Roman domination in Cartesian product graphs and strong product graphs

Ismael G. Yero\(^1\) and Juan A. Rodríguez-Velázquez\(^2\)

\(^1\)Departamento de Matemáticas, Escuela Politécnica Superior de Algeciras
Universidad de Cádiz, Av. Ramón Puyol, s/n, 11202 Algeciras, Spain.
ismael.gonzalez@uca.es

\(^2\)Departament d’Enginyeria Informàtica i Matemàtiques
Universitat Rovira i Virgili, Av. Països Catalans 26, 43007 Tarragona, Spain.
juanalberto.rodriguez@urv.cat

April 26, 2021

Abstract

A set \(S\) of vertices of a graph \(G\) is a dominating set for \(G\) if every vertex outside of \(S\) is adjacent to at least one vertex belonging to \(S\). The minimum cardinality of a dominating set for \(G\) is called the domination number of \(G\). A map \(f : V \rightarrow \{0, 1, 2\}\) is a Roman dominating function on a graph \(G\) if for every vertex \(v\) with \(f(v) = 0\), there exists a vertex \(u\), adjacent to \(v\), such that \(f(u) = 2\). The weight of a Roman dominating function is given by \(f(V) = \sum_{u \in V} f(u)\). The minimum weight of a Roman dominating function on \(G\) is called the Roman domination number of \(G\). In this article we study the Roman domination number of Cartesian product graphs and strong product graphs. More precisely, we study the relationships between the Roman domination number of product graphs and the (Roman) domination number of the factors.

Keywords: Domination number; Roman domination number; Cartesian product graphs; strong product graphs.

AMS Subject Classification Numbers: 05C69; 05C70; 05C76.

1 Introduction

Nowadays the study of the behavior of several graph parameters in product graphs have become an interesting topic of research \([10][11]\). For instance, we emphasize the Shannon capacity of a graph \([12]\), which is a certain limiting value involving the vertex independence number of strong product powers of a graph, and the Hedetniemi’s coloring conjecture for the categorical product \([8][11]\).
The article. Hereafter G[3, 4, 5, 9, 16]. In this article we obtain Vizing-like results for the Roman domination number of Roman domination introduced first by Steward in [13] and studied further by other authors [3, 4, 5, 9, 13]. Nevertheless, the quantity of works about the conjecture results obtained around the conjecture. Also, in these surveys appear some references to similar open problems on product graphs. Nevertheless, the quantity of works about the conjecture have not been enough to finally prove or disprove it. One variant of domination is the concept of Roman domination introduced first by Steward in [13] and studied further by other authors [3, 4, 5, 9, 16]. In this article we obtain Vizing-like results for the Roman domination number of Cartesian product graphs and strong product graphs.

We begin by establishing the principal terminology and notation which we will use throughout the article. Hereafter G = (V, E) denotes a finite simple graph. For two adjacent vertices u and v of G we use the notation u ∼ v and, in this case, we say that uv is an edge of G, i.e., uv ∈ E. For a vertex v of G, N(v) = {u ∈ V : u ∼ v} denotes the set of neighbors that v has in G. N(v) is called the open neighborhood of v and the close neighborhood of v is defined as N[v] = N(v) ∪ {v}. For a set D ⊆ V, the open neighborhood is N(D) = ∪v∈DN(v) and the closed neighborhood is N[D] = N(D) ⊆ D. A set D is a dominating set if N[D] = V. The domination number γ(G) is the minimum cardinality of a dominating set in G. We say that a set S is a γ(G)-set if it is a dominating set and |S| = γ(G).

A map f : V → {0, 1, 2} is a Roman dominating function for a graph G if for every vertex v with f(v) = 0, there exists a vertex u ∈ N(v) such that f(u) = 2. The weight of a Roman dominating function is given by f(V) = ∑u∈V f(u). The minimum weight of a Roman dominating function on G is called the Roman domination number of G and it is denoted by γR(G).

Any Roman dominating function f on a graph G induces three sets B0, B1, B2, where B_{i} = \{v ∈ V : f(v) = i\}. Thus, we will write f = (B0, B1, B2). It is clear that for any Roman dominating function f = (B0, B1, B2) on a graph G = (V, E) of order n we have that f(V) = \sum_{u∈V} f(u) = 2|B2| + |B1| and |B0| + |B1| + |B2| = n. We say that a function f = (B0, B1, B2) is a γR(G)-function if it is a Roman domination function and f(V) = γR(G).

Several results about the Roman dominating sets have been obtained in the last years, [3, 4, 5, 9, 13, 16], and it is natural to try to relate the Roman domination number with the standard domination number. For instance, in [3, 9] was obtained the following result, which we will use as a tool in this article.

**Lemma 1.** [3, 9] For any graph G, γ(G) ≤ γR(G) ≤ 2γ(G).

In this article we study the Roman domination number of Cartesian product graphs and strong product graphs. More precisely, we study the relationships between the Roman domination number of product graphs and the domination number (Roman domination number) of the factors.

