Note on the classical solutions of Friedmann’s equation

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Graphical representations of classical Friedmann’s models are often misleading when one considers the age of the universe. Most textbooks disregard conceptua l differences in the representations, as far as ages are concerned. We discuss the details of the scale-factor versus time function for Friedmann’s solutions in the time range that includes the ages of model universes.

I. INTRODUCTION

Modern Big Bang cosmological models are described by modified Friedmann’s models, with the inclusion of a cosmological constant (e.g., [1], p. 403). Hence there is always renewed interest in all aspects of Friedmann’s models.

Classical Friedmann’s models are fully described by the expansion rate of the scale factor \( R(t) \), \( H_0=\frac{R'}{R} \) — the so-called Hubble parameter, evaluated now \((t=t_o, \text{the age of a given model})\) —, and the density parameter \( \Omega_0 = \frac{\rho_0}{\rho_{cr}} \), where \( \rho_0 \) is the model mass density and \( \rho_{cr} = \frac{3H_0^2}{8\pi G} \) is the critical density — the density of the flat model —, both on \( t=t_o \).

The classical Friedmann’s equation, i.e., without the cosmological constant, is written as (see, e.g., [2], chap. 27 and [3], chap. 2):

\[
\left(\frac{dR}{dt}\right)^2 - \frac{H_0^2\Omega_0}{R} = -H_0^2(\Omega_0 - 1),
\]

where \( R(t_o) \) is, conventionally, set to unity. The solution for the flat model is readily obtained inserting \( \Omega_0 = 1 \) in eq. 1:

\[
R(t) = \left(\frac{t}{t_o}\right)^{2/3},
\]

with \( t_o=2/(3H_0) \) being the flat model’s age. For the closed model \((\Omega_0 > 1)\) the solution is expressed in the parametric form (see [2], eqs. 27.24 and 27.26):

\[
R(x) = \frac{1}{2\Omega_0} \left[ \cosh(x) - 1 \right],
\]

\[
t(x) = \frac{1}{2H_0(\Omega_0 - 1)^{3/2}} \left[ x - \sinh(x) \right],
\]

where the parameter is \( x \geq 0 \). Likewise, the solution for the open model \((\Omega_0 < 1)\) is given by:

\[
R(x) = \frac{1}{2} \frac{\Omega_0}{1 - \Omega_0} \left[ \cosh(x) - 1 \right],
\]

\[
t(x) = \frac{1}{2H_0(1 - \Omega_0)^{3/2}} \left[ \sinh(x) - x \right].
\]

These solutions for the open \((\Omega_0=0.5)\), flat or critical \((\Omega_0=1)\) and closed \((\Omega_0=2)\) models are depicted in Fig. 1.

It is worthwhile stressing that the subsequent discussion does not depend on the \( \Omega_0 \) values chosen for the close and open models.

Most textbooks, or qualitative papers on modern cosmology, present diagrams like this one, when describing
Friedmann’s models. With rare exceptions, they differ from our Fig. 1 in two main features. First, there is no quantitative axes, which are generically labeled “scale factor” and “time”. Second, the fine features at the small R(t) range are not considered, let alone plotted (see R(t)<2 in Fig. 1). Such differences lead to a wrong conceptual apprehension of the models, concerning their ages, i.e., the times corresponding to R=1.

In the next section, we show why it is important to present a precise graphical representation of Friedmann’s models, in the light of the models’ ages. The last section concludes with a quantitative account on the relative differences between classical Friedmann’s models as functions of cosmic time.

II. AGES OF FRIEDMANN’S UNIVERSES

The ages of Friedmann’s universes are obtained by making R = 1 in eqs. 2, 3 and 5. Then, with the aid of eqs. 4 and 6, one gets the ages of the open, flat and closed models as functions of the density parameter $\Omega_0$. The result is plotted in Fig. 2. The age of the open model is the greatest amongst Friedmann’s classical solutions. This is not clear when one examines Fig. 1 because the relevant range of the scale factor R(t)≈ 1 is not immediately apparent in the figure. This sort of graphical representation is predominant in most textbooks. For example, it is seen in Harrison [1], Fig. 18.6, p. 360, Carroll & Ostlie [2], Fig. 27.4, p. 1230, Rindler [1], Fig. 18.2, p. 402, Shu [3], Fig. 15.7, p. 362 and Box 15.4, p. 368. Harrison discusses ages after presenting Fig. 18.6. He calls the reader’s attention to their differences, but the conclusion remains conceptually inconsistent with the diagram shown in Fig. 18.6. In Carroll & Ostlie there is a hint of fine features in the small R(t) range. In fact, they comply with both desired features mentioned in the introductory section, but do not discuss the age-related issue. Nevertheless, they constitute a rare exception, in the cosmological literature, when plotting solutions of Friedmann’s equations.

