ABSTRACT. We introduce to the context of multigraded modules the methods of modules and spaces over categories from algebraic topology and homotopy theory. The main application is to study the Betti poset $B = B(I, k)$ of a monomial ideal $I$ in the polynomial ring $R = k[x_1, \ldots, x_m]$ over a field $k$, which consists of all degrees in $\mathbb{Z}^m$ of the homogeneous basis elements of the free modules in the minimal free $\mathbb{Z}^m$-graded resolution of $I$ over $R$. We show that the order simplicial complex of $B$ supports a free resolution of $I$ over $R$. As consequences we give a formula for the Betti numbers of $I$ in terms of Betti numbers of certain open intervals of $B$, and we show that the isomorphism class of $B$ controls the structure of the minimal free resolution of $I$ in the following sense: if $J$ is any monomial ideal in a polynomial ring $S = k[y_1, \ldots, y_n]$ such that the Betti poset of $J$ is also isomorphic to $B$, then the minimal free resolution of $J$ over $S$ is obtained by applying a certain canonical functor to the minimal free resolution of $I$. This generalizes and provides a new proof of the main results of Gasharov, Peeva, and Welker [GPW99].

INTRODUCTION

Monomial ideals are fundamental objects that provide a gateway for interaction between commutative algebra, combinatorics, symbolic computation, algebraic geometry, and algebraic topology. A main outstanding problem, dating back to the 1960’s, and an area of very active current research, is to give an explicit description in closed form of the minimal free resolution of a monomial ideal. A substantial progress was made by Gasharov, Peeva, and Welker, who introduced in [GPW99] the lcm-lattice $\mathcal{L}$ of a monomial ideal $I$ in a polynomial ring $R = k[x_1, \ldots, x_m]$ and showed that $\mathcal{L}$ is a combinatorial object that encodes the structure of the minimal free resolution of $I$ in the following precise sense: if two monomial ideals have isomorphic lcm-lattices then the minimal free resolution of one over any given field can be obtained from the minimal free resolution of the other by a process called “relabeling”. In addition, [GPW99] gave a formula for the Betti numbers of a monomial ideal as the Betti numbers of certain open intervals in the lcm-lattice.

In this paper we adapt to the setting of monomial ideals and, more generally, $\mathbb{Z}^m$-graded $R$-modules the language and ideas of modules and spaces over categories, which are familiar tools in algebraic topology and homotopy theory. As an application of our newly acquired methods, we show that if one fixes the base field $k$, then the results of [GPW99] can be substantially sharpened. We consider the Betti poset $B$ of the monomial ideal $I$ (over $k$), which consists of the $\mathbb{Z}^m$-degrees of the basis elements of the free modules in the minimal free $\mathbb{Z}^m$-graded resolution of $I$ over $R$. In our main result, Theorem 7.1 we show that the order simplicial complex $\Delta(B)$ of the Betti poset supports a free resolution of $I$ over $R$. We should
note that, since $L$ is a lattice, it is not hard to see that its order complex $\Delta(L)$ supports a free resolution of $I$. However, proving the same for $\Delta(B)$ becomes a complicated matter. As a first consequence of our main theorem, we show in Theorem 7.2 and Theorem 7.4 that the lcm-lattice in the results from [GPW99] can be replaced by the Betti poset of the monomial ideal. In particular, when the field $k$ is fixed the isomorphism class of the Betti poset encodes the structure of the minimal free resolution of a significantly wider class of ideals. We include a simple example of two ideals that do not have isomorphic lcm-lattices, but have isomorphic Betti posets over every field $k$. As a second consequence, our results also allow us to provide in Theorem 8.4 a necessary and sufficient condition for a finite poset to be the Betti poset of some monomial ideal.

The technique we use in our proofs is new in this setting and we believe that it is of independent interest. As in [Mil00, Car09, CZ09], we adopt the point of view that a monomial ideal, and more generally a $Z^m$-graded $R$-module $M$, is a collection of $k$-vector spaces $M_\gamma$ indexed by the elements of $Z^m$, together with $k$-linear maps $x^\alpha : M_\gamma \to M_{\gamma+\alpha}$ for all $\alpha \in N^m$ and $\gamma \in Z^m$ that satisfy $x^\alpha x^\beta = x^{\alpha+\beta}$ and $x^0 = \text{id}$. Using more formal language, we view $M$ as a functor from the small category of the poset $Z^m$ to the category $k$-mod of $k$-vector spaces. In particular if $P$ is any $Z^m$-graded poset, i.e., we are given a morphism of posets $\text{gr} : P \to Z^m$, then composition yields a functor $M^P$, that we call the $P$-sample of $M$, from the small category of $P$ to $k$-mod. The usefulness of this approach comes from the fact that the collection $kP$-mod of all functors from the category of $P$ to $k$-mod is an abelian category, whose objects are called $kP$-modules, and therefore we can do homological algebra there. Such categories of functors from small categories to categories of modules have been extensively studied in the context of algebraic topology. In particular they have proven to be invaluable tools in studying equivariant phenomena, i.e., studying spaces with group actions. There one typically considers functors defined on the so-called orbit category of the acting group, see for example [tD87, Section I.11, Lück89, Section 9, DL98, LRV03].

The “sampling” process of passing from a $Z^m$-graded module $M$ to its $P$-sample $M^P$ is an exact functor that remembers only those homogeneous components of $M$ that are indexed by elements of $Z^m$ coming from $P$ and then replaces these indices with the corresponding elements of $P$. Applying sampling to a free $Z^m$-graded resolution allows us to avoid the combinatorial complications that arise when using standard dehomogenization methods such as the one in [PV11], where choosing a basis is a necessary but highly undesirable (for us) step. The natural way to recover some or all of the information that is lost during sampling is through an appropriate notion of tensor product. Using this idea we provide in Section 5 a functorial “homogenization” procedure for producing (possibly new) homogeneous components and $Z^m$ indexes for them. It is a substantial functorial generalization of the standard homogenization techniques used to turn a complex of $k$-vector spaces into a complex of free $Z^m$-graded $R$-modules, see e.g. [BPS98, BS98, BW02, CT03, Tch07, PV11]. The interplay between sampling and homogenization is a main key to the proof of our results, and we investigate it in Section 6. As one should expect, composing the two operations can be used to give a functorial generalization of the “relabeling” processes from [GPW99] and [PV11]. Furthermore, these functors allow us to study closely the relationship between the homological properties of $M$ and its $P$-sample $M^P$. To put this relationship to work
we also need a reasonable understanding of the homological algebra of the category \( \mathcal{P}\text{-mod} \). This is the second main key to our proofs, and we accumulate the needed results in Sections 4 and 5 where we study projective and free \( \mathcal{P}\text{-modules} \) and a functorial free resolution of the constant \( \mathcal{P}\text{-module} \) with value \( k \).

We would like to thank Amanda Beecher and Timothy Clark for useful conversations related to the material in Section 8.

1. Preliminaries

Throughout this paper \( k \) is a commutative, associative, unital ring, modules are unitary, unadorned tensor products are over \( k \), and unless otherwise specified all functors are understood to be covariant. If \( W \) is any set, we write \( k[W] \) for the free \( k\text{-module} \) with basis the set \( W \). We denote by \( k\text{-mod} \) the category of \( k\text{-modules} \).

Given an additive category \( \mathcal{A} \) (like for example \( k\text{-mod} \)) we denote by \( \text{ch}(\mathcal{A}) \) the additive category of chain complexes in \( \mathcal{A} \) and chain homomorphisms. For us chain complexes are always understood to be indexed over the integers \( \mathbb{Z} \), and the differentials in a complex \( X \) decrease degree. Given any poset \( P \) we view it as a small category whose objects are exactly the elements of \( P \) and in which there is exactly one morphism from \( p \) to \( q \) if and only if \( p \leq q \), and none otherwise; when \( p \leq q \) we abuse notation and denote the unique morphism from \( p \) to \( q \) by \( p \leq q \).

Notice that order-preserving functions between posets, i.e., morphisms of posets, correspond exactly to functors between the associated categories. We always use the same notation for a poset and its corresponding category. For each \( a \in P \) we write \( P_{\leq a} \) for the filter \( \{ x \in P \mid x \leq a \} \) in \( P \). The filter \( P_{\leq a} \) is defined analogously. We denote by \( \mathbb{N} \) the set of all non-negative integers, and for every \( n \in \mathbb{N} \) we write \([n]\) for the totally ordered set \( \{ 0 < 1 < \cdots < n \} \). We consider \( \mathbb{N}^m \) and \( \mathbb{Z}^m \) as posets via the coordinatewise partial order: \((a_1, \ldots, a_m) \leq (b_1, \ldots, b_m)\) if and only if \( a_i \leq b_i \) for all \( i \). In particular, both posets are lattices with joins given by taking componentwise maximums. A \( \mathbb{Z}^m\text{-graded poset} \) is a poset \( P \) together with a morphism of posets \( \text{gr}: P \to \mathbb{Z}^m \) called the grading morphism. When \( P \) is a subposet of \( \mathbb{Z}^m \) then we always consider it as \( \mathbb{Z}^m\text{-graded} \) with grading morphism the inclusion map.

Now let \( R = k[x_1, \ldots, x_m] \) be a polynomial ring over \( k \) in the \( m \) variables \( x_1, \ldots, x_m \). We consider the ring \( R \) with the canonical \( \mathbb{Z}^m\text{-grading} \), called multigrading. For each \( \alpha = (a_1, \ldots, a_m) \in \mathbb{N}^m \) we write \( x^\alpha \) for the monomial \( x_1^{a_1} \cdots x_m^{a_m} \).

Thus we have \( R = \bigoplus_{\alpha \in \mathbb{Z}^m} R_{\alpha} \), where

\[
R_{\alpha} = \begin{cases} 
  kx^\alpha & \text{if } \alpha \in \mathbb{N}^m; \\
  0 & \text{otherwise}.
\end{cases}
\]

Let \( M = \bigoplus_{\alpha \in \mathbb{Z}^m} M_{\alpha} \) be a multigraded \( R\text{-module} \). We denote by \( \text{mgR-mod} \) the category of multigraded \( R\text{-modules} \) (also called monomial graded \( R\text{-modules} \)) and homogeneous \( R\text{-linear homomorphisms} \) of degree 0. We set

\[
\deg(M) = \{ \alpha \in \mathbb{Z}^m \mid M_{\alpha} \neq 0 \}.
\]

If \( \gamma \in \mathbb{Z}^m \) we write \( M(\gamma) \) for the corresponding degree-shifted \( R\text{-module} \), i.e., \( M(\gamma)_\alpha = M_{\alpha+\gamma} \). In particular, \( R(-\gamma) \) stands for the free multigraded \( R\text{-module} \) of rank one generated by a single free generator \( e = 1_R \in R(-\gamma)_{\gamma} \) of (multi)degree \( \gamma \).

