Symmetric and Non-symmetric Macdonald Polynomials

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The symmetric Macdonald polynomials are able to be constructed out of the non-symmetric Macdonald polynomials. This allows us to develop the theory of the symmetric Macdonald polynomials by first developing the theory of their non-symmetric counterparts. In taking this approach we are able to obtain new results as well as simpler and more accessible derivations of some of the known fundamental properties of both kinds of polynomials.

1 Introduction

The symmetric Macdonald polynomial $P_\kappa := P_\kappa(x; q, t)$ is a polynomial of $n$ variables $x = (x_1, \ldots, x_n)$ having coefficients in the field $\mathbb{Q}(q, t)$ of rational functions in $q$ and $t$. The symbols $q$ and $t$ can be interpreted either as indeterminants or as parameters ranging over $0 < q, t < 1$. The symmetric Macdonald polynomial $P_\kappa(x; q, t)$ is labeled by a partition of length $\leq n$ and can be defined as the unique eigenfunction of the operator

$$D^1_n(q, t) = \sum_{i=1}^{n} \sum_{i \neq j} t x_i - x_j \tau_i$$

which is of the form

$$P_\kappa(x; q, t) = m_\kappa(x) + \sum_{\mu < \kappa} u_{\kappa \mu} m_\mu(x)$$

In (1.3), $m_\kappa(x)$ is the monomial symmetric function in variables $x_1, \ldots, x_n$ and the sum is over the partitions $\mu$ which have the same modulus as $\kappa$, but are smaller in dominance ordering. The $q$-shift operator $\tau_i$ in (1.3) acts on functions so that

$$(\tau_i f)(x_1 \ldots x_n) = f(x_1, \ldots, qx_i, \ldots x_n)$$

The symmetric Macdonald polynomials have been the subject of much recent study, both for their mathematical properties [5],[17],[23] and their applications to the trigonometric Ruijsenaars-Schneider quantum many body model [18]. They can be viewed as a $q$-generalisation of the symmetric Jack polynomials, the latter being obtained from the former in the limit $q \to 1$ with $q = t^\alpha$ and $\alpha$ fixed. In this paper we will develop the theory of the Macdonald polynomials by generalising the approach taken by Baker and Forrester [6] towards the Jack polynomials.

The strategy is to first develop the theory of non-symmetric Macdonald polynomials. These polynomials were first introduced [1],[19] some time after the seminal work of Macdonald [20] on the symmetric Macdonald polynomials. The symmetric polynomials can be constructed from their non-symmetric counterparts. This opens the way to using the theory of the non-symmetric Macdonald polynomials to develop the theory of the symmetric Macdonald polynomials. In taking this approach we will obtain new results as well as new and simpler derivations of known results. In the latter case references will be provided to the original contributors.

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2 Preliminaries

In this section we will revise the basic definitions and results of the non-symmetric, symmetric and q-
antisymmetric polynomials. Following Macdonald [19], the symmetric and q-antisymmetric polynomials
will be constructed in terms of their non-symmetric counterparts, rather than as an independent entity as
would stem from making (1.1) and (1.2) the starting point. In addition, dual non-symmetric Macdonald
polynomials will be defined and related to the symmetric and q-antisymmetric Macdonald polynomials.
The results presented on this topic are for the most part new.

The non-symmetric Macdonald polynomials are defined in terms of operators which generate an
extended affine Hecke algebra (see e.g. [16]). Let $s_{ij}$ be the operator which acts on functions of $x := (x_1, \ldots, x_n)$
by interchanging the variables $x_i$ and $x_j$. The Demazure-Lustig operators are defined by

$$T_i := t + \frac{t x_i - x_{i+1}}{x_i - x_{i+1}} (s_i - 1) \quad i = 1, \ldots, n - 1 \quad \text{and}$$

$$T_0 := t + \frac{q t x_n - x_1}{q x_n - x_1} (s_0 - 1)$$

where $s_i := s_{i+1}$ and $s_0 := s_1 \tau_1 \tau_n^{-1}$. The operators $T_i$ have the following action on the monomial
$x_i^a x_{i+1}^b$ for $1 \leq i \leq n - 1$ (see e.g [16]).

$$T_i x_i^a x_{i+1}^b = \begin{cases} (1 - t)x_i^{a-1}x_{i+1}^{b+1} + \cdots + (1 - t)x_i^{b+1}x_{i+1}^{a-1} + x_i^{b}x_{i+1}^a & a > b \\ tx_i^a x_{i+1}^b & a = b \\ (t - 1)x_i^a x_{i+1}^{b+1} + \cdots + (t - 1)x_i^{b-1}x_{i+1}^{a+1} + tx_i^{b}x_{i+1}^a & a < b \end{cases}$$

The operator $\omega$ is defined by

$$\omega := s_{n-1} \cdots s_2 s_1 \tau_1 = s_{n-1} \cdots s_is_{i+1} \cdots s_1$$

The extended affine Hecke algebra is then generated by elements $T_i$, $0 \leq i \leq n - 1$ and $\omega$, satisfying the
relations

$$(T_i - t) (T_i + 1) = 0$$

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$$

$$T_i T_j = T_j T_i \quad |i - j| \geq 2$$

$$\omega T_i = T_{i-1} \omega$$

where the indices $0, 1, \ldots, n - 1$ are understood as elements of $\mathbb{Z}_n$. From the quadratic relation (2.3), we
have the identity

$$T_i^{-1} = t^{-1} - 1 + t^{-1} T_i$$

Given a permutation $\sigma$ with reduced word decomposition $\sigma := s_{i_1} \cdots s_{i_p}$ we define

$$T_\sigma := T_{i_1} \cdots T_{i_p}$$

The Cherednik operators [3, 10] are defined by

$$Y_i = t^{-n+i} T_i \cdots T_{n-1} \omega T_{n-1}^{-1} \cdots T_{i-1}^{-1}, \quad 1 \leq i \leq n$$

The fact that the Cherednik operators commute with each other, along with the triangularity of their
action on $x^\eta := x_{n_1} \cdots x_{n_n}$, implies that they possess a set of simultaneous eigenfunctions. These are the
non-symmetric Macdonald polynomials $E_\eta$ which can be defined by the conditions

$$E_\eta(x; q, t) = x^\eta + \sum_{\nu < \eta} b_{\eta \nu} x^\nu$$

$$Y_i E_\eta(x; q, t) = t^{\eta_i} E_\eta(x; q, t) \quad 1 \leq i \leq n$$

where

$$\eta_i := \lambda^{-1} \eta_i - \ell'_\eta(i) \quad \ell'_\eta(i) := \# \{ k < i \mid \eta_k \geq \eta_i \} + \# \{ k > i \mid \eta_k > \eta_i \}$$

(2.14)
with $\lambda$ is parameter such that $t = q^\lambda$. Let $\eta^+$ be the unique partition obtained by permuting $\eta$. The partial order $\prec$ is defined on compositions having the same modulus so that

$$\nu \prec \eta \iff \nu^+ < \eta^+ \text{ or in the case } \nu^+ = \eta^+ \nu < \eta$$

(2.15)

where $\prec$ is the usual dominance ordering for $n$-tuples, that is, $\nu \prec \eta$ iff $\sum_{i=1}^{p}(\eta_i - \nu_i) \geq 0$, for all $1 \leq p \leq n$.

Following Sahi $[24]$ $\lambda'(s) := \lambda(i)$ is called the leg colength of the node $s = (i, j)$ in the composition $\eta$. The arm length $a(s)$, arm colength $a'(s)$ and leg length $l(s)$ are defined by

$$a(s) = \eta_i - j \quad l(s) = \#\{k > i \mid j \leq \eta_k \leq \eta_i\} + \#\{k < i \mid j \leq \eta_k + 1 \leq \eta_i\}$$

$$a'(s) = j - 1$$

(2.16)

The associated quantities occur frequently in the theory of the Macdonald polynomials.

$$d_\eta(q, t) := \prod_{s \in \eta} \left(1 - q^{a(s)+1}t^{l(s)+1}\right) \quad d'_\eta(q, t) := \prod_{s \in \eta} \left(1 - q^{a(s)+1}t^{l(s)}\right)$$

$$e_\eta(q, t) := \prod_{s \in \eta} \left(1 - q^{a(s)+1}t^{n-l'(s)}\right) \quad e'_\eta(q, t) := \prod_{s \in \eta} \left(1 - q^{a(s)+1}t^{n-1-l'(s)}\right)$$

$$b_\eta(q, t) := \prod_{s \in \eta} \left(1 - q^{a(s)}t^{n-l'(s)}\right) \quad \lambda(\eta) := \sum_{s \in \eta} \lambda(s)$$

$$l'(\eta) := \sum_{s \in \eta} l'(s)$$

(2.17)

All these quantities are equal to one if $\eta = 0$. For future reference some properties of these quantities, easily derivable from $[24]$, are listed.

**Lemma 2.1** Let $\Psi_\eta := (\eta_2 \ldots \eta_n, \eta_1 + 1)$ and $\delta_{i, \eta} := \bar{\eta}_i - \bar{\eta}_{i+1}$. We have

$$\frac{d_{\Psi_\eta}(q, t)}{d_{\eta}(q, t)} = \frac{e_{\Psi_\eta}(q, t)}{e_{\eta}(q, t)} = 1 - qt^{n+\bar{\eta}_1} \quad \frac{d'_{\Psi_\eta}(q, t)}{d'_{\eta}(q, t)} = \frac{e'_{\Psi_\eta}(q, t)}{e'_{\eta}(q, t)} = 1 - qt^{n-1+\bar{\eta}_1}$$

$$l(\Psi_\eta) = l(\eta) + \#\{k > 1, \eta_k \leq \eta_1\} \quad l'(\Psi_\eta) = l'(\eta) + n - 1 - \#\{k > 1, \eta_k \leq \eta_1\}$$

If $\eta_i > \eta_{i+1}$ we have

$$\frac{d_{s_i\eta}(q, t)}{d_{\eta}(q, t)} = \frac{1 - t^{\delta_{i, \eta}+1}}{1 - t^{\delta_{i, \eta}}} \quad \frac{d'_{s_i\eta}(q, t)}{d'_{\eta}(q, t)} = \frac{1 - t^{\delta_{i, \eta}}}{1 - t^{\delta_{i, \eta}+1}}$$

$$e_{s_i\eta} = e_\eta \quad e'_{s_i\eta} = e'_\eta \quad l(s_i\eta) = l(\eta) + 1 \quad l'(s_i\eta) = l'(\eta)$$

The $q$-gamma function is defined by

$$\Gamma_q(x) := (q; q)_x(1 - q)^{1-x} \quad 0 < q < 1$$

(2.18)

where

$$(q; q)_x := \frac{(q; q)_\infty}{(q^x; q)_\infty} \quad (b; q)_\infty := \prod_{i=0}^{\infty}(1 - bq^i)$$

(2.19)

We remark that with the generalised factorial defined by

$$[q^x]_{\eta,q} := \prod_{s \in \eta^+} \left(t^{l'(s)} - q^{a(s)+x}\right)$$

$$= t^{l'(\eta^+)}(1 - q)\prod_{i=1}^{n} \frac{\Gamma_q(x - \lambda(i - 1) + \eta_i^+)}{\Gamma_q(x - \lambda(i - 1))}$$

(2.20)
we have
\[ e_\eta(q, t) = t^{-l'(\eta^+)}[q^{1+\lambda_{i+1}}]_{\eta^+}, \quad e_\eta'(q, t) = t^{-l'(\eta^+)}[q^{1+\lambda_{(n-1)}}]_{\eta^+}, \quad b_\eta(q, t) = t^{-l'(\eta^+)}[q^{\lambda}q_{\eta^+}^t] \]  
(2.21)

Given a Laurent polynomial \( f \) let \( CT(f) \) denote the constant term in the Laurent expansion of \( f \) with respect to the variables \( x_1, \ldots, x_n \). In the more general case where \( f \) is not a Laurent polynomial let \( CT(f) \) denote the corresponding Fourier integral. The non-symmetric Macdonald polynomials have the following orthogonality property, which can be deduced from (2.13).

