THE BLOW-UP PROBLEM FOR A SEMILINEAR PARABOLIC EQUATION WITH A POTENTIAL

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Abstract. Let $\Omega$ be a bounded smooth domain in $\mathbb{R}^N$. We consider the problem $u_t = \Delta u + V(x)u^p$ in $\Omega \times [0,T)$, with Dirichlet boundary conditions $u = 0$ on $\partial \Omega \times [0,T)$ and initial datum $u(x,0) = M\varphi(x)$ where $M \geq 0$, $\varphi$ is positive and compatible with the boundary condition. We give estimates for the blow up time of solutions for large values of $M$. As a consequence of these estimates we find that, for $M$ large, the blow up set concentrates near the points where $\varphi^{p-1}V$ attains its maximum.

1. INTRODUCTION

In this paper we study the blow-up phenomena for the following semilinear parabolic problem with a potential

$$u_t = \Delta u + V(x)u^p \quad \text{in} \quad \Omega \times (0,T),$$

$$u(x,t) = 0 \quad \text{on} \quad \partial \Omega \times (0,T),$$

$$u(x,0) = M\varphi(x) \quad \text{in} \quad \Omega.$$  

First, let us state our basic assumptions. They are: $\Omega$ is a bounded, convex, smooth domain in $\mathbb{R}^N$ and the exponent $p$ is subcritical, that is, $1 < p < (N+2)/(N-2)$. The potential $V$ is Lipschitz continuous and there exists a constant $c > 0$ such that $V(x) \geq c$ for all $x \in \Omega$. As for the initial condition we assume that $M \geq 0$ and that $\varphi$ is a smooth positive function compatible with the boundary condition. Moreover, we impose that

$$M\Delta \varphi + \min_{x\in\Omega} V(x) \geq 0.$$  

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We note that (1.2) holds for $M$ large if $\Delta \varphi$ is nonnegative in a neighborhood of the set where $\varphi$ vanishes.

It is known that, and we will prove it later for the sake of completeness, once $\varphi$ is fixed the solution to (1.1) blows up in finite time for any $M$ sufficiently large. By this we understand that there exists a time $T = T(M)$ such that $u$ is defined in $\Omega \times [0, T)$ and
\[
\lim_{t \to T} \|u(\cdot, t)\|_{L^\infty(\Omega)} = +\infty.
\]

The study of the blow-up phenomena for parabolic equations and systems has attracted considerable attention in recent years, see for example, [B], [BB], [GK1], [GK2], [GV], [HV1], [HV2], [M], [Z] and the corresponding references. A good review in the topic can be found in [GV2]. When a large or small diffusion is considered, see [IY], [MY].

Important issues in a blow-up problem are to obtain estimates for the blow-up time, $T(M)$, and determine the spatial structure of the set where the solution becomes unbounded, that is, the blow-up set. More precisely, the blow-up set of a solution $u$ that blows up at time $T$ is defined as
\[
B(u) = \{x/ \text{there exist } x_n \to x, t_n \nearrow T, \text{ with } u(x_n, t_n) \to \infty\}.
\]

The problem of estimating the blow-up time and the description and location of the blow-up set has proved to be a subtle problem and has been addressed by several authors. See for example [SGKM], [GV2] and the corresponding bibliographies.

Our interest here is the description of the asymptotic behavior of the blow-up time, $T(M)$, and of the blow-up set, $B(u)$, as $M \to \infty$. It turns out that their asymptotics depend on a combination of the shape of both the initial condition, $\varphi$, and the potential $V$. Roughly speaking one expects that if $\varphi \equiv 1$ then the blow-up set should concentrate near the points where $V$ attains its maximum. On the other hand if $V \equiv 1$ the blow-up set should be near the points where $\varphi$ attains its maximum. Just to see what to expect, if we drop the laplacian, we get the ODE $u_t = V(x) u^p$ with initial condition $u(x, 0) = M \varphi(x)$. Here $x$ plays the role of a parameter. Direct integration gives $u(x, t) = C(T - t)^{-1/(p-1)}$ with
\[
T = \frac{M^{1-p}}{(p - 1)V(x)\varphi^{p-1}(x)}.
\]

