Perturbation Resilient Clustering for $k$-Center and Related Problems via LP Relaxations*

(Full Version)

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March 14, 2022

Abstract

We consider clustering in the perturbation resilience model that has been studied since the work of Bilu and Linial [16] and Awasthi, Blum and Sheffet [7]. A clustering instance $I$ is said to be $\alpha$-perturbation resilient if the optimal solution does not change when the pairwise distances are modified by a factor of $\alpha$ and the perturbed distances satisfy the metric property — this is the metric perturbation resilience property introduced in [3] and a weaker requirement than prior models. We make two high-level contributions.

- We show that the natural LP relaxation of $k$-center and asymmetric $k$-center is integral for $2$-perturbation resilient instances. We believe that demonstrating the goodness of standard LP relaxations complements existing results [11, 3] that are based on new algorithms designed for the perturbation model.
- We define a simple new model of perturbation resilience for clustering with outliers. Using this model we show that the unified MST and dynamic programming based algorithm proposed in [3] exactly solves the clustering with outliers problem for several common center based objectives (like $k$-center, $k$-means, $k$-median) when the instances is $2$-perturbation resilient. We further show that a natural LP relaxation is integral for $2$-perturbation resilient instances of $k$-center with outliers.

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1 Introduction

Clustering is an ubiquitous task that finds applications in numerous areas since it is a basic primitive in data analysis. Consequently, clustering methods are extensively studied in many scientific communities and there is a vast literature on this topic. In a typical clustering problem the input is a set of points with a notion of similarity (also called distance) between every pair of points, and a parameter $k$, which specifies the desired number of clusters. The goal is to partition the points into $k$ clusters such that points assigned to the same cluster are similar. One way to obtain this partition is to select $k$ centers and then assign each point to the nearest center. The quality of the clustering can be measured in terms of an objective function. Some of the popular and commonly studied ones are $k$-median (sum of distances of points to nearest center), $k$-means (sum of squared distances of points to nearest center), and $k$-center (maximum distance between a point to its nearest center). These are center-based objective functions. Unlike some applications in Operations Research, in many clustering problems in data analysis, the objective function is a proxy to identify the clusters and the actual value of the objective function is not necessarily meaningful. Clustering is often considered in the presence of outliers. In this setting the goal is to find the best clustering of the input after removing (at most) a specified number (or fraction) of points — this is useful in practice when the input data is noisy.

Most of the natural optimization problems that arise in clustering turn out to be NP-Hard. Extensive work exists on approximation algorithm design as well as heuristics. Although clustering and its variants are intractable in the worst case, various heuristic based algorithms like Lloyd’s, K-Means++ perform very well in practice and are routinely used — at the same time some of these heuristics have poor worst-case approximation performance. On the other hand algorithms designed for worst-case approximation bounds may not work well in practice or may not be sufficiently fast for large data sets. To bridge this gap between theory and practice, there has been an increasing emphasis on beyond worst case analysis. Several models have been proposed to understand real-world instances and why they may be computationally easier. One such model is based on the notion of instance stability. This is based on the assumption that typical instances have a clear underlying optimal clustering (also known as ground-truth clustering) which is significantly better than all other clusterings, and remains the same under small perturbations.

The notion of stability/perturbation resilience was formalized in the work of Bilu and Linial [16] initially for Max Cut, and by Awasthi, Blum and Sheffet [7] for clustering. For clustering problems, an instance $I$ is said to be $\alpha$-perturbation resilient for some $\alpha > 1$ if the optimum clustering remains the same even if pairwise distances between points are altered by a multiplicative factor of at most $\alpha$. Intuitively, $\alpha$ determines the degree of resilience of the instance, with a higher value translating to more structured, and separable instances. In the past few years, there has been increasing interest in understanding stable/perturbation-resilient instances. After several papers [7, 12, 11], a recent result by Angelidakis, Makarychev and Makarychev [3] showed that $2$-perturbation resilient instances of several clustering problems with center based objectives (which includes $k$-median, $k$-center, $k$-means) can be solved exactly in polynomial time. For $k$-center finding the optimum solution for $(2-\delta)$-perturbation resilient instances is NP-HARD [11]. One criticism of perturbation resilience for clustering was the assumption in some earlier works that the optimum clustering remains stable under perturbation of the original metric $d$ even when perturbed distance $d'$ itself may not be a metric. Interestingly the results of [3] hold even under the weaker assumption of metric perturbation resilience, which constrains the perturbed pairwise distances to be a metric. The results in [3] are based on a simple and unified algorithm that computes the MST $T$ of the given set of points and then applies dynamic programming on $T$ to find the clusters; it is only in the second step that the specific objective function is used. We believe that empirically evaluating the
performance of this algorithm, and related heuristics, on real-world data is an interesting avenue and plan to study it. Our work in this paper is motivated by the existing work and several interrelated questions on theoretical concerns, that we discuss next.

One of the objectives in beyond-worst-case analysis is to explain the empirical success of existing algorithms and mathematical programming formulations. For stable instances of Max-Cut and Minimum Multiway Cut, convex relaxations are known to be integral for various bounds on the perturbation parameter [30, 3]. In the context of $k$-median and $k$-means Awasthi et al. [8] showed that if the data is generated uniformly at random from $k$ unit balls with well-separated centers, convex relaxations (linear and semi-definite) give an optimal integral solution under appropriate separation conditions on the centers. However, for perturbation resilient clustering instances not much is known about the the natural LP relaxations. This raises a natural question.

**Question 1.** Are the natural LP relaxations for 2-metric perturbation resilient instances of clustering problems integral?

There are several advantages in proving that well-known relaxations are integral. First, they provide evidence of the goodness of the relaxation; often these relaxations also have worst-case approximation bounds. Second, when the relaxation does not give an integral solution for a given instance we can deduce that the instance is not perturbation resilient.

As we remarked, one major takeaway from the paper of Angelidakis et al. [3], apart from its strong theoretical results, is the simple and unified algorithm that they propose which may lead to an effective heuristic. In real-world data there is often noise, and it would be useful to develop algorithms in the more general setting of clustering with outliers. This leads us to the question,  

**Question 2.** Is there any stability model under which the algorithm proposed by [3] gives optimal solution for the problem of clustering with outliers?

We remark that even for instances without outliers, removing a small fraction of the points can lead to a residual instance which has better stability parameters than the initial one. Thus, clustering with outlier removal is relevant even when there is no explicit noise.

### 1.1 Our Results

In this paper we address the preceding questions and obtain the following results.

- We show that a natural LP relaxation for $k$-center has an optimum integral solution for 2-metric-perturbation resilience instances\(^1\). Thus, when running the LP on a clustering instance, either we are guaranteed to have found the optimal solution (if the LP solution is integral), or we are guaranteed the solution is not 2-perturbation resilient (if the LP solution is not integral). The previous algorithms of Angelidakis et al. [3], and Balcan et al. [11] do not have this guarantee, and could be arbitrarily bad if the instance is not 2-PR.

- Motivated by the work of [11] we consider the asymmetric $k$-center (ASYM-$k$-CENTER) problem. We show that a natural LP relaxation has an optimum integral solution for 2-metric-perturbation resilient instances\(^2\). For ASYM-$k$-CENTER the worst-case integrality gap of the LP relaxation is known to be $\Theta(\log^* k)$ [4, 22]. Previously [11] described a specific combinatorial algorithm that outputs an optimum solution for 2-perturbation resilient instances. We obtain it via the LP relaxation in the weaker metric perturbation model.

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\(^1\) Although the LP provides a 2-approximation it is not immediate that it would be exact for perturbation-resilience instances

\(^2\) In the asymmetric setting the perturbed distances should satisfy triangle inequality but symmetry is not required.
• We define a simple model of perturbation resilience for clustering with outliers. It is a clean extension of the existing perturbation resilience model. We show that under this new model, a modification of the algorithm of Angelidakis et al. [3] gives an exact solution for the outliers problem (for $k$-median, $k$-means, $k$-center and outer $\ell_p$ based objectives). This algorithm may lead to an interesting heuristic for clustering (noisy) real-world instances. We also show that for a 2-perturbation resilient instance of $k$-center with outliers, a natural LP relaxation has an optimum integral solution.

Our results show the efficacy of LP relaxations for $k$-center and its variants. We also demonstrate, via a natural model, that the interesting algorithm from [3] extends to handle outliers. Perturbation resilience appears to be a simple definition but it is hard to pin down its precise implications. Prior work demonstrates that observations and algorithms that appear simple in retrospect have not been easy to find. For $k$-center and ASYM-$k$-CENTER we work with notion of perturbation resilience under Voronoi clusterings as was done in [11]; this is the more restrictive version. See Section 2 for the formal definitions.

We would like to understand the integrality gap of the natural LP relaxations for perturbation resilient instances of $k$-median and $k$-means. We believe that the following open question is quite interesting to resolve.

**Question 3.** Is there a fixed constant $\alpha$ such that the natural LP relaxation for $k$-median (similarly $k$-means) has an integral optimum solution for every $\alpha$-perturbation resilient instance?\(^3\)

### 1.2 Related Work

There is extensive related work on clustering topics. Here we only mention some closely related work.

**Clustering.** For both $k$-center and asymmetric $k$-center tight approximation bounds are known. For $k$-center, already in the mid 1980’s Gonzales [24] and Hochbaum & Shmoys [25] had developed remarkably simple 2-approximation algorithms, which are in fact tight. Approximating asymmetric $k$-center is significantly harder. Panigrahy and Vishwanathan [33] designed an elegant $O(\log^* n)$ approximation algorithm, which was subsequently improved by Archer [4] to $O(\log^* k)$. Interestingly, the result is asymptotically tight [22].

For $k$-means and $k$-median— arguably the two most popular clustering problems — there is a long line of research (see [17] for a survey on $k$-means). The first constant factor approximation for the $k$-median problem was given by Charikar et al. [19], and the current best-known is a 2.675 approximation by Byrka et al. [18]; and it is NP-HARD to do better than $1 + 2/e \approx 1.736$ [26]. For $k$-means the best approximation known is 6.357 [2]. The $k$-means problem is widely used in practice as well, and the commonly used algorithm is Lloyd’s algorithm, which is a special case of the EM algorithm [29]. While there is no explicit approximation guarantee of the algorithm, it performs remarkably well in practice with careful seeding [5] (this heuristic is called K-Means++).

**Clustering with Outliers.** The influential paper by Charikar et al. [20] initiated the work on clustering with outliers and other robust clustering problems. For $k$-center with outliers, they gave a greedy 3-approximation algorithm. Further, for $k$-median with outliers they gave a bicriteria approximation algorithm, which achieves an approximation ratio of $4(1 + \epsilon)$, violating the number\(^3\) It may be possible to answer this question in the positive if we additionally assume that the optimum clusters are balanced in terms of number of points. However, we feel that such an assumption does not shed light on the structure of perturbation resilient instances that are not balanced.

\(^3\)It may be possible to answer this question in the positive if we additionally assume that the optimum clusters are balanced in terms of number of points. However, we feel that such an assumption does not shed light on the structure of perturbation resilient instances that are not balanced.
of outliers by a factor of $(1 + \epsilon)$. The first constant factor approximation algorithm for this problem was given by Chen (the constant is not explicitly computed) [21]. Very recently, Krishnaswamy et al. [27] proposed a generic framework for clustering with outliers. It improves the results of Chen and gives the first constant factor approximation for $k$-means with outliers. However, the algorithm does not appear suitable for practice in its current form (See [1] for details on algorithms used in practice for clustering with outliers).

**Perturbation Resilience.** The notion of perturbation resilience was introduced by Bilu and Linial [16]. They originally considered it for the Max Cut problem, designing an exact polynomial time algorithm for $O(n)$-stable instances of Max Cut. It was later improved to $O(\sqrt{n})$-stable instances [15], and finally Makarychev et al. gave a polynomial time exact algorithm for $O(\sqrt{\log n \cdot \log \log n})$-stable instances [30].

