GLOBAL $H^s, s > 0$ LARGE DATA SOLUTIONS OF 2D DIRAC EQUATION WITH HARTREE TYPE INTERACTION

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Abstract. Local and global well - posedness of the solution to the two space dimensional Dirac equation with Hartree type nonlinearity is established with the initial datum in the space $H^s(\mathbb{R}^2, \mathbb{C}^2)$ with $s > 0$.

1. Introduction

The 2D Dirac type models have manifested an increasing role in the last years in connection with the new hypothetical field called anionic Dirac matter that is a type of quasiparticle that can only occur in two-dimensional systems (see for example [Wehling et al. (2014)]).

The interaction between fermions (a typical example is the electron) confined to a plane with massive bosons particles can be interpreted as a 2D nonlinear Dirac with cubic nonlinearity of Hartree type (see [Chadam et al.(1976)], [Alves et al.(2018)]). A General overview on the application of variational methods for this type of models can be found in [Alves(2018)].

In this paper we study the 2D Dirac equation with a non-local interaction term of Hartree type. To be more precise, we plan to study local and global well - posedness of the Cauchy problem

\[
\begin{cases}
  -i\gamma^\mu \partial_\mu \psi + m \psi = ((b - \Delta)^{-1}|\psi|^2)\psi, \\
  \psi(0, x) = \psi_0 \in H^s(\mathbb{R}^2),
\end{cases}
\]

where $m > 0$ is the mass of the spinor, $s > 0$, $b > 0$, $\psi : \mathbb{R}^{1+2} \to \mathbb{C}^2$, the Dirac matrices $\gamma^\mu$ are given by

\[
\begin{align*}
  \gamma^0 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \\
  \gamma^1 &= \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \\
  \gamma^2 &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}
\end{align*}
\]

and $\langle \cdot, \cdot \rangle$ is the inner product in $\mathbb{C}^2$. We use the standard summation rule for repeated indices as well the classical rule of raising and lowering indices by the aid of the metric diag(1, $-1$) so that

\[
\partial^0 = \partial_0 = \partial_t, -\partial^j = \partial_j = \partial_{x_j}, j = 1, 2.
\]

This Cauchy problem was treated in [Tesfahun(2018)] and the main result of that work is the global existence and scattering with an initial datum which has sufficiently small Sobolev norm $H^s(\mathbb{R}^2)$ with $s > 0$.

Our main goal is to improve this result and establish local and global well - posedness with initial data in $H^s(\mathbb{R}^2)$ without the smallness assumption.

For the case of the local interaction of the type $((\gamma^0 \psi, \psi)) \psi$ one can see the corresponding results in [Bejenaru et al.(2016)] where the initial datum is in $H^\frac{1}{2}$.

In order to state our main result, we can rewrite the Cauchy problem (1) in the equivalent form

\[
\begin{cases}
  i\partial_t \psi = D_m \psi - ((b - \Delta)^{-1}|\psi|^2)\gamma^0 \psi, \\
  \psi(0, x) = f \in H^s(\mathbb{R}^2),
\end{cases}
\]

where

\[
D_m = -i\alpha^j \partial_j + m\gamma^0
\]
is a self-adjoint operator with the domain $H^1(\mathbb{R}^2)$ due to the fact that the matrices

$$
\alpha^1 = \gamma^0 \gamma^1 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad \alpha^2 = \gamma^0 \gamma^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},
$$

and $\gamma^0$ are self-adjoint. Thus, we look for a solution $\psi \in C([0, \infty); H^s(\mathbb{R}^2))$ satisfying the integral equation

$$
\psi(t) = e^{-itD_m} \psi_0 + \int_0^t e^{-i(t-s)D_m} ((b - \Delta)^{-1} |\psi|^2) \gamma^0 \psi(s) ds,
$$

(3)

**Theorem 1.** [Global existence] For any $s > 0$ and any $\psi_0(x) \in H^s(\mathbb{R}^2)$ there exists a unique solution $\psi(t, x) \in C([0, \infty); H^s(\mathbb{R}^2))$ to integral equation (3). Moreover, for any $t \in [0, \infty)$,

$$
\|\psi(t, \cdot)\|_{L^2(\mathbb{R}^2)} = \|\psi_0\|_{L^2(\mathbb{R}^2)}.
$$

The key point in the proof is the following Brezis-Gallouët Type Inequality (see [Brezis et al. (1980)]):