Hence, blow-up takes place at points $x_0$ that satisfy $V(x_0)\varphi^{p-1}(x_0) = \max_x V(x)\varphi^{p-1}$. Therefore, we expect that the quantity that plays a major role is $(\max_x V(x)\varphi^{p-1}(x))$. 
Theorem 1.1. There exists \( \bar{M} > 0 \) such that if \( M \geq \bar{M} \) the solution of (1.1) blows up in a finite time that we denote by \( T(M) \). Moreover, let

\[
A = A(\varphi, V) := \frac{1}{(\max_x \varphi^{p-1}(x)V(x))}
\]

then there exist two positive constants \( C_1, C_2 \), such that, for \( M \) large enough,

\[
(1.3) \quad -\frac{C_1}{M^{\frac{p-1}{4}}} \leq T(M)M^{p-1} - \frac{A}{p-1} \leq \frac{C_2}{M^{\frac{p-1}{4}}},
\]

and the blow-up set verifies,

\[
(1.4) \quad \varphi^{p-1}(a)V(a) \geq \frac{1}{A} - \frac{C}{M^{\gamma}}, \quad \text{for all } a \in B(u),
\]

where \( \gamma = \min\left(\frac{p-1}{4}, \frac{1}{3}\right) \).

Note that this result implies that

\[
\lim_{M \to \infty} T(M)M^{p-1} = \frac{A}{p-1}.
\]

Moreover, it provides precise lower and upper bounds on the difference \( T(M)M^{p-1} - \frac{A}{p-1} \).

We also observe that (1.4) shows that the set of blow-up points concentrates for large \( M \) near the set where \( \varphi^{p-1}V \) attains its maximum.

If in addition the potential \( V \) and the initial datum \( \varphi \) are such that \( \varphi^{p-1}V \) has a unique non-degenerate maximum at a point \( \bar{a} \), then there exist constants \( c > 0 \) and \( d > 0 \) such that

\[
\varphi^{p-1}(\bar{a})V(\bar{a}) - \varphi^{p-1}(x)V(x) \geq c|\bar{a} - x|^2 \quad \text{for all } x \in B(\bar{a}, d).
\]

Therefore, according to our result, if \( M \) is large enough one has

\[
|\bar{a} - a| \leq \frac{C}{M^{\gamma}} \quad \text{for any } a \in B(u),
\]

with \( \gamma = \min\left(\frac{p-1}{4}, \frac{1}{3}\right) \).

Throughout the paper we will denote by \( C \) a constant that does not depend on the relevant parameters involved but may change at each step.

2. Proof of Theorem 1.1

We begin with a lemma that provides us with an upper estimate of the blow-up time. This upper estimate gives the upper bound for \( T(M)M^{p-1} \) in (1.3) and will be crucial in the rest of the proof of Theorem 1.1.
Lemma 2.1. There exist a constant $C > 0$ and $M_0 > 0$ such that for every $M \geq M_0$, the solution of (1.1) blows up in a finite time that verifies

$$T(M) \leq A \frac{1}{M^{p-1}(p-1)} + C \frac{1}{M^{\frac{p-1}{3}}M^{p-1}}. \tag{2.1}$$

Proof: Let $\bar{a} \in \Omega$ be such that

$$\varphi^{p-1}(\bar{a})V(\bar{a}) = \max_x \varphi^{p-1}(x)V(x),$$

$L$ the constant of Lipschitz continuity of $V$, and $K$ an upper bound for the first derivatives of $\varphi$ and $L$.

