Cut Limits on Hyperbolic Extensions

Pedro Ontaneda*

Abstract

Hyperbolic extensions were defined and studied in [4]. Cut limits of families of metrics were introduced in [5]. In this paper we show that if a family of metrics \( \{ h_\lambda \} \) has cut limits then the family of hyperbolic extensions \( \{ E_k(h_\lambda) \} \) also has cut limits.

The results in this paper are used in the problem of smoothing Charney-Davis strict hyperbolizations [2], [3].

Section 0. Introduction.

This paper deals with the relationship between two concepts: “hyperbolic extensions”, which were studied in [4], and “cut limits of families of metrics”, which were defined in [5]. Before stating our main result we first introduce these concepts here.

A. Hyperbolic extensions. Recall that hyperbolic \( n \)-space \( \mathbb{H}^n \) is isometric to \( \mathbb{H}^k \times \mathbb{H}^{n-k} \) with warped metric \( (\cosh^2 r) \sigma_{\mathbb{H}^k} + \sigma_{\mathbb{H}^{n-k}} \), where \( \sigma_{\mathbb{H}^l} \) denotes the hyperbolic metric of \( \mathbb{H}^l \), and \( r: \mathbb{H}^{n-k} \to [0, \infty) \) is the distance to a fixed point in \( \mathbb{H}^{n-k} \). For instance, in the case \( n = 2 \), since \( \mathbb{H}^1 = \mathbb{R}^1 \) we have that \( \mathbb{H}^2 \) is isometric to \( \mathbb{R}^2 = \{(u, v)\} \) with warped metric \( \cosh^2 v \, du^2 + dv^2 \). The concept of “hyperbolic extension” is a generalization of this construction; we explain this in the next paragraph.

Let \( (M^n, h) \) be a complete Riemannian manifold with center \( o = o_M \in M \), that is, the exponential map \( \exp_o: T_oM \to M \) is a diffeomorphism. The warped metric

\[
f = (\cosh^2 r) \sigma_{\mathbb{H}^k} + h
\]

on \( \mathbb{H}^k \times M \) is the hyperbolic extension (of dimension \( k \)) of the metric \( h \). Here \( r \) is the distance-to-\( o \) function on \( M \). We write \( \mathcal{E}_k(M) = (\mathbb{H}^k \times M, f) \), and \( f = \mathcal{E}_k(h) \). We also say that \( \mathcal{E}_k(M) \) is the hyperbolic extension (of dimension \( k \)) of \( (M, h) \) (or just of \( M \)). Hence, for instance, we have \( \mathcal{E}_k(\mathbb{H}^l) = \mathbb{H}^{k+l} \). Also write \( \mathbb{H}^k = \mathbb{H}^k \times \{ o_M \} \subset \mathcal{E}_k(M) \) and we have that any \( p \in \mathbb{H}^k \) is a center of \( \mathcal{E}_k(M) \) (see [4]).

Remarks 0.1.

1. Let \( M^n \) have center \( o \). Using a fixed orthonormal basis on \( T_oM \) and the exponential map we can identify \( M \) with \( \mathbb{R}^n \), and \( M - \{ o \} \) with \( \mathbb{R}^n - \{ 0 \} = S^{n-1} \times (0, \infty) \). Hence the spheres \( S^{n-1} \times \{ r \} \subset S^{n-1} \times (0, \infty) \) are geodesic spheres, and the rays \( t \mapsto tv = (v, t) \in S^{n-1} \times (0, \infty) = [1]

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$M - \{o\}$, are geodesics rays emanating from the center.

2. Let $g'$ be other metric on $M$. Suppose we can write $g' = g' + dr^2$ on $\mathbb{S}^{n-1} \times (0, \infty) = M - \{o\}$, (this last identification is done using $g$). Then the geodesic spheres around $o$, and the geodesics emanating from $o$ for $g$ and $g'$ coincide.

**B. Cut limits.** Before we talk about “cut limits” we need some preliminary definitions and facts. Let $(M^n, g)$ have center $o$. Then the metric $g$ (outside the center) has the form $g = g_r + dr^2$. Here we are identifying (see 0.1) the space $M - \{o\}$ with $\mathbb{S}^{n-1} \times (0, \infty)$, thus each $g_r$ is a metric on the sphere $\mathbb{S}^{n-1}$.

**Examples.**

1. The Euclidean metric $\sigma_{\mathbb{R}^n}$ on $\mathbb{R}^n$ can be written as $\sigma_{\mathbb{R}^n} = g_r + dr^2$ with $g_r = r^2\sigma_{\mathbb{S}^{n-1}}$, where $\sigma_{\mathbb{S}^{n-1}}$ is the round metric on the sphere $\mathbb{S}^{n-1}$.

2. The hyperbolic metric $\sigma_{\mathbb{H}^n}$ on $\mathbb{R}^n$ can be written as $\sigma_{\mathbb{H}^n} = g_r + dr^2$ with $g_r = sinh^2(r)\sigma_{\mathbb{S}^{n-1}}$.

