MULTIPARAMETER QUANTUM GROUPS,
BOSONIZATIONS AND COCYCLE DEFORMATIONS

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Abstract. We study cocycle deformations on bosonizations of braided Hopf algebras $R$ over Hopf algebras with bijective antipode $H$. We first recall how a Hopf 2-cocycle on $H$ gives rise to a Hopf 2-cocycle on $A = R \# H$ inducing a section of the restriction map $Z^2(A, k) \to Z^2(H, k)$. Then we apply the result to multiparameter quantum groups. We first describe the quantum groups of $[HPR]$ given by multiparametric deformations of enveloping algebras of Kac-Moody algebras as a family of pointed Hopf algebras from $[ARS]$, which are quotients of bosonizations of pre-Nichols algebras, and show that under some hypothesis, these quantum groups depend only on one parameter on each connected component of the Dynkin diagram; in particular, we obtain in this way a known result of $[HPR]$.

1. Introduction

Let $\mathbb{k}$ be an algebraically closed field of characteristic zero and $\pi : A \to H$ a Hopf algebra projection that admits a Hopf algebra section $\iota : H \to A$. Then $A \simeq R \# H$, the Radford-Majid product or bosonization of $R$ over $H$, where $R = A^\co \pi$ is a braided Hopf algebra in the category of Yetter-Drinfeld modules $H_H^D$ over $H$. Conversely, given a braided Hopf algebra $R$ in $H_H^D$, its bosonization $R \# H$ is an ordinary Hopf algebra with a projection to $H$, see $[R]$. This fact play a crucial role in the classification of finite-dimensional pointed Hopf algebras $[AS4]$ and in the description of quantum groups arising from deformation of enveloping algebras of semisimple Lie algebras, in particular their quantum Borel subalgebras.

If $A$ is a Hopf algebra such that its coradical $A_0$ is a Hopf subalgebra, then the graded object $\text{gr} A$ associated to the coradical filtration is again a Hopf algebra and $\text{gr} A \simeq R \# H$. A key point in the lifting method to classify pointed Hopf algebras, where $A_0$ is a group algebra, relies on the description of $R$ as a Nichols algebra, a graded connected braided Hopf algebra generated by the elements of degree one. If $A$ is a pointed Hopf algebra such that $\text{gr} A \simeq R \# H$, it is said that $A$ is a lifting of $R$ over $H$. This procedure can be generalized for other types of filtrations, such as the standard filtration, see $[AC]$.

An open question concerning this problem is whether all possible liftings of a bosonization $R \# H$ can be obtained by a (left) 2-cocycle deformation on the multiplication. Positive answers were obtained for $H = \mathbb{k} \Gamma$ a group algebra by different authors using different methods, among them, $[GrM]$, $[MR]$, $[Mk2]$ for the case $\Gamma$ is abelian and $R$ is a quantum linear space, $[GM]$ for $\Gamma = \mathbb{S}_n$ a symmetric group, $[GM]$ for $\Gamma = \mathbb{D}_m$ a family of dihedral groups and $[GIV]$ for pointed Hopf algebras associated to affine racks. Moreover, in $[AAGMV]$ a systematic procedure is described to construct liftings as cocycle deformations via Hopf-Galois objects. Variations of this problem were also studied by other authors, see for example $[ABM1]$ $[ABM2]$ and references therein.

Hence, a natural question is to describe the set $Z^2(A, \mathbb{k})$ of Hopf 2-cocycles on a bosonization $A = R \# H$ for $H$ a Hopf algebra with bijective antipode and $R$ a braided Hopf algebra in $H_H^D$. Since $H$ is a Hopf subalgebra of $A$, the restriction of any Hopf 2-cocycle on $A$ gives a Hopf 2-cocycle on $H$. This restriction admits a section that gives an injective map $Z^2(H, \mathbb{k}) \hookrightarrow Z^2(A, \mathbb{k})$. In particular, any Hopf 2-cocycle $\sigma$ on $H$ defines a Hopf 2-cocycle $\tilde{\sigma}$ on $A$. This result was proved by J. Cuadra and F. Panaite $[CP]$ for right 2-cocycles. Besides, in loc. cit. the authors study lazy cohomology on Drinfeld...
doubles and bosonizations and relate the lazy cohomology of \( R \) to that of \( A \). We give in Proposition 3.1 a direct proof for left 2-cocycles and describe several applications:

First we combine this construction with the Hopf 2-cocycles given in [GM] and [GM]. These cocycles arise as exponentiations of \( H \)-invariant Hochschild 2-cocycles on \( R \) and are trivial on \( H \). In case the convolution product \( e^0 * \tilde{\sigma} \) is well-defined (which is always the case if the cocycles are lazy), we obtain a Hopf 2-cocycle on \( A \) from a Hochschild 2-cocycle \( \eta \) on \( R \) and a Hopf 2-cocycle \( \sigma \) on \( H \), see [CP] for a similar result. We apply this to the case where \( H = kS_n \) is the group algebra of the symmetric group in four letters, where we show in Proposition 3.10 that the liftings of a family of Nichols algebras associated to a constant rack 2-cocycle are Hopf 2-cocycle deformations of bosonizations of Fomin-Kirillov algebras. Therefore, the description of the Hochschild cohomology of \( R \) could play an important role in the classification of Hopf algebras. Several results were obtained in this direction in [BNPP], [MPSW], [PS], [SV].

Then we turn our attention to the multiparameter quantum groups \( U_q(\mathfrak{g}_A) \) associated to a generalized Cartan matrix \( A \) defined in [HPR]. In order to treat them in a unified way, we first describe them explicitly in Theorem 4.9 as a family of reductive pointed Hopf algebras given in [ARS]. A similar description is given in [H] using bicharacters and the Lusztig isomorphisms are described. This allows us to study them as quotients of bosonizations and therefore as pre-Nichols algebras. Finally, using the results on 2-cocycles deformations, we show that in case the braiding is generic, they are cocycle deformations of multiparameter quantum groups that depends only on one parameter on each connected component of the Dynking diagram associated to \( A \), obtaining in case \( \mathfrak{g}_A \) is simple, a result of Hu, Pei and Rosso [HPR Thm. 28], see Theorem 4.7 and Corollaries 4.10, 4.12. This relation was previously described in [AS3], [R] as a twist-equivalence between the matrices associated to the braiding; here we re-interpret them as 2-cocycle deformations.

The paper is organized as follows. In Section 2 we fix notation and recall some known facts on Hopf 2-cocycles, Yetter-Drinfeld modules, Nichols algebras and bosonizations. In Section 3 we study cocycle deformations of Hopf algebras given by bosonizations \( A = R \# H \) and describe the relation between \( Z^2(A, k) \) and \( Z^2(H, k) \). If \( H \) is cocommutative, then \( A_{\tilde{\sigma}} = R_{\tilde{\sigma}} \# H \) and \( \sigma \) deforms the Yetter-Drinfeld module structure of \( R \), see Proposition 3.2 which coincides with similar results in [MO].

Finally, in Section 4 we explicitly describe the multiparameter quantum groups given in [HPR] as a family of pointed Hopf algebras associated to reduced Yetter-Drinfeld data of Cartan type and prove that in case the braiding is generic, these Hopf algebras depend only on one parameter for each connected component of the Dynkin diagram.

Acknowledgements. Research of this paper was begun when the author was visiting Università di Roma “Tor Vergata” under the support of the GNSAGA and CONICET. He thanks F. Gavarini and the people of the Math. Department for the warm hospitality. These notes are intended to contribute to a joint project with F. Gavarini on multiparameter quantum groups; the author wishes to thank all conversations and comments which helped to improve the paper. He also wishes to thank A. García Iglesias and L. Vendramin for interesting discussions, and F. Panaite for drawing him the attention to the results on [CP].

2. Preliminaries

2.1. Conventions. We work over an algebraically closed field \( k \) of characteristic zero and by \( k^x \) we denote the group of units of \( k \). If \( \Gamma \) is a group, we denote by \( \widehat{\Gamma} \) the character group. By convention, \( \mathbb{N} = \{0, 1, \ldots\} \). If \( A \) is an algebra and \( g \in A \) is invertible, then \( g \triangleright a = gag^{-1} \), \( a \in A \), denotes the inner automorphism defined by \( g \).

Our references for the theory of Hopf algebras are [Mo], [Sw] and [R]. We use standard notation for Hopf algebras; the comultiplication is denoted \( \Delta \) and the antipode \( S \). The left adjoint representation of \( H \) on itself is the algebra map \( \text{ad}: H \to \text{End}(H) \), \( \text{ad}_x(y) = x(1)yS(x(2)) \), \( x, y \in H \); we shall write \( \text{ad} \) for \( \text{ad}_t \), omitting the subscript \( t \) unless strictly needed. There is also a right adjoint action given by \( \text{ad}_r \), \( x(y) = S(x(1))yx(2) \). Note that both \( \text{ad}_r \) and \( \text{ad}_l \) are multiplicative. The set of group-like elements of a coalgebra \( C \) is denoted by \( G(C) \). We also denote by \( C^+ = \ker \varepsilon \) the augmentation ideal of \( C \).
where $\varepsilon : C \to k$ is the counit of $C$. Let $g, h \in G(H)$, the set of $(g, h)$-primitive elements is given by $P_{g,h}(H) = \{ x \in H : \Delta(x) = x \otimes g + h \otimes x \}$. We call $P_{1,1}(H) = P(H)$ the set of primitive elements.

Let $A \xrightarrow{\pi} H$ be a Hopf algebra map, then $$A^{\text{co}H} = A^{\text{co}\pi} = \{ a \in A | (\text{id} \otimes \pi)\Delta(a) = a \otimes 1 \},$$
denotes the subalgebra of right coinvariants and $^{\text{co}H}A = ^{\text{co}\pi}A$ denotes the subalgebra of left coinvariants.

2.1.1. Deforming cocycles. Let $A$ be a Hopf algebra. Recall that a convolution invertible linear map $\sigma$ in $\text{Hom}_k(A \otimes A, k)$ is a normalized (or unitary) Hopf 2-cocycle if
$$\sigma(b(1), c(1))\sigma(a, b(2)c(2)) = \sigma(a(1), b(1))\sigma(a(2)b(2), c)$$
and $\sigma(a, 1) = \varepsilon(a) = \varepsilon(1, a)$ for all $a, b, c \in A$, see [Mo, Sec. 7.1]. We will simply say a 2-cocycle if no confusion arises.

Using a 2-cocycle $\sigma$ it is possible to define a new algebra structure on $A$ by deforming the multiplication, which we denote by $A_\sigma$. Moreover, $A_\sigma$ is indeed a Hopf algebra with $A = A_\sigma$ as coalgebras, deformed multiplication $m_\sigma = \sigma * m * \sigma^{-1} : A \otimes A \to A$ given by
$$m_\sigma(a, b) = a \cdot_\sigma b = \sigma(a(1), b(1))a(2)b(2)\sigma^{-1}(a(3), b(3))$$
and antipode $S_\sigma = \sigma * S * \sigma^{-1} : A \to A$ given by (see [D] for details)
$$S_\sigma(a) = \sigma(a(1), S(a(2)))S(a(3))\sigma^{-1}(S(a(4)), a(5))$$
for all $a \in A$.

We denote by $Z^2(A, k)$ the set of normalized Hopf 2-cocycles on $A$. Let $\tau, \sigma : K \otimes K \to k$ be two linear maps. We denote by $\tau * \sigma : K \otimes K \to k$ the linear map given by the convolution, that is
$$(\tau * \sigma)(a, b) = \tau(a(1), b(1))\sigma(a(2), b(2))$$
for all $a, b \in K$.

Remark 2.1. Assume $A = k\Gamma$, with $\Gamma$ a group. Then a normalized group 2-cocycle on $A$ is equivalent to a 2-cocycle $\varphi \in Z^2(\Gamma, k)$, that is a map $\varphi : \Gamma \times \Gamma \to k^\times$ such that
$$\varphi(g, h)\varphi(gh, t) = \varphi(h, t)\varphi(g, ht)$$
and $\varphi(g, e) = \varphi(e, g) = 1$ for all $g, h, t \in \Gamma$.

For $n > 0$ and $q \in k^\times$, define
$$\begin{align*}
(n)_q &= \frac{q^n - 1}{q - 1} = q^{n-1} + \cdots + q + 1, \\
(n)_q! &= (n)_q(n-1)_q \cdots (2)_q(1)_q \quad \text{and} \quad (0)_q = 1, \\
\binom{n}{k}_q &= \frac{(n)_q}{(k)_q(n-k)_q}.
\end{align*}$$
It is well-known that
$$\binom{n}{k}_q = q^k \binom{n-1}{k}_q + \binom{n-1}{k-1}_q = \binom{n-1}{k}_q + q^{n-k} \binom{n-1}{k-1}_q.$$

2.2. Yetter-Drinfeld modules, Nichols algebras and bosonization. Let $H$ be a Hopf algebra with bijective antipode. A Yetter-Drinfeld module over $H$ is a left $H$-module and a left $H$-comodule with comodule structure denoted by $\delta : V \to H \otimes V$, $v \mapsto v(-1) \otimes v(0)$, such that $\delta(h \cdot v) = h(1)v(1)S(h(3))\otimes h(2)v(0)$ for all $v \in V, h \in H$. Let $H^\text{YD}$ be the category of Yetter-Drinfeld modules over $H$ with $H$-linear and $H$-colinear maps as morphisms. The category $H^\text{YD}$ is monoidal and braided. If $V, W \in H^\text{YD}$, then $V \otimes W$ is the tensor product over $k$ with the diagonal action and coaction of $H$ and braiding $c_{V,W} : V \otimes W \to W \otimes V$, $v \otimes w \mapsto v(-1) \cdot w \otimes v(0)$ for all $v \in V, w \in W$.

