Syntactic characterizations of classes of first-order structures in mathematical fuzzy logic

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Abstract

This paper is a contribution to graded model theory, in the context of mathematical fuzzy logic. We study characterizations of classes of graded structures in terms of the syntactic form of their first-order axiomatization. We focus on classes given by universal and universal–existential sentences. In particular, we prove two amalgamation results using the technique of diagrams in the setting of structures valued on a finite MTL-algebra, from which analogues of the Łoś–Tarski and the Chang–Łoś–Suszko preservation theorems follow.

Keywords

Graded model theory · Mathematical fuzzy logic · Universal classes · Universal-existential classes · Amalgamation theorems · Preservation theorems

1 Introduction

Graded model theory is the generalized study, in mathematical fuzzy logic (MFL), of the construction and classification of graded structures. The field was properly started in Hájek and Cintula (2006) and has received renewed attention in recent years (Badia and Noguera 2018a; Bagheri and Moniri 2013; Cintula and Metcalfe 2013; Cintula et al. 2015; Costa and Dellunde 2017; Dellunde 2011, 2014). Part of the programme of graded model theory is to find non-classical analogues of results from classical model theory (e.g. Hodges 1993; Sacks 1972; Chang and Keisler 1973). This will not only provide generalizations of classical theorems but will also provide insight into what avenues of research are particular to classical first-order logic and do not make sense in a broader setting.

On the other hand, classical model theory was developed together with the analysis of some very relevant mathematical structures. In consequence, its principal results provided a logical interpretation of such structures. Thus, if we want model theory’s idiosyncratic interaction with other disciplines to be preserved, the redefinition of the fundamental notions of graded model theory cannot be obtained from directly fuzzifying every classical concept. Quite the contrary, the experience acquired in the study of different structures, the results obtained using specific classes of structures, and the potential overlaps with other areas should determine the light the main concepts of graded model theory
have to be defined in. It is in this way that several fundamen-
tal concepts of the model theory of mathematical fuzzy logic
have already appeared in the literature.

The goal of this paper is to give syntactic character-
izations of classes of graded structures; more precisely,
we want to study which kinds of formulas can be used
to axiomatize certain classes of structures based on finite
(expansions of) MTL-chains. Traditional examples of such
sort of results are preservation theorems in classical model
theory, which, in general, can be obtained as consequences
of certain amalgamation properties (cf. Hodges 1993). We
provide some amalgamation results using the technique of
diagrams which will allow us to establish analogues of the
Łoś–Tarski preservation theorem (Hodges 1993, Theorem
6.5.4) and the Chang–Łoś–Suszko theorem (Hodges 1993,
Theorem 6.5.9).

This is not the first work that addresses a model-theoretic
study of the preservation and characterization of classes of
fuzzy structures. Indeed, Bagheri and Moniri (2013) have
obtained results for the particular case of continuous model
theory by working over the standard MV-algebra $[0, 1]_\ell$, and
with a predicate language enriched with a truth-constant for
each element of $[0, 1]_\ell$. In that context, they characterize
universal theories in terms of the preservation under sub-
structures (Bagheri and Moniri 2013, Prop. 5.1) and prove
versions of the Tarski–Vaught theorem (Bagheri and Moniri
2013, Prop. 4.6) and of the Chang–Łoś–Suszko theorem
(Bagheri and Moniri 2013, Prop. 5.5).

The connection between classical model theory and the
study of classes of fuzzy structures needs to be clarified.
Namely, as explained and developed in previous papers (Cin-
tula et al. 2009; Dellunde et al. 2016, 2018), there is a
translation of fuzzy structures into classical many-sorted
structures, more precisely, two-sorted structures with one sort
for the first-order domain and another accounting for truth-
values in the algebra. Such connection certainly allows to
directly import to the fuzzy setting several classical results,
but, as already noted in the mentioned papers, it does not go
as a long way. Indeed, the translation does not preserve the syn-
tactical complexity of sentences (regarding quantifiers), and
hence, it cannot be used for syntactically sensitive results,
such as those studied in the present paper.

The paper is structured as follows: in Sect. 1, we intro-
duce the syntax and semantics of fuzzy predicate logics.
In Sect. 2, several fuzzy model-theoretic notions such as
homomorphisms or the method of diagrams are presented.
In Sect. 3, we study the preservation of universal formu-
las, obtain an existential form of amalgamation and derive
from it an analogue of the Łoś–Tarski theorem. In Sect. 4,
we study classes given by universal-existential sentences by
showing that such formulas are preserved under unions of
chains, obtaining another corresponding amalgamation result
and a version of Chang–Łoś–Suszko preservation theorem.

We end with some concluding remarks and suggestions for
lines of further research.

2 Preliminaries

In this section, we introduce the syntax and semantics
of fuzzy predicate logics, and recall the basic results on
diagrams we will use in the paper. We use the notation
and definitions of the Handbook of Mathematical Fuzzy
Logic (Cintula et al. 2011).

Definition 1 (Syntax of Predicate Languages) A predicate
language $P$ is a triple $(\text{Pred}_P, \text{Func}_P, \text{Ar}_P)$, where $\text{Pred}_P$ is a non-empty set of predicate symbols, $\text{Func}_P$ is a set
of function symbols (disjoint from $\text{Pred}_P$), and $\text{Ar}_P$ repre-
sents the arity function, which assigns a natural number to
each predicate symbol or function symbol. We call this nat-
ural number the arity of the symbol. The predicate symbols
with arity zero are called truth-constants, while the func-
tion symbols whose arity is zero are named object constants
(constants for short).

