Effects of Surface Tension and Uneven Bottom on Surface Solitary Waves

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Abstract. The flow of an incompressible inviscid fluid with surface tension (Bond number is closed to 1/3) on the uneven bottom is studied. The effects of different cases of flow on the nonlinear surface waves were analyzed. The numerical simulation for the fifth-order equation was made by pseudo-spectral method, and the waterfall plot of the wave was draw with software. The results show that the fifth-order dispersive item did not change the developing tendency of the surface wave, only affected the number and position of the solitary wave, however, when the flow condition was changed, there appeared greatly variation on the surface wave due to fifth-order dispersive item.

1. Introduction
When flow interacts with a bottom feature and the fluid can support wave propagation, then there is the potential for solitary waves to be generated upstream and or downstream, which is related to the oceanography, ocean engineering, but also to thermal power engineering and engineering thermo physics closely.

In such cases, when the bottom feature has small amplitude, the situation can be successfully described using a linear theory, and any nonlinear effects are determined as a small perturbation on the linear theory. However, when the flow is critical, then typically the linear theory fails and it is necessary to develop an intrinsically nonlinear theory. It is now known that in many cases such a transcritical, weakly nonlinear and weakly dispersive theory leads to an fKdV equation [1, 2]. Allen Chwang discuss the wave profile due to bottom with a finite different method based on the Euler equations [3]. Wu showed that the forced Boussinesq or the forced KdV equation, which is confirmed by a number of experiments at least in the early stage of wave development, could describe the free surface wave excited by a bottom obstacle near resonance [4]. Porter showed the interaction of linearized surface gravity water waves with three-dimensional periodic topography [5]. Davis thinks that there is a Bragg resonance between the surface waves and the ripple, which is associated with the reflection of incident wave energy [6]. The effect of the nonlinear waving bottom on the surface solitary wave has been analyzed theoretically by Zhong [7]. In recent years, many methods were suggested to search for wave solutions for the KdV equation and its generalizations, e.g. the pseudo-spectral method [8], He’s Homotopy Perturbation Method [9-11], Homotopy Analysis Method [12].

In the paper, the flows of an incompressible, inviscid fluid with surface tension and bottoms were considered. The effects of the different flow state on the nonlinear surface waves were researched. The
wave equation was reduced from the potential flow theory with the little parameter perturbation technique, and then was solved with pseudo-spectral method. The waterfall of the surface wave was simulated with software.

2. The Deduction of the Fifth-order fKdV Equation
We shall consider a two-dimensional flow of an inviscid irrotational incompressible fluid on which surface tension acts and with a bottom topography, and Bond number is closed to 1/3. The coordinate system is sketched in figure 1.

![Figure 1. a schematic diagram of the flow](image)

A velocity potential function \( \phi(x, y, t) \) was introduced, the control equation, initial condition and boundary condition as follows:

\[
\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \quad f(x) < y < \eta(x, t) \tag{1a}
\]

\[
\frac{\partial \eta}{\partial t} + \frac{\partial \phi}{\partial x} \frac{\partial \eta}{\partial x} - \frac{\partial \phi}{\partial y} = 0 \quad y = \eta(x, t) \tag{1b}
\]

\[
\frac{\partial \phi}{\partial t} + \frac{(\partial \phi / \partial x)^2 + (\partial \phi / \partial y)^2}{2} + g\eta - \frac{T}{\rho} \frac{\partial^2 \eta}{\partial x^2} \left[1 + \left(\frac{\partial \eta}{\partial x}\right)^2\right]^{\frac{3}{2}} = 0 \quad y = \eta(x, t) \tag{1c}
\]

\[
\frac{\partial \phi}{\partial y} = \frac{\partial f}{\partial x} \frac{\partial \phi}{\partial x} \quad y = f(x) \tag{1d}
\]

Where \( \eta(x, t) \) is the free surface function, \( f(x) \) is the bottom function, \( T \) is the coefficient of surface tension, \( \rho \) is the density of the fluid, \( U_0 \) is the basic uniform flow. It is convenient to introduce the transformation for \( \phi \) and non-dimension variable based on

\[
\varphi = \phi - U_0 x + \frac{1}{2} U_0^2 t, \quad x^* = \frac{x}{l}, \quad y^* = \frac{y}{h_0}, \quad T^* = \frac{T}{\rho g h_0^2}, \quad t^* = \frac{c_0 t}{l}
\]
\[ \eta^* = \frac{\eta}{a}, \quad f^* = \frac{f}{h_0}, \quad \varphi^* = \frac{c_0}{gla}, \quad F = \frac{U_0}{c_0} \]

Where \( l \) is a typical horizontal length, \( h_0 \) the typical mean depth of fluid, \( a \) the typical vertical amplitude of the free surface, \( c_0 \) is the linear typical speed \( c_0 = \sqrt{gh_0} \), and \( F \) is Froude number.