Let $(M, g)$ have center $o$ and write $g = g_r + dr^2$. Let $r_o > 0$. We can think of the metric $g_{r_o}$ as being obtained from $g = g_r + dr^2$ by “cutting” $g$ along the sphere of radius $r_o$, so we call the metric $g_{r_o}$ on $\mathbb{S}^{n-1}$ the warped spherical cut of $g$ at $r_o$. Let

$$
\hat{g}_{r_o} = (\frac{1}{sinh^2(r_o)})g_{r_o}
$$

We call the metric $\hat{g}_{r_o}$ on $\mathbb{S}^{n-1}$ given by (0.2) the (unwarped) spherical cut of $g$ at $r_o$. In the particular case that $g = g_r + dr^2$ is a warped-by-sinh metric we have $\hat{g}_{r_o} = sinh^2(r)g_r'$ for some fixed $g'$ independent of $r$. In this case the warped spherical cut of $g = sinh^2(r)g_r' + dr^2$ at $r_o$ is $sinh^2(r_0)g_r'$, and the the spherical cut at $r_0$ is $\hat{g}_{r_o} = g_r'$. Hence the terms “warped” and “unwarped” (usually we will omit the term “unwarped”).

**Example.** If $g = \sigma_{\mathbb{H}^n} = sinh^2(r)\sigma_{\mathbb{S}^{n-1}} + dr^2$, the spherical cut at $r_o$ is $(\sigma_{\mathbb{H}^n})_{r_o} = \sigma_{\mathbb{S}^{n-1}}$. And the warped spherical cut at $r_o$ is $sinh^2(r_o)\sigma_{\mathbb{S}^{n-1}}$.

Let $(M^n, g)$ have center $o$. We now consider families of metrics $\{g_\lambda\}_{\lambda > \lambda_0}$ on $M$ of the form $g_\lambda = (g_\lambda)_r + dr^2$. Here $\lambda_0 > 0$, and the identification $M - \{o\} = \mathbb{S}^{n-1} \times (0, \infty)$ is done using $g$; see Remark 0.1. The reason we are interested in these families is that they are key ingredients in Riemannian Hyperbolization [3] (also see [5]). Moreover, the Main Theorem in this paper is used in [3].

Let $b \in \mathbb{R}$. By cutting each $g_\lambda$ at $b + \lambda$ we obtain a one-parameter family $\{(g_\lambda)_{\lambda+b}\}_\lambda$ of metrics on the sphere $\mathbb{S}^{n-1}$. (The metric $(g_\lambda)_{\lambda+b}$ is the spherical cut of $g_\lambda$ at $\lambda + b$). Here $\lambda > \lambda_0 - b$, if $-b \geq \lambda_0$, so that the definition makes sense. We say that the $\{g_\lambda\}$ has cut limit at $b$ if this family $C^2$-converges, as $\lambda \to \infty$. That is, there is a $C^2$ metric $\hat{g}_{\infty+b}$ on $\mathbb{S}^{n-1}$ such that

$$
| (g_\lambda)_{\lambda+b} - \hat{g}_{\infty+b} |_{C^2} \to 0 \quad \text{as} \quad \lambda \to \infty
$$

Here the arrow means convergence in the $C^2$-norm on the space of $C^2$ metrics on $\mathbb{S}^{n-1}$. See
remark 1 in 1.1 of [5] for more details. Note that the concept of cut limit at \( b \) depends strongly on the indexation of the family.

Let \( I \subset \mathbb{R} \) be an interval (compact or noncompact). We say the \( \circ \)-family \( \{g_\lambda\}_\lambda \) has cut limits on \( I \) if the convergence in (0.3) is uniform in \( b \in I \). Explicitly this means: for every \( \epsilon > 0 \), and \( b \in I \) there are \( \lambda_* \) and neighborhood \( U \) of \( b \) in \( I \) such that \( |(g_\lambda)_{\lambda+\epsilon} - \hat{g}_{\lambda+\epsilon} | < \epsilon \), for \( \lambda > \lambda_* \) and \( \lambda' \in U \). In particular \( \{g_\lambda\}_\lambda \) has a cut limit at \( b \), for every \( b \in I \). Finally we say that \( \{g_\lambda\}_\lambda \) has cut limits (everywhere) if \( \{g_\lambda\}_\lambda \) has a cut limits on \( \mathbb{R} \).

Remark 0.4. If \( \{g_\lambda\}_\lambda \) is a family of metrics then \( \{g_{\lambda(\lambda')}\}_{\lambda'} \) is a reparametrization of \( \{g_\lambda\}_\lambda \), where \( \lambda' \mapsto \lambda(\lambda') \) is a change of variables. For instance, if we use translations, the following holds: \( \{g_\lambda\}_\lambda \) has cut limits at \( b \) if and only if \( \{g_{\lambda+a}\}_{\lambda'} \) has cut limits at \( b + a \); here the change of variables is \( \lambda = \lambda' + a \).

C. Statement of main result. Here is a natural question:

Question. If \( \{h_\lambda\}_\lambda \) has cut limits, does \( \{\mathcal{E}_k(h_\lambda)\}_\lambda \) have cut limits?

Remark. More generally we can ask whether \( \{\mathcal{E}_k(h_\lambda)\}_\lambda' \) has cut limits, where \( \lambda = \lambda(\lambda') \). Of course the answer would depend on the change of variables \( \lambda = \lambda(\lambda') \).

Our main result gives an affirmative answer to this question provided the family \( \{h_\lambda\}_\lambda \) is, in some sense, nice near the origin. Explicitly, we say that \( \{h_\lambda\}_\lambda \) is hyperbolic around the origin if there is a \( B \in \mathbb{R} \) such that

\[
\hat{(h_\lambda)}_{\lambda+b} = \sigma_{g^{n-1}}
\]

for every \( b \leq B \) and every (sufficiently large) \( \lambda \). Note that this implies that each \( h_\lambda \) is canonically hyperbolic on the ball of radius \( \lambda + B \). Examples of \( \circ \)-families that are hyperbolic around the origin are families obtained using hyperbolic forcing [5].