In case $H = k\Gamma$ is a group algebra of a group $\Gamma$, a Yetter-Drinfeld module over $k\Gamma$ is a $\Gamma$-graded vector space $V = \bigoplus_{g \in \Gamma} V_g$ which is a left $\Gamma$-module such that each homogeneous component $V_g$, $g \in \Gamma$, is stable under the action of $\Gamma$. Here, the $\Gamma$-grading gives the left $k\Gamma$-comodule structure by $\delta : V \to k\Gamma \otimes V, \delta(v) = g \otimes v$ if $v$ is homogeneous of degree $g \in \Gamma$. For $V, W \in k^\text{YD}$, the braiding is given by $c_{V,W}(v \otimes w) = g \cdot w \otimes v$, for all $v \in V_g, w \in W$ and $g \in \Gamma$. We denote this category simply by $k^\text{YD}$. If $\Gamma$ is finite, then $k^\text{YD}$ is a semisimple category.
Let $V \in \mathcal{YD}$ and $g \in \Gamma$, $\chi \in \hat{\Gamma}$. We denote by

$$V^x_g = \{v \in V : \delta(v) = g \otimes v, h \cdot v = \chi(h)v \ \forall h \in \Gamma\},$$

the Yetter-Drinfeld submodule given by the $g$-homogeneous elements with diagonal action of $\Gamma$ given by $\chi$. In case $\Gamma$ is finite abelian, the pairs $(g, \chi)$ with $g \in \Gamma$ and $\chi \in \hat{\Gamma}$ parametrize the simple modules and for all $V \in \mathcal{YD}$ we have that $V = \bigoplus_{g \in \Gamma, \chi \in \hat{\Gamma}} V^x_g$.

Since $\mathcal{YD}$ is a braided monoidal category, we may consider Hopf algebras in this category. For $V \in \mathcal{YD}$, the tensor algebra $T(V) = \oplus_{n \geq 0} T^n(V)$ is an $\mathbb{N}$-graded algebra and coalgebra in the braided category $\mathcal{YD}$ where the elements of $V = T(V)(1)$ are primitive. It is a Hopf algebra in $\mathcal{YD}$ since $T(V)(0) = \mathbb{k}$.

Let $(V, c)$ be a finite-dimensional braided vector space. We say that the braiding $c : V \otimes V \to V \otimes V$ is diagonal [AS3, Def. 1.1] if there exists a basis $x_1, \ldots, x_\theta$ of $V$ and non-zero scalars $q_{ij}$ such that $c(x_i \otimes x_j) = q_{ij} x_j \otimes x_i$ for all $1 \leq i, j \leq \theta$. The braiding is called generic if it is diagonal and $q_{ii}$ is not a root of unity for all $1 \leq i \leq \theta$, and it is called positive if it is generic and $q_{ii}$ is a positive real number for all $1 \leq i \leq \theta$. We say that two finite-dimensional braided vector spaces of diagonal type $(V, c)$ and $(W, d)$ with matrices $(q_{ij})$ and $(\hat{q}_{ij})$ are twist-equivalent [AS2, Def. 3.8] if $\dim V = \dim W$, $q_{ii} = \hat{q}_{ii}$ and

$$q_{ij} \hat{q}_{ji} = q_{ji} \hat{q}_{ij} \quad \text{for all } 1 \leq i, j \leq \theta.$$

We are particularly interested in one class of braided Hopf algebras in these categories, which turn out to be crucial in the theory, the (pre-) Nichols algebras.

**Definition 2.2.** Let $V \in \mathcal{YD}$ and $I(V) \subseteq T(V)$ the largest $\mathbb{N}$-graded ideal and coideal such that $I \cap V = 0$. We call $\mathfrak{B}(V) = T(V)/I(V)$ the Nichols algebra of $V$. In particular, $\mathfrak{B}(V) = \bigoplus_{n \geq 0} \mathfrak{B}^n(V)$ is an $\mathbb{N}$-graded Hopf algebra in $\mathcal{YD}$.

Given a braided vector space $(V, c)$, one may construct the Nichols algebra $\mathfrak{B}(V, c) = \mathfrak{B}(V)$ in a way similar to the construction above, by taking a quotient of the tensor algebra $T(V)$ by the homogeneous two-sided ideal given by the kernel of a homogeneous symmetrizer: Let $\mathbb{B}_n$ be the braid group of $n$ letters. Since $c$ satisfies the braid equation, it induces a representation of $\mathbb{B}_n$, $\rho_n : \mathbb{B}_n \to \text{GL}(V^{\otimes n})$ for each $n \geq 2$. Consider the morphisms

$$Q_n = \sum_{\sigma \in \mathbb{S}_n} \rho_n(M(\sigma)) \in \text{End}(V^{\otimes n}),$$

where $M : \mathbb{S}_n \to \mathbb{B}_n$ is the Matsumoto section corresponding to the canonical projection $\mathbb{B}_n \to \mathbb{S}_n$. Then the Nichols algebra $\mathfrak{B}(V)$ is the quotient of the tensor algebra $T(V)$ by the two-sided ideal $\mathcal{J} = \bigoplus_{n \geq 2} \text{Ker} Q_n$.

A pre-Nichols algebra is an intermediate graded braided Hopf algebra between $T(V)$ and $\mathfrak{B}(V)$, see [Mk] and Definition 4.3 later on.

Let $R$ be a Hopf algebra in $\mathcal{YD}$ with multiplication $m_R$. For $x, y \in \mathcal{P}(R)$, we define the braided adjoint action of $x$ on $y$ by

$$\text{ad}_c(x)(y) = m_R(x \otimes y) - m_R \circ c_{R \otimes R}(x \otimes y) = xy - (x_{(1)} \cdot y)x_{(0)}.$$  

This element is also called the braided commutator of $x$ and $y$.

2.2.1. Bosonization and Hopf algebras with a projection. Let $R$ be a Hopf algebra in $\mathcal{YD}$. The procedure to obtain a usual Hopf algebra from the braided Hopf algebra $R$ and $H$ is called bosonization or Radford-Majid product, and it is usually denoted by $R\# H$. As a vector space, $R\# H = R \otimes H$ and the multiplication and comultiplication are given by the smash-product and smash-coproduct, respectively. That is, for all $r, s \in R$ and $g, h \in H$, we have

$$(r \# g)(s \# h) = r(g_{(1)} \cdot s) \# g_{(2)} h,$$

$$\Delta(r \# g) = r_{(1)} \# (r_{(2)})(-1)g_{(1)} \otimes (r_{(2)})_{(0)} \# g_{(2)},$$

$$S(r \# g) = (1 \# S_H(r_{(-1)}g))(S_R(r_{(0)}) \# 1),$$

where $S_H$ and $S_R$ are the antipodes of $H$ and $R$, respectively.
where $\Delta_R(r) = r^{(1)} \otimes r^{(2)}$ denotes the comultiplication in $R \in H_H \mathcal{YD}$ and $S_R$ the antipode. Clearly, the map $\iota : H \to R \# H$ given by $\iota(h) = 1 \# h$ for all $h \in H$ is an injective Hopf algebra map, and the map $\pi : R \# H \to H$ given by $\pi(r \# h) = \varepsilon_R(r)h$ for all $r \in R, h \in H$ is a surjective Hopf algebra such that $\pi \circ \iota = \text{id}_H$. Moreover, it holds that $R = (R \# H)^{co \pi}$.

Conversely, let $A$ be a Hopf algebra with bijective antipode and $\pi : A \to H$ a Hopf algebra epimorphism admitting a Hopf algebra section $\iota : H \to A$ such that $\pi \circ \iota = \text{id}_H$. Then $R = A^{co \pi}$ is a braided Hopf algebra in $H_H \mathcal{YD}$ and $A \simeq R \# H$ as Hopf algebras. See [R 11.6] for more details.

3. ON COCYCLE DEFORMATIONS AND BOSONIZATIONS

In this section we collect some results on the construction of 2-cocycles on bosonizations of Hopf algebras and give an example on finite-dimensional pointed Hopf algebras over the symmetric group $S_4$. These results will be also applied in the next section on multiparameter quantum groups.

3.1. COCYCLES ON BOSONIZATIONS. Let $H$ be a Hopf algebra with bijective antipode, $R$ a braided Hopf algebra in $H_H \mathcal{YD}$ and $A = R \# H$ its bosonization. To avoid confusion, in this section we denote by $\to : H \otimes R \to R$ the action of $H$ on $R$.

The following proposition was proved by J. Cuadra and F. Panaite for right 2-cocycles [CP Prop. 4.2]. It shows that a Hopf 2-cocycle on $R$ may deform the module and consequently the braided structure of $R$. For another version of the same result see [GIM Lemma 4.1]. We give the proof for completeness.

**Proposition 3.1.** Let $\sigma \in \mathcal{Z}^2(H,k)$. Then the map $\tilde{\sigma} : A \otimes A \to k$ given by

$$
\tilde{\sigma}(r \# h, s \# k) = \sigma(h, k) \varepsilon_R(r) \varepsilon_R(s) \quad \text{for all } r, s \in R, h, k \in H,
$$
is a normalized Hopf 2-cocycle such that $\tilde{\sigma}|_{H \otimes H} = \sigma$. Moreover, $H_\sigma$ is a Hopf subalgebra of $A_\sigma$ and the map $\mathcal{Z}^2(H,k) \to \mathcal{Z}^2(A,k)$ given by $\sigma \mapsto \tilde{\sigma}$ gives a section of the map $\mathcal{Z}^2(A,k) \to \mathcal{Z}^2(H,k)$ induced by the restriction; in particular, it is injective.

**Proof.** Clearly $\tilde{\sigma}(r \# h, 1 \# 1) = \varepsilon_R(r) \varepsilon_R(h) = \tilde{\sigma}(1 \# 1, r \# h)$, because $\sigma$ is normalized. Moreover, since $\varepsilon_R$ is an $H$-comodule map we have that $\varepsilon_R(r) = r_{(-1)} \varepsilon_R(r_{(0)})$ for all $r \in R$ and therefore

$$
\tilde{\sigma}((s \# k)(1), (t \# \ell)(1)) \tilde{\sigma}(r \# h, (s \# k)(2)(t \# \ell)(2)) =
$$

$$
= \tilde{\sigma}(s^{(1)} \# (s^{(2)})(1)k_{(1)}, t^{(1)} \# (t^{(2)})(1)\ell_{(1)}) \tilde{\sigma}(r \# h, [(s^{(2)})(0) \# k_{(2)}][t^{(2)}(0) \# \ell_{(2)}])
$$

$$
= \tilde{\sigma}(s^{(1)} \# (s^{(2)})(1)k_{(1)}, t^{(1)} \# (t^{(2)})(1)\ell_{(1)}) \tilde{\sigma}(r \# h, (s^{(2)})(0)k_{(2)} \rightarrow (t^{(2)})(0) \# \ell_{(2)})
$$

$$
= \varepsilon_R(s^{(1)})) \varepsilon_R(t^{(1)}) \sigma((s^{(2)})(1)k_{(1)}, (t^{(2)})(0)\ell_{(1)}) \varepsilon_R(r) \varepsilon_R((s^{(2)})(0)(h_{(2)} \rightarrow (t^{(2)})(0))) \sigma(h, k_{(3)} \ell_{(2)})
$$

$$
= \varepsilon_R(r) \varepsilon_R(s^{(1)}) \varepsilon_R(t^{(1)}) \varepsilon_R((t^{(2)})) \sigma(k_{(1)}, \ell_{(1)}) \sigma(h, k_{(2)} \ell_{(2)})
$$

$$
= \varepsilon_R(r) \varepsilon_R(s) \varepsilon_R(t) \sigma(h_{(1)}, \ell_{(1)}) \sigma(h, k_{(2)} \ell_{(2)}) = \varepsilon_R(r) \varepsilon_R(s) \varepsilon_R(t) \sigma(h_{(1)}, \ell_{(1)}) \sigma(h, k_{(2)} \ell_{(2)}),
$$

On the other hand,

$$
\tilde{\sigma}((r \# h)(1), (s \# k)(2)) \tilde{\sigma}(r \# h)(2)(s \# k)(2), t \# \ell) =
$$

$$
= \tilde{\sigma}(r^{(1)} \# (r^{(2)})(1)h_{(1)}, s^{(1)} \# (s^{(2)})(1)k_{(1)}) \tilde{\sigma}((r^{(2)})(0)h_{(2)} \rightarrow (s^{(2)})(0) \# h_{(3)}k_{(2)}), t \# \ell)
$$

$$
= \varepsilon_R(r^{(1)}) \sigma((r^{(2)})(1)h_{(1)}, (s^{(2)})(1)k_{(1)}) \varepsilon_R(s^{(1)}) \varepsilon_R(r^{(2)})(0) \varepsilon_R((h_{(2)} \rightarrow (s^{(2)})(0))) \sigma(h_{(3)}k_{(2)}), t \# \ell)
$$

$$
= \varepsilon_R(r^{(1)}) \sigma((r^{(2)})(1)h_{(1)}, (s^{(2)})(1)k_{(1)}) \varepsilon_R(s^{(1)}) \varepsilon_R(r^{(2)})(0) \varepsilon_R((h_{(2)} \rightarrow (s^{(2)})(0))) \sigma(h_{(3)}k_{(2)}), t \# \ell)
$$

$$
= \varepsilon_R(r^{(1)}) \varepsilon_R(s^{(1)}) \varepsilon_R(t) \sigma(h_{(1)}, k_{(1)}) \sigma(h, k_{(2)} \ell_{(2)}),
$$

