Dualities and Phase Transitions for Calabi-Yau Threefolds and Fourfolds

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Abstract

We review several aspects of heterotic, type II, F-theory, and M-theory compactifications on Calabi-Yau threefolds and fourfolds. In the context of dualities we focus on the heterotic gauge structure determined by the various types of fibration relevant in the framework of heterotic/type II duality in D=4 as well as 4D F-theory. We also consider transitions between Calabi-Yau manifolds in both three and four dimensions and review some of the consequences for the behavior of the superpotential.

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1 Introduction

In the last two years many dualities between perturbatively different string theories have been discovered. An important step was the discovery of a duality

\[ \text{Het}(T^4) \leftrightarrow \text{IIA}(K3) \]  

in six dimensions between the heterotic string compactified on a four-torus \( T^4 \) and the type IIA string compactified on a K3 surface \( [1, 2, 3, 4] \), extending an earlier observation of the equivalence of their moduli spaces \( [5] \). The non-abelian gauge structure of the heterotic string is related to the ADE singularities of the K3 surface. A four-dimensional generalization of this duality

\[ \text{Het}(K3 \times T^2) \leftrightarrow \text{IIA}(\text{CY}_3) \]  

was formulated in \( [6] \) by considering IIA on Calabi-Yau threefolds \( \text{CY}_3 \). Subsequently it was understood that the structure of the heterotic couplings requires the \( \text{CY}_3 \) on the type II side to be K3-fibrations

\[ \text{K3} \rightarrow \text{CY}_3 \rightarrow \mathbb{P}_1 \]

with generic fibers K3 and base \( \mathbb{P}_1 \sim S^2 \). This suggests that to a certain extent the four-dimensional duality relation should be induced by a fiberwise application of the 6D duality \( [8] \). The detailed structure of the heterotic gauge sector turns out to depend not only on the singularity structure of the fibers but also on the particular way these fibers are embedded into the threefold. The tool used in \( [9] \) to determine the gauge group from the singularities and fibration structure is a bundle twist map, which constructs K3-fibered threefolds from a given K3 surface and an algebraic curve determined by the automorphism group of this surface. From the threefold point of view the twist map shows how the A\( _n \) singularities of the K3 surfaces are tied together in the ADE Dynkin diagrams of the corresponding threefolds by singular curves on the threefold. We will review the salient features of this construction in Section 2 and describe some applications in Section 3.

Extending the net of dualities further by considering T-duality transformations between IIA and IIB theories as well as SL(2,\( \mathbb{Z} \)) duality in IIB, and taking into account the old relation between D=11 supergravity and IIA string theory in conjunction with the relation between IIA branes and M-branes and Kaluza-Klein magnetic monopoles, it appears that many, if not all, dualities can be traced back to a few ancestors in 11D M-theory \( [1, 2, 10, 11, 12] \) and 12D F-theory \( [13, 14] \). Thus a picture arises in which all these different theories describe different regions of the moduli space of an underlying parent theory.
M-theory is expected to be the eleven dimensional strong coupling limit of type IIA string theory. This has been emphasized in [1, 2, 10, 11, 12] not only on the basis of the D=11 Kaluza-Klein interpretation of the IIA effective field theory but also because the IIA brane structure follows from the electric M2-brane and its dual magnetic M5-brane induced by the 3-form of D=11 supergravity. Further compactification of M-theory induces a number of known dualities. In six dimensions the relation (1) follows from the duality $\text{M}(\text{K3} \times \text{S}^1) \leftrightarrow \text{Het}(\text{T}^4)$ on the one hand and the relation of M-theory to IIA on the other. We will describe more general compactifications of M-theory in Section 3. Some aspects of this unification of dualities have been reviewed in [15].

F-theory is a theory with a 12-dimensional origin which so far has been considered mostly through its compactifications to lower dimensions. The strong-weak coupling symmetry of type IIB theory was a motivation to consider not only M-theory but also F-theory as a candidate for a unifying theory. Recall that the $\text{SL}(2, \mathbb{Z})$ symmetry of the type IIB string transforms the combination $\lambda = a + i e^{-\phi}$ of the RR-scalar $a$ and the dilaton $\phi$ as the modular parameter of a torus. In the context of M-theory this symmetry appears when M-theory is compactified on a torus and the modular parameter of this torus is compared with the coupling $\lambda$ of a type IIB string on a circle [16]. F-theory provides us with a more geometric interpretation of this symmetry. The compactifications of F-theory can be interpreted as special kinds of type IIB compactifications where the coupling $\lambda$ is allowed to vary over the internal manifold and to undergo $\text{SL}(2, \mathbb{Z})$ monodromies. More precisely, the internal manifold $X$ on which F-theory is compactified, has to admit an elliptic fibration

$$T^2 \longrightarrow X \longrightarrow \mathcal{B}. \quad (3)$$

The modular parameter of the fiber is identified with the coupling of the IIB string, so that F-theory on $X$ is type IIB on $\mathcal{B}$ with varying coupling constant. Because of the relation between M-theory on $T^2$ and IIB on $S^1$ there is a duality between F-theory on $X \times S^1$ and M-theory on $X$.

Consistent compactifications of M-theory and F-theory to three and four dimensions respectively can be obtained via Calabi-Yau fourfolds $\text{CY}_4$. The resulting theories are N=1 supersymmetric theories. Again the fibration structure of these fourfolds is important for possible duality relations in lower dimensions. For F-theory in particular the general fibration structure (3) implies that the fourfold $\text{CY}_4$ has to admit an elliptic fibration. If the base space $\mathcal{B}$ in turn is a fibration of the form

$$\mathbb{P}_1 \longrightarrow \mathcal{B} \longrightarrow S$$
then type IIB on $B$ can be conjectured to be dual to a heterotic string on a Calabi-Yau threefold which is elliptically fibered over $S$ \cite{17}. This conjecture is suggested by fiberwise application of the duality between $F_{12}(K3)$ and the heterotic string on $T^2$ \cite{13}. Altogether we then have the structure:

$$
\begin{array}{c}
T^2 \rightarrow \text{CY}_4 \\
\downarrow \\
\mathbb{P}_1 \rightarrow B \\
\downarrow \\
S,
\end{array}
$$

Here again the twist map is useful to construct fibrations from scratch by generalizing the constructions of \cite{1} to Calabi-Yau fourfolds \cite{18}. Particularly simple are CY$_3$-fibered fourfolds for which the fibers themselves are K3 fibrations of elliptically fibered K3 surfaces. The iterative fibrations of such manifolds show a nested structure which can be summarized in the diagram:

$$
\begin{array}{c}
T^2 \rightarrow \text{K3} \rightarrow \text{CY}_3 \rightarrow \text{CY}_4 \\
\downarrow \downarrow \downarrow \\
\mathbb{P}_1 \rightarrow \mathbb{P}_1 \rightarrow \mathbb{P}_1.
\end{array}
$$

We will describe the twist map for Calabi-Yau $n$-fold fibered Calabi-Yau $(n+1)$-folds in Section 2. In Section 3 we apply the twist map to the discussion of heterotic/type II and F & M theory duality relations involving the relation between the cohomology of Calabi-Yau threefolds \cite{5} and fourfolds and their dual heterotic gauge groups \cite{18}.