**Theorem 2.** For any $b > 0$ and $s > 0$ there exists a constant $C = C(b, s) > 0$ such that, for any $f \in B^s_{1, \infty}(\mathbb{R}^2)$ the following inequality is true

$$
\|(b - \Delta)^{-1} f\|_{L^\infty(\mathbb{R}^2)} + \|(b - \Delta)^{-1} (1 - \Delta)^{s/2} f\|_{L^2(\mathbb{R}^2)} \leq C \|f\|_{L^1(\mathbb{R}^2)} \log \left(2 + \frac{\|f\|_{B^s_{1, \infty}(\mathbb{R}^2)}}{\|f\|_{L^1(\mathbb{R}^2)}}\right).
$$

(5)

Concerning the behavior of energy type norm $\|\psi(t)\|_{H^s(\mathbb{R}^2)}$ we can show double exponential growth for the Dirac case, namely we have the following.

**Theorem 3.** [Growth of $H^{1/2}$ norm] For any $\psi_0(x) \in H^{1/2}(\mathbb{R}^2)$ there exist $C_1(\|\psi_0\|_{H^s}) > 0$ and $C_2 > 0$ such that the solution $\psi(t, x) \in C([0, \infty); H^{1/2}(\mathbb{R}^2))$ to integral equation (3) satisfies the estimate

$$
\|\psi(t)\|_{H^{1/2}(\mathbb{R}^2)} \leq e^{C_1 e^{C_2 t}}.
$$

(6)

Since the kinetic energy of the Dirac equation is determined by the indefinite form

$$
\langle D_m \psi(t), \psi(t) \rangle_{L^2}
$$

we can obtain the following better exponential bound.

**Theorem 4.** [Growth of the kinetic energy] For any $\psi_0(x) \in H^{1/2}(\mathbb{R}^2)$, there exists a constant $C = C(\|\psi_0\|_{H^{1/2}}) > 0$ such that the solution $\psi(t, x) \in C([0, \infty); H^{1/2}(\mathbb{R}^2))$ to integral equation (3) satisfies the estimate

$$
|\langle D_m \psi(t), \psi(t) \rangle_{L^2}| \leq e^{C t}.
$$

(7)

We will use the following notations:

- $f(t) \lesssim g(t)$ means that there exists a constant $C > 0$, possibly depending on some fixed values but independent of $t$, such that $f(t) \leq C g(t)$;
- $H^s := H^s(\mathbb{R}^2, \mathbb{C}^2)$ with $s > 0$. In the same way $L^p := L^p(\mathbb{R}^2, \mathbb{C}^2)$;
- Throughout the equations, we use $C$ for positive constants coming from various known inequalities. With abuse of notation, $C$ can change.
- We define the Fourier transform and the Anti-Fourier transform in the usual way: $\hat{\psi}(\xi) = \int e^{-ix \cdot \xi} \psi(x) dx$ and $\mathcal{F}^{-1}(\psi(x)) = \frac{1}{(2\pi)^d} \int e^{ix \cdot \xi} \hat{\psi}(\xi) d\xi$.
- Let $\hat{\rho}(\xi)$ be a radial, positive, Schwartz function, equal to 1 in $1 \leq |\xi| \leq 2$, to 0 for $|\xi| \leq 1 - \frac{1}{4}$ and $|\xi| \geq 2 + \frac{1}{4}$ and such that $\sum_{j \in \mathbb{Z}} \hat{\rho}(2^{-j} \xi) = 1$. We define $\hat{\rho}_{(0)}(\xi) = \sum_{j \leq 0} \hat{\rho}(2^{-j} \xi)$ Let $\psi$ be a Schwartz function. We define $\psi_{(j)} = \mathcal{F}^{-1}(\hat{\rho}(2^{-j} \xi) \hat{\psi}(\xi))$ and
\[ \psi_{(0)} = \mathcal{F}^{-1}(\hat{\rho}_{(0)}(\xi)\hat{\psi}(\xi)) \] The Besov space \( B^s_{p,q} \) is defined as the semi-normed space of functions such that
\[ \| \psi \|_{B^s_{p,q}} = \| \psi_{(0)} \|_{L^p} + \left( \sum_{j=1}^{\infty} (2^j s \| \psi_{(j)} \|_{L^p})^q \right)^{\frac{1}{q}} < \infty, \]
following substantially the definition found in [Grafakos (2014)].