which shows that $\tilde{\sigma} \in \mathcal{Z}^2(A,k)$. Since $\tilde{\sigma}|_{H \otimes H} = \sigma$, it is clear that $H_\sigma$ is a Hopf subalgebra of $A_\sigma$. The last assertion follows by construction. \qed
Proposition 3.2. Let $\sigma \in \mathcal{Z}^2(H, \mathbb{k})$ and $\tilde{\sigma} \in \mathcal{Z}^2(A, \mathbb{k})$ the 2-cocycle given by Proposition 3.1. Then $A_{\tilde{\sigma}} = R_{\sigma} \# H_{\sigma}$, where $R_{\sigma} = R$ as coalgebras, and the product is given by
\[
(2) \quad a \cdot_{\sigma} b = \sigma(a_{(-1)}, b_{(-1)})a_{(0)}b_{(0)} \text{ for all } a, b \in R.
\]
Moreover, $R_{\sigma} \in H^*_H \mathcal{Y} \mathcal{D}$ with the action of $H_{\sigma}$ given by
\[
(3) \quad h \rightarrow_{\sigma} a = \sigma(h_{(1)}, a_{(-1)})(h_{(2)} \rightarrow a_{(0)})\sigma^{-1}((h_{(2)} \rightarrow a_{(0)})(-1), h_{(3)}) \text{ for all } h \in H_{\sigma}, a \in R_{\sigma}.
\]
Proof. Since $A = R \# H$, there exist a Hopf algebra epimorphism $\pi : A \to H$ admitting a Hopf algebra section $\iota : H \to A$ such that $\pi \circ \iota = \text{id}_H$ and $R = A^{co \pi} \in H^*_H \mathcal{Y} \mathcal{D}$. Since $A = A_{\tilde{\sigma}}$ as coalgebras, we have a coalgebra map $\pi : A_{\tilde{\sigma}} \to H_{\sigma}$, which is indeed a Hopf algebra surjection that admits a section. Indeed, since for all $a \# h \in A$ we have that
\[
\Delta^{(2)}(a \# h) = a^{(1)} \# (a^{(2)})(-1)(a^{(3)})(-2)h_{(1)} \otimes (a^{(2)})(0)\#(a^{(3)})(-1)h_{(2)} \otimes (a^{(3)})(0)\#h_{(3)},
\]
and it follows that $\pi$ is an algebra map since
\[
\pi((a \# h) \cdot_{\sigma} (b \# k)) = \\
= \tilde{\sigma}(a^{(1)})(-1)(a^{(3)})(-2)h_{(1)}, b^{(1)} \# (b^{(2)})(-1)(b^{(3)})(-2)k_{(1)}).
\]
\[
\cdot \pi \left( \left( [a^{(2)})(0)\#(a^{(3)})(-1)h_{(2)}][b^{(2)})(0)\#(b^{(3)})(-1)k_{(2)}] \right) \tilde{\sigma}^{-1}((a^{(3)})(0)\#h_{(3)}, (b^{(3)})(0)\#k_{(3)})
\]
\[
= \varepsilon(a^{(1)})\varepsilon(b^{(1)})\sigma((a^{(2)})(-1)(a^{(3)})(-2)h_{(1)}, (b^{(2)})(-1)(b^{(3)})(-2)k_{(1)}).
\]
\[
\cdot \pi \left( \left( [a^{(2)})(0)\#(a^{(3)})(-1)h_{(2)}][b^{(2)})(0)\#(b^{(3)})(-1)k_{(2)}] \right) \varepsilon((a^{(3)})(0)\#(b^{(3)})(0))\varepsilon^{-1}(h_{(3)}, k_{(3)})
\]
\[
= \varepsilon(a^{(1)})\varepsilon(b^{(1)})\sigma((a^{(2)})(-1)h_{(1)}, (b^{(2)})(-1)k_{(1)}),
\]
\[
\pi((a \# h) \cdot_{\sigma} (b \# k) = \pi(a \# h) \cdot_{\sigma} \pi(b \# k),
\]
where the third and fifth equality hold since $\varepsilon = \varepsilon_R$ is an $H$-comodule map, that is $a_{(-1)}\varepsilon(a_{(0)}) = \varepsilon(a)$ for all $a \in R$. Clearly, the inclusion $H_{\sigma} \hookrightarrow A_{\tilde{\sigma}}$ splits $\pi$ and we have that $A_{\tilde{\sigma}} = R_{\sigma} \# H_{\sigma}$ with $R_{\sigma} = A^{co \pi} = R$ as coalgebras. For the product we have
\[
(a \# 1) \cdot_{\sigma} (b \# 1) = \tilde{\sigma}(a^{(1)})(-1)(a^{(3)})(-2), b^{(1)} \# (b^{(2)})(-1)(b^{(3)})(-2).
\]
\[
\cdot \left( (a^{(2)})(0)\#(a^{(3)})(-1)(b^{(2)})(0)\#(b^{(3)})(-1) \right) \tilde{\sigma}^{-1}((a^{(3)})(0)\#1, (b^{(3)})(0)\#1)
\]
\[
= \varepsilon(a^{(1)})\varepsilon(b^{(1)})\sigma((a^{(2)})(-1)(a^{(3)})(-2), (b^{(2)})(-1)(b^{(3)})(-2)),
\]
\[
\cdot \left( (a^{(2)})(0)\#(a^{(3)})(-1)(b^{(2)})(0)\#(b^{(3)})(-1) \right) \varepsilon((a^{(3)})(0)\#(b^{(3)})(0))
\]
\[
= \varepsilon(a^{(1)})\varepsilon(b^{(1)})\sigma((a^{(2)})(-1), (b^{(2)})(-1))(a^{(2)})(0)\#1][(b^{(2)})(0)\#1] \varepsilon((a^{(3)})(0))
\]
\[
= \varepsilon(a^{(1)})\varepsilon(b^{(1)})\sigma((a^{(2)})(-1), (b^{(2)})(-1))(a^{(2)})(0)\#1
\]
\[
= \sigma(a_{(-1)}, b_{(-1)})a_{(0)}b_{(0)}\#1.
\]
Finally, we describe the action of $H_{\sigma}$ on $R_{\sigma}$. By the definition of $\tilde{\sigma}$, the proof follows the same lines as [MO] Thm. 2.7. For completeness, we give the proof also here. Let $h \in H_{\sigma}$ and $a \in R_{\sigma}$, then
\[
h \cdot_{\sigma} a = (1 \# h) \cdot_{\sigma} (a \# 1)
\]
\[
= \tilde{\sigma}(1 \# h_{(1)}, a^{(1)})(-1)(a^{(3)})(-2)[1 \# h_{(2)}][(a^{(2)})(0)\#(a^{(3)})(-1)]\tilde{\sigma}^{-1}(1 \# h_{(3)}, (a^{(3)})(0)\#1)
\]
\[
= \varepsilon(a^{(1)})\sigma(h_{(1)}, (a^{(2)})(-1)(a^{(3)})(-2))[1 \# h_{(2)}][(a^{(2)})(0)\#(a^{(3)})(-1)]\varepsilon((a^{(3)})(0))
\]
\[
= \varepsilon(a^{(1)})\sigma(h_{(1)}, (a^{(2)})(-1))[1 \# h_{(2)}][(a^{(2)})(0)\#1]
\]
\[
= \sigma(h_{(1)}, a_{(0)})(h_{(2)} \rightarrow a_{(0)})\#h_{(3)}.
\]
Thus, by the definition of the action on the biproduct we have
\[
h \mapsto_a a = (1\#h(1)) \cdot \tilde{\sigma} (a\#1) \cdot \tilde{\sigma} S_\tilde{\sigma}(1\#h(2)) = \sigma(h(1), a(\cdot 1))(h(2) \rightarrow a(0))\#h(3) \cdot \tilde{\sigma} [1\#S_\sigma(h(4))]
\]
\[
= \sigma(h(1), a(\cdot 1))(h(2) \rightarrow a(0))\#h(3) \cdot \tilde{\sigma} [\sigma(h(4), S(h(5)))1\#S(h(6))\sigma^{-1}(S(h(7)), h(8))]
\]
\[
= \sigma(h(1), a(\cdot 1)) \cdot \tilde{\sigma}((h(2) \rightarrow a(0))(1)\#((h(2) \rightarrow a(0))(2))(-1)((h(2) \rightarrow a(0))(3))(-2)h(3), 1\#S(h(10)))
\]
\[
\cdot \tilde{\sigma}^{-1}(((h(2) \rightarrow a(0))(3)(0)\#h(5)))(1\#S(h(6)))\sigma(h(6), S(h(7)))\sigma^{-1}(S(h(11)), h(12))
\]
\[
\sigma(h(1), a(\cdot 1)) \cdot \tilde{\sigma}((h(2) \rightarrow a(0))(1)\#((h(2) \rightarrow a(0))(2))(-1)h(3), 1\#S(h(6)))
\]
\[
\cdot \tilde{\sigma}^{-1}(((h(2) \rightarrow a(0))(3)(0)\#h(5))(1\#S(h(6)))\sigma(h(6), S(h(7)))\sigma^{-1}(S(h(11)), h(12))
\]
\[
\sigma(h(1), a(\cdot 1)) \cdot \tilde{\sigma}((h(2) \rightarrow a(0))(1)\#((h(2) \rightarrow a(0))(2))(-1)h(3), 1\#S(h(6)))
\]
\[
\cdot \tilde{\sigma}^{-1}(((h(2) \rightarrow a(0))(3)(0)\#h(5))(1\#S(h(6)))\sigma(h(6), S(h(7)))\sigma^{-1}(S(h(11)), h(12))
\]
\[
\sigma(h(1), a(\cdot 1)) \cdot \tilde{\sigma}((h(2) \rightarrow a(0))(1)\#((h(2) \rightarrow a(0))(2))(-1)h(3), 1\#S(h(6)))
\]
\[
\cdot \tilde{\sigma}^{-1}(((h(2) \rightarrow a(0))(3)(0)\#h(5))(1\#S(h(6)))\sigma(h(6), S(h(7)))\sigma^{-1}(S(h(11)), h(12))
\]
where in the last sixth equality we used the 2-cocycle condition on \(\sigma\).

**Remark 3.3.** Assume \(H = k\Gamma\) and \(\sigma \in \mathcal{Z}^2(\Gamma, k)\). Let \(h \in \Gamma\) and \(a \in R\) be a homogeneous element of degree \(g \in \Gamma\); in particular, \(\delta(a) = g \otimes a\) and \(\Delta_a(a) = a \otimes 1 + g \otimes a\). Then \([\mathfrak{M}]\) yields
\[
(4) \quad h \mapsto_a a = h \cdot \sigma a \cdot \sigma h^{-1} = \sigma(h, g)\sigma^{-1}(gh^{-1}, h)h \rightarrow a
\]

**Remark 3.4.** In [AST], the authors introduced another type of cocycle deformation on a Hopf algebra, which is closely related to the one given above. We describe it shortly. Let \(\Gamma\) be an abelian group and \(A\) a Hopf algebra that is \(\Gamma \times \Gamma\)-graded. Given any \(\varphi \in \mathcal{Z}^2(\Gamma, k)\), define a new product on \(A\) by
\[
(5) \quad h * k := \varphi(\eta, \kappa)\varphi(\eta', \kappa')^{-1} h \cdot k
\]
for all homogeneous \(h, k \in H\) with degrees \((\eta, \eta'), (\kappa, \kappa') \in \Gamma \times \Gamma\). With this multiplication, the new algebra \(A^{(\varphi)}\) is a Hopf algebra with the same coalgebraic structure and unit as \(A\). Assume \(A = R\#k\Gamma\) is given by a bosonization over an abelian group \(\Gamma\). Then, the coaction of \(\Gamma\) on the elements of \(R\) induces a \(\Gamma \times \Gamma\) grading on \(A\) with \(\deg g = (g, g)\) for all \(g \in \Gamma\) and \(\deg(x) = (g, 1)\) if \(\delta(x) = g \otimes x\) with \(x \in R\) a homogeneous element and we have that \(A^{(\varphi)} = A_{\varphi}\), where \(\varphi\) is the Hopf 2-cocycle on \(A\) induced by \(\varphi\). In particular, for \(x, y\) homogeneous elements of \(R\) of degree \(g\) and \(h\) respectively, we have that
\[
x \cdot y = \varphi(g, h)\varphi(1, 1)^{-1} xy = \varphi(g, h)\varphi(1, 1)^{-1} xy = \varphi(x(0), y(0)) x(1) y(1),
\]
which coincides with formula (2).

We recall now a result from [AAGMV] for the case that \( R \) is a (pre-) Nichols algebra; as stated in loc. cit, the first two items are [MO] 2.7, 3.4. It implies in particular that under the hypothesis of the proposition above, if \( R \) is a (pre-) Nichols algebra, then \( R_\sigma \) is also a (pre-) Nichols algebra and the action is described by (3).

Let \( \mathfrak{B} \) be a pre-Nichols algebra over \( V, A = \mathfrak{B} \# H \) and \( \sigma : A \otimes A \to k \) a Hopf 2-cocycle. Let \( \mathfrak{F} = (F_n)_{n \geq 0} \) be the filtration of \( A_\sigma \) induced by the graduation of \( A \). Then \( \text{gr}_\mathfrak{F} A_\sigma = A_\sigma = A \) as coalgebras. Notice that, if \( \mathfrak{B} \) is a Nichols algebra, then \( \mathfrak{F} \) coincides with the coradical filtration.

**Proposition 3.5.** [AAGMV] Prop. 4.14

(a) There is an isomorphism of graded Hopf algebras \( \text{gr}_F A_\sigma \cong \mathfrak{B} \# H_\sigma \), for \( \mathfrak{B}' \) a pre-Nichols algebra over \( V' \in H_\sigma \mathcal{YD} \). Here \( V' \) is the \( H_\sigma \)-comodule \( V \) with action

\[
h \to \sigma v = \sigma(h(1), v((-1))h(2) \to v(0))(0)\sigma^{-1}((h(2) \to v(0))(-1), h(3)),
\]

for \( h \in H_\sigma, v \in V \); here \( V \) is identified with a subspace of \( T(V) \# H \), with \( H \)-action given by the adjoint. Furthermore, the product in \( \mathfrak{B}' \) is given by \( x \cdot y = \sigma(x(-1), y(-1))x(0)y(0), \) for \( x, y \in \mathfrak{B}' \) homogeneous.

(b) With the notation in (a), if \( \mathfrak{B} = \mathfrak{B}(V) \) is the Nichols algebra of \( V \), then \( \mathfrak{B}' = \mathfrak{B}(V') \).

