Analysis of thermoelastic damping in the clamped-free microbeam with linearly tapered circular cross-section

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Abstract. An accurate estimation of thermoelastic damping (TED) is of significance and importance in the design for the microbeam resonators with variable cross-section operated in vacuum. TED determines the up limit of quality factor (Q-factor) used for evaluating the microresonator performance. However, in the past, the previous works focus on the TED modelling of uniform microbeam resonators with rectangular cross-section. There is lack of works aiming at the TED modelling of non-uniform microbeam resonators with circular cross-section. In addition, the previous TED model developed by Zener for uniformly circular cross-section beams is not suitable for calculating TED values of the non-uniform beams. Therefore, the TED model for clamped–free microbeam resonators with linearly tapered circular cross-section is meaningful to be derived in this work. The radius and length dependences of TED are investigated. The results of finite element method (FEM) model are calculated for purpose of comparison. Results indicate that for large aspect ratio ($l/r_0$), TED values of linearly tapered microbeams with circular cross-section are larger than those of uniform microbeams, while for small aspect ratio ($l/r_0$), TED values of linearly tapered microbeams are lower than those of uniform microbeams. Moreover, the maximum values of TED called Debye peaks in linearly tapered microbeams are lower than those of uniform microbeams.

1. Introduction

Due to the enhanced properties, non-uniformly variable microbeam resonators have been considered in real applications [1-2]. Quality factor (Q-factor) is an important parameter used for evaluating the microresonator performance. It is well-known that the up limit of $Q$-factor is determined by thermoelastic damping (TED). Therefore, the accurate estimation of TED is significant and important in the design for the microbeam resonators with variable cross-section operated in vacuum. TED is a fundamental mechanism of energy dissipation and is impossible to be eliminated completely by improved fabrication.

Zener [3] firstly developed the theory of TED in the uniform beam with circular cross-section. The TED model derived by Zener is expressed as follow

$$Q^{-1}_{Zener} = \frac{ET\alpha^2}{C} \frac{\omega \tau}{1 + (\omega \tau)^2}, \quad \tau = 0.295 \frac{r_0^2 C}{\kappa}$$

(1)

where $r_0$ is beam radius. The other symbols are as follows: $T$ - equilibrium temperature, $E$ –Young’s elastic modulus, $\alpha$ - the coefficient of thermal expansion, $C$ - the specific heat, $\omega$ - the vibrational frequency, $\tau$ - the thermal relaxation time, and $\kappa$ - the coefficient of thermal conductivity. Zener’s model is a popular model utilized to compute TED values of uniform microbeams. However, Zener’s model cannot be applied for non-uniform microbeams directly.
With the rapid development of manufacturing, the micro- and nano devices with circular cross-section have been produced, such as micropillar resonator [4] and cone-shaped nano wire resonator [5]. In 2010, Tunvir et al. investigated the TED behaviors of hollow and circular cross-section microbeam resonators [6]. In 2016, considering three-dimensional heat conduction, Li et al. derived an analytical model for TED in microring resonators with circular cross-section [7]. However, Tunvir [8] and Zhou et al. [9] all pointed out a fact that Zener’s model is not suitable for non-uniform microbeam resonators. In this study, a TED model is developed for microbeam resonators with linearly tapered circular cross-section, which is a typical beam with continuous variable cross-section. TED behaviors are investigated for beams operated in the fundamental mode under the boundary condition of clamped-free.

2. TED model for clamped-free microbeam with linearly tapered cross-section

Consider two types of clamped-free microbeam resonators with circular cross-section as shown in figure 1. Figure 1(a) exhibits a uniform beam with constant radius \( r_0 \). The beam length is denoted by \( l \). Figure 1(b) exhibits a non-uniform beam with linearly tapered radius. The radii of beam ends are \( r_0 \) and \( 0.5r_0 \), accordingly. Hence the function of radius in the tapered circular cross-section beam is

\[
r(x) = r_0 \left(1 - \frac{x}{2l}\right)
\]

(2)

\( y \)
\( x \)
\( r_0 \)
\( l \)
\( (a) \)

\( y \)
\( x \)
\( r_0 \)
\( l \)
\( (b) \)

**Figure 1.** Schematic drawings of clamped-free microbeams with circular cross-section: (a) uniform microbeam and (b) linearly tapered microbeam.