The definition of perturbation resilience naturally extends to clustering problems. Awasthi, Blum, and Sheffet [7] presented an exact algorithm for solving 3-perturbation resilient clustering problems with *separable center based objectives* (s.c.b.o) — this includes $k$-median, $k$-means, $k$-center. This result was later improved by Balcan and Liang [12], who gave an exact algorithm for clustering with s.c.b.o under $(1 + \sqrt{2})$-perturbation resilience. Specifically for $k$-center and asymmetric $k$-center, Balcan, Hagtalab, and White [11] designed an algorithm for 2-perturbation resilient instances. In fact, for $k$-center they gave a stronger result, that any 2-approximation algorithm for $k$-center can give an optimal solution for 2-perturbation resilient instances. They also showed the results are essentially tight unless $NP = RP$.

Recently, Angelidakis et al. [3], gave an unifying algorithm which gives exact solution for 2-perturbation resilient instances of clustering problems with center based objectives. In fact, their algorithms work under metric perturbation resilience, which is a weaker assumption. Perturbation resilience has also been studied in various other contexts, like TSP, Minimum Multiway Cut, Clustering with min-sum objectives [12, 30, 31].

**Robust Perturbation Resilience.** Perturbation resilience requires optimal solution to remain unchanged under any valid perturbation. Balcan and Liang [12] relaxed this condition slightly, and defined $(\alpha, \epsilon)$-perturbation resilience (or robust perturbation resilience), in which at most $\epsilon$ fraction of the points can change their cluster membership under any $\alpha$-perturbation. They gave a near optimal solution for $k$-median under $(4, \epsilon)$-perturbation resilience, when the clusters are not too small. Further, for $k$-center and asymmetric $k$-center efficient algorithms are known for $(3, \epsilon)$-perturbation resilient instances, assuming mild size lower bound on optimal clusters [11].

**Other Stability Notions.** Several other stability models, and separation conditions have also been studied to better explain real-world instances. In a seminal paper Ostrovsky, Rabani, Schulman, and Swamy [32] considered $k$-means instances where the cost of clustering using $k$ is clusters is much lower than $k - 1$ clusters. They showed, that popular K-Means++ algorithm achieves an $O(1)$-approximation for these instances. Subsequently there has been series of work many other models like approximation stability [10], agnostic clustering [13], distribution stability [6, 23], spectral separability [28, 9, 23], and more recently on additive perturbation stability [34].

**Organization:** The rest of the paper is organized as follows: in Section 2 we formally define the clustering problems and perturbation resilience; in Section 3 we prove that any 2-approximation algorithm gives optimal solution for 2-perturbation resilient $k$-center instance, further we show

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4 They used this name to denote perturbation resilient instances of Max Cut

5 They showed, unless $NP = RP$, no polynomial-time algorithm can solve $k$-center under $(2 - \epsilon)$-approximation stability, a notion that is stronger than perturbation resilience
that the natural LP is integral; in Section 4 we show that even for asymmetric $k$-center the natural LP relaxation is integral under 2-perturbation resilience; in Section 5 we prove the integrality of LP for 2-perturbation resilient $k$-center with outliers instance; finally in Section 6 we show present a dynamic programming based algorithm which exactly solves $k$-median with outliers (and also $k$-CENTER-OUTLIER, $k$-MEANS-OUTLIER) under 2-perturbation resilience.

2 Preliminaries

2.1 Definitions & Notations

In this section we formally define the clustering problems and perturbation resilience.

Clustering. An instance $I$ of a clustering problem is defined by the tuple $(V, d, k)$, where $V$ is a set of $n$ points, $d : V \times V \rightarrow \mathbb{R}_{\geq 0}$ is a metric distance function, and $k$ is an integer parameter. The goal is to find a set of $k$ distinct points $S = \{c_1, \ldots, c_k\} \subseteq V$ called centers such that an objective function defined over the points is optimized. The objective function, also known as clustering cost, can be defined in various ways, and depends on the problem in hand. Here, we are interested in the $k$-median, and $k$-center objectives. Given a set of centers $S = \{c_1, \ldots, c_k\}$ these objectives are defined as follows:

\begin{align*}
(k\text{-median}) \quad & \text{cost}_d(S) = \sum_{u \in V} d(S, u) \\
(k\text{-means}) \quad & \text{cost}_d(S) = \sum_{u \in V} d^2(S, u) \\
(k\text{-center}) \quad & \text{cost}_d(S) = \max_{u \in V} d(S, u)
\end{align*}

where $d(S, u) = \min_{i \in \{1, \ldots, k\}} d(c_i, u)$.

The Voronoi partition induced by the centers, gives a natural way of clustering the input point set. In fact, the inherent goal of clustering is to uncover the underlying partitioning of points, and one expects with correct choice of distance modeling, "$k"", and objective function, the Voronoi partition induced by the optimal set of centers will reveal the underlying clustering. Throughout this paper, whenever we mention optimal clustering, we indicate the Voronoi partition corresponding to the optimal set of centers. Thus with this dual view of the clustering problem, given a set of centers $S = \{c_1, \ldots, c_k\}$, and corresponding Voronoi partition $C = \{C_1, \ldots, C_k\}$, the clustering cost can be rewritten as:

\begin{align*}
(k\text{-median}) \quad & \text{cost}_d(C, S) = \sum_{i=1}^{k} \sum_{u \in C_i} d(c_i, u) \\
(k\text{-means}) \quad & \text{cost}_d(C, S) = \sum_{i=1}^{k} \sum_{u \in C_i} d^2(c_i, u) \\
(k\text{-center}) \quad & \text{cost}_d(C, S) = \max_{i \in \{1, \ldots, k\}} \max_{u \in C_i} d(c_i, u)
\end{align*}

So far, in the clustering problem instance, we considered the distance function $d$ to be a metric. However, this may not always be the case. Specifically, for the $k$-center objective, a generalization which is also studied is the Asymmetric $k$-center problem (ASYM-$k$-CENTER), where the distance function $d$ in the input instance $I = (V, d, k)$ is an asymmetric distance function. In other words, $d$
obey triangle inequality, but not symmetry. That is, \( d(u, v) \leq d(u, w) + d(w, v) \) for all \( u, v, w \in V \). However \( d(u, v) \) may be not be same as \( d(v, u) \). The objective is the \( k \)-center objective, but because the distance is assymmetric, order matters - we define the cost in terms of distance from the center to the points i.e. given a center \( c \) and a point \( u \), \( d(c, u) \) is used to define cost. To reiterate, given a set of centers \( S = \{c_1, \ldots, c_k\} \) and corresponding Voronoi partition (w.r.t \( d(c_i, u) \) \( \mathcal{C} = \{C_1, \ldots, C_k\} \), the clustering cost is:

\[
\text{(ASYM-}k\text{-CENTER)} \quad \text{cost}_d(\mathcal{C}, S) = \max_{i \in \{1, \ldots, k\}} \max_{u \in C_i} d(c_i, u)
\]

**Clustering with Outliers.** An instance \( \mathcal{I} \) of a clustering with outliers problem is defined by the tuple \((V, d, k, z)\), where \( V \) is a set of \( n \) points, \( d : V \times V \to \mathbb{R}_{\geq 0} \) is a metric distance function, and \( k, z \) are integer parameters. The goal is to identify \( z \) points \( Z \subseteq V \) as outliers and partition the remaining \( V \setminus Z \) points into \( k \) clusters such that the clustering cost is minimized. Formally, given a set of outliers \( Z \), a set of centers \( S = \{c_1, \ldots, c_k\} \subseteq V \setminus Z \), and a Voronoi partition of \( V \setminus Z \), \( \mathcal{C} = \{C_1, \ldots, C_k\} \) induced by \( S \), the clustering cost is defined as:

\[
\begin{align*}
\text{(k-MEDIAN-OUTLIER)} & \quad \text{cost}_d(\mathcal{C}, S; Z) = \sum_{i=1}^{k} \sum_{u \in C_i} d(c_i, u) \\
\text{(k-MEANS-OUTLIER)} & \quad \text{cost}_d(\mathcal{C}, S; Z) = \sum_{i=1}^{k} \sum_{u \in C_i} d^2(c_i, u) \\
\text{(k-CENTER-OUTLIER)} & \quad \text{cost}_d(\mathcal{C}, S; Z) = \max_{i \in \{1, \ldots, k\}} \max_{u \in C_i} d(c_i, u)
\end{align*}
\]

**Perturbation Resilience.** A clustering instance \( \mathcal{I} = (V, d, k) \) is \( \alpha \)-metric perturbation resilient (\( \alpha \)-PR) for a given objective function, if for any metric distance function \( d' : V \times V \to \mathbb{R}_{\geq 0} \), such that for all \( u, v \in V \), \( \frac{d(u, v)}{\alpha} \leq d'(u, v) \leq d(u, v) \), the unique optimal clustering of \( \mathcal{I}' = (V, d', k) \) is identical to the unique optimal clustering of \( \mathcal{I} \).

Note that after perturbation the optimal centers may change, however for the instance to be perturbation resilient, the optimal clustering i.e. Voronoi partition induced by the optimal centers must stay the same. Unless otherwise noted, for the rest of the paper \( \alpha \)-perturbation resilient indicates metric perturbation resilience.

**Outlier Perturbation Resilience.** A clustering with outliers instance \( \mathcal{I} = (V, d, k, z) \) is \( \alpha \)-metric outlier perturbation resilient (\( \alpha \)-OPR) for a given objective function, if for any metric distance function \( d' : V \times V \to \mathbb{R}_{\geq 0} \), such that for all \( u, v \in V \), \( \frac{d(u, v)}{\alpha} \leq d'(u, v) \leq d(u, v) \), the unique optimal clustering and outliers of \( \mathcal{I}' = (V, d', k, z) \) are identical to the optimal solution of \( \mathcal{I} \).

It is easy to see, if a clustering with outliers instance \( (V, d, k, z) \) with unique optimal clusters \( \mathcal{C} \) and outliers \( Z \) is \( \alpha \)-OPR, then the clustering instance \( (V \setminus Z, d, k) \) is \( \alpha \)-PR.

**Notation.** For integer, \( k \), let \( [k] = \{1, \ldots, k\} \). Throughout, we use \( V \) to denote the input set of points, and \( n \) is the number of points. For any clustering instance (including outlier instances), \( S = \{c_1, \ldots, c_k\} \) denotes an optimal set of centers, and \( \mathcal{C} = \{C_1, \ldots, C_k\} \) denotes the corresponding Voronoi partition, which we call optimal clusters. Further, for a point \( p \in C_i \), we often interchangeably use the terms, \( p \) is assigned/belongs to center \( c_i \) or cluster \( C_i \). For a clustering with

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\(^6\)In case of Asym-\( k \)-center, we consider perturbations in which \( d' \) obeys triangle inequality, but not symmetry
outlier instance, $Z$ denotes the optimal set of outliers. In case of $k$-center, we refer to the optimal clustering cost as optimal radius, and denote it as $R^*_d$.

2.2 Some useful lemmas

Here we state some intuitive and useful lemmas regarding $k$-center and and Asym-$k$-CENTER instances. The proofs of these lemmas are fairly simple and can be found in Appendix A.

Recall, in the definition of perturbation resilience, we insisted that the optimal $k$ clustering of the perturbed instance $I'$ has to be same as the optimal $k$ clustering of the original instance. It is not hard to show, that for Asym-$k$-CENTER (and also for $k$-center), if a $k-1$ clustering of $I'$ exists whose cost ist at most the optimal cost of $k$ clustering, then the instance is not perturbation resilent. Formally,

**Lemma 2.1.** Consider any Asym-$k$-CENTER instance $I = (V,d,k)$. Let $S = \{c_1,\ldots,c_k\}$ be an optimal set of centers, and $C = \{C_1,\ldots,C_k\}$ be the corresponding optimal clustering. The optimal radius is $R^*_d$. Suppose there exists a set of $k−1$ centers $S' = \{c'_1,\ldots,c'_{k−1}\}$, inducing the Voronoi partition $C' = \{C'_1,\ldots,C'_{k−1}\}$, with cost $\text{cost}_d(C',S') \leq R^*_d$. Then, the optimal clustering $C$ is not unique.

