Global geometrised Rankin-Selberg method for GL(n)

Sergey Lysenko

Abstract
We propose a geometric interpretation of the classical Rankin-Selberg method for GL(n) in the framework of the geometric Langlands program. We show that the geometric Langlands conjecture for an irreducible unramified local system $E$ of rank $n$ on a curve implies the existence of automorphic sheaves corresponding to the universal deformation of $E$. Then we calculate the ‘scalar product’ of two automorphic sheaves attached to this universal deformation.

Introduction

0.1. In this paper, which is a continuation of [14], we propose a geometric interpretation of the global Rankin-Selberg method for GL(n) in the framework of the geometric Langlands program. The corresponding classical result computes the scalar product of two cuspidal nonramified automorphic forms on GL(n) over a function field.

Our first motivation is the following result of G. Laumon ([11]) and M. Rothstein ([16]) in the case of GL(1). Let $X$ be a smooth, projective, connected curve of genus $g \geq 1$ over $\mathbb{C}$. Let $M'$ be the Picard scheme classifying invertible $\mathcal{O}_X$-modules $L$ of degree zero. Denote by $M$ the coarse moduli space of invertible $\mathcal{O}_X$-modules $L$ with connection $\nabla : L \to L \otimes_{\mathcal{O}_X} \Omega_X$. This is an abelian group scheme over $\mathbb{C}$ (for the tensor product), which has a natural structure of a $H^0(X, \Omega_X)$-torsor over $M'$. In [11, 16] a certain invertible $\mathcal{O}_{M \times M'}$-module $\text{Aut}$ with connection (relative to $M$) is considered as a kernel of two Fourier transforms $F : D^b_{\text{qcoh}}(D_{M'}) \to D^b_{\text{qcoh}}(\mathcal{O}_M)$ and $F' : D^b_{\text{qcoh}}(\mathcal{O}_M) \to D^b_{\text{qcoh}}(D_{M'})$. The theorem of G. Laumon and M. Rothstein claims that these functors are quasi-inverse to each other.

This result can be obtained as a formal consequence of two orthogonality relations. One of them states that the complex

$$R(\text{pr}_{12})_* (\text{pr}_{13}^* \text{Aut} \otimes \text{pr}_{23}^* \text{Aut})$$

is canonically isomorphic to $\Delta_* \mathcal{O}_M$ in $D^b_{\text{qcoh}}(\mathcal{O}_{M \times M})$ (up to a shift and a sign), where $\text{pr}_{13}, \text{pr}_{23} : M \times M \times M' \to M \times M'$ and $\text{pr}_{12} : M \times M \times M' \to M \times M$ are the projections, $\Delta : M \to M \times M$ is the diagonal, and the functor $R(\text{pr}_{12})_*$ is understood in the $D$-modules sense.

0.2. An $\ell$-adic analogue of this orthogonality relation is the following. Let $X$ be a smooth, projective, connected curve of genus $g \geq 1$ over an algebraically closed field $k$. Fix a prime $\ell$ invertible in $k$ and an algebraic closure $\overline{\mathbb{Q}}_\ell$ of $\mathbb{Q}_\ell$. Let $E_0$ be a smooth $\overline{\mathbb{Q}}_\ell$-sheaf on $X$ of rank 1.
The moduli space of \(\ell\)-adic local systems on \(X\) is not known to exist. However, we can consider the deformations of \(E_0\) over \(\mathbb{Q}_\ell\) (cf. Sect. 3.1). The local system \(E_0\) admits a universal deformation \(E\) (cf. Proposition 3). Let \(\text{Spf}(R)\) be the base of this deformation. In fact, \(R\) is isomorphic to the ring of formal power series over \(\mathbb{Q}_\ell\) of dimension 2\(g\).

For a positive integer \(d\) we have the \(R\)-sheaf \(E^{(d)}\) on the symmetric power \(X^{(d)}\) of \(X\) (cf. Sect. 1.5). Let \(\text{Pic}^d X\) denote the Picard scheme of \(X\) parametrizing the isomorphism classes of invertible \(\mathcal{O}_X\)-modules of degree \(d\). According to the geometric abelian class field theory, \(E^{(d)}\) descends to a smooth \(R\)-sheaf \(E^d\) of rank 1 on \(\text{Pic}^d X\). Denote by \(E^d_1, E^d_2\) the two liftings of \(E^d\) to \(\text{Spf}(R \otimes \mathbb{Q}_\ell R)\). Essentially, we show that \(\text{there is a canonical isomorphism of } R \otimes R\)-modules \(H^2g(\text{Pic}^d X, \mathcal{H}om(E^d_1, E^d_2)) \cong R(-g),\)

where the \(R \otimes R\)-module structure on \(R\) is given by the diagonal map \(R \otimes R \to R\). Besides, \(H^i(\text{Pic}^d X, \mathcal{H}om(E^d_1, E^d_2)) = 0\) for \(i \neq 2g\). Actually, a bit different statement is proved (cf. the discussion at the end of Sect. 0.4).

This is the particular case \(n = 1\) of our Main Global Theorem (cf. Sect. 4.1), which is an analogue of this orthogonality relation for \(\text{GL}(n)\).

Already in the case of \(\text{GL}(1)\) we observe the important role of deformations. Applying the base change theorem for the above result, we get the scalar square of \(E^d_0\), namely

\[ \text{R} \Gamma(\text{Pic}^d X, \mathcal{E}nd(E^d_0, E^d_0)) \cong \text{R} \otimes \text{R} \otimes \mathbb{Q}_\ell(-g)[-2g] \]

As is easy to see, this complex has cohomology groups in all degrees \(0, 1, \ldots, 2g\). For \(\text{GL}(n)\) the answer will also be simpler when considered as an object on the moduli of parameters.

0.3. Let \(X\) be a smooth, projective, geometrically connected curve over \(\mathbb{F}_q\). Let \(\ell\) be a prime invertible in \(\mathbb{F}_q\). According to the Langlands correspondence for \(\text{GL}(n)\) over function fields (proved in full generality by L. Lafforgue), to any smooth geometrically irreducible \(\mathbb{Q}_\ell\)-sheaf \(E\) on \(X\) is associated a (unique up to a multiple) cuspidal automorphic form \(\varphi_E : \text{Bun}_n(\mathbb{F}_q) \to \mathbb{Q}_\ell\), which is a Hecke eigenvector with respect to \(E\). The function \(\varphi_E\) is defined on the set \(\text{Bun}_n(\mathbb{F}_q)\) of isomorphism classes of rank \(n\) vector bundles on \(X\).

The global Rankin-Selberg for \(\text{GL}(n)\) allows to calculate for any integer \(d\) the scalar product of two (appropriately normalized) automorphic forms

\[ \sum_{L \in \text{Bun}_n^d(\mathbb{F}_q)} \frac{1}{\# \text{Aut} L} \varphi_{E_1}(L) \varphi_{E_2}(L), \]

where \(\text{Bun}_n^d(\mathbb{F}_q)\) is the set of isomorphism classes of vector bundles \(L\) on \(X\) of rank \(n\) and degree \(d + n(n - 1)(g - 1)\), and \(\# \text{Aut} L\) stands for the number of elements in \(\text{Aut} L\). More precisely, this scalar product vanishes if and only if \(E_1\) and \(E_2\) are non isomorphic. In the case \(E_1 \cong E_2 \cong E\) the answer is expressed in terms of the action of the geometric Frobenius endomorphism on \(\text{H}^1(X \otimes \mathbb{F}_q, \mathcal{E}nd E)\).
Assume that $\varphi_E$ is canonically normalized (cf. footnote 1). The local part of the classical Rankin-Selberg method for $GL(n)$ may be stated as the equality of formal series (cf. \[\text{14}\])

$$
\sum_{d \geq 0} \sum_{(\Omega^{n-1} \hookrightarrow L) \in \mathcal{M}_d(F_q)} \frac{1}{\# Aut(\Omega^{n-1} \hookrightarrow L)} \varphi_{E_1^*}(L) \varphi_{E_2}(L) t^d = q^{n^2(1-g)} L(E_1^* \otimes E_2, t)
$$

(2)

Here $\mathcal{M}_d(F_q)$ is the set of isomorphism classes of pairs $(\Omega^{n-1} \hookrightarrow L)$, where $L$ is a vector bundle on $X$ of rank $n$ and degree $d + n(n - 1)(g - 1)$, and $\Omega$ is the canonical invertible sheaf on $X$ ($\Omega^{n-1}$ is embedded in $L$ as a subsheaf, i.e., the quotient is allowed to have torsion). We have denoted by

$$L(E_1^* \otimes E_2, t) = \sum_{d \geq 0} \sum_{D \in X^{(d)}(F_q)} \text{tr}(Fr_1(E_1^* \otimes E_2)^{(d)}_D) t^d$$

the $L$-function attached to the local system $E_1^* \otimes E_2$ (here $Fr$ is the geometric Frobenius endomorphism).

The calculation of (1) is an asymptotic argument. First, rewrite (2) as

$$
\sum_{d \geq 0} \sum_{L \in Bun^d_1(F_q)} \frac{1}{\# \text{Aut} L} (q^{\dim \text{Hom}(\Omega^{n-1}, L)} - 1) \varphi_{E_1^*}(L) \varphi_{E_2}(L) t^d = q^{n^2(1-g)} L(E_1^* \otimes E_2, t)
$$

The cuspidality of $\varphi_E$ implies that if $\varphi_E(L) \neq 0$ and $\deg L$ is large enough then $\text{Ext}^1(\Omega^{n-1}, L) = 0$, and $\dim \text{Hom}(\Omega^{n-1}, L) = d - n^2(g - 1)$. To conclude, it remains to study the asymptotic behaviour of the above series when $t$ goes to $q^{-1}$, using the cohomological interpretation

$$L(E_1^* \otimes E_2, t) = \prod_{r=0}^2 \det(1 - Fr t, H^r(X \otimes \overline{\mathbb{F}}_q, E_1^* \otimes E_2))^{(-1)^{r+1}}$$

of the $L$-function.

0.4. Let $X$ be as in 0.2. Let $E_0$ be a smooth irreducible $\overline{\mathbb{Q}}_l$-sheaf on $X$ of rank $n$. Let $(E, R)$ be a universal deformation of $E_0$ (cf. Sect. 3.1). The base $R$ is, in fact, isomorphic to the ring of formal power series over $\overline{\mathbb{Q}}_l$ of dimension $2 + (2g - 2)n^2$. We assume the geometric Langlands conjecture for $GL(n)$ (Conjecture \[\text{2}\] Sect. 2.4), so that to $E$ is associated a perverse $R$-sheaf $\text{Aut}_E$ on the moduli stack $\text{Bun}_n$ of rank $n$ vector bundles on $X$, which is a geometric analogue of the automorphic form $\varphi_E$. Denote by $\text{Aut}_E^d$ the restriction of $\text{Aut}_E$ to $\text{Bun}^d_1$, the connected component of $\text{Bun}_n$ classifying vector bundles on $X$ of rank $n$ and degree $d + n(n - 1)(g - 1)$. One can calculate the cohomology

$$\text{R}\Gamma_c(\text{Bun}^d_1, \text{pr}_1^* \text{Aut}_E^d \otimes_{R \otimes_{R} R} \text{pr}_2^* \text{Aut}_E^d),$$

\[\text{1}\]We normalize $\text{Aut}_E$ as in Conjecture \[\text{2}\], this also gives a normalization of $\varphi_E$ as the function trace of Frobenius of $\text{Aut}_E$. 

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which is a geometric analogue of \((\square)\). (Here \(\text{pr}_1 : \text{Spf}(R \hat{\otimes} R) \to \text{Spf} R\) are the two projections.) But the answer turns out to be ‘bad’: this complex has nontrivial cohomology groups in infinitely many degrees (bounded from above). To get the desired answer, we modify the problem as follows.

Scalar automorphisms of vector bundles provide an action of \(G_m\) on \(\text{Bun}_n\) by 2-automorphisms of stacks. We introduce the stack \(\text{Bun}_n\) (cf. Sect. 2.5), the quotient of \(\text{Bun}_n\) under this action. There exists a perverse \(R\)-sheaf \(\text{Aut}_E\) on \(\text{Bun}_n\) such that the inverse image of \(\text{Aut}_E[-1](-\frac{1}{2})\) under the projection \(\text{Bun}_n \to \text{Bun}_n\) is identified with \(\text{Aut}_E\).

Our Main Global Theorem essentially says that if Conjecture 1 is true then for any integers \(i\) and \(d\) there is a canonical isomorphism of \(R\hat{\otimes}R\)-modules

\[
H^i_c(\text{Bun}_d, \text{pr}_1^* \text{Aut}_E \otimes R \hat{\otimes} \text{pr}_2^* \text{Aut}_E) \cong \begin{cases} R, & \text{if } i = 0 \\ 0, & \text{if } i \neq 0, \end{cases}
\]

where the \(R \hat{\otimes} R\)-module structure on \(R\) is given by the diagonal map \(R \hat{\otimes} R \to R\).

Actually a bit different statement is proved. Conceptually, we should work in the derived categories \(\text{D}_b^c(\text{Y}, R)\) of complexes of \(R\)-sheaves in étale topology on \(\text{Y}\), where \(\text{Y}\) is a \(k\)-scheme of finite type and \(R\) is the ring of formal power series over \(\mathbb{Q}_\ell\). (For \(R = \mathbb{Q}_\ell\) this is the derived category of \(\ell\)-adic sheaves on \(\text{Y}\)). However, for \(\dim R > 0\) the definition of this derived category is not known. We impose an additional condition: the reduction of \(E_0\) modulo \(\ell\) has no nontrivial endomorphisms. This allows to work in another triangulated category denoted \(\text{D}_b^c(\text{Y}, \mathbb{Q}_\ell)\) (cf. Sect. 1.4), which ‘approximates’ the desired one. Our Main Global Theorem is an isomorphism in such triangulated category.

0.5. The paper is organized as follows. In Sect. 1 we partially establish some formalism of \(R\)-sheaves on schemes. We adopt the point of view that an appropriate formalism of \(R\)-sheaves holds for algebraic stacks locally of finite type over \(k\). In Sect. 2 we formulate the geometric Langlands conjecture relative to \(R\) (Conjecture \((\square)\)), and essentially show that if this conjecture is true over \(\mathbb{Q}_\ell\) then it is also true over the rings of formal power series over \(\mathbb{Q}_\ell\) (actually a bit different statement is proved, cf. Proposition 2).

In Sect. 3.1-3.3 we study the deformations of local systems on \(X\). In Sect. 3.4 we calculate the cohomology of some natural sheaves arising from the universal deformations. The proof of Main Global Theorem is given in Sect. 4.

We refer the reader to Sect. 0.2 of [14] for our conventions.

1 \(R\)-sheaves

1.1 Let \(E\) be a finite extension field of \(\mathbb{Q}_\ell\). Denote by \(\mathcal{C}_E\) the category of local Artin \(E\)-algebras with residue field \(E\) (the morphisms are local homomorphisms of \(E\)-algebras).

Let \(A \in \text{Ob}(\mathcal{C}_E)\) and \(Y\) be a \(k\)-scheme of finite type. The category \(\text{Sh}(Y, A)\) of constructable \(A\)-sheaves on \(Y\) is the category of pairs \((E, \rho)\), where \(E\) is a constructable \(E\)-sheaf on \(Y\), and \(\rho : A \to \text{End}(E)\) is an action of \(A\) on \(E\).
A constructible $A$-sheaf $(E, \rho)$ is smooth of rank $m$, if $E$ is a smooth $\mathbb{E}$-sheaf, and the fibres of $(E, \rho)$ are free $A$-modules of rank $m$.

Let $R$ be a complete local noetherian $\mathbb{E}$-algebra with residue field $\mathbb{E}$ and maximal ideal $m$. We let $\text{Sh}(Y, R)$ be the projective 2-limit of $\text{Sh}(Y, R/m^n)$. This is a category whose objects are projective systems $(F_n, \psi_n)_{n \in \mathbb{N}}$, where $F_n \in \text{Ob}(\text{Sh}(Y, R/m^n))$ and $\psi_n : F_{n+1} \otimes_{R/m^{n+1}} R/m^n \to F_n$ is an isomorphism. Morphisms are defined as the morphisms of the corresponding projective systems.

Conceptually, we should work in the derived category $D_\mathbb{C}^b(Y, R)$, where $R$ is the ring of formal power series in several variables over $\mathbb{Q}_\ell$, but its definition is not known. We ‘approximate’ it by another triangulated category defined below.

1.2 Let $A$ be a commutative ring, $S \subset A$ be a multiplicative system. We denote by $A_S$ the localization of $A$ in $S$. Given an $A$-additive category $\mathcal{A}$, denote by $\mathcal{A} \otimes_A A_S$ the category such that $\text{Ob}(\mathcal{A} \otimes_A A_S) = \text{Ob}(\mathcal{A})$, and morphisms in $\mathcal{A} \otimes_A A_S$ from $K$ to $K'$ are $\text{Hom}_A(K, K') \otimes_A A_S$. This is an $A_S$-additive category. The following statement is left to the reader (the points i) and iii) follow from [8], Proposition 3.6.1).

**Lemma 1.** 1) Let $D$ be an $A$-additive triangulated category. Then $D \otimes_A A_S$ is triangulated (its distinguished triangles are those isomorphic to images of distinguished triangles in $D$). If $F : D \to D'$ is an $A$-additive triangulated functor between $A$-additive triangulated categories then the induced functor $D \otimes_A A_S \to D' \otimes_A A_S$ is triangulated.

2) Assume, in addition, that $D$ is equipped with a $t$-structure $(D^{\leq 0}, D^{> 0})$ such that any $K \in \text{Ob}(D)$ is of finite amplitude, i.e., $D = \bigcup_i D^{\leq i} = \bigcup_i D^{> i}$. Let $A$ denote the core of this $t$-structure. Let $S$ be the full-subcategory of $D$ consisting of $K$ with $H^i(K) \otimes_A A_S = 0$ in $\mathcal{A} \otimes_A A_S$ for all $i$. Then

i) $S$ is a thick subcategory of $D$;

ii) the localization of $D$ in $S$ is canonically equivalent to $D \otimes_A A_S$;

iii) $D \otimes_A A_S$ has a $t$-structure $(D_S^{\leq 0}, D_S^{> 0})$, where $D_S^{\leq 0}$ (resp., $D_S^{> 0}$) is the essential image of those $K \in D$ that $H^i(K) \otimes_A A_S = 0$ for $i > 0$ (resp., for $i < 0$). The core of this $t$-structure is an abelian category equivalent to $A \otimes_A A_S$. □

**Remark 1.** We will use the following trivial observation. Let $c-A-mod$ denote the category of finite type $A$-modules. If $A$ is noetherian then the natural functor $(c-A-mod) \otimes_A A_S \to c-A_S-mod$ is fully faithful.

The following statement is left to the reader (point 3) follows from [2], 4.3 and 4.5.1).

**Lemma 2.** Let $\Lambda$ be a local noetherian ring with residue field $\kappa_\Lambda$.

1) If $K$ is a perfect complex of $\Lambda$-modules then it can be represented as a direct sum $K = K_1 \oplus K_2$ of two perfect complexes, where $K_1$ is acyclic, and the differential in $K_2 \otimes_\Lambda \kappa_\Lambda$ is zero.

