Transitively-saturated property, Banach recurrence and Lyapunov regularity

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Abstract
The topological entropy of upper recurrent, lower recurrent points and Birkhoff regularity were considered in Tian (2016 Adv. Math. 288 464–526) but Banach upper recurrence points are not considered. In this paper we mainly consider whether the points that are Banach upper recurrent but not upper recurrent can carry full topological entropy. Moreover, we use statistical ω-limit sets to obtain entropy results on the refined orbit distribution of Banach recurrence. In this process, we strengthen Pfister and Sullivan’s result of Pfister and Sullivan (2007 Ergod. Theor. Dynam. Syst. 27 929–56) from saturated property to transitively-saturated property (and from single-saturated property to transitively-convex-saturated property). We also establish an abstract framework on the combination of Banach recurrence and multifractal analysis of general observable functions and in particular, how these results can be applied in the combination of Banach recurrence Lyapunov regularity to obtain a generalized and refined theory of Barreira and Doutor (2009 J. Math. Pures Appl. 92 1–17); Barreira and Gelfert (2006 Commun. Math. Phys. 267 393–418); Barreira and Saussol (2001 Trans. Am. Math. Soc. 353 3919–44); Feng and Huang (2010 Commun. Math. Phys. 297 1–43); Feng (2003 J. Math. 138 353–76); Feng (2009 Israel J. Math. 170 355–94); Feng and Lau (2002 Math. Res. Lett. 9 363–78) etc, on mixed Lyapunov multifractal analysis such as irregular sets and level sets of Lyapunov exponents.

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1. Introduction

The study of dimension theory for systems with specification property or weaker versions has drawn the attention of many researchers (Pesin et al, see [3, 4, 47, 57] for example). The general concept of dimension theory or multi-fractal analysis is to decompose the phase space in subsets of points that have similar dynamical behaviour and describe the size of each of the subsets from a geometrical or topological viewpoint. Sets with similar dynamical behaviour include basin sets of an invariant measure or general saturated sets [12, 50], recurrent and transitive sets [62], non-dense sets [26, 64], level sets and irregular sets of Birkhoff ergodic average [9, 39, 41, 47, 48, 56, 57, 60], level sets and irregular sets of Lyapunov exponents [61], which have been studied a lot by using measures of Hausdorff dimension, topological entropy or pressure, Lebesgue measure and distributional chaos etc.

The main goal of this paper is to continue the work of [62], turning to study the dynamical complexity of Banach recurrence (for which the set of recurrent visiting time has positive Banach density). Firstly, let us describe the topic more precisely. In the present paper, a dynamical system \((X, T)\) means always that \(X\) is a compact metric space and \(T : X \to X\) is a continuous map. Given \(x \in X\), let \(\omega_T(x)\) denote the \(\omega\)-limit set of \(x\). Define the set of recurrent points and the set of transitive points by

\[
\text{Rec} := \{x \in X | x \in \omega_T(x)\} \quad \text{and} \quad \text{Tran} := \{x \in X | \omega_T(x) = X\}.
\]

For \(S \subset \mathbb{N}\), the upper density and lower density of \(S\) are defined by

\[
\bar{d}(S) := \limsup_{n \to \infty} \frac{|S \cap \{0, 1, \ldots, n-1\}|}{n}, \quad d(S) := \liminf_{n \to \infty} \frac{|S \cap \{0, 1, \ldots, n-1\}|}{n},
\]

respectively, where \(|Y|\) denotes the cardinality of the set \(Y\). The Banach upper density and Banach lower density of \(S\) are defined by

\[
B^*(S) := \limsup_{|I| \to \infty} \frac{|S \cap I|}{|I|}, \quad B_*(S) := \liminf_{|I| \to \infty} \frac{|S \cap I|}{|I|},
\]

respectively. Here, \(I \subseteq \mathbb{N}\) is taken from finite continuous integer intervals. Let \(U, V \subseteq X\) be two nonempty open subsets and \(x \in X\). Define sets of visiting time

\[
N(U, V) := \{n \geq 1 | U \cap T^{-n}(V) \neq \emptyset\} \quad \text{and} \quad N(x, U) := \{n \geq 1 | T^n(x) \in U\}.
\]

**Definition 1.1.** A point \(x \in X\) is called the Banach upper recurrent, if for any \(\varepsilon > 0\), \(N(x, B_\varepsilon(x))\) has a positive Banach upper density where \(B_\varepsilon(x)\) denotes the ball centred at \(x\) with radius \(\varepsilon\). Similarly, one can define the Banach lower recurrent, upper recurrent, and lower recurrent.

Let \(BR\) denote a set of all Banach upper recurrent points and let \(QW, W\) denote a set of upper recurrent points and lower recurrent points, respectively. Let \(AP\) denote a set of points contained in minimal sets (called minimal points or almost periodic points). Note that \(AP\) coincides with the set of all Banach lower recurrent points. From [65, 68, 71] \(W, QW, BR, \text{Rec}\) all have full measure for any invariant measure but may not for \(AP\). Note that
Definition 1.2. For \( x \in X \) and \( \xi = \overline{d}, d, B^*, B_+ \), a point \( y \in X \) is called \( x - \xi\)-accessible, if for any \( \epsilon > 0 \), \( N(x, B_\epsilon(y)) \) has positive density w. r. t. \( \xi \). Let
\[
\omega_\xi(x) := \{ y \in X | y \text{ is } x - \xi\text{-accessible} \}.
\]
For convenience, it is called the \( \xi - \omega\)-limit set of \( x \).
Given a measure $m$, let $S_m$ denote the support of $m$. It is worth mentioning that $\omega_B(x)$ coincides with the measure centre of the subsystem $f_{|\omega(x)}$, that is $\omega_B(x) = \bigcup_{\mu \in M(f_{|\omega(x)})} S_\mu$, see [26]. If $\omega_B(x) = \omega_T(x)$, we can say $x$ has a full measure centre. For any $x \in X$, if $\omega_B(x) = \emptyset$ and $\omega_B(x) = \omega_T(x)$, then from [26] we know that $x$ satisfies only one of the following six cases:

Case (1) $\emptyset = \omega_B(x) \subseteq \omega_T(x) = \omega_B(x) = \omega_T(x)$;
Case (2) $\emptyset = \omega_B(x) \subseteq \omega_T(x) = \omega_B(x) \subseteq \omega_T(x)$;
Case (3) $\emptyset = \omega_B(x) = \omega_T(x) \subseteq \omega_B(x) = \omega_T(x)$;
Case (4) $\emptyset = \omega_B(x) \subseteq \omega_T(x) = \omega_B(x) = \omega_T(x)$;
Case (5) $\emptyset = \omega_B(x) = \omega_T(x) \subseteq \omega_B(x) \subseteq \omega_T(x)$;
Case (6) $\emptyset = \omega_B(x) \subseteq \omega_T(x) \subseteq \omega_B(x) \subseteq \omega_T(x)$;

Now let us state the next main result for the entropy estimate on the orbit distribution by different statistical behaviour.

**Theorem 1.3.** Suppose that $(X, T)$ has $g$-almost product property. If $(X, T)$ is not uniquely ergodic and there is an invariant measure with full support, then

$$\{x \in \text{Tran} | x \text{ satisfies Case i} \} \neq \emptyset, \quad \{x \in BR | x \text{ satisfies Case i} \} \neq \emptyset$$

and they all have full topological entropy, $i = 1, 2, \cdots, 6$.

The main idea in [62] uses the saturated property established by Pfister and Sullivan in [50], but here it is not enough for the results of this paper. We find that if we strengthen the saturated property to a new version that requires the points to be transitive, then the difficulties in Banach recurrence can be dealt with. Thus, we introduce a new concept called transitivity-saturated property and fortunately we observe that this stronger property is still valid for systems with specification-like property and uniform separation. Let us introduce these more precisely.

Let $M(X)$, $M(T, X)$, $M_{eq}(T, X)$ denote the space of all probability measures, $T$ – invariant measures, $T$ – ergodic measures, respectively. Given $x \in X$, let $M_x$ be the limit set of the empirical measures of the Birkhoff average

$$\frac{1}{n} \sum_{i=0}^{n-1} \delta_{T^i(x)}.$$ 

$M_x$ gives the description of asymptotic behaviour of the orbit of $x$ in the measure space. For a compact connected nonempty set $K \subseteq M(T, X)$, define $G_K = \{x \in X | M_x = K \}$ (called a saturated set). The existence of saturated sets is proved by Sigmund [53] (or see [22]) for systems with uniform hyperbolicity or specification and generalized to non-uniformly hyperbolic systems in [40]. $K$ is called to have saturated property, if

$$G_K \neq \emptyset \quad \text{and} \quad h_{\text{top}}(T, G_K) = \inf \{h_\mu(T) | \mu \in K \}, \quad (1.1)$$

where $h_\mu(T)$ denotes its metric entropy of $\mu$. It was first established by Pfister and Sullivan in [50] that for any compact connected nonempty set $K \subseteq M(T, X)$, $K$ has saturated property if the dynamical system has $g$–almost product property (which is weaker than specification) and uniform separation property (which is weaker than expansiveness). Topological mixing uniformly hyperbolic systems have specification and expansiveness so that they have saturated property. Recently, saturated property is generalized to non-uniformly hyperbolic systems.
in [41] and non-uniformly expanding maps in [63]. For a compact connected nonempty set \( K \subseteq M(T, X) \), define the recurrently-saturated set and transitively-saturated set by

\[
G^R_K = \text{Rec} \cap G_K \quad \text{and} \quad G^T_K = \text{Tran} \cap G_K.
\]

Note that \( G^T_K \subseteq G^R_K \subseteq G_K \). We say \( K \) has \textit{recurrently-saturated} property if (1.1) holds replacing \( G_K \) by \( G^R_K \). In parallel, \( K \) has \textit{transitively-saturated} property if (1.1) holds replacing \( G_K \) by \( G^T_K \).

**Definition 1.3.** We say that the system \( T \) has \textit{transitively-saturated} property or \( T \) is transitively-saturated, if for any compact connected nonempty set \( K \subseteq M(T, X) \), \( K \) has transitively-saturated property. If transitively-saturated property holds only for the \( K \) that is contained in some convex sum of finite invariant measures, we say the system \( T \) is transitively-convex-saturated. If transitively-saturated property holds only for the \( K \) that consists of one invariant measure, we say the system \( T \) is transitively-single-saturated.

Now we start to state the result on transitively-saturated systems.

**Theorem 1.4.** Suppose that \((X, T)\) satisfies the \( g \)-almost product property and there is an invariant measure with full support. Then,

1. \( T \) is transitively-convex-saturated, the set \( \{ \mu \in M(T, X) \mid \mu \text{ is ergodic, } S_\mu \text{ is minimal} \} \)

is dense in \( M(T, X) \) and the almost periodic set \( \text{AP} \) is dense in \( X \).
2. If further \( T \) has uniform separation, then \( T \) is transitively-saturated.

**Remark 1.2.**

1. If \( \bigcup_{\mu \in K} S_\mu = X \), remark that \( \mu \) has saturated property and it is equivalent that \( \mu \) has transitively-saturated property, since naturally \( \bigcup_{\mu \in M, S_\mu \subseteq \omega_T(x)} \mu \) is dense in \( X \). It is worth recalling a classical result from [12] to consider \( K \) to be a single ergodic measure.
2. From [12] we know every \( \mu \in M_{\text{erg}}(T, X) \) has saturated property and transitively-saturated property. Let us give an explanation. By [12, theorem 3]

\[
G_\mu \neq \emptyset, \quad \mu(G_\mu) = 1, \quad \text{and} \quad h_{\text{top}}(T, G_\mu) = h_\mu(T).
\]

On the other hand, it is known that \( \mu(\text{Rec}) = 1 \) and by [12, theorem 3] \( h_{\text{top}}(T, \Gamma) \geq h_\mu(T) \) for any \( \Gamma \) with \( \mu(\Gamma) = 1 \) so that \( h_{\text{top}}(T, G^\mu_\Gamma) = h_\mu(T) \). However, it is not expected that \( \mu \) also naturally has transitively-saturated property. For example, if \( T \) is not transitive with positive topological entropy (e.g. a product map by identity map on \([0, 1]\) and a system with positive topological entropy), then for any ergodic \( \mu \) with positive metric entropy, \( G^\mu_\Gamma \) is empty so that \( h_{\text{top}}(T, G^\mu_\Gamma) = 0 < h_\mu(T) \). But we can show that any invariant measure \( \mu \) with \( S_\mu \neq X \) also has transitively-saturated property if the system has \( g \)-almost product property (by theorem 1.4). In other words, transitively-saturated property gives much information about generic points of invariant measures outside the support even if the measure is ergodic.

It was proven in [53] (or see [22]) that periodic measures are dense in the space of invariant measures for systems with Bowen’s periodic specification. Here, \( g \)-almost product property does not require periodic information so that periodic measures are replaced by ergodic measures supported on minimal sets.
The proof idea of theorem 1.4 in section 3.2 modifies Pfister and Sullivan’s proof of [50] in the pseudo-orbits construction to ensure that the tracing orbit can be dense in the whole space but the limit set of the empirical measures along the tracing orbit does not change. For example, if K is composed of one periodic measure μ, one may choose an orbit dense in the whole space (from the topological or geometric perspective) but still guarantee that the limit set of the empirical measures along the tracing orbit coincides with the given periodic measure μ (from a probabilistic or statistical perspective).

Organization of this paper. In section 2 we will recall some definitions used in our main results and in section 3 we give the proof of theorem 1.4. In section 4 we provide an abstract framework of multifractal analysis on general observable functions and in section 5 we study some basic properties of Banach recurrence and Transitive points. In section 6 we study Banach recurrent gap-sets and multifractal analysis of general observable functions together, and then in section 7 these results are used to prove theorem 1.2 and theorem 1.3. In section 8 we give some examples, applications to multifractal analysis on Lyapunov exponents, comments and further questions.

2. Some definitions

2.1. Entropy

Now let us to recall the definition of topological entropy in [12] by Bowen. Let \( x \in X \). The dynamical ball \( B_{n}(x, \varepsilon) \) is the set

\[
B_{n}(x, \varepsilon) := \{ y \in X | \max \{ d(T^{j}(x), T^{j}(y)) | 0 \leq j \leq n - 1 \} \leq \varepsilon \}.
\]

Let \( E \subseteq X \), and \( \mathcal{F}_{n}(E, \varepsilon) \) be the collection of all finite or countable covers of \( E \) by sets of the form \( B_{m}(x, \varepsilon) \) with \( m \geq n \). We set

\[
C(E; t, n, \varepsilon, T) := \inf \{ \sum_{B_{n}(x, \varepsilon) \in \mathcal{C}} 2^{-tm} : \mathcal{C} \in \mathcal{F}_{n}(E, \varepsilon) \}, \quad \text{and}
\]

\[
C(E; t, \varepsilon, T) := \lim_{n \to \infty} C(E; t, n, \varepsilon, T). \quad \text{Then} \quad h_{\text{top}}(E, \varepsilon, T) := \inf \{ t : C(E; t, \varepsilon, T) = 0 \} = \sup \{ t : C(E; t, \varepsilon, T) = \infty \} \quad \text{and the topological entropy of} \ E \ \text{is defined as}
\]

\[
h_{\text{top}}(T, E) := \lim_{\varepsilon \to 0} h_{\text{top}}(E, \varepsilon, T).
\]

In particular, if \( E = X \), we also denote \( h_{\text{top}}(T, X) \) by \( h_{\text{top}}(T) \). It is known from [12] that if \( E \) is an invariant compact subset, then the topological entropy \( h_{\text{top}}(T, E) \) is same as the classical definition (for classical definition of topological entropy, see chapter 7 in [65]). Let us recall some basic facts about topological entropy. From [12] we know for any \( Y \subseteq X \),

\[
h_{\text{top}}(T, TY) = h_{\text{top}}(T, Y), \quad \text{(2.1)}
\]

and for any subsets \( Y_{1} \subseteq Y_{2} \subseteq X \),

\[
h_{\text{top}}(T, Y_{1}) \leq h_{\text{top}}(T, Y_{2}). \quad \text{(2.2)}
\]

If one considers a collection of subsets of \( X \): \( \{ Y_{i} \}_{i=1}^{+\infty} \), from [12] we know that the topological entropy satisfies

\[
h_{\text{top}}\left( T, \bigcup_{i=1}^{+\infty} Y_{i} \right) = \sup_{i \geq 1} h_{\text{top}}(T, Y_{i}). \quad \text{(2.3)}
\]
Let $\xi = \{V_i| i = 1, 2, \ldots, k\}$, be a finite partition of measurable sets of $X$. The entropy of a probability measure $\nu \in M(X)$ with respect to $\xi$ is $H(\nu, \xi) := -\sum_{V_i \in \xi} \nu(V_i) \log \nu(V_i)$. We write $T^{\nu, n} \xi := \vee_{k=0}^{n-1} T^{-k} \xi$. The entropy of an invariant measure $\nu \in M(T, X)$ with respect to $\xi$ is $h(T, \nu, \xi) := \lim_{n \to \infty} \frac{1}{n} H(\nu, T^{\nu, n} \xi)$, and the metric entropy of $\nu$ is

$$h_\nu(T) := \sup_{\xi} h(T, \nu, \xi).$$

For more information on metric entropy, see chapter 4 of [65].

