BLOCH-WIGNER THEOREM OVER RINGS WITH MANY UNITS

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INTRODUCTION

The Bloch-Wigner theorem appears in different areas of mathematics. It appears in Number theory, in relation with the dilogarithm function [3]. It has a deep connection with the scissor congruence problem in three dimensional hyperbolic geometry [6], [15]. In algebraic K-theory, it appears in connection with the third K-group [21], etc.

This theorem was proved by Bloch and Wigner independently and in a somewhat different form. It asserts the existence of the exact sequence

\[ 0 \to \mathbb{Q}/\mathbb{Z} \to H_3(\text{SL}_2(F), \mathbb{Z}) \to p(F) \to \bigwedge^2 F^\times \to K_2(F) \to 0, \]

where \( F \) is an algebraically closed field of \( \text{char}(F) = 0 \) and \( p(F) \) is the pre-Bloch group of \( F \). A precise description of the groups and the maps involved will be explained in this paper. See also [6] or [17] for an alternative source.

In his remarkable paper [21], Suslin generalized this theorem to all infinite fields. He proved that for an infinite field \( F \), there is an exact sequence

\[ 0 \to \text{Tor}_1^\mathbb{Z}(\mu_F, \mu_F) \to K_3(F) \text{ind} \to p(F) \to (F^\times \otimes_{\mathbb{Z}} F^\times)_\sigma \to K_2(F) \to 0, \]

where \( \text{Tor}_1^\mathbb{Z}(\mu_F, \mu_F) \) is the unique nontrivial extension of \( \text{Tor}_1^\mathbb{Z}(\mu_F, \mu_F) \) by \( \mathbb{Z}/2 \) and \( K_3(F) \text{ind} := \text{coker}(K_3^M(F) \to K_3(F)) \) is the indecomposable part of \( K_3(F) \).

It is very desirable to have such a nice result for a much wider class of rings, such as the class of commutative rings with many units (see for example [5]). The purpose of this article is to provide a version of Bloch-Wigner theorem over the class of rings with many units. To do this, we shall replace \( K_3(R) \text{ind} \) or \( H_0(R^\times, H_3(\text{SL}_2(R), \mathbb{Z})) \) with a group “close” to them. Here we establish the exact sequence

\[ \text{Tor}_1^\mathbb{Z}(\mu_R, \mu_R) \to \tilde{H}_3(\text{SL}_2(R), \mathbb{Z}) \to p(R) \to (R^\times \otimes_{\mathbb{Z}} R^\times)_\sigma \to K_2(R) \to 0, \]

where \( \tilde{H}_3(\text{SL}_2(R), \mathbb{Z}) \) is the group

\[ H_3(\text{GL}_2(R), \mathbb{Z})/\text{im}(H_3(R^\times, \mathbb{Z}) + R^\times \otimes H_2(R^\times, \mathbb{Z})). \]

Here \( \text{im}(H_3(R^\times, \mathbb{Z}) + R^\times \otimes H_2(R^\times, \mathbb{Z})) \subseteq H_3(\text{GL}_2(R), \mathbb{Z}) \) is induced by the diagonal inclusion of \( R^\times \times R^\times \) in \( \text{GL}_2(R) \). Moreover when \( R \) is a domain with many units, e.g. an infinite field, then the left hand side map in the above exact sequence is injective.
When we kill two torsion elements in $\tilde{H}_3(\text{SL}_2(R), \mathbb{Z})$ or when $R^\times = R^\times_2$, we get a familiar group, that is

$$\tilde{H}_3(\text{SL}_2(R), \mathbb{Z}[1/2]) \simeq H_0(R^\times, H_3(\text{SL}_2(R), \mathbb{Z}[1/2]))$$

or $\tilde{H}_3(\text{SL}_2(R), \mathbb{Z}) \simeq H_3(\text{SL}_2(R), \mathbb{Z})$, respectively. We should also mention that,

$$\tilde{H}_3(\text{SL}_2(R), \mathbb{Q}) \simeq H_0(R^\times, H_3(\text{SL}_2(R), \mathbb{Q})) \simeq K_3(R)_{\text{ind}} \otimes \mathbb{Q}$$

(see [7] or [14] for the second isomorphism).

Let $k$ be an algebraically closed field of char($k$) $\neq 2$. If $R$ is the ring of dual numbers $k[\epsilon]$ or a local henselian ring with residue field $k$, then we have the Bloch-Wigner exact sequence

$$\text{Tor}_1^\mathbb{Z}(\mu_R, \mu_R) \rightarrow H_3(\text{SL}_2(R), \mathbb{Z}) \rightarrow \mathfrak{p}(R) \rightarrow (R^\times \otimes \mathbb{Z} R^\times)_{\sigma} \rightarrow K_2(R) \rightarrow 0.$$ 

Moreover when $R$ is a henselian domain or a $k$-algebra with char($k$) $= 0$, then the left-hand side map in this exact sequence is injective. Furthermore if char($k$) $= 0$, $H_3(\text{SL}_2(R), \mathbb{Z})$ reduces to $K_3(R)_{\text{ind}}$.

In some cases, we also establish a ‘relative’ version of Bloch-Wigner exact sequence.

To prove our main theorem, we introduce and study a spectral sequence similar to the one introduced in [12]. This spectral sequence has a close connection to the one introduced in [21, §2], but it is somewhat easier to study. Our main result comes out of careful analysis of this spectral sequence.

**Notation.** In this paper by $H_i(G)$ we mean the homology of group $G$ with integral coefficients, namely $H_i(G, \mathbb{Z})$. By $\text{GL}_n$ (resp. $\text{SL}_n$) we mean the general (resp. special) linear group $\text{GL}_n(R)$ (resp. $\text{SL}_n(R)$). If $A \rightarrow A'$ is a homomorphism of abelian groups, by $A'/A$ we mean $\text{coker}(A \rightarrow A')$. We denote an element of $A'/A$ represented by $a' \in A'$ again by $a'$. If $Q = \mathbb{Z}[1/2]$ or $\mathbb{Q}$, by $A_Q$ we mean $A \otimes \mathbb{Q} Q$.

1. Bloch groups over rings with many units

Let $R^\times$ be the multiplicative group of invertible elements of a commutative ring $R$. Define the *pre-Bloch group* $\mathfrak{p}(R)$ of $R$ as the quotient of the free abelian group $Q(R)$ generated by symbols $[a]$, $a, 1 - a \in R^\times$, to the subgroup generated by elements of the form

$$[a] - [b] + \left[ \frac{b}{a} \right] - \left[ \frac{1 - a^{-1}}{1 - b^{-1}} \right] + \left[ \frac{1 - a}{1 - b} \right],$$

where $a, 1 - a, b, 1 - b, a - b \in R^\times$. Define

$$\lambda' : Q(R) \rightarrow R^\times \otimes R^\times, \quad [a] \mapsto a \otimes (1 - a).$$

**Lemma 1.1.** If $a, 1 - a, b, 1 - b, a - b \in R^\times$, then

$$\lambda'([a] - [b] + \left[ \frac{b}{a} \right] - \left[ \frac{1 - a^{-1}}{1 - b^{-1}} \right] + \left[ \frac{1 - a}{1 - b} \right]) = a \otimes \left( \frac{1 - a}{1 - b} \right) + \left( \frac{1 - a}{1 - b} \right) \otimes a.$$

**Proof.** This follows from a direct computation. \( \square \)
Let \((R^\times \otimes R^\times)_\sigma := R^\times \otimes R^\times / \langle a \otimes b + b \otimes a : a, b \in R^\times \rangle\). We denote the elements of \(p(R)\) and \((R^\times \otimes R^\times)_\sigma\) represented by \([a]\) and \(a \otimes b\) again by \([a]\) and \(a \otimes b\), respectively. By Lemma 1.1, we have a well-defined map

\[ \lambda : p(R) \to (R^\times \otimes R^\times)_\sigma, \quad [a] \mapsto a \otimes (1 - a). \]

**Definition 1.2.** The kernel of \(\lambda : p(R) \to (R^\times \otimes R^\times)_\sigma\) is called the \textit{Bloch group} of \(R\) and is denoted by \(B(R)\).

