A linear bound on the number of scalarizations needed to solve discrete tricriteria optimization problems

Kerstin Dächert, Kathrin Klamroth

June 20, 2013

General multi-objective optimization problems are often solved by a sequence of parametric single objective problems, so-called scalarizations. If the set of nondominated points is finite, and if an appropriate scalarization is employed, the entire nondominated set can be generated in this way. In the bicriteria case it is well known that this can be realized by an adaptive approach which, given an appropriate initial search space, requires the solution of at most $2|N| - 1$ subproblems, where $N$ denotes the nondominated set of the underlying problem. For higher dimensional problems, the best known bound is $O(|N|^{m-1})$ with $m$ being the number of objectives. We present a new procedure for finding the entire nondominated set of tricriteria optimization problems for which the number of scalarized subproblems to be solved is bounded by $3|N| - 2$. The approach includes an iterative update of the search space that, given a (sub-)set of nondominated points, describes the area in which additional nondominated points may be located. In particular, we show that the number of boxes, into which the search space is decomposed, depends linearly on the number of nondominated points.

Keywords: Discrete tricriteria optimization; scalarization; box algorithm

1 Introduction

The determination of the nondominated set is the basis for a multitude of methods in multiple criteria decision making. In multiple objective combinatorial optimization the nondominated set is discrete and, assuming that some natural bounds are given for the objective values, also finite. In this situation, a complete enumeration of all nondominated points can be realized by the successive solution of a series of appropriately formulated scalarized subproblems. Similarly, if an approximation of
the nondominated set of a discrete or of a continuous multiple objective optimization problem is sought, this is usually computed based on the generation of a finite number of nondominated points using a series of scalarized subproblems. Thereby, the number of these subproblems is of high interest, because it influences the computational time. In the bicriteria case approaches are known which, given an appropriate initial search space, require the solution of at most \( 2|N| - 1 \) subproblems, where \( N \) denotes the finite nondominated set of the underlying problem. Thereby, \(|N|\) subproblems are solved to generate all points in \( N \), and the additional \(|N| - 1\) subproblems are needed to ensure that no further nondominated points exist between the already generated ones (see, e.g., [Ralphs et al., 2006]). Up to now, for higher dimensional problems the best known approach has a theoretical bound of \( \mathcal{O}(|N|^{m-1}) \) subproblems for problems with \( m \) objectives ([Laumanns et al., 2006]). In this paper, we present a new procedure for finding the entire nondominated set of tricriteria optimization problems which the number of scalarized subproblems to be solved is bounded by \( 3|N| - 2 \). This is achieved by the definition of a new split criterion which allows to exclude redundant parts of the search space. It can then be shown that the number of boxes, into which the search space is decomposed, depends linearly on the number of nondominated points.

### 1.1 Terminology and Definitions

We consider multiple criteria optimization problems

\[
\begin{align*}
\min \quad & f(x) = (f_1(x), \ldots, f_m(x))^\top \\
\text{s.t.} \quad & x \in X
\end{align*}
\]

with \( m \geq 2 \) objective functions \( f_i : X \rightarrow \mathbb{R}, i = 1, \ldots, m \), and with feasible set \( X \neq \emptyset \). Throughout this paper we assume that \( X \) is a discrete set and that its image set \( f(X) \) is finite.

We use the Pareto concept of optimality: A solution \( \bar{x} \in X \) is called Pareto optimal or efficient if there does not exist a feasible solution \( x \in X \) such that \( f_i(x) \leq f_i(\bar{x}) \) for all \( i = 1, \ldots, m \) and \( f_j(x) < f_j(\bar{x}) \) for at least one \( j \in \{1, \ldots, m\} \). The corresponding objective vector \( f(x) \in \mathbb{R}^m \) is called nondominated in this case. If, on the other hand, \( f(x) \leq f(\bar{x}) \) for some feasible \( x \in X \), i.e., \( f_i(x) \leq f_i(\bar{x}) \) for all \( i = 1, \ldots, m \) and \( f_j(x) < f_j(\bar{x}) \) for at least one \( j \in \{1, \ldots, m\} \), we say that \( f(x) \) dominates \( f(\bar{x}) \), and \( x \) dominates \( \bar{x} \). If strict inequality holds for all \( m \) components, i.e., if \( f_i(x) < f_i(\bar{x}) \) for all \( i = 1, \ldots, m \), then \( x \) strictly dominates \( \bar{x} \). If there exists no feasible solution \( x \in X \) that strictly dominates \( \bar{x} \), then \( \bar{x} \) is called weakly Pareto optimal or weakly efficient. We denote the set of efficient solutions of (I) by \( X_E \) and refer to it as the efficient set.

The image set of the feasible set \( X \) in the outcome space is denoted by \( Z := f(X) \) where, based on the above assumptions on \( X \) and \( f(X) \), \( Z \) is a finite set of distinct points in \( \mathbb{R}^m \). The image set of the set of efficient solutions is denoted by \( N := f(X_E) \) and is called the nondominated set of problem (I). To simplify notation, we will often refer to the points in \( Z \) without relating them back to their preimages in the feasible set. Consequently, we equivalently formulate problem (I) in the outcome space as

\[
\begin{align*}
\min \quad & z = (z_1, \ldots, z_m)^\top \\
\text{s.t.} \quad & z \in Z,
\end{align*}
\]
where $Z$ is a discrete set of points in $\mathbb{R}^m$. For two vectors $z, \bar{z} \in Z$ we write

\[ z < \bar{z} \text{ if } z_i < \bar{z}_i \text{ for all } i = 1, \ldots, m, \]

\[ z \leq \bar{z} \text{ if } z_i \leq \bar{z}_i \text{ for all } i = 1, \ldots, m \text{ and } \exists j \in \{1, \ldots, m\} : z_j < \bar{z}_j, \]

and

\[ z \approx \bar{z} \text{ if } z_i \leq \bar{z}_i \text{ for all } i = 1, \ldots, m. \]

The symbols $>$, $\geq$, and $\approx$ are used accordingly.

**Definition 1.1** (Nondominance). A point $\bar{z} \in Z$ is called nondominated if and only if there exists no point $z \in Z$ such that $z \leq \bar{z}$.

A lower bound on the nondominated points of (2) is given by the ideal point which we denote by $z^I$. The $i$-th component of the ideal point is defined as the minimum of the $i$-th objective, i.e. $z^I_i := \min\{z_i : z \in Z\}$ for all $i = 1, \ldots, m$. A point $z^U$ that strictly dominates $z^I$ is called a utopia point. Note that in general $z^I \notin Z$. On the other hand, the nadir point $z^N$ with components $z^N_i := \max\{z_i : z \in N\}$ for all $i = 1, \ldots, m$ provides an upper bound on the nondominated set of (2). While it can be easily determined for biobjective problems, this is in general not the case for higher dimensional problems. However, any upper bound on the nondominated set is sufficient for our purpose. Therefore, we typically use upper bounds on $Z = f(X)$ which can be determined also in the presence of more than two criteria.

A common technique to solve problems of the form (1) is to iteratively transform the original multiple objective problem into a series of parametric single objective problems, so called scalarizations (see, e.g., Ehrgott, 2005; Miettinen, 1999). A variety of different scalarization methods exists which differ, among other things, with respect to their theoretical properties. Of particular importance is the question whether the solutions generated by a specific method always correspond to nondominated points of (1) and whether all nondominated points of (1) can be generated by appropriately varying the involved parameters. In this article we do not focus on a specific scalarization, but assume to have an arbitrarily parameterized scalarization method at hand, which for any feasible parameter choice either generates a nondominated point or detects infeasibility.

**1.2 Literature Review**

The idea of solving parameterized single-objective optimization problems in order to generate a set of nondominated points is well known in the literature. Especially for bicriteria problems, the literature is rich: Already in the late seventies Aneja and Nair (1979) used a parametric weighted sum method in order to find the extreme nondominated points of linear problems. Later, scalarization methods which are suitable for nonlinear or discrete problems replace the weighted sum. We only review some of the more recent approaches in the following. In Sayin and Kouvelis (2005), the lexicographic weighted Tchebycheff method and a variant of it serve to solve bicriteria discrete optimization problems. Ralphs et al. (2006) present an improved algorithm, also based on the Tchebycheff scalarization method. Their algorithm is shown to find
all nondominated points by solving $|N| - 1$ subproblems, in addition to the two subproblems needed for generating the lexicographic optimal points. An approach based on an augmented ε-constraint method is presented by Özpeynirci and Köksalan (2010).

Besides the generation of the entire nondominated set, the generation of a representative system or an approximation of the nondominated set is considered. Therefore, quality criteria are formulated and studied. The diagonal algorithm of Klamroth et al. (2002) aims at refining the approximation where the error is maximal with respect to a problem dependent error measure. The box algorithm of Hamacher et al. (2007), which is designed for bicriteria problems, uses parametric lexicographic ε-constraint problems for the determination of nondominated points and always chooses the box with the biggest volume for further refinement. The adaptive algorithm based on sensitivity information of Eichfelder (2009) addresses the problem of finding a representative system with equidistant spacing. Equidistant representations especially for bicriteria programs are studied in Faulkenberg and Wiecek (2012).

An approach for finding the entire nondominated set of finite problems with an arbitrary number of criteria is proposed in Laumanns et al. (2006). They use parametric lexicographic ε-constraint problems to solve the subproblems. The parameter values are taken as grid points of an $(m-1)$-dimensional projection of the outcome space. The authors show that at most $(|N| + 1)^{m-1}$ scalarized subproblems need to be solved, where the generation of each nondominated point is counted as one subproblem. In the bicriteria case, this yields a total number of only $|N| + 1$ subproblems, which is smaller than the upper bound of Ralphs et al. (2006) due to the special scalarization employed. The numerical experiments for a knapsack problem with three objectives reveal that the number of subproblems needed is considerably less than $(|N| + 1)^2$. The authors state that it is an open question whether a better upper bound can be obtained. Özlem and Azizzoğlu (2009) also study the problem of generating the entire nondominated set of multicriteria problems. Their derived bound on the number of iterations is again $O(|N|^{m-1})$. In Lokman and Köksalan (2012), two improved algorithms based on an augmented ε-constraint scalarization are presented to find all nondominated points for multicriteria integer problems. While the numerical study of the authors suggests a linear bound on the number of subproblems to be solved in the tricriteria case, only an upper bound of $O(|N|^2)$ is derived for $m = 3$. Dhaenens et al. (2010) use the approach of Laumanns et al. (2006), but aim at generating the true nadir point in the first stage of their algorithm. Their numerical experiments show that the determination of the nadir point is very expensive regarding computational time. Przybylski et al. (2010) also tackle the problem of describing the search space in their generalization of the two phase method to integer problems with more than two objectives, however, without deriving an upper bound on the number of generated search areas, respectively, subproblems. In the first phase, all supported nondominated points are computed and iteratively inserted into an initial search space. The updated search space is described with the help of appropriate lower and upper bounds. ‘Dominated’ upper bounds are deleted, wherefore the upper bounds induced by the current nondominated point are compared to all upper bounds of the current search space. The non-supported nondominated points which are detected in the second phase are inserted in the same way. While we follow a similar algorithmic approach in this paper, we will show that the
filtering for ‘dominated’ upper bounds (and, equivalently, redundant search areas) can be avoided in the tricriteria case by explicitly identifying the non-redundant search areas (and, equivalently, ‘nondominated’ upper bounds). In this way we can show that the number of search areas grows only linearly with the number of nondominated points.

1.3 Goals and Outline

We present an algorithm that generates the entire nondominated set of a general tricriteria problem by solving at most \(3|N| - 2\) subproblems, if \(|N| \geq 3\) and if an appropriate initial search space is given. Thereby, to the best of our knowledge, a linear bound with respect to the number of nondominated points is given for the first time for tricriteria problems. Our algorithm does not depend on a specific scalarization, but can be used with any suitable scalarization method. It is also applicable if a subset of nondominated points is already known and the search space potentially containing further nondominated points shall be generated.

