Approximation to the stable law by Lindeberg principle

Peng Chen and Lihu Xu
University of Macau

Abstract: By Lindeberg principle, we develop in this paper an approximation to one dimensional (possibly) asymmetric \( \alpha \)-stable distributions with \( \alpha \in (0, 2) \) in smooth Wasserstein distance, which implies the stable central limit theorem. Our main tools are Taylor-like expansion and Dynkin’s formula of stable process.

Key words: asymmetric \( \alpha \)-stable distribution; Lindeberg principle; Taylor-like expansion; Dynkin’s formula of stable process; stable central limit theorem.

Contents

1. Motivation and main results
2. A short proof of Theorem 1.4 in a special case to illustrate the main idea
3. Preliminaries of stable processes and nonlocal operators
   3.1. Estimates for the operator \( \mathcal{L}^{\alpha, \beta} \)
   3.2. Truncation for asymmetric \( \alpha \)-stable process \( \hat{Y} \)
   3.3. Truncation for random variable \( X \)
4. Taylor-like expansions
   4.1. Taylor-like expansions for Theorem 1.4
   4.2. Taylor-like expansions for Theorem 1.7
5. Proof of Theorem 1.4 and Theorem 1.7
6. A more difficult example: Proof of (1.7)

References

1. Motivation and main results

Let \( S_n = X_1 + \ldots + X_n \) be a sum of i.i.d. random variables whose common distribution is heavy tailed. The stable central limit theorem (CLT) tells us that under some condition \[19\], there exists some \( c_n \) such that \( n^{-1/\alpha}(S_n - c_n) \) converges to an \( \alpha \)-stable distribution \( \mu \) with \( \alpha \in (0, 2) \) as \( n \to \infty \). Moreover, there have been many works studying the convergence rate of stable CLT in Kolmogorov distance. Recently, several works considered the convergence rate in Wasserstein-1 distance or smooth Wasserstein distance by Stein’s method \[43, 11, 10, 33\] or Tikhomirov-Stein’s method \[2, 8\].

Lindeberg’s proof \[30\] avoids the use of characteristic functions and gives a new and easy-following way to prove the normal CLT, it is now well known and has been well developed to study other limit theorems. Chatterjee \[9\] first applied Lindeberg principle to identify the limiting spectral distribution of Wigner matrices with exchangeable entries, then Tao and Vu \[39\] generalized this idea to prove the long standing conjecture that the university of local eigenvalue of random matrices is determined by the first four moments of the distribution of entries. By Lindeberg principle again, Caravenna et. al. \[8, 7\] obtained a general scaling of disordered system by expanding polynomial chaos. Besides a lot of applications in random matrices \[4, 6, 20, 11, 29, 42, 1, 5\], Lindeberg principle has also been applied to other research.
areas such as high dimensional regressions \cite{14, 15, 20}, time series \cite{21, 41, 32, 31}, bootstrap \cite{35, 15}, statistical learning \cite{27, 44} and so on.

Although stable CLT is one of the most important limit theorems in probability theory, to the best of our knowledge, surprisingly there have not been any works which apply Lindeberg principle to prove stable CLT. One motivation of this paper is to fill this gap. Note that a Lindeberg’s condition, which is completely different from Lindeberg principle, was proposed in \cite{25} to prove stable CLT.

The main contribution of this paper is that we first prove general stable CLT with $\alpha \in (0, 2)$ by Lindeberg principle, and that our results further provide convergence rates in smooth Wasserstein distance, which match the best known rates in Kolmogorov distance. Note that there is no subordination between the above two distances. Although \cite{25} proved symmetric stable CLT for $\alpha \in (0, 2)$ in Mallow distance by a maximal coupling argument, its argument seems to heavily depend on the symmetry assumption and the related convergence rate is far from the best one. To the best of our knowledge, when $\alpha \in (0, 1]$, our paper is the first try which succeeds proving asymmetric stable CLT by a method other than characteristic function.

Let us give a brief discussion on our main results. Theorem \ref{thm.gen_stable_CLT} below provides a general convergence rates in smooth Wasserstein distance for stable CLT with $\alpha \in (0, 2)$ when $X_1$ has a distribution which falls in the domain of normal attraction of stable law, while Theorem \ref{thm.app_gen_stable_CLT} further improves the rate for the case $\alpha \in (0, 1]$ under a slightly stronger condition. The convergence rate in Theorem \ref{thm.app_gen_stable_CLT} matches the optimal one in Kolmogorov distance found by Hall \cite{22}, see more details in Remark \ref{rem.app_gen_stable_CLT}. When $X_1$ is out of the scope of normal attraction of stable law, we also found a convergence rate, which is in the same order as the best rate reported in \cite{24}.

To apply Lindeberg principle to prove normal CLT, one only needs to use a third order Taylor expansion and control the remainder, but it does not work for stable CLT. Alternatively, we develop a Taylor-like expansion and use the Dynkin’s formula of stable process to handle the remainder. When $\alpha \in (1, 2)$, this expansion is similar to that in \cite{11}. As $\alpha \in (0, 1]$ due to the lack of first moment, we need to use a truncation technique and estimate the remainder in a much more delicate way.

To describe our results in a more explicit way, we first start with a definition of an $\alpha$–stable distribution.

**Definition 1.1.** Let $\alpha \in (0, 2)$, $\sigma > 0$ and $\beta \in [-1, 1]$ be real numbers. We say that $Y$ is an $\alpha$-stable distributed random variable with parameters $\sigma$ and $\beta$, writing $Y \sim S_\alpha(\sigma, \beta)$, if for all $\lambda \in \mathbb{R},$

$$E[e^{i\lambda Y}] = \begin{cases} \exp\left\{ -\sigma^\alpha|\lambda|^\alpha(1 - i\beta \text{sign}(\lambda) \tan \frac{\pi \alpha}{2}) \right\} & \text{if } \alpha \neq 1 \\ \exp\left\{ -\sigma|\lambda|(1 + i\beta \frac{2}{\pi} \text{sign}(\lambda) \log \lambda) \right\} & \text{if } \alpha = 1 \end{cases}.$$  

In particular, when $\beta = 0$, we say that $Y$ is distributed according to the symmetric $\alpha$-stable law of parameter $\sigma$, and write $Y \sim S_\alpha S(\sigma)$.

It is immediate to check that $Y/\sigma \sim S_\alpha(1, \beta)$ if $Y \sim S_\alpha(\sigma, \beta)$. So, starting from now and without loss of generality, we will only consider stable distributions with $\sigma = 1$.

Let $(\hat{Y}_t)_{t \geq 0}$ be a one-dimensional $\alpha$–stable process with $\hat{Y}_0 = 0$ such that the distribution of $\hat{Y}_t$ has a density $p(t, x)$ satisfying

$$E\left[e^{i\lambda \hat{Y}_t}\right] = e^{-t \psi(\lambda)},$$  

where

$$\psi(\lambda) = \begin{cases} |\lambda|^\alpha(1 - i\beta \text{sign}(\lambda) \tan \frac{\pi \alpha}{2}) & \text{if } \alpha \neq 1 \\ |\lambda|(1 + i\beta \frac{2}{\pi} \text{sign}(\lambda) \log \lambda) & \text{if } \alpha = 1 \end{cases}.$$
We denote $C_b^k(\mathbb{R}) = \{ f : \mathbb{R} \to \mathbb{R}; f, f', \ldots, f^{(k)} \text{ are all continuous and bounded functions} \}$. The infinitesimal generator $\mathcal{L}^{\alpha,\beta}$ of $(Y_t)_{t \geq 0}$ is

**Definition 1.2.** Let $\alpha \in (0, 2)$ and $\beta \in [-1, 1]$. For $f : \mathbb{R} \to \mathbb{R}$ in $C_b^2(\mathbb{R})$, we define

$$\mathcal{L}^{\alpha,\beta} f(x) = d_\alpha \int_{\mathbb{R}} \frac{f(y + x) - f(x) - y^{(\alpha)} f'(x)}{2y^{1+\alpha}} \left[ (1 + \beta) \mathbf{1}_{(0,\infty)}(y) + (1 - \beta) \mathbf{1}_{(-\infty,0)}(y) \right] \, dy,$$

where $d_\alpha = \left( \int_0^\infty \frac{1 - \cos y}{y^{1+\alpha}} \, dy \right)^{-1}$ and

$$y^{(\alpha)} = \begin{cases} y, & \alpha \in (1, 2), \\ y \mathbf{1}_{(-1,1)}(y), & \alpha = 1, \\ 0, & \alpha \in (0, 1). \end{cases}$$

When $\beta = 0$, $\mathcal{L}^{\alpha,0}$ reduces to the fractional Laplacian $\Delta_0^\alpha$ of order $\alpha/2$.

Recall that Wasserstein-1 distance between two probability measures $\mu_1$ and $\mu_2$ is defined by

$$W_1(\mu_1, \mu_2) = \sup_{h \in \text{Lip}(1)} |\mu_1(h) - \mu_2(h)|,$$

where $\text{Lip}(1) = \{ h : \mathbb{R} \to \mathbb{R}; |h(y) - h(x)| \leq |y - x| \text{ and } \mu_i(h) = \int h(x) \mu_i(dx) \text{ for } i = 1, 2 \}$ and the Kolmogorov distance is

$$d_{Kol}(\mu_1, \mu_2) = \sup_{x \in \mathbb{R}} |\mu_1((\infty, x]) - \mu_2((\infty, x])|.$$

The smooth Wasserstein distance of order $k \in \mathbb{N}$ ([2, (4.1)]) is

$$d_{W_k}(\mu_1, \mu_2) = \sup_{h \in \mathcal{H}_k} |\mu_1(h) - \mu_2(h)|,$$

where $\mathcal{H}_k$ is the set of all bounded $k$-th order differentiable functions $h$ such that $||h^{(j)}|| \leq 1$ for $j = 0, 1, \ldots, k$.

**Definition 1.3.** [23, Theorem 2.6.7] Let $F_X$ denote the distribution of a random variable $X$. A necessary and sufficient condition for $F_X$ to lie in the domain of normal attraction of a stable law of exponent $\alpha$ is that it admits the representations

$$F_X(x) = \left(1 - \frac{A + \epsilon(x)}{x^\alpha}(1 + \beta)\right) \mathbf{1}_{(0,\infty)}(x) + \frac{A + \epsilon(x)}{|x|^\alpha}(1 - \beta) \mathbf{1}_{(-\infty,0)}(x),$$

where $\alpha \in (0, 2)$, $A > 0$, $\beta \in [-1, 1]$ and $\epsilon : \mathbb{R} \to \mathbb{R}$ is a measurable function vanishing at $\pm \infty$.

It is obvious from the Definition 1.3 that there exist some constant $K$ and some $\gamma \geq 0$ such that

$$|\epsilon(x)| \leq \frac{K}{|x|^\gamma}, \quad x \neq 0.$$  

(1.4)

Before stating our main result, let us have a discussion on the case $\alpha = 1$. It is known that when $\beta \neq 0$ and $\alpha = 1$, the random variable is not strictly stable because it does not have the scaling property (see e.g., [38, Theorem 14.15]). So, to rule out this singularity, we always assume $\beta = 0$ as $\alpha = 1$.

**Theorem 1.4.** Let $X_1, X_2, \ldots$ be independent and identically distributed random variables defined on a common probability space, and suppose that $X_1$ has a distribution of the form (1.3) with $\epsilon(x)$ satisfying (1.4). Set $\sigma = \left( A \alpha \int_\mathbb{R} \frac{1 - \cos y}{y^{1+\alpha}} \, dy \right)^{\frac{1}{\alpha}}$ and

$$S_n = \frac{1}{\sigma} n^{-\frac{1}{\alpha}} \begin{cases} X_1 + \cdots + X_n - n \mathbb{E}[X_1], & \alpha \in (1, 2), \\ X_1 + \cdots + X_n - n \mathbb{E}[X_1] \mathbf{1}_{(0,\sigma n^\frac{1}{\alpha})}(|X_1|), & \alpha = 1, \\ X_1 + \cdots + X_n, & \alpha \in (0, 1). \end{cases}$$

(1.5)
Consider \( Y \sim S_\alpha(1, \beta) \) and assume \( \beta = 0 \) in the case \( \alpha = 1 \). Then for any \( f \in C^3_b(\mathbb{R}) \), there exists \( c_{\alpha, \beta, \gamma} \) (that can be made explicit) depending only on \( \alpha, \beta \) and \( \gamma \) such that

i) When \( \alpha \in (1, 2) \), we have

\[
|E[f(S_n)] - E[f(Y)]| \leq c_{\alpha, \beta, \gamma}(\|f\|_\infty + \|f''\|_\infty + \|f''\|_\infty + \|f(3)\|_\infty) \\
\cdot \left[ n^{\frac{2-\alpha}{2}} \left( 1 + \int_{-\sigma n^{\frac{1}{n}} n}^{\sigma n^{\frac{1}{n}} n} \frac{|\epsilon(x)|}{|x|^{n-1}} dx \right) + \sup_{|x| \geq \sigma n^{\frac{1}{n}}} |\epsilon(x)| \right].
\]

ii) When \( \alpha = 1, \beta = 0 \) and \( \gamma > 0 \), we have

\[
|E[f(S_n)] - E[f(Y)]| \leq c_{\alpha, \beta, \gamma}(\|f\|_\infty + \|f''\|_\infty + \|f''\|_\infty + \|f(3)\|_\infty) \\
\cdot \left\{ n^{-1} \log n, \quad \gamma \in [1, \infty), \\
n^{-\gamma}, \quad \gamma \in (0, 1) \right\}.
\]

iii) When \( \alpha \in (0, 1) \), we have

\[
|E[f(S_n)] - E[f(Y)]| \leq \|f\|_\infty \mathcal{R}_{\alpha, \beta, \gamma}(n) + c_{\alpha, \beta, \gamma}(\|f\|_\infty + \|f''\|_\infty + \|f''\|_\infty) \mathcal{R}_{\alpha, \beta, \gamma}(n),
\]

where

\[
\mathcal{R}_{\alpha, \beta, \gamma}(n) = \frac{n^{\frac{\alpha-1}{\alpha}}}{\alpha} \left( 1 + \beta \right) \int_0^{\sigma n^{\frac{1}{n}}} \frac{\epsilon(x)}{x^\alpha} dx - (1 - \beta) \int_0^{\sigma n^{\frac{1}{n}}} \frac{\epsilon(-x)}{x^\alpha} dx
\]

