas for option prices are introduced for a popular class of SVJ models - models utilizing a variance process postulated by Heston (1993). In particular, we inspect in detail the approximation formula for the Bates (1996) model with log-normal jump sizes and we provide a numerical comparison with the industry standard - Fourier transform pricing methodology. For this model, we also reformulate the approximation formula in terms of implied volatilities. The main advantages of the introduced pricing approximations are twofold. Firstly, we are able to significantly improve computation efficiency (while preserving reasonable approximation errors) and secondly, the formula can provide an intuition on the volatility smile behaviour under a specific SVJ model.

Keywords: option pricing; stochastic volatility models; jump diffusion models; implied volatility

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1 Introduction

The main problem of the Black-Scholes option pricing model is the assumption of constant volatility for the underlying stock price process. In practice, this model is used as a marking model to quote implied volatilities instead of traded option prices. Contrary to the model assumptions, the implied volatilities observed in the vanilla option markets are not flat - they typically exhibit a non-zero skew and a convex smile-like shape in the moneyness dimension. To correctly capture the shape of implied volatility surfaces, various stochastic volatility (SV) models were developed. These models assume that not only the spot prices are stochastic, but also their volatility is driven...
by a suitable stochastic process. Another way how to deal with drawbacks of the Black-Scholes model is to add a jump term to the stock price process. This results into the jump diffusion setting, which was originally studied by Merton (1976). In this article, we build an option price approximation framework for a popular class of financial models that utilize both of the aforementioned ideas. Hence, the main objects of our study are stochastic volatility jump diffusion (SVJ) models.

The first SVJ model is credited to Bates (1996) who incorporated a stochastic variance process postulated by Heston (1993) alongside Merton (1976) - style jumps. The variance of stock prices follows a CIR process (Cox, Ingersoll, and Ross 1985) and the stock prices themselves are assumed to be of a jump diffusion type with log-normal jump sizes. In particular, this model should improve the market fit for short-term maturity options, while the original Heston (1993) approach would often need unrealistically high volatility of variance parameter to fit reasonably well the short-term smile (Bayer, Friz, and Gatheral 2016; Mrázek, Pospíšil, and Sobotka 2016). An SVJ model with a non-constant interest rate was introduced by Scott (1997). Several other authors studied SVJ models that have a different distribution for jump sizes, e.g. Yan and Hanson (2006) utilized log-uniform jump amplitudes.

Naturally, one can extend SVJ models by adding jumps into the variance process (e.g. a model introduced by Duffie, Pan, and Singleton (2000)). However, based on several empirical studies, these models tend to overfit market prices and despite having more parameters than the original Bates (1996) model they might not provide a better calibration errors (see e.g. Gatheral (2006)). Another way to improve standard SV models might be to introduce time-dependent model parameters. The Heston (1993) model with time-dependent parameters was studied by Mikhailov and Nögel (2003) for piece-wise constant parameters, by Elices (2008) for a linear dependence and a more general modification was introduced by Benhamou, Gobet, and Miri (2010). These approaches involve several additional parameters and might also suffer from overfitting. Moreover, Bayer, Friz, and Gatheral (2016) mentioned that these models do not fully comply with properties of observable market data - a general overall shape of the volatility surface typically does not change in time and hence the option prices should be derived using a time-homogeneous stochastic process.

The valuation of derivatives under these more complex models is, of course, a more elaborate task compared to the standard Black-Scholes model. Many authors have introduced semi-closed form formulas using various transformation techniques of the pricing partial (integro) differential equations, to name a few: Heston (1993), Bates (1996), Scott (1997), Lewis (2000), Albrecher, Mayer, Schoutens, and Tistaert (2007), Baustian, Mrázek, Pospíšil, and Sobotka (2017) and many others. Although transform pricing methods are typically efficient tools to evaluate non-path dependent derivatives, they do not provide any intuition on the smile behavior. Moreover, calibration routines utilizing these methods lead typically to non-convex optimization problems (see e.g. Mrázek, Pospíšil, and Sobotka (2016)).

Other authors considered approximation techniques that were pioneered by Hull and White (1987). In the last years, the Hull and White (1987) pricing formula was reinvented using techniques of the Malliavin calculus, because a future average volatility that is used in the formula is a non adapted stochastic process. In Alós (2006), Alós, León, and Vives (2007) and Alós, León, Pontier, and Vives (2008), a general jump diffusion model with no prescribed volatility process is analyzed. There have been several extensions thereof, e.g. by assuming Lévy processes in Jafari and Vives (2013), see also the survey in Vives (2016).

In Alós (2012), a new approach of dealing with the Hull and White formula and the Heston model has been proposed. The main idea of this approach is to use an adapted projection for the future volatility. The formula provides a valuable intuition on the behavior of smiles and term structures under the Heston model. This is not a purely theoretical result - it can significantly fasten/improve the calibration process by providing a good initial guess by analytical calibration or by specifying a region where calibrated parameters should lie in as it is done in Alós, de Santiago, and Vives (2015). In Merino and Vives (2015), the idea of Alós (2012) has been used to find a general decomposition formula for any stochastic volatility process satisfying basic integrability conditions.
In the present paper, we apply the same set of ideas and we extend them to the domain of SVJ models with finite activity jumps. This should serve not only to find a more efficient way to price vanilla options compared to transform pricing methods (see Section 5), but as a side product we provide a similar intuition of the smile behavior for the studied SVJ model.

In particular, we start by finding a generic decomposition formula for a vanilla call option price and an approximation for both the price and implied volatility under a specific SVJ model. Explicit pricing formulas are provided for one of the most popular SVJ models - Heston (1993) type models with compound Poisson process in the stock price evolution. To assess the accuracy and efficiency of the newly derived solution, we perform a numerical comparison for the Bates (1996) model (i.e. log-normal jump sizes) alongside its Fourier transform pricing formula introduced by Baustian, Mrázek, Pespašil, and Sobotka (2017).

The structure of the paper is as follows. In Section 2, we give basic preliminaries and our notation related to SVJ models. This notation will be used throughout the paper without being repeated in particular theorems, unless we find useful to do so in order to guide the reader through the results. In Sections 3 and 4, we derive decomposition formulas for SV and SVJ models, respectively, generalizing the decomposition formula obtained by Alòs (2012). Newly obtained decomposition is rather versatile since it does not need to specify the underlying volatility process. Particular approximation formulas for several SVJ models are presented in Section 5 alongside the numerical comparison for the Bates (1996) model. The decomposition result in terms of implied volatilities is introduced in Section 6. A discussion of the results is provided in Section 7 and technical error estimates are presented in A.

2 Preliminaries and notation

Let \( S = \{S_t, t \in [0, T]\} \) be a strictly positive price process under a market chosen risk neutral probability that follows the model:

\[
dS_t = rS_t dt + \sigma_t S_t \left( \rho dW_t + \sqrt{1 - \rho^2} d\tilde{W}_t \right) + S_t dZ_t,
\]

where \( S_0 \) is the current price, \( W \) and \( \tilde{W} \) are independent Brownian motions, \( r \) is the interest rate, \( \rho \in (-1, 1) \) is the correlation between the two Brownian motions and

\[
Z_t = \int_0^t \int_\mathbb{R} (e^y - 1) \hat{N}(ds, dy)
\]

where \( N \) and \( \hat{N} \) denote the Poisson measure and the compensated Poisson measure, respectively. We can associate to measure \( N \) a compound Poisson process \( J \), independent of \( W \) and \( \tilde{W} \), with intensity \( \lambda \geq 0 \) and jump amplitudes given by random variables \( Y_i \), independent copies of a random variable \( Y \) with law given by \( Q \). Recall that this compound Poisson process can be written as

\[
J_t := \int_0^t \int_\mathbb{R} yN(ds, dy) = \sum_{i=1}^{n_t} Y_i,
\]

where \( n_t \) is a \( \lambda \)-Poisson process. Denote by \( k := \mathbb{E}_Q(e^Y - 1) \).

Without any loss of generality, it will be convenient in the following sections, to use as underlying process, the log-price process \( X_t = \log S_t, t \in [0, T] \), that satisfies

\[
dX_t = \left( r - \lambda k - \frac{1}{2} \sigma_t^2 \right) dt + \sigma_t \left( \rho dW_t + \sqrt{1 - \rho^2} d\tilde{W}_t \right) + dJ_t.
\]

We introduce also the corresponding continuous process,

\[
d\hat{X}_t = \left( r - \lambda k - \frac{1}{2} \sigma_t^2 \right) dt + \sigma_t \left( \rho dW_t + \sqrt{1 - \rho^2} d\tilde{W}_t \right).
\]
The volatility process $\sigma$ is a square-integrable process assumed to be adapted to the filtration generated by $W$ and $J$ and its trajectories are assumed to be a.s. square integrable, càdlàg and strictly positive a.e.

**Remark 2.1.** Observe that this is a very general stochastic volatility model. We can consider the following particular cases:

- If $\sigma$ is constant and we have finite activity jumps, we have a generic jump-diffusion model as for example the Merton model. In the particular case of $\sigma = 0$ we have an exponential Lévy model.

- If we assume no jumps, that is $\lambda = 0$, we have a generic stochastic volatility diffusion model. This is the case treated in Merino and Vives (2015).

- If in addition $\rho = 0$ we have a generalization of different non correlated stochastic volatility diffusion models as Hull and White (1987), Scott (1987), Stein and Stein (1991) or Ball and Roma (1994).

- If we assume no correlation but presence of jumps we cover for example the Heston-Kou model (e.g. see Gulisashvili and Vives (2012)), or any uncorrelated model with the addition of finite activity Lévy jumps on the price process.

- Finally, if we have no jumps and $\sigma$ is constant, we have the classical Osborne-Samuelson-Black-Scholes model.

The following notation will be used throughout the paper:

- We denote by $\mathcal{F}^W$, $\mathcal{F}^\tilde{W}$ and $\mathcal{F}^N$ the filtrations generated by the independent processes $W$, $\tilde{W}$ and $J$ respectively. Moreover, we define $\mathcal{F} := \mathcal{F}^W \lor \mathcal{F}^\tilde{W} \lor \mathcal{F}^N$.

