WEIGHTS FOR $\ell$-LOCAL COMPACT GROUPS

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Abstract. In this note, we initiate the study of $F$-weights for an $\ell$-local compact group $F$ over a discrete $\ell$-toral group $S$ with discrete torus $T$. Motivated by Alperin’s Weight Conjecture for simple groups of Lie-type, we conjecture that when $T$ is the unique maximal abelian subgroup of $S$ up to $F$-conjugacy and every element of $S$ is $F$-fused into $T$, the number of weights of $F$ is bounded above by the number of ordinary irreducible characters of its Weyl group. By combining the structure theory of $F$ with the theory of blocks with cyclic defect group, we are able to give a proof of this conjecture in the case when $F$ is simple and $|S:T| = \ell$. We also propose and give evidence for an analogue of the height zero case of Robinson’s Ordinary Weight conjecture in this setting.

1. Introduction

An $\ell$-compact group is, broadly speaking, the $\ell$-local homotopy theoretic analogue of a compact Lie group with a ‘Weyl group’ $W$ which is a $\mathbb{Z}_\ell$-reflection group. In [14], the authors introduce and prove a version of Alperin’s Weight Conjecture (AWC) for fusion systems associated to homotopy fixed point spaces of connected $\ell$-compact groups under the action of unstable Adams operations. For such a fusion system $F$ associated to a simply connected $\ell$-compact group with Weyl group $W$, under some mild hypotheses this conjecture asserts that $w(F) = |\text{Irr}(W)|$ where $w(F)$ is the number of weights (defect zero characters of $F$-automorphism groups of $F$-centric radical subgroups up to $F$-conjugacy).

One consequence is that the number of weights appears to be an invariant of the underlying $\ell$-compact group, as opposed to the space of fixed points. In an attempt to understand this, we introduce the study of weights for $\ell$-compact groups by appealing to the more general theory of $\ell$-local compact groups [3]. Here, we view an $\ell$-local compact group as a fusion system $F$ on an $\ell$-group $S$ which contains a finite index infinite torus $T$ (see Definition 2.2). Since $F$ has finitely many classes of $F$-centric radical subgroups, each with a finite $F$-outer automorphism group (see Proposition 2.3), we obtain an integer invariant $w(F)$ which, by analogy with the finite case, we call the ‘number of weights’ of $F$ (see Definition 2.4).

An $\ell$-local compact group $F$ is connected if $T$ is the unique maximal abelian subgroup of $S$ up to $F$-conjugacy and every element of the underlying $\ell$-group is $F$-conjugate to an element of $T$. In this case we refer to the group $W = \text{Aut}_F(T)$ as the Weyl group of $F$. Based on [14, Theorem 1] for finite fusion systems, and observations made in the present paper, we make the following conjecture which may be regarded as a weak analogue of Alperin’s Weight Conjecture for connected $\ell$-local compact groups.

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Conjecture 1.1 (Weak analogue of AWC). Let $\mathcal{F}$ be a connected $\ell$-local compact group with Weyl group $W$. Then

$$w(\mathcal{F}) \leq |\text{Irr}(W)|.$$ 

Markus Linckelmann has asked for conditions on a fusion system $\mathcal{G}$ which imply the existence of a subgroup $T$ with $w(\mathcal{G}) = |\text{Irr}(\text{Out}_c(T))|$ so Conjecture 1.1 would provide a partial answer to this. Letting $v_\ell(-)$ denote the $\ell$-adic valuation, it is a straightforward observation that equality holds in Conjecture 1.1 when $v_\ell(|W|) = 0$. Indeed, $\mathcal{F}$ is automatically connected in this case (since $S = T$) and hence $T$ is the unique $\mathcal{F}$-centric radical subgroup in $\mathcal{F}$. Thus $w(\mathcal{F})$ is the number of defect zero characters of $W$, which is $|\text{Irr}(W)|$ since $v_\ell(|W|) = 0$. Our main result concerns the case $v_\ell(|W|) = 1$.

Theorem 1.2. Suppose $\mathcal{F}$ is a simple $\ell$-local compact group and $v_\ell(|W|) = 1$. Then $\mathcal{F}$ is connected and

$$w(\mathcal{F}) = |\text{Irr}(W)|.$$ 

Here, $\mathcal{F}$ is simple if it possesses no non-trivial normal subsystems. Theorem 1.2 initially inspired the author to conjecture that an equality $w(\mathcal{F}) = |\text{Irr}(W)|$ always holds for connected $\ell$-local compact groups, but this was later shown to be false in joint work with Kessar and Malle [15]. For example if $\mathcal{F} = \mathcal{F}_2(\text{Sp}(2))$, then $4 = w(\mathcal{F}) \leq |\text{Irr}(D_8)| = 5$ (see [15, Example 4.2]). On the other hand, [15, Theorem 1] shows that that $w(\mathcal{F}) = |\text{Irr}(W)|$ where $\mathcal{F} = \mathcal{F}_l(G)$ for any compact connected Lie group $G$ for which the prime $\ell$ is good. Thus an equality-predicting refinement of Conjecture 1.1 might be obtained from an appropriate generalisation of “good” to connected $\ell$-local compact groups; we do not pursue that here.

Next, we introduce a version of the height zero case of Robinson’s Ordinary Weight Conjecture (OWC) for connected $\ell$-local compact groups. Via deep results in modular representation theory, Conjecture 1.3 below may be understood as a generalisation of height zero OWC for Lie-type groups with Weyl group $W$ when $\ell$ is a very good prime and $q \equiv 1 \pmod{\ell}$. This connection is made formally at the end of Section 2.

Conjecture 1.3 (Analogue of height zero OWC). Let $\mathcal{F}$ be a connected $\ell$-local compact group on $S$ with Weyl group $W$ and set

$$\mathcal{M}(\mathcal{F}) = \{(\psi, \chi) \mid \psi \in \text{Irr}(S^{ab}), \chi \in \text{Irr}(C_{\text{Out}_\mathcal{F}(S)}(\psi))\}/\sim_{\mathcal{F}}$$

$$\mathcal{P}(\mathcal{F}) = \{(s, \chi) \mid s \in Z(S), \chi \in \text{Irr}_0(W(s))\}/\sim_{\mathcal{F}}$$

where for each $s \in Z(S)$, $W(s)$ denotes the Weyl group of $C_{\mathcal{F}}(s)$. Then there is an $\text{Out}(\mathcal{F})$-equivariant bijection

