Abstract

In this work we show that it is possible to extend analytically, and with the use of tempered ultradistributions, the pseudonorm defined by T. Berggren for Gamow states. We define this pseudonorm for all states determined by the zeros of the Jost function for any short range potential.

As an example we study the s-states corresponding to the square well potential.

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1 Introduction

Resonant states play a central role in the quantum description of decaying nuclear states. In the ordinary formulation of Quantum Mechanics these states appear as complex energy poles of the $S$ scattering matrix. In turn, these states can be defined as solutions of the time-independent Schrödinger equation with purely out-going waves at large distances [5]. Several attempts have been performed to handle adequately resonant states, the main obstacle being the divergent behaviour of the corresponding wave functions at large distances, which makes it impossible to normalize them in an infinite volume with the conventional mathematical tools. The first successful attempt to handle resonant wave functions has been made by Tore Bergreen [6] using a regularization method first suggested by Zel’dovich. In his work Bergreen has shown that at least for finite range potentials it is possible to define an orthogonality criteria among bound and resonant states, and also a pseudonorm can be evaluated using the general analysis of Newton [3].

A proper inclusion of resonant states within the general framework of Quantum Mechanics has been done through the Rigged Hilbert Space (RHS) or Gelfand’s Triplet (GT) formulation [2]. Resonant states are described, within the RHS, as generalized complex energy solutions of a self-adjoint Hamiltonian. The structure of the RHS guarantees that any matrix element involving resonant states is a well defined quantity, provided the topology in the GT has been properly choosen to handle the exponential growing of Gamow States at large distances.

The literature concerning the application of RHS to resonant states is extensive [9]-[12]. Among these works we shall mention, for instance, those of Bohm [9], Gadella [10] and also reference [11], where resonant states are introduced using a RHS of entire Hardy-class functions defined in a half complex energy-plane. This allows to extend analytically the concept of a resonant state as an antilinear complex functional over the intersection of Schwartz test functions with Hardy class.

A more general theory of resonant states follows if the RHS is built up on tempered ultradistributions [13]. In this case resonant states arise as continuous linear functionals over rapidly decreasing entire analytical test functions. This can be obtained by using the Dirac’s formula, which allows a more direct determination of these states. Another advantage of using tempered ultradistributions is that only the physical spectrum appears in
the definition of complex-energy states \[13\].

In the present paper we want to show that it is possible to define a complex pseudonorm for resonant states in the sense of Bergreen using tempered ultradistributions. With this pseudonorm we generalize the Bergreen’s result \[6\], and hence it can be considered as the proper analytical extension of a pseudoscalar product for resonant states.

We give an introduction of tempered ultradistributions and Gelfand Triplet in section 2. In section 3 we define resonant states starting from the Schrödinger equation, and then we focus our attention on the calculus of the pseudonorm of a complex-energy state. We apply in section 4 the results of the previous section to the case of a square well potential. We give a resume in section 5.

2 The Tempered Ultradistributions

2.1 The Triplet \((H, \mathcal{H}, \Lambda_\infty)\)

We define the space \(H\) of test functions \(\phi(x)\) such that \(e^{p|x|}\|D^q\phi(x)\|\) is bounded for any \(p\) and \(q\) by means of the set of countably norms (ref.[4]):

\[
\|\hat{\phi}\|''_p = \sup_{0 \leq q \leq p, x} e^{p|x|}\|D^q\hat{\phi}(x)\| ; \quad p = 0, 1, 2, ...
\]  

(2.1. 1)

According to the ref.[3] \(H\) is a space \(K\{\mathbb{M}_p\}\) with:

\[
\mathbb{M}_p(x) = e^{(p-1)|x|} ; \quad p = 1, 2, ...
\]  

(2.1. 2)

\[
\|\hat{\phi}\|_p = \sup_{0 \leq q \leq p} \mathbb{M}_p(x)\|D^q\hat{\phi}(x)\|
\]  

(2.1. 3)

\(K\{e^{(p-1)|x|}\}\) satisfies condition \((\mathcal{N})\) of Guelfand ( ref.[3] ). Then if we define:

\[
<\hat{\phi}, \hat{\psi}>_p = \int_{-\infty}^{\infty} e^{2(p-1)|x|} \sum_{q=0}^{p} D^q\overline{\phi}(x)D^q\psi(x) \, dx ; \quad p = 1, 2, ...
\]  

