Linear stability of Landau jet: non-parallel effects

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Abstract. We perform linear stability analysis of Landau jet using quasi-parallel approximation and self-similar approach. While the former detects only sinusoidal unstable modes, the latter also captures axisymmetric disturbances in agreement with previous results. The direct comparison of dispersion curves and eigenmode shapes for $m = 1$ and $Re_D \approx 56$ indicates that low frequencies are poorly described within quasi-parallel approximation making this approach questionable for a laminar-turbulent transition model.

1. Introduction

Jet-like flows play a central role in many technological applications representing one of the basic research objects in hydrodynamics. Flow instabilities are intimately linked to the laminar-turbulent transition making that area of the research essential. Batchelor & Gill [1] followed by others [2–5] were the first to use local linear stability analysis of the ‘top-hat’ and fully developed velocity profiles showing that the first one is unstable to disturbances with azimuthal wavenumbers $m = 0$ and $m = 1$, while the Gaussian-like profile is unstable with respect to $m = 1$ only. For the latter case, in the far-jet region the theoretical value of the critical Reynolds number is $Re_{D,cr} \approx 38$, based on the diameter $D$ of the supplying orifice, and corresponds to the appearance of temporally or spatially growing disturbances [4, 5]. However, experiments typically observe unstable motion at higher $Re_D$ [6, 7].

Our present approach relies on self-similar features of the base flow. The exact solution of Navier–Stokes equations, derived by Slezkin [8], Landau [9] and Squire [10], represents a steady axisymmetric unbounded jet due to a point force. This solution well approximates the numerical results and experimental measurements of the laminar far-jet [11–13]. The conservation of momentum leads to the linear decay of centreline velocity $V_c x$ with the streamwise coordinate $x$. The main hypothesis we employ is the assumption that a disturbance traveling downstream is continuously adjusting to the local characteristics of the jet, i.e. the local values of length and time scales are $\delta \propto x$ and $\tau = \delta/V_c^2 \propto x^2$, respectively. Firstly, within WKBJ approach [14] the growth of a disturbance amplitude is typically expressed as $\exp\left(\int i\alpha(x)dx\right)$ together with the self-similar behaviour of the local axial wavenumber $\alpha \propto \delta^{-1} \propto x^{-1}$ leading to $\log x$ in the exponent and the overall power law dependence. Secondly, instead of considering a wavepacket with constant frequency $\omega$, we track a disturbance with constant self-similar frequency $\Omega \propto x^2 \omega$ according to our hypothesis mentioned above. These ideas have been successfully applied by Likhachev [15] and Shtern & Hussain [16] showing that, besides $m = 1$, the disturbance with
m = 0 appears to be unstable although with a much lower growth rate, explaining some earlier experimental observations by Reynolds [6].

In the present paper we discuss the results of the quasi-parallel approximation applied to Landau jet when the effects of flow non-parallelism and deceleration become important. An alternative self-similar approach described above provides reference eigensolutions.

2. Theoretical framework and governing equations

2.1. Base flow

In this section we give a brief description of the base flow in spherical \((R, \theta, \phi)\) and cylindrical \((x, r, \phi)\) coordinate systems, Fig. 1. We consider Landau–Squire solution corresponding to a non-swirling jet \((V_\phi = 0)\):

\[
V_R = -\frac{\nu y'(\psi)}{R}, \quad V_\theta = -\frac{\nu y(\psi)}{R\sqrt{1 - \psi^2}}, \quad y(\psi) = \frac{2(1 - \psi^2)}{A - \psi}, \tag{1}
\]

where \((V_R, V_\theta, V_\phi)\) are the base flow velocity components, \(\psi = \cos \theta\) and \(\nu\) denotes the kinematic viscosity of the fluid, while \(A\) is connected to the Reynolds number as discussed below. This solution describes a jet with a zero mass source at \(R = 0\) and non-zero momentum [9]

\[
P_x = 16\pi \rho \nu^2 A \left(1 + \frac{4}{3(A^2 - 1)} - \frac{A}{2} \ln \frac{A + 1}{A - 1}\right), \tag{2}
\]

where \(\rho\) is the fluid density. Several definitions of the Reynolds number can be introduced emerging in the quasi-parallel approximation, present self-similar analysis or experiments:

\[
Re_\delta = \frac{V_R^c \delta}{\nu}, \quad Re_a = \frac{V_R^c R}{\nu} = -y'(1), \quad Re_D = \frac{V_\delta D}{\nu}, \tag{3}
\]

where \(V_R^c\) is the centreline jet velocity, \(V_\delta\) the bulk velocity in the supplying pipe of diameter \(D\) with the assumed parabolic velocity profile, \(\delta\) the half-width of the jet. The connection between \(Re_\delta, Re_D\) and \(Re_a\) for large \(Re_a\) is \(Re_\delta = (\sqrt{2} - 1)Re_D\) and \(Re_D = \sqrt{8}Re_a\). Let us introduce the velocity components in cylindrical coordinates:

\[
V_x = V_R \cos \theta - V_\theta \sin \theta, \quad V_r = V_R \sin \theta + V_\theta \cos \theta. \tag{4}
\]