For convenience, we write $h_{\text{top}}$ and $h_\nu$ to denote $h_{\text{top}}(T)$ and $h_\nu(T)$.

### 2.2. Specification property and product property

First we recall the definition of specification property, which is stronger than $g -$ almost product property, see [11, 14, 15, 22, 52, 59].

**Definition 2.1.** We say that the dynamical system $T$ satisfies specification property, if the following holds: for any $\epsilon > 0$ there exists an integer $M(\epsilon)$ such that for any $k \geq 2$, any $k$ points $x_1, \ldots, x_k$ in $X$ such that

$$d(T^i(x_j), T^i(x_k)) \leq \epsilon, \quad \text{for } a_i \leq j \leq b_i, \quad 1 \leq i \leq k.$$  

(2.4)

In other words, the set $B = \cap_{i=1}^k I^{-a_i}_{-b_i} B_{h_{\text{top}}(f^a x_i) \epsilon}$ is nonempty.

(2.5)

The original definition of specification, due to Bowen, was stronger.

**Definition 2.2.** We say that the dynamical system $T$ satisfies Bowen’s specification property, if under the assumptions of definition 2.1 and for any integer $p \geq M(\epsilon) + b_k - a_1$, there exists a point $x \in X$ with $T^p(x) = x$ satisfying (2.4).

Now we start to recall the concept of $g -$ almost product property in [50] (there is a slightly weaker variant, called almost specification, see [60]). It is weaker than specification property (see proposition 2.1 in [50]). A striking and typical example of $g -$ almost product property (and almost specification) is that it applies to every $\beta -$ shift [50, 60]. In sharp contrast, the set of $\beta$ for which the $\beta -$ shift has specification property has zero Lebesgue measure [16, 55]. Let $\Lambda_n = \{0, 1, 2, \ldots, n-1\}$. The cardinality of a finite set $\Lambda$ is denoted by $\# \Lambda$.

**Definition 2.3.** Let $g : \mathbb{N} \to \mathbb{N}$ be a given nondecreasing unbounded map with the properties

$$g(n) < n \quad \text{and} \quad \lim_{n \to \infty} \frac{g(n)}{n} = 0.$$  

The function $g$ is called the blowup function. Let $x \in X$ and $\epsilon > 0$. The $g -$ blowup of $B_n(x, \epsilon)$ is the closed set $B_n(g; x, \epsilon) := \left\{ y \in X \mid \exists \Lambda \subseteq \Lambda_n, \#(\Lambda_n \setminus \Lambda) \leq g(n) \text{ and } \max\{d(T^j(x), T^j(y)) | j \in \Lambda\} \leq \epsilon \right\}$. 

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Definition 2.4. We say that the dynamical system $T$ satisfies the $g$-almost product property with the blowup function $g$, if there is a nonincreasing function $m : \mathbb{R}^+ \to \mathbb{N}$, such that for any $k \geq 2$, any $k$ points $x_1, \ldots, x_k \in X$, any positive $\varepsilon_1, \ldots, \varepsilon_k$ and any integers $n_1 \geq m(\varepsilon_1), \ldots, n_k \geq m(\varepsilon_k)$,

$$\bigcap_{j=1}^{k} T^{-M_{j-1}} B_0(g; x_j, \varepsilon_j) \neq \emptyset,$$

where $M_0 := 0, M_i := n_1 + \cdots + n_i, i = 1, 2, \ldots, k - 1$.

2.3. Entropy-dense and uniform separation

Definition 2.5. We say $T$ satisfies the entropy-dense property if for any $\mu \in M(T, X)$, for any neighbourhood $G$ of $\mu$ in $M(X)$, and for any $\eta > 0$, there exists a closed $T$-invariant set $\Lambda_\mu \subseteq X$ such that $M(T, \Lambda_\mu) \subseteq G$ and $h_{\text{top}}(T, \Lambda_\mu) > h_\mu - \eta$. By the classical variational principle, it is equivalent that for any neighbourhood $G$ of $\mu$ in $M(X)$, and for any $\eta > 0$, there exists a $\nu \in M_{\text{seq}}(T, X)$ such that $h_\nu > h_\mu - \eta$ and $M(T, S_\nu) \subseteq G$.

From [49, proposition 2.3 (1)], entropy-dense property holds for dynamical systems with $g$-almost product property.

Now we recall the definition of uniform separation property [50]. For $x \in X$, define $\Upsilon_\mu(x) := \frac{1}{n} \sum_{i=0}^{n-1} \delta_{\tau_i(x)}$ where $\delta_y$ is the Dirac probability measure supported at $y \in X$. For $\delta > 0$ and $\varepsilon > 0$, two points $x$ and $y$ are $(\delta, n, \varepsilon)$-separated if

$$\# \{ j : d(T^j x, T^j y) > \varepsilon, j \in \Lambda_n \} \geq \delta n.$$

A subset $E$ is $(\delta, n, \varepsilon)$-separated if any pair of different points of $E$ is $(\delta, n, \varepsilon)$-separated. Let $F \subseteq M(X)$ be a neighbourhood of $\nu \in M(T, X)$. Define $X_{n, F} := \{ x \in X | \Upsilon_n(x) \in F \}$, and define

$$N(F; \delta, n, \varepsilon) := \text{maximal cardinality of a } (\delta, n, \varepsilon) \text{-separated subset of } X_{n, F}.$$

Definition 2.6. We say that the dynamical system $T$ satisfies uniform separation property, if the following holds. For any $\eta > 0$, there exists $\delta^* > 0, \varepsilon^* > 0$ such that for $\mu$ ergodic and any neighbourhood $F \subseteq M(X)$ of $\mu$, there exists $n^*_F(\mu, \eta)$ such that for $n \geq n^*_F(\mu, \eta)$,

$$N(F; \delta^*, n, \varepsilon^*) \geq 2^{n(h_\mu(T) - \eta)}.$$

3. Proof of theorem 1.4: minimal set and saturated property

3.1. Full support and minimal set

Lemma 3.1. Suppose that a subset $B'$ of $B := \{ \omega \in M(T, X)| S_\omega \neq X \}$ is dense in $M(T, X)$. Then there is some invariant measure $\mu$ with full support (i.e. $S_\mu = X$) $\iff \bigcup_{\omega \in B} S_\omega = X$.

Proof. $\Rightarrow$ By assumption there is a sequence of invariant measures $\mu_i \in B'$ with $S_{\mu_i} \neq X$ converging to $\mu$. Then $1 = \limsup_{n \to \infty} \mu_n(\bigcup_{\omega \in B} S_\omega) \leq \mu(\bigcup_{\omega \in B} S_\omega)$. It follows that $X = S_{\mu} \subseteq \bigcup_{\omega \in B} S_\omega$. 

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\[ \sum_{\alpha} X \mathcal{G} \subseteq X \text{ and so } X = \mathcal{G} \sum_{\alpha} \mathcal{G} \text{ dense in } \mathcal{G} \subseteq \mathcal{G} \text{ with full support} \]

\[ \text{Proof.} \quad \text{By entropy-dense property, the set } B \text{ is dense in } M(T, X). \text{ Then by lemma 3.1 } \bigcup_{\omega \in B \mathcal{G}} = X. \]

\[ \text{Lemma 3.3.} \quad \text{Suppose that } \{ \mu \in M(T, X) \mid S_{\mu} \text{ is minimal} \} \text{ is dense in } M(T, X). \text{ Then there is some invariant measure } \mu \text{ with full support (i.e. } S_{\mu} = X) \Leftrightarrow \text{ the almost periodic set } \text{AP is dense in } X. \]

\[ \text{Proof.} \quad \Rightarrow \text{ By assumption there is a sequence of invariant measures } \mu_{n} \text{ with } S_{\mu_{n}} \subseteq \mathcal{A} \text{ converging to } \mu. \text{ Then } 1 = \limsup_{n \to \infty} \mu_{n}(A) \leq \limsup_{n \to \infty} \mu_{n}(\overline{A}) \leq \mu(\overline{A}). \text{ It follows that } X = S_{\mu} \subseteq \overline{A}. \]

\[ \Leftarrow \text{ Take a sequence of points } \{ x_{i} \} \subseteq A \text{ dense in } X. \text{ For any } i, \text{ take } \mu_{n} \text{ to be an invariant measure on } \omega_{T}(x_{i}). \text{ Then } x_{i} \in \omega_{T}(x_{i}) = S_{\mu_{n}} \text{ and so } \bigcup_{i \geq 1} S_{\mu_{n}} = X. \text{ Let } \mu = \sum_{n \geq 1} \frac{1}{\overline{A}} \mu_{n}. \text{ Then } \mu(\bigcup_{i \geq 1} S_{\mu_{n}}) = 1 \text{ so that } S_{\mu} = X. \]

\[ \text{Lemma 3.4.} \quad \text{Suppose that } T \text{ has } g \text{-almost product property. Then ergodic measures supported on minimal sets are dense in } M(T, X). \]

\[ \text{Proof.} \quad \text{Let } \nu \in M(T, X) \text{ and } G \subseteq M(X) \text{ be a neighbourhood of } \nu. \text{ Take an open ball } G' \subseteq M(X) \text{ such that } \nu \in G' \subseteq \overline{G' \subseteq G}. \text{ From the proof of [49, proposition 2.3 (1)], one constructs a closed invariant set } Y \text{ and there exists } n_{G'} \in \mathbb{N} \text{ such that for any } y \in Y \text{ and any } n \geq n_{G'}, \text{ } \mathcal{T}_{n}(y) \in G'. \text{ So for any } m \in M_{\text{erg}}(T, Y), \text{ by Birkhoff's ergodic theorem there is } y \in Y \text{ such that } \mathcal{Y}_{n}(y) \text{ converge to } m \text{ in weak* topology and thus } m \in \overline{G}. \text{ In other words, } M_{\text{erg}}(T, Y) \subseteq \overline{G}. \text{ By convex property of the ball } G' \text{ and the ergodic decomposition theorem, } M(T, Y) \subseteq \overline{G}. \text{ Take an ergodic measure } \mu \text{ supported on a minimal subset of } Y, \text{ then } \mu \in \overline{G} \subseteq G. \]

\[ \text{Lemma 3.5.} \quad \text{Suppose that } T \text{ has } g \text{-almost product property and there is some invariant measure } \mu \text{ with full support (i.e. } S_{\mu} = X). \text{ Then ergodic measures supported on minimal sets are dense in } M(T, X) \text{ and the almost periodic set } \text{AP is dense in } X. \]

\[ \text{Proof.} \quad \text{By lemma 3.4, ergodic measures supported on minimal sets are dense in } M(T, X). \text{ Combining with lemma 3.3, the almost periodic set } \text{AP is dense in } X. \]

3.2. Transitivity saturated: proof of theorem 1.4

By lemma 3.5 the set
\[ \{ \mu \in M(T, X) \mid \mu \text{ is ergodic, } S_{\mu} \text{ is minimal} \} \]
is dense in $M(T, X)$ and the almost periodic set $AP$ is dense in $X$. So we only need to prove the following theorems 3.1 and 3.2, which imply theorem 1.4.

Let $\{\varphi_j\}_{j \in \mathbb{N}}$ be a dense subset of $C(X, \mathbb{R})$, which is the space of continuous functions, then

$$d(\xi, \tau) = \sum_{j=1}^{\infty} \left| \int \varphi_j d\xi - \int \varphi_j d\tau \right|^{1/2}$$

defines a metric on $M(X)$ for the weak* topology [65], where $||\varphi|| = \max\{||\varphi(x)|| : x \in X\}$. Note that

$$d(\xi, \tau) \leq 1 \ $\ \text{for any } \xi, \tau \in M(X). \ \ \ (3.1)$$

It is well known that the natural projection $x \mapsto \delta_x$ is continuous and if we define operator $P_f$ on $M(X)$ by formula $P_f(\mu)(A) = \mu(T^{-1}(A))$, then we can identify $(X, T)$ with $P_T$, restricted to the set of Dirac measures (these systems are conjugate). Therefore, without loss of generality we will assume that $d(x, y) = d(\delta_x, \delta_y)$. Denote a ball in $M(X)$ by

$$B(\nu, \zeta) := \{\mu \in M(X) : d(\nu, \mu) \leq \zeta\}.$$ 

**Theorem 3.1.** Let $T : X \to X$ be a continuous map of a compact metric space $X$ with $g$-almost product property and uniform separation. Suppose that there is an invariant measure with full support. Then $T$ is transitively-saturated.

**Proof.** From [50, theorem 4.1 (3)] $h_{top}(f, G_K) \leq \inf\{h_\mu(T) | \mu \in K\}$. Since $G_K^T$ is contained in $G_K$, then $h_{top}(f, G_K^T) \leq h_{top}(f, G_K) \leq \inf\{h_\mu(T) | \mu \in K\}$.

The difficult part of the proof is to obtain a lower bound for $h_{top}(T, G_K^T)$. One can modify the construction in the proof of [50, theorem 1.1] to obtain a closed subset $F \subseteq G_K$ such that the entropy of $F$ is close to $\inf\{h_\mu(T) | \mu \in K\}$ and simultaneously we can require that the chosen points in $F$ are transitive. For the convenience of readers, we give a precise construction as follows.

By connectedness of $K$, one has

**Lemma 3.6 ([50, Page 944] or [22, page 202]).** There exists a sequence $\{\alpha_1, \alpha_2, \cdots\}$ in $K$ such that

$$\overline{\{\alpha_j : j \in \mathbb{N}, j > n\}} = K, \ \forall \ n \in \mathbb{N} \ \text{and} \ \lim_{j \to \infty} d(\alpha_j, \alpha_{j+1}) = 0.$$

Let $\eta > 0$ and

$$h^* := \inf\{h_\mu(T) | \mu \in K\} - 2\eta, \ H^* := \inf\{h_\mu(T) | \mu \in K\} - \eta.$$

Given this sequence of measures $\{\alpha_k\}$, we will construct a subset $G$ such that for each $x \in G$, $\{Y_n(x)\}$ has the same limit-point set as the sequence $\{\alpha_k\}$, and $h_{top}(T, G) \geq h^*$. The construction of $G$ is the core of the proof, which is also used in the proof of the theorem 3.2 below.

**Lemma 3.7 ([50, Corollary 3.1]).** Assume that $(X, d)$ has uniform separation property, and entropy-dense property. For any $\eta$, there exist $\delta^* > 0, \epsilon^* > 0$ such that for $\mu \in M(T, X)$ and any neighbourhood $F \subseteq M(X)$ of $\mu$, there exists $n^*_{F, \delta^*, \eta}$, such that for $n \geq n^*_{F, \delta^*, \eta}$,

$$N(F; \delta^*, n, \epsilon^*) \geq 2^{n(h_\mu(T) - \eta)}.$$
By lemma 3.7, we can find $\delta^* > 0, \epsilon^* > 0$ such that for $\mu \in M(T, X)$ and any neighbourhood $F \subseteq M(X)$ of $\mu$, there exists $n^*_{\delta^*, \mu, \eta}$, such that for $n \geq n^*_{\delta^*, \mu, \eta}$,

$$N(F; \delta^*, n, \epsilon^*) \geq 2^{n(t_0(T) - \eta)}.$$  \hspace{1cm} (3.2)

Let $m : \mathbb{R}^+ \to \mathbb{N}$ be the nonincreasing function by the $g$-almost product property. Let $\{g_k\}$ and $\{\epsilon_k\}$ be two strictly decreasing sequences so that $\lim_{k \to \infty} \epsilon_k = 0 = \lim_{k \to \infty} \epsilon_k$ with $\epsilon_1 < \frac{1}{2} \epsilon^*$. By lemma 3.5 the almost periodic set $AP$ is dense in $X$. Thus, for any fixed $k$ there is a finite set $\Delta_k := \{x_1^k, x_2^k, \ldots, x_l^k\} \subseteq AP$ and $L_k \in \mathbb{N}$ such that $\Delta_k$ is $\epsilon_k$-dense in $X$ and for any $1 \leq i \leq L_k$, there is $n \in [l, l + L_k]$ such that $T^n(x_i^k) \in B(x_i^k, \epsilon_k)$. This implies that any $1 \leq i \leq L_k$,

$$\frac{\# \{0 \leq n \leq LL_k : d(T^n x_i^k, x_i^k) \leq \epsilon_k\}}{LL_k} \geq \frac{1}{L_k}.$$  \hspace{1cm} (3.3)

Take $l_k$ large enough such that

$$l_k L_k \geq m(\epsilon_k), \quad \frac{g(l_k L_k)}{l_k L_k} < \frac{1}{4L_k}.$$  \hspace{1cm} (3.4)

We may assume that the two sequences of $\{t_k\}, \{l_k\}, \{L_k\}$ are strictly increasing.

