INVISCID, INCOMPRESSIBLE AND SEMICLASSICAL LIMITS OF QUANTUM NAVIER-STOKES EQUATIONS

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Abstract. In the paper, we consider the inviscid, incompressible and semiclassical limits of the barotropic quantum Navier-Stokes equations of compressible flows in a periodic domain. We show that the limit solutions satisfy the incompressible Euler system based on the relative entropy inequality and on the detailed analysis for general initial data. The rate of convergence is estimated in terms of the Mach number.

1. Introduction

This paper is concerned with the following non-dimensional compressible Quantum-Navier-Stokes (QNS) system on \((0, T) \times \Omega\),

\[
\partial_t n + \text{div}(nu) = 0, \quad \text{(1.1)}
\]

\[
\partial_t (nu) + \text{div}(nu \otimes u) + \nabla p(n) - 2\kappa^2 n \nabla \left( \frac{\Delta \sqrt{n}}{\sqrt{n}} \right) = 2\nu \text{div}(nD(u)). \quad \text{(1.2)}
\]

The domain \(\Omega\) we consider is the 2-dimensional torus \(T^2\). There are two unknowns, the density \(n = n(t, x)\) and the velocity field \(u = u(t, x)\) of the fluid. \(\nu\) and \(\kappa\) are positive constants and they are called the viscosity and the dispersive coefficients. \(p(n)\) is the pressure, and in this paper, we consider the case of isentropic flows with \(p(n) = n^\gamma\) for \(\gamma > 1\). In \([12]\), \(D(u) = (\nabla u + \nabla u^T)/2\). The term \(2\kappa^2 n \nabla (\Delta \sqrt{n}/\sqrt{n})\) can be interpreted as the quantum Bohm potential term, or as a quantum correction to the pressure. Moreover, the following relation holds

\[
2n \nabla \left( \frac{\Delta \sqrt{n}}{\sqrt{n}} \right) = \nabla \Delta n - 4\text{div}(\nabla \sqrt{n} \otimes \nabla \sqrt{n}), \quad \text{(1.3)}
\]

which can avoid using too high regularities of the density \(n\). Brull and Méhats \([3]\) utilized a moment method and a Chapman-Enskog expansion around the quantum equilibrium to derive \([11]-[12]\) from a Wigner equation.

The main purpose of this paper is to rigorously prove the combined incompressible and inviscid limits in the framework of the global weak solutions to \([11]-[12]\).

To begin with, we introduce the scaling

\[ t \mapsto \epsilon t, \quad u \mapsto \epsilon u, \]

and set

\[ \kappa = \epsilon^2, \quad \nu = \epsilon^2, \]

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where $0 < \epsilon < 1$ is a small parameter proportional to the Mach number. With such scalings, the quantum Navier-Stokes equations (1.1)-(1.2) read

$$\partial_t n + \text{div}(nu) = 0, \quad (1.4)$$

$$\partial_t (nu) + \text{div}(nu \otimes u) + \frac{1}{\epsilon^2} \nabla p(n) - 2\epsilon^2 \nabla \left( \frac{\Delta \sqrt{n}}{\sqrt{n}} \right) = 2\epsilon \text{div}(nD(u)). \quad (1.5)$$

Our aim is to identify the scaled QNS system (1.4)-(1.5) in the limit $\epsilon \to 0$, meaning the incompressible, inviscid and semiclassical limit. More precisely, we want to prove that the weak solution of the QNS system (1.4)-(1.5) converges to the classical solution of the corresponding incompressible Euler system, namely

$$\partial_t v + (v \cdot \nabla)v + \nabla \Pi = 0, \quad \text{div}v = 0. \quad (1.6)$$

Multiplying equation (1.5) by $u$ and integrating by parts, we obtain the energy inequality of the QNS system (1.4)-(1.5) in its integral form

$$E(t) + 2\epsilon \int_0^t \int_{\mathbb{T}^2} n|D(u)|^2 dx dt \leq E(0), \quad (1.7)$$

where the total energy $E$ is given by the sum of the kinetic, internal and quantum energy:

$$E(t) = \int \left( \frac{1}{2} n|u|^2 + \frac{1}{\epsilon^2} H(n) + 2\epsilon^2 |\nabla \sqrt{n}|^2 \right) dx,$$

with

$$E(0) = \int \left( \frac{1}{2} n_0|u_0|^2 + \frac{1}{\epsilon^2} H(n_0) + 2\epsilon^2 |\nabla \sqrt{n_0}|^2 \right) dx$$

the initial energy, and

$$H(n) = \frac{1}{\gamma - 1} (n^\gamma - \gamma(n - 1) - 1) = \frac{1}{\gamma - 1} (p(n) - p'(1)(n - 1) - p(1)) \quad (1.8)$$

the Helmholtz free energy.

Without quantum effects, system (1.4)-(1.5) reduces to the Navier-Stokes equations, whose incompressible inviscid limit was investigated by Lions-Masmoudi [16] in the case of well-prepared initial data and by Masmoudi [17] in the case of ill-prepared initial data in the whole space case and also in the periodic case. In [4], Caggio-Neasov consider the inviscid incompressible limits of the rotating compressible Navier-Stokes system for a barotropic fluid and show that the limit system is represented by the rotating incompressible Euler equation on the whole space. The present paper will extend the results in [17] to the quantum Navier-Stokes equations. In comparison with [4][16][17], the present problem features some additional mathematical difficulties related to the third-order derivative term in the momentum equations. Next, the viscosity we consider here is dependent on the density. Finally, we use the refined energy analysis to obtain the desired rate of convergence. Therefore new techniques and ideas are introduced to treat them. Our approach is based on the existence of global in time finite energy weak solutions [1] for the Cauchy problem of QNS system (1.4)-(1.5) and the recently discovered relative entropy inequality [5][9] which gives us a very powerful and concise tool for the purpose of measuring the stability of a solution compared to another solution with a better regularity. We will prove that the convergence of QNS system to the
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limit problem (incompressible Euler system) takes place on any time interval $[0, T]$ on which the Euler system (1.6) possesses a regular solution by introducing a new relative entropy functional.

Recently, we saw a publication [13] by Kwon-Li in which a similar result was proved on the convergence of the degenerate quantum compressible Navier-Stokes equations with damping to the incompressible Navier-Stokes/Euler equations. Here we state the main differences between two papers. First, the relative entropy functional we introduced is very different from that of [13]. Second, the initial condition which is needed in our main results is different from that of their main results in [13]. Finally, The rate of convergence in two papers is also different. We also mention that the incompressible limits of the compressible Navier-Stokes system and related models are very interesting and there are a lot of references on this topic. See Alazard [2] for Navier-Stokes equations, Feireisl-Novotny [3,7] for the Navier-Stokes-Fourier system and the Euler-Boussinesq System, Jiang-Ou [12] for the non-isentropic Navier-Stokes equations, Hu-Wang [10] for the viscous compressible magnetohydrodynamic flows, Jiang-Ju-Li [11] for the compressible magnetohydrodynamic equations with vanishing viscosity coefficients, Ukai [19] for the Euler equations, Wang-Yu [20] for the compressible flow of Liquid crystals and references therein.

In this present paper, we denote by $1$ the characteristics function and $C$ the generic positive constants independent of $\epsilon$.

The rest of this paper is organized as follows. In the next section, we state some useful known results and our main results. Finally, Section 3 is devoted to the proof of our main result.

2. Main Results

In this section, we state our main results. For this, we first recall the following classical result on the existence of sufficiently regular solutions of the incompressible Euler system (1.6) with the initial data $v(0) = v_0$.

**Proposition 2.1.** (Ref. [14,18]) Let $v_0 \in C^\infty(T^2)$ satisfying $\text{div}v_0 = 0$. Then, for any $T > 0$, the incompressible Euler system (1.6) exists a unique classical solution $v$ satisfying

$$v, \Pi \in C^\infty([0,T] \times T^2).$$

(2.1)

Recently, in [1] Antonelli-Spirito consider the QNS system (1.4)-(1.5) both in two and in three space dimensions and prove the following result on the global existence of finite energy weak solutions for large initial data.

