MAPPING COALGEBRAS II
OPERADS

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Abstract. In this article, we describe how coalgebraic structures on operads induce algebraic structures on their categories of algebras and coalgebras.

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Introduction

This is the second article of a series about enrichment of categories of algebras or coalgebras by categories of coalgebras. The first one dealt with algebras over a monad and coalgebras over a comonad; this one deals with algebras and coalgebras over an operad.

Let us fix a ground symmetric monoidal category $E$. The 2-category of $E$-enriched coloured operads has the canonical structure of a symmetric monoidal 2-category (monoidal context) given by the Hadamard tensor product $\mathcal{P} \otimes \mathcal{H} \mathcal{Q}$ defined as

$$\text{Ob}(\mathcal{P} \otimes \mathcal{H} \mathcal{Q}) = \text{Ob}(\mathcal{P}) \times \text{Ob}(\mathcal{Q})$$

$$(\mathcal{P} \otimes \mathcal{H} \mathcal{Q})( (o_1, o'_1), \ldots, (o_n, o'_n); (o, o')) = \mathcal{P}(o_1, \ldots, o_n; o) \otimes \mathcal{Q}(o'_1, \ldots, o'_n; o').$$

The comonoids for this tensor product are called Hopf operads.

Besides, for any symmetric monoidal enriched category $\mathcal{C}$, one can produce an enriched operad $\mathcal{E}\text{nd}(\mathcal{C})$, whose colours are the objects of $\mathcal{C}$ and so that for any such objects $x_1, \ldots, x_n, x$, then

$$\mathcal{E}\text{nd}(\mathcal{C})(x_1, \ldots, x_n; x) = \mathcal{C}(x_1 \otimes \cdots \otimes x_n, x)$$

Such a construction is part of a monoidal strict 2-functor (strong context functor) from the 2-category of symmetric monoidal enriched categories and lax functors to the 2-category of enriched operads.

Actually, the enriched operad $\mathcal{E}\text{nd}(\mathcal{C})$ has the canonical structure of a $E_{\infty}$-monoid for the Hadamard tensor product. Using the fact that the Yoneda 2-functor $\mathcal{D}^{op} \times \mathcal{D} \rightarrow \text{Cats}$ for a monoidal 2-category $\mathcal{D}$ is lax monoidal, one gets the fact that for any cocommutative Hopf enriched operad $\mathcal{Q}$, the category of $\mathcal{Q}$-algebras in $\mathcal{C}$ and the category of the category $\mathcal{Q}$-coalgebras in $\mathcal{C}$

$$\text{Alg}_C(\mathcal{Q}) = \text{Operad}_E(\mathcal{Q}, \mathcal{E}\text{nd}(\mathcal{C}))$$

$$\text{Cog}_C(\mathcal{Q}) = \text{Operad}_E(\mathcal{Q}, \mathcal{E}\text{nd}(\mathcal{C}^{op}))^{op}$$

are symmetric monoidal categories. More generally, one has the following result.

Date: August 31, 2022.
Theorem (1). Let $B, B'$ be two categorical operads. Then, the lax context functor

$$\text{Alg}_E(-) : \text{Operad}_{E, \text{small}}^\text{op} \times \text{CMon}_{lax}(\text{Cat}_E) \to \text{Cats}_I$$

induces a lax context functor that is natural with respect to $B, B'$:

$$\text{ALG}_{\text{Operad}_{E, \text{small}}^\text{op}}(B)_{\text{lax}} \times \text{ALG}_{\text{CMon}_{lax}(\text{Cat}_E)}(B')_{\text{lax}} \to \text{ALG}_{\text{Cats}_I}(B \times B')_{\text{lax}}.$$ 

This result has further consequences. For instance, if $P$ is an enriched operad that has the structure of a left comodule over a Hopf enriched operad $Q$ then the category $\text{Alg}_E(P)$ of $P$-algebras in $\mathcal{C}$ is tensored over that of $Q$-algebras. Similarly the category $\text{Cog}_E(P)$ of $P$-coalgebras in $\mathcal{C}$ is tensored over that of $Q$-coalgebras.

Furthermore, one can consider the case where a symmetric monoidal enriched category $\mathcal{C}$ is cotensored in some sense over another symmetric monoidal $\mathcal{D}^{\text{op}}$. An example of this situation is provided by the ground symmetric monoidal category $E$ if it is closed. Then the $E$-enriched category $\mathcal{E}$ associated to $E$ is cotensored over $\mathcal{E}^{\text{op}}$ through the internal hom

$$E \times E^{\text{op}} \to E$$

$$y, x \mapsto [x, y].$$

In such a context, for any Hopf enriched operad $Q$ and any enriched operad $P$ equipped with the structure of a right $Q$-comodule, then the category $\text{Alg}_E(P)$ of $P$-algebras in $\mathcal{C}$ is cotensored over the category $\text{Cog}_D(Q)$ of $Q$-coalgebras in $\mathcal{D}$.

Finally, in many contexts, the cotensorisation functor

$$\text{Alg}_E(P) \times \text{Cog}_D(Q)^{\text{op}} \to \text{Alg}_E(P)$$

has a left adjoint with respects to both variables. Then the category $\text{Alg}_E(P)$ is not only cotensored over $\text{Cog}_D(Q)$ but also tensored and enriched over this monoidal category. One has actually the same phenomenon mutatis mutandis for the tensorisation functor mentioned above

$$\text{Cog}_E(Q) \times \text{Cog}_E(P) \to \text{Cog}_E(P).$$

Universes. As in [Grib], we consider three universes $\mathcal{U} \in \mathcal{V} \subset \mathcal{W}$. A set is called $\mathcal{U}$-small if it is an element of $\mathcal{U}$ and it is called $\mathcal{U}$-large if it is a subset of $\mathcal{U}$. The notion of smallness and largeness are defined similarly for the other universes. We thus have a hierarchy of sizes of sets

$$\mathcal{U} \text{ small sets } \subset \mathcal{U} \text{ large sets } \subset \mathcal{V} \text{ small sets } \subset \mathcal{V} \text{ large sets } \subset \mathcal{W} \text{ small sets } \subset \mathcal{W} \text{ large sets}.$$ 

Besides, a $\mathcal{U}$-category is a category whose set of objects is $\mathcal{U}$-large and whose hom sets are all $\mathcal{U}$-small. Such a $\mathcal{U}$-category is called $\mathcal{U}$-small if its set of object is $\mathcal{U}$-small. We have similar notions for $\mathcal{V}$ and $\mathcal{W}$.

Finally, we will use the following aliases:

- a set, also called small set will be a $\mathcal{W}$-small set;
- a large set will be a $\mathcal{W}$-large set;
- a category will be a $\mathcal{W}$-category;
- a small category will be a $\mathcal{W}$-small category.

Some notations.

- For any natural integer $n$, the set of permutations of the set

$$\{1, \ldots, n\}$$

will be denoted $S_n$.

- Let $S$ be the groupoid of permutations whose objects are integers $n \in \mathbb{N}$ and whose morphisms are:

$$\begin{cases}
\text{hom}_S(n, m) = \emptyset & \text{if } n \neq m \\
\text{hom}_S(n, n) = S_n & \text{otherwise}.
\end{cases}$$

- For any natural integer $n$, we denote $[n]$ the poset $0 < 1 < \cdots < n$.

- For any natural integer $n$, any permutation $\sigma \in S_n$ and any category $\mathcal{C}$, we denote $\sigma^*$ the following functor

$$\mathcal{C}^n = \text{Fun}(\mathbb{N}, \mathcal{C}) \xrightarrow{\sigma^*} \text{Fun}(\mathbb{N}, \mathcal{C}) = \mathcal{C}^n.$$
For a monad \( M \) on a category \( E \), the induced monadic adjunction relating \( M \)-algebras to \( E \) will be denoted \( T_M \dashv U_M \). Similarly, for a comonad \( Q \), the induced comonadic adjunction relating \( Q \)-coalgebras to \( E \) will be denoted \( U_Q \dashv L_M \).

Let \( X \) be an object of a monoidal category \( E \). Then we will call an element of \( X \) a morphism from the monoidal unit to \( X \). Then, the notation \( x \in X \) is a substitute for \( x \in \text{hom}_E(1, X) \).

Let \( X \) be a finite set and, for any object \( x \in X \), let \( V_x \) be an object of a symmetric monoidal category \((E, \otimes, 1)\). Then we will denote

\[
\bigotimes_{x \in X} V_x := \left( \prod_{\phi: \{1, \ldots, n\} \to X} V_{\phi(1)} \otimes \cdots \otimes V_{\phi(n)} \right)_{S_n}.
\]

1. The monoidal context of Pseudo-commutative monoids

Let \( C \) be a \( W \)-small monoidal context.

1.1. **Pseudo commutative monoids within pseudo commutative monoids.** Let \( Q \) be a categorical monochromatic operad so that for any natural integer \( n \), \( Q(n) \) is a contractible groupoid. Recall that the 2-category \( \text{ALG}_C(Q)_{lax} \) of \( Q \)-algebras and lax morphisms has the structure of a monoidal context. The tensor product of two algebras \( A, A' \) is the object \( A \otimes A' \in C \) equipped with the morphism of categorical operads

\[
Q \to Q \times Q \xrightarrow{A \otimes A'} \text{End}(C) \times \text{End}(C) \to \text{End}(C)
\]

where the last map is induced by the tensor product of \( C \). In particular, the map \( (A \otimes A')(q) \) is the composition

\[
(A \otimes A')^{\otimes n} \simeq A^{\otimes n} \otimes A^{\otimes n} \xrightarrow{A(q) \otimes A'(q)} A \otimes A'
\]

for any \( q \in Q(n) \).

**Lemma 1.** Let \( A \) be an \( Q \)-algebra and let \( q \in Q(n) \) be an operation. Then the map

\[
A(q) : A^{\otimes n} \to A
\]

has the canonical structure of a strong morphism of \( Q \)-algebras. Then,

- for any morphism \( \phi : q \to q' \) in \( Q(n) \), the map
  \[
  A(\phi) : A(q) \to A(q')
  \]
  is an invertible 2-morphism in the 2-category of \( Q \)-algebras and strong morphisms;
- for any \( q'' \in Q(m) \), the following diagram of \( Q \)-algebras and strong morphisms
  \[
  A^{\otimes n} \xrightarrow{A(q) \otimes A(q'')} A^{\otimes n+m-1} \xrightarrow{A(q'' \circ q)} A
  \]
  is strictly commutative;
- for any permutation \( \sigma \in S_n \), the following diagram of \( Q \)-algebras and strong morphisms
  \[
  A^{\otimes n} \xrightarrow{\sigma^*} A^{\otimes n} \xrightarrow{A(q)} A
  \]
  is strictly commutative;
- the strong morphism of \( Q \)-algebras \( A(1) : A \to A \) is the identity of \( A \) (and is thus a strict morphism).
Proof. Let us consider an element \( p \in Q(m) \). The action of \( p \) on the \( Q \)-algebra \( A^{\otimes n} \) is the map

\[
A^{\otimes n}(p) : (A^{\otimes n})^{\otimes m} \to (A^{\otimes m})^{\otimes n} \xrightarrow{\sigma^*} A^{\otimes n},
\]

where \( \sigma \) is the permutation

\[
\sigma(m+i+j+1) = n+i+j+1, \quad 0 \leq i < n, \quad 0 \leq j < m.
\]

One has a unique isomorphism

\[
\phi : p \otimes q^{\otimes m} \simeq (q \otimes p^{\otimes n})^\sigma
\]
in \( Q(n \ast m) \). We thus get a 2-isomorphism

\[
A(p) \circ A(q)^{\otimes n} = A(p \otimes q^{\otimes m}) \xrightarrow{A(\phi)} A((q \otimes p^{\otimes n})^\sigma) = A(q) \circ A(p)^{\otimes n} \circ \sigma^* = A(q) \circ A^{\otimes n}(p).
\]

The data of this 2-isomorphism \( A(\phi) \) for any operation \( p \) makes \( A(q) \) a strong morphism of \( Q \)-algebras. Indeed, the diagram commutates required by the definition of a strong morphism of \( Q \)-algebras just follow from the fact that the category \( Q(k) \) is a contractible groupoid for any natural integer \( k \).

The proofs of the remaining statements follow from the same fact that the category \( Q(k) \) is contractible groupoid for any natural integer \( k \). \( \square \)

Let \( f : A \to A' \) be a lax morphism of \( Q \)-algebras in the monoidal context \( C \). The lax structure of \( f \) is the data of a 2-morphism in \( C \)

\[
A'(q) \circ f^{\otimes n} \to f \circ A(q)
\]
for any \( s \in Q(n) \). At the same time, both maps \( A'(q) \circ f^{\otimes n} \) and \( f \circ A(q) \) have structures of a lax morphism of \( Q \)-algebras by the previous Lemma 1.

Lemma 2. For any lax morphism \( f : A \to A' \) of \( Q \)-algebras and any operation \( q \in Q(n) \), the map

\[
A'(q) \circ f^{\otimes n} \to f \circ A(q)
\]
is a 2-morphism in the 2-category \( \text{ALG}_{\mathbb{C}}(Q)_{lax} \).

Proof. This is a consequence of the definition of a lax morphism. Precisely, for any operation \( p \in Q(m) \), the following diagram commutes

\[
\begin{array}{ccc}
A'(p) \circ (A'(q) \circ f^{\otimes n})^{\otimes m} & \to & A'(p) \circ (f \circ A'(q))^{\otimes m} \\
\downarrow & & \downarrow \\
A'(p) \circ A'(q)^{\otimes m} \circ f^{\otimes n+m} & \to & A'(p) \circ f^{\otimes m} \circ A(q)^{\otimes m} \\
\downarrow & & \downarrow \\
A'(p \otimes q^{\otimes m})^{\otimes m} & \to & f \circ A'(p \otimes q^{\otimes m}) \\
\downarrow & & \downarrow \\
A'((q \otimes p^{\otimes n})^\sigma)^{\otimes m} & \to & f \circ A((q \otimes p^{\otimes n})^\sigma) \\
\downarrow & & \downarrow \\
A'(q) \circ A'(p)^{\otimes n} \circ f^{\otimes n+m} \circ \sigma^* & \to & A'(q) \circ f^{\otimes n} \circ A(p)^{\otimes n} \circ \sigma^* \\
\downarrow & & \downarrow \\
(f \circ A'(q)) \circ A^{\otimes n}(p) & \to & (f \circ A(q)) \circ A^{\otimes n}(p)
\end{array}
\]

where \( \sigma \in S_{n+m} \) is the permutation defined in the proof of Lemma 1. \( \square \)

For any monoidal context \( D \), one has a canonical strict context functor

\[
\text{ALG}_{\mathbb{D}}(Q)_{lax} \to D.
\]

Taking \( D = \text{ALG}_{\mathbb{C}}(Q)_{lax} \), this gives a strict context functor

\[
\text{ALG}_{\mathbb{ALG}_{\mathbb{C}}(Q)_{lax}}(Q)_{lax} \to \text{ALG}_{\mathbb{C}}(Q)_{lax}.
\]
Moreover, for any strict context functor $D \to D'$, we get another strict context functor $\text{ALG}_D(Q)_{\text{lax}} \to \text{ALG}_{D'}(Q)_{\text{lax}}$. Taking the strict context functor to be the forgetful 2-functor

$$\text{ALG}_C(Q)_{\text{lax}} \to C,$$

we get another strict context functor

$$\text{ALG}_{\text{ALG}_C(Q)_{\text{lax}}}(Q)_{\text{lax}} \to \text{ALG}_C(Q)_{\text{lax}}.$$

**Proposition 1.** The two forgetful strict context functors described just above

$$\text{ALG}_{\text{ALG}_C(Q)_{\text{lax}}}(Q)_{\text{lax}} \rightleftharpoons \text{ALG}_C(Q)_{\text{lax}}.$$

have a canonical common section in the category of monoidal contexts and strict context functors.

**Proof.** Such a section sends

- a $Q$-algebra $A$ in $C$ to the $Q$-algebra in $\text{ALG}_C(Q)_{\text{strong}}$ (subset of $\text{ALG}_C(Q)_{\text{lax}}$) whose structural morphisms and 2-morphisms are described in Lemma 1. This same Lemma 1 ensures us that such maps do describe the structure of a $Q$-algebra;
- a lax morphism of $Q$-algebras $f : A \to A'$ to the lax morphism of $Q$-algebras in $Q$-algebras whose underlying morphism of $Q$-algebras is $f$ itself and whose lax structure is given by the lax structure of $f$ as in Lemma 2;
- finally any 2-morphism $a : f \to g$ between lax morphisms of $Q$-algebras in $C$ induces canonically a 2-morphism between their images in $Q$-algebras in $Q$-algebras.

One can check in a straightforward way that these constructions do define a strict context functor. □

**Corollary 1.** The restrictions of the section

$$\text{ALG}_C(Q)_{\text{lax}} \to \text{ALG}_{\text{ALG}_C(Q)_{\text{lax}}}(Q)_{\text{lax}}$$

from Proposition 1 to $\text{ALG}_C(Q)_{\text{strong}}$ factorises through

$$\text{ALG}_{\text{ALG}_C(Q)_{\text{lax}}}(Q)_{\text{strong}}.$$

Moreover, its restriction to $\text{ALG}_C(Q)_{\text{strict}}$ factorises through

$$\text{ALG}_{\text{ALG}_C(Q)_{\text{lax}}}(Q)_{\text{strict}}.$$

**Proof.** Straightforward with the definitions. □

**Remark 1.** The restriction of this 2-functor to $\text{ALG}_C(Q)_{\text{strict}}$ does not factorise in general through

$$\text{ALG}_{\text{ALG}_C(Q)_{\text{str}}}(Q)_{\text{str}}.$$

### 1.2. A model of pseudo-commutative monoids

Remember that a pseudo-commutative monoid in the monoidal context $C$ is an algebra over the categorical operad $E_{\infty,\text{cat}}$ with one colour and so that $E_{\infty,\text{cat}}(n)$ is the contractible groupoid whose objects are pairs $(t, \sigma)$ of a (equivalence class of) planar tree $t$ whose node have arity 0 or 2 and a permutation $\sigma \in S_n$.

Mac Lane’s coherence tells us that a pseudo-commutative monoid is concretely the data of an object $A$ equipped with morphisms $\gamma : A \otimes A \to A$ and $\eta : 1 \to A$ and natural transformations

$$\gamma \circ (\gamma \otimes \text{id}) \simeq \gamma \circ (\text{id} \otimes \gamma) \quad \gamma \circ \kappa(C, C) \simeq \gamma \quad \gamma \circ (\eta \otimes \text{id}) \simeq \text{id}_C \simeq \gamma \circ (\text{id} \otimes \eta)$$

called the associator, the commutator, the left unitor and the right unitor of $A$, and that satisfies coherence conditions (see [LG17]). A lax morphism between two pseudo-commutative monoids $A, B$ is the data of a morphism $f : A \to B$ and 2-morphisms

$$\gamma_B \circ (f \otimes f) \to f \circ \gamma_A \quad \eta_B \to f \circ \eta_A$$

called the associator, the commutator, the left unitor and the right unitor of $A$, and that satisfies coherence conditions. A strong morphism of pseudo-commutative monoids is a lax morphism whose structural 2-morphisms are 2-isomorphisms. A strict morphism of pseudo-commutative monoids is a lax morphism whose structural 2-morphisms are identities.
**Definition 1.** Let us denote

\[ \text{CMon}_{\text{lax}}(C) = \text{Alg}_C(E_{\infty, \text{cat}})_{\text{lax}} \]

the monoidal context of pseudo-commutative monoids whose objects are \( E_{\infty, \text{cat}} \)-algebras, whose morphism are lax \( E_{\infty, \text{cat}} \)-morphisms and whose 2-morphisms are \( E_{\infty, \text{cat}} \)-2-morphisms. Similarly, we denote

\[ \text{CMon}_{\text{oplax}}(C) = \text{Alg}_C(E_{\infty, \text{cat}})_{\text{oplax}}; \]
\[ \text{CMon}_{\text{strong}}(C) = \text{Alg}_C(E_{\infty, \text{cat}})_{\text{strong}}; \]
\[ \text{CMon}_{\text{strict}}(C) = \text{Alg}_C(E_{\infty, \text{cat}})_{\text{strict}}. \]

The structure of a monoidal context on the 2-category of pseudo-commutative monoids and lax morphisms \( \text{CMon}_{\text{lax}}(C) \) is given by underlying monoidal structure on \( C \); indeed, given two pseudo-commutative monoids \( A, A' \) their tensor product \( A \otimes A' \in C \) has the structure of a pseudo-commutative monoid given by the product

\[ (A \otimes A') \otimes (A \otimes A') \simeq (A \otimes A) \otimes (A' \otimes A') \rightarrow A \otimes A' \]

and the unit

\[ 1_C \simeq 1_C \otimes 1_C \rightarrow A \otimes A'. \]

2. The Hadamard tensor product of enriched coloured operads

Let \( (E, \otimes) \) be a \( \mathcal{U} \)-cocomplete symmetric monoidal \( \mathcal{U} \)-category. We assume that it is bilinear in the sense that the tensor product commutes with colimits in each variable.

We describe here the 2-category of coloured operads enriched in \( E \) and its Hadamard tensor product. We also describe monoidal properties of the Boardman–Vogt construction. The reader may refer to [GL19], [LV12] or the articles [BM06] and [BM03] for definitions of operads.

From now on, a small set or just a set is a \( \mathcal{U} \)-small set and a large set is a \( \mathcal{U} \)-large set.

2.1. Categories of trees.

**Definition 2.** Let \( C \) be a (possibly large) set called the set of colours. The inclusion \( n \mapsto \{1, \ldots, n\} \) of the groupoid of permutations \( S \) into the category of sets and functions gives us the following comma category

\[ S_C := S \downarrow C \]

whose objects \( c \in (n, \phi) \) are the data of a natural integer \( n \) and a function \( \phi : \{1, \ldots, n\} \rightarrow C \) (or equivalently a tuple of colours \((c_1, \ldots, c_n)\)) and whose morphisms from \( c \in (n, \phi) \) to \( c' \in (n, \psi) \) are permutations \( \sigma \in S_n \) so that \( \psi = \psi \circ \sigma \).

Notation. For any object \( c \) of \( S_C \), \( |c| \) will denote the length of \( c \) while \( c[i] \) will denote the \( i \)th colour of \( c \). For instance if \( c = (c_1, \ldots, c_n) \), then \( |c| = n \) and \( c[i] = c_i \). Moreover for any permutation \( \sigma \in S_n \), \( c^\sigma \) will denote the \( n \)-tuple

\[ c^\sigma = (c_{\sigma(1)}, \ldots, c_{\sigma(n)}). \]

For any \( c' = (c'_1, \ldots, c'_{m}) \in C^m \), let us define

\[ c \triangleleft c' := (c_1, \ldots, c_{i-1}, c'_i, c_{i+1}, \ldots, c_n); \]
\[ c \triangleright c' := (c_1, \ldots, c_n, c'_1, \ldots, c'_m). \]

**Definition 3.** [MW09] We denote \( \Omega \) the dendroidal category:

\( \triangleright \) its objects are trees;
\( \triangleright \) any tree induces a coloured operad in sets whose set of colours is the set of edges and whose operations are generated by vertices. Then a morphology of trees is a morphism of the induced coloured operads in sets. Any morphism \( f \) induces in particular a function \( \text{edge}(f) \) between the sets of edges.