The paper is organized as follows: First we present a decomposition of the search space based on nondominance and develop a first generic box algorithm. Then we show that this generic algorithm may produce redundant boxes, which makes the algorithm inefficient. Under the technical assumption that all nondominated points differ pairwise in every component, we show how to construct a decomposition in the tricriteria case that only contains non-redundant boxes. The number of boxes is proven to be at most \(3|N| - 2\) for \(|N| \geq 3\). Finally, we show that the algorithm can also be applied if the nondominated points are in arbitrary position, i.e., every pair of points may have up to \(m - 2\) equal components for \(m = 3\). The upper bound \(3|N| - 2\) is also valid in this general case.

2 Split of the search space for multicriteria problems

Let \(B_0\) denote an initial search space of the form \(B_0 := \{x \in \mathbb{R}^m : l_j \leq x_j < u_j, j = 1, \ldots, m\}\) with \(l, u \in \mathbb{R}^m, l \leq u\). As lower and upper bound of \(B_0\) we choose a global lower and upper bound on the nondominated set, for example, \(l := z^I\) and \(u := z^M\), where \(z^I\) is the ideal point and \(z^M_j := \max\{z_j : z \in Z\} + \delta\) for all \(j = 1, \ldots, m\) with \(\delta > 0\) is an upper bound on the set \(Z\). Alternatively, explicit bounds on the search space as provided, for example, by a decision maker can be used to specify \(l\) and \(u\).

If no special scalarization method is employed, then the iterative reduction of the search space can solely be based on nondominance. Therefore, every generated nondominated point allows to restrict the search space, as, by Definition [13], for any \(z^* \in N\), the two sets

\[ S_1(z^*) := \{z \in B_0 : z \leq z^*\} \quad \text{and} \quad S_2(z^*) := \{z \in B_0 : z \geq z^*\} \]

do not contain any nondominated points besides \(z^*\), i.e., \(S_1(z^*) \cap N = S_2(z^*) \cap N = \{z^*\}\). Moreover, \(S_1(z^*) \cap Z = \{z^*\}\), thus \(S_1(z^*) \setminus \{z^*\}\) contains no feasible points.
In the following, we decompose a given initial search space \( B_0 \) iteratively into subsets \( B \subset B_0 \) of the same form, i.e., into sets \( B = \{ x \in \mathbb{R}^m : l_j \leq x_j < u'_j, j = 1, \ldots, m \} \) with \( u' \in \mathbb{R}^m, l \leq u' \leq u \). As the initial search space that potentially contains nondominated points of (1) as well as each subset \( B \) as defined above describe rectangular subsets of \( \mathbb{R}^m \) with sides parallel to the coordinate axes, we call these sets boxes in the following. The search space is always represented as the union of certain boxes \( B \).

With the generation of every new nondominated point we replace some of the boxes of the current search space by appropriate new boxes such that the whole search space is covered. This property is called correctness in the following.

**Definition 2.1 (Correct decomposition).** Let \( B_0 \) denote the starting box, let \( B_s \) denote the set of boxes at the beginning of iteration \( s \geq 1 \), where \( B_1 := \{ B_0 \} \), and let \( z^p \in N, p = 1, \ldots, s-1 \), be already determined nondominated points. We call \( B_s \) correct with respect to \( z^1, \ldots, z^{s-1} \), if

\[
B_0 \setminus \left( \bigcup_{B \in B_s} B \right) = \bigcup_{p=1,\ldots,s-1} S_2(z^p)
\]

holds, where \( S_2(z^p) := \{ z \in B_0 : z \geq z^p \} \) denotes that subset of the box \( B_0 \) that is dominated by the point \( z^p \in N, p = 1, \ldots, s-1 \).

Any split presented in the following maintains a correct decomposition of the search space at any time. Under this basic condition, we try to generate as few boxes as possible, as for every generated box a scalarized subproblem needs to be solved. Our aim is to keep the number of subproblems low. The simplest split decomposes a box \( B \) which contains a new outcome \( z^* \in (B \cap N) \) into \( m \) subboxes:

**Definition 2.2 (Full \( m \)-split).** Let a nondominated point \( z^* \in (B \cap N) \) be given. We call the replacement of \( B \) by the \( m \) sets

\[
B_i := \{ z \in B : z_i < z^*_i \} \forall i = 1, \ldots, m
\]

a full \( m \)-split of \( B \).

Recursively applying the full \( m \)-split to every box which contains the current nondominated point yields a correct decomposition, as the following lemma shows:

**Lemma 2.3 (Correctness of the full \( m \)-split).** Let \( B_s, s \geq 1 \), with \( B_1 := \{ B_0 \} \) be a correct decomposition with respect to the nondominated points \( z^1, \ldots, z^{s-1} \), and let \( z^* \in N \). If a full \( m \)-split is applied to all boxes \( B \in B_s \) with \( z^* \in B \), then the resulting decomposition is correct.

**Proof.** By induction on \( s \).

\( s = 1 \): Let \( B_1 := \{ B_0 \} \) be correct, \( z^1 \in N \). Then, by definition of the full \( m \)-split, \( B_0 \) is replaced by \( m \) boxes. It holds that

\[
B_0 \setminus \left( \bigcup_{B \in B_2} B \right) = B_0 \setminus \left( \bigcup_{i=1,\ldots,m} \{ z \in B_0 : z_i < z^1_i \} \right) = S_2(z^1),
\]
thus, $B_2$ is correct.

$s \rightarrow s + 1$: Let $B_s$ be correct and let $z^* \in N$. Let $\overline{B}_s \subset B_s$ denote the set of all boxes $B \in B_s$ for which $z^* \in B$ holds. Let $I$ be the index set of these boxes and let $Q := |\overline{B}_s|$. Now, let a full $m$-split with respect to $z^*$ be applied to all $B \in \overline{B}_s$, i.e. each of the boxes $B^{I(q)}, q = 1, \ldots, Q$, is replaced by $m$ new boxes $B^{I(q)}_i, q = 1, \ldots, Q$ and

$$\bigcup_{i=1}^{m} B^{I(q)}_i = \bigcup_{B \in \overline{B}_s} B \setminus S_2(z^*)$$

holds. Then

$$B_0 \setminus \left( \bigcup_{B \in \overline{B}_{s+1}} B \right) = B_0 \setminus \left( \left( \bigcup_{B \in \overline{B}_s} B \right) \cup \left( \bigcup_{i=1}^{m} B^{I(q)}_i \right) \right)$$

$$= B_0 \setminus \left( \left( \bigcup_{B \in \overline{B}_s} B \right) \cup \left( \bigcup_{B \in \overline{B}_s} B \setminus S_2(z^*) \right) \right) = B_0 \setminus \left( \bigcup_{B \in \overline{B}_s} B \setminus S_2(z^*) \right)$$

$$= \left( B_0 \setminus \left( \bigcup_{B \in \overline{B}_s} B \right) \right) \cup S_2(z^*) = \bigcup_{p=1}^{s} S_2(z^p).$$

Note that all new boxes $B \in \overline{B}_{s+1}, s \geq 2$, obtained from boxes in $\overline{B}_s$, are defined as sets with open upper boundary, as we need to exclude $z^*$ from the search space in order to prevent it from further generation. In practical applications, it will often be useful to replace the boxes by closed subsets and exclude $z^*$ by using, for example, appropriate scalarization approaches.

Also note that we describe the search boxes by their upper bound $u$ only and that the lower bound of all boxes is kept constant. This means that our decomposition contains the union of the sets $S_1(z^p), 1 \leq p \leq s$, for all nondominated points $z^p \in N$ which have already been generated by the algorithm, even if these sets do not contain any feasible points besides the already known points $z^p, 1 \leq p \leq s$. However, the splitting operation is simplified by including these sets, since a box is never split into more than $m$ new boxes in this case.

3 A generic algorithm based on the full $m$-split

Algorithm 1 shows a basic algorithm using the full $m$-split. Due to Lemma 2.3, the algorithm is correct as it does not exclude regions from the search space which might contain further nondominated points. As long as $B_s$ contains unexplored boxes, a box $B$ is selected according to some rule as specified, for example, by an error measure or by a decision maker. However, as we are interested in generating the entire nondominated set, no special rule is employed in the following, and we may, for example, always take
the first box in the list $B_s$. The upper bounds of the chosen box $B$ are used to determine the parameters of the selected scalarization. Thereby, the scalarization method can be chosen freely, as long as it is guaranteed that the method finds a nondominated point in $B$ whenever there exists one. Note that, for example, the augmented weighted Tchebycheff scalarization is an appropriate method, see, e.g., [Dächert et al., 2012] for the bicriteria case. The result of the parametric subproblem is either a nondominated point $z^*$ or the detection of infeasibility. In the latter case, $B$ is removed from the list $B_s$ and the iteration is finished. Otherwise, $z^*$ is saved and all boxes $\hat{B} \in B_s$ are identified that contain $z^*$. All these boxes are split with respect to all $i \in \{1, \ldots, m\}$ for which $z^*_s > z^*_I$ holds and replaced by the new boxes. The algorithm iterates until all boxes are explored. Then the entire nondominated set has been detected.

### 3.1 Bicriteria case

For $m = 2$, Algorithm 1 is not only correct but also efficient, in the sense that the number of subproblems needed to be solved depends linearly on the number of nondominated points. As the decomposition does not contain redundant boxes, an upper bound on the number of boxes can easily be derived, which can be seen as follows: Let $B_0$ denote the starting box and let $z^1 \in B_0 \cap N$ be the first generated point. Consider the two new boxes $B_1, B_2$ replacing $B_0$ in the first iteration. It holds that

\[
B_1 \cap Z = \{z \in B_0 : z_1 < z_1^1\} \cap Z = (\{z \in B_0 : z_1 < z_1^1\} \cap Z) \setminus S_1(z^1)
\]

\[
= \{z \in B_0 : z_1 < z_1^1, z_2 > z_2^1\} \cap Z
\]

and, analogously,

\[
B_2 \cap Z = \{z \in B_0 : z_2 < z_2^1\} \cap Z = \{z \in B_0 : z_2 < z_2^1, z_1 > z_1^1\} \cap Z,
\]
Algorithm 1 Algorithm with full $m$-split

**Input:** Image of the feasible set $Z \subset \mathbb{R}^m$

1: $N := \emptyset; \delta > 0$
2: \textbf{InitStartingBox}(Z, δ)
3: $s := 1$; \hfill // Initialize starting box
4: \textbf{while} $B_s \neq \emptyset$ \textbf{do}
5: \hspace{1em} Choose $B \in B_s$;
6: \hspace{1em} $z^* := \text{opt}(Z, u(B))$ \hfill // Solve scalarized subproblem
7: \hspace{1em} \textbf{if} $z^* = \emptyset$ \textbf{then}
8: \hspace{2em} $B_{s+1} := B_s \setminus \{B\}$ \hfill // Subproblem infeasible
9: \hspace{1em} \textbf{else}
10: \hspace{2em} $N := N \cup \{z^*\}$ \hfill // Save nondominated point
11: \hspace{2em} $B_{s+1} := B_s$; \hfill // Copy set of current boxes
12: \hspace{2em} \textbf{GenerateNewBoxes}(B_s, z^*, z^I, B_{s+1})
13: \hspace{1em} \textbf{end if}
14: \hspace{1em} $s := s + 1$
15: \textbf{end while}

**Output:** Set of nondominated points $N$

16: \textbf{procedure} \textbf{InitStartingBox}(Z, δ)
17: \hspace{1em} \textbf{for} $j = 1$ \textbf{to} $m$ \textbf{do}
18: \hspace{2em} $z^I_j := \min\{z_j : z \in Z\}$ \hfill // Compute bounds on $Z$
19: \hspace{2em} $z^M_j := \max\{z_j : z \in Z\} + \delta$
20: \hspace{2em} $u_j(B_0) := z^M_j$
21: \hspace{1em} \textbf{end for}
22: \hspace{1em} $B_1 := \{B_0\}$; \hfill // Initialize set of boxes
23: \hspace{1em} return $B_1$
24: \textbf{end procedure}

