Anderson Transition and Generalized Lyapunov Exponents
(comment on comment by P. Markos, L. Schweitzer and M. Weyrauch, cond-mat/0402068)

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The generalized Lyapunov exponents describe the growth of the second moments for a particular solution of the quasi-1D Schroedinger equation with initial conditions on the left end. Their possible application in the Anderson transition theory became recently a subject for controversy in the literature. The approach to the problem of the second moments advanced by Markos et al (cond-mat/0402068) is shown to be trivially incorrect. The difference of approaches by Kuzovkov et al (cond-mat/0212036, 0501446) and the present author (cond-mat/0504557, 0512708) is discussed.

Recently Markos et al have published a comment [1] on the paper by Kuzovkov et al [2] where a growth of the second moments for a particular solution of the quasi-1D Schroedinger equation was related with the problem of Anderson localization. It was stated in [2] that the Anderson transition is of the first order and exists not only for space dimensions \( d > 2 \) [3] but also in the 2D case. These statements look wild and the authors of [1] are right in not believing them. They are also right in statement that the growth of the second moment of wave function does not mean the growth of its typical value, which is governed by the average logarithm. However, Markos et al [1] go further and claim that analysis of the second moments presented by Kuzovkov et al [2, 3] is qualitatively incorrect and cannot provide any evidence for the metallic phase. These conclusions contradict the recent papers by the present author [4, 5] and are shown below to be trivially incorrect. The difference of approaches suggested in [2, 3] and [4, 5] is also discussed.

Consider the 2D Anderson model described by the discrete Schroedinger equation

\[
\psi_{n+1,m} + \psi_{n-1,m} + \psi_{n,m+1} + \psi_{n,m-1} + V_{n,m} \psi_{n,m} = E \psi_{n,m}
\]  

and interprete it as a recurrence relation in the variable \( n \), which we accept as a longitudinal coordinate. Initial conditions are assumed to be fixed on the left end of the system, while the periodic boundary conditions are accepted in the transverse direction, \( \psi_{n,m+L} = \psi_{n,m} \). Cite energies \( V_{n,m} \) are considered as uncorrelated random quantities with the first two moments

\[
\langle V_{n,m} \rangle = 0, \quad \langle V_{n,m} V_{n',m'} \rangle = W^2 \delta_{n,n'} \delta_{m,m'}.
\]

The growth of the second moments for this problem can be studied using the old idea by Thouless [6] based on the observation that variables \( \psi_{n,m} \) are statistically independent of
The main quantity of interest is \( \langle \psi_{n,m}^2 \rangle \); solving (1) for \( \psi_{n+1,m} \) and averaging its square, we can relate it with the pair correlators containing lower values of \( n \). Deriving analogous equations for the pair correlators, we end with the closed system of difference equations for the quantities

\[
\begin{align*}
x_{m,m'}(n) &\equiv \langle \psi_{n,m} \psi_{n,m'} \rangle, \\
y_{m,m'}(n) &\equiv \langle \psi_{n,m} \psi_{n-1,m'} \rangle, \\
z_{m,m'}(n) &\equiv \langle \psi_{n-1,m} \psi_{n,m'} \rangle,
\end{align*}
\]

which for \( E = 0 \) has a form [4]

\[
x_{m,m'}(n+1) = W^2 \delta_{m,m'} x_{m,m'}(n) + x_{m+1,m'}+1(n) + x_{m-1,m'}-1(n) + x_{m+1,m'}+1(n) + x_{m-1,m'}-1(n) + x_{m,m'}(n-1) + y_{m+1,m'}(n) + y_{m-1,m'}(n) + z_{m,m'+1}(n) + z_{m,m'-1}(n)
\]

\[
y_{m,m'}(n+1) = -x_{m+1,m'}(n) - x_{m-1,m'}(n) - z_{m,m'}(n)
\]

\[
z_{m,m'}(n+1) = -x_{m,m'+1}(n) - x_{m,m'-1}(n) - y_{m,m'}(n).
\]

