ON SOME NONCOMMUTATIVE ALGEBRAS RELATED TO K-THEORY OF FLAG VARIETIES, PART I

ANATOL N. KIRILLOV AND TOSHIAKI MAENO

Dedicated to Adriano Garsia on the occasion of his 75-th birthday

Abstract

For any Lie algebra of classical type or type $G_2$ we define a $K$-theoretic analog of Dunkl’s elements, the so-called truncated Ruijsenaars-Schneider-Macdonald elements, RSM-elements for short, in the corresponding Yang-Baxter group, which form a commuting family of elements in the latter. For the root systems of type $A$ we prove that the subalgebra of the bracket algebra generated by the RSM-elements is isomorphic to the Grothendieck ring of the flag variety. In general, we prove that the subalgebra generated by the images of the RSM-elements in the corresponding Nichols-Woronowicz algebra is canonically isomorphic to the Grothendieck ring of the corresponding flag varieties of classical type or of type $G_2$. In other words, we construct the “Nichols-Woronowicz algebra model” for the Grothendieck Calculus on Weyl groups of classical type or type $G_2$, providing a partial generalization of some recent results by Y. Bazlov. We also give a conjectural description (theorem for type $A$) of a commutative subalgebra generated by the truncated RSM-elements in the bracket algebra for the classical root systems. Our results provide a proof and generalization of recent conjecture and result by C. Lenart and A. Yong for the root system of type $A$.

1 Introduction

In the paper [3] S. Fomin and the first author have introduced a model for the cohomology ring of flag varieties of type $A$ as a commutative subalgebra generated by the so-called truncated Dunkl elements in a certain (noncommutative) quadratic algebra. This construction has been generalized to other root systems in [7]. The main purpose of the present paper is to construct a $K$-theoretic analog of these constructions. More specifically, we introduce certain families of pairwise commuting elements in the Yang-Baxter group $\mathcal{YB}(B_n)$ or in the bracket algebra $\mathcal{BE}(B_n)$, which conjecturally generate commutative subalgebras in the bracket algebra $\mathcal{BE}(B_n)$ isomorphic to the Grothendieck ring of the flag varieties of type $B_n$. The corresponding results/conjectures for the flag varieties of other classical type root systems can be obtained from those for the type $B$ after certain specializations. There exists the natural
surjective homomorphism \(^1\) from the algebra \(\mathcal{BE}(B_n)\) to the Nichols-Woronowicz algebra \(\mathcal{B}_B\) of type \(B\). One of our main results of the present paper states that the \textit{image} of our construction in the Nichols-Woronowicz algebra \(\mathcal{B}_B\) is indeed isomorphic to the Grothendieck ring of the flag variety of type \(B_n\). We also present a similar construction for the root system of type \(G_2\). These results can be viewed as a multiplicative analog/generalization for classical root systems and for \(G_2\) of the “Nichols-Woronowicz algebra model” for the cohomology ring of flag varieties which has been constructed recently by Y. Bazlov \([1]\).

In a few words the main idea behind the constructions of the paper can be described as follows. As it was mentioned, in \([3]\) for type \(A\) and in \([7]\) for other root systems, a realization of the (small quantum) cohomology ring of flag varieties has been invented. More specifically, the papers mentioned above present a model for the cohomology ring of flag varieties as a commutative subalgebra generated by the so-called Dunkl elements in a certain (noncommutative) algebra. The main ingredient of this construction is based on some very special solutions to the \textit{classical} Yang-Baxter equation (for type \(A\)) and classical reflection equations (for types \(B, C\) and \(G_2\)).

Our original motivation was to study the related algebras and groups which correspond to the “quantization” of the solutions to the classical Yang-Baxter type equations mentioned above, in connection with classical and quantum Schubert and Grothendieck Calculi. In more detail, we define the group of “local set-theoretical solutions” to the quantum Yang-Baxter equations of type \(B_n\) or of type \(G_2\), together with the distinguished set of pairwise commuting elements in the former, the so-called truncated \textit{Ruijsenaars-Schneider-Macdonald elements}. The latter is a relativistic or multiplicative generalization of the Dunkl elements. For applications to the \(K\)-theory, we specialize the general construction to the bracket algebra \(\mathcal{BE}(B_n)\) and algebra \(\mathcal{BE}(G_2)\).

Summarizing, the main construction of our paper presents a conjectural description of the Grothendieck ring \(K(G/B)\) corresponding to flag varieties \(G/B\) of classical types (or \(G_2\)-type) to be a commutative subalgebra in the corresponding bracket algebra generated by the truncated RSM-elements. To be more specific, we construct in the algebra \(\mathcal{BE}(B_n)\) a pairwise commuting family of elements, \textit{multiplicative or relativistic Dunkl elements}, and state a conjecture about the complete list of relations among the latter.

Using some properties of the Chern homomorphism, we prove our conjecture for the root systems of type \(A\). To our best knowledge, for the root systems of type \(A\) a similar description of the Grothendieck ring was given by C. Lenart \([13]\), Lenart and Sottile \([14]\), and Lenart and Yong \([15]\), however without reference to the Yang-Baxter theory.

The main problem to prove relations between the RSM-elements in the bracket algebra \(\mathcal{BE}(B_n)\) appears to be that the intersection of kernels of all the “braided derivations” \(\Delta_{ij}, \Delta_{ij}^{-1}\), \(1 \leq i < j \leq n\), and \(\Delta_i, 1 \leq i \leq n\), acting on the algebra \(\mathcal{BE}(B_n)\) contains only constants, see Section 5. At this point we pass to the Nichols-Woronowicz algebra \(\mathcal{B}_B\), where the corresponding property of the braided derivations is guaranteed, \([1]\). Since as mentioned above, there exists the natural epimorphism of braided Hopf algebras \(\mathcal{B}_B \to \mathcal{BE}(B_n)\),

\(^1\)It is believed that for a simply–laced (finite) Coxeter system \((W, S)\) the corresponding bracket algebra \(BE(W, S)\) and the Nichols–Woronowicz algebra \(\mathcal{B}_{W, S}\) are isomorphic as braided Hopf algebras. However, this is not the case in a non simply–laced case. For example, if \(n \geq 3\), the natural epimorphism \(\mathcal{BE}(B_n) \to \mathcal{B}_B\) has a non-trivial kernel in degree 6. In fact, \(\text{Hilb}(\mathcal{BE}(B_3), t) - \text{Hilb}(\mathcal{B}_B, t) = 4t^6 + \cdots\).
to check the corresponding relations in the Nichols-Woronowicz algebra $B_{B_n}$ seems to be a good step to confirm our conjectures. To prove the needed relations in the algebra $B_{B_n}$, we develop a multiplicative analog/generalization of the Nichols-Woronowicz algebra model for cohomology ring of flag varieties recently introduced by Y. Bazlov [1].

In the fundamental papers [8] and [9] by B. Kostant and S. Kumar a description of the cohomology ring $H^*(G/B)$ and the $T$-equivariant $K$-theory $K_T(G/B)$ of a generalized flag variety $G/B$ has been obtained. The description of the cohomology ring $H^*(G/B)$ and that $K_T(G/B)$ given by Kostant and Kumar is based on the use of certain noncommutative algebras. The latter are suitable generalizations of the familiar nilCoxeter $NC(W)$ and nilHecke $NH(W)$ algebras corresponding to a finite Weyl group $W$, to the case of Weyl groups corresponding to generalized Kac-Moody algebras. Note that the generators of the algebra $NC(W)$ (resp. $NH(W)$) are parametrized by the set of simple roots in the corresponding Lie algebra $\text{Lie}(G)$.

The main results obtained in [8, 9] are a far generalization of the well-known results in a finite dimensional case where there exists a natural non-degenerate pairing between the cohomology ring $H^*(G/B)$ (resp. the Grothendieck ring $K(G/B)$) of a flag variety $G/B$ and the nilCoxeter algebra $NC(W)$ (resp. nilHecke algebra) of the Weyl group $W$ in question, see e.g. [2], [11], [1]. In a few words, the pairing mentioned corresponds to a natural action of the Demazure operators on the cohomology ring $H^*(G/B)$ (resp. the Grothendieck ring $K(G/B)$). With respect to this pairing the basis consisting of the Schubert polynomials in $H^*(G/B)$ (resp. the Grothendieck polynomials in $K(G/B)$) is in a duality with the standard basis $\{e_w, w \in W\}$, in the nilCoxeter algebra $NC(W)$ (resp. nilHecke algebra $NH(W)$). Under this approach the Pieri formula for Schubert (resp. Grothendieck) polynomials is an easy consequence of the Leibniz formula for the divided difference operators.

Our approach has its origins in the study of “formal” properties of the Pieri rules for Schubert (resp. Grothendieck) polynomials. To be more specific, the generators of our algebra corresponds to the set of positive roots in the algebra $\text{Lie}(G)$, and the relations are chosen in such a way that at first to guarantee the commutativity of the so-called Dunkl elements which are “formal” analog of the Pieri formulas, and secondly, to guarantee the existence of the so-called Bruhat representation of an algebra we would like to construct. The existence of the Bruhat representation is a key point which connects our algebras with the Schubert and Grothendieck Calculi. But we would like to point out that our algebras have some other interesting representations as well, see e.g. [7].

The above program was initiated and realized in [3] for the type $A$ flag varieties, and has been generalized for arbitrary finite Weyl groups in [2]. Another approach which is based on the theory of braided Hopf algebras, and comes up with the so-called Nichols-Woronowicz model for Schubert Calculus on Coxeter groups, has been developed by Y. Bazlov [1]. One of the main motivations and purposes of the present paper is to construct the Nichols-Woronowicz model for the Grothendieck Calculus for classical Weyl groups and $G_2$, as well as to generalize some results from our previous paper [7] to the case of $K$-theory.

Now we want to point out the main differences between noncommutative algebras which have appeared in the papers by B. Kostant and S. Kumar [8,9] and those in the present paper. First of all, the results of the present paper are proved only in the special case of classical root
systems and $G_2$. The results of [8] and [9] has been proved in much greater generality. On the other hand, in the case of type $B_n$, our algebra contains as (dual) subalgebras both the nilCoxeter algebra $NC(B_n)$ and a commutative subalgebra which is canonically isomorphic to the cohomology ring of $B_n$-type flag variety. Even more, the braided cross product of our algebra and its dual contains also as dual subalgebras the nilHecke algebra $NH(B_n)$ and a commutative subalgebra which is canonically isomorphic to the Grothendieck ring of $B_n$-type flag variety. Furthermore, an easily described deformation of our algebra contains commutative subalgebras one of which is isomorphic to the small quantum cohomology ring of type $B_n$-flag variety, and another one is conjecturally isomorphic to the quantum $K$-theory (theorem for type $A$). In subsequent papers we are going to introduce (quantum) “degenerate affine Fomin-Kirillov” algebra together with a commutative subalgebra which is isomorphic to the (quantum) $T$-equivariant $K$-theory of type $B_n$ flag variety. We expect that our constructions can be extended to the case of generalized flag varieties corresponding to (symmetrizable) Kac-Moody algebras.

Let us describe briefly the content of our paper.

Section 2 is devoted to a general construction of commuting family of elements in the group $\mathcal{YB}(B_n)$ generated by local set-theoretical solutions to the family of quantum Yang-Baxter equations of type $B$, see Definition 2.1 for precise formulation. This construction lies at the heart of our approach. In the case of type $A$ (i.e. if $g_{ij} = h_i = 1$ for all $i$ and $j$) and the Calogero-Moser representation (i.e. $h_{ij} = 1 + \partial_{ij}$) of the bracket algebra $BE(A_{n-1})$, the elements $\Theta_1^{A_{n-1}}, \ldots, \Theta_n^{A_{n-1}}$ correspond to the (rational) truncated (i.e. without differential part) Ruijsenaars-Schneider-Macdonald operators. It seems an interesting problem to classify all irreducible finite dimensional representations of the groups $\mathcal{YB}(X)$, $(X = A_{n-1}, B_n, ...)$ together with a simultaneous diagonalization of the operators $\Theta_1^X, \ldots, \Theta_n^X$ in these representations.

In Section 3 we apply the result of Section 2 (Key Lemma) to construct some distinguished multiplicative analogue $\Theta_j^{A_{n-1}} := \Theta_j^{A_{n-1}}(x)$ of the Dunkl elements $\theta_j^{A_{n-1}}$, $1 \leq j \leq n$, in the bracket algebra $BE(A_{n-1})$. It happened that our elements $\Theta_j^{A_{n-1}}$ coincide with the $K$-theoretic Dunkl elements $1 - \kappa_j$ introduced by C. Lenart and A. Yong in [13] and [15]. Proof of the statement that the elements $\kappa_1, \ldots, \kappa_n$ form a family of pairwise commuting elements in the algebra $BE(A_{n-1})$ given in [13], appears to be quite long and involved. On the other hand, our “Yang-Baxter approach” enables us to give a simple and transparent proof that the elements $\Theta_j^{A_{n-1}}$ mutually commute, as well as to describe relations among these commuting elements in the algebra $BE(A_{n-1})$. On this way we come to the main result of Section 3, namely

**Theorem A** The subalgebra in $BE(A_{n-1})$ generated by the elements $\Theta_j^{A_{n-1}}, 1 \leq j \leq n$, is isomorphic to the Grothendieck ring of the flag varieties of type $A$.

In particular,

**Theorem B** The following identity in the algebra $BE(A_{n-1})$ holds:

$$\sum_{j=1}^{n} (\Theta_j^{A_{n-1}}(x))^k = n$$
for any $k \in \mathbb{Z}$.