**Remark 1.3.** This version of the pre-Bloch group and the Bloch group of a commutative ring is due to Suslin [21, §1]. We refer to [11] and [17] for more information in this direction.

In this article we will study Bloch groups over the class of rings with many units, which seems the appropriate class of rings to study them.

**Definition 1.4.** We say that a commutative ring \(R\) is a \textit{ring with many units} if for any \(n \geq 2\) and for any finite number of surjective linear forms \(f_i : R^n \to R\), there exists a \(v \in R^n\) such that, for all \(i\), \(f_i(v) \in R^\times\).

For a commutative ring \(R\), the \(n\)-the Milnor \(K\)-group \(K_n^M(R)\) is defined as an abelian group generated by symbols

\[ \{a_1, \ldots, a_n\}, \quad a_i \in R^\times, \]

subject to multilinearity and the relation \(\{a_1, \ldots, a_n\} = 0\) if there exits \(i, j\), \(i \neq j\), such that \(a_i + a_j = 0\) or 1.

**Lemma 1.5.** Let \(R\) be a commutative ring with many units. Then

\[ K_2^M(R) \simeq R^\times \otimes R^\times / \langle a \otimes (1 - a) : a, 1 - a \in R^\times \rangle. \]

**Proof.** See Proposition 3.2.3 in [9].

Therefore for a ring \(R\) with many units we obtain the exact sequence

\[ 0 \to B(R) \to p(R) \xrightarrow{\lambda} (R^\times \otimes R^\times)_\sigma \to K_2^M(R) \to 0. \]

The study of rings with many units is originated by W. van der Kallen in [23], where he showed that \(K_2\) of such rings behaves very much like \(K_2\) of fields. In fact he proved that when \(R\) is a ring with many units, then \(K_2(R) \simeq K_2^M(R)\).

Here we give a new proof of this fact (Corollary 4.2). See [16, Corollary 4.3] for another proof of this fact.

Important examples of rings with many units are semilocal rings which their residue fields are infinite. In particular for an infinite field \(F\), any commutative finite dimensional \(F\)-algebra is a semilocal ring and so it is a ring with many units.

**Remark 1.6.** Let \(R\) be a commutative ring with many units. It is easy to see that for any \(n \geq 1\), there exist \(n\) elements in \(R\) such that the sum of each nonempty subfamily belongs to \(R^\times\) [9, Proposition 1.3]. Rings with this
property are considered by Nesterenko and Suslin in [16]. For more about rings with many units we refer to [23], [16], [9] and [13].

In the rest of this article we always assume that $R$ is a commutative ring with many units, unless it is mentioned otherwise.

2. Homological interpretation of Bloch group

Let $C_l(R^2)$ be the free abelian group with a basis consisting of $(l+1)$-tuples $\langle v_0, \ldots, v_l \rangle$, where every $\min \{l + 1, 2\}$ of $v_i \in R^2$ is a basis of a direct summand of $R^2$. Here by $\langle v_i \rangle$ we mean the submodule of $R^2$ generated by $v_i$. We consider $C_l(R^2)$ as a left $GL_2(R)$-module in a natural way. If it is necessary we convert this action to the right action by the definition $m.g := g^{-1}m$.

Let us define a differential operator

$$\partial_l : C_l(R^2) \to C_{l-1}(R^2), \ l \geq 1,$$ 

as an alternating sum of face operators $d_i$, which throw away the $i$-th component of generators and let $\partial_0 : C_0(R^2) \to \mathbb{Z}$, $\sum_i n_i(\langle v_i \rangle) \mapsto \sum_i n_i$. The complex

$$C_* : \ldots \to C_2(R^2) \xrightarrow{\partial_2} C_1(R^2) \xrightarrow{\partial_1} C_0(R^2) \xrightarrow{\partial_0} \mathbb{Z} \to 0$$

is exact. (This follows from [13, Lemma 1]). By applying the functor $H_0$ to

$$C_4(R^2) \to C_3(R^2) \to \partial_3(C_3(R^2)) \to 0,$$

we get the exact sequence

$$C_4(R^2)_{GL_2} \to C_3(R^2)_{GL_2} \to H_0(GL_2, \partial_3(C_3(R^2))) \to 0.$$

For simplicity we use the following standard notation

$$\infty := \langle e_1 \rangle, \ 0 := \langle e_2 \rangle, \ 1 := \langle e_1 + e_2 \rangle, \ b^{-1} := \langle e_1 + be_2 \rangle.$$

Denote the orbit of the frame $\langle \infty, 0, 1, a^{-1} \rangle \in C_3(R^2)$ by $p(a)$ and the orbit of the frame $\langle \infty, 0, 1, a^{-1}, b^{-1} \rangle \in C_4(R^2)$ by $p(a, b)$, where $a, 1-a, b, 1-b, a-b \in R^\times$. Then

$$C_3(R^2)_{GL_2} = \coprod_a \mathbb{Z}.p(a), \ C_4(R^2)_{GL_2} = \coprod_{a,b} \mathbb{Z}.p(a, b).$$

Now a direct computation shows that

$$(1) \quad H_0(GL_2, \partial_3(C_3(R^2))) \simeq p(R).$$

From the short exact sequence $0 \to \partial_1(C_1(R^2)) \to C_0(R^2) \xrightarrow{\partial_0} \mathbb{Z} \to 0$ we obtain the connecting homomorphism $H_3(GL_2) \to H_2(GL_2, \partial_1(C_1(R^2)))$. By iterating this process, we get a homomorphism

$$\eta' : H_3(GL_2) \to H_0(GL_2, \partial_3(C_3(R^2))) \simeq p(R).$$

Since $C_0(R^2) \to \mathbb{Z}$ has a $(R^\times \times R^\times)$-equivariant section $m \mapsto m(0)$, the restriction of $\eta'$ to $H_3(R^\times \times R^\times)$ is zero. Thus we obtain a homomorphism

$$\eta : H_3(GL_2)/H_3(R^\times \times R^\times) \to p(R).$$
Proposition 2.1. Let $R$ be a ring with many units. Then $\eta$ is injective and $B(R)$ is its homomorphic image. That is

$$B(R) \simeq H_3(\text{GL}_2)/H_3(R^\times \times R^\times).$$

We will postpone the proof of this proposition to Section 4.

Remark 2.2. Let $\text{GM}_2$ denote the subgroup of monomial matrices in $\text{GL}_2$. By a well known result of Suslin, for an infinite field $F$,

$$B(F) \simeq H_3(\text{GL}_2(F))/H_3(\text{GM}_2(F)),$$

[21, Theorem 2.1]. In [21, Remark 2.1], Suslin claims that most of the times $\text{im}(H_3(\text{GM}_2(F)) \rightarrow H_3(\text{GL}_2(F)))$ is strictly larger than $\text{im}(H_3(F^\times \times F^\times) \rightarrow H_3(\text{GL}_2(F)))$. It appears that this claim is not true due to Proposition 2.1 above.

3. The spectral sequence

Consider the complex

$$0 \rightarrow H_1(X) \xrightarrow{j} C_1(R^2) \rightarrow C_0(R^2) \rightarrow 0,$$

where $H_1(X) := \ker(\partial_1)$. (See Remark 1.1 in [12] for an explanation for the choice of the notation). Set $L_i = C_i(R^2)$ for $i = 0, 1$, and $L_2 = H_1(X)$.