The governing equation of thermoelectricity with two-dimensional heat conduction can be expressed in the cylindrical coordinate system as follows

\[
\frac{\partial \theta(r, \varphi, t)}{\partial t} = \chi \nabla^2 \theta(r, \varphi, t) + \frac{\Delta_E}{\alpha} r \sin(\varphi) \frac{\partial}{\partial t} \left( \frac{d^2 w(x, t)}{dx^2} \right)^2
\]

(3)

where \( \nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial \varphi^2} \), \( w(x, t) \) is the transverse deflection function and \( \theta(r, \varphi) \) is the fluctuating temperature with respect to \( T \). \( \chi = \kappa/C \) is the thermal diffusivity, and \( \Delta_E \) is equal to \( \frac{E \sigma^2 T}{C} \).

The beam resonator works in the resonant mode, hence the response functions can be written in the following forms.
\[ w(x,t) = Y(x)e^{i\omega t} \quad \text{and} \quad \theta(r,\varphi,t) = \Theta(r,\varphi)e^{i\omega t} \]  \hspace{1cm} (4)

Substituting equation (4) into equation (3) yields

\[ i\omega \frac{\partial \theta(r,\varphi)}{\partial t} = \chi T^2 \theta(r,\varphi) + i\omega \frac{\Delta E}{\alpha} \cdot r \sin(\varphi) \left( \frac{d^2Y(x)}{dx^2} \right)^2 \]  \hspace{1cm} (5)

In the paper [3], \( \Theta(r,\varphi) \) is obtained in the form of thermal modes summation. Note that thermoelastic energy dissipation is dominated by the first-order thermal mode. Thereby, \( \Theta(r,\varphi) \) of clamped-free beam with linearly tapered circular cross-section is reasonable to expressed by the first-order thermal mode. In addition, assume that the beam surfaces are adiabatic, i.e., \( \partial \Theta/\partial r = 0 \) at \( r_0 = 0 \). Finally, one can obtain

\[ \Theta(r,\varphi) = \frac{2\alpha \Delta E}{\alpha} \sin(\varphi) (\gamma_X(x)) \frac{i\omega r}{1 + i\omega r^2} \frac{1}{1} J_1 \left( \frac{a - r}{r_0} \right) \]  \hspace{1cm} (6)

where \( \gamma_X(x) = \frac{d^2Y(x)}{dx^2} \), \( a = 1.841 \) and \( J_1 \) is first order Bessel function of first kind.

The lost energy dissipation in a period for the beam dented by \( \Delta W \) is computed by [7]

\[ \Delta W = \int_0^x \int_0^{\gamma(x)} r E \sin(\varphi) \gamma_Y(x) \text{Im}(\alpha \partial \theta(r,\varphi)) \cdot r d\varphi dx dr = 2\pi^2 E \Delta E \int_0^x \frac{\omega \gamma(x)}{1 + [\omega \gamma(x)]} r^4(x)(\gamma_X(x))^2 dx \]  \hspace{1cm} (7)

where \( \text{Im} \) represents the imaginary part.

The maximum stored energy in a period denoted by \( W \) is calculated by [7]

\[ \Delta W = \frac{E}{2} \int_0^x \int_0^{\gamma(x)} r^2 (\sin(\varphi))^2 (\gamma_X(x))^2 dx dr = \frac{E \epsilon}{8} \int_0^x r^4(x)(\gamma_X(x))^2 dx \]  \hspace{1cm} (8)

Basing on the energy definition [3], the TED model for microbeam with linearly tapered circular cross-section is expressed as

\[ Q_{TED}^{-1} = \frac{1}{2\pi} \frac{\Delta W}{W} = \Delta E \int_0^x \frac{\omega \gamma(x)}{1 + [\omega \gamma(x)]} r^4(x)(\gamma_X(x))^2 dx \]  \hspace{1cm} (9)

Substituting \( \gamma(x) = 0.295 r^2(\gamma(x)) \) into equation (9) yields

\[ Q_{TED}^{-1} = \Delta E \int_0^x \frac{0.295 \omega \gamma(x)}{\gamma^2 + (0.295 \omega)^2} r^4(x)(\gamma_X(x))^2 dx \]  \hspace{1cm} (10)

Obviously, to calculate TED needs the solution of \( \gamma(x) \). In the section, the solution of \( \gamma(x) \) will be obtained by the modified Adomian decomposition method [10].

