Convergence of Ricci Flows with Bounded Scalar Curvature

Richard Bamler

May 5, 2016

We will start with an elementary problem in Ricci flow. Consider a smooth family of metrics $g(t)$ with $t \in [0,T)$ on a compact manifold $M^n$. The Ricci flow equation reads $\frac{\partial}{\partial t} g(t) = -2Ric_g(t)$ and we are guaranteed solutions to this equation on some maximal time interval $[0,T)$ after which we may reach a singularity. Assume $T < \infty$. The result from Hamilton ’82 is that at a maximal time $T$ we have $\max_M \|Rm\|_{(\cdot,t)} \to \infty$ as $t \to T$. A more refined result by Sesum ’03 shows that the norm of the Ricci tensor $\|Ric\|$ becomes unbounded on $M \times [0,T)$. This yields a natural question/conjecture?

**Conjecture 1** Does the scalar curvature $R$ become unbounded on $M \times [0,T)$?

A remark is that this conjecture is true when $n \in \{2,3\}$ where $n=3$ is proved by the Hamilton-Ivey pinching, and the result is also true for Kähler manifolds. We also have

$$\frac{\partial}{\partial t} R = \Delta R + 2 \|Ric\|^2 \geq \Delta R$$

hence by the maximum principle we have that $R$ is bounded from below, so $R > -C$. It’d be interesting as well to consider the contrapositive of this conjecture, or rather

1. Assuming that $R < C$ on $M \times [0,T)$, what is the behavior of the metric $g(t)$ as $t \to T$?
2. Assume that $R(\cdot,t) < \frac{C}{T-t}$ like in a type 1 singularity, and that $\text{diam}_t M < C\sqrt{T-t}$, what is the behavior of the rescaled metric $g(t)_{T-t}$ as $t \to T$?

Question 2 here is interesting here since Perelman proved that if $M$ is a Fano manifold an $g$ is a Kähler Ricci flow, then both the inequalities listed are true. This yields another conjecture

**Conjecture 2 (Hamilton-Tian)** If $g(t)$ is a Kähler-Ricci flow and $M$ is Fano, then $(T-t)^{-1}g(t)$ subconverges to a gradient shrinking soliton away from codimension $\geq 4$ as $t \to T$.

We will discuss for now the case that $n = 4$. Last year there was a theorem proved in this direction:

**Theorem 1 (Bamler-Zhang ’15, Simon ’15)** If $n = 4$ and $R < C$ on $M \times [0,T)$, then

1. $\int_M |Rm|^2(\cdot,t) < C$ for all $t \in [0,T)$,
2. $\int_M |Ric|^{4-\epsilon}(\cdot,t) < C_\epsilon$ for all $t \in [0,T)$ and $\epsilon > 0$,
3. $\int_0^T \int_M |Ric|^4 < \infty$,
4. $(M,g(t))$ in the limit as $t \to T$ approaches a $C^0$ orbifold $(M_T,g(T))$, and
5. the Ricci flow can be continued from $(M_T,g(T))$, hence we obtain a smooth orbifold structure.
We can gain some intuition behind this theorem by picking an example. Recall the Kummer construction where we have a surface

\[ K3^4 = (\mathbb{T}^4/\mathbb{Z}_2) \#_{\mathbb{RP}^3} 16EH^4. \]

Cross-sections around the corners of \( \mathbb{T}^4/\mathbb{Z}_2 \) are diffeomorphic to \( \mathbb{RP}^3 \), so we can glue in Eguchi-Hanson metrics along these copies of \( \mathbb{RP}^3 \). We therefore obtain an almost Ricci flat metric on \( K3^4 \). In this gluing process, we can choose the size of the the Eguchi-Hanson metrics arbitrarily small. So our gluing process generates a family of almost Ricci flat metrics that degenerate to the original orbifold \( \mathbb{T}^4/\mathbb{Z}_2 \).

A third question naturally arises: **Can such a degeneration occur in a Ricci flow?** In higher dimensions we will look at a Ricci flow \( g(t) \) on \( M \times [0,T) \) where \( T < \infty \).

We now move to higher dimensions \( n \geq 4 \). In the setting of Question 1, we know the following:

**Theorem 2 (Bamler-Zhang ’15)** If we have an upper bound \( C \) on our scalar curvature \( R < C \) on \( M \times [0,T) \) then \( d_T = \lim_{t \to T} d_t \) exists and is a pseudometric.

**Theorem 3 (Bamler ’15)** If \( R < C \) on \( M \times [0,T) \) then there exists an open subset \( \mathcal{R} \subset M \) such that

1. \( g(t) \to g(T) \) in a \( C^\infty \) manner as \( t \to T \), when restricted to \( \mathcal{R} \).
2. given the relation \( x \sim y \) if and only if \( d(x,y) = 0 \), we have that

   \[ (M_T := M/\sim, d_T) \cong (\mathcal{R}, g(T)) \]

   where the overbar indicates the completion of the submanifold \( \mathcal{R} \).
3. \( \dim_M(M_T \setminus \mathcal{R}) \leq n - 4 \), where \( \dim_M \) denotes the Minkowski dimension.

