Exact time evolution in harmonic quantum Brownian motion

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We consider a particular (exactly soluble) model of the one discussed in a previous work. We show numerical results for the time evolution of the main dynamical quantities and compare them with analytical results.

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I. INTRODUCTION

In Ref. [1] (hereafter referred as I) we have obtained a generalized and exact form of the master and Langevin equations. In this work we consider a particular soluble model and numerical results related to such equations. We show that from the microscopic quantum-mechanical laws a sort of “irreversible” behavior emerges under the following conditions: a privileged initial condition (no correlation between the Brownian oscillator and the bath), a relevance criterion arising from the form of the Hamiltonian, which distinguishes the Brownian particle from the bath, and a natural extra-dynamical hypothesis in order to interpret the exact dynamical evolution. That is, to consider such a process during a time scale of observation smaller than the recurrence time and with a minimum resolution such that fluctuations cannot be seen (see figures of Sec. V).

In Sec. II we exactly solve the particular model, while in Secs. III and IV we use this solution in order to obtain the coefficients of the master and Langevin equations. Sec. V contains the numerical results.

II. THE EXACT SOLUTION OF THE MODEL

Let $h$ be the following Hamiltonian,

$$h = \Omega \langle \Omega | + \sum_{n=1}^{N} \omega_n \langle \omega_n | + \sum_{n=1}^{N} g_n (|\Omega \rangle \langle \omega_n | + |\omega_n \rangle \langle \Omega |) .$$

The eigenvalue problem, $h |\alpha_\nu \rangle = \alpha_\nu |\alpha_\nu \rangle$ , is reduced to the algebraic system

$$\Omega \langle \Omega |\alpha_\nu \rangle + \sum_{n=1}^{N} g_n \langle \omega_n |\alpha_\nu \rangle = \alpha_\nu \langle \Omega |\alpha_\nu \rangle ,$$

$$g_n \langle \Omega |\alpha_\nu \rangle + \omega_n \langle \omega_n |\alpha_\nu \rangle = \alpha_\nu \langle \omega_n |\alpha_\nu \rangle .$$

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From the second of Eqs. (2) we have
\[
\langle \omega_n | \alpha_\nu \rangle = \frac{g_n}{\alpha_\nu - \omega_n} \langle \Omega | \alpha_\nu \rangle ,
\]
for \( \alpha_\nu \neq \omega_n, \forall \nu, n \). Replacing Eq. (3) in the first equation of (2) we obtain the secular equation \( \alpha_\nu - \Omega = N \sum_{n=1}^{N} \frac{g_n^2}{\alpha_\nu - \omega_n} \). Finally, by using Eq. (3) and the completeness relation \( |\Omega \rangle \langle \Omega| + \sum_{n=1}^{N} |\omega_n \rangle \langle \omega_n| = I \), we have
\[
|\langle \Omega | \alpha_\nu \rangle|^2 = \frac{1}{1 + \sum_{n=1}^{N} \left( \frac{g_n}{\alpha_\nu - \omega_n} \right)^2},
\]
where we have taken into account that \( \langle \alpha_\nu | \alpha_\nu \rangle = 1 \).

III. THE EXACT MASTER EQUATION

As was showed in I the general solution of the master equation is given by
\( (n = 0, ..., N) \langle N_n(t) \rangle = \sum_n P_{nm}(t) \langle N_m(0) \rangle \), which in the particular model given by Eq. (1) reads
\[
\langle N_\Omega(t) \rangle = P_{\Omega \Omega}(t) \langle N_\Omega(0) \rangle + \sum_{n=1}^{N} P_{\Omega n}(t) \langle N_n(0) \rangle ,
\]
\[
\langle N_n(t) \rangle = P_{n \Omega}(t) \langle N_\Omega(0) \rangle + \sum_{m=1}^{N} P_{nm}(t) \langle N_m(0) \rangle .
\]
From Eq. (13) of I we have
\[
P_{nm}(t) = \sum_{\mu,\nu=0}^{N} \frac{e^{-i(\alpha_\mu - \alpha_\nu)t}}{\langle \Omega | \alpha_\mu \rangle} \langle \alpha_\mu | \psi_n \rangle \langle \alpha_\nu | \psi_n \rangle \langle \psi_m | \alpha_\mu \rangle \langle \psi_m | \alpha_\nu \rangle .
\]
By taking into account the exact solution of the model [Eqs. (3) and (4)] these probabilities are given by
\[
P_{\Omega \Omega}(t) = 2 \sum_{\mu,\nu=0}^{N} \frac{\cos \left[ (\alpha_\mu - \alpha_\nu) t \right]}{|\langle \Omega | \alpha_\mu \rangle|^2 |\langle \Omega | \alpha_\nu \rangle|^2} + \sum_{\nu=0}^{N} \frac{1}{|\langle \Omega | \alpha_\nu \rangle|^4},
\]
\[
P_{\Omega n}(t) = P_{n \Omega}(t) = 2 \sum_{\mu,\nu=0}^{N} \frac{g_n^2 \cos \left[ (\alpha_\mu - \alpha_\nu) t \right]}{|\langle \Omega | \alpha_\mu \rangle|^2 |\langle \Omega | \alpha_\nu \rangle|^2 (\alpha_\mu - \omega_n)(\alpha_\nu - \omega_n)},
\]
The coefficients result to be

\[ X(t) = \sum_{\nu=1}^{N} \langle \Omega \rangle_{\nu}^2 \left( \frac{\sin(\alpha_{\nu} t)}{M} + f(\nu) \right) \]

which together with \( \langle \text{A}_0(\nu) \rangle = \sum_{\nu=0}^{\infty} \langle \Omega \rangle_{\nu}^2 \left( \frac{\sin(\alpha_{\nu} t)}{M} + f(\nu) \right) \)

yields \( X(t) = \sum_{\nu=1}^{N} \langle \Omega \rangle_{\nu}^2 \left( \frac{\sin(\alpha_{\nu} t)}{M} + f(\nu) \right) \)

