EIGENVALUES OF SCHRÖDINGER OPERATORS ON FINITE AND INFINITE INTERVALS

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Abstract. We consider a Sturm-Liouville operator $a$ with integrable potential $q$ on the unit interval $I = [0, 1]$. We consider a Schrödinger operator with a real compactly supported potential on the half line and on the line, where this potential coincides with $q$ on the unit interval and vanishes outside $I$. We determine the relationships between eigenvalues of such operators and obtain estimates of eigenvalues in terms of potentials.

1. Introduction and main results

In this paper we discuss eigenvalues of Schrödinger operators defined by

\begin{align*}
\text{Case 1 :} \quad Ty &= -y'' + qy \quad \text{on} \quad L^2(\mathbb{R}_+), \quad \text{with} \quad y(0) = 0, \\
\text{Case 2 :} \quad \tilde{T}y &= -y'' + qy \quad \text{on} \quad L^2(\mathbb{R}_+), \quad \text{with} \quad y'(0) = 0, \\
\text{Case 3 :} \quad Ty &= -y'' + qy \quad \text{on} \quad L^2(\mathbb{R}),
\end{align*}

and defined by (1.3)-(1.5) on the unit interval. We assume that the potential $q$ satisfies

\begin{equation}
\text{supp} \quad q \in [0, 1], \quad q \in L^1_{\text{real}}(0, 1). \tag{1.2}
\end{equation}

It is well known ([7], [8], [23]) that their spectrum consists of an absolutely continuous part $[0, \infty)$ plus a finite number of simple negative eigenvalues given by

\begin{align*}
\sigma_{ac}(T) &= [0, \infty), \quad \sigma_{d}(T) = \{E_1 < \cdots < E_m < 0\}, \\
\sigma_{ac}(\tilde{T}) &= [0, \infty), \quad \sigma_{d}(\tilde{T}) = \{\tilde{E}_1 < \cdots < \tilde{E}_N < 0\}, \\
\sigma_{ac}(T) &= [0, \infty), \quad \sigma_{d}(T) = \{E_1 < \cdots < E_N < 0\}.
\end{align*}

We introduce Sturm-Liouville operators on the interval $[0, 1]$ with the Dirichlet and Neumann boundary conditions:

\begin{align*}
H_0f &= -f'' + qf, \quad f(0) = f(1) = 0, \\
H_1f &= -f'' + qf, \quad f'(0) = f'(1) = 0,
\end{align*}

and with so-called mixed boundary conditions:

\begin{align*}
H_{01}f &= -f'' + qf, \quad f(0) = f'(1) = 0, \\
H_{10}f &= -f'' + qf, \quad f'(0) = f(1) = 0,
\end{align*}

and the operator $H_{\pi}$ on $[0, 2]$ with 2-periodic boundary conditions:

\begin{equation}
H_{\pi}f = -f'' + q_{\pi}f, \quad q_{\pi} = \begin{cases} q(x), & x \in [0, 1] \\ q(x-1), & x \in [1, 2] \end{cases}. \tag{1.5}
\end{equation}

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Let $\mu_n$ and $\nu_n, n \geq 1$ be eigenvalues of $H_0$ and $H_1$ respectively. Let $\tau_n$ and $\varrho_n, n \geq 1$ be eigenvalues of $H_{01}$ and $H_{10}$ respectively. All these eigenvalues are simple. It is well known, that the spectrum of $H_0$ discrete and its eigenvalues $\lambda_0^+, \ldots, \lambda_n^+, \ldots$ satisfy

$$
\lambda_n^+, \nu_n, \mu_n = (\pi n)^2 + c_0 + o(1), \quad \varrho_n, \tau_n = \pi^2(n - \frac{1}{2})^2 + c_0 + o(1) \quad \text{as} \ n \to \infty,
$$

where $\overline{u,v}$ denotes $\min\{u,v\} \leq \max\{u,v\}$ for shortness and $c_0 = \int_0^1 q dx$.

Given a self-adjoint, bounded from below operator $A$ whose negative spectrum is discrete, we denote by $n_-(A)$ the number of its negative eigenvalues, counted with multiplicities.

**Theorem 1.1.** i) The eigenvalues $E_1 < \cdots < E_m < 0$ of $T$ for the Case 1 satisfy

$$
n_-(T) = n_-(H_{01}) \geq n_-(H_0),
$$

$$
\tau_1 < E_1 < \mu_1 < \tau_2 < E_2 < \mu_2 < \cdots < \tau_m < E_m < \min\{0, \mu_m\}.
$$

ii) The eigenvalues $\tilde{E}_1 < \cdots < \tilde{E}_N < 0$ of $\tilde{T}$ for the Case 2 satisfy

$$
n_-(\tilde{T}) = n_-(H_1),
$$

and

$$
\tilde{E}_1 < E_1 < \tilde{E}_2 < E_2 < \cdots < \tilde{E}_N < E_N, \quad \left\{
\begin{array}{ll}
\text{if} & \tau_N \geq 0 \Rightarrow m = N - 1 \\
\text{if} & \tau_N < 0 \Rightarrow m = N \quad \text{and} \quad \tilde{E}_N < E_N.
\end{array}
\right.
$$

iii) The operator $T$ for the Case 3 has the eigenvalues $\mathcal{E}_1 < \ldots < \mathcal{E}_N$, which satisfy

$$
n_-(T) = n_-(\tilde{T}) = n_-(H_1),
$$

$$
\nu_0 < \tilde{E}_1 < \mathcal{E}_1 < E_1 < \nu_1 < \tilde{E}_2 < \mathcal{E}_2 < \cdots < \nu_{N-1} < \tilde{E}_N < \mathcal{E}_N < \min\{0, E_N\}.
$$