3. Modal parameters of clamped-free microbeam with linearly tapered cross-section

The governing equation of transverse motion for the clamped-free microbeam with linearly tapered cross-section is written as

\[ \frac{d^4Y(X)}{DX^4} = \frac{4}{(1 - 0.5X)} \frac{d^3Y(X)}{DX^3} + \frac{3}{(1 - 0.5X)^2} \frac{d^2Y(X)}{DX^2} - \frac{5}{(1 - 0.5X)^2} Y(X) = 0 \]  \hspace{1cm} (11)
where $X = \frac{x}{l}$, $\xi = \omega^2 \frac{4\rho l^4}{E_0^2}$, and $\rho$ is the material density.

Utilizing the modified Adomian decomposition method [10], the solution of $Y(X)$ can be obtained in the form of finite series with respect to $X$. For instance, the first-order modal shape of $Y(X)$ can be expressed as follows

$$Y(X) = 0.1022638611e-12X^{49}+0.1989050793e-12X^{48}+0.3865893892e-12X^{47}+0.7507889985e-12X^{46}+0.1456907813e-11X^{45}+0.3953395823e-10X^{44}+0.7626069969e-10X^{43}+0.1058895220e-10X^{42}+0.1969281595e-9X^{41}+0.3827150218e-9X^{40}+0.5432438963e-8X^{39}+0.1042321647e-7X^{38}+0.1996752236e-7X^{37}+0.381868077e-7X^{36}+0.728963358e-7X^{35}+0.1388846504e-6X^{34}+0.2640448242e-6X^{33}+0.500843115e-5X^{32}+0.9476315985e-5X^{31}+0.1788100046e-4X^{30}+0.336325972e-4X^{29}+0.637075898e-4X^{28}+0.1178502103e-5X^{27}+0.219301550e-5X^{26}+0.4062659145e-5X^{25}+0.7488639674e-5X^{24}+0.1372594920e-4X^{23}+0.2499749776e-4X^{22}+0.451920013e-4X^{21}+0.8100999560e-4X^{20}+0.1437800434e-3X^{19}+0.2521896312e-3X^{18}+0.4360456495e-3X^{17}+0.7405783759e-3X^{16}+0.1228820204e-2X^{15}+0.1973276265e-2X^{14}+0.3003643745e-2X^{13}+0.4065446810e-2X^{12}+0.585269830e-2X^{11}+0.873171909e-2X^{10}+0.1159828913e0X^{9}+0.713105217e-1X^{8}+0.1786894783e0X^{5}+0.2945628976e3; \quad (12)
$$

The natural frequency $\omega$ can be calculated by

$$\omega = \xi \frac{r_0}{l^2} \sqrt{\frac{E}{4\rho}} \quad (13)$$

where $\xi = 4.6252, 19.5476, 48.5789…$

Finally, inserting equation (12) into equation (10) and integrating the equation with respect to $x$ can get TED values of interest.

4. Validation and discussions

In this section, the length and radius dependence of TED behaviours are examined. The values of properties of silicon in simulation, which is a popular material used in MEMS/NEMS area, are listed as follows: $T = 300K$, $E = 160$ GPa, $\kappa = 120$ W·m⁻¹·K⁻¹, $\chi = 7.4033\times10^{-5}$ m²/s, $\rho = 2330$ Kg·m⁻³, $\alpha = 2.6\times10^{6}$ K⁻¹, and $C = 1.6\times10^{6}$ J·m⁻３·K⁻¹. In fact, TED in the fundamental mode is of interest [6-9]. The results of the present model for clamped-free microbeam with linearly tapered cross-section are compared with those of FEM in ANSYS. The element of Solid226 is utilized in the coupled thermal-structural component to calculate TED values.

