STABILITY PROPERTIES OF THE STEADY STATE FOR THE ISENTROPIC COMPRESSIBLE NAVIER-STOKES EQUATIONS WITH DENSITY DEPENDENT VISCOSITY IN BOUNDED INTERVALS

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Abstract. We prove existence and asymptotic stability of the stationary solution for the compressible Navier-Stokes equations for isentropic gas dynamics with a density dependent diffusion in a bounded interval. We present the necessary conditions to be imposed on the boundary data which ensure existence and uniqueness of the steady state, and we subsequently investigate its stability properties by means of the construction of a suitable Lyapunov functional for the system. The Saint-Venant system, modeling the dynamics of a shallow compressible fluid, fits into this general framework.

1. Introduction

In this paper we study existence and stability properties of the steady state for the one dimensional compressible Navier-Stokes equations with density-dependent viscosity, which describes the isentropic motion of compressible viscous fluids in a bounded interval. In terms of the variables mass density and velocity of the fluid \((ρ, w)\), the problem reads as

\[
\begin{aligned}
ρ_t + (ρ w)_x &= 0 \\
(ρ w)_t + (ρ w^2 + P(ρ))_x &= ε(ν(ρ)w_x)_x,
\end{aligned}
\]

(1.1)

to be complemented with boundary conditions

\[
ρ(−ℓ) = ρ_−, \quad w(±ℓ) = w_± > 0,
\]

and initial data \((ρ, w)(x, 0) = (ρ_0, w_0)\).

We restrict our analysis to the barotropic regime, where the pressure \(P \in C^2(\mathbb{R}^+)\) is a given function of the density \(ρ\) satisfying the following assumptions

\[
P(0) = 0, \quad P(+∞) = +∞, \quad P'(s), P''(s) > 0 \quad \forall s > 0.
\]

(1.2)

As concerning the viscosity term \(ν\), we require \(ν(ρ) > 0\) for all \(ρ > 0\).

A prototype for the term of pressure is given by the power law \(P(ρ) = Cρ^γ\), \(γ > 1\), while the case \(ν(ρ) = ρ^α\) with \(α > 0\) is known as the Lamé viscosity coefficient. In particular, the well known viscous Saint-Venant system, describing the motion of a shallow compressible fluid, corresponds to the choice \(P(ρ) = \frac{1}{2}κρ^2\) and \(ν(ρ) = ρ\).

By considering the variables density and momentum \((u, v) = (ρ, ρw)\), system (1.1) becomes

\[
\begin{aligned}
υ_t + (υu)_x &= 0 \\
υ_t + \left(\frac{υ^2}{u} + P(u)\right)_x &= ε\left(ν(u)\left(\frac{υ}{υ}\right)_x\right)_x,
\end{aligned}
\]

(1.3)

together with boundary conditions

\[
w_±u(−ℓ) − v(±ℓ) = 0, \quad v(−ℓ) = ρ_-w_-.
\]

(1.4)

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In the following, we shall use both the formulations (1.1) and (1.3), depending on what is necessary; as an example, when studying the stationary problem, the variables \((u,v)\) appear to be more appropriate since in this case the second component of the steady state turns to be a constant.

**Remark 1.1.** Given an initial datum \((\rho_0, w_0) \in H^1(I) \times H^1(I)\), throughout the paper we will consider solutions to (1.1) \((\rho, w) \in L^\infty([0,T], H^1(I)) \times L^\infty([0,T], H^1(I))\). We refer the readers to Theorem A.1 in Appendix A for the proof of the existence of such a solution. We stress that, of course, also the solution \((u,v)\) to (1.3) belongs to the same functional space, since \((u,v) = (\rho, \rho w)\).

Depending on assumptions and approximations, the Navier-Stokes system may also contain other terms and gives raise to different types of partial differential equations. Indeed, natural modifications of the model emerge when additional physical effects are taken into account, like viscosity, friction or Coriolis forces; far from being exhaustive but only intended to give a small flavor of the huge number of references, see, for instance, [12, 15, 16, 24, 10] and the references therein for existence results of global weak and strong solutions, [20, 28, 34] for the problem with a density-dependent viscosity vanishing on vacuum, [6] for the full Navier-Stokes system for viscous compressible and heat conducting fluids. More recently, the interesting phenomenon of metastability (see, among others, [4, 9, 37] and the references therein) has been investigated both for the incompressible model [5, 27] and for the 1D compressible problem [31, 39].

As concerning system (1.1), there is a vast literature in both one and higher dimensions. Global existence results and asymptotic stability of the equilibrium states are obtained from Kawashima's theory of parabolic-hyperbolic systems in [22], D. Bresch, B. Desjardins and G. Métivier in [8], P.L. Lions in [29] and W. Wang in [42] for the viscous model, and C.M. Dafermos (see [11]) for the inviscid model. As the compressible Navier-Stokes equations with density-dependent viscosity are suitable to model the dynamics of a compressible viscous flow in the appearance of vacuum [18], there are many literatures on the well-posedness theory of the solutions for the 1D model (see, for instance, [17, 21, 36, 43, 44] and the references therein). However, most of these results concern with free boundary conditions. Recently, the analysis of the dynamics in bounded domains has also been investigated (see, for instance, [7]): the initial-boundary value problem with \(\nu(\rho) = \rho^\alpha, \alpha > 1/2\), has been studied by H.L. Li, J. Li and Z. Xin in [25]: here the authors are concerned with the phenomena of finite time vanishing of vacuum. We also quote the analysis performed in [26], where a particular attention is devoted to the dynamical behavior close to equilibrium configurations.

The existence of stationary solutions for system (1.1) and, in particular, for shallow water's type systems, and the subsequent investigation of their stability properties has also been considered in the literature. To name some of these results, we recall here [1] and [13], where the authors are concerned with the inviscid case; in particular, in [13] the authors address the issue of stating sufficient boundary conditions for the exponential stability of linear hyperbolic systems of balance laws (for the investigation of the nonlinear problem, we refer to [2, 3]).

The case with real viscosity has been addressed, for example, in [32]; we mention also the recent contributions [23, 33], where the authors investigate asymptotic stability of the steady state in the half line. We point out that when dealing with the open channel case (i.e. \(x \in \mathbb{R}\)), the study of the stationary problem presents less difficulties than the case of bounded domains, where one has to handle compatibility conditions on the boundary values coming from the study of the formal hyperbolic limit \(\varepsilon = 0\). In this direction, we quote the papers [35, 40], where the authors address the problem of the long time behavior of solutions for the Navier-Stokes system in one dimension and with Dirichlet boundary conditions (see also [41] for the extension of the results to the case of a density dependent diffusion). Finally, a recent contribution in the study of the stationary problem for the viscous model in a bounded interval and with nonhomogeneous Dirichlet boundary
conditions is the paper [38], where the author considers (1.1) in the special case of \( P(\rho) = \kappa \rho^2/2 \) and \( \nu(\rho) = \rho \), corresponding to the viscous shallow water system. Being the literature on the subject so vast, we are aware that this list of references is far from being exhaustive.

Our aim in the present paper is at first to prove existence and uniqueness of a stationary solution to (1.3)-(1.4). Because of the discussion above, this results is likely to be achieved only if some appropriate assumptions on the boundary values are imposed; precisely, following the line of [38] where this problem has been addressed for the case of a linear diffusion \( \nu(u) = u \), our first main contribution (for more details, see Section 3), is the following theorem.

**Theorem 1.2.** Given \( \ell > 0 \) and \( v_-,u_+ > 0 \), let us consider the problem

\[
\begin{align*}
& u_t + v_x = 0 \quad \quad \quad \quad \quad \quad x \in (-\ell,\ell), \ t \geq 0 \\
& v_t + \left\{ \frac{v^2}{u} + P(u) - \varepsilon \nu(u) \left( \frac{v}{u} \right)_x \right\} = 0 \\
& u(\pm \ell, t) = u_\pm, \quad v(-\ell, t) = v_-, \quad t \geq 0 \\
& u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x) \quad \quad x \in (-\ell, \ell),
\end{align*}
\]

and let us suppose that the following assumptions are satisfied:

**H1.** The term of pressure \( P(u) \in C^2(\mathbb{R}^+) \) and the viscosity term \( \nu(u) \) verify, for all \( u > 0 \)

\[
P(0) = 0, \quad P(+) = +\infty, \quad P'(u) \quad \text{and} \quad P''(u) > 0, \quad \nu(u) > 0;
\]

**H2.** Setting \( f(u) := u \int_0^u \frac{P(s)}{s^2} \, ds \), there hold

\[
v_*^2 (u_+ - u_-) = u_- u_+ \left[ P(u_+) - P(u_-) \right] \quad \text{and} \quad \left[ \frac{v^3}{2u^2} + vf'(u) \right] \leq 0,
\]

where \( v_* := v_- \equiv v_+ \). Then there exists a unique stationary solution \((\bar{u}(x), \bar{v}(x))\) to (1.5), i.e. a unique solution \((\bar{u}, \bar{v})\) independent on the time variable \( t \) to the following boundary value problem

\[
\begin{align*}
& v_x = 0 \\
& v_t + \left\{ \frac{v^2}{u} + P(u) - \varepsilon \nu(u) \left( \frac{v}{u} \right)_x \right\} = 0, \\
& u(\pm \ell) = u_\pm, \quad v(-\ell) = v_-.
\end{align*}
\]

**Remark 1.3.** We stress the the choice of the variables \((u, v)\) (instead of the most common choice density/velocity), is dictated by the fact that the second component of the steady state turns to be a constant, and this constant value is univocally determined once the boundary data are imposed.

Once the existence of a unique steady state for system (1.5) is proved, we devote the second part of this paper to investigate its stability properties. Precisely, we prove stability of the steady state in the sense of the following definition.

**Definition 1.4.** A stationary solution \((\bar{u}, \bar{v})\) to (1.5) is stable if for any \( \varepsilon_0 > 0 \) there exists \( \delta_0 = \delta_0(\varepsilon_0) \) such that, if \( |(u_0, v_0)(x) - (\bar{u}, \bar{v})(x)|_{L^2} < \delta_0 \), then, for all \( T > 0 \), it holds

\[
\sup_{0 \leq t \leq T} |(u, v)(t) - (\bar{u}, \bar{v})|_{L^2} \leq \varepsilon_0,
\]

where \((u, v)(t)\) is the solution to (1.5).

Our second main result is stated in the next theorem, showing that the stability of the stationary solution constructed in Theorem 1.2 can be proved only if some additional assumptions on the boundary data are imposed.
Theorem 1.5. Let the assumptions of Theorem 1.2 be satisfied, and let us also assume the following additional hypotheses.

**H3.** There hold, for all $t \geq 0$

$$u_x(\pm \ell, t) = \bar{u}_x(\pm \ell) \quad \text{and} \quad v_x(\pm \ell, t) = \bar{v}_x(\pm \ell),$$

being $(\bar{u}, \bar{v})$ the unique steady state of (1.5) given in Theorem 1.2.

**H4.** The boundary values $u(\pm \ell, t) = u_\pm$ are chosen such that

$$|u_+ - u_-| < \delta,$$

for some positive $\delta$ small enough.

Then the steady state $(\bar{u}, \bar{v})$ is stable in the sense of Definition (1.4).

**Remark 1.6.** It is worth notice that Theorem 1.5 prove stability of the steady state for all time (since the constant $\varepsilon_0$ in Definition 1.4 in independent on $T > 0$). We also point out that the strategy used here do not provide stability of $(\bar{u}, \bar{v})$ in the case the boundary values $u_{\pm}$ do not satisfy any smallness condition, while its existence is assured also in this setting (cfr Theorem 1.2); however, our guess is that “large” solutions are not stable (see also the analysis of [33] and [45], where a similar smallness condition has been imposed in order to have stability of the steady state to a Navier-Stokes system in the half line), and this will be the object of further investigations.

We close this Introduction with a short plan of the paper. In Section 2 we study the inviscid problem, obtained formally by setting $\varepsilon = 0$ in (1.3); we show that, at the hyperbolic level, some compatibility conditions on the boundary data are needed in order to ensure the existence of a weak solution. In particular, such conditions follow from the definition of a couple entropy/entropy flux which, in the present setting, are given by

$$E(u, v) := \frac{v^2}{2u} + f(u) \quad \text{and} \quad Q(u, v) := \frac{v}{u} \left[ \frac{v^2}{2u} + uf'(u) \right],$$

being $f(u) := u \int_0^u P(z)/z^2 \, dz$.