After dropping the stars, we have the following dimensionless equations:

\[ \beta^2 \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0 \quad f(x) < y < \eta(x,t). \quad (2a) \]

\[ \frac{\partial \eta}{\partial t} + (F + \alpha \frac{\partial \varphi}{\partial x}) \frac{\partial \eta}{\partial x} - \beta^2 \frac{\partial \varphi}{\partial y} = 0 \quad y = \eta(x,t). \quad (2b) \]

\[ \frac{\partial \varphi}{\partial t} + \frac{1}{2} \alpha \left( \frac{\partial \varphi}{\partial x} \right)^2 + \frac{1}{2} \alpha \beta^2 \left( \frac{\partial \varphi}{\partial y} \right)^2 + F \frac{\partial \varphi}{\partial x} + \eta - \beta^2 \frac{\partial^2 \eta}{\partial x^2} \left[ 1 + \alpha^2 \beta^2 \left( \frac{\partial \eta}{\partial x} \right)^2 \right]^{3/2} = 0 \]

\[ y = \eta(x,t) \quad (2c) \]

\[ \frac{\partial \varphi}{\partial y} = \beta^2 \frac{\partial f}{\alpha \frac{\partial \varphi}{\partial x}}(F + \alpha \frac{\partial \varphi}{\partial x}) \quad y = f(x). \quad (2d) \]

Here, \( \alpha = a/h_0 \) is the dimensionless wave amplitude which measures the non-linearity of the system, and \( \beta = h_0/l \) is the dimensionless wavelength, and measures the dispersive effect of the system.

In the paper, weakly nonlinear and dispersion with the assumption of small boundary variation were considered. So, \( \alpha \) and \( \beta \) are assumed to be small, and we further require a balance between nonlinearity and dispersion, so some assumption were made, as follows:

\[ f(x) = -1 + \gamma W(x), \quad T = \frac{1}{3} \beta^2 T, \quad F = 1 + \Delta \varepsilon, \quad \alpha = \beta^4 = \gamma^{3/2} = \varepsilon, \quad \tau = \alpha T \]

Where \( W(x) \) is a continuous function, in the assumption of small bottom variation, \( \gamma \) and \( \varepsilon \) is small parameter, \( \Delta \) a constant, \( T_\gamma \) positive constant, \( \tau \) new time scale.

According to the perturbation technique, the suitable changeable boundary condition was used. So, we can derive a fifth-order fKdV equation which has positive dispersion and is valid for Bond \( \approx 1/3 \), as follows.

\[ -(\eta_{0r} + \Delta \eta_{0x}) + \frac{3}{2} \eta_0 \eta_{0x} + \frac{T_1}{2} \eta_{0xxx} + \frac{1}{90} \eta_{0xxxx} + \frac{1}{2} W_x = 0 \quad (3) \]

Rescaling above equation by

\[ A^*(x^*, \tau^*) = \frac{1}{4 \delta} \eta_0(x, \tau), \quad x^* = x \left( \frac{2 \delta}{T_1} \right)^{1/2} \]
where $\delta$ is an arbitrary positive parameter, omitting the asterisk $\ast$, we have the rescaled equation as follows:

$$-(A_t + \Delta A_x) + 6A A_x + A_{xxx} + k_i A_{xxxx} + G_x = 0 \quad k_i = \frac{2\delta}{45 T_i^2}$$

We note this equation for surface wave is different from the fKdV equation [2] in which the dispersion term is negative. Here, $\Delta$ is a parameter which denote the different flow case, $\Delta = 0$ is corresponding to the case of exact resonance, and $\Delta > 0$ (or $\Delta < 0$) to the supercritical (or subcritical). $A(x,t)$, the amplitude of the surface wave, the function $G$ describes the bottom topography. For the fixed boundary, the $G$ maybe supposed as $G = \pm G_0 \exp[-\theta^2 (x-x_0)^2]$, $G_0$ is the maximum height of the boundary. And $x_0$ presents the central position of the forcing (obstacle), $\theta$ expresses the size of the forcing. In this paper, $G_0 = 0.5, \theta = 0.3 , x_0 = 110$.

