Calculus of a kind of improper integral

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Abstract. The calculation of improper integral is of great importance. In this paper, we consider a kind of real improper integral using the method of complex analysis and prove the equation
\[ \int_0^{\infty} \frac{x^\alpha}{(1+x^\beta)} \, dx = \frac{\pi}{\beta \sin\frac{\pi\alpha}{\beta}} \] Firstly we give the condition for the convergence of the improper integral \( \int_0^{\infty} \frac{x^\alpha}{(1+x^\beta)} \, dx \) when \( 0 < \alpha < \beta \), so we can see the integrand as a complex function.

This function is defined on a simply connected set not containing zero to ensure that the function is holomorphic. We use three methods to prove the equation. In the first method, we choose a certain closed path as the boundary of the simply connected set, compute the integral along the path by the Residue Formula according to Cauchy’s Theorem, and obtain the value of the real improper integral. In the second method, we use Mellin transform, while the idea is similar to that of the Residue Formula. In the third method, we find that the path we choose becomes simpler by a variable substitution. The function that is integrated along the new path does not have the problem of multivaluedness, so we do not have to define it on a simply connect set. Moreover, using the method of complex analysis, we prove that the equation holds when \( \alpha, \beta \) are complex numbers, and the condition is about the real parts of \( \alpha \) and \( \beta \), i.e., \( 0 < \Re \alpha < \Re \beta \). The real improper integral is just a particular case where \( \alpha, \beta \) are real numbers. We compute the value of such kind of complex improper integral by some calculation and simplification, which is exactly \( \frac{\pi}{\beta \sin\frac{\pi\alpha}{\beta}} \). We find the relationship between this integral and gamma function. The equation can prove a property of gamma function. We prove that the equation holds when \( \alpha, \beta \) are complex numbers, and the condition becomes \( 0 < \Re \alpha < \Re \beta \). We hope that this can be used in the research of more properties of the gamma function. The proof of the equation reminds us of a way of calculating such kinds of real improper integrals.

1. Introduction
There are many ways to compute improper integral. Vigirdas Mackevičius talked about ways to compute different kinds of improper integrals [1]. Miklós Laczkovich and Vera T. Sós gave ways to solve problems whose solutions require us to integrate functions on unbounded intervals [2]. Sometimes rules for the integral of real functions can not be used to compute improper integral, and we can solve this kind of problem by means of complex integral.

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Before computing, we first emphasize the convergence of the real improper integral. Thus we have to give the necessary condition for the convergence of the real improper integral. Gray Walls researched the divergence of improper integral and gave its condition [3]. Xiangfeng Yang proved the convergence under some restrictions on real-valued functions [4]. Qinpeng Cai discussed the convergence problem for fuzzy complex integrals concerning several convergence theories [5]. Azizullah Paeyndah showed that the methods and techniques used in the integration of real-valued functions are not applicable in complex integral. The first and foremost method of the complex integral is residue [6], so it is necessary to compute the real improper integral via complex integral along a closed path. Serge Lvovski studied integrals over piecewise smooth paths [7]. We mention the book, where some theories of complex analysis of integral along paths were presented [8-12]. Integrals that appear in singular integral equations are usually principal value integrals [13-15], while those in hyper-singular integral equations are usually finite part integrals [16-18]. Ricardo Estrada showed that a suitable variation of the well-known residue formula holds when an analytic function has isolated essential singularities along the integration contour [19]. Masahiko Yamazaki, Hiroshi Yamazaki, Katsumi Wasaki, and Yasunari Shidama proved that the complex integration of complex curve’s connection is the sum of each complex integral of the individual complex curve [20]. So we use the Risedue formula to compute the complex integral along the closed contour, which is the sum of the integral of four parts of the contour.

We find that this real integral can be interpreted as Mellin integral transform. There have been researches on Mellin transform. Urs Graf established the connection of the classical Mellin transform with the two-sided Laplace transform [21]. Yuri Luchko and Virginia Kiryakova demonstrated the role of the Mellin integral transform in Fractional Calculus [22]. Yonggong Peng, Yixian Wang, Xianguwu Zuo, and Lihua Gong formulated a similar version of Mellin transform by referring to the Parseval theorem of Fourier transform [23]. Here we introduce Mellin also transform by complex integral along particular contours.

Helene Barucq, Vanessa Mattesi, and Sébastien Tordeux used a rectangular closed path when expliciting the relation between the behavior of a function and the domain of analyticity of its Mellin transform [24]. By changes in variables, Yury et al. turned the Mellin transform into the Fourier and Laplace transforms [25]. Inspired by their work, we use a variable substitution and find a rectangular closed path, avoiding multivaluedness.

In this paper, we consider the equation when $\alpha, \beta$ are complex numbers under the condition $0 \leq \Re \alpha \leq \Re \beta$. This work has not been done before. Hussam et al. investigated the properties of Mellin transform and the relationship between Mellin transform of some special functions and gamma function [26]. Choi et al. developed a variety of integral representations for some gamma functions [27]. Derkachov et al. discussed complex Gamma function integrals [28]. Their work reminds us that there may be some relationship with the real improper integral and gamma function.

2. Mechanism

This section should recall some definitions and present some auxiliary theorems and lemmas, which will play important roles in our proofs. We begin with recalling the definition of the Residue.

2.1. Definition 2.1

Laurent expansion of a function at a point is expressed as

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

where $a_{-1}$ is called the residue of $f(z)$ at $z_0$. We denote it as $a_{-1} = \text{Res}_{z_0} f$.

Without a doubt, the most basic and the most important theorem for us is the Residue Formula. For proving the Residue Formula, we should first present the Cauchy Integral Theorem and its lemma [29]. We notice that the Mellin Transform can make our proof easier.