As it was shown in I the solution of the Langevin equation is

\[ P_n(t) = 2 \sum_{m=0}^{\infty} \frac{g_m^2}{\langle \Omega \rangle_{\nu}^2 (\langle \Omega \rangle_{\nu}^2 - \langle \Omega \rangle_{\nu}^2)} \frac{g_m^2}{\langle \Omega \rangle_{\nu}^2 (\langle \Omega \rangle_{\nu}^2 - \langle \Omega \rangle_{\nu}^2)} \]

where \( \{\langle \Omega \rangle_{\nu}^2\} = \{9, \langle \nu, \nu \rangle\} (\nu = 1, \ldots, N) \).
bath, which is in thermal equilibrium at temperature $\beta^{-1}$ at $t = 0$. The initial population for the Brownian oscillator is fixed to the unity and its natural frequency $\Omega = \beta^{-1}$. The frequencies of the bath oscillators are spaced with a constant step $A$ and centered around $\Omega$ according to $\omega_n = \Omega + A\left(n - \frac{N}{2}\right)$, where $A = \omega_{n+1} - \omega_n; \ n = 1, ..., N$. The coupling function is a Lorentzian given by $g_n = \frac{Aa^2}{a^2 + (\omega_n - \Omega)^2}$, where $a = \frac{A(N-2)}{2}$. A more detailed discussion about this choice can be found in Ref. [2].

All the figures below correspond to $N = 100$, $A = 0.018$, $\Omega = 1$.

A. Master equation

In Fig. 1 we have the time evolution of $\langle N_\Omega \rangle$ [given in Eq. (5)] departing from an initial population $\langle N_\Omega(0) \rangle = 1$. We can see that after a short non-exponential regime (known as Zeno period [3]) the decay profile fits a decreasing exponential until it reaches the equilibrium value $\langle N_\Omega(t \gg \Omega^{-1}, \gamma^{-1}) \rangle = \left(e^{\beta\Omega} - 1\right)^{-1} \approx 0.58$ (where the Brownian oscillator thermalizes with the bath).

It is the asymptotic value reached by $\langle N_\Omega \rangle$ for times smaller than the recurrence time $t_r \approx \frac{2\pi}{\min(\alpha_{n+1} - \alpha_n)} \approx 37,311$, which is several orders of magnitude greater than the oscillator period $\tau_\Omega = 2\pi/\Omega = 2\pi$ (see a detailed explanation in Ref. [2]). Fig. 2 shows the contributions to $\langle N_\Omega \rangle$ stemming from the survival probability $P_{\Omega\Omega}(t) = \left|\langle \Omega | e^{-i\hbar t} | \Omega \rangle \right|^2$ of the initially prepared unstable state $|\Omega\rangle$, which reaches for $t < t_r$ a vanishing asymptotic value, and the contribution coming from the bath $\sum_n P_{\Omega n}(t) \langle N_n(0) \rangle$, which provides the equilibrium value to $\langle N_\Omega \rangle$.

B. Langevin equation

In Fig. 3 we show the damped oscillations of the Brownian particle comparing it with the reconstruction and decay of $\langle N_\Omega \rangle$ in the peak around the
recurrence time. Fig. 4 gives the time behavior of the “damping” coefficient \( \Gamma(t) \) re-scaled with respect to its asymptotic value \( \gamma \) obtained in I through second order perturbation theory. It can be shown that \( \Gamma(t) \) essentially follows \(-2\text{Re}\frac{\Omega_{n\Omega}}{\Omega_{n\Omega}}\) at all times. We firstly see that \( \Gamma(t) \) grows from 0 to \( \gamma \), corresponding to the Zeno regime. Secondly, it oscillates around \( \Gamma = \gamma \) as a consequence of the presence of fluctuations which modulates the exponential decay. After that, the amplitude of these oscillations increases in time because of the relative variations between fluctuations and the exponential decay \((-2\text{Re}\frac{\Omega_{n\Omega}}{\Omega_{n\Omega}})\) become more important when the system reaches equilibrium. Finally, we see the effect of recurrences also in \( \Gamma(t) \). It is interesting to stress the fact that \( \Gamma(t) \) allows us to visualize the presence of fluctuations which, at the same time scale, are hidden in the observable macroscopic profiles.

[1] E.T. Garcia Alvarez and F.H. Gaioli, Exact derivation of the master and Langevin equations for harmonic quantum Brownian motion, Physica A 257 (1998, in press).

[2] F.H. Gaioli, E.T. Garcia Alvarez, and J. Guevara, Int. J. Theor. Phys. 36 (1997) 2167.

[3] B. Misra and E.C.G. Sudarshan, J. Math. Phys. 18 (1977) 756.
Figure captions:

1. Population of the Brownian oscillator vs. $t$
2. Survival probability vs. $t$
3. Mean position of the Brownian oscillator vs. $t$
4. Damping factor of the Langevin equation vs. $t$
Figure 1
Figure 2
Figure 3
Figure 4