Let $F_* \rightarrow \mathbb{Z}$ be the standard resolution of $\mathbb{Z}$ over $\text{GL}_2$ [4, Chapter I, §5]:

$$\cdots \rightarrow F_2 \xrightarrow{\delta_2} F_1 \xrightarrow{\delta_1} F_0 \xrightarrow{\epsilon} \mathbb{Z} \rightarrow 0.$$

From the double complex $C_{*,*} := F_* \otimes_{\text{GL}_2} L_*$, $C_{p,q} := F_q \otimes_{\text{GL}_2} L_p$, one obtains a first quadrant spectral sequence

$$E^1_{p,q} = \begin{cases} H_q(\text{GL}_2, C_p(R^2)) & \text{if } p = 0, 1 \\ H_q(\text{GL}_2, H_1(X)) & \text{if } p = 2 \Rightarrow H_{p+q}(\text{GL}_2). \\ 0 & \text{if } p \geq 3 \end{cases}$$

Using the Shapiro lemma [4, Chapter III, Proposition 6.2] and a theorem of Suslin [20, Theorem 1.9], [9, 2.2.2],

$$E^1_{p,q} \simeq H_q(R^\times \times R^\times), \text{ for } p = 0, 1.$$

It is not difficult to see that

$$d^1_{1,q} = H_q(\alpha) - H_q(\text{id}),$$

where $\alpha : R^\times \times R^\times \rightarrow R^\times \times R^\times, (a, b) \mapsto (b, a)$.

Lemma 3.1. $E^2_{1,q} = 0$. 

Proof. Consider the following commutative diagram with exact columns

\[
\begin{array}{cccc}
F_{q+1} \otimes_{\text{GL}_2} H_1(Y) & \rightarrow & F_q \otimes_{\text{GL}_2} H_1(Y) & \rightarrow & F_{q-1} \otimes_{\text{GL}_2} H_1(Y) \\
\downarrow & & \downarrow & & \downarrow \\
F_{q+1} \otimes_{\text{GL}_2} C_1(R^2) & \rightarrow & F_q \otimes_{\text{GL}_2} C_1(R^2) & \rightarrow & F_{q-1} \otimes_{\text{GL}_2} C_1(R^2) \\
\downarrow & & \downarrow & & \downarrow \\
F_{q+1} \otimes_{\text{GL}_2} C_0(R^2) & \rightarrow & F_q \otimes_{\text{GL}_2} C_0(R^2) & \rightarrow & F_{q-1} \otimes_{\text{GL}_2} C_0(R^2).
\end{array}
\]

Let \(x \otimes (\infty,0) \in F_q \otimes_{\text{GL}_2} C_1(R^2)\) represents an element of \(H_q(\text{GL}_2, C_1(R^2)) \simeq H_q(R^\times \times R^\times)\) such that \(d^1_{1,q}(x \otimes (\infty,0)) = 0\). So there exists \(y \in F_{q+1}\) such that \(\delta_{q+1}(y) \otimes (\infty) = x \otimes \partial_1(\infty,0) = (xw - x) \otimes (\infty)\), where \(w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\).

From the isomorphism

\[F_q \otimes_{\text{Stab}_{\text{GL}_2}}(\infty) \mathbb{Z} \xrightarrow{\simeq} F_q \otimes_{\text{GL}_2} C_0(R^2), \quad s \otimes 1 \mapsto s \otimes (\infty),\]
we obtain \(\delta_{q+1}(y) \otimes 1 = (xw - x) \otimes 1 \in F_q \otimes_{\text{Stab}_{\text{GL}_2}}(\infty) \mathbb{Z}\). If \(z = x \otimes \partial_2(\infty,0,1)\), then

\[(id_{F_q} \otimes j)(z) = x \otimes (\infty,0) + (xw - x) \otimes (\infty,0),\]

where \(j : H_1(Y) \hookrightarrow C_1(R^2)\) and \(g = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\). Consider the natural map

\[F_q \otimes_{\text{Stab}_{\text{GL}_2}}(\infty) \mathbb{Z} \rightarrow F_q \otimes_{\text{GL}_2} C_0(R^2), \quad s \otimes 1 \mapsto s \otimes (\infty,0)\]

Since \((xw - x) \otimes 1 = (xw - xg) \otimes 1\) in \(F_q \otimes_{\text{Stab}_{\text{GL}_2}}(\infty) \mathbb{Z}\), by the above map we have \(\delta_{q+1}(y) \otimes (\infty,0) = (xw - xg) \otimes (\infty,0) \in F_q \otimes_{\text{GL}_2} C_1(R^2)\). Therefore

\[(id_{F_q} \otimes j)(z) = x \otimes (\infty,0) + \delta_{q+1}(y) \otimes (\infty,0),\]

This completes the proof of the triviality of \(E^2_{1,q}\). \(\square\)

From the short exact sequence \(0 \rightarrow \partial_3(C_3(R^2)) \rightarrow C_2(R^2) \xrightarrow{\partial_2} H_1(X) \rightarrow 0\) one obtains the long exact sequence

\[
\cdots \rightarrow H_1(\text{GL}_2, C_2(R^2)) \xrightarrow{H_1(\partial_2)} H_1(\text{GL}_2, H_1(X)) \rightarrow H_0(\text{GL}_2, \partial_3(C_3(R^2))) \\
\rightarrow H_0(\text{GL}_2, C_2(R^2)) \xrightarrow{H_0(\partial_2)} H_0(\text{GL}_2, H_1(X)) \rightarrow 0.
\]

It is easy to see that \(H_0(\text{GL}_2, H_1(X)) \simeq \mathbb{Z}\) (see formula (3) in [12]) and \(d^1_{2,0} = \text{id}\). Thus \(E^2_{2,0} = 0\). The composition

\[R^\times \simeq H_1(\text{GL}_2, C_2(R^2)) \xrightarrow{H_1(\partial_2)} H_1(\text{GL}_2, H_1(X)) \xrightarrow{H_1(j)} H_1(\text{GL}_2, C_1(R^2)) \simeq R^\times \times R^\times\]
is given by $H_1(j) \circ H_1(\partial_2)(a) = (a, a)$. Thus $H_1(\partial_2)$ is injective. Since $H_0(\text{GL}_2, C_2(R^2)) \simeq \mathbb{Z}$ and $H_0(\partial_2) = \text{id}$, the above results, together with (1), imply the exact sequence

$$0 \to R^\times \to H_1(\text{GL}_2, H_1(X)) \to p(R) \to 0.$$  

By Lemma 3.1, $H_1(\text{GL}_2, H_1(X)) \to \ker(d_{1,1}^1) \simeq R^\times$ is surjective and one can see without any difficulty that this map splits the exact sequence (2). Therefore

$$H_1(\text{GL}_2, H_1(X)) \simeq p(R) \oplus R^\times.$$

Thus we proved the following.

