The Running Coupling from SU(3) Lattice Gauge Theory

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Abstract

From an accurate determination of the inter-quark potential, one can study the running coupling constant for a range of $R$-values and hence estimate the scale $\Lambda_{\overline{MS}}$. Detailed results are presented for SU(3) pure gauge theory.

1 Introduction

In the continuum the potential between static quarks is known perturbatively to two loops in terms of the scale $\Lambda_{\overline{MS}}$. For SU(3) colour, the continuum force is given by

$$\frac{dV}{dR} = \frac{4}{3} \frac{\alpha(R)}{R^2}$$

with the effective coupling $\alpha(R)$ defined as

$$\alpha(R) = \frac{1}{4\pi \left[b_0 \log(R \Lambda_R)^{-2} + (b_1/b_0) \log \log(R \Lambda_R)^{-2}\right]}$$

where $b_0 = 11/16\pi^2$ and $b_1 = 102b_0^2/121$ are the usual coefficients in the perturbative expression for the $\beta$-function and, neglecting quark loops in the
vacuum, \( \Lambda_R = 1.048 \Lambda_{\text{MS}} \). Note that the usual lattice regularisation scale \( \Lambda_L = 0.03471 \Lambda_{\text{MS}} \).

At large separation \( R \), the potential behaves as \( KR \) where \( K \) is the string tension. Thus in principle knowledge of the potential \( V(R) \) serves to determine the dimensionless ratio \( \sqrt{K}/\Lambda \) which relates the perturbative scale \( \Lambda \) to a non-perturbative observable such as the string tension \( K \). This is the basis of the method \([2]\) used for \( SU(2) \) which we extend here to \( SU(3) \) colour.

For \( SU(3) \) as for \( SU(2) \), the bare lattice coupling proves to be a poor guide to physical behaviour in that asymptotic scaling to two loops is not yet manifest. However, the weaker requirement of scaling is well satisfied: the dimensionless ratios of physical quantities are found to be independent of \( \beta \). For example in \( SU(2) \), the potential \( V(R) \) scales \([3]\) over a range of lattice spacing of a factor of 4 (from \( \beta = 2.4 \) to 2.85). That scaling but not asymptotic scaling is valid implies that the bare coupling constant derived from \( \beta \) is inappropriate and that an effective coupling constant derived from some physical quantity is a better choice. This has been emphasized by Lepage and Mackenzie \([4]\). It is also the basis of the method proposed by Lüscher et al. \([5]\) to extract the running coupling constant.

Here we choose to determine the running coupling constant from the interquark potential between static quarks at small distance \( R \). This quantity can be determined in a straightforward way from lattice simulation on large volume lattices. Although we require small \( R \) and hence large energy \( 1/R \) to make most precise contact with the perturbative expression, the lattice method implies the presence of lattice artefacts when \( R \approx a \). Thus we need to work on the largest spatial lattice available, consistent with avoiding finite size effects. We present results from a \( 36^4 \) lattice at \( \beta = 6.5 \) to achieve this. These results are compared with previous UKQCD data \([6]\) from \( 24^3 \times 48 \) lattices at \( \beta = 6.2 \) to check scaling.

## 2 Lattice potentials

Our main aim is to explore the interquark potential at as large a \( \beta \)-value as feasible. We wish to remain in the large volume region so memory constraints are limiting. On our Meiko Computing Surface with 64 i860 processors, each having 16 megabytes main memory, we are able to update at most a \( 36^4 \) \( SU(3) \) lattice efficiently. Previous work \([6, 8]\) has shown that at \( \beta = 6.5 \) such a lattice should be suitable.
The difficult part of the analysis is to determine the string tension accurately on such lattice configurations. In order to measure the largest number of configurations within our data communication constraints, we use rather similar methods to those we used to determine the string tension in SU(2) [3]. We first combine the spatial links of the lattice once using a sum of straight and U-bends of length 2 to an effective $18^3 \times 36$ lattice. The spatial links of these lattices are then further smeared repeatedly (APE smearing [11] with SU(3) projection of $2.5 \times$ straight link plus four spatial U-bends) to build up paths between the static sources. Building up spatial paths from these links, we measure generalised Wilson loops to determine the potentials at on-axis separations with $R/a = 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24$. We also measure at planar off-axis separations with $R/a$-vectors $(2,2,0), (4,2,0), (4,4,0), (6,2,0), (6,4,0)$ and $(6,6,0)$. We use a recursive smearing with 40 and 10 levels — this provides a $2 \times 2$ basis for the standard variational technique of finding the optimum combination of these two spatial paths at each $R$-value.