One common technique we use in multiple arguments, is perturbing the input instance in a structured way. The next two lemmas are related to that.

**Lemma 2.2.** Consider a set of points $V$, and let $d$ be an asymmetric distance function defined over $V$. Let $G$ be a complete directed graph on vertices $V$. The edge lengths in graph $G$ are given by the function $\ell$, where for any edge $(u,v)$, $\frac{d(u,v)}{2} \leq \ell(u,v) \leq d(u,v)$. Then the distance function $d'$, defined as the shortest path distance in graph $G$ using $\ell$, is a metric\footnote{Satisfies triangle inequality, and not necessarily symmetry} 2-perturbation of $d$.

**Lemma 2.3.** Consider an Asym-$k$-CENTER instance $I = (V,d,k)$, and let $C$ be the optimal clustering and $R^*_d$ be the optimal radius. Let $G$ be a complete directed graph on vertex set $V$. The edge lengths in graph $G$ are given by the function $\ell$, where (1) for a subset of edges $E'$, $\ell(u,v) = \min\{d(u,v),R^*_d\}$; (2) for every other edge, $\ell(u,v) = d(u,v)$. Suppose $d'$ is defined as the shortest path distance in graph $G$ using $\ell$. Consider the Asym-$k$-CENTER instance $I' = (V,d',k)$, let $R^*_{d'}$ be the optimal radius. If $C$ is an optimal clustering in $I'$, then $R^*_d = R^*_{d'}$.

The undirected versions of Lemma 2.2 and Lemma 2.3 are as follows:

**Lemma 2.4.** Consider a set of points $V$, and let $d$ be a metric defined over $V$. Let $G$ be a complete undirected graph on vertices $V$. The edge lengths in graph $G$ are given by the function $\ell$, where for any edge $(u,v)$, $\frac{d(u,v)}{2} \leq \ell(u,v) \leq d(u,v)$. Then the distance function $d'$, defined as the shortest path distance in graph $G$ using $\ell$, is a metric 2-perturbation of $d$.

**Lemma 2.5.** Consider a $k$-center instance $I = (V,d,k)$, and let $C$ be the optimal clustering and $R^*_d$ be the optimal radius. Let $G$ be a complete undirected graph on vertex set $V$. The edge lengths in graph $G$ are given by the function $\ell$, where (1) for a subset of edges $E'$, $\ell(u,v) = \min\{d(u,v),R^*_d\}$; (2) for every other edge, $\ell(u,v) = d(u,v)$. Suppose $d'$ is defined as the shortest path distance in graph $G$ using $\ell$. Consider the $k$-center instance $I' = (V,d',k)$, let $R^*_{d'}$ be the optimal radius. If $C$ is an optimal clustering in $I'$, then $R^*_d = R^*_{d'}$. 
3 $k$-center under Perturbation Resilience

In this section, we show that the natural LP relaxation for a 2-perturbation resilient $k$-center has an integral optimum solution. To this end consider the result of Balcan et al. [11] — any 2-approximation algorithm for $k$-center finds the optimal clustering for a 2-perturbation resilient instance. They proved this result under the stronger definition of non-metric perturbation resilience, which was subsequently extended to metric perturbation resilience in an unpublished follow-up paper [14]. Formally, the result is as follows:

**Theorem 3.1.** Let $A$ be an arbitrary 2-approximation algorithm for $k$-center. Consider a 2-perturbation resilient $k$-center instance $I = (V, d, k)$. Let $C = \{C_1, \ldots, C_k\}$ be the unique optimum clustering. Suppose $B$ is the set of centers returned by algorithm $A$ when invoked on $I$. Then, the Voronoi partition induced by $B$ gives the optimal clustering $C$.

**Proof:** Let $R_d^*$ denote the optimum solution value for the given instance. Let $C'$ be a Voronoi partition induced by $B$. In clustering $C'$, for each point $p \in V$, let $c(p)$ be the center in $B$ it is assigned to i.e. $d(c(p), p) \leq d(B \setminus \{c(p)\}, p)$. We define a distance function $d'$ which is a metric 2-perturbation of $d$: consider the complete graph $G$ on vertices $V$. Let $E' = \{(c(p), p) : p \in V\}$. The edge lengths in graph $G$ are given by the function $\ell$, where for any edge $(u, v)$,

$$\ell(u, v) = \begin{cases} \min\{d(u, v), R_d^*\} & (u, v) \in E' \\ d(u, v) & \text{otherwise} \end{cases}$$

For any pair of points $u, v$, the distance $d'(u, v)$ is the shortest path distance between $u$ and $v$ in graph $G$, using $\ell$.

**Observation 3.1.** $d'$ is a metric 2-perturbation of $d$.

**Proof:** Since algorithm $A$ returns a 2-approximate solution, for each point $p \in V$, $d(c(p), p) \leq 2 \cdot R_d^*$. Therefore, for any $(u, v) \in E'$, $\ell(u, v) = \min\{d(u, v), R_d^*\} \geq \frac{d(u, v)}{2}$. For any other $(u, v)$, by definition $\ell(u, v) = d(u, v)$. In other words, for any $u, v$, we have $\frac{d(u, v)}{2} \leq \ell(u, v) \leq d(u, v)$. As stated in Lemma 2.4, $d'$ defined as the shortest path distance in graph $G$ with edge lengths satisfying $\frac{d(u, v)}{2} \leq \ell(u, v) \leq d(u, v)$, is a metric 2-perturbation of $d$. $\blacksquare$

**Observation 3.2.** For any $p \in V$, $d'(c(p), p) \leq \min\{d(c(p), p), R_d^*\}$. Further, $d'(p, B \setminus \{c(p)\}) \geq \min\{d(c(p), p), R_d^*\}$.

**Proof:** The first claim follows immediately from the fact $d'(c(p), p) \leq \ell(c(p), p)$. For the second claim, consider any $s \in B \setminus \{c(p)\}$. Let $P$ be an arbitrary $s \leadsto p$ path. If $P \cap E' = \emptyset$, then by triangle inequality $\ell(P) = \sum_{e \in P, d(e) \geq d(s, p) \geq d(c(p), p)} + \sum_{e \in P \cap E'} \min\{d(e), R_d^*\} \geq \min\{d(s, p), R_d^*\} \geq \min\{d(c(p), p), R_d^*\}$. Therefore $d'(s, p) = \min_P \ell(p) \geq \min\{d(c(p), p), R_d^*\}$. $\blacksquare$

Consider the instance $I' = (V, d', k)$. Since, $I'$ is a 2-perturbed instance, the optimal clustering is given by $C = \{C_1, \ldots, C_k\}$, and let $R_{d'}^*$ denote the cost of optimal solution. Using Lemma 2.5 we get, $R_{d'}^* = R_d^*$. Now, Observation 3.2 implies, $\text{cost}_p(B) = \max_{p \in V} d'(B, p) \leq \max_{p \in V} d'(c(p), p) \leq R_d^* = R_{d'}^*$. Therefore, $B$ is a set of optimal centers for $I'$. By perturbation resilience $B$ induces the unique Voronoi partition $C$ in $I'$. For any $p \in V$, $d'(c(p), p) \leq d'(p, B \setminus \{c(p)\})$ (follows
from Observation 3.2). Since $B$ induces a unique Voronoi partition in $\mathcal{I}'$, it must be the case $d'(c(p), p) < d'(p, B \setminus \{c(p)\})$. This also implies, that in clustering $\mathcal{C}$, every $p \in V$ belongs to the same cluster as $c(p)$. Recall the definition of $c(p)$: $p$ was assigned to $c(p)$ in the clustering induced by $B$ in the original instance $\mathcal{I}$. Thus the clusters in $\mathcal{C}'$ and $\mathcal{C}$ are identical. This also proves that the Voronoi clustering induced by $B$ in $\mathcal{I}$ is unique.

**Properties of 2-perturbation resilient $k$-center instance:** Angelidakis et al. [3] showed that in the optimal clustering of a 2-perturbation resilient $k$-center instance, every point is closer to its assigned center than to any point in a different cluster. In fact they show this property for general center based objectives, not just $k$-center. Here we observe that Theorem 3.1 implies stronger structural properties for $k$-center: (1) any point is closer to a point in its own cluster, than to a point in a different cluster; (2) the distance between two points in two different clusters is at least the optimal radius (see Figure 1a). Rest of this section is devoted to proving these properties.

**Lemma 3.1.** Consider a 2-perturbation resilient $k$-center instance $\mathcal{I} = (V, d, k)$. Let $S = \{c_1, \ldots, c_k\}$ be an optimal set of centers, and $\mathcal{C} = \{C_1, \ldots, C_k\}$ be the corresponding unique optimal clustering. Consider any cluster $C_i$ with $|C_i| \geq 2$, and let $p, w$ be any two points in $C_i$. For any point $q$ in a different cluster $C_j$ ($i \neq j$), we have $d(p, q) > d(p, w)$.

**Proof:** Note that the set of centers $B = S \setminus \{c_i, c_j\} \cup \{w, q\}$ gives a 2-approximation. Therefore Theorem 3.1 immediately implies $d(w, p) < d(q, p)$. \hfill $\Box$

**Lemma 3.2.** Consider a 2-perturbation resilient $k$-center instance $\mathcal{I} = (V, d, k)$. Let $S = \{c_1, \ldots, c_k\}$ be an optimal set of centers, and $\mathcal{C} = \{C_1, \ldots, C_k\}$ be the corresponding unique optimal clustering. The optimal radius is $R^*_d$. Consider any point $p \in V$, and let $p \in C_i$. For any point $q$ in a different cluster $C_j$ ($i \neq j$), we have $d(p, q) > R^*_d$.

**Proof:** Assume for the sake of contradiction that the claim is not true, that is, there exists two points $p \in C_i$, and $q \in C_j$ such that $d(p, q) \leq R^*_d$. We claim both $C_i$ and $C_j$ cannot have cardinality 1; if it is the case then $k - 1$ centers $S \setminus \{c_i\}$ will give solution of cost $R^*_d$, which would imply $\mathcal{I}$ is not 2-perturbation resilient (follows from Lemma 2.1). Assume without loss of generality that $|C_i| > 1$. Now Lemma 3.1, coupled with our assumption $d(p, q) \leq R^*_d$ indicates, $\forall u \in C_i$, $d(u, p) < R^*_d$. First,
note that this implies \(|C_j| > 1\), as otherwise the \(k - 1\) centers \(S \setminus \{c_j\} \cup \{p\}\) will give an optimal solution, which cannot happen for a 2-PR instance. Second, by triangle inequality, we have \(\forall u \in C_i, d(q, u) < 2R_d^*\). Let \(q'\) be any arbitrary point in \(C_j \setminus \{q\}\). The set of centers \(B = S \setminus \{c_j\} \cup \{q, q'\}\) gives a 2-approximation since every point in \(C_i \cup C_j\) is within \(2R_d^*\) of \(\{q, q'\}\). However, the Voronoi partition induced by \(B\) is clearly different from \(C\), as \(C_i\) is no longer a cluster. This contradicts Theorem 3.1.