As mentioned before our main result answers affirmatively the question above. Moreover it also says that some reparametrized families \( \{\mathcal{E}_k(h_\lambda)\}_\lambda' \), for certain change of variables \( \lambda = \lambda(\lambda') \), have cut limits as well. Write \( \lambda = \lambda(\lambda', \theta) = \sinh^{-1}(\sinh(\lambda') \sin \theta) \), for fixed \( \theta \). We say that \( \{\mathcal{E}_k(h_\lambda)\}_\lambda' \) is the \( \theta \)-reparametrization of \( \{\mathcal{E}_k(h_\lambda)\}_\lambda \). Note that if we consider an hyperbolic right triangle with one angle equal to \( \theta \) and side (opposite to \( \theta \)) of length \( \lambda \), then \( \lambda' \) is the length of the hypothenuse of the triangle. All \( \theta \)-reparametrizations, in the limit \( \lambda' \to \infty \), differ just by translations. We are now ready to state our Main result.

Main Theorem. Let \( M \) have center \( o \). Let \( \{h_\lambda\}_\lambda \) be \( \circ \)-family of metrics on \( M \). Assume \( \{h_\lambda\}_\lambda \) is hyperbolic around the origin. If \( \{h_\lambda\}_\lambda \) has cut limits, then the \( \theta \)-reparametrization \( \{\mathcal{E}_k(h_\lambda)\}_\lambda' \) has cut limits, where \( \theta \in (0, \pi/2] \).

Note that \( \theta = \pi/2 \) gives \( \lambda = \lambda' \). The paper is structured as follows. In Section 1 we review some facts about hyperbolic extensions. In Section 2 we introduce useful coordinates on the spheres of a hyperbolic extension. In Section 3 we study spherical cuts on hyperbolic extensions. Finally in Section 4 we deal with cut limits in a bit more detail and prove the Main Theorem.
Section 1. Hyperbolic Extensions.

Notational convention: we will denote all centers on manifolds by the same letter “o”. If the manifold $M$ needs to be specified we will write $o = o_M$, which means that $o$ is a center in $M$.

Let $(M^n, h)$ have center $o$. Recall that in the Introduction we called the metric $f = E_k(g) = (\cosh^2 r)\sigma_{\eta^k} + h$ on $\mathbb{H}^k \times M$ a hyperbolic extension of $h$. Here $r$ is the distance-to-$o$ function on $M$. We write $E_k(M) = (\mathbb{H}^k \times M, f)$, and call it a hyperbolic extension of $M$. Also write $\mathbb{H}^k = \mathbb{H}^k \times \{o_M\} \subset E_k(M)$ and we have that any $p \in \mathbb{H}^k$ is a center of $E_k(M)$ (see [4], p.23).

Note that $\mathbb{H}^k$ and every $\{y\} \times M$ are convex in $E_k(M)$ (see [4], p.23). Let $\eta$ be a complete geodesic line in $M$ passing though $o$ and let $\eta^+$ be one of its two geodesic rays (beginning at $o$). Then $\eta$ is a totally geodesic subspace of $M$ and $\eta^+$ is convex (see [4]). Also, let $\gamma$ be a complete geodesic line in $\mathbb{H}^k$. The following two results are proved in [4].

**Lemma 1.1.** We have that $\gamma \times \eta^+$ is a convex subspace of $E_k(M)$ and $\gamma \times \eta$ is totally geodesic in $E_k(M)$.

**Corollary 1.2.** We have that $\mathbb{H}^k \times \eta^+$ and $\gamma \times M$ are convex in $E_k(M)$. Also $\mathbb{H}^k \times \eta$ is totally geodesic in $E_k(M)$.

**Remark 1.3.** Note that $\mathbb{H}^k \times \eta$ (with metric induced by $E_k(M)$) is isometric to $\mathbb{H}^k \times \mathbb{R}$ with warped metric $\cosh^2 v \sigma_{\eta^k} + dv^2$, which is just hyperbolic $(k+1)$-space $\mathbb{H}^{k+1}$. Also $\gamma \times \eta$ is isometric to $\mathbb{R} \times \mathbb{R}$ with warped metric $\cosh^2 v du^2 + dv^2$, which is just hyperbolic 2-space $\mathbb{H}^2$. In particular every point in $\mathbb{H}^k = \mathbb{H}^k \times \{o\} \subset E_k(M)$ is a center point.

As before we use $h$ to identify $M - \{o\}$ with $S^{n-1} \times \mathbb{R}^+$. Sometimes we will denote a point $v = (u, r) \in S^{n-1} \times \mathbb{R}^+ = M - \{o\}$ by $v = ru$. Fix a center $o \in \mathbb{H}^k \in E_k(M)$. Then, for $y \in \mathbb{H}^k - \{o\}$ we can also write $y = tw$, $(w, t) \in S^{k-1} \times \mathbb{R}^+$. Similarly, using the exponential map we can identify $E_k(M) - \{o\}$ with $S^{k+n-1} \times \mathbb{R}^+$, and for $p \in E_k(M) - \{o\}$ we can write $p = sx$, $(x, s) \in S^{k+n-1} \times \mathbb{R}^+$.

We denote the metric on $E_k(M)$ by $f$ and we can write $f = f_s + ds^2$. Since $\mathbb{H}^k$ is convex in $E_k(M)$ we can write $\mathbb{H}^k - \{o\} = S^{k-1} \times \mathbb{R}^+ \subset S^{k+n-1} \times \mathbb{R}^+$ and $S^{k-1} \subset S^{k+n-1}$. A point $p \in E_k(M) - \mathbb{H}^k$ has two sets of coordinates: the polar coordinates $(x, s) = (x(p), s(p)) \in S^{k+n-1} \times \mathbb{R}^+$ and the hyperbolic extension coordinates $(y, v) = (y(p), v(p)) \in \mathbb{H}^k \times M$. Write
Remark 1.5. Note that the right geodesic triangle follows from 1.1 (see 2.1 in [4]).