Moreover, let \( \Omega_{\text{core}} \) the maximal subgroupoid of \( \Omega \) and let \( \Omega_{\text{act}} \) the subcategory of active morphisms, that is the subcategory of \( \Omega \) that contains all objects and all morphisms that sends leaves to leaves and the root to the root.
Remark 2. The morphisms of $\Omega^{\text{act}}$ are actually generated by isomorphisms, inner cofaces and codegeneracies.

**Definition 4.** Let $C$ be a set. The mapping $Y \mapsto \text{edge}(Y)$ induces a functor from $\Omega$ to $\text{Sets}$. This gives us the comma category $\Omega_{\downarrow} C = \Omega \downarrow C$.

that we call the $C$-coloured dendroidal category. The categories $\Omega^{\text{core}}_{\downarrow} C$ and $\Omega^{\text{act}}_{\downarrow} C$ are defined similarly.

### 2.2. Coloured symmetric sequences.

**Definition 5.** A coloured symmetric sequence $M$ is the data of a (possibly large) set of colours $\text{Ob}(M)$ and a left $\text{S}^{\text{op}} \times \text{Ob}(M)$-module in the category $E$, that is the data of objects $M(c ; c)$ for any tuple of colours $c$ and any colour $c$, together with maps $\sigma^* : M(c ; c) \to M(c^\sigma ; c)$ for any permutations $\sigma \in \text{S}_n$ (where $|c| = n$), so that $\sigma^* \circ \mu^* = (\mu \circ \sigma)^*$ , $\mu^* = \text{Id}$.

**Definition 6.** A morphism of coloured symmetric sequences from $M$ to $N$ is the data of a function $f(-) : \text{Ob}(M) \to \text{Ob}(N)$ and morphisms in $E$ $f(c ; c) : M(c ; c) \to N(c ; c)$ that commute with the action of the symmetric groups, that is $\sigma^* \circ f(c ; c) = f(c^\sigma ; c) \circ \sigma^*$.

**Definition 7.** The data of coloured symmetric sequences and their morphisms form a $V$-small category that we denote $\text{S-mod}$.

### 2.3. Enriched operads.

**Definition 8.** An enriched operad $P$ is the data of a coloured symmetric sequence $P$ together with maps $\gamma_i : \text{P}(c ; c) \otimes \text{P}(c' ; c[i]) \to \text{P}(c \triangleleft i ; c)$ and morphisms in $E$ $\eta_c : 1 \to \text{P}(c ; c)$; for any $c \in \text{Ob}(P), c, c' \in \text{S}(P)$ and $1 \leq i \leq |c|$, that satisfy associativity and unitality conditions and that are coherent with respect to the action of the symmetric groups (see for instance [LV12], [Grec]). If the set of colours $\text{Ob}(P)$ is large (resp. small), we often say that $P$ is a large (resp. small) enriched operad.

**Definition 9.** Given two enriched operads $P$ and $Q$, a morphism of enriched operads between them is a morphism of coloured symmetric sequences that commutes with the composition and the units.

**Definition 10.** Given two morphisms of enriched operads $f, g : P \to Q$, a 2-morphism $A$ between them is the data of elements $A(c) \in Q(f(c); g(c))$, $\forall c \in \text{Ob}(P)$; so that the following diagram commutes

\[
\begin{array}{ccc}
\text{P}(c ; c) & \xrightarrow{f} & \text{Q}(f(c); f(c)) \\
g \downarrow & & \downarrow \\
\text{Q}(g(c); g(c)) & \xrightarrow{g} & \text{Q}(f(c); g(c))
\end{array}
\]

for any $c, c \in \text{Ob}(M)^n \times \text{Ob}(M)$.

**Definition 11.** Let us denote $\text{Operad}_E$ the $V$-small strict 2-category of enriched operads and let us denote $\text{Operad}_{E, \text{small}}$ its full sub 2-category spanned by small enriched operads. Actually $\text{Operad}_{E, \text{small}}$ is a $U$-2-category in the sense that its set of objects is $U$-large and any of its mapping categories is $U$-small.
2.4. The Hadamard tensor product.

**Definition 12.** For any two coloured-symmetric sequences \( M \) and \( N \), the Hadamard tensor product \( M \otimes_H N \) is the symmetric sequence whose objects are

\[
\text{Ob}(M \otimes_H N) = \text{Ob}(M) \times \text{Ob}(N)
\]

and so that

\[
(M \otimes_H N)((c_1, c'_1), \ldots, (c_n, c'_n); (c, c')) = M(c_1, \ldots, c_n; c) \otimes N(c'_1, \ldots, c'_n; c).
\]

The right action of \( \sigma \in S_n \) is diagonal

\[
(M \otimes_H N)((c_1, d_1), \ldots, (c_n, d_n); (c, d))
\]

\[
\Downarrow \sigma \otimes \sigma
\]

\[
M(c; c) \otimes N(d; d)
\]

\[
\Downarrow \eta_\otimes \eta
\]

\[
M(c; c) \otimes N(d; d)
\]

\[
(M \otimes_H N)((c_1, d_1), \ldots, (c_n, d_n); (c, d))
\]

This defines the structure of a monoidal category on the category of coloured-symmetric sequences whose unit is the underlying symmetric sequence of the operad \( u\text{Com} \), whose associator, unitor and commutator are given by those of \( E \).

**Definition 13.** Let \( \mathcal{P} \) and \( \mathcal{Q} \) be two enriched operads. Then, their Hadamard tensor product \( \mathcal{P} \otimes_H \mathcal{Q} \) is the operad whose underlying symmetric sequence is the Hadamard tensor product of the underlying symmetric sequences of \( \mathcal{P} \) and \( \mathcal{Q} \). The operad structure is given by the following maps

\[
(\mathcal{P} \otimes_H \mathcal{Q})((c, d); (c, d)) \otimes (\mathcal{P} \otimes_H \mathcal{Q})((c', d'); (c[i], d[i])) \xrightarrow{1 \otimes 1} (\mathcal{P} \otimes_H \mathcal{Q})((c, d); (c, d))
\]

\[
\mathcal{P}(c; c) \otimes \mathcal{P}(c'; c[i]) \otimes \mathcal{Q}(d; d) \otimes \mathcal{Q}(d'; d[i]) \xrightarrow{\eta_\otimes \eta} (\mathcal{P} \otimes_H \mathcal{Q})((c, d); (c, d)).
\]

**Proposition 2.** The Hadamard tensor product defines the structure of a monoidal context on the 2-category of enriched operads that lifts that of coloured symmetric sequences, in the sense that the forgetful functor

\[
\text{sk(Operad}_E) \to \text{S-mod}
\]

is a strict monoidal functor.

**Proof.** It is straightforward to show that the Hadamard tensor product induces a monoidal structure on the category \( \text{sk(Operad}_E) \). It is then straightforward to see that it extends canonically to the 2-category \( \text{Operad}_E \). Indeed, given morphisms \( f, f' : \mathcal{P}_1 \to \mathcal{P}_2 \) and \( g, g' : \mathcal{Q}_1 \to \mathcal{Q}_2 \) and 2-morphisms \( A : f \to f' \) and \( B : g \to g' \) we obtain a 2-morphism \( A \otimes B : f \otimes g \to f' \otimes g' \) given by the elements

\[
A(c) \otimes B(d) \in (\mathcal{P}_2 \otimes_H \mathcal{Q}_2)(f \otimes_H g(c, d); f' \otimes_H g'(c, d)) = \mathcal{P}_2(f(c); f'(c)) \otimes \mathcal{Q}_2(g(d); g'(d)).
\]

for any \( (c, d) \in \text{Ob}(\mathcal{P}_1) \times \text{Ob}(\mathcal{Q}_1) \). \( \square \)

**Definition 14.** A Hopf operad is a comonoid in the category \( \text{sk(Operad}_E) \) with respect to the Hadamard tensor product. A comodule of such a comonoid is called a Hopf comodule.
Remark 3. Usually in the literature, a Hopf operad is a cocommutative comonoid in the category of enriched operads with respect to the Hadamard tensor product. The reason why we change a standard name is that many operads that we deal with are non cocommutative comonoids: for instance, if the ground category E is that of chain complexes, then the Boardman–Vogt constructions of a cocommutative Hopf operad is in general a non-cocommutative Hopf operad.

Remark 4. Hopf operads may be seen as operads enriched in coalgebras.

2.5. Operads with a fixed set of colours. Let us consider the following commutative diagram of monoidal categories and strict monoidal functors

\[ \text{sk(Operad}_E\text{)} \longrightarrow S\text{-mod} \]

\[ \text{Ob}(-) \downarrow \quad \downarrow \text{Ob}(-) \]

\[ \text{Set.} \]

Proposition 3. The two \( \text{Ob}(-) \) functors are symmetric monoidal fibrations in the sense that

\[ \triangleright \text{ they are strict symmetric monoidal functors;} \]

\[ \triangleright \text{ they are cartesian fibrations;} \]

\[ \triangleright \text{ the tensor product preserves cartesian liftings.} \]

Moreover, the forgetful functor from enriched operads to coloured symmetric sequences is a strict functor of symmetric monoidal fibrations in the sense that

\[ \triangleright \text{ it is a strict symmetric monoidal functor;} \]

\[ \triangleright \text{ it sends cartesian morphisms to cartesian morphisms.} \]

Proof. Let us consider a function \( \phi : C \to C' \) and an operad \( P \) whose set of colours is \( C' \). Then let \( \phi^*(P) \) be the \( C \)-coloured operad so that

\[ \phi^*(P)(c; c') = P(\phi(c); \phi(c')). \]

Then, the canonical morphism of enriched operads \( \phi^*(P) \to P \) above the function \( \phi \) is a cartesian lifting of \( \phi \).

Moreover, we have a canonical isomorphism

\[ \phi^*(P) \otimes_H \psi^*(Q) = (\phi \times \psi)^*(P \otimes_H Q) \]

for any operads \( P, Q \) and any functions \( \phi, \psi \) towards respectively \( \text{Ob}(P) \) and \( \text{Ob}(Q) \). Hence, the tensor product of two cartesian maps is cartesian.

The same constructions and arguments apply mutatis mutandis to the case of coloured symmetric sequences. In particular, the strict symmetric monoidal forgetful functor from enriched operads to coloured symmetric sequences sends cartesian maps to cartesian maps.

□

Definition 15. Let \( C \) be a set. Let us denote

\[ \triangleright \text{S}_C\text{-mod the category of } C \text{-coloured symmetric sequences that is the fiber of the colours functor } S\text{-mod}\to \text{Set over the set } C; \]

\[ \triangleright \text{Operad}_{E,C} \text{ the category of } C \text{-coloured operads that is the fiber of the functor } \text{sk(Operad}_E\text{)} \to \text{Set over the set } C. \]

Proposition 4. [Shu08 Theorem 12.7] For any set \( C \), the category \( \text{Operad}_{E,C} \) inherit from \( \text{Operad}_E \) the structure of a symmetric monoidal category. Moreover, for any function \( \phi : C \to D \), the functor

\[ \phi^* : \text{Operad}_{E,D} \to \text{Operad}_{E,C} \]

has the canonical structure of a strong symmetric monoidal functor. Finally, for any two composable functions \( \phi, \psi \), the natural isomorphism \( (\phi \circ \psi)^* \simeq \psi^* \circ \phi^* \) is a monoidal natural transformation. The same phenomenon holds if we replace enriched operads by coloured symmetric sequences.

Let \( C \) be a set. Let us describe the structure of a symmetric monoidal category on \( \text{Operad}_{E,C} \).

The tensor product is

\[ P \otimes_C Q = \text{diag}_C(P \otimes_H Q) \]

where \( \text{diag}_C \) is the diagonal map of \( C \). The tensor unit is \( \pi_C^*(uCom) \) where \( \pi_C \) is the unique function from \( C \) to \( * \).
Definition 16. Let $M$ be a coloured symmetric sequence whose set of colours is $C$. Then, for any $C$-coloured tree $Y$ we define
\[
\bigotimes_Y M = \bigotimes_{v \in \text{vert}(Y)T} M(v)
\]
where
\[
M(v) = \lim_{(\xi; c) \in Y} M(\xi; c).
\]
This defines a bifunctor
\[
S_C-\text{mod} \times (\Omega_C^\text{core})^{\text{op}} \to E.
\]

Proposition 5. The construction $\mathcal{P}, Y \mapsto \bigotimes_Y \mathcal{P}$ defines a bifunctor
\[
\text{Operad}_{E,C} \times (\Omega_C^{\text{larg}})^{\text{op}} \to E.
\]

Proof. The inner face maps correspond to composition within the operad, while the degeneracies correspond to the units. 

One can also describe $C$-coloured operads as monoids for a monoidal structure on $C$-coloured symmetric sequences. Let $M$ and $N$ be two $C$-coloured symmetric sequences. Their composite product $M \triangleleft N$ is the symmetric sequence given by the following formula
\[
M \triangleleft N(\xi; c) = \int_{\xi \in C} M(\xi; c) \otimes \left( \bigotimes_{i=1}^{I_\xi} M(\xi_i'; \xi_i') \right) \otimes \text{hom}_C(\xi, \cup_i \xi_i').
\]

Proposition 6. \cite{LV12,Gnc} The bifunctor $- \triangleleft -$ is the tensor product of a monoidal structure on symmetric sequences whose unit $I_C$ is
\[
I_C(\xi; c) = \begin{cases} 1 & \text{if } \xi = c; \\ 0 & \text{otherwise}. \end{cases}
\]
Moreover, the category of $C$-coloured enriched operads is canonically isomorphic to the category of $\triangleleft$-monoids.

Remark 5. The fact that the Hadamard tensor product on $C$-coloured symmetric sequences extends to $C$-coloured enriched operads is linked to the following facts.

- The monad that encodes $C$-coloured enriched operads within $C$-coloured symmetric sequences is a Hopf monad with respect to the Hadamard tensor product on symmetric sequences; see \cite{Moe02} and \cite{Grib} for the definition of a Hopf monad.
- The composition tensor product of $C$-coloured symmetric sequences $\triangleleft$ distributes the Hadamard tensor product.

2.6. Segments and intervals.

Definition 17 (Segments and intervals). \cite{BM06} A segment of $E$ is an object $H$ of $E$ together with maps
\[
1 \sqcup 1 \xrightarrow{(\delta^H, \delta^H_1)} H \xrightarrow{\sigma^H} 1
\]
that factorise the morphism $1 \sqcup 1 \to 1$ together with a map $\gamma_H : H \otimes H \to H$ such that
- the product $\gamma_H$ is associative, that is $\gamma_H(1d_H \otimes \gamma_H) = \gamma_H(\gamma_H \otimes 1d_H)$ and $\gamma_H \circ \tau = \gamma_H$,
- the product has a unit given by $\delta^H_0 : 1 \to H$,
- the morphism $\sigma^H$ is a morphism of monoids (that is an augmentation),
- the morphism $\delta^H_1 : 1 \to H$ is absorbing, that is the following diagram commutes
\[
\begin{array}{ccc}
H & \xrightarrow{\delta^H_1 \otimes 1d} & H \otimes H \\
\downarrow{\gamma^H} & & \downarrow{\gamma^H} \\
1 & \xrightarrow{\delta^H_1} & H
\end{array}
\]
\[
\begin{array}{ccc}
H & \xleftarrow{1d \otimes \delta^H_1} & H \otimes 1 \\
\downarrow{\gamma^H} & & \downarrow{\gamma^H} \\
1 & \xleftarrow{\delta^H_1} & 1.
\end{array}
\]

A morphism of segments from $(H, \delta^H_0, \delta^H_1, \sigma^H, \gamma^H)$ to $(H', \delta^{H'}_0, \delta^{H'}_1, \sigma^{H'}, \gamma^{H'})$ is a morphism of $f : H \to H'$ that commutes with all the structures. We denote $\text{Seg}$ the category of segments.
Definition 18. A segment \((H, \delta^0_H, \delta^1_H, \sigma_H, \gamma_H)\) is said to be commutative if the product \(\gamma_H\) is commutative.

Definition 19. If \(E\) is a monoidal model category, an interval is a segment such that the map \((\delta^0_H, \delta^1_H)\) is a cofibration and the map \(\sigma_H\) is a weak equivalence.

Remark 6. Note that these are exactly the notions of segments and intervals introduced in [BM06] but they differ from the notions of segments and intervals that I used in [Gria].

The tensor product of \(E\) induces a monoidal structure on the category of segments as well as on the full subcategory of commutative segments. For instance, the tensor product of a segment \((H, \delta^0_H, \delta^1_H, \sigma_H, \gamma_H)\) with a segment \((H', \delta^0_{H'}, \delta^1_{H'}, \sigma_{H'}, \gamma_{H'})\) is the segment

\[
1 \sqcup 1 \simeq 1 \otimes 1 \sqcup 1 \otimes 1 \xrightarrow{(\delta^0_H \otimes \delta^0_{H'}, \delta^1_H \otimes \delta^1_{H'})} H \otimes H' \xrightarrow{\sigma_H \otimes \sigma_{H'}} 1 \otimes 1 \simeq 1;
\]

Definition 20. A Hopf segment in \(E\) is a comonoid in the monoidal category of segments. In a monoidal model category, a Hopf interval is a Hopf segment whose underlying segment is an interval.

Remark 7 ([Mal09] [Gria]). Since the monoidal category \(E\) is a bilinear, the category of segments is canonically equivalent to the category of left adjoint functors from cubical sets with connections to \(E\). We refer the reader to [Mal09] and [Cis14] for a definition of the category of cubical sets with connections.

2.7. The Boardman–Vogt construction.

Definition 21. Let \(H\) be a segment in \(E\) and let \(Y\) be a coloured tree. Then let

\[
Y \otimes H := \bigotimes_{\text{inner}(Y)} H,
\]

where \(\text{inner}(Y)\) is the set of inner edges of \(Y\). In particular, if \(Y\) has no internal edge, then \(Y \otimes H := 1\).

Proposition 7. Let \(C\) be a set. The formula \(Y \otimes H\) induces a bifunctor

\[
\bigotimes_C \colon \Omega_C^{\text{core}} \times \text{Seg} \to E.
\]

Proof. It is clear that it induces a bifunctor \(\Omega_C^{\text{core}} \times \text{Seg} \to E\). Besides, let \(\delta : Y \to Y'\) be an inner coface. The map

\[
Y \otimes H \simeq 1 \otimes Y \otimes H \xrightarrow{\delta \otimes \text{Id}} H \otimes Y \otimes H \simeq Y' \otimes H,
\]

gives us the functoriality of the formula \(Y \otimes H\) with respect to this inner coface. Let \(\sigma : Y \to Y'\) be a codegeneracy. If the two edges \(e\) and \(e'\) (with \(e\) below \(e'\)) of \(Y\) that will merge in \(Y'\) are internal, then the functor \(\bigotimes C \sigma\) sends \(\sigma\) to the map

\[
Y \otimes H = Y \otimes H \simeq Y \otimes H \otimes Y' \otimes H \xrightarrow{m \otimes \text{Id}^{-2}} H \otimes Y' \otimes H \simeq Y' \otimes H = Y' \otimes H.
\]

If one of these two edges is external and the other one is internal (for instance \(e\) is internal), the functor \(\bigotimes C \sigma\) sends \(\sigma\) to the map

\[
Y \otimes H = Y \otimes H \simeq Y \otimes H \otimes (Y')^{-e} \otimes H \xrightarrow{\sigma \otimes \text{Id}^{-1}} Y \otimes H \simeq Y \otimes H = Y' \otimes H.
\]

Finally, if these two edges are external, which is only possible if \(Y\) is the corolla with one leaf, then the functor \(\bigotimes C \sigma\) sends \(\sigma\) to the map \(1 \simeq 1\). \qed

Remark 8. For codegeneracies that will merge two inner edges \(e\) below \(e'\) to a new inner edge \(e''\), the labelling of \(e''\) is obtained from that of \(e\) and \(e'\) using the product \(m\). The first input of \(m\) refers to the labelling of \(e\) while the second input of \(m\) refers to the labelling of \(e'\). This is a convention and we could have made the other choice.
Definition 22 ([BM06]). For any enriched operad $P$ and any segment $H$, the Boardman–Vogt construction of $P$ with respect to $H$ is an enriched operad with the same set of colours and whose underlying coloured symmetric sequence is given by the following formula:

$$W_H P(c; c) := \int_{Y \in \Omega_{\text{Op}}} \left( \bigotimes_Y H \otimes \bigotimes_Y P \right) \otimes \text{Iso}_{\text{Set}}(c, \text{leaves}(Y)).$$

The unit maps are

$$\eta_c : 1 \to H \otimes \bigotimes_P \to W_H P(c; c), \quad c \in \text{Ob}(P).$$

The composition maps $\gamma_i$ are given on generators by the following composition

$$\bigotimes_Y H \otimes \bigotimes_Y P \otimes \{\phi\} \otimes \bigotimes_Y P \otimes \{\psi\} \xrightarrow{=} \bigotimes_Y H \otimes 1 \otimes \bigotimes_Y P \otimes \{\phi \circ_i \psi\} \xrightarrow{\text{Id} \otimes \delta \otimes \text{Id}} \bigotimes_Y H \otimes H \otimes \bigotimes_Y P \otimes \{\phi \circ_i \psi\} \xrightarrow{\bigotimes Y \sqcup_{\phi(i)} Y'} H \otimes \bigotimes_{\phi(i)} Y', P \otimes \{\phi \circ \psi\}$$

where the tree $Y \sqcup_{\phi(i)} Y'$ is the tree obtained from $Y$ and $Y'$ by gluing the root edge of $Y'$ with the leaf $\phi(i)$ of $Y$. Moreover, the map

$$\phi \circ \psi : c \circ_i c' \to \text{leaves}(Y \sqcup_{\phi(i)} Y')$$

sends an element $c[k]$ to its image through $\phi$ for $k \neq i$ and an element $c[j]$ to its image through $\psi$.

