Stochastic Approximation of the Law of the Paths of Killed Markov Processes in Finite State Spaces Conditioned on Survival

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Abstract

Reinforced processes are known to provide a stochastic representation for the quasi-stationary distribution of a given killed Markov process. In this paper we shall adapt the construction to provide a stochastic representation for the law of the paths of a given killed Markov process conditioned on survival, proving this in discrete time with the state space being finite.

1 Introduction

In sections 1 and 2 we shall not specify whether time is discrete or continuous nor shall we make assumptions on the state space, but we shall restrict our attention to the discrete time case with the state space being finite in sections 3 and 4.

We consider in discrete or continuous time a killed (càdlàg in continuous time) Markov process \((X_t)_{t < \tau_\partial}\) on the state space \(\chi \cup \partial\), evolving in \(\chi\) until the killing time \(\tau_\partial = \inf\{t > 0 : X_t \in \partial\}\), after which time it remains in the cemetery state (we assume without loss of generality that \(\partial\) is a one-point set). In general, one is interested in the law of the killed Markov process (with initial condition \(X_0 \sim \mu \in \mathcal{P}(\chi)\)) conditioned on survival,

\[\mathcal{L}_\mu(X_t|\tau_\partial > t),\]

and the long-time limits of this law,

\[\mathcal{L}_\mu(X_t|\tau_\partial > t)(\cdot) \to \pi(\cdot) \quad \text{as} \quad t \to \infty.\]

General conditions for the existence and uniqueness of these limits are given by [6] and [7]. In general, these limits correspond to quasi-stationary distributions (QSDs) \(\pi\),

\[\mathcal{L}_\pi(X_t|\tau_\partial > t)(\cdot) = \pi(\cdot) \quad \text{for all} \quad 0 \leq t < \infty.\]

Aldous, Flannery and Palacios [1] introduced a method for simulating QSDs based on reinforced processes. The reinforced process, \((Y_t)_{0 \leq t < \infty}\), is obtained by running a copy of \(X_t\) until it is killed,

\[(Y_t)_{0 \leq t < \tau_\partial} = (X_t)_{0 \leq t < \tau_\partial}.\] (1.1)

At this killing time, \(Y_t\) jumps to a point sampled independently from the empirical measure of the history,

\[Y_{\tau_\partial} \sim \frac{1}{\tau_\partial} \int_0^{\tau_\partial} \delta_{Y_s}(\cdot) ds.\] (1.2)
This is then repeated inductively, so that at the \( n \)th killing time \( \tau_n \), \( Y_t \) jumps to a point sampled independently from the empirical measure of the history of \( Y_t \) up to that killing time,

\[
Y_{\tau_n} \sim \frac{1}{\tau_n} \int_0^{\tau_n} \delta_{Y_s}(\cdot)ds,
\]

and continues evolving like a copy of \( X_t \) as before. In [1] they established that these provide an approximation method for the QSDs of irreducible killed Markov chains when the state space is finite and time is discrete, proving that

\[
\int_0^t \delta_{Y_s}(\cdot)ds \xrightarrow{a.s.} \pi(\cdot) \quad \text{as} \quad t \to \infty.
\]

This was made quantitative for a more general version of this algorithm by Benaim and Cloez [3], and has since been extended to a quite general setting in [9] and [2].

One may also be interested in obtaining pathwise information, so we may seek to approximate

\[
\mathcal{L}((X_s)_{0 \leq s \leq t | \tau_0 > t})(\cdot)
\]

for finite \( t \). One may also consider the \( Q \)-process, which provides a pathwise description of \((X_t)_{0 \leq t < \tau_0}\) conditioned never to be killed, a definition of which is given in [6, Theorem 3.1].

A second method for approximating QSDs is given by the Fleming-Viot process, a particle system introduced by Burdzy, Holyst and March in [5]. They considered the case whereby the killed Markov process is Brownian motion in an open, bounded domain, killed instantaneously upon contact with the boundary. They established in [5] that this particle system provides an approximation method for both the distribution of this killed Brownian motion conditioned on survival at fixed instants of time, and the corresponding QSD. This was later extended to a general setting by Villemonais [11]. In [4], Bieniek and Burdzy established that the Fleming-Viot process also provides for the distribution of the path of a killed Markov process conditioned to survive over a fixed time interval. Since then, the present author established in [10, Corollary 5.1] that the Fleming-Viot process also provides a representation for the \( Q \)-process when the killed Markov process is a normally reflected diffusion in a compact domain, killed at position-dependent Poisson rate (it should be possible to extend this result to a more general setting, subject to overcoming additional difficulties).

Thus, whilst both reinforced processes and the Fleming-Viot process are known to provide an approximation method for the QSDs of killed Markov processes, prior to the present paper only the Fleming-Viot process was known to provide an approximation method for the distribution conditioned on survival of killed Markov processes at fixed instants of time, the distribution of the path conditioned on survival over finite time horizons, or the law of the \( Q \)-process. In the present paper, we shall show how the construction of reinforced processes may be adapted to provide an approximation method for the law of the path of a killed Markov process conditioned on survival over a finite time horizon (thus, by extension, the law conditioned on survival at fixed instants of time). In the future, we hope to extend this to obtain an approximation method for the \( Q \)-process.