**Theorem A** was stated as Conjecture 3.4 in [13] and Conjecture 3.5 in [15]. We also state a positivity conjecture as Conjecture 3.13, which relates the elements $\Theta_j^{A_{n-1}}$ to the Grothendieck Calculus on the group $GL_n$. This conjecture is a restatement of non-negativity conjectures from [3], Conjecture 8.1, and [13], Conjecture 3.2., in our setting.

It should be emphasized that there are a lot of possibilities to construct a mutually commuting family of elements in the algebra $BE(A_{n-1})$ which generate a subalgebra isomorphic to the Grothendieck ring $K(Fl_n)$. For example one can take the elements $E_1 := \exp(\theta_1^{A_{n-1}}), \ldots, E_n := \exp(\theta_n^{A_{n-1}})$. It is easy to see that $E_j \neq \Theta_j$ for all $j$, however connections between Grothendieck polynomials and the elements $E_1, \ldots, E_n$ are not clear for the authors.

Our method to describe the relations between the elements $\Theta_j^{A_{n-1}}$ is based on the study of the Chern homomorphism which relates the $K$-theory to the cohomology theory of flag varieties, and moreover, on description of the commutative quotient of the algebra $BE(A_{n-1})$, see Subsection 3.2.

In Section 4 we study the $B_n$-case. First of all we introduce a modified version $BE(B_n)$ of the algebra $BE(B_n)$, which was introduced in our paper [1]. Namely, we add additional relations in degree four, see Definition 4.1, (6). In fact we have no need to use these relations in order to describe relations between Dunkl elements $\theta_1^{B_n}, \ldots, \theta_n^{B_n}$ in the algebra $BE(B_n)$. However, to ensure that the $B_2$ Yang-Baxter relations $h_{ij} h_i g_{ij} h_j = h_j g_{ij} h_i h_{ij}$ are indeed satisfied, the relations (6) are necessary. Another reason to add relations (6) is that these relations are satisfied in the Nichols-Woronowicz algebra $B_{B_n}$. However, we would like to repeat again that if $n \geq 3$, then the natural homomorphism of algebras $BE(B_n) \rightarrow B_{B_n}$ has a non-trivial kernel.

The main results of Section 4 are:

1. construction of a multiplicative analog $\Theta_j^{B_n}$ of the $B_n$-Dunkl elements $\theta_j^{B_n}$, see Definition 4.4;

2. proof of the fact that the RSM-elements $\Theta_j^{B_n}, 1 \leq j \leq n$, form a pairwise commuting family of elements in the algebra $BE(B_n)$.

Finally we give a conjectural description of all relations between the elements $\Theta_j^{B_n}$. Here we state this conjecture in the following form.

**Conjecture** The following identity in the algebra $BE(B_n)$ holds:

$$\sum_{j=1}^n (\Theta_j(x, y)^{B_n} + (\Theta_j(x, y)^{B_n})^{-1})^k = n \cdot 2^k$$

for all $k \in \mathbb{Z}_{\geq 0}$.

In Section 5 we discuss on a model for the Grothendieck ring of flag varieties in terms of the Nichols-Woronowicz algebra for the classical root systems. Our construction is a $K$-theoretic analog of Bazlov’s result [4]. Our second main result proved in Section 5 is:

**Theorem C** Let $\varphi : BE(B_n) \rightarrow B_{B_n}$ be a natural homomorphism of algebras. Then

$$\varphi(F(\Theta_1^{B_n}(x, y), \ldots, \Theta_n^{B_n}(x, y))) = 0$$
for any Laurent polynomial $F$ from the defining ideal of the Grothendieck ring of the flag variety of type $B_n$.

**Theorem C** implies the corresponding results for other classical root systems after some specializations. The Nichols-Woronowicz algebra $\mathcal{B}_X$ treated in this paper is a quotient of the algebra $YB(\mathcal{X})$ for the classical root system $\mathcal{X}$. In particular, the result for $A_{n-1}$ is a consequence of Theorem A, but the argument in Section 5 is another approach based on the property of the Nichols-Woronowicz algebra, which works well for the root systems other than those of type $A_{n-1}$. The idea of the proof is to construct the operators on the Nichols-Woronowicz algebra which induce isobaric divided difference operators on the commutative subalgebra generated by the RSM-elements.

The main interest of this paper is concentrated on the classical root systems, for which we can use advantages of explicit handling, particularly in order to construct the RSM-elements. Though most of the ideas in this paper are expected to be applicable to an arbitrary root system, to develop the general framework including the exceptional root systems is a matter of concern for the forthcoming work. However, the simplest exceptional root system $G_2$ can be dealt with in similar manner to the case of the classical root systems. In the last section, we formulate the Yang-Baxter relations and define the RSM-elements for the root system of type $G_2$. The argument in Section 5 again works well, so the Nichols-Woronowicz model for the Grothendieck ring of the flag variety of type $G_2$ is presented.

### 2 Key Lemma

**Definition 2.1** Let $\mathcal{YB}(B_n)$ be a group generated by the elements \{h_{ij}, g_{ij} \mid 1 \leq i \neq j \leq n\} and \{h_i \mid 1 \leq i \leq n\}, subject to the following set of relations:

- $g_{ij} = g_{ji}$, $h_{ij} = h_{ji}^{-1}$,
- $h_{ij} h_{kl} = h_{kl} h_{ij}$, $g_{ij} g_{kl} = g_{kl} g_{ij}$, $h_k h_{ij} = h_{ij} h_k$, $h_k g_{ij} = g_{ij} h_k$,
  if all $i, j, k, l$ are distinct;
- $h_i h_j = h_j h_i$, if $1 \leq i, j \leq n$; $h_{ij} g_{ij} = g_{ij} h_{ij}$, if $1 \leq i < j \leq n$;
- $(A_2$ Yang–Baxter relations)
  \begin{align*}
  (1) & \quad h_{ij} h_{ik} h_{jk} = h_{jk} h_{ik} h_{ij}, \\
  (2) & \quad h_{ij} g_{ik} g_{jk} = g_{jk} g_{ik} h_{ij}, \\
  (3) & \quad h_{ik} g_{ij} g_{jk} = g_{jk} g_{ij} h_{ik}, \\
  (4) & \quad h_{jk} g_{ij} g_{ik} = g_{ik} g_{ij} h_{jk},
  \end{align*}
  if $1 \leq i < j < k \leq n$;
- $(B_2$ quantum Yang-Baxter relation)
  \[ h_{ij} h_i g_{ij} h_j = h_j g_{ij} h_i h_{ij}, \]
  if $1 \leq i < j \leq n$. 


**Definition 2.2** Define the following elements in the group $\mathcal{YB}(B_n)$:

\[
\Theta_j = (\prod_{i=j}^{1} h_{ij}^{-1}) \cdot h_j \cdot (\prod_{i=1,i\neq j}^{n} g_{ij}) \cdot h_j \cdot (\prod_{k=n}^{j+1} h_{jk}),
\]

for $1 \leq j \leq n$.

**Theorem 2.3** (Key Lemma)

$$\Theta_i \Theta_j = \Theta_j \Theta_i \text{ for all } 1 \leq i, j \leq n.$$ 

Our proof is based on induction plus a masterly use of the Yang-Baxter relations, see defining relations in the definition of the group $\mathcal{YB}(B_n)$. See the proof of Corollary 3.3 and Example 2.5 (2) below. A complete proof of Theorem 2.3 one can find in Appendix.

**Remark 2.4** It's not difficult to see that

\[
\prod_{1 \leq j \leq k} \Theta_j = \prod_{j=1}^{k} (\prod_{s=j+1}^{n} g_{js}) \cdot h_{jk} \prod_{j=1}^{k} (\prod_{s=n}^{j} h_{js}).
\]

In particular,

\[
\prod_{j=1}^{n} \Theta_j = (\prod_{k=1}^{n} (\prod_{j \leq k} g_{jk}) \cdot h_{k})^2.
\]

**Example 2.5** (1) Take $n = 2$. Then $\Theta_1 = h_1 g_{12} h_1 h_2$ and $\Theta_2 = h_{12}^{-1} h_2 g_{12} h_2$. Let us check that $\Theta_1$ and $\Theta_2$ commute. Indeed, using the $B_2$-quantum Yang-Baxter relation $h_{12} h_1 g_{12} h_2 = h_2 g_{12} h_1 h_{12}$ and the commutativity relation $h_1 h_2 = h_2 h_1$, we see that

\[
\Theta_1 \Theta_2 = h_1 g_{12} h_1 h_2 g_{12} h_2 = h_{12}^{-1} (h_{12} h_1 g_{12} h_2) h_1 g_{12} h_2
\]

\[
= h_{12}^{-1} h_2 g_{12} h_1 (h_{12} h_1 g_{12} h_2) = h_{12}^{-1} h_2 g_{12} h_1 h_2 g_{12} h_1 h_{12} = \Theta_2 \Theta_1 = (h_1 g_{12} h_2)^2.
\]

(2) Take $n = 3$. Then we have

\[
\Theta_1 = h_1 g_{12} g_{13} h_1 h_{13} h_{12}, \quad \Theta_2 = h_{12}^{-1} h_2 g_{12} g_{23} h_2 h_{23}, \quad \Theta_3 = h_{23}^{-1} h_{13}^{-1} h_3 g_{13} g_{23} h_3,
\]

and

\[
\Theta_1 \Theta_2 \Theta_3 = (h_1 g_{12} h_2 g_{13} g_{23} h_3)^2.
\]

Let us illustrate the main ideas behind the proof of Key Lemma by the following example.

\[
\Theta_1 \Theta_3 \Theta_1^{-1} = h_1 g_{12} g_{13} h_1 h_{13} h_1 h_{13} h_1 h_{13} h_3 g_{13} g_{23} h_3 g_{13} h_{12}^{-1} h_{13}^{-1} h_{13}^{-1} g_{13} g_{12}^{-1} h_1^{-1}
\]

\[
= h_1 g_{12} g_{13} h_1 h_{23}^{-1} h_3 g_{23} g_{13} h_3 h_{13}^{-1} h_{13}^{-1} g_{13} g_{12}^{-1} h_1^{-1} \quad \text{(by (1) and (2))}
\]

\[
= h_1 h_{23}^{-1} g_{13} g_{12} h_2 g_{13} h_3 h_{13}^{-1} h_3 g_{12}^{-1} h_1^{-1} \quad \text{(by (4) and $B_2$-YBE)}
\]

\[
= h_1 h_{23}^{-1} g_{13} h_3 h_{13}^{-1} g_{23} h_3 h_1^{-1} \quad \text{(by (3))}
\]

\[
= \Theta_3 \quad \text{(by $B_2$-YBE)}.
\]
We define the groups $\mathcal{YB}(A_{n-1})$ and $\mathcal{YB}(D_n)$ to be the quotients of that $\mathcal{YB}(B_n)$ by the normal subgroups generated respectively by the elements $\{h_i, g_{ij}, 1 \leq i < j \leq n\}$ and $\{h_i, 1 \leq i \leq n\}$. The group $\mathcal{YB}(G_2)$ will be defined in Section 6. We expect that the subgroup in $\mathcal{YB}(B_n)$ generated by the elements $\Theta_1^{B_n}, \ldots, \Theta_n^{B_n}$ is isomorphic to the free abelian group of rank $n$. It seems an interesting problem to construct analogs of the group $\mathcal{YB}(B_n)$ and the elements $\Theta_i^{B_n}, \ldots, \Theta_n^{B_n}$ for any (finite) Coxeter group.

**Question 2.6** Does there exist a finite-dimensional *faithful* representation of the group $\mathcal{YB}(X)$, $X = A_{n-1}, B_n, \ldots$?

### 3 Algebras $\mathcal{YB}(A_{n-1})$ and $\mathcal{BE}(A_{n-1})$

#### 3.1 Definitions and main results

(i) **Algebra $\mathcal{YB}(A_{n-1})$**

**Definition 3.1** Let $R$ be a $\mathbb{Q}$-algebra. Define the algebra $\mathcal{YB}_R(A_{n-1})$ as an associative algebra over $R$ generated by the elements $h_{ij}(x), 1 \leq i \neq j \leq n, x \in R$, subject to the relations (0) – (4):

1. $h_{ij}(x)h_{ji}(x) = 1$,
2. $h_{ij}(x)h_{ij}(y) = h_{ij}(x + y)$; in particular, $h_{ij}(x)h_{ij}(-x) = 1$,
3. $h_{ij}(x)h_{kl}(y) = h_{kl}(y)h_{ij}(x)$, if $i, j, k, l$ are distinct,
4. $h_{ij}(x)h_{jk}(y) + h_{ik}(x + y) = h_{jk}(y)h_{ik}(x) + h_{ik}(y)h_{ij}(x), h_{jk}(y)h_{ij}(x) + h_{ik}(x + y) = h_{ik}(x)h_{jk}(y) + h_{ij}(x)h_{ik}(y), if 1 \leq i < j < k \leq n$.

For any element $z \in R$ we denote by $\mathcal{YB}(A_{n-1})[z]$ (resp. $\mathcal{YB}(A_{n-1})$) the algebra over $\mathbb{Q}$ generated by the elements $h_{ij}(z)$ and $h_{ij}(-z)$, (resp. $h_{ij}(1)$ and $h_{ij}(-1)$), $1 \leq i \neq j \leq n$.

**Lemma 3.2** (Quantum Yang-Baxter equation)

The following relations in the algebra $\mathcal{YB}(A_{n-1})[z]$

$$h_{ab}(z)h_{ac}(z)h_{bc}(z) = h_{bc}(z)h_{ac}(z)h_{ab}(z), \quad 1 \leq a < b < c \leq n, \quad (3.2)$$

in the algebra $\mathcal{YB}(A_{n-1})[z]$ are a consequence of the relations (0) – (4).