25: \textbf{procedure} \textbf{GenerateNewBoxes}(B_s, z^*, z^I, B_{s+1})
26: \hspace{1em} \textbf{for all} $\hat{B} \in B_s$ \textbf{do}
27: \hspace{2em} \textbf{if} $z^* < u(\hat{B})$ \textbf{then}
28: \hspace{3em} $B_{s+1} := B_{s+1} \setminus \{\hat{B}\}$; \hfill // Point is contained in box
29: \hspace{3em} \textbf{for} $i = 1$ \textbf{to} $m$ \textbf{do}
30: \hspace{4em} \textbf{if} $z^I_i > z^*_i$ \textbf{then}
31: \hspace{5em} $B' := \emptyset$; \hfill // Apply full $m$-split
32: \hspace{5em} $u_i(B') := z^*_i$; $u_j(B') := u_j(\hat{B}) \forall j \neq i$; \hfill // Create new box
33: \hspace{5em} $B_{s+1} := B_{s+1} \cup B'$; \hfill // Update upper bound
34: \hspace{4em} \textbf{end if}
35: \hspace{3em} \textbf{end for}
36: \hspace{2em} \textbf{end if}
37: \hspace{1em} \textbf{end for}
38: \hspace{1em} return $B_{s+1}$
39: \textbf{end procedure}
Figure 1: Boxes $B_{i,i} = 1, 2, 3$, obtained by a full $m$-split of the initial search space.

thus, $(B_1 \cap Z) \cap (B_2 \cap Z) = \emptyset$. Therefore, the second generated point $z^2 \in N$ is contained in exactly one of the two boxes $B_1, B_2$. This box is again split into two new boxes whose intersections with $Z$ are disjoint among themselves as well as from the box (intersected with $Z$) which has not been changed in the current iteration. Repeating this argument, we see that for $m = 2$, no redundancy occurs. Therefore, we can easily indicate the running time of Algorithm 1 in the bicriteria case based on the knowledge that a new nondominated point lies in exactly one box: In the initialization phase, $z^I$ and $z^M$ are computed in order to define $B_0$. In every iteration, either a (new) nondominated point is generated or a box is discarded from the search space. For every new point $z^s > z^I$, two new boxes replace the currently investigated box, and for each of the two lexicographic optimal points (defining the ideal point) the current search box is replaced by one new box. So, the total number of iterations is $2|Y| - 1$, cf. Ralphs et al. (2006).

In Figure 2, we illustrate the search space after four nondominated points $z^1, z^2, z^3, z^4$ have been generated. If we assume that these solutions build the entire nondominated set then Algorithm 1 terminates after seven iterations.

### 3.2 Multicriteria case ($m \geq 3$)

When dealing with more than two criteria, the full $m$-split can also be applied, see Dhaenens et al. (2010) and Figure 2 for an illustration for $m = 3$. However, for $m \geq 3$, a nondominated point may lie in the intersection of multiple boxes. If we perform the full $m$-split to every box which contains the current nondominated point, we typically create nested and, thus, redundant subboxes. This is illustrated in the following example.

**Example 3.1.** Let $m = 3$ and let the initial search space be given by

$$B_0 := \{z \in Z : 0 \leq z_i \leq 5 \forall i = 1, 2, 3\}.$$

Assume that the first nondominated point that is generated is $z^1 = (2, 2, 2)^\top$. Performing a full $3$-split of $B_0$ with respect to $z^1$ replaces the search space $B_0$ by the three sets

$$B_{1,i} := \{z \in B_0 : z_i < 2\}, \quad i = 1, 2, 3.$$
Let $z^2 = (1,1,4)^\top$ be the next nondominated point that is generated. It holds that $z^2 \in B_{11}$ as well as $z^2 \in B_{12}$, but $z^2 \notin B_{13}$. Performing a full 3-split in $B_{11}$ yields

\[ B_{21} := \{ z \in B_0 : z_1 < 1 \} \]
\[ B_{22} := \{ z \in B_0 : z_1 < 2, z_2 < 1 \} \]
\[ B_{23} := \{ z \in B_0 : z_1 < 2, z_3 < 4 \} \].

Performing a full 3-split in $B_{12}$ yields

\[ B'_{21} := \{ z \in B_0 : z_1 < 1, z_2 < 2 \} \]
\[ B'_{22} := \{ z \in B_0 : z_2 < 1 \} \]
\[ B'_{23} := \{ z \in B_0 : z_2 < 2, z_3 < 4 \} \].

It holds that $B'_{21} \subset B_{21}$ and $B_{22} \subset B'_{22}$, thus, the boxes $B_{22}$ and $B'_{21}$ are redundant in the decomposition of $B_0$.

If redundant boxes are kept in the decomposition, this typically increases the running time of the algorithm, as additional, unnecessary scalarized subproblems are solved. Depending on the given problem, this may be time-consuming. Thus, redundant boxes should be detected and removed immediately. In the following, we will analyze under which conditions redundant boxes can occur. We first define our notion of non-redundancy:

**Definition 3.2** (Non-redundant decomposition). Let $B_0$ denote the starting box and let $B_s$ be a correct decomposition at the beginning of iteration $s \geq 1$. We call $B_s$ (and every $B \in B_s$) non-redundant, if for every pair of boxes $B, \tilde{B} \in B_s, B \neq \tilde{B}$, it holds:

\[ \exists i \in \{1,\ldots,m\} : u_i(B) < u_i(\tilde{B}) \text{ and } \exists j \in \{1,\ldots,m\} : u_j(B) > u_j(\tilde{B}). \]

In the case that $u(B) \leq u(\tilde{B})$ we say that box $\tilde{B}$ dominates $B$, and, conversely, that $B$ dominates $\tilde{B}$. If $u(B) \geq u(\tilde{B})$.

For simplicity, we make a technical assumption concerning the values of the nondominated points that will be removed later. Moreover, we define our general setting:

**Assumption 3.3.** Let the following hold:

1. For all nondominated points $z^p \in N, p = 1, \ldots, s$, generated up to iteration $s \geq 1$, it holds that $z^j \neq z^q$ for all $j = 1, \ldots, m$ and $1 \leq q < p$.

2. The starting box $B_0$ is non-empty, and $B_1 := \{B_0\}$ denotes the initial decomposition of the search space.

3. For every $s \geq 1$, $B_s$ is a correct, non-redundant decomposition of the search space. By $\overline{B}_s := \{B \in B_s : z^s \in B\}$ we denote the subset of boxes in iteration $s$ containing $z^s$. 

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Lemma 3.4 (Generation of redundant boxes). Let Assumption 3.3 be satisfied. If we apply a full \( m \)-split to every box \( B \in \mathcal{B}_s \), then redundancy can only occur among the ‘descendants’ of two different boxes which have been split with respect to the same component in this iteration.

Proof. We first show that no redundancy occurs between two boxes if at least one of the boxes has not been changed in the current iteration. Therefore, consider two arbitrary boxes \( B, \hat{B} \in \mathcal{B}_s \) where \( \hat{B} \in \mathcal{B}_s \setminus \mathcal{B}_s \):

1. If \( B \in \mathcal{B}_s \setminus \mathcal{B}_s \) then both boxes remain unchanged in the current iteration and, thus, due to the assumption are non-redundant.
2. If \( B \in \mathcal{B}_s \) then none of the boxes obtained from a split in \( B \) can dominate \( \hat{B} \), as \( B \) does not dominate \( \hat{B} \) and the upper bound of \( B \) is only decreased by the split. Conversely, \( \hat{B} \) cannot dominate any of the boxes obtained from a split in \( B \), as \( z^s_i = u_i(B_i) \) and \( z^s_j < u_j(B_i) \) for all \( j \neq i \). Thus, \( \hat{B} \) dominates \( B_i \) if and only if \( z^s_i = u_i(\hat{B}) \) holds. This, however, is excluded by Assumption 3.3 (1).

Therefore, redundancy can only occur among newly generated boxes. Consider two boxes \( B_i \neq \hat{B}_j \) obtained from \( B, \hat{B} \in \mathcal{B}_s \) (the case \( B = \hat{B} \) is included) that are split with respect to components \( i \neq j \). Then \( u_i(B_i) = z^s_i < u_i(\hat{B}_j) \) and \( u_j(B_i) > z^s_j = u_j(\hat{B}_j) \), thus, none of the boxes can dominate the other one. It follows that redundancy can only occur among the descendants of two different boxes split with respect to the same component.

Corollary 3.5. Let Assumption 3.3 hold. If only one box is split in some iteration, then all \( m \) resulting subboxes are non-redundant. In particular, the boxes obtained in the first iteration are always non-redundant.

Corollary 3.6. Let Assumption 3.3 hold. Let two boxes \( B, \hat{B} \in \mathcal{B}_s \) be split with respect to the same component \( i = 1, \ldots, m \). Then the resulting boxes \( B_i, \hat{B}_i \) are non-redundant if and only if there exists an index \( p \neq i \) such that \( u_p(B_i) < u_p(\hat{B}_i) \) and there exists an index \( q \neq i \) such that \( u_q(B_i) > u_q(\hat{B}_i) \).

These observations allow to detect redundant boxes by checking specific boxes of the current decomposition. Translating this to an algorithm for arbitrary \( m \geq 3 \), we apply, in every iteration, a full \( m \)-split to every box containing the current solution. If more than one box is split in one iteration, we compare the upper bounds of those new boxes that were generated with respect to the same component pairwise to detect redundancy. The respective boxes are then removed from the decomposition.

A corresponding algorithm can be improved further if redundant boxes are already detected before their creation. In the next section we develop such an explicit criterion for tricriteria problems that indicates already before the split is performed if the resulting box is redundant or not, and, thus, allows to maintain only non-redundant boxes.
Figure 3: Individual subsets $V(B_i), i = 1, 2, 3$, in $\mathbb{R}^3$ obtained by a full 3-split of the initial search space with respect to $z^1 \in N$.

in the decomposition. We prove that the number of non-redundant boxes or, equivalently, the number of subproblems to be solved in the course of an algorithm based on such an improved split operation depends linearly on the number of nondominated points.

4 A split criterion to avoid redundant boxes for $m = 3$

According to Definition 3.2, a non-redundant box can be characterized as follows: A box is non-redundant if and only if it contains a non-empty subset which is not part of any other box of the decomposition. These subsets are studied in the following.

Definition 4.1 (Individual subsets). Let $B_s, s \geq 1$ be a non-redundant decomposition. For every $B \in B_s$, the set

$$V(B) := B \setminus \bigcup_{\tilde{B} \in B \setminus \{B\}} \tilde{B}$$

is called individual subset of $B$.

Obviously, for every $B \in B_s, s \geq 1$, it holds that $V(B) \subseteq B$ and $V(B) \cap V(\tilde{B}) = \emptyset$ for every $\tilde{B} \in B_s, \tilde{B} \neq B$. Figure 3 shows the (closures of the) individual subsets of the three boxes $B_i, i = 1, 2, 3$, in $\mathbb{R}^3$ obtained by a full 3-split of the initial search box, which are depicted in Figure 2.

Now, maintaining only non-redundant boxes in the decomposition of the search space is equivalent to maintaining boxes with non-empty individual subsets. An explicit split criterion should indicate already before performing the split if a given box will have a non-empty individual subset after having performed the split. To this end, we have to describe the individual subsets explicitly. For $m = 3$, we observe that the individual subset of a box is bounded by the neighbours of that box. After defining the neighbour of a box with respect to a certain component, we show its existence and indicate the respective neighbouring boxes by a constructive proof.
Definition 4.2 (Neighbour of a box). Let $B_s$, $s \geq 1$ be a non-redundant decomposition of the search space, and let $u_i := \min\{u_i(B) : B \in B_s\}$. Let any $B \in B_s$ be given. For every $i \in \{1, 2, 3\}$, for which $u_i(B) > u_i$, we call a box $\tilde{B} \in B_s \setminus \{B\}$ that satisfies

$$u_i(\tilde{B}) < u_i(B)$$

$$u_j(\tilde{B}) > u_j(B) \quad \text{for some} \ j \neq i$$

$$u_k(\tilde{B}) \geq u_k(B) \quad \text{for} \ k \neq i, j$$

and

$$u_i(\tilde{B}) = \max\{u_i(B) : B \in B_s \setminus \{\tilde{B}\}, u_i(B) < u_i(\tilde{B})\}$$

the neighbour of $\tilde{B}$ with respect to $i$ in iteration $s$, denoted by $B_i^s(\tilde{B})$.

