On Mittag–Leffler $d$-Orthogonal Polynomials

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Abstract. This paper presents a first result of a long-term research project dealing with the construction of $d$-orthogonal polynomials with Hahn’s property. We shall show that the latter class could be characterized by expanding a polynomial as a finite sum of first derivatives of the elements of the sequence and we shall explain how this characterization could be used to construct Hahn-classical $d$-orthogonal polynomials as well. In this paper, we look for solutions of linear combinations of the first derivatives of two consecutive elements of the sequence by considering the derivative operator and Delta (discrete) operator. The resulting polynomials constitute a particular class of Laguerre $d$-orthogonal polynomials and a generalization of Mittag–Leffler polynomials, respectively.

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1. Introduction

We aim to start a construction of $d$-orthogonal polynomials of Hahn type, that is to say, the $d$-orthogonal polynomials with $d$-orthogonal derivatives (towards Askey tableaux). Our idea is based on the fact that both of the latter sequences are $d$-orthogonal. This means that we can express any polynomial from the sequence as linear combination in terms of the derivative sequence’s elements. Moreover, the linear combination should be finite to guarantee the $d$-orthogonality of the derivative sequence. In the present paper, we shall look at this type of polynomials as solutions of a linear combination by considering the first two consecutive terms from the linear combination in below and this will be our starting paper

$$P_n(x) = \sum_{\nu=0}^{d+1} \lambda_{n,\nu} Q_{n-\nu}(x), \quad \forall n \geq 0,$$

(1.1)

where $Q_n = (n+1)^{-1} LP_{n+1}$, $n \geq 0$, and $L$ is a lowering operator, that is, a linear operator that decreases in one unit the degree of a polynomial and such that $L(1) = 0$. The aim of this paper is to consider the case $L := \Delta_w$. 

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According to Hahn property, orthogonal polynomials of Hahn type (classical when \( d = 1 \)) are referred to as the orthogonal polynomials with orthogonal derivatives. As mentioned above, in this paper, we shall look at the solutions of (1.1) when \( \lambda_{n,\nu} = 0 \) for \( 2 \leq \nu \leq d + 1 \). The resulting polynomials have the following generating functions

\[
G(x, t) = \exp \left\{ \frac{xt}{1 - \beta t} + b_0 + b_1 t + \cdots + b_{d-1} t^{d-1} \right\}, \quad (1.2)
\]

\[
K(x, t) = \begin{cases} 
1 - \beta t^{x/(\alpha - \beta)} & \text{if } \alpha = -\beta = 1 \\
\frac{j(x)}{\alpha - \beta} & \text{if } \alpha = 1, \beta = 0 
\end{cases}
\exp \left\{ \sum_{i=0}^{d-1} b_i t^i \right\}, \quad (1.3)
\]

corresponding to the case \( \mathcal{L} := d/dx \) and \( \mathcal{L} := \Delta_w \), respectively.

We would like to mention on one hand that the \( d \)-orthogonal polynomials generated by (1.2) are of Laguerre type [10, p. 10]. On the other hand, we will call the polynomials generated by (1.3) the Mittag–Leffler \( d \)-orthogonal polynomials since they reduce to classical Mittag–Leffler orthogonal polynomials with \( d = 1 \) and \( \alpha = -\beta = 1 \). Furthermore, if \( \alpha \) or \( \beta \) is zero, this generating function yields Charlier \( d \)-orthogonal polynomials studied by Ben Cheikh and Zaghouani in [9]. Furthermore, it will be of interest to consider the same problem with different operators such as \( D_{q, w} \) and \( D_{p, q} \) and, for instance, to look at \( d \)-orthogonal polynomials at the quadratic lattice.

The main new results of the manuscript are presented in Sects. 3 and 4. In Sect. 2, we present basic concepts related with the \( d \)-quasi-orthogonality. First, we present a characterization of the definition introduced by Maroni occupied with an example of Laguerre \( d \)-orthogonal polynomials. Then, we shall show that there is a gap in the definition of Maroni which leads us to distinguish between the \( d \)-quasi-orthogonality of order exactly \( l \) (which agrees with Maroni’s definition) and at most \( l \). The first part of Sect. 3 contains an explicit expression of the first structure relation corresponding to the first two consecutive terms of the second structure relation. Then, we specify the latter structure relation by considering the case \( \mathcal{L} = d/dx \) in the second part 3.1. The resulting family of polynomials constitutes a subclass of Laguerre \( d \)-orthogonal polynomials. The third part 3.2 deals with the case \( \mathcal{L} = \Delta_w \). In this case, the exponential generating function constitutes a generalization of the generating function of Mittag–Leffler polynomials. Interesting properties of this family, structure relations as well as difference equations are presented. In the last Sect. 4, we use the quasi-monomiality to determine the dual sequence only for the discrete case (for the Laguerre case it suffices to replace \( \Delta_w \) by \( D := d/dx \) and repeat the same process or just take the limit as \( w \) goes to zero).

2. Quasi-Orthogonality and Linear Combinations

The generalized rising factorial is defined by \((x|w)_0 = 1\) and

\[
(x|w)_n = x(x + w)(x + 2w) \cdots (x + (n - 1)w), \quad n \geq 1,
\]
and generalized falling factorial is defined by \( \langle x|w \rangle_0 = 1 \) and
\[
\langle x|w \rangle_n = x(x - w)(x - 2w) \ldots (x - (n - 1)w), \quad n \geq 1.
\]

Let us remark that the rising factorial and the falling factorial are connected through \( \langle x|w \rangle_n = \langle x+(n-1)w|w \rangle_n \) and \( \langle x|w \rangle_n = (x-(n-1)w|w)_n \). Notice also that when \( w = 1 \), the rising factorial reduces to Pochhammer symbol, i.e. \( (x|1)_n := (x)_n = x(x+1)(x+2) \ldots (x+n-1) \).

The generating function of the generalized falling factorials can be obtained directly from the binomial series as follows
\[
(1 + wt)^{x/w} = \sum_{n=0}^{\infty} \frac{t^n}{n!} \langle x|w \rangle_n.
\] (2.1)

Now, let \( \{P_n\}_{n \geq 0} \) be a sequence of monic polynomials with \( \text{deg } P_n = n \), \( n \geq 0 \). The dual sequence \( \{u_n\}_{n \geq 0} \), \( u_n \in P'_n \), of \( \{P_n\}_{n \geq 0} \) is defined by duality bracket denoted throughout as \( \langle u_n, P_m \rangle := \delta_{n,m}, \ n, m \geq 0 \). The latter equality can be regarded as a bi-orthogonality between two sequences.

Before we dive into the \( d \)-orthogonality, let us briefly recall the standard orthogonality. A sequence \( \{P_n\}_{n \geq 0} \) is said to be orthogonal with respect to a linear functional \( u \) in the linear space of polynomials with complex coefficients, if
\[
\langle u, P_m P_n \rangle := r_n \delta_{n,m}, \quad n, m \geq 0, \quad r_n \neq 0, \ n \geq 0.
\]
In this case, necessarily \( u \) is proportional to \( u_0 \), i.e., \( u = \lambda u_0 \), \( \lambda \neq 0 \).

For a generalization of the above standard orthogonality, we will deal with the concept of \( d \)-orthogonality. Let us recall the definition and some characterizations which will be needed in the sequel. Throughout this work, all the sequences of polynomials are supposed to be monic.