**Corollary 3.3** Define elements $\Theta_j^{A_{n-1}}(z), \quad j = 1, \ldots, n$, in the algebra $\mathcal{YB}(A_{n-1})[z]$ as follows:

$$\Theta_j^{A_{n-1}}(z) = h_{j-1,j}^{-1}(z) \cdots h_{i,j}^{-1}(z) h_{jn}(z) \cdots h_{j,j+1}(z), \quad 1 \leq j \leq n. \quad (3.3)$$

Then

$$\Theta_j^{A_{n-1}}(z)\Theta_k^{A_{n-1}}(z) = \Theta_k^{A_{n-1}}(z)\Theta_j^{A_{n-1}}(z), \quad \text{for all } 1 \leq j, k \leq n.$$
This Corollary is a particular case of Key Lemma above. We would like to include a separate proof of this special case to show the main ideas behind the usage of the Yang-Baxter relations, and since in this case the proof is much easy.

Proof. It is enough to check that if \( 1 \leq i \leq j \leq n \), then

\[
\Theta_i \Theta_j \Theta_i^{-1} = \Theta_j.
\]

By definition,

\[
\Theta_i \Theta_j \Theta_i^{-1} = h_{i-1,i}^{-1} \cdots h_{i,i+1}^{-1} h_{i+1,j}^{-1} \cdots h_{j-1,j}^{-1} h_{j+1,i}^{-1} \cdots h_{i,i}^{-1}
\]

Using local commutativity relations, see Definition 3.1 (2), we can move the factor \( h_{i,i+1}^{-1} \) to the left till we have touched on the factor \( h_{i,j}^{-1} \). As a result, we will come up with the triple product:

\[
h_{i+1,i}^{-1} h_{i,j}^{-1} h_{i,j+1}^{-1},
\]

which is equal, according to the Yang-Baxter relation (3.2), to the product

\[
h_{i,j}^{-1} h_{i,j+1}^{-1} h_{i,j+2}^{-1}.
\]

Now we can move the factor \( h_{i,j+1}^{-1} \) to the left to cancel it with the term \( h_{i,i+1} \), which comes from the rightmost factor in the element \( \Theta_i \).

As a result, we will have

\[
\Theta_i \Theta_j \Theta_i^{-1} = h_{i-1,i}^{-1} \cdots h_{i,i+2}^{-1} h_{i+2,j}^{-1} \cdots h_{j-1,j}^{-1} h_{j+2,i}^{-1} \cdots h_{i,i}^{-1}.
\]

Now we can move to the left the factor \( h_{i+1,i}^{-1} \) till we have touched on the factor \( h_{i,j}^{-1} \) to give the triple product

\[
h_{i+2,j}^{-1} h_{i,j}^{-1} h_{i,j+2}^{-1},
\]

which is equal to \( h_{i,j+2}^{-1} h_{i,j}^{-1} h_{i,j+2}^{-1} \). Now we can move the factor \( h_{i,j+2}^{-1} \) to the left to cancel it with the corresponding factor \( h_{i,i+2} \), and so on.

It is readily seen that finally we will come to the element \( \Theta_j \).

It is clear that \( \prod_{j=1}^{n} \Theta_j^{A_{n-1}}(z) = 1 \).

Remark 3.4 Let \( \Theta_j(z) := \Theta_j^{A_{n-1}}(z) \). Then it is not true that \( \Theta_j(x) \Theta_k(y) = \Theta_k(y) \Theta_j(x) \), if \( j \neq k \), \( x \neq y \).

Remark 3.5 Though the algebra \( YB(A_{n-1})[z] \) can be constructed as a quotient of the group algebra \( \mathbb{Q}[YB(A_{n-1})] \), they are not isomorphic.

Theorem 3.6 (Main theorem, the case of algebra \( YB(A_{n-1})[z] \))

\[
\prod_{j=1}^{n} (1 + (1 - \Theta_j^{A_{n-1}}(z)) t) = 1. \quad \text{Equivalently,} \quad \prod_{j=1}^{n} (1 + \Theta_j^{A_{n-1}}(z) t) = (1 + t)^n. \quad (3.4)
\]
This theorem is equivalent to:

**Theorem 3.7** Let $G_j^{A_{n-1}} = \Theta_j^{A_{n-1}}(z) - 1$, $1 \leq j \leq n$. Then, after the substitution $z = 1$, 

$$e_j(G_1^{A_{n-1}}, \ldots, G_n^{A_{n-1}}) = 0, \quad 1 \leq j \leq n$$

is the complete list of relations in the algebra $YB(A_{n+1})$ among the elements $G_1^{A_{n-1}}, \ldots, G_n^{A_{n-1}}$. Here, $e_j$ is the $j$-th elementary symmetric polynomial.

The proof is given in Subsection 3.2. It is based on the properties of the Chern homomorphism.

**Corollary 3.8** The algebra over $Z$ generated by the elements $G_i^{A_{n-1}}|_{z=1}, \ldots, G_n^{A_{n-1}}|_{z=1}$, is canonically isomorphic to the integral Grothendieck ring $K(\mathcal{F}l_n)$ of the flag manifold of type $A_{n-1}$.

(ii) **Algebra $BE(A_{n-1})$**

**Definition 3.9** \(3\) Define algebra $BE(A_{n-1})$ (denoted by $E_n$ in \(3\)) as an associative algebra over $Z$ with generators $x_{ij}$, $1 \leq i \neq j \leq n$, subject to the following relations

\[
\begin{align*}
(0) \quad & x_{ij} + x_{ji} = 0, \quad 1 \leq i \neq j \leq n, \\
(1) \quad & x_{ij}^2 = 0, \quad 1 \leq i \neq j \leq n, \\
(2) \quad & x_{ij} x_{jk} + x_{jk} x_{ki} + x_{ki} x_{ij} = 0, \quad \text{if all } i, j, k \text{ are distinct.}
\end{align*}
\]

The Dunkl elements form a pairwise commuting family of elements in the algebra $BE(A_{n-1})$, \(3\), and generate a commutative subalgebra in $BE(A_{n-1})$, which is canonically isomorphic to the cohomology ring $H^*(\mathcal{F}l_n)$ of the flag variety $\mathcal{F}l_n$ of type $A_{n-1}$, \(3\).

For an element $t$ of a $\mathbb{Q}$-algebra $R$, define $h_{ij}(t) = 1 + tx_{ij} = \exp(tx_{ij}) \in BE(A_{n-1}) \otimes R$.

**Lemma 3.10** The elements $h_{ij}(t)$, $1 \leq i, j \leq n$, satisfy the all relations (0) – (4) of the definition of the algebra $YB(A_{n-1})$.

We will use the same notation $\Theta_j^{A_{n-1}}$, $1 \leq j \leq n$, to denote the elements in the algebra $BE(A_{n-1})$ defined by the formula (3.3). It follows from Corollary 3.3 that they form a pairwise commuting family of elements in the algebra $BE(A_{n-1})$.

It’s clear that $\Theta_j^{A_{n-1}}(z) = 1 + z \Theta_j^{A_{n-1}} + \cdots$, and the product in the RHS of (3.3) may be written as follows:

$$\Theta_j^{A_{n-1}}(z) = \sum (-1)^s x_{b_1,j} x_{b_2,j} \cdots x_{b_s,j} x_{j,a_1} x_{j,a_2} \cdots x_{j,a_r} z^{r+s}, \quad (3.5)$$

where the sum runs over the all sequences of integers $(a_1 > a_2 > \cdots > a_r)$ and $(b_1 > b_2 > \cdots > b_s)$ such that $n \geq a_1 > a_r > j > b_1 > b_s \geq 1$; cf. \(3\) Section 2.

Remember that $G_j^{A_{n-1}} := \Theta_j^{A_{n-1}} - 1, \quad 1 \leq j \leq n.$
Definition 3.11 Let \( w \in S_n \) be a permutation. Define the Grothendieck polynomial \( G_w(X_n) \in \mathbb{Z}[X_n] \) to be a unique polynomial of the form \( G_w(X_n) = \sum_{a \in \delta_n} c_a(w) x^a \) such that

\[
G_w(G_1^{A_{n-1}}, \ldots, G_n^{A_{n-1}}) \cdot \text{id} = w
\]

in the Bruhat representation of the algebra \( BE(A_{n-1}) \) (see [3] Section 3.1), where \( \delta_n := (n-1, n-2, \ldots, 1, 0) \) and \( X_n := (x_1, \ldots, x_n) \).

It is not difficult to see that the Grothendieck polynomials defined here coincide with those introduced in [10], see also [14].

Corollary 3.12 Let \( u, v \in S_n \) be two permutations. Assume that in the group ring \( \mathbb{Z}[S_n] \) of the symmetric group \( S_n \) one has the following equality:

\[
G_u(G_1^{A_{n-1}}, \ldots, G_n^{A_{n-1}}) \cdot v = \sum_{w \in S_n} c_{u,v}^w w.
\]

Then the coefficient \( c_{u,v}^w \) is equal to the multiplicity of the Grothendieck polynomial \( G_w(X_n) \) in the product of \( G_u(X_n) \) and \( G_v(X_n) \):

\[
G_u(X_n) G_v(X_n) = \sum_{w \in S_n} c_{u,v}^w G_w(X_n)
\]

in the Grothendieck ring \( K(\mathcal{F}l_n) \) of the flag manifold of type \( A_{n-1} \).

Conjecture 3.13 For any permutation \( w \in S_n \) the value of the Grothendieck polynomial \( G_w(x_1, \ldots, x_n) \) after the substitution \( x_1 := G_1^{A_{n-1}}, \ldots, x_n := G_n^{A_{n-1}} \), and \( z = 1 \), can be written as a linear combination of monomials in \( x_{ij} \)'s, \( 1 \leq i < j \leq n \), with non-negative integer coefficients.

Example 3.14 (Grothendieck-Pieri formula in the algebra \( BE(A_{n-1}) \), cf [14])

\[
1 + G_{(k,k+1)}(G_1, \ldots, G_n) = \prod_{1 \leq j \leq k} \Theta_j = \prod_{j=1}^{k+1} \prod_{s=n}^{j} h_{js} = \sum_{j=1}^{r} \prod_{a_j,b_j} x_{a_j,b_j},
\]

where the sum runs over all sequences of integers \( (1 \leq a_1 \leq \cdots \leq a_r \leq k) \) and \( (b_1, \ldots, b_r) \) such that \( k < b_j \leq n, j = 1, \ldots, r, \) and \( a_i = a_{i+1} \Rightarrow b_i > b_{i+1} \).

Our methods allow to obtain a subtraction free formula in the algebra \( BE(A_{n-1}) \) for the value of the Grothendieck polynomials \( G_{(k,k+1,\ldots,k+r)}(G_1, \ldots, G_n) \), \( 1 \leq k \leq n-r-1, \) as well. We hope to report on our results in a separate publication.

Example 3.15 Take \( n = 3 \), then

\[
\Theta_1 := \Theta_1^{A_{3}}(1) = h_{13}(1) h_{12}(1) = 1 + x_{12} + x_{13} + x_{12} x_{13},
\]

\[
\Theta_2 := \Theta_2^{A_{3}}(1) = h_{12}^{-1}(1) h_{23}(1) = 1 - x_{13} + x_{23} - x_{13} x_{12} - x_{23} x_{13},
\]

\[
\Theta_3 := \Theta_3^{A_{3}}(1) = h_{23}^{-1}(1) h_{13}^{-1}(1) = 1 - x_{13} - x_{23} + x_{23} x_{13}.
\]
As a preliminary step, we compute the elementary symmetric polynomials \( e_k(\Theta_1, \Theta_2, \Theta_3) \), \( k = 1, 2, 3 \). Indeed, it’s easily seen from the formulae above that \( \Theta_1 + \Theta_2 + \Theta_3 = 3 \) and \( \Theta_1 \Theta_2 \Theta_3 = 1 \). To compute \( e_2(\Theta_1, \Theta_2, \Theta_3) \), all one has to do is to apply the following relation

\[
h_{12} h_{23}^{-1} = h_{23}^{-1} h_{13} + h_{13}^{-1} h_{12} - 1,
\]

where we put by definition \( h_{ij} := h_{ij}(1) \). The former equality follows from the relation (3) in Definition 3.1. Hence,

\[
e_2(\Theta_1, \Theta_2, \Theta_3) = h_{13} h_{23} + h_{13} h_{12} h_{23}^{-1} h_{13}^{-1} + h_{12} h_{13}^{-1} - 1 + h_{12}^{-1} h_{13}^{-1} = 2h_{13} + 2h_{13}^{-1} - 1 = 3.
\]

To continue, let us list the Grothendieck polynomials \( G \) corresponding to the symmetric group \( S_3 \):

\[
G_{id}(x) = 1, \ G_{s_1}(x) = x_1, \ G_{s_2}(x) = x_1 + x_2 + x_1 x_2, \\
G_{s_1 s_2}(x) = x_1 x_2, \ G_{s_2 s_1}(x) = x_1^2, \ G_{w_0}(x) = x_1^2 x_2.
\]

Now let us consider the substitution \( x_j = G_j = \Theta_j(1) - 1, \ j = 1, 2, 3 \). More explicitly, \( G_1 = x_{12} + x_{13} + x_{12} x_{12} \) and \( G_2 = -x_{12} + x_{23} - x_{13} x_{12} - x_{23} x_{13} \). Therefore,

\[
G_{s_2}(G_1, G_2) = x_{13} + x_{23} + x_{13} x_{23}, \ G_{s_1 s_2}(G_1, G_2) = x_{13} x_{23} + x_{23} x_{13}, \\
G_{s_2 s_1}(G_1, G_2) = x_{12} x_{13} + x_{13} x_{12}, \\
G_{w_0}(G_1, G_2) = x_{12} x_{13} x_{23} + x_{13} x_{12} x_{13} x_{13} x_{13} x_{13} x_{13} x_{23}.
\]

Finally, let us consider the commutative subalgebra in \( \mathcal{E}(A_2) \otimes \mathbb{Q} \) generated by the elements \( E_j := \exp(\Theta_j), \ j = 1, 2, 3 \). It’s not difficult to check that

\[
2E_1 = h_{13} h_{12} + h_{12} h_{13}, \ 2E_2 = h_{12}^{-1} h_{23} + h_{23} h_{12}^{-1}, \ 2E_3 = h_{23}^{-1} h_{13}^{-1} + h_{13}^{-1} h_{23}^{-1}.
\]

It is an easy matter as well to see that the subalgebra in \( \mathcal{E}(A_2) \otimes \mathbb{Q} \) generated over \( \mathbb{Q} \) by the elements \( E_i, \ i = 1, 2, 3 \), is isomorphic to the algebra \( \mathbb{Q}[\Theta_1, \Theta_2, \Theta_3] \). In particular, for all symmetric polynomials \( f(x_1, x_2, x_3) \) we have

\[
f(1 - E_1, 1 - E_2, 1 - E_3) = 0.
\]

**Proposition 3.16** The subalgebra in \( \mathcal{E}(A_{n-1}) \otimes \mathbb{Q} \) generated by the elements \( E_i := \exp(\Theta_i), \ 1 \leq i \leq n \), is isomorphic to the algebra over \( \mathbb{Q} \) generated by the elements \( \Theta_j^{A_{n-1}}, \ 1 \leq j \leq n \).