- We will denote by $BS(t, x, y)$ the price of a plain vanilla European call option under the classical Black-Scholes model with constant volatility $y$, current log stock price $x$, time to maturity $\tau = T - t$, strike price $K$ and interest rate $r$. In this case,

$$BS(t, x, y) = e^{x \Phi(d_+) - K e^{-r\tau} \Phi(d_-)},$$

where $\Phi(\cdot)$ denotes the cumulative distribution function of the standard normal law and

$$d_+ = \frac{x - \ln K + (r + \frac{y^2}{2})\tau}{y\sqrt{\tau}}.$$

- In our setting, the call option price is given by

$$V_t = e^{-r\tau} \mathbb{E}_t[(e^{X_T} - K)^+].$$

- Recall that from the Feynman-Kac formula for the model (3), the operator

$$\mathcal{L}_\sigma := \partial_t + \frac{1}{2} \sigma^2_t \partial^2_x + \left(r - \lambda k - \frac{1}{2} \sigma^2_t\right) \partial_x - r$$

satisfies $\mathcal{L}_\sigma BS(t, X_t, \sigma_t) = 0$.

- We define the operators $\Lambda := \partial_x$, $\Gamma := (\partial^2_x - \partial_x)$ and $\Gamma^2 = \Gamma \circ \Gamma$. In particular, for the Black-Scholes formula we obtain:

$$\Gamma BS(t, x, y) := \frac{e^x}{y\sqrt{2\pi\tau}} \exp\left(-\frac{d_+^2(y)}{2}\right),$$

$$\Lambda \Gamma BS(t, x, y) := \frac{e^x}{y\sqrt{2\pi\tau}} \exp\left(-\frac{d_+^2(y)}{2}\right) \left(1 - \frac{d_+^2(y)}{y\sqrt{\tau}}\right),$$

$$\Gamma^2 BS(t, x, y) := \frac{e^x}{y\sqrt{2\pi\tau}} \exp\left(-\frac{d_+^2(y)}{2}\right) \frac{d^2_x(y - y d_+(y)\sqrt{\tau} - 1)}{y^2\tau}.$$
• We define \( p_n(\lambda T) \) as the Poisson probability mass function with intensity \( \lambda T \). I.e. \( p_n \) takes the following form:

\[
p_n(\lambda T) := \frac{e^{-\lambda T}(\lambda T)^n}{n!}.
\]

3 A generic SV decomposition formula

In this section, following the ideas of Alòs (2012), see also Merino and Vives (2015), we extend the decomposition formula to a generic stochastic volatility model. We recall that the formula is valid without having to specify the underlying volatility process explicitly, which enables us to obtain a very flexible decomposition formula. The formula proved in Alòs (2012) is the particular case of the Heston model.

It is well known that if the stochastic volatility process is independent of the price process, then the pricing formula of a plain vanilla European call is given by

\[
V_t = \mathbb{E}_t[BS(t, S_t, \bar{\sigma}_t)]
\]

where \( \bar{\sigma}_t^2 \) is the so called average future variance and it is defined by

\[
\bar{\sigma}_t^2 := \frac{1}{T-t} \int_t^T \sigma_s^2 ds.
\]

Naturally, \( \bar{\sigma}_t \) is called the average future volatility, see Fouque, Papanicolaou, and Sircar (2000), page 51.

The idea used in Alòs (2012) consists of using an adapted projection of the average future variance

\[
v_t^2 := \mathbb{E}_t(\bar{\sigma}_t^2) = \frac{1}{T-t} \int_t^T \mathbb{E}_s[\sigma_s^2] ds
\]

to obtain a decomposition of \( V_t \) in terms of \( v_t \). This idea switches an anticipative problem related with the anticipative process \( \bar{\sigma}_t \) into a non-anticipative one related to the adapted process \( v_t \).

We define

\[
M_t = \int_0^T \mathbb{E}_t[\sigma_s^2] ds,
\]

and hence

\[
dv_t^2 = \frac{1}{T-t} \left[ dM_t + (v_t^2 - \sigma_t^2) dt \right].
\]

Recall that \( M \) is a martingale with respect the filtration generated by \( W \) and \( J \).

The following processes will play an important role in a generic decomposition formula that will be introduced in this section. Let

\[
R_t = \frac{1}{8} \mathbb{E}_t \left[ \int_t^T d[M, M]_u \right]
\]

and

\[
U_t = \frac{\rho^2}{2} \mathbb{E}_t \left[ \int_t^T \sigma_u d[W, M]_u \right],
\]

where \( [\cdot, \cdot] \) denotes the quadratic covariation process.

Now we prove a generic version of Theorem 2.2 in Alòs (2012) which will be useful for our problem.
Theorem 3.1 (Generic decomposition formula). Let $B_t$ be a continuous semimartingale with respect to the filtration $\mathcal{F}_t$, let $A(t, x, y)$ be a $C^{1,2,2}([0, T] \times [0, \infty) \times [0, \infty))$ function and let $v^2, M_t$ be defined as above. Then we are able to formulate the expectation of $e^{-rt}A(T, \tilde{X}_T, v^2_T)B_T$ in the following way:

$$
E \left[ e^{-rt}A(T, \tilde{X}_T, v^2_T)B_T \right] = A(0, \tilde{X}_0, v^2_0)B_0
$$

$$
+ E \left[ \int_0^T e^{-ru} \partial_y A(u, \tilde{X}_u, v^2_u)B_u \frac{1}{T-u} \left( v^2_u - \sigma^2_u \right) du \right]
$$

$$
+ E \left[ \int_0^T e^{-ru} A(u, \tilde{X}_u, v^2_u)dB_u \right]
$$

$$
+ \frac{1}{2} E \left[ \int_0^T e^{-ru} \left( \partial^2_x - \partial_u \right) A(u, \tilde{X}_u, v^2_u)B_u \left( \sigma^2_u - v^2_u \right) du \right]
$$

$$
+ \frac{1}{2} E \left[ \int_0^T e^{-ru} \partial^2_y A(u, \tilde{X}_u, v^2_u)B_u \frac{1}{(T-u)^2} d[M, M]_u \right]
$$

$$
+ \rho E \left[ \int_0^T e^{-ru} \partial^2_{x,y} A(u, \tilde{X}_u, v^2_u)B_u \frac{\sigma_u}{T-u} d[W, M]_u \right]
$$

$$
\sqrt{1 - \rho^2} E \left[ \int_0^T e^{-ru} \partial^2_{x,y} A(u, \tilde{X}_u, v^2_u)B_u \frac{\sigma_u}{T-u} d[W, B]_u \right]
$$

$$
+ \rho E \left[ \int_0^T e^{-ru} \partial_x A(u, \tilde{X}_u, v^2_u)\sigma_u d[W, B]_u \right]
$$

$$
\sqrt{1 - \rho^2} E \left[ \int_0^T e^{-ru} \partial_x A(u, \tilde{X}_u, v^2_u)\sigma_u d[W, B]_u \right]
$$

$$
+ E \left[ \int_0^T e^{-ru} \partial_x A(u, \tilde{X}_u, v^2_u) \frac{1}{T-u} d[M, B]_u \right].
$$

Proof. Applying the Itô formula to the process $e^{-rt}A(t, \tilde{X}_t, v^2_t)B_t$ we obtain:

$$
e^{-rt}A(T, \tilde{X}_T, v^2_T)B_T = A(0, \tilde{X}_0, v^2_0)B_0
$$

$$
- r \int_0^T e^{-ru}A(u, \tilde{X}_u, v^2_u)B_u du
$$

$$
+ \int_0^T e^{-ru} \partial_y A(u, \tilde{X}_u, v^2_u)B_u du
$$

$$
+ \int_0^T e^{-ru} \partial_x A(u, \tilde{X}_u, v^2_u)B_u d\tilde{X}_u
$$

$$
+ \int_0^T e^{-ru} \partial_y A(u, \tilde{X}_u, v^2_u)B_u dv^2_u
$$

$$
+ \int_0^T e^{-ru} A(u, \tilde{X}_u, v^2_u)dB_u
$$

$$
+ \frac{1}{2} \int_0^T e^{-ru} \partial^2_x A(u, \tilde{X}_u, v^2_u)B_u d[X, \tilde{X}]_u
$$

$$
+ \frac{1}{2} \int_0^T e^{-ru} \partial^2_{x,y} A(u, \tilde{X}_u, v^2_u)B_u d[v^2, v^2]_u
$$

$$
+ \int_0^T e^{-ru} \partial^2_{x,y} A(u, \tilde{X}_u, v^2_u)B_u d[\tilde{X}, v^2]_u
$$
In the next step we apply the Feynman-Kac operator with volatility \( \nu_t \), alongside the definition of \( M_t \). After algebraic operations, we retrieve

\[
e^{-rT}A(T, \tilde{X}_T, v^2_T)B_T = A(0, \tilde{X}_0, v^2_0)B_0 + \int_0^T e^{-ru} \partial_x A(u, \tilde{X}_u, v^2_u)B_u(v^2_u - \sigma^2_u)du + \int_0^T e^{-ru} \partial_y A(u, \tilde{X}_u, v^2_u)B_u(\frac{1}{T-u})dM_u + \int_0^T e^{-ru} \partial_y A(u, \tilde{X}_u, v^2_u)B_u(\sigma_v u T - u)du + \int_0^T e^{-ru} \partial_y A(u, \tilde{X}_u, v^2_u)B_u d[Bu]_u.
\]

After applying expectations on both sides of the equation, we end up with the statement of the theorem. \( \square \)

4 A decomposition formula for SVJ models.

In the previous section, we have given a general decomposition formula that can be used for stochastic volatility models with continuous sample paths. In this section, we are going to extend the previous decomposition to the case of a general jump diffusion model with finite activity jumps. The main idea, like the one used in Merino and Vives (2017), is to adapt the pricing process in a way to be able to apply the decomposition technique effectively. In our case, this would translate into conditioning on the finite number of jumps \( n_T \). If we denote \( J_n = \sum_{i=0}^{n} Y_i \), using the integrability of Black-Scholes function, we can obtain the following conditioning formula for European options with payoff at maturity \( T \):\( BS(T, X_T, v_T) \).

\[
V_0 = e^{-rT}E[B_S(T, X_T, v_T)]
\]
\[
\begin{align*}
&= e^{-rT} \sum_{n=0}^{\infty} p_n(\lambda T) \mathbb{E} \left[ BS \left( T, \tilde{X}_T + \sum_{i=0}^{n_T} Y_i, v_T \right) \right]_{n_T = n} \\
&= e^{-rT} \sum_{n=0}^{\infty} p_n(\lambda T) \mathbb{E} \left[ BS \left( T, \tilde{X}_T + J_n, v_T \right) \right] \\
&= e^{-rT} \sum_{n=0}^{\infty} p_n(\lambda T) \mathbb{E} \left[ J_n \mathbb{E} \left[ BS(T, \tilde{X}_T + J_n, v_T) \right] \right] \\
&= e^{-rT} \sum_{n=0}^{\infty} p_n(\lambda T) \mathbb{E} \left[ G_n(T, \tilde{X}_T, v_T) \right],
\end{align*}
\]

where
\[
G_n(T, \tilde{X}_T, v_T) := \mathbb{E}_n \left[ BS(T, \tilde{X}_T + J_n, v_T) \right].
\]

We have switched our problem from a jump diffusion model with stochastic volatility to another one with no jumps. Combining the generic SV decomposition formula (from Theorem 3.1) and conditioning on the number of jumps we obtain a corner-stone for our approximation.