We will write \( H \) copy of hyperbolic 2-space and \( \sinh \) between the geodesic segment \( \Delta o \) from \( y \) emanating from \( (0) \in E \). (These geodesic segments are unique and well-defined because: (1) \( \mathbb{H}^k \) is convex in \( E \), (2) \( (y, o) = o \in \mathbb{H}^k \) and \( o \) are centers in \( \{y\} \) of \( M \) and \( \mathbb{H}^k \subset E \), respectively.)

Let \( \partial_r \) and \( \partial_s \) be the gradient vector fields of \( r \) and \( s \), respectively. Since the \( M \)-fibers \( M_y = \{y\} \times M \) are convex the vectors \( \partial_r \) are the velocity vectors of the speed one geodesics of the form \( a \mapsto (y, a u), u \in \mathbb{S}_{n-1} \subset M \). These geodesics emanate from (and orthogonally to) \( \mathbb{H}^k \subset E \). Also the vectors \( \partial_s \) are the velocity vectors of the speed one geodesics emanating from \( o \in \mathcal{E}_k(M) \). For \( p \in \mathcal{E}_k(M) \), denote by \( \Delta = \Delta(p) \) the right triangle with vertices \( o, y = y(p), p \) and sides the geodesic segments \( [o, p] \in \mathcal{E}_k(M), [o, y] \in \mathbb{H}^k, [p, y] \in \{y\} \times M \subset \mathcal{E}_k(M) \). (These geodesic segments are unique and well-defined because: (1) \( \mathbb{H}^k \) is convex in \( \mathcal{E}_k(M) \), (2) \( (y, o) = o \in \mathbb{H}^k \) and \( o \) are centers in \( \{y\} \times M \) and \( \mathbb{H}^k \subset \mathcal{E}_k(M) \), respectively.)

Let \( \alpha : \mathcal{E}_k(M) - \mathbb{H}^k \to \mathbb{R} \) be the angle between \( \partial_s \) and \( \partial_r \) (in that order), thus \( \cos \alpha = f(\partial_r, \partial_s), \alpha \in [0, \pi] \). Then \( \alpha = f(p) \) is the interior angle, at \( p = (y, v) \), of the right triangle \( \Delta = \Delta(p) \). We call \( \beta(p) \) the interior angle of this triangle at \( o \), that is \( \beta(p) = \beta(x) \) is the spherical distance between \( x \in \mathbb{S}_{k+n-1} \) and the totally geodesic sub-sphere \( \mathbb{S}^{k-1} \). Alternatively, \( \beta \) is the angle between the geodesic segment \( [o, p] \subset \mathcal{E}_k(M) \) and the convex submanifold \( \mathbb{H}^k \). Therefore \( \beta \) is constant on geodesics emanating from \( o \in \mathcal{E}_k(M) \), that is \( \beta(sx) = \beta(x) \). The following corollary follows from 1.1 (see 2.1 in [3]).

**Corollary 1.4.** Let \( \eta^+ \) (or \( \eta \)) be a geodesic ray (line) in \( M \) through \( o \) containing \( v = v(p) \) and \( \gamma \) a geodesic line in \( \mathbb{H}^k \) through \( o \) containing \( y = y(p) \). Then \( \Delta(p) \subset \gamma \times \eta^+ \subset \gamma \times \eta \).

**Remark 1.5.** Note that the right geodesic triangle \( \Delta(p) \) has sides of length \( r = r(p), t = t(p) \) and \( s = s(p) \). By Lemma 1.1 and Remark 1.3 we can consider \( \Delta \) as contained in a totally geodesic copy of hyperbolic 2-space \( \mathbb{H}^2(p) \). The plane \( \mathbb{H}^2(p) \) is well defined for \( p \) outside \( \mathbb{H}^k \cup \{o\} \times M \). We will write \( \mathbb{H}^2(p) = \gamma_w \times \eta_u \), where \( p = (y, v) \in \mathbb{H}^k \times M, y = tw, v = ru \).

Hence, by 1.5, using hyperbolic trigonometric identities we can find relations between \( r, t, s, \alpha \) and \( \beta \). For instance, using the hyperbolic law of sines we get:

\[
\sinh(r) = \sin(\beta) \sinh(s)
\]
In Section 3 we will need the following result.

**Proposition 1.7.** We have the following identity defined outside $\mathbb{H}^k \cup \{o\} \times M$

\[(\sinh^2(s)) \, d\beta^2 + ds^2 = \cosh^2(r) \, dt^2 + dr^2\]

**Proof.** First a particular case. Take $M = \mathbb{R}$ and $k = 1$, hence $\mathcal{E}_k(M) = \mathcal{E}_1(\mathbb{R}) = \mathbb{H}^2$. In this case the left-hand side of the identity above is the expression of the metric of $\mathbb{H}^2$ in polar coordinates $(\beta, s)$, and right hand side of the equation is the expression of the same metric in the hyperbolic extension coordinates $(r, t) = (v, y)$. (Here $r$ and $t$ are “signed” distances.) Hence the equation holds in this particular case. Now, for the general case note that, by 1.1 and 1.5, the functionals $d\beta, ds, dt, dr$, at some $p$, are zero on vectors perpendicular to $\mathbb{H}^2(p)$. Hence the general case can be reduced to the particular. This proves the proposition.