**Definition 2.1.** [15] A sequence of monic polynomials \( \{P_n\}_{n \geq 0} \) is said to be a \( d \)-orthogonal polynomial sequence, in short a \( d \)-OPS, with respect to the \( d \)-dimensional vector of linear forms \( U = (u_0, \ldots, u_{d-1})^T \) if
\[
\begin{cases}
\langle u_r, x^m P_n (x) \rangle = 0, & n \geq md + r + 1, \quad m \geq 0, \\
\langle u_r, x^m P_{md+r} (x) \rangle \neq 0, & m \geq 0,
\end{cases}
\] (2.2)
for each \( 0 \leq r \leq d - 1 \).

The first and second conditions of (2.2) are called, respectively, the \( d \)-orthogonality conditions and the \( d \)-regularity conditions. In this case, the \( d \)-dimensional vector form \( U \) is called regular. Notice further that if \( d = 1 \), then we meet again the notion of usual (standard) orthogonality.

The following characterization constitutes an analog of Favard’s theorem.

**Theorem 2.2.** [15] Let \( \{P_n\}_{n \geq 0} \) be a monic sequence of polynomials, then the following statements are equivalent.

(a) The sequence \( \{P_n\}_{n \geq 0} \) is \( d \)-OPS with respect to \( U = (u_0, \ldots, u_{d-1}) \).
(b) The sequence \( \{ P_n \}_{n \geq 0} \) satisfies a \( (d + 2) \)-term recurrence relation

\[
P_{m+d+1}(x) = (x - \beta_{m+d}) P_{m+d}(x) - \sum_{\nu=0}^{d-1} \gamma_{m+d-\nu}^{d-1} P_{m+d-1-\nu}(x), \quad m \geq 0,
\]

with the initial data

\[
\begin{cases}
P_0(x) = 1, & P_1(x) = x - \beta_0, \\
P_m(x) = (x - \beta_{m-1}) P_{m-1}(x) - \sum_{\nu=0}^{m-2} \gamma_{m-1-\nu}^{d-1} P_{m-2-\nu}(x), & 2 \leq m \leq d,
\end{cases}
\]

and the regularity conditions \( \gamma_{m+1}^0 \neq 0, m \geq 0 \).

Now, we recall the concept of quasi-orthogonality and some characterizations.

**Definition 2.3.** [15] A sequence \( \{ P_n \}_{n \geq 0} \) is said to be \( d \)-quasi-orthogonal of order \( s \) with respect to the form \( U = (u_0, \ldots, u_{d-1})^T \), if for every \( 0 \leq r \leq d - 1 \), there exist \( s_r \geq 0 \) and \( \sigma_r \geq s_r \) integers such that

\[
\begin{align*}
\langle u_r, P_m P_n \rangle &= 0, & n \geq (m + s_r) d + r + 1, & m \geq 0, \\
\langle u_r, P_{\sigma_r} P_{(\sigma_r + s_r)d+r} \rangle &\neq 0, & m \geq 0,
\end{align*}
\]

with \( s = \max_{0 \leq r \leq d-1} s_r \).

If the linear form \( U \) in the definition above is regular, then there exists another sequence of polynomials, say \( \{ Q_n \}_{n \geq 0} \), \( d \)-orthogonal with respect to \( U \). The question to think about now is: what is the connection between these two sequences? Maroni in his papers does not provide any information on the latter connection. Unfortunately, in our work, we shall use this hard stage. Next, we recall two characterizations of the \( d \)-quasi-orthogonality rely only on the polynomials. First, we have the following

**Proposition 2.4.** [16,18] Suppose that \( \{ P_n \}_{n \geq 0} \) is \( d \)-OPS with respect to \( U \). Then, a sequence of polynomials \( \{ Q_n \}_{n \geq 0} \) is strictly \( d \)-quasi-orthogonal of order \( l \) with respect to \( U \) if and only if the following relation holds

\[
Q_n(x) = \sum_{i=n-dl}^{n} a_{n,i} P_i(x), \quad n \geq dl,
\]

with \( a_{n,n-dl} \neq 0 \).

We shall give a motivating example of the latter proposition. For this end, we want to give a generalization of the hypergeometric polynomials discussed in [13]. Indeed, in that paper, the authors proved the following lemma which is given explicitly here
Lemma 2.5. Let \( n \in \mathbb{N}, k \in \{1, 2, \ldots, n-1\} \) and \( \alpha_2, \ldots, \alpha_p, \beta_1, \ldots, \beta_q \in \mathbb{R} \) such that \( \alpha_2, \ldots, \alpha_p, \beta_1, \ldots, \beta_q \notin \{0, -1, -2, \ldots, -n\} \) and \( \alpha_2 \notin \{0, 1, 2, \ldots, k-1\} \). Then

\[
pFq{-n, \alpha_2 + 1, \ldots, \alpha_p}{\beta_1, \ldots, \beta_q}{\alpha_2 - k + 1, \ldots, \alpha_p}{z} = \sum_{i=0}^{k} (-1)^i \binom{k}{i} \frac{\langle n \rangle_i \langle n + \alpha_2 - i \rangle_{k-i}}{\langle \alpha_2 \rangle_k} pFq{-n+i, \alpha_2 - k + 1, \ldots, \alpha_p}{\beta_1, \ldots, \beta_q}{z}.
\]

Accordingly, we could give a linear combination of some hypergeometric type d-OPS.

Example. The Laguerre d-OPS denoted \( L_n^{(\alpha_1, \ldots, \alpha_d)}(x) \) are defined in terms of the generalized hypergeometric function \( _1F_d \) as follows

\[
L_n^{\alpha_d}(x) := L_n^{(\alpha_1, \ldots, \alpha_d)}(x) = _1F_d \left( \begin{array}{c} -n \\ \alpha_1 + 1, \ldots, \alpha_d + 1 \end{array} \right| x \right),
\]

\( \alpha_i \neq -1, -2, \ldots, \ i = 1, \ldots, d. \)

By taking \( p = 1, q = d \) and \( \beta_i = \alpha_i + 1 \) with \( \alpha_i \neq -1, -2, \ldots \) in (2.7), we readily get the following representation

\[
\sum_{k=0}^{dl} (-1)^k \binom{dl}{k} \langle n \rangle_k (\beta + dl + 1)_{n-k} (\beta + 1)_n L_n^{(\alpha_1, \ldots, \alpha_d)}(x)
\]

\[
= _2F_{d+1} \left( \begin{array}{c} -n, \beta + dl + 1 \\ \alpha_1 + 1, \ldots, \alpha_d + 1, \beta + 1 \end{array} \right| x \right) := P_n^{(\alpha_1, \ldots, \alpha_d, \beta)}(x)
\]

with \( \beta \neq -1, -2, \ldots \).

The latter linear combination (2.9) together with Proposition 2.4 shows that the polynomial \( P_n^{\alpha_d, \beta}(x) \) is \( d \)-quasi-orthogonal of order \( l \) with respect to \( P_n^{\alpha_d}(x) \).

Notice also that if there exists \( i \), say \( i = 1 \) such that \( \alpha_1 = \beta + dl \), then the expansion (2.9) reduces to

\[
\sum_{k=0}^{l} (-1)^k \binom{l}{k} \langle n \rangle_k (\alpha_1 + 1)_{n-k} (\beta + 1)_n L_n^{(\alpha_1, \ldots, \alpha_d)}(x) = L_n^{(\beta, \alpha_2, \ldots, \alpha_d)}(x).
\]

Remark that (2.10) is a connection formula between two sequences of Laguerre d-OPS such that they differ in the first parameter, which shows, in turn, the possibility of expanding the Laguerre d-OPS in terms of Laguerre d-OPS. The latter expansion shows further that a linear combination of d-OPS could be again d-OPS [14]. It can be used also to construct semi-classical d-orthogonal polynomials examples [17] (of hypergeometric type).

