Euclidean Jordan Algebras and Inequalities over the Spectrum of a Strongly Regular Graph

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Abstract.
Let $G$ be a primitive strongly regular graph of order $n$ with three distinct eigenvalues and $A$ its adjacency matrix. In this paper we associate to $A$ the 3-dimensional real Euclidean Jordan algebra $\mathcal{A}$ spanned by $I_n$ and the natural powers of $A$ equipped with the Jordan product of matrices and with inner product of two matrices being the trace of their Jordan product and next, by the spectral analysis of some elements of $\mathcal{A}$ we establish some inequalities over the spectrum and the parameters of a strongly regular graph.

1. Introduction
Several Scientists had recurred to Euclidean Jordan algebras to develop theory on various branches of Mathematics, namely on the generalization of the properties of symmetric matrices to simple Euclidean Jordan algebras, [1–3], on the establishment of the formalism of quantum mechanics [4], on the deduction of inequalities over the spectra of strongly regular graphs in the environment of Euclidean Jordan algebras, [5–9], on the establishment of theory for interior-point methods [10–12] and on applications to statistics, [13–15].

In this work we establish some conditions over the parameters of a strongly regular graph recurring to their Generalized Krein parameters as they had being defined in the publication [9] in the environment of Euclidean Jordan algebras and finally we present the main result of this paper in the establishment of an admissible condition for the parameters of a strongly regular graph in the Theorem 4.2.

We assume that the reader of this paper is familiar with the preliminar results on Euclidean Jordan algebras. Herein we we reproduce the principal results on Euclidean Jordan algebras. Most of these can be found in [16].

This paper is organized as follows. In section 2 we expose the principal results on Euclidean Jordan algebras. Next in section 3 we present the principal concepts on strongly regular graphs necessary for a clear exposition of this paper. Finally, in section 4 we associate a three dimensional real Euclidean Jordan algebra $\mathcal{A}$ to a strongly regular graph and next we establish some conditions on the generalized Krein parameters of a strongly regular graph, see the inequalities (17) of Theorem 4.1 and (21) of Corollary 4.1, and finally we establish the inequalities (32) and (33) of Theorems 4.2 and 4.3.
2. Main Results of Euclidean Jordan Algebras

In this section we present the more relevant concepts and results about Jordan algebras and in particular about Euclidean Jordan algebras, which can be seen, for instance in [17] and in [16]. For a very readable text about Euclidean Jordan algebras one should read the chapter "an introduction to formally real Jordan algebras and their applications in optimization" in the review edited by Jean B. Lasserre and Michael F. Anjos, see [18].

Let’s consider a natural number $n$ and the field $\mathbb{K}$, and the $n$ dimensional algebra $\mathcal{A}$ over the field $\mathbb{K}$ equipped with the bilinear mapping $(u,v) \mapsto u \ast v$ from $\mathcal{A} \times \mathcal{A}$ to $\mathcal{A}$. If $\mathcal{A}$ is an algebra such that for each $u$ in $\mathcal{A}$ the algebra spanned by $u$ is associative then we say that algebra $\mathcal{A}$ is power associative. If $\mathcal{A}$ contains an element $e$, such that for all $u$ in $\mathcal{A}$, $e \ast u = u \ast e = u$, then $e$ is called the unit element of $\mathcal{A}$. If for all $u$ and $v$ in $\mathcal{A}$ we have $(J_1)\ u \ast v = v \ast u$ and $(J_2)\ u \ast (u^{2*} \ast v) = u^{2*} \circ (u \ast v)$, with $u^{2*} = u \ast u$, then $\mathcal{A}$ is called a Jordan algebra. If $\mathcal{A}$ is a Jordan algebra with unit element then $\mathcal{A}$ is a power associative algebra (cf. [16]).

From now on we suppose that the field $\mathbb{K}$ is the field of the reals and we suppose also that the unit element of a Jordan $\mathcal{A}$ is denoted by $e$. So, from now on, we only consider real finite dimensional Jordan algebras whose unit element is denoted by $e$.

