Scaling of transfer functions in vehicular platoons: the role of asymmetry disputed

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Abstract—This paper investigates disturbance propagation in vehicular platoons. In particular, we investigate how the $H\infty$ norm of some chosen transfer functions scale with the number of vehicles in the platoon. Dependency on the number of integrators in open-loop transfer functions is also investigated. The transfer functions are presented in a new and convenient product form. A simple test for bidirectional string stability is presented as well as conditions for undesired exponential scaling in the frequency domain. Although some recent results in communication-free vehicular platooning suggest that introducing asymmetry between the front and the rear spacing is beneficial for scaling, we show that there are actually only a few cases when asymmetry can help. It either scales badly in the $H\infty$-infinity norm or is outperformed by the predecessor following control scheme. The distinction is based on number of integrators in the open loop.

Index Terms—Vehicular platoons, string stability, harmonic instability, eigenvalues uniformly bounded from zero, asymmetric control.

I. INTRODUCTION

Vehicular platoons are chains of automatic vehicles which are supposed to travel with close spacing in a highway lane. Platooning has a potential to increase the safety and capacity of highways and at the same time can allow the driver to relax.

Nowadays, there are a great number of research results available in the literature. Some are theoretical while the others focus on the experimental aspects. Experiments in which a small vehicular platoon with standard vehicles equipped with distance measurement devices and communication were built were described in [1]–[3], to name just a few. For an overview of recent experimental platforms see [4]. Most of them rely on intervehicular communication (abbreviated as V2V), in which the vehicles share information about their states or control efforts. This allows the platoon to be easily scalable. The most commonly adopted approach is Cooperative Adaptive Cruise Control (CACC) [1], [5], in which the preceding vehicle sends its control effort to the succeeding vehicle which then uses it within a feedforward control scheme. Some other algorithms are the leader following or leader’s velocity transmission. However, the transmission or reception can be suspected to delay. Furthermore, it can also be disturbed or even denied by an intruder. The effect of communication delays was analyzed in [6].

In the absence of intervehicular communications, only the information obtained through onboard sensors is available. A number of strategies for communication-free platoon control was proposed. There are two basic types: fixed or variable desired distance to neighboring vehicle. The constant-time headway spacing policy changes the distance to the preceding vehicle based on the required speed of the platoon. This strategy is known to be scalable [7] but it increases the platoon’s length with the speed of the leader, therefore it does not increase the capacity much.

The simplest to implement among the fixed-distance approaches is predecessor following, in which the vehicle only measures the distance to the immediately preceding vehicle. When the vehicle is also equipped with a sensor measuring the distance to the immediately following vehicle, bidirectional—symmetric or asymmetric—control can be implemented.

In all such strategies, scalability in terms of disturbance propagation is an issue. A phenomenon known as string instability should be avoided. String instability occurs when a disturbance or measurement error acting at a given vehicle is amplified as it is propagated along the platoon (also called string) of vehicles. String instability has been shown to occur if two integrators are in the open loop of a vehicle model [8].

In a bidirectional scheme, if the weights of front and rear measurements differ, a nonzero lower bound on the formation eigenvalues can be achieved [9]. This would guarantee controllability [10] of the formation of arbitrary size. On the other hand, scaling of the $H\infty$ norm of a particular transfer function in the platoon was analyzed in [11] and they prove that the $H\infty$ norm grows exponentially in number of vehicles in the platoon when asymmetric control is used. They coined the term harmonic instability. Later this was generalized in [12] by showing that for arbitrary platoon with uniformly bounded eigenvalues, the norm grows exponentially as long as there are two integrators in an open loop or the vehicle model satisfies some simple properties. Hence, the uniform boundedness of eigenvalues must be payed for by very bad scaling in the frequency domain.

On the other hand, symmetric bidirectional control, while being harmonically stable [13], suffers from very long transients. The effect of noise acting on the platoon scales polynomially with length of the platoon and also qualitatively depends on the number of integrators in the open loop [14]. The transient time can be improved using a clever wave-absorbing control of the leader [15]. The scheme also extends to heterogeneous platoons [16]. Finally, in [17], heterogeneous symmetric bidirectional formation was formulated in port-Hamiltonian framework, which allowed proving passivity and boundedness of the $L_2$ norm of the system. Measurement errors can be incorporated in the framework too [18].

In this paper we deal with scaling in asymmetric bidirectional vehicular platoons. As the main results we further generalize the criteria for string stability and harmonic instability.
stability. We show that there is a very simple test for bidirectional string stability involving only a closed-loop transfer function of an individual vehicle. On the other hand, sufficient conditions for exponential scaling of the $\mathcal{H}_\infty$ norm of this transfer function with the distance in a platoon are shown as well. Both conditions are not more difficult than those for the predecessor following control strategy. In addition, we show that for asymmetric control the steady-state gain is bounded and the bound does not depend on the number of vehicles. Finally, we show that there is almost no need to use asymmetric control with a small asymmetry since the predecessor following algorithm (being an extreme version of asymmetry) outperforms it in most cases.

The paper is structured as follows. First we present the model and the setting for the analysis. Then some fundamental properties of transfer functions in a vehicular platoon are derived in the third section. In the fourth section we show the steady-state gains and in the fifth we analyze the harmonic instability. In sixth section the conditions for string stability are proved. We finish the paper by examining the situations when the asymmetric control scheme should be used.

II. VEHICLE AND PLATOON MODELLING

Consider $N$ identical vehicles travelling along a straight line. They are indexed as $i = 1, 2, \ldots, N$, with $i = 1$ corresponding to the platoon leader. The leader drives independently of the platoon.

The vehicles have identical transfer function models of arbitrary type and order

$$ G(s) = \frac{b(s)}{a(s)}. \quad (1) $$

The input to the vehicle model is produced by the dynamic controller

$$ R(s) = \frac{q(s)}{p(s)}. \quad (2) $$

The object of concern is the open-loop model

$$ M(s) = R(s)G(s) = \frac{b(s)q(s)}{a(s)p(s)}. \quad (3) $$

The input $u_i$ to the controller is a regulation error consisting of two parts: the inter-vehicular coupling $c_i$ and external command signal $r_i$. Usually, the external command is related to reference distances $d_{ref}$. The controller input is then

$$ u_i = \mu_i(y_{i-1} - y_i - d_{ref}) - \mu_i \epsilon_i(y_i - y_{i+1} - d_{ref}). \quad (4) $$

The weight $\mu_i$ is a proportional gain of the controller and $\epsilon_i$ is a weight of the rear spacing error—we will call it the level of asymmetry. If $\epsilon_i < 1$, the weight is stronger towards the predecessor, if $\epsilon_i = 1$ the car uses symmetric bidirectional control and when $\epsilon_i > 1$ the gain is stronger towards the follower. The controller input parts are: $c_i = \mu_i(y_{i-1} - y_i) - \mu_i \epsilon_i(y_i - y_{i+1})$ and $r_i = -\mu_i d_{ref} + \mu_i \epsilon_i d_{ref}$.

The leader’s input is just $r_1$ and the trailing vehicle has the input $u_N = \mu_N(y_{N-1} - y_N - d_{ref})$. The regulation error is given in a vector form as

$$ u = Lx + r \quad (5) $$

with $u = [u_1, \ldots, u_N]^T$, $x = [x_1, \ldots, x_N]^T$ and $r = [r_1, \ldots, r_N]^T$. The matrix $L = [l_{ij}] \in \mathbb{R}^{N \times N}$ is a graph Laplacian, which describes the communication structure of the platoon.

A. Laplacian properties

The Laplacian in (5) is a Laplacian of a path graph and has the following structure

$$ L = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots \\ -1 & 1 + \epsilon_2 & -\epsilon_2 & 0 & \cdots \\ 0 & -1 & 1 + \epsilon_3 & -\epsilon_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & -1 & 1 + \epsilon_{N-1} & -\epsilon_{N-1} \\ 0 & \cdots & 0 & -1 & 1 \end{bmatrix} \quad (6) $$

Note that since the controller gains $\mu_i$ only shift all eigenvalues, they were omitted from $L$ and also from the following development. Now we summarize some basic properties of the Laplacian.