Figures 2 and 3 show the variation of TED against the beam length in the cases of $r_0 = 5 \mu m$ and $10 \mu m$, respectively. The range of beam length is considered from 50 $\mu m$ to 500 $\mu m$. The results of Zener’s model for uniform beams are also plotted in the figures. As observed from figure 2 and 3, there exists only one peak damping called Debye peak in each TED curve. However, Debye peak of linearly tapered beam is smaller than that of uniform beam. In addition, the value relationship of the two models varies with the beam length or aspect ratio ($l/r_0$) increasing. For instance, in the case of $r_0 = 10 \mu m$ shown in figure 3, TED values of the present model are lower than those of Zener’s model below approximately 250 $\mu m$; TED values of the present model are greater than those of Zener’s model above 250 $\mu m$. The FEM results are shown for comparison in figure 2 and 3. As expected, the differences between the results of the present model and FEM model are negligible, indicating that the present model is valid for calculation of TED in linearly tapered microbeams.

Figure 4 exhibits the variation of TED against the $r_0$ in the cases of $l = 100, 200, 500$, and $1000 \mu m$. Obviously, Debye peaks of the present model and Zener’s model retain constant in these cases. As mentioned previously, Debye peak of uniform beam is larger than that of linearly tapered beam. In fact, Debye peak of uniform beam is equal to $0.5\Delta\epsilon$, while Debye peak of linearly tapered beam is equal to $0.68\epsilon$.
approximately $0.489\Delta \varepsilon$. In order to achieve a larger value of $Q$-factor by avoiding the large TED value, the structural parameters of the tapered beam should be selected carefully.

Figure 2. Variation of TED against the beam length with fixed $r_0 = 5 \, \mu\text{m}$.

Figure 3. Variation of TED against the beam length with fixed $r_0 = 10 \, \mu\text{m}$.
Figure 4. Variation of TED against the $r_0$ in the cases of $l = 100$, 200, 500, and 1000 $\mu$m.

Figure 5 shows FEM results of a representative beam with $l = 100$ $\mu$m and $r_0 = 10$ $\mu$m: (a) FEM element model and (b) distribution of temperature imaginary part. As shown in figure 5(b), the maximum value of temperature imaginary part occurs in the clamped end of the beam. It can be found that the expanded regions of the clamped-free microbeam with linearly tapered circular cross-section lose temperature while the compression regions produce temperature. The process of thermoelastic damping right causes this interesting phenomenon.

(a)  
(b)

Figure 5. FEM results of a representative beam with $l = 100$ $\mu$m and $r_0 = 10$ $\mu$m: (a) FEM element model and (b) distribution of temperature imaginary part.

5. Conclusions

In this work, the analysis of thermoelastic damping (TED) in the clamped-free microbeam with linearly tapered circular cross-section are investigated. The results of the present model are consistent with those of FEM model. To achieve a high value of quality factor, the structural parameters of the
beam with linearly tapered circular cross-section should be selected carefully. The present model may be useful in the design of non-uniform beam resonators.

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References
[1] Mohanty P, Harrington D A, Ekinci K L, et al 2002 Phys. Rev. B 66(8) 085416
[2] Matova S P, Renaud M, Jambunathan M, et al 2013 Smart Mater. Struct 22(7) 075015
[3] Zener C 1938 Phys. Rev 53(1) 90-99
[4] Kahl M, Thomay T, Kohnle V, et al 2007 Nano. Lett 7 (9) 2897-2900
[5] Yeo I, de Assis P L, Gloppe A, et al 2014 Nat. Nanotechnol 9(2) 106-110
[6] Tunvir K, Ru C Q and Miaduchowski A 2010 Physica E 42(9) 2341-2352
[7] Li P, Fang Y M and Zhang J R 2016 J. Sound Vib 361 341-354
[8] Tunvir K 2013 Microsyst. Technol 19(5) 721-731
[9] Zhou H Y, Li P and Zuo W L 2016 13th IEEE International Conference on Mechatronics and Automation 1590-1595
[10] Hsu J C, Lai H Y and Chen C K 2008 J. Sound Vib 318 965-981