We measure potentials for the $R$-separations listed above and $T$-separations 0 to 7. The lattice is well equilibrated with 1200 heat-bath sweeps and configurations are measured every 150 sweeps thereafter, using 4 pseudo-overrelaxation sweeps for every heatbath sweep. ‘Pseudo-overrelaxation’ refers to performing SU(2) overrelaxation steps in 3 subgroups of SU(3). We are able to perform a complete updating step in 27 seconds for pseudo-relaxation and in 30 seconds for the heatbath. The similarity between these timings may seem surprising, but one major reason is that message-passing between processors is the most significant factor in both cases. All the CPU-intensive parts of our programme are written in well-optimised assembler code. We obtain an average action $S = 0.63835(2)$.

We measure the on-axis potentials on 50 configurations. With such limited statistics an accurate determination of the autocorrelation is not possible, but our measurements are consistent for all observables with an autocorrelation time of less than 225 sweeps. Our error analysis is obtained from bootstrap analysis of these 50 measurements which we treat as being statistically independent since we find equivalent results when using 25 blocks of 2 measurements.

The potentials are given by the extrapolation in $T$ of the ratio of generalised Wilson loops,

$$ V(R) = \lim_{T \to \infty} V(R, T), $$
where

$$V(R, T) = -\log \frac{W(R, T)}{W(R, T - a)}$$

We take the optimum linear combination of paths to determine the ground state potential where we use the $2/1$ $T$-ratio for this purpose since the statistical errors are quite small at those $T$-values. We find that this path combination gives an overlap of 85% or more. Now there is a monotonic decrease with $T$ of the effective potential $V(R, T)$ evaluated from the correlations $(W(R, T))$ in that path eigenvector. One way to select the limit at large $T$ is to identify a plateau where $V(R, T)$ and $V(R, T + a)$ analyses give consistent values within the error on the difference. We find that the $4/3$ $T$-ratio is consistent with the start of such a plateau. As a check on possible systematic errors arising from this $T$-extrapolation we can try to fit the decrease of $V(R, T)$ with $T$ using the energy gap to the first excited state. For the larger $R$ potentials where this extrapolation in $T$ may introduce sizeable errors, we obtain estimates of this energy gap from our variational method in the 2:1 $T$-ratio basis and these estimates agree with the string model estimate $2\pi/R$. We then use those estimates to obtain a systematic error associated with the $T$-extrapolation. We find consistency between the effective potential based on $T$-values 2-4 and on $T$-values 3-5 which confirms the stability of the method. In order to represent the data in table form it is better to give the force since the errors there are less correlated between different $R$-values. The force derived from our potential measurements of adjacent $R$-values with $T$-ratio 4/3 is shown in table 1.

We fit the $R$-dependence of the potential at large $R$ to obtain the string tension $K$ with

$$V(R) = C - \frac{E}{R} + KR$$

where for the Coulomb component we use a lattice one gluon exchange propagator (see below) even though at large $R$ this is very close to the continuum expression $1/R$. When we take full account in the fit of the statistical correlations between Wilson loops at different $R$-values, we get an acceptable $\chi^2$ but some sign of instability in inverting the $15 \times 15$ correlation matrix. Fitting instead to the force with diagonal errors gives similar results which are more stable. These results to a fit for $R \geq 4a$ are shown in table 2.