Example 4.3. Consider Figure 2. It holds that $B_1^1(B_2) = B_1$, as $B_1$ is the unique box satisfying $[\bar{B}_1, \bar{B}_2]$ for $\tilde{B} := B_2$. Analogously, $B_2^1(B_2) = B_3$ holds. A neighbour $B_2^1(B_2)$ is not defined as $u_2(B_2) = u_2$. The following lemma shows that, under appropriate assumptions, for every box $\tilde{B} \in B_s$ and every component $i \in \{1, 2, 3\}$ for which $u_i(\tilde{B}) > u_i$, there exists a unique neighbour $B_i^s(\tilde{B})$ satisfying $[\bar{B}_1, \bar{B}_2]$ of Definition 4.2. These neighbours $B_i^s(\tilde{B}), i \in \{1, 2, 3\}$, which will be indicated with the help of a constructive proof will turn out to be the boxes that define the individual subset of $\tilde{B}$.

Assumption 4.4. Let the following hold:

1. For all nondominated points $z^p \in N, p = 1, \ldots, s$, generated up to iteration $s \geq 1$, it holds that $z_j^p \neq z_j^q$ for all $j \in \{1, 2, 3\}$ and $1 \leq q < p$.

2. The starting box $B_0$ is non-empty, and $B_1 := \{B_0\}$ denotes the initial decomposition of the search space.

3. For every $s \geq 1$, the set $B_{s+1}$ is obtained from $B_s$ by applying a full 3-split to every $B \in B_s$, where $B_s := \{B \in B_s : z^s \in B\}$. All redundant boxes are removed from $B_{s+1}$ at the end of the respective iteration $s$.

Note that Assumption 4.4 substantiates Assumption 3.3 by specifying that the correct, non-redundant decompositions are obtained by iterative full 3-splits and that redundant boxes are removed.

Lemma 4.5. Let Assumption 4.4 be satisfied. Then, for every $s \geq 2$, every $\tilde{B} \in B_s$ and every $i \in \{1, 2, 3\}$, for which $u_i(\tilde{B}) > u_i := \min\{u_i(B) : B \in B_s\}$ holds, there exists a unique neighbour $B_i^s(\tilde{B}) \in B_s$ satisfying $[\bar{B}_1, \bar{B}_2]$. Particularly, $u_k(B_i^s(\tilde{B})) = u_k(\tilde{B})$ holds, i.e. $B_i^s(\tilde{B})$ satisfies

$$u_i(B_i^s(\tilde{B})) < u_i(\tilde{B})$$

$$u_j(B_i^s(\tilde{B})) > u_j(\tilde{B}) \quad \text{for some} \ j \neq i$$

$$u_k(B_i^s(\tilde{B})) \geq u_k(\tilde{B}) \quad \text{for} \ k \neq i, j$$

$$u_i(\tilde{B}) = \max\{u_i(B_i^s(\tilde{B})) : B_i \in B_s \setminus \{\tilde{B}\}, u_i(B_i) < u_i(\tilde{B})\}$$
\[ u_k(B_i^z(\bar{B})) = u_k(\bar{B}) \quad \text{for } k \neq i, j, \]  
\( (12) \)

and
\[ u_i(B_i^z(\bar{B})) = \max\{u_i(B) : B \in \mathcal{B}_s, u_i(B) < u_i(\bar{B})\}. \]
\( (13) \)

If \( u_i(\bar{B}) = u_i \), then we set \( B_i^z(\bar{B}) := \emptyset \).

Proof. By induction on \( s \).

(i) \( B_i^{z_i+1}(\bar{B}) = B_i^z(\bar{B}) \):

\( B_i \) is the only new box \( B \) with \( u_i(B) = z_i^s \) and there is no other new box \( B \) satisfying \( u_i(B) < z_i^s \). Hence, \( B_i^{z_i+1}(\bar{B}) \notin \{B_1, B_2, B_3\} \). If \( B_i^z(\bar{B}) = \emptyset \) then \( B_i^{z_i+1}(\bar{B}) = \emptyset \). Otherwise, i.e. if \( B_i^z(\bar{B}) \) exists, \( u_i(B_i^z(\bar{B})) < u_i(\bar{B}) \), \( u_i(B_i^z(\bar{B})) > u_i(\bar{B}) \) for some \( j \neq i \) and \( u_k(B_i^z(\bar{B})) = u_k(\bar{B}) \) for \( k \neq i, j \) hold due to the induction hypothesis. Now \( u_i(B_i^z(\bar{B})) < z_i^s \) must be satisfied, as otherwise \( B_i^z(\bar{B}) \in \mathcal{B}_s \) would hold, a contradiction to the assumption that \( \mathcal{B}_s = \{\bar{B}\} \). Moreover, \( u_i(B_i^z(\bar{B})) = z_i^s \) is excluded due to Assumption 4.4 (1). Thus, \( 10 \)-\( 13 \) holds for \( B_i^{z_i+1}(\bar{B}) = B_i^z(\bar{B}) \). The uniqueness of \( B_i^z(\bar{B}) \) follows from the induction hypothesis.

(ii) \( B_j^{z_j+1}(\bar{B}) = B_j \) for all \( j \neq i \):

\( B_j \) is the only new box \( B \) with \( u_j(B) = z_j^s \) and there is no other new box \( B \) satisfying \( u_j(B) < z_j^s \). Furthermore, \( B_j \) satisfies \( 10 \)-\( 12 \), as \( u_j(B_j) < u_j(\bar{B}) \), \( u_i(B_j) > u_i(\bar{B}) \) and \( u_k(B_j) = u_k(\bar{B}) \) for \( k \neq i, j \) hold. Moreover, \( u_j(B_j) \) is maximal, as \( u_j(\bar{B}) = z_j^s \) if \( B_j^z(\bar{B}) \) is not \( \emptyset \) and \( u_j(B_j^z(\bar{B})) \) maximal due to the induction hypothesis. As \( z_j^s = u_j(B_j^z(\bar{B})) \) is excluded due to Assumption 4.4 (1), the uniqueness of \( B_j^{z_j+1}(\bar{B}) \) follows.

Now consider an arbitrary box \( B \neq \bar{B} \). Then the following holds:
(iii) If \( B_i^* (B) \neq \hat{B} \) for some \( i \in \{1, 2, 3\} \), then \( B_i^{*+1} (B) \) remains unchanged:

Assume that \( B_i^{*+1} (B) \) changes due to the split of \( \hat{B} \). Then the only candidate for \( B_i^{*+1} (B) \) is \( \hat{B}_i \) and only in case that \( u_i (\hat{B}_i) < u_j (B) \) as otherwise \( B_i^* (B) = \hat{B} \) would have been valid. Now suppose that \( B_i^{*+1} (B) = \hat{B}_i \). As \( u_i (\hat{B}_i) = u_j (\hat{B}_i) \) for all \( j \neq i \) and, by definition of \( B_i^* (B) \), \( u_i (\hat{B}_i) \geq u_j (B) \) for all \( j \neq i \), we have that \( u_i (\hat{B}_i) \geq u_j (B) \) for all \( j \neq i \) and hence \( u_i (\hat{B}_i) \geq u_j (B) \) for all \( l \in \{1, 2, 3\} \), a contradiction to \( B_s \) being non-redundant. Thus, \( B_i^{*+1} (B) \) remains unchanged.

(iv) If \( B_i^* (B) = \hat{B} \) for some \( i = 1, \ldots, m \), then \( B_i^{*+1} (B) = \hat{B}_j \) with \( j \) being the unique index for which \( u_j (\hat{B}_j) > u_j (B) \) holds:

By the induction hypothesis, \( u_i (\hat{B}_i) < u_i (B) \), \( u_j (\hat{B}_i) > u_j (B) \) for some \( j \neq i \) and \( u_k (\hat{B}_i) = u_k (B) \) for \( k \neq i, j \). As \( z_i^* = u_i (\hat{B}_i) < u_i (B) \) and \( u_i (\hat{B}_i) = u_k (B) \) for all \( l \neq i \), \( \hat{B}_j \) is a candidate for \( B_i^{*+1} (B) \). As \( B \notin B_s \) and \( z_i^* < u_i (B) \) for all \( l \neq j \) it follows that \( z_i^* \geq u_j (B) \), and, due to Assumption 4.4 (1), \( z_i^* > u_j (B) \). Thus, \( u_i (\hat{B}_j) = u_i (\hat{B}_i) < u_i (B) \), \( u_j (\hat{B}_j) = z_i^* > u_j (B) \) and \( u_k (\hat{B}_j) = u_k (B) \) hold.

Therefore, \( \hat{B}_j \) is the unique other candidate for \( B_i^{*+1} (B) \) besides \( \hat{B}_i \). As \( u_i (\hat{B}_i) < u_i (\hat{B}_j) = u_i (\hat{B}) \), \( \hat{B}_j \) is the unique neighbour \( B_i^{*+1} (B) \) after the split.

Case 2: \(|B_s| > 1\).

By definition of \( B_s \), it holds that \( z_i^* < u_i (B) \) for all \( i \in \{1, 2, 3\} \) and \( B \notin B_s \), thus, \( z_i^* < \min \{ u_i (B) : B \in B_s \} \) for all \( i \in \{1, 2, 3\} \). According to Lemma 3.4 and Corollary 3.6 redundancy occurs only for boxes \( B, \hat{B} \in B_s \) which are split with respect to the same component \( i \in \{1, 2, 3\} \) (i.e., \( u_i (\hat{B}_i) = u_i (B_i) = z_i^* \)) and for which \( u_i (\hat{B}_i) \geq u_i (B_i) \) or \( u_i (\hat{B}_i) < u_i (B_i) \) holds for all \( l \neq i \). By assumption, those boxes are removed, i.e. \( B_{i+1} \) contains only non-redundant boxes.

Let \( B_s = \{ \hat{B}_1, \ldots, \hat{B}_P \} \) with \( P \in N, P \geq 2 \). For every \( i \in \{1, 2, 3\} \), let \( I_i \subseteq \{1, \ldots, P\} \) be the index set of the boxes from \( B_s \) whose split with respect to \( i \) yields a non-redundant box. Note that \( I_i \neq \emptyset \) for every \( i \in \{1, 2, 3\} \), which can be seen as follows: Consider an arbitrary box \( B \in B_s \). Applying the full 3-split to \( B \) results in three new boxes. Now any of the resulting boxes is removed if and only if there exists another box that dominates it. According to Lemma 3.2, the dominating box must have been created by a split with respect to the same component as the dominated box. Therefore, \( I_i \neq \emptyset \) for every \( i \in \{1, 2, 3\} \). We set \( Q_i := |I_i| \geq 1 \) for every \( i \in \{1, 2, 3\} \). Furthermore, let \( \hat{u}_i := \max \{ u_i (B) : B \in B_s \} \) for all \( i \in \{1, 2, 3\} \) in the following, which is well defined as \( B_s \neq \emptyset \).

Consider now \( i \) arbitrary but fixed. Let \( \hat{B}_i^{(1)}, \ldots, \hat{B}_i^{(Q_i)} \) denote the boxes whose split with respect to component \( i \) yields a non-redundant box. As \( u_i (\hat{B}_i^{(q)}) = z_i^* \) holds for all \( \hat{B}_i^{(q)} \in B_{i+1}, q = 1, \ldots, Q_i, m = 3 \) and as we assume non-redundancy, we can order the boxes with respect to their upper bounds increasingly by some component \( j \neq i \) and decreasingly by component \( k \neq i, j \), i.e.

\[
\begin{align*}
  z_j^* &< u_j (\hat{B}_i^{(1)}) < u_j (\hat{B}_i^{(2)}) < \cdots < u_j (\hat{B}_i^{(Q_i)}) \\
  u_k (\hat{B}_i^{(1)}) &> u_k (\hat{B}_i^{(2)}) > \cdots > u_k (\hat{B}_i^{(Q_i)}) > z_k^*.
\end{align*}
\]
Thereby, \( u_i(\hat{B}_i^{(Q)}) = u_j(\hat{B}_j^{(Q)}) = \bar{u}_j \) holds, since in the other case, i.e., if there was some \( \hat{B} \in \mathcal{B}_s \) with \( u_j(\hat{B}) > u_j(\hat{B}_j^{(Q)}) \), either \( \hat{B} \) would have been the last box in \( Q \) with index \( I_q(Q) \) or \( \hat{B}_j^{(Q)} \) would have been dominated by \( \hat{B} \), both in contradiction to the construction. Analogously, \( u_k(\hat{B}_k^{(1)}) = u_k(\hat{B}_k^{(1)}) = \bar{u}_k \) must hold.