2.2. Theorem 2.1

We assume that the following $f(z)$ is analytic except for a finite number of poles, none of which lies on the positive real axis. Then
\[ \int_0^\infty f(x)x^{\alpha-1}dx = -\frac{\pi e^{-i\pi\alpha}}{\sin \pi \alpha} \sum \text{residues of } f(z)z^{\alpha-1} \text{ at the poles of } f, \text{ excluding the residue at 0} \]  

(2)

Proof:

The contour C consists of four paths, as is shown in Figure 1:

\[ \eta: re^{i(2\pi - \phi)}, \epsilon \leq r \leq R, \]
\[ \gamma: re^{i\phi}, \epsilon \leq r \leq R, \]
\[ C_e: e^{i(2\pi - \phi)}, \phi \leq \theta \leq 2\pi - \phi, \]
\[ C_R: Re^{i\theta}, \phi \leq \theta \leq 2\pi - \phi \]  

(3)

Here, \( \phi \) is the angle between the straight path and the positive direction of the x axis.

According to the Residue Formula,

\[ \int_C f(z)z^{\alpha-1}dz = 2\pi i \sum \text{residues of } f(z)z^{\alpha-1} \text{ at the poles of } f, \text{ excluding the residue at 0} \]  

(4)

which is equivalent to

\[ \int_{C_e} f(z)z^{\alpha-1}dz + \int_{\gamma} f(z)z^{\alpha-1}dz + \int_{C_R} f(z)z^{\alpha-1}dz + \int_{\eta} f(z)z^{\alpha-1}dz \]

\[ = 2\pi i \sum \text{residues of } f(z)z^{\alpha-1} \text{ at the poles of } f, \text{ excluding the residue at 0}. \]  

(5)

We have

\[ |\int_{C_e} f(z)z^{\alpha-1}dz| \leq 2\pi \epsilon |z|^{\alpha-1} \leq 2\pi \epsilon^\alpha \]  

(6)

\[ |\int_{C_R} f(z)z^{\alpha-1}dz| \leq 2\pi R^{\alpha-\beta} \]  

(7)

Hence

\[ \int_{C_e} f(z)z^{\alpha-1}dz \to 0, \text{ when } \epsilon \to 0 \]

(8)

\[ \int_{C_R} f(z)z^{\alpha-1}dz \to 0, \text{ when } R \to \infty \]  

(9)

We also have when \( \phi \to 0 \)
\begin{align}
\int_{\gamma} f(z)z^{a-1} \, dz &\rightarrow \int_{\varepsilon}^{R} f(x)x^{a-1} \, dx \\
\int_{\eta} f(z)z^{a-1} \, dz &\rightarrow \int_{R}^{\infty} f \left( re^{i(2\pi-\varphi)} \right) r^{a-1} e^{i(2\pi-\varphi)(a-1)} \, dr \left( re^{i2\pi} \right) \\
&\rightarrow -e^{i2\pi a} \int_{\varepsilon}^{R} f(r) r^{a-1} e^{i2\pi a} \, dr 
\end{align}

(10)

Using the equation (5), when \( \varepsilon \rightarrow 0, R \rightarrow \infty, \varphi \rightarrow 0 \) we can get

\[ 0 + \int_{0}^{\infty} f(x)x^{a-1} \, dx + 0 + \left( -e^{i2\pi a} \int_{0}^{\infty} f(x)x^{a-1} \, dx \right) \]

\[ = 2\pi i \sum \text{residues of } f(z)z^{a-1} \text{ at the poles of } f \text{, excluding the residue at 0} \]

Actually, the above equation (4) is similar to

\[ (1 - e^{i2\pi a}) \int_{0}^{\infty} f(x)x^{a-1} \, dx \]

\[ = 2\pi i \sum \text{residues of } f(z)z^{a-1} \text{ at the poles of } f \text{, excluding the residue at 0} \]

At last, we obtain

\[ \int_{0}^{\infty} f(x)x^{a-1} \, dx \]

\[ = -\frac{\pi e^{-i\pi a}}{\sin \pi a} \sum \text{residues of } f(z)z^{a-1} \text{ at the poles of } f \text{, excluding the residue at 0} \]

Q.E.D.

2.3. Lemma 2.1

The improper integral

\[ \int_{0}^{\infty} f(x)x^{a} \, dx \]

is valid under the conditions:

- There exists a number \( b > \alpha \) s.t.

\[ |f(z)| \leq \frac{1}{|z|^b} \text{ for } |z| \rightarrow \infty \]

(16)

- There exists a number \( 0 < b' < \alpha \) s.t..

\[ |f(z)| \leq \frac{1}{|z|^{b'}} \text{ for } |z| \rightarrow 0 \]

(17)

2.4. Theorem 2.2

Let \( f \) be continuous on an open set \( U \), and suppose that \( f \) has a primitive \( g \), that is, \( g \) is holomorphic and \( g' = f \). Let \( \alpha \) and \( \beta \) be two points of \( U \), and let \( \gamma \) be a path in \( U \) joining \( \alpha \) to \( \beta \). Then

\[ \int_{\gamma} f = g(\beta) - g(\alpha) \]

(18)

and in particular, this integral depends only on the beginning and endpoint of the path. It is independent of the path itself.

