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\textbf{Abstract.} We improve the upper bound on the Ramsey number $R(5, 5)$ from $R(5, 5) \leq 49$ to $R(5, 5) \leq 48$. We also complete the catalogue of extremal graphs for $R(4, 5)$.

1. Introduction

The Ramsey number $R(s, t)$ is defined to be the smallest $n$ such that every graph of order $n$ contains either a clique of $s$ vertices or an independent set of $t$ vertices.

\textbf{Theorem 1.1.} The Ramsey number $R(5, 5)$ is less than or equal to 48.

The history of $R(5, 5)$ is provided in \cite{4}. The lower bound of 43 established constructively by Exoo \cite{1} is still the best. The previous best upper bound of 49 was proved by McKay and Radziszowski \cite{4}. By Theorem 1.1 we now have $43 \leq R(5, 5) \leq 48$.

The actual value of $R(5, 5)$ is widely believed to be 43, because a lot of computer resources have been expended in an unsuccessful attempt to construct a Ramsey(5,5)-graph of order 43 \cite{4}. As additional evidence, we can report that, in unpublished 2014 work, Lieby and the second author proved that any Ramsey(5,5) graph on 42 vertices other than the 656 reported in \cite{4} do not share a 37-vertex subgraph with any of the 656.

The proof of Theorem 1.1 is via computer verification, checking approximately two trillion separate cases. We wrote two independent programs to carry out the calculation, to minimise the chance of any computer bugs affecting our results.

2. Outline of the proof of Theorem 1.1

Let $R(s, t, n)$ denote the set of isomorphism classes of graphs of order $n$ without an $s$-clique or independent $t$-set, and $R(s, t) = \bigcup_{n} R(s, t, n)$. The main idea is that given a graph $F \in R(5, 5, 48)$, a large subgraph of it must be obtained by gluing together two graphs in $R(4, 5, 24)$ along a graph in $R(3, 5, d)$ for some $d$.

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A list of 350,904 graphs in $\mathcal{R}(4, 5, 24)$ was compiled by McKay and Radziszowski \[3\] in 1995, and our first task was to complete their list. This was actually the most time-consuming part of the project.

**Theorem 2.1.** $|\mathcal{R}(4, 5, 24)| = 352,366.$

Figure 1. $G$ and $H$ are given graphs from $\mathcal{R}(4, 5, 24)$ that overlap in a graph $K$. The problem is to choose the edges between $A$ and $B$ so that the whole is in $\mathcal{R}(5, 5)$.

We now explain the main proof idea in more detail. For a graph $F$, $V F$ is the vertex-set of $F$, $N_F(w)$ is the neighbourhood of vertex $w \in VF$ and $F[W]$ is the subgraph of $F$ induced by $W \subseteq VF$. First, note that because $\mathcal{R}(4, 5) = 25 \[3\]$, every vertex in a graph $F \in \mathcal{R}(5, 5, 48)$ must have degree 23 or 24. By replacing $F$ by its complement if necessary we can assume that $F$ has at least 24 vertices of degree 24. Hence $F$ must have two adjacent vertices $a, b$ of degree 24. Define

\[
\begin{align*}
G &= F[N_F(b)], \\
H &= F[N_F(a)], \\
K &= F[V G \cap V H].
\end{align*}
\]

In words, $G$ is the subgraph of $F$ induced by the 24 vertices adjacent to $b$ (this includes $a$ but not $b$), $H$ is the subgraph induced by the vertices adjacent to $a$, and $K$ is the intersection of $G$ and $H$. Please see Figure 1. Note that $G, H \in \mathcal{R}(4, 5, 24)$ and that $K \in \mathcal{R}(3, 5, d)$ for some $d$. Because $\mathcal{R}(3, 5) = 14$ we must have $d \leq 13$, and $d$ is also equal to the degree of $a$ in $G$ and the degree of $b$ in $H$.