2. Local and Global Existence

**Theorem 2.1** (Local Existence). For any \( s > 0 \) and any \( \psi_0 \in H^s(\mathbb{R}^2) \), there exists a time \( T = T(\| \psi_0 \|_{H^s}) > 0 \) and a unique solution \( \psi \in C([0, T); H^s(\mathbb{R}^2)) \) to the equation (3).

**Proof.** Let \( R := \| \psi_0 \|_{H^s} \) and take \( b = 1 \). We use a contraction principle in the space \( X_T = \{ \psi \in L^\infty([0, T]; H^s(\mathbb{R}^2)); \| \psi \|_{L^\infty((0, T), H^s(\mathbb{R}^2))} \leq 2R \} \) equipped with the distance
\[ d(u,v) = \| u - v \|_{L^\infty((0, T), H^s(\mathbb{R}^2))}, \]
with \( T = T(R) \) to be chosen later. Observe that \( (X_T, d) \) is a complete metric space. We define the map
\[ (8) \quad S(\psi) = e^{-itD_m} \psi_0 + i \int_0^t e^{-i(t-\tau)D_m} ((1 - \Delta)^{-1} |\psi(\tau)|^2) \gamma_0 \psi(\tau) d\tau, \]
and we are going to prove that \( S \) is a contraction in the space \( X_T \). Indeed, taking an element \( \psi \in X_T \), one can see that
\[ (9) \quad \| S(\psi)(t) \|_{H^s(\mathbb{R}^2)} \leq \| \psi_0 \|_{H^s(\mathbb{R}^2)} + \int_0^t \| (1 - \Delta)^{s/2} \left[ ((1 - \Delta)^{-1} |\psi(\tau)|^2) \gamma_0 \psi(\tau) \right] \|_{L^2(\mathbb{R}^2)} d\tau \]
\[ \leq \| \psi_0 \|_{H^s(\mathbb{R}^2)} + C \int_0^t \| (1 - \Delta)^{-1+s/2} |\psi(\tau)|^2 \|_{L^2} \| \psi(\tau) \|_{L^\infty(\mathbb{R}^2)} \| \Delta \|_{L^\infty(\mathbb{R}^2)} \| \gamma_0 \psi(\tau) \|_{L^2(\mathbb{R}^2)} d\tau \]
\[ \leq \| \psi_0 \|_{H^s(\mathbb{R}^2)} + C \int_0^t \| \psi(\tau) \|^2_{L^2} \log \left( 2 + \frac{\| \psi(\tau) \|^2_{H^s}}{\| \psi(\tau) \|^2_{L^2(\mathbb{R}^2)}} \right) \| \psi(\tau) \|_{H^s(\mathbb{R}^2)} d\tau, \]
where we used (3), the Sobolev’s embedding \( \| \psi \|_{L^1(\mathbb{R}^2)} \leq C \| \psi \|_{H^s(\mathbb{R}^2)} \) and that \( \| \psi \|^2_{B^1_{1,\infty}} \leq C \| \psi \|^2_{B^2_{1,2}} \). Indeed one has that, for any \( \phi, \psi \in H^s \),
\[ (10) \quad \sup_{j \geq 0} 2^{js} \| (\phi \psi)(j) \|_{L^1} \leq \sup_{j \geq 0} 2^{js} \sum_{m \leq j} \| \phi_{(m)} \psi_{(j-m)} \|_{L^1} \leq \sup_{j \geq 0} \sum_{m \leq j} 2^{ms} \| \phi_{(m)} \|_{L^2} 2^{j-m} \| \psi_{(j-m)} \|_{L^2} \]
\[ \leq \| \phi \|_{B^2_{1,2}} \| \psi \|_{B^2_{1,2}}, \]
It follows that
\[ (11) \quad \| S(\psi) \|_{L^\infty(0, T), H^s} \leq R + CT \| \psi(\tau) \|^2_{L^\infty(0, T), L^2} \| \psi(\tau) \|^2_{L^\infty(0, T), H^s} \log \left( 2 + \sup_{\tau \in (0, T)} \left( \frac{\| \psi(\tau) \|^2_{H^s}}{\| \psi(\tau) \|^2_{L^2}} \right) \right) \]
\[ \leq R + C_R T, \]