Proof:

Assume first that the path is a curve. Then

\[ \int_{\gamma} f(z) \, dz = \int_{a}^{b} g'(\gamma(t))\gamma'(t) \, dt \]

(19)
By the chain rule, the expression under the integral sign is the derivative
\[
\frac{d}{dt} g(\gamma(t))
\]  
(20)

Hence by ordinary calculus, the integral is equal to
\[
g(\gamma(t))\big|_a^b = g(\gamma(b)) - g(\gamma(a))
\]  
(21)

which proves the theorem in this case. In general, if the path consists of curves \( \gamma_1, \ldots, \gamma_n \), and \( z_j \) is the endpoint of \( \gamma_j \), then by the case we have just settled, we find
\[
\int_{\gamma} f = g(z_1) - g(z_0) + g(z_2) - g(z_1) + \cdots + g(z_n) - g(z_{n-1}) = g(z_n) - g(z_0)
\]  
(22)

which proves the theorem.

Q.E.D.

The theorem has a corollary as follows.

2.5. Lemma 2.2

Let \( f \) be holomorphic on \( U \). Let \( \gamma_i: [a_i, a_{i+1}] \to U \) be the restriction of \( \gamma \) to the smaller interval \( [a_i, a_{i+1}] \). Then
\[
\int_{\gamma} f = \sum_{i=0}^{n-1} \int_{\gamma_i} f
\]  
(23)

We omit the proof as it is straightforward.

2.6. Theorem 2.3

Let \( D \) be a closed disc of positive radius, and the contour \( C \) the boundary of \( \overline{D} \). The function \( f \) is holomorphic in \( D \) except \( a_1, a_2, \ldots, a_n \) and is continuous on \( \overline{D} \) except \( a_1, a_2, \ldots, a_n \). Then we can obtain
\[
\int_C f(z)\,dz = 2\pi i \sum \text{Res } f(z)
\]  
(24)

Proof:

Let \( \Gamma_k: |z - a_k| = \rho_k \) be the circle of sufficiently small radius \( \rho_k \) centered at \( a_k \) \( (k=1, 2, \ldots, n) \). Besides, we shall have all of these circles contained in \( D \) and isolated. Applying Lemma 2.1, we obtain
\[
\int_C f(z)\,dz = \sum_{k=1}^{n} \int_{\Gamma_k} f(z)\,dz
\]  
(25)

According to Definition 2.1, we have
\[
\int_{\Gamma_k} f(z)\,dz = 2\pi i \text{Res }_{z=a_k} f(z)
\]  
(26)

After substituting the above equation (9), we can prove
\[
\int_C f(z)\,dz = 2\pi i \sum \text{Res } f(z)
\]  
(27)

Q.E.D.

We would use the gamma function on a special occasion, so we present its definition and one of its important identities \([30]\).

2.7. Lemma 2.2

Let \( \gamma \) be the Euler constant, that is
\[
\gamma = \lim_{n \to \infty} \left( 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln(n) \right)
\]  
(28)

By the general theory of Weierstrass products, there is an entire function \( g(z) \) whose zeros are the negative integers and 0, having the Weierstrass product
We define the gamma function to be

\[ \Gamma(z) = \lim_{n \to \infty} \left( \frac{n!}{n^n} \right)^{1/n} \]

so that

\[ \frac{1}{\Gamma(z)} = e^{-\gamma z} \sum_{n=0}^{\infty} \frac{z^n}{n!} \]

(30)

2.8. Theorem 2.4

\[ \Gamma(z) \Gamma(1-z) = \frac{\pi}{\sin \pi z} \]

(31)

The gamma function has poles of order 1 at the negative integers. We record at once its logarithmic derivative

\[ -\frac{\Gamma'}{\Gamma(z)} = \frac{g'}{g(z)} = \frac{1}{z} + \gamma + \sum_{n=1}^{\infty} \left( \frac{1}{z+n} - \frac{1}{n} \right) \]

(32)

The Euler constant is the unique constant such that the Weierstrass product for \( g \) satisfies the property

\[ g(z+1) = \frac{1}{z} g(z) \]

(33)

and therefore, the gamma function satisfies the property

\[ \Gamma(z+1) = z \Gamma(z) \]

(34)

To prove this, let \( g_1 = z^{-1} g(z) \), so that \( g_1(0) = 1 \). Taking logarithmic derivatives, we find immediately that

\[ \frac{g'}{g(z+1)} = \frac{1}{z+1} + \gamma + \sum_{n=1}^{\infty} \left( \frac{1}{z+n} - \frac{1}{n} \right) = \frac{g_1'}{g_1(z)} \]

(35)

Next comes the identity

\[ \Gamma(z) \Gamma(1-z) = \frac{\pi}{\sin \pi z} \]

(36)

To prove this, we note that from the Weierstrass products,

\[ \Gamma(z) \Gamma(-z) = -\frac{1}{z^2} \prod_{n=1}^{\infty} \left( 1 - \frac{z^2}{n^2} \right) = -\frac{\pi}{z \sin \pi z} \]

(37)

Using the equation (35), we finally obtain the equation (32).

3. Results

In this section, we prove the equation

\[ \int_{0}^{\infty} \frac{x^\alpha}{(1+x^\beta)} dx = \frac{\pi}{\beta \sin \pi \alpha}, (\alpha, \beta \in \mathbb{R}) \]

(38)

Let

\[ f(z) = \frac{1}{1+z^\beta} \]

(39)

Now that

\[ |f(z)| = \left| \frac{1}{1+z^\beta} \right| \leq \frac{1}{|z|^\beta} \text{ for } |z| \to \infty \]

(40)

For the convergence of \( \int_{0}^{1} f(x)x^\alpha dx \), there must exist a number \( 0 < b' < \alpha \) s.t. \( |f(z)| \leq \frac{1}{|z|^{b'}} \) for \( |z| \to 0 \) as long as \( \alpha > 0 \).
We get the condition for the convergence of the improper integral

\[ 0 < \alpha < \beta \]  

(41)

The integrand \( f(z)z^{\alpha-1} \) is not holomorphic on the whole complex plane because

- Obviously \( f(z)z^{\alpha-1} \) is not holomorphic at zero.
- We define an exponential function in complex field as:

\[ z^\alpha = e^{a \log z} = e^{a \left( \log|z| + i2\pi k \right)}, k \text{ is arbitrary integer} \]  

(42)

which shows that \( z^\alpha \) is multivalued on the whole complex plane.