**Corollary 4.1 (SVJ decomposition formula).** Let \( X_t \) be a log-price process (2), \( G_n \) be the previously defined function. Then we can express the call option fair value \( V_0 \) using the Poisson mass function \( p_n \) and a martingale process \( M_t \) (defined by (5)). In particular,

\[
V_0 = \sum_{n=0}^{\infty} p_n(\lambda T) G_n(0, \tilde{X}_0, v_0) \\
+ \frac{1}{8} \sum_{n=0}^{\infty} p_n(\lambda T) \mathbb{E} \left[ \int_0^T e^{-ruT} g_n(u, \tilde{X}_u, v_u) dM_u \right] \\
+ \frac{\rho}{2} \sum_{n=0}^{\infty} p_n(\lambda T) \mathbb{E} \left[ \int_0^T e^{-ruT} \Lambda g_n(u, \tilde{X}_u, v_u) dW_u \right].
\]

**Proof.** We apply Theorem 3.1 to \( A(t, \tilde{X}_t, v_t^2) := G_n(t, \tilde{X}_t, v_t) \) and \( B_t \equiv 1 \). Note that

\[
\partial_x^2 BS(t, x, \sigma) = \frac{(T-t)}{2} \left( \partial_x^2 - \partial_x \right) BS(t, x, \sigma)
\]

and

\[
\partial_x^2 BS(t, x, \sigma) = \frac{(T-t)^2}{4} \left( \partial_x^2 - \partial_x \right)^2 BS(t, x, \sigma).
\]

Then, the corollary follows immediately. Note that in order to apply the Itô formula to function \( G_n \) we need to use a mollifier argument as it is done in Merino and Vives (2015). \( \square \)

**Remark 4.2.** For clarity, in the following we will refer to terms of the previous decomposition as

\[
V_0 = \sum_{n=0}^{\infty} p_n(\lambda T) G_n(0, \tilde{X}_0, v_0) + \sum_{n=0}^{\infty} p_n(\lambda T) [ (I_n) + (II_n) ].
\]

To compute the above expression can be cumbersome. The main idea is to find an alternative formula such that the main terms are easier to be computed while paying the price by having more terms in the formula. Fortunately, in many cases these new terms can be neglected as approximation error. The size of the error depends on the model and whether we are focusing on short or long time dynamics.

The following lemma is proved in Alós (2012), p. 406; and will help us to derive bounds on the error terms that appear in the main result of this paper - a computationally suitable decomposition formula for generic finite activity SVJ models.
Lemma 4.3. Let $0 \leq t \leq s \leq T$ and $\mathcal{G}_t := \mathcal{F}_t \vee \mathcal{F}_W^t$. For every $n \geq 0$, there exists $C = C(n)$ such that
\[
\left| \mathbb{E} \left( \Lambda^n \Gamma B S \left( s, \tilde{X}_s, v_s \right) \mid \mathcal{G}_t \right) \right| \leq C \left( \int_s^T E_s \left( \sigma_0^2 \right) \, d\theta \right)^{-\frac{1}{2}(n+1)}.
\]

Theorem 4.4 (Computationally suitable SVJ decomposition). Let $X_t$ be a log-price process (2) and $G_n$ be the previously defined function. Then we can express the call option fair value $V_0$ using the Poisson probability mass function $p_n$ and processes $R_t, U_t$ defined by (6) and (7), respectively. In particular,
\[
V_0 = \sum_{n=0}^{\infty} p_n(\lambda T) G_n(0, \tilde{X}_0, v_0)
+ \sum_{n=0}^{\infty} p_n(\lambda T) \Gamma^2 G_n(0, \tilde{X}_0, v_0) R_0
+ \sum_{n=0}^{\infty} p_n(\lambda T) \Lambda \Gamma G_n(0, \tilde{X}_0, v_0) U_0
+ \sum_{n=0}^{\infty} p_n(\lambda T) \Omega_n
\]
where $\Omega_n$ are error terms fully derived in Appendix A.1.

Proof. We use Theorem 3.1 iteratively for the following choices of $A(t, X_t, v_t^2)$:

(I):
\[
A(t, X_t, v_t^2) := \Gamma^2 G_n(t, \tilde{X}_t, v_t)
\]
and
\[
B_t := R_t = \frac{1}{8} \mathbb{E}_t \left[ \int_t^T d[M, M]_u \right].
\]

(II):
\[
A(t, X_t, v_t^2) := \Lambda \Gamma G_n(t, \tilde{X}_t, v_t)
\]
and
\[
B_t := U_t = \frac{\bar{\sigma}}{2} \mathbb{E}_t \left[ \int_t^T \sigma_u d[W, M]_u \right].
\]
and then the statement follows immediately. See also the terms in Appendix A.1.

Remark 4.5. In particular, we have a closed formula for a log-normal jump diffusion model (e.g. Bates (1996) SVJ model):
\[
G_n(0, \tilde{X}_0, v_0) = B S \left( 0, \tilde{X}_0, \sqrt{v_0^2 + n \frac{\sigma_T^2}{T}} \right)
\]
where we modified the risk-free rate used in the Black-Scholes formula to
\[
r^* = r - \lambda \left( e^{\mu J} + \frac{1}{2} \sigma_j^2 - 1 \right) + n \frac{\mu J + \frac{1}{2} \sigma_j^2}{T}.
\]
A very similar formula for the Merton case is deduced by Hanson (2007). More details will follow in the next sections. Under general (finite-activity) jump diffusion settings, we will need to solve
\[
\int_{\mathbb{R}} BS \left( 0, \tilde{X}_0 + y, v_0 \right) f_{J_n}(y) dy
\]
where \(f_{J_n} = (f_{J_n}^\circ)^n(y)\) is the convolution of the law of \(n\) jumps.

Here we provide a list of known results for various popular models.

- **Kou (2002) double exponential model:**
  \[
f^{(n)}(u) = e^{-q u} \sum_{k=1}^{n} P_{n,k} \eta_1^k \frac{1}{(k-1)!} u^{k-1} 1_{\{u \geq 0\}} + e^{-q u} \sum_{k=1}^{n} Q_{n,k} \eta_2^k \frac{1}{(k-1)!} (-u)^{k-1} 1_{\{u < 0\}}
\]
  where
  \[
P_{n,k} = \sum_{i=k}^{n-1} \left( \begin{array}{c} n-k-1 \\ i-k \end{array} \right) \left( \begin{array}{c} n \\ i \end{array} \right) \left( \begin{array}{c} \eta_1 \\ \eta_1 + \eta_2 \end{array} \right)^i \left( \begin{array}{c} \eta_2 \\ \eta_1 + \eta_2 \end{array} \right)^{n-i} p^i q^{n-i}
\]
  for all \(1 \leq k \leq n-1\), and
  \[
Q_{n,k} = \sum_{i=k}^{n-1} \left( \begin{array}{c} n-k-1 \\ i-k \end{array} \right) \left( \begin{array}{c} n \\ i \end{array} \right) \left( \begin{array}{c} \eta_1 \\ \eta_1 + \eta_2 \end{array} \right)^n \left( \begin{array}{c} \eta_2 \\ \eta_1 + \eta_2 \end{array} \right)^{i-k} p^{n-i} q^i
\]
  for all \(1 \leq k \leq n-1\). In addition, \(P_{n,n} = p^n\) and \(Q_{n,n} = q^n\).

- **Yan and Hanson (2006) model** uses log-uniform jump sizes and hence the density is of the form (Killmann and von Collani 2001):
  \[
f^{(n)}(u) = \begin{cases} \frac{b^{(n,a)}(-1)^{n} (u - na - (b-a))^n}{(n-1)! (b-a)^n} & \text{if } na \leq u \leq nb \\ 0 & \text{otherwise} \end{cases}
\]
  where \(\tilde{n}(n,u) := \left\lceil \frac{u - na}{b-a} \right\rceil \) is the largest integer less than \(\frac{u - na}{b-a}\).

5 **SVJ models of the Heston type**

In this section, we apply the previous generic results to derive a pricing formula for SVJ models with the Heston variance process. The aim is not to provide pricing solution for all known/studied models, but rather to detail the derivation for a selected model and comment on possible extension to different models. I.e. we focus on models with dynamics satisfying the following stochastic differential equations
\[
\begin{align*}
    dX_t &= \left( r - \lambda k - \frac{1}{2} \sigma_t^2 \right) dt + \sigma_t \left( \rho dW_t + \sqrt{1 - \rho^2} d\tilde{W}_t \right) + dJ_t \\
    d\sigma_t^2 &= \kappa (\theta - \sigma_t^2) dt + \nu \sqrt{\sigma_t^2} dW_t
\end{align*}
\]
where \(\sigma_0, \kappa, \theta, \nu\) are positive constants satisfying the Feller condition \(2\kappa\theta \geq \nu^2\). The process \(\sigma_t^2\) represents an instantaneous variance of the price at time \(t\), \(\theta\) is a long run average level of the variance, \(\kappa\) is a rate at which \(\sigma_t\) reverts to \(\theta\) and, last but not least, \(\nu\) is a volatility of volatility parameter. We will distinguish between the two cases:

- either jump amplitudes follow a Gaussian process (Bates (1996) model),
- or they are driven by other models, e.g. a log-uniform process (Yan and Hanson (2006) model).
5.1 Approximation of the SVJ models of the Heston type

For a standard Heston model, we have the following results, see Alòs, de Santiago, and Vives (2015):

**Lemma 5.1.** Assume the standard notation from the previous sections alongside specific definitions. Define \( \varphi(t) := \int_t^T e^{-\kappa(z-t)}dz \). We have the following results:

