The Lax Operator Approach for the Virasoro and the W-Constraints in the Generalized KdV Hierarchy

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Abstract

We show directly in the Lax operator approach how the Virasoro and W-constraints on the $\tau$-function arise in the $p$-reduced KP hierarchy or Generalized KdV hierarchy. In particular, we consider the KdV and the Boussinesq hierarchy to show that the Virasoro and the W-constraints follow from the string equation by expanding the “additional symmetry” operator in terms of the Lax operator. We also mention how this method could be generalized for higher KdV hierarchies.

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I. Introduction:

It is by now a well recognized fact that the integrable models play very interesting role in the matrix model formulation of two dimensional (2D) quantum gravity [1-5], 2D topological gravity [6-8] and the intersection theory on the moduli space of Riemann surfaces [9-11]. In the matrix model approach of the 2D gravity, one employs the method of orthogonal polynomials and makes use of the operators $Q$ and $P$ corresponding to the insertions of the spectral parameter and a derivative with respect to it in the matrix integral [12,13]. As Douglas argued, $Q$ and $P$ can be realized in terms of some finite order differential operators and can be recognized as the Lax-pair of an associated integrable hierarchy [14-16]. Since the operators $P$ and $Q$ are conjugate to each other, they satisfy the so-called “string equation” [13] $[P,Q] = 1$. Once the pair of operators $(P,Q)$ is recognized as the Lax pair and we set their commutator to be one, it puts a very stringent condition on the coefficient functions of the Lax operator. This in turn implies an infinite number of additional symmetries of the integrable hierarchies and can be recognized as the Virasoro and W-constraints. In the usual integrable models, the symmetries arise from the isospectral deformation of the Lax operator, but these additional “time”-dependent symmetries originate from a general Galilean transformation of the evolution parameters as emphasized in [17]. The origin and the geometry of the string equation and its connection with the Sato-Grassmanian can be found in the recent literature [18-21].

By formulating the Hermitian one matrix model [22] and two matrix model [23], with specific interaction, in terms of the continuum Schwinger-Dyson equations it is shown that they give rise to a semi-infinite set of Virasoro (for 1-matrix model) and $W^{(3)}$-constraints (for 2-matrix model) on the square root of the partition function. Through an identification of the square root of the partition function with the $\tau$-function of the corresponding integrable hierarchy it has been conjectured [22] that the whole set of Virasoro and W-constraints follow as a consequence of the string equation itself. By making use of the string equation and the associated biHamiltonian structure of the KdV hierarchy it has been shown (although in a very indirect way) that this is indeed true [7]. Goeree [24] has also shown using the vertex operator techniques of KP hierarchy developed in [25,26], that
not only the Virasoro constraints but also the W-constraints follow from the string equation. However, the additional symmetry operator $M$ as used there, does not reproduce the correct W-constraints and needed to be modified. This has been reported by us in a recent letter [27].

Taking into account the above modification and to make the structure more transparent we develop a direct approach in this paper to show how the generators of the additional symmetries [28-31] give rise to the Virasoro and W-constraints. We use the method of $(L,M)$ pair of the $p$-reduced KP hierarchy by which one can construct the generators of the additional symmetries associated with such integrable systems. We then expand the operator $M$ as a power series of the Lax operator. In this way the residues of the generators of the additional symmetries can be recognized as the Virasoro and the W-constraints found in the matrix model approach.

The paper is organized as follows. In section II, we consider the 2-reduced (KdV) KP hierarchy and explain our method. The 3-reduced (Boussinesq) KP hierarchy is considered in details in section III. In section IV, we discuss the generalization of the method for higher KdV hierarchy. Our conclusions are drawn in Section V.

II. KdV Hierarchy and the Virasoro Constraints:

The 2-reduced KP hierarchy or KdV hierarchy is described in terms of the following Lax equation,

$$\frac{\partial L}{\partial t_{2k+1}} = [L_{+}^{2k+1}, L] \quad k = 0, 1, 2, 3, \ldots$$

Here, $L = \frac{\partial^2}{\partial x^2} + u(x,t_3,t_5,\ldots) \equiv \partial^2 + u(x,t)$ is the Lax operator of the KdV hierarchy and $L_{+}^{2k+1}$ is the non-negative differential part of the $(2k+1)\text{th}$ power of the formal pseudo-differential operator $L_{+}^{\frac{1}{2}}$. Together they are known as the Lax-pair. $L_{+}^{\frac{1}{2}}$ is the two reduced KP Lax operator and has a formal expansion in the form

$$L_{+}^{\frac{1}{2}} = \partial + \frac{u}{2} \partial^{-1} - \frac{u'}{4} \partial^{-2} + \left(\frac{u''}{8} - \frac{u^2}{8}\right) \partial^{-3} + \left(\frac{3uu'}{8} - \frac{u''}{16}\right) \partial^{-4}$$

$$+ \left(\frac{uu''}{32} - \frac{11u'^2}{32} + \frac{u^3}{16} - \frac{7uu''}{16}\right) \partial^{-5} + \cdots$$

(2.2)
such that we have
\[(L^\frac{1}{2}) = L = \partial^2 + u\] (2.3)

Here, ‘prime’ denotes the differentiation with respect to \(x\). \(t_{2k+1}, \, k = 0, 1, 2, \ldots\) are the infinite number of evolution parameters associated with the KdV hierarchy. From (2.1), one can identify \(t_1 \equiv x\). Also note that the differential part of the even powers of \(L^\frac{1}{2}\) will commute with \(L\). In order to evaluate various powers of \(L^\frac{1}{2}\), one makes use of the Liebnitz rule
\[
\partial^{-i} f = \sum_{j=0}^{\infty} (-)^j \binom{i+j-1}{j} f^{(j)} \partial^{-i-j}
\] (2.4)

where we have denoted \(f^{(j)} = \frac{\partial^j f}{\partial x^j}\).

According to Douglas, the string equation [13] (for one matrix model) corresponding to \(k\)-th critical point is given by
\[
[L_{2k+1}^\frac{1}{2}, L] = 1 \tag{2.5}
\]

An arbitrary massive model which interpolates between various critical points can be written by generalizing (2.5) as follows,
\[
\sum_{k=1}^{\infty} (k + \frac{1}{2}) t_{2k+1} \left[ L, L_{2k+1}^{\frac{2(k-1)+1}{2}} \right] = 1 \tag{2.6a}
\]

where we have introduced an infinite number of evolution parameters \(t_{2k+1}\) proportional to \(-1/(k + \frac{1}{2})\). Note that (2.6a) can also be expressed in an equivalent form as
\[
\sum_{k=1}^{\infty} (k + \frac{1}{2}) t_{2k+1} \text{res} L^{\frac{2(k-1)+1}{2}} + \frac{1}{2} x = 0 \tag{2.6b}
\]

We have integrated (2.6a) once with respect to \(x\) in order to derive (2.6b). Also “\text{res}” here simply means the coefficient of \(\partial^{-1}\) term in the pseudo-differential operator. Equation (2.6) can be written in a different form given by
\[
\left[ L, (ML^{-\frac{1}{2}})_+ \right] = 1 \tag{2.7}
\]