In order to get the upper estimate let $M$ be fixed and $\varepsilon = \varepsilon(M) > 0$ to be defined latter, small enough so all functions involved are well defined. Pick

$$\delta = \frac{\varepsilon}{2K},$$

then

$$V(x) \geq V(\bar{a}) - \frac{\varepsilon}{2} \quad \text{and} \quad \varphi(x) \geq \varphi(\bar{a}) - \varepsilon \quad \text{for all } x \in B(\bar{a}, \delta).$$

Let $w$ be the solution of

$$w_t = \Delta w + \left(V(\bar{a}) - \frac{\varepsilon}{2}\right)w^p \quad \text{in } B(\bar{a}, \delta) \times (0, T_w),$$

$$w = 0 \quad \text{on } \partial B(\bar{a}, \delta) \times (0, T_w),$$

$$w(x, 0) = M(\varphi(\bar{a}) - \varepsilon), \quad \text{in } B(\bar{a}, \delta)$$

and $T_w$ its corresponding blow up time. A comparison argument shows that $u \geq w$ in $B(\bar{a}, \delta) \times (0, T)$ and hence

$$T \leq T_w.$$

Our task now is to estimate $T_w$ for large values of $M$. To this end, let $\lambda_1(\delta)$ be the first eigenvalue of $-\Delta$ in $B(\bar{a}, \delta)$ and let $\phi_1$ be the corresponding positive eigenfunction normalized so that

$$\int_{B(\bar{a}, \delta)} \phi_1(x) \, dx = 1.$$
Then $\Phi(t)$ satisfies $\Phi(0) = M(\varphi(\bar{a}) - \varepsilon)$ and
\[
\Phi'(t) = \int_{B(\bar{a}, \delta)} w(x, t) \phi_1(x) \, dx \\
= \int_{B(\bar{a}, \delta)} \left( \Delta w(x, t) \phi_1(x) + \left( V(x_1) - \frac{\varepsilon}{2} \right) w^p(x, t) \phi_1(x) \right) \, dx \\
\geq -\lambda_1(\delta) \int_{B(\bar{a}, \delta)} w(x, t) \phi_1(x) \, dx \\
+ \left( V(\bar{a}) - \frac{\varepsilon}{2} \right) \left( \int_{B(\bar{a}, \delta)} w(x, t) \phi_1(x) \, dx \right)^p \\
= -\lambda_1(\delta) \Phi(t) + \left( V(\bar{a}) - \frac{\varepsilon}{2} \right) \Phi(t)^p.
\]

Let us recall that there exists a constant $D$, depending on the dimension only, such that the eigenvalues of the laplacian scale according to the rule $\lambda_1(\delta) = D\delta^{-2}$.

Now, we choose $\varepsilon$ such that
\[
\lambda_1(\delta) = D\delta^{-2} = D \left( \frac{\varepsilon}{2K} \right)^{-2} = \frac{\varepsilon}{2}(M(\varphi(\bar{a}) - \varepsilon))^{p-1}.
\]
So, $\varepsilon$ is of order
\[
\varepsilon \sim \frac{C}{M^{\frac{p-1}{p}}}
\]
Choose $M_0$ such that for $M \geq M_0$ the resulting $\varepsilon$ is small enough. Then for any $M \geq M_0$ we have that
\[
(2.2) \quad \Phi'(t) \geq (V(\bar{a}) - \varepsilon) \Phi(t)^p,
\]
for all $t \geq 0$ for which $\Phi$ is defined.

Since $\Phi(0) = M(\varphi(\bar{a}) - \varepsilon)$ and $T_w$ is less or equal than the blow up time of $\Phi$ integrating (2.2) it follows that
\[
T_w \leq \frac{1}{M^{p-1}(p-1)(V(\bar{a}) - \varepsilon)(\varphi(\bar{a}) - \varepsilon)^{p-1}} \\
\leq \frac{1}{M^{p-1}(p-1)V(\bar{a})\varphi(\bar{a})^{p-1}} + \frac{C}{M^{\frac{p-1}{p}} M^{p-1}},
\]
for all $M \geq M_0$. $\square$

Now we prove a lemma that provides us with an upper bound for the blow up rate. We observe that this is the only place where we use hypothesis (1.2).

