Homogenization of Fractional Kinetic Systems with Random Initial Data

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Abstract

Let \( w(t, x) := (u, v)(t, x), \) \( t > 0, \ x \in \mathbb{R}^n, \) be the \( \mathbb{R}^2 \)-valued spatial-temporal random field \( w = (u, v) \) arising from a certain two-equation system of fractional kinetic equations of reaction-diffusion type, with given random initial data \( u(0, x) \) and \( v(0, x) \). The space-fractional derivative is characterized by the composition of the inverses of the Riesz potential and the Bessel potential. We discuss two scaling limits, the macro and the micro, for the homogenization of \( w(t, x) \), and prove that the rescaled limit is a singular field of multiple Itô-Wiener integral type, subject to suitable assumptions on the random initial conditions. In the two scaling procedures, the Riesz and the Bessel parameters play distinctive roles. Moreover, since the component fields \( u, v \) are dependent on the interactions present within the system, we employ a certain stochastic decoupling method to tackle this components dependence. The time-fractional system is also considered, in which the Mittag-Leffler function is used.

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1 Introduction

The purpose of this paper is to present a certain homogenization theory associated with the following linear fractional kinetic system of reaction-diffusion type

\[
\begin{align*}
\frac{\partial^\beta}{\partial t^\beta} u(t, x) &= -\mu_1 (I - \Delta)^\frac{\gamma}{2} (-\Delta)^\frac{\alpha}{2} u(t, x) + b_{11} u + b_{12} v, \\
\frac{\partial^\beta}{\partial t^\beta} v(t, x) &= -\mu_2 (I - \Delta)^\frac{\gamma}{2} (-\Delta)^\frac{\alpha}{2} v(t, x) + b_{21} u + b_{22} v,
\end{align*}
\]

(1.1)

in the above, \(\mu_i > 0, \ 0 < \beta \leq 1, \ 0 < \alpha \leq 2, \ 0 \leq \gamma, \) and \(t > 0, x \in \mathbb{R}^n.\) The parameter \(\beta\) denotes the time-fractional index, and \(\alpha, \gamma\) denote the space-fractional indices, for which we refer as the Riesz parameter and the Bessel parameter respectively (see [28, V.1 and V.3]).

When \((\beta, \alpha, \gamma) = (1, 2, 0),\) the system (1.1) is reduced to a classical reaction-diffusion system. The time-fractional index \(\beta < 1\) means sub-diffusive (super-diffusive in case \(\beta > 1,\) which we do not study in this paper; see the remark in Section 6). The spatial-fractional Riesz index \(\alpha\) means the jumps of the evolution, and Bessel index \(\gamma\) means the tempering of large jumps; see the now-classic book of Stein [28, V.1 and V.3] for precise mathematical explanations. To our knowledge, fractional kinetic equations of Riesz-Bessel type appear firstly in Anh and Leonenko [2, 3]; abundant subsequent works in this direction by the authors and collaborators can be seen in [4, 5, 6, 7, 14, 17] and the references therein. The two papers [7, 17] with external potentials are particularly related to the study of this paper. We should mention that fractional operators with two fractional parameters are natural mathematical objects to describe long-range dependence and/or intermittency; one can find data exhibiting such characteristics in
a large number of fields including economics, finance, telecommunications, turbulence, and hydrology.

In this paper, we consider the system (1.1) with $\mu_1 = \mu_2 = \mu > 0$ and with the random initial data $u_0$ and $v_0$, of which are independent and each one has a certain long-range dependence in its random structure. This paper is along the [3] on single fractional kinetic equation; yet our results in this paper are with the following novel features. Firstly and most importantly, we study two scaling procedures, the macro and the micro, of the homogenization of the associated spatial-temporal random solution-field, in which the Riesz and the Bessel parameters play distinctive roles; our result on the micro-scaling is new, even for the single equation case; this micro-scaling makes use of both the Riesz and the Bessel indices and also needs the rescaling on the initial data. We feel that our result in this micro-scaling may capture the proclaimed feature of the intermittency of the random motions. Moreover, due to the interactions present within the system, the components fields $u, v$ are dependent (even we have assumed the independence of the initial data $u_0, v_0$), and we employ a certain stochastic decoupling method to tackle this components dependence. Our study may show how the theory of Riesz potentials and Bessel potentials, as in the Chapter V of Stein [28], may appear significantly in the homogenization of random fields.

The study on the single P.D.E. with random initial condition can be traced back to [15] and [26], and then has a long active development; we refer to the citations in the above and the references therein. There also has very significant progress on Burgers’ equation with random initial data; see the monograph of Woyczyński [30] and the Chapter 6 of Bertoin [8]. Whilst, to our knowledge, relevant study on P.D.E. system with random initial data seems few in previous literatures, except the works of Leonenko and Woyczyński [19, 20] on multi-dimensional Burgers’ random fields (Burgers’ Turbulence).

The results of this paper show that, for the fractional kinetic system (1.1) with suitable random initial data, the rescaled field $w^\varepsilon(t, x)$ in the micro-scaling (which we mean $\varepsilon t$ and $\varepsilon \downarrow 0$) of homogenization both the Riesz parameter $\alpha$ and the Bessel
parameter $\gamma$ play their roles; while in the macro-scaling (which we mean $\frac{1}{\varepsilon}$ and $\varepsilon \downarrow 0$) only the Riesz parameter $\alpha$ plays the role. Nevertheless, in either case the limiting field is a singular field of multiple Itô-Wiener integral type. Furthermore, the component fields $u, v$ are dependent, due to the interactions present within the system, we employ a certain stochastic decoupling method to tackle this components dependence. The method itself could be potentially important in the future study on some random systems, for example the gradient system of Hamilton-Jacobi equation with random initial data, as in \cite{30} p.173; we notice that the decoupling has been traditionally used in solving differential equation systems.

The underlying idea in this paper is motivated by those works in \cite{2, 3, 7, 17, 18} and the references therein. Namely, we use the spectral representations to describe the sample field arising from the initial data, and the relations between Hermite polynomials and homogeneous chaos associated with the initial data, to get representations for the limit field in terms of multiple Itô-Wiener integrals. From limit theorems point-of-view, our results, also those in the above citations, belong to the realm of non-central limit theorems for convolution type integrals, in which the papers \cite{29, 12} are pioneering; see also the monograph of Major \cite{25} and survey papers in the special volume edited by Doukhan, Oppenheim and Taqqu \cite{10}.

The paper is organized as follows. In Section 2 we give the explicit solution of the system (1.1). In Section 3 the initial data are assumed to be stationary random fields, and we discuss the covariance structure of the resulting solution-vector random field of (1.1), subject to the specified random initial condition; we show that the spectral method is suitable in describing our random field relative to the space-time parameter. We also introduce the initial field to be a certain subordinated Gaussian random field generated by a class of non-random functions whose variables are relative to the spatial parameter. In the main Sections 4 and 5, we consider (1.1) with the usual time-derivative and the fractional spatial-derivative characterized by the Riesz and the Bessel parameters. We present the homogenization of micro-scaling in Section 4 and the less subtle macro-
scaling in section 5, respectively. In Section 6 we provide extensions of the results in Sections 4 and 5 to the time-fractional $\beta < 1$, in which we need to use the Mittag-Leffler function. The proofs of all our results are given in Section 7.

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## 2 Preliminaries

To begin with, we rewrite the system (1.1), with $\mu_1 = \mu_2 = \mu > 0$, $\alpha > 0$, $\gamma \geq 0$ and $\beta = 1$, in the matrix form as follows:

$$
\frac{\partial}{\partial t} \begin{pmatrix} u \\ v \end{pmatrix} = -\mu(I - \Delta)\frac{\partial}{\partial t} (-\Delta)^\frac{\beta}{2} \begin{pmatrix} u \\ v \end{pmatrix} + B \begin{pmatrix} u \\ v \end{pmatrix},
$$

subject to some initial conditions

$$
\begin{pmatrix} u(0, x) \\ v(0, x) \end{pmatrix} = \begin{pmatrix} u_0(x) \\ v_0(x) \end{pmatrix}, \quad x \in \mathbb{R}^n,
$$

where $u = u(t, x), v = v(t, x), t > 0, x \in \mathbb{R}^n$, $\Delta$ is the $n$-dimensional Laplacian, and $B$ is a $2 \times 2$ matrix.