(c) Suppose that \( H = k \Gamma, \Gamma \) a finite group; in particular, \( H_\sigma = H \). Let \( \{x_1, ..., x_\theta \} \) be a basis of \( V \) with \( x_i \in V^{yi}, g_i \in \Gamma, 1 \leq i \leq \theta \). If

\[
\sigma(g_ig_i) = \sigma(gg^{-1}, g), g \in \Gamma, 1 \leq i \leq \theta,
\]

then \( V' = V \in \Gamma \mathcal{YD} \).

The next lemma follows by a straightforward computation.

**Lemma 3.6.** Let \( \sigma \in \mathcal{Z}^2(A, k) \) and \( \tau \in \mathcal{Z}^2(A_\sigma, k) \). Then \( \tau \ast \sigma \in \mathcal{Z}^2(A, k) \).

**Proof.** Since \( \tau \ast \sigma(a, b) = \tau(a(1), b(1))\sigma(a(2), b(2)) \) and \( \tau, \tilde{\sigma} \) are both normalized 2-cocycles, it is clear that \( (\tau \ast \sigma)(a, 1) = \varepsilon(a) = (\tau \ast \sigma)(1, 1) \) for all \( a \in A \). Since \( \tau \in \mathcal{Z}^2(A_\sigma, k) \) we have that

\[
\tau(b(1), c(1))\tau(a, b(2) \cdot \sigma c(2)) = \tau(a(1), b(1))\tau(a(2) \cdot \sigma b(2), c),
\]

which implies that

\[
\tau(b(1), c(1))\sigma(b(2), c(2))\tau(a, b(3)c(3))^{-1}(b(4), c(4)) = \tau(a(1), b(1))\sigma(a(2), b(2))\tau(a(3)b(3), c)^{-1}(a(4), b(4)),
\]

for all \( a, b \in A \). Thus

\[
(\tau \ast \sigma)(b(1), c(1))(\tau \ast \sigma)(a, b(2)c(1)) = \tau(b(1), c(1))\sigma(b(2), c(2))\tau(a(1), b(3)c(3))\sigma(a(2), b(4)c(4))
\]

\[
= \tau(a(1), b(1))\sigma(a(2), b(2))\tau(a(3)b(3), c(1))\sigma(a(4), b(4)c(3))
\]

\[
= \tau(a(1), b(1))\sigma(a(2), b(2))\tau(a(3)b(3), c(1))\sigma(a(4)b(4), c(2))
\]

\[
= (\tau \ast \sigma)(a(1), b(1))(\tau \ast \sigma)(a(2)b(2), c).
\]

Remark 3.7. Let \( A \) be a Hopf algebra. As is known, the set \( \mathcal{Z}^2(A, k) \) is not a group in general. It is the case if the cocycles are lazy, see for example [CP]. Nevertheless, the lemma above suggests a groupoid structure on the set of all Hopf 2-cocycles.

Let \( \mathcal{Z} \) be the groupoid whose objects are Hopf algebras \( A \) and arrows labelled by the set of 2-cocycles \( \{\alpha_\sigma : A \to A_\sigma : \sigma \in \mathcal{Z}^2(A, k)\} \). The source and target maps are given by \( s(\alpha_\sigma) = A, t(\alpha_\sigma) = A_\sigma \) and the composition by \( \alpha_\sigma \circ \alpha_\sigma = \alpha_{\tau \ast \sigma} \) for \( \sigma \in \mathcal{Z}^2(A, k) \) and \( \tau \in \mathcal{Z}^2(A_\sigma, k) \), which by Lemma 3.6 is well-defined.

Clearly, the identity arrow is given by \( \text{id}_A = \alpha_{\varepsilon_A} \), and since \( (A_\sigma)_{\sigma^{-1}} = A_{\sigma^{-1}} = A = A_{\sigma^{-1}}^{-1} = (A_{\sigma^{-1}})_{\sigma} \), each arrow is invertible with inverse \( \alpha_{\sigma^{-1}} : A \to A_\sigma \).
3.2. Examples on pointed Hopf algebras over $S_4$. In this subsection we apply the results described above on examples of finite-dimensional pointed Hopf algebras over $S_4$. They give the relation between two non-isomorphic families related to different Nichols algebras. The 2-cocycles are given by the product of a Hochschild 2-cocycle on a Nichols algebra and a group 2-cocycle founded by Venden- 
ramin [V] which serves as twisting of the 2-cocycles associated to the rack of transpositions in $S_n$. For this purpose we need to introduce first some terminology, see [AGr2] Def. 1.1 for more details.

3.2.1. Racks and Nichols algebras. A rack is a pair $(X, \triangleright)$, where $X$ is a non-empty set and $\triangleright: X \times X \to X$ is a function, such that $q_i = i \triangleright (\cdot): X \to X$ is a bijection for all $i \in X$ satisfying that $i \triangleright (j \triangleright k) = (i \triangleright j) \triangleright (i \triangleright k)$ for all $i, j, k \in X$. A group $G$ is a rack with $x \triangleright y = xyx^{-1}$ for all $x, y \in G$. If $G = S_n$, then we denote by $O^n_j$ the conjugacy class of all $j$-cycles in $S_n$.

Let $(X, \triangleright)$ be a rack. A rack 2-cocycle $q: X \times X \to k^*$, $(i, j) \mapsto q_{ij}$ is a function such that

$$q_{ij} = q_{ij} = q_{ij} = q_{ij}, \text{ for all } i, j, k \in X.$$  

It determines a braiding $\psi$ on the vector space $kX$ with basis $\{x_i\}_{i \in X}$ by $\psi(x_i \otimes x_j) = q_{ij}x_i \otimes x_j$ for all $i, j \in X$. We denote this braided vector space $(kX, \psi)$ by $M(X, q)$ and the Nichols algebra associated with it by $\mathcal{B}(X, q)$.

Let $X$ be a subrack of a conjugacy class $O$ in $K$, $q$ a rack 2-cocycle on $X$ and $\varphi \in Z^2(K, k)$. Then the map $q^\varphi: X \times X \to k^*$ given by

$$(6) \quad q_{xy}^\varphi = \varphi(x, y)\varphi^{-1}(x \triangleright y, x)q_{xy}, \text{ for all } x, y \in X,$$

is a rack 2-cocycle.

If $X$ is any rack, $q$ a rack 2-cocycle on $X$ and $\varphi: X \times X \to k^*$, then define $q^\varphi$ by (6). It can be shown that $q^\varphi$ is a 2-cocycle if and only if

$$\varphi(x, z)\varphi(x \triangleright y, x \triangleright z)\varphi(x \triangleright (y \triangleright z), x)\varphi(y \triangleright z, y) = \varphi(y, z)\varphi(x, y \triangleright z)\varphi(x \triangleright (y \triangleright z), x \triangleright y)\varphi(x \triangleright z, x)$$

for any $x, y, z \in X$. Thus, if $X$ is a subrack of a group $\Gamma$ and $\varphi \in Z^2(\Gamma, k)$, then $\varphi|_{X \times X}$ satisfies the equation above.

Definition 3.8. Let $q, q': X \times X \to k^*$ be rack 2-cocycles on $X$. We say that $q$ and $q'$ are twist equivalent if there exists $\varphi: X \times X \to k^*$ such that $q' = q^\varphi$ as in (6).

3.2.2. Nichols algebras over $S_n$. Let $X = O^n_2$ with $n \geq 3$ or $X = O^n_4$ and consider the cocycles:

$$(-1): O^n_2 \times O^n_2 \to k^*, \quad (j, i) \mapsto \text{sg}(j) = -1;$$

$$\chi: O^n_2 \times O^n_2 \to k^*, \quad (j, i) \mapsto \chi(i, j) = \begin{cases} 1, & \text{if } i = (a, b) \text{ and } j(a) < j(b), \\ -1, & \text{if } i = (a, b) \text{ and } j(a) > j(b). \end{cases}$$

for all $i, j \in O^n_2$. By [MS, Ex. 6.4], [AGr2] Thm. 6.12, the Nichols algebras are given by

(a) $\mathcal{B}(O^n_2, -1)$; generated by the elements $\{x_{(km)}\}_{1 \leq m \leq n}$ satisfying for all $1 \leq a < b < c \leq n, 1 \leq e < f \leq n, \{a, b\} \cap \{e, f\} = \emptyset$ the identities

$$0 = x_{(ab)}^2 = x_{(ab)}x_{(ef)} + x_{(ef)}x_{(ab)} = x_{(ab)}x_{(bc)} + x_{(bc)}x_{(ac)} + x_{(ac)}x_{(ab)}.$$

(b) $\mathcal{B}(O^n_2, \chi)$; generated by the elements $\{x_{(km)}\}_{1 \leq m \leq n}$ satisfying for all $1 \leq a < b < c \leq n, 1 \leq e < f \leq n, \{a, b\} \cap \{e, f\} = \emptyset$ the identities

$$0 = x_{(ab)}^2 = x_{(ab)}x_{(ef)} - x_{(ef)}x_{(ab)} = x_{(ab)}x_{(bc)} - x_{(bc)}x_{(ac)} - x_{(ac)}x_{(ab)};$$

$$0 = x_{(bc)}x_{(ab)} - x_{(ac)}x_{(bc)} - x_{(ab)}x_{(ac)}.$$  

(c) $\mathcal{B}(O^n_4, -1)$; generated by the elements $x_{i}, i \in O^n_4$ satisfying for $ij = ki, j \neq i \neq k \in O^n_4$ that

$$0 = x_{i}^2 = x_{i}x_{i-1} + x_{i-1}x_{i} = x_{i}x_{j} + x_{k}x_{i} + x_{j}x_{k}.$$  

For $3 \leq n \leq 5$ the first two families of Nichols algebras are finite-dimensional. If $n > 5$ it is not known if this is the case. Moreover, the cocycles associated to them are twist equivalent.

Theorem 3.9. [V] Thm. 3.8 Let $n \geq 4$. The rack 2-cocycles $\chi$ and $-1$ associated to $O^n_2$ are twist equivalent. Hence the Hilbert series of $\mathcal{B}(O^n_2, -1)$ and $\mathcal{B}(O^n_2, \chi)$ are equal. □
Remark 3.10. The twist given by Theorem 3.9 is defined using a group 2-cocycle \( \varphi \in \mathcal{Z}^2(S_n, k) \). In particular, \(-1 = \varphi(x, y)\varphi^{-1}(x \triangleright y, x)\chi(x, y)\) for all \( x, y \in O_2^n \).

3.2.3. Cocycles on pointed Hopf algebras over \( S_4 \). The following theorem summarizes the classification of finite dimensional pointed Hopf algebras over \( S_4 \), see [AHS, GC].

Theorem 3.11. Let \( H \) be a nontrivial finite dimensional pointed Hopf algebra with \( G(H) = S_4 \). Then either \( H \simeq \mathfrak{B}(X, q) \# kS_n \) with \( (X, q) = (O_2^n, 1), (O_2^n, 1) \) or \( (O_2^n, \chi) \), or \( H \simeq \mathcal{H}(Q_4^n[1[t]]) \), or \( H \simeq \mathcal{H}(Q_4^n[1]) \), or \( H \simeq \mathcal{H}(D[1]) \) with \( t \in \mathbb{P}_k^1 \).

Let \( \varphi \in \mathcal{Z}^2(S_n, k) \) be the group 2-cocycle given in Remark 3.10. Denote again by \( \varphi \) the associated Hopf 2-cocycle in \( \mathcal{Z}^2(kS_n, k) \) and by \( \sigma \in \mathcal{Z}^2(A, k) \) the Hopf 2-cocycle on the bosonization \( A = \mathfrak{B}(O_2^n, \chi) \# kS_n \). Then by Propositions 3.7 and 3.5 we have that

\[
\mathfrak{B}(O_2^n, 1) \# kS_n \simeq (\mathfrak{B}(O_2^n, \chi) \# kS_n)_{\sigma \varphi}.
\]

On the other hand, we know that the family \( \mathcal{H}(Q_n^{-1}[(2\lambda, 3\lambda)]) \) of pointed Hopf algebras with diagram \( \mathfrak{B}(O_2^n, 1) \) is a cocycle deformation of the bosonization \( \mathfrak{B}(O_2^n, 1) \# kS_n \). The explicit cocycle is given in the theorem below; it was also shown in [GM] by other methods.

Shorty, let \( X \) be a rack, \( q \) a rack 2-cocycle and \( \{x_\tau \}_{\tau \in X} \) be homogeneous elements in \( V = M(X, q) \in S_n \mathcal{YD} \). Then the linear combination of tensor products of linear functionals \( \delta_\tau \) given by \( \delta_\tau(x_\mu) = \delta_{\tau, \mu} \) for all \( \mu, \tau \in X \) give rise to a Hochschild 2-cocycle \( \eta = \sum_{\tau, \mu \in X} \alpha_{\tau, \mu} d_\tau \otimes \mu \) by defining it via

\[
\eta(\mathfrak{B}^m(V) \otimes \mathfrak{B}^n(V)) = 0 \text{ if } (m, n) \neq (1, 1).
\]

If this cocycle is invariant under the action of \( S_n \), i.e., \( \eta^h(x, y) = \eta(h(1) \rightarrow x, h(2) \rightarrow y) = \eta(x, y) \) for all \( x, y \in \mathfrak{B}(V) \) and \( h \in S_n \), then one may define a Hochschild 2-cocycle on \( \mathfrak{B}(V) \# kS_n \) by

\[
\tilde{\eta}(x \# h, y \# k) = \eta(x, h \rightarrow y) \varepsilon(k) \text{ for all } x, y \in \mathfrak{B}(V), h, k \in S_n.
\]

Moreover, by [GM] Cor. 2.6 the exponentiation \( \sigma = e^{\tilde{\eta}} \) is a Hopf 2-cocycle on \( \mathfrak{B}(V) \# kS_n \). These families of cocycles give the desired deformation for two of the three families of pointed Hopf algebras over \( S_4 \). See [GM] 2.1 for more details.