To reconstruct $F[V G \cup V H]$, which is a graph with $48 - d$ vertices, from $G$, $H$ and $K$, it suffices to specify how $K$ is a subgraph of $G$ and $H$, and
whether or not we have an edge between \( x \) and \( y \) for \( x \in VG - VK - \{a\} \) and \( y \in VH - VK - \{b\} \); i.e. between parts labelled \( A \) and \( B \) in Figure 1. We call this procedure gluing. For each inclusion of \( K \) into \( G \) and \( H \) there are \( 2^{(23 - d)^2} \) ways of gluing \( G \) and \( H \) along \( K \), but we will only consider gluings that could give a graph in \( R(5, 5, 48 - d) \).

For \( K \in R(3, 5, d) \), define \( R(4, 5, 24, K) = \{(G, a) \mid G \in R(4, 5, 24), a \in VG, G[N_G(a)] \cong K \} \).

We will call \((G, a)\) a pointed graph of type \( K \). Our proof of Theorem 1.1 consists of the following steps.

**Step 1:** We completed the list of graphs in \( R(4, 5, 24) \) compiled by McKay and Radziszowski, thereby proving Theorem 2.1. This was done by a straightforward (but computationally expensive) extension of the method in [3]. While that calculation would have taken too long in 1995, it was doable in 2016.

**Step 2:** For each \( K \in R(3, 5, d) \) with \( d \leq 11 \) and for each pair \((G, a), (H, b) \in R(4, 5, 24, K)\), we used a computer program to calculate all ways of gluing \( G \) and \( H \) along \( K \). Note that this consisted of one gluing problem for each automorphism of \( K \).

**Step 3:** For each graph generated in Step 2, we used another program which attempts in all possible ways to add one vertex while staying within \( R(5, 5) \). Since this was never possible, none of the graphs generated in Step 2 are subgraphs of a graph in \( R(5, 5, 48) \).

**Lemma 2.2.** Execution of Steps 1–3 is sufficient to prove Theorem 1.1.

**Proof.** Suppose \( F \in R(5, 5, 48) \). We first prove that either \( F \) or its complement has a vertex of degree 24 adjacent to at least 12 other vertices of degree 24. Suppose that \( F \) is a counterexample to this claim, and let \( W \subseteq VF \) be its vertices of degree 24. Since \( F[W] \) has maximum degree 11, there are at least \( e_1 = 13 |W| \) edges between \( W \) and \( VF \setminus W \) in \( F \). Similarly, there are at least \( e_2 = 13 (48 - |W|) \) edges between \( W \) and \( VF \setminus W \) in \( F \). However, this is impossible since \( e_1 + e_2 = 13 \times 48 = 624 \) and \( |W| (48 - |W|) \leq 24^2 = 576 \).

So let \( b \) be a vertex of \( F \) of degree 24 that is adjacent to at least 12 other vertices of degree 24 and define \( G = F[N_F(b)] \). From the \( R(4, 5, 24) \) catalogue we find that \( G \) has at most 8 vertices of degree more than 11, so we can choose \( a \in N_F(b) \) that has degree 24 in \( F \) and degree at most 11 in \( G \). Define \( H = F[N_F(a)] \). Then the gluing of \((G, a)\) and \((H, b)\) in Step 2 will find a subgraph of \( F \) and the failure of one point extension in Step 3 will show that \( F \) doesn’t exist. \( \square \)
3. Step 1: Completing the list of graphs in $\mathcal{R}(4, 5, 24)$

McKay and Radziszowski \cite{McKay89} produced a list of 350,904 such graphs, and proved that the list contains all graphs in $\mathcal{R}(4, 5, 24)$ with minimum degree is 6, 7 or 8, or maximum degree 12 or 13, or if the graph is regular of degree 11.