The operator \(M\) for the 2-reduced KP hierarchy is defined as
\[
M \equiv \frac{1}{2} K \left( \sum_{n=1}^{\infty} nt_n \partial^{n-1} \right) K^{-1} \tag{2.8}
\]
We would like to point out here that it not necessary to remove the coordinates $t_n$ where $n = 0(\text{mod}2)$ in the definition of $M$ in order to get the correct Virasoro constraints. But, this becomes necessary in order to get right $W$-constraints [27]. We, therefore, define $M$ in this way from the beginning so that we do not face any problem later. Also, here, $K = 1 + \sum_{i=1}^{\infty} a_i(x,t) \partial^{-i}$ is a pseudo-differential operator known as the Zakharov-Shabat dressing operator and satisfies the relation [25]

\[ L^{\frac{1}{2}} = K \partial K^{-1} \]  \hspace{1cm} (2.9)

This fixes the coefficients of $K$ in terms of $u(x,t)$ and their derivatives. Using (2.9), we rewrite $M$ as

\[ M = \frac{1}{2} K x K^{-1} + \frac{1}{2} \sum_{n=3}^{\infty} nt_n L^{\frac{n-1}{2}} \]  \hspace{1cm} (2.10)

Thus, we have

\[ (ML^{-\frac{1}{2}})_{+} = \frac{1}{2} \sum_{n=3}^{\infty} nt_n L^{\frac{n+1}{2}} \]  \hspace{1cm} (2.11)

Substituting (2.11) into (2.7) we recover (2.6a). So, (2.7) is indeed an equivalent form of (2.6) i.e. Douglas’ string equation. Using the definition of $M$ in (2.10) we can show that [30]

\[ [ L^{\frac{1}{2}}, M ] = \frac{1}{2} \]  \hspace{1cm} (2.12)

and therefore,

\[ [ L, ML^{-\frac{1}{2}} ] = 1 \]  \hspace{1cm} (2.13)

In view of the string equation (2.7), one concludes that $(ML^{-\frac{1}{2}})_-$ which is purely pseudo-differential part of $ML^{-\frac{1}{2}}$ should commute with $L$. Using (2.12) one can derive that

\[ [ M, L^{-\frac{1}{2}} ] = \frac{1}{2} L^{-1} \]  \hspace{1cm} (2.14)

Since $(ML^{-\frac{1}{2}})_-$ commutes with $L$ and it satisfies (2.14), therefore, it must be proportional to $L^{-1}$. We set,

\[ (ML^{-\frac{1}{2}})_- = \alpha L^{-1} \]  \hspace{1cm} (2.15)
where $\alpha$ is an arbitrary constant which can not be determined just from the string equation (2.7) as mentioned in ref.[24]. From this it follows that for $n \geq 0$ we have

$$(ML^{n + \frac{1}{2}})_- = ((ML^{-\frac{1}{2}})_- L^{n + 1})_- = 0 \quad (2.16)$$

The second expression is because of the fact that $L^n$ does not contain any negative power of $\partial$ for $n \geq 0$. It has been noted in ref.[31] that the particular combination of $L$ and $M$ (2.15) and (2.16) are the generators of the additional symmetries of the KdV hierarchy in the sense that they satisfy, for $n \geq -1$,

$$\frac{\partial L}{\partial t_{2n+1,1}} = [L, (ML^{n + \frac{1}{2}})_-] = 0 \quad (2.17)$$

These flows commute with the original KdV hierarchy flows given in (2.1), but they do not commute among themselves and have nice interpretation in terms of the Sato-Grassmannian [31].

We first show that (2.15) does indeed imply the string equation (2.6b) and then work out the consequences of (2.16). The operator $M$ has an expansion in the power series of the Lax operator in the form [32] (see appendix)

$$M = \frac{1}{2} \sum_{n=1}^{\infty} \sum_{n \neq 0 \mod 2} nt_n L^{n-\frac{1}{2}} + \frac{1}{2} \sum_{i=1}^{\infty} V_{i+1}(x,t) L^{n+\frac{1}{2}} \quad (2.18)$$

The functions $V_{i+1}(x,t)$ can be be determined in terms of the coefficient functions of the dressing operators $K$ and $K^{-1}$ as follows

$$V_{i+1}(x,t) = -(ia_i + \sum_{j=1}^{i-1} ja_j \tilde{a}_{i-j}) \quad (2.19)$$

The operator $K^{-1}$ is chosen to have the form

$$K^{-1} = 1 + \sum_{i=1}^{\infty} \tilde{a}_i(x,t) \partial^{-i} \quad (2.20)$$

By requiring $KK^{-1} = 1$, $\tilde{a}_i$’s can be fixed in terms of $a_i$’s from the relation,

$$a_i + \tilde{a}_i + \sum_{k=1}^{i} \sum_{j=1}^{k-1} \frac{(-)^{i-k}(i + j - k - 1)}{i - k} a_j \tilde{a}_{k-j}^{(i-k)} = 0 \quad i = 1, 2, 3, \ldots \quad (2.21)$$
Taking the residue of (2.15) we get

$$res\ (ML^{-\frac{1}{2}}) = 0 \quad (2.22)$$

Inserting the expression of $M$ as given in (2.10) in the above we obtain

$$\frac{1}{2}x + \frac{1}{2} \sum_{n \geq 3, n \neq 0 \mod 2} nt_n resL^{\frac{n-2}{2}} = 0 \quad (2.23)$$

This is precisely equation (2.6b). This, therefore, establishes the equivalence between (2.6), (2.7) and (2.15). One defines the $\tau$-function of the KdV hierarchy as a function depending on the coefficient functions of the Lax operator (in the present case $u$) and their derivatives and is given by (see appendix for the derivation),

$$res\ L^{\frac{2k+1}{2}} = \frac{\partial}{\partial x} \frac{\partial \log \tau}{\partial t_{2k+1}} \quad (2.24)$$

Now, using (2.24) in (2.23) and performing an integration with respect to $x$, and multiplying by $\tau$ we find that

$$L_{-1} \tau = 0 \quad (2.25)$$

where we have defined the operator $L_{-1}$ as

$$L_{-1} \equiv \sum_{k=1}^{\infty} (k + \frac{1}{2})t_{2k+1} \frac{\partial}{\partial t_{2(k-1)+1}} + \frac{1}{4}x^2 \quad (2.26)$$