Our final subject will be the unification of vacua and some applications to the generation of superpotentials in fourfold compactifications. Having been led to a new class of compactifications in terms of Calabi-Yau fourfolds an immediate question arises whether the recent understanding of the connectedness of string vacua generalizes to the context of F-theory and M-theory. Much progress has been achieved recently in the interpretation of the earlier observation \cite{19, 20} that (weighted) projective complete intersection Calabi-Yau manifolds are connected via singular varieties involving only nodes. Such conifold transitions have now been understood physically in type II string theory via an effective field theory interpretation \cite{21} along the lines of Seiberg-Witten theory. The connectedness of the space of Calabi-Yau threefolds immediately implies the connectedness of certain regions of the moduli space of Calabi-Yau fourfolds. For fibered fourfolds for which the generic fiber is a Calabi-Yau threefold we can use the known conifold transitions, or more severe transitions, to induce a transition in the fourfold by degenerating the fibers. Concrete ways to implement such fiber induced transitions have been described in \cite{22}. More general transitions are possible and in Section 4 we will review \cite{22} how complete intersection fourfolds can be connected by transitions which are similar to the threefold conifold transitions of \cite{19, 20} but involve more severe degenerations of the va-
erties. Similar to the conifold transitions of [19] the constructions of [22] are independent of any fibration structure of the fourfold. As in the case of the threefold splitting transition which connects fibered threefolds with nonfibered manifolds, the splitting transitions between fourfolds also connects different types of Calabi-Yau spaces. In the final Section we show that it is possible in the process of such splitting transitions between fourfolds to generate nonvanishing superpotentials in M-theory compactifications [22].

Several other aspects of M-theory and F-theory have been discussed recently in [12] and [23].

2 The twist map in arbitrary dimensions

2.1 The twist map

In order to understand the way lower dimensional dualities can be inherited from the higher dimensional ones, and in particular to see to what extent this is possible at all, it is useful to have a tool which constructs the necessary fibrations explicitly. One way to do this is by generalizing the orbifold construction of [9] to arbitrary dimensions [18]. In the following we will call this generalized map the twist map. Our starting point is a Calabi-Yau $n$-fold with an automorphism group $\mathbb{Z}_\ell$ whose action we denote by $m_\ell$. Furthermore we choose a curve $C_\ell$ of genus $g = (\ell - 1)^2$ with projection $\pi_\ell : C_\ell \to \mathbb{P}_1$. The twist map then fibers Calabi-Yau $n$-folds into Calabi-Yau $(n+1)$-folds

$$C_\ell \times CY_n / \mathbb{Z}_\ell \ni \pi_\ell \times m_\ell \to CY_{n+1}. \quad (5)$$

For the class of weighted hypersurfaces

$$\mathbb{P}_{(k_0,k_1,...,k_{n+1})}[k] \ni \{y_0^{k/k_0} + p(y_1, ..., y_{n+1}) = 0\} \quad (6)$$

for odd $k_0$ with $\ell = k/k_0 \in \mathbb{N}$ and $k = \sum_{i=0}^{n+1} k_i$, the cyclic action can be defined as

$$\mathbb{Z}_\ell \ni m_\ell : (y_0, y_1, ..., y_{n+1}) \to (\alpha y_0, y_1, ..., y_{n+1}), \quad (7)$$

where $\alpha$ is the $\ell$th root of unity. An algebraic representation of the curve $C_\ell$ is provided by

$$\mathbb{P}_{(2,1,1)}[2\ell] \ni \{x_0^\ell - (x_1^{2\ell} + x_2^{2\ell}) = 0\} \quad (8)$$
with action $x_0 \mapsto \alpha x_0$ and the remaining coordinates are invariant. The twist map in this weighted context takes the form

$$\mathbb{P}_{(2,1,1)}[2\ell] \times \mathbb{P}_{(k_0,k_1,\ldots,k_{n+1})}[k]/\mathbb{Z}_\ell \rightarrow \mathbb{P}_{(k_0,k_0,2k_1,\ldots,2k_{n+1})}[2k]$$

(9)

and is defined as

$$((x_0, x_1, x_2), (y_0, y_1, \ldots, y_{n+1})) \rightarrow \left(x_1 \sqrt{\frac{y_0}{x_0}}, x_2 \sqrt{\frac{y_0}{x_0}}, y_1, \ldots, y_{n+1}\right).$$

(10)

This map therefore embeds the orbifold of the product on the lhs into the weighted $(n+2)$-space as a hypersurface of degree $2k$.

For $\ell = 2$ the constructions of [9, 18] reduce to the Voisin-Borcea case [24] and provide a more general class of fibrations for which the curve need not be the torus.

### 2.2 Properties of the twist map

The map (10) shows that the structure of the $(n+1)$-fold is determined by a single $n$-dimensional fiber. This is of importance in the context of duality because it suggests that lower dimensional dualities can be derived from known higher dimensional dualities [8]. It indicates in particular that the degeneration structure of the fibers will play an important part in the determination of the gauge structure of the dual heterotic theories.

It is evident from (10) however that the twist introduces new singularities on the $(n+1)$-fold fibration: the action of $\pi_\ell \times m_\ell$ has fixed point sets which have to be resolved. This resolution introduces new cohomology and therefore the heterotic gauge structure is not completely determined by the $n$-fold fiber.

In the weighted category this aspect has two manifestations:

1. If $k_0 = 1$, the action of $\mathbb{Z}_\ell$ generates a singular $(n-1)$-fold on the $n$-fold fiber. This $(n-1)$-fold singular set, which is not present in the original $n$-fold, plays different roles in different dimensions. In the case of K3-fibered threefolds the effect of the resulting singular curve is to introduce additional branchings in the resolution diagram of the Calabi-Yau threefold. It is this branching which determines the final gauge structure of the heterotic dual of the IIA theory on the threefold.

For CY$_3$-fibered Calabi-Yau fourfolds such additional branchings do not necessarily occur.
2. More generally one encounters \( k_0 > 1 \), considered for threefolds in [20] and for fourfolds in [18, 22]. In such a situation the orbifolding \( \mathbb{Z}_\ell \) generates further singularities on the threefold and the fourfold. Depending on the structure of the weights these additional singularities on \( n \)-folds can have dimensions 0, ..., \((n - 1)\). Thus for threefolds we can have additional singular points or curves and for fourfolds there can be points, curves or surfaces.

If we start from a particular \( n \)-fold there are in general several possibilities to pick an automorphism and the corresponding curve and to construct a twist map out of this. The resulting images of the twist maps will have different singularities and therefore different cohomology. We will illustrate this for threefolds and fourfolds in the following Section.

3 Applications of the twist map

In this Section we will apply the twist map to derive the gauge structure of fibered Calabi-Yau threefolds and fourfolds in the context of heterotic/type II duality and 4D F-theory respectively.

3.1 4D heterotic/type II duality

The dualities (1) and (2) are rather brief short hand notation for relations that involve somewhat more elaborate constructions. For 4D dualities what is meant by Het\((K3 \times T^2)\) in the case of the \(E_8 \times E_8\) string is a two step compactification in which the string is first compactified on a torus \(T^2\). This leads to a grand gauge group of \(E_8 \times E_8 \times G \times U(1)^2\) where \(G\) is the group induced by the torus. Generically \(G = U(1)^2\) but at special radii this symmetry becomes enhanced. The second step involves further compactification on K3 and the choice of a vector bundle \(\oplus_i V_i \twoheadrightarrow K3\) such that anomaly cancellation holds \(\sum_i c_2(V_i) = c_2(TK3)\). Embedding the structure group into the gauge group provides the starting point for a whole cascade of models obtained by repeated Higgsing [6]. More precisely the relations then take the form

\[
\text{Het}_{E_8 \times E_8}(\oplus_i V_i \twoheadrightarrow K3 \times T^2)_{\text{Higgsed}} \leftrightarrow \text{IIA}(\text{CY}_3).
\]

We will describe some examples further below.