In particular, the complete list of relations among the elements \( 1 - E_1, \ldots, 1 - E_n \) in the quadratic algebra \( \mathcal{E}(A_{n-1}) \) is given by

\[
e_i(1 - E_1, \ldots, 1 - E_n) = 0,
\]

for \( i = 1, \ldots, n \). Thus the commutative subalgebra generated by the elements \( \exp(\Theta_1), \ldots, \exp(\Theta_n) \) is isomorphic to the rational Grothendieck ring \( K(\mathcal{F}_{l_n}) \otimes \mathbb{Q} \) of the flag manifold \( \mathcal{F}_{l_n} \) of type \( A_{n-1} \).
However, it seems that there are no direct connections of the elements $E_j$’s with the Grothendieck Calculus.

**Remark 3.17** More generally, let $Q(t) \neq 0$ be a polynomial such that $Q(0) = 0$. Define the elements $q_i := 1 + Q(\theta_i)$, $1 \leq i \leq n$, in the algebra $BE(A_{n-1})$. It’s clear that the elements $q_1, \ldots, q_n$ pairwise commute, and

$$c_i(q_1 - 1, \ldots, q_n - 1) = 0, \quad 1 \leq i \leq n.$$  

**Remark 3.18** *(Quantum Grothendieck Calculus)*

It is easy to see that the relations in Definition 3.1 are still true, if we replace the condition (1) in Definition 3.9 by the following one

$$(1') \quad x_{ij}^k = q_{ij}, \quad 1 \leq i < j \leq n,$$

where the parameters $q_{ij}$ are assumed to commute with all the generators $x_{kl}$, $1 \leq k < l \leq n$.

The algebra over $\mathbb{Z}[q_{ij} | 1 \leq i < j \leq n]$ generated by the elements $x_{ij}$, $1 \leq i \neq j \leq n$, subject to the relations $(0)$, $(1')$ and $(2)$, is called the *quantized bracket algebra* and denoted by $qBE(A_{n-1})$, cf. [3, Section 15] and [5].

As a corollary we see that the elements $\Theta^q_j$, $1 \leq j \leq n$, defined by the formula (3.2), form a pairwise commuting family of elements in the algebra $qBE(A_{n-1})$.

After the specialization

$$q_{ij} = \begin{cases} q_i, & \text{if } j = i + 1, \\ 0, & \text{otherwise,} \end{cases}$$

the multiplicative Dunkl elements generate the quantum Grothendieck ring in the sense of Givental and Lee [4]. The generalization to the equivariant $K$-theory is an open problem.

**Problem 3.19** Describe the commutative subalgebras in the quantized algebra $qYB(A_{n-1})$ generated by

$$(1) \quad \Theta^q_i(1), \ldots, \Theta^q_n(1),$$

$$(2) \quad \tilde{E}_1 := \exp(\theta_1), \ldots, \tilde{E}_n := \exp(\theta_n).$$

### 3.2 Chern homomorphism

Denote by $\mathcal{H} := BE(A_{n-1})^{ab} \otimes \mathbb{Q}$ the quotient of the algebra $BE(A_{n-1})$ by its commutant. It is known, [3, Proposition 4.2], that the algebra $BE(A_{n-1})^{ab}$ has dimension $n!$, and its Hilbert polynomial is given by

$$Hilb(BE(A_{n-1})^{ab}, t) = (1 + t)(1 + 2t) \cdots (1 + (n - 1)t).$$

Denote by $1 + \mathcal{H}^+$ the multiplicative monoid generated by the elements of the form $1 + h$, where $h \in \mathcal{H}$ does not have the term of degree zero.

**Proposition 3.20** Let $R^{(n-1)}$ be the subspace of the commutative subalgebra $R = \mathbb{Q}[\theta_1, \ldots, \theta_n] \subset BE(A_{n-1}) \otimes \mathbb{Q}$ whose elements are of degree $\leq n - 1$. Then the subspace $R^{(n-1)}$ is injectively mapped into $\mathcal{H}$ by the quotient homomorphism $BE(A_{n-1}) \otimes \mathbb{Q} \to \mathcal{H}$. 

13
Proof. Since the algebra $R$ is isomorphic to the coinvariant algebra of the symmetric group, the monomials

$$\theta^1_i \cdots \theta^{n-1}_{n-1}, \quad 0 \leq i_k \leq n - k,$$

form a linear basis of $R$. The linear map $R^{(n-1)} \to \mathcal{H}$ induced by the quotient homomorphism is a homomorphism between $S_n$-modules. Hence, it is enough to show the images of the monomials $\theta^1_i \cdots \theta^{n-1}_{n-1}$ do not vanish in $\mathcal{H}$ for $(i_1, \ldots, i_{n-1})$ such that $\sum_{k=1}^{n-1} i_k = n - 1$ and $i_1 \geq i_2 \geq \cdots \geq i_{n-1}$. We expand the monomials $\theta^1_i \cdots \theta^{n-1}_{n-1}$ of this form in the algebra $BE(A_{n-1}) \otimes \mathbb{Q}$ by using the Pieri formula proved by Postnikov [16], (first conjectured in [3]). The Pieri formula shows that

$$e_k(\theta_1, \ldots, \theta_m) = \sum [i_1 j_1] \cdots [i_k j_k],$$

where $\sum$ stands for the multiplicity-free sum, and $(i_1, j_1), \ldots, (i_k, j_k)$ run over all pairs such that $i_a \leq m < j_a \leq n$, $a = 1, \ldots, k$, and all $i_a$'s are distinct.

On the other hand, the monomials of form

$$[i_1 j_1] \cdots [i_k j_k], \quad i_a < j_a \ (a = 1, \ldots, k), \quad j_1 < j_2 < \cdots < j_k,$$

give a linear basis of $\mathcal{H}$ ([16 Corollary 10.3]). By the involution $\omega : [ij] \mapsto [n+1-j \ , \ n+1-i]$, we have a linear basis of form

$$[i_1 j_1] \cdots [i_k j_k], \quad i_a < j_a \ (a = 1, \ldots, k), \quad i_1 < i_2 < \cdots < i_k. \quad (3.7)$$

For each monomial expression $[i_1 j_1] \cdots [i_k j_k]$ in $\mathcal{H}$, we define

$$\mu([i_1 j_1] \cdots [i_k j_k]) := \sum_{m=1}^{k} (j_m - i_m).$$

Every element in $\mathcal{H}$ can be expressed as a linear combination of the monomials listed in (3.7) by repeatedly applying the substitution $[ab][ac] \to [ab][bc] - [ac][bc]$ with $a < b < c$. On each step of the procedure, the monomials of minimal $\mu$ appearing in the expression of $\theta^1_i \cdots \theta^{n-1}_{n-1}$ with $i_1 + \cdots + i_{n-1} = n - 1$ are not cancelled or are replaced by new ones. So one can check the image of $\theta^1_i \cdots \theta^{n-1}_{n-1}$ in $\mathcal{H}$ is not zero. □

**Definition 3.21** Define the Chern homomorphism (to the commutative quotient)

$$c' : YB(A_{n-1}) \to 1 + \mathcal{H}^+$$

by the following rules:

- $c'(f + g) = c'(f)c'(g)$, if $f, g \in YB(A_{n-1})$,
- $c'(\prod_{i<j} h_{ij}^{n_{ij}}) = 1 + \sum_{i<j} n_{ij} x_{ij}.$

It is clear that $c'(\Theta_j) = 1 + \theta_j$, $\forall j$.  

14
Remark 3.22 We can also define the homomorphism
\[ c : \mathbb{Q}[[\Theta_1, \ldots, \Theta_n]] \to \mathbb{Q}[[\theta_1, \ldots, \theta_n]] \]
by the conditions \( c(f + g) = c(f)c(g) \) and \( c(\Theta_j) = 1 + \theta_j, \ j = 1, \ldots, n \), which is compatible with the Chern homomorphism (in the usual sense)
\[ c : K(Fl_n) \to 1 + H^+(Fl_n). \]
However, the homomorphism \( c' \) defined above does not coincide with \( c \) in the part of degree \( \geq n \). Indeed, the maximal degree of the commutative quotient \( H \) is \( n - 1 \).

Proposition 3.23 (cf. [11, Section 5]) For any permutation \( w \in S_n \),
\[ c(1 + G_w(G_1, \ldots, G_n)) = 1 - (-1)^{l(w)}(l(w) - 1)! \mathfrak{S}_w(\theta_1, \ldots, \theta_n) + \sum_u a_u(w) \mathfrak{S}_u(\theta_1, \ldots, \theta_n), \]
where the sum ranges over all permutations \( u \in S_n \) such that \( l(u) > l(w) \), and \( a_u(w) \) is a constant in \( \mathbb{Z} \) determined by \( u \) and \( w \).

Proof of Theorem 3.7. Note that the commutative quotient \( YB(A_{n-1})^a \) is isomorphic to the algebra \( H \). Moreover, The subspace of polynomials of degree \( \leq n - 1 \) in the RSM-elements \( \Theta_1^{A_{n-1}}, \ldots, \Theta_n^{A_{n-1}} \) in \( YB(A_{n-1}) \) is also injectively mapped into \( H \) from Proposition 3.20. We regard \( 1 + H^+ \) as an \((n! - 1)\)-dimensional \( \mathbb{Q} \)-linear space so that the homomorphism \( \bar{c} : H^+ \to 1 + H^+ \) induced by the Chern homomorphism \( c' \) is a \( \mathbb{Q} \)-linear map. The image of the linear basis (3.7) of \( H^+ \) by the homomorphism \( \bar{c} \) is linearly independent. Hence, \( \bar{c} : H^+ \to 1 + H^+ \) is an isomorphism between linear spaces. Since it is easy to see
\[ c'(e_j(\Theta_1^{A_{n-1}}, \ldots, \Theta_n^{A_{n-1}})) = 1 \in 1 + H^+, \ 1 \leq j \leq n - 1, \]
one can conclude that
\[ e_j(G_1^{A_{n-1}}, \ldots, G_n^{A_{n-1}}) = 0, \ 1 \leq j \leq n - 1. \]
The equality
\[ \prod_{i=1}^{n} \Theta_i^{A_{n-1}} = 1 \]
in the algebra \( YB(A_{n-1}) \) can be obtained by direct computation.

Problem 3.24 Construct a lift of \( c' \) to
\[ YB(A_{n-1}) \to 1 + BE(A_{n-1})^+ \]
in some suitable sense.
4 Algebras $\mathcal{BE}(B_n)$ and $YB(B_n)$

(i) Algebra $\mathcal{BE}(B_n)$ (cf. [4])

**Definition 4.1** Define the algebra $\mathcal{BE}(B_n)$ as the algebra (say, over $\mathbb{Q}$) with generators 

$$[i,j], \ [i,j], \ 1 \leq i \neq j \leq n, \text{ and } [i], \ 1 \leq i \leq n,$$

subject to the following relations:

1. $[i,j] = -[j,i], \ [i,j] = [j,i]$
2. $[i,j]^2 = 0, \ [i,j]^2 = 0, \ 1 \leq i < j \leq n,$ and $[i]^2 = 0, \ 1 \leq i \leq n,$
3. $[i,j][k,l] = [k,l][i,j], \ [i,j][k,l] = [k,l][i,j], \ [i,j][k,l] = [k,l][i,j], \ \text{if } \{i,j\} \cap \{k,l\} = \emptyset$
4. $[i,j][i,k] + [j,k][i,j] + [k,i][i,j] = 0,$
5. $[i,j][i,j][j,i] + [i,j][i,j][i,j] + [i,j][i,j][i,j] + [i,j][i,j][i,j] = 0, \ \text{if } i < j,$
6. $[i,j][i,j][j,i] = [j][i,j][i,j], \ \text{if } i < j.$

**Remark 4.2** (a) In the definition of the algebra $BE(B_n)$, see [7], Section 9.1, the condition (6) is absent. In fact, there is no need to use the latter condition for the purposes of [7]. However, we need the condition (6) to ensure the $B_2$ quantum Yang-Baxter relation, which is necessary for our construction of a commutative family of elements in the algebra $YB(B_n)$, see (ii) below.