where \( C_R = C \| \psi(\tau) \|^2_{L^\infty(0, T), L^2} \| \psi(\tau) \|^2_{L^\infty(0, T), H^s} \log \left( 2 + \sup_{\tau \in (0, T)} \left( \frac{\| \psi(\tau) \|^2_{H^s}}{\| \psi(\tau) \|^2_{L^2}} \right) \right) \) which is finite. Indeed \( \psi \in X_T \) implies that \( \| \psi(\tau) \|^2_{L^\infty(0, T), L^2} \in [0, 2R] \), and \( \lim_{M \to 0} M \log(2 + \frac{1}{M}) = 0 \). A posteriori, once the local existence is established and the conservation of the \( L^2 \) norm is proved, we have that \( C_R \leq C \| \psi_0 \|^2_{L^2} R \log(2 + \frac{R}{\| \psi_0 \|^2_{L^2}}) \). In any case, we choose \( T \leq \frac{R}{C_R} \) so that \( S : X_T \to X_T \).
Now we show that (8) is a contraction in \( X_T \). Note that for any \( \psi, \phi \in X_T \), one has
\[
\| \psi - \phi \|_{L^\infty(0,T),H^s} = \\
\left\| \int_0^t e^{i(t-\tau)D_m}((1-\Delta)^{-1}|\psi|^2)\gamma_0\psi(\tau) - ((1-\Delta)^{-1}|\phi|^2)\gamma_0\phi(\tau)d\tau \right\|_{L^\infty(0,T),H^s} \\
\leq CT \left( \|((1-\Delta)^{-1}(|\psi|^2 + |\phi|^2))(\psi - \phi)\|_{L^\infty(0,T),H^s} + \right) \\
+ CT \left( \|((1-\Delta)^{-1}(|\psi|^2 - |\phi|^2))(\psi + \phi)\|_{L^\infty(0,T),H^s} \right).
\]
From the estimate (3) it follows that
\[
\|((1-\Delta)^{-1}(|\psi|^2 + |\phi|^2))(\psi - \phi)\|_{L^\infty(0,T),H^s} \\
\leq C \|\psi - \phi\|_{L^\infty(0,T),H^s} \sup_{\tau \in (0,T)} \left( \|\psi\|^2 + \|\phi\|^2 \right) \log \left( 2 + \frac{\|\psi\|^2 + \|\phi\|^2}{\|\psi\|^2 + \|\phi\|^2} \right) \\
\leq C_R \|\psi - \phi\|_{L^\infty(0,T),H^s},
\]
and
\[
\|((1-\Delta)^{-1}(|\psi|^2 - |\phi|^2))(\psi + \phi)\|_{L^\infty(0,T),H^s} \\
\leq C \|\psi + \phi\|_{L^\infty(0,T),H^s} \sup_{\tau \in (0,T)} \left( \|\psi\|^2 - \|\phi\|^2 \right) \log \left( 2 + \frac{\|\psi\|^2 - \|\phi\|^2}{\|\psi\|^2 - \|\phi\|^2} \right) \\
\leq CR \|\psi + \phi\|_{L^\infty(0,T),H^s} \sup_{\tau \in (0,T)} \left( \|\psi\|^2 - \|\phi\|^2 \right) \log \left( 2 + \frac{\|\psi\|^2 - \|\phi\|^2}{\|\psi\|^2 - \|\phi\|^2} \right) \\
\leq CR \|\psi + \phi\|_{L^\infty(0,T),H^s} \sup_{\tau \in (0,T)} \left( \|\psi\|^2 - \|\phi\|^2 \right) \log \left( 2 + \frac{4R}{\|\psi\|^2 - \|\phi\|^2} \right) \\
\leq C_R \|\psi - \phi\|_{L^\infty(0,T),H^s},
\]
where we used (10) to have
\[
\|\psi\|^2 - \|\phi\|^2 \leq \|\psi - \phi\|^2 + \gamma_0(\psi - \phi) \leq 4R \|\psi - \phi\|_{H^s}
\]
and that \( \|\psi\|^2 - \|\phi\|^2 \leq C_R \) since \( \psi \neq \phi \). We conclude that (8) is a contraction in \( X_T \). The conservation of the \( L^2 \) norm is shown in the appendix (see Lemma 5.1). \( \square \)

**Theorem 2.2** (Global Existence). For any \( s > 0 \) and any \( \psi_0 \in H^s(\mathbb{R}^2) \), there exists a unique solution \( \psi \in C([0, +\infty); H^s(\mathbb{R}^2)) \) to the equation (3).

