ON THE ASSOCIATIVITY OF GLUING

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Abstract. This paper studies the associativity of gluing of trajectories in Morse theory. We show that the associativity of gluing follows from the existence of compatible manifold with face structures on the compactified moduli spaces. Using our previous work, we obtain the associativity of gluing in certain cases.

In particular, associativity holds when the ambient manifold is compact and the vector field is Morse-Smale.

1. Introduction

In order to develop his homology theory, Floer invented two techniques in Morse theory (see e.g. [15]). One is the compactification of the moduli spaces of negative gradient trajectories. The other one is the gluing of broken trajectories. These two arguments have continuously impacted Morse theory since then. For example, moduli spaces have extensive applications in geometry and topology (see e.g. [16], [18], [1]-[3] and [5]-[13]).

Due to this influence, there is a folklore theorem or rather a philosophy as follows. Under certain conditions of compactness, a moduli space of trajectories can be compactified to be a manifold with corners. There has been some progress on this topic in the literature as it was interpreted and proved in certain cases. For example, see [18, Proposition 2.11], [3, Theorem 1], [2, Appendix], [20, Theorem 3.3] and [21, Theorem 7.5].

Another related problem is the so-called “associativity of gluing” that is alluded to in the title. We first learned of this problem in the paper of Cohen, Jones and Segal [7].

This paper shows that the associativity of gluing is a direct consequence of the existence of compatible manifold structures on the compactified moduli spaces. We will in fact see that there is a general result along these lines in which Morse theory occurs as a special case.

Suppose $p_1$, $p_2$ and $p_3$ are critical points, $\gamma_1$ is a trajectory from $p_1$ to $p_2$ and $\gamma_2$ is a trajectory from $p_2$ to $p_3$. In a strict sense, the pair $(\gamma_1, \gamma_2)$ of consecutive trajectories is not a trajectory. We consider $(\gamma_1, \gamma_2)$ as a broken trajectory from $p_1$ to $p_3$. A gluing of $(\gamma_1, \gamma_2)$ is a smooth family

$$\gamma_1 \# \lambda \gamma_2,$$

where $\lambda \in [0, \epsilon)$ is the gluing parameter, $\gamma_1 \#_0 \gamma_2 = (\gamma_1, \gamma_2)$ and $\gamma_1 \#_\lambda \gamma_2$ is an unbroken trajectory when $\lambda \neq 0$.

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Suppose now that $\gamma_1$, $\gamma_2$ and $\gamma_3$ are three consecutive trajectories. Then one can form two families according to the various ways of associating pairs:

$$(\gamma_1 \# \lambda_1 \gamma_2) \# \lambda_2 \gamma_3$$

and

$$\gamma_1 \# \lambda_1 (\gamma_2 \# \lambda_2 \gamma_3).$$

If these families coincide, one says that associativity gluing is satisfied.

The manifold structure of a compactified moduli space is actually related to the associativity of gluing. One can derive the manifold structure from the associativity of gluing because the latter provides nice coordinate charts for the former. However, this is not the only way to get the manifold structure. The papers [18], [3], [20] and [21] do not use any gluing arguments.

In this paper, we shall strengthen the above relationship by working in the opposite direction: we will show that the associativity of gluing is a consequence of the existence of a certain kind of manifold structure. More precisely, Theorems 3.2 and 3.3 show that, if the manifold structures satisfy Assumption 3.1, then one will get the associativity of gluing for free. In fact, we reformulate a gluing of broken trajectories as parametrizations of collar neighborhoods of the strata of the compactified moduli spaces. Then associativity of gluing will be seen to be equivalent to a choice of compatible collar structure. The above theorems will be generalized to Theorem 4.4 which is a statement about the compatible collar structures of manifold with faces.

In short, these theorems convert the problem of the associativity of gluing to the problem of manifold structures. By the results we proved about manifold structures in [20] and [21], we get Propositions 3.4 and 3.5. They show the associativity of gluing in Morse theory in two contexts. An informal restatement of these results is given by

**Corollary A.** Suppose $M$ is a compact Riemannian manifold and $f$ is a Morse function on $M$. Suppose $-\nabla f$ satisfies the Morse-Smale condition.

Then there exists an associative gluing rule.

**Corollary B.** Suppose $M$ is a complete Hilbert-Riemannian manifold. Assume $f$ satisfies Condition (C) and has finite indices. Suppose $-\nabla f$ satisfies the Morse-Smale condition. Assume that the metric on $M$ is locally trivial (see [20, Definition 2.16]).

Then there exists an associative gluing rule.

A byproduct of our work is Proposition 7.1 which is also about compatible collar structures. Theorem 4.4 is about a family of manifolds with faces (see Assumption 4.1), while Proposition 7.1 is about a single one. However, the assumption of Proposition 7.1 is more general.

The outline of this paper is as follows. Section 2 reviews the definition of moduli spaces of trajectories. Section 3 gives our main results on the associativity of gluing. Section 4 generalizes the theorems in the previous section. The proof of our main theorem occupies Sections 5 and 6. We conclude this paper by presenting the byproduct in Section 7.

### 2. Moduli Spaces

In this section, we review the definition of the moduli spaces of trajectories of negative gradient vector fields. (See [22] or [20] for more details.)
Suppose $M$ is a Hilbert-Riemannian manifold and $f$ is a Morse function on $M$. Let $-\nabla f$ be the negative gradient of $f$.

**Definition 2.1.** Let $\phi_t(x)$ be the flow generated by $-\nabla f$ with initial value $x$. Suppose $p$ is a critical point. Define the descending manifold of $p$ as $D(p) = \{ x \in M \mid \lim_{t \to -\infty} \phi_t(x) = p \}$. Define the ascending manifold of $p$ as $A(p) = \{ x \in M \mid \lim_{t \to +\infty} \phi_t(x) = p \}$. Both $D(p)$ and $A(p)$ are smoothly embedded submanifolds in $M$.

**Definition 2.2.** If the descending manifold $D(p)$ and the ascending manifold $A(q)$ are transversal for all critical points $p$ and $q$, then we say $-\nabla f$ satisfies the transversality or Morse-Smale condition.

If $-\nabla f$ satisfies transversality, then $D(p) \cap A(q)$ is an embedded submanifold which consists of points on trajectories (or flow lines) from $p$ to $q$. Since a trajectory has an $\mathbb{R}$-action, we may take the quotient of $D(p) \cap A(q)$ by this $\mathbb{R}$-action, i.e. consider its orbit space acted upon by the flow. This leads to the following definition.

**Definition 2.3.** Suppose $-\nabla f$ satisfies transversality. Define $W(p, q) = D(p) \cap A(q)$. Define the moduli space $\mathcal{M}(p, q)$ as the orbit space $W(p, q)/\mathbb{R}$.

We assume transversality all through this paper. It’s well known that, when $f$ has finite indices, $\mathcal{M}(p, q)$ is a finitely dimensional manifold of dimension $\text{ind}(p) - \text{ind}(q) - 1$, where $\text{ind}(\ast)$ is the Morse index of $\ast$.

**Definition 2.4.** Suppose $p$ and $q$ are two critical points. We define the relation $p \succeq q$ if there is a trajectory from $p$ to $q$. We define the relation $p \succ q$ if $p \succeq q$ and $p \neq q$.

The transversality implies that “$\succeq$” is a partial order. To guarantee this, it suffices to show the transitivity of “$\succeq$”. The best proof is probably to use the $\lambda$-Lemma (see [13], p. 85, Corollary 1). It is valid even if $M$ is a Banach manifold and the vector field is a general one (not necessarily a negative gradient) with hyperbolic singularities. In Floer theory (see e.g. [15], p. 529), this can be proved by a gluing argument.

**Definition 2.5.** An ordered set $I = \{ r_0, r_1, \cdots, r_{k+1} \}$ is a critical sequence if $r_i$ ($i = 0, \cdots, k+1$) are critical points and $r_0 \succ r_1 \succ \cdots \succ r_{k+1}$. We call $r_0$ the head of $I$, and $r_{k+1}$ the tail of $I$. The length of $I$ is $|I| = k$.

Suppose $I = \{ r_0, r_1, \cdots, r_{k+1} \}$ is a critical sequence. We define the following product manifold

\[
\mathcal{M}_I = \prod_{i=0}^{k} \mathcal{M}(r_i, r_{i+1}).
\]

Each element in $\mathcal{M}_I$ stands for a (un)broken trajectory from $r_0$ to $r_{k+1}$ which is broken at exactly the points $r_i$ ($i = 1, \cdots, k$).

3. **Main Theorems**

In this section, we state our results on the associativity of gluing.

Theorems 3.2 and 3.3 will be based on the following assumption. For the definitions of manifold with faces and the $k$-stratum, see Definitions 4.3 and 4.2.
Assumption 3.1. Suppose $\Omega$ is the set of critical points of $f$. Assume $\Omega$ is countable. The relation " $\succ$ " (see Definition 2.3) defined on $\Omega$ is a partial order. Suppose $\mathcal{M}(p, q)$ is a finite dimensional manifold for each $p, q \in \Omega$ such that $p \succ q$ (see Remark 3.2). Suppose $\mathcal{M}(p, q)$ can be compactified to $\overline{\mathcal{M}(p, q)}$ having the structure of a compact smooth manifold with faces. In addition, assume each $\mathcal{M}(p, q)$ satisfies the following conditions:

1. We have $\overline{\mathcal{M}(p, q)} = \bigsqcup I \mathcal{M}_I$, where the disjoint union is over all critical sequences with head $p$ and tail $q$. The $k$-stratum of $\overline{\mathcal{M}(p, q)}$ is $\bigsqcup_{|I|=k} \mathcal{M}_I$, and each $\mathcal{M}_I$ is an open subset of the $k$-stratum. The smooth structure of $\overline{\mathcal{M}(p, q)}$ is compatible with those of $\mathcal{M}_I$.

2. Suppose $p \succ r \succ q$, then the natural inclusion $\overline{\mathcal{M}(p, r)} \times \overline{\mathcal{M}(r, q)} \rightarrow \overline{\mathcal{M}(p, q)}$ is a smooth embedding.