Thus, we have to define the integrand on a simply connected set not containing zero. We shall prove the equation using three methods, method(i), method(ii) and method(iii).

3.1. Method (i)

If we choose such a particular closed contour \( C \) (as shown in Figure 2), then the function \( f(z) \) is holomorphic inside the contour \( C \) except for a simple pole \( z_0 = e^{i\frac{\pi}{\beta}} \)

![Figure 2. Contour C consisting four paths.](image)

According to the Residue Formula,

\[ \oint_C f(z)z^{\alpha-1}dz = 2\pi i \text{Res}(f, z_0) \]  

(44)

which is equivalent to

\[ \int_{C_\epsilon} f(z)z^{\alpha-1}dz + \int_\gamma f(z)z^{\alpha-1}dz + \int_{C_R} f(z)z^{\alpha-1}dz + \int_\eta f(z)z^{\alpha-1}dz = 2\pi i \text{Res}(f, z_0) \]  

(45)

We have

\[ \int_{C_\epsilon} f(z)z^{\alpha-1}dz = \int_0^{2\pi} e^{(\alpha-1)i\theta} e^{i\theta} d\theta = \int_0^{2\pi} e^{\alpha e^{i\theta}} d\theta \]  

(46)

Hence

\[ \left| \int_{C_\epsilon} f(z)z^{\alpha-1}dz \right| = \left| \int_0^{2\pi} e^{\alpha e^{i\theta}} d\theta \right| \leq \int_0^{2\pi} e^{\beta e^{i\theta}} + 1 d\theta \leq \frac{2\pi}{\beta} e^{\alpha} \]  

(47)
\[ \int_{C_{\epsilon}} f(z)z^{\alpha-1}dz \to 0 \text{ when } \epsilon \to 0, (\text{due to } \alpha > 0) \quad (48) \]

Similarly
\[ \int_{C_{R}} f(z)z^{\alpha-1}dz = \int_{0}^{2\pi} \frac{R^{\alpha-1}e^{i(\alpha-1)\theta}}{R^{\beta}e^{i\beta\theta} + 1} i\epsilon e^{i\theta} \, d\theta = i \int_{0}^{2\pi} \frac{R^{\alpha}e^{i\alpha\theta}}{R^{\beta}e^{i\beta\theta} + 1} \, d\theta \]

Hence
\[ \left| \int_{C_{R}} f(z)z^{\alpha-1}dz \right| = \left| \int_{0}^{2\pi} \frac{R^{\alpha}e^{i\alpha\theta}}{R^{\beta}e^{i\beta\theta} + 1} \, d\theta \right| \leq \int_{0}^{2\pi} \frac{R^{\alpha}e^{i\alpha\theta}}{R^{\beta}e^{i\beta\theta} + 1} \, d\theta \leq \frac{2\pi}{\beta} \frac{R^{\alpha}e^{i\alpha\theta}}{R^{\beta}e^{i\beta\theta} + 1} \leq \frac{2\pi}{\beta} R^{\alpha-\beta} \]

\[ \int_{C_{R}} f(z)z^{\alpha-1}dz \to 0 \text{ when } R \to \infty, (\text{due to } \alpha < \beta) \quad (51) \]

We also have
\[ \int_{\gamma} f(z)z^{\alpha-1}dz = \int_{\epsilon}^{R} f(x)x^{\alpha-1}dx = \int_{\epsilon}^{R} \frac{x^{\alpha-1}}{x^{\beta} + 1} dx \]

\[ \int_{\eta} f(z)z^{\alpha-1}dz = \int_{R}^{\epsilon} f \left( r e^{i\frac{2\pi}{\beta}} \right) r^{\alpha-1}e^{\frac{2(\alpha-1)\pi}{\beta}} d \left( r e^{i\frac{2\pi}{\beta}} \right) \]

\[ = \int_{R}^{\epsilon} r^{\alpha-1}e^{\frac{2\alpha\pi}{\beta}} d r = -e^{\frac{2\alpha\pi}{\beta}} \int_{\epsilon}^{R} \frac{x^{\alpha-1}}{x^{\beta} + 1} dx \]

Hence
\[ \int_{\gamma} f(z)z^{\alpha-1}dz \to \int_{0}^{\infty} \frac{x^{\alpha-1}}{x^{\beta} + 1} dx \text{ when } \epsilon \to 0, R \to \infty \quad (54) \]

\[ \int_{\eta} f(z)z^{\alpha-1}dz \to -e^{\frac{2\alpha\pi}{\beta}} \int_{0}^{\infty} \frac{x^{\alpha-1}}{x^{\beta} + 1} dx \text{ when } \epsilon \to 0, R \to \infty \quad (55) \]

Then, we calculate the residue of the function at the point according to the Residue Formula
\[ \text{Res}(f, z_0) = \lim_{z \to z_0} (z - z_0)f(z) = \lim_{z \to e^{i\frac{\pi}{\beta}}} (z - e^{i\frac{\pi}{\beta}}) f \left( e^{i\frac{\pi}{\beta}} \right) = -e^{\frac{i\alpha\pi}{\beta}} \]

(56)

Since
\[ \int_{C_{\epsilon}} f(z)z^{\alpha-1}dz + \int_{\gamma} f(z)z^{\alpha-1}dz + \int_{C_{R}} f(z)z^{\alpha-1}dz + \int_{\eta} f(z)z^{\alpha-1}dz = 2\pi i \text{Res}(f, z_0) \]