The age of the universe is assigned to R(t=)$t_0$=1. In Fig. 3 we plot R(t) in the range appropriate to show the ages of the models. Contrary to the representation in Fig. 1 it is now clear that the closed model has the largest R(t), and the smallest age. Around t=2, the closed model performs two crossovers, first with the flat model and then with the open model. Notice also a third crossover, shortly later, between the open and flat models. Following the crossovers, the behavior of R(t) is the usual one, as shown in Fig. 1.

If one does not carefully examine the numerical scale of the vertical axis in Fig. 1 conclusions about the ages of the models are confusing. Things may get worse because, usually, as mentioned above, the plot is shown without a numerical scale in both axes.

III. CLOSED AND OPEN MODELS VERSUS THE FLAT MODEL

In the very small R(t), the term on the right side of Friedmann’s equation becomes negligible, when compared to the second term on the left side. This holds in
FIG. 4: The relative differences between the closed and open models and the flat model as a function of time. The differences at \( t \to 0 \) are easily calculated from eq. 7. Note the crossover times with the flat model, marked by the horizontal dotted line. The vertical dotted line points to the age of the flat model (\( t=1 \)).

the range \( R(t) \ll \Omega_0 / (|\Omega_0 - 1|) \), i.e., \( R(t) \ll 1 \), for the models studied here. The approximate solution is given by

\[
R(t) \equiv \Omega_0^{1/3} \left( \frac{t}{t_\circ} \right)^{2/3} = \Omega_0^{1/3} R_{\text{flat}}(t), \tag{7}
\]

where \( t_\circ = 2/(3H_0) \) is the flat model's age. Such an approximation confirms the fact that, for \( t \to 0 \), \( R(t) \) is larger for the closed model, as plotted in Fig. 4.

The relative differences among the models may be investigated by the percentage function \( f = 100 \times (R - R_{\text{flat}}) / R_{\text{flat}} \). With the help of eq. 4 one gets \( f = 100 \times (\Omega_0^{1/3} - 1) \), which means \( f = +26\% \), for the closed model, and \( f = -21\% \), for the open model, at \( t \approx 0 \). This clearly shows that the models are quite different early on in the cosmic history, being the relative differences larger than around \( t=t_\circ (|f| \approx 9\% \), for the closed and open models).

Fig. 4 shows the exact function \( f(t) \). It is worthwhile noticing the location of the crossover times mentioned in section II, and that they occur after the ages of all models.

FIG. 5: Open (\( \Omega_0=0.5 \)), flat and closed (\( \Omega_0=2 \)) Friedmann’s models have the same slope — or Hubble’s constant — at \( R = 1 \) (now). Different ages \( t_\circ \) appear along the time axis: \( t_\circ = 1.0^{+0.13}_{-0.14} \) for the closed (−) and open (+) models (see also Fig. 2).

IV. CONCLUSIONS

We have shown that the ages of Friedmann’s classical universes are better appreciated when the fine features of the scale-factor function \( R(t) \), present in the range of times from \( t = 0 \) to \( t \approx 3 \times 2/(3H_0) \), are represented in detail (see Fig. 3). This is an alternative way to that adopted by Harrison ([4], Fig. 18.7, p. 360) and Linder ([7], Fig. 2.3, p. 32). These authors choose to stress the fact that all models have the same Hubble’s parameter now (\( R = 1 \)), as remarked in the legend of our Fig. 3. Those two figures are equivalent to our Fig. 3 just by sliding the closed model curve forwards and the open model backwards, along the time axis, until they touch the flat model curve at \( t = 1 \) (see Fig. 5).

We calculate the relative differences between the closed and open models and the flat model, as a function of time (Fig. 4). The models are quantitatively different right from the beginning, pass through crossovers around \( t \approx 2 \times 2/(3H_0) \) before diverging for \( t \gg 2/(3H_0) \).

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