A free multigraded \( R\text{-module} \) is then just a direct sum \( \bigoplus_{\alpha \in \mathbb{Z}^m} R(-\alpha)^{b_{\alpha}} \). Let

\[
F_* = 0 \leftarrow F_0 \leftarrow F_1 \leftarrow \cdots \leftarrow F_d \leftarrow \cdots
\]
be a free multigraded chain complex over $R$, i.e., the free modules $F_k$ are free multigraded and the differentials of $F_\bullet$ are morphisms of multigraded modules. For each $k$ let $B_k$ be a homogeneous basis of $F_k$, and for $\alpha \in \mathbb{Z}^m$ write $B_{k,\alpha}$ for the set of basis elements in $B_k$ of multidegree $\alpha$. Write $F_{\bullet, \alpha}$ for the multigraded strand
\[
F_{\bullet, \alpha} = \quad 0 \leftarrow (F_0)_{\alpha} \leftarrow (F_1)_{\alpha} \leftarrow \cdots \leftarrow (F_d)_{\alpha} \leftarrow \cdots
\]
of $F_\bullet$ in degree $\alpha$. It is straightforward to notice that each $k$-module $(F_d)_{\alpha}$ is free over $k$ with basis the set $\prod_{\gamma \leq \alpha}(x^{\alpha - \gamma} b \mid b \in B_{d, \gamma})$. Since the differentials in $F_\bullet$ preserve multidegrees the chain complex $F_\bullet$ decomposes into a direct sum of strands $F_\bullet = \bigoplus_{\alpha \in \mathbb{Z}^m} F_{\bullet, \alpha}$, and is a free resolution of $M$ if and only if each strand $F_{\bullet, \alpha}$ is a free resolution of $M_\alpha$ over $k$. Finally, for any $\alpha \in \mathbb{Z}^m$ and $\beta \in \mathbb{N}^m$ we have a canonical injective morphism of chain complexes
\[
x^\beta : F_{\bullet, \alpha} \rightarrow F_{\bullet, \alpha + \beta}
\]
via multiplication by the monomial $x^\beta$.

Our main case of interest will be when $k$ is a field, $M = I$ is a monomial ideal in the ring $R$, and $F_\bullet$ is the minimal free resolution (or MFR for short) of $I$ over $R$. In that case the integer $\beta_{d, \alpha} = \beta_{d, \alpha}(I, k) = |B_{d, \alpha}|$ is called the $d$th Betti number of $I$ in multidegree $\alpha$ over $k$ and
\[
\beta_{d, \alpha} = \dim_k (F_{\bullet, \alpha} / F_{\bullet, \alpha} \cap mF_\bullet)_\alpha = \dim_k \Tor^R_d (I, k)_\alpha
\]
where as usual $m = (x_1, \ldots, x_m)$ is the maximal ideal generated by the variables in the polynomial ring $R$. An important combinatorial object associated with $I$ is the lcm-lattice $L = L(I)$, which is the subposet of $\mathbb{Z}^m$ join-generated in $\mathbb{N}^m$ by the multidegrees of the minimal generators of $I$. Our main object of study in this paper is the following smaller poset.

Definition 1.1. Let $k$ be a field and let $I$ be a monomial ideal in $R$. The set
\[
B(I, k) = \{ \alpha \in \mathbb{Z}^m \mid \beta_{d, \alpha}(I, k) \neq 0 \text{ for some } d \}
\]
is called the set of Betti degrees of $I$ over $k$. We consider it as a poset, and call it then the Betti poset of $I$ over $k$, with respect to the partial ordering induced by the partial ordering on $\mathbb{Z}^m$.

It is well known that the Betti poset is a suposet of the lcm-lattice. We would like to emphasize that as a consequence of the definition the Betti poset of $I$ does not in general contain a smallest element. Its minimal elements are exactly the multidegrees of the minimal generators of $I$.

The significance of posets other than the lcm-lattice for the study of free resolutions of monomial ideals has recently become more apparent, and Betti posets have accordingly received more attention, see e.g. [CM11, CM13]. We will show in our main results that for the purposes of describing the minimal free resolution of $I$ over a fixed field $k$, the corresponding Betti poset encodes all the required information.

2. Modules over a category

In this section we introduce the fundamental concept of modules over a category and the sampling process, and we describe several examples. We denote by $\mathcal{J}$ a small category, e.g., the category associated with a poset.
Definition 2.1. A $k\mathcal{J}$-module is a functor $M : \mathcal{J} \to k\text{-mod}$. A homomorphism of $k\mathcal{J}$-modules is a natural transformation. So the category of $k\mathcal{J}$-modules, denoted $k\mathcal{J}\text{-mod}$, is just the category of functors from $\mathcal{J}$ to $k\text{-mod}$; in symbols, $k\mathcal{J}\text{-mod} = \text{fun}(\mathcal{J}, k\text{-mod})$.

Example 2.2. Here are some trivial examples. If $\mathcal{J} = \{0\}$ is the category with exactly one object and one (identity) morphism, then obviously $k\{0\}\text{-mod} = k\text{-mod}$. If $G$ is a group and $\underline{G}$ is the category with only one object and one (invertible) morphism for every element of $G$, with composition defined by multiplication in $G$, then $k\underline{G}\text{-mod}$ is the category of left modules over the group ring $k[G]$ and $k\underline{G}\text{-mod}^\text{op}$ is the category of right $k[G]$-modules. This explains the notation and terminology.

Notice that $k\mathcal{J}\text{-mod}$ is an abelian category. Kernels and images are computed objectwise. A sequence of $k\mathcal{J}$-modules $L \to M \to N$ is exact if and only if $L(j) \to M(j) \to N(j)$ is exact for every $j \in \text{obj} \mathcal{J}$. In particular it makes sense to speak of projective $k\mathcal{J}$-modules, for example, and to consider chain complexes of $k\mathcal{J}$-modules. Notice that a chain complex of $k\mathcal{J}$-modules can equivalently be thought of as functor from $\mathcal{J}$ to $\text{ch}(k\text{-mod})$; in symbols, $\text{ch}(k\mathcal{J}\text{-mod}) = \text{fun}(\mathcal{J}, \text{ch}(k\text{-mod}))$.

Definition 2.3. Let $\mathcal{P}$ be a $\mathbb{Z}^m$-graded poset with grading map $\text{gr} : \mathcal{P} \to \mathbb{Z}^m$. Let $M$ be a multigraded $R$-module.

(a) The $\mathcal{P}$-sample of $M$ is the $k\mathcal{P}$-module $M^\mathcal{P}$ given by

$$M^\mathcal{P}(a) = M_{\text{gr}(a)} \quad \text{and} \quad M^\mathcal{P}(a \leq b) = x^{\text{gr}(b) - \text{gr}(a)} : M_{\text{gr}(a)} \to M_{\text{gr}(b)}.$$

(b) We refer to a functor of the form $(-)^\mathcal{P}$ as a sampling (functor). Clearly a sampling is an exact functor

$$(-)^\mathcal{P} : \text{mod}R \to k\mathcal{P}\text{-mod}$$

from the abelian category of multigraded $R$-modules to $k\mathcal{P}\text{-mod}$.

We specialize to the case of greatest interest to us.

Example 2.4. Let $k$ be a field, and let $\mathcal{B}$ be the Betti poset over $k$ of a monomial ideal $I$. The minimal free resolution $F_\bullet$ of $I$ over $R$ yields a $k\mathcal{B}$-chain complex $F_\bullet^\mathcal{B}$ given by

$$F_\bullet^\mathcal{B}(a) = F_\bullet(a) \quad \text{and} \quad F_\bullet^\mathcal{B}(a \leq \beta) = x^{\beta - a} : F_\bullet(a) \to F_\bullet(\beta)$$

and for each $n \geq 0$ a $k\mathcal{B}$-module $F_n^\mathcal{B}$ given by

$$F_n^\mathcal{B}(a) = (F_n)_a \quad \text{and} \quad F_n^\mathcal{B}(a \leq \beta) = x^{\beta - a} : (F_n)_a \to (F_n)_\beta.$$

Similarly, the ideal $I$ induces a $k\mathcal{B}$-module $I^\mathcal{B}$, given by

$$I^\mathcal{B}(\alpha) = I_\alpha = kx^\alpha \quad \text{and} \quad I^\mathcal{B}(\alpha \leq \beta) = x^{\beta - \alpha} : I_\alpha \to I_\beta.$$

Remark 2.5. With $I$ as in the previous example, and $\mathcal{P}$ a subposet of $\mathbb{Z}^m$ such that $\mathcal{P} \subseteq \text{deg}(I)$, then the $k\mathcal{P}$-module $I^\mathcal{P}$ is in fact isomorphic to the constant $k\mathcal{P}$-module with value $k$. Indeed, the isomorphism $\iota$ is given objectwise for each $\alpha$ by the isomorphisms $\iota(\alpha) : k \to I_\alpha$ via the obvious formula $c \mapsto cx^\alpha$.

Next we show how standard topological and combinatorial constructions used in the literature to study free resolutions of multigraded $R$-modules can be interpreted as giving rise to $k\mathcal{P}$-chain complexes.
Example 2.6. (Simplicial chain complexes) Let $\Delta$ be a simplicial complex, and let $\mathcal{P} = \mathcal{P}(\Delta)$ be its face poset. For each face $H$ let $\Delta_H$ be the subcomplex formed by taking all faces contained in $H$. Clearly $H_1 \subseteq H_2$ if and only if $\Delta_{H_1} \subseteq \Delta_{H_2}$. By taking simplicial chain complexes with coefficients in $k$ we obtain a $k\mathcal{P}(\Delta)$-chain complex $S_* = S_*(\Delta, k)$ given by

$$S_*(H) = C_*(\Delta_H; k),$$

where for $H_1 \leq H_2$ the morphism

$$S_*(H_1 \leq H_2): C_*(\Delta_{H_1}; k) \longrightarrow C_*(\Delta_{H_2}; k)$$

is the morphism of simplicial chain complexes induced by the inclusion $\Delta_{H_1} \subseteq \Delta_{H_2}$.