From (3.2) we get the existence of $n_k$ and a $(\delta^*, \epsilon^*)-$separated subset $\Gamma_k \subseteq X_{m, B(\alpha_k, \zeta_k)}$ with

$$\# \Gamma_k \geq 2^{n^*H^*}. \hspace{1cm} (3.5)$$

We may assume that $n_k$ satisfies

$$n_k > m(\epsilon_k), \quad \frac{t_k l_k L_k}{n_k} \leq \zeta_k, \quad \delta^* n_k > 2g(n_k) + 1 \quad \text{and} \quad \frac{g(n_k)}{n_k} \leq \epsilon_k \quad \text{and} \hspace{1cm} (3.6)$$

$$2^H n_k \geq 2^{\eta(n_k + n_k L_k)}.$$  \hspace{1cm} (3.7)

We choose a strictly increasing $\{N_k\}$, with $N_k \in \mathbb{N}$, so that

$$n_{k+1} + t_{k+1} l_{k+1} L_{k+1} \leq \sum_{j=1}^{k} (n_j N_j + t_j L_j) \quad \text{and} \hspace{1cm} (3.8)$$

$$\sum_{j=1}^{k-1} (n_j N_j + t_j L_j) \leq \sum_{j=1}^{k} (n_j N_j + t_j L_j).$$  \hspace{1cm} (3.9)

Now we define the sequences $\{n'_j\}, \{\epsilon'_j\} \text{ and } \{\Gamma'_j\}$, by setting for

$$j = N_1 + N_2 + \cdots + N_{k-1} + t_1 + \cdots + t_{k-1} + q \text{ with } 1 \leq q \leq N_k,$$

$$n'_j := n_k, \quad \epsilon'_j := \epsilon_k, \quad \Gamma'_j := \Gamma_k \quad \text{and for} \hspace{1cm} (3.10)$$

$$j = N_1 + N_2 + \cdots + N_k + t_1 + \cdots + t_{k-1} + q \text{ with } 1 \leq q \leq t_k,$$

$$n'_j := l_k L_k, \quad \epsilon'_j := \epsilon_k, \quad \Gamma'_j := \{x^k_i\}. \hspace{1cm} (3.11)$$

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Let
\[ G_k := \bigcap_{j=1}^{k} \left( \bigcup_{i \in \Gamma_j'} T^{-M-1} B_{\delta_j'}(g; x_j, \epsilon_j') \right) \text{ with } M_j := \sum_{i=1}^{j} n_j', \]

Note that \( G_k \) is a non-empty closed set. We can label each set obtained by developing this formula by the branches of a labelled tree of height \( k \). A branch is labelled by \( (x_1, \cdots, x_k) \) with \( x_j \in \Gamma_j' \). Then theorem 3.1 can be deduced by the following lemma.

**Lemma 3.8.** Let \( \epsilon = \epsilon^* \) and let \( G := \cap_{k \geq 1} G_k \). Then we have the following:

1. Let \( x_j, y_j \in \Gamma_j' \) with \( x_j \neq y_j \). If \( x \in B_{\delta_j'}(g; x_j, \epsilon_j') \) and \( y \in B_{\delta_j'}(g; y_j, \epsilon_j') \), then
   \[ \max \{ d(T^m x, T^m y) : m = 0, \cdots, n_j - 1 \} > 2\epsilon. \]

2. \( G \) is a closed set that is the disjoint union of non-empty closed sets \( G(x_1, x_2, \cdots) \). Labelled by \( (x_1, x_2, \cdots) \) with \( x_j \in \Gamma_j' \). Two different sequences label two different sets.

3. \( G \subseteq \overline{G} \).

4. \( h_{\text{top}}(T, G) \geq H^* - \eta = h^* \).

5. \( G \subseteq \text{Tran.} \)

**Proof.** Different to [50, lemma 5.1], our new construction implies item (5). We can modify the proof of [50, lemma 5.1] to be adaptable to our new construction and simultaneously the new construction guarantees item (5).

1. Let \( x \in B_{\delta_j'}(g; x_j, \epsilon_j') \) and \( y \in B_{\delta_j'}(g; y_j, \epsilon_j') \). Since \( x_j \) and \( y_j \) are \((\delta^*, n_j, \epsilon^*)\)-separated and (3.6) holds, there exists \( m \in \Lambda_{\delta_j} \) so that
   \[ d(T^m x_j, T^m y_j) > \epsilon^* = 4\epsilon, \quad d(T^m y_j, T^m x) \leq \epsilon_j', \quad d(T^m y_j, T^m y) \leq \epsilon_j'. \]

   However,
   \[ d(T^m x, T^m y) \geq d(T^m x_j, T^m y_j) - d(T^m x_j, T^m x) - d(T^m y_j, T^m y) > 2\epsilon. \]

2. Note that \( G \) is the intersection of closed sets. Let \( (x_1, x_2, \cdots) \) be a sequence with \( x_j \in \Gamma_j' \). By the \( g \) - almost product property and compactness
   \[ \bigcap_{j \geq 1} T^{-M-1} B_{\delta_j'}(g; x_j, \epsilon_j') \]

   is nonempty and closed. By item (1) the two sets of \( B_{\delta_j'}(g; x_j, \epsilon_j') \) and \( B_{\delta_j'}(g; y_j, \epsilon_j') \) are disjoint when \( x_j \neq y_j \). So two different sequences label two different sets.

3. Define the stretched sequence \( \{\alpha'_m\} \) by
   \[ \alpha'_m := \alpha_k \quad \text{if} \quad \sum_{j=1}^{k-1} (n_j N_j + t_j L_j) + 1 \leq m \leq \sum_{j=1}^{k} (n_j N_j + t_j L_j). \]

   Then the sequence \( \{\alpha'_m\} \) has the same limit-point set as the sequence of \( \{\alpha_k\} \). If
   \[ \lim_{n \to \infty} d(T_n(y), \alpha'_m) = 0, \]

   then the two sequences \( \{T_n(y)\}, \{\alpha'_m\} \) have the same limit-point set. By (3.8) \( \lim_{n \to \infty} \frac{M_{\alpha'_m}}{M_{\alpha_k}} = 1 \). So from the definition of \( \{\alpha'_m\} \), we only need to prove that for any \( y \in G \), one has
\[
\lim_{l \to \infty} d(Y_{M_l}(y), \alpha'_{M_l}) = 0.
\]

**Lemma 3.9 ([50, lemma 2.1]).** Assume that \((X, T)\) satisfies the \(g\)-almost product property. Let \(x_1, \ldots, x_k \in X, \varepsilon_1, \ldots, \varepsilon_k\) and \(q_1 \geq m(\varepsilon_1), \ldots, q_k \geq m(\varepsilon_k)\) be given. Assume that
\[
Y_{q_j}(x_j) \in B(v_j, \zeta_j), \quad j = 1, 2, \ldots, k.
\]

Then for any \(z \in \bigcap_{j=1}^{k} T^{-t_{j}L_{j}}B_{g}(z, \varepsilon_{j})\) and any probability measure \(\alpha\)
\[
d(Y_{Q_0}(z), \alpha) \leq \sum_{j=1}^{k} \frac{n_j}{Q_k} (\varepsilon_j + g(q_j))
\]
where \(Q_0 = 0, Q_i = q_1 + \cdots + q_i\).

Assume that \(\sum_{j=1}^{k} (n_j N_j + t_j L_j) + 1 \leq M_l \leq \sum_{j=1}^{k} (n_j N_j + t_j L_j)\). If
\[
M_l \leq \sum_{j=1}^{k} (n_j N_j + t_j L_j) + n_{k+1} N_{k+1}, \quad \text{by lemma 3.9 and (3.6)}
\]
\[
d(Y_{M_l}, \alpha) \leq \frac{n_{k+1} N_{k+1}}{M_l - \sum_{j=1}^{k} (n_j N_j + t_j L_j)} \times \left( \sum_{j=1}^{k} (n_j N_j + t_j L_j) \right) \leq \frac{\zeta_{k+1} + 2 \varepsilon_{k+1}}{M_l - \sum_{j=1}^{k} (n_j N_j + t_j L_j)}
\]
\[
\leq 2 \zeta_{k+1} + 2 \varepsilon_{k+1}.
\]

By lemma 3.9 and (3.6),
\[
d(Y_{n_{k+1}}(T^{\sum_{j=1}^{k-1} (n_j N_j + t_j L_j)} y), \alpha_{k+1}) \leq \zeta_k + 2 \varepsilon_k + d(\alpha_k, \alpha_{k+1})
\]
Thus, by (3.1), (3.9) and (3.6),
\[
d(Y_M(y), \alpha_{k+1}) \leq \frac{\sum_{j=1}^{k-1} (n_j N_j + t_j L_j) M_l}{M_l} d(Y_{n_{k+1}}(T^{\sum_{j=1}^{k-1} (n_j N_j + t_j L_j)} y), \alpha'_{M_l})
\]
\[
+ \frac{n_{k} N_{k}}{M_l} d(Y_{n_{k+1}}(T^{\sum_{j=1}^{k-1} (n_j N_j + t_j L_j)} y), \alpha_{k+1}) + \frac{t_k L_k}{M_l} \times 1
\]
\[
+ d(Y_{M_l}, \alpha) \leq \frac{\sum_{j=1}^{k-1} (n_j N_j + t_j L_j) M_l}{M_l} \times 1 + 1 \times (\zeta_k + 2 \varepsilon_k + d(\alpha_k, \alpha_{k+1})) + \frac{t_k L_k}{n_k}
\]
\[
+ 2 \zeta_{k+1} + 2 \varepsilon_{k+1}
\]
\[
\leq \zeta_k + \zeta_k + 2 \varepsilon_k + d(\alpha_k, \alpha_{k+1}) + \zeta_k + 2 \zeta_{k+1} + 2 \varepsilon_{k+1}.
\]
Since $\zeta_k, \epsilon_k, d(\alpha_k, \alpha_{k+1})$ all converge to zero as $k$ goes to zero, this proves item (3).

(4) As stated in the proof of [50, lemma 5.1, item 4] on page 946, the details of the construction are unimportant and in fact Pfister and Sullivan proved that:

**Lemma 3.10** If $\{n_p\}$ is a strictly increasing sequence of natural numbers such that $\lim_{p \to \infty} \frac{M_{n_p+1}}{M_{n_p}} = 1$ and $\#\Gamma_{n_p+1} \times \#\Gamma_{n_p+2} \times \cdots \times \#\Gamma_{n_p+1} \geq 2^{h^*(M_{n_p+1} - M_p)}$, then $h_{\text{top}}(T, G) \geq h^*$.

For $k \geq 1, i = 0, 1, 2, \ldots, N_k - 1$, let

$$n_{N_1 + \cdots + N_{k-1} + i} := N_1 + \cdots + N_{k-1} + t_1 + \cdots + t_{k-1} + i$$

Then for any $p \geq 1$, there is some $k$ so that $N_1 + \cdots + N_{k-1} + t_1 + \cdots + t_{k-1} \leq n_p \leq N_1 + \cdots + N_{k-1} + t_1 + \cdots + t_k + 1$. By (3.8)

$$1 \leq \frac{M_{n_p+1}}{M_{n_p}} \leq \frac{M_{n_p} + \max\{n_k, n_k + t_kL_k\}}{M_{n_p}} = 1 + \frac{n_k + t_kL_k}{M_{n_p}} \leq 1 + \frac{n_k + t_kL_k}{\sum_{j=1}^{k-1} (n_jN_j + t_jL_j)} \leq 1 + \zeta_k.$$

By (3.7)

$$\#\Gamma_{n_p+1} \times \#\Gamma_{n_p+2} \times \cdots \times \#\Gamma_{n_p+1} \geq \#\Gamma_k \geq 2^{h^*(n_k + t_kL_k)} \geq 2^{h^*(M_{n_p+1} - M_p)}.$$

By lemma 3.10 we finish the proof of item (4).

(5) Fix $x \in G$. By construction, for any fixed $k \geq 1$, there is $a = a_k$ such that for any $j = 1, \cdots, t_k$, there is $\Lambda^j \subseteq A_k L_k$

$$\max\{d(T^{a+(j-1)L_k} x, T^l x^j) \mid l \in \Lambda^j\} \leq \epsilon_k.$$ 

By (3.4)

$$\frac{\#\Lambda^j}{L_k} \geq 1 - \frac{g(l_j L_k)}{L_k} \geq 1 - \frac{1}{4L_k}.$$ 

Together with (3.3) we get that for any $j = 1, \cdots, t_k$ there is $p_j \in [0, l_jL_k - 1]$ such that

$$d(T^{a+(j-1)L_k} x, T^{P_j} x^j) \leq \epsilon_k$$

and

$$d(T^l T^{p_j} x^j, T^l x^j) \leq \epsilon_k.$$ 

This implies $d(T^{a+(j-1)L_k} x, T^{p_j} x^j) \leq 2\epsilon_k$ so that the orbit of $x$ is $3\epsilon_k$-dense in $X$. By the arbitrariness of $k$, one has $x \in \text{Tran}_\mu$.

\[ \square \]

**Remark 3.2.** For any periodic measure $\mu$, our above result tells us there exists a transitive point whose empirical measure is exactly equal to $\mu$, which can not be deduced from Sigmund’s result [53].
If we only have the $g$-almost product property but do not know the property of uniform separation, then we still have the following characterization.

**Theorem 3.2.** Let $T : X \to X$ be a continuous map of a compact metric space $X$ with $g$-almost product property. Suppose that there is an invariant measure with full support. Then $T$ is transitively-convex-saturated and in particular transitively-single-saturated.

**Proof.** Similar to the construction of transitive points in theorem 3.1, one can adapt the proof of [25, theorem 1.5] (or [50, theorem 1.2]) to complete the proof for which the ergodic decomposition theorem replaces the role of uniform separation. Here we omit the details. □

3.3. Locally saturated property

For any compact connected $K \subseteq M(T, X)$, let $G_K = \{ x \in X | M_x = K \}$, $G_K^T = \{ x \in \text{Tran} | M_x = K \}$. They are a saturated set of $K$ and transitively-saturated set of $K$, respectively.

**Definition 3.1.** We say that the system $T$ has saturated property or $T$ is saturated (simply, S), if for any compact connected nonempty set $K \subseteq M(T, X)$,

$$h_{\text{top}}(T, G_K) = \inf \{ h_\mu(T) | \mu \in K \}.$$  \hspace{1cm} (3.10)

We say that the system $T$ has locally-saturated property or $T$ is locally-saturated (simply, LS), if for any compact connected nonempty set $K \subseteq M(T, X)$, any nonempty open set $U \subseteq X$,

$$h_{\text{top}}(T, G_K \cap U) = \inf \{ h_\mu(T) | \mu \in K \}.$$  \hspace{1cm} (3.11)

In parallel, one can define locally-transitively-saturated just by replacing $G_K$ by $G_K^T$ in (3.11). On the other hand, one can define single-saturated, locally-single-saturated, locally-transitively-single-saturated when $K$ is a singleton and define convex-saturated, locally-convex-saturated, locally-transitively-convex-saturated, respectively, when $K$ consists of a convex sum of two invariant measures.

Note that locally-saturated is stronger than saturated (just taking $U = X$), saturated is stronger than convex-saturated and the latter is stronger than single-saturated. We will provide a basic discussion on the relations of these concepts in lemma 3.11 (see below). For convenience, we use Sat, Loc-Sat, Tran-Sat, Loc-Tran-Sat to denote saturated, locally-saturated, transitively-saturated, locally-transitively-saturated; Conv-Sat, Loc-Conv-Sat, Tran-Conv-Sat, Loc-Tran-Conv-Sat to denote convex-saturated, locally-convex-saturated, transitively-convex-saturated, locally-transitively-convex-saturated; and Sing-Sat, Loc-Sing-Sat, Tran-Sing-Sat, Loc-Tran-Sing-Sat to denote single-saturated, locally-single-saturated, transitively-single-saturated, and locally-transitively-single-saturated, respectively.

**Lemma 3.11.** Let $T : X \to X$ be a continuous map of a compact metric space $X$. Then the above saturated properties have the following relation:

| Tran-Sat | ⇔ | Loc-Tran-Sat | ⇒ | Loc-Sat | ⇒ | Sat |
| Tran-Conv-Sat | ⇔ | Loc-Tran-Conv-Sat | ⇒ | Loc-Conv-Sat | ⇒ | Conv-Sat |
| Tran-Sing-Sat | ⇔ | Loc-Tran-Sing-Sat | ⇒ | Loc-Sing-Sat | ⇒ | Sing-Sat |

Before proving Lemma 3.11 we need a following basic result, which is also useful for the proof of our main theorems.
Lemma 3.12. Let $T : X\to X$ be a continuous map of a compact metric space $X$, $B\subseteq X$ be invariant and $U \subseteq X$ be a nonempty open set. Then $h_{\text{top}}(T, B \cap \text{Tran} \cap U) = h_{\text{top}}(T, B \cap U)$. 

Proof. By (2.2), the part $\prime \leq$ is obvious. Now we start to consider the part $\prime \geq$.