**Remark.** Thus eigenvalues $\{\tau_n\}$ control $\{E_n\}$; $\{\nu_n\}$ control both $\{\tilde{E}_n\}$ and $\{\mathcal{E}_n\}$.

We describe eigenvalues for $T_{xy} = -y'' + \varrho(x)$ on $L^2(\mathbb{R})$ with even potentials $\varrho(x) = q(|x|)$ for all $x \in \mathbb{R}$. Using Theorem 1.1 and the known identity (2.23) we obtain

**Corollary 1.2.** The eigenvalues $\mathcal{E}_1^r < \cdots < \mathcal{E}_N^r < 0$ of $T_{xy}$ coincide with eigenvalues of the operators $T$ and $\tilde{T}$ given by (1.1) and satisfy

$$
\sigma_d(T_e) = \sigma_d(T) \cup \sigma_d(\tilde{T}), \quad n_e = n_-(T_e) = N + m = \begin{cases} 
2N - 1 & \text{if} \ \tau_N \geq 0 \\
2N & \text{if} \ \tau_N < 0
\end{cases}, \quad (1.14)
$$

$$
\nu_0 < \mathcal{E}_1^r < \varrho_1; \quad \tau_1 < \mathcal{E}_2^r < \mu_1; \quad \nu_1 < \mathcal{E}_3^r < \varrho_2; \quad \tau_2 < \mathcal{E}_4^r < \mu_2; \ldots
$$

**Localization of resonances.** We discuss resonances for the class compactly supported potentials $\mathcal{P} = \{ f \in L^1_{\text{real}}(\mathbb{R}_+) : \sup \text{sup} f = 1 \}$. We define the Jost solutions $f_+(x, k)$ of the equation

$$
-f'' + q(x)f = k^2 f, \quad k \in \mathbb{C} \setminus \{0\},
$$

under the conditions: $f_+(x, k) = e^{ixk}$ for $x > 1$. The Jost function $\psi = f_+(0, \cdot)$ is entire and satisfies

$$
\psi(k) = 1 + O(1/k) \quad \text{as} \quad |k| \to \infty, \quad k \in \mathbb{C}_+,
$$

(1.17)
uniformly in \( \arg k \in [0, \pi] \). The function \( \psi(k) \) has \( m \) simple zeros \( k_j, j \in \mathbb{N}_m = \{1, \ldots, m\} \) in \( \mathbb{C}_+ \), given by \( k_j = i E_j |^{2}, j \in \mathbb{N}_m \), possibly one simple zero at 0, and an infinite number (so-called resonances) in \( \mathbb{C}_- : 0 \leq |k_{m+1}| \leq |k_{m+2}| \leq \ldots \ldots \) The function \( \psi \) has an odd number \( \geq 1 \) of zeros on any interval \( I_j := (-k_j, -k_{j+1}), j \in \mathbb{N}_{m-1} \) and an even number \( \geq 0 \) of zeros on the interval \( I_m := (-i|k_m|, 0) \) counted with multiplicity. Let \( \#(E_j) \) be the number of zeros of \( \psi \) (counted with multiplicity) in the set \( E \subset \mathbb{C} \). We describe the localization of resonances on the interval \([-k_1, 0]\) in terms of eigenvalues.

**Theorem 1.3.** i) Let \( q \in \mathcal{P} \) and let a resonance \( k_0 \in I_j \subset i\mathbb{R}_- \) for some \( j \in \mathbb{N}_m \). Then
\[
\mu_j < k_0^2 < \tau_{j+1}, \quad j \in \mathbb{N}_m = \{1, \ldots, m\}. \tag{1.18}
\]

ii) For any finite sequence \( p_1, p_2, \ldots, p_m \in i\mathbb{R}_+ \) (labeled by \( |p_1| > |p_2| > \ldots > |p_m| > 0 \)) and any odd integers \( s_1, \ldots, s_{m-1} \in \mathbb{N} \) and even integer \( s_m \geq 0 \) there exists a potential \( q \in \mathcal{P} \), such that each its eigenvalue \( E_j = p_j^2 < 0 \) and \( \#(I_j) = s_j, j = \mathbb{N}_m \).

**Remark.** 1) Thus eigenvalues \( \mu_j, \tau_{j+1} \) control resonances on the interval \( I_j \).

2) If \( \mu_m \geq 0 \), then there are no resonances on the interval \((-k_m, 0]\).

3) Jost and Kohn \[14\] adapted the method of Gelfand and Levitan to determine a potential (exponentially decaying) of a Schrödinger operator by the spectral data. Theorem 1.3 is used in the paper \[21\] to show that Jost and Kohn solution is not complete and there exists another solution, which gives a compactly supported potential.