2) If $K_1$ and $K_2$ are perfect complexes of $\Lambda$-modules such that the differential in $K_i \otimes_\Lambda \kappa_\Lambda$ is zero ($i = 1, 2$) and $f : K_1 \to K_2$ is a homotopical equivalence then $f$ is an isomorphism of complexes.
3) Let $D_{\text{par}}(\Lambda)$ be the category whose objects are perfect complexes of $\Lambda$-modules, and the morphisms are morphisms of complexes of $\Lambda$-modules modulo homotopical equivalence. Let $D_{\text{coh}}^b(\Lambda)$ be the full subcategory of the bounded derived category of complexes of $\Lambda$-modules, consisting of complexes whose cohomologies are of finite type. If $\Lambda$ is regular then the natural functor $D_{\text{par}}(\Lambda) \rightarrow D_{\text{coh}}^b(\Lambda)$ is an equivalence of categories. □

1.3 Let $\mathcal{O} \subset \mathbb{E}$ be the ring of integers of $\mathcal{O}$, $\kappa$ be the residue field of $\mathcal{O}$. Let $\bar{R}$ be a local noetherian $\mathcal{O}$-algebra with maximal ideal $\bar{m}$ and residue field $\kappa$. Assume that $\bar{R}$ is complete in the $\bar{m}$-adic topology. Let $\text{Sh}(Y, \bar{R})$ denote the category of $\bar{m}$-adic constructable $\bar{R}$-sheaves (SGA5, V, 3.1.1). We say that $F \in \text{Sh}(Y, \bar{R})$ is smooth of rank $m$ if for each $k > 0$ the sheaf $F \otimes_{\bar{R}} \bar{R}/\bar{m}^k$ is locally constant and its fibres are free $\bar{R}/\bar{m}^k$-modules of rank $m$.

Assume, in addition, that $\bar{R}$ is regular. Denote by $D_{c}^b(Y, \bar{R})$ the triangulated category defined in (4), Theorem 6.3 (and denoted $D_{c}^b(Y, \bar{R})$ in loc. cit.). So, $D_{c}^b(Y, \bar{R})$ is $\bar{R}$-additive, has the natural $t$-structure, whose core is equivalent to $\text{Sh}(Y, \bar{R})$, and the perverse $t$-structure $p_{1/2}$ obtained by the gluing procedure (4, 2.2.14). So, the usual definition of the perverse $t$-structure $p_{1/2}$ is applicable:

**Definition 1.** An object $K \in D_{c}^b(Y, \bar{R})$ lies in $pD_{c}^0(Y, \bar{R})$ (resp., in $pD_{c}^{>0}(Y, \bar{R})$) if and only if for any irreducible closed subscheme $Y' \subset Y$ there is a nonempty open subscheme $U \subset Y'$ such that $H^j(i_U^* K) = 0$ for $j > -\dim U$ (resp., $H^j(i_U^* K) = 0$ for $j < -\dim U$), where $i_U : U \hookrightarrow Y$ is the inclusion.

We let $\text{Perv}(Y, \bar{R}) = pD_{c}^0(Y, \bar{R}) \cap pD_{c}^{>0}(Y, \bar{R})$. Any object of $D_{c}^b(Y, \bar{R})$ has a finite amplitude with respect to both the natural and the perverse $t$-structure $p_{1/2}$.

The perverse $t$-structure $p_{1/2}$ on $D_{c}^b(Y, \bar{R})$ is not preserved by Verdier duality (unless $\bar{R} = \kappa$). This is already observed for $Y = \text{Spec} \, k$. We have an equivalence of triangulated categories that preserves natural $t$-structures as well as tensor products and internal Hom’s (4, 7.2)

$$D_{c}^b(\text{Spec} \, k, \bar{R}) \rightarrow D_{c}^b(\bar{R}),$$

and $D(D_{c}^0(\bar{R})) \subset D_{c}^{>0}(\bar{R})$. However, the image of $D_{c}^{>0}(\bar{R})$ under the Verdier duality functor $\mathbb{D}$ is strictly bigger than $D_{c}^0(\bar{R})$ (this image is described by 3, 6.3.2).

**Lemma 3.** If $K \in pD_{c}^0(Y, \bar{R})$ then $\mathbb{D}K \in pD_{c}^{>0}(Y, \bar{R})$.

**Proof.** Let $Y' \subset Y$ be an irreducible closed subscheme. Choose a nonempty open subscheme $U \subset Y'$ such that $i_U^* K$ is placed in usual degrees $\leq -\dim U$ and all $H^j(i_U^* K)$ are locally constant (by this we mean that $H^j(i_U^* K) \otimes_{\bar{R}} \bar{R}/\bar{m}^k$ are locally constant for all $k > 0$). Then $\mathbb{D}(i_U^* K)$ is placed in usual degrees $\geq -\dim U$. □

We have a conservative triangulated functor of extension of scalars (4, 6.3)

$$\kappa \otimes_{\bar{R}}^L : D_{c}^b(Y, \bar{R}) \rightarrow D_{c}^b(Y, \kappa)$$

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Lemma 4. 1) For $K \in \mathcal{D}_c^b(Y, \bar{R})$ we have $K \in \mathcal{D}_c^{\leq 0}(Y, \bar{R})$ if and only if $\kappa \otimes_{\bar{R}} K \in \mathcal{D}_c^{\leq 0}(Y, \kappa)$. Besides, if $\kappa \otimes_{\bar{R}} K \in \mathcal{D}_c^{\geq 0}(Y, \kappa)$ then $K \in \mathcal{D}_c^{\geq 0}(Y, \bar{R})$.

2) Let $f : Y \to Y'$ be a morphism of $k$-schemes of finite type. If $f$ is affine then $f_*$ is right exact with respect to $p_{1/2}$. If $f$ is an open immersion then $f_*$ (resp., $f_!$) is left exact (resp., right exact) with respect to $p_{1/2}$ (the functors are from $\mathcal{D}_c^b(Y, \bar{R})$ to $\mathcal{D}_c^b(Y', \bar{R})$).

Proof 1) By ([4], 6.3), the functors $\mathcal{P}erv(\mathcal{D}_c^b(Y, \bar{R}))$ commute with $\kappa \otimes_{\bar{R}}$. To conclude, apply ([4], 3.1).

2) By (loc.cit., 6.3), $f_*, f_!$ commute with $\kappa \otimes_{\bar{R}}$. It remains to use 1) and ([4], 4.1.1). □

Definition 2. We say that $K \in \mathcal{D}_c^b(Y, \bar{R})$ is a $\bar{R}$-flat perverse $\bar{R}$-sheaf on $Y$ if it satisfies one of the following equivalent conditions:

i) $K \in \mathcal{D}_c^b(Y, \bar{R})$ and $\mathbb{D}K \in \mathcal{D}_c^{\leq 0}(Y, \bar{R})$;

ii) $K \in \mathcal{P}erv(Y, \bar{R})$ and $\mathbb{D}K \in \mathcal{P}erv(Y, \bar{R})$;

iii) $\kappa \otimes_{\bar{R}} K \in \mathcal{P}erv(Y, \kappa)$

We denote by $\mathcal{P}erv_{fl}(Y, \bar{R})$ the full subcategory of $\mathcal{P}erv(Y, \bar{R})$ consisting of $\bar{R}$-flat perverse $\bar{R}$-sheaves on $Y$.

Note that the Verdier duality functor preserves $\bar{R}$-flat perverse $\bar{R}$-sheaves and induces an autoequivalence of $\mathcal{P}erv_{fl}(Y, \bar{R})$.

1.4 Given a local homomorphism of $\mathcal{O}$-algebras $\sigma : \bar{R} \to \mathcal{O}$, denote by $m_\sigma$ the kernel of the induced map $\bar{R} \otimes_{\mathcal{O}} \mathcal{E} \to \mathcal{E}$, and by $\bar{R}_\sigma$ the localization of $\bar{R} \otimes_{\mathcal{O}} \mathcal{E}$ in $m_\sigma$. By Lemma 4, we have a triangulated category $\mathcal{D}_c^b(Y, \bar{R}) \otimes_{\bar{R}} \bar{R}_\sigma$, which will also be denoted $\mathcal{D}_c^b(Y, \bar{R})_\sigma$. It is equipped with the natural and perverse $t$-structures induced by those of $\mathcal{D}_c^b(Y, \bar{R})$. So, by definition, $\mathcal{P}erv(Y, \bar{R})_\sigma$ is the essential image of $\mathcal{P}erv_{fl}(Y, \bar{R})_\sigma$.

The core of the perverse $t$-structure on $\mathcal{D}_c^b(Y, \bar{R})_\sigma$ will be denoted $\mathcal{P}erv(Y, \bar{R})_\sigma$. The category $\mathcal{P}erv(Y, \mathcal{O})_\sigma$ will also be denoted by $\mathcal{P}erv(Y, \mathcal{E})$. We also write $\mathcal{S}h(Y, \mathcal{E})$ for $\mathcal{S}h(Y, \mathcal{O}) \otimes_{\mathcal{O}} \mathcal{E}$.

By 1) of Lemma 4, the functors defined for $\mathcal{D}_c^b(Y, \bar{R})$ in 4 extend trivially to $\mathcal{D}_c^b(Y, \bar{R})_\sigma$. So, for a separated morphism $f : Y \to Y'$ of schemes of finite type over $k$ we have functors

$$f_! : \mathcal{D}_c^b(Y, \bar{R})_\sigma \to \mathcal{D}_c^b(Y', \bar{R})_\sigma$$

and

$$f^! : \mathcal{D}_c^b(Y', \bar{R})_\sigma \to \mathcal{D}_c^b(Y, \bar{R})_\sigma$$

functorial in $f$. We also have triangulated functors

$$(-) \otimes_{\bar{R}_\sigma} (-) : \mathcal{D}_c^b(Y, \bar{R})_\sigma \times \mathcal{D}_c^b(Y, \bar{R})_\sigma \to \mathcal{D}_c^b(Y, \bar{R})_\sigma$$

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and
\[ \mathcal{R} \text{Hom}(\cdot, \cdot) : D^b_c(Y, \bar{R})^\text{op} \times D^b_c(Y, \bar{R}) \to D^b_c(Y, \bar{R}) \]

The next lemma claims that the perverse $t$-structure on $D^b_c(Y, \bar{R})$ defined by Lemma 5 is, in fact, obtained by a gluing procedure as in (I, 2.2).

**Lemma 5.** $K \in \mathcal{D}^b_c(Y, \bar{R})$ (resp., $K \in \mathcal{D}^{b>0}_c(Y, \bar{R})$) if and only if for any irreducible closed subscheme $Y' \subset Y$ there is a nonempty open subscheme $U \subset Y'$ such that $\mathcal{H}^j(i_U^*K) = 0$ for $j > -\dim U$ (resp., $\mathcal{H}^j(i_U^*K) = 0$ for $j < -\dim U$), where $i_U : U \hookrightarrow Y$ is the inclusion, and the cohomologies are calculated with respect to the usual $t$-structure on $D^b_c(U, \bar{R})$.

**Proof.** The only if part is trivial. The if part follows from (I, 1.4.12). □

From Lemma 5 and 2) of Lemma 4 we immediately get the next corollary.

**Corollary 1. 1)** If $K \in \mathcal{D}^b_c(Y, \bar{R})$ then $\mathcal{D}K \in \mathcal{D}^b_c(Y, \bar{R})$.

2) Let $f : Y \to Y'$ be a morphism of $k$-schemes of finite type. If $f$ is affine then $f_*$ is right exact with respect to $p_{1/2}$. If $f$ is an open immersion then $f_*$ (resp., $f_!$) is left exact (resp., right exact) with respect to $p_{1/2}$ (the functors are from $D^b_c(Y, \bar{R})$ to $D^b_c(Y', \bar{R})$). □

Let $\bar{R} \to \bar{R}'$ be a local homomorphism of complete local noetherian regular $\mathcal{O}$-algebras with residue fields $\kappa$. It induces a conservative triangulated functor of extension of scalars (I, A.1)

\[ \bar{R}' \otimes_{\bar{R}} : D^b_c(Y, \bar{R}) \to D^b_c(Y, \bar{R}') \]

By (loc.cit., A.1.5), the functors $f_!, f_* , f^*$ commute with $(\mathcal{E} \otimes \mathcal{R}_\sigma)$. Assume that $\sigma$ is the restriction to $\bar{R}$ of a local homomorphism of $\mathcal{O}$-algebras $\sigma' : \bar{R} \to \mathcal{O}$.

Then $\bar{R}' \otimes_{\bar{R}}$ factors to give a triangulated functor

\[ D^b_c(Y, \bar{R}) \to D^b_c(Y, \bar{R}') \]

that we denote by $\bar{R}' \otimes_{\bar{R}} \sigma$. The map $\sigma$ itself yields the functor of extension of scalars

\[ \mathcal{E} \otimes_{\bar{R}} : D^b_c(Y, \bar{R}) \to D^b_c(Y, \bar{E}) \]

where, by definition, $D^b_c(Y, \bar{E})$ is the triangulated category $D^b_c(Y, \mathcal{O}) \otimes_{\mathcal{O}} \mathcal{E}$ (I, 2.2.18).

**Lemma 6.** i) The functor $\mathcal{E} \otimes_{\bar{R}} : D^b_c(Y, \bar{R}) \to D^b_c(Y, \bar{E})$ is triangulated and conservative.

ii) If for an object $M$ of $D^b_c(Y, \bar{R})$ we have $H^r(\mathcal{E} \otimes_{\bar{R}} M) = 0$ for some $r$, then $H^r(M) = 0$, where the cohomologies are calculated with respect to the usual $t$-structures.

\[ ^2 \text{The author does not know if } f' \text{ and } \mathcal{D} \text{ commute with (I).} \]
Lemma 7. \textit{Assume that $Y$ is connexe, and $y \to Y$ is a geometric point. Then the functor that sends $K$ to $K_y$ is an equivalence between the category of smooth $\bar{R}_\sigma$-sheaves of rank $m$ on $Y$ and...}
and the category of pairs \((M, \rho)\), where \(M\) is a free \(\mathbb{R}_\sigma\)-module of rank \(m\), \(\rho : \pi_1(Y, y) \to \text{Aut} M\) is a homomorphism such that there exists a finite type \(\mathbb{R}\)-submodule \(M' \subset M\) generating \(M\), invariant under \(\pi_1(Y, y)\), and such that the homomorphism \(\rho : \pi_1(Y, y) \to \text{Aut} M'\) is continuous (note that we do not require \(M'\) to be free over \(\mathbb{R}\)).

2) Let \(U \subset Y\) be an open subscheme and \(K \in D^b_c(Y, R)_\sigma\). If both \(K \otimes_{\mathbb{R}_\sigma} E\) and \((\mathbb{D}K) \otimes_{\mathbb{R}_\sigma} E\) are perverse and the Goresky-MacPherson extensions from \(U\) to \(Y\) then \(K \in \text{Perv}_f(Y, R)_\sigma\) is the Goresky-MacPherson extension from \(U\) to \(Y\).

Proof 1) Argue as in SGA5, VI, 1.2.

2) Let \(M \in D^b_c(Y, R)_\sigma\) the condition \(M \otimes_{\mathbb{R}_\sigma} E \in p D_*^{\leq 0}(Y, E)\) implies \(M \in p D_*^{\leq 0}(Y, R)_\sigma\). So, our assumption implies \(K \in \text{Perv}_f(Y, R)_\sigma\). Let \(i : S \to Y\) be a closed subscheme whose complement is \(U\). The complexes \(i^*(K \otimes_{\mathbb{R}_\sigma} E)\) and \(i^*((\mathbb{D}K) \otimes_{\mathbb{R}_\sigma} E)\) lie in \(p D_*^{\leq 0}(S, E)\). So, \(i^*K\) and \(\mathbb{D}i^!K\) lie in \(p D_*^{> 0}(S, R)_\sigma\). By Corollary [1], \(i^!K \in p D_*^{> 0}(S, R)_\sigma\). \(\square\)

1.5 Laumon’s sheaf \(L^d_E\)

Let \(\bar{R}\) be as in Sect. 1.3. Assume that \(\bar{R}\) is of characteristic zero. Let \(\bar{E}\) be a smooth \(\bar{R}\)-sheaf on \(X\). Denote by \(\text{sym} : X^d \to X(d)\) the natural map. Consider the smooth \(\bar{R}\)-sheaf \(\bar{E}^{\otimes d}\) on \(X^d\) (the tensoring is taken over \(\bar{R}\)). Set \(\bar{E}^{(d)} = (\text{sym}_!(\bar{E}^{\otimes d}))^{S_d}\). This is a direct summand of the constructible \(\bar{R}\)-sheaf \(\text{sym}_!(\bar{E}^{\otimes d})\) on \(X^d\). Since \(\text{sym}_!(\bar{E}^{\otimes d})[d]\) is a \(\bar{R}\)-flat perverse \(\bar{R}\)-sheaf, which is the Goresky-MacPherson extension of its restriction to any nonempty open subscheme, the same holds for \(\bar{E}^{(d)}[d]\).

Following [2], we associate to \(\bar{E}\) a perverse \(\bar{R}\)-sheaf \(L^d_E\) on \(\text{Sh}^d_0\) that we call Laumon’s sheaf. Denote by \(\mathcal{F}l^1\cdots^1\) (1 occurs \(d\) times) the stack of complete flags \((F_1 \subset \ldots \subset F_d)\), where \(F_i\) is a coherent torsion sheaf on \(X\) of length \(i\). The morphism \(p_0 : \mathcal{F}l^1\cdots^1 \to \text{Sh}^d_0\) that sends \((F_1 \subset \ldots \subset F_d)\) to \(F_d\) is representable and proper. The morphism \(q_0 : \mathcal{F}l^1\cdots^1 \to \text{Sh}^0_0 \times \ldots \times \text{Sh}^0_0\) that sends \((F_1 \subset \ldots \subset F_d)\) to \((F_1, F_2/F_1, \ldots, F_d/F_{d-1})\) is a generalized affine fibration. This, in particular, implies that \(\mathcal{F}l^1\cdots^1\) is smooth.

Springer’s sheaf \(\mathcal{S}pr^d_E\) is defined as \(\mathcal{S}pr^d_E = (p_0)!((q_0)^*((\text{div} \times d)^*)(\bar{E}^{\otimes d}))\). Laumon’s theorem claims that \(p_0\) is small. It follows that \(\mathcal{S}pr^d_E\) is a \(\bar{R}\)-flat perverse \(\bar{R}\)-sheaf, which is the Goresky-MacPherson extension of its restriction to any nonempty open substack. It also carries a natural action of the symmetric group \(S_d\) ([2], 3.3.1). Set \(L^d_E = (\mathcal{S}pr^d_E)^{S_d}\), where the invariants are taken in \(\text{Perv}(\text{Sh}^d_0, \bar{R})\). As a direct summand of \(\mathcal{S}pr^d_E\), the perverse \(\bar{R}\)-sheaf \(L^d_E\) is \(\bar{R}\)-flat and coincides with the Goresky-MacPherson extension of its restriction to any nonempty open substack. Besides, the formation of \(L^d_E\) commutes with extension of scalars ([3]), and the Verdier dual of \(L^d_E\) is canonically isomorphic to \(L^d_E^{\vee}\).
2 Automorphic sheaves $\text{Aut}_E$

2.1 Let $\bar{R}, E$ be as in Sect. 1.5. Assume that $\mathbb{E}$ contains the group of $p$-th roots of unity, so that we can fix a nontrivial additive character $\psi : \mathbb{F}_p \to \mathcal{O}^*$. Then the Artin-Schreier sheaf $L_\psi$ associated to $\psi$ can be viewed as a smooth $\mathcal{O}$-sheaf (or smooth $\bar{R}$-sheaf) of rank 1 on $\mathbb{A}^1$.

Let $n > 0, d \geq 0$. In Sect 2.1 and 4.1 of [14] we introduced the diagram

\[
\begin{array}{ccc}
\leq n \text{Sh}_d & \xrightarrow{\beta} & n \mathcal{Q}_d \\
\downarrow \psi & \xrightarrow{\mu} & \mathbb{A}^1 \\
\n \end{array}
\]

Definition 4. We associate to $E$ an object $n\mathcal{F}^d_{E,\psi} \in \text{Perv}_{fl}(n\mathcal{Q}_d, \bar{R})$ given by

\[n\mathcal{F}^d_{E,\psi} = \beta^* L^d_E \otimes_R \mu^* L_\psi[b](\frac{b}{2}),\]

where $b = \dim n\mathcal{Q}_d$. We also define $n\mathcal{P}^d_{E,\psi} \in \mathcal{D}^b_{c}(n\mathcal{Y}_d, \bar{R})$ and $n\mathcal{K}^d_E \in \mathcal{D}^b_{c}E,\psi \in \mathcal{D}^b_{c}(n\mathcal{M}_d, \bar{R})$ by $n\mathcal{P}^d_{E,\psi} = \varphi((n\mathcal{F}^d_{E,\psi}))$ and $n\mathcal{K}^d_E = \zeta((n\mathcal{F}^d_{E,\psi}))$.