The virtue of heterotic/type IIA duality is that it allows an exact computation of the vector multiplet couplings in the heterotic theory. This result is not obvious because the dilaton of the heterotic theory sits in a vector multiplet and therefore the vector multiplet moduli space
receives corrections from spacetime instantons. Under the dualities (1) and (2) it is however identified with the vector multiplet moduli space of the type IIA string. In type IIA the dilaton sits in a hyper multiplet and therefore its vector multiplet does not receive corrections by spacetime instantons. Compactifying IIA on a Calabi-Yau threefold associates the vector multiplets with Kähler deformations living in $H^{(1,1)}(CY_3)$ and therefore these couplings are corrected by worldsheet instantons. Via mirror symmetry we can however analyze the vector couplings of a type IIA$(CY_3)$ theory by computing the exact hyper multiplet couplings of IIB compactified on the mirror of $CY_3$. Thus 4D heterotic/type IIA duality in tandem with mirror symmetry makes it possible to determine nonperturbative heterotic corrections by doing a tree-level computation in the type II theory.

For type IIA string theory compactified on Calabi-Yau threefolds $CY_3$ we are interested in the twist map for $n=2$. Even though the possibilities are somewhat limited compared to higher dimensions because there is just one two-dimensional Calabi-Yau space, the $CY_2$ =K3 surface, it is possible to construct a great many K3-fibered threefolds by choosing different realizations of this surface with their associated automorphisms. Starting from the heterotic/type IIA duality (1) in six dimensions we know that the heterotic gauge structure in D=4, determined by the vector multiplets in type IIA$(CY_3)$ (4), is parametrized by the second cohomology group $H^{(1,1)}(CY_3)$ of the K3-fibered threefolds.

From (4) with $n=2$ we see that this group is determined by a single K3 surface and the action of the automorphism $ZZ_{\ell}$ on this surface and the curve $C_{\ell}$. In the simplest instances when the orbifold resolution of the resulting threefold does not introduce new cohomology off the fibers the gauge group is specified by the invariant part of the Picard lattice with respect to the action defining the fibration.

**Example I:** We start with the K3 Fermat surface $K$ in the weighted configuration $IP_{(1,6,14,21)}[42]$. $K$ has an automorphism group $ZZ_{42}$ and the associated curve is $C_{42} = IP_{(2,1,1)}[84]$. The image of the twist map is $IP_{(1,1,12,28,42)}[84]$. The twist has introduced a $ZZ_2$-singular curve $C = IP_{(6,14,21)}[42]$. On top of the curve $C$ one finds a $ZZ_2$, a $ZZ_3$ and a $ZZ_7$ fixed point, leading to 1, 2 and 6 new (1,1)-forms, respectively. Hence we have a total of $h^{1,1} = 11$. Each of these
resolutions leads to an $A_n$ resolution diagram with $n = 1, 2, 6$ respectively, i.e.

![Resolution Diagram with $n = 1, 2, 6$]

This is the blow-up structure as it appears on the fiber. On the threefold these exceptional divisors all sit on the curve $C$ and therefore $C$ links up the $A_n$ branches into the tree diagram

![Tree Diagram with $C$]

The resolution diagram is given by Dynkin diagram of $E_8 \times U(1)^2$. The heterotic dual of this model is an $E_8 \times E_8$ string compactified on $K3 \times T^2$ by embedding all 24 instantons in the first $E_8$ factor. This $E_8$-factor is completely higgsed whereas the second $E_8$ remains unbroken. The radii of the torus are not fixed at some particular symmetric point. Altogether, we obtain a gauge group $E_8 \times U(1)^4$. The heterotic spectrum contains 11 vectormultiplets plus the graviphoton and 492 hypermultiplets, in agreement with the Hodges $(h^{(1,1)}, h^{(2,1)})$ of the threefold. Also the resolution diagram agrees with the heterotic gauge group. It is entirely determined by the Picard lattice [7].

This manifold was also considered in the context of F-theory compactifications to six dimensions. It is the element $n = 12$ of the series $\mathbb{P}_{(1,1,2,2n+4,3n+6)}[6(n+2)]$ which was considered in [14].

**Example II:** In our second example we will illustrate that the same K3-fiber can lead to different Calabi-Yau threefolds, depending on the twist map chosen to construct the fibration. In general a single K3 surface can lead to many threefolds with different Hodge diamonds.

Consider first the Fermat surface in the K3 configuration $\mathbb{P}_{(1,1,1,3)}[6]$. The Fermat polynomial has an automorphism $\mathbb{Z}_6 : (y_0, y_1, y_2, y_3) \mapsto (\alpha y_0, y_1, y_2, y_3)$ where $\alpha$ is a $6^{th}$ root of unity. The associated curve is $C_6 = \mathbb{P}_{(2,1,1)}[12]$. Applying the twist map leads to the well-known
two-parameter Calabi-Yau manifold in the threefold configuration \( \mathbb{P}^{(1,1,2,2,6)}[12] \) considered by Kachru and Vafa \[6\]. This manifold has a \( \mathbb{Z}_2 \)-singular curve \( \mathbb{P}^{(1,1,3)}[6] \) whose resolution leads to an additional \((1,1)\)-form. There are no further singularities and therefore this manifold has Hodge numbers \((h^{(1,1)}, h^{(2,1)}) = (2, 128)\).

A different threefold can be obtained by starting again from the Fermat hypersurface in \( \mathbb{P}^{(3,1,1,1)}[6] \) but alternatively using the automorphism \( \mathbb{Z}_2 \) : \((y_0, y_1, y_2, y_3) \mapsto (\alpha y_0, y_1, y_2, y_3)\), where \( \alpha \in \{1, -1\} \). The corresponding curve is the torus \( C_2 = \mathbb{P}^{(2,1,1)}[4] \). Using this curve we obtain a different twist map, resulting in the manifold \( \mathbb{P}^{(3,3,2,2,2)}[12] \) with Hodge numbers \((h^{(1,1)}, h^{(2,1)}) = (6, 60)\), which therefore is topologically distinct from the degree twelve hypersurface considered above. Again the image of the twist map has a singular \( \mathbb{Z}_2 \) curve, giving one \((1,1)\)-form. But in addition there are four \( \mathbb{Z}_3 \) points \( \mathbb{P}_1[4] \) which do not lie on the singular curve. The resolution of these additional singularities leads to four additional \((1,1)\)-forms.

The change in the Hodge diamond is even more pronounced in the case of Calabi-Yau fourfolds in the context of M-theory and F-theory to which we will turn now.

### 3.2 4D F-theory and 3D M-theory on Calabi-Yau fourfolds

In the context of F-theory we are interested in pushing down the 8-dimensional duality \[13\]

\[
F_{12}(K3) \leftrightarrow \text{Het}(T^2) \tag{11}
\]

on elliptically fibered K3s to D=6 compactification

\[
F_{12}(CY_3) \leftrightarrow \text{Het}(K3) \tag{12}
\]

on elliptic threefolds and, finally, to D=4 compactifications

\[
F_{12}(CY_4) \leftrightarrow \text{Het}(CY_3) \tag{13}
\]

on elliptic Calabi-Yau fourfolds. Compactifying these relations further on a torus we obtain the diagram

\[
\begin{array}{cccccc}
F_{12}(K3 \times T^2) & \leftrightarrow & M_{11}(K3 \times S^1) & \leftrightarrow & \text{IIA}(K3) & \leftrightarrow & \text{Het}(T^4) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
F_{12}(CY_3 \times T^2) & \leftrightarrow & M_{11}(CY_3 \times S^1) & \leftrightarrow & \text{IIA}(CY_3) & \leftrightarrow & \text{Het}(K3 \times T^2) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
F_{12}(CY_4 \times T^2) & \leftrightarrow & M_{11}(CY_4 \times S^1) & \leftrightarrow & \text{IIA}(CY_4) & \leftrightarrow & \text{Het}(CY_3 \times T^2)
\end{array}
\]

Consider the first line of the diagram, describing dual pairs in six dimensions. We have learned that the gauge group on the IIA side is determined by the Picard lattice of the K3. Suppose
we apply the twist map to get a threefold with $k_0 = 1$, so that the structure of the gauge group descends from 6d. We have seen that the threefold contains a singular curve which induces a branching in the resolution diagram. To push the duality further down, we apply the twist map to this threefold. We see from the structure of the twist map for $n = 3$ that for the subclass of fourfolds with $k_0 = 1$ the orbifolding essentially embeds the singular curve $C$ of the threefold fiber into a singular surface on the fourfold. This leads to a prediction for the rank of the group $H^2$ of the fourfold because from our previous discussion of the rank of the gauge groups of the models dual to IIA(CY$_3$) we expect the rank of $H^2$(CY$_4$) to be determined by the rank of these gauge groups [18]. In order to perform this test of the dualities we need to compute the Hodge numbers of the fourfolds.