3.1 LP Integrality

Now, we show that as a consequence of Lemma 3.2, the LP relaxation for \(k\)-center is integral. Given an instance \(I = (V, d, k)\) of \(k\)-center and a parameter \(R \geq 0\), we define the graph (also called threshold graph) \(G_R = (V, E_R)\), where \(E_R = \{(u, v) : u, v \in V, d(u, v) \leq R\}\). For a vertex \(v\), let \(\text{Nbr}[v] = \{u : (u, v) \in E_R\} \cup \{v\}\) be the neighbors (including itself). Observe, for any \(R \geq R_d^*\), where \(R_d^*\) is the optimal solution cost of \(I\), there exists a set of \(k\) centers \(S \subseteq V\), such that \(S\) covers \(V\) in \(G_R\), i.e. \(\bigcup_{c \in S} \text{Nbr}[c] = V\). Given a parameter \(R\), we can define the following LP on graph \(G_R\). We use \(y_v\) as an indicator variable for open centers, and \(x_{uv}\) to denote if \(v\) is assigned to \(u\).

\[
\begin{align*}
\sum_{u \in V} y_u &\leq k \quad \text{(kc-LP)} \\
x_{uv} &\leq y_u \quad \forall v \in V, u \in V \\
\sum_{u \in \text{Nbr}[v]} x_{uv} &\geq 1 \quad \forall v \in V \\
y_v, x_{uv} &\geq 0
\end{align*}
\]

The minimum \(R\) for which \(\text{kc-LP}\) is feasible provides a lower bound on the optimum solution, and is the standard relaxation for \(k\)-center. It easy to see for all \(R \geq R_d^*\) \(\text{kc-LP}\) is feasible. Further, it is well-known that the integrality gap is 2, that is, for all \(R < R_d^*/2\), the LP is infeasible. However, if the \(k\)-center instance is 2-perturbation resilient, we can show that LP has no integrality gap.

**Theorem 3.2.** Consider a 2-perturbation resilient instance \(I = (V, d, k)\) of \(k\)-center. Let \(R_d^*\) be the cost of the optimal solution. Then, for any \(R < R_d^*\), \(\text{kc-LP}\) is infeasible.

**Proof:** Let \(C_1, \ldots, C_k\) be the unique optimal clustering of instance \(I = (V, d, k)\), with optimal radius \(R_d^*\). Consider an arbitrary \(R < R_d^*\), and let \(G_R\) denote the corresponding threshold graph. Recall, in graph \(G_R\) the vertex set is \(V\), and the edge set \(E_R = \{(u, v) : d(u, v) \leq R\}\). According to Lemma 3.2, in a 2-PR instance, two points belonging to two different optimal clusters are separated by a distance of strictly more than \(R_d^*\). Since \(I\) is 2-PR, graph \(G_R\) has a simple structure — for any \(v \in C_i, i \in [k]\), \(\text{Nbr}[v] \subseteq C_i\). Or in other words, the connected components of \(G_R\) are subsets of the optimal clusters (see Figure 1b).

Suppose, the \(k\)-center LP (\(\text{kc-LP}\)) defined over graph \(G_R\) is feasible, and \((x, y)\) is the feasible fractional solution. Since every point is fully covered, and it can be covered only by its neighbors in \(G_R\), we have, for all \(C_i, \sum_{u \in C_i} y_u \geq 1\). Since, \(\sum_{v \in V} y_v \leq k\), and the clusters \(C_1, \ldots, C_k\) are disjoint, we have \(\sum_{u \in C_i} y_u = 1\), for each \(i\).

From the definition of \(R_d^*\), there is an optimum cluster \(C_i\) such that \(\min_{v \in C_i} \max_{v \in C_i} d(c, v) = R_d^*\). Let \(C'_i = \{u \in C_i : y_u > 0\}\). As we argued earlier, \(\sum_{u \in C_i} y_u = \sum_{u \in C'_i} y_u = 1\). Further, since for every \(v \in C_i\), \(\text{Nbr}[v] \subseteq C_i\), and \(v\) needs to be covered, we must have \(C'_i \subseteq \text{Nbr}[v]\). Consider any \(c \in C'_i\). Note that for every \(v \in C_i, c\) is a neighbor of \(v\) in graph \(G_R\), i.e. \(d(c, v) \leq R < R_d^*\). This implies, \(\max_{v \in C_i} d(c, v) < R_d^*\) which is a contradiction. 

\(\square\)
4 LP Integrality of ASYM-\(k\)-CENTER under Perturbation Resilience

We start with an LP relaxation for ASYM-\(k\)-CENTER problem by considering an unweighted directed graph on node set \(V\). Specifically, for a parameter \(R \geq 0\), we define the directed graph \(G_R = (V, E_R)\), where \(E_R = \{(u, v) : u, v \in V, d(u, v) \leq R\}\). For a node \(v\), let \(\text{Nbr}^-[v] = \{u : (u, v) \in E_R\}\) denote the in-neighbors, and \(\text{Nbr}^+[v] = \{u : (v, u) \in E_R\}\) be the out-neighbors (including itself). Observe, for any \(R \geq R^*_d\), there exists a set of \(k\) centers \(S \subseteq V\), such that \(S\) covers \(V\) in \(G_R\), i.e. \(\bigcup_{c \in S} \text{Nbr}^+[c] = V\). Thus, given a parameter \(R\), we can define the following LP relaxation on graph \(G_R\). We use \(y_v\) as an indicator variable for open centers, and \(x_{uv}\) to denote if \(v\) is assigned to \(u\).

\[
\begin{align*}
\sum_{u \in V} y_u & \leq k & \text{(asym-kc-LP)} \\
x_{uv} & \leq y_u & \forall v \in V, u \in V \\
\sum_{u \in \text{Nbr}^-[v]} x_{uv} & \geq 1 & \forall v \in V \\
y_v, x_{uv} & \geq 0
\end{align*}
\]

For ASYM-\(k\)-CENTER, Archer [4] showed that the integrality gap is atmost \(O(\log^\ast k)\), in fact it is tight within a constant factor [22]. The main result of the section is captured by the following theorem.

**Theorem 4.1.** Let \(\mathcal{I} = (V, d, k)\) be a 2-perturbation resilient instance of ASYM-\(k\)-CENTER and let \(R^*_d\) be the cost of the optimal solution. Then, for any \(R < R^*_d\), asym-kc-LP is infeasible.

4.1 Properties of 2-perturbation resilient ASYM-\(k\)-CENTER instance

In Section 3 we showed that the clusters in an optimal solution to a 2-PR \(k\)-center instance have a strong separation property: \(d(p, q) > R^*_d\) if \(p, q\) are in different clusters. For ASYM-\(k\)-CENTER the asymmetry in the distances does not permit a such a strong and simple separation property. However, we can show slightly weaker properties: (1) every optimal center is separated from any point in a different cluster by at least \(R^*_d\); (2) points in a cluster which are far off from core points (these points have small distance "to" corresponding cluster centers) in the cluster, are well-separated from core points of other clusters as well (See Figure 2a, Figure 2b). These properties suffice to prove our desired theorem. The rest of the section is dedicated to proving these properties.

**Lemma 4.1.** Consider a 2-perturbation resilient ASYM-\(k\)-CENTER instance \(\mathcal{I} = (V, d, k)\). Let \(C = \{C_1, \ldots, C_k\}\) be the unique optimal clustering, induced by a set of centers \(S = \{c_1, \ldots, c_k\}\). Let the optimal radius be \(R^*_d\). Consider any center \(c_i\). Then for any point \(q\) in a different cluster \(C_j\) (\(i \neq j\)), we have \(d(q, c_i) > R^*_d\).

**Proof:** Assume for the sake of contradiction, that the claim is false. That is, there exists a point \(q \in C_j\), such that \(d(q, c_i) \leq R^*_d\). We construct a distance function \(d'\), which is a metric 2-perturbation of \(d\). And show that in the instance thus constructed, the optimal clustering is not unique, which contradicts the definition of perturbation resilience.

We define \(d'\) as follows: consider the complete directed graph \(G\) on vertices \(V\). Let \(E' = \{(q, v) : v \in C_i\}\). The edge lengths in graph \(G\) are given by the function \(\ell\), where for any edge \((u, v)\),

\[
\ell(u, v) = \begin{cases}
\min\{d(u, v), R^*_d\} & (u, v) \in E' \\
d(u, v) & \text{otherwise}
\end{cases}
\]

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For any pair of points $u, v$, the distance $d'(u, v)$ is the shortest path distance between $u$ and $v$ in graph $G$, using $\ell$.

**Observation 4.1.** $d'$ is a metric 2-perturbation of $d$.

**Proof:** For any $v \in C_i$, by triangle inequality, $d(q, v) \leq d(q, c_i) + d(c_i, v) \leq 2 \cdot R_d^*$. Therefore, for any $(u, v) \in E'$, $\ell(u, v) = \min\{d(u, v), R_d^*\} \geq \frac{d(u, v)}{2}$. For any other $(u, v)$, by definition $\ell(u, v) = d(u, v)$. That is, for any edge $(u, v)$ we have, $\frac{d(u, v)}{2} \leq \ell(u, v) \leq d(u, v)$. The shortest path distance function $d'$, defined on such a graph $G$, can be easily shown is a metric 2-perturbation of $d$ (details are chalked out in proof Lemma 2.2).

Consider the instance $I' = (V, d', k)$. Since, $I'$ is a 2-perturbed instance, the optimal clustering is given by $C = \{C_1, \ldots, C_k\}$. Let $S' = \{c'_1, \ldots, c'_k\}$ be the optimal set of centers. Further, $R^*_d$ denotes the cost of optimal solution. And we can show, $R^*_d = R^*_{d'}$ (follows from Lemma 2.3).

**Case 1:** $q \neq c'_j$. Consider the set of centers $S'' = S' \setminus \{c'_j\} \cup \{q\}$. Let $C''$ be a Voronoi partition induced by $S''$. For any point $u \in C_i$, where $\ell \neq i$, $d'(S'', u) \leq d'(c'_i, u) \leq R_{d'}^*$. For any point $u \in C_i$, $d'(S'', u) \leq d'(q, u) \leq R_{d'}^*$. That is, for any point $u \in V$, $d'(S'', u) \leq R_{d'}^*$. Therefore $\text{cost}_{d'}(C'', S'') \leq R_{d'}^*$. Now, in clustering $C''$, the points $q$ and $c'_j$ are in different clusters, which is not true for $C$. Thus the optimal clustering is not unique, and this leads to contradiction.

**Case 2:** $q = c'_j$. Consider the set of $k - 1$ centers $S'' = S' \setminus \{c'_j\}$. Let $C''$ be a Voronoi partition induced by $S''$. As in the previous case, we can show $d(S'', u) \leq R_{d'}^*$, for any $u \in V$, implying $\text{cost}_{d'}(C'', S'') \leq R_{d'}^*$. Therefore, we have a $k - 1$ clustering of $I'$ with cost at most the optimal cost. Then, by Lemma 2.1, the optimal clustering of $I'$ is not unique. This contradicts the definition of perturbation resilience.

The next lemma formalizes the notion of core points and the property they enjoy.
Lemma 4.2. Consider a 2-perturbation resilient ASYM-k-CENTER instance $I = (V,d,k)$. Let $C = \{C_1, \ldots, C_k\}$ be the unique optimal clustering induced by a set of centers $S = \{c_1, \ldots, c_k\}$. Let the optimal radius is $R^*_d$. Suppose $p \in C_i$ and $q \in C_j$ where $i \neq j$ and $d(p,c_i) \leq R^*_d$ and $d(q,c_j) \leq R^*_d$. Then for any $w \in C_i$ such that $d(p,w) \geq R^*_d$ we have $d(q,w) > R^*_d$.

Proof: Consider a triplet of points $p, w \in C_i$ and $q \in C_j$, where $d(p,c_i), d(q,c_j) \leq R^*_d$, and $d(p,w) \geq R^*_d$. Assume for the sake of contradiction, $d(q,w) \leq R^*_d$. We construct a distance function $d'$, which is a metric 2-perturbation of $d$. Next we show that in the ASYM-k-CENTER instance constructed using $d'$, the optimal clustering is not unique, which contradicts the definition of perturbation resilience.