Let $B^T = B \cap \text{Tran}$. Notice that by (2.1) and (2.2) for any $n \geq 1$,

$$h_{\text{top}}(T, T^{-n}(U \cap B^T)) = h_{\text{top}}(T, T^\alpha T^{-n}(U \cap B^T)) \leq h_{\text{top}}(T, U \cap B^T)$$

and by the definition of transitivity and invariance of $B$ and $\text{Tran}$

$$B^T = B^T \cap (\bigcup_{n \geq 0} T^{-n}U) \subseteq \bigcup_{n \geq 0} T^{-n}(U \cap B^T).$$

Thus by (2.3)

$$h_{\text{top}}(T, B^T) \leq h_{\text{top}}(T, \bigcup_{n \geq 0} T^{-n}(U \cap B^T))$$

$$= \sup_{n \geq 0} h_{\text{top}}(T, T^{-n}(U \cap B^T)) \leq h_{\text{top}}(T, U \cap B^T).$$

Proof of lemma 3.11. Given a compact and connected subset $K \subseteq M(X, T, \cdot)$ we only need to prove that

1. $h_{\text{top}}(T, G_K^T) = \inf \{h_{\mu}(T) \mid \mu \in K\} \iff h_{\text{top}}(T, G_K^T \cap U) = \inf \{h_{\mu}(T) \mid \mu \in K\}$ holds for any nonempty open set $U \subseteq X$.
2. $h_{\text{top}}(T, G_K^T \cap U) = \inf \{h_{\mu}(T) \mid \mu \in K\}$ holds for any nonempty open set $U \subseteq X$ 

$\Rightarrow h_{\text{top}}(T, G_K \cap U) = \inf \{h_{\mu}(T) \mid \mu \in K\}$ holds for any nonempty open set $U \subseteq X$ 

$\Rightarrow h_{\text{top}}(T, G_K) = \inf \{h_{\mu}(T) \mid \mu \in K\}.$

For part (1), letting $B = G_K$, then it is invariant and by lemma 3.12 we know that $h_{\text{top}}(T, G_K^T) = h_{\text{top}}(T, G_K^T \cap U)$. This implies part (1).

For part (2), on one hand, if $h_{\text{top}}(T, G_K^T \cap U) = \inf \{h_{\mu}(T) \mid \mu \in K\}$ holds for any nonempty open set $U \subseteq X$, then $h_{\text{top}}(T, G_K \cap U) \geq h_{\text{top}}(T, G_K^T \cap U) = \inf \{h_{\mu}(T) \mid \mu \in K\}$. Recall that in general $h_{\text{top}}(T, G_K) \leq \inf \{h_{\mu}(T) \mid \mu \in K\}$ (see [49, theorem 4.1 (3)]), thus $h_{\text{top}}(T, G_K \cap U) = \inf \{h_{\mu}(T) \mid \mu \in K\}$. On the other hand, if $h_{\text{top}}(T, G_K \cap U) = \inf \{h_{\mu}(T) \mid \mu \in K\}$ holds for any nonempty open set $U \subseteq X$, then taking $U = X$, we have $h_{\text{top}}(T, G_K) = h_{\text{top}}(T, G_K^T \cap U) = \inf \{h_{\mu}(T) \mid \mu \in K\}$. 

Remark 3.3. If $G_K^T \neq \emptyset$, then there is a transitive point so that the system is transitive and thus is surjective. So for $n \geq 1$,

$$h_{\text{top}}(T, T^{-n}(U \cap G_K^T)) = h_{\text{top}}(T, T^\alpha T^{-n}(U \cap G_K^T)) = h_{\text{top}}(T, U \cap G_K^T).$$

By definition and the system being surjective, one also sees that $T^{\pm 1}G_K = G_K$, $T^{\pm 1}\text{Tran} = \text{Tran}, T^{\pm 1}G_K^T = G_K^T$.

We also note that sometimes single-saturated implies the transitivity of the system.

Lemma 3.13. Suppose that $T$ is single-saturated and there is some invariant measure $\mu$ with full support (i.e. $S_\mu = X$). Then the system $T$ is transitive.

Proof. By single-saturated property, $G^T_{\mu} \neq \emptyset$. Take $x \in G^T_{\mu}$, then $\omega_T(x) = X$. 

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4. Abstract framework on an irregular set and level sets

Before giving an abstract framework on multifractal analysis, let us recall the history of multifractal analysis of the Birkhoff ergodic average. A point $x \in X$ is generic for some invariant measure $\mu$ means that $x \in G_\mu$ (or equivalently, Birkhoff averages of all continuous functions converge to the integral of $\mu$). Let $QR = \bigcup_{\mu \in M(X)} G_\mu$. The points in $QR$ are called quasiregular points of $T$ in [22, 44]. Let $IR = X \setminus QR$.

For a continuous function $\varphi$ on $X$, define the $\varphi$-irregular set as

$$I_\varphi := \left\{ x \in X : \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(T^i x) \text{ diverges} \right\}.$$ 

$\varphi$-regular set and the irregular set, the union of $I_\varphi(T)$ over all continuous functions of $\varphi$ (coinciding with $IR$), arise in the context of multifractal analysis. The irregular points are also called points with historic behaviour, see [51, 56]. From Birkhoff’s ergodic theorem, the irregular set is not detectable from the point of view of any invariant measure. However, the irregular set may have strong dynamical complexity in the sense of the Hausdorff dimension, Lebesgue positive measure, topological entropy and topological pressure, etc. Pesin and Pitskel [48] are the first to notice the phenomenon of the irregular set carrying full topological entropy in the case of the full shift on two symbols. There are lots of advanced results to show that the irregular points can carry full entropy in symbolic systems, hyperbolic systems, non-uniformly expanding or hyperbolic systems, and systems with specification-like or shadowing-like properties, for example, see [3, 8, 17, 24, 41, 60, 63] and references therein.

On the other hand, level set is a natural concept to slice points with convergent the Birkhoff ergodic average operated by some continuous function, regarded as the multifractal decomposition, for example, see [4, 18, 28]. For a continuous function $\varphi$ on $X$, denote

$$L_\varphi = \left[ \inf_{\mu \in M(X)} \int \varphi \, d\mu, \sup_{\mu \in M(X)} \int \varphi \, d\mu \right]$$ and $\text{Int}(L_\varphi) = \left( \inf_{\mu \in M(X)} \int \varphi \, d\mu, \sup_{\mu \in M(X)} \int \varphi \, d\mu \right)$.

Note that if $I_\varphi(T) \neq \emptyset$, then $\text{Int}(L_\varphi) \neq \emptyset$. The inverse is also true if the system has specification property, see [59] (see [60] for the case of almost specification), and it is easy to check that the continuous functions with $\text{Int}(L_\varphi) \neq \emptyset$ form an open and dense subset in the space of continuous functions; so do the functions with $I_\varphi(T) \neq \emptyset$ if the system has specification property or almost specification. For any $a \in L_\varphi$, consider the level set

$$R_\varphi(a) := \left\{ x \in X : \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(T^i x) = a \right\}.$$ 

Denote $R_\varphi = \bigcup_{a \in L_\varphi} R_\varphi(a)$, then $R_\varphi$ represents the regular points for $\varphi$. Many authors have considered the entropy of the $R_\varphi(a)$. For example, Barreira and Saussol proved in [9] that the following properties for a dynamical system $(X, T)$ whose function of metric entropy is upper semi-continuous. Consider an Hölder continuous function $\varphi$ (see [4, 6] for almost additive functions with tempered variation) which has a unique equilibrium measure, then for any constant $a \in \text{Int}(L_\varphi)$,

$$h_{\text{top}}(T, R_\varphi(a)) = \sup_{\mu \in M(X)} \left\{ h_\mu : \int \varphi \, d\mu = a \right\},$$

(4.1)
where \( h_{\text{top}}(R_\varphi(a)) \) denotes the entropy of \( R_\varphi(a) \), \( h_\mu \) denotes the measure entropy of \( \mu \). For \( \varphi \) being an arbitrary continuous function (hence there may exist more than one equilibrium measure), (4.1) was established by Takens and Verbitski [57] under the assumption that \( f \) has the specification property. This result was further generalized by Pfister and Sullivan [50] to dynamical systems with \( g \)-almost product property (see [58, 62] for more related discussions). The method used in [6, 9] mainly depends on thermodynamic formalism such as differentiability of pressure function while the method in [50, 57] is a direct approach by constructing fractal sets.

Motivated by Kingman’s ergodic theorem, it is also interesting to study multifractal analysis of subadditive functions. Thus, for possibly applicability to more general functions, we introduce an abstract version of multifractal analysis.

Let \( \alpha : M(T,X) \to \mathbb{R} \) be a continuous function. Let \( L_\alpha = [\inf_{\mu \in M} \alpha(\mu), \sup_{\mu \in M} \alpha(\mu)] \) and \( \text{Int}(L_\alpha) \) denote its interior interval. Define

\[
I_\alpha := \{ x \in X : \inf_{\mu \in M} \alpha(\mu) < \sup_{\mu \in M} \alpha(\mu) \}; \quad R_\alpha := \{ x \in X : \inf_{\mu \in M} \alpha(\mu) = \sup_{\mu \in M} \alpha(\mu) \};
\]

\[
R_\alpha(a) := \{ x \in X : \inf_{\mu \in M} \alpha(\mu) = \sup_{\mu \in M} \alpha(\mu) = a \};
\]

\[
H_\alpha(a) := \sup \{ h_\rho | \rho \in M(T,X) \text{ and } \alpha(\rho) = a \}.
\]

We list three conditions for \( \alpha \):

[A.1]. For any \( \mu, \nu \in M(T,X) \), \( \beta(\theta) := \alpha(\theta \mu + (1-\theta)\nu) \) is strictly monotonic on \([0,1]\) when \( \alpha(\mu) \neq \alpha(\nu) \).

[A.2]. For any \( \mu, \nu \in M(T,X) \), \( \beta(\theta) := \alpha(\theta \mu + (1-\theta)\nu) \) is constant on \([0,1]\) when \( \alpha(\mu) = \alpha(\nu) \).

[A.3]. For any \( \mu, \nu \in M(T,X) \), \( \beta(\theta) := \alpha(\theta \mu + (1-\theta)\nu) \) is not constant over any subinterval of \([0,1]\) when \( \alpha(\mu) \neq \alpha(\nu) \) (Note that [A.1] implies [A.3]).

One has a following abstract result on multifractal analysis of an irregular set and level sets for which the variational principle on the level set does not require any condition on \( \alpha \).

**Theorem 4.1.** Let \( (X,T) \) be a dynamical system and \( \alpha : M(T,X) \to \mathbb{R} \) be a continuous function.

1. If \( (X,T) \) is transitively-single-saturated, then for any real number \( a \in L_\alpha \), the set \( R_\alpha(a) \cap \text{Tran} \) is not empty and

\[
h_{\text{top}}(T, R_\alpha(a)) = h_{\text{top}}(T, R_\alpha(a) \cap \text{Tran}) = H_\alpha(a).
\]

If further \( T \) has positive topological entropy and \( \text{Int}(L_\alpha) \neq \emptyset \), then for any real number \( a \in \text{Int}(L_\alpha) \),

\[
h_{\text{top}}(T, R_\alpha(a)) = h_{\text{top}}(T, R_\alpha(a) \cap \text{Tran}) = H_\alpha(a) > 0.
\]

2. Suppose that \( (X,T) \) is transitively-convex-saturated and \( \inf_{\mu \in M(T,X)} \alpha(\mu) < \sup_{\mu \in M(T,X)} \alpha(\mu) \). Then the set \( I_\alpha \cap \text{Tran} \) is not empty and moreover,

2.1. If \( \alpha : M(T,X) \to \mathbb{R} \) satisfies [A.3], then \( h_{\text{top}}(T, I_\alpha) = h_{\text{top}}(T, I_\alpha \cap \text{Tran}) = h_{\text{top}}(T) \).

2.2. If \( T \) has positive topological entropy, then \( h_{\text{top}}(T, I_\alpha) \geq h_{\text{top}}(T, I_\alpha \cap \text{Tran}) > 0 \).

3. If \( a \in L_\alpha \setminus \text{Int}(L_\alpha) \), then the set \( R_\alpha(a) \) is not empty and
\( h_{\top}(T, R_\alpha(a)) = H_\alpha(a). \)

(4) \( h_{\top}(T) = h_{\top}(T, R_\alpha). \) If further \( \text{Int}(L_\alpha) \neq \emptyset \) and \( \alpha \) satisfies [A.3], then

\[
 h_{\top}(T) = \sup_{a \in \text{Int}(L_\alpha)} H_\alpha(a) = h_{\top}(T, R_\alpha).
\]

**Remark 4.1.** The second part in item (1) sometimes is not true for \( a \in L_\alpha \setminus \text{Int}(L_\alpha) \), since maybe \( H_\alpha(a) = 0 \). For example, letting \( T : S^1 \to S^1 \) be an expanding map defined as \( T(x) = 2x \text{mod} 1 \) and \( \mu_1, \mu_2 \) be two different periodic measures, define a continuous function \( \phi \) such that

\[
 \phi|_{S_{\mu_1}} = 0, \phi|_{S_{\mu_2}} = 1
\]

and \( 0 < \phi(x) < 1 \) for other points of \( x \). For \( \alpha(\mu) := \int \phi d\mu \) and \( a = 0 \) or 1, by the first part in item (1) one has

\[
 h_{\top}(T, R_\alpha(a)) = h_{\top}(T, R_\alpha(a) \cap \text{Tran}) = H_\alpha(a) = h_{\mu_1} = 0 \text{ or } = h_{\mu_2} = 0.
\]

It is worth mentioning that

\[
 \{ \phi : \phi \text{ is Lipschitz, } H_\alpha(a) = 0, \forall a \in L_\alpha \setminus \text{Int}(L_\alpha) \text{ where } \alpha(\mu) := \int \phi d\mu \}
\]

is an open and dense subset in the space of Lipschitz functions, see [20].

**Remark 4.2.** Let \( \phi, \psi \) be two continuous functions on \( X \) and \( \psi \) required to be positive. Define \( \alpha(\mu) = \int \phi d\mu / \int \psi d\mu \) which satisfies [A.1]–[A.3]. In this case, recall that the part for \( R_\alpha(a) \) has been discussed in [9] for hyperbolic systems by Barreira and Saussol. Here we get a generalized version for \( R_\alpha(a) \cap \text{Tran} \) and the studied system is more general, not just including hyperbolic systems but also including mixing interval maps, \( C^0 \) generic conservative systems, symbolic systems and non-hyperbolic smooth dynamics etc, see section 8.2. For more refinement, see item (2) of theorem 6.1 below. For results on mixed Lyapunov exponents of asymptotically additive functions, see item (2) of theorem 8.1.

**Remark 4.3.** Let \( \phi, \psi \) be two continuous functions on \( X \). Define \( \alpha(\mu) = \int \phi d\mu \int \psi d\mu \), which may not satisfy any one of conditions [A.1]–[A.3]. Define

\[
 R_{\phi, \psi}(a) = \{ x \in X : \lim_{n \to +\infty} \frac{1}{n^2} \sum_{i=0}^{n-1} \phi(T^i x) \sum_{i=0}^{n-1} \psi(T^i x) = a \}.
\]

Then \( R_{\phi, \psi}(a) = R_\alpha(a) \) and thus from item (1) of theorem 4.1, we have

\[
 h_{\top}(T, R_{\phi, \psi}(a) \cap \text{Tran}) = h_{\top}(T, R_{\phi, \psi}(a)) = \sup\{ h_\rho : \rho \in M(T, X) \text{ and } \alpha(\rho) = a \}.
\]

However, it is not certain that

\[
 I_{\phi, \psi} = \{ x \in X : \lim_{n \to +\infty} \frac{1}{n^2} \sum_{i=0}^{n-1} \phi(T^i x) \sum_{i=0}^{n-1} \psi(T^i x) \text{ does not converge } \}
\]
has full topological entropy, since it is unknown whether \([A,3]\) holds so that item (2.1) of theorem 4.1 cannot apply. But by item (2.2) of theorem 4.1 it has positive entropy provided that the system has positive entropy.

**Proof of theorem 4.1.**

1. For any real number \(t \geq 0\), define the (maybe empty) set
   \[
   Q(t) := \{x : \exists \mu \in M, \text{ s.t. } h_\mu(T) \leq t\}.
   \]

   From [12, theorem 2]: \(h_{\text{top}}(f, Q(t)) \leq t\). Let \(t = H_\alpha(a)\), notice that \(R_\alpha(a) \subseteq Q(t)\) and thus \(h_{\text{top}}(T, R_\alpha(a) \cap \text{Tran}) \leq h_{\text{top}}(T, R_\alpha(a)) \leq h_{\text{top}}(T, Q(t)) \leq t\).

   On the other hand, for any invariant measure \(\mu\) with \(\alpha(\mu) = a\), note that \(G_\rho \cap \text{Tran}\) is nonempty, since it is unknown whether \([A,3]\).

   Suppose that further \(T\) has positive topological entropy and \(\text{Int}(L_B) \neq \emptyset\), fix \(a \in \text{Int}(L_B)\).

   By the classical variational principle, we can take an ergodic measure \(\mu_1\) such that \(h_{\mu_1} > 0\). If \(\alpha(\mu_1) = a\), then \(H_\alpha(a) > 0\). If \(\alpha(\mu_1) \neq a\), without loss of generality, we may assume that \(\alpha(\mu_1) < a\). Since \(a \in \text{Int}(L_B)\), we can take another invariant measure \(\mu_2\) such that \(\alpha(\mu_2) > a\). Then one can take suitable \(\theta \in (0, 1)\) such that \(\rho := \theta \mu_1 + (1 - \theta)\mu_2\) satisfies that \(\alpha(\rho) = a\). Note that \(h_\rho \geq \theta h_{\mu_1} > 0\), then \(H_\alpha(a) > 0\).