**Proposition 2.2**  \[ 20 \] Given any two polynomials \( f(x; q, t) \) and \( g(x; q, t) \) define the scalar product
\[
(f, g)_{q,t} := CT \left( f(x; q, t)g(\frac{1}{x}; \frac{1}{q}, \frac{1}{t})W(x) \right)
\]
where
\[
W(x) := W(x; q, t) := \prod_{1 \leq i < j \leq n} \left( \frac{x_i}{x_j}; q \right)_\lambda \left( \frac{x_j}{x_i}; q \right)_\lambda
\]  
(2.23)

The polynomials \( E_\eta(x; q, t) \) form an orthogonal set with respect to \( \langle \cdot, \cdot \rangle_{q,t} \).

A consequence of this is that the non-symmetric Macdonald polynomials are able to be constructed by means of a Gram-Schmidt procedure. Let \( \eta^{(1)} < \cdots < \eta^{(p)} \) be a chain of compositions satisfying
\[
\text{If } \eta^{(i)} \prec \mu \prec \eta^{(i+1)} \text{ then } \mu = \eta^{(i)} \text{ or } \mu = \eta^{(i+1)}
\]
(2.24)
The non-symmetric Macdonald polynomial \( E_{\eta^{(i)}}(q, t) \) can be determined as the unique polynomial satisfying (2.12) which is orthogonal to all \( E_{\eta^{(i)}}(q, t) \) with \( i < p \).

The non-symmetric Macdonald polynomials are elements of the ring of \( n \) variable polynomials whose coefficients are elements of the field \( \mathbb{C}(q, t) \) of rational functions in \( q \) and \( t \). As in the symmetric case the symbols \( q \) and \( t \) can be interpreted as indeterminants or as parameters ranging over \( 0 < q, t < 1 \). Let the hat symbol \( \hat{\cdot} \) denote the involution on this ring which sends \( x_i \mapsto x_{n-i+1} \), \( q \mapsto q^{-1} \) and \( t \mapsto t^{-1} \). Extend this operator to act on operators so that for any operator \( T \) and polynomial \( f \), \( \hat{T}f = (\hat{T}f) \). We define the dual non-symmetric Macdonald polynomial by \( E_{\eta}(x; q, t) := E_{\eta}(xq^{-1}, t^{-1}) \) where \( x := (x_n, \ldots, x_1) \).

These polynomials are uniquely determined by the conditions
\[
\hat{E}_\eta(x; q, t) = x^2 + \sum_{\nu \prec \eta} c_{\nu, \eta} x^\nu
\]
(2.25)
\[
\hat{Y}_i \hat{E}_\eta(x; q, t) = t^{-\eta^{-1}} \hat{E}_{\eta}(x; q, t) \quad 1 \leq i \leq n
\]
(2.26)
where \( \hat{\eta} := (\eta_n, \ldots, \eta_1) \) and the partial order \( \prec \) is defined on compositions so that
\[
\nu \prec \eta \text{ iff } \nu^+ < \eta^+ \text{ or in the case } \nu^+ = \eta^+ \text{ } \nu > \eta
\]
(2.27)
Note that if \( \nu^+ = \eta^+ \) then \( \nu \prec \eta \) iff \( \nu \triangleright \eta \).

The dual non-symmetric Macdonald polynomials are simply related to the non-symmetric Macdonald polynomials by means of the Demazure-Lustig operators.

**Lemma 2.3**

a) \[ T_{(n, \ldots, 1)} \hat{E}_\eta(x; q, t) = t^{\# \{ (i, j) | i < j, \eta_i \geq \eta_j \}} \hat{E}_\eta(x; q, t) \]
(2.28)
b) \[ T_{(n, \ldots, 1)} E_{\eta}(x; q, t) = t^{\# \{ (i, j) | i < j, \eta_i \leq \eta_j \}} \hat{E}_{\eta}(x; q, t) \]
(2.29)

**Proof.** We shall only consider (a) as the proof of (b) is similar. A direct calculation using (2.3) reveals that
\[
T_{(n, \ldots, 1)} x^\eta = t^{\# \{ (i, j) | i < j, \eta_i \leq \eta_j \}} x^\eta + \sum_{\mu \prec \eta} a_{\mu} x^\mu
\]
(2.30)
It follows that

\[ T_{\{n,\ldots,1\}} \hat{E}_\eta(x; q, t) = t^\# \{(i,j)|i<j,n \geq \eta\} \left( x^\eta + \sum_{\mu<\eta} a'_\mu x^\mu \right) \]  

(2.31)

It suffices then to show that given a chain \( \eta^{(1)} \prec \cdots \prec \eta^{(p)} = \eta \) satisfying (2.24), \( T_{\{n,\ldots,1\}} \hat{E}_{\eta^{(p)}} \) is orthogonal to \( E_{\eta^{(i)}} \) for all \( i < p \). This will be done by induction. If \( \mu \) is a minimal composition under the partial ordering \( \prec \) then \( E\mu(x; q, t) = x^\mu \). It follows from (2.31) that (2.28) is true for the composition \( \eta^{(1)} \). Suppose (2.28) is true for \( \eta^{(1)}, \ldots, \eta^{(r-1)} \). Then for any \( k < r \)

\[ \langle E_{\eta^{(k)}}, T_{\{n,\ldots,1\}} \hat{E}_{\eta^{(r)}} \rangle_{q,t} = t^{-\# \{(i,j)|i<j,n \geq \eta\}} \langle T_{\{n,\ldots,1\}} \hat{E}_{\eta^{(k)}}, T_{\{n,\ldots,1\}} \hat{E}_{\eta^{(r)}} \rangle_{q,t} \]  

(2.32)

Since \( T_i^{-1} \) is the adjoint operator of \( T_i \) with respect to \( \langle \cdot, \cdot \rangle_{q,t} \) and \( \langle \hat{f}, \hat{g} \rangle_{q,t} = \langle f, g \rangle_{q,t} \) we have

\[ \langle E_{\eta^{(k)}}, T_{\{n,\ldots,1\}} \hat{E}_{\eta^{(r)}} \rangle_{q,t} = t^{-\# \{(i,j)|i<j,n \geq \eta\}} \langle T_{\{n,\ldots,1\}} \hat{E}_{\eta^{(k)}}, \hat{E}_{\eta^{(r)}} \rangle_{q,t} = t^{-\# \{(i,j)|i<j,n \geq \eta\}} \langle E_{\eta^{(k)}}, E_{\eta^{(r)}} \rangle_{q,t} \]  

(2.33)

Next we revise the construction of the symmetric and \( q \)-antisymmetric Macdonald polynomials from the non-symmetric Macdonald polynomials. This requires introducing \( q \)-analogues of the symmetrization and antisymmetrization operators defined by [19]

\[ U^+ := \sum_{\sigma \in S_n} T_\sigma \quad U^- := \sum_{\sigma \in S_n} (-t)^{-l(\sigma)} T_\sigma \]  

(2.34)

where \( l(\sigma) := \# \{(i,j)|i < j, \sigma_i > \sigma_j\} \) is the length of the permutation \( \sigma \). These operators have the following properties

\[ T_i^\pm U^\pm = U^\pm T_i^\pm = t^\pm U^\pm \]  

(2.35)

\[ T_i^\pm U^- = U^- T_i^\pm = -U^- \]  

(2.36)

From these properties it can be deduced that

\[ U^- s_i = -U^- \]  

(2.37)

from which it follows that

\[ U^- x^\mu = 0 \quad \text{if} \quad \mu_i = \mu_j \quad \text{for} \quad i \neq j \]  

(2.38)

For \( \delta := (n-1, \ldots, 1, 0) \) it is also the case that

\[ t^{n(n-1)/2} U^- x^\delta = \prod_{1 \leq i < j \leq n} (tx_i - x_j) := \Delta_\delta(x) \]  

(2.39)

Now, when acting on symmetric functions, the Macdonald operator \( D^1_n(q,t) \) can be decomposed in terms of the Cherednik operators according to [16]

\[ D^1_n(q,t) = t^{n-1} \sum_{i=1}^n Y_i \]  

(2.40)

Since the operator \( U^+ \) commutes with \( \sum_{i=1}^n Y_i \) it follows from (2.12) and (2.13) that there exist unique symmetric polynomials indexed by partitions which satisfy

\[ \left( \sum_{i=1}^n Y_i \right) P_\kappa(x;q,t) = \left( \sum_{i=1}^n t^{\ell_i^i} \right) P_\kappa(x;q,t) \]  

(2.41)

\[ P_\kappa(x;q,t) = m_\kappa(x) + \sum_{\mu<\kappa} u_{\kappa\mu} m_\mu(x) \]  

(2.42)
From Section 1 these are the symmetric Macdonald polynomials. One has the relation

\[ P_{\eta^+}(x; q, t) = \frac{1}{\alpha_\eta(q, t)} U^+ E_\eta(x; q, t) \]  

(2.43)

for scalars \( \alpha_\eta(q, t) \). We can also define the \( q \)-antisymmetric Macdonald polynomials \[19\]. The \( q \)-antisymmetric monomial \( m'_\kappa \), indexed by the partition \( \kappa \) with non-repeating parts, is

\[ m'_\kappa := U^+ x^\kappa \] 

(2.44)

A function \( f \) is \( q \)-antisymmetric if for all \( i \),

\[ T_i f = -f. \]

The \( q \)-antisymmetric monomials are a basis for the analytic \( q \)-antisymmetric functions. The \( q \)-antisymmetric Macdonald polynomials \( S_\kappa(x; q, t) \) are indexed by partitions with non-repeating parts and can be defined by the following conditions.

\[ \left( \sum_{i=1}^{n} Y_i \right) S_\kappa(x; q, t) = \left( \sum_{i=1}^{n} t^{\kappa_i} \right) S_\kappa(x; q, t) \] 

(2.45)

\[ S_\kappa(x; q, t) = m'_\kappa(x) + \sum_{\mu < \kappa} v_{\kappa\mu} m'_\mu(x) \] 

(2.46)

Analogous to the derivation of (2.43) we have

\[ S_{\eta^+}(x; q, t) = \frac{1}{\beta_\eta(q, t)} U^- E_\eta(x; q, t) \] 

(2.47)

The symmetric and \( q \)-antisymmetric Macdonald polynomials can also be expressed as linear combinations of the non-symmetric Macdonald polynomials.

**Lemma 2.4** \[19\]

\[ a) \quad P_\kappa(x; q, t) = \sum_{\eta: \eta^+ = \kappa} \frac{d'_{\eta^+}(q, t)}{d'_\eta(q, t)} E_\eta(x; q, t) \] 

(2.48)

\[ b) \quad S_\kappa(x; q, t) = \sum_{\sigma \in S_\kappa} (-t)^{-l(\sigma)} \frac{d_{\sigma(\kappa)}(q, t)}{d_\kappa(q, t)} E_{\sigma(\kappa)}(x; q, t) \] 

(2.49)

**Proof.** A simple generalisation of the derivation of the analogue results in the case of the Jack polynomials \[8\]. \( \square \)

It immediately follows from the orthogonality of the non-symmetric Macdonald polynomials and Lemma 2.4 that

**Proposition 2.5** \[19\]

Both the symmetric Macdonald polynomials \( \{ P_\kappa(x; q, t) \} \) and the \( q \)-antisymmetric Macdonald polynomials \( \{ S_\kappa(x; q, t) \} \) form orthogonal sets with respect to \( \langle \cdot, \cdot \rangle_{q,t} \).

It follows that both the symmetric and \( q \)-antisymmetric Macdonald polynomials are able to be constructed by means of a Gram-Schmidt procedure similar to that in the case of the non-symmetric polynomials.

The dual non-symmetric Macdonald polynomials share many properties with the non-symmetric Macdonald polynomials. In particular they are equally able to serve as building blocks for the symmetric and \( q \)-antisymmetric Macdonald polynomials. This is explained by the following results.