If \( u_i(\hat{B}_i^{(1)}) = \max\{u_i(B) : B \in \mathcal{B}_s, u_k(B) = \bar{u}_k\} =: \bar{u}_{i,k} \) holds, then the split of \( \hat{B}_i^{(1)} \) with respect to \( j \) generates a non-redundant box, too, and, depending on the chosen enumeration, \( I_q(1) \) either equals \( I_j(1) \) or \( I_j(Q) \). Wlog we can set \( I_j(1) = I_j(1) \). Otherwise, i.e. if \( u_i(\hat{B}_i^{(1)}) < \bar{u}_{i,k} \) holds, then \( \hat{B}_i^{(1)} \) is dominated by a unique box \( \hat{B} \in \mathcal{B}_s \) with \( u_k(\hat{B}) = \bar{u}_k \) and \( u_i(\hat{B}) = \bar{u}_{i,k} \). Then \( \hat{B} = \hat{B}_i^{(1)} \) holds.

Analogously, if \( u_i(\hat{B}_i^{(Q)}) = \max\{u_i(B) : B \in \mathcal{B}_s, u_j(B) = \bar{u}_j\} =: \bar{u}_{i,j} \) holds, then the split of \( \hat{B}_i^{(Q)} \) with respect to \( k \) generates a non-redundant box, too, and, wlog, we can identify \( I_q(Q) = I_k(Q) \). Otherwise, i.e. if \( u_i(\hat{B}_i^{(Q)}) < \bar{u}_{i,j} \) holds, then \( \hat{B}_i^{(Q)} \) is dominated by a unique box \( \hat{B} \in \mathcal{B}_s \) with \( u_j(\hat{B}) = \bar{u}_j \) and \( u_i(\hat{B}) = \bar{u}_{i,j} \). Then \( \hat{B} = \hat{B}_i^{(Q)} \) holds.

Note that if \( Q_i = 1 \), then \( \hat{B}_i^{(1)} = \hat{B}_i^{(Q)} = \hat{B} \) and \( u_j(\hat{B}) = \bar{u}_j \) as well as \( u_k(\hat{B}) = \bar{u}_k \) hold. In this case, \( u_i(\hat{B}) < \bar{u}_i \) must be satisfied, as otherwise \( \hat{B} \) would dominate any other box in \( \mathcal{B}_s \), a contradiction to \( |\mathcal{B}_s| > 1 \) and \( \mathcal{B}_s \) being non-redundant.

Analogously to Case 1, we will now indicate the neighbour boxes explicitly. Therefore, consider \( \hat{B}_i^{(q)} \in \mathcal{B}_{s+1} \) for fixed \( i \in \{1, 2, 3\} \), \( q \in \{1, \ldots, Q_i\} \). It holds that

(i) \( B_j^{q+1}(\hat{B}_i^{(q)}) = B_j^*(\hat{B}_i^{(q)}) \);

Assume \( B_j^*(\hat{B}_i^{(q)}) \in \mathcal{B}_s \). By definition of \( B_j^* \), \( u_i(B_j^*(\hat{B}_i^{(q)})) < u_i(\hat{B}_i^{(q)}) \) and \( u_j(B_j^*(\hat{B}_i^{(q)})) \geq u_j(\hat{B}_i^{(q)}) \) for all \( l \neq i \) hold. But then, by an \( i \)-split of \( \hat{B}_i^{(q)} \), the box \( \hat{B}_i^{(q)} \) would be redundant. Therefore, \( B_j^*(\hat{B}_i^{(q)}) \notin \mathcal{B}_s \) must hold. Analogously to Case 1(i), we obtain \( B_j^{q+1}(\hat{B}_i^{(q)}) = B_j^*(\hat{B}_i^{(q)}) \).

(ii) Determination of \( B_j^{q+1}(\hat{B}_i^{(q)}) \) and \( B_k^{q+1}(\hat{B}_i^{(q)}) \) for \( j, k \neq i \):

Consider all \( \hat{B}_i^{(q)} \in \mathcal{B}_{s+1}, q = 1, \ldots, Q_i \), ordered as in [14] and [15]. It holds that

\[ B_j^{q+1}(\hat{B}_i^{(q)}) = \hat{B}_i^{(q-1)} \]

for all \( q = 2, \ldots, Q_i \), as for all other boxes \( \hat{B}_i^{(q)} \), \( p \neq q - 1 \), either \( u_j(\hat{B}_i^{(q)}) < u_j(\hat{B}_i^{(q-1)}) \) or \( u_j(\hat{B}_i^{(q)}) > u_j(\hat{B}_i^{(q-1)}) \) holds. Moreover, all new boxes split with respect to \( j \) have component \( u_j \) smaller than \( u_j(\hat{B}_i^{(q-1)}) \) and all new boxes split with respect to \( k \) have component \( u_k \) smaller than \( u_k(\hat{B}_i^{(q-1)}) \), and, thus, do not satisfy [12]. For all boxes \( B \notin \mathcal{B}_s \), it holds that \( u_i(B) < \min\{u_i(B) : B \in \mathcal{B}_s\} \) for some \( i \), so either \( u_j(B) < u_j(\hat{B}_i^{(q-1)}) \) or [12] is not satisfied.

Next, we determine \( B_j^{q+1}(\hat{B}_i^{(1)}) \): As \( u_j(\hat{B}_i^{(q)}) > u_j(\hat{B}_i^{(1)}) \) for all \( q = 2, \ldots, Q_i \), no box split with respect to \( i \) can be the neighbour \( B_j^{q+1}(\hat{B}_i^{(1)}) \). Furthermore, as \( u_k(\hat{B}_i^{(1)}) > \bar{u}_k \), \( B_j^{q+1}(\hat{B}_i^{(1)}) \) cannot be found among the new boxes split.
with respect to \( k \). Therefore, \( B^{s+1}_j(\hat{B}^{I_i(1)}_i) \) can only be found among the boxes split with respect to component \( j \). Now, as shown above, \( u_k(\hat{B}^{I_i(1)}_i) = \hat{u}_k \) holds, which implies that \( u_k(B^{s+1}_j(\hat{B}^{I_i(1)}_i)) = \hat{u}_k \) must be satisfied. Therefore, the unique candidate for \( B^{s+1}_j(\hat{B}^{I_i(1)}_i) \) is \( \hat{B}^{I_i(1)}_j \), which, as explained above, either equals the box obtained from \( \hat{B}^{I_i(1)}_i \) by a split with respect to \( j \) or the unique box dominating it.

Analogously, it can be shown that

\[
B^{s+1}_k(\hat{B}^{I_i(q)}) = \hat{B}^{I_i(q+1)}_k \quad \text{for all } q = 1, \ldots, Q_i - 1,
\]

and

\[
B^{s+1}_k(\hat{B}^{I_i(Q_i)}) = \hat{B}^{I_k(Q_k)}_k,
\]

where \( \hat{B}^{I_k(Q_k)}_k \) either equals the box obtained from \( \hat{B}^{I_k(Q_k)}_i \) by a split with respect to \( k \) (then \( I_i(Q_i) = I_k(Q_k) \)) or the unique box dominating it.

Finally, for all \( B \notin \mathcal{B}_s \) we obtain the following results which are equivalent to Case 1:

(iii) If \( B^*_i(B) \notin \mathcal{B}_s \) for some \( i \in \{1, 2, 3\} \), then \( B^{s+1}_i(B) \) remains unchanged.

(iv) If \( B^*_i(B) =: \hat{B} \in \mathcal{B}_s \) for some \( i \in \{1, 2, 3\} \), then, following the same argumentation as in Case 1(iv), \( z_j^s > u_j(B) \) for one unique index \( j \neq i \) and, thus, the correct candidate for \( B^{s+1}_i(B) \) would be \( \hat{B}_j \). It remains to show that \( \hat{B}_j \) exists and that \( u_i(\hat{B}_j) = \max \{ u_i(\hat{B}) : \hat{B} \in \mathcal{B}_{s+1}, u_i(\hat{B}) < u_i(B) \} \).

Assume that \( \hat{B}_j \) does not exist, i.e., it is redundant in \( \mathcal{B}_{s+1} \). Then there exists \( \hat{B} \in \mathcal{B}_s \) with \( u_i(\hat{B}) \geq u_i(\hat{B}) \) and \( u_k(\hat{B}) \geq u_k(\hat{B}) \). As \( \hat{B}, \hat{B} \in \mathcal{B}_s \) and \( \mathcal{B}_s \) by induction, is non-redundant, \( u_j(\hat{B}) < u_j(\hat{B}) \) must hold. As \( \hat{B} \in \mathcal{B}_s \) it follows that \( z_j^s < u_j(\hat{B}) \), so \( u_j(\hat{B}) < z_j^s < u_j(\hat{B}) \) and \( u_k(\hat{B}) = u_k(\hat{B}) \leq u_k(\hat{B}) \) hold.

If \( u_i(\hat{B}) \geq u_i(B) \), \( B \) would have been redundant in \( \mathcal{B}_s \). Thus, \( u_i(\hat{B}) < u_i(B) \) must hold. However, as \( B^*_i(B) = \hat{B} \), the induction hypothesis then implies that \( u_i(\hat{B}) < u_i(\hat{B}) \), a contradiction to the assumption on \( \hat{B} \). Thus, \( \hat{B}_j \) is non-redundant and \( u_i(\hat{B}_j) = u_i(\hat{B}) = \max \{ u_i(\hat{B}) : \hat{B} \in \mathcal{B}_{s+1}, u_i(\hat{B}) < u_i(B) \} \) holds.

In the following corollary we summarize the properties of the neighbours of all new boxes obtained in the constructive proof of Lemma 4.5.

**Corollary 4.6.** Let Assumption 4.4 be satisfied. For every \( i \in \{1, 2, 3\} \), let \( I_i \subseteq \{1, \ldots, P\} \), \( I_i \neq \emptyset \), \( P \in \mathbb{N} \), \( |I_i| = Q_i \), be the index set of the boxes of \( \mathcal{B}_s \) whose split with respect to \( i \in \{1, 2, 3\} \) yields a non-redundant box. Then for all new boxes \( B^*_i(q) \), \( q = 1, \ldots, Q_i \), it holds that

\[
B^{s+1}_i(\hat{B}^{I_i(q)}) = B^*_i(\hat{B}^{I_i(q)}) \quad \forall q = 1, \ldots, Q_i,
\]

(16)
Lemma 4.7. Let Assumption 4.4 hold. Then, for \( m = 3 \), the individual subsets \( V(B), B \in \mathcal{B}_s \), which are introduced in Definition 4.1, can be represented as

\[
V(B) = \{ z \in B_0 : v(B) \leq z < u(B) \}
\]

with

\[
v_i(B) := \begin{cases} 
  u_i(B^*_i(B)), & \text{if } B^*_i(B) \neq \emptyset \\
  z_i^0, & \text{otherwise}
\end{cases}, \quad i \in \{1, 2, 3\}.
\]

Proof. For \( \bar{B} \in \mathcal{B}_s, s \geq 1 \), by definition,

\[
V(B) := \bar{B} \setminus \left( \bigcup_{\tilde{B} \in \mathcal{B}_s \setminus \{B\}} \tilde{B} \right)
\]

holds. We consider the sets \( \mathcal{B}_{s,i} := \{ B \in \mathcal{B}_s : u_i(B) < u_i(\bar{B}) \} \) for \( i = 1, 2, 3 \). For fixed \( i \in \{1, 2, 3\} \), the following two cases can occur: If \( \mathcal{B}_{s,i} \neq \emptyset \), then, as shown in Lemma 4.5, \( B^*_i(\bar{B}) \neq \emptyset \) and \( B^*_i(\bar{B}) \in \mathcal{B}_{s,i} \), where \( u_i(B^*_i(\bar{B})) = \max \{ u_i(B) : B \in \mathcal{B}_{s,i} \} \). Furthermore, as \( u_l(B^*_l(\bar{B})) \geq u_l(\bar{B}) \) for all \( l \neq i \),

\[
\bar{B} \setminus \left( \bigcup_{\tilde{B} \in \mathcal{B}_{s,i}} \tilde{B} \right) = \{ z \in \bar{B} : z_i \geq u_i(B^*_i(\bar{B})) \}.
\]

Otherwise, i.e. if \( \mathcal{B}_{s,i} = \emptyset \), then, obviously, \( \bar{B} \setminus \left( \bigcup_{\tilde{B} \in \mathcal{B}_{s,i}} \tilde{B} \right) = \bar{B} \). So, in both cases, it holds that

\[
\bar{B} \setminus \left( \bigcup_{\tilde{B} \in \mathcal{B}_{s,i}} \tilde{B} \right) = \{ z \in B : z_i \geq v_i(\bar{B}) \}
\]

with

\[
v_i(\bar{B}) := \begin{cases} 
  u_i(B^*_i(\bar{B})), & \text{if } B^*_i(\bar{B}) \neq \emptyset \\
  z_i^0, & \text{otherwise}
\end{cases}.
\]
As every box $B \in \mathcal{B}_s \setminus \{\bar{B}\}$ belongs, due to the assumption of non-redundancy, at least to one set $\mathcal{B}_{s,i}, i \in \{1,2,3\},$ there does not exist any other box which can reduce $V(B)$ further. Thus, we obtain the desired representation.