We recall that given two graphs G and H with set of vertices V_1 = \{v_1, v_2, ..., v_{n_1}\} and
\( V_2 = \{u_1, u_2, ..., u_{n_2}\} \), respectively, the Cartesian product of \( G \) and \( H \) is the graph \( G \Box H = (V, E) \), where \( V = V_1 \times V_2 \) and two vertices \((v_i, u_j)\) and \((v_k, u_l)\) are adjacent in \( G \Box H \) if and only if

- \( v_i = v_k \) and \( u_j \sim u_l \), or
- \( v_i \sim v_k \) and \( u_j = u_l \).

The strong product \( G \boxtimes H \) of the graphs \( G \) and \( H \) is defined on the Cartesian product of the vertex sets of the factors. Two distinct vertices \((v_i, u_j)\) and \((v_k, u_l)\) of \( G \boxtimes H \) being adjacent with respect to the strong product if and only if

- \( v_i = v_k \) and \( u_j \sim u_l \), or
- \( v_i \sim v_k \) and \( u_j = u_l \), or
- \( v_i \sim v_k \) and \( u_j \sim u_l \).

So, the Cartesian product graph \( G \Box H \) is a subgraph of the strong product graph \( G \boxtimes H \).

## 2 Cartesian product graphs

Currently there are few known results on the Roman domination number of Cartesian product graphs. As far as we know, the only works on this topic are as follows. The Roman domination number of \( C_{5t} \Box C_{5k} \) was studied in [16] and the Roman domination number of some grid graphs was studied in [3, 4]. Also, the following general relationship between the Roman domination number of Cartesian product graphs and the domination number of its factors was obtained in [15]:

\[
\gamma_R(G \Box H) \geq \gamma(G) \gamma(H). \tag{1}
\]

The following lemma will be helpful in obtaining the results reported here.

**Lemma 2.** Let \( G \) be a graph. For any \( \gamma_R(G) \)-function \( f = (B_0, B_1, B_2) \),

(i) \( |B_2| \leq \gamma_R(G) - \gamma(G) \).

(ii) \( |B_1| \geq 2\gamma(G) - \gamma_R(G) \).

**Proof.** Since \( B_2 \cup B_1 \) is a dominating set for \( G \) and \( B_1 \cap B_2 = \emptyset \), we have \( \gamma(G) \leq |B_2| + |B_1| \). So, (i) is deduced as \( \gamma(G) = 2|B_2| + |B_1| = \gamma_R(G) - |B_2| \), and (ii) is obtained as \( 2\gamma(G) \leq 2|B_2| + 2|B_1| = 2|B_2| + |B_1| + |B_1| = \gamma_R(G) + |B_1| \).

**Theorem 3.** For any graphs \( G \) and \( H \),

(i) \( \gamma_R(G \Box H) \geq \frac{2\gamma(G) \gamma_R(H)}{3} \).

(ii) \( \gamma_R(G \Box H) \geq \frac{\gamma(G) \gamma_R(H) + \gamma(G \Box H)}{2} \).

3
Proof. Let $V_1$ and $V_2$ be the vertex sets of $G$ and $H$, respectively. Let $f = (B_0, B_1, B_2)$ be a $\gamma_R(G \Box H)$-function. Let $S = \{u_1, u_2, \ldots, u_{\gamma(G)}\}$ be a dominating set for $G$. Let $\{A_1, A_2, \ldots, A_{\gamma(G)}\}$ be a vertex partition of $G$ such that $u_i \in A_i$ and $A_i \subseteq N[u_i]$. Let $\{\Pi_1, \Pi_2, \ldots, \Pi_{\gamma(G)}\}$ be a vertex partition of $G \Box H$, such that $\Pi_i = A_i \times V_2$ for every $i \in \{1, \ldots, \gamma(G)\}$.

For every $i \in \{1, \ldots, \gamma(G)\}$, let $f_i : V_2 \to \{0, 1, 2\}$ be a function such that $f_i(v) = \max\{f(u, v) : u \in A_i\}$. For every $j \in \{0, 1, 2\}$, let $X_j^{(i)} = \{v \in V_2 : f_i(v) = j\}$. Now, let $Y_j^{(i)} \subseteq X_j^{(i)}$ such that for every $v \in Y_j^{(i)}$, $N(v) \cap X_2^{(i)} = \emptyset$. Hence, we have that $f_i' = (X_0^{(i)} - Y_0^{(i)}, X_1^{(i)} + Y_0^{(i)}, X_2^{(i)})$ is a Roman dominating function on $H$. Thus,

$$\gamma_R(H) \leq 2|X_2^{(i)}| + |X_1^{(i)}| + |Y_0^{(i)}|$$
$$\leq 2|B_2 \cap \Pi_i| + |B_1 \cap \Pi_i| + |Y_0^{(i)}|.$$

Hence,

$$\gamma_R(G \Box H) = 2|B_2| + |B_1|$$
$$= \sum_{i=1}^{\gamma(G)} (2|B_2 \cap \Pi_i| + |B_1 \cap \Pi_i|)$$
$$\geq \sum_{i=1}^{\gamma(G)} (\gamma_R(H) - |Y_0^{(i)}|)$$
$$= \gamma(G)\gamma_R(H) - \sum_{i=1}^{\gamma(G)} |Y_0^{(i)}|.$$