Theorem 3.12. [GM] Thm. 4.10] Let \( A = \mathfrak{B}(X, -1) \# kS_n \) and \( \sigma_\lambda = e^{\tilde{\eta}_\lambda} \) a Hopf 2-cocycle with \( \eta_\lambda = \frac{1}{3} \sum_{\mu, \tau \in O_2^n} d_\tau \otimes d_\mu \) and \( \lambda \in k \).

(i) If \( X = O_2^n \) then \( A_{\sigma_\lambda} \simeq \mathcal{H}(Q_n^{-1}[(2\lambda, 3\lambda)]) \) for \( n \geq 4 \).

(ii) If \( X = O_2^4 \) then \( A_{\sigma_\lambda} \simeq \mathcal{H}(D[(\lambda, 3\lambda)]) \).

We end this section with the following result.

Theorem 3.13. Let \( \sigma_\lambda = e^{\tilde{\eta}_\lambda} \in \mathcal{Z}^2(\mathfrak{B}(O_2^n, -1) \# kS_n, k) \) and \( \sigma_{\varphi} \in \mathcal{Z}^2(\mathfrak{B}(O_2^n, \chi) \# kS_n, k) \) be the Hopf 2-cocycles defined above. Then

\[
\mathcal{H}(Q_n^{-1}[(2\lambda, 3\lambda)]) \simeq (\mathfrak{B}(O_2^n, \chi) \# kS_4)_{\sigma_\lambda \star \sigma_{\varphi}}.
\]

Proof. Since \( \sigma_\lambda = e^{\tilde{\eta}_\lambda} \) is a Hopf 2-cocycle on \( \mathfrak{B}(O_2^n, -1) \# kS_n \) and this algebra is isomorphic to \( (\mathfrak{B}(O_2^n, \chi) \# kS_4)_{\sigma_{\varphi}} \) for \( \varphi \in \mathcal{Z}^2(S_4, k) \), the claim follows by Lemma 3.16.

4. Multiparameter quantum groups, liftings of pre-Nichols algebras and cocycle deformations

In this section we show explicitly the probably well-known fact that multiparameter quantum groups can be described using the theory of pointed Hopf algebras developed by Andruskiewitsch and Schneider [AS4, ARS].

First we describe these families of multiparameter pointed Hopf algebras and show that they are cocycle deformations of certain (one-parameter) families of pointed Hopf algebras by using the results of Section 3.
4.1. On pointed Hopf algebras associated to generalized Cartan matrices. We begin with the following definition.

**Definition 4.1.** [ARS] Def. 3.2] A reduced YD-datum
\[ D_{\text{red}} = D(\Gamma, (L_i)_{1 \leq i \leq \ell}, (K_i)_{1 \leq i \leq \theta}, (\chi_i)_{1 \leq i \leq \theta}) \]
consists of an abelian group \( \Gamma \), a positive integer \( \theta \), \( K_i, L_i \in \Gamma \) and characters \( \chi_i \in \hat{\Gamma} = \text{Hom}(\Gamma, k^\times) \) for all \( 1 \leq i \leq \theta \) satisfying
\[ \chi_j(K_i) = \chi_i(L_j) \quad \text{for all } 1 \leq i, j \leq \theta, \]
\[ K_iL_i \neq 1 \quad \text{for all } 1 \leq i \leq \theta. \]
A reduced YD-datum \( D_{\text{red}} \) is called generic if for all \( 1 \leq i \leq \theta \), \( \chi_i(K_i) \) is not a root of unity. A linking parameter \( \ell \) for a reduced YD-datum \( D_{\text{red}} \) is a family \( \ell = (\ell_i)_{1 \leq i \leq \theta} \) of non-zero elements in \( k \).

Let \( I = \{1, 2, \ldots, \theta \} \) and define \( q_{ij} = \chi_j(K_i) \) for all \( i, j \in I \). We have an equivalence relation on \( I \): for \( i \neq j \in I \) we say that \( i \sim j \) if and only if there are \( i_1, \ldots, i_t \in I \) with \( t \geq 2 \), \( i_1 = i, i_t = j \) and \( q_{i_1i_2}q_{i_2i_3}q_{i_3i_4} \cdots q_{i_{t-1}i_t} \neq 1 \) for all \( 1 \leq k < t \). We denote by \( \mathcal{X} \) the set of equivalence classes.

Let \( (a_{ij})_{1 \leq i,j \leq \theta} \) be a generalized Cartan matrix, that is, \( (a_{ij})_{1 \leq i,j \leq \theta} \) is a matrix with integer entries such \( a_{ii} = 2 \) for all \( 1 \leq i \leq \theta \), and for all \( 1 \leq i, j \leq \theta \), \( i \neq j \), \( a_{ij} \leq 0 \), and if \( a_{ij} = 0 \), then \( a_{ji} = 0 \).

**Definition 4.2.** (i) [AS3] Def. 2.1] A datum of Cartan type
\[ D = D(\Gamma, (g_i)_{1 \leq i \leq \theta}, (\chi_i)_{1 \leq i \leq \theta}, (a_{ij})_{1 \leq i,j \leq \theta}) \]
consists of an abelian group \( \Gamma \), elements \( g_1, \ldots, g_\theta \in \Gamma \), \( \chi_1, \ldots, \chi_\theta \in \hat{\Gamma} \) and a general Cartan matrix \( (a_{ij})_{1 \leq i,j \leq \theta} \) satisfying
\[ q_{ij}q_{ji} = q_{ij}^{a_{ij}}, \quad q_{ii} \neq 1 \quad \text{with } q_{ij} = \chi_j(g_i) \quad \text{for all } 1 \leq i, j \leq \theta. \]

(ii) [ARS] Def. 3.17] A reduced YD-datum of Cartan type
\[ D = D(\Gamma, (L_i)_{1 \leq i \leq \theta}, (K_i)_{1 \leq i \leq \theta}, (\chi_i)_{1 \leq i \leq \theta}, (a_{ij})_{1 \leq i,j \leq \theta}) \]
is a reduced YD-datum \( D(\Gamma, (L_i)_{1 \leq i \leq \theta}, (K_i)_{1 \leq i \leq \theta}, (\chi_i)_{1 \leq i \leq \theta}) \) such that
\[ q_{ij}q_{ji} = q_{ij}^{a_{ij}}, \quad q_{ii} \neq 1, \quad 0 \leq -a_{ij} < \text{ord}(q_{ii}) \leq \infty, \quad \text{for all } 1 \leq i, j \leq \theta. \]

(iii) [AS3] A (reduced) YD-datum of Cartan type is said of DJ-type (Drinfeld-Jimbo type) if the Cartan matrix is symmetrizable, i.e. there exist \( d_i \in k^\times \) such that \( d_ia_{ij} = d_ja_{ji} \) for all \( 1 \leq i, j \leq \theta \), and for all \( I \in \mathcal{X} \) there exists \( q_I \in k \) such that
\[ q_{ij} = q_I^{d_{aij}}, \quad \text{for all } i \in I, 1 \leq j \leq \theta. \]
In particular, \( q_{ij} = 1 \) if \( a_{ij} = 0 \). We denote this datum by
\[ D_q = D(\Gamma, (L_i)_{1 \leq i \leq \theta}, (K_i)_{1 \leq i \leq \theta}, (q_i)_{I \in \mathcal{X}}, (a_{ij})_{1 \leq i,j \leq \theta}). \]

As before, a reduced YD-datum of Cartan type is called generic if \( q_{ii} \) is not a root of unity for all \( i \in I \). The equivalence relations above can be described as usual in terms of the Cartan matrix. Indeed, for all \( 1 \leq i, j \leq \theta \), \( i \sim j \) if and only if there are \( i_1, \ldots, i_t \in I \) with \( t \geq 2 \) with \( i_1 = i, i_t = j \), and \( a_{i_1i_2}a_{i_2i_3} \cdots a_{i_{t-1}i_t} \neq 0 \) for all \( 1 \leq k < t \).

Let \( D_{\text{red}} = D(\Gamma, (L_i)_{1 \leq i \leq \theta}, (K_i)_{1 \leq i \leq \theta}, (\chi_i)_{1 \leq i \leq \theta}) \) be a reduced YD-datum. Let \( V, W \) be vector spaces with basis \( \{v_i\}_{1 \leq i \leq \theta} \) and \( \{\omega_i\}_{1 \leq i \leq \theta} \), respectively. Then \( D_{\text{red}} \) defines on \( V \) and \( W \) a Yetter-Drinfeld module structure over \( k\Gamma \) given by \( v_i \in V_{K_i}^\times \) and \( \omega_i \in W_{L_i}^{\times -1} \) for all \( 1 \leq i \leq \theta \), that is,
\[ \delta(v_i) = K_i \otimes v_i, \quad g \cdot v_i = \chi_i(g)v_i, \]
\[ \delta(\omega_i) = L_i \otimes \omega_i, \quad g \cdot \omega_i = \chi_i^{-1}(g)\omega_i, \]
for all \( g \in \Gamma \) and \( 1 \leq i \leq \theta \). The braiding \( c_V = c_{V,V} \) of \( V \) and \( c_W \) of \( W \) are given by
\[ c_V(v_i \otimes v_j) = K_i \cdot v_j \otimes v_i = \chi_j(K_i)v_j \otimes v_i = q_{ij}v_j \otimes v_i, \quad 1 \leq i, j \leq \theta, \]
\[ c_W(\omega_i \otimes \omega_j) = L_i \cdot \omega_j \otimes \omega_i = \chi_j^{-1}(L_i)\omega_j \otimes \omega_i = q_{ji}^{-1}\omega_j \otimes \omega_i, \quad 1 \leq i, j \leq \theta, \]
in particular, they are of diagonal type, and the corresponding adjoint actions are given by
\[
\text{ad}_c(v_i)(v_j) = v_i v_j - \chi_j(K_i) v_j v_i = v_i v_j - q_{ij} v_j v_i, \quad 1 \leq i, j \leq \theta, \\
\text{ad}_c(\omega_i)(\omega_j) = \omega_i \omega_j - \chi_j^{-1}(L_i) \omega_j \omega_i = \omega_i \omega_j - q_{ji}^{-1} \omega_j \omega_i, \quad 1 \leq i, j \leq \theta.
\]

**Definition 4.3.** [AS1] Def. 2.4] The pre-Nichols algebras \( R(D) \), \( R(D, V) \) and \( R(D, W) \) associated to the reduced YD-datum described above are given by the quotient (braided) Hopf algebras
\[
R(D) = T(V \oplus W)/ (\text{ad}_c(v_i))^{1-\alpha_{ij}}(v_j), \quad \text{ad}_c(\omega_i)^{1-\alpha_{ij}}(\omega_j), 1 \leq i \neq j \leq \theta), \\
R(D, V) = T(V) / (\text{ad}_c(v_i))^{1-\alpha_{ij}}(v_j), 1 \leq i \neq j \leq \theta), \\
R(D, W) = T(W) / (\text{ad}_c(\omega_i))^{1-\alpha_{ij}}(\omega_j), 1 \leq i \neq j \leq \theta).
\]

Since \( c_{W,V}c_{V,W} = \text{id} \), we have that \( R(D) \simeq R(D, V) \otimes R(D, W) \), see [MK2]. By abuse of notation, we denote the images of the elements \( v_i, \omega_j \) in \( R(D) \) again by \( v_i, \omega_j \). It is well-known that the elements \( \text{ad}_c(x_i)^{1-\alpha_{ij}}(x_j), \quad 1 \leq i \neq j \leq \theta \) are primitive in the free algebra \( T(V) \) (see for example [AS1, A.1]), hence they generate a Hopf ideal.

**Remark 4.4.** In case the braiding is positive and generic, the pre-Nichols algebras \( R(D, V) \) and \( R(D, W) \) coincide with the Nichols algebras \( \mathcal{B}(V) \) and \( \mathcal{B}(W) \) respectively, that is, the ideals \( I(V) \) and \( I(W) \) are generated by the quantum Serre relations \( \text{ad}_c(v_i)^{1-\alpha_{ij}}(v_j), \quad \text{ad}_c(\omega_i)^{1-\alpha_{ij}}(\omega_j) \) associated to the braided commutators, see [AS3] Thm. 4.3] and references therein. These Serre relations are not enough to define the ideal \( I(V \oplus W) \), see Remark 4.6.

**Definition 4.5.** [AS4] Def. 2.4] Let \( D_{\text{red}} = D(\Gamma, (L_i)_{1 \leq i \leq \theta}, (K_i)_{1 \leq i \leq \theta}, (\chi_i)_{1 \leq i \leq \theta}, (a_{ij})_{1 \leq i,j \leq \theta}) \) be a reduced YD-datum of Cartan type and \( \ell \) a linking parameter. We define \( \bar{U}(D_{\text{red}}, \ell) \) as the quotient Hopf algebra of the bosonization \( R(D)\#k\Gamma \) modulo the ideal generated by the elements
\[
v_i \omega_j - \chi_j^{-1}(K_i) \omega_j v_i - \delta_{ij} \ell_i (K_i L_i - 1) \quad \text{for all } 1 \leq i, j \leq \theta,
\]
where we identify \( v_i = v_i \# 1, \omega_i = \omega_i \# 1 \) and \( K_i = 1 \# K_i, L_i = 1 \# L_i \) for all \( 1 \leq i \leq \theta \).

**Remark 4.6.** In case the braiding is positive and generic, the ideal \( I(V \oplus W) \subseteq T(V \oplus W) \) is generated by \( I(V), I(W) \) and \( v_i \omega_j - \chi_j^{-1}(K_i) \omega_j v_i \) for all \( 1 \leq i, j \leq \theta \), see [ARS, Rmk. 1.10]. In particular, we have that \( \bar{U}(D_{\text{red}}, 0) \simeq \mathcal{B}(V \oplus W)\# \mathbb{Z}^{2\theta} \).

We end this subsection with a result that translates the notion of twist-equivalence of matrices of diagonal braiding [AS3] Prop. 2.2], [Ro] Thm. 2.1] to cocycle deformations. It states that, under some assumptions, these families of pointed Hopf algebras depend only on one parameter for each connected component, up to cocycle deformations.