$$\mathcal{M}(\mathcal{F}) \longleftrightarrow \mathcal{P}(\mathcal{F}).$$

Here $\text{Out}(\mathcal{F})$ denotes the quotient $\text{Aut}(\mathcal{F})/\text{Aut}(S)$, where $\text{Aut}(\mathcal{F}) \leq \text{Aut}(S)$ is the group of $\mathcal{F}$-fusion preserving automorphisms of $S$. The above equivalences $\sim_{\mathcal{F}}$ are defined precisely in Section 2 as is the natural action of $\text{Out}(\mathcal{F})$ on the equivalence classes $\mathcal{M}(\mathcal{F})$ and $\mathcal{P}(\mathcal{F})$. The conjectured existence of an $\text{Out}(\mathcal{F})$-equivariant bijection is partially inspired by the inductive McKay condition for simple groups due to Isaacs–Malle–Navarro [11]; indeed it seems likely that the natural map $\text{Out}(G) \to \text{Out}(\mathcal{F})$ (see [11, Section 4]) can be used to exhibit a precise connection with that condition when $\mathcal{F}$ is realisable by a simple group $G$, but we do not attempt that here. As with Conjecture 1.1 it is
relatively straightforward to prove Conjecture 1.3 when \( v_\ell(|W|) = 0 \) (see Proposition 2.9). Our second main result supplies some evidence for the case \( v_\ell(|W|) = 1 \). We first recall the fact, due to González [10, Theorem 1] that any \( \ell \)-local finite group \( F \) on \( S \) can be approximated as a direct limit of a sequence of (categorical) inclusions of saturated fusion systems on finite \( \ell \)-groups

\[
(F_1, S_1) \rightarrow (F_2, S_2) \rightarrow (F_3, S_3) \rightarrow \cdots
\]

with \( S = \lim_{\longrightarrow} S_n \). We use this to establish a ‘truncated’ version of Conjecture 1.3 in the case \( v_\ell(|W|) = 1 \).

**Theorem 1.4.** Let \( F \) be a simple \( \ell \)-local compact group with Weyl group \( W \) and assume \( v_\ell(|W|) = 1 \). Then there is a sequence of fusion system inclusions \((F_n, S_n)_{n\geq 1}\) for which \( F = \lim_{\longrightarrow} F_n \) where for each \( n \geq 1 \), \( F_n \) is a finite fusion system on the \( \ell^n \)-torsion subgroup \( S_n \) of \( S \). Moreover, for each \( n \geq 1 \), there is a bijection

\[
\mathcal{M}(F_n) \leftrightarrow P(F_n).
\]

We prove Theorems 1.2 and 1.4 in Section 5. The proofs only require the character theory of finite groups with Sylow \( \ell \)-subgroups of order \( \ell \) and particular information concerning the \( F \)-automorphisms of \( F \)-centric radical subgroups mostly established in [16] (see Theorems 4.1 and 4.5). The connectedness of \( F \) is crucial to ensuring that there are a sufficiently small number of classes of \( F \)-centric radical subgroups.

We expect Theorem 1.4 to follow from Conjecture 1.3. In [12], Junod, Levi and Libman prove that for any \( \ell \)-local compact group \( F \) with torus \( T \) and any \( \zeta \in \mathbb{Z}_\ell^\times \) with \( v_\ell(\zeta - 1) > 0 \), \( \text{Aut}(F) \) contains an unstable Adams operation \( \Psi = \Psi(\zeta) \) with the property that \( \Psi|_T(g) = g^\zeta \) for each \( g \in T \). Given such an operation \( \Psi \), computations of González [9] seem to indicate that the homotopy fixed points of powers of \( \Psi \) stratify \( F \) as a direct limit of finite fusion systems \( \{F_n | n \geq 1\} \) where \( F_n \) is the fusion subsystem of \( F \) fixed by \( \Psi^n \) (so \( F = \lim_{\longrightarrow} F_n \)). Thus the \( \text{Aut}(F) \)-equivariance in Conjecture 1.3 should imply the existence of bijections between sets fixed by \( \Psi^n \in \text{Aut}(F) \) as in Theorem 1.4.

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2. \( \ell \)-LOCAL COMPACT GROUPS AND WEIGHTS

For the rest of the paper, \( \ell \) is a prime and \( k \) is an algebraically closed field of characteristic \( \ell \). We refer the reader to [2] for basic notation and results in the theory of fusion systems. We begin by introducing the class of \( \ell \)-groups on which we focus our attention.
Definition 2.1. Write $\mathbb{Z}/\ell^\infty$ for the infinite union of the cyclic $\ell$-groups $\mathbb{Z}/\ell^n$ under the obvious inclusions. A group isomorphic to a direct product of finitely many copies of $\mathbb{Z}/\ell^\infty$ is called a discrete $\ell$-torus. A discrete $\ell$-toral group is a group $S$ which contains a discrete $\ell$-torus $T$ as a normal subgroup of finite $\ell$-power index.

As observed in [3, Lemma 1.3], any quotient, subgroup or extension of an infinite discrete $\ell$-toral group $S$ is a discrete $\ell$-toral group and $S$ contains only finitely many conjugacy classes of subgroups of order $\ell^n$ for every $n \geq 0$ ([3, Lemma 1.4]). By [3, Lemma 1.9] there is an increasing sequence $S_1 \leq S_2 \leq S_3 \leq \cdots$ of $\text{Aut}(S)$-invariant $\ell^n$-torsion subgroups of $S$, whose direct limit is $S = \lim_{n \geq 1} S_n$. When $S$ is abelian, the sequence
\[
\cdots \to \text{Hom}(S_3, \mathbb{C}^\times) \to \text{Hom}(S_2, \mathbb{C}^\times) \to \text{Hom}(S_1, \mathbb{C}^\times) \to \text{Hom}(S_1, \mathbb{C}^\times)
\]

is $\text{Aut}(S)$-invariant, so that $\text{Aut}(S)$ acts naturally on the corresponding inverse limit $\text{Irr}(S) = \lim_{n \geq 1} \text{Hom}(S_n, \mathbb{C}^\times)$.

A fusion system $\mathcal{F}$ over $S$ with discrete torus $T$ is defined by analogy with the finite case [3, Definition 2.1]. What it means for $\mathcal{F}$ to be saturated is almost analogous, provided one makes an additional assumption that morphisms behave well with respect to filtrations of subgroups (see [3, Definition 2.2 (III)]).

Definition 2.2. Let $S$ be a discrete $\ell$-toral group with discrete torus $T$. An $\ell$-local compact group is a saturated fusion system on $S$. $\mathcal{F}$ is, in addition, connected if $T$ is the unique maximal abelian subgroup of $S$ up to $\mathcal{F}$-conjugacy and every element of $S$ is $\mathcal{F}$-conjugate to an element of $T$. In this case we refer to the group $\text{Aut}_\mathcal{F}(T)$ as the Weyl group of $\mathcal{F}$.