**Proof.** Fix a \( s > 0 \) let \( \psi_0 \in H^s \). From the integral form (3) and (9), one can easily derive that for any \( t \in [0, T) \), where \( T > 0 \) is given by (2.1)
\[
\frac{d}{dt} \|\psi(t)\|_{H^s} \leq C \log (2 + \|\psi(t)\|_{H^s}) \|\psi(t)\|_{H^s}.
\]
Since one has
\[
\frac{d}{dt} \|\psi(t)\|_{H^s} \leq \left\{ \begin{array}{ll}
C \log(4) \|\psi(t)\|_{H^s} & \text{if } \|\psi(t)\|_{H^s} \leq 2 \\
C \log(2 \|\psi(t)\|_{H^s}) \|\psi(t)\|_{H^s} & \text{if } \|\psi(t)\|_{H^s} \leq 2C \log(\|\psi\|_{H^s}) \|\psi(t)\|_{H^s} \text{ if } \|\psi(t)\|_{H^s} \geq 2
\end{array} \right.
\]
it follows that
\[
\|\psi(t)\|_{H^s} \leq \left\{ \begin{array}{ll}
C_1 \log(2) \|\psi(t)\|_{H^s} & \text{if } \|\psi(t)\|_{H^s} \leq 2 \\
\|\psi(t)\|_{H^s} & \text{if } \|\psi(t)\|_{H^s} \geq 2
\end{array} \right.
\]
where \( C_1 \) depends on \( \|\psi_0\|_{H^s} \) and \( C_2 \) depends on \( C \). \( \square \)
In this section we prove

**Theorem 5.** For any \( b > 0 \) and \( s > 0 \) there exists a constant \( C = C(b, s) > 0 \) such that, for any \( \psi \in B_{t,\infty}^s(\mathbb{R}^2) \) the following inequality is true

\[
\| (b - \Delta)^{-1}\psi \|_{L^\infty} + \| (b - \Delta)^{-1}(1 - \Delta)^{s/2}\psi \|_{L^2} \leq C \| \psi \|_{L^1(\mathbb{R}^2)} \ln \left( 2 + \frac{\| \psi \|_{B_{t,\infty}^1}}{\| \psi \|_{L^1}} \right).
\]

Recall the definition of \( \psi(j) \) given in the introduction. We will use the following lemmas (see e.g. [Bergh et al. 1976])

**Lemma 3.1.** For any \( j \in \mathbb{Z} \) and any \( q, r \) such that \( 1 \leq q < r \leq \infty \), there exists a constant \( C > 0 \) such that

\[
\| \psi(j) \|_{L^r} \lesssim 2^{2j\left(\frac{1}{r} - \frac{1}{q}\right)} \| \psi(j) \|_{L^q}.
\]

**Lemma 3.2.** Assume \( \psi \) to be a Schwartz function and assume \( \psi(j) \in L^p \) for some \( p \in [1, \infty] \). Then, for any \( b > 0, s \in \mathbb{R} \) and \( j \geq 1 \), there exists a constant \( C > 0 \) such that

\[
\| (b - \Delta)^{-\frac{1}{2}}\psi(j) \|_{L^p} \leq C 2^{2j} \| \psi(j) \|_{L^p}.
\]

**Proof.** Let \( \psi \in B_{t,\infty}^1(\mathbb{R}^2) \) and observe that \( \psi \in L^1 \). Let \( M \in \mathbb{R} \) a constant such that \( sM = \ln \left( 2 + \frac{\| \psi \|_{L^\infty(\mathbb{R}^2)}}{\| \psi \|_{L^1(\mathbb{R}^2)}} \right) \), and let \([M]\) be the integer part of \( M \). Observe that \( ((b - \Delta)^{-1}\psi)(j) = (b - \Delta)^{-1}\psi(j) \) and so

\[
\| (b - \Delta)^{-1}\psi \|_{L^\infty} = \left\| (b - \Delta)^{-1} \left( \sum_{j \in \mathbb{Z}} \psi(j) \right) \right\|_{L^\infty} \lesssim \| (b - \Delta)^{-1}\psi(0) \|_{L^\infty} + \sum_{j > 0} \| (b - \Delta)^{-1}\psi(j) \|_{L^\infty}
\]