**Proposition 2.2** (Ref. [1]). Let $\alpha$ be a small fixed positive number and the initial data $n_0, u_0$ satisfying

$$0 < \frac{1}{\pi_0} < n_0 \leq \pi_0, n_0 \in L^1(T^2) \cap L^\gamma(T^2), \nabla \sqrt{n_0} \in L^2(T^2) \cap L^{2+\alpha}(T^2),$$

(2.2)

$$\sqrt{n_0}u_0 \in L^2(T^2) \cap L^{2+\alpha}(T^2)$$

(2.3)

for a positive constant $\pi_0$. Then for any $0 < T < +\infty$, there exists a finite energy weak solutions $(n', u')$ of the compressible QNS system (1.4)-(1.5) with initial data $n_0, u_0$ on $(0, T) \times T^2$ satisfying

- **Integrability conditions:**
  $$n \in L^\infty(0,T;L^\gamma(T^2)), \sqrt{n}u \in L^\infty(0,T;L^2(T^2)), \sqrt{n} \in L^\infty(0,T;H^1(T^2)).$$
• **Continuity equation:**

\[
\int_{\mathbb{T}^2} n_0 \phi(0) dx + \int_0^T \int_{\mathbb{T}^2} n \phi_t + \sqrt{n} \sqrt{n} u \nabla \phi dx dt = 0,
\]

for any \( \phi \in C^\infty_0([0,T] \times (\mathbb{T}^2)) \).

• **Momentum equation:**

\[
\int_{\mathbb{T}^2} n_0 u_0 \psi(0) + \int_0^T \int_{\mathbb{T}^2} \sqrt{n} \sqrt{n} u \nabla \psi dx dt + \frac{1}{\epsilon^2} \int_0^T \int_{\mathbb{T}^2} p(n) \text{div} \psi dx dt - 2e \int_0^T \int_{\mathbb{T}^2} (\nabla \sqrt{n} \otimes \nabla \sqrt{n}) \nabla \psi dx dt
\]

\[
- 2e \int_0^T \int_{\mathbb{T}^2} (\nabla \sqrt{n} \otimes \nabla \sqrt{n}) \nabla \psi dx dt + 2e^2 \int_0^T \int_{\mathbb{T}^2} \sqrt{n} \nabla \nabla \nabla \psi dx dt = 0,
\]

for any \( \psi \in C^\infty_0([0,T] \times (\mathbb{T}^2)) \).

• **Energy inequality:** if

\[
E(t) = \int_{\mathbb{T}^2} \left( \frac{1}{2} n |u|^2 + \frac{1}{\epsilon^2} H(n) + 2e^2 |\nabla \sqrt{n}|^2 \right) dx,
\]

then the following energy inequality is satisfied for a.e. \( t \in [0,T] \)

\[
E(t) \leq E(0).
\]

**Remark 2.1.** The scaling we introduced for QNS system satisfies the requirement to the viscosity and the dispersive coefficients.

Motivated by [5], we introduce the following relative entropy functional

\[
E(t) = \int_{\mathbb{T}^2} \left[ \frac{1}{2} n |u - v - \nabla \Psi|^2 + 2e^2 |\nabla \sqrt{n} - \nabla \sqrt{1 + \epsilon \sigma}|^2 \right.
\]

\[
- \frac{1}{\epsilon^2} [H(n) - H'(1 + \epsilon \sigma)(n - 1 - \epsilon \sigma) - H(1 + \epsilon \sigma)] \right] dx, \tag{2.4}
\]

where \( \sigma, \Psi \) are the solution of the following acoustic system related to the QNS system (1.4)-(1.5) by the following linear relations [8]

\[
\partial_t \sigma + \frac{1}{\epsilon} \Delta \Psi = 0, \tag{2.5}
\]

\[
\partial_t \nabla \Psi + \frac{p'(1)}{\epsilon} \nabla \sigma = 0 \tag{2.6}
\]

supplemented with the initial data

\[
\sigma(0) = \sigma_0 = n_0^1, \quad \nabla \Psi(0) = \nabla \Psi_0 = u_0 - v_0, \tag{2.7}
\]

where \( v_0 = H[n_0] \) denotes the Helmholtz projection into the space of solenoidal functions. Similarly to [6], our goal is to apply a Gronwall-type argument to the
relative entropy inequality \([3.13]\) to deduce the strong convergence to the limit system claimed in Theorem [2.3]. The initial data \((n_0^1, \nabla \Psi_0)\) can be regularized in the following way
\[
n_0^1 = n_{0,\eta}^1 = \chi_\eta \ast (\psi_\eta n_0^1), \quad \nabla \Psi_0 = \nabla \Psi_{0,\eta} = \chi_\eta \ast (\psi_\eta \nabla \Psi_0), \quad \eta > 0, \tag{2.8}
\]
where \(\{\chi_\eta\}\) is a family of regularizing kernels and \(\psi_\eta \in C^\infty_0(T^2)\) is standard cut-off function. The total change in energy of the fluid caused by acoustic wave is given by
\[
\frac{1}{2} \int_{T^2} (p'(1)|\sigma|^2 + |\nabla \Psi|^2) dx,
\]
which is conserved in time, namely
\[
\frac{1}{2} \int_{T^2} (p'(1)|\sigma|^2 + |\nabla \Psi|^2) dx = \frac{1}{2} \int_{T^2} (p'(1)|\sigma_0|^2 + |\nabla \Psi_0|^2) dx. \tag{2.9}
\]
In addition, for any \(t > 0\), the dispersive estimates hold [6]
\[
\|\nabla \Psi\|_{W^{k,p}} + \|\sigma\|_{W^{k,p}} \leq C \left(1 + \frac{t}{\epsilon}\right)^{\frac{k-\frac{1}{q}}{2}} \left(\|\nabla \Psi_0\|_{W^{k,p}} + \|n_0^1\|_{W^{k,p}}\right), \tag{2.10}
\]
where
\[
2 \leq p \leq +\infty, \quad \frac{1}{p} + \frac{1}{q} = 1, \quad k = 0, 1, \ldots .
\]

The main result of this paper can be stated as follows.

**Theorem 2.3.** Let \(M > 0\) be a constant. Let the initial data of the compressible QNS system \([1.4, 1.5]\) be of the following form
\[
n(0) = n_{0,\epsilon}, \quad u(0) = u_{0,\epsilon}, \tag{2.11}
\]
and satisfying \(2.8\) for fixed \(\eta\). Let all requirements of Proposition 2.1 be satisfied with the initial data \(n_0 = H[u_0]\) for the Euler system \([1.7]\). Let \(\sigma, \Psi\) be the solution of the acoustic system \([2.5, 2.6]\) with the initial data \([2.3]\). Then, for any \(t \in [0, T]\) and any weak solutions \((n, u)\) of compressible QNS system \([1.4, 1.5]\), we have
\[
\|\sqrt{n}(u - v - \nabla \Psi)\|_{L^2(T^2)} + \left\|\frac{n - 1 - \epsilon \sigma}{\epsilon}\right\|_{L^2(T^2)}
\]
\[
+ \epsilon^2 \|\nabla \sqrt{n} - \nabla 1 + \epsilon \sigma\|_{L^2(T^2)}^2
\]
\[
\leq C \left(\|n_{0,\epsilon} - u_0\|_{L^2(T^2)}^2 + \|n_{0,\epsilon} - n_0^1\|_{H^1(T^2)}^2 + \epsilon^{\min\{1-\frac{1}{q} - \frac{1}{p}\}}\right), \tag{2.13}
\]
for a.e. \(t \in [0, T]\).

For any vector field \(\phi\), we use \(P \) and \(Q\) to denote the divergence-free part of \(\phi\), and the gradient part of \(\phi\), respectively, i.e. \(P = I - Q\) and \(Q = \nabla - \nabla \Delta^{-1} \text{div}\).

The following corollary is a consequence of the above theorem.

