Quasi-Static Solution of a Problem of Thermal Shock Acting on a Plate

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Abstract. In most research works related to stress analysis of thin structures subjected to temperature change it is assumed that variability of temperature in time is limited. In classical theory of thermoelastic thin structures the case of strong dynamical effects is excluded from consideration. Under this assumption, from three-dimensional equations of thermoelasticity the equations of thermoelastic plates and shells of Kirchhoff-type are derived. In this paper we focus on the analysis of stress-strain state of a plate in strongly dynamical temperature field, for which the classical Kirchhoff theory is not suitable. The analysis is carried out by a mathematical method without making any hypotheses. A numerical example is considered.

1. General Formulation of the 3D ThermoElasticity Problem
In research works related to classical linear problem of thermoelastic plates and shells, thermoelastic stress-strain state of a plate in strongly dynamical temperature field is typically not considered [1–4]. In this work we deal with this very case.

For simplicity, it is assumed that investigations are carried out for a temperature range within which the material properties are varied weakly. Otherwise, physically nonlinear constitutive equations must be used and the results obtained with the help of linear theory should be regarded only as approximate. For construction of the theory of thermoelastic plates, the asymptotic method of reducing three-dimensional problems of elasticity to two-dimensional ones is used [5–11]. This method allows us to construct two-dimensional theories with a prescribed accuracy for each particular class of problems. Since linear three-dimensional theory of elasticity has a unique solution, it has a corresponding unique asymptotic representation for the unknown functions. If the asymptotic representation is chosen correctly, then the approximate equations, obtained by using this asymptotic representation, allow us to satisfy the given three-dimensional equations and boundary conditions with a specified accuracy.

Let us choose the Cartesian coordinate system in the following way: the axes \(x_1\) and \(x_2\) lie in the middle plane of the plate, and the axis \(x_3\) is orthogonal to this plane. In what follows the indices \(i\) and \(j\) typically take the values 1 or 2, and the index 3 is always specified explicitly. Let \(\sigma_i, i = 1, 2\), and \(\sigma_3\) denote the normal stress components directed along the respective axes, \(\sigma_{ij}\) is the shear stress acting in the plane \(x_1 - x_2\), with \(i \neq j\), \(\sigma_{13}, \sigma_{23}\) are out-of-plane shear stress components. Components of the strain tensor follow the same notation. Also, we denote the in-plane...
displacement components by \( v_i, \), \( i = 1, 2, \) and \( v_3 \) is the transverse displacement. Let \( T \) be the temperature field. The Young’s modulus is denoted by \( E \), the Poisson’s ratio by \( \nu \), the density of the material by \( \rho \). The coefficient of thermal diffusivity is \( \alpha^2 \) and the thermal expansion coefficient is \( \alpha \).

The governing equations of three-dimensional thermoelasticity theory can then be written as:

- **constitutive equations**
  \[
  \frac{\partial v_i}{\partial x_i} = \frac{1}{E} \sigma_i - \frac{\nu}{E} \sigma_1 - \frac{\nu}{E} \sigma_2 + \alpha T, \quad \frac{\partial v_i}{\partial x_3} = -\frac{\partial v_3}{\partial x_i} + \frac{2(1+\nu)}{E} \sigma_{i3} \quad (1.1)
  \]

- **equations of motion**
  \[
  \frac{\partial \sigma_i}{\partial x_i} + \frac{\partial \sigma_{ij}}{\partial x_j} + \frac{\partial \sigma_{ij}}{\partial x_j} = \rho \frac{\partial^2 v_i}{\partial t^2}, \quad \frac{\partial \sigma_{13}}{\partial x_1} + \frac{\partial \sigma_{23}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3} = \rho \frac{\partial^2 v_3}{\partial t^2}. \quad (1.3)
  \]

Usual strain-displacement relationships are assumed to hold

\[
\varepsilon_i = \frac{\partial v_i}{\partial x_i}, \quad \varepsilon_{ij} = \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}. \quad (1.4)
\]

Each equality with indices \( i \) and \( j \) should be regarded as dual: when \( i = 1, \ j = 2 \) we obtain one equation, when \( i = 2, \ j = 1 \) we obtain another equation.