**Lemma 2.2.** Assume (1.2). Then there exists a constant $C$ independent of $M$ such that
\[
u(x, t) \leq C(T - t)^{-\frac{1}{p-1}}.
\]
Proof: Let $m = \min_{x \in \Omega} V$. Following ideas of [FMc], set

$$v = u_t - \frac{m}{2} u^p.$$  

Then $v$ verifies

$$v_t - \Delta v - V(x)pu^{p-1}v = \frac{m}{2} p(p - 1)u^{p-2}|\nabla u|^2 \geq 0 \quad \text{in } \Omega \times (0, T),$$

$$v = 0 \quad \text{on } \partial \Omega \times (0, T),$$

$$v(x, 0) = M\Delta \varphi + \left( V(x) - \frac{m}{2} \right) M^p \varphi^p \geq 0 \quad \text{in } \Omega.$$  

Therefore $v \geq 0$ and hence

$$u_t \geq \frac{m}{2} u^p.$$  

Integrating this inequality from 0 to $T$ we get

$$u(x, t) \leq 2^{\frac{1}{p-1}} \frac{1}{(m(p-1)(T-t))^{\frac{1}{p-1}}} \equiv C(T-t)^{-\frac{1}{p-1}},$$

as we wanted to prove. \qed

We are now in a position to prove Theorem 1.1.

Proof of Theorem 1.1: The idea of the proof is to combine the estimate of the blow-up time proved in Lemma 2.1 with local energy estimates near a blow-up point $a$, like the ones considered in [GK1] and [GK2], to obtain an inequality that forces $\varphi^{p-1}(a) V(a)$ to be close to $\max_x \varphi^{p-1} V$.

Let us now proceed with the proof of the estimates on the blow-up set. We fix for the moment $M$ large enough such that $u$ blows up in finite time $T = T(M)$ and let $a = a(M)$ be a blow up point. As in [GK2], for this fixed $a$ we define

$$w(y, s) = (T - t)^{\frac{1}{p-1}} u(a + y(T - t)^{\frac{1}{p}}, t)|_{t = T(1 - e^{-s})}.$$  

Then $w$ satisfies

$$(2.3) \quad w_s = \Delta w - \frac{1}{2} y \cdot \nabla w - \frac{1}{p - 1} w + V(a + ye^{-s})w^p,$$

in $\cup_{s \in [0, \infty)} \Omega(s) \times \{s\}$ where $\Omega(s) = \Omega_d(s) = \{y : a + ye^{-s} \in \Omega\}$ with $w(y, 0) = T^{\frac{1}{p-1}} \varphi(a + yT^{\frac{1}{2}})$. The above equation can rewritten as

$$w_s = \frac{1}{r} \nabla (\rho \nabla w) - \frac{1}{p - 1} w + V(a + ye^{-s})w^p$$

where $\rho(y) = \exp(-\frac{|y|^2}{4})$. 


Consider the energy associated with the "frozen" potential 

\[ V \equiv V(a), \]

that is

\[ E(w) = \int_{\Omega(s)} \left( \frac{1}{2} |\nabla w|^2 + \frac{1}{2(p-1)} w^2 - \frac{1}{p+1} V(a) w^{p+1} \right) \rho(y) dy. \]

Then, using the fact that \( \Omega \) is convex, we get

\[ \frac{dE}{ds} \leq - \int_{\Omega(s)} (w_s)^2 \rho(y) dy + \int_{\Omega(s)} (V(a + y Te^{-\frac{s}{T}}) - V(a)) w^p w_s \rho(y) dy. \]

Since \( V(x) \) is Lipschitz and \( w \) is bounded due to Lemma 2.2, then there exists a constant \( C \) depending only on \( N, p \) and \( V \), recall that the constant in Lemma 2.2 does not depend on \( M \), such that

\[ \frac{dE}{ds} \leq - \int (w_s)^2 \rho(y) dy + Ce^{-\frac{s}{T}} T \left( \int (w_s)^2 \rho(y) dy \right)^{1/2}. \]

Maximizing the right hand side of the above expression with respect to \( \int (w_s)^2 \rho(y) dy \) we obtain

\[ \frac{dE}{ds} \leq Ce^{-s} T^2 \]

and integrating is \( s \) we get

(2.4) \[ E(w) \leq E(w_0) + CT^2. \]

Since \( w \) is bounded and satisfies (2.3), following the arguments given in [GK1] and [GK2], one can prove that \( w \) converges as \( s \to \infty \) to a non trivial bounded stationary solution of the limit equation

(2.5) \[ 0 = \Delta z - \frac{1}{2} y \cdot \nabla z - \frac{1}{p-1} z + V(a) z^p \]

in the whole \( \mathbb{R}^N \).

