Spiral waves: linear and nonlinear theory

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Spiral waves

- CO oxidation on platinum [Nettesheim, von Oertzen, Rotermund, Ertl]
- Belousov-Zhabotinsky reaction [Swinney et al.]
- Calcium waves in Xenopus oocytes [Clapham et al.]
- cAMP signalling of amoebae [Newell]
Spiral waves

[Dynamics of core / spiral tip]
[Modulations of wave trains in far field]

[Li, Ouyang, Petrov, Swinney]

VOLUME 77, NUMBER 10 PHYSICAL REVIEW LETTERS 2 SEPTEMBER 1996

Transition from Simple Rotating Chemical Spirals to Meandering and Traveling Spirals

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(Received 8 May 1996)

Experiments on the Belousov-Zhabotinsky reaction unfold the bifurcation from simple (temporally periodic) rotating spirals to meandering (quasiperiodic) spirals in the neighborhood of a codimension-2 point. There are two types of meandering spirals, inward-petal (epicycloid) spirals and outward-petal (hypocycloid) spirals. These two types of meandering regimes are separated in the phase diagram by a line of traveling spirals that terminates at the codimension-2 point. The observations are in good accord with theory.

PACS numbers: 82.40.Bj, 82.20.Mj, 87.90.+y

Rotating spiral waves are ubiquitous in systems ranging from excitable reaction-diffusion media [1,2] to aggregating slime-mold cells [3] to cardiac muscle tissue [4]. Winfree discovered that under certain conditions a spiral tip meanders rather than follows a periodic circular orbit [5]. Meandering spirals have been subsequently extensively studied experimentally [6–9] and theoretically [10–16]. Experiments [8,9] and theoretical analyses [12,13,16] have shown that the meandering is often not an erratic motion; rather, the spiral tip moves in epicycloid-like [17] orbits (flowerlike orbits with inward petals) or hypocycloid-like orbits [17] (with outward petals) that are quasiperiodic in time [8]. Meandering is of interest in part because of its predicted relation to defect mediated turbulence [18–20]. It may also provide a clue to the cause of cardiac arrythmias, which can lead to ventricular fibrillation [21].

There have been few definitive experimental results on meandering other than the observation that the onset of meandering is a periodic-quasiperiodic transition [8,22]. We present here experiments on the Belousov-Zhabotinsky (BZ) reaction that reveal, as predicted by Barkley [16], an unfolding of the bifurcation to meandering about a codimension-2 point that is the terminus of a line of traveling spirals.

Figure 1 shows the orbit of a spiral tip for the two types of meandering motion [23]: (a) an outward-petal meandering spiral and (b) an inward-petal meandering spiral. We take the spiral tip to be the point with maximum local curvature on the wave front. Hypocycloid motion with outward petals is illustrated in Fig. 1(c), where the primary circle (radius $r_1$) orbits the secondary circle (radius $r_2$) in one direction with frequency $f_2$ and spins about its center in the opposite direction with frequency $f_1$; epicycloid motion with both rotations in the same sense and inward petals is illustrated in Fig. 1(d).

Figure 2 illustrates all four types of spiral motion that we have observed. Figure 2(a) is a simple periodic rotating spiral, which becomes unstable as a parameter is varied. Depending on the control parameter, the system then chooses outward-petal meandering, as in Fig. 2(b), or inward-petal meandering, as in Fig. 2(d). The tip of a meandering spiral emits waves that are compressed in front of the tip and dilated behind the tip. This produces superspirals, as can be seen in Fig. 2(b), which has a retrograde super spiral, and in Fig. 2(d), which has a prograde super spiral. At the transition from outward-petal to inward-petal spirals, the spiral tip travels in a straight line; see Fig. 2(c).

FIG. 1. Meandering spiral with (a) outward and (b) inward petals. The white lines in the images show the trajectories of spiral tip. (c) and (d) illustrate, respectively, a hypocycloid and an epicycloid, analogous to the motion in (a) and (b). The only parameter different in (a) and (b) is the concentration of sulfuric acid in reservoir B: (a) 0.46 M, (b) 0.40 M. The other control parameter in the present experiments is the concentration of malonic acid in reservoir B, which is fixed in this figure and Figs. 2 and 3 at 0.8 M but varied in Fig. 4. Other conditions are fixed in our experiments at the values given in Ref. [23]. The pictures in (a) and (b) are 1.7 mm$^2$. 0031-9007 96 77(10) 2105(4)$10.00 © 1996 The American Physical Society 2105.
Spiral waves in cardiac tissue

- Cardiac arrhythmias can be caused by spiral waves pinned to inhomogeneities
- Low-energy excitation can remove spiral waves [Luther et al.]
- Proposed strategies are sensitive to the relative phase difference between excitation waves and externally induced stimuli

![Canine heart (in vitro) [Luther et al.]](image1)
![Heart model [Fenton et al.]](image2)
Spiral waves in cardiac tissue

**Ventricular Tachycardia (VT)**
Abnormally fast heart rate

**Ventricular Fibrillation (VF)**
Fatal unless treated immediately

Alternans rhythm

Arrhythmic pattern
Spatio-temporal period-doubling of spiral waves

Alternans in cardiac tissue (subcritical?)