It is assumed that the stress-strain state in a plate is caused only by dynamic temperature field and the surface pressure is absent. Then the boundary conditions on the upper and lower surfaces of the plate can be written as

\[
\sigma_3 \mid_{x_1=\pm h} = 0, \quad \sigma_{i3} \mid_{x_3=\pm h} = 0 \quad (1.5)
\]

Here \( h \) denotes the semi-thickness of the plate. Dual sign \( \pm \) is introduced for brevity of the notation: each equation that contains \( \pm \) actually constitutes two equations – the first equation is obtained by taking the plus sign and the second one is obtained by taking the minus sign.

We will solve a coupled thermoelastic problem, i.e., heat conduction equation is solved simultaneously with the equations of thermoelasticity. The heat conduction equation is given by

\[
\frac{\partial^2 T}{\partial x_1^2} + \frac{\partial^2 T}{\partial x_2^2} + \frac{\partial^2 T}{\partial x_3^2} - \frac{1}{\alpha^2} \frac{\partial T}{\partial t} = 0, \quad (1.6)
\]

where \( \alpha^2 \) is the coefficient of thermal diffusivity.

The temperature must satisfy the initial condition

\[
T \mid_{t=0} = T_0 \quad (1.7)
\]

and the following boundary conditions on the upper and lower surfaces of the plate

\[
T \mid_{x_1=\pm h} = T_1, \quad T \mid_{x_3=\pm h} = T_2. \quad (1.8)
\]

In this formulation, as in classical theory of thermoelasticity, this problem is subdivided into two problems. The first problem is the problem of heat conduction \((1.7)-(1.8)\); the second problem, which is solved after the solution of the first problem has been obtained, constitutes in the solution of the thermoelastic problem \((1.1)-(1.5)\), in which the temperature is already known.
2. Solution of the 3D ThermoElasticity Problem by Asymptotic Method

In classical theory of thermoelasticity of thin structures the variability index of stress-strain state along the thickness coordinate is equal to 1 [5,6]. However, for a strongly dynamic thermoelastic problem the variability index along the thickness is greater than 1 and the variability index in time is greater than 2.

As is customary in asymptotic methods, in equations (1.1)–(1.8) let us change the scale for independent space variables \( x_i \) and \( x_3 \) and time \( t \)

\[
x_i = L \eta^{s} \xi, \quad x_3 = L \eta^{r/2} \zeta, \quad t = \eta^r \frac{L^2}{a^2} \tau, \quad \eta = \frac{h}{L}, \quad 0 \leq s < 1.
\] (2.1)

Here \( h \) is semi-thickness of the plate, \( L \) is its characteristic size, \( s \) is the variability index along the coordinates \( x_i \), \( r \) is the variability index for the time variable \( t \), \( \eta \) is the dimensionless small parameter equal to the ratio of semi-thickness of the plate to its characteristic size \( L \). As is customary in asymptotic methods, dimensionless variables are chosen in such a way that differentiation with respect to these variables does not result in strong decrease or increase of the unknown quantities. Usually, in classical theory of plates, for the coordinate \( x_3 \), the stretch \( x_3 = L \eta \zeta \) is employed. If the variability index in time is greater than 2 (strongly dynamic problem), the variability index of the thermal-stress-strain state along the thickness coordinate \( x_3 \) becomes greater than 1 and the classical Kirchhoff theory is not suitable.