Given a Jordan algebra $\mathcal{A}$ with unit element $e$, if there is an inner product $\langle \cdot , \cdot \rangle$ that verifies the equality $\langle u \ast v, w \rangle = \langle v, u \ast w \rangle$, for any $u, v, w$ in $\mathcal{A}$, then $\mathcal{A}$ is called an Euclidean Jordan algebra.

Let’s consider a real finite dimensional Euclidean Jordan algebra $\mathcal{A}$ whose unit element is denoted by $e$. An element $u$ of $\mathcal{A}$, is an idempotent if $u^{2*} = u$. Two idempotents $u$ and $v$ are orthogonal if $u \ast v = 0$ where 0 is the zero vector of $\mathcal{A}$. Let $k$ be a natural number. We call the set $\{u_1, u_2, \ldots, u_k\}$ of nonzero idempotents of $\mathcal{A}$ a complete system of orthogonal idempotents if (i) $u_i^{2*} = u_i$, $\forall i \in \{1, \ldots, k\}$; (ii) $u_i \ast u_j = 0$, $\forall i \neq j$ and (iii) $u_1 + u_2 + \ldots + u_k = e$. The rank of an element $u$ in $\mathcal{A}$ is the least natural number $l$, such that the set $\{e, u^{1*}, u^{2*}, \ldots, u^{l*}\}$ is linearly dependent (where $u^{2*} = u \ast u$ and $u^{l*} = u \ast u^{(l-1)*}$), and we write rank($u$) = $l$. We define the rank of the algebra $\mathcal{A}$ as the natural number rank($\mathcal{A}$) = max{rank($u$) : $u$ $\in$ $\mathcal{A}$}. Herein we must say, that the rank of the Euclidean Jordan algebra $\mathcal{A}$ is well defined since we are considering that the Euclidean Jordan algebra is a real finite dimensional Euclidean Jordan algebra. The regular elements of $\mathcal{A}$ are the elements of $\mathcal{A}$ with rank equal to the rank of $\mathcal{A}$. This set of the regular elements is an open and dense subset in $\mathcal{A}$. Let’s suppose that rank($\mathcal{A}$) = $r$. If $v$ is a regular element of $\mathcal{A}$, then $r$ = rank($v$), and therefore the set $\{e, v^{1*}, v^{2*}, \ldots, v^{r*}\}$ is a linearly dependent set and the set $\{e, v^{1*}, v^{2*}, \ldots, v^{(r-1)*}\}$ is linearly independent. Thus we may conclude that there exist unique real numbers $\gamma_1(v), \ldots, \gamma_r(v)$, such that $v^{r*} - \gamma_1(v)v^{(r-1)*} + \ldots + (-1)^{r-1}\gamma_{r-1}(v)e = 0$, where 0 is the zero vector of $\mathcal{A}$. Making the necessary adjustments we obtain the polynomial $p$ in $\lambda$

$$p(v, \lambda) = \lambda^r - \gamma_1(v)\lambda^{r-1} + \ldots + (-1)^r\gamma_r(v),$$

(1)

that is called the characteristic polynomial of $v$, where each coefficient $a_i$ is a homogeneous polynomial of degree $i$ in the coordinates of $v$ in a fixed basis of $\mathcal{A}$. Although the characteristic polynomial $p$ is defined for a regular element of $\mathcal{A}$, we can extend this definition to all the elements of $\mathcal{A}$, since each polynomial $a_i$ is an homogeneous polynomial of degree $i$ in the coordinates of $v$ in a fixed basis of $\mathcal{A}$ and the set of regular elements of $\mathcal{A}$ is dense in $\mathcal{A}$. The roots of the characteristic polynomial $p(v,-)$, $\lambda_1, \lambda_2, \ldots, \lambda_r$, are called the eigenvalues of $v$. Furthermore, the coefficients $a_1(v)$ and $a_r(v)$ of the characteristic polynomial of $v$, are called the trace and the determinant of $v$, respectively.