To complete the catalogue it suffices to find those graphs with minimum degree 9 or 10. We did this using the well-tested code from \cite{McKay89} to glue together graphs of type $\mathcal{R}(3, 5, 9)$ and $\mathcal{R}(4, 4, 14)$, and of types $\mathcal{R}(3, 5, 10)$ and $\mathcal{R}(4, 4, 13)$. Although this requires a very large number of graph pairs to be glued, it is feasible when the graphs of type $\mathcal{R}(3, 5, 9)$ and $\mathcal{R}(3, 5, 10)$ are arranged in a tree structure that exhibits common subgraphs and symmetries. See \cite{McKay89} for details. All graphs in $\mathcal{R}(4, 5, 24)$ with a vertex of degree 9 or 10 were found, to increase the overlap with \cite{McKay89} for checking purposes. This took about 1.5 core-years of computer time and discovered 1462 new graphs in $\mathcal{R}(4, 5, 24)$; recall that the search in \cite{McKay89} was not intended to be complete.

Then we devoted another 6 core-months to sanity-checking of the completed catalogue. As an example, let $\mathcal{A}'$ be the set of all neighbourhoods of a vertex of degree 9 or 10 in the 1462 new graphs, and let $\mathcal{B}'$ be the set of all complementary neighbourhoods of the same vertices in those graphs. Then, using a completely separate program, we constructed all graphs in $\mathcal{R}(4, 5, 24)$ with a vertex having a neighbourhood in $\mathcal{A}'$ and a complementary neighbourhood in $\mathcal{B}'$. Only known graphs appeared. We also proved, with a separate computation, that if there are any graphs in $\mathcal{R}(4, 5, 24)$ but

| $e$ | $i_3$ ($-$) | $i_4$ ($-$) | $c_3$ ($-$) | $\delta$ ($-$) | $\Delta$ ($-$) | count |
|-----|-------------|-------------|-------------|-------------|-------------|-------|
| 116 | 356–368     | 225–262     | 123–128     | 8–9         | 10–11       | 9     |
| 117 | 346–362     | 216–253     | 122–132     | 8–9         | 10–11       | 90    |
| 118 | 340–360     | 206–251     | 120–136     | 6–9         | 10–12       | 806   |
| 119 | 332–356     | 198–247     | 124–140     | 6–9         | 10–13       | 4358  |
| 120 | 324–352     | 186–243     | 127–144     | 6–10        | 10–13       | 16346 |
| 121 | 319–344     | 181–232     | 130–146     | 6–10        | 11–13       | 43457 |
| 122 | 314–337     | 178–223     | 133–149     | 6–10        | 11–13       | 79678 |
| 123 | 310–330     | 171–215     | 136–152     | 6–10        | 11–13       | 92504 |
| 124 | 304–324     | 163–208     | 140–154     | 6–10        | 11–13       | 67209 |
| 125 | 302–318     | 161–201     | 144–157     | 6–10        | 11–13       | 31996 |
| 126 | 296–312     | 155–195     | 147–160     | 7–10        | 11–12       | 11485 |
| 127 | 291–301     | 152–177     | 152–162     | 8–10        | 11–12       | 3401  |
| 128 | 286–296     | 149–171     | 156–164     | 8–10        | 11–12       | 843   |
| 129 | 281–290     | 146–165     | 162–166     | 9–10        | 11–12       | 147   |
| 130 | 276–282     | 143–155     | 166–169     | 9–10        | 11–12       | 32    |
| 131 | 270–270     | 143–149     | 172–172     | 10–10       | 11–11       | 3     |
| 132 | 264–264     | 138–144     | 176–176     | 11–11       | 11–11       | 2     |

| all | 264–368     | 138–262     | 120–176     | 6–11        | 10–13       | 352366 |

Table 1. Statistics for all $(4, 5, 24)$-graphs
not in the catalogue, they do not share any 21-vertex subgraph with a graph in the catalogue.

Summary statistics of the catalogue, to complete [3, Table 4], are provided in Table 1; \(e\) is the number of edges, \(i_k\) is the number of independent sets of size \(k\), \(c_3\) is the number of triangles, and \(\delta, \Delta\) are the minimum and maximum degrees. The graphs themselves are available at [2].