![Figure 1. Regions of strongly dynamic problem and a thin plate theory problem](image)

Let us remark again on the difference between this method and the known asymptotics for the theory of thin structures: in classical theory of thin structures [1,2] it was assumed that \( 0 \leq r \leq 2 \) since for \( r > 2 \) the known equations of plate theory are not valid. For a strongly dynamic temperature field considered in this paper, we assume that \( r > 2 \). In this case a new asymptotic representation of the dependent variables is required. Fig. 1 shows a diagram in which a total time scale is divided into three regions. The largest values of time correspond to \( r < 2 \) and this is the region of thin plate theory validity. When to \( r > 2 \) the problem becomes strongly dynamic. In this strongly dynamic region, there is a sub-region in which a quasi-static solution can be constructed as is shown below.

Let us introduce a new dimensionless quantity \( \mu \) that incorporates the effect of thermal conduction, density, elastic modulus of the material

\[
\frac{\rho a^2}{EL} = \eta^c \mu,
\] (2.2)

where the exponent \( c \) of the small parameter \( \eta \) is chosen in such a way that the parameter \( \mu \) of the order equal to 1. Depending on the value of the power \( c \), as is shown below, the response of the plate can be either dynamic or static.

First, let us investigate the heat conduction equation. We introduce the dimensionless temperature

\[\alpha T = T^*,\]
and perform change of variables according to (2.1). Then the heat conduction equation (1.6) takes the form

$$\eta^{-2s} \left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right) T_s + \frac{\partial^2 T_s}{\partial \tau} - \frac{\partial T_s}{\partial \tau} = 0.$$ 

With the order of accuracy equal to $\eta^{-2s}$ this equation takes a simpler form

$$\frac{\partial^2 T_s}{\partial \tau} - \frac{\partial T_s}{\partial \tau} = 0. \tag{2.3}$$

Now let us make a new asymptotic representation for other dependent variables. Depending on the values of the parameters $r$, $c$, $s$ we obtain different equations and corresponding stress-strain states. First, we consider the case when $r - 2s > 0$ and $c - 2r + 2s > 0$. Then since $c > 2r - 2s = r + (r - 2s)$ it automatically follows that $c > r$ and therefore $c - r > 0$. Equivalently, we can assume that $r - 2s > 0$, $c - r > 0$ and $2s < c - r$. Then it automatically follows that $c - 2r + 2s > 0$. For this case, a unique asymptotic representation exists and can be written as follows (here again dimensionless unknown quantities are denoted by letters with stars)

$$\sigma_i / E = \sigma_{i\ast}, \quad \sigma_{j\ast} / E = \eta^{-2s} \sigma_{j\ast}, \quad \sigma_3 / E = \eta^{-2s} \sigma_{3\ast}, \quad \sigma_{i3} / E = \eta^{12s} \sigma_{i3\ast} \tag{2.4}$$

$$v_s / L = \eta^{1/2} v_{s\ast}, \quad v_s / L = \eta^{-1/2} v_{s\ast}.$$ 

After substitution of (2.1)–(2.4) into (1.1)–(1.4), we obtain

$$\frac{\partial \sigma_{i\ast}}{\partial \xi_i} + \eta^{-r} \frac{\partial \sigma_{j\ast}}{\partial \xi_j} + \frac{\partial \sigma_{j\ast}}{\partial \zeta_j} = \eta^{-r} \frac{\partial^2 v_{s\ast}}{\partial \zeta^2}, \quad \frac{\partial \sigma_{i3\ast}}{\partial \xi_i} + \frac{\partial \sigma_{j3\ast}}{\partial \zeta_j} = \eta^{-2r+2s} \mu \frac{\partial^2 v_{s\ast}}{\partial \tau^2} \tag{2.5}$$

$$\frac{\partial v_{s\ast}}{\partial \zeta} = \eta^{-2s} \sigma_{s\ast} - \nu(\sigma_{i\ast} + \sigma_{j\ast}) + T_s, \quad \frac{\partial v_{s\ast}}{\partial \xi_i} = \frac{\eta^{-2s}}{1 - \nu} \sigma_{s\ast} - \frac{1}{1 - \nu} T_s, \quad \sigma_{i\ast} = \frac{1}{2(1 + \nu)} e_{i\ast}, \quad \sigma_{j\ast} = \frac{1}{2(1 + \nu)} e_{j\ast}.$$ 

