Hidden connection between
general relativity and Finsler geometry

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Abstract

Modern formulation of Finsler geometry of a manifold $M$ utilizes the equivalence between this geometry and the Riemannian geometry of $VTM$, the vertical bundle over the tangent bundle of $M$, treating $TM$ as the base space. We argue that this approach is unsatisfactory when there is an indefinite metric on $M$ because the corresponding Finsler fundamental function would not be differentiable over $TM$ (even without its zero section) and therefore $TM$ cannot serve as the base space. We then make the simple observation that for any differentiable Lorentzian metric on a smooth space-time, the corresponding Finsler fundamental function is differentiable exactly on a proper subbundle of $TM$. This subbundle is then used, in place of $TM$, to provide a satisfactory basis for modern Finsler geometry of manifolds with Lorentzian metrics. Interestingly, this Finslerian property of Lorentzian metrics does not seem to exist for general indefinite Finsler metrics and thus, Lorentzian metrics appear to be of special relevance to Finsler geometry. We note that, in contrast to the traditional formulation of Finsler geometry, having a Lorentzian metric in the modern setting does not imply reduction to pseudo-Riemannian geometry because metric and connection are entirely disentangled in the modern formulation and there is a new indispensable non-linear connection, necessary for construction of Finsler tensor bundles. It is concluded that general relativity—without any modification—has a close bearing on Finsler geometry and a modern Finsler formulation of the theory is an appealing idea. Furthermore, in any such attempt, the metric should probably be left unchanged (not generalized) or the newly discovered property, which provides a satisfactory basis for the corresponding Finsler geometry, would be lost.

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1 Introduction

Finsler geometry is widely considered as the most natural generalization of Riemannian geometry or even closer (just a “more developed” form \([1]\)). On the other hand, general relativity is the first and by far the most important physical application of (pseudo-) Riemannian geometry. It is therefore natural to seek for a viable Finsler formulation of general relativity. Indeed, there is a long and extensive history of research in this area and the field has been steadily growing to this date. See, e.g., \([2, 3, 4, 5, 6]\) and the references therein.

Finsler geometry has originated from two simple innovations in Riemannian geometry, namely\(^1\):

(a) Supplementation of the position parameter in geometric quantities with a new independent vector variable. Here, this is given the name “Finsler parameter”.

(b) Use of a norm, here called “Finsler fundamental function” (a scalar distance function of position and the Finsler parameter) in order to implement the first technique.

As we shall see, a Finsler fundamental function is equivalent to a metric tensor (see equations (3) and (4)) and some authors use the term metric to mean a fundamental function, however, for more clarity, the term metric is used in this article only to mean a metric tensor.

The range of Finsler parameter is usually assumed to be all non-zero tangent vectors and skipped over quickly, however, the subject merits more attention, particularly, in the case of metrics with indefinite signatures. Finsler parameter can be present in intrinsic geometric quantities such as connections and curvatures, which in a differential geometric context, all need to be differentiable, albeit, not infinitely. It is therefore necessary that this parameter takes only values for which all such quantities are well-behaved. The most natural and practical way to determine the range of Finsler parameter is evidently through Finsler fundamental function. Domain of differentiability of this function seems the best (and the only available) candidate for the purpose. However, life is not that simple. As we shall see, the range of Finsler parameter has to be a fibre bundle in order to obtain a vertical bundle, absolutely necessary in the modern formulation. When the metric is positive definite this requirement is easily satisfied because the corresponding fundamental function is differentiable for all non-zero tangent vectors, which form a fibre bundle. This is the prevailing situation in most studies of Finsler geometry and its applications. However, for an indefinite metric, domain of differentiability of the fundamental function is more restricted and it is not clear if it forms a fibre bundle in general.