**Section 2. Coordinates On The Spheres** $S_s(\mathcal{E}_k(M))$.

Let $N^n$ have center $o$. The geodesic sphere of radius $r$ centered at $o$ will be denoted by $S_r = S_r(N)$ and we can identify $S_r$ with $S^{n-1} \times \{r\}$.

Let $M$ have center $o$ and metric $h$. Consider the hyperbolic extension $\mathcal{E}_k(M)$ of $M$ with center $o \in \mathbb{H}^k = \mathbb{H}^k \times \{o\} \subset \mathcal{E}_k(M)$ and metric $f$. Since $\mathbb{H}^k \subset \mathcal{E}_k(M)$ is convex, we can write $S_s(\mathcal{E}_k(M)) \cap \mathbb{H}^k = S_s(\mathbb{H}^r)$. Equivalently $(S^{n+k-1} \times \{s\}) \cap \mathbb{H}^k = S^{k-1} \times \{s\}$. Write $M_o = \{o\} \times M$. Also write

$$E_k(M) = \mathcal{E}_k(M) - (\mathbb{H}^k \coprod M_o)$$

and

$$S_s(\mathcal{E}_k(M)) = S_s(\mathcal{E}_k(M)) \cap E_k(M) = S_s(\mathcal{E}_k(M)) - (\mathbb{H}^k \coprod M_o)$$

Note that the functions $\alpha$ and $\beta$ are well-defined and smooth on $E_k(M)$, and $0 < \beta(p) < \pi/2$. Moreover, by 1.5, the plane $\mathbb{H}^2(p) = \gamma_w \times \eta_u$ is well defined for $p \in E_k(M)$. As in 1.5, here $p = (y, v) \in \mathbb{H}^k \times M$, $y = tw, v = ru$. Recall that $\triangle(p) \subset \mathbb{H}^2(p)$ (see 1.4 and 1.5).

By the identification between $S^{n+k-1} \times \{s\}$ with $S_s(\mathcal{E}_k(M))$ and Lemma 1.1 we have that $\mathbb{H}^2(p) \cap S_s(\mathcal{E}_k(M))$ gets identified with a geodesic circle $S^1(p) \subset S^{n+k-1}$. Moreover, since $\mathbb{H}^2(p)$ and $\mathbb{H}^k$ intersect orthogonally on $\gamma_w$, we have that the spherical geodesic segment $[x(p), w(p)]_{S^{n+k-1}}$ intersects $S^{k-1} \subset S^{n+k-1}$ orthogonally at $w$. This together with the fact that $\beta < \pi/2$ imply that $[x(p), w(p)]_{S^{n+k-1}}$ is a length minimizing spherical geodesic in $S^{k+n-1}$ joining $x$ to $w$. Consequently $\beta = \beta(p)$ is the length of $[x(p), w(p)]_{S^{n+k-1}}$.

We now give a set of coordinates on $S_s(\mathcal{E}_k(M))$. For $p \in S_s(\mathcal{E}_k(M))$ define

$$\Xi(p) = \Xi_s(p) = (w, u, \beta) \in S^{k-1} \times S^{n-1} \times (0, \pi/2)$$

where $w = w(p), u = u(p), \beta = \beta(p)$. Note that $\Xi$ is constant on geodesics emanating from $o \in \mathcal{E}_k(M)$, that is $\Xi(sz) = \Xi(z)$.

Using hyperbolic trigonometric identities we can find well defined and smooth functions $r = r(s, \beta)$ and $t = t(s, \beta)$ such that $r$, $s$, $t$ are the lengths of the sides of a right geodesic triangle
on $\mathbb{H}^2$ with angle $\beta$ opposite the the side with length $r$. With these functions we can construct explicitly a smooth inverse to $\Xi$.

2.1 Remarks.
1. For $(w, u) \in S^{k-1} \times S^{n-1}$ we have

$$\Xi \left( (\gamma_w \times u) \cap S_s(\mathcal{E}_k(M)) \right) = \{ \pm w \} \times \{ \pm u \} \times (0, \pi/2)$$

By Lemma 1.1 the paths $a \mapsto (\pm w, \pm u, a)$ four spherical (open) geodesic segments emanating orthogonally from $S^{k-1}$.

2. For $w \in S^{k-1}$ we have

$$\Xi \left( (\gamma_w \times M) \cap S_s(\mathcal{E}_k(M)) \right) = \{ w \} \times S^{n-1} \times (0, \pi/2)$$

By Lemma 1.2 we have that this set is a spherical geodesic ball of radius $\pi/2$ and of dimension $n$ (with its center deleted) intersecting $S^{k-1}$ orthogonally at $w$. Note that the geodesic segments on this ball emanating from $w$ are the spherical geodesic segments of item 1, for all $u \in S^{n-1}$.

3. For $w \in S^{k-1}$ and $r$ with $0 < r < s$ we have

$$\Xi \left( (\gamma_w \times S_r(M)) \cap S_s(\mathcal{E}_k(M)) \right) = \{ w \} \times S^{n-1} \times \beta(r)$$

where $\beta(r)$ is the angle of the right geodesic hyperbolic triangle with sides of length $s$ (opposite to the right angle) and $r$, opposite to $\beta$. We have $\beta = \sin^{-1} \left( \frac{\sinh(r)}{\sinh(s)} \right)$; see 1.6.