We define $d'$ as follows: consider the complete directed graph $G$ on vertices $V$. Let $E' = \{(p,v) : v \in C_i\} \cup \{(q,v) : v \in C_j\}$. The edge lengths in graph $G$ are given by the function $\ell$, where for any edge $(u,v)$,

$$\ell(u,v) = \begin{cases} \min\{d(u,v), R^*_d\} & (u,v) \in E' \\ d(u,v) & \text{otherwise} \end{cases}$$

For any pair of points $u,v$, the distance $d'(u,v)$ is the shortest path distance between $u$ and $v$ in graph $G$, using $\ell$.

Observation 4.2. $d'$ is a metric 2-perturbation of $d$.

Proof: For any $v \in C_i$, by triangle inequality, $d(p,v) \leq d(p,c_i) + d(c_i,v) \leq 2 \cdot R^*_d$. Similarly, for any $v \in C_j$, by triangle inequality, $d(q,v) \leq 2 \cdot R^*_d$. Therefore, for any $(u,v) \in E'$, $\ell(u,v) = \min\{d(u,v), R^*_d\} \geq \frac{d(u,v)}{2}$. For any other $(u,v)$, by definition $\ell(u,v) = d(u,v)$. As we previously stated (in Lemma 2.2), $d'$ defined on graph $G$, satisfying the property $\frac{d(u,v)}{2} \leq \ell(u,v) \leq d(u,v)$ is a metric 2-perturbation of $d$.

Observation 4.3. For any point $v \in C_i$, we have $d'(p,v) \leq R^*_d$. Similarly for any $v \in C_j$, $d'(q,v) \leq R^*_d$. In particular, for point $w$, $d'(p,w) = R^*_d$ and $d'(q,w) \leq R^*_d$.

Proof: For any point $v \in C_i$, by definition, $\ell(p,v) \leq R^*_d$. Since, $d'(p,v) \leq \ell(p,v)$ the claim follows. Similarly, for any $v \in C_j$, it is easy to see $d'(q,v) \leq R^*_d$.

Now, consider point $w$. Let $P$ be any directed path from $p \sim w$ in graph $G$, excluding the single edge path $(p,w)$. If $P$ includes an edge from $E'$, by triangle inequality, $\ell(P) = \sum_{e \in P} \ell(e) \geq d(p,w) \geq R^*_d$. The last inequality follows from our choice of $p, w$ at the outset. Otherwise, if $P$ includes at least one edge from $E'$, $\ell(P') = \sum_{e \in P \cap E'} \ell(e) + \sum_{e \in P' \setminus E'} \ell(e) \geq \sum_{e \in P \cap E'} \min\{d(e), R^*_d\} + \sum_{e \in P' \setminus E'} \ell(e) = \min\{d(p,w), R^*_d\} \geq R^*_d$. Finally, by our definition $\ell(p,w) = R^*_d$. Therefore, $d'(p,w) = R^*_d$. The final observation, $d'(q,w) \leq R^*_d$, follows from our assumption $d(q,w) \leq R^*_d$.

Consider the instance $I' = (V,d',k)$. Since, $I'$ is a 2-perturbed instance, the optimal clustering is given by $C = \{C_1, \ldots, C_k\}$. Let $S' = \{c'_1, \ldots, c'_k\}$ be the optimal set of centers. Further, $R^*_{d'}$ denotes the cost of optimal solution. We can show, $R^*_{d'} = R^*_d$.

Consider the set of centers $S'' = \left(S' \setminus \{c'_i, c'_j\}\right) \cup \{p, q\}$. Let $C''$ be a Voronoi partition induced by $S''$. For any point $u \in C_i$, $d'(S'', u) \leq d'(S'', u) \leq R^*_d$. For any point $u \in C_i$, $d'(S'', u) \leq d(p,u) \leq R^*_d$. Similarly, for any point $u \in C_j$, $d'(S'', u) \leq d'(q,u) \leq R^*_d$. That is, for any point $u \in V$, $d'(S'', u) \leq R^*_d$. Therefore $\text{cost}_{d'}(C'', S'') \leq R^*_d$. Recall by Observation 4.3, $d'(p,w) = R^*_d$, while $d'(q,w) \leq R^*_d$. Therefore, we can assume without loss of
generality, in $C''$, $w$ and $p$ are not in same cluster. This however is not true for $C$, implying $C'' \neq C$. Thus the optimal clustering of $\mathcal{T}'$ is not unique, and this contradicts the definition of perturbation resilience.

\[
\sum
\]

4.2 Proof of Theorem 4.1

Let $C_1, \ldots, C_k$ be the unique optimal clustering of instance $\mathcal{I} = (V, d, k)$, with optimal radius $R_d^*$. Consider an arbitrary $R < R_d^*$, and let $G_R$ denote the corresponding threshold graph. Recall, graph $G_R$ is a directed graph defined over vertex set $V$, and the edge set $E_R = \{(u, v) : d(u, v) \leq R\}$. Suppose, the ASYM-k-CENTER LP (ASYM-kc-LP) defined over graph $G_R$ is feasible, and $(x, y)$ is a feasible fractional solution.

From Lemma 4.1, in a 2-PR instance, we have the following: if $q \notin C_i$ then $d(q, c_i) > R_d^*$. This implies that, in the graph $G_R$, for any $c_i, i \in [k], \text{Nbr}^-[c_i] \subseteq C_i$. Let $C_i^t = \{u \in \text{Nbr}^-[c_i] : y_u > 0\}$. Since $(x, y)$ is a feasible solution, we must have $\sum_{u \in C_i^t} y_u \geq 1$. Since, $\sum_{v \in V} y_v \leq k$, and the clusters $C_1, \ldots, C_k$ are disjoint, we have $\sum_{u \in C_i} y_u = \sum_{u \in C_i^t} y_u = 1$, for all $i \in [k]$.

From the definition of $R_d^*$ there must be a cluster $C_t$ such that $\min_{c \in C_t} \max_{v \in C_t} d(c, v) = R_d^*$. Consider its center $c_t$ and let $p \in C_t$. Clearly $d(p, c_t) \leq R < R_d^*$. Furthermore, since $C_t$ is the largest radius cluster, there exists $w \in C_t$, such that $d(p, w) = R_d^*$. Therefore in graph $G_R$, $p \notin \text{Nbr}^-[w]$. For any other cluster $C_j$, by Lemma 4.2, for any point $q \in C_j^t$, we have $d(q, w) > R_d^*$. That is, $\text{Nbr}^-[w] \cap C_j^t = \emptyset$, for any $j \neq t$. This implies $w$ can be covered only by points that belong to $C_t^t$. Therefore $\sum_{u \in \text{Nbr}^-[w]} x_{uw} \leq \sum_{u \in C_t^t - p} y_u < 1$ since $y_p > 0$. This contradicts feasibility of $(x, y)$.

5 LP Integrality of k-CENTER-OUTLIER under Perturbation Resilience

In this section we consider the k-CENTER-OUTLIER problem. Recall that an instance $\mathcal{I} = (V, d, k, z)$ consists of a finite metric space $(V, d)$ an integer $k$ specifying the number of centers and an integer $z < |V|$ specifying the number of outliers that are allowed. One can write a natural LP relaxation for this problem as follows. As before, for a parameter $R \geq 0$, we define the graph $G_R = (V, E_R)$, where $E_R = \{(u, v) : u, v \in V, d(u, v) \leq R\}$. For a node $v$, let $\text{Nbr}[v] = \{u : (u, v) \in E_R \cup \{v\}\}$ be the neighbors (including itself). Observe, for any $R \geq R_d^*$, there exists a set of $k$ centers $S \subseteq V$, and a set of outliers $Z$ with $|Z| \leq z$, such that $S$ covers $V \setminus Z$ in $G_R$, i.e. $\cup_{c \in S} \text{Nbr}[c] = V \setminus Z$. Thus, given a parameter $R$, we can define the following LP relaxation on graph $G_R$. We use $y_v$ as an indicator variable for open centers, and $x_{uv}$ to denote if $v$ is assigned to $u$.

\[
\begin{align*}
\sum_{u \in V} y_u & \leq k & \text{(kco-LP)} \\
x_{uv} & \leq y_u & \forall v \in V, u \in V \\
\sum_{u \in V} x_{uv} & \leq 1 & \forall v \in V \\
\sum_{v \in V} \sum_{u \in V} x_{uv} & \geq n - z \\
x_{uv} & = 0 & \forall v \in V, u \notin \text{Nbr}[v] \\
y_v, x_{uv} & \geq 0
\end{align*}
\]

The kco-LP is feasible for all $R \geq R_d^*$. The main theorem we prove in this section is as follows:
Theorem 5.1. Given a 2-perturbation resilient instance $I = (V, d, k, z)$ of $k$-CENTER-OUTLIER with optimal cost $R_d^*$, kco-LP is infeasible for any $R < R_d^*$.

5.1 Properties of 2-perturbation resilient $k$-CENTER-OUTLIER instance

For $k$-CENTER-OUTLIER we extend the properties from Section 3 that hold for 2-perturbation resilient instances. The first property shows that if $p$ is a non-outlier point $p$ and $q$ is any point not in the same cluster as $p$ ($q$ could be an outlier) then $d(p, q) > R_d^*$. The second property is that for any outlier point $q$, the number of outliers in a ball of radius $2R_d^*$ is small. Specifically the number of points is strictly smaller than the size of the smallest cluster in the optimum clustering. This property makes intuitive sense, for otherwise $q$ can define another cluster with outlier points and contradict the uniqueness of the clustering in after perturbation. We formally state them below after setting up the required notation.

Let $I = (V, d, k, z)$ be a 2-outlier perturbation resilient $k$-CENTER-OUTLIER instance. Let $C = \{C_1, \ldots, C_k\}$ be the optimum clustering, and $Z$ be the set of outliers in the optimal solution of $I$. Further, let $S = \{c_1, \ldots, c_k\}$ be the optimal centers inducing the clustering $C$. Let the optimal cost be $R_d^*$. For each optimal cluster $C_i$, $n_i = |C_i|$ denotes its cardinality. Additionally, given a point $u \in Z$, and radius $R$, let $\text{Ball}_d(u, R) = \{v \in V : d(u, v) \leq R\}$ be the set of points in a ball of radius $R$ centered at $u$.

The two main structural properties of an 2-OPR $k$-CENTER-OUTLIER instance we show are as follows (See Figure 3a, Figure 3b):

Lemma 5.1. Consider any non-outlier point $p \in V \setminus Z$, and let $p \in C_i$. For all $q \notin C_i$, $d(p, q) > R_d^*$.

Lemma 5.2. For any outlier $p \in Z$, we have $|\text{Ball}_d(p, 2 \cdot R_d^*) \cap Z| < \min\{n_1, \ldots, n_k\}$.

We observe that a much weaker version of the preceding lemma suffices for our proof of LP integrality. The weaker version states that $|\text{Ball}_d(p, R_d^*) \cap Z| < \min\{n_1, \ldots, n_k\}$. For if the statement is false, we could replace the smallest cluster with the cluster $\text{Ball}_d(p, R_d^*)$; this gives an alternate clustering with at most $z$ outliers and the same optimum radius contradicting the uniqueness of the optimum solution.

We now prove the two lemmas.
5.1.1 Proof of Lemma 5.1

We prove Lemma 5.1 by splitting it into two cases. We first show that the lemma holds true for all \( q \in Z \). Next we show that the lemma holds true, even when \( q \in C_j \) (\( j \neq i \)).

**Lemma 5.3.** Consider any point \( p \in V \setminus Z \), and let \( p \in C_i \). For all \( q \in Z \), \( d(p, q) > R^*_d \).

**Proof:** Assume that the claim is not true, that is, there exists \( q \in Z \) such that \( d(p, q) \leq R^*_d \). Since \( p \in C_i \), we have \( d(c_i, p) \leq R^*_d \). Therefore by triangle inequality, \( d(c_i, q) \leq 2 \cdot R^*_d \).