**Estimates.** Recall the Bargmann inequality \[3\]: let in general, \( xq(x) \) belong to \( L^1(0, \infty) \), then
\[
n_-(T) \leq \frac{1}{2} \int_{0}^{\infty} xq_-(x)dx, \tag{1.19}
\]
where \( q_- = \max\{-q, 0\} \). The case of distributions was discussed in \[1\], and more operators was studied in \[2\]. Recall the Calogero-Cohn inequality \[4, 5\]: if \( q, |q|^2 \in L^1(0, \infty) \) and if \( q \leq 0 \) is monotone, then
\[
n_-(T) \leq \frac{2}{\pi} \int_{0}^{\infty} |q(x)|^2dx. \tag{1.20}
\]
We recall the estimates from \[19, 20\]: Consider the operator \( H_\pi \) with the potential \( q \in \mathcal{K}_0 = \{q \in L^2_{\text{real}}(0, 1) : \int_{0}^{1} q(x)dx = 0 \} \) having eigenvalues \( \lambda^+_0, \lambda^\pm_n, n \geq 1 \). Let \( \gamma_n = \lambda^+_n - \lambda^-_n \geq 0, n \geq 1 \) and \( \gamma = (\sum_{n \geq 1} \gamma^2_n)^{1/2} \geq 0 \). Then
\[
||q|| \leq 2\gamma \max\{1, \gamma^{1/2} \},
\gamma \leq 2\||q|| \max\{1, ||q||^{1/2} \}. \tag{1.21}
\]

**Corollary 1.4.** Let the operators \( H_0, H_{01}, H_\pi \) given by (1.3)–(1.5).

i) Let \( \mathcal{G} = \sum_{\mu_n < 0} |\mu_n|^2 \) or \( \mathcal{G} = \frac{1}{2} \sum_{n > 0, \lambda_n^+ < 0} |\lambda_n^\pm|^2 \). Then
\[
\frac{1}{2}(n_-(H_\pi) - 1) \leq n_-(H_0) \leq n_-(H_{01}) \leq \int_{0}^{1} xq_-(x)dx, \tag{1.22}
\]
\[
\mathcal{G} \leq \frac{1}{2} \int_{0}^{1} q_-(x)dx, \tag{1.23}
\]
Let \( q(x) \) be monotone. Then
\[
n_-(H_0) \leq n_-(H_{01}) \leq \frac{2}{\pi} \int_0^1 q_-(x)^{1/2} dx.
\] (1.24)

ii) Let \( q \in L^2(0,1) \) and \( \int_0^1 q(x) dx = 0 \). Then
\[
\frac{1}{2}(n_-(H_{\pi}) - 1) \leq n_-(H_0) \leq n_-(H_{01}) \leq 2\gamma \max\{1, \gamma^\frac{1}{4}\},
\] (1.25)

Remark. Instead of \( q \) in Corollary 1.3 we can use \( q - \varepsilon \) for \( \varepsilon \in \mathbb{R} \)

There are a lot of results devoted to estimates of negative eigenvalues of dim \( d = 1 \) Schrödinger operators, see [4], [5], [3], [7], [11], [25], [26] and references therein.

There exist many results about Sturm-Liouville operators on the unit interval, see [6], [12], [22], [23], [24] and on the circle, see [10], [18], [23] and references therein. Unfortunately there only few results about estimates for such Sturm-Liouville operators, for example, two-sided estimates of periodic potentials in terms of gap-lengths see (1.21).

2. Proof

2.1. Fundamental solutions. Let \( \varphi(x, \lambda), \vartheta(x, \lambda) \) be the solutions of the equation
\[-y'' + qy = \lambda y, \quad \lambda \in \mathbb{C}, \] (2.1)

under the conditions \( \varphi'(0, \lambda) = \vartheta(0, \lambda) = 1 \) and \( \varphi(0, \lambda) = \vartheta(0, \lambda) = 0 \). If \( q = 0 \), then the corresponding fundamental solutions are given by \( \varphi_0(x, \lambda) = e^{\frac{\varphi(x, \lambda)}{\sqrt{\lambda}}} \) and \( \vartheta_0(x, \lambda) = \cos x \sqrt{\lambda} \).

Recall the well-known results.

Lemma 2.1. Let \( q \in L^1(0,1) \). Then the functions \( \varphi(1, \lambda), \varphi'(1, \lambda), \vartheta(1, \lambda), \vartheta'(1, \lambda) \) are entire and satisfy:
\[
\varphi(1, \lambda) - \frac{\sin \sqrt{\lambda}}{\sqrt{\lambda}} = e^{\text{Im} \sqrt{\lambda}} O(\lambda^{-1}),
\] (2.2)
\[
\varphi'(1, \lambda) - \cos \sqrt{\lambda} = e^{\text{Im} \sqrt{\lambda}} O(\lambda^{-\frac{3}{4}}),
\] (2.3)

and
\[
\vartheta(1, \lambda) - \cos \sqrt{\lambda} = e^{\text{Im} \sqrt{\lambda}} O(\lambda^{-\frac{1}{4}}),
\] (2.4)
\[
\vartheta'(1, \lambda) + \sqrt{\lambda} \sin \sqrt{\lambda} = e^{\text{Im} \sqrt{\lambda}} O(1),
\]
as \( |\lambda| \to \infty \), uniformly in \( \text{arg} \lambda \in [0, 2\pi] \), and, in particular,
\[
\varphi(1, \lambda), \varphi'(1, \lambda), \vartheta(1, \lambda), \vartheta'(1, \lambda) \to +\infty \quad \text{as} \quad \lambda \to -\infty.
\]