$P$-terms, $P$-formulas, $\forall_n$ and $\exists_n P$-formulas, and the
notions of free occurrence of a variable, open formula, sub-
stitutability, and sentence are defined as in classical predicate
logic. A theory is a set of sentences. When it is clear from the
context, we will refer to $P$-terms and $P$-formulas simply as
terms and formulas.

Let MTL stand for the monoidal t-norm based logic intro-
duced by Esteva and Godo (2001). Throughout the paper, we
consider the predicate logic $\text{MTL}\forall$ [for a definition of the
axiomatic system for $\text{MTL}\forall$ we refer the reader to Cintula et al.
(2011, Def. 5.1.2, Ch. I)]. Let us recall that the deduction
rules of $\text{MTL}\forall$ are those of MTL and the rule of general-
ization: from $\varphi$ infer $(\forall x)\varphi$. The definitions of proof and
provability are analogous to the classical ones. We denote by
$\varphi \vdash_{\text{MTL}\forall} \psi$ the fact that $\varphi$ is provable in $\text{MTL}\forall$ from the set
of formulas $\varphi$. For the sake of clarity, when it is clear from
the context we will write $\vdash$ to refer to $\vdash_{\text{MTL}\forall}$. The algebraic
semantics of $\text{MTL}\forall$ are based on $\text{MTL}$-algebras (Esteva and
Godo 2001).

$A$ is called an $\text{MTL}$-chain if its underlying lattice is lin-
early ordered. Since it is customary to consider fuzzy logics
in languages expanding that of MTL, henceforth, we will
confine our attention to algebras which are expansions of
MTL-chains of such kind and just call them $\text{chains}$.

Definition 2 (Semantics of Predicate Fuzzy Logics [Cintula
et al. 2011, Def. 5.2.1, Ch. I]) Consider a predicate language
$P = (\text{Pred}_P, \text{Func}_P, \text{Ar}_P)$ and let $A$ be a chain. We define
an $A$-structure $M$ for $P$ as a pair $\mathfrak{M} = \langle A, M \rangle$ where
$M = \langle M, (P_M)_{P \in \text{Pred}}, (F_M)_{F \in \text{Func}} \rangle$. 
where $M$ is a non-empty domain. $P_M$ is an $n$-ary fuzzy relation for each $n$-ary predicate symbol, i.e. a function from $M^n$ to $A$, identified with an element of $A$ if $n = 0$; and $F_M$ is a function from $M^n$ to $M$, identified with an element of $M$ if $n = 0$. As usual, if $M$ is an $A$-structure for $\mathcal{P}$, an $M$-evaluation of the object variables is a mapping $v$ assigning to each object variable an element of $M$. The set of all object variables is denoted by $\text{Var}$. If $v$ is an $M$-evaluation, $x$ is an object variable, and $d \in M$, we denote by $v[x \mapsto d]$ the $M$-evaluation so that $v[x \mapsto d](x) = d$ and $v[x \mapsto d](y) = v(y)$ for $y$ an object variable such that $y \neq x$. If $M$ is an $A$-structure and $v$ is an $M$-evaluation, we define the values of terms and the truth-values of formulas in $M$ for an evaluation $v$ recursively as follows:

$$\|x\|_{M,v} = v(x);$$
$$\|F(t_1, \ldots, t_n)\|_{M,v} = F_M(\|t_1\|_{M,v}, \ldots, \|t_n\|_{M,v}), \text{ for } F \in \text{Func};$$
$$\|P(t_1, \ldots, t_n)\|_{M,v} = P_M(\|t_1\|_{M,v}, \ldots, \|t_n\|_{M,v}), \text{ for } P \in \text{Pred};$$
$$\|c\|_{M,v} = c \quad \text{for every constant } c \in A;$$
$$\|\psi\|_{M,v} = \|\phi\|_{M,v} \quad \text{for every } \phi \in \text{Fun}(\psi);$$
$$\|\exists x \psi\|_{M,v} = \inf_{d \in M} \|\psi\|_{M,v[x \mapsto d]};$$
$$\|\forall x \psi\|_{M,v} = \sup_{d \in M} \|\psi\|_{M,v[x \mapsto d]}.$$

For a set of formulas $\Phi$, we write $\Phi^A_{M,v} = 1$, if $\|\phi\|_{M,v} = 1$ for every $\phi \in \Phi$. We denote by $\|\phi\|_{M,v} = 1$ the fact that $\|\phi\|_{M,v} = 1$ for all $M$-evaluations $v$. We say that $(A, M)$ is a model of a set of formulas $\Phi$, if $\|\phi\|_{M,v} = 1$ for any $\phi \in \Phi$. Sometimes, we will denote by $\overrightarrow{x}$ a sequence of variables $x_1, \ldots, x_n$ (and the same with sequences $\overrightarrow{d}$ of elements of the domain). Given a structure $(A, M)$ and a formula $\psi(\overrightarrow{x})$, we say that $\overrightarrow{d} \subseteq M$ satisfies $\psi(\overrightarrow{x})$ (or that $\psi(\overrightarrow{x})$ is satisfied by $\overrightarrow{d}$) if $\|\psi(\overrightarrow{x})\|_{M,v[\overrightarrow{x} \mapsto \overrightarrow{d}]} = 1$ for any $M$-evaluation $v$ (also written $\|\overrightarrow{d}\|_M$ for the sake of clarity, we will use also the notation $(A, M) \models \psi(\overrightarrow{d})$ when is needed. Two theories $T$ and $U$ are said to be $I$-equivalent if a structure is a model of $T$ if it is also a model of $U$ (in the case where $T$ and $U$ are singletons of formulas, we say that these formulas are $I$-equivalent).

