A sub-modular receding horizon solution for mobile multi-agent persistent monitoring

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Abstract—We study the problem of persistent monitoring of finite number of inter-connected geographical nodes for event detection via a group of heterogeneous mobile agents. We use Poisson process model to capture the probability of the events occurring at the geographical nodes. We then tie a utility function to the probability of detecting an event in each point of interest and use it in our policy design to incentivize the agents to visit the geographical nodes with higher probability of event occurrence. We show that the design of an optimal monitoring policy to maximize the utility of event detection over a mission horizon is an NP-hard problem. By showing that the reward function is a monotone increasing and submodular function, we then proceed to propose a suboptimal dispatch policy design with a known optimality gap. To reduce the time complexity of constructing the feasible search set and also to induce robustness to changes in event occurrence and other operational factors, we preform our suboptimal policy design in a receding horizon setting. Our next contribution is to add a new term to our optimization problem to compensate for the short-sightedness of the receding horizon approach. This added term provides a measure of importance for nodes beyond the receding horizon’s sight, and is meant to give the policy design an intuition to steer the agents towards areas with higher importance on the global map. Finally, we discuss how our proposed algorithm can be implemented in a decentralized manner. We demonstrate our results through a simulation study.

I. INTRODUCTION

In extension of cities and technology there is always a need for surveillance to monitor for incidences of interest. Traditionally, the surveillance systems are stationary and usually cover limited areas. The cost and communication bandwidth limitation bounds the number of the stationary sensors that can be deployed. To solve the coverage within the limits of the system, use of mobile sensors, e.g. aerial sensors, which the infrastructure can move within the urban area is of interest. In this paper, we study designing a dispatch policy that orchestrates the topological distribution of a set of mobile sensors such that the ‘best’ service for a global monitoring task is obtained with a reasonable computational cost. Our work is related to the problem of multi-agent persistent monitoring/patrolling of a set of geographical points in an area to gather information, e.g., by capturing a photo of the point or communicating with a local agent in that point. Long term multi-agent patrolling of an area offers a low cost and effective monitoring solution for applications such as discovering forest fires [2] and oil spillage in their early stages [3], and locating endangered animals in a large habitat [4].

When the routes between the geographical points of interest are not pre-specified, the optimal monitoring design includes also finding the optimal inter-point trajectories that the agents can follow without violating their controllers bounds. Computing such optimal routes requires the solution of the two-point boundary value problem [5]. Except for monitoring scenarios over a one dimensional space, for which the closed form of the optimal trajectories can be calculated [6], solving for optimal trajectory is generally infeasible [7]. Sub-optimal solutions however can be found by limiting the trajectory space to elliptical or Fourier series functions [5]. Another approach is to divide the mission horizon into smaller time spans and find the optimal trajectory over them [8]. In many applications, however, the geographical points are connected via a set of pre-specified known corridors (each referred henceforth as an edge), and the mobile agents are confined to travel through these edges in order to traverse from one geographical point (referred henceforth as geographical node or simply node) to another. For example, in a smart city setting regulations can restrict the admissible routes between the geographical nodes. In this paper, we consider a monitoring scenario over such a setting, see Fig. 1.

In a single agent patrolling of a set of inter-connected nodes in an area, the complexity of finding the ‘best’ route is the same as the complexity of Traveling Salesman problem, and grows exponentially with the number of the nodes [9]. The problem of optimal multi-agent patrolling is however more complex, since each agent’s patrolling scheme depends on other agents’ policy. This problem is formalized in earlier studies such as [10], [11]. Generally, when there are multiple edges to travel between every two nodes or when each node is connected to multiple other nodes, finding an optimal long term patrolling scheme is not tractable. Constraining the agents to travel through specific routes to traverse among the geographical nodes allows seeking optimal solutions for the problem. For example, when the connection topology between the geographical nodes is a line or a circular graph, optimal solutions for the problem are proposed in [12]–[15]. To overcome the complexity issue on generic graphs, [16] suggest forming different cycles in the graph and assigning agents to these cycles to patrol periodically and seek to minimize the time that a node stays un-visited. Alternatively, [17] proposes agents to move to the most rewarding neighboring node based on their current location.

In this paper, we propose a suboptimal solution for the problem of monitoring events over an inter-connected set of geographical nodes, such as the ones in Fig. 1. Instead of using the customary idle time of the nodes, which comes with the underlying assumption that the rate of events happening at all the nodes is uniform, we base our utility function on...
the probability of detecting an event in each point of interest. We use a Poisson random variable as the model for arrival of valuable information at the nodes. Poisson process is a widely-used counting process for scenarios where we are counting the occurrences of certain events that happen at a certain rate, but completely at random (without a certain structure) [18].

We show that the design of an optimal monitoring policy to maximize the utility of event detection over a mission horizon is an NP-hard problem. Next, we show that the reward function is a monotone increasing submodular set function. To establish our proof, we develop a set of auxiliary results based on the Karamata’s inequality [19]. Given the submodularity of the reward function, we propose a receding horizon sequential greedy algorithm to compute a suboptimal dispatch policy with a polynomial computation cost and guaranteed bound on optimality. The receding horizon nature of our solution induces robustness to uncertainties of the environment. Our next contribution is to add a new term to our optimization problem in order to compensate for the short-sightedness of the receding horizon approach, see Fig. 2. This added term becomes useful when the agents are supposed to patrol a large field by giving them an intuition of existing value in long distances.