Note that \( \{\mu_n\}, \{\nu_n\}, \{\tau_n\}, \{\vartheta_n\} \) are the zeros of \( \varphi(1, \lambda), \varphi'(1, \lambda), \vartheta(1, \lambda), \vartheta'(1, \lambda) \) respectively.

2.2. The periodic case. We consider the 2-periodic operator \( H_\pi = -\frac{d^2}{dx^2} + q_\pi(x) \) on \([0,2] \) given by (1.5). It is well known, that the spectrum of \( H_\pi \) discrete and is eigenvalues \( \lambda^+_n, \lambda^-_n, n \geq 1 \), which satisfy \( \lambda^-_{n-1} < \lambda^-_n \leq \lambda^+_n, \ n \geq 1 \). The sequence \( \lambda^+_0 < \lambda^-_1 \leq \lambda^+_1 < \ldots \) is the spectrum of equation
\[-y'' + q_\pi y = \lambda y \] (2.5)

with 2-periodic boundary conditions, i.e. \( y(x + 2) = y(x), x \in \mathbb{R} \). If \( \lambda^-_n = \lambda^+_n \) for some \( n \), then this number \( \lambda^+_n \) is a double eigenvalue of Eq. (2.5) with 2-periodic boundary conditions. The lowest eigenvalue \( \lambda^-_0 \) is always simple, and the corresponding eigenfunction is 1-periodic. The
eigenfunctions corresponding to the eigenvalue $\lambda_n^\pm$ are 1-periodic, when $n$ is even and they are antiperiodic, i.e. $y(x+1) = -y(x)$, $x \in \mathbb{R}$, when $n$ is odd. We introduce the Lyapunov function $\Delta = \frac{1}{2}(\varphi'(1, \lambda) + \vartheta(1, \lambda))$ It is well known that
\[
\Delta(\lambda_n^+) = 1, \quad \Delta(\lambda_n^-) = (-1)^n, \quad (-1)^n \Delta([\lambda_n^-, \lambda_n^+]) \subset [1, \infty), \quad n \geq 1. \tag{2.6}
\]

2.3. Jost functions. We recall well-known result about the Jost function from [9, 23, 8]. The Schrödinger equation
\[ -f'' + q(x)f = k^2 f, \quad k \in \mathbb{C} \setminus \{0\}, \tag{2.7} \]
has unique solutions $f_\pm(x, k)$ such that $f_+(x, k) = e^{ikx}$ for $x > 1$ and $f_-(x, k) = e^{-ikx}$ for $x < 0$. Outside the support of $q$ any solutions of (2.7) have to be combinations of $e^{\pm ikx}$. The Wronskian $w$ for Case 3 is given by
\[ w(k) = \{f_-(\cdot, k), f_+(\cdot, k)\} = ikf_+(0, k) + f'_+(0, k), \tag{2.8} \]
where $\{f, g\} = fg' - f'g$. The functions $f_+(0, k)$, $f'_+(0, k)$, $w(k)$ are entire, real on the imaginary line and satisfy as $|k| \to \infty$, $k \in \mathbb{C}_+$:
\[ f_+(0, k) = 1 + O(1/k), \quad f'_+(0, k) = ik + O(1), \quad w(k) = 2ik + O(1), \tag{2.9} \]
uniformly in $\arg \in [0, \pi]$. Let $F$ be one of the functions $f_+(0, \cdot)$, $f'_+(0, \cdot)$ or $w$. The function $F$ has only finite simple number of zeros in the upper half-plane $\mathbb{C}_+$ given by
\[ \begin{align*}
\text{Case 1:} & \quad i|E_1|^{\frac{1}{2}}, \ldots, i|E_m|^{\frac{1}{2}} \in i\mathbb{R}_+, \\
\text{Case 2:} & \quad i|\tilde{E}_1|^{\frac{1}{2}}, \ldots, i|\tilde{E}_N|^{\frac{1}{2}} \in i\mathbb{R}_+, \\
\text{Case 3:} & \quad i|E_1|^{\frac{1}{2}}, \ldots, i|E_N|^{\frac{1}{2}} \in i\mathbb{R}_+, 
\end{align*} \tag{2.10} \]
and an infinite number of zeros, so-called resonances, in $\mathbb{C}_-$ and possibly one simple zero at 0 (for $q \neq 0$). By definition, a zero of $F$ is called a resonance of the corresponding Schrödinger operator. The multiplicity of the resonance is the multiplicity of the corresponding zero of $F$ and it can be any number, see [15]. Introduce the norm $\|u\|^2 = \int_0^1 |u(x)|^2 dx$ in $L^2(0, 1)$.