2. Take \(\mu_1, \mu_2\) with \(\alpha(\mu_1) < \alpha(\mu_2)\) and let \(K = \{t \mu_1 + (1 - t)\mu_2 : t \in [0, 1]\}\). Since \(T\) is transitivity-convex-saturated, \(G_\alpha \subseteq \text{Int}\) and thus \(I_\alpha \subseteq \text{Tran}\) is not empty. If \(h_{\text{top}} = 0\), then the result of item (2.1) is trivial. Now we suppose that \(h_{\text{top}} > 0\) and start to show (2.2) and (2.1) one by one.

   Fix \(\epsilon \in (0, h_{\text{top}})\).

   By the classical variational principle, we can take an ergodic measure \(\nu\) such that \(h_\nu > h_{\text{top}} - \epsilon > 0\). By assumption we can take another invariant measure \(\nu_1\) such that \(\alpha(\nu_1) \neq \alpha(\nu)\). Then by continuity of \(\alpha\) there is \(\theta \in (0, 1)\) such that \(\rho := \theta \nu + (1 - \theta)\nu_1\) satisfies that \(\alpha(\rho) = \alpha(\nu)\). Remark that \(h_\rho \geq \theta h_\nu > 0\). Let \(K = \{t \nu + (1 - t)\rho : t \in [0, 1]\}\). Since \(T\) is transitivity-convex-saturated, one can get \(h_{\text{top}}(T, G_\rho) = \min\{h_\rho, \mu_2\} > 0\). Note that \(G_\rho \subseteq I_\alpha \cap \text{Tran}\) is not empty. If \(h_{\text{top}} = 0\), then the result of item (2.1) is trivial. Now we suppose that \(h_{\text{top}} > 0\) and start to show (2.2) and (2.1) one by one.

   If further \(\alpha : (M, T) \rightarrow \mathbb{R}\) satisfies [A,3], then above \(\theta \in (0, 1)\) can be chosen very close to \(1\) such that \(\rho := \theta \nu + (1 - \theta)\nu_1\) satisfies that \(h_\rho \geq \theta h_\nu > h_{\text{top}} - \epsilon\). Then one has \(h_{\text{top}}(T, I_\alpha \cap \text{Tran}) \geq h_{\text{top}}(T, G_\rho) = \min\{h_\rho, \rho_\nu\} > h_{\text{top}} - \epsilon\). This ends the proof of item (2.2).

3. The \(\epsilon \leq \) part is the same as the paragraph in the proof of item (1). For the \(\epsilon > 0\) part, fix an invariant measure \(\mu\) with \(\alpha(\mu) = a\). By the ergodic decomposition theorem, for any \(\epsilon > 0\), there exists an ergodic measure \(\nu\) (as one ergodic component) such that \(\alpha(\nu) = a\) and \(h_\nu > h_\mu - \epsilon\). Note that \(\nu(G_\rho) = 1\) so that \(h_{\text{top}}(G_\rho) \geq h_\nu\) by [12, theorem 3]. Note that \(G_\rho \subseteq R_\alpha(a)\) and thus \(h_{\text{top}}(R_\alpha(a)) \geq h_{\text{top}}(G_\rho) \geq h_\nu > h_\mu - \epsilon\). We now complete the proof of item (3).

4. Note that \(\rho \in R_\alpha\) so that \(\mu(R_\alpha) = 1\) for any invariant measure \(\mu\). By [12, theorem 3] \(h_{\text{top}}(T, \Gamma) \geq h_\mu(T)\) for any \(\Gamma\) with \(\mu(\Gamma) = 1\). Thus, \(h_{\text{top}}(R_\alpha) = \sup_{\mu \in M(T, X)} h_\mu(T) \geq h_{\text{top}}(T) \geq \sup_{\nu \in \text{Int}(L_\alpha)} H_\alpha(a)\). Now we only need to prove \(h_{\text{top}}(T) \leq \sup_{\nu \in \text{Int}(L_\alpha)} H_\alpha(a)\).

   By the classical variational principle, there is an ergodic measure \(\mu\) such that \(h_\mu > h_{\text{top}} - \epsilon\). If \(\alpha(\mu) \in \text{Int}(L_\alpha)\), take \(\omega = \mu\). Otherwise, take an invariant measure \(\nu\) such that
\(\alpha(\nu) \neq \alpha(\mu)\) and \(\alpha(\nu) \in \text{Int}(L_\alpha)\). By condition [A.3], one can choose \(\theta \in (0, 1)\) close to 1 such that \(\omega = \theta \mu + (1 - \theta) \nu\) satisfies \(\alpha(\omega) \in \text{Int}(L_\alpha)\) and \(h_\omega \geq \theta h_\mu > h_{\text{top}} - \epsilon\). Thus, \(\sup_{\nu \in \text{Int}(L_\alpha)} H_\alpha(a) \geq H_\alpha(\alpha(\omega)) > h_{\text{top}} - \epsilon\).

\(\square\)

**Remark 4.4.** By the proof of theorem 4.1, one can see that

(a) the first part of item (1) of theorem 4.1 is still true for each \(a \in \{\alpha(\mu) \mid \mu \in M(T, X)\}\) if the continuity assumption of \(\alpha\) is omitted.

(b) the second part of item (1) of theorem 4.1 is still true for each \(a \in \text{Int}(L_\alpha)\) and item (2) of theorem 4.1 is also true if the continuity assumption of \(\alpha\) is weakened as \(\alpha\) satisfies the intermediate value property, that is, for any \(\mu, \nu \in M(T, X)\), the set of \(\{\alpha(\theta \mu + (1 - \theta) \nu) \mid \theta \in [0, 1]\}\) contains \(\{\theta \alpha(\mu) + (1 - \theta) \alpha(\nu) \mid \theta \in [0, 1]\}\).

5. Basic facts on transitive and Banach recurrence

A point \(x\) is called quasi-generic for some measure \(\mu\), if there are two sequences of positive integers \(\{a_k\}\) and \(\{b_k\}\) with \(b_k > a_k\), \(\lim_{k \to \infty} b_k - a_k = \infty\) such that

\[
\lim_{k \to \infty} \frac{1}{b_k - a_k + 1} \sum_{j=a_k}^{b_k} \delta_{T^j(x)} = \mu
\]

in weak* topology. This concept is from [32, page 65]. Let \(M^*_\alpha = \{\mu \in M(T, X) : x \text{ is quasi-generic for } \mu\}\) and \(C^*_\alpha = \bigcup_{\mu \in M^*_\alpha} S_\mu\). Let \(C_\alpha = \bigcup_{\mu \in M^*_\alpha} S_\mu\). Note that \(M_\alpha \subseteq M^*_\alpha \subseteq M(T, X)\) and \(C_\alpha \subseteq C^*_\alpha \subseteq X\). From [32, proposition 3.9] it is known that \(M^*_\alpha\) is always nonempty, compact and connected. It is not difficult to show that (see [37, 69–72]):

**Lemma 5.1.** For \(\forall x \in X\),

\[
C_x \subseteq C^*_x \subseteq \omega_T(x);
\]

(5.1)

\[
x \in QW \iff x \in C_x \iff x \in \omega(x) = C_x
\]

(5.2)

\[
x \in BR \iff x \in C^*_x \iff x \in \omega_T(x) = C^*_x.
\]

(5.3)

**Lemma 5.2.** For \(x \in X\), if \(C^*_x = X\), then \(x \in BR \cap \text{Tran}\).

**Proof.** By (5.1) \(\omega_T(x) = X\) and \(x \in X = C^*_x\) so that by (5.3), \(x \in BR \cap \text{Tran}\). \(\square\)

**Lemma 5.3.** For any \(x \in X\), \(C_x = \bigcup_{\nu \in M(T, \omega_T(x))} S_\nu = \bigcup_{\nu \in M(T, \omega_T(x))} S_\nu\). Moreover, there is an invariant measure \(\mu \in M(T, \omega_T(x))\) so that \(S_\mu = C_x^*_\alpha\). (We emphasize that it is unknown whether there is an invariant measure \(\mu \in M^*_\alpha\) so that \(S_\mu = C^*_x\)).

**Proof.** It is obvious that \(\bigcup_{\nu \in M(T, \omega_T(x))} S_\nu \subseteq \bigcup_{\nu \in M(T, \omega_T(x))} S_\nu\). By the ergodic decomposition theorem, we know that for any invariant measure \(\mu\),

\[
\mu(\bigcup_{\nu \in M(T, \omega_T(x))} S_\nu) = 1
\]

so that \(S_\mu \subseteq \bigcup_{\nu \in M(T, \omega_T(x))} S_\nu\) and thus \(\bigcup_{\nu \in M(T, \omega_T(x))} S_\nu \subseteq \bigcup_{\nu \in M(T, \omega_T(x))} S_\nu\).
From [32, proposition 3.9, page 65] we know that for a point $x_0$ and an ergodic measure $\mu_0 \in M(\omega_T(x_0), T)$, $x_0$ is quasi-generic for $\mu_0$. This implies that for any $x \in X$, $M_{\text{erg}}(T, \omega_T(x)) \subseteq M^*_x$. So $\bigcup_{\mu \in M_{\text{erg}}(T, \omega_T(x))} S_{\mu} \subseteq C^*_x \subseteq \bigcup_{\mu \in M(T, \omega_T(x))} S_{\mu}$.

Take a sequence of invariant measures $\mu_i \in M(T, \omega_T(x))$ dense in $M(T, \omega_T(x))$ such that $\bigcup_{i \geq 1} S_{\mu_i} = \bigcup_{\mu \in M(T, \omega_T(x))} S_{\mu}$. Let $\mu = \sum_{i \geq 1} \frac{1}{i} \mu_i$. Then $\mu(\bigcup_{i \geq 1} S_{\mu_i}) = 1$ so that $S_{\mu} = \bigcup_{\mu \in M(T, \omega_T(x))} S_{\mu} = C^*_x$. $\square$

**Lemma 5.4.** For any $x \in \text{Rec}$,

$x \in \text{BR} \iff$ there is an invariant measure $\mu \in M(T, \omega_T(x))$ so that $S_{\mu} = \omega_T(x)$.

**Proof.** The part $\Rightarrow$: By (5.3) $x \in \omega_T(x) = C^*_x$. By lemma 5.3, there is an invariant measure $\mu \in M(T, \omega_T(x))$ so that $S_{\mu} = C^*_x$. So $S_{\mu} = C^*_x = \omega_T(x)$.

The part $\Leftarrow$: If there is an invariant measure with $S_{\mu} = \omega_T(x) \ni x$, then by (5.1) $C^*_x \subseteq \omega_T(x)$ and by lemma 5.3, $C^*_x = \bigcup_{\nu \in M(T, \omega_T(x))} S_{\nu} \supseteq S_{\mu} = \omega_T(x)$. By (5.3) $x \in \text{BR}$. $\square$

By lemma 5.4, if $x \in \text{Tran}$, then

$x \in \text{BR} \iff$ there is $\mu \in M(T, X)$ so that $S_{\mu} = X$.

Thus, we have

**Lemma 5.5.** Suppose $(X, T)$ is transitive. Then $\text{Tran} \subseteq \text{BR} \iff \text{Tran} \cap \text{BR} \neq \emptyset \iff$ there is an invariant measure with full support.

Let $\text{BV} := \{x \in \text{BR} | \exists \mu \in M^*_x \text{ s.t. } S_{\mu} = C^*_x\}$.

**Lemma 5.6.** If ergodic measures are dense in the space of invariant measures (or $T$ has entropy-dense property), then for any $x \in \text{Tran}$, $M^*_x = M(T, X)$. If further $T$ has an invariant measure with full support, then $\text{Tran} \subseteq \text{BV}$.

**Proof.** From [32, proposition 3.9, page 65] we know that for a point $x_0$ and an ergodic measure $\mu_0 \in M(\omega_T(x_0), T)$, $x_0$ is quasi-generic for $\mu_0$. This implies that for any $x \in \text{Tran}$, $M_{\text{erg}}(T, X) \subseteq M^*_x$. By assumption of the density of ergodic measures, $M^*_x = M(T, X)$. If further $T$ has an invariant measure $\mu$ with full support, then $\mu \in M^*_x$ and $C^*_x = S_{\mu} = X = \omega_T(x)$ and by (5.3) $x \in \text{BV}$.

For a set $K \subseteq M(T, X)$, define $C_K = \bigcup_{x \in K} S_x$. Recall the notions that $G_K = \{x \in X | M_x = K\}$, $G^K = \{x \in \text{Tran} | M_x = K\}$. Let $V^* = \{x \in \text{Rec} | \exists \mu \in M^*_x \text{ such that } S_{\mu} = C^*_x\}$, $W^* = \{x \in \text{Rec} | S_{\mu} = C^*_x \text{ for every } \mu \in M_x\}$.

**Lemma 5.7.** Let $K$ be a compact connected subset of $M(T, X)$.

1. If for any $\omega \in K$, $S_\omega = C_K$, then $G_K \subseteq W^*$.
2. If there are two measures $\omega_1 \in K$ ($i = 1, 2$), $S_{\omega_1} \subseteq S_{\omega_2} = C_K$, then $G_K \subseteq (V^* \setminus W^*)$.
3. If for any $\omega \in K$, $S_\omega \neq C_K$, then $G_K \subseteq X \setminus V^*$.
4. If $T|_{C_K}$ is not a transitive subsystem, then $G_K \subseteq X \setminus QW$.
5. If $C_K \neq X$, then $G^K \subseteq \text{Tran} \setminus QW$.

**Proof.** The proofs of (1)–(3) are not difficult. We only give the proof of (4) and (5).
For case (4), by contradiction there is \( x \in G_K \cap QW \), then \( C_x = C_K \) and by (5.2) \( x \in \omega_T(x) = C_x \) so that \( x \in \omega_T(x) = C_x \). It means that \( T|_{C_x} \) is transitive, and it contradicts the assumption.

For case (5), by contradiction there is \( x \in G_K^T \cap QW \), then \( x \in G_K^T \) implies that \( C_x = C_K \neq X \), \( \omega_T(x) = X \) so that \( C_x \neq \omega_T(x) \) and by (5.2) \( x \in QW \) implies that \( x \in \omega_T(x) = C_x \). It is a contradiction.

\[ \Box \]

6. Combination of multifractal analysis, transitive and Banach recurrence

As a refined version of theorem 1.2, we consider the topological entropy on refined Banach recurrent gap-sets which will be used to prove theorem 1.3. Given \( x \in X \), let \( C_x = \bigcup_{\mu \in M_x} S_{\mu} \).

Let \( BR^# := BR \setminus QW \).

\[
W^# := \{ x \in BR^# \mid S_\mu = C_x \text{ for every } \mu \in M_x \},
\]

\[
V^# := \{ x \in BR^# \mid \exists \mu \in M_x \text{ such that } S_\mu = C_x \},
\]

\[
S^# := \{ x \in X \} \cap \{ x \in M_x, S_\mu \neq \emptyset \}.
\]

More precisely, in the present paper we mainly consider \( BR^# \), which is divided into the following several levels with different asymptotic behaviour:

\[
BR_1 := W^#, \\
BR_2 := V^# \cap S^#, \\
BR_3 := V^#, \\
BR_4 := V^# \cup (BR^# \cap S^#), \\
BR_5 := BR^#.
\]

Note that \( BR_1 \subseteq BR_2 \subseteq BR_3 \subseteq BR_4 \subseteq BR_5 \).

**Definition 6.1.** For a collection of subsets \( Z_1, Z_2, \ldots, Z_k \subseteq X (k \geq 2) \), we say \( \{Z_i\} \) has **full entropy gaps** with respect to \( Y \subseteq X \) (simply, FEG w.r.t. \( Y \)) if

\[
h_{\text{hop}}(T, (Z_{i+1} \setminus Z_i) \cap Y) = h_{\text{hop}}(T, Y) \text{ for all } 1 \leq i < k,
\]

where \( h_{\text{hop}}(T, Z) \) denotes the topological entropy of a set \( Z \subseteq X \).

Often, but not always, the sets \( Z_i \) are nested (\( Z_i \subseteq Z_{i+1} \)). Note that for any system with zero topological entropy, it is obvious that any collection \( \{Z_i\} \) has full entropy gaps with respect to any \( Y \subseteq X \).

**Theorem 6.1.** Suppose that \((X, T)\) has transitivity-convex-saturated property and entropy-dense property, and there is an invariant measure with full support. Let \( \alpha : M(T, X) \rightarrow \mathbb{R} \) be a continuous function.

(1) If \( \alpha \) satisfies [A.3] and \( \text{Int}(L_\alpha) \neq \emptyset \), then \( \{QR, BR_1, BR_2, BR_3, BR_4, BR_5\} \) has full entropy gaps w.r.t. \( I_\alpha \cap \text{Tran} \).

(2) If \( \alpha \) satisfies [A.1], [A.2] and \( \text{Int}(L_\alpha) \neq \emptyset \), then for any \( a \in \text{Int}(L_\alpha) \), \( \{0, QR \cap BR_1, BR_1, BR_2, BR_3, BR_4, BR_5\} \) has full entropy gaps w.r.t. \( R_\alpha(a) \cap \text{Tran} \).

(3) If \((X, T)\) is not uniquely ergodic, then \( \{0, QR \cap BR_1, BR_1, BR_2, BR_3, BR_4, BR_5\} \) has full entropy gaps w.r.t. \( \text{Tran} \).