Given a set of sentences $\Sigma$, and a sentence $\phi$, we denote by $\Sigma \models \phi$ the fact that every $A$-model of $\Sigma$ is also an $A$-model of $\phi$. We focus on classes of structures over a fixed finite chain $A$ whose set of elements is denoted by $\{a_1, \ldots, a_k\}$. Such restriction is due to the fact that dropping finiteness can cause to lose compactness, which is an essential element of our proofs. However, the results will still be quite encompassing in practice. Indeed, for instance, prominent examples of weighted structures in computer science are valued over finite chains. Structures over a fixed finite chain $A$ have two important properties: they are witnessed (the values of the quantifiers are maxima and minima achieved in particular instances) and have the compactness property, both for satisfiability and for consequence (see e.g. Dellunde 2014).

**Proposition 1** Let $A$ be a fixed finite chain. For every set of sentences $\Sigma \cup \{\alpha\}$, the following holds:

1. If every finite subset $\Sigma_0 \subseteq \Sigma$ has a model $\langle A, M_{\Sigma_0} \rangle$, then $\Sigma$ has a model $\langle A, M \rangle$.
2. If $\Sigma \models \alpha$, then there is a finite subset $\Sigma_0 \subseteq \Sigma$ such that $\Sigma_0 \models \alpha$.

From now on, we refer to $A$-structures simply as structures (or as $\mathcal{P}$-structures if we need to specify the language). For the remainder of the article, let us assume that we have a crisp identity $\approx$ in the language.

**Definition 3** Let $\mathcal{P}$ be a predicate language, $(A, M)$ and $(B, N)$ structures for $\mathcal{P}$, $f$ a mapping from $A$ to $B$ and $g$ a mapping from $M$ to $N$. The pair $(f, g)$ is said to be a strong homomorphism from $(A, M)$ to $(B, N)$ if $f$ is an algebraic homomorphism and for every $n$-ary function symbol $F \in \mathcal{P}$ and $d_1, \ldots, d_n \in M$,

$$g(F_M(d_1, \ldots, d_n)) = F_N(g(d_1), \ldots, g(d_n))$$

and for every $n$-ary predicate symbol $P \in \mathcal{P}$ and $d_1, \ldots, d_n \in M$,

$$f(\|P(d_1, \ldots, d_n)\|_M) = \|P(g(d_1), \ldots, g(d_n))\|_N.$$
Definition 4 Given a predicate language $\mathcal{P}$, we expand it by adding an individual constant symbol $c_m$ for every $m \in M$, and denote it by $\mathcal{P}^M$. If $\langle A, M \rangle$ is a $\mathcal{P}^M$-structure, we denote by $\langle A, M^F \rangle$ the expansion of the structure $\langle A, M \rangle$ to $\mathcal{P}^M$, where for every $m \in M$, $(c_m)^{MF} = m$.

Definition 5 Given a predicate language $\mathcal{P}$, we expand it by adding a truth-constant symbol $\bar{a}$ for every $a \in A$, and denote it by $\mathcal{P}^A$. When we expand the language $\mathcal{P}^A$ further by adding an individual constant symbol $c_m$ for every $m \in M$, we will denote it by $\mathcal{P}^{(A,M)}$.

Definition 6 Let $\mathcal{P}$ be a predicate language and $\langle A, M \rangle$ a $\mathcal{P}$-structure. We define the following sets of $\mathcal{P}^{(A,M)}$-sentences:

$$ElDiag(A, M) = \{ \sigma \leftrightarrow \bar{a} \mid \sigma \text{ is a sentence of } \mathcal{P}^M, a \in A \text{ and } \|\sigma\|^A_{MF} = a \},$$

whereas $Diag(A, M)$ is the subset of $ElDiag(A, M)$ containing all formulas $\sigma \leftrightarrow \bar{a}$ where $\sigma$ is quantifier-free.

Following the same lines of the proof of Dellunde (2011, Prop. 32), we can obtain a characterization of strong and elementary embeddings between two $\mathcal{P}$-structures over a chain $A$.

Corollary 2 Let $\langle A, M \rangle$ and $\langle A, N \rangle$ be two $\mathcal{P}$-structures for $\mathcal{P}^A$. The following are equivalent:

1. There is an expansion of $\langle A, N \rangle$ that is a model of $Diag(A, M)$ (ElDiag(A, M), respectively).
2. There is a mapping $g: M \to N$ such that $\langle Id_A, g \rangle$ is a strong (elementary, respectively) embedding from $\langle A, M \rangle$ into $\langle A, N \rangle$.

3 Universal classes

In this section, we prove a result on existential amalgamation (Proposition 4) from which we extract a Łoś–Tarski preservation theorem for universal theories (Theorem 5) and a characterization of universal classes of structures (Theorem 6). Relevant structures in computer science are axiomatized by sets of universal formulas; one prominent example is the class of weighted graphs. Particular versions of the above-mentioned results appeared for Łoś in Spada (2009). In the context of fuzzy logic programming, Gerla (2005) studied universal formulas with relation to Herbrand interpretations.

For the upcoming results, we need to recall the notion of substructure.

Definition 7 (Substructure) Let $\langle A, M \rangle$ and $\langle B, N \rangle$ be $\mathcal{P}$-structures. We say that $\langle A, M \rangle$ is a substructure of $\langle B, N \rangle$ if:

(1) $A$ is a subalgebra of $B$;
(2) $M \subseteq N$;
(3) for any $n$-ary function symbol $F \in \mathcal{P}$ and elements $d_1, \ldots, d_n \in M$, we have $F_M(d_1, \ldots, d_n) = F_N(d_1, \ldots, d_n)$;
(4) for any $n$-ary predicate symbol $P \in \mathcal{P}$ and elements $d_1, \ldots, d_n \in M$, we have $P_M(d_1, \ldots, d_n) = P_N(d_1, \ldots, d_n)$.

