Dynamics of quantum discord in the purification process

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We investigate the dynamics of quantum discord during the purification process. In the case of Werner states, it is shown that quantum discord is increased after a round of purification protocol. We call this process quantum correlation purification in analogy with entanglement purification. Furthermore, quantum mutual information and classical correlation is also increased during this process. We also give an analytic expression for a class of higher dimensional states which have additive quantum discord.

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It is widely recognized that entanglement is a key resource and ingredient in the field of quantum information science. However, entanglement is not the only aspect of quantum correlations, and it has been found that there are quantum nonlocality without entanglement and quantum speedup with unentangled states. Therefore, it is important and interesting to investigate such quantum correlations. In this context, several measures have been proposed to quantify the nonclassical nature of a quantum state such as quantum discord and quantum speedup with unentangled states. Therefore, it is important and interesting to investigate such quantum correlations. In this context, several measures have been proposed to quantify the nonclassical nature of a quantum state such as quantum discord and quantum speedup with unentangled states. Furthermore, Modi et al. provides an unified view of quantum and classical correlations recently. These measures can be interpreted as indicating the nonclassical correlations in a state, while entanglement is only a special kind of the correlations. Among these measures, quantum discord introduced by Oliver and Zurek is widely used to quantify the quantumness of the correlation contained in a bipartite quantum state. A closely related measure concerning classical correlation was proposed by Henderson and Vedral. Quantum discord is defined as the difference between the total mutual information and the classical correlation contained in a bipartite quantum state. It is more general than entanglement in the sense that separable mixed states can have nonzero quantum discord. It has been shown that almost all quantum states have a nonvanishing quantum discord and offers exponential speed up in the model of computation known as deterministic quantum computation with one qubit.

On the other hand, quantum discord will degrade due to unavoidable interaction between a quantum system and its environment. It has been shown that quantum discord decays exponentially, whereas the entanglement may suffer from sudden death under Markovian noises. The dynamics of quantum discord has also been investigated in a non-Markovian environment and has been demonstrated experimentally under both Markovian and non-Markovian environments. In the field of quantum entanglement theory, it is well known that there are two qualitatively different types of entanglement. One is free, which means that the state can be distilled under local operations and classical communication (LOCC); the other is bound, which means that no LOCC strategy is able to extract pure-state entanglement from the state even if many copies are available. Since entanglement is only a special kind of quantum correlations, we present the following question. The two observers, Alice and Bob, each have N quantum systems with nonzero quantum correlation prepared in a given state \( \rho \). Each one can perform local operations with her/his N particles, and exchange classical information with the other. The question is whether they can in this way obtain a pair of states with higher quantum correlation. We call this process quantum correlation purification in analogy with entanglement purification. The physical meaning of this process is to find a way to extract a subset of states with higher quantum correlation from a large set of states.

Now we want to know whether any mixed state with nonzero quantum correlation can be purified into a new state with higher quantum correlation. In the following discussions we use the quantum discord to measure the quantum correlation. As an example, we shall investigate the dynamics of the quantum discord during the original BBPSSW purification protocol. In the case of Werner states, we find that quantum discord can also be increased after performing the original purification protocol. During one round of purification process, the quantum discord decreases at first and then increases when the intermediate state is transformed into the final state under LOCC. Suppose the noisy Werner state pairs \( \rho^{(0)} \) and \( \rho^{(1)} \) are given as two copies in the following form:

\[
\rho^{(0)} = \rho^{(1)} = \rho = F |\beta_{11}\rangle \langle \beta_{11}| + \frac{1 - F}{3} (|\beta_{01}\rangle \langle \beta_{01}| + |\beta_{10}\rangle \langle \beta_{10}| + |\beta_{00}\rangle \langle \beta_{00}|)
\]