(4) If \((X, T)\) is not uniquely ergodic and \( \alpha \) satisfies [A.1], [A.2], then \( \{0, QR \cap BR_1, BR_1, BR_2, BR_3, BR_4, BR_5\} \) has full entropy gaps w.r.t. \( R_\alpha \cap \text{Tran} \).
Remark 6.1. If \( a \in L_\alpha \setminus \text{Int}(L_\alpha) \) and \( H_\alpha(a) > 0 \), then item (2) of theorem 6.1 may not be true, though by the first part of item (1) of theorem 4.1, \( h_{\text{top}}(R_\alpha(a) \cap \text{Tran}) = H_\alpha(a) > 0 \). Sometimes it may appear that \( h_{\text{top}}(R_\alpha(a) \cap \text{Tran} \cap BR_{i+1} \setminus BR_i) = 0 \) (even \( R_\alpha(a) \cap \text{Tran} \cap BR_{i+1} \setminus BR_i \) is empty). For example, letting \( T \) be a full shift over finite symbols and \( \Lambda_1, \Lambda_2 \) be two different compact invariant subsets, define a continuous function \( \phi \) such that

\[
\phi|_{\Lambda_1} = 0, \phi|_{\Lambda_2} = 1
\]

and \( 0 < \phi(x) < 1 \) for other points of \( x \). By [34] one can choose \( \Lambda_i \) to be two minimal subsets on which just a unique invariant measure \( \mu_i \) with positive metric entropy is supported, respectively. For \( \alpha(\mu) := \int \phi d\mu \) and \( a = 0 \) or 1, it is easy to check that \( R_\alpha(a) = G_{\mu_{i+1}}^+ \) and \( G_{\mu_i}^+ \subseteq BR_1 \) so that \( R_\alpha(a) \cap \text{Tran} = R_\alpha(a) \cap \text{Tran} \cap BR_1 \). Thus, \( R_\alpha(a) \cap \text{Tran} \cap BR_{i+1} \setminus BR_i = \emptyset, i = 1, 2, 3, 4 \).

Remark 6.2. For a general continuous \( \alpha \) without assuming condition [A,3], by slight modification in the proof one can get that \( \{QR, BR_1, BR_2, BR_3, BR_4, BR_5\} \) has positive entropy gaps w.r.t. \( I_\alpha \cap \text{Tran} \), that is \( h_{\text{top}}((BR_1 \setminus QR) \cap I_\alpha \cap \text{Tran}) > 0 \) and \( h_{\text{top}}((BR_{i+1} \setminus BR_i) \cap I_\alpha \cap \text{Tran}) > 0, i = 1, 2, 3, 4 \). For example, if \( \alpha \) is the function as in remark 4.3 and assume that

\[
\inf_{\mu \in M(T,X)} \int \phi d\mu \int \psi d\mu < \sup_{\mu \in M(T,X)} \int \phi d\mu \int \psi d\mu,
\]

then in this case

\[
I_\alpha = I_{\phi,\psi} = \{ x \in X : \lim_{n \to +\infty} \frac{1}{n!} \sum_{i=0}^{n-1} \phi(T^i x) \sum_{i=0}^{n-1} \psi(T^i x) \text{ does not converge} \}.
\]

But it is unknown whether \( \{QR, BR_1, BR_2, BR_3, BR_4, BR_5\} \) has full entropy gaps w.r.t. \( R_\alpha, R_\alpha(a) \).

Remark 6.3. We say \( \{Z_i\} \) has locally full entropy gaps with respect to \( Y \subseteq X \) (simply, LFEG w.r.t. \( Y \)) if for any nonempty open set \( U \subseteq X \),

\[
h_{\text{top}}(T, (Z_{i+1} \setminus Z_i) \cap Y \cap U) = h_{\text{top}}(T, Y) \quad \text{for all } 1 \leq i < k.
\]

Since the results in this paper are all restricted on transitive points so that by lemma 3.12 all results on full entropy gaps can be stated as locally full entropy gaps.

Before proving theorem 6.6.1, we need to show some technique results.

Lemma 6.1. Suppose that \( T \) has entropy-dense property. Let \( \alpha : M(T,X) \to \mathbb{R} \) be a continuous function satisfying [A,3] and \( \text{Int}(L_\alpha) \neq \emptyset \). Then for any \( \epsilon > 0 \), any integer \( k \geq 2 \), there exist ergodic measures \( \mu_1, \mu_2, \ldots, \mu_k \), such that

1. \( h_{\mu_i} > h_{\text{top}} - \epsilon, i = 1, 2, \ldots, k \),
2. \( S_{\mu_i} \cap S_{\mu_j} = \emptyset, 1 \leq i < j \leq k \),
3. \( \alpha(\mu_1) < \alpha(\mu_2) < \cdots < \alpha(\mu_k) \).

Proof. Fix \( \epsilon > 0 \). By the classical variational principle, we can take an ergodic measure \( \nu \) such that \( h_{\nu} > h_{\text{top}} - \frac{\epsilon}{k^2} \). By assumption we can choose another invari-
ant measure $\nu_1$ such that $\alpha(\nu_1) \neq \alpha(\nu)$. Then there is $\theta_0 \in (0, 1)$ very close to 1 such that $h_{\theta_0 \nu + (1-\theta_0)\nu_1} \geq \theta_0 h_\nu \geq h_{\text{top}} - \frac{1}{2} \epsilon$ hold for any $\theta \in (\theta_0, 1)$. By condition [A.3] and continuity of $\alpha$ there are $\theta_0 < \theta_1 < \theta_2 < \cdots < \theta_k < 1$ such that $\tau_i := \theta_i \nu + (1 - \theta_i)\nu_1$ satisfies that $h_{\tau_i} \geq \theta_i h_\nu \geq h_{\text{top}} - \frac{1}{2} \epsilon, i = 1, 2, \cdots, k$ and $\alpha(\tau_1) < \alpha(\tau_2) < \cdots < \alpha(\tau_k)$ or $\alpha(\tau_1) > \alpha(\tau_2) > \cdots > \alpha(\tau_k)$. We may assume that $\alpha(\tau_1) < \alpha(\tau_2) < \cdots < \alpha(\tau_k)$ (Otherwise, by reverse order to change).

By continuity of $\alpha$, take neighbourhood $G_i$ of $\tau_i$ in $M(T, X) (i = 1, 2, \cdots, k)$ such that $G_i \cap G_j = \emptyset, \sup_{\tau \in G_i} \alpha(\tau) < \inf_{\tau \in G_j} \alpha(\tau), 1 \leq i < j \leq k$. By entropy-dense property, there exist $\rho_i \in M_{\text{erg}}(T, X) (i = 1, 2, \cdots, k)$ such that $h_{\rho_i} > h_{\tau_i} - \frac{1}{2} \epsilon$ and $M(T, S_{\rho_i}) \subseteq G_i$. Then these ergodic measures $\rho_1, \rho_2, \cdots, \rho_k$, are required.

**Lemma 6.2.** Suppose that $T$ has entropy-dense property. Let $\alpha : M(T, X) \to \mathbb{R}$ be a continuous function satisfying [A.1] and $\text{Int}(L_\alpha) \neq \emptyset$. Then for any $\epsilon > 0$, any $a \in \text{Int}(L_\alpha)$, any integers $k \geq 2, l \geq 2$ there exist ergodic measures $\rho_1, \rho_2, \cdots, \rho_{k+l}$, such that

1. $h_{\rho_i} > H_\alpha(a) - \epsilon, i = 1, 2, \cdots, k + l$,
2. $S_{\rho_i} \cap S_{\rho_j} = \emptyset, 1 \leq i < j \leq k + l$,
3. $\alpha(\rho_i) < \alpha(\rho_2) < \cdots < \alpha(\rho_k) < a < \alpha(\rho_{k+1}) < \cdots < \alpha(\rho_{k+l})$.

**Proof.** Fix $\epsilon > 0$ and $a \in \text{Int}(L_\alpha)$. Take an ergodic measure $\nu$ with $\alpha(\nu) = a$ such that $h_\nu > H_\alpha(a) - \frac{1}{2} \epsilon$. By assumption we can choose another two invariant measures $\nu_1, \nu_2$ such that $\alpha(\nu_1) < a = \alpha(\nu) < \alpha(\nu_2)$. Then there is $\theta_0 \in (0, 1)$ very close to 1 such that $h_{\theta_0 \nu + (1-\theta_0)\nu_1} \geq \theta_0 h_\nu \geq h_{\text{top}} - \frac{1}{2} \epsilon$ hold for $i = 1, 2$ and any $\theta \in (\theta_0, 1)$. By condition [A.1] and continuity of $\alpha$ there are $\theta_0 < \theta_1 < \theta_2 < \cdots < \theta_k < 1$ and $\theta_0 < \theta_{k+1} < \theta_{k+2} < \cdots < \theta_{k+l} < 1$ such that $\tau_i := \theta_i \nu + (1 - \theta_i)\nu_1$ satisfies that $h_{\tau_i} \geq \theta_i h_\nu > H_\alpha(a) - \frac{1}{2} \epsilon, i = 1, 2, \cdots, k + l$ and $\alpha(\tau_1) < \alpha(\tau_2) < \cdots < \alpha(\tau_k) < a < \alpha(\tau_{k+1}) < \cdots < \alpha(\tau_{k+l})$.

By continuity of $\alpha$, take neighbourhood $G_i$ of $\tau_i$ in $M(T, X) (i = 1, 2, \cdots, k + l)$ such that $G_i \cap G_j = \emptyset, \sup_{\tau \in G_i} \alpha(\tau) < \inf_{\tau \in G_j} \alpha(\tau), 1 \leq i < j \leq k + l$. By the entropy-dense property, there exist $\rho_i \in M_{\text{erg}}(T, X) (i = 1, 2, \cdots, k + l)$ such that $h_{\rho_i} > h_{\tau_i} - \frac{1}{2} \epsilon$ and $M(T, S_{\rho_i}) \subseteq G_i$. Then these ergodic measures $\rho_1, \rho_2, \cdots, \rho_{k+l}$, are required.

**Lemma 6.3.** Suppose that $T$ has entropy-dense property. Let $\alpha : M(T, X) \to \mathbb{R}$ be a continuous function satisfying [A.1] and $\text{Int}(L_\alpha) \neq \emptyset$. Then for any $\epsilon > 0$, any $a \in \text{Int}(L_\alpha)$, any integers $k \geq 2$, there exist invariant measures $\mu_1, \mu_2, \cdots, \mu_k$, such that

1. $h_{\mu_i} > H_\alpha(a) - \epsilon, i = 1, 2, \cdots, k$,
2. $S_{\mu_i} \cap S_{\mu_j} = \emptyset, 1 \leq i < j \leq k$,
3. $\alpha(\mu_i) = a, i = 1, 2, \cdots, k$.

**Proof.** Fix $\epsilon > 0$ and $a \in \text{Int}(L_\alpha)$. Take ergodic measures $\rho_1, \rho_2, \cdots, \rho_{2k}$ (in the case $l = k$) satisfying lemma 6.2. By continuity of $\alpha$, take suitable $\theta_i \in (0, 1)$ such that $\mu_i = \theta_i \rho_i + (1 - \theta_i)\rho_{2k+1-i}$ satisfies item (3), $i = 1, 2, \cdots, k$. Then it is easy to check that such $\mu_1, \mu_2, \cdots, \mu_k$ are required.

Now we are ready to prove theorem 6.1.
Proof of theorem 6.1. Let $\text{Tran}^\#: = \{ x \in \text{Tran} \mid C_x \subseteq C_x^* \},$

$W_T^\#: = \{ x \in \text{Tran}^\# \mid S_\mu = C_x \text{ for every } \mu \in M_3 \},$

$V_T^\#: = \{ x \in \text{Tran}^\# \mid \exists \mu \in M_3 \text{ such that } S_\mu = C_x \},$

$S := \{ x \in X \mid \cap_{\mu \in M_3} S_\mu \neq \emptyset \};$

$\text{Tran}_1 := W_T^\#, \text{Tran}_2 := V_T^\# \cap S, \text{Tran}_3 := V_T^\#$,

$\text{Tran}_4 := V_T^\# \cup (\text{Tran}^\# \cap S), \text{Tran}_5 := \text{Tran}^\#.$

Since by lemma 5.5, $\text{Tran} \subseteq \text{BR}$ so that $\text{Tran}_1 \subseteq \text{BR}_c.$ Thus we only need to show that

$\{\emptyset, \text{Tran}_1, \text{Tran}_2, \text{Tran}_3, \text{Tran}_4, \text{Tran}_5 \}$

has full entropy gaps w.r.t. $I_\alpha$ and $R_\alpha(a),$ respectively. From definition 6.1, note that we only need to prove the $\{\emptyset, \text{Tran}_1, \text{Tran}_2, \text{Tran}_3, \text{Tran}_4, \text{Tran}_5 \}$ part.

(1) Fix $\epsilon > 0.$ Take $\rho_1, \rho_2, \rho_3, \rho_4$ satisfying lemma 6.1. By condition [A.3] of $\alpha,$ one can take suitable $\theta_i \in (0, 1)$ ($i = 1, 2$) such that $\nu_i = \theta_i \rho_1 + (1 - \theta_i) \rho_2$ satisfies that $\alpha(\nu_i) \neq \alpha(\rho_2).$ Note that $S_{\nu_i} = S_{\rho_1} \cup S_{\rho_2}$ and $h_{\nu_i} = \theta_i h_{\rho_1} + (1 - \theta_i) h_{\rho_2} > h_{\top} - \epsilon.$ Take suitable $\theta \in (0, 1)$ such that $\omega = \theta \rho_1 + (1 - \theta) \rho_2$ satisfies that $\alpha(\omega) \neq \alpha(\rho_1).$ Note that $S_{\omega} = S_{\rho_1} \cup S_{\rho_2}$ and $h_{\omega} = \omega h_{\rho_1} + (1 - \omega) h_{\rho_2} > h_{\top} - \epsilon.$ Let

$K_1 := \{ t\nu_1 + (1 - t)\nu_2 \mid t \in [0, 1] \}$,

$K_2 := \{ t\rho_1 + (1 - t)\omega \mid t \in [0, 1] \}$,

$K_3 := \{ t\rho_2 + (1 - t)\omega \mid t \in [0, 1] \}$,

$K_4 := \{ t\rho_3 + (1 - t)\nu_1 \mid t \in [0, 1] \}$,

$K_5 := \{ t\rho_4 + (1 - t)\nu_2 \mid t \in [0, 1] \}$.

By transitivity-convex-saturated property,

$h_{\top}(G^T_K) = \inf_{\mu \in K_i} h_{\mu} \geq \min\{h_{\rho_1}, h_{\rho_2}, h_{\rho_3}, h_{\nu_1}, h_{\nu_2}, h_{\omega} \} > h_{\top} - \epsilon.$

Note that $K_i \subseteq I_\alpha$ and $C_{K_i} \subseteq \cup_{i=1}^3 S_\mu \subseteq \cup_{i=1}^4 S_\mu \subseteq X.$ By lemma 5.7 $G_K^T \subseteq \text{Tran} \setminus \text{QW}$ and $G_K^T \subseteq \text{Tran}_{i-1} \setminus \text{Tran}_i,$ where $\text{Tran}_i = \emptyset, i = 1, 2, \cdots, 5.$ Then we complete the proof of item (1).

(2) Fix $\epsilon > 0.$ Take $\mu_1, \mu_2, \mu_3, \mu_4$ satisfying lemma 6.3. By condition [A.2] of $\alpha,$ one can take different $\delta_i \in (0, 1)$ ($i = 1, 2$) such that $\nu_1 = \theta\mu_1 + (1 - \theta)\mu_2$ satisfies that $\nu_1 \neq \nu_2,$ $\alpha(\nu_1) = \alpha(\nu_2) = \alpha.$ Note that $S_{\nu_1} = S_{\mu_1} \cup S_{\mu_2}$ and $h_{\nu_1} = \theta h_{\mu_1} + (1 - \theta) h_{\mu_2} > h_{\alpha}(a) - \epsilon.$ Take $\theta \in (0, 1)$ such that $\omega = \theta \mu_1 + (1 - \theta) \mu_3$ satisfies that $\alpha(\omega) = \alpha(\mu_1) = \alpha$ by condition [A.2]. Note that $S_{\omega} = S_{\mu_1} \cup S_{\mu_3}$ and $h_{\omega} = \omega h_{\mu_1} + (1 - \omega) h_{\mu_3} > h_{\alpha}(a) - \epsilon.$ Let

$K_0 := \{ \mu_1 \}$,

$K_1 := \{ t\nu_1 + (1 - t)\nu_2 \mid t \in [0, 1] \}$,

$K_2 := \{ t\mu_1 + (1 - t)\omega \mid t \in [0, 1] \}$,

$K_3 := \{ t\mu_3 + (1 - t)\omega \mid t \in [0, 1] \}$,

$K_4 := \{ t\mu_1 + (1 - t)\nu_1 \mid t \in [0, 1] \}$,

$K_5 := \{ t\mu_3 + (1 - t)\nu_1 \mid t \in [0, 1] \} \cup \{ t\mu_1 + (1 - t)\omega \mid t \in [0, 1] \}.$
By the transitivity-convex-saturated property,
\[ h_{\text{top}}(G_K^i) = \inf_{\mu \in K_i} h_{\mu} \geq \min\{h_{\mu_1}, h_{\mu_2}, h_{\nu_1}, h_{\nu_2}, h_{\omega_1}\} > H_\alpha(a) - \epsilon. \]

Note that \( K_i \subseteq R_\alpha(a) \) and \( C_K \subseteq \bigcup_{i=1}^5 S_\mu \subseteq \bigcup_{i=1}^4 S_\mu \subseteq X \). By lemma 5.7, \( G_K^i \subseteq \text{Tran} \setminus QW \), \( i = 0, 1, \ldots, 5 \) and \( G_K^5 \subseteq QW \setminus \text{Tran}. \) Then we complete the proof of item (2).