In recent years, submodular optimization has been widely used in sensor and actuator placement problems [20–25]. In comparison to the sensor/actuator placement, the challenge in our work is that the assigned policy per each mobile agent over the receding horizon is a vector rather than a point. To deal with this challenge, we use the matroid constraint [26] to design our suboptimal submodular-based policy. Finally, we discuss how the final algorithm can be implemented in a decentralized manner. We demonstrate our results through a simulation study.

II. Preliminaries

Notation: $2^A$ is the set of all the subsets of a set $A$. For a set $B$ and a singleton set $\{a\}$, for simplicity we represent $B \cup \{a\}$ by $B \cup a$. Given an event set $\mathcal{V}$ and $e \in \mathcal{V}$, $P(e) : \mathcal{V} \to [0, 1]$ denotes the probability of event $e$ happening. We denote a sequence of $m$ increasing real numbers $(t_1, \ldots, t_m)$ (i.e., $t_k \leq t_{k+1}$ for $k \in \{1, \ldots, m\}$) by $(t)^m$. Given $(t)^n$ and $(v)^n$, we denote by $(t)^n \oplus (v)^n$ their concatenated increasing sequence, i.e., for $(u)^{1+m} = (t)^n \oplus (v)^m$ we have that any $u_k$, $k \in \{1, \ldots, n+m\}$ is either in $(t)^n$ or $(v)^m$ or is in both of $(t)^n$ and $(v)^m$. We assume that $(u)^{1+m}$ preserves the relative labeling of $(t)^n$ or $(v)^m$, i.e., if $t_k$ and $t_k$, $k \in \{1, \ldots, n-1\}$ (resp. $v_i$ and $v_{i+1}$, $k \in \{1, \ldots, m-1\}$) correspond to $u_i$ and $u_j$ in $(u)^{1+m}$, then $i < j$.

We formulate our optimal mobile sensor dispatch problem via an optimization problem of the form

$$\max_{\mathcal{P} \subseteq \mathcal{P}} g(\mathcal{P}), \quad \text{s.t.,} \quad |\mathcal{P} \cap \mathcal{P}_i| \leq 1, \ i \in \{1,\ldots,\mathcal{M}\},$$

where $\mathcal{P}_1, \ldots, \mathcal{P}_\mathcal{M}$ are collection of $\mathcal{M}$ disjoint sets that satisfy $\bigcup_{i=1}^{\mathcal{M}} \mathcal{P}_i = \mathcal{P}$. The constraint condition (1b) is in the so-called partition matroid form [27] and restricts the choice of the optimal solution for (1) to be a set that contains at most one member from each disjoint sets $\mathcal{P}_1, \ldots, \mathcal{P}_\mathcal{M}$. Problem (1) is known to be NP-hard [28]. When the objective function $g$ is submodular, however, a sequential greedy member selection algorithm described below results in a suboptimal solution with a guaranteed bound on optimality.

A set function $g : 2^\mathcal{P} \to \mathbb{R}$ is monotone decreasing if for all $\mathcal{P}_1, \mathcal{P}_2 \subseteq \mathcal{P}$ we have $\mathcal{P}_1 \subseteq \mathcal{P}_2$ if and only if $g(\mathcal{P}_1) \geq g(\mathcal{P}_2)$ [27]. A set function $g : 2^\mathcal{P} \to \mathbb{R}$ is submodular if and only if for two sets $\mathcal{P}_1, \mathcal{P}_2$ satisfying $\mathcal{P}_1 \subseteq \mathcal{P}_2 \subseteq \mathcal{P}$, and for $\mathcal{P} \not\subseteq \mathcal{P}_2$ we have [27]

$$\Delta_g(\mathcal{P} \cup \mathcal{P}_1) \geq \Delta_g(\mathcal{P} \cup \mathcal{P}_2).$$

Here, $\Delta_g : (\mathcal{P}, \mathcal{P} \to \mathbb{R}$ is

$$\Delta_g(\mathcal{P} \cup \mathcal{P}_1) = g(\mathcal{P} \cup \mathcal{P}_1) - g(\mathcal{P}), \quad \forall \mathcal{P} \in 2^\mathcal{P}, \ \forall \mathcal{P} \subseteq \mathcal{P},$$

which shows the increase in value of the set function $g$ going from set $\mathcal{P}$ to $\mathcal{P} \cup \mathcal{P}_1$. Equation (2) shows that when a new member is added to a smaller set the gain is going to be greater than or equal to when the same member is added to a larger set where the smaller set is a subset of larger set. Therefore, submodularity is a property of a set functions that shows diminishing reward as new members are being introduced to the system.