**Theorem 4.7.** Assume \( \Gamma \) is a free abelian group of rank \( 2\theta \) with generators \( (L_i)_{1 \leq i \leq \theta}, (K_i)_{1 \leq i \leq \theta} \). Let \( D_{\text{red}} = D(\Gamma, (L_i)_{1 \leq i \leq \theta}, (K_i)_{1 \leq i \leq \theta}, (\chi_i)_{1 \leq i \leq \theta}, (a_{ij})_{1 \leq i,j \leq \theta}) \) be a reduced YD-datum and \( \ell \) a linking parameter. If \( 1 \neq q_{ii} \) is a positive real number for all \( i \in \mathbb{I} \), then \( \bar{U}(D_{\text{red}}, \ell) \) is a cocycle deformation of a Hopf algebra \( \mathcal{U}(D_{\mathcal{Q}}, \ell) \) associated with a reduced YD-datum of DJ-type.

**Proof.** By [Ro] Thm. 2.1] we have that the Cartan matrix \( (a_{ij})_{1 \leq i,j \leq \theta} \) is symmetrizable, with symmetrizing diagonal matrix \( (d_i)_{1 \leq i \leq \theta} \) and there is a collection of positive numbers \( (q_I)_{I \in \mathcal{X}} \) such that \( (q_{ij}) \) is twist-equivalent to \( (q_{ij}) \), where \( q_{ij} = q_I^{d_{ij}} \) for all \( i, j \in I \).

If we order the group generators by \( L_1, \ldots, L_\theta, K_1, \ldots, K_\theta \) and take the corresponding characters \( \chi_1^{-1}, \ldots, \chi_\theta^{-1}, \chi_1, \ldots, \chi_\theta \), the matrix of the braiding in \( V \oplus W \) is given by
\[
p_{ij} = \begin{cases}
q_{ij}^{-1} & \text{if } 1 \leq i, j \leq \theta, \\
q_{ij}^{-1} & \text{if } 1 \leq i \leq \theta, \; \theta + 1 \leq j \leq 2\theta, \\
q_{ij} & \text{if } \theta + 1 \leq i \leq 2\theta, \; 1 \leq j \leq \theta, \\
q_{ij} & \text{if } \theta + 1 \leq i, j \leq 2\theta.
\end{cases}
\]
Let \( D_q = D(\Gamma, (L_i)_{1 \leq i \leq \theta}, (K_i)_{1 \leq i \leq \theta}, (\eta_i)_{i \in \mathbb{L}}, (a_{ij})_{1 \leq i,j \leq \theta}) \) be the reduced YD-datum of DJ-type associated with \( (\hat{p}_{ij}) \). Denote by \( \hat{V}, \hat{W} \) the braided vector spaces associated with this datum and by \( (\hat{p}_{ij})_{i,j} \) the matrix of the braiding in \( \hat{V} \oplus \hat{W} \). Let \( \hat{U}(D_q, \ell) \) be the corresponding pointed Hopf algebra with linking parameter \( \ell \).

If we set \( g_i = L_i \) and \( g_{i+\theta} = K_i \) for all \( 1 \leq i \leq \theta \), then by [AS3 Prop. 2.2], the map \( \sigma : \Gamma \times \Gamma \to \mathbb{k}^x \) given by

\[
\sigma(g_i, g_j) = \begin{cases} \hat{p}_{ij}^{-1} & \text{if } i \leq j, \\ 1 & \text{otherwise,} \end{cases}
\]

is a group 2-cocycle. Denote by \( \tilde{\sigma} \in \mathbb{Z}^2((V \oplus W)^\# \Gamma, \mathbb{k}) \) the Hopf 2-cocycle given by Proposition \( \ref{prop:2-cocycle} \). Then \( \tilde{\sigma} \) induces a Hopf 2-cocycle on \( \hat{U}(D_{\text{red}}, \ell)_{\tilde{\sigma}} \) and we have that \( \hat{U}(D_{\text{red}}, \ell)_{\tilde{\sigma}} \simeq \hat{U}(D_q, \ell) \). Indeed, by Propositions \( \ref{prop:2-cocycle} \) and \( \ref{prop:2-cocycle} \) and the proof of [AS2 Prop. 3.9] we have that \( (T(V \oplus W)^\# \Gamma)_{\tilde{\sigma}} = T(V \oplus W)_{\sigma} \# \Gamma = T(V \oplus W)_{\# \Gamma} \). For example, for \( i \leq j \in I \)

\[
K_i \cdot v_j = \sigma(K_i, K_j)\sigma^{-1}(K_j, K_i)K_i \cdot v_j = \hat{q}_{ij}^{-1}q_{ij}v_j = \hat{q}_{ij}v_j = q_{ij}d_{a_{ij}} = v_j.
\]

Let \( J \) be the ideal of \( T(V \oplus W) \) generated by the elements \( v_i \omega_j - q_{ij}^{-1} \omega_j v_i - \delta_{ij} \ell_i(L_i - 1) \) for all \( 1 \leq i,j \leq \theta \). To prove the claim it suffices to show that the corresponding ideal in \( T(V \oplus W)_{\sigma} \) coincides with the ideal \( J \) generated by the elements \( v_i \omega_j - q_{ij}^{-1} \omega_j v_i - \delta_{ij} \ell_i(L_i - 1) \). But by Proposition \( \ref{prop:2-cocycle} \) and the definition of \( \sigma \) we have for all \( 1 \leq i,j \leq \theta \) that

\[
v_i \cdot \omega_j - q_{ij}^{-1} \omega_j \cdot v_i - \delta_{ij} \ell_i(L_i - 1) = \sigma(K_i, L_j)v_i \omega_j - q_{ij}^{-1}\sigma(L_j, K_i)\omega_j v_i - \delta_{ij} \ell_i(L_i - 1)
\]

\[
= v_i \omega_j - q_{ij}^{-1}(\hat{q}_{ij}^{-1})\omega_j v_i - \delta_{ij} \ell_i(L_i - 1)
\]

\[
= v_i \omega_j - q_{ij}^{-1} \omega_j v_i - \delta_{ij} \ell_i(L_i - 1).
\]

\( \square \)

4.2. Multiparameter quantum groups as quotients of bosonizations of pre-Nichols algebras. In this subsection we show how the multiparameter quantum groups \( U_q(g_A) \), or multiparameter quantum groups for short, associated with a symmetrizable generalized Cartan matrix, introduced by Hu, Pei and Rosso [HPR], can be described using reduced data. These multiparameter quantum groups describe in a unified way families of quantum groups introduced by other authors, see [HPR] and references therein. Notice that in [AS3], Andruskiewitsch and Schneider characterized all pointed Hopf algebras that can be constructed using a generic datum of finite Cartan type for a free group of finite rank.

Let \( g_A \) be a symmetrizable Kac-Moody algebra with \( A = (a_{ij})_{i,j \in I} \) the associated generalized Cartan matrix, with \( I \) a finite set. Let \( d_i \) be relatively prime positive integers such that \( d_i a_{ij} = d_j a_{ji} \) for all \( i,j \in I \). Let \( \Phi \) be a finite root system with \( \Pi = \{ \alpha_i : i \in I \} \) a set of simple roots, \( Q = \bigoplus_{i \in I} \mathbb{Z} \alpha_i \) the root lattice, \( \Phi^+ \) the set of positive roots with respect to \( \Pi \), \( Q^+ = \bigoplus_{i \in I} \mathbb{Z}^+ \alpha_i \) the positive root lattice, \( \Lambda \) the weight lattice and \( \Lambda^+ \) the set of dominant weights. For \( i,j \in I \), let \( q_{ij} \) be indeterminates over \( \mathbb{Q} \) and \( \mathbb{Q}(q_{ij} : i,j \in I) \) the fraction field of the polynomial ring \( \mathbb{Q}[q_{ij} : i,j \in I] \) satisfying

\[
q_{ij}q_{ji} = q_{ii}^{-a_{ij}} \quad \text{for all } i,j \in I.
\]

Without loss of generality, we may assume that \( k \supset \mathbb{Q}(q_{ij} : i,j \in I) \) and it contains all \( m \)th roots of unity of \( q_{ii} \) for all \( m \in \mathbb{N} \) and \( i \in I \). Denote \( q = (q_{ij})_{ij \in I} \).
Definition 4.8. [HPR] Def. 7] \( U_q(\mathfrak{g}_A) \) is the unital associative algebra over \( k \) generated by elements \( e_i, f_i, \omega_i^{\pm 1} \) and \( \omega_i^{\prime \pm 1} \) with \( i \in I \) satisfying the following relations:

\[
\begin{align*}
(R1) \quad & \omega_i^{\pm 1} \omega_j^{\pm 1} = \omega_j^{\pm 1} \omega_i^{\pm 1}, \quad \omega_i^{\pm 1} \omega_i^{\mp 1} = \omega_i^{\prime \pm 1} \omega_i^{\prime \mp 1} = 1, \\
(R2) \quad & \omega_i^{\pm 1} \omega_j^{\pm 1} = \omega_j^{\pm 1} \omega_i^{\pm 1}, \quad \omega_i^{\pm 1} \omega_j^{\mp 1} = \omega_j^{\prime \pm 1} \omega_i^{\prime \mp 1}, \\
(R3) \quad & \omega_i e_j \omega_i^{-1} = q_{ij} e_j, \quad \omega_i e_j \omega_i^{-1} = q_{ji}^{-1} e_j, \\
(R4) \quad & \omega_i f_j \omega_i^{-1} = q_{ij}^{-1} f_j, \quad \omega_i f_j \omega_i^{-1} = q_{ji} f_j, \\
(R5) \quad & [e_i, f_j] = \delta_{ij} - q_{ii}^{-1} (\omega_i - \omega_i^{\prime}), \\
(R6) \quad & \sum_{k=0}^{1-a_{ij}} (-1)^k \left( 1 - a_{ij} \right) \frac{k!}{k!} q_{ij}^{-\frac{i}{2}} e_i^{-a_{ij}-k} e_i^k = 0 \quad (i \neq j), \\
(R7) \quad & \sum_{k=0}^{1-a_{ij}} (-1)^k \left( 1 - a_{ij} \right) \frac{k!}{k!} q_{ij}^{-\frac{i}{2}} f_i^{-a_{ij}-k} f_i^k = 0 \quad (i \neq j).
\end{align*}
\]

Moreover, \( U_q(\mathfrak{g}_A) \) is a Hopf algebra with its coproduct, counit and antipode determined for all \( i, j \in I \) by (see [HPR Prop. 7]):

\[
\begin{align*}
\Delta(e_i) &= e_i \otimes 1 + \omega_i \otimes e_i, \quad \epsilon(e_i) = 0, \quad S(e_i) = -\omega_i^{-1} e_i, \\
\Delta(f_i) &= f_i \otimes \omega_i^{-1} + 1 \otimes f_i, \quad \epsilon(f_i) = 0, \quad S(f_i) = -f_i \omega_i^{-1}, \\
\Delta(\omega_i^{\pm 1}) &= \omega_i^{\pm 1} \otimes \omega_i^{\pm 1}, \quad \epsilon(\omega_i^{\pm 1}) = 1, \quad S(\omega_i^{\pm 1}) = \omega_i^{\mp 1}, \\
\Delta(\omega_i^{\prime \pm 1}) &= \omega_i^{\prime \pm 1} \otimes \omega_i^{\prime \pm 1}, \quad \epsilon(\omega_i^{\prime \pm 1}) = 1, \quad S(\omega_i^{\prime \pm 1}) = \omega_i^{\prime \mp 1}.
\end{align*}
\]

Next we prove that this multiquantum group can be described using reduced data.

Definition of \( \bar{U}(\mathcal{D}_{\text{red}}, \ell) \). Let

- \( \theta = |I| \),
- \( \Gamma = \mathbb{Z}^{|I|} \) and denote \( K_i, L_i \) with \( i \in I \) two (commuting) generators,
- \( \chi_i \in \mathbb{F} \) given by \( \chi_i(K_j) = q_{ij} \) and \( \chi_i(L_j) = q_{ij} \) for all \( i, j \in I \).

In particular, we have that \( \chi_i(L_j) = \chi_j(K_i) \) for all \( i, j \in I \). Since by assumption \( K_i L_i \neq 1 \) and by [7], \( q_{ij} q_{ji} = q_{ii}^a \) with \( q_{ii} \neq 1 \), we have that \( \mathcal{D}_{\text{red}} = \mathcal{D}(\Gamma, (K_i), (L_i), (\chi_i), (a_{ij})) \) is a reduced YD-datum of Cartan type.

Let \( V, W \) be the vector spaces linearly generated by the elements \( x_i \) and \( y_i \) for all \( 1 \leq i \leq \theta \). Following the definition of reduced data, both have a Yetter-Drinfeld module structure. In this case, it is given for all \( i, j \in I \) by

\[
\begin{align*}
\delta(x_j) &= K_j \otimes x_j, \quad K_i \cdot x_j = \chi_j(K_i)x_j = q_{ij} x_j, \quad L_i \cdot x_j = q_{ji} x_j, \\
\delta(y_j) &= L_j \otimes y_j, \quad K_i \cdot y_j = \chi_j^{-1}(K_i)y_j = q_{ij}^{-1} y_j, \quad L_i \cdot y_j = q_{ji}^{-1} y_j.
\end{align*}
\]

Recall that for a linking parameter \( \ell = (\ell_i)_{1 \leq i \leq \theta} \), the pointed Hopf algebra \( \bar{U}(\mathcal{D}_{\text{red}}, \ell) \) associated with these data is given by the quotient Hopf algebra of the bosonization \( R(\mathcal{D}) \# k\mathbb{Z}^{2\theta} \) modulo the ideal generated by

\[ x_i y_j - q_{ij}^{-1} y_j x_i - \delta_{ij} \ell_i (K_i L_i - 1) \quad \text{for all } i, j \in I. \]

In particular, in \( \bar{U}(\mathcal{D}_{\text{red}}, \ell) \), \( x_i \) is a \((1, K_i)\)-primitive and \( y_i \) is a \((1, L_i)\)-primitive. Indeed, for \( x_i \in R(\mathcal{D}) \# k\mathbb{Z}^{2\theta} \) we have \( \Delta(x_i) = x_i^{(1)} \otimes x_i^{(2)} = x_i \otimes 1 + 1 \otimes x_i \) and

\[ \Delta(x_i) = x_i^{(1)} \# (x_i^{(2)})_{(-1)} \otimes (x_i^{(2)})_{(0)} \# 1 = (x_i \# 1) \otimes (1 \# 1) + (1 \# K_i) \otimes (x_i \# 1) = x_i \otimes 1 + K_i \otimes x_i. \]

Now we proceed to prove that \( U_q(\mathfrak{g}_A) \) can be described using the reduced datum of Cartan type defined above.
Theorem 4.9. $U_q(\mathfrak{g}_A) \simeq \widetilde{U}(\mathcal{D}_{\text{red}}, \ell)$ with linking parameter $\ell_i = \frac{q_{ii}}{q_{ii} - 1}$ for all $1 \leq i \leq \theta$.