(3) Since the system is not uniquely ergodic, then there exist two different invariant measures \( \mu, \nu \). Then by weak" topology, there is a continuous function \( \phi : X \rightarrow \mathbb{R} \) such that \( \int \phi d\mu \neq \int \phi d\nu \). Define \( \alpha : M(T, X) \rightarrow \mathbb{R}, \tau \mapsto \int \phi d\tau \). By the convex-saturated property, there is \( x \in X \) such that \( M_x = \{t\mu + (1-t)\nu| t \in [0, 1]\} \). Then \( x \in L_\alpha \) so that one can use item (1) of theorem 6.1 to end the proof of item (3).

(4) If \( R_\alpha = X \), then one can use item (3) to complete the proof. Otherwise \( L_\alpha \neq \emptyset \) so that \( \text{Int}(L_\alpha) \neq \emptyset \). For convenience, we denote the sets of \( 0, QW \cap Br, Br_1, Br_2, Br_3, Br_4, Br_5 \) by \( Z_1, Z_2, \ldots, Z_7 \), respectively. By the second part in item (4) of theorem 4.1 and item (2) of theorem 6.1, for any \( i = 1, 2, \ldots, 6, \)
\[
\inf_{a \in \text{Int}(L_\alpha)} h_{\text{top}}(R_\alpha \cap Z_{i+1} \setminus Z_i) = \sup_{a \in \text{Int}(L_\alpha)} h_{\text{top}}(R_\alpha(a) \cap Z_{i+1} \setminus Z_i)
\]

We complete the proof of item (4).

\[ \square \]

Remark 6.4. By lemmas 3.4 and 5.6, \( \text{Tran} \subseteq BV \) so that theorem 6.1 can be stated w.r.t. \( BV \).

7. Proof of theorems 1.2 and 1.3

7.1. Proof of theorem 1.2

Note that \( BR \cap \text{Tran} \subseteq (BR \setminus QW) \cap \text{Tran} \subseteq \text{Tran} \). Thus, one can use item (3) of theorems 1.4 and 6.1 to end the proof of theorem 1.2.

\[ \square \]

7.2. Statistical perspective of recurrence: proof of theorem 1.3

To prove theorem 1.3, we also need to recall the results on \( QW \). Let
\[
W := \{ x \in QW | S_\mu = C_\tau \text{ for every } \mu \in M_\tau \}, \\
V := \{ x \in QW | \exists \mu \in M_\tau \text{ such that } S_\mu = C_\tau \}, \\
S := \{ x \in X | \cap_{\mu \in M_\tau} S_\mu \neq \emptyset \}; \\
QW_1 := W, \quad QW_2 := V \cap S, \quad QW_3 := V, \\
QW_4 := V \cup (QW \cap S), \quad QW_5 := QW.
\]

Lemma 7.1. For any \( K \subseteq M(T, X) \), if \( C_K = X \), then \( G_K = G_K^T \) and
\[
h_{\text{top}}(T, G_K) = \inf_{\mu \in K} \{h_{\mu}(T) | x \in K \} \iff h_{\text{top}}(T, G_K^T) = \inf_{\mu \in K} \{h_{\mu}(T) | x \in K \}.
\]

Proof. We only need to show \( G_K \subseteq G_K^T \). Fix \( x \in G_K \). Then \( C_x = C_K ) X \). By (5.1) \( \omega_f(x) = X \) so that \( x \in \text{Tran} \cap G_K = G_K^T \).

\[ \square \]
Theorem 7.1. Suppose that \((X, T)\) has saturated property and entropy-dense property. If \((X, T)\) is not uniquely ergodic and there is an invariant measure with full support, then
\[
\{\emptyset, QR \cap QW_1, QW_1, QW_2, QW_3, QW_4, QW_5\}
\] has full entropy gaps w.r.t. Tran. Similar arguments hold w.r.t. \(\text{Tran} \cap I_\alpha, \text{Tran} \cap R_\alpha, \text{Tran} \cap R_\alpha(a)\).

Proof. All constructed \(K\) in [62] satisfy \(C_k = X\) so that by lemma 7.1 transitively-saturated property is not necessary for this result and saturated property is enough.

For \(\{\emptyset, QR \cap QW_1, QW_1, QW_2, QW_3\}\), one can follow the constructed \(K\) in [62] that consists of a convex sum of finite measures. On the other hand, for \(QW_3 \setminus QW_4\) and \(QW_4 \setminus QW_3\), the constructed \(K\) in [62] is not contained in the convex sum of finite measures so that convex-saturated is not enough. Thus here we assume the system to be saturated. □

Remark 7.1. The assumptions of uniform separation in theorem 1.1 can be omitted, since by theorem 1.4 the \(g\)-almost product property is enough to get transitively-convex-saturated and from the proof of theorem 7.1 one can see that convex-saturated is enough, since \(\text{Tran} \cap QW_1 \subseteq W \setminus \text{AP}\) and \(QW_2 \setminus QW_1 \subseteq QW \setminus W\).

Remark 7.2. The results in theorems 6.1 and 7.1 are all restricted on transitive points. By their assumption the system is not minimal so that the almost periodic case is not contained in these results. The almost periodic case will be discussed in [27]. On the other hand, here \(BR \setminus \text{Tran}\) is not considered, which will be studied in [26].

Proof of theorem 1.3. For any \(x \in X\), from [26] we know \(\omega_d(x) = \bigcap_{\mu \in M^*} S_{\mu}^*, \omega_f(x) = \bigcap_{\mu \in M^*} S_{\mu}^*, \omega_B(x) = \bigcap_{\mu \in M^*} S_{\mu}^*\). Thus,
\[
x \in BR \iff x \in \omega_d(x) \text{ and } x \in QW \iff x \in \omega_f(x).
\]
The construction of \(x\) in the proof of the results for \(BR_i\) of the present paper and \(QW_i\) in [62] always satisfies that \(x \in \text{Tran} \cap BR, \omega_{BR_i}(x) = C^*_x = X = \omega_f(x)\) and \(M^*(x) = M(T, X)\). Since the dynamical systems of the main theorems are not minimal, but minimal points are dense in the whole space, so that for any \(x \in \text{Tran}\), \(\omega_B(x) = \bigcap_{\mu \in M^*} S_{\mu}^* = \bigcap_{\mu \in M(T, X)} S_{\mu}^* = X\). Thus, one can check that

(a) \(\text{Tran} \cap BR_1\) belongs to Case (2),
(b) \(\text{Tran} \cap BR_2 \setminus BR_1\) and \(\text{Tran} \cap BR_4 \setminus BR_3\) belong to Case (6),
(c) \(\text{Tran} \cap BR_3 \setminus BR_2\) and \(\text{Tran} \cap BR_4 \setminus BR_4\) belong to Case (5),
(d) \(\text{Tran} \cap QW_1\) belongs to Case (1),
(e) \(\text{Tran} \cap QW_2 \setminus QW_1\) belong to Case (4),
(f) \(\text{Tran} \cap QW_3 \setminus QW_2\) belong to Case (3). So the results of theorem 1.3 can be deduced from theorems 6.1 and 7.1. Only one point is worth mentioning: that the convex-saturated property is enough to show that the gap-sets in (d)–(f) have full topological entropy, as stated in the proof of theorem 7.1. □

Remark 7.3. Suppose that \((X, T)\) has \(g\)-almost product property. If \((X, T)\) is not uniquely ergodic and there is an invariant measure with full support, then
\[
\{x \in \text{Rec} | x \text{ satisfies Case } i\}
\]
has full topological entropy w.r.t. \( \text{Tran} \), \( i = 1, 2, \cdots, 6 \). Similar arguments hold w.r.t. \( \text{Tran} \cap I_\alpha \), \( \text{Tran} \cap R_\alpha \), \( \text{Tran} \cap R_\alpha (a) \). Here it is not necessary to assume uniform separation, since only the constructed \( K \) for the gap-sets \( QW_4 \setminus QW_3 \) and \( QW_3 \setminus QW_4 \) used saturated property, but transitively-convex-saturated is enough for the others.

**Remark 7.4.** The case of \( \omega_{B_{\ast}} (x) \neq \emptyset \) can also be classified into many cases (including the almost periodic case) restricted on non-recurrent points and recurrent points, which will be studied in [27], respectively.

**Remark 7.5.** For any \( x \in X \), if \( \omega_{B_\ast} (x) = \emptyset \) and \( \omega_{B_{\ast}} (x) \neq \omega_T (x) \), then from [26] we know that \( x \) satisfies only one of the following six cases:

\[
\begin{align*}
\text{Case (1')} & : \emptyset = \omega_{B_{\ast}} (x) \subseteq \omega_T (x) = \omega_{B_{\ast}} (x) \subseteq \omega_T (x); \\
\text{Case (2')} & : \emptyset = \omega_{B_{\ast}} (x) \subseteq \omega_T (x) \subseteq \omega_{B_{\ast}} (x) \subseteq \omega_T (x); \\
\text{Case (3')} & : \emptyset = \omega_{B_{\ast}} (x) = \omega_T (x) \subseteq \omega_{B_{\ast}} (x) \subseteq \omega_T (x); \\
\text{Case (4')} & : \emptyset = \omega_{B_{\ast}} (x) \subseteq \omega_T (x) \subseteq \omega_{B_{\ast}} (x) = \omega_T (x); \\
\text{Case (5')} & : \emptyset = \omega_{B_{\ast}} (x) = \omega_T (x) \subseteq \omega_{B_{\ast}} (x) \subseteq \omega_T (x); \\
\text{Case (6')} & : \emptyset = \omega_{B_{\ast}} (x) \subseteq \omega_T (x) \subseteq \omega_{B_{\ast}} (x) \subseteq \omega_T (x).
\end{align*}
\]

Cases (1)–(6) and Cases (1’)–(6’) restricted on non-recurrent points are considered in [26]. Cases (1)–(6) restricted on the set \( B_R \setminus \text{Tran} \) are also considered in [26].

**8. Application to Lyapunov exponents, examples, comments and further questions**

**8.1. Combination of Banach recurrence and Lyapunov exponents**

Now let us recall the concept of ‘asymptotically additive’ introduced in [28], which helps us to study the multifractal behaviour of Lyapunov exponents.

**Definition 8.1.** A sequence of functions \( \phi_n : X \to \mathbb{R} \) is said to be asymptotically additive if for each \( \epsilon > 0 \), there exists a continuous function \( \phi : X \to \mathbb{R} \) such that

\[
\limsup_{n \to \infty} \frac{1}{n} \sup_{x \in X} |\phi_n(x) - S_n \phi(x)| \leq \epsilon,
\]

where \( S_n \phi = \sum_{k=0}^{n-1} \phi \circ T^k \).

Given an asymptotically additive sequence of functions \( \Phi := \{ \phi_n : X \to \mathbb{R} \}_{n \geq 1} \), the limit \( \lim_{n \to \infty} \frac{1}{n} \phi_n (x) \) (if it exists) is called the Lyapunov exponent of \( \Phi \) at \( x \), see [28].

Given two asymptotically additive sequences of functions \( \Phi := \{ \phi_n : X \to \mathbb{R} \}_{n \geq 1} \) and \( \Psi := \{ \psi_n : X \to \mathbb{R} \}_{n \geq 1} \) where \( \psi_n \) are all positive functions, define \( (\Phi, \Psi) \)-irregular set, \( (\Phi, \Psi) \)-regular set and \( (\Phi, \Psi) \)-level set, respectively, as follows:

\[
I^\Phi_\Psi := \{ x \in X : \lim_{n \to +\infty} \frac{\phi_n(x)}{\psi_n(x)} \text{ does not converge } \}, \quad R^\Phi_\Psi := X \setminus I^\Phi_\Psi \quad \text{and} \quad R^\Phi_\Psi (a) := \{ x \in X : \lim_{n \to +\infty} \frac{\phi_n(x)}{\psi_n(x)} = a \}.
\]
There are lots of classical results for $R^n_F(a)$ where $\psi_n \equiv n$ for any $n$, for example, see [5, 28] (see [43, 50, 57] for additive functions and a review article [19] and references therein). In the present paper we are going to study the topological entropy of more general level sets $R^n_F(a)$ and moreover we combine Banach recurrence for study simultaneously. Thus, the results of the present paper can be viewed as refined generalizations of [5, 28, 43, 50, 57]. The concept of asymptotically sub-additive potentials of [28] is mainly motivated by some works on the Lyapunov exponents of matrix products [29–31] and the Lyapunov exponents of differential maps on nonconformal repellers [7] so that the results of the present paper are suitable for the cases of [7, 29–31].

Now we start to state the results of studying the dynamical complexity on Banach recurrence, irregular set and level sets of Lyapunov exponents together.

**Theorem 8.1.** Suppose that $(X, T)$ has transitivity-convex-saturated property and entropy-dense property, and there is an invariant measure with full support. Let $\Phi := \{\phi_n : X \to \mathbb{R}\}_{n \geq 1}$ and $\Psi := \{\psi_n : X \to \mathbb{R}\}_{n \geq 1}$ be two asymptotically additive sequences of functions where $\psi_n$ are all positive functions. Then

1. either $I_n^\Phi = \emptyset$ or $I_n \cap (\text{Tran} \setminus QW) \neq \emptyset$, $I_n^\Psi \cap (\text{BR} \setminus QW) \neq \emptyset$ and they all have full topological entropy, that is,
   \[
   h_{\text{top}}(I_n^\Phi \cap (\text{Tran} \setminus QW)) = h_{\text{top}}(I_n^\Psi \cap (\text{BR} \setminus QW)) = h_{\text{top}}(I_n^\Phi) = h_{\text{top}}(T).
   \]

2. for any $a \in \mathbb{R}$, either $R_n^\Phi(a) = \emptyset$ or $R_n^\Phi(a) \cap (\text{Tran} \setminus QW) \neq \emptyset$, $R_n^\Psi(a) \cap (\text{BR} \setminus QW) \neq \emptyset$ and they all have same topological entropy as the set $R_n^\Phi(a)$ and there is a variational principle, that is,
   \[
   h_{\text{top}}(R_n^\Phi(a) \cap (\text{Tran} \setminus QW)) = h_{\text{top}}(R_n^\Psi(a) \cap (\text{BR} \setminus QW)) = h_{\text{top}}(R_n^\Phi(a))
   \]
   \[
   = \sup\{h_\mu(T) \mid \lim_{n \to +\infty} \frac{1}{n} \int \phi_n(x) d\mu = a, \mu \in M(T, X)\}.
   \]

3. if $(X, T)$ is not uniquely ergodic, then $R_n^\Phi \cap (\text{Tran} \setminus QW) \neq \emptyset$, $R_n^\Psi \cap (\text{BR} \setminus QW) \neq \emptyset$ and they all have full topological entropy, that is,
   \[
   h_{\text{top}}(R_n^\Phi \cap (\text{Tran} \setminus QW)) = h_{\text{top}}(R_n^\Psi \cap (\text{BR} \setminus QW)) = h_{\text{top}}(R_n^\Phi) = h_{\text{top}}(T).
   \]

**Proof of theorem 8.1.** Given an asymptotically additive sequence of functions $\Phi := \{\phi_n : X \to \mathbb{R}\}_{n \geq 1}$, the limit
   \[
   \lim_{n \to \infty} \frac{1}{n} \phi_n(x)
   \]
   (if exists) is called the Lyapunov exponent of $\Phi$ at $x$, see [28]. Define
   \[
   \beta(\mu) := \liminf_{n \to \infty} \int \frac{1}{n} \phi_n(x) d\mu, \; \mu \in M(T, X).
   \]
   It is not difficult to see that for any invariant measure $\mu$, the limit $\beta(\mu) := \lim_{n \to \infty} \frac{1}{n} \int \phi_n(x) d\mu$ exists and the function $\beta(\cdot) : M(T, X) \to \mathbb{R}$ is affine and continuous, for example, see [28]. Thus $\beta$ satisfies conditions [A.1]–[A.3]. Given two asymptotically additive sequences of functions $\Phi := \{\phi_n : X \to \mathbb{R}\}_{n \geq 1}$ and $\Psi := \{\psi_n : X \to \mathbb{R}\}_{n \geq 1}$ where $\psi_n$ are all positive functions, define
\[ \alpha(\mu) := \liminf_{n \to \infty} \frac{1}{\liminf_{n \to \infty} \int \varphi_n \, d\mu} \int 1_n \varphi_n \, d\mu, \quad \mu \in M(T, X). \]

Note that the above \( \alpha \) is continuous and satisfies conditions [A.1]–[A.3]. We emphasize that here \( \alpha \) may be not affine. It is easy to check that

\[ I_\alpha = I^0_\Phi \cdot R_\alpha = R^0_\Phi \text{ and } R_\alpha(a) = R^0_\Phi(a). \]

Note that for \( \xi = I_\alpha, R_\alpha(a), R_\alpha, \) onchas \( \xi \cap BR \cap \text{Tran} \subseteq \xi \cap (BR \setminus QW) \cap \text{Tran} \subseteq \xi \cap \text{Tran}. \) Thus, one can use theorem 6.1 to end the proof of theorem 8.1.