Proposition 8. ([BM06]) The construction $H, P \mapsto W_H P$ defines a functor

$$W_- : \text{Seg} \times \text{sk}(\text{Operad}_E) \to \text{sk}(\text{Operad}_E).$$

Proof. Such a functor sends a pairs of morphisms $(u, f) : (H, P) \to (H', P')$ to the morphism of enriched operads induced by the maps

$$\{\phi\} \otimes \bigotimes_Y H \otimes \bigotimes_Y P \xrightarrow{1 \otimes \delta \otimes \text{Id}} \{\phi\} \otimes \bigotimes_Y H' \otimes \bigotimes_Y Q$$

where $f(Y)$ is the $\text{Ob}(Q)$-coloured tree whose underlying tree is the same as $Y$ and whose colouring is the map

$$\text{edge}(Y) \to \text{Ob}(P) \xleftarrow{f} \to \text{Ob}(Q).$$

\hfill $\Box$

2.8. Monoidal structure of the Boardman–Vogt construction.

Proposition 9. The functor $W_- : \text{Seg} \times \text{sk}(\text{Operad}_E) \to \text{sk}(\text{Operad}_E)$ is oplax symmetric monoidal for the symmetric monoidal structure $\otimes \times \otimes_\text{ht}$ on the product category $\text{Seg} \times \text{sk}(\text{Operad}_E)$. 
Proof. For any two segments \( H, H' \), any two enriched operads \( \mathcal{P}, \mathcal{P}' \) and any \( \text{Ob}(\mathcal{P}) \times \text{Ob}(\mathcal{P}) \)-coloured tree \( Y \), let us consider the following map

\[
\left( \bigotimes^Y (H \otimes H') \otimes \bigotimes_Y (\mathcal{P} \otimes_{\mathcal{H}} \mathcal{P}') \right) \otimes \text{Iso}_{\text{Set}/(\text{Ob}(\mathcal{P}) \times \text{Ob}(\mathcal{P}'))}((c_1, d_1), \ldots, (c_n, d_n)), \text{leaves}(Y))
\]

\[
\downarrow
\]

\[
\left( \bigotimes^Y H \otimes \bigotimes_Y (\mathcal{P} \otimes_{\mathcal{H}} \mathcal{P}') \otimes \text{Iso}_{\text{Set}/(\text{Ob}(\mathcal{P}'))}((d_1, \ldots, d_n), \text{leaves}(Y)) \right)
\]

\[
\downarrow
\]

\[
W_H \mathcal{P}((c_1, \ldots, c_n; c) \otimes W_H \mathcal{P}'(d_1, \ldots, d_n; d))
\]

\[
(W_H \mathcal{P} \otimes_{\mathcal{H}} W_H \mathcal{P}')(\text{leaves}(Y), (d_1), \ldots, (c_n, d_n); (c, d))
\]

Taking the coend over varying trees, we get a morphism of \( \text{Ob}(\mathcal{P}) \times \text{Ob}(\mathcal{P}') \)-coloured symmetric sequences

\[
W_{H \otimes H'}(\mathcal{P} \otimes_{\mathcal{H}} \mathcal{P}') \rightarrow W_H \mathcal{P} \otimes_{\mathcal{H}} W_H \mathcal{P}'
\]

This is actually of morphism of \( \text{Ob}(\mathcal{P}) \times \text{Ob}(\mathcal{P}') \)-coloured enriched operads. Besides, we have a canonical isomorphism

\[
W_{\mathcal{H}} \text{uCom} \simeq \text{uCom}
\]

A long but straightforward checking shows that this defines the structure of an oplax symmetric monoidal functor on \( \mathcal{W}_{\mathcal{H}} \). \( \square \)

Remark 9. If one works with planar operads, the Boardman–Vogt construction becomes a strong symmetric monoidal functor.

Corollary 2. Let \( \mathcal{H} \) be a Hopf segment. Then the endofunctor \( W_{\mathcal{H}} - \) of \( \text{sk}(\text{Operad}_E) \) is oplax monoidal.

Proof. The oplax monoidal structure is given by the following natural map

\[
W_{\mathcal{H}}(\mathcal{P} \otimes_{\mathcal{H}} \mathcal{P}') \rightarrow W_{\mathcal{H} \otimes \mathcal{H}}(\mathcal{P} \otimes_{\mathcal{H}} \mathcal{P}') \rightarrow W_{\mathcal{H}} \mathcal{P} \otimes_{\mathcal{H}} W_{\mathcal{H}} \mathcal{P}'
\]

\( \square \)

Corollary 3. Let \( \mathcal{Q} \) be a Hopf operad and let \( \mathcal{H} \) be a Hopf segment. Then \( W_{\mathcal{H}} \mathcal{Q} \) inherits the canonical structure of a Hopf operad. Moreover, for any \( \mathcal{Q} \)-left comodule \( \mathcal{P} \), \( W_{\mathcal{H}} \mathcal{P} \) inherits the structure of a \( W_{\mathcal{H}} \mathcal{Q} \)-comodule. The same holds for right comodules and bi-comodules.

2.9. Planar operads.

Definition 23. A planar enriched operad \( \mathcal{P} \) is the data of a set of colours \( \text{Ob}(\mathcal{P}) \) and elements \( \mathcal{P}(\xi; c) \) of the ground category \( E \) for any tuple of colours \( \xi \) and any colour \( c \in \text{Ob}(\mathcal{P}) \) together with maps

\[
\gamma_i : \mathcal{P}(\xi; c) \otimes \mathcal{P}(\xi'; \xi[i]) \rightarrow \mathcal{P}(\xi \land_i \xi'; c);
\]

\[
\eta_c : 1 \rightarrow \mathcal{P}(c; c);
\]

that satisfies the same associativity and unitality conditions as enriched operads. A morphisms of planar enriched operads from \( \mathcal{P} \) to \( \mathcal{Q} \) is the data of a function \( f : \text{Ob}(\mathcal{P}) \rightarrow \text{Ob}(\mathcal{Q}) \) together with morphisms

\[
\mathcal{P}(\xi; c) \rightarrow \mathcal{Q}(f(\xi); f(c))
\]

that commutes with the structural maps \( \gamma_i \) and \( \eta_c \). We denote \( \text{plOperad}_E \) the category of planar enriched operads.

We have an adjunction relating coloured planar operad to coloured operad

\[
\text{plOperad}_E \xleftarrow{\text{sk}} \xrightarrow{\text{uS}} \text{sk(Operad}_E)\]
whose right adjoint is the forgetful functor and whose left adjoint sends a planar enriched operad \( P \) to the enriched operad \( P_S \) with the same set of colours and defined as

\[
P_S(\mathcal{C}; c) = \coprod_{\mathcal{C} \in \mathcal{C}_n} P(\mathcal{C}'; c) \otimes \text{hom}_{S_n}(\mathcal{C}, \mathcal{C}')
\]

\[
= \prod_{\sigma \in S_n} P(\mathcal{C}^{\sigma^{-1}}; c) \otimes \{\sigma\},
\]

where \( n = |\mathcal{C}| \). The action \( \mu^* : P_S(\mathcal{C}; c) \to P_S(\mathcal{C}'; c) \) of the symmetric groups is defined by maps of the form

\[
P(\mathcal{C}^{\sigma^{-1}}; c) \otimes \{\sigma\} \to P(\mathcal{C}^{\sigma^{-1}}; c) \otimes \{\sigma \circ \mu\} = P((\mathcal{C}^{\mu})^{(\sigma \circ \mu)^{-1}}; c) \otimes \{\sigma \circ \mu\},
\]

while the operadic composition \( \gamma_i \) is defined as

\[
\left(P(\mathcal{C}^{\sigma^{-1}}; c) \otimes \{\sigma\}\right) \otimes \left(P(\mathcal{C}^{\sigma^{-1}}; c) \otimes \{\mu\}\right)
\]

\[
\xrightarrow{\gamma_{\sigma(i)}}
\]

\[
P(\mathcal{C}^{\sigma^{-1}}; c) \otimes P(\mathcal{C}^{\sigma^{-1}}; c) \otimes \{\sigma \circ \mu\}
\]

\[
\xrightarrow{\gamma_{\sigma \circ \mu}}
\]

\[
P((\mathcal{C} \circ \mathcal{C}')^{(\sigma \circ \mu)^{-1}}; c) \otimes \{\sigma \circ \mu\},
\]

where \( \gamma^\text{pl}_{\sigma(i)} \) denotes the planar operadic composition in the planar operad \( P \) and where \( \sigma \circ \mu \) is the permutation

\[
\{1, \ldots, n + m - 1\}
\]

\[
\{1, \ldots, i - 1\} \cup \{i, \ldots, i + m - 1\} \cup \{i + m, \ldots, n + m - 1\}
\]

\[
\cong
\]

\[
\{1, \ldots, i - 1\} \cup \{1, \ldots, m\} \cup \{i + 1, \ldots, n\}
\]

\[
\{1, \ldots, n\} \setminus \{i\} \cup \{1, \ldots, m\}
\]

\[
\cong
\]

\[
\{1, \ldots, n\} \setminus \{\sigma(i)\} \cup \{1, \ldots, m\}
\]

\[
\{1, \ldots, n + m - 1\}
\]

where \( n = |\mathcal{C}| \) and \( m = |\mathcal{C}'| \).

3. Enriched symmetric monoidal categories and algebras

In this section, we describe symmetric monoidal categories enriched over the ground category \( E \). We show how they are related to enriched operads and we describe how coalgebraic structures on enriched operads induce monoidal structures on their categories of algebras in a symmetric monoidal enriched category \( \mathcal{C} \).
3.1. The monoidal context of enriched categories.

**Definition 24.** An enriched category $C$ is the data of a set (possibly large) called the set of objects (or colours) and denoted $\text{Ob}(C)$ together with, for any two colours $x, y$, an object $C(x, y) \in E$ and with an associative composition

$C(y, z) \otimes C(x, y) \rightarrow C(x, z)$

with units $\text{Id}_x \in C(x, x)$.

**Definition 25.** An enriched category $C$ is called small if its set of objects is small.

**Definition 26.** A functor of enriched categories from $C$ to $D$ is the data of

$\mathcal{F} \, \text{a function } f(\cdot) : \text{Ob}(C) \rightarrow \text{Ob}(D)$;

$\mathcal{F}$ for any two colours of $C$ $x, y$, a morphism

$f(x, y) : C(x, y) \rightarrow C(f(x), f(y))$

that commutes with units and compositions.

**Definition 27.** Given two functors of enriched categories $f, g : C \rightarrow D$, a natural transformation between them is the data of elements

$A(x) \in D(f(x), g(x)), \, x \in \text{Ob}(C)$,

so that the following square is commutative

$C(x, y) \xrightarrow{g(x, y) \otimes A(x)} D(g(x), g(y)) \otimes D(f(x), g(x))$

$\downarrow A(y) \otimes f(x, y)$

$D(f(y), g(y)) \otimes D(f(x), f(y)) \rightarrow D(f(x), g(y))$

for any objects $x, y \in \text{Ob}(C)$.

**Definition 28.** We denote $\text{Cat}_E$ the strict 2-category of enriched categories. This a $V$ strict 2-category and hence a $W$-small strict 2-category.

**Definition 29.** Let $C, D$ be two enriched categories. Their tensor product $C \otimes D$ is the enriched category so that

$\mathcal{F} \, \text{Ob}(C \otimes D) = \text{Ob}(C) \times \text{Ob}(D)$;

$(C \otimes D)((x, x'), (y, y')) = C(x, y) \otimes D(x', y')$;

$\mathcal{F}$ the composition is given as follows

$(C \otimes D)((y, y'), (z, z')) \otimes (C \otimes D)((x, x'), (y, y'))$

$\downarrow \cong$

$C(y, z) \otimes D(y', z') \otimes C(x, y) \otimes D(x', y')$ \quad \xrightarrow{\cong} \quad C(x, z) \otimes D(x', z')$

$\downarrow$

$C(x, z) \otimes D(x', z')$

$(C \otimes D)((y, y'), (z, z'))$;

$\mathcal{F}$ the unit of the object $(x, x')$ is $\text{Id}_x \otimes \text{Id}_{x'}$.

**Definition 30.** Let $I$ be the enriched category with one object $*$ and so that

$I(*, *) = 1_E$. 

Proposition 10. The tensor product described just above induces the structure of a monoidal context on the 2-category $\mathcal{C} \text{at}_{E}$ whose unit is $I$.

Proof. This tensor product induces the structure of a monoidal category on $\text{sk}(\mathcal{C} \text{at}_{E})$. Moreover, this monoidal structure extends to the 2-category $\mathcal{C} \text{at}_{E}$. For instance, given two enriched functors $f, f' : C \to C'$ and $g, g' : D \to D'$ and natural transformations $A : f \to f'$ and $B : g \to g'$, the natural transformation $A \otimes B : f \otimes g \to f' \otimes g'$ is given by the elements $A(c) \otimes B(d)$ for $(c, d) \in \text{Ob}(C) \times \text{Ob}(D)$.

Definition 31. For an enriched category $C$, we denote $\text{cat}(C)$ the underlying category, that is the category whose set of objects is $\text{Ob}(C)$ and so that $\text{hom}_{\text{cat}(C)}(x, y) = \text{hom}_{E}(1_{E}, C_{x, y})$.

This defines a lax context functor

$$\text{cat} : \mathcal{C} \text{at}_{E} \to \text{Cats}_{U}.$$  

3.2. The monoidal context of enriched symmetric monoidal categories.

Definition 32. Let us recall that $\text{CMon}_{\text{lax}}(\mathcal{C} \text{at}_{E})$ is the monoidal context of pseudo-commutative monoids and lax morphisms in the monoidal context of $E$-enriched categories. We will call its objects, morphisms and 2-morphisms respectively symmetric monoidal enriched categories, lax symmetric monoidal functors and monoidal enriched natural transformations.

Unravelling the definitions, the data of a symmetric monoidal enriched category is equivalent to the data of

- an enriched category $C$;
- a functor of enriched categories $\gamma : C \otimes C \to C$
  $$x, y \mapsto x \otimes_{C} y;$$
- an element $1_{C}$;
- invertible enriched natural transformations $x \otimes_{C} (y \otimes_{C} z) \simeq (x \otimes_{C} y) \otimes_{C} z$, $x \otimes_{C} y \simeq y \otimes_{C} x$, $x \otimes_{C} 1_{C} \simeq x$;
- so that the induced bifunctor $\text{cat}(C) \times \text{cat}(C) \to \text{cat}(C \otimes C) \xrightarrow{\text{cat}(\gamma)} \text{cat}(C)$.

the element $1_{C}$ and the induced natural transformations involving $\text{cat}(C)$ instead of $C$ make $\text{cat}(C)$ a symmetric monoidal category.

Moreover, the data of a lax symmetric monoidal functor between symmetric monoidal enriched categories $C$ and $D$ is equivalent to the data of

- an enriched functor $f : C \to D$;
- an enriched natural transformation $l : \gamma_{D} \circ (f \otimes f) \to f \circ \gamma_{C}$;
- a map $1_{D} \to f(1_{C})$;
- so that the induced functor $\text{cat}(f)$, the induced natural transformation $\text{cat}(l)$ and the previous map form a lax symmetric monoidal functor between symmetric monoidal categories.

Finally, a 2-morphism in $\text{CMon}_{\text{lax}}(\mathcal{C} \text{at}_{E})$ between two lax symmetric monoidal functors $f$ and $f'$ is the data of an enriched natural transformation $A : f \to f'$ so that the following diagrams commute

$$\begin{align*}
\gamma \circ (f \otimes f) & \xrightarrow{A_{1}} f \circ \gamma \\
\gamma \circ (f' \otimes f') & \xrightarrow{A_{1}} f' \circ \gamma
\end{align*}$$

$$f \xrightarrow{1_{D}} f(1_{C}) \xrightarrow{A(1_{C})} f'(1_{C}).$$
3.3. Multiple tensors. Let \( \mathcal{C} \) be a symmetric monoidal enriched category and let \( n \geq 3 \). In the same way as in [Grib] Definition 23 and Definition 24, one gets an enriched functor from \( \mathcal{C}^{\otimes n} \) to \( \mathcal{C} \) for any binary planar tree with \( n \)-leaves: roughly, one composes operadically the tensor product \( \otimes : \mathcal{C} \otimes \mathcal{C} \rightarrow \mathcal{C} \) along this tree. The associators induces 2-isomorphisms between these functors so that we get a contractible groupoid (by Mac Lane’s coherence, see [ML63, Grib, Proposition 22]).

**Definition 33.** The colimit in \( \text{Cat}_E(\mathcal{C}^{\otimes n}, \mathcal{C}) \) of this contractible groupoid is the \( n \)-tensor product of \( \mathcal{C} \)

\[
\otimes^n : \mathcal{C}^{\otimes n} \rightarrow \mathcal{C}
\]

\[
(x_1, \ldots, x_n) \mapsto x_1 \otimes \cdots \otimes x_n.
\]

We also define the 2-tensor product of \( \mathcal{C} \) to be the tensor product, the 1-tensor product to be the identity of \( \mathcal{C} \) and the 0-tensor product to be the unit \( I \rightarrow \mathcal{C} \).

Again, for any planar tree \( t \) with \( n \) leaves and any permutation \( \sigma \in S_n \), one obtains an enriched functor from \( \mathcal{C}^{\otimes n} \) to \( \mathcal{C} \) by first permuting the tensors using \( \sigma^* \) and then applying the operadic composition of multi-tensors along the planar tree \( t \). Then, one can consider the groupoid whose objects are such pairs \((t, \sigma)\) and whose morphisms are those of \( \text{Cat}_E(\mathcal{C}^{\otimes n}, \mathcal{C}) \) between the induced enriched functors and that are generated by operadic compositions of the associator, the commutator, the unitors and the reduction of binary subtrees into multi-tensors. One can check after Mac Lane’s coherence that such a groupoid is contractible.

3.4. The End 2-functor.

**Definition 34.** For any symmetric monoidal \( E \)-enriched category \( \mathcal{C} \), let \( \mathcal{E}\text{nd}(\mathcal{C}) \) be the enriched operad whose colours are \( \text{Ob}(\mathcal{C}) \) and so that

\[
\mathcal{E}\text{nd}(\mathcal{C})(x_1, \ldots, x_n; x) = \mathcal{C}(x_1 \otimes \cdots \otimes x_n, x).
\]

The operadic composition is induced by the composition in \( \mathcal{C} \) and the action of symmetric groups is induced by the commutator of \( \mathcal{C} \).

**Proposition 11.** The \( \mathcal{E}\text{nd} \) construction induces a strict 2-functor from the 2-category of symmetric monoidal enriched categories and lax enriched functors to the 2-category of enriched operads:

\[
\mathcal{E}\text{nd} : \text{CMon}_{\text{lax}}(\text{Cat}_E) \rightarrow \text{Operad}_E.
\]

**Proof.** For any lax monoidal functor between symmetric monoidal enriched categories \( f : \mathcal{C} \rightarrow \mathcal{D} \), we obtain a morphism of enriched operads

\[
\mathcal{E}\text{nd}(f) : \mathcal{E}\text{nd}(\mathcal{C}) \rightarrow \mathcal{E}\text{nd}(\mathcal{D})
\]

whose underlying function on colours is the underlying function of \( f \) on objects, and which is given by the maps

\[
\mathcal{C}(x_1 \otimes \cdots \otimes x_n, x) \rightarrow \mathcal{D}(f(x_1 \otimes \cdots \otimes x_n), f(x)) \rightarrow \mathcal{D}(f(x_1) \otimes \cdots \otimes f(x_n), f(x)).
\]

For any monoidal natural transformation \( A : f \rightarrow g \) between lax symmetric monoidal functors \( f, g : \mathcal{C} \rightarrow \mathcal{D} \), we obtain a 2-morphism of enriched operads given by the same elements \( A(x) \in \mathcal{D}(f(x), g(x)) \) for \( x \in \text{Ob}(\mathcal{C}) \).

A straightforward check shows that this defines a 2-functor. \( \square \)

**Proposition 12.** The 2-functor \( \mathcal{E}\text{nd} \) is strictly fully faithful.

**Proof.** This amounts to prove that for any symmetric monoidal enriched categories \( \mathcal{C}, \mathcal{D} \), the functor

\[
\text{CMon}_{\text{lax}}(\text{Cat}_E)(\mathcal{C}, \mathcal{D}) \rightarrow \text{Operad}_E(\mathcal{E}\text{nd}(\mathcal{C}), \mathcal{E}\text{nd}(\mathcal{D}))
\]

is an isomorphism of categories. Let us describe its inverse. Let us consider a morphism of operads \( f : \mathcal{E}\text{nd}(\mathcal{C}) \rightarrow \mathcal{E}\text{nd}(\mathcal{D}) \). By restriction to arity 1 elements, we obtain a functor \( r(f) \) between the underlying enriched categories of \( \mathcal{E}\text{nd}(\mathcal{C}) \) and \( \mathcal{E}\text{nd}(\mathcal{D}) \) that are just respectively \( \mathcal{C} \) and \( \mathcal{D} \). This functor \( r(f) \) has the structure of a lax functor given by the images through

\[
\mathcal{E}\text{nd}(\mathcal{C})(x_1, \ldots, x_n; x_1 \otimes \cdots \otimes x_n) \rightarrow \mathcal{E}\text{nd}(\mathcal{D})(f(x_1), \ldots, f(x_n); f(x_1) \otimes \cdots \otimes x_n))
\]
of the identity of $x_1 \otimes \cdots \otimes x_n$. Besides, for any 2-morphism of enriched operads $A : f \to g$, it is straightforward to check that the induced enriched natural transformation from $r(f)$ to $r(g)$ is monoidal. This construction $f \mapsto r(f)$ actually defines a functor
\[ r : \text{Operad}_E(\text{End}(C), \text{End}(D)) \to \text{CMonlax}((\text{Cat}_E)(C, D)). \]

It is clear that $r \circ \text{End} = \text{Id}$. The fact that $\text{End} \circ r = \text{Id}$ follows from the commutation of the squares
\[
\begin{array}{ccc}
C(x_1 \otimes \cdots \otimes x_n, x) & \xrightarrow{f} & D(f(x_1 \otimes \cdots \otimes x_n), f(x)) \\
-\text{Id}(x_1 \otimes \cdots \otimes x_n) & & -\text{Id}(f(x_1 \otimes \cdots \otimes x_n)) \\
\text{End}(C)(x_1, \ldots, x_n, x) & \xrightarrow{r} & \text{End}(D)(f(x_1), \ldots, f(x_n); f(x))
\end{array}
\]
for any morphism of enriched operads $f : \text{End}(C) \to \text{End}(D)$. □

**Proposition 13.** The $\text{End}$ 2-functor has the canonical structure of a strong context functor from the monoidal context $\text{CMonlax}((\text{Cat}_E)(C, D))$ of symmetric monoidal enriched categories and lax enriched functors to the monoidal context of enriched operads $\text{Operad}_E$.