So,

$$\sum_{i=1}^{\gamma(G)} |Y_0^{(i)}| \geq \gamma(G)\gamma_R(H) - \gamma_R(G \Box H). \quad (2)$$

Now, for every $v \in V_2$, let $Z^v \in \{0, 1\}^{\gamma(G)}$ be a binary vector associated to $v$ as follows: $Z_i^v = 1$ if $v \in Y_i^{(i)}$ and $Z_i^v = 0$ if $v \not\in Y_i^{(i)}$. So, $t_v = \|Z^v\|^2$ counts the number of components of $Z^v$ equal to one. Hence,

$$\sum_{v \in V_2} t_v = \sum_{i=1}^{\gamma(G)} |Y_0^{(i)}|. \quad (3)$$

Notice that, if $Z_i^v = 1$ and $u \in A_i$, then vertex $(u, v)$ belongs to $B_0$. Moreover, $(u, v)$ is not adjacent to vertices of $B_2 \cap \Pi_i$. So, since $B_0$ is dominated by $B_2$, there exists $u' \in X_v = \{x \in V_1 : (x, v) \in B_2\}$ which is adjacent to $u$. Hence, $S_v = (S - \{u_i \in S : Z_i^v = 1\}) \cup X_v$ is a dominating set for $G$.

\[\text{\footnotesize Notice that this partition always exists, and it could be not unique.}\]
Now, if \( t_v > |X_v| \), then we have

\[
|S_v| = |S| - t_v + |X_v| \\
= \gamma(G) - t_v + |X_v| \\
< \gamma(G) - t_v + t_v \\
= \gamma(G),
\]

which is a contradiction. So, we have \( t_v \leq |X_v| \) and we obtain

\[
\sum_{v \in V_2} t_v \leq \sum_{v \in V_2} |X_v| = |B_2|,
\]

which leads to,

\[
2 \sum_{v \in V_2} t_v \leq 2|B_2| + |B_1| = \gamma_R(G \boxtimes H).
\]

Thus, by (2), (3) and (5) we deduce

\[
\gamma_R(G \boxtimes H) \geq \gamma(G) \gamma_R(H) - \frac{\gamma_R(G \boxtimes H)}{2},
\]

and, as a consequence, (i) follows.

Now, By Lemma 2 (i) and (4) we have

\[
\gamma_R(G \boxtimes H) \geq \gamma(G) \gamma_R(H) - \frac{\gamma_R(G \boxtimes H)}{2},
\]

which leads to,

\[
2 \sum_{v \in V_2} t_v \leq 2|B_2| + |B_1| = \gamma_R(G \boxtimes H).
\]

Thus, by (2), (3) and (6) we obtain (ii).

Thus, by (2), (3) and (6) we obtain (ii).

Lemma 1 and Theorem 3 lead to the following result.

**Corollary 4.** For any graphs \( G \) and \( H \),

(i) \( \gamma_R(G \boxtimes H) \geq \frac{\gamma_R(G) \gamma_R(H)}{3} \).

(ii) \( \gamma(G \boxtimes H) \geq \frac{\gamma(G) \gamma_R(H)}{3} \).

Note that if there exists a graph that satisfies the above equalities, then Vizing’s conjecture is false.

The following inequality related to Vizing’s conjecture was obtained in [2]:

\[
\gamma(G \boxtimes H) \geq \frac{\gamma(G) \gamma(H)}{2}.
\]

As the following Remark shows, if \( \gamma_R(H) > \frac{3 \gamma(H)}{2} \), then Corollary 4 (ii) leads to a result which improves the above inequality.
Remark 5. Let $G$ and $H$ be two graphs. If $\gamma_R(H) > \frac{3\gamma(H)}{2}$, then

$$\gamma(G \Box H) \geq \frac{\gamma(G)\gamma(H)}{2} + \frac{\gamma(G)}{3}.$$ 

A graph $H$ is a Roman graph if $\gamma_R(H) = 2\gamma(H)$. Roman graphs were introduced in [3] where the authors presented some classes of Roman graphs and they proposed some open problems. Theorem 3(ii) leads to the following result.

Corollary 6. For any graph $G$ and any Roman graph $H$,

(i) $\gamma_R(G \Box H) \geq \frac{4}{3}\gamma(G)\gamma(H)$.

(ii) $\gamma(G \Box H) \geq \frac{2}{3}\gamma(G)\gamma(H)$.

Let $\mathcal{F}$ be the class of all graphs having a dominating set $S = \{u_1, u_2, ..., u_{\gamma(G)}\}$ such that $N[u_i] \cap N[u_j] = \emptyset$, for every $i, j \in \{1, ..., \gamma(G)\}$, $i \neq j$. In this case the set $S$ is called an efficient dominating set. Notice that $\mathcal{F}$ is the family of all graphs having a perfect code. Examples of graphs belonging to $\mathcal{F}$ are the path graphs $P_n$, the cycle graphs $C_{3k}$ and the cube graph $Q_3 = K_2 \Box K_2 \Box K_2$. Examples of Roman graphs belonging to $\mathcal{F}$ are $C_{3k}$, $P_{3k+1}$, $P_{3k+2}$ and $Q_3$. Note that $P_{3k+1} \in \mathcal{F}$ but $P_{3k+2}$ are not Roman paths, while $C_{3k+2}$ are Roman cycles but $C_{3k+2} \not\in \mathcal{F}$.