It is worthy to notice, as you can see in the definition 2.3, Maroni defines the \( d \)-quasi-orthogonality as \( \langle u_t, P_n \rangle = 0 \) for \( n > dl + t \), the question now is what happens for values between \( d(l - 1) + 1 \) and \( dl - 1 \)? Indeed,
characterizations of the above definition are given by only considering the case \( n > dl + t \) which is equivalent to assume the quasi-orthogonality of order exactly \( l \). Next, we shall distinguish between the \( d \)-quasi-orthogonality of order exactly \( l \) and at most \( l \).

**Definition 2.6.** [18] A sequence \( \{P_n\}_{n \geq 0} \) is \( d \)-quasi-orthogonal of order at most \( l \) with respect to the form \( \mathcal{U} = (u_0, \ldots, u_{d-1})^T \), if there exists an integer \( 1 \leq r \leq d \) such that for every \( 0 \leq t \leq d - 1 \), there exist integer numbers \( l_t \geq 0 \) and \( s_t \geq l_t \) such that

\[
\begin{align*}
\langle u_t, P_m P_n \rangle &= 0, \quad n \geq (m + l_t - 1)d + r + t + 1, \quad m \geq 0, \\
\langle u_t, P_s P(s_t + l_t - 1)d + r + t \rangle &\neq 0, \quad m \geq 0.
\end{align*}
\tag{2.11}
\]

The latter helps us to extract as well as to close the implication between the first and the second structure relation (interested reader on quasi-orthogonality and Hahn’s property could look at [18] for more details). For instance, in this paper, we shall be interested to the following characterizations

**Proposition 2.7.** Let \( \{P_n\}_{n \geq 0} \) be a \( d \)-OPS with respect to \( \mathcal{U} \). The following properties are equivalent:

(i) \( \{P_n\}_{n \geq 0} \) is a sequence of Hahn-classical \( d \)-orthogonal polynomials.

(ii) There exist complex numbers \( \lambda_{n,\nu} \) not all zero, such that [14,18]

\[
P_n(x) = \sum_{i=0}^{d+1} \lambda_{n,i} Q_{n-i}(x), \quad \forall n \geq 0.
\tag{2.12}
\]

Now, we would like to mention that the connection (2.12) could be used to enumerate all the \( \mathcal{L} \)-classical \( d \)-OPS of Hahn type. Therefore, we shall focus next, on the linear combination (2.12) and we shall look for its solutions by considering, in this paper, a linear combination of the first two consecutive terms. Hence if we assume that \( d = 1 \), then with an appropriate choice of the operator we should obtain some families of Askey scheme.

3. Constructing OPS Classical in The Hahn Sense

To start with, let us remark from (2.12) that if \( \lambda_{n,\nu} = 0 \) for \( 1 \leq \nu \leq d + 1 \), then the solutions of these equations are \( \mathcal{L} \)-Appell \( d \)-orthogonal polynomials (see for instance [9,11,22] the respective cases of \( \mathcal{L} \)).

Now suppose that \( \lambda_{n,\nu} = 0 \) for \( 2 \leq \nu \leq d + 1 \), i.e.,

\[
P_n(x) = Q_n(x) - \lambda_n Q_{n-1}(x), \quad \lambda_n := \lambda_{n,1}
\tag{3.1}
\]

To determine all classical \( d \)-OPS satisfying (3.1), we shall explicitly determine the corresponding generating functions. For this end, it is more convenient to transform (3.1) to certain initial value problem.

First of all, besides structure relations (2.12) we have one more interesting structure relation inspired by [14]
Proposition 3.1. For \( d \geq 2 \) the polynomials generated by (3.1) satisfy the following structure relation
\[
(x-c)Q_{n-1}(x) = P_n(x) + (\lambda_n + \xi_{n-1} - c)P_{n-1}(x) - \sum_{i=2}^{d} \sum_{j=i}^{d} \frac{\eta_{n-i}^{d-j}}{\lambda_{n-i} \cdots \lambda_{n-j}} P_{n-i}(x).
\]
(3.2)

To prove the latter proposition, we need the following lemma based on the paper’s results [14].

Lemma 3.2. The recurrence coefficients of the two sequences of polynomials generated by (3.1) satisfy the following
\[
\lambda_{n-1} \left[ \frac{\eta_{n-d+1}^{0}}{\lambda_n \cdots \lambda_{n-d+1}} + \frac{\eta_{n-d+2}^{0}}{\lambda_n \cdots \lambda_{n-d+2}} + \cdots + \frac{\eta_{n-2}^{d-3}}{\lambda_n \cdots \lambda_{n-2}} + \frac{\gamma_{n-1}^{d-3}}{\lambda_{n-1}} \right] = \frac{\eta_{n-d}^{0}}{\lambda_n \cdots \lambda_{n-d}} + \frac{\eta_{n-d+1}^{0}}{\lambda_n \cdots \lambda_{n-d+1}} + \cdots + \frac{\eta_{n-2}^{d-3}}{\lambda_{n-2}}.
\]
(3.3)

Proof. If \( \{P_n\}_{n \geq 0} \) and \( \{Q_n\}_{n \geq 0} \) are two \( d \)-OPS connected through (3.1), then [14, Eq. (25)]
\[
\lambda_{n+d} \eta_n^0 = \lambda_n \gamma_{n+1}^0, \quad n \geq 1,
\]
\[
\eta_n^k = \gamma_n^k + \lambda_{n+d-k-1} \eta_{n+1}^{k+1} - \lambda_n \gamma_{n+1}^{k+1}, \quad 0 \leq k \leq d - 2, \quad n \geq 1.
\]
(3.4)

Now use the second equality in (3.4) to replace \( f(\eta_{n+1}^k, \gamma_{n+1}^k) \) by \( g(\eta_n^k, \gamma_n^k) \).

Substitute recursively the latter fact in the left-hand side of (3.3) to get its right-hand side.

Proof of Proposition 3.1. We have from [14, Eq.(31)] using also the first equality of (28) that
\[
\beta_n = c - \lambda_n - \frac{\eta_{n-1}^{d-1}}{\lambda_n} - \frac{\eta_{n-1}^{d-2}}{\lambda_n \cdots \lambda_{n-d+2}} - \cdots - \frac{\eta_{n-1}^{1}}{\lambda_n \cdots \lambda_{n-d+1}} - \frac{\eta_{n-1}^{0}}{\lambda_n \cdots \lambda_{n-1}}.
\]
\[
\gamma_{n-1}^{d-1} = c - \lambda_n - \xi_{n-1} - \frac{\eta_{n-1}^{d-2}}{\lambda_n \cdots \lambda_{n-d+2}} - \cdots - \frac{\eta_{n-1}^{0}}{\lambda_n \cdots \lambda_{n-1}}.
\]