**Lemma 3.2.** $E_{2,1}^2 \simeq p(R)$.

In order to describe a map $p(R) \to H_1(\text{GL}_2, H_1(X))$ that splits the exact sequence (2), we look at the following commutative diagram with exact rows

$$
\begin{array}{cccccc}
0 & \to & F_1 \otimes_{\text{GL}_2} \partial_3(C_3(R^2)) & \to & F_1 \otimes_{\text{GL}_2} C_2(R^2) & \to & F_1 \otimes_{\text{GL}_2} H_1(X) & \to 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & F_0 \otimes_{\text{GL}_2} \partial_3(C_3(R^2)) & \to & F_0 \otimes_{\text{GL}_2} C_2(R^2) & \to & F_0 \otimes_{\text{GL}_2} H_1(X) & \to 0.
\end{array}
$$

The element $[a] \in p(R)$ comes from

$$x_a := (1) \otimes \partial_3(\infty, 0, 1, a^{-1}) \in F_0 \otimes_{\text{GL}_2} \partial_3(C_3(R^2)) = \left[(g_1) - (g_2) + (g_3) - (1)\right] \otimes (\infty, 0, 1) \in F_0 \otimes_{\text{GL}_2} C_2(R^2),$$

where

$$g_1 = \begin{pmatrix} 0 & 1 \\ a - 1 & 1 \end{pmatrix}, \quad g_2 = \begin{pmatrix} 1 - a & a \\ 0 & a \end{pmatrix}, \quad g_3 = \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}.$$

The element

$$y_a := \left[(g_2, g_1) - (g_3, 1)\right] \otimes (\infty, 0, 1) \in F_1 \otimes_{\text{GL}_2} C_2(R^2)$$

maps to $x_a$. And finally $y_a$ maps to

$$z_a := \left[(g_2, g_1) - (g_3, 1)\right] \otimes \partial_2(\infty, 0, 1) \in F_1 \otimes_{\text{GL}_2} H_1(X).$$

Thus a splitting map $p(R) \to H_1(\text{GL}_2, H_1(X))$ can be given by sending $[a]$ to the element of $H_1(\text{GL}_2, H_1(X))$ represented by $z_a$.

Putting all these together, the $E^2$-terms of our spectral sequence look as follow

$$
\begin{array}{cccc}
E_{0,2}^2 & 0 & * & * \\
E_{3,3}^2 & 0 & \times & 0 \\
R^\times & 0 & p(R) & 0 \\
\mathbb{Z} & 0 & 0 & 0 & 0
\end{array}
$$
By the Künneth theorem, \( H_2(R^x \times R^x) \cong H_2(R^x) \oplus H_2(R^x) \oplus R^x \otimes R^x \). An easy calculation shows
\[
d_{1,2}^1 : E_{1,2}^1 = H_2(R^x \times R^x) \rightarrow E_{0,2}^1 = H_2(R^x \times R^x)
\]
\[
(r, s, a \otimes b) \mapsto (-r + s, r - s, -b \otimes a - a \otimes b).
\]
Therefore \( E_{0,2}^1 \cong H_2(R^x) \oplus (R^x \otimes R^x)_\sigma \). On the other hand, since for an abelian group \( A \), \( H_2(A) \cong \wedge^2 A \) (see Lemma 3.3 below), we have
\[
E_{0,2}^2 \cong \wedge^2 (R^x \times R^x)/K,
\]
where \( K = \langle (b, a) \wedge (d, c) - (a, b) \wedge (c, d) | a, b, c, d \in R^x \rangle \). Now an isomorphism
\[
H_2(R^x) \oplus (R^x \otimes R^x)_\sigma \xrightarrow{\cong} \wedge^2 (R^x \times R^x)/K
\]
can be given by
\[
(a \wedge b, c \otimes d) \mapsto (a, 1) \wedge (b, 1) + (c, 1) \wedge (1, d).
\]

Here we introduce a notion that will be used later. Let \( G \) be a group and let
\[
c(g_1, g_2, \ldots, g_n) := \sum_{\sigma \in \Sigma_n} \text{sign}(\sigma)[g_{\sigma(1)}|g_{\sigma(2)}| \cdots |g_{\sigma(n)}] \in H_n(G),
\]
where \( g_i \in G \) pairwise commute and \( \Sigma_n \) is the symmetric group of degree \( n \). Here we use the bar resolution of \( G \) [4, Chapter I, Section 5] to define the homology of \( G \).

**Lemma 3.3.** Let \( G \) and \( G' \) be two groups.

(i) If \( h_1 \in G \) commutes with all the elements \( g_1, \ldots, g_n \in G \), then
\[
c(g_1 h_1, g_2, \ldots, g_n) = c(g_1, g_2, \ldots, g_n) + c(h_1, g_2, \ldots, g_n).
\]

(ii) For every \( \sigma \in \Sigma_n \),
\[
c(g_{\sigma(1)}, \ldots, g_{\sigma(n)}) = \text{sign}(\sigma)c(g_1, g_2, \ldots, g_n).
\]

(iii) The shuffle product of \( c(g_1, \ldots, g_p) \in H_p(G, \mathbb{Z}) \) and \( c(g'_1, \ldots, g'_q) \in H_q(G', \mathbb{Z}) \) is \( c((g'_1, 1), \ldots, (g_p, 1), (1, g'_1), \ldots, (1, g'_q)) \in H_{p+q}(G \times G') \).

(iv) If \( A \) is an abelian group, then the map \( \wedge^2 A \rightarrow H_2(A) \) given by \( a \wedge b \mapsto c(a, b) \) is an isomorphism.

*Proof.* The proof is standard. \hfill \( \square \)

4. **Proof of Proposition 2.1**

**Lemma 4.1.** The differential map \( d_{2,1}^2 : p(R) \rightarrow H_2(R^x) \oplus (R^x \otimes R^x)_\sigma \) is given by \( [a] \mapsto (a \wedge (1 - a), -a \otimes (1 - a)) \).

*Proof.* Here we argue as in [10, pp. 189-190] or [6, pp 192-193]. Consider the following commutative diagram with exact rows
\[
0 \longrightarrow F_2 \otimes GL_2 \rightarrow H_1(X) \longrightarrow F_2 \otimes GL_2 \rightarrow C_1(R^2) \xrightarrow{1 \otimes \partial \Sigma} F_2 \otimes GL_2 \rightarrow C_0(R^2)
\]
\[
0 \longrightarrow F_1 \otimes GL_2 \rightarrow H_1(X) \xrightarrow{1 \otimes j} F_1 \otimes GL_2 \rightarrow C_1(R^2) \rightarrow F_1 \otimes GL_2 \rightarrow C_0(R^2).
\]
The map \( p(R) \to H_1(\text{GL}_2, H_1(X)) \), constructed in the previous section, sends \([a]\) to the element represented by \(z_a = [(g_2, g_1) - (g_3, 1)] \otimes \partial_2(\infty, 0, 1)\). Then \((1 \otimes j)(z_a)\) is equal to

\[
\left[g_2g_1, g_1^2 - (g_3g_1, g_1) - (g_2g_1, g_2) + (g_3g_2, g_2) + (g_2, g_1) - (g_3, 1)\right] \otimes (\infty, 0).
\]

Since \(a^{-1}g_2g_1 = g_1^2g_3^{-1}, (a^{-1}g_2g_3^2, g_1g_2) = (g_3g_2, g_3g_1)(a^{-1}(1-a) \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix})\), it is easy to see that \(\delta_2(u_a) \otimes (\infty, 0) = (1 \otimes j)(z_a)\), where

\[
u_a = + (g_3g_1, g_2, g_1) - (g_3g_2, g_3g_1, g_2) - (a^{-1}g_2g_3^2, g_2, g_1g_2) + (a^{-3}g_2g_3^2, a^{-2}g_2^2, 1) - (a^{-3}g_2g_3^2, a^{-2}g_3^2, 1) + (a^{-1}g_2g_1, a^{-1}g_2^2, 1) - (g_1^2g_3^{-1}, a^{-1}g_3^{-1}, 1) + (g_3, a^{-1}g_3^2, 1).
\]