A 2-packing of a graph $G$ is a set of vertices in $G$ that are pair-wise at distance more than two. The 2-packing number $P_2(G)$ of a graph $G$ is the size of a largest 2-packing in $G$. The 2-packing number is a graph invariant closely related to the domination number. In fact, it is well known that $P_2(G) \leq \gamma(G)$, cf. \[10\] \[11\].

Let $G \in \mathcal{F}$. Since every efficient dominating set $S = \{u_1, u_2, ..., u_{\gamma(G)}\}$ is a 2-packing, we have $\gamma(G) \leq P_2(G)$. So, we conclude that if $G \in \mathcal{F}$, then $P_2(G) = \gamma(G)$ (The converse is not true). We recall that if $P_3(G) = \gamma(G)$, then Vizing’s conjecture holds for $G$ \[11\]. As a consequence, by Theorem 3(ii) we deduce the following result which improves the inequality \[11\] when $G \in \mathcal{F}$.

Corollary 7. Let $G$ and $H$ be two graphs. If $G \in \mathcal{F}$, then

$$\gamma_R(G \Box H) \geq \frac{1}{2}\max \{\gamma(G) (\gamma_R(H) + \gamma(H)), \gamma(H) (\gamma_R(G) + \gamma(G))\}.$$ 

Theorem 8. Let $G$ and $H$ be two graphs. If $G \in \mathcal{F}$, then

$$\gamma_R(G \Box H) \geq \gamma(G)\gamma_R(H).$$

Proof. Let $V_1$ and $V_2$ be the vertex sets of $G$ and $H$, respectively. Let $S = \{u_1, u_2, ..., u_{\gamma(G)}\}$ be an efficient dominating set for $G$, i.e., \{\$N[u_1], N[u_2], ..., N[u_{\gamma(G)}]\} is a vertex partition of $G$ and, as a consequence, \{\$\Pi_1', \Pi_2', ..., \Pi_{\gamma(G)}'\} is a vertex partition of $G \Box H$, where $\Pi_i' = N[u_i] \times V_2$ for every $i \in \{1, ..., \gamma(G)\}$.

Proceeding analogously to the proof of Theorem 3 we consider a $\gamma_R(G \Box H)$-function $f = (B_0, B_1, B_2)$ and, for every $i \in \{1, ..., \gamma(G)\}$, we define the function $f_i : V_2 \to \{0, 1, 2\}$ as $f_i(v) = \begin{cases} 0 & \text{if } v \in B_0, \\ 1 & \text{if } v \in B_1, \\ 2 & \text{if } v \in B_2. \end{cases}$
max\{f(u, v) : u \in N[u_i]\}. In addition, for every j \in \{0, 1, 2\} we define X_j^{(i)} = \{v \in V_2 : f_i(v) = j\}.

Now, if v \in X_0^{(i)} then for every u \in N[u_i] we have that (u, v) \in B_0. Hence, since \( u_i \) has no neighbors in \( V_1 - N[u_i] \) and \( B_2 \) dominates \( B_0 \), there exists \( (u_i, v') \in B_2 \) such that \( v' \) is adjacent to \( v \). We conclude that every \( v \in X_0^{(i)} \) has a neighbor \( v' \in X_2^{(i)} \) and, as a consequence, \( f_i = (X_0^{(i)}, X_1^{(i)}, X_2^{(i)}) \) is a Roman dominating function on \( H \), for every \( i \in \{1, ..., \gamma(G)\} \). Therefore, the result is deduced as follows:

$$\gamma_R(G \square H) = 2|B_2| + |B_1|$$

$$= \sum_{i=1}^{\gamma(G)} (2|B_2 \cap \Pi_i| + |B_1 \cap \Pi_i|)$$

$$\geq \sum_{i=1}^{\gamma(G)} \left( 2|X_2^{(i)}| + |X_1^{(i)}| \right)$$

$$\geq \gamma(G)\gamma_R(H).$$

An interesting consequence of Theorem 8 is the following result.