Thus, (2.25) can also be called as the string equation since it is equivalent to (2.6). Next, we can work out the residue of $ML^{n+\frac{1}{2}}$ for $n \geq 0$ by using the expression for $M$ as in (2.18). Note that for these cases, the last term in (2.18) will contribute for odd values of $i$. Thus, we need to express the functions $V_{i+1}$ in terms of the $\tau$-function. In (2.19), we have expressed $V_{i+1}$ in terms of $a_i$ and $\tilde{a}_i$ which are some meromorphic functions and can be expressed in terms of $\tau$-function. It can be shown that $V_{i+1}$ has the following form (see appendix)

$$V_{i+1} = -i \sum_{\alpha_1+3\alpha_3+5\alpha_5+\ldots = i} (-)^{\alpha_1+\alpha_3+\alpha_5+\ldots} (\frac{\partial t_1}{\alpha_1})(\frac{\partial t_3}{\alpha_3})(\frac{\partial t_5}{\alpha_5}) \ldots \log \tau \quad (2.27)$$
After integrating $\text{res } (ML^{\frac{1}{2}}) = 0$ once with respect to $x$ and multiplying by $\tau$, we obtain for $n = 0$ that
\[
\left[ \sum_{k=0}^{\infty} (k + \frac{1}{2}) t_{2k+1} \frac{\partial}{\partial t_{2k+1}} + C \right] \tau = 0
\] (2.28)
where $C$ is an arbitrary integration constant which appears here unlike in (2.26) is because the scaling dimension of (2.28) is zero. This constant will be fixed later. For $n \geq 1$, the residue of $ML^{n+\frac{1}{2}}$ could also be calculated in an identical way and the result is that
\[
\left[ \sum_{k=0}^{\infty} (k + \frac{1}{2}) t_{2k+1} \frac{\partial}{\partial t_{2k+1}} + \frac{1}{4} \sum_{i,j=0}^{2n \mod 2} \frac{\partial}{\partial t_i} \frac{\partial}{\partial t_j} \right] \tau = 0
\] (2.29)
The appearance of the second term in (2.29) comes from the contribution of $V_{i+1}$ functions contained in $M$. In fact, it can be shown for $n \geq 0$ that these functions satisfy the following relation;
\[
\sum_{k=0}^{n} V_{2k+2} \text{res } L^{\frac{2n-2k-1}{2}} = \frac{\partial}{\partial t_{2n+1}} + \frac{1}{2} \frac{\partial}{\partial x} \left[ \frac{1}{\tau} \sum_{i \neq 0, j \neq 0}^{2n \mod 2} \frac{\partial}{\partial t_i} \frac{\partial}{\partial t_j} \right]
\] (2.30)
where we have used the simple identity
\[
\frac{\partial^2 \log \tau}{\partial t_i \partial t_j} + \frac{\partial \log \tau}{\partial t_i} \frac{\partial \log \tau}{\partial t_j} = \frac{1}{\tau} \frac{\partial^2 \tau}{\partial t_i \partial t_j}
\] (2.31)
To check the validity of (2.30), we give the expressions for few of $V_{2k+2}$, for example,
\[
V_2 = \frac{\partial \log \tau}{\partial x}
\]
\[
V_4 = \frac{1}{2} \left( \frac{\partial^3}{\partial x^3} + 2 \frac{\partial}{\partial t_3} \right) \log \tau
\]
\[
V_6 = \left( \frac{1}{4!} \frac{\partial^5}{\partial x^5} + \frac{5}{6} \frac{\partial^2}{\partial x^2} \frac{\partial}{\partial t_3} + \frac{\partial}{\partial t_5} \right) \log \tau
\]
\[
V_8 = \left( \frac{1}{6!} \frac{\partial^7}{\partial x^7} + \frac{\partial}{\partial t_7} + \frac{7}{10} \frac{\partial^2}{\partial x^2} \frac{\partial^2}{\partial t_5} + \frac{7}{144} \frac{\partial^4}{\partial x^4} \frac{\partial}{\partial t_5} + \frac{7}{18} \frac{\partial}{\partial x} \frac{\partial^2}{\partial t_3^2} \right) \log \tau
\]
and the $\tau$-function satisfies the following identities,
\[
\frac{1}{48} \frac{\partial^5 \log \tau}{\partial x^5} - \frac{1}{12} \frac{\partial^2 \log \tau}{\partial x^2} \frac{\partial^3 \log \tau}{\partial x^3} + \frac{1}{4} \frac{\partial^3 \log \tau}{\partial x^3} \frac{\partial^2 \log \tau}{\partial x^2} = 0
\]
\[
\frac{1}{1440} \frac{\partial^7 \log \tau}{\partial x^7} - \frac{3}{20} \frac{\partial^2 \log \tau}{\partial x^2} \frac{\partial \log \tau}{\partial t_5} + \frac{7}{144} \frac{\partial^4 \log \tau}{\partial x^4} \frac{\partial \log \tau}{\partial t_3} - \frac{1}{18} \frac{\partial^2 \log \tau}{\partial x \partial t_3^2} (2.33)
\]
Equations (2.32) and (2.33) would check that the relation (2.30) is true up to \( n = 3 \) and for higher values of \( n \), it becomes more tedious.

It is customary in the matrix model to identify the operators appearing in (2.28) and (2.29) as \( L_0 \) and \( L_n \) respectively for the reason that they satisfy a centerless semi-infinite Virasoro algebra. In general for arbitrary value of \( C \) in (2.28), the Virasoro algebra will not be satisfied. So if we insist that these operators along with (2.25) satisfy the Virasoro algebra in analogy with the matrix model result we have to fix the integration constant \( C \) to be \( \frac{1}{16} \). In this situation, we have the standard form of Virasoro constraints on the \( \tau \)-function

\[
L_n \, \tau = 0 \quad n \geq -1
\]

where

\[
\begin{align*}
L_{-1} &= \sum_{k=1}^{\infty} (k + \frac{1}{2}) t_{2k+1} \frac{\partial}{\partial t_{2(k-1)+1}} + \frac{1}{4} x^2 \\
L_0 &= \sum_{k=0}^{\infty} (k + \frac{1}{2}) t_{2k+1} \frac{\partial}{\partial t_{2k+1}} + \frac{1}{16} \\
L_n &= \sum_{k=0}^{\infty} (k + \frac{1}{2}) t_{2k+1} \frac{\partial}{\partial t_{2(n+k)+1}} + \frac{1}{4} \sum_{i+j=2n \atop i,j \neq 0 (mod 2)} \frac{\partial}{\partial t_i} \frac{\partial}{\partial t_j} 
\end{align*}
\] (2.34)

### III. Boussinesq Hierarchy and \( W_3 \)-Constraints:

The Boussinesq hierarchy can also be described in terms of the Lax equation given by

\[
\frac{\partial L}{\partial t_{3n+i}} = [L_{3n+1}^{3n+1}, L] \quad n = 0, 1, 2, \ldots \quad \text{and} \quad i = 1, 2
\] (3.1)