(2.1. 4)

\[
\|\hat{\phi}\|'_p = \sqrt{<\hat{\phi}, \hat{\phi}>_p}
\]  

(2.1. 5)

\(K\{e^{(p-1)|x|}\}\) is a countable Hilbert and nuclear space.

\[
K\{e^{(p-1)|x|}\} = H = \bigcap_{p=1}^{\infty} H_p
\]  

(2.1. 6)
where $H_\rho$ is the completed of $H$ by the norm (2.1.5). Let

$$<\hat{\phi}, \hat{\psi} > = \int_{-\infty}^{\infty} \overline{\phi(x)} \psi(x)$$ (2.1. 7)

Then, the completed of $H$ by (2.1.7) is $\mathcal{H}$, the Hilbert space of square integrable functions. Now

$$<\hat{\phi}, \hat{\psi} > \leq C \|\hat{\phi}\|_1 \|\hat{\psi}\|_1$$ (2.1. 8)

and according to ref. [3] the triplet

$$(H, \mathcal{H}, \Lambda_\infty)$$ (2.1. 9)

is a Rigged Hilbert space or Gelfand’s Triplet. Here $\Lambda_\infty$ is the dual of $H$ and it consist of distributions of exponential type $T$ ( ref.[1] ): $T = D^p \left[ e^{p|x|} f(x) \right] ; \ p = 0, 1, 2...$ (2.1. 10)

where $f(x)$ is bounded continuous.

### 2.2 The Triplet $(h, \mathcal{H}, \mathcal{U})$

The space $h = \mathcal{F}\{H\}$ ( $\mathcal{F}$= Fourier transform ) consist of entire analytic rapidly decreasing test functions given by the countable set of norms :

$$\|\phi\|_{pm} = \sup_{|f_m(z)| \leq n} (1 + |z|)^p |\phi(z)|$$ (2.2. 1)

Then $h$ is a $\mathcal{Z}\{M_p\}$ space, complete and countable normed ( Frechet ) with:

$$M_p(z) = (1 + |z|)^p$$ (2.2. 2)

If we define:

$$<\phi(z), \psi(z) >_p = <\hat{\phi}(x), \hat{\psi}(x) >_p$$ (2.2. 3)

then, $\mathcal{Z}\{(1 + |z|)^p\}$ is a countable Hilbert and nuclear space. Let be:

$$\psi(z) = \int_{-\infty}^{\infty} e^{izx} \hat{\psi}(x) dx$$ (2.2. 4)
\[ \phi(z) = \int_{-\infty}^{\infty} e^{ixz} \hat{\phi}(x) dx \] (2.2. 5)

\[ \phi_1(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ixz} \phi(x) dx \] (2.2. 6)

Then we define:

\[ \langle \phi(z), \psi(z) \rangle = \int_{-\infty}^{\infty} \phi_1(z) \psi(z) dz = \int_{-\infty}^{\infty} \overline{\phi(x)} \overline{\psi(x)} dx \]

\[ = \langle \hat{\phi}(x), \hat{\psi}(x) \rangle \] (2.2. 7)

The completed of \( h \) by this last scalar product is the Hilbert space \( \mathcal{H} \) of square integrable functions and the dual of \( h \) is the space \( \mathcal{U} \) of tempered ultradistributions (ref.[1]). Then \( (h, \mathcal{H}, \mathcal{U}) \) is a Gelfand’s triplet.

The space \( \mathcal{U} \) can be characterized as follow (ref.[1]). Let be \( A_\omega \) the space of all functions \( F(z) \) such that:

(i) \( F(z) \) is analytic in \( \{ z \in \mathbb{C} : |Im(z)| > p \} \).