Note that the definition of connected here differs slightly to that found in [3, Definition 3.1.4]. The definitions of fully $\mathcal{F}$-centralised, fully $\mathcal{F}$-normalised, $\mathcal{F}$-centric and $\mathcal{F}$-radical are unchanged from those for fusion systems on finite groups. We write $\mathcal{F}^{\text{cr}}$ for the set of $\mathcal{F}$-centric radical subgroups. As in [16, Definition 1.4], we say that $\mathcal{F}$ is simple if it possesses no non-trivial normal subsystems. Using a certain “bullet construction” first considered by Benson (see [3, Definition 3.1]) it is possible to prove the following result.

Proposition 2.3. Let $\mathcal{F}$ be an $\ell$-local compact group on $S$. Then $\text{Out}_\mathcal{F}(P)$ is finite for all $P \leq S$. Moreover, $\mathcal{F}$ has only finitely many conjugacy classes of $\mathcal{F}$-centric, $\mathcal{F}$-radical subgroups.

Proof. See [3, Proposition 2.3 and Corollary 3.5].

In particular the Weyl group of $\mathcal{F}$ is finite. We are interested in the following invariant.

Definition 2.4. Let $\mathcal{F}$ be an $\ell$-local compact group on $S$. The number of weights $w(\mathcal{F})$ of $\mathcal{F}$ is given by:
\[
w(\mathcal{F}) := \sum_{P \in \mathcal{F}^{\text{cr}}/\mathcal{F}} z(k\text{Out}_\mathcal{F}(P)),
\]
where the sum runs over a set of $\mathcal{F}$-conjugacy class representatives of $\mathcal{F}$-centric, $\mathcal{F}$-radical subgroups and for a group $H$, $z(kH)$ denotes the number of projective simple $kH$-modules up to isomorphism.
By Proposition 2.3, \( w(F) \) is finite. For the remainder of this section we assume that \( F \) be a connected \( \ell \)-local compact group on \( S \).

**Definition 2.5.** The set \( \mathcal{M}(F) \) of ordinary weights of height 0 is the set of equivalence classes
\[
\mathcal{M}(F) = \{ (\psi, \chi) \mid \psi \in \text{Irr}(S^{\text{ab}}), \chi \in \text{Irr}(C_{Out_F}(\psi)) \}/ \sim_F,
\]
where \( (\psi, \chi) \sim_F (\psi', \chi') \) if there is \( \rho \in \text{Aut}_F(S) \) with \( \psi^\rho = \psi' \) and \( \chi^{\rho^{-1}} = \chi' \); we write \([[(\psi, \chi)]]\) for the class of \((\psi, \chi)\).

Recall that an element \( \phi \in \text{Aut}(S) \) is an automorphism of \( F \) if it induces an automorphism \( \Phi : F \to F \) where:

1. for each \( P \in \text{Ob}(F) \), \( \Phi(P) = \phi(P) \);
2. for each \( \psi \in \text{Hom}_F(P, Q) \),
\[
\Phi(\psi) = \phi|_Q \circ \psi \circ (\phi|_{\phi(P)})^{-1} : \phi(P) \to \phi(Q)
\]
is an element of \( \text{Hom}_F(\phi(P), \phi(Q)) \).

Write \( \text{Aut}(F) \subseteq \text{Aut}(S) \) for the set of all automorphisms of \( S \) which induce automorphisms of \( F \). Since \( \text{Aut}_F(S) \subseteq \text{Aut}(F) \), we define \( \text{Out}(F) := \text{Aut}(F)/\text{Aut}_F(S) \) to be the outer automorphism group of \( F \) and denote by \([\phi] \in \text{Out}(F)\) the class of an element \( \phi \) of \( \text{Aut}(F) \). If \( s \in Z(S) \) then \( s \) is fully \( F \)-centralised, so \( C_F(s) \) is also a saturated fusion system on \( S \) and we write \( W(s) := C_W(s) \).

**Lemma 2.6.** For any \( s \in Z(S) \) and \( \phi \in \text{Aut}(F) \), the induced automorphism \( \Phi \) restricts to an isomorphism \( C_F(s) \to C_F(\phi(s)) \) which sends \( W(s) \) to \( W(\phi(s)) \).

**Proof.** Let \( P \leq S \), and \( \psi \in \text{Hom}_{C_F(s)}(P, S) \), then by definition of \( C_F(s) \), \( \psi \) extends to a map \( \overline{\psi} \in \text{Hom}_F(P(s), S) \) which fixes \( s \); hence
\[
\phi \circ \overline{\psi} \circ \phi^{-1} \in \text{Hom}_F(\phi(P(s)), S) = \text{Hom}_F(\phi(P), \phi(s), S)
\]
and this latter map clearly fixes \( \phi(s) \) and restricts to \( \phi \circ \psi \circ \phi^{-1} \in \text{Hom}_F(\phi(P), S) \). We deduce that \( \phi \circ \psi \circ \phi^{-1} \in \text{Hom}_{C_F(\phi(s))}(\phi(P), S) \). Moreover, since \( \Phi \) fixes \( T \), \( \phi|_T \circ W \circ (\phi|_T)^{-1} = W \) and hence \( \phi|_TW(s)(\phi|_T)^{-1} = W(\phi(s)) \).

**Definition 2.7.** The set \( \mathcal{P}(F) \) is the set of equivalence classes
\[
\mathcal{P}(F) := \{ (s, \chi) \mid s \in Z(S), \chi \in \text{Irr}_0(W(s)) \}/ \sim_F,
\]
where \( (s, \chi) \sim_F (s', \chi') \) if and only if there is \( \phi \in \text{Aut}_F(S) \) with \( s' = \phi(s) \) and \( \chi' = \chi^{\phi^{-1}|_{W(\phi(s))}} \).

**Proposition 2.8.** \( \text{Out}(F) \) acts on \( \mathcal{P}(F) \) via
\[
[\phi] \cdot [(s, \chi)] = [(\phi(s), \chi^{\phi^{-1}|_{W(\phi(s))}})],
\]
for each \([\phi] \in \text{Out}(F)\) and class \([(s, \chi)] \in \mathcal{P}(F)\).

**Proof.** It suffices to check the action of \([\phi] \) on \( \chi \) is independent of the choice of representative \( \phi \). If \( \phi' \in \text{Aut}(F) \) is such that \([\phi'] = [\phi] \) then \( \gamma \circ \phi' = \phi \) for some \( \gamma \in \text{Aut}_F(S) \) so
\[
[\phi'] \cdot [(s, \chi)] = [(\phi'(s), \chi^{\phi^{-1}})] \sim_F [(\phi(s), \chi^{\phi^{-1}})]
\]
since \((\gamma \circ \phi')(s) = \phi(s)\) and \(\chi^{e^{-1} \Gamma^{-1}} = \chi^{e^{-1}}\) where \(\Gamma\) is the functor \(C_{\mathcal{F}}(\phi'(s)) \to C_{\mathcal{F}}(\phi(s))\) induced by \(\gamma\).