Remark that $\langle A, M \rangle$ is a substructure of $\langle B, N \rangle$ if and only if conditions (1)–(3) are satisfied and, instead of (4), the following condition holds: for every quantifier-free formula $\varphi(x_1, \ldots, x_n)$ and any elements $d_1, \ldots, d_n \in M$,

$$\|\varphi(d_1, \ldots, d_n)\|^A_M = \|\varphi(d_1, \ldots, d_n)\|^B_N.$$

With this notion at hand, we can define a corresponding closure property for classes of structures.

Definition 8 (Class Closed Under Substructures) Let $K$ be a class of $\mathcal{P}$-structures. We say that $K$ is closed under substructures if, for any structure $\langle A, M \rangle \in K$,

if $\langle B, N \rangle$ is a substructure of $\langle A, M \rangle$, then $\langle B, N \rangle \in K$.

Since our characterizations will be based on axiomatizability of classes, we need to recall the definition of elementary class of structures.

Definition 9 [Elementary Class (Burris and Sankappanavar 1981, Def. 2.15)] A class $K$ of $\mathcal{P}$-structures is an elementary class (or a first-order class) if there is a set $\Sigma$ of sentences such that for every $\langle A, M \rangle$,

$$\langle A, M \rangle \in K \text{ if and only if } \langle A, M \rangle \models \Sigma.$$  

In this case, $K$ is said to be axiomatized (or defined) by $\Sigma$.

Using a predicate language with only one binary relation $R$, the class of weighted undirected graphs is axiomatized by the following set of universal sentences:

$$\{(\forall x)(R(x, x) \to \bar{0}), (\forall x)(\forall y)(R(x, y) \to R(y, x))\}.$$  

Notice that the notion of induced weighted undirected subgraph corresponds to the model-theoretic notion of substructure used in MFL.

Definition 10 Let $\mathcal{P}$ be a predicate language. We say that a $\mathcal{P}$-formula $\varphi(x_1, \ldots, x_n)$ is preserved under substructures if for any $\mathcal{P}$-structure $\langle A, M \rangle$ and any substructure $\langle B, N \rangle$, if $\|\varphi(d_1, \ldots, d_n)\|^A_M = \mathbf{1}^A$ for some $d_1, \ldots, d_n \in N$, then $\|\varphi(d_1, \ldots, d_n)\|^B_N = \mathbf{1}^B$.
The following lemma can be easily proved by induction on the complexity of universal formulas.

**Lemma 3** Let \( \varphi(x_1, \ldots, x_n) \) be a universal formula. Then, \( \varphi(x_1, \ldots, x_n) \) is preserved under substructures.

In classical model theory, amalgamation properties are often related in elegant ways to preservation theorems (see e.g. Hodges 1993). We will try an analogous approach to often related in elegant ways to preservation theorems (see e.g. Hodges 1993). We will try an analogous approach to

Let \( \psi(\bar{x}) \) be a set of formulas \( \text{Diag}_g^*(A, M_2) \) such that:

\[
\text{ElDiag}(A, M_1) \models (\exists \bar{x}) \left( \bigwedge \text{Diag}_g^*(A, M_2) \right) \rightarrow \sigma.
\]

Quantifying away the new individual constants, we obtain a set of formulas \( \text{Diag}_g^*(A, M_2) \) such that:

\[
\text{ElDiag}(A, M_1) \models (\exists \bar{x}) \left( \bigwedge \text{Diag}_g^*(A, M_2) \right) \rightarrow \sigma.
\]

Since \( \langle A, M_2, \bar{d} \rangle \models \exists \bar{x} \langle A, M_1, \bar{d} \rangle \), then

\[
\langle A, M_2 \rangle \not\models (\exists \bar{x}) \left( \bigwedge \text{Diag}_g^*(A, M_2) \right),
\]

which is a contradiction. Note, moreover, that if

\[
\langle A, M_2, \bar{d} \rangle \models \exists \bar{x} \langle A, M_1, \bar{d} \rangle,
\]

we also have that whenever \( \varphi(\bar{x}) \) is quantifier-free formula of \( \mathcal{P}^A \), \( \langle A, M_1 \rangle \models \varphi[\bar{d}] \) iff \( \langle A, M_1 \rangle \models \varphi[\bar{d}] \). Left-to-right is clear; the contrapositive of the right-to-left direction follows easily: if \( \langle A, M_1 \rangle \models \varphi[\bar{d}] \), then \( \langle A, M_2 \rangle \models \varphi \leftrightarrow \varphi[\bar{d}] \) for some \( a \in \bar{d} \), so \( \langle A, M_2 \rangle \models \varphi \leftrightarrow \varphi[\bar{d}] \), which means that \( \langle A, M_2 \rangle \models \varphi[\bar{d}] \).

Observe that the proof can be similarly carried out, *mutatis mutandi*, when \( \langle A, M_1 \rangle \) and \( \langle A, M_2 \rangle \) have no common part as well. \( \square \)

Now, we have the elements to establish an exact analogue of Theorem 5 from Łoś (1955), Łoś–Tarski preservation theorem.

**Theorem 5** (Łoś–Tarski preservation theorem) Let \( T \) be a \( \mathcal{P}^A \)-theory and \( \Phi(\bar{x}) \) a set of formulas in \( \mathcal{P}^A \). Then, the following are equivalent:

(i) For any models of \( T \), \( \langle A, M \rangle \subseteq \langle A, N \rangle \), we have: if \( \langle A, N \rangle \models \Phi \), then \( \langle A, M \rangle \models \Phi \).

(ii) There is a set of universal \( \mathcal{P}^A \)-formulas \( \Theta(\bar{x}) \) such that: \( T, \Phi \models \Theta \) and \( T, \Theta \models \Phi \).