Remark 3.1. By Definition 2.3 $\mathcal{M}(p, q)$ has a natural smooth structure induced from those of $\mathcal{D}(p)$ and $\mathcal{A}(q)$ (see e.g. [22, 18, 3, 20] and [21]). However, in order to make Assumption 3.1 hold, we may give the following definitions of the tuples induced from $\Lambda^I$. Here we consider $\Lambda^I$ by $\Lambda^I = \{\lambda_{1}, \cdots, \lambda_{|I|}\} \in \prod_{i=1}^{|I|} [0, +\infty)$ and $\Lambda^I_1 = \{\lambda_{1}, \cdots, \lambda_{|I|}\}$. By the relation between $I_1$ and $I_2$, we introduce the following definitions of the tuples induced from $\Lambda^I$. Define $\Lambda^I_1, \Lambda^I_2 \in [0, +\infty)^{|I|}$ as

$$\Lambda^I_1, \Lambda^I_2 = (\lambda_{1}, \cdots, \lambda_{|I|})$$

Here we consider $\Lambda^I_1, \Lambda^I_2$ as a gluing parameter for $\mathcal{M}^I_2$. Define $\lambda_1, \lambda_2 \in [0, +\infty)^{|I|}$ as

$$\Lambda^I_1, \Lambda^I_2 = \begin{cases} 0 & \text{if } r_i \in I_2, \\
_i & \text{otherwise} \end{cases}$$

For example, suppose $I_1 = \{r_0, r_1, r_2, r_3, r_4\}$, $I_2 = \{r_0, r_2, r_3\}$ and $\Lambda^I_1 = (5, 6, 7)$, then $\Lambda^I_1 \cdot \Lambda^I_2 = (6)$ and $\Lambda^I_1, (I_1 - I_2) = (5, 0, 7)$.

Suppose $I_1 = \{r_0, \cdots, r_{k+1}\}$, $I_2 = \{r_0', \cdots, r'_{k+1}\}$ and $r_{k+1} = r_0'$. Define

$$\Lambda^I_1 \cdot \Lambda^I_2 = \{r_0, \cdots, r_{k+1}, r_1', \cdots, r'_{k+1}\}.$$ 

If $x_1 = (a_1, \cdots, a_{k+1}) \in \mathcal{M}^I_1$ and $x_2 = (a_1', \cdots, a'_{k+1}) \in \mathcal{M}^I_2$, then define

$$x_1 \cdot x_2 = (a_1, \cdots, a_{k+1}, a_1', \cdots, a'_{k+1}) \in \mathcal{M}^I_1 \times \mathcal{M}^I_2 = \mathcal{M}^I_1 \cdot \mathcal{M}^I_2.$$ 

Suppose $\Lambda^I_1 = (\lambda_{1}, \cdots, \lambda_{|I_1|})$ and $\Lambda^I_2 = (\lambda_{1}', \cdots, \lambda_{|I_2|})$, define

$$\Lambda^I_1 \cdot \Lambda^I_2 = (\lambda_{1}, \cdots, \lambda_{|I_1|}, 0, \lambda_{1}', \cdots, \lambda_{|I_2|})$$

In particular, if $|I_1| = 0$, then $\Lambda^I_1 \cdot \Lambda^I_2 = (0, \lambda_{|I_2|}, \cdots, \lambda_{|I_2|})$. If $|I_2| = 0$, then $\Lambda^I_1 \cdot \Lambda^I_2 = (\lambda_{1}, \cdots, \lambda_{|I_1|}, 0)$. If $|I_1| = |I_2| = 0$, then $\Lambda^I_1 \cdot \Lambda^I_2 = (0, 0)$.

Suppose $I = \{r_0, r_1, \cdots, r_{k+1}\}$ is a critical sequence. Recall that an element $x \in \mathcal{M}$ is a (un)broken trajectory which is broken at the points $r_i$ ($i = 1, \cdots, k$).
A gluing should be a map \( G_I : M_I \times [0, \epsilon_I)^{|I|} \rightarrow \overline{M(r_0, r_{I+1})} \) for some \( \epsilon_I > 0 \). For all \((x, \Lambda_I) \in M_I \times [0, \epsilon_I)^{|I|}\), we have \( \Lambda_I = (\lambda_1, \cdots, \lambda_{|I|}) \) is a parameter of gluing, and \( G_I(x, \Lambda_I) \) is the (un)broken trajectory glued from \( x \). We expect that \( G_I(x, \Lambda_I) \) is not broken at \( r_i \) if and only if \( \lambda_i > 0 \). Thus we can interpret the gluing map as a collaring map, which leads to the following definition.

**Definition 3.1.** A map \( G_I : M_I \times [0, \epsilon_I)^{|I|} \rightarrow \overline{M(r_0, r_{I+1})} \) for some \( \epsilon_I > 0 \) is a gluing map if it satisfies the following properties. (1). It is a smooth embedding. In particular, if \(|I| = 0\), \( G_I : M_I = M(r_0, r_1) \rightarrow \overline{M(r_0, r_1)} \) is the inclusion. (2). It satisfies the stratum condition, i.e., suppose \( I = \{r_0, r_1, \cdots, r_{k+1}\} \), \( \Lambda_I = (\lambda_1, \cdots, \lambda_{|I|}) \in [0, \epsilon_I)^{|I|} \), \( I_1 \subseteq I \), and \( \lambda_i = 0 \) if and only if \( r_i \in I_1 \), then for all \( x \in M_I \), we have \( G_I(x, \Lambda_I) \in M_{I_1} \).

Now we give two examples to illustrate the compatibility issue of gluing.

Suppose the gluing maps are defined for all critical sequences. Suppose \( I_1 = \{r_0, r_1, r_2, r_3, r_4\}, I_2 = \{r_0, r_2, r_4\} \), \( \Lambda_{I_1} = (\lambda_1, \lambda_2, \lambda_3) \), \( \lambda_1 > 0, \lambda_3 > 0 \), and \( x \in M_{I_1} \). Gluing \( x \) at the points \( r_1 \) and \( r_3 \) at first, we get \( y = G_{I_1}(x, \lambda_1, 0, \lambda_3) \in M_{I_2} \). Do we have \( G_{I_2}(y, \lambda_2) = G_{I_1}(x, \lambda_1, \lambda_2, \lambda_3) \)? This is a question about the compatibility for a fixed critical pair \((r_0, r_2)\).

Suppose \( I_1 = \{r_0, r_1, r_2\}, I_2 = \{r_2, r_3, r_4\} \), \( \Lambda_{I_1} = (\lambda_1), \Lambda_{I_2} = (\lambda_2), x_1 \in M_{I_1} \) and \( x_2 \in M_{I_2} \). Gluing \( x_1 \) and \( x_2 \), we get \( y_1 = G_{I_1}(x_1, \lambda_1) \in M(r_0, r_2) \) and \( y_2 = G_{I_2}(x_2, \lambda_2) \in M(r_2, r_4) \). Do we have \( G_{I_1 \cdot I_2}(x_1 \cdot x_2, \lambda_1, 0, \lambda_2) = (y_1, y_2) \)? This is a question about the compatibility for different critical pairs.

The following theorem answers the above two questions.

**Theorem 3.2.** Under Assumption 3.1, the gluing maps (see Definition 3.1) can be defined for all critical sequences. They satisfy the following compatibility:

(1). (Compatibility for one critical Pair). Suppose \( I_2 \subseteq I_1 \), let \( \epsilon = \min\{\epsilon_{I_1}, \epsilon_{I_2}\} \). Then, for all \( x \in M_{I_1} \) and \( \Lambda_{I_1} = (\lambda_1, \cdots, \lambda_{|I_1|}) \in [0, \epsilon)^{|I_1|} \) such that \( \lambda_i > 0 \) when \( r_i \notin I_2 \), we have

\[
G_{I_1}(x, \Lambda_{I_1}) = G_{I_2}(G_{I_1}(x, \Lambda_{I_1}(I_1 - I_2)), \Lambda_{I_1 \cdot I_2}).
\]

(2). (Compatibility for Critical Pairs). Suppose \( I_1 = \{r_0, \cdots, r_k + 1\} \) and \( I_2 = \{r_{k+1}, \cdots, r_n\} \). Let \( \epsilon = \min\{\epsilon_{I_1}, \epsilon_{I_2}, \epsilon_{I_1 \cdot I_2}\} \), then for all \( x_1 \in M_{I_1} \), \( x_2 \in M_{I_2} \), \( \Lambda_{I_1} \in [0, \epsilon)^{|I_1|} \), and \( \Lambda_{I_2} \in [0, \epsilon)^{|I_2|} \), we have

\[
G_{I_1 \cdot I_2}(x_1 \cdot x_2, \Lambda_{I_1} \cdot \Lambda_{I_2}) = (G_{I_1}(x_1, \Lambda_{I_1}), G_{I_2}(x_2, \Lambda_{I_2})) \in \overline{M(r_0, r_{k+1}) \times M(r_{k+1}, r_n)}.
\]

Theorem 3.2 will follow from a more general Theorem 4.4.

We introduce a traditional notation of gluing as in the Introduction (see e.g., p. 529]). Suppose \( \gamma_1 \in M(p, r) \) and \( \gamma_2 \in M(r, q) \) are two trajectories. We denote the gluing map \( G_{(p, r, q)}(\gamma_1, \gamma_2, \lambda) \) by \( \gamma_1 \#_{\lambda} \gamma_2 \). From Theorem 3.2 we immediately derive the following.

**Theorem 3.3.** Under Assumption 3.1 there exist \( \epsilon_I > 0 \) for all critical sequences \( I \) with \(|I| = 1 \) or \(|I| = 2\). For all \( \{r_0, r_1, r_2\} \), the gluing \( \gamma_1 \#_{\lambda} \gamma_2 \) can be defined for \( (\gamma_1, \gamma_2) \in M(r_0, r_1) \times M(r_1, r_2) \) and \( \lambda \in [0, \epsilon_{(r_0, r_1, r_2)}] \). The gluing satisfies the following associativity:

For all \( \gamma_1 \in M(p_1, p_2), \gamma_2 \in M(p_1, p_2), \gamma_3 \in M(p_2, p_3), \) and \( \lambda_1, \lambda_2 \in (0, \epsilon) \), where \( \epsilon = \min\{\epsilon_{(p_0, p_1, p_2)}, \epsilon_{(p_1, p_2, p_3)}, \epsilon_{(p_0, p_1, p_2, p_3)}\} \), we have

\[
(\gamma_1 \#_{\lambda_1} \gamma_2) \#_{\lambda_2} \gamma_3 = \gamma_1 \#_{\lambda_1} (\gamma_2 \#_{\lambda_2} \gamma_3).
\]
Proof.

\[
(\gamma_1 \#_{\lambda_1} \gamma_2) \#_{\lambda_2} \gamma_3 = G_{\{p_0, p_2, p_3\}} \left( G_{\{p_0, p_1, p_2\}}(\gamma_1, \gamma_2, \lambda_1), \gamma_3, \lambda_2 \right) = G_{\{p_0, p_2, p_3\}} \left( G_{\{p_0, p_1, p_2, p_3\}}(\gamma_1, \gamma_2, \gamma_3, 0), \lambda_2 \right) = G_{\{p_0, p_1, p_2, p_3\}}(\gamma_1, \gamma_2, \gamma_3, \lambda_1, \lambda_2).
\]

Here we have used the (2) of Theorem 3.2 in the second equality and the (1) of Theorem 3.2 in the third equality.