The fundamental idea is to, at each killing time, jump onto both the spatial and temporal location of a point sampled from the history or, with some probability, sample a killed Markov process started from some fixed initial distribution at time 0 instead. We refer to the resultant constructions as reinforced path processes. A more precise definition is given by the following.

**Definition 1.1 (Reinforced Path Processes).** Reinforced path processes are constructed by inductively sampling triples \( u_n = (t_b^n, f^n, t_d^n) \), whereby \( t_b^n \) is the \( n \)th birth time, \( t_d^n \) is the \( n \)th killing time, and \( f^n \) is the \( n \)th path from time 0 to time \( t_d^n \). Whilst \( u_n \) is considered to be "alive" only between
times \( t^n_b \) and \( t^n_d \), its path \( f^n \) is defined prior to time \( t^n_b \) - it includes an "ancestral path". The first triple \( u_1 = (t^1_b, f^1, t^1_d) \) is defined by taking a copy of the given killed Markov process \( (X_t)_{0 \leq t < \tau_0} \) and setting \( t^1_b \) to be 0, \( t^1_d \) to be \( \tau_0 \), and \( f^1 \) to be the path taken by \( X_t \) during its lifetime. We then inductively assume we have defined \( u_1, \ldots, u_n \). With a certain probability which may be dependent upon \( u_1, \ldots, u_n \), the next iteration \( u_{n+1} \) is defined in the same manner as \( u_1 \): we take an independent copy of the given killed Markov process \( (X_t)_{0 \leq t < \tau_0} \) and setting \( t^{n+1}_b \) to be 0, \( t^{n+1}_d \) to be \( \tau_0 \), and \( f^{n+1} \) to be the path taken by \( X_t \) during its lifetime. Otherwise, we sample the index \( m \in \{1, \ldots, n\} \) and the time \( t^m_b \leq t' < t^m_d \) according to a prescribed sampling rule, dependent upon \( u_1, \ldots, u_n \). Whilst different sampling rules may be chosen, it is essential that, conditional on sampling from the paths alive at time \( t \), we choose from amongst these paths uniformly at random. We then take a copy of the killed Markov process started from time \( t' \) at position \( f^m(t') \). The triple \( u_{n+1} = (t^{n+1}_b, f^{n+1}, t^{n+1}_d) \) is defined by taking \( t^{n+1}_b \) to be \( t' \), \( t^{n+1}_d \) to be the killing time of the sampled killed Markov process, and \( f^{n+1} \) to be \( f^m \) up to time \( t^{n+1}_b \) followed by the path of the sampled killed Markov process between the times \( t^{n+1}_b \) and \( t^{n+1}_d \). This may be defined over an infinite time horizon, or over a finite time horizon \( T < \infty \) (in which case we kill particles upon reaching time \( T \)).

There are many possible reinforced path processes for a given killed Markov process and initial condition, corresponding to different rules for sampling from the history and different probabilities for starting a new path from time 0. A simple reinforced path process is depicted in Figure 1.

![Figure 1](image-url)

Figure 1: The killed Markov process here is Brownian motion in \((-1, 1)\), killed instantaneously at \((-1, 1)\). We begin with a Brownian motion with some given initial distribution \( \mu \), run until it hits the boundary. This forms the first path. We then inductively assume we have sampled the first \( n \) paths for \( n \geq 1 \). We independently flip a biased coin, which turns up heads with fixed probability \( p \). If the coin turns up heads, we sample another independent copy of the Brownian motion with the same initial distribution \( \mu \) and run it until it hits the boundary. Otherwise, if the coin turns up tails, we build a new path by sampling a temporal and spatial location uniformly at random from the previous paths, starting a copy of the Brownian motion from the sampled temporal and spatial location, and running the Brownian until it's killed. The third and fourth iteration are shown above, with the first, second, third and fourth paths in green, blue, red and black respectively. We may see that in the second and third iteration we have a tails, whilst with the fourth we have a heads. While each path is considered to be alive only between the time it is born (the sampled time) and the time it is killed at the boundary, it also carries with it information of its ancestral path from time 0 to the time it is born. Thus the three paths corresponding to time 2 after 4 iterations are shown below (the first path is not alive at time 2), with the path while they are alive in blue and the ancestral path before they are born in green.
In general, if the renormalised empirical measure of the paths at time \( t \) converges to a limit for all times \( t \) (which it may not do, see Remark 3.3), we expect the time-\( t \) limit to correspond to \( \mathcal{L}((X_s)_{0 \leq s \leq \tau_\theta > t}) \) (multiplied by some constant dependent upon \( t \)). In the main theorem of this paper, Theorem 3.2, we prove that a discrete time, finite state space version of the reinforced path process in Figure 1 (defined in Definition 3.1) provides a representation for \( \mathcal{L}((X_s)_{0 \leq s \leq \tau_\theta > t}) \):

\[
\frac{1}{N} \sum_{n=1}^{N} \mathbb{1}(t_n^a \leq t < t_n^b) \delta_{f_n^a} \rightarrow \gamma_t \mathcal{L}((X_s)_{0 \leq s \leq \tau_\theta > t}) \quad \text{as} \quad N \to \infty \quad (1.5)
\]

for some constants \( \gamma_t \) we determine, whereby \( \mu \) is some given initial distribution.