2.4. Proof Theorem 1.1 i-ii) Dirichlet and Neumann boundary conditions. Recall the identity from [15]:
\[ e^{-ik}f'_+(0, k) = ik\vartheta(1, k) - \vartheta'(1, k). \tag{2.11} \]
From (2.11), (2.4) and (1.6) we have that $\tilde{E}_1 > \nu_0$. Thus if $\nu_0 \geq 0$, then the operator $\tilde{T}$ has no eigenvalues. We fix a real potential $q_0 \in L^1(\mathbb{R}_+)$ such that supp $q_0 \in [0, 1]$ and $\nu_0(q_0) = 0$. Define the potential $q_\varepsilon$, $\varepsilon \in \mathbb{R}$ by
\[ \text{supp } q_\varepsilon \subset [0, 1], \quad q_\varepsilon(x) = q_0(x) - \varepsilon, \quad x \in [0, 1]. \]
We sometimes write $f_+(0, k, \varepsilon), \mu_n(\varepsilon), \ldots$ instead of $f_+(0, k), \mu_n, \ldots$ when several potentials $q_\varepsilon$ are being dealt with. The operators in (1.13)–(1.14) and (1.11) with the potential $q_\varepsilon$ we denote by $H_0(\varepsilon), H_1(\varepsilon), \ldots$ and $T(\varepsilon), \ldots$ and the corresponding eigenvalues by $\mu_n(\varepsilon), \nu_n(\varepsilon), \ldots$ and $E_n(\varepsilon), \ldots$. Note that the eigenvalues $\mu_n(\varepsilon), \nu_n(\varepsilon), \ldots$ satisfy
\[ \mu_n(\varepsilon) = \mu_n(0) - \varepsilon, \quad \nu_n(\varepsilon) = \nu_n(0) - \varepsilon, \quad \tau_n(\varepsilon) = \tau_n(0) - \varepsilon, \quad \varrho_n(\varepsilon) = \varrho_n(0) - \varepsilon. \tag{2.12} \]
Firstly we consider the operator $\widetilde{T}$ with the Neumann boundary condition. Due to (2.11) the Jost function for $\widetilde{T}(\varepsilon)$ for $k = it, t \in \mathbb{R}$ satisfies
\[
e^{-ik}f^+_+(0, k, \varepsilon) = -\vartheta'(1, k^2, \varepsilon) + ik\vartheta(1, k^2, \varepsilon) = ik\vartheta(1, \varepsilon - t^2, 0) - \vartheta'(1, \varepsilon - t^2, 0).
\]
(2.13)

Define functions
\[
F(t, \varepsilon) = -f(t^2, \varepsilon) + tg(t^2, \varepsilon), \quad f(t^2, \varepsilon) = \vartheta'(1, \varepsilon - t^2, 0), \quad g(t^2, \varepsilon) = \vartheta(1, \varepsilon - t^2, q_0).
\]
These functions are entire in $t, \varepsilon$ and satisfy
\[
f(0, 0) = 0, \quad g(0, 0) \neq 0, \quad F(0, 0) = 0, \quad F_t(0, 0) \neq 0.
\]
(2.14)

Then due to the Implicit Function Theorem there exists a function $t(\varepsilon)$, analytic in small disk $\{|\varepsilon| < \tau\}$ such that $F(t(\varepsilon), \varepsilon) = 0$ in the disk $\{|\varepsilon| < \tau\}$ (and here $\widetilde{E}_1(\varepsilon) = -t^2(\varepsilon)$ for $\varepsilon > 0$).

We have $t(\varepsilon) = t_1(\varepsilon + O(\varepsilon^2))$. Then from (2.13) we obtain for $t = t(\varepsilon)$ as $\varepsilon \to 0$:
\[
\vartheta'(1, \varepsilon - t^2, 0) = -t(\varepsilon)\vartheta(1, \varepsilon - t^2, 0) = t_1(\varepsilon)\vartheta(1, 0, 0) + O(\varepsilon^2),
\]
\[
\vartheta'(1, \varepsilon - t^2, 0) = \varepsilon\vartheta'(1, 0, 0) + O(\varepsilon^2),
\]
(2.15)

which yields $t_1 = -\vartheta'(1, 0, 0) / \vartheta(1, 0, 0) > 0$, where $\vartheta = \partial u / \partial t$.

If $\varepsilon > 0$ is small enough, then $\nu_0(\varepsilon) = -\varepsilon$ and $g_1(\varepsilon) = g_1(q_0) - \varepsilon > 0$, since there is the basic relation (1.6). Thus we obtain $\nu_0(\varepsilon) < \widetilde{E}_1(\varepsilon) < 0 < g_1(\varepsilon)$. If $\varepsilon$ is increasing then all eigenvalues $\nu_0(\varepsilon) < \widetilde{E}_1(\varepsilon) < 0 < g_1(\varepsilon)$ move monotonically to left and at $\varepsilon_1 = \tau_1(0)$ we have $\tau_1(\varepsilon) = 0$.