Similarly,

\[
\gamma_1 \#_{\lambda_1} (\gamma_2 \#_{\lambda_2} \gamma_3) = G_{\{p_0, p_1, p_2, p_3\}}(\gamma_1, \gamma_2, \gamma_3, \lambda_1, \lambda_2).
\]

This completes the proof. \(\square\)

Remark 3.2. Suppose \(I = \{r_0, \cdots, r_{n+1}\}\) is a critical sequence. Let \(\epsilon = \min \{\epsilon_J | J \subseteq I, \text{ and } |J| = 1 \text{ or } 2\}\). Then, for \((\gamma_1, \gamma_2) \in \mathcal{M}(r_i, r_j) \times \mathcal{M}(r_j, r_k)\), the gluing \(\gamma_1 \#_{\lambda} \gamma_2\) in Theorem 3.3 can be defined for \(\lambda \in [0, \epsilon)\). And the gluing satisfies the associativity. Thus we can define \(G_J\) on \(\mathcal{M}_I \times (0, \epsilon)^{|J|}\) for any \(J \subseteq I\) by inductive gluing of pairs of trajectories. The definition of \(G_J\) does not depend on the order of the pairwise gluing.

By [20, Theorem 3.3] and [21, Theorem 7.5], Assumption 3.1 holds in certain cases. Thus Theorems 3.2 and 3.3 lead to the following two propositions. See [20, Definition 2.16] for the definition of a locally trivial metric.

Proposition 3.4. Suppose \(M\) is a complete Hilbert-Riemannian manifold equipped with Morse function \(f\) satisfying Condition (C) and having finite indices. Assume that the metric on \(M\) is locally trivial and \(-\nabla f\) satisfies transversality. Give \(M(p, q)\) the smooth structure induced from \(D(p)\) and \(A(q)\). Then there exist smooth structures on \(\mathcal{M}(p, q)\) and gluing maps which satisfy the compatibility and associativity in Theorems 3.2 and 3.3.

Proposition 3.5. Suppose \(M\) is a compact Riemannian manifold equipped with Morse function \(f\). Assume \(-\nabla f\) satisfies transversality. Then there exist smooth structures on \(\mathcal{M}(p, q)\) and \(\mathcal{M}(p, q)\) and gluing maps which satisfy the compatibility and associativity in Theorems 3.2 and 3.3.

Remark 3.3. Proposition 3.5 is based on [21]. In the case of a compact \(M\), it has the advantage that the metric is allowed to be general. However, the smooth structure on \(\mathcal{M}(p, q)\) may be different from the natural one when the metric is not locally trivial.

4. Generalization

The proof of Theorem 3.2 actually does not directly depend on the speciality of Morse theory. Therefore, we will generalize the results to Theorem 4.4 which is about collaring maps of manifolds with faces.

Definition 4.1. An \(n\)-dimensional smooth manifold with corners is a space defined in the same way as a smooth manifold except that its atlases are open subsets of \([0, +\infty)^n\).
If $L$ is a smooth manifold with corners, $x \in L$, a neighborhood of $x$ is diffeomorphic to $(0, \epsilon)^{n-k} \times [0, \epsilon)^{k}$, then define $c(x) = k$. Clearly, $c(x)$ does not depend on the choice of atlas.

**Definition 4.2.** Suppose $L$ is a smooth manifold. We call \{ $x \in L$ | $c(x) = k$ \} the $k$-stratum of $L$. Denote it by $\partial^k L$.

Clearly, $\partial^k L$ is a submanifold without corners inside $L$, its codimension is $k$.

**Definition 4.3.** (c.f. [17]). A smooth manifold $L$ with faces is a smooth manifold with corners such that each $x$ belongs to the closures of $c(x)$ different components of $\partial^1 L$.

Now we introduce the notation $\Omega$, “$\succeq$”, $I$ and $\mathcal{M}(p, q)$ as in Section 3. However, in the present context they are generalizations: they are independent of Morse theory.

Suppose $\Omega$ is a partially ordered set with a partial order “$\succeq$”. Suppose $I = \{r_0, r_1, \cdots, r_{k+1}\}$ is a finite chain of $\Omega$, i.e., $I \subseteq \Omega$ and $r_i \succ r_{i+1}$. We call $r_0$ the head of $I$ and $r_{k+1}$ the tail of $I$. Define the length of $I$ as $|I| = k$. If $J \subseteq I$, $J = \{r'_0, \cdots, r'_{k+1}\}$, $r'_0 = r_0$ and $r'_{k+1} = r_{k+1}$, i.e. $J = \{r_0, r_1, \cdots, r_i, r_{k+1}\}$, denote them by $J \preceq I$. Suppose $I_1 = \{r_0, \cdots, r_k\}$ and $I_2 = \{r_{k+1}, \cdots, r_n\}$ are two chains. Define $I_1 \cdot I_2 = \{r_0, \cdots, r_n\}$, which is also a chain.

Suppose a finite dimensional manifold $\mathcal{M}(p, q)$ is defined for each pair $(p, q) \subseteq \Omega$ such that $p \succ q$. For the above chain $I$, define $\mathcal{M}_I = \prod_{i=0}^{|I|} \mathcal{M}(r_i, r_{i+1})$.

**Assumption 4.1.** The partially ordered set $\Omega$ is countable. The finite dimensional manifolds $\mathcal{M}(p, q)$ can be compactified to be $\overline{M}(p, q)$ which are compact smooth manifolds with faces. These $\overline{M}(p, q)$ satisfy the following conditions:

1. We have $\overline{M}(p, q) = \bigcup_{I} \mathcal{M}_I$, where the disjoint is over all finite chains $I$ with head $p$ and tail $q$. The $k$-stratum of $\overline{M}(p, q)$ is $\bigcup_{|I|=k} \mathcal{M}_I$, and each $\mathcal{M}_I$ is an open subset of the $k$-stratum. The smooth structure of $\overline{M}(p, q)$ is compatible with those of $\mathcal{M}_I$.

2. Suppose $p \succ r \succ q$, then the natural inclusion $\overline{M}(p, r) \times \overline{M}(r, q) \rightarrow \overline{M}(p, q)$ is a smooth embedding.

We introduce the following definitions similar to Section 3. Use $\Lambda_I = (\lambda_1, \cdots, \lambda_{|I|})$ to represent the collaring parameter for $\mathcal{M}_I$. Define $\Lambda_{I_1}(I_1 - I_2), \Lambda_{I_1} I_2$ and $\Lambda_{I_1} \Lambda_{I_2}$. Also for $x_1 \in \mathcal{M}_{I_1}$ and $x_2 \in \mathcal{M}_{I_2}$, define $x_1 \cdot x_2 \in \mathcal{M}_{I_1, I_2}$.

Define the collaring map $G_I : \mathcal{M}_I \times [0, \epsilon)^{|I|} \rightarrow \overline{M}(r_0, r_{|I|+1})$ as Definition 3.1.

The proof of the following theorem is given in Section 3.

**Theorem 4.4.** Under Assumption 4.1, the collaring maps $G_I$ can be defined for all finite chains $I$ of $\Omega$. These maps satisfy the following compatibility:

1. Suppose $I_2 \leq I_1$, let $\epsilon = \min\{\epsilon_{I_1}, \epsilon_{I_2}\}$. Then, for all $x \in \mathcal{M}_{I_1}$ and $\Lambda_{I_1} \in [0, \epsilon)^{|I_1|}$ such that $\lambda_i > 0$ when $v_i \not\in I_2$, we have

$$G_{I_1}(x, \Lambda_{I_1}) = G_{I_2}(G_{I_1}(x, \Lambda_{I_1}(I_1 - I_2)), \Lambda_{I_1} I_2).$$

2. Suppose $I_1 = \{r_0, \cdots, r_{k+1}\}$ and $I_2 = \{r_{k+1}, \cdots, r_n\}$. Let $\epsilon = \min\{\epsilon_{I_1}, \epsilon_{I_2}, \epsilon_{I_1} I_2\}$, then for all $x_1 \in \mathcal{M}_{I_1}, x_2 \in \mathcal{M}_{I_2}, \Lambda_{I_1} \in [0, \epsilon)^{|I_1|}$, and $\Lambda_{I_2} \in [0, \epsilon)^{|I_2|}$, we have

$$G_{I_1} I_2(x_1 \cdot x_2, \Lambda_{I_1} \cdot \Lambda_{I_2}) = (G_{I_1}(x_1, \Lambda_{I_1} I_2), G_{I_2}(x_2, \Lambda_{I_2})).$$
5. Face Structures

In order to prove Theorem [4,4] we first study the face structures.

Suppose $L$ is manifold with faces. The closure of a component of $\partial^1 L$ (see Definition [4,2]) is still connected. Following the terminology of [17], we have the following definition.

**Definition 5.1.** We call the closure of a component of $\partial^1 L$ a (closed) face of $L$. We call any union of pairwise disjoint connected faces a face of $L$.

Thus, if $F$ is a face of $L$, then $F = \bigcup_{\alpha \in \mathbb{N}} C_\alpha$, where $C_\alpha$ is the closure of $C_\alpha^\circ$ and $C_\alpha^\circ$ is a component of $\partial^1 L$. As pointed in [17], $F$ is still a manifold with corners. We have the following result which is trivial when $\mathbb{N}$ is a finite set.

**Lemma 5.2.** Using the notation as the above, we have that $F$ is a smoothly embedded submanifold with corners inside $L$. The components of $F$ are $C_\alpha$. The interior of $F$ (i.e. $\partial^0 F$) is $\bigcup_{\alpha \in \mathbb{N}} C_\alpha^\circ$ and $F$ is a closed subset of $L$.