**Lemma 1.** Let $\lambda_i$ be the eigenvalues of Laplacian $L$.

1. The eigenvalues $\lambda_i$ are all real and nonnegative, i.e., $\lambda_i \in \mathbb{R}$, $\lambda_i \geq 0 \forall i$.
2. Let the eigenvalues be ordered as $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_N$. Then $\lambda_1 = 0$ and this eigenvalue is simple.
3. The eigenvalues are upper bounded by $\lambda_i \leq 2 \max(l_i)$
4. Let $L_c$ be the matrix obtained from $L$ by deleting the first row and the first column (both correspond to the leader). Then $\lambda_i(L_c) = \lambda_i(L_c)$ for all $\lambda_i \neq 0$.

The proof of realness of the eigenvalues is in [19, Lemma 0.1.1]. The simplicity of the zero eigenvalue follows from the presence of a directed spanning tree in the platoon [20]. The upper bound and nonnegativity are due to Gershgorin theorem [21]. The equality of eigenvalues follows from the fact that the first row of $L$ has only zeros, which only adds zero eigenvalues.

We will often use the following fact of uniform boundedness of the nonzero eigenvalues of the Laplacian.

**Definition 1** (Uniform boundedness). The nonzero eigenvalues $\lambda_i$ of a Laplacian matrix $L \in \mathbb{R}^{N \times N}$ are uniformly bounded from zero if there exists a constant $\lambda_{\min} > 0$ such that $\lambda_i \geq \lambda_{\min}$ for $i = 2, \ldots, N$ and $\lambda_{\min}$ does not depend on $N$.

**Theorem 2.** Suppose that $\epsilon_i \leq \epsilon_{\max} < 1 \forall i$ and $\forall N$. Then the eigenvalues of $L$ are upper-bounded by $\lambda_{\max} = 2(1 + \epsilon_{\max})$ and lower-bounded by

$$ \lambda_{\min} \geq \frac{1 - (1 - \epsilon_{\max})^2}{2(1 + \epsilon_{\max})} \quad (7) $$

and these bounds do not depend on the number of vehicles $N$.

The proof was given in [12]. The Lemma implies that whenever some asymmetry towards the leader is achieved for all vehicles in a platoon of arbitrary length, the eigenvalues are uniformly bounded from zero. There is then no convergence of eigenvalues of $L$ to zero, as was incorrectly stated in [10].
lower boundedness might suggest good scaling properties, but it is not the case for practical applications, as we see further.

As we will show later, Laplacian of a platoon is in a close connection to the totally nonnegative matrices.

**Definition 2** ([19]). A matrix, which has all minors nonnegative, is called totally nonnegative matrix.

**Lemma 3** ([19, page 6.7]). A Jacobi (tridigonal) matrix has the following properties:

- It is similar to the totally nonnegative Jacobi matrix with transformation by a signature matrix. The signature matrix is a diagonal matrix with ±1 on the diagonal.
- Diagonally dominant Jacobi matrix with positive entries is totally nonnegative.

Let $A(k)$ be the matrix, which was obtained from $A$ by deleting $k$th row and column.

**Lemma 4** ([19, Theorem 5.5.6]). Let $A$ be an $n$-by-$n$ totally nonnegative matrix. Suppose the eigenvalues of $A$ are given by $\lambda_1 > \lambda_2 > \ldots > \lambda_n > 0$, and the eigenvalues of $A(k)$, $1 < k < n$, be $\mu_1 > \mu_2 > \ldots > \mu_{n-1}$. Then

$$\lambda_{j-1} \geq \mu_j \geq \lambda_{j+1}. \quad (8)$$

If $k = 1$ or $k = n$, we get $\lambda_j \geq \mu_j \geq \lambda_{j+1}$.

Applying this successively, the same result is obtained for arbitrary principal submatrix. Because of the fact that Laplacian (6) is a Jacobi matrix, it is similar to a totally nonnegative matrix and interlacing in the sense of Lemma 4 occurs.

### III. Transfer Functions in Graphs

We state here important results for transfer functions in graphs with higher order dynamics. Suppose the external input $r_C$ is applied to the control vehicle (also called control node) denoted $C$ and we measure the position $y_O$ of the observer vehicle (or observer node) denoted $O$. Let $T_{CO}(s)$ be the transfer function from the input at vehicle $C$ to the position (output) of vehicle $O$. Further, let $d_{CO}$ be the graph distance from $C$ to $O$ and $w_{CO}$ be the weight of the shortest paths between them. The weight is defined as

$$w_{CO} = \sum_{i=1}^{p} \prod_{j=1}^{d_{CO}} w_{j,j+1}, \quad (9)$$

where $w_{j,j+1}$ is the weight of the edge from $j$th node on the path to the $(j+1)$th node and $p$ is the number of paths with distance $d_{CO}$. In our case $p = 1$ and $w_{j,j+1}$ is either 1 when the the control node is closer to the leader than the observer node or $\epsilon$ when the observer node is closer to the leader.

#### A. Single integrator dynamics

Although this paper deals with vehicles with arbitrary identical models, consider for a while a single integrator dynamics of individual agent. The state $x_i$ of an individual agent is then a scalar, i.e., $x_i \in \mathbb{R}$ hence $R(s) = 1$ and $G(s) = \frac{1}{s}$. The state-space model with a scalar input $r_C \in \mathbb{R}$ and a scalar output $y_O \in \mathbb{R}$ is

$$\dot{x} = -Lx + e_C r_C, \quad y_O = e_O x \quad (10)$$

Calculating the transfer function from $C$ to $O$ we get a fraction

$$T_{CO}(s) = \frac{Q(s)}{P(s)}. \quad (11)$$

Let $\gamma_i$ be the roots of $Q(s)$—transfer function zeros—and $\lambda_i$ be the roots of $P(s)$—transfer function poles and also the eigenvalues of the Laplacian.

#### B. General models

Having defined $\lambda_i$ and $\gamma_i$, we can work with general single agent dynamics.

**Lemma 5.** The transfer function from a control node to an observer node is given as

$$[b(s)q(s)]^{d_{CO}+1} \prod_{i=1}^{N-d_{CO}-1} [a(s)p(s) + \gamma_i b(s)q(s)]$$

$$T_{CO}(s) = w_{CO} \prod_{i=1}^{N-d_{CO}-1} [a(s)p(s) + \lambda_j b(s)q(s)] \quad (12)$$

The Lemma 5 holds in general graphs and is not limited to vehicular platoons. The proofs are shown in [22] in Theorem 5 and Proposition 6.

The product form can be rewritten into a more compact form

$$T_{CO}(s) = w_{CO} \prod_{j=1, j \notin J}^{N} \prod_{i=1, i \in J}^{N-d_{CO}-1} Z_{ij}(s) \quad (13)$$

with

$$T_j(s) = \frac{b(s)q(s)}{a(s)p(s) + \lambda_j b(s)q(s)}, \quad (14)$$

$$Z_{ij}(s) = \frac{a(s)p(s) + \gamma_i b(s)q(s)}{a(s)p(s) + \lambda_j b(s)q(s)}. \quad (15)$$

Since there is always more terms of the form $a(s)p(s) + \lambda_j b(s)q(s)$ in the denominator of (12) than terms $a(s)p(s) + \gamma_i b(s)q(s)$ in the numerator, we picked $N - d_{CO} - 1$ terms $a(s)p(s) + \lambda_j b(s)q(s)$ and paired them with $a(s)p(s) + \gamma_i b(s)q(s)$ to obtain transfer functions $Z_{ij}(s)$ with desired properties. The indices of the selected terms with $\lambda_i$ form the set $J$ with cardinality $N - d_{CO} - 1$. The other unused denominator terms $a(s)p(s) + \lambda_j b(s)q(s)$ with $\lambda_i \notin J$ form the transfer functions $T_j(s)$. The possibility to pick $\lambda_i$ for $Z_{ij}(s)$ to have some prescribed properties will be used in subsequent sections.

In further development we assume that all transfer functions $Z_{ij}(s)$ and $T_j(s)$ are stable for all $N$. Stabilization is not a difficult task, since all eigenvalues $\lambda_i$ are real and bounded. A root-locus approach can be used to design a stable formation.

The numerator of $Z_{ij}(s)$ has the same structure as its denominator and the role of the roots of polynomials in single
integrator dynamics — $\lambda_j$ as a root of $P(s)$ and $\gamma_i$ as a root of $Q(s)$ — is the same. Also both numerator and denominator of $Z_{ij}(s)$ have the structure of the denominator of a closed-loop output feedback system, therefore $\lambda_j$ and $\gamma_i$ can be thought of as gains in the root-locus. Moreover, the transfer functions $T_j(s)$ are, up to the steady-state gain, a standard unit feedback closed-loops.