**Proof** Let us prove the difficult direction (the converse direction is clear by Lemma 3). Consider \( (T \cup \Phi(\bar{x}))_{\forall_1} \), the collection of all \( \forall_1 \) logical consequences of \( T \cup \Phi(\bar{x}) \). We need to establish that the only models of \( (T \cup \Phi(\bar{x}))_{\forall_1} \) among the models of \( T \) are the substructures of models of \( \Phi(\bar{x}) \). Let \( \langle A, M \rangle \) be a model of \( (T \cup \Phi(\bar{x}))_{\forall_1} \). All we need to do is find a model \( \langle A, N \rangle \) of the theory \( T \cup \Phi(\bar{x}) \) such that \( \langle A, M \rangle \models \exists \bar{x} \langle A, N \rangle \) and then quote the existential amalgamation theorem.

Let \( U \) be all \( \exists \bar{x} \)-formulas that hold in \( \langle A, M \rangle \). We claim then that \( T \cup \Phi(\bar{x}) \cup U \) has a model. Otherwise, by compactness, for

\[
\{ (\exists x_0)\phi_0(x_0), \ldots, (\exists x_n)\phi_n(x_n) \} \subseteq U
\]
we have that in all models of \( T \cup \Phi(\overline{\tau}) \), it holds that
\[
(\exists \overline{x}_0)(\exists \overline{x}_n)(\psi_0(\overline{x}_0) \land \cdots \land (\exists \overline{x}_n)(\psi_n(\overline{x}_n)) \rightarrow \overline{a},
\]
where \( a \) is the immediate predecessor of \( \overline{T}^A \), and by basic manipulations,
\[
(\exists \overline{x}_0, \ldots, \overline{x}_n)(\psi_0(\overline{x}_0) \land \cdots \land \psi_n(\overline{x}_n)) \rightarrow \overline{a},
\]
which is just equivalent to
\[
(\forall \overline{x}_0, \ldots, \overline{x}_n)(\psi_0(\overline{x}_0) \land \cdots \land \psi_n(\overline{x}_n) \rightarrow \overline{a}).
\]
The latter formula must be in \( (T \cup \Phi(\overline{\tau}))_{V_1} \) then, which is a contradiction. \( \square \)

Following a similar proof, we can obtain an algebraic characterization equivalent to Theorem 5 (as long as we have truth-constants around).

**Theorem 6** Let \( \mathbb{K} \) be a class of \( \mathcal{P}^A \)-structures. Then, the following are equivalent:

(i) \( \mathbb{K} \) is closed under isomorphisms, substructures, and ultraproducts.

(ii) \( \mathbb{K} \) is axiomatized by a set of universal \( \mathcal{P}^A \)-sentences.

The following corollary can be obtained because in our setting, two forms of compactness (that are generally distinct, in, say, Łukasiewicz logic) collapse, namely (1) the compactness of the consequence relation and (2) the compactness of the satisfiability relation. (1) clearly implies (2) in the presence of \( \bar{0} \) in our language. To see the converse, say that \( T \models \varphi \), which amounts to say that \( T \cup \{ \varphi \rightarrow \overline{a} \} \) (where \( a \) is the predecessor of \( \bar{T}^A \)) does not have a model. Hence, by (2), there is a finite \( T_0 \subseteq T \) such that \( T_0 \cup \{ \varphi \rightarrow \overline{a} \} \) has no model, so, in fact, \( T_0 \models \varphi \).

**Corollary 7** Let \( T \cup \{ \varphi \} \) be a set of \( \mathcal{P}^A \)-sentences. Then, \( \varphi \) is preserved under substructures of models of \( T \) if, and only if, \( \varphi \) is 1-equivalent to a universal \( \mathcal{P}^A \)-sentence modulo \( T \).

**Proof** Apply Theorem 5 for \( \Phi = \{ \varphi \} \). Consequently, \( \varphi \) is axiomatized by a set of universal \( \mathcal{P}^A \)-sentences. Then, bring it down to a single such formula using \( A \)-compactness for consequence. \( \square \)

A natural question is whether Corollary 7 can be strengthened to strong equivalence in terms of \( \leftrightarrow \), that is, whether one can find a universal formula that agrees with \( \varphi \) on each value in every structure (not just on value \( \bar{T}^A \)). Following the lines of the above proof, this would require to show something like, for an arbitrary model \( \langle A, M \rangle \),
\[
\| \varphi \|^A_M \leq \inf_{A} \| \psi \|^A_M \quad \text{for all } \psi \text{ s.t. } \phi \rightarrow \psi.
\]

Then, one would expect to reduce the left side of the inequality to a finite set \( \Psi \) such that
\[
\inf_{A} \| \psi \|^A_M | \psi \in \Psi \| A \leq \inf_{A} \| \phi \|^A_M.
\]

However, this reduction would come from compactness in the usual argument, but it does not in this one. This is because compactness is about consequence as opposed to implication, which are different in a setting without a deduction theorem such as this. In fact, in Spada (2009) similar results in the framework Łukasiewicz logic are obtained only for 1-equivalence as well.

Needless to say, the previous results, in particular, allow to conclude that a class of \( \mathcal{P} \)-structures (that is, structures for a language without additional truth-constants) closed under substructures can be axiomatized by universal \( \mathcal{P} \)-sentences. One might wonder, of course, if it is really necessary to resort to a universal axiomatization in the expanded language.