(b) In [7], the authors has introduced the quantum deformation $qBE(B_n)$ of the bracket algebra. Similarly we introduce the quantum deformation of the algebra $q\mathcal{BE}(B_n)$ which is generated by the same symbols as in $\mathcal{BE}(B_n)$ and is obtained by replacing the relation in (1) corresponding to the simple roots by

$$[i,i + 1]^2 = q_i, \ 1 \leq i \leq n - 1, \text{ and } [n]^2 = q_n.$$ 

In the subsequent construction, we can work in the quantum bracket algebra $q\mathcal{BE}(B_n)$ instead of $\mathcal{BE}(B_n)$. The RSM-elements in Definition 4.4 also form a commuting family of elements in $q\mathcal{BE}(B_n)$. Though it is expected that the RSM-elements in the quantum setting should describe the quantum Grothendieck Calculus in $B_n$-case, the relations satisfied by them in the algebra $q\mathcal{BE}(B_n)$ are not clearly seen.

The Dunkl elements [7] are given by

$$\theta_i := \theta_i^{B_n} = \sum_{j \neq i} ([i,j] + [i,j]) + 2[i], \ 1 \leq i \leq n. \quad (4.8)$$
Note that the Dunkl elements $\tilde{\theta}_i$ correspond to the Monk type formula in the cohomology ring of the flag variety of type $B$.

(ii) Algebra $YB(B_n)$

Let $x$ and $y$ be elements in a $\mathbb{Q}$-algebra $R$. Define the algebra $YB(B_n)$ as a subalgebra in $\mathcal{B}\mathcal{E}(B_n) \otimes R$ generated over $R$ by the elements:

$$h_{ij} := \exp(x[i,j]) = 1 + x[i,j], \quad g_{ij} := \exp(x[i,j]) = 1 + x[i,j], \quad 1 \leq i < j \leq n,$$

and $h_j := \exp(y[j]) = 1 + y[j], \quad 1 \leq j \leq n$.

**Proposition 4.3** The elements $h_{ij}, g_{ij}$ and $h_k, 1 \leq i < j \leq n, 1 \leq k \leq n$, satisfy the all relations listed in Definition 2.1.

**Definition 4.4** Define

$$\Theta^B_n(x, y) = (\prod_{i=j-1}^{1} h_{ij}(x)^{-1} h_j(y) \prod_{i=1, i \neq j}^{n} g_{ij}(x)) h_j(y) \prod_{k=n}^{j+1} h_{jk}(x),$$

for $1 \leq j \leq n$.

**Corollary 4.5** The elements $\Theta^B_n(x, y)$ commute pairwise.

**Remark 4.6** It is not difficult to see that

$$\Theta^B_n(1, 1) \neq \exp(\theta^B_n),$$

where $\theta^B_n, 1 \leq j \leq n$, denote the $B_n$-Dunkl elements in the algebra $\mathcal{B}\mathcal{E}(B_n)$. The commuting family of elements $\exp(\theta^B_n), 1 \leq j \leq n$, also generate a (finite dimensional) commutative subalgebra in $\mathcal{B}\mathcal{E}(B_n) \otimes \mathbb{Q}$. However, we don’t know the complete list of relations among these elements.

**Conjecture 4.7** (The case of algebra $YB(B_n)$)

In the algebra $YB(B_n)$ we have the following identity

$$\prod_{j=1}^{n}(1 + (\Theta^B_n(x, y) + (\Theta^B_n(x, y))^{-1})t) = (1 + 2t)^n. \quad (4.9)$$

Equivalently,

$$\prod_{j=1}^{n}(1 + \Theta^B_n(x, y)t(1 + (\Theta^B_n(x, y))^{-1}t) = (1 + t)^{2n}. \quad (4.10)$$

This conjecture is equivalent to:

**Conjecture 4.8** Let $G^B_{j,\alpha} = (\Theta^B_n(x, y))^\alpha - (\Theta^B_n(x, y))^{-\alpha}, 1 \leq j \leq n, \quad \alpha \in \mathbb{Q}$. Then

$$e_j((G^B_{1,\alpha})^2, \ldots, (G^B_{n,\alpha})^2) = 0, \quad 1 \leq j \leq n. \quad (4.11)$$
Remark 4.9 Theorem 3.6, i.e. the equality
\[ \prod_{j=1}^{n} (1 + \Theta_j^{A_{n-1}} t) = (1 + t)^n. \] (4.12)
follows from Conjecture 4.7.

Proof. The multiplicative Dunkl elements \( \Theta_j^{A_{n-1}}(x) \) can be obtained from those \( \Theta_j^{B_n}(x, y) \) after the specialization \( y := 0 \) and \( g_{ij} := 1 \). Since \( \prod_{j=1}^{n} \Theta_j^{A_{n-1}} = 1 \), it follows from Conjecture 4.7 that if we denote by \( P_n(t) \) the LHS of (4.11) then \( P_n(t)P_n(t^{-1}) = (1 + t)^n(1 + t^{-1})^n \). Therefore, \( P_n(t) = (1 + t)^n \). \( \blacksquare \)

Remark 4.10 The algebra \( YB(C_n) \) can be naturally identified with the algebra \( YB(B_n) \). The corresponding RSM-elements relate via \( \Theta_j^{C_n}(x, y) = \Theta_j^{B_n}(x, y/2) \).

5 Nichols-Woronowicz model for Grothendieck ring of flag varieties

In the preceding sections, we have tried to construct the models of the Grothendieck ring \( K(G/B) \) in the algebras \( BE(A_{n-1}) \) and \( BE(B_n) \) for the corresponding root systems respectively. The algebras \( BE(A_{n-1}) \) and \( BE(B_n) \) have braided Hopf algebra structures. In particular, \( BE(A_{n-1}) \) is conjecturally isomorphic to the so-called Nichols-Woronowicz algebra. Bazlov \[ \| \] has constructed the model of the coinvariant algebra of the finite Coxeter group from this viewpoint. In this section, we construct a model of the Grothendieck ring of the flag variety in terms of the Nichols-Woronowicz algebra associated to a Yetter-Drinfeld module over the Weyl group \( W \) for the classical root systems.

The Nichols-Woronowicz algebra \( \mathcal{B}(V) \) is a braided Hopf algebra determined by a given braided vector space \( V = (V, \psi) \). The braided vector space \( (V, \psi) \) is a finite-dimensional vector space \( V \) equipped with the braiding \( \psi : V \otimes V \rightarrow V \otimes V \) that is a canonically given linear endomorphism satisfying the braid relation
\[ \psi_{12} \psi_{23} \psi_{12} = \psi_{23} \psi_{12} \psi_{23} : V \otimes V \otimes V \rightarrow V \otimes V \otimes V, \]
where \( \psi_{ij} : V \otimes V \otimes V \rightarrow V \otimes V \otimes V \) is obtained by applying \( \psi \) on the \( i \)-th and \( j \)-th components. The Nichols-Woronowicz algebra \( \mathcal{B}(V) \) is a braided analog of the symmetric tensor algebra, which is defined by replacing the symmetrizer by the braided symmetrizer. For an element \( w \in S_n \) with a reduced decomposition \( w = s_{i_1} \cdots s_{i_l} \), the linear endomorphism \( \psi_w := \psi_{i_1 i_1+1} \cdots \psi_{i_l i_l+1} \) on \( V^\otimes n \) is well-defined from the braid relation. The Woronowicz symmetrizer \( \sigma_n(\psi) : V^\otimes n \rightarrow V^\otimes n \) is given by the formula
\[ \sigma_n(\psi) := \sum_{w \in S_n} \psi_w. \]
The Nichols-Woronowicz algebra $\mathcal{B}(V)$ is the quotient of the tensor algebra $\bigoplus_n V^\otimes n$ by the kernels of the braided symmetrizers $\sigma_n(\psi)$:

$$\mathcal{B}(V) = \bigoplus_n V^\otimes n / \bigoplus_n \ker(\sigma_n(\psi)).$$

The Nichols-Woronowicz algebra $\mathcal{B}(V)$ provides a natural framework to perform the braided differential calculus.

Let us consider the Nichols-Woronowicz algebra $\mathcal{B}_X$ obtained from the Yetter-Drinfeld module

$$V = \bigoplus_{\alpha \in \Psi} Q[\alpha]/([\alpha] + [-\alpha])_{\alpha \in \Psi}$$

for the root system $\Psi$ of classical type $X$, $(X = A_{n-1}, B_n, C_n, D_n)$. Let $W(X)$ be the corresponding Weyl group. The $W(X)$-action on $V$ is given by $w([\alpha]) = [w(\alpha)]$, and the $W(X)$-degree of $[\alpha]$ is the reflection $s_\alpha$. The structure of the braided vector space on $V$ is given by the braiding $\psi([\alpha] \otimes [\beta]) = [s_\alpha(\beta)] \otimes [\alpha]$. For the details on the definition of the algebra $\mathcal{B}_X$, see [1]. The algebra $\mathcal{B}_X$ is a quotient of the algebra $YB(X)$.

The Weyl group $W(B_n)$ acts on the algebra $YB(B_n)$. Denote by $s_1 = s_{12}, \ldots, s_{n-1} = s_{n-1}^n$, and $s_n$ the simple reflections. The subgroup $S_n = W(A_{n-1}) \subset W(B_n)$ acts on $YB(B_n)$ via the permutation of the indices of $h_{ij}, g_{ij}$ and $h_i$. The action of the simple reflection $s_n$ is given as follows:

$$s_n(h_{ij}) = h_{ij}, \quad s_n(g_{ij}) = g_{ij}, \quad s_n(h_i) = h_i, \quad \text{for } i, j \neq n,$$

$$s_n(h_{in}) = h_{in}, \quad s_n(g_{in}) = h_{in}, \quad s_n(h_n) = h_n^{-1}.$$

Define the twisted derivations $\Delta_{ij} (i < j)$ and $\Delta_i$ on $YB(B_n)$ by

$$\Delta_{ij}(h_{kl}) = \begin{cases} 1, & \text{if } i = k \text{ and } j = l, \\ 0, & \text{otherwise}, \end{cases}$$

$$\Delta_{ij}(g_{ij}) = \Delta_{ij}(h_k) = 0,$$

$$\Delta_i(h_j) = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{otherwise}, \end{cases}$$

$$\Delta_i(h_{jk}) = \Delta_i(g_{jk}) = 0,$$

and the twisted Leibniz rule

$$\Delta_{ij}(xy) = \Delta_{ij}(x)y + s_{ij}(x)\Delta_{ij}(y),$$

$$\Delta_i(xy) = \Delta_i(x)y + s_i(x)\Delta_i(y).$$

Let us consider the operators $Q_i := h_{ii+1}^{-1} \circ \Delta_{i+1} (i < n)$ and $Q_n := h_n^{-1} \circ \Delta_n$ on $YB(B_n)$. 

---

19
Lemma 5.1 Let $\Theta_j := \Theta_j^{B_n}(1, 1)$. One has

$$Q_i(\Theta_j) = \begin{cases} 
\Theta_{i+1}, & \text{if } j = i, \\
-\Theta_{i+1}, & \text{if } j = i + 1, \\
0, & \text{otherwise},
\end{cases}$$

for $i < n$, and

$$Q_n(\Theta_j) = \begin{cases} 
1 + \Theta_n^{-1}, & \text{if } j = n, \\
0, & \text{otherwise}.
\end{cases}$$

Proof. It is clear that $Q_i(\Theta_j) = h_{i+1}^{-1} \Delta_i(\Theta_j) = 0$ ($i < n$) for $j \neq i, i + 1$ and $Q_n(\Theta_j) = h_n^{-1} \Delta_n(\Theta_j) = 0$ for $j \neq n$. We have by direct computation

$$Q_i(\Theta_i) = h_{i+1}^{-1} \Delta_i \left( \prod_{k=i}^{n} h_{k_i}^{-1} \cdot h_i \prod_{k=1, k \neq i}^{n} g_{k_i} \cdot h_i \cdot \prod_{k=n}^{i+1} h_{i+1}^{i+2} \right) = \Theta_{i+1},$$

$$Q_i(\Theta_{i+1}) = h_{i+1}^{-1} \Delta_i \left( \prod_{k=i}^{n} h_{k_i}^{-1} \cdot h_{i+1} \prod_{k=1, k \neq i+1}^{n} g_{k_i+1} \cdot h_{i+1} \cdot \prod_{k=n}^{i+2} h_{i+1+k} \right) = -\Theta_{i+1}.$$ 

Similarly,

$$Q_n(\Theta_n) = h_n^{-1} \Delta_n \left( \prod_{k=n-1}^{1} h_{k_n}^{-1} \cdot h_n \prod_{k=1}^{n-1} g_{k_n} \cdot h_n \right)$$

$$= h_n^{-1} \left( h_n + \prod_{k=n-1}^{1} g_{k_n} \cdot h_n \right) = 1 + \Theta_n^{-1}. \quad \blacksquare$$

Lemma 5.2 The simple reflections act on the elements $\Theta_1, \ldots, \Theta_n$ as follows.

$$h_{i+1}^{-1} \cdot s_i(\Theta_j) \cdot h_{i+1} = \Theta_{s_i(j)}, \quad \text{for } i = 1, \ldots, n - 1,$$

$$h_n^{-1} \cdot s_n(\Theta_j) \cdot h_n = \begin{cases} 
\Theta_n^{-1}, & \text{if } j = n, \\
\Theta_j, & \text{otherwise}.
\end{cases}$$

Proof. If $j \neq i, i + 1$, then the equality

$$h_{i+1}^{-1} \cdot s_i(\Theta_j) \cdot h_{i+1} = \Theta_j$$

20
follows from the relations

\[ h^{-1}_{i+1} h_{ij} h_{ji} h_{i+1} = h_{j+1} h_{ji} \]

and

\[ h^{-1}_{i+1} g_{i+1} g_{ji} h_{i+1} = g_{i} g_{i+1} \cdot \]

We also have

\[ h^{-1}_{i+1} \cdot s_i(\Theta_i) \cdot h_{i+1} \]

\[ = h^{-1}_{i+1} \left( \prod_{k=i-1}^{i} h^{-1}_{k+1} \cdot h_{i+1} \prod_{k=1, k \neq i+1}^{n} g_{k+1} \cdot h_{i+1} \prod_{k=n, k \neq i+1}^{i+2} h_{i+1} \cdot h_{i+1} \right) \cdot h_{i+1} \]

\[ = \Theta_{i+1}, \]

and this completes the proof of the first equality.