Now in (2.5) we can neglect the terms of the order $\varepsilon = O(\eta^{-2r+2s} + \eta^{-2s} + \eta^{-r})$. After returning to original variables, the following simpler equations for determining the unknown quantities can be obtained from (2.5)

$$\frac{\partial^2 T}{\partial x_3^2} - \frac{1}{a^2} \frac{\partial T}{\partial t} = 0, \quad \sigma_i = -\frac{E\alpha}{1 - \nu} T, \quad \frac{\partial v}{\partial x_3} = \frac{\nu}{E} (\sigma_i + \sigma_{j3}) + \alpha T, \tag{2.6}$$

$$\frac{\partial \sigma_{i3}}{\partial x_3} = \frac{\partial \sigma_{i3}}{\partial x_3}, \quad \frac{\partial v}{\partial x_3} = \frac{2(1 + \nu)}{E} \sigma_{i3},$$

$$\frac{\partial \sigma_{i3}}{\partial x_3} = \frac{\partial \sigma_{i3}}{\partial x_3} - \frac{\partial \sigma_{i3}}{\partial x_3} = \frac{\partial v}{\partial x_3} + \frac{2(1 + \nu)}{E} \sigma_{i3}, \quad \sigma_{ij} = \frac{1}{2(1 + \nu)} e_{ij}, \quad e_{ij} = \frac{\partial v}{\partial x_3} + \frac{\partial v}{\partial x_3}.$$ 

It is important to note that the resulting thermal-stress-strain state is static or quasi-static since there is no inertia term in (2.6).

We can also consider a dynamic case, which we call dynamic case I. This case is realized when we increase $r$ to the extent that $c - 2r + 2s \leq 0$ but, as before, $c - r > 0$, $r - 2s > 0$. If we assume that $r - 2s < c - r$, as before, then we obtain $c - 2r + 2s > 0$, which contradicts our assumption about the negative sign of $c - 2r + 2s$. Thus, the only possible situation is when $0 < c - r \leq r - 2s$, which leads to $c - 2r + 2s \leq 0$. For the dynamic case I we cannot neglect the term $\eta^{-2r+2s}$ because of
negative power. Thus, a new asymptotic representation is required which will be the focus of our future research.

Finally, consider a special case when $c - 2r + 2s = 0$ and still $c - r > 0$. Then in (2.5) we must retain the inertia term which becomes $O(1)$. Therefore, instead of (2.6) we will end up with the following system of equations:

$$\frac{\partial^2 T}{\partial x_2^2} - \frac{1}{a^2} \frac{\partial T}{\partial t} = 0, \quad \sigma_i = -\frac{E \alpha}{1 - \nu} T, \quad \frac{\partial v_{i_1}}{\partial x_3} = -\frac{v}{E} (\sigma_1 + \sigma_2) + \alpha T,$$

$$\frac{\partial \sigma_{i_3}}{\partial x_3} = -\frac{\partial \sigma_i}{\partial x_1}, \quad \frac{\partial v_{i_1}}{\partial x_3} = -\frac{\partial v_{i_1}}{\partial x_1} + \frac{2(1 + \nu)}{E} \sigma_{i_3},$$

$$\frac{\partial \sigma_{i_1}}{\partial x_3} = \rho \frac{\partial^2 v_{i_1}}{\partial t^2} - \frac{\partial \sigma_{i_3}}{\partial x_1} - \frac{\partial \sigma_{i_3}}{\partial x_1}, \quad \sigma_{i_1} = \frac{1}{2(1 + \nu)} v_{i_1}, \quad e_{i_1} = \frac{\partial v_{i_1}}{\partial x_1} + \frac{\partial v_{i_1}}{\partial x_1}$$

In spite of the fact that (2.7) contains an inertia term, the problem is still quasi-static because the inertia term can be expressed from the first three equations of (2.7) in terms of the temperature, which has already been found.