Proof. Let $\varphi : U_q(\mathfrak{g}_A) \to \widetilde{U}(\mathcal{D}_{\text{red}}, \ell)$ be the algebra map defined by

$$
\varphi(\omega_i) = K_i, \quad \varphi(\omega_i') = L_i^{-1}, \quad \varphi(e_i) = x_i, \quad \varphi(f_i) = y_i L_i^{-1}
$$

for all $1 \leq i \leq \theta$.

Clearly, the map $\varphi$ is an epimorphism, if it is well-defined. To prove that it is well-defined, we show that the relations in $U_q(\mathfrak{g}_A)$ are mapped to $0$ by $\varphi$. First, notice that the action of $K_i$ and $L_i$ on $x_j$ and $y_j$ yields a commutation relation in $T(V \oplus W)^{\# \mathbb{Z}^{2\theta}}$; for example, $(1#K_i)(x_j\#1)(1#K_i^{-1}) = [(K_i \cdot x_j \# K_i)(1#K_i^{-1})] = K_i \cdot x_j \# 1$. Clearly, we need to verify only relations (R3) - (R7). For (R3), we have

$$
\varphi(\omega_i e_j \omega_i^{-1} - q_{ij} e_j) = K_i x_j K_i^{-1} - q_{ij} x_j = K_i \cdot x_j - q_{ij} x_j = 0,
$$

$$
\varphi(\omega_i' e_j \omega_i'^{-1} - q_{ij} e_j) = L_i^{-1} x_j L_i - q_{ij}^{-1} x_j = L_i^{-1} \cdot x_j - q_{ij}^{-1} x_j = 0.
$$

The proof for (R4) follows the same lines. Since $D$ is a reduced datum, for (R5) we have

$$
\varphi([e_i, f_j]) = x_i y_j L_i - y_j L_i^{-1} x_i = x_i y_j L_i - \chi_i(L_j^{-1}) y_j x_i L_j^{-1} = x_i y_j L_i - q_{ij}^-1 y_j x_i L_j^{-1} = (x_i y_j - q_{ij}^{-1} y_j x_i)L_i^{-1} = \delta_{ij} \ell_i (K_i L_i - 1)L_i^{-1} = \delta_{ij} \frac{q_{ii}}{q_{ii} - 1} (K_i - L_i^{-1})
$$

$$
= \varphi \left( \delta_{ij} \frac{q_{ii}}{q_{ii} - 1} (\omega_i - \omega_i') \right).
$$

To verify (R6) and (R7) one could note that their images under $\varphi$ are the quantum Serre relations in $T(V \oplus W)$, e.g. $\text{ad}_c(x_i)(1-x_j)(x_i) = (x_i - (K_i \cdot x_j)) x_i = x_i x_j - q_{ij} x_j x_i$.

one may prove by induction that

$$
\text{ad}_c(x_i)^n(x_j) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} q_{ii}^{k(k-1)} q_{ij}^k x_i^{n-k} x_j x_i^k \quad \text{for all } n \in \mathbb{N}.
$$

Assuming that the equality holds for $n \in \mathbb{N}$ and using (11) we have

$$
\text{ad}_c(x_i)^{n+1}(x_j) = \text{ad}_c(x_i)(\text{ad}_c(x_i)(x_j)) = \text{ad}_c(x_i) \left( \sum_{k=0}^{n} (-1)^k \binom{n}{k} q_{ii}^{k(k-1)} q_{ij}^k x_i^{n-k} x_j x_i^k \right)
$$

$$
= \sum_{k=0}^{n} (-1)^k \binom{n}{k} q_{ii}^{k(k-1)} q_{ij}^k \left[ x_i^{n+1-k} x_j x_i^k - K_i \cdot (x_i^{n-k} x_j x_i^k) x_i \right]
$$

$$
= \sum_{k=0}^{n} (-1)^k \binom{n}{k} q_{ii}^{k(k-1)} q_{ij}^k \left[ x_i^{n+1-k} x_j x_i^k - q_{ii} q_{ij} x_i^{n-k} x_j x_i^{k+1} \right]
$$

$$
= \sum_{k=0}^{n} (-1)^k \binom{n}{k} q_{ii}^{k(k-1)} q_{ij}^k x_i^{n+1-k} x_j x_i^k + \sum_{k=0}^{n} (-1)^{k+1} \binom{n}{k} q_{ii}^{k(k-1)} q_{ij}^{k+1} x_i^{n-k} x_j x_i^{k+1}
$$

$$
= x_i^{n+1} x_j + (-1)^{n+1} n q_{ii}^{(n+k-2)} q_{ij}^k x_i^{n+1-k} x_j x_i^k + \sum_{k=1}^{n} (-1)^k q_{ij}^k \left[ \binom{n}{k} q_{ii}^{k(k-1)} + \binom{n}{k-1} q_{ii}^{(k-1)(k-2)} q_{ij}^2 \right] x_i^{n+1-k} x_j x_i^k
$$

$$
= x_i^{n+1} x_j + (-1)^{n+1} \binom{n}{k} q_{ii}^{n(k-1)k} q_{ij}^{n+1} x_i^{n+1-k} x_j x_i^k + \sum_{k=1}^{n} (-1)^k q_{ij}^k \left[ \binom{n}{k} q_{ii}^{k(k-1)} + \binom{n}{k-1} q_{ii}^{n+1-k} \right] x_i^{n+1-k} x_j x_i^k
$$
\begin{align*}
&= x_i^{n+1} x_j + (-1)^{n+1} q_i^{\frac{n(n+1)}{2}} q_j^{n+1} x_j a_i^{n+1} + \sum_{k=1}^{n} (-1)^k q_{ij}^k q_{ii}^{\frac{k(k-1)}{2}} \left( \begin{array}{c}
n + 1 \\
k
\end{array} \right) q_{ii} x_i^{n+1-k} x_j x_i^k \\
&= \sum_{k=0}^{n+1} (-1)^k q_{ij}^k q_{ii}^{\frac{k(k-1)}{2}} \left( \begin{array}{c}
n + 1 \\
k
\end{array} \right) q_{ii} x_i^{n+1-k} x_j x_i^k.
\end{align*}

Since \( \text{ad}_c(x_i)^{1-a_{ij}}(x_j) = 0 \) in \( R(D) \), and
\[
\varphi \left( \sum_{k=0}^{n} (-1)^k \left( \begin{array}{c}1-a_{ij} \\
k
\end{array} \right) q_{ii}^{\frac{k(k-1)}{2}} q_{ij}^k q_{ji}^k \right) = \text{ad}_c(x_i)^{1-a_{ij}}(x_j),
\]
the assertion about (R6) follows. Analogously,
\[
\text{ad}_c(y_i)(y_j) = y_i y_j - (L_i \cdot y_j) y_i = y_i y_j - q_{ji}^{-1} y_j y_i = q_{ji}^{-1} (y_j y_i - q_{ji} y_i y_j),
\]
and one may prove by induction that
\[
\text{ad}_c(y_i)^n(y_j) = (-1)^n q_{ji}^{-n} q_{ii}^{\frac{n(n+1)}{2}} \left( \sum_{k=0}^{n} (-1)^k \left( \begin{array}{c}n \\
k
\end{array} \right) q_{ii}^{\frac{k(k-1)}{2}} q_{ji}^k q_{ji}^k y_j y_i y_i^{n-k} \right).
\]

Hence,
\[
\varphi \left( \sum_{k=0}^{n} (-1)^k \left( \begin{array}{c}1-a_{ij} \\
k
\end{array} \right) q_{ii}^{\frac{k(k-1)}{2}} q_{ij}^k q_{ji}^k \right) = \sum_{k=0}^{n} (-1)^k \left( \begin{array}{c}1-a_{ij} \\
k
\end{array} \right) q_{ii}^{\frac{k(k-1)}{2}} q_{ij}^k q_{ji}^k (y_i y_j)^{n-k} = \sum_{k=0}^{n} (-1)^k \left( \begin{array}{c}1-a_{ij} \\
k
\end{array} \right) q_{ii}^{\frac{k(k-1)}{2}} q_{ij}^k q_{ji}^k (y_i y_j)^{n-k}.
\]

But
\[
\frac{k(k-1)}{2} + \frac{(1-a_{ij}-k)(a_{ij}-k)}{2} + k(1-a_{ij}-k) = \frac{k(k-1)}{2} + \frac{a_{ij}(1-a_{ij}-k) - k(1-a_{ij}-k)}{2} + k(1-a_{ij}-k) = \frac{k(k-1)}{2} + \frac{-a_{ij}(1-a_{ij}-k) + k(1-a_{ij}-k)}{2} = \frac{1}{2} [k^2 - k^2 - a_{ij} + a_{ij}^2 + a_{ij} k + k - ka_{ij} - k^2] = \frac{1}{2} a_{ij}(a_{ij} - 1).
\]

Thus, \( \varphi \) of (R7) equals
\[
\left( \sum_{k=0}^{n} (-1)^k \left( \begin{array}{c}1-a_{ij} \\
k
\end{array} \right) q_{ii}^{\frac{k(k-1)}{2}} q_{ij}^k q_{ji}^k q_{ij}^k q_{ji}^k y_j y_i y_i^{a_{ij} - k} \right) (L_j - 1)^{1-a_{ij}} L_i^{1-a_{ij}} = q_{ij}^{1-a_{ij}}(-1)^{1-a_{ij}} q_{ji}^{1-a_{ij}} \text{ad}_c(y_i)^{1-a_{ij}}(y_j) L_j^{1-a_{ij}} L_i^{1-a_{ij}}
\]

Since \( \text{ad}_c(y_i)^{1-a_{ij}}(y_j) = 0 \) in \( R(D) \), the claim about (R7) follows.

Hence, \( \varphi \) is a well-defined algebra map. Moreover, it is a Hopf algebra map, since \( \omega_i, \omega'_i \) and \( K_i, L_i \) are grouplike elements, \( e_i \) is \((1, \omega_i)\)-primitive and \( f_i \) is \((\omega'_i, 1)\)-primitive, and the elements \( x_i \) and \( y_i \) are \((1, K_i)\)-primitive and \((1, L_i)\)-primitive, respectively, for all \( i \in I \).
Now we show $\varphi$ is an isomorphism. Let $\tilde{\psi} : R(D)\# k\mathbb{Z}^g \to U_q(\mathfrak{g}_A)$ be the algebra map given by

$$\tilde{\psi}(1#K_i) = \omega_i, \quad \tilde{\psi}(1#L_i) = \omega_i^{-1}, \quad \tilde{\psi}(x_i#1) = e_i, \quad \tilde{\psi}(y_i#1) = f_i\omega_i^{-1},$$

for all $i \in \mathbb{I}$. Again, $\tilde{\psi}$ is clearly a Hopf algebra epimorphism, if it is well-defined. For example, it preserves the algebra structure, if $i \in \mathbb{I}$ we have

$$\tilde{\psi}((1#K_i)(x_j#1)(1#K_i^{-1})) = \omega_i e_j \omega_i^{-1} = q_{ij} e_j = \chi_j(K_i)e_j = \tilde{\psi}(K_i \cdot x_j#1),$$

and the coalgebra structure with $\varepsilon(\omega_i) = 1 = \varepsilon(1#K_i)$, $\varepsilon(\omega_i^{-1}) = 1 = \varepsilon(1#L_i)$, $\varepsilon(e_i) = 0 = \varepsilon(x_i#1)$, $\varepsilon(f_i\omega_i^{-1}) = 1 = \varepsilon(y_i#1)$ and

$$\Delta(\tilde{\psi}(y_i#1)) = \Delta(f_i\omega_i^{-1}) = f_i\omega_i^{-1} \otimes 1 + \omega_i^{-1} \otimes f_i\omega_i^{-1} = \tilde{\psi}(y_i#1) \otimes 1 + \tilde{\psi}(1#L_i) \otimes \tilde{\psi}(y_i#1) = (\tilde{\psi} \otimes \tilde{\psi}) \Delta(y_i#1).$$

To see that $\tilde{\psi}$ is indeed well-defined, we have to check that the quantum Serre relations are mapped to 0. For $i \neq j \in I$ we have by (R5) and (R6) that

$$\tilde{\psi}(\text{ad}_c(x_i)^{1-a_{ij}}(x_j)) = \sum_{k=0}^{1-a_{ij}} (-1)^k \binom{1-a_{ij}}{k} q_{ii}^{k(k-1)/2} q_{ij}^{k} q_{ij}^{1-a_{ij}-k} e_i e_j e_k = 0.$$

Analogously, by (R7) and the same calculation with the exponents as above we have

$$\tilde{\psi}(\text{ad}_c(y_i)^{1-a_{ij}}(y_j)) = (-1)^{1-a_{ij}} q_{jj}^{-1-a_{ij}} q_{ii}^{a_{ij}(1-a_{ij})/2} \cdot \left( \sum_{k=0}^{1-a_{ij}} (-1)^k \binom{1-a_{ij}}{k} q_{ii}^{k(k-1)/2} q_{jj}^{k} q_{ij}^{1-a_{ij}-k} \right).$$