**Lemma 2.6**

\[ a) \quad \hat{U}^+ = t^{-\frac{n(n-1)}{2}} U^+ \] 

(2.50)

\[ b) \quad \hat{U}^- = t^{\frac{n(n-1)}{2}} U^- \] 

(2.51)

**Proof.** It is well known \[26\] that given a Hecke algebra \( \mathcal{H}_n(t) \) generated by \( T_1, \ldots, T_{n-1} \), with \( t \) not a root of unity, there exist unique elements (up to scalar multiplication) \( \alpha, \beta \in \mathcal{H}_n(t) \) such that for all \( i = 1, \ldots, n-1 \),

\[ T_i \alpha = t\alpha \quad \text{and} \quad T_i \beta = -\beta \] 

(2.52)
It follows from the definitions that \( \hat{T}_i = T_{n-i}^{-1}, \hat{U}^+ \) and \( \hat{U}^- \) are then both elements of \( \mathcal{H}_n(t) \). Using (2.35) and (2.36) we have

\[
T_i \hat{U}^+ = \left( T_i \hat{U} \right)^+ = \left( T_{n-i}^{-1} \hat{U}^+ \right) = t \hat{U}^+ \tag{2.53}
\]

\[
T_i \hat{U}^- = \left( T_i \hat{U} \right)^- = \left( T_{n-i}^{-1} \hat{U}^- \right) = - \hat{U}^- \tag{2.54}
\]

Hence \( \hat{U}^+ = c_1 U^+ \) and \( \hat{U}^- = c_2 U^- \). Equating coefficients of the basis \( \{ T_\sigma : \sigma \in S_n \} \) of \( \mathcal{H}_n(t) \) reveals the coefficients \( c_1 \) and \( c_2 \) to be \( t^{-n(n-1)/2} \) and \( t^{n(n-1)/2} \) respectively. \( \Box \)

**Lemma 2.7**

a) \( P_\kappa(x; q, t) = \hat{P}_\kappa(x; q, t) = P_\kappa(x; q^{-1}, t^{-1}) \) \hspace{1cm} (2.55)

b) \( S_\kappa(x; q, t) = (-t)^{-\frac{n(n-1)}{2}} \hat{S}_\kappa(x; q, t) \) \hspace{1cm} (2.56)

**Proof.** We shall consider only the second identity as (a) is well known and is proven in a similar way as (b). It follows from Lemma 3.6 that \( m'_\lambda(x) = (-t)^{n(n-1)/2} m'_\lambda(x) \). Using the defining property (2.46) we then have

\[
\hat{S}_\kappa(x; q, t) = (-t)^{-\frac{n(n-1)}{2}} \left( m'_\kappa(x) + \sum_{\mu < \kappa} \hat{v}_{\kappa\mu} m'_\mu(x) \right) \tag{2.57}
\]

Since \( \{ (-t)^{-n(n-1)/2} \hat{S}_\kappa(x; q, t) \} \) is orthogonal with respect to \( \langle \cdot, \cdot \rangle_{q,t} \) and possesses the triangular structure (2.46) (b) must be true. \( \Box \)

Using the above two lemmas in conjunction with (2.43), (2.47) and lemma 2.4 we obtain the following two lemmas.

**Lemma 2.8**

a) \( P_{\eta^+}(x; q, t) = \frac{t^{-\frac{n(n-1)}{2}}}{\alpha_{\eta}(q^{-1}, t^{-1})} U^+ \hat{E}_{\eta}(x; q, t) \) \hspace{1cm} (2.58)

b) \( S_{\eta^+}(x; q, t) = (-1)^{n(n-1)/2} \frac{\beta_{\eta}(q^{-1}, t^{-1})}{\alpha_{\eta}(q^{-1}, t^{-1})} U^- \hat{E}_{\eta}(x; q, t) \) \hspace{1cm} (2.59)

**Lemma 2.9**

a) \( P_\kappa(x; q, t) = \sum_{\eta, \eta^+ = \kappa} \frac{d_{\eta^+}(q^{-1}, t^{-1})}{d_{\eta}(q^{-1}, t^{-1})} \hat{E}_{\eta}(x; q, t) \) \hspace{1cm} (2.60)

b) \( S_\kappa(x; q, t) = \sum_{\sigma \in S_n} (-t)^{|\sigma|} \frac{d_{\sigma(\kappa)}(q^{-1}, t^{-1})}{d_{\kappa}(q^{-1}, t^{-1})} \hat{E}_{\sigma(\kappa)}(x; q, t) \) \hspace{1cm} (2.61)

where for any permutation \( \sigma, \sigma := (\sigma_n, \ldots, \sigma_1) \).

### 3 Non-symmetric Macdonald Polynomial Theory

In this section we will derive some of the basic properties of the non-symmetric Macdonald polynomials independently of the theory of the symmetric Macdonald polynomials. A required preliminary result is the Cauchy type formula for the non-symmetric Macdonald polynomials.

**Proposition 3.1** \(^2\)

\[
\Omega(x, y; q, t) = \sum_{\eta} \frac{1}{w_{\eta}(q, t)} E_{\eta}(x; q, t) E_{\eta}(y; q^{-1}, t^{-1}), \quad \Omega(x, y; q, t) := \prod_{i=1}^{n} \frac{1}{(x_i y_i; q)_{\lambda+1}} \prod_{1 \leq i < j \leq n} \frac{1}{(x_i y_j; q)(x_j y_i; q)} \tag{3.1}
\]
Proof. Hence the $q$-analogue of these polynomials have the following stability property.

Define a scalar product by Corollary 3.2

Applying this property to (3.1) shows that the scalars $u_n(q,t)$ are independent of $n$.

Dunkl has introduced a family of multivariable polynomials which allow a workable treatment of some important constructions and has a close relationship to the theory of the non-symmetric Macdonald polynomials. The $q$-analogue of these polynomials are the polynomials $q_n(x; q, t)$ defined by

$$
\Omega(x, y; q, t) := \sum_{\eta} q_n(x; q, t) y^{\eta}
$$

(3.3)

Corollary 3.2 Define a scalar product by $\langle E_\nu(x; q, t), (E_\nu(x; q^{-1}, t^{-1}))_q := u_\nu(q, t) \delta_{\nu \eta}$. We have

$$
\langle q_\nu(x; q, t), x^{\eta}\rangle_q = \delta_{\nu \eta}
$$

(3.4)

Hence the $q_n(x; q, t)$ are a basis for the multivariable polynomials with coefficients in $\mathbb{Q}(q, t)$.

Proof. From the triangular structure of the non-symmetric Macdonald polynomials, $\{\frac{1}{u_n(q, t)} E_n(x; q, t)\}$ and $\{E_n(x; q^{-1}, t^{-1})\}$ are basis for the multivariable polynomials. The scalar product $\langle \cdot, \cdot \rangle_q$ is then well defined. An argument similar to Macdonals [20, p310-11] can now be used to show that (3.2) is equivalent to (3.1).

The non-symmetric Macdonald polynomials can be computed recursively by just two kinds of operators. The first are the Demazure-Lustig operators $T_i$, $1 \leq i \leq n - 1$. The second, introduced by Baker and Forrester [3], is the raising-type operator

$$
\Phi_q := x_n T_{n-1}^{-1} \cdots T_2^{-1} T_1^{-1} = t^{i-n} T_{n-1} \cdots T_1 x_i T_{i-1}^{-1} \cdots T_1^{-1}
$$

(3.5)

These operators have the following action on the non-symmetric Macdonald polynomials [3, 21]

$$
\Phi_q E_n(x; q, t) = t^{-\#(i | i > n, \eta \leq n)} E_{\phi \eta}(x; q, t)
$$

(3.6)

and

$$
T_i E_\eta = \begin{cases}
  \left(\frac{t^{-1}}{1-t^{-1} e_{i-1}^{\eta} e_{i, \eta}}\right) E_\eta + t E_{s_i \eta} & \eta_i < \eta_{i+1} \\
  t E_\eta & \eta_i = \eta_{i+1} \\
  \left(\frac{t^{-1}}{1-t e_{i-1}^{\eta}}\right) E_\eta + \frac{(1-t^{i+1})(1-t^{i, \eta})}{(1-t^{i, \eta})^2} E_{s_i \eta} & \eta_i > \eta_{i+1}
\end{cases}
$$

(3.7)

Using these operators, it is simple to derive the following two identities by verifying that the respective quantities satisfy the same recursion relationships.

Proposition 3.3 [11] Let $t^\omega := (1, t, \ldots, t^{n-1})$. We have

$$
E_\eta(t^\omega; q, t) = t^{\nu(\eta)} e_\eta(q, t) d_\eta(q, t)
$$

(3.8)

Proof. Noting that for any function $f = f(x)$

$$
(T_i f)(t^\omega) = tf(t^\omega)
$$

(3.9)

shows that

$$
(\Phi_q E_n(x; q, t))\big|_{x=t^\omega} = (t^{1-n} T_{n-1} \cdots T_1 x_1 E_n(x; q, t))\big|_{x=t^\omega} = E_n(t^\omega; q, t)
$$

(3.10)
The relations (3.11) and (3.12) uniquely determine $E_i \eta$. Supposing $\eta > \eta_{i+1}$ and applying (3.6) to (3.7) and rearranging, we also obtain

$$E_{s_i \eta}(t^\delta \eta, q, t) = \frac{1 - t^{\delta_i \eta}}{1 - t^{\delta_{i+1} \eta}} E_\eta(t^\delta \eta, q, t)$$

(3.12)

The relations (3.11) and (3.12) uniquely determine $E_i \eta(t^\delta \eta, q, t)$ given $E_0(t^\delta \eta, q, t)$. Since Proposition 3.3 is obviously true for the case $\eta = 0$ all that remains is to show that the right hand side of (3.8), $\text{RHS}(\eta)$ say, obeys these relations. Using Lemma 2.1 we have

$$\text{RHS}(\Phi_\eta) \equiv t^\# \{i| i > \eta_i \leq \eta \}$$

(3.13)

While supposing $\eta_i > \eta_{i+1}$ and again using Lemma 2.1 we have

$$\text{RHS}(s_i \eta) = \frac{1 - t^{\delta_i \eta}}{1 - t^{\delta_{i+1} \eta}} \text{RHS}(\eta)$$

(3.14)

**Proposition 3.4** Write $\mathcal{N}_\eta^{(E)}(q, t) := \langle E_\eta, E_\eta \rangle_{q, t}$. We have

$$\mathcal{N}_\eta^{(E)}(q, t) = \frac{d'_{\eta}(q, t) e_\eta(q, t)}{d_{\eta}(q, t)e'_{\eta}(q, t)}$$

(3.15)

**Remark.** Macdonald [19] and Cherednik [11] have derived (3.15) although in a different form.

**Proof.** Using (3.4) we have

$$\langle E_\eta, E_\eta \rangle_{q, t} = \langle \Phi_\eta E_\eta, \Phi_\eta E_\eta \rangle_{q, t} = \text{CT} \left( t^\# \{i| i > \eta_i \leq \eta \} \eta \left( T_{\eta-1} \cdots T_1 E_\eta(x; q, t) \right) \right)$$

$$= \langle T_{\eta-1} \cdots T_1 E_\eta(x; q, t) \rangle_{q, t}$$

(3.16)

In the last line we have used the fact that $T_i^{-1}$ is the adjoint operator of $T_i$ with respect to $\langle \cdot, \cdot \rangle_{q, t} [19]$. Supposing $\eta_i < \eta_{i+1}$ and using (3.7) we have

$$\langle E_{s_i \eta}, E_{s_i \eta} \rangle_{q, t} = \langle t^{-1} T_i E_\eta - \frac{1 - t^{\delta_i \eta}}{1 - t^{\delta_{i+1} \eta}} E_0, t^{-1} T_i E_\eta - \frac{1 - t^{-1}}{1 - t^{\delta_{i+1} \eta}} E_\eta \rangle_{q, t}$$

$$= \langle T_i E_\eta, T_i E_\eta \rangle_{q, t} - t^{-1} \frac{1 - t}{1 - t^{\delta_{i+1} \eta}} \langle T_i E_\eta, E_\eta \rangle_{q, t}$$

$$= \frac{1 - t^{-1}}{1 - t^{\delta_{i+1} \eta}} \langle E_\eta, T_i E_\eta \rangle_{q, t} + \frac{(1 - t)(1 - t^{-1})}{1 - t^{\delta_{i+1} \eta}} \langle E_\eta, E_\eta \rangle_{q, t}$$

(3.17)

Consider the right hand side of this expression. The first term simplifies by again using the fact that $T^{-1}$ and $T_i$ are adjoint operators, while the second and third terms simplify by making further use of (3.7) and then noting that for $\eta_i \neq \eta_{i+1}$, $E_\eta$ and $E_{s_i \eta}$ are orthogonal. We obtain after rearranging

$$\langle E_{s_i \eta}, E_{s_i \eta} \rangle_{q, t} = \frac{(1 - t^{\delta_{i+1} \eta})(1 - t^{\delta_{i+1} \eta}^{-1})}{(1 - t^{\delta_{i+1} \eta})^2} \langle E_\eta, E_\eta \rangle_{q, t}$$

(3.18)

By replacing $\eta_i$ by $s_i \eta$ and noting that if $\eta_i \neq \eta_{i+1}$ $\delta_{i, s_i \eta} = -\delta_{i \eta}$ we see that in the case $\eta_i > \eta_{i+1}$

$$\langle E_{s_i \eta}, E_{s_i \eta} \rangle_{q, t} = \frac{(1 - t^{\delta_{i+1} \eta})^2}{(1 - t^{\delta_{i+1} \eta})(1 - t^{\delta_{i+1} \eta}^{-1})} \langle E_\eta, E_\eta \rangle_{q, t}$$

(3.19)
Using Lemma 2.1 it is clear that the right hand side of (3.15) satisfies both the recursion relations (3.17) and (3.18). Since (3.15) is true in the trivial case $\eta = 0$ Proposition 3.4 is true by induction. \[ \Box \]

We shall now show that the multivariable $q$-binomial theorem involving the non-symmetric Macdonald polynomials can be deduced using Propositions 2.1 and 3.3.