Secondly, we consider the operator $T$ with the Dirichlet boundary condition. The function $f_+(0, k)$ is expressed in terms of the fundamental solutions $\varphi, \vartheta$ by
\[
e^{-ik}f_+(0, k, \varepsilon) = \varphi'(1, k^2) - ik\varphi(1, k^2) \quad \forall k \in \mathbb{C}.
\]
(2.16)

Note that if the operator $H_{01}$ have eigenvalue $\tau_1 \geq 0$, then from (2.16), (2.4) and (1.6) we deduce that the operator $T$ has not any eigenvalue. Let $\varepsilon = \varepsilon_1 + z$, where $\varepsilon_1 = \tau_1(0)$. Then due to (2.16) the Jost function $f_+(0, k, \varepsilon)$ for the operator $T(\varepsilon)$ with satisfies at $k = it, t \in \mathbb{R}$:
\[
e^{-ik}f_+(0, k, \varepsilon) = \varphi'((1, z - t^2, \varepsilon_1) + t\varphi(1, z - t^2, \varepsilon_1).
\]
(2.17)

We rewrite the rhs of the last identity in the form:
\[
F_o(t, \varepsilon) = f_o(t^2, \varepsilon) + tg_o(t^2, \varepsilon), \quad f_o(t^2, z) = \varphi'(1, z - t^2, \varepsilon_1), \quad g_o(t^2, z) = \varphi(1, z - t^2, \varepsilon_1).
\]
These functions are entire in $t, z$ and satisfy
\[
f_o(0, 0) = 0, \quad g_o(0, 0) \neq 0, \quad F_o(0, 0) = 0, \quad \partial F_o(0, 0) \neq 0.
\]
(2.18)

Then the Implicit Function Theorem gives that there exists a function $t_o(z)$, analytic in small disk $\{|z| < \delta\}$ such that $F(t_o(z), z) = 0$ in the disk $\{|z| < \delta\}$. Here we have $E_1(\varepsilon) = t_o^2(z), z = \varepsilon - \varepsilon_1$. We have $t(z) = t_1 z + O(z^2)$ as $z \to 0$. Then from $f_+(0, k, \varepsilon) = 0$ and (2.17) we obtain
\[
\varphi'(1, z - t^2, \varepsilon_1) = -t(z)\varphi(1, z - t^2, \varepsilon_1) = -t_1 z\varphi(1, 0, \varepsilon_1) + O(z^2),
\]
\[
\varphi'(1, z - t^2, \varepsilon_1) = \varphi'(1, 0, \varepsilon_1) z + O(z^2),
\]
(2.19)

which yields $t_1 = -\varphi'(1, 0, \varepsilon_1) / \varphi(1, 0, \varepsilon_1) > 0$, since $\varphi(1, \lambda, \varepsilon_1) \to \infty$ as $\lambda \to -\infty$ and $\mu_1 > 0$. 

If $z > 0$ is small enough, then $\tau_1(\varepsilon) = -z$ and $\mu_1(\varepsilon) = \mu_1(0) - \varepsilon > 0$, since there is the basic relation (1.6). Thus we obtain

$$\tau_1(\varepsilon) < E_1(\varepsilon) < 0 < \mu_1(\varepsilon), \quad \text{and} \quad \bar{E}_1(\varepsilon) < E_1(\varepsilon).$$

It is important that $\bar{E}_1(\varepsilon) < E_1(\varepsilon)$ for any $\varepsilon$ since $\sigma_d(T) \cap \sigma_d(\bar{T}) = \emptyset$. If $\varepsilon$ is increasing then all eigenvalues $\tau_1(\varepsilon) < E_1(\varepsilon) < \mu_1(\varepsilon)$ and $\bar{E}_1(\varepsilon) < E_1(\varepsilon)$ move monotonically to left. If $\varepsilon$ is increasing more, then we have $\mu_1(\varepsilon_1) = 0$ at $\varepsilon_1 = \mu_1(0)$, but this eigenvalues does not "create" eigenvalues for the operators $T(\varepsilon), \bar{T}(\varepsilon)$. If $\varepsilon$ is increasing again then we have $\nu_1(\varepsilon_2) = 0$ at $\varepsilon_2 = \nu_1(q_0)$. Using the above arguments for the case $\nu_0 = 0$ we obtain (1.7) and (1.8), since we can take any $q_0$ and $\varepsilon$.

iii) We consider the Schrödinger operator $T$ on the real line. Note that if the entire function $f'(0, k) < 0$ for $k \in \mathbb{R}^+$ and has the zero $k = 0$, then using $w = ikf_+(0, \cdot) + f'_+(0, \cdot)$ from (2.20) we deduce that the operator $T$ has not any eigenvalue.