8.2. Examples

8.2.1. Dynamics with specification. From [21], we know that for any dynamical system with specification property (not necessarily Bowen’s strong version), the almost periodic points are dense in \( X \) and the invariant measures supported on minimal sets are dense in the space of invariant measures. By lemma 3.1 (see below) there is some invariant measure with full support. From [50, proposition 2.1] we know that specification implies \( g \)-almost product property so that the assumptions of the \( g \)-almost product property and the existence of a measure with full support in theorem 1.4 can be replaced by specification. From [49, proposition 2.3 (1)], entropy-dense property holds for dynamical systems with \( g \)-almost product property.

**Corollary 8.1.** Suppose that \((X, T)\) satisfies the specification property and is not uniquely ergodic. Then item (1) of theorem 1.4, theorems 1.2 and 1.3 hold.

**Corollary 8.2.** For non-uniquely ergodic dynamical systems with specification and uniform separation, all the results of theorems 1.4, 1.2 and 1.3 hold.

Corollary 8.1 applies in the following examples: (1) It is known from [10, 16] that any topologically mixing interval map satisfies Bowen’s specification but may not have uniform separation. For example, Jakobson [38] showed that there exists a set of parameter values \( \Lambda \subseteq [0, 4] \) of positive Lebesgue measure such that if \( \lambda \in \Lambda \), then the logistic map \( f_\lambda(x) = \lambda x(1-x) \) is topologically mixing. (2) Recently we learned from [35] that \( C^0 \) generic volume-preserving dynamical systems have specification property. (3) In [1] Aoki constructs a zero-dimensional ergodic automorphism without densely periodic property that obeys specification, but not Bowen’s specification. For the class of all solenoidal automorphisms, it is proved in [2] that the class of automorphisms with specification is wider than the class of automorphisms with Bowen’s specification. (4) In [50], Pfister and Sullivan gave an example of a dynamical system with finite topological entropy, for which the entropy density of ergodic measures is true (specification property is true), but the uniform separation property and the upper semi-continuity of the entropy map fail. This example is a subshift of the shift space \( Y := [-1, 1]^2 \), see [50, pages 952–953] for more details.

Corollary 8.2 applies in the following examples: (1) all topological mixing subshifts of finite type and mixing soft subshifts (in particular, all full shifts on finite alphabets); (2) all subsystems restricted on topological mixing a locally maximal expanding set or hyperbolic set (in particular, all topological mixing expanding maps or topological mixing hyperbolic diffeomorphisms, called Anosov). This is because such examples all satisfy Bowen’s
specification (see [13, 67] for item (1) and see [11] for item (2), also see [22]) and are expansive, which is stronger than uniform separation, see [50]. For example, if \( d \geq 2 \) is an integer, then \( T : S^1 \rightarrow S^1, T(x) = dx \pmod{1} \) is topological mixing and expanding. (3) The results of this paper are also applicable to some dynamical systems beyond uniform hyperbolicity. From [33] we know that non-hyperbolic diffeomorphism \( f \) with \( C^{1+\alpha} \) smoothness, conjugated to a transitive Anosov diffeomorphism \( g \), exists and even the conjugation and its inverse is Hölder continuous. (4) From [59, section 4.3] we know the time-1 map of a transitive Anosov flow satisfies specification. In this case, \( f \) is partially hyperbolic with a one-dimension central bundle. Then \( f \) is entropy-expansive (see [23, 45]). Recall that from [42] entropy-expansive implies asymptotically \( h - \) expansive and from theorem 3.1 of [50] any expansive or asymptotically \( h - \) expansive system satisfies uniform separation property.

8.2.2. \( \beta - \)shifts.

**Corollary 8.3.** Theorems 1.4, 1.2 and 1.3 hold for any \( \beta - \)shift.

Let us explain why corollary 8.3 holds. We know that any \( \beta - \)shift is expansive (which is stronger than uniform separation, see [50]) and from [50] it always satisfies the \( g \)-almost product property. Furthermore, from [54] we know that periodic points are dense in the whole space and the periodic measures are dense in the space of invariant measures (i.e. \( \hat{M}_g(T, X) = M(T, X) \)). By lemma 3.1 (see below) there is some invariant measure with full support (from [66, theorem 13 (ii)] we can also learn that the unique maximal entropy measure of \( \beta - \)shifts always carries full support). Thus, theorems 1.4, 1.2 and 1.3 apply in any \( \beta - \)shift. It is worth mentioning that from [16] the set of parameters of \( \beta \), for which specification holds, is dense in \((1, +\infty)\) but has Lebesgue zero measure.

It is well-known that any subshift of finite type and any \( \beta - \)shift has unique maximal entropy measure with full support. From [46] there exist subshifts with multiple measures of maximal entropy with disjoint support that have \( g \)-almost product property and a measure with full support. Thus, theorems 1.4, 1.2 and 1.3 also apply in these subshifts [46].

8.3. Examples without \( g \)-almost product property

In this subsection we give a dynamical system as an example such that it has uniform separation, entropy-dense property, positive topological entropy and transitivity-saturated property, is not uniquely ergodic, and there is an invariant measure, but it does not have \( g \)-almost product property.

Let \( s \) be a full shift of finite symbols and \( h \) be an irrational rotation on the circle, which is minimal and uniquely ergodic. Let \( f := s \times h \). Since \( s \) has entropy-dense property, positive topological entropy and transivity-saturated property, it is expansive, not uniquely ergodic, and there is an invariant measure measure, then it is not difficult to check that \( f \) has entropy-dense property, positive topological entropy and transivity-saturated property, is not uniquely ergodic, there is an invariant measure, \( f \) is asymptotically entropy-expansive so that it has uniform separation. Thus, all the conclusions in the theorems of the present paper hold for this \( f \), since we can observe that the roles of \( g \)-almost product property and uniform separation can be replaced by transivity-saturated property and entropy-dense property.

Now we show that \( f \) does not have \( g \)-almost product property. Since \( f := s \times h \), one only needs to show that \( h \) does not have \( g \)-almost product property as in following:

**Theorem 8.2.** If \( h \) is an irrational rotation over the circle \( S^1 \), then \( h \) does not have \( g \)-almost product property.
Proof. By contraction, we assume \( h \) has \( g \)-almost product property. Let \( \theta > 0 \) such that \( d(x, hx) = \theta \) for any \( x \in S^1 \). Since \( g(n) < n \) and \( d(h^n x, h^n y) = d(x, y) \) for any \( i \in \mathbb{Z} \), then for any \( n \geq 1 \), \( B_n(g; x, \epsilon) = B_n(x, \epsilon) = B(x, \epsilon) \).

Take \( \epsilon_1 = \epsilon_2 = \frac{\theta}{10} \) and fix \( x_1, x_2 \in S^1 \). Then by the \( g \)-almost product property, for \( n_1 > m(\epsilon_1) \) and \( n_2 > m(\epsilon_2) \), there exist two points \( y, z \) such that

\[
y \in B_{n_1}(g; x_1, \epsilon_1) \cap h^{-n_1}B_n(g; x_2, \epsilon_2) = B(x_1, \epsilon_1) \cap h^{-n_1}B(x_2, \epsilon_2) = B(x_1, \epsilon_1) \cap B(h^{-n_1}x_2, \epsilon_2),
\]

\[
z \in B_{n_2+1}(g; x_1, \epsilon_1) \cap h^{-n_2}B_n(g; x_2, \epsilon_2) = B(x_1, \epsilon_1) \cap h^{-n_2}B(x_2, \epsilon_2) = B(x_1, \epsilon_1) \cap B(h^{-n_2}x_2, \epsilon_2).
\]

Then \( d(y, z) \leq d(y, x_1) + d(x_1, x_2) \leq 2\epsilon_1 \) and \( d(y, hz) \leq d(y, h^{-n_1}x_2) + d(hz, h^{-n_1}x_2) \leq \epsilon_2 + d(z, h^{-n_1}x_2) \leq 2\epsilon_2 \). Thus, \( 0 < \theta = d(z, hz) \leq d(z, y) + d(hz, y) \leq 2\epsilon_1 + 2\epsilon_2 = \frac{2}{10} \theta < \theta \), which is a contraction.

8.4. Regularity and quasiregularity

In [44] Oxtoby also introduced several concepts (for the quasiregular point, also see [22]):

\[
QR_{\text{reg}} := \{ \text{points generic for ergodic measures} \} = \bigcup_{\mu \in \mathcal{M}(T, X)} G_{\mu},
\]

\[
QR_{\text{d}} := \{ \text{points of density in } QR \} = \bigcup_{\mu \in \mathcal{M}(T, X)} (G_{\mu} \cap S_\mu),
\]

\[
R := \{ \text{regular points of } T \} = QR_{\text{d}} \cap QR_{\text{reg}} = \bigcup_{\mu \in \mathcal{M}(T, X)} (G_{\mu} \cap S_\mu).
\]

Such sets are all \( T \)-invariant and remark that

\[
R \subseteq QR_{\text{d}} \cup QR_{\text{reg}} \subseteq QR.
\]

In [62] \( R \) and \( QR_{\text{d}} \setminus R = QR_{\text{d}} \setminus QR_{\text{reg}} \) were considered but \( QR_{\text{reg}} \setminus R = QR_{\text{reg}} \setminus QR_{\text{d}} \) and \( QR \setminus (QR_{\text{reg}} \cup QR_{\text{d}}) \) are not considered.

Theorem 8.3. Suppose that \((X, T)\) has \( g \)-almost product property. Let \( \alpha : M(T, X) \to \mathbb{R} \) be a continuous function with \( \text{Int}(L_{\alpha}) \neq \emptyset \). If \((X, T)\) is not uniquely ergodic and there is an invariant measure with full support, then

1. if \( \alpha \) satisfies [A.1], then

\[
h_{\text{top}}(BR_1 \cap R_\alpha(a) \cap QR \setminus (QR_{\text{reg}} \cup QR_{\text{d}})) = h_{\text{top}}(R_\alpha(a)) \quad \text{where } a \in \text{Int}(L_{\alpha})
\]

and

\[
h_{\text{top}}(QR \setminus (QR_{\text{reg}} \cup QR_{\text{d}})) = h_{\text{top}}(BR_1 \cap R_\alpha \cap QR \setminus (QR_{\text{reg}} \cup QR_{\text{d}})) = h_{\text{top}}(T).
\]

2. \( h_{\text{top}}(QR_{\text{reg}} \setminus R) = h_{\text{top}}(BR_1 \cap R_\alpha \cap (QR_{\text{reg}} \setminus R)) = h_{\text{top}}(T) \).

Proof. In the proof of item (2) of theorem 6.1, \( G_{K_\delta} \) is in fact contained in \( \text{Tran}_1 \cap R_\alpha(a) \cap QR \setminus (QR_{\text{reg}} \cup QR_{\text{d}}) \) and [A.2] is not used. By lemma 5.5, \( \text{Tran}_1 \subseteq BR \) so that \( \text{Tran}_1 \subseteq BR_1 \). Thus, one can follow the proof of item (2) of theorem 6.1 to get the first result in item (1).

By item (1) and item (4) of theorem 4.1,

\[
\geq \sup_{a \in \text{Int}(L_{\alpha})} h_{\text{top}}(BR_1 \cap R_\alpha(a) \cap QR \setminus (QR_{\text{reg}} \cup QR_{\text{d}}))
\]

\[
\geq h_{\text{top}}(BR_1 \cap R_\alpha(\alpha) \cap QR \setminus (QR_{\text{reg}} \cup QR_{\text{d}})).
\]
we have

\[ h_{\text{top}}(R_\alpha(a)) = \sup_{a \in \text{int}(I_\alpha)} h_{\text{top}}(R_\alpha). \]

On the other hand, using Tran \(_1 \subseteq BR_\alpha\), entropy-dense and transitivity-single-saturated, we have

\[ h_{\text{top}}(BR_1 \cap R_\alpha \cap QR_{\text{erg}} \setminus R) \geq \sup \{ h_{\text{top}}(\text{Tran}_1 \cap I_\mu \cap G_{\mu} \cap QR_{\alpha}) | \mu \in M_{\text{erg}}(T, X), S_\mu \neq X \} \]

\[ = \sup \{ h_{\text{top}}(G_{\mu} \cap S_\mu) | \mu \in M_{\text{erg}}(T, X), S_\mu \neq X \} \]

\[ = \sup \{ h_\mu(T) | \mu \in M_{\text{erg}}(T, X), S_\mu \neq X \} = h_{\text{top}} = h_{\text{top}}(R_\alpha). \]

**Theorem 8.4.** Suppose that \((X, T)\) has entropy-dense property. Then \(h_{\text{top}}(R \setminus \text{Tran}) = h_{\text{top}}(T)\).

**Proof.** By the variational principle and entropy-dense property, for any \(\epsilon > 0\), there is an ergodic measure \(\mu\) with \(S_\mu \neq X\) such that \(h_\mu(T) > h_{\text{top}} - \epsilon\). By [12, theorem 3] \(h_{\text{top}}(G_\mu \cap S_\mu) = h_\mu(T)\) and note that \(G_\mu \cap S_\mu \subseteq R \setminus \text{Tran}\) so that \(h_{\text{top}}(R \setminus \text{Tran}) > h_{\text{top}} - \epsilon\). \(\square\)

### 8.5. Topological entropy on \(\text{Rec} \setminus BR\)

Let \(\text{Rec}^\# := \text{Rec} \setminus BR\),

\(\text{Rec}_i := \{ x \in \text{Rec}^\# | C_i = C_i \}\), \(\text{Rec}_2 := \{ x \in \text{Rec}^\# | C_i \subseteq C_i \}\),

\(S_i := \{ x \in X | \cap_{\mu \in M_i} S_\mu \neq \emptyset \}\),

\(W_i := \{ x \in \text{Rec}^\# | S_\mu = C_i \text{ for every } \mu \in M_i \}\),

\(V_i := \{ x \in \text{Rec}^\# | \exists \mu \in M_i \text{ such that } S_\mu = C_i \}\),

\(\text{Rec}_{i,1} := W_i\), \(\text{Rec}_{i,2} := V_i \cup S_i\), \(\text{Rec}_{i,3} := V_i\),

\(\text{Rec}_{i,4} := V_i \cup (\text{Rec}^\# \cap S_i)\), \(\text{Rec}_{i,5} := \text{Rec}^\#\), where \(i = 1, 2\).

**Question 8.1.** Suppose that \((X, T)\) has \(g\)-almost product property and uniform separation. If \((X, T)\) is not uniquely ergodic and there is an invariant measure with full support, then does \(\text{Rec} \setminus BR\) have full topological entropy and moreover, does

\[ \{ \emptyset, \text{Rec}_{1,1}, \text{Rec}_{1,2}, \text{Rec}_{1,3}, \text{Rec}_{1,4}, \text{Rec}_{1,5} \} \]

have full entropy gaps w.r.t. \(X\) for any \(i = 1, 2\)? Similar questions can be asked w.r.t. \(I_\alpha, R_\alpha, \text{Rec}_\alpha(a)\).

**Remark 8.1.** Under the assumption of question 8.1, Tran \(\subseteq BR\) so that \(\text{Rec}_{i,j} \cap \text{Tran} = \emptyset\).

The sets of \(\text{Rec}_{i,j} \cap \{ x | \omega_{I_\mu}(x) = \emptyset \}\) are related to Cases (1')–(6') restricted on recurrent points. So a similar question can be asked:

**Question 8.2.** Suppose that \((X, T)\) has \(g\)-almost product property and uniform separation. If \((X, T)\) is not uniquely ergodic and there is an invariant measure with full support, then do Cases (1')–(6') restricted on recurrent points all carry full topological entropy?
Also, another open question is whether the sets of \( \text{Rec}_{ij} \) and their gap-sets are nonempty and in which kind of dynamical system? For \( \text{Rec}_{1,1} \), it is nonempty in full shifts over finite symbols by the following theorem. Note that any full shift over finite symbols satisfies the assumption of question 8.1 so that question 8.1 is possibly meaningful.

**Theorem 8.5.** Let \( \sigma : \{0, 1, \ldots, k-1\}^\mathbb{N} \rightarrow \{0, 1, \ldots, k-1\}^\mathbb{N} \) be a full shift where \( k \geq 2 \). Then \( \text{Rec}_{1,1} \cap \{x | \omega_B,(x) \neq \emptyset \} \neq \emptyset \) and \( \text{Rec} \setminus BR \neq \emptyset \).

**Proof.** We only need to consider \( k = 2 \). We learned an example from [36] that there is topological mixing but a uniquely ergodic subshift \( \Lambda \) of two symbols for which the unique invariant measure is supported on a fixed point so that in this example \( \emptyset \neq \text{Tran}(\sigma|_{\Lambda}) \subseteq \text{Rec}_{1,1} \cap \{x | \emptyset \neq \omega_B,(x) = \omega_{\bar{d}}(x) = \omega_T(x) \subseteq \omega_T(x) \}. \)

For the discussion of \( \{x | \omega_B,(x) \neq \emptyset \} \), up to now just some entropy results on minimal points and some cases restricted on non-recurrent points are obtained, see [27] for more details.

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