**Example I:** We consider an example of the type ([1]) for which we choose the fiber to be the threefold $\mathbb{P}_{(1,1,12,28,42)}[84]$ of our first example in the previous Section. The threefold is a K3-fibration with an elliptic K3 surface. The twist map now simply embeds the singular curve $C = \mathbb{P}_{(6,14,21)}[42]$ on the threefold into the $\mathbb{Z}_2$-singular K3 surface $\mathbb{P}_{(1,6,14,21)}[42]$ of the resulting fourfold $\mathbb{P}_{(1,1,2,24,56,84)}[168]$. Therefore we expect the gauge structure of the fourfold to be determined completely by the threefold and we expect

$$h^{(1,1)}(\text{CY}_4) = h^{(1,1)}(\text{CY}_3) + 1.$$  

Computing the Hodge diamond [18]

```
1
0 0 12 0 0
0 0 27548 110284 27548 0
1 27548 110284 27548 1
```

(14)

confirms this expectation.

**Example II:** For fourfolds a single fiber with different automorphisms can lead to topologically distinct fibrations, similar to the case of fibered threefolds. To illustrate this consider the Calabi-Yau threefold configuration $\mathbb{P}_{(1,1,3,3,4)}[12]$ with spectrum $(h^{(1,1)}, h^{(2,1)}) = (5, 89)$. In this configuration we pick the Fermat hypersurface and first consider the image of the twist map obtained from the automorphism $\mathbb{Z}_{12} : (y_0, y_1, y_2, y_3, y_4) \mapsto (\alpha y_0, y_1, y_2, y_3, y_4)$, where $\alpha$ is a $12^{th}$ root of unity. The resulting fourfold lives in the configuration $\mathbb{P}_{(1,1,2,6,6,8)}[24]$ and has the Hodge (half-) diamond
This fourfold contains the usual singular $\mathbb{Z}_2$-surface $\mathbb{P}_{(1,3,3,4)}$, whose resolution gives one $(1,1)$-form. On this surface there are four $\mathbb{Z}_4$-points in $\mathbb{P}_{(1,1)}[4]$ introducing four further $(1,1)$-forms. Together with the $(1,1)$-form from the ambient space we get $h^{(1,1)} = 6$.

An alternative possibility to obtain a fourfold with the fiber $\mathbb{P}_{(3,1,1,3,4)}[12]$ is to use the automorphism $\mathbb{Z}_4 : (y_0, y_1, y_2, y_3, y_4) \mapsto (\alpha y_0, y_1, y_2, y_3, y_4)$, with $\alpha$ a $4^{th}$ root of unity. Twisting with the appropriate curve $\mathcal{C}_4$ leads to the manifold $\mathbb{P}_{(3,3,2,2,6,8)}[24]$ with Hodge diamond

$$
\begin{array}{cccccc}
  & & & 1 & & \\
  & & 0 & & 0 & \\
  & 0 & 3 & 0 & 0 & \\
 1 & 254 & 1054 & 254 & 1.
\end{array}
$$

Again we have a singular $\mathbb{Z}_2$-surface $\mathbb{P}_{(1,1,3,4)}$, introducing a $(1,1)$-form. Furthermore there is a $\mathbb{Z}_3$ curve $\mathbb{P}_{(1,1,2)}[8]$ introducing the $(2,1)$-forms and a $(1,1)$-form.

**Example III:** In our final example we illustrate the complexity of the singularity structure of the fourfolds which can occur in configurations with $k_0 > 1$, depending on the choice of automorphism which is being used to construct the fibered fourfold. We start with the threefold $\mathbb{P}_{(1,2,3,6,6)}[18]$, with $(h^{(1,1)}, h^{(2,1)}) = (7, 79)$. Here we have a singular $\mathbb{Z}_3$-curve $\mathbb{P}_{(1,2,2)}[6]$ and a singular $\mathbb{Z}_2$ curve $\mathbb{P}_{(1,3,3)}[9]$ which intersect in the three points $\mathbb{P}_1[3]$. A fibration with this threefold fiber is $\mathbb{P}_{(1,1,4,6,12,12)}[36]$ with Hodge numbers $(h^{(1,1)} = 8, h^{(2,1)} = 3, h^{(3,1)} = 899, h^{(2,2)} = 3666)$.

We have the usual $\mathbb{Z}_2$-surface on top of which we find the threefold singularities. One other possibility to obtain a fibration with this threefold is to consider $\mathbb{P}_{(3,3,2,4,12,12)}$ with Hodge numbers $(h^{(1,1)} = 8, h^{(2,1)} = 1, h^{(3,1)} = 321, h^{(2,2)} = 358)$. Here again we have a $\mathbb{Z}_2$-surface with a $\mathbb{Z}_4$ curve, but because this time $k_0 = 3$ we have an additional $\mathbb{Z}_3$ surface instead of a curve. This time the fact that $k_0 > 1$ has lead to a higher dimensional singular set because of the particular form of the weights.

From these results we see [18] that the prediction via the twist map for the $h^{(1,1)}$-cohomology of the Calabi-Yau fourfolds from the dual heterotic gauge groups are confirmed.
In the next Section we will further generalize the twist map to complete intersection spaces of higher codimension. Such manifolds occur in the process of connecting the moduli spaces of different Calabi-Yau manifolds via the splitting type transition in arbitrary complex dimensions. First however we will describe these transitions and the resulting unification of vacua.

4 Unification of vacua

4.1 Connecting threefolds

A longstanding problem in string theory is the issue of the vacuum degeneracy. Soon after the discovery of anomaly free low energy field theories of various types of strings the existence of a plethora of consistent ground states was shown to exist among different compactification schemes and 4D string constructions. This appears to create a difficulty which in the early days of the first string revival led to some disenchantment. Namely, if all these consistent vacua are disjoint then this raises the question whether even in principle the string could ever determine its own ground state and eventually make some detailed predictions. If on the other hand vacua with different spectra are connected then this implies that a singularity must occur somewhere. It is then not a priori clear that the string can consistently propagate in the background of the singular configuration. Thus one has to face the issues of connecting different vacua and providing a physical interpretation of the resulting singularities. Both of these problems admit a solution in the context of Calabi-Yau compactifications.

The first of these problems was solved in [19] by showing that complete intersection Calabi-Yau manifolds with different moduli spaces can indeed be connected. In the simplest instance the Calabi-Yau spaces degenerate at the transition locus at a number of nodes, leading to conifold configurations. The resulting conifold transitions have been shown to connect all complete intersection Calabi-Yau manifolds [25] and have also been shown to generalize to the class of weighted complete intersection spaces [20]. In the simplest class of transitions hypersurfaces in weighted $\mathbb{P}_4$ [26, 27] are connected with codimension two varieties. Other possible ways of connecting different manifolds involve the extremal transitions of the toric framework [14, 28].

At the time when conifold transitions [19] were introduced it was unclear what the correct physical interpretation is of the divergences associated to the conifold transition [25]. Only recently has it been understood through the work of Greene, Morrison and Strominger [21] in type II compactifications, following similar ideas in Seiberg-Witten theory, that the divergences
in the low energy effective action arise because of new massless states which are generated at the conifold configuration when the volume of vanishing cycles degenerates. This then shows that at least in type II string theory the conifold transitions of \cite{n.b} admit a physically reasonable interpretation.