Moreover, identifying $x_i = x_i#1, y_i = y_i#1$ and $K_i = 1#K_i, L_i = 1#L_i$, by (R5) we have that

$$\tilde{\psi}(x_iy_j - q_{ij}^{-1}y_jx_i) = e_i f_j\omega_j^{-1} - q_{ij}^{-1}f_j\omega_j^{-1} e_i = (e_i f_j - q_{ij}^{-1}f_j e_i)\omega_j^{-1} = [e_i, f_j]\omega_j^{-1} = \delta_{ij} q_{ii}^{-1}(\omega_i - \omega_j)\omega_j^{-1} = \delta_{ij} q_{ii}^{-1}(\omega_i\omega_j^{-1} - 1) = \tilde{\psi}(\delta_{ij} \ell_i(K_i L_i - 1)),$$

for all $i, j \in \mathbb{I}$. Thus $\tilde{\psi}$ induces a Hopf algebra epimorphism $\psi : \tilde{U}(D_{\text{red}}, \ell) \to U_q(\mathfrak{g}_A)$ such $\varphi \circ \psi = \text{id} = \psi \circ \varphi$, implying that $\varphi$ is an isomorphism.
Remark 4.13. Similarly, we shall also write group 2-cocycle \( \sigma \) Hopf algebra associated to a reduced YD-datum of DJ-type. For, the proof of Theorem 4.7 gives a braiding is positive and generic, Theorem 4.7 implies that 
\[
\Gamma = U_q^\mathbb{Z}^{\hat{U}(\mathfrak{g}_A)} \bigg/ (K_i - L_i^{-1}) \bigg/ (\mathcal{G})
\]
Moreover, by Theorem 4.9 we know that Jimbo type. Let Corollary 4.10. Then by \([HPR, 27,28]\), Remark 4.11. Then we have \( U_q(\mathfrak{g}_A) \) is a two-parameter quantum group, see \([HPR, Rmk. 9]\). Then Corollary 4.12. Proof. By Theorem 4.9 we know that \( \tilde{U}(\mathfrak{D}_q, \ell) \) is the pointed Hopf algebra associated to the reduced YD-datum of DJ-type given by \( \mathcal{D}_q = \mathcal{D}(\Gamma, (L_i)_{1 \leq i \leq \theta}, (K_i)_{1 \leq i \leq \theta}, (q_{ij})_{j \in \mathcal{X}, (a_{ij})_{1 \leq i,j \leq \theta}} \) and linking parameter \( \ell \).

Using this language we translate \([HPR, Thm. 2.8]\).

Corollary 4.12. There exists a group 2-cocycle \( \sigma \in Z^2(\Gamma, \mathbb{k}) \) such that \( \tilde{U}(\mathfrak{D}_q, \ell) \simeq U_q(\mathfrak{g}_A) \). In particular, if \( \mathfrak{g}_A \) is simple and \( q_{ij} = q^{d_{a_{ij}}} \) for all \( i, j \in I \), we have that \( U_{q,q^{-1}}(\mathfrak{g}_A) \simeq U_q(\mathfrak{g}_A) \).

Proof. By Theorem 4.9 we know that \( U_q(\mathfrak{g}_A) \simeq \tilde{U}(\mathfrak{D}_q, \ell) \) with \( \ell_i = \frac{q_{ij}}{q_{ii} - 1} \) for all \( i \in I \). Since the braiding is positive and generic, Theorem 4.7 implies that \( U_q(\mathfrak{g}_A) \) is a 2-cocycle deformation of a pointed Hopf algebra associated to a reduced YD-datum of DJ-type. For, the proof of Theorem 4.7 gives a group 2-cocycle \( \sigma \) such that \( \tilde{U}(\mathfrak{D}_q, \ell) \simeq U_{q,q^{-1}}(\mathfrak{g}_A) \). Taking the corresponding 2-cocycle induced by the isomorphism we have the assertion.

If \( \mathfrak{g}_A \) is simple and \( q_{ij} = q^{d_{a_{ij}}} \) for all \( i, j \in I \), by Remark 4.11 we have that \( \tilde{U}(\mathfrak{D}_q, \ell) \simeq U_{q,q^{-1}}(\mathfrak{g}_A) \).

Remark 4.13. Assume \( \mathfrak{g}_A \) is simple. The result above was previously obtained by \([HPR]\) where a Hopf 2-cocycle \( \sigma \) is defined in \( U_{q,q^{-1}}(\mathfrak{g}_A) \). We show that this cocycle comes from a group 2-cocycle on \( \Gamma \).

First we fix the notation \( \omega_\lambda := \prod_{i \in I} \omega_\lambda^i \) and \( \omega_\lambda' := \prod_{i \in I} \omega_\lambda'^i \) for every \( \lambda = \sum_{i \in I} \lambda_i \alpha_i \in \mathbb{Q} \). Similarly, we shall also write 
\[
q_{\mu \nu} := \prod_{i,j \in I} q_{ij}^{d_{a_{ij}}} \quad \forall \, \mu = \sum_{i \in I} \mu_i \alpha_i , \, \nu = \sum_{j \in I} \nu_j \alpha_j \in \mathbb{Q}
\]
Let \( \sigma : U_{q,q^{-1}}(\mathfrak{g}_A) \otimes U_{q,q^{-1}}(\mathfrak{g}_A) \to \mathbb{k} \) be the unique \( \mathbb{k} \)-linear form such that 
\[
\sigma(x,y) := \begin{cases} q_{\mu \nu} & \text{if } x = \omega_\mu \text{ or } x = \omega_\mu' , \, y = \omega_\nu \text{ or } y = \omega_\nu' \\ 0 & \text{otherwise.} \end{cases}
\]
Then by \([HPR, 27, 28]\), \( \sigma \in Z^2(U_{q,q^{-1}}(\mathfrak{g}_A), \mathbb{k}) \) and \( U_q(\mathfrak{g}_A) \simeq U_{q,q^{-1}}(\mathfrak{g}_A) \).

On the other hand, we know that \( U_{q,q^{-1}}(\mathfrak{g}_A) \) is a quotient of a bosonization \( T(V \oplus W) \# k\Gamma \) with \( \Gamma = \mathbb{Z}^{2d} \). As in Remark 3.3 we have a \( \Gamma \times \Gamma \) grading on \( T(V \oplus W) \) induced by the coaction on the
Yetter-Drinfeld module; for example, $\omega_i$ have degree $(\omega_i, \omega_i)$, $e_i$ has degree $(\omega_i, 1)$ and $f_i$ has degree $(1, \omega_i^{-1})$ for all $i \in I$. Consider now the group 2-cocycle $\varphi \in Z^2(\Gamma, k)$ given by $\varphi = \sigma|_{\Gamma \times \Gamma}$, that is,

$$\varphi(h, k) := q_{h, k}^{\varphi} \quad \text{if} \quad h = \omega'_\mu \text{ or } h = \omega'_\nu, \quad k = \omega_\nu \text{ or } k = \omega'_\nu,$$

and let $\tilde{\varphi}$ be the 2-cocycle defined on $T(V \oplus W)\#k\Gamma$. Since the group is abelian and $e_i \cdot \tilde{\varphi}_j f_j = e_i f_j$ for all $i, j \in I$, we have that $e_i \cdot \tilde{\varphi}_j f_j - f_j \cdot e_i = [e_i, f_j]$ and consequently $\tilde{\varphi}$ defines a Hopf 2-cocycle on $U_{q, q^{-1}}(g_A)$. Since $\sigma(x, y) = 0 = \varepsilon(x)\varepsilon(y)$ if $x, y \notin \Gamma$, it follows that $\sigma = \tilde{\varphi}$ and whence $U_q(g_A) = U_{q, q^{-1}}(g_A)\sigma$.

References

[AAGMV] N. Andruskiewitsch, I. Angioni, A. García Iglesias, A. Masuoka and C. Vay, Lifting via cocycle deformation. J. Pure and Applied Algebra, to appear. Preprint: arXiv:1212.5279.

[AC] N. Andruskiewitsch and J. Cuadra, On the structure of (co-Frobenius) Hopf algebras, J. Noncommutative Geometry, Volume 7, Issue 1, pp. 83–104.

[AGr] N. Andruskiewitsch and M. Graña, From racks to pointed Hopf algebras, Adv. Math. 178 (2003), 177–243.

[AGr2] Andruskiewitsch, N. and Graña, M., Examples of liftings of Nichols algebras over racks, Théories d’homologie, représentations et algèbres de Hopf, Algebra Montp. Announc. 2003, Paper 1, 6 pp. (electronic).

[AHS] N. Andruskiewitsch, I. Heckenberger, and H.J. Schneider The Nichols algebra of a semisimple Yetter-Drinfeld module, Ann. J. Math. 132, 6, (2010) pp. 1493–1547.

[ARS] N. Andruskiewitsch, D. Radford and H-J. Schneider, Complete reducibility theorems for modules over pointed Hopf algebras, J. Algebra 324 (2010) 2932–2970.

[AS1] N. Andruskiewitsch and H-J. Schneider Finite quantum groups and Cartan matrices, Adv. Math. 154 (2000), no. 1, 1–45.

[AS2] , Pointed Hopf algebras, MSRI series Cambridge Univ. Press 48 (2002), 1–68.

[AS3] A. Ardizzoni, M. Beattie and C. Menini, Cocycle deformations for liftings of quantum linear spaces, Comm. Algebra 39 (2011), no. 12, 4518–4535.

[ABM1] A. Ardizzoni, M. Beattie and C. Menini, Cocycle deformations for liftings of quantum linear spaces, Comm. Algebra 39 (2011), no. 12, 4518–4535.

[ABM2] , Gauge deformations for Hopf algebras with the dual Chevalley property, J. Algebra Appl. 11 (2012), no. 3, 1250051, 37 pp.

[AST] M. Artin, W. Schelter and J. Tate, Quantum deformations of $GL_n$, Comm. Pure Appl. Math. 44 (1991), no. 8-9, 879–895.

[BNPP] C. Bendel, D. Nakano, B. Parshall and C. Pillen, Cohomology for quantum groups via the geometry of the Nullcone, Memoire of the AMS, 2010.

[CP] J. Cuadra and F. Panaite, Extending lazy 2-cocycles on Hopf algebras and lifting projective representations afforded by them, J. Algebra 313 (2007), no. 2, 695–723.

[D] Y. Doi, Braided bialgebras and quadratic bialgebras, Comm. Algebra 21 (1993), no. 5, 1731–1749.

[DT] Y. Doi and M. Takeuchi, Multiplication alteration by two-cocycles the quantum version, Comm. Algebra. 22 (1994), 5715–5732.

[GG] G. A. García and A. García Iglesias, Finite dimensional pointed Hopf algebras over $S_4$, Israel J. Math. 183 (2011), 417–444.

[GM] G. A. García and M. Mastnak, Deformation by cocycles of pointed Hopf algebras over non-abelian groups, Preprint: arXiv:1203.0967.

[GM2] G. A. García and M. Mombelli, Representations of the category of modules over pointed Hopf algebras over $S_3$ and $S_4$, Pacific J. Math. 252 (2) (2011), pp. 343–378.

[GIV] A. García Iglesias and C. Vay, Finite-dimensional Pointed or Copointed Hopf algebras over affine racks, J. Algebra, to appear. Preprint: arXiv:1210.6396.

[GrM] L. Grunenfelder and M. Mastnak, Pointed Hopf algebras as cocycle deformations. Preprint: arXiv:1010.4976v1.

[H] I. Heckenberger, Lusztig isomorphisms for Drinfeld doubles of bosonizations of Nichols algebras of diagonal type, J. Algebra 323 (2010), 2130–2180.

[HLR] N. Hu, Y. Li and M. Rosso, Multi-parameter quantum groups via quantum quasi-symmetric algebras, arXiv:1307.1351 (2013).

[HPR] N. Hu, Y. Pet and M. Rosso, Multi-parameter quantum groups and quantum shuffles. I. Quantum affine algebras, extended affine Lie algebras, and their applications, 145–171, Contemp. Math., 506, Amer. Math. Soc., Providence, RI, 2010.

[MQ] S. Majid and R. Oeckl, Twisting of Quantum Differentials and the Planck Scale Hopf Algebra, Commun. Math. Phys. 205 (1999), 617–655.

[MPSW] M. Mastnak, J. Pevtsova, P. Schauenburg, and S. Witherspoon, Cohomology of finite dimensional pointed Hopf algebras, Proc. London Math. Soc. 100, no. 2 (2010), 377–404.
[Mk] A. Masuoka, Abelian and non-abelian second cohomologies of quantized enveloping algebras, J. Algebra 320 (2008), 1–47.

[Mk2] _______, Construction of quantized enveloping algebras by cocycle deformation, Arab. J. Sci. Eng. Sect. C Theme Issues 33 (2008), no. 2, 387–406.

[MS] A. Milinski and H.J. Schneider, Pointed indecomposable Hopf algebras over Coxeter groups, Contemp. Math. 267 (2000), 215–236.

[Mo] S. Montgomery, Hopf Algebras and their Actions on Rings, CBMS Reg. Conf. Ser. Math. 82, Amer. Math. Soc. (1993).

[PS] F. Panaite and D. Ştefan, Deformation cohomology for Yetter-Drinfel’d modules and Hopf (bi)modules Comm. Algebra 30 (2002), 331–345.

[R] D. E. Radford, Hopf algebras, Series on Knots and Everything, 49. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2012.

[Ro] N. Rosso, Quantum groups and quantum shuffles. Invent. Math. 133 (1998), 399–416.

[ȘV] D. Ştefan and C. Vay, The cohomology ring of the 12-dimensional Fomin-Kirillov algebra. Preprint: arXiv:1404.5101v1.

[Sw] M. Sweedler, Hopf algebras, Benjamin, New York, 1969.

[V] L. Vendramin, Nichols algebras associated to the transpositions of the symmetric group are twist-equivalent, Proc. Amer. Math. Soc. 140 (2012), no. 11, 3715–3723.