**Proposition 3.5** \[ 21 \]

\[ \prod_{i=1}^{n} \frac{1}{(x_i; q)_r} = \sum_{\eta} \frac{[q^r]_{\eta^+}}{u_\eta(q,t) d_\eta(q,t)} E_\eta(x; q, t) \]  

(3.20)

**Remark.** The expression on the right hand side of (3.20) will be able to be simplified using (4.40).

**Proof.** In (3.1) first replace $n$ by $kn$ for some $k \in \mathbb{Z}_{>0}$ and then substitute $y_j = t^{kn-j}$ and let $x_{n+1} = \cdots = x_{kn} = 0$. Since $E_\eta(cx) = e^{\eta} E_\eta(x)$ we can use Proposition 3.3 to obtain

\[ \prod_{i=1}^{n} \frac{1}{(x_i; q)_{kn\lambda+1}} = \sum_{\eta} \frac{t^{(kn-1)\eta - \eta(1)}}{u_\eta(q,t) d_\eta(q,t)} E_\eta(x_1, \ldots, x_n, 0, \ldots, 0; q, t) \]  

(3.21)

Making use of (2.21), Lemma 2.1 the stability property (3.2) and the identity

\[ e_\eta(q^{-1}, t^{-1}) |_{n \rightarrow kn} = \frac{t^{(kn-1)(\eta-\eta)}}{d_\eta(q,t)^{kn}} \frac{e_\eta(q,t) |_{n \rightarrow kn}}{d_\eta(q,t)} \]  

(3.22)

we obtain for $k \in \mathbb{Z}_{>0}$

\[ \prod_{i=1}^{n} \frac{1}{(x_i; q)_{kn\lambda+1}} = \sum_{\eta} \frac{[q^{kn\lambda+1}]_{\eta^+}}{u_\eta(q,t) d_\eta(q,t)} E_\eta(x; q, t) \]  

(3.23)

To show that (3.23) is true for all $k \in \mathbb{R}$ we require $u_\eta(q,t)$ to be able to be written as a power series in $q$ and $t$ for all $0 < q, t < 1$. This falls out of the proof of Proposition 3.1 in [21] by using expansions (2.3),(2.4) and by noting that the coefficients of the non-symmetric Macdonald polynomials can be written as power series in $q$ and $t$ for all $0 < q, t < 1$. Both sides of (3.23) are then power series in $x, q, t$ and $t^k$. Equating the coefficients with respect to $q$ and $x$ we can apply the following lemma to show that the $q$-binomial theorem (3.20) is true for all $k \in \mathbb{R}$. \[ \Box \]

**Lemma 3.6** \[ 21 \]

Let $F(z, q)$ and $G(z, q)$ be formal power series in $z$ and $q$. If $F(q^k, q) = G(q^k, q)$ for infinitely many integers $k \geq 0$ then $F = G$.

## 4 A Generalisation of the q-Selberg Integral

The $q$-Selberg Integral, as formulated by Askey \[ 4 \] and subsequently proved by Kadell \[ 12 \] and Habseiger \[ 12 \], has been extended by Kadell \[ 13 \] and Kaneko \[ 14 \] to involve the symmetric Macdonald polynomial as a factor in the integrand. An equivalent formulation of this result is as a constant term identity which generalises the $q$-Morris identity \[ 15 \]. Here this result will itself be extended in that the symmetric Macdonald polynomial will be replaced by the non-symmetric Macdonald polynomial. The derivation of this identity will also yield a new derivation of the $q$-Selberg integral as well as allowing us to specify the constant $u_\eta(q,t)$ appearing in (3.4). The derivation is based on the multivariable $q$-binomial theorem (3.20).

Since $\{E_\eta(x;q,t)\}$ is an orthogonal basis for multivariable analytic functions with respect to $\langle\cdot, \cdot\rangle_{q,t}$ we can write

\[ \prod_{i=1}^{n} \frac{1}{(x_i; q)_r} = \sum_{\eta} \frac{\prod_{i=1}^{n} \frac{1}{(x_i; q)} \cdot E_\eta(x; q, t) |_{q,t}}{(E_\eta(x; q, t) E_\eta(x; q, t))_{q,t}} E_\eta(x; q, t) \]  

(4.1)
Comparing (4.1) with Proposition 3.3 we have
\[ \text{CT} \left( \prod_{i=1}^{n} \frac{1}{(x_i; q)_r} E_{\eta}(x^{-1}; q^{-1}, t^{-1})W(x) \right) = \frac{[q^r]_{\eta}^{q, t}}{u_{\eta}(q, t)d_{\eta}(q, t)} \mathcal{A}_{\eta}^{(E)}(q, t) \] (4.2)

Letting \( x_i \mapsto x_i^{-1} \) inside the argument of the constant term function, an operation that leaves it’s value unchanged, we obtain
\[ \text{CT} \left( \prod_{i=1}^{n} \frac{1}{(x_i^{-1}; q)_r} \hat{E}_{\eta}(x; q, t)W(x) \right) = \frac{[q^r]_{\eta}^{q, t}}{u_{\eta}(q, t)d_{\eta}(q, t)} \mathcal{A}_{\eta}^{(E)}(q, t) \] (4.3)

Our first task is to manipulate (4.3) so that \( \prod_{i=1}^{n}(x_i^{-1}; q)_r^{-1} \) is replaced by \( \prod_{i=1}^{n}(x_i; q)_a(qx_i^{-1}; q)_b \). We require

**Lemma 4.1** We have
\[ x^p E_{\eta}(x; q, t) = E_{\eta+p}(x; q, t) \] (4.4)
where \( \eta + p = (\eta_1 + p, \ldots, \eta_n + p) \) and \( x^p = (x_1 \ldots x_n)^p \)

**Proof.** From the definition of \( Y_i \) we have
\[ Y_i \left( x^p E_{\eta}(x; q, t) \right) = q^p x^p Y_i E_{\eta}(x; q, t) \] (4.5)
Using (2.13) we obtain
\[ Y_i \left( x^p E_{\eta}(x; q, t) \right) = q^p t^{\eta_i} x^p E_{\eta}(x; q, t) \]
\[ = t^{(\eta_1 + p)_i} x^p E_{\eta}(x; q, t) \] (4.6)
From the defining properties (2.12), (2.13) we then have the required conclusion. \( \square \)

**Corollary 4.2**
\[ x^p \hat{E}_{\eta}(x; q, t) = \hat{E}_{\eta+p}(x; q, t) \] (4.8)
Using the above proof we can extend the non-symmetric Macdonald polynomials to include Laurent polynomials. The defining properties of these Laurent polynomials \( E_{\eta} \) are the same as for the ordinary non-symmetric Macdonald polynomials except that they are indexed by compositions which can have negative parts. The non-symmetric Macdonald Laurent polynomials can be expressed in terms of the ordinary non-symmetric Macdonald polynomials by use of (4.4). The dual non-symmetric Macdonald polynomials can be similarly extended to include Laurent polynomials.

Consider (4.3) with \( \eta \) replaced by \( \eta + a \). Using Lemma 4.1 we can write \( \hat{E}_{\eta + a} = x^a \hat{E}_\eta \). Set \( r = -a - b \) with \( -a, r \in \mathbb{Z}_{\leq 0} \). A brief calculation shows that
\[ x^a \prod_{i=1}^{n} \frac{1}{(x_i; q)_r} = (-1)^{na} q^{-\frac{na}{2}(2b+1)} \prod_{i=1}^{n} (x'_i; q)_a(\frac{q}{x'_i}; q)_b \] (4.9)
where \( x'_i = q^{b+1}x_i \). Substituting into (4.3) we obtain
\[ \text{CT} \left( \prod_{i=1}^{n} (x_i; q)_a(\frac{q}{x_i}; q)_b \hat{E}_{\eta}(q^{-(b+1)}x_i; q, t)W(x) \right) = (-1)^{na} q^{-\frac{na}{2}(2b+1)} \frac{[q^r]_{\eta}^{q, t}}{u_{\eta+a}(q, t)d_{\eta+a}(q, t)} \mathcal{A}_{\eta+a}^{(E)}(q, t) \] (4.10)
Since \( \hat{E}_\eta(cx) = c|\eta| \hat{E}_\eta(x) \) and \( \mathcal{A}_{\eta+a}^{(E)}(q, t) = \mathcal{A}_{\eta}^{(E)}(q, t) \) we get
\[ \text{CT} \left( \prod_{i=1}^{n} (x_i; q)_a(\frac{q}{x_i}; q)_b \hat{E}_{\eta}(x_i; q, t)W(x) \right) = (-1)^{na} q^{-\frac{na}{2}(2b+1)+(b+1)|\eta|} \frac{[q^r]_{\eta}^{q, t}}{u_{\eta+a}(q, t)d_{\eta+a}(q, t)} \mathcal{A}_{\eta}^{(E)}(q, t) \] (4.11)
The dependence on \( a \) in \( 1/u_{\eta+a}d_{\eta+a} \) can be determined using
Theorem 4.3 We have
\[ E_q\left(\frac{1}{x}; q, t\right) = E_{-\frac{q}{x}}(x; q, t) \] (4.12)

Proof. Let the star symbol * denote the involution on the ring of n-variable polynomials with coefficients in \( \mathbb{Q}(q, t) \) which sends \( x_i \rightarrow x_i^{-1}, \quad q \rightarrow q^{-1} \) and \( t \rightarrow t^{-1} \). Extend this operator to act on operators so that for any operator \( T \) and polynomial \( f, \quad T^* f^* = (T f)^* \). From the relations \( T_i^* = T_i^{-1} \), \( w^* = w \), \( \tilde{T}_i = T_{n-i}^{-1} \) and \( \tilde{w} = w^{-1} \) it follows that
\[ (Y_{n-i+1}^*)^{-1} = t^{1-n} \tilde{Y}_i \] (4.13)

From (2.13)
\[ Y_{i}^{-1} E_q(x; q, t) = t^{-n} E_q(x; q, t) \] (4.14)

Applying the * operator and replacing \( i \) with \( n - i + 1 \) we get
\[ (Y_{n-i+1}^*)^{-1} E_q(x^{-1}; q^{-1}, t^{-1}) = t^{n-i} E_q(x^{-1}; q^{-1}, t^{-1}) \] (4.15)

Using (4.13) we obtain
\[ \tilde{Y}_i E_q(x^{-1}; q^{-1}, t^{-1}) = t^{n-1+i} E_q(x^{-1}; q^{-1}, t^{-1}) \] (4.16)

From the defining properties (2.25) and (2.26) it follows that \( E_q(x^{-1}; q^{-1}, t^{-1}) \) is a dual non-symmetric Macdonald polynomial. Since \( E_q(x^{-1}; q^{-1}, t^{-1}) \) has the same leading term as \( \tilde{E}_{-\frac{q}{x}}(x; q, t) \)
\[ E_q\left(\frac{1}{x}; q^{-1}, t^{-1}\right) = E_{-\frac{q}{x}}(x; q^{-1}, t^{-1}) \] (4.17)

The conclusion follows. \( \Box \)

Corollary 4.4 We have
\[ \tilde{E}_q\left(\frac{1}{x}; q, t\right) = \tilde{E}_{-\frac{q}{x}}(x; q, t) \] (4.18)

Now
\[ \text{CT} \left( \prod_{i=1}^{n} (x_i; q)_{a(i)} \frac{q}{x_i} q \tilde{E}_q(x; q, t) W(x) \right) = \text{CT} \left( \prod_{i=1}^{n} \frac{q}{x_i} q \tilde{E}_q(x; q, t) W(x) \right) \]
\[ = q^{\sum_{i=1}^{n} i} \text{CT} \left( \prod_{i=1}^{n} \frac{q}{x_i} q \tilde{E}_q(x; q, t) W(x) \right) \] (4.19)

To obtain the first equality we have used the invariance of the constant term identity under \( x_i \rightarrow \frac{q}{x_i} \), \( x_{n-i+1} \)
while to get the second equality we have used Corollary 4.4 and \( W(\frac{q}{x}) = W(x) \).