An idempotent $v$ of $\mathcal{A}$ is primitive if it is a nonzero idempotent of $\mathcal{A}$ and if it can’t be written as a sum of two non-zero orthogonal idempotents. We say that $\{v_1, v_2, \ldots, v_k\}$ is a Jordan frame of $\mathcal{A}$ if $\{v_1, v_2, \ldots, v_k\}$ is a complete system of orthogonal idempotents of $\mathcal{A}$ such that each idempotent is primitive.
Theorem 2.1. (\cite{16}, p. 43). Let $A$ be a real Euclidean Jordan algebra. Then for $u$ in $A$ there exist unique real numbers $\lambda_1(u), \lambda_2(u), \ldots, \lambda_n(u)$, all distinct, and a unique complete system of orthogonal idempotents $\{e_1, e_2, \ldots, e_k\}$ such that

$$u = \lambda_1(u)e_1 + \lambda_2(u)e_2 + \cdots + \lambda_k(u)e_k.$$  \hfill (2)

The numbers $\lambda_j(u)$'s of (2) are the eigenvalues of $u$ and the decomposition (2) is the first spectral decomposition of $u$.

Theorem 2.2. (\cite{16}, p. 44). Let $A$ be a real Euclidean Jordan algebra with rank($A$) = $r$. Then for each $v$ in $A$ there exists a Jordan frame $\{c_1, c_2, \cdots, c_r\}$ and real numbers $\lambda_1(v), \ldots, \lambda_{r-1}(v)$ and $\lambda_r(v)$ such that

$$v = \lambda_1(v)c_1 + \lambda_2(v)c_2 + \cdots + \lambda_r(v)c_r.$$  \hfill (3)

The decomposition (3) is called the second spectral decomposition of $v$.

Now let suppose that $A$ is a n-finite dimensional real Euclidean Jordan algebra such that rank($A$) = $n$. Since the set of regular elements of $A$ is dense in $A$ then let $u$ be one of his regular elements then, then there exists a unique complete system of orthogonal idempotents $S = \{u_1, u_2, \cdots, u_n\}$ such that $x = \sum_{i=1}^{n} \lambda_i e_i$. But, now we have that $S$ is a linear independent set of $A$, therefore $S$ is a basis of $A$.

An Euclidean Jordan algebra $A$ is equipped also with the inner product $x\cdot y = \text{trace}(x \times y)$ for any $x$ and $y$ elements of $A$ and therefore with the norm $||x|| = \sqrt{\text{trace}(x \times y)}$.

An Euclidean Jordan algebra is simple if it doesn’t have a non trivial ideal, these is equivalent to say that it is not a direct sum of Euclidean Jordan algebras.

3. Some Notions on Strongly Regular Graphs

Along this paper we only consider non-empty, simple and non-complete graphs. By simple graphs we mean graphs without loops and parallel edges. Strongly regular graphs were firstly introduced by R. C. Bose in the paper \cite{19}. Let $G$ be a graph of order $n$. We denote its vertices set by $V(G)$ and its edge set by $E(G)$. An edge whose endpoints are $u$ and $v$ is denoted by $uv$ and, in such case, the vertices $u$ and $v$ are adjacent or neighbors. The numbers of vertices of $G$, $|V(G)|$, is called the order of $G$. If all vertices of $G$ have $k$ neighbors, then $G$ is called a $k$-regular graph. $G$ is called a $(n, k; \lambda, \mu)$ strongly regular graph if is $k$-regular and any pair of adjacent vertices have $\lambda$ common neighbors and any pair of non-adjacent vertices have $\mu$ common adjacent vertices.

A strongly regular graph $G$ is a $(n, k; \lambda, \mu)$ strongly regular graph if and only if its complements $\overline{G}$ is a $(\overline{n}, k; \overline{\lambda}, \overline{\mu})$ strongly regular graph where $\overline{n} = n, \overline{k} = n - k - 1, \overline{\lambda} = n - 2 - 2k + \mu$ and $\overline{\mu} = n - 2k + \lambda$.