(1.6)

and

\[
\mathcal{R}_{\alpha, \beta, \gamma}(n) = \begin{cases} 
  n^{-1}, & \gamma \in [2 - \alpha, \infty), \\
  n^{-1} + n^{-\alpha}, & \gamma \in (1 - \alpha, 2 - \alpha), \\
  n^{-1} + n^{-\alpha} \log n, & \gamma = 1 - \alpha, \\
  n^{-1} + n^{-\alpha} \gamma, & \gamma \in (0, 1 - \alpha), \\
  n^{-1} + n^{-\alpha} \int_{-\sigma n^{\frac{1}{n}}}^{\sigma n^{\frac{1}{n}}} \frac{|\epsilon(x)|}{|x|^{n-1}} dx + \left( \sup_{|x| \geq \sigma n^{\frac{1}{n}}} |\epsilon(x)| \right) \alpha, & \gamma = 0.
\end{cases}
\]

Remark 1.5. From Theorem 1.1 and the definition of the distance \( d_{W_2} \) above, we see that

\[
d_{W_2}(S_n, Y) \leq c_{\alpha, \beta} \left[ n^{\frac{2-\alpha}{\alpha}} \left( 1 + \int_{-\sigma n^{\frac{1}{n}}}^{\sigma n^{\frac{1}{n}}} \frac{|\epsilon(x)|}{|x|^{\alpha-1}} dx \right) + \sup_{|x| \geq \sigma n^{\frac{1}{n}}} |\epsilon(x)| \right], \quad \text{for } \alpha \in (1, 2);
\]

\[
d_{W_2}(S_n, Y) \leq c_{\alpha, \beta} \left\{ n^{-1} \log n, \quad \gamma \in [1, \infty), \quad \text{for } \alpha = 1, \beta = 0; \\
n^{-\gamma}, \quad \gamma \in (0, 1), \quad \text{for } \alpha = 1, \beta = 0;
\right.
\]

\[
d_{W_2}(S_n, Y) \leq c_{\alpha, \beta, \gamma} [\mathcal{R}_{\alpha, \beta, \gamma}(n) + \mathcal{R}_{\alpha, \beta, \gamma}(n)], \quad \text{for } \alpha \in (0, 1).
\]

We can see that our convergence rate in the distance \( d_{W_2} \) matches the optimal rate in the Kolmogorov distance found by Hall [22], but we do not need ultimately monotonicity condition. Moreover, when \( \alpha \in (0, 1) \) and \( X_1 \) is symmetric, it is easy to see that \( \mathcal{R}_{\alpha, \beta, \gamma}(n) = 0 \) and thus the convergence rate is \( n^{-1} \), this is consistent with the result in [28].

If \( \epsilon(x) \to 0 \) as \( x \to \pm \infty \), then we have \( |E[f(S_n)] - E[f(Y)]| \to 0 \) from the previous theorem. Moreover, by the same argument as the proof of [13 Corollary I.1], we get the following stable CLT.

**Corollary 1.6.** When \( \epsilon(x) \to 0 \) as \( x \to \pm \infty \), we have

\[
|P(S_n \leq x) - P(Y \leq x)| \to 0, \quad \text{as } n \to \infty.
\]

Our next result gives an improved upper bound on \( |E[f(S_n)] - E[f(Y)]| \) for \( \alpha \in (0, 1] \), under slightly more restrictive conditions (see, e.g., [22, Theorem 2]).
Theorem 1.4. In addition, we further assume \( \frac{d(x)}{x} \mathbf{1}_{(0, \infty)}(x) \) and \( \frac{d(x)}{|x|} \mathbf{1}_{(-\infty, 0)}(x) \) are ultimately monotone (that is, there exists \( x_0 > 0 \) such that \( \frac{d(x)}{x} \mathbf{1}_{(0, \infty)}(x) \) and \( \frac{d(x)}{|x|} \mathbf{1}_{(-\infty, 0)}(x) \) are monotone for any \( |x| > x_0 \)). Then there exists \( \hat{c}_{\alpha, \beta}(\epsilon) \) that can be made explicit depending only on \( \alpha \) and \( \beta \) such that

i) When \( \alpha = 1 \) and \( \beta = 0 \), we have

\[
\begin{align*}
|\mathbb{E}[f(S_n)] - \mathbb{E}[f(Y)]| & \leq \hat{c}_{\alpha, \beta}(\|f\|_\infty + \|f'\|_\infty + \|f''\|_\infty + \|f^{(3)}\|_\infty) \\
& \quad \cdot \left(n^{-1} \log n + n^{-1} \int_{-\sigma_n}^{\sigma_n} |\epsilon(x)| dx + \sup_{|x| \geq \sigma_n} |\epsilon(x)|\right).
\end{align*}
\]

ii) When \( \alpha \in (0, 1) \), we have

\[
|\mathbb{E}[f(S_n)] - \mathbb{E}[f(Y)]| \leq \|f'\|_\infty R_{\alpha, \beta}(n) + \hat{c}_{\alpha, \beta}(\|f\|_\infty + \|f'\|_\infty + \|f''\|_\infty)
\]

\[
\begin{align*}
& \quad \cdot \left(n^{-1 + \frac{\alpha - 2}{\alpha}} \left(1 + \int_{-\sigma_n}^{\sigma_n} \frac{|\epsilon(x)|}{|x|} dx\right) + \sup_{|x| \geq \sigma_n} |\epsilon(x)|\right),
\end{align*}
\]

where \( R_{\alpha, \beta}(n) \) is defined by (1.9).

Observing in Corollary 1.6 we obtain stable CLT by assuming that the function \( \epsilon \) satisfies \( \epsilon(x) \to 0 \) as \( x \to \pm \infty \). But that \( \epsilon \) vanishes is not a necessary condition for stable CLT. By slightly modifying the approach leading to Theorem 1.4, we can also consider the case where \( \epsilon \) is a slowly varying function diverging at infinity. Because it would be too technical to state such result at a great level of generality, we prefer to illustrate an explicit situation for which our methodology still allows to conclude. Here we give a simpler proof that relies on the density function and the distribution function is similarly available.

**Example:** Pareto multiplied by a slowly varying function. We consider

\[
x \sim \frac{\alpha^2 \epsilon^\alpha}{2(1 + \alpha)} \log \frac{1}{|x|^{\alpha + 1}} \mathbf{1}_{[0, \infty)}(|x|), \quad \text{with } \alpha \in (0, 2).
\]

For the partial sums \( S_n \) to converge to the symmetric \( \alpha \)-stable distribution, we must modify the normalization given in (1.3) (observe that \( \mathbb{E}[X_1] = 0 \) here). Define the sequence \( (\gamma_n)_{n \geq 1} \) implicitly by \( \gamma_n = (n \log \gamma_n)^{\frac{1}{\alpha}} \) and set \( \sigma = \left( \frac{\alpha^2 \gamma_\alpha}{(1 + \alpha) \gamma_\alpha} \right)^{\frac{1}{\alpha}} \). Consider \( \tilde{Y} \sim \alpha S(1) \). We can deduce from a suitable modification of Theorem 1.4 (see Section 4) that for any \( f \in C^4_b(\mathbb{R}) \),

\[
|\mathbb{E} \left[ f \left( \frac{1}{\gamma_n} (X_1 + \ldots + X_n) \right) \right] - \mathbb{E} \left[ f(\tilde{Y}) \right]| = O\left( (\log n)^{-1} \right).
\]

The rest of the paper is organized as follows. In Section 2 we give a short proof of the Theorem 1.4 in a special case to illustrate the main idea. In Section 3 we give some useful properties of the operator \( \mathcal{L}_{\alpha, \beta} \) and asymmetric \( \alpha \)-stable process. In Section 4 we develop the Taylor-like expansion. In Section 5 we extend the Lindeberg principle to the asymmetric \( \alpha \)-stable distributions and provide the proof of Theorem 1.4 and Theorem 1.7. In Section 6 we will focus on the proof of (1.7).

In order to prove Theorem 1.4 and Theorem 1.7 we need the following classical Dynkin’s formula (Chapter 1, section 3)

\[
\mathbb{E}_X \left[ f(X_t) \right] - f(X) = \mathbb{E}_X \left[ \int_0^t \mathcal{G} f(X_s) ds \right]
\]

for \( f \) from some appropriate class of functions and here \( X_0 = X, \mathcal{G} \) is understood as an infinitesimal operator of the process.
2. A SHORT PROOF OF THEOREM 1.4 IN A SPECIAL CASE TO ILLUSTRATE THE MAIN IDEA

As mentioned in the introduction, we shall use Taylor-like expansion and Dynkin’s formula of stable process to prove Theorem 1.3. Before we go into the details, let us give its short proof in the symmetric Pareto distribution case (see, e.g., [17]) to illustrate how these two tools will work.

Assume that $X_1, X_2, \ldots$ are independent copies drawn from the Pareto law of index $\alpha \in (0, 2)$, that is, suppose that the common density is

$$p(x) = \frac{\alpha}{2} |x|^{-(1+\alpha)}1_{[1,\infty)}(|x|).$$

Consider $\sigma = \left( \frac{\alpha}{2} \int_{\mathbb{R}} \frac{1-\cos y}{|y|^{1+\alpha}} dy \right)^{\frac{1}{\alpha}}$ and $S_n = \frac{1}{\sigma} n^{-\frac{1}{\alpha}} (X_1 + \ldots + X_n)$. According to Lindeberg principle, for any fixed $n$, set

$$Z_i = Y_1 + \cdots + Y_{i-1} + \frac{1}{\sigma} X_{i+1} + \cdots + \frac{1}{\sigma} X_n, \quad 1 \leq i \leq n,$$

where $Y_1, Y_2, \ldots$ are independent copies of symmetric $\alpha$-stable law $Y$ and we know $\frac{X_1 + \ldots + X_n}{n^{\alpha}} \overset{d}{=} Y$. Then

$$\mathbb{E}[f(S_n)] - \mathbb{E}[f(Y)] = \sum_{i=1}^{n} \mathbb{E} \left[ f \left( \frac{X_1 + \cdots + X_n}{\sigma n^{\alpha}} \right) - f \left( \frac{Y_1 + \cdots + Y_n}{\sigma n^{\alpha}} \right) \right]$$

$$= \sum_{i=1}^{n} \mathbb{E} \left[ f \left( \frac{Z_i}{n^{\alpha}} + \frac{X_i}{\alpha n^{\alpha}} \right) - f \left( \frac{Z_i}{n^{\alpha}} \right) \right] - \sum_{i=1}^{n} \mathbb{E} \left[ f \left( \frac{Z_i}{n^{\alpha}} + \frac{Y_i}{n^{\alpha}} \right) - f \left( \frac{Z_i}{n^{\alpha}} \right) \right].$$

For the first term, since $Z_i$ and $X_i$ are independent, we have

$$\mathbb{E} \left[ f \left( \frac{Z_i}{n^{\alpha}} + \frac{X_i}{\alpha n^{\alpha}} \right) - f \left( \frac{Z_i}{n^{\alpha}} \right) \right] = \mathbb{E} \left[ \frac{\alpha}{2} \int_{|x| \geq 1} \frac{f \left( \frac{Z_i}{n^{\alpha}} + \frac{x}{\alpha n^{\alpha}} \right) - f \left( \frac{Z_i}{n^{\alpha}} \right)}{|x|^{\alpha+1}} dx \right]$$

$$= n^{-1} \mathbb{E} \left[ \frac{d\alpha}{2} \int_{|y| > n^{-1} - \frac{1}{\alpha}} \frac{f \left( \frac{Z_i}{n^{\alpha}} + y \right) - f \left( \frac{Z_i}{n^{\alpha}} \right)}{|y|^{\alpha+1}} dy \right]$$

$$= n^{-1} \mathbb{E} \left[ \Delta \frac{\alpha}{2} \int_{-\sigma^{-1} n^{-\frac{1}{\alpha}}}^{\sigma^{-1} n^{-\frac{1}{\alpha}}} f \left( \frac{Z_i}{n^{\alpha}} + y \right) - f \left( \frac{Z_i}{n^{\alpha}} \right) \frac{\alpha}{|y|^{\alpha+1}} dy \right],$$

and

$$\int_{-\sigma^{-1} n^{-\frac{1}{\alpha}}}^{\sigma^{-1} n^{-\frac{1}{\alpha}}} \frac{f \left( \frac{Z_i}{n^{\alpha}} + y \right) - f \left( \frac{Z_i}{n^{\alpha}} \right)}{|y|^{\alpha+1}} dy = \int_{-\sigma^{-1} n^{-\frac{1}{\alpha}}}^{\sigma^{-1} n^{-\frac{1}{\alpha}}} \frac{f \left( \frac{Z_i}{n^{\alpha}} + y \right) - f \left( \frac{Z_i}{n^{\alpha}} \right) - y f' \left( \frac{Z_i}{n^{\alpha}} \right)}{|y|^{\alpha+1}} dy$$

$$= \int_{-\sigma^{-1} n^{-\frac{1}{\alpha}}}^{\sigma^{-1} n^{-\frac{1}{\alpha}}} \int_{0}^{1} y \left[ f' \left( \frac{Z_i}{n^{\alpha}} + y t \right) - f' \left( \frac{Z_i}{n^{\alpha}} \right) \right] \frac{dt}{|y|^{\alpha+1}} dy$$

$$= \int_{-\sigma^{-1} n^{-\frac{1}{\alpha}}}^{\sigma^{-1} n^{-\frac{1}{\alpha}}} \int_{0}^{1} ty^{2} f'' \left( \frac{Z_i}{n^{\alpha}} + y t \theta \right) \frac{d\theta}{|y|^{\alpha+1}} \frac{dy}{|y|^{\alpha+1}},$$

which further gives

$$n^{-1} \mathbb{E} \left[ \frac{d\alpha}{2} \int_{|y| < \sigma^{-1} n^{-\frac{1}{\alpha}}} \frac{f \left( \frac{Z_i}{n^{\alpha}} + y \right) - f \left( \frac{Z_i}{n^{\alpha}} \right)}{|y|^{\alpha+1}} dy \right] \leq n^{-1} \frac{d\alpha}{\sigma^{2-\alpha} (2-\alpha)} \alpha^{-\frac{2}{\alpha}}.$$
For the second term, notice that $Z_i$ and $Y_i$ are independent, we can consider $Y_i$ as a symmetric $\alpha$-stable process $\hat{Y}_i$, then we have by Dynkin’s formula

$$
\mathbb{E}\left[f\left(\frac{Z_i}{n^{\alpha}} + \frac{Y_i}{n^{\alpha}}\right) - f\left(\frac{Z_i}{n^{\alpha}}\right)\right] = \mathbb{E}\left[\int_0^1 \Delta_\frac{\alpha}{Y_i} f\left(\frac{Z_i}{n^{\alpha}} + \frac{\hat{Y}_i}{n^{\alpha}}\right)\right] = n^{-1}\mathbb{E}\left[\int_0^1 \Delta_\frac{\alpha}{Y_i} f\left(\frac{Z_i}{n^{\alpha}} + \frac{\hat{Y}_i}{n^{\alpha}}\right)\right].
$$