Inserting the latter expressions in the recurrence relation of \( P_n(x) \) written as follows
\[
\gamma_{n-1}^{d-1} P_{n-1} = (x - \beta_n)P_n - \sum_{\nu=0}^{d-1} \gamma_{n-\nu}^{d-1} P_{n-\nu} - P_{n+1}
\]
and then replacing each term of \( P_k \) using (3.1), we get
\[
\begin{aligned}
&\left[ c - \lambda_n - \xi_{n-1} - \frac{\eta_{n-1}^{d-2}}{\lambda_n \cdots \lambda_{n-d+2}} - \cdots - \frac{\eta_{n-1}^{0}}{\lambda_n \cdots \lambda_{n-1}} \right] P_{n-1} = P_n \\
&+ \left[ c - x - \frac{\eta_{n-1}^{d-2}}{\lambda_n \cdots \lambda_{n-d+2}} - \cdots - \frac{\eta_{n-1}^{0}}{\lambda_n \cdots \lambda_{n-1}} \right] Q_{n-1} \\
&+ \left( \eta_{n-1}^{d-1} - \frac{\lambda_{n-1} \gamma_{n-1}^{d-1}}{\lambda_n} \right) Q_{n-2} + \sum_{i=2}^{d} \eta_{n-i}^{d-i} Q_{n-1-i},
\end{aligned}
\]
which can be written using again (3.1), lemma 3.2 and (3.4) as follows
\[(c - \lambda_n - \xi_{n-1})P_{n-1} = P_n + (c - x)Q_{n-1}\]
\[-\left[\frac{\eta_{n-d}^0}{\lambda_{n-2}...\lambda_{n-d}} + \frac{\eta_{n-d+1}^1}{\lambda_{n-2}...\lambda_{n-d+1}} + \cdots + \frac{\eta_{n-2}^{d-2}}{\lambda_{n-2}}\right]Q_{n-2} + \sum_{i=2}^{d} \eta_{n-i}^{d-i}Q_{n-1-i},\]
and also in the following expression
\[(c - \lambda_n - \xi_{n-1})P_{n-1} = P_n + (c - x)Q_{n-1}\]
\[-\frac{\eta_{n-2}^{d-2}}{\lambda_{n-2}} (Q_{n-2} - \lambda_{n-2}Q_{n-3}) - \frac{\eta_{n-2}^{d-2}}{\lambda_{n-2}\lambda_{n-3}} (Q_{n-2} - \lambda_{n-2}\lambda_{n-3}Q_{n-4})\]
\[\cdots - \frac{\eta_{n-d}^0}{\lambda_{n-2}\lambda_{n-3}...\lambda_{n-d}} (Q_{n-2} - \lambda_{n-2}\lambda_{n-3}...\lambda_{n-d}Q_{n-d-1}).\]

Thus, on account of (3.1) with some rearrangement, we get the structure relation (3.2).

**Remark 3.3.** Let us remark that if we multiply (3.2) by \(\lambda_n\) and replace \(\lambda_n Q_{n-1}(x)\) using (3.1), then the structure relation (3.2) can also be written as
\[(x - c)Q_n(x) = (x + \lambda_n - c)P_n(x) + \lambda_n(\lambda_n + \xi_{n-1} - c)P_{n-1}(x)\]
\[-\lambda_n \sum_{i=2}^{d} \sum_{j=i}^{d} \frac{\eta_{n-j}^{d-j}}{\lambda_{n-i}...\lambda_{n-j}} P_{n-i}(x).\]

### 3.1. Differential Operator: Laguerre Type Polynomials

Let us begin with \(L = \frac{d}{dx}\). This case has been given as an example for the regularity of linear combination of \(d\)-orthogonal polynomials in [14]. First, let us remark that exp \(\{\frac{xt}{(1 - at)}\}\) is the unique solution of the following parametric first-order differential equation \((1 - at)y'(x) = ty(x)\) with \(y(0) = 1\).

Next, we shall denote by \(R(x,t)\) the exponential generating function corresponding to the sequence of polynomials generated by (3.1). Then, it is straightforward to transform (3.1) to the following initial value problem
\[tR(x,t) = (1 - at)\frac{\partial}{\partial x} R(x,t).\] (3.5)

Therefore, according to the previous paragraph, the general solution of (3.5) takes the following form
\[R(x,t) = A(t) \exp \{xt / (1 - at)\}, \quad A(0) = 1.\] (3.6)

Now, by taking the first derivative with respect to the variable \(t\) and set
\[A'(t)/A(t) = \sum_{n \geq 0} \alpha_k \frac{t^k}{k!},\]
we get
\[
P_{n+1}(x) = (x + 2an + \alpha_0) P_n(x) - n \left[ a^2(n - 1) + 2a\alpha_0 - \alpha_1 \right] P_{n-1}(x) \\
+ \sum_{k=2}^{n} \binom{n}{k} \left( \alpha_k - 2ak\alpha_{k-1} + a^2k(k-1)\alpha_{k-2} \right) P_{n-k}(x).
\]

On the other hand, since the sequence of polynomials \( \{P_n\}_{n \geq 0} \) is \( d \)-orthogonal, then we should have \( \alpha_k = 0 \) for \( k \geq n - 2 \). Accordingly, we obtain
\[
A(t) = \exp \left( \sum_{k=0}^{d-1} b_k \frac{t^k}{k!} \right).
\]

The expression of \( A(t) \) shows that the generating function (3.6) is a subclass of the exponential generating function of Laguerre type \( d \)-OPS \([10, \text{p.10}]\). Then, it is more convenient to consider the following generating function
\[
G(x, t) = \sum_{n=0}^{\infty} P_n(x) \frac{t^n}{n!}
= (1 - at)^e \exp \left\{ \frac{xt + \theta}{1 - at} + b_0 + b_1 t + \cdots + b_{d-1} \frac{t^{d-1}}{(d-1)!} \right\}. \tag{3.8}
\]

In this case, assuming that \( b_i \equiv 0 \) if \( i \geq d \), we have
\[
P_{n+1}(x) = (x + a(\theta - \beta + 2n) + b_1) P_n(x)
- n \left[ a^2(n - \beta - 1) + 2ab_1 - b_2 \right] P_{n-1}(x)
+ \sum_{i=2}^{d} \left[ \frac{b_{i+1}}{i!} - 2a \frac{b_i}{(i-1)!} + a^2 \frac{b_{i-1}}{(i-2)!} \right] \langle n \rangle_i P_{n-i}(x),
\]
and
\[
Q_{n+1}(x) = (x + a(\theta - \beta + 2n + 1) + b_1) Q_n(x)
- n \left[ a^2(n - \beta) + 2ab_1 - b_2 \right] Q_{n-1}(x)
+ \sum_{i=2}^{d} \left[ \frac{b_{i+1}}{i!} - 2a \frac{b_i}{(i-1)!} + a^2 \frac{b_{i-1}}{(i-2)!} \right] \langle n \rangle_i Q_{n-i}(x).
\]

The above generating function allows us to present further linear combination of this multiple Laguerre type polynomials in terms of Multiple Laguerre polynomials analogue of (2.10). The Laguerre type \( d \)-orthogonal polynomials given in the Example 2 have been studied and evoked in many places see for instance \([6–8]\). Thereof, the generating function shows that the family of polynomials generated by (3.8) is quite different from that given in the Example 2 as well as from the example studied in \([21]\).