We can obtain

\[ h^{-1}_{n} h_{jn} g_{jn} h_{n} = g_{jn} h_{j} \]

for \( j \neq n \). Since

\[ s_{n}(\Theta_{n}) = \prod_{k=n-1}^{n} g_{kn}^{-1} \cdot h_{n}^{n-1} \prod_{k=1}^{n} h_{kn} \cdot h_{n}^{-1}, \]

we have

\[ h^{-1}_{n} \cdot s_{n}(\Theta_{n}) \cdot h_{n} = \Theta_{n}^{-1}. \]

Consider the action of \( W(B_n) \) on the ring of Laurent polynomials \( \mathbb{Q}[X_1^{\pm 1}, \ldots, X_n^{\pm 1}] \) via

\[ (w f)(X_1, \ldots, X_n) := f(X_{w(1)}, \ldots, X_{w(n)}), \ w \in S_n = W(A_{n-1}), \]

and

\[ (s_{n} f)(X_1, \ldots, X_n) := f(X_1, \ldots, X_{n-1}, X_n^{-1}). \]

**Lemma 5.3** Let \( F(\Theta) \) and \( G(\Theta) \) be Laurent polynomials in \( \Theta_1, \ldots, \Theta_n \). Then,

\[ Q_i(F(\Theta)G(\Theta)) = Q_i(F(\Theta))G(\Theta) + (s_i F)(\Theta)Q_i(G(\Theta)), \ i = 1, \ldots, n. \]

**Proof.** The equalities in Lemma 5.2 imply

\[ h^{-1}_{i+1} \cdot s_i(F(\Theta)) \cdot h_{i+1} = (s_i F)(\Theta), \]

so

\[ Q_i(F(\Theta)G(\Theta)) = h^{-1}_{i+1} \Delta_{i+1}(F(\Theta)G(\Theta)) \]

\[ = h^{-1}_{i+1} \Delta_{i+1}(F(\Theta))G(\Theta) + h^{-1}_{i+1} s_i(F(\Theta))h_{i+1} \cdot h^{-1}_{i+1} \Delta_{i+1}(G(\Theta)) \]

\[ = Q_i(F(\Theta))G(\Theta) + (s_i F)(\Theta)Q_i(G(\Theta)) \]
for \( i < n \). The equality

\[
Q_n(F(\Theta)G(\Theta)) = Q_n(F(\Theta))G(\Theta) + (s_n F)(\Theta)Q_n(G(\Theta))
\]

is proved in the same way. \( \blacksquare \)

Define the operators \( \tau_1, \ldots, \tau_{n-1} \) and \( \tau_n := \tau_n^B \) on \( \mathbb{Q}[X_1^\pm, \ldots, X_n^\pm] \) by

\[
(\tau_i f)(X) := X_{i+1} \frac{f(X) - (s_i f)(X)}{X_i - X_{i+1}}, \quad i = 1, \ldots, n - 1,
\]

\[
(\tau_n f)(X) := \frac{f(X) - (s_n f)(X)}{X_n - 1}.
\]

The operator corresponding to \( \tau_n \) in the case of type \( C_n \) is given by

\[
(\tau_n^{C_n} f)(X) := \frac{f(X) - (s_n f)(X)}{X_n^2 - 1}.
\]

We consider the group \( W(D_n) \) as the subgroup of \( W(B_n) \). Let \( \tau_n^{D_n} := \tau_n^B \tau_{n-1} \tau_n^B \). Then we have

\[
(\tau_n^{D_n} f)(X_1, \ldots, X_{n-1}, X_n) = \frac{f(X_1, \ldots, X_{n-1}, X_n) - f(X_1, \ldots, X_{n-1}, X_{n-1}^1)}{X_{n-1} X_n - 1}.
\]

**Proposition 5.4** Let \( \Theta_j := \Theta_j^B (1, 1), \ 1 \leq j \leq n \), then

\[
Q_n(F(\Theta_1, \ldots, \Theta_n)) = (\tau_i F)(\Theta_1, \ldots, \Theta_n).
\]

**Proof.** This follows from Lemmas 5.1 and 5.3. \( \blacksquare \)

**Remark 5.5** One can obtain the corresponding results for \( A_{n-1} \) (resp. \( D_n \)) after specialization \( g_{ij} = h_i = 1 \) (resp. \( h_i = 1 \)), \( \forall i, j \).

**Remark 5.6** All the construction in this section till Proposition 5.4 can be done on the level of the group algebra \( \mathbb{Q} \langle \mathcal{Y}B(B_n) \rangle \).

We have the homomorphisms

\[
\varphi : \mathcal{Y}B(A_{n-1}) \to \mathcal{BE}(A_{n-1}) \to \mathcal{B}_{A_{n-1}},
\]

\[
\varphi : \mathcal{Y}B(D_n) \to \mathcal{BE}(D_n) \to \mathcal{B}_{D_n},
\]

\[
\varphi : \mathcal{Y}B(B_n) \to \mathcal{BE}(B_n) \to \mathcal{B}_{B_n},
\]

given by \( h_{ij} \mapsto 1 + [ij], \ g_{ij} \mapsto 1 + [ij] \) and \( h_{i} \mapsto 1 + [i] \).

Conjecturally, the quadratic algebras \( \mathcal{BE}(A_{n-1}) \) and \( \mathcal{BE}(D_n) \) \( \footnote{22} \) are isomorphic respectively to the Nichols-Woronowicz algebras \( \mathcal{B}_{A_{n-1}} \) and \( \mathcal{B}_{D_n} \).
The Nichols-Woronowicz algebra is equipped with the duality pairing

$$\langle \cdot, \cdot \rangle : B_X \otimes B_X \rightarrow \mathbb{Q}$$

and naturally defined braided derivations acting on it. Here we are interested in the derivations $\overline{D}_{[\alpha]}$ given by the formula

$$\overline{D}_{[\alpha]}(\xi) = (\text{id}_B \otimes (\cdot, \cdot))(\psi_B \otimes \text{id}_B)([\alpha] \otimes \xi(1) \otimes \xi(2)),$$

where $\psi_B : V \otimes B \rightarrow B \otimes V$ is the braiding induced by $\psi$, and we use Sweedler’s notation $\Delta(\xi) = \xi(1) \otimes \xi(2)$ for the coproduct $\Delta$ of the Nichols-Woronowicz algebra. The twisted derivations $\Delta_{ij}$, $\Delta_{ij}$, and $\Delta_i$ are corresponding to the derivations on the Nichols algebras, namely $\varphi(\Delta_{ij}(x)) = \overline{D}_{ij}(\varphi(x))$, $\varphi(\Delta_{ij}(x)) = \overline{D}_{ij}(\varphi(x))$, $\varphi(\Delta_i(x)) = \overline{D}_i(\varphi(x))$. Note that the intersection of the kernels of all the derivations $\overline{D}_{[\alpha]}$ coincides with the degree zero part $B^0_X = \mathbb{Q}$. This is the essential property of the Nichols-Woronowicz algebra which will be used in the subsequent argument.

Let $P$ be the weight lattice associated to some root system and $\mathbb{Q}[P] = \mathbb{Q}[\epsilon^\lambda | \lambda \in P]$ its group algebra. Denote by $\epsilon : \mathbb{Q}[P] \rightarrow \mathbb{Q}$ the algebra homomorphism given by $\epsilon \lambda \mapsto 1$, $\forall \lambda \in P$. The Grothendieck ring of the corresponding flag variety can be expressed as a quotient algebra $\mathbb{Q}[P]/I$, where the ideal $I$ is generated by the $W$-invariant elements of form $f - \epsilon(f)$.

**Theorem 5.7** Let $F$ be a Laurent polynomial in the defining ideal of the Grothendieck ring of the flag variety of classical type $X$, and $\Theta_j := \Theta^X_j$. Then,

$$\varphi(F(\Theta_1, \ldots, \Theta_n)) = 0$$

in the corresponding Nichols-Woronowicz algebra $B_X$ ($X = A_n$, $B_n$, $C_n$ or $D_n$).

**Proof.** In the following, we consider the root system of type $B_n$. The cases of type $A, C, D$ can be obtained from this case by a certain specialization. For simplicity, we use the same symbol $\Theta_i$ for the corresponding element to the RSM-elements in $B_{B_n}$. Let $\epsilon_j(X) := \epsilon_j(X_1 + X_1^{-1}, \ldots, X_n + X_n^{-1}) - \epsilon_j(2, \ldots, 2)$. Proposition 5.4 implies that

$$\varphi(Q_i(\epsilon_j(\Theta))) = 0.$$

Hence, we have $\overline{D}_{i+1}(\epsilon_j(\Theta)) = 0$ and $\overline{D}_n(\epsilon_j(\Theta)) = 0$. From the $W$-invariance of the polynomial $\epsilon_j$ and Lemma 5.2, it follows that

$$s_k(\epsilon_j(\Theta)) = h_{k+1}^{-1} \epsilon_j(h_{k+1}) h_{k+1}^{-1} \epsilon_{k+1}(\Theta) h_{k+1}^{-1}$$

and

$$s_n(\epsilon_j(\Theta)) = h_n^{-1} \epsilon_j(h_n) h_n^{-1} \epsilon_j(h_n).$$

Thus, for $k \neq i$,

$$\Delta_{i+1}(s_k(\epsilon_j(\Theta))) = \Delta_{i+1}(h_{k+1}^{-1} \epsilon_j(h_{k+1}) h_{k+1}^{-1})$$

$$= s_i(h_{k+1}) \Delta_{i+1}(\epsilon_j(\Theta)) h_{k+1}^{-1} = 0.$$

For $k = i$,

$$\Delta_{i+1}(s_i(\epsilon_j(\Theta))) = \Delta_{i+1}(h_{i+1}^{-1} \epsilon_j(h_{i+1}) h_{i+1}^{-1})$$

23
More generally, one can show that if $\Delta_{kl}(\epsilon_j(\Theta)) = 0$, then $\Delta_{kl}(s_i(\epsilon_j(\Theta))) = 0$. Since $w \circ \Delta_{kl} \circ w^{-1} = \Delta_{w(k),w(l)}$ for $w \in W$, we can conclude that $\overline{D}_{kl}(\epsilon_j(\Theta)) = \overline{D}_{kl}(\epsilon_j(\Theta)) = 0$, for all $k,l$. Similarly, $\overline{D}_k(\epsilon_j(\Theta)) = 0$, for all $k$. Since the constant term of $\epsilon_j(\Theta)$ considered as a polynomial in $[ab]'s$, $[ab]'s$ and $[a]'s$ is zero, it follows that $\epsilon_j(\Theta) = 0$ in $B_{\mathcal{B}_n}$. 

Let us remark that it follows from the above considerations that in the case of $D_n$ we have relations $e_k((\Theta^1_1)^n + (\Theta^1_n)^{-1}) \cdots , (\Theta^n_1)^n + (\Theta^n_n)^{-1} = 0$ for $1 \leq k < n$ and the additional relation $\prod_{j=1}^n((\Theta^j_1)^{1/2} - (\Theta^j_n)^{-1/2}) = 0$ in $B_{\mathcal{D}_n}$.

For any Laurent polynomial $F$ that is not in the ideal generated by $\epsilon_1, \ldots, \epsilon_n$, one can find a sequence of indices $i_1, \ldots, i_r$ such that $\tau_{i_1} \cdots \tau_{i_r}F(X) \in \mathbb{Q} \setminus \{0\}$. Hence we have the following.

**Corollary 5.8** The RSM elements $\Theta_i$ generate the algebra isomorphic to the Grothendieck ring of the corresponding flag variety of classical type as a commutative subalgebra in $\mathcal{B}_X$.

**Remark 5.9** The operators $\mathcal{Q}_1, \ldots, \mathcal{Q}_n$ satisfy the relations

$$\mathcal{Q}_i^2 = \mathcal{Q}_i, \quad i = 1, \ldots, n,$$

$$\mathcal{Q}_i \mathcal{Q}_{i+1} \mathcal{Q}_i = \mathcal{Q}_{i+1} \mathcal{Q}_i \mathcal{Q}_{i+1}, \quad i = 1, \ldots, n - 2,$$

$$\mathcal{Q}_{n-1} \mathcal{Q}_n \mathcal{Q}_{n-1} \mathcal{Q}_n = \mathcal{Q}_n \mathcal{Q}_{n-1} \mathcal{Q}_n \mathcal{Q}_{n-1}.$$

### 6 The case of root system of type $G_2$

Let us consider the root system of type $G_2$. Let

$$\Psi_+ = \{a, \ b, \ c, \ d, \ e, \ f\}$$

be the set of positive roots, where $a$ and $f$ are the simple roots and $b = 3a + f$, $c = 2a + f$, $d = 3a + 2f$, $e = a + f$.

**Definition 6.1** Denote by $\mathcal{YB}(G_2)$ the group generated by six elements $h_a, h_b, h_c, h_d, h_e, h_f$ subject to the following relations:

- $h_a \ h_d = h_d \ h_a, \ h_b \ h_c = h_c \ h_b, \ h_c \ h_f = h_f \ h_c$;
- $(A_2 - Yang-Baxter \ relation) \quad h_b \ h_d \ h_f = h_f \ h_d \ h_b$;
- $(G_2 - Yang-Baxter \ relation) \quad h_a \ h_b \ h_c \ h_d \ h_e \ h_f = h_f \ h_e \ h_d \ h_c \ h_b \ h_a$.