(ii) \( F(z)/z^p \) is bounded continuous in \( \{ z \in \mathbb{C} : |Im(z)| \geq p \} \) where \( p \) depends of \( F(z) \). Here \( p = 0, 1, 2, ... \)

Let \( \Pi \) be the set of all \( z \)-dependent polynomials \( P(z), z \in \mathbb{C} \). Then \( \mathcal{U} \) is the quotient space:

\[ \mathcal{U} = \frac{A_\omega}{\Pi} \] (2.2. 8)

Due to these properties any ultradistribution can be represented as a linear functional where \( F(z) \in \mathcal{U} \) is the indicatrix of this functional (ref.[1]):

\[ F(\phi) = \langle F(z), \phi(z) \rangle = \oint_{\Gamma} F(z)\phi(z) dz \] (2.2. 9)

where the path \( \Gamma \) runs parallel to the real axis from \(-\infty \) to \( \infty \) for \( Im(z) > \rho \), \( \rho > p \) and back from \( \infty \) to \(-\infty \) for \( Im(z) < -\rho \), \(-\rho < -p \) (\( \Gamma \) lies outside a horizontal band of width \( 2p \) that contain all the singularities of \( F(z) \)).

Formula (2.2.9) will be our fundamental representation for a tempered ultradistribution. An interesting property, according to “Dirac formula” for ultradistributions (ref.[4]),

\[ F(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} dt \frac{f(t)}{t-z} \] (2.2. 10)
is that the indicatrix \( f(t) \) satisfies:

\[
\oint_{\Gamma} dz\, F(z)\phi(z) = \int_{-\infty}^{\infty} dt\, f(t)\phi(t) \quad (2.2.11)
\]

While \( F(z) \) is analytic on \( \Gamma \), the density \( f(t) \) is in general singular, so that the r.h.s. of (2.2.11) should be interpreted in the sense of distribution theory.

The representation (2.2.9) makes evident that the addition of a polynomial \( P(z) \) to \( F(z) \) do not alter the ultradistribution:

\[
\oint_{\Gamma} dz\, \{F(z) + P(z)\}\phi(z) = \oint_{\Gamma} dz\, F(z)\phi(z) + \oint_{\Gamma} dz\, P(z)\phi(z)
\]

But:

\[
\oint_{\Gamma} dz\, P(z)\phi(z) = 0
\]

as \( P(z)\phi(z) \) is entire analytic (and rapidly decreasing),

\[
\therefore \oint_{\Gamma} dz\, \{F(z) + P(z)\}\phi(z) = \oint_{\Gamma} dz\, F(z)\phi(z) \quad (2.2.12)
\]

In the Rigged Hilbert spaces \((\Phi, H, \Phi^*)\) is valid the following very important property:

Every symmetric operator \( A \) acting on \( \Phi \), that admit a self-adjoint prolongation operating on \( H \), has in \( \Phi^* \) a complete set of generalized eigenvectors (or proper distributions) that correspond to real eigenvalues (ref.[3]).

This property is then valid in \((H, \mathcal{H}, \Lambda_\infty)\) and in \((h, \mathcal{H}, \mathcal{U})\).

3 The pseudonorm of eigenstates of short range potentials

In this paragraph we describe the main properties of the solutions of the Schrödinger equation for a central short range potential ref.[5]. According to this reference the regular (\( \phi_l(k, r) \)) and irregular (\( f_l(k, r) \)) solutions for this equation satisfy, respectively, the following boundary conditions

\[
\lim_{r \to 0} (2l + 1)!! \, r^{-l-1}\phi_l(k, r) = 1 \quad (3.1)
\]
\[ \lim_{r \to \infty} e^{ikr} f_i(k, r) = i^l \]  

Both solutions are related by

\[ \phi_l(k, r) = \frac{1}{2} ik^{-l-1} \left[ f_i(-k) f_i(k, r) - (-1)^l f_i(k) f_i(-k, r) \right] \]  \hspace{1em} (3.3)

In (3.3) \( f_i(k) \) is the Jost function defined by:

\[ f_i(k) = k^l W [f_i(k, r), \phi_l(k, r)] \]  \hspace{1em} (3.4)

where \( W[f, \phi] \) is the Wronskian of the two solutions. The zeros of the Jost function \( f_i(k) \) are the bound (\( \text{Re}(k) = 0, \text{Im}(k) \leq 0 \)), virtual (\( \text{Re}(k) = 0, \text{Im}(k) > 0 \)) and resonant states (\( \text{Re}(k) \neq 0, \text{Im}(k) > 0 \)) (refs.\([5, 6]\)). With these definitions we are now in position to calculate the pseudonorm of the above states. According to ref.\([5]\) the derivative of the Jost function with respect to the variable \( k \), \( \dot{f}_i(k) \), satisfies

\[ \dot{f}_i(k) = lk^{l-1} W [f_i(k, r), \phi_l(k, r)] + k^l W [\dot{f}_i(k, r), \dot{\phi}_l(k, r)] \]  