1. For \( s \geq t \) we have
   \[
   E_t(\sigma_s^2) = \theta + (\sigma_t^2 - \theta) e^{-\kappa(s-t)} = \sigma_t^2 e^{-\kappa(s-t)} + \theta(1 - e^{-\kappa(s-t)}),
   \]
   so, in particular, this quantity is bounded below by \( \sigma_t^2 \wedge \theta \) and above by \( \sigma_t^2 \vee \theta \).
2. \( E_t \left( \int_t^T \sigma_s^2 ds \right) = \theta (T-t) + \frac{\sigma_t^2 - \theta}{\kappa} \left( 1 - e^{-\kappa(T-t)} \right) \).
3. \( dM_t = \nu \sigma_t \left( \int_t^T e^{-\kappa(u-t)} du \right) dW_t = \frac{\kappa}{\nu} \sigma_t \left( 1 - e^{-\kappa(T-t)} \right) dW_t. \)
4. \( U_t := \frac{\rho}{2} E_t \left( \int_t^T \sigma_s d\langle M, W \rangle_s \right) = \frac{\rho}{4} \sigma_t^2 \left( \int_t^T e^{-\kappa(u-t)} du \right) ds \\
   = \frac{\rho \mu}{2 \kappa^2} \left\{ \frac{\theta \kappa (T-t) - 2 \theta + \sigma_t^2 + e^{-\kappa(T-t)} (2 \theta - \sigma_t^2) - \kappa (T-t) e^{-\kappa(T-t)} (\sigma_t^2 - \theta)}{1 - e^{-\kappa(T-t)}} \right\}.
   \)
5. \( R_t := \frac{1}{8} E_t \left( \int_t^T d\langle M, M \rangle_s \right) = \frac{1}{8} \nu^2 \left( \int_t^T E_t \left( \sigma_s^2 \right) \left( \int_t^T e^{-\kappa(u-t)} du \right) ds \right) \\
   = \frac{\nu^2}{8 \kappa^2} \left\{ \frac{\theta (T-t) + \frac{\sigma_t^2 - \theta}{\kappa} \left( 1 - e^{-\kappa(T-t)} \right)}{1 - e^{-2\kappa(T-t)}} \right\} \\
   - \frac{2 \theta}{\kappa} \left( 1 - e^{-\kappa(T-t)} \right) \left( \sigma_t^2 - \theta \right) (T-t) e^{-\kappa(T-t)} \\
   + \frac{\theta}{2 \kappa} \left( 1 - e^{-2\kappa(T-t)} \right) + \frac{\sigma_t^2 - \theta}{\kappa} \left( e^{-\kappa(T-t)} - e^{-2\kappa(T-t)} \right) \right\}.
   \)
6. \( dU_t = \frac{\omega^2}{4} \left( \int_t^T e^{-\kappa(z-t)} \varphi(z) dz \right) \sigma_t dW_t - \frac{\omega^2}{4} \varphi(t) \sigma_t^2 dt, \)
7. \( dR_t = \frac{\omega^2}{4} \left( \int_t^T e^{-\kappa(z-t)} \varphi(z)^2 dz \right) \sigma_t dW_t - \frac{\omega^2}{4} \varphi(t) \sigma_t^2 dt. \)

Furthermore, the following lemma is proved in Alòs, de Santiago, and Vives (2015).

**Lemma 5.2.** Let all the objects be well defined as above, then for a standard Heston model we have that

(i) \( \int_s^T E_s(\sigma_u^2) du \geq \frac{\theta \mu}{4} \left( \int_s^T e^{-\kappa(u-s)} du \right)^2, \)

(ii) \( \int_s^T E_s(\sigma_u^2) du \geq \sigma_s^2 \left( \int_s^T e^{-\kappa(u-s)} du \right). \)

**Remark 5.3.** We can utilize these equalities to get analogue results for Theorem 4.4. The \( \Omega_n \) terms can be founded in Appendix A.2.

Now we have all the tools needed to introduce the main practical result - pricing formula
Corollary 5.4 (Heston-type SVJ pricing formula). Let \( G_n(0, \tilde{X}_0, v_0) \) takes the expression as in Remark 4.5 for a particular jump-type setting, let

\[
R_0 = \frac{\nu^2}{8\kappa^2} \left\{ \frac{\theta}{\kappa} (1 - e^{-\kappa T}) - \frac{2\theta}{\kappa} (1 - e^{-\kappa T}) - 2 (\sigma_0^2 - \theta) T e^{-\kappa T} \right. \\
+ \left. \frac{\theta}{2\kappa} (1 - e^{-2\kappa T}) + \frac{(\sigma_0^2 - \theta)}{\kappa} (e^{-\kappa T} - e^{-2\kappa T}) \right\}
\]

and let

\[
U_0 = \frac{\rho\nu}{2\kappa^2} \left\{ \theta \kappa T - 2\theta + \sigma_0^2 + e^{-\kappa T} (2\theta - \sigma_0^2) - \kappa T e^{-\kappa T} (\sigma_0^2 - \theta) \right\}.
\]

Then the European option fair value is expressed as

\[
V_0 = \sum_{n=0}^{\infty} p_n(\lambda T) G_n(0, \tilde{X}_0, v_0)
+ \sum_{n=0}^{\infty} p_n(\lambda T) \Gamma^2 G_n(0, \tilde{X}_0, v_0) R_0
+ \sum_{n=0}^{\infty} p_n(\lambda T) \Lambda \Gamma G_n(0, \tilde{X}_0, v_0) U_0
+ \sum_{n=0}^{\infty} p_n(\lambda T) \Omega_n,
\]

where \( \Omega_n \) are error terms detailed in Appendix A.2. The upper bound for any \( \Omega_n \) is given by

\[
\Omega_n \leq \nu^2 (|\rho| + \nu)^2 \left( \frac{1}{T} \wedge (T-t) \right) \Pi(\kappa, \theta)
\]

where \( \Pi(\kappa, \theta) \) is a positive function. Therefore, the total error

\[
\Omega = \sum_{n=0}^{\infty} p_n(\lambda T) \Omega_n
\]

is bounded by the same constant.

Proof. We plug-in the Heston volatility model dynamics into Theorem 4.4. Using the integrability of the Black-Scholes function, Fubini Theorem and the fact that the upper bound of Lemma 4.3 does not depend on the log spot price, the upper bound can be used for every \( G_n \) function. Using Lemma 5.1 and Lemma 5.2 we prove the corollary. The whole proof is in Appendix A.3.

Remark 5.5 (Approximate fractional SVJ model). For the model introduced by Pospíšil and Sobotka (2016) one can derive a very similar decomposition as in Corollary 5.4. In fact, only the terms \( R_0 \) and \( U_0 \) have to be changed while the other terms remain the same.

5.2 Numerical analysis of the SVJ models of the Heston type

In this section, we compare the newly obtained approximation formula for option prices under Bates (1996) model (i.e. log-normal jump sizes alongside Heston model’s instantaneous variance) with the market standard approach for pricing European options under SVJ models - the Fourier-transform based pricing formula. The comparison is performed with two important aspects in mind:
Call option prices under Bates (1996) model

![Graph of call option prices]

| Strike price | Option price |
|--------------|-------------|
| 50           | 60          |
| 100          | 50          |
| 150          | 40          |

| Strike price | Absolute errors in log_{10} scale |
|--------------|----------------------------------|
| 50           | 10^{-4}                           |
| 100          | 10^{-5}                           |
| 150          | 10^{-6}                           |

Figure 1: Approximation and reference prices for \( \rho = -0.2, \nu = 5\% \) and \( \tau = 0.3 \).

- practical precision of the pricing formula when neglecting the total error term \( \Omega \),
- efficiency of the formula expressed in terms of the computational time needed for particular pricing tasks.

In particular, we utilize a semi-closed form solution with one numerical integration as a reference price (Baustian, Mrázek, Pospíšil, and Sobotka 2017) alongside a classical solution derived by Bates (1996). The numerical integration errors according to Baustian, Mrázek, Pospíšil, and Sobotka (2017) should be typically well beyond \( 10^{-10} \), hence we can take the numerically computed prices as the reference prices for the comparison.

Due to the theoretical properties of the total error term \( \Omega \), we illustrate the approximation quality for several values of \( \rho \) and \( \nu \) while keeping other parameters fixed.

In Figure 1, we inspect a mode of low volatility of the spot variance \( \nu \) and low absolute value of the instantaneous correlation \( \rho \) between the two Brownian motions. The errors for an option price smile that corresponds to \( \tau = 0.3 \) are within \( 10^{-4} - 10^{-6} \) range, while slightly better absolute errors were obtained at-the-money. Increasing either the absolute value of \( \rho \) or volatility \( \nu \) should, in theory, worsen the computed error measures. However, if only one of the values is increased we are still able to keep the errors below \( 10^{-3} \) in most of the cases, see Figure 2.

Last but not least, we illustrate the approximation quality for parameters that are not well suited for the approximation. This is done by setting \( \nu = 50\% \), correlation \( \rho = -0.8 \) and a smile with respect to \( \tau = 3 \). The obtained errors are depicted by Figure 3. Despite the values of parameters, the shape of the option price curve remains fairly similar to the one obtained by a more precise semi-closed formula.

Main advantage of the proposed pricing approximation lies in its computational efficiency – which might be advantageous for many tasks in quantitative finance that need fast evaluation of derivative prices. To inspect the time consumption we set up three pricing tasks. We use

1With a slight modification mentioned in Gatheral (2006) to not suffer the "Heston trap" issues.
2The considered model and market parameters take the following values: \( S_0 = 100; r = 0.001; \tau = 0.3; \nu_0 = 0.25; \kappa = 1.5; \theta = 0.2; \lambda = 0.05; \mu_J = -0.05; \sigma_J = 0.5 \).
Figure 2: Approximation and reference prices for $\rho = -0.8$, $\nu = 5\%$ and $\tau = 0.3$.

6 The approximated implied volatility surface for SVJ models of the Heston type

In the above section, we have computed a bound for the error between the exact price and the approximated pricing formula for the SVJ models of the Heston type. Now, we are going to derive an approximation of the implied volatility surface alongside the corresponding ATM implied volatility profiles. These approximations can help us to understand the volatility dynamics of studied models in a better way.