In Section 2 we shall propose a general strategy for proving that a given reinforced path process converges to this limit, which motivates why we should expect to obtain \( \mathcal{L}((X_s)_{0 \leq s \leq \tau_\theta > t}) \) in the limit. Thereafter, we restrict our attention to the case of discrete time killed Markov processes on a finite state space. In Section 3 we shall provide a statement of Theorem 3.2. We shall then prove this theorem in Section 4.

**Notation**

For any topological space \( \mathcal{T} \), we write \( \mathcal{M}(\mathcal{T}) \) and \( \mathcal{P}(\mathcal{T}) \) for the set of Borel measures (respectively Borel probability measures) on \( \mathcal{T} \), equipped with the topology of weak convergence of measures. When we state and prove precise results, we shall do so for finite state spaces, so all notions of convergence of measures are the same. Thus we shall not specify under which metric our notion of convergence shall be referring to. When referring to intervals of discrete time, we abuse notation by writing \([a, b]\) for \([a, b] \cap \mathbb{N}\), for all \(a, b \in \mathbb{N}\). For paths \( f \) and \( g \) on the time intervals \( I \) and \( J \) respectively such that \( f = g \) on \( I \cap J \), we define

\[
f \oplus g : I \cup J \ni t \mapsto \begin{cases} f(t), & t \in I \\ g(t), & t \in J \end{cases} \quad (1.6)
\]

**2 General Strategy**

We propose the following:

1. Reinforced path processes \((u_n)_{n \in \mathbb{N} > 0} = ((t_n^a, f_n^a, t_n^b))_{n \in \mathbb{N} > 0}\) may often be formulated as measure-valued Polya processes. Thus if one can establish an appropriate theorem on convergence of Polya urns (such as [9, Theorem 1]), one obtains convergence of \( \frac{1}{N} \sum_{n=1}^{N} \delta_{u_n} \) to a deterministic limit as \( N \to \infty \). Alternatively it may be possible to obtain convergence using stochastic approximation ideas as in [3].

2. From the previous step, the random empirical measure \( \frac{1}{N} \sum_{n=1}^{N} \delta_{u_n} \) converges as \( N \to \infty \) to the deterministic measure \( \mathcal{L}(u)(\cdot) \) in an appropriate sense, for some random variable \( u = (t_b, f, t_d) \). This gives rise to a fixed point equation as follows.

For all times \( t \), the empirical measure of the paths alive at time \( t \) rescaled by the number of iterations, \( \frac{1}{N} \sum_{n=1}^{N} \mathbb{1}(t_n^a \leq t < t_n^b) \delta_{f_n^a} \), converges to

\[
m_t(\cdot) = \mathcal{L}(f_{\lceil t \rceil}; \mathbb{1}(t_b \leq t < t_d)(\cdot)) \mathbb{P}(t_b \leq t < t_d) \in \mathcal{M}([0, t]; \chi)). \quad (2.7)
\]

We set \( m = (m_t)_{0 \leq t < \infty} \) (or \( (m_t)_{0 \leq t < T} \) if the time horizon is finite). Then the probability of sampling in each additional iteration an independent copy of the killed Markov process
starting from time 0 with initial distribution $\mu$ converges to some probability $p^m$, while the distribution of the sampled time if we sample from amongst the previous paths converges to some distribution $\xi^m$ (both possibly dependent upon the limit $m$). These are obtained by applying the sampling rule to the limit. Thus by applying the sampling rule to the limit, we see that $E[1(t_1^m \leq t < t_2^m)\delta_{f^m_{[0,t]}}(\cdot)] = P(t_1^m \leq t < t_2^m, f^m_{[0,t]} \in \cdot)$ should converge to

$$p^m\mathcal{L}_\mu((X_u)_{0 \leq u \leq t})\mathcal{L}_f(s)(f \oplus (X_{u-s})_{s \leq u \leq t})|D(0,t;\chi)}(\cdot)
\frac{m_s(df)}{|m_s|}\xi^m(ds) \quad (2.8)$$

as $n \to \infty$. The first term on the right-hand side corresponds to sampling a new path started from time 0, whilst the second term corresponds to sampling from amongst the previous paths. Note that restricting the law of the path of a killed process to $D(0,t;\chi)$ corresponds to restricting to the event of surviving for time $t$. Therefore $E[\frac{1}{N}\sum_{n=1}^N 1(t_1^m \leq t < t_2^m)\delta_{f^m_{[0,t]}}(\cdot)]$ should converge to (2.8) as $N \to \infty$. However since $\frac{1}{N}\sum_{n=1}^N 1(t_1^m \leq t < t_2^m)\delta_{f^m_{[0,t]}}(\cdot)$ converges to the deterministic limit $m_t(\cdot)$, $m$ must be a fixed point of

$$m_t(\cdot) = \left( p^m\mathcal{L}_\mu((X_u)_{0 \leq u \leq t})\mathcal{L}_f(s)(f \oplus (X_{u-s})_{s \leq u \leq t})|D(0,t;\chi)}(\cdot)
\frac{m_s(df)}{|m_s|}\xi^m(ds) \right) |D(0,t;\chi)}.$$