**Proof.** Such a structure is given by the canonical isomorphisms
\[ \text{End}(C) \otimes \text{End}(D) \simeq \text{End}(C \otimes D) \]
given by the maps
\[
(\text{End}(C) \otimes \text{End}(D))((x_1, y_1), \ldots, (x_n, y_n); (x, y))
\]
\[
\text{End}(C)(x_1, \ldots, x_n; x) \otimes \text{End}(D)(y_1, \ldots, y_n; y)
\]
\[
C(x_1 \otimes \cdots \otimes x_n, x) \otimes D(y_1 \otimes \cdots \otimes y_n, y)
\]
\[
(C \otimes D)((x_1 \otimes \cdots \otimes x_n, x_1 \otimes \cdots \otimes y_n), (x, y))
\]
\[
\simeq
\]
\[
(C \otimes D)((x_1, y_1) \otimes \cdots \otimes (x_n, y_n), (x, y))
\]
\[
\text{End}(C \otimes D)((x_1, y_1), \ldots, (x_n, y_n); (x, y)).
\]

□

### 3.5. The Yoneda 2-functor and algebras over an operad.

**Definition 35.** Let $C$ be a $V$ strict 2-category. The canonical strict 2-functor
\[
C^\text{op} \times C \to \text{Cat}_V\text{--small}
\]
\[
X, Y \mapsto C(X, Y)
\]
is called the Yoneda 2-functor of $C$.

**Lemma 3.** Let $C$ be a $V$ monoidal context. Then $C \times C^\text{op}$ gets the canonical structure of a $V$ monoidal context and the Yoneda strict 2-functor
\[
X, Y \mapsto C(X, Y)
\]
has the canonical structure of a lax context functor.

**Proof.** The lax structure is given by the naturality of the tensor product, that is the map
\[
C(X, Y) \times C(X', Y') \to C(X \otimes X', Y \otimes Y'),
\]
and by the identity of the monoidal unit of $C$:
\[
* \to C(1_C, 1_C).
\]
**Definition 36.** Let $\text{Alg}_\mathcal{C} (-)$ be the lax context functor that is the composition of the strong context functor $\text{Id} \times \mathcal{E} \text{nd}$ with Yoneda 2-functor of $\text{Operad}_E^\text{op}$ (that is a lax context functor):

$$\text{Alg}_\mathcal{C} (-) : \text{Operad}_E^\text{op} \times \text{CMon}_{\text{lax}}(\text{Cat}_E) \xrightarrow{\text{Id} \times \mathcal{E} \text{nd}} \text{Operad}_E^\text{op} \times \text{Operad}_E \rightarrow \text{Cats}_{\mathcal{C}}^\mathcal{V}.$$

For any symmetric monoidal enriched category $\mathcal{C}$ and any enriched operad $\mathcal{P}$, the $\mathcal{V}$-small category $\text{Alg}_\mathcal{C} (\mathcal{P})$ is called the category of $\mathcal{P}$-algebras in $\mathcal{C}$.

**Definition 37.** For any symmetric monoidal enriched category $\mathcal{C}$ and any morphism of enriched operads $f : \mathcal{P} \rightarrow \mathcal{P}'$, we denote $f^*$ the functor

$$f^* = \text{Alg}_\mathcal{C} (f) : \text{Alg}_\mathcal{C} (\mathcal{P}') \rightarrow \text{Alg}_\mathcal{C} (\mathcal{P}).$$

**Proposition 14.** Let $\mathcal{P}$ be a $\mathcal{U}$-small enriched operad and let $\mathcal{C}$ be a symmetric monoidal enriched category. Then the $\mathcal{V}$-small category $\text{Alg}_\mathcal{C} (\mathcal{P})$ is actually a $\mathcal{U}$-category.

**Sketch of the proof.** A $\mathcal{P}$-algebra $A$ in $\mathcal{C}$ is a tuple

$$(\{A_o\}_{o \in \text{Ob} (\mathcal{P})}, (\{\gamma (o_1, \ldots, o_n; o)\}_{o_1, \ldots, o_n \in \text{Ob} (\mathcal{P})})_{n \in \mathbb{N}})$$

that satisfies some conditions, where

- $A_o$ is an object of $\mathcal{C}$ and thus an element of $\mathcal{U}$;
- $\gamma (o_1, \ldots, o_n; o)$ is an element of

$$\text{hom}_\mathcal{E} (\mathcal{P} (o_1, \ldots, o_n; o), \mathcal{C} (A_{o_1} \otimes \cdots \otimes A_{o_n}, A_o))$$

and thus an element of $\mathcal{U}$.

Hence, such an algebra is a $\mathcal{U}$-small collection of elements of $\mathcal{U}$. Thus it is an element of $\mathcal{U}$. Hence the set of morphisms of enriched operads from $\mathcal{P}$ to $\mathcal{E} \text{nd} (\mathcal{C})$ is $\mathcal{U}$-large.

Besides, for any two $\mathcal{P}$-algebras $A, A'$ in $\mathcal{C}$, the set of morphisms of algebras from $A$ to $A'$ is a subset of

$$\prod_{o \in \text{Ob} (\mathcal{P})} \text{hom}_{\text{cat} (\mathcal{C})} (A_o, A'_o);$$

which is small. It is thus small.

**Corollary 4.** The restriction of the lax context functor $\text{Alg}_\mathcal{C} (-)$ to $\text{CMon}_{\text{lax}}(\text{Cat}_E) \times \text{Operad}_E^\text{op}$ factorises through $\text{Cats}_{\mathcal{U}}$

$$\xymatrix{ \text{Operad}_E^\text{op} \times \text{CMon}_{\text{lax}}(\text{Cat}_E) \ar[r]^-{\text{Alg}_\mathcal{C} (-)} \ar@{^(->}[d] & \text{Cats}_{\mathcal{C}}^\mathcal{V} \ar@{^(->}[d] \\
\text{Operad}_{E, \text{small}}^\text{op} \times \text{CMon}_{\text{lax}}(\text{Cat}_E) \ar[r] & \text{Cats}_{\mathcal{U}}.}$$

Moreover, since the vertical arrows are strictly fully faithful strict context functors, then the horizontal bottom arrow is also a lax context functor.

### 3.6. Monoidal structures on categories of algebras

Recall from [Grib, Proposition 12] that the functor between $\mathcal{W}$-small categories:

$$\text{ALG}_\mathcal{C} (-)_{lax} : \text{Operad}_{\text{cat}}^\text{op} \times \text{Context}_{\text{lax}} \rightarrow \text{Context}_{\text{lax}}$$

is lax monoidal. Hence, for any two categorical operads $\mathcal{B}, \mathcal{B}'$ and any two monoidal contexts $\mathcal{M}, \mathcal{M}'$, we get a canonical 2-functor

$$\text{ALG}_M (\mathcal{B})_{lax} \times \text{ALG}_{M'} (\mathcal{B}')_{lax} \rightarrow \text{ALG}_{M \times M'} (\mathcal{B} \times \mathcal{B})_{lax},$$

that sends a $\mathcal{B}$-algebra $A$ in $\mathcal{M}$ and a $\mathcal{B}'$-algebra $A'$ in $\mathcal{M}'$ to the $\mathcal{B} \times \mathcal{B}'$-algebra in $\mathcal{M} \times \mathcal{M}'$ given by the composition

$$\mathcal{B} \times \mathcal{B}' \xrightarrow{\mathcal{A} \times \mathcal{A}'} \text{End} (\mathcal{M}) \times \text{End} (\mathcal{M}') \simeq \text{End} (\mathcal{M} \times \mathcal{M}').$$

Applying this result to $\mathcal{M} = \text{Operad}_{E, \text{small}}^\text{op}$ and $\mathcal{M}' = \text{CMon}_{\text{lax}}(\text{Cat}_E)$, we get the following result.
**Theorem 1.** Let $B, B'$ be two categorical operads. Then, the lax context functor

$$\text{Alg}_-(-) : \text{Operad}^\text{op}_{\text{E,small}} \times \text{CMon}_{\text{lax}}(\text{Cat}_E) \to \text{Cats}_U$$

induces a lax context functor that is natural with respect to $B, B'$:

$$\text{ALG}_{\text{Operad}^\text{op}_{\text{E,small}}} (B)_{\text{lax}} \times \text{ALG}_{\text{CMon}_{\text{lax}}(\text{Cat}_E)} (B')_{\text{lax}} \to \text{ALG}_{\text{Cats}_U} (B \times B')_{\text{lax}}.$$ 

**Proof.** This lax context functor is the composition

$$\text{ALG}_{\text{Operad}^\text{op}_{\text{E,small}}} (B)_{\text{lax}} \times \text{ALG}_{\text{CMon}_{\text{lax}}(\text{Cat}_E)} (B')_{\text{lax}} \to \text{ALG}_{\text{Cats}_U} (B \times B')_{\text{lax}}.$$ 

The first arrow proceeds from the structure of a lax monoidal functor on $\text{ALG}_-(-)_{\text{lax}}$ as described above and the second arrow is the image through the functor $\text{ALG}_- (B \times B')_{\text{lax}}$ of the lax context functor $\text{Alg}_-(-)_{\text{lax}}$. □

**Corollary 5.** Let us use the notations of Theorem 1. The 2-functor

$$\text{ALG}_{\text{Operad}^\text{op}_{\text{E,small}}} (B)_{\text{lax}} \times \text{ALG}_{\text{CMon}_{\text{lax}}(\text{Cat}_E)} (B')_{\text{lax}} \to \text{ALG}_{\text{Cats}_U} (B \times B')_{\text{lax}}.$$ 

sends a pair $(f, g)$ of a strict morphism of $B$-algebras $f$ and a strict morphism of $B'$-algebras $g$ to a strict morphism of $B \times B'$-algebras. The same results holds for strong morphisms. One thus gets a commutative diagram

$$
\begin{array}{ccc}
\text{ALG}_{\text{Operad}^\text{op}_{\text{E,small}}} (B)_{\text{lax}} \times \text{ALG}_{\text{CMon}_{\text{lax}}(\text{Cat}_E)} (B')_{\text{lax}} & \to & \text{ALG}_{\text{Cats}_U} (B \times B')_{\text{lax}} \\
\uparrow & & \uparrow \\
\text{ALG}_{\text{Operad}^\text{op}_{\text{E,small}}} (B)_{\text{strong}} \times \text{ALG}_{\text{CMon}_{\text{lax}}(\text{Cat}_E)} (B')_{\text{strong}} & \to & \text{ALG}_{\text{Cats}_U} (B \times B')_{\text{strong}} \\
\uparrow & & \uparrow \\
\text{ALG}_{\text{Operad}^\text{op}_{\text{E,small}}} (B)_{\text{strict}} \times \text{ALG}_{\text{CMon}_{\text{lax}}(\text{Cat}_E)} (B')_{\text{strict}} & \to & \text{ALG}_{\text{Cats}_U} (B \times B')_{\text{strict}}.
\end{array}
$$

**Proof.** This is a direct consequence of [GriB, Proposition 12]. □

### 3.7. The op construction and coalgebras.

**Definition 38.** For any enriched category $\mathcal{C}$, let $\mathcal{C}^{\text{op}}$ be the enriched category with the same set of objects and so that

$$\mathcal{C}^{\text{op}}(x, y) = \mathcal{C}(y, x), \quad x, y \in \text{Ob}(\mathcal{C}).$$

The units and compositions are given by those of $\mathcal{C}$. This defines two inverse strict context functors

$$-^{\text{op}} : \text{Cat}_E \to \text{Cat}_E^{\text{op}}; \quad -^{\text{op}} : \text{Cat}_E^{\text{op}} \to \text{Cat}_E.$$

**Definition 39.** Let

$$\text{Cog}_-(-) : \text{Operad}^{\text{op}_{\text{E,small}}} \times \text{CMon}_{\text{plax}}(\text{Cat}_E) \to \text{Cats}_U$$

be the lax context functor defined as the composition

\[
\begin{array}{ccc}
\text{Operad}_{E,\text{small}}^{\text{coop}} & \times & \text{CMon}_{\text{oplax}}(\text{Cat}_E) \\
\downarrow & & \downarrow \\
\text{Operad}_{E,\text{small}}^{\text{coop}} & \times & \text{CMon}_{\text{lax}}(\text{Cat}_E)_{\text{opp}} \\
\downarrow & & \downarrow \\
(\text{Operad}_{E,\text{small}}^{\text{op}} & \times & \text{CMon}_{\text{lax}}(\text{Cat}_E))_{\text{opp}} \\
\downarrow & & \downarrow \\
\text{Cats}_{\text{opp}} & \rightarrow & \text{Cats}_{\text{opp}} \\
\end{array}
\]

In particular, for any enriched operad and any symmetric monoidal enriched category, we have

\[\text{Cog}_{\mathcal{C}}(\mathcal{P}) = \text{Alg}_{\mathcal{C}}(\mathcal{P})_{\text{opp}}.\]

**Definition 40.** For any symmetric monoidal enriched category \(\mathcal{C}\) and any morphism of operads \(f : \mathcal{P} \rightarrow \mathcal{P}'\), we denote \(f^*\) the functor

\[f^* = \text{Cog}_{\mathcal{C}}(f) : \text{Cog}_{\mathcal{C}}(\mathcal{P}') \rightarrow \text{Cog}_{\mathcal{C}}(\mathcal{P}).\]

**Corollary 6.** Let \(B, B'\) be two categorical operads. Then, the lax context functor

\[\text{Cog}_{-}(-) : \text{Operad}_{E,\text{small}}^{\text{coop}} \times \text{CMon}_{\text{oplax}}(\text{Cat}_E) \rightarrow \text{Cats}_{\text{opp}}\]

induces a lax context functor that is natural with respect to \(B, B'\):

\[\text{ALG}_{\text{Operad}_{E,\text{small}}^{\text{coop}}} (B)_{\text{lax}} \times \text{ALG}_{\text{CMon}_{\text{oplax}}(\text{Cat}_E)} (B')_{\text{lax}} \rightarrow \text{ALG}_{\text{Cats}_{\text{opp}}} (B \times B')_{\text{lax}}.\]

Moreover, this 2-functor sends pairs of strict (resp. strong) morphisms of algebras to strict (resp. strong) morphisms of algebras.

**Proof.** This follows from the same arguments as those used in the proof of Theorem 1. \(\square\)

### 3.8. Structure of pseudo-commutative algebras on endomorphisms operads.

The 2-functor \(\mathcal{E}\text{nd}\) factorises as

\[\text{CMon}_{\text{lax}}(\text{Cat}_E) \rightarrow \text{CMon}_{\text{lax}}\text{CMon}_{\text{lax}}(\text{Cat}_E) \xrightarrow{\text{CMon}_{\text{lax}}(\mathcal{E}\text{nd})} \text{CMon}_{\text{lax}}(\text{Operad}_E) \rightarrow \text{Operad}_E\]

where the first component is the 2-functor described in Proposition 1 and the last component is just the forgetful 2-functor.

In particular, for any symmetric monoidal enriched category \(\mathcal{C}\), the enriched operad \(\mathcal{E}\text{nd}(\mathcal{C})\) is a pseudo commutative monoid in the monoidal context \(\text{Operad}_E\). Unfolding the definitions, its product is the morphism of enriched operads from \(\mathcal{E}\text{nd}(\mathcal{C}) \otimes_{\mathcal{H}} \mathcal{E}\text{nd}(\mathcal{C})\) to \(\mathcal{E}\text{nd}(\mathcal{C})\) whose underlying function
sends a pair \((x, y)\) to \(x \otimes y\) and whose structural morphisms in \(E\) are the maps

\[
\mathcal{E}\text{nd}(C) \otimes_{H} \mathcal{E}\text{nd}(C) \left( ((x_1, y_1), \ldots, (x_n, y_n)), (x_0, y_0) \right)
\]

\[
\mathcal{C}(x_1 \otimes \cdots \otimes x_n) \otimes (y_1 \otimes \cdots \otimes y_n)
\]

\[
\mathcal{C}( (x_1 \otimes y_1) \otimes \cdots \otimes (x_i \otimes y_i), x_0 \otimes y_0)
\]

\[
\mathcal{E}\text{nd}(C) \left( (x_1 \otimes y_1), \ldots, (x_n \otimes y_n); (x_0 \otimes y_0) \right).
\]

The unit, associator, commutator and unitors of \(\mathcal{E}\text{nd}(C)\) are just given by those of \(C\).

Besides, for any lax functor between two symmetric monoidal enriched categories \(f: C \to D\), the morphism \(\mathcal{E}\text{nd}(f)\) has the canonical structure of a lax morphism of pseudo-commutative monoids. This lax structure is actually just given by that of \(f\).

4. Algebras and coalgebras in a symmetric monoidal enriched category

Let \(C\) be a symmetric monoidal enriched category. The goal of this section is to describe monoidal structures of categories of algebras (or coalgebras) in \(C\) over a small enriched operad \(P\) induced by some coalgebraic structures on \(P\).

From now on, \(\text{Cats}\) will denote the \(\mathcal{U}\)-categories (that were previously denoted \(\text{Cats}_{\mathcal{U}}\)).

4.1. Monoidal structure on algebras and coalgebras. Remember from Corollary 1 that \(C\) has the canonical structure of an \(E_{\infty,\text{cat}}\)-algebra (and in particular the structure of an \(A_{\infty,\text{cat}}\)-algebra) within the monoidal context of symmetric monoidal enriched categories and strong monoidal functors.

**Corollary 7.** Let \(B\) be an operad in sets (thus it is a categorical operad). Then, the 2-functor from \(\text{Operad}_{E,\text{small}}^\text{op}\) to \(\text{Cats}\) that sends an enriched operad \(Q\) to its category of algebras in \(C\) yields a functor

\[
\text{sk} \left( \text{COG}_{\text{Operad}_{E,\text{small}}} (B)_{\text{strict}} \right)^{\text{op}} = \text{sk} \left( \text{ALG}_{\text{Operad}_{E,\text{small}}} (B)_{\text{strict}} \right) \to \text{sk} \left( \text{ALG}_{\text{Cats}} \left( E_{\infty,\text{cat}} \times B \right)_{\text{strict}} \right);
\]

\[
(P_{o})_{o \in \text{Ob}(B)} \mapsto (\text{Alg}_{C} (P_{o}))_{o \in \text{Ob}(B)}
\]

that is natural with respect to \(B\).
Proof. Such a functor may be actually decomposed as

\[
\text{sk} \left( \text{ALG}_{\text{Operad}^\text{op}_{\text{E,small}}} (B)_{\text{strict}} \right) \\
\downarrow \\
\text{sk} \left( \text{ALG}_{\text{Operad}^\text{op}_{\text{E,small}}} (B)_{\text{strict}} \right) \times \ast \\
\downarrow \text{id} \times C \\
\text{sk} \left( \text{ALG}_{\text{Operad}^\text{op}_{\text{E,small}}} (B)_{\text{strict}} \right) \times \text{sk} (\text{CMon}_{\text{strict}}(\text{Cat}_E)) \\
\downarrow \\
\text{sk} \left( \text{ALG}_{\text{Operad}^\text{op}_{\text{E,small}}} (B)_{\text{strict}} \right) \times \text{sk} (\text{ALG}_{\text{CMod}_{\text{str}}}(\text{Cat}_E)(E_{\infty,\text{cat}})_{\text{strict}}) \\
\downarrow \\
\text{sk} (\text{ALG}_{\text{Cats}}(E_{\infty,\text{cat}} \times B)_{\text{strict}}) \\
\text{Id} \times C
\]

The third map is described in Corollary 1 and the last map is described in Theorem 1.

One can work with the opposite symmetric monoidal enriched category $C^{\text{op}}$.

**Corollary 8.** Let $B$ be an operad in sets. Then, the 2-functor from $\text{Operad}^\text{op}_{\text{E,small}}$ to $\text{Cats}$ that sends an enriched operad $Q$ to its category of coalgebras in $C$ yields a functor

\[
\text{sk} \left( \text{COG}_{\text{Operad}^\text{op}_{\text{E,small}}} (B)_{\text{strict}} \right)^{\text{op}} \\
\downarrow \\
\text{sk} \left( \text{ALG}_{\text{Operad}^\text{op}_{\text{E,small}}} (B)_{\text{strict}} \right) \\
\downarrow \\
\text{sk} (\text{ALG}_{\text{Cats}}(E_{\infty,\text{cat}} \times B)_{\text{strict}})_{\text{op}} \\
\text{(P_o)}_{o \in \text{Ob}(B)} \\
\text{sk} \left( \text{ALG}_{\text{Cats}}(E_{\infty,\text{cat}} \times B)_{\text{strict}} \right)_{\text{op}} \\
\text{(Cag}_{\text{C}}(P))_{o \in \text{Ob}(B)}
\]

that is natural with respect to $B$.

**Corollary 9.** The statements of Corollary 7 and Corollary 8 remain valid if $B$ is a planar categorical operad and if we replace the categorical operad $E_{\infty,\text{cat}}$ by the planar categorical operad $A_{\infty,\text{cat}}$.

4.2. Consequences. We draw here some consequences of Corollary 7 and Corollary 8.

4.2.1. Hopf operads. Remember that by a Hopf operad, we mean a counital coassociative comonoid in the category $\text{sk}(\text{Operad}_E)$ of enriched operads for the Hadamard tensor product.

Applying Corollary 7 to the planar categorical operad $u\text{As}$ that encodes unital associative algebras, one gets that the construction

\[
P \in \text{sk}(\text{Operad}_{E,\text{small}}) \mapsto \text{Alg}_C(P)
\]

induces a functor

\[
\text{HopfOperad}^\text{op}_{E,\text{small}} = \text{sk}(\text{ALG}_{\text{Operad}^\text{op}_{\text{E,small}}} (u\text{As})_{\text{strict}})^{\text{op}} \rightarrow \text{sk}(\text{ALG}_{\text{Operad}^\text{op}_{\text{E,small}}} (A_{\infty,\text{cat}})_{\text{strict}}) = \text{MonoidalCategories}.
\]
More concretely, for any small Hopf operad $Q$, the category $\text{Alg}_C(Q)$ has a structure of a monoidal category whose tensor product $A \otimes A'$ is given by

$$Q \to Q \otimes_H Q \to \text{End}(C) \otimes \text{End}(C) \simeq \text{End}(C \otimes C) \to \text{End}(C).$$

Moreover, for any morphism of Hopf operads $f : Q \to Q'$, the induced functor

$$f^* : \text{Alg}_C(Q') \to \text{Alg}_C(Q)$$

is strict monoidal.

One has the same results mutatis mutandis for coalgebras over a Hopf operad using Corollary 

4.2.2. Cocommutative Hopf operad. Pursuing the link between Hopf operads and monoidal structures, for any cocommutative Hopf operad $Q$, the monoidal category of $Q$-algebras in $C$ is actually a symmetric monoidal category. For any morphism of cocommutative Hopf operads $f : Q \to Q'$, the induced functor

$$f^* : \text{Alg}_C(Q) \to \text{Alg}_C(P)$$

remains a strict symmetric monoidal functor. Again, one has the same results mutatis mutandis for coalgebras over a cocommutative Hopf operad.