4. THE STRUCTURE OF \(\mathcal{R}(4, 5, 24, K)\)

The neighbourhood of a vertex \(a\) of degree \(d\) in a pointed graph \((G, a) \in \mathcal{R}(4, 5, 24, K)\) is the graph \(K \in \mathcal{R}(3, 5, d)\). However not all graphs in \(\mathcal{R}(3, 5)\) appear in pointed graphs. In Table 2 we show the number of graphs \(K\) which occur at least once and the total number of pointed graphs for each \(d\). Note that we have not used the automorphism group of \(G\), so some of the pointed graphs are isomorphic. The great majority of graphs in \(\mathcal{R}(4, 5, 24)\) have trivial automorphism group, so we gave up the small available speedup (estimated at 3%) in order to have fewer steps in the computation. The total of 8,456,784 in the table is \(24 \times |\mathcal{R}(4, 5, 24)|\).

| \(d\) | \(|\mathcal{R}(3, 5, d)|\) occurring | count |
|---|---|---|
| 1–5 | 21 | 0 | 0 |
| 6 | 32 | 2 | 1979 |
| 7 | 71 | 11 | 7497 |
| 8 | 179 | 88 | 64395 |
| 9 | 290 | 240 | 832288 |
| 10 | 313 | 294 | 4651124 |
| 11 | 105 | 103 | 2800499 |
| 12 | 12 | 11 | 97968 |
| 13 | 1 | 1 | 1034 |
| all | 1029 | 750 | 8456784 |

Table 2. Counts of pointed graphs

The number of pointed graphs in \(\mathcal{R}(4, 5, 24, K)\) for \(K \in \mathcal{R}(3, 5, \leq 11)\) varies greatly: from 0 to 526,073, the latter from a rather irregular graph of order 11 and 21 edges. For Step 2 we take two pointed graphs \((G, a), (H, b) \in \mathcal{R}(4, 5, 24, K)\) and overlap them so that their common subgraph \(K\) coincides. This can be done in one distinct way for each automorphism of \(K\) (again ignoring some small reductions arising from automorphisms of \(G\) and \(H\)). Most graphs \(K\) have only trivial automorphisms but some have large automorphism groups, the largest having order 1152 (a vertex-transitive quartic graph of order 8).

Taking the wildly varying sizes of \(\mathcal{R}(4, 5, 24, K)\) as well as the automorphism groups of the various \(K\) into account we needed to solve approximately 2 trillion gluing problems. While that is certainly a lot, we were able to perform hundreds of thousands of such gluings per second per core. The whole
calculation took approximately six core-months for one implementation and
two core-months for the other.

5. Step 2. Finding all ways to glue

In order to ensure correctness, the list of pointed graphs was prepared
independently by the two authors and all the gluings were performed by
two programs written independently using different methods. The decision
to use two different methods rather than identifying the fastest method and
implementing it twice was based on the long-established axiom of software
engineering that different programmers tend to make the same errors when
faced with the same task.

Now we will describe the two different methods for gluing \((G, a), (H, b)\)
\(\in \mathcal{R}(4, 5, 24, K)\) after they are overlapped at the common subgraph \(K\). Be-

cause of the large number of calculations needed, the naive approach of
deciding one unknown adjacency at a time takes far too long.

Define \(d' = 23 - d\). Suppose \(K\) has vertices \(v_0, ..., v_{d-1}\), \(G\) has vertices \(v_0, ..., v_{d-1}, a, a_1, ..., a_d\) and \(H\) has vertices \(v_0, ..., v_{d-1}, b, b_1, ..., b_d\). Note
that the vertices \(a\) and \(b\) cannot participate in any 5-cliques or independent
5-sets by the construction. To specify a gluing it suffices to specify whether
or not \(a_i\) and \(b_j\) are connected by an edge for \(1 \leq i, j \leq d'\). We will record
this data in a \(d' \times d'\) matrix \(M\) with entries 0 (for no edge) and 1 (for edge).