The formation of all these complexes commutes with extension of scalars [3]. The complex $n\mathcal{K}^d_E$ does not depend on $\psi$ in the following sense.

Lemma 8. For any two nontrivial additive characters $\psi, \psi' : \mathbb{F}_p \to \mathcal{O}^*$ there is a canonical isomorphism $\zeta((n\mathcal{F}^d_{E,\psi})) \xrightarrow{\sim} \zeta((n\mathcal{F}^d_{E,\psi}'))$ in $\mathcal{D}^b_{c}(n\mathcal{M}_d, \bar{R})$.

Proof. There is a unique $a \in \mathbb{F}_p^*$ such that $\psi'(x) = \psi(ax)$ for $x \in \mathbb{F}_p$. So, $L_\psi \xrightarrow{\sim} a^* L_\psi$, where $a : \mathbb{A}^1 \xrightarrow{\sim} \mathbb{A}^1$ denotes the multiplication by $a$. Let $\alpha$ denote the automorphism of $n\mathcal{Q}_d$ that multiplies $s_i : \Omega^{n-i} \to L_i / L_{i-1}$ by $a^{i-1}$ for $i = 2, \ldots, n$, where $(L_1 \subset \ldots \subset L_n \subset L, (s_i))$ is a point of $n\mathcal{Q}_d$. Then $a^* ((n\mathcal{F}^d_{E,\psi})) \xrightarrow{\sim} n\mathcal{F}^d_{E,\psi}'$, and our assertion follows. $\square$

2.2 Fix $\sigma : \bar{R} \to \mathcal{O}$ as in Sect. 1.4. Let $E_\sigma$ be the image of $E \otimes_R \mathcal{O}$ in $\text{Sh}(X, \mathbb{E})$. It was shown in ([7], 7.3 and 7.5) that $n\mathcal{P}^d_{E_\sigma,\psi}$ (resp., $n\mathcal{K}^d_{E_\sigma}$) satisfies Hecke property with respect to $E_\sigma$. Let us show that this Hecke property still holds in the corresponding categories $\mathcal{D}^b_{c}(\sigma, \bar{R})$.

As for $n\mathcal{P}^d_{E,\psi}$ (Proposition 5, Corollary 2 and Lemma 12, [14]) hold with $E_\sigma$ replaced by $E$, if the maps and isomorphisms are understood as such in $\mathcal{D}^b_{c}(\sigma, \bar{R})$. Indeed, one constructs the morphisms in the same way and checks that they are isomorphisms applying the conservative functor $\mathbb{E} \otimes_R$. We will use this method in Sect. 4.1.2. The Hecke property of $n\mathcal{K}^d_E$ is formulated as follows. Let $\text{Mod}_d$ denote the stack classifying modifications $(L \subset L')$ of rank $n$ vector bundles on $X$ with $\deg (L'/L) = d$. Let $\text{supp} : \text{Mod}_d \to X^{(d)}$ be the map that sends $(L \subset L')$ to $\text{div}(L'/L)$. For $d' \geq 0$ denote by

\[p_M : n\mathcal{M}_d \times_{\text{Bun}_n} \text{Mod}_{d'} \to n\mathcal{M}_{d+d'}\]

the map that sends $(\Omega^{n-1} \hookrightarrow L \hookrightarrow L')$ to the composition $(\Omega^{n-1} \hookrightarrow L')$. It is representable and proper. Let also $q_M : n\mathcal{M}_d \times_{\text{Bun}_n} \text{Mod}_{d'} \to n\mathcal{M}_d$ denote the projection.

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Proposition 1. For any smooth $\tilde{R}$-sheaf $\tilde{E}$ on $X$ and any $d \geq 0$ there is a natural morphism

$$(q_M \times \text{supp})! \tilde{p}_M^*(n\mathcal{K}_E^{d+1}) \to n\mathcal{K}_E^d \boxtimes \tilde{E}(\frac{2-n}{2}[2-n])$$

in $D_c(n\mathcal{M}_d \times X, \tilde{R})$, which is an isomorphism if $\text{rk} \tilde{E} \leq n$.

Let $\tilde{\text{supp}} : n\mathcal{M}_d \to X^d$ be the map that sends $(L_0 \subset \ldots \subset L_d)$ to $(\text{div}(L_1/L_0), \ldots, \text{div}(L_d/L_{d-1}))$.

Lemma 9. The map (5) restricted to $n\mathcal{M}_d \times \text{supp}$ is $\text{supp}$-equivariant. □

As in ([14], Sect. 6.6) we write $\text{rss}X^d$ for the open subscheme of $X^d$ that parametrizes pairwise different points $(x_1, \ldots , x_d)$ in $X^d$ (‘rss’ stands for ‘regular semisimple’).

Corollary 2. For any smooth $\tilde{R}$-sheaf $\tilde{E}$ on $X$ and any $d, d' \geq 0$ there is a natural morphism

$$(q_M \times \text{supp})! \tilde{p}_M^*(n\mathcal{K}_E^{d+d'}) \to n\mathcal{K}_E^d \boxtimes \tilde{E}^{\boxtimes d'}(\frac{2d' - nd'}{2}[2d' - nd'])$$

in $D_c(n\mathcal{M}_d \times X^{d'}, \tilde{R})$, which is an isomorphism if $\text{rk} \tilde{E} \leq n$. □

2.3 In this subsection we prove Proposition 1.

Denote by $n\mathcal{Y}_d$ the stack of collections $((\nu_i), L, x \in X)$, where $L$ is a rank $n$ vector bundle on $X$ with $\text{deg} L = d + \text{deg}(\Omega_{(n-1)+ \ldots +(n-n)})$ and

$$t_i : \Omega_{(n-1)+ \ldots +(n-n)} \leftrightarrow (\wedge^i L)(x)$$
are sections satisfying Plücker’s relations as in (\cite{I}, Sect. 4.1). We have a closed immersion $n\mathcal{Y}_d \times X \hookrightarrow n\mathcal{Y}_d^{1}$ given by the condition: every $t_i$ factors as $\Omega^{(n-1)+\cdots+(n-i)} \hookrightarrow \wedge^i L \subset \wedge^i L(x)$.

If $(L \subset L') \in n\text{Mod}_1$ with $x = \text{div}(L'/L)$ then $(\wedge^i L')(-x) \subset \wedge^i L \subset \wedge^i L'$ for all $i$. This allows to define a map

$$\tilde{q}_{Y} : n\mathcal{Y}_{d+1} \times \text{Bun}_n \text{ n Mod}_1 \to n\tilde{\mathcal{Y}}^{1}_d$$

that sends $((t'_i, L')) \in n\mathcal{Y}_{d+1}, L \subset L'$ to $((t_i), L, x)$, where $x = \text{div}(L'/L)$ and $t_i$ is the composition

$$t_i : \Omega^{(n-1)+\cdots+(n-i)} \hookrightarrow \wedge^i L' \subset (\wedge^i L(x)$$

The map $\tilde{q}_Y$ is representable and proper.

**Definition 5.** For any smooth $\mathcal{R}$-sheaf $\mathcal{E}$ on $X$ set $n\tilde{P}_{E,\psi}^{d,1} = (\tilde{q}_Y)^*(n\tilde{P}_{E,\psi}^{d+1} \boxtimes \tilde{R}(\frac{1}{2})[n]$.

**Remark 2.** i) It may be shown that $n\tilde{P}_{E,\psi}^{d,1}$ lies in $\text{Perv}(n\tilde{\mathcal{Y}}^{1}_d, \mathcal{R})$ and coincides with the Goresky-MacPherson extension of its restriction to any nonempty open substack. Besides, the Verdier dual to $n\tilde{P}_{E,\psi}^{d,1}$ is canonically isomorphic to $n\tilde{P}_{E,\psi}^{d,1}$, We will not need these facts.

ii) The following square is cartesian

$$\begin{array}{ccc}
n\mathcal{Y}_d \times \text{Bun}_n \text{ n Mod}_1 & \hookrightarrow & n\mathcal{Y}_{d+1} \times \text{Bun}_n \text{ n Mod}_1 \\
\downarrow q_{Y} \times \text{supp} & & \downarrow \tilde{q}_{Y} \\
n\mathcal{Y}_d \times X & \hookrightarrow & n\tilde{\mathcal{Y}}^{1}_d
\end{array}$$

So, the restriction of $n\tilde{P}_{E,\psi}^{d,1}$ to $n\mathcal{Y}_d \times X$ is described by (\cite{I}, Prop. 5).

The only property of $n\tilde{P}_{E,\psi}^{d,1}$ we need is the following. Let $n\tilde{\mathcal{Y}}^{1}_d \hookrightarrow n\mathcal{Y}_d^{1}$ be the closed substack given by: $t_1$ factors as $\Omega^{n-1} \hookrightarrow L \hookrightarrow L(x)$. So, $n\mathcal{Y}_d \times X \hookrightarrow n\tilde{\mathcal{Y}}^{1}_d$ is a closed substack. Proposition 3 will follow from the next observation.

**Lemma 10.** For any smooth $\mathcal{R}$-sheaf $\mathcal{E}$ on $X$ the restriction of $n\tilde{P}_{E,\psi}^{d,1}$ to $n\tilde{\mathcal{Y}}^{1}_d$ vanishes outside $n\mathcal{Y}_d \times X$.

**Proof** In the case $\mathcal{R} = \mathcal{O}$ this follows from (\cite{I}, 7.5) (the assumption $\text{rk} \mathcal{E} = n$ required in loc.cit. is, in fact, not used for this particular statement). Applying the conservative functor $\mathcal{E} \boxtimes \mathcal{R}$, one reduces the general case to $\mathcal{R} = \mathcal{O}$. $\square$

**Proof of Proposition 3** Define the closed substack $\mathcal{T}$ of $n\mathcal{Y}_{d+1} \times \text{Bun}_n \text{ n Mod}_1$ from the cartesian square

$$\begin{array}{ccc}
\mathcal{T} & \hookrightarrow & n\mathcal{Y}_{d+1} \times \text{Bun}_n \text{ n Mod}_1 \\
\downarrow & & \downarrow \tilde{q}_{Y} \\
n\tilde{\mathcal{Y}}^{1}_d & \hookrightarrow & n\tilde{\mathcal{Y}}^{1}_d
\end{array}$$

Then we have a commutative diagram, where the right square is cartesian

$$\begin{array}{ccc}
n\mathcal{Y}_d \times X & \hookrightarrow & n\tilde{\mathcal{Y}}^{1}_d \\
\downarrow \xi & & \downarrow \xi_{Y} \\
n\mathcal{M}_d \times X & \hookrightarrow & n\mathcal{M}_d \times \text{Bun}_n \text{ n Mod}_1 \\
\downarrow \eta & & \downarrow \eta \\
n\mathcal{M}_d \times X & \hookrightarrow & n\mathcal{M}_d \times \text{Bun}_n \text{ n Mod}_1
\end{array}$$
We have denoted here by $\xi_Y$ the natural forgetful map. Denote for brevity by $F$ the restriction of $n_!\mathcal{P}_{E,\psi}^{d,1}$ to $n_!\mathcal{Y}_d^{1}$. By the base change theorem,

$$(q_{\mathcal{M}} \times \text{supp})^! p_{\mathcal{M}}^* (n_!\mathcal{K}_E^{d+1}) \to (\xi_Y)_! F[-n](\frac{n-1}{2})$$

By Lemma [10], $F$ is the extension by zero from $n_!\mathcal{Y}_d \times X$. By ([14], Prop. 5), we get a morphism $F \to n_!\mathcal{P}_{E,\psi}^{d} \otimes \bar{E}(1)[2]$, which is an isomorphism when $\text{rk} \bar{E} \leq n$. Our assertion follows. □

2.4. Recall the definition of a Hecke-eigensheaf ([7], 1.1). Consider the correspondence

$$\text{Bun}_n \times X \to \text{Mod}_1 \overset{\mathcal{P}}{\to} \text{Bun}_n,$$

where $\mathcal{P}$ sends $(L \subset L') \in n_!\text{Mod}_1$ to $L'$, $q$ sends $(L \subset L')$ to $L$, and $\text{supp} : n_!\text{Mod}_1 \to X$ is the map defined in Sect. 2.2.

The Hecke functor $H^1_n : D^b_c(\text{Bun}_n, \bar{R})_\sigma \to D^b_c(\text{Bun}_n, \bar{X}, \tilde{R})_\sigma$ is defined by

$$H^1_n(K) = (q \times \text{supp})^! p^*(K)(\frac{n-1}{2})[n-1]$$

(8)

Consider the $d$-th iteration of $H^1_n$:

$$(H^1_n)^{\otimes d} : D^b_c(\text{Bun}_n, \bar{R})_\sigma \to D^b_c(\text{Bun}_n, X^d, \tilde{R})_\sigma$$

For any $K \in D^b_c(\text{Bun}_n, \bar{R})_\sigma$ the restriction of $(H^1_n)^{\otimes d}(K)$ to $\text{Bun}_n \times \text{rss}X^d$ is naturally equivariant with respect to the action of the symmetric group $S_d$ on $\text{rss}X^d$.

**Definition 6.** Let $\bar{E}$ be a smooth $\bar{R}$-sheaf of rank $n$ on $X$. A Hecke eigensheaf with respect to $\bar{E}$ is a nonzero object $K \in D^b_c(\text{Bun}_n, \bar{R})_\sigma$ equiped with an isomorphism $H^1_n(K) \to K \otimes \bar{E}$ such that the resulting map

$$(H^1_n)^{\otimes d}(K) \mid_{\text{Bun}_n \times \text{rss}X^d} \to K \otimes E^{\otimes d} \mid_{\text{Bun}_n \times \text{rss}X^d}$$

is $S_d$-equivariant.

Following [7], pick a line bundle $L^{\text{est}}$ on $X$ such that for any vector bundle $M$ on $X$ of rank $k < n$, $\text{Hom}(M, L^{\text{est}}) = 0$ implies that

a) $\deg(M) > nk(2g-2)$,

b) $\text{Ext}^1(\Omega^{k-1}, M) = 0$.

For example, $L^{\text{est}}$ of degree $> 2n(2g-2)$ satisfies this property. Let us denote by $n_!\mathcal{M}_d^{\text{est}} \subset n_!\mathcal{M}_d$ the open substack consisting of $(\Omega^{n-1} \to L) \in n_!\mathcal{M}_d$ such that $\text{Hom}(L, \Omega^{n-1}, L^{\text{est}}) = 0$. Set $n_!\mathcal{M} = \cup_{d \geq 0} n_!\mathcal{M}_d$. Denote by $\phi : n_!\mathcal{M} \to \text{Bun}_n$ the natural projection.

**Notational Convention.** For notational convenience, in what follows by degree of a coherent sheaf on $X$ of generic rank $n$ we will understand its usual degree $-n(n-1)(g-1)$, so that
\( \mathcal{O} \oplus \Omega \oplus \ldots \oplus \Omega^{n-1} \) is of degree zero. We write \( \text{Bun}_n^d \) for the connected component of \( \text{Bun}_n \) classifying vector bundles of rank \( n \) and degree \( d \) on \( X \).

Recall that \( \bar{R} \) is a local complete noetherian regular \( \mathcal{O} \)-algebra with residue field \( \kappa \), and \( \sigma : \bar{R} \to \mathcal{O} \) is a local homomorphism of \( \mathcal{O} \)-algebras (in particular, \( \bar{R} \) is of characteristic zero). The geometric Langlands conjecture (relative to \( \bar{R} \)) may be formulated as follows (cf. [9, 5, 7]).

**Conjecture 1.** Let \( \bar{E} \) be a smooth \( \bar{R} \)-sheaf on \( X \) of rank \( n \) such that \( E_0 \otimes \bar{Q}_\ell \) is irreducible. Then for any \( d \geq 0 \) the natural map \( \zeta^! \left( nF^d \bar{E}, \psi \right) \to \zeta_* \left( nF^d \bar{E}, \psi \right) \) is an isomorphism over \( n\text{M}_{\text{est}}^d \) in \( D^b_{\text{c}}(n\text{M}_{\text{est}}^d, \bar{R}) \sigma \), and the restriction of \( n\mathcal{K}_E^d \) to \( n\mathcal{M}_{\text{est}}^d \) lies in \( \text{Perv}_{\text{fl}}(n\text{M}_{\text{est}}^d, \bar{R}) \sigma \). Moreover, there exist the following data:

- a perverse sheaf \( \text{Aut}_E \) \( \in \text{Perv}_{\text{fl}}(\text{Bun}_n, \bar{R}) \sigma \) equipped with the structure of a Hecke-eigensheaf with respect to \( \bar{E} \);
- for each \( d \geq 0 \) an isomorphism in \( D^b_{\text{c}}(n\mathcal{M}_d, \bar{R}) \sigma \) between \( n\mathcal{K}_E^d \) and the inverse image \( \varphi^* \text{Aut}_E^d[d - n^2(g - 1)](\frac{d - n^2(g - 1)}{2}) \) under \( \varphi : n\mathcal{M}_d \to \text{Bun}_n^d \) (we write \( \text{Aut}_E^d \) for the restriction of \( \text{Aut}_E \) to \( \text{Bun}_n^d \)).

These data satisfy the following properties:

1) the Hecke properties of \( \text{Aut}_E \) and of \( n\mathcal{K}_E^d \) are compatible.

2) \( \text{Aut}_E^d \) is the Goresky-MacPherson extension from any nonempty open substack of \( \text{Bun}_n^d \).

3) \( \text{Aut}_E \) is cuspidal in the sense that for any nontrivial partition \( \bar{n} = (n_1, \ldots, n_k) \) of \( n \) determining the correspondence

\[
\begin{align*}
\text{Flag}_{\bar{n}} & \quad \xrightarrow{\text{p}_{\bar{n}}} \text{Bun}_n \\
\downarrow \text{q}_{\bar{n}} & \quad \\
\text{Bun}_{n_1} \times \ldots \times \text{Bun}_{n_k} & ,
\end{align*}
\]

we have \( (\text{q}_{\bar{n}})(\text{p}_{\bar{n}})^* \text{Aut}_E = 0 \). Here \( \text{Flag}_{\bar{n}} \) is the stack of flags \( (M_1 \subset \ldots \subset M_k) \), where \( M_i/M_{i-1} \in \text{Bun}_{n_i} \). The map \( \text{p}_{\bar{n}} \) sends this flag to \( M_k \), and \( \text{q}_{\bar{n}} \) sends this flag to \( (M_1, M_2/M_1, \ldots, M_k/M_{k-1}) \).

**Remarks.** 1) The property i) means the following. For any \( d \geq 0 \) we have a commutative diagram, where the left square is cartesian

\[
\begin{array}{ccc}
n\mathcal{M}_d \times X & \xleftarrow{q^\times \text{supp}} & n\mathcal{M}_d \times \text{Bun}_n \times \text{Mod}_1 \\
\downarrow & & \downarrow \\
\text{Bun}_n \times X & \xleftarrow{q^\times \text{supp}} & \text{Mod}_1 \\
\end{array}
\]

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It is required that the restriction of the isomorphism $H^1_{\overline{\mathcal{E}}} (\text{Aut}_{\overline{\mathcal{E}}}) \xrightarrow{\sim} \text{Aut}_{\overline{\mathcal{E}}} \boxtimes \overline{\mathcal{E}}$ under the left vertical arrow in the above diagram coincides with the isomorphism (5) (up to a cohomological shift and a Tate twist).

2) From main result of [7] it follows that if certain vanishing conjecture (Conjecture 2.3, loc.cit) is true then Conjecture 1 is true for $R = \mathcal{O}$. Indeed, the first assertion follows from (3.6, loc.cit.) combined with the properties of the Fourier transform ([10], 1.3.1.1, 1.3.2.3).

3) The perverse sheaf $\text{Aut}_{\overline{\mathcal{E}}}$ in Conjecture 1 is defined up to a canonical isomorphism.