3. We now observe for $0 \leq t_1 \leq t_2$ that if $(Y_u)_{0 \leq u \leq t_1} \sim \mathcal{L}_\mu((X_u)_{0 \leq u \leq t_1}|\tau_\theta > t_1)$ and, conditional on $(Y_u)_{0 \leq u \leq t_1}, (Z_u)_{t_1 \leq u \leq t_2} \sim \mathcal{L}_{Y_\tau}(L_{X_{t_1-t}}), (X_{t-\tau_\theta} \in [0,t_2], )$, then

$$Y \oplus Z \sim \mathcal{L}_\mu((X_u)_{0 \leq u \leq t_2}|\tau_\theta > t_1). \quad (2.10)$$

This may be seen, for instance, by considering the finite-dimensional distributions. We then consider the Ansatz

$$m_t(\cdot) = \gamma_t\mathcal{L}_\mu((X_u)_{0 \leq u \leq t}|\tau_\theta > t)(\cdot). \quad (2.11)$$

If we plug (2.11) into (2.9) and apply (2.10), we obtain an equation for $\gamma_t$ which may or may not be able to solve (dependent on $\xi^m$ and $p^m$). If we are able to solve the resultant equation for $\gamma_t$, we obtain a solution to (2.9) of the form (2.11).

4. One must then show that this solution is the one corresponding to the limit (2.7), for instance by showing that solutions to (2.8) are unique.

In the proof of Theorem 3.2, we will obtain uniqueness of the corresponding fixed point equation, (4.23), by interpreting such fixed points as the QSDs of an irreducible killed Markov process on a finite state space (which are unique by [8]).

3 Theorem Statement

Henceforth we assume that $\chi$ is a finite metric space and time is discrete. For (discrete and finite) time intervals $[t_1, t_2] \subseteq \mathbb{N}$, we write $F([t_1, t_2]; \chi)$ for the set of functions $[t_1, t_2] \to \chi$, which we equip with the discrete metric. We define $K : \chi \to \mathcal{M}(\chi)$ to be a submarkovian transition kernel, which defines the discrete-time killed Markov process $(X_t)_{0 \leq t < \tau_\theta}$. We assume throughout that
Theorem 3.2. There exists a unique solution,  

\[ \Pr(\tau_0 < \infty) = 1 \text{ for all } x \in \chi. \]

We fix an initial condition \( \mu \in \mathcal{P}(\chi) \) and (possibly infinite) time horizon \( T \in \mathbb{N} \cup \{\infty\}. \)

We seek to obtain a representation for the path of \((X_s)_{0 \leq s \leq \tau_0}\) with initial condition \( X_0 \sim \mu \), conditioned to survive for finite time \( t < \infty \), \( \mathcal{L}_\mu((X_u)_{0 \leq u \leq \tau_0} > t) \). We define (replace \([0, T]\) and \([0, T + 1]\) with \( N \) in the following definition when \( T = \infty \))

\[ \varepsilon = \{(t_b, f, t_d) \in [0, T] \times F([0, T]; \chi) \times [0, T + 1] : t_b < t_d \text{ and } f_{[t_d-1, t]} \equiv f(t_d-1)\}. \]

We fix \( 0 < p \leq 1 \) and construct the \( \varepsilon \)-valued reinforced path process \((u_n)_{n \in \mathbb{N} \geq 0} = (t^n_b, f^n, t^n_d)_{n \in \mathbb{N} \geq 0}\) as follows.

**Definition 3.1** (Simple reinforced path process on a finite state space in discrete time). We take a copy \((X_t)_{0 \leq t \leq \tau_0}\) of the killed Markov process with initial condition \( X_0 \sim \mu \), and define \( u_1 = (t_1^b, f^1, t_1^d) \) to be \((0, (X_{\tau_0-1})_{0 \leq t \leq \tau_0} \wedge (T+1)). \) Given \( u_1, \ldots, u_n \) we inductively define \( u_{n+1} = (t^n_b, f^{n+1}, t^n_d) \) as follows. With probability \( p \), we take another independent copy \((X_t)_{0 \leq t \leq \tau_0}\) of the killed Markov process with initial condition \( X_0 \sim \mu \), and define \( u_{n+1} = (t^n_b, f^{n+1}, t^n_d) \) to be \((0, (X_{\tau_0-1})_{0 \leq t \leq \tau_0} \wedge (T+1)). \) Otherwise, with probability \( 1-p \), we choose \( m \in [1, n] \) independently with probability

\[ \frac{t^n_d - t^n_b}{\sum_{1 \leq i \leq n}(t_i^d - t_i^b)}. \]

Given our choice of \( m \), we then choose \( t' \in [t^n_b, t^n_d-1] \) independently at random. Given the choice of \( m \) and \( t' \), we independently take \((X_t)_{0 \leq t \leq \tau_0}\) to be a copy of the killed Markov process with initial condition \( X_0 = f^n(t') \), and set

\[ u_{n+1} = (t^n_b, f^{n+1}, t^n_d) := (t', f^n_{[t', \tau_0]} \oplus (X_{(t'-\tau_0)}_{0 \leq s \leq \tau_0} \wedge (T+1))). \]