Let us present a counterexample showing that, in general, the base language \( \mathcal{P} \) does not suffice for Theorem 6. Let \( \mathcal{P} \) be the language with only one monadic predicate \( P \) and take two structures over the standard Gödel chain, \( \langle [0, 1], G \rangle, M \rangle \) and \( \langle [0, 1], G \rangle, N \rangle \). The domain in both cases is the set of all natural numbers \( \mathbb{N} \), and the interpretation of the predicate is, respectively, defined as: \( P_M(n) = \frac{3}{2} \), and \( P_N(n) = \frac{1}{2} \), for every \( n \in \mathbb{N} \). First, we show that \( \langle [0, 1], G \rangle, M \rangle \models \langle [0, 1], G \rangle, N \rangle \). This is because, for any \( \mathcal{P} \)-sentence \( \varphi \), \( \langle [0, 1], G \rangle, M \rangle \models \varphi \) iff \( \langle [0, 1], G \rangle, N \rangle \models \varphi \). Take \( f \) as any non-decreasing bijection from \([0, 1]\) to \([0, 1]\) such that \( f\left( \frac{3}{2} \right) = \frac{1}{2} \), \( f(1) = 1 \), \( f(0) = 0 \). It is easy to check that \( f \) is a G-homomorphism preserving suprema and infima. Then, we can consider the \( \sigma \)-mapping \( (f, Id) \) and apply (Dellunde et al. 2018, Lemma 11) to obtain that \( \langle [0, 1], G \rangle, M \rangle \models \langle [0, 1], G \rangle, N \rangle \). Consider now the finite subalgebra \( A \) of \( [0, 1]_G \) generated by the subset \( \{ 0, 2 \} \). Clearly, the structures \( \langle [0, 1], G \rangle, M \rangle \) and \( \langle [0, 1], G \rangle, N \rangle \) can be regarded as structures over \( A \). Thus, we have
\[
\langle A, M \rangle \equiv \langle A, N \rangle.
\]

Observe that \( \| (\forall x)P(x) \|^A_M = \frac{3}{2} \) and \( \| (\forall x)P(x) \|^A_N = \frac{1}{2} \). Consider the expanded language \( \mathcal{P}_A \) obtained by adding a constant symbol \( \overline{a} \) for every element \( a \in A \). Let \( \mathbb{K} \) be the class of \( \mathcal{P} \)-structures valued on \( A \), whose natural extension to \( \mathcal{P}_A \) (that is, the expansion in which every constant \( \overline{a} \) is interpreted as the corresponding element \( a \)) satisfies the sentence
\[
\frac{3}{4} \rightarrow (\forall x)P(x).
\]

Clearly, \( \mathbb{K} \) is closed under substructures and \( \langle A, M \rangle \in \mathbb{K} \). However, \( \langle A, N \rangle \notin \mathbb{K} \), because \( \| (\forall x)P(x) \|^A_N = \frac{1}{2} \). Therefore, \( \mathbb{K} \) cannot be axiomatized by \( \mathcal{P} \)-sentences, and a fortiori, by a set of universal \( \mathcal{P} \)-sentences, since it contains \( \langle A, M \rangle \).
but not the elementary equivalent \( \langle A, N \rangle \). Hence, we have produced an example of a class of \( \mathcal{P} \)-structures closed under substructures (and, obviously, under isomorphisms and ultraproducts) which is not axiomatizable with universal \( \mathcal{P} \)-sentences.

Whether constants are necessary for Theorem 5 in general is an open question, we conjecture that they are.

### 4 Universal–existential classes

This section runs quite parallel to the previous one. We recall the notion of elementary chain of structures and its corresponding Tarski-Vaught theorem and prove that universal–existential formulas are preserved under unions of chains (Lemma 9). After that, we prove a result on existential–universal amalgamation (Proposition 10) and derive from it a Chang–Łoś–Suszko preservation theorem (Theorem 11).

Consider the class \( \mathbb{K} \) of all structures in a signature with a binary function symbol +, a unary function symbols −1, an individual constant 1, and a unary predicate \( G \) satisfying the following axioms:

\[
(\forall x)(\exists y)(y^n \approx x) \text{ for each } n \geq 2.
\]

\[
(\forall x, y)((x \cdot y) \cdot z \approx x \cdot (y \cdot z)).
\]

\[
(\forall x)(x \cdot 1 \approx x).
\]

\[
(\forall x)(x \cdot x^{-1} \approx 1).
\]

\[
(\forall x, y)(x \cdot y \approx y \cdot x).
\]

\[
(\forall x, y)((Gx \land Gy) \rightarrow G(xy)).
\]

\[
(\forall x)(Gx \rightarrow G(x^{-1})).
\]

This is the class of divisible Abelian groups with a fuzzy subgroup defined by the predicate \( G \) (following the definition of Rosenfeld 1971). By our Chang–Łoś–Suszko preservation theorem below, \( \mathbb{K} \) is a class closed under unions of chains.

Another example of such class be provided by the class of all weighted graphs where the formula

\[
(\forall x)(\exists y, z)(y \not\approx z \land Rx y \land Rzx)
\]

holds, that is, every vertex has at least two incident edges. This axiomatizes the class of graphs where every vertex has degree \( \geq 2 \).

Given an ordinal \( \gamma \), a sequence \( \{ \langle A_i, M_i \rangle \mid i < \gamma \} \) of models is called a chain when for all \( i < j < \gamma \) we have that \( \langle A_i, M_i \rangle \) is a substructure of \( \langle A_j, M_j \rangle \). If, moreover, these substructures are elementary, we speak of an elementary chain. The union of the chain \( \{ \langle A_i, M_i \rangle \mid i < \gamma \} \) is the structure \( \langle A, M \rangle \) where \( M \) is defined by taking as its domain \( \bigcup_{i < \gamma} M_i \), interpreting the constants of the language as they were interpreted in each \( M_i \) and similarly with the relational symbols of the language. Observe as well that \( M \) is well defined given that \( \{ \langle A_i, M_i \rangle \mid i < \gamma \} \) is a chain.