**Proposition 6.2** Define the RSM-elements of type $G_2$ in $\mathcal{YB}(G_2)$ as follows

$$\Theta_1^{G_2} := h_d \ h_b \ h_c \ h_d \ h_e \ h_f, \quad \Theta_2^{G_2} := h_f^{-1} \ h_b \ h_d \ h_c \ h_b \ h_a.$$ 

Then we have $\Theta_1^{G_2} \cdot \Theta_2^{G_2} = \Theta_2^{G_2} \cdot \Theta_1^{G_2}$.
Let us consider the group algebra \( \mathbb{Q}(\mathcal{B}(G_2)) \). The Weyl group \( W(G_2) \) naturally acts on the algebra \( \mathbb{Q}(\mathcal{B}(G_2)) \). The twisted derivations \( \Delta_a \) and \( \Delta_f \) determined by the conditions

\[
\Delta_a(h_i) = \begin{cases} 
1, & \text{if } i = a, \\
0, & \text{otherwise},
\end{cases}
\]

\[
\Delta_f(h_i) = \begin{cases} 
1, & \text{if } i = f, \\
0, & \text{otherwise},
\end{cases}
\]

and the twisted Leibniz rule are well-defined on \( \mathbb{Q}(\mathcal{B}(G_2)) \). Let \( Q_a := h_a^{-1} \circ \Delta_a \) and \( Q_f := h_f^{-1} \circ \Delta_f \). The action of the simple reflections \( s_a \) and \( s_f \) on the Laurent polynomial ring \( \mathbb{Q}[X_1^{\pm 1}, X_2^{\pm 1}] \) is given by

\[
s_a(X_1) = X_1, \quad s_a(X_2) = X_1X_2^{-1},
\]

\[
s_f(X_1) = X_2, \quad s_f(X_2) = X_1.
\]

Define the operators \( \tau_a^{G_2} \) and \( \tau_f^{G_2} \) acting on \( \mathbb{Q}[X_1^{\pm 1}, X_2^{\pm 1}] \) by

\[
(\tau_a^{G_2}F)(X_1, X_2) := X_1 \frac{F(X_1, X_2) - (s_a F)(X_1, X_2)}{X_2^2 - X_1},
\]

\[
(\tau_f^{G_2}F)(X_1, X_2) := X_2 \frac{F(X_1, X_2) - (s_f F)(X_1, X_2)}{X_1 - X_2}.
\]

The arguments as in the previous section show the following.

**Proposition 6.3**

\[
Q_a F(\Theta_1, \Theta_2) = (\tau_a^{G_2}F)(\Theta_1, \Theta_2), \quad Q_f F(\Theta_1, \Theta_2) = (\tau_f^{G_2}F)(\Theta_1, \Theta_2).
\]

**Proposition 6.4** There exists a natural homomorphism from \( \mathbb{Q}(\mathcal{B}(G_2)) \) to the Nichols algebra \( \mathcal{B}_{G_2} \) obtained by \( h_a \mapsto 1 + [\alpha], \alpha \in \Psi_+ \). In other words, the \( G_2 \) Yang-Baxter relation holds in \( \mathcal{B}_{G_2} \).

**Proof.** The Yang-Baxter relations give a set of relations among \( [a], \ldots, [f] \) up to degree six. It is easy to check the compatibility of the quadratic relations and those from subsystems of type \( A_2 \). The rest of cubic relations and the ones of higher degree can be verified by direct computation with help of the factorization of the braided symmetrizer, \[ \square \].

The independent \( W(G_2) \)-invariant Laurent polynomials are given by

\[
\phi_1(X_1, X_2) = X_1 + X_1^{-1} + X_2 + X_2^{-1} + X_1X_2^{-1} + X_1^{-1}X_2,
\]

\[
\phi_2(X_1, X_2) = X_1X_2 + X_1^{-1}X_2^{-1} + X_1X_2^{-1} + X_1^{-1}X_2 + X_1X_2^{-2} + X_1^{-2}X_2 + X_1X_2^{-2}.
\]

The propositions above imply:

**Theorem 6.5** We have \( \phi_1(\Theta_1, \Theta_2) = \phi_2(\Theta_1, \Theta_2) = 6 \) in the Nichols algebra \( \mathcal{B}_{G_2} \), so the subalgebra of \( \mathcal{B}_{G_2} \) generated by the images of the RSM-elements \( \Theta_1^{G_2} \) and \( \Theta_2^{G_2} \) is isomorphic to the Grothendieck ring of the flag variety of type \( G_2 \).
Definition 6.6 Define the algebra $\mathcal{B}\mathcal{E}(G_2)$ as an associative algebra over $\mathbb{Q}$ with generators \{a, b, c, d, e, f\} subject to the relations

- (Commutativity) $ad = da$, $be = eb$, $cf = fc$;
- (Quadratic relations) $ae = ec + ca$, $ea = ce + ac$, $fb = df + bd$, $bf = fd + db$;

\[ af = ba + cb + dc + ed + fe, \quad fa = ab + bc + cd + de + ef; \]

- (Quartic relations)

\[ abac + acab + acbc = baca + cbca + caba, \quad dfef + dedf + efdf = fded + fdef + efed, \]
\[ abde + bced + becf + ecdb = cdbc + cdcd + decd + fdca, \]
\[ bdec + edcb + edba + fceb = cbdc + dcd + dced + acdf, \]

- ($G_2$ Yang–Baxter relation) $abcdef = fedcba$.

Conjecture 6.7 The relations $\phi_1(\Theta_1, \Theta_2) = \phi_2(\Theta_1, \Theta_2) = 6$ are still valid in the algebra $\mathcal{B}\mathcal{E}(G_2)$.

Remark 6.8 One can show that there exists the natural epimorphism of algebras $\mathcal{B}\mathcal{E}(G_2) \to \mathcal{B}_{G_2}$, which has a non-trivial kernel, however.

Appendix

Definition A.1 Let $\mathcal{YB}(B_n)$ be a group generated by the elements $\{h_{ij}, g_{ij} \mid 1 \leq i \neq j \leq n\}$ and $\{h_i \mid 1 \leq i \leq n\}$, subject to the following set of relations:

- $g_{ij} = g_{ji}$, $h_{ij} = h_{ji}^{-1}$;
- $h_{ij} h_{kl} = h_{kl} h_{ij}$, $g_{ij} g_{kl} = g_{kl} g_{ij}$, $h_k h_{ij} = h_{ij} h_k$, $h_k g_{ij} = g_{ij} h_k$, if all $i, j, k, l$ are distinct;
- $h_i h_j = h_j h_i$, if $1 \leq i, j \leq n$; $h_{ij} g_{ij} = g_{ij} h_{ij}$, if $1 \leq i < j \leq n$;
- (A$_2$ Yang-Baxter relations)

\[ (I) \quad h_{ij} h_{ik} h_{jk} = h_{jk} h_{ik} h_{ij}, \]
\[ (II) \quad h_{ij} g_{ik} g_{jk} = g_{jk} g_{ik} h_{ij}, \]
\[ (III) \quad h_{ik} g_{ij} g_{jk} = g_{jk} g_{ij} h_{ik}, \]
\[ (IV) \quad h_{jk} g_{ij} g_{ik} = g_{ik} g_{ij} h_{jk}, \]

if $1 \leq i < j < k \leq n$;

- (B$_2$ quantum Yang-Baxter relation)

\[ h_{ij} h_i g_{ij} h_j = h_j g_{ij} h_i h_{ij}, \]

if $1 \leq i < j \leq n$. 

26
**Definition A.2** Define the following elements in the group $\mathcal{YB}(B_n)$:

$$\Theta_j = (\prod_{i=j-1}^1 h_{ij}^{-1}) h_j (\prod_{i=1, i\neq j}^n g_{ij}) h_j (\prod_{k=n}^{j+1} h_{jk}),$$

for $1 \leq j \leq n$.

Proof of Theorem 2.3 (key lemma). We have to prove that $\Theta_i \Theta_j \Theta_i^{-1} = \Theta_j$. It is enough to consider the case $i < j$.

To begin with, it is convenient to introduce a bit of notation:

$$A_i = \prod_{a=i-1}^1 h_{ai}^{-1}, \quad B_i = \prod_{a=1}^n g_{ai}, \quad C_i = \prod_{a=n}^{i+1} h_{ia},$$

and in a similar way, we define $A_j, B_j$, and $C_j$:

$$A_j' = \prod_{c=j-1}^1 h_{cj}^{-1}, \quad B_j' = \prod_{a=1}^n g_{aj}, \quad B_j' = \prod_{c=1}^n g_{cj}, \quad C_j' = \prod_{a=n}^{i+1} h_{ja}.$$

Using this notation we can write

$$\Theta_i \Theta_j \Theta_i^{-1} = A_i h_i B_i h_i C_j A_j h_j B_j h_j C_j C_i^{-1} h_i^{-1} B_i'^{-1} h_i^{-1} A_i^{-1}.$$  

The Lemma below describes commutation relations between elements we have introduced.

**Lemma A.3** (1a) $C_i A_j = A_j' C_i'$;

(2a) $C_j C_i^{-1} = (C_i')^{-1} h_{ij} C_j$;

(3a) $C_i' B_j = B_j' g_{ij} C_i'$;

(4a) $B_i A_i' = A_j' g_{ij} B_i'$;

(5a) $C_j B_i'^{-1} = g_{ij}^{-1} (B_i')^{-1} C_j$;

(6a) $B_i' h_{ij}^{-1} (B_i')^{-1} = B_i' h_{ij}^{-1} (B_i')^{-1}$;

(7a) $A_i A_j' h_{ij}^{-1} = A_j A_i$;

(8a) $g_{ij} B_j A_i'^{-1} = A_i'^{-1} B_j$;

(9a) $[h_i, B_j] = 0, \quad [h_j, B_i] = 0, \quad [h_j, A_i] = 0, \quad [h_i, C_j] = 0, \quad [h_i, A_i'] = 0, \quad [A_i, C_j'] = 0$.

Using relations (1a) and (2a), and commutativity relations (9a), we can write

$$\Theta_i \Theta_j \Theta_i^{-1} = A_i h_i B_i h_i C_i A_i' h_j C_i' B_j (C_i')^{-1} h_j h_{ij}^{-1} h_i^{-1} C_j B_i'^{-1} h_i^{-1} A_i^{-1}.$$  

Now we are going to apply the relations (3a), (4a), and (5a) respectively to the market terms to reduce the above expression to the following form

$$A_i A_j' h_i g_{ij} B_i h_j B_j' g_{ij} h_j B_i'^{-1} g_{ij}^{-1} (B_i')^{-1} C_j h_i^{-1} A_i^{-1}.$$  

27
To the market terms we can apply the $B_2$-Yang-Baxter relation presented in an equivalent form $g_{ij} h_j h_{ij}^{-1} h_i^{-1} = h_i^{-1} h_{ij}^{-1} h_j g_{ij}$, and after that do cancellations of $h_i$ and $g_{ij}$. As a result we will have

$$\Theta_i \Theta_j \Theta_i^{-1} = A_i A_j^{-1} h_i g_{ij} B_i^{-1} h_j B_j^{-1} h_i^{-1} B_i^{-1} h_j B_j^{-1} h_i^{-1} A_i^{-1}. $$

The next step is to apply to the bold terms the relation (6a), and rewrite the above expression in the following form:

$$A_i A_j^{-1} h_i g_{ij} h_{ij}^{-1} B_j h_j C_j h_i^{-1} A_i^{-1}. $$

Now we can apply to the market terms the $B_2$-Yang-Baxter relation again, but this time written in the form $h_i g_{ij} h_j h_{ij}^{-1} = h_{ij}^{-1} h_j g_{ij} h_i$, and after the cancellation of $h_i$, to obtain

$$A_i A_j^{-1} h_{ij}^{-1} g_{ij} B_j h_j C_j A_i^{-1}. $$

Now applying the relation (7a) to the market terms, and using the fact that $C_j$ and $A_i$ commute, see relations (9a), we can write

$$\Theta_i \Theta_j \Theta_i^{-1} = A_j A_i h_j g_{ij} B_j^{-1} A_i^{-1} h_j C_j. $$

Finally, applying the relation (8a) to the market terms, after cancellations we will have

$$\Theta_i \Theta_j \Theta_i^{-1} = A_j h_j B_j h_j C_j = \Theta_j. $$

It remains to prove the commutativity relations listed in Lemma A.3.

- **Proof of (1a):**

  Note that if $j = i + 1$, then $C_i A_j = 1 = A_j A_i^{-1} C_i'$. So we will assume that $j - i \geq 2$. Under this assumption, we can write

  $$C_i A_j = h_{i,n} \cdots h_{i,j} h_{i,j-1} \cdots h_{i,i} h_{j-1,j}^{-1} \cdots h_{i,j}^{-1} \cdots h_{i,j}^{-1}. $$

  Using local commutativity relations, see Definition 3.1, (2), we can move the factor $h_{i,j-1}^{-1}$ to the left until we have touched on the factor $h_{i,j-1}$, as a result, we will come up with the triple product $h_{i,j} h_{i,j-1} h_{j-1,j}^{-1}$. Now we can apply the $A_2$-Yang-Baxter relation, see Definition A.1, (I), in an equivalent form $h_{j,k}^{-1} h_{j,k} h_{i,j} h_{i,k} = h_{i,k} h_{i,j} h_{j,k}^{-1}$, and move the factor $h_{j-1,j}$ to the left most position, to obtain

  $$C_i A_j = h_{j-1,i}^{-1} h_{i,n} \cdots h_{i,j-1} h_{i,j} h_{i,j-2} \cdots h_{i,i+1} h_{j-2,i}^{-1} \cdots h_{j,i}^{-1} \cdots h_{1,j}^{-1}. $$

  In a similar fashion as above, we can move the factor $h_{j-2,i}^{-1}$ to the left until we have touched on the factor $h_{ij-2}$. Now we can apply the $A_2$-Yang-Baxter relation (I) in the form presented above, to the triple product $h_{ij} h_{i,j-2} h_{j-2,i}^{-1}$, and move to the left the factor $h_{j-2,i}^{-1}$. We can continue this procedure until the factor $h_{ij}^{-1}$ will touch the factor $h_{i,j}^{-1}$. After cancellation and moving to the left the product $h_{i-1,j}^{-1} \cdots h_{i,j}^{-1}$, we will come to the product $A_j C_i.$
• Proof of (2a) is similar to that of (1a).