4.2.3. Comodules and tensorisation. Let $\text{LMod}$ the planar coloured operad in sets that encodes a unital associative algebra and a left module of this algebra. It has two colours $(a, m)$ so that

- $\text{LMod}(a, \ldots, a, a) = *$;
- $\text{LMod}(a, \ldots, a, m; m) = *$;
- $\text{LMod}(\ldots, -) = \emptyset$ otherwise.

A $\text{LMod}$-coalgebra in $\text{sk(Operad}_E_{small})$ is a pair $(Q, P)$ of a small Hopf operad $Q$ and small left $Q$-comodule $P$. Given such a pair, the pair of categories

$$(\text{Alg}_C(Q), \text{Alg}_C(P))$$

has the structure of a $A_{\infty, \text{cat}} \times \text{LMod}$-algebra in the cartesian monoidal context of categories.

We recover in particular, the structure of a $A_{\infty, \text{cat}}$-algebra (that is the structure of a monoidal category) on $\text{Alg}_C(Q)$ described above. Moreover, the whole structure of a $A_{\infty, \text{cat}} \times \text{LMod}$-algebra on the pair $(\text{Alg}_C(Q), \text{Alg}_C(P))$ induces a tensorisation of the category $\text{Alg}_C(P)$ over $Q$-algebras. Such a tensorisation is given by the map

$$P \to Q \otimes_H P \xrightarrow{\Delta \otimes \text{id}} \text{End}(C) \otimes_H \text{End}(C) \to \text{End}(C)$$

for any $Q$-algebra $A$ and any $P$-algebra $A'$. One can notice that the structural natural transformations of the tensorisation are invertible since they proceed from the associators and unitors of $C$.

Similarly, the category $\text{Cog}_C(P)$ is tensored over $Q$-coalgebras.

4.3. The Grothendieck construction perspective. Since $C$ is a pseudo-commutative monoid in $\text{CMon}_{\text{lax}}(\text{Cat}_E)$, it should correspond to some weak notion of a lax $E_{\infty, \text{cat}}$-morphism

$$* \to \text{CMon}_{\text{lax}}(\text{Cat}_E)$$

that is less tight than our notion of lax $E_{\infty, \text{cat}}$-morphism. Indeed, we can notice that a lax $E_{\infty, \text{cat}}$-morphism as defined in \cite{Gr} would not give a pseudo-commutative monoid but a strictly commutative monoid. Then, the composite map

$$\text{Operad}_{E_{small}}^{op} \simeq \text{Operad}_{E_{small}}^{op} \times * \xrightarrow{\text{id} \times C} \text{Operad}_{E_{small}}^{op} \times \text{CMon}_{\text{lax}}(\text{Cat}_E) \xrightarrow{\text{Alg}(\cdot)} \text{Cat}_{ul}$$

should have the same structure of a weak lax $E_{\infty, \text{cat}}$-morphism. In particular the composite 2-functor

$$\text{sk(Operad}_{E_{small}}^{op}) \to \text{Operad}_{E_{small}}^{op} \to \text{Cat}_{ul}$$

should also have this structure of a weak lax $E_{\infty, \text{cat}}$-morphism. Such a structure is related by the Grothendieck construction to the notion of a symmetric monoidal fibration as described in \cite{Shu08}. Let us first describe the Grothendieck construction $\text{OpAlg}_C$ of the 2-functor $\text{Alg}_C(\cdot) : \text{sk(Operad}_{E_{small}}^{op}) \to \text{Cat}_{ul}$. 


**Definition 41.** Let $\text{OpAlg}_C$ be the category whose objects are pairs $(\mathcal{P}, A)$ of a small enriched operad $\mathcal{P}$ and a $\mathcal{P}$-algebra $A$ in $\mathcal{C}$. A morphism from $(\mathcal{P}, A)$ to $(\mathcal{P}', A')$ is the data of a morphism of enriched operads $f : \mathcal{P} \to \mathcal{P}'$ and a morphism of $\mathcal{P}$-algebras,

$$g : A \to f^* A'.$$

In particular, the fiber of the forgetful functor $\text{OpAlg}_C \to \text{sk}(\text{Operad}_{E, \text{small}})$ above an enriched operad $\mathcal{P}$ is the category $\text{Alg}_C(\mathcal{P})$.

**Proposition 15.** The category $\text{OpAlg}_C$ admits the structure of a symmetric monoidal category so that the forgetful functor towards $\text{sk}(\text{Operad}_{E, \text{small}})$ is a symmetric monoidal cartesian fibration (Prop. 17) in the sense that

> it is a cartesian fibration;

> it is a symmetric monoidal functor;

> for any two cartesian lifting maps in $\text{OpAlg}_C$, their tensor product is also cartesian.

**Proof.** First, the forgetful functor towards $\text{sk}(\text{Operad}_{E, \text{small}})$ is a cartesian fibration since, for any morphism of enriched operads $f : \mathcal{P} \to \mathcal{P}'$ and any $\mathcal{P}'$-algebra $A'$, a cartesian lifting of $f$ at $A'$ is given by the pair $(f, \text{Id}_{A'})$.

The tensor product $(P_1, A_1) \hat{\otimes} (P_2, A_2)$ is the pair $(P_1 \hat{\otimes} P_2, A_1 \hat{\otimes} A_2)$, where $A_1 \hat{\otimes} A_2$ is the algebra given by the map

$$P_1 \hat{\otimes} P_2 \rightarrow A_1 \hat{\otimes} A_2, \quad \text{End}(C) \hat{\otimes} \text{End}(C) \simeq \text{End}(C \otimes C) \xrightarrow{\text{End}(- \otimes C)} \text{End}(C).$$

The unit is the pair $(\text{uCom}, 1_C)$, where the monoidal unit $1_C$ of $\mathcal{C}$ is equipped with its canonical structure of a commutative algebra. The associator, commutator and unitors proceed from those of $\mathcal{C}$ and from those of $\text{Operad}_{E, \text{small}}$. It is clear that the forgetful functor towards operads is a strict symmetric monoidal functor. Finally, one can notice that cartesian morphism are pairs $(f, g)$ so that $g$ is an isomorphism. It is then clear that the tensor product of two cartesian morphisms is cartesian. □

The monoidal structures on categories of algebras described above (Corollary 7) may be rethought in terms of the symmetric monoidal cartesian fibration $\text{OpAlg}_C \rightarrow \text{Operad}_{E, \text{small}}$. For instance, let us consider a Hopf enriched operad $\mathcal{Q}$. We know from Corollary 7 that the category of $\mathcal{Q}$-algebras in $\mathcal{C}$ has the structure of a monoidal category. Then, denoting $- \otimes \mathcal{Q}$ — the underlying tensor product and $1_{\mathcal{Q}}$ the monoidal unit we have:

> For any two $\mathcal{Q}$-algebras $A_1, A_2$ in $\mathcal{C}$, we have a cartesian map in $\text{OpAlg}_C$

$$(\mathcal{Q}, A_1) \hat{\otimes} (\mathcal{Q}, A_2) \rightarrow (\mathcal{Q}, A_1) \otimes (\mathcal{Q}, A_2) = (\mathcal{Q} \hat{\otimes} \mathcal{Q}, A_1 \hat{\otimes} A_2)$$

above the coproduct map $\mathcal{Q} \rightarrow \mathcal{Q} \hat{\otimes} \mathcal{Q}$. This actually defines the tensor product $A_1 \hat{\otimes} A_2$ up to a unique isomorphism.

> We have a cartesian map

$$(\mathcal{Q}, 1_{\mathcal{Q}}) \rightarrow (\text{uCom}, 1_C) = 1_{\text{OpAlg}_C}$$

above the counit map $\mathcal{Q} \rightarrow \text{uCom}$. This actually defines the monoidal unit $1_{\mathcal{Q}}$ up to a unique isomorphism.

> For any three $\mathcal{Q}$-algebras $A_1, A_2, A_3$ in $\mathcal{C}$, the horizontal maps of the following diagram

$$(\mathcal{Q}, (A_1 \hat{\otimes} A_2) \otimes (Q, A_3)) \rightarrow ((\mathcal{Q}, A_1) \otimes (\mathcal{Q}, A_2)) \otimes (\mathcal{Q}, A_3)$$

are cartesian since cartesian maps are stable through composition and tensor product. Moreover, the right vertical map is an isomorphism. Hence, the two maps

$$(\mathcal{Q}, (A_1 \hat{\otimes} A_2) \otimes (Q, A_3)) \rightarrow (\mathcal{Q}, A_1) \otimes ((\mathcal{Q}, A_2) \otimes (\mathcal{Q}, A_3))$$
are cartesian liftings of the same morphism of operads. Thus, we get a unique isomorphism in $\mathbf{OpAlg}_C$ from $(Q, (A_1 \otimes_Q A_2) \otimes_Q A_3)$ to $(Q, A_1 \otimes_Q (A_2 \otimes_Q A_3))$ above the identity of $Q$; this is equivalently an isomorphism in $\mathbf{Alg}_C(Q)$ which is actually the associator of the monoidal structure.

The unitors may be recovered in a similar way from the fact that the forgetful functor $\mathbf{OpAlg}_C \to \mathbf{Operad}_{E,small}$ is a symmetric monoidal cartesian fibration.

One can make the same work for coalgebras.

**Definition 42.** Let $\mathbf{OpCog}_C$ be the category whose objects are pairs $(P, V)$ of a small enriched operad $P$ and a $P$-coalgebra $V$ in $C$. A morphism from $(P, V)$ to $(P', V')$ in this category is the data of a morphism of enriched operads $f : P \to P'$ and a morphism of $P$-coalgebras $V \to f^*(V')$.

**Corollary 10.** The category $\mathbf{OpCog}_C$ admits the structure of a symmetric monoidal category so that the forgetful functor towards $\mathbf{Operad}_{E,small}$ is a symmetric monoidal cartesian fibration (Proposition [15] and [Shul8]).

**Proof.** This follows from the same arguments as those used to prove Proposition [15]. □

Again, the monoidal structures on categories of coalgebras described above (Corollary [8]) may be rethought using the symmetric monoidal cartesian fibration $\mathbf{OpCog}_C \to \mathbf{Operad}_{E,small}$.  

4.4 The tensored case and the cotensored context. Let us suppose that for any object $x \in C$ the functor

$$\mathbf{cat}(C) \to E$$

$$y \mapsto C(x, y)$$

has a left adjoint $z \mapsto z \boxtimes x$. In this context, we get natural maps

$$(z \otimes_E z') \boxtimes (x \otimes_C x') \to (z \boxtimes x) \otimes_C (z' \boxtimes x'),$$

as the adjoint of the composition

$$z \otimes z' \to C(x, z \boxtimes x) \otimes C(x', z' \boxtimes x') \to C(x \otimes x', (z \boxtimes x) \otimes (z' \boxtimes x')).$$

Moreover, for any $x \in \mathbb{Ob}(C)$, the map

$$1_E \xrightarrow{\mathbf{Id}} C(x, x)$$

gives a morphism $1_E \boxtimes x \to x$.

Let $P$ be a small enriched operad. A $P$-algebra $A$ in $C$ is the data of objects $A_0 \in C$, $o \in \mathbb{Ob}(P)$ and morphisms in $\mathbf{cat}(C)$

$$P(o_1, \ldots, o_n; o) \boxtimes (A_{o_1} \otimes \cdots \otimes A_{o_n}) \to A_o$$

that satisfies conditions with respect to the structure of an operad on $P$. In the context where the forgetful functor $\mathbf{Alg}_C(P) \to \mathbf{cat}(C)^{\mathbb{Ob}(P)}$ is monadic, with monad $M_P$, then the following diagram is commutative

$$\xymatrix{ P(o_1, \ldots, o_n; o') \boxtimes (A_{o_1} \otimes \cdots \otimes A_{o_n}) \ar[d] \ar[r] & P(o_1, \ldots, o_n; o) \boxtimes (M_P(A)_{o_1} \otimes \cdots \otimes M_P(A)_{o_n}) \ar[d] \\
A_o & M_P(A)_o. \ar[l] \ar[u] }$$

Let $P'$ be another small enriched operad and let $A'$ be a $P'$-algebra in $C$. The tensor product in the category $\mathbf{OpAlg}_C$ of $(P, A)$ with $(P', A')$ gives the pair

$$(P, A) \otimes (P', A') = (P \otimes_B P', A \otimes A')$$

where $A \otimes A'$ is the $P \otimes_B P'$-algebra so that:

$$(A \otimes A')((o, o')) = A_o \otimes A'_{o'} \quad \forall (o, o') \in \mathbb{Ob}(P) \times \mathbb{Ob}(P') = \mathbb{Ob}(P \otimes_B P').$$
Moreover, the structure of an algebra is given by the maps
\[
(P(o_1, \ldots, o_n, o) \otimes P'(o'_1, \ldots, o'_n, o')) \cong ((A_{o_1} \otimes A'_{o'_1}) \otimes \cdots \otimes (A_{o_n} \otimes A'_{o'_n}))
\]
\[
(P(o_1, \ldots, o_n, o) \otimes P'(o'_1, \ldots, o'_n, o')) \cong ((A_{o_1} \otimes \cdots \otimes A_{o_n}) \otimes (A'_{o'_1} \otimes \cdots \otimes A'_{o'_n}))
\]
\[
(P(o_1, \ldots, o_n, o) \otimes (A_{o_1} \otimes \cdots \otimes A_{o_n})) \otimes (P'(o'_1, \ldots, o'_n, o') \otimes (A'_{o'_1} \otimes \cdots \otimes A'_{o'_n}))
\]
\[
A_o \otimes A'_o.
\]

The structure of a uCom-algebra on the monoidal unit $1_C$ of $\mathcal{C}$ (that yields the monoidal unit of $\text{OpAlg}_C$) is given by the map
\[
u \text{Com}(n) \boxtimes 1_C^{\otimes n} = 1_E \boxtimes 1_C^{\otimes n} \rightarrow 1_C^{\otimes n} \rightarrow 1_C.
\]

One sees, mutatis mutandis the same phenomena for coalgebras over small enriched operads in the context where for any object $y \in \mathcal{C}$ the functor
\[
\text{cat}(\mathcal{C}) \rightarrow E
\]
\[
x \mapsto \mathcal{C}(x, y)
\]
has a left adjoint $z \mapsto (y, z)$.

4.5. **Mapping coalgebras.** Let us consider two small enriched operads $P_0, P_1$. We have a bifunctor
\[
\text{Cog}_C(P_0) \times \text{Cog}_C(P_1) \rightarrow \text{Cog}_C(P_0 \otimes_H P_1)
\]
which is just the restriction to $\text{Cog}_C(P_0) \times \text{Cog}_C(P_1)$ of the tensor product of $\text{OpCog}_C$.

**Proposition 16.** Let us suppose that :
- the symmetric monoidal category $\text{cat}(\mathcal{C})$ is a closed symmetric monoidal category with internal hom object denoted $[-, -]$;
- the category $\text{cat}(\mathcal{C})$ has small products;
- the functors
\[
\text{Cog}_C(P_0) \rightarrow \text{cat}(\mathcal{C})^{\text{Ob}(P_0)}
\]
\[
\text{Cog}_C(P_0 \otimes_H P_1) \rightarrow \text{cat}(\mathcal{C})^{\text{Ob}(P_0 \otimes_H P_1)}
\]
are comonadic;
- the category $\text{Cog}_C(P_0)$ has coreflexive equalisers.

Then, for any $P_1$-coalgebra $V$, the functor
\[
V_0 \in \text{Cog}_C(P_0) \mapsto V_0 \otimes V \in \text{Cog}_C(P_0 \otimes_H P_1)
\]
has a right adjoint. The result still holds mutatis mutandis if we swap $P_0$ and $P_1$ in the proposition.

**Proof.** This is consequence of the adjoint lifting theorem ([Joh75], [Grib Theorem 4]). Indeed, let us denote $Q_0$ and $Q_{0,1}$ the comonad related to the enriched operads respectively $P_0$ and $P_0 \otimes_H P_1$, let us denote $U_{Q_0} : L^{Q_0}$ and $U_{Q_{0,1}} : L^{Q_{0,1}}$ the corresponding comonadic adjunctions, and let us consider the following commutative square diagram of categories

\[
\begin{array}{ccc}
\text{Cog}_C(P_0) & \xrightarrow{V \mapsto V \otimes V_0} & \text{Cog}_C(P_0 \otimes_H P_1) \\
U_{Q_0} \downarrow & & \downarrow U_{Q_{0,1}} \\
\text{cat}(\mathcal{C})^{\text{Ob}(P_0)} & \xrightarrow{x \mapsto (X_0 \otimes V_0, o^0 \otimes_{\text{Ob}(P_0)} o^0_{\otimes P_1})} & \text{cat}(\mathcal{C})^{\text{Ob}(P_0) \times \text{Ob}(P_1)}
\end{array}
\]
The functor on the bottom horizontal arrow has a right adjoint \( R \) defined as:
\[
R(Y)_o = \prod_{o' \in \text{Ob}(\mathcal{P}_1)} [V_{1,o'}, Y_{o,o'}].
\]

Then, the functor that sends a \( \mathcal{P}_0 \otimes_{\mathcal{H}} \mathcal{P}_1 \)-coalgebra \( W \) to the coreflexive equaliser of the pair of maps of \( \mathcal{P}_0 \)-coalgebras
\[
L^{Q_0} R_{Q_0,1} (W) \Rightarrow L^{Q_0} R_{Q_0,1} (W)
\]
is right adjoint to the functor on the top horizontal arrow of the diagram. □

Let \( f : \mathcal{P} \rightarrow \mathcal{P}' \) be a morphism of small enriched operads.

**Proposition 17.** Let us suppose that

- The category \( \text{cat}(\mathcal{C}) \) is complete;
- The functors
  \[
  \text{Cog}_c(\mathcal{P}) \rightarrow \text{cat}(\mathcal{C})^{\text{Ob}(\mathcal{P})}
  \quad \text{Cog}_c(\mathcal{P}') \rightarrow \text{cat}(\mathcal{C})^{\text{Ob}(\mathcal{P}')}
  \]
  are comonadic and the related comonad preserve coreflexive equalisers.

Then the functor \( f^* : \text{Cog}_c(\mathcal{P}) \rightarrow \text{Cog}_c(\mathcal{P}') \) has a right adjoint \( f^! \). Moreover, \( f^* \) is comonadic if the function \( f : \text{Ob}(\mathcal{P}) \rightarrow \text{Ob}(\mathcal{P}') \) is surjective.

**Proof.** The functor
\[
f^* : \text{cat}(\mathcal{C})^{\text{Ob}(\mathcal{P})} \rightarrow \text{cat}(\mathcal{C})^{\text{Ob}(\mathcal{P})}
\]
\[
X \mapsto \left( \text{Ob}(\mathcal{P}) \xrightarrow{f} \text{Ob}(\mathcal{P}') \xrightarrow{X} \mathcal{C} \right)
\]
has a right adjoint \( R \) given by
\[
R(Y)_o = \prod_{o' \in \text{Ob}(\mathcal{P}) | f(o) = o'} Y_{o,o'}
\]
Moreover, the category \( \text{Cog}_c(\mathcal{P}') \) has coreflexive equalisers ([Grib, Corollary 15]). The right adjoint of \( f^* : \text{Cog}_c(\mathcal{P}) \rightarrow \text{Cog}_c(\mathcal{P}') \) is then built using the adjoint lifting theorem ([Joh75, Grib Theorem 4]).

Besides, let us assume that the underlying function of \( f \) on colours is surjective. In that context, the functor \( f^* : \text{Cog}_c(\mathcal{P}) \rightarrow \text{Cog}_c(\mathcal{P}') \) is conservative and preserves coreflexive equalisers. Since it is left adjoint, then it is comonadic. □

**Corollary 11.** Let us assume that the symmetric monoidal category \( \text{cat}(\mathcal{C}) \) is closed and that it is a complete category. Moreover, let us assume that for any small enriched operad \( \mathcal{P} \), the functor
\[
\text{Cog}_c(\mathcal{P}) \rightarrow \text{cat}(\mathcal{C})^{\text{Ob}(\mathcal{P})}
\]
is comonadic and that the related comonad preserves coreflexive equalisers. Then for any morphism of small enriched operads
\[
f : \mathcal{P}_2 \rightarrow \mathcal{P}_0 \otimes_{\mathcal{H}} \mathcal{P}_1
\]
any \( \mathcal{P}_0 \)-coalgebra \( V \) in \( \mathcal{C} \) and any \( \mathcal{P}_1 \)-coalgebra \( W \) in \( \mathcal{C} \), the functors
\[
V' \in \text{Cog}_c(\mathcal{P}_0) \mapsto f^*(V' \otimes W) \in \text{Cog}_c(\mathcal{P}_2)
\]
\[
W' \in \text{Cog}_c(\mathcal{P}_1) \mapsto f^*(V \otimes W') \in \text{Cog}_c(\mathcal{P}_2)
\]
are both left adjoint functors.

**Proof.** This is just a consequence of Proposition 16 and Proposition 17 □

**Corollary 12.** Let us suppose that

- The category \( \text{cat}(\mathcal{C}) \) is cocomplete;
The functors
\[ \text{Alg}_C(P) \to \text{cat}(\text{Ob}(P)) \]
\[ \text{Alg}_C(P') \to \text{cat}(\text{Ob}(P')) \]
are monadic and the related monads preserve reflexive coequalisers.

Then the functor \( f^* : \text{Alg}_C(P) \to \text{Alg}_C(P') \) has a left adjoint \( f_! \). Moreover, \( f^* \) is monadic if the function \( f : \text{Ob}(P) \to \text{Ob}(P') \) is surjective.

**Proof.** This is Proposition 17 written for \( C^{\text{op}} \) instead of \( C \). \( \square \)

Now, let us consider a small Hopf enriched operad \( Q \) and a small left \( Q \)-comodule \( P \). We know that the category of \( Q \)-coalgebras in \( C \) has the structure of a monoidal category and that the category of \( P \)-coalgebras in \( C \) is tensored over that of \( Q \)-coalgebras. We denote as follows the tensorisation functor:

\[ \text{Cog}_C(Q) \times \text{Cog}_C(P) \to \text{Cog}_C(P) \]
\[ (Z,V) \mapsto Z \boxtimes V. \]

**Theorem 2.** Let us suppose that the functor
\[ V \in \text{Cog}_C(P) \mapsto Z \boxtimes V \in \text{Cog}_C(P) \]
has a right adjoint \( W \mapsto \langle W,Z \rangle \) for any \( Q \)-coalgebra \( Z \). Then, the construction
\[ (W,Z) \in \text{Cog}_C(P) \times \text{Cog}_C(Q)^{\text{op}} \mapsto \langle W,Z \rangle \in \text{Cog}_C(P) \]
is natural. Moreover, the category of \( P \)-coalgebras is cotensored over the monoidal category of \( Q \)-coalgebras through this bifunctor.