It follows that there are two types of zeros in the transfer functions in graphs. The first type $b(s)q(s)$ is given by numerator of the open-loop of individual agent dynamics and the designer can affect it by choosing the controller. The second type $a(s)p(s) + \gamma_i b(s)q(s)$ stems from the interconnection and the designer cannot change it independently of the poles.

A simple corollary shows that for the transfer function from the leader’s position to position of the last vehicle the transfer function is

$$T_N(s) = \prod_{j=2}^{N} \frac{b(s)p(s)}{a(s)p(s) + \lambda_j b(s)q(s)}, \quad (16)$$

We started from index $j = 2$ because the leader’s position acts as an input to the second vehicle.

Since the platoon is represented by a path graph, we can use the following Lemma, which allows us to find the zeros of the transfer functions from principal submatrices of $L$.

Lemma 6 ([22, Theorem 10] [23]). If there is only one path between the control and observer nodes, then the coefficients $\gamma_i$ are given as eigenvalues of matrix $\hat{L}$ with

$$Q(s) = w_{CO} \prod_{i=1}^{N-d_{CO}-1} (s - \gamma_i) = w_{CO} \det(sI_{N-d_{CO}-1} - \hat{L}), \quad (17)$$

where $\hat{L}$ was obtained from $L$ by deleting all rows and columns corresponding to the vertices on the path from the input of the control node to the output of the observer node.

The use of principal submatrices allows us to state the following result, which order $\lambda_j$ and $\gamma_i$ in platoons.

Lemma 7. For each eigenvalue $\gamma_i$ of $\hat{L} \in \mathbb{R}^{N-d_{CO}-1 \times N-d_{CO}-1}$ there is an eigenvalue $\lambda_i$ of $L$ in (6) such that $\gamma_i \geq \lambda_i$. Alternatively, for each $\gamma_i$, $i \leq N-d_{CO}-2$ there is $\lambda_{i+2}$ such that $\gamma_i \leq \lambda_{i+2}$. No $\lambda_i$ is taken twice in the pairs.

Proof: To transform $L$ to totally nonnegative matrix, we use the matrix $S_N = \text{diag}[1, -1, 1, \ldots, 1]$. Then we obtain $\hat{L}_\text{tn} = S_N^1 L S_N = |L|$ with the absolute value taken elementwise. The matrix $\hat{L}_\text{tn}$ has the same eigenvalues as $L$ and is still diagonally dominant, so it is by Lemma 3 a totally nonnegative matrix. We can apply the same transformation with $S_z$ to the matrix $\hat{L}$ from which zeros of the transfer function are calculated. We again obtain $\hat{L}_\text{tn} = S_z^{-1} \hat{L} S_z = |L|$. So $\hat{L}_\text{tn}$ is a totally nonnegative matrix having the same eigenvalues as $\hat{L}$.

In addition, $\hat{L}_\text{tn}$ is a principal submatrix of $L_{11n}$. Therefore Lemma 4 must hold for its eigenvalues and, by similarity, also for the eigenvalues of $L$ and $L$. Then, when we sort the eigenvalues of $L$ as $\lambda_1 < \lambda_2 < \ldots \lambda_N$ and those of $\hat{L}$ as $\gamma_1 < \gamma_2 < \ldots \gamma_r$, we can choose $\lambda_i$ for each $\gamma_i$ such that

$$\gamma_1 \geq \lambda_1, \gamma_2 \geq \lambda_2, \ldots, \gamma_r \geq \lambda_r \quad (18)$$

or, alternatively as

$$\gamma_1 \leq \lambda_3, \gamma_2 \leq \lambda_4, \ldots, \gamma_{r-1} \leq \lambda_{r+1}, \quad (19)$$

which proves the the matching in the Lemma. The eigenvalue $\gamma_i \leq \lambda_{r+1}$, but in this case $\lambda_{r+1}$ would be taken twice. ■

Corollary 8. The zeros of and poles of $T_{CO}(s)$ interlace (in a sense of Lemma 4) each other on the root-locus-like curve.

Proof: The eigenvalues $\lambda_i$ and $\gamma_i$ are all real (they are eigenvalues of tri-diagonal matrices) and by Lemma 7, they interlace each other. Since both numerator and denominator have the structure of denominator of closed loop system $a(s)p(s) + kb(s)q(s)$ with gains $k = \gamma_i$ or $k = \lambda_i$, respectively, root-locus rules hold for them and the poles and zeros of (12) lie on the root-locus curve. ■

This is illustrated in Fig 1 for $R(s) = \frac{s+1}{s}$ and $G(s) = \frac{1}{s^2}$.

Corollary 9. If the platoon is stable for an arbitrary number of vehicles (which we assume throughout the paper), then the zeros of the type $a(s)p(s) + \gamma_i b(s)q(s)$ of all transfer functions are stable.

Proof: If the platoon is stable for an arbitrary number of vehicles, then $a(s)p(s) + \lambda_i b(s)q(s)$ must be stable for all $\lambda_i \in [\lambda_{\min}, \lambda_{\max}]$. From Lemma 4, the smallest numerator gain satisfies $\gamma_{\min} \geq \lambda_{\min}$ and similarly for the largest numerator gain $\gamma_{\max} \leq \lambda_{\max}$, hence all other numerator gains $\gamma_i$ must be within the range and $a(s)p(s) + \gamma_i b(s)q(s)$ has stable roots.

The only “unstable” zeros in the platoon control can be those in $b(s)q(s)$, which either stem from the vehicle model or the designer’s choice. But their location is not affected by the interconnection.

IV. STEADY-STATE GAIN OF TRANSFER FUNCTIONS

One of the requirements for platoon control is that the steady-state gain of the transfer function from the leader’s
position to the position of any other vehicle is equal to one. Nevertheless, as shown below, this is not the case for any other vehicle acting as a control node.

At least one integrator in the open loop is a necessary condition for a good performance because it allows tracking of the leader’s (or desired) velocity. This follows from the internal model principle [24]. In the paper we assume that at least one integrator is present in the model of each vehicle.

It this section we will analyze the steady-state gain of an arbitrary transfer function in the platoon.

**Lemma 10.** Let $T_{CO}(s)$ be the transfer function between a control node $C$ and an observer node $O$. The steady-state gain is $T_{CO}(0) = w_{CO} \prod_{i=1}^{N-d_{CO}-1} \lambda_i$.

**Proof:** For at least one integrator in the open-loop we get $a(0)p(0) = 0$. Then (12) reduces to

$T_{CO}(0) = w_{CO} [b(0)q(0)]^{d_{CO}+1} \prod_{i=1}^{N-d_{CO}-1} \lambda_i b(0) q(0)] \prod_{j=1}^{N} \frac{\lambda_j b(0) q(0)}{\prod_{j=1}^{N} \lambda_j}.$

(20)

The lemma shows that the steady-state gain does not depend on the dynamic model of an individual agent, it is only a function of the structure of the network because $\lambda_i$ and $\gamma_i$ are both obtained from the Laplacian. In other words, the gain is identical for any and all open-loop models and is given by a formation with single-integrator dynamics (10).

We can apply the previous Lemma to get the steady-state gain of the transfer function $T_{CO}(s)$ from $C$ to $O$ in vehicular platoons, i.e., in path graphs.

**Theorem 11.** The steady-state gain $T_{CO}(0)$ of a transfer function in platoon is given by

$T_{CO}(0) = w_{CO} \left( 1 + \sum_{i=1}^{C-2} \prod_{j=1}^{i} \epsilon_{C-j} \right)$

(21)

with the weight of the path

$w_{CO} = \begin{cases} 1 & \text{for } C \leq O \\ \prod_{i=O}^{C} \epsilon_i & \text{for } C > O. \end{cases}$

(22)

**Proof:** We will start the products in Lemma 10 from the index of 2 in order to ignore zeros and poles at the origin. This is due to the fact that leader is uncontrollable from the platoon and it keeps the front end of the platoon fixed. In the rest of the proof we will work with the reduced Laplacian $L_r$.