From the two lemmas, one can obtain that

\[
\sum_{j > 0} \| (b - \Delta)^{-1}\psi(j) \|_{L^\infty} \lesssim \sum_{j > 0} 2^{-2j} \| \psi(j) \|_{L^\infty} \lesssim \sum_{j > 0} \| \psi(j) \|_{L^1} = \sum_{0 \leq j \leq [M]} \| \psi(j) \|_{L^1} + \sum_{j > [M] + 1} \| \psi(j) \|_{L^1} \lesssim [M] \| \psi \|_{L^1} + \sum_{j \geq [M] + 1} 2^{-js} 2^j \| \psi(j) \|_{L^1} \lesssim [M] \| \psi \|_{L^1} + 2^{-s([M] + 1)} \| \psi \|_{B_{t,\infty}^1},
\]

Moreover, since \( \| \mathcal{F}^{-1}(\rho(\hat{\psi})) \|_{L^\infty} \leq C \), one has

\[
\| (b - \Delta)^{-1}\rho(\hat{\psi}) \|_{L^\infty} \leq C \| \mathcal{F}^{-1}(\rho(\hat{\psi})) \|_{L^\infty} \| (b - \Delta)^{-1}\psi \|_{L^1} \leq C \| \psi \|_{L^1}.
\]

Moreover, if \( b = 1 \), then

\[
\| (1 - \Delta)^{-\frac{1}{2}}\psi(j) \|_{L^\infty} \lesssim \| (1 - \Delta)^{-\frac{1}{2}}\psi(0) \|_{L^\infty} + \sum_{j > 0} \| (1 - \Delta)^{-\frac{1}{2}}\psi(j) \|_{L^\infty}
\]

and using the lemmas before, one has

\[
\sum_{j > 0} \| (1 - \Delta)^{-1+j/2}\psi(j) \|_{L^\infty} \lesssim \sum_{0 \leq j \leq [M]} 2^{(2-2s)j} \| \psi(j) \|_{L^\infty} + \sum_{j > [M]} 2^{-js} 2^{2j} \| \psi(j) \|_{L^1} \lesssim [M] \| \psi \|_{L^1} + 2^{-s([M] + 1)} \| \psi \|_{B_{t,\infty}^1},
\]

and, as before

\[
\| (1 - \Delta)^{-\frac{1}{2}}\psi(0) \|_{L^\infty} \leq C \| \mathcal{F}^{-1}(\rho(\hat{\psi})) \|_{L^\infty} \| \psi \|_{L^1} \leq C \| \psi \|_{L^1}.
\]

The result follows easily from the choice of \( M \). \( \square \)
4. Exponential Bound of the Kinetic Energy

This section is dedicated to find a result for the growth rate of the kinetic energy of a solution to the equation (2) with initial data in $H^\frac{1}{2}$. Note that we have already proved a super exponential growth of the norm for any $s > 0$. In particular, if $\psi \in C([0, \infty); H^\frac{1}{2}(\mathbb{R}^2))$, then

$$\|\psi(t)\|_{H^\frac{1}{2}(\mathbb{R}^2)} \lesssim e^{C_1 e^{C_2 t}}.$$  

As soon as $\psi(t) \in H^\frac{1}{2}$, we gain the total energy conservation, that is, for any $t \in [0, \infty)$,

$$E(\psi_0) = E(\psi(t)) := \frac{1}{2} \langle Dm \psi(t), \psi(t) \rangle_{L^2(\mathbb{R}^2)} - \frac{1}{4} \| (1 - \Delta)^{-\frac{1}{2}} |\psi(t)|^2 \|^2_{L^2}.$$  

So there exists a constant $C > 0$ such that

$$\langle Dm \psi(t), \psi(t) \rangle_{L^2} \leq C \| (1 - \Delta)^{-\frac{1}{2}} |\psi(t)|^2 \|^2_{L^2}.$$  

**Lemma 4.1.** For any $\varepsilon > 0$, let $p = \frac{2 + 2\varepsilon}{1 + 3\varepsilon} \in (\frac{2}{3}, 2)$. Then the following inequality is true

$$\| (1 - \Delta)^{-\frac{1}{2}} |\psi(t)|^2 \|_{L^2} \lesssim \left( \frac{2\varepsilon}{\varepsilon} \right)^{\frac{1}{p}} \| \psi(t) \|_{H^\frac{1}{2}}.$$  