Example 2.7. (Frames) Let $U_*= (U_k, \partial_k)$ be a chain complex of based free $k$-modules; in particular this includes the case of a frame as defined in [PV11] and therefore also covers the simplicial chain complex case from Example 2.6 and the case of a cellular chain complex of a CW-complex. Let $B_k$ be the fixed basis of $U_k$ and let $B = \prod_k B_k$. Let $\mathcal{P}$ be any poset structure on $B$ such that if $b \in B_k$ and $\partial_k(b) = \sum_{c \in B_{k-1}, a_c} a_c$ with $a_c \neq 0$ then $b > c$ in $\mathcal{P}$. For any $b \in \mathcal{P}$ we write $U_k(b)$ for the free submodule of $U_k$ with basis the set

$$B_k(b) = \{ c \in B_k | c \leq b \}.$$

Then clearly $U_*(b) = (U_k(b), \partial_k)$ is a subcomplex of $U_*$. Therefore we obtain a $k\mathcal{P}$-chain complex $F_* = F_*(U_*)$ given by

$$F_*(b) = U_*(b),$$

where for $c \leq b$ in $\mathcal{P}$ the morphism

$$F_*(c \leq b): U_*(c) \longrightarrow U_*(b)$$

is just the inclusion $U_*(c) \subseteq U_*(b)$. We will revisit in much more detail the case of a cellular chain complex of a CW-complex in Section 5.

The previous two examples deal with cases where one must have a basis chosen for the free modules involved. The next example is of a slightly different nature.

Example 2.8. (T-complexes) Given a representation $\phi$ of a matroid $M$ over a field $k$, in [Tch07] a canonical vector space $T_\mathcal{T}$ is constructed for each element $A$ in the lattice $\mathcal{T}$ of T-flats of $M$. These are then assembled into a canonical chain complex $T_*(\phi)$, and it is shown that for $A \leq B$ in $\mathcal{T}$ one has a canonical inclusion map of chain complexes

$$T_*(\phi|A) \longrightarrow T_*(\phi|B)$$

where $\phi|X$ denotes the representation induced by $\phi$ of the restriction $M|X$ of $M$ to $X$. It is clear that this produces naturally a $k\mathcal{T}$-chain complex $T_*$ given by $T_*(A) = T_*(\phi|A)$ where for $A \leq B$ in $\mathcal{T}$ the morphism $T_*(A \leq B)$ is just the morphism (2.9).

The examples given so far are by no means exhaustive. For instance Tchernev’s poset construction [Cla10] can also be interpreted in the framework of chain complexes over a poset. We conclude this section with a very important for us example.
Example 2.10. Let $\mathcal{P}$ be a poset, and let $\Delta(\mathcal{P})$ be the so-called order simplicial complex of $\mathcal{P}$. The $n$-faces of $\Delta(\mathcal{P})$ are all strictly increasing chains

$$A = \{a_0 < \cdots < a_n\}$$

in $\mathcal{P}$. We will call the maximal element $a_n = \max A$ of such a chain the apex of the face $A$. Now we obtain a natural $\mathbb{k}\mathcal{P}$-chain complex $E_\bullet = E_\bullet(\mathcal{P}, \mathbb{k})$ by taking simplicial chain complexes as follows:

$$E_\bullet(a) = C_\bullet(\Delta(\mathcal{P}_{\leq a}); \mathbb{k})$$

where $\Delta(\mathcal{P}_{\leq a})$ is the subcomplex of $\Delta(\mathcal{P})$ with faces those chains $A$ such that $\max A \leq a$. Whenever $a \leq b$ in $\mathcal{P}$, the morphism

$$E_\bullet(a \leq b) : C_\bullet(\Delta(\mathcal{P}_{\leq a}); \mathbb{k}) \rightarrow C_\bullet(\Delta(\mathcal{P}_{\leq b}); \mathbb{k})$$

is defined to be the morphism of simplicial chain complexes induced by the inclusion $\mathcal{P}_{\leq a} \subseteq \mathcal{P}_{\leq b}$. By taking simplicial $n$-chains, we also obtain for each $n$ a $\mathbb{k}\mathcal{P}$-module $E_n = E_n(\mathcal{P}, \mathbb{k})$ with

$$E_n(a) = C_n(\Delta(\mathcal{P}_{\leq a}); \mathbb{k}).$$

Our main goal for the next three sections will be to show that this produces a canonical projective resolution of the constant $\mathbb{k}\mathcal{P}$-module with value $\mathbb{k}$. This fact is an important ingredient in the proofs of our main results.

3. Tensor Products

One should think of the sampling functors $(-)^\mathcal{P}$ and $(-)^\mathcal{B}$ from the previous section as sophisticated dehomogenization tools. Applying them forgets about those homogeneous components of our multigraded modules whose degrees are not coming from $\mathcal{P}$ and $\mathcal{B}$, respectively. Furthermore, they have the effect of stripping from the remaining homogeneous components their actual multidegrees. The canonical way to recover some or all of that lost information is through an appropriate notion of a tensor product.

Definition 3.1. Let $N$ be a $\mathbb{k}\mathcal{J}^{\text{op}}$-module and $M$ be a $\mathbb{k}\mathcal{J}$-module. Define their tensor product over $\mathbb{k}\mathcal{J}$ to be the $\mathbb{k}$-module

$$N \otimes_{\mathbb{k}\mathcal{J}} M = \left( \bigoplus_{j \in \text{obj } \mathcal{J}} N(j) \otimes M(j) \right) / S$$

where $S$ is the submodule generated by

$$\{ nu \otimes m - n \otimes um \mid m \in M(j), n \in N(i), u \in \text{mor}_\mathcal{J}(j, i) \}$$

and we abbreviate $um = M(u)(m)$ and $nu = N(u)(n)$. It is a straightforward consequence of the definition that the functor $N \otimes_{\mathbb{k}\mathcal{J}} (-)$ preserves epimorphisms and direct sums.

Example 3.2. Let $G$ be a group, $M$ a left $\mathbb{k}[G]$-module, and $N$ a right $\mathbb{k}[G]$-module. As explained in Example 2.22 we can think of $N$ as an $\mathbb{k}[G]^{\text{op}}$-module and of $M$ as an $\mathbb{k}[G]$-module. Then $N \otimes_{\mathbb{k}[G]} M = N \otimes_{\mathbb{k}[G]} M$.

Now we explain that for any $\mathbb{k}\mathcal{J}^{\text{op}}$-module $N$ the functor $N \otimes_{\mathbb{k}\mathcal{J}} (-)$ is right exact. The proof of this proceeds as in the classical case when $\mathcal{J} = [0]$, by adjointness.
Definition 3.3. Let $N$ be a $k\mathcal{J}^{\text{op}}$-module and $T$ a $k$-module. Define the $k\mathcal{J}$-module $\text{hom}_k(N, T)$ by sending $j$ to $\text{hom}_k(N(j), T)$. Notice that $\text{hom}_k(N, T)$ is covariant in $\mathcal{J}$ since $N$ is contravariant.

From the definitions and the usual tensor-hom adjunction for modules one sees that the functor

$$\text{hom}_k(N, -): k\text{-mod} \rightarrow k\mathcal{J}\text{-mod}$$

is right adjoint to

$$N \otimes_{k\mathcal{J}} (-): k\mathcal{J}\text{-mod} \rightarrow k\text{-mod},$$

i.e., for all $k\mathcal{J}$-modules $M$ and all $k$-modules $T$ there are natural isomorphisms

$$\text{hom}_k(N \otimes_{k\mathcal{J}} M, T) \cong \text{hom}_{k\mathcal{J}}(M, \text{hom}_k(N, T)).$$

From this one concludes as usual (see for example [Wei94, Theorem 2.6.1 on page 51]) that $N \otimes_{k\mathcal{J}} (-)$ is right exact, and $\text{hom}_k(N, -)$ is left exact.

Notice that we can repeat everything done so far in this section when $N$ is a multigraded $R\mathcal{J}^{\text{op}}$-module, i.e., a functor $N: \mathcal{J}^{\text{op}} \rightarrow \text{mgR-mod}$. If $N$ is multigraded $R\mathcal{J}^{\text{op}}$-module and $M$ is a $k\mathcal{J}$-module, then their tensor product $N \otimes_{k\mathcal{J}} M$ is a multigraded $R$-module. Moreover, if $T$ is a multigraded $R$-module, then $\text{hom}_R(N, T)$ is a $k\mathcal{J}$-module, where $\text{hom}_R$ denotes homogeneous $R$-linear homomorphisms of degree 0. Then the functor

$$\text{hom}_R(N, -): \text{mgR-mod} \rightarrow k\mathcal{J}\text{-mod}$$

is right adjoint to

$$N \otimes_{k\mathcal{J}} (-): k\mathcal{J}\text{-mod} \rightarrow \text{mgR-mod},$$

i.e., for all $k\mathcal{J}$-modules $M$ and all multigraded $R$-modules $T$ there are natural isomorphisms

$$\text{hom}_R(N \otimes_{k\mathcal{J}} M, T) \cong \text{hom}_{k\mathcal{J}}(M, \text{hom}_R(N, T)).$$

The fundamental example for us of a multigraded $R\mathcal{J}^{\text{op}}$-module is defined next.

Definition 3.4. Let $\mathcal{P}$ be a $\mathbb{Z}^m$-graded poset with grading map $\text{gr}: \mathcal{P} \rightarrow \mathbb{Z}^m$. It induces in a natural way a $k\mathcal{P}^{\text{op}}$-module $\mathcal{R}_{\text{gr}}$ via the degree-shift operation by setting

$$\mathcal{R}_{\text{gr}}(a) = R(-\text{gr}(a)) \quad \text{and} \quad \mathcal{R}_{\text{gr}}(a \geq b) = x^{\text{gr}(a)-\text{gr}(b)}: R(-\text{gr}(a)) \rightarrow R(-\text{gr}(b))$$

for any $a \geq b$ in $\mathcal{P}$. We call $\mathcal{R}_{\text{gr}}$ the multidegree shift functor. Notice that the values of $\mathcal{R}_{\text{gr}}$ are not just $k$-modules but also free multigraded $R$-modules, and $\mathcal{R}_{\text{gr}}$ sends the morphisms of $\mathcal{P}^{\text{op}}$ to morphisms of multigraded $R$-modules; i.e., $\mathcal{R}_{\text{gr}}$ is a multigraded $R\mathcal{P}^{\text{op}}$-module. When $\mathcal{P}$ is a subposet of $\mathbb{Z}^m$ and gr is the inclusion map, we omit the subscript gr from the notation, and write just $\mathcal{R}$.

When $\mathcal{P}$ is a $\mathbb{Z}^m$-graded poset the sampling process introduced in the previous section is a special case of the above described hom-construction. Indeed, for every multigraded $R$-module $T$, we have that

$$\text{hom}_R(\mathcal{R}_{\text{gr}}, T) = T^\mathcal{P}.$$

Hence the sampling functor

$$(-)^\mathcal{P}: \text{mgR-mod} \rightarrow k\mathcal{P}\text{-mod}$$
is right adjoint to
\[ R_{\text{gr}} \otimes (-): kP\text{-mod} \to mgR\text{-mod}. \]
This latter construction is the main tool that we will use to recover information lost during sampling.