2 A brief review of modern Finsler geometry

In order to have a closer look at modern formulation of Finsler geometry, let \(M\) be a manifold and \(TM\) its tangent bundle. Also let \(x\) represent a point of \(M\)

\(^1\)For a lucid exposition to modern Finsler geometry, see \([7]\) and for a recent account of the classical treatment, consult \([8]\).
with coordinates \( \{x^\mu\} \), \((x, y)\) a point of \( TM \), and \( \{y^\mu\} \) coordinates of the tangent vector \( y \) with respect to the natural basis \( \{\partial/\partial x^\mu\} \). Einstein summation convention is used throughout. Being merely a tangent vector to \( M \), Finsler parameter has classically no proper geometrical basis to work with. This is clearly illustrated by the classical “Finsler vector field” \( X^\mu(x, y)(\partial/\partial x^\mu)_M \) on \( M \), which is not in fact a true vector field on \( M \) (an assignment of at most one vector to each point \( x \)). The most direct and natural attempt to accommodate Finsler parameter geometrically has been to consider \( TM \) as the base space and Finsler vector fields as sections of \( \pi^*TM \) (the pull back of \( TM \) to itself by its own projection \( \pi \)). Although some authors may be happy with such an improvement, this is still unsatisfactory and cumbersome to work with because sections of \( \pi^*TM \) are not tangent vector fields to \( TM \). And similarly, sections of the dual bundle cannot be considered as differential forms on \( TM^2 \). The natural isomorphism between \( \pi^*TM \) and \( VTM \) (the vertical bundle over \( TM \)) \[11, p. 18\] helps to remedy the situation by mapping sections of \( \pi^*TM \) to sections \( X^\mu(x, y)(\partial/\partial y^\mu)_TM \) of \( VTM \), which are tangent vector fields to \( TM \). Riemannian geometry of \( VTM \) would thus provide a lucid and satisfactory framework for Finsler geometry of \( M \). This is what is meant by modern formulation of Finsler geometry in this article. There is a minor variation of this formulation in which the zero section of \( TM \) is removed \[12\].

The basic limitation in this modern \( VTM \)–formulation, which we wish to point out, is due to the implicit assumption that: All (non-zero) tangent vectors of \( M \) are admissible values for the Finsler parameter. As explained in the introduction, this assumption is justified when we have a positive definite metric or none at all. However, for any space with an indefinite metric, the corresponding Finsler fundamental function is not differentiable over \( TM \) or \( TM \setminus \{\text{zero section}\} \) and hence none of these would be a suitable basis for formulating the corresponding Finsler geometry. Our new simple result (see next section) offers exactly the further improvement that we need for the important case of Lorentzian metrics.

### 3 A new result

We need only a few basic relations in Finsler geometry. To collect these, let \( N \) be some open submanifold of tangent bundle \( TM \). A Finsler fundamental function is defined as a map \( F: N \to \mathbb{R} \), satisfying a varying set of conditions. Naturally, first-degree homogeneity in \( y \) is nearly always among these conditions,

\[
F(x, ky) = kF(x, y), \quad \forall k > 0, \quad \forall (x, y) \in N,
\]  

where, it is implicitly assumed that if \((x, y) \in N\) then so is \((x, ky) \forall k > 0\). Some authors restrict choice of \( N \) to only \( TM \), however, important classes of Finsler spaces would be lost by this restriction \[7, p. 13\]. Applying Euler theorem on homogeneous functions to \( F \) yields:

\[
F^2 = \frac{1}{2} \frac{\partial^2 F^2}{\partial y^\mu \partial y^\nu} y^\mu y^\nu.
\]

\[1\] Fibres of \( \pi^*TM \) are spanned by basis vectors \((\partial/\partial x^\mu)_x \), which are tangent only to \( M \). These vectors should not be confused with similar looking objects \((\partial/\partial x^\mu)(x, y) \), which transform differently under a coordinate transformation of \( M \). See, e.g., \[7, p. 11\].