---

It includes OTM, ATM, ITM options with short-, mid- and long-term times to maturities

For both Baustian, Mrázek, Pospíšil, and Sobotka (2017) and Gatheral (2006) formulas we use an adaptive Gauss-Kronrod(7,15) quadrature.
6.1 Deriving an approximated implied volatility surface for SVJ models of the Heston type

The price of an European call option with strike $K$ and maturity $T$ is an observable quantity which will be referred to as $P_{obs}^0 = P_{obs}(K, T)$. Recall that the implied volatility is defined as the value $I(T, K)$ that satisfies

$$BS(0, S_0, I(T, K)) = P_{obs}. $$

Using the results from the previous section, we are going to derive an approximation to the implied volatility as in Fouque, Papanicolaou, Sircar, and Solna (2003). We use the idea to expand the implied volatility function $I(T, K)$ with respect to two scales. For illustration of the idea, we recall that according to asymptotic sequences $\{\delta_k\}_{k=0}^\infty, \{\epsilon_k\}_{k=0}^\infty$ converging to 0 are considered. Thus, we can write

$$f = f_{0,0} + \delta f_{1,0} + \epsilon f_{0,1} + O((\delta + \epsilon)^2),$$

for a particular function $f$. Let $\epsilon = \rho \nu$ and $\delta = \nu^2$, then we expand $I(T, K)$ with respect to these two scales as

$$I(T, K) = v_0 + \rho \nu I_1(T, K) + \nu^2 I_2(T, K) + O((\rho \nu + \nu^2)).$$

We will denote by $\hat{I}(T, K) = v_0 + \rho \nu I_1(T, K) + \nu^2 I_2(T, K)$ the approximation to the implied volatility and by $\hat{V}(0, x, v_0)$ the approximation to the option price which was obtained in Corollary 5.4. We know that according to Corollary 5.4:

$$\hat{V}(0, x, v_0) = \sum_{n=0}^\infty p_n(\lambda T)BS(0, x + J_n, v_0)$$

$$+ \sum_{n=0}^\infty p_n(\lambda T)\Gamma^2 BS(0, x + J_n, v_0)R_0$$

$$+ \sum_{n=0}^\infty p_n(\lambda T)\Lambda BS(0, x + J_n, v_0)U_0.$$
Table 1: Efficiency of the Bates SVJ pricing formulas

| Pricing approach | Task     | Time [sec] | Speed-up factor |
|------------------|----------|------------|-----------------|
| Approximation formula | #1      | 0.97       | 3.23×           |
|                   | #2      | 10.03      | 2.94×           |
|                   | #3      | 99.67      | 2.83×           |
| Baustian, Mrázek, Pospůšil, and Sobotka (2017) | #1      | 2.09       | 1.52×           |
|                   | #2      | 17.28      | 1.71×           |
|                   | #3      | 135.95     | 2.01×           |
| Gatheral (2006)   | #1      | 3.18       | -               |
|                   | #2      | 29.48      | -               |
|                   | #3      | 281.72     | -               |

† The results were obtained on a PC with Intel Core i7-6500U CPU and 8 GB RAM.

To simplify the notation, we define

\[ \gamma_n := \frac{d_2^2(x, r, \sigma) - d_2^2(x + J_n, r, \sigma)}{2} \]

and

\[ D_1(x, J_n, \sigma, T) := \mathbb{E}_{J_n} \left[ e^{J_n + \gamma_n} \left( 1 - \frac{d_+^2(x + J_n, r, \sigma)}{\sigma \sqrt{T}} \right) \right] \]

\[ D_2(x, J_n, \sigma, T) := \mathbb{E}_{J_n} \left[ e^{J_n + \gamma_n} \left( d_2^2(x + J_n, r, \sigma) - \sigma d_+(x + J_n, r, \sigma) \sqrt{T} - 1 \right) \right] \]

Using the fact that

\[ \partial_{\sigma} BS(t, x, \sigma) = \frac{e^x e^{-d_1^2(t, \sigma)/2} \sqrt{T-t}}{\sqrt{2\pi}} \]

we can re-write the approximated price as

\[ \hat{V}(0, x, v_0) = \sum_{n=0}^{\infty} p_n(\lambda T) BS(0, x + J_n, v_0) + \partial_{\sigma} BS(v_0) \sum_{n=0}^{\infty} p_n(\lambda T) D_1(x, J_n, \sigma, T) U_0 + \partial_{\sigma} BS(v_0) \sum_{n=0}^{\infty} p_n(\lambda T) D_2(x, J_n, v_0, T) R_0. \]

where we write \( BS(v_0) \) as a shorthand for \( BS(0, x, I(T, K)) \) around \( v_0 \):

\[ BS(0, x, I(T, K)) = BS(v_0) + \partial_{\sigma} B(v_0)(\rho v I_1(T, K) + \nu^2 I_2(T, K) + \cdots) + \frac{1}{2} \partial_{\sigma}^2 BS(v_0)(\rho v I_1(T, K) + \nu^2 I_2(T, K) + \cdots)^2 + \cdots \]

Noticing that

\[ BS(v_0) = \sum_{n=0}^{\infty} p_n(\lambda T) BS(0, x + J_n, v_0) \]
and equating
\[ \hat{V}(0, x, v_0) = BS(0, x, I(T, K)), \]
we obtain
\begin{align*}
\hat{I}_1(T, K) &:= \rho I_1(T, K) = U_0 \sum_{n=0}^{\infty} p_n(\lambda T) D_1(x, J_n, v_0, T), \quad (10) \\
\hat{I}_2(T, K) &:= \nu^2 I_2(T, K) = R_0 \sum_{n=0}^{\infty} p_n(\lambda T) D_2(x, J_n, v_0, T).
\end{align*}

Hence, we have the following approximation of implied volatility
\begin{align*}
\hat{I}(T, K) &= v_0 + U_0 \sum_{n=0}^{\infty} p_n(\lambda T) D_1(x, J_n, v_0, T) \\
&\quad + R_0 \sum_{n=0}^{\infty} p_n(\lambda T) D_2(x, J_n, v_0, T).
\end{align*}

In particular, when we look at the ATM curve, we have that
\[ I_{ATM}(T) = v_0 + U_0 \sum_{n=0}^{\infty} p_n(\lambda T) E_{J_n} \left[ e^{J_n + \gamma n} \frac{1}{v_0 T} \left( \frac{1}{2} - J_n T v_0^2 \right) \right] - R_0 \sum_{n=0}^{\infty} p_n(\lambda T) E_{J_n} \left[ e^{J_n + \gamma n} \frac{1}{v_0 T^2} \left( 1 + 1 + J_n^2 v_0^4 T^2 \right) \right]. \]

**Remark 6.1.** When \( T \) converges to 0, the dynamics of the model is the same as in the Heston model. This is due to the behavior of the Poisson process when \( T \downarrow 0 \).

### 6.2 Deriving an approximated implied volatility surface for Bates model

The Bates model is a particular example of SVJ model of the Heston type. The fact that jumps are also log-normal makes the model more tractable. In this section, we will adapt the generic formulas to this particular case. In this model, after each jump, the drift- and volatility-like parameters will change. We define
\[ \tilde{v}_0^{(n)} = \sqrt{v_0^2 + \frac{n \sigma^2_J}{T}} \]
as the new volatility and
\[ \tilde{r}_n = r - \lambda \left( e^{\mu_J + \frac{1}{2} \sigma^2_J} - 1 \right) + n \frac{\mu_J + \frac{1}{2} \sigma^2_J}{T} \]
as the new drift. The parameter \( n \) is the number of realized jumps, \( \mu_J \) and \( \sigma_J \) are the jump-size parameters and \( \lambda \) is the jump intensity. For simplicity, we denote:
\[ c_n := -\lambda \left( e^{\mu_J + \frac{1}{2} \sigma^2_J} - 1 \right) + n \frac{\mu_J + \frac{1}{2} \sigma^2_J}{T}. \]

As a consequence, we have that
\[ d_\pm \left( x, \tilde{r}_n, \tilde{v}_0^{(n)} \right) = \frac{x - \ln K + \tilde{r}_n T}{\tilde{v}_0^{(n)} \sqrt{T}} \pm \frac{\tilde{v}_0^{(n)} \sqrt{T}}{2}. \]

Following the steps done in the generic formula, we can define the variables
\[ D_{B,1} \left( x, \tilde{r}_n, \tilde{v}_0^{(n)}, T \right) = \frac{\tilde{v}_0^{(n)} \sqrt{T}}{2} \left( 1 - \frac{d_+ \left( x, \tilde{r}_n, \tilde{v}_0^{(n)} \right)}{\tilde{v}_0^{(n)} \sqrt{T}} \right), \]

17
\[
D_{B,2}(x, \tilde{r}_n, \tilde{v}_b^{(n)}, T) = \frac{e^{\gamma_n}}{\tilde{v}_0^n T} \left( \frac{d_+^2(x, \tilde{r}_n, \tilde{v}_b^{(n)}) - \tilde{v}_0^n d_+^2(x, \tilde{r}_n, \tilde{v}_0^{(n)})}{\sqrt{T} - 1} \right).
\]

It follows that
\[
\hat{I}_{B,1}(T, K) = \rho \nu I_{B,1}(T, K) = U_0 \sum_{n=0}^{\infty} p_n(\lambda T) D_{B,1}(x, \tilde{r}_n, \tilde{v}_b^{(n)}, T),
\]
\[
\hat{I}_{B,2}(T, K) = \nu^2 I_{B,2}(T, K) = R_0 \sum_{n=0}^{\infty} p_n(\lambda T) D_{B,2}(x, \tilde{r}_n, \tilde{v}_b^{(n)}, T).
\]

The approximation of the implied volatility surface has the following shape
\[
\hat{I}_B(T, K) = v_0 + U_0 \sum_{n=0}^{\infty} p_n(\lambda T) \frac{e^{\gamma_n}}{\tilde{v}_0^n T} \left( 1 - \frac{d_+^2(x, \tilde{r}_n, \tilde{v}_b^{(n)})}{\sqrt{T}} \right)
+ R_0 \sum_{n=0}^{\infty} p_n(\lambda T) \frac{e^{\gamma_n}}{\tilde{v}_0^n T} \left( \frac{d_+^2(x, \tilde{r}_n, \tilde{v}_b^{(n)}) - \tilde{v}_0^n d_+^2(x, \tilde{r}_n, \tilde{v}_0^{(n)})}{\sqrt{T} - 1} \right).
\]

In particular, the ATM implied volatility curve under the studied model takes the form:
\[
\hat{I}_{B,1}^{\text{ATM}}(T) = v_0 + U_0 \sum_{n=0}^{\infty} p_n(\lambda T) \frac{e^{\gamma_n}}{\tilde{v}_0^n T} \left( \frac{1}{2} - \frac{\epsilon_n}{\tilde{v}_0^n} \right)
- R_0 \sum_{n=0}^{\infty} p_n(\lambda T) \frac{e^{\gamma_n}}{\tilde{v}_0^n T} \left( \frac{1}{4} + \frac{1}{\tilde{v}_0^n T} - \frac{\epsilon_n^2}{\tilde{v}_0^n} \right)
\]

where
\[
\gamma_n^{\text{ATM Bates}} = -\frac{1}{2} \left( c_n T + \frac{\epsilon_n^2 T}{\tilde{v}_0^n} \right).
\]

### 6.3 Numerical analysis of the approximation of the implied volatility for the Bates case

In the previous section we have compared the approximation and semi-closed form formulas for option prices under Bates (1996) model. For this model, we also illustrate the approximation quality in terms of implied volatilities.