**Proof.** First, we show that $C_\alpha$ is a submanifold with corners and its 0-stratum is $C_\alpha^\circ$. It suffices to show that, for each $x \in C_\alpha$, there exists an open neighborhood $U_x$ of $x$ such that $F_x \cap C_\alpha$ has the desired corner structure.

We can choose $U_x$ such that it has the chart $(-\epsilon, \epsilon)^{n-i} \times \{0, \epsilon \}^l$ and $x$ has the coordinate $(0, \cdots, 0)$. Clearly,

$$U_x \cap C_\alpha^\circ \subseteq \bigcup_{i=1}^l \left[ (-\epsilon, \epsilon)^{n-i} \times \{0, \epsilon \}^{l-i} \times \{0\} \times \{0, \epsilon \}^{l-i} \right],$$

and $U_x \cap C_\alpha^\circ \neq \emptyset$. We may assume $[(-\epsilon, \epsilon)^{n-i} \times \{0\} \times \{0, \epsilon \}^{l-i}] \cap C_\alpha^\circ \neq \emptyset$. Since $(-\epsilon, \epsilon)^{n-i} \times \{0\} \times \{0, \epsilon \}^{l-i}$ is connected and contained in $\partial^1 L$, and $C_\alpha^\circ$ is a component of $\partial^1 L$, we infer that $(-\epsilon, \epsilon)^{n-i} \times \{0\} \times \{0, \epsilon \}^{l-i} \subseteq C_\alpha^\circ$. By Definition [4,3] it's easy to see $U_x \cap C_\alpha = (-\epsilon, \epsilon)^{n-i} \times \{0\} \times \{0, \epsilon \}^{l-i}$. Since, $U_x$ is open, we have $U_x \cap C_\alpha$ is the relative closure of $U_x \cap C_\alpha^\circ$ in $U_x$. In other words, $U_x \cap C_\alpha = (-\epsilon, \epsilon)^{n-i} \times \{0\} \times \{0, \epsilon \}^{l-i}$ and the 0-stratum of $U_x \cap C_\alpha$ is contained in $C_\alpha^\circ$. Thus we get the desired corner structure.

Second, we show that $F$ is a manifold with corners.

Since $C_\alpha$ has no intersection with other $C_\beta$, by the above argument, we can see that the above open neighborhood $U_x$ has no intersection with other $C_\beta$. Thus $\bigcup_{x \in C_\alpha} U_x$ is an open neighborhood of $C_\alpha$ which has no intersection with other $C_\beta$. So $C_\alpha^\circ$ is relatively open in $F$. This verifies the manifold structure of $F$.

Finally, we show that $F$ is a closed subset of $L$. Suppose $x$ is in the closure of $F$, then $x$ can be approximated by points in $F$ and thus by points in $\bigcup_{\alpha \in \mathbb{N}} C_\alpha^\circ$. By the above argument, it's easy to see that $x$ belongs to some $C_\alpha$.

**Lemma 5.3.** Suppose $L$ is an $n$ dimensional manifold with faces. Suppose $F_i$ ($i = 1, \cdots, k$) are faces of $L$ such that their interiors are pairwise disjoint and $\bigcap_{i=1}^k F_i$ is nonempty. Then $\bigcap_{i=1}^k F_i$ is an $n-k$ dimensional smoothly embedded submanifold with corners inside $L$.

**Proof.** Let $x$ be an arbitrary point in $\bigcap_{i=1}^k F_i$. It suffices to prove that there exists an open neighborhood $U$ of $x$ such that $U \cap \bigcap_{i=1}^k F_i$ has a corner structure.

For each $i$, $x$ belongs to an unique component of $F_i$. Since this component is relatively open in $F_i$, we can choose $U$ small enough such that $U$ has no intersection with other components. Thus we may assume $F_i$ is connected.
By the proof of Lemma 5.2 we can choose \( U \) such that it has a chart \((-\epsilon, \epsilon)^{n-k} \times [0, \epsilon)\), \( x \) has the coordinate \((0, \cdots, 0)\) and \( U \cap F_i = (-\epsilon, \epsilon)^{n-k} \times \{0\} \times (0, \epsilon)^{k-1} \). Since the interior of \( F_i \) are pairwise disjoint, repeating this argument, we get \( U \cap F_i = (-\epsilon, \epsilon)^{n-k} \times [0, \epsilon)^{i-1} \times \{0\} \times (0, \epsilon)^{k-i} \). Thus \( U \cap \bigcap_{i=1}^{k} F_i = (-\epsilon, \epsilon)^{n-k} \times \{0\}^k \times [0, \epsilon)^{k-i} \). This verifies the corner structure. \(\square\)

We introduce some other concepts following [13].

**Definition 5.4.** Suppose \( L \) is a manifold with corners. For all \( x \in L \),

\[
A_x L = \{ v \in T_x L \mid v = \gamma'(0) \text{ for some smooth curve } \gamma : [0, \epsilon) \to L. \}
\]

is the tangent sector of \( L \) at \( x \).

Definition 5.4 is equivalent to the *secteur tangent* in [13, p. 3].

**Definition 5.5.** Suppose \( L_1 \) is a submanifold without corners inside \( L \) and \( x \in L_1 \), we define the normal sector \( A_x(L_1, L) = A_x L / T_x L_1 \).

By [13], \( A_x(L_1, L) \) is called *secteur transverse*.

Define the tangent sector bundle \( AL \) as the subbundle of \( TL \) with fibers \( A_x L \). Define the normal bundle \( N(L_1, L) \) as the bundle whose fibers are the normal space \( N_x(L_1, L) = T_x L_1 / T_x L_1 \). Define the normal sector bundle \( A(L_1, L) \) as the subbundle of \( N(L_1, L) \) with fiber \( A_x(L_1, L) \) and \( A_L L_1, L \) as the restriction of \( AL \) to \( L_1 \).

**Lemma 5.6.** Under the assumption of Lemma 5.3 assume that \( L_1 \) is an open subset of \( \partial^k L \) and \( L_1 \subseteq \bigcap_{i=1}^{k} F_i \). Then there exist smooth sections \( e_i \) of \( A_{L_1} L \) \( (i = 1, \cdots, k) \) satisfying the following stratum condition: (1) \( e_i \in A_{L_1}((\bigcap_{j \neq i} F_j)) \); (2) \( \{ \pi e_1, \cdots, \pi e_k \} \) is linearly independent everywhere and all elements in \( A_{L_1} L \) can be linearly represented by \( \{ \pi e_1(x), \cdots, \pi e_k(x) \} \) with nonnegative coefficients, where \( \pi : A_{L_1} L \to A(L_1, L) \) is the natural projection.

**Proof.** Suppose \( x \in L_1 \), by the proof of Lemma 5.3, there exists a neighborhood \( U \) of \( x \) such that \( U \) has a chart \( (-\epsilon, \epsilon)^{n-k} \times [0, \epsilon) \), \( x \) has the coordinate \((0, \cdots, 0)\), \( U \cap L_1 = (-\epsilon, \epsilon)^{n-k} \times \{0\}^{k} \) and \( U \cap F_i = (-\epsilon, \epsilon)^{n-k} \times [0, \epsilon)^{i-1} \times \{0\} \times (0, \epsilon)^{k-i} \). Thus \( U \cap \bigcap_{j \neq i} F_j = (-\epsilon, \epsilon)^{n-k} \times [0, \epsilon)^{i-1} \times [0, \epsilon) \times \{0\}^{k-i} \). Obviously, for any vector \( e_i(x) \in A_x((\bigcap_{j \neq i} F_j) - T_x L_1 \), we have \( \{ \pi e_1(x), \cdots, \pi e_k(x) \} \) satisfies the desired property in \( A_{L_1} L_1 \).

Since \( L_1 \) is an open subset of the 1-stratum of \( \bigcap_{j \neq i} F_j \), we can choose a smooth inward normal section \( e_i \) along \( L_1 \). \(\square\)

In the case of Assumption 4.1 it’s easy to see that \( \overline{M(p, q)} \) is a manifold with faces \( \overline{M(p, r)} \times \overline{M(r, q)} \). The interiors of these faces are \( M(p, r) \times M(r, q) \) which are pairwise disjoint. Suppose \( I = \{ p, r_1, \cdots, r_k, q \} \) is a chain of \( \Omega \). Let \( I_i = \{ p, r_1, \cdots, r_{i-1}, r_{i+1}, \cdots, r_k, q \} \). Then \( M_I \) is the interior of \( \bigcap_{i=1}^{k} \overline{M(p, r_i)} \times \overline{M(r_i, q)} \), and \( \bigcap_{j \neq i} \overline{M(p, r_j)} \times \overline{M(r_j, q)} = M_{I_i} \). By Lemma 5.6 we have the following corollary.

**Corollary 5.7.** There exists a smooth frame \( \{ e_1, \cdots, e_k \} \) along \( M_I \) satisfying the following stratum condition: (1) \( e_i \in A_{M_I} M_{I_i} \); (2) \( \{ \pi e_1, \cdots, \pi e_k \} \) is linearly independent everywhere and all elements in \( A_x(M_I, \overline{M(p, q)}) \) can be linearly represented by \( \{ \pi e_1(x), \cdots, \pi e_k(x) \} \) with nonnegative coefficients, where \( \pi : A_{M_I} \overline{M(p, q)} \to A(M_I, \overline{M(p, q)}) \) is the natural projection.
For a manifold \( L \) with corners, [14] p. 8 shows that there exists a connection on \( L \) such that all strata are totally geodesic. (See [4] Chapter 4 for a detailed treatment of connections.) Suppose \( L_1 \) is a stratum of \( L \). Then by the above connection and the exponential map, [14] shows that an open neighborhood of \( L_1 \) in \( A(L_1, L) \) is diffeomorphic to an open neighborhood of \( L_1 \) in \( L \). Thus by the frame in Corollary 5.7, we get the following lemma.

**Lemma 5.8.** There is a smooth embedding \( \varphi_I : \mathcal{M}_I \times [0, 1)^{|I|} \rightarrow \mathcal{M}(p, q) \) satisfying the stratum condition (See (2) in Definition 3.1).

In order to prove Theorem 4.4 we need some connections even better than the above one. This leads to the definition of the product connection. There are several ways to define a connection on a manifold. One is as follows. A connection is to assign each smooth curve \( \gamma : [0, 1] \rightarrow L \) a parallel transport (or displacement) \( \gamma_t : T_{\gamma(0)}L \rightarrow T_{\gamma(t)}L \) which is a linear isomorphism. Suppose \( L_1 \) and \( L_2 \) are two manifolds with corners. Clearly, \( \gamma \) pulls back the connection on \( L_1 \times L_2 \) as follows.