Applying (4.11) with \( \eta \) replaced by \(-\frac{q}{x}\) and \( a \) interchanged with \( b \) gives
\[ \text{CT} \left( \prod_{i=1}^{n} (x_i; q)_{a(i)} \frac{q}{x_i} q \tilde{E}_q(x; q, t) W(x) \right) = (-1)^{n} q^{\left( \frac{n(n+1)}{2} \right)} \left( \frac{[q^{-1}]_{a(i)} [q^{-1}]_{i} q^{t}}{x_{-a+b}} \right) \tilde{E}_{-\frac{q}{x}}(x; q, t) \] (4.20)

We write \( \eta \leq c \) if \( \eta_i \leq c \) for all \( i = 1, \ldots, n \). The equation (4.20) is valid for \( \eta \leq b \) while (4.11) is valid for \( \eta \geq -a \). Equating the right hand sides of (4.11) and (4.20) and setting \( a = 0 \) we obtain for \( 0 \leq \eta \leq b \)
\[ \frac{1}{u_{-a+b}(q, t) d_{-a+b}(q, t)} = (-1)^{n} q^{\left( \frac{n(n+1)}{2} \right)} \left( \frac{[q^{-1}]_{a(i)} [q^{-1}]_{i} q^{t}}{x_{-a+b}} \right) \] (4.21)
We can use (4.21) to define \( 1/u_{-\eta+b}d_{-\eta+b} \) for \(-\eta+b \leq 0\). It then follows that (4.21) is true for all \( \eta \in \mathbb{Z}^n \) which in turn can be used to show that (4.21) is true for all \( \eta \in \mathbb{Z}^n \). Substituting (4.21) into (4.20) we then obtain for \( a, b \in \mathbb{Z}_{\geq 0} \)

\[
\text{CT} \left( \prod_{i=1}^{n} (x_i; q)_{\alpha} \frac{q}{x_i} ; q \right) \hat{E}_\eta(x; q) W(x) = \frac{q^{(n\alpha + (b+1)-a)\alpha}[q^{-b}]_{\eta} [q^{-a+b} - \eta + b]}{u_{\eta}(q,t) d_{\eta}(q,t) [q^{-a+b} - \eta + b]} N_0^\eta(q,t) \quad (4.22)
\]

To extend this result to all \( a, b \in \mathbb{R} \) we note that both sides of (4.22) are series in \( q, q^a, q^b \) and \( q^\lambda \). We then apply Lemma 3.3 twice, once with respect to \( q^a \) and once with respect to \( q^b \).

The identity (4.22) can be simplified by taking the limit \( a \to \infty \) with \( r = -a - b \) remaining constant.

For this purpose it is convenient to first take the ratio of (4.22) to that obtained with \( \eta = 0 \), thus obtaining

\[
\frac{\text{CT} \left( \prod_{i=1}^{n} (x_i; q)_{\alpha} \frac{q}{x_i} ; q \right) \hat{E}_\eta(x; q) W(x)}{\text{CT} \left( \prod_{i=1}^{n} (x_i; q)_{\alpha} \frac{q}{x_i} ; q \right) \hat{E}_0(x; q) W(x)} = \frac{q^{(b+1-a)\alpha}[q^{-b}]_{\eta} [q^{-a+b} - \eta + b]}{u_{\eta}(q,t) d_{\eta}(q,t) [q^{-a+b} - \eta + b]} N_0^\eta(q,t) \quad (4.23)
\]

where we used the facts that \( [q^{-b}]_0 = d_0 = u_0 = 1 \). Computing the the asymptotics requires

**Lemma 4.5** For a general Laurent polynomial \( f(s_1, \ldots, s_n) \) we have

\[
\left( \Gamma_q \frac{a+1}{q^{-a}} \right) \prod_{i=1}^{n} \int_0^1 d_q s_i s_i^{-b-1} \frac{(q s_i; q)^\infty}{(q^{-a+b+1} s_i; q)^\infty} f(s_1, \ldots, s_n) = \left( \frac{(q,q)_a (q,q)_b}{(q,q)_{a+b}} \right)^n \text{CT}(s) \prod_{i=1}^{n} (s_i; q)_a \left( \frac{q}{s_i} ; q \right)_b f(q^{-b+1} s_1, \ldots, q^{-b+1} s_n) \quad (4.24)
\]

where \( \int_0^1 f(s) d_q s := (1 - q) \sum_{i=0}^{\infty} f(q^i) q^i \) is the \( q \)-integral.

**Remark.** There is a typing error in the statement of the above lemma in [3].

**Lemma 4.6** Letting \( \lambda \in \mathbb{Z}_{\geq 0} \) and \( a + b = \text{const} \) we have

\[
\lim_{a \to \infty} \frac{\text{CT} \left( \prod_{i=1}^{n} (x_i; q)_{\alpha} \frac{q}{x_i} ; q \right) \hat{E}_\eta(x; q) W(x)}{\text{CT} \left( \prod_{i=1}^{n} (x_i; q)_{\alpha} \frac{q}{x_i} ; q \right) \hat{E}_0(x; q) W(x)} = q^{(b+1)\alpha} \hat{E}_\eta(t^b, q, t) \quad (4.25)
\]

**Proof.** Fixing \( r = -a - b \) and applying Lemma (4.3) we obtain

\[
\frac{\text{CT} \left( \prod_{i=1}^{n} (x_i; q)_{\alpha} \frac{q}{x_i} ; q \right) \hat{E}_\eta(x; q) W(x)}{\text{CT} \left( \prod_{i=1}^{n} (x_i; q)_{\alpha} \frac{q}{x_i} ; q \right) \hat{E}_0(x; q) W(x)} = q^{(b+1)\alpha} \prod_{i=1}^{n} \int_0^1 d_q s_i s_i^{-b-1} \frac{(q s_i; q)^\infty}{(q^{-a+b+1} s_i; q)^\infty} \hat{E}_\eta(s; q) W(s) \quad (4.26)
\]

Using the definition of the \( q \)-integral we have

\[
\prod_{i=1}^{n} \int_0^1 d_q s_i s_i^{-b-1} \frac{(q s_i; q)^\infty}{(q^{-a+b+1} s_i; q)^\infty} \hat{E}_\eta(s; q) W(s) = (1 - q)^n \sum_{k_i \in \mathbb{Z}_{\geq 0}} q^{-b} \sum_{k_i=1}^{n} \prod_{k_i=1}^{n} \frac{(q^{k_i+1} s_i; q)^\infty}{(q^{(2-r)k_i}; q)^\infty} W(q^{k_1}, \ldots, q^{k_n}) \hat{E}_\eta(q^{k_1}, \ldots, q^{k_n}; q, t) \quad (4.27)
\]

Suppose \( \lambda \in \mathbb{Z}_{\geq 0} \). Then

\[
W(q^{k_1}, \ldots, q^{k_n}) = 0 \quad \text{if} \quad k_{i+1} = k_i - \lambda, \ldots, k_i + \lambda - 1 \quad \text{while} \quad W(1, q^\lambda, \ldots, q^{\lambda(n-1)}) \neq 0 \quad (4.29)
\]
It follows that in the limit \( b \to \infty \) with \( r = -a - b \) fixed

\[
\prod_{i=1}^{n} \int_{0}^{1} dq_i s_i^{-b-1} \frac{(q s_i; q)_{\infty}}{(q^{-r-i}s_i; q)_{\infty}} \hat{E}_q(s; q) W(s)
\]

\[
\sim (1 - q)^n q^{-b} \sum_{i=1}^{n} \lambda(i-1) \prod_{i=1}^{n} \frac{(q^{\lambda(i-1)}; q)_{\infty}}{(q^{1+r-i}\lambda(i-1); q)_{\infty}} \hat{E}_q(t^2; q) W(t^2)
\]

(4.30)

Substituting (4.30) in to (4.26) gives (4.25).

Lemma 4.6 gives the asymptotics of the left hand side of (4.23). We now seek the asymptotics of the right hand side of (4.23). It follows from the property

\[
\Gamma_q[x + 1] = [x]_q \Gamma_q[x]
\]

where \([x]_q := \frac{1 - q^x}{1 - q}\) (4.31)

that for all \( k \in \mathbb{Z} \)

\[
\frac{\Gamma_q[x + k]}{\Gamma_q[x]} = (-1)^k q^{kx + \frac{k(k-1)}{2}} \frac{\Gamma_q[1 - x]}{\Gamma_q[1 - (x + k)]}
\]

(4.32)

Using these properties along with

\[
\frac{\Gamma_q[x + a]}{\Gamma_q[x]} \sim [x]_q^n \quad \text{as} \quad x \to \infty
\]

(4.33)

shows that in the limit \( a \to \infty \) with \( a + b \) fixed

\[
[q^{-b}; q]_b^{-t} \sim t^l (q^{-1})^{(|l|)} [a]_q^{-|l|}
\]

(4.34)

\[
\frac{[q^{-a-b}; q]_b^{-t} + b}{[q^{-a-b}; q]_b^{-t} - a + b} \sim (-1)^{|l|} (1 - q)^{-|l|} q^{l(t'+(n+1))} \frac{[a]_q^{-|l|}}{[a]_q^{1+\lambda(n-1)} [q^{-1}; q]_q^{t'}}
\]

(4.35)

\[
\frac{[q^{-b}; q]_b^{-t} + b}{[q^{-b}; q]_b^{-t} - a + b} \sim (-1)^{|l|} q^{-\frac{1}{2} \sum_{i=1}^{n} n_i(n_i+1)} q^{-l t'(n^+)} [a]_q^{1+\lambda(n-1)}
\]

(4.36)

Substituting these results in to the right hand side of (4.23) and using Lemma 4.6 we have in the limit \( a \to \infty \) with \( a + b \) fixed and \( \lambda \in \mathbb{Z}_{\geq 0} \)

\[
\hat{E}_q(t^2; q, t) = \frac{[q^{1+\lambda(n-1)}]_q^{t'} N^E_{\lambda}(q, t)}{u_q(q, t) d_q(q, t) N^E_{\lambda}(q, t)}
\]

(4.37)

Since both sides of this expression can be written as power series in \( q \) and \( t \) for \( 0 < q, t < 1 \) we can apply Lemma 3.4 to extend the validity of this result to all \( \lambda > 0 \). Using this result, (4.36) and

\[
\frac{[q^{-a-b}; q]_b^{-t} + b}{[q^{-a-b}; q]_b^{-t} - a + b} = (-1)^{|l|} q^{-\frac{1}{2} \sum_{i=1}^{n} n_i(n_i+1)}
\]

we can simplify (4.23) to obtain

\[
\frac{CT \left( \prod_{i=1}^{n} (x_i; q) a_q^{(\frac{4}{q}; q), (\frac{2}{q}; q)\hat{E}_q(x; q, t) W(x) \right)}{CT \left( \prod_{i=1}^{n} (x_i; q) a_q^{(\frac{4}{q}; q), (\frac{2}{q}; q) W(x) \right)} = q^{(b+1)|l|} \hat{E}_q(t^2; q, t) \frac{[q^{-b}; q]_q^{t'}}{[q^{1+a+\lambda(n-1)}]_q^t}
\]

(4.39)

It follows from Lemma 2.3 and (3.7) that \( \{E_q\}_{q^{+} = \kappa} \) and \( \{\hat{E}_q\}_{q^{+} = \kappa} \) span the same set of functions. In particular we can write

\[
E_{\mu}(x; q, t) = \sum_{\{q^{+} = \mu^{+}\}} c_{\mu q} \hat{E}_q(x; q, t)
\]

(4.40)

for scalars \( c_{\mu q} \). Multiplying both sides of (4.39) by \( c_{\mu q} \) and summing over distinct permutations of \( \mu^{+} \) we obtain
Proposition 4.7

\[
\frac{\text{CT} \left( \prod_{i=1}^{n} (x_i; q)_{a_i} \left( \frac{q}{x_i}; q \right)_b W(x) \right)}{\text{CT} \left( \prod_{i=1}^{n} (x_i; q)_{a_i} \left( \frac{q}{x_i}; q \right)_b W(x) \right)} = q^{\binom{n}{2}} E_\eta(t; q, t) \frac{[q^{-b}]_{\eta^+}}{[q^{1+a+n(i-1)}]_{\eta^+}} \tag{4.41}
\]

Note that by multiplying both sides of (4.41) by \(d^a_+ (q, t) d^a_+(q, t)\), summing over distinct permutations of \(\kappa = \eta^+\) and applying (2.48) we get back Proposition 4.7 with \(E_\eta\) replaced by the symmetric Macdonald polynomial \(P_\kappa\). Restraining \(\lambda\) to be a non-negative integer we can use Lemma 1.5 to transform (4.41) into a generalisation of the \(q\)-Selberg integral.