Because there is no exact closed formula for implied volatilities under the studied model, we take as a reference price the one obtained by means of the complex Fourier transform (Baustian, Mrázek, Pospíšil, and Sobotka 2017). Once we have computed the prices we use a numerical inversion to obtain the desired implied volatilities.

As previously, we start by comparing implied volatilities for well-suited parameter sets. The illustration in Figure 4 is obtained by setting \( \rho = -0.1 \), \( \nu = 5\% \) and other parameters as in Section 5.2. Typically, for a well-suited parameter set, the absolute approximation errors stay within the range \( 10^{-5} - 10^{-7} \). Even for not entirely well-suited parameters we are able to obtain reasonable errors especially for ATM options, see Figures 5 and 6. In the mode of high volatility \( \nu \) of the variance process and high absolute value of the instantaneous correlation \( \rho \), the curvature of the smile is not fully captured. However, the errors are typically well below \( 10^{-2} \) even in this adverse setting.
Figure 4: Approximation and reference implied volatilities for $\rho = -0.2$, $\nu = 5\%$ and $\tau = 0.3$.

Figure 5: Approximation and reference implied volatilities for $\rho = -0.8$, $\nu = 5\%$ and $\tau = 0.3$. 
The aim of the paper was to derive a generic decomposition formula for SVJ option pricing models with finite activity jumps. In Section 4 we derived this decomposition by extending the results obtained by Alòs (2012) for Heston (1993) SV model. Newly obtained decomposition is rather versatile since it does not need to specify the underlying volatility process and only common integrability and specific sample path properties are required.

Particular approximation formulas for several SVJ models were presented in Section 5 together with the numerical comparison for the Bates (1996) model for which we showed that the newly proposed approximation is typically three times faster compared to the classical two integral semi-closed pricing formula. Moreover, its computational time does not depend on the model parameters nor on market data. The biggest advantage of the proposed pricing approximation therefore lies in its computational efficiency, which is advantageous for many tasks in quantitative finance such as calibration to real market data that can lead to an extensive number of formula evaluations for SVJ models. On the other hand, general decomposition formula allowed us to understand the key terms contributing to the option fair value under specific models and hence this theoretical result has also its practical impact.

In Section 6, we have obtained an approximated volatility surface under SVJ models and we provided a boundary case simplification for ATM options. In particular, we have studied the approximation in the Bates (1996) model case. A numerical comparison of this approximation is also presented.

Although the generic approach covers various interesting SVJ models, there are other models that do not fit into the general structure described in Section 2. For these models, such as Barndorff-Nielsen and Shephard (2001) model or infinite activity jumps models, we still might be able to derive a similar decomposition, that was beyond the scope of the present paper. Newly obtained results therefore give suggestions on how to derive approximation formulas for other models.
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A Appendices

In the following appendices we obtain the error terms of the decomposition in Theorem 4.4 (Appendix A.1), the same formulas for the SVJ model of the Heston type (Appendix A.2) and upper bounds for those terms using Corollary 5.4 (Appendix A.3).

A.1 Decomposition formulas in the general model

In this section, we obtain the error terms for a general model.

A.1.1 Decomposition of the term \((I_n)\)

The term I can be decomposed by

\[
\frac{1}{8} \mathbb{E} \left[ \int_0^T e^{-ru} \Gamma^2 G_n(u, \tilde{X}_u, v_u) d[M, M]_u \right] - \Gamma^2 G_n(0, \tilde{X}_0, v_0) R_0
\]

\[
= \frac{1}{8} \mathbb{E} \left[ \int_0^T e^{-ru} \Gamma^4 G_n(u, \tilde{X}_u, v_u) R_u d[M, M]_u \right]
\]

\[
+ \frac{\rho}{2} \mathbb{E} \left[ \int_0^T e^{-ru} \Gamma^3 G_n(u, \tilde{X}_u, v_u) R_u \sigma_u d[W, M]_u \right]
\]

\[
+ \rho \mathbb{E} \left[ \int_0^T e^{-ru} \Gamma^2 G_n(u, \tilde{X}_u, v_u) \sigma_u d[W, R]_u \right]
\]

\[
+ \frac{1}{2} \mathbb{E} \left[ \int_0^T e^{-ru} \Gamma^3 G_n(u, \tilde{X}_u, v_u) d[M, R]_u \right].
\]

A.1.2 Decomposition of the term \((II_n)\)

The term II can be decomposed by

\[
\frac{\rho}{2} \mathbb{E} \left[ \int_0^T e^{-ru} \Gamma G_n(u, \tilde{X}_u, v_u) \sigma_u d[W, M]_u \right] - \Gamma G_n(0, \tilde{X}_0, v_0) U_0
\]

\[
= \frac{1}{8} \mathbb{E} \left[ \int_0^T e^{-ru} \Gamma^3 G_n(u, \tilde{X}_u, v_u) R_u d[M, M]_u \right]
\]

\[
+ \frac{\rho}{2} \mathbb{E} \left[ \int_0^T e^{-ru} \Gamma^2 G_n(u, \tilde{X}_u, v_u) R_u \sigma_u d[W, M]_u \right]
\]

\[
+ \rho \mathbb{E} \left[ \int_0^T e^{-ru} \Gamma G_n(u, \tilde{X}_u, v_u) \sigma_u d[W, U]_u \right]
\]

\[
+ \frac{1}{2} \mathbb{E} \left[ \int_0^T e^{-ru} \Gamma^2 G_n(u, \tilde{X}_u, v_u) d[M, U]_u \right].
\]
A.2 Decomposition formulas in the general model for the SVJ models of the Heston type

In this section, we obtain the error terms for the SVJ models of the Heston type.

A.2.1 Decomposition of the term ($I_n$) in the SVJ models of the Heston type

The term $I$ can be decomposed by

$$1/8 \mathbb{E} \left[ \int_0^T e^{-ru} \Gamma^2 G_n(u, \tilde{X}_u, v_u) d[M, M]_u \right] - \nu^2/8 \Gamma^2 G_n(0, \tilde{X}_0, v_0) \left( \int_0^T \mathbb{E} (\sigma_u^2) \varphi(s)^2 ds \right) \]

$$

$$= \nu^4/64 \mathbb{E} \left[ \int_0^T e^{-ru} \Gamma^4 G_n(u, \tilde{X}_u, v_u) \left( \int_u^T \mathbb{E} (\sigma_s^2) \varphi(s)^2 ds \right) \sigma_u^2 \varphi(u) du \right] + \rho \nu^3/16 \mathbb{E} \left[ \int_0^T e^{-ru} \Lambda \Gamma^3 G_n(u, \tilde{X}_u, v_u) \left( \int_u^T \mathbb{E} (\sigma_s^2) \varphi(s)^2 ds \right) \sigma_u^2 \varphi(u) du \right] + \rho^2 \nu^2/4 \left[ \int_0^T e^{-ru} \Lambda^2 \Gamma^2 G_n(u, \tilde{X}_u, v_u) \left( \int_u^T \mathbb{E} (\sigma_s^2) \varphi(s)^2 ds \right) \sigma_u^2 du \right] + \nu^4/16 \mathbb{E} \left[ \int_0^T e^{-ru} \Gamma^3 G_n(u, \tilde{X}_u, v_u) \left( \int_u^T e^{-\kappa(z-u)} \varphi(z)^2 dz \right) \sigma_u^2 \varphi(u) du \right].$$

A.2.2 Decomposition of the term ($II_n$) in the SVJ models of the Heston type

The term II can be decomposed by

$$\frac{\rho}{2} \mathbb{E} \left[ \int_0^T e^{-ru} \Lambda G_n(u, \tilde{X}_u, v_u) \sigma_u d[W, M]_u \right] - \frac{\rho \nu}{2} \Lambda G_n(0, \tilde{X}_0, v_0) \left( \int_0^T \mathbb{E} (\sigma_u^2) \varphi(s) ds \right)$$

$$= \frac{\rho \nu^3}{16} \left[ \int_0^T e^{-ru} \Lambda \Gamma^3 G_n(u, \tilde{X}_u, v_u) \left( \int_u^T \mathbb{E} (\sigma_s^2) \varphi(s) ds \right) \sigma_u^2 \varphi(u)^2 du \right] + \rho^2 \nu^2/4 \left[ \int_0^T e^{-ru} \Lambda^2 \Gamma^2 G_n(u, \tilde{X}_u, v_u) \left( \int_u^T \mathbb{E} (\sigma_s^2) \varphi(s) ds \right) \sigma_u^2 \varphi(u) du \right] + \rho^2 \nu^3/8 \left[ \int_0^T e^{-ru} \Lambda^2 \Gamma G_n(u, \tilde{X}_u, v_u) \left( \int_u^T e^{-\kappa(z-u)} \varphi(z) dz \right) \sigma_u^2 du \right] + \frac{\rho \nu^3}{4} \left[ \int_0^T e^{-ru} \Lambda^2 \Gamma^2 G_n(u, \tilde{X}_u, v_u) \left( \int_u^T e^{-\kappa(z-u)} \varphi(z) dz \right) \sigma_u^2 \varphi(u) du \right].$$

A.3 Upper-Bound of decomposition formulas in the SVJ models of the Heston type

In this section, we obtain the upper-bounds for the SVJ models of the Heston type.