\[2\] Differential 1-forms on \( TM \) are sections of only \( T^*TM \).
Finsler metric tensor is classically defined by:
\[ G_{\mu\nu}(x, y) = \frac{1}{2} \frac{\partial^2 F^2}{\partial y^\mu \partial y^\nu}, \]  
(3)
where, the \( y \)-Hessian of \( F^2 \) is assumed to be of maximal rank and hence of fixed signature. \( G_{\mu\nu} \) may be definable only on a subset of \( N \) because \( F \) may be not differentiable on the whole of \( N \). Combining equations (2) and (3) yields the important relation:
\[ F^2(x, y) = G_{\mu\nu}(x, y) y^\mu y^\nu. \]  
(4)
Alternative to the classical approach, given any arbitrary zero-degree \( y \)-homogeneous Finsler metric tensor, we can consider equation (4) as the definition of the Finsler fundamental function corresponding to the given metric. Property (2) ensures that equations (3) and (4) are not only consistent, but equivalent, given only that the metric tensor in equation (4) is zero-degree \( y \)-homogeneous. Needless to say, any (pseudo-) Riemannian metric is also a \( y \)-homogeneous Finsler metric, albeit a special one. In what follows, “differentiable” shall mean differentiable of class \( C^k \) with \( k \) as large as necessary, and “Lorentzian metric”, a pseudo-Riemannian metric of signature \((+---)\). We can now state an intriguing result which has a simple proof:

**Proposition 1** Given any differentiable Lorentzian metric on a smooth space-time, the corresponding Finsler fundamental function is differentiable exactly on a fibre bundle over the space-time.

**Proof** Let \( M \) be a \( C^\infty \) space-time manifold with a differentiable Lorentzian metric \( g_{\mu\nu}(x) \) and assume that the corresponding Finsler fundamental function \( F(x, y) := (g_{\mu\nu} y^\mu y^\nu)^{1/2} \) is defined over the largest possible domain \( N := \{ (x, y) \in TM \mid g_{\mu\nu} y^\mu y^\nu \geq 0 \} \). Clearly, \( N \) has a boundary in \( TM \) given by \( \{ (x, y) \in TM \mid g_{\mu\nu} y^\mu y^\nu = 0 \} \) at which \( F \) cannot, by definition, be differentiable\(^3\). Elsewhere in \( N \), \( F \) can easily be seen to be differentiable. Hence, \( F \) is differentiable exactly on:
\[ LM := \{ (x, y) \in TM \mid g_{\mu\nu} y^\mu y^\nu > 0 \}. \]  
(5)
We see that \( LM \) is simply made of all timelike vectors, yet surprisingly, no proof or statement to the effect that \( LM \) is in fact a fibre bundle, is found in the literature. Here is, therefore, a detailed proof of that, wherein, the definition for a fibre bundle is taken from [11] and followed closely:

(i) Three \( C^\infty \) manifolds are needed to start with. The base manifold is already given and the total space \( LM \) is an open subset of the \( C^\infty \) manifold \( TM \) and hence, a \( C^\infty \) manifold [2, p. 7]. As for the standard fibre, let \( V \) be a real four-dimensional vector space with an inner product \( \eta \) of signature \((+---)\) and define \( L \) to be the open subset \( \{ v \in V \mid \eta(v, v) > 0 \} \). Being a real vector space, \( V \) is also a \( C^\infty \) manifold [3] p. 7] and hence so is \( L \), which would be our standard fibre for \( LM \).

\(^3\) Clearly, at no boundary point, can all directional derivatives of a function exist, see, e.g., [9] p. 5] or [10] p. 349].
(ii) Given the projection map $\pi: TM \to M$, its restriction $\chi := \pi|_{LM}$ serves as the projection for $LM$.