**Definition 5.9.** Let \( \gamma = (\gamma_1, \gamma_2) : [0, 1] \rightarrow L_1 \times L_2 \) be a smooth curve. Define the parallel transport \( \gamma_t : T_{\gamma(0)}(L_1 \times L_2) \rightarrow T_{\gamma(t)}(L_1 \times L_2) \) as \( \gamma_t(v_1, v_2) = (\gamma_t(v_1, v_2), \gamma_t(v_2), v_2) \), where \( \gamma_t \) is the parallel transport along \( \gamma_t \). The connection assigning \( \gamma_t \) is the product connection.

For a product connection, a curve \( \gamma \) in \( L_1 \times L_2 \) is a geodesic if and only if both \( \gamma_1 \) and \( \gamma_2 \) are geodesics. By Lemma 5.8, \( \varphi_I \) pulls back the connection on \( \mathcal{M}(p, q) \) to \( \mathcal{M}_I \times [0, 1)^{|I|} \). Let \( \gamma \) be a curve in \( \mathcal{M}_I \times [0, 1)^{|I|} \) such that \( \gamma(t) = (x, \sigma(t)) \), where \( x \in \mathcal{M}_I \) and \( \sigma \) is a straight line in \( [0, 1)^{|I|} \). If \( \sigma \) passes through the origin, then \( \gamma \) is a geodesic because \( \varphi_I \) is defined by the exponential map. Since \( \mathcal{M}_I \) is totally geodesic in \( \mathcal{M}(p, q) \), we infer that \( \mathcal{M}_I \) has a connection. Moreover, \( [0, 1)^{|I|} \) also has its standard flat connection. We can define the product connection of \( \mathcal{M}_I \times [0, 1)^{|I|} \). The product connection coincides with the old one on \( T(\mathcal{M}_I \times \{0\}^{|I|}) \), and \( \varphi_I \) is still given by the exponential map under the new connection. This new connection has its advantage over the old one. In particular, for every straight line \( \sigma \) in \( [0, 1)^{|I|} \), not necessarily passing through the origin, \( \gamma(t) = (x, \sigma(t)) \) is a geodesic of the new connection. This is important in the proof of Theorem 4.4.

### 6. Proof of Theorem 4.4

Before proving Theorem 4.4 we shall introduce some definitions and notation.

**Definition 6.1.** Suppose \( (p, q) \in \Omega \), where \( \Omega \) is the set defined in Assumption 4.1. If \( p \neq q \), then define the length of \( (p, q) \) as \( |p, q| = -1 \). Otherwise, define the length of \( (p, q) \) as \( |p, q| = \sup\{|I| : I \text{ is a chain with head } p \text{ and tail } q\} \).

By (1) of Assumption 4.1 we know that \( |p, q| \leq \dim(\mathcal{M}(p, q)) < +\infty \).

By the compactness of \( \mathcal{M}(p, q) \) and (1) of Assumption 4.1 there are only finitely many chains \( I \) with head \( p \) and tail \( q \).

Suppose \( I_1 = \{r_0, \ldots, r_{k+1}\} \) and \( I_2 = \{r_0, r_i, \ldots, r_{i}, r_{k+1}\} \) are two chains of \( \Omega \) such that \( I_2 \preceq I_1 \). Like Section 3 if \( \Lambda_{I_2} = (\lambda_{i_1}, \ldots, \lambda_{i_2}) \in [0, +\infty)^{|I_2|} \) is a collaring parameter for for \( \mathcal{M}_{I_2} \), then define \( \Lambda_{I_2, I_1}(i) \) as

\[
\Lambda_{I_2, I_1}(i) = \begin{cases} 
\lambda_i & r_i \in I_2, \\
0 & r_i \notin I_2.
\end{cases}
\]
Here we consider $\Lambda_{I_2, I_1}$ as a collaring parameter for $\mathcal{M}_{I_1}$.

If $I_1 \prec I$ ($i = 1, \ldots, n$), then define

$$(6.2) \quad \Lambda_I + \Lambda_{I_1} + \cdots + \Lambda_{I_n} = \Lambda_I + \Lambda_{I_1, I} + \cdots + \Lambda_{I_n, I}. $$

Clearly,

$$\Lambda_{I_1} = \Lambda_{I_1} (I_1 - I_2) + \Lambda_{I_1, I_2}$$

For example, suppose $I_1 = \{r_0, r_1, r_2, r_3, r_4\}$, $I_2 = \{r_0, r_2, r_4\}$ and $\Lambda_{I_1} = (5, 6, 7)$, then $\Lambda_{I_1, I_2} = (6)$, and

$$\Lambda_{I_1} (I_1 - I_2) + \Lambda_{I_1, I_2} = (5, 0, 7) + (6) = (5, 0, 7) + (0, 6, 0) = (5, 6, 7) = \Lambda_{I_1}.$$ 

If $\Lambda_{I_2} = (8)$, then $\Lambda_{I_1, I_2} = (0, 8, 0)$ and

$$\Lambda_{I_1} + \Lambda_{I_2} = (5, 6, 7) + (8) = (5, 6, 7) + (0, 8, 0) = (5, 14, 7).$$

Proof of Theorem 4.4. We shall define $G_I$ by exponential maps. This requires two things. First, we need a frame satisfying the stratum condition (See Corollary 5.7 in $\mathcal{A}(\mathcal{M}_{I_1}, \mathcal{M}(p, q))$). Second, we need a connection on $\mathcal{M}(p, q)$. The proof is to construct the above two things by a double induction. The outer induction is on the length $\left[ p, q \right]$. We construct the desired $G_I$ in the case of $\left[ p, q \right] = n$ based on the hypothesis that all $G_I$ have been constructed and satisfy (1.1) and (4.2) for all $\left[ p, q \right] < n$. The inner induction is the process to construct $G_I$ for a fixed pair $(p, q)$.

1. The first step of the outer induction (the induction on $\left[ p, q \right]$).

When $\left[ p, q \right] = 0$, then $\mathcal{M}_I = \mathcal{M}(p, q)$, define $G_I : \mathcal{M}_I \to \mathcal{M}(p, q)$ as the identity.

2. The second step of the outer induction (the induction on $\left[ p, q \right]$).

Suppose we have constructed the desired $G_I$ for all pair $(p, q)$ such that $\left[ p, q \right] < n$. We shall construct $G_I$ in the case of $\left[ p, q \right] = n$. The construction is the inner induction. Let $X_k$ be the union of all $l$-strata of $\mathcal{M}(p, q)$ with $l \geq k$. Clearly, $X_{k+1} \subseteq X_k$, $X_1$ is the full boundary of $\mathcal{M}(p, q)$. We shall construct a family of open sets $U_k$ such that $U_{k+1} \subseteq U_k$ and $X_k \subseteq U_k$ by a downward induction on $k$. In other words, we construct $U_k$ after having constructed $U_{k+1}$. For each $k$, we shall construct $G_I : (\mathcal{M}_I \cap U_k) \times [0, e)^{|I|} \to \mathcal{M}(p, q)$ such that $\text{Im} G_I \subseteq U_k$ and all $G_I$ satisfy (1.1) and (4.2). We call such a map $G_I$ in $U_k$ denote it by $G_I|_{U_k}$. Extend $G_I$ with the step of the inner induction. Clearly, $U_1$ contains all $\mathcal{M}_I$ such that $|I| > 0$. If the construction of $G_I|_{U_1}$ is finished, we shall complete the proof by defining $G_{(p, q)}$ as the inclusion.

Since $\left[ p, q \right] = n$, the stratum with the lowest dimension is the $n$-stratum.

1. The first step of the inner induction (the induction on $U_k$).

We shall construct $U_n$, $G_I|_{U_n}$, frames for $\mathcal{M}_I \cap U_n$ and a connection providing all $G_I$ via the exponential map. Moreover, $(\mathcal{M}_I \cap U_n) \times [0, e)^{|I|}$ will also have a product connection (see Definition 5.8 and the comment following it) if we pull back the connection on $U_n$ via $G_I$.

We know that $X_n = \bigcup_{|J|=n} \mathcal{M}_J$. By Lemma 5.8 we can construct a smooth embedding $\varphi_J : \mathcal{M}_J \times [0, e_0)^{|J|} \to \mathcal{M}(p, q)$ satisfying the stratum condition (See (2) in Definition 3.1). Furthermore, $\mathcal{M}_J$ is compact because it is closed (also open) in the lowest dimensional stratum. Choose $e_0$ small enough so that $\text{Im} \varphi_J$ are pairwise disjoint for all $J$ such that $|J| = n$. Fix $J = \{p, r_1, \ldots, r_n, q\}$. Suppose $J_l = \{p, r_1, \ldots, r_l\}$ and $J_l^* = \{r_l, \ldots, r_n, q\}$. Clearly, $|p, r_l| < n$ and $|r_l, q| < n$. By the outer induction on $\left[ p, q \right]$, $G_{J_l^*}$ and $G_{J_l}$ have been defined.
Lemma 6.2. There exists $\epsilon > 0$. And $\varphi_j$ can be modified to be defined on $M_J \times [0, \epsilon)^{|J|}$ such that for all $l \in \{1, \ldots, n\}$, we have

$$\varphi_j(x_1 \cdot x_2, \Lambda_{j_1 \cdot \Lambda_{j_1}}) = (G_{j_1}(x_1, \Lambda_{j_1}), G_{j_1'}(x_2, \Lambda_{j_1})).$$

Proof. For small $\epsilon$, $G_{j_1} \times G_{j_1'}(M_J \times \prod_{i=1, i \neq l}^{|J|} [0, \epsilon]) \subseteq \text{Im} \varphi_j$, where $G_{j_1} \times G_{j_1'}(x_1 \cdot x_2, \Lambda_{j_1}, \Lambda_{j_1'}) = (G_{j_1}(x_1, \Lambda_{j_1}), G_{j_1'}(x_2, \Lambda_{j_1'}))$.