Proposition 4.8

\[
\frac{n!}{\prod_{i=1}^{n} \Gamma_{q} \left[ a_i + \lambda(n-1) / 4 \right]} \prod_{i<j} \frac{\Gamma_{q} \left[ a_i - \lambda(n-1) / 4 \right]}{\Gamma_{q} \left[ a_i + \lambda(n-1) / 4 \right]} \prod_{i=1}^{n} \frac{\Gamma_{q} \left[ \lambda n + 1 \right]}{\Gamma_{q} \left[ \lambda + 1 \right]} \tag{4.44}
\]

\[
W(s) = (-1)^{n-1} q^{n(n-1)/2} \prod_{i=1}^{n} \Gamma_{q} \left[ a_i + \lambda(n-1) / 4 \right] \prod_{i<j} \frac{\Gamma_{q} \left[ a_i - \lambda(n-1) / 4 \right]}{\Gamma_{q} \left[ a_i + \lambda(n-1) / 4 \right]} \prod_{i=1}^{n} \frac{\Gamma_{q} \left[ \lambda n + 1 \right]}{\Gamma_{q} \left[ \lambda + 1 \right]} \tag{4.43}
\]

Then let \(x = -b - \lambda(n-1), y = a + b + 1\).

The above derivation of Proposition 4.8 has some further consequences in relation to the general theory. First, it allows new derivations of the \(q\)-Morris identity and the \(q\)-Selberg integral.

Proposition 4.9

\[
\frac{n!}{\prod_{i=1}^{n} \Gamma_{q} \left[ a_i + \lambda(n-1) / 4 \right]} \prod_{i<j} \frac{\Gamma_{q} \left[ a_i - \lambda(n-1) / 4 \right]}{\Gamma_{q} \left[ a_i + \lambda(n-1) / 4 \right]} \prod_{i=1}^{n} \frac{\Gamma_{q} \left[ \lambda n + 1 \right]}{\Gamma_{q} \left[ \lambda + 1 \right]} \tag{4.44}
\]

\[
\frac{n!}{\prod_{i=1}^{n} \Gamma_{q} \left[ a_i + \lambda(n-1) / 4 \right]} \prod_{i<j} \frac{\Gamma_{q} \left[ a_i - \lambda(n-1) / 4 \right]}{\Gamma_{q} \left[ a_i + \lambda(n-1) / 4 \right]} \prod_{i=1}^{n} \frac{\Gamma_{q} \left[ \lambda n + 1 \right]}{\Gamma_{q} \left[ \lambda + 1 \right]} \tag{4.43}
\]

The \(q\)-Morris identity (4.44) is then obtained by using the properties (4.31), (4.32) and the evaluation 1

\[
\lambda_0^{(E)}(q, t) = \frac{\Gamma_{q} \left[ \lambda n + 1 \right]}{\Gamma_{q} \left[ \lambda + 1 \right]} \tag{4.46}
\]

The \(q\)-Selberg integral can be evaluated as in \(\boxed{\text{3}}\) by applying Lemma 4.5 to the \(q\)-Morris identity and making some manipulations.

Proposition 4.10

\[
\frac{n!}{\prod_{i=1}^{n} \Gamma_{q} \left[ a_i + \lambda(n-1) / 4 \right]} \prod_{i<j} \frac{\Gamma_{q} \left[ a_i - \lambda(n-1) / 4 \right]}{\Gamma_{q} \left[ a_i + \lambda(n-1) / 4 \right]} \prod_{i=1}^{n} \frac{\Gamma_{q} \left[ \lambda n + 1 \right]}{\Gamma_{q} \left[ \lambda + 1 \right]} \tag{4.44}
\]

\[
\frac{n!}{\prod_{i=1}^{n} \Gamma_{q} \left[ a_i + \lambda(n-1) / 4 \right]} \prod_{i<j} \frac{\Gamma_{q} \left[ a_i - \lambda(n-1) / 4 \right]}{\Gamma_{q} \left[ a_i + \lambda(n-1) / 4 \right]} \prod_{i=1}^{n} \frac{\Gamma_{q} \left[ \lambda n + 1 \right]}{\Gamma_{q} \left[ \lambda + 1 \right]} \tag{4.43}
\]

We can use (4.47) to simplify Proposition 4.8.
Proposition 4.11
\[
\prod_{i=1}^{n} \int_{0}^{1} d_{q} s_{i} s_{i}^{-1} \frac{(q s_{i}; q\infty)}{(q^{a+b+1} s_{i}; q\infty)} E_{q}(s; q, t) \prod_{i<j} s_{i}^{2 \lambda(q^{1-\lambda} s_{i}; q)2\lambda}
= q^{\lambda x(\tau)+2\lambda(x)} E_{q}(t^{5}; q, t) \prod_{i=1}^{n} \frac{\Gamma_{q}[\lambda+1] \Gamma_{q}[x + \lambda(n-i) + \eta_{+}]}{\Gamma_{q}[\lambda+1] \Gamma_{q}[x + \lambda(2n-i-1) + \eta_{+}]}
\]  
(4.48)

This formula is a generalisation of the integration formula of Kadell \[13\] and Kaneko \[14\]. The formula of \[14\] can be reclaimed by multiplying both sides of (4.48) by \(d_{q}\) by \(d_{q}\) and summing over distinct permutations of \(\kappa\) using (2.60).

The second consequence of the derivation of Proposition 4.8 is that it allows us to calculate the normalisation constant \(u_{q}(q, t)\) appearing in (3.1).

**Proposition 4.12** \[21\]

\[
u _{q}(q, t) = \frac{d'_{q}(q, t)}{d_{q}(q, t)}
\]  
(4.49)

**Proof.** Using (2.21), (3.8), (3.22) and \(E_{q}(cx) = c^{n} E_{q}(x)\) we obtain \(u_{q}(q, t) = d'_{q}(q, t)/d_{q}(q, t)\) for \(\lambda \in \mathbb{Z}_{>0}\). Since both sides of this expression can be written as formal power series in \(q\) and \(t\) if \(0 < q, t < 1\), we can apply Lemma 3.6 to show that this result is true for all \(\lambda > 0\). \(\square\)

5 Symmetric Macdonald Polynomial Theory

In this section we will deduce analogues of Propositions 3.1 – 3.5 and Proposition 4.12 for the symmetric Macdonald polynomials. This will be done by exploiting the relationships between the symmetric, \(q\)-antisymmetric and non-symmetric Macdonald polynomials.

In order to deduce the analogue of Proposition 3.1 we need to derive the following two results. The first reveals the relationship between the symmetric and \(q\)-antisymmetric Macdonald polynomials.

**Lemma 5.1**

\[
S_{\kappa+\delta}(x; q, t) = t^{-\frac{n(n-1)}{2}} \Delta_{t}(x) P_{\kappa}(x; q, qt)
\]  
(5.1)

**Proof.** Kadell’s Lemma \[13\] gives for any antisymmetric function \(h(x)\)

\[
CT \left( \prod_{i<j}(x_{i} - a x_{j}) h(x) \right) = \frac{[n]_{a}}{n!} CT \left( \prod_{i<j}(x_{i} - x_{j}) h(x) \right)
\]  
(5.2)

Consider

\[
(\Delta_{t}(x) P_{\kappa}(x; q, qt), \Delta_{t}(x) P_{\lambda}(x; q, qt))_{q, t} = CT \left( \prod_{i<j}(x_{i} - \frac{1}{t} x_{j}) h(x) \right)
\]  
(5.3)

where

\[
h(x) := \prod_{i<j}(1 - \frac{1}{x_{i} - x_{j}})(q, x_{i} x_{j}; q)_{\lambda}(q x_{i} x_{j}; q)_{\lambda} P_{\kappa}(x; q, qt) P_{\lambda}(x; q, qt)\]
(5.4)

is an antisymmetric polynomial. Applying Kadell’s Lemma twice to the left hand side of (5.3) gives

\[
(\Delta_{t}(x) P_{\kappa}(x; q, qt), \Delta_{t}(x) P_{\lambda}(x; q, qt))_{q, t} = \frac{[n]_{q, t}}{[n]_{q, t}} CT \left( \prod_{i<j}(x_{i} - qt x_{j}) h(x) \right)
\]  
(5.5)

\[
= \frac{[n]_{q, t}}{[n]_{q, t}} (P_{\kappa}(x; q, qt), P_{\lambda}(x; q, qt))_{q, qt}
\]  
(5.6)

\[
= \frac{[n]_{q, t}}{[n]_{q, t}} (P_{\kappa}(x; q, qt), P_{\kappa}(x; q, qt))_{q, qt} \delta_{\kappa \lambda}
\]  
(5.7)
The polynomials \( t^{-n(n-1)/2} \Delta_t(x) P_\kappa(x; q, qt) \) then form an orthogonal set with respect to \( \langle \cdot, \cdot \rangle_{q,t} \). Since they also satisfy (2.46) with leading term \( m'_{\kappa+\delta} \) we obtain (5.1).

For the second result define an equivalence relationship \( \sim \) such that

\[
  f(x) \sim g(x) \quad \text{iff} \quad f(x) - g(x) = \sum_i x^{\eta(i)} \quad \text{where all the } \eta(i) \text{ have repeated parts.} \tag{5.8}
\]

Note that if \( f(x) \sim g(x) \) then it follows from (2.38) that \( U^{-f}(x) = U^{-g}(x) \). The sought identity is a partial confirmation of a \( q \)-generalisation of the Cauchy double alternant formula.

**Lemma 5.2**

\[
  U^{-}(x) \left( \prod_{i=1}^{n} \frac{1}{1 - tx_i y_i} \prod_{j<i}^{n} \frac{1 - x_i y_j}{1 - tx_i y_j} \right) = \frac{F(y) \Delta_t(x)}{\prod_{i,j} (1 - tx_i y_j)} \tag{5.9}
\]

where \( F(y) \sim \Delta_{t-1}(y) \).

**Remark.** We shall see later (5.55) that \( F(y) = \Delta_{t-1}(y) \).

**Proof.** We shall first show that

\[
  \Delta_t(y) \sim (-1)^{n(n-1)/2} \sum_{\sigma \in S_n} (-t)^{\ell(\sigma)} y^{\sigma^{-1}} \tag{5.10}
\]

where for the permutation \( \sigma = (\sigma(1), \ldots, \sigma(n)) \)

\[
  y^{\sigma^{-1}} := y_{\sigma(1)}^{\sigma(1)-1} \cdots y_{\sigma(n)}^{\sigma(n)-1} = y_{\sigma^{-1}(1)}^{\sigma^{-1}(1)} \cdots y_{\sigma^{-1}(n)}^{\sigma^{-1}(n)} \tag{5.11}
\]

It is clear that the only terms of \( \Delta_t(y) \) with non-repeating parts are \( \{ y^{\sigma^{-1}} \} \).

\[
  \Delta_t(y) \sim \sum_{\sigma \in S_n} a_{\sigma} y^{\sigma^{-1}} \tag{5.12}
\]

Also, given a permutation \( \sigma \), we can write

\[
  \prod_{i<j} (ty_i - x_j) \sim (-1)^{n(n-1)/2} \prod_{i=1}^{n} \left( \prod_{k: \sigma^{-1}(k) \neq \sigma^{-1}(i)}^{\sigma^{-1}(i)-1} (y_{\sigma^{-1}(i)} - ty_k) \right) \left( \prod_{k: \sigma^{-1}(k) = \sigma^{-1}(i)+1}^{\sigma^{-1}(i)+1} (y_k - ty_{\sigma^{-1}(i)}) \right) \tag{5.13}
\]

It is then apparent that the coefficient of \( y^{\sigma^{-1}} - 1 \) is

\[
  a_{\sigma} = (-1)^{n(n-1)/2} \prod_{i=1}^{n} (-t)^{\ell(\sigma) - \# \{(j: \sigma^{-1}(i) < \sigma^{-1}(j), j < i) \}}
  = (-1)^{n(n-1)/2} (-t)^{n(n-1)/2 - \# \{(i,j): \sigma(i) < \sigma(j), i < j \}}
  = (-1)^{n(n-1)/2} (-t)^{\ell(\sigma)} \tag{5.14}
\]

so (5.10) follows.