Similarly, \(\text{Out}(\mathcal{F})\) acts on \(\mathcal{M}(\mathcal{F})\) via \([\phi] \cdot [(\psi, \chi)] = [(\psi^\phi, \chi^{\phi^{-1}})]\) for each \([\phi] \in \text{Out}(\mathcal{F})\) and class \([(\psi, \chi)] \in \mathcal{M}(\mathcal{F})\) and thus the statement of Conjecture \([1.3]\) makes sense. In particular we can prove:

**Proposition 2.9.** Conjecture \([1.3]\) holds if \(S\) is abelian.

**Proof.** Since \(|W|\) is coprime with \(\ell\), by the Glauberman correspondence, there exists an infinite family of \(W\)-equivariant bijections \(S_n \to \text{Irr}(S_n)\) which, by the remarks following Definition \([2.2]\) induces a \(W\)-equivariant bijection \(S \to \text{Irr}(S)\). The result follows immediately from this.

We conclude this section by explaining the sense in which Conjecture \([1.3]\) may be regarded as an analogue of OWC for height zero characters. Let \(B\) be the principal \(\ell\)-block of a finite group \(G\), \(S \in \text{Syl}_l(G)\) be its defect group and \(\mathcal{F} = \mathcal{F}_S(G)\) be its fusion system. The maximal defect case of OWC for \(B\) (see \([14]\) Section 2) predicts an equality

\[
(1) \quad m(\mathcal{F}, d) = \text{Irr}^d(B),
\]

for \(d = v_\ell(|S|)\) where \(\text{Irr}^d(B)\) denotes the set of characters of \(\ell\)-defect \(d\) in \(B\). We have

\[
(2) \quad m(\mathcal{F}, v_\ell(|S|)) = \sum_{\chi \in \text{Irr}_0(S)/\text{Out}_G(S)} |\text{Irr}(\text{Out}_\mathcal{F}(S)(\chi))| = |\text{Irr}(S_\text{ab} : \text{Out}_\mathcal{F}(S))| = |\mathcal{M}(\mathcal{F})|,
\]

where the last equality follows from Lemma \([3.2]\). If \(G\) is of Lie-type we have the following result.

**Lemma 2.10.** Suppose that \(B = B_0(G)\) is the principal \(\ell\)-block of \(G = G^F\) with defect group \(S\), where \(G\) is a semisimple algebraic group defined over \(\mathbb{F}_q\) (for which \(\ell\) is very good) with respect to a Frobenius endomorphism \(F : G \to G\) and let \(\mathcal{F}\) be the \(\ell\)-fusion system of \(G\) on \(S\). If \(q \equiv 1 \pmod{\ell}\) then \(|\text{Irr}^d(B)| = |\mathcal{P}(\mathcal{F})|\).

**Proof.** The proof of \([14]\) Proposition 6.8] shows that there is a degree-preserving bijection between \(\text{Irr}(B_0(G))\) and the set of irreducible characters in the principal \(\ell\)-block \(B_0\) of the associated \(\mathbb{Z}_\ell\)-spts \(G(q)\) (see \([14]\) Definition 6.7]). The number of characters of defect \(d\) in the latter set is computed in \([14]\) Proposition 6.11]. In the notation of that result, we see that for each \(s \in S\), \(u_s = 0\) by \([14]\) Proposition 5.7] and \(W(s)^{\phi^{-1}_s \cdot \xi^{-1}} = W(s)\) since \(q \equiv 1 \pmod{\ell}\). Since maximal defect characters only occur when \(s \in Z(S)\), the result follows.

Thus, in the situation of Lemma \([2.10]\) by \([2]\) we see that height zero OWC for \(B\) is exactly the equality \(|\mathcal{M}(\mathcal{F})| = |\mathcal{P}(\mathcal{F})|\).

### 3. Some Character Theory

We need an elementary consequence of the character theory of groups with a Sylow \(\ell\)-subgroup of order \(\ell\).

**Lemma 3.1.** Let \(W\) be a finite group with a cyclic Sylow \(\ell\)-subgroup \(U\) of order \(\ell\). Then

\[
|\text{Irr}(W)| - z(kW) = |\text{Irr}(N_W(U))| = |\text{Irr}_0(N_W(U))| = |\text{Irr}_0(W)|.
\]
Proof. This follows from the existence of a bijection between blocks of $W$ and $N_W(U)$ with non-trivial defect which preserves the number of irreducible characters (see [7, VII.2.12]). Hence summing over all such blocks we obtain,
\[ |\text{Irr}(W)| - z(kW) = |\text{Irr}(N_W(U))| - z(kN_W(U)) = |\text{Irr}(N_W(U))|, \]
since $z(kN_W(U)) = 0$, as required.

The following is sometimes referred to as the “Method of Little Groups.”

Lemma 3.2. Let $H \trianglelefteq G$ be finite groups and $\Theta := \text{Irr}(H)/G$ be a set of $G$-orbit representatives for $\text{Irr}(H)$. Suppose that each $\theta \in \text{Irr}(H)$ extends to a character $\bar{\theta} \in \text{Irr}(I_G(\theta))$, where $I_G(\theta)$ denotes the inertia subgroup of $\theta$ in $G$. Then there is a bijection
\[ \{(\theta, \beta) \mid \theta \in \Theta, \beta \in \text{Irr}(I_G(\theta)/H)\} \longrightarrow \text{Irr}(G), \text{ given by } (\theta, \beta) \mapsto (\bar{\theta} \bar{\beta})_{I_G(\theta)}. \]

Proof. See, for example, [5, Theorem 11.5].

The next lemma concerning characters of the extension of a normal subgroup by an $\ell'$-group is well-known.

Lemma 3.3. Let $G$ be a finite group with $N \trianglelefteq G$. Suppose that $V$ is an inertial projective simple $kN$-module and that $G/N$ is a cyclic $\ell'$-group. Then $G$ has exactly $|G:N|$ projective simple modules lying over $V$. In particular if $N = \text{SL}_2(\ell) \leq G \leq \text{GL}_2(\ell)$, then $z(kG) = |G: \text{SL}_2(\ell)|$.

Proof. See, for example, [13, Lemma 4.12].

The following technical result is used in our calculation of $w(\mathcal{F})$ in Section 4.