Theorem 2.1 (See [27]) Let $g$ in the optimization problem (1) be a monotone submodular set function. Suppose $g(\mathcal{P}^*)$ is the global maximum of (1). Also, let $\mathcal{P}_\mathcal{S}\mathcal{G}(\mathcal{M}) \subseteq \mathcal{P}$ be the output of sequential greedy policy selection below started at $i = 1$ and $\mathcal{P}_\mathcal{S}\mathcal{G}(0) = \emptyset$, and repeated until $i = \mathcal{M}$,

$$\mathcal{P}_i^* = \arg\max_{\mathcal{P} \subseteq \mathcal{P}} \Delta_g(\mathcal{P} | \mathcal{P}_\mathcal{S}\mathcal{G}(i-1)),$$

where $\mathcal{P}_\mathcal{S}\mathcal{G}(i) = \mathcal{P}_\mathcal{S}\mathcal{G}(i-1) \cup \mathcal{P}_i^*$. Then, $g(\mathcal{P}_\mathcal{S}\mathcal{G}(\mathcal{M})) \geq \frac{1}{2} g(\mathcal{P}^*)$.

Next, we develop a set of auxiliary results that we use later in the proof of our main theorem (Theorem 2.1).

Definition I: The non-increasing sequence $(\delta t)^n_1$ majorizes the non increasing sequence $(\delta v)^n_1$ if
\[ \delta t_1 \geq \delta t_2 \geq \cdots \geq \delta t_n \text{ and } \delta v_1 \geq \delta v_2 \geq \cdots \geq \delta v_n \]
\[ \delta t_1 + \cdots + \delta t_i \geq \delta v_1 + \cdots + \delta v_i \text{ for } \forall i \in \{1, \cdots, n-1\} \]
\[ \delta t_1 + \cdots + \delta t_n = \delta v_1 + \cdots + \delta v_n \]

**Theorem 2.2 (Karamata’s inequality [19])** Let \( f : \mathbb{R} \to \mathbb{R} \) be a concave function. Then, if sequence \((\delta t)_1^n\) maximizes \((\delta v)_1^n\), then the following holds
\[ f(\delta t_1) + \cdots + f(\delta t_n) \leq f(\delta v_1) + \cdots + f(\delta v_n). \]

**Corollary 2.1:** Let \( f : \mathbb{R} \to \mathbb{R} \) be a concave function with \( f(0) = 0 \). If sequences \((\delta t)_1^n\) and \((\delta v)_1^n\) with \( n \leq m \) satisfy
\[ \delta t_1 + \cdots + \delta t_i \geq \delta v_1 + \cdots + \delta v_i, \quad \forall i \in \{1, \cdots, n-1\} \]
\[ \delta t_1 + \cdots + \delta t_n = \delta v_1 + \cdots + \delta v_m, \]
then the following holds,
\[ f(\delta t_1) + \cdots + f(\delta t_n) \leq f(\delta v_1) + \cdots + f(\delta v_m). \quad (3) \]

**Proof:** We can define the sequence \((\delta u)_1^n\) such that \( \delta u_i = \delta t_i, \quad i \in \{1, \cdots, n\} \text{ and } \delta u_i = 0, \quad i \in \{n+1, \cdots, m\} \) there for \((\delta u)_1^n\) majorizes \((\delta v)_1^n\), following the fact that \( f(0) = 0 \) and as a consequence of Theorem 2.2 we conclude (3). \( \square \)

**Corollary 2.2:** Let \( f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \) be a monotone increasing and concave function. Then for any \( a, b, c, d \in \mathbb{R}_{\geq 0} \) such that \( 0 \leq a \leq c \) and \( 0 \leq b \leq d \), the following holds,
\[ f(c) + f(d) - f(c+d) \geq f(a) + f(b) - f(a+b). \]

**Proof:** Having \( a \leq c \) and \( b \leq d \) and \( a + b \leq c + d \), \( \square \)

**Lemma 2.1:** For any \((q)_1^n\), let
\[ g((q)_{1}^{i]) = \sum_{i=1}^{l-1} f(\Delta q_i), \quad (5) \]
where \( \Delta q_i = q_{i+1} - q_i \) and \( f \in K \) be a concave and positive and increasing function with \( f(0) = 0 \). Now, consider two increasing sequences \((t)_1^n\) and \((u)_1^n\), and concatenation \((a)_1^{n+l} = (t)_1^n \oplus (u)_1^n\). Then,
\[ g((a)_1^{n+l}) - g((t)_1^n) \geq 0. \]

**Proof:** Since \((a)_1^{n+l}\) is the result of concatenation of \((t)_1^n\) and \((u)_1^n\), therefore \( \exists p, q \in \mathbb{Z} \) such that \( a_p = t_1 \) and \( a_q = t_n \). We take a sub-sequence of \((a)_1^{n+l}\) ranging from index \( p \) to \( q \) and name it \((v)_1^m\) where \( m \geq n \). Taking \( \Delta v_i = v_{i+1} - v_i \) and \( \Delta t_i = t_{i+1} - t_i \), by rearranging \( \Delta u_i \)’s and \( \Delta t_i \)’s in a descending manner, we form the sequences \((\delta v)_1^{n-1}\) and \((\delta t)_1^{n-1}\). Since \( a_p = t_1 \) and \( a_q = t_n \) then
\[ \sum_{i=1}^{m-1} \Delta v_i = \sum_{i=1}^{m-1} \delta v_i = \sum_{i=1}^{n-1} \delta t_i = t_n - t_1. \quad (6) \]