Define a potential \(r, s, t\)-clique to be \(r\) vertices \(w_1, ..., w_r\) in \(V_K\), \(s\) ver-
tsices \(x_1, ..., x_s\) in \(V_G - V_K - \{a\}\), and \(t\) vertices \(y_1, ..., y_t\) in \(V_H - V_K - \{b\}\) such that
\[
\{w_1, ..., w_r, x_1, ..., x_s\}
\]
is an \((r + s)\)-clique in \(G\) and
\[
\{w_1, ..., w_r, y_1, ..., y_t\}
\]
is an \((r + t)\)-clique in \(H\). Define a potential independent \(r, s, t\)-set similarly.
The following lemma is immediate.

**Lemma 5.1.** A \(d' \times d'\) 0-1 matrix \(M = (m_{ij})\) defines a gluing if and only if

1. For each potential \((r, s, t)\)-clique with \(r + s + t = 5\), \(m_{x_0y_0} = 0\) for
some \(1 \leq i \leq s, 1 \leq j \leq t\). (This is needed for \((1, 2, 2)\), \((0, 2, 3)\) and
\((0, 3, 2)\).)
2. For each potential independent \((r, s, t)\)-set with \(r + s + t = 5\), \(m_{x_0y_0} =
1\) for some \(1 \leq i \leq s, 1 \leq j \leq t\). (This is needed for \((3, 1, 1)\), \((2, 1, 2)\),
\((2, 2, 1)\), \((1, 1, 3)\), \((1, 2, 2)\), \((1, 3, 1)\), \((0, 2, 3)\) and \((0, 3, 2)\).)

**Proof.** Please refer to Figure 1 and consider a set \(W\) of size 5. For \(W\) to be
a clique in the completed graph, it must overlap both \(K \cup A\) and \(K \cup B\), and
the pairs of vertices in each those intersections must be edges. That implies
it is one of the potential \((r, s, t)\)-cliques listed in part (1), and to prevent \(W\)
from being a clique in the completed graph we need to include a non-edge.
The case of an independent set is similar. \(\square\)
The two gluing methods are logically similar but implemented very differently. The first gluing method expands on the method in \[3\]. Define an interval to be a set of the form \(I = \{X \mid B \subseteq X \subseteq T\}\), where \(B\) and \(T\) are subsets of \(\{a_1, \ldots, a_d\} \times \{b_1, \ldots, b_d\}\). We write \(I = [B, T]\). We represent \(I\) by two \(d' \times d'\) matrices with coefficients in \(\{0, 1\}\).

Given an interval \([B, T]\), we define collapsing rules as follows. There are 11 in total, one for each of the triples in Lemma 5.1 above. The special event FAIL means that there is no \(X \in [B, T]\) which corresponds to a proper gluing.

**Rule \(K_{1,2,2}\).** Suppose \(\{w_1, x_1, x_2, y_1, y_2\}\) is a potential \((1, 2, 2)\)-clique.

\[
\begin{align*}
\text{if } (x_1, y_1), (x_1, y_2), (x_2, y_1), (x_2, y_2) \in B & \text{ then FAIL} \\
\text{else if } (x_1, y_1), (x_1, y_2), (x_2, y_1) \in B & \text{ then } T := T - (x_2, y_2) \\
\text{else if } (x_1, y_1), (x_1, y_2), (x_2, y_2) \in B & \text{ then } T := T - (x_2, y_1) \\
\text{else if } (x_1, y_1), (x_2, y_1), (x_2, y_2) \in B & \text{ then } T := T - (x_1, y_2) \\
\text{else if } (x_1, y_2), (x_2, y_1), (x_2, y_2) \in B & \text{ then } T := T - (x_1, y_1)
\end{align*}
\]

The collapsing rules for \(K_{0,2,3}\) and \(K_{0,3,2}\) are similar. In each case, the rule says that if 5 vertices include 9 edges, then the remaining vertex pair must not be an edge.