Lemma 3.4. Let $X_1$ be a finite $\ell'$-group and $X_2$ be a finite group and suppose $H$ is a normal subgroup of $X_1 \times X_2$ with $(X_1 \times X_2)/H$ cyclic of order $e | \ell - 1$. The following hold:

1. If $X_2 = \text{GL}_2(\ell)$ and $O^{\ell'}(H) = \text{SL}_2(\ell)$ then $z(kO^{\ell'}(H)X_1) = |\text{Irr}(X_1)| \cdot (\ell - 1)/e$.
2. If $X_2 = N_{\text{GL}_2(\ell)}(U)$ for some Sylow $\ell'$-subgroup $U$ of $\text{SL}_2(\ell)$ and $O^{\ell'}(H) = U$ then $|\text{Irr}(H)| = |\text{Irr}(X_1)| \cdot \ell \cdot (\ell - 1)/e$.

Proof. Suppose the hypotheses of (1) hold. Since $X_1$ is an $\ell'$-group, $z(k(X_1 \times X_2)) = |\text{Irr}(X_1)| \cdot (\ell - 1)/e$ by Lemma 3.3. On the other hand, Lemma 3.3 applied to $N = O^{\ell'}(H)X_1$ and $G = X_1 \times X_2$ yields $z(k(X_1 \times X_2)) = z(kO^{\ell'}(H)X_1) \cdot e$ and (1) follows from this. Part (2) is proved similarly: in this case $X_1 \times X_2 \cong X_1 \times (C_\ell \times (C_{\ell-1})^2)$, and this group has $|\text{Irr}(X_1)|\ell(\ell - 1)$ irreducible characters. Now Lemma 3.2 implies that $|\text{Irr}(X_1)|\ell(\ell - 1) = |\text{Irr}(H)|e$, as needed.

4. FUSION SYSTEMS ON $\ell$-GROUPS WITH AN ABELIAN MAXIMAL SUBGROUP

In this section we assume that $\mathcal{F}$ is a simple $\ell$-local compact group on $S$ with discrete torus $T$ of index $\ell$. We adopt the notation of [16] to describe the structure of $\mathcal{F}$. Set
\[ \mathcal{H} = \{Z(S)(x) \mid x \in S\setminus T\} \text{ and } \mathcal{B} = \{Z_2(S)(x) \mid x \in S\setminus T\}, \]
where $Z(S)$ and $Z_2(S)$ denote the first and second centres of $S$ respectively. We have the following result from [16] when $T$ is not $\mathcal{F}$-centric radical.
Theorem 4.1. Assume that $T$ is not $\mathcal{F}$-centric radical. Then $\mathcal{F}$ is connected, $T$ has rank $\ell - 2$, $\mathcal{F}^\sigma = \{S\} \cup \mathcal{H}$ and for any prime $q \neq \ell$, $\mathcal{F}$ is isomorphic to the unique subsystem of the $\ell$-fusion system of $T = \text{PSL}_2(\mathbb{F}_q)$ with

1. $\text{Out}_\mathcal{F}(S) = \text{Out}_\mathcal{T}(S) \cong C_{\ell-1}$;
2. $\text{Aut}_\mathcal{F}(T) = \text{Aut}_\mathcal{T}(S) \cong C_\ell \times C_{\ell-1}$, where $C_{\ell-1}$ is acting faithfully; and
3. $\text{Aut}_F(P) = \text{Aut}_\mathcal{T}(P) \cong \text{SL}_2(\ell)$, for $P \in \mathcal{H}$.

Proof. Parts (1), (2) and (3) follow from [16] Theorem 5.12. If $x \notin T$ then $x$ is $\text{Aut}_\mathcal{F}(P)$-conjugate to an element of $Z(S) \leq T$ where $P = Z(S)(x) \in \mathcal{H}$ and $\mathcal{F}$ is connected. \qed

In particular, for $\mathcal{F}$ as described in Theorem 4.1 we have $\mathcal{H} = P^\mathcal{F}$ for any $P \in \mathcal{H}$, and Lemma 3.4(1) applied in the case $\text{Out}^{\mathcal{F}}$ them. Define:

$$w(\mathcal{F}) = z(k\text{Out}_\mathcal{F}(S)) + z(k\text{Out}_\mathcal{F}(P)) = (\ell - 1) + z(k\text{SL}_2(\ell)) = \ell = |\text{Irr}(\text{Aut}_\mathcal{F}(T))|,$$

so Theorem 1.2 holds in this case. Thus for the remainder of this subsection we may assume that $T$ is an $\mathcal{F}$-centric radical subgroup of $S$.

The following notation from [16] provides a convenient way to describe the actions of $\text{Out}_\mathcal{F}(S)$ and $\text{Out}_\mathcal{F}(T)$ on $S/T$ and $Z(S) \cap [S,S]$, as well as the two-way traffic between them. Define:

$$\text{Aut}^\mathcal{F}(S) := \{\alpha \in \text{Aut}(S) \mid [\alpha, Z(S)] \leq Z(S) \cap [S,S]\} \text{ and } \text{Aut}^\mathcal{F}(T) = \{\alpha|_\mathcal{T} \mid \alpha \in \text{Aut}^\mathcal{F}(S)\}.$$

and let

$$\text{Aut}^\mathcal{F}(S) := \text{Aut}^\mathcal{F}(S) \cap \text{Aut}_\mathcal{F}(S) \text{ and } \text{Aut}^\mathcal{F}(T) := \text{Aut}^\mathcal{F}(T) \cap \text{Aut}_\mathcal{F}(T).$$

Hence,

$$\text{Aut}^\mathcal{F}(T) = \{\beta \in N_{\text{Aut}_\mathcal{T}(\text{Aut}_\mathcal{S}(T))} \mid [\beta, Z(S)] \leq Z(S) \cap [S,S]\}$$

(see [16] Notation 2.9). Observe that $\text{Inn}(S) \leq \text{Aut}^\mathcal{F}(S)$ and $\text{Aut}^\mathcal{F}(S) \leq \text{Aut}_\mathcal{F}(S)$ (it is the kernel of a homomorphism to $\text{Aut}(Z(S)/(Z(S) \cap [S,S]))$), so we may define $\text{Out}^\mathcal{F}(S) := \text{Aut}_\mathcal{F}(S)/\text{Inn}(S)$. Now set,

$$\Delta := (\mathbb{Z}/\ell)x \times (\mathbb{Z}/\ell)x, \text{ and } \Delta_i := \{(r, r^i) \mid r \in (\mathbb{Z}/\ell)x\} \leq \Delta \text{ for } i \in \mathbb{Z},$$

define

$$\mu : \text{Aut}^\mathcal{F}(S) \to \Delta \text{ via } \mu(\alpha) = (r, s) \text{ if } \begin{cases} x\alpha \in x'^T \text{ for } x \in S \setminus T \\ g\alpha = g^s \text{ for } g \in Z(S) \cap [S,S], \end{cases}$$

and let

$$\mu_T : \text{Aut}^\mathcal{F}(T) \to \Delta \text{ and } \hat{\mu} : \text{Out}^\mathcal{F}(S) \to \Delta$$

be given by $\mu_T(\alpha|_\mathcal{T}) = \mu(\alpha)$ and $\hat{\mu}([\alpha]) = \mu(\alpha)$ if $\alpha \in \text{Aut}^\mathcal{F}(S)$ (here $[\alpha]$ denotes the class of $\alpha$ in $\text{Out}^\mathcal{F}(S)$).