**Definition 3.5.** Let \( P \) be a \( \mathbb{Z}^n \)-graded poset. Let \( M \) be a \( kP \)-module. The multigraded \( R \)-module
\[ R_{\text{gr}} \otimes M \]
is called the \( P \)-homogenization of \( M \).

The process of \( P \)-homogenization is a functorial generalization of the standard homogenization technique used to convert a chain complex of free \( k \)-modules into a complex of free multigraded \( R \)-modules. We briefly demonstrate how this works in the standard example of the Taylor resolution.

**Example 3.6.** Let \( I \) be a monomial ideal in \( R \) with a set of minimal generators \( \{ x^{a_0}, \ldots, x^{a_n} \} \). Let \( \Delta \) be the full simplex on the set \( [n] \), let \( P \) be the its face poset, and let \( S_* \) be the corresponding \( kP \)-chain complex from Example 2.6. Let \( \text{gr}: P \to \mathbb{Z}^m \) be the grading morphism given by the formula \( \text{gr}(\{i_1, \ldots, i_k\}) = \alpha_{i_1} \lor \cdots \lor \alpha_{i_k} \), where \( \lor \) denotes join in the lattice \( N^m \). Then it is clear that \( \text{gr} \) is a morphism of posets and unravelling the definitions shows that the corresponding \( P \)-homogenization \( T_* = R_{\text{gr}} \otimes_{kP} S_* \) is exactly the Taylor resolution of the ideal \( I \).

Composing an appropriate sampling with an appropriate homogenization yields a functorial description of the “relabeling” procedures from [GPW99] and [PV11]. Here is an example of this in the case of the lcm-lattice relabeling. One can clearly produce in a similar manner the corresponding examples in the more general setting of frames considered in [PV11].

**Example 3.7.** Let \( I, R, \Delta, P, \text{gr}, S_* \), and \( T_* \) be as in Example 3.6. Notice that the image of \( \text{gr} \) is precisely the lcm-lattice \( \mathcal{L} \) of \( I \). Suppose \( f: \mathcal{L} \to \mathbb{Z}' \) is a map of posets. Let \( \text{gr}' = f \circ \text{gr} \), let \( S = k[y_1, \ldots, y_t] \) be a polynomial ring with the standard \( \mathbb{Z}' \)-grading, and let \( \mathcal{J}_{\text{gr}'} \) and \( \mathcal{J}_f \) be the corresponding multidegree-shift functors. Thus \( \mathcal{J}_f \) is a multigraded \( S\mathcal{L}^{op} \)-module, and \( \mathcal{J}_{\text{gr}'} \) is a multigraded \( S\mathcal{P}^{op} \)-module. Then the homogenization \( T_*' = \mathcal{J}_{\text{gr}'} \otimes_{k\mathcal{P}} S_* \) is exactly the complex of free multigraded \( S \)-modules obtained from the Taylor resolution of \( f \) by applying the relabeling procedure from [GPW99] using the relabeling map \( f \). Furthermore, one can check directly from the definitions, or use Proposition 3.8 below, that also \( T_*' = \mathcal{J}_f \otimes_{k\mathcal{L}} (T_*)^\mathcal{L} \).

The adjunction between homogenization and sampling yields for each \( kP \)-module \( M \) a natural homomorphism of \( kP \)-modules
\[ \eta: M \to \left( R_{\text{gr}} \otimes_{kP} M \right)^P, \]
the unit of the adjunction, and for every multigraded \( R \)-module \( T \) a natural degree 0 homogeneous homomorphism
\[ \epsilon: R_{\text{gr}} \otimes_{kP} (T^P) \to T, \]
the counit of the adjunction.
Proposition 3.8. Let $\mathcal{P}$ be a subposet of $\mathbb{Z}^m$, let $M$ be a $k\mathcal{P}$-module, and let $F$ be a free multigraded $R$-module such that $\mathcal{P}$ contains the degrees of the elements of some (hence every) homogeneous basis of $F$. Then:

1. The unit of adjunction $\eta: M \to (\mathcal{R} \otimes_{k\mathcal{P}} M)^{\mathcal{P}}$ is an isomorphism of $k\mathcal{P}$-modules.

2. The counit of adjunction $\epsilon: \mathcal{R} \otimes_{k\mathcal{P}} (F^{\mathcal{P}}) \to F$ is an isomorphism of multigraded $R$-modules.

Proof. Since $X = \mathcal{R} \otimes_{k\mathcal{P}} M$ is just the quotient of the multigraded $R$-module $G = \bigoplus_{\alpha \in \mathcal{P}} R(-\alpha) \otimes M(\alpha)$ by the multigraded submodule $H$ spanned over $R$ by all elements of the form $x^{\beta-\alpha} \otimes m - 1 \otimes M(\alpha \leq \beta)(m)$ with $m \in M(\alpha)$ and $\alpha \leq \beta$ in $\mathcal{P}$, it is straightforward to see that for each $\gamma \in \mathcal{P}$ we get

$$X_\gamma = G_\gamma / H_\gamma$$

$$= \left( \bigoplus_{\alpha \leq \gamma} kx^{\gamma-\alpha} \otimes M(\alpha) \right) / k(x^{\gamma-\alpha} \otimes m - 1 \otimes M(\alpha \leq \gamma)(m) \mid m \in M(\alpha)).$$

Since in $X$ we have $x^{\gamma-\beta}(1 \otimes m) = x^{\gamma-\beta} \otimes m = 1 \otimes M(\beta \leq \gamma)(m)$ whenever $\beta \leq \gamma$ and $m \in M(\beta)$, the isomorphism (1) is immediate.

Since tensoring by $\mathcal{R}$ and applying $(-)^{\mathcal{P}}$ both preserve direct sums, and since $\mathcal{P}$ contains the degrees of the elements of any homogenous basis of $F$, to prove (2) we just need to show that if $\alpha \in \mathcal{P}$ then

$$\epsilon: \mathcal{R} \otimes_{k\mathcal{P}} (R(-\alpha)^{\mathcal{P}}) \to R(-\alpha)$$

is an isomorphism. However, it is immediate from the definition that the source of $\epsilon$ is the quotient of the multigraded $R$-module $Y = \bigoplus_{\alpha \leq \gamma} R(-\gamma) \otimes_k kx^{\gamma-\alpha}$ by the multigraded $R$-submodule $Z$ generated over $R$ by all elements of the form $x^{\gamma-\beta} \otimes x^{\beta-\alpha} - 1 \otimes x^{\gamma-\alpha}$ with $\alpha \leq \beta \leq \gamma$ in $\mathcal{P}$, hence $Y/Z = R(-\alpha) \otimes k$. □

We should note that both (1) and (2) in the proposition above will fail for some $M$ and some $F$ if the grading morphism $\text{gr}: \mathcal{P} \to \mathbb{Z}^m$ does not map $\mathcal{P}$ isomorphically onto a subposet of $\mathbb{Z}^m$.

We conclude this section with two important consequences of the previous proposition.

Corollary 3.9. With $F$ and $\mathcal{P}$ as in the previous proposition, $F^{\mathcal{P}}$ is a projective $k\mathcal{P}$-module.

Proof. Let $\mathcal{F} = F^{\mathcal{P}}$. Consider an epimorphism $L \to N$ of $k\mathcal{P}$-modules and a morphism $f: \mathcal{F} \to N$. Since tensoring by $\mathcal{R}$ preserves epimorphisms, we obtain a lifting $\mathcal{F} \to L$ of $f$ by taking $g^P$, where $g: \mathcal{R} \otimes_{k\mathcal{P}} \mathcal{F} = F \to \mathcal{R} \otimes_{k\mathcal{P}} L$ is a lifting of $\mathcal{R} \otimes f$. □

Corollary 3.10. Let $I$ be a monomial ideal with Betti poset $\mathcal{B}$ over a field $k$. Let $F^\bullet$ be a minimal free multigraded resolution of $I$ over $R$. Let $\mathcal{P}$ be a subposet of $\mathbb{Z}^m$ such that $\mathcal{B} \subseteq \mathcal{P}$. Then the $k\mathcal{P}$-chain complex $F^\bullet$ is a resolution of $I^{\mathcal{P}}$ by projective $k\mathcal{P}$-modules. In particular, if $\mathcal{B} \subseteq \mathcal{P} \subseteq \text{deg}(I)$ then it is a projective resolution of the constant $k\mathcal{P}$-module with value $k$. □
4. Free and projective $k\mathcal{J}$-modules

In this section, for any small category $\mathcal{J}$ we want to define free $k\mathcal{J}$-modules and bases for them, in such a way that free $k\mathcal{J}$-modules are projective. To this end we first need to define what the underlying “object” of a $k\mathcal{J}$-module $M$ is, and the most convenient way of doing so is by forgetting not just the $k$-module structure on each $M(j)$ but also the homomorphisms $M(j) \to M(i)$ that we have for any morphism $j \to i$ in $\mathcal{J}$. So the underlying object of a $k\mathcal{J}$-module $M$ is just the collection of sets $M(j)$ indexed by the objects of $\mathcal{J}$. This data can conveniently be encoded into a functor from the discrete category $\text{obj} \mathcal{J}$ (i.e., the subcategory of $\mathcal{J}$ where the only morphisms are the identities) to sets, i.e., an $(\text{obj} \mathcal{J})$-set. This defines a forgetful functor

$$U : k\mathcal{J}\text{-mod} \to (\text{obj} \mathcal{J})\text{-sets}.$$ 

The key observation is that $U$ has a left adjoint

$$L : (\text{obj} \mathcal{J})\text{-sets} \to k\mathcal{J}\text{-mod},$$

that is defined by sending an $(\text{obj} \mathcal{J})$-set $B$ to the $k\mathcal{J}$-module

$$LB = \bigoplus_{j \in \text{obj} \mathcal{J}} \bigoplus_{B(j)} k[\text{mor}_\mathcal{J}(j, -)].$$

Notice that there is a natural morphism of $(\text{obj} \mathcal{J})$-sets

$$\eta : B \to ULB$$

(which will be the unit of the adjunction) that for every $j \in \text{obj} \mathcal{J}$ sends $b \in B(j)$ to $\text{id}_j$ in the corresponding summand $k[\text{mor}_\mathcal{J}(j, j)] \subseteq LB(j)$ indexed by $b$.

**Example 4.1.** Fix an object $j_0 \in \text{obj} \mathcal{J}$ and consider the $(\text{obj} \mathcal{J})$-set $B$ given by

$$B(j) = \begin{cases} \emptyset & \text{if } j \neq j_0, \\ \text{pt} & \text{if } j = j_0. \end{cases}$$

Then $LB = k[\text{mor}_\mathcal{J}(j_0, -)]$, and $\eta : B \to ULB$ sends pt to $\text{id}_{j_0}$.