4. Since the $M$-fibers $\{ y \} \times M$ are orthogonal in $\mathcal{E}_k(M)$ to the $\mathbb{H}^k$-fibers $\mathbb{H}^k \times \{ v \}$, items 1, 2, and 3 above imply that the $S^{k-1}$-fibers, the $S^{n-1}$-fibers and $(0, \pi/2)$-fibers are mutually orthogonal in $S^{k-1} \times S^{n-1} \times (0, \pi/2)$ with the metric $\Xi_s f$.

5. The map

$$\Xi' = (\Xi, s) : E_k(M) \to S^{k-1} \times S^{n-1} \times (0, \pi/2) \times \mathbb{R}^+$$

gives coordinates on $E_k(M)$.

Section 3. Spherical Cuts on Hyperbolic Extensions.

Let $(N^m, g)$ have center $o$. Recall from the Introduction that the metric $g_r$ on $S_r$ is called the warped (by sinh) spherical cut of $g$ at $r$, and the metric $\hat{g}_r = \left( \frac{1}{\sinh^2(r)} \right) g_r$ is the spherical cut of $g$ at $r$.

Now, let $(M^n, h)$ have center $o$. Thus we can write $h = h_r + ds^2$, where each $h_r$ is a metric on $S^{n-1}$. As before we denote by $f = \mathcal{E}_k(h)$ the hyperbolic extension of $h$, and we write $f = f_s + ds^2$ on $\mathcal{E}_k(M) - \{ o \}$; each $f_s$ is a metric on $S^{n+k-1}$. We use the map $\Xi = \Xi_s$ of Section 2 that gives coordinates on $S_s(\mathcal{E}_k(M))$. Note that the metric $\Xi s f_s$ is a metric on $S^{k-1} \times S^{n-1} \times (0, \pi/2)$, and it is the expression of $f_s$ in the $\Xi$-coordinates.

Proposition 3.1 We have that

$$\Xi_s f_s = \left( \sinh^2(s) \cos^2(\beta) \right) \sigma_{g_{k-1}} + h_r + \left( \sinh^2(s) \right) d\beta^2$$

7
where \( r = \sinh^{-1}(\sin(\beta) \sinh(s)) \) (see 1.6).

**Proof.** By item 4 in 2.1 we have that \( \Xi_f \) has the form \( A + B + C \), where \( A(u, \beta) \) is a metric on \( S^{k-1} \times \{ u \} \times \{ \beta \} \), \( B(w, \beta) \) is a metric on \( \{ w \} \times S^{n-1} \times \{ \beta \} \) and \( C(u, \beta) \) is a metric on \( \{ w \} \times \{ u \} \times (0, \pi/2) \), i.e \( C = f(w, u, \beta) d\beta^2 \), for some positive function \( f \).

Now, by definition we have

\[
f = \cosh^2(r) \sigma_{hk} + h_r + dr^2 = \cosh^2(r) \left( \sinh^2(t) \sigma_{hk-1} + dt^2 \right) + h_r + dr^2
\]

By Proposition 1.7 and the identity \( \cosh(r) \sinh(t) = \sinh(s) \cos(\beta) \) (which follows from the law of sines and the second law of cosines, also see 1.6) we can write

\[
f_s + ds^2 = f = \left( \sinh^2(s) \cos^2(\beta) \right) \sigma_{hk-1} + h_r + \left( \sinh^2(s) \right) d\beta^2 + ds^2
\]

This proves the proposition.

Hence Proposition 3.1 gives the expression of the the warped spherical cut, at \( s \), of the metric \( f = \xi_k(h) \) in the \( \Xi \)-coordinates. The next corollary does the same for the (unwarped) spherical cuts \( \hat{f} \) of \( f \) at \( s \).

**Corollary 3.2** We have that

\[
\Xi_s(\hat{f}_s) = \cos^2(\beta) \sigma_{hk-1} + \sin^2(\beta) \hat{h}_r + d\beta^2
\]

where \( r \) as in Proposition 3.1.

**Proof.** Since \( \sinh^2(r) \hat{h}_r = h_r \), and \( \sinh^2(s) \hat{f}_s = f_s \), the corollary follows from Proposition 3.1 and the identity 1.6.

**Section 4. Cut Limits and Proof of The Main Theorem.**

First a bit of notation. Let \((N^m, g)\) have center \( o \). Recall that we can write the metric on \( N \setminus \{ o \} = S^{m-1} \times \mathbb{R}^+ \) as \( g = g_r + dr^2 \), where \( r \) is the distance to \( o \). Let \( A \subset S^{n-1} \) and denote by \( C_A \) the open cone \( A \times \mathbb{R}^+ \subset S^{m-1} \times \mathbb{R}^+ \subset M \). We write \( A_r = C_A \cap S_r(M) = A \times \{ r \} \). We will use the definition of cut limit at \( b \), cut limit on \( I \), and cut limit (everywhere), given in the Introduction. We also have “partial” versions of these definitions. Let \( A \subset S^n \). We say that \( \{ g_\lambda \}_\lambda \) is a \( \odot \)-family of metrics on \( A \) if each \( g_\lambda = (g_\lambda)_r + dr^2 \) is a metric defined (at least) on \( C_A \). The definitions of cut limit over \( A \), at \( b \), cut limits over \( A \), on \( I \), and cut limits over \( A \) (everywhere) are similar.

**Remark 4.1.** In the case of cut limits on an interval \( I \) (including \( I = \mathbb{R} \) the \( C^2 \)-convergence is uniform \( C^2 \)-convergence with compact supports in the \( I \)-direction and uniform \( C^2 \)-convergence over \( A \) in the \( S^{n-1} \)-direction.