The ratio of \(V_x\) and \(V_r\) to the centreline velocity \(V_x^c\) is

\[
\frac{V_x}{V_x^c} = \frac{A - 1}{2\sqrt{1 + b^2 \eta^2}} \frac{A(2 + b^2 \eta^2)}{(A\sqrt{1 + b^2 \eta^2} - 1)^2}, \quad \frac{V_r}{V_x^c} = \frac{(A - 1) \eta}{2(1 + b^2 \eta^2)} \frac{A\sqrt{1 + b^2 \eta^2} - 1 - b^2 \eta^2}{(A\sqrt{1 + b^2 \eta^2} - 1)^2}. \tag{5}
\]
Asymptotic expression for the half-width of the jet $\left( V_x/V^c_x = 1/2 \at \eta = 1 \right)$ gives $b = \sqrt{8(\sqrt{2} - 1)/Re_a}$.

2.2. Linearized Navier–Stokes equations

The Navier–Stokes equations linearized around the base flow $V_i$ described in the previous subsection provide governing equations for a small amplitude velocity disturbance with the components $v_i$:

$$\frac{\partial v_i}{\partial t} + V_j \frac{\partial v_i}{\partial x_j} + v_j \frac{\partial V_i}{\partial x_j} = -\frac{1}{\rho} \frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 v_i}{\partial x_j^2}, \quad (7)$$

$$\frac{\partial v_j}{\partial x_j} = 0. \quad (8)$$

2.3. Quasi-parallel approximation

Within this approximation the base flow is assumed to slowly vary in the axial direction allowing to consider Fourier modes with respect to $x$. The governing equations (7), (8) are non-dimensionalized using $V^c_x(x_0)$ and $\delta(x_0)$ at some fixed $x_0$. In the vicinity of $x_0$ the velocity and pressure disturbances have the following form:

$$(v_x, v_r, v_\phi, p) = (H, iF, G, P)e^{i\alpha x + im\phi - i\omega t}, \quad (9)$$

where $\alpha$ and $\omega$ are the non-dimensional axial wavenumber and frequency, $m$ the azimuthal wavenumber, the dependence on $\eta$ is kept only within amplitude functions $H, F, G, P$. The governing equations simplify to

$$F' + \frac{F}{\eta} + \frac{mG}{\eta} + \alpha H = 0, \quad (10)$$

$$-\frac{H''}{Re_\delta} - \frac{H'}{Re_\delta} + \left(-i\omega + i\alpha V_x + \frac{m^2}{Re_\delta \eta^2} + \frac{\alpha^2}{Re_\delta}\right)H + iV_x' F + i\alpha P = 0, \quad (11)$$

$$-\frac{iF''}{Re_\delta} - \frac{iF'}{Re_\delta} + \left(\omega - \alpha V_x + \frac{i(m^2 + 1)}{Re_\delta \eta^2} + \frac{i\alpha^2}{Re_\delta}\right)F + \frac{2im}{\eta^2 Re_\delta} G + P' = 0, \quad (12)$$

$$-\frac{G''}{Re_\delta} - \frac{G'}{Re_\delta} + \left(-i\omega + i\alpha V_x + \frac{m^2 + 1}{Re_\delta \eta^2} + \frac{\alpha^2}{Re_\delta}\right)G + \frac{2m}{\eta^2 Re_\delta} F + \frac{imP}{\eta} = 0, \quad (13)$$

where the prime denotes the derivative with respect to $\eta$.

With the appropriate boundary conditions at $\eta = 0$ and $\infty$ the system of equations (10)-(13) describes an eigenvalue problem where we seek a non-zero solution for $(H, F, G, P)$ with a specific values of $(\alpha, \omega)$ satisfying the dispersion relation $D(\alpha, \omega, m, Re_\delta) = 0$. We consider the spatial stability analysis where the branches $\alpha(\omega)$ are obtained by solving for complex axial wavenumbers when $\omega$ and $m$ are real. The flow is considered unstable if the disturbance grows with $x$, i.e. when the imaginary part of the eigenvalue $\alpha$ is negative. The numerical solution is performed by a spectral collocation method using Chebyshev polynomials as described in [17,18].

2.4. Fully non-parallel framework

Using the self-similar features of the base flow in the spherical coordinate system, a general disturbance can be reduced to the following form [15,16]:

$$(v_x, v_r, v_\phi, p) = \frac{\nu}{R} (f, -\frac{g}{\sqrt{1-\psi^2}}, \frac{ih}{\sqrt{1-\psi^2}} \sqrt{\rho \nu q \frac{R}{\rho}} e^{i(k \xi + im\phi - i\omega t)}), \quad (14)$$
all expressed by dimensional quantities while the amplitudes \( f, g, h, q \) are non-dimensional functions of \( \psi = \cos \theta \). The variable \( \xi = \log(R/R_0) \) is introduced to keep the common exponential form of a disturbance with \( R_0 \) being a constant. The wavenumber \( \alpha \) in quasi-parallel approach is replaced with non-dimensional \( k \).