We begin by calculating the product in the denominator. The product equals the determinant of the reduced Laplacian $L_r = [l_{ij}] \in \mathbb{R}^{N-1 \times N-1}$ (obtained from $L$ by deleting the row and the column corresponding to the leader). The recursive rule to calculate the determinant of tridiagonal matrix is [21, Lem. 0.9.10]

$D_n = l_{n,n}D_{n-1} - l_{n,n+1}l_{n+1,n}D_{n-2},$

(23)

where $D_n$ is the determinant of the submatrix of size $n$. We begin from the bottom right corner of $L_r$. The determinant of $D_1$ is equal to one (the element $(N-1,N-1)$ in $L_r$). The determinant of $D_2$ is also equal to one. Then $D_3$ can be calculated as

$D_3 = (1 + \epsilon_{N-3})D_2 - \epsilon_{N-3}D_1 = 1.$

(24)

By induction the determinant of $L_r$, which can be obtained as a product of nonzero eigenvalues $\lambda_i$, is

$\det L_r = \prod_{j=2}^{N} \lambda_j = 1.$

(25)

Now we calculate the product in the numerator. Suppose the control node is located before the observer node in the platoon, i.e. $C \leq O$. If the index of the observer node is smaller than that of the control node, then the indices $C$ and $O$ are only swapped and the weight of the path is different. There is always only one path in platoon, so we can use the Lemma 6. The reduced Laplacian $\hat{L}$ obtained by removing from $L_r$ the rows and columns corresponding to the vertices on the path has the form

$\hat{L} = \begin{bmatrix} 1 & 0 \\ 0 & \hat{L}_2 \end{bmatrix}$

(26)

with

$L_1 = \begin{bmatrix} 1 + \epsilon_2 & -\epsilon_2 & 0 & \ldots & 0 \\ -1 & 1 + \epsilon_3 & -\epsilon_3 & \ldots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & -1 & 1 + \epsilon_{C-1} \end{bmatrix}$

(27)

and

$L_2 = \begin{bmatrix} 1 + \epsilon_{O+1} & -\epsilon_{O+1} & 0 & \ldots & 0 \\ -1 & 1 + \epsilon_{O+2} & -\epsilon_{O+2} & \ldots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & -1 & 1 \end{bmatrix}$

(28)

The dimensions are $L_1 \in \mathbb{R}^{(C-2) \times (C-2)}$ and $L_2 \in \mathbb{R}^{(N-O-1) \times (N-O-1)}$.

Since $\hat{L}$ is a block diagonal matrix, $\det \hat{L} = \det L_1 \det L_2$. The matrix $L_2$ has the same structure as $L_1$, hence $\det L_2 = 1$. The determinant of $L_1$ of size $n \times n$ can be recursively calculated as $\det L_{1,n} = (1 + \epsilon_n) \det L_{1,n-1} - \epsilon_n \det L_{1,n-2}$. Let us start from the bottom right corner again. Then $\det L_{1,1} = 1 + \epsilon_{C-1}$ and $\det L_{1,2} = 1 + \epsilon_{C-1} + \epsilon_{C-1} \epsilon_{C-2}$. The determinant

$\det L_{1,3} = (1 + \epsilon_{C-3}) \det L_{1,2} - \epsilon_{C-3} \det L_{1,1}$

$= 1 + \epsilon_{C-1} + \epsilon_{C-1} \epsilon_{C-2} + \epsilon_{C-1} \epsilon_{C-2} \epsilon_{C-3}.$

(29)

The pattern is now apparent and the determinant of $L_1$ is

$\det L_1 = 1 + \sum_{i=1}^{C-2} \prod_{j=1}^{i} \epsilon_{C-j}.$

(30)

The sum goes from 1 to $C-2$ because we excluded the leader from the formation and the vehicle $C$ is already a part of the
path from $C$ to $O$, so $C - 2$ vehicles remain. The steady state gain is then

$$T_{CO}(0) = w_{CO} \prod_{i=1}^{N-dco} \gamma_i = w_{CO} \det L_1 \det L_2 \det L_i$$

$$= w_{CO} \left( 1 + \sum_{i=1}^{C-2} \prod_{j=1}^{C-1} \epsilon_{C-j} \right). \quad (31)$$

The weight is a product of the weights of the edges on the path form $C$ to $O$, so it is one when the control node is closer to the leader and $\prod_{i=0}^{C-1} \epsilon_i$ if it is further from the leader than the observer node.

Clearly, the steady-state gain is only a function of the asymmetries and the control node.

This theorem makes sense, since the input at control vehicle acts as a desired distance and when we increase the desired distance for a given vehicle, all the vehicles in front of it must increase their distances too. The worst case is when the input is applied to the trailing vehicle, which causes all the vehicles in the platoon to increase their distance. Therefore, the only vehicle which has a steady-state gain equal to one is the leader. The maximum steady-state gain depends only on the distance of the control vehicle from the leader.

**Corollary 12.** If there is in $L$ a maximum asymmetry $\epsilon_{\max}$ such that $\epsilon_i \leq \epsilon_{\max} < 1 \forall i$ then the steady-state gain is bounded as

$$T_{CO}(0) \leq \frac{1}{1 - \epsilon_{\max}} \forall C, O. \quad (32)$$

**Proof:** We can bound the product in Theorem 11 as

$$\prod_{i=1}^{C-2} \epsilon_{C-j} \leq \epsilon_{i_{\max}}. \quad (33)$$

Then we get the the bound on the steady state gain,

$$T_{CO}(0) = w_{CO} \left( 1 + \sum_{i=1}^{C-2} \epsilon_{i_{\max}} \right) \leq w_{CO} \frac{1}{1 - \epsilon_{\max}}, \quad (34)$$

since $\sum_{i=0}^{\infty} \epsilon_{i_{\max}} = \frac{1}{1 - \epsilon_{\max}}$. If $C \leq O$, then all the weights are 1 and $w_{CO} = 1$. If $C \leq O$, then $w_{CO} = \prod_{i=0}^{C-1} \epsilon_i \leq \epsilon_{dco} < 1$. Therefore,

$$T_{CO}(0) \leq w_{CO} \frac{1}{1 - \epsilon_{\max}} \leq \frac{1}{1 - \epsilon_{\max}}. \quad (35)$$

The bound for the predecessor-following control strategy is one, which is the lowest amidst all control strategies.

**Corollary 13.** In symmetric bidirectional control the steady-state gain is $C - 1$.

As a consequence, the gain grows without bound with $N$. The steady-state gain for a fixed control node and a varying observer node is in Fig. 2b, while the gain from $C$ to $C$ is in Fig. 2a. Although the gain grows with $C$, for fixed $C$ the gain does not grow with the number of agents $N$.

### V. Exponential Scaling of $H_\infty$ Norm with Distance

In [12] it was proved that the response of the last vehicle in the platoon to the movement of the leader grows exponentially in the number of vehicles, which was attributed to the presence of a uniform lower bound on eigenvalues. But all the analysis was done only for one particular transfer function in the platoon—the transfer function from the leader to the trailing vehicle. Here we extend the results to an arbitrary transfer function in a finite platoon.

First we will restate the definition of harmonic instability, the term coined by [11]. Let $T_N(s)$ be the transfer function from the leader’s position (second vehicle’s input) to the position of the last vehicle in the platoon.

**Definition 3** (Harmonic stability [11]). Let $\gamma_N \equiv \sup_{\omega \in \mathbb{R}} |T_N(j\omega)|$, where $j = \sqrt{-1}$. The platoon is called harmonically stable if it is asymptotically stable and if $\lim sup_{N \to \infty} \gamma_N^{1/N} \leq 1$. Otherwise it is harmonically unstable.

An interpretation of harmonic instability is that the growth of the magnitude frequency response is exponential in $N$. In order words, the $H_\infty$ norm of $T_N(s)$ grows exponentially with $N$. Loosely speaking, whereas string stability guarantees that the amplitude is not magnified as the number of vehicles grows, harmonic stability guarantees that the magnification does not grow too fast with the number of vehicles.

Here we use the same assumptions on the transfer functions in the platoon as in [12]. Let $\lambda_{min}(s)$ be the transfer function of the closed-loop system with a lower bound $\lambda_{min}$ on nonzero
eigenvalues acting as a proportional gain,
\[ T_{\text{min}}(s) = \frac{\lambda_{\text{min}} b(s) q(s)}{a(s) p(s) + \lambda_{\text{min}} b(s) q(s)} = \lambda_{\text{min}} T_{\text{min}}(s). \] (36)

We also denote \( T_{\lambda_j}(s) = T_j(s) \lambda_j \).

First we need two technical lemmas describing amplifications at a frequency \( \omega_0 \).

**Lemma 14.** If \( |T_{\text{min}}(\omega_0)| > 1 \) at some frequency \( \omega_0 \), then \( |T_{\lambda_j}(\omega_0)| > 1 \) at \( \omega_0 \) for all \( \lambda_j \geq \lambda_{\text{min}} \).