**Proof.** This follows from the fact that the structure of a tensorisation on \( - \boxtimes - \) is equivalent to the structure of a cotensorisation on \( \langle -, - \rangle \). \( \square \)

**Theorem 3.** Let us suppose that the functor
\[ Z \in \text{Cog}_C(Q) \mapsto Z \boxtimes V \in \text{Cog}_C(P) \]
has a right adjoint \( W \mapsto \{ V,W \} \) for any \( P \)-coalgebra \( V \). Then, the construction
\[ (V,W) \in \text{Cog}_C(P)^{\text{op}} \times \text{Cog}_C(Q) \mapsto \{ V,W \} \in \text{Cog}_C(Q) \]
is natural. Moreover, the category of \( P \)-coalgebras is enriched over the monoidal category of \( Q \)-coalgebras through this bifunctor.

**Proof.** This follows from the fact that the structure of a tensorisation on \( - \boxtimes - \) is equivalent to the structure of an enrichment on \( \langle -, - \rangle \). \( \square \)

**Remark 10.** Why are we talking about mapping coalgebras and not about mapping algebras? After all, coalgebras are just algebras in the opposite category. So talking about coalgebras instead of algebras is just a matter of conventions. Actually, when dealing with usual categories (sets, simplicial sets, chain complexes, ...), one encounters much more mapping coalgebras than mapping algebras (see for instance Section 5). One explanation for this is the fact that the operation
\[ x \in C \mapsto x \otimes y \]
is often left adjoint.
5. Algebras, coalgebras and convolution

5.1. Oplax right modules. Remember from [Grib] that $\mathsf{RM}_{\text{oplax}}$ is the planar categorical operad that encodes a pair of a pseudo-monoid $A$ and an oplax right module $M$ of $A$. For instance, a $\mathsf{RM}_{\text{oplax}}$-algebra in the monoidal context of categories is the data of a monoidal category together with a category cotensored over the opposite of this monoidal category.

It has two colours $(a, m)$ and is generated by operations

\[
\begin{align*}
\mu &\in \mathsf{RM}_{\text{oplax}}(a, a, a) \\
\iota &\in \mathsf{RM}_{\text{oplax}}(; a) \\
\kappa &\in \mathsf{RM}_{\text{oplax}}(m, a, m)
\end{align*}
\]

together with isomorphisms

\[
\begin{align*}
\mu \triangleleft (\mu, 1_a) &\simeq \mu \triangleleft (1_a, \mu) \\
\mu \triangleleft (\iota, 1_a) &\simeq 1_a \simeq \mu \triangleleft (1_a, \iota)
\end{align*}
\]

that satisfy the same relation as in the definition of $A_{\infty, \text{cat}}$ (the pentagon identity and and triangle identities), and morphisms

\[
\begin{align*}
\kappa \triangleleft (\kappa, 1_a) &\to \kappa \triangleleft (1_m, \mu) \\
1_m &\to \kappa \triangleleft (1_m, \iota)
\end{align*}
\]

so that the following diagrams are commutative

\[
\begin{array}{ccc}
(\kappa \triangleleft (\kappa, 1_m)) \triangleleft (\kappa, 1_a, 1_a) & \quad & \kappa \triangleleft (\kappa \triangleleft (\kappa, 1_a), 1_a) \\
\downarrow & & \downarrow \\
(\kappa \triangleleft (1_m, \mu)) \triangleleft (\kappa, 1_a, 1_a) & \quad & \kappa \triangleleft (1_a, \kappa \triangleleft (1_m, \mu)) \\
\downarrow & & \downarrow \\
(\kappa \triangleleft (\kappa, 1_a)) \triangleleft (1_m, 1_a, \mu) & \quad & (\kappa \triangleleft (\kappa, 1_a)) \triangleleft (1_m, \mu, 1_a) \\
\downarrow & & \downarrow \\
(\kappa \triangleleft (1_m, \mu)) \triangleleft (1_m, 1_a, \mu) & \quad & (\kappa \triangleleft (1_m, \mu)) \triangleleft (1_m, \mu, 1_a) \\
\kappa \triangleleft (1_m, \mu \triangleleft (1_a, \mu)) & \quad & \simeq \quad \kappa \triangleleft (1_m, \mu \triangleleft (\mu, 1_a))
\end{array}
\]

\[
\begin{array}{ccc}
\kappa & \quad & \\
\kappa \triangleleft (1_m, 1_a) & \quad & \kappa \triangleleft (1_m, \mu \triangleleft (\iota, 1_a), 1_m) \\
\downarrow & & \downarrow \\
\kappa \triangleleft (\kappa \triangleleft (1_m, \iota), 1_a) & \quad & \kappa \triangleleft (\kappa \triangleleft (1_m, \iota), 1_a) \\
\kappa \triangleleft (\kappa, 1_a) \triangleleft (1_m, \iota, 1_a) & \quad & \kappa \triangleleft (1_m, \mu) \triangleleft (1_m, \iota, 1_a)
\end{array}
\]
Moreover, an oplax right module $M$ of a pseudo-monoid $A$ is called a strong right module if the canonical 2-morphisms

$$
\begin{align*}
M \otimes (A \otimes A) & \xrightarrow{\simeq} (M \otimes A) \otimes A \\
M \otimes A & \xrightarrow{\simeq} M \\
M & \xrightarrow{\simeq} M
\end{align*}
$$

are invertible. The categorical operad $RM_{\text{strong}}$ that encodes a pseudo-monoid $A$ and a strong right module $M$ is built from $RM_{\text{oplax}}$ by imposing that the morphisms

$$
\begin{align*}
\kappa \triangleleft (\kappa, 1_a) & \rightarrow \kappa \triangleleft (1_m, \mu) \\
1_m & \rightarrow \kappa \triangleleft (1_m, \iota)
\end{align*}
$$

are invertible.

**Lemma 4.** Let us suppose that the symmetric monoidal category $E$ is closed and let us denote $[-, -]$ its internal hom. In that context, the enriched category $\mathcal{E}$ induced by $E$ has the canonical structure of a symmetric monoidal enriched category.

**Proof.** The tensor product of $\mathcal{E}$ is the morphism of enriched categories $\mathcal{E} \otimes \mathcal{E} \rightarrow \mathcal{E}$ given by the function on objects

$$(x, y) \in \text{Ob}(\mathcal{E} \otimes \mathcal{E}) \mapsto x \otimes_E y \in \text{Ob}(\mathcal{E})$$

and the structural morphisms in $E$

$$(\mathcal{E} \otimes \mathcal{E})(x, x', y, y') = [x, y] \otimes_E [x', y'] \rightarrow [x \otimes_E x', y \otimes_E y'] = \mathcal{E}(x \otimes_E x', y \otimes_E y').$$

Moreover, its monoidal unit, its associators, unitors and commutators are given by those of the symmetric monoidal category $E$. They satisfy the conditions to form a symmetric monoidal enriched category since they proceed from the structural maps making $E$ a symmetric monoidal category. □

Thus, in the context where the symmetric monoidal category $E$ is closed, and using Proposition 4 and Corollary 4, the symmetric monoidal enriched categories $\mathcal{E}$ and $\mathcal{E}^{\text{op}}$ have canonical structures of $E_{\infty, \text{cat}}$-algebras within $\text{CMon}_{\text{strong}}(\text{Cat}_E) \subset \text{CMon}_{\text{lax}}(\text{Cat}_E)$. In particular, they are $A_{\infty, \text{cat}}$-algebras.

One has an enriched functor

$$
\langle -, - \rangle : \mathcal{E} \otimes \mathcal{E}^{\text{op}} \rightarrow \mathcal{E}
$$
whose underlying function sends a pair of objects \((x, x')\) to \([x', x]\) and whose structural morphisms in \(E\) are the maps

\[ (E \otimes E^{\text{op}})((x, x'), (y, y')) \]

\[ [x, y] \otimes [y', x'] \]

\[ [[x', x], [y', y]] \]

\(\mathcal{E}([x', x], [y', y])\).

that are adjoint to the maps

\[ (\{x, y\} \otimes [y', x']) \otimes [x', x] \simeq [x, y] \otimes [x', x] \otimes [y', x'] \rightarrow [y', y]. \]

**Lemma 5.** The natural maps

\[ [x, y] \otimes [x', y'] \rightarrow [x \otimes x', y \otimes y'] \]

\(I_E \rightarrow [I_E, I_E].\)

form two 2-morphisms in \(\text{Cat}_E\)

\[ (E \otimes E^{\text{op}}) \otimes (E \otimes E^{\text{op}}) \xrightarrow{(-,-) \otimes (-,-)} E \otimes E \]

\[ (E \otimes E) \otimes (E^{\text{op}} \otimes E^{\text{op}}) \xrightarrow{(-,-) \otimes (-,-)} E \otimes E^{\text{op}} \]

\[ E \otimes E^{\text{op}} \xrightarrow{(-,-)} E \]

\[ I \rightarrow E \]

**Proof.** For the first map, this amounts to prove that the following diagram is commutative

\[ [y_0, y_0'] \otimes [x_0, x_0] \otimes [y_1, y_1'] \otimes [x_1, x_1] \rightarrow [[x_0, y_0], [x_0', y_0']] \otimes [[x_1, y_1], [x_1', y_1']] \]

\[ \simeq \]

\[ [y_0, y_0'] \otimes [y_1, y_1'] \otimes [x_0, x_0] \otimes [x_1, x_1] \]

\[ \rightarrow \]

\[ [y_0 \otimes y_1, y_0' \otimes y_1'] \otimes [x_0 \otimes x_1, x_0' \otimes x_1'] \]

\[ \rightarrow \]

\[ [[x_0 \otimes x_1, y_0 \otimes y_1], [x_0' \otimes x_1', y_0' \otimes y_1']] \rightarrow [[x_0, y_0], [x_1, y_1], [x_0', y_0'], [x_1', y_1']]. \]

for any objects \(x_i, x'_i, y_i, y'_i\) (for \(i \in \{0, 1\}\)). This follows from the fact that both maps are adjoint to the same morphism

\[ [y_0, y_0'] \otimes [x_0, x_0] \otimes [y_1, y_1'] \otimes [x_1, x_1] \rightarrow [x_0 \otimes x_1, y_0 \otimes y_1]. \]

The naturality of the second map is clear. \(\square\)

**Proposition 18.** The enriched functor \(\langle - , - \rangle : \mathcal{E} \otimes \mathcal{E}^{\text{op}} \rightarrow \mathcal{E}\) has the structure of a lax monoidal enriched functor whose structural maps are those of Lemma 5.
Proof. The results amounts the commutation of the following diagrams

\[
\begin{array}{c}
([x, y] \otimes [x', y']) \otimes [x'', y'] \\
\xrightarrow{\simeq} [x, y] \otimes ([x', y'] \otimes [x'', y'])
\end{array}
\]

which is straightforward to prove.

\[\square\]

**Proposition 19.** The maps

\[ [x \otimes x', y] \simeq [x' \otimes x, y] \simeq [x', [x, y]] \]

\[ [1_E, x] \simeq x \]

form two 2-morphisms in the 2-category of symmetric monoidal enriched categories and lax morphisms

\[
\begin{array}{c}
\mathcal{E} \otimes \mathcal{E}^{op} \otimes \mathcal{E}^{op} \\
\xrightarrow{\simeq} \mathcal{E} \otimes \mathcal{E}^{op}
\end{array}
\]

\[
\begin{array}{c}
\mathcal{E} \otimes I \\
\xrightarrow{\text{Id} \otimes \text{Id}} \mathcal{E} \otimes \mathcal{E}^{op}
\end{array}
\]

\[
\begin{array}{c}
\mathcal{E} \otimes \mathcal{E}^{op} \\
\xrightarrow{\simeq} \mathcal{E}
\end{array}
\]

\[
\begin{array}{c}
\mathcal{E}^{op} \\
\xrightarrow{\simeq} \mathcal{E}
\end{array}
\]

Proof. Let us consider the first set of maps \([x \otimes x', y] \simeq [x, [x', y]]\). On the one hand, it form a 2-morphism in \(\text{Cat}_E\) since the following diagram is commutative

\[
\begin{array}{c}
[y, y'] \otimes [x_0, x_0] \otimes [x_1, x_1] \\
\xrightarrow{[x_0 \otimes x_1, y], [x_0' \otimes x_1', y']}
\end{array}
\]

\[
\begin{array}{c}
[[x_0 \otimes x_1, y], [x_0', x_1', y']] \\
\xrightarrow{[[x_0 \otimes x_1, y], [x_0', x_1', y']]}[[x_0, [x_0, y]], [x_0', [x_0', y']]]
\end{array}
\]

for any objects \(x_0, x_0', x_1, x_1', y, y'\) of \(E\). Indeed, both maps from the top left to the bottom right of the square are adjoint to the same map

\[
[y, y'] \otimes [x_0, x_0] \otimes [x_1, x_1] \otimes [x_0 \otimes x_1, y] \otimes [x_0' \otimes x_1', y'] \rightarrow y'.
\]

On the other hand, this set of maps form a 2-morphism in \(\text{CMon}_{\text{ax}}(\text{Cat}_E)\) since the following square diagrams are commutative

\[
\begin{array}{c}
[x_0 \otimes x_1, y] \otimes [x_0' \otimes x_1', y'] \\
\xrightarrow{[x_0 \otimes x_1, [x_0', x_1'], y \otimes y']}
\end{array}
\]

\[
\begin{array}{c}
[[x_0 \otimes x_1, y], [x_0', x_1', y']] \\
\xrightarrow{[[x_0 \otimes x_1, y], [x_0', x_1', y']]}[[x_1, [x_0, y]], [x_1', [x_0', y']]]
\end{array}
\]

\[
\begin{array}{c}
[x_1, [x_0, y]] \otimes [x_0', y'] \\
\xrightarrow{[x_1, [x_0, y]], [x_0', y']}
\end{array}
\]

\[
\begin{array}{c}
[[x_1, [x_0, y]], [x_0', y']] \\
\xrightarrow{[[x_1, [x_0, y]], [x_0', y']]}[[x_1 \otimes x_1', [x_0, y] \otimes [x_0', y']], [x_1 \otimes x_1', [x_0 \otimes x_0', y \otimes y']]]
\end{array}
\]

\[
\begin{array}{c}
1_E \\
\xrightarrow{[1_E, 1_E]}
\end{array}
\]

\[
\begin{array}{c}
1_E \\
\xrightarrow{[1_E, 1_E]}
\end{array}
\]

for any objects \(x_0, x_0', x_1, x_1', y, y'\) of \(E\).

The fact that the second set of maps \([1_E, x] \simeq x\) forms a 2-morphism in the 2-category of symmetric monoidal enriched categories and lax morphisms follows from similar arguments.

\[\square\]

**Theorem 4.** If the symmetric monoidal category \(E\) is closed, then the cotensorisation of \(E\) over itself canonically makes \(E\) a strong right module over the pseudo-monoid \(E^{op}\) in the monoidal context \(\text{CMon}_{\text{ax}}(\text{Cat}_E)\).
**Proof.** Proposition 18 provides us with a lax morphism

\[ (-, -) : \mathcal{E} \otimes \mathcal{E}^{\text{op}} \to \mathcal{E} \]

and Proposition 19 provides us with invertible 2-morphisms

\[ (-, -) \circ \left( (-, -) \otimes \text{Id} \right) \to (-, -) \circ \left( \text{Id} \otimes (- \otimes_{\mathcal{E}^{\text{op}}} -) \right) \]

\[ \text{Id} \to (-, -) \circ \left( \text{Id} \otimes 1_{\mathcal{E}} \right) \]

The fact that they define the structure of a strong right module on \( \mathcal{E} \) over the pseudo-monoid \( \mathcal{E}^{\text{op}} \) within the monoidal context \( \text{CMon}_{\text{lax}}(\text{Cat}_E) \) follows from the commutation of the following diagrams

\[
\begin{array}{c}
\left[ x'', [x', [x, y]] \right] \\
\downarrow \\
\left[ x' \otimes x'', [x, y] \right] \\
\downarrow \\
\left[ (x \otimes x') \otimes x'', y \right] \\
\downarrow \\
[x, [1_E, y]]
\end{array}
\]

\[
\begin{array}{c}
\left[ x, [1_E, y] \right] \\
\downarrow \\
\left[ x \otimes 1_E, y \right] \\
\downarrow \\
\left[ 1_E \otimes x, y \right]
\end{array}
\]

for any objects \( x, x', x'', y \) of \( \mathcal{E} \).

**Corollary 13.** Let us suppose that the symmetric monoidal category \( \mathcal{E} \) is closed and let us denote \( [-, -] \) its internal hom. In that context the cotensorisation of \( \mathcal{E} \) over itself canonically makes \( \mathcal{E} \) a strong right module over the pseudo-monoid \( \mathcal{E}^{\text{op}} \) in the monoidal context \( \text{CMon}_{\text{lax}}(\text{Cat}_E) \).

5.2. **Operads and cotensorisation.** Let \( \mathcal{D} \) be a symmetric monoidal enriched category and let us equip it with its canonical structure of an \( A_{\infty, \text{cat}} \)-algebra within the monoidal context of symmetric monoidal enriched categories and lax functors \( \text{CMon}_{\text{lax}}(\text{Cat}_E) \) (Proposition 1). Then let \( \mathcal{C} \) be a symmetric monoidal enriched category equipped with the structure of a \( \mathcal{D}^{\text{op}} \)-oplax right module within the monoidal context \( \text{CMon}_{\text{lax}}(\text{Cat}_E) \).

**Corollary 14.** Let \( \mathcal{Q} \) be a Hopf operad and let \( \mathcal{P} \) be a right \( \mathcal{Q} \)-comodule. Then the category \( \text{Alg}_{\mathcal{C}}(\mathcal{P}) \) is cotensored over the monoidal category \( \text{Cog}_{\mathcal{D}}(\mathcal{P}) = \text{Alg}_{\mathcal{D}^{\text{op}}}(\mathcal{P})^{\text{op}} \). Moreover, for any morphism of \( \text{RM} \)-coalgebras

\[ (f, g) : (\mathcal{Q}, \mathcal{P}) \to (\mathcal{Q}', \mathcal{P}') \]

the functor \( g^* : \text{Alg}_\mathcal{D}(\mathcal{P}') \to \text{Alg}_\mathcal{D}(\mathcal{P}) \) commutes strictly with the cotensorisation structures.

**Proof.** Such a construction is given by the following 2-functor

\[
\begin{array}{c}
\text{sk} \left( \text{COG}_{\text{Operad}_{\text{enr}}}^{\text{op}} \right) \\
\downarrow_{\simeq} \\
\text{sk} \left( \text{COG}_{\text{Operad}_{\text{enr}}}^{\text{op}} \right) \times * \\
\downarrow_{\text{Id} \times (\mathcal{C}, \mathcal{D}^{\text{op}})} \\
\text{sk} \left( \text{COG}_{\text{Operad}_{\text{enr}}}^{\text{op}} \right) \times \text{sk} \left( \text{ALG}_{\text{CMon}_{\text{lax}}(\text{Cat}_E)}^{\text{op}} \right) \\
\downarrow \\
\text{sk} \left( \text{ALG}_{\text{Cats}}^{\text{op}} \right) \\
\downarrow \\
\text{sk} \left( \text{ALG}_{\text{Cats}}^{\text{op}} \right)
\end{array}
\]
where the third map proceeds from Theorem 1 and the last map proceeds from the canonical morphism of categorical operads
\[ \text{RM}_{\text{oplax}} \to \text{RM} \times \text{RM}_{\text{oplax}}. \]

5.3. The Grothendieck construction perspective.

**Proposition 20.** The category \( \text{OpAlg}_C \) is canonically cotensored over the monoidal category \( \text{OpAlg}_{D^{\text{op}}} \).

**Proof.** This is equivalent to the fact that \( \text{OpAlg}_C \) is an oplax right module over the pseudo-monoid \( \text{OpAlg}_{D^{\text{op}}} \) within the monoidal context of categories. For any two elements \( X = (P, A) \in \text{OpAlg}_C \) and \( Y = (P', V) \in \text{OpAlg}_{D^{\text{op}}} \), let us denote \( \langle X, Y \rangle \) the element
\[(P \otimes H P', \langle A, V \rangle) \in \text{OpAlg}_C \]
where \( \langle A, V \rangle \) is the \( P \otimes H P' \)-algebra in \( C \) given by the composition
\[ P \otimes H P' \to \text{End}(C) \otimes H \text{End}(D^{\text{op}}) \cong \text{End}(C \otimes D^{\text{op}}) \xrightarrow{\text{End}((-,-) \otimes \text{Id})} \text{End}(C). \]

This construction is natural and thus defines a functor
\[ \text{OpAlg}_C \times \text{OpAlg}_{D^{\text{op}}} \to \text{OpAlg}_C. \]

Moreover, one has natural maps
\[ (P, A) \to \langle (P, A), 1_{\text{OpAlg}_{D^{\text{op}}}} \rangle \langle (P, A), (Q, V), (Q', V'') \rangle \to \langle (P, A), (Q, V) \otimes \text{OpAlg}_{D^{\text{op}}}(Q', V'') \rangle \]
given by the following 2-morphisms in the 2-category of enriched operads

\[ \begin{array}{ccc}
\mathcal{P} & \xrightarrow{\Lambda} & \text{End}(C) \\
\downarrow & & \downarrow \\
\mathcal{P} \otimes H \text{uCom} & \xrightarrow{\text{End}(C) \otimes H \text{uCom}} & \text{End}(C) \otimes H \text{End}(D^{\text{op}}) \\
\downarrow & & \\
\text{End}(C \otimes D^{\text{op}}) & \xrightarrow{\text{End}((-,-) \otimes \text{Id})} & \text{End}(C)
\end{array} \]

that proceeds from the structure of an oplax right module of \( C \) over \( D^{\text{op}} \).

This functor and these two natural transformations define a cotensorisation of \( \text{OpAlg}_C \) over \( \text{OpAlg}_{D^{\text{op}}} \) since these natural transformations proceeds from the structure of an oplax right module of \( C \) over \( D^{\text{op}} \).

Besides, using the canonical morphism of categorical operads
\[ \text{RM}_{\text{oplax}} \to A_{\infty, \text{cat}} \]
and the monoidal structure on \( \text{sk}(\text{Operad}_{E, \text{small}}) \), we get that the pair of categories \( (\text{sk}(\text{Operad}_{E, \text{small}}), \text{sk}(\text{Operad}_{E, \text{small}})) \) has the structure of a \( \text{RM}_{\text{oplax}} \)-algebra in the monoidal context of categories.
Corollary 15. The forgetful functors
\[ \text{OpAlg}_C \to \text{sk} (\text{Operad}_{E, \text{small}}) \]
\[ \text{OpAlg}_{D^{op}} \to \text{sk} (\text{Operad}_{E, \text{small}}) \]
are cartesian fibrations that form a strict morphism of \( RM_{\text{oplax}-\text{algebras}} \). Moreover, the bifunctor
\[ \left\langle -, - \right\rangle : \text{OpAlg}_C \times \text{OpAlg}_{D^{op}} \to \text{OpAlg}_C \]
sends pairs of cartesian maps to cartesian maps.