It is important to note that the boundary conditions at the edges of the plate are still not satisfied since the system of equations (2.6) does not include free parameters that can be chosen to satisfy these boundary conditions. At this point, the problem is solved fully only for special cases such as infinite elastic layer or closed shell. In order to satisfy boundary conditions in a more general case, it is required to solve the problem with boundary layer.

3. Infinite Layer
Consider a layer of infinite length and of thickness $2h$. Let us first solve the heat conduction equation for the following initial and boundary conditions

$$T \mid_{t=0} = T_0, \quad T \mid_{x_1=h} = A_1, \quad T \mid_{x_1=-h} = A_2.$$

For simplicity, we assume that $T_0, A_1, A_2$ are constants. Since the temperature field is independent of the coordinates $x_1$ and $x_2$ it follows that all other dependent variables are independent of these variables. From the symmetry of the problem, it is also obvious that the in-plane displacements $v_{i_1}$ are equal to zero. Thus, as it follows from (1.1) and (1.2), the shear stresses $\sigma_{i_1}, \sigma_{23}$ and strains $e_{i_1}, e_{23}$ also vanish. From (1.4) it is clear that $e_{i_2} = 0$ and from the symmetry of the problem $\sigma_{i_2} = 0$.

Therefore, the equations (1.1)–(1.3) and (1.6) are reduced to

- constitutive equations

$$\frac{\partial v_{i_1}}{\partial x_3} = \frac{1}{E} \sigma_{i_1} - \frac{v}{E} \sigma_{i_1} - \frac{v}{E} \sigma_{i_2} + \alpha T, \quad \sigma_{i_1} = \frac{v}{1 - \nu} \sigma_{i_1} - \frac{E \alpha}{1 - \nu} T, \quad e_{i_1} = \frac{\partial v_{i_1}}{\partial x_1} + \frac{\partial v_{i_1}}{\partial x_1}$$

- equation of motion

$$\frac{\partial \sigma_{i_1}}{\partial x_3} = \rho \frac{\partial^2 v_{i_1}}{\partial t^2},$$

- heat conduction equation

$$\frac{\partial T}{\partial x_2} = \frac{1}{a^2} \frac{\partial T}{\partial t}.$$
\[
\frac{\partial^2 T}{\partial x^2} - \frac{1}{\alpha^2} \frac{\partial T}{\partial t} = 0.
\] (3.4)

For convenience, let us introduce dimensionless quantities
\[
\alpha T = T_r, \quad \alpha T_0 = T_{r0}, \quad \alpha A_1 = A_{r1}, \quad \alpha A_2 = A_{r2}, \quad \sigma_1 / E = \sigma_{r1}, \quad \sigma_3 / E = \sigma_{r3}, \quad v_3 / h = v_{r3},
\]
and dimensionless independent variables \( \zeta \) and \( \tau \) such that
\[
x_3 = h \zeta, \quad t = \frac{h^2}{\alpha^2} \tau.
\]
Note that definitions of dimensionless variables introduced here are different from (2.1) and (2.4) and the parameter \( L \) is no longer involved because of infinite extent of the layer.

In terms of these dimensionless quantities, the problem (3.1)-(3.4) can be written as
\[
T_r \bigg|_{x=0} = T_{r0}, \quad T_r \bigg|_{x=1} = A_{r1}, \quad T_r \bigg|_{x=-1} = A_{r2},
\]
\[
\frac{\partial T_r}{\partial \zeta} - \frac{\partial T_r}{\partial \tau} = 0, \quad \frac{\partial \sigma_{r3}}{\partial \zeta} = \frac{\rho \alpha^4}{E h^2} \frac{\partial^2 v_{r3}}{\partial \tau^2},
\]
\[
\sigma_{r1} = \frac{\nu}{1-\nu} - \frac{1}{1-\nu} T_r, \quad \frac{\partial v_{r3}}{\partial \zeta} = -v (\sigma_{r1} + \sigma_{r3}) + T_r.
\] (3.5)

Now let us assume that
\[
\varepsilon = \frac{\alpha \lambda^4}{E h^2}
\] (3.6)

is a small parameter. Let us make an estimate of this parameter for steel. Suppose \( h = 1/100 \) m.