Here the Lax operator \( L \) for the three reduced KP hierarchy or Boussinesq hierarchy is a third order differential operator,

\[
L = \partial^3 + 4u \partial + (2u' + w)
\] (3.2)

where \( u(x,t) \) and \( w(x,t) \) are the coefficient functions of the Lax operator. The particular form of \( L \) in (3.2) ensures that \( L \) transforms covariantly under conformal transformation.
Note that we have an infinite number of evolution parameters $t_k$ where $k \neq 0(\text{mod}3)$ because, for those values the commutator in (3.1) will vanish. Also, the formal pseudodifferential operator $L^\frac{1}{3}$ has the expansion of the form

$$L^\frac{1}{3} = \partial + \frac{4}{3}u\partial^{-1} + \frac{1}{3}(w - 2u')\partial^{-2} - \frac{1}{3}(w' - \frac{2}{3}u' + \frac{16}{3}u^2)\partial^{-3}$$

$$- \frac{1}{3}(-\frac{2}{3}w'' + \frac{8}{3}uw - 16uw')\partial^{-4} - \frac{1}{3}(\frac{1}{3}w''' + \frac{2}{9}u''' - 16u'w$$

$$- \frac{16}{3}uw' + \frac{44}{3}(u')^2 + 16uu'' + \frac{1}{3}w^2 - \frac{320}{27}u^3)\partial^{-5}$$

$$- \frac{1}{3}(-\frac{1}{9}w''' - \frac{2}{9}u''' + \frac{70}{9}u''w + \frac{20}{3}uw'' + \frac{110}{9}u'w' - \frac{100}{3}u'u'' - \frac{40}{3}uu'''$$

$$- \frac{5}{3}w'u' + \frac{800}{9}u^2u' - \frac{80}{9}u^2w)\partial^{-6} + \ldots$$

such that $(L^\frac{1}{3})^3 = L$

The string equation for the $k$-th critical point can now be written as

$$[L^\frac{3k+i}{3}, L] = 1 \quad (3.4)$$

In terms $u(x, t)$ and $w(x, t)$, this means

$$\frac{\partial u}{\partial t_{3k+i}} = 0 \quad k = 0, 1, 2, \ldots$$

$$\frac{\partial w}{\partial t_{3k+i}} = 1 \quad i = 1, 2 \quad (3.5)$$

Again $t_1$ can be identified with $x$ from (3.1) and for general massive model we can write down the string equation in the form

$$\sum_{k=1}^{\infty} (k + \frac{i}{3})t_{3k+i} [L, L^\frac{3(k-1)+i}{3}] = 1 \quad (3.6)$$

As in the KdV case we have introduced infinite number of evolution parameters $t_{3k+i}$ proportional to $-1/(k + \frac{i}{3})$. In terms of the Gelfand-Dikii polynomials eq.(3.6) can be rewritten as

$$\sum_{k=1}^{\infty} 3(k + \frac{i}{3})t_{3k+i} \frac{\partial R_k^{(i)}}{\partial x} + 1 = 0 \quad (3.7)$$

and

$$\sum_{k=1}^{\infty} 6(k + \frac{i}{3})t_{3k+i} \frac{\partial \tilde{R}_k^{(i)}}{\partial y} + 1 = 0 \quad (3.8)$$
Here we have identified $t_2$ with $y$ and also note that we have got two equations from the string equation (3.6). This is because that the Gelfand-Dikii polynomials satisfy the relation

$$\frac{\partial R_k^{(i)}}{\partial x} = 2 \frac{\partial \tilde{R}_k^{(i)}}{\partial y} \quad k = 1, 2, \ldots \quad i = 1, 2$$

(3.9)

So, (3.7) and (3.8) are really two equivalent forms of the string equation (3.6) and we will work only with (3.7). The Gelfand-Dikii polynomials can be calculated from the Lax equation which we write in terms of a $2 \times 2$ matrix notation as follows,

$$\left(\begin{array}{c}
\frac{\partial u}{\partial \xi_{3n+1}} \\
\frac{\partial w}{\partial \xi_{3n+1}}
\end{array}\right) = 3K_1 \left(\begin{array}{c}
R_{n+1}^{(i)} \\
\tilde{R}_{n+1}^{(i)}
\end{array}\right) \quad n = 0, 1, 2, \ldots \quad i = 1, 2$$

(3.10)

and they satisfy the following recursion relation

$$K_2 \left(\begin{array}{c}
R_{n}^{(i)} \\
\tilde{R}_{n}^{(i)}
\end{array}\right) = 3K_1 \left(\begin{array}{c}
R_{n+1}^{(i)} \\
\tilde{R}_{n+1}^{(i)}
\end{array}\right)$$

(3.11)

where $K_1$ and $K_2$ are the biHamiltonian structures associated with the Boussinesq hierarchy and they are given below,

$$K_1 = \begin{pmatrix}
0 & \partial \\
\partial & 0
\end{pmatrix}$$

and

$$K_2 = \begin{pmatrix}
\frac{1}{2} \partial^3 + 2u \partial + w' & 3w \partial + 2w' \\
3w \partial + w' & -\frac{2}{3}(\partial^5 + 20u \partial^3 + 30u' \partial^2 + 18u'' \partial + 64u^2 \partial + 4u'' + 64uu')
\end{pmatrix}$$

(3.12)
The first few Gelfand-Dikii polynomials are listed here,

\[
R_0^{(1)} = 1 \\
R_1^{(1)} = \frac{1}{3} w \\
R_2^{(1)} = -\frac{2}{27} u''' - \frac{16}{9} uu'' - \frac{8}{9} (u')^2 + \frac{2}{9} w^2 - \frac{256}{81} u^3 \\
\tilde{R}_0^{(1)} = 0 \\
\tilde{R}_1^{(1)} = \frac{1}{3} u \\
\tilde{R}_2^{(1)} = \frac{1}{18} w'' + \frac{4}{9} uw \\
R_0^{(2)} = 0 \\
R_1^{(2)} = -\frac{2}{9} u'' - \frac{16}{9} u^2 \\
R_2^{(2)} = -\frac{2}{27} w''' - \frac{80}{27} u^2 w - \frac{20}{27} uw'' - \frac{10}{27} u'w' - \frac{10}{27} uu'' \\
\tilde{R}_0^{(2)} = \frac{1}{4} \\
\tilde{R}_1^{(2)} = \frac{1}{6} w \\
\tilde{R}_2^{(2)} = -\frac{1}{27} u''' - \frac{20}{27} uu'' - \frac{5}{9} (u')^2 + \frac{5}{36} w^2 - \frac{80}{81} u^3
\]

(3.13)

The Gelfand-Dikii polynomials appeared in (3.7) could be related to the \(\tau\)-function of the Boussinesq hierarchy by the following relation (see appendix for the derivation),

\[
R_k^{(i)} = (L^{3(k-1)+i})_{-2} + \frac{1}{2}(L^{3(k-1)+i})_{-1} = \frac{1}{2} \partial^2 \partial t_{3(k-1)+i} \log \tau \quad k = 1, 2, \ldots
\]

(3.14)

where the subscript ‘-2’ and ‘-1’ refer to the coefficient of \(\partial^{-2}\) and \(\partial^{-1}\) terms of the expansion of the formal pseudodifferential operators. Using (3.14) in eq.(3.7) and then integrating with respect to \(x\) and \(y\) once and the multiplying by \(\tau\) we get,

\[
\left[ \sum_{k=1,2}^{\infty} \left( k + \frac{i}{3} \right) t_{3k+1} \frac{\partial}{\partial t_{3(k-1)+i}} + \frac{2}{3} xy \right] \tau = 0
\]

(3.15)

This, therefore, is an equivalent form of the string equation (3.6). Following the procedure described in sec.II, it is easy to show that eq.(3.6) can also be written in another form

\[
[L, (ML^{-\frac{3}{2}})_+] = 1
\]

(3.16)
where the operator $M$ is now defined as

$$M = \frac{1}{3} K x K^{-1} + \frac{1}{3} \sum_{n=2}^{\infty} nt \frac{L^{n-1}}{n \neq 0 (\text{mod} 3)}$$  \hspace{1cm} (3.17)$$

We note that, when $M$ is multiplied by $L^{-\frac{2}{3}}$, then the first term and $n = 2$ term in the sum in (3.17) will not contribute in $(ML^{-\frac{2}{3}})_+$. So, (3.16) precisely matches with (3.6).