Hence, when $\alpha \in (1,2)$, we have by Proposition 3.2 below (whose proof in this setting is much simpler)

$$
|\mathbb{E}[f(S_n)] - \mathbb{E}[f(Y)]| \leq c_\alpha \left(\|f\|_\infty + \|f''\|_\infty + \|f''\|_\infty\right)n^{-\frac{\alpha-2}{\alpha}},
$$

when $\alpha \in (0,1]$, we have by truncated Lemma 3.4 below (whose proof in this setting is much simpler)

$$
|\mathbb{E}[f(S_n)] - \mathbb{E}[f(Y)]| \leq c_\alpha \left(\|f\|_\infty + \|f''\|_\infty + \|f''\|_\infty\right)n^{-1}\log n, \quad \alpha = 1,
$$

$$
|\mathbb{E}[f(S_n)] - \mathbb{E}[f(Y)]| \leq c_\alpha \left(\|f\|_\infty + \|f''\|_\infty + \|f''\|_\infty\right)n^{-1}, \quad \alpha \in (0,1).
$$

3. Preliminaries of stable processes and nonlocal operators

Let us first recall the following heat kernel estimates from [12 Theorem 1.1 (iii)], which will be used in the analysis in this section.

**Lemma 3.1.** Let $(\hat{Y}_t)_{t \geq 0}$ be defined by (1.1). Then the distribution of $\hat{Y}_t$ has a density $p(t,x)$ for all $t > 0$. Moreover, for any $t \in (0,1)$, there exists a constant $C_\alpha > 1$ such that

$$
C_\alpha^{-1}t^{-\frac{1}{\alpha}}\left(1 \wedge t^{\frac{\alpha+1}{\alpha}}\right) \leq p(t,x) \leq C_\alpha t^{-\frac{1}{\alpha}}\left(1 \wedge t^{\frac{\alpha+1}{\alpha}}\right). \quad (3.1)
$$

3.1. Estimates for the operator $L^{\alpha,\beta}$. By the definition of operator $L^{\alpha,\beta}$, we can first get the following Proposition.

**Proposition 3.2.** For any $f \in C^3_b(\mathbb{R})$, and $x, z \in \mathbb{R}$, we have

$$
|L^{\alpha,\beta}f(x+z) - L^{\alpha,\beta}f(x)| \leq D_\alpha|z|,
$$

where

$$
D_\alpha = \begin{cases} 
\frac{2d_\alpha\|f''\|_\infty + d_\beta\|f^{(3)}\|_\infty}{\alpha-1} + \frac{d_\beta\|f^{(3)}\|_\infty}{2(1-\alpha)}, & \alpha \in (1,2), \\
2d_\alpha\|f''\|_\infty + d_\beta\|f^{(3)}\|_\infty, & \alpha = 1, \\
\frac{\alpha}{2d_\alpha}\|f''\|_\infty + \frac{1}{1-\alpha}d_\beta\|f''\|_\infty, & \alpha \in (0,1). 
\end{cases}
$$

**Proof.** For convenience, we denote

$$
I^\beta(y) = (1+\beta)1_{(0,1]}(y) + (1-\beta)1_{[-1,0)}(y), \quad I^\beta(y) = (1+\beta)1_{(1,\infty)}(y) + (1-\beta)1_{(-\infty,-1)}(y).
$$

1. When $\alpha \in (1,2)$, we have by (1.2)

$$
\frac{1}{d_\alpha}L^{\alpha,\beta}f(x) = \int_{-\infty}^{\infty} \frac{f(y+x) - f(x) - yf'(x)}{2|y|^{1+\alpha}}I^\beta(y)dy + \int_{-\infty}^{\infty} \frac{f(y+x) - f(x) - yf'(x)}{2|y|^{1+\alpha}}I^\beta(y)dy
$$

$$
= \int_{-\infty}^{\infty} \int_0^1 yf'(x+ty) - yf'(x)I^\beta(y)dtidy + \int_{-\infty}^{\infty} \int_0^1 yf''(x+ut)I^\beta(y)dudty,
$$

it follows that

$$
\frac{1}{d_\alpha}|L^{\alpha,\beta}f(x+z) - L^{\alpha,\beta}f(x)|
$$

$$
\leq \left| \int_{-\infty}^{\infty} \int_0^1 yf'(x+z+ty) - f'(x+z) - f'(x+ty) + f'(x)I^\beta(y)dtidy \right|
$$

$$
+ \left| \int_{-\infty}^{\infty} \int_0^1 tf''(x+z+ut) - tf''(x+ut)I^\beta(y)dudty \right|
$$

$$
+ \left| \int_{-\infty}^{\infty} \int_0^1 tf''(x+z+ut) - tf''(x+ut)I^\beta(y)dudty \right|
$$

$$
7
$$
\[ \leq |z| \left[ \int_{-\infty}^{\infty} \frac{\|f''\|_{\infty}}{|y|^\alpha} I^\beta(y)dy + \int_{-\infty}^{\infty} \int_0^1 \frac{t\|f^{(3)}\|_{\infty}}{2|y|^{\alpha-1}} I^\beta(y)dt
dy \right] = \left( \frac{2\|f''\|_{\infty}}{\alpha - 1} + \frac{\|f^{(3)}\|_{\infty}}{2(2-\alpha)} \right) |z|. \]

2. When \( \alpha = 1 \), we have by (1.2)
\[
\frac{1}{d_\alpha} L^{1,\beta} f(x) = \int_{-\infty}^{\infty} \frac{f(y+x) - f(x)}{2|y|^2} I^\beta(y)dy + \int_{-\infty}^{\infty} \frac{f(y+x) - f(x) - yf'(x)}{2|y|^2} I^\beta(y)dy
\]
\[
= \int_{-\infty}^{\infty} \frac{f(y+x) - f(x)}{2|y|^2} I^\beta(y)dy + \int_{-\infty}^{\infty} \int_0^1 \frac{tf''(x + uty)}{2} I^\beta(y)dudy, \quad (3.2)
\]
then by the same argument as above, we have
\[
\frac{1}{d_\alpha} |L^{1,\beta} f(x + z) - L^{1,\beta} f(x)| \leq \left( \frac{2\|f''\|_{\infty}}{2} + \frac{1}{2}\|f^{(3)}\|_{\infty} \right) |z|.
\]

3. When \( \alpha \in (0,1) \), we have by (1.2)
\[
\frac{1}{d_\alpha} L^{\alpha,\beta} f(x) = \int_{-\infty}^{\infty} \frac{f(y+x) - f(x)}{2|y|^{1+\alpha}} I^\beta(y)dy + \int_{-\infty}^{\infty} \int_0^1 \frac{yf'(x + ty)}{2|y|^{1+\alpha}} I^\beta(y)dydt, \quad (3.3)
\]
it follows that
\[
\frac{1}{d_\alpha} |L^{\alpha,\beta} f(x + z) - L^{\alpha,\beta} f(x)| \leq \left( \frac{2\|f''\|_{\infty}}{\alpha} + \frac{1}{1-\alpha}\|f''\|_{\infty} \right) |z|.
\]

By (3.2) and (3.3), we can immediately obtain the following proposition:

**Proposition 3.3.** Let \( \alpha \in (0,1] \). For any \( f \in C^2_0 \), we have
\[
\|L^{\alpha,\beta} f\|_{\infty} \leq \hat{D}_\alpha,
\]
where
\[
\hat{D}_\alpha = \begin{cases} 
2d_\alpha \|f\|_{\infty} + \frac{d_\alpha}{2}\|f''\|_{\infty}, & \alpha = 1, \\
\frac{2d_\alpha}{\alpha}\|f\|_{\infty} + \frac{d_\alpha}{1-\alpha}\|f''\|_{\infty}, & \alpha \in (0,1).
\end{cases}
\]

In addition, it is easy to verify by the definition of \( L^{\alpha,\beta} \) that if \( z = x - y \), then
\[
L^{\alpha,\beta}_x f(x - y) = L^{\alpha,\beta}_z f(z), \quad (3.4)
\]
where \( L^{\alpha,\beta}_x \) means that the operator \( L^{\alpha,\beta} \) acts on the variable \( x \). Similarly, for \( z = cx \) for some constant \( c > 0 \), we have
\[
L^{\alpha,\beta}_x f(cx) = c^\alpha L^{\alpha,\beta}_z f(z). \quad (3.5)
\]

### 3.2. Truncation for asymmetric \( \alpha \)-stable process \( \hat{Y} \)

When \( \alpha \in (0,1] \), we have by (3.1) that \( \mathbb{E} |\hat{Y}_s| = \infty \) for any \( s > 0 \), we need the following lemma for the analysis in the next section.

**Lemma 3.4.** Consider \( \alpha \in (0,1] \). Let \( \hat{Y} \) be the one-dimensional asymmetric \( \alpha \)-stable process, then for any \( 0 < a < 1 \), \( z \in \mathbb{R} \) and \( f \in C^3_0(\mathbb{R}) \), we have
\[
\mathbb{E} \left[ \int_0^1 |L^{\alpha,\beta} f(z) - L^{\alpha,\beta} f(a\hat{Y}_s + z)|ds \right] \leq C_\alpha \begin{cases} 
(\hat{D}_\alpha + D_\alpha)a - D_\alpha a \log a, & \alpha = 1, \\
(\hat{D}_\alpha + D_\alpha)a^\alpha, & \alpha \in (0,1),
\end{cases} \quad (3.6)
\]
where \( \hat{D}_\alpha \) and \( D_\alpha \) are defined as above.

**Proof.** Observe
\[
\mathbb{E} \left[ \int_0^1 |L^{\alpha,\beta} f(z) - L^{\alpha,\beta} f(a\hat{Y}_s + z)|ds \right]
\]
\[
= \mathbb{E} \left[ \int_0^1 |L^{\alpha,\beta} f(z) - L^{\alpha,\beta} f(a\hat{Y}_s + z)| \left[ I_{(0,1]}(|\hat{Y}_s|) + I_{(0,a^{-1})}(|\hat{Y}_s|) \right] ds \right] := I + II.
\]

8
By Proposition 3.3 and Lemma 3.1 we have
\[ I \leq 2 \hat{D}_a \int_0^1 \mathbb{P}(\hat{Y}_s \geq a^{-1}) ds \leq C_\alpha \hat{D}_a \int_0^1 \int_{a^{-1}}^\infty \frac{s}{y^{\alpha+1}} dy ds \leq C_\alpha \hat{D}_a a^\alpha. \]

By Proposition 3.2 and Lemma 3.1 when \( \alpha = 1 \) and \( \beta = 0 \), we have
\[ II \leq D_\alpha a \mathbb{E}\left[ \int_0^1 |\hat{Y}_s| I_{|\hat{Y}_s| \leq a^{-1}} ds \right] \leq C_\alpha D_\alpha a \int_0^1 \int_0^{a^{-1}} ys^{-\frac{1}{2}}(1 \wedge \frac{s}{y^2}) dy ds \leq C_\alpha (D_\alpha a - C_\alpha D_\alpha a \log a); \]
when \( \alpha \in (0, 1) \), we have
\[ II \leq D_\alpha a \mathbb{E}\left[ \int_0^1 |\hat{Y}_s| I_{|\hat{Y}_s| \leq a^{-1}} ds \right] \leq C_\alpha D_\alpha a \int_0^1 \int_0^{a^{-1}} ys^{-\frac{1}{2}}(1 \wedge \frac{s^{\alpha+1}}{y^{\alpha+1}}) dy ds \leq C_\alpha D_\alpha a^\alpha, \]
which get the desired results. The proof is complete. \( \square \)

**Remark 3.5.** In the above lemma, because of \( \mathbb{E}|\hat{Y}_s| < \infty \) in the case \( \alpha \in (1, 2) \), we have by Proposition 3.2 that
\[ \mathbb{E}\left[ \int_0^1 |\mathcal{L}^{\alpha, \beta} f(z) - \mathcal{L}^{\alpha, \beta} f(a\hat{Y}_s + z)| ds \right] \leq D_\alpha a \int_0^1 \mathbb{E}|\hat{Y}_s| ds \leq C_\alpha D_\alpha a. \]

### 3.3. Truncation for random variable \( X \)

Let \( X \) have a distribution of the form (1.3) with \( \epsilon(x) \) satisfying (1.4), then it is obvious that \( \mathbb{E}|X| = \infty \) in the case \( \alpha \in (0, 1] \). However, we can use a truncation technique to handle the problem. Before giving the truncation Lemma, we need

**Lemma 3.6.** Let \( X \geq 0 \) be a random variable defined on \( \Omega \) and \( t > 0 \), then
\[ \mathbb{E}[X1_{(0,t)}] = \int_0^t \mathbb{P}(X > r) dr - t \mathbb{P}(X > t). \]

**Proof.** Using the definition of expected value, Fubini’s theorem and then calculating the resulting integrals gives
\[ \int_0^t \mathbb{P}(X > r) dr = \int_0^t \int_\Omega 1_{(r, \infty)}(X) dPdr \]
\[ = \int_\Omega \int_0^t 1_{(0, \infty)}(r) dr dP = \int_\Omega (X \wedge t) dP = \mathbb{E}[X1_{(0,t)}] + t \mathbb{P}(X > t), \]
from which we immediately obtain the equality in the lemma, as desired. \( \square \)

Now, we are at the position to give the truncation lemma.