Also since, \((a)_1^{n+l}\) is the result of concatenation of \((t)_1^n\) and \((u)_1^n\). Therefore, \( \forall i \in \{1, \cdots, n\} \) there exists a \( S_i \subset \{1, \cdots, m\} \) such that \( \sum_{j \in S_i} \Delta v_j = \delta t_i \), where \( S_i \cap S_k = \emptyset, \ i \neq k \). Consequently, for \( r \in \{1, \cdots, m\} \) we have
\[ \sum_{i=1}^{r} \delta v_i = \sum_{j \in S} \delta t_j, \quad \text{where } S \subset \{1, \cdots, n\} \text{ and } |S| \leq r. \]
Since \((\delta t)_1^{n-1}\) is and decreasing sequence, we can write
\[ \sum_{i=1}^{r} \delta v_i \leq \sum_{i=1}^{r} \delta t_i. \quad (7) \]

Considering equations (6) and (7) we conclude that the conditions of Corollary 2.1 are met then,
\[ f(\delta t_1) + \cdots + f(\delta t_{n-1}) \leq f(\delta v_1) + \cdots + f(\delta v_{m-1}). \quad (8) \]

Given that \( f(\delta t_1) + \cdots + f(\delta t_{n-1}) = \sum_{i=1}^{n-1} f(\Delta t_i) \)
and \( f(\delta v_1) + \cdots + f(\delta v_{m-1}) = \sum_{i=1}^{m-1} f(\Delta v_i) \leq \sum_{i=1}^{n-1} f(\Delta a_i) \), we have \( \sum_{i=1}^{n-1} f(\Delta t_i) \leq \sum_{i=1}^{n-1} f(\Delta a_i) \).

**Lemma 2.2:** For any \((q)_1^n\), let
\[ g((q)_{1}^{i]) = \sum_{i=1}^{l} f(\Delta q_i), \quad (9) \]
where \( \Delta q_i = q_{i+1} - q_i \) and \( f \in K \) is a concave and increasing function with \( f(0) = 0 \). Now, consider three increasing sequences \((t)_1^n\) and \((v)_1^n\) and concatenations \((a)_1^{n+l} = (t)_1^n \oplus (u)_1^n\) and \((b)_1^{n+l} = (v)_1^n \oplus (u)_1^n\) where \((v)_1^n\) is a sub-sequence of \((t)_1^n\), then
\[ g((b)_1^{n+l}) - g((v)_1^n) \geq 0. \]

**Proof:** We define the sequence \((u)_1^n\) to be the first \( p \) elements of \((u)_1^n\) sequence. Then we can form
\[ \Delta S_p = g((v)_1^n \oplus (u)_1^n) - g((v)_1^n \oplus (u)_1^{p-1}) - g((t)_1^n \oplus (u)_1^n) - g((t)_1^n \oplus (u)_1^{p-1})), \]
with \((u)_1^n\) to be an empty sequence with no members. Since \((v)_1^{m}\) is a sub-sequence of \((t)_1^n\) and \((u)_1^n\) having one member more over \((u)_1^{p-1}\), then we have
\[ \Delta S_p = (f(\Delta S_1) + f(\Delta S_2) - f(\Delta S_1 + \Delta S_2)) - (f(\Delta S_1) + f(\Delta S_2) - f(\Delta S_1 + \Delta S_2)), \]
with \( 0 \leq \Delta S_3 \leq \Delta S_1 \) and \( 0 \leq \Delta S_4 \leq \Delta S_2 \), from Corollary 2.2 we can conclude that \( \Delta S_p \geq 0 \). Given that
\[ \sum_{p=1}^{l} \Delta S_p = g((b)_1^{n+l}) - g((v)_1^n) - g((a)_1^{n+l}) - g((v)_1^n), \]
we can conclude the proof. \( \square \)

### III. PROBLEM DEFINITION

We consider a persistent patrolling/monitoring for events detection, where a set of \( A = \{1, \cdots, M\} \) mobile sensors (each referred henceforth as mobile agent or simply agent) are deployed to monitor a set of \( V = \{1, \cdots, N\} \) geographical nodes on a \( \mathbb{R}^2 \) or \( \mathbb{R}^3 \) space. At each geographical node
There is an event that takes place in random, whose probability of occurrence follows a Poisson random process \( C_0(t, t_{v,i}) \sim \text{Poisson}(\lambda_0(t - t_{v,i})) \), where \( t_{v,i} \in \mathbb{R}_{\geq 0} \) is the last time node \( i \) is visited by a mobile agent \( j \in A \). The rate of event occurrence \( \lambda_0 \in \mathbb{R}_{\geq 0} \) can be specified from the historical data, and can change based on new data that has become available from other sources. The rate also can be manipulated to steer the agents in certain direction on the map. Finally, \( \lambda_0 \in \mathbb{R}_{\geq 0} \) can be time varying. However to simplify the exposition, we demonstrate our results for fix occurrence rate. The results can be generalized to time-varying case by using \( C_0(t, t_{v,i}) \sim \text{Poisson}(\int_{t_{v,i}}^{t} \lambda_0(\tau) d\tau) \).