Recall that $q_0 \in L^1(\mathbb{R}^+)$ is such that $\text{supp } q_0 \in [0, 1]$ and $\nu_0(q_0) = 0$. We need the identity from (1.7):

$$e^{-ikw(k)} = 2ik\Delta(\lambda) + \lambda \varphi(1, \lambda) - \varphi'(1, \lambda).$$

Due to (2.20) the Wronskian for $q_\varepsilon$ satisfies at $k = it$:

$$\begin{align*}
e^{-ikw(k, \varepsilon)} & = 2ik\Delta(\lambda, \varepsilon) + \lambda \varphi(1, \lambda, \varepsilon) - \varphi'(1, \lambda, \varepsilon), \\
e^iw(it, \varepsilon) & = -2i(\Delta(\varepsilon - t^2, 0) - t^2\varphi(1, \varepsilon - t^2, 0) - \varphi'(1, \varepsilon - t^2, 0). 
\end{align*}$$

We rewrite the rhs of the last identity for $k = it, t \in \mathbb{R}$ in the form:

$$F = -f - g, \quad f(t, \varepsilon) = \varphi'(1, \varepsilon - t^2, 0), \quad g(t, \varepsilon) = 2t\Delta(\varepsilon - t^2, 0) + t^2\varphi(1, \varepsilon - t^2, 0).$$

These functions are entire in $t, \varepsilon$ and satisfy at $\varepsilon = t = 0$:

$$f(0, 0) = 0, \quad f_0(0, 0) = 0, \quad g(0, 0) = 0, \quad g_0(0, 0) = 2\Delta(0, 0) \geq 2,$$

since $\Delta(\lambda, 0) \geq 1$ for any $\lambda \leq \nu_0(g_0)$. We show that exists exactly one eigenvalue for small $\varepsilon > 0$. From (2.20) we have the simple fact: $w(0, 0) = 0$ iff $\varphi'(1, 0, 0) = 0$. Then the Implicit Function Theorem gives that there exists a function $t_\varepsilon(\varepsilon)$, analytic in small disk $\{\varepsilon : \delta < \delta\}$ such that $F(t_\varepsilon(\varepsilon), \varepsilon) = 0$ in the disk $\{\varepsilon : \delta < \delta\}$. Moreover, from (2.21) we have $t_\varepsilon(\varepsilon) = -C\varepsilon + O(\varepsilon^2)$, where $C = \frac{\varphi'(1, 0, 0)}{\Delta(0, 0)} < 0$. Here we have $E_1(\varepsilon) = -t_\varepsilon^2(\varepsilon).$ Thus the eigenvalues $\nu_0(\varepsilon)$ at $\varepsilon = 0$ creates two eigenvalues $\nu_0(\varepsilon) < \bar{E}_1(\varepsilon) < E_1(\varepsilon)$ for small $\varepsilon > 0$. If $\varepsilon$ is increasing then repeating the arguments for the case of the operator $\bar{T}$ we have (1.12), (1.13). Here we need the fact: if $\mathcal{E}$ is an eigenvalue of $T$, then $\mathcal{E} \notin \sigma_d(T) \cup \sigma_d(\bar{T})$. ■

**Proof of Corollary 1.2** The Schrödinger equation $-f'' + q_\varepsilon(x)f = k^2f$, $k \in \mathbb{C} \setminus \{0\}$, has the Jost solutions $\psi_+(x, k)$ such that $\psi_+(x, k) = e^{ixk}, \quad x \geq 1$ and $\psi_-(x, k) = e^{-ixk}, \quad x \leq -1$.

Note that the symmetry of the potential $q_\varepsilon$ yields

$$\psi_+(x, k) = f_+(x, k) = \psi_-(x, k) \quad \forall x \in [0, 1].$$

This implies that the Wronskian $w_\varepsilon(k)$ for the potential $q_\varepsilon$ satisfies

$$w_\varepsilon(k) = \{\psi_+(x, k), \psi_-(x, k)\}|_{x=0} = 2f_+(0, k)f'_+(0, k),$$

where $f_+(0, k)$ and $f'_+(0, k)$ are the Jost functions for the Cases 1 and 2 respectively with the potential $q$. This identity and Theorem 1.1 yield the proof of Corollary 1.2. ■

In order to discuss resonances we define the class of all Jost functions from [13]:
Definition 1. By \( J \) we mean the class of all entire functions \( f \) having the form
\[
f(k) = 1 + \frac{\hat{f}(k) - \hat{f}(0)}{2\text{i}k}, \quad k \in \mathbb{C},
\] (2.24)
where \( \hat{f}(k) = \int_{0}^{1} F(x)e^{2\pi k x}dx \) is the Fourier transformation of \( F \in \mathcal{P} \) and a set of zeros \( K = \{ k_n, n \in \mathbb{N} \} \) (counted with multiplicity) of \( f \) satisfy:

1) The set \( K = \overline{K} \) and has not zeros from \( \mathbb{R} \setminus \{ 0 \} \) and has possibly one simple zero at 0.
2) Let \( \# E \) be the number of zeros of \( K \) on the set \( E \subset \mathbb{C} \). The set \( K \) has finite number elements \( k_1, ..., k_m \) from \( \mathbb{C}_+ \), which are simple, belong to \( i\mathbb{R}_+ \) and if they are labeled by \( |k_1| > |k_2| > ... > |k_m| > 0 \), then intervals on \( i\mathbb{R}_+ \) defined \( I_j = (-k_j, -k_{j+1}), j \in \mathbb{N}_{m-1} \) and \( I_m = (-k_m, 0] \) satisfy
\[-k_j \notin K, \quad j \in \mathbb{N}_m := \{ 1, 2, ..., m \}, \quad \begin{cases}
\#I_j \geq 1 & \text{is odd, } j \in \mathbb{N}_{m-1} \\
\#I_m \geq 0 & \text{is even}
\end{cases}.
\] (2.25)

We recall the basic result about inverse problem (including the characterization) for compactly supported potentials from [15].