Consider the following map $\phi_l = \varphi_j^{-1} \circ (G_{j_1} \times G_{j_1'})$,

$$\phi_l : M_J \times \prod_{i=1, i \neq l}^{|J|} [0, \epsilon) \to \text{Im} \varphi_j \to M_J \times [0, \epsilon_0)^{|J|}.$$

We only need to prove that $\phi_l$ can be modified such that for all $l$,

$$\phi_l(x, \lambda_1, \cdots, \lambda_{l-1}, \lambda_{l+1}, \cdots, \lambda_n) = (x, \lambda_1, \cdots, \lambda_{l-1}, 0, \lambda_{l+1}, \cdots, \lambda_n).$$

Denote $(\lambda_1, \cdots, \lambda_n)$ by $\Lambda_{j_1}$, $(\lambda_1, \cdots, \lambda_{l-1}, \lambda_{l+1}, \cdots, \lambda_n)$ by $\Lambda_{j_1}$, and $\Lambda_{j_1'}$ by $\Lambda_{j_1}$. Since $\text{Im}(G_{j_1} \times G_{j_1'}) \subseteq M(p, r_1) \times M(r_2, q)$ and $\varphi_j$ satisfies the stratum condition, we have

$$\phi_l(x, \Lambda_{j_1\cdot j_{l-1}}) = (a, c_1, \cdots, c_{l-1}, 0, c_{l+1}, \cdots, c_n)$$

where $a$ and $c_i$ are smooth functions of $x$ and $\Lambda_{j_1\cdot j_{l-1}}$.

Define $\theta_l : M_J \times [0, \epsilon)^{|J|} \to M_J \times [0, \epsilon_0)^{|J|}$ as

$$\theta_l(x, \Lambda_{j_1}) = (a, \cdots, c_{l-1}, \lambda_l, c_{l+1}, \cdots, c_n).$$

Since $\phi_l$ is a smooth embedding, so is $\theta_l$. Since $M_J$ is compact, shrink $\epsilon_0$ if necessary, we may assume $\theta_l^{-1}$ can be defined on $M_J \times [0, \epsilon_0)^{|J|}$. Thus

$$(\varphi_j \circ \theta_l)^{-1} \circ (G_{j_1} \times G_{j_1'})(x, \Lambda_{j_1\cdot j_{l-1}})$$

$$= \theta_l^{-1} \circ \phi_l(x, \Lambda_{j_1\cdot j_{l-1}})$$

$$= (x, \lambda_1, \cdots, \lambda_{l-1}, 0, \lambda_{l+1}, \cdots, \lambda_n)$$

$$= (x, \Lambda_{j_1\cdot j_{l-1}}).$$

Modify $\varphi_j$ to be $\varphi_j \circ \theta_l$, we get (6.3) is true for a fixed $l$ and some $\epsilon > 0$.

In general, suppose we have proved (6.3) is true for $l \in \{1, \cdots, j - 1\}$, we shall modify $\varphi_j$ such that (6.3) is true for all $l \in \{1, \cdots, j\}$. Let $x = x_1 \cdot x_2 \cdot x_3$, where $x_1 = (a_0, \cdots, a_{l-1}), x_2 = (a_l, \cdots, a_{j-1})$ and $x_3 = (a_j, \cdots, a_n)$. Denote $\{r_1, \cdots, r_j\}$ by $J_{l(i,j)}$ and $(\lambda_{l+1}, \cdots, \lambda_{j-1})$ by $\Lambda_{J_{l(i,j)}}$.

$$\phi_j(x, \Lambda_{j_1\cdot j_{l(i,j)}}, \Lambda_{j_1'}) = \varphi_j^{-1}(G_{j_1}(x_1 \cdot x_2, \Lambda_{j_1\cdot j_{l(i,j)}}, \Lambda_{j_1'}(x_3, \Lambda_{j_1'})).$$

Since $|p, r_1| < n$, by the outer inductive hypothesis, $G_{j_1}$ satisfies (4.2). Shrink $\epsilon$ if necessary, we have

$$G_{j_1}(x_1 \cdot x_2, \Lambda_{j_1\cdot j_{l(i,j)}}) = (G_{j_1}(x_1, \Lambda_{j_1}), G_{j_1}(x_2, \Lambda_{j_1})),$$

Similarly, $G_{j_1'}(x_2 \cdot x_3, \Lambda_{j_1\cdot j_{l(i,j)}}, \Lambda_{j_1'}) = (G_{j_1\cdot j_{l(i,j)}}(x_2, \Lambda_{j_1\cdot j_{l(i,j)}}, \Lambda_{j_1'}(x_3, \Lambda_{j_1'})).$

Thus

$$G_{j_1} \times G_{j_1'}(x, \Lambda_{j_1 \cdot j_{l(i,j)}}, \Lambda_{j_1'}) = G_{j_1} \times G_{j_1'}(x, \Lambda_{j_1 \cdot j_{l(i,j)}}, \Lambda_{j_1'}).$$
Then
\[ \phi_j(x, \Lambda_{j, l} \cdot \Lambda_{j, l+1}, \Lambda_{j, r}) \]
\[ = \varphi_j^{-1} \circ (G_{j, l} \times G_{j, r})(x, \Lambda_{j, l} \cdot \Lambda_{j, l+1}, \Lambda_{j, r}) \]
\[ = \varphi_j^{-1} \circ (G_{j, l} \times G_{j, r})(x, \Lambda_{j, l}, \Lambda_{j, l+1} \cdot \Lambda_{j, r}) \]
\[ = \phi_l(x, \Lambda_{j, l}, \Lambda_{j, l+1} \cdot \Lambda_{j, r}). \]

Since \( \phi_l \) satisfies (6.3), we have \( \phi_l(x, \Lambda_{j, l}, \Lambda_{j, l+1} \cdot \Lambda_{j, r}) = (x, \Lambda_{j, l} \cdot \Lambda_{j, l+1} \cdot \Lambda_{j, r}) \), or
\[ \phi_j(x, \Lambda_{j, l}, \Lambda_{j, l+1} \cdot \Lambda_{j, r}) = (x, \lambda_{j, l}, \cdot, \lambda_{j, l+1}, \cdot, \lambda_{j, r}). \]

Define \( \theta_j : M_J \times [0, \epsilon]^{\# J} \rightarrow M_J \times [0, \epsilon_0]^{\# J} \) as (6.4), we have
\[ \theta_j(x, \lambda_{j, l}, \cdot, \lambda_{j, l+1}, \cdot, \lambda_{j, r}) = (x, \lambda_{j, l}, \cdot, \lambda_{j, l+1}, \cdot, \lambda_{j, r}). \]

The operation of \( \theta_j \) on \( M_J \times \prod_{l=1, l \neq l}^{\# J} [0, \epsilon] \times \{ 0 \} \) is the identity. Thus
\[ (\varphi_j \circ \theta_j)^{-1} \circ (G_{j, l} \times G_{j, r}) \]
\[ = \theta_j^{-1} \circ (\varphi_j^{-1} \circ (G_{j, l} \times G_{j, r})) \]
\[ = \varphi_j^{-1} \circ (G_{j, l} \times G_{j, r}) \]
\[ = \phi_l. \]

So if we modify \( \varphi_j \) to be \( \varphi_j \circ \theta_j \), then \( \phi_l (l < j) \) will not change and still satisfy (6.3). However, \( \phi_j \) may change and must satisfy (6.3) now. Thus we get a new \( \varphi_j \) such that (6.3) is true for \( l \in \{ 1, \cdots, j \} \).

By repeating this process, we finish the proof of this lemma. \( \Box \)

Now we define \( G_I \) in \( \text{Im} \varphi_I \). If \( I \neq J \), then \( \text{Im} \varphi_I \cap M_J = \emptyset \), we don’t need to consider it. We assume \( I \leq J \).

For all \( y \in \text{Im} \varphi_I \cap M_J \), there exist \( x \in M_J \) and \( \Lambda_I \in [0, \epsilon]^n \) such that \( y = \varphi_I(x, \Lambda_I) \) where \( x \) and \( \Lambda_I \) are unique and \( \lambda_I = 0 \) if and only if \( r_I \in I \). Define \( G_I(y, \Lambda_I) = \varphi_I(x, \Lambda_I + \Lambda_I) \). Since \( \varphi_I \) is a smooth embedding, so is \( G_I \). (Actually, if we identify \( \text{Im} \varphi_I \) with \( M_J \times [0, \epsilon]^{\# J} \) via \( \varphi_I \), then \( G_I \) has the form \( G_I((x, \Lambda_I), \Lambda_I) = (x, \Lambda_I + \Lambda_I) \).)

**Lemma 6.3.** The maps \( G_I \) satisfy (4.1) in \( \text{Im} \varphi_I \).

Proof. Suppose \( I_2 \leq I_1 \leq J \) and \( y \in \text{Im} \varphi_I \cap M_J \), we need to show that \( G_{I_1}(y, \Lambda_{I_1}) = G_{I_2}(G_{I_1}(y, \Lambda_{I_1}(I_1 - I_2)), \Lambda_{I_1, I_2}). \)

Suppose \( y = \varphi_I(x, \Lambda_I) \), we have \( G_{I_1}(y, \Lambda_{I_1}) = \varphi_I(x, \Lambda_I + \Lambda_I), G_{I_2}(y, \Lambda_{I_2}(I_1 - I_2)) = \varphi_I(x, \Lambda_I + \Lambda_I(I_1 - I_2)) \), and
\[ G_{I_2}(G_{I_1}(y, \Lambda_{I_1}(I_1 - I_2)), \Lambda_{I_1, I_2}) \]
\[ = G_{I_2}(\varphi_I(x, \Lambda_I + \Lambda_I(I_1 - I_2)), \Lambda_{I_1, I_2}) \]
\[ = \varphi_I(x, \Lambda_I + \Lambda_I(I_1 - I_2)) + \Lambda_{I_1, I_2} \]
\[ = \varphi_I(x, \Lambda_I + \Lambda_{I_1}) = G_{I_1}(y, \Lambda_{I_1}). \]

This completes the proof of the lemma. \( \Box \)

**Lemma 6.4.** The maps \( G_I \) satisfy (4.2) in \( \text{Im} \varphi_I \).
Suppose \( I \preceq J \), \( I = I_1 \cdot I_2 \), \( y_1, y_2 \in M_{I_1} \), and \( y_1 \cdot y_2 \in \text{Im} \varphi_J \). We need to show that \( G_I(y_1, y_2, \Lambda_{I_1} \cdot \Lambda_{I_2}) = (G_{I_1}(y_1, \Lambda_{I_1}), G_{I_2}(y_2, \Lambda_{I_2})) \).