In most cases, since the above polynomials and their derivatives are both \( d \)-orthogonal, then one can explicitly determine the respective measures of the \( d \)-orthogonality using either Pearson equation \([12]\) or the quasi-monomiality principle \([10]\). Notice that the latter idea does not require any information on the derivative sequence, for this end we shall use the latter idea in the next family which converges toward the above polynomials as \( w \) goes to zero and the computations are almost the same (see \([10, \text{Lemma 2.7}]\)). But this does not preclude mentioning some properties, compared by classical Laguerre polynomials, we shall take \( \alpha + 1 = -\beta \) and denote \( \pi_{d-1}(t; b_i) = \)}
$$b_0 + b_1 t + b_2 \frac{t^2}{2!} + \cdots + \frac{b_{d-1} t^{d-1}}{(d-1)!}$$. In this case, since $t \pi_d' (t; b_i) = \pi_d (t; ib_i)$ we can show that

$$x \frac{\partial}{\partial x} G (x, t) + (1 - at) \pi_d (t; ib_i) G (x, t) = t \frac{\partial}{\partial t} G (x, t) - at^{1-\alpha} \frac{\partial}{\partial t} \{ t^{\alpha+1} G (x, t) \}$$

from which we deduce the structure relation (3.2) in closed form

$$x P_n' (x) = n P_n (x) - n \{ b_1 + a(n + \alpha) \} P_{n-1} (x) + \sum_{i=2}^{d} \left\{ \frac{a b_{i-1}}{(i-2)!} - \frac{b_i}{(i-1)!} \right\} \langle n \rangle_i P_{n-i} (x)$$

which reduces in turn to classical Laguerre with $d = 1$, i.e., when $\pi_d (t; b_i) = 0$. We would like to mention further that the differential equation satisfied by these types of polynomials is completely ignored in literature except for Appell case. In our point of view, differential equations can be constructed using the linear combination (2.12) as well as some structure relations (further results of this idea will be presented in forthcoming papers). For the above Laguerre case, it can be obtained simply by taking $w = 0$ in (3.18) and replace $\Delta_w$ by $d/dx$ (see next subsection for more details).

### 3.2. A Discrete Solution: Mittag–Leffler Type Polynomials

Now let us suppose that $L = \Delta_w$. The $\Delta_w$ difference operator is defined as follows

$$\Delta_w f(x) = \frac{f(x + w) - f(x)}{w}.$$

Next, we shall prove that the discrete solution of (3.1) is a generalization of Mittag–Leffler polynomials [3] which seems to be new.

Let denote by $K(x, t) = \sum_{n \geq 0} P_n (x) \frac{t^n}{n!}$ the respective exponential generating function of $\{ P_n \}_{n \geq 0}$. According to (2.12), in this paper we shall assume that $\lambda_{n,1} = n \alpha$ and $w = \alpha - \beta$. It is straightforward to transform (3.1) to the following initial value problem

$$K(x, 0) = 1, \quad (1 - at) \Delta_w K(x, t) = tK(x, t).$$

It is not difficult to show that the unique solution of the above equation is

$$K(x, t) = \left( \frac{1 - \beta t}{1 - \alpha t} \right)^{x/w} A(t), \quad A(0) = 1. \quad (3.9)$$

Now assume that $A'(t)/A(t) = \sum_{k \geq 0} b_k \frac{t^k}{k!}$. Therefore, the partial derivative of (3.9) with respect to $t$ gives

$$(1 - \alpha t)(1 - \beta t) K'_i (x, t) = [x + (1 - \alpha t)(1 - \beta t) A'(t)/A(t)] K(x, t),$$

from which it follows

$$P_{n+1} (x) = [x + (\alpha + \beta)n + b_0] P_n (x) - n \left[ (n - 1)\alpha \beta + (\alpha + \beta)b_0 - b_1 \right] P_{n-1} (x) + \sum_{k=2}^{n} \binom{n}{k} \left[ b_k - (\alpha + \beta)kb_{k-1} + \alpha \beta k(k - 1)b_{k-2} \right] P_{n-k} (x). \quad (3.10)$$
On the other hand, since \( \{P_n\}_{n \geq 0} \) is an \( d \)-OPS, then we must have \( b_k = 0 \) for \( k \geq d - 1 \).

It is worthy to notice that the generating function (3.9) reduces to \( \Delta_w \)-Appell \( d \)-OPS (Charlier \( d \)-OPS [9]) if \( \alpha \) or \( \beta \) is zero, and to Mittag–Leffler’s generating function [3] in the case \( d = 1 \) with \( \alpha = -\beta = 1 \).

Moreover, since sequences generated by the above generating function and their derivatives are both \( d \)-OPS, then it is more convenient to write down the corresponding recurrence of the derivative sequence. Then by acting the operator \( \Delta_w \) on (3.10) \((w = \alpha - \beta)\) and making use of (3.1), we obtain upon writing \( b_k \equiv 0 \) for \( k \geq d - 1 \) the recurrence of the derivative sequence

\[
Q_{n+1}(x) = [x + (\alpha + \beta)n + b_0 + \alpha]Q_n(x) - n[n\alpha\beta + (\alpha + \beta)b_0 - b_1]Q_{n-1}(x)
+ \sum_{k=2}^{n} \binom{n}{k} [b_k - (\alpha + \beta)kb_{k-1} + \alpha\beta k(k-1)b_{k-2}]Q_{n-k}(x).
\]

(3.11)

Let us now mention some properties of the obtained polynomials.

**Proposition 3.4.** The above family of polynomials satisfies the following recurrences

\[
P_n(x + w) = Q_n(x) - n\beta Q_{n-1}(x),
\]

(3.12)

\[
P_n(x) - \betanP_{n-1}(x) = P_n(x + w) - \alphanP_{n-1}(x + w),
\]

(3.13)

\[
wQ_n(x) = \alpha P_n(x + w) - \beta P_n(x),
\]

(3.14)

\[
\Delta_w \{P_{n+1}(x)P_n(x)\} = (n + 1)P_n(x + w)Q_n(x) + nP_{n+1}(x)Q_{n-1}(x)
\]

\[
= (n + 1)Q_n^2(x) + nP_{n+1}(x)Q_{n-1}(x) - n(n + 1)
\]

\[
\beta Q_n(x)Q_{n-1}(x),
\]

\[
(x - c)Q_n = P_{n+1}(x) - (c + b_0 + \beta n)P_n(x)
\]

\[- \sum_{i=1}^{d-1} \binom{n}{i} (\beta b_{i-1} - b_i) P_{n-i}(x).
\]

(3.15)

(3.16)

**Proof.** The recurrence coefficients of (3.10) and (3.11) show that the second structure relation in (2.12) reduces to (3.12) while (3.2) takes the form (3.16).

Let us remark further that from

\[
(1 - \beta t)K(x, t) = (1 - \alpha t)K(x + w, t)
\]

we deduce (3.13). Accordingly, we have

\[
w \Delta_w P_n(x) = nwQ_{n-1}(x) = P_n(x + w) - P_n(x)
\]

\[
= \alpha nP_{n-1}(x + w) - \beta nP_{n-1}(x),
\]

from which (3.14) follows.

Let us now prove (3.15). Remark that from the following fact

\[
w \Delta_w \{P_{n+1}(x)P_n(x)\} = P_{n+1}(x + w)P_n(x + w) - P_{n+1}(x)P_n(x),
\]
we can eliminate the factor $P_{n+1}(x+w)P_n(x+w)$ by multiplying both sides of (3.13) by $P_{n-1}(x+w)$ and $n \rightarrow n + 1$ then, replace the obtained result in the latter equality above and use also (3.14) to deduce the desired result which in turn could be simplified to the second equality using (3.12). □

Let us now turn to the difference equation satisfied by the above polynomials. We would like to mention that property (3.1) makes the construction of the respective differential/difference equation very simple. Indeed, it suffices to apply $d + 1$ times the operator $\Delta_w$ to the recurrence relation satisfied by the polynomials and use in each time the connection (3.1) to move from $n$ to $n - 1$. For convenience, let us denote the recurrence coefficients of (3.10) satisfied by the polynomials $\{P_n\}_{n \geq 0}$ by $\beta_n$ and $\gamma_n^i$. Then, the latter sequence satisfies the following difference equation which can be easily proved by induction on $k$ using (3.1) and the binomial property $\binom{n}{i} + \binom{n}{i+1} = \binom{n+1}{i+1}$ with some computations.