3. Numerical solution for Different Flow Case

According to the different flow case, we use a Pseudo-spectral method to numerically integrate the rescaled equation (4). The method uses a Fourier transform treatment of the space dependence together with a leap-frog scheme in time. And it is different from VIM, HPM, and HAM. The waterfall as follows,
The figure 2 is the results for the resonant case, the waves system include upstream solitary wave train, a plat concave section and a train of downstream dispersive waves. Form the figure, we can get that the number of the solitary wave upstream deceases with the increase of the $k_1$ value, and the position of the solitary move to the obstacle, the generating period of the solitary wave increase. Compared with the normal fKdV equation, the number of the solitary wave is more within the same developing time, but the concave section and the dispersive wave downstream keep fixed.
Figure 3. Numerical solution of the fifth-order fKdV for supercritical case
The figure 3 is the simulation result for the supercritical case, in which, (a) and (b) is the result that $k_1$ value keep fixed but $\Delta$ value change. We can find, from the figure, the generating period of the upstream solitary wave increasing with the changing of the $\Delta$ value, then the appearing time of the upstream solitary like wave delay and the wave number reduce, but its amplitude did not change. Simultaneously, the concave section turns wide, and amplitude of the downstream dispersive wave decrease, the generating filed reduced, and there appear some fluctuate in some filed, which did not influence on the generation of the solitary wave, only for whole surface wave.

In figure 3, (c) and (d) is the result that $\Delta$ value fixed but $k_1$ value change. From which, the results show, when the $\Delta$ value fixed, the field for the upstream solitary wave and downstream dispersive wave are all turn narrow with $k_1$ value increase, and within the same time, the generation period of solitary wave increasing, but the flat field keep unchangeable almost.

Figure 4. Numerical solution of the fifth-order fKdV for subcritical case
The figure 4 is the simulation result for the supercritical case, in which, $k$ value keep fixed but $\Delta$ value change. The results show, with the deceasing of the $\Delta$ value, the amplitude of the upstream solitary wave decreasing gradually, and the period of the downstream dispersion wave reduce, the number of the dispersive wave obviously increasing, and the middle flat field turn narrow. The tendency of the surface wave developing keep consistence with the fKdV equation.

Moreover, the solution is connected with the parameter $G_0, \theta$ and $x_0$, work on these cases will be published separately.

4. Conclusion
The different flows case of an incompressible, inviscid fluid with surface tension and bottom has been considered. The wave equation was reduced from the potential flow theory with the little parameter perturbation technique, and then was solved with pseudo-spectral method. The conclusion as follows, if the flow case was confirmed, the fifth-order dispersive item did not change the tendency of the surface wave, only affect the number and position of the solitary wave or the dispersive wave, but, if the flow case change, there appear greatly variation on the surface wave. It should be pointed out that some new findings from the analysis of this case are lack of experiment evidence. Therefore, they need to be verified by experiment evidence.

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References
[1] Roger Grimshaw 2002 4th Int. Conf. on Nonlinear Mechanics (Shanghai: Shanghai University Press)
[2] Wu Z R, Cheng Y L, Wang S L and Lv Y K 2006 Appl. Math. Mech. 27 409
[3] Zhang D H and Chwang A T 2001 J. Fluid Mech 432 341
[4] Wu T Y 1984 J. Fluid Mech 184 75
[5] Porter R and Porter D 2001 J. Fluid Mech 434 301
[6] Davis A G and Heatthershaw A D 1984 J. Fluid Mech 144 419
[7] Zhong J X and Yan X H 1992 Natural Sci. J. Xiangtan Univ. 14 40
[8] Bengt. Fornberg 1996 A Practical Guide to Pseudospectral Methods (New York: Cambridge University Press)
[9] He J H 2006 Phys Lett A 350 87
[10] Ozis T and Yildirim A 2007 Int. J. Nonlinear Sci. Numer. Simul. 8 239
[11] Rafiei M and Ganji DD 2006 Int. J. Nonlinear Sci. Numer. Simul. 7 321
[12] Liao Shijun 2003 Beyond Perturbation: Introduction to the homotopy analysis method (Boca Raton: Chapman & Hall/CRC press)