In order to prove that $L$ is left adjoint to $U$, i.e., that there are natural bijections

$$\text{mor}_\mathcal{J}(B, UM) \cong \text{hom}_{k\mathcal{J}}(LB, M)$$

for all $(\text{obj} \mathcal{J})$-sets $B$ and $k\mathcal{J}$-modules $M$, we are going to explain in which sense $B$ is a basis for $LB$.

**Definition 4.2.** Let $M$ be a $k\mathcal{J}$-module and let $B$ be an $(\text{obj} \mathcal{J})$-set together with a morphism of $(\text{obj} \mathcal{J})$-sets $\mu : B \to UM$. We say that $M$ is free with basis $B$ if for every $k\mathcal{J}$-module $N$ and every morphism of $(\text{obj} \mathcal{J})$-sets $g : B \to UN$ there is a unique homomorphism of $k\mathcal{J}$-modules $G : M \to N$ such that $(UG) \circ \mu = g$.

Notice that the definition implies that $\mu$ is objectwise injective.

The following three lemmas follow easily from the definitions and Yoneda’s lemma.

**Lemma 4.3.** If $B$ is an $(\text{obj} \mathcal{J})$-set then the $k\mathcal{J}$-module $LB$ is free with basis $\eta : B \to ULB$. Therefore the functor $L$ is left adjoint to the forgetful functor $U$.

**Lemma 4.4.** Free $k\mathcal{J}$-modules are projectives.
Lemma 4.5. Fix an object \( j_0 \in \text{obj} \mathcal{J} \). For any \( k \mathcal{J}^{\text{op}} \)-module \( M \) there is a natural isomorphism
\[
M \otimes k[\text{mor}_\mathcal{J}(j_0,-)] \cong M(j_0).
\]

Proposition 4.6. Let \( \mathcal{P} \) be a subposet of \( \mathbb{Z}^m \).
(a) Let \( F \) be a free multigraded \( R \)-module with homogeneous basis \( B \) such that \( \mathcal{P} \) contains the degrees of all the elements of \( B \). Then the \( k \mathcal{P} \)-module \( \mathcal{F} = F^\mathcal{P} \) is free with basis \( B^\mathcal{P} \rightarrow U \mathcal{F} \), where the \( \{\text{obj} \mathcal{P}\} \)-set \( B^\mathcal{P} \) is given by
\[
B^\mathcal{P}(\alpha) = B_\alpha = \{ b \in B | \deg(b) = \alpha \} \subset F_\alpha = \mathcal{F}(\alpha) = UF(\alpha),
\]
and \( \mu(\alpha) \) is the inclusion map.
(b) Let \( \mathcal{G} \) be a free \( k \mathcal{P} \)-module with basis \( C \rightarrow U \mathcal{G} \). Then \( \mathcal{A} \otimes_{k \mathcal{P}} \mathcal{G} \) is a free multigraded \( R \)-module with homogeneous basis \( B = \coprod_{\alpha \in \mathcal{P}} \{ 1 \otimes \mu(\alpha) | c \in C(\alpha) \} \).

Proof. Part (b) follows from part (a). Indeed, let \( G \) be the free multigraded \( R \)-module with homogeneous basis the set \( B \) from part (b). Then it is straightforward from the definitions in part (a) that \( B^\mathcal{P} \cong C \) as \( \{\text{obj} \mathcal{P}\} \)-sets, hence the free by part (a) \( k \mathcal{P} \)-module \( G^\mathcal{P} \) is isomorphic to \( \mathcal{G} \). Therefore \( G \cong \mathcal{A} \otimes_{k \mathcal{P}} G^\mathcal{P} \cong \mathcal{A} \otimes_{k \mathcal{P}} \mathcal{G} \).

We proceed with the proof of part (a). Let \( \mathcal{G} \) be any \( k \mathcal{P} \)-module and let \( g: B^\mathcal{P} \rightarrow U \mathcal{G} \) be a morphism of \( \{\text{obj} \mathcal{P}\} \)-sets. We need to show that there is a unique morphism \( \mathcal{G}: \mathcal{F} \rightarrow \mathcal{G} \) of \( k \mathcal{P} \)-modules that extends \( g \). We note that any such extension \( \mathcal{G} \) of \( g \) has to satisfy for any \( \gamma \leq \beta \) and any \( b \in B_\gamma \), the equality
\[
(4.7) \quad G(\beta)(x^{\beta-\gamma}b) = \mathcal{G}(\gamma \leq \beta)(g(\gamma)(b)).
\]

Let \( \alpha \in \mathcal{P} \). Note that \( \mathcal{F}(\alpha) = F_\alpha \) is a free \( k \)-module with basis \( \coprod_{\gamma \leq \alpha} x^{\alpha-\gamma}B_\gamma \). In particular, if \( \alpha \) is a minimal element of \( \mathcal{P} \) then \( B_\alpha \) is a basis of the free \( k \)-module \( \mathcal{F}(\alpha) \), hence the map of sets \( g(\alpha): B_\alpha \rightarrow \mathcal{G}(\alpha) \) extends uniquely to a map of free \( k \)-modules \( G(\alpha): \mathcal{F}(\alpha) \rightarrow \mathcal{G}(\alpha) \).

Suppose by induction that \( \alpha \) is not minimal and that for all \( \beta < \alpha \) there is a unique system of maps \( G(\beta) \) that extend \( g \) and satisfies (4.7) for all \( \gamma \leq \beta < \alpha \). Since \( G(\alpha) \) must also satisfy (4.7) with \( \beta = \alpha \) we must have for each \( \gamma \leq \alpha \) and for each \( b \in B_\gamma \), that
\[
G(\alpha)(x^{\alpha-\gamma}b) = \mathcal{G}(\gamma \leq \alpha)(g(\gamma)(b)).
\]

But the elements \( x^{\alpha-\gamma}b \) are a basis of \( \mathcal{F}(\alpha) \), hence there exists exactly one such \( k \)-modules map \( G(\alpha): \mathcal{F}(\alpha) \rightarrow \mathcal{G}(\alpha) \).

As an immediate consequence, we see that Corollary 3.10 can be strengthen by replacing projective with free.

Corollary 4.8. Let \( I \) be a monomial ideal with Betti poset \( \mathcal{B} \) over a field \( k \). Let \( F_\bullet \) be a minimal free multigraded resolution of \( I \) over \( R \). Let \( \mathcal{P} \) be a subposet of \( \mathbb{Z}^m \) such that \( \mathcal{B} \subseteq \mathcal{P} \). Then the \( k \mathcal{P} \)-chain complex \( F^\mathcal{P}_\bullet \) is a resolution of \( I^\mathcal{P} \) by free \( k \mathcal{P} \)-modules. In particular, if \( \mathcal{B} \subseteq \mathcal{P} \subseteq \text{deg}(I) \) then it is a free resolution of the constant \( k \mathcal{P} \)-module with value \( k \).

5. Spaces over a category

Notice that the constant \( k \mathcal{J} \)-module with value \( k \) is in general not free. Our next goal is to explain how to construct a resolution of the constant \( k \mathcal{J} \)-module with value \( k \) by an acyclic chain complex of free \( k \mathcal{J} \)-modules. In order to do this we need to talk about spaces and CW-complexes over a category.
We denote by spaces the category of compactly generated weak Hausdorff spaces (simply spaces from now on) and continuous maps. We introduce the notion of spaces over a small category $\mathcal{J}$ in complete analogy to the Definition 2.1 of $\mathbb{k}\mathcal{J}$-modules.

**Definition 5.1.** A space over $\mathcal{J}$ (or a $\mathcal{J}$-space) is a functor $X: \mathcal{J} \to$ spaces. A map of spaces over $\mathcal{J}$ is a natural transformation. Thus the category $\mathcal{J}$-spaces of spaces over $\mathcal{J}$ is just the category of functors $\text{fun}(\mathcal{J}, \text{spaces})$.

Since the category spaces is complete and cocomplete (i.e., all limits and colimits indexed over small categories exist), the same is true for $\mathcal{J}$-spaces, and all limits and colimits are computed objectwise.

Given an object $j_0 \in \text{obj} \mathcal{J}$ and a natural number $n \in \mathbb{N}$ we consider the $\mathcal{J}$-spaces $\text{mor}_\mathcal{J}(j_0, -) \times S^{n-1}$ and $\text{mor}_\mathcal{J}(j_0, -) \times D^n$ obtained by taking the product of the discrete $\mathcal{J}$-space $\text{mor}_\mathcal{J}(j_0, -): \mathcal{J} \to \text{sets} \subset \text{spaces}$ with the constant $\mathcal{J}$-spaces with values $S^{n-1}$ and $D^n$, respectively. We denote by $\iota$ the map of $\mathcal{J}$-spaces

$$\iota: \text{mor}_\mathcal{J}(j_0, -) \times S^{n-1} \to \text{mor}_\mathcal{J}(j_0, -) \times D^n$$

obtained by taking the product of the identity and the inclusion $S^{n-1} \subset D^n$.

**Definition 5.2.** Let $\mathcal{X}$ and $\mathcal{Y}$ be spaces over $\mathcal{J}$, and let $n \in \mathbb{N}$ and $j_0 \in \text{obj} \mathcal{J}$. We say that $\mathcal{Y}$ is obtained from $\mathcal{X}$ by attaching a free $n$-cell based at $j_0$ if there is a map of $\mathcal{J}$-spaces $\varphi: \text{mor}_\mathcal{J}(j_0, -) \times S^{n-1} \to \mathcal{X}$ (called attaching map) such that the following square is a pushout.

$$\begin{array}{ccc}
\text{mor}_\mathcal{J}(j_0, -) \times S^{n-1} & \xrightarrow{\varphi} & \mathcal{X} \\
\downarrow{\iota} & & \downarrow{\tau} \\
\text{mor}_\mathcal{J}(j_0, -) \times D^n & \xrightarrow{\varphi} & \mathcal{Y}
\end{array}$$

Notice that if $\mathcal{Y}$ is obtained from $\mathcal{X}$ by attaching a free $n$-cell then for every $j \in \text{obj} \mathcal{J}$ there is a pushout square

$$\begin{array}{ccc}
\text{mor}_\mathcal{J}(j_0, j) \times S^{n-1} & \xrightarrow{\varphi} & \mathcal{X}(j) \\
\downarrow{\iota} & & \downarrow{\tau} \\
\text{mor}_\mathcal{J}(j_0, j) \times D^n & \xrightarrow{\varphi} & \mathcal{Y}(j)
\end{array}$$

i.e., the space $\mathcal{Y}(j)$ is obtained from $\mathcal{X}(j)$ by attaching $n$-cells, one for every morphism $j_0 \to j$ in $\mathcal{J}$. Notice also that then $\tau: \mathcal{X} \to \mathcal{Y}$ is objectwise a cofibration, since the same true for $\iota$.