Substituting (14) to (7), (8) after a little algebra we arrive to

\[
(1 + ik)f + g' - \frac{mh}{1 - \psi^2} = 0, \tag{15}
\]

\[
(1 - \psi^2)f'' - (2\psi + y)f' + \left(i\Omega - 2 - 2y' - ik + iky' - \frac{m^2}{1 - \psi^2}\right)f - 2g' + \left(\frac{2y}{1 - \psi^2} + y''\right)g + (2 - ik)q + \frac{2m}{1 - \psi^2}h - k^2f = 0, \tag{16}
\]

\[
(1 - \psi^2)q'' - y'g' + \left(i\Omega - ik + iky' - y' - \frac{2xy}{1 - x^2} - \frac{m^2}{1 - x^2}\right)g + 2(1 - x^2)f' - (1 - x^2)q' - \frac{2mx}{1 - \psi^2}h - k^2g = 0, \tag{17}
\]

\[
(1 - \psi^2)h'' - y'h' + \left(i\Omega - ik + iky' - \frac{m^2}{1 - \psi^2}\right)h + 2mf - mq - \frac{2m\psi}{1 - \psi^2}g - k^2h = 0, \tag{18}
\]

where \( \Omega = \omega R^2/\nu \). According to (14) the appropriate boundary conditions imply:

\[
g(\pm 1) = h(\pm 1) = 0. \tag{19}
\]

As within the quasi-parallel approximation we consider the spatial stability analysis where the branches \( k(\Omega) \) are obtained by solving for complex axial wavenumbers when \( \Omega \) is real. The flow is considered unstable if the imaginary part of the eigenvalue \( k_i \) is negative. The numerical solution of (15)-(18) is performed using finite difference discretization and the shooting method similar to the one described in [19].

2.5. Comments on non-dimensionalization

The connection between frequency in quasi-parallel approximation with a self-similar approach comes from

\[
\frac{V_x^c}{\delta} \omega = \frac{\nu}{R^2} \Omega, \quad \omega = \frac{\nu \delta}{V_x^c R^2} \Omega = -\frac{\nu b R}{\nu y'(1) R} \Omega = -\frac{b}{y'(1)} \Omega, \tag{20}
\]

thus, \( \omega = bRe_\alpha/\Omega \). The connection between \( \alpha \) and \( k \) are as follows

\[
\frac{\alpha}{\delta} = \frac{k}{R}, \tag{21}
\]

thus, \( \alpha = bk \). Note that the spreading rate of the jet \( b \) introduced above is a function of the Reynolds number.

3. Results

The results concerning neutral modes computed within self-similar approach are in close agreement with [15,16] where, besides \( m = 1, m = 0 \) has been detected. Below we discuss the comparison of results between quasi-parallel approximation and self-similar approach for \( m = 1 \).
Figure 2. Real (left) and imaginary (right) parts of $\alpha$ as a function of $\omega$. Blue line corresponds to the quasi-parallel approximation, while the red one to self-similar one. Vertical dashed lines show $\omega = 0.1, 0.3, 0.8, 1.5$ with its eigenmodes shown in Fig. 3. $Re_a = 400$ ($Re_D = 55.98$, $Re_\delta = 36.29$) and $m = 1$.

Figure 3. Comparison of the modulus of axial (blue), radial (orange) and azimuthal (red) amplitude of the perturbation. Solid line corresponds to quasi-parallel analysis, dashed – self-similar. The eigenmodes are shown for $\omega = 0.1, 0.3, 0.8, 1.5$, $Re_a = 400$ ($Re_D = 55.98$, $Re_\delta = 36.29$) and $m = 1$.

and $Re_D \approx 56$. Figure 2 shows a dispersion curve $\alpha(\omega)$ for both approaches. The real part of $\alpha$ responsible for the value of the perturbation axial wavelength is in close agreement especially for high enough $\omega$. The imaginary part describing the growing instability for $-\alpha_{im} > 0$ shows a significant destabilization for self-similar approach compared to quasi-parallel approximation.

Figure 3 shows the modulus of different components of the perturbation amplitude obtained within quasi-parallel approximation and self-similar approach for the same $Re_D \approx 56$ and $m = 1$. For low frequencies ($\omega = 0.1$ and $0.3$) there is a significant mismatch between eigenfunction profiles especially for the radial component. For higher frequencies ($\omega = 0.8$ and $1.5$) all components behave similarly within a good quantitative agreement. However, the modes for $\omega = 0.2 - 0.3$ have the highest growth rate, thus, representing the main interest for the future study of the laminar-turbulent transition based on the present linear stability model within
self-similar approach.

4. Conclusion
In this paper we studied the linear stability of Landau jet using the quasi-parallel approximation and self-similar approach. While the former detected only sinusoidal unstable modes, the latter also captured axisymmetric disturbances in agreement with previous results. The direct comparison of dispersion curves and eigenmode shapes for $m = 1$ and $Re_D \approx 56$ indicated that low frequencies are poorly described within quasi-parallel approximation making this approach questionable for a laminar-turbulent transition model.

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