We assume that it takes \( \delta_0 \in \mathbb{R}_{\geq 0} \) time for each mobile agent \( j \in A \) to process its sensor measurement collected upon its arrival time at a geographical node. For example, \( \delta_0 \) can be the time a mobile agent needs to process and detect an event from an image shot upon its arrival time. To incentivize agents to visit and scan a geographical node \( i \in V \), we assign the following reward function to node \( i \in V \):

\[
R_i(t) = \begin{cases} 0, & t = t_{s,i} \\ \psi_i(t - t_{s,i}), & t > t_{s,i} \end{cases},
\]

where \( t_{s,i} \) is the latest scan time of node \( i \) and

\[
\psi_i(t - t_{s,i}) = 1 - e^{-\lambda_i(t - t_{s,i})},
\]

is the probability of at least one event taking place at time interval \((t_{s,i}, t)\). \( R_i(t) \) resets to 0 after an agent arrives and scans the node, and monotonically increases with a rate exponentially proportional to \( \lambda_i \) afterward. Upon arrival of any agent \( j \in A \) at time \( t_v \in \mathbb{R}_{>0} \) at node \( i \in V \), the agent immediately scans for the events (e.g., takes a picture) and the reward \( R_i(t_v) \) is scored for the patrolling team \( A \) and \( t_{s,i} \) is set to \( t_v \).

**Assumption 1:** If more than one agent arrive at node \( i \in V \) and scan it at the same time \( t_v \), the reward collected for the team is still \( R_i(t_v) \) (note that after the first scan \( t_{s,i} \) sets to \( t_v \)). Furthermore, every agent should spend \( \delta_0 \) amounts of time at the node to complete its measurement processing and event detection task. During the processing time the agent cannot scan for events.

To traverse from one node to another, the mobile agents are confined to use a set of pre-specified edges (corridors), see Fig. 1. Depending on their vehicle type, agents may have to take different paths going from one node to another. We let \( \mathcal{E}_{v,w} \) be the set of edges between nodes \( v, w \in V \). We assume that each geographical node is connected at least through one edge to another geographical node. We also add a self-loop to each node. i.e., \( |\mathcal{E}_{v,v}| = 1 \) for any \( v \in V \). The self-loops are introduced to allow our motion planning policy to let agents to stay put and continue scanning at a node. For every node \( v \in V \), we let \( \mathcal{N}_v \) be the set of its neighboring nodes that are connected to it via an edge. We assume that the travel time \( \tau_{v,w}(k) \in \mathbb{R}_{\geq 0} \) between every node \( v, w \in V \) for agent \( i \in A \) along any edge \( k \in \mathcal{E}_{v,w} \) is known, and \( \tau_{v,v} = 0 \) (travel time along a self-loop edge is zero). We assume that the set of mobile robots is heterogeneous, therefore the travel time differs for different agents on the same edge. The travel time at each edge also can change during the mission time. We assume that the travel time along every edge and for each agent is proportional to the length of the edge, and time to go from a point along the edge to its end nodes is also known.

Our ultimate goal is to detect maximum number of events over a given mission horizon. By definition of the reward function (10), this objective is equivalent to designing a patrolling policy (what sequence of nodes to visit at what times by which agent) so that the group of mobile agents \( A \) collects maximum possible reward over the mission horizon.

In the policy designs (optimal and suboptimal) we consider the following assumption.

**Assumption 2:** An agent \( i \in A \) can only move to the neighboring nodes \( \mathcal{N}_i \) of the previously visited node \( j \in V \). Moreover, every agent scans the node that it visits.

The optimal monitoring policy, under Assumptions 1 and 2, over the given mission time should assign a sequence of \( \mathcal{N}_j \) nodes to each agent \( i \in A \). Let \( n^i = [n^i(0), n^i(1), \cdots, n^i(\mathcal{N}_j)]^T \) be the sequence of the nodes visited by agent \( i \in A \), with \( n^i(0) \) being the first node that agent \( i \) visits (starts from). Because of the Assumption 2 we have \( n^i(j + 1) \in \mathcal{N}_n(i(j)) \), for all \( j \in \{1, 2, \cdots, \mathcal{N}_j - 1\} \). Let \( T^i = [T^i(0), T^i(1), \cdots, T^i(\mathcal{N}_j)] \) be the visiting time associated with visiting sequence \( n^i \) of agent \( i \in A \). Given the starting location, let \( P^i \) be the set of all the feasible tuples \( p^i = (n^i, T^i, i) \) over the mission horizon for agent \( i \in A \) and let \( \mathcal{P} = \bigcup_{i=1}^{M} P^i \). Then, given any \( P \in \mathcal{P} \), the collected reward function \( R : 2^\mathcal{P} \rightarrow \mathbb{R}_{>0} \) is

\[
R(P) = \sum_{p \in P} \sum_{j=1}^{\lvert n^i \rvert} R_{n^i(j)}(T_p(j)),
\]

with \( p = (n_p, T_p, i_p) \). Then, given (12), the optimal policy to maximize the team collected reward over a given mission horizon is given by

\[
\mathcal{P}^* = \arg\max_{P \in \mathcal{P}} R(P), \quad \text{s.t.} \quad \left(13a\right) \quad \lvert P \cap \mathcal{P}^* \rvert \leq 1, \quad i \in \{1, \ldots, M\}.
\]

Here, the partition matroid constraint (13b) ensures that the optimal policy chooses only one member of \( P^i \) for each agent \( i \in A \) from the collective feasible set \( \mathcal{P} \). The optimization problem is in the standard form of (1), which is known to be NP hard [29]. The following result gives the cost of constructing the feasible set \( \mathcal{P} \) and time complexity of solving optimization problem (13).