Section 3 is devoted to the study of the stationary problem for (1.5), and in particular to the proof of Theorem (1.2); to this aim we will state and prove several Lemmas showing that, once the boundary conditions are imposed and assumption **H2** is satisfied, there exists a unique positive connection for (1.5), i.e. a unique stationary solution connecting the boundary data. Such analysis deeply relies on the strategy firstly performed in [38], where the author addresses the same problem in the easiest case of a linear diffusion, namely $\nu(u) = u$.

In Section 4, we turn our attention to the stability properties of the steady state, proving that it is stable in the sense of Definition 1.4; the key point to achieve such result is the construction of a Lyapunov functional, which, in the present setting, is defined as

$$L(u(t), v(t), \bar{u}, \bar{v}) := \int_{-\ell}^\ell \frac{(v - \bar{v})^2}{2u} + u \psi(u, \bar{u}) \, dx, \quad \psi(u, \bar{u}) = \int_{\bar{u}}^u \frac{P(z) - P(\bar{u})}{z^2} \, dz.$$

It is easy to check that $L(u, v, \bar{u}, \bar{v})$ is positive defined and null only when computed on the steady state; the tricky part will be the computation of the sign of its time derivative along the solutions, needed in order to apply a Lyapunov type stability theorem.

Finally, in Appendix A we prove the existence of a solution to (1.1) belonging to $L^\infty([0, T], H^1(I)) \times L^\infty([0, T], H^1(I))$; part of the computations are inspired by [19].

As stressed in the introduction, results relative to the existence and stability properties of the steady state for the Navier-Stokes equations in bounded intervals appear to be rare; the study of the stationary problem for (1.5) (with generic pressure $P(u)$ and viscosity $\nu(u)$) and, mostly,
the subsequent investigation of the stability properties of the steady state are, to the best of our knowledge, new.

It is also worth noticing that this analysis is meaningful in view of the possible investigation of the phenomenon of metastability for the one dimensional compressible Navier-Stokes system; indeed, all the informations on the stability properties of the steady state can be useful for the study of the slow motion of the corresponding time dependent solution (see [39, Section 3.1]).

2. The inviscid problem

We start our analysis by studying the limiting regime $\varepsilon \to 0$, obtained formally by putting $\varepsilon = 0$ in (1.1); we obtain the following hyperbolic system for unviscous isentropic fluids

$$\begin{cases}
\rho_t + (\rho w)_x = 0, \\
(\rho w)_t + (\rho w^2 + P(\rho))_x = 0.
\end{cases}$$

System (2.1) is complemented with the same boundary and initial conditions of (1.1). We recall that the usual setting where such a system is studied is given by the entropy formulation, hence non classical discontinuous solutions can appear; thus, we primarily concentrate on the problem of determining the entropy jump conditions for the hyperbolic system (2.1). As previously done in [38] (see also [30]), such conditions are dictated by the choice of a couple entropy/entropy flux $E = E(\rho, w)$ and $Q = Q(\rho, w)$ such that

- the mapping $(\rho, w) \to E$ is convex;
- $E_t + Q_x = 0$ in any region where $(\rho, w)$ is a solution to (2.1).

In particular, $E_t + Q_x \equiv 0$ if and only if

$$\begin{cases}
Q_\rho = w E_\rho + \frac{P'}{\rho} E_w \\
Q_w = \rho E_\rho + w E_w.
\end{cases}$$

To start with, let us consider the easiest case of a power law type of pressure, i.e. $P(\rho) = \kappa \rho^\gamma$ with $\kappa > 0$ and $\gamma > 1$. In this case (see, for instance, [14]) the entropy corresponds to the physical energy of the system and it is defined as

$$E(\rho, w) := \frac{1}{2} \rho w^2 + \frac{\kappa}{\gamma - 1} \rho^\gamma.$$  

By solving (2.2), it turns out that $Q$ is defined as

$$Q(\rho, w) = w \left[\frac{1}{2} \rho w^2 + \frac{\kappa_\gamma}{\gamma - 1} \rho^\gamma\right].$$

In the case of a general term of pressure $P(\rho)$ satisfying assumptions (1.2), the couple entropy/entropy flux is given by

$$E(\rho, w) = \frac{1}{2} \rho w^2 + f(\rho) \quad \text{and} \quad Q(\rho, w) = w \left[\frac{1}{2} \rho w^2 + \rho f'(\rho)\right],$$

being

$$f(\rho) = \rho \int_0^\rho \frac{p(z)}{z^2} \, dz.$$  

Of course we observe that, when $P(\rho) = \kappa \rho^\gamma$, we recover (2.3)-(2.4).
Following the line of [38], given $\rho_{\pm} > 0$, $w_{\pm} > 0$ and $c \in \mathbb{R}$, let $(\rho_-, w_-)$ and $(\rho_+, w_+)$ be an entropic discontinuity of (2.1) with speed $c$, that is we assume the function

\begin{equation}
(\Upsilon, W)(x, t) := \begin{cases}
(\rho_-, w_-) & \text{for } x < ct \\
(\rho_+, w_+) & \text{for } x > ct
\end{cases}
\end{equation}

(2.6) to be a weak solution to (2.1) satisfying, in the sense of distributions, the entropy inequality

\begin{equation}
\frac{\partial \mathcal{E}}{\partial t} + \frac{\partial \mathcal{Q}}{\partial x} \leq 0.
\end{equation}

On one side, with the change of variable $\xi = x - ct$, system (2.1) reads

\begin{equation}
\begin{cases}
-c \rho \xi + (\rho w) \xi = 0, \\
-c(\rho w) \xi + (\rho w^2 + P(\rho)) \xi = 0,
\end{cases}
\end{equation}

and the request of weak solution translates into the Rankine-Hugoniot conditions, that read

\begin{equation}
[\rho(w - c)] = 0 \quad \text{and} \quad [\rho w(w - c) + P(\rho)] = 0.
\end{equation}

On the other side, the entropy condition (2.7) reads $[\mathcal{Q} - c \mathcal{E}] \leq 0$, where

\[\mathcal{Q} - c \mathcal{E} = \frac{1}{2} \rho w^3 + f'(\rho)w\rho - c \rho w^2 - cf(\rho).\]

Setting $w - c = z$, we have

\[\mathcal{Q} - c \mathcal{E} = \frac{1}{2} \rho z^3 + f'(\rho)\rho z + c [\rho v^2 - f(\rho) + f'(\rho)\rho] + \frac{1}{2} c^2 \rho v,
\]

so that, recalling

\[f'(\rho) = \int_0^\rho \frac{P(z)}{z^2} \, dz + \frac{P(\rho)}{\rho} \quad \Rightarrow \quad f'(\rho)\rho = P(\rho) + f(\rho),
\]

and by using (2.8), the entropy condition translates into

\begin{equation}
\left[\frac{1}{2} \rho z^3 + \rho z f'(\rho)\right] \leq 0.
\end{equation}

By squaring the first condition in (2.8), we obtain a system for the quantities $z_{\pm}^2$, whose solutions are given by

\begin{equation}
z_+^2 = \frac{\rho_+ [P(\rho_+ - P(\rho_-)]}{\rho_+ - (\rho_+ - \rho_-)} , \quad z_-^2 = \frac{\rho_- [P(\rho_+ - P(\rho_-)]}{\rho_- - (\rho_+ - \rho_-)} .
\end{equation}

When looking for the stationary solutions to (2.1), i.e. $c = 0$, (2.10) translates into the following conditions for the boundary values

\begin{equation}
w_+^2 = \frac{\rho_- [P(\rho_+ - P(\rho_-)]}{\rho_+ - (\rho_+ - \rho_-)} \quad \text{and} \quad w_-^2 = \frac{\rho_+ [P(\rho_+ - P(\rho_-)]}{\rho_- - (\rho_+ - \rho_-)} ,
\end{equation}

that, together with (2.9), univocally determine the possible choices of the boundary data for the jump solution (2.6) with $c = 0$ to be an admissible steady state for the system.

In particular, for all $x_0 \in (-\ell, \ell)$, we can state that the one-parameter family

\[(\Upsilon, W)(x) = (\rho_-, w_-)\chi_{(-\infty, x_0)} + (\rho_+, w_+)\chi_{(x_0, \infty)}
\]

is a family of stationary solutions to (2.1) if and only if both (2.9) and (2.11) are satisfied.

Finally, we point out that, in terms of the variables density/momentum, conditions (2.9)-(2.11) read as

\[v_+^2 = v_-^2 = \frac{u_+ u_- [P(u_+) - P(u_-)]}{(u_+ - u_-)} \quad \text{and} \quad \left[\frac{v}{u} \left(\frac{v^2}{2u} + uf'(u)\right)\right] \leq 0.
\]
Example 2.1. In the case of the scalar Saint-Venant system, i.e. \( P(u) = \frac{1}{2} \kappa u^2 \), stationary solutions to

\[
\begin{align*}
    u_t + v_x &= 0, \\
    v_t + \left( \frac{v^2}{u} + P(u) \right)_x &= 0,
\end{align*}
\]

to be considered with boundary data \( u(\pm \ell, t) = u_{\pm} \) and \( v(-\ell) = v_* \), solve

\[
    v = v_*, \quad \frac{1}{2} \kappa u^3 - \alpha u + v_*^2 = 0,
\]

where

\[
    v_*^2 = \frac{1}{2} \kappa u_- u_+(u_+ + u_-) \quad \text{and} \quad \alpha = \frac{1}{2} \kappa (u_*^2 + u_+ u_- + u_-^2).
\]

Moreover, only entropy solutions are admitted, so that, from (2.9)

\[
    \frac{v_+}{u_+} (u_+ - u_-) \geq 0.
\]

Since \( v_+, u_+ > 0 \), then \( u_- < u_+ \), and this condition describes the realistic phenomenon of the hydraulic jump consisting in an abrupt rise of the fluid surface and a corresponding decrease of the velocity.

3. Stationary solutions for the viscous problem

This section is devoted to the study of the existence and uniqueness of a stationary solution for the Navier-Stokes system (1.3). As stressed in the introduction, we here prefer to use the variables density/momentum \((u, v)\) rather than the most common choice density/velocity since in this case the second component of steady state turns to be a constant, which is univocally determined by the boundary values. We are thus left with a single equation for the variable \( u \) which can be integrated with respect to \( x \), by paying the price of the appearance of an integration constant.

For \( \varepsilon > 0 \), the stationary equations read

\[
(3.1) \quad v_x = 0, \quad \left\{ \frac{v^2}{u} + P(u) - \varepsilon \nu(u) \left( \frac{v}{u} \right)_x \right\} = 0,
\]

which is a couple of ordinary differential equations; by integrating in \( x \) we can lower the order of the system obtaining the following stationary problem for the couple \((u, v)\):

\[
\begin{align*}
    v &= v_* \\
    v_* \varepsilon \frac{\nu(u)}{u} u_x &= -P(u)u + \alpha u - v_*^2 \\
    u(\pm \ell) &= u_{\pm}, \quad v(-\ell) = v_-,
\end{align*}
\]

being \( \alpha \) an integration constant that depends on the values of the solution and its derivative on the boundary, while \( v_* \) is univocally determined by the boundary datum \( v(-\ell) \); indeed, since the component \( v \) of the steady state turns to be constant, the values \( v(\ell) \) and \( v(-\ell) \) are forced to be equal to a common value, named here \( v_* \).

Let us define \( \Phi(u) = \int_0^u \frac{\nu(s)}{s} ds \); since \( \nu > 0 \) there hold

\[
\Phi(u) > 0 \quad \text{and} \quad \Phi'(u) = \frac{\nu(u)}{u} > 0, \quad \forall u > 0,
\]

where \( \Phi' \) indicates the derivative of \( \Phi \) with respect to \( u \). Thus, the second equation in (3.2) can be rewritten as

\[
    v_* \varepsilon \left[ \Phi(u) \right]_x = -P(u)u + \alpha u - v_*^2.
\]
Setting \( f(u) := -P(u)u + \alpha u - v_*^2 \), with the change of variable \( w = \Phi(u) \), and since \( \Phi(u) \) is invertible, we have
\[
(3.3) \quad v_* \epsilon \Phi^{-1}(w) = (f \circ \Phi^{-1})(w) \equiv g(w).
\]
We thus end up with an autonomous first order differential equation of the form \( w' = g(w) \); in this case it is not possible to obtain an explicit expression for the solution, and in order to provide qualitative properties of the solution we have to study the function \( g(w) \).