where the four Bell states \( \beta_{ab} \) are...
\[ \frac{1}{\sqrt{2}} (|0, b \rangle + (-1)^a |1, 1 \oplus b \rangle) \]. We begin with a brief outline of the basic process of BBPSSW purification protocol. Suppose Alice and Bob share two identical noisy Werner state pairs \( \rho^{(0)} \) and \( \rho^{(1)} \). Firstly, one of them perform \( \sigma_y \) operations on her/his particles and transform the Werner state pairs into the form: 
\[ F |\beta_{00}\rangle \langle \beta_{00}| + \frac{1}{4F} (|\beta_{10}\rangle \langle \beta_{10}| + |\beta_{11}\rangle \langle \beta_{11}| + |\beta_{01}\rangle \langle \beta_{01}|) \]. 
Next they apply a bilateral C-Not gate for \( \rho^{(0)} \) and \( \rho^{(1)} \) as the control and target qubits, respectively. Then they bilaterally measure \( \rho^{(1)} \) in the computational basis \( \{|0\rangle, |1\rangle\} \) and communicate the measurement outcomes to each other. They keep \( \rho^{(0)} \) if the measurement outcomes coincide. Otherwise, they discard \( \rho^{(0)} \). After performing the measurement, the original mixed Werner state \( \rho^{(0)} \) will be mapped into a new mixed state \( \rho' \) described as:

\[
\rho' = \frac{10F^2 - 2F + 1}{8F^2 - 4F + 5} |\beta_{00}\rangle \langle \beta_{00}| + \frac{6F - 6F^2}{8F^2 - 4F + 5} |\beta_{10}\rangle \langle \beta_{10}| + \frac{2F^2 - 4F + 2}{8F^2 - 4F + 5} |\beta_{01}\rangle \langle \beta_{01}| + \frac{2F^2 - 4F + 2}{8F^2 - 4F + 5} |\beta_{11}\rangle \langle \beta_{11}|.
\]

In fact, the mixed state \( \rho' \) is only an intermediate state and hence it cannot be the initial state in the second round of BBPSSW purification. We perform \( \sigma_y \) operations firstly and then twirl the state applying at random one of the four operations \( \{\sigma_x \otimes \sigma_x, \sigma_y \otimes \sigma_y, \sigma_z \otimes \sigma_z, \mathcal{I} \otimes \mathcal{I}\} \) locally to each party of the pair. In this way we transform the mixed state \( \rho' \) into the Werner state:

\[ \chi = F' |\beta_{11}\rangle \langle \beta_{11}| + \frac{1-F'}{2} |\beta_{01}\rangle \langle \beta_{01}| + \frac{1-F'}{2} |\beta_{10}\rangle \langle \beta_{10}| + \frac{1-F'}{2} |\beta_{00}\rangle \langle \beta_{00}|, \]

with \( F' = \frac{10F^2 - 2F + 1}{8F^2 - 4F + 5} \). The mixed state \( \chi \) is the input state in the next round of BBPSSW purification. Because \( F' > F \) in the range of \( \frac{1}{2} < F < 1 \), using the iteration of the protocol we can probabilistic extract a mixed state pair with a relatively high degree of entanglement from two entangled copies using only LOCC. In order to compare the quantum correlation during the purification process, we compute the quantum discord of the initial mixed state \( \rho \), the intermediate state \( \rho' \) and the final state \( \chi \), respectively.

Firstly, we recall the definitions of quantum discord. To define quantum discord, we start with the quantum mutual information defined as 
\[ I(\rho_{AB}) = S(\rho_{A}) + S(\rho_{B}) - S(\rho_{AB}), \]
where \( S(\rho) = -Tr(\rho \log_2 \rho) \) is the von Neumann entropy. The quantum mutual information is regarded as quantifying the total correlation in the mixed state \( \rho_{AB} \), then the quantum discord is defined as