We begin by appealing to results in [4] [16] to describe particular groups of automorphisms of subsystems which turn out to be $\mathcal{F}$-centric radical. The following result is crucial to our analysis.

Lemma 4.2. Let $A$ be an abelian discrete $\ell$-toral group and $G \leq \text{Aut}(A)$ be finite. Assume $\nu_\ell([G]) = 1$, $O_\ell(G) = 1$ and that $|[x,A]| = \ell$ for every element $x \in G$ of order $\ell$. Then, there is a factorization $A = A_1 \times A_2$ with $A_2 \cong C_\ell \times C_\ell$ such that $\text{Aut}_G(A_2)$ contains $\text{SL}_2(\ell)$ and $G$ is a normal subgroup of index dividing $\ell - 1$ in $\text{Aut}_G(A_1) \times \text{Aut}_G(A_2)$.
We first deal with the subgroups \( Q \in \mathcal{B} \cup \mathcal{H} \) described above.

**Lemma 4.3.** Let \( Q \in \mathcal{B} \cup \mathcal{H} \). There are unique subgroups \( \tilde{Z} \leq Z(S) \) and \( \tilde{Q} \geq Q \cap [S,S] \) such that

1. If \( Q \in \mathcal{H} \) then \( Q = \tilde{Z} \times \tilde{Q} \) and \( \tilde{Q} \cong C_\ell \times C_\ell \); and
2. If \( Q \in \mathcal{B} \) then \( Z = Z(S) \), \( Q = \tilde{Z} \tilde{Q} \) and \( \tilde{Q} \cong \ell^{1+2} \).

In either case if \( Q \) is \( \mathcal{F} \)-centric radical there is a unique subgroup \( \Theta \leq \text{Aut}(Q) \) containing \( \text{Inn}(Q) \) which acts trivially on \( \tilde{Z} \), normalises \( \tilde{Q} \) and is such that \( \Theta/\text{Inn}(Q) \cong \text{SL}_2(\ell) \). Moreover, there is an \( \ell' \)-subgroup \( X \) of \( \text{Out}_\mathcal{F}(Q) \) such that

\[
\text{Out}_\mathcal{F}(Q) = X(\Theta/\text{Inn}(Q)) \quad \text{and} \quad \text{N}_{\text{Out}_\mathcal{F}(Q)}(\text{Out}_S(Q)) = X(C_\ell \rtimes C_{\ell-1}).
\]

**Proof.** Parts (1) and (2) are shown in \cite{16} Lemma 5.9. The existence of the subgroup \( \Theta \) with the stated properties is argued in the proof of \cite{16} Theorem 5.11 where it is also shown that \( \Theta \) has \( \ell' \)-index in \( \text{Aut}_\mathcal{F}(Q) \). From Lemma 4.2 applied with \( Q \) (case (1)) or \( Q/(Z(S) \cap [S,S]) \) (case (2)) and \( \Theta \) in the roles of \( A \) and \( G \) respectively we have

\[
X_1(\Theta/\text{Inn}(Q)) \leq \text{Out}_\mathcal{F}(Q) \leq X_1 \times X_2
\]

where \( X_2 \cong \text{GL}_2(\ell) \) (see also \cite{16} Lemma 5.9(b)(iv)). This description of \( \text{Out}_\mathcal{F}(Q) \) implies that there must exist a group \( X \) with the stated properties. The lemma follows. \( \square \)

We next deal with automorphisms of \( S \) and \( T \).

**Lemma 4.4.** Suppose \( Q \in \mathcal{B} \cup \mathcal{H} \) is \( \mathcal{F} \)-centric radical and \( X \) is as in Lemma 4.3. For some \( t \in \{0,-1\} \) the following hold:

1. There is an \( \ell' \)-subgroup \( Y \leq \text{Out}_\mathcal{F}(S) \) with \( Y \cong X \) such that
\[
\text{Out}_\mathcal{F}(S) = Y \times \text{Out}_\mathcal{F}(S) \cap \tilde{\mu}^{-1}(\Delta_i),
\]
   where \( \text{Out}_\mathcal{F}(S) \cap \tilde{\mu}^{-1}(\Delta_i) \cong C_{\ell-1} \).
2. There is an \( \ell' \)-subgroup \( Z \leq \text{N}_{\text{Out}_\mathcal{F}(T)}(\text{Aut}_S(T)) \) where \( Z \cong X \) such that
\[
\text{N}_{\text{Out}_\mathcal{F}(T)}(\text{Aut}_S(T)) = Z(\text{Aut}_\mathcal{F}(T) \cap \tilde{\mu}^{-1}(\Delta_i)),
\]
   with \( \text{Aut}_\mathcal{F}(T) \cap \tilde{\mu}^{-1}(\Delta_i) \cong C_\ell \rtimes C_{\ell-1} \).