A.3.1 Upper-Bound of the term ($I_n$) in the SVJ models of the Heston type

We can re-write the decomposition formula as

$$\frac{1}{8} \mathbb{E} \left[ \int_0^T e^{-r(u-t)} \Gamma^2 G_n(u, \tilde{X}_u, v_u) d[M, M]_u \right] - \frac{\nu^2}{8} \Gamma^2 G_n(0, \tilde{X}_0, v_0) \left( \int_0^T \mathbb{E} (\sigma_u^2) \varphi(s)^2 ds \right)$$

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\[ \begin{align*}
&= \frac{\nu^4}{64} \mathbb{E} \left[ \int_0^T e^{-ru} \left( \partial_x^6 - 3 \partial_x^5 + 3 \partial_x^2 - \partial_x \right) \Gamma G_n(u, \bar{X}_u, v_u) \left( \int_u^T \mathbb{E}_u \left( \sigma_u^2 \right)^2 ds \right) \sigma_u^2 \varphi^2(u) du \right] \\
&+ \frac{\nu^2}{16} \mathbb{E} \left[ \int_0^T e^{-ru} \left( \partial_x^5 - 2 \partial_x^3 + \partial_x \right) \Gamma G_n(u, \bar{X}_u, v_u) \left( \int_u^T \mathbb{E}_u \left( \sigma_u^2 \right) \varphi(s) ds \right) \sigma_u^2 \varphi(u) du \right] \\
&+ \frac{\nu^2}{8} \mathbb{E} \left[ \int_0^T e^{-ru} \left( \partial_x^4 - \partial_x^2 \right) \Gamma G_n(u, \bar{X}_u, v_u) \left( \int_u^T e^{-\kappa(z-u)} \varphi(z)^2 dz \right) \sigma_u^2 \right] \\
&+ \frac{\nu^4}{16} \mathbb{E} \left[ \int_0^T e^{-ru} \left( \partial_x^4 - 2 \partial_x^2 + \partial_x \right) \Gamma G_n(u, \bar{X}_u, v_u) \left( \int_u^T e^{-\kappa(z-u)} \varphi(z)^2 dz \right) \varphi(u) \sigma_u^2 du \right].
\end{align*} \]

Applying Lemma 4.3 and defining \( a_u := v_u \sqrt{T - u} \), we obtain

\[ \begin{align*}
&\leq \frac{\nu^4}{64} \mathbb{E} \left[ \int_0^T e^{-ru} \Gamma^2 G_n(u, \bar{X}_u, v_u) d[M, M]_u \right] - \frac{\nu^2}{8} \Gamma^2 G_n(0, \bar{X}_0, v_0) \left( \int_0^T \mathbb{E}_u \left( \sigma_u^2 \right)^2 ds \right) \\
&+ \frac{\nu^2}{16} \mathbb{E} \left[ \int_0^T e^{-ru} \left( \partial_x^6 - 3 \partial_x^5 + 3 \partial_x^2 - \partial_x \right) \Gamma^2 G_n(u, \bar{X}_u, v_u) \left( \int_u^T \mathbb{E}_u \left( \sigma_u^2 \right)^2 ds \right) \sigma_u^2 \varphi^2(u) du \right] \\
&+ \frac{\nu^2}{8} \mathbb{E} \left[ \int_0^T e^{-ru} \left( \partial_x^5 - 2 \partial_x^3 + \partial_x \right) \Gamma^2 G_n(u, \bar{X}_u, v_u) \left( \int_u^T \mathbb{E}_u \left( \sigma_u^2 \right) \varphi(s) ds \right) \sigma_u^2 \varphi(u) du \right] \\
&+ \frac{\nu^4}{16} \mathbb{E} \left[ \int_0^T e^{-ru} \left( \partial_x^4 - \partial_x^2 \right) \Gamma^2 G_n(u, \bar{X}_u, v_u) \left( \int_u^T e^{-\kappa(z-u)} \varphi(z)^2 dz \right) \sigma_u^2 \right] \\
&+ \frac{\nu^4}{16} \mathbb{E} \left[ \int_0^T e^{-ru} \left( \partial_x^4 - 2 \partial_x^2 + \partial_x \right) \Gamma^2 G_n(u, \bar{X}_u, v_u) \left( \int_u^T e^{-\kappa(z-u)} \varphi(z)^2 dz \right) \varphi(u) \sigma_u^2 du \right].
\end{align*} \]

Now, using Lemma 5.2 (ii), we have

\[ \begin{align*}
&\leq \frac{\nu^4}{64} \mathbb{E} \left[ \int_0^T e^{-ru} \Gamma^2 G_n(u, \bar{X}_u, v_u) d[M, M]_u \right] - \frac{\nu^2}{8} \Gamma^2 G_n(0, \bar{X}_0, v_0) \left( \int_0^T \mathbb{E}_u \left( \sigma_u^2 \right)^2 ds \right) \\
&+ \frac{\nu^2}{16} \mathbb{E} \left[ \int_0^T e^{-ru} \left( \partial_x^6 - 3 \partial_x^5 + 3 \partial_x^2 - \partial_x \right) \Gamma^2 G_n(u, \bar{X}_u, v_u) \left( \int_u^T \mathbb{E}_u \left( \sigma_u^2 \right)^2 ds \right) \sigma_u^2 \varphi^2(u) du \right] \\
&+ \frac{\nu^2}{8} \mathbb{E} \left[ \int_0^T e^{-ru} \left( \partial_x^5 - 2 \partial_x^3 + \partial_x \right) \Gamma^2 G_n(u, \bar{X}_u, v_u) \left( \int_u^T \mathbb{E}_u \left( \sigma_u^2 \right) \varphi(s) ds \right) \sigma_u^2 \varphi(u) du \right] \\
&+ \frac{\nu^4}{8} \mathbb{E} \left[ \int_0^T e^{-ru} \left( \partial_x^4 - \partial_x^2 \right) \Gamma^2 G_n(u, \bar{X}_u, v_u) \left( \int_u^T e^{-\kappa(z-u)} \varphi(z)^2 dz \right) \sigma_u^2 \right] \\
&+ \frac{\nu^4}{16} \mathbb{E} \left[ \int_0^T e^{-ru} \left( \partial_x^4 - 2 \partial_x^2 + \partial_x \right) \Gamma^2 G_n(u, \bar{X}_u, v_u) \left( \int_u^T e^{-\kappa(z-u)} \varphi(z)^2 dz \right) \varphi(u) \sigma_u^2 du \right].
\end{align*} \]

Finally, applying Lemma 5.2 (i), we find that

\[ \begin{align*}
&\leq \frac{\nu^4}{64} \mathbb{E} \left[ \int_0^T e^{-ru} \Gamma^2 G_n(u, \bar{X}_u, v_u) d[M, M]_u \right] - \frac{\nu^2}{8} \Gamma^2 G_n(0, \bar{X}_0, v_0) \left( \int_0^T \mathbb{E}_u \left( \sigma_u^2 \right)^2 ds \right) \\
&+ \frac{\nu^2}{16} \mathbb{E} \left[ \int_0^T e^{-ru} \left( \partial_x^6 - 3 \partial_x^5 + 3 \partial_x^2 - \partial_x \right) \Gamma^2 G_n(u, \bar{X}_u, v_u) \left( \int_u^T \mathbb{E}_u \left( \sigma_u^2 \right)^2 ds \right) \sigma_u^2 \varphi^2(u) du \right] \\
&+ \frac{\nu^2}{8} \mathbb{E} \left[ \int_0^T e^{-ru} \left( \partial_x^5 - 2 \partial_x^3 + \partial_x \right) \Gamma^2 G_n(u, \bar{X}_u, v_u) \left( \int_u^T \mathbb{E}_u \left( \sigma_u^2 \right) \varphi(s) ds \right) \sigma_u^2 \varphi(u) du \right] \\
&+ \frac{\nu^4}{16} \mathbb{E} \left[ \int_0^T e^{-ru} \left( \partial_x^4 - \partial_x^2 \right) \Gamma^2 G_n(u, \bar{X}_u, v_u) \left( \int_u^T e^{-\kappa(z-u)} \varphi(z)^2 dz \right) \sigma_u^2 \right] \\
&+ \frac{\nu^4}{16} \mathbb{E} \left[ \int_0^T e^{-ru} \left( \partial_x^4 - 2 \partial_x^2 + \partial_x \right) \Gamma^2 G_n(u, \bar{X}_u, v_u) \left( \int_u^T e^{-\kappa(z-u)} \varphi(z)^2 dz \right) \varphi(u) \sigma_u^2 du \right].
\end{align*} \]
We can re-write the decomposition formula as
\[ \Pi = \int_0^T e^{-ru} \left( \frac{2}{\theta K} + \frac{\sqrt{2}}{\kappa \sqrt{\theta K}} \right) du \]
\[ + \frac{\nu^2}{8} \Gamma^2 G_n(0, \bar{X}_0, v_0) \left( \int_0^T E (\sigma^2) \varphi(s)^2 ds \right) \]
\[ + \frac{\nu^2}{16} \Gamma^2 G_n(0, \bar{X}_0, v_0) \left( \int_0^T E (\sigma^2) \varphi(s)^2 ds \right) \]

Then we have that
\[ \left| \frac{1}{8} \mathbb{E} \left[ \int_0^T e^{-ru} \Gamma^2 G_n(u, \bar{X}_u, v_u) d[M, M]_u \right] \right| - \frac{\nu^2}{8} \Gamma^2 G_n(0, \bar{X}_0, v_0) \left( \int_0^T E (\sigma^2) \varphi(s)^2 ds \right) \]
\[ \leq \frac{\nu^3}{16} \left( \frac{2 + 2\sqrt{2}}{\theta K \sqrt{\theta K} + \frac{3\sqrt{2}}{\kappa^2 \sqrt{\theta K}} + \frac{1}{\kappa^3}} \right) \left( \int_0^T e^{-ru} du \right) \]
\[ + \frac{\nu^3}{8} \left( \frac{2}{\theta K \sqrt{\theta K} + \frac{1}{\kappa^3}} \right) \left( \int_0^T e^{-ru} du \right) \]
\[ + \frac{\nu^3}{16} \left( \frac{2 + 2\sqrt{2}}{\theta K \sqrt{\theta K} + \frac{1}{\kappa^3}} \right) \left( \int_0^T e^{-ru} du \right) \]
\[ + \frac{\nu^3}{16} \left( \frac{2 + 2\sqrt{2}}{\theta K \sqrt{\theta K} + \frac{1}{\kappa^3}} \right) \left( \int_0^T e^{-ru} du \right) \]

Using the fact that \( \int_0^T e^{-ru} du \leq \frac{1}{r} \wedge T \), we conclude that
\[ \left| \frac{1}{8} \mathbb{E} \left[ \int_0^T e^{-ru} \Gamma^2 G_n(u, \bar{X}_u, v_u) d[M, M]_u \right] \right| - \frac{\nu^2}{8} \Gamma^2 G_n(0, \bar{X}_0, v_0) \left( \int_0^T E (\sigma^2) \varphi(s)^2 ds \right) \]
\[ \leq \nu^3 \left( \frac{1}{r} \wedge T \right) \Pi_1(\kappa, \theta) \]
where \( \Pi_1 \) is a positive function.