The problem of studying properties of the right hand side of (3.3) has been previously addressed in [38] in the case of a linear diffusion \( \nu(u) = u \) (that is, \( \Phi(u) \equiv \text{Id} \)). Precisely, the author states and proves a set of results describing the behavior of the function \( f(u; \alpha) \) both with respect to \( u \) and with respect to \( \alpha \).

We recall here some of these results for completeness, since they will be useful to describe the qualitative properties of the function \( g(w) := (f \circ \Phi^{-1})(w) \); for more details we refer to [38, Lemma 3.1, Lemma 3.2]. From now on, we will always suppose the pressure term \( P(u) \) to satisfy assumptions (1.2). We also recall that, by definition
\[
(3.4) \quad f(u) := -P(u)u + \alpha u - v_*^2.
\]

**Lemma 3.1.** For every \( v_* > 0 \), there exists at least a value \( \alpha \) such that there exist two positive solutions to the equation \( f(u) = 0 \).

**Remark 3.2.** As enlightened in the proof of [38, Lemma 3.1], a sufficient condition on the constant \( \alpha \) for the existence of two positive solutions to \( f(u) = 0 \) is given by
\[
(3.5) \quad v_* < \sqrt{P'(u^*)u^{*2}},
\]
where \( u^* = u^*(\alpha) \) solves \( f'(u) = 0 \), while \( v_* = v(\pm \ell) \). Indeed, since \( f(0) = -v_*^2 \) and \( f(+\infty) = -\infty \), if \( u^* \) is such that
\[
f(u^*) = \max_{\mathbb{R}^+} f, \quad f(u^*) > 0,
\]
then the thesis follows. By exploiting the conditions \( f(u^*) > 0 \) and \( f'(u^*) = 0 \), we end up with (3.5).

**Lemma 3.3.** Let \( \alpha \) be such that there exist two positive solutions \( u_1 < u_2 \) to the equation \( f(u) = 0 \). Hence, given \( u_\pm > 0 \), the set \( \mathcal{A} \) defined as
\[
\mathcal{A} := \{ \alpha > 0 : u_1 < u_- < u_+ < u_2 \}
\]
is such that \( \mathcal{A} = [\bar{\alpha}, +\infty) \), for some \( \bar{\alpha} > 0 \).

Lemmas 3.1-3.3 assure that, once the boundary conditions \( u_\pm \) are imposed, there always exists a value for the integration constant \( \alpha \) such that there exist two positive solutions \( u_{1,2} \) to the equation \( f(u) = 0 \) satisfying
\[
(u_-, u_+) \subset (u_1, u_2).
\]
This is of course a necessary condition for the existence of an increasing positive connection between \( u_- \) and \( u_+ \), as enlightened in Figure 1 in the specific example of \( P(u) = \kappa u^2 \) and \( \nu(u) = u \).

### 3.1. The stationary problem.

By taking advantage of the already known properties of the function \( f(u) \), we now study the function \( w \mapsto g(w) \). We first notice that the function \( f \) is increasing for \( u \in [0, u^*] \) and decreasing for \( u \in (u^*, +\infty) \), where \( u^* \) is implicitly defined as
\[
P(u^*) = \alpha - P'(u^*)u^*
\]
is such that \( f'(u^*) = 0 \). Moreover, as already stressed in Remark 3.2, if \( \alpha \) is such that \( f(u^*) > 0 \), that is
\[
P'(u^*)u^{*2} > v_*^2,
\]
Figure 1. Plot of the solutions to $\varepsilon v_x u = f(u)$. The choice for $u_-$ and $u_+$ is such that $u_+ > u_2$. In the plane $(x, u)$ we can see that the solution starting from $u(-\ell) = u_-$ can not reach $u_+$, since $u_2$ is an equilibrium solution for the equation. The same holds if $u_- < u_1$.

then there exist two positive solutions to the equation $f(u) = 0$. Given $\nu(u) > 0$, since $\Phi(u) > 0$ and $\Phi'(u) > 0$, we have

$$\Phi^{-1}(w) > 0, \quad (\Phi^{-1})'(w) = \frac{1}{\Phi'(u)} > 0,$$

proving that $\Phi^{-1}$ is a positive increasing function as well.

Let us now consider $g(w) = (f \circ \Phi^{-1})(w)$; we prove the following lemma.

**Lemma 3.4.** Let $g(w) = (f \circ \Phi^{-1})(w)$, with $f$ defined in (3.4). For every $\nu > 0$ there exist $w_1, w_2 > 0$ such that $g(w_1) = g(w_2) = 0$. Moreover, the function $g$ is increasing in the interval $[0, w^*)$, and decreasing in the interval $(w^*, +\infty)$, being $w^* := \Phi(u^*)$.

**Proof.** Lemma 3.1 assures the existence of two positive values $u_1$ and $u_2$ such that $f(u_1) = f(u_2) = 0$ and, as a consequence, $w_1$ and $w_2$ has to be defined as

$$(3.6) \quad \Phi^{-1}(w_1) = u_1 \quad \text{and} \quad \Phi^{-1}(w_2) = u_2.$$  

Since $\Phi^{-1}(0) = 0$ and $(\Phi^{-1})' > 0$, there exist and they are unique $w_1$ and $w_2$ such that (3.6) holds. Hence, $g(w)$ has exactly two positive zeros for all the choices of $\nu(u) > 0$. Furthermore

$$g'(w) = [f(\Phi^{-1}(w))]' = f'(\Phi^{-1}(w)) \cdot (\Phi^{-1})'(w),$$

so that the sign of $g'$ is univocally determined by the sign of $f'$. Therefore, if $w^*$ is such that $\Phi^{-1}(w^*) = u^*$, then

$$g'(w^*) = 0, \quad g'(w) > 0 \text{ for } w \in [0, w^*), \quad g'(w) < 0 \text{ for } w \in (w^*, +\infty).$$

We finally notice that condition (3.5) for the existence of two positive solutions to the equation $f(u) = 0$, also assures that $g(w)$ has to positive zeros. Indeed

$$g'(w) = f'(\Phi^{-1}(w)) \cdot (\Phi^{-1})'(w) = \frac{f'(\Phi^{-1}(w))}{\Phi'(w)},$$

so that $g'(w) = 0$ if and only if $w = w^*$, where $w^*$ is such that $\Phi^{-1}(w^*) = u^*$. Furthermore,

$$g(w^*) > 0 \iff f(\Phi^{-1}(w^*)) > 0 \iff f(u^*) > 0,$$

which is exactly (3.5).
Example 3.5 (The Saint-Venant system with density dependent viscosity). When \( P(u) = \frac{1}{2} \kappa u^2 \), the stationary equation (3.2) for \( u \) reads
\[
v_* \varepsilon [\Phi(u)]_x = -\frac{1}{2} \kappa u^3 + \alpha u - v_*^2.
\]
Let us consider the simplest case \( \nu(u) = Cu^\gamma \), \( \gamma > 0 \), and let us plot the function \( g(w) = (f \circ \Phi^{-1})(w) \). We have
\[
\Phi(u) = C \int_0^u s^{\gamma-1} ds = \frac{C}{\gamma} u^{\gamma} \quad \text{and} \quad \Phi^{-1}(w) = \left( \frac{\gamma}{C} u \right)^{\frac{1}{\gamma}},
\]
so that
\[
g(w) = -\frac{1}{2} \kappa \left( \frac{C}{\gamma} \right)^{\frac{3}{\gamma}} w^{\frac{3}{\gamma}} + \left( \frac{C}{\gamma} \right)^{\frac{1}{\gamma}} \alpha w^{\frac{1}{\gamma}} - v_*^2.
\]

Figure 2. Plots of different \( g(w) \) with \( \kappa = 1 \), \( \alpha = 400 \) and \( v_*^2 = 1000 \). The dashed line plots \( g(w) = -\frac{1}{2} w^6 + 400w^3 - 1000 \), the dashed point line plots \( g(w) = -\frac{1}{2} w^{3/2} + 400\sqrt{w} - 1000 \), while the black line plots \( f(w) = -\frac{1}{2} w^3 - 400w - 1000 \).

Figure 2 shows the plot of \( g(w) \) for different choice of \( \nu(u) \), compared with the plot of \( f(w) \) (where \( \nu(u) = u \)); the dashed line and the dashed point line plot \( g(w) \) with \( \nu(u) = \sqrt{s} \) and \( \nu(u) = 2s^2 \) respectively. As proved in Lemma 3.4, we can see that the monotonicity properties of the function \( g \) are preserved, as well as the existence of two positive zeros.

3.2. Existence and uniqueness of a positive connection. Let us go back to the problem of the existence and uniqueness of the solution to the stationary problem (3.1). As already shown, once the boundary conditions for the function \( v \) are imposed, problem (3.1) reads
\[
\begin{cases}
v = v_*, \\
v_* \varepsilon w_x = g(w), \quad w(\pm \ell) = \Phi(u_\pm)
\end{cases}
\]
where \( v_* = v(-\ell) \) and \( g(w) = (f \circ \Phi^{-1})(w) \), being \( f(u) = -P(u)u + \alpha u - v_*^2 \).

Hence, the equation for the variable \( w := \Phi(u) \) is an equation on the form
\[
w' = g(w; \alpha), \quad w(\pm \ell) = w_\pm,
\]
where \( \alpha \) is an integration constant depending on the boundary data. Once the boundary conditions are imposed, a positive connection between \( \Phi(u_-) \) and \( \Phi(u_+) \) (i.e. a positive solution to \( \varepsilon v_* w_x = \))
g(w) connecting \( \Phi(u_-) \) and \( \Phi(u_+) \) exists only if
\[
(\Phi(u_-), \Phi(u_+)) \subset (w_1, w_2),
\]
being \( w_1 \) and \( w_2 \) such that \( g(w_1) = g(w_2) = 0 \).

The following Lemma (to be compared with Lemma 3.3) aims at showing some properties of the function \( g(w; \alpha) \) as a function of \( \alpha \); precisely, we describe how the distance between the two zeroes of the function changes with respect to this parameter.

**Lemma 3.6.** Let \( g(w) = (f \circ \Phi^{-1})(w) \) with \( f \) defined in (3.4), and let \( \alpha \) be such that (3.5) holds, so that there exist two positive solutions \( w_1 < w_2 \) to the equation \( g(w) = 0 \). Given \( u_\pm > 0 \), the set \( A \) defined as
\[
A := \{ \alpha > 0 : w_1 < \Phi(u_-) < \Phi(u_+) < w_2 \},
\]
is such that \( A = [\bar{\alpha}, +\infty) \), for some \( \bar{\alpha} > 0 \).

**Proof.** Since \( w_1 = w_1(\alpha) \) and \( w_2 = w_2(\alpha) \), we want to show that \( g(w; \alpha) \) is an increasing function with respect to \( \alpha \). Indeed, this would imply that, if there exists a value \( \alpha \) such that
\[
w_1 < w_- < w_+ < w_2,
\]
then, for all \( \alpha' > \alpha \)
\[
w_1' < w_- < w_+ < w_2',
\]
being \( w_1' \) and \( w_2' \) the two positive zeroes of \( g(w; \alpha') \).

Since \( g(w; \alpha) = f(\phi^{-1}(w; \alpha)) \) and \( \Phi^{-1}(w) \) is an increasing function that does not depend on \( \alpha \),
g(\( w, \alpha \)) is an increasing function in the variable \( \alpha \) if so it is for \( f(u; \alpha) \). We have
\[
f(u; \alpha) - f(u; \alpha') = (\alpha - \alpha')u,
\]
so that, since \( u > 0 \), \( f(u, \alpha') - f(u, \alpha) > 0 \) when \( \alpha' > \alpha \).