Now we define the appropriate notion of CW-complex in the context of $\mathcal{J}$-spaces, following [DL98, Section 3].

**Definition 5.3.** A free $\mathcal{J}$-CW-complex is a space $\mathcal{X}$ over $\mathcal{J}$ such that there exist spaces over $\mathcal{J}$ and maps

$$\mathcal{X}_{-1} = \emptyset \to \mathcal{X}_0 \to \mathcal{X}_1 \to \cdots \to \mathcal{X}_{n-1} \to \mathcal{X}_n \to \cdots$$

satisfying the following two conditions.
(1) For every $n \in \mathbb{N}$, the $\mathcal{J}$-space $\mathcal{X}_n$ is obtained from $\mathcal{X}_{n-1}$ by attaching some free $n$-cells, i.e., there is a set $A_n$ and a collection of objects $j_a \in \text{obj} \mathcal{J}$ indexed by $a \in A_n$, together with a map of $\mathcal{J}$-spaces

$$\varphi = (\varphi_a)_{a \in A_n} : \coprod_{a \in A_n} \text{mor}_\mathcal{J}(j_a, -) \times S^{n-1} \to \mathcal{X}_{n-1}$$

such that the square

$$\begin{array}{ccc}
\coprod_{a \in A_n} \text{mor}_\mathcal{J}(j_a, -) \times S^{n-1} & \xrightarrow{\varphi} & \mathcal{X}_{n-1} \\
\downarrow & & \downarrow \\
\coprod_{a \in A_n} \text{mor}_\mathcal{J}(j_a, -) \times D^n & \xrightarrow{\varphi_a} & \mathcal{X}_n
\end{array}$$

is a pushout.

(2) $\mathcal{X} \cong \colim \mathcal{X}_n$.

Notice that in the special case when $\mathcal{J}$ is the trivial category $[0]$ with only one object and only one morphism, Definition 5.3 specializes to the usual definition of a CW-complex. Notice also that if $\mathcal{X}$ is a free $\mathcal{J}$-CW-complex, then $\mathcal{X}(i)$ is a CW-complex for every $i \in \text{obj} \mathcal{J}$, but the converse is not necessarily true: free $\mathcal{J}$-CW-complexes are not just functors from $\mathcal{J}$ to CW-complexes. For example, the constant space over $\mathcal{J}$ with value the one-point space pt is in general not a free $\mathcal{J}$-CW-complex.

We recall the definition of the cellular chain complex of an usual CW complex $X$ and fix some notation. Given the pushout diagram

$$\begin{array}{ccc}
\coprod_{a \in A_n} S^{n-1} & \xrightarrow{(\varphi_a)} & X_{n-1} \\
\downarrow & & \downarrow \\
\coprod_{a \in A_n} D^n & \xrightarrow{(\varphi_a)} & X_n
\end{array}$$

we denote by $e_a = \epsilon_n^a = \varphi_a(D^n - S^{n-1})$ the open $n$-cells of $X$, and we write $\tau_a = \tau_a(D^n)$ for their closures and $\epsilon_a = \epsilon_a(S^{n-1})$ for their boundaries. We write $C^\text{cell}_n(X) = C^\text{cell}_n(X; \mathbb{k})$ for the cellular chain complex of $X$, where for each $n \geq 0$

$$C^\text{cell}_n(X) = H_n(X_n, X_{n-1}; \mathbb{k}) \cong \bigoplus_{a \in A_n} H_n(\tau_a, \epsilon_a; \mathbb{k}) = \bigoplus_{a \in A_n} \mathbb{k}[e_a],$$

where we write $[e_a]$ for the generator of $H_n(\tau_a, \epsilon_a; \mathbb{k})$ defined as the image of the canonical generator of $H_n(D^n, S^{n-1}; \mathbb{k})$ under the attaching map $\varphi_a$. Then for the cellular differential $d_n : C^\text{cell}_n(X) \to C^\text{cell}_{n-1}(X)$ we have

$$d_n[e_a] = \sum_{b \in A_{n-1}} [e_a : e_b][e_b],$$

where the coefficients $[e_a : e_b]$ are the so-called incidence numbers of $X$.

Now let $\mathcal{X}$ be a free $\mathcal{J}$-CW-complex. Since, as observed above, each space $\mathcal{X}(i)$ is a CW-complex, we can make the following definition.

**Definition 5.4.** The **cellular chain complex** of a $\mathcal{J}$-CW-complex $\mathcal{X}$ is the chain complex of $\mathbb{k}[\mathcal{J}]$-modules given by the functor

$$C^\bullet(\mathcal{X}) = C^\bullet(\mathcal{X}(\mathcal{J}); \mathbb{k}) : \mathcal{J} \to \text{ch}(\mathbb{k}\text{-mod}), \quad i \mapsto C^\bullet(\mathcal{X}(i); \mathbb{k}).$$
The following lemma captures the good homological properties of free \( J \)-CW-complexes.

**Lemma 5.5.** If \( X \) is a free \( J \)-CW-complex then \( C_\bullet^{\text{cell}}(X) \) is a chain complex of free (and hence projective) \( kJ \)-modules.

**Proof.** If \( Y \) and \( Z \) are \( J \)-spaces and \( Z \) is obtained from \( Y \) by attaching a free \( n \)-cell based at \( j_0 \) (see Definition 5.2) then

\[
H_n(Z, Y; k) \cong k[\text{mor}_J(j_0, -)]
\]

is a free \( kJ \)-module. The statement then follows from the definition of cellular chain complex. \( \square \)

**Definition 5.6.** An *universal space over* \( J \) is a free \( J \)-CW-complex \( E_J \) such that \( E_J(i) \) is contractible for every object \( i \) of \( J \).

Universal spaces over categories exist and are unique up to the appropriate notion of homotopy. We refer to [DL98, Section 3] for more information on universal spaces. Here we explain an explicit functorial construction of \( E_J \) in the special case when \( J = P \) is a poset, but in a way that immediately generalizes to arbitrary small categories.

Given any \( a \in P \) consider the filter \( P_{\leq a} \), whose classifying space \( B(P_{\leq a}) = |N(P_{\leq a})| = \Delta(P_{\leq a}) \) is contractible since \( P_{\leq a} \) has a greatest (i.e., terminal) element (here \( N \) denotes the nerve of a category, and \(|-|\) the geometric realization of a simplicial set). If \( a \leq a' \) then there is a map of posets \( P_{\leq a} \to P_{\leq a'} \) and therefore we get a space over \( P \)

\[
E_P = B(P_{\leq -}): P \to \text{spaces}
\]

such that \( E_P(a) \) is contractible for every \( a \in P \).

We now want to prove that \( E_P \) is a free \( P \)-CW-complex.

First notice that for any fixed \( a \in P \), the space \( E_P(a) \) has a natural CW-structure (since it is the geometric realization of the simplicial set \( N(P_{\leq a}) \) with one \( n \)-cell for every non-degenerate \( n \)-simplex of \( N(P_{\leq a}) \). The non-degenerate \( n \)-simplices of \( N(P_{\leq a}) \) are exactly the strictly increasing chains \( x_0 \preceq x_1 \preceq \cdots \preceq x_{n-1} \preceq x_n \) of length \( n + 1 \).

More explicitly, there is a filtration

\[
\cdots \subseteq \text{sk}_{n-1} B(P_{\leq a}) \subseteq \text{sk}_n B(P_{\leq a}) \subseteq \cdots \subseteq B(P_{\leq a}) = \bigcup_{n \in \mathbb{N}} \text{sk}_n B(P_{\leq a})
\]

such that for every \( n \in \mathbb{N} \) there is a pushout of spaces over \( P \)

\[
\begin{array}{ccc}
\prod_{N_n^d(P_{\leq a})} S^{n-1} & \longrightarrow & \text{sk}_{n-1} B(P_{\leq -}) \\
\downarrow & & \downarrow \\
\prod_{N_n^u(P_{\leq a})} D^n & \longrightarrow & \text{sk}_n B(P_{\leq -})
\end{array}
\]

where \( N_n^d(P_{\leq a}) \) is the set of non-degenerate \( n \)-simplices of \( N(P_{\leq a}) \), i.e., strictly increasing chains \( x_0 \preceq x_1 \preceq \cdots \preceq x_{n-1} \preceq x_n \) whose apex \( x_n \) satisfies \( x_n \leq a \).
Therefore there is a pushout
\[
\begin{array}{ccc}
\prod_{x \in N_n^c(P)} \text{mor}_P(x_n, -) \times S^{n-1} & \longrightarrow & \text{sk}_{n-1} B(P_{\leq a}) \\
\downarrow & & \downarrow \\
\prod_{x \in N_n^d(P)} \text{mor}_P(x_n, -) \times D^n & \longrightarrow & \text{sk}_n B(P_{\leq a})
\end{array}
\]
and this shows that \( E \mathcal{P} \) is a free \( \mathcal{P} \)-CW-complex, as claimed.

Now we can consider the the cellular chain complex (see Definition 5.4) of the just constructed universal space \( E \mathcal{P} \) over \( \mathcal{P} \). By Lemma 5.5, \( C_{\text{cell}}^\bullet(\mathcal{E} \mathcal{P}) \) is a chain complex of free (and hence projective) \( k \mathcal{P} \)-modules. Unraveling the definitions one sees that \( C_{\text{cell}}^\bullet(\mathcal{E} \mathcal{P}) \cong \mathcal{E} \bullet \), the chain complex defined in Example 2.10. Therefore we obtain the following result.

**Proposition 5.7.** The \( k \mathcal{P} \)-chain complex \( \mathcal{E} \bullet(\mathcal{P}, k) \) defined in Example 2.10 is a free (and hence projective) resolution of the constant \( k \mathcal{P} \)-module with value \( k \).

### 6. CW-complexes and their posets

In this section we explain how, given a usual CW-complex \( X \), we can make the set of its open cells into a poset, and how \( X \) yields a universal free CW-complex over this poset. We then use the opportunity to rephrase the definition of cellular resolution in this new language.

**Definition 6.1.** Let \( X \) be a CW-complex, and let \( P_X \) be the set of all open cells of \( X \). There are several natural ways to make the set \( P_X \) into a poset.

1. Let \( P_X \) be the poset generated by the relations \( e_b \leq e_a \) if \( e_b \cap \overline{e_a} \neq \emptyset \).
2. Let \( P_X^{\text{cl}} \) be the poset given by \( e_b \leq e_a \) if and only if \( e_b \subseteq \overline{e_a} \). This is commonly called the *closure poset* of \( X \).
3. Let \( P_X^{\text{inc}} \) be generated by \( e_b \leq e_a \) if \( [e_a : e_b] \neq 0 \in k \). This we will call the *incidence poset* of \( X \) (over \( k \)).