Let $G$ be a $(n, k; \lambda, \mu)$ strongly regular graph. The adjacency matrix of $G$, $A = [a_{ij}]$, is a binary matrix of order $n$ such that $a_{ij} = 1$, if the vertex $i$ is adjacent to $j$ and 0 otherwise. The adjacency matrix of $G$ satisfies the equation $A^2 = kI_n + \lambda A + \mu(J_n - A - I_n)$, where $J_n$ is the all one matrix of order $n$. It is well known (see, for instance, \cite{20}) that the eigenvalues of $G$ are $k, \theta$ and $\tau$, where $\theta$ and $\tau$ are given by the equalities (4) and (5).

$$\theta = \frac{(\lambda - \mu + \sqrt{(\lambda - \mu)^2 + 4(k - \mu)^2})}{2}$$ \hfill (4)

$$\tau = \frac{(\lambda - \mu - \sqrt{(\lambda - \mu)^2 + 4(k - \mu)^2})}{2},$$ \hfill (5)

, see \cite{20}. A $(n, k; \lambda, \mu)$— strongly regular graph $G$ satisfies the admissibility conditions (6),...
where \( m_\theta \) is the multiplicity of the eigenvalue \( \theta \) and \( m_\tau \) is the multiplicity of the eigenvalue \( \tau \) of the strongly regular graph \( G \). The conditions (7) and (8) are known as the Krein conditions of the strongly regular graph \( G \) and the conditions (9) and (10) are known as the absolute bound conditions of the strongly regular graph \( G \).

J. H. Van Lint and R. M Wilson in Theorem 21.6, page 272, presented in the publication [21] establishes that if, \( G \) is a strongly regular graph such that \( \mu \neq \tau^2 \) and \( \mu \neq (\tau+1)(\mu+1) \). This property is known as the claw bound. The sequence of parameters \((n,k;\lambda,\mu) = (2058,242;91,20)\) satisfy all the admissibility conditions (6), (7), (8), (9) and (10), but fails the claw property, since in this case we have \( \theta = 74 \) and \( \tau = -3 \).

A strongly regular graph \( G \) is primitive if and only if \( G \) and its complement \( \overline{G} \) are connected. A strongly regular graph that is not primitive is called an imprimitive strongly regular graph. A \((n,k;\lambda,\mu)\)-strongly regular graph is imprimitive if and only if \( \mu = 0 \) or \( \mu = k \).

In the next section we only consider primitive strongly regular graphs.

4. Inequalities over the Spectrum of a Strongly Regular Graph in the Environment of Euclidean Jordan Algebras

In this section we first present the description of the Generalized Krein parameters of a primitive \((n,k;\lambda,\mu)\)-strongly regular graph and next we establish conditions on the Generalized Krein parameters of a \((n,k;\lambda,\mu)\)-strongly regular graph are deduced. These conditions are generalizations of the Krein conditions (see in [20, Theorem 21.3]). Finally, we will establish an admissibility condition that permits us to conclude that doesn’t exist a strongly regular graph with parameters \((n,k;\lambda,\mu) = (4761,490;1,56)\). Herein, we must say that this sequence satisfy all necessary conditions for the existence of a strongly regular graph presented in this paper.