(57)

When \( \epsilon \to 0, R \to \infty \)
\[ 0 + \int_{0}^{\infty} \frac{x^{\alpha-1}}{x^{\beta} + 1} dx + 0 + \left( -e^{\frac{2\alpha\pi}{\beta}} \int_{0}^{\infty} \frac{x^{\alpha-1}}{x^{\beta} + 1} dx \right) = 2\pi i \text{Res}(f, z_0) \]

(58)

which is equivalent to
\[ \left( 1 - e^{\frac{2\alpha\pi}{\beta}} \right) \int_{0}^{\infty} \frac{x^{\alpha-1}}{x^{\beta} + 1} dx = 2\pi i \left( -e^{\frac{\alpha\pi}{\beta}} \right) \]

(59)

Finally, we obtain
\[ \int_{0}^{\infty} \frac{x^{\alpha-1}}{x^{\beta} + 1} dx \]
\[ 2\pi i \left( -\frac{e^{\frac{\alpha \pi}{\beta}}}{1 - e^{\frac{2\alpha \pi}{\beta}}} \right) = \frac{\pi}{\beta \left( e^{\frac{\alpha \pi}{\beta}} - e^{-\frac{\alpha \pi}{\beta}} \right) / 2i} = \frac{\pi}{\beta \sin \frac{\alpha \pi}{\beta}} \] (60)

Q.E.D.

3.2. Method (ii)
Evidently, \( f(z) \) does not have any pole which lies on the positive real axis. Hence we can use Theorem 2.1, and we have \( \text{Res}(f(z)z^{\alpha - 1}, e^{\theta i}) = -\frac{1}{\beta} e^{\alpha \theta i} \), then

\[ \text{Res}(f(z)z^{\alpha - 1}, e^{\theta i}) = -\frac{1}{\beta} e^{\alpha \theta i} \] (61)

To simplify this equation, we assume \( T = e^{\frac{\alpha \pi}{\beta}} \) and obtain

\[ \int_{0}^{\infty} f(x)x^{\alpha - 1} \, dx = \frac{\pi}{\beta} \cdot \frac{2i}{T^{2\beta} - 1} \cdot \frac{T - T^{2\beta + 1}}{1 - T^2} = \frac{\pi}{\beta \sin \frac{\alpha \pi}{\beta}} \] (62)

Q.E.D.

3.3. Method (iii)
Actually, we can use a simple variable substitution \( e^\frac{\alpha \pi}{\beta} = x, \beta a = \alpha \), to prove Eq. (35). Then we only need to prove the following equation:

\[ \int_{-\infty}^{\infty} e^{\alpha t} \frac{dt}{1 + e^t} = \frac{\pi}{\sin \alpha \pi}, \quad 0 < \alpha < 1 \] (63)

This form becomes straightforward to prove because:

- The function \( f(t) = \frac{e^{\alpha t}}{1 + e^t} \) is analytic and holomorphic on the whole complex plane except for its only singularity \( z_0 = \pi i \), avoiding the problem that function \( \frac{z^{\alpha - 1}}{1 + z^\beta} \) is multivalued.
- We do not have to compute integrals along curves but only need to compute integrals along straight lines.

At first, we should choose the rectangular contour C, the length of which is 2R and the width of which is \( 2\pi \).

It consists of four paths, as shown in Figure 3:

\[ \gamma: -R \to R, \]
\[ \eta: R + 2\pi i \to -R + 2\pi i, \]
\[ A: R \to R + 2\pi i, \]
\[ B: -R + 2\pi i \to -R \] (64)

in addition, the only singularity \( z_0 = \pi i \).
According to the Residue Formula

\[ \int f(z)dz = 2\pi i \text{Res}(f, z_0) \]  

which is equivalent to

\[ \int f(z)dz + \int_A f(z)dz + \int_B f(z)dz + \int_C f(z)dz = 2\pi i \text{Res}(f, z_0) \]  

Hence

\[ \left| \int_A f(z)dz \right| \leq \int_0^{2\pi} \frac{e^{a(R+it)}}{1 + e^{R+it}} dt \leq 2\pi e^{(a-1)R} \]  

\[ \int_A f(z)dz \to 0\ , \text{when } R \to \infty (\text{due to } \alpha < 1) \]  

Similarly,

\[ \int_B f(z)dz = \int_0^{2\pi} \frac{e^{a(-R+it)}}{1 + e^{-R+it}} dt \]  

\[ \left| \int_B f(z)dz \right| \leq \int_0^{2\pi} \frac{e^{a(-R+it)}}{1 + e^{-R+it}} dt \leq 2\pi e^{-aR} \]  

\[ \int_B f(z)dz \to 0\ , \text{when } R \to \infty (\text{due to } \alpha > 0) \]  

At the same time, we also have

\[ \int f(z)dz = \int_R^{\infty} \frac{e^{a(t)}}{1 + e^t} dt \]  

\[ \int f(z)dz = \int_{-\infty}^{-R} \frac{e^{a((t+2\pi)i)}}{1 + e^{(t+2\pi)i}} dt = -e^{a2\pi i} \int_{-\infty}^{R} \frac{e^{a(t)}}{1 + e^t} dt. \]  

Then, we calculate the residue of the function at the point \( z_0 \) according to the Residue Formula

\[ \text{Res}(f, z_0) = \lim_{z \to z_0} (z - z_0)f(z) = e^{-a\pi i} \]  