It is clear that the identity map \( \text{id}_{P_X} \) induces morphisms of posets
\[
P_X^{\text{inc}} \to P_X^{\text{cl}} \to P_X
\]
and it is well known that when \( X \) is a regular CW-complex then all these posets are equal.

**Definition 6.2.** (a) We define the functor \( \mathcal{X} : P_X \to \text{top} \) by the assignment
\[
e_a \mapsto \bigcup_{e_b \leq e_a} e_b
\]
with \( \mathcal{X}(e_b \leq e_c) \) the obvious inclusion map.

(b) We define the functor \( \mathcal{X}^{\text{cl}} : P_X^{\text{cl}} \to \text{top} \) by the assignment \( e_a \mapsto \overline{e_a} \), with \( \mathcal{X}(e_b \leq e_c) \) again the obvious inclusion map.

Notice that \( \mathcal{X} \) can be restricted (along the inclusion \( P_X^{\text{cl}} \to P_X \)) to a functor \( \mathcal{X}^{\text{cl}} : P_X^{\text{cl}} \to \text{top} \), but in general \( \mathcal{X}^{\text{cl}} \) cannot be extended to a functor \( P_X \to \text{top} \). Notice also that if \( X \) is a normal CW-complex (see, e.g., [LW69, Definition I.1.7]), then \( P_X^{\text{cl}} = P_X \) and \( \mathcal{X}^{\text{cl}} = \mathcal{X} \).

It is clear that the functor \( \mathcal{X}^{\text{cl}} : P_X^{\text{cl}} \to \text{top} \) is objectwise contractible. On the other hand we have the following lemma.
Lemma 6.3. The space $X$ over $\mathcal{P}_X$ is a free $\mathcal{P}_X$-CW-complex.

Proof. For each $n \in \mathbb{N}$ and $e_a \in \mathcal{P}_X$ let $X_n(e_a) = X_n \cap X(e_a)$, where $X_n$ denotes the $n$-skeleton of $X$. Now, with $A_n$ the indexing set of all $n$-cells of $X$, we obtain immediately the desired pushout diagram

$$\bigsqcup_{a \in A_n} \text{mor}_{\mathcal{P}_X}(e_a, -) \times S^{n-1} \longrightarrow X_{n-1}$$

$$\bigsqcup_{a \in A_n} \text{mor}_{\mathcal{P}_X}(e_a, -) \times D^n \longrightarrow X_n.$$ 

Combining this lemma with the observation immediately preceding it we obtain the following corollary.

Corollary 6.4. If $X$ is a normal CW-complex then $\mathcal{X}: \mathcal{P}_X \to \text{top}$ is a universal space over $\mathcal{P}_X$.

Now consider the chain complex of $k\mathcal{P}_X$-modules $C_X = C_{\text{cell}}(\mathcal{X})$, and denote by $C^{\text{cl}}_X$ and $C^{\text{inc}}_X$ its restriction to the posets $\mathcal{P}^{\text{cl}}_X$ and $\mathcal{P}^{\text{inc}}_X$ respectively. Combining Lemmas 5.5 and 6.3 we obtain that $C_X$ is a complex of free $k\mathcal{P}_X$-modules. But by inspecting the definitions we also see that $C^{\text{cl}}_X$ and $C^{\text{inc}}_X$ are complexes of free $k\mathcal{P}^{\text{cl}}_X$- and $k\mathcal{P}^{\text{inc}}_X$-modules respectively.

Now let $I$ be a monomial ideal in the polynomial ring $R = k[x_1, \ldots, x_m]$ over the field $k$. Let

$$F_\bullet = \cdots \leftarrow F_d \leftarrow F_{d-1} \leftarrow \cdots \leftarrow F_0 \leftarrow 0$$

be a multigraded free resolution of $I$ over $R$.

Definition 6.5. The CW-complex $X$ supports the free resolution $F_\bullet$ of $I$ over $R$ if there exists a function

$$\text{gr}: \mathcal{P}_X \to \mathbb{Z}^m$$

such that the following conditions are satisfied:

1. The function $\text{gr}$ is a morphism of posets $\mathcal{P}^{\text{inc}}_X \to \mathbb{Z}^m$.
2. There is an isomorphism of chain complexes of free multigraded $R$-modules

$$F_\bullet \cong \mathcal{R}_{\text{gr}} \otimes_{k\mathcal{P}^{\text{inc}}_X} C^{\text{inc}}_X.$$

Remarks 6.6. (a) When $X$ is a regular CW-complex, this is exactly the definition of cellular resolution given in [BS98].

(b) The definition of a CW-complex supporting the free resolution of a monomial ideal given in [BW02] replaces condition (1) with the condition that $\text{gr}$ is a morphism of posets $\mathcal{P}^{\text{cl}}_X \to \mathbb{Z}^m$. Every such morphism is also a morphism of posets $\mathcal{P}^{\text{inc}}_X \to \mathbb{Z}^m$, hence every free resolution supported on $X$ in the sense of [BW02] is also supported on $X$ in our sense. However, in general it is not clear whether our definition covers a larger class of ideals, i.e., we do not know if there is an ideal $I$ whose minimal free resolution is supported on a CW-complex $X$ in our sense but cannot be supported on any CW-complex in the sense of [BW02]. From our point of view, incidence posets are more robust—they do not change when the attaching maps of the cells are deformed, thus a definition based on them seems more flexible.

(c) In Velasco’s paper [Vel08] it is shown that there is a monomial ideal $I$ whose minimal free resolution cannot be supported on any CW-complex in the sense
of [BW02]. The exact same proof shows that Velasco’s ideal \( I \) cannot be supported on any CW-complex in our sense either. The only change that is needed is in the statement of [Ve08, Lemma 2.2] where the assumption on \( gr \) needs to be relaxed to state that \( gr \) is only a morphism of posets \( P^m_X \to \mathbb{Z}^m \).

7. Main results

Throughout this section \( k \) is a field, \( R = k[x_1, \ldots, x_m] \), and \( I \) is a monomial ideal in \( R \) with Betti poset \( B \) over \( k \).

**Theorem 7.1.** Let \( P \) be any subposet of \( \mathbb{Z}^m \) such that \( B \subseteq P \subseteq \text{deg}(I) \). Then the order complex \( \Delta(P) \) supports a free resolution of \( I \).

**Proof.** Recall from Definition 3.4 that \( \mathcal{R} \) is the multidegree shift functor

\[
\mathcal{R} : P^{\text{op}} \to \text{mgR-mod}
\]

given objectwise by \( \mathcal{R}(\alpha) = R(-\alpha) \) and for \( \alpha \leq \beta \) in \( P \) the corresponding morphism \( R(-\beta) \to R(-\alpha) \) is given by multiplication by \( x^{\beta - \alpha} \). Let \( E_* = E_*(P, k) \) be the free resolution of the constant \( kP \)-module with value \( k \) constructed in Example 2.10. We need to show that \( \mathcal{R} \otimes_{kP} E_* \) is a free multigraded resolution of the monomial ideal \( I \) over \( R \).

Let \( F_* \) be the minimal free resolution of \( I \) over \( R \), and let \( G_* = \mathcal{R} \otimes_{kP} E_* \). Since both \( F_* \) and \( E_* \) are projective resolutions of the constant \( kP \)-module with value \( k \), they are chain homotopy equivalent and hence so are \( F_* = \mathcal{R} \otimes_{kP} F_* \) and \( G_* \). Therefore \( G_* \) is a multigraded resolution of \( I \) over \( R \). Finally, since in each homological degree \( n \) the \( kP \)-module \( E_n = E_n(P, k) \) is free, the corresponding \( R \)-module \( G_n = \mathcal{R} \otimes_{kP} E_n \) is free multigraded.

As an important standard consequence we obtain that the Betti numbers of \( I \) over \( k \) are completely determined by the topology of \( P \), in particular by the topology of \( B \). In the special case when \( P \) is the lcm-lattice \( L \) this gives a new proof of [GPW99, Theorem 2.1].

**Theorem 7.2.** Let \( P \) be a subposet of \( \mathbb{Z}^m \) such that \( B = B(I, k) \subseteq P \subseteq \text{deg}(I) \).

(a) For any \( \alpha \in \mathbb{Z}^m \) and any \( d \geq 0 \) we have

\[
\beta_{d, \alpha}(I, k) = \begin{cases} \dim_k \bar{H}_{d-1}(\Delta(P_{\leq \alpha}); k) & \text{if } \alpha \in P; \\ 0 & \text{otherwise.} \end{cases}
\]

(b) In particular, \( B(I, k) = \{ a \in P \mid \bar{H}_k(\Delta(P_{\leq \alpha}); k) \neq 0 \text{ for some } k \} \).

(c) For any \( \alpha \in B(I, k) \) the inclusion \( \Delta(B_{\leq \alpha}) \subseteq \Delta(P_{\leq \alpha}) \) is a homology isomorphism over \( k \).

**Proof.** (a) Since the multidegrees of the basis elements of the free modules in the resolution \( G_* = \mathcal{R} \otimes_{kP} E_*(P, k) \) are all in \( P \), it is immediate that \( \beta_{d, \alpha}(I, k) = 0 \) for \( \alpha \notin P \). If \( \alpha \in P \) then we have

\[
\text{Tor}_d^R(I, k)_\alpha = H_d(G_{\alpha}/mG_{\alpha} \cap G_{\alpha}) = H_d\left(C_*(\Delta(P_{\leq \alpha}); k) / C_*(\Delta(P_{\leq \alpha}); k) \right).
\]

Since \( \Delta(P_{\leq \alpha}) \) is a cone with apex \( \alpha \) hence contractible, we obtain \( \text{Tor}_d^R(I, k)_\alpha \cong \bar{H}_{d-1}(\Delta(P_{\leq \alpha}); k) \).