The Green function $G(t, x; \alpha, \gamma)$ associated with the operator $\partial_t + \mu(I - \Delta)\frac{\partial}{\partial t} (-\Delta)^\frac{\beta}{2}$ is represented via the spatial Fourier transform as follows; see, [25, Chapter 5] or [3, Section 2].

$$
\int_{\mathbb{R}^n} e^{i < x, \lambda >} G(t, x; \alpha, \gamma) dx = \exp[-\mu t |\lambda|^{\alpha}(1 + |\lambda|^2)^\frac{\beta}{2}], \quad \lambda \in \mathbb{R}^n,
$$

where $< \cdot, \cdot >$ denotes the inner product on $\mathbb{R}^n$.

In order to get a explicit representation for the solution of (2.1), we impose the following assumption on the matrix $B$.

**Condition A.** Suppose the matrix $[b_{ij}]_{1 \leq i,j \leq 2}$ is diagonalizable, i.e., the matrix $B$ can
be written as
\[
B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = PD^{-1}, \quad \text{with} \quad P = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix},
\] (2.4)
where \( P \) is a real-valued non-degenerate eigenvector matrix associated with the matrix \( B \), and \( D = \text{diag}(d_1, d_2) \), \( d_1, d_2 \in \mathbb{R} \), where \( d_j \) is the eigenvalue associated with the eigenvector \( (p_{1,j}, p_{2,j})^T \) (here and henceforth, \( T \) denotes the transpose). Without loss of generality, we suppose that \( \det(P) = 1 \).

Under Condition A, the Cauchy problem (2.1) (2.2) has the unique solution given by
\[
\begin{pmatrix} u(t, x) \\ v(t, x) \end{pmatrix} = Q(t; d_1, d_2) \begin{pmatrix} U(t, x) \\ V(t, x) \end{pmatrix}, \quad t > 0, \ x \in \mathbb{R}^n,
\] (2.5)
where
\[
Q(t; d_1, d_2) := P \begin{pmatrix} e^{d_1 t} & 0 \\ 0 & e^{d_2 t} \end{pmatrix} P^{-1},
\] (2.6)
and \( U(t, x), V(t, x) \) are determined by
\[
\begin{pmatrix} U(t, x) \\ V(t, x) \end{pmatrix} = \int_{\mathbb{R}^n} G(t, y; \alpha, \gamma) \begin{pmatrix} u_0(x - y) \\ v_0(x - y) \end{pmatrix} dy,
\] (2.7)
where the Green function \( G(t, y; \alpha, \gamma) \) is defined in (2.3).

For completeness, we give the sketchy proofs of (2.5). Firstly, by taking the spatial Fourier transform on both sides of (2.1), under Condition A, we have
\[
\frac{\partial}{\partial t} \begin{pmatrix} \hat{u} \\ \hat{v} \end{pmatrix} (t, \lambda) = (-\mu |\lambda|^\alpha (1 + |\lambda|^2) \hat{\pi} + B) \begin{pmatrix} \hat{u} \\ \hat{v} \end{pmatrix} (t, \lambda)
= P \begin{pmatrix} -\mu |\lambda|^\alpha (1 + |\lambda|^2) \hat{\pi} + d_1 & 0 \\ 0 & -\mu |\lambda|^\alpha (1 + |\lambda|^2) \hat{\pi} + d_2 \end{pmatrix} P^{-1} \begin{pmatrix} \hat{u} \\ \hat{v} \end{pmatrix} (t, \lambda)
\]
Thus,
\[
\begin{pmatrix} \hat{u} \\ \hat{v} \end{pmatrix} (t, \lambda) = \exp \left\{ tP \begin{pmatrix} -\mu |\lambda|^\alpha (1 + |\lambda|^2) \hat{\pi} + d_1 & 0 \\ 0 & -\mu |\lambda|^\alpha (1 + |\lambda|^2) \hat{\pi} + d_2 \end{pmatrix} P^{-1} \right\} \begin{pmatrix} \hat{u} \\ \hat{v} \end{pmatrix} (0, \lambda)
= P \begin{pmatrix} e^{d_1 t} & 0 \\ 0 & e^{d_2 t} \end{pmatrix} P^{-1} e^{-\mu |\lambda|^\alpha (1 + |\lambda|^2) \hat{\pi}} \begin{pmatrix} \hat{u} \\ \hat{v} \end{pmatrix} (0, \lambda).
\] (2.8)
Finally, (2.5) and (2.7) are followed by taking the inverse Fourier transform on both sides of (2.8) and using the representation (2.3). Additionally, by (2.3) we can also observe that

$$\int_{\mathbb{R}^n} G(t, x; \alpha, \gamma) \, dx = 1, \text{ for any } t \geq 0. \quad (2.9)$$

## 3 Correlated random structures

### 3.1 general random initial data

Firstly, we set $(\Omega, \mathcal{F}, P)$ to be an underlying probability space, such that all random element appeared in this paper are measurable with respect to it. The following condition is imposed on the initials, in which and henceforth.

**Condition B.** Let $u_0(x) = \eta_1(x, \omega)$ and $v_0(x) = \eta_2(x, \omega)$, $x \in \mathbb{R}^n$, $\omega \in \Omega$, be independent, and distributed as two real, mean-square continuous, homogeneous and isotropic random fields defined on the underlying complete probability space $(\Omega, \mathcal{F}, P)$. We assume that $E\eta_j(x) \equiv 0$, $\text{Var}(\eta_j(x)) \equiv 1$, and covariance functions

$$R_{\eta_j}(x) = \tilde{R}_{\eta_j}(|x|) := \text{Cov}(\eta_j(0), \eta_j(x)) = \int_{\mathbb{R}^n} e^{i<\lambda, x>} F_j(d\lambda), \quad j \in \{1, 2\},$$

where the last equality is guaranteed by Bochner-Khintchine theorem and $F_j(\cdot)$ is the spectral measure corresponding to the field $\eta_j(\cdot)$ for $j \in \{1, 2\}$, respectively.

Under Condition B, in view of Karhunen’s Theorem (see, for example, Gihman and Skorokhod [13], pp. 208-230), there exist complex-valued orthogonally scattered random measures $Z_{F_j}$, $j \in \{1, 2\}$, such that the random fields $\eta_j(x)$, $j \in \{1, 2\}$, have the following spectral representations

$$\eta_j(x) = \int_{\mathbb{R}^n} e^{i<\lambda, x>} Z_{F_j}(d\lambda), \quad j \in \{1, 2\}, \quad (3.1)$$

where $E Z_{F_k}(\Delta_1) = 0$, $E Z_{F_k}(\Delta_1) Z_{F_j}(\Delta_2) = \delta_k^j F_j(\Delta_1 \cap \Delta_2)$, for any $j, k \in \{1, 2\}$ and $\Delta_1, \Delta_2 \in \mathcal{B}(\mathbb{R}^n)$ ($\delta_k^j$ is the Kronecker symbol).
By the above spectral representation for the initial data (2.2), we can describe the vector-solution \(\{(u(t, x), v(t, x)), t > 0, x \in \mathbb{R}^n\}\) by stochastic integration:

**Proposition 1** Let \(w(t, x; w_0(\cdot)) := (u(t, x; u_0(\cdot)), v(t, x; v_0(\cdot)), w_0(\cdot) = (u_0(\cdot), v_0(\cdot)))\) be the vector-solution of the initial value problem (2.1) (2.2), of which satisfies Condition A and B, then

\[
w(t, x; w_0(\cdot)) = Q(t; d_1, d_2) \int_{\mathbb{R}^n} e^{i<\lambda, x>} e^{-\mu(t+|\lambda|^2)} \frac{\tilde{Z}}{2} \begin{pmatrix} Z_{F_1}(d\lambda) \\ Z_{F_2}(d\lambda) \end{pmatrix}, \quad (3.2)
\]

with the inter-relative covariance structure

\[
Ew(t, x; w_0(\cdot))w^T(t', x'; w_0(\cdot)) = \int_{\mathbb{R}^n} e^{i<\lambda, x-x'>} e^{-\mu(t+t')|\lambda|^2} \frac{\tilde{Z}}{2} Q(t; d_1, d_2) \begin{pmatrix} F_1(d\lambda) & 0 \\ 0 & F_2(d\lambda) \end{pmatrix} Q(t'; d_1, d_2)^T, \quad (3.3)
\]

where \(Q(t; d_1, d_2)\) is defined in (2.6).