**Proof.** By \cite{16} Corollary 2.6, we may choose \( x \in S \setminus T \) of order \( \ell \) and \( Q \in \mathcal{B} \cup \mathcal{H} \) so that \( x \in Q \). Plainly, each element of \( \text{Aut}_\mathcal{F}(S) \) normalises \( S/T \cong \langle x \rangle \), so in particular the restriction to \( Q \) of any \( \ell' \)-element in \( \text{Aut}_\mathcal{F}(S) \) lies in \( N_{\text{Aut}_\mathcal{F}(Q)}(\text{Aut}_S(Q)) \). By the Schur–Zassenhaus theorem, \( \text{Inn}(S) \) and \( \text{Aut}_S(Q) \) are complemented by \( \ell' \)-groups \( Y_1 \) and \( Y_2 \) in \( \text{Aut}_\mathcal{F}(S) \) and \( N_{\text{Aut}_\mathcal{F}(Q)}(\text{Aut}_S(Q)) \) respectively. We claim that the restriction map \( \psi : \text{Aut}_\mathcal{F}(S) \to N_{\text{Aut}_\mathcal{F}(Q)}(\text{Aut}_S(Q)) \) restricts to an isomorphism from \( Y_1 \) to \( Y_2 \). If \( \alpha \in \ker(\psi_{|Y_1}) \) then \( \alpha \) acts trivially on \( Q \) so \( \alpha \in \text{Aut}_\mathcal{F}(S) \), \( \tilde{\mu}(\langle \alpha \rangle) = (1,1) \) and hence \( |\alpha| = 1 \) since \( \tilde{\mu}_{|\text{Out}_\mathcal{F}(S)} \) is injective (by \cite{16} Lemma 4.3(a)). We conclude that \( \alpha = 1 \) whence \( \psi_{|Y_1} \) is injective. Conversely, since \( \mathcal{F} \) is saturated, every element of \( Y_2 \) must extend to an element of \( Y_2 \) and \( \psi_{|Y_1} \) is surjective. We conclude that \( Y_1 \cong Y_2 \). By \cite{16} Lemma 5.9(b)(iii), the restriction to \( Q \) of elements in \( \text{Aut}_\mathcal{F}(S) \cap \tilde{\mu}^{-1}(\Delta_i) \) lies in \( N_{\Theta}(\text{Aut}_S(Q)) \cong C_\ell : C_{\ell-1} \). Hence, by Lemma 4.3 we must have
\[
\text{Out}_\mathcal{F}(S) = Y \times \text{Out}_\mathcal{F}(S) \cap \tilde{\mu}^{-1}(\Delta_i) \text{ where } X \cong Y := \{ \alpha \in \text{Out}_\mathcal{F}(S) \mid [\alpha, Z(S)] \not\subseteq Z(S) \cap [S,S] \}. 
\]
(with $X$ is as in Lemma [4.3]). Hence (1) holds. We next prove (2). Since every element of $N_{\text{Aut}_{\mathcal{F}}(T)}(\text{Aut}_{S}(T))$ extends to an automorphism of $S$, the description of this group follows from (1) with $Z = \{\alpha T \mid \alpha \in Y\}$. Moreover, $\text{Aut}_{S}(T) \leq \text{Aut}_{\mathcal{F}}(T), \mu_{\mathcal{F}}(\Delta_t)$ since $[x, Z(S)] \leq Z(S)\cap[S, S]$ for $x \in S \setminus T$ so $\text{Aut}_{\mathcal{F}}(T)\cap\mu_{\mathcal{F}}^{-1}(\Delta_t) \cong C_{\ell} \times C_{\ell-1}$ and (2) holds. □

The structure of $\mathcal{F}$ is now given as follows.

**Theorem 4.5.** Let $\mathcal{F}$ be a simple $\ell$-local compact group on $S$ with discrete torus $T$ of index $\ell$ which is $\mathcal{F}$-centric radical. Then $\mathcal{F}$ is connected and for some $t \in \{0, -1\}$ one of the following holds:

1. $\dim(\Omega_1(T)) = \ell - 1, \mu_T(\text{Aut}_{\mathcal{F}}(T)) \geq \Delta_t$ and $\text{Aut}_{\mathcal{F}}(T) = O_{\ell}(\text{Aut}_{\mathcal{F}}(T))\mu_T^{-1}(\Delta_t)$;
2. $\dim(\Omega_1(T)) \geq \ell, t = 0, \mu_T(\text{Aut}_{\mathcal{F}}(T)) = \Delta_0$ and $\text{Aut}_{\mathcal{F}}(T) = O_{\ell}(\text{Aut}_{\mathcal{F}}(T))\text{Aut}_{\mathcal{F}}(T)$.

In either case,

$$\mathcal{F}_{\text{cr}} = \begin{cases} \{S, T\} \cup \mathcal{H} & \text{if } t = -1 \\ \{S, T\} \cup \mathcal{B} & \text{if } t = 0. \end{cases}$$

where the sets $\mathcal{H}$ and $\mathcal{B}$ each form a single $S$-conjugacy class and there is an $\ell'$-group $X$ such that for each $Q \in \mathcal{H}$ (if $t = -1$) or $Q \in \mathcal{B}$ (if $t = 0$),

Out$_{\mathcal{F}}(S) \cong C_{\ell-1} \times X, \ N_{\text{Aut}_{\mathcal{F}}(T)}(\text{Aut}_{S}(T)) \cong (C_{\ell} : C_{\ell-1})X, \text{ and } \ Out_{\mathcal{F}}(Q) \cong \text{SL}_2(p)X.$

*Proof.* Parts (1) and (2) and the description of $\mathcal{F}_{\text{cr}}$ follow from [16] Theorem B(a). $\mathcal{H}$ and $\mathcal{B}$ each consist of one $S$-conjugacy class by [16] Lemma 5.4. The descriptions of Out$_{\mathcal{F}}(Q)$, Out$_{\mathcal{F}}(S)$ and $N_{\text{Aut}_{\mathcal{F}}(T)}(\text{Aut}_{S}(T))$ follow from more precise descriptions in Lemmas [4.3] and [4.4]. By [16] Corollary 5.2, $T$ is the unique abelian subgroup of index $\ell$ in $S$, and moreover, any $x \in S \setminus T$ is $\mathcal{F}$-conjugate to an element of $Q \cap S$ (as an $\mathcal{F}$-automorphism of $Q = Z(S)\langle x \rangle$ (if $t = -1$) or $Q = Z_2(S)\langle x \rangle$ (if $t = 0$). Hence $\mathcal{F}$ is connected. □

We end this section by showing that $\mathcal{F}$ can be expressed as direct limit of finite fusion systems.

**Proposition 4.6.** Let $\mathcal{F}$ be a simple fusion system over an infinite non-abelian discrete $\ell$-toral group $S$ with discrete torus $T$ of index $\ell$. Then $\mathcal{F} = \varinjlim_{n \geq 1} \mathcal{F}_n$ for saturated fusion systems $\mathcal{F}_n$ on $S_n \leq S$ with $S = \varprojlim_{n \geq 1} S_n$.

*Proof.* If $T$ is not $\mathcal{F}$-centric radical then for any prime $q$ with $\nu_q(q - 1) = 1$ the $\ell$-fusion system of $\text{PSL}_q(q)$ is a direct limit

$$\varinjlim_{n \geq 1} \mathcal{F}_{S_n}(\text{PSL}_q(q^{\ell^n-1})), \text{ where } S = \varprojlim_{n \geq 1} S_n \text{ and } S_n \in \text{Syl}_\ell(\text{PSL}_q(q^{\ell^n-1})).$$

Setting $\mathcal{F}_n$ to be the unique subsystem of $\mathcal{F}_{S_n}(\text{PSL}_q(q^{\ell^n-1}))$ with

$$\text{Aut}_{\mathcal{F}_n}(S_n) = C_{\ell-1}, \text{ Aut}_{\mathcal{F}_n}(T_n) \cong C_{\ell} \times C_{\ell-1} \text{ and } \text{Aut}_{\mathcal{F}_n}(P_n) \cong \text{SL}_2(\ell) \text{ for } P_n \in \mathcal{H},$$

we realise $\mathcal{F}$ as a direct limit of fusion systems.