**Theorem 3.2.** There exists a unique solution, \( Z \), to

\[ z = p \sum_{s=0}^{T} \Pr(\tau_0 > s) \left[1 - \frac{1-p}{z}\right]^{-(s+1)}, \quad z \in [1, \infty). \]   \hspace{1cm} (3.12)

We define the coefficients \((\gamma_t)_{0 \leq t \leq T} \) (or \((\gamma_t)_{0 \leq t < \infty} \) if \( T = \infty \)) to be

\[ \gamma_t = p \Pr(\tau_0 > t) \left[1 - \frac{1-p}{Z}\right]^{-(t+1)}, \]   \hspace{1cm} (3.13)

whereby \( Z \) is the unique solution to (3.12). Then for any \( t \in [0, T] \) (or \( t \in \mathbb{N} \) if \( T = \infty \)) we have

\[ \frac{1}{N} \sum_{n=1}^{N} 1(t^n_b \leq t < t^n_d)\delta_{(0, t)}(\cdot) \overset{a.s.}{\rightarrow} \gamma_t \mathcal{L}_\mu((X_s)_{0 \leq s \leq t} > \tau_0) \) as \( N \to \infty. \] \hspace{1cm} (3.14)

We note that for \( T = \infty \), the right-hand side of (3.12) is a divergent infinite sum for \( 1 \leq z \leq \frac{1-p}{1-\lambda_0} \), whereby \( 0 \leq \lambda_0 < 1 \) is the maximal real eigenvalue of \( K \) (see the proof of Proposition 4.1 on Page 10).

**Remark 3.3.** Clearly (3.12) doesn’t have any solutions when \( p = 0 \). In fact we don’t expect convergence of the reinforced path process to a deterministic limit when \( p = 0 \), except for the trivial cases whereby \( \Pr(f^1(0)\tau_0 = 1) = 1 \text{ or } T = 0 \). To see this note that any deterministic limit \( m = (m_t)_{0 \leq t \leq T} \) corresponds to a solution of (2.9) with \( p^m = 0 \) and \( \xi^m(t) = \sum_{t=0}^{\lfloor m_t \rfloor} \). By considering \( |m_0| \) we see that \( \sum_{t=0}^{T} |m_t| = 1 \). Then by taking the mass of both sides of (2.9) and summing over \( t \), we see that \( \Pr(f^1(0)\tau_0 = 1) = 1 \text{ or } T = 0 \).
If a killed Markov process is reducible, it is an open problem to determine which QSDs (if any at all) will be obtained from the reinforced processes \((Y_t)_{0 \leq t < \infty}\) described in the introduction, by taking the limit (1.4). We suppose that for some \(\mu \in \mathcal{P}(\chi)\) we have
\[
\mathcal{L}_\mu(X_t|\tau_0 > t) \to \pi(\cdot) \quad \text{as} \quad t \to \infty.
\]
The following corollary, which came from a discussion of the author with Michel Benaim, allows us to obtain \(\pi(\cdot)\). We construct for \(0 < p \leq 1\) a reinforced process with renewal \((Y^p_t)_{0 \leq t < \infty}\) as follows. We firstly take a copy of the killed Markov process \((X_t)_{0 \leq t < \tau_0}\) with initial condition \(X_0 \sim \mu\), and set \((Y^p_t)_{0 \leq t < \tau_0}\) as in (1.1). At this killing time, with probability \(p\) we "renew", sampling
\[
Y_{\tau_0} \sim \mu(\cdot). \tag{3.15}
\]
Otherwise, with probability \(1 - p\), we sample from the empirical measure of the history,
\[
Y^p_{\tau_0} \sim \frac{1}{\tau_0} \sum_{t=0}^{\tau_0-1} \delta_{Y_t}(\cdot), \tag{3.16}
\]
as in (1.2). The process \(Y^p_t\) then continues evolving like a copy of \((X_t)_{0 \leq t < \tau_0}\) up to its next killing time. This is then repeated inductively, so that at each killing time we sample from \(\mu\) with probability \(p\), otherwise sampling from the empirical measure of the history as in (1.3). We are then able to obtain the quasi-limiting distribution \(\pi(\cdot)\) from \((Y^p_t)_{0 \leq t < \infty}\), for small \(0 < p \ll 1\).

**Corollary 3.4.** For given \(0 < p \leq 1\) we let \(Z^p\) be the unique solution to (3.12) for \(T = \infty\) and \((\gamma^p_t)_{0 \leq t < \infty}\) be the coefficients thereby defined in (3.13). Then we have
\[
\frac{1}{t} \sum_{s=0}^{t-1} \delta_{Y^p_s}(\cdot) \xrightarrow{a.s.} \pi^p(\cdot) \quad \text{as} \quad t \to \infty, \tag{3.17}
\]
whereby
\[
\pi^p(\cdot) := \sum_{t=0}^{\infty} \gamma^p_t \mathcal{L}_\mu(X_t|\tau_0 > t)(\cdot) \to \pi(\cdot) \quad \text{as} \quad p \to 0. \tag{3.18}
\]

## 4 Proof of Theorem 3.2

We defer for later the proof of the following proposition.