**Lemma 3.7.** Consider \( \alpha \in (0, 1] \). Let \( X \) have a distribution of the form (1.3) with \( \epsilon(x) \) satisfying (1.4), then for any \( 0 < a < 1 \), \( z \in \mathbb{R} \) and \( f \in C^2_\alpha(\mathbb{R}) \), we have
\[ \mathbb{E}|f'(z + aX) - f'(z)| \leq \begin{cases} 2 \|f''\|_\infty a + 2(A + K)(2\|f''\|_\infty + \|f''\|_\infty \log a^{-1})a, \quad \alpha = 1, \\ 2\|f''\|_\infty a + 2(A + K)(2\|f''\|_\infty + \|f''\|_\infty \log a^{-1})a^\alpha, \quad \alpha \in (0, 1). \end{cases} \]

**Proof.** Observe
\[ \mathbb{E}|f'(z + aX) - f'(z)| = \mathbb{E}\left[ f'(z + aX) - f'(z) \left| (1_{(a^{-1}, \infty)}(|X|) + 1_{(0,a^{-1}]}(|X|)) \right. \right] \]
\[ \leq 2\|f''\|_\infty \mathbb{P}(|X| > a^{-1}) + \|f''\|_\infty a \mathbb{E}\left[ |X| 1_{(0,a^{-1}]}(|X|) \right]. \]
By (1.4), we know \( |\epsilon(x)| \leq K \) for \( |x| \geq 1 \), so
\[ \mathbb{P}(|X| > a^{-1}) = a^\alpha (A + \epsilon(a^{-1}))(1 + \beta) + a^\alpha (A + \epsilon(-a^{-1}))(1 - \beta) \leq 2(A + K)a^\alpha. \]
By (3.8),
\[
\mathbb{E}[|X|1_{(0, a^{-1})}(|X|)] \leq \int_0^{a^{-1}} \mathbb{P}(|X| > r)dr = (1 + \beta) \int_0^{a^{-1}} \frac{A + \epsilon(x)}{x^\alpha} dx + (1 - \beta) \int_0^{a^{-1}} \frac{A + \epsilon(-x)}{x^\alpha} dx,
\]
1.) when \( \alpha = 1 \), we have
\[
(1 + \beta) \int_0^{a^{-1}} \frac{A + \epsilon(x)}{x} dx \leq 1 + (1 + \beta) \int_1^{a^{-1}} \frac{A + K}{x} dx = 1 + (1 + \beta)(A + K) \log a^{-1}.
\]
2.) when \( \alpha \in (0, 1) \), we have
\[
(1 + \beta) \int_0^{a^{-1}} \frac{A + \epsilon(x)}{x^\alpha} dx \leq 1 + (1 + \beta) \int_0^{a^{-1}} \frac{A + K}{x^{\alpha}} dx \leq 1 + \frac{(1 + \beta)(A + K)}{1 - \alpha} a^{-1}.
\]
Since similar bounds hold true for \( (1 - \beta) \int_0^{a^{-1}} \frac{A + \epsilon(-x)}{x^\alpha} dx \), the desired conclusion follows. \( \square \)

**Remark 3.8.** From the proof of Lemma 3.7, we immediately have
\[
\mathbb{E}[|X|1_{(0, a^{-1})}(|X|)] \leq 2 \begin{cases} 1 + (A + K) \log a^{-1}, & \alpha = 1, \\ 1 + \frac{A + K}{1 - \alpha} a^{-1}, & \alpha \in (0, 1). \end{cases} \tag{3.9}
\]

4. **Taylor-like expansions**

In this section, we develop the following Taylor-like expansions, which can be taken as replacements of Taylor expansions in the Lindeberg’s approach to proving the normal CLT.

4.1. **Taylor-like expansions for Theorem 1.4**

- \( \alpha \in (1, 2) \):

**Lemma 4.1.** Consider \( \alpha \in (1, 2) \). Let \( X \) have a distribution \( F_X \) with the form (1.3), \( X \) and \( Z \) are independent. We have, for any \( 0 < a \leq (2A)^{-\frac{1}{\alpha}} \) and \( f \in C^2_0(\mathbb{R}) \):

\[
\left| \mathbb{E}[f(Z + aX)] - \mathbb{E}[f(Z)] - \mathbb{E}[aX]\mathbb{E}[f'(Z)] - \frac{2A\alpha}{d_\alpha} a^\alpha \mathbb{E}[L^{\alpha, \beta} f(Z)] \right| \\
\leq \frac{4\|f''\|_\infty}{2 - \alpha} (2A)^{\frac{2}{\alpha} - 2} a^2 + \frac{8\|f''\|_\infty}{\alpha - 1} a^\alpha \sup_{|x| > a^{-1}} |\epsilon(x)| + 2\|f''\|_\infty a^2 \int_{a^{-1}}^{a^{-1}} \frac{\|\epsilon(x)\|}{x^\alpha} dx.
\]

**Proof.** For convenience, we denote \( 1_\beta(y) = (1 + \beta)1_{(0, \infty)}(y) + (1 - \beta)1_{(-\infty, 0)}(y) \). Then we have by (1.2)

\[
\frac{2A\alpha}{d_\alpha} a^\alpha \mathbb{E}[L^{\alpha, \beta} f(Z)] = A\alpha a^\alpha \mathbb{E} \left[ \int_\mathbb{R} \frac{f(Z + y) - f(Z) - y f'(Z)}{|y|^{1+\alpha}} 1_\beta(y) dy \right] \\
= A\alpha \left[ \int_\mathbb{R} \frac{f(Z + ax) - f(Z) - ax f'(Z)}{|x|^{1+\alpha}} 1_\beta(x) dx \right] \\
= A\alpha \left[ \int_{|x| > (2A)^{\frac{1}{2}}} \frac{f(Z + ax) - f(Z) - ax f'(Z)}{|x|^{1+\alpha}} 1_\beta(x) dx \right] + \mathcal{R},
\]

where the second equality is by taking \( y = ax \) and

\[
\mathcal{R} = A\alpha \left[ \int_{|x| < (2A)^{\frac{1}{2}}} \frac{f(Z + ax) - f(Z) - ax f'(Z)}{|x|^{1+\alpha}} 1_\beta(x) dx \right]. \tag{4.1}
\]

Since \( A\alpha \int_{|x| > (2A)^{\frac{1}{2}}} \frac{1}{|x|^{1+\alpha}} 1_\beta(x) dx = 1 \), we can consider a random variable \( \tilde{X} \) which is independent of \( Z \) and satisfies

\[
\mathbb{P}(\tilde{X} > x) = \frac{A(1 + \beta)}{|x|^{\alpha}}, \quad x \geq (2A)^{\frac{1}{2}}, \quad \mathbb{P}(\tilde{X} \leq x) = \frac{A(1 - \beta)}{|x|^{\alpha}}, \quad x \leq -(2A)^{\frac{1}{2}}, \tag{4.2}
\]
it follows that
\[ \frac{2A\alpha}{d_{\alpha}} a^{\alpha} \mathbb{E}[\mathcal{L}^{\alpha,\beta} f(Z)] = \mathbb{E}[f(Z + a\bar{X}) - f(Z) - a\bar{X} f'(Z)] + \mathcal{R}. \]

As a result, denote the distribution function of \( \bar{X} \) by \( F_{\bar{X}} \), then
\[
\left| \mathbb{E}[f(Z + aX)] - \mathbb{E}[f(Z)] - \mathbb{E}[aX] \mathbb{E}[f'(Z)] - \frac{2A\alpha}{d_{\alpha}} a^{\alpha} \mathbb{E}[\mathcal{L}^{\alpha,\beta} f(Z)] \right|
\leq \mathbb{E} \left| \int_{-\infty}^{\infty} \left[ f(Z + ax) - ax f'(Z) \right] d[F_{X}(x) - F_{\bar{X}}(x)] \right| + |\mathcal{R}|. \tag{4.3}
\]

By (1.3) and (1.2), we have
\[
F_{X}(x) - F_{\bar{X}}(x) = \left( \frac{1}{2} - \frac{A + \epsilon(x)}{|x|^\alpha} \right) (1 + \beta) 1_{(0,2A) \frac{\alpha}{2}}(x) - \frac{\epsilon(x)}{|x|^\alpha} (1 + \beta) 1_{(2A,\infty)}(x)
\]
\[
+ \left( \frac{A + \epsilon(x)}{|x|^\alpha} - \frac{1}{2} \right) (1 - \beta) 1_{(2A,0)}(x) + \frac{\epsilon(x)}{|x|^\alpha} (1 - \beta) 1_{(-\infty,-(2A) \frac{\alpha}{2})}(x),
\]
using integration by parts, we have
\[
\mathbb{E} \left| \int_{-\infty}^{\infty} \left[ f(Z + ax) - ax f'(Z) \right] d[F_{X}(x) - F_{\bar{X}}(x)] \right|
\leq 2 \mathbb{E} \left| \int_{-\infty}^{(2A) \frac{\alpha}{2}} [a f'(Z + ax) - a f'(Z)] dx \right|
+ 2 \mathbb{E} \left| \int_{(2A) \frac{\alpha}{2}}^{\infty} [a f'(Z + ax) - a f'(Z)] \frac{\epsilon(x)}{|x|^\alpha} dx \right| \tag{4.4}
\]
and
\[
2 \mathbb{E} \left| \int_{-\infty}^{(2A) \frac{\alpha}{2}} [a f'(Z + ax) - a f'(Z)] dx \right| \leq 2(2A)^{\frac{\alpha}{2}} \|f''\|_\infty a^2.
\]

For the remainder, one has
\[
\mathbb{E} \left[ \int_{a^{-1}}^{\infty} |a f'(Z + ax) - a f'(Z)| \frac{\epsilon(x)}{|x|^\alpha} dx \right] \leq 2 \|f''\|_\infty a^2 \sup_{x \geq a^{-1}} |\epsilon(x)|,
\]
whereas
\[
\mathbb{E} \left[ \int_{(2A) \frac{\alpha}{2}}^{a^{-1}} |a f'(Z + ax) - a f'(Z)| \frac{\epsilon(x)}{|x|^\alpha} dx \right] \leq \|f''\|_\infty a^2 \int_{0}^{a^{-1}} \frac{|\epsilon(x)|}{|x|^\alpha-1} dx.
\]

Since similar bounds hold true for \( \mathbb{E} \left[ \int_{(2A) \frac{\alpha}{2}}^{\infty} [a f'(Z) - a f'(Z)] \frac{\epsilon(x)}{|x|^\alpha} dx \right] \) and
\[
|\mathcal{R}| \leq 2Aa \|f''\|_\infty a^2 \int_{|x|<(2A) \frac{\alpha}{2}} \frac{1}{|x|^\alpha-1} dx = \frac{4Aa \|f''\|_\infty}{2 - \alpha} (2A)^{2 - \alpha} a^2. \tag{4.5}
\]
the desired conclusion follows.

\[ \bullet \quad \alpha = 1 \text{ and } \beta = 0 : \]

**Lemma 4.2.** Consider \( \alpha = 1, \beta = 0 \) and \( \gamma \in (0, \infty) \). Let \( X \) have a distribution of the form (1.3) with \( \epsilon(x) \) satisfying (1.4). \( X \) and \( Z \) are independent. We have, for any \( 0 < a \leq (2A)^{-1} \wedge 1 \) and \( f \in C_0^2(\mathbb{R}) \), denote
\[
T_1 := \left| \mathbb{E}[f(Z + aX)] - \mathbb{E}[f(Z)] - \mathbb{E}[aX 1_{(-1,1)}(aX)] \mathbb{E}[f'(Z)] - \frac{2A}{d_1} a \mathbb{E}[\mathcal{L}^{1,0} f(Z)] \right|
\]
then we have
\[ T_1 \leq 12A^2\|f''\|_{\infty}a^2 + 2K(2\|f\|_{\infty} + \frac{\gamma + 1}{\gamma} \|f'\|_{\infty})a^{1+\gamma} + 2K\|f''\|_{\infty} \left\{ \begin{array}{ll} \frac{(2A)^{1-\gamma}}{\gamma-1}a^2, & \gamma \in (1, \infty), \\
\frac{a^2\log a^{-1}}{1-\gamma}a^{1+\gamma}, & \gamma \in (0, 1). \end{array} \right. \]

**Proof.** By the same argument as (4.3), we have
\[ T_1 \leq \mathbb{E}\left| \int_{-\infty}^{\infty} [f(Z + ax) - axf'(Z)]d[F_X(x) - F_{\tilde{X}}(x)] \right| + |\mathcal{R}|, \quad (4.6) \]
where \( F_{\tilde{X}} \) and \( \mathcal{R} \) is defined by (4.2) and (4.1) with \( \alpha = 1, \beta = 0 \), respectively. What’s more, by (4.5), we know \( |\mathcal{R}| \leq 8A^2\|f''\|_{\infty}a^2 \).