By definition, \( C_i C_j^{-1} = h_{j,n} \cdots h_{j,i+1} h_{i,j}^{-1} \cdots h_{i,n} \). Using local commutativity relations, see Definition 3.1, (2), we can move the factor \( h_{j,j+1} \) to the right until we have touched on the factor \( h_{i,j}^{-1} \). As a result, we will come up with the triple product

\[ h_{i,j+1} h_{j,i}^{-1} h_{i,j+1}^{-1}. \]

Now we can apply the A2-Yang-Baxter relation (I), see Definition A.1, in an equivalent form \( h_{j,k} h_{i,j}^{-1} h_{i,k}^{-1} = h_{i,k} h_{j,i}^{-1} h_{j,k} \), and move the factor \( h_{j,j+1} \) to the right most position and \( h_{i,j+1}^{-1} \) to the left most position, to obtain

\[ C_j C_i^{-1} = h_{i,j+1}^{-1} h_{i,j+2} h_{i,n} \cdots h_{i,j+3} h_{i,i+1}^{-1} \cdots h_{i,j}^{-1} h_{i,j+2}^{-1} \cdots h_{i,n}^{-1} h_{j,j+1}. \]

Now we can move the factor \( h_{j,j+2} \) to the right until we have touched on the factor \( h_{i,j}^{-1} \), and apply the A2-Yang-Baxter relation (I) in the form mentioned above, to the triple product

\[ h_{j,j+2} h_{j,i}^{-1} h_{j,j+2}^{-1}. \]

Just as before, we will come to the equality

\[ C_j C_i^{-1} = h_{i,j+1}^{-1} h_{i,j+2} h_{i,n} \cdots h_{i,j+3} h_{i,i+1}^{-1} \cdots h_{i,j}^{-1} h_{i,j+3}^{-1} \cdots h_{i,n}^{-1} h_{j,j+2} h_{j,j+1}. \]

Repeating this procedure we will come to

\[ \prod_{a=i+1}^{n} h_{i,a}^{-1} h_{i,j}^{-1} \prod_{c=n}^{j+1} h_{j,c}. \]

• Proof of (3a) is similar to that of (1a) and (2a), but this time we have to use A2-Yang-Baxter relation (II).

By definition, \( C_i' B_j = h_{i,n} \cdots h_{i,j} \cdots h_{i,i+1} g_{1,j} \cdots g_{i,j} g_{i,j+1} \cdots g_{n,j} \). We can move the factor \( h_{i,i+1} \) to the right until we have touched on the factor \( g_{ij} \). Now we can apply the A2-Yang-Baxter relation II to the triple product \( h_{i,i+1} g_{ij} g_{i+1,j} \). As a result, we will come to an equality

\[ C_i' B_j = h_{i,n} \cdots h_{i,j+2} g_{1,j} \cdots g_{i-1,j} g_{i+1,j} g_{ij} g_{i,j+2} \cdots g_{n,j} h_{i,i+1}. \]

Now we can again move the factor \( h_{i,i+2} \) to the right until we have touched on the factor \( g_{ij} \), and then apply the A2-Yang-Baxter relation II to the triple product \( h_{i,i+2} g_{ij} g_{i+2,j} \). The result can be written as follows

\[ C_i' B_j = h_{i,n} \cdots h_{i,j} \cdots h_{i,i+3} g_{1,j} \cdots g_{i-1,j} g_{i+1,j} g_{i+2,j} g_{ij} g_{i,j+3} \cdots g_{n,j} h_{i,i+2} h_{i,i+1}. \]

Repeating this procedure we can move the all \( h_{i,k} \), \( k \neq j \), to the right through the factor \( g_{ij} \).

• Proof of (4a) is a very similar to that of (3a), but this time we have to use A2-Yang-Baxter relation (IV) in the form

\[ g_{ik} g_{ij} h_{jk}^{-1} = h_{jk}^{-1} g_{ij} g_{ik}. \]
By definition
\[ B_i' A'_j = g_{i,j} \cdots g_{i,j-1} g_{ij} \cdots g_{i,n} h_{j-1,j}^{-1} \cdots h_{ij}^{-1} \cdots h_{1,j}^{-1}. \]

We can move the factor \( h_{j-1,j}^{-1} \) to the left until we have touched on the factor \( g_{ij} \). Then we can apply the \( A_2 \)-Yang-Baxter relation (IV) in the form presented above, and transform the result to the following form
\[
\begin{aligned}
&h_{j-1,j}^{-1} g_{i,j} \cdots g_{i,j-2} g_{ij} \cdots g_{i,n} h_{j-2,j}^{-1} \cdots h_{ij}^{-1} \cdots h_{1,j}^{-1} g_{i,j-1}.
\end{aligned}
\]

Now we can move the factor \( h_{j-2,j}^{-1} \) to the left until we have touched on the factor \( g_{ij} \), and apply the Yang-Baxter relation (IV) to the triple product \( g_{ij} g_{ij} h_{j-2,j}^{-1} \), and so on. As a final result we will come to the RHS of the equality (4a). \( \blacksquare \)

- **Proof of (5a)** is a very similar to that of (4a), but this time we have to use \( A_2 \)-Yang-Baxter relation (IV) in the following form
\[
h_{jk} g_{ik}^{-1} g_{ij}^{-1} = g_{ij}^{-1} g_{ik}^{-1} h_{jk}.
\]

We will give only an outlook of the proof and leave details to the reader.

By definition
\[ C_j B_i^{-1} = h_{jn} \cdots h_{j,j+1} g_{m} \cdots g_{i,j+1} g_{ij}^{-1} \cdots g_{i,1}^{-1}. \]

Therefore we can apply to the market terms the Yang-Baxter relation (IV) in the form presented above and write
\[
C_j B_i^{-1} = h_{jn} \cdots h_{j,j+2} g_{m}^{-1} \cdots g_{i,j+2} g_{ij}^{-1} \cdots g_{i,j+1}^{-1} \cdots g_{i,1}^{-1} h_{j,j+1}.
\]

We can continue by applying the Yang-Baxter relation (IV) to the market factors, and so on. As a final result we obtain the RHS of the equality (5a). \( \blacksquare \)

- **Proof of (6a)** runs in the same way as that of (5a), but this time we have to use \( A_2 \)-Yang-Baxter relation (II) in the following form
\[
g_{jk} h_{ij}^{-1} h_{ik}^{-1} = g_{ik}^{-1} h_{ij}^{-1} g_{jk}.
\]

Again we will give only an outlook of the proof and leave details to the reader.

By definition
\[ B'_i h_{ij}^{-1} (B'_i)^{-1} = g_{i,j} \cdots g_{i,j} \cdots g_{i,n} h_{ij}^{-1} g_{i,n}^{-1} \cdots g_{i,j} \cdots g_{i,1}^{-1}. \]

Now we can apply the Yang-Baxter relation (II) in the form presented above to the market terms to conclude that
\[
B'_j h_{ij}^{-1} (B'_i)^{-1} = g_{i,n}^{-1} g_{i,j} \cdots g_{i,n}^{-1} \cdots g_{i,j} \cdots g_{i,1}^{-1} h_{ij}^{-1} g_{i,n}^{-1} \cdots g_{i,j} \cdots g_{i,1}^{-1} g_{j,n}.
\]

Now we can continue and apply the Yang-Baxter relation (II) to the market terms, and so on. As a final step we have come to the RHS of the equality (6a). \( \blacksquare \)
• Proof of (7a) runs in the same way as that of the previous one, but this time we have to use $A_2$-Yang-Baxter relation (I) in the following form

$$h^{-1}_{ij} h^{-1}_{ik} h^{-1}_{jk} = h^{-1}_{jk} h^{-1}_{ik} h^{-1}_{ij}.$$ 

Again we will give only an outlook on the proof and leave details to the reader. By definition

$$A_i A_j h^{-1}_{ij} = h_{i-1,i}^{-1} \cdots h_{1,i}^{-1} h_{j-1,j}^{-1} \cdots \hat{h}_{ij}^{-1} \cdots h_{1,j}^{-1} h_{1,i}^{-1}.$$ 

Therefore we can apply to the market terms the Yang-Baxter relation (I) in the form presented above and write

$$A_i A_j h^{-1}_{ij} = h_{i-1,i}^{-1} \cdots h_{2,i}^{-1} h_{j-1,j}^{-1} \cdots \hat{h}_{ij}^{-1} \cdots h_{2,j}^{-1} h_{1,j}^{-1} h_{1,i}^{-1}.$$ 

Now we can continue and apply the Yang-Baxter relation (I) to the market terms, and so on. As a final step we have come to the RHS of the equality (7a).

• Proof of (8a) runs in the same way as that of the previous one, but this time we have to use $A_2$-Yang-Baxter relation (II). We will give only an outlook on the proof and leave details to the reader. By definition

$$A^{-1} B = h_{1,i} \cdots h_{i-1,i} g_{1,j} \cdots g_{i-1,j} g_{ij} \cdots g_{nj}.$$ 

Therefore we can apply to the market terms the Yang-Baxter relation (II) in the form presented above and write

$$A^{-1} B = h_{1,i} \cdots h_{i-1,i} g_{1,j} \cdots g_{i-1,j} g_{ij} \cdots g_{nj} h_{i-1,i}.$$ 

Now we can continue and apply the Yang-Baxter relation (II) to the market terms, and so on. As a final step we have come to the RHS of the equality (8a).

Acknowledgements

The authors would like to thank Yuri Bazlov for fruitful discussions and help with computation of the Hilbert series of the Nichols-Woronowicz algebras $B_{B_3}$ and that $B_{G_2}$. Both of the authors were supported by Grant-in-Aid for Scientific Research.

References

[1] Bazlov Y., Nichols-Woronowicz algebra model for Schubert calculus on Coxeter groups, preprint, math.QA/0409206.

[2] Bernstein I.N, Gelfand I.M. and Gelfand S.I., Schubert cells and cohomology of the space $G/P$, Russian Math. Survey 28 (1973), 1-26.
[3] Fomin S.V. and Kirillov A.N., Quadratic algebras, Dunkl elements, and Schubert calculus, Advances in Geometry, 147-182, Progress in Math. 172, Birkhauser, Boston, 1998.

[4] Givental A. and Lee Y.-P., Quantum K-theory on flag manifolds, finite-difference Toda lattices and quantum groups, Invent. Math., 151 (2003), 193-219.

[5] Kirillov A.N., On some quadratic algebras: Jucys-Murphy and Dunkl elements, Calogero-Moser-Sutherland models, (Montreal, PQ, March 10-15, 1997), CRM Series in Mathematical Physics, Springer, New York, 2000, 231-248.

[6] Kirillov A.N., Quantum Grothendieck polynomials, Algebraic Methods and q-Special Functions (Montreal, PQ, May 13-17, 1996), CRM Proc. Lecture Notes 22, AMS, 1999, 215-226; q-alg/9610034.

[7] Kirillov A.N. and Maeno T., Noncommutative algebras related with Schubert calculus on Coxeter groups, European J. of Combin. 25, (2004), 1301-1325.

[8] Kostant B. and Kumar S., The nil Hecke ring and cohomology of $G/\mathcal{K}$ for a Kac-Moody group $G$, Advances in Math. 62 (1986), 187-237.

[9] Kostant B. and Kumar S., T-equivariant K-theory of generalized flag varieties, Journ. Differential Geometry 32 (1990), 549-603.

[10] Lascoux A., Anneau de Grothendieck de la variété de drapeaux, The Grothendieck Festschrift, Vol. III, 1–34, Progress in Math., 88, Birkhäuser Boston, Boston, MA, 1990.

[11] Lascoux A. and Schützenberger M.-P., Symmetry and flag manifolds, Invariant theory (Montecatini, 1982), 118–144, Lecture Notes in Math., 996, Springer, Berlin, 1983.

[12] Lascoux A. and Schützenberger M.-P., Polynomes dé Schubert, C. R. Acad. Sci. Paris Sér. I Math. 294 (1982), 13, 447–450.

[13] Lenart C., The K-theory of the flag variety and the Fomin-Kirillov quadratic algebra, J. Algebra 285 (2005), 120-135.

[14] Lenart C. and Sottile F., A Pieri-type formula for the K-theory of a flag manifold, preprint, math.CO/0407412.

[15] Lenart C. and Yong A., Lecture notes on the K-theory of the flag variety and the Fomin-Kirillov quadratic algebra, http://www.math.umn.edu/~ayong/

[16] Postnikov, A., On a quantum version of Pieri’s formula, Advances in Geometry, 371-383, Progress in Math. 172, Birkhauser, Boston, 1998.

Research Institute for Mathematical Sciences, Kyoto University, Sakyo-ku, Kyoto 606-8502, Japan
e-mail: kirillov@kurims.kyoto-u.ac.jp

Department of Mathematics,
Kyoto University,
Sakyo-ku, Kyoto 606-8502, Japan

e-mail: maeno@math.kyoto-u.ac.jp