Instead to follow this natural procedure, Markos et al [1] invent their own approach. They rewrite (1) using the transfer matrix and construct the tensor product of two such matrices. After averaging, they arrive to a linear system of equations determined by a matrix

\[
T = \begin{pmatrix}
W^2 1 \otimes 1 + D_0 \otimes D_0 & -D_0 \otimes 1 & -1 \otimes D_0 & 1 \otimes 1 \\
D_0 \otimes 1 & 0 & -1 \otimes 1 & 0 \\
1 \otimes D_0 & -1 \otimes 1 & 0 & 0 \\
1 \otimes 1 & 0 & 0 & 0
\end{pmatrix},
\]

where \( D_0 = E - H_0 \) and \( H_0 \) is the Hamiltonian of the \( n \)-th slice for a pure system, 1 is the unit matrix of the size \( L \times L \). Even if being correct, this system is practically untractable due to sophisticated matrix constructions. The eigenvalues \( \lambda = \exp(iq) \) of the matrix (5) are declared to be determined by equation

\[
2 \cos 2q - 2\kappa_i \kappa_j \cos q + (\kappa_i^2 + \kappa_j^2 - 2) = 2W^2 i \sin q
\]

where \( \kappa_i = E - 2 \cos p_i \) and \( p_i \) are allowed values of the transverse momentum \( p \). Equation (6) has no resemblance with the corresponding Eq. 45 in [4]. The main difference is the absence of functions with the argument \( qL \), which inevitably arise due to the boundary conditions and can be absent only in trivial cases. One can suspect, that this difference is related with incorrect treatment of the disorder term (like \( W^2 1 \times 1 \) in (5)), without which the problem is trivial. This term is local in \( m - m' \) (see (4)) and nondiagonal in the momentum representation, which is seemingly used in (6). It looks likely that the local nature of this term was neglected and it was replaced by a suitable constant. We can try such thing for the system (4), replacing \( \delta_{m,m'} \) by unity. Then (4) is solved trivially

\[
x_{m,m'}(n) = xe^{ipm+ip'm'} e^{\beta n}, \quad y_{m,m'}(n) = ye^{ipm+ip'm'} e^{\beta n}, \quad z_{m,m'}(n) = ze^{ipm+ip'm'} e^{\beta n},
\]
where allowed values for \( p \) and \( p' \) \((2\pi s/L, s = 0, 1, \ldots, L - 1)\) are determined by the boundary conditions and the quantities \( x, y, z \) satisfy the equation

\[
\begin{pmatrix}
W^2 + 4 \cos p \cos p' - 2 \sinh \beta & 2 \cos p & 2 \cos p' \\
2 \cos p & e^\beta & 1 \\
2 \cos p' & 1 & e^\beta
\end{pmatrix}
\begin{pmatrix}
 x \\
y \\
z
\end{pmatrix}
= 0.
\] (8)

The determinant vanishes under condition

\[
2 \cosh 2\beta - 2\kappa \kappa' \cosh \beta + (\kappa^2 + \kappa'^2 - 2) = 2W^2 \sinh \beta
\] (9)

where \( \kappa = -2 \cos p, \kappa' = -2 \cos p' \). In the case \( E = 0 \) (when Eq.4 holds) this equation is identical to (6), if correspondence \( \beta = iq \) is taken into account. We see that, indeed, the disorder term in (5) was treated inadequately and its local nature was neglected. Physically, the equations (6, 9) correspond not to the true Anderson model but to its degenerate version when cite energies \( V_{n,m} \) are independent of \( m \).

In fact, the error is present already in the matrix (5). To obtain the term with disorder, one needs to produce averaging of the kind

\[
\begin{pmatrix} V_1 & 0 \\ 0 & V_2 \end{pmatrix} \otimes \begin{pmatrix} V_1 & 0 \\ 0 & V_2 \end{pmatrix} = \begin{pmatrix} V_1V_1 & 0 & 0 & 0 \\ 0 & V_1V_2 & 0 & 0 \\ 0 & 0 & V_2V_1 & 0 \\ 0 & 0 & 0 & V_2V_2 \end{pmatrix} \rightarrow W^2 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\] (10)

and the result cannot be represented as \( W^2 1 \otimes 1 \). The latter form is valid for \( V_1 = V_2 \) in Eq.10, while in the general case it corresponds to the model (1) with \( V_{n,m} \) being independent of \( m \), in accordance with the previous analysis. Such model is of no interest and all conclusions made in [1] are irrelevant for the problem under consideration.

We see that Markos et al [1] became victims of their own formalism and were unable to make a way through the jungles of the tensor algebra.