For the first term, using an integration by parts similar to (4.4) and (4.4), we have
\[ \mathbb{E}\left| \int_{-\infty}^{\infty} f(Z + ax)d[F_X(x) - F_{\tilde{X}}(x)] \right| \leq \|f\|_{\infty}a|\epsilon(a^{-1})| + a\|f'\|_{\infty}\int_{-\infty}^{\infty} \frac{|\epsilon(x)|}{x} \, dx \leq K(\|f\|_{\infty} + \frac{\|f'\|_{\infty}}{\gamma})a^{1+\gamma}, \]
and in the same way
\[ \mathbb{E}\left| \int_{-\infty}^{\infty} [f(Z + ax) - axf'(Z)]d[F_X(x) - F_{\tilde{X}}(x)] \right| \leq K(\|f\|_{\infty} + \frac{\|f'\|_{\infty}}{\gamma})a^{1+\gamma}, \]
whereas
\[ \mathbb{E}\left| \int_{-\infty}^{\infty} [f(Z + ax) - axf'(Z)]d[F_X(x) - F_{\tilde{X}}(x)] \right| \leq \|f\|_{\infty} + \|f'\|_{\infty} + a^2\|f''\|_{\infty} \left\{ \begin{array}{ll} \frac{(2A)^{1-\gamma}}{\gamma-1}a^2, & \gamma \in (1, \infty), \\
\frac{a^2\log a^{-1}}{1-\gamma}a^{1+\gamma}, & \gamma \in (0, 1), \end{array} \right. \]
the desired conclusion follows. \( \square \)

- \( \alpha \in (0, 1) \): For any \( \beta \in [-1, 1] \), we have
\[ \int_{\mathbb{R}} \frac{y1_{(-1,1)}(y)}{2|y|^{1+\alpha}} \left[ (1 + \beta)1_{(0,\infty)}(y) + (1 - \beta)1_{(-\infty,0)}(y) \right] dy = \frac{\beta}{\Gamma(1-\alpha)}, \]
which follows that
\[ \frac{1}{d_\alpha}L^{\alpha,\beta}f(x) - \frac{\beta f(x)}{1-\alpha} = \int_{\mathbb{R}} \frac{f(y + x) - f(x) - y1_{(-1,1)}(y)f'(x)}{2|y|^{1+\alpha}} \left[ (1 + \beta)1_{(0,\infty)}(y) + (1 - \beta)1_{(-\infty,0)}(y) \right] dy. \quad (4.7) \]
According to (1.7), we have the following Taylor-like expansion lemma.

**Lemma 4.3.** Consider \( \alpha \in (0, 1) \). Let \( X \) have a distribution \( F_X \) with the form (1.3) satisfying (1.4), \( X \) and \( Z \) are independent. We have, for any \( 0 < a \leq (2A)^{-\frac{1}{\alpha}} \wedge 1 \) and \( f \in C_0^2(\mathbb{R}) \), denote
\[ T_2 := \left| \mathbb{E}[f(Z + aX)] - \mathbb{E}[f(Z)] - \mathbb{E}[aX1_{(-1,1)}(aX)] \mathbb{E}[f'(Z)] - \frac{2A\alpha}{d_\alpha}a^\alpha \mathbb{E}[L^{\alpha,\beta}f(Z)] - \frac{\beta d_\alpha f'(Z)}{1-\alpha} \right|. \]

a.) When \( \gamma \in (1 - \alpha, \infty) \), we have
\[ T_2 \leq \frac{2 + \alpha}{2 - \alpha} (2A)^{\frac{2}{\alpha}} \|f''\|_{\infty}a^2 + 2K \left( 3\|f\|_{\infty} + \frac{2(\alpha + \gamma) - 1}{\alpha + \gamma - 1}\|f'\|_{\infty} \right) a^{\alpha + \gamma} \]
\[ + 4K\|f''\|_\infty \begin{cases} (2A)^{\frac{2-a-\gamma}{a+\gamma-2}}a^2, & \gamma \in (2 - \alpha, \infty), \\ a^2 \log a^{-1}, & \gamma = 2 - \alpha, \\ \frac{1}{2-a-\gamma}a^{\alpha+\gamma}, & \gamma \in (1 - \alpha, 2 - \alpha). \end{cases} \]

b.) When \( \gamma \in (0, 1 - \alpha] \), we have
\[
T_2 \leq \frac{2 + \alpha}{2 - \alpha} (2A)^{\frac{2}{\alpha}} \|f''\|_\infty a^2 + 4K (\|f\|_\infty + \|f'\|_\infty + \frac{\|f''\|_\infty}{2 - \alpha - \gamma}) a^{\alpha+\gamma}
\]
\[
+ (4A + 6K)\|f\|_\infty a^{\frac{\alpha}{1 - \gamma}} + 4K\|f'\|_\infty a^{\frac{\alpha}{1 - \gamma}} \begin{cases} \log a^{-1}, & \gamma = 1 - \alpha, \\ \frac{1}{1 - \alpha - \gamma}, & \gamma \in (0, 1 - \alpha). \end{cases}
\]

c.) When \( \gamma = 0 \), we have
\[
T_2 \leq \frac{2 + \alpha}{2 - \alpha} (2A)^{\frac{2}{\alpha}} \|f''\|_\infty a^2 + 2\|f''\|_\infty a^2 \int_{a^{-1}}^{a^{-1}} \frac{\epsilon(x)}{|x|^{a-1}} dx + 4(\|f\|_\infty + \|f'\|_\infty) a^\alpha \sup_{|x| \geq a^{-1}} |\epsilon(x)|
\]
\[
+ \left[(4A + 6K)\|f\|_\infty + \frac{4\|f'\|_\infty}{1 - \alpha}\right] a^\alpha \left( \sup_{|x| \geq a^{-1}} |\epsilon(x)| \right)^{\alpha}.
\]

**Proof.** By the same argument as (1.3), we have
\[
T_2 \leq \mathbb{E} \left[ \int_{-\infty}^{\infty} \left( f(Z + ax) - ax \mathbf{1}_{(-1,1)}(ax)f(Z) \right) d[F_X(x) - F_{\hat{X}}(x)] \right] + |\mathcal{R}|, \quad (4.8)
\]
where \( F_{\hat{X}} \) and \( \mathcal{R} \) is defined by (1.2) and (4.1) with \( \alpha \in (0, 1) \), respectively. What’s more, by (1.5), we know \( |\mathcal{R}| \leq \frac{4A\|f''\|_\infty}{2 - \alpha} (2A)^{\frac{2}{\alpha}} a^2 \).

For the first term, according to (1.4),
1. When \( \gamma \in (1 - \alpha, \infty) \), using an integration by parts similar to (1.4) and (1.4), we have
\[
\mathbb{E} \left[ \int_{-\infty}^{\infty} f(Z + ax) d[F_X(x) - F_{\hat{X}}(x)] \right] \leq \|f\|_\infty a \|\epsilon(a^{-1})\| + a \|f'\|_\infty \int_{a^{-1}}^{\infty} \frac{\epsilon(x)}{x^\alpha} dx
\]
\[
\leq K \left( \|f\|_\infty + \frac{\|f'\|_\infty}{\alpha + \gamma - 1} \right) a^{\alpha+\gamma}.
\]
Similarly, we get
\[
\mathbb{E} \left[ \int_{-\infty}^{a^{-1}} f(Z + ax) d[F_X(x) - F_{\hat{X}}(x)] \right] \leq K \left( \|f\|_\infty + \frac{\|f'\|_\infty}{\alpha + \gamma - 1} \right) a^{\alpha+\gamma},
\]
and
\[
\mathbb{E} \left[ \int_{a^{-1}}^{\infty} [f(Z + ax) - ax f'(Z)] d[F_X(x) - F_{\hat{X}}(x)] \right]
\]
\[
\leq 4K \left( \|f\|_\infty + \|f'\|_\infty \right) a^{\alpha+\gamma} + a^2 \|f''\|_\infty \left( \int_{-(2A)^{\frac{1}{\alpha}}}^{(2A)^{\frac{1}{\alpha}}} |x| dx + 2 \int_{a^{-1}}^{a^{-1}} \frac{\epsilon(x)}{|x|^{a-1}} 1_{(2A, \infty)}(|x|^\alpha) dx \right)
\]
\[
\leq 4K \left( \|f\|_\infty + \|f'\|_\infty \right) a^{\alpha+\gamma} + (2A)^{\frac{2}{\alpha}} \|f''\|_\infty a^2 + 4K\|f''\|_\infty \begin{cases} (2A)^{\frac{2-a-\gamma}{a+\gamma-2}}a^2, & \gamma \in (2 - \alpha, \infty), \\ a^2 \log a^{-1}, & \gamma = 2 - \alpha, \\ \frac{1}{2-a-\gamma}a^{\alpha+\gamma}, & \gamma \in (1 - \alpha, 2 - \alpha). \end{cases}
\]

2. When \( \gamma \in [0, 1 - \alpha] \), we choose a number \( N > a^{-1} \). One has by \( |\epsilon(x)| \leq K \) for \( |x| > N \),
\[
\mathbb{E} \left[ \int_{|x| > N} f(Z + ax) d[F_X(x) - F_{\hat{X}}(x)] \right] \leq \|f\|_\infty \left[ \int_{|x| > N} dF_X(x) + \int_{|x| > N} dF_{\hat{X}}(x) \right]
\]
\[
\leq (4A + 2K)\|f\|_\infty N^{-\alpha},
\]
whereas by integration by parts
\[
\mathbb{E}\left(\int_{-a}^{-1} + \int_{a}^{-1} \left[ f(Z + ax) - ax 1_{(-1,1)}(ax) f'(Z) \right] d\left[F_X(x) - F_X(x)\right]\right) \\
\leq a^2 \|f''\|_{\infty} (2A)^2 + 2 \int_{-a}^{-1} \frac{|\epsilon(x)|}{|x|^{\alpha-1}} 1_{(2A, \infty)}(|x|) \, dx + 4\|f\|_{\infty} + \|f'\|_{\infty} a^\alpha \sup_{|x| \geq a^{-1}} |\epsilon(x)| \\
+ 4K \|f\|_{\infty} N^{-\alpha} + 2\|f'\|_{\infty} a \left( \int_{a}^{N} \frac{|\epsilon(x)|}{x^{\alpha}} \, dx + \int_{a}^{N} \frac{|\epsilon(-x)|}{x^{\alpha}} \, dx \right).
\]

For \(\gamma \in (0, 1 - \alpha]\), one has
\[
4\|f\|_{\infty} + \|f'\|_{\infty} a^\alpha \sup_{|x| \geq a^{-1}} |\epsilon(x)| \leq 4K \|f\|_{\infty} + \|f'\|_{\infty} a^{\alpha + \gamma},
\]
and
\[
\int_{-a}^{-1} \frac{|\epsilon(x)|}{|x|^{\alpha-1}} 1_{(2A, \infty)}(|x|) \, dx \leq \frac{2K}{2 - \alpha - \gamma} a^{\alpha + \gamma - 2},
\]
and
\[
\int_{a}^{N} \frac{|\epsilon(x)|}{x^{\alpha}} \, dx + \int_{a}^{N} \frac{|\epsilon(-x)|}{x^{\alpha}} \, dx \leq 2K \begin{cases} \log N, & \gamma = 1 - \alpha, \\
\frac{N^{1-\alpha-\gamma}}{1-\alpha-\gamma}, & \gamma \in (0, 1 - \alpha). \end{cases}
\]

For \(\gamma = 0\), we have
\[
\int_{a}^{N} \frac{|\epsilon(x)|}{x^{\alpha}} \, dx + \int_{a}^{N} \frac{|\epsilon(-x)|}{x^{\alpha}} \, dx \leq \frac{2}{2 - \alpha} \sup_{|x| \geq a^{-1}} |\epsilon(x)| N^{1-\alpha}.
\]
Hence, we can consider
\[
N^{-\alpha} = \begin{cases} aN^{1-\alpha-\gamma}, & \gamma \in (0, 1 - \alpha], \\
aN^{1-\alpha} \sup_{|x| \geq a^{-1}} |\epsilon(x)|, & \gamma = 0, \end{cases}
\]
which implies
\[
N = \begin{cases} \frac{1}{\gamma}, & \gamma \in (0, 1 - \alpha], \\
\frac{1}{\gamma} \left( \sup_{|x| \geq a^{-1}} |\epsilon(x)| \right)^{-1}, & \gamma = 0, \end{cases}
\]
the desired conclusion follows. \(\square\)

4.2. Taylor-like expansions for Theorem 1.7

**Lemma 4.4.** Consider \(\alpha \in (0, 1]\) and assume \(\beta = 0\) in the case \(\alpha = 1\). Let \(X\) have a distribution \(F_X\) with the form (1.3), \(X\) and \(Z\) are independent. In addition, we further assume \(\epsilon(x) 1_{(0, \infty)}(x)\) and \(\epsilon(x) 1_{(-\infty, 0)}(x)\) are ultimately monotone. We have, for any \(0 < \alpha \leq (2A)^{-\frac{1}{\alpha}} \wedge 1\) such that \(\epsilon(x) 1_{(0, \infty)}(x)\) and \(\epsilon(x) 1_{(-\infty, 0)}(x)\) are monotone for any \(|x| > a^{-1}\) and \(f \in C^2_b(\mathbb{R})\):

a.) when \(\alpha = 1\) and \(\beta = 0\), we have
\[
\left| \mathbb{E}[f(Z + aX)] - \mathbb{E}[f(Z)] - \mathbb{E}[aX 1_{(-1,1)}(aX) \mathbb{E}[f'(Z)]] - \frac{2A}{d_1} a \mathbb{E}[\mathcal{L}^{1,0} f(Z)] \right| \\
\leq \left( \|f''\|_{\infty} a^2 \left( 12A^2 + \int_{-a}^{-1} |\epsilon(x)| \, dx \right) + 2\|f\|_{\infty} + \|f'\|_{\infty} a \sup_{|x| \geq a^{-1}} |\epsilon(x)| \right) .
\]

b.) when \(\alpha \in (0, 1)\), we have
\[
\left| \mathbb{E}[f(Z + aX)] - \mathbb{E}[f(Z)] - \mathbb{E}[aX 1_{(-1,1)}(aX) \mathbb{E}[f'(Z)]] - \frac{2A}{da} a \mathbb{E}[\mathcal{L}^{1,0} f(Z) - \frac{\beta d_a f'(Z)}{1 - \alpha}] \right| \
\]
\[ \frac{2 + \alpha}{2 - \alpha} (2A)^{\alpha} \|f''\|_\infty a^2 + 4(\|f\|_\infty + \|f'\|_\infty) a^\alpha \sup_{|x| \geq a^{-1}} |\epsilon(x)| + 2 \|f''\|_\infty a^2 \int_{a^{-1}}^{a^{-1}} |\epsilon(x)| dx. \quad (4.10) \]

**Proof.** According to (4.7), the proofs of inequalities (4.9) and (4.10) are similar, so here we only give the proof of (4.9) and the proof of (4.10) is similar.