Using the expression of $M$ in (3.17), we can show that the operators $L$ and $M$ satisfy the following commutation relations,

$$[L^{\frac{1}{3}}, M] = \frac{1}{3}$$ \hspace{1cm} (3.18a)$$

$$[L, ML^{-\frac{2}{3}}] = 1$$ \hspace{1cm} (3.18b)$$

and

$$[M, L^{-\frac{2}{3}}] = \frac{2}{3} L^{-1}$$ \hspace{1cm} (3.18c)$$

From the relations (3.18) and (3.16), we conclude that

$$(ML^{-\frac{2}{3}})_- = \alpha L^{-1}$$ \hspace{1cm} (3.19)$$

where $\alpha$ is again some arbitrary constant. The residue of eq.(3.19) can be shown to be the string equation (3.6). As before, eq.(3.19) now implies,

$$(ML^{n+\frac{1}{3}})_- = ((ML^{-\frac{2}{3}})_- L^{n+1})_+ = 0 \quad n \geq 0$$ \hspace{1cm} (3.20)$$

The expansion of $M$ in terms of Boussinesq Lax operator has the following form

$$M = \frac{1}{3} \sum_{n=1}^{\infty} nt L^{n-1} + \frac{1}{3} \sum_{i=1}^{\infty} V_{i+1} L^{-\frac{i-3}{3}}$$ \hspace{1cm} (3.21)$$

The functions $V_{i+1}$ can be expressed in terms of the $\tau$ function of Boussinesq hierarchy and is given as,

$$V_{i+1} = -i \sum_{\alpha_1 + 2\alpha_2 + 4\alpha_4 + 5\alpha_5 + \cdots = i} (-)^{\alpha_1 + \alpha_2 + \cdots} \left( \frac{\partial_{t_1}}{\alpha_1!} \right)^{\alpha_1} \left( \frac{\partial_{t_2}}{\alpha_2!} \right)^{\alpha_2} \left( \frac{\partial_{t_4}}{\alpha_4!} \right)^{\alpha_4} \left( \frac{\partial_{t_5}}{\alpha_5!} \right)^{\alpha_5} \cdots \log \tau$$ \hspace{1cm} (3.22)$$
For \( n = 0 \), we get,

\[
\text{res } (ML^\frac{n}{3}) = 0 = \frac{1}{3} \sum_{n=1 \atop n \neq 0 (\mod 3)} nt_n \text{ res } L_n^\frac{n}{3} + \frac{1}{3} \sum_{i=1}^{\infty} V_{i+1} \text{ res } L^{-\frac{i}{3}} \tag{3.23}
\]

The second term will contribute only for \( i = 1 \) and therefore, (3.23) can be expressed after integrating with respect to \( x \) and multiplying by \( \tau \) as,

\[
\left[ \sum_{k=0}^{\infty} (k + \frac{i}{3}) t_{3k+i} \frac{\partial}{\partial t_{3k+i}} + C \right] \tau = 0 \tag{3.24}
\]

where ‘\( C \)’ is an integration constant which can not be fixed at this stage.

For \( n \geq 1 \), using the expression of \( M \) in (3.21) we get from \( \text{res } (ML^{n+\frac{1}{3}}) = 0 \) after an integration with respect to \( x \) and multiplying by \( \tau \),

\[
\left[ \sum_{k=0}^{\infty} (k + \frac{i}{3}) t_{3k+i} \frac{\partial}{\partial t_{3k+i}} + \frac{1}{6} \sum_{i+j=3n \atop i,j \neq 0 (\mod 3)} \frac{\partial^2}{\partial t_i \partial t_j} \right] \tau = 0 \tag{3.25}
\]

The functions \( V_{i+1} \) in this case satisfy the identity

\[
\sum_{k=0 \atop i=1,2}^{n} V_{3k+i+1} \text{ res } L_{\frac{3n-3k-i}{3}} = \frac{\partial \log \tau}{\partial t_{3n+i}} + \frac{1}{2} \frac{\partial}{\partial x} \left[ \frac{1}{\tau} \sum_{i+j=3n \atop i,j \neq 0 (\mod 3)} \frac{\partial^2 \tau}{\partial t_i \partial t_j} \right] \tag{3.26}
\]

The eqs. (3.15), (3.24) and (3.25) can be written combinedly as,

\[
L_n \tau = 0 \quad n \geq -1 \tag{3.27}
\]

If we now impose the condition that \( L_n \) would satisfy a centerless Virasoro algebra, then the integration constant in (3.24) can be fixed to be \( \frac{1}{9} \) and the forms of the Virasoro generators would be

\[
\begin{align*}
L_{-1} &= \sum_{k=1 \atop i=1,2}^{\infty} (k + \frac{i}{3}) t_{3k+i} \frac{\partial}{\partial t_{3(k-1)+i}} + \frac{2}{3} xy \\
L_0 &= \sum_{k=0 \atop i=1,2}^{\infty} (k + \frac{i}{3}) t_{3k+i} \frac{\partial}{\partial t_{3k+i}} + \frac{1}{9} \\
L_n &= \sum_{k=0 \atop i=1,2}^{\infty} (k + \frac{i}{3}) t_{3k+i} \frac{\partial}{\partial t_{3(k+n)+i}} + \frac{1}{6} \sum_{i+j=3n \atop i,j \neq 0 (\mod 3)} \frac{\partial^2}{\partial t_i \partial t_j} \quad n \geq 0
\end{align*}
\]
In order to show that \( \tau \)-function of the Boussinesq hierarchy also satisfies \( W^{(3)} \) constraints, we will make use of the relations (3.19) and (3.20). First, we note that the relation (3.18a) implies

\[
[M, L^{n+\frac{2}{3}}] = -(n + \frac{1}{3})L^n
\]  

(3.29)

Therefore,

\[
[(ML^{n+\frac{2}{3}})(ML^{m+\frac{2}{3}})]_- = (M^2L^{m+n+\frac{2}{3}})_- = 0 \quad for \quad m \geq 0, n \geq 0 \quad (3.30)
\]

For \( m + n = -1 \), we can show using (3.19) that,

\[
res \ (ML^{-\frac{1}{3}}) = 0 \quad (3.31)
\]

and finally we have using (3.19) and (3.18c)

\[
\left( M^2L^{-\frac{2}{3}} - (2\alpha + \frac{2}{3})ML^{-\frac{2}{3}} + (\alpha + \alpha^2)L^{-2} \right)_- = 0 \quad (3.32)
\]

We would like to point out here that the particular combination (3.32) produces correct \( W^{(3)}_{-2} \) constraint only for \( \alpha = \frac{1}{3} \) [24]. As we mentioned in the KdV hierarchy case, that \( \alpha \) can not be fixed from the relations (3.18) and the string equation (3.16). It can be fixed to this particular value \( \frac{1}{3} \), by making use of the residual symmetry of the Zakharov-Shabat dressing operator as noted in ref.[34].