Lemma 4.7 shows that for $m = 3$ the individual subset of a box can be represented as a box itself. As the upper bound of $V(B)$ and $B$ are the same, $V(B)$ can be described by its lower bound $v(B) \in \mathbb{R}^m$ only. Next we show, using Corollary 4.6 how the lower bounds $v(B)$ can be updated in an iterative algorithm.

Lemma 4.8. Let Assumption 4.4 be satisfied. We use the notation of Corollary 4.6. Let $\hat{B}_i^{l(q)}, q = 1, \ldots, Q_i,$ be the non-redundant boxes obtained from $\hat{B}_i^{l(q)} \in \mathcal{B}_s,$ by a split with respect to $i \in \{1,2,3\}.$ Then the lower bound vectors $v(B) \in \mathbb{R}^m$ of these new boxes in $\mathcal{B}_{s+1}$ are determined by

\[
\begin{align*}
    v_i(\hat{B}_i^{l(q)}) &= v_i(\hat{B}_i^{l(q)}) \quad \forall q = 1, \ldots, Q_i \\
    v_j(\hat{B}_i^{l(q)}) &= \begin{cases} 
        u_j(\hat{B}_i^{l_1(1)}) = x_j^s & q = 1 \\
        u_j(\hat{B}_i^{l_1(q-1)}) & \forall q = 2, \ldots, Q_i
    \end{cases} \\
    v_k(\hat{B}_i^{l(q)}) &= \begin{cases} 
        u_k(\hat{B}_k^{l_i(q+1)}) \quad \forall q = 1, \ldots, Q_i - 1 \\
        u_k(\hat{B}_k^{l_i(Q_i)}) = z_k^s & q = Q_i.
    \end{cases}
\end{align*}
\]

All individual subsets $V(B)$ of all $B \notin \mathcal{B}_s$ remain unchanged.

Proof. The update of $v(\hat{B}_i^{l_i(q)})$ of all new boxes $\hat{B}_i^{l_i(q)}, q = 1, \ldots, Q_i,$ for some fixed $i \in \{1,2,3\}$ is derived directly from Corollary 4.6. The individual subsets of all boxes which are not split in the current iteration do not change, as, according to the proof of Lemma 4.5, either $B_i^{s+1}(B)$ remains unchanged (Case (iii)) or $B_i^{s+1}(B) = \hat{B}_i$ (Case (iv)), i.e. $u_i(B)$ remains unchanged.

Recall that we want to split a box $B \in \mathcal{B}_s$ with respect to a component $i \in \{1,2,3\}$ if and only if the individual subset $V(B_i)$ of the resulting box $B_i$ is non-empty, which is equivalent to $B_i$ being non-redundant. With the vector $v(B) \in \mathbb{R}^m$ at hand, this can be easily checked, as the following lemma shows.

Lemma 4.9. Let Assumption 4.4 hold up to iteration $s - 1$ for $s \geq 2,$ i.e. let $\mathcal{B}_s$ be a correct, non-redundant decomposition of the search space obtained by iterative 3-splits. Let $z^* \in N$ satisfy Assumption 4.4 (1), and let $B_i$ be the box obtained from $B \in \mathcal{B}_s$ by a split with respect to component $i \in \{1,2,3\}.$ Then $B_i$ is non-redundant if and only if $z^*_i > v_i(B)$ holds.

Proof. Consider a fixed $i \in \{1,2,3\}$.

“⇒”: Let $B_i$ be non-redundant and assume that $z^*_i < v_i(B)$ holds. (The case $z^*_i = v_i(B)$ does not occur due to Assumption 4.4 (1).) Then $v_i(B) > z^*_i$ and, thus, $v_i(B) = u_i(B_i^*(B))$ with $B_i^*(B) \neq \emptyset.$ As $u_l(B_i^*(B)) \geq u_l(B)$ for all $l \neq i,$ $z^*_i \in B_i^*(B)$ must hold. But then, $B_i$ would be redundant as it would be dominated by the box obtained from $B_i^*(B)$ by a split with respect to $i,$ a contradiction to the assumption of non-redundancy.
⇐: Let \( z^*_s > v_i(B) \). A split of \( B \) with respect to \( i \) yields \( B_i = \{ z \in B : z_i < z^*_s \} \).

Assume that there exists \( \tilde{B}_i \neq B_i \) which dominates \( B_i \). As \( B_s \) is non-redundant and due to Lemma 3.4, \( \tilde{B}_i \) must result from a split with respect to \( i \) from some box \( B \in B_s \), i.e. \( z^* \in \tilde{B} \) must hold. As \( B \) and \( \tilde{B} \) are split with respect to \( i \), \( u_i(B) = u_i(\tilde{B}) \) holds, and, due to the assumption that \( \tilde{B}_i \) dominates \( B_i \), \( u_l(B) \geq u_l(\tilde{B}) \) for all \( l \neq i \). Now \( B, \tilde{B} \in B_s \), \( B \) and \( \tilde{B} \) being non-redundant imply that \( u_i(B) < u_i(\tilde{B}) \). This in turn means that \( v_i(B) \geq u_i(\tilde{B}) \). But then \( z^*_s > u_i(\tilde{B}) \), a contradiction to \( z^*_s \in \tilde{B} \). It follows that \( B_i \) is non-redundant.

Lemma 4.9 provides a tool for defining a split operation for tricriteria problems which generates all boxes that are necessary for maintaining the correctness of a decomposition, but avoids the generation of redundant boxes. We call the split based on the individual subsets \( V(B) \) a \( v \)-split in the following.

Definition 4.10 (\( v \)-split). Let Assumption 4.4 hold up to iteration \( s - 1 \) for \( s \geq 2 \), i.e. let \( B_s \) be a correct, non-redundant decomposition of the search space obtained by iterative 3-splits, and let \( z^*_s \in N \). We call the split of a box \( B \in B_s \) with respect to components \( i \in \{1, 2, 3 \} \), for which

\[
    z^*_s \geq v_i(B)
\]

holds, a \( v \)-split of \( B \).

Note that equality in (20) does not occur due to Assumption 4.4 (1). However, as Assumption 4.4 (1) will be removed in Section 4.3 we present the \( v \)-split already at this point in this general form.

Lemma 4.11. Let Assumption 4.4 (1),(2) hold. Then the iterative application of a \( v \)-split to every \( B \in B_s \) in every iteration \( s \geq 1 \) yields a correct, non-redundant decomposition.

Proof. Due to Assumption 4.4 (1), \( z^*_s \geq v_i(B) \) is equivalent to \( z^*_s > v_i(B) \). According to Lemma 4.9 the \( v \)-split avoids exactly the generation of redundant boxes and, therefore, yields a correct, non-redundant decomposition.

4.1 An algorithm for tricriteria problems based on the \( v \)-split

Algorithm 2 implements the \( v \)-split. As in Algorithm 1, an initial box \( B_0 \) is computed, which is represented by its upper bound \( u(B_0) \). Additionally, for \( B_0 \) as well as for all other boxes \( B \) which are generated in the course of the algorithm, the lower bound of the individual subset \( v(B) \) is saved. Analogously to Algorithm 1, as long as the decomposition contains unexplored boxes, a box is selected and a scalarized subproblem is solved. If the problem is infeasible, then the selected box is deleted from the list of unexplored boxes. Otherwise, the nondominated point \( z^* \) is saved and all boxes are determined that contain \( z^* \). Now, different to Algorithm 1, \( z^* \) is compared component-wise to \( v(B) \) for every \( B \in B_s \). A split with respect to component \( i \) is performed if
and only if $z_i^s \geq v_i(B)$ and $z_i^s > z_i^I$ hold. If $v_i(B) > z_i^s$ for all $i \in \{1, 2, 3\}$, then $B$ is deleted. Finally, the vectors $v$ of all new boxes are updated according to Lemma 4.8 and a new iteration starts.

Note that according to the proof of Lemma 4.5 we can order all newly generated, non-redundant boxes resulting from a split with respect to component $i$ such that their upper bound values $u$ are increasing in one component $j \neq i$ and decreasing in the remaining component $k \neq i, j$. In Algorithm 2 Line 47, the strict inequalities are replaced by inequalities, which leads to an equivalent expression due to Assumption 4.4 (1). In this more general form, the algorithm is also applicable when Assumption 4.4 (1) is relaxed, see Section 4.3 below.

Algorithm 2: Algorithm with $v$-split for $m = 3$

**Input:** Image of the feasible set $Z \subset \mathbb{R}^m$

1: $N := \emptyset; \delta > 0$
2: INITSTARTINGBOXVsplit($Z, \delta$)
3: $s := 1$
4: while $B_s \neq \emptyset$ do
5: Choose $\bar{B} \in B_s$; // Solve scalarized subproblem
6: $z^s := opt(Z, u(\bar{B}))$
7: if $z^s = \emptyset$ then // No nondominated point found
8: $B_{s+1} := B_s \backslash \{\bar{B}\}$
9: else
10: $N := N \cup \{z^s\}$; // Add point to nondominated set
11: $B_{s+1} := B_s$; // Copy set of current boxes
12: GENERATENEWBOXESVsplit($B_s, z^s, z^I, B_{s+1}$)
13: UPDATEINDIVIDUALSUBSETS($S_1, S_2, S_3, B_{s+1}$)
14: end if
15: $s := s + 1$
16: end while
17: return Set of nondominated points $N$

Example 4.12 (Application of Algorithm 2). Consider again the tricriteria problem of Example 3.7 with initial search space

$$B_0 := \{z \in Z : 0 \leq z_i \leq 5 \forall i = 1, 2, 3\}$$

and $V(B_0) = B_0$, thus, $v(B_0) = (0, 0, 0)^\top$. Consider $z^1 = (2, 2, 2)^\top$. The $v$-split applied to the initial box equals a full $3$-split and, thus, results in

$$B_{1,i} := \{z \in B_0 : z_i < 2\}, i = 1, 2, 3.$$ The corresponding individual subsets are

$$V(B_{1,i}) := \{z \in B_{1,i} : z_j \geq 2 \forall j \neq i\}, i = 1, 2, 3,$$
18: procedure InitStartingBoxVsplit($Z, \delta$)
19: \hspace{1em} for $j = 1$ to $3$ do
20: \hspace{2em} $z_j^I := \min\{ z_j : z \in Z \}$
21: \hspace{2em} $z_j^M := \max\{ z_j : z \in Z \} + \delta$
22: \hspace{2em} $v_j(B_0) := z_j^I; u_j(B_0) := z_j^M$
23: \hspace{1em} end for
24: \hspace{1em} $B_1 := \{ B_0 \}$
25: \hspace{1em} return $B_1$
26: end procedure