4.2 Making fourfolds, a mirror symmetry test for dualities

The same problem arises in even more pronounced form in F-theory and M-theory. One consistent compactification type of both of these theories involves Calabi-Yau fourfolds, of which there are many more than there are threefolds. The class of Fermat type hypersurface threefolds embedded in weighted $\mathbb{P}_4$, for instance, consists of only 147 configurations, whereas the number of Fermat hypersurface fourfolds is 3462, more than an order of magnitude larger. Preliminary results show that the number of all weighted hypersurfaces consists of at least several hundred thousand configurations \cite{n.2}. This suggests that there are millions of complete intersection Calabi-Yau fourfolds, providing possible vacua for 4D F-theory.

A complete construction of this class is not known at present and even the enumeration along the lines of \cite{n.2, n.3} of the subset of all hypersurfaces embedded in weighted $\mathbb{P}_5$ has not been achieved at this point. Some information can however be derived by indirect means. Via the dualities in Section 2, for instance, we are lead to expect mirror symmetry for the space of Calabi-Yau fourfolds because of its relation to (0,2) vacua. For the space of (0,2) ground states of the heterotic string recent results have established mirror symmetry for a large set of Landau-Ginzburg type theories \cite{m.2}. Turning this observation around we see that the same dualities support the expectation that the space of heterotic (0,2) vacua is indeed vastly larger than the number of (2,2) symmetric ground states and that there are many classes of (0,2) theories that remain to be discovered (see \cite{m.2} and references therein).

The cohomology of Calabi-Yau threefolds is characterized by only two distinct Hodge numbers, leading to the mirror plot \cite{n.2, n.3} of the combinations \( (h^{(1,1)} + h^{(2,1)}, \frac{1}{2} = h^{(1,1)} - h^{(2,1)}). \) For fourfolds the cohomology leads to four independent Hodge numbers \( (h^{(1,1)}, h^{(2,1)}, h^{(3,1)}, h^{(2,2)}) \) with the duality relations \( h^{(p,q)} = h^{(4-p,4-q)} = h^{(q,p)}. \) A proper Hodge type plot thus would be four-dimensional, somewhat difficult to visualize.

Mirror symmetry among Calabi-Yau fourfolds however distinguishes again the combinations \( (h^{(3,1)} - h^{(1,1)}, h^{(3,1)} + h^{(1,1)}) \) because both \( h^{(2,1)} \) and \( h^{(2,2)} \) remain invariant under the mirror flip \( h^{(p,q)}(M) = h^{(4-p,4-q)}(M') \) of a mirror pair \( (M, M') \). The standard mirror plot \cite{n.2} therefore is a measure of mirror symmetry in the present class as well. In Figure 1 we have plotted these
combinations within a lower range of these variables, cutting off the Hodge numbers at about 40k. This restriction cuts off those hypersurfaces whose degree is in the range of millions for which the Hodge numbers can easily reach the x00k range.

Figure 1: Plot of $(h^{(3,1)} - h^{(1,1)})$ vs. $h^{(3,1)} + h^{(1,1)}$ for some tens of thousands of Calabi-Yau fourfolds.

A zoom-in of Figure 1 into a smaller range of Hodge numbers with a cut-off for the Hodge numbers at about 1k reveals the expected mirror symmetric structure, as shown in Figure 2.

Figure 2: The CY$_4$ plot for a smaller region of Hodge numbers.

Comparing Figs. 1 and 2 suggests that the construction still has to go some distance.
Results in this direction have also been obtained in refs. [31, 32].

As mentioned above we expect the moduli space of CY$_3$-fibered fourfolds to be connected via degenerations of the fibers over the base space. By using the twist map described in Section 2 it is possible to construct such transitions explicitly [22]. We will see however that the connectedness is in fact more general property, independent of any fibration properties. In the present Section we will discuss a particular type of transition generalizing the splitting type transition to fourfolds. Whereas in the case of threefolds this transition leads to varieties which degenerate at a configuration of nodes in the case of fourfolds the splitting type transition leads to more severe singular varieties which degenerate at a configuration of singular curves [22]. Extremal types of transitions have been generalized to fourfolds in refs. [33, 31].

4.3 The splitting transition between weighted complete intersections

In arbitrary dimensions it is natural to consider configurations of the type

$$\mathbb{P}(k^1_1, ..., k^1_{n_1+1}), \mathbb{P}(k^2_1, ..., k^2_{n_2+1}), ..., \mathbb{P}(k^F_1, ..., k^F_{n_F+1})$$

$$\begin{bmatrix}
  d^1_1 & d^1_2 & \cdots & d^1_N \\
  d^2_1 & d^2_2 & \cdots & d^2_N \\
  \vdots & \vdots & \ddots & \vdots \\
  d^F_1 & d^F_2 & \cdots & d^F_N
\end{bmatrix} = X.$$  \hspace{1cm} (17)

Such configurations describe the intersection of the zero locus of $N$ polynomials embedded in a product of weighted projective spaces, where $N = \left(\sum_{i=1}^F n_i - D\right)$ is the number of polynomials $p_a$ of F-degree $(d^a_1, \ldots, d^a_F)$ and $D$ is the dimension of the manifold described by this degree matrix. Even though our considerations can be applied to general intersection spaces our main interest is in manifolds for which the first Chern class

$$c_1(X) = \sum_{i=1}^F \left[\sum_{l=1}^{n_i+1} k^i_l - \sum_{a=1}^N d^i_a \right] h_i$$ \hspace{1cm} (18)

vanishes. Here we denote by $h_i, i = 1, ..., F$ the pullback of the generators of $H^2(\mathbb{P}(k^i_1, ..., k^i_{n_1+1}))$. For simplicity we will first write down a split between intersections in ordinary projective spaces.

Introducing two vectors $u, v$ such that

$$(u^i + v^i) = d^i_1$$

and denoting the remaining $(F \times (N - 1))$-matrix by $M$, we write the manifolds (17) as $Y[(u + v) \ M]$. The simplest kind of transition is the $\mathbb{P}1$-split which is defined by

$$X = Y[(u + v) \ M] \leftrightarrow \mathbb{P}1_Y \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} = X_{\text{split}}.$$ \hspace{1cm} (19)
The split variety of the rhs is described by the polynomials of the original manifold and two additional polynomials, which we can write as

\[ p_1 = x_1 Q(y_i) + x_2 R(y_i) \]
\[ p_2 = x_1 S(y_i) + x_2 T(y_i), \] (20)

where \( Q(y_i) \), \( R(y_i) \) are of multi-degree \( u \) and \( S(y_i) \), \( T(y_i) \) are of degree \( v \). In (20) we collectively denote the coordinates of the space \( Y \) by \( y_i \) whereas the \( x_i \) are the coordinates of the projective line \( \mathbb{P}_1 \). To understand the relation between the two manifolds in (19) we consider (20) in more detail. We can regard the vanishing locus of (20) as a linear equation system in \( x_1 \) and \( x_2 \). These equations only have nontrivial solutions if the determinant

\[ p_{\text{det}} = QT - RS \]

vanishes. Together with the original polynomials the determinant defines a determinantal variety \( X^2 \). The polynomial \( p_{\text{det}} \) is of multidegree \( u + v \) in the variables of \( Y \) and therefore \( X^2 \in Y[(u+v)M] \) where

\[ X^2 = \{ p_{\text{det}} = QT - RS = 0, \ p_a = 0, \ a = 2, ..., N \} \] (21)

The space \( X^2 \) however is not a smooth manifold. It is singular on the locus where the determinant has a double zero because the polynomials \( Q, R, S, T \) vanish simultaneously. The singular set is described by

\[ \Sigma = Y[u \ u \ v \ v M]. \] (22)

The manifold on the lhs in (19) is \( D \)-dimensional, therefore \( \Sigma \) has dimension \( (D - 3) \).