**Proof:** Denote a complex number \( \alpha + j\beta = \lambda_{\text{min}} M(\omega_0) \), with \( \alpha, \beta \) real. Then the frequency response at \( \omega_0 \) of \( T_{\text{min}}(\omega_0) \) is
\[ T_{\text{min}}(\omega_0) = \frac{\alpha + j\beta}{1 + \alpha + j\beta} = \frac{\alpha^2 + \alpha + \beta^2 + j\beta}{(\alpha + 1)^2 + \beta^2}. \] (37)

The modulus of this frequency response is
\[ |T_{\text{min}}(\omega_0)| = \sqrt{\frac{\alpha^2 + \beta^2}{(\alpha + 1)^2 + \beta^2}}. \] (38)

This modulus must be greater than 1. Since the argument of square root is positive, we can write
\[ |T_{\text{min}}(\omega_0)| > 1 \iff \frac{\alpha^2 + \beta^2}{(\alpha + 1)^2 + \beta^2} > 1. \] (39)

We can multiply by the denominator to get the condition
\[ \alpha < -\frac{1}{2}. \] (40)

A similar approach is used for any closed-loop \( T_{\lambda_j}(s) \) with a feedback gain \( \lambda_j > \lambda_{\text{min}} \). Denote \( \zeta_i = \frac{\lambda_j}{\lambda_{\text{min}}} > 1 \). Then the frequency response of \( T_{\lambda_j}(s) \) is
\[ T_{\lambda_j}(\omega_0) = \frac{\zeta_i (\alpha + j\beta)}{1 + \zeta_i (\alpha + j\beta)}. \] (41)

Its modulus is equal to
\[ |T_{\lambda_j}(\omega_0)| = \sqrt{\frac{\zeta_i^2 (\alpha^2 + \beta^2)}{(\alpha \zeta_i + 1)^2 + \beta^2 \zeta_i^2}}. \] (42)

We can find conditions under which this modulus is greater than one
\[ |T_{\lambda_j}(\omega_0)| > 1 \iff \alpha < -\frac{1}{2 \zeta_i}. \] (43)

Since \( \zeta_i > 1 \), the last inequality is always satisfied for \( \alpha < -\frac{1}{2} \).

We established that whenever the closed-loop for the lower bound has \( |T_{\text{min}}(\omega_0)| > 1 \), then all other \( T_{\lambda_j}(s) \)'s with greater feedback gain have also gain greater than one at the same frequency.

The next lemma describes the modulus of \( Z_{ij}(\omega_0) \).

**Lemma 15.** Let \( \lambda_j M(\omega_0) = \alpha_j + j\beta_j \) for some \( \omega_0 \), \( \alpha_j, \beta_j \in \mathbb{R} \). Let \( Z_{ij}(s) = \frac{a(s) p(s) + \gamma_j b(s) q(s)}{a(s) p(s) + \lambda_j(b(s) q(s))} \) and its steady-state gain be \( |Z_{ij}(0)| = \frac{1}{\lambda_j} \). Then
a) \( |Z_{ij}(\omega_0)| \geq |Z_{ij}(0)| \) for \( \{ \alpha_j \leq -1 \text{ and } \gamma_i \geq \lambda_j \} \)
b) \( |Z_{ij}(\omega_0)| \geq |Z_{ij}(0)| \) for \( \{ -1 < \alpha_j \leq -\frac{1}{2} \text{ and } \gamma_i \leq \lambda_j \} \)
c) \( |Z_{ij}(\omega_0)| \leq |Z_{ij}(0)| \) for \( \{ \alpha_j > -\frac{1}{2} \text{ and } \gamma_i \geq \lambda_j \} \)

**Proof:** First note that the frequency response of the scaled open loop \( \lambda_j M(\omega_0) = \alpha_j + j\beta_j \) is evaluated in this lemma.

The transfer function \( Z_{ij}(s) \) can be written as
\[ Z_{ij}(s) = \frac{a(s) p(s) + \gamma_j b(s) q(s)}{a(s) p(s) + \lambda_j b(s) q(s)} = \frac{1 + \gamma_j M(s)}{1 + \lambda_j M(s)}. \] (44)

Evaluating its frequency response at \( \omega_0 \), we get
\[ |Z_{ij}(\omega_0)| = \frac{1 + \kappa_{ij}(\alpha_j + j\beta_j)}{1 + (\alpha_j + j\beta_j)} \] (45)

with \( \kappa_{ij} = \frac{\gamma_i}{\lambda_j} \). Its squared modulus is
\[ |Z_{ij}(\omega_0)|^2 = \frac{(\alpha_j \kappa_{ij} + 1)^2 + \kappa_{ij}^2 \beta_j^2}{(\alpha_j + 1)^2 + \beta_j^2} \]
\[ = \kappa_{ij}^2 \left[ 1 + \frac{1}{(\alpha_j + 1)^2 + \beta_j^2} \right]. \] (46)

The square of the steady-state gain of \( Z_{ij}(s) \) is \( |Z_{ij}(0)| = \kappa_{ij}^2 \).

The fraction \( m_{ij} = \frac{1}{\alpha_j + 1 + \frac{1}{\kappa_{ij}}} \) will be analyzed. If it is positive, then \( |Z_{ij}(\omega_0)|^2 > \kappa_{ij}^2 \). Let us analyze the statements in the Lemma.

a) \( \alpha_j \leq -1 \) and \( \gamma_i \geq \lambda_j \), so \( \kappa_{ij} = \frac{\gamma_i}{\lambda_j} \geq 1 \). Then \( \left( \frac{1}{\kappa_{ij}} - 1 \right) \leq 0 \) and also \( 2(\alpha_j + 1 + \frac{1}{\kappa_{ij}}) \leq 0 \). Therefore both numerator and denominator of \( m_{ij} \) are non-negative, making \( m_{ij} \geq 0 \), so \( |Z_{ij}(\omega_0)|^2 \geq \kappa_{ij}^2 \), which proves the statement.

b) \( \alpha_j \leq -\frac{1}{2} \) and \( \gamma_i \leq \lambda_j \), so \( \kappa_{ij} \leq 1 \). Then \( \left( \frac{1}{\kappa_{ij}} - 1 \right) \geq 0 \) and also \( 2(\alpha_j + 1 + \frac{1}{\kappa_{ij}}) \geq 0 \). Therefore both numerator and denominator of \( m_{ij} \) are again non-negative, so \( |Z_{ij}(\omega_0)|^2 \geq \kappa_{ij}^2 \).

c) \( \alpha_j > -\frac{1}{2} \) and \( \gamma_i \geq \lambda_j \), so \( \kappa_{ij} \geq 1 \). Then \( \frac{1}{\kappa_{ij}} - 1 \leq 0 \) and \( 2(\alpha_j + 1 + \frac{1}{\kappa_{ij}}) \geq 0 \). Therefore, the numerator \( m_{ij} \) is non-positive, the denominator is positive and \( m_{ij} \leq 0 \). Then \( |Z_{ij}(\omega_0)|^2 \leq \kappa_{ij}^2 \).

The next theorem shows that the \( H_\infty \) norm of the transfer function in platoon grows exponentially with the graph distance between the control and observer nodes.

**Theorem 16.** If \( \|T_{\text{min}}(s)\|_\infty > 1 \) and the eigenvalues of \( L \) are uniformly bounded from zero, then there is a constant \( \zeta > 1 \) such that the norm of the transfer function \( \|T_{CO}(s)\|_\infty > T_{CO}(0) \zeta^2 \zeta^{\text{deg}O} \) when \( 0 < \xi \leq 1 \) depends only on \( \lambda_{\text{min}}, \lambda_{\text{max}} \) and \( M(s) \), and is therefore bounded.

**Proof:** Since the leader is uncontrollable from the platoon vehicle, \( \lambda_1 = \gamma_1 = 0 \) and by removing the leader, we get rid of both of them. That’s why in the proof we work with reduced Laplacian \( L \) and smallest index of \( \lambda_j \) and \( \gamma_i \) is two. The leader can be included afterwards by multiplying \( T_{CO}(s) \) by \( M(s) \). The transfer function from \( C \) to \( O \) given in (13) can be slightly modified to emphasize some properties,
\[ T_{CO}(0) = w_{CO} N_{\text{deg}O} \prod_{i=2,j \in \mathcal{J}}^{N_{\text{deg}O}} Z_{ij}(s) \prod_{j=2,j \notin \mathcal{J}}^{N} \frac{1}{\lambda_j} \prod_{j=2,j \notin \mathcal{J}}^{N} T_{\lambda_j}(s). \] (47)
The terms \( w_{CO} \prod_{j=2,j \notin J}^{N} \frac{1}{\lambda_j} \) only affect the steady-state gain, not the shape of the magnitude frequency response. The steady-state gain was analyzed in section IV.