**Corollary 2.4.** In Theorem 2.3, we assume that
\[
\|n_{0,\epsilon} - u_0\|_{L^2(T^2)} + \|n_{0,\epsilon} - n_0^1\|_{H^1(T^2)} \leq C \epsilon^{\min\{1-\frac{1}{q}, \frac{1}{p}\}}. \tag{2.14}
\]
Then, for any weak solutions \((n, u)\) of compressible QNS system \([1.4, 1.5]\), we have
\[
\|P(\sqrt{n}u) - v\|_{L^2(T^2)} + \left\|\frac{n - 1}{\epsilon}\right\|_{L^2(T^2)}^2 + \epsilon^2 \|\nabla \sqrt{n}\|_{L^2(T^2)}^2 \leq C \epsilon^{\min\{1-\frac{1}{q}, \frac{1}{p}\}}. \tag{2.15}
\]
for a.e. $t \in [0, T]$. 

3. Proof of Theorem

We consider the class of finite energy weak solutions of the compressible QNS system (1.4) - (1.5) satisfying, besides the standard weak formulation of the equations, the energy inequality

$$
\int_{\mathbb{T}^2} \left\{ \frac{1}{2} |n| u|^2 + \frac{1}{\epsilon^2} H(n) + 2\epsilon^2 |\nabla \sqrt{n}|^2 \right\} dx + 2\epsilon \int_0^t \int_{\mathbb{T}^2} |n| D(u)|^2 dx dt 
\leq \int_{\mathbb{T}^2} \left\{ \frac{1}{2} n_0 |u_{0,e}|^2 + \frac{1}{\epsilon^2} H(n_0,e) + 2\epsilon^2 |\nabla \sqrt{n_0,e}|^2 \right\} dx 
\leq C.
$$

(3.1)

Therefore, we have the following properties:

- $\sqrt{n}u$ is bounded in $L^\infty([0, T]; L^2(\mathbb{T}^2))$, 
- $\frac{1}{\epsilon^2} H(n)$ is bounded in $L^\infty([0, T]; L^1(\mathbb{T}^2))$, 
- $\sqrt{\epsilon n} D(u)$ is bounded in $L^2([0, T]; L^2(\mathbb{T}^2))$. 

(3.2)
(3.3)
(3.4)

Lemma 3.1. Let $(n, u)$ be the weak solution to quantum Navier-Stokes equations (1.4) - (1.5) on $[0, T]$. Then there exists a constant $C > 0$ such that for all $\epsilon \in (0, 1)$ and $\gamma > 1$,

$$
\|(n - 1)1_{\{|n-1|<1\}}\|_{L^\infty([0, T]; L^2(\mathbb{T}^2))} \leq C\epsilon, 
\|(n - 1)1_{\{|n-1|\geq 1\}}\|_{L^\infty([0, T]; L^\gamma(\mathbb{T}^2))} \leq C\epsilon^{\frac{\lambda}{\gamma}}.
$$

(3.5)
(3.6)

Furthermore, we have

$$
\|n^\epsilon - 1\|_{L^\infty([0, T]; L^\gamma(\mathbb{T}^2))} \leq C\epsilon^{\frac{\lambda}{\gamma}} \quad \text{and} \quad \|n^\epsilon - 1\|_{L^\infty([0, T]; L^\gamma(\mathbb{T}^2))} \leq C\epsilon,
$$

(3.7)

where, $\lambda = \min\{2, \gamma\}$. 

Proof. In view of the Lemma 5.3 in [15], there exist two positive constants $c_1 \in (0, 1)$ and $c_2 \in (1, +\infty)$ independent of $n$ such that the following inequality

$$
c_1 \int_{\mathbb{T}^2} (|n - 1|^2 1_{\{|n-1|<1\}} + |n - 1|^\gamma 1_{\{|n-1|\geq 1\}}) dx 
\leq \int_{\mathbb{T}^2} H(n) dx \leq c_2 \int_{\mathbb{T}^2} (|n - 1|^2 1_{\{|n-1|<1\}} + |n - 1|^\gamma 1_{\{|n-1|\geq 1\}}) dx,
$$

(3.8)

which implies the Lemma 3.1 holds due to (3.3). 

With the help of the smoothness of the limit system (1.6), we take $v + \nabla \Psi$ as the test function in the weak formulation of equation (1.5) to yield the following equality for almost all $t$,

$$
\mathcal{E}(t) + 2\epsilon \int_0^t \int_{\mathbb{T}^2} n|D(u-v-\nabla \Psi)|^2 dx dt 
= \int_{\mathbb{T}^2} \left( \frac{1}{2} |n| u|^2 + 2\epsilon^2 |\nabla \sqrt{n}|^2 + \frac{1}{\epsilon^2} H(n) \right) dx + 2\epsilon \int_0^t \int_{\mathbb{T}^2} n|D(u)|^2 dx dt 
\leq \int_{\mathbb{T}^2} \left[ \frac{1}{2} n_0 |u_{0,e}|^2 + 2\epsilon^2 |\nabla \sqrt{n_0,e}|^2 + \frac{1}{\epsilon^2} H(n_0,e) \right] dx
$$

\[
\leq C_2 \left[ \frac{1}{2} n_0 |u_{0,e}|^2 + 2\epsilon^2 |\nabla \sqrt{n_0,e}|^2 + \frac{1}{\epsilon^2} H(n_0,e) \right] dx
\]
We use $v + \nabla \Psi$ as a test function in the weak formulation of momentum equation (1.5) to yield the following equality for almost all $t$:

\[
- \int_{T_2} n u \cdot (v + \nabla \Psi) dx = - \int_{T_2} n_0, \epsilon u_0, \epsilon \cdot (v_0 + \nabla \Psi_0) dx - \int_0^t \int_{T_2} n u \cdot \partial_t (v + \nabla \Psi) dxd\tau
- \int_0^t \int_{T_2} n u \cdot \nabla (v + \nabla \Psi) \cdot u dxd\tau
- \frac{1}{\epsilon^2} \int_0^t \int_{T_2} p(n) \Delta \Psi dxd\tau + \epsilon^2 \int_0^t \int_{T_2} (n - 1) \Delta \Psi dxd\tau
+ 4\epsilon^2 \int_0^t \int_{T_2} (\nabla \sqrt{n} \otimes \nabla \sqrt{n}) : \nabla (v + \nabla \Psi) dxd\tau
+ 2\epsilon \int_0^t \int_{T_2} n D(u) : D(v + \nabla \Psi) dxd\tau,
\]

(3.10)

where we have used the equality (1.3). By a direct computation, we have

\[
\frac{1}{2} \int_{T_2} n(v + \nabla \Psi)^2 dx
= - \frac{1}{2} \int_{T_2} n_0, \epsilon (v_0 + \nabla \Psi_0)^2 dx + \int_0^t \int_{T_2} n u \cdot \nabla (v + \nabla \Psi) \cdot (v + \nabla \Psi) dxd\tau
+ \int_0^t \int_{T_2} n(v + \nabla \Psi) \partial_t (v + \nabla \Psi) dxd\tau
\]

(3.11)

and

\[
- \frac{1}{\epsilon^2} \int_{T_2} [H'(1 + \epsilon \sigma)(n - 1 - \epsilon \sigma) + H(1 + \epsilon \sigma)] dx
= - \frac{1}{\epsilon^2} \int_{T_2} [\epsilon H'(1 + \epsilon n_0^1) (n_0^1, \epsilon - n_0^1) + H(1 + \epsilon n_0^1)] dx
\]