**Proposition 4.1.** There exists a unique solution, \(Z\), to (3.12).

We will establish convergence by formulating the reinforced path process as an urn process. We use the terminology given in [9, Section 1.1] throughout. We must distinguish between the \(T < \infty\) and \(T = \infty\) cases. If \(T = \infty\) we fix arbitrary \(\bar{T} \in \mathbb{N}\). We define \((\ast)\) is a point distinguished from \(N\)
\[
E_t := \{(t, f) \in \mathbb{N} \times F([0, t]) : f \in \text{supp}(\mathcal{L}_\mu((X_s)_{0 \leq s \leq t}))\}, \quad t \in \mathbb{N},
\]
\[
E_\ast := \{\ast\} \times \text{supp}(\mathcal{L}_\mu(X_{\bar{T}+1})), \quad E := \begin{cases} \bigcup_{t \in [0, T]} E_t, & T < \infty \\ \bigcup_{t \in [0, T]} E_t \cup E_\ast, & T = \infty \end{cases},
\]
which we equip with the discrete metric. We then define
\[
m_N(\cdot) := \sum_{n=1}^{N} v_n(\cdot) \in \mathcal{M}(E) \quad \text{whereby}
\]
\[
v_n(\cdot) := \begin{cases} \sum_{t=0}^{\bar{T}-1} \mathbf{1}(t \leq \bar{T}) \delta_{(t, f^n_{(t, \ast)})}(\cdot), & T < \infty \\ \sum_{t=0}^{\bar{T}-1} (\mathbf{1}(t \leq \bar{T}) \delta_{(t, f^n_{(t, \ast)})}(\cdot) + \mathbf{1}(t > \bar{T}) \delta_{(\ast, f^n_{(t, \ast)})}(\cdot)), & T = \infty \end{cases}.
\]

We define $\Gamma := \sup_{x \in \chi} E_x[\tau_0]$ and observe that $(m^N)_{N \in \mathbb{N} \geq 0}$ is a measure-valued Polya process with:

- Initial composition

$$m_1 = \begin{cases} \sum_{s=0}^{t_0-1} \delta_{(s,X_s^{(0,t)})_{0 \leq s \leq t}}, & B = 1 \\ \sum_{s=0}^{t_0-1} \delta_{(s,X_s^{(T)})_{0 \leq s \leq t}}, & B = 0 \end{cases}$$

whereby $(X_t^{(0,t)})_{0 \leq t < \tau_0}$ is an independent copy of the killed Markov process with submarkovian transition kernel $K$ and initial condition $X_0^0 \sim \mu$.

- Independent and identically distributed random replacement kernels

$$E \ni (t, f) \mapsto R^n((t, f); : \cdot) \in \mathcal{M}(E),$$

defined as follows. We take for each $n$, independently of each other and everything else, a Bernoulli random variable $B \sim \text{Ber}(p)$, a copy $(X_t^{(0,t)})_{0 \leq t < \tau_0}$ of the killed Markov process with submarkovian transition kernel $K$ and initial condition $X_0 \sim \mu$, and a family of copies $\{(X_t^{(0,t)})_{0 \leq t < \tau_0} : x \in \chi\}$ of the same killed Markov process with initial conditions $X_0^x = x$. When $T < \infty$ we define the random kernel

$$R^n((t, f); \cdot) = \begin{cases} \sum_{s=0}^{t_0-1} \delta_{(s,X_s^{(0,t)})_{0 \leq u \leq s}}(\cdot), & B = 1 \\ \sum_{s=0}^{t_0-1} \delta_{(t+s, f \oplus (X_{t+s-1}^{(0,t)})_{0 \leq u \leq t+s})}(\cdot), & B = 0 \end{cases}$$

We adopt the convention that $s + \tau := \tau > T$ for $s \in \mathbb{N}$ and $f(\tau) := f$ for $(\tau, f) \in E_*$. When $T = \infty$ we define the random kernel $R^n$ as

$$R^n((t, f); \cdot) := \begin{cases} \sum_{s=0}^{t_0-1} \left\{ \delta_{(s,X_s^{(0,t)})_{0 \leq u \leq s}}(\cdot) + \delta_{(t+s, f \oplus (X_{t+s-1}^{(0,t)})_{0 \leq u \leq t+s})}(\cdot) \right\}, & B = 1 \\ \delta_{(t+s, f \oplus (X_{t+s}^{(0,t)})_{0 \leq u \leq t+s})}(\cdot), & B = 0 \end{cases}$$

These random kernels $R^n$ have common expectation given by the (deterministic) kernel $R : E \to \mathcal{M}(E)$, which in the $T < \infty$ case is given by

$$R((t, f); \cdot) = \mathbb{E}[R^n((t, f); \cdot)] = p \sum_{s=0}^{T-t} \mathcal{L}_{\mu}(s, (X_u)_{0 \leq u \leq s} \mid \tau_0 > s)(\cdot) \mathbb{P}_{\mu}(\tau_0 > s)$$

$$+ (1 - p) \sum_{s=0}^{T-t} \mathcal{L}_{f(t)}((t+s, f \oplus (X_{t+s-1})_{0 \leq u \leq t+s}) \mid \tau_0 > s)(\cdot) \mathbb{P}_{f(t)}(\tau_0 > s).$$