By the same argument as (4.3), we have

\[
\left| E[f(Z + aX)] - E[f(Z)] - E[aX1_{(-1,1)}(aX)|E[f'(Z)] - \frac{2A}{d_1} aE[L^{1,0} f(Z)] \right|
\leq E \left| \int_{\infty}^{-\infty} f(Z + ax) - ax1_{(-1,1)}(ax)f'(Z) \right| d[F_X(x) - F_{\tilde{X}}(x)] + |R|,
\]

where \( F_X \) and \( R \) is defined by (4.12) and (4.1) with \( \alpha = 1, \beta = 0 \), respectively. What’s more, by (4.5), we know \( |R| \leq 8A^2 \|f''\|_\infty a^2 \).

For the first term, one has

\[
E \left| \int_{\infty}^{-\infty} f(Z + ax) d[F_X(x) - F_{\tilde{X}}(x)] \right| \leq \|f\|_\infty \int_{a^{-1}}^{\infty} \left| d[F_X(x) - F_{\tilde{X}}(x)] \right| \leq \|f\|_\infty a \epsilon(a^{-1}),
\]

and in the same way

\[
E \left| \int_{-\infty}^{-a^{-1}} f(Z + ax) d[F_X(x) - F_{\tilde{X}}(x)] \right| \leq \|f\|_\infty a \epsilon(-a^{-1}),
\]

whereas by integration by parts similar to (4.4),

\[
E \left| \int_{-a^{-1}}^{a^{-1}} \left[ f(Z + ax) - axf'(Z) \right] d[F_X(x) - F_{\tilde{X}}(x)] \right|
\leq \left( \|f\|_\infty + \|f''\|_\infty a (|\epsilon(a^{-1})| + |\epsilon(-a^{-1})|) + \|f''\|_\infty a^2 (2A^2 + \int_{-a^{-1}}^{a^{-1}} |\epsilon(x)| dx) \right),
\]

the desired conclusion follows. \( \square \)

### 5. Proof of Theorem 1.4 and Theorem 1.7

In this section, with the help of the Taylor-like expansion in the previous section, we extend the celebrated Lindeberg principle of normal approximation (see, e.g., [16 pages 211-212]) to prove main results.

**Proof of Theorem 1.4.** Observe \( Y \sim S_\alpha(1, \beta) \) in Theorem 1.4 let \( Y_1, Y_2, \cdots \) be independent copies of \( Y \), then it is well known \( \frac{Y_1 + \cdots + Y_n}{n^{\alpha}} \sim S_\alpha(1, \beta) \). Recall the definition of \( S_n \), we denote

\[ \hat{X}_i = \left\{ \begin{array}{ll}
X_i - E[X_i], & \alpha \in (1, 2), \\
X_i - E[X_i1_{(0, \sigma n^{\frac{1}{\alpha}})}(|X_i|)], & \alpha = 1, \\
X_i, & \alpha \in (0, 1),
\end{array} \right. \]

where \( i = 1, 2, \cdots \), then for any fixed \( n \), set

\[ Z_i = Y_1 + \cdots + Y_{i-1} + \hat{X}_{i+1} + \cdots + \hat{X}_n, \quad 1 \leq i \leq n, \]

and obviously agree \( Z_1 = \hat{X}_2 + \cdots + \hat{X}_n, Z_n = Y_1 + \cdots + Y_{n-1} \). Then clearly

\[
E[f(S_n)] - E[f(Y)] = E \left[ f \left( \frac{\hat{X}_1 + \cdots + \hat{X}_n}{n^{\alpha}} \right) \right] - E \left[ f \left( \frac{Y_1 + \cdots + Y_n}{n^{\alpha}} \right) \right]
= \sum_{i=1}^{n} E \left[ f \left( \frac{\hat{X}_i + Z_i}{n^{\alpha}} \right) - f \left( \frac{Y_i + Z_i}{n^{\alpha}} \right) \right]
= \sum_{i=1}^{n} E \left[ f \left( \frac{\hat{X}_i + Z_i}{n^{\alpha}} \right) - f \left( \frac{Z_i}{n^{\alpha}} \right) \right] - \sum_{i=1}^{n} E \left[ f \left( \frac{Y_i + Z_i}{n^{\alpha}} \right) - f \left( \frac{Z_i}{n^{\alpha}} \right) \right]. \quad (5.1)
\]
For the first term, by observing \( X_i \) and \( Z_i \) are independent, we have
\[
\sum_{i=1}^{n} \mathbb{E} \left[ f \left( \frac{X_i + Z_i}{n^\alpha} \right) - f \left( \frac{Z_i}{n^\alpha} \right) \right] = \frac{1}{n} \sum_{i=1}^{n} \mathbb{E} \left[ \mathcal{L}^{\alpha, \beta} f \left( \frac{Z_i}{n^\alpha} \right) \right] + \mathbf{I} + \mathbf{II},
\]
where in the case \( \alpha \in [1, 2) \),
\[
\mathbf{I} = \sum_{i=1}^{n} \left\{ \mathbb{E} \left[ f \left( \frac{Z_i}{n^\alpha} + \frac{n^{-\frac{1}{\alpha}}}{\sigma} X_i \right) \right] - \mathbb{E} \left[ f \left( \frac{Z_i}{n^\alpha} \right) \right] - \mathbb{E} \left[ \frac{n^{-\frac{1}{\alpha}}}{\sigma} X_i \right) \right] \mathbb{E} \left[ f' \left( \frac{Z_i}{n^\alpha} \right) \right] \right\} - 2A\alpha \left( \frac{n^{-\frac{1}{\alpha}}}{\sigma} \right) \mathbb{E} \left[ \mathcal{L}^{\alpha, \beta} f \left( \frac{Z_i}{n^\alpha} \right) \right],
\]
\[
\mathbf{II} = \sum_{i=1}^{n} \left\{ \mathbb{E} \left[ f \left( \frac{Z_i}{n^\alpha} + \frac{X_i}{\sigma n^\alpha} \right) - \mathbb{E} \left[ \frac{n^{-\frac{1}{\alpha}}}{\sigma} X_i \right) \right] \mathbb{E} \left[ f' \left( \frac{Z_i}{n^\alpha} \right) \right] \right\} + \mathbb{E} \left[ \frac{n^{-\frac{1}{\alpha}}}{\sigma} X_i \right) \right] \mathbb{E} \left[ f' \left( \frac{Z_i}{n^\alpha} \right) \right],
\]
with
\[
\left( \frac{n^{-\frac{1}{\alpha}}}{\sigma} X_i \right) (\alpha) = \left\{ \frac{n^{-\frac{1}{\alpha}}}{\sigma} X_i, \quad \alpha \in (1, 2), \right. \\
\left. \frac{n^{-\frac{1}{\alpha}}}{\sigma} X_i 1_{(0, \sigma n^\alpha)} (|X_i|), \quad \alpha = 1, \right.
\]
and when \( \alpha \in (0, 1) \), we have
\[
\mathbf{I} = \sum_{i=1}^{n} \left\{ \mathbb{E} \left[ f \left( \frac{Z_i}{n^\alpha} + \frac{n^{-\frac{1}{\alpha}}}{\sigma} X_i \right) \right] - \mathbb{E} \left[ f \left( \frac{Z_i}{n^\alpha} \right) \right] - \mathbb{E} \left[ \frac{n^{-\frac{1}{\alpha}}}{\sigma} X_i \right) \right] \mathbb{E} \left[ f' \left( \frac{Z_i}{n^\alpha} \right) \right] \right\} - 2A\alpha \left( \frac{n^{-\frac{1}{\alpha}}}{\sigma} \right) \mathbb{E} \left[ \mathcal{L}^{\alpha, \beta} f \left( \frac{Z_i}{n^\alpha} \right) \right],
\]
\[
\mathbf{II} = \sum_{i=1}^{n} \left\{ \mathbb{E} \left[ \frac{n^{-\frac{1}{\alpha}}}{\sigma} X_i 1_{(0, \sigma n^\alpha)} (|X_i|) \right] \mathbb{E} \left[ f' \left( \frac{Z_i}{n^\alpha} \right) \right] \right\} - \frac{2A\alpha}{1 - \alpha} \left( \frac{n^{-\frac{1}{\alpha}}}{\sigma} \right) \mathbb{E} \left[ f' \left( \frac{Z_i}{n^\alpha} \right) \right].
\]
For the second term, notice that \( Z_i \) and \( Y_i \) are independent, we can consider \( Y_i \) as an asymmetric \( \alpha \)-stable process \( \hat{Y}_1 \), then we have by Dynkin’s formula
\[
\sum_{i=1}^{n} \mathbb{E} \left[ f \left( \frac{Y_i + Z_i}{n^\alpha} \right) - f \left( \frac{Z_i}{n^\alpha} \right) \right] = \sum_{i=1}^{n} \mathbb{E} \left[ \int_0^1 \mathcal{L}^{\alpha, \beta} f \left( \frac{Z_i + \hat{Y}_s}{n^\alpha} \right) ds \right] = \frac{1}{n} \sum_{i=1}^{n} \mathbb{E} \left[ \int_0^1 \mathcal{L}^{\alpha, \beta} f \left( \frac{Z_i + \hat{Y}_s}{n^\alpha} \right) ds \right],
\]
where the second equality thanks to (3.31) and (3.5).
Therefore, we have
\[
\mathbb{E}[f(S_n)] - \mathbb{E}[f(Y)] = \frac{1}{n} \sum_{i=1}^{n} \mathbb{E} \left[ \int_0^1 \left( \mathcal{L}^{\alpha, \beta} f \left( \frac{Z_i}{n^\alpha} \right) - \mathcal{L}^{\alpha, \beta} f \left( \frac{Z_i + \hat{Y}_s}{n^\alpha} \right) \right) ds \right] + \mathbf{I} + \mathbf{II},
\]
and using (3.7) and (3.6) respectively, we have
\[
\frac{1}{n} \sum_{i=1}^{n} \mathbb{E} \left[ \int_0^1 \left| \mathcal{L}^{\alpha, \beta} f \left( \frac{Z_i}{n^\alpha} \right) - \mathcal{L}^{\alpha, \beta} f \left( \frac{Z_i + \hat{Y}_s}{n^\alpha} \right) \right| ds \right] \leq C_{\alpha} \left\{ \begin{array}{ll}
D_\alpha n^{-\frac{1}{\alpha}}, & \alpha \in (1, 2), \\
(D_\alpha + D_\alpha n^{-1} \log n, & \alpha = 1,
\end{array} \right.
\]
\[
\left( D_\alpha + D_\alpha n^{-1}, \quad \alpha \in (0, 1). \right.
\]
Now, let us bound the \( \mathbf{I} \) and \( \mathbf{II} \).
i) When \( \alpha \in (1, 2) \), one has by Lemma 4.11

\[
I \leq \frac{4}{(2 - \alpha)\sigma^2} (2A)^{\frac{2-\alpha-\gamma}{\alpha}} n^{\frac{\alpha-2}{\alpha}} + \frac{8}{(\alpha - 1)\sigma^2} \sup_{x \geq \sigma n^{\frac{1}{\alpha}}} |\epsilon(x)| + \frac{2}{\sigma^2} n^{\frac{-\alpha}{\alpha}} \int_{-\sigma n^{\frac{1}{\alpha}}}^{\sigma n^{\frac{1}{\alpha}}} |\epsilon(x)| \frac{dx}{x^{\alpha-1}},
\]

whereas

\[
|II| \leq \|f''\|_{\infty} \sigma^2 n^{\frac{-\alpha}{\alpha}} \sum_{i=1}^{n} |E[X_i]|(|E|X_i| + |E[X_i]|) \leq \frac{2}{\sigma^2} \|E|X_i||E|X_i||f''\|_{\infty} n^{\frac{-\alpha}{\alpha}}.
\]

ii) When \( \alpha = 1, \beta = 0 \) and \( \gamma \in (0, \infty) \), one has by Lemma 4.2

\[
I \leq \frac{12A^2}{\sigma^2} n^{-1} + \frac{2K}{\sigma^{1+\gamma}} (3\|f\|_{\infty} + \sigma n^{\frac{1}{\alpha}}) n^{-\gamma} + 2K \|f''\|_{\infty} \begin{cases} \frac{(2A)^{\frac{1}{\alpha}}}{(\gamma-1)\sigma^2} n^{-\gamma}, & \gamma \in (1, \infty), \\ \frac{n}{\sigma^2} \log(\sigma n^{\frac{1}{\alpha}}), & \gamma = 1, \\ \frac{(\sigma^{-1} - \gamma n^{-\gamma}}, & \gamma \in (0, 1), \end{cases}
\]

whereas by (3.8) and Lemma 5.1,

\[
|II| \leq \sum_{i=1}^{n} \left\{ \frac{E}{\sigma} \int_{0}^{1} E[X_{i1}(0,\sigma_{n})(|X_i|)] \left[ f'(\frac{Z_i}{\sigma}) - f'(\frac{Z_i}{\sigma} + \frac{X_i}{\sigma}) \right] dt \right\} 
+ \frac{4}{(\sigma^2 (\gamma + k + 1)) \left[ \|f''\|_{\infty} (\frac{K}{\gamma} + K + 2) + (A + K) (2\|f\|_{\infty} + \|f''\|_{\infty} \log(\sigma n^{\frac{1}{\alpha}})) \right] n^{-1}.
\]

iii) When \( \alpha \in (0, 1) \), on the one hand, using Lemma 4.3

a.) When \( \gamma \in (1 - \alpha, \infty) \), we have

\[
I \lessapprox \frac{2 + \alpha}{(2 - \alpha)\sigma^2} (2A)^{\frac{2-\alpha-\gamma}{\alpha}} n^{\frac{\alpha-2}{\alpha}} + \frac{2}{(\sigma^{1+\gamma})} \left[ 2\|f\|_{\infty} + \frac{2(\alpha + \gamma) - 1}{\alpha + \gamma - 1} \|f''\|_{\infty} \right] n^{-\frac{\alpha}{\alpha}} + 4K \|f''\|_{\infty} \left\{ \begin{cases} \frac{(2A)^{\frac{2-\gamma}{\alpha}}}{(\alpha + \gamma - 2)\sigma^2} n^{\frac{\alpha-2}{\alpha}}, & \gamma \in (2 - \alpha, \infty), \\ \frac{\sigma^{1-\gamma}}{\sigma^2} \log(\sigma n^{\frac{1}{\alpha}}), & \gamma = 2 - \alpha, \\ \frac{\sigma^{-1} - \gamma n^{-\gamma}}, & \gamma \in (1 - \alpha, 2 - \alpha). \end{cases} \right.
\]