(3.12)
Then, putting (3.10)–(3.12) into (3.9), we have

\begin{align*}
\mathcal{E}(t) + 2\epsilon \int_0^t \int_{\mathbb{T}^2} n |D(u - v - \nabla \Psi)|^2 \, dx \, d\tau \\
\leq \mathcal{E}(0) - \int_0^t \int_{\mathbb{T}^2} n [\partial_t (v + \nabla \Psi) + u \cdot \nabla (v + \nabla \Psi)] \cdot (u - v - \nabla \Psi) \, dx \, d\tau \\
- \frac{1}{\epsilon} \int_0^t \int_{\mathbb{T}^2} (n - 1 - \epsilon \sigma) \partial_t H'(1 + \epsilon \sigma) \, dx \, d\tau \\
- \frac{1}{\epsilon} \int_0^t \int_{\mathbb{T}^2} n u \cdot \nabla H'(1 + \epsilon \sigma) \, dx \, d\tau \\
- \frac{1}{\epsilon^2} \int_0^t \int_{\mathbb{T}^2} p(n) \Delta \Psi \, dx \, d\tau + \epsilon^2 \int_0^t \int_{\mathbb{T}^2} (n - 1) \Delta \Psi \, dx \, d\tau \\
+ 4\epsilon^2 \int_0^t \int_{\mathbb{T}^2} \left( \nabla \sqrt{n} \otimes \nabla \sqrt{n} \right) : \nabla (v + \nabla \Psi) \, dx \, d\tau \\
- 4\epsilon^2 \int_{\mathbb{T}^2} \nabla \sqrt{n} \cdot \nabla \sqrt{1 + \epsilon \sigma} \, dx + 2\epsilon^2 \int_{\mathbb{T}^2} |\nabla \sqrt{1 + \epsilon \sigma}|^2 \, dx \\
+ 4\epsilon^2 \int_{\mathbb{T}^2} \nabla \sqrt{n_0} \cdot \nabla \sqrt{1 + \epsilon \sigma_0} \, dx - 2\epsilon^2 \int_{\mathbb{T}^2} |\nabla \sqrt{1 + \epsilon \sigma_0}|^2 \, dx \\
- 2\epsilon \int_0^t \int_{\mathbb{T}^2} n D(u - v - \nabla \Psi) : D(v + \nabla \Psi) \, dx \, d\tau \\
= \mathcal{E}(0) + I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7 = I_8 + I_9 + I_{10} + I_{11}. \tag{3.13}
\end{align*}

Now, we begin to treat \(\mathcal{E}(0)\) and the integrals \(I_k (k = 1, 2, \cdots, 11)\) term by term.

For the \(\mathcal{E}(0)\), we have

\begin{align*}
\mathcal{E}(0) = & \frac{1}{2} \int_{\mathbb{T}^2} n_{0, \epsilon} |u_{0, \epsilon} - u_0|^2 \, dx + 2\epsilon^2 \int_{\mathbb{T}^2} |\nabla \sqrt{n_{0, \epsilon}} - \nabla \sqrt{1 + \epsilon \sigma_0}|^2 \, dx \\
+ & \frac{1}{\epsilon^2} \int_{\mathbb{T}^2} [H(1 + n_{0, \epsilon}) - \epsilon H'(1 + \epsilon n_{0, \epsilon}) (n_{0, \epsilon} - n_0) - H(1 + \epsilon n_0)] \, dx, \tag{3.14}
\end{align*}

where \(u_0 = v_0 + \nabla \Psi_0 = H[u_0] + \nabla \Psi_0\).

From (2.11)–(2.12), the first term on the right hand side of (3.14) can be estimated as

\begin{align*}
\frac{1}{2} \int_{\mathbb{T}^2} n_{0, \epsilon} |u_{0, \epsilon} - u_0|^2 \, dx \\
\leq & \frac{1}{2} \int_{\mathbb{T}^2} |1 + \epsilon n_{0, \epsilon}||u_{0, \epsilon} - u_0|^2 \, dx \\
\leq & \frac{1}{2} \int_{\mathbb{T}^2} |u_{0, \epsilon} - u_0|^2 \, dx + \frac{\epsilon}{2} \int_{\mathbb{T}^2} |n_{0, \epsilon}||u_{0, \epsilon} - u_0|^2 \, dx \\
\leq & \frac{1}{2} \int_{\mathbb{T}^2} |u_{0, \epsilon} - u_0|^2 \, dx + \frac{\epsilon}{2} \|n_{0, \epsilon}\|_{L^\infty(\mathbb{T}^2)} \int_{\mathbb{T}^2} |u_{0, \epsilon} - u_0|^2 \, dx \\
\leq & C(1 + \epsilon)\|u_{0, \epsilon} - u_0\|_{L^2(\mathbb{T}^2)}^2. \tag{3.15}
\end{align*}

Using (2.12) and the Cauchy-Schwarz’s inequality, the second term on the hand of (3.14) can be estimated by

\begin{align*}
2\epsilon^2 \int_{\mathbb{T}^2} |\nabla \sqrt{n_{0, \epsilon}} - \nabla \sqrt{1 + \epsilon \sigma_0}|^2 \, dx.
\end{align*}
\[\frac{1}{2} \epsilon^4 \int_{T^2} \left| \nabla n_{0,e}^{1} \frac{1}{\sqrt{1 + \epsilon n_{0,e}^{1}}} - \frac{\nabla n_{0}^{1}}{\sqrt{1 + \epsilon n_{0}^{1}}} \right|^2 dx\]

\[\leq \epsilon^4 \int_{T^2} \left| \nabla \left(n_{0,e}^{1} - n_{0}^{1}\right) \right|^2 dx + \epsilon^4 \int_{T^2} \left| \nabla n_{0}^{1} \left( \frac{1}{\sqrt{1 + \epsilon n_{0,e}^{1}}} - \frac{1}{\sqrt{1 + \epsilon n_{0}^{1}}} \right) \right|^2 dx\]

\[\leq C \epsilon^4 \left\| n_{0,e}^{1} - n_{0}^{1} \right\|^2_{H^1(T^2)} + C \epsilon^4. \quad (3.16)\]

For the third term on the right hand side of (3.13), using the Taylor formula one gets

\[\frac{1}{\epsilon^2} \int_{T^2} \left[ H(1 + n_{0,e}^{1}) - \epsilon H'(1 + n_{0}^{1})(n_{0,e}^{1} - n_{0}^{1}) - H(1 + n_{0}^{1}) \right] dx\]

\[\leq \frac{C}{\epsilon^2} \int_{T^2} \epsilon (n_{0,e}^{1} - n_{0}^{1})^2 dx\]

\[\leq C \epsilon^4 \left\| n_{0,e}^{1} - n_{0}^{1} \right\|^2_{L^2(T^2)}. \quad (3.17)\]

So, we can get

\[\mathcal{E}(0) \leq C \left( \|u_{0,e} - u_0\|^2_{L^2(T^2)} + \|n_{0,e}^{1} - n_{0}^{1}\|^2_{H^1(T^2)} + C \epsilon^4 \right). \quad (3.18)\]

For \( I_1 \), we have

\[I_1 = - \int_0^t \int_{T^2} n(u - v - \nabla \Psi) \cdot \nabla (v + \nabla \Psi)(u - v - \nabla \Psi) dx d\tau\]

\[+ \int_0^t \int_{T^2} n [\partial_t (v + \nabla \Psi) + (v + \nabla \Psi) \cdot \nabla (v + \nabla \Psi)] \cdot (u - v - \nabla \Psi) dx d\tau\]

\[= - \int_0^t \int_{T^2} n(u - v - \nabla \Psi) \cdot \nabla (v + \nabla \Psi) \cdot (u - v - \nabla \Psi) dx d\tau\]

\[- \int_0^t \int_{T^2} n u \partial_t v + v \cdot \nabla v dx d\tau + \int_0^t \int_{T^2} n v \partial_t \nabla \Psi dx d\tau + \frac{1}{2} \int_0^t \int_{T^2} n \nabla \partial_t |\nabla \Psi|^2 dx d\tau\]

\[- \int_0^t \int_{T^2} n(u - v - \nabla \Psi) \otimes \nabla \Psi : \nabla v dx d\tau\]

\[- \int_0^t \int_{T^2} n(u - v - \nabla \Psi) \otimes v : \nabla^2 \Psi dx d\tau\]

\[- \frac{1}{2} \int_0^t \int_{T^2} n(u - v - \nabla \Psi) \cdot \nabla |\nabla \Psi|^2 dx d\tau\]

\[= I_{1,1} + I_{1,2} + I_{1,3} + I_{1,4} + I_{1,5} + I_{1,6} + I_{1,7} + I_{1,8} + I_{1,9}. \quad (3.19)\]