### 3.2 Subordinated Gaussian initial data

In this subsection, we assume further that the initials are subordinated fields, as follows:

**Condition C.** We consider the random initial data (2.2) \(w_0(x) := (u_0(x), v_0(x)) = (\eta_1(x), \eta_2(x)), x \in \mathbb{R}^n\), satisfies Condition B and each component has the following form

\[
\eta_j(x) := h_j(\zeta_j(x)), \ x \in \mathbb{R}^n, \ j \in \{1, 2\}. \quad (3.4)
\]

The \(\zeta_1(x)\) and \(\zeta_2(x)\) are independent, mean-square continuous, homogeneous and isotropic Gaussian random fields, each is of mean zero and of variance 1, and for each the spectral measure \(F_j(d\lambda)\) has the (spectral) density \(f_j(\lambda)\), \(\lambda \in \mathbb{R}^n\), and \(f_j(\lambda)\) is decreasing for \(|\lambda| > \lambda_0\) for some \(\lambda_0 > 0\) and continuous for all \(\lambda \neq 0, j \in \{1, 2\}\), respectively. Moreover, we assume that \(h_j(\cdot), j \in \{1, 2\}\), are real non-random Borel functions satisfy

\[
Eh_j^2(\zeta_j(0)) < \infty, \ j \in \{1, 2\}. \quad (3.5)
\]
Under Condition C, we have the spectral representations for the sample paths of \( \zeta_j(x) \), \( j \in \{1, 2\} \), as below:

\[
\zeta_j(x) = \int_{\mathbb{R}^n} e^{i <x, \lambda>} \sqrt{f_j(\lambda)} W_j(d\lambda), \quad x \in \mathbb{R}^n, \quad j \in \{1, 2\},
\]

(3.6)

where \( W_j(A) \) is a Gaussian noise measure, \( W_j(A), A \in \mathcal{B}(\mathbb{R}^n) \), are centered Gaussian with \( EW_j(d\lambda)W_j(d\mu) = \delta_j^2 \delta(\lambda - \mu) d\lambda \, d\mu \).

Due to (3.5) in Condition C, we can consider the following orthogonal expansions of \( h_j(u) \) in the Hilbert space \( L^2(\mathbb{R}, p(u)du) \) with \( p(u) = \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} \):

\[
h_j(u) = C_0^{(j)} + \sum_{\sigma=1}^{\infty} C_{\sigma}^{(j)} \frac{H_\sigma(u)}{\sqrt{\sigma!}}p(u)du, \quad j \in \{1, 2\},
\]

(3.7)

where

\[
C_{\sigma}^{(j)} = \int_{\mathbb{R}} h_j(u) \frac{H_\sigma(u)}{\sqrt{\sigma!}} p(u)du, \quad j \in \{1, 2\},
\]

(3.8)

and \( \{H_\sigma(u), \sigma = 0, 1, 2, \ldots\} \) are the Hermite polynomials, that is,

\[
H_\sigma(u) = (-1)^\sigma e^{\frac{u^2}{2}} \frac{d^\sigma}{du^\sigma} e^{-\frac{u^2}{2}}, \quad \text{for} \quad \sigma \in \{0, 1, 2, \ldots\}.
\]

It is known that the following two important properties hold (see, for example, Major [25], Corollary 5.5 and p. 30):

\[
E[H_{\sigma_1}(\zeta_j(y_1))H_{\sigma_2}(\zeta_j(y_2))] = \delta_{\sigma_1}^j \delta_{\sigma_2}^{j'} \sigma_1! \sigma_2! R_{\zeta_j}^{\sigma_1}(y_1 - y_2), \quad y_1, y_2 \in \mathbb{R}^n
\]

(3.9)

and

\[
H_\rho(\zeta_j(x)) = \int_{\mathbb{R}^n} e^{i <x, \lambda_1 + \ldots + \lambda_\rho>} \prod_{\sigma=1}^{\rho} \sqrt{f_j(\lambda_\sigma)} W_j(d\lambda_\sigma).
\]

(3.10)

In the integral representation (3.10), the integration \( \int' \) means that it excludes the diagonal hyperplanes \( z_i = \mp z_j, i, j = 1, \ldots, \rho, i \neq j \).

The \textit{Hermite rank} of the functions \( h_j(\cdot) \) is defined by

\[
m_j := \inf\{\sigma \geq 1 : C_{\sigma}^{(j)} \neq 0\}, \quad j \in \{1, 2\}.
\]

Specializing Proposition 1 in Subsection 3.1 to the present subordinated Gaussian initials, we have
Proposition 2 Let $w(t, x; w_0(\cdot)) := (u(t, x; u_0(\cdot)), v(t, x; v_0(\cdot))), t > 0 \ x \in \mathbb{R}^n$ be the vector-solution (2.5) of the initial value problem (2.1) (2.2), of which satisfies Condition A and C, then all the statements in Proposition 1 remain valid, with (3.2) is expressed as

$$w(t, x; w_0(\cdot)) = Q(t; d_1, d_2) \left\{ \begin{pmatrix} C_{01}^{(1)} \\ C_{02}^{(2)} \end{pmatrix} + \sum_{\rho \in \mathbb{N}^n} \int_{\mathbb{R}^n \times \rho} e^{i <x, \xi_1 + \cdots + \xi_\rho> - \mu |\xi_1 + \cdots + \xi_\rho|^\alpha (1 + |\xi_1 + \cdots + \xi_\rho|^2)^{\gamma} } \left( \begin{pmatrix} Z_{F_1}^{(\rho)}(d\lambda) \\ Z_{F_2}^{(\rho)}(d\lambda) \end{pmatrix} \right) \right\},$$

where

$$\left( \begin{pmatrix} Z_{F_1}^{(\rho)}(d\lambda) \\ Z_{F_2}^{(\rho)}(d\lambda) \end{pmatrix} \right) := \left( \begin{pmatrix} C_{01}^{(1)} \sqrt{\rho} \prod_{\sigma=1}^{\rho} \sqrt{f_1(\lambda_\sigma)W_1(d\lambda_\sigma)} \\ C_{02}^{(2)} \sqrt{\rho} \prod_{\sigma=1}^{\rho} \sqrt{f_2(\lambda_\sigma)W_2(d\lambda_\sigma)} \end{pmatrix} \right),$$

and the coefficient $C_{\rho}^{(j)}, j \in \{1, 2\}$ is defined in (3.8).

We also impose the following assumption which is related to the long-range dependence of the underlying Gaussian fields $\zeta_j(x), j \in \{1, 2\}$; we refer to [1, 10] for the notion and the literatures of long-range dependence. In the following and henceforth, the notation $f(\cdot) \sim g(\cdot)$ means that the ratio $f(\cdot)/g(\cdot)$ tends to 1, as the indicated variable “·” tends to $\infty$ or tends to 0, according to the context.

**Condition D.** The Gaussian random fields $\zeta_j(x), j \in \{1, 2\}$, in Condition C, have their covariance functions to be regular varying at infinity in the sense that:

$$R_{\zeta_j}(x) \sim \frac{L(|x|)}{|x|^{\kappa_j}}, \quad \text{as } |x| \to \infty, \ 0 < \kappa_j < \frac{n}{m_j}, \ j \in \{1, 2\},$$

where $L : (0, \infty) \to (0, \infty)$ is a slowly varying function at infinity and is bounded on each finite interval; recall that $L$ is said to be slowly varying at infinity if $\lim_{y \to \infty} [L(cy)/L(y)] = 1$ uniformly for any $c \in (a, b), \ 0 < a < b < \infty$. Here, the relation “~” in (3.11) means that

$$\lim_{|x| \to \infty} |x|^{\kappa_j} R_{\zeta_j}(x)/L(|x|) = 1;$$

the similar notation will be used in this section and also in Section 7.