The key idea of the proof is to form \( T_{\lambda_i}(s) \) and \( Z_{ij}(s) \) from (13) in a suitable way. Let us pair each term \( a(s)p(s) + \gamma_i b(s)q(s) \) from the numerator with \( a(s)p(s) + \lambda_j b(s)q(s) \) from the denominator to form \( Z_{ij}(s) \) such that either Lemma 15 a) or b) is satisfied. The indices \( j \) of eigenvalues \( \lambda_j \) then form the set \( J \). The remaining unpaired \( a(s)p(s) + \lambda_j b(s)q(s) \) are used to form \( T_{\lambda_i}(s) \) and there are \( d_{CO} + 1 \) of them.

Suppose that the suitable \( Z_{ij}(s) \) and \( T_{\lambda_i}(s) \) are already formed (the approach is given below). Since \( \|T_{\lambda_{\min}}(s)\|_\infty > 1 \), there is a frequency \( \omega_0 \) such that \( |T_{\lambda_{\min}}(j\omega_0)| > 1 \). Then it follows from Lemma 14 that all transfer functions \( T_{\lambda_i}(s) \) have a gain greater than one at \( \omega_0 \), so \( |T_{\lambda_i}(j\omega_0)| > 1 \). By (47) these blocks form a product. Due to the lower and upper bounds on eigenvalues, there is a minimum \( \zeta \) of modulus frequency response \( |T_{\lambda_i}(j\omega)| \), attained for some \( \lambda_j \) with \( \lambda_{\min} \leq \lambda_j \leq \lambda_{\max} \). There are \( d_{CO} + 1 \) terms of the form \( T_{\lambda_i}(s) \) in (47). Then we get the lower bound on the magnitude frequency response of second part of the product as

\[
\prod_{j=2,j \notin J}^{N} |T_{\lambda_i}(j\omega)| \geq \zeta^{d_{CO}+1}.
\]

(48)

Clearly, this part of (47) scales exponentially with \( d_{CO} \).

Now let us analyze the transfer functions \( Z_{ij}(s) \) in (47),

\[
Z_{ij}(s) = \frac{a(s)p(s) + \gamma_i b(s)q(s)}{a(s)p(s) + \lambda_j b(s)q(s)}.
\]

(49)

Denote the ratio \( \kappa_{ij} = \frac{\gamma_i}{\lambda_j} \) and denote \( \alpha_j + j\beta_j = \lambda_j M(s) \). We know that the blocks \( \frac{a(s)p(s) + \lambda_j b(s)q(s)}{a(s)p(s) + \lambda_j b(s)q(s)} \) have their \( H_\infty \) norm greater than one at \( \omega_0 \), so \( \alpha_j < -\frac{1}{2} \).

The steady-state gain of \( Z_{ij}(s) \) is \( \kappa_{ij} \) and can also be incorporated in the steady-state gain analyzed in the previous section. Calculating the frequency response as before we get

\[
Z_{ij}(j\omega_0) = \frac{1 + \kappa_{ij}(\alpha_j + j\beta_j)}{1 + \alpha_j + j\beta_j}.
\]

(50)

Its modulus is in (46). Now we describe how to pair the numerator with the denominator to get the required \( Z_{ij}(s) \):

1. Take each term \( a(s)p(s) + \lambda_j b(s)q(s) \) from the denominator of (12). Recall that \( \alpha_j + j\beta_j = \lambda_j M(j\omega_0) \) and \( \alpha_j < -\frac{1}{2} \).

2. If \( \alpha_j \leq -1 \), then find a numerator term \( a(s)p(s) + \gamma_i b(s)q(s) \) such that \( \gamma_i \geq \lambda_j \). Form \( Z_{ij}(s) \) from them. Then by Lemma 15 a) this transfer function has gain greater than \( \kappa_{ij} \) at \( \omega_0 \).

3. If \( -1 < \alpha_j \leq -\frac{1}{2} \), then find a numerator term \( a(s)p(s) + \gamma_i b(s)q(s) \) such that \( \gamma_i \leq \lambda_j \). Form a block \( Z_{ij}(s) \) from them. Then by Lemma 15 b) \( Z_{ij}(s) \) has gain greater than \( \kappa_{ij} \) at \( \omega_0 \), so it amplifies at such frequency.

4. Form as much \( Z_{ij}(s) \)'s as possible using the steps 2) and 3).

5. Use \( d_{CO} + 1 \) remaining terms \( a(s)p(s) + \lambda_j b(s)q(s) \) to form \( T_{\lambda_i}(s) \).

The pairing is shown in Fig. 2. Unfortunately, we can only guarantee to form \( (N-d_{CO}-3) \) suitable \( Z_{ij}(s) \)'s using 2) and

3), although there are \( (N-d_{CO}-1) \) terms in the numerator. Therefore, at most two terms \( Z_{ij}(s) \) might not satisfy either a) or b) in Lemma 15. The reason is that neither there is a \( \gamma_i \) smaller than \( \lambda_3 \) nor can we guarantee that \( \gamma_2 < \lambda_3 \). That's why the smallest two poles do not have to be paired. Then we might have \( |Z_{ij}(j\omega_0)| \leq 1 \) for them. On the other hand, since \( \gamma_i \) and \( \lambda_j \) are bounded from below and above, also \( |Z_{ij}(j\omega_0)| \) is bounded by some \( 0 < \xi \leq 1 \) as \( \xi \leq |Z_{ij}(j\omega_0)| \leq 1 \).

All but two blocks \( Z_{ij}(s) \) amplify at \( \omega_0 \), so the total gain of all \( |Z_{ij}(j\omega_0)| \)'s is

\[
\prod_{i=1}^{N-d_{CO}-1} |Z_{ij}(j\omega_0)| \geq \xi^2
\]

(51)

and the norm of the transfer function \( T_{CO}(s) \) is

\[
\|T_{CO}(s)\|_\infty = \left\| T_{CO}(0) \prod_{i=1}^{N-d_{CO}-1} Z_{ij}(s) \right\|_\infty
\]

(52)

\[
\|T_{CO}(s)\|_\infty \geq \xi^2 T_{CO}(0) \rho_{d_{CO}}.
\]

The constant \( \xi^2 \) only scales the total gain at \( \omega_0 \) and does not change qualitatively the exponential scaling with the growing distance in the platoon.

We remark that for two and more integrators in the open loop we can always find a frequency for which \( \alpha_j < -1 \) for all \( \lambda_j \), see the Nyquist plot of \( M(s) \). Then the matching is complete and \( \xi = 1 \).

In other words, this theorem states that the peak in magnitude frequency response scales exponentially with the distance between the control and observer nodes. If the control signal propagates in the direction from the leader \( (C \leq O) \), then \( T_{CO}(0) = 1 \) and the \( H_\infty \) norm grows exponentially. If the signal propagates towards the leader \( (C > O) \), then the steady-state gain can decrease with the distance faster than \( \zeta^{d_{CO}} \) grows, so the \( H_\infty \) norm can be bounded. This depends on the asymmetry.

However, the \( H_\infty \) norm remains approximately fixed when also the control and observer nodes are fixed regardless of the platoon length. The simulation result showing the frequency response from \( C = 3 \) to \( O \) is shown in Fig. 3.

The scaling of the transfer function from the leader's position to the position of the last car in the platoon was used in examining harmonic instability. Since the distance between \( C \) and \( O \) increases with \( N \), we get the following corollary, which was stated and proved as the main result in [12].
Corollary 17. If the (nonzero) eigenvalues of $L$ are uniformly bounded from zero and $\|T_{\lambda}(s)\|_{\infty} > 1$, then the formation is harmonically unstable.

Next we emphasize the role of number of integrators, which relates our work to the results in [14].

Corollary 18. If each vehicle has at least two integrators in the open loop and there is a uniform lower bound on the eigenvalues, then $\|T_{\lambda}(s)\|_{\infty} \geq \zeta^{d_{CO}}$.

Proof: For at least two integrators in the open-loop, $\|T_{\lambda}(s)\|_{\infty} > 1$ [8]. Then the conditions for Theorem 16 are satisfied. Moreover, as discussed in the proof of Theorem 16, $\xi = 1$.