Next, we begin to estimate \( I_{1,i} (i = 1, 2, \ldots, 9) \). Using the Sobolev imbedding theorem, the Minkowski inequality, (2.1) and the estimate (2.10), one gets

\[I_{1,1} \leq \int_0^t \mathcal{E}(\tau) \|\nabla (v + \nabla \Psi)\|_{L^\infty(T^2)} d\tau\]

\[\leq \int_0^t \mathcal{E}(\tau) \|\nabla v\|_{L^\infty(T^2)} d\tau + \int_0^t \mathcal{E}(\tau) \|\nabla^2 \Psi\|_{L^\infty(T^2)} d\tau\]
\[ \leq C \int_0^t \mathcal{E}(\tau) d\tau \]  
(3.20)

and

\[ I_{1,2} = \int_0^t \int_{T^2} \partial_\tau (n - 1) \Pi dx d\tau \]
\[ = \int_0^t \int_{T^2} (n - 1) \Pi dx - \int_0^t \int_{T^2} (n - 1) \partial_\tau \Pi dx d\tau \]
\[ \leq \|(n - 1) 1_{\{ |n - 1| < 1 \}} \|_{L^2(T^2)} \|\Pi\|_{L^2(T^2)} \]
\[ + C \|\partial_\tau\Pi\|_{L^\infty([0,T];L^2(T^2))} + C \epsilon \]
\[ = C \epsilon \min(1, \frac{1}{\epsilon}). \]  
(3.21)

Using (2.10) and the interpolation inequality, we have

\[ \|\nabla \psi \cdot \nabla \Pi\|_{L^{\frac{2}{1}}(T^2)} \leq \|\nabla \psi \cdot \nabla \Pi\|_{L^2(T^2)} \]
\[ \leq C \left( 1 + \frac{t}{\epsilon} \right) \frac{1}{\epsilon}. \]  
(3.22)

So, by using (3.3) - (3.6), the acoustic equations (2.5) - (2.6) and \( \text{div} = 0 \), we have

\[ I_{1,3} = - \int_0^t \int_{T^2} n (v + \nabla \psi) \cdot \nabla \Pi dx d\tau \]
\[ = - \int_0^t \int_{T^2} (n - 1) (v + \nabla \psi) \cdot \nabla \Pi dx d\tau - \int_0^t \int_{T^2} (v + \nabla \psi) \cdot \nabla \Pi dx d\tau \]
\[ \leq C \|\nabla \psi\|_{L^2(T^2)} \|\nabla \Pi\|_{L^2(T^2)} \]
\[ + C \|\partial_\tau\Pi\|_{L^\infty([0,T];L^2(T^2))} \]
\[ \leq C \epsilon - \epsilon \int_0^t \int_{T^2} \partial_\tau s \cdot \Pi dx d\tau \]
\[ + C \|\nabla \psi\|_{L^2(T^2)} \|\nabla \Pi\|_{L^{\frac{2}{1}}(T^2)} \]
\[ = C \epsilon - \epsilon \int_{T^2} s \cdot \Pi dx - \int_0^t n_0^1 \cdot \Pi_0 dx + \epsilon \int_0^t \int_{T^2} s \cdot \partial_\tau \Pi dx d\tau \]
\[ \leq C \epsilon + C \epsilon \]  
(3.23)
\[ \leq C\epsilon + C\epsilon^{\frac{2}{5}} \cdot \frac{\gamma \epsilon}{\gamma - 1} \left[ \left( 1 + \frac{T}{\epsilon} \right)^{1 - \frac{1}{5}} - 1 \right] \]
\[ \leq C\epsilon^{\min\{1, \frac{2}{5}\}}, \quad (3.23) \]

where, we used
\[ \lim_{\epsilon \to 0} \epsilon \left[ \left( 1 + \frac{T}{\epsilon} \right)^{1 - \frac{1}{5}} - 1 \right] = T^{1 - \frac{1}{5}}. \quad (3.24) \]

The term \( I_{1,4} \) will be canceled by its counterpart in \( I_3 \). Using the acoustic equations (2.4)–(2.6), with the help of (2.11), (2.13), (3.5), and \( \text{div} v = 0 \), we have
\[ I_{1,5} = \int_0^t \int_{T^2} (n - 1)v \partial_{\tau} \nabla \Psi dx d\tau + \int_0^t \int_{T^2} v \partial_{\tau} \nabla \Psi dx d\tau \]
\[ = - \frac{p'(1)}{\epsilon} \int_0^t \int_{T^2} (n - 1)v \cdot \nabla \sigma dx d\tau \]
\[ \leq \frac{p'(1)}{\epsilon} \int_0^t \left[ \|(n - 1)1_{\{n-1<1\}}\|_{L^2(T^2)} \|v\|_{L^2(T^2)} \|
abla \sigma\|_{L^\infty(T^2)} d\tau \]
\[ + \frac{p'(1)}{\epsilon} \int_0^t \left[ \|(n - 1)1_{\{n-1\geq 1\}}\|_{L^\gamma(T^2)} \|v\|_{L^\gamma(T^2)} \|
abla \sigma\|_{L^\infty(T^2)} d\tau \right] \]
\[ \leq C\epsilon^{\min\{1, \frac{2}{5}\}} [\ln(\epsilon + T) - \ln \epsilon] \]
\[ \leq C\epsilon^{\min\{1 - \frac{4}{5}, \frac{2}{5}\}}, \quad (3.25) \]

where we used
\[ \lim_{\epsilon \to 0} \frac{\ln(\epsilon + T) - \ln \epsilon}{\epsilon^{-\frac{4}{5}}} = 0. \quad (3.26) \]

From the acoustic equations (2.5)–(2.6), we have
\[ I_{1,6} = \frac{1}{2} \int_0^t \int_{T^2} (n - 1)\partial_{\tau} \nabla \Psi^2 dx d\tau + \frac{1}{2} \int_0^t \int_{T^2} \partial_{\tau} |\nabla \Psi|^2 dx d\tau \quad (3.27) \]
\[ = \frac{p'(1)}{\epsilon} \int_0^t \int_{T^2} (n - 1)\nabla \Psi \cdot \nabla \sigma dx d\tau + \frac{1}{2} \int_{T^2} |\nabla \Psi|^2 dx - \int_{T^2} |\nabla \Psi_0|^2 dx \]
\[ \leq \frac{p'(1)}{\epsilon} \int_0^t \left[ \|(n - 1)1_{\{n-1<1\}}\|_{L^2(T^2)} \|
abla \Psi\|_{L^2(T^2)} \|
abla \sigma\|_{L^\infty(T^2)} d\tau \]
\[ + \frac{p'(1)}{\epsilon} \int_0^t \left[ \|(n - 1)1_{\{n-1\geq 1\}}\|_{L^\gamma(T^2)} \|
abla \Psi\|_{L^\gamma(T^2)} \|
abla \sigma\|_{L^\infty(T^2)} d\tau \right] \]
\[ + \frac{1}{2} \left[ \int_{T^2} |\nabla \Psi|^2 dx - \int_{T^2} |\nabla \Psi_0|^2 dx \right] \]
\[ \leq C \int_0^t \left( 1 + \frac{T}{\epsilon} \right)^{-1} d\tau + C\epsilon^{\frac{2}{5}} - 1 \int_0^t \left( 1 + \frac{T}{\epsilon} \right)^{-1 - \frac{1}{5}} d\tau \]
\[ + \frac{1}{2} \left[ \int_{T^2} |\nabla \Psi_0|^2 dx - \int_{T^2} |\nabla \Psi|^2 dx \right] \]
\[ \leq C \epsilon [\ln(\epsilon + T) - \ln \epsilon] + C\epsilon^{\frac{2}{5}} \left[ 1 - \left( \frac{\epsilon}{\epsilon + T} \right)^{\frac{1}{5}} \right] \]
where we have used the following interpolation inequality for $\nabla \Psi$

$$\|\nabla \Psi\|_{L^\infty(T^2)} \leq C \|\nabla \Psi\|_{L^1(T^2)} \|\nabla \Psi\|_{L^2(T^2)}^{\frac{1}{2}} \|\nabla \Psi\|_{L^\infty(T^2)}^{\frac{1}{2}} \|\nabla \Psi\|_{L^\infty(T^2)}$$

$$\leq C \|\nabla \Psi\|_{L^1(T^2)} \|\nabla \Psi\|_{L^\infty(T^2)} \|\nabla \Psi\|_{L^\infty(T^2)}$$

$$\leq C \left(1 + \frac{t}{\epsilon}\right)^{-\frac{1}{4}}.$$  (3.29)