Under Condition D, by a Tauberian theorem (see, for example, the book of Leonenko...
the spectral density functions of the random fields $\zeta_j(x)$, $j \in \{1, 2\}$, are regular varying near the origin as follows:

$$f_j(\lambda) \sim K(n, \kappa_j)|\lambda|^{\rho_j-n}L(|\frac{1}{\chi}|), \text{ as } \lambda \to 0, \ j \in \{1, 2\}, \quad (3.12)$$

where the Tauberian constant $K(n, \kappa_j) = \frac{\Gamma(n-\rho_j/2)}{2^{n-\rho_j/2}\pi^{n/2}\Gamma(\rho_j/2)}$.

We note that, for each natural number $\rho \geq 2$, the power of the covariance function $(R_{\zeta_j}(x))^\rho$ itself is still the covariance function of some random field, for which there exists the corresponding spectral density function $(f_j)^{*\rho}(\lambda)$. Indeed, the function $(f_j)^{*\rho}(\lambda)$, $\lambda \in \mathbb{R}^n$, is the $\rho-$th convolution of $f_j(\lambda)$ defined as

$$(f_j)^{*\rho}(\lambda) = \int_{\mathbb{R}^n \times (\rho-1)} f_j(\lambda - \lambda_1)f_j(\lambda_1 - \lambda_2)\cdots f_j(\lambda_{\rho-2} - \lambda_{\rho-1})f_j(\lambda_{\rho-1})^{\rho-1}d\lambda_i. \quad (3.13)$$

for $\rho \geq 2$. Since $L^\rho(|x|)$ is still a slowly varying function for any $\rho$, when the $\rho$ satisfies $0 < \rho \kappa_j < n$, we can apply the Tauberian theorem again to get

$$(f_j)^{*\rho}(\lambda) \sim K(n, \rho \kappa_j)|\lambda|^{\rho \kappa_j-n}L^\rho(|\frac{1}{\chi}|), \text{ as } \lambda \to 0, \ 0 < \rho \kappa_j < n, \quad (3.14)$$

for $j \in \{1, 2\}$. While, if $\rho \kappa_j > n$ then the covariance function $(R_{\zeta_j}(x))^\rho$ belongs to the class $L^1(\mathbb{R}^n)$; thus the corresponding spectral density function is everywhere continuous and satisfies

$$(2\pi)^n(f_j)^{*\rho}(0) = \int_{\mathbb{R}^n} (R_{\zeta_j}(x))^\rho dx \leq \int_{\mathbb{R}^n} |R_{\zeta_j}(x)|^\rho dx \leq \int_{\mathbb{R}^n} |R_{\zeta_j}(x)|^{\rho^*} dx < \infty, \quad (3.15)$$

where $\rho^* := \inf\{\rho \in \mathbb{N} \mid \rho \kappa_j > n\}$; we note that $|R_{\zeta_j}(\cdot)| \leq 1$.

The displays (3.13), (3.14) and (3.15) will be used in the proofs in Section 7.

## 4 Micro-scalings for the solution vector-field

In this section, we present the main result of this paper, which concerns with the micro-scaling of the homogenization of the spatial-temporal random field associated with (2.1), with the initial data (2.2) subject to the conditions in Section 3. We show that both the
Riesz parameter $\alpha$ and the Bessel parameter $\gamma$ play their roles in the scaling procedure. The results in this section are more subtle than the macro-scaling discussed in the next section; see the remark below Theorem 1 for the interpretation.

Firstly, we prove the following micro-scaling of homogenization for a single fractional kinetic equation, subject to the random initial data.

$$\frac{\partial s}{\partial t}(t, x) = -\mu(I - \Delta)^{\frac{\alpha}{2}}(-\Delta)^{\frac{\gamma}{2}}s(t, x), \quad s(0, x) = h(\zeta(x)). \quad (4.1)$$

To our knowledge, the homogenization present in the below is a completely new type. In (4.2) below, the notation imposed on $\zeta$ wants to mean that the variable of $\zeta$ is under the indicated dilation factor.

**Theorem 1** Let $s := s(t, x; s_0(\cdot)), \ t > 0, \ x \in \mathbb{R}^n$, be a solution of (4.1), which satisfies the above Condition B, C and D with $\kappa \in (0, \frac{n}{m})$, where $m$ denotes the Hermite rank of the non-random function $h(\cdot)$ on $\mathbb{R}$, which has the Hermite coefficients $C_i(h), \ i = 0, 1, \ldots$ (i.e., $h_1(x) = h(x), \ \zeta_1(x) = \zeta(x), \ f_1(\lambda) = f(\lambda)$ and $\kappa_1 = \kappa$, etc. in Section 3). Then, for any fixed parameter $\chi > 0$,

1. The behaviour of the covariance function of the rescaled random field $s^\varepsilon(t, x), \ t > 0, \ x \in \mathbb{R}^n$,

$$s^\varepsilon(t, x) := [\varepsilon^m L^m(\varepsilon^{-x}) - 1\{s(\varepsilon t, \varepsilon \frac{1}{\alpha + \gamma} x; h(\varepsilon^{-\frac{1}{\alpha + \gamma}} \cdot)) - C_0(h)\}, \quad (4.2)$$

is given by:

$$\lim_{\varepsilon \to 0} \text{Cov}(s^\varepsilon(t, x) s^\varepsilon(t', x')) = (C_m(h))^2 K(n, m \kappa) \int_{\mathbb{R}^n} e^{i<x-x'|\tau>} \frac{e^{-\mu(t+t')|\tau|^{\alpha+\gamma}}}{|\tau|^{n-m \kappa}} d\tau. \quad (4.3)$$

2. When $\varepsilon \to 0$, the rescaled random field $s^\varepsilon(t, x), \ t > 0, \ x \in \mathbb{R}^n$, converges to the limiting spatial-temporal random field $s_m(t, x), \ t > 0, \ x \in \mathbb{R}^n$, in the finite dimensional distribution sense, and $s_m(t, x)$ is represented by the Multiple-Wiener integrals

$$s_m(t, x) := \frac{C_m(h)}{\sqrt{m!}} K(n, \kappa)^\frac{m}{2} \int_{\mathbb{R}^{n \times m}} e^{i<x,z_1+\cdots+z_m>} \frac{e^{-\mu|z_1+\cdots+z_m|^{\alpha+\gamma}}}{(|z_1| \cdots |z_m|)^{\frac{n-\kappa}{2}}} m \prod_{i=1}^{m} W(dz_i), \quad (4.4)$$

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where $\int' \cdots$ denotes a $m$-fold Wiener integral with respect to the complex Gaussian white noise $W(\cdot)$ on $\mathbb{R}^n$.

**Remark.** To compare with Proposition 4 in the next Section 5, Theorem 1 has the features that it involves both the Riesz and the Bessel parameters, and that it also needs to rescale the initial condition. The intuitive meaning behind the latter situation is that, while the micro-scaling enforces to “freeze down” both the time $t$ and the space $x$, besides the overall renormalization we also have to “heat up” the initial data, in order to get a non-degenerate (though singular) limiting field.

Now, the micro-scaling of the system is

**Theorem 2** Let $w(t,x;w_0(\cdot)) := (u(t,x;u_0(\cdot)), v(t,x;v_0(\cdot)))$, $t > 0$, $x \in \mathbb{R}^n$, be the solution-vector of the initial value problem (2.1) and (2.2), satisfying the Condition A, B, C and D. In the following, $\chi$ is a positive parameter, $Q(t;d_1,d_2)$ is a matrix defined in (2.6), and the two Gaussian noise fields $W_j$, $j \in \{1,2\}$ are totally independent. Additionally, $m_1$, $m_2$, $\kappa_1$ and $\kappa_2$ denote the parameters contained in Condition C and D for $u_0$ and $v_0$.