**Theorem 3.5.** The d-OPS solution of (3.1) with $\lambda_{n,1} = n\alpha$ satisfies

$$P_{n-k+1}(x) = [x + kw - k\alpha(n - k + 2) - \beta_n]P_{n-k}(x)$$

$$+ \sum_{i=1}^{k} \left\{ \alpha^i \left[ \binom{k}{i} (x + kw + \alpha - \beta_n) - \alpha \binom{k+1}{i+1} (n - k + i + 2) \right]$$

$$- \sum_{j=0}^{i-1} \binom{k-1-j}{i-1-j} \alpha^{i-1-j} \gamma^{d-1-j}_{n-j} \binom{n-j}{j+1} \right\} \Delta_w^i P_{n-k}(x)$$

$$- \sum_{i=k}^{d-1} \gamma^{d-1-i}_{n-i} \Delta_w^k P_{n-i-1}(x).$$

(3.17)

From the latter result, we merely deduce the following difference equation

**Corollary 3.6.** The d-OPS solution of (3.1) satisfies the following (d+1)-order difference equation

$$(n-d)P_{n-d}(x)$$

$$= [x + (d+1)w - (d+1)\alpha(n - d + 1) - \beta_n] \Delta_w P_{n-d}(x)$$

$$+ \sum_{i=1}^{d} \left\{ \alpha^i \left[ \binom{d}{i} (x + (d+1)w - \beta_n) - \alpha \binom{d+1}{i+1} (n - d + i + 1) \right]$$

$$- \sum_{j=0}^{i-1} \binom{d-1-j}{i-1-j} \alpha^{i-1-j} \gamma^{d-1-j}_{n-j} \binom{n-j}{j+1} \right\} \Delta_w^{i+1} P_{n-d}(x).$$

(3.18)

**Proof.** Take first $k = d - 1$ in (3.17), then apply two times $\Delta_w$ on both sides of (3.17) together with (3.1) in each time to deduce explicitly, after some straightforward calculations, the difference equation.

In this case, the explicit form of the polynomials generated by (1.3) may be written in terms of generalized falling factorial. Let us denote $b_{i-1}/i! = a_i$ and exp $\{a_0\} = 1$, then
Theorem 3.7. We have
\[ P_n(x) = \sum_{s=0}^{n} k_1 \cdots + (d-1)k_{d-1} = m \left( k_1, \ldots, k_{d-1}, n-s, s-m, m \right) \]
\[ \times (a_1)^{k_1} (a_2)^{k_2} \cdots (a_{d-1})^{k_{d-1}} \left( \frac{\beta}{\alpha} \right)^s (-\beta)^m \]
\[ x \langle x + (n-s-1)w \mid w \rangle_{n-m-1} \tag{3.19} \]

The explicit form of the polynomial is a direct consequence of the following result together with the Cauchy product of power series

Lemma 3.8. With \( w = \alpha - \beta \), we have
\[ \left( \frac{1 - \beta t}{1 - \alpha t} \right)^{x/w} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} \left( \frac{\beta}{\alpha} \right)^k x \langle x + (n-k-1)w \mid w \rangle_{n-1} \left( \frac{-\alpha t}{n!} \right)^n \]
\[ \tag{3.20} \]

Proof. Taking into account the following power series
\[ (1 - \beta t)^{x/w} = \sum_{n=0}^{\infty} (-\beta)^n \langle x \mid w \rangle_n \frac{t^n}{n!} \]
\[ (1 - \alpha t)^{-x/w} = \sum_{n=0}^{\infty} (-\alpha)^n \langle x(n-1)w \mid w \rangle_n \frac{t^n}{n!} \]
the convolution follows from the Cauchy product of two series together with
\[ \langle x \mid w \rangle_m \langle x + (n-m-1)w \mid w \rangle_{n-m} = x \langle x + (n-m-1)w \mid w \rangle_{n-1}, \]
or \[ \langle x \mid w \rangle_{n-m} \langle x + (m-1)w \mid w \rangle_m = x \langle x + (m-1)w \mid w \rangle_{n-1}. \]
\[ \square \]

4. The Dual Sequence

In the applications, it might be useful sometimes to have an explicit expression for the moments to interpret, combinatorially or physically, the corresponding family of polynomials in one hand. For this end, we shall give here some information about the moments at first. Therefore, starting from the generating function, we can identify the expression of polynomials as well as their inversion formulas by comparing the coefficients of \( t \).

On the other hand, since these polynomial sequences and their derivatives are both \(d\)-OPS, then it might be possible to use Pearson equation \([12]\) to determine the dual sequence (i.e., \(d\)-dimensional vector of linear forms) with respect to which the polynomials are \(d\)-orthogonal. Unfortunately, to determine the dual sequence’s elements \( \{ \varphi_r, \ 0 \leq r \leq d-1 \} \), the latter fact leads, in general, to look at solutions of linear differential equations of order exactly \( d \)
\[ \sum_{i=0}^{d} \pi_{i+d+r,i}(x) \frac{d^i}{dx} \varphi_r(x) = 0, \ \text{for each} \ 0 \leq r \leq d-1, \]
with polynomial coefficients but of degrees greater than \( i \) (at most \( d + r + i \)). The above differential equation comes from Pearson equation by direct computations.

Moreover, if the generating function is of Brenke type (the lucky and the faster case), then we can use the Laplace, \( h \)-Laplace, \( q \)-Laplace transformations (discrete time scales) and their inverse to compute the measures of orthogonality.

Besides, a practical technique is the quasi-monomiality principle which has been developed by Ben Cheikh et al. \([4,5]\) to determine the dual sequence of polynomials mainly in the discrete case \([10]\).

To start with, let us take \( -c_i = b_{i-1}/(i - 1) \) and \( c_0 = 0 \), then from the generating function \((1.3)\) we have

\[
\left(1 - \beta t \over 1 - \alpha t\right) x^w = \exp\left\{ \sum_{i=1}^{d-1} c_i t^i \right\} \sum_{n \geq 0} P_n(x) t^n \over n!,
\]

now expand the right-hand side of \((4.1)\) in powers of \( t \) and then identify the coefficient of \( t^n \) in both sides, we deduce using \((3.20)\) the following

\[
\sum_{k=0}^{n} \left( n \over k \right) \left( \beta \over \alpha \right)^k \langle x + (n - k - 1)w|w \rangle_n \over n - 1 \rangle (\alpha)^n.
\]

If we denote by \( \{\varphi_r\}_{r \geq 0}\), the dual sequence of Mittag–Leffer \( d \)-orthogonal polynomials \( \{P_n\}_{n \geq 0}\), we infer that

\[\text{Proposition 4.1.}\] The moments satisfy \( \langle \varphi_r, x^n \rangle = 0 \) for \( n < r \) and the following finite linear recursion for \( n \geq r \)

\[
\begin{aligned}
(n_1 + 2n_2 + \cdots + (d - 1)n_{d-1}) c_1^{n_1} \cdots c_{d-1}^{n_{d-1}} = \sum_{k=r}^{n} \left( n \over k \right) \left( \beta \over \alpha \right)^k (\alpha)^n \langle \varphi_r, x^k \rangle.
\end{aligned}
\]