Thus, we only need to prove that there exist a value \( \bar{\alpha} \) such that \( w_1 < \Phi(u_-) < \Phi(u_+) < w_2 \). We know that \( g(0) = -\frac{v^2}{\alpha} < 0 \) and \( g'(w) > 0 \) for all \( w \in (0, w^*) \). Moreover
\[
g(w^*) = f(\Phi^{-1}(w^*)) = f(u^*) > 0,
\]
so that \( w_1 \in (0, w^*) \). Furthermore, if we ask for
\[
g\left(\frac{2v^2}{\alpha}\right) = f\left(\Phi^{-1}\left(\frac{2v^2}{\alpha}\right)\right) > 0
\]
we have \( w_1 < \frac{2v^2}{\alpha} \). Condition (3.7) can be rewritten as
\[
f\left(\Phi^{-1}\left(\frac{2v^2}{\alpha}\right)\right) > f(u_1) = 0,
\]
that is, since \( \Phi^{-1}(w^*) = u_1 \)
\[
f\left(\Phi^{-1}\left(\frac{2v^2}{\alpha}\right)\right) > f\left(\Phi^{-1}(w_1)\right).
\]
Since \( f \) and \( \Phi^{-1} \) are increasing function in the interval \( [0, u^*) \) and \( [0, w^*) \) respectively, we obtain the following condition for the constant \( \alpha \)
\[
\frac{2v^2}{\alpha} > w_1.
\]
If this condition holds, then
\[
0 < w_1 < \frac{2v^2}{\alpha},
\]
showing that \( w_1 \to 0 \) as \( \alpha \to +\infty \). On the other hand we know that \( u_2 > u^* \) where \( u^* \) is such that \( f(u^*) = \max_\alpha f \). Hence
\[
\Phi^{-1}(u_2) > \Phi^{-1}(w^*) \quad \Rightarrow \quad w_2 > \Phi(u^*).
Since \( u^* \to +\infty \) as \( \alpha \to +\infty \), and since \( \Phi \) is an increasing and continuous function, we know that 
\[
\Phi(u^*) \to \Phi(+\infty) = +\infty \text{ as } \alpha \to +\infty,
\]
implying \( u_2 \to +\infty \) as \( \alpha \to +\infty \).

We have thus proved that, if we choose \( \bar{\alpha} \) large enough, then \((\Phi(u_-), \Phi(u_+)) \subset (w_1, w_2)\) for every choice of \( u_\pm > 0 \). More precisely, \( \bar{\alpha} \) is chosen in such a way that 
\[
\bar{\alpha} > \max\{\alpha^*, \alpha^{**}\},
\]
where \( \alpha^* \) and \( \alpha^{**} \) are such that either \( g(\Phi(u_-), \alpha^*) = 0 \) or \( g(\Phi(u_+), \alpha^{**}) = 0 \). 

\[
\square
\]

**Definition 3.7.** We define the region \( \Sigma \) of admissible values \( \alpha \) as the set of all the values \( \alpha \) such that there exists two positive solutions to the equation \( g(w) = 0 \) and Lemma 3.6 holds. In the plane \( \{v^*, \alpha\}, \) \( \Sigma \) is determined by the equations
\[
v^*_2 < P'(u^*)u^{*2}, \quad g(\Phi(u_\pm)) > 0, \quad \alpha < \frac{2v_*}{w_1},
\]
We recall that \( u^* \) is such that \( f'(u^*) = 0 \) and \( v_* = v(\pm \ell) \).

**Proposition 3.8.** The region \( \Sigma \) is the epigraph of an increasing function \( h: \mathbb{R} \to \mathbb{R} \), i.e.
\[
\Sigma := \text{epi}(h) = \{(v^*, \alpha) : v_* \in \mathbb{R}, \alpha \in \mathbb{R}, \alpha \geq h(x)\} \subset \mathbb{R} \times \mathbb{R}.
\]

**Proof.** Setting \( \varphi(\alpha) := \sqrt{P'(u^*)}u^* \), we have
\[
\lim_{\alpha \to 0} \varphi(\alpha) = 0, \quad \lim_{\alpha \to +\infty} \varphi(\alpha) = +\infty, \quad \varphi'(\alpha) > 0,
\]
meaning that \( v_* = \varphi(\alpha) \) is an increasing function in the plane \((v_*, \alpha)\). Moreover, the condition \( g(\Phi(u_\pm)) > 0 \) is equivalent to
\[
g(\Phi(u_\pm)) = (f \circ \Phi^{-1})(\Phi(u_\pm)) = f(u_\pm) > 0,
\]
and we get
\[
\alpha > \frac{1}{u_-} v^*_2 + P(u_-), \quad \alpha > \frac{1}{u_+} v^*_2 + P(u_+)
\]
whose equality defines two parabolas. Finally, the function \( \Psi(\alpha) = \frac{2v_*}{w_1(\alpha)} \) is such that
\[
\lim_{\alpha \to +\infty} \Psi(\alpha) = +\infty \quad \text{and} \quad \Psi'(\alpha) = -\frac{2v_*}{w_1^2} w'_1 > 0,
\]
since \( w_1(\alpha) \) is a decreasing function. Hence \( \frac{dh}{dv_*} > 0 \), since \( h \) is obtained by matching increasing functions. 

\[
\square
\]

We now prove the existence of a \( 2\ell \)-connection, i.e. we prove the existence of a solution to 
\[
\varepsilon v_* w_x = g(w), \quad w(\pm \ell) = \Phi(u_\pm),
\]
satisfying
\[
2\ell = \varepsilon v_* \int_{\Phi(u_-)}^{\Phi(u_+)} \frac{dw}{(f \circ \Phi^{-1})(w)} := G(\alpha).
\]
We first notice that \( G|_{\partial \Sigma} = +\infty \). From the study of \( G(\alpha) \), we can prove that there exists a unique value \( \alpha^* \) such that \( G(\alpha^*) = 2\ell \). Indeed, we can easily see that
\[
\lim_{\alpha \to +\infty} G(\alpha) = 0, \quad \lim_{\alpha \to 0} G(\alpha) = +\infty \quad \text{and} \quad \frac{dG}{d\alpha} < 0,
\]
for \( \bar{\alpha} \in \partial \Sigma \) and for all \( \alpha > 0 \).

We are finally able to prove Theorem 1.2, which we recall here for completeness.
**Theorem 3.9.** Given \( \ell > 0 \) and \( u_+, v_- > 0 \), let us consider the following problem

\[
\begin{align*}
u_t + v_x &= 0 & x \in (-\ell, \ell), \ t \geq 0 \\
v_t + \left\{ \frac{v^2}{u} + P(u) - \varepsilon \nu(u) \left( \frac{v}{u} \right)_x \right\} &= 0 \\
u(\pm \ell, t) &= u_{\pm}, \ v(-\ell, t) = v_-, \ t \geq 0 \\
u(x, 0) &= u_0(x), \ v(x, 0) = v_0(x) & x \in (-\ell, \ell),
\end{align*}
\]

where \( P(u) \) and \( \nu(u) \) satisfy hypothesis \( \text{H1} \). If \( u_\pm \) and \( v_- \) verify

\[
\text{H2.} \quad v_+^2 (u_+ - u_-) = u_- u_+ [P(u_+) - P(u_-)] \quad \text{and} \quad \left[ \frac{v^3}{2u^2} + vf'(u) \right] \leq 0,
\]

being \( f(u) := u \int P(z)/z^2 dz \) and \( v^* = v_- > 0 \), then there exists a unique stationary solution \((\bar{u}(x), \bar{v}(x))\) to (3.8).

**Proof.** As already mentioned, the second component of the steady state is univocally determined once the boundary conditions are imposed, that is \( \bar{v}(x) \equiv v_\ast \).

Going further, Lemma 3.6 assures that, for any choice of \( u_\pm \), there exists at least a value \( \alpha \in \Sigma \) such that \( w_1 < \Phi(u_-) < \Phi(u_+) < w_2 \), so that there exists a *positive connection* satisfying the boundary conditions. Moreover, from the study of the function \( G(\alpha) \), we know that there exists a unique value \( \alpha^* \in \Sigma \) such that \( G(\alpha^*) = 2\ell \), so that there exists a unique *positive connection* \( \bar{w}(x) \) between \( \Phi(u_-) \) and \( \Phi(u_+) \) of “length” \( 2\ell \). Since \( \Phi \) is invertible, \( \bar{w}(x) := \Phi^{-1}(\bar{w}) \) is the unique positive connection between \( u_- \) and \( u_\ast \).

\[ \square \]

### 3.3. The Saint-Venant system.

An interesting case where we can explicitly develop computations is the Saint-Venant system, already studied in [38]; here the term of pressure \( P(u) \) is given by the quadratic formula \( P(u) = \frac{1}{2}\kappa u^2 \), \( \kappa > 0 \) and the viscosity \( \nu(u) = u \).

In this case, stationary solutions solve

\[
v = v_- \quad \text{and} \quad \varepsilon v_{\ast} u_x = -\frac{1}{2}\kappa u^3 + \alpha u - v_{\ast}^2 := f(u),
\]

where, as usual, \( v_\ast = v_- \). The condition (3.5) for the existence of two positive solution \( u_1 \) and \( u_2 \) enlightened in Remark 3.2 reads \( \alpha^3 > 27/8 \kappa v_\ast^8 \) (which is exactly the Cardano condition for the existence of three real solutions to third order equations in the form \( u^3 + pu + q = 0 \)). Moreover, since \( f(0) = -v_{\ast}^2 \) and \( \alpha > 0 \), we can explicitly show that \( u_0 < 0 < u_1 < u_2 \), where \( u_0 \) is the third (negative) root of the equation \( f(u) = 0 \).

Figure 3 plots the function \( f(u) \) for different choices of the constant \( \alpha \). The picture explicitly shows how the first positive zero \( u_1 \) remains close to zero while \( u_2 \) becomes bigger as \( \alpha \to +\infty \). Figure 3 also shows that the interval \( (u_-, u_+) \) is included or not inside \( (u_1, u_2) \), depending on the choice of \( \alpha \).

### 4. Stability properties of the steady state

In this Section we study the stability properties of the unique steady state \((\bar{u}, \bar{v})\) to the Navier-Stokes equations

\[
\begin{align*}
u_t + v_x &= 0 & x \in (-\ell, \ell), \ t \geq 0 \\
u_t + \left\{ \frac{v^2}{u} + P(u) - \varepsilon \nu(u) \left( \frac{v}{u} \right)_x \right\} &= 0 \\
u(\pm \ell, t) &= u_{\pm}, \ v(-\ell, t) = v_-, \ t \geq 0 \\
u(x, 0) &= u_0(x), \ v(x, 0) = v_0(x) & x \in (-\ell, \ell),
\end{align*}
\]
Figure 3. Plot of $f(u) = -\frac{1}{2}\kappa u^3 + \alpha u - v_*^2$ for fixed $v_*$ and multiple choices of $\alpha$.

which is known to exist and to be unique thanks to Theorem 3.9. As stated in the Introduction, the key tool we are going to use is the construction of a Lyapunov functional for (4.1); a similar strategy has been already used in [23], where the authors prove asymptotic stability for the steady state of the Navier-Stokes system in the half line. We here prove stability in the sense of Definition 1.4, and our goal is to prove Theorem 1.5, providing an estimate on the $L^2$-norm of the difference $(u,v) - (\bar{u},\bar{v})$, being $(u,v)$ the solution to (4.1).

4.1. Construction of the Lyapunov functional. System (4.1) admits a mathematical entropy which is also a physical energy

$$E(u,v) := \frac{v^2}{2u} + u \phi(u), \quad \phi(u) = \int_0^u \frac{P(z)}{z^2} \, dz,$$

where we recall that $u \phi(u) + P(u) = u f'(u)$, being

$$f(u) = u \int_0^u \frac{P(z)}{z^2} \, dz.$$

In the present setting, since $\varepsilon > 0$, the entropy flux is given by

$$Q(u,v) := \frac{v}{u} \left[ \frac{v^2}{2u} + u \phi(u) + P(u) - \frac{v}{u} \left( \nu(u) \left( \frac{v}{u} \right)_x \right) \right],$$

and the energy equality thus becomes

$$\frac{1}{2} \left( \frac{v^2}{2u} + u \phi(u) \right)_t + \left[ \frac{v^3}{2u^2} + v \phi(u) + P(u) - \frac{v^2}{u^2} \left( \nu(u) \left( \frac{v}{u} \right)_x \right) \right]_x = -\varepsilon \nu(u) \left[ \left( \frac{v}{u} \right)_x \right]^2.$$

Inspired by the results in [23], we introduce the new energy form

$$\mathcal{E}_L(u,v,\bar{u},\bar{v}) := \frac{(v - \bar{v})^2}{2u} + u \psi(u,\bar{u}), \quad \psi(u,\bar{u}) = \int_\bar{u}^u \frac{P(z) - P(\bar{u})}{z^2} \, dz,$$

and we claim that a good candidate to be a Lyapunov functional for the system is

$$L(u,v,\bar{u},\bar{v}) := \int_{-\ell}^{\ell} \mathcal{E}_L(u,v,\bar{u},\bar{v}) \, dx.$$
Indeed, (4.3) is of course null as computed on the steady state \((u, v) = (\bar{u}, \bar{v})\) and positive defined since \(P'(u) > 0\).