(1) If $m_2\kappa_2 > m_1\kappa_1$, then the finite-dimensional distributions of the rescaled random field

$$[\varepsilon^{m_1\kappa_1}L^{m_1}(\varepsilon^{-\chi})]^{-\frac{1}{2}} \left\{ w(\varepsilon t, \varepsilon^{-\frac{1}{m_1+\gamma}} x; \varepsilon^{-\frac{1}{m_1+\gamma}} x) - Q(\varepsilon t; d_1, d_2) \begin{pmatrix} C_0^{(1)} \\ C_0^{(2)} \end{pmatrix} \right\}, \quad t > 0, \ x \in \mathbb{R}^n,$$

converge weakly, as $\varepsilon \to 0$, to the finite-dimensional distributions of the random field

$$ \begin{pmatrix} Y_1^*(t,x) \\ Y_2^*(t,x) \end{pmatrix} := \begin{pmatrix} \tilde{X}_{m_1}^{(1)}(t,x) \\ 0 \end{pmatrix}, \quad t > 0, \ x \in \mathbb{R}^n,$$

where

$$\tilde{X}_{m_1}^{(1)}(t,x) := \frac{C_{m_1}^{(1)}}{\sqrt{m_1!}} K(n, \kappa_1)^{\frac{m_1}{2}} \int_{\mathbb{R}^{n \times m_1}} e^{i \langle x, z_1 + \cdots + z_{m_1} \rangle - \mu \| z_1 + \cdots + z_{m_1} \|^\alpha + \gamma} \prod_{i=1}^{m_1} W_1(dz_i), \quad (4.5)$$

with $W_1(\cdot)$ is a complex Gaussian white noise on $\mathbb{R}^n$ (i.e., (4.4) with $m$, $\kappa$ and $W$ replaced by $m_1$, $\kappa_1$ and $W_1$, respectively).
(2) If $m_1 \kappa_1 > m_2 \kappa_2$, then the finite-dimensional distributions of the rescaled random field

$$[e^{m_2 \kappa_2 x} L^{m_2}(e^{-\chi})]^{-\frac{1}{2}} \left\{ w(\varepsilon t, \varepsilon^{\frac{1}{m_2}} x; \varepsilon^0(\varepsilon^{\frac{1}{m_2}} \chi)) - Q(\varepsilon t; d_1, d_2) \right\} \left( \begin{array}{c} \mathcal{C}_{c_1}^{(1)} \\ \mathcal{C}_{c_2}^{(2)} \end{array} \right), \quad t > 0, \ x \in \mathbb{R}^n,$$

converge weakly, as $\varepsilon \to 0$, to the finite-dimensional distributions of the random field

$$\left( \begin{array}{c} Y_{1*}^{**}(t, x) \\ Y_{2*}^{**}(t, x) \end{array} \right) := \left( \begin{array}{c} 0 \\ \tilde{X}_m^{(2)}(t, x) \end{array} \right), \quad t > 0, \ x \in \mathbb{R}^n,$$

where

$$\tilde{X}_m^{(2)}(t, x) := \frac{C_{m_2}^{(2)}}{\sqrt{m_2}} K(n, \kappa_2) \int_{\mathbb{R}^n \times m_2} e^{i < x, z_1 + \ldots + z_{m_2} > - \mu l_{z_1 + \ldots + z_{m_2}}^{\alpha} - \gamma} \prod_{l=1}^{m_2} W_2(dz_l), \quad (4.6)$$

and $W_2(\cdot)$ is a complex Gaussian white noise on $\mathbb{R}^n$ (i.e., (4.4) with $m, \kappa$ and $W$ replaced by $m_2, \kappa_2$ and $W_2$, respectively).

(3) If $m_1 = m_2 := m, \ k_1 = k_2 := \kappa$, then the finite-dimensional distributions of the rescaled random field

$$[e^{m_1 \chi} L^m(e^{-\chi})]^{-\frac{1}{2}} \left\{ w(\varepsilon t, \varepsilon^{\frac{1}{m}} x; \varepsilon^0(\varepsilon^{\frac{1}{m}} \chi)) - Q(\varepsilon t; d_1, d_2) \right\} \left( \begin{array}{c} \mathcal{C}_{c_1}^{(1)} \\ \mathcal{C}_{c_2}^{(2)} \end{array} \right), \quad t > 0, \ x \in \mathbb{R}^n,$$

converge weakly, as $\varepsilon \to 0$, to the finite-dimensional distributions of the random field

$$\left( \begin{array}{c} Y_{1*}^{***}(t, x) \\ Y_{2*}^{***}(t, x) \end{array} \right) := \left( \begin{array}{c} \tilde{X}_m^{(1)}(t, x) \\ \tilde{X}_m^{(2)}(t, x) \end{array} \right), \quad t > 0, \ x \in \mathbb{R}^n, \quad (4.7)$$

where $\tilde{X}_m^{(1)}$ and $\tilde{X}_m^{(2)}$, are defined in (4.3) and (4.4) with $m_1 = m_2 = m$ and $\kappa_1 = \kappa_2 = \kappa$.

To understand the stochastic structure of the limiting fields, we state, for instance, the following covariance result of $(Y_{1*}^{***}(t, x), Y_{2*}^{***}(t, x))$.

**Proposition 3** For each fixed $t > 0$, the limiting vector field \( \left( Y_{1*}^{***}(t, x), Y_{2*}^{***}(t, x) \right) \) in the case (3) of Theorem 2 is spatial-homogeneous and its covariance matrix has the following spectral representation

$$E \left( \begin{array}{c} Y_{1*}^{***}(t, x) \\ Y_{2*}^{***}(t, x) \end{array} \right) \left( \begin{array}{c} Y_{1*}^{***}(t', x') \\ Y_{2*}^{***}(t', x') \end{array} \right) = \int_{\mathbb{R}^n} e^{i < x - x', \lambda >} S(\lambda; \alpha, \gamma) d\lambda,$$

where \( S(\lambda; \alpha, \gamma) := K(n, m\kappa)e^{-\mu(t+t')}|\lambda|^{\alpha} + \gamma} \left( \begin{array}{cc} (C_m^{(1)})^2 & 0 \\ 0 & (C_m^{(2)})^2 \end{array} \right). \)

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Remark. In view of the singularity of the spectral matrix near the origin, we may conclude that, for limiting vector field in the case (3), the long-range dependence (LRD) not only exists for each component field but also exists between the two component fields; this is a rather new phenomena for LRD, to our knowledge. Similar situation happens for other cases, too.

5 Macro-scalings for the solution vector-field

In this section, we present the macro-scaling limits for the solution of the fractional kinetic systems (2.1) and (2.2), in which only the Riesz parameter $\alpha$ plays its role in the scaling.

We again begin with the following single-equation case, which is adapted from [3, Theorems 2.2 and 2.3].

Proposition 4 Let $s := s(t, x; \mathfrak{s}_0(\cdot)), t > 0, x \in \mathbb{R}^n$, satisfies (4.1), which satisfies the above Condition B, C and D with $\kappa \in (0, n/m)$, where $m$ denotes the Hermite rank of the non-random function $h(\cdot)$ on $\mathbb{R}$, which has the Hermite coefficients $C_i(h), i = 0, 1, \ldots$ (i.e., $h_1(\cdot) = h(\cdot), \zeta_1(x) = \zeta(x), f_j(\lambda) = f(\lambda)$ and $\kappa_1 = \kappa$, etc.). Then,

1. The behaviour of the covariance function of the rescaled random field $s^\varepsilon(t, x), t > 0, x \in \mathbb{R}^n,$ is given by:

$$
\lim_{\varepsilon \to 0} \text{Cov}(s^\varepsilon(t, x)s^\varepsilon(t', x')) = (C_m(h))^{2K} (n, m\kappa) \int_{\mathbb{R}^n} e^{i<x-x', \lambda>-\mu(t+t')}|\lambda|^{n-m\kappa} d\lambda.
$$

(5.1)

2. Moreover, the finite dimensional distributions of the rescaled field $s^\varepsilon(t, x)$ converge weakly, as $\varepsilon \to 0$, to the finite-dimensional distributions of the random field

$$
s_m(t, x) := \frac{C_m(h)}{\sqrt{m!} K(n, \kappa)^{m/2}} \int_{\mathbb{R}^{n \times m}} e^{i<x, z_1+\cdots+z_m>-\mu(z_1+\cdots+z_m)\kappa} \prod_{l=1}^{m} W(dz_l),
$$

(5.2)
\[ x \in \mathbb{R}^n, \ t > 0, \text{ where } \int' \cdots \text{ denotes a } m\text{-fold Wiener integral with respect to the complex Gaussian white noise } \mathbf{W}() \text{ on } \mathbb{R}^n. \]

**Remark.** The above \((5.1)\) is expressed on the “Fourier-domain”, which is more suitable for we will need; while that \((2.40)\) in [3] is in term of the variable domain.