**Lemma 3.1** (Time complexity of problem 13) The cost of constructing the feasible set \( \mathcal{P} \) for optimization problem (13) is of order \( O(MD^{N_f}) \), where \( D = \max\{\mathcal{N}_1, \ldots, \mathcal{N}_N\} \), \( N_f = \max\{\lvert n^i \rvert \}_{n^i \in \mathcal{P}}^{M} \). Furthermore, the time complexity

\[1\]With a slight abuse of notation, here we use \( n^i \in \mathcal{P}^i \) in place of \( (n^i, T^i, i) \in \mathcal{P} \).
of solving optimization problem \((13)\) is \(O(\prod_{i=1}^{N_f} D^{N_f})\) where \(\hat{N}_f = \max\{|n^i|\}_{n^i \in \mathcal{P}}\).

Proof: To find the optimum solution, we have to find all the feasible sets containing \(p^j = (n^{i}, T^j, j)\) for every agent \(j \in \mathcal{A}\) over the mission horizon. The time complexity of constructing \(\mathcal{P}\) then is of order \(O(D^{N_f})\). To find the optimal policy, we need to evaluate the reward function for each \(p^j \in \mathcal{P}^j\) for each agent \(j \in \mathcal{A}\), requiring a cost of \(O(D^{N_f})\). The policy \(p^j \in \mathcal{P}^j\) chosen for agent \(i \in \mathcal{A}\) affects visited time \(\tau_{v,k}\) of geographical node \(k \in \mathcal{V}\), as well as the policy \(p^j \in \mathcal{P}^j\) chosen for agent \(j \in \mathcal{A}\)\{i\}. Therefore, policy of different agents are not independent but tied together toward value of \((12)\) and should be evaluated in conjunction with each other. Therefore, the time complexity of finding \(\mathcal{P}^*\) in \((13)\) is of order \(O(\prod_{i=1}^{N_f} D^{N_f})\).

If the system parameters change after the optimal policy design (e.g., number of the mobile agents or the nodes, or the rate of event detection at the nodes), the operation should be stopped and the optimal design policy should be repeated for the remainder of the mission horizon under the new conditions. Our objective in this paper is to construct a suboptimal policy to solve the persistent patrolling problem described above with a time complexity that is ‘reasonable’ and robust to changes that can happen during the mission horizon. We note here that because of the changes that happen to various operational aspect of the problem, periodical solutions are not suitable.

## IV. Suboptimal Policy Design

Lemma \([1,1]\) indicates that the time complexity of finding an optimal patrolling policy increases exponentially by the horizon of the agents’ policy \(N_f\) and also the number of exploring agents \(M\). To reduce the computational cost, a rational suboptimal strategy is to trade in optimality and divide the policy making horizon to multiple shorter horizon of length \(N_H\), and design an optimal policy for each horizon. In other words, in the policy making optimization problem \((13)\), the search space \(\mathcal{P}\) of the optimal policy is limited to sub-policies with the length of \(N_H\). To provide a suboptimal policy making that is also robust to the online changes that can occur during the mission time, we propose a receding horizon policy making algorithm with planning horizon of \(N_H\) and execution horizon of \(N_e < N_H\). However, this approach comes with two shortcomings. First, receding horizon design suffers from shortsightedness. That is, in a large, field, this design is oblivious to the reward distribution of the area that do not appear in the planning horizon. Therefore, the optimal policy over \(N_H\) can inadvertently steer the agents away from the distant points with high probability of event occurrence that are not in horizon \(N_H\), see Fig. 2. To compensate for this shortcoming, we introduce the notion of nodal importance and augment the reward function \((12)\) over the design horizon of \(N_H\) with an additional term that takes into account how close an agent will end up after \(N_H\) visits to an area in a map with high intensity of reward. Another shortcoming of receding horizon design is that even though design over a smaller horizon of \(N_H\) cuts down on the time complexity of optimization problem \((13)\), the time complexity of finding an optimal policy still increases exponentially with the number of the agents. To reduce this computational cost, we show that our cost function in the receding horizon design is submodular, and propose to implement a greedy algorithm to obtain a suboptimal policy with known optimality gap given in Theorem \([2,1]\). The greedy algorithm also has the potential for decentralized implementation as we discuss below.

We define the nodal importance of a node \(v \in \mathcal{V}\) at time \(\tau\) as

\[
L(v, \tau, r) = \sum_{i \in \mathcal{N}_e^r} R_i(\tau),
\]

(14)

where \(\mathcal{N}_e^r\) is the set of nodes (containing \(v\)) that can be reached started from node \(v\) using at most \(r\) edges in \(\mathcal{E}\). Nodal importance \((14)\) is a measure that indicates the concentration of reward at some neighborhood around node \(v\). As the nodal importance of node \(v\) in \(\mathcal{V}\) is discounted by the travel time of agent \(i \in \mathcal{A}\) from its current location to node \(v\), we define the projected nodal importance of a node \(v\) with respect to an agent \(i\) as

\[
L^i(v, \tau, r) = \frac{L(v, \tau, r)}{\tau_i^r(\tau) - \tau},
\]

where \(\tau_i^r(\tau)\) is the time that agent \(i\) can reach node \(v\) using the fastest route and \(\tau\) is the current time.