4.2. Computation of the time derivative of \(L\) along the solutions. We want to compute the time derivative of (4.3), showing that it is negative along the solutions to (4.1). We have

\[
L(u, v, \bar{u}, \bar{v}) = \int_{-\ell}^{\ell} \left\{ \frac{v^2}{2u} + \frac{\bar{v}^2}{2\bar{u}} - \frac{v\bar{v}}{u} + u \psi(u, \bar{u}) \right\} \, dx,
\]

and we observe that

\[
\psi(u, \bar{u}) = \int_{\bar{u}}^{u} \frac{P(z) - P(\bar{u})}{z^2} \, dz = \int_{\bar{u}}^{u} \frac{P(z)}{z^2} \, dz - \int_{\bar{u}}^{u} \frac{P(\bar{u})}{z^2} \, dz.
\]

In particular, \(L\) can be rewritten as

\[
(4.4) \quad L(u, v, \bar{u}, \bar{v}) = \int_{-\ell}^{\ell} \left\{ \frac{v^2}{2u} + u \phi(u) + \frac{\bar{v}^2}{2\bar{u}} - \frac{v\bar{v}}{u} - u \int_{\bar{u}}^{u} \frac{P(z)}{z^2} \, dz + P(\bar{u}) \frac{\bar{u} - u}{\bar{u}} \right\} \, dx.
\]

Notations: Throughout this section, we shall write \(A \lesssim B\) if there exists a positive constant \(C\) such that

\[
A \leq C B.
\]

Also, for the sake of shortness, we will omit the dependence of \((u, v)\) from the variables \((x, t)\).

Finally, given two functional spaces \(X\) and \(Y\) and a function \(f\) of the two variables \((x, t) \in [-\ell, \ell] \times [0, T]\) such that \(f(x, \cdot) \in X\) and \(f(\cdot, t) \in Y\), we will denote with

\[
|f|_{XY} := |f|_{X([0,T],Y(t))}.
\]

Hypotheses: As stated in the Introduction, in order to prove the stability of the steady state, we need to state some additional assumptions on the boundary values. Precisely, we make the following hypotesis on the derivative of the solution \((u, v)\) at the boundary:

\[\text{H3.}\] Let us suppose that, for all \(t \geq 0\)

\[
u_x(\pm \ell, t) = \bar{u}_x(\pm \ell) \quad \text{and} \quad v_x(\pm \ell, t) = (v_*)_x \equiv 0.
\]

Also, we require that \(u(\pm \ell, t) = u_\pm\) satisfy the following smallness condition:

\[\text{H4.}\] There exists a positive constant \(\delta\) such that \(|u_+ - u_-| < \delta\).

As already remarked, similar requests as the one in \textbf{H4}, providing a smallness condition on the density \(u\), have already been stated in [33, 45] in order to have stability of the steady state for a Navier-Stokes system in the half line.

Proposition 4.1. Let assumptions \textbf{H3-4} be satisfied; then, for \(\delta\) sufficiently small, we have

\[
\frac{d}{dt} L(u, v, \bar{u}, \bar{v}) \leq 0,
\]

for all \((u, v)\) solutions to (4.1), being \((\bar{u}, \bar{v})\) the unique stationary solution to (4.1) and \(L\) as in (4.3).

Remark 4.2. Throughout the proof, we will extensively make use of the positivity, for all \(t \geq 0\), of the function \(u(x, t)\) and its space derivative in the interval \((-\ell, \ell)\). Also, we will use the results of Appendix A, providing the existence of a solution \((u, v) \in L^\infty([0, T], H^1(I)) \times L^\infty([0, T], H^1(I))\); in particular, we will use the fact that the quantities \(|(u, v)|_{L^\infty H^1} \) and \(|(u, v)|_{L^\infty L^\infty}\) are finite.
Proof. By taking advantage of the energy equality (4.2), from (4.4) we get
\[
\frac{d}{dt}L = \int_{-\ell}^{\ell} \left\{ -\frac{v^3}{2u} + v f'(u) - \varepsilon \frac{v^2}{u} \left( \nu(u) \left( \frac{v}{u} \right)_x \right) \right\} \, dx \\
+ \frac{d}{dt} \int_{-\ell}^{\ell} \left\{ \left[ \frac{\bar{v}^2}{2u} - \frac{v \bar{v}}{u} - u \int_0^u \frac{P(z)}{z^2} \, dz + P(\bar{u}) \frac{\bar{u} - u}{u} \right] \right\} \, dx \\
= -\left[ \frac{v}{u} \left( \frac{v^2}{2u} + u f'(u) \right) - \varepsilon \frac{v^2}{u^2} \left( \nu(u) \left( \frac{v}{u} \right)_x \right) \right] \bigg|_{-\ell}^{\ell} + \frac{d}{dt} \int_{-\ell}^{\ell} \left[ \frac{\bar{v}^2}{2u} - \frac{v \bar{v}}{u} - u \int_0^u \frac{P(z)}{z^2} \, dz + P(\bar{u}) \frac{\bar{u} - u}{u} \right] \, dx \\
= A_x(u, v, u_x, v_x) + B_x(u, v, u_x, v_x)
\]
with notation
\[
A_x(u, v, u_x, v_x) := -\left[ \frac{v}{u} \left( \frac{v^2}{2u} + u f'(u) \right) - \varepsilon \frac{v^2}{u^2} \left( \nu(u) \left( \frac{v}{u} \right)_x \right) \right], \\
B_x(u, v, u_x, v_x) := -\varepsilon \int_{-\ell}^{\ell} \nu(u) \left( \left( \frac{v}{u} \right)_x \right)^2 \, dx.
\]
The term \(B_x\) is negative. In order to check the sign of \(A_x\), we recall that the steady state \((\bar{u}, \bar{v})\) satisfies
\[
\bar{v} = v_s, \quad -\varepsilon v_s \nu(\bar{u}) \bar{v} = v_s^2 \bar{u} + P(\bar{u}) \bar{u}^2 - \alpha \bar{u}^2
\]
where the constants \(v_s\) and \(\alpha\) verify
\[
(4.6) \quad v_s = v_\pm \quad \text{and} \quad \left[ \frac{v_s^2}{u} + P(\bar{u}) - \varepsilon \left( \nu(\bar{u}) \left( \frac{v_s}{u} \right)_x \right) \right] \bigg|_{\pm \ell} = \alpha.
\]

We now use (4.6) to compute \(A_x|_{x=\ell}\). We preliminary recall that
\[
u \phi(u) + P(u) = f(u) + P(u) = u f'(u),
\]
with notations introduced before
\[
f(u) = u \int_0^u \frac{p(z)}{z^2} \, dz \quad \text{and} \quad \phi(u) = \int_0^u \frac{p(z)}{z^2} \, dz.
\]
We have
\[
A_x \bigg|_{x=\ell} = -\left[ \frac{v_s}{u_+} \left( \frac{v_s^2}{2u_+} + u_+ f'(u_+) \right) - \varepsilon \frac{v_s^2}{u_+^2} \left( \nu(u_+) \left( \frac{v_s}{u_+} \right)_x \right) \right].
\]

By using assumption \textbf{H3}, (4.7) becomes
\[
A_x \bigg|_{x=\ell} = -\left[ \frac{v_s}{u_+} \left( \frac{v_s^2}{2u_+} + u_+ f'(u_+) \right) + \varepsilon \frac{v_s^2}{u_+^2} \nu(u_+) \frac{v_s u_x(\ell)}{u_+^2} \right],
\]
and we take advantage of the fact that \(\bar{u}\) solves the stationary problem; from (4.6) it follows
\[
\varepsilon \nu(u_+) \frac{v_s u_x(\ell)}{u_+^2} = \alpha - \frac{v_s^2}{u_+} - P(u_+),
\]
and (4.8) becomes
\[
A_x \bigg|_{x=\ell} = -\left[ \frac{v_s}{u_+} \left( \frac{v_s^2}{2u_+} + u_+ f'(u_+) \right) + \frac{v_s^2}{u_+^2} \left( \alpha - \frac{v_s^2}{u_+} - P(u_+) \right) \right].
\]
By using the same arguments for \( x = -\ell \), we obtain

\[
-A_\varepsilon \bigg|_{x = -\ell} = \left[ \frac{v_s}{u_-} \left( \frac{v_s^2}{2u_-} + u_- f'(u_-) \right) + \frac{v_s^2}{u_-^2} \left( \alpha - \frac{v_s^2}{u_-} - P(u_-) \right) \right],
\]

so that, summing (4.9) and (4.10) and recalling that \( u_+ f'(u_+) = P(u_+) + f(u_+) \), we end up with

\[
A_\varepsilon \bigg|_{-\ell} = \left[ \frac{v_s^3}{2} + \alpha v_s^2 \right] \left( \frac{1}{u_-} - \frac{1}{u_+} \right) + v_s^2 \left( \frac{P(u_+)}{u_+} - \frac{P(u_-)}{u_-} \right) + v_s \left[ \frac{P(u_+)}{u_+} - \frac{P(u_-)}{u_-} \right] + v_s \left[ f(u_-) - f(u_+) \right].
\]

Since \( v_s^3/2 + \alpha v_s^2 > 0 \), the first term of the above sum can be bounded via the difference \( u_+ - u_- \); the second term is negative since \( u_- < u_+ \).

Concerning the third and fourth terms, on one side, since \( P \in C^2(\mathbb{R}^+) \), there exists a constant \( C \) such that \( P'(u) \leq C \) for all \( u > 0 \), implying that \( P(v) - P(w) < C(v - w) \) for all \( v, w > 0 \). We thus have

\[
P(u_+)u_- - P(u_-)u_+ = P(u_+)u_- - P(u_-)u_- + P(u_-)u_- - P(u_+)u_-
\]

\[
= P(u_-)(u_+ - u_-) + u_-(P(u_-) - P(u_+)),
\]

which can be bounded from above via the difference \( u_+ - u_- \) since \( P(u_-) < P(u_+) \).

For the last term, we recall that, by definition

\[
\frac{f(u_-)}{u_-} - \frac{f(u_+)}{u_+} = \int_0^{u_-} \frac{P(s)}{s^2} \, ds - \int_0^{u_+} \frac{P(s)}{s^2} \, ds,
\]

which is negative since \( u_- < u_+ \).

In order to compute the sign of the time derivative of \( L(u, v, \bar{u}, \bar{v}) \) given in (4.5), we are thus left with evaluating the sign of

\[
\frac{d}{dt} \int_{-\ell}^{\ell} \left[ \frac{v_s^2}{2u} - \frac{vv_s}{u} - u \int_0^{u} \frac{P(z)}{z^2} \, dz - P(\bar{u}) - \frac{\bar{u}}{\bar{u}} \right] \, dx :=
\]

\[
[A(u, v, v_s) + B(u, v, v_s) + C(u, \bar{u}) + D(u, \bar{u})].
\]

**Computation of the sign of \( A(u, v, v_s) \).** We have, by integration by parts

\[
A(u, v, v_s) = -\frac{1}{2} \int_{-\ell}^{\ell} \frac{v_s^2}{u} \, dx = \frac{v_s^2}{2} \int_{-\ell}^{\ell} \frac{v_s}{u} \, dx
\]

\[
= \frac{v_s^2}{2} \left[ \frac{v}{u} \bigg|_{-\ell}^{\ell} + 2 \int_{-\ell}^{\ell} \frac{u v}{u^2} \, dx \right]
\]

\[
\leq \frac{v_s^2}{2} \left[ v_s \left( \frac{1}{u_+^2} - \frac{1}{u_-^2} \right) + 2 \frac{|v|_{L^\infty L^\infty}}{|u|^3} (u_+ - u_-) \right].
\]

where we used (recall that \( u_+ > 0 \))

\[
\int_{-\ell}^{\ell} \frac{u v}{u^3} \, dx \leq \int_{-\ell}^{\ell} \frac{|u| v}{|u|^3} \, dx \leq \frac{|v|_{L^\infty L^\infty}}{|u|} \int_{-\ell}^{\ell} |u| \, dx = \frac{|v|_{L^\infty L^\infty}}{|u_+|^2} \int_{-\ell}^{\ell} u_+ \, dx \leq u_+ - u_-.
\]