Coefficient of thermal conductivity for steel is taken as \( k = 20 \frac{W}{m \cdot K} \), density is \( \rho = 7800 \frac{kg}{m^3} \), and specific heat is \( c = 490 \frac{J}{kg \cdot K} \). Thus the coefficient of thermal diffusivity is
\[
a^2 = \frac{k}{\rho c} = 5.23 \times 10^{-6}.
\]

Taking Young’s modulus of steel equal to \( E = 210 \times 10^9 \) Pa, we can obtain that \( \varepsilon \approx 1 \times 10^{-14} \). Indeed, it is a very small number. Thus, to solve the differential equation of motion (3.5) we can use the following asymptotic representation for the unknown fields
\[
v_{r3} = v_{r3}^{(0)} + \alpha v_{r3}^{(1)}, \quad \sigma_{r3} = \sigma_{r3}^{(0)} + \alpha \sigma_{r3}^{(1)}
\] (3.7)

where the quantities with subscripts are all unknown functions and the subscript itself signifies the order of approximation. Substitution of these quantities into the differential equation of motion gives
\[
\frac{\partial \sigma_{r3}^{(0)}}{\partial \zeta} + \varepsilon \frac{\partial \sigma_{r3}^{(1)}}{\partial \zeta} = \varepsilon \frac{\partial^2 v_{r3}^{(0)}}{\partial \tau^2} + \varepsilon^2 \frac{\partial^2 v_{r3}^{(1)}}{\partial \tau^2}
\] (3.8)

Collecting the most significant terms in (3.8) we obtain that
\[
\frac{\partial \sigma_{r3}^{(0)}}{\partial \zeta} = 0.
\] (3.9)

But the stress \( \sigma_{r3} \) must be equal to zero on the upper and lower surfaces of the plate, and thus we conclude that
\[
\sigma_{r3}^{(0)} = 0.
\] (3.10)

Now the zeroth order approximation for the unknown quantities immediately follows from (3.5)
\[ \sigma_r^{(0)} = -\frac{1}{1-\nu} T_r, \quad \frac{\partial v_\zeta^{(0)}}{\partial \zeta} = -v \left( \sigma_r^{(0)} + \sigma_\zeta^{(0)} \right) + T_r. \tag{3.11} \]

Note that it is possible to solve for the temperature field \( T_r \) exactly. Below we show how the asymptotic solution of the zeroth order can be obtained for the present problem.

### 3.1. Zeroth Order Solution

Solution of the heat conduction equation can be found as

\[ T_r = \sum_{n=1}^{\infty} C_n \sin \left( \frac{n\pi}{2} (\zeta + 1) \right) \exp \left( -\frac{n^2 \pi^2 \tau}{4} \right) + A_r \left( \frac{\zeta + 1}{2} \right) + A_{\zeta} \left( \frac{1-\zeta}{2} \right), \]

where the constant \( C_n \) is given by

\[ C_n = \frac{2}{n\pi} \left( 1 - (-1)^n \right) (T_{0r} - A_{\zeta}) - \frac{4(-1)^n}{n\pi} \left( \frac{A_r}{2} + \frac{A_{\zeta}}{2} \right). \tag{3.13} \]