As an estimate of the systematic error coming from $T$-extrapolation, we fitted potential values from several different prescriptions. Using $T$-ratio 5/4 gave $Ka^2 = 0.0110(3)$; extrapolating (as discussed above) the $T$ 2-4 potentials gave 0.0111(4) while extrapolating 3-5 potentials gave 0.0107(8).
Table 1: The force $\Delta V/\Delta R$ at average separation $R$ derived from $T$-ratio 4/3.

| $R/a$ | $\Delta V/\Delta R$  |
|-------|-----------------------|
| 2.4142 | 0.0667(4)             |
| 3.4142 | 0.0344(4)             |
| 4.2361 | 0.0286(8)             |
| 5.0645 | 0.0226(5)             |
| 5.8284 | 0.0196(19)            |
| 6.1623 | 0.0190(16)            |
| 6.7678 | 0.0174(9)             |
| 7.6056 | 0.0161(10)            |
| 8.2426 | 0.0161(11)            |
| 9.0000 | 0.0149(3)             |
| 11.0000| 0.0130(4)             |
| 13.0000| 0.0134(5)             |
| 15.0000| 0.0121(5)             |
| 17.0000| 0.0132(6)             |
| 19.0000| 0.0117(6)             |
| 21.0000| 0.0125(8)             |
| 23.0000| 0.0126(7)             |

Table 2: The fitted parameters to the potential for $R \geq 4a$.

| $E$      | $K a^2$   | $\chi^2$/d.o.f. |
|----------|-----------|-----------------|
| 0.278(7) | 0.0114(2) | 9.1/13          |

Hence the systematic error from $T$-extrapolation appears comparable to the statistical error and we quote $K a^2 = 0.0114(4)$ as an overall determination.

As well as such a high statistics study of the potential at large $R$, we determine the small $R$ potential at a variety of on- and off-axis $R$-values. These have separations with $R/a$ vectors (1,0,0), (1,1,0), (2,0,0), (2,1,0), (2,2,0), (3,0,0), (3,1,0), (3,2,0) and (3,3,0). For this analysis we use the full spatial configuration and a recursive smearing [10] with $c = 2.0$ and 30 iterations. Since this small $R$-potential is easily measured, we use lower statistics. We evaluate 4 configurations (as 8 half-configurations for error analysis). The force obtained from these results using $T$-ratio 4/3 is shown in table 3. Results for the potential at common $R$-values between the two methods are consistent, with the latter method giving a larger overlap (greater than 97%).
\[ \Delta \frac{V}{\Delta R} / \Delta \frac{R}{\Delta R} \]

| \( R/a \) | \( \Delta V / \Delta R \) | \( \Delta V_c / \Delta R \) | \( \alpha(R) \) |
|---|---|---|---|
| 1.2071 | 0.2067(7) | 0.1607 | 0.170(1)(5) |
| 1.7071 | 0.0750(7) | 0.0930 | 0.197(1)(4) |
| 2.1180 | 0.0959(19) | 0.0664 | 0.223(6)(10) |
| 2.5322 | 0.0541(5) | 0.0523 | 0.248(2)(1) |
| 2.9142 | 0.0263(40) | 0.0424 | 0.270(26)(10) |
| 3.0811 | 0.0471(39) | 0.0368 | 0.262(28)(7) |
| 3.3839 | 0.0391(12) | 0.0371 | 0.317(10)(2) |
| 3.9241 | 0.0292(5) | 0.0290 | 0.333(6)(1) |

Table 3: The force \( \Delta V / \Delta R \) and lattice artefact corrected force \( \Delta V_c / \Delta R \) at average separation \( R \). The running coupling \( \alpha(R) \) derived from the corrected force is shown as well. The second error shown on \( \alpha \) is 10\% of the lattice artefact correction.

| \( A \) | \( f \) | \( K a^2 \) | \( B/a \) |
|---|---|---|---|
| 0.311(14) | 0.64(6) | 0.01138 | 0.067(13) |

Table 4: Fit to force from table 3 for \( R > a \).

The potential shows a lack of rotational invariance at small \( R \). To lowest order this can be attributed to the difference \( \delta G(R) \) between the lattice one gluon exchange expression and the continuum expression.

\[
\delta G(R) = \frac{4\pi}{a} \int_{-\pi}^{\pi} \frac{d^3 k}{(2\pi)^3} \frac{e^{ik.R/a}}{4 \sum_{i=1}^{3} \sin^2(k_i/2)} - \frac{1}{R}
\]

On a lattice, the next order of perturbation has been calculated \[11\] and the dominant effect is a change from the bare coupling to an effective coupling \[4\]. In that case, using the difference above but with an adjustable strength will correct for the small \( R/a \) lack of rotational invariance. A test of this will be that a smooth interpolation of \( V(R) \) versus \( R \) is obtained with this one free parameter to the 6 off-axis potential values.