27: procedure GenerateNewBoxesVsplit($B_s, z_s, z_I^i, B_{s+1}$)
28: \hspace{1em} $S_i := \emptyset \forall i = 1, 2, 3$; \hspace{1em} // Initialize set for each component $i \in \{1, 2, 3\}$
29: \hspace{1em} for all $B \in B_s$ do
30: \hspace{2em} if $z_s^* < u(B)$ then \hspace{1em} // Point is contained in box
31: \hspace{3em} $B_{s+1} := B_s \setminus \{ B \}$ \hspace{1em} // Remove $B$
32: \hspace{3em} for $i = 1$ to 3 do \hspace{1em} // Apply $v$-split
33: \hspace{4em} if $z_s^i \geq v_i(B)$ and $z_s^i > z_I^i$ then \hspace{1em} // Create new box
34: \hspace{5em} $B' := \emptyset$
35: \hspace{5em} $u_i(B') := z_s^i$
36: \hspace{5em} $u_j(B') := u_j(B) \forall j \neq i$
37: \hspace{5em} $S_i := S_i \cup \{ B' \}$ \hspace{1em} // Save new box in respective set $S_i$
38: \hspace{4em} end if
39: \hspace{3em} end for
40: \hspace{1em} end if
41: \hspace{1em} end for
42: \hspace{1em} return $B_{s+1}, S_1, S_2, S_3$
43: end procedure

44: procedure UpdateIndividualSubsets($S_1, S_2, S_3, B_{s+1}$)
45: \hspace{1em} for $i = 1$ to 3 do
46: \hspace{2em} $Q := |S_i|$
47: \hspace{2em} Sort all boxes $B_i^{I_q}(q), q = 1, \ldots, Q_i$, in $S_i$ such that for $j, k \neq i$
48: \hspace{2em} $u_j(B_i^{I_q}(1)) \leq u_j(B_i^{I_q}(2)) \leq \cdots \leq u_j(B_i^{I_q}(Q_i));$
49: \hspace{2em} and $u_k(B_i^{I_q}(1)) \geq u_k(B_i^{I_q}(2)) \geq \cdots \geq u_k(B_i^{I_q}(Q_i));$
50: \hspace{2em} Set $v_j(B_i^{I_q}(q)) := z_s^j; v_k(B_i^{I_q}(q)) := z_s^k$ \hspace{1em} // Update $v$
51: \hspace{2em} for $q = 2$ to $Q_i$ do
52: \hspace{3em} $v_j(B_i^{I_q}(q)) := u_j(B_i^{I_q}(q-1)); v_k(B_i^{I_q}(q-1)) := u_k(B_i^{I_q}(q));$
53: \hspace{2em} end for
54: \hspace{2em} $B_{s+1} := B_{s+1} \cup S_i$ \hspace{1em} // Append new boxes
55: \hspace{1em} end for
56: \hspace{1em} return $B_{s+1}$
57: end procedure
thus, $v(B_{11}) = (0, 2, 2)^\top$, $v(B_{12}) = (2, 0, 2)^\top$ and $v(B_{13}) = (2, 2, 0)^\top$. Let $z^2 = (1, 1, 4)^\top$. It holds that $z^2 \in B_{11}$ as well as $z^2 \notin B_{12}$, but $z^2 \notin B_{13}$. Consider first the $v$-split in $B_{11}$: As $z^2_1 \geq v_1(B_{11})$, $z^2_2 \notin v_2(B_{11})$ and $z^2_3 \geq v_3(B_{11})$, $B_{11}$ is split with respect to the first and third component into $$B_{21} := \{ z \in B_{11} : z_1 < 1 \} \quad \text{and} \quad B_{23} := \{ z \in B_{11} : z_3 < 4 \}$$ and $S_1 = \{ B_{21} \}$, $S_2 = \emptyset$ and $S_3 = \{ B_{23} \}$. Applying the $v$-split to $B_{12}$ results in a split with respect to the second and third component into $$B_{22} := \{ z \in B_{12} : z_2 < 1 \} \quad \text{and} \quad B'_{23} := \{ z \in B_{12} : z_3 < 4 \}$$ and $S_1 = \{ B_{21} \}$, $S_2 = \{ B_{22} \}$ and $S_3 = \{ B_{23}, B'_{23} \}$. Note that the redundant boxes which were obtained with the full 3-split in Example 3.4 are not generated by the $v$-split.

Finally, the individual subsets of the new boxes of each set $S_i$, $i \in \{ 1, 2, 3 \}$, are updated: Box $B_{21}$ is the only box generated for $i = 1$, box $B_{22}$ the only one for $i = 2$. Therefore, $v(B_{21}) = (v_1(B_{11}), z^2_1, z^2_3)^\top = (0, 1, 4)^\top$ and $v(B_{22}) = (z^2_1, v_2(B_{12}), z^2_3)^\top = (1, 0, 4)^\top$. Boxes $B_{23}$ and $B'_{23}$ are both generated by a split with respect to the third component $i = 3$. We can order the upper bounds of the boxes $u(B_{23}) = (2, 5, 4)^\top$ and $u(B'_{23}) = (5, 2, 4)^\top$ increasingly with respect to component $j = 1$ and, at the same time, decreasingly with respect to $k = 2$, thus, $B^{(1)}_3 := B_{23}$ and $B^{(2)}_3 := B'_{23}$ and then set

$$v_1(B_{23}) = z^2_1 = 1, \quad v_3(B_{23}) = z^2_3 = 1,$$

$$v_2(B_{23}) = v_2(B'_{23}) = 2, \quad v_1(B'_{23}) = u_1(B_{23}) = 2.$$

The third component is not changed, so $v(B_{23}) = (1, 2, 2)^\top$ and $v(B'_{23}) = (2, 1, 2)^\top$.

Note that, according to Lemma 4.11, Algorithm 2 maintains a correct, non-redundant decomposition at each iteration. In this sense, the non-redundant representation of the search space based on the upper bounds $u(B)$ is equivalent to the construction presented in Przybylski et al. (2010). While the algorithm of Przybylski et al. (2010) requires the deletion of ‘nondominated’ upper bounds in each iteration, this is not needed in Algorithm 2 since the $v$-split explicitly constructs only ‘nondominated’ upper bounds in each iteration.

### 4.2 A linear bound on the number of subproblems for $m = 3$

In the following, we will bound the number of boxes generated in the course of the algorithm with the help of the $v$-split. If a box $B \in \mathcal{B}_s$ contains the current point $z^*$, i.e. if $B \in \mathcal{B}_s$, then we can make the following assertion concerning the neighbours of $B$:

**Lemma 4.13.** Let Assumption 4.4 hold. Consider any $B \in \mathcal{B}_s$. We denote by $J_B \subseteq \{1, 2, 3\}$ the index set of all components with respect to which $B$ is split, and by $\bar{J}_B := \{1, 2, 3\} \setminus J_B$. Then the following holds:

1. If $\bar{J}_B \neq \emptyset$, then for every $j \in \bar{J}_B$, the neighbour $B^*_j(B)$ exists and contains $z^*$, i.e. $B^*_j(B) \neq \emptyset$ and $B^*_j(B) \in \mathcal{B}_s$ holds for every $j \in \bar{J}_B$.

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2. If $\bar{J}_B = \emptyset$, then $\overline{B}_s = \{B\}$ holds.

Proof. Let $B \in \overline{B}_s$. By definition of the $v$-split, it holds that $z^s < u(B), z^s_j \geq v_j(B)$ for every $j \in J_B$ and $z^s_j < v_j(B)$ for every $j \in J_B$. Thus, $v_j(B) > z^s_j$ holds for every $j \in \bar{J}_B$. This, however, implies that $B^\ast_j(B) \neq \emptyset$ for every $j \in \bar{J}_B$, see the update of $v$ in (19).

First, let $\bar{J}_B \neq \emptyset$. Then, for fixed $j \in \bar{J}_B$ and according to Definition 4.2, $u_j(B^\ast_j(B)) \leq u_j(B)$ and $u_l(B^\ast_j(B)) \geq u_l(B)$ for all $l \neq j$. As $u_j(B^\ast_j(B)) = v_j(B) > z^s_j$, $B^\ast_j(B) \in \overline{B}_s$ holds.

Now, consider the case $\bar{J}_B = \emptyset$. Then, due to Lemma 4.5, every box $\tilde{B} \in B_s \setminus \{B\}$ has upper bound $u(B) \leq u_l(B)$ for at least one $l \in \{1, 2, 3\}$. This implies that $z^s \notin \tilde{B}$ for any $B \in B_s \setminus \{B\}$, thus, $\overline{B}_s = \{B\}$.

With the help of Lemma 4.13, we can bound the number of new boxes that are generated in each iteration of Algorithm 2.

Lemma 4.14. Let Assumption 4.4 hold. Then, in every iteration $s \geq 1$ in which a new nondominated point $z^s$ is found, the number of boxes in the decomposition increases by at most two.

Proof. If there exists a box $B \in B_s$ which is split with respect to all three components, then, using Lemma 4.13, $\overline{B}_s = \{B\}$ holds, thus, $|\overline{B}_s| = 1$. In this case, the box $B$ is removed and replaced by three new boxes in the decomposition, and, thus, the number of boxes in the decomposition increases by two.

It follows that if $|\overline{B}_s| > 1$, then every $B \in \overline{B}_s$ is split with respect to at most two components. Let $|\overline{B}_s| > 1$ and let $B \in \overline{B}_s$ be split with respect to two components $i, j \in \{1, 2, 3\}, j \neq i$. Then, for all other boxes $B \in B_s \setminus \{B\}$ it holds that $u_l(B) \leq v_l(B)$ for some $l \in \{1, 2, 3\}$. If $l = i$, then $u_i(B) \leq v_i(B) \leq z^s_i$, thus the box is not split with respect to $i$. Analogously, if $l = j$, then $u_j(B) \leq v_j(B) \leq z^s_j$, thus the box is not split with respect to $j$. If $l = k$ (with $k \neq i, j$), then, for any $B$ satisfying $u_k(B) \leq v_k(B)$ it holds that $v_i(B) \geq u_i(B)$ or $v_j(B) \geq u_j(B)$, thus, $B$ can not be split with respect to both components $i$ and $j$.

Therefore, if two boxes are split with respect to two components, then these components must differ in one component. This implies that in one iteration, at most three boxes are split with respect to two components. Any other boxes in $B_s$ are split with respect to at most one component.

In the case that three boxes are split with respect to two components, six new boxes would replace three old ones, thus, the number of boxes would increase by three. So it remains to show that in this case, at least one box $B \in \overline{B}_s$ is removed without being split, i.e. $v(B) > z^s$ holds for at least one $B \in \overline{B}_s$. In other words, we have to prove the existence of a '0-box', i.e. a box, which is contained in $\overline{B}_s$, but is not split with respect to any component.

To this end, we assume to the contrary that $\overline{B}_s$ contains three boxes which are split with respect to two components ('2-boxes'), respectively, but that no '0-box' exists. From Lemma 4.13 we see that a '2-box' has exactly one neighbour in $\overline{B}_s$, as $J_B$ contains exactly one index. A '1-box' has exactly two neighbours in $\overline{B}_s$, while all
three neighbours are contained in $\mathcal{B}_s$ in case of a '0-box'. Now, starting from a '2-box', one uniquely defined neighbour of it must be in $\mathcal{B}_s$. If that box is also a '2-box' (see Figure 4 on the left), then no neighbour of the latter box is in $\mathcal{B}_s$. The third '2-box' would require a neighbour in $\mathcal{B}_s$, but only '1-boxes' are available, which require a second neighbour in turn. Thus, a fourth '2-box' would be needed, which, however, does not exist. Therefore, the three '2-boxes' must all be connected by one structure of neighbours. But this implies that there exists exactly one '0-box' connecting the three branches emerging from each '2-box' (see Figure 4 on the right).

**Theorem 4.15.** For a finite set of nondominated points and a given appropriate starting box which includes all nondominated points and has the ideal point as lower bound, Algorithm 2 requires the solution of at most $3|N| - 2$ subproblems in order to generate the entire nondominated set.