\[ D(\rho_{AB}) = I(\rho_{AB}) - C(\rho_{AB}) \]  
where \( C(\rho_{AB}) \) denote the classical correlation of the state \( \rho_{AB} \) and it is defined as 
\[ C(\rho_{AB}) = \max_{\{\Pi_k\}} [S(\rho_{A}) - S(\rho_{AB} | \{\Pi_k\})], \]
where the maximum is taken over the set of projective measurements \( \{\Pi_k\} \) and 
\[ S(\rho_{AB} | \{\Pi_k\}) = \sum_{k} p_k S(\rho_k) \]
is the conditional entropy of system A, with 
\[ \rho_k = T_{TB}(\rho_k \rho_{AB} \Pi_k) \]
and 
\[ p_k = T_{TB} (\rho_{AB} \Pi_k) \]. Here, we only consider projective measures because Hamieh et al. have shown that for a two-qubit system the projective measurement is the positive operator-valued measure (POVM) which maximizes classical correlation. Consider the Bell-diagonal states of the form:

\[ \rho_{AB} = \frac{1}{4} \left( I + \sum_{j=1}^{3} c_j |\sigma_j^A\rangle \otimes |\sigma_j^B\rangle \right) = \sum_{a,b} \lambda_{ab} |\beta_{ab}\rangle \langle \beta_{ab}| \]  
with eigenvalues

\[ \lambda_{ab} = \frac{1}{4} \left( 1 + (-1)^a c_1 - (-1)^{a+b} c_2 + (-1)^b c_3 \right) \]  
An analytical formula of quantum discord have been obtained for Bell-diagonal states and can be expressed as:

\[ Q = \frac{1}{4} \left[ (1 - c_1 - c_2 - c_3) \log_2 (1 - c_1 - c_2 - c_3) + (1 - c_1 + c_2 + c_3) \log_2 (1 - c_1 + c_2 + c_3) + (1 + c_1 - c_2 + c_3) \log_2 (1 + c_1 - c_2 + c_3) + (1 + c_1 + c_2 - c_3) \log_2 (1 + c_1 + c_2 - c_3) \right] - \frac{1 - c}{2} \log_2 (1 - c) - \frac{1 + c}{2} \log_2 (1 + c) \]  
where \( c \equiv \max \{|c_1|, |c_2|, |c_3|\} \). For the mixed state of Eq.(2), we have

\[ c_1 = \frac{16F^2 - 8F + 1}{8F^2 - 4F + 5} \quad c_2 = -c_1 \quad c_3 = \frac{12F - 3}{8F^2 - 4F + 5} \]  
It is directly to see that \( c_1 \) is always a nonnegative value, and the maximum value among them is \( |c_3| \). Thus,
we can calculate the quantum discord of $\rho'$ according to the formula in Eq. (6). In Fig. 1 we plot the discord $D(\rho')$ as a function of $F$. In order to make a comparison with the original discord before the purification process, we also plot $D(\rho)$ in Fig. 1. We can see that the quantum discord of the intermediate state always decreases for arbitrary $F$ compared to the initial state. Using the same method, we can also compute $D(\chi)$ and plot it in the same figure. It is shown that quantum discord of the final state is increased after one round of purification process, which indicates that the purification protocol can also purify the quantum discord at the same time. To summarize the above discussions, the quantum discord of the original Werner states experiences two phases during one round of purification protocol. During the first process the quantum discord decreases when the initial Werner states are transformed into the intermediate state. During the second process quantum discord increases when the intermediate state is transformed into the final state under LOCC. For the special case $F = \frac{1}{2}$, the initial Werner state has a zero discord. In this case, it has the form $\rho = \sum_i p_i |i\rangle \langle i| \otimes \rho_i$ where $|i\rangle$ is the orthonormal bases. We can increase its discord to a nonzero value by transforming this classical state into the nonclassical form under LOCC. Thus, we conclude that LOCC can be used to increase discord of a bipartite quantum state both for the zero and non-zero cases.