Since \( I \preceq J \), we have \( J = J_1 \cdot J_2, I_1 \preceq J_1 \) and \( I_2 \preceq J_2 \) for some \( J_1 = \{p, r_1, \ldots, r_l \} \) and \( J_2 = \{r_1, \ldots, r_n, q \} \). Since \( y_1 \cdot y_2 \in M_{I_1} \times M_{I_2} \) and \( y_1 \cdot y_2 = \varphi_J(x, \Lambda_J) \), we have \( x = x_1 \cdot x_2 \) for some \( x_1 \in M_{I_1} \) and \( x_2 \in M_{J_2} \), and \( \Lambda_J = \Lambda_{J_1} \cdot \Lambda_{J_2} \) for some \( \Lambda_{J_1} \) and \( \Lambda_{J_2} \). Thus \( y_1 \cdot y_2 = \varphi_J(x_1 \cdot x_2, \Lambda_{J_1} \cdot \Lambda_{J_2}) \). By Lemma 6.2, \( y_1 = G_{J_1}(x_1, \Lambda_{J_1}) \) and \( y_2 = G_{J_2}(x_2, \Lambda_{J_2}) \). Furthermore,

\[
\begin{align*}
G_I(y_1, y_2, \Lambda_{I_1} \cdot \Lambda_{I_2}) &= \varphi_J(x_1, x_2, \Lambda_{J_1} \cdot \Lambda_{J_2} + \Lambda_{I_1} \cdot \Lambda_{I_2}) \\
&= \varphi_J(x_1, x_2, (\Lambda_{J_1} + \Lambda_{I_1}) \cdot (\Lambda_{J_2} + \Lambda_{I_2})).
\end{align*}
\]

By Lemma 6.2,

\[
\varphi_J(x_1 \cdot x_2, (\Lambda_{J_1} + \Lambda_{I_1}) \cdot (\Lambda_{J_2} + \Lambda_{I_2})) = (G_{J_1}(x_1, \Lambda_{J_1} + \Lambda_{I_1}), G_{J_2}(x_2, \Lambda_{J_2} + \Lambda_{I_2})).
\]

Since \( |p, r_l| < n \) and \( |r_l, q| < n \), by the outer inductive hypothesis, \( G_{J_1}, G_{J_2}, G_{I_1} \) and \( G_{I_2} \) satisfy (4.1). Thus

\[
\begin{align*}
(G_{I_1}(x_1, \Lambda_{J_1} + \Lambda_{I_1}), G_{I_2}(x_2, \Lambda_{J_2} + \Lambda_{I_2})) &= (G_{I_1}(G_{J_1}(x_1, \Lambda_{J_1}), \Lambda_{I_1}), G_{I_2}(G_{J_2}(x_2, \Lambda_{J_2}), \Lambda_{I_2})) \\
&= (G_{I_1}(y_1, \Lambda_{I_1}), G_{I_2}(y_2, \Lambda_{I_2})).
\end{align*}
\]

This completes the proof of the lemma.

We have defined the desired \( G_I \) in \( \text{Im} \varphi_J \) for all \( I \) such that \( M_I \cap \text{Im} \varphi_J \neq \emptyset \). Clearly, \( (M_I \cap \text{Im} \varphi_J) \times [0, \epsilon)^{|I|} \) has a frame \( \{ \frac{\partial}{\partial \lambda_1}, \ldots, \frac{\partial}{\partial \lambda_{|I|}} \} \). Then

\[
\{ \mathcal{N}_1(I), \cdots, \mathcal{N}_{|I|}(I) \} = dG_I|_{\lambda=0} \cdot \left\{ \frac{\partial}{\partial \lambda_1}, \ldots, \frac{\partial}{\partial \lambda_{|I|}} \right\}
\]

serves a desired frame of \( A((M_I \cap \text{Im} \varphi_J), \mathcal{M}(p, q)) \). Identify \( \text{Im} \varphi_J \) with \( M_I \times [0, \epsilon)^{|I|} \) via \( \varphi_J \), give \( \text{Im} \varphi_J \) the product connection (See Definition 5.9 and the comment following it.). Again, \( G_I(y, \Lambda_I) = \varphi_J(x, \Lambda_J + \Lambda_I) \), and \( \Lambda_J + t \Lambda_I \) for \( t \in [0, 1] \) is a line segment in \( [0, \epsilon)^{|I|} \). Then \( G_I(y, t \Lambda_I) \) is a geodesic segment. Thus \( G_I(y, \Lambda_I) = \exp(y, \sum_{i=1}^{|I|} \lambda_i \mathcal{N}_i(I)) \) and this connection is the desired one.

Do the above construction for each \( J \) such that \( |J| = n \). Clearly, \( G_J = \varphi_J \) when \( |J| = n \). Let \( U_n = \bigcup_{|J|=n} \text{Im} \varphi_J \), then \( U_n \supseteq X_n \). This completes the first step of the inner induction.

(II). The second step of the inner induction (the induction on \( U_k \)).

Suppose we have constructed \( U_{k+1} = \bigcup_{|I| \geq k+1} \text{Im} G_I \). Suppose, for all \( I \), we have constructed \( G_I|_{U_{k+1}} \), the frames on \( M_I \cap U_{k+1} \) and the connection on \( U_{k+1} \) which provides \( G_I \) via exponential maps. Moreover, \( (M_I \cap U_{k+1}) \times [0, \epsilon)^{|I|} \) has a product connection if we pull back the connection on \( U_{k+1} \) via \( G_I \). We shall extend the above things to those on \( U_k \).

The construction shares many details with the first step. The essential point is that the definition of \( G_I|_{U_k} \) should be an extension of \( G_I|_{U_{k+1}} \).

Let \( U_{k+1}(\delta) = \bigcup_{|I| \geq k+1} G_I|_{U_{k+1}}(M_I \times [0, \delta)^{|I|}) \) for \( \delta \in (0, \epsilon) \). It’s an open set such that \( X_{k+1} \subset U_{k+1}(\delta) \subset U_{k+1} \). Let \( \bar{U}_{k+1}(\delta) = \bigcup_{|I| \geq k+1} G_I|_{U_{k+1}}(M_I \times [0, \delta)^{|I|}) \).

**Lemma 6.5.** The set \( \bar{U}_{k+1}(\delta) \) is closed.
Proof. For each $I_0$ such that $|I_0| \geq k + 1$, we have $\overline{M}_{I_0} = \bigcup_{I_0 \subseteq J} M_J$ is compact. Moreover, $G_J|_{U_{k+1}} : M_J \times [0, \epsilon_0]^{|J|} \to \overline{M}(p, q)$ has been defined.

Define $G_{I_0} : M_{I_0} \times [0, \epsilon_0]^{|I_0|} \to \overline{M}(p, q)$ as $G_{I_0}(x, \Lambda_{I_0}) = G_I|_{U_{k+1}}(x, \Lambda_{I_0}, i)$ for $(x, \Lambda_{I_0}) \in M_I \times [0, \epsilon_0]^{I_0}$. Since the maps $G_I|_{U_{k+1}}$ satisfy (4.1), we infer that $G_{I_0}$ is well-defined and is a smooth embedding.

Thus $U_{k+1}(\delta) = \bigcup_{|I_0| \geq k+1} G_{I_0}(M_{I_0} \times [0, \delta]^{I_0})$ is compact. □

As the first step, by Lemma 5.3. for each $J$ such that $|J| = k$, there is a smooth embedding $\varphi_J : M_J \times [0, \epsilon_0]^{|J|} \to \overline{M}(p, q)$ satisfying the stratum condition. Thus $d\varphi_J \cdot \{ \frac{\partial}{\partial \lambda_1}, \cdots, \frac{\partial}{\partial \lambda_{|J|}} \}$ is a frame satisfying the stratum condition (See Corollary 5.7). By the inner inductive hypothesis, $M_J \cap U_{k+1}$ already has a frame $\{ N_i(J), \cdots, N_{|J|}(J) \}$ satisfying the stratum condition. Both $N_i(J)$ and $d\varphi_J \frac{\partial}{\partial \lambda_i}$ represent nonzero elements in the same $A(M_J, M_I) \cong [0, +\infty)$ for some $I \not\subseteq J$ such that $|I| = |J| - 1$. Thus, for all $\alpha(x) \geq 0$, $\{ \alpha(x) N_i(J) + (1 - \alpha(x)) \frac{\partial}{\partial \lambda_i} \mid i = 1, \cdots, n \}$ is also a frame satisfying the stratum condition. By Lemma 5.3. and the partition of unity $\xi$, there is a frame satisfying the stratum condition and coinciding with the old one in $U_{k+1}(\delta)$ for some $\delta > 0$. Also by the same reason, there is a connection in $U_{k+1} \cup \Im \varphi_J$ such that it coincides with the old one in $U_{k+1}(\delta)$.

Then, by the above frame and connection, we can modify $\varphi_J$ such that it coincides with $G_J|_{U_{k+1}}$ in $U_{k+1}(\delta)$. Since $|M_J - U_{k+1}(\delta)| = |M_J - U_{k+1}(\delta)|$ is compact, and $G_J|_{U_{k+1}}$ is an embedding, by Lemma 5.3. we infer $\varphi_J$ is an embedding defined on $M_J \times [0, \epsilon_0]^{J_I}$ for some $\epsilon_0 \in (0, \delta)$. Just as the first step, we can modify $\varphi_J$ further, such that it satisfies the conclusion of Lemma 5.2. Since originally $\varphi_J$ and $G_J|_{U_{k+1}}$ coincide in $U_{k+1}(\delta)$ and $G_I|_{U_{k+1}}$ satisfies (4.2), the modification does not change $\varphi_J|_{U_{k+1}}(\delta)$. Thus the modified $\varphi_J$ still coincides with $G_J|_{U_{k+1}}(\delta)$.