2.5 The quotient of $\text{Bun}_n$ by a 2-action of $\mathbb{G}_m$

Consider the $k$-prestack whose category fibre at a scheme $S$ is the following groupoid. Its objects are vector bundles $L$ on $S \times X$ of rank $n$. A morphism from $L_1$ to $L_2$ is an equivalence class $\in \{(A,u) \}/ \sim$, where $A$ is an invertible sheaf on $S$, $u : L_1 \xrightarrow{\sim} L_2 \otimes A$ is an isomorphism of $\mathcal{O}_{S \times X}$-modules, and the pairs $(A,u)$ and $(A',u')$ are equivalent if there exists an isomorphism $A \xrightarrow{\sim} A'$ making commute the diagram

\[
\begin{array}{ccc}
L_1 & \xrightarrow{u} & L_2 \otimes A \\
\downarrow u' & & \downarrow ? \\
L_2 \otimes A'
\end{array}
\]

We define $\text{Bun}_n$ as the stack associated to this prestack. Then the natural morphism $\tau : \text{Bun}_n \to \text{Bun}_n$ is a $\mathbb{G}_m$-gerb.

**Lemma 11.** $\text{Bun}_n$ is an algebraic stack locally of finite type and smooth of pure dimension $n^2(g-1)+1$.

**Proof** Let $S$ be a noetherian scheme, $L_1$ and $L_2$ be vector bundles on $S \times X$ of rank $n$. Consider the morphism $\text{Isom}(L_1, L_2) \to S$ obtained by the base change $S \xrightarrow{(L_1, L_2)} \text{Bun}_n \times \text{Bun}_n$ from the diagonal mapping $\text{Bun}_n \to \text{Bun}_n \times \text{Bun}_n$. Recall that $\text{Isom}(L_1, L_2)$ is an open subscheme of some affine $S$-scheme $V(F)$, where $F$ is a coherent $\mathcal{O}_{S \times X}$-module, and $\text{Isom}(L_1, L_2)$ is of finite type over $S$ ([12], the proof of 4.6.2.1). We have a free action of $\mathbb{G}_m$ on $\text{Isom}(L_1, L_2)$, and the square is cartesian

\[
\begin{array}{ccc}
\text{Bun}_n \times \text{Bun}_n & \to & \text{Bun}_n \\
\text{Isom}(L_1, L_2)/\mathbb{G}_m & \to & S
\end{array}
\]

It follows that $\text{Isom}(L_1, L_2)/\mathbb{G}_m$ is an open subscheme of $\mathbb{P}(F)$, in particular it is separated over $S$. So, the diagonal mapping $\text{Bun}_n \to \text{Bun}_n \times \text{Bun}_n$ is representable, separated and quasi-compact.

If $Y \to \text{Bun}_n$ is a presentation of $\text{Bun}_n$ then the composition $Y \to \text{Bun}_n \to \text{Bun}_n$ is a presentation of $\text{Bun}_n$. □
The connected components of \( \text{Bun}_n \) are numbered by \( d \in \mathbb{Z} \): the component \( \text{Bun}_d^1 \) is the image of \( \text{Bun}_d^1 \) under \( r: \text{Bun}_n \to \text{Bun}_n^d \). The morphism \( r: \text{Bun}_d^1 \to \text{Bun}_d^1 \) is, in fact, the canonical map \( \text{Pic}^d X \to \text{Pic}^d X \), where \( \text{Pic}^d X \) is the Picard scheme of \( X \).

Let \( \text{mult}: \text{Bun}_n \times X \to \text{Bun}_n \) be the map that sends \((L,x)\) to \( L(x) \). There is a unique map \( \text{mult}: \text{Bun}_n \times X \to \text{Bun}_n \) making commute the diagram

\[
\begin{array}{ccc}
\text{Bun}_n \times X & \xrightarrow{\text{mult}} & \text{Bun}_n \\
\downarrow r \times \text{id} & & \downarrow r \\
\text{Bun}_n \times X & \xrightarrow{\text{mult}} & \text{Bun}_n
\end{array}
\] (9)

The composition \( 1_{\text{M}_d} \to \text{Bun}_d^1 \to \text{Bun}_d^1 \) is, in fact, the Abel-Jacobi map \( X^d(\text{d}) \to \text{Pic}^d X \) sending a divisor \( D \) to the isomorphism class of \( \mathcal{O}(D) \). More generally, the composition \( n_{\text{M}_d} \to \text{Bun}_d^1 \to \text{Bun}_d^1 \) is representable. Given a scheme \( S \) and an \( S \)-point \( S \xrightarrow{L} \text{Bun}_d^1 \), consider the scheme \( Z_S = n_{\text{M}_d} \times \text{Bun}_n S \). This is an \( S \)-scheme that classifies nonzero sections \( s: \Omega^{-1} \to L \). The group \( \mathbb{G}_m \) acts freely on \( Z_S \) (over \( S \)), multiplying \( s \) by a scalar. One checks that \( n_{\text{M}_d} \times \text{Bun}_n \) is identified with the quotient \( Z_S / \mathbb{G}_m \). In particular, for \( S = \text{Spec} k \) the scheme \( Z_S \) is the projective space \( (\text{Hom}(\Omega^{-1}, L) - \{0\}) / \mathbb{G}_m \).

Now we are able to prove the following result.

**Proposition 2.**
1) If Conjecture [?] is true then \( \mathbb{D}(\text{Aut}_E^\sim) \xrightarrow{\sim} \text{Aut}_E^\sim \), canonically.
2) If Conjecture [?] is true for \( \bar{\mathcal{R}} = \mathcal{O} \) then it is true in full generality. Moreover, there exists a perverse sheaf \( \text{Aut}_E^\sim \in Perv_{fl}(\text{Bun}_n, \bar{\mathcal{R}}) \) together with an isomorphism

\[
\text{Aut}_E^\sim \xrightarrow{\sim} r^* \text{Aut}_E^\sim[-1](\frac{1}{2})
\] (10)

By this condition \( \text{Aut}_E^\sim \) is defined up to a canonical isomorphism. The formation of \( \text{Aut}_E^\sim \) commutes with extension of scalars [4]. We also have canonically

\[
\text{mult}^* \text{Aut}_E^\sim \xrightarrow{\sim} \text{Aut}_E^\sim \otimes \wedge^n \tilde{E}
\] (11)

**Proof**
1) Let \( \text{Bun}_n^{\text{est}} \subset \text{Bun}_n \) be the open substack consisting of \( L \) such that \( \deg L > n^2(g-1) \) and \( \text{Hom}(L, \mathcal{L}^{\text{est}}) = 0 \). Let \( n\mathcal{M}^{\text{est}} = \cup_{d \geq 0} n\mathcal{M}_d^{\text{est}} \). By Conjecture [?], over \( n\mathcal{M}^{\text{est}} \) there is a canonical isomorphism

\[
\mathbb{D}(nK_{\tilde{E}}) \xrightarrow{\sim} nK_{\tilde{E}}
\] (12)

Over \( \text{Bun}_n^{\text{est}} \cap \text{Bun}_d^1 \) the map \( \varrho: n\mathcal{M}_d \to \text{Bun}_d^1 \) is a vector bundle of rank \( d - n^2(g-1) \) with removed zero section. Since the preimage of \( \text{Bun}_n^{\text{est}} \cap \text{Bun}_d^1 \) under this map is contained in \( n\mathcal{M}_d^{\text{est}} \), the isomorphism (12) descends to give an isomorphism over \( \text{Bun}_n^{\text{est}} \)

\[
\mathbb{D}(\text{Aut}_{\tilde{E}}) \xrightarrow{\sim} \text{Aut}_{\tilde{E}}
\] (13)
By ([7], 1.5), Hecke property of $\text{Aut}_E$ yields an isomorphism

$$\text{mult}^* \text{Aut}_E \cong \text{Aut}_E \otimes \wedge^n \bar{E} \quad (14)$$

For any open substack of finite type $U \subset \text{Bun}_n$ there exists an integer $d'$ such that for any $x \in X$ the morphism $\text{mult}_{d',x} : \text{Bun}_n \to \text{Bun}_n$ sending $L$ to $L(d'x)$ maps $U$ isomorphically onto a substack of $\text{Bun}^\text{est}_n$. We get

$$\mathcal{D}(\text{Aut}_E)_{|U} \cong \mathcal{D}((\wedge^n \bar{E}_x)^{\otimes -d'} \otimes \text{mult}^*_{d',x} \text{Aut}_E) \cong$$

$$(\wedge^n \bar{E}_x)^{\otimes -d'} \otimes \text{mult}_{d',x} \text{Aut}_E \cong (\wedge^n \bar{E}_x)^{\otimes -d'} \otimes \text{mult}_{d',x} \text{Aut}_{E^*} \cong \text{Aut}_{E^*} \ | U$$

According to (14), this gives a well-defined isomorphism (13) over the entire $\text{Bun}_n$, which coincides with the old one over $\text{Bun}^\text{est}_n$.

2) **Step 1.** Applying the conservative functor $\mathbb{E} \otimes \bar{R}_\sigma$, one checks that $\xi(t(\mathbb{F}_{d',\psi})) \to \xi(t(\mathbb{F}_{d',\psi}))$ is an isomorphism over $n\mathcal{M}^\text{est}_d$. This yields the isomorphism (12) over $n\mathcal{M}^\text{est}_d$.

Since both $n\mathcal{K}^d_{B_\sigma} \otimes \bar{R}_\sigma \otimes \mathbb{E}$ and $(\mathcal{D}(n\mathcal{K}^d_{B_\sigma}) \otimes \bar{R}_\sigma \otimes \mathbb{E})$ restricted to $n\mathcal{M}^\text{est}_d$ are irreducible perverse sheaves, by 2) of Lemma [7], the restriction of $n\mathcal{K}^d_{B_\sigma}$ to $n\mathcal{M}^\text{est}_d$ is an object of $\text{Perv}_{fl}(n\mathcal{M}^\text{est}_d, \bar{R}_\sigma)$, which is the Goresky-MacPherson extension from any nonempty open substack of $n\mathcal{M}^\text{est}_d$.

**Step 2.** As explained in ([7], 7.6), we have a notion of a Hecke eigensheaf on $\text{Bun}^\text{est}_n$. Indeed, for the diagram (6) we have

$$(q \times \text{supp})^{-1}(\text{Bun}^\text{est}_n \times X) \subset p^{-1}(\text{Bun}^\text{est}_n)$$

Define the functor $\mathcal{D}(\text{Bun}^\text{est}_n, \bar{R})_{\sigma} \to \mathcal{D}(\text{Bun}^\text{est}_n \times X, \bar{R})_{\sigma}$ by formula (8) and denote it again by $H^1_{\sigma}$. Considering its iterations $(H^1_{\sigma})_{\otimes n}^{\otimes n}$, one can repeat Definition 6 in this context.

Let $\text{Bun}^\text{est}_n$ be the image of $\text{Bun}^\text{est}_n$ under $\tau : \text{Bun}_n \to \text{Bun}_n$.

**Lemma 12.** 1) There exists a perverse sheaf $\text{Aut}^\text{est}_E \in \text{Perv}_{fl}(\text{Bun}^\text{est}_n, \bar{R})_{\sigma}$ and for each $d \geq 0$ an isomorphism

$$\varrho^* \tau^* \text{Aut}^\text{est}_E[d - 1 - n^2(g - 1)](d - 1 - n^2(g - 1)) \otimes \wedge^n \mathcal{K}^d_E \cong \tilde{\text{Aut}^\text{est}_E}$$

over $\varrho^{-1}(\text{Bun}^d_n \cap \text{Bun}^\text{est}_n)$. These data are defined up to a canonical isomorphism. Besides, over each connected component of $\text{Bun}^\text{est}_n$, $\text{Aut}^\text{est}_E$ is the Goresky-MacPherson extension from any nonempty open substack, and the formation of $\text{Aut}^\text{est}_E$ commutes with extension of scalars (4).

2) Set $\text{Aut}^\text{est}_E$ to be $\tau^* \text{Aut}^\text{est}_E[-1](-\frac{1}{2})$ over $\text{Bun}^\text{est}_n$. Then $\text{Aut}^\text{est}_E$ has a unique structure of a Hecke eigensheaf with respect to $\mathcal{E}$, which is compatible with the Hecke property of $n\mathcal{K}^d_E$.

**Proof.** Let $d > n^2(g - 1)$ be such that $\text{Bun}^d_n \cap \text{Bun}^\text{est}_n$ is nonempty. Let $U \subset \text{Bun}^d_n \cap \text{Bun}^\text{est}_n$ be a nonempty open substack such that $\text{Aut}_{E_0}$ is an (appropriately shifted) smooth $\mathcal{E}$-sheaf over $U$. Let $U' = \varrho^{-1}(U)$. So, over $U'$, $n\mathcal{K}^d_E$ is a smooth $\bar{R}_\sigma$-sheaf (appropriately shifted).
Recall that $U' \to U$ is a vector bundle of rank $d - n^2(g - 1)$ with removed zero section. Let $ar{U}$ be the image of $U$ under $r : \text{Bun}_d \to \text{Bun}_d$. Then $U' \to \bar{U}$ is a projectivization of a vector bundle. By 1) of Lemma [3], there exists an (appropriately shifted) smooth $\bar{R}_\sigma$-sheaf $V_0$ on $\bar{U}$ and an isomorphism in $D^b_c(U', \bar{R})$

$$g^*r^*V_0[d - 1 - n^2(g - 1)](\frac{d - 1 - n^2(g - 1)}{2})^{-1}K^n_E$$

Define the restriction of $\text{Aut}^\text{est}_E$ to $\text{Bun}_d \cap \text{Bun}_d^\text{est}$ as the Goresky-MacPherson extension of $V_0$ from $\bar{U}$. Since $g^{-1}(\text{Bun}_d^\text{est}) \subset n^*M^\text{est}$, our first assertion follows. All the other assertions follow from the fact that the restriction $g^{-1}(\text{Bun}_d^\text{est}) \to \text{Bun}_d^\text{est}$ of $g$ is smooth and surjective with connected fibres. □

**Step 3.** As in ([7], 1.5), Hecke property of $\text{Aut}^\text{est}_E$ yields the isomorphism (14) over $\text{Bun}_d^\text{est} \times X$. This isomorphism descends to give the isomorphism (11) over $\text{Bun}_d^\text{est} \times X$. Then one extends $\text{Aut}^\text{est}_E$ to the entire $\text{Bun}_d$ as follows.

For any open substack of finite type $U \subset \text{Bun}_d$ there exists an integer $d'$ such that for any $x \in X$ the morphism $\text{mult}_{d'x} : \text{Bun}_d \to \text{Bun}_d$ sending $L$ to $L(d'x)$ maps $U$ isomorphically onto a substack of $\text{Bun}_d^\text{est}$. Set $\text{Aut}^\text{est}_E|_U$ to be

$$(\wedge^n E_x)^{\otimes d'} \otimes \text{mult}_{d'x}^* \text{Aut}^\text{est}_E$$

This gives a well-defined perverse sheaf $\text{Aut}^\text{est}_E \in \text{Perv}_{fl}(\text{Bun}_d)$ together with the isomorphism (11) over the entire $\text{Bun}_d \times X$. One concludes the proof as in (loc.cit., 7.8 and 7.9).

□(Proposition 2)

**Remark 3.** 1) Denote by $\text{Bun}_\text{PGL}_n$ the moduli stack of $\text{PGL}_n$-bundles on $X$. There exists a morphism $\tilde{r} : \text{Bun}_d \to \text{Bun}_\text{PGL}_n$ such that the composition $\text{Bun}_d \xrightarrow{\tilde{r}} \text{Bun}_d^\text{est} \xrightarrow{\tilde{r}} \text{Bun}_\text{PGL}_n$ is the canonical map $\text{Bun}_d \to \text{Bun}_\text{PGL}_n$. The morphism $\tilde{r}$ is representable, smooth and separated. Let $t_X : \text{Bun}_d^\text{est} \times \text{Pic}^0 X \to \text{Bun}_d^\text{est}$ be the map that sends $(L, A)$ to $L \otimes A$. We have a map $t_X : \text{Bun}_d^\text{est} \times \text{Pic}^0 X \to \text{Bun}_d^\text{est}$ such that the diagram

$$
\begin{array}{ccc}
\text{Bun}_d^\text{est} \times \text{Pic}^0 X & \xrightarrow{t_X} & \text{Bun}_d^\text{est} \\
\downarrow & & \downarrow \\
\text{Bun}_d^\text{est} \times \text{Pic}^0 X & \xrightarrow{t_X} & \text{Bun}_d^\text{est}
\end{array}
$$

commutes, and the following two squares are cartesian

$$
\begin{array}{ccc}
\text{Bun}_d^\text{est} \times \text{Pic}^0 X & \xrightarrow{pr_1} & \text{Bun}_d^\text{est} \\
\uparrow t_X & & \uparrow \text{tr}_X \\
\text{Bun}_d^\text{est} \times \text{Pic}^0 X & \xrightarrow{pr_1} & \text{Bun}_d^\text{est}
\end{array}
$$

$$
\begin{array}{ccc}
\text{Bun}_d^\text{est} \times \text{Pic}^0 X & \xrightarrow{pr_2} & \text{Bun}_d^\text{est} \\
\uparrow t_X & & \uparrow \text{tr}_X \\
\text{Bun}_d^\text{est} \times \text{Pic}^0 X & \xrightarrow{pr_2} & \text{Bun}_d^\text{est}
\end{array}
$$

2) Let $\Lambda$ be a noetherian ring such that the characteristic of $\Lambda$ is invertible in $k$. Then $\text{Bun}_d^\text{est}$ is a Bernstein-Lunts stack with respect to $\Lambda$ in the sense of ([12], 18.7.4).
Indeed, if $\mathcal{X}_1 \to \mathcal{X}_2$ is a representable separated morphism of algebraic stacks, and $\mathcal{X}_2$ is a Bernstein-Lunts stack then $\mathcal{X}_1$ is also. Apply this for $\mathfrak{r}$. The stack $\text{Bun}_{\text{PGL}_n}$ is a Bernstein-Lunts stack, because it is of the form $M/G$, where $M$ is a separated algebraic space with an action of an affine algebraic group $G$ (cf. [12], 18.7.5).

3 Deformations of local systems and cohomology of $\mathcal{H}om(\widetilde{E}_1, \widetilde{E}_2)$

3.1 Let $\mathbb{E}$ be a finite extension field of $\mathbb{Q}_\ell$. Fix a smooth $\mathbb{E}$-sheaf $E_0$ on $X$ of rank $m$. First, we recall the structure of the universal deformation of $E_0$ over $\mathbb{E}$. This construction is standard (cf. [17] for the definition of pro-representability, etc.).

Let $\eta \in X$ be the generic point of $X$ and $\bar{\eta} \to \eta \to X$ be a geometric point over $\eta$. Set $G = \pi_1(X, \bar{\eta})$. Let $A \in \text{Ob}(C_\mathbb{E})$. Recall that the functor that sends $E$ to $E_0$ is an equivalence between the category of smooth $A$-sheaves of rank $m$ on $X$ and the category of pairs $(E, \rho)$, where $E$ is a free $A$-module of rank $m$ and $\rho : G \to \text{Aut}_A E$ is a representation continuous in the $\ell$-adic topology.

Definition 7. An $A$-deformation of $E_0$ is a pair $(E, h)$, where $E$ is a smooth $A$-sheaf on $X$ of rank $m$ and $h : E \otimes_A \mathbb{E} \overset{\sim}{\to} E_0$ is an isomorphism of $\mathbb{E}$-sheaves on $X$.

Define the functor $F_{E_0} : C_\mathbb{E} \to \text{Sets}$ by $F_{E_0}(A) =$ the set of isomorphism classes of $A$-deformations of $E_0$.