Line defects in light-sensitive BZ-reaction [Yoneyama, Fujii, Maeda] (supercritical?)
Outline

- Wave trains: spectral stability and modulation equations

- Planar spiral waves: spiral spectra on the plane and on bounded disks

- One-dimensional defects: anticipated dynamics, and nonlinear stability results
Slowly varying modulations

\[ u(x, t) = u_*(kx - \omega_*(k)t; k) \]

Harmonic modulations

\[ u_*(kx - \omega_*(k)t + \epsilon e^{\lambda t} \cos(\gamma x)) \approx u_*(kx - \omega_*(k)t) + \epsilon e^{\lambda t} \cos(\gamma x) u'_{wt}(kx - \omega_*(k)t) \]

Linear temporal response to spatial modulation with wavenumber \( \gamma \) is given by \( \lambda = \lambda*(i\gamma) \)

Spectrum of wave trains

\[ \lambda*(i\gamma) = -ic_g \gamma - d\gamma^2 + O(|\gamma|^3) \]

\[ e^{\lambda t} \cos(\gamma x) \approx e^{-ic_g \gamma t} e^{i\gamma x} = e^{i\gamma(x-c_g t)} = \cos(\gamma(x - c_g t)) \]
Slowly varying modulations

Viscous Burgers equation: \[ q_T = \lambda''(0)q_{XX} - \omega''(k)(q^2)_X \]

for slowly varying wavenumber modulations \( q(X,T) \)

on scale \( X=\varepsilon(x-c_gt) \) and \( T=\varepsilon^2t/2 \) with \( 0<\varepsilon \ll 1 \)

- Formal derivation: [Howard & Kopell], [Kuramoto]
- Validity over natural time scale \( 1/\varepsilon^2 \): [Doelman, S., Scheel, Schneider]
- Stability of wave trains: [S., Scheel, Schneider, Uecker], [Johnson, Zumbrun], [Iyer, S.]
Outline

- Wave trains: spectral stability and modulation equations
- Planar spiral waves: spiral spectra on the plane and on bounded disks
- One-dimensional defects: anticipated dynamics, and nonlinear stability results
Planar spiral waves

Reaction-diffusion systems: \( u_t = D \Delta u + f(u) \)

Rotating spiral waves: \( u(x, t) = u_*(r, \phi - \omega_* t) \)

Linearization in co-rotating frame: \( \mathcal{L}_* v = D \Delta v + f_u(u_*) v + \omega_* v_\phi \)

Archimedean spiral waves:

\[
\begin{align*}
\mathcal{L}_* & \approx D \partial_{rr} + f_u(u_{wt}(\kappa r + \phi)) + \omega_* \partial_\phi \\
u_*(r, \phi) & \approx u_{wt}(\kappa r + \phi), \quad r \gg 1
\end{align*}
\]

Theorem [S., Scheel] Assume that \( u_*(r, \phi) \) is a generic spiral wave so that the asymptotic wave train has \( c_g > 0 \), then there is an \( a \neq 0 \) such that \( |u_*(r, \phi) - u_{wt}(\kappa r + \phi - a \log r)| \to 0 \) as \( r \to \infty \).

Goals:
- Understand the structure of the spectrum of the linearization about a spiral wave
- Relate the spectra of asymptotic wave train and spiral wave
Spectra of planar spiral waves

Linear dispersion relation of asymptotic wave train in laboratory frame: $\lambda(i\gamma)$

Far field eigenfunctions: $v(r, \phi) \approx e^{i\gamma r} e^{i\ell \phi} v_\infty(\kappa r + \phi)$
Spectra of planar spiral waves

Spectrum on spaces with exponentially decaying weights $e^{-r}$

- Adjoint eigenfunctions associated with eigenvalues $\lambda = 0, \pm i\omega^*$ are exponentially localized
- $\Sigma_{\text{ext}}(\text{core}) :=$ point spectrum in weighted spaces
Spiral waves on large disks

Planar spiral wave

Boundary sink

Spiral wave on large disk:
- Glue planar spiral wave and boundary sink together
- Time (sink) = angle (spiral)

Neumann boundary conditions at $r=R$
Spectra of spiral waves on disks of radius \( R \)

Absolute spectrum: depends on wave train only (\( 1/R \) convergence)

Discrete eigenvalues: depend on spiral wave (\( \exp(-R) \) convergence)