Since

\[ \int f(z)dz + \int_A f(z)dz + \int_{\eta} f(z)dz + \int_B f(z)dz = 2\pi i \text{Res}(f, z_0) \]  

when \( R \to \infty \)
\[
\int_{-\infty}^{\infty} \frac{e^{at}}{1 + e^t} dt + 0 + \left( -e^{a2\pi i} \right) \int_{-\infty}^{\infty} \frac{e^{at}}{1 + e^t} dt + 0 = 2\pi i (e^{-a\pi i})
\] (77)

which is equivalent to

\[
\left(1 - e^{a2\pi i}\right) \int_{-\infty}^{+\infty} \frac{e^{at}}{1 + e^t} dt = 2\pi i (-e^{a\pi i})
\] (78)

At last, we can get the result

\[
\int_{-\infty}^{\infty} \frac{e^{at}}{1 + e^t} dt = \frac{2\pi i e^{a\pi i}}{e^{2a\pi i} - 1} = \frac{\pi}{\sin \pi a}
\] (79)

Q.E.D.

3.4. Complex integral

Consider the situation that \(\alpha, \beta\) are complex numbers. We aim to prove the equation

\[
\int_0^\infty \frac{x^{\alpha}}{(1 + x^\beta)x} dx = \frac{\pi}{\beta \sin \frac{\pi \alpha}{\beta}}, \quad (\alpha, \beta \in \mathbb{Z})
\] (80)

Then the equation we proved is just a case when \(\Im \alpha = \Im \beta = 0\)

The value of \(|x^\alpha|\ (x \in \mathbb{R})\) is only determined by \(\Re \alpha\) and has nothing to do with \(\Im \alpha\), so according to Lemma 2.1, the condition for the convergence of the improper integral \(\int_0^\infty \frac{x^{\alpha}}{(1 + x^\beta)x} dx\ (\alpha, \beta \in \mathbb{Z})\) is

\[0 < \Re \alpha < \Re \beta\]

Let \(\alpha = a + bi, \beta = c + di, \ f(z) = \frac{1}{1 + z^\beta}\).

The convergence condition is equivalent to

\[0 < a < c\] (81)

If we choose such a particular closed contour \(C\) (as shown in Figure 4), then the function \(f(z)\) is holomorphic inside the contour \(C\) except for a simple pole \(z_0 = e^{i\pi\beta}\)

Figure 4. Contour \(C\) consisting four paths.

According to the Residue Formula,

\[
\int_C f(z) z^{\alpha-1} dz = 2\pi i \text{Res}(f, z_0)
\] (83)

which is equivalent to
\[
\int_{C_c} f(z)z^{a-1}dz + \int_{C_R} f(z)z^{a-1}dz + \int_{C_R} f(z)z^{a-1}dz + \int_{\eta} f(z)z^{a-1}dz = 2\pi i \text{Res}(f, z_0) \tag{84}
\]

We have
\[
\int_{C_c} f(z)z^{a-1}dz = \int_0^{2\pi} \frac{z^{a-1}}{\beta + 1} i e^{i \theta} d\theta = i \int_0^{2\pi} \frac{z^a}{\beta + 1} d\theta \tag{85}
\]

Hence
\[
\left| \int_{C_c} f(z)z^{a-1}dz \right| = \left| \int_0^{2\pi} \frac{z^{a-1}}{\beta + 1} d\theta \right| \leq \int_0^{2\pi} \frac{z^a}{\beta + 1} d\theta \leq \frac{2\pi c}{c^2 + d^2} |z^a| \tag{86}
\]

\[
\int_{C_c} f(z)z^{a-1}dz \rightarrow 0 \text{ when } \epsilon \rightarrow 0 \text{ (due to } a > 0) \tag{87}
\]

Similarly
\[
\int_{C_R} f(z)z^{a-1}dz = \int_0^{2\pi} \frac{z^{a-1}}{\beta + 1} i Re^{i \theta} d\theta = i \int_0^{2\pi} \frac{z^a}{\beta + 1} d\theta \tag{88}
\]

Hence
\[
\left| \int_{C_R} f(z)z^{a-1}dz \right| = \left| \int_0^{2\pi} \frac{z^{a-1}}{\beta + 1} d\theta \right| \leq \int_0^{2\pi} \frac{z^a}{\beta + 1} d\theta \leq \frac{2\pi c}{c^2 + d^2} |z^a| \tag{89}
\]

\[
\int_{C_R} f(z)z^{a-1}dz \rightarrow 0 \text{ when } R \rightarrow \infty \text{ (due to } a < c) \tag{90}
\]

We also have
\[
\int_{C_R} f(z)z^{a-1}dz = \int_0^R f(x)x^{a-1}dx = \int_0^R \frac{x^{a-1}}{\beta + 1} dx \tag{91}
\]

\[
\int_{\eta} f(z)z^{a-1}dz = \int_{r \epsilon}^R e^{2\pi d e^{i \epsilon^{c^2+d^2}}} e^{2\pi d (a-1+b)} d \left( re^{c^2+d^2} e^{i \epsilon^{c^2+d^2}} \right) \tag{92}
\]

\[
= \int_{r \epsilon}^R e^{2\pi d (a-1+b)} e^{2\pi d (c+bi)} d \left( re^{c^2+d^2} e^{i \epsilon^{c^2+d^2}} \right) + 1 \tag{93}
\]
\[
\begin{align*}
\int \frac{e^{a+bi}}{c+di} \, dr & = e^{\frac{2\pi a}{\beta}} \int e^{\frac{2\pi a}{\beta} \frac{r^\alpha}{r^\beta + 1}} \, dr \\
& = -e^{\frac{2\pi a}{\beta}} \int e^{\frac{2\pi a}{\beta} \frac{r^\alpha}{r^\beta + 1}} \, dr
\end{align*}
\]