Proposition 3. If $\text{End}(E_0) = \mathbb{E}$ then $F_{E_0}$ is pro-representable by a pro-pair $(R, E)$, where $R$ is (non canonically) isomorphic to the ring of formal power series over $\mathbb{E}$ in $2 + (2g - 2)m^2$ variables. Let $m \subset R$ be the maximal ideal of $R$. The $\mathbb{E}$-dual of $m/m^2$ is canonically identified with $H^1(X, \text{End}E_0)$. If, in addition, $\mathbb{E} \subset \mathbb{E}'$ is a finite extension field and $\text{End}(E_0 \otimes_\mathbb{E} \mathbb{E}') = \mathbb{E}'$ then the pro-pair $(R \otimes_\mathbb{E} \mathbb{E}', E \otimes_\mathbb{E} \mathbb{E}')$ pro-represents the functor $F_{E_0 \otimes_\mathbb{E} \mathbb{E}'}$.

Lemma 13. Suppose that $\text{End}(E_0) = \mathbb{E}$

1) If $E$ is an $A$-deformation of $E_0$ then $\text{End}(E) = A$.

2) Let $A' \to A, A'' \to A$ be two morphisms in $C_\mathbb{E}$. Suppose that $A'' \to A$ is surjective then the natural morphism $F_{E_0}(A' \times_A A'') \to F_{E_0}(A') \times_{F_{E_0}(A)} F_{E_0}(A'')$ is a bijection.

Proof 2) The surjectivity is easy. To prove the injectivity use point 1) and Corollary 3.6, p.217 of [17]. □

Lemma 14. Let $A' \to A$ be a surjection in $C_\mathbb{E}$, whose kernel is a 1-dimensional $\mathbb{E}$-vector space. Let $V'$ be a free $A'$-module of rank $m$. Put $V = V' \otimes_{A'} A$. Then the natural map

$$\text{GL}(V') \to \text{PGL}(V') \times_{\text{PGL}(V)} \text{GL}(V) \times_{\text{GL}(\text{det} V)} \text{GL}(\text{det} V')$$

is an isomorphism of groups. □
Theorem 2.11 of [17] we get the pro-representability of $F$ that $V$ is identified with $H^1(X, \mathcal{E}ndE_0)$. Now combining Lemma [4] with Theorem 2.11 of [17] we get the pro-representability of $F_{E_0}$ by a pro-pair $(R, E)$.

Let us show that the morphism of functors associating to an $A$-deformation $E$ of $E_0$ the $A$-deformation $\det E$ of $\det E_0$ is a formally smooth morphism from the universal deformation of $E_0$ to the universal deformation of $\det E_0$. Let $A' \to A$, $V'$ and $V$ be as in Lemma [4]. Suppose that $V$ is equipped with a structure of an $A$-deformation of $E_0$. Let $\rho : G \to \text{Aut}_A V$ be the corresponding representation of $G$. Since $H^2(X, \mathcal{E}nd_{E_0}E_0) = 0$, we get $H^2(X, \mathcal{E}nd_{E_0}E_0) = 0$ (we write $\mathcal{E}nd_{E_0}E$ for the sheaf of traceless endomorphisms of $E$). It follows that the corresponding representation of $G$ in $\text{PGL}(V)$ can be lifted to a representation $\rho' : G \to \text{PGL}(V')$. Now our assertion follows from Lemma [4].

The universal deformation of $\det E_0$ is formally smooth, because it is isomorphic to the universal deformation of the trivial 1-dimensional local system, which is an infinitesimal formal $\mathbb{E}$-group (cf. SGA3, t.I.VIII, 3.3). So, $R$ is formally smooth, i.e., by ([17], 2.5), is isomorphic to the ring of formal power series over $\mathbb{E}$.

Since $\chi(\mathcal{E}nd_{E_0}E_0) = (2 - 2g)(m^2 - 1)$, we have $\dim H^1(X, \mathcal{E}nd_{E_0}E_0) = (2g - 2)m^2 + 2$.

If $E \subset E'$ is a finite extension with $\text{End}(E_0 \otimes \mathcal{E} E') = E'$ then $(R \otimes E', E \otimes E')$ is a pro-pair for the functor $F_{E_0 \otimes E'} : \mathcal{C}_{E'} \to \text{Sets}$, which defines a morphism of functors $h_{R \otimes E'} : F_{E_0 \otimes E'}$, where $h_{R \otimes E'} : \mathcal{C}_{E'} \to \text{Sets}$ denotes the functor represented by $R \otimes E'$, that is, $h_{R \otimes E'}(B) = \text{Hom}_{\text{local} \mathcal{E} \text{-alg}}(R \otimes E', B)$. We must show that this is an isomorphism of functors. Since $F_{E_0 \otimes E'}$ can be represented by a ring of formal power series over $E'$, our assertion follows from the fact that the induced map on the tangent spaces is an isomorphism. □

By definition, $E = (E_k)_{k \in \mathbb{N}}$, $E_k \in F_{E_0}(R/m^k)$ are such that the image of $E_{k+1}$ under $F_{E_0}(R/m^{k+1}) \to F_{E_0}(R/m^k)$ is $E_k$. Fix a $R/m^k$-deformation of $E_0$ in the isomorphism class $E_k$ and denote it by the same symbol $E_k$. For each $k$ fix an isomorphism of $R/m^k$-deformations of $E_0$: $E_{k+1} \otimes_R m^{k+1} \to E_k$. Then the projective system $(E_k)_{k \in \mathbb{N}}$ is an object of $\text{Sh}(X, R)$, equipped with an isomorphism $\alpha : E \otimes_R \mathcal{E}_X E_0$ of $\mathcal{E}$-sheaves on $X$. Notice that $R$ is defined up to a canonical isomorphism, whence the $R$-sheaf $E$ is defined up to a non-canonical isomorphism.

3.2 Let $\mathcal{O} \subset \mathbb{E}$ be the ring of integers of $\mathbb{E}$, $\kappa$ be the residue field of $\mathcal{O}$, and $\omega \in \mathcal{O}$ be a uniformizing parameter. A smooth $\mathcal{O}$-sheaf $E'_0$ on $X$ together with an isomorphism $E'_0 \otimes_{\mathcal{O}} \mathcal{E} \to E_0$ can be viewed as a $G$-invariant $\mathcal{O}$-lattice in $(E_0)_\eta$. Set $E_0 = E'_0 \otimes_{\mathcal{O}} \kappa$. It is easy to see that, though $E_0$ is not defined up to an isomorphism by $E_0$, the image of $E_0$ in the Grothendieck group of the category of smooth $\kappa$-sheaves on $X$ is uniquely defined by $E_0$.

In this subsection we compare the universal deformation $(R, E)$ of $E_0$ and the universal deformation of $E_0$ over $\mathcal{O}$.

Let $\mathcal{C}_0$ be the category of local Artin $\mathcal{O}$-algebras with residue field $\kappa$ (the morphisms are local homomorphisms of $\mathcal{O}$-algebras). For $A \in \text{Ob}(\mathcal{C}_0)$ one defines a notion of an $A$-deformation of $E_0$ and a functor $F_{E_0} : \mathcal{C}_0 \to \text{Sets}$ as in Sect. 3.1. The proof of the next result is similar to that of Proposition [3].

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Proposition 4. If $\text{End}(\bar{E}_0) = \kappa$ then $F_{\bar{E}_0}$ is pro-representable by a pro-pair $(\bar{R}, \bar{E})$, where $\bar{R}$ is a complete local noetherian $\mathcal{O}$-algebra with residue field $\kappa$. If $\bar{m}$ is the maximal ideal of $\bar{R}$ then the $\kappa$-dual of $\bar{m}/(\bar{m}^2, \omega)$ is canonically identified with $H^1(X, \mathcal{E}nd\bar{E}_0)$. $\square$

As in Sect. 3.1, we may and will view $\bar{E}$ as an object of $\text{Sh}(X, \bar{R})$ equipped with an isomorphism $\bar{\alpha}: \bar{E} \otimes_{\bar{R}} \kappa \rightarrow \bar{E}_0$. (Notice that $\bar{R}$ is defined by $\bar{E}_0$ up to a canonical isomorphism, whence $\bar{E}$ is defined up to a non canonical isomorphism.)

Since $E'_0$ is an $\mathcal{O}$-deformation of $E_0$, it defines a local homomorphism of $\mathcal{O}$-algebras $\beta : \bar{E} \otimes_{\bar{R}} \mathcal{O} \rightarrow E'_0$ in $\text{Sh}(X, \mathcal{O})$ compatible with $\bar{\alpha}$ (so, $\beta$ is defined up to multiplication by an element of $1 + \omega \mathcal{O}$). Let $m_\sigma$ denote the kernel of the induced map $\bar{R} \otimes_{\mathcal{O}} \mathbb{E} \rightarrow \mathbb{E}$. We denote by $R_\sigma$ the localization of $\bar{R} \otimes_{\mathcal{O}} \mathbb{E}$ in $m_\sigma$, and by $\hat{R}_\sigma$ the $m_\sigma$-adic completion of $\bar{R} \otimes_{\mathcal{O}} \mathbb{E}$.

Remark 4. The rings $\bar{R} \otimes_{\mathcal{O}} \mathbb{E}$ (and, hence, $\bar{R}_\sigma$ and $\hat{R}_\sigma$) are noetherian. Indeed, $\bar{R}$ is isomorphic to a quotient of the ring of formal power series $\mathcal{O}[\mathbb{E}]$ for some $N$, so that $\bar{R} \otimes_{\mathcal{O}} \mathbb{E}$ is isomorphic to a quotient of $\mathcal{O}[[t_1, \ldots, t_N]] \otimes_{\mathcal{O}} \mathbb{E}$, which is noetherian (cf. $\mathbb{S}$, Appendix A.2).

The isomorphism $\beta$ induces on $\mathbb{E} \otimes_{\bar{R}} \hat{R}_\sigma$ a structure of a $\hat{R}_\sigma$-deformation of $E_0$. This defines a local homomorphism of $\mathbb{E}$-algebras $\tau : R \rightarrow \hat{R}_\sigma$ that does not depend on the choice of $\beta$.

Proposition 5. Assume that $\text{End}(E_0) = \mathbb{E}$ and $\text{End}(\bar{E}_0) = \kappa$. Then $\mathcal{O} \rightarrow \bar{R}$ is formally smooth, that is, $\bar{R}$ is (non canonically) isomorphic to the ring of formal power series over $\mathcal{O}$ in $2 + (2g - 2)m^2$ variables. Besides, the natural map $\tau : R \rightarrow \hat{R}_\sigma$ is an isomorphism of $\mathbb{E}$-algebras.

Remark 5. One easily checks that if $\bar{E}_0$ is an irreducible $\kappa$-sheaf then $E_0$ is an irreducible $\mathbb{E}$-sheaf on $X$. However, the converse is not true.

We start with the following observation. Let $A \in \mathcal{C}_g$. Denote by $n \subset A$ the maximal ideal. If $A' \subset A$ is an $\mathcal{O}$-subalgebra, which is a finite $\mathcal{O}$-module with $A' \otimes_{\mathcal{O}} \mathbb{E} \rightarrow A$, then $A' = \mathcal{O} \oplus n'$, where $n' = A' \cap n$ is an $\mathcal{O}$-lattice in the $\mathbb{E}$-vector space $n$. Conversely, if $n'$ is an $\mathcal{O}$-submodule of finite type in $n$ with $n' \otimes_{\mathcal{O}} \mathbb{E} \rightarrow n$ then the $\mathcal{O}$-subalgebra $A' \subset A$ generated by $n'$ satisfies $A' \otimes_{\mathcal{O}} \mathbb{E} \rightarrow A$, and $A'$ is a finite $\mathcal{O}$-module.

Lemma 15. 1) Let $A' \subset A$ be an $\mathcal{O}$-subalgebra, which is a finite $\mathcal{O}$-module with $A' \otimes_{\mathcal{O}} \mathbb{E} \rightarrow A$. Let $M$ be a free $A$-module of rank $m$, $M' \subset M$ be an $A'$-submodule of finite type with $M' \otimes_{\mathcal{O}} \mathbb{E} \rightarrow M$. Then there exists an $\mathcal{O}$-subalgebra $A' \subset A'' \subset A$ such that $A''$ is a finite $\mathcal{O}$-module, and $A''M'$ is a free $A''$-module.

2) Let, in addition, $M'_0$ denote the image of $M' \rightarrow M/nM$ then the image of $A''M' \rightarrow M/nM$ also equals $M'_0$, and the latter map induces an isomorphism of $\mathcal{O}$-modules $A''M' \otimes_{A''} \mathcal{O} \rightarrow M'_0$.

Proof Notice that $M'_0 = M'/\langle nM \cap M' \rangle$ is a free $\mathcal{O}$-submodule of rank $m$ (an $\mathcal{O}$-lattice) in the $m$-dimensional $\mathbb{E}$-vector space $M/nM$. Pick $e_1, \ldots, e_m \in M'$ that define an $\mathcal{O}$-basis in $M'_0$. Then $e_1, \ldots, e_m$ is an $A$-basis in $M$. Set $L = A'e_1 + \ldots + A'e_m \subset M'$.

Any $x \in M'$ is written uniquely as $x = a_1e_1 + \ldots + a_me_m$ with $a_i \in A$. Reducing modulo $n$ we learn that every $a_i$ lies in $\mathcal{O} \oplus n$. 

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Let \( u_1, \ldots, u_r \) generate \( M' \) over \( A' \). Write \( u_i = \sum_j a_{ij} e_j \) with \( a_{ij} \in \mathcal{O} \otimes \mathfrak{n} \). Let \( A'' \) be the \( A' \)-subalgebra of \( A \) generated by all \( a_{ij} \). Then \( M' \subset A''L \), so that \( A''M' = A''L \), and \( A''L \) is a free \( A'' \)-module. The second statement is clear. \( \square \)

**Lemma 16.** 1) Let \( V \) be a smooth \( A \)-sheaf on \( X \). Then there exists an \( \mathcal{O} \)-subalgebra \( A' \subset A \), which is a finite \( \mathcal{O} \)-module with \( A' \otimes_\mathcal{O} E \rightarrow A \), a smooth \( A' \)-sheaf \( V' \) on \( X \), and an isomorphism \( \nu : V' \otimes_{A'} A \rightarrow V \) of \( A \)-sheaves on \( X \).

2) Given, in addition, a smooth \( \mathcal{O} \)-sheaf \( V'_0 \) and an isomorphism \( V'_0 \otimes \mathcal{O} E \rightarrow V \otimes_\mathcal{O} E \), one may choose \( A', V', \nu \) such that the isomorphism \((V' \otimes_{A'} \mathcal{O}) \otimes \mathcal{O} E \rightarrow V'_0 \otimes \mathcal{O} E \) induced by \( \nu \) is obtained by extension of scalars from an isomorphism of \( \mathcal{O} \)-sheaves \( V' \otimes_{A'} \mathcal{O} \rightarrow V'_0 \).

**Proof** 1) View \( V \) as a free \( A \)-module with a continuous representation of \( G \). Pick a \( G \)-invariant \( \mathcal{O} \)-lattice \( V_1 \) in \( V \). Pick any \( \mathcal{O} \)-subalgebra \( A' \subset A \), which is a finite \( \mathcal{O} \)-module with \( A' \otimes_\mathcal{O} E \rightarrow A \), and apply Lemma 15 for the \( \mathcal{O} \)-submodule \( A'V_1 \) of \( V \).

2) Recall that \( \mathfrak{n} \subset A \) denotes the maximal ideal. Chose \( V_1 \) with an additional property: the image of \( V_1 \) in \( \mathcal{O} / \mathfrak{n} \mathcal{O} \) is the \( \mathcal{O} \)-lattice \( V'_0 \). Then for any \( \mathcal{O} \)-subalgebra \( A'' \subset A \), which is a finite \( \mathcal{O} \)-module with \( A'' \otimes \mathcal{O} E \rightarrow A \), we still have the same property for \( V' \) replaced by \( A''V_1 \). So, our assertion follows from ii) of Lemma 15. \( \square \)

**Proof of Proposition 3**

Let \( D' \rightarrow D \) be a surjection in \( \mathcal{C}_\mathcal{O} \), whose kernel is a principal ideal \( (t) \subset D' \) with \( t \mathfrak{m}_{D'} = 0 \), where \( \mathfrak{m}_{D'} \) denotes the maximal ideal of \( D' \). We must show that any local homomorphism of \( \mathcal{O} \)-algebras \( \bar{R} \rightarrow D \) can be lifted to \( \bar{R} \rightarrow D' \).

Pick \( k > 0 \) with \( \mathfrak{m}_D^k = 0 \), where \( \mathfrak{m}_D \subset D \) is the maximal ideal. Then \( \mathfrak{m}_D^{k+1} = 0 \). Set \( A = R/\mathfrak{m}_D^{k+1} \), and denote by \( \mathfrak{n} \subset A \) the maximal ideal. By Lemma 16, there is an \( \mathcal{O} \)-subalgebra \( A' \subset A \), which is a finite \( \mathcal{O} \)-module with \( A' \otimes_\mathcal{O} E \rightarrow A \), a smooth \( A' \)-sheaf \( V \) on \( X \) and an isomorphism \( \delta : V \otimes_{A'} E \rightarrow E \otimes_R A \) of \( A \)-sheaves on \( X \) such that the induced isomorphism

\[
(V \otimes_{A'} \mathcal{O}) \otimes \mathcal{O} E \rightarrow E_0' \otimes \mathcal{O} E
\]

of \( \mathcal{E} \)-sheaves is obtained by extension of scalars from an isomorphism \( \gamma : V \otimes_{A'} \mathcal{O} \rightarrow E_0' \) of \( \mathcal{O} \)-sheaves on \( X \). Further, \( \gamma \) endows \( V \) with a structure of an \( A' \)-deformation of \( E_0 \). This defines a local homomorphism of \( \mathcal{O} \)-algebras \( \bar{R} \rightarrow A' \) such that \( E \otimes_R A' \) and \( V \) are isomorphic as \( A' \)-deformations of \( E_0 \).

Notice that the composition \( \bar{R} \rightarrow A' \rightarrow A \) coincides with \( \sigma \). Set \( B = (\bar{R} \otimes \mathcal{E}) / \mathfrak{m}_\sigma^{k+1} \), so that \( \bar{R} \rightarrow A' \) yields a morphism \( B \rightarrow A \) in \( \mathcal{C}_\mathcal{E} \).

**Lemma 17.** The composition \( R \rightarrow \bar{R}_\sigma \rightarrow B \rightarrow A \) coincides with the natural map \( R \rightarrow R/\mathfrak{m}^{k+1} \).

**Proof** Let \( f_i : R \rightarrow C \) be two local homomorphisms of \( \mathcal{E} \)-algebras with \( C \in \text{Ob}(\mathcal{C}_\mathcal{E}) \). If \( h : E \otimes_R f_1 C \rightarrow E \otimes_R f_2 C \) is an isomorphism of \( C \)-sheaves on \( X \) then, for a suitable \( a \in \mathcal{E}^* \), \( ah \) is an isomorphism of \( C \)-deformations of \( E_0 \), which implies \( f_1 = f_2 \). In our case the corresponding \( A \)-sheaves on \( X \) are isomorphic by definition. Indeed, by definition of \( \tau : R \rightarrow \bar{R}_\sigma \), we have \((E \otimes_R \bar{R}_\sigma) \otimes_{\bar{R}_\sigma} B \rightarrow E \otimes_R B \). Finally,

\[
(E \otimes_R B) \otimes_B A \rightarrow (E \otimes_R A') \otimes_{A'} A \rightarrow V \otimes_{A'} A \rightarrow E \otimes_R R/\mathfrak{m}^{k+1}
\]
By Lemma [7], the image $A_0$ of $\tilde{R} \to A'$ generates $A$ as a $\mathbb{E}$-vector space. Replacing $A'$ by $A_0$ and $V$ by $E \otimes_{\tilde{R}} A_0$, we may assume that $\tilde{R} \to A'$ is surjective.

Consider the composition $\tilde{R} \to A' \to A \to A/n^2$. Its image is an $\mathcal{O}$-subalgebra $\mathcal{O} \oplus \tilde{n}$ of $A/n^2$ such that $\tilde{n}$ is an $\mathcal{O}$-lattice in $n/n^2$. Let $r$ denote the dimension of $n/n^2$ over $\mathbb{E}$. Pick $e_1, \ldots, e_r \in \tilde{R}$ whose images in $A/n^2$ define an $\mathcal{O}$-basis of $\tilde{n}$. Let $\rho : \mathcal{O}[t_1, \ldots, t_r] \to \tilde{R}$ be the continuous homomorphism of $\mathcal{O}$-algebras that takes $t_i$ to $e_i$.