Let’s \( G \) be a \((n,k;\lambda,\mu)\)-primitive strongly regular graph, with the distinct eigenvalues \( k \), and the restricted eigenvalues \( \theta \) and \( \tau \), given in (4) and (5). \( A \) its adjacency matrix and let’s consider the Euclidean Jordan Subalgebra \( \mathcal{A} \) spanned by the natural powers of \( A \) and \( I_n \) of the Euclidean Jordan algebra of the real symmetric matrices of order \( n \) equipped with the Jordan product \( \ast \) such that \( x \ast y = \frac{xy+yx}{2} \) for any two symmetric \( x \) and \( y \), where \( xy \) and \( yx \) represents the usual product of \( x \) by \( y \) and of \( y \) by \( x \), and equipped with the inner product \( x \mid y = \text{trace}(x \ast y) \). Since \( \mathcal{A} \) is a three dimensional Euclidean Jordan algebra and \( A \) has three distinct eigenvalues then we have that \( \text{rank}(\mathcal{A}) = 3 \). Let \( S = \{E_0,E_1,E_2\} \) be the unique complete system of orthogonal idempotents of \( \mathcal{A} \) associated to \( A \), with \( E_0 = (A^2 - (\theta + \tau)A + \theta \tau I_n)/((k-\theta)(k-\tau)) = J_n/n, E_1 = (A^2 - (k + \tau)A + k \tau I_n)/((\theta - \tau)(\theta - k)) \) and \( E_2 = (A^2 - (k + \theta)A + k \theta I_n)/((\tau - \theta)(\tau - k)) \), where \( J_n \) is the matrix whose entries are all equal to \( 1 \). Since \( \mathcal{A} \) is an Euclidean Jordan algebra that is closed for the Hadamard product of matrices (see, for instance, [22]), denoted by \( \bullet \) and \( S \) is a basis of \( \mathcal{A} \), then there exist real numbers \( q_{\alpha,2}^\prime \) and \( q_{\alpha,3,1}^\prime \), \( 0 \leq \alpha, \beta \leq 2, \alpha \neq \beta \), such that
\[ E_\alpha \bullet E_\alpha = \sum_{j=0}^{2} q^j_{\alpha \alpha} E_j, \quad (11) \]

\[ E_\alpha \bullet E_\beta = \sum_{j=0}^{2} q^j_{\alpha \beta 11} E_j. \quad (12) \]

These numbers (whose notation will be clarified later) \( q^j_{\alpha \alpha} \) and \( q^j_{\alpha \beta 11} \), \( 0 \leq \alpha, \beta \leq 2, \alpha \neq \beta, \) are called the “classical” Krein parameters of the graph \( X \) (cf. [?]). Since \( q^1_{11} \geq 0 \) and \( q^2_{22} \geq 0 \), the “classical” Krein admissibility conditions \( \theta \tau^2 - 2\theta^2 \tau - \theta^2 - k\theta + k\tau^2 + 2k\tau \geq 0 \), and \( \theta^2 \tau - 2\theta \tau^2 - \tau^2 - k\tau + k\theta^2 + 2k\theta \geq 0 \) (presented in [20, Theorem 21.3] ) can be deduced. In what follows we generalize the Krein parameters in order to obtain new generalized admissibility conditions on the parameters of strongly regular graphs.

Firstly, considering \( S = \{E_0, E_1, E_2\} \) and rewriting the idempotents under the new basis \( \{I_n, A, J_n - A - I_n\} \) of \( \mathcal{A} \) we obtain

\[ E_0 = \frac{\theta - \tau}{n(\theta - \tau)} I_n + \frac{\theta - \tau}{n(\theta - \tau)} (J_n - A - I_n), \]
\[ E_1 = \frac{\tau |n(\theta - \tau) - (n - k)}{n(\theta - \tau)} I_n + \frac{n + 2\tau - k}{n(\theta - \tau)} (J_n - A - I_n), \]
\[ E_2 = \frac{\tau |n(\theta - \tau) - (n - k)}{n(\theta - \tau)} I_n + \frac{n + 2\tau - k}{n(\theta - \tau)} (J_n - A - I_n). \]

Let \( l \) be a natural number and let \( M_l(\mathbb{R}) \) be the set of real square matrices of order \( l \) with real entries. For \( x \in M_l(\mathbb{R}) \), we denote by \( x^1 \) and \( x^{\otimes l} \) the Hadamard power of order \( l \) of \( x \) and the Kronecker power of order \( l \) of \( x \), respectively, with \( x^1 = x \) and \( x^{\otimes 1} = x \) (see, for instance, [22] for the definition of Kronecker product).