Now let us discuss the difference of approaches suggested by Kuzovkov et al [2, 3] and the present author [4, 5]. The initial system of equations (4) and its higher dimensional analogue (Eq.5 in [5]) coincide with those used in [2, 3]. However, the quantity \( z_{mm'}(n) \) was not introduced in [2, 3] and its role was played by \( y_{mm'}(n) \). As a result, the system of equations had no complete difference form and could not be solved in the natural manner with evaluation of full spectrum of exponents \( \beta_s \). The \( Z \)-transform exploited in [2, 3] allow to find a solution only in the thermodynamical limit \( L \rightarrow \infty \), using a questionable procedure of averaging over translations in the transversal direction. Translational invariance takes no place for the solution of (4) (see Eq. 10 in [5]) and the latter procedure probably eliminates all terms with the transverse momentum different from \( p = 0 \) and \( p = G/2 \), if effectively only squares of these terms are relevant (\( G \) is a vector of a reciprocal lattice corresponding to the main diagonal of the Brillouin zone). As a result, a zero value for the critical
disorder was obtained for \( d = 3 \) in the band center \( E = 0 \) [3], in a striking contrast with [5]. Fortunately for the authors of [2, 3], a condition \( p = G/2 \) corresponds to the minimal exponent \( \beta_{\text{min}} \) for \( d = 2 \) and \( d \geq 4 \), so the critical values \( \sigma_0' \) (corresponding to \( W_c \) in [4, 5]) were found correctly for these cases. As for the second critical point \( \sigma_0 \) for higher dimensionalities, we see no evidence for it in the spectrum of \( \beta_s \). It looks that a filter function \( H(z) \) used in [2, 3] has not only poles corresponding to eigenvalues of the transfer matrix but also another singularities, which are physically irrelevant. Correspondingly, we see no evidence of a special role of dimensionality \( d = 6 \), which is surely absent in the exact field theory approach [7].

Relation of the generalized exponents \( \beta_s \) with the Anderson transition was established in [4, 5] using the conventional variant of finite-size scaling [8]. Contrary, the papers [2, 3] were formulated in the engineer’s language (using the concepts of signals, filters etc.), which has no direct relation to the Anderson transition. The limit \( L \to \infty \) was taken in [2, 3] from the very beginning and the finite-size scaling approach could not be used for interpretation of results. As a consequence, a transition in the 2D case was interpreted as being of the first order, in evident contradiction with all available information. In fact, this transition is of the Kosterlitz-Thouless type [4] and there is no need for a serious revision in the weak localization region. The possibility of power-law localization is mentioned in both approaches [2, 3] and [4, 5], but in completely different contexts.

Contrary to [2, 3], a clear distinction is made in [4, 5] between the generalized exponents \( \beta_s \) and the true Lyapunov exponents \( \gamma_s \): the latter are self-averaging quantities and surely have a more fundamental character. Fortunately, the knowledge of \( \beta_s \) provides essential information on \( \gamma_s \): (a) inequality \( \beta_s \geq 2\gamma_s \) can be rigorously proven; (b) the order of magnitude relation \( \beta_s \sim \gamma_s \) takes place on the physical level of rigorousness; (c) \( \beta_s \) and \( \gamma_s \) are practically equivalent from viewpoint of finite-size scaling philosophy. Relation of \( \beta_s \) and \( \gamma_s \) with the parameters of the log-normal distribution is also of great importance. In fact, inequality \( \beta_s \geq 2\gamma_s \) is sufficient for the most responsible statements, such as existence of the 2D phase transition and absense of one-parameter scaling for \( \gamma_s \). Relation \( \beta_s \sim \gamma_s \) is used with great precaution and only in the cases when it does not contradict to numerical results.

Finally, we do not consider as indisputable the conventional variant of finite-size scaling based on relation of the Anderson transition with the minimal Lyapunov exponent \( \gamma_{\text{min}} \). In the general case, some effective exponent \( \gamma_{\text{eff}} \) should be used instead \( \gamma_{\text{min}} \) [4]. Such modification restores one-parameter scaling in the weak localization region and eliminates the 2D transition from roughly half of models.

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References

[1] P. Markos, L. Schweitzer and M. Weyrauch, J. Phys.: Condens. Matter 16, 1679 (2004); cond-mat/0402068.
[2] V. N. Kuzovkov, W. von Niessen, V. Kashcheyevs, O. Hein, J. Phys.: Condens. Matter 14, 13777 (2002); cond-mat/0212036.

[3] V. N. Kuzovkov, W. von Niessen, Eur. Phys. J. B 42, 529 (2004); cond-mat/0501446.

[4] I. M. Suslov, Zh. Eksp. Teor. Fiz. 128, 768 (2005) [JETP 101, 661 (2005)]; cond-mat/0504557.

[5] I. M. Suslov, Zh. Eksp. Teor. Fiz. 129, 1064 (2006) [JETP 102, 938 (2006)]; cond-mat/0512708.

[6] D. J. Thouless, Phys. Rep. 13, 92 (1974).

[7] I. M. Suslov, Usp. Fiz. Nauk 168, 503 (1998) [Physics-Uspekhi 41, 441 (1998)]; cond-mat/9912307.

[8] J.L. Pichard, G. Sarma, J.Phys.C: Solid State Phys. 14, L127 (1981); 14, L617 (1981).