Hence
\[
\int f(z)z^{\alpha-1} \, dz \to \int_0^{\infty} \frac{x^{\alpha-1}}{x^\beta + 1} \, dx \text{ when } \epsilon \to 0, R \to \infty
\]
\[
\int f(z)z^{\alpha-1} \, dz \to -e^{\frac{2\pi a}{\beta}} \int_0^{\infty} \frac{x^{\alpha-1}}{x^\beta + 1} \, dx \text{ when } \epsilon \to 0, R \to \infty
\]

The residue of the function at any point is not affected by whether \(\alpha, \beta\) are real numbers or not, so the residue of \(f(z)z^{\alpha-1}\) at the point \(z_0\) is still \(-e^{\frac{\alpha \pi}{\beta}}\). Since
\[
\int_{c_R} f(z)z^{\alpha-1} \, dz + \int_{c_R} f(z)z^{\alpha-1} \, dz + \int_{c_R} f(z)z^{\alpha-1} \, dz + \int_{c_R} f(z)z^{\alpha-1} \, dz = 2\pi i \text{Res}(f, z_0)
\]
When \(\epsilon \to 0, R \to \infty\)
\[
0 + \int_0^{\infty} \frac{x^{\alpha-1}}{x^\beta + 1} \, dx + 0 \left( -e^{\frac{2\pi a}{\beta}} \int_0^{\infty} \frac{x^{\alpha-1}}{x^\beta + 1} \, dx \right) = 2\pi i \text{Res}(f, z_0)
\]
which is equivalent to
\[
\left( 1 - e^{\frac{2\pi a}{\beta}} \right) \int_0^{\infty} \frac{x^{\alpha-1}}{x^\beta + 1} \, dx = 2\pi i \left( -e^{\frac{\alpha \pi}{\beta}} \right)
\]
Finally, we obtain
\[
\int_0^{\infty} \frac{x^{\alpha-1}}{x^\beta + 1} \, dx = \frac{2\pi i \left( -e^{\frac{\alpha \pi}{\beta}} \right)}{1 - e^{\frac{2\pi a}{\beta}}} = \frac{\pi}{\beta \left( e^{\frac{\alpha \pi}{\beta}} - e^{-\frac{\alpha \pi}{\beta}} \right)} = \frac{\pi}{\beta \sin \frac{\alpha \pi}{\beta}}
\]

Q.E.D.

3.5. Gamma function
When setting \(\beta = 1\), we can obtain something special. We can define Gamma function as
\[
\Gamma(s) = \int_0^{\infty} e^{-t} t^{s-1} \, dt, \text{ for } s > 0
\]
and Gamma function has an interesting identity as we present in Theorem 2.4
\[
\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}
\]
To prove this identity, except for using Weierstrass product, we also can use the conclusion above.
Let \(\beta = 1\), that we get the form
\[
\int_0^{\infty} \frac{x^\alpha}{(1 + x)x} \, dx = \frac{\pi}{\sin \pi \alpha}, \text{ for } 0 < \alpha < 1
\]
Obviously
\[ \Gamma(s) = \int_0^\infty e^{-t} t^{s-1} dt \]  

(102)

use a variable substitution \( v t = u \) to work on \( \Gamma(1 - s) \)

\[ \Gamma(1 - s) = \int_0^\infty e^{-u} u^{-s} du = t \int_0^\infty e^{-vt} (vt)^{-s} dv \]  

(103)

Use the form (35) in the condition that \( \beta = 1 \)

\[ \Gamma(s) \Gamma(1 - s) = \left( \int_0^\infty e^{-t} t^{s-1} dt \right) \left( t \int_0^\infty e^{-vt} (vt)^{-s} dv \right) = \int_0^\infty v^{-s} \frac{\pi}{1 + v} dv = \frac{\pi}{\sin \pi(1 - s)} = \frac{\pi}{\sin \pi s} \]  

(104)

Q.E.D.

4. Conclusion

This formula \( \int_0^\infty \frac{x^\alpha}{(1+x^\beta)x} dx = \frac{\pi}{\beta \sin \frac{\pi \alpha}{\beta}} \) was proved in three ways. Firstly, we used the complex integral along a closed curve. This proof was done by making \( f(z) = \frac{x^\alpha}{(1+z^\beta)z} \) on a simply connected set not containing zero, and by determining that the function \( f(z) \) is holomorphic inside the Contour except for a simple pole consisting of four paths. The calculation of the Residue Formula completed the last step. The second way was carried out by using the Merlin transform, assuming the function \( f(z) = \int_0^\infty f(x)x^\alpha dx \) is analytic with no poles located on the positive real axis except for a finite number of poles. Then we determined Contour and proved the Merlin formula. Therefore, it is straightforward to complete the second way using Merlin's formula. The third way simplified the original formula by using variable substitution, that is, we brought \( x = e^t \) and \( \alpha = \beta \alpha \) into the original formula \( \int_0^\infty \frac{x^\alpha}{(1+x^\beta)x} dx \) and obtained \( \int_0^\infty \frac{e^{\alpha t}}{1+e^t} dt \) which is equal to \( \frac{\pi}{\sin (\alpha \pi)} \) for \( 0 < \alpha < 1 \).

The formula can be proved by proving that \( \int_0^\infty \frac{e^{\alpha t}}{1+e^t} dt = \frac{\pi}{\sin (\alpha \pi)} \). All three ways succeed in proving that \( \int_0^\infty \frac{x^\alpha}{(1+x^\beta)x} dx = \frac{\pi}{\beta \sin \frac{\pi \alpha}{\beta}} \), and on this basis, we have shown that the formula still holds under the condition of broadening from real numbers to complex numbers. As a byproduct, we defined that a Gamma function as \( \Gamma(s) = \int_0^\infty e^{-t} t^{s-1} dt \) for all \( s < 0 \) and we got \( \Gamma(1 - s) = t \int_0^\infty e^{-vt} (vt)^{-s} dv \). Then we obtained the result \( \Gamma(s) \Gamma(1 - s) = \frac{\pi}{\sin (\pi s)} \). This study shows the relationship between a kind of complex improper integral and the gamma function and is of great value. This study can be used in the research of more properties of the gamma function.