We evaluate \( \delta G(R) \) numerically using the limit of a very large lattice since we are not here concerned with long-range effects. Fixing the string tension \( K \) at the value found in the large \( R \) fit, we find the following empirical expression provides a good fit to the data of table 3 for \( R > a \),

\[
aV(R) = C - \frac{A}{R} + \frac{B}{R^2} + KR - Af\delta G(R),
\]
with $\chi^2$ per degree of freedom 4.1/4. The fit parameters are shown in table 4. For our present purposes the detailed form of this fit at small $R$ is not relevant — what is needed is a confirmation that a good fit can be obtained. This then supports our prescription to correct the lattice artefacts responsible for the lack of rotational invariance. What is more difficult is to assign errors to this correction procedure. We follow the $SU(2)$ analysis \cite{3} in using as an illustration a 10% systematic error on the artefact correction itself with the proviso that for the lowest $R$ value ($R = a$) the smooth interpolation is less of a constraint so that we disregard that datum in the analysis. We then assume that an improved estimate for the continuum potential $V_c$ will be obtained by correcting the measured lattice values $V$ by $\delta G$ with the fitted coefficient. These values are shown in table 3.

3 Running Coupling

It is now straightforward to extract the running coupling constant by using

$$\alpha\left(\frac{R_1 + R_2}{2}\right) = \frac{3}{4} R_1 R_2 \frac{V_c(R_1) - V_c(R_2)}{R_1 - R_2}$$

where the error in using a finite difference is here negligible. This is shown in table 3 and is plotted in the figure versus $R\sqrt{K}$ where $K$ is taken from the fit - see table 2. The interpretation of $\alpha$ as defined above as an effective running coupling constant is only justified at small $R$ where the perturbative expression dominates. Also shown are the two-loop perturbative results for $\alpha(R)$ for different values of $\Lambda_R$.

The figure clearly shows a running coupling constant. Moreover the result is consistent with the expected perturbative dependence on $R$ at small $R$. There are systematic errors, however. At larger $R$, the perturbative two-loop expression will not be an accurate estimate of the measured potentials, while at smaller $R$, the lattice artefact corrections are relatively big. Setting the scale using $\sqrt{K} = 0.44$ GeV implies $1/a(\beta = 6.5) = 4.13$ GeV, so $R < 4a(6.5)$ corresponds to values of $1/R > 1$ GeV. This $R$-region is expected to be adequately described by perturbation theory. Another indication that perturbation theory is accurate at such $R$-values is that $\Delta V_c/\Delta R$ at small $R$ is found to be very much greater than the non-perturbative value $K$ at large $R$.

Even though the lattice artefact correction of all 6 off-axis points by one parameter is very encouraging, the only way to be certain that lattice
Figure 1: The effective running coupling constant \( \alpha(R) \) obtained from the force between static quarks at separation \( R \). The scale is set by the string tension \( K \). Data at \( \beta = 6.5 \) are from table (diamonds) and at \( \beta = 6.2 \) (triangles). The dotted error bars represent an estimate of the systematic error due to lattice artefact correction as described in the text. The curves are the two-loop perturbative expression with \( a(6.5)\Lambda_R = 0.060 \) (dotted) and 0.070 (continuous).
artefacts are eliminated is by the comparison of different $R/a$ values at the same physical $R$ value and this can be achieved by using different $\beta$ values. Now this test was satisfied in an $SU(2)$ study [4, 3]. Even so we can check independently in $SU(3)$ and we use UKQCD data [5] at $\beta = 6.2$. This data comes from measuring 30 well separated $24^3 \times 48$ lattice configurations with smearing $c = 4.0$ and 28 and 40 recursive iterations for a 2 path basis. Both on- and off-axis potentials are evaluated. We fitted the potentials from $a < R \leq 12a$ with a 4-parameter expression to take account of the lattice artefacts. From $T$-ratio $4/3$ we get $K a^2 = 0.0251(5)$, while the $5/4$ $T$-ratio yields $0.0252(8)$. The $3/2$ and $4/3$ $T$ extrapolation method gave $K a^2 = 0.0239(11)$. These results are consistent and we use a compromise value of $0.0251(8)$. This corresponds to a ratio of lattice spacings $a(6.2)/a(6.5) = 1.484(35)$ to be compared with the two-loop perturbative ratio of 1.404. Setting the scale from the measured string tensions, we also show the $\beta = 6.2$ results for the effective running coupling in the figure. There is excellent agreement with the results from $\beta = 6.5$.