The first term on the right hand side of the last line in (4.12) is negative, while the last one can be bounded via the difference \( u_+ - u_- \).
Computation of the sign of \( B(u,v) \). Such term needs more care; we have

\[
-v_+ \int_{-\ell}^{\ell} \left( \frac{w}{u} \right)_t \, dx = v_+ \int_{-\ell}^{\ell} \frac{vu_t - v_t u}{u^2} \, dx
\]

\[
= -v_+ \int_{-\ell}^{\ell} \frac{vv_x}{u^2} \, dx
\]

\[
+ v_+ \int_{-\ell}^{\ell} \frac{1}{u} \left[ \frac{v^2}{u} + P(u) - \varepsilon v(u) \left( \frac{v}{u} \right)_x \right] \, dx
\]

(4.13)

\[
= v_+ \int_{-\ell}^{\ell} \left\{ \frac{vv_x}{u^2} + \frac{1}{u} \left( \frac{v^2}{u} \right)_x \right\} \, dx
\]

\[
+ v_+ \int_{-\ell}^{\ell} \frac{1}{u} \left[ P(u) - \varepsilon v(u) \left( \frac{v}{u} \right)_x \right] \, dx
\]

\[
= v_+ \int_{-\ell}^{\ell} B_1(u,v) + B_2(u,v) \, dx.
\]

We start by computing \( B_1(u,v) \); we get

\[
\int_{-\ell}^{\ell} B_1(u,v) \, dx = \int_{-\ell}^{\ell} \left\{ \frac{1}{u} \left( \frac{v^2}{u} \right)_x - \frac{vv_x}{u^2} \right\} \, dx
\]

\[
= \int_{-\ell}^{\ell} \frac{1}{u} \left( \frac{2v v_x u - v^2 u_x}{u^2} \right) \, dx - \int_{-\ell}^{\ell} \frac{v^2 u_x}{u^3} \, dx
\]

\[
= 2 \int_{-\ell}^{\ell} \frac{vv_x}{u^2} \, dx - 2 \int_{-\ell}^{\ell} \frac{v^2 u_x}{u^3} \, dx
\]

\[
= \int_{-\ell}^{\ell} \left( \frac{v^2}{u^2} \right)_x \, dx + \int_{-\ell}^{\ell} \frac{v^2}{u^2} \, dx
\]

\[
= \left( \frac{2v^2}{2u^2} \right) \bigg|_{-\ell}^{\ell} + \int_{-\ell}^{\ell} \frac{u_x v^2}{u^3} \, dx + \int_{-\ell}^{\ell} \frac{u_x v^2}{u^3} \, dx
\]

\[
= \frac{2v^2}{u_x^2} (u^2 - u_+^2) + \frac{2v^2}{u_x^2} (u_+ - u_-),
\]

where in the fifth equality we integrated by parts both terms; the first term in the above sum is negative, while the second can be bounded via the difference \( u_+ - u_- \).

We turn our attention to \( B_2(u,v) \); by integrating by parts

\[
\int_{-\ell}^{\ell} B_2(u,v) \, dx =
\]

\[
= \left\{ \frac{1}{u} \left[ P(u) - \varepsilon v(u) \left( \frac{v}{u} \right)_x \right] \right\} \bigg|_{-\ell}^{\ell} + \int_{-\ell}^{\ell} \frac{u_x}{u^2} \left[ P(u) - \varepsilon v(u) \left( \frac{v}{u} \right)_x \right] \, dx
\]

\[
= \frac{1}{u_+} \left[ P(u_+) - \varepsilon v(u_+) \left( \frac{v}{u_+} \right)_x \right] - \frac{1}{u_-} \left[ P(u_-) - \varepsilon v(u_-) \left( \frac{v}{u_-} \right)_x \right]
\]

\[
+ \int_{-\ell}^{\ell} \frac{u_x}{u^2} \left[ P(u) - \varepsilon v(u) \left( \frac{v}{u} \right)_x \right] \, dx.
\]

As concerning the terms at the boundary, we use again (4.6); since

\[
\varepsilon \left( v(u_{\pm}) v_x \left( \frac{u_{\pm}}{u_{\mp}} \right) \right) = \alpha - \frac{v^2}{u_{\pm}} - P(u_{\pm}),
\]
we obtain
\[
\left\{ \frac{1}{u} \left[ P(u) - \varepsilon \nu(u) \left( \frac{v}{u} \right)_x \right] \right\}|_{-\ell}^{\ell} = \frac{1}{u_+} \left[ \alpha - \frac{v_+^2}{u_+} \right] - \frac{1}{u_-} \left[ \alpha - \frac{v_-^2}{u_-} \right] - \frac{1}{u_+} \left[ \alpha - \frac{v_+^2}{u_+} \right] + \frac{1}{u_-} \left[ \alpha - \frac{v_-^2}{u_-} \right].
\]
Again, the first term in the above sum is negative, while the second one can be bounded via the difference \( u_+ - u_- \). As concerning the last term in \( B_2(u, v) \), we first observe that
\[
\int_{-\ell}^{\ell} \frac{u_x}{u^2} P(u) \, dx \lesssim u_+ - u_-,
\]
and we are thus left with
\[
- \int_{-\ell}^{\ell} \frac{u_x}{u^2} \left[ \varepsilon \nu(u) \left( \frac{v}{u} \right)_x \right] \, dx = \varepsilon \int_{-\ell}^{\ell} \left( \frac{u_x \nu(u)}{u^3} - \frac{u_x^2 \nu(u)}{u^4} \right) v \, dx
\]
\[
= \int_{-\ell}^{\ell} \frac{u_x x \nu(u)}{u^3} v \, dx - \int_{-\ell}^{\ell} \frac{u_x^2 \nu(u)}{u^4} v \, dx
\]
\[
= B_{21}(u, v) + B_{22}(u, v).
\]
For \( B_{22}(u, v) \), by taking advantage of the positive sign of the functions \( \nu(u) \), \( u \) and its first derivative, we can state
\[
B_{22}(u, v) \lesssim c_1 \int_{-\ell}^{\ell} u_x \, dx
\]
where the positive constant \( c_1 \) depends, among others, on \( u_\pm \), \( \|v\|_{L^\infty} \) and \( \|u\|_{L^\infty} \). We recall that these quantities are finite because of Theorem A.1 (see, in particular, Lemma A.5 and Lemma A.9). As concerning \( B_{21}(u, v) \) we have, again by integration by parts
\[
B_{21}(u, v) = - \int_{-\ell}^{\ell} u_x \left( \frac{\nu(u)}{u^3} v \right)_x \, dx
\]
\[
= - \int_{-\ell}^{\ell} u_x \left( \frac{\nu'(u) v}{u^3} + \frac{\nu(u) v_x}{u^3} - \frac{3u_x \nu(u) v}{u^4} \right) \, dx
\]
\[
\lesssim c_2 \int_{-\ell}^{\ell} u_x \, dx,
\]
where, as before, the positive constant \( c_2 \) depends, among others, on \( u_\pm \), \( \|v\|_{L^\infty} \) and \( \|u\|_{L^\infty} \). Hence \( B_{21}(u, v) \) and \( B_{22}(u, v) \) can be bounded via the difference \( u_+ - u_- \).

**Computation fo the sign of** \( C(u, \bar{u}) \) **and** \( D(u, \bar{u}) \). **We finally compute the last two terms in (4.11):** on one side we have
\[
C(u, \bar{u}) = \frac{d}{dt} \int_{-\ell}^{\ell} u \left( \int_0^{\bar{u}} \frac{P(z)}{z^2} \, dz \right) \, dx = - \int_{-\ell}^{\ell} v_x \left( \int_0^{\bar{u}} \frac{P(z)}{z^2} \, dz \right) \, dx;
\]
on the other side
\[
D(u, \bar{u}) = \frac{d}{dt} \int_{-\ell}^{\ell} P'(\bar{u}) \left( 1 - \frac{u}{\bar{u}} \right) \, dx = \int_{-\ell}^{\ell} u_t \frac{P'(\bar{u})}{\bar{u}} \, dx = - \int_{-\ell}^{\ell} v_x \frac{P'(\bar{u})}{\bar{u}} \, dx.
\]
In both cases, by integration by parts and by taking advantage of the positivity of \( P(s) \), \( \bar{u} \) and their derivatives, we can bound these terms from above with \( \int \bar{u}_x \), i.e. via the difference \( u_+ - u_- \).
Conclusion. Summing up, we have shown that

\[
\frac{d}{dt} \int_{-\ell}^{\ell} E_L(u, v, \bar{u}, \bar{v}) \, dx \leq C_- + C_+(u_+ - u_-),
\]

where \( C_- < 0 \) is a negative constant collecting all the negative terms appearing in the previous computations, while \( C_+ > 0 \) is a positive constant.

By taking advantage of hypothesis \( \text{H4} \), we can thus choose \( \delta \) in such a way that the right hand side in (4.14) is negative, and the proof is completed.

\[ \Box \]

As a consequence of Proposition 4.1, we are finally able to prove Theorem 1.5; we recall the result for completeness.

**Theorem 4.3.** Let assumptions \( \text{H1-2-3-4} \) be satisfied. Then \((\bar{u}, \bar{v})\), the unique steady state of system (4.1), is stable in the following sense: for every \( T > 0 \) it holds

\[
\sup_{0 \leq t \leq T} \|(u, v) - (\bar{u}, \bar{v})\|_{L^2} \leq \|(\bar{u}, \bar{v}) - (u_0, v_0)\|_{L^2}.
\]

Moreover

\[
\|(u, v) - (\bar{u}, \bar{v})\|_{L^1 H^1} \leq C_T,
\]

with \( 0 < C_T \to +\infty \) if and only if \( T \to +\infty \).

**Proof.** We make use of Proposition 4.1; recalling that in (4.5) we erased the positive term \( B_\varepsilon \), by integrating in time the relation \( \frac{d}{dt} L(t) \leq 0 \) we have

\[
L(t) + \varepsilon \int_0^t \int_{-\ell}^{\ell} \nu(u) \left[ \frac{v}{u} \right]_x^2 \, dx \, dt \leq L(0).
\]

On one side, by using the very definition of \( L(t) \), the inequality \( L(t) \leq L(0) \) implies

\[
|u - \bar{u}|_{L^2} + |v - \bar{v}|_{L^2} \leq C \left( |\bar{u} - u_0|_{L^2} + |\bar{v} - v_0|_{L^2} \right).
\]

On the other side, we have the inequality

\[
C \int_0^t \int_{-\ell}^{\ell} \left[ \frac{v}{u} \right]_x^2 \, dx \, dt \leq \varepsilon \int_0^t \int_{-\ell}^{\ell} \nu(u) \left[ \frac{v}{u} \right]_x^2 \, dx \, dt \leq L(0)
\]

and

\[
\int_{-\ell}^{\ell} \left[ \frac{v}{u} \right]_x^2 \, dx = \int_{-\ell}^{\ell} \left\{ \frac{v_x^2}{u^2} + \frac{v^2 u_x^2}{u^4} - \frac{(v^2)_x u_x}{u^3} \right\} \, dx
\]

\[
= \int_{-\ell}^{\ell} \left\{ \frac{v_x^2}{u^2} + \frac{v^2 u_x^2}{u^4} + v^2 \left( \frac{u_x}{u^2} \right)_x \right\} \, dx
\]

\[
= \int_{-\ell}^{\ell} \left\{ \frac{v_x^2}{u^2} + \frac{v^2 u_x^2}{u^4} + \frac{3v^2 u_x^2}{u^4} + \frac{v^2 u_{xx}}{u^3} \right\} \, dx.
\]

Hence, (4.16) becomes

\[
\int_0^t \int_{-\ell}^{\ell} \frac{v_x^2}{u^2} \, dx \, dt \leq 2 \int_0^t \int_{-\ell}^{\ell} \frac{v^2 u_x^2}{u^4} \, dx \, dt \leq L(0) + \int_0^t \int_{-\ell}^{\ell} \frac{v^2}{u^3} |u_{xx}| \, dx \, dt \leq L(0) + C_T.
\]

where we used Remark A.7. In particular, recalling that \( \bar{v}_x = 0 \), we can thus state that

\[
|v_x - \bar{v}_x|_{L^1 L^2} = \int_0^t \int_{-\ell}^{\ell} v_x^2 \, dx \, dt \leq (|\bar{u} - u_0|_{L^2} + |\bar{v} - v_0|_{L^2}) + C_T.
\]
Moreover
\[\int_0^t \int_{-\ell}^{\ell} (u_x - \bar{u}_x)^2 \, dx \, dt \leq 2 \int_0^t \int_{-\ell}^{\ell} u_x^2 + \int_0^t \int_{-\ell}^{\ell} \bar{u}_x^2 \, dx \, dt \leq L(0) + CT\]
implying
\[|u_x - \bar{u}_x|_{L^1 L^2} \leq \left( |\bar{u} - u_0|_{L^2} + |\bar{v} - v_0|_{L^2} \right) + CT.\]

The proof is now complete. □

We point out that estimate (4.15) implies stability of the steady state in the sense that initial data \((u_0, v_0)\) close to the steady state in the \(L^2\)-norm will generate a solution \((u, v)\) to (4.1) which is still close to \((\bar{u}, \bar{v})\) in the \(L^2\)-norm, for all \(t \geq 0\).