Then, the macro-scaling of the system is

**Theorem 3** Let \( w(t, x; w_0(\cdot)) := (u(t, x; u_0()), v(t, x; v_0(\cdot)), \ t > 0, \ x \in \mathbb{R}^n, \) be the solution-vector of the initial value problem \((2.1)\) and \((2.2)\), satisfying the Condition A, B, C and D. In the following, \( Q(t; d_1, d_2) \) is the matrix defined in \((2.6)\), \( p_{ij} \) is the entry in \((2.4)\), and the two Gaussian noise fields \( \mathbf{W}_j, \ j \in \{1, 2\} \) are totally independent.

Additionally, \( m_1, m_2, \kappa_1 \) and \( \kappa_2 \) denote the parameters contained in Condition C and D for \( u_0 \) and \( v_0 \).

1. If \( m_2\kappa_2 > m_1\kappa_1 \) and \( d_1 > d_2 \), then the finite-dimensional distributions of the rescaled random field

\[
[\varepsilon^{-\frac{1}{m_2}} L^{m_1}(\varepsilon^{-\frac{1}{m_2}})]^{-\frac{1}{2}} e^{-d_1 t} \bigg\{ w(t, x; w_0(\cdot)) - Q(t; d_1, d_2) \begin{pmatrix} C_0^{(1)} \\ C_0^{(2)} \end{pmatrix} \bigg \}, \ t > 0, \ x \in \mathbb{R}^n,
\]

converge weakly, as \( \varepsilon \to 0 \), to the finite-dimensional distributions of the random field

\[
T_{m_1}^{(1)}(t, x) := \begin{pmatrix} p_{11}p_{22}X^{(1)}_{m_1}(t, x) \\ p_{21}p_{22}X^{(1)}_{m_1}(t, x) \end{pmatrix}, \ t > 0, \ x \in \mathbb{R}^n,
\]

where

\[
X^{(1)}_{m_1}(t, x) := \frac{C^{(1)}}{\sqrt{m_1!}} K(n, \kappa_1) \frac{m_1}{2} \left( \int_{\mathbb{R}^n \times m_1} e^{\sum_{i=1}^{m_1} (|z_1| + \cdots + |z_{m_1}|)^{n-\frac{n-1}{2}} \frac{\prod_{l=1}^{m_1} W_1(\text{d}z_l)}{2}} \right)
\]

with \( W_1(\cdot) \) is a complex Gaussian white noise on \( \mathbb{R}^n \) \( \text{i.e., } (5.2) \) with \( m, \kappa \) and \( W \) replaced by \( m_1, \kappa_1 \) and \( W_1 \), respectively

2. If \( m_1\kappa_1 > m_2\kappa_2 \) and \( d_1 > d_2 \), then the finite-dimensional distributions of the rescaled random field

\[
[\varepsilon^{-\frac{1}{m_1}} L^{m_2}(\varepsilon^{-\frac{1}{m_1}})]^{-\frac{1}{2}} e^{-d_1 t} \bigg\{ w(t, x; w_0(\cdot)) - Q(t; d_1, d_2) \begin{pmatrix} C_0^{(1)} \\ C_0^{(2)} \end{pmatrix} \bigg \}, \ t > 0, \ x \in \mathbb{R}^n,
\]
converge weakly, as $\varepsilon \to 0$, to the finite-dimensional distributions of the random field

$$T^{(2)}_{m_2}(t, x) := \begin{pmatrix} -p_{11}p_{12}X^{(2)}_{m_2}(t, x) \\ -p_{21}p_{12}X^{(2)}_{m_2}(t, x) \end{pmatrix}, \quad t > 0, \ x \in \mathbb{R}^n,$$

where

$$X^{(2)}_{m_2}(t, x) := \frac{C^{(2)}_{m_2}}{\sqrt{m_2!}} K(n, \kappa_2) \sqrt{m_2^2} \int_{\mathbb{R}^n \times m_2} e^{i\langle x, z_1 + \ldots + z_{m_2} \rangle - \mu t |z_1 + \ldots + z_{m_2}|^2} \prod_{l=1}^{m_2} W_2(dz_l), \quad (5.4)$$

and $W_2(\cdot)$ is a complex Gaussian white noise on $\mathbb{R}^n$ (i.e., (5.2) with $m, \kappa$ and $W$ replaced by $m_2, \kappa_2$ and $W_2$, respectively).

(3) If $m_1 = m_2 := m$, $\kappa_1 = \kappa_2 := \kappa$, and $d_1 > d_2$, then the finite-dimensional distributions of the rescaled random field

$$\left[ \varepsilon^\frac{m_1}{m_2} L^m \left( \varepsilon^{-\frac{1}{2}} \right) \right]^{-\frac{1}{2}} e^{-d_1 t} \left\{ \mathbf{w} \left( \frac{t}{\varepsilon}, \frac{x}{\sqrt{\varepsilon}} ; \mathbf{w}_0 \left( \cdot \right) \right) - Q \left( \frac{t}{\varepsilon} ; d_1, d_2 \right) \begin{pmatrix} C^{(1)}_0 \\ C^{(2)}_0 \end{pmatrix} \right\}, \quad t > 0, \ x \in \mathbb{R}^n,$$

converge weakly, as $\varepsilon \to 0$, to the finite-dimensional distributions of the random field

$$T^{(3)}_m(t, x) := T^{(1)}_m(t, x) + T^{(2)}_m(t, x), \quad (5.5)$$

where $T^{(1)}_m(t, x)$ and $T^{(2)}_m(t, x)$, are defined in the case (1) and the case (2) with $m_1 = m_2 = m$ and $\kappa_1 = \kappa_2 = \kappa$.

Remark: In the above, we assume that $d_1 > d_2$. In case $d_1 < d_2$, all the corresponding assertions hold, by interchanging the roles of $m_1$, $\kappa_1$ and $m_2$, $\kappa_2$, etc. As for $d_1 = d_2$, it is reduced to the uncoupled case and the result is induced from Proposition 4 directly.

6 Time-fractional systems

We extend the above results to the time-fractional derivative $\frac{\partial^\beta}{\partial t^\beta}$, $\beta \in (0, 1)$, in the system (2.1), that is,

$$\frac{\partial^\beta}{\partial t^\beta} \begin{pmatrix} u \\ v \end{pmatrix} = -\mu (I - \Delta)^{\frac{\gamma}{2}} (-\Delta)^{\frac{\alpha}{2}} \begin{pmatrix} u \\ v \end{pmatrix} + B \begin{pmatrix} u \\ v \end{pmatrix}, \quad \mu, \ \alpha, \ \gamma > 0, \quad (6.1)$$
We recall that the time-fractional derivative \( \frac{\partial^\beta}{\partial t^\beta} \) is defined (see, for example, the book of Djrbashian [9]) by, for any \( \beta > 0 \),
\[
\frac{d^\beta f}{dt^\beta}(t) = \begin{cases} f^{(m)}(t) & \text{if } \beta = m \in \mathbb{N} \\ \frac{1}{\Gamma(m-\beta)} \int_0^t \frac{f^{(m)}(\tau)}{(t-\tau)^{\beta+1}} d\tau & \text{if } \beta \in (m-1, m), \end{cases}
\]
(6.2)
where \( f^{(m)}(t) \) denotes the ordinary derivative of order \( m \) of a causal function \( f(t) \) (i.e., \( f \) is vanishing for \( t < 0 \)).