\hspace{1em} (3.5)

In particular, when \( k_0 \) is a zero of the Jost function then (3.5) takes the form:

\[ \dot{f}_i(k_0) = k_0^l W [\dot{f}_i(k_0, r), \phi_l(k_0, r)] + k_0^l W [f_i(k_0, r), \phi_l(k_0, r)] \]  \hspace{1em} (3.6)

Due to eq.(3.3) at \( k = k_0 \) we have the equality:

\[ f_i(k_0, r) = C(k_0) \phi_l(k_0, r) \hspace{1em} \text{;} \hspace{1em} C(k_0) = \frac{-2i k_0^{l+1}}{f_i(-k_0)} \]  

\hspace{1em} (3.7)

and following the procedure of ref.\([5]\) we get:

\[ \dot{f}_i(k_0) = k_0^l \lim_{\beta \to \infty} \left\{ W [\dot{f}_i(k_0, \beta), \phi_l(k_0, \beta)] - 2k_0 C(k_0) \int_0^\beta \phi_l^2(k_0, r) \, dr \right\} \]  

\hspace{1em} (3.8)
From (3.8) we deduce immediately:

\[
\lim_{\beta \to \infty} \int_{0}^{\beta} \phi_I^2(k_0, r) \, dr = - \lim_{\beta \to \infty} \frac{f_i(-k_0)}{4i k_0^{4+2}} \mathcal{W} \left[ \hat{f}_i(k_0, \beta), \phi_I(k_0, \beta) \right]
\]

\[
+ \frac{\hat{f}_i(k_0) f_i(-k_0)}{4i k_0^{4+2}}
\]

(3.9)

Now, we want to show that: i) the integral appearing in (3.9) can be defined as an ultradistribution in the variable \(k_0\) and ii) in the limit \(\beta \to \infty\), as an ultradistribution in \(k_0\), the Wronskian \(\mathcal{W}\) vanishes. With this purpose and according to ref.[5] we note that \(k^l f_i(k_0, r) = h_i(k_0, r)\) is an entire analytic function of the variable \(k_0\) and therefore \(k^{l+1} f_i(k_0, r)\) is also too. Hence \(k^{l+1} \hat{f}_i(k_0, r) = g_i(k_0, r)\) is an entire analytic function of \(k_0\). Moreover it has been shown in ref.[5] that \(h_i(0, r) = C \phi_I(0, r)\). And as a consequence we have \(h_i(0, 0) = g_i(0, 0) = 0\) because \(\phi_I\) has the property \(\phi_I(0, 0) = 0\).

We can write now (3.8) in terms of \(g_i(k, r)\) as:

\[
\hat{f}_i(k_0) = \lim_{\beta \to \infty} \left\{ \frac{\mathcal{W} \left[ g_i(k_0, \beta), \phi_I(k_0, \beta) \right]}{k_0} \right\}
\]

\[
- 2k_0^{l+1} C(k_0) \int_{0}^{\beta} \phi_I^2(k_0, r) \, dr \right\}
\]

(3.10)

But:

\[
\lim_{\beta \to \infty} \int_{\Gamma} \mathcal{W} \left[ g_i(k_0, \beta), \phi_I(k_0, \beta) \right] \phi(k_0) \, dk_0 =
\]

\[
\lim_{\beta \to \infty} \frac{\mathcal{W} \left[ g_i(0, \beta), \phi_I(0, \beta) \right] \phi(0)}{k_0} \]

(3.11)

where \(\phi(k_0) \in h\) is an entire analytic test function and the path \(\Gamma\) runs parallel to the real axis from \(-\infty\) to \(\infty\) for \(Im(k_0) > \rho, \rho > 0\) and back from \(\infty\) to \(-\infty\) for \(Im(k_0) < -\rho, -\rho < 0\) (\(\Gamma\) lies outside a horizontal band that contains the singularity in the origin). Taking into account that \(f_i\) satisfies:

\[
\frac{d}{dr} \mathcal{W} \left[ \hat{f}_i(k, r), f_i(k, r) \right] = 2k f_i^2(k, r)
\]