Next, we recall a useful theorem that has been established and used to construct saturated models in the context of mathematical fuzzy logic in Badia and Noguera (2018b).

**Theorem 8** (Badia and Noguera 2018b) (Tarski–Vaught) Let \( \langle A, M \rangle \) be the union of the elementary chain \( \{ \langle A_i, M_i \rangle \mid i < \gamma \} \). Then, for every sequence \( \vec{d} \) of elements of \( M_i \) and formula \( \varphi \), \( \mathcal{M} \models \varphi(\vec{d}) \) if and only if \( \mathcal{M}_i \models \varphi(\vec{d}) \) for all \( i < \gamma \).

**Lemma 9** \( \forall_2 \)-formulas are preserved under unions of chains.

**Proof** Let \( (\forall \vec{x})(\exists \vec{y}) \varphi \) be a \( \forall_2 \)-formula, \( \langle A, M \rangle \) be the union of a chain \( \{ \langle A_i, M_i \rangle \mid i < \gamma \} \), and \( \vec{c} \) some sequence of elements of \( M_0 \). Assume that for every \( \vec{e} \), \( \mathcal{M}_i \models \varphi(\vec{e}) \) for some sequence \( \vec{d} \) of elements of \( M_0 \), then \( \mathcal{M} \models \varphi(\vec{d}) \).

Let \( \vec{a} \) be the element of \( A \) immediately above \( \vec{0} \).

Next, we provide the amalgamation result that will allow us to prove a version of Chang–Łoś–Suszko theorem for graded model theory.
Proposition 10 (̂∃₂-amalgamation) Let \( \langle A, M_1 \rangle \) and \( \langle A, M_2 \rangle \) be two structures for \( \mathcal{P}^A \) with a common part \( \langle A, M \rangle \) with domain generated by a sequence of elements \( \vec{d} \). Moreover, suppose that

\[
\langle A, M_2, \vec{d} \rangle \models_{\exists_2} \langle A, M_1, \vec{d} \rangle.
\]

Then, there is a structure \( \langle A, N \rangle \) into which \( \langle A, M_2 \rangle \) can be strongly embedded by \( \langle f, g \rangle \) preserving all \( \forall_1 \)-formulas, while \( \langle A, M_1 \rangle \) is \( \mathcal{P}^A \)-elementarily strongly embedded (taking isomorphic copies, we may assume that \( \langle A, M_1 \rangle \) is just a \( \mathcal{P}^A \)-elementary substructure). The situation is described by the following picture:

\[
\begin{array}{ccc}
\langle A, N \rangle & \models_{\exists_2} & \langle A, M_2, \vec{d} \rangle \\
\subseteq & \Downarrow & \subseteq \\
\langle A, M_2, \vec{d} \rangle & \models_{\exists_2} & \langle A, M_1, \vec{d} \rangle \\
\end{array}
\]

Moreover, the result is also true when \( \langle A, M_1 \rangle \) and \( \langle A, M_2 \rangle \) have no common part.

Proof Let \( \text{Diag}_{\forall_1}(A, M_2) \) be the collection of all \( \forall_1 \)-formulas in the language of the domain of \( \langle A, M_2 \rangle \) (where we let the elements of the domain serve as constants to name themselves) that hold in said structure. It is not difficult to show that \( \text{ElDiag}(A, M_1) \cup \text{Diag}_{\forall_1}(A, M_2) \) (where again we let the elements of the domain serve as constants to name themselves) has a model, which suffices for the purposes of the result. For suppose otherwise, that is, for some finite

\[
\text{Diag}_{\forall_1}(A, M_2) \subseteq \text{Diag}_{\forall_1}(A, M_2).
\]

we have that

\[
\text{ElDiag}(A, M_1) \models \left( \bigwedge \text{Diag}_{\forall_1}(A, M_2) \right) \rightarrow \mathfrak{a}
\]

for some \( a \neq \mathbb{T}^A \) (the supremum of all the values taken by \( \bigwedge \text{Diag}_{\forall_1}(A, M_2) \) in \( A \)). Quantifying away the new individual constants,

\[
\text{ElDiag}(A, M_1) \models (\forall \mathfrak{r})(\left( \bigwedge \text{Diag}_{\forall_1}(A, M_2) \right) \rightarrow \mathfrak{a}),
\]

so

\[
\text{ElDiag}(A, M_1) \models (\exists \mathfrak{r})(\left( \bigwedge \text{Diag}_{\forall_1}(A, M_2) \right)) \rightarrow \mathfrak{a}.
\]

Since \( \langle A, M_2, \vec{d} \rangle \models_{\exists_2} \langle A, M_1, \vec{d} \rangle \), then

\[
\langle A, M_2, \vec{d} \rangle \not\models (\exists \mathfrak{r})(\left( \bigwedge \text{Diag}_{\forall_1}(A, M_2) \right)) \rightarrow \mathfrak{a},
\]

which is a contradiction. \( \square \)

Now, we are ready to prove the promised analogue of Robinson (1959, Theorem 1.2).

Theorem 11 (Chang–Loś–Suszko preservation theo.) Let \( T \) be a theory and \( \Phi(\vec{x}) \) a set of formulas in \( \mathcal{P}^A \). Then, the following are equivalent:

(i) \( \Phi(\vec{x}) \) is preserved under unions of chains of models of \( T \).

(ii) \( \Phi(\vec{x}) \) is \( 1 \)-equivalent modulo \( T \) to a set of \( \forall_2 \)-formulas.