The operator \( M^2 \) can be expressed as a power series in the Boussinesq Lax operator (see appendix) as,

\[
M^2 = \frac{1}{9} \left[ \sum_{n=1 \atop n \neq 0, (m \text{ or } 3)}^{\infty} n(n-1)t_nL^{\frac{n-2}{3}} + \sum_{n,m=1 \atop n, m \neq 0, (m \text{ or } 3)}^{\infty} nmt_n t_mL^{\frac{n+m-2}{3}} 
\right.
\]

\[
+ 2 \sum_{n,i=1 \atop n \neq 0, (m \text{ or } 3)}^{\infty} nt_n V_{i+1}L^{\frac{n-i-2}{3}} - \sum_{i=1}^{\infty} (i+1)V_{i+1}L^{\frac{-i-2}{3}}
\]

\[
\left. + \sum_{i,j=1}^{\infty} V_{i+1}V_{j+i}L^{\frac{-i-j-2}{3}} \right] 
\]

(3.33)

Using the expression of \( M \) in (3.21) and \( M^2 \) in (3.33), it is quite straightforward to calculate the residue of (3.32) and we get after an integration with respect to \( x \) and then multiplying...
where $C_6$ is an integration constant which does not depend on $x$ and has scaling dimension 6. We have also made use of the expression of $V_{i+1}$ given in (3.22) and noted that they satisfy the following relation

$$
\sum_{n \geq 7 \atop n \neq 0 \text{ (mod 3)}} n t_n \left[ (n-5) \frac{\partial^2 \log \tau}{\partial x \partial t_{n-6}} - 2 \frac{\partial \log \tau}{\partial t_{n-5}} + 2V_{n-5} \right] + 2 \sum_{n-m \geq 7 \atop n \neq 0 \text{ (mod 3)}} n t_n V_{m+1} \frac{\partial^2 \log \tau}{\partial x \partial t_{n-m-6}}
$$

Similarly, we calculate for $p = -1$

$$
\text{res} \left( M^2 L^{p+\frac{2}{3}} \right) = \frac{1}{9} \left[ \sum_{n \geq 7 \atop n \neq 0 \text{ (mod 3)}} n(n-1) t_n L^{-\frac{n-3}{3}} + \sum_{n, m = 1 \atop n \neq 0 \text{ (mod 3)}} \sum_{n, m = 1 \atop n \neq 0 \text{ (mod 3)}} n m t_n t_m L^{-\frac{n+m-3}{3}} + 2 \sum_{n, m = 1 \atop n \neq 0 \text{ (mod 3)}} n t_n V_{m+1} L^{-\frac{n-m-3}{3}} \right]
$$

Again after integration with respect to $x$ and multiplying by $\tau$ the above expression reduces to,

$$
\frac{1}{9} \left[ \sum_{n \geq 7 \atop n \neq 0 \text{ (mod 3)}} n m t_n t_m \frac{\partial}{\partial t_p} + \sum_{n \geq 7 \atop n \neq 0 \text{ (mod 3)}} n m t_n t_m \frac{\partial^2}{\partial t_n \partial t_p} + \frac{1}{3} \frac{x^3}{L} + C_3 \right] \tau = 0 \quad (3.37)
$$

Here $C_3$ is the integration constant independent of $x$ and has scaling dimension 3. We have used the explicit form of $V_{i+1}$ and made use of the first Virasoro constraint $L_{-1} \tau = 0$. Proceeding in a similar way for higher values of $p$, we recover the higher $W^{(3)}$ constraints in the form

$$
W_n^{(3)} \tau = 0 \quad n \geq 0
$$
where

\[ W_n^{(3)} = \frac{1}{9} \left[ \sum_{p+q+r=3n} \sum_{p,q,r \neq 0 (\text{mod} 3)} pqt_p t_q \frac{\partial}{\partial t_r} + \sum_{p+q+r=3n} \sum_{p,q,r \neq 0 (\text{mod} 3)} pt_p \frac{\partial^2}{\partial t_q \partial t_r} + \frac{1}{3} \sum_{p+q+r=3n} \frac{\partial^3}{\partial t_p \partial t_q \partial t_r} \right] \]

(3.38)

In obtaining this we have to use the Virasoro constraints and also we note the simple identity of the form

\[ \frac{\partial^3 \log \tau}{\partial t_m \partial t_n \partial t_p} + \left( \frac{\partial^2 \log \tau}{\partial t_m \partial t_n} \frac{\partial \log \tau}{\partial t_p} + \frac{\partial^2 \log \tau}{\partial t_n \partial t_p} \frac{\partial \log \tau}{\partial t_m} + \frac{\partial^2 \log \tau}{\partial t_m \partial t_p} \frac{\partial \log \tau}{\partial t_n} \right) + \frac{\partial \log \tau}{\partial t_m} \frac{\partial \log \tau}{\partial t_n} \frac{\partial \log \tau}{\partial t_p} = \frac{1}{\tau} \left( \frac{\partial^3 \tau}{\partial t_m \partial t_n \partial t_p} \right) \]

(3.39)

Note that we have conditions \( p, q, r \neq 0 (\text{mod} 3) \) unlike in ref.[34], because in our definition of the operator \( M \), we have removed the coordinates \( t_{3k}, k = 1, 2, \ldots \) Eqs.(3.34), (3.37) and (3.38) can be written combinedly in the form

\[ W_n^{(3)} \tau = 0 \quad n \geq -2 \]

(3.40)

It is easy to check that the constraints \( L_n \tau = 0 \) for \( n \geq -1 \) and \( W_n^{(3)} \tau = 0 \) for \( n \geq -2 \) satisfy a closed \( W^{(3)} \)-algebra provided the integration constant appearing in (3.36) is \( \frac{8}{27} t_2^3 \) and \( C_3 \) in (3.37) is zero. In that case, we have the standard matrix-model form of the \( W^{(3)} \) constraints

\[ W_n^{(3)} = \frac{1}{9} \left[ \frac{1}{3} \sum_{p+q+r=3n} \sum_{p,q,r \neq 0 (\text{mod} 3)} pqr t_p t_q t_r + \sum_{p+q+r=3n} \sum_{p,q,r \neq 0 (\text{mod} 3)} pqt_p t_q \frac{\partial}{\partial t_r} + \sum_{p+q+r=3n} \sum_{p,q,r \neq 0 (\text{mod} 3)} pt_p \frac{\partial^2}{\partial t_q \partial t_r} + \frac{1}{3} \sum_{p+q+r=3n} \frac{\partial^3}{\partial t_p \partial t_q \partial t_r} \right] \quad n \geq -2 \]

(3.41)

and \( p, q, r \) in all the terms are not \( 0 (\text{mod} 3) \).