Fig. 2: A scenario where an agent has two possible routes over the designated receding horizon. The nodes’ color intensity shows their reward value. The route in blue offers a higher reward over the receding horizon but it puts the agent close to an area with lower amount of reward, while the route in red results in lower total reward over the receding horizon but puts the agent near an area with higher amount of reward.
that incentivizes moving closer to the rewarding areas outside the planning horizon, we set the reward function to be

$$\hat{R}(\bar{P}) = R(\bar{P}) + \alpha \sum_{p \in P} L^*(p), \quad \alpha \in \mathbb{R}_{\geq 0}, \quad (15)$$

where $R(\bar{P})$ is defined in (12) and

$$L^*(p) = L^*(v_i^p, T_p(N_H), r), \quad v_i^p = \arg \max_{v \in \bar{V}} L^*(v, T_p(N_H), r).$$

Given the augmented reward function (15) then the optimal policy design over each receding horizon is

$$P^* = \arg \max_{\bar{P} \in P} R(\bar{P}), \quad \text{s.t.} \quad |\bar{P} \cap P^i| \leq 1, \quad (16)$$

where $P = \bigcup_{i=1}^M P^i$ is the set of the union of the feasible policies of the agents, $P^i$, $i \in \{1, \ldots, M\}$, over the horizon $N_H$. Next, we show that the reward function (15) is submodular over any given feasible policy set $P$.

**Theorem 4.1 (Submodularity of the reward function (15))**

Let $P = \bigcup_{i=1}^M P^i$ be the set of the union of the feasible policies of the agents, $P^i$, $i \in \{1, \ldots, M\}$, over the horizon $N_H$. Then, for any $\alpha \in \mathbb{R}_{\geq 0}$, the reward function (15) is a monotone increasing and submodular set function over $P$, i.e., $R : 2^P \rightarrow \mathbb{R}_{\geq 0}$ is monotone increasing and submodular.

**Proof:** By a change in labeling of geographical nodes, we denote the set $\bar{V}(\bar{P}) = \{1, \ldots, v(\bar{P})\} \subset \bar{V}$ be the set of the nodes being visited given policy $\bar{P}$, without loss of generality we can rewrite the reward function $\hat{R}$ as a summation of gathered rewards on visit times of the nodes in the set $\bar{V}(\bar{P})$.

$$\hat{R}(\bar{P}) = R(\bar{P}) + \alpha \sum_{p \in P} L^*(p) = \sum_{l \in \bar{V}(\bar{P})} \left( \sum_{j=1}^{c(l, \bar{P})} \psi_j(\Delta t_j(l, \bar{P})) \right) + \alpha \sum_{p \in \bar{P}} L^*(p),$$

where $\Delta t_j(l, \bar{P}) = t_j(l, \bar{P}) - t_{j-1}(l, \bar{P})$ and $c(l, \bar{P}) \in \mathbb{Z}_{\geq 0}$, $l \in \bar{V}(\bar{P})$ denotes the number of visits to the geographical node $l$ and $\{t_j(l, \bar{P})\}_{j \in c(l, \bar{P})} = \{t_1(l, \bar{P}), t_2(l, \bar{P}), \ldots, t_{c(l, \bar{P})}(l, \bar{P})\}$ is the sequence of time that the same node was visited by an agent given policy $\bar{P}$, where $0 \leq t_1(l, \bar{P}) \leq t_2(l, \bar{P}) \leq \cdots \leq t_{c(l, \bar{P})}(l, \bar{P})$ and $t_0(l, \bar{P}) = 0$.

Given the monitoring policies $P_1, P_2, q$ where $P_1 \subset P_2 \subset P$ and $q \notin P_1$, $q \notin P_2$, we know that $\{t_j(P_1)\}_{j \in c(l, P_1)}$ is a subsequence of $\{t_j(P_2)\}_{j \in c(l, P_2)}$. Taking $c(l, P_1)_{c(l, \bar{P})}$ to be visiting sequence of node $l$ given policy $q$ then using Lemma 4 and the fact that $\Psi(.)$ is a normalized increasing concave function, then $\forall l \in \bar{V}(\bar{P})$