Proof Once more, we only deal with the non-trivial direction of the equivalence. Consider \( (T \cup \Phi(\vec{x}))_{\forall_2} \). We want to show that

\[
T \cup (T \cup \Phi(\vec{x}))_{\forall_2} \models \Phi(\vec{x}),
\]

which will suffice to establish the theorem. The strategy is to establish that any model of \( T \cup (T \cup \Phi(\vec{x}))_{\forall_2} \) has an elementary extension which is a union of \( \omega \)-many models of \( \Phi(\vec{x}) \), so by hypothesis, \( \Phi(\vec{x}) \) will hold there, and hence back in our original model of \( T \cup (T \cup \Phi(\vec{x}))_{\forall_2} \).

So, we start with \( \langle A, M_0 \rangle \) being an arbitrary model of \( T \cup (T \cup \Phi(\vec{x}))_{\forall_2} \). Now, assuming that we have \( \langle A, M_i \rangle \) which is an elementary extension of \( \langle A, M_0 \rangle \). We first need to find a model \( \langle A, M'_i \rangle \) of the theory \( T \cup \Phi(\vec{x}) \) such that \( \langle A, M_i \rangle \models_{\exists_2} \langle A, M'_i \rangle \) and then quote the \( \exists_2 \)-amalgamation theorem to obtain a model \( \langle A, M_i' \rangle \cup \Phi(\vec{x}) \) into which \( \langle A, M_i \rangle \) can be strongly embedded in such a way that all \( \forall_1 \)-formulas are preserved by such strong embedding.

Let \( U \) be all \( \exists_2 \)-formulas that hold in \( \langle A, M_i \rangle \). We claim then that \( T \cup \Phi(\vec{x}) \cup U \) has a model. Otherwise by compactness, for

\[
\{ (\exists \mathfrak{x}_0)(\forall \mathfrak{y}_0)\phi_0(\mathfrak{x}_0, \mathfrak{y}_0), \ldots, (\exists \mathfrak{x}_n)(\forall \mathfrak{y}_n)\phi_n(\mathfrak{x}_n, \mathfrak{y}_n) \} \subseteq U
\]

we have that in all models of \( T \), it holds that

\[
(\exists \mathfrak{x}_0)(\forall \mathfrak{y}_0)\phi_0(\mathfrak{x}_0, \mathfrak{y}_0) \land \cdots \land
(\exists \mathfrak{x}_n)(\forall \mathfrak{y}_n)\phi_n(\mathfrak{x}_n, \mathfrak{y}_n) \rightarrow \mathfrak{a}
\]

where \( a \) is the immediate predecessor of \( \mathbb{T}^A \) and by basic manipulations,

\[
(\exists \mathfrak{x}_0, \ldots, \mathfrak{x}_n)(\forall \mathfrak{y}_0, \ldots, \mathfrak{y}_n)\phi_0(\mathfrak{x}_0, \mathfrak{y}_0) \land \cdots \land
\phi_n(\mathfrak{x}_n, \mathfrak{y}_n) \rightarrow \mathfrak{a}.
\]

The latter formula must be in \( (T \cup \Phi(\vec{x}))_{\forall_2} \) then, which is a contradiction.
Now, \( \langle A, N_i \rangle \) is also such that for a listing \( \vec{d} \) of all the elements of \( \langle A, M_i \rangle \), \( \langle A, N_i, \vec{d} \rangle \supseteq_1 \langle A, M_i, \vec{d} \rangle \). To prove the contrapositive, suppose that

\[
\langle A, M_i, \vec{d} \rangle \models (\exists \vec{x} \phi(\vec{x}, \vec{d}) \rightarrow \vec{a})
\]

where \( a \) is the immediate predecessor of \( \vec{a} \) in the linear order of \( A \). But then

\[
\langle A, M_i, \vec{d} \rangle \models (\forall \vec{x} \phi(\vec{x}, \vec{d}) \rightarrow \vec{a}),
\]

so indeed,

\[
\langle A, N_i, \vec{d} \rangle \models (\exists \vec{x} \phi(\vec{x}, \vec{d}) \rightarrow \vec{a}),
\]

and hence,

\[
\langle A, N_i, \vec{d} \rangle \models (\exists \vec{x} \phi(\vec{x}, \vec{d}) \rightarrow \vec{a}).
\]

Now, using the existential amalgamation theorem we can obtain a structure \( \langle A, M_{i+1} \rangle \) as an elementary extension of \( \langle A, M_i \rangle \) into which \( \langle A, N_i \rangle \) can be strongly embedded. Now, just take the union \( \langle A, \bigcup_{i \in \omega} M_i \rangle = \langle A, \bigcup_{i \in \omega} N_i \rangle \) and apply Theorem 8.

\[\square\]

As a consequence, we can again obtain a result for single formulas, using the compactness of the consequence relation.

**Corollary 12** Let \( T \) be a theory in \( \mathcal{P}^A \) and \( \varphi \) a formula. Then, the following are equivalent:

(i) \( \varphi \) is preserved under unions of chains of models of \( T \).

(ii) \( \varphi \) is 1-equivalent modulo \( T \) to a set of \( \forall_2 \)-formulas.

**5 Conclusions**

In this paper, we have provided some necessary steps in the systematic study of syntactic characterizations of classes of graded structures and their corresponding preservation theorems. Work in progress in the same line includes the study of the universal Horn fragment of predicate fuzzy logics and the classes axiomatized by sets of Horn clauses. Moreover, in the general endeavour of graded model theory, we believe that, among others, future works should focus on the study of types, with the construction of saturated models and type-omission theorems, the study of particular kinds of graded structures that are relevant for computer science applications and, also, the development of Lindström-style characterization theorems for predicate fuzzy logics that may lead to the creation of a non-classical abstract model theory.

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**Compliance with ethical standards**

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