The easiest way to describe the value of the running coupling constant $\alpha$ is in terms of a scale or $\Lambda$ value with the understanding that we are only determining $\alpha$ for a range of energy scales $1/R$ - namely 1 to 3 GeV. The final estimate of $\Lambda$ is made from the figure, weighting smaller $R$ more heavily since the perturbative expression is more accurate as $\alpha(R)$ becomes smaller. We exclude the lowest $R$ point since the lattice artefact correction is for $R > a$. Remembering that the systematic errors due to lattice artefact correction are estimates only and since these systematic errors are dominant, we do not attempt a fit but we can conclude that our results are consistent with values of $\Lambda$ lying in the range shown by the two curves plotted. From the data at $\beta = 6.5$, these curves have $a(6.5)\Lambda_R = 0.070$ and 0.060. Using the value of the string tension from the fit, we get $\sqrt{K}/\Lambda_L = 49.6(3.8)$. Moreover, this value is consistent with the evaluation at both $\beta = 6.5$ and 6.2.

4 Conclusions

Using the bare coupling $g$ derived from $\beta = 6/g^2$ and the two-loop perturbative relationship $a(g)$ in terms of the scale $\Lambda_L$ gave [6, 2, 3] the following slowly decreasing values of $\sqrt{K}/\Lambda_L = 93.0(7)$ and $96.7(1.6)(2.6)$ at $\beta = 6.0$; 85.9(1.5) and 86.4(1.0)(1.9) at $\beta = 6.2$ and 82.3(8)(1.7) at $\beta = 6.4$. Our present analysis at $\beta = 6.5$ yields $\sqrt{K}/\Lambda_L = 80.0(1.4)$. Clearly, the $\beta \to \infty$ limit lies below these values. Moreover the statistically significant decrease
of these values is evidence that two-loop perturbative scaling is not obtained in terms of the bare coupling. Our present method which does not rely on the bare coupling gives the scaling result which should be independent of $\beta$. Our estimate is $\sqrt{K}/\Lambda_L = 49.6(3.8)$. This is sufficiently far below the values extracted from the bare coupling to imply that asymptotic scaling to two-loop perturbation theory is not “just around the corner” but will only be satisfied accurately at larger $\beta$-values than those currently accessible to lattice simulation.

Our result for the running coupling $\alpha_R(R)$ given in the figure and table 3 can be read directly as $\alpha_{\overline{\text{MS}}}(q)$ with $q = 1/R$ since these schemes are so close to each other. Since we obtain results consistent with the perturbative evolution, We can estimate the continuum ratio $\sqrt{K}/\Lambda_{\overline{\text{MS}}} = 1.72(13)$ for pure SU(3) theory. Setting the scale using $\sqrt{K} = 0.44$ GeV, then gives $\Lambda_{\overline{\text{MS}}} = 256(20)$ MeV. These results are obtained for rather modest energies ($1/R \approx 1–3$ GeV) but there is evidence from studies in SU(2) where higher energies have been reached [3] that the method is stable as the energy scale is increased somewhat. From lattice results for ratios of other non-perturbative quantities (glueball masses, critical temperature, etc) to the string tension, one can then determine their value in terms of $\Lambda_{\overline{\text{MS}}}$ as well.

Even though the scales probed in this work are relatively small (i.e. only 3 GeV), the agreement with the perturbative evolution of the coupling constant implies that this is a reasonable way to determine the coupling constant in terms of non-perturbative physical quantities. We are able to determine $\Lambda_{\overline{\text{MS}}}$ relatively accurately compared to experiment. Of course experiment has full QCD with dynamical light quarks included while precise lattice simulation of full QCD is still a considerable challenge.

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