Again by the results of [GK1] and [GK2], since \( p \) is subcritical, \( 1 < p < (N + 2)/(N - 2) \), the only non trivial bounded positive solution of (2.5) with \( V(a) = 1 \) is the constant \((p - 1)^{-\frac{1}{p-1}} \). A scaling argument gives that the only non trivial bounded positive solution of (2.5) is the constant \( k = k(a) \) given by

\[ k(a) = \frac{1}{(V(a)(p-1))^{\frac{1}{p-1}}}. \]

Therefore, we conclude that

\[ \lim_{s \to \infty} w = k(a) \]
if $a$ is a blow-up point. Also by the results of [GK1], [GK2] we have

$$E(w(\cdot, s)) \rightarrow E(k(a)) \quad \text{as} \quad s \rightarrow \infty,$$

where

$$E(k(a)) = \int \left( \frac{1}{2(p-1)}(k(a))^2 - \frac{1}{p+1} V(a)(k(a))^{p+1} \right) \rho(y) \, dy$$

$$= (k(a))^2 \left( \frac{1}{2(p-1)} - \frac{1}{(p+1)(p-1)} \right) \int \rho(y) \, dy.$$

By (2.4) and (2.6) we obtain that, if $a$ is a blow-up point, then

$$E(k(a)) \leq E(w_0) + CT^2.$$

where $w_0(y) = w(y, 0) = T^{\frac{1}{p-1}} M \varphi(a + y T^{\frac{1}{2}})$.

As $\varphi$ is smooth, $y \rho(y)$ integrable, and $T^{\frac{1}{p-1}} M$ is bounded by Lemma 2.1, there are constants $C$ independent of $a$ such that for $M \geq M_0$

$$E(w(\cdot, 0)) = \int_{\Omega(0)} \left( \frac{1}{2} |\nabla w_0(y)|^2 + \frac{1}{2(p-1)} w_0^2(y) \right) \rho(y) \, dy$$

$$- \int_{\Omega(0)} \left( \frac{1}{p+1} V(a)w_0^{p+1}(y) \right) \rho(y) \, dy$$

$$\leq \int_{\Omega(0)} \left( \frac{1}{2} (T^{\frac{1}{p-1}} M)^2 T |\nabla \varphi(a)|^2 \right) \rho(y) \, dy$$

$$+ \int_{\Omega(0)} \left( \frac{1}{2(p-1)} (T^{\frac{1}{p-1}} M \varphi(a))^2 \right) \rho(y) \, dy$$

$$- \int_{\Omega(0)} \left( \frac{1}{p+1} V(a)(T^{\frac{1}{p-1}} M \varphi(a))^{p+1} \right) \rho(y) \, dy$$

$$+ CT^{\frac{1}{2}} + CT^\frac{1}{2}.$$

Therefore, since $|\nabla \varphi|$ is bounded,

$$E(w(\cdot, 0)) \leq \int_{\Omega(0)} \left( \frac{1}{2(p-1)} (T^{\frac{1}{p-1}} M \varphi(a))^2 \right) \rho(y) \, dy$$

$$- \int_{\Omega(0)} \left( \frac{1}{p+1} V(a)(T^{\frac{1}{p-1}} M \varphi(a))^{p+1} \right) \rho(y) \, dy$$

$$+ CT^{\frac{1}{2}} + CT^\frac{1}{2}.$$

Or, since $T \leq 1$ for $M$ large

$$E(w(\cdot, 0)) \leq E(T^{\frac{1}{p-1}} M \varphi(a)) + CT^\frac{1}{2}.$$

Hence we arrive to the following bound for $E(k(a))$

(2.7) \hspace{1cm} E(k(a)) \leq E(w(\cdot, 0)) + CT^2 \leq E(T^{\frac{1}{p-1}} M \varphi(a)) + CT^\frac{1}{2}.
Observe that if \( b \) is a constant then the energy can be written as
\[
E(b) = \Gamma F(b),
\]
where \( \Gamma \) is the constant
\[
\Gamma = \int \rho(y) \, dy
\]
and \( F \) is the function
\[
F(z) = \left( \frac{1}{2(p-1)} z^2 - \frac{1}{p+1} V(a) z^{p+1} \right).
\]