**Proof.** Let $G$ be the Kernel of the operator $(1 - \Delta)^{-\frac{1}{2}}$. Then, from the Young inequality, for any $\varepsilon > 0$, it follows that

$$\| (1 - \Delta)^{-\frac{1}{2}} |\psi(t)|^2 \|_{L^2} = \| G * |\psi(t)|^2 \|_{L^2} \lesssim \| G \|_{L^p} \| |\psi(t)|^2 \|_{L^{1+p}} = \| G \|_{L^p} \| \psi(t) \|_{L^{2+2\varepsilon}},$$

where $p = \frac{2 + 2\varepsilon}{1 + 3\varepsilon} < 2$. It is known (see e.g. [Grafakos(2014)]) that the kernel of the Bessel operator $(1 - \Delta)^{-\frac{1}{2}}$ can be estimated as

$$G(x) \leq C e^{-\frac{|x|^2}{4(T^2 + 1)}} + C |x|^{-1} 1_{|x| \leq 2},$$

and so, for any $\varepsilon > 0$, one has

$$\| G(x) \|_{L^p}^p \lesssim \int_0^\infty e^{-\frac{2 + 2\varepsilon}{1 + 3\varepsilon} r} dr + \int_0^2 r^{-\frac{2 + 2\varepsilon}{1 + 3\varepsilon} + 1} dr \lesssim \left( \frac{2 + 2\varepsilon}{1 + 3\varepsilon} + 1 \right) \left( \frac{1 + \varepsilon}{1 + 3\varepsilon} \right)^{-2} e^{-\frac{2 + 2\varepsilon}{1 + 3\varepsilon} + \frac{2\varepsilon}{\varepsilon}} \lesssim \frac{2\varepsilon}{\varepsilon},$$

and by interpolation and the conservation of the $L^2$ norm, we get that

$$\| \psi(t) \|_{L^{2+2\varepsilon}} \lesssim \| \psi(t) \|_{L^2} \| \psi(t) \|_{L^\infty} \lesssim \| \psi(t) \|_{H^\frac{1}{2}}.$$  

Thus, for any $\varepsilon \in (0, 1)$, it follows that

$$\| (1 - \Delta)^{-\frac{1}{2}} |\psi(t)|^2 \|_{L^2} \lesssim \left( \frac{2\varepsilon}{\varepsilon} \right)^{\frac{1}{p}} \| \psi(t) \|_{H^\frac{1}{2}}.$$  

For any $T > 0$ and any $t \in [0, T]$, from (16), (17) and (15), one has

$$\langle Dm \psi(t), \psi(t) \rangle_{L^2} \lesssim 2^{\frac{2\varepsilon}{p}} \varepsilon^{-\frac{2}{p}} \| \psi(t) \|_{H^\frac{1}{2}}^{\frac{8\varepsilon}{p}} \lesssim 2^{\frac{2\varepsilon}{p}} \varepsilon^{-\frac{2}{p}} \varepsilon^{\frac{8\varepsilon}{1 + 3\varepsilon}} \lesssim 2^{\frac{2\varepsilon}{p}} \varepsilon^{-\frac{2}{p}} \varepsilon^{\frac{8\varepsilon}{1 + 3\varepsilon}} C_1 e^{C_2 T},$$

and since for $\varepsilon > 0$, $\frac{8\varepsilon}{1 + 3\varepsilon} = \frac{2}{p} \in (1, 3)$, it easily follows that there exists a constant $C > 0$, independent of $\varepsilon$, such that

$$\sup_{t \in [0, T]} |\langle Dm \psi(t), \psi(t) \rangle_{L^2}| \leq C e^{\varepsilon^{\frac{2}{p}} \varepsilon^{\frac{8\varepsilon}{1 + 3\varepsilon}} C_1 e^{C_2 T}}.$$
We choose $\varepsilon = \frac{1}{C_1} e^{-C_2 T}$ with $T$ sufficiently large so that $\varepsilon \in (0, 1)$ and we get
\[
\sup_{t \in [0,T]} |\langle D_m \psi(t), \psi(t) \rangle| \leq C e^{\frac{1}{C_1} e^{-C_2 T}} \left( C_1 e^{-C_2 T} \right)^{-\frac{p}{1+p}} e^{\frac{8}{1+p}} \leq e^{C_2 \frac{8}{1+p} T},
\]
where $C_2$ depends on $\|\psi_0\|_{H^\frac{1}{2}}$ and not on $T$. Since $p < 2$ we get (7).