Proof. We already know from Proposition 15 that they are cartesian fibrations. The fact that they form a strict morphism of \( RM_{\text{oplax}-\text{algebras}} \) follows from the definitions.

Finally the bifunctor \( \left\langle -, - \right\rangle \) sends pairs of cartesian maps to cartesian maps because for any two morphisms of enriched operads
\[ f : P \to P' \]
\[ g : Q \to Q', \]
any \( P' \)-algebra \( A \) in \( C \) and any \( Q' \)-coalgebra \( V \) in \( D \), then
\[ \left\langle f^*(A), g^*(V) \right\rangle = (f \otimes_H g)^*(\langle A, V \rangle). \]

\[ \square \]

As in Section 4.3, the monoidal structures relating categories of coalgebras in \( D \) and categories of algebras in \( C \) described above (Corollary 14) may be rethinked in terms of the fibrations
\[ \text{OpAlg}_C \to \text{sk} (\text{Operad}_{E, \text{small}}); \]
\[ \text{OpAlg}_{D^{op}} \to \text{sk} (\text{Operad}_{E, \text{small}}). \]

For instance, let us consider a small Hopf enriched operad \( Q \) and an small enriched operad \( P \) that has the structure of right \( Q \)-comodule. We know that the category of \( P \)-algebras in \( C \) is cotensored over the monoidal category of \( Q \)-coalgebras. Then, denoting \( \langle -, - \rangle_Q \) the underlying cotensorisation bifunctor, we have:

\[ \blacktriangleright \] For any \( P \)-algebra \( A \) in \( C \) and any \( Q \)-coalgebra \( V \) in \( D \) we have a cartesian map in \( \text{OpAlg}_C \)
\[ \langle P, \langle A, V \rangle_Q \rangle \to (\langle P, A \rangle, \langle Q, V \rangle) \]
above the decomposition map \( P \to P \otimes_H Q \). This actually defines the cotensor \( \langle A, V \rangle_Q \) up to a unique isomorphism.

\[ \blacktriangleright \] For any \( P \)-algebra \( A \) in \( C \) and any two \( Q \)-coalgebras \( V_1, V_2 \) in \( D \), the horizontal maps of the following diagram
\[ (P, \langle \langle A, V_1 \rangle_Q, V_2 \rangle_Q) \to (\langle \langle P, A \rangle, \langle Q, V_1 \rangle, (Q, V_2) \rangle \]
\[ (P, \langle A, V_1 \otimes_Q V_2 \rangle_Q) \to (\langle P, A \rangle, \langle Q, V_1 \otimes_Q V_2 \rangle) \]
are cartesian since cartesian maps are stable through composition, cotensorisation and tensor product. Thus, we get a canonical map in \( \text{OpAlg}_C \)
\[ (P, \langle \langle A, V_1 \rangle_Q, V_2 \rangle_Q) \to (P, \langle A, V_1 \otimes_Q V_2 \rangle_Q) \]
above the identity of \( P \). This gives us the structural map
\[ (\langle A, V_1 \rangle_Q, V_2 \rangle_Q \to \langle A, V_1 \otimes_Q V_2 \rangle_Q. \]

\[ \blacktriangleright \] The structural map
\[ A \to \langle A, 1_{\text{Cog}_C(Q)} \rangle_Q \]
may be recovered in a similar way.
5.4. **Mapping coalgebras.** Let us consider two small enriched operads $P, Q$. We have a bifunctor

$$\text{Alg}_C(P) \times \text{Cog}_D(Q)^{\text{op}} \longrightarrow \text{Alg}_C(P \otimes H Q)$$

which is just the restriction to $\text{Alg}_C(P) \times \text{Cog}_D(Q)$ of the cotensorisation bifunctor

$$\langle -, - \rangle : \text{OpAlg}_C \times \text{OpAlg}_D^{\text{op}} \rightarrow \text{OpAlg}_C.$$ 

**Corollary 16.** Let us suppose that:

- For any object $y \in \text{Ob}(C)$, the functor
  $$\langle y, - \rangle : \text{cat}(D)^{\text{op}} \rightarrow \text{cat}(C)$$
  $$x \mapsto \langle y, x \rangle$$

  has a left adjoint $z \mapsto \|z, y\|;$
- the category $\text{cat}(D)$ has small products;
- the functor
  $$\text{Alg}_C(P) \rightarrow \text{cat}(C)^{\text{Ob}(P) \times \text{Ob}(Q)}$$

  is monadic and the functor
  $$\text{Cog}_D(Q) \rightarrow \text{cat}(D)^{\text{Ob}(Q)}$$

  is comonadic;
- the category $\text{Cog}_C(Q)$ has coreflexive equalisers.

Then, for any $P$-algebra $A$ the functor

$$V \in \text{Cog}_C(Q)^{\text{op}} \mapsto \langle A, V \rangle \in \text{Alg}_C(P \otimes H Q)$$

has a left adjoint.

**Proof.** This follows from the same arguments as those used to prove Proposition 16. $\square$

**Corollary 17.** Let us suppose that:

- For any object $x \in \text{Ob}(D)$, the functor
  $$\langle -, x \rangle : \text{cat}(C) \rightarrow \text{cat}(C)$$
  $$y \mapsto \langle y, x \rangle$$

  has a left adjoint $z \mapsto x \boxplus z$;
- the category $\text{cat}(C)$ has small coproducts;
- the functors
  $$\text{Alg}_C(P \otimes H Q) \rightarrow \text{cat}(C)^{\text{Ob}(P) \times \text{Ob}(Q)}$$
  $$\text{Alg}_C(P) \rightarrow \text{cat}(C)^{\text{Ob}(P)}$$

  are monadic;
- the category $\text{Alg}_C(P)$ has reflexive coequalisers.

Then, for any $Q$-coalgebra $V$ in $D$ the functor

$$A \in \text{Alg}_C(P) \mapsto \langle A, V \rangle \in \text{Alg}_C(P \otimes H Q)$$

has a left adjoint.

**Proof.** Again, this follows from the same arguments as those used to prove Proposition 16. $\square$

**Corollary 18.** Let us assume that the category $\text{cat}(C)$ is cocomplete, that the category $\text{cat}(D)$ is complete and that both functors

$$\langle y, - \rangle : \text{cat}(D)^{\text{op}} \rightarrow \text{cat}(C)$$
$$\langle -, x \rangle : \text{cat}(C) \rightarrow \text{cat}(C)$$

have a left adjoint for any $x \in \text{Ob}(D), y \in \text{Ob}(C)$. Moreover, let us assume that for any small enriched operad $P$, the functor

$$\text{Alg}_C(P) \rightarrow \text{cat}(C)^{\text{Ob}(P)}$$
is monadic and that the related monad preserves reflexive coequalisers and that the functor
\[ \text{Cog}_D(P) \to \text{cat}(\text{Ob}(P)) \]
is comonadic and that the related comonad preserves coreflexive equalisers. Then for any morphism of small enriched operads
\[ f : P_2 \to P_0 \otimes_H P_1 \]
any \( P_1 \)-coalgebra \( V \) in \( D \) and any \( P_0 \)-algebra \( A \) in \( C \), the functors
\[ A' \in \text{Alg}_C(P_0) \mapsto f^*(\langle A', V \rangle) \in \text{Alg}_C(P_2) \]
\[ V' \in \text{Cog}_D(P_1)^{\text{op}} \mapsto f^*(\langle A, V' \rangle) \in \text{Alg}_C(P_2) \]
both have a left adjoint functor.

Proof. This is just a consequence of Corollary 16, Corollary 17 and Corollary 12.

Now, let us assume that \( Q \) is a small Hopf enriched operad and that \( P \) is a small right \( Q \)-comodule.
We know that the category of \( Q \)-coalgebras in \( D \) has the structure of a monoidal category and that the category of \( P \)-algebras in \( C \) is cotensored over that of \( Q \)-coalgebras through the bifunctor
\[ \langle -,- \rangle : \text{Alg}_C(P) \times \text{Cog}_D(Q)^{\text{op}} \to \text{Alg}_C(P) \]

Theorem 5. Let us suppose that the functor
\[ V \in \text{Cog}_D(Q)^{\text{op}} \mapsto \langle A, V \rangle \in \text{Alg}_C(P) \]
has a left adjoint \( A' \mapsto \{ A', A \} \) for any \( P \)-algebra \( A \) in \( C \). Then, the construction
\[ (A', A) \in \text{Alg}_C(P)^{\text{op}} \times \text{Alg}_C(P) \mapsto \{ A', A \} \in \text{Cog}_D(Q) \]
is natural. Moreover, the category of \( P \)-algebras in \( C \) is enriched over the monoidal category of \( Q \)-coalgebras in \( D \) through this bifunctor.

Proof. This follows from the fact that the structure of an enrichment on \( \langle -,- \rangle \) is equivalent to the structure of a cotensorisation on \( \langle -,- \rangle \).

Theorem 6. Let us suppose that the functor
\[ A \in \text{Alg}_C(P) \mapsto \langle A, V \rangle \in \text{Alg}_C(P) \]
has a left adjoint \( A \mapsto V \boxtimes A \) for any \( Q \)-coalgebra \( V \) in \( D \). Then, the construction
\[ (V, A) \in \text{Cog}_D(Q) \times \text{Alg}_C(P) \mapsto V \boxtimes A \in \text{Alg}_C(P) \]
is natural. Moreover, the category of \( P \)-algebras in \( C \) is tensored over the monoidal category of \( Q \)-coalgebras in \( D \) through this bifunctor.

Proof. This follows from the fact that the structure of a tensorisation on \( - \boxtimes - \) is equivalent to the structure of a cotensorisation on \( \langle -,- \rangle \).

5.5. The case of the ground category. Let us suppose that the ground symmetric monoidal category \( E \) is closed with induced enriched category denoted \( \mathcal{E} \). Let us also suppose that \( C = D = \mathcal{E} \) and that the structure of a \( \text{RM}_{\text{op}} \)-algebra on the pair \( \langle \mathcal{E}^{\text{op}}, \mathcal{E} \rangle \) is the one induced by the internal hom \( [-,-] \) of \( E \) (see Theorem 4).

Let \( A \) be a \( P \)-algebra and let \( V \) be a \( Q \)-coalgebra. In that context, \( \langle A, V \rangle \) is the \( P \otimes_H Q \)-algebra whose underlying objects are
\[ [V_\circ, A_\circ], \quad (\circ, \circ') \in \text{Ob}(P \otimes_H Q) = \text{Ob}(P) \times \text{Ob}(Q) \]
and whose structural maps are adjoint to the maps
\[
P(o_1, \ldots, o_n; o) \otimes Q(o'_1, \ldots, o'_n; o') \otimes ([V_{o_1}^0, A_{o_1}] \otimes \cdots \otimes [V_{o_n}^0, A_{o_n}]) \otimes V_{o'}
\]
\[
P(o_1, \ldots, o_n; o) \otimes [V_{o_1}^0 \otimes \cdots \otimes V_{o_n}^0, A_{o_1} \otimes \cdots \otimes A_{o_n}] \otimes Q(o'_1, \ldots, o'_n; o') \otimes V_{o'}
\]
\[
P(o_1, \ldots, o_n; o) \otimes [V_{o_1}^0 \otimes \cdots \otimes V_{o_n}^0, A_{o_1} \otimes \cdots \otimes A_{o_n}] \otimes V_{o'}^0 \otimes \cdots \otimes V_{o_n}^0
\]
\[
P(o_1, \ldots, o_n; o) \otimes A_{o_1} \otimes \cdots \otimes A_{o_n}
\]

6. Hopf operads in action

In the final section of this part, we suppose that the ground symmetric monoidal category \((E, \otimes, 1)\) is a closed symmetric monoidal category and that is complete and cocomplete. We denote \(E\) the induced \(E\)-enriched category. Moreover, we suppose that for any small enriched operad \(P\) the functor
\[
\text{Alg}_E(P) \to E^{\text{Ob}(P)}
\]
is monadic and that the related monad preserves reflexive coequalisers. We also suppose that the functor
\[
\text{Cog}_E(P) \to E^{\text{Ob}(P)}
\]
is comonadic and that the related comonad preserves coreflexive equalisers. In that context, for any morphism of small enriched operads
\[
f : P_2 \to P_0 \otimes H P_1
\]
the functors
\[
A \in \text{Alg}_E(P_0) \mapsto f^*(A, V_1) \in \text{Alg}_E(P_2)
\]
\[
V \in \text{Cog}_E(P_1)^{\text{op}} \mapsto f^*(A_0, V) \in \text{Alg}_E(P_2)
\]
have a left adjoint for all \(V_1 \in \text{Cog}_E(P_1)\) and all \(A_0 \in \text{Alg}_E(P_0)\) and the functors
\[
V \in \text{Cog}_E(P_0) \mapsto f^*(V \odot V_1) \in \text{Cog}_E(P_2)
\]
\[
V \in \text{Cog}_E(P_1)^{\text{op}} \mapsto f^*(V_0 \odot V) \in \text{Cog}_E(P_2)
\]
have a right adjoint for all \(V_1 \in \text{Cog}_E(P_1)\) and all \(V_0 \in \text{Cog}_E(P_0)\).

The goal of this section is to draw some more concrete consequences of these results for algebras and coalgebras in \(E\).

6.1. Mapping cocommutative coalgebras and linear duality. Any small enriched operad \(P\) is a right and a left comodule over the cocommutative Hopf operad \(u\text{Com}\).

**Corollary 19.** For any \(u\text{Com}\)-algebra \(A\), one has a linear duality functor
\[
\text{Cog}_E(P)^{\text{op}} \to \text{Alg}_E(P)
\]
\[
V \mapsto \langle A, V \rangle
\]
that has a left adjoint.

**Corollary 20.** The category of \(u\text{Com}\)-coalgebras is closed symmetric monoidal. Moreover, the category of \(P\)-algebras in \(E\) and the category of \(P\)-coalgebras in \(E\) are both tensored-cotensored and enriched over \(u\text{Com}\)-coalgebras.

**Corollary 21.** The monoidal category of \(u\text{Com}\)-coalgebras is cartesian.
Proof. Let $V$, $W$ and $Z$ be three $\text{ucom}$-coalgebras. Given two morphisms $f : Z \to V$ and $g : Z \to W$, we can build a morphism $f + g$ as follows

$$Z \to Z \otimes Z \xrightarrow{f \otimes g} V \otimes W.$$ 

Moreover, given a morphism $t = Z \to V \otimes W$, we can build two morphisms

- $d_1(t) : Z \xrightarrow{t} V \otimes W \to V \otimes 1 \simeq V$;
- $d_2(t) : Z \xrightarrow{t} V \otimes W \to 1 \otimes W \simeq W$.

It is clear that $d_1(f + g) = f$ and $d_2(f + g) = g$. Moreover, the fact that the following diagram commutes

$$
\begin{array}{ccc}
Z & \xrightarrow{t} & V \otimes W \\
\downarrow & & \downarrow \\
Z \otimes Z & \xrightarrow{t \otimes t} & V \otimes W \otimes V \otimes W & \simeq & V \otimes W
\end{array}
$$

implies that $t = d_1(t) + d_2(t)$. Thus, we have a bijection

$$\text{hom}_{\text{Cog}_{\text{ucom}}}(Z, V) \times \text{hom}_{\text{Cog}_{\text{ucom}}}(Z, W) \simeq \text{hom}_{\text{Cog}_{\text{ucom}}}(Z, V \otimes W).$$

So, $V \otimes W$ is the product of $V$ with $W$. Finally, $1$ is the final object and the associators, unitors and commutators are those of the cartesian monoidal structure. $\square$

6.2. Planar operads are $u_{\text{As}}$-comodules. For any planar enriched operad $P$, the induced enriched operad $P_\Sigma$ has the natural structure of a $u_{\text{As}}$-comodule as follows

$$P(c^{\sigma^{-1}}; c) \otimes \{\sigma\} \to P(c^{\sigma^{-1}}; c) \otimes \{\sigma\} \to P_\Sigma(c; c) \otimes u_{\text{As}}(n).$$

Thus the functor $-_\Sigma : \text{sk}(\text{plOperad}_E) \to \text{sk}(\text{Operad}_E)$ factorises through a functor

$$\text{sk}(\text{plOperad}_E) \to \text{Comodules}_{\text{sk}(\text{Operad}_E)}(u_{\text{As}}),$$

followed by the forgetful functor.

**Theorem 7.** Let us suppose that coproducts in $E$ are disjoint and pullback-stable. Then, this functor from planar enriched operads to $u_{\text{As}}$-comodules in enriched operads is an equivalence of categories.

Remark 11. Recall that the fact that pullbacks are disjoint means that any map $X \to X \sqcup Y$ is a monomorphism and that any square of the form

$$
\begin{array}{ccc}
\emptyset & \to & X \\
\downarrow & & \downarrow \\
Y & \to & X \sqcup Y
\end{array}
$$

is a pullback.

**Proof.** It is clear that this functor is faithful. Let us show that it is full. Consider two planar enriched operads $P$, $P'$ and a morphism $f$ of comodules from $P_\Sigma$ to $P'_\Sigma$. Let us denote $\phi$ the underlying function of $f$ from $\text{Ob}(P)$ to $\text{Ob}(P')$. Then, $f$ is given by maps of the form

$$f(c; c) : \coprod_{\sigma \in S_n} P(c^{\sigma^{-1}}; c) \otimes \{\sigma\} \to \coprod_{\sigma \in S_n} P'(\phi(c)^{\sigma^{-1}}; \phi(c)) \otimes \{\sigma\}$$
Since this is a morphism of comodules, then the following diagram commutes
\[
\begin{array}{ccc}
P(\mathcal{E}; c) \otimes \{\sigma\} & \longrightarrow & \coprod_{\mu} P(\mathcal{E}; c^\mu; \phi(c)) \otimes \{\mu\} \\
\downarrow & & \downarrow \\
P(\mathcal{E}; c) \otimes \{\sigma\} \otimes \{\sigma\} & \longrightarrow & \coprod_{\mu, \nu} P(\mathcal{E}; c^\mu; \phi(c)) \otimes \{\mu\} \otimes \{\nu\}
\end{array}
\]
Thus the top horizontal map factors through the pullback of the span
\[
\left(\coprod_{\mu} P(\mathcal{E}; c^\mu; \phi(c)) \otimes \{\mu\}\right) \otimes \{\sigma\} \rightarrow \coprod_{\mu, \nu} P(\mathcal{E}; c^\mu; \phi(c)) \otimes \{\mu\} \otimes \{\nu\} \leftarrow \coprod_{\mu} P(\mathcal{E}; c^\mu; \phi(c)) \otimes \{\mu\}
\]
which is by assumption \(P(\mathcal{E}; c^\mu; \phi(c)) \otimes \{\sigma\}\). Subsequently we have maps
\[
g(\mathcal{E}; c; \sigma) : P(\mathcal{E}; c) \rightarrow P(\mathcal{E}; c)
\]
so that \(f(\mathcal{E}; c) = \coprod_{\mu} g(\mathcal{E}; c; \sigma)\). The fact that \(f\) is a natural with respect to the actions of symmetric groups implies that \(g(\mathcal{E}; c; \sigma) = g(\mathcal{E}; c; \sigma \circ \mu)\) for any two permutations \(\sigma, \mu\). Thus, if we denote \(g(\mathcal{E}; c) := g(\mathcal{E}; c; 1)\), we have for any permutation \(\sigma\)
\[
g(\mathcal{E}; c; \sigma) = g(\mathcal{E}; c).
\]
A straightforward check shows that the maps \(g(\mathcal{E}; c) : P(\mathcal{E}; c) \rightarrow P(\mathcal{E}; c)\) form a morphism of planar operads whose image in the category of \(u_4\)-comodules is \(f\). So, the functor from planar enriched operads to \(u_4\)-comodules is full.

Let us show now that this functor is essentially surjective. Let \(P\) be a \(u_4\)-comodule in enriched operads. Then, for any \(n\)-tuple of colours \(\mathcal{E}\) and any colour \(c\), let \(Q(\mathcal{E}; c; \sigma)\) be the following pullback (which is actually a coreflexive equaliser) in \(E\)
\[
\begin{array}{ccc}
Q(\mathcal{E}; c; \sigma) & \longrightarrow & P(\mathcal{E}; c) \otimes \{\sigma\} \\
\downarrow & & \downarrow \\
P(\mathcal{E}; c) & \longrightarrow & P(\mathcal{E}; c) \otimes u_4(n) \longrightarrow \coprod_{\mu} P(\mathcal{E}; c) \otimes \{\mu\}.
\end{array}
\]
and let us denote \(Q(\mathcal{E}; c) := Q(\mathcal{E}; c; 1)\). Since coproducts are pullback stable, we have
\[
P(\mathcal{E}; c) = \coprod_{\sigma} Q(\mathcal{E}; c; \sigma).
\]
Then, one can check that the operadic composition \(\gamma_i\) in \(P\) decomposes into maps
\[
Q(\mathcal{E}; c; \sigma) \otimes Q(\mathcal{E}; c_i; \mu) \rightarrow Q(\mathcal{E}; c; \sigma; u_4; \mathcal{E}_i; \mu).
\]
where \(\sigma \triangleleft \mu = \gamma^{u_4}(\{\sigma\} \otimes \{\mu\})\). In particular, the collection of objects \(Q(\mathcal{E}; c)\) inherits from this operadic composition the structure of a planar operad. Our goal now is to exhibit an isomorphism of enriched operads between \(P\) and \(Q_5\). The commutativity of the following diagram
\[
\begin{array}{ccc}
P(\mathcal{E}; c) & \longrightarrow & \coprod_{\mu} P(\mathcal{E}; c) \otimes \{\mu\} \\
\downarrow & & \downarrow \\
P(\mathcal{E}'; c) & \longrightarrow & \coprod_{\mu} P(\mathcal{E}; c) \otimes \{\mu \circ \nu\}
\end{array}
\]
whose vertical arrows are all isomorphisms induces an isomorphism \(Q(\mathcal{E}; c; \sigma) \simeq Q(\mathcal{E}'; c; \sigma \circ \nu)\) for any two permutations \(\sigma, \nu\). So in particular \(Q(\mathcal{E}; c; \sigma) \simeq Q(\mathcal{E}; c)\). Thus, we get an isomorphism of objects indexed by tuples of colours
\[
P(\mathcal{E}; c) \simeq \coprod_{\sigma} Q(\mathcal{E}; c; \sigma) = \coprod_{\sigma} Q(\mathcal{E}; c; \sigma) \otimes \{\sigma\} = Q_5(\mathcal{E}; c).
\]
Actually, this map relating $P$ to $Q$ commutes with the action of $S$ and is thus an isomorphism of coloured symmetric sequences. Finally, one can check that it commutes with the units and the operadic composition. Hence this is an isomorphism of operads. So the functor from planar enriched operads to $uA$-comodules is essentially surjective. □

\textbf{Corollary 22.} For any planar enriched operad $P$, the category of $P$-algebras in $E$ and the category of $P$-coalgebras in $E$ are both tensoired-cotensored and enriched over $uA$-comodules.