**Corollary 9.** Let \( G \) and \( H \) be two graphs. If \( G \in \mathcal{F} \) and \( H \) is a Roman graph, then

$$\gamma_R(G \square H) \geq 2\gamma(G)\gamma(H).$$

**Theorem 10.** For any graphs \( G \) and \( H \) of order \( n_1 \) and \( n_2 \), respectively,

$$\gamma_R(G \square H) \leq \min\{n_1\gamma_R(H), n_2\gamma_R(G)\}.$$ 

**Proof.** Let \( f_1 \) be a \( \gamma_R(G) \)-function. Let \( f : V_1 \times V_2 \rightarrow \{0, 1, 2\} \) be a function defined by \( f(u, v) = f_1(u) \). If there exists a vertex \( (x, y) \in V_1 \times V_2 \) such that \( f(x, y) = 0 \), then \( f_1(x) = 0 \). Since \( f_1 \) is Roman, there exists \( u \in V_1 \), adjacent to \( x \), such that \( f_1(u) = 2 \). Hence, we obtain that \( f(u, y) = 2 \) and \( (x, y) \) is adjacent to \( (u, y) \). So, \( f \) is a Roman dominating function. Therefore,

$$\gamma_R(G \square H) \leq \sum_{(u,v) \in V_1 \times V_2} f(u, v) = \sum_{v \in V_2} \sum_{u \in V_1} f_1(u) = \sum_{v \in V_2} \gamma_R(G) = n_2\gamma_R(G).$$

Analogously we obtain that \( \gamma_R(G \square H) \leq n_1\gamma_R(H) \) and the result follows.

The above inequality is tight. It is achieved, for instance, for \( G = P_n \), a path graph of order \( n \), and \( H = S_{1,r} \), a star graph with \( r \geq 2 \) leaves. In this case we have \( \gamma_R(S_{1,r}) = 2 = 2\gamma(S_{1,r}) \), \( \gamma(P_n) = \left\lceil \frac{n}{3} \right\rceil \), \( \gamma(R(P_n)) = \frac{2n+1}{3} \) if \( n \equiv 1(3) \) and \( \gamma(R(P_n)) = 2\left\lceil \frac{n}{3} \right\rceil \) if \( n \not\equiv 1(3) \). So, \( \gamma_R(G \square H) = 2n = n\gamma_R(H) \).

**Corollary 11.** For any graphs \( G \) and \( H \) of order \( n_1 \) and \( n_2 \), respectively,

$$\gamma_R(G \square H) \leq 2\min\{n_1\gamma(H), n_2\gamma(G)\}. $$
Lemma 12. [3] A graph $G$ is Roman if and only if it has a $\gamma_R(G)$-function $f = (A_0, A_1, A_2)$ with $|A_1| = 0$.

Theorem 13. Let $G$ be a graph of order $n$ and let $H$ be a graph.

(i) If $G$ has at least one connected component of order greater than two, then

$$\gamma_R(G\Box H) \leq (n+1)\gamma_R(H) - 2\gamma(H).$$

(ii) If $G$ is a Roman graph, then

$$\gamma_R(G\Box H) \leq 2n(\gamma_R(H) - \gamma(H)) + 2\gamma(G)(2\gamma(H) - \gamma_R(H)).$$

Proof. Let $f_1 = (A_0, A_1, A_2)$ be a $\gamma_R(G)$-function and let $f_2 = (B_0, B_1, B_2)$ be $\gamma_R(H)$-function. We define the map $f : V_1 \times V_1 \rightarrow \{0, 1, 2\}$ as follows.

- $f(u, v) = f_2(v)$ for every $(u, v) \notin (A_0 \cup A_2) \times B_1$.
- If $(u, v) \in A_0 \times B_1$, then $f(u, v) = 0$.
- If $(u, v) \in A_2 \times B_1$, then $f(u, v) = 2$.

Since every vertex from $A_0 \times B_1$ has a neighbor in $A_2 \times B_1$ and every vertex of $V_1 \times V_0$ has a neighbor in $V_1 \times B_2$, we have that $f$ is a Roman dominating function on $G\Box H$. Thus,

$$\gamma_R(G\Box H) \leq n\gamma_R(H) - |A_0||B_1| + |A_2||B_1| = n\gamma_R(H) - |B_1|(|A_0| - |A_2|).$$

Since $G$ has at least one connected component of order greater than two, it is satisfied that $|A_0| \geq |A_2| + 1$ and, by Lemma [2] (ii), $|B_1|(|A_0| - |A_2|) \geq 2\gamma(H) - \gamma_R(H)$. Therefore, by (8) we deduce (i).

Now, if $G$ is a Roman graph, then by Lemma [12] there exists a $\gamma_R(G)$-function $f = (A_0, A_1, A_2)$ with $|A_1| = 0$. Thus, $|A_0| + |A_2| = n$ and, as a consequence, $|A_0| - |A_2| = n - 2\gamma(G)$. Therefore, by (8) we deduce (ii):

$$\gamma_R(G\Box H) \leq n\gamma_R(H) - |B_1|(|A_0| - |A_2|)
\leq n\gamma_R(H) - (2\gamma(H) - \gamma_R(H))(n - 2\gamma(G))
= 2n(\gamma_R(H) - \gamma(H)) + 2\gamma(G)(2\gamma(H) - \gamma_R(H)).$$

For any Roman graph $H$, Theorem [13] leads to $\gamma_R(G\Box H) \leq 2n\gamma(H)$. Now, for any non-Roman graph $H$ we have $\gamma_R(H) - 2\gamma(H) \leq -1$ and, as a consequence, Theorem [13] leads to the following result.