Let us now consider the left hand side of (5.9). We can write

\[
  \prod_{i=1}^{n} \frac{1}{1 - tx_i y_i} \prod_{j<i}^{n} \frac{1 - x_i y_j}{1 - tx_i y_j} = \left( \prod_{i,j=1}^{n} \frac{1}{1 - tx_i y_j} \right) \prod_{i,j} (1 - t^{\theta(j-i)} x_i y_j) \tag{5.15}
\]

where

\[
  \theta(s) = \begin{cases} 1 & s > 0 \\ 0 & s < 0 \end{cases} \tag{5.16}
\]
Since symmetric functions commute with $U$ the left hand side of (5.4) can be written

$$ (\prod_{i,j=1}^{n} \frac{1}{1-tx_{i}y_{j}}) U^{-} (x) \prod_{i,j} (1 - t^{\theta(j-i)}x_{i}y_{j}) $$

(5.17)

For the power series $f = \sum c_{n}x^{n}$ let $[x^{n}]f$ denote $c_{n}$. Since the only terms of $\prod_{i\neq j} (1 - t^{\theta(j-i)}x_{i}y_{j})$ with $x^{n}$ having non-repeating parts are $\{ x^{\sigma-1} \}$ we have

$$ U^{-}(x) \prod_{i\neq j} (1 - t^{\theta(j-i)}x_{i}y_{j}) = \sum_{\sigma \in S_{n}} \left[ x^{\sigma-1} \right] \prod_{i\neq j} (1 - t^{\theta(j-i)}x_{i}y_{j}) U^{-}(x) x^{\sigma-1} $$

$$ = (-t)^{-n(n-1)/2} \Delta_{t}(x) \sum_{\sigma \in S_{n}} (-1)^{1-\ell(\sigma)} [x^{\sigma-1}] \prod_{i\neq j} (1 - t^{\theta(j-i)}x_{i}y_{j}) $$

(5.18)

In the second line we have used the properties (2.37) and (2.39).

We need to determine $[x^{\sigma-1}] \prod_{i\neq j} (1 - t^{\theta(j-i)}x_{i}y_{j})$ up to equivalence under $\sim_{y}$, which means finding the coefficient of $x^{\sigma-1}$ neglecting any $y^{n}$ terms with repeated parts. This coefficient must be a linear combination of $y^{\sigma-1}$ with $\sigma \in S_{n}$. Write

$$ \prod_{i\neq j} (1 - t^{\theta(j-i)}x_{i}y_{j}) = (1 - x_{\sigma-1}(n)y_{1}) \ldots (1 - x_{\sigma-1}(n)y_{\sigma-1}(n) - 1)(1 - tx_{\sigma-1}(n)y_{\sigma-1}(n) + 1) \ldots $$

$$ \ldots (1 - tx_{\sigma-1}(n)y_{n}) \prod_{i \neq j, \sigma-1(n)} (1 - t^{\theta(j-i)}x_{i}y_{j}) $$

(5.19)

It is clear that the coefficient of $x^{\sigma-1} = x_{\sigma-1}(n) \ldots x_{\sigma-1}(1)$ must be a linear combination of $y^{\sigma}$ with $\eta_{j} \geq 1$ for all $j$ except $j = \sigma^{-1}(n)$. Hence the coefficient of $x^{\sigma-1}$ must be a linear combination of $y^{\sigma}$ with $\eta_{\sigma^{-1}(n)} = 0$. Continuing in this vein we reach the conclusion that the coefficient of $x^{\sigma-1}$ up to equivalence under $\sim_{y}$ is a scalar multiple of $y^{\sigma-1}$. Hence

$$ [x^{\sigma-1}] \prod_{i\neq j} (1 - t^{\theta(j-i)}x_{i}y_{j}) \sim_{y} [x^{\sigma-1}] \prod_{i\neq j} (1 - t^{\theta(j-i)}x_{i}y_{j}) y^{\sigma-1} $$

$$ = (-1)^{n(n-1)/2} n^{n-\sigma-1}(n) \times t^{n-\sigma-1(n)-1}(\sigma^{-1}(n) - \sigma^{-1}(n-1)) \ldots $$

$$ \ldots \times t^{n-\sigma-1(1) - \sum_{i=1}^{n} \theta(\sigma^{-1}(i) - \sigma^{-1}(1))} y^{\sigma-1} $$

$$ = (-t)^{n(n-1)/2} \prod_{i<j} (1)^{\sigma^{-1}(i)-\sigma^{-1}(j)} y^{\sigma-1} $$

$$ = (-t)^{n(n-1)/2} \prod_{i<j} (1)^{\sigma^{-1}(i)-\sigma^{-1}(j)} y^{\sigma-1} $$

(5.20)

The stated result now follows after substituting (5.20) into (5.18), making use of (5.10), and substituting the resulting identity in (5.17). 

We can now give a new derivation of the symmetric analogue of Proposition 3.1

**Proposition 5.3** [2] We have

$$ \Pi(x, y; q, t) = \sum_{\kappa} \frac{1}{\nu_{\kappa}(q, t)} P_{\kappa}(x; q, t) P_{\kappa}(y; q, t), \quad \Pi(x, y; q, t) := \prod_{i,j=1}^{n} \frac{1}{(x_{i}y_{j}; q)_{\lambda}} $$

(5.21)

for scalars $\nu_{\kappa}(q, t)$ independent of $n$.

**Proof.** To derive (5.21) we apply $U^{-}(x)$ followed by $U^{-}(y)|_{t \rightarrow t^{-1}}$ to both sides of (3.1). Write

$$ \Omega(x, y; q, t) = \prod_{i,j=1}^{n} \frac{1}{(x_{i}y_{j}; q)_{\lambda}} \prod_{i=1}^{n} \frac{1}{1 - tx_{i}y_{i}} \prod_{j<i}^{n} \frac{1 - x_{i}y_{j}}{1 - t x_{i}y_{j}} $$

(5.22)

Applying $U^{-}(x)$ to the left hand side of (3.1) and using Lemma 5.2 we get

$$ U^{-}(x) \Omega(x, y; q, t) = F(y) \Delta_{t}(x) \Pi(x, y; q, qt) $$

(5.23)
Corollary 5.4

Lemma 5.5

Noting that \( U^{-}U^- = [n]_{t^{-1}}! U^- \) and using (2.39) we have

\[
U^{-}(y)|_{t^{-1}} F(y) = U^{-}(y)|_{t^{-1}} \Delta_{t^{-1}}(y) = [n]_t! \Delta_{t^{-1}}(y)
\]  
(5.24)

So applying \( U^{-}(y)|_{t^{-1}} \) to (5.23) we obtain

\[
U^{-}(y)|_{t^{-1}} U^{-}(x) \Omega(x, y; q, t) = [n]_t! \Delta_t(x) \Delta_{t^{-1}}(y) \Pi(x; y; q, qt)
\]  
(5.25)

Applying \( U^{-}(y)|_{t^{-1}} U^{-}(x) \) now to the right hand side of (3.4) and using (2.47) and Lemma 5.1 gives

\[
\sum_{\rho} \frac{1}{u_{\rho}(q, t)} U^{-}(x) P_{\rho}(x; q, t) U^{-}(y)|_{t^{-1}} E_{\rho}(y; q^{-1}, t^{-1})
\]

\[
= \sum_{\rho} \frac{\beta_{\rho}(q,t)\beta_{\rho}^{-1}(q^{-1}, t^{-1})}{u_{\rho}(q, t)} \Delta_{t}(x) P_{\rho^{-}}(x; q, qt) \Delta_{t^{-1}}(y) P_{\rho^{-}}(x; \frac{1}{q}, \frac{1}{qt})
\]  
(5.26)

where the * denotes that the sum is restricted to \( \rho \) with distinct parts. Equating (5.25) and (5.26), using (5.25) and letting \( qt \to t \) we obtain (5.21). The stability property of the symmetric Macdonald polynomials [20] \( P_{\kappa}(x_1, \ldots, x_{n-1}; q, t) \) shows that the \( \nu_n(q, t) \) are independent of \( n \).

Corollary 5.4 cf. [20] p310-11,313] Define an scalar product by \( (\langle P_{\kappa}(x; q, t), P_{\mu}(x; q, t) \rangle_g := \nu_n(q, t) \delta_{\mu\kappa} \). We have

\[ \langle g_\mu(x; q, t), m_{\kappa}(x) \rangle_g = \delta_{\mu\kappa} \]

and hence the \( g_\mu(x; q, t) \) are a basis for the multivariable polynomials with coefficients in \( \mathbb{Q}(q, t) \).

Proof. Similar to the proof of Corollary 3.2. \( \square \)

In order to proceed further with the development of the symmetrization theory we require the following symmetrization formulas.

Lemma 5.5 Let \( \eta^R := (\eta^+) \) and \( f_j := f_j(\eta) := \# \{ i : \eta_i = j \} \). Then

\[
a) \quad P_{\eta^+}(x; q, t) = t^{-n(n-1)/2} \prod_{j=0}^{n-1} \frac{1}{(f_j)_t!} U^+ E_{\eta^R}(x; q, t)
\]  
(5.30)

\[
b) \quad S_{\eta^+}(x; q, t) = (-1)^{n(n-1)/2} U^- E_{\eta^R}(x; q, t)
\]  
(5.31)

Remark. Using the theory of the symmetric Macdonald polynomials Baker and Forrester [3] (5.8),(5.18) have derived a more general formula for the constant relating \( U^+ E_{\eta} \) and \( P_{\eta^+} \). Their expression is not in the same form as (5.30), although they can be shown to be equal using the first equality of (5.30).

Proof. We shall only consider (a) as the proof of (b) is similar. From the triangular structure of \( E_{\eta^R}(x; q, t) \)

\[ U^+ E_{\eta^R}(x; q, t) = U^+ x^\eta^R + \sum_{\nu < \eta^+} a_{\nu} m_{\nu}(x) \]

(5.32)
From (2.43) we know that \( U^+ E_{\eta^+} \) is a scalar multiple of \( P_{\eta^+} \). To find the scalar multiple we need to determine \( [m_{\eta^+}] U^+ x^{\eta^+} \). Suppose \( \sigma = s_i \ldots s_p \) is the reduced decomposition of the permutation \( \sigma \). For all \( k = 1, \ldots, p \)

\[
(s_i, \ldots, s_p, \eta^R) \leq (s_i, \ldots, s_p, \eta^R)_{k+1}
\]

From the action of \( T_\iota \) (3.7) we then have

\[
T_\sigma x^{\eta^R} = t^{(\sigma)} x^{\eta^R} + \sum_{\mu \leq \sigma^R} b_\mu x^\mu
\]

It follows that

\[
[x^{\eta^+}] U^+ x^{\eta^R} = \sum_{\sigma^R = \eta^+} T_\sigma x^{\eta^R} = \sum_{\sigma^R = \eta^+} t^{(\sigma)} = \sum_{\sigma^R = \eta^+} t^{(\sigma)} = t^{n(n-1)/2} \sum_{\sigma^R = \eta^R} t^{-l(\sigma)}
\]

Since \( U^+ x^{\eta^R} \) is symmetric this shows that the coefficient of \( m_{\eta^+} \) in \( U^+ x^{\eta^R} \) is given by the right hand side of (3.35) as required by (5.30).

We can now deduce the symmetric analogue of Proposition 3.8.