As \( F \) attains a unique maximum at \( k(a) \) and \( F''(k(a)) = -1 \) there are \( \alpha \) and \( \beta \) such that if \( |z - k(a)| \leq \alpha \) then
\[
F''(z) \leq -\frac{1}{2},
\]
and if \( |F(z) - F(k(a))| \leq \beta \) then
\[
|z - k(a)| \leq \alpha.
\]

From (2.7) we obtain
\[
F(k(a)) \leq F(T^{\frac{1}{p-1}} M \varphi(a)) + CT^{\frac{1}{2}}.
\]
If \( M_1 \) is such that \( C(T(M_1))^{\frac{1}{2}} = \beta \) then for \( M \geq \max(M_0, M_1) \)
\[
\beta \geq CT^{\frac{1}{4}} \geq F(k(a)) - F(T^{\frac{1}{p-1}} M \varphi(a)).
\]
Hence by the properties of \( F \),
\[
|k(a) - T^{\frac{1}{p-1}} M \varphi(a)| \leq \alpha.
\]
Therefore
\[
CT^{\frac{1}{4}} \geq F(k(a)) - F(T^{\frac{1}{p-1}} M \varphi(a)) \geq \frac{1}{4}(T^{\frac{1}{p-1}} M \varphi(a) - k(a))^2.
\]
So, using Lemma 2.1
\[
k(a) - CT^{\frac{1}{4}} \leq T^{\frac{1}{p-1}} M \varphi(a)
\]
(2.8)
\[
\leq \frac{\varphi(a)}{(p-1)^{\frac{1}{p-1}} V^{\frac{1}{p-1}}(\tilde{a}) \varphi(\tilde{a})} + C \varphi(a) M^{\frac{1}{2}} - \frac{1}{M^{\frac{1}{2}}}.
\]
where
\[
\theta(a) = \left( \frac{\varphi(a) V(a) r_{\tilde{a}}^{\frac{1}{p-1}}}{\varphi(\tilde{a}) V(\tilde{a}) r_{\tilde{a}}^{\frac{1}{p-1}}} \right)
\]
and \( \bar{a} \) is such that

\[
\varphi^{p-1}(\bar{a}) V(\bar{a}) = \max_x \varphi^{p-1}(x) V(x).
\]

Recall that

\[
T \leq \frac{C}{M^{p-1}}.
\]

Therefore, we get

\[
k(a)(1 - \theta(a)) \leq \frac{C\varphi(a)}{M^{\frac{1}{p}}} + \frac{C}{M^{\frac{p-1}{4}}} \leq \frac{C}{M^{\gamma}},
\]

with \( \gamma = \min(\frac{p-1}{4}, \frac{1}{3}) \).

As \( V \) is bounded we have that \( k(a) \) is bounded from below, hence

\[
(1 - \theta(a)) \leq \frac{C}{M^{\gamma}},
\]

that is,

\[
\theta(a) \geq 1 - \frac{C}{M^{\gamma}}
\]

and we finally obtain

\[
(2.9) \quad \varphi(a)V(a)^{\frac{1}{p-1}} \geq \varphi(\bar{a})V(\bar{a})^{\frac{1}{p-1}} - \frac{C}{M^{\gamma}}.
\]

This proves (1.4).

To obtain the lower estimate for the blow-up time observe that from (2.9) and the fact that \( V(a) \geq c > 0 \) we get

\[
(2.10) \quad \varphi(a) \geq \frac{\varphi(\bar{a})V(\bar{a})^{\frac{1}{p-1}}}{V(a)^{\frac{1}{p-1}}} - \frac{C}{V(a)^{\frac{1}{p-1}}M^{\gamma}}
\]

with \( \gamma = \min(\frac{p-1}{4}, \frac{1}{3}) \).