**Rule \(E_{3,1,1}\).** Suppose \(\{w_1, w_2, w_3, x_1, y_1\}\) is a potential independent \((3, 1, 1)\)-set.

\[
\begin{align*}
\text{if } (x_1, y_1) \notin T & \text{ then FAIL} \\
\text{else } B := B \cup (x_1, y_1).
\end{align*}
\]

The collapsing rules for the other potential independent sets from Lemma 5.1 are once again similar.

We start the search with a single interval \(I = [B, T]\) with \(B = \emptyset\) and \(T = \{a_1, \ldots, a_d\} \times \{b_1, \ldots, b_d\}\), and we note that the collapsing rule \(E_{3,1,1}\) can be applied even in this case. Each time we add an edge to \(B\) or remove an edge from \(T\) the number of possible gluings is cut in half.

After applying these collapsing rules repeatedly, we must eventually encounter either FAIL or a stable situation. The discussion in \[3\] applies, and the final state is independent of the order of the application of the collapsing rules.

If we do not encounter FAIL, we pick some \((a_i, b_j)\) with \((a_i, b_j) \notin B\) and \((a_i, b_j) \in T\), and consider the cases \(I = [B, T - (a_i, b_j)]\) and \(I = [B \cup (a_i, b_j), T]\) separately.

The second method applies an equivalent procedure using data structures familiar from the constraint satisfaction area. Each entry \(m_{ij}\) of \(M\) is a variable, with value FALSE, TRUE or UNKNOWN, while each set \(\{x_1, \ldots, x_s\} \times \{y_1, \ldots, y_t\}\) is a clause. Clauses from potential \((r, s, t)\)-cliques can’t have all their variables TRUE, while clauses from potential independent \((r, s, t)\)-sets can’t have all their variables FALSE. Each variable \(\alpha\) has
a list \( C(\alpha) \) of the clique clauses which contain \( \alpha \), and a list \( I(\alpha) \) of the independent set clauses which contain \( \alpha \). There is also a stack \( S \) which maintains a set of distinct variables on a last-in first-out basis. Informally, at each moment \( S \) contains those variables which have been assigned FALSE or TRUE, but their clause lists have not yet been scanned.

Initially, variables are set to TRUE if required by independent \((3,1,1)\)-set clauses, and UNKNOWN otherwise. The variables equal to TRUE are put onto \( S \). Then we execute the following until it terminates.

\[
\text{while } S \neq \emptyset \text{ do } \\
\quad \text{Pop the top variable } \alpha \text{ off } S \\
\quad \text{if } \alpha = \text{FALSE then } \\
\quad \quad \text{for each clause } C \in I(\alpha) \text{ do } \\
\quad \quad \quad \text{if all variables in } C \text{ are FALSE then} \\
\quad \quad \quad \quad \text{exit FAIL} \\
\quad \quad \quad \text{else if all variables in } C \text{ are FALSE except for } \beta = \text{UNKNOWN then} \\
\quad \quad \quad \quad \quad \text{Set } \beta := \text{TRUE and push } \beta \text{ onto } S \\
\quad \quad \quad \text{end if} \\
\quad \quad \text{end for} \\
\text{else} \\
\quad \text{for each clause } C \in C(\alpha) \text{ do } \\
\quad \quad \text{if all variables in } C \text{ are TRUE then} \\
\quad \quad \quad \text{exit FAIL} \\
\quad \quad \quad \text{else if all variables in } C \text{ are TRUE except for } \beta = \text{UNKNOWN then} \\
\quad \quad \quad \quad \quad \text{Set } \beta := \text{FALSE and push } \beta \text{ onto } S \\
\quad \quad \quad \text{end if} \\
\quad \text{end for} \\
\text{end if} \\
\end{while}

For good efficiency it is essential that variables be assigned values as they enter the stack and not when they leave it. Also, a good optimization is for clauses to remember how many UNKNOWN variables they have. If the algorithm terminates with “exit FAIL”, there is no solution. Otherwise, all the variables with value FALSE or TRUE have those values in all solutions. If there is any variable with value UNKNOWN, we can choose one such variable and try FALSE and TRUE separately with \( S \) initialised to that variable only. And so on, recursively.