Now, we introduce the following compact notation for the Hadamard and the Kronecker powers of the elements of \( S \). Let’s consider the natural numbers \( x, y, \alpha, \beta \) and \( \gamma \) such that \( 0 \leq \alpha, \beta, \gamma \leq 2, \alpha \geq 2 \) and \( \alpha < \beta \) necessary to recover the definition of the Generalized Krein parameters as defined in [9].

Now, we consider some notation, for defining the matrices \( E_{\alpha}^{xx} \) and \( E_{\alpha, \beta}^{yz} \):

\[ E_{\alpha}^{xx} = (E_{\alpha})^{xx} \quad \text{and} \quad E_{\alpha}^{\otimes x} = (E_{\alpha})^{\otimes x}, \]
\[ E_{\alpha, \beta}^{yz} = (E_{\alpha})^{yz} \bullet (E_{\beta})^{yz} \quad \text{and} \quad E_{\alpha, \beta}^{\otimes yz} = (E_{\alpha})^{\otimes yz} \otimes (E_{\beta})^{\otimes yz}. \]

Since the Euclidean Jordan algebra \( \mathcal{A} \) is closed under the Hadamard product and \( S \) is a basis of \( \mathcal{A} \), then there exist real numbers \( q^i_{\alpha x} \) and \( q^i_{\alpha, \beta y z} \) such that the equalities (13) and (14) are verified.

\[ E_{\alpha}^{xx} = \sum_{i=0}^{2} q^i_{\alpha x} E_i, \quad (13) \]
\[ E_{\alpha, \beta}^{yz} = \sum_{i=0}^{2} q^i_{\alpha, \beta y z} E_i \quad (14) \]

These parameters \( q^i_{\alpha x} \) and \( q^i_{\alpha, \beta y z} \) are the Generalized Krein parameters of the strongly regular graph \( G \) with \( x \geq 3 \) and \( y + z \geq 3 \), see [9].

We observe that \( q^i_{\alpha 2} \) and \( q^i_{\alpha, \beta 11} \) are precisely the Krein parameters of \( G \).
We must note that since the matrices $E_{\alpha x}^\otimes$ and $E_{\alpha y}^\otimes$ are idempotents matrices then the Generalized Krein parameters $q_{\alpha x}^i$ and $q_{\alpha y}^i$ are such that $0 \leq q_{\alpha x}^i \leq 1$ and $0 \leq q_{\alpha y}^i \leq 1$.

Throughout this paper we use a slight different notation from classical books like [23], since, by this way, the connections between the ‘classical’ and the generalized parameters are better understood. Now, we will analyze the Generalized Krein parameters $q_{3x}^0$ and $q_{2y}^0$. Now, we have the following calculations for the development of the parameter $q_{3x}^0$.

\[
q_{23}^0 = \left( \frac{\theta n + k - \theta}{n(\theta - \tau)} \right)^3 + \left( \frac{n + k - \theta}{n(\theta - \tau)} \right)^3 k + \left( \frac{k - \theta}{n(\theta - \tau)} \right)^3 (n - k - 1)
\]

Therefore since $q_{23}^0 \geq 0$ then we obtain the inequality (15).

\[(\theta^3 - k)n^2 + (3\theta^2(k - \theta) + 3(k - \theta)k)n^2 + (3\theta(k - \theta)^2 - 3(k - \theta)^2k + (k - \theta)^3)n \geq 0 \quad (15)\]

Now, dividing both sides of the inequality (15) by $n$ we obtain the inequality (16).

\[(\theta^3 - k)n^2 + (3\theta^2(k - \theta) + 3(k - \theta)k)n + 3\theta(k - \theta)^2 - 3(k - \theta)^2k + (k - \theta)^3 \geq 0. \quad (16)\]

So, we consider the Theorem 4.1.