(b) is immediate from (a) by the definition of Betti poset.
(c) The inclusion \( \mathcal{B} \subset \mathcal{P} \) induces canonically a morphism of \( \mathbb{Z}^m \)-graded free resolutions over \( R \)

\[
E_\bullet = E_\bullet \otimes \mathcal{E}(\mathcal{B}, k) \longrightarrow E_\bullet \otimes \mathcal{E}(\mathcal{P}, k) = G_\bullet
\]

which is an isomorphism in homology, hence a chain homotopy equivalence. Therefore it stays a chain homotopy equivalence after tensoring by the \( \mathbb{Z}^m \)-graded \( R \)-module \( R/\mathfrak{m} \), hence induces an isomorphism in homology between the corresponding multigraded strands in degree \( \alpha \) for each \( \alpha \in \mathcal{B} \). But for these graded strands we have

\[
(E_\bullet \otimes R/m)_\alpha = E_\bullet \alpha / (mE_\bullet \cap E_\bullet \alpha) = C_\bullet(\Delta(\mathcal{B}_{\leq \alpha}); k) / C_\bullet(\Delta(\mathcal{B}_{< \alpha}); k)
\]

\[
(G_\bullet \otimes R/m)_\alpha = G_\bullet \alpha / (mG_\bullet \cap G_\bullet \alpha) = C_\bullet(\Delta(\mathcal{P}_{\leq \alpha}); k) / C_\bullet(\Delta(\mathcal{P}_{< \alpha}); k).
\]

Therefore the map of pairs \( (\Delta(\mathcal{B}_{\leq \alpha}), \Delta(\mathcal{B}_{< \alpha})) \longrightarrow (\Delta(\mathcal{P}_{\leq \alpha}), \Delta(\mathcal{P}_{< \alpha})) \) induced by the inclusion \( \mathcal{B} \subseteq \mathcal{P} \) is an isomorphism in relative homology over \( k \). Since \( \Delta(\mathcal{B}_{\leq \alpha}) \) and \( \Delta(\mathcal{P}_{\leq \alpha}) \) are cones with apex \( \alpha \), the desired conclusion is immediate. \( \square \)

**Remark 7.3.** Part (c) of Theorem 7.2 can be derived from parts (a) and (b) also by using the results of [CM13], where entirely different methods of proof are used.

Finally, we show that the isomorphism class of the Betti poset completely determines not just the Betti numbers, but also the structure of the minimal free resolution of \( I \). The following theorem also generalizes, with a new proof, [GPW99, Theorem 3.3].

**Theorem 7.4.** Let \( \mathcal{P} \) be a subposet of \( \mathbb{Z}^m \) such that \( \mathcal{B}(I, k) \subseteq \mathcal{P} \subseteq \deg(I) \). Let \( F_\bullet \) be a minimal free multigraded resolution of \( I \) over \( R \), and let \( F_\bullet = F_\bullet^\mathcal{P} \) be the \( \mathcal{P} \)-sample of \( F_\bullet \). Let \( S = k[y_1, \ldots, y_t] \) be another polynomial ring over the field \( k \), and let \( J \) be a monomial ideal of \( S \) such that \( \mathcal{P} \) is isomorphic to a subposet \( Q \) of \( \mathbb{Z}^t \) with \( B(J, k) \subseteq Q \subseteq \deg(J) \). Fix one such isomorphism \( \text{gr}: \mathcal{P} \longrightarrow Q \).

(a) The isomorphism \( \text{gr} \) maps \( B(I, k) \) isomorphically onto \( B(J, k) \).

(b) Viewing \( \text{gr} \) as morphism \( \mathcal{P} \longrightarrow \mathbb{Z}^t \), consider the multidegree-shift functor \( \mathcal{I}_\text{gr}: \mathcal{P} \longrightarrow \text{mg}S\text{-mod} \). Then the homogenization \( \mathcal{I}_\text{gr} \otimes_{k\mathcal{P}} F_\bullet \) is a minimal free multigraded resolution of \( J \) over \( S \).

**Proof.** Part (a) is immediate from Theorem 7.2 and we proceed with the proof of part (b). Identifying \( \mathcal{P} \) with \( Q \), we consider \( \text{gr} \) as the inclusion map and write \( I \) for \( \mathcal{I}_\text{gr} \). Let \( G_\bullet \) be a minimal free resolution of \( J \) over \( S \). Then the \( k\mathcal{P} \)-chain complexes \( F_\bullet \) and \( G_\bullet^\mathcal{P} \) are free resolutions of the constant \( k\mathcal{P} \)-module with value \( k \), hence are chain homotopy equivalent. Hence so are the the complexes \( \mathcal{H}_\bullet = \mathcal{I}_\text{gr} \otimes_{k\mathcal{P}} F_\bullet \) and \( G_\bullet^\mathcal{P} \cong G_\bullet \). Therefore \( \mathcal{H}_\bullet \) is a free resolution of \( J \) over \( S \), and since it has by Theorem 7.2 and Proposition 4.6 the correct ranks for its free modules, it is a minimal free resolution. \( \square \)

We conclude this section with an example of two monomial ideals that do not have isomorphic lcm-lattices, but do have isomorphic Betti posets.

**Example 7.5.** Let \( R = k[a, b, c, d, e] \) and let \( I = (ac, ae, bd, de) \). The Hasse diagram of \( \mathcal{L}(I) \setminus \hat{0} \) is given below. (We always write \( \hat{0} \) for the minimal element of a lattice).
It easy to see by examining the filters \((L(I) \setminus 0)_{x<z}\) for all \(x \in L(I) \setminus 0\) that for any field \(k\) the Betti poset \(B(I, k)\) consists of the non-circled elements.

When we investigate in a similar manner the monomial ideal \(J = (wx, xy, wz, yz)\) in the polynomial ring \(S = k[w, x, y, z]\), we see that for every field \(k\) we have \(B(J, k) = L(J) \setminus 0\), with the following Hasse diagram.

Thus we have \(L(I) \not\cong L(J)\) but \(B(I, k) \cong B(J, k)\) for every field \(k\).

8. When is a poset the Betti poset of an ideal?

Our results allow us now to answer this question. Let \(P\) be any poset. For every \(x \in P\) we write \(A_x\) for the set of all maximal elements of \(P\) that are less than or equal to \(x\). The poset \(P\) is called atomic if each \(x \in P\) is the join (the unique least upper bound) in \(P\) of the elements of \(A_x\). We consider every atomic poset \(P\) with set of minimal elements \(A\) naturally as a subposet of the Boolean lattice \(\Sigma(A)\) (the set of all subsets of \(A\), ordered by inclusion) via the embedding map \(\sigma: P \rightarrow \Sigma(A)\) given by \(\sigma(x) = A_x\); in particular we will not distinguish between an element \(a \in A\) and the singleton \(\{a\} \in \Sigma(A)\). It as an easy exercise for the reader to check that \(\sigma\) preserves meets (unique greatest lower bounds, whenever they exist) of elements in \(P\). When \(P\) is atomic, we also write \(M(P)\) for the subposet of \(\Sigma(A)\) with elements all meets in \(\Sigma(A)\) of elements of \(P\). (We say that \(M(P)\) is meet-generated in \(\Sigma(A)\) by \(P\).)

Lemma 8.1. Let \(P\) be a finite atomic poset with set of minimal elements \(A\). Then \(M(P)\) is a finite atomic lattice with set of atoms \(A\).

Proof. \(M(P)\) is a finite meet-semilattice by construction, hence a finite lattice. Let \(Z \in M(P)\). Since \(Z \subseteq A\) and for each \(a \in A\) the set \(\{a\}\) is an atom of \(M(P)\), it is
Let $I$ be a monomial ideal in the polynomial ring $R = \mathbb{k}[x_1, \ldots, x_m]$ over the field $\mathbb{k}$, and let $B$ be its Betti poset over $\mathbb{k}$. Thus, the set $A$ of minimal elements of $B$ is exactly the set of degrees in $\mathbb{Z}^m$ of the minimal generators of $I$. Recall that the lcm-lattice of $I$ is the set $L = L(I)$ of all joins in $\mathbb{Z}^m$ of elements of $A$. In particular, the poset $L \setminus \hat{0}$ is atomic with $A$ as its set of minimal elements. (We write $\hat{0}$ for the unique minimal element of a lattice.) It is well known that $B$ is a subposet of $L$, and that an element $y \in L \setminus \hat{0}$ is not in $B$ exactly when the relative homology $\tilde{H}_n(\Delta(L_{<y} \setminus \hat{0}); \mathbb{k}) = 0$ for all $n$.

**Lemma 8.2.** The Betti poset $B$ is an atomic poset.

**Proof.** Let $x \in B$, then clearly $x$ is an upper bound for the set $A_x$. Let $y \in B$ be any other upper bound for $A_x$. Then $y$ is also an upper bound for $A_x$ in the lcm-lattice $L$. Therefore $y$ is greater than or equal to the join in $L$ of the elements in $A_x$. Since $L$ is join-generated by the elements of $A$ it follows that $z = \bigvee A_x$ for each $z \in L$; in particular $\bigvee A_x = x$. Therefore $y \geq x$. □

**Proposition 8.3.** Let $I$ be a monomial ideal with Betti poset $B$. Let $A$ be the set of minimal elements of $B$ and let $M(B)$ be the subposet of $\Sigma(A)$ meet-generated in $\Sigma(A)$ by the elements of $B$.

Then for each $x \in M(B) \setminus \hat{0}$ we have that the element $x$ is not in $B$ precisely when $\tilde{H}_n(\Delta(M(B)_{<x} \setminus \hat{0}); \mathbb{k}) = 0$ for all $n$.

**Proof.** Let $L$ be the lcm-lattice of $I$. Since $L$ is an atomic lattice, the embedding of $L$ inside $\Sigma(A)$ preserves meets and therefore $M(L) = L$. It follows that $B \subseteq M(B) \setminus \hat{0} \subseteq L \setminus \hat{0} \subseteq \deg(I)$. Now the assertion of the proposition is immediate from Theorem 8.4 applied to $P = M(B) \setminus \hat{0}$.

**Theorem 8.4.** Let $P$ be a finite atomic poset with set of minimal elements $A$, and let $M(P)$ be the subposet of $\Sigma(A)$ meet-generated by $P$ in $\Sigma(A)$.

The poset $P$ is the Betti poset of a monomial ideal over a field $\mathbb{k}$ if and only if an element $x \in M(P) \setminus \hat{0}$ is not in $P$ precisely when $\tilde{H}_n(\Delta(M(P)_{<x} \setminus \hat{0}); \mathbb{k}) = 0$ for all $n$.

**Proof.** The “only if” direction of the theorem is Proposition 8.3. Suppose now for each $x \in M(P) \setminus \hat{0}$ that $x \notin P$ exactly when $\tilde{H}_n(\Delta(M(P)_{<x} \setminus \hat{0}); \mathbb{k}) = 0$ for each $n$. Since $M(P)$ is an atomic lattice by Lemma 8.2, it is the LCM-lattice of some monomial ideal $J$ [Phan06]. Thus by [GPW99] $P$ is the Betti poset of $J$ over $\mathbb{k}$. □
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Department of Mathematics and Statistics, University at Albany, SUNY, USA

E-mail address: atchernev@albany.edu
URL: http://www.albany.edu/~tchernev/

E-mail address: mvvarisco@albany.edu
URL: http://www.albany.edu/~mv312143/