### 4.4 Splitting and contracting Calabi-Yau fourfolds

In the following we will focus on varieties of complex dimension three and four. For a threefold split (22) describes a number of points, i.e. a conifold configuration, whereas for a fourfold split the singular set is an algebraic curve, i.e. a real two-dimensional surface, with in general several components. Finally, to move from the rhs of (19) to the lhs we have to smooth out the singularity. This can be done by adding a transverse piece to the nontransverse polynomial \( p_{\text{det}} \)

\[ p_{\text{det}} = p_{\text{det}} + t \cdot p_{\text{trans}}. \] (23)

The process to start from the rhs of (19) and to come to the lhs in the way described above is referred to as contraction [19]. The important point however is that the singular set also
admits a small resolution which takes us via the singular variety $X^\sharp$ from the lhs to the rhs. This means that the singular set is smoothed out using an exceptional set of codimension two. For threefolds this involves the projective line $\mathbb{P}_1$ and for fourfolds the projective plane $\mathbb{P}_2$. Performing such a small resolution leads to the higher codimension split manifold. To support this picture a relation between the Euler numbers of $X$ and $X^\sharp$ was proven in [19] for threefolds and in [22] for fourfolds. For threefolds the result is (neglecting the phenomenon of colliding singularities for the moment)

$$\chi(X_{\text{split}}) = \chi(X) + 2n,$$ (24)

where $n$ denotes the number of singular points and the factor 2 originates from the Euler number of the $\mathbb{P}_1$ used to smooth out the manifold. The analogous formula for fourfolds

$$\chi(X_{\text{split}}) = \chi(X) + 3\chi(\Sigma)$$ (25)

describes the resolution involving a $\mathbb{P}_2$ whose Euler number is three.

Thus we arrive at the same singular space by degenerating two distinct manifolds in different ways

$$X \rightarrow X^\sharp \leftarrow X_{\text{split}}.$$

Put differently, we can start from a determinantal variety and smooth out the singularities in two distinct ways

$$X \leftarrow X^\sharp \rightarrow X_{\text{split}}.$$

A simple example for a split between two threefolds is

$$\mathbb{P}_4[5]_{-200} \leftrightarrow \mathbb{P}_4 \begin{bmatrix} 1 & 1 \\ 1 & 4 \end{bmatrix}_{-168}.$$ (26)

This conifold transition connects the quintic in $\mathbb{P}_4$ with Hodge numbers $(h^{(1,1)}, h^{(2,1)}) = (1, 101)$ to the codimension two configuration on the rhs with Hodge $(h^{(1,1)}, h^{(2,1)}) = (2, 86)$. The physical interpretation in the context of type II string theory of this transition has been discussed in [21].

The perhaps simplest example of a splitting transition involving fourfolds is the split of the sextic

$$\mathbb{P}_5[6]_{2610} \leftrightarrow \mathbb{P}_5 \begin{bmatrix} 1 & 1 \\ 1 & 5 \end{bmatrix}_{2160},$$ (27)

where the subscripts denote the Euler numbers. The smooth hypersurface can be defined by the Fermat polynomial $p = \sum_i z_i^6$ and a transverse choice of the split configuration is provided.
by

\[ p_1 = x_1 y_1 + x_2 y_2 \]
\[ p_2 = x_1 \left( y_2^6 + y_4^6 + y_6^6 \right) + x_2 \left( y_1^6 + y_3^6 + y_5^6 \right). \]

The singular set of the determinantal variety is given by the genus \( g = 76 \) curve \( \Sigma = \mathbb{P}_3[5, 5] \). Thus we can verify the relation (23). More precisely the split (27) connects the Hodge diamond of the sextic hypersurface

\[
\begin{array}{cccccc}
1 &  &  &  &  & \\
0 & 0 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & \\
1 & 426 & 1752 & 426 & 1 & \\
\end{array}
\]

with the Hodge diamond

\[
\begin{array}{cccccc}
1 &  &  &  &  & \\
0 & 0 & 0 & 0 & 0 & \\
0 & 0 & 2 & 0 & 0 & \\
1 & 350 & 1452 & 350 & 1 & \\
\end{array}
\]

of the codimension two complete intersection manifold of (27).

It is of interest to generalize splitting for three- and four-dimensional manifolds to complete intersections in weighted projective spaces. In the weighted context it will also be possible to apply the twist map, which helps to investigate transitions between fibered manifolds.

### 4.5 Weighted transitions between K3-fibered threefolds

The weighted conifold transitions between threefolds of [20] are of the general form:

\[
\mathbb{P}(k_1, k_2, k_3, k_4)[d] \leftrightarrow \mathbb{P}(1, 1)_{(k_1, k_2, k_3, k_4)} \left[ \begin{array}{cc} 1 & 1 \\ k_1 & (d - k_1) \end{array} \right] \]

with \( d = 2k_1 + k_2 + k_3 + k_4 \).

In general conifold transitions connect K3-fibered manifolds with spaces which are not fibrations. An example is the transition from the quasismooth octic \( \mathbb{P}(1,1,2,2,2)_8 \) to the quintic \( \mathbb{P}_4[5] \). It can be shown, however, that certain types of weighted conifold transitions exist which do connect K3 fibered manifolds.

A simple class of such conifold transitions between fibered manifolds is provided by the weighted splits summarized in the diagram [20]

\[
\mathbb{P}(2l, 2l, 2m, 2k - 1, 2k - 1)[2(d + l)] \leftrightarrow \mathbb{P}(1, 1)_{(2l, 2l, 2m, 2k - 1, 2k - 1)} \left[ \begin{array}{cc} 1 & 1 \\ 2l & 2d \end{array} \right],
\]
where \( d = (2k - 1 + l + m) \). Here the hypersurfaces, containing the K3 surfaces \( \mathbb{P}_{(2k-1,l,l,m)}[2k-1+2l+m] \), split into codimension two manifolds which contain the K3 manifolds

\[
\begin{bmatrix}
\mathbb{P}^{(1,1)}_{(l,l,m,2k-1)} & \mathbb{P}^{(1,1)}_{(l,l,m,2k-1)} \\
[1 & d] & [l & d]
\end{bmatrix}
\]

of codimension two.

### 4.6 Twist map for split manifolds

In order to see the detailed fiber structure of the above varieties of codimension two it is useful to generalize the twist map we described in Section 2 to complete intersection manifolds. Consider the K3 surfaces of the type \( \mathbb{P}^{(1,1)}_{(2k-1,l,l,m)}[2k-1+2l+m] \), split into codimension two manifolds which contain the K3 manifolds

\[
\mathbb{P}^{(1,1)}_{(l,l,m,2k-1)}[2k-1+2l+m]
\]

via

\[
((x_0, x_1, x_2), (u_0, u_1), (y_0, \ldots, y_3)) \rightarrow \left( (u_0, u_1), (x_1 \sqrt{\frac{y_0}{x_0}}, x_2 \sqrt{\frac{y_0}{x_0}}, y_1, y_2, y_3) \right).
\]

Again we see that the quotienting introduces additional singular sets and the remarks of the previous Section apply in the present context as well. In particular we see the new singular curve

\[
Z_2 : \quad C = \mathbb{P}^{(1,1)}_{(l,l,m)}[1 & l & d]
\]

which emerges on the threefold image of the twist map.