(iii) Let $\mathcal{C}$ be an open covering of $M$ such that there is a complete set of orthonormal basis vector fields defined on each $U \in \mathcal{C}$. Denote one such frame on $U \in \mathcal{C}$ by $\{e_a\}$ and the corresponding co-frame by $\{e^a\}$: $e^a(e_b) = \delta^a_b$, $a, b = 0, 1, 2, 3$. Define $\varphi: \chi^{-1}U \to L$ by $\varphi(x, y) = e^a(y)f_a$, where, $\{f_a\}$ is an orthonormal basis for $V$ with $\eta(f_a, f_b) = g(e_a, e_b)$. For any $x \in U$, the map $\varphi|_x: \chi^{-1}(x) \to L$ is clearly a diffeomorphism, and hence, so is $(\chi, \varphi): \chi^{-1}U \to U \times L$. Local triviality of $LM$ is thus established. □

4 Modern Finsler geometry for Lorentzian metrics

To distinguish Finsler parameter from an arbitrary tangent vector, let us denote it from now on by $z$ rather than $y$. Given any Lorentzian metric, the crucial step we now take is to let bundle $LM$, defined by equation (5), be our “Finsler base space”: the space of admissible values of $z$ or more precisely $(x, z)$. The justification is that the corresponding fundamental function is differentiable (and non-zero) exactly on $LM$. More importantly, being a fibre bundle, $LM$ is endowed with a natural vertical bundle $VLM$, which has all the crucial properties of $VTM$:

Proposition 2 Fibres of $VLM$, $VTM$ and $TM$ are isomorphic as vector spaces and have the same coordinate transformations under changes of coordinates on $M$.

Proof $VTM$ and $TM$ already have isomorphic fibres with the same coordinate transformations [7, p. 19]. To prove that $VLM$ is also in this category, it suffices to note that fibres of $VLM$ and $VTM$ are tangent spaces to fibres of $LM$ and $TM$ respectively, and that, fibres of $LM$ are open subsets of fibres of $TM$. Therefore, fibres of $VLM$ are in fact also fibres of $VTM$ and have the same properties. □

Through this simple modification, Finsler parameter has been effectively raised to the status of a coordinate parameter and sections $X^\mu(x, z)(\partial/\partial y^\mu)_{TM}$ are traded for the more natural ones $X^\mu(x, z)(\partial/\partial z^\mu)_{LM}$. It is easy to verify that further constructions, such as non-linear connection, Finsler connection, etc., can all be obtained for $VLM$ in a straightforward manner. See, e.g., [7, p. 109] for a general setup.

An important feature of modern Finsler geometry is that the treatments of connection and metric are conveniently decoupled and in such a way that the geometry can never be reduced to Riemannian geometry of the original manifold even when the metric is Riemannian: There is a new indispensable non-linear connection, which is vital for the construction of dual spaces to fibres of the above vertical bundles [7, p. 111] and hence also essential in the construction of the necessary vertical tensor bundles. This feature of modern Finsler geometry is in direct contrast to the widely used classical formulation, where every thing hinges on a Finsler metric tensor such that reduction of this to a Riemannian metric implies reduction to
Riemannian geometry \[1, 3, 5\].

It is interesting to note that, the above approach yields a satisfactory basis for modern Finsler geometry only for $z$-independent metrics. While, for any $z$-dependent indefinite metric, a corresponding fundamental function exists, such a function would not in general be differentiable on the whole of $TM$, nor on any other known fibre bundle. For example, assuming that $F$ is differentiable for $G_{\mu\nu}(x, z) z^\mu z^\nu > 0$, the spaces $\{(x, z) \in T_x M \mid G_{\mu\nu}(x, z) z^\mu z^\nu > 0\}$ would depend so non-trivially on $z$ that it is hard to imagine how they can form a fibre bundle in general. Trying to remedy this situation by resorting to the classical strategy of starting with a general indefinite $F$ instead, would not help because we still need a Finsler base space, in the form of a fibre bundle over which $F$ is differentiable, in order to obtain the necessary vertical bundles. Consequently, an appropriate framework for modern Finsler geometry of general indefinite metrics has yet to be found.