The solution of (6.1) can be obtained, under Condition A, by applying the Laplace and the Fourier transforms (see, for example, [23, 24]), as follows.
\[
w(t, x; w_0(\cdot)) = \int_{\mathbb{R}^n} P \left( \begin{array}{cc} G_\beta(t, x - y; d_1) & 0 \\ 0 & G_\beta(t, x - y; d_2) \end{array} \right) P^{-1} \left( \begin{array}{c} u_0(y) \\ v_0(y) \end{array} \right) dy,
\]
(6.3)
with the fractional Green function \( G_\beta(t, x; d_j) \) is defined by the transformation
\[
E_\beta(-\mu|\lambda|^\alpha(1 + |\lambda|^2)^{\frac{\beta}{2}} t^\beta + d_j t^\beta) = \int_{\mathbb{R}^n} e^{i<x,\lambda>} G_\beta(t, x; d_j) dx,
\]
\( j \in \{1, 2\} \),
where \( E_\beta(\cdot) \) is the Mittag-Leffler function defined by (see, for example, [9] or [9, Chapter 1])
\[
E_\beta(z) = \sum_{p=0}^{\infty} \frac{z^p}{\Gamma(\beta p + 1)}, \quad z \in \mathbb{C},
\]
(6.4)
and we shall use the following basic properties about the Mittag-Leffler functions: they are entire functions on the complex plane and their asymptotic behavior, when \( \beta \in (0, 2], \beta \neq 1, 2 \), has the inverse power law as follows:
\[
|E_{\beta,\gamma}(z)| \sim O\left(\frac{1}{|z|^\gamma}\right), \quad |z| \to \infty \text{ with } |\arg(-z)| < \pi(1 - \beta/2), \quad \forall \gamma > 0,
\]
(6.5)
where \( \arg: \mathbb{C} \to (-\pi, \pi) \) and the notation \( f(z) \sim O(g(z)) \) means that \( f(z)/g(z) \) remains bounded as \( z \) approaches the indicated limit point; see, for example, the classic book by Erdélyi et.al. [11] (pp. 206-212, in particular p. 206 (7) and p. 210 (21)).

We note that, from [31] (45)], for \( \beta \in (0, 1) \), there is another representation for the fractional Green function
\[
G_\beta(t, x; d_j) = t^{-\beta} \int_0^\infty f_\beta(t^{-\beta}s) G(s, x; d_j) ds, \quad t > 0, \quad x \in \mathbb{R}^n,
\]
(6.6)
where
\[ G(s, x; d_j) = \left( \frac{1}{4\pi \mu s} \right)^{\frac{n}{2}} e^{-\frac{|x|^2}{4\mu s}} e^{d_j s}, \quad s > 0, \quad x \in \mathbb{R}^n, \] (6.7)

while \( f(p), p \geq 0, \) is a probability density which can be represented by the H-function (see, for example, [31, Section 3] and [27, p. 284]) and its Laplace transform is given by
\[ \int_0^\infty e^{-qs} f_\beta(s) ds = E_\beta(-q), \quad q \geq 0. \] (6.8)

Hence,
\[ \int_{\mathbb{R}^n} G_\beta(t, x; d_j) dx = \int_{\mathbb{R}^n} t^{-\beta} \int_0^\infty f_\beta(t^{-\beta} s) G(s, x; d_j) ds dx \]
\[ = t^{-\beta} \int_0^\infty f_\beta(t^{-\beta} s) \int_{\mathbb{R}^n} G(s, x; d_j) ds dx, \quad \text{(by Tonelli theorem)} \]
\[ = t^{-\beta} \int_0^\infty f_\beta(t^{-\beta} s) e^{d_j s} ds \int_{\mathbb{R}^n} G(s, x; d_j) dx, \]
(6.9)

where the convergence of the integral in (6.9) is guaranteed by the asymptotic behavior of the H-function (see, for example, [27, (3.7)]).

From the above discussion we know \( G_\beta(t, x; d_j) \in L^1(\mathbb{R}^n, dx) \) and \( \int_{\mathbb{R}^n} G_\beta(t, x; d_j) dx = E_\beta(d_j t^\beta) \) for any \( t > 0 \) and \( j \in \{1, 2\} \). Therefore, if the initial data \( u_0(x) \) and \( v_0(x) \) satisfy the special form (3.4), then by the representation (6.3) we have
\[ C(t; B) := Ew(t, x; w_0(\cdot)) = P \begin{pmatrix} E_\beta(d_1 t^\beta) & 0 \\ 0 & E_\beta(d_2 t^\beta) \end{pmatrix} P^{-1} \begin{pmatrix} C_0^{(1)} \\ C_0^{(2)} \end{pmatrix}, \] (6.10)

where \( C_0^{(j)}, j \in \{1, 2\} \) are the Hermite coefficients defined in (3.8).

The following Theorems 4 and 5 are time-fractional versions those in Theorems 3 and 2, respectively. However, it is now needed to add the feature of the sub-diffusive property, which is the reflection of time-fractional \( \beta < 1 \) (see Section 1), into consideration of the macro-scaling of homogenization. We need to take an additional scaling on the matrix \( B \) in the system (6.1) in order to compromise the effect of this sub-diffusivity upon the interaction between \( u \) and \( v \). To emphasize this situation, we denote the vector solution \( w(t, x; w_0(\cdot)) \) by \( w(t, x; w_0(\cdot), B) \) in the following formulation of macro-scaling of homogenization of a spatial-temporal fractional kinetic system.
Theorem 4 Let \( \{w(t, x; w_0(\cdot), B), t > 0, x \in \mathbb{R}^n\} \) be the solution-vector of the initial value problem (6.1) and (2.2), satisfying Condition A, B, C and D. Moreover, the LRD parameter \( \kappa_j \) and the Hermite rank \( m_j \) satisfy \( m_j \kappa_j < \min\{2\alpha, n\} \) and the Gaussian noise fields \( W_{(j)} \) are totally independent for \( j \in \{1, 2\} \).

(1) If \( m_1 \kappa_1 < m_2 \kappa_2 \), then the finite-dimensional distributions of the rescaled random field

\[
T_e^{(1)}(t, x) := \left( \epsilon^{-\frac{m_1}{\alpha}} L^{m_1}(\epsilon^{-\frac{\beta}{\alpha}}) \right)^{-\frac{1}{2}} \left\{ w(\epsilon^{-1} t, \epsilon^{-\frac{\beta}{\alpha}} x; w_0(\cdot), \epsilon^{\beta} B) - C(\epsilon^{-1} t; \epsilon^{\beta} B) \right\},
\]

\( t > 0, x \in \mathbb{R}^n \), converge weakly, as \( \epsilon \to 0 \), to the finite-dimensional distributions of the random field

\[
T^{(1)}(t, x) = \left( p_{11} p_{22} T^{(1)}(t, x; d_1) - p_{12} p_{21} T^{(1)}(t, x; d_2) \right), \quad t > 0, x \in \mathbb{R}^n,
\]

where for \( j \in \{1, 2\} \)

\[
T^{(1)}(t, x; d_j) := \frac{C_{m_1}}{\sqrt{m_1!}} K(n, \kappa_1) \frac{m_1}{n \lambda_1} \int_{\mathbb{R}^n} e^{i < x, \lambda_1 + \cdots + \lambda_{m_1} >} \frac{E_\beta(\epsilon^{-1} t, \epsilon^{-\frac{\beta}{2}} x; w_0(\cdot), \epsilon^{\beta} B) - C(\epsilon^{-1} t; \epsilon^{\beta} B)}{(|\lambda_1| \cdots |\lambda_{m_1}|)^{-\frac{n - m_1}{2}}} \prod_{t=1}^{m_1} W_1(d\lambda_t).
\]