Inequality (2.8) gives us

\[
\frac{1}{(V(a)(p-1))^{\frac{1}{p-1}}} - CT^{\frac{1}{4}} \leq T^{\frac{1}{p-1}}M\varphi(a).
\]

Hence

\[
\frac{1}{\varphi(a)(V(a)(p-1))^{\frac{1}{p-1}}} - \frac{CT^{\frac{1}{4}}}{\varphi(a)} \leq T^{\frac{1}{p-1}}M.
\]

By (2.10) and \( \varphi^{p-1}(\bar{a})V(\bar{a}) = \max_x \varphi^{p-1}(x)V(x) \) we get

\[
\frac{1}{\varphi(\bar{a})(V(\bar{a})(p-1))^{\frac{1}{p-1}}} - CT^{\frac{1}{4}} \leq T^{\frac{1}{p-1}}M
\]
and using 

\[ T \leq \frac{C}{M^{p-1}} \]

we obtain

\[ \frac{1}{\varphi(\bar{a})V(\bar{a})(p-1))^{\frac{1}{p-1}}} - \frac{C}{M^{\frac{p-1}{c}}} \leq T^{\frac{1}{p-1}}M \]

as we wanted to prove. \qed

References

[B] J. Ball. Remarks on blow-up and nonexistence theorems for nonlinear evolution equations. Quart. J. Math. Oxford, Vol. 28, (1977), 473–486.

[BB] C. Bandle and H. Brunner. Blow-up in diffusion equations: a survey. J. Comp. Appl. Math. Vol. 97, (1998), 3–22.

[FMc] A. Friedman and J. B. McLeod. Blow up of positive solutions of semilinear heat equations. Indiana Univ. Math. J., Vol. 34, (1985), 425–447.

[GV] V. A. Galaktionov and J. L. Vázquez. Continuation of blow-up solutions of nonlinear heat equations in several space dimensions. Commun. Pure Applied Math. 50, (1997), 1–67.

[GV2] V. A. Galaktionov and J. L. Vázquez. The problem of blow-up in nonlinear parabolic equations. Discrete Contin. Dynam. Systems A. Vol 8, (2002), 399–433.

[GK1] Y. Giga and R. V. Kohn. Nondegeneracy of blow up for semilinear heat equations. Comm. Pure Appl. Math. Vol. 42, (1989), 845–884.

[GK2] Y. Giga and R. V. Kohn. Characterizing blow-up using similarity variables. Indiana Univ. Math. J. Vol. 42, (1987), 1–40.

[HV1] M. A. Herrero and J. J. L. Velazquez. Flat blow up in one-dimensional, semilinear parabolic problems. Differential Integral Equations. Vol. 5(5), (1992), 973–997.

[HV2] M. A. Herrero and J. J. L. Velazquez. Generic behaviour of one-dimensional blow up patterns. Ann. Scuola Norm. Sup. di Pisa, Vol. XIX (3), (1992), 381–950.

[IY] K. Ishige and H. Yagisita. Blow-up problems for a semilinear heat equation with large diffusion. J. Differential Equations. Vol. 212(1), (2005), 114–128.

[M] F. Merle. Solution of a nonlinear heat equation with arbitrarily given blow-up points. Comm. Pure Appl. Math. Vol. XLV, (1992), 263–300.

[MY] N. Mizoguchi and E. Yanagida. Life span of solutions for a semilinear parabolic problem with small diffusion. J. Math. Anal. Appl. Vol. 261(1), (2001), 350–368.

[SGKM] A. Samarski, V. A. Galaktionov, S. P. Kurdyumov and A. P. Mikhailov. Blow-up in quasilinear parabolic equations. Walter de Gruyter, Berlin, (1995).

[Z] H. Zaag. One dimensional behavior of singular N dimensional solutions of semilinear heat equations. Comm. Math. Phys. Vol. 225 (3), (2002), 523–549.
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