Corollary 14. Let $G$ be a graph of order $n$ and let $H$ be a graph. If $G$ has at least one connected component of order greater than two and $H$ is not Roman, then

$$\gamma_R(G\Box H) \leq n\gamma_R(H) - 1.$$
Proposition 15. \cite{3} If \( G \) is a connected graph of order \( n \), then \( \gamma_R(G) = \gamma(G) + 1 \) if and only if there exists a vertex of \( G \) of degree \( n - \gamma(G) \).

From Proposition 15 and Theorem 13 we derive the following result.

Proposition 16. If \( G \) is a graph of order \( n_1 \) having at least one connected component of order greater than two and \( H \) is a connected graph of order \( n_2 \) having a vertex of degree \( n_2 - \gamma(H) \), then

\[
\gamma_R(G \square H) \leq n_1(\gamma(H) + 1) - \gamma(H) + 1.
\]

The above inequality is tight. For instance, if \( G \) is a path graph of order three and \( H \) is the star \( K_{1,3} \) with one of its edges subdivided, then we have \( \gamma(H) = 2 \) and \( \gamma_R(G \square H) = 8 \). So, Proposition 16 leads to the exact value of \( \gamma_R(G \square H) \).

Theorem 17. For any graphs \( G \) and \( H \) of order \( n_1 \) and \( n_2 \), respectively,

\[
\gamma_R(G \square H) \leq 2\gamma(G)\gamma(H) + (n_1 - \gamma(G))(n_2 - \gamma(H)).
\]

Proof. Let \( S_1 \) be a \( \gamma(G) \)-set and let \( S_2 \) be a \( \gamma(H) \)-set. Let \( B_2 = S_1 \times S_2 \), \( B_1 = (V_1 - S_1) \times (V_2 - S_2) \) and \( B_0 = S_1 \times (V_2 - S_2) \cup (V_1 - S_1) \times S_2 \). Since \( B_2 \) dominates \( B_0 \), the map \( f : V_1 \times V_2 \to \{0, 1, 2\} \) defined by \( f(u, v) = i \), for every \( (u, v) \in B_i \), is a Roman function on \( G \square H \). Therefore, the result is obtained as follows,

\[
\gamma_R(G \square H) \leq 2|B_2| + |B_1|
= 2|S_1||S_2| + |V_1 - S_1||V_2 - S_2|
= 2\gamma(G)\gamma(H) + (n_1 - \gamma(G))(n_2 - \gamma(H)).
\]

\[\square\]

We know that \( \gamma_R(P_{3k+2}) = 2\gamma(P_{3k+2}) = 2(k + 1) \), \( \gamma_R(P_{3k+1}) = 2k + 1 \) and \( \gamma(P_{3k+1}) = k + 1 \). So, Theorem 17 leads to \( \gamma_R(P_{3k+1} \square P_{3k+2}) \leq 6k^2 + 6k + 2 \), while by Theorem 13 we only get \( \gamma_R(P_{3k+1} \square P_{3k+2}) \leq 6k^2 + 7k + 2 \) and by Theorem 13 we only get \( \gamma_R(P_{3k+2} \square P_{3k+1}) \leq 6k^2 + 7k + 1 \).

From the above results we have that bounds on the Roman domination number and the domination number of the factor graphs lead to bounds on the Roman domination number of Cartesian product graphs. For example, it is well-known that for any graph \( G \) of order \( n \) and maximum degree \( \Delta \) it follows \( \gamma(G) \geq \frac{n}{\Delta + 1} \), cf. \cite{7}. The following straightforward result allow us to derive several bounds on \( \gamma_R(G \square H) \).

Remark 18. For any graph \( G \in \mathcal{G} \) of order \( n \) and minimum degree \( \delta \), \( \gamma(G) \leq \frac{n}{\delta + 1} \). As a consequence, for any \( \delta \)-regular graph \( G \in \mathcal{G} \) it follows, \( \gamma(G) = \frac{n}{\delta + 1} \).

An example of result derived from the above remark, Theorem 8 and Theorem 10, is the following one.

Proposition 19. For any \( \delta \)-regular graph \( G \in \mathcal{G} \) of order \( n \),

\[
\frac{2n}{\delta + 1} \leq \gamma_R(G \square K_2) \leq \frac{4n}{\delta + 1}.
\]
3 Strong product graphs

In this section we obtain some results on the Roman domination number of strong product graphs. To begin with, we recall the following well-known result, cf. [11].

**Theorem 20.** [11] For any graphs $G$ and $H$,

$$\max\{P_2(G)\gamma(H), \gamma(G)P_2(H)\} \leq \gamma(G \boxtimes H) \leq \gamma(G)\gamma(H).$$

One immediate consequence of Theorem 20 is the following result.

**Corollary 21.** For any graph $G \in \mathcal{G}$ and any graph $H$, $\gamma(G \boxtimes H) = \gamma(G)\gamma(H)$.

The next result follows from Lemma 1 and Theorem 20.