(Hear by \(ag, a \in R^\times\) and \(g \in \text{GL}_2\), we mean \(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\).) We have

\[
u_a \otimes \partial_1(\infty, 0) = (u_a w - u_a) \otimes (\infty) \in F_2 \otimes_{\text{GL}_2} C_0(R^2),
\]

where \(w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\). The element of \(E_{0,2}^2 = H_2(R^\times \times R^\times)/\text{im}(d_{1,2}^1)\) represented by \((u_a w - u_a) \otimes (\infty)\) is \(d_{2,1}^1([a])\). We recall the isomorphism

\[
H_2(\text{GL}_2, C_0(R^2)) \xrightarrow{\sim} H_2(B),
\]

where \(B := \text{Stab}_{\text{GL}_2}(\infty) = \left(\begin{array}{cc} R^\times & R \\ 0 & R^\times \end{array}\right)\). This is described on the chain level by

\[
F_* \otimes_{\text{GL}_2} C_0(R^2) \to F_* \otimes_B \mathbb{Z}, \quad y \otimes (\infty) \mapsto y \otimes 1.
\]

Let \(F_*(B) \to \mathbb{Z}\) be the standard resolution of \(\mathbb{Z}\) over \(B\). An augmented preserving chain map of \(B\)-resolutions \(\phi_* : F_* \to F_*(B)\) is obtained as follows: Let \(s : \text{GL}_2/B \to \text{GL}_2\) be any (set-theoretic) section of the canonical projection \(\pi : \text{GL}_2 \to \text{GL}_2/B\). For \(g \in \text{GL}_2\), set \(\overline{g} = (s \circ \pi(g))^{-1} g\). Then

\[
\phi_n(g_0, \ldots, g_n) = (\overline{g_0}, \ldots, \overline{g_n}).
\]

By choosing the section

\[
s(gB) = \begin{cases} 1 & \text{if } g(\infty) = \infty \\ w & \text{if } g(\infty) = 0 \\ \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} & \text{if } g(\infty) = b^{-1}, \end{cases}
\]
By an easy computation we see that under the map we see that

\[ g = \begin{cases} 
  g & \text{if } g(\infty) = \infty \\
  wg & \text{if } g(\infty) = 0 \\
  \begin{pmatrix}
  1 & 0 \\
  -b & 1 
  \end{pmatrix} g & \text{if } g(\infty) = b^{-1}.
\end{cases} \]

Thus on the chain level we have the following map

\[ F_n \otimes_{GL_2} C_0(R^2) \rightarrow F_n(B) \otimes_B \mathbb{Z}, \quad (g_0, \ldots, g_n) \otimes (\infty) \mapsto (\overline{g}_0, \ldots, \overline{g}_n) \otimes 1, \]

which induces the isomorphism (5) on homology. The diagonal inclusion $R^\times \times R^\times \rightarrow B$ splits by the map $p : B \rightarrow R^\times \times R^\times$, \( \begin{pmatrix} a & c \\ 0 & b \end{pmatrix} \) \mapsto \((a, b)\).

This induces an isomorphism $H_2(B) \simeq H_2(R^\times \times R^\times)$, which is inverse to the isomorphism $H_2(R^\times \times R^\times) \simeq H_2(B)$, obtained by a theorem of Suslin [20, Theorem 1.9].

Let $B_\ast(G) \rightarrow \mathbb{Z}$ and $F_\ast(G) \rightarrow \mathbb{Z}$ be the bar and the standard resolutions of $\mathbb{Z}$ over a group $G$, respectively. Then the map

\[ F_\ast(G) \rightarrow B_\ast(G), \quad (g_0, \ldots, g_n) \mapsto [g_0g_1^{-1} | g_1g_2^{-1} | \cdots | g_{n-2}g_{n-1}^{-1} | g_{n-1}g_n^{-1}] \]

induces the identity map of $H_\ast(G)$. Now if we follow the isomorphisms

\[ H_2(GL_2, C_0(F^2)) \xrightarrow{\sim} H_2(F_\ast(B) \otimes_B \mathbb{Z}) \xrightarrow{\sim} H_2(F_\ast(R^\times \times R^\times) \otimes (R^\times \times R^\times) \mathbb{Z}) \]

then by a direct computation one sees that $(u_w - u_a) \otimes (\infty)$ maps to

\[ X_a = -[(a^{-1}, a^{-1})|(-1, a)] + [(a^{-1}, a)|(-1, a)] + [(-a^{-1}, a)|(-1, a) - (a^{-1}, a)|(-1, a)] \]

\[ -[(a^{-1}, a)|(-1, a)] + [(a^{-1}, a)|(-1, a)] - [(a^{-1}, a)|(-1, a)] + [(-a^{-1}, a)|(-1, a) - (a^{-1}, a)|(-1, a)] \]

\[ +[(a^{-2}(1 - a)^2, 1)|(-1, a)] - [(a^{-1}, a^{-1}|(-1, a)] - [(a^{-1}, a^{-1}|(-1, a)] + [(a^{-1}, a^{-1}|(-1, a)] \]

\[ +[(a^{-1}(1 - a), a^{-1}(1 - a))|(-1, a)] - [(a^{-1}, a^{-1}|(-1, a)] - [(a^{-1}, a^{-1}|(-1, a)] \]

\[ +[(a^{-1}(1 - a), a^{-1}(1 - a))|(-1, a)] - [(a^{-1}, a^{-1}|(-1, a)] - [(a^{-1}, a^{-1}|(-1, a)] \]

Using the fact that

\[ \delta_{a}[((a^{-1}, a)|(-1, a)|(-1, a))] = [(-a^{-1}, a]|(-1, a) - [(a^{-1}, a)|(-1, a)] - [(a^{-1}, a)|(-1, a)] - [(a^{-1}, a)|(-1, a)] \]

we see that

\[ X_a = +c((-1, a), (-a, a^{-1}) + 2c((-1, a), (a, a^{-1})) \]

\[ +c((a^{-2}(1 - a)^2, 1), (a^{-1}, a)) + c((a^{-1}, a^{-1}), (a^{-1}, a^{-1}(1 - a)^2)) \]

\[ +c((a^{-1}(1 - a), a^{-1}(1 - a)), (a, a^{-1})] \]

By an easy computation we see that under the map

\[ H_2(R^\times \times R^\times) \rightarrow E_{0,2}^2 \simeq \wedge^2(R^\times \times R^\times)/K, \]

$X_a$ maps to $(a, 1) \wedge (1 - a, 1) - (a, 1) \wedge (1, 1 - a)$ (see Lemma 3.3). Now the claim follows from the isomorphism (4). \qed
As a corollary, we give a homological proof of a theorem of W. van der Kallen [23].

**Corollary 4.2.** Let $R$ be a commutative ring with many units. Then

$$K_2(R) \simeq K_2^M(R).$$

**Proof.** By an easy analysis of the above spectral sequence, and Lemma 1.5, one sees that

$$H_2(\text{GL}_2) \simeq E_\infty^{0,2} \simeq H_2(R^\times) \oplus (R^\times \otimes R^\times)_{\alpha} / \langle (a \wedge (1-a), -a \otimes (1-a)) | a \in R^\times \rangle.$$

From the maps

- $H_2(R^\times) \twoheadrightarrow E_\infty^{0,2}$, $x \mapsto (x,0)$,
- $E_\infty^{0,2} \twoheadrightarrow H_2(R^\times)$, $(x, c \otimes d) \mapsto x + c \wedge d$,
- $K_2^M(R) \twoheadrightarrow E_\infty^{0,2}$, $\{a, b\} \mapsto (a \wedge b, -a \otimes b)$,
- $E_\infty^{0,2} \twoheadrightarrow K_2^M(R)$, $(x, c \otimes d) \mapsto -\{c, d\}$,

we obtain the decomposition

$$H_2(\text{GL}_2) \simeq H_2(R^\times) \oplus K_2^M(R).$$

From the corresponding Lyndon-Hochschild-Serre spectral sequence of the extension $1 \to \text{SL} \to \text{GL} \xrightarrow{\text{det}} R^\times \to 1$, using the fact that $SK_1(R) := \text{SL}/E(R) = 0$ [13, Proposition 2], we obtain the decomposition

$$H_2(\text{GL}) \simeq H_2(R^\times) \oplus K_2(R).$$

Here $E(R)$ is the elementary subgroup of GL. To obtain the above decomposition we use the fact that $E(R)$ is a perfect group, and $K_2(R) = H_2(E(R))$. Now the claim follows from the homology stability theorem $H_2(\text{GL}_2) \simeq H_2(\text{GL})$ [9, Theorem 1]. □

Now we are ready to prove Proposition 2.1.