Since any platoon with stronger gain towards the leader for each vehicle achieves uniform bound on eigenvalues, it is harmonically unstable with at least two integrators in the open loop.

One might ask what happens if $\|T_{\lambda}(s)\|_{\infty} \leq 1$, but $\|T_{\lambda}(s)\|_{\infty} > 1$? Thus, for some $\lambda_j$ such that $\lambda_{\min} < \lambda_j \leq \lambda_{\max}$ we have $\|T_{\lambda}(s)\|_{\infty} > 1$. For such case we have the following condition.

Lemma 19. If there is $\lambda_j \leq 1$ such that $\|T_{\lambda}(s)\|_{\infty} \geq 1$, then there exists asymmetry $\epsilon_{\max} < 1$ such that the formation is harmonically unstable.

Proof: If we choose $\epsilon_j = 0$ for all $j$, then we get predecessor following algorithm. The eigenvalues $\lambda_j = 1$ for all $j \geq 2$ and from (12) we get a series connection of identical blocks. By the fact that $\|T_{\lambda}(s)\|_{\infty} \geq 1$ and Lemma 14 all $T_{\lambda}(s)$ have $\mathcal{H}_\infty$ norm greater than one, which proves harmonic instability.

VI. DESIGN OF A STRING STABLE CONTROLLER

So far we have discussed harmonic instability, which is an undesired phenomenon appearing in vehicular platooning. String (in)stability is another important phenomenon and in this section we provide a very simple sufficient test for it.

Lemma 20. Let $T_{\lambda}(s) = \frac{\lambda_{\max}(s)q(s)}{s(s)p(s) + \lambda_{\max}(s)q(s)}$ be the closed loop corresponding to the upper bound $\lambda_{\max}$ on the eigenvalues of the Laplacian. If $\|T_{\lambda}(s)\|_{\infty} \leq 1$, then $\|T_{\lambda}(s)\|_{\infty} \leq 1 \quad \forall \lambda_j \leq \lambda_{\max}$.

Proof: Lemma 14 shows that when a transfer function $T_{\lambda}$ has $\mathcal{H}_\infty$ norm greater than one, then all closed loops $T_{\lambda}(s)$, $\lambda_j > \lambda_1$ have gain greater than 1. Therefore, when the closed loop with the highest possible gain has a norm $\|T_{\lambda}(s)\|_{\infty} \leq 1$, no system with a lower gain can have its norm greater than 1. It follows that $\|T_{\lambda}(s)\|_{\infty} \leq 1, \forall \lambda_i \in [\lambda_{\min}, \lambda_{\max}]$.

Theorem 21. If $\|T_{\lambda}(s)\|_{\infty} \leq 1$, then for all transfer functions in the platoon the amplification is bounded by the steady-state gain, that is $\|T_{\lambda}(s)\|_{\infty} = |T_{\lambda}(0)|$.

Proof: Any transfer function in the platoon is given by (47). We will again exclude the terms $w_{CO}$ and all $\frac{1}{\lambda_j}$, because they only affect the steady-state gain, which was already analyzed. The key idea of the proof is to establish that when $T_{\lambda}(s)$ has its $\mathcal{H}_\infty$ norm less than or equal to one, then $\|Z_{ij}(s)\|_{\infty} \leq \frac{1}{\lambda_j}$ for $\gamma_i \geq \lambda_j$. Using Lemma 20, by the assumption in this theorem all $T_{\lambda}(s)$'s have their $\mathcal{H}_\infty$ norm less than or equal to one.

Let $\alpha_j + j\beta_j = \lambda_j M(\omega_0)$, i.e., frequency response of the scaled open loop at frequency $\omega_0$. From the modulus of the frequency response of $T_{\lambda}(\omega_0)$ and the fact that $\|T_{\lambda}(\omega_0)\|_1$ follows that $\alpha_j \geq -\frac{1}{2}$. Since by Lemma 20 $\|T(\omega_0)\|_1 \leq 1 \forall \omega_0$, the real part $\alpha_j \geq -\frac{1}{2}$ for all $\omega_o$.

Let us now check the modulus of $Z_{ij}(s)$ at the same frequency. Using Lemma 7 we can always choose $\lambda_j$ and $\gamma_i$ such that $\gamma_i \geq \lambda_j$. Then Lemma 15(c) can be applied to establish that $|Z_{ij}(\omega_0)| \leq |Z_{ij}(0)|$ for all $i, j$. Since $\alpha_j \geq -\frac{1}{2}$ for all $\omega_0$, we have that $\|Z_{ij}(s)\|_{\infty} \leq |Z_{ij}(0)|$ for all pairs $\gamma_i \geq \lambda_j$. Hence $\mathcal{H}_\infty$ norm of every transfer function in the platoon is upper bounded by its steady-state gain, which only depends on the vehicle’s position in the platoon, not on the platoon length.

A simple consequence is that there must be at most one integrator in the open loop. The Theorem 21 provides a simple condition on tuning of a SISO controller for a vehicle model in a platoon of arbitrary size, namely that $\|T_{\lambda}(s)\|_{\infty} \leq 1$. Root-locus approach (rttool in Matlab, for example) can be used to achieve that. The upper bound on the eigenvalues is easily known using Gershgorin disks.

A. Bidirectional string stability

There are many definitions of string stability in literature, see [2], [25]. One of the most common string stability condition in vehicular platoons is

$$\frac{\|y_i(s)\|_{\infty}}{\|y_{i-1}(s)\|_{\infty}} \leq 1 \quad \forall i.$$  \hspace{1cm} (53)

In other words, the effect of disturbance at one vehicle when measured using $L_2$ (signal) norm must be attenuated when propagated along the platoon. However, in a bidirectional platoon the signal can propagate in both directions. Thus, we have to generalize the condition.

Definition 4 (Bidirectional string stability). The bidirectional platoon is string-stable if for an input acting at the vehicle $C$...
the output (at a vehicle \( O \)) satisfies
\[
\begin{align*}
\left\| \frac{y_{O}(s)}{y_{O-1}(s)} \right\|_{\infty} & \leq 1 \quad \forall O \geq C, \\
\left\| \frac{y_{O-1}(s)}{y_{O}(s)} \right\|_{\infty} & \leq 1 \quad \forall O < C
\end{align*}
\]
(54) (55)

Using Theorem 21, we can state the condition for bidirectional string stability.

**Theorem 22.** The platoon is bidirectionally string-stable if
\[
\left\| T_{\lambda_{\text{max}}}(s) \right\|_{\infty} \leq 1.
\]
(56)

**Proof:** Consider \( O > C \) and let \( r_{C} \) be the input at the control node. Then the transfer function in (54) can be written
\[
\begin{align*}
\frac{y_{O}(s)}{y_{O-1}(s)} &= \frac{r_{C}(s)T_{C,O}(s)}{r_{C}(s)T_{C,O-1}(s)} = \frac{T_{C,O}(s)}{T_{C,O-1}(s)} \\
&= \frac{b(s)q(s)\prod_{j=1}^{N-d_{C,O}} a(s)p(s) + \gamma_{j,O}b(s)q(s)}{\prod_{j=1}^{N-d_{C,O}} a(s)p(s) + \gamma_{j,O-1}b(s)q(s)}.
\end{align*}
\]
(57)

Let \( \hat{L}_{O-1} \) and \( \hat{L}_{O} \) be the submatrices of \( L \) corresponding to the paths from \( C \) to \( O - 1 \) and from \( C \) to \( O \), respectively. Their eigenvalues are \( \gamma_{j,O-1} \) and \( \gamma_{j,O} \), respectively. Clearly, \( \hat{L}_{O} \) is also a submatrix of \( \hat{L}_{O-1} \), where the row and column corresponding to the node \( O \) were deleted. Therefore, the eigenvalues of \( \hat{L}_{O-1} \) and \( \hat{L}_{O} \) must interlace in a sense of Lemma 4. We can pair \( \gamma_{j,O-1} \) and \( \gamma_{j,O} \) by Lemma 7 such that \( \gamma_{j,O-1} \leq \gamma_{j,O} \) and form \( \hat{Z}_{i}(s) \) using the approach in the proof of Theorem 21. Then,
\[
\left\| \frac{a(s)p(s) + \gamma_{j,O}b(s)q(s)}{a(s)p(s) + \gamma_{j,O-1}b(s)q(s)} \right\|_{\infty} \leq 1 \forall j.
\]
(58)

There is now only one term in the numerator with a form
\[
\frac{b(s)q(s)}{a(s)p(s) + \gamma_{j,O-1}b(s)q(s)},
\]
which cannot be paired. Up to the gain, this block has its \( H_{\infty} \) norm smaller or equal to one (Lemma 20). Nevertheless, its steady-state gain \( \frac{y_{O}(0)}{y_{O-1}(0)} \) is one, since by Theorem 11 the steady-state gain is identical for all the vehicles behind the control node, so
\[
\frac{y_{O}(0)}{y_{O-1}(0)} = \prod_{j=1}^{N-d_{C,O}} \frac{a(s)p(s) + \gamma_{j,O}b(s)q(s)}{a(s)p(s) + \gamma_{j,O-1}b(s)q(s)} = 1.
\]
(60)

Hence, all terms in the product (57) have \( H_{\infty} \) norm less than one, so bidirectional string stability was proved in the direction from the control node towards the end of the platoon.