**Corollary 22.** For any graphs $G$ and $H$,

$$\max\{P_2(G)\gamma(H), \gamma(G)P_2(H)\} \leq \gamma_R(G \boxtimes H) \leq 2\gamma(G)\gamma(H).$$

**Theorem 23.** Let $f_1 = (A_0, A_1, A_2)$ be a $\gamma_R(G)$-function and let $f_2 = (B_0, B_1, B_2)$ be a $\gamma_R(H)$-function. Then,

$$\gamma_R(G \boxtimes H) \leq \gamma_R(G)\gamma_R(H) - 2|A_2||B_2|.$$  

**Proof.** We define the function $f$ on $G \boxtimes H$ as follows:

$$f(u, v) = \begin{cases} 
2, & (u, v) \in (A_1 \times B_2) \cup (A_2 \times B_1) \cup (A_2 \times B_2), \\
1, & (u, v) \in A_1 \times B_1, \\
0, & \text{otherwise}.
\end{cases}$$

Note that the set $(A_0 \times B_0) \cup (A_0 \times B_2) \cup (A_2 \times B_0)$ is dominated by $A_2 \times B_2$, the set $A_1 \times B_0$ is dominated by $A_1 \times B_2$, and $A_0 \times B_1$ is dominated by $A_2 \times B_1$. Then we have that $f$ is a Roman dominating function on $G \boxtimes H$.

Therefore,

$$\gamma_R(G \boxtimes H) \leq 2|A_2||B_2| + 2|A_1||B_2| + 2|A_2||B_1| + |A_1||B_1| - 2|A_2||B_2|$$

$$= 4|A_2||B_2| + 2|A_1||B_2| + 2|A_2||B_1| + |A_1||B_1| - 2|A_2||B_2|$$

$$= 2|A_2|(2|B_2| + |B_1|) + |A_1||B_2| + |B_1| - 2|A_2||B_2|$$

$$= (2|A_2| + |A_1|)(2|B_2| + |B_1|) - 2|A_2||B_2|$$

$$= \gamma_R(G)\gamma_R(H) - 2|A_2||B_2|.$$

Now we present some interesting consequences of Theorem 23.

**Corollary 24.** For any non-empty graphs $G$ and $H$, $\gamma_R(G \boxtimes H) \leq \gamma_R(G)\gamma_R(H) - 2.$
The above inequality is achieved, for instance, if $G$ and $H$ are graphs of order $n_1$ and $n_2$, containing a vertex of degree $n_1 - 1$ and $n_2 - 1$, respectively. In such a case, we have $\gamma_R(G \Box H) \leq \gamma_R(G)\gamma_R(H) - 2 = 2 \cdot 2 - 2 = 2$.

In order to show one example where Corollary 24 leads to better result than Corollary 22 we take a graph $G$ such that $\gamma_R(G) = \gamma(G) + 1 > 3$ (see Proposition 15). In this case Corollary 24 leads to $\gamma_R(G \Box G) \leq (\gamma(G))^2 + 2\gamma(G)$, while Corollary 22 leads to $\gamma_R(G \Box G) \leq 2(\gamma(G))^2$.

If $H = P_n$ or $H = C_n$, then we have that for any $\gamma_R(H)$-function $f = (B_0, B_1, B_2)$, $|B_2| = \left\lceil \frac{n}{3} \right\rceil$. Hence, Theorem 23 leads to the following result.

**Corollary 25.** Let $G$ be a non-empty graph. If $H = P_n$ or $H = C_n$, then

$$\gamma_R(G \Box H) \leq \begin{cases} 
\frac{2n+1}{3} \gamma_R(G) - 2 \left\lceil \frac{n}{3} \right\rceil, & n \equiv 1(3) \\
2 \left\lceil \frac{n}{3} \right\rceil \gamma_R(G) - 2 \left\lceil \frac{n}{3} \right\rceil, & n \not\equiv 1(3).
\end{cases}$$

Every star graph $G = K_{1,r}$ satisfies the above equality for $n \neq 2(3)$. In such a case we have $\gamma_R(C_n \Box K_{1,r}) = \gamma_R(P_n \Box K_{1,r}) = 2 \left\lceil \frac{n}{3} \right\rceil$. Note that $C_n \Box K_{1,r}$ and $P_n \Box K_{1,r}$ are Roman graphs for $n \neq 2(3)$.

**Theorem 26.** Let $G$ and $H$ be two graphs. If $G \in \mathbb{R}$, then $\gamma_R(G \Box H) \geq \gamma(G)\gamma_R(H)$

**Proof.** Let $V_1$ and $V_2$ be the vertex sets of $G$ and $H$, respectively. Let $S = \{u_1, u_2, ..., u_{\gamma(G)}\}$ be an efficient dominating set for $G$, i.e., $\{N_G[u_1], N_G[u_2], ..., N_G[u_{\gamma(G)}]\}$ is a vertex partition for $G$. Let $\{\Pi_1, \Pi_2, ..., \Pi_{\gamma(G)}\}$ be the vertex partition of $G \Box H$ defined as $\Pi_i = N_G[u_i] \times V_2$, for every $i \in \{1, ..., \gamma(G)\}$.