Let \( M^n \) have metric \( h \) and center \( o \). As always we identify \( M \setminus \{ o \} \) with \( S^{n-1} \times \mathbb{R}^+ \) and \( M \) with \( \mathbb{R}^n \). Choose a center \( o \in \mathbb{R}^k \subset E_k(M) \). Let \( \{ h_\lambda \}_\lambda \) be a \( \odot \)-family of metrics on \( M \), thus \( o \) is a center for all \( h_\lambda \). Denote by \( f_\lambda = E_k(h_\lambda) \) the hyperbolic extension of \( h_\lambda \). We have that \( \{ f_\lambda \}_\lambda \) is a \( \odot \)-family on \( E_k(M) \) (see 1.3 [4]). We now \( \theta \)-reparametrize \( \{ f_\lambda \}_\lambda \), that is, we use the change of variables \( \lambda = \lambda(\lambda') = \sinh^{-1}(\sinh(\lambda') \sin \theta) \). Here \( \theta \in (0, \pi/2] \) is a fixed constant. (Note that
\( \lambda' \) plays the role of the variable \( s \) in 1.6, and \( \lambda \) plays the role of \( r \).) We obtain in this way the \( \odot \)-family \( \{ f_{\lambda(\lambda')} \}_{\lambda'} \). Write \( S = \mathbb{S}^{n+k-1} - \{ \mathbb{S}^{k-1} \sqcup \mathbb{S}^{n-1} \} \). In the next proposition we fix \( c \in \mathbb{R} \). Also we use concept of “hyperbolic around the origin” given in the Introduction.

**Proposition 4.2.** Assume that \( \{ h_{\lambda} \} \) has cut limits on the interval \( J_c = (-\infty, c) \), and that it is hyperbolic around the origin. Then \( \{ f_{\lambda(\lambda')} \}_{\lambda'} \) has cut limits on \( J_c \) over \( S \), where where \( c' < c + \ln \sin(\theta) \).

**Proof.** By hypothesis \( \{ h_{\lambda} \} \) is hyperbolic around the origin. Hence there is \( B \) such that

\[
(\hat{h}_{\lambda})_{\lambda+b} = \sigma_{\mathbb{S}^{n-1}} \quad \text{for all} \quad b \leq B
\]

(1)

Hence the metrics \( h_{\lambda} \) are canonically hyperbolic on the ball of radius \( \lambda + B \). Also, since we are assuming \( \{ h_{\lambda} \} \) has cut limits on \( J_c \) we have that

\[
(\hat{h}_{\lambda})_{\lambda+b} \rightarrow \hat{h}_{\infty+b}
\]

(2)

uniformly with compact supports on \( b \in J_c \).

As mentioned before we can write \( f_{\lambda} = (f_{\lambda})_s + ds^2 \). We have to compute the limit of \( (f_{\lambda(\lambda')})_{\lambda' \rightarrow \infty} \), as \( \lambda' \rightarrow \infty \). Let the \( \Xi \)-coordinates be as defined in Section 2 (for the space \((\mathcal{E}_k(M), f)\)).

From Corollary 3.2. we can express \((\hat{f}_{\lambda})_s\) in \( \Xi \)-coordinates:

\[
\Xi_s\left( (f_{\lambda(\lambda')})_{\lambda' \rightarrow \infty} \right) = \cos^2(\beta) \sigma_{\mathbb{S}^{k-1}} + \sin^2(\beta) (\hat{h}_{\lambda(\lambda')})_{r(\lambda' + b, \beta)} + d\beta^2
\]

where \( r = r(s, \beta) \) is given by 1.6. Therefore we want to find the limit of \( (\hat{h}_{\lambda(\lambda')})_{r(\lambda' + b, \beta)} \) as \( \lambda' \rightarrow \infty \). To do this take the inverse of \( \lambda = \lambda(\lambda') \), and we get \( \lambda' = \lambda(\lambda') = \sinh^{-1}\left( \frac{\sinh(\lambda)}{\sin(\theta)} \right) \). Hence

\[
\lim_{\lambda' \rightarrow \infty} (\hat{h}_{\lambda(\lambda')})_{r(\lambda' + b, \beta)} = \lim_{\lambda \rightarrow \infty} (\hat{h}_{\lambda})_{\vartheta(\lambda, \beta, b)}
\]

where

\[
\vartheta(\lambda, \beta, b) = r(\lambda(\lambda') + b, \beta) = \sinh^{-1}\left( \sinh\left\{ b + \sinh^{-1}\left( \frac{\sinh(\lambda)}{\sin(\theta)} \right) \right\} \sin(\beta) \right)
\]

and a straightforward calculation shows

\[
\lim_{\lambda \rightarrow \infty} \left( \vartheta(\lambda, \beta, b) - \lambda \right) = b + \ln\left( \frac{\sin(\beta)}{\sin(\theta)} \right)
\]

(3)

And this convergence is uniform with compact supports in the \( C^2 \)-topology. Hence from (2) and (3) we get

\[
\lim_{\lambda' \rightarrow \infty} (\hat{h}_{\lambda(\lambda')})_{r(\lambda' + b, \beta)} = \hat{h}_{\infty+b+\ln\left( \frac{\sin(\beta)}{\sin(\theta)} \right)}
\]

Caveat. The limit (3) is uniform with compact supports in the \( \beta \) direction, but not uniform in the \( \beta \) direction. The problem is when \( \beta \rightarrow 0 \).

Before solving the problem mentioned in the caveat we deal with the condition on \( c' \) stated in Proposition 4.2. Note that we want
\[
\lambda + \left( b + \ln \left( \frac{\sin(\beta)}{\sin(\theta)} \right) \right) < \lambda + c
\]
where we are using equation (3). This inequality implies that we want \( b \leq c' < c - \ln \left( \frac{\sin(\pi/2)}{\sin(\theta)} \right) = c + \ln \sin(\theta) \).