We now define a metric distance function \( d' \), which is 2-perturbation of \( d \). To this end, consider the complete undirected graph \( G \) on vertex set \( V \). The edge lengths in graph \( G \) are given by the function \( \ell \), where for any edge \( (u, v) \),

\[
\ell(u, v) = \begin{cases} 
\min\{d(u, v), R^*_d\} & u = c_i, v = q \\
 2d(u, v) & \text{otherwise}
\end{cases}
\]

For any pair of points \( u, v \), the distance \( d'(u, v) \) is the shortest path distance between \( u \) and \( v \) in graph \( G \), using \( \ell \). The following observation is easy to see since \( d(u, v)/2 \leq \ell(u, v) \leq d(u, v) \) for every pair \((u, v)\) (follows from Lemma 2.4).

**Observation 5.1.** \( d' \) is a metric 2-perturbation of \( d \).

Consider the instance \( \mathcal{I}' = (V, d', k, z) \). Since, \( \mathcal{I}' \) is a 2-perturbed instance, the unique optimal solution is given by the clusters \( \mathcal{C} = \{C_1, \ldots, C_k\} \), and outliers \( Z \). Let \( S' = \{c'_1, \ldots, c'_k\} \) be the optimal set of centers. Let \( R^*_d \) denote the optimal radius of \( \mathcal{I}' \). We will construct an alternate solution (clustering and outliers) for \( \mathcal{I}' \) with cost at most \( R^*_d \). This contradicts the uniqueness of the optimal solution, and thus fails to satisfy the definition of perturbation resilience.

The following claim is also easy to establish (refer Lemma 2.5).

**Claim 5.1.** \( R^*_d' = R^*_d \)

Next, we show the existence of an alternate solution of cost at most \( R^*_d \). Consider the set of outliers \( Z' = Z \setminus \{q\} \). Let \( \mathcal{C}' \) be the Voronoi partition of \( V \setminus Z' \) induced by \( S' \). Clearly the clustering \( \mathcal{C}' \) is different from \( \mathcal{C} \). Further, since \( d'(c_i, q) \leq R^*_d = R^*_d' \), we have, \( \text{cost}_{d'}(\mathcal{C}', S'; Z') = \max_{u \in V \setminus Z'} d'(S', u) \leq R^*_d \). This contradicts the uniqueness of the optimal clustering and outliers of \( \mathcal{I}' \).

**Lemma 5.4.** Consider any point \( p \in V \setminus Z \), let \( p \in C_i \). For all \( q \in C_j \), \( d(p, q) > R^*_d \).

**Proof:** Follows from the fact, that instance \((V \setminus Z, d, k)\) is a 2-perturbation resilient instance for \( k \)-center and **Lemma 3.2**.

Lemma 5.1 follows immediately from Lemma 5.3 and Lemma 5.4.

5.1.2 Proof of Lemma 5.2

Let \( C_i \) be the smallest cardinality cluster. Assume for the sake of contradiction that the claim is false, i.e., there exists \( p \in Z \), such that \( |\text{Ball}_d(p, 2 \cdot R^*_d) \cap Z| \geq n_i \).

We construct a distance function \( d' \) which is a metric 2-perturbation of \( d \). Consider the complete graph \( G \) with edge lengths \( \ell \). Let \( E' = \{(p, v) : v \in \text{Ball}_d(p, 2 \cdot R^*_d) \cap Z\} \). The edge lengths are defined as follows:

\[
\ell(u, v) = \begin{cases} 
\min\{d(u, v), R^*_d\} & (u, v) \in E' \\
d(u, v) & \text{otherwise}
\end{cases}
\]
For any pair of points \( u, v \), the distance \( d'(u, v) \) is the shortest path distance between \( u \) and \( v \) in graph \( G \), using \( \ell \). Note that, for all \( u, v \in V \), \( \ell(u,v) \geq \frac{d(u,v)}{2} \). We can immediately make the following observation about \( d' \).

**Observation 5.2.** \( d' \) is a metric 2-perturbation of \( d \).

Consider the instance \( I' = (V, d', k, z) \). Since, \( I' \) is a 2-perturbed instance, the optimal clustering and outliers are \( C = \{C_1, \ldots, C_k\} \) and \( Z \) respectively. Let \( S' = \{c'_1, \ldots, c'_k\} \) be the optimal set of centers inducing the \( C \). Further, \( R'^*_d \) denotes the cost of optimal solution. Again note that \( R'^*_d = R'^*_d \).

Now, consider the set of outliers \( Z'' = (Z \setminus \text{Ball}_d(p, 2 \cdot R'^*_d)) \cup C_i \). Let \( S'' = S' \setminus \{c_i\} \cup \{p\} \) be a set of \( k \) centers, and \( C'' \) is a Voronoi partition of \( V \setminus Z'' \) induced by \( S'' \). For any point \( u \in C_\ell \), where \( \ell \neq i \), \( d'(S'', u) \leq d'(c'_\ell, u) \leq R'^*_d \). For any point \( u \in \text{Ball}_d(p, 2 \cdot R'^*_d) \cap Z \), we have, \( d'(S'', u) \leq d'(p, u) \leq R'^*_d \). Therefore, for any point \( u \in V \setminus Z'' \), \( d'(S'', u) \leq R'^*_d \). This implies \( \text{cost}_{d'}(C'', S''; Z'') \leq R'^*_d \). Clearly the clustering \( C'' \) is different from \( C \). Since \( Z \cap C_i = \emptyset \), therefore \( |Z''| = |Z| - |\text{Ball}_d(p, 2 \cdot R'^*_d) \cap Z| + |n_i| \leq z \). Thus, \( C'', Z'' \) is another solution for instance \( I' \) having cost at most the optimal. In other words, the optimal solution of \( I' \) is not unique, and this leads to contradiction.

## 5.2 Integrality Gap and Proof of Theorem 5.1

In this section, we show that \textbf{kco-LP} is infeasible for \( R < R'^*_d \). Recall in Lemma 5.1, we showed that the optimal clusters are well-separated from each other and also from the outliers. Therefore, in graph \( G_R \), the connected components are either subsets of optimal clusters or outliers (See Figure 4). As a consequence, in a fractional solution, non-outlier points can only be covered by points inside the cluster, and similarly outliers can be covered by outliers only. However, unlike \( k \)-center, here the tricky part is, the fractionally open outliers can potentially cover a lot of points. We show that this in fact is not possible because of the sparsity of an outlier’s neighborhood.

Suppose the claim is not true, that is for some \( R < R'^*_d \), \textbf{kco-LP} has a feasible solution \((x^*, y^*)\). Let \( C = \{C_1, \ldots, C_k\} \) be the set of clusters and \( Z \) be the outliers in the unique optimal solution of \( I \).

First, let us consider the simpler case when \( y^*(Z) = 0 \). Recall Lemma 5.1, for every \( p \in C_i \), \((i \in [k])\), the distance to any \( q \notin C_i \) is more than \( R'^*_d \). In other words, for any \( p \in C_i \), \( \text{Nbr}[p] \subseteq C_i \), and for any \( w \in Z \), \( \text{Nbr}[w] \cap V \setminus Z = \emptyset \). Therefore, \( y^*(Z) = 0 \) and the LP constraint \( x_{uw} = 0, \forall v \in V, u \notin \text{Nbr}[v] \) implies (1) for any \( w \in Z \), \( x_{uw}^* = 0 \) for all \( u \in V \); (2) for any \( v \in V \setminus Z \), and \( w \in Z \), \( x_{uw}^* = 0 \). Therefore, \((x^*, y^*)\) [restricted to \( V \setminus Z \)] is a feasible fractional solution for \textbf{kco-LP} defined for the \( k \)-center instance \( I' = (V \setminus Z, d, k) \), and parameter \( R \). The optimal radius of \( I' \) is also \( R'^*_d \).
Therefore by Theorem 3.2, we cannot have a feasible fractional solution for \( R < R^*_d \), leading to a contradiction.

We focus on the case \( y^*(Z) > 0 \). Without loss of generality assume that the optimum clusters are numbered such that \( n_1 \leq n_2 \leq \ldots \leq n_k \). For \( i \in [k] \) let \( a_i = y(C_i) \) and let \( b = y^*(Z) \). For a point \( p \) let \( \gamma_p = \sum u^*_p x^*_u \) be the amount to which \( p \) is covered. For a set of points \( S \) we let \( \gamma(S) \) denote \( \sum_{p \in S} \gamma_p \).

**Claim 5.2.** Total coverage of outlier points, that is, \( \gamma(Z) = \sum_{p \in Z} p < bn_1 \).

**Proof:** Recall that an outlier point can only be covered by an outlier point. Further a point \( q \) can cover point \( p \) only if \( p \) is in the ball of radius \( R \) around \( q \). Thus we have

\[
\sum_{p \in Z} \gamma_p \leq \sum_{q \in Z} |\text{Ball}_d(q,R)| \cdot y_q < n_1 \sum_{q \in Z} y_q = bn_1
\]

where we used Lemma 5.2 to strictly upper bound \( |\text{Ball}_d(q,R)| \) by \( n_1 \).

**Claim 5.3.** Let \( C_i \) be an optimum cluster such that \( a_i < 1 \). Then \( \gamma(C_i) \leq n_i a_i \).

**Proof:** Only points in \( C_i \) can cover any given point \( p \in C_i \). Therefore \( \gamma_p \leq a_i \) for each \( p \in C_i \), and hence \( \gamma(C_i) \leq n_i a_i \).

Let \( A = \{ i \in [k] \mid a_i < 1 \} \) be the indices of the clusters whose total \( y \) value is strictly less than 1. Since \( y(V) = k \), we have \( b \leq \sum_{i \in A} (1 - a_i) \). Using the preceding two claims we have the following:

\[
\gamma(V) = \gamma(Z) + \sum_{i \in A} \gamma(C_i) + \sum_{j \notin A} \gamma(C_i) \\
\leq \gamma(Z) + \sum_{i \in A} \gamma(C_i) + \sum_{j \notin A} n_j \\
\leq \gamma(Z) + \sum_{i \in A} n_i a_i + \sum_{j \notin A} n_j \\
\leq \gamma(Z) - \sum_{i \in A} n_i (1 - a_i) + \sum_{j=1}^k n_j \\
\leq \gamma(Z) - \sum_{i \in A} n_i (1 - a_i) + (n - z) \\
< bn_1 - n_1 \sum_{i \in A} (1 - a_i) + (n - z) \\
< n - z.
\]

This is a contradiction to the feasibility of the LP solution.

### 6 Algorithm for \( k \)-MEDIAN-OUTLIER under Perturbation Resilience

In this section, we present a dynamic programming based algorithm for \( k \)-MEDIAN-OUTLIER, which gives an optimal solution when the instance is 2-perturbation resilient. First, we prove some structural properties of a 2-OPR \( k \)-MEDIAN-OUTLIER instance. They serve as the key ingredient in showing that our algorithm will return exact solution for 2-OPR instances.

This section is essentially a straightforward extension of the ideas in [3] once the model is set up. In a sense the model justifies the natural extension of the algorithm from [3] to the outlier setting.
6.1 Properties of 2-perturbation resilient $k$-MEDIAN-OUTLIER instance

Angelidakis et al. [3] proved that in the optimal solution of a 2-perturbation resilient $k$-median instance, every point is closer to its assigned center than to any point in a different cluster. In the optimal solution of $k$-MEDIAN-OUTLIER, points are not only assigned to clusters, some points are identified as outliers as well. Here, we extend the result of [3] to show that the optimal solution of a 2-OPR $k$-MEDIAN-OUTLIER instance satisfies the property: any non-outlier point is closer to its assigned center than to any point outside the cluster.