An example for such a transition between manifolds which are K3-fibered as well as elliptically fibered is given by

\[
\mathbb{P}^{(1,1)}_{(1,1,2,4,12)} \leftrightarrow \mathbb{P}^{(1,1)}_{(4,4,1,1,2)}[1 & 1 & 8]
\]

where the Hodge numbers of the hypersurface are \( (h^{(1,1)}, h^{(2,1)}) = (5, 101) \) while those of the codimension two threefold \( (h^{(1,1)}, h^{(2,1)}) = (6, 70) \). Here the transverse codimension two variety of the rhs configuration is chosen to be

\[
p_1 = x_1y_1 + x_2y_2
\]
\[
p_2 = x_1(y_1^2 + y_4^2 + y_5^2 + y_2) + x_2(y_1^2 + y_3^2 + y_4^2)
\]

which leads to the determinantal variety in the lhs hypersurface configuration

\[
p_{\text{det}} = y_1^3 - y_2^3 + (y_1y_5^2 - y_2y_4^2) - (y_1 + y_2)y_5^4.
\]
This variety is singular at $\mathbb{P}_{(4,4,1,1,2)}[4 \ 4 \ 8 \ 8] = 32$ nodes, which can be resolved by deforming the polynomial.

Contained in these 2 CY-fibrations are the K3 configurations

$$\mathbb{P}_{(2,2,1,1)}[6] \leftrightarrow \mathbb{P}_{(1,1)} \begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix},$$

where the left-right arrow indicates that these two are indeed related by splitting. The K3 of the rhs is obtained by considering the divisor

$$D_\theta = \{y_4 = \theta y_3\}$$

which leads to

$$\begin{bmatrix} 1 & 1 \\ 4 & 8 \end{bmatrix},$$

with

\begin{align*}
p_1 &= x_1 y_1 + x_2 y_2 \\
p_2 &= x_1 (y_2^2 + \theta^8 y_3^8 + y_5^4) + x_2 (y_1^2 + y_3^8 - y_5^4).
\end{align*}

Because of the weights in the weighted $\mathbb{P}_3$ this is equivalent to the rhs of (39) with

\begin{align*}
p_1 &= x_1 y_1 + x_2 y_2 \\
p_2 &= x_1 (y_2^2 + \theta^8 y_3^8 + y_5^4) + x_2 (y_1^2 + y_3^4 - y_5^4).
\end{align*}

The determinantal variety following from this space is

$$p_s = y_1^3 - y_2^3 + (y_1 - \theta y_2) y_3^4 - (y_1 - y_2) y_5^4.$$ But this is precisely what one gets by considering the divisor in the determinantal 3-fold variety (38) and thus we see that the conifold transitions take place in the fiber of the CY 3-fold.

### 4.7 Weighted fourfold transitions via fiber degenerations

We will now look at fourfold transitions originating from threefold transitions via the twist map. We want to apply the twist map to the transitions (30) in order to obtain fibered fourfolds.
Let \( \ell = d/k_1 \in 2\mathbb{N} + 1 \). For the hypersurfaces of \((30)\) this amounts to choosing the curve \( C_\ell = \mathbb{P}_{(2,1,1)}[2\ell] \) and applying the twist map

\[
\mathbb{P}_{(2,1,1)}[2\ell] \times \mathbb{P}_{(k_1,k_1,k_2,k_3,k_4)}[d] \longrightarrow \mathbb{P}_{(2k_1,2k_2,2k_3,k_4)}[2d]
\]

(45)
defined as

\[
((x_1, x_2, x_3), (y_1, y_2, y_3, y_4, y_5)) \mapsto \left( y_1, y_2, y_3, y_4, \sqrt[3]{\frac{y_5}{x_1}}, x_3 \sqrt[3]{\frac{y_5}{x_1}} \right).
\]

(46)

For the codimension two threefold in \((30)\) the twist map produces the complete intersection fourfolds

\[
\mathbb{P}_{(2,1,1)}[2\ell] \times \mathbb{P}_{(k_1,k_1,k_2,k_3,k_4)}[1] \bigg[ \frac{1}{k_1} \bigg] \mathbb{P}_{(2k_1,2k_2,2k_3,k_4)}[1] \bigg[ \frac{1}{2k_1} 2(d - k_1) \bigg].
\]

(47)

From this we see that the twist map applied to threefolds which are connected via conifold transitions induces splitting transitions between fibered fourfolds

\[
\mathbb{P}_{(2k_1,2k_2,2k_3,k_4,k_4)}[2d] \Longleftrightarrow \mathbb{P}_{(8,8,4,2,1,1)}[24] \bigg[ \frac{1}{8} \bigg] \mathbb{P}_{(k_1,k_1,k_2,k_3,k_4)}[2k_1] \bigg[ \frac{1}{2k_1} 2(d - k_1) \bigg].
\]

(48)

In this way the twist map maps the threefold split \((30)\) in the previous Section to the fourfold split

\[
\mathbb{P}_{(8,8,4,2,1,1)}[24] \bigg[ \frac{1}{8} \bigg] \mathbb{P}_{(8,8,4,2,1,1)}[24] \bigg[ \frac{1}{16} \bigg],
\]

(49)

where the lhs manifold is defined by the zero locus of the polynomial

\[
p = z_0^3 + z_1^3 + z_2^6 + z_3^{12} + z_4^{24} + z_5^{24}
\]

and the rhs by the equations

\[
p_1 = x_1 y_1 + x_2 y_2
\]
\[
p_2 = x_1 (y_2^2 + y_4^4 + y_6^{16}) + x_2 (y_1^4 + y_3^4 + y_5^{16}).
\]

The determinantal variety determined by

\[
p_{det} = y_1 (y_2^2 + y_3^4 + y_5^{16}) - y_2 (y_2^2 + y_4^4 + y_6^{16})
\]
is singular on the locus \( \Sigma = \mathbb{P}_{(4,2,1,1)}[16 16], \) describing a smooth curve of genus \( g = 385. \)

Altogether, the fibration structure of the lhs of this example can be summarized by several applications of the twist map as

\[
\mathbb{P}_2[3] \longrightarrow \mathbb{P}_{(2,2,1,1)}[6] \longrightarrow \mathbb{P}_{(4,4,2,1,1)}[12] \longrightarrow \mathbb{P}_{(8,8,4,2,1,1)}[24],
\]

(50)
whereas the codimension two space leads to the iterative structure

\[
\begin{align*}
\mathbb{P}_1 &\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \rightarrow \mathbb{P}_{(1,1)} &\begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix} \rightarrow \mathbb{P}_{(2,2,1,1)} &\begin{bmatrix} 1 & 1 \\ 4 & 8 \end{bmatrix} \rightarrow \mathbb{P}_{(4,4,2,1,1)} &\begin{bmatrix} 1 & 1 \\ 8 & 16 \end{bmatrix} \rightarrow \mathbb{P}_{(8,8,4,2,1,1)} &\begin{bmatrix} 1 & 1 \\ 8 & 16 \end{bmatrix}.
\end{align*}
\]

Both sides of this split provide examples for the nested fibration structure (4) mentioned in the introduction.

\section{Superpotentials}

In the previous Section we have seen that some regions of M- and F-theory vacua are connected by transitions. In the present Section we will see that these vacua can be distinguished in an intrinsic manner by the property that some of them lead to a nonperturbatively generated superpotential. In \[\textbf{17}\] it was shown that a nonperturbative superpotential is generated for compactifications of M-theory on certain Calabi-Yau fourfolds. An example which shows modular behaviour for the superpotential was described in \[\textbf{34}\]. In \[\textbf{22}\] the variety of \[\textbf{34}\] was connected to a manifold which does not generate a superpotential by a splitting transition. Before we describe this transition we will briefly review the generation of superpotentials in three-dimensional field theory and M-theory compactifications to three dimensions.

\subsection{The superpotential in 3D field theory}

The generation of a superpotential by instantons in \(d = 3\), \(N = 2\) field theory was considered in \[\textbf{33}\] via dimensional reduction of an \(N = 1\) supersymmetric gauge theory with gauge group \(SU(2)\) from four dimensions. It was shown that due to instanton effects there are potential terms for the scalar \(\varphi\) arising in the three-dimensional theory as a mode of the 4-dimensional gauge field.