**Proof of Proposition 2.1.** The spectral sequence studied in the above, implies the exact sequence

$$E_2^{0,3} \to H_3(\text{GL}_2) \to B(R) \to 0.$$

One can show without difficulty that the map $H_3(\text{GL}_2) \to B(R)$ coincides with the one obtained in Section 2. The rest follows from the fact that $E_2^{0,3} = H_3(R^\times \times R^\times) / \text{im}(d_{1,3}^1)$. □

5. Bloch-Wigner exact sequence

We are now ready to prove our main theorem, which can be considered as a generalization of the known Bloch-Wigner theorem.

**Theorem 5.1.** Let $R$ be a commutative ring with many units. We have the exact sequence

$$\text{Tor}_1^Z(\mu_R, \mu_R) \to \tilde{H}_3(\text{SL}_2(R), \mathbb{Z}) \to B(R) \to 0,$$
where \( \tilde{H}_3(\text{SL}_2(R), \mathbb{Z}) := H_3(\text{GL}_2)/\text{im}(H_3(\mathbb{Z}) \oplus R^\times \otimes H_2(\mathbb{Z})). \)

When \( R \) is an integral domain, then the left hand side map in the above exact sequence is injective.

**Proof.** Consider the exact sequence

\[
E^2_{0,3} \rightarrow H_3(\text{GL}_2) \rightarrow B(R) \rightarrow 0
\]

(see the proof of Proposition 2.1 in the previous section). To describe \( E^2_{0,3} \), let \( M = H_3(\mathbb{Z}) \oplus H_3(\mathbb{Z}) \oplus R^\times \otimes H_2(\mathbb{Z}) \oplus H_2(\mathbb{Z}) \otimes R^\times \). We have the following commutative diagram with exact rows

\[
\begin{array}{cccc}
0 & \rightarrow & M & \rightarrow & H_3(\mathbb{Z}) \times R^\times & \rightarrow & \text{Tor}_1^{\mathbb{Z}}(\mu_R, \mu_R) & \rightarrow & 0 \\
& & \downarrow d_{1,3}|M & & \downarrow d_{1,3} & & \downarrow \nu & \downarrow & \\
0 & \rightarrow & M & \rightarrow & H_3(\mathbb{Z}) \times R^\times & \rightarrow & \text{Tor}_1^{\mathbb{Z}}(\mu_R, \mu_R) & \rightarrow & 0.
\end{array}
\]

Here the exact rows follow from the Künneth theorem and \( \nu \) is induced in a natural way. We also have

\[
d_{1,3}|M : (r, s, a \otimes r', s' \otimes b) \mapsto (-r + s, r - s, -a \otimes r' + b \otimes s', r' \otimes a - s' \otimes b).
\]

So the Snake lemma implies the exact sequence

\[
(6) \quad N \rightarrow E^2_{0,3} \rightarrow \text{Tor}_1^{\mathbb{Z}}(\mu_R, \mu_R)/\text{im}(\nu) \rightarrow 0,
\]

where \( N \simeq H_3(\mathbb{Z}) \oplus R^\times \otimes H_2(\mathbb{Z}). \) From the commutative diagram

\[
\begin{array}{ccc}
N & \rightarrow & N \\
\downarrow & & \downarrow \\
E^2_{0,3} & \rightarrow & H_3(\text{GL}_2) \rightarrow B(R) \rightarrow 0,
\end{array}
\]

we obtain the exact sequence

\[
E^2_{0,3}/N \rightarrow H_3(\text{GL}_2)/N \rightarrow B(R) \rightarrow 0.
\]

Now the first claim follows from the exact sequence (6). To complete the proof, for an integral domain \( R \) we must prove the injectivity of

\[
\text{Tor}_1^{\mathbb{Z}}(\mu_R, \mu_R) \rightarrow \tilde{H}_3(\text{SL}_2).
\]

Denote by \( \overline{F} \) the algebraic closure of the quotient field \( F \) of \( R \). Since the homomorphism

\[
\text{Tor}_1^{\mathbb{Z}}(\mu_R, \mu_R) \rightarrow \text{Tor}_1^{\mathbb{Z}}(\mu_\overline{F}, \mu_\overline{F})
\]

is injective, we may replace \( R \) with the field \( \overline{F} \). We know that for an algebraically closed field \( F \), \( H_3(\text{SL}_2(F)) \simeq \tilde{H}_3(\text{SL}_2(F)) \) (see the proof of Corollary 5.4 below) and \( K_3(F)^{\text{ind}} \simeq H_3(\text{SL}_2(F)) \) [14, Proposition 6.4]. But torsion elements of \( K_3(F)^{\text{ind}} \) are known by a result of Suslin [18], [19]. \( \square \)
Remark 5.2. (i) The main results of this paper, Proposition 2.1 and Theorem 5.1, remain true over non-commutative rings with many units.

Let $R$ be a non-commutative ring with many units (see [14, §2], [9, §1]). Let $p(R)$ be the quotient of the free abelian group $Q(R)$, defined as in Section 1, to the subgroup generated by

$$[aba^{-1}] - [b],$$

$$[a] - [b] + [a^{-1}b] - [(1 - b^{-1})^{-1}(1 - a^{-1})] + [(1 - b)^{-1}(1 - a)],$$

where $a, 1 - a, b, 1 - b, a - b \in R^\times$. We define $B(R)$ as the kernel of the map

$$\lambda : p(R) \to (K_1(R) \otimes K_1(R))_\sigma, \quad [a] \mapsto a \otimes (1 - a).$$

Since $K_2^M(R) \simeq (K_1(R) \otimes K_1(R))_\sigma/(a \otimes (1 - a)|a, 1 - a \in K_1(R))$, we have the exact sequence

$$0 \to B(R) \to p(R) \xrightarrow{\lambda} (K_1(R) \otimes K_1(R))_\sigma \to K_2^M(R) \to 0.$$

Note that by the homology stability theorem [9, Theorem 1], $K_1(R) \simeq R^\times/[R^\times, R^\times]$. Moreover Lemmas 3.1 and 3.2 are valid, $E_{p,0}^2 = 0$, $p \geq 0$, and

$$H_1(GL_2, H_1(X)) \simeq p(R) \oplus K_1(R).$$

We also have $E_{0,2}^2 \simeq H_2(R^\times) \oplus (K_1(R) \otimes K_1(R))_\sigma$ and

$$d_{1,1}^2 : p(R) \to H_2(R^\times) \oplus (K_1(R) \otimes K_1(R))_\sigma$$

is given by $[a] \mapsto \left(c(a, 1 - a), -a \otimes (1 - a)\right)$. Thus we have the isomorphism

$$B(R) \simeq H_3(GL_2)/H_3(R^\times \times R^\times)$$

and the Bloch-Wigner exact sequence

$$\text{Tor}_1^R(K_1(R), K_1(R)) \to \tilde{H}_3(SL_2(R), \mathbb{Z}) \to B(R) \to 0.$$

(ii) For a non-commutative ring $R$, $K_2^M(R)$ and $K_2(R)$ are not isomorphic in general. For more information in this direction see [9, Theorem 4.2.1].