(3.12)
it is easy to show that:
\[
\frac{d}{dr} \mathcal{W}[g_t(k,r), h_t(k,r)] = 2k^2 h_t^2(k,r) \tag{3.13}
\]
and then
\[
\frac{d}{dr} \mathcal{W}[g_t(0,r), h_t(0,r)] = 0 \tag{3.14}
\]
Eq. (3.14) implies that:
\[
\mathcal{W}[g_t(0,r), h_t(0,r)] = \text{constant} \tag{3.15}
\]
and from \( h_t(0,0) = g_t(0,0) = 0 \) we obtain:
\[
\mathcal{W}[g_t(0,r), h_t(0,r)] = 0 \tag{3.16}
\]
This implies that
\[
\mathcal{W}[g_t(0,r), \phi_t(0,r)] = 0 \tag{3.17}
\]
and then we have:
\[
\lim_{\beta \to \infty} \int_{\Gamma} \frac{\mathcal{W}[g_t(k_0, \beta), \phi_t(k_0, \beta)]}{k_0} \phi(k_0) \, dk_0 = 0 \tag{3.18}
\]
As a consequence of (3.18) it results that:
\[
\lim_{\beta \to \infty} \frac{\mathcal{W}[g_t(k_0, \beta), \phi_t(k_0, \beta)]}{k_0} = P(k_0) \tag{3.19}
\]
where \( P(k_0) \) is an arbitrary polynomial in the variable \( k_0 \).

Now we have the freedom to select \( P(k_0) \equiv 0 \), and in this case (3.9) takes the form:
\[
\lim_{\beta \to \infty} \int_0^\beta \phi_t^2(k_0, r) \, dr = \frac{\dot{f}_t(k_0) f_t(-k_0)}{4i k_0^{2t+2}} \tag{3.20}
\]
where the limit is taken in the sense of ultradistributions. By definition the pseudonormalized state is:
\[
\psi_t(k_0, r) = \left[ \frac{4i k_0^{2t+2}}{f_t(k_0) f_t(k_0)} \right]^{1/2} \phi_t(k_0, r) \tag{3.21}
\]
and it can be thought as a tempered ultradistribution in the variable \( k_0 \).
4 The square well potential

We start from the Schrödinger equation for the radial component $\mathcal{R}_l(r)$ (ref. [7, 8]):

$$\mathcal{R}_l''(r) + \frac{2}{r} \mathcal{R}_l'(r) + \left[ q^2 - \frac{l(l+1)}{r^2} \right] \mathcal{R}_l(r) = 0$$

(4.1)

(‘ denotes the derivative $d/dr$) and with

$$q^2 = \frac{2m}{\hbar^2} [E - \mathcal{V}(r)] = k^2 - \frac{2m}{\hbar^2} \mathcal{V}(r)$$

(4.2)

where

$$\mathcal{V}(r) = \begin{cases} 0 & \text{for } r > a \\ -\mathcal{V}_0 & \text{for } r \leq a \end{cases}$$

(4.3)

and

$$k^2 = \frac{2mE}{\hbar^2}$$

(4.4)

The regular solution is:

$$\phi_l(k, r) = \begin{cases} q^{-l} r j_l(qr) & \text{for } r < a \\ r [A_l \tilde{j}_l(kr) + B_l n_l(kr)] & \text{for } r > a \end{cases}$$

(4.5)

where $j_l$ and $n_l$ are respectively the spherical Bessel and Newmann functions. The constants $A_l$ and $B_l$ in (4.5) are:

$$A_l = ka q^{-l} \left[ k j_l(qa) n_l'(ka) - q j'_l(qa) n_l(ka) \right]$$

$$B_l = ka q^{-l} \left[ q j_l(ka) j'_l(qa) - k j'_l(ka) j_l(qa) \right]$$

(4.6)

The irregular solution $f_l(k, r)$ is given by:

$$f_l(k, r) = \begin{cases} r [C_l j_l(qr) + D_l n_l(qr)] & \text{for } r < a \\ -ikr h^+_l(kr) & \text{for } r > a \end{cases}$$