We claim that $\rho$ is surjective. To see this it suffices to show that the reduction $\kappa[[t_1, \ldots, t_r]] \to \tilde{R}/\omega \tilde{R}$ of $\rho$ is surjective. Now, $\tilde{R} \to \mathcal{O} \oplus \tilde{n}$ induces a $\kappa$-linear surjective map $\tilde{n}/(\tilde{n}^2, \omega) \to \tilde{n}/\omega \tilde{n}$. The latter is an isomorphism, because the dimension of $\tilde{n}/(\tilde{n}^2, \omega)$ equals $r$ by Proposition [3]. So, $\rho$ is surjective and induces an isomorphism $\mathcal{O}[[t_1, \ldots, t_r]]/(t_1, \ldots, t_r)^{k+1} \cong A'$.

Clearly, $\tilde{R} \to D$ factors as $\tilde{R} \to A' \to D$ and $A' \to D$ can be lifted to $A' \to D'$, so that $\mathcal{O} \to \tilde{R}$ is formally smooth. By Lemma [7], $\tau : R \to \tilde{R}_\sigma$ is injective. Since $\tau$ induces an isomorphism on tangent spaces, it is an isomorphism.

\[ \square \]

(Proposition [3])

Remark 6. If we do not assume that $\text{End}(\tilde{E}_0) = \kappa$ then $F_{\tilde{E}_0}$ has a hull $(\tilde{R}, \tilde{E})$ in the sense of Schlessinger [17], where $\tilde{R}$ is a complete local noetherian $\mathcal{O}$-algebra (defined up to a non canonical isomorphism). One still can define the maps $\sigma : \tilde{R} \to \mathcal{O}$ and $\tau : R \to \tilde{R}_\sigma$ as above (they are no more unique) and show that $\tau$ is injective. We conjecture that this map $\tau : R \to \tilde{R}_\sigma$ is formally smooth, that is, $\tilde{R}_\sigma$ is a ring of formal power series over $R$.

3.3 Cohomology of $\text{Hom}(\tilde{E}_1, \tilde{E}_2)$

In the rest of Sect. 3 we assume the conditions of Proposition [3] satisfied.

Recall that $(\tilde{R}, \tilde{E})$ denotes the universal deformation of $\tilde{E}_0$ over $\mathcal{O}$. Notice that $\tilde{R} \otimes_\mathcal{O} \tilde{R}$ is isomorphic to the ring of formal power series over $\mathcal{O}$ in $2r$-variables, where $r = 2 + (2g - 2)m^2$. Put $\tilde{E}_i = \tilde{E} \otimes_{\tilde{R}} (\tilde{R} \otimes_\mathcal{O} \tilde{R})$, where the homomorphisms $\tilde{R} \to \tilde{R} \otimes \tilde{R}$ correspond to two projections $\text{Spf}(\tilde{R} \otimes \tilde{R}) \to \text{Spf} R$. So, $\text{Hom}(\tilde{E}_1, \tilde{E}_2)$ is a smooth $\tilde{R} \otimes \tilde{R}$-sheaf on $X$. Consider the map

$$
\text{Hom}(\tilde{E}_1, \tilde{E}_2) \to \text{Hom}(\tilde{E}_1, \tilde{E}_2) \otimes_{\tilde{R} \otimes \tilde{R}} \tilde{R} \to \text{End}(\tilde{E}) \to \tilde{R}
$$

Applying the functor $H^2(X, \cdot)$, we get a canonical map $H^2(X, \text{Hom}(\tilde{E}_1, \tilde{E}_2)) \to H^2(X, \tilde{R} \otimes \tilde{R} \sim \tilde{R}(-1))$.

**Proposition 6.** 1) $\Gamma(X, \text{Hom}(\tilde{E}_1, \tilde{E}_2))$ is an object of $D^b_{\text{coh}}(\tilde{R} \otimes \tilde{R})$ that can be represented by a complex $(V^0 \to V^1 \to V^2)$ of free $\tilde{R} \otimes \tilde{R}$-modules with $\text{rk} V^0 = \text{rk} V^2 = 1$, $\text{rk} V^1 = (2g - 2)m^2 + 2$ such that the differential in $V$ is zero modulo the maximal ideal of $\tilde{R} \otimes \tilde{R}$. The complex $V$ is defined up to a non canonical isomorphism of complexes.

2) The canonical map $H^2(X, \text{Hom}(\tilde{E}_1, \tilde{E}_2)) \to \tilde{R}(-1)$ is an isomorphism of $\tilde{R} \otimes \tilde{R}$-modules.

**Proof** 1) By ([1], 6.3), $\Gamma(X, \text{Hom}(\tilde{E}_1, \tilde{E}_2))$ lies in $D^b_{\text{coh}}(\tilde{R} \otimes \tilde{R})$, and we have

$$
\Gamma(X, \text{End}(\tilde{E}_0)) \cong \Gamma(X, \text{Hom}(\tilde{E}_1, \tilde{E}_2)) \otimes_{\tilde{R} \otimes \tilde{R}} \kappa
$$

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By Lemma 18, $\Gamma(X, \mathcal{H}om(\bar{E}_1, \bar{E}_2))$ can be represented by a perfect complex $V$ of $\bar{R} \otimes \bar{R}$-modules whose differential is zero modulo the maximal ideal of $\bar{R} \otimes \bar{R}$, and $V$ is defined up to a non-canonical isomorphism. However, the complex $V \otimes_{\bar{R} \otimes \bar{R}} \kappa$ is defined up to a canonical isomorphism. More precisely, $\bar{V} \otimes_{\bar{R} \otimes \bar{R}} \kappa \rightarrow \Gamma^i(X, \mathcal{E}nd(\bar{E}_0))$ canonically for every $i$. Our first assertion follows.

2) By the projection formulae (8, ii) of Proposition A.1.5,

$$\Gamma(X, \mathcal{H}om(\bar{E}_1, \bar{E}_2)) \otimes_{\bar{R} \otimes \bar{R}} \bar{R} \rightarrow \Gamma(X, \mathcal{E}nd\bar{E}) \rightarrow \Gamma(X, \bar{R}) \oplus \Gamma(X, \mathcal{E}nd0\bar{E})$$

Since $\Gamma(X, \mathcal{E}nd0\bar{E}) = 0$ for $i \neq 1$, the differential in $V \otimes_{\bar{R} \otimes \bar{R}} \bar{R}$ vanishes. Let $A \in \text{Ob}\mathcal{O}$. Let $\bar{R} \rightarrow A$ be a surjective local homomorphism of $\mathcal{O}$-algebras, $I \subset A \otimes \mathcal{O} A$ be the ideal of the diagonal, and $J \subset I$ be another ideal.

The next assertion is an immediate consequence of the universal property of $(\bar{R}, \bar{E})$.

**Lemma 19.** If the images of $\bar{E}_1 \otimes_{\bar{R} \otimes \bar{R}} (A \otimes A)$ and $\bar{E}_2 \otimes_{\bar{R} \otimes \bar{R}} (A \otimes A)$ in $F_{E_0}(A \otimes A/J)$ coincide then $J = I$. □

**Lemma 20.** If the differential $d^0 : V^0 \otimes_{\bar{R} \otimes \bar{R}} B \rightarrow V^1 \otimes_{\bar{R} \otimes \bar{R}} B$ vanishes then $J = I$.

**Proof.** Consider the $B$-deformations $M_i = E_i \otimes_{\bar{R} \otimes \bar{R}} B$ of $\bar{E}_0$ ($i = 1, 2$). By our assumption, $\Gamma(X, \mathcal{H}om(M_1, M_2)) \rightarrow V^0 \otimes_{\bar{R} \otimes \bar{R}} B$ is a free $B$-module of rank 1. Suppose that $J \neq I$ then $M_1$ and $M_2$ are non-isomorphic by Lemma 18. Denote by $\mathfrak{n}$ the maximal ideal of $B$ and set $\text{Ann} \mathfrak{n} = \{ b \in B \mid bn = 0 \}$. By Lemma 19, Ann $\mathfrak{n}$ annihilates $H^0(X, \mathcal{H}om(M_1, M_2))$. Since Ann $\mathfrak{n} \neq 0$, we get a contradiction. □

Consider the complex $V \otimes_{\bar{R} \otimes \bar{R}} (A \otimes A)$. Combining Lemma 20 with the Poincaré duality, one proves that the image of the differential $d^1 : V^1 \otimes_{\bar{R} \otimes \bar{R}} (A \otimes A) \rightarrow V^2 \otimes_{\bar{R} \otimes \bar{R}} (A \otimes A)$ is $I(V^2 \otimes_{\bar{R} \otimes \bar{R}} (A \otimes A))$. In other words, the natural map

$$H^2(X, \mathcal{H}om(\bar{E}_1, \bar{E}_2) \otimes_{\bar{R} \otimes \bar{R}} (A \otimes A)) \rightarrow A(-1)$$

is an isomorphism. Passing to the limit we get the desired assertion. □(Proposition 3)

3.4 **Cohomology of** $\mathcal{H}om(\bar{E}_1, \bar{E}_2)$

Recall that we write Pic$^d X$ for the Picard stack classifying invertible sheaves of degree $d$ on $X$. Let Pic$^d X$ be the corresponding Picard scheme of $X$, so that the natural map $\tau : \text{Pic}^d X \rightarrow$
Pic\(_d\) \(X\) is a \(\mathbb{G}_m\)-gerbe. Chose a closed point \(x \in X\). It defines a section \(\alpha_x : \text{Pic}_d X \to \text{Pic}_d X\) of \(\tau\). Namely, if one considers \(\text{Pic}_d X\) as the moduli scheme of pairs \((A, t)\), where \(A \in \text{Pic}_d X\) and \(t : A \to k\) is a trivialization of the geometric fibre at \(x\) then \(\alpha_x\) sends \((A, t)\) to \(A\).

Define the \(\mathbb{G}_m\)-torsor \(\alpha'_x : xX(d) \to X(d)\) by the cartesian square

\[
\begin{array}{ccc}
xX(d) & \to & X(d) \\
\downarrow & & \downarrow \\
\text{Pic}_d X & \xrightarrow{\alpha_x} & \text{Pic}_d X,
\end{array}
\]

where the right vertical arrow sends a divisor \(D \in X(d)\) to \(\mathcal{O}_X(D)\).

Recall that \(\mathcal{H}om(\bar{E}, \bar{E})\) is a smooth \(\bar{R} \otimes \bar{R}\)-sheaf on \(X\), so that \((\mathcal{H}om(\bar{E}, \bar{E}))^d\) is a constructable \(\bar{R} \otimes \bar{R}\)-sheaf on \(X(d)\) (cf. Sect. 1.4). Let \(x(\mathcal{H}om(\bar{E}, \bar{E}))^d\) denote its inverse image to \(xX(d)\). In this subsection we prove the following result.

**Proposition 7.** For \(d > 0\) there is a canonical isomorphism of \(\bar{R} \otimes \bar{R}\)-modules

\[
\Pi^d_{i=0} \left( xX(d), x(\mathcal{H}om(\bar{E}, \bar{E}))^d \right) \cong \left\{ \begin{array}{ll}
\bar{R}(-d-1), & \text{if } i = 0 \\
0, & \text{if } 0 < i < d,
\end{array} \right.
\]

where the \(\bar{R} \otimes \bar{R}\)-module structure on \(\bar{R}\) is given via the diagonal mapping \(\bar{R} \otimes \bar{R} \to \bar{R}\).

This will be done using Proposition 3 and Appendices A and B.

3.4.1 To prove Proposition 7 we need the following linear algebra lemma.

Let \(A\) be a (commutative) ring of characteristic 0. Consider a complex of \(A\)-modules \(M = (A \to M^{-1} \to A)\), where \(M^{-1}\) is a free \(A\)-module of rank \(r\). (So, \(M^{-2} = M^0 = A\) and \(M^i = 0\) for \(i < -2\).) Suppose that there exists a basis \(e_1, \ldots, e_r \in M^{-1}\) such that \(\bar{\partial}(e_1), \ldots, \bar{\partial}(e_r)\) is a regular sequence for \(A\). Denote by \(I\) the image of \(\bar{\partial}\). Let \(\xi_1 : M \to M[2]\) be a morphism of complexes such that the induced map \(M^{-2} \to M^0\) is an isomorphism. Define the morphism \(\xi : \otimes_{i=1}^d M \to (\otimes_{i=1}^d M)[2]\) as \(\xi_1 \otimes \text{id} \cdots \otimes \text{id} + \ldots + \text{id} \otimes \cdots \otimes \text{id} \otimes \xi_1\). Then there is a (unique) morphism \(\xi_d : \text{Sym}^d(M) \to \text{Sym}^d(M)[2]\) such that the diagram commutes

\[
\begin{array}{ccc}
\otimes_{i=1}^d M & \xrightarrow{\xi} & \otimes_{i=1}^d M[2] \\
\cup & \uparrow & \cup \\
\text{Sym}^d(M) & \xrightarrow{\xi_d} & \text{Sym}^d(M)[2]
\end{array}
\]

Notice that \(\text{Sym}^d(M)\) is a bounded complex of free \(A\)-modules of finite type.

**Lemma A.1.** Define the object \(K \in D_{\text{parf}}(A)\) from the distinguished triangle \(K \to \text{Sym}^d(M) \xrightarrow{\xi_d} \text{Sym}^d(M)[2]\). Then \(\mathcal{H}^0(K) \to A/I\) and \(\mathcal{H}^i(K) = 0\) for \(-d < i < 0\) and for \(i > 0\).
The proof is given in Appendix A.

3.4.2 Denote by $Y_x : X^{(d-1)} \hookrightarrow X^{(d)}$ the closed immersion that sends $D$ to $D + x$. We consider $Y_x$ as a divisor on $X^{(d)}$ and write sometimes $Y_x \hookrightarrow X^{(d)}$ for the same closed subscheme. Denote by $Y_x'$ the inverse image of $Y_x$ under sym : $X^d \to X^{(d)}$. (In other words, the closed immersion $Y_x' \hookrightarrow X^d$ is obtained from $Y_x \hookrightarrow X^{(d)}$ by the base change $\text{sym} : X^d \to X^{(d)}$). Denote by $Y_x''$ the inverse image of $x$ under $\text{pr}_i : X^d \to X$. So, $Y_x'$ and $Y_x''$ are divisors on $X^d$, and we have $Y_x' = Y_x'' + \ldots + Y_x''$.

Consider the invertible sheaf $O(Y_x)$ on $X^{(d)}$.

Lemma 21. $x X^{(d)}$ is naturally isomorphic to the total space of $O(Y_x)$ with removed zero section.

Proof Denote by $Y^{univ} \hookrightarrow X^{(d)} \times X$ the universal divisor. Clearly, the inverse image of $Y^{univ}$ under the closed immersion $X^{(d)} \times x \hookrightarrow X^{(d)} \times X$ is the divisor $Y_x \times x$ on $X^{(d)} \times x$ with some multiplicity $r > 0$. It is enough to show that $r = 1$, i.e., to show that the following square is cartesian

$$
\begin{array}{c}
\uparrow & \uparrow \\
Y^{univ} \hookrightarrow X^{(d)} \times X & Y_x \times \text{id} \hookrightarrow X^{(d)} \times x \\
X^{(d-1)} \times x & X^{(d-1)} \times x
\end{array}
$$

To do so, denote by $Y^{univ}$ the inverse image of $Y^{univ}$ under $X^d \times X \xrightarrow{\text{sym} \times \text{id}} X^{(d)} \times X$. Since the inverse image of $Y^{univ}$ under the closed immersion $X^d \times x \hookrightarrow X^{(d)} \times x$ is $Y_x \times x$ with multiplicity one, our assertion follows. □

Proof of Proposition 7

Let for brevity $\mathcal{R} = \hat{R} \otimes_O \hat{R}$. By Lemma 3.1, we have a distinguished triangle

$$(\alpha_x')_! \mathcal{R} \to \mathcal{R}(-1)[2] \to \mathcal{R}$$

in $\mathcal{D}^b(X^{(d)}, \mathcal{R})$, where $c \in H^2(X^{(d)}, \mathcal{R}(1))$ is the Chern class of $O(Y_x)$. By Künneth's formulae,

$$H^2(X^{(d)}, \mathcal{R}) = \bigoplus_{i_1 + \ldots + i_d = 2} H^{i_1}(X, \mathcal{R}) \otimes \cdots \otimes H^{i_d}(X, \mathcal{R})$$

and $H^2(X^{(d)}, \mathcal{R}) = H^2(X^d, \mathcal{R}^{Sd})$. Denote by $c'$ the image of $c$ in $H^2(X^d, \mathcal{R}(1))$. The construction of the Chern class is functorial, so that $c'$ is the Chern class of $O(Y_x')$. Since $Y_x' = Y_x'' + \ldots + Y_x''$, we get

$$c' = c_1 \otimes 1 \otimes \cdots \otimes 1 + \ldots + 1 \otimes \cdots \otimes 1 \otimes c_1 \in H^2(X, \mathcal{R}(1)) \otimes H^0(X, \mathcal{R}) \otimes \cdots \otimes H^0(X, \mathcal{R}) \otimes \cdots \otimes H^0(X, \mathcal{R}) \otimes H^2(X, \mathcal{R}(1)) \subset H^2(X, \mathcal{R}(1)),$$

where $c_1 \in H^2(X, \mathcal{R}(1))$ is the Chern class of the invertible sheaf $O(x)$ on $X$. Since $O(x)$ is of degree 1, we have $c_1 \neq 0$. On $X^{(d)}$ we get a distinguished triangle

$$(\alpha_x')^! (\mathcal{H}om(\tilde{E}_1, \tilde{E}_2))^{(d)}(1)[2] \to (\mathcal{H}om(\tilde{E}_1, \tilde{E}_2))^{(d)}(1)[2] \to (\mathcal{H}om(\tilde{E}_1, \tilde{E}_2))^{(d)}(1)[2]$$

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Let $\xi_d$ be the morphism obtained from $(\mathcal{H}om(\tilde{E}_1, \tilde{E}_2))^{(d)} \xrightarrow{\zeta} (\mathcal{H}om(\tilde{E}_1, \tilde{E}_2))^{(d)(1)[2]}$ by applying the functor $R\Gamma(X^{(d)}, \_ \_ )$. We get the distinguished triangle in $D^b_{coh}(\mathcal{R})$

$$R\Gamma_c(x^{(d)}, x(\mathcal{H}om(\tilde{E}_1, \tilde{E}_2))^{(d)(1)[2]} \to R\Gamma(x^{(d)}, (\mathcal{H}om(\tilde{E}_1, \tilde{E}_2))^{(d)}) \xrightarrow{\xi_d} R\Gamma(x^{(d)}, (\mathcal{H}om(\tilde{E}_1, \tilde{E}_2))^{(d)(1)[2]}$$

Since $\text{sym}_1(\mathcal{H}om(\tilde{E}_1, \tilde{E}_2))^{\mathbb{Z}_d}$ is a direct sum over the irreducible representations of $S_d$, the same holds for $R\Gamma(x^{(d)}, (\mathcal{H}om(\tilde{E}_1, \tilde{E}_2))^{\mathbb{Z}_d}) = \bigotimes_{i=1}^{d} R\Gamma(x, \mathcal{H}om(\tilde{E}_1, \tilde{E}_2))$, and we have naturally

$$R\Gamma(x^{(d)}, (\mathcal{H}om(\tilde{E}_1, \tilde{E}_2))^{(d)}) \xrightarrow{\zeta} \bigotimes_{i=1}^{d} R\Gamma(x, \mathcal{H}om(\tilde{E}_1, \tilde{E}_2))$$

Denote also by $\xi : \bigotimes_{i=1}^{d} R\Gamma(x, \mathcal{H}om(\tilde{E}_1, \tilde{E}_2)) \to \bigotimes_{i=1}^{d} R\Gamma(x, \mathcal{H}om(\tilde{E}_1, \tilde{E}_2))(1)[2]$ the morphism obtained from $\text{sym}_1(\mathcal{H}om(\tilde{E}_1, \tilde{E}_2))^{\mathbb{Z}_d} \xrightarrow{\zeta} \text{sym}_1(\mathcal{H}om(\tilde{E}_1, \tilde{E}_2))^{\mathbb{Z}_d}(1)[2]$ by applying the functor $R\Gamma(x^{(d)}, \_ \_ )$.