In the $T = \infty$ case the kernel $R : E \to \mathcal{M}(E)$ is given by

$$R((t, f); \cdot) = \mathbb{E}[R^n((t, f); \cdot)] = p \sum_{s=0}^{\infty} \left\{ \delta_{(s \leq T)} \mathcal{L}_{\mu}(s, (X_u)_{0 \leq u \leq s} \mid \tau_0 > s)(\cdot) \mathbb{P}_{\mu}(\tau_0 > s) \right\}$$

$$+ (1 - p) \sum_{s=0}^{\infty} \left\{ \delta_{(s > T)} \mathcal{L}_{f(t)}((t+s, f \oplus (X_{t+s-1})_{0 \leq u \leq t+s}) \mid \tau_0 > s)(\cdot) \mathbb{P}_{f(t)}(\tau_0 > s) \right\}$$

$$+ \sum_{s=0}^{\infty} \left\{ \delta_{(s \leq T)} \mathcal{L}_{f(t)}((t+s, f \oplus (X_{t+s-1})_{0 \leq u \leq t+s}) \mid \tau_0 > s)(\cdot) \mathbb{P}_{f(t)}(\tau_0 > s) \right\}.$$
• Non-negative weight kernel

\[ P : E \ni (t, f) \mapsto \frac{1}{\Gamma} \delta_{(t,f)}(\cdot) \in \mathcal{M}(E). \] (4.19)

We therefore define the (deterministic) submarkovian kernel \( Q \) as

\[ Q : E \ni (t, f) \mapsto \int_E P(z; \cdot) R((t, f); dz) = \frac{1}{\Gamma} R((t, f); \cdot) \in \mathcal{M}(E). \] (4.20)

We remark here that we could instead prove Theorem 3.2 by formulating \( \sum_{n=1}^N \delta_{\omega_n} \) as an urn process, but the formulation we use here seems to lead to a simpler proof.

Since \( \chi \) and \( E \) are finite, it is trivial to verify assumptions [9, T>0, (A1), (A2) and (A4)]. Moreover the \( E \)-valued continuous-time killed Markov process \( (Y_t)_{t<\tau^Y_0} \) with submarkovian infinitesimal generator \( Q - I \) is an irreducible killed Markov process on a finite state space, hence [8, (3.3)] implies that \( (Y_t)_{t<\tau^Y_0} \) admits a unique QSD, \( \nu \), such that

\[ \mathcal{L}_\xi(Y_t|_{\tau^Y_0}>t)(\cdot) \overset{TY}{\underset{\xi \in \mathcal{P}(E)}{\rightarrow}} \nu(\cdot) \quad \text{as} \quad t \to \infty \quad \text{uniformly in} \quad \xi \in \mathcal{P}(E). \] (4.21)

Thus we have verified Assumption [9, (A3)], hence we may invoke [9, Theorem 1], giving that

\[ \frac{m_N}{N}(\cdot) \to \nu R(\cdot) \quad \text{almost surely}. \] (4.22)

Since QSDs of \( (Y_t)_{t<\tau^Y_0} \) correspond to solutions of

\[ \alpha(\cdot) = \frac{\alpha Q(\cdot)}{\alpha Q(E)} = \frac{\alpha R(\cdot)}{\alpha R(E)}, \quad \alpha \in \mathcal{P}(E), \] (4.23)

\( \nu \) is the unique solution to (4.23).

We assume for the time being that \( T < \infty \) and write

\[ \tilde{\nu}(\cdot) := \sum_{t = 0}^T \sum_{s = 0}^T \gamma_t \mathcal{L}_\mu((t, (X_u)_{0 \leq u \leq t})|\tau_0 > s) \in \mathcal{P}(E). \] (4.24)

We use (2.10) and straightforward algebra to calculate

\[ \tilde{\nu} R(\cdot) = p \sum_{s = 0}^T \mathcal{L}_\mu((s, (X_u)_{0 \leq u \leq s})|\tau_0 > s) + (1 - p) \sum_{s = 0}^T \sum_{t = 0}^T \mathcal{L}_f((s, f \oplus (X_u-t)_{t \leq u \leq s})|\tau_0 > s)(\cdot) \mathcal{P}_f(t)(\tau_0 > s) \mathcal{L}_\mu((X_u)_{0 \leq u \leq t}|\tau_0 > t)(df) \]

\[ \overset{(2.10)}{=} p \sum_{s = 0}^T \mathcal{L}_\mu((s, (X_u)_{0 \leq u \leq s})|\tau_0 > s) \]

\[ + (1 - p) \sum_{s = 0}^T \sum_{t = 0}^T \mathcal{L}_\mu((s, (X_u)_{0 \leq u \leq s})|\tau_0 > s)(\cdot) \]

\[ \overset{(2.10)}{=} p \sum_{s = 0}^T \mathcal{L}_\mu((s, (X_u)_{0 \leq u \leq s})|\tau_0 > s) \]

\[ + (1 - p) \sum_{s = 0}^T \sum_{t = 0}^T \mathcal{L}_\mu((s, (X_u)_{0 \leq u \leq s})|\tau_0 > s)(\cdot) \]