To further understanding of the evolution of quantum discord under LOCC. Suppose we perform a random bilateral SU(2) rotation locally on each side of $\rho'$ directly and transform it into the following Werner state: $\chi' = F'' |\beta_{11}\rangle \langle \beta_{11}| + \frac{1-f''}{3} |\beta_{01}\rangle \langle \beta_{01}| + \frac{1-f''}{3} |\beta_{10}\rangle \langle \beta_{10}| + \frac{1-f''}{3} |\beta_{00}\rangle \langle \beta_{00}|$, with $F'' = \frac{(F''-1)^2}{2(F''+2)}$. We also plot the quantum discord of $\chi'$ in Fig. 1. We can see that the quantum discord is decreased in this case. It indicates that we can decrease the quantum discord by LOCC. In Fig. 2 and Fig. 3 we also plot the dynamics of quantum mutual information and classical correlation during the purification process, respectively. We find that they are both increased after a round of purification protocol for $\frac{1}{2} < F < 1$. Under the LOCC operation from the intermediate state $\rho'$ to the final state $\chi$ or $\chi'$, we find that the evolution of quantum mutual information and classical correlation is always nonincreasing.

In the above discussions we have investigated the dynamics of quantum discord during the purification process. It is found that LOCC is able to increase or decrease the quantum discord. We show that the original BBPSSW purification can also purify the quantum discord at the same time for the Werner state. This result is reasonable since Bell states have the maximum discord among all Bell-diagonal states. Therefore, we conclude that the quantum discord of arbitrary two-qubit state can be purified by the original purification protocol. However, for the higher-dimensional case, we do not know whether there exist bounded quantum discord in analogy with bound entangled states yet. For bound entangled state, it is true that its quantum correlation quantified by quantum discord is also bounded? At present, we cannot answer this question due to the lack of analytical formula of quantum discord for the higher-dimensional case. Before ending this paper, we want to provide a class of higher-dimensional mixed states with computable quantum discord of the form: $\rho_{AB} = \sum_{ij} p_{ij} |a_i\rangle \langle a_j| \otimes |ji\rangle$, where $|a_i\rangle$ is not restricted to the orthonormal bases. For these class of state, we have $C(\rho_{AB}) = S(\rho_A)$, because the minmum is obtained by using the projector $|i\rangle \langle i|$ to get $|a_i\rangle \langle a_i|$ as the resulting state. Thus we have $D(\rho_{AB}) = S(\rho_B) - S(\rho_{AB})$. For the general case, the classical correlations are super-additive and thus the quantum discord is subadditive. However, for the special mixed state $\rho_{AB}$ mentioned above, we can prove its quantum discord is additive. Recalling the Koashi-Winter monogamy relation for quantum correlations within a pure tripartite state $|\psi\rangle_{ABC}$:

$$D(\rho_{AB}) = E_F(\rho_{AC}) + S(A|C),$$

where $S(A|C) = S(AC) - S(C)$ denotes the quantum conditional entropy of $\rho_{AC}$ and $E_F$ denote the entanglement of formation. We can further define its regularized version $|\psi\rangle_{ABC}$ entanglement cost $E_C(\rho) = \lim_{n\to\infty} \frac{1}{n} E_F(\rho^{\otimes n})$ and $J(\rho_{AB}) = \lim_{n\to\infty} \frac{1}{n} D(\rho^{\otimes n})$, then we have:
\[ \mathcal{J}(\rho_{AB}) = E_C(\rho_{AC}) + S(A|C) \]  

(9)

If we choose \( \rho_{AB} \) as the above mixed states, then we can write its purified form as: 
\[ |\psi\rangle_{ABC} = \sum_i \lambda_i |a_i\rangle |i\rangle |b_i\rangle, \]
where \( |b_i\rangle \) may not be orthogonal for different \( i \). It is directly to see that \( \rho_{AC} \) is a separable state with zero entanglement formation and entanglement cost. Thus by Eq. (8) and (9) we have that quantum discord of 
\[ \rho_{AB} = \sum_{ij} p_{ij} |a_i\rangle |i\rangle \langle a_j| \]  
is additive.

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