For $I_{1,7}$, noting that

$$\|{nu}\|_{L^{\frac{2}{\gamma - 1}}(T^2)} \leq \|\sqrt{n}\|_{L^2(T^2)} \|\sqrt{n}\|_{L^2(T^2)} \leq C,$$  (3.30)

we have

$$I_{1,7} = -\int_0^t \int_{T^2} n \nabla \Psi : \nabla v dx d\tau + \int_0^t \int_{T^2} v \nabla \Psi : \nabla v dx d\tau$$

$$+ \int_0^t \int_{T^2} n \nabla \Psi \otimes \nabla \Psi : \nabla v dx d\tau$$

$$\leq C \int_0^t \|nu\|_{L^{\frac{2}{\gamma - 1}}(T^2)} \|\nabla \Psi\|_{L^2(T^2)} \|\nabla v\|_{L^\infty(T^2)} d\tau$$

$$+ C \int_0^t \|n\|_{L^1(T^2)} \|v\|_{L^\infty(T^2)} \|\nabla \Psi\|_{L^\infty(T^2)} \|\nabla v\|_{L^\infty(T^2)} d\tau$$

$$+ C \int_0^t \|n\|_{L^1(T^2)} \|\nabla \Psi\|_{L^\infty(T^2)}^2 \|\nabla v\|_{L^\infty(T^2)} d\tau$$

$$\leq C \int_0^t \left(1 + \frac{T}{\epsilon}\right)^{-\frac{1}{4}} d\tau + C \int_0^t \left(1 + \frac{T}{\epsilon}\right)^{-\frac{1}{4}} d\tau + C \int_0^t \left(1 + \frac{T}{\epsilon}\right)^{-\frac{3}{4}} d\tau$$

$$\leq C \left[\frac{\gamma \epsilon}{\gamma - 1} \left(1 + \frac{T}{\epsilon}\right)^{-\frac{1}{4}} - 1\right] + C \epsilon \ln(\epsilon + T) - \ln \epsilon + \frac{\epsilon T}{\epsilon + T}$$

$$\leq C \epsilon^\frac{1}{4},$$  (3.31)

where, we used (3.28) and (3.29). Similarly to the estimate of $I_{1,7}$, we have

$$I_{1,8} = -\int_0^t \int_{T^2} n \nabla \Psi \otimes v : \nabla^2 \Psi dx d\tau + \int_0^t \int_{T^2} n \nabla \Psi \otimes v : \nabla^2 \Psi dx d\tau$$

$$+ \int_0^t \int_{T^2} n \nabla \Psi \otimes v : \nabla^2 \Psi dx d\tau$$

$$\leq C \left[\frac{\gamma \epsilon}{\gamma - 1} \left(1 + \frac{T}{\epsilon}\right)^{-\frac{1}{4}} - 1\right] + C \epsilon \ln(\epsilon + T) - \ln \epsilon + \frac{\epsilon T}{\epsilon + T}$$

$$\leq C \epsilon^\frac{1}{4}$$  (3.32)
and
\[
I_{1,9} = -\frac{1}{2} \int_0^t \int_{T^2} nu \cdot \nabla |\nabla \Psi|^2 dx d\tau + \frac{1}{2} \int_0^t \int_{T^2} n v \cdot \nabla |\nabla \Psi|^2 dx d\tau \\
+ \frac{1}{2} \int_0^t \int_{T^2} n \nabla \Psi \cdot \nabla |\nabla \Psi|^2 dx d\tau \\
\leq C \left[ \frac{\gamma}{\gamma - 1} \left( 1 + \frac{T}{\epsilon} \right)^{1 - \frac{s}{2}} - 1 \right] + \frac{\epsilon T}{\epsilon + T} + \frac{\epsilon}{2}
\]
\[
\leq C \epsilon^{\frac{s}{2}}.
\]
(3.33)

Therefore, we obtain that
\[
I_1 \leq C \int_0^t \mathcal{E}(\tau) d\tau + C \epsilon^{\min(1-\frac{s}{2}, \frac{s}{2})}
\]
\[
+ \frac{1}{2} \left[ \int_0^t |\nabla \Psi|^2 dx - \int_{T^2} |\nabla \Psi_0|^2 dx \right] - \int_0^t \int_{T^2} nu \partial_t \nabla \Psi dx d\tau.
\]
(3.34)

For \( I_2 \), using \( H''(1) = p'(1) \) and observing that
\[
H''(1 + \epsilon \sigma) - H''(1) = \epsilon H''(\xi) \sigma, \quad \xi \in (1, 1 + \epsilon \sigma) \text{ or } (1 + \epsilon \sigma, 1),
\]
(3.35)
once gets
\[
I_2 = -\frac{1}{\epsilon} \int_0^t \int_{T^2} (n - 1 - \epsilon \sigma) H''(1 + \epsilon \sigma) \partial_z \sigma dx d\tau
\]
\[
= -\frac{1}{\epsilon^2} \int_0^t \int_{T^2} (n - 1)(H''(1 + \epsilon \sigma) - H''(1)) \Delta \Psi dx d\tau
\]
\[
+ \frac{1}{\epsilon^2} \int_0^t \int_{T^2} (n - 1)H''(1) \Delta \Psi dx d\tau
\]
\[
+ \int_0^t \int_{T^2} \sigma[H''(1 + \epsilon \sigma) - H''(1)] \partial_z \sigma dx d\tau + \int_0^t \int_{T^2} \sigma H''(1) \partial_z \sigma dx d\tau
\]
\[
= -\frac{1}{\epsilon} \int_0^t \int_{T^2} (n - 1)H''(\xi) \sigma \Delta \Psi dx d\tau + \frac{p'(1)}{\epsilon^2} \int_0^t \int_{T^2} (n - 1) \Delta \Psi dx d\tau
\]
\[
- \int_0^t \int_{T^2} H''(\xi) \sigma^2 \Delta \Psi dx d\tau + p'(1) \int_0^t \int_{T^2} \sigma \partial_z \sigma dx d\tau
\]
\[
\leq \frac{1}{\epsilon} \int_0^t \left[ \|(n - 1) \mathbf{1}_{\{n-1<1\}} \|_{L^2(T^2)} \|H''(\xi)\|_{L^2(T^2)} \|\sigma\|_{L^2(T^2)} \|\Delta \Psi\|_{L^2(T^2)} \right] d\tau
\]
\[
+ \frac{1}{\epsilon} \int_0^t \left[ \|(n - 1) \mathbf{1}_{\{n-1\geq1\}} \|_{L^2(T^2)} \|H''(\xi)\|_{L^2(T^2)} \|\sigma\|_{L^2(T^2)} \|\Delta \Psi\|_{L^2(T^2)} \right] d\tau
\]
\[
+ C \int_0^t \|\sigma\|_{L^2(T^2)}^2 \|\Delta \Psi\|_{L^2(T^2)} d\tau + \frac{p'(1)}{\epsilon^2} \int_0^t \int_{T^2} (n - 1) \Delta \Psi dx d\tau
\]
\[
+ \frac{1}{2} \left[ \int_{T^2} p'(1) |\sigma|^2 dx - \int_{T^2} p'(1) |\sigma_0|^2 dx \right]
\]
\[
\leq C \int_0^t \left( 1 + \frac{T}{\epsilon} \right)^{-2} d\tau + C \epsilon^{\frac{s}{2} - 1} \int_0^t \left( 1 + \frac{T}{\epsilon} \right)^{-2} d\tau + C \int_0^t \left( 1 + \frac{T}{\epsilon} \right)^{-3} d\tau
\]
\[
+ \frac{p'(1)}{\epsilon^2} \int_0^t \int_{T^2} (n - 1) \Delta \Psi dx d\tau + \frac{1}{2} \left[ \int_{T^2} p'(1) |\sigma|^2 dx - \int_{T^2} p'(1) |\sigma_0|^2 dx \right]
\[
\leq C \left( \frac{cT}{\varepsilon + T} + \frac{2T^2}{\varepsilon + T} + \frac{2T^2 + T^2\varepsilon}{2(\varepsilon + T)^2} \right) + \frac{p'(1)}{\varepsilon^2} \int_0^t \int_{\mathbb{T}^2} (n - 1)\nabla \Psi dxd\tau \\
+ \frac{1}{2} \left[ \int_{\mathbb{T}^2} p'(1)|\sigma|^2 dxd - \int_{\mathbb{T}^2} p'(1)|\sigma_0|^2 dxd \right]
\leq C\varepsilon^{\text{min}(1, \hat{\phi})} + \frac{p'(1)}{\varepsilon^2} \int_0^t \int_{\mathbb{T}^2} (n - 1)\nabla \Psi dxd\tau \\
+ \frac{1}{2} \left[ \int_{\mathbb{T}^2} p'(1)|\sigma|^2 dxd - \int_{\mathbb{T}^2} p'(1)|\sigma_0|^2 dxd \right].
\] (3.36)