The other direction \( (C \geq O) \) has the ratio of outputs
\[
\begin{align*}
\frac{y_{O-1}(s)}{y_{O}(s)} &= w_{CO} \frac{b(s)q(s)\prod_{j=1}^{N-d_{C,O}-1} a(s)p(s) + \gamma_{j,O-1}b(s)q(s)}{\prod_{j=1}^{N-d_{C,O}} a(s)p(s) + \gamma_{j,O}b(s)q(s)}.
\end{align*}
\]
(61)

which has the same form as (57), the only difference is the steady-state gain. It follows from Theorem 11 that the steady-state gain is given as
\[
\begin{align*}
\frac{T_{C,O-1}(0)}{T_{C,O}(0)} &= \frac{w_{C,O-1}}{w_{C,O}} \prod_{j=1}^{N-d_{c,O}} \frac{\gamma_{j,O-1}}{\gamma_{j,O}} \\
&= \epsilon_{O-1} \frac{1 + \sum_{i=1}^{O-2} \epsilon_{i}}{1 + \sum_{i=1}^{O-1} \epsilon_{i}} \leq 1,
\end{align*}
\]
(62)

Since the steady-state gain is always smaller than 1 and all the terms in the product have \( H_{\infty} \) norm less than or equal to 1, bidirectional string stability was proved.

If the condition in the theorem is satisfied and the upper bound on eigenvalues does not depend on the number of vehicles, then the string stability holds for a platoon of an arbitrary size and asymmetry. Moreover, by successive application of the theorem, we can guarantee attenuation of the disturbance on the path from vehicle \( i \) to vehicle \( j \) with \( |j - i| > 1 \).

**VII. WHEN IS ASYMMETRY BENEFICIAL?**

It is clear from the previous sections that a crucial role is played by the number of integrators in the open loop of each agent. With two or more integrators in the open loop, harmonic instability occurs (asymmetric platoon) or the transient is very long (symmetric platoon). By internal model principle, two integrators are necessary for tracking of the leader, just note that the leader driving at a constant speed generates a ramp reference in the position signal for the platoon. The necessary condition for string stability is the presence of at most one integrator. From [9], [10] it might seem that asymmetric control is much better than symmetric.

We will now analyze the case proposed in [10], where asymmetric control outperformed its symmetric counterpart. The authors show that the transients of the step response decay faster for an asymmetric platoon than for a symmetric one. The beneficial effects of such control strategy were proved in [26]. The control error was given as
\[
e_{t} = k_{t}(y_{i-1} - y_{i} - d_{\text{ref}}) - k_{h}(y_{i} - y_{i+1} - d_{\text{ref}}) - b(v_{i} - v_{\text{ref}}).
\]
(63)

Note that that the knowledge of the desired speed \( v_{\text{ref}} \) is assumed. The vehicle model used in the paper was a double integrator. Using Laplace transform one obtains
\[
(s^{2} + bs)Y(s) = \hat{E}(s)
\]
(64)

with \( \hat{E}(s) = k_{l}(Y_{i-1}(s) - Y_{i}(s) - d_{\text{ref}}(s)) - k_{h}(Y_{i}(s) - Y_{i+1}(s) - d_{\text{ref}}(s)) + bv_{\text{ref}} \) being a spacing error. The model of the open loop is then \( M(s) = \frac{1}{s^{2} + bs} \). Since there is only one integrator on the left-hand side, it is possible to tune a dynamical controller such that the \( H_{\infty} \) norm of each subsystem in (13) is smaller than 1. However, note that it is the knowledge of the desired speed that enables the platoon to follow the leading vehicle. Without this knowledge, the vehicles will not achieve zero spacing errors, although they will reach the desired speed.
In order to make the reference speed available to every vehicle, the leader must perpetually transmit its own velocity. As a consequence, this control scheme requires establishing a permanent communication network and the graph topology changes to a star graph.

A. Comparison

Let us compare three bidirectional control schemes—asymmetric bidirectional control, predecessor following and symmetric control. We will use the following model \( G(s) \), which is identical to the model used in [10],

\[
G(s) = \frac{1}{s(s + a)},
\]

(65)

where \( a \) used in that paper was 0.5. To compare the performance, we will use two output-feedback controllers, one is just a proportional controller, and the other one is a dynamic controller of a lead-type,

\[
R_a(s) = 1, \quad R_d(s) = \frac{b s + a}{a s + b},
\]

(66)

with \( b > a \). The normalization factor \( \frac{3}{4} \) is used to obtain the same steady-state gain in both open loops. The controller \( R_a(s) \) was used in [10].

The simulations show the case when all cars travel at the desired speed \( \lambda_1 = 5 \) and have the desired spacing \( d_i = 1 \). The distance of the leader to the following vehicle is greater, \( d_1 = 2 \), which simulates the step response of the platoon.

The simulation results for \( R_a(s) \) are in Fig. 4. It is clear that in this case asymmetric controller performs the best because it has the shortest settling time. But when we start increasing the level of asymmetry (see Fig. 5), string instability appears. This could be expected since by using asymmetry we only weight the behavior between symmetric control and predecessor following and performance should be continuous function of the asymmetry. The Figure 6 shows the frequency responses of \( T_{\lambda_1}(s) \) for different \( \lambda_1 \). They show that the conditions for Lemma 19 hold, so for sufficiently small asymmetry harmonic instability occurs. Therefore asymmetric control without additional design measures such as in Theorem 21 does not guarantee string stability, even for a single integrator in the open loop.

On the other hand, as can be seen from Fig. 7, when the dynamic controller designed in accordance with Theorem 21 used pole-zero cancellation, the settling time for the predecessor following strategy outperforms any other strategy. Therefore, using a dynamic controller with a known desired velocity, one can easily obtain the best performance with the predecessor following strategy without the need to implement the measurement of the distance to the vehicle behind. Purely from a dynamic response perspective, there is no need to use an asymmetric bidirectional controller!

VIII. CONCLUSION

In this paper we investigated asymmetric control approaches for vehicular platoons. We proved that for more than one integrator in the open loop, the algorithm is not scalable, because its \( H_\infty \) norm scales exponentially with the graph distance. If we allow the vehicles to know the desired (reference) velocity of the platoon, only one integrator in the open loop can be present. Then we provide a simple design method for tuning the controller to achieve bidirectional string stability. Moreover, string stability does not depend on the level of asymmetry.

We proved that the steady-state gain grows unbounded with the size of platoon for a symmetric bidirectional control scheme, while it stays bounded in presence of asymmetry. We conclude that asymmetric control is a reasonable solution only in very specific cases where backward measurement is strictly required for some other reasons then dynamical response (safety, fault tolerance, . . . ) and the knowledge of the leader’s (or reference) velocity is granted. Otherwise the simple predecessor following scheme (being the extreme version of an asymmetric control scheme) can always perform better with a properly tuned controller.

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(a) Asymmetric control, $\epsilon = 0.9$

(b) Pred. following

(c) Symmetric control

Fig. 4: Comparison of three control strategies for a static controller, $N = 30$. 

(a) Asymmetric control, $\epsilon = 0.9$

(b) Asymmetric control, $\epsilon = 0.5$

(c) Asymmetric control, $\epsilon = 0.1$

Fig. 5: Comparison of different levels of asymmetry for a static controller, $N = 30$. 

(a) Asymmetric control, $\epsilon = 0.9$

(b) Pred. following

(c) Symmetric control

Fig. 7: Comparison of three control strategies for a dynamic controller, $N = 30$. 

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