We now deal with the problem mentioned in the caveat. Let \( c' \) as above, i.e. such that \( c' < c + \ln \sin \theta \). We will need the following claim.

**Claim.** There is \( \beta_1 > 0 \) such that \( r(\lambda' + c', \beta_1) \leq \lambda(\lambda') + B \), for every \( \lambda' \) sufficiently large.

**Proof of the claim.** A calculation shows that taking \( \beta_1 = \sin^{-1}(e^{2(B-c-1)}) \) works. (Find the limit \( \lambda' \to \infty \) of both terms in the inequality, and use the fact that \( c' < c + \ln \sin \theta \).) This proves the claim.

Since the function \( r = r(s, \beta) \) is increasing in both variables, the claim implies that \( r(\lambda + b, \beta) \leq \lambda(\lambda') + B \), for every \( b \leq c' \), \( \beta \leq \beta_1 \) and \( \lambda' \) sufficiently large (how large not depending on \( b \), nor \( \beta \)). This together with (1) imply that for every \( b \leq c' \), \( \beta \leq \beta_1 \) and \( \lambda' \) sufficiently large we have

\[
\hat{h}(\lambda') r(\lambda' + b, \beta) = \sigma_{\epsilon_0}^{-1}
\]
Hence for every \( b \in J_c \) and \( \beta \leq \beta_1 \) we have

\[
\lim_{\lambda' \to \infty} \hat{h}(\lambda') r(\lambda' + b, \beta) = \sigma_{\epsilon_0}^{-1}
\]
Since \( \beta_1 > 0 \) the problem mentioned in the caveat (i.e., when \( \beta \to 0 \)) has been removed. This proves the proposition.

Taking \( c \to \infty \) in Proposition 4.2 gives the following corollary.

**Corollary 4.3.** Assume that \( \{h_{\lambda}\} \) has cut limits, and that it is hyperbolic around the origin. Then \( \{f_{\lambda(\lambda')}\}_{\lambda'} \) has cut limits over \( S \).

**Proof of the Main Theorem.** Note that the only difference between Corollary 4.3 and the Main Theorem is that in the corollary the cut limits exist over \( S \subset S^{n-1} \). Hence we have to show that the existence of cut limits over \( S \) imply the existence of cut limits on the whole of \( S^{n-1} \). Corollary 4.3 and 0.3 in the Introduction imply

\[
\left\| (f_{\lambda})|S \right\|_{\lambda' + b} - \hat{f}_{\infty + b} \right\|_{C^2} \to 0 \quad \text{as} \quad \lambda' \to \infty
\]
where \( \hat{f}_{\infty + b} \) is a metric on \( S \). In particular the one-parameter family \( \{(f_{\lambda})|S \}_{\lambda' + b} \) (\( b \) is fixed) is Cauchy, that is

\[
\left\| (f_{\lambda(\lambda')})|S \right\|_{\lambda' + b} - \left\| (f_{\lambda(\lambda'')})|S \right\|_{\lambda' + b} \right\|_{C^2} \to 0 \quad (4)
\]
uniformly as \( \lambda'_1, \lambda''_2 \to \infty \). But since \( S \) is dense in \( S^{n-1} \) we get for any \( C^2 \) (pointwise) bilinear form \( g \) on \( S^{n-1} \) we have \( \|g|S\|_{C^2} = \|g|C^2 \). Therefore we can drop the restriction “\( g \)” in (4) to get

\[
\left\| (f_{\lambda(\lambda'_1)})|S \right\|_{\lambda'_1 + b} - \left\| (f_{\lambda(\lambda'_2)})|S \right\|_{\lambda'_2 + b} \right\|_{C^2} \to 0
\]
This implies that the family \( (\hat{f}_\lambda)_{\lambda' + b} \) is Cauchy, therefore converges to some \( \hat{f}_{\infty + b} \). Note that \( \hat{f}_{\infty + b} \) is a symmetric bilinear form on \( S^{n-1} \), and it is positive definite on \( S \). It remains to prove that \( \hat{f}_{\infty + b} \) is also positive definite outside \( S \). Recall \( S = S^{n+k-1} - (S^{k-1} \cup S^{n-1}) \). But it is straightforward to verify that we have \( \hat{f}_{\infty + b}|_{S^{k-1}} = \sigma_{\tilde{k}-1} + \sigma_{\tilde{n}}(0) \). On the other hand on \( S^{n-1} \) we have \( \beta = \pi/2 \), hence \( \lambda = \lambda' \). Also by definition we have \( f_\lambda = \cosh^2(\lambda)\sigma_{\tilde{k}} + h_\lambda = \cosh^2(\lambda)\sigma_{\tilde{k}} + (h_\lambda)r + dr^2 \). But on \( M_b \) we get \( r = s \). Therefore

\[
((f_\lambda)|_{S^{n-1}})_{\lambda' + b} = \left( (f_\lambda)|_{S^{n-1}} \right)_{\lambda' + b} = \cosh^2(\lambda)\sigma_{\tilde{k}}(0) + (h_\lambda)_{\lambda + b} \rightarrow \cosh^2(\lambda)\sigma_{\tilde{k}}(0) + \hat{h}_{\infty + b}
\]

Consequently \( \hat{f}_{b + \infty} \) is positive definite on \( S^{n-1} \). Thus it is positive definite outside \( S \). This proves the Main Theorem.

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Pedro Ontaneda
SUNY, Binghamton, N.Y., 13902, U.S.A.