b.) When \( \gamma \in (0, 1 - \alpha) \), we have

\[
I \lessapprox \frac{2 + \alpha}{(2 - \alpha)\sigma^2} (2A)^{\frac{2-\alpha-\gamma}{\alpha}} n^{\frac{\alpha-2}{\alpha}} + \frac{4K}{(\sigma^{1+\gamma})} \left( \|f\|_{\infty} + \|f''\|_{\infty} + \frac{\|f''\|_{\infty}}{(2 - \alpha - \gamma)} \right) n^{-\frac{\alpha}{\alpha}} + \frac{4A + 6K}{\sigma^{1-\gamma}} \|f''\|_{\infty} n^{-\frac{\alpha}{\alpha}} + \frac{4K}{(\sigma^{1+\gamma})} \|f''\|_{\infty} n^{-\frac{\alpha}{\alpha}} \left\{ \begin{cases} \frac{\log(\sigma n^{\frac{1}{\alpha}})}{\alpha}, & \gamma = 1 - \alpha, \\ \frac{1}{1 - \alpha - \gamma}, & \gamma \in (0, 1 - \alpha). \end{cases} \right.
\]

c.) When \( \gamma = 0 \), we have

\[
I \lessapprox \frac{2 + \alpha}{(2 - \alpha)\sigma^2} (2A)^{\frac{2-\alpha-\gamma}{\alpha}} n^{\frac{\alpha-2}{\alpha}} + \frac{2}{\sigma^2} \|f''\|_{\infty} n^{\frac{\alpha-2}{\alpha}} \int_{-\sigma n^{\frac{1}{\alpha}}}^{\sigma n^{\frac{1}{\alpha}}} |\epsilon(x)| \frac{dx}{x^{\alpha-1}}
+ \frac{4}{\sigma^2} \left( \|f\|_{\infty} + \|f''\|_{\infty} \right) \sup_{x \geq \sigma n^{\frac{1}{\alpha}}} |\epsilon(x)| + \frac{1}{\sigma^2} \left( 4A + 6K \right) \|f\|_{\infty} + \frac{4}{1 - \gamma} \left( \sup_{x \geq \sigma n^{\frac{1}{\alpha}}} |\epsilon(x)| \right)^{\alpha}.
\]

On the other hand, by Lemma 3.8 we have

\[
E[X_{i1}(0,\sigma_{n})(|X_i|)] = \frac{2A \alpha \beta}{1 - \alpha} \left( \frac{n^{\frac{1}{\alpha}}}{\sigma} \right)^{\alpha-1} + (1 + \beta) \int_{0}^{\sigma_{n}^{\frac{1}{\alpha}}} \epsilon(x) \frac{dx}{x^\alpha} - (1 - \beta) \int_{0}^{\sigma_{n}^{\frac{1}{\alpha}}} \epsilon(-x) \frac{dx}{x^\alpha}
+ \left( \frac{n^{\frac{1}{\alpha}}}{\sigma} \right)^{\alpha-1} \left[ (1 - \beta) \epsilon(-\sigma n^{\frac{1}{\alpha}}) - (1 + \beta) \epsilon(\sigma n^{\frac{1}{\alpha}}) \right],
\]
which follows that
\[\mathbf{I} \leq \|f\|_\infty \left( \frac{4}{\sigma^\alpha} \sup_{|x| \geq \sigma n^{\frac{1}{\alpha}}} |\epsilon(x)| + \frac{n^{\frac{\alpha-1}{\alpha}}}{\sigma} (1 + \beta) \int_0^{\frac{\sigma n}{\alpha}} \frac{\epsilon(x)}{x^\alpha} dx - (1 - \beta) \int_0^{\frac{\sigma n}{\alpha}} \frac{\epsilon(-x)}{x^\alpha} dx \right).\]

Combining all of above, we get the desired conclusion of Theorem 1.4. \(\square\)

**Proof of Theorem 1.7** It suffices to bound the I in the proof of Theorem 1.4

i) When \(\alpha = 1\), using (4.9), we have
\[I \leq \frac{\|f''\|_\infty}{\sigma^2} n^{-1} \left( 12A^2 + \int_{-\sigma n^{\frac{1}{\alpha}}}^{\sigma n^{\frac{1}{\alpha}}} |\epsilon(x)| dx \right) + \frac{4}{\sigma^\alpha} (2\|f\|_\infty + \|f''\|_\infty) \sup_{|x| \geq \sigma n^{\frac{1}{\alpha}}} |\epsilon(x)|.

ii) When \(\alpha \in (0, 1)\), using (4.10), we have
\[I \leq \frac{2 + \alpha}{(2 - \alpha)\sigma^2} \|f''\|_\infty n^{\frac{\alpha-2}{\alpha}} + \frac{4}{\sigma^\alpha} (2\|f\|_\infty + \|f''\|_\infty) \sup_{|x| \geq \sigma n^{\frac{1}{\alpha}}} |\epsilon(x)|
+ \frac{2}{\sigma^2} \|f''\|_\infty n^{\frac{\alpha-2}{\alpha}} \int_{-\sigma n^{\frac{1}{\alpha}}}^{\sigma n^{\frac{1}{\alpha}}} |\epsilon(x)| dx,

the desired conclusion follows. \(\square\)

6. A more difficult example: Proof of (1.7)

In this section, we prove the estimate (1.7). Consider independent copies \(X_1, \ldots, X_n\) of a random variable with density \(p_X(x) = \frac{\alpha^2 e^{\alpha x}}{2(1+\alpha) |x|^{\alpha+1}} 1_{|x| \leq \infty}(|x|)\) and define the sequence \((\gamma_n)_{n \geq 1}\) implicitly by \(\gamma_n = (n \log \gamma_n)^{\frac{1}{\alpha}}\). We set \(\sigma = \frac{\alpha^2 e^{\alpha}}{2(1+\alpha)\alpha n^{\frac{1}{\alpha}}} \), \(\tilde{X}_i = \frac{\alpha}{\sigma \gamma_i} X_i\), \(\tilde{S}_n = n^{\frac{1}{\alpha}} (\tilde{X}_1 + \ldots + \tilde{X}_n)\), and \(\tilde{Z}_i = \tilde{Y}_i + \ldots + \tilde{Y}_{i-1} + \tilde{X}_{i+1} + \ldots + \tilde{X}_n\) for \(1 \leq i \leq n\), where \(\tilde{Y}_1, \ldots, \tilde{Y}_n\) are independent copies of \(\tilde{Y}\).

By the same argument as (6.1), for any \(f \in C_b^0(\mathbb{R})\), we have
\[\mathbb{E}[f(\tilde{S}_n)] - \mathbb{E}[f(\tilde{Y})] = \sum_{i=1}^n \mathbb{E} \left[ f \left( \frac{\tilde{X}_i + \tilde{Z}_i}{n^{\frac{1}{\alpha}}} \right) \right] - \sum_{i=1}^n \mathbb{E} \left[ f \left( \frac{\tilde{Y}_i + \tilde{Z}_i}{n^{\frac{1}{\alpha}}} \right) \right].

Using among other that \(n^{\gamma_n - \frac{\alpha}{\alpha}} = \frac{1}{\log \gamma_n}\), we have
\[\mathbb{E} \left[ f \left( \frac{\tilde{Z}_i}{n^{\frac{1}{\alpha}}} \right) \right] - f \left( \frac{\tilde{Z}_i}{n^{\frac{1}{\alpha}}} \right)\]
\[= \frac{\alpha^2 e^{\alpha}}{2(1+\alpha)} \mathbb{E} \left\{ \int_{\mathbb{R}} f \left( \frac{\tilde{Z}_i}{n^{\frac{1}{\alpha}}} + \frac{n^{\frac{1}{\alpha}} u}{\sigma} \right) - f \left( \frac{\tilde{Z}_i}{n^{\frac{1}{\alpha}}} \right) \log \left( n^{\frac{1}{\alpha}} \gamma_n |u| \right) \log \gamma_n |u|^{-\frac{1}{\alpha+1}} 1_{|u| \leq \infty} (n^{\frac{1}{\alpha}} \gamma_n |u|) du \right\}
\[= \frac{\alpha^2 e^{\alpha}}{2(1+\alpha)} \mathbb{E} \left\{ \int_{\mathbb{R}} f \left( \frac{\tilde{Z}_i}{n^{\frac{1}{\alpha}}} + \frac{n^{\frac{1}{\alpha}} u}{\sigma} \right) - f \left( \frac{\tilde{Z}_i}{n^{\frac{1}{\alpha}}} \right) \log \left( n^{\frac{1}{\alpha}} \gamma_n |u| \right) \log \gamma_n |u|^{-\frac{1}{\alpha+1}} 1_{|u| \leq \infty} (n^{\frac{1}{\alpha}} \gamma_n |u|) du \right\}
\[+ \frac{\alpha^2 e^{\alpha}}{2(1+\alpha)} \mathbb{E} \left\{ \int_{\mathbb{R}} f \left( \frac{\tilde{Z}_i}{n^{\frac{1}{\alpha}}} + \frac{n^{\frac{1}{\alpha}} u}{\sigma} \right) - f \left( \frac{\tilde{Z}_i}{n^{\frac{1}{\alpha}}} \right) \log \left( n^{\frac{1}{\alpha}} \gamma_n |u| \right) \log \gamma_n |u|^{-\frac{1}{\alpha+1}} 1_{|u| \leq \infty} (n^{\frac{1}{\alpha}} \gamma_n |u|) du \right\}.

On the one hand, we have
\[\frac{1}{n} \mathbb{E} \left[ \mathcal{L}^{\alpha,0} f \left( \frac{\tilde{Z}_i}{n^{\frac{1}{\alpha}}} \right) \right] = \frac{\alpha^2 e^{\alpha}}{2(1+\alpha)} \mathbb{E} \left\{ \int_{\mathbb{R}} f \left( \frac{\tilde{Z}_i}{n^{\frac{1}{\alpha}}} + \frac{n^{\frac{1}{\alpha}} u}{\sigma} \right) - f \left( \frac{\tilde{Z}_i}{n^{\frac{1}{\alpha}}} \right) \right\} \frac{1}{|u|^{\alpha+1}} du \right\},
\[\mathbb{E} \left[ f \left( \frac{\tilde{Y}_i + \tilde{Z}_i}{n^{\frac{1}{\alpha}}} \right) - f \left( \frac{\tilde{Z}_i}{n^{\frac{1}{\alpha}}} \right) \right] = \frac{1}{n} \mathbb{E} \left[ \int_0^1 \mathcal{L}^{\alpha,0} f \left( \frac{\tilde{Z}_i + \tilde{Y}_s}{n^{\frac{1}{\alpha}}} \right) ds \right].
\]
As a result,

\[
|\mathbb{E}[f(S_n)] - \mathbb{E}[f(\tilde{Y})]| \leq \frac{1}{n} \sum_{i=1}^{n} \mathbb{E} \left[ \int_{0}^{1} \left| \mathcal{L}^{\alpha,0} f \left( \frac{\tilde{Z}_i}{n^{\alpha}} \right) - \mathcal{L}^{\alpha,0} f \left( \frac{\tilde{Z}_i + \hat{Y}_i}{n^{\alpha}} \right) \right| ds \right] + \mathcal{I} + \mathcal{II},
\]

where

\[
\mathcal{I} := \frac{\alpha^2 e^{\alpha}}{2(1 + \alpha)} \sum_{i=1}^{n} \mathbb{E} \left[ \int_{\mathbb{R}} \left| f \left( \frac{\tilde{Z}_i}{n^{\alpha}} + \frac{n^{-\frac{1}{\alpha}}}{\sigma} u \right) - f \left( \frac{\tilde{Z}_i}{n^{\alpha}} \right) \right| \frac{1}{|u|^{\alpha + 1}} 1_{[0,\infty)}(n^{-\frac{1}{\alpha}} \gamma u |u|) du \right],
\]

\[
\mathcal{II} := \frac{\alpha^2 e^{\alpha}}{2(1 + \alpha)} \sum_{i=1}^{n} \mathbb{E} \left[ \int_{\mathbb{R}} \left| f \left( \frac{\tilde{Z}_i}{n^{\alpha}} + \frac{n^{-\frac{1}{\alpha}}}{\sigma} u \right) - f \left( \frac{\tilde{Z}_i}{n^{\alpha}} \right) \right| \log \left( n^{-\frac{1}{\alpha}} |u| \right) \frac{1}{\log \gamma u |u|^{\alpha + 1}} 1_{[e, \infty)}(n^{-\frac{1}{\alpha}} \gamma u |u|) du \right].
\]

By (5.2) with \( \beta = 0 \) and \( Z_i \) replaced by \( \tilde{Z}_i \), we know

\[
\frac{1}{n} \sum_{i=1}^{n} \mathbb{E} \left[ \int_{0}^{1} \left| \mathcal{L}^{\alpha,0} f \left( \frac{\tilde{Z}_i}{n^{\alpha}} \right) - \mathcal{L}^{\alpha,0} f \left( \frac{\tilde{Z}_i + \hat{Y}_i}{n^{\alpha}} \right) \right| ds \right] \leq C_{\alpha} \begin{cases} D_{\alpha} n^{-\frac{1}{\alpha}}, & \alpha \in (1, 2), \\ (D_{\alpha} + D_{\alpha}) n^{-1} \log n, & \alpha = 1, \\ (D_{\alpha} + D_{\alpha}) n^{-1}, & \alpha \in (0, 1). \end{cases}
\]