(4.7)

where $h^+_l = j_l - in_l$ is the spherical Hankel function and the constants $C_l$ and $D_l$ are given by

$$C_l = -ik q a^2 \left[ q h^+_l(ka) n_l'(qa) - k h^+_l'(ka) n_l(qa) \right]$$
\[ D_l = \frac{ikqa^2}{2} \left[ q h^{-}_l(ka) \ j'_l(qa) - k h^{-'}_l(ka) \ j_l(qa) \right] \]  

(4.8)

Using eqs.(3.4),(4.5) and (4.7) we can evaluate the corresponding Jost function \( f_l(k) \):

\[ f_l(k) = \left( \frac{k}{q} \right)^l ika^2 \left[ k \ j_l(qa) \ h^{-}_l(ka) - q \ j'_l(qa) \ h^{-'}_l(ka) \right] \]  

(4.9)

We wish to calculate eq.(3.20) for this example in the case \( l = 0 \). With this purpose we need the expressions of \( f_0(-k_0) \) and \( \dot{f}_0(k_0) \). For this purpose we take into account that \( f_0(k_0) = 0 \). From (4.9) we obtain for \( l = 0 \):

\[ f_0(k_0) = e^{-ik_0a} \left( ik_0 \frac{\sin q_0a}{q_0} + \cos q_0a \right) = 0 \]  

(4.10)

and

\[ \dot{f}_0(k_0) = i \frac{q_0^2 - k_0^2}{q_0^3} e^{-ik_0a} \left( \sin q_0a - q_0a \cos q_0a \right) \]  

(4.11)

where

\[ q_0^2 = k_0^2 \ + \ \frac{2m}{\hbar^2} \gamma_0 \]

Therefore we deduce from (4.10) and (4.11) that:

\[ f_0(-k_0) = -\frac{2ik_0}{q_0} e^{ik_0a} \sin q_0a \]  

(4.12)

\[ \dot{f}_0(k_0) = i \frac{q_0^2 - k_0^2}{q_0^3} \left( 1 + ik_0a \right) e^{-ik_0a} \sin q_0a \]  

(4.13)

If we replace eqs.(4.12) and (4.13) into eq.(3.20) we obtain finally:

\[ \int_0^\infty \phi_0^2(k_0, r) dr = \frac{1 + ik_0a}{2ik_0} \frac{q_0^2 - k_0^2}{q_0^4} \sin^2 q_0a \]  

(4.14)

It should be noted that when \( k_0 \) corresponds to a bound state the integral (4.14) is real and positive. When \( k_0 \) corresponds to a virtual state or a resonant state \( \Re k_0 \neq 0, \Im k_0 > 0 \) the integral (4.14) is in general a complex number. It is not surprising since (4.14) is an analytical extension.
in the sense of ultradistributions of the habitual Lebesgue integral. In fact for a bound state \((k_0 = -i\kappa_0, \kappa_0 > 0)\) we have

\[
\int_0^\infty \phi_0^2(k_0, r)dr = \frac{1 + \kappa_0 a}{2\kappa_0} \frac{q_0^2 + \kappa_0^2}{q_0^4} \sin^2 q_0 a \tag{4.15}
\]

which is the well-known norm of the \(l = 0\) bound state of the square well. For the \(l = 0\) virtual state \((k_0 = i\kappa_0, \kappa_0 > 0)\) we have:

\[
\int_0^\infty \phi_0^2(k_0, r)dr = \frac{\kappa_0 a - 1}{2\kappa_0} \frac{q_0^2 + \kappa_0^2}{q_0^4} \sin^2 q_0 a \tag{4.16}
\]

and in this case the integral is real.

5 Discussion

We have shown here that tempered ultradistributions allow to perform a general treatment of complex-energy states, incorporating in a natural way bound and continuum states as well as resonant and virtual states together, within a more general framework of Quantum Mechanics, based on the Rigged Hilbert Space formulation. In this work we have applied this formulation to the specific evaluation of the complex pseudonorm, showing that the results come out in a more transparent way, since they are free from regularization schemes.

As an example of the goodness of the procedure introduced in this paper we give the evaluation of the pseudonorm of virtual and resonant s-states for the square-well potential.
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