**Appendix A. Existence and regularity of the solution**

We here discuss existence and regularity of the solutions to the Navier-Stokes equations (1.1); we write the problem in terms of the variables mass density and velocity of the fluid \((\rho, w)\)
(A.1)
\[
\begin{cases}
\rho_t + (\rho w)_x = 0 \\
(\rho w)_t + (\rho w^2 + P(\rho))_x = \varepsilon (\nu(\rho) w_x)_x
\end{cases}
\]
and, for simplicity, we consider \(P(\rho) = \kappa \rho^\gamma, \gamma > 1\) and \(\nu(\rho) = 1\). We recall that system (A.1) is considered in the bounded interval \(I = (-\ell, \ell)\) with boundary conditions
(A.2)
\[
\rho(-\ell) = \rho_-, \quad w(\pm \ell) = w_\pm > 0,
\]
and it is subject to the initial datum \((\rho, w)(x, 0) = (\rho_0, w_0)\).

**Notations:** As before, we will denote with \(|f|_{X,Y} := |f|_{X([0,T],Y(I))}\). If not specified otherwise, we will denote with
\[
\int f := \int_0^t f(x, t) \, dx.
\]

**Theorem A.1.** Let us consider the Cauchy problem for (A.1), and let assume \(\rho_0(x) := \rho(x, 0) \in H^1(I)\) and \(w_0(x) := w(x, 0) \in H^1(I)\). Then there exists a unique solution \((\rho, w)\) to (A.1) satisfying the boundary conditions (A.2) and such that
\[
\rho \in L^\infty([0,T], H^1(I)) \quad \text{and} \quad w \in L^\infty([0,T], H^1(I)).
\]

The proof of Theorem A.1 follows from the proof of several Lemmas.

**Lemma A.2.** There exists \(T > 0\) such that, for all \(t \in [0,T]\), there holds
\[
|\sqrt{\rho} w|_{L^2 L^2}^2 + |\rho|_{L^\infty L^\gamma}^\gamma + |w_x|_{L^1 L^2} \leq CT.
\]

**Proof.** By combining the two equations in (A.1) we get
\[
\rho w_t + (\rho w)_x + P_x = \varepsilon w_{xx}.
\]
By integrating in space over \(I\) and by multiplying by \(w\)
\[
\frac{1}{2} \int \rho(w^2)_t - \frac{1}{2} \int w^2(\rho w)_x + \rho w^3|_1 + \int P_x w - \int \varepsilon w_{xx},
\]
where we also integrated by parts. We now observe that
\[
\int P_x w = \gamma \kappa \rho^{-1} (-\rho w_x - \rho_t)
\]
\[
= -\frac{d}{dt} \int \kappa \rho \gamma w_x - \int \kappa \gamma \rho \gamma w_x
\]
\[
= -\frac{d}{dt} \int \rho \gamma + \int \gamma P_x w - \gamma P w |_I,
\]
implying
\[
\int P_x w = \frac{\kappa}{\gamma - 1} \frac{d}{dt} \int \rho \gamma - \int \kappa \gamma w_x.
\]
Summing up, recalling the energy formula
\[
E(\rho(x, t), w(x, t)) = \frac{1}{2} \rho w^2 + \frac{\gamma}{\gamma - 1} \rho \gamma,
\]
we have
\[
\frac{d}{dt} \int E + \varepsilon \int w_x^2 = C.
\]
Finally, integrating in time we get
\[
\int (E(t) - E(\rho(0), w(0))) dt + \varepsilon \int_0^t \int w_x^2 dt = Ct,
\]
implying
\[
(A.3) \quad \sup_{t \in [0, T]} |\sqrt{\rho} w|_{L^2}^2 + \sup_{t \in [0, T]} |\rho|_{L^\gamma} + \varepsilon \int_0^T \int w_x^2 dt \leq C_T.
\]

Lemma A.3. There holds
\[
(A.4) \quad |\rho|_{L^\infty} \leq C.
\]
Proof. Consider the Lagrangian flow \(X = X(x, t)\) of \(w\), defined as
\[
(A.5) \quad \begin{cases}
\frac{\partial X}{\partial t} = w(X(x, t), x), \\
X(x, 0) = x \in [-\ell, \ell].
\end{cases}
\]
In order to prove (A.4) we thus need to prove that
\[
\rho(X(x, t), x) \leq C,
\]
for any \((x, t) \in (0, T) \times (\ell, \ell)\) and for some constant \(C \geq 0\). Fixed \(t_0 \in (0, T)\) and given the initial mass \(\int \rho_0 := m_0 \leq C\), because of the conservation of the mass and from the very definition of the Lagrangian flow we can find \(x_1 \in (-\ell, \ell)\) such that
\[
\rho_0(x_1) \geq C^{-1} \quad \text{and} \quad \rho(X(x_1, t_0), t_0) \leq C.
\]
We now want to prove that, for any \(x_2 \in (-\ell, \ell)\), \(\rho(x_2, X(x_2, t_0)) \leq C\); we let \(X_j := X(x_j, t)\) for \(j = 1, 2\) and we define
\[
F(t) = \log(\rho(X_2, t)) - \log(\rho(X_1, t)).
\]
By using (A.5) we have

\[
\frac{dF}{dt} = \frac{1}{\rho(X_2, t)} \left( \rho_t(X_2, t) + \rho x(X_2, t) \frac{dX_2}{dt} \right) - \frac{1}{\rho(X_1, t)} \left( \rho_t(X_1, t) + \rho x(X_1, t) \frac{dX_1}{dt} \right)
\]

\[
= \frac{1}{\rho(X_2, t)} \left( -\rho x(X_2, t) w(X_2, t) - \rho(x_2, t) w_x(X_2, t) + \rho x(X_2, t) w(X_2, t) \right)
\]

\[
- \frac{1}{\rho(X_1, t)} \left( -\rho x(X_1, t) w(X_1, t) - \rho(x_1, t) w_x(X_1, t) + \rho x(X_1, t) w(X_1, t) \right)
\]

(A.6)

\[
= -w_x(X_2, t) + w_x(X_1, t)
\]

\[
= - \int_{X_1}^{X_2} w_{xx} \, dx
\]

\[
= - \frac{1}{\varepsilon} \int_{X_1}^{X_2} (\rho w_t + \rho w w_x + P_x) \, dx.
\]

Let now

\[
V(t) = \int_{X_1}^{X_2} \rho w \, dx,
\]

so that

\[
\frac{dV}{dt} = (\rho w)(X_2, t) \frac{dX_2}{dt} - (\rho w)(X_1, t) \frac{dX_1}{dt} + \int_{X_1}^{X_2} (\rho w)_t \, dx
\]

\[
= (\rho w^2)(X_2, t) - (\rho w^2)(X_1, t) + \int_{X_1}^{X_2} [- (\rho w)_x w + \rho w_t] \, dx
\]

(A.7)

\[
= \int_{X_1}^{X_2} [(\rho w^2)_x - (\rho w)_x w + \rho w_t] \, dx
\]

\[
= \int_{X_1}^{X_2} [\rho w w_x + \rho w_t] \, dx.
\]

Substituting (A.7) into (A.6) we get

(A.8)

\[
\varepsilon \frac{dF}{dt} + \frac{dV}{dt} = - \int_{X_1}^{X_2} P_x \, dx.
\]

By setting

\[
\alpha(t) = \frac{P(\rho(X_1, t)) - P(\rho(X_2, t))}{\varepsilon F(t)} \geq 0,
\]

equation (A.8) can be rewritten as

\[
\varepsilon \frac{dF}{dt} + \frac{dV}{dt} = - \alpha(\varepsilon F + V) + \alpha V,
\]

with solution given by

\[
\varepsilon F(t) + V(t) = e^{- \int_0^t \alpha(s) \, ds} (\varepsilon F(0) + V(0)) + \int_0^t e^{- \int_0^\tau \alpha(\tau) \, d\tau} \alpha(s) V(s) \, ds.
\]
Since $F(0) \leq |\rho_0|_{L^\infty}$, we have, for any $t \geq 0$
\[
\varepsilon F(t) \leq \varepsilon C + V(0) + |V(t)| + \int_0^t e^{-\int_0^s \alpha(r) \, dr} \alpha(s)|V(s)| \, ds
\]
\[
\leq \varepsilon C + V(0) + |V(t)| + \sup_{0 \leq \tau \leq t_0} |V(t)| \int_0^t e^{-\int_0^s \alpha(r) \, dr} \, ds
\]
where we used the fact that $e^{-\int_0^s \alpha(s) \, ds} \leq 1$ for all $t \geq 0$. We now observe that
\[
|V(0)| \leq \int_{X_1}^{X_2} \rho_0 |w_0| \, dx \leq C,
\]
\[
|V(t)| \leq \int_{X_1}^{X_2} \rho |w| \, dx \leq \left( \int_{X_1}^{X_2} \rho \, dx \right)^{1/2} \left( \int_{X_1}^{X_2} \rho w^2 \, dx \right)^{1/2},
\]
implying
\[
\sup_{0 \leq t \leq t_0} |V(t)| \leq \sup_{0 \leq t \leq t_0} \left( \int_\rho \right)^{1/2} \left( \int \rho w^2 \right)^{1/2} \leq C,
\]
where the quantity on the right hand side is bounded by a constant because of (A.3). Combining the above estimates with (A.9), we finally obtain $\varepsilon F(t_0) \leq C$, implying
\[
\log(\rho(X(t_0, x_2), t_0)) = \log(\rho(X(t_0, x_1), t_0)) + F(t_0) \leq C.
\]
The thesis then follows because of the arbitrariness of $x_2$ and $t_0$. \qed

**Remark A.4.** We point out that, because of assumption $H4$ on the boundary data, it necessary has to be $|\rho_0| < \delta$. In particular this implies that the constant $C$ in (A.4) is less than $\delta > 0$.