On the other hand,

\[
\mathcal{I} = \frac{\alpha^2 e^{\alpha}}{2(1 + \alpha)} \sum_{i=1}^{n} \mathbb{E} \left[ \int_{-\infty}^{\infty} \left| f \left( \frac{\tilde{Z}_i}{n^{\alpha}} + \frac{n^{-\frac{1}{\alpha}}}{\sigma} u \right) - f \left( \frac{\tilde{Z}_i}{n^{\alpha}} \right) \right| \frac{1}{|u|^{\alpha + 1}} \right]
\]

\[
= \frac{\alpha^2 e^{\alpha}}{2(1 + \alpha)} \sum_{i=1}^{n} \mathbb{E} \left[ \int_{-\infty}^{\infty} 1 \left| \frac{n^{-\frac{1}{\alpha}}}{\sigma} u \right| f' \left( \frac{\tilde{Z}_i}{n^{\alpha}} + \frac{n^{-\frac{1}{\alpha}}}{\sigma} u \right) - f' \left( \frac{\tilde{Z}_i}{n^{\alpha}} \right) \right| \frac{1}{|u|^{\alpha + 1}} \right]
\]

\[
\leq C_{\alpha} \begin{cases} \|f''\|_{\infty} n^{\frac{\alpha - 1}{\alpha}} \int_{0}^{\frac{1}{\gamma n}} \frac{du}{u^{\alpha}}, & \alpha \in (0, 1), \\ \|f''\|_{\infty} n^{\frac{\alpha - 2}{\alpha}} \int_{0}^{\frac{1}{\gamma n}} \frac{du}{u^{\alpha - 1}}, & \alpha \in [1, 2), \\ O(\gamma_{n}^{\alpha - 1}), & \alpha \in (0, 1), \\ O(\gamma_{n}^{\alpha - 2}), & \alpha \in [1, 2). \end{cases}
\]

Finally,

\[
\mathcal{II} \leq C_{\alpha} \sum_{i=1}^{n} \mathbb{E} \left[ \int_{\mathbb{R}} \left| f \left( \frac{\tilde{Z}_i}{n^{\alpha}} + \frac{n^{-\frac{1}{\alpha}}}{\sigma} u \right) - f \left( \frac{\tilde{Z}_i}{n^{\alpha}} \right) \right| \log \left( n^{-\frac{1}{\alpha}} |u| \right) \frac{1}{\log \gamma u |u|^{\alpha + 1}} 1_{[e, \infty)}(n^{-\frac{1}{\alpha}} \gamma u |u|) du \right]
\]

\[
+ C_{\alpha} \sum_{i=1}^{n} \mathbb{E} \left[ \int_{\mathbb{R}} \left| f \left( \frac{\tilde{Z}_i}{n^{\alpha}} + \frac{n^{-\frac{1}{\alpha}}}{\sigma} u \right) - f \left( \frac{\tilde{Z}_i}{n^{\alpha}} \right) \right| \log \left( n^{-\frac{1}{\alpha}} |u| \right) \frac{1}{\log \gamma u |u|^{\alpha + 1}} 1_{[e, \infty)}(n^{-\frac{1}{\alpha}} \gamma u |u|) du \right]
\]

\[
:= \mathcal{II}_1 + \mathcal{II}_2.
\]

One has

\[
\mathcal{II}_1 = C_{\alpha} \sum_{i=1}^{n} \mathbb{E} \left[ \int_{0}^{1} \frac{n^{-\frac{1}{\alpha}}}{\sigma} u \left| f' \left( \frac{\tilde{Z}_i}{n^{\alpha}} + \frac{n^{-\frac{1}{\alpha}}}{\sigma} u \left| u \right| \right) - f' \left( \frac{\tilde{Z}_i}{n^{\alpha}} \right) \right| \log \left( n^{-\frac{1}{\alpha}} |u| \right) \frac{1}{\log \gamma u |u|^{\alpha + 1}} 1_{[e, \infty)}(n^{-\frac{1}{\alpha}} \gamma u |u|) du \left| \right| dt du \right]
\]

\[
\leq C_{\alpha} \begin{cases} n^{\frac{\alpha - 1}{\alpha}} \int_{0}^{\frac{1}{\gamma n}} \frac{\log \left( \log \frac{n^{\alpha}}{\gamma} \right)}{\log \gamma u |u|^{\alpha}} du, & \alpha \in (0, 1), \\ n^{\frac{\alpha - 2}{\alpha}} \int_{0}^{\frac{1}{\gamma n}} \frac{\log \left( \log \frac{n^{\alpha}}{\gamma} \right)}{\log \gamma u u^{\alpha - 1}} du, & \alpha \in [1, 2), \\ = C_{\alpha} \frac{1}{\log \gamma} \int_{0}^{\frac{1}{\gamma n}} \frac{\log \left( \log \frac{n^{\alpha}}{\gamma} \right)}{\log \gamma u^{\alpha} u} du, & \alpha \in (0, 1), \\ = C_{\alpha} \frac{1}{\log \gamma} \int_{0}^{\frac{1}{\gamma n}} \frac{\log \left( \log \frac{n^{\alpha}}{\gamma} \right)}{\log \gamma u^{\alpha} u} du, & \alpha \in [1, 2), \\ = O((\log n)^{-1}). \end{cases}
\]
whereas

$$
\mathcal{I}_2 \leq C_\alpha \begin{cases}
\frac{1}{n} \int_1^n \frac{1}{\log \gamma_n} \left| \log \left( \frac{n - \frac{1}{2} u}{\gamma_n u^{\alpha}} \right) \right| du, & \alpha \in (0, 1], \\
\frac{1}{n} \int_1^n \frac{1}{\log \gamma_n} \left| \log \left( \frac{n - \frac{1}{2} u}{\gamma_n u^{\alpha}} \right) \right| du, & \alpha \in (1, 2),
\end{cases}
$$

$$
= O((\log n)^{-1}).
$$

Putting everything together, we get that

$$
|E[f(\tilde{S}_n)] - E[f(\tilde{Y})]| = O((\log n)^{-1}).
$$

**Acknowledgements:** We would like to gratefully thank Persi Diaconis and Elton Hsu for very helpful discussions and suggestions. This work was partially supported by the grant 346300 for IMPAN from the Simons Foundation and the matching 2015-2019 Polish MNiSW fund. This research is also partly supported by the following grants: Macao S.A.R. (FDCT 038/2017/A1, FDCT 030/2016/A1, FDCT 025/2016/A1), NNSFC 11571390, University of Macau MYRG (2016-00025-FST, 2018-00133-FST).

**References**

[1] R. Adamczak, D. Chafa and P. Wolff (2016): Circular law for random matrices with exchangeable entries. *Random Structures Algorithms*. [J]. 48(3), pp. 454-479.

[2] B. Arras and C. Houdré (2018): On Stein’s method for infinitely divisible laws with finite first moment. Preprint, arXiv: 1712.10051.

[3] B. Arras, G. Mijoule, G. Poly and Y. Swan (2017): A new approach to the Stein-Tikhomirov method: with applications to the second Wiener chaos and Dickman convergence. Preprint, arXiv: 1605.06819

[4] V. Bally, L. Caramellino and G. Poly (2018): Convergence in distribution norms in the CLT for non identical distributed random variables. *Electron. J. Probab.*. [J]. 23(45), pp. 1-51.

[5] M. Banna, F. Merlevède and M. Peligrad (2015): On the limiting spectral distribution for a large class of symmetric random matrices with correlated entries. *Stochastic Process. Appl.*. [J]. 125(7), pp. 2700-2726.

[6] A. Basak, N. Cook and O. Zeitouni (2018): Circular law for the sum of random permutation matrices. *Electron. J. Probab.*. [J]. 23(33), pp. 1-51.

[7] F. Caravenna, R. Sun and N. Zygouras (2016): The continuum disordered pinning model. *Probab. Theory Related Fields*. 164, pp. 17-59.

[8] F. Caravenna, R. Sun and N. Zygouras (2017): Polynomial chaos and scaling limits of disordered systems. *Eur. Math. Soc. (JEMS)*. [J]. 19(1), pp. 1-65.

[9] S. Chatterjee (2012): A generalization of the Lindeberg principle. *Annals of Probability*. [J]. 34(6), pp. 2061-2076.

[10] L.H.Y. Chen, L. Goldstein and Q.-M. Shao (2011): *Normal approximation by Stein’s method*. Springer-Verlag Berlin Heidelberg.

[11] P. Chen, I. Nourdin, L. Xu (2018): Stein’s method for asymmetric $\alpha$-stable distributions, with application to the stable CLT. Preprint, arXiv: 1808.02405.

[12] Z.Q. Chen, X.C. Zhang (2018): Heat kernels for time-dependent non-symmetric stable-like operators. *Journal of Mathematical Analysis and Applications*. [J]. 465, pp. 1-21.

[13] V. Chernozhukov, D. Chetverikov, K. Kato (2013): Gaussian approximations and multiplier bootstrap for maxima of sums of high-dimensional random vectors. *Annals of Statistics*. [J]. 41(6), pp. 2786-2819.

[14] V. Chernozhukov, D. Chetverikov and K. Kato (2014): Gaussian approximation of suprema of empirical processes. *Ann. Statist.*. [J]. 42(4), pp. 1564-1597.

[15] V. Chernozhukov, D. Chetverikov and K. Kato (2017): Central limit theorems and bootstrap in high dimensions. *Ann. Probab.*. [J]. 45(4), pp. 2309-2352.

[16] K.L Chung (2010): A course in probability theory 3rd ed[M]. China Machine Press.

[17] Y. Davydov, A. V. Nagaev (2002): On Two Approaches to Approximation of Multidimensional Stable Laws. *Journal of Multivariate Analysis*. [J]. 82(1), pp. 210-239.

[18] E.B. Dynkin (1965): Markov processes, Vol. 1. Springer, Berlin et al.
[19] R. Durrett (2010): Probability: theory and examples. Fourth edition. Cambridge Series in Statistical
and Probabilistic Mathematics, 31. Cambridge University Press, Cambridge, x+428 pp.
[20] F. Götze, A. Naumov and V. Ulyanov (2017): Asymptotic analysis of symmetric functions. Journal
of Theoretical Probability. [J]. 30(3), pp. 876-897.
[21] Graham, S. Bryan (2017): An econometric model of network formation with degree heterogeneity. Econometrica. [J]. 85(4), pp. 1033-1063.
[22] P. Hall (1981): Two-sided bounds on the rate of convergence to a stable law. Probability Theory and
Related Fields. [J]. 57(3), pp. 349-364.
[23] I.A. Ibragimov, Yu.V. Linnik (1971): Independent and Stationary Sequences of Random Variables. Groningen: Wolters-Noordhoff
[24] A. Juozulynas, V. Paulauskas (1998): Some remarks on the rate of convergence to stable laws. Lithuanian Mathematical Journal. [J]. 38(4), pp. 335-347.
[25] O. Johnson, R. Samworth (2005): Central limit theorem and convergence to stable laws in Mallows
distance. Bernoulli. [J]. 11(5), pp. 829-845.
[26] A. Knowles, J. Yin (2017): Anisotropic local laws for random matrices. Probab. Theory and Related Fields. [J]. 169(1-2), pp. 257-352.
[27] S.B. Korada, A. Montanari (2011): Applications of the Lindeberg principle in communications and statistical learning. IEEE Transactions on Information Theory. [J]. 57(4), pp. 2440-2450.
[28] R. Kuske, J. B. Keller (2000): Rate of Convergence to a Stable Law. Siam Journal on Applied
Mathematics. [J]. 61(4), pp. 1308-1323.
[29] J.O. Lee, K. Schnelli (2016): Tracy-Widom distribution for the largest eigenvalue of real sample
covariance matrices with general population. Ann. Appl. Probab.. [J]. 26(6), pp. 3786-3839.
[30] J.W. Lindeberg (1922): Eine neue Herleitung des Exponentialgesetzes in der Wahrscheinlichkeits-
rechnung. Mathematische Zeitschrift. [J]. 15(1), pp. 211-225.
[31] H. Liu, A. Aue and D. Paul (2015): On the Marčenko-Pastur law for linear time series. Ann. Statist..
[J]. 43(2), pp. 675-712.
[32] F. Merlevde, M. Peligrad (2016): On the empirical spectral distribution for matrices with long
memory and independent rows. Stochastic Process. Appl. [J]. 126(9), pp. 2734-2760.
[33] I. Nourdin and G. Peccati (2012): Normal approximations with Malliavin calculus: from Stein’s
method to universality. Cambridge Tracts in Mathematics 192. Cambridge University Press.
[34] V. Paulauskas, A. Račkauskas (1989): Approximation theory in the central limit theorem[M]. Springer Netherlands.
[35] D. Pouzo (2015): Bootstrap consistency for quadratic forms of sample averages with increasing
dimension. Electron. J. Stat. [J]. 9(2), pp. 3046-3097.
[36] Y. X. Ren, R. Song, Renming and R. Zhang (2017): Central limit theorems for supercritical branching
nonsymmetric Markov processes. Ann. Probab.. [J]. 45(1), pp. 564-623.
[37] V.I. Rotar (2008): Limit theorems for polynorphic forms. Journal of Multivariate Analysis. [J]. 94(4),
pp. 511-530.
[38] K.I. Sato ( 1999): Levy processes and infinitely divisible distributions[M]. Cambridge University Press.
[39] T. Tao, V. Vu (2011): Random matrices: Universality of local eigenvalue statistics. Acta Mathematica.
[J]. 206(1), pp. 127-204.
[40] H.F. Trotter, (1959): An elementary proof of the central limit theorem. Archiv Der Mathematik. [J].
10(1), pp. 226-234.
[41] L. Wang, A. Aue and D. Paul (2017): Spectral analysis of sample autocovariance matrices of a class
of linear time series in moderately high dimensions. Bernoulli. [J]. 23 (2017)(4A), pp. 2181-2209.
[42] P.M. Wood (2016): Universality of the ESD for a fixed matrix plus small random noise: a stability
approach. Ann. Inst. Henri Poincar Probab. Stat.. [J]. 52(4), pp. 1877-1896.
[43] L. Xu (2017): Approximation of stable law in Wasserstein-1 distance by Stein’s method. Accepted
by Annals of Applied Probability, [arXiv:1709.00805].
[44] J. Zhang, C.K. Wen, S. Jin, X. Gao and K.K. Wong (2013): On capacity of large-scale MIMO
multiple access channels with distributed sets of correlated antennas. IEEE Journals on Selected
Areas in Communications. [J]. 31(2), pp. 133-148.