We have an additional theorem in the setting of Question 2;
Theorem 4 (Bamler’15) If \( R(t) < \frac{C}{T-t} \) and \( \text{diam}_t M < C \sqrt{T-t} \) then for any \( t_i \to T \) there is a subsequence such that

\[
\left( M, \frac{g(t_i)}{T-t_i} \right) \to_{i \to \infty} (X, d_X)
\]

and \( X = \mathcal{R} \cup \mathcal{S} \) such that

1. \( d_X |_{\mathcal{R}} \) is isometric to the length metric of a smooth Riemannian manifold \( g_\infty \).
2. \( \frac{g(t_i)}{T-t_i} \to g_\infty \) on \( \mathcal{R} \) in a \( C^\infty \) manner as \( i \to \infty \).
3. There exists \( f \in C^\infty(\mathcal{R}) \) such that \( \text{Ric}_{g_\infty} + \nabla^2 f = \frac{1}{2} g_\infty \) on \( \mathcal{R} \).
4. \( \dim_{M, d_X}(\mathcal{S}) \leq n - 4 \).

There is a corollary to this theorem, which is

Corollary 1 The Hamilton-Tian conjecture is true.

A remark about this is that there is progress in the complex geometric setting currently being addressed by Tian-Zhang and Chen-Wang. Further results include a compactness theorem (Bamler’15). Particularly, if \( (M \times [-2, 0], g_t) \) are Ricci flows with \( R_{g_t(t)} < C \) on \( M \times [-2, 0] \), and \( \mu[g_t(\cdot), \frac{1}{2}] > -C \). We also have that there exists a subsequence so that

\[
(M, g(0)) \to (X, d_X, \mathcal{R}, g_\infty)
\]

just as before with \( \dim_{M}(X \setminus \mathcal{R}) \leq n - 4 \).

Definition 1 (Curvature Radius) Let \( (M, g) \) be a Riemannian manifold. We define the curvature radius to be the quantity

\[
r_{\text{Rm}}(x) := \sup\{r > 0 : |\text{Rm}| < \frac{1}{r^2} \text{ on } B(x, r)\}.
\]

There is additionally a curvature bound result (Bamler’15): Let \( (M \times [-2, 0], g_t) \) be a Ricci flow with

1. \( R < A \) on \( M \times [-2, 0] \), and
2. \( \mu(g(-2), \frac{1}{2}) > -A \).

then with \( 0 < r < 1 \) we have

\[
\int_{B(x,0,r)} |\text{Rm}|^{2-\epsilon}(\cdot,0) \leq \int_{B(x,0,r)} r_{\text{Rm}}^{-1+2\epsilon}(\cdot,0) < C(A, \epsilon)r^{n-4+2\epsilon}.
\]

There are some ingredients that are used in setting up this proof. Suppose for the remaining discussion that \( R < 1 \) on \( M \times [-2, 0] \), and that we have \( \mu(g(-2), \frac{1}{2}) > -A \), where \( A \) is a lower entropy bound.

A result by Perelman, Zhang, and Chen-Wang shows that for all \( (x,t) \in M \times [-1, 0] \) with \( 0 < r < 1 \) we have that

\[
k_1(A)r^n < |B(x,t,r)| < k_2(A)r^n.
\]

A result by Bamler-Zhang shows that if \( t_1, t_2 \in [-2, 0] \) and \( d_{t_1}(x,y) < D \) then

\[
|d_{t_1}(x,y) - d_{t_2}(x,y)| < C(D)\sqrt{|t_1-t_2|}.
\]

Another result by Bamler-Zhang is that if \( -1 \leq s < t \leq 0 \) with \( x,y \in M \) we have lower and upper Gaussian bounds:
\[
\frac{1}{C(t-s)^{n/2}} \exp \left( \frac{C d^2_s(x,y)}{t-s} \right) < k(x,t,y,s) < \frac{C}{(t-s)^{n/2}} \exp \left( \frac{d^2_s(x,y)}{C(t-s)} \right).
\]

where \( k \) is a heat kernel of the heat equation \( \partial_t - \Delta = 0 \) coupled with Ricci flow. Lastly there is a backwards pseudo-locality bound (Bamler-Zhang) where if \( 0 < r < 1 \) and \( (x,t) \in M \times [-1,0] \) with the conditions that \( |Rm|(<,t) < \frac{1}{r^2} \) on \( B(x,t,r) \) then this implies that \( |Rm| < \frac{2}{r^2} \) on \( B(x,t,\frac{r}{2}) \times \{ t-er^2, t \} \).

This yields a corollary:

**Corollary 2** If \( |Rm|(,t) < \frac{1}{r^2} \) on \( B(x,t,r) \) then \( |Ric|(x,t) < \frac{C}{r^2} \).

We also provide a quick idea of the proof of the compactness theorem mentioned earlier. Suppose a priori
\[
\int_{B(x,t,r)} r^{-3.1}_{RM}(<,t) < Er^{n-3.1}
\]
We can deduce that under this a priori assumption, every blowup of a Ricci flow has the form \((X, d_X, R, g)\), as in the previously mentioned compactness theorem. Moreover, this blowup is Ricci flat on its regular subset. Using the results of Cheeger-Colding-Naber, it is then possible to deduce \( L^p \) bounds on \( r^{-1}_{RM} \), which improve the a priori assumptions.