The map $\xi$ is a cup product by an element $c \in H^2(x^{(d)}, \mathcal{R}(1))$. Replacing the cup product on $X^{(d)}$ by that on $X^d$, we get

$$\xi = \xi_1 \otimes \text{id} \otimes \ldots \otimes \text{id} + \ldots + \text{id} \otimes \ldots \otimes \text{id} \otimes \xi_1$$

Pick a perfect complex $M$ of $\mathcal{R}$-modules that represents $R\Gamma(x, \mathcal{H}om(E_1, E_2))$. We assume that $M$ is chosen as in 1) of Proposition 3. Pick a morphism $\tilde{\xi}_1 : M \to M(1)[2]$ that represents $\xi_1$ in $D_{parf}(\mathcal{R})$ (so, $\tilde{\xi}_1$ is defined up to a homotopy). Since $c_1 \neq 0$, it follows that $\tilde{\xi}_1$ is given by the diagram

$$M^0 \to M^1 \to M^2$$

where the vertical arrow is an isomorphism of $\mathcal{R}$-modules.

Let $r = 2 + (2g - 2)m^2$. By Proposition 3, $M^1$ is a free $\mathcal{R}$-module of rank $r$. Further, any $r$ elements in $\mathcal{R}$, which generate the ideal of the diagonal, form a regular sequence in $\mathcal{R}$. So, combining 2) of Proposition 3 with Lemma 3 we get the desired assertion. □

4 Main Global Theorem

4.1 Let $\mathbb{E}$ be a finite extension of $\mathbb{Q}_\ell$ that contains the group of $p$-th roots of unity, $\mathcal{O} \subset \mathbb{E}$ be its ring of integers, and $\kappa$ be the residue field of $\mathcal{O}$. Let $E_0$ be a smooth $\mathbb{E}$-sheaf of rank $n$ on $X$ such that $E_0 \otimes_{\mathbb{E}} \mathbb{Q}_\ell$ is irreducible (this, in particular, implies $\text{End}(E_0) = \mathbb{E}$).

Choose a smooth $\mathcal{O}$-sheaf $E'_0$ on $X$ together with an isomorphism $E'_0 \otimes_{\mathcal{O}} \mathbb{E} \xrightarrow{\sim} E_0$ and let $\tilde{E}_0 = E'_0 \otimes_{\mathcal{O}} \kappa$. The local system $\tilde{E}_0$ may not be irreducible. We impose an additional assumption $\text{End}(\tilde{E}_0 \otimes_{\kappa} \tilde{\kappa}) = \tilde{\kappa}$, where $\tilde{\kappa}$ is an algebraic closure of $\kappa$. This automatically implies $\text{End}(\tilde{E}_0) = \kappa$. 

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Let \((\bar{R}, \bar{E})\) be the universal deformation of \(\bar{E}_0\) over \(\mathcal{O}\). It is equipped with a local homomorphism of \(\mathcal{O}\)-algebras \(\sigma : \bar{R} \rightarrow \mathcal{O}\) (cf. Sect. 3.2).

Put \(\bar{E}_i = E_i \otimes_{R} (\bar{R} \otimes \mathcal{O})\) \((i = 1, 2)\), where the homomorphisms \(\bar{R} \rightarrow \bar{R} \otimes \bar{R}\) correspond to two projections \(\text{Spf}(\bar{R} \otimes \bar{R}) \rightarrow \text{Spf}(\bar{R})\). These are smooth \(\bar{R} \otimes \bar{R}\)-sheaves on \(X\) of rank \(n\).

By abuse of notation, the composition of the diagonal map \(\bar{R} \otimes \bar{R} \rightarrow \bar{R}\) will also be denoted by \(\sigma\). So, we have the categories \(\text{D}^b_c(\cdot, \bar{R} \otimes \bar{R})_{\sigma}\) (cf. Sect. 1.4).

Our main result is the following.

**Main Global Theorem.** Suppose that Conjecture \([3]\) is true. Then for any integer \(d\) there is a canonical isomorphism in \(\text{D}^b_c(\text{Spec} \, k, \bar{R} \otimes \bar{R})_{\sigma}\)

\[
\text{RG}_{i}((\text{Bun}_n^d, \text{Aut}_E^d \otimes (\bar{R} \otimes \bar{R})_{\sigma}) \rightarrow \bar{R})
\]

where \(\bar{R}\) is considered as a \(\bar{R} \otimes \bar{R}\)-module via the diagonal map \(\bar{R} \otimes \bar{R} \rightarrow \bar{R}\).

**Remarks.**

i) As in Sect. 1.4, let \((\bar{R} \otimes \bar{R})_{\sigma}\) denote the localization of \(\bar{R} \otimes \bar{R}\) in the multiplicative system \(\{x \in \bar{R} \otimes \bar{R} \mid \sigma(x) \neq 0\}\). Notice that \(\bar{R}_{\sigma}\) is the localization of \(\bar{R}\) in \(\{x \in \bar{R} \mid \sigma(x) \neq 0\}\).

By Proposition \([3]\),

\[
\text{Aut}_{\bar{E}_i} \rightarrow \text{Aut}_{\bar{E}} \otimes_{\bar{R}_{\sigma}} (\bar{R} \otimes \bar{R})_{\sigma}
\]

is an object of \(\text{Perv}_{fl}(\text{Bun}_n^d, \bar{R} \otimes \bar{R})_{\sigma}\). By Remark \([3]\), the core of the natural \(t\)-structure on \(\text{D}^b_c(\text{Spec} \, k, \bar{R} \otimes \bar{R})_{\sigma}\) is a full subcategory of the category of \((\bar{R} \otimes \bar{R})_{\sigma}\)-modules. Since

\[
\bar{R} \otimes_{\bar{R} \otimes \bar{R}} (\bar{R} \otimes \bar{R})_{\sigma} \rightarrow \bar{R}_{\sigma},
\]

Main Global Theorem can be reformulated as follows: for any integers \(i, d\) there is a canonical isomorphism of \((\bar{R} \otimes \bar{R})_{\sigma}\)-modules

\[
\text{H}^i_c((\text{Bun}_n^d, \text{Aut}_E^d \otimes (\bar{R} \otimes \bar{R})_{\sigma}) \rightarrow \bar{R}_{\sigma}, \quad \text{if } i = 0,
\]

\[
0, \quad \text{if } i \neq 0,
\]

where \(\bar{R}_{\sigma}\) is viewed as a \((\bar{R} \otimes \bar{R})_{\sigma}\)-module via the localized diagonal map \((\bar{R} \otimes \bar{R})_{\sigma} \rightarrow \bar{R}_{\sigma}\).

ii) The stack \(\text{Bun}_n^d\) is not of finite type (for \(n > 1\)). However, the cuspidality condition of Conjecture \([4]\) implies that \(\text{Aut}_E^d\) is the extension by zero from a substack of finite type of \(\text{Bun}_n^d\).

So, in fact, we calculate the cohomology of a stack of finite type.

iii) The definition of G. Laumon and L. Moret-Bailly ([14], 18.8) of the cohomology with compact support of a stack is applicable here, because \(\text{Bun}_n^d\) is a Bernstein-Lunts stack (cf. Remark \([3]\)).

4.2 Essentially, the idea is to derive Main Global Theorem from Main Local Theorem ([14]). Actually, instead of using Main Local Theorem, we will replace it by the following statement, which is easier to prove as soon as Conjecture \([4]\) is assumed true.

Let \(\eta : \text{Bun}_n^d \rightarrow \text{Pic}^d X\) be the map that sends \(L\) to \((\det L) \otimes \Omega^{1-n}+(2-n)+\ldots+(n-n)\). Let us write \(\gamma : X^{(d)} \rightarrow \text{Pic}^d X\) for the map that sends \(D \in X^{(d)}\) to \(\mathcal{O}_X(D)\).
Proposition 8. Assume that Conjecture [4] is true. Then for each \( d \geq 0 \) there is a canonical isomorphism in \( D^b_c(\text{Pic}^d X, R) \),

\[
\eta(\text{Aut}^d_{E_1} \otimes \mathfrak{r}(n\mathcal{K}^d_{E_2})) \sim \gamma_!\text{Hom}(\bar{E}_1, \bar{E}_2)(d)[d + n^2(g - 1)](\frac{d + n^2(g - 1)}{2})
\]

The complex \( n\mathcal{K}^0_E \) does not depend on \( \bar{E} \) and will be denoted \( n\mathcal{K}^0 \). To prove the above proposition, we will only use the particular case \( d = 0 \) of Main Local Theorem under the following form.

Lemma 22. There is a canonical isomorphism \( \eta(\mathfrak{r}(n\mathcal{K}^0) \otimes n\mathcal{K}^0) \sim \gamma R \) in \( D^b_c(\text{Pic}^0 X, R) \).

Proof We have \( R\Gamma_c(\mathcal{K}^1, \mathcal{L}_\psi) = 0 \) not only in \( D^b_c(\text{Spec} k, \mathbb{Q}_\ell) \) but also in \( D^b_c(\text{Spec} k, R) \). This can be seen, for example, from ([13], Lemma 3.3). Therefore, the proof given in ([14], Lemma 6) holds for \( \mathbb{Q}_\ell \)-sheaves replaced by \( R \)-sheaves. □

Proof of Proposition 8. By Proposition 2, \( D(\text{Aut}_E) \sim \text{Aut}_E \). Therefore,

\[
\eta(\text{Aut}^d_{E_1} \otimes \mathfrak{r}(n\mathcal{K}^d_{E_2})) \sim \eta \otimes \text{RHom}(\mathfrak{r}(n\mathcal{K}^d_{E_2}), \text{Aut}^d_{E_1})
\]

Recall the map \( \text{supp} : n \text{Mod}_d \to X^{(d)} \) (cf. Sect. 2.2). Let \( \pi : n \text{Mod}_d \to \text{Sh}^d_0 \) be the map that sends \( (L \subset L') \) to \( L'/L \). Consider the diagram

\[
\begin{array}{ccc}
\text{Bun}_n & \xleftarrow{h} & \text{Bun}_n \\
\hphantom{\text{Bun}_n} & \searrow \text{supp} & \\
\text{n Mod}_d & \xrightarrow{h} & \text{Bun}_n,
\end{array}
\]

where \( h^- \) (resp., \( h^+ \)) sends \( (L \subset L') \in n \text{ Mod}_d \) to \( L \) (resp., to \( L' \)). Following [4], consider the averaging functor \( \mathcal{H}^d_{n,E} : D(\text{Bun}_n) \to D(\text{Bun}_n) \) given by

\[
\mathcal{H}^d_{n,E}(K) = h^+((h^-)^*K \otimes \pi^*\mathcal{L}_E^d)[dn](\frac{dn}{2})
\]

We have \( \mathcal{H}^d_{n,E}(\mathfrak{r}(n\mathcal{K}^0)) \sim \mathfrak{r}(n\mathcal{K}^d_E) \). Define the functor \( \mathcal{H}^{-d}_{n,E} : D(\text{Bun}_n) \to D(X^{(d)} \times \text{Bun}_n) \) by

\[
\mathcal{H}^{-d}_{n,E}(K') = (\text{supp} \times h^-)((\pi^*\mathcal{L}_E^d \otimes h^+K')[dn](\frac{dn}{2})
\]

Let \( \epsilon : X^{(d)} \times \text{Pic}^0 X \to X^{(d)} \times \text{Pic}^d X \) be the isomorphism that sends \( (D \in X^{(d)}, \mathcal{A} \in \text{Pic}^0 X) \) to \( (D, \mathcal{A}(D)) \). Denote by \( p : X^{(d)} \times \text{Bun}^0_n \to \text{Bun}^0_n \) and \( p' : X^{(d)} \times \text{Pic}^d X \to \text{Pic}^d X \) the projections. The next result is a straightforward application of the formalism of six functors.

Lemma 23. For \( K \in D(\text{Bun}^0_n) \) and \( K' \in D(\text{Bun}^d_n) \) we have

\[
\eta_* \text{RHom}^{(d)}_{n,E}(K, K') \sim p'\epsilon_*(\text{id} \times \eta)_* \text{RHom}(p^*K, \mathcal{H}^{-d}_{n,E}(K'))
\]

□
Applying this lemma we get
\[ \eta_* \mathcal{R} \text{Hom}(\mathcal{O}((n\mathcal{K}^d_{E_2})), \text{Aut}_d^{E_1}) \cong p'_* \epsilon_* (\text{id} \times \eta)_* \mathcal{R} \text{Hom}(\mathcal{O}((n\mathcal{K}^0)), \mathcal{H}_{n,E_2}^{-d}(\text{Aut}_d^{E_1})) \]
By (7. 9.5), we have
\[ \mathcal{H}_{n,E_2}^{-d}(\text{Aut}_d^{E_1}) \cong (E_1 \otimes E_2^*)(d) \boxtimes \text{Aut}_d^{E_1}[d](\frac{d}{2}) \]
This yields an isomorphism
\[ \mathcal{R} \text{Hom}(\mathcal{O}((n\mathcal{K}^0)), \mathcal{H}_{n,E_2}^{-d}(\text{Aut}_d^{E_1})) \cong (E_1 \otimes E_2^*)(d) \boxtimes \mathcal{R} \text{Hom}(\mathcal{O}((n\mathcal{K}^0)), \text{Aut}_d^{E_1}[d](\frac{d}{2}) \]
Applying Lemma 22, we get
\[ \eta_* \mathcal{R} \text{Hom}(\mathcal{O}((n\mathcal{K}^0)), \text{Aut}_d^{E_1}) \cong \eta_* \mathcal{D}(\mathcal{O}((n\mathcal{K}^0)) \otimes \text{Aut}_d^{E_1}) \cong \mathcal{D}(\mathcal{O}((n\mathcal{K}^0)) \otimes \text{Aut}_d^{E_1}) \cong \]
\[ \mathcal{D}(\mathcal{O}((n\mathcal{K}^0)) \otimes \text{Aut}_d^{E_1})\]
Since the diagram commutes
\[ X(d) \times X(0) \xrightarrow{\cong} X(d) \]
\[ \downarrow \text{id} \times \gamma \]
\[ X(d) \times \text{Pic}^0 X \xleftarrow{\cong} X(d) \times \text{Pic}^d X \xrightarrow{p'_*} \text{Pic}^d X, \]
we get
\[ \eta_* \mathcal{R} \text{Hom}(\mathcal{O}((n\mathcal{K}^d_{E_2})), \text{Aut}_d^{E_1}) \cong p'_* \epsilon_* ((E_1 \otimes E_2^*)(d) \boxtimes \gamma_* \mathcal{R})[d + n^2(1 - g)](\frac{d + n^2(1 - g)}{2}) \]
Applying the Verdier duality functor \(\mathcal{D}\), we get the desired assertion.
\(\square\)(Proposition 8)

4.3 Pick a closed point \(x \in X\). Recall that it defines the map \(\alpha_x : \text{Pic}^d X \rightarrow \text{Pic}^d X\) (cf. Sect. 3.4). Define the stacks \(\mathcal{M}_d\) from the cartesian squares
\[ \begin{array}{ccc}
\mathcal{M}_d & \rightarrow & n\mathcal{M}_d \\
\downarrow \varepsilon & & \downarrow \varepsilon \\
\mathcal{M}_n^d & \xrightarrow{\beta} & \mathcal{M}_n^d \\
\downarrow & & \downarrow \\
\text{Pic}^d X & \xrightarrow{\alpha} & \text{Pic}^d X
\end{array} \]
Note that the composition \(\mathcal{M}_n^d \xrightarrow{\beta} \mathcal{M}_n^d \xrightarrow{\alpha} \mathcal{M}_d\) is a \(\mu_n\)-gerbe, so that \((\alpha \circ \beta) : \mathcal{M}_d \rightarrow \mathcal{M}_d\). Thus, in Main Global Theorem we may and will replace the calculation of cohomologies of \(\mathcal{M}_n^d\) by that of \(\mathcal{M}_n^d\).
Definition 8. Let $x \text{Aut}^d_E$ be the pull-back of $\text{Aut}^d_E$ under $x \text{Bun}^d_n \to \text{Bun}^d_n$. Let also $x_nK^d$ be the pull-back of $nK^d_E[1](\frac{1}{2})$ under $x_n\mathcal{M}_d \to n\mathcal{M}_d$.

Proposition 8 admits the following immediate corollary.

Corollary 3. Assume that Conjecture 4 is true. Then for any $d \geq 0$ there is a canonical isomorphism in $D^b_c(\text{Spec} k, \hat{R} \otimes \hat{R})$

$$R\Gamma_c(x \text{Bun}^d_n, x \text{Aut}^d_{E_1} \otimes \eta_x(\frac{x}{n}K^d_{E_2})) \cong R\Gamma_c(xX^{(d)}, x\text{Hom}(\tilde{E}_1, \tilde{E}_2)^{(d)}[d + n^2(g - 1) + 2](\frac{d + n^2(g - 1) + 2}{2}) \square$$

For $d \geq 0$ define the complex $N^d$ on $x \text{Bun}^d_n$ as $\text{Hom}(\tilde{R} \otimes \tilde{R})[2d - 2n^2(g - 1)](d - n^2(g - 1))$. Combining Corollary 3 with Proposition 7, we get the following result.

Corollary 4. Assume that Conjecture 4 is true. Then for any $d > 0$ there is a canonical isomorphism in $D^b_c(\text{Spec} k, \hat{R} \otimes \hat{R})$

$$\tau_{\geq 1-d} R\Gamma_c(x \text{Bun}^d_n, x \text{Aut}^d_{E_1} \otimes x \text{Aut}^d_{E_2} \otimes N^d) \cong \hat{R},$$

where the $\hat{R} \otimes \hat{R}$-module structure on $\hat{R}$ is given via the diagonal map $\hat{R} \otimes \hat{R} \to \hat{R}$. \square

Proof of Main Global Theorem

Recall that in 4 a vector bundle $M$ is called very unstable if it can be represented as a direct sum of two vector bundles $M \supset M_1 \oplus M_2$ with $\text{Ext}^1(M_1, M_2) = 0$. Let $\text{Bun}^\text{vuns}_n \subset \text{Bun}_n$ denote the substack of very unstable vector bundles. By the cuspidality property, $*$-restriction of $\text{Aut}_E$ to $\text{Bun}^\text{vuns}_n$ vanishes.

Recall that we have fixed a line bundle $L^{\text{est}}$ on $X$ (cf. Sect. 2.4). There is a constant $c_{g,n}$ such that for $d \geq c_{g,n}$ and $M \in \text{Bun}^d_n$ the condition $\text{Hom}(M, L^{\text{est}}) \neq 0$ implies that $M$ is very unstable.

Pick $c' \in \mathbb{Z}$ such that for each $d \in \mathbb{Z}$ all the nontrivial cohomology sheaves of $\text{Aut}^d_E$ with respect to the usual $t$-structure are places in degrees $\leq c'$, the existence of such constant follows from formulae 11.

Given a pair of integers $i$ and $d$, calculate $\Pi^i_c(\text{Bun}^d_n, \text{Aut}^d_{E_1} \otimes \text{Aut}^d_{E_2})$ as follows. Pick $k \in \mathbb{Z}$ large enough, so that if we put $d' = d + kn$ then the following conditions are satisfied:

A) $d' > 0$, $d' \geq c_{g,n}$, $d' > n^2(g - 1)$

B) $i \geq 1 - d'$, $i > -2d' + 2n^2(g - 1) + 2c' + 2\text{dim}(\text{Bun}^d_n)$

We have used the fact that the dimension of $x \text{Bun}^d_n$ does not depend on $d'$.