Let $R^\times \otimes H_2(R^\times) \to H_3(GL_2)$, $a \otimes (b \wedge c) \mapsto a \cup (b \wedge c)$ be the shuffle product. Note that

$$a \cup (b \wedge c) = c(\text{diag}(a, 1), \text{diag}(1, b), \text{diag}(1, c)).$$

Let $\Phi : R^\times \otimes K_2^M(R) \to H_3(GL_2)$ be defined by

$$a \otimes \{b, c\} \mapsto c(\text{diag}(a, a), \text{diag}(b, 1), \text{diag}(c, c^{-1})).$$

Lemma 5.3. Let inc : $R^\times \to GL_2$ be the natural inclusion. Then in $H_3(GL_2)$ we have

$$\Phi(a \otimes \{b, c\}) = H_3(\text{inc}) \left(c(a, b, c) - c \cup (a \wedge b) + a \cup (b \wedge c) + b \cup (a \wedge c),
2 a \cup (b \wedge c) = 2 H_3(\text{inc}) \left(c(a, b, c)\right) + \Phi(b \otimes \{a, c\} - c \otimes \{a, b\}).$$

Proof. These are direct computations. \qed

The following corollary justifies why we call Theorem 5.1 a generalization of the Bloch-Wigner theorem.
Corollary 5.4. Let $R$ be a commutative ring with many units.

(i) Then
\[ \text{Tor}_1^Z(\mu_R, \mu_R)_\mathbb{Z} \to H_3(\text{SL}_2, \mathbb{Z}[1/2]) \to B(\mathbb{R}) \to 0 \]
is exact.

(ii) If $R^\times = R^{\times 2}$, then
\[ \text{Tor}_1^Z(\mu_R, \mu_R) \to H_3(\text{SL}_2) \to B(\mathbb{R}) \to 0 \]
is exact.

Moreover when $R$ is a domain, then the left hand side maps in the above exact sequences are injective.

Proof. From the corresponding Lyndon-Hochschild-Serre spectral sequence of the extension $1 \to \text{SL}_2 \to \text{GL}_2 \xrightarrow{\text{det}} R^\times \to 1$, we obtain the exact sequence
\[ H_2(R^\times, H_2(\text{SL}_2)) \xrightarrow{\partial} H_3(\text{SL}_2) \to H_1(R^\times, H_2(\text{SL}_2)) \to 0. \]

(i) The map $\gamma : R^\times \times \text{SL}_2 \to \text{GL}_2$, $(a, g) \mapsto ag$ induces an isomorphism of homology groups $H_3(R^\times \times \text{SL}_2, \mathbb{Z}[1/2]) \xrightarrow{\sim} H_3(\text{GL}_2, \mathbb{Z}[1/2])$ (see the proof of Theorem 6.1 in [14] or proof of Corollary 1 in [13]). So by the Künneth theorem, $H_3(\text{SL}_2, \mathbb{Z}[1/2]) \to H_3(\text{GL}_2, \mathbb{Z}[1/2])$. One can also see that
\[ H_1(R^\times, H_2(\text{SL}_2, \mathbb{Z}[1/2])) \simeq H_1(R^\times, H_2(\text{SL}, \mathbb{Z}[1/2])) \simeq (R^\times \otimes K^M_2(R))_{\mathbb{Z}[1/2]}. \]

See [13, Lemma 2] or the proof of Corollary 6.2 in [14] for a proof of the first isomorphism. Note that we use Corollary 4.2 for the second isomorphism. Thus we obtained the exact sequence
\[ 0 \to H_3(\text{SL}_2, \mathbb{Z}[1/2]) \to H_3(\text{GL}_2, \mathbb{Z}[1/2]) / H_3(R^\times, \mathbb{Z}[1/2]) \to (R^\times \otimes K^M_2(R))_{\mathbb{Z}[1/2]} \to 0. \]

We show that the map
\[ (R^\times \otimes K^M_2(R))_{\mathbb{Z}[1/2]} \to H_3(\text{GL}_2, \mathbb{Z}[1/2]) / H_3(R^\times, \mathbb{Z}[1/2]), \]
\[ a \otimes \{b, c\} \mapsto \frac{1}{2} \epsilon(\text{diag}(a, a), \text{diag}(b, 1), \text{diag}(c, c^{-1})), \]
splits the above exact sequence. In order to verify this, consider the following commutative diagram with exact rows
\[ \begin{array}{ccc}
0 & \to & H_3(\text{SL}_2, \mathbb{Z}[1/2]) \to S_2 \to (R^\times \otimes K^M_2(R))_{\mathbb{Z}[1/2]} \to 0 \\
\downarrow & & \downarrow \\
0 & \to & H_3(\text{SL}, \mathbb{Z}[1/2]) \to S_3 \to (R^\times \otimes K^M_2(R))_{\mathbb{Z}[1/2]} \to 0,
\end{array} \]
where $S_i := H_3(\text{GL}_i, \mathbb{Z}[1/2]) / H_3(R^\times, \mathbb{Z}[1/2])$. The second exact sequence can be obtained similar to the first one. Consider the homotopy equivalence
\[ B\text{GL}^+ \sim B\mathbb{R}^\times \times B\text{SL}^+ \]
By an easy analysis of this spectral sequence, one sees that, the kernel of $H_3(\text{GL}_3) \simeq H_3(\text{GL}) \simeq H_3(\text{SL}) \oplus K_2^M(R) \oplus H_3(R^\times)$. It is easy to see that the embedding $R^\times \otimes K_2^M(R) \rightarrow H_3(\text{GL})$ is induced by $a \otimes \{b, c\} \mapsto c(\text{diag}(a, 1, 1), \text{diag}(1, b, 1), \text{diag}(1, c, c^{-1}))$, and that it splits the second exact sequence in the above diagram. The image of $c(\text{diag}(a, a), \text{diag}(b, 1), \text{diag}(c, c^{-1}))$ in $H_3(\text{GL}_3)$ is equal to
c
\[
\begin{align*}
& c(\text{diag}(a, 1, a^{-1}), \text{diag}(b, b^{-1}, 1), \text{diag}(c, 1, c^{-1})) \\
& + c(\text{diag}(a^2, 1, 1), \text{diag}(b, 1, 1), \text{diag}(1, c, c^{-1})). \\
\end{align*}
\]

The rest is easy to verify. Now using Lemma 5.3, it is easy to see that $H_3(\text{SL}_2, \mathbb{Z}[1/2])_{R^\times} \simeq H_3(\text{GL}_2, \mathbb{Z}[1/2]) / \text{im}(L_{\mathbb{Z}[1/2]})$, where $L := H_3(R^\times) \oplus R^\times \otimes H_2(R^\times)$.

(ii) The map $\gamma$ induces the short exact sequence $1 \rightarrow \mu_2, R \rightarrow R^\times \times \text{SL}_2 \rightarrow \text{GL}_2 \rightarrow 1$, and we consider its corresponding spectral sequence,

\[E^2_{p,q} = H_p(\text{GL}_2, H_q(\mu_2)) \Rightarrow H_{p+q}(R^\times \times \text{SL}_2).\]

Using the fact that $H_2(R^\times)$ and $K_2^M(R)$ are uniquely 2-divisible (see the proof of Proposition 1.2 in [2]), we have the following $E^2$-terms:

\[
\begin{array}{cccc}
\ast & H_3(\mu_2) & 0 & \ast \\
\ast & H_2(\mu_2) & 0 & \ast \\
\ast & \mu_2 & 0 & \text{Tor}^\mathbb{Z}(\mu_R, \mu_2, R) \\
\ast & \mathbb{Z} & R^\times & H_2(\text{GL}_2) \\
\ast & & H_3(\text{GL}_2) & H_4(\text{GL}_2).
\end{array}
\]