On the other hand, for Lorentzian metrics, the simple proof of proposition 1 allows the conditions of global space-time smoothness and metric differentiability to be reduced to local ones. Thus, the above approach based on $VLM$, is generalizable to even local Lorentzian metrics. The generalized form of proposition 1 would be:

**Proposition 3** Given a Lorentzian metric, differentiable over a smooth open subset $U$ of space-time, the corresponding Finsler fundamental function, restricted to $U$, is differentiable exactly on a fibre bundle over $U$. □

5 Conclusions

In conclusion, we see that general relativity—without any modification—has a close bearing on Finsler geometry. Propositions 1-3 provide some “mathematical” evidence to support this conclusion. The generality and inherent simplicity of these propositions indicate that the connection between general relativity and Finsler geometry is not artificial or lightly dispensable. Accordingly, searching for a viable modern Finsler formulation of general relativity would now seem more appealing and hopeful than before. It seems as though general relativity has some special built-in provisions for Finsler geometry and it would be interesting to see if there are any further clues in this avenue yet to be discovered.

The second conclusion is that, in any viable Finsler formulation of general relativity, the Lorentzian metric should probably not be generalized to a $z$-dependent one or the newly discovered connection, which provides a satisfactory basis for the corresponding Finsler geometry, would be lost. Naturally, this point can greatly simplify the search for a viable Finsler formulation of general relativity, albeit a great deal of work still remains.

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References

[1] S. S. Chern, Not. Am. Math. Soc. 43 (1996) 959.

[2] G. Randers, Phys. Rev. 59 (1941) 195; G. Stephenson and C. W. Kilmister, Nuovo Cimento 10 (1953) 230; G. S. Asanov, Nuovo Cimento 49B (1979) 221; A. A. Coley, Gen. Rel. Grav. 14 (1982) 1107; H. Rund, Found. Phys. 13 (1983) 93; G. S. Asanov, Found. Phys. 13 (1983) 501; A. K. Aringazin and G. S. Asanov, Gen. Rel. Grav. 17 (1985) 900.

[3] J. D. Bekenstein, Phys. Rev. D 48 (1993) 3641.

[4] S. F. Rutz, Gen. Rel. Grav. 25 (1993) 1139; J. P. Hsu, Nuovo Cimento 108B (1993) 183 and 109B (1994) 645; H. Akbar-Zadeh, J. Geom. Phys. 17 (1995) 342; S. I. Vacaru, Nucl. Phys. B 494 (1997) 590; M. Panahi and M. Mehrafarin, J. Geom. Phys. 25 (1998) 346.

[5] G. Yu. Bogoslovsky and H. F. Goenner, Phys. Lett. A 244 (1998) 222.

[6] H. F. Goenner and G. Yu. Bogoslovsky, Gen. Rel. Grav. 31 (1999) 1383; G. Yu. Bogoslovsky and H. F. Goenner, Gen. Rel. Grav. 31 (1999) 1565; S. S. De, Int. J. Theor. Phys. 38 (1999) 2419; J. G. Vargas and D. G. Torr, Found. Phys. 29 (1999) 145; T. P. Storer, Int. J. Theor. Phys. 39 (2000) 1351; H. E. Brandt, Found. Phys. Lett. 13 (2000) 307 and 13 (2000) 581; F. Giannoni, A. Masiello and P. Piccione, J. Geom. Phys. 35 (2000) 1; J. G. Vargas and D. G. Torr, Int. J. Theor. Phys. 40 (2001) 275.

[7] A. Bejancu, Finsler Geometry and Applications (Ellis Horwood, London) 1990.

[8] G. S. Asanov, Finsler Geometry, Relativity and Gauge Theories (D. Reidel, Dordrecht) 1985.

[9] F. W. Warner, Foundations of Differentiable Manifolds and Lie Groups (Scott, Foresman and company, Glenview, Illinois) 1971.

[10] R. G. Bartle, The Elements of Real Analysis (John Wiley & Sons, New York) 1976.

[11] W. A. Poor, Differential Geometric Structures (McGraw-Hill, New York) 1981.

[12] M. Abate and G. Patrizio, Finsler metrics—A global approach (Springer-Verlag, Berlin) 1994.