The big difference between this step and the first step is as follows. In the first step, $\Im \varphi_J$ are pairwise disjoint for $|J| = n$. Thus there is no contradiction of the definition when $G_J$ is defined in each $\Im \varphi_J$. Now it's impossible to make $\Im \varphi_J$ pairwise disjoint. We shall control their pair-wise intersections. Suppose $J_1 \not\subseteq J_2$ and $|J_1| = |J_2| = k$. Then $(M_{J_1} - U_{k+1}(\delta)) \cap (M_{J_2} - U_{k+1}(\delta)) \subseteq M_{J_1} \cap M_{J_2} = \emptyset$. Since $M_{J_1} - U_{k+1}(\delta)$ is compact, shrink $\epsilon_0$ if necessary, we have

$$\varphi_{J_1}((M_{J_1} - U_{k+1}(\delta)) \times [0, \epsilon_0]^{J_1}) \cap \varphi_{J_2}((M_{J_2} - U_{k+1}(\delta)) \times [0, \epsilon_0]^{J_2}) = \emptyset.$$ 

Since

$$\varphi_{J_1}((M_{J_1} \cap U_{k+1}(\delta)) \times [0, \epsilon_0]^{J_1}) \subseteq U_{k+1}(\delta),$$

we get $\Im \varphi_{J_1} \cap \Im \varphi_{J_2} \subseteq U_{k+1}(\delta)$.

Now we define $G_I$ in each $\Im \varphi_J$. We only need to consider $I$ such that $I \not\subseteq J$. For all $y \in M_J \cap \Im \varphi_J$, if $y = \varphi_J(x, \Lambda_J)$, define $G_I(J)(y, \Lambda_I) = \varphi_J(x, \Lambda_J + \Lambda_I)$. Given $\varphi_J = G_J|_{U_{k+1}}$ in $U_{k+1}(\delta)$, similarly to the argument in the first step, we get $G_I(J) = G_I|_{U_{k+1}}$ in $U_{k+1}(\delta)$. Since $\Im \varphi_{J_1} \cap \Im \varphi_{J_2} \subseteq U_{k+1}(\delta)$, $G_I(J_1)$ coincides with $G_I(J_2)$ in their common domains. Define $G_I|_{\Im \varphi_J} = G_I(J)$. Then $G_I$ is well-defined on $U_{k+1}(\delta) \cup \bigcup_{|J| = k} \Im \varphi_J$ and it coincides with $G_I|_{U_{k+1}}$ in $U_{k+1}(\delta)$.

Similarly to the first step, the maps $G_I|_{\Im \varphi_J}$ satisfy (4.1) and (4.2). Shrink $U_{k+1}$ to be $U_{k+1}(\epsilon_0)$. Again, $G_J = \varphi_J$ when $|J| = k$. Let

$$U_k = U_{k+1} \cup \bigcup_{|J| = k} G_J(M_J \times [0, \epsilon_0]^k).$$

ASSOCIATIVITY OF GLUING 15
The desired $G_I|_{U_k}$ is defined in the above. Shrink $\epsilon_I$ to be $\epsilon_0$ for all $I$. Give frames to $M_I \cap U_k$ as the first step. For $|J| = k$, give $\text{Im}G_J$ the product connection via $G_J$. The old connection in $U_{k+1}$ is the product connection. Thus the new connection in $\text{Im}G_J$ coincides with the old one in $U_{k+1}$. This completes the second step of the inner induction.

(III). The completion of the second step of the outer induction (the induction on $|p,q|$).

For the fixed pair $(p,q)$, the construction in $U_k$ requires a shrink of $\epsilon_I$ for all $I$ with head $p$ and tail $q$. However, the inner induction stops in a finite number of steps. Eventually, we have $\epsilon_I > 0$ which are the same for all $I$ with head $p$ and tail $q$. And if $I_1 \cdot I_2 = I$, then $\epsilon_I \leq \epsilon_{I_1}$. Thus we have constructed the desired $G_I$ for the pair $(p,q)$ with length $n$. This completes the second step of the outer induction and also the proof of this theorem.

\[\square\]

7. A Byproduct

The argument for Theorem [4] already gives the following Proposition 7.1 which gives a compatible collar structure for an arbitrary compact manifold with faces.

Suppose $L$ is a smooth manifold with faces. Suppose $F_i$ $(i = 1, \ldots, n)$ are its faces such that $\bigcup_{i=1}^n F_i = \bigcup_{k \geq 0} \partial^k L$. In other words, $\bigcup_{i=1}^n F_i$ is the full boundary of $L$. Suppose the interiors of $F_i$ are pairwise disjoint.

Let $I = \{i_1, \ldots, i_k\}$ be a subset of $\{1, \ldots, n\}$. Define $|I| = k$. Define $F_I = \bigcap_{i \in I} F_i$. In particular, when $I = \emptyset$, define $F_\emptyset = L$. Then, by Lemma [5,3] $F_I$ is either empty or an $n-k$ dimensional smoothly embedded submanifold with corners inside $L$. Denote the interior of $F_I$ by $F_I^\circ$.

Let $V_I = \prod_{i \in I} [0, +\infty)$ be a factor space of $[0, +\infty]^n$. In other words, $V_I$ is the product of the $i$th coordinate spaces of $[0, +\infty)^n$ such that $i \in I$. In particular, $V_\emptyset$ consists of one point. Let $V_I(\epsilon) = \prod_{i \in I} [0, \epsilon)$.

Let $\Lambda_I = \{\lambda_{i_1}, \ldots, \lambda_{i_k}\} \in V_I$ represent the collaring parameter for $F_I^\circ$. Suppose $J \subseteq I$. Define $\Lambda_I(I-J) \in V_I$ as

$$\Lambda_I(I-J)(i) = \begin{cases} 0 & i \in J, \\ \lambda_i & i \in I-J. \end{cases}$$

Define $\Lambda_{I,J} \in V_J$ as $\Lambda_{I,J}(i) = \lambda_i$ for $i \in J$.

**Proposition 7.1.** Suppose $L$ is compact. Then collaring maps $G_I : F_I^\circ \times V_I(1) \to L$ can be defined for all $I$ such that $F_I^\circ \neq \emptyset$. These maps satisfy the following conditions:

1. They are smooth embeddings which satisfy the following stratum condition. If $J \subseteq I = \{i_1, \ldots, i_k\}$, $\Lambda_I = \{\lambda_{i_1}, \ldots, \lambda_{i_k}\} \in V_I(1)$, and $\lambda_i = 0$ if and only if $i \in J$, then $G_I(x, \Lambda_I) \in F_I^\circ$ for all $x \in F_I^\circ$. In particular, $G_\emptyset : F_\emptyset^\circ = \partial^0 L \to L$ is the inclusion.

2. They satisfy the following compatibility. If $J \subseteq I$ and $\lambda_i > 0$ when $i \notin J$, then, for all $x \in F_I^\circ$, we have

$$G_I(x, \Lambda_I) = G_J(G_I(x, \Lambda_I(I-J)), \Lambda_{I,J}).$$

The assumption of Proposition [7,1] is more general than that of Theorem [4,3] in some sense. However, this proof is actually even easier than that one because we only deal with one manifold with faces. It only requires that [4,3] is true in a more general setting. We don’t need any more the arguments related to [4,2] such as
Lemmas 6.2 and 6.4. Instead of a double induction, it suffices to repeat the inner induction in the proof of Theorem 4.4. Since there are only finitely many set $I$, we can find $\epsilon > 0$ such that $\epsilon_I = \epsilon$ for all $I$. By a scaling of parameter, we get $\epsilon = 1$, which finishes the proof.

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REFERENCES

[1] J. Barraud and O. Cornea, Homotopic dynamics in symplectic topology, Morse theoretic methods in nonlinear analysis and in symplectic topology, 109-148, NATO Sci. Ser. II Math. Phys. Chem., 217, Springer, Dordrecht, 2006.
[2] J. Barraud and O. Cornea, Lagrangian intersections and the Serre spectral sequence, Ann. of Math., 166 (2007), 657–722.
[3] D. Burghelea and S. Haller, On the topology and analysis of a closed one form (Novikov’s theory revisited), Essays on geometry and related topics, 133-175, Monoigr. Enseign. Math., 38, Enseignement Math., Geneva, 2001.
[4] S. Chern, W. Chen and K. Lam, Lectures on Differential Geometry, Series on University Mathematics, 1, World Scientific, 1998.
[5] R. Cohen, Floer homotopy theory, realizing chain complexes by module spectra, and manifolds with corners, Algebraic topology, 39C59, Abel Symp., 4, Springer, Berlin, 2009.
[6] R. Cohen, The Floer homotopy type of the cotangent bundle, Pure Appl. Math. Q., 6 (2010), 391-438.
[7] R. Cohen, J. Jones, and G. Segal, Morse theory and classifying spaces, Warwick University preprint, 1995.
[8] R. Cohen, J. Jones, and G. Segal, Floer’s infinite dimensional Morse theory and homotopy theory, The Floer Memorial Volume, 297-325, Progress in Mathematics, 133, Birkhäuser Verlag, 1995.
[9] O. Cornea, Homotopical dynamics: suspension and duality, Ergod. Th. & Dynam. Sys., 20 (2002), 379-391.
[10] O. Cornea, Homotopical dynamics, II: Hopf invariants, smoothings and the Morse complex, Ann. Sci. cole Norm. Sup. (4) 35 (2002), 549-573.
[11] O. Cornea, Homotopical dynamics, III: real singularities and Hamiltonian flows, Duke Math. J., 109 (2001), 183C204
[12] O. Cornea, Homotopical dynamics, IV: Hopf invariants and Hamiltonian flows, Comm. Pure Appl. Math. 55 (2002), 1033-1088.
[13] O. Cornea, New obstructions to the thickening of CW-complexes, Proc. Amer. Math. Soc. 132 (2004), 2769-2781.
[14] A. Douady, Variétés à bord anguleux et voisinages tubulaires, Séminaire Henri Cartan, 14 (1961-1962), 1-11.
[15] A. Floer, Morse theory for Lagrangian intersections, J. Differential Geom. 28 (1988), 513-547.
[16] J. Franks, Morse-Smale flows and homotopy theory, Topology, 18 (1979), 199-215.
[17] K. Jünich, On the classification of $O(n)$-manifolds, Math. Annalen, 176 (1968), 53-76.
[18] F. Latour, Existence de 1-formes fermées non singulières dans une classe de cohomologie de de Rham, Publications mathématiques de l'I.H.É.S., 80 (1994), 135-194.
[19] J. Palis and W. de Melo, Geometric Theorey of Dynamical Systems, Springer-Verlag, 1982.
[20] L. Qin, On moduli spaces and CW structures arising from Morse theory on Hilbert manifolds, Journal of Topology and Analysis, 2 (2010), 409-526.
[21] L. Qin, An application of topological equivalence to Morse theory, Preprint, (2010) (see arXiv:1102.2838).
[22] M. Schwarz, Morse Homology, Progress in Mathematics, \textbf{111}, Birkhäuser Verlag, 1993.

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