**Lemma A.5.**
\[
\int_0^T |\sqrt{\rho}w_t|_{L^2}^2 \, dt + |w_x|_{L^2}^2 \leq C_T \quad \text{and} \quad \sup_{0 \leq t \leq T} (|w|_{L^\infty} + |w_x|_{L^2}) \leq C_T
\]

**Proof.** We multiply the second equation in (A.1) by $w_t$ and we integrate in space. We get
\[
\int \rho_t w w_t + \rho w_t^2 + (\rho w^2)_x w_t + P_x w_t = \int \rho w_t^2 + \int \rho w w_x w_t + \int P_x w_t
\]
\[= \int \varepsilon w_{xx} w_t.
\]
We also have
\[
(\rho w_t) w w_x = (-\rho_t w + \varepsilon w_{xx} - (\rho w^2)_x - P_x) w w_x
\]
\[= ((\rho w)_x w + \varepsilon w_{xx} - (\rho w^2)_x - P_x) w w_x
\]
\[= \rho w^2 w_x^2 + \varepsilon w_{xx} w w_x - P_x w w_x,
\]
\[= \rho w^2 w_x^2 + G_x w w_x + (\varepsilon - 1) w_{xx} w w_x,
\]
where $G := w_x - P$. The previous equality implies

$$\int \rho w_t^2 + \frac{\varepsilon}{d} \frac{d}{dt} \int \frac{1}{2} w_x^2 \leq \varepsilon w_x w_t |_I + \int P_{xt} - P w_t |_I$$

(A.10)

$$+ \int \rho w_t^2 w_x^2 + \int G_x w w_x + (\varepsilon - 1) \int w_{xx} w w_x$$

$$\leq C + \int \rho w_t^2 w_x^2 + \int P_{xt} + \int G_x w w_x.$$ 

Going further

$$\int G_x w w_x \leq \left( \int G_x^2 \right)^{1/2} \left( \int w_x^2 \right)^{1/2} \leq |G_x|_{L^2} |w|_{L^\infty} \left( \int w_x^2 \right)^{1/2}$$

$$\leq |G_x|_{L^2} |w_x|_{L^2}$$

Moreover, a straightforward computation shows that

$$\int P_{xt} = \frac{d}{dt} \int Pw_x - \int \int Pw(G_x + P_x) + (\gamma - 1) \int (G + P)^2 P,$$

$$\frac{1}{2(2\gamma - 1)} \frac{d}{dt} \int P^2 = \int PwP_x + C,$$

and

$$(\gamma - 1) \int P_t w_x^2 = (\gamma - 1) \int PG^2 - 4(\gamma - 1) \int PP_x w - (\gamma - 1) \int P^3 + C,$$

where we used the explicit expression for the pressure $P(\rho) = \kappa \rho^\gamma$. By using the previous identities we finally obtain

$$\int \int P_{xt} = C + \frac{d}{dt} \int Pw_x - \int \int PwG_x + (\gamma - 1) \int P(G^2 - P^2) - \frac{4\gamma - 3}{2(2\gamma - 1)} \frac{d}{dt} \int P^2.$$

We can thus integrate in time (A.10), obtaining

(A.11)

$$\int_0^t \int \rho w_t^2 dt + \frac{\varepsilon}{d} \frac{d}{dt} \int_0^t \int w_x^2 dt \leq C + \int_0^t \int \rho w_t^2 w_x^2 dt$$

$$+ \int_0^t |G_x|_{L^2} |w_x|^2_{L^2} dt$$

$$+ (\gamma - 1) \int_0^t \int PG^2 dt - (\gamma - 1) \int_0^t \int P^3 dt$$

$$+ \int_0^t \int P |w| |G_x| dt$$

$$+ \int_0^t \int (Pw_x(t) - Pw_x(0)) - \frac{4\gamma - 3}{2(2\gamma - 1)} \int [P^2(t) - P^2(0)].$$
There hold the following estimates for the terms appearing on the right hand side of (A.11)

\[ \int_0^t |G_x|_{L^2} |w_x|^2_{L^2} dt \leq \left( \int_0^t |G_x|^2_{L^2} dt \right)^{1/2} \left( \int_0^t |w_x|^4_{L^2} dt \right)^{1/2} \]

\[ \leq \left( \int_0^t |G_x|^2_{L^2} dt \right) + \left( \int_0^t |w_x|^4_{L^2} dt \right) ; \]

\[-(\gamma - 1) \int_0^t \int P^3 dt < 0 ; \]

\[ \int [P^2(t) - P^2(0)] = \int (\rho^2 - \rho_0^2) \leq C \quad \text{for Lemma A.3} ; \]

\[ \int P w_x(0) \leq C \quad \text{for the regularity of the initial data} ; \]

\[ \int P w_x(t) \leq |\rho|^{\gamma} \int |w_x|^2 . \]

Inequality (A.11) can be thus rewritten as

\[ \int_0^t |\sqrt{\rho} w|_{L^2}^2 dt + \varepsilon |w_x|_{L^2}^2 \leq C + \int_0^t \int (\rho w^2 w_x^2 + PG^2 + P |w||G_x|) dt \]

\[ \leq C + (|\rho|_{L^\infty} + 1) \int_0^t |w_x|^4_{L^2} dt \]

\[ + |\rho|_{L^\infty} |w|_{L^\infty} |w|^2_{L^2} \]

\[ + \sup_t |\sqrt{\rho} w|_{L^2} \int_0^t |w_x|^2_{L^2} dt \]

\[ + |\rho|_{L^\infty} |w|_{L^\infty} |w|^2_{L^2} , \]

where we used

\[ \int_0^t \int \rho w^2 w_x^2 dt \leq \int_0^t |\rho|_{L^\infty} |w|_{L^\infty} |w|^2_{L^2} \leq |\rho|_{L^\infty} \int_0^t |w_x|^4_{L^2} dt ; \]

\[ \int_0^t \int P G^2 dt \leq \int_0^t \int P (w_x - P)^2 dt \leq \int_0^t \int P (w_x^2 + P^2) dt \leq \int_0^t \int P^3 dt + \int_0^t \int P^2 w_x^2 dt \]

\[ \leq |P|_{L^\infty} + |P|^2_{L^\infty} \int_0^t w_x^2 dt \leq C \quad \text{for Lemmas A.2 and A.3} ; \]

\[ |G_x|_{L^2} = |\sqrt{\rho} (\sqrt{\rho} w_t + \sqrt{\rho} w w_x)|_{L^2} \]

\[ \leq |\rho|_{L^\infty}^{1/2} \left( |\sqrt{\rho} w_t|_{L^2} + |\sqrt{\rho} w w_x|_{L^2} \right) \]

\[ \leq |\rho|_{L^\infty}^{1/2} \left( |\sqrt{\rho} w_t|_{L^2} + |\rho|_{L^\infty}^{1/2} |w_x|^2_{L^2} \right) ; \]

\[ \int_0^t |G_x|^2_{L^2} \leq |\rho|_{L^\infty} \int_0^t |\sqrt{\rho} w_t|^2_{L^2} dt + |\rho|_{L^\infty} \int_0^t |w_x|^4_{L^2} dt ; \]

\[ \int_0^t \int P |w||G_x| dt \leq \int_0^t |\rho|_{L^\infty}^{\gamma - 1/2} |\sqrt{\rho} w|_{L^2} |G_x|_{L^2} \]

\[ \leq |\rho|_{L^\infty}^{\gamma - 1/2} \sup_t |\sqrt{\rho} w|_{L^2} \int_0^t |\sqrt{\rho} w_t|_{L^2} dt + |\rho|_{L^\infty}^{\gamma + 1/2} \sup_t |\sqrt{\rho} w|_{L^2} \int_0^t |w_x|^2_{L^2} dt . \]
Recalling that $|\rho|_{L^\infty} < \delta$ by Remark A.4, (A.12) becomes

\begin{equation}
(1 - \varepsilon) \int_0^t |\sqrt{\rho} w_t|^2 \, dt + (\varepsilon - \delta^\gamma) |w_x|^2 \, dt \leq C + (|\rho|_{L^\infty} + 1) \int_0^t |w_x|^4 \, dt
\end{equation}

(A.13)

\begin{align*}
&+ |\rho|^{\gamma + 1/2} \sup_t \left|\sqrt{\rho} w\right|_{L^2} \int_0^t |w_x|^2 \, dt \\
&+ |\rho|_{L^\infty} \sup_t \left|\sqrt{\rho} w\right|_{L^2} \int_0^t \left|\sqrt{\rho} w_t\right|_{L^2} \, dt.
\end{align*}

By applying the Gronwall's inequality to (A.13) we thus end up with

\begin{equation}
\int_0^t \left|\sqrt{\rho} w_t\right|_{L^2}^2 \, dt + |w_x|_{L^2}^2 \leq C_T,
\end{equation}

(A.14)

providing $\delta < \varepsilon^{\gamma/\gamma}$. Moreover, by passing to the sup in time for $t \in [0,T]$ in (A.14) we also have

\begin{equation}
\sup_{0 \leq t \leq T} |w_x|_{L^2}^2 \leq C_T,
\end{equation}

(A.15)

and, recalling that $|w|_{L^\infty} \leq C |w_x|_{L^2}$ by Sobolev embedding, the proof is complete. □

**Lemma A.6.**

\[ \int_0^T \left( |w_x|^2_{L^\infty} + |G_x|^2_{L^2} \right) \, dt \leq C_T, \]

where $G_x := w_{xx} - P_x$.

**Proof.** From the very definition of $G$ we have

\[ |w_x|^2_{L^\infty} \leq 2 |G|^2_{L^\infty} + 2 |P|^2_{L^\infty} \leq 2 |G_x|^2_{L^2} + 2 |P|^2_{L^\infty}. \]

Moreover, on one side

\[ \int_0^T |G|^2_{L^2} \, dt \leq \int_0^T \int |w_x|^2 \, dt + \int_0^T \int |P|^2 \, dt \leq C_T, \]

because of Lemma A.2. On the other side

\[ G_x = w_{xx} - P_x = (\rho w)_t + (\rho w)_x = \rho w_t + \rho_t w + \rho_x w^2 + 2 \rho w w_x = \rho w_t - \rho_x w^2 + \rho w w_x + \rho_x w^2 + 2 \rho w w_x, \]

implying

\[ \int_0^T |G_x|^2_{L^2} \, dt \leq \int_0^T \left( |\sqrt{\rho} w_t|^2_{L^2} + |w w_x|^2_{L^2} \right) \, dt \leq C + \int_0^T |w|^2_{L^\infty} |w_x|^2_{L^2} \, dt \leq C_T, \]

again because of Lemma A.2 and Lemma A.5. The thesis then follows. □

**Remark A.7.** For further notice, we observe that, since $w_{xx} = G_x + P_x$, Lemma A.6 also implies $|w_{xx}|_{L^1, L^2} \leq C_T$.

**Lemma A.8.**

\[ \sup_{0 \leq t \leq T} |\rho|_{L^2} \leq C_T. \]
Proof. Multiplying by $\rho$ the first equation in (A.1) and integrating in space we get

$$\int \rho_t \rho = - \int \rho_x \rho w - \int \rho w_x \rho = \int \rho (w\rho)_x - \int \rho^2 w_x + C$$

$$= \int \rho w \rho_x + C.$$ 

Hence

$$\frac{1}{2} \frac{d}{dt} \int \rho^2 \leq \frac{1}{2} \int \rho^2 |w_x| + C,$$

implying, after integrating in time

$$\int \rho^2 \leq \int_0^t \int |\rho^2||w_x| \, dt + C t$$

$$\leq \int_0^t \left[ \left( \int \rho^4 \right)^{1/2} \left( \int w_x^2 \right)^{1/2} \right] + C t$$

$$\leq |\rho|_{L^4 L^\infty} \int_0^t |w_x|_{L^2} + C t,$$

and the thesis follows from Lemma A.5.

□

Lemma A.9.

$$\sup_{0 \leq t \leq T} |\rho_x|_{L^2} \leq C_T.$$ 

Proof. Let us differentiate with respect to $x$ the first equation in (A.1); by multiplying it by $\rho_x$ and integrating over $I$ we get

(A.16)  $$\frac{1}{2} \frac{d}{dt} \int \rho_x^2 + \int (\rho_x w + \rho w_x) \rho_x = 0.$$  

Moreover

(A.17)  $$\int (\rho_x w)_x \rho_x = \int \rho_{xx} \rho_x w + \int \rho_x^2 w_x$$ 

and

$$\int \rho_{xx} \rho_x w = - \int \rho_x (\rho_x w)_x + C = C - \int \rho_x \rho_{xx} w - \int \rho_x^2 w_x,$$

implying

$$\int \rho_{xx} \rho_x w = \frac{C}{2} - \frac{1}{2} \int \rho_x^2 w_x.$$  

Hence, (A.17) becomes

$$\int (\rho_x w)_x \rho_x = \frac{C}{2} + \frac{1}{2} \int \rho_x^2 w_x.$$  

Going further

$$\int (\rho w_x)_x \rho_x = \int \rho_x^2 w_x + \int \rho w_{xx} \rho_x,$$

and

$$\int \rho w_{xx} \rho_x = \int \rho (G_x + P_x) \rho_x \leq \int \rho^2 (G_x + P_x)^2 + \int \rho_x^2$$

$$\leq |\rho|_{L^4}^2 \left( \int G_x^2 + \int P_x^2 \right) + \int \rho_x^2$$

$$\leq |\rho|_{L^4}^2 \int G_x^2 + \int P_x^2 + \int \rho_x^2.$$
Collecting all the above estimates, (A.16) thus becomes
\[
\frac{1}{2} \frac{d}{dt} \int \rho_x^2 \leq (|w_x|_{L^\infty} + |ho_x|_{L^\infty}^{2\gamma} + 1) \int \rho_x^2 + \int G_x^2,
\]
and the thesis follows after integrating in time from an application of Gronwall's inequality, recalling that \( \int_0^t |w_x|_{L^\infty}^2 \) and \( \int_0^t |G_x|_{L^2}^2 \) are bounded because of Lemma A.6.

Combining Lemmas A.8 and A.9 we get \( \rho \in L^\infty([0,T],H^1(I)) \); this, together with estimate (A.15), completes the proof of Theorem A.1.

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