$$\sum_{j=1}^{c(l, P_1 \cup q)} \psi_j(\Delta(t_j(P_2 \cup q))) - \sum_{j=1}^{c(l, P_2)} \psi_j(\Delta(t_j(P_2))) \geq 0.$$
copy of Algorithm 1 locally. Although reasonable for small-size networks, the communication and storage costs of this approach scale poorly with the network size. The sequential structure of Algorithm 1, however, offers an opportunity for a communicationally and computationally more efficient decentralized implementations, as described in Algorithm 2. To implement Algorithm 2, we assume that the agents \( A \) can form a bidirectional connected communication graph \( G^a = (A, E^a) \), i.e., there is a path from every agent to every other agent on \( G^a \). Then, there always exists a route \( \text{SEQ} = a_1 \to \cdots \to a_i \to \cdots \to a_K, a_k \in A, k \in \{1, \ldots, K\}, K \geq M \), that visits all the agents (not necessarily only one time), see Fig. 3(a). The agents follow SEQ to share their information while implementing Algorithm 2. The communication cost to execute Algorithm 2 can be optimized by picking SEQ to be the shortest path \( \text{PATH} \) that visits all the agents over graph \( G^a \). If \( G^a \) has a Hamiltonian path \( \text{PATH} \), the optimal choice for SEQ is a Hamiltonian path. When, there is a SEQ that visits every agent on \( G^a \), the directed information graph \( G^I = (A, E^I) \) of Algorithm 2 which shows the information access of each agent while implementing of Algorithm 2 is full, see the bottom plot in Fig. 3(a). That is, each agent in SEQ is aware of the previous agents’ decision. Therefore, the solution obtained by Algorithm 2 is an exact sequential greedy algorithm and its optimality gap is \( 1/2 \). We recall that the labeling order of the mobile agents does not have an effect on the optimality gap guaranteed by Theorem 4.2 [26]. If an agent \( i \in A \) appears repeatedly in SEQ (e.g., the blue agent in Fig. 3(a)), with a slight increase in computation cost we can modify Algorithm 2 to allow agent \( i \) to redesign and improve its suboptimal policy \( p^i \) by re-executing step 5 of Algorithm 2. The communication topology of the graph is directed or there is a message dropout while executing Algorithm 2, the directed information sharing graph \( G^I = (A, E^I) \) may not be full, see Fig. 3(b). Then, the decentralized Algorithm is deviated from the exact sequential greedy algorithm structure. For such cases, [26] shows that the optimality gap is \( 1/\omega(G^I) + 2 \), where \( \omega(G^I) \) is the clique number of \( G^I \) [26].

Another form of decentralization implementation of Algorithm 1 which may be more relevant in urban environments, is through a client-server framework. In this framework, agents (clients) connect to a shared memory or a cloud (server) to download or upload information, or use the cloud’s computing power asynchronously. Let \( \{T^1, \ldots, T^j\} \) be the sequence of time slots that is allotted respectively to agents \( A \), see Fig. 4. To implement Algorithm 1, agent \( j \in A \) connects to the server at the beginning of \( T^j \) to check out \( \mathcal{P} \). Then, it completes steps 5 and 6, and checks in the updated \( \mathcal{P} \) to the server before \( T^j \) elapses fully. Since the time slots assigned to the agents do not overlap, agent \( i \) has access to policy \( p^k \) of all agents \( k \) with \( k < j \). Therefore, the information graph \( G^I \) is full, and the optimality gap of \( 1/2 \) holds. In scenarios where agent \( j \) takes a longer time than \( T^j \) to complete and check back in the \( \mathcal{P} \) to the server, the information graph is going to be incomplete. For example, in the scenario shown in Fig. 4, the information sharing graph \( G^I \) is the same as the one shown in Fig. 3(b). Therefore, the optimally gap of executing Algorithm 2 is determined by \( \frac{1}{M - \omega(G^I) + 2} \).

V. NUMERICAL EXAMPLE

In this section, we provide a numerical example to highlight the main features of our algorithm. Consider three agents patrolling an area, which is divided into 20 by 20 grid map as shown in Fig. 5(a). Each agent can fly to 4 of the neighboring grid cells (above-below-right-left) without any barriers between the cells. The region shown by hollow blue

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2 Hamiltonian path is a path that visits every agent on \( G^a \) only once [31].
3 The clique number of a graph is equal to the number of the nodes in the largest sub-graph such that adding an edge will cause a cycle [32].

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We proposed a solution for persistent monitoring of a finite number of inter-connected geographical nodes with the purpose of maximizing the expected value of event detection. We modeled the probability of discovering at least one event in each geographical node as a Poisson distribution and tied this with trajectory scheduling of the agents via a utility function. We showed that maximizing the utility function is NP-hard. By showing that the reward function is a monotone increasing and submodular set function, we laid the ground to propose a suboptimal solution with a known optimality gap for this NP-hard problem. To induce robustness to the changes in the problem parameters, we proposed our suboptimal solution in a receding horizon setting. Next, to compensate for the shortsightedness of the receding horizon approach, we added a new term, called the projected nodal importance, to the reward function as a measure to incorporate a notion of importance of the regions beyond the feasible solution set of the receding horizon optimization problem. Finally, we discuss how our suboptimal solution can be implemented in a decentralized manner. Our future work is to investigate decentralized algorithms that allow agents to communicate synchronously with each other in order to have a consensus on a policy with known optimality gap.

VI. Conclusion

We proposed a solution for persistent monitoring of a finite number of inter-connected geographical nodes with the purpose of maximizing the expected value of event detection. We modeled the probability of discovering at least one event in each geographical node as a Poisson distribution and tied this with trajectory scheduling of the agents via a utility function. We showed that maximizing the utility function is NP-hard. By showing that the reward function is a monotone increasing and submodular set function, we laid the ground to propose a suboptimal solution with a known optimality gap for this NP-hard problem. To induce robustness to the changes in the problem parameters, we proposed our suboptimal solution in a receding horizon setting. Next, to compensate for the shortsightedness of the receding horizon approach, we added a new term, called the projected nodal importance, to the reward function as a measure to incorporate a notion of importance of the regions beyond the feasible solution set of the receding horizon optimization problem. Finally, we discuss how our suboptimal solution can be implemented in a decentralized manner. Our

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