**Lemma 6.1.** Consider a 2-perturbation resilient $k$-MEDIAN-OUTLIER instance $I = (V, d, k, z)$. Let $C = \{C_1, \ldots, C_k\}$, and $Z$ be the unique optimal clustering and outliers resp. Consider any point $p \in V \setminus Z$, and let $p \in C_i$. For all $q \notin C_i$, we have $d(c_i, p) < d(p, q)$.

To prove Lemma 6.1, we split it into two cases: we show that it holds true for (1) all outlier points $q$; (2) all non-outlier points $q$ belonging to a different optimal cluster.

**Lemma 6.2.** Consider a 2-perturbation resilient $k$-MEDIAN-OUTLIER instance $I = (V, d, k, z)$. Let $C = \{C_1, \ldots, C_k\}$, and $Z$ be the unique optimal clustering and outliers resp. Consider any point $p \in V \setminus Z$, and let $p \in C_i$. Then, for any outlier $q \in Z$, we have $d(c_i, p) < d(p, q)$.

**Proof:** Let $S = \{c_1, \ldots, c_k\}$ be an optimal set of centers, inducing $C$. Without loss of generality we assume, $p \neq c_i$, since $c_i, q$ being distinct points $d(c_i, q) > 0 = d(c_i, c_i)$.

Assume for the sake of contradiction, the claim is false, that is, for some $q \in Z$, $d(p, c_i) \geq d(p, q)$. To prove the contradiction, we construct a distance function $d'$, which is a metric 2-perturbation of $d$. And show that in the instance thus constructed, the optimal solution is not unique — that is there exists an optimal clustering and outliers different from $C; Z$. This contradicts the definition of perturbation resilience.

We define $d'$ as follows: consider the complete graph $G$ on vertices $V$. The edge lengths in graph $G$ are given by the function $\ell$, where for any edge $(u, v)$,

$$\ell(u, v) = \begin{cases} 
  d(c_i, p) & (u, v) = (c_i, q) \\
  d(u, v) & \text{otherwise}
\end{cases}$$

For any pair of points $u, v$, the distance $d'(u, v)$ is the shortest path distance between $u$ and $v$ in graph $G$, using $\ell$. We can make some simple observations about $d'$:

**Observation 6.1.** $d'$ has the following properties:

1. For any $u, v \in V$,
   $$d'(u, v) = \min\{\ell(u, v), \ell(u, q) + \ell(q, c_i) + \ell(c_i, v), \ell(u, c_i) + \ell(c_i, q) + \ell(q, v)\}$$
   $$= \min\{d(u, v), d(u, q) + d(c_i, p) + d(c_i, v), d(u, c_i) + d(c_i, p) + d(q, v)\}$$
2. $d'(c_i, p) = d'(c_i, q) = d(c_i, p)$.

**Observation 6.2.** $d'$ is a metric 2-perturbation of $d$.

**Proof:** By triangle inequality, $d(c_i, q) \leq d(c_i, p) + d(p, q) \leq 2 \cdot d(c_i, p)$ — the last inequality follows from our assumption. Therefore $\frac{d(c_i, q)}{2} \leq d(c_i, p) = \ell(c_i, q)$. Further, note that $d(c_i, p) < d(c_i, q)$. Indeed, as otherwise we can swap $p$ and $q$ in the optimal solution, i.e. identify $p$ as an outlier and assign $q$ to the nearest center in $S$. Therefore, $\frac{d(c_i, q)}{2} \leq \ell(c_i, q) < d(c_i, q)$. For any other edge $(u, v)$, $\ell(u, v) = d(u, v)$. As we claimed in Lemma 2.2, $d'$ defined as the shortest path metric on an undirected graph $G$ with edge lengths $\ell$ satisfying the property $\frac{d(u, v)}{2} \leq \ell(u, v) \leq d(u, v)$, is a metric 2-perturbation of $d$. 

□
Consider the instance $I' = (V, d', k, z)$. Since, $I'$ is a 2-perturbation of $I$ instance, the unique optimal solution is given by the clusters $C = \{C_1, \ldots, C_k\}$, and outliers $Z$. Let $S' = \{c'_1, \ldots, c'_k\}$ be an optimal set of centers. We show that we can construct an alternate solution of cost at most the optimal solution cost by swapping $q$ with a non-outlier point. To this end we consider two case:

**Case 1:** $c'_i = c_i$. Consider a solution for $I'$, with set of outliers $Z' = Z \setminus \{q\} \cup \{p\}$, and centers $S'$. Let $C'$ be a Voronoi partition of $V \setminus Z'$ induced by $S'$. The cost of the clustering $C'$ is,

$$\text{cost}_{d'}(C', S'; Z) = \sum_{u \in V \setminus Z'} d'(S', u) = d'(S', q) + \sum_{u \in C_i \{p\}} d'(c'_i, u) + \sum_{s=1}^{k} \sum_{s \neq i} d'(c'_s, u)$$

$$\leq d'(c'_i, q) - d'(c'_i, p) + \sum_{s=1}^{k} \sum_{u \in C_s} d'(c'_s, u)$$

$$= d'(c_i, q) - d'(c_i, p) + \text{cost}_{d'}(C', S'; Z) \quad (\because c'_i = c_i)$$

$$= \text{cost}_{d'}(C, S'; Z) \quad (\because d'(c_i, p) = d'(c_i, q) \text{ by Observation 6.1})$$

**Case 2:** $c'_i \neq c_i$. We can assume without loss of generality, $S' \setminus \{c'_i\} \cup \{c_i\}$ is not an optimal set of centers for $I'$, otherwise the the argument is same as Case 1. In particular, this implies, $\sum_{u \in C_i} d'(c'_i, u) > \sum_{u \in C_i} d'(c'_i, u)$. Recall, for any two points $u, v \in V$, $d'(u, v) \leq d(u, v)$. We claim there must be a point $r \in C_i$, such that $d'(c'_i, r) < d(c'_i, r)$ (that is the distance between $c'_i$ and $r$ becomes strictly smaller after perturbation). Indeed this is true, as otherwise,

$$\sum_{u \in C_i} d'(c'_i, u) = \sum_{u \in C_i} d(c'_i, u) \geq \sum_{u \in C_i} d(c_i, u) \geq \sum_{u \in C_i} d'(c_i, u)$$

where the first inequality uses the fact that $c_i$ is a center in the optimal solution of $I$. Now, $d'(c'_i, r) < d(c'_i, r)$ implies couple of things: (1) $c'_i \neq r$, as in that case $d'(c'_i, r) = d(c'_i, r) = 0$; (2) $d'(c'_i, r) = \min\{\ell(r, q) + \ell(q, c_i) + \ell(c_i, c'_i), \ell(r, c_i) + \ell(c_i, q) + \ell(q, c'_i)\}$. Also, $d'(q, c'_i) = \min\{\ell(q, c'_i), \ell(q, c_i) + \ell(c_i, c'_i)\}$. Putting it together, we get $d'(c'_i, r) \geq d'(c'_i, q)$.

Consider a solution for $I'$, with set of outliers $Z' = Z \setminus \{q\} \cup \{r\}$, and centers $S'$. Let $C'$ be a Voronoi partition of $V \setminus Z'$ induced by $S'$. The cost of the solution is,

$$\text{cost}_{d'}(C', S'; Z') = \sum_{u \in V \setminus Z'} d'(S', u) = d'(S', q) + \sum_{u \in C_i \{p\}} d'(c'_i, u) + \sum_{s=1}^{k} \sum_{s \neq i} d'(c'_s, u)$$

$$\leq d'(c'_i, q) - d'(c'_i, r) + \sum_{s=1}^{k} \sum_{u \in C_s} d'(c'_s, u) \quad (\because c'_i = c_i)$$

$$\leq \text{cost}_{d'}(C, S'; Z) \quad (\because d'(c'_i, r) \geq d'(c'_i, q))$$

In both cases, we constructed a solution for $I'$ which is different from the optimal solution $C; Z$, and has cost less than or equal to the optimal cost. This contradicts the uniqueness of the optimal solution. \[\square\]

Next we show that Lemma 6.1 holds true for all non-outliers points $q$ belonging to an optimal cluster different from $C_i$. The proof is same as the one given in [3], we briefly sketch it here for completeness.
Lemma 6.3. Consider a 2-perturbation resilient $k$-median-outlier instance $\mathcal{I} = (V, d, k, z)$. Let $C = \{C_1, \ldots, C_k\}$, and $Z$ be the unique optimal clustering and outliers resp. Let $S = \{c_1, \ldots, c_k\}$ be optimal centers inducing $C$. Let $p \in V \setminus Z$ be an arbitrary point, and $c_i$ be the center it is assigned to. For any other center $c_j$ ($c_j \neq c_i$), it follows $2 \cdot d(p, c_i) < d(p, c_j)$.

Proof Sketch: Suppose the claim is not true, that is, for some $c_j \neq c_i$, $2 \cdot d(p, c_i) \geq d(p, c_j)$. Similar to Lemma 6.2, we construct a distance function $d'$ which is a metric 2-perturbation of $d$. To this end, consider the complete graph $G$ defined on the vertex set $V$, with edge lengths $\ell$, where (1) $\ell(c_j, p) = d(c_i, p)$; (2) for every other edge $(u, v)$, $\ell(u, v) = d(u, v)$. We define $d'$, as the shortest path distance (using $\ell$) between vertices in graph $G$.

Observation 6.3. $d'$ has the following properties:

i) for any $u, v \in V$, such that $(u, v) \neq (c_j, p), (c_i, p)$,

$$d'(u, v) = \min\{\ell(u, v), \ell(u, c_j) + \ell(c_j, p) + \ell(p, v), \ell(u, p) + \ell(p, c_j) + \ell(c_j, v)\}$$

ii) $d'(c_j, p) = d'(c_i, p) = d(c_i, p)$

iii) $d'$ is a metric 2-perturbation of $d$

Since instance $\mathcal{I}$ is 2-OPR for $k$-median-outlier, even for the perturbed instance $\mathcal{I}' = (V, d', k, z)$ the unique optimal clustering is $C$ and outliers is $Z$. We can further show that for any two points $u, v \in C_i$ (and $C_j$), $d'(u, v) = d(u, v)$. Thus $c_i, c_j$ are cluster centers in the optimal solution of $\mathcal{I}'$. Now, consider a solution for $\mathcal{I}'$ with clustering $C' = C \setminus \{C_i, C_j\} \cup \{C_i \setminus \{p\}, C_j \cup \{p\}\}$ and outliers $Z$. We can show that the cost of this solution $C'; Z$ is at most the cost of the optimal solution $C; Z$. Thus contradicting the fact that the optimal solution is unique.

Corollary 6.1. Consider any point $p \in V \setminus Z$, and let $C_i$ be the optimal cluster $p$ is assigned to. Then, for any other point $q$ from a different cluster $C_j$ ($i \neq j$), $d(p, c_i) < d(p, q)$.

Lemma 6.1 follows immediately from Corollary 6.1 and Lemma 6.2.

6.2 Algorithm

In the previous section, we showed that in the optimal solution of a 2-perturbation resilient $k$-median-outlier instance, any non-outlier point is closer to its assigned center than to any point outside the cluster. This gives a nice structure to the optimal solution. In particular, the optimal clusters form subtrees in the minimum spanning tree over input point set. We leverage this property to design a dynamic programming based algorithm to identify the optimal clusters and outliers. In what follows, we interchangeably use the terms point and vertex.

Lemma 6.4. Let $\mathcal{I} = (V, d, k, z)$ be a 2-perturbation resilient instance of the $k$-median-outlier problem. Let $T$ be a minimum spanning tree on $V$. The optimal clusters of $\mathcal{I}$, $C_1, \ldots, C_k$ are subtrees in $T$ i.e. for any two points $p, q \in C_i$, all the points along the unique tree path between $p$, and $q$ belongs to cluster $C_i$.