**IV. Generalization to Higher KdV Hierarchies:**

Generalization of the method described in sections II and III can be most easily done if we note that the operator \( M \) for the \( p \)-reduced KP hierarchy satisfies

\[ [L^\frac{1}{p}, M] = \frac{1}{p} \]

(4.1)

17
The Lax operator for the $p$-reduced KP hierarchy is defined as

$$L = \partial^p + u_p - 2\partial^{p-2} + \cdots + u_0$$  \hspace{1cm} (4.2)

and the additional symmetry operator $M$ has the form

$$M = \frac{1}{p} K \left( \sum_{n=1}^{\infty} n t_n \partial^{n-1} \right) K^{-1}$$  \hspace{1cm} (4.3)

Douglas’ string equation for the general massive model in this case can be written as,

$$\sum_{k=1}^{\infty} \left( k + \frac{i}{p} \right) t_{kp+i} \left[ L, L_{\pm}^{p(k-1)+i} \right] = 1$$  \hspace{1cm} (4.4)

Since the operators $L$ and $M$ satisfies the fundamental relation (4.1), we can show that

$$[L, M L^{p-1}] = 1$$  \hspace{1cm} (4.5)

Using the definition of $M$ in (4.3), it is an easy exercise to check that (4.4) can also be written equivalently as,

$$[L, (ML^{p-1})_{+}] = 1$$  \hspace{1cm} (4.6)

The string equation (4.6) and the relation (4.5) together therefore implies,

$$[L, (ML^{p-1})_{-}] = 0$$  \hspace{1cm} (4.7)

It is therefore clear that the operator $(ML^{p-1})_{-}$ should be some negative powers of $L$. Using the relation (4.1) we can also show that

$$[M, L^{p-1}] = \frac{p-1}{p} L^{-1}$$  \hspace{1cm} (4.8)

From (4.8) we conclude that

$$(ML^{p-1})_{-} = \alpha L^{-1}$$  \hspace{1cm} (4.9)

where $\alpha$ is an arbitrary constant. Evaluating the residue of (4.9) we can easily show that this is equivalent to the string equation (4.4) or (4.6). Eq.(4.9) gives an infinite set of relations of the form

$$(ML^{n+\frac{1}{p}})_{-} = ((ML^{\frac{1}{p}-1})_{-} L^{n+1})_{-} = 0 \quad n \geq 0$$  \hspace{1cm} (4.10)
The consequences of (4.9) and (4.10) together can be worked out using the expression of $M$ in (4.3) and the Lax operator (4.2) of the $p$-reduced KP hierarchy. Calculating the residue and after an integration with respect to $x$ and then multiplying by $\tau$, they can be shown to be equivalent to the semi-infinite set of Virasoro constraints following the discussions in sections II and III,

$$L_n \tau = 0 \quad n \geq -1$$

where $L_n$’s would have the form

$$L_n = \frac{1}{2p} \sum_{i+j=-pm} ij t_i t_j + \frac{1}{p} \sum_{i-j=-pm} it_i \frac{\partial}{\partial t_j}$$

$$+ \frac{1}{2p} \sum_{i+j=pm} \frac{\partial^2}{\partial t_i \partial t_j} + \frac{p^2 - 1}{24p} \delta_{n,-1}$$

for $n \geq -1$ and $i, j$ do not take values $0(\text{mod}p)$.

In order to get higher $W$-constraints, one has to take various powers of the operator $(ML_1^p - 1)$. They could be simplified using the basic relation (4.1) as explained for the case of $W^{(3)}$ in section III. Also, we point out that $\alpha$ in (4.9) could remain arbitrary in order to get correct Virasoro constraints as we just explained but this no longer remains to be true for obtaining $W$ constraints. We saw in section III that the constant has to be fixed to a particular value for obtaining correct $W^{(3)}$ constraints. In this case the constant has to be chosen as $\frac{p-1}{2p}$. This is usually done by using the residual symmetry of the operator $K$ noted in ref.[34]. In order to obtain higher powers of $M$ as a power series expansion of the Lax operator, we make use of the relation

$$M \psi = \frac{1}{p} \frac{\partial \psi}{\partial \lambda}$$

where $\psi$ is known as the Baker-Akhiezer function of the $p$-reduced KP hierarchy [26] defined as

$$\psi = K \exp \left( \sum_{n=1 \atop n \neq 0(\text{mod}p)} t_n \lambda^n \right)$$

and $\lambda$ is the spectral parameter. We note that the Baker-Akhiezer function for $p$-reduced KP hierarchy does not depend on the coordinates $t_{kp}, k = 1, 2, \ldots$. By successively applying
$M$ to (4.13) one can obtain

$$M^n \psi = \frac{1}{p^n} \frac{\partial^n \psi}{\partial \lambda^n} \quad n < p \quad (4.15)$$

One can easily express $(\frac{1}{\psi} \frac{\partial^n \psi}{\partial \lambda^n})$ in terms of log $\psi$ using the recursion relation of the form

$$A_n = \frac{\partial A_{n-1}}{\partial \lambda} + \frac{\partial \log \psi}{\partial \lambda} A_{n-1} \quad n \geq 1 \quad (4.16)$$

where we defined $A_n \equiv \frac{1}{\psi} \frac{\partial^n \psi}{\partial \lambda^n}$. Expressing log $\psi$ in powers of $\lambda$ as,

$$p \log \psi = \sum_{n=1}^{\infty} t_n \lambda^n - \sum_{i=1}^{\infty} \frac{1}{t} V_{i+1} \lambda^{-i} \quad (4.17)$$

and then plugging (4.17) and (4.16) in (4.15) we obtain an expression of $M^n$ in power series of $L$ after replacing $\lambda$ by $L^\frac{1}{p}$. With this procedure, it is quite straightforward to calculate various powers of $(ML^\frac{1}{p} - 1)$. Thus using the procedure described in sections II and III, we can get the full set of $W^{(p)}$ constraints.

**V. Conclusions:**

By expanding the additional symmetry operator associated with the $p$-reduced KP hierarchy as a power series in the Lax operator we have shown directly that the Virasoro and the $W$-constraints do indeed follow from the string equation for such integrable systems. Our method also clarifies the reason for the appearance of the constant term in $L_0$ constraint and the appearance of $t_2^3$ term in $W^{(3)}_{-2}$ constraint which are not clear in other approaches. We noted that these terms appear as integration constants and can be fixed from the algebra in analogy with the matrix model results. We have emphasized that the correct Virasoro constraints may be obtained without any restriction on the additional symmetry operator $M$, but the correct $W$- constraints follow only if we put an additional condition $\frac{\partial M}{\partial t_k} = 0$, $k = 1, 2, \ldots$. We have noticed that the generators of the additional symmetries $(M^n L^{m+\frac{1}{p}})$, $1 \leq n \leq p - 1$, $m \geq -1$ give rise to the Virasoro and the $W$-constraints if we consider only the residue of this operator. There are infinite other conditions corresponding to the coefficients of $\partial^{-2}, \partial^{-3}, \ldots$ terms which remain unexplored.
It would be interesting to see whether they correspond to further constraints. It is possible that they are nothing but the Virasoro and $W$-constraints in various guises. Also, it is not clear what sort of constraints one would get if one considers the operators $(ML^{m+n/p})$ for $n \geq p$. These questions are presently under investigation.