Using (3.25), (3.35) and the acoustic equations (2.25) - (2.26), we have

\[
I_3 = -\frac{1}{\varepsilon} \int_0^t \int_{\mathbb{T}^2} nu \cdot H''(1 + \varepsilon \sigma)\nabla \sigma dxd\tau \\
= -\frac{1}{\varepsilon} \int_0^t \int_{\mathbb{T}^2} nu \cdot (H''(1 + \varepsilon \sigma) - H''(1))\nabla \sigma dxd\tau \\
- \frac{1}{\varepsilon} \int_0^t \int_{\mathbb{T}^2} nu \cdot H''(1)\nabla \sigma dxd\tau \\
= -\int_0^t \int_{\mathbb{T}^2} nu \cdot H''(\xi)\nabla \sigma dxd\tau + \int_0^t \int_{\mathbb{T}^2} nu \partial_\tau \nabla \Psi dxd\tau \\
\leq C \int_0^t \|nu\|_{L^2(\mathbb{T}^2)} \|\nabla \sigma\|_{L^2(\mathbb{T}^2)} d\tau + \int_0^t \int_{\mathbb{T}^2} nu \partial_\tau \nabla \Psi dxd\tau \\
\leq C \int_0^t \left( 1 + \frac{T}{\varepsilon} \right)^{-\frac{\hat{\phi}}{\varepsilon}} d\tau + \int_0^t \int_{\mathbb{T}^2} nu \partial_\tau \nabla \Psi dxd\tau \\
\leq C \left[ \frac{\gamma \varepsilon}{\gamma - 1} \left( \left( 1 + \frac{T}{\varepsilon} \right)^{-1 - \frac{\hat{\phi}}{\varepsilon}} - 1 \right) \right] + \int_0^t \int_{\mathbb{T}^2} nu \partial_\tau \nabla \Psi dxd\tau \\
\leq C\varepsilon^{\hat{\phi}} + \int_0^t \int_{\mathbb{T}^2} nu \partial_\tau \nabla \Psi dxd\tau.
\] (3.37)

In view of (1.8), we deduce that

\[
I_4 = -\frac{1}{\varepsilon^2} \int_0^t \int_{\mathbb{T}^2} \left[ p(n) - p'(1)(n - 1) - p(1) \right] \nabla \Psi dxd\tau \\
- \frac{p'(1)}{\varepsilon^2} \int_0^t \int_{\mathbb{T}^2} (n - 1)\nabla \Psi dxd\tau - \frac{1}{\varepsilon^2} \int_0^t \int_{\mathbb{T}^2} p(1)\nabla \Psi dxd\tau \\
= -\frac{\gamma - 1}{\varepsilon^2} \int_0^t \int_{\mathbb{T}^2} H(n)\nabla \Psi dxd\tau - \frac{p'(1)}{\varepsilon^2} \int_0^t \int_{\mathbb{T}^2} (n - 1)\nabla \Psi dxd\tau \\
= -\frac{\gamma - 1}{\varepsilon^2} \int_0^t \int_{\mathbb{T}^2} |H(n) - H'(1 + \varepsilon \sigma)(n - 1 - \varepsilon \sigma) - H(1 + \varepsilon \sigma)|\nabla \Psi dxd\tau \\
+ \frac{\gamma - 1}{\varepsilon^2} \int_0^t \int_{\mathbb{T}^2} |H'(1 + \varepsilon \sigma)(n - 1)|\nabla \Psi dxd\tau \\
+ \frac{\gamma - 1}{\varepsilon^2} \int_0^t \int_{\mathbb{T}^2} H'(1 + \varepsilon \sigma)\sigma \nabla \Psi dxd\tau - \frac{\gamma - 1}{\varepsilon^2} \int_0^t \int_{\mathbb{T}^2} H(1 + \varepsilon \sigma)\nabla \Psi dxd\tau
\]
\[-\frac{p'(1)}{e^2} \int_0^t \int_{\mathbb{T}^2} (n-1) \Delta \Psi dx d\tau\]

\[= I_{4,1} + I_{4,2} + I_{4,3} + I_{4,4} - \frac{p'(1)}{e^2} \int_0^t \int_{\mathbb{T}^2} (n-1) \Delta \Psi dx d\tau. \]  

(3.38)

For \(I_{4,1}\), we have

\[I_{4,1} \leq C \int_0^t E(\tau) d\tau. \]  

(3.39)

Noting that

\[(1 + \epsilon \sigma)^{\gamma - 1} = (\gamma - 1) \epsilon \zeta^{\gamma - 2} \sigma, \quad \zeta \in (1, 1 + \epsilon \sigma) \text{ or } (1 + \epsilon \sigma, 1), \]  

(3.40)

we have

\[I_{4,2} = -\frac{\gamma}{e^2} \int_0^t \int_{\mathbb{T}^2} \left[(1 + \epsilon \sigma)^{\gamma - 1} - 1\right](n-1) \Delta \Psi dx d\tau \]

\[= -\frac{\gamma}{e} \int_0^t \int_{\mathbb{T}^2} \zeta^{\gamma - 2}(n-1) \sigma \Delta \Psi dx d\tau \]

\[\leq \frac{C}{e} \int_0^t \left\|(n-1)I_{\{|n-1|<1\}} \|_{L^2(\mathbb{T}^2)} \|\sigma\|_{L^2(\mathbb{T}^2)} \|\Delta \Psi\|_{L^\infty(\mathbb{T}^2)} d\tau \]

\[+ \frac{C}{e} \int_0^t \left\|(n-1)I_{\{|n-1|\geq 1\}} \|_{L^2(\mathbb{T}^2)} \|\sigma\|_{L^{\frac{\gamma}{\gamma-1}}(\mathbb{T}^2)} \|\Delta \Psi\|_{L^\infty(\mathbb{T}^2)} d\tau \]

\[\leq C\epsilon^{\min\{1 - \frac{\gamma}{2}, \frac{\gamma}{2}\}}. \]  

(3.41)

Using (2.10), (3.26) and (3.40), we get

\[I_{4,3} = -\frac{\gamma}{e} \int_0^t \int_{\mathbb{T}^2} \left[(1 + \epsilon \sigma)^{\gamma - 1} - 1\right] \sigma \Delta \Psi dx d\tau \]

\[= -\frac{\gamma}{e} \int_0^t \int_{\mathbb{T}^2} \zeta^{\gamma - 2} \sigma^2 \Delta \Psi dx d\tau \]

\[\leq C \int_0^t \left\|\sigma\|^2_{L^2(\mathbb{T}^2)} \|\Delta \Psi\|_{L^\infty(\mathbb{T}^2)} d\tau \]

\[\leq C\epsilon^{1 - \frac{\gamma}{2}}. \]  