By an easy analysis of this spectral sequence, one sees that, the kernel of $H_3(\text{SL}_2) \rightarrow H_3(\text{GL}_2)$ is torsion of exponent 4. On the other hand

\[H_2(R^\times, H_2(\text{SL}_2)) \simeq H_2(R^\times) \otimes H_2(\text{SL}_2) \oplus \text{Tor}^\mathbb{Z}_1(H_1(R^\times), H_2(\text{SL}_2)) \simeq H_2(R^\times) \otimes K_2^M(R) \oplus \text{Tor}^\mathbb{Z}_1(\mu_R, K_2^M(R)).\]

Since $H_2(R^\times) \otimes K_2^M(R)$ and $\text{Tor}^\mathbb{Z}_1(\mu_R, K_2^M(R))$ are uniquely 2-divisible, $\beta$ must be a trivial map. This implies that $H_3(\text{SL}_2) \rightarrow H_3(\text{GL}_2) / H_3(R^\times)$ is injective. The rest of the proof is similar to the proof of the part (i), since

\[H_1(R^\times, H_2(\text{SL}_2)) \simeq R^\times \otimes K_2^M(R).\]

Here a splitting map $R^\times \otimes K_2^M(R) \rightarrow H_3(\text{GL}_2) / H_3(R^\times)$ can be given by the formula

\[a \otimes \{b, c\} \mapsto c(\text{diag}(\sqrt{a}, \sqrt{a}), \text{diag}(1, b, 1), \text{diag}(c, c^{-1})).\]
Corollary 5.5. Let \( k \) be an algebraically closed field of \( \text{char}(k) \neq 2 \). Let \( R \) be the ring of dual numbers \( k[\epsilon] \) or a henselian local ring with residue field \( k \).

(i) Then we have the following exact sequence

\[
\text{Tor}_1^n(\mu_R, \mu_R) \to H_3(\text{SL}_2) \to B(R) \to 0.
\]

If \( R \) is a henselian domain, then the left hand side map in this exact sequence is injective.

(ii) If \( \text{char}(k) = 0 \), then \( H_3(\text{SL}_2) \cong K_3(R)^\text{ind} \). Furthermore, if \( R \) is a \( k \)-algebra, then we have the exact sequence

\[
0 \to \text{Tor}_1^n(\mu_R, \mu_R) \to K_3(R)^\text{ind} \to B(R) \to 0.
\]

Proof. Since \( \text{char}(k) \neq 2 \), \( R^\times = R^{\times 2} \) (use Hensel’s lemma, when \( R \) is henselian). Thus the first part follows from Theorem 5.1 and Corollary 5.4. The proof of the isomorphism \( H_3(\text{SL}_2) \cong K_3(R)^\text{ind} \) is similar to the proof of Proposition 6.4(iii) in [14]. To prove Proposition 6.4(iii) in [14], we used Theorems 5.4(iii) and 6.1(iii) in [14]. Note that these theorems are also valid for rings considered here, with almost the same proofs, once we make sure that Proposition 4.1 in [14] is valid as well. But the proof of this proposition follows \textit{mutatis mutandis} as in [14], noting that here \( \mu_R \cong \mu_k \). If \( R \) is the ring of dual numbers \( k[\epsilon] \), the isomorphism \( \mu_R \cong \mu_k \) follows from a direct computation. If \( R \) is a local henselian ring, this follows from Hensel’s lemma. In both cases we need the condition \( \text{char}(k) = 0 \). To complete the proof, we have to show that if \( R \) is a \( k \)-algebra, then

\[
\text{Tor}_1^n(\mu_k, \mu_k) \to K_3(R)^\text{ind}
\]

is injective. In this case, \( K_3(k)^\text{ind} \) is a direct summand of \( K_3(R)^\text{ind} \) and the above map factors through \( K_3(k)^\text{ind} \). Thus the injectivity claim can be proved as in the last paragraph of page 189 in [6]. This also follows from a result of Suslin in [21] or [19]. \( \square \)

Remark 5.6. Corollary 5.5 remains true if \( k \) is an infinite field such that \( k^\times = k^{\times 2} \) and \( \text{char}(k) \neq 2 \).

6. Relative bloch-wigner exact sequence

For a functor \( \mathcal{F} : \text{Rings} \to \underline{Ab} \) from the category of rings to the category of abelian groups, we set \( \mathcal{F}(R[\epsilon], \langle \epsilon \rangle) := \ker(\mathcal{F}(R[\epsilon]) \xrightarrow{\epsilon=0} \mathcal{F}(R)) \).

Proposition 6.1. (i) Let \( R[\epsilon] \) be the ring of dual numbers, where \( R \) is a ring with many units. Then we have the relative Bloch-Wigner exact sequence

\[
0 \to K_3(R[\epsilon], \langle \epsilon \rangle)^\text{ind}_{\mathbb{Q}} \to p(R[\epsilon], \langle \epsilon \rangle)_{\mathbb{Q}} \to \bigwedge^2(R[\epsilon], \langle \epsilon \rangle)^\mathbb{Q} \to K_2(R[\epsilon], \langle \epsilon \rangle)_{\mathbb{Q}} \to 0.
\]

(ii) Let \( k \) be an algebraically closed field of \( \text{char}(k) = 0 \). Then we have the relative Bloch-Wigner exact sequence

\[
0 \to K_3(k[\epsilon], \langle \epsilon \rangle)^\text{ind} \to p(k[\epsilon], \langle \epsilon \rangle) \to \bigwedge^2(k[\epsilon], \langle \epsilon \rangle)^\mathbb{Q} \to K_2(k[\epsilon], \langle \epsilon \rangle) \to 0.
\]
Proof. The ring $R[\epsilon]$ is a ring with many units [8, Proposition 2.5]. Consider the following diagram with exact rows

$$
0 \rightarrow K_3(R[\epsilon])_{\text{ind}} \rightarrow p(R[\epsilon])_Q \rightarrow \mathcal{K}_2(R[\epsilon])_Q \rightarrow 0
$$

The exact rows follow from Corollary 5.4 and [14, Proposition 6.4], [7, Theorem 2.2]. The vertical maps are surjective. We can break the above diagram into two diagrams with rows that are short exact sequences. Now the part (i) follows from the Snake lemma. The proof of (ii) is analogue. Here one has to use Corollary 5.5(ii).

Remark 6.2. (i) For a commutative ring $R$, the relative group $K_2(R[\epsilon], \langle \epsilon \rangle)$ is studied by W. van der Kallen in terms of generators and relations [22]. As a special case, he proves that when $\frac{1}{2} \in R$, then $K_2(R[\epsilon], \langle \epsilon \rangle) \simeq \Omega_{R/Z}^1$.

(ii) The relative group $K_3(R[\epsilon], \langle \epsilon \rangle)_{\text{ind}}$ is studied in [8] in terms of homology of linear groups.

(iii) For a field $k$, the groups $p(k[\epsilon], \langle \epsilon \rangle)$ and $K_3(k[\epsilon], \langle \epsilon \rangle)_{\text{ind}}$ are studied by Cathelineau in [5].

Corollary 6.3. Let $k$ be an algebraically closed field of $\text{char}(k) = 0$. Let $R$ be a henselian local $k$-algebra with residue ring $R/m \simeq k$. For a functor $\mathcal{F} : \text{Rings} \rightarrow \text{Ab}$, define the relative group $\mathcal{F}(R, m) := \ker(\mathcal{F}(R) \rightarrow \mathcal{F}(R/m))$. Then we have the relative Bloch-Wigner exact sequence

$$
0 \rightarrow K_3(R, m)^{\text{ind}} \rightarrow p(R, m) \rightarrow \mathcal{K}_2(R, m)^{\text{ind}} \rightarrow 0.
$$

Proof. The proof is similar to the proof of Proposition 6.1.

Acknowledgements. Part of this work was done during my stay at IHÉS and ICTP. I would like to thank them for their support and hospitality.

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