References

[1] V. Mackevičius, Improper Integrals, in: V. Mackevičius (Ed.), Integral and Measure: From Rather Simple to Rather Complex, John Wiley & Sons, Inc., New Jersey, 2014, pp. 79-90.

[2] M. Laczkovich, V.T. Sós, The Improper Integral, in: M. Laczkovich, V.T. Sós, Real Analysis, Undergraduate Texts in Mathematics, Springer, New York, 2015, pp. 417-437.

[3] G. Walls, Divergence of an improper integral, Mathematics and Computer Education, 21 (1987) 28-30.

[4] X. Yang, Integral convergence related to weak convergence of measures, Applied Mathematical Sciences (Ruse), 5 (2011).

[5] Q. Cai, The convergence of fuzzy complex integral, International Journal of Pure and Applied Mathematics, 65 (2010).
[6] A. Paeyndah, Amazement of Complex Integration, International Journal of Scientific and Research Publications (IJSRP), 10 (2020) 63-69.

[7] S. Lvovski, Complex Integrals, in: S. Lvovski, Principles of Complex Analysis, Moscow Lectures, vol 6, Springer, Cham, 2020, pp. 37-48.

[8] Steven G. Krantz, Complex Variables, first ed., Chapman and Hall/CRC, New York, 2005, pp. 225-230.

[9] Lars V. Ahlfors, Complex Analysis, third ed., McGraw-Hill Companies, Inc., New York, 1979.

[10] James W. Brown, Ruel V. Churchill, Complex Variables and Applications, eighth ed., McGraw-Hill Companies, New York, 2009.

[11] S. Ponnusamy, Herb Silverman, Complex Variables with Applications, first ed., Birkhäuser, Boston, 2006.

[12] Murray R. Spiegel, Seymour Lipschutz, John J. Schiller, Dennis Spellman, Complex Variables, second ed., McGraw-Hill Companies, Inc., New York, 2009.

[13] R. Estrada and R.P. Kanwal, Singular Integral Equations, Birkhäuser, Boston, 2000.

[14] F.D. Gakov, Boundary Value Problems, Pergamon Press, Oxford, 1996.

[15] N.I. Mushkelishvili, Singular Integral Equations, P. Noordhoff, Groning, Netherlands, 1953.

[16] S.K. Bose, Finite part representation of hyper singular integral equations of acoustic scattering and radiation by open surfaces, Proc. Indian Acad. Sci. (Math. Sci.), 106 (1996) 271-280.

[17] A.C. Kaya and F. Erdogan, On the solution of integral equations with strong singular kernels, Quart. Appl. Math., XLV (1987) 105-122.

[18] P.A. Martin, Endpoint behaviors of solutions to hypersingular equations, Proc. Roy. Soc. London A, 432 (1991) 301-320.

[19] Ricardo Estrada, A Generalization of the Residue Formula, Bulletin of the Malaysian Mathematical Sciences Society (Second Series), 25(2002) 39-52.

[20] M. Yamazaki, H. Yamazaki, K. Wasaki, Y. Shidama, Complex Integral, Formalized Mathematics, 17(2009) 233-236.

[21] U. Graf, Mellin Transforms, in: Introduction to Hyperfunctions and Their Integral Transforms, Birkhäuser, Basel, 2010, pp.309-336.

[22] Yuri Luchko, Virginia Kiryakova, The mellin integral transform in fractional calculus, Fractional Calculus and Applied Analysis, Vol. 16, No 2 (2013), pp. 405-430.

[23] Y. Peng, Y. Wang, X. Zuo, L. Gong, Properties of Fractional Mellin Transform, INTERNATIONAL JOURNAL ON Advances in Information Sciences and Service Sciences, 5 (2013) 90-96.

[24] Hélène Barucq, Vanessa Mattesi, Sébastien Tordeux, The Mellin Transform, (Research Report) RR-8743, INRIA Bordeaux; INRIA, 2015, pp. 16.

[25] Yu. A. Brychkov, O. I. Marichev, N. V. Savischenko, Handbook of Mellin Transforms, first ed., Chapman and Hall/CRC, Boca Raton, 2018.

[26] Hussam M. Gubara, Alshaikh. A. Shokeralla, Mellin Transform of Mittag-Leffler Density and its Relationship with Some Special Functions, International Journal of Research in Engineering and Science (IJRES), Volume 9 Issue 3 (2021) pp. 08-13.

[27] Junesang Choi, H. M. Srivastava, Integral representations for the Gamma function, the Beta function, and the Double Gamma function, Integral Transforms and Special Functions. Vol. 20 No. 11 (2009) pp. 859-869.

[28] Sergey É. Derkachov, Alexander N. Manashov, On complex Gamma function integrals, Symmetry, Integrability and Geometry: Methods and Applications (SIGMA), 16 (2020), 003, 20 pages.

[29] Serge Lang, Cauchy’s Theorem, First Part, in: Serge Lang, Complex Analysis, fourth ed., Springer Science+Business Media, New York, 1999, pp. 86-125.

[30] Serge Lang, Gamma and Zeta Functions, in: Serge Lang, Complex Analysis, fourth ed., Springer Science+Business Media, New York, 1999, pp. 408-433.