Now suppose $T$ is $\mathcal{F}$-centric radical, and for each $P \in \mathcal{F}_{\text{cr}}$ and $n \in \mathbb{N}$, let $P_n$ be the $\ell^n$-torsion subgroup of $P$. Then $P = \varinjlim_{n \geq 1} P_n$ and we may define

$$\mathcal{F}_n := \langle \{\varphi|P_n \mid \varphi \in \text{Out}_\mathcal{F}(P)\} \mid P \in \mathcal{F}_{\text{cr}} \rangle.$$
By \cite[Proposition 2.3]{3}, $\Out_{\mathcal{F}_n}(P_n) = \Out_{\mathcal{F}}(P_n)$ is finite. Moreover $T_n$ is the unique abelian maximal subgroup of $S_n$ (since $\mathcal{F}$ is connected). Now by \cite[Theorem 2.8]{4}, $\mathcal{F}_n$ is a saturated fusion system on $S_n$ uniquely determined by $\Aut_{\mathcal{F}_n}(T_n)$, $T_n$, $\Aut_{S_n}(T_n)$ and $\mathcal{F}_n^{cr}\setminus\{S_n, T_n\} \subseteq H_n \cup B_n$ where

$$H_n = \{Z(S_n)\langle x \rangle \mid x \in S_n\setminus T_n\} \text{ and } B_n = \{Z_2(S_n)\langle x \rangle \mid x \in S_n\setminus T_n\}.$$ 

Moreover $\mathcal{F}_n \subseteq \mathcal{F}_{n+1}$ for all $n \geq 1$ by construction. The result follows. \qed

5. PROOFS OF THEOREMS 1.2 AND 1.4

Combining the results in Sections 3 and 4 we can now prove:

**Theorem 5.1.** Let $\mathcal{F}$ be a simple $\ell$-local compact group on $S$ with discrete torus $T$ of index $\ell$ and Weyl group $W = \Aut_{\mathcal{F}}(T)$ Then $\mathcal{F}$ is connected and

$$\text{w}(\mathcal{F}) = |\Irr(W)|.$$ 

**Proof.** By the remarks following Theorem 4.1 we may assume that $T$ is $\mathcal{F}$-centric radical. By Theorem 4.5 $\mathcal{F}$ is connected and either $\mathcal{F}^{cr} = \{S, T\} \cup H$ or $\mathcal{F}^{cr} = \{S, T\} \cup B$. Set $U = \Aut_{S}(T)$ for short. By Theorem 4.5 and Lemma 3.1 for any representative $Q \in H$ or $Q \in B$ we have,

$$|\Irr(W)| - \text{w}(\mathcal{F}) = |\Irr(W)| - (z(kW) + z(k\Out_{\mathcal{F}}(Q)) + z(k\Out_{\mathcal{F}}(S)))$$

$$= |\Irr(N_{W}(U))| - |\Irr(\Out_{\mathcal{F}}(S))| - z(k\Out_{\mathcal{F}}(Q)).$$

The descriptions of the groups $N_{W}(U)$, $\Out_{\mathcal{F}}(S)$ and $\Out_{\mathcal{F}}(Q)$ in Theorem 4.5 combined with Lemma 3.4 then yield:

$$|\Irr(W)| - \text{w}(\mathcal{F}) = \ell|\Irr(X)| - (\ell - 1)|\Irr(X)| - z(k(\text{SL}_2(\ell)X))$$

$$= |\Irr(X)| - z(k(\text{SL}_2(\ell)X))$$

$$= 0,$$

where $X$ is as in Lemma 4.3. The result follows from this. \qed

**Theorem 5.2.** Let $\mathcal{F}$ be a simple $\ell$-local compact group with $v_{\ell}(|W|) = 1$ and write $\mathcal{F} = \lim_{n\geq 1}\mathcal{F}_n$ for saturated fusion systems $\mathcal{F}_n$ on $S_n \leq S$ as described in Proposition 4.6. For each $n$ we have,

$$|\mathcal{M}(\mathcal{F}_n)| = |\mathcal{P}(\mathcal{F}_n)|.$$ 

**Proof.** Let $n \in \mathbb{N}$ be fixed and set $T_n := T \cap S_n$, $W = \Aut_{\mathcal{F}}(T_n) = \Aut_{\mathcal{F}_n}(T_n)$ and $U = \Aut_{S_n}(T_n) \leq W$. We have two bijections

$$\mathcal{M}(\mathcal{F}_n) \longleftrightarrow \Irr(S_n^{ab} : \Out_{\mathcal{F}}(S_n)) \longleftrightarrow \Irr(T_n N_{W}(U)),$$

the first from the discussion before Lemma 2.10 and the second from the isomorphism $S_n^{ab} : \Out_{\mathcal{F}}(S_n) \cong T_n N_{W}(U)$ (implied by Theorems 4.1 and 4.5). By Lemma 3.2 this latter set is in bijection with equivalence classes of pairs

$$\{ (\psi, \chi) \mid \psi \in \Irr(T_n), \chi \in \Irr(C_{N_{W}(U)}(\psi)) \mid [S_n, S_n] \leq \ker(\psi) \} / \sim_{\mathcal{F}_n}$$

where $(\psi, \chi) \sim_{\mathcal{F}_n} (\psi', \chi')$ if there is $\rho \in N_{W}(U)$ with $\psi^\rho = \psi'$ and $\chi'^{-1} = \chi$. By the Glauberman-Isaacs correspondence there exists a $N_{W}(U)$-equivariant bijection between
$T_n$ and $\text{Irr}(T_n)$ which maps $Z(S_n) = C_{T_n}(U)$ to $\{ \psi \in \text{Irr}(T_n) \mid [S_n, S_n] \leq \ker(\psi) \}$, and hence $\mathcal{M}(\mathcal{F}_n)$ is in bijection with

$$\{(s, \chi) \mid s \in Z(S_n), \chi \in \text{Irr}(C_{N_W(U)}(s)) \}/\sim_{\mathcal{F}_n}$$

where $(s, \chi) \sim_{\mathcal{F}_n} (s', \chi')$ if and only if there is $\phi \in N_W(U)$ with $s' = \phi(s)$ and $\chi' = \chi^{\phi^{-1}}$. Now since every morphism between elements of $Z(S_n)$ extends to an $\mathcal{F}_n$-automorphism of $S_n$ and $|\text{Irr}(C_{N_W(U)}(s))| = |\text{Irr}_0(C_{N_W(U)}(s))| = |\text{Irr}_0(C_W(s))|$ for each $s \in Z(S_n)$ (by Lemma 6.31), we have $|\mathcal{M}(\mathcal{F}_n)| = \sum_{s \in Z(S_n)/N_W(U)} |\text{Irr}(C_{N_W(U)}(s))| = \sum_{s \in Z(S_n)/\mathcal{F}_n} |\text{Irr}_0(C_W(s))| = |\mathcal{P}(/\mathcal{F}_n)|$, as required.

\[\square\]

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