**Theorem 4.1.** Let $X$ be a $(n, k; \lambda, \mu)$-primitive strongly regular graph such that $0 < \mu < k < n - 1$, whose adjacency matrix $A$ has the eigenvalues $\theta$, $\tau$ and $k$. Then, the inequality (17) is verified.

\[(\theta^3 - k)n^2 + (3\theta^2(k - \theta) + 3(k - \theta)k)n + 3\theta(k - \theta)^2 - 3(k - \theta)^2k + (k - \theta)^3 \geq 0. \quad (17)\]

Now we present in Table 1. some numerical results for the value of the parameter $n^3(r-s)^3q_{23}^0$.

| $n$ | $k$ | $\lambda$ | $\mu$ | $\theta$ | $\tau$ | $n^3(r-s)^3q_{23}^0$ |
|-----|-----|-----------|-------|---------|-------|-----------------|
| 28  | 9   | 0         | 4     | 1       | -5    | -576.0          |
| 63  | 22  | 1         | 11    | 1       | -11   | -10584.0        |
| 144 | 65  | 16        | 40    | 1       | -25   | -26624.0        |
| 154 | 51  | 8         | 21    | 2       | -15   | -9996.0         |
| 300 | 92  | 10        | 36    | 2       | -28   | -1.2 \times 10^6|


Now, supposing that $k \neq 1$ and that $\theta = 1$ in the inequality (18).

$$(1 - k)n^2 + (3(k - 1) + 3(k - 1)k)n + (3(k - 1)^2 - 3(k - 1)^2k + (k - 1)^3) \geq 0$$

we obtain the inequality (19).

$$-n^2 + (3 + 3k)n + (3(k - 1) - 3(k - 1)k + (k - 1)^2) \geq 0.$$  

(19)

After rewriting the first side of (19) we obtain the inequality (20).

$$-n^2 + (3 + 3k)n - 2k^2 + 4k - 2 \geq 0.$$  

(20)

Now, we analyse the parameter $q$ then, we have the following calculations for the development of $q_{13}$.

$$-n^2 + (3 + 3k)n - 2k^2 + 4k - 2 \geq 0$$

(21)

Now, dividing both sides of (23) by $n$ we obtain the inequality (24).

$$\frac{|\tau|^3n^3 + 3|\tau|^2n^2(\tau - k) + 3|\tau|n(\tau - k)^2 + (\tau - k)^3}{n^3(\tau - k)^3} +$$

$$+ \frac{n^3 + 3n^2(\tau - k) + 3n(\tau - k)^2 + (\tau - k)^3}{(n(\tau - k))^3}k +$$

$$+ \frac{(\tau - k)^3}{(n(\tau - k))^3}(n - k - 1)$$

$$= \frac{|\tau|^3n^3 + 3|\tau|^2n^2(\tau - k) + 3|\tau|n(\tau - k)^2}{(n(\tau - k))^3} +$$

$$+ \frac{n^3 + 3n^2(\tau - k) + 3n(\tau - k)^2}{(n(\tau - k))^3}k +$$

$$+ \frac{(\tau - k)^3}{(n(\tau - k))^3}n.$$
Now, we will establish an admissibility condition over the spectra of the inverse of the adjacency matrix a strongly regular graph recurring to the Frobenius norm and using the basis $S = \{E_1, E_2, E_3\}$ on the Euclidean Jordan $A$ and recurring to the inequality of Cauchy-Schwarz, and considering the inner product $x|y = \text{trace}(x \ast y)$.