Now, let $f = (B_0, B_1, B_2)$ be a $\gamma_R(G \Box H)$-function and, for every $i \in \{1, ..., \gamma(G)\}$, let the function $f^{(i)} : V_2 \rightarrow \{0, 1, 2\}$ defined by $f^{(i)}(v) = \max\{f(u, v) : (u, v) \in \Pi_i\}$. Let $\{B^{(i)}, B^{(i)}_1, B^{(i)}_2\}$ such that $B^{(i)}_j = \{v \in V_2 : f^{(i)}(v) = j\}$ with $j \in \{0, 1, 2\}$ and $i \in \{1, ..., \gamma(G)\}$.

If there is a vertex $y$ of $H$ such that $f^{(i)}(y) = 0$ and $N_H[y] \cap B^{(i)}_2 = \emptyset$, then $f(u_i, y) = 0$ and $(u_i, y)$ is not adjacent to any vertex $(a, b)$ of $G \Box H$ with $f(a, b) = 2$, a contradiction. Thus, $f^{(i)} = (B^{(i)}, B^{(i)}_1, B^{(i)}_2)$ is a Roman dominating function on $H$ for every $i \in \{1, ..., \gamma(G)\}$. As a consequence,

$$\gamma_R(G \Box H) = 2|B_2| + |B_1|$$

$$= \sum_{i=1}^{\gamma(G)} (2|B_2 \cap \Pi_i| + |B_1 \cap \Pi_i|)$$

$$\geq \sum_{i=1}^{\gamma(G)} (2|B^{(i)}_2| + |B^{(i)}_1|)$$

$$\geq \sum_{i=1}^{\gamma(G)} \gamma_R(H)$$

$$= \gamma(G)\gamma_R(H).$$

Therefore, the proof is complete. \qed
Cockayne et al. [3] gave some classes of Roman graphs and they posed the following question: Can you find other classes of Roman graphs? The next result is an answer to this question.

**Theorem 27.** If $G \in \mathcal{F}$ and $H$ is a Roman graph, then $G \boxtimes H$ is a Roman graph.

**Proof.** If $G \in \mathcal{F}$ and $H$ is Roman, then Theorem 26 leads to $\gamma_R(G \boxtimes H) \geq 2\gamma(G)\gamma(H)$. So, by Corollary 22 we obtain $\gamma_R(G \boxtimes H) = 2\gamma(G)\gamma(H)$. Hence, by Corollary 21 we conclude the proof. \[\square\]

**References**

[1] B. Brešar, P. Dorbec, W. Goddard, B. L. Hartnell, M. A. Henning, S. Klavžar, D. F. Rall, Vizing’s conjecture: a survey and recent results, *Journal of Graph Theory*, to appear. www.imfm.si/preprinti/PDF/01099.pdf.

[2] W. E. Clark, S. Suen, An inequality related to Vizing’s conjecture, *The Electronic Journal of Combinatorics* 7 (2000), no. 1, Note 4, 3 pp.

[3] E. J. Cockayne, P. A. Dreyer, S M. Hedetniemi, S. T. Hedetniemi, Roman domination in graphs, *Discrete Mathematics* 278 (1-3) (2004) 11–22.

[4] P. A. Dreyer, *Applications and variations of domination in graphs*. Ph. D. Thesis. New Brunswick, New Jersey, 2000.

[5] O. Favaron, H. Karamic, R. Khoeilar, S. M. Sheikholeslami, On the Roman domination number of a graph, *Discrete Mathematics* 309 (2009) 3447–3451.

[6] B. Hartnell, D. F. Rall, Domination in Cartesian products: Vizing’s conjecture. In *Domination in graphs*, volume 209 of *Monography Textbooks Pure Applied Mathematics*, pages 163–189. Marcel Dekker, New York, 1998.

[7] T. W. Haynes, S. T. Hedetniemi, P. J. Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker, Inc. New York, 1998.

[8] S. H. Hedetniemi, Homomorphisms of graphs and automata, University of Michigan Technical Report 03105-44-T, 1966.

[9] M. A. Henning, Defending the Roman Empire from multiple attacks *Discrete Mathematics* 271 (2003) 101–115.

[10] W. Imrich, S. Klavžar, D. F. Rall. *Topics in Graph Theory*. A K Peters Ltd., Wellesley, MA, 2008.

[11] W. Imrich, S. Klavžar. *Product Graphs: structure and recognition*. Wiley-Interscience, New York, USA, 2000.

[12] C. E. Shannon, The zero-error capacity of a noisy channel, *IRE Transactions on Information Theory* 2 (3) (1956) 8–19.
[13] I. Stewart, Defend the Roman Empire, *Scientific American*, December (1999) 136–138.

[14] V. G. Vizing, The Cartesian product of graphs, *Vyčisl. Sistemy* 9 (1963) 30–43.

[15] Y. Wu, An Improvement on Vizing’s conjecture, manuscript. [http://arxiv.org/PS_cache/arxiv/pdf/0909/0909.3695v1.pdf](http://arxiv.org/PS_cache/arxiv/pdf/0909/0909.3695v1.pdf)

[16] F. Xuejiang, Y. Yuansheng, J. Bao, Roman domination in regular graphs, *Discrete Mathematics* 309 (6) (2009) 1528–1537.