\textbf{Corollary 23.} For any planar enriched operad $P$ and any $uA$-algebra $A$, one has a linear duality functor

$$
\text{Cog}_E(P)^{\text{op}} \to \text{Alg}_E(P)
$$

$$V \mapsto \langle A, V \rangle$$

that has a left adjoint.

6.3. \textit{Categories and their modules as operads and their algebras.} The category $\text{sk}(\text{Cat}_E)$ of enriched-categories is a coreflexive localisation of the category of enriched operads $\text{sk}(\text{Operad}_E)$. Indeed, one can see an enriched category $A$ as an operad by declaring

$$A(c; c') = \emptyset$$

whenever $|c| \neq 1$. The right adjoint to this inclusion functor just sends an enriched operad $P$ to its underlying enriched category.

\textbf{Proposition 21.} The category of enriched categories is canonically isomorphic to the category of Hopf $I$-comodules within enriched operads.

\textit{Proof.} Straightforward. □

Under this point of view, the inclusion $\text{sk}(\text{Cat}_E) \hookrightarrow \text{sk}(\text{Operad}_E)$ is the functor that forgets the structure of a $I$-comodule and its right adjoint is the cofree $I$-comodule functor.

\textbf{Proposition 22.} For any small enriched category $A$, we have a canonical isomorphism of categories

$$\text{Alg}_E(A) \simeq \text{Cog}_E(A^{\text{op}}).$$

that is natural is the sense that for any morphism of small enriched categories $f : A \to B$, the following diagram of categories commutes

$$
\begin{array}{ccc}
\text{Alg}_E(B) & \xrightarrow{f^*} & \text{Alg}_E(A) \\
\downarrow & & \downarrow \\
\text{Cog}_E(B^{\text{op}}) & \xrightarrow{f^*} & \text{Cog}_E(A^{\text{op}}).
\end{array}
$$

\textit{Proof.} Straightforward. □

Subsequently, for any morphism of small enriched categories $f : A \to B$ the functor $f^* : \text{Alg}_E(B) \to \text{Alg}_E(A)$ has a left adjoint and a right adjoint. The left adjoint is $f_!$ while the right adjoint is the composite functor

$$\text{Alg}_E(A) \simeq \text{Cog}_E(A^{\text{op}}) \xrightarrow{(f^*)^*} \text{Cog}_E(B^{\text{op}}) \to \text{Alg}_E(B).$$

In particular, given a small enriched category $A$, the forgetful functor $U^A : \text{Alg}_E(A) \to E^{\text{Ob}(A)}$ has a right adjoint

$$X \mapsto \left( \prod_{c'} \langle A(c', c), X_c \rangle \right)_{c \in \text{Ob}(P)}$$

and a left adjoint

$$X \mapsto \left( \prod_{c'} A(c', c) \otimes X_c \right)_{c \in \text{Ob}(P)}.$$
6.4. **The cartesian case and beyond.** In this section, we suppose that \((E, \times, \ast)\) is a cartesian closed monoidal category.

**Lemma 6.** In the cartesian context, the two forgetful functors

\[
\text{Cog}_E (\mathbf{uCom}) \to \text{Cog}_E (\mathbf{uAs}) \to E
\]

are isomorphisms of categories. The inverse functors with source \(E\) send an object \(X\) to itself equipped with the diagonal coalgebra structure.

**Proof.** Straightforward. □

In that case:

- Any enriched operad \(P\) has a unique structure of a cocommutative Hopf operad. Thus, the category of \(P\)-algebras and the category of \(P\)-coalgebras inherit the structures of closed symmetric monoidal categories.
- For any enriched operad \(P\), the category of \(P\)-algebras and the category of \(P\)-coalgebras are tensored-cotensored-enriched over the category \(E\).

Moreover, for any enriched operad \(Q\), a \(Q\)-comodule (left or right, for the Hadamard tensor product) is just an enriched operad \(P\) together with a morphism \(f : P \to Q\).

**Proposition 23.** Since \(E\) is cartesian, for any enriched operad \(P\) there exists an enriched category \(P_{\text{cart}}\), and a canonical isomorphism of categories

\[
\text{Cog}_E (P) \simeq \text{Cog}_E (P_{\text{cart}}) = \text{Alg}_E (P_{\text{cart}})
\]

that commutes with the forgetful functor towards \(E^{\mathbf{Ob}(P)}\).

**Proof.** A \(P\)-coalgebra is the data of objects \(V(c) \in E\) for any \(c \in \mathbf{Ob}(P)\) and of maps

\[
a(c, c; i) : P(c, c) \times V(c) \to V(c[i])
\]

for \(|c| \geq 1\), so that the following diagram commutes

\[
\begin{array}{ccc}
V(c) \times P(c, c) \times P(c', c[i]) & \xrightarrow{id \times \gamma} & V(c[i]) \times P(c', c[i]) \\
\downarrow{\text{id} \times \gamma} & & \downarrow{\alpha(c, c, c[i])} \\
V(c) \times P(c, c) & \xrightarrow{\text{proj}} & V(c[i])
\end{array}
\]

for any \(1 \leq i \leq |c|\) and any \(1 \leq j \leq |c'|\), the following diagram commutes

\[
\begin{array}{ccc}
V(c) \times P(c, c) \times P(c', c[j]) & \xrightarrow{id \times \gamma} & V(c[j]) \times P(c', c[j]) \\
\downarrow{\text{id} \times \gamma} & & \downarrow{\alpha(c, c, c[j])} \\
V(c) \times P(c, c) & \xrightarrow{\text{proj}} & V(c[j])
\end{array}
\]

for \(1 \leq i < k \leq |c|\) \((c' \text{ may be empty})\), the following diagram commutes

\[
\begin{array}{ccc}
V(c) \times P(c, c) \times P(c', c[k]) & \xrightarrow{id \times \gamma} & V(c[k]) \times P(c', c[k]) \\
\downarrow{\text{id} \times \gamma} & & \downarrow{\alpha(c, c, c[k])} \\
V(c) \times P(c, c) & \xrightarrow{\text{proj}} & V(c[k])
\end{array}
\]

for \(1 \leq k < i \leq |c|\) \((c' \text{ may be empty})\), and the following diagrams commute

\[
\begin{array}{ccc}
V(c) & \xrightarrow{\text{id} \times \eta} & V(c) \times P(c, c) \\
\downarrow{a(c, c, 1)} & & \downarrow{\alpha(c, c, c[i])} \\
V(c) & \xrightarrow{\text{proj}} & V(c)
\end{array}
\]

\[
\begin{array}{ccc}
V(c) \times P(c, c) & \xrightarrow{\text{id} \times \sigma^{-1}(i)} & V(c) \times P(c, c[i]) \\
\downarrow{a(c, c, i)} & & \downarrow{\alpha(c, c, c[i])} \\
V(c) \times P(c, c) & \xrightarrow{\text{proj}} & V(c)
\end{array}
\]

\[
\begin{array}{ccc}
V(c) \times P(c, c) & \xrightarrow{\text{id} \times \sigma^{-1}(i)} & V(c) \times P(c, c[i]) \\
\downarrow{a(c, c, i)} & & \downarrow{\alpha(c, c, c[i])} \\
V(c) \times P(c, c) & \xrightarrow{\text{proj}} & V(c)
\end{array}
\]
for any permutation $\sigma \in S_n$. For any two colours $c', c$, let $X(c, c')$ be the quotient in $E$ of
$$
\prod_{n>0} \prod_{1 \leq i \leq n} \prod_{x \mid i \mid c = n \cdot x \mid c'} \mathcal{P}(c; c')
$$
by the following relations
$$
\prod_{c \mid i \mid c} \sigma \mathcal{P}(c; c) \rightarrow \prod_{c \mid i \mid c} \mathcal{P}(c; c)
$$
$$
\prod_{c \mid i \mid c \neq c'} \mathcal{P}(c; c) \rightarrow \prod_{c \mid i \mid c} \mathcal{P}(c; c) \times \mathcal{P}(c'; [k]).
$$
Then, let $\mathcal{P}_{\text{cart}}$ be the quotient in enriched categories of the enriched category $TX$ freely generated by $X$ by the following relations
$$
\mathcal{P}(c; c) \times \mathcal{P}(c'; [c]) \rightarrow X(c; [c]) \times X(c' ; [c]) \rightarrow X(c; [c])
$$
$$
\mathcal{P}(c; c) \rightarrow X(c; c) \rightarrow TX(c; c).
$$
The small enriched category $\mathcal{P}_{\text{cart}}$ having such a presentation, one gets a canonical isomorphism
$$
\text{Cog}_E(\mathcal{P}) \simeq \text{Alg}_E(\mathcal{P}_{\text{cart}}).
$$

**Corollary 24.** The forgetful functor from $\mathcal{P}$-coalgebras to $E^{\text{Ob}(\mathcal{P})}$ has a right adjoint,
$$
X \mapsto \left( \prod_{c'} \left[ \mathcal{P}_{\text{cart}}(c', c'), X(c') \right] \right)_{c \in \text{Ob}(\mathcal{P})},
$$
and a left adjoint
$$
X \mapsto \left( \prod_{c'} \mathcal{P}_{\text{cart}}(c', c) \times X(c') \right)_{c \in \text{Ob}(\mathcal{P})}.
$$

For any symmetric monoidal category $(F, \otimes, 1_F)$, any oplax symmetric monoidal functor from the cartesian monoidal category $E$ to $F$ factorises essentially uniquely through $u\text{Com}$-coalgebras in $F$. Hence, in some sense $u\text{Com}$-coalgebras in $F$ form the best cartesian approximation of $F$.

**6.5. Changing the ground category.** Consider two bilinear symmetric monoidal categories $(E, \otimes, 1_E)$ and $(F, \otimes, 1_F)$ and a functor $G : E \rightarrow F$. It induces a functor from $E$-enriched coloured symmetric sequences to $F$-enriched coloured symmetric sequences.

**Proposition 24.** Suppose that $G$ is lax symmetric monoidal. Then, for any $E$-enriched operad $\mathcal{P}$, the coloured symmetric sequence $G(\mathcal{P})$ inherits the structure of a $F$-enriched operad. This gives us a functor from $E$-enriched operads to $F$-enriched operads.

**Proof.** Straightforward.

□
Proposition 25. Suppose that $G$ is both lax symmetric monoidal and is opm lax monoidal so that the following diagrams commute

$$
\begin{align*}
G(X \otimes Y) \otimes G(U \otimes V) & \longrightarrow G(X \otimes Y \otimes U \otimes V) \\
G(X) \otimes G(Y) \otimes G(U) \otimes G(V) & \longrightarrow G(X \otimes U \otimes Y \otimes V) \\
G(X) \otimes G(U) \otimes G(X) \otimes G(V) & \longrightarrow G(X \otimes U \otimes Y \otimes V)
\end{align*}
$$

Then, the induced functor from operads enriched in $E$ to operads enriched in $F$ is opm lax monoidal. If $G$ is actually lax symmetric monoidal, then so is the induced functor.

Proof. This amounts to prove that for any $E$-enriched operads $P$ and $Q$, the structural maps that make $G$ lax monoidal

$$
G(P(c; c) \otimes Q(c; c)) \rightarrow G(P(c; c)) \otimes G(Q(c; c))
$$

form morphisms of operads. This follows from the commutation of the above diagrams. \qed

Corollary 25. Under the assumption of Proposition 25, the functor $G$ induces a functor from Hopf operads enriched in $E$ to Hopf operads enriched in $F$ and a functor from Hopf $Q$-comodules to Hopf $G(Q)$-comodules, for any $E$-enriched Hopf operad $Q$. If $G$ is lax symmetric monoidal, the induced functor sends cocommutative Hopf operads to cocommutative Hopf operads.

Example 1. For any bilinear symmetric monoidal category $(E, \otimes, 1_E)$, the essentially unique cocontinuous functor from sets to $E$ that sends the one element set $*$ to $1_E$ is symmetric monoidal. Hence, the image through this functor of any operad in sets in an enriched cocommutative Hopf operad.

Let $R$ be a commutative ring. The normalised Moore complex $N$ is the left adjoint functor from simplicial sets to chain complexes of $R$-modules so that

$$
N(X)_k := K \cdot \{\text{non degenerate } k \text{ simplicies of } X\} = (K \cdot X_k) / \{\text{degenerate } k \text{ simplicies}\}.
$$

Moreover, $d(x) = \sum_{i=0}^{k}(-1)^i d_i(x)$, for any $x \in X_k$.

On the one hand, $N$ has the structure of a lax symmetric monoidal functor given by the Eilenberg-Zilber shuffle map that may be defined as follows. An $k$-simplex of $\Delta[n] \times \Delta[m]$ is a function $f = (f_1, f_2)$ from $\{0, \ldots, k\}$ to $\{0, \ldots, n\} \times \{0, \ldots, m\}$ whose two projections $f_1$ and $f_2$ are nondecreasing. It is degenerate if and only if there exists $1 \leq i \leq k$ so that $f(i-1) = f(i)$. Subsequently a $n+m$-simplex $f = (f_1, f_2)$ is nondegenerate if and only if for any $1 \leq i \leq n+m$ either

- $f_1(i) = f_1(i+1) + 1$ and $f_2(i) = f_2(i) - 1$;
- or $f_1(i) = f_1(i-1) + 1$ and $f_2(i) = f_2(i) - 1$.

Such a simplex is determined by the subset of $\{1, \ldots, n+m\}$ spanned by elements $i$ so that $f_1(i) = f_1(i-1) + 1$. So, the set of nondegenerate $n+m$-simplicies of $\Delta[n] \times \Delta[m]$ is isomorphic to the set of subsets $a \subset \{1, \ldots, n+m\}$ of cardinal $n$ which is isomorphic to the set of subsets $\mathfrak{I} \subset \{1, \ldots, n+m\}$ of cardinal $m$. So, we have $1_n$ and $1_m$ are the top nondegenerate simplicies of respectively $\Delta[n]$ and $\Delta[m]$, then

$$
\text{sh}(1_n \otimes 1_m) = \sum_{a \subset \{1, \ldots, n+m\} : \# a = n} \text{sign}(a) \cdot a,
$$
indeed, elements of lower degrees are image of some top degrees elements through a map such as
are subsets of respectively
Moreover, it suffices to prove the result for top degree elements of the form
one can check that
one can check that
these two maps are natural and since
from the initial object to the final object of the first diagram of Proposition 25 are equal. Since
the commutation of the diagrams that involve the unit is clear. Thus, let us prove that the
Proof. The commutation of the diagrams that involve the unit is clear. Thus, let us prove that the
first diagram commutes, that is the two maps
\[ f, g : N(X \times Y) \otimes N(U \times V) \to N(X \times U) \otimes N(Y \times V) \]
from the initial object to the final object of the first diagram of Proposition 25 are equal. Since
these two maps are natural and since \( N \) preserves colimits, it suffices to prove the result for objects
\( X, Y, U, V \) of the form
\[
\begin{cases}
X = \Delta[n]; \\
Y = \Delta[m]; \\
U = \Delta[l]; \\
V = \Delta[k].
\end{cases}
\]
Moreover, it suffices to prove the result for top degree elements of the form \( a \otimes b \) where \( a \) and \( b \) are subsets of respectively \( \{1, \ldots, n + m\} \) and \( \{1, \ldots, l + k\} \) and of cardinals respectively \( n \) and \( k \); indeed, elements of lower degrees are image of some top degrees elements through a map such as
\[ \Delta[n'] \times \Delta[m'] \times \Delta[l] \times \Delta[k'] \to \Delta[n] \times \Delta[m] \times \Delta[l] \times \Delta[k]. \]
One can check that \( f(a \otimes b) = 0 = g(a \otimes b) \) whenever \( a \neq \{1, \ldots, n\} \) and \( b \neq \{1, \ldots, k\} \). Otherwise, one can check that
\[ f(a \otimes b) = (-1)^{mk}\{1, \ldots, n\} \otimes \{1, \ldots, m\} = g(a \otimes b). \]
Since the monoidal category of simplicial sets is cartesian, then all simplicial operads are cocommutative Hopf. Moreover, Proposition 26 tells us that the images under the functor \( N \) of simplicial operads are Hopf dg operads. However, they are not in general cocommutative Hopf operads. This is the reason why we deal so much about non necessarily cocommutative Hopf operads. Actually, these dg operads should be cocommutative Hopf up to homotopy.

6.6. Mapping operadic structures. Monochromatic enriched operads (as well as operads over a
fixed set of colours) are themselves algebras over a set-theoretical operad \( \text{Op} \). Hence, the category \( \text{Alg}_{\text{SE}}(\text{Op}) \) of monochromatic enriched operads have the structure of a symmetric monoidal category ; this is the Hadamard tensor product. Moreover, the category the category \( \text{Alg}_{\text{SE}}(\text{Op}) \) of \( \text{Op} \)-coalgebras endow a closed symmetric monoidal structure and operads are enriched-tensored-cotensored over \( \text{Op} \)-coalgebras. We call these coalgebras monochromatic enriched co-operads. The case of operads and co-operads in chain complexes is for instance treated in [RiL22].

Definition 43. A monochromatic enriched co-operad is a coalgebra in \( E \) over the operad \( \text{Op} \), that is a sequence \( (D(n))_{n \in \mathbb{N}} \) of objects of \( E \) together with maps
\[
\begin{align*}
\delta_i : D(n + m - 1) & \to D(n) \otimes D(m) \\
\tau & : D(1) \to 1 \\
\sigma_* & : D(n) \to D(n)
\end{align*}
\]
for any $n \geq 1$, $m \geq 0$, any $1 \leq i \leq n$ and any $\sigma \in S_m$, that satisfy relations that are dual to those defining an operad. We denote coOperad$_{E^*}$, and Operad$_{E^*}$, respectively the category of monochromatic enriched co-operads and the category of monochromatic enriched operads.

**Corollary 26.** The category coOperad$_{E^*}$ of monochromatic co-operads admits a closed symmetric monoidal structure given by the Hadamard monoidal structure on monochromatic symmetric sequences. Moreover, the category of monochromatic operads is tensored-cotensored-enriched over the category of monochromatic co-operads.

**Definition 44.** Let Perm be the monochromatic operad in sets generated by $m \in \text{Perm}(2)$ that satisfies the relations
\[
\begin{align*}
  m \circ (1, m) &= m \circ (m, 1); \\
  m \circ (1, m) &= m \circ (1, m^n).
\end{align*}
\]

**Remark 12.** A Perm algebra that admits a unit for the product is actually a unital commutative algebra. Indeed, if $u$ is a unit we have for any element $x, y$
\[
m(x, y) = m(u, m(x, y)) = m(u, m(y, x)) = m(y, x).
\]
I was communicated this fact that makes purposeless the study for themselves of unital Perm algebras by Joost Nuiten.

Let us denote nuOp the operad that encodes nonunital monochromatic operads. Its set of colours is $\mathbb{N}$ and it is generated by composition products $\gamma \in \nu\text{Op}((n, m); n + m - 1)$ and permutations $\sigma^* \in \nu\text{Op}(n; n)$. We have a morphism of set theoretical operads from nuOp to Perm that sends $\sigma^*$ to $1$ and sends any element $\gamma$, to $m$.

**Corollary 27.** The category of monochromatic enriched non unital operads is tensored-cotensored-enriched over Perm-coalgebras in $E$.

**Definition 45.** A nonunital wheeled PROP is the data of objects $P(n, m) \in E$ together with an action of $S^o_n \times S_m$, with horizontal and vertical composition
\[
m_v : P(n, m) \otimes P(k, n) \to P(k, m) \\
m_h : P(n, m) \otimes P(k, l) \to P(n + k, m + l)
\]
and together with contraction operators
\[
\xi_{ij} : P(n, m) \to P(n - 1, m - 1), \quad 1 \leq i \leq n, \quad 1 \leq j \leq m
\]
that satisfy coherence conditions. See [MM09] for more details.

We know from [MM09] that wheeled PROPs are encoded by a set-theoretical operad that we denote WP. This operad is and coloured by $\mathbb{N} \times \mathbb{N}$. The set
\[
\text{WP}((n_1, m_1), \ldots, (n_k, m_k); (n, m))
\]
is spanned by the elements $(G, \phi, \psi)$ where

- $G$ is an equivalence class of nonempty wheeled oriented graphs with $n$ input edges and $m$ output edges, with vertices counted from $1$ to $k$ and with a counting of the inputs (resp. outputs) of the $i^{th}$-vertex from $1$ to $n_i$ (resp. from $1$ to $m_i$);
- $\phi$ is a counting of the inputs of $G$ from $1$ to $n$;
- $\psi$ is a counting of the outputs of $G$ from $1$ to $m$.

The group $S_k$ acts by changing the order of the vertices. Moreover, $\gamma_i((G, \phi, \psi); (G', \phi', \psi'))$ is the element $(G'', \phi'', \psi'')$ defined as follows.

- First $G''$ is the graph obtained by replacing the $i^{th}$ vertex of $G$ by $G'$ and identifying the input (resp. output) $j$ of this vertex to the input (resp. output) $j$ of $G'$. The vertices of $G''$ are counted by inserting the vertices of $G'$ in the ordered set of vertices of $G$ at place $i$. The counting of the inputs and the outputs of each vertex of $G''$ follows from that of $G$ and $G'$.

- The way we have built the graph $G''$ induces an isomorphisms relating respectively the inputs of $G$ and $G''$ and the outputs of $G$ and $G''$. Then the maps $\phi''$ and $\psi''$ are defined respectively as the compositions of these isomorphisms with $\phi$ and $\psi$. 
**Definition 46.** The wheeled commutative operad $\text{WCom}$ is the monochromatic set-theoretical operad generated by $m \in \text{WCom}(2)$ and $l \in \text{WCom}(1)$ that satisfy the following relations

\[
\begin{align*}
m \triangleleft (m, 1) &= m \triangleleft (1, m); \\
m^{(01)} &= m; \\
m \triangleleft (l, 1) &= l \triangleleft m = m.
\end{align*}
\]

We have a morphism of set theoretical operads from $\text{WP}$ to $\text{WCom}$ that sends a wheeled graph with a single vertex with $k$ wheels to $l^k$ and a graph with $n \geq 2$ vertices to $m^{n-1}$.

**Corollary 28.** The category of wheeled props is tensored-cotensored-enriched over $\text{WCom}$-coalgebras.

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