**Proof.** In every iteration of Algorithm 2, one scalarized subproblem is solved, and, thus, the number of subproblems to be solved equals the number of iterations. When a nondominated point is generated, then the number of boxes increases by at most two according to Lemma 4.14. As every nondominated point is generated exactly once, and since every empty box is investigated exactly once to verify that no further nondominated points are contained, at most $3|N|$ boxes are explored in the course of the algorithm. Together with the initial search box, at most $3|N| + 1$ boxes are explored, which corresponds to the number of subproblems to be solved for a given appropriate initial search box containing all nondominated points.

As we additionally assume that the ideal point is given, we can reduce this bound further: in every iteration in which the current nondominated point equals the ideal point in at least one component, one box per component equal to the ideal point can be directly discarded. For each component $i \in \{1, 2, 3\}$, there must exist at least one nondominated point whose $i$-th component equals $z_i^l$. Therefore, the total number of subproblems to be solved is $3|N| - 2$. 

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Figure 4: Possible neighbourhood structures of boxes in $\mathcal{B}_s$. Left figure: $\mathcal{B}_s$ contains two '2-boxes'; right figure: $\mathcal{B}_s$ contains three '2-boxes'.
4.3 Applying the $v$-split to arbitrary nondominated sets for tricriteria problems

For the construction of the $v$-split we assumed that no pair of nondominated points has an identical value in at least one component, i.e. that all values are pairwise different (Assumption 4.4 (1)). Under this assumption, the individual subsets of all (non-redundant) boxes are boxes themselves, which is the basis for the $v$-split criterion. In practice, Assumption 4.4 (1) may be violated as arbitrary nondominated points may coincide in up to $m - 2$ components, i.e. in one component for $m = 3$. In this case, additional redundant boxes may occur as the following example shows.

**Example 4.16.** Let $z^1 = (3, 1, 4)\top$, $z^2 = (3, 2, 1)\top$ and let the initial search space be given as $B_0 := \{z \in Z: 0 \leq z_i \leq 5 \forall i = 1, 2, 3\}$. If we insert $z^1$ into $B_0$, we obtain the three subboxes $B_{1,i} := \{z \in B_0 : z_i < z^1_i\}$, $i = 1, 2, 3$, with respective upper bounds

$$u(B_{11}) = (3, 5, 5)\top, u(B_{12}) = (5, 1, 5)\top, u(B_{13}) = (5, 5, 4)\top.$$

The second point $z^2 = (3, 2, 1)\top$ is only contained in $B_{13}$. Thus, $B_{13}$ is replaced by the three subboxes $B_{2,i} := \{z \in B_{13} : z_i < z^2_i\}$, $i = 1, 2, 3$, with respective upper bounds

$$u(B_{21}) = (3, 5, 4)\top, u(B_{22}) = (5, 2, 4)\top, u(B_{23}) = (5, 5, 1)\top.$$

It holds that $B_{23} \subseteq B_{11}$.

Note that under Assumption 4.4 (1) no redundancy appears if $|\mathcal{B}| = 1$, which is, as shown in the example above, no longer true for arbitrary nondominated points. If the redundant box $B_{21}$ is removed from the decomposition, i.e. if we set $B_3 := \{B_{11}, B_{12}, B_{22}, B_{23}\}$, then, however, the individual subset $V(B_{11})$ does not have the structure of a box anymore, as

$$V(B_{11}) := B_{11} \setminus \bigcup_{\tilde{B} \in B_3 \setminus \{B_{11}\}} \tilde{B} \cap B_{11}$$

$$= \{z \in B_{11} : z \geq (0, 2, 1)\top\} \cup \{z \in B_{11} : z \geq (0, 1, 4)\top\}.$$

However, the box format of the individual subsets can be preserved if we maintain the redundant box $B_{21}$ in the decomposition. Then, $V(B_{11}) = \{z \in B_{11} : z \geq v(B_{11})\}$ with $v(B_{11}) = (0, 1, 4)\top$ holds. Also the individual subset of $B_{21}$ has box format with $v(B_{21}) = (3, 2, 1)\top$. However, as $B_{21} \subseteq B_{11}$, it holds that $V(B_{21}) = \emptyset$.

In order to distinguish these redundant boxes that appear in the case that a point equals a previously generated point in one component from the actual redundant boxes, we call the former boxes *quasi non-redundant* boxes. Evidently, Lemma 4.5 does not hold if quasi non-redundant boxes are part of the decomposition. However, the neighbourhood structure which was obtained under Assumption 4.4 (1) can be preserved even in the presence of quasi non-redundant boxes if we adapt the definition of $B_i^s(B)$ appropriately in this case. Then, the boxes $B_i^s(B)$ can be set as derived in Corollary 4.6. This in turn means that the $v$-split as well as Algorithm 2 do not need to
be changed, but can be applied also when Assumption 4.4 (1) is removed. Thus, Theorem 4.15 which shows that the number of subproblems is bounded by $3|N| - 2$ holds independently from Assumption 4.4 (1). Finally, we revisit the previous example to illustrate how Algorithm 2 is applied in the presence of quasi non-redundant boxes.

**Example 4.17.** Let again $z^1 = (3, 1, 4)^\top$, $z^2 = (3, 2, 1)^\top$ and let the initial search space be given as $B_0 := \{ z \in Z : 0 \leq z_i \leq 5 \ \forall \ i = 1, 2, 3 \}$. In the first iteration, $B_0$ is replaced by the three subboxes

$u(B_{11}) = (3, 5, 5)^\top, u(B_{12}) = (5, 1, 5)^\top, u(B_{13}) = (5, 5, 4)^\top$

with respective individual subsets defined by

$v(B_{11}) = (0, 1, 4)^\top, v(B_{12}) = (3, 0, 4)^\top, v(B_{13}) = (3, 1, 0)^\top$.

The second point $z^2 = (3, 2, 1)^\top$ is only contained in $B_{13}$. Thus, $B_{13}$ is replaced by the three subboxes

$u(B_{21}) = (3, 5, 4)^\top, u(B_{22}) = (5, 2, 4)^\top, u(B_{23}) = (5, 5, 1)^\top$

with respective individual subsets defined by

$v(B_{21}) = (3, 2, 1)^\top, v(B_{22}) = (3, 1, 1)^\top, v(B_{23}) = (3, 2, 0)^\top$.

The individual subsets of all non-redundant boxes of the decomposition are depicted in Figure 5. The (empty) individual subset of the quasi non-redundant box $B_{21}$ is illustrated by the set

$\{ z \in B_0 : v(B_{21}) \leq z \leq u(B_{21}) \}$,
Let now a third nondominated point \( z^3 = (2, 3, 3)^T \) be given. As \( z^3 \) is contained in \( B_{11} \) and \( B_{21} \), we consider \( v(B_{11}) \) and \( v(B_{21}) \): Comparing \( z^3 \) with these two vectors reveals that \( B_{11} \) is split with respect to the first and the second component, and \( B_{21} \) is split with respect to the second and the third component, which yields

\[
u(B_{31}) = (2, 5, 5)^T, \quad u(B_{32}) = (3, 3, 5)^T, \quad u(B'_{32}) = (3, 3, 4)^T, \quad u(B_{33}) = (3, 5, 3)^T.
\]

For the update of \( v(B_{32}) \) and \( v(B_{32}') \), the respective upper bound vectors need to be ordered increasingly with respect to one component \( j \neq 2 \) and decreasingly with respect to the remaining component \( k \neq j, k \neq 2 \). As \( B'_{32} \) is quasi non-redundant, we have no strict inequality in both components, but can, nevertheless, order the boxes as required, e.g., decreasingly with respect to \( k = 3 \). Therefore,

\[
v(B_{31}) = (0, 3, 3)^T, \quad v(B'_{32}) = (2, 1, 4)^T, \quad v(B'_{33}) = (3, 2, 3)^T, \quad v(B_{33}) = (2, 3, 1)^T
\]

holds. Figure 6 shows the respective sets \( V(B) \).

### 4.4 Numerical results

We implemented Algorithm 2 in MATLAB. A series of test problems with discrete point sets was conducted to verify our theoretical findings, i.e. to show that the algorithm is correct as well as that the bound \( 3|N| - 2 \) on the number of subproblems to be solved is valid. For all test problems, all nondominated points were generated reliably. Furthermore, the number of subproblems to be solved was exactly \( 3|N| - 2 \), i.e. the upper bound turned out to be sharp in all test cases.
Figure 7: Individual subsets $V(B)$ of the final decomposition of an example with 21 nondominated points

Figure 7 illustrates an example with 21 nondominated points from the ideal point as a viewpoint. After having determined the initial search box, 61 subproblems were solved until the termination criterion of Algorithm 2 was reached.

5 Conclusion

In this paper, we positively answer the question if there exists an algorithm which generates the entire nondominated set of a problem with three objectives by solving a number of subproblems, which depends linearly on the number of nondominated points. This is achieved by avoiding the generation of redundant boxes and by using neighbourhood properties between the boxes. Further research should analyze if and how the concept of individual subsets can be transferred to problems with more than three criteria. Furthermore, the presented algorithm can be improved further by using the neighbourhood properties also for identifying all boxes containing a current point. Thereby, no exhaustive search is needed in each iteration and the boxes can be updated more efficiently. If only a representative subset of the nondominated set shall be generated, the presented algorithm can be easily modified such that a specific box is selected in each iteration. For example, one could always choose the box with the largest volume. Then, however, the choice of the initial search box is crucial. One should keep in mind that if the true nadir point is not known, a simple upper bound on the feasible set might cause problems in this respect.
References

Y. P. Aneja and K. P. K. Nair. Bicriteria Transportation Problem. *Management Science*, 25:73–78, 1979.

K. Dächert, J. Gorski, and K. Klamroth. An augmented weighted tchebycheff method with adaptively chosen parameters for discrete bicriteria optimization problems. *Computers and Operations Research*, 39(12):2929 – 2943, 2012.

C. Dhaenens, J. Lemesre, and E.-G. Talbi. K-PPM: A new exact method to solve multi-objective combinatorial optimization problems. *European Journal of Operational Research*, 200:45–53, 2010.

M. Ehrgott. *Multicriteria Optimization*. Springer Verlag, Berlin, Heidelberg, 2005.

G. Eichfelder. An Adaptive Scalarization Method in Multi-Objective Optimization. *SIAM Journal on Optimization*, 19(4):1694–1718, 2009.

S. L. Faulkenberg and M. M. Wiecek. Generating equidistant representations in biobjective programming. *Computational Optimization and Applications*, 51(3):1173–1210, 2012.

H. W. Hamacher, C. R. Pedersen, and S. Ruzika. Finding representative systems for discrete bicriteria optimization problems by box algorithms. *Operations Research Letters*, 35:336–344, 2007.

K. Klamroth, J. Tind, and M. M. Wiecek. Unbiased approximation in multicriteria optimization. *Mathematical Methods of Operations Research*, 56:413–437, 2002.

M. Laumanns, L. Thiele, and E. Zitzler. An efficient, adaptive parameter variation scheme for metaheuristics based on the epsilon-constraint method. *European Journal of Operational Research*, 169:932–942, 2006.

B. Lokman and M. Köksalan. Finding all nondominated points of multi-objective integer programs. *Journal of Global Optimization*, 2012. doi: 10.1007/s10898-012-9955-7.

K. Miettinen. *Nonlinear Multiobjective Optimization*. Kluwer Academic Publishers, Boston, 1999.

M. Özlön and M. Azizoğlu. Multi-objective integer programming: A general approach for generating all non-dominated solutions. *European Journal of Operations Research*, 199:25–35, 2009.

Ö. Özpeynirci and M. Köksalan. Pyramidal tours and multiple objectives. *Journal of Global Optimization*, 48:569–582, 2010.

A. Przybylski, X. Gandibleux, and M. Ehrgott. A two phase method for multi-objective integer programming and its application to the assignment problem with three objectives. *Discrete Optimization*, 7:149–165, 2010.
T. Ralphs, M. Saltzman, and M. M. Wiecek. An improved algorithm for solving biobjective integer programs. *Annals of Operations Research*, 147:43–70, 2006.

S. Sayin and P. Kouvelis. The multiobjective discrete optimization problem: A weighted min-max two-stage optimization approach and a bicriteria algorithm. *Management Science*, 51(10):1572–1581, 2005.