For the dual sequence, it has been proved in the discrete case via the latter could be obtained via

\[
\langle \varphi_r, f \rangle = \frac{\sigma^r}{r! A(\sigma)} f(0) \times 1 \over r! \sigma^r \exp\left\{ \sum_{i=1}^{d-1} a_i \sigma^i \right\} f(0) \quad \text{for} \quad 0 \leq r \leq d - 1,
\]

where \( \sigma := \sigma_x \) is the lowering operator, i.e., \( \sigma G(x, t) = tG(x, t) \) with \( G(x, t) = G_0(x, t)^\lambda(t) \) and where we have denoted by \( a_k = -b_{k-1} \). Therefore, according to \([5,10]\), the operator \( \sigma \) is given by

\[
\sigma := \frac{e^{wD} - 1}{\alpha e^{wD} - \beta} = \frac{\Delta_w}{1 + \alpha \Delta_w},
\]

and by the binomial theorem, we have

\[
\langle \varphi_r, f \rangle = \frac{1}{r!} \sum_{n_1, \ldots, n_{d-1} \geq 0} \sum_{a_1, \ldots, a_{d-1} \geq 0} a_1^{n_1} \cdots a_{d-1}^{n_{d-1}} \sum_{k=0}^{\infty} \frac{\Delta_w^{l+r+k}}{k!} \langle \varphi_r, f \rangle(0),
\]

where \( \Delta_w \) is the general difference operator.
with \(l := n_1 + 2n_2 + \cdots + (d - 1)n_{d-1}\) which can be written using the expansion

\[
\Delta^m w f (0) = \left(\frac{-1}{w}\right)^m \sum_{j=0}^{m} \binom{m}{j} (-1)^j f (wj),
\]

in the following form

\[
r! \langle \varphi_r, f \rangle = \sum_{k,n_1,\ldots,n_{d-1} \geq 0}^{\infty} \frac{a_{n_1}^1 \cdots a_{n_d-1}^d}{n_1! \cdots n_{d-1}!} \left(\frac{-1}{w}\right)^{l+r+k} \frac{(l+r)_k (-\alpha)^k}{k!}
\]

by writing the sum from 0 to \(l+r+k\) as two sums, the first ends at \(l+r-1\) and the second starts from \(l+r\), using also series manipulation \([19, \text{p. 100–102}]\), we get the following

**Proposition 4.2.** The elements of the dual sequence of Mittag–Leffler d-OPS satisfy

\[
\langle \varphi_r, f \rangle = \frac{1}{r!} \sum_{k,n_1,\ldots,n_{d-1} \geq 0}^{\infty} \frac{a_{n_1}^1 \cdots a_{n_d-1}^d}{n_1! \cdots n_{d-1}!} \left(\frac{-1}{w}\right)^{l+r+k} \frac{(l+r)_k (-\alpha)^k}{k!}
\]

\[
\times \left[ \sum_{s=0}^{k} \frac{(s+l+r)(-1)^{s+l+r} f(w(s+l+r))}{(l+r)_s} \right] + \sum_{s=0}^{l+r-1} \frac{(l+r+k)(-1)^{s} f(ws)}{(l+r)_s}.
\]

To specify the latter sums, let us denote the first and second sum by \(A\) and \(B\), respectively. Taking into account the form of \(l\), we shall write \(B\) as a \(d - 1\) partial sums each one from \((i - 1)n_{i-1}\) to \(in_i - 1\) as bellow. Therefore, using the following

\[
\sum_{n=0}^{\infty} \sum_{k=0}^{da+r-1} A(k,n) = \sum_{n,k=0}^{\infty} A(k,n + [(k + 1 - r) /d])
\]

with \(l_0 = 0\) and \(l_i := n_1 + 2n_2 + \cdots + in_i\), for \(1 \leq i \leq d - 2\), we have

\[
r!B_i := \sum_{k,n_1,\ldots,n_{d-1} \geq 0}^{\infty} \frac{a_{n_1}^1 \cdots a_{n_d-1}^d}{n_1! \cdots n_{d-1}!} \left(\frac{-1}{w}\right)^{l+r+k} \frac{(l+r)_k (-\alpha)^k}{k!}
\]

\[
\times \sum_{s=0}^{in_i - 1} \frac{(s+l_{i-1})(-1)^{s+l_{i-1}} f(w(s+l_{i-1}))}{(l+r)_s}
\]

\[
= \sum_{s,k,n_1,\ldots,n_{d-1} \geq 0}^{\infty} \frac{a_{n_1}^1 \cdots a_{n_d-1}^d}{n_1! \cdots n_{d-1}!} \left(\frac{-1}{w}\right)^{l+r+k+[(s+1)/d]} \frac{(l+r+k+[(s+1)/d])}{s+l_{i-1}!}
\]

\[
\times \frac{(l+r+k+[(s+1)/d])}{k!} (-\alpha)^{(l+r+k+[(s+1)/d])} \left(\frac{-1}{w}\right)^{s+l_{i-1}} f(w(s+l_{i-1})).
\]
and

$$\begin{align*}
r!B_{d-1} & := \sum_{k,n_1,\ldots,n_{d-1} \geq 0} \frac{a^{n_1}_{1}}{n_1!} \cdots \frac{a_{d-1}^{n_{d-1}}}{n_{d-1}!} \left( \frac{-1}{w} \right)^{l+r+k} \frac{(l+r+k)(l+r)_{k}(\alpha)^k}{k!} \\
& \times \sum_{s=0}^{(d-1)n_{d-1}+r+1} \frac{a_{n_{d-1}}}{n_1! \cdots n_{d-1}!} \frac{a^{n_{d-2}}_{d-2}}{a^{n_{d-1}}_{d-1}} \frac{a_{d-2}^{n_{d-2}}}{(l+r+k)(s+l_{d-2})} (l+r+k)(s+1) f(w(s+l_{d-2})) \\
& = \sum_{s,k,n_1,\ldots,n_{d-1} \geq 0} \frac{a^{n_1}_{1}}{n_1!} \cdots \frac{a_{d-1}^{n_{d-1}}}{n_{d-1}!} \frac{a^{n_{d-2}}_{d-2}}{a^{n_{d-1}}_{d-1}} \frac{a^{n_{d-2}}_{d-2}}{(l+r+k)(s+1)} \frac{a_{d-2}^{n_{d-2}}}{(l+r+k)(s+1)} \frac{a_{d-1}^{n_{d-1}}}{(s+l_{d-2})} f(w(s+l_{d-2})) \\
& \times \left( \frac{-1}{w} \right)^{l+r+k+(d-1)((s+1)-(d-1))} \frac{(l+r+k)(s+1)}{(s+l_{d-2})} f(w(s+l_{d-2})).
\end{align*}$$

Let us now return to the first sum denoted by $A$. We have, using $(k+s)! = (s+1)_{k} s!$, that

$$\begin{align*}
r!A & := \sum_{s,k,n_1,\ldots,n_{d-1} \geq 0} \frac{a^{n_1}_{1}}{n_1!} \cdots \frac{a_{d-1}^{n_{d-1}}}{n_{d-1}!} \left( \frac{1}{w} \right)^{l+r+k+s} \frac{(-1)^s (s+1)_{k} (l+r+k+1)}{(s+l+r)!} f(w(s+l+r)) \\
& = \sum_{s,k,n_1,\ldots,n_{d-1} \geq 0} \frac{a^{n_1}_{1}}{n_1!} \cdots \frac{a_{d-1}^{n_{d-1}}}{n_{d-1}!} \left( \frac{1}{w} \right)^{l+r+s} \frac{(-1)^s}{(s+1)!} f(w(s+l+r)) \\
& \times 2F1 \left( l+r+s, \quad l+r+s+1 \middle| \frac{\alpha}{w} \right),
\end{align*}$$

and

$$\begin{align*}
r!B_1 & = \sum_{s,k,n_1,\ldots,n_{d-1} \geq 0} \frac{a_{n_1 + s + 1}}{(n_1 + s)!} \cdots \frac{a_{d-1}^{n_{d-1} + s}}{a^{n_{d-1}}_{d-1}} \left( \frac{-1}{w} \right)^{l+r+s+1} \frac{(l+r+s+2)}{s!} f(w s) \\
& \times 2F1 \left( l+r+s+1, \quad l+r+s+2 \middle| \frac{\alpha}{w} \right).
\end{align*}$$