Theorem 2.2. The mapping \( \psi : \mathcal{P} \to J \) given by \( q \to \psi \) is one-to-one and onto, where \( \psi(k) = f_+(0, k) \).

Thus if \( q \in \mathcal{P} \), then \( \psi \in J \). Conversely, for each \( f \in J \) exists a unique potential \( q \in \mathcal{P} \) such that the Jost function \( \psi = f \). In fact using this result we reformulate the problem for the differential operator as the problem of the entire function theory. We need Theorem 3 from [16]:

Theorem 2.3. Let \( \psi^o \in J \) have zeros \( \{ k_n^o \}_{1}^{\infty} \) and let the sequence complex points \( \{ k_n \}_{1}^{\infty} \) satisfy 1) and 2) in Condition J and \( \sum_{n=1}^{\infty} |k_n - k_n^o|^2 < \infty \). Then \( \{ k_n \}_{1}^{\infty} \) is a sequence of zeroes of unique \( \psi \in J \), which is the Jost function for unique \( q \in \mathcal{P} \).

Proof of Theorem 2.3 i) Let a resonance \( k_n = -ir \in I_j \). Then from (2.16) we obtain \( \varphi'(1, k_n^o) = r \varphi'(1, k_n^o) \), which yields sign \( \varphi'(1, k_n^o) \) = sign \( \varphi'(1, k_n^o) \). From (1.8) we deduce that \( \mu_j < r_o < \tau_{j+1} \), since \( \varphi'(1, \lambda) \to +\infty \) and \( \varphi(1, \lambda) \to +\infty \) as \( \lambda \to -\infty \).

ii) We take any potential \( q^o \in \mathcal{P} \) such that the Jost function \( \psi^o \in \mathcal{P} \) has zeros \( \{ k_n^o \}_{1}^{\infty} \) only from \( \mathbb{C}_- \). We take the half-disc \( D_r := \{ k \in \mathbb{C}_- : |k| < r \} \) for a radius \( r > 3 + |p_1| \) large enough such that the number \( N_r \) of zeros \( \{ k_n^o \} \) in \( D_r \) satisfy \( N_r > m + (s_1 + .. + s_m) \), since we have the following result from [27]: \( N_r = \frac{r}{2\pi}(1 + o(1)) \ ar \to \infty \). We construct the new sequence of zeros \( \{ k_n \} \):
- if \( k_n^o \notin D_r \), then \( k_n = k_n^o \).
- Consider \( N_r \) zeros \( \{ k_n^o \} \) in \( D_r \). We have three cases:
  a) We remove \( m \) zeros from \( \mathbb{D}_-(r) \) on the points \( k_j = p_j \in i\mathbb{R}_+ \) for each \( j \in \mathbb{N}_m \).
  b) We remove \( s_j \) zeros from \( \mathbb{D}_-(r) \) on each interval \( I_j = (-k_j, -k_{j+1}), j \in \mathbb{N}_{m-1} \) and \( I_m = (-k_m, 0] \) at \( j = m \).
  c) We remove all remaining zeros \( n_r = N_r - m - (s_1 + .. + s_m) > 0 \) from \( \mathbb{D}_-(r) \) on point \( -p_1 - i \) with the multiplicity \( n_r \). Recall that \( r \) is large enough and \( r > 3 + |p_1| \).

Thus the sequence \( \{ k_n \} \) satisfy 1) and 2) in Condition J. Then due to Theorem 2.3 \( (k_n)_{1}^{\infty} \) is a sequence of zeroes of unique \( f \in J \), which is the Jost function for unique \( q \in \mathcal{P} \).
Proof of Corollary 1.4  

i) We need an inequality from [11], [26]: if \( q \in L^1(\mathbb{R}) \), then

\[
\sum_{\varepsilon_n<0} |\mathcal{E}_n|^\frac{3}{2} \leq \frac{1}{2} \int_{\mathbb{R}} q_-(x)dx.
\]

From Theorem 1.1 and inequalities (1.19), (1.20) and (2.26) we obtain (1.22), (1.24) and (1.23) for \( \mathcal{S} = \sum_{\mu_n<0} |\mu_n|^\frac{3}{2} \). Applying (2.26) to the operator \( T_\varepsilon \) and estimates from (1.6), Corollary 1.2 \(|\lambda^+_n| \leq |\mathcal{E}_n|\) for negative eigenvalues) we obtain

\[
\sum_{n>0,\lambda^+_n<0} |\lambda^+_n|^\frac{3}{2} \leq \sum_{\varepsilon_n<0} |\mathcal{E}_n|^\frac{3}{2} \leq \frac{1}{2} \int_{\mathbb{R}} q_-(x)dx = \int_0^1 q_-(x)dx.
\]

ii) Substituting the estimate \( \|q\| \leq 2\gamma \max\{1, \gamma^4\} \) from (1.21) into (1.22), (1.23) we obtain (1.25).  

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