Recall that there are two types of supersymmetric invariants in 4 dimensions (and this is also valid for the 3D gauge theories under consideration).

\[ I_1 = \int d^4x\, d^2\theta\, d^2\overline{\theta} \, Q(x, \theta, \overline{\theta}) \quad \text{(52)} \]
\[ I_2 = \int d^4x\, d^2\theta \, R(x, \theta), \quad \text{(53)} \]

where \(Q\) and \(R\) are superfields. Because \(R\) does not depend on \(\overline{\theta}\) we can build a supersymmetric invariant without integration over \(\overline{\theta}\). The invariants of type \(I_1\) contain the kinetic terms, whereas
those of type $I_2$ contain the mass terms and Yukawa couplings. There are nonrenormalization theorems saying that to any finite order in perturbation theory quantum corrections can only affect operators which can be written in the form $I_1$. Therefore we have to take into account nonperturbative effects if we want to generate a superpotential. The instantons in the 3D gauge theory are monopoles in the BPS-limit. They are invariant under half of the four supercharges. Two supercharges generate two fermion zero modes. Instanton corrections lead to a factor $e^{-I}$ in the effective action, where $I$ is the one-instanton action. In the BPS-case it is given by

$$I = \frac{4\pi \varphi}{e}$$

In 3D we are in the situation that the gauge field is dual to a scalar $\phi$. If one computes the effective action in 3D it can be seen that the scalar field $\varphi$ combines with the field $\phi$ to a complex field $Z = \varphi + i\phi$. The instanton correction becomes

$$e^{-(I+i\phi)}$$ (54)

Actually, $\varphi$ and $\phi$ have combined to give the scalar component of a chiral superfield.

The effect of the fermion zero modes is that the function (54) must be integrated over chiral superspace

$$\int d^2\theta \ e^{-(I+i\phi)}$$ (55)

and is a superpotential rather than an ordinary potential.

### 5.2 Superpotentials in M-theory compactifications

Consider now the compactification of M-theory to three dimensions on a Calabi-Yau fourfold. It was shown in [17] that a superpotential can be generated by wrapping 5-branes over certain divisors in the fourfold, resulting in the M-theory analogs of the field theory quantities reviewed in the previous Section.

Gauge fields in three dimensions are obtained as modes of the 3-form potential $C$

$$C_{\mu ij}(x,y) = \sum_{\Lambda=1}^{h^{(1,1)}} A^{A}_\mu(x) \omega_{ij}^{(A)}.$$ (56)

Here, $\mu$ is the three-dimensional spacetime index and $i, j$ are internal indices. $\omega_{ij}^{(A)}$ are a basis of (1,1)-forms of the fourfold. Again, the $A^{A}_\mu$'s are dual to scalars. The next thing we have to look for is the analog of the BPS-monopoles of the field theory. The magnetic source for $C$ is
the M-theory M5-brane. Thus, we can produce something that looks like an instanton in 3D by wrapping the world-volume of the M5-brane around 6-cycles in the Calabi-Yau. As we have seen in the previous Section we need the property that the instanton must be invariant under two of the four supersymmetries, so the cycle must be a complex divisor. The field theory analysis of the previous Section leads us to expect a superpotential of the form

\[ e^{-(V_D + i\phi_D)} P, \quad (57) \]

where \( V_D \) is the instanton-action determined by the volume of the divisor \( D \) around which the M5-brane wraps, and \( \phi_D \) is a linear combination of dual scalars. \( P \) is a one-loop determinant of world-volume fields, whose zeroes determine the vacua of the theory. In [17] the fermion zero modes were discussed. Using an anomaly-cancellation argument it was derived that the divisor has to fulfill additional properties to give a nonvanishing contribution to the superpotential. A particularly simple case is the situation where the only fermionic zero modes are the two fermion zero modes coming from the supercharges. Here a superpotential is generated (see Section 5.1). One way to guarantee that there are no further fermion zero modes is to require that the divisor has to satisfy the conditions

\[ \dim \Omega^{(0,1)} = \dim \Omega^{(0,2)} = \dim \Omega^{(0,3)} = 0. \]

(58)

The reason for this is that on a Kähler manifold the zero modes of the Dirac operator are given by the Dolbeault-cohomology. We then obtain a superpotential similar to (55).

The condition (58) implies that the arithmetic genus of the divisor equals one.

\[ \chi(D, \mathcal{O}_D) = \sum_{n=0}^{3} (-1)^n \dim \Omega^{(0,n)} = 1. \]

(59)

The precise anomaly cancellation argument shows that (59) has to hold also in more general situations where a superpotential is generated, i.e. (59) is a necessary (but not sufficient) condition for superpotential generation, whereas (58) is sufficient but not necessary.

### 5.3 Generating a superpotential via splitting

We will now consider the behaviour of the superpotential under splitting transitions. Consider the manifold

\[ X = \begin{bmatrix} \mathbb{P}_1 \\ \mathbb{P}_2 \\ \mathbb{P}_2 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}. \]

(60)
From Lefschetz’ hyperplane theorem we know that $h^{(1,1)} = 3$ and $h^{(2,1)} = 0$. Furthermore we can determine $h^{(3,1)} = 280$ by counting complex deformations. Plugging all this into the Euler number leads to the complete Hodge half-diamond

$$
\begin{array}{cccc}
1 & 0 & 0 \\
0 & 3 & 0 \\
280 & 1176 & 280 & 1
\end{array}
$$

It has been shown in [17] that manifolds of this type, i.e. hypersurfaces embedded in products of ordinary projective spaces, do not lead to nonvanishing superpotential. However the manifold above can be split into one that does contain divisors which generate a superpotential, namely the manifold studied in [34].

$$
\begin{bmatrix}
\mathbb{P}_1 & 2 \\
\mathbb{P}_2 & 3 \\
\mathbb{P}_2 & 3
\end{bmatrix} \leftrightarrow \begin{bmatrix}
\mathbb{P}_1 & 1 & 1 \\
\mathbb{P}_1 & 2 & 0 \\
\mathbb{P}_2 & 3 & 0 \\
\mathbb{P}_2 & 0 & 3
\end{bmatrix} = X_{\text{split}}.
$$

Both of these spaces are elliptic fibrations and the split manifold is also a K3-fibration with generic elliptic K3 fibers.

The determinantal hypersurface

$$X^\sharp = \{p_{\text{det}} = QT - RS = 0\} \in \begin{bmatrix}
\mathbb{P}_1 & 2 \\
\mathbb{P}_2 & 3 \\
\mathbb{P}_2 & 3
\end{bmatrix}$$

is singular at the locus

$$
\begin{bmatrix}
2 & 2 & 0 & 0 \\
3 & 3 & 0 & 0 \\
0 & 0 & 3 & 3
\end{bmatrix} = 9 \times \Sigma,
$$

where $\Sigma = \begin{bmatrix}
2 & 2 \\
3 & 3
\end{bmatrix}$ and $\mathbb{P}_2[3 \ 3] = 9 \text{pts}$. The curve $\Sigma$ has Euler number $\chi(\Sigma) = -54$ and hence genus $g(\Sigma) = 28$. Thus the singular set has 9 different components and the splitting formula (25) becomes

$$\chi(X_{\text{split}}) = \chi(X) + 3 \cdot 9 \chi(\Sigma) = 288.$$

We see from this that it is precisely the small resolution of the curve $\Sigma$ which introduces the divisors in $X_{\text{split}}$ which are responsible for the superpotential. But splitting does not necessarily change the superpotential. Starting from a manifold with vanishing superpotential there can also be splits which connect this manifold with another manifold with no superpotential. An example for this is the sextic split (27).

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