Now we use, to get the value of $B_i$, for $2 \leq i \leq d - 1$, the fact that

$$(dk+s)! = sl^{dk} \left( \frac{s+1}{d} \right)_k \left( \frac{s+2}{d} \right)_k \cdots \left( \frac{s+d}{d} \right)_k$$

and properties of factorial to obtain for $2 \leq i \leq d - 2$

$$\begin{align*}
r!B_i & = \sum_{s,k,n_1,\ldots,n_{d-1} \geq 0} \frac{a^{n_1}_{1}}{n_1!} \cdots \frac{a_{d-1}^{n_{d-1} + [s+1/i]}}{(n_1 + [s+1/i])!} \frac{a_{d-2}^{n_{d-2} + [s+1/i]}}{a^{n_{d-1} + [s+1/i]}_{d-1}} \frac{a^{n_{d-2} + [s+1/i]}_{d-2}}{(n_1 + [s+1/i])!} \frac{a^{n_{d-1} + [s+1/i]}_{d-1}}{(n_1 + [s+1/i])!} \\
& \times \left( \frac{-1}{w} \right)^{l+r+i(s+1/i)} f(w(s+l_{d-1})) \left( l+r+i(s+1/i)! \right) \\
& \times 2F1 \left( l+r+i(s+1/i), \quad l+r+i(s+1/i) \right)_k \left( l+i(s+1/i) - s \middle| \frac{\alpha}{w} \right).
\end{align*}$$
and

\[ r!B_{d-1} = \sum_{s,k,n_1,\ldots,n_{d-1} \geq 0} \frac{a_{n_1}^{n_1}}{n_1!} \cdots \frac{a_{n_{d-1}}^{n_{d-1}}}{(n_{d-1}+\sum_{s=1}^{r} i_{i_{d-1}} - 1)!} \frac{(l+r+(d-1)\frac{s+1-r}{d-1})(-1)^{r+i_{d-1}}}{(s+l_{i_{d-1}})!((s+l_{i_{d-1}}+r+(d-1))[(s+l_{i_{d-1}}+r+(d-1))-(s+1-r)/d-1])} f(w(s+l_{i_{d-1}})) \times \frac{2F_1}{(s,k,n_1,\ldots,n_{d-1})} \left( \begin{array}{c} l + r + (d-1) \frac{s+1-r}{d-1}, \ l + r + 1 + (d-1) \frac{s+1-r}{d-1} \end{array} \right) \left( \begin{array}{c} \alpha \end{array} \right) \right) .

We can simplify the expression of \( B_i \) using the properties of the integer part. Indeed, for \( 2 \leq i \leq d-2 \), we can write

\[ r!B_i = \sum_{p=0}^{i-2} \sum_{s,k,n_1,\ldots,n_{d-1} \geq 0} \frac{a_{n_1}^{n_1}}{n_1!} \cdots \frac{a_{n_{d-1}}^{n_{d-1}}}{(n_{d-1}+\sum_{s=1}^{r} i_{i_{d-1}} - 1)!} \frac{(l+r+is)(l+r+is+1)(l+r+is+2)\cdots(l+r+is+i)}{(l+i_{d-1}+r+is+p)} f(w(is+p+l_{i_{d-1}})) \times \frac{2F_1}{(l+r+is)} \left( \begin{array}{c} l + r + is, \ l + r + 1 + is \end{array} \right) \left( \begin{array}{c} \alpha \end{array} \right) \right) .

For \( B_{d-1} \), since \( 2 - d \leq p + 1 - r \leq d - 1 \), we have

\[ \left[ \frac{(d-1)s+p+1-r}{d-1} \right] = \begin{cases} s-1, & \text{for } 2 - d \leq p + 1 - r < 0, \\ s, & \text{for } 0 \leq p + 1 - r \leq d - 2, \\ s+1, & \text{for } p + 1 - r = d - 1, \end{cases} \]

then, for \( 2 - d \leq p + 1 - r < 0 \), we have

\[ r!B_{d-1} = \sum_{s,k,n_1,\ldots,n_{d-1} \geq 0} \frac{a_{n_1}^{n_1}}{n_1!} \cdots \frac{a_{n_{d-1}}^{n_{d-1}}}{(n_{d-1}+\sum_{s=1}^{r} i_{i_{d-1}} - 1)!} \frac{(l+r+(d-1)s+1)(-1)^{r+i_{d-1}}}{(s+l_{i_{d-1}})!((s+l_{i_{d-1}}+r+(d-1)s+1) - (s+1-r)/d-1))} f(w((d-1)s+p+l_{i_{d-1}})) \times \frac{2F_1}{((d-1)s+p+l_{i_{d-1}})!((l+i_{d-1}+r+(d-1)s+1)-p)} \left( \begin{array}{c} l + r + (d-1)(s-1), \ l + r + 1 + (d-1)(s-1) \end{array} \right) \left( \begin{array}{c} \alpha \end{array} \right) \right) .

\]
and for \(0 \leq p + 1 - r \leq d - 2\), we obtain

\[
\begin{align*}
  r! B_{d-1} &= \sum_{s,k,n_1,\ldots,n_{d-1}}^{\infty} \frac{a_1^{n_1}}{n_1!} \ldots \frac{a_{d-1}^{n_{d-1}+s}}{(n_{d-1}+s)!} \left( \frac{-1}{w} \right)^{(l+r+(d-1)s)} \\
  &\times f(w \left( (d-1)s + p + l_{i-1} \right)) \left( \frac{(l+r+(d-1)s)!}{((d-1)s+p+l_{i-1})!(l-l_{i-1}+r-p)!} \right) \\
  &\times 2F_1 \left( \begin{array}{c}
  l + r + (d-1)s, \quad l + r + 1 + (d-1)s \\
  l - l_{i-1} + r + 1 - p
  \end{array} \bigg| \frac{\alpha}{w} \right)
\end{align*}
\]

and, finally, for \(p + 1 - r = d - 1\), this means that \(r = 0\) and \(p = d - 2\), we get

\[
\begin{align*}
  B_{d-1} &= \sum_{s,k,n_1,\ldots,n_{d-1}}^{\infty} \frac{a_1^{n_1}}{n_1!} \ldots \frac{a_{d-1}^{n_{d-1}+s+1}}{(n_{d-1}+s+1)!} \left( \frac{-1}{w} \right)^{(l+(d-1)(s+1))} \\
  &\times f(w \left( (d-1)s + d - 2 + l_{i-1} \right)) \left( \frac{(l+(d-1)(s+1))!}{(s+l_{i-1})!(l-l_{i-1}+1)!} \right) \\
  &\times 2F_1 \left( \begin{array}{c}
  l + (d-1)(s+1), \quad l + 1 + (d-1)(s+1) \\
  l - l_{i-1} + 2
  \end{array} \bigg| \frac{\alpha}{w} \right).
\end{align*}
\]

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