**Proposition 5.6**

\[
P_{\eta^+}(t^\iota q, t) = \frac{t^{(\iota^R)} |n|_{\iota-1}! e_{\eta^R}(q, t)}{\prod_{t=0}^{|n|_{\iota}!} [f_j]_{\iota-1}! d_{\eta^R}(q, t)} = \frac{t^{(\eta^+)} b_{\eta^+}(q, t)}{h_{\eta^+}(q, t)}
\]

where

\[
h_{\eta}(q, t) := \prod_{sl \in \iota} (1 - q^{a(s)} t^{l(s)+1})
\]

**Proof.** Applying Lemma 5.5(a) we have

\[
P_{\eta^+}(t^\iota q, t) = \frac{t^{-n(n-1)/2} \sum_{\sigma \in S_n} (T_\sigma E_{\eta^R}(x; q, t))|_{x = t^\iota}}{\prod_{j=0}^{|n|_{\iota}!} [f_j]_{\iota-1}!}
\]

Using (3.9) we obtain

\[
P_{\eta^+}(t^\iota q, t) = \frac{t^{-n(n-1)/2} \sum_{\sigma \in S_n} t^{l(\sigma)} E_{\eta^R}(t^\iota q, t)}{\prod_{j=0}^{|n|_{\iota}!} [f_j]_{\iota-1}!}
\]

Since \( t^{-n(n-1)/2} \sum_{\sigma \in S_n} t^{l(\sigma)} = \sum_{\sigma \in S_n} t^{l(\sigma)} = |n|_{\iota-1}! \) we obtain the first equality in (5.36) by using Proposition 3.3.

The second equality follows immediately from the identities

\[
\frac{|n|! e_{\eta^R}(q, t)}{\prod_{j=0}^{|n|_{\iota}!} [f_j]_{\iota}! d_{\eta^R}(q, t)} = \frac{b_{\eta^+}(q, t)}{h_{\eta^+}(q, t)}
\]

\[
t^{(\eta^R)-l(\eta^+)} = \frac{|n|!}{|n|_{\iota-1}!} \prod_{j=0}^{|n|_{\iota}!} [f_j]_{\iota-1}! [f_j]_{\iota!}
\]

For the first identity we use (2.21) and (4.31) to obtain

\[
\frac{e_{\eta^R}(q, t)}{b_{\eta^+}(q, t)} = \frac{1}{|n|_{\iota}!} \prod_{i=1}^n [\lambda^{-1} \eta^+_i + n - i + 1]_t 
\]

\[
= \frac{[f_0(\eta)]_{\iota}!}{|n|_{\iota}!} \prod_{i=1}^{n-f_0(\eta)} [\lambda^{-1} \eta^+_i + n - i + 1]_t
\]

\[
= \prod_{i=1}^{n-f_0(\eta)} [\lambda^{-1} \eta^+_i + n - i + 1]_t
\]
It suffices then to show that
\[
\frac{\eta^+}{n^{(n-1)/2} [f_0]_{t!}!} = \frac{\delta_{x,q}(q,t)}{\prod_{i=1}^{n-1}[\lambda^{-1}\eta_i + n - i + 1]} \tag{5.44}
\]
This is an easy consequence of a natural \(q\)-generalisation of the argument used in [1] to prove the corresponding identity in the Jack polynomial theory.

We now turn to the second identity. Noting that \(\frac{[m]!}{[m]_{t}!} = t^{m(m-1)/2}\) we have
\[
\frac{[n]!}{[n]_{t-1}!} \prod_{j=0}^{n-1} [f_j]_{t!}! = t^{n(n-1)/2 - \sum_{j=0}^{n-1} f_j (f_j - 1)/2} \tag{5.45}
\]
It follows from Lemma 2.1 that
\[
l(q^R) = l(\sigma) + l(\eta^+) \tag{5.46}
\]
where \(\sigma\) is the permutation of minimum length for which \(\eta^+ = \sigma(q^R)\). Since the minimum such length is \(l(\sigma) = n(n-1)/2 - \sum_{j=0}^{n-1} f_j (f_j - 1)/2\) we obtain (5.41).

**Proposition 5.7**[2]

Let \(N^{(P)}_\kappa(q,t) := \langle P_\kappa(x; q,t), P_\kappa(x; q,t) \rangle_{q,t}\). With \(\eta^+ = \kappa\) we have
\[
\frac{N^{(P)}_\eta^+(q,t)}{N^{(P)}_0(q,t)} = \prod_{j=0}^{n-1} [f_j]_{t!}! \frac{\delta_{x,q}(q,t) \epsilon_{\eta^+}(q,t)}{\delta_{x,q}(q,t) \epsilon_{\eta^+}(q,t) - \delta_{x,q}(q,t) \epsilon_{\eta^+}(q,t)} \tag{5.47}
\]

**Proof.** We have
\[
\langle U^+ E_{\eta^+}, U^+ E_{\eta^+} \rangle_{q,t} = \sum_{\sigma \in S_n} \langle T_{\sigma}^{-1} U^+ E_{\eta^+}, E_{\eta^+} \rangle_{q,t} = \sum_{\sigma \in S_n} t^{-l(\sigma)} \langle U^+ E_{\eta^+}, E_{\eta^+} \rangle_{q,t} = [n]_{t-1} \langle U^+ E_{\eta^+}, E_{\eta^+} \rangle_{q,t} \tag{5.48}
\]
In the second equality we have used the fact that \(T_{\sigma}^{-1}\) is the adjoint operator of \(T_{\sigma}\) while in the third equality we have used (2.33). Multiplying each side of (5.48) by \(\prod_{j=0}^{n-1} [f_j]_{t!}! / [f_0]_{t!}!\) and using Proposition 5.3 we obtain
\[
\langle P_{\eta^+}, P_{\eta^+} \rangle_{q,t} = \frac{t^{n(n-1)/2} [n]_{t-1}!}{\prod_{j=0}^{n-1} [f_j]_{t!}!} \langle P_{\eta^+}, E_{\eta^+} \rangle_{q,t} \tag{5.49}
\]
Using (2.48) and the orthogonality of the non-symmetric Macdonald polynomials we get
\[
\langle P_{\eta^+}, P_{\eta^+} \rangle_{q,t} = \frac{[n]_{t!}!}{\prod_{j=0}^{n-1} [f_j]_{t!}!} \frac{\delta_{x,q}(q,t)}{\delta_{x,q}(q,t)} \langle E_{\eta^+}, E_{\eta^+} \rangle_{q,t} \tag{5.50}
\]
Dividing each side by \(N^{(P)}_0(q,t) = N^{(E)}_0(q,t)\) and using Proposition 3.4 we obtain the equality on the right hand side of (5.47). The second identity follows from using the identity (5.40).

It remains to establish the analogue of Proposition 3.3 and to specify the constant \(v_{\kappa}(q,t)\) appearing in Proposition 5.3. We proceed as in the derivation of Proposition 2.3 using (5.21), (5.36) and the identity
\[
\frac{b_{\eta^+}(\frac{1}{n-1})}{h_{\eta^+}(\frac{1}{n-1})} = t^{l(\eta^+) + l(\eta^+) - (n-1)[n]} \frac{b_{\eta^+}(q,t)}{h_{\eta^+}(q,t)} \tag{5.51}
\]
We obtain
\[
\sum_{i=1}^{n} \frac{1}{[x_i; q]_r} = \sum_{\eta^+} \frac{[q]^+}{v_{\eta^+}(q,t) h_{\eta^+}(q,t)} P_{\eta^+}(x; q, t) \tag{5.52}
\]
Now substituting (2.48) for \(P_{\eta^+}\) and comparing the results with (3.20) we can read off the value of \(v_{\eta^+}(q,t)\).
Proposition 5.8 [21]

\[ v_\kappa(q, t) = \frac{d'_\kappa(q, t)}{h_\kappa(q, t)} \]  

(5.53)

Substituting this result back into (5.52) we obtain the \( q \)-binomial theorem involving the symmetric Macdonald polynomials.

Proposition 5.9 [14]

\[ \prod_{i=1}^{n} \frac{1}{(x_i; q)_r} = \sum_{\kappa} [q^n]_{\kappa} P_\kappa(x; q, t) \]  

(5.54)

To tie things up we shall prove that \( F(y) \) appearing in (2.9) is equal to \( \Delta_{\lambda^{-1}}(y) \) and hence derive a \( q \)-generalisation of the Cauchy double alternant formula. The derivation will also yield the value of the constant \( \beta_\eta(q, t) \) appearing in (2.47).

Lemma 5.10

\( a) \quad F(y) = \Delta_{\lambda^{-1}}(y) \)  

\( b) \quad \beta_\sigma(\eta^+)(q, t) = (-1)^l(q') \frac{d'_\sigma(\eta^+)(q, t)}{d_\eta^+(q, t)} \)  

(5.55)

(5.56)

Proof.

As in the proof of Proposition 5.3 apply \( U^+ \) to (5.1) and cancel the factor \( \Delta_\lambda(x) \) from the resulting expression. Substituting the right hand side of (5.1) with \( t \to qt \), using (5.55) and multiplying by \( \Delta_{\lambda^{-1}}(y) \), we obtain

\[ F(y) \sum_\kappa \frac{1}{v_\kappa(q, qt)} P_\kappa(x; q, qt) \Delta_{\lambda^{-1}}(y) P_\kappa(y; \frac{1}{q}, \frac{1}{qt}) = \Delta_{\lambda^{-1}}(y) \sum_\eta \frac{t^{n(\eta^{-1})/2} \beta_\eta(q, t)}{u_\eta(q, t)} P_{\eta^+}(x; q, qt) E_{\eta^+}(y; \frac{1}{q}, \frac{1}{l}) \]  

(5.57)

Using (5.1) and (2.48) we can write

\[ \Delta_{\lambda^{-1}}(y)P_\kappa(y; \frac{1}{q}, \frac{1}{qt}) = t^{n(\eta^{-1})/2} \sum_{\sigma \in S_n} (-t)^{l(\sigma)} \frac{d_\sigma(\eta^+)(\frac{1}{q}, \frac{1}{l})}{d_{\eta^+}(\frac{1}{q}, \frac{1}{l})} E_{\sigma(\eta^+)}(y; \frac{1}{q}, \frac{1}{l}) \]  

(5.58)

Substituting into the left hand side of (5.57) and equating the coefficients of \( P_\kappa(y; \frac{1}{q}, \frac{1}{qt}) \) we obtain for \( \eta \) with no repeated parts

\[ F(y) \sum_{\sigma \in S_n} (-t)^{l(\sigma)} u_{\sigma(\eta^-)}(q, qt) d_{\eta^+}(\frac{1}{q}, \frac{1}{l}) E_{\sigma(\eta^+)}(y; \frac{1}{q}, \frac{1}{l}) = \Delta_{\lambda^{-1}}(y) \sum_{\sigma \in S_n} \beta_\sigma(\eta^+)(q, t) u_{\sigma(\eta^-)}(q, qt) E_{\sigma(\eta^+)}(y; \frac{1}{q}, \frac{1}{l}) \]  

(5.59)

Let \( \overline{x}^\mu = \begin{cases} x^\mu & \text{if } \mu \text{ has no repeated parts} \\ 0 & \text{else} \end{cases} \)  

(5.60)

Apply this operator to (5.59). Since \( F(y) = \overline{\Delta_{\lambda^{-1}}(y)} \), equating the coefficients of the linearly independent \( \{ \overline{\Delta_{\lambda^{-1}}(y)} E_{\sigma(\eta^+)}(y; q^{-1}, l^{-1}) \} \) results in

\[ \beta_\sigma(\eta^+)(q, t) = (-t)^{l(\sigma)} \frac{u_{\sigma(\eta^-)}(q, qt)}{v_{\eta^+}(q, qt)} \frac{d_{\eta^+}(\frac{1}{q}, \frac{1}{l})}{d_{\sigma(\eta^+)}(\frac{1}{q}, \frac{1}{l})} \]  

(5.61)

Substituting back into (5.59) we obtain (5.58). To obtain (5.56) we simplify (5.61) using the identity

\[ \frac{d_{\eta^+}(q, t)}{d_{\sigma(\eta^+)}(q, t)} = t^{-l(\sigma)} \]  

(5.62)

Remark. The expression for the constant \( \beta_\sigma(\eta^+)(q, t) \) in Lemma 5.10 can be simplified using a natural \( q \)-generalisation of the argument in [3]. The simplification gives

\[ \beta_\sigma(\eta^+)(q, t) = (-1)^{l(\sigma)} \frac{d_{\sigma(\eta^+)}(q, t)}{d_{\sigma(\eta^+)}(q, t)} \]  

(5.63)
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