\[ = \sum_{s = 0}^T \mathcal{P}_\mu(\tau_0 > s) \left[ 1 - \frac{1 - p}{Z} \right]^{-(s + 1)} \mathcal{L}_\mu((s, (X_u)_{0 \leq u \leq s})|\tau_0 > s)(\cdot) = Z \tilde{\nu}(\cdot) \]
Therefore \( \nu(\cdot) \) is the solution to (4.23), hence \( \nu = \hat{\nu} \). Therefore

\[
\nu R(\cdot) = Z \hat{\nu}(\cdot) = \sum_{t=0}^{T} \gamma_t \mathcal{L}_\mu((t, (X_u)_{0 \leq u \leq t})| \tau_0 > t).
\]

Combining this with (4.22) we have (3.14) in the \( T < \infty \) case. In the \( T = \infty \) case we consider

\[
\hat{\nu}(\cdot) := \sum_{t=0}^{\infty} \gamma_t \mathcal{L}_\mu((t, (X_u)_{0 \leq u \leq t})| \tau_0 > t)(\cdot) + \sum_{t=T+1}^{\infty} \gamma_t \mathcal{L}_\mu(X_t| \tau_0 > t)(\cdot)
\]

and repeat the above calculation to obtain (3.14) for all \( t \leq \bar{T} \). Since \( \bar{T} \in \mathbb{N} \) was arbitrary, we have (3.14) for all \( t \in \mathbb{N} \).

We have left only to prove Proposition 4.1.

**Proof of Proposition 4.1.** We consider for \( T < \infty \) the continuous function

\[
f_T : [1, \infty) \ni z \mapsto z - \sum_{s=0}^{T} \mathbb{P}_\mu(\tau_0 > s)p\left[1 - \frac{1-p}{z}\right]^{-(s+1)} \in \mathbb{R}.
\]

(4.25)

Solutions to \( f_T(z) = 0, z \in [1, \infty) \) correspond to solutions to (3.12). We calculate for \( T < \infty \),

\[
f_T(1) \leq 1 - \mathbb{P}_\mu(\tau_0 > 0) = 0, \quad f_T'(z) \geq 1 \quad \text{for all} \quad z \in [1, \infty) \quad \text{and} \quad f_T(z) \to \infty \quad \text{as} \quad z \to \infty,
\]

so that by the intermediate value theorem there exists a unique solution to (3.12).

We now consider the \( T = \infty \) case. Without loss of generality, we assume that all of \( \chi \) is accessible from \( \mu \) (otherwise we may restrict to those states which are accessible). The state space \( \chi \) may be decomposed into communicating classes \( \chi_1, \ldots, \chi_n \), ordered so that \( \chi_j \) is accessible from \( \chi_i \) only if \( j > i \). Therefore, reordering the indices of the elements of \( \chi \) if necessary, \( K \) can be written

\[
K = \begin{pmatrix}
K_1 & \cdots & \cdots \\
0 & \cdots & \cdots \\
0 & 0 & K_n
\end{pmatrix}
\]

so that

\[
K' = \begin{pmatrix}
K_1' & \cdots & \cdots \\
0 & \cdots & \cdots \\
0 & 0 & K_n'
\end{pmatrix},
\]

whereby \( K_1, \ldots, K_n \) are the submarkovian transition matrices of irreducible killed Markov chains on \( \chi_i \) which are almost surely killed. We let \( 0 \leq \lambda_i < 1 \) be the Perron-Frobenius eigenvalue of \( K_i \) and write \( \lambda_0 := \max_{1 \leq i \leq n} \lambda_i \). Simple linear algebra gives that \( (if \lambda_0 = 0 \text{ we define } \frac{1}{\lambda_0} := +\infty) \)

\[
\liminf_{t \to \infty} k^t \mathbb{P}_\mu(\tau_0 > t) > 0 \quad \text{for} \quad k \geq \frac{1}{\lambda_0}, \quad \lim_{t \to \infty} k^t \mathbb{P}_\mu(\tau_0 > t) = 0 \quad \text{for} \quad k < \frac{1}{\lambda_0}.
\]

(4.26)

Thus the infinite sum \( \sum_{s=0}^{\infty} \mathbb{P}_\mu(\tau_0 > s)p\left[1 - \frac{1-p}{z}\right]^{-(s+1)} \) is well-defined for

\[
z \in (z_0, \infty) \cap [1, \infty) \quad \text{whereby} \quad z_0 := \frac{1-p}{1-\lambda_0} < z < \infty,
\]

so that we may define \( f_\infty \) on \( (z_0, \infty) \cap [1, \infty) \) similarly to (4.25). Moreover (4.26) gives that \( f_\infty \) is continuous and strictly increasing. It also gives that \( f_\infty(z) \to -\infty \) as \( z \downarrow z_0 \) if \( z_0 \geq 1 \). If \( z_0 < 1 \) (which is the case if \( \lambda_0 = 0 \)) we observe that \( f_\infty(1) < 0 \) as with \( T < \infty \). The same argument as with \( T < \infty \) gives the existence of a unique solution to (3.12).

This concludes the proof of Theorem 3.2.

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