Both methods were very fast for \( d \geq 8 \), often performing 100,000 gluings per second per core, primarily because failure occurred early most of the time.

For \( d \leq 7 \), the methods as described could take much longer since extremely large search trees with many useless branches could be generated. For those values of \( d \) we used additional techniques.
For the first method, two techniques were used. First, for each pair \((a_i, b_j) \in T - B\) we applied the collapsing rules to both \([B, T - (a_i, b_j)]\) and \([B \cup (a_i, b_j), T]\). If for some pair \((a_i, b_j)\) we arrived at FAIL in both cases we then concluded that there were no gluings. If \([B, T - (a_i, b_j)]\) led to FAIL then we replaced \([B, T]\) by \([B \cup (a_i, b_j), T]\), and if \([B \cup (a_i, b_j), T]\) led to FAIL then we replaced \([B, T]\) by \([B, T - (a_i, b_j)]\). This is of course more expensive than the original algorithm at each node of the search tree, but we found that for \(6 \leq d \leq 7\) it was worth it.

Second, we ordered the pairs \((a_i, b_j)\) according to how many independent sets of type \((2, 2, 1)\) and \((2, 1, 2)\) they were contained in and started the binary search with a pair \((a_i, b_j)\) which was maximal in this sense. The advantage is that when considering \([B, T - (a_i, b_j)]\) the collapsing rules \(E_{2,2,1}\) and \(E_{2,1,2}\), which require only a single edge to be missing from \(T\) in order to modify \(B\), come into play as much as possible.

For the second method, instead of choosing an arbitrary UNKNOWN variable to branch on, we used an UNKNOWN variable which occurred in the greatest number of clique clauses with all TRUE variables except two UNKNOWN variables, or independent set clauses with all FALSE variables except two UNKNOWN variables. This is a heuristic for how beneficial it is to assign FALSE or TRUE to the variable.

In both cases, these enhancements made the cost per node of the search tree much greater but, due to the smaller number of pointed graphs for small \(d\), the computation finished quickly enough.

6. Step 3. Empirical results

For \(6 \leq d \leq 9\), no gluings produced any output graphs, so Step 3 was unnecessary. For \(d = 10\) we found a total of 647,424 graphs (81,936 non-isomorphic) in \(\mathcal{R}(5, 5, 38)\), all of them from a single \(K \in \mathcal{R}(3, 5, 10)\). For \(d = 11\) we found a total of 15,244 graphs in \(\mathcal{R}(5, 5, 37)\), with 15,152 graphs (14,412 nonisomorphic) coming from one \(K \in \mathcal{R}(3, 5, 11)\) and 92 graphs (84 nonisomorphic) coming from another \(K\). An example is shown in Figure 2. None of these graphs could be extended by one more vertex while staying within \(\mathcal{R}(5, 5)\), so Step 3 was completed successfully.

By Step 2, we do not need gluings for \(d \geq 12\), which is fortunate since the number of successful gluings is around 57 billion for \(d = 12\) and perhaps even larger for \(d = 13\). This would make Step 3 very onerous. Of course, these considerations are the reason we sought to eliminate \(d \geq 12\) theoretically (Lemma 2).

We wish to acknowledge useful comments from Staszek Radziszowski.
Figure 2. An example of $G, H \in \mathcal{R}(4, 5, 24)$ glued along $K \in \mathcal{R}(3, 5, 11)$ (the square in the centre), to make $F \in \mathcal{R}(5, 5, 37)$. 
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