Proof: Let $c_i$ denote the center of cluster $C_i$. To prove the lemma, it is sufficient to show that every point on the unique tree path between $p$ and $c_i$ belongs to cluster $C_i$. We prove this via induction on the length of path between $p$ and $c_i$. Let $u$ be the vertex after $p$ along this path. Since $(p, u)$ is an MST edge, we have $d(p, u) \leq d(p, c_i)$. By Lemma 6.1, $u$ must belong to $C_i$. The proof then follows by applying induction on $u$ to $c_i$ path.
Lemma 6.4 implies that we can find the optimal solution of \( I \) by solving the following optimization problem, which we call tree-partition: Partition the MST \( T \) into \( k \) subtrees \( P_1, \ldots, P_k \), with centers \( c_1, \ldots, c_k \) (each \( c_i \in P_i \)) and identify remaining \( Z \) vertices of the tree as outliers, where \( |Z| \leq z \). The goal is to minimize the following objective function,

\[
\sum_{i=1}^{k} \sum_{u \in P_i} d(c_i, u)
\]

Solving tree-partition on a general tree is complicated. We simplify it by transforming \( T \) into a binary tree \( T' \) with dummy vertices. The procedure is as follows: while there is a vertex \( v \) with more than two children, pick any two children of \( v \) — \( v_1 \) and \( v_2 \); create a new child (dummy vertex) \( u \) of \( v \); reattach subtrees rooted at \( v_1 \) and \( v_2 \) as children of \( u \). At the end of this process, let \( U \) be the set of dummy vertices added. For each dummy vertex \( u \in U \), set \( d(u, v) = 0 \), for every \( v \in U \cup V \).

Now consider the following optimization problem (bin-tree-partition): Partition binary tree \( T' \) into \( k \) subtrees \( P_1', \ldots, P_k' \), with centers \( c_1', \ldots, c_k' \) (each \( c_i' \in P_i' \cap V \)) and identify remaining \( Z' \) vertices of the tree as outliers, where \( |Z' \cap V| \leq z \). The cost function we want to minimize is,

\[
\sum_{i=1}^{k} \sum_{u \in P_i'} d(c_i, u)
\]

It is not hard to show, that given a solution to tree-partition, we can construct a solution for bin-tree-partition of equal cost, and vice-versa. Thus, to solve k-median-outlier it is sufficient to solve bin-tree-partition on the binary tree \( T' \) with dummy nodes. Given an optimal solution \( P_1', \ldots, P_k'; Z' \) for bin-tree-partition, the optimal clusters of the corresponding k-median-outlier instance is \( P_1' \cap V, \ldots, P_k' \cap V \) and outliers is \( Z' \cap Z \).

To optimally solve bin-tree-partition we use dynamic programming. For the rest of the section, we consider \( T \) to be the input binary tree, with \( V \) being the vertices corresponding to points, and \( U \) denotes the dummy vertices. Let \( T_u \) denote the subtree rooted at \( u \). Further let \( \ell_u, r_u \) respectively denote the left child, right child of \( u \).

Let \( \text{opt}(u, j, t, c) \) be the minimum cost of partitioning the points in subtree \( T_u \) into \( j \) clusters after discarding \( t \) points as outliers. Here \( c \) can be any vertex in \( V \) or it can be the null (denoted using \( \emptyset \)). The clustering satisfies the following constraints:

- if \( c = \emptyset \), then \( u \) is marked as an outlier.
- if \( c \neq \emptyset \), then the cluster in which \( u \) belongs has center \( c \).
- Each cluster forms a subtree in \( T_u \).

We can define \( \text{opt}(u, j, t, c) \) using the following recursive formula.

\[
c = \emptyset, u \in V. \text{ Here } u \text{ is an outlier. Hence, } \ell_u \text{ and } r_u \text{ are assigned to centers } c' \in T_{\ell_u} \text{ and } c'' \in T_{r_u} \text{ respectively. Further, since } u \text{ is already being marked as an outlier, there can be } t - 1 \text{ outliers between } T_{\ell_u} \text{ and } T_{r_u}.
\]

\[
\text{opt}(u, j, t, c) = \min \{ \text{opt}(\ell_u, j', t', c') + \text{opt}(r_u, j'', t'', c'') : j' + j'' = j, t' + t'' = t - 1, c' \in T_{\ell_u} \cup \emptyset, c'' \in T_{r_u} \cup \emptyset \}
\]

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c = ∅, u ∈ V. Here u is an outlier. However, since it is a dummy vertex we do not count it as one of t outliers in T_u.

\[
\text{opt}(u,j,t,c) = \min \left\{ \text{opt}(\ell_u, j', t', c') + \text{opt}(r_u, j'', t'', c'') : j' + j'' = j, t' + t'' = t, c' \in T_{\ell_u} \bigcup \emptyset, c'' \in T_{r_u} \bigcup \emptyset \right\}
\]

\(c \notin T_{\ell_u} \bigcup T_{r_u}\). The recursive formula is defined by 4 cases (lines 1-4 in the formula). The explanation for each case is as follows: (1) Neither \(l_u\) nor \(r_u\) is assigned to the same cluster as \(u\). They are either outliers, or they are assigned to centers \(c', c''\) in subtree \(T_{\ell_u}, T_{r_u}\) resp. (2) \(r_u\) is assigned to the same cluster as \(u\) but not \(\ell_u\). It is either an outlier or assigned to a center \(c' \in T_{\ell_u}\) (3) \(\ell_u\) is assigned to the same cluster as \(u\) but not \(r_u\). It is either an outlier or assigned to a center \(c'' \in T_{r_u}\) (4) Both \(\ell_u\) and \(r_u\) are assigned to the same cluster as \(u\).

\[
\text{opt}(u,j,t,c) = d(u,c) + \min \left( \begin{array}{l}
\min \left\{ \text{opt}(\ell_u, j', t', c') + \text{opt}(r_u, j'', t'', c'') : j' + j'' = j - 1, t' + t'' = t, c' \in T_{\ell_u} \bigcup \emptyset, c'' \in T_{r_u} \bigcup \emptyset \right\},
\min \left\{ \text{opt}(\ell_u, j', t', c') + \text{opt}(r_u, j'', t'', c) : j' + j'' = j, t' + t'' = t, c' \in T_{\ell_u} \bigcup \emptyset \right\},
\min \left\{ \text{opt}(\ell_u, j', t', c) + \text{opt}(r_u, j'', t'', c'') : j' + j'' = j, t' + t'' = t, c' \in T_{\ell_u} \bigcup \emptyset \right\},
\min \left\{ \text{opt}(\ell_u, j', t', c) + \text{opt}(r_u, j'', t'', c) : j' + j'' = j - 1, t' + t'' = t \right\} \end{array} \right)
\]

\(c \in T_{\ell_u}\). The recursive formula in this case is obtained by removing lines (1), (2) from the above formula.

\(c \in T_{r_u}\). The recursive formula in this case is obtained by removing lines (1), (3) from the above formula.

Remark. The algorithm we presented easily generalizes to give exact solution for 2-perturbation resilient instances of other clustering with outliers problems like k-CENTER-OUTLIER, k-MEANS-OUTLIER, and more general \(\ell_p\) objectives.

Acknowledgements: CC thanks Mohit Singh for initial discussions on the integrality of the LP relaxation for 2-perturbation-resilient instances of k-median. We thank Yury Makarychev for comments on Voronoi clustering for k-center.

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A.1 Proof of Lemma 2.1

Since, \(|C'| = k - 1\), there must be a cluster \(C'_i \in C'\), such that \(C'_i \cap S \geq 2\). Let \(c_i, c_j \in S\) be the cluster centers which belong to \(C'_i\). Wlog, assume \(c_i\), the center of cluster \(C'_i\) does not belong to cluster \(C_j\).

**Case 1:** \(V \setminus S' \nsubseteq C_j\). Consider any point \(q \in V \setminus (S' \cup C_j)\). Consider the set of \(k\) centers \(S'' = S' \cup \{q\}\). Let \(C''\) be a corresponding Voronoi partition. Clearly \(\text{cost}_d(C'', S'') \leq \text{cost}_d(C', S') \leq R_d^*\), as adding a new center can not increase the clustering cost. Now, for any \(c \in S'' \setminus \{c'_i, q\}\), we have \(d(c, c_j) \geq d(c'_i, c_j)\). Therefore, in the Voronoi partition \(C''\), we can assume wlog, either \(c'_i\) and \(c_j\) are in the same cluster, or \(q\) and \(c_j\) are in same cluster. However both \(c'_i, q \notin C_j\). Thus, \(C''\) is a different \(k\) clustering of \(V\) of cost at most \(R_d^*\).

**Case 2:** \(V \setminus S' \subseteq C_j\). In this case, we have \(S' = S \setminus \{c_j\}\). Further, for any \(\ell \neq j\), \(|C_\ell| = 1\). Therefore, there exists a point \(q \in C_j\), such that \(d(c_j, q) = R_d^*\). Since, \(\text{cost}_d(C', S') \leq R_d^*\), we have, \(d(S', q) \leq R_d^*\). Therefore, there exists a different Voronoi partition \(C''\) induced by \(S = S' \cup \{c_j\}\), where the points \(q\) and \(c_j\) do not belong in the same cluster.

A.2 Proof of Lemma 2.2

Since \(d'\) is defined as the shortest path distance (over non-negative edge lengths) in graph \(G\), it satisfies triangle inequality. Also, as mentioned in the lemma statement \(\ell(u, v) \leq d(u, v)\). Therefore \(d'(u, v) \leq \ell(u, v) \leq d(u, v)\). Consider any two points \(u, v \in V\). Let \(P = (u, u_1, \ldots, v)\) be an arbitrary directed \(u \leadsto v\) path in graph \(G\). The length of path \(P\) is given by \(\ell(P) = \ell(u, u_1) + \ldots + \ell(u_t, v) \geq 1/2 \cdot (d(u, u_1) + \ldots + d(u_t, v)) \geq \frac{d(u,v)}{2}\). Here the last inequality uses the fact that \(d\) satisfies triangle inequality. Therefore, \(d'(u, v) = \min_P \ell(P) \geq \frac{d(u,v)}{2}\).

A.3 Proof of Lemma 2.3

Since for any pair of points \(u, v \in V\), we have \(d'(u, v) \leq d(u, v)\), clearly \(R_d^* \leq R_d^*\). Let \(C_t\) be the largest radius optimal cluster in \(I\), i.e., \(\max_{u \in C_t} d(c_t, u) = R_d^*\). Therefore, for every \(c \in C_t\), there exists a point \(r(c) \in C_t\), such that \(d(c, r(c)) \geq R_d^*\). Now, for any path \(P\) between \(c\) and \(r(c)\) in graph \(G\), which does not include an edge from \(E'\), \(\ell(P) = \sum_{e \in P} \ell(e) = \sum_{e \in P} d(e) \geq d(c, r(c)) \geq R_d^*\).
by triangle inequality. Further, for any path \( P' \) between \( c \) and \( r(c) \) in graph \( G \), which includes at least one edge from \( E' \),

\[
\ell(P') = \sum_{e \in P' \cap E'} \ell(e) + \sum_{e \in P' \setminus E'} \ell(e) \geq \sum_{e \in P' \cap E'} \min\{\ell(e), R_d^*\} + \sum_{e \in P' \setminus E'} \ell(e) \geq \min\{\ell(P'), R_d^*\} \geq R_d^* .
\]

Therefore, \( d'(c, r(c)) \geq R_d^* \). Now recall, we assumed \( C \) is an optimal clustering in \( I' \), then \( C_t \) is an optimal cluster in \( I' \). Therefore,

\[
R_{d'} = \text{cost}_{d'}(C, S') \geq \min \max_{e \in C_t} d'(c, u) \geq \min_{e \in C_t} d'(c, r(c)) \geq R_d^* .
\]

Therefore, \( R_{d'}^* = R_d^* . \)

The proofs of Lemma 2.4, and Lemma 2.5 are similar to the above.