**Theorem [S., Scheel]**

\[ \Sigma(\text{bounded disk}) \rightarrow \Sigma_{\text{abs}} \cup \Sigma_{\text{ext}(\text{core})} \cup \Sigma_{\text{ext}(\text{boundary sink})} \text{ as } R \rightarrow \infty. \]
Case study: Period doubling of spiral waves

Line defects in Rössler model

Alternans in Karma model

**Goal:**
- What causes these instabilities?
- Core, boundary, or absolute spectrum?
Case study: Period doubling of spiral waves

Line defects in Rössler model

Alternans in Karma model

\[ \lambda - 3i\omega/2 \]

\[ \lambda - i\omega/2 \]
Case study: Period doubling of spiral waves

Methodology: Compute spectra separately for
- Spiral wave on disk with Neumann boundary conditions: \( u_r = 0 \) at \( r=R \)
- Spiral wave on disk with non-reflecting boundary conditions: \( u_r = \kappa u_\phi \) at \( r=R \)
- Boundary sink on \((-L,0]\) with Neumann conditions at \( x = 0 \)

- **Neumann spiral**
  - ✔ Absolute spectrum
  - ✔ Core spectrum
  - ✔ Boundary spectrum

- **Non-reflecting spiral**
  - ✔ Absolute spectrum
  - ✔ Core spectrum

- **Boundary sink**
  - ✔ Absolute spectrum
  - ✗ Core spectrum
  - ✔ Boundary spectrum
Case study: Period doubling of spiral waves

Line defects in Rössler model

Alternans in Karma model

Result:
- Line defect structure caused by Neumann boundary conditions
- Alternans caused by spiral core: should arise independently of the boundary conditions
Zero-diffusion limit of spiral spectra

\[ u_t = \Delta u + f(u, v) \]

\[ v_t = \delta \Delta v + g(u, v) \]
Zero-diffusion limit of spiral spectra

\[ u_t = \Delta u + f(u, v) \]
\[ v_t = \delta \Delta v + g(u, v) \]

Far field eigenfunctions: \( v(r, \phi) \approx e^{i\gamma r} e^{i\epsilon \phi} v_\infty(\kappa r + \phi) \)
Zero-diffusion limit of spiral spectra

\[ u_t = \Delta u + f(u, v) \]
\[ v_t = \delta \Delta v + g(u, v) \]

Theorem [Dodson, S.]:
Assume that \( g(u, v) = h(u) + av \) and that \( u_\ast (r, \phi; \delta) \) is a nondegenerate planar spiral wave that depends smoothly on \( \delta \), then the essential spectrum of the linearization about the spiral wave is discontinuous in the limit \( \delta \to 0 \) near the points \( \lambda = a + i\ell \).

[Rademacher]: Wave-train spectrum is continuous in the limit \( \delta \to 0 \)
Outline

- Wave trains: spectral stability and modulation equations
- Planar spiral waves: spiral spectra on the plane and on bounded disks
- Nonlinear stability of one-dimensional spiral waves
$\frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2} + f(u), \quad x \in \mathbb{R}, \quad u \in \mathbb{R}^n$

$u_*(x, t)$ is periodic in time (possibly in moving frame) and spatially asymptotic to two (possibly different) wave trains.

Standing time-periodic structure
Spectra of sources

\[ \frac{\partial v}{\partial t} = D \frac{\partial^2 v}{\partial x^2} + f(u_*(x, t))v \]

Defect core

\[ L^2_{\text{exp}} := \{ v : \|v(x)e^{-|x|}\|_{L^2} < \infty \} \]
Nonlinear stability of sources

- Defect core converges exponentially to new position
- Modulation interface is localized and travels with speed given by the group velocity
Nonlinear stability of sources

Unperturbed source

Difference of perturbed and unperturbed source
Nonlinear stability of sources

**Theorem [Beck, Nguyen, S., Zumbrun]:**
Assume \( u_*(x, t) \) is a spectrally stable source, and let 
\( u(x,0) = u_*(x,0) + v_0(x) \) with \( v_0(x) \) small in a weighted norm. Then there are small constants \( p_\infty \) and \( \varphi_\infty \) such that

\[
|u(x, t) - u_*(x - p_\infty, t - \varphi_\infty)| < \epsilon C e^{-\eta t} \text{ in } \Omega_1
\]
\[
|u(x, t) - u_*(x, t)| < \epsilon C e^{-\eta t} \text{ in } \Omega_2
\]
Summary and outlook

Summary:
- stability of wave trains under classes of localized perturbations
- robustness and structure of planar spiral waves and target patterns
- structure of spectra of spiral waves on the plane and on bounded disks
- nonlinear stability of 1d sources

Open problems:
- stability of wave trains under perturbations of the wave number
- linear stability of planar spiral waves
- interaction of spirals (or other waves)