**Acknowledgements:**

The authors would like to thank Professor Abdus Salam, International Atomic Energy Agency and UNESCO at the International Centre for Theoretical Physics, Trieste for hospitality and support.
Appendix

Here we list a few useful formulas for the KP hierarchy. The corresponding formulas for \( p \)-reduced KP hierarchy which were used in the text can be obtained easily using the reduction condition \( (L_{KP}^p)_- = 0 \) and \( \frac{\partial \tau}{\partial t_{kp}} = 0 \) for \( k = 1, 2, \ldots \). Here \( L_{KP} \) is the KP Lax operator given as

\[
L_{KP} = \partial + \sum_{i=1}^{\infty} u_{i+1}(t) \partial^{-i}
\]  

(A.1)

where \( t \) denotes the infinite set of evolution parameters \( (t_1, t_2, t_3, \ldots) \) and \( \tau \) is the \( \tau \)-function of the \( p \)-reduced KP hierarchy. The KP hierarchy is described in terms of the Lax equation

\[
\frac{\partial L_{KP}}{\partial t_n} = [(L_{KP})_+, L_{KP}]
\]  

(A.2)

From the Lax equation \( t_1 \) can be identified with \( x \). The Baker-Akhiezer function \( \psi \) satisfies the eigenvalue equation

\[
L_{KP}\psi = \lambda \psi
\]  

(A.3)

and has a solution of the form

\[
\psi(t) = K(t) \exp \left( \sum_{n=1}^{\infty} t_n \lambda^n \right)
\]  

(A.4)

Where the Zakharov-Shabat dressing operator is defined as

\[
K(t) = 1 + \sum_{i=1}^{\infty} a_i \partial^{-i}
\]  

(A.5)

The additional symmetry operator \( M \) is defined as

\[
M = K(\sum_{n=1}^{\infty} nt_n \partial^{n-1})K^{-1}
\]  

(A.6)

\[
= \sum_{n=1}^{\infty} nt_n L^{n-1} + \sum_{i=1}^{\infty} V_{i+1} L^{-i-1}
\]

where \( V_{i+1} = -(ia_i + \sum_{j=1}^{i-1} ja_j \tilde{a}_{i-j}) \). The first two \( V \)'s have the form

\[
V_2 = -a_1
\]  

(A.7a)
\[ V_3 = -2a_2 + a_1^2 \]  
(A.7b)

The operators \( L_{KP} \) and \( M \) satisfies

\[ [L_{KP}, M] = 1 \]  
(A.8)

From the form of \( M \) in (A.6) we derive,

\[ M\psi = \frac{\partial \psi}{\partial \lambda} \]  
(A.9)

which gives

\[ \sum_{n=1}^{\infty} nt_n \lambda^{n-1} + \sum_{i=1}^{\infty} V_{i+1} \lambda^{-i-1} = \frac{\partial \log \psi}{\partial \lambda} \]  
(A.10)

In terms of the \( \tau \)-function

\[ \psi = \left( \frac{\exp(\sum_{n=1}^{\infty} t_n \lambda^n) - \exp(-\sum_{n=1}^{\infty} \frac{1}{n} \lambda^{-n} \frac{\partial}{\partial t_n})}{\tau(t)} \right) \tau(t) \]  
(A.11)

So,

\[ \log \psi = \sum_{n=1}^{\infty} t_n \lambda^n + \sum_{N=1}^{\infty} \frac{1}{N!} \left( -\sum_{n=1}^{\infty} \frac{1}{n} \lambda^{-n} \frac{\partial}{\partial t_n} \right)^N \log \tau(t) \]  
(A.12)

Comparing (A.12) and (A.10) we obtain

\[ V_{i+1} = -i \sum_{\alpha_1+2\alpha_2+3\alpha_3+\cdots = i} (-)^{\alpha_1+\alpha_2+\alpha_3+\cdots} \frac{(\partial t_1)^{\alpha_1}}{\alpha_1!} \frac{(\frac{1}{2}\partial t_2)^{\alpha_2}}{\alpha_2!} \frac{(\frac{1}{3}\partial t_3)^{\alpha_3}}{\alpha_3!} \cdots \log \tau(t) \]  
(A.13)

The first two \( V \)'s calculated from above have the form

\[ V_2 = \frac{\partial \log \tau}{\partial t_1} \]  
(A.14a)

\[ V_3 = \frac{\partial \log \tau}{\partial t_2} - \frac{\partial^2 \log \tau}{\partial t_1^2} \]  
(A.14b)

From the evolution equation (A.2), we get the evolution equation for the Zakharov-Shabat dressing operator as

\[ \frac{\partial K}{\partial t_n} = -(L_{KP}^n)_- K \]  
(A.15)

where we made use of the fact that the Zakharov-Shabat dressing operator satisfies

\[ L_{KP} = K \partial K^{-1} \]  
(A.16)
From (A.15) we get
\[ \frac{\partial a_1}{\partial t_n} = -\text{res } L_{KP}^n \] (A.17a)
and
\[ \frac{\partial a_2}{\partial t_n} = -(\text{res } L_{KP}^n) a_1 - (L_{KP}^n)_{-2} \] (A.17b)

Since \( V_2 = a_1 = \frac{\partial \log \tau}{\partial t_1} \), so, from (A.17a) we get
\[ \frac{\partial^2 \log \tau}{\partial t_1 \partial t_n} = \text{res } L_{KP}^n \] (A.18)

Similarly from (A.7b)
\[ a_2 = -\frac{1}{2} \frac{\partial \log \tau}{\partial t_2} + \frac{1}{2} \frac{\partial^2 \log \tau}{\partial t_1^2} + \frac{1}{2} \left( \frac{\partial \log \tau}{\partial t_1} \right)^2 \] (A.19)

Using (A.19) in (A.17b) we get,
\[ \frac{\partial^2 \log \tau}{\partial t_2 \partial t_n} = (\text{res } L_{KP}^n)' + 2(L_{KP}^n)_{-2} \] (A.20)

We also show how \( M^2 \) can be calculated using (A.9). Since \( M^2 \psi = \frac{\partial^2 \psi}{\partial \lambda^2} \) so,
\[ M^2 \psi = \left[ \frac{\partial^2 \log \psi}{\partial \lambda^2} + \left( \frac{\partial \log \psi}{\partial \lambda} \right)^2 \right] \psi \] (A.21)

Using (A.10) we get,
\[ M^2 \psi = \left[ \sum_{n=1}^{\infty} n(n-1)t_n \lambda^{n-2} + \sum_{i=1}^{\infty} (-i-1)V_{i+1} \lambda^{-i-1} \right. \\
+ \left( \sum_{n=1}^{\infty} nt_n \lambda^{n-1} + \sum_{i=1}^{\infty} V_{i+1} \lambda^{-i-1} \right) \left( \sum_{m=1}^{\infty} m t_m \lambda^{m-1} + \sum_{j=1}^{\infty} V_{j+1} \lambda^{-j-1} \right) \right] \] (A.22)

So, in terms of the Lax operator \( M^2 \) can be expressed as,
\[ M^2 = \sum_{n=1}^{\infty} n(n-1)t_n L_{KP}^{n-2} + \sum_{i=1}^{\infty} (-i-1)V_{i+1} L_{KP}^{i-2} + \sum_{n,m=1}^{\infty} n m t_n t_m L_{KP}^{n+m-2} \]
\[ + 2 \sum_{n,i=1}^{\infty} n t_n V_{i+1} L_{KP}^{n-i-2} + \sum_{i,j=1}^{\infty} V_{i+1} V_{j+1} L_{KP}^{i-j-2} \] (A.23)

24
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