The constant \( C_n \) was found from the condition that the initial temperature is equal to \( T_{0r} \). To find the constant \( C_n \), it is convenient to make the change of variables \( x \leftarrow \zeta + 1 \) in (3.12) and afterwards it is required to evaluate the following integrals

\[ \int_0^2 \sin \left( \frac{n\pi}{2} x \right) dx = \frac{2}{n\pi} \left( 1 - (-1)^n \right), \quad \int_0^2 \sin \left( \frac{n\pi}{2} x \right) x dx = \frac{4(-1)^n}{n\pi}. \]

After finding the in-plane stresses from (3.11), we can find the transverse strain as

\[ \frac{\partial v_\zeta^{(0)}}{\partial \zeta} = \frac{1+\nu}{1-\nu} T_r. \tag{3.14} \]

The transverse displacement can be found after integration of (3.14), which gives

\[ v_{\zeta}^{(0)} = \frac{1+\nu}{1-\nu} \left[ -\frac{2}{n\pi} \sum_{n=1}^{\infty} C_n \cos \left( \frac{n\pi}{2} (\zeta + 1) \right) \exp \left( -\frac{n^2 \pi^2 \tau}{4} \right) + A_r \left( \frac{\zeta^2 / 2 + \zeta}{2} + A_{\zeta} \frac{\zeta - \zeta^2 / 2}{2} \right) \right] + V, \tag{3.15} \]

where \( V \) is a constant of integration. Assume now that the initial temperature is equal to zero, i.e.,

\[ T_{0r} = 0. \tag{3.16} \]

Then the constant of integration can be found from the condition that at time \( t = 0 \) the displacement is also equal to zero, i.e.,

\[ v_{\zeta}^{(0)}(\tau = 0) = \frac{1+\nu}{1-\nu} \left[ -\frac{2}{n\pi} \sum_{n=1}^{\infty} C_n \cos \left( \frac{n\pi}{2} (\zeta + 1) \right) + A_r \left( \frac{\zeta^2 / 2 + \zeta}{2} + A_{\zeta} \frac{\zeta - \zeta^2 / 2}{2} \right) \right] + V = 0. \tag{3.17} \]

The constant \( V \) can be found from (3.17) by noting that the bracketed expression in (3.17) converges to a constant, i.e.,

\[ \left[ -\frac{2}{n\pi} \sum_{n=1}^{\infty} C_n \cos \left( \frac{n\pi}{2} (\zeta + 1) \right) + A_r \left( \frac{\zeta^2 / 2 + \zeta}{2} + A_{\zeta} \frac{\zeta - \zeta^2 / 2}{2} \right) \right] \rightarrow \text{const} \]

Indeed, we can show this by changing the variables \( x = \zeta + 1 \) in the expression above

\[ \left[ -\frac{2}{n\pi} \sum_{n=1}^{\infty} C_n \cos \left( \frac{n\pi}{2} x \right) + A_r \frac{x^2 - 1}{4} + A_{\zeta} \frac{-x^2 + 2x - 3}{2} \right] \rightarrow \text{const} \tag{3.18} \]
and then using the Fourier series expansions for $x$ and $x^2$, which are valid for $x \in [0,2]$,

$$
x = 1 + \sum_{n=1}^{\infty} \left( \frac{4(-1)^n}{\pi^2 n^2} - \frac{4}{\pi^2 n^2} \right) \cos \left( \frac{n\pi}{2} x \right),
$$

$$
x^2 = \frac{4}{3} + \sum_{n=1}^{\infty} \frac{16(-1)^n}{\pi^2 n^2} \cos \left( \frac{n\pi}{2} x \right).
$$

Substituting these expansions into (3.18) and using the expression for $C_n$ given by (3.13), with $T_{0r} = 0$, we can show that the constant to which (3.18) converges is given by

$$
\text{const} = \frac{A_{0r}}{12} - \frac{A_{1r}}{12}.
$$

Therefore, the constant $V$ in (3.15) becomes

$$
V = \frac{1 + \nu \left( \frac{A_{0r}}{12} - \frac{A_{1r}}{12} \right)}{1 - \nu}.
$$

(3.20)

This completes determination of the zeroth order approximation for the transverse displacement (3.15). The temperature field (3.12) was found exactly.