But firstly, we will express $A^{-1}$ on the basis $B' = \{I_n, A, J_n - A - I_n\}$ of $A$. Since $A^2 = kI_n + \lambda A + \mu(J_n - A - I_n)$ then we have $A^2 = (k - \mu)I_n + (\lambda - \mu)A + \mu J_n$, and therefore, multiplying both sides of this equality by $A^{-1}$ we obtain the equality (25).

\[ A = (k - \mu)A^{-1} + (\lambda - \mu)I_n + \frac{\mu}{k}J_n. \]  

(25)

So, rewriting the inequality (25) we obtain (26)

\[ Ak = (k - \mu)kA^{-1} + (\lambda - \mu)kI_n + \mu J_n. \]  

(26)

Then, since $J_n = J_n - A - I_n + A + I_n$ we deduce that $A(k - \mu)A^{-1} + (\lambda - \mu)kI_n + \mu(J_n - A - I_n + A + I_n)$, this is, we conclude that $(k - \mu)A = (k - \mu)kA^{-1} + ((\lambda - \mu)k + \mu)I_n + \mu(J_n - A - I_n)$. Hence, dividing both sides of this equality by $(k - \mu)k$ we deduce the equality (27).

\[ \frac{1}{k}A = A^{-1} + \frac{\lambda - \mu}{(k - \mu)k}I_n + \frac{\mu}{(k - \mu)k}(J_n - A - I_n). \]  

(27)

Finally, we have that

\[ A^{-1} = -\frac{(\lambda - \mu)k + \mu}{(k - \mu)k}I_n + \frac{1}{k}A - \frac{\mu}{(k - \mu)k}(J_n - A - I_n). \]  

(28)

Now, recurring to the inner product on $A$, $x|y = \text{trace}(xy)$ and therefore to the norm $||x|| = \sqrt{\text{trace}(x^2)}$ and since $\text{trace}(I_3) = 3, \text{trace}(A) = k + \lambda - \mu, \text{trace}(J_n - A - I_n) = n - (k + \lambda - \mu) - 3$ in the Cauchy-Schwarz inequality (29) to the elements $A^{-1}$ and $I_n$ of $A$

\[ |\text{trace}(A^{-1} I_n)| \leq ||A^{-1}|| ||I_n||, \]  

(29)

supposing that $\lambda > \mu$ we obtain the inequality (30).

\[ \left| -\frac{3(\lambda - \mu)k + \mu}{(k - \mu)k} \right| \leq \sqrt{\frac{3}{||\tau||^2}} \sqrt{3.0}. \]  

(30)

And, from (30) we obtain (31)

\[ |\tau| \leq \left| \frac{(k - \mu)k}{(\lambda - \mu)k + \mu} \right|. \]  

(31)

So, we have established the inequality (32) of Theorem 4.2.

**Theorem 4.2.** Let $G$ be a $(n, k; \lambda, \mu)$-primitive strongly regular graph such that $0 < \mu < k < n - 1$ and such that $\lambda > \mu$ with the distinct eigenvalues $\theta, \tau$ and $k$. Then

\[ |\tau| \leq \frac{(k - \mu)k}{(\lambda - \mu)k + \mu}. \]  

(32)
In the same way, if $\mu > \lambda$ then we deduce the inequality (33) of Theorem (4.3).

**Theorem 4.3.** Let $G$ be a $(n, k; \lambda, \mu)$ − primitive strongly regular graph such that $0 < \mu < k < n - 1$ and such that $\mu > \lambda$ with the distinct eigenvalues $\theta, \tau$ and $k$. Then

$$\theta \leq \left|\frac{(k - \mu)k}{(\lambda - \mu)k + \mu}\right|$$

The results obtained in Theorems 4.1, 4.2 and in Theorem 4.3 furnish some admissibility conditions for the existence of an imprimitive strongly regular graph. In these admissibility conditions we develop majorants for the eigenvalues of the adjacency matrix of a $(n, k; \lambda, \mu)$ − strongly regular graph in function of its regularity and the parameters $\lambda$ and $\mu$. In the future, we will analyse other elements of the Euclidean Jordan algebra associated to the adjacency matrix of a strongly regular graph and use some properties of the positive definite symmetric matrices to develop better majorants for eigenvalues of the spectrum of a strongly regular graph. Some relevant studies can be found in paper [24].

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