A.3.2 Upper-Bound of the term \( (II_n) \) in the SVJ models of the Heston type

We can re-write the decomposition formula as
\[ \frac{\rho^2}{2} \left[ \int_0^T e^{-ru} \Lambda \Gamma G_n(u, \bar{X}_u, v_u) \sigma_u d[W, M]_u \right] - \frac{\rho}{2} \Lambda \Gamma G_n(0, \bar{X}_0, v_0) \left( \int_0^T E (\sigma^2) \varphi(s) ds \right) \]
\[ = \frac{\rho^2}{8} \left[ \int_0^T e^{-ru} (\bar{X}^3 - 3\bar{X}^2 \hat{\varphi}(z) \sigma^2 \varphi^2(u)) du \right] \]
\[ + \frac{\rho^2}{4} \left[ \int_0^T e^{-ru} (\bar{X}^3 - 3\bar{X}^2 \hat{\varphi}(z) \sigma^2 \varphi^2(u)) du \right] \]
\[ + \frac{\rho^2}{2} \left[ \int_0^T e^{-ru} (\bar{X}^3 - 3\bar{X}^2 \hat{\varphi}(z) \sigma^2 \varphi^2(u)) du \right] \]
\[ + \frac{\rho^2}{4} \left[ \int_0^T e^{-ru} (\bar{X}^3 - 3\bar{X}^2 \hat{\varphi}(z) \sigma^2 \varphi^2(u)) du \right] \]

Applying Lemma 4.3 and defining \( a_u := v_u \sqrt{T - u} \), we obtain
\[ \left| \frac{\rho}{2} \left[ \int_0^T e^{-ru} \Lambda \Gamma G_n(u, \bar{X}_u, v_u) \sigma_u d[W, M]_u \right] - \frac{\rho}{2} \Lambda \Gamma G_n(0, \bar{X}_0, v_0) \left( \int_0^T E (\sigma^2) \varphi(s) ds \right) \right| \]
\begin{align*}
& \leq C_1 \nu^3 \left\lfloor \int_0^T \! e^{-ru} \left( \frac{1}{a_0^2} + \frac{2}{a_0^2} + \frac{1}{\rho^2} \right) \left( \int_u^T \! \mathbb{E}_u (\sigma_s^2) \varphi(s) \, ds \right) \sigma_u^2 \varphi(u)^2 \, du \right\rfloor \\
& + C_2 \nu^2 \mathbb{E} \left[ \int_0^T \! e^{-ru} \left( \frac{1}{a_0^2} + \frac{1}{\rho^2} \right) \left( \int_u^T \! \mathbb{E}_u (\sigma_s^2) \varphi(s) \, ds \right) \sigma_u^2 \varphi(u) \, du \right] \\
& + C_3 \nu^2 \mathbb{E} \left[ \int_0^T \! e^{-ru} \left( \frac{1}{a_0^2} \right) \left( \int_u^T \! e^{-\kappa(z-u)} \varphi(z) \, dz \right) \sigma_u^2 \varphi(u) \, du \right] \\
& + C_4 \nu^3 \left\lfloor \int_0^T \! e^{-ru} \left( \frac{1}{a_0^2} + \frac{1}{\rho^2} \right) \left( \int_u^T \! e^{-\kappa(z-u)} \varphi(z) \, dz \right) \sigma_u^2 \varphi(u) \, du \right\rfloor .
\end{align*}

Using Lemma 5.2 (ii), then

\begin{align*}
& \left\lfloor \frac{\nu}{2} \mathbb{E} \left[ \int_0^T \! e^{-ru} \Lambda \Gamma \sigma_n(u, \tilde{X}_u, \nu_0) \sigma_u d[W, M]_u \right] \right\rfloor - \frac{\nu}{2} \Lambda \Gamma \sigma_n(0, \tilde{X}_0, \nu_0) \left( \int_0^T \! \mathbb{E} (\sigma_s^2) \varphi(s) \, ds \right) \\
& \leq C_1 \nu^3 \left\lfloor \int_0^T \! e^{-ru} \left( \frac{1}{a_0^2} + \frac{2}{a_0^2} + \frac{1}{\rho^2} \right) v_u^6 (T-u)^2 \varphi(u)^2 \, du \right\rfloor \\
& + C_2 \nu^2 \mathbb{E} \left[ \int_0^T \! e^{-ru} \left( \frac{1}{a_0^2} + \frac{1}{\rho^2} \right) v_u^4 (T-u)^2 \varphi(u) \, du \right] \\
& + C_3 \nu^2 \mathbb{E} \left[ \int_0^T \! e^{-ru} \left( \frac{1}{a_0^2} \right) \varphi(u) v_u^2 (T-u) \, du \right] \\
& + C_4 \nu^3 \left\lfloor \int_0^T \! e^{-ru} \left( \frac{1}{a_0^2} + \frac{1}{\rho^2} \right) \varphi(u)^2 v_u^2 (T-u) \, du \right\rfloor .
\end{align*}

Finally, applying Lemma 5.2 (i), we find that

\begin{align*}
& \left\lfloor \frac{\nu}{2} \mathbb{E} \left[ \int_0^T \! e^{-ru} \Lambda \Gamma \sigma_n(u, \tilde{X}_u, \nu_0) \sigma_u d[W, M]_u \right] \right\rfloor - \frac{\nu}{2} \Lambda \Gamma \sigma_n(0, \tilde{X}_0, \nu_0) \left( \int_0^T \! \mathbb{E} (\sigma_s^2) \varphi(s) \, ds \right) \\
& \leq C_1 \nu^3 \left\lfloor \int_0^T \! e^{-ru} \left( \frac{2}{\theta \kappa} + \frac{2 \sqrt{2}}{\kappa \sqrt{\theta \kappa}} + \frac{1}{\kappa^2} \right) \, du \right\rfloor \\
& + C_2 \nu^2 \mathbb{E} \left[ \int_0^T \! e^{-ru} \left( \frac{\sqrt{2}}{\sqrt{\theta \kappa}} + \frac{1}{\kappa} \right) \, du \right] \\
& + C_3 \nu^2 \mathbb{E} \left[ \int_0^T \! e^{-ru} \left( \frac{\sqrt{2}}{\sqrt{\theta \kappa}} \right) \, du \right] \\
& + C_4 \nu^3 \left\lfloor \int_0^T \! e^{-ru} \left( \frac{2}{\theta \kappa} + \frac{\sqrt{2}}{\kappa \sqrt{\theta \kappa}} \right) \, du \right\rfloor .
\end{align*}

Then we have that

\begin{align*}
& \left\lfloor \frac{\nu}{2} \mathbb{E} \left[ \int_0^T \! e^{-ru} \Lambda \Gamma \sigma_n(u, \tilde{X}_u, \nu_0) \sigma_u d[W, M]_u \right] \right\rfloor - \frac{\nu}{2} \Lambda \Gamma \sigma_n(0, \tilde{X}_0, \nu_0) \left( \int_0^T \! \mathbb{E} (\sigma_s^2) \varphi(s) \, ds \right) \\
& \leq C_1 \nu^3 \left( \frac{2}{\theta \kappa} + \frac{2 \sqrt{2}}{\kappa \sqrt{\theta \kappa}} + \frac{1}{\kappa^2} \right) \left( \int_0^T \! e^{-ru} \, du \right) \\
& + C_2 \nu^2 \left( \frac{\sqrt{2}}{\sqrt{\theta \kappa}} + \frac{1}{\kappa} \right) \left( \int_0^T \! e^{-ru} \, du \right) \\
& + C_3 \nu^2 \left( \frac{\sqrt{2}}{\sqrt{\theta \kappa}} \right) \left( \int_0^T \! e^{-ru} \, du \right) \\
& + C_4 \nu^3 \left( \frac{2}{\theta \kappa} + \frac{\sqrt{2}}{\kappa \sqrt{\theta \kappa}} \right) \left( \int_0^T \! e^{-ru} \, du \right) .
\end{align*}
\[ + C \frac{\rho^2 \nu^2}{2} \sqrt{\frac{2}{\theta \kappa}} \left( \int_0^T e^{-ru} du \right) \]
\[ + C \frac{\rho^3}{4} \left( \frac{2}{\theta \kappa} + \frac{\sqrt{2}}{\kappa \sqrt{\theta \kappa}} \right) \left( \int_0^T e^{-ru} du \right). \]

Using the fact that \( \int_t^T e^{-rs} ds \leq \frac{1}{r} \wedge T \), we conclude that
\[
\left| \frac{\rho}{2} \mathbb{E} \left[ \int_0^T e^{-ru} \Gamma G_n(u, \tilde{X}_u, v_u) \sigma_u d[M, M]_u \right] - \rho \nu^2 \Gamma G_n(0, \tilde{X}_0, v_0) \left( \int_0^T \mathbb{E} (\sigma^2_s) \varphi(s) ds \right) \right| \leq \nu^2 (|\rho| + \nu) \left( \frac{1}{r} \wedge T \right) \Pi(\kappa, \theta)
\]

where \( \Pi \) is a positive function.

**A.3.3 Upper-Bound for the terms \((I_n)\) and \((II_n)\) in the SVJ models of the Heston type**

We have that
\[
\left| \frac{1}{8} \mathbb{E} \left[ \int_0^T e^{-ru} \Gamma^2 G_n(u, \tilde{X}_u, v_u) \sigma_u d[M, M]_u \right] - \frac{\nu^2}{8} \Gamma^2 G_n(0, \tilde{X}_0, v_0) \left( \int_0^T \mathbb{E} (\sigma^2_s) \varphi(s)^2 ds \right) \right| \leq \nu^3 (|\rho| + \nu) \left( \frac{1}{r} \wedge T \right) \Pi_1(\kappa, \theta) + |\rho| \nu^2 (|\rho| + \nu) \left( \frac{1}{r} \wedge T \right) \Pi_2(\kappa, \theta)
\]
\[
\leq \nu^2 (|\rho| + \nu)^2 \left( \frac{1}{r} \wedge T \right) \Pi(\kappa, \theta).
\]

where function \( \Pi \) is the maximum of functions \( \Pi_1 \) and \( \Pi_2 \).
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