(3.42)

Observing that

\[(1 + \epsilon \sigma)^{\gamma} = 1 + \gamma \epsilon \sigma + \frac{\gamma(\gamma - 1)}{2} \omega^{\gamma - 2} \epsilon^2 \sigma^2; \quad \omega \in (1, 1 + \epsilon \sigma) \text{ or } (1 + \epsilon \sigma, 1), \]  

(3.43)

we have

\[I_{4,4} = -\frac{1}{e^2} \int_0^t \int_{\mathbb{T}^2} \left[(1 + \epsilon \sigma)^{\gamma} - 1 - \gamma \epsilon \sigma\right] \Delta \Psi dx d\tau \]

\[= -\frac{\gamma}{2} \int_0^t \int_{\mathbb{T}^2} \omega^{\gamma - 2} \sigma^2 \Delta \Psi dx d\tau \]

\[\leq C \int_0^t \left\|\sigma\|^2_{L^2(\mathbb{T}^2)} \|\Delta \Psi\|_{L^\infty(\mathbb{T}^2)} d\tau \]

\[\leq C\epsilon^{1 - \frac{\gamma}{2}}. \]  

(3.44)
Then, we have
\[ I_4 \leq C \int_0^t E(\tau) d\tau + C\epsilon_{\min(1-\frac{k}{2})} \cdot \frac{p'(1)}{\epsilon^2} \int_0^t \int_{T^2} (n-1) \Delta \Psi \, dx \, d\tau. \] (3.45)

For \( I_5 \), we get
\[
I_5 \leq C\epsilon^2 \int_0^t \left\| (n-1) \mathbf{1}_{(n-1)<1} \right\|_{L^2(T^2)} \left\| \Delta \Psi \right\|_{L^2(T^2)} d\tau \\
+ C\epsilon^2 \int_0^t \left\| (n-1) \mathbf{1}_{(n-1) \geq 1} \right\|_{L^{\gamma(T^2)}} \left\| \Delta \Psi \right\|_{L^{\frac{2}{\gamma-1}(T^2)}} d\tau \\
\leq C\epsilon^3 + C\epsilon^{2+\frac{2}{\gamma}} \int_0^t \left( 1 + \frac{t}{\epsilon} \right)^{-\frac{1}{\gamma}} d\tau \\
\leq C\epsilon^3 + C\epsilon^{2+\frac{2}{\gamma}},
\] (3.46)

where we used (3.24) and the following interpolation inequality for \( \Delta \Psi \)
\[
\left\| \Delta \Psi \right\|_{L^{\frac{2}{\gamma-1}(T^2)}} \leq \left\| \frac{\Delta \Psi}{\left\| \Psi \right\|_{L^1(T^2)}} \right\|_{L^{\gamma(T^2)}} \left\| \Delta \Psi \right\|_{L^{\frac{2}{\gamma-1}(T^2)}} \\
\leq C \left\| \frac{\Delta \Psi}{\left\| \Psi \right\|_{L^1(T^2)}} \right\|_{L^\infty(T^2)} \\
\leq C \left( 1 + \frac{t}{\epsilon} \right)^{-\frac{1}{\gamma}}.
\] (3.47)

For \( I_6 \), we have
\[ I_6 \leq C \int_0^t E(\tau) d\tau. \] (3.48)

Using (1.7), (2.10) and the Young inequality, we have
\[ I_7 = -2\epsilon^3 \int_{T^2} \frac{\nabla \sqrt{n} \cdot \nabla \sigma}{\sqrt{1 + \epsilon \sigma}} \, dx \leq C\epsilon^3 \left( \int_{T^2} |\nabla \sqrt{n}|^2 \, dx + \int_{T^2} |\nabla \sigma|^2 \, dx \right) \leq C\epsilon. \] (3.49)

For \( I_8 \), we get
\[ I_8 = 2\epsilon^4 \int_{T^2} \left| \frac{\nabla \sigma}{\sqrt{1 + \epsilon \sigma}} \right|^2 \, dx \leq C\epsilon^4. \] (3.50)

Using (2.10) and the Young inequality, we have
\[ I_9 = \epsilon^4 \int_{T^2} \frac{\nabla n_{0,\epsilon} \cdot \nabla n_0}{\sqrt{(1 + \epsilon n_{0,\epsilon})(1 + \epsilon n_0)}} \, dx \leq C\epsilon^4 \left( \int_{T^2} |\nabla n_{0,\epsilon}|^2 \, dx + \int_{T^2} |\nabla n_0|^2 \, dx \right) \leq C\epsilon^4. \] (3.51)

For \( I_{10} \), we have
\[ I_{10} = -2\epsilon^4 \int_{T^2} \left| \frac{\nabla n_{0,\epsilon}}{\sqrt{1 + \epsilon n_{0,\epsilon}}} \right|^2 \, dx \leq C\epsilon^4. \] (3.52)

For the last term \( I_{11} \), using the Young inequality with \( \epsilon \), one gets
\[ I_{11} = -2\epsilon \int_0^t \int_{T^2} nD(u - v - \nabla \Psi) : D(v + \nabla \Psi) \, dx \, d\tau \]
\[ C \epsilon \int_0^\tau \int_{T^2} |\sqrt{n}xD\Psi| dx \text{d}\tau + \epsilon \int_0^\tau \int_{T^2} |\sqrt{n}D(u-v-\nabla \psi)|^2 dx \text{d}\tau \]
\[ \leq C \epsilon + \epsilon \int_0^\tau \int_{T^2} n|D(u-v-\nabla \psi)|^2 dx \text{d}\tau. \]  

Using (3.31) and adding the estimates for \( E(0), I_1, I_2, \ldots, I_{11}, \) some integrals cancel, and we end up with

\[ E(t) \leq C \left( \|u_{0,c} - u_0\|^2_{L^2(T^2)} + \|n_{0,c} - n_0^1\|^2_{H^1(T^2)} + C \epsilon^{\min\{1-\frac{1}{4}, \frac{1}{2}\}} \right) + C \int_0^\tau E(\tau) d\tau, \]  

where we have used the following fact

\[ \epsilon^a > \epsilon^b, \quad 0 < a < b \]

for any \( \epsilon \in (0, 1). \) The integral form of the Gronwall inequality gives

\[ E(t) \leq C \left( \|u_{0,c} - u_0\|^2_{L^2(T^2)} + \|n_{0,c} - n_0^1\|^2_{H^1(T^2)} + \epsilon^{\min\{1-\frac{1}{4}, \frac{1}{2}\}} \right), \quad t \in [0, T], \]

which implies that we have proved Theorem 2.3. Finally, we conclude by observing that Corollary 2.4 follows from the inequality (2.13), combined with the estimates (3.34). In fact, using the Young inequality and Lemma 3.1 we have

\[ \|P(\sqrt{n}u) - v\|^2_{L^2(T^2)} = \|P(\sqrt{n}u - v - \nabla \psi)\|^2_{L^2(T^2)} \]
\[ \leq \|\sqrt{n}u - v - \nabla \psi\|^2_{L^2(T^2)} = \|\sqrt{n}(u - v - \nabla \psi) + (\sqrt{n} - 1)(v + \nabla \psi)\|^2_{L^2(T^2)} \]
\[ \leq 2\|\sqrt{n}(u - v - \nabla \psi)\|^2_{L^2(T^2)} + 2\|\sqrt{n} - 1)(v + \nabla \psi)\|^2_{L^2(T^2)} \]
\[ \leq CE(t) + 2\|\sqrt{n} - 1)(v + \nabla \psi)\|^2_{L^2(T^2)} \leq CE(t) + C \|[u| - 1|_{|n-1|<1}\|^2_{L^2(T^2)} \]
\[ + C \|[u| - 1|_{|n-1|\geq 1}\|^2_{L^2(T^2)} \]
\[ \leq CE^{\min\{1-\frac{1}{4}\}}, \]

we have used the following elementary inequality

\[ |\sqrt{n} - 1|^2 \leq C|x - 1|^k, \quad k \geq 1 \]

for some positive constant \( C \) and any \( x \geq 0. \)

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