**Remark 4.1.** In [Tesfahun(2018)], it was proved that the solution to (3) scatters for every $s > 0$, whenever the $H^s$ norm of the initial datum is small enough. In particular, as soon as the $\psi_0 \in H^\frac{1}{2}$ and there exists $\sigma \in (0, \frac{1}{2}]$, such that $\psi$ scatters in $H^\sigma$ (for example when $\|\psi_0\|_{H^\sigma}$ is small enough), one can also obtain, by Sobolev’s embedding, the following estimate
\[
\|(1 - \Delta)^{-\frac{1}{2}} |\psi(t)|^2\|_{L^2} \lesssim \|G\|_{L^p} \|\psi(t)\|^2_{L^2} \lesssim C_\sigma \|\psi(t)\|^2_{H^\sigma},
\]
where we have chosen $\varepsilon = \frac{\sigma}{1-\sigma}$. This means that, eventually, in this case, the exponential growth estimate (19) is not sharp and the kinetic energy will actually be bounded by a constant for all the times $t \in [0, \infty)$.

5. Appendix: Conservation of Mass

**Lemma 5.1.** If $\psi \in C([0,T); L^2(\mathbb{R}^2))$ is a solution to the equation (3) with initial datum $\psi_0$ then for any $t \in [0, T)$ we have
\[
\|\psi(t)\|_{L^2} = \|\psi_0\|_{L^2}.
\]

**Proof.** We follow the idea from [Ozawa(2006)]. For the purpose we rewrite (3) in the form
\[
e^{itD_m} \psi(t) = \psi_0 + i \int_0^t e^{i s D_m} F(\psi)(s) ds,
\]
where

\[
F(\psi)(s) = (V * (\gamma^0 \psi(s), \psi(s))) \gamma^0 \psi(s),
\]

so taking the square in $L^2$, we find
\[
\|\psi(t)\|^2_{L^2} = \|\psi_0\|^2_{L^2} + \int_0^t e^{isD_m} F(\psi)(s) ds \left( \int_0^t e^{isD_m} F(\psi)(s) ds \right) \quad + \quad 2 \text{Im} \int_0^t \langle \psi_0, \int_0^s e^{isD_m} F(\psi)(s) ds \rangle_{L^2}.
\]

For any function $g(s) \in C([0, T]; H)$ with $H$ a Hilbert space we can use the relation
\[
\left\| \int_0^t g(s) ds \right\|^2_{L^2} = 2 \text{Re} \int_0^t \left( \int_{t>s} \langle g(s), g(s') \rangle_H ds' ds \right) = 2 \text{Re} \int_0^t \langle g(s), \int_0^s g(s') ds' \rangle_H ds.
\]
Then we can write
\[
\left\| \int_0^t e^{isD_m} F(\psi)(s) ds \right\|^2_{L^2} = 2 \text{Re} \int_0^t \langle e^{isD_m} F(\psi)(s), e^{isD_m} F(\psi)(s') ds' \rangle_{L^2} ds
\]

and we are in position to use the integral equation
\[
\int_0^s e^{-i(s-s')D_m} F(\psi)(s') ds' = -i\psi(s) + ie^{-isD_m} \psi_0
\]
so we have
\[
\left\| \int_0^t e^{isD_m} F(\psi)(s) ds \right\|^2_{L^2} = 2 \text{Re} \int_0^t \langle F(\psi)(s), -i\psi(s) \rangle_{L^2} ds + 2 \text{Re} \int_0^t \langle F(\psi)(s), ie^{-isD_m} \psi_0 \rangle_{L^2} ds.
\]
Now we can take advantage of the fact that \( \langle F(\psi(s)), \psi(s) \rangle_{L^2} \) is purely real and hence
\[
\left\| \int_0^t e^{isD_m} F(\psi(s)) \, ds \right\|_{L^2}^2 = 2 \text{Re} \int_0^t \langle F(\psi(s)), ie^{-isD_m} f \rangle_{L^2} \, ds =
\]
\[
= 2 \text{Im} \int_0^t \langle e^{-isD_m} F(\psi(s)), f \rangle_{L^2} \, ds = -2 \text{Im} \int_0^t \langle f, e^{-isD_m} F(\psi(s)) \rangle_{L^2} \, ds
\]
and from (22) we arrive at the desired identity (20). □

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