Consider the map $\text{mult}_{kx} : \text{Bun}^d_n \to \text{Bun}^d_n$ that sends $L$ to $L(kx)$. By Proposition 2 we have

$$\text{mult}_{kx} \text{Aut}^d_E \cong \text{Aut}^d_E \otimes (\wedge^n \tilde{E})^\otimes x.$$
Therefore,

\[ H^1_c(Bun^d_n, Aut^d_{E_1} \otimes Aut^d_{E_2}) \otimes (\wedge^n E_1 \otimes \wedge^n E_2) \overset{\otimes k}{\rightarrow} H^1_c(Bun^d_n, Aut^d_{E_1} \otimes Aut^d_{E_2}) \overset{\otimes}{\rightarrow} H^1_c(x \text{Bun}^d_n, x \text{Aut}^d_{E_1} \otimes x \text{Aut}^d_{E_2}) \]

Recall the stack \( \text{Bun}^{est}_n \) defined in the proof of Proposition 3. The condition A) implies that \( \text{Aut}^d_E \) is the extension by zero from \( \text{Bun}^{est}_n \cap \text{Bun}^d_n \). Let \( U \) denote the preimage of \( \text{Bun}^{est}_n \cap \text{Bun}^d_n \) under \( \beta_x : x \text{Bun}^d_n \to \text{Bun}^d_n \). Then over \( U \) the map \( g_x : x_n M^d \to x \text{Bun}^d_n \) is a vector bundle of rank \( d' - n^2(g - 1) \) with removed zero section. So, over \( U \) we have

\[ \tau_{\geq -2d' + 2n^2(g-1) + 2} N^{d'} \cong \hat{R} \hat{\otimes} \hat{R} \]

Now, using condition B), from Corollary 3 we conclude that

\[ H^1_c(x \text{Bun}^d_n, x \text{Aut}^d_{E_1} \otimes x \text{Aut}^d_{E_2}) \overset{\otimes}{\rightarrow} \begin{cases} \hat{R} \sigma, & \text{if } i = 0 \\ 0, & \text{if } i \neq 0, \end{cases} \]

as \((\hat{R} \hat{\otimes} \hat{R})_\sigma\)-modules. Since \( (\wedge^n E_1 \otimes \wedge^n E_2) \) is a free \( \hat{R} \hat{\otimes} \hat{R} \)-module of rank 1, and

\[ (\wedge^n E_1 \otimes \wedge^n E_2) \hat{\otimes}_{\hat{R} \hat{\otimes} \hat{R}} \hat{R} \overset{\otimes}{\rightarrow} \hat{R} \]

canonically, we are done. \( \Box \)

### A Some linear algebra

In this appendix we prove Lemma A.1 (cf. Sect. 3.4.1). Let \( A \) be a commutative ring of characteristic 0. Let \( M \) be a bounded complex of \( A \)-modules. Then \( S_d \) acts on the complex \( \bigotimes_{i=1}^d M \), and we put \( \text{Sym}^d(M) = (\bigotimes_{i=1}^d M)^{S_d} \). Clearly, \( \text{Sym}^{k+l}(M) \) is a direct summand of \( \text{Sym}^k(M) \otimes_A \text{Sym}^l(M) \), so that we have both natural maps

\[ \text{Sym}^k(M) \otimes_A \text{Sym}^l(M) \to \text{Sym}^{k+l}(M) \quad \text{and} \quad \text{Sym}^{k+l}(M) \to \text{Sym}^k(M) \otimes_A \text{Sym}^l(M) \]

We will be interested in complexes \( M \) of the form \( \cdots \to M^{-2} \to M^{-1} \to A \to 0 \), i.e., we suppose that \( M^0 = A \) and \( M^i = 0 \) for \( i > 0 \). Denote by \( \sigma_{\leq k}, \sigma_{\geq k} \) the 'foolish' truncation functors. Put \( V = \sigma_{\leq -1} M \), so the sequence of complexes

\[ 0 \to A \to M \to V \to 0 \]

is exact. Define a morphism \( f_d : \text{Sym}^d(M) \to \text{Sym}^{d+1}(M) \) as the composition \( A \otimes \text{Sym}^d(M) \to M \otimes \text{Sym}^d(M) = \text{Sym}^1(M) \otimes \text{Sym}^d(M) \to \text{Sym}^{d+1}(M) \). We get an inductive system of complexes \( (\text{Sym}^d(M), f_d)_{d \in \mathbb{N}} \). Put \( \text{Sym}^\infty(M) = \varinjlim \text{Sym}^d(M) \).
Lemma A.2. For any $d \geq 0$ the sequence of complexes

$$0 \to \text{Sym}^d(M) \xrightarrow{f_d} \text{Sym}^{d+1}(M) \to \text{Sym}^{d+1}(V) \to 0$$

is exact, where the second arrow is defined by functoriality from the natural morphism $M \to V$.

Proof If $k \leq 0$ then $(\bigotimes_{i=1}^{d+1} M)^k$ is the direct sum

$$\left[ \bigotimes_{i_1 + \cdots + i_{d+1} = k} M^{i_1} \otimes \cdots \otimes M^{i_{d+1}} \right] \oplus \left[ (\bigotimes_{i=1}^{d+1} V)^k \right]$$

$i_j = 0$ for some $j$

The group $S_{d+1}$ acts on every summand in square brackets. It is easy to understand that the invariants

$$\left[ \bigotimes_{i_1 + \cdots + i_{d+1} = k} M^{i_1} \otimes \cdots \otimes M^{i_{d+1}} \right]_{S_{d+1}}$$

are identified with $(\text{Sym}^d(M))^k$.

From the above lemma it follows that the natural map $\text{Sym}^d(M) \to \text{Sym}^\infty(M)$ is injective and $\sigma_{\geq -d} \text{Sym}^d(M) \to \sigma_{\geq -d} \text{Sym}^\infty(M)$ is an isomorphism for $d \geq 0$. So, $\text{Sym}^\infty(M)$ is a filtered complex with the filtration $(\text{Sym}^d(M))_{d \in \mathbb{N}}$. Since the morphisms $\text{Sym}^k(M) \otimes \text{Sym}^l(M) \to \text{Sym}^{k+l}(M)$ are compatible with $f_d$, by passing to the limit we get the morphism of multiplication $\text{Sym}^\infty(M) \otimes \text{Sym}^\infty(M) \to \text{Sym}^\infty(M)$ (compatible with the above filtration).

Lemma A.3. Suppose that $M = (M^{-1} \to A)$, i.e., $M^i = 0$ for $i < -1$. Then $\text{Sym}^\infty(M)$ is the Koszul complex for $M$. In addition,

$$\text{Sym}^d(M) \to \sigma_{\geq -d} \text{Sym}^\infty(M)$$

is an isomorphism for any $d \geq 0$.

Now we impose on $M$ the additional condition: $M^i = 0$ for $i < -2$ and $M^{-2} = A$, so $M = (A \to M^{-1} \to A)$. Then we have

Lemma A.4. Let $d \geq 0$.

1) For any $k$ we have $(\text{Sym}^d(M))^k = (\text{Sym}^d(M))^{-2d-k}$ canonically.

2) If $0 \leq k \leq d$ then $(\text{Sym}^d(M))^{-k} = \wedge^k(M^{-1}) \oplus \wedge^{k-2}(M^{-1}) \oplus \wedge^{k-4}(M^{-1}) \oplus \ldots$

Set $W = \sigma_{\geq -1} M$. Pick $c_1 \in A^*$ and consider the exact sequence $0 \to W \to M \to A[2] \to 0$, where $M \to A[2]$ is given by $M^{-2} \to A$. Let us define a morphism $g_d : \text{Sym}^{d+1}(M) \to \text{Sym}^d(M)[2]$ as the composition $\text{Sym}^{d+1}(M) \to \text{Sym}^1(M) \otimes \text{Sym}^d(M) = M \otimes \text{Sym}^d(M) \to (A[2]) \otimes \text{Sym}^d(M)$. The proof of the next result is analogous to that of Lemma A.2.

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Lemma A.5. For any $d \geq 0$ the sequence
\[
0 \to \text{Sym}^{d+1}(W) \to \text{Sym}^{d+1}(M) \xrightarrow{\varphi} \text{Sym}^d(M)[2] \to 0
\]
is exact, where the first arrow is defined by functoriality from the natural map $W \to M$. □

Remark 7. i) The map $(\text{Sym}^{d+1}(M))^k \xrightarrow{g} (\text{Sym}^d(M))^{k+2}$ is described as follows. For $-d-1 \leq k \leq 0$ this is the morphism
\[
\wedge^{-k}(M^{-1}) \oplus \wedge^{-k-2}(M^{-1}) \oplus \ldots \to \wedge^{-k-2}(M^{-1}) \oplus \ldots
\]
that sends $\wedge^{-k}(M^{-1})$ to zero and induces isomorphisms on the others direct summands. For $k < -d-1$ this is an isomorphism preserving the direct sum decomposition.

ii) Passing to the limit we get an exact sequence
\[
0 \to \text{Sym}^{\infty}(W) \to \text{Sym}^{\infty}(M) \xrightarrow{g} \text{Sym}^{\infty}(M)[2] \to 0
\]

Denote by $\xi_d$ the composition $\text{Sym}^d(M) \xrightarrow{f_A} \text{Sym}^{d+1}(M) \xrightarrow{\varphi} \text{Sym}^d(M)[2]$. In particular, $\xi_1$ is given by the diagram
\[
\begin{array}{ccc}
A & \to & M^{-1} \to A \\
\downarrow & & \downarrow \ c_1 \\
A & \to & M^{-1} \to A
\end{array}
\]

Define the morphism $\xi : \bigotimes_{i=1}^d M \to \bigotimes_{i=1}^d M[2]$ as $\xi_1 \otimes \text{id} \otimes \ldots \otimes \text{id} + \ldots + \text{id} \otimes \ldots \otimes \text{id} \otimes \xi_1$. Then the diagram commutes
\[
\begin{array}{ccc}
\bigotimes_{i=1}^d M & \xrightarrow{\xi} & \bigotimes_{i=1}^d M[2] \\
\cup & & \cup \\
\text{Sym}^d(M) & \xrightarrow{\xi} & \text{Sym}^d(M)[2]
\end{array}
\]

Proof of Lemma A.1
$\text{Sym}^{\infty}(W)$ is the Koszul complex for $\partial(e_1), \ldots, \partial(e_r) \in A$, so that the natural map $\text{Sym}^{\infty}(W) \to A/I$ is a quasi-isomorphism. Our assertion follows now from Lemmas A.5 and A.2. □

B Chern classes

Let $S$ be a smooth separated scheme of finite type, $\mathcal{A}$ an invertible $\mathcal{O}_S$-module, $f : Y \to S$ the total space of $\mathcal{A}$ with removed zero section. Fix $n > 0$ invertible as a function on $S$ (we don’t need here $S$ to be defined over $k$).

Lemma B.1. The complex $f_! \mu_n$ is included into a distinguished triangle $f_! \mu_n \to \mathbb{Z}/n\mathbb{Z}[-2] \xrightarrow{c(\mathcal{A})} \mu_n$ on $S$, where $c(\mathcal{A}) \in H^2(S, \mu_n)$ is the Chern class of $\mathcal{A}$.

To prove this we need two lemmas.
Lemma B.2. Let $\mathcal{A}$ be an abelian category, $D(\mathcal{A})$ be its derived category. Let $K' \xrightarrow{\alpha} K \rightarrow K''$ be a distinguished triangle in $D(\mathcal{A})$. Suppose that the morphism $H^{i}(K'') \rightarrow H^{i+1}(K')$ is surjective. Then there is a unique morphism $\tau_{\leq i+1}K' \rightarrow \tau_{\leq i+1}K$ such that the composition $\tau_{\leq i+1}K' \rightarrow \tau_{\leq i}K \rightarrow \tau_{\leq i+1}K$ is obtained from $\alpha$ by applying the functor $\tau_{\leq i+1}$, and the triangle $\tau_{\leq i+1}K' \rightarrow \tau_{\leq i}K \rightarrow \tau_{\leq i}K''$ is distinguished. □

Lemma B.3. The complex $f_{*}\mu_{n}$ is included into a distinguished triangle $f_{*}\mu_{n} \rightarrow \mathbb{Z}/n\mathbb{Z}[-1] \xrightarrow{c(\mathcal{A})} \mu_{n}[1]$, where $c(\mathcal{A}) \in H^{2}(S, \mu_{n})$ is the Chern class of $\mathcal{A}$.

Proof The Kummer exact sequence $1 \rightarrow \mu_{n} \rightarrow \mathbb{G}_{m} \xrightarrow{x \mapsto x^{n}} \mathbb{G}_{m} \rightarrow 1$ on $S$ defines a distinguished triangle $\mathbb{G}_{m}[1] \rightarrow \mathbb{G}_{m}[1] \xrightarrow{\delta} \mu_{n}[2]$, and the Chern class of $\mathcal{A} \in \text{Pic} S = \text{Hom}_{D}(\mathbb{Z}, \mathbb{G}_{m}[1])$ is the composition $\delta \circ \mathcal{A} \in \text{Hom}_{D}(\mathbb{Z}, \mu_{n}[2])$ of morphisms in the derived category.

Consider now the Kummer exact sequence on $Y$. It provides a distinguished triangle $f_{*}\mu_{n} \rightarrow f_{*}\mathbb{G}_{m} \rightarrow f_{*}\mathbb{G}_{m}$ on $S$. As is easily seen, the morphism $(R^{0}f_{*})\mathbb{G}_{m} \rightarrow (R^{1}f_{*})\mu_{n}$ is surjective (the question is local for the ¨etale topology on $S$, and one can assume $f$ to be the projection $S \times \mathbb{G}_{m} \rightarrow S$). Since $\tau_{\leq i}f_{*}\mu_{n} \cong f_{*}\mu_{n}$, by Lemma B.2 we get a distinguished triangle

$$f_{*}\mu_{n} \rightarrow (R^{0}f_{*})\mathbb{G}_{m} \xrightarrow{\kappa} (R^{0}f_{*})\mathbb{G}_{m}$$

Let us now construct a morphism $(R^{0}f_{*})\mathbb{G}_{m} \rightarrow \mathbb{Z}$. If $U$ is a smooth scheme then $H^{0}(U \times \mathbb{G}_{m}, \mathbb{G}_{m}) \rightarrow H^{0}(U, \mathbb{G}_{m}) \times H^{0}(U, \mathbb{Z})$ canonically. Let $S' \rightarrow S$ be an ¨etale morphism. Put $Y' = S' \times_{S}Y$ and denote by $g : \mathbb{G}_{m} \times Y' \rightarrow Y'$ the action of $\mathbb{G}_{m}$ on $Y'$. Let $s \in H^{0}(S', (R^{0}f_{*})\mathbb{G}_{m})$. Since $Y'$ is smooth, to $s \circ g$ there corresponds an element of $H^{0}(Y', \mathbb{Z}) = H^{0}(S', \mathbb{Z})$. This provides a morphism $(R^{0}f_{*})\mathbb{G}_{m} \rightarrow \mathbb{Z}$. It is included into an exact sequence

$$0 \rightarrow \mathbb{G}_{m} \rightarrow (R^{0}f_{*})\mathbb{G}_{m} \rightarrow \mathbb{Z} \rightarrow 0,$$

where the first arrow comes from the natural morphism $\mathbb{G}_{m} \rightarrow f_{*}f^{*}\mathbb{G}_{m}$. Using Chech coverings one proves that the corresponding element of $\text{Ext}_{S}(\mathbb{Z}, \mathbb{G}_{m}) = \text{Pic} S$ is $\mathcal{A}$. In other words, we get a distinguished triangle $(R^{0}f_{*})\mathbb{G}_{m} \rightarrow \mathbb{Z} \xrightarrow{\mathcal{A}} \mathbb{G}_{m}[1]$ on $S$.

The morphism $\kappa$ yeilds a morphism of exact sequences

$$\begin{array}{cccccc}
0 & \rightarrow & \mathbb{G}_{m} & \rightarrow & (R^{0}f_{*})\mathbb{G}_{m} & \rightarrow & \mathbb{Z} & \rightarrow & 0 \\
\downarrow x & \mapsto & x^{n} & \downarrow \kappa & \downarrow n & & & \\
0 & \rightarrow & \mathbb{G}_{m} & \rightarrow & (R^{0}f_{*})\mathbb{G}_{m} & \rightarrow & \mathbb{Z} & \rightarrow & 0
\end{array}$$

The latter provides a commutative diagram, where the rows and columns are distinguished triangles

$$\begin{array}{cccccccc}
(R^{0}f_{*})\mathbb{G}_{m} & \rightarrow & \mathbb{Z} & \xrightarrow{\mathcal{A}} & \mathbb{G}_{m}[1] \\
\downarrow \kappa & & \downarrow n & & \downarrow \\
(R^{0}f_{*})\mathbb{G}_{m} & \rightarrow & \mathbb{Z} & \xrightarrow{\mathcal{A}} & \mathbb{G}_{m}[1] \\
\downarrow \kappa & & \downarrow n & & \downarrow \delta \\
f_{*}\mu_{n}[1] & \rightarrow & \mathbb{Z}/n\mathbb{Z} & \xrightarrow{\mu_{n}[2]}
\end{array}$$

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So, the morphism $\mathbb{Z}/n\mathbb{Z} \to \mu_n [2]$ in the lowest row is $c(A)$. □

Lemma [3.1] follows from Lemma [3.3] by Verdier duality.

References

[1] A. Beilinson, J. Bernstein, P. Deligne, Faisceaux pervers, Astérisque 100 (1983), p. 1-172
[2] P. Deligne, Rapport sur la formule des traces, in: SGA4 ½
[3] V. Drinfeld, Two-dimensional $\ell$-adic representations of the fundamental group of a curve over a finite field and automorphic forms on GL(2), Amer. J. Math., 105, p. 85-114 (1983)
[4] T. Ekedahl, On the adic formalism, in: The Grothendieck Festschrift, vol. 2, Progress in Math., 87 (1990), p. 197-218
[5] E. Frenkel, D. Gaitsgory, D. Kazhdan, K. Vilonen, Geometric realization of Whittaker functions and the Langlands conjecture, J. Amer. Math. Soc., 11 (1998) 451-484
[6] E. Frenkel, D. Gaitsgory, K. Vilonen, Whittaker patterns in the geometry of moduli spaces of bundles on curves, math.AG/9907133
[7] E. Frenkel, D. Gaitsgory, K. Vilonen, On the geometric Langlands conjecture, preprint math.AG/0012255
[8] O. Gabber, F. Loeser, Faisceaux pervers $\ell$-adiques sur un tore, Duke Math. J., 83 (1996), no. 3, p. 501-606.
[9] G. Laumon, Correspondance de Langlands géométrique pour les corps de fonctions, Duke Math. J., 54 (1987) vol.54, No.2, p.309-359.
[10] G. Laumon, Transformation de Fourier, constantes d’équations fonctionnelles et conjecture de Weil, Publ. Math. IHES (1987), No. 65, p. 131-210.
[11] G. Laumon, Transformation de Fourier généralisée, alg-geom/9603004
[12] G. Laumon, L. Moret-Bailly, Champs algébriques, Springer, A series of modern surveys in math., vol. 39 (2000)
[13] S. Lysenko, Orthogonality relations between the automorphic sheaves attached to 2-dimensional irreducible local systems on a curve, PhD thesis (1999) http://www.math.u-psud.fr/~biblio/th/1999/2111/the_1999_2111.html
[14] S. Lysenko, Local geometrized Rankin-Selberg method for GL(n), to appear in Duke Math. J.
[15] B. Ngô, Preuve d’une conjecture de Frenkel-Gaitsgory-Kazhdan-Vilonen pour les groupes linéaires généraux, Israel J. Math., 120 (2000), part A, p. 259-270.
[16] M. Rothstein, Sheaves with connections on abelian varieties, Duke Math. J. 84 (1996), no. 3, p. 565-598.

[17] M. Schlessinger, Functors of Artin rings, Trans.AMS, vol.130, n.2 (1968).

Université Paris-Sud, bât. 425, Mathématiques, 91405 Orsay France
e-mail: Sergey.Lysenko@math.u-psud.fr