Fig. 2 shows through-the-thickness distribution of temperature field in an infinite layer when the prescribed (normalized) temperature at the upper surface $\zeta = 1$ is equal to 1, i.e., $\alpha T(\zeta = 1) = 1$. The temperature is shown for three values of normalized time $\tau$: 0.001, 0.01 and 0.1.

Fig. 3 shows similar distribution for the (normalized) transverse displacement $v_z^{(0)} / h$. The Poisson’s coefficient of the layer is taken equal to 0.3. It is seen that the displacement of the upper surface is positive and the displacement of the lower surface is negative which suggests an overall increase in the thickness of the plate. As time increases the displacement grows but the displacement of the upper surface always remains larger (in magnitude) than the displacement of the lower surface.

![Figure 2. Through-the-thickness distribution of temperature in an infinite layer; prescribed temperature at the upper surface is equal to 1](image)

steel plate
$L=1$ m, $h=0.01$ m,
\nu=0.3;
temperature at upper surface $\alpha T=1$
Figure 3. Through-the-thickness distribution of transverse displacement in an infinite layer; prescribed temperature at the upper surface is equal to 1

4. Estimates of Validity of Quasi-Static Solution

In this section we again consider a general case with parameter \( L \) included. Let us make an estimate of the time range \( t_{\text{mn}} < t < t_{\text{max}} \) for which the quasi-static solution can be valid for various geometries of the plate. Consider a plate made of steel with properties indicated in the previous section. Depending on the values of parameters \( L \) and \( h \) the parameter \( \eta \) can be estimated according to (2.1).

Then from (2.2) the estimate of the power \( c \) can be obtained assuming that \( \mu = O(1) \).

The lower bound on the value of time (or the upper bound on the value of \( r \)) can be obtained from the inequality

\[
c - 2r + 2s > 0,
\]

which must hold for quasi-static solution. Assuming now that \( s = 0 \), (4.1) gives

\[
r < c/2.
\]

The upper bound on the value of time (or the lower bound on the value of \( r \)) can be simply evaluated from the requirement that \( r > 2 \). Therefore, if \( s = 0 \)

\[
2 < r < c/2.
\]

Using now definition of the dimensionless time \( \tau \) given by (2.1), the desired bounds can be obtained from

\[
t = \eta \frac{L^2}{d^2} \tau
\]

and the inequality (4.3) assuming that \( \tau = 1 \). Table 1 presents computed time bounds for various geometries of steel plate.
Table 1. Estimates of time range for quasi-static solution validity for various geometries of steel plate

| h [m] | L [m] | \( \eta \) | \( c \) | \( t_{\text{min}} \) [sec] | \( t_{\text{max}} \) [sec] |
|-------|-------|-----------|---------|-----------------|-----------------|
| 1/100 | 1     | 1/100     | 9       | 1.91 \times 10^{-4} | 19.11           |
| 1/100 | 10    | 0.001     | 6.66    | 0.0020          | 19.11           |
| 1/1000| 1     | 0.001     | 6       | 1.91 \times 10^{-4} | 0.191           |
| 1/1000| 10    | 0.0001    | 5       | 0.0019          | 0.191           |

The solution for the indicated time ranges can be obtained from the solution presented in the previous section for the infinite layer. The infinite layer solution can be adjusted to take into consideration different definitions of dimensionless time and coordinate in Sections 2 and 3. Using the following connections

\[ t = \eta \frac{L^2}{a^2} \tau \leftrightarrow \frac{h^2}{a^2} \tau, \quad x_3 = L \eta^{1/2} \zeta \leftrightarrow h \zeta \]

and replacing one dimensionless variable with another, the solution presented in Section 3 can be properly adjusted.

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