(2) If \( m_2 \kappa_2 < m_1 \kappa_1 \), then the finite-dimensional distributions of the rescaled random field

\[
T_e^{(2)}(t, x) := \left( \epsilon^{-\frac{m_2}{\alpha}} L^{m_2}(\epsilon^{-\frac{\beta}{\alpha}}) \right)^{-\frac{1}{2}} \left\{ w(\epsilon^{-1} t, \epsilon^{-\frac{\beta}{\alpha}} x; w_0(\cdot), \epsilon^{\beta} B) - C(\epsilon^{-1} t; \epsilon^{\beta} B) \right\},
\]

\( t > 0, x \in \mathbb{R}^n \), converge weakly, as \( \epsilon \to 0 \), to the finite-dimensional distributions of the random field

\[
T^{(2)}(t, x) = \left( \frac{-p_{11} p_{12} T^{(2)}(t, x; d_1) + p_{11} p_{21} T^{(2)}(t, x; d_2)}{2} + p_{12} p_{21} T^{(2)}(t, x; d_1) + p_{11} p_{22} T^{(2)}(t, x; d_2) \right), \quad t > 0, x \in \mathbb{R}^n,
\]

where for \( j \in \{1, 2\} \)

\[
T^{(2)}(t, x; d_j) := \frac{C_{m_2}}{\sqrt{m_2!}} K(n, \kappa_2) \frac{m_2}{n \lambda_1} \int_{\mathbb{R}^n} e^{i < x, \lambda_1 + \cdots + \lambda_{m_2} >} \frac{E_\beta(\epsilon^{-1} t, \epsilon^{-\frac{\beta}{2}} x; w_0(\cdot), \epsilon^{\beta} B) - C(\epsilon^{-1} t; \epsilon^{\beta} B)}{(|\lambda_1| \cdots |\lambda_{m_2}|)^{-\frac{n - m_2}{2}}} \prod_{t=1}^{m_2} W_2(d\lambda_t).
\]
If $m_1 = m_2 = m$ and $\kappa_1 = \kappa_2 = \kappa$, then the finite-dimensional distributions of the rescaled random field

$$T^{(3)}(t, x) := (\varepsilon \frac{m\kappa}{n} L^m \frac{\varepsilon^{-\theta}}{\alpha})^{-\frac{1}{2}} \left\{ \mathbf{w}(\varepsilon^{-1}t, \varepsilon^{-\theta}x; \mathbf{w}_0(\cdot), \varepsilon^{\beta} B) - C(\varepsilon^{-1}t; \varepsilon^{\beta} B) \right\},$$

t > 0, \ x \in \mathbb{R}^n,$

converge weakly, as $\varepsilon \to 0$, to the finite-dimensional distributions of the random field

$$T^{(3)}(t, x) = \tilde{T}^{(1)}(t, x) + \tilde{T}^{(2)}(t, x), \ t > 0, \ x \in \mathbb{R}^n,$$

where the random field $\tilde{T}^{(j)}(t, x)$ is the same as the limiting random field $T^{(j)}(t, x)$ in (6.11) and (6.13) by replacing $m_j \to m$ and $\kappa_j \to \kappa$ for $j \in \{1, 2\}$.

**Remark.** The restriction $m_j\kappa_j < \min\{2\alpha, n\}$ in the above Theorem 4, together with the power law decay of Mittag-Leffler functions, guarantee that the random fields $T^{(j)}(t, x)$, $j \in \{1, 2, 3\}$, are indeed defined as $L^2(\Omega, \mathcal{F}, \mathbb{P})$ stochastic integrals.

As for the micro-scaling, the sub-diffusivity has no influence, and the same micro-scaling procedure as Theorem 2 applies.

**Theorem 5** Let $\{\mathbf{w}(t, x; \mathbf{w}_0(\cdot), B), t > 0, \ x \in \mathbb{R}^n\}$ be the solution-vector of the initial value problem (6.1) and (2.2), satisfying Condition A, B, C and D. Moreover, the LRD parameter $\kappa_j$ and the Hermite rank $m_j$ satisfy $m_j\kappa_j < \min\{2(\alpha+\gamma), n\}$ and the Gaussian noise fields $W^{(j)}$ are totally independent for $j \in \{1, 2\}$.

(1) If $m_1\kappa_1 < m_2\kappa_2$, then the finite-dimensional distributions of the rescaled random field

$$M^{(1)}_\varepsilon := \left[ \varepsilon^{m_1\kappa_1} \frac{L^{(1)}(\varepsilon^-)}{\alpha} \right]^{-\frac{1}{2}} \left\{ \mathbf{w}(\varepsilon t, \varepsilon^{\frac{\theta}{\alpha+\gamma}} x; \mathbf{w}_0(\varepsilon^{-\theta} \varepsilon^{\frac{\gamma}{\alpha+\gamma}} x)) - \left( \begin{array}{c} C^{(1)}_0 \\ C^{(2)}_0 \end{array} \right) \right\}, \ t > 0, \ x \in \mathbb{R}^n,$

converge weakly, as $\varepsilon \to 0$, to the finite-dimensional distributions of the random field

$$M^{(1)}(t, x) = \left( \begin{array}{c} M^{(1)}_0(t, x) \\ 0 \end{array} \right), \ t > 0, \ x \in \mathbb{R}^n,$$
where

\[ M^{(1)} := C^{(1)}_m \sqrt{m_1} K(n, \kappa_1) \int_{\mathbb{R}^n \times m_1} e^{i<x, \lambda_1 + \cdots + \lambda_m>} \frac{E_\beta(-\mu|\lambda_1 + \cdots + \lambda_m|^{\alpha+\gamma t^\beta})}{(|\lambda_1| \cdots |\lambda_m|)^{\frac{n-\kappa_1}{2}}} \prod_{l=1}^{m_1} W_1(d\lambda_l). \]

(2) If \( m_2 \kappa_2 < m_1 \kappa_1 \), then the finite-dimensional distributions of the rescaled random field

\[ M^{(2)}_\varepsilon := \varepsilon m_2 \kappa_2 X \left[ m^{\varepsilon(-\chi)} \right]^{-\frac{1}{2}} \left\{ w(\varepsilon t, \varepsilon^{-\frac{n}{\alpha+t^\gamma}} x; w_0(\varepsilon^{-\frac{n}{\alpha+t^\gamma}} x)) - \left( C^{(1)}_0 \right) \right\}, \]

converge weakly, as \( \varepsilon \to 0 \), to the finite-dimensional distributions of the random field

\[ M^{(2)}(t, x) = \begin{pmatrix} 0 \\ M^{(2)}(t, x) \end{pmatrix}, \]

where

\[ M^{(2)} := C^{(2)}_m \sqrt{m_2} K(n, \kappa_2) \int_{\mathbb{R}^n \times m_2} e^{i<x, \lambda_1 + \cdots + \lambda_m>} \frac{E_\beta(-\mu|\lambda_1 + \cdots + \lambda_m|^{\alpha+\gamma t^\beta})}{(|\lambda_1| \cdots |\lambda_m|)^{\frac{n-\kappa_2}{2}}} \prod_{l=1}^{m_2} W_2(d\lambda_l). \]

(3) If \( m_1 = m_2 = m \) and \( \kappa_1 = \kappa_2 = \kappa \), then the finite-dimensional distributions of the rescaled random field

\[ M^{(3)}_\varepsilon := [\varepsilon m^\kappa X^{\varepsilon(-\chi)}]^{-\frac{1}{2}} \left\{ w(\varepsilon t, \varepsilon^{-\frac{n}{\alpha+t^\gamma}} x; w_0(\varepsilon^{-\frac{n}{\alpha+t^\gamma}} x)) - \left( C^{(1)}_0 \right) \right\}, \]

converge weakly, as \( \varepsilon \to 0 \), to the finite-dimensional distributions of the random field

\[ M^{(3)}(t, x) = \tilde{M}^{(1)}(t, x) + \tilde{M}^{(2)}(t, x), \]

where the random field \( \tilde{M}^{(j)}(t, x) \) is the same as the limiting random field \( M^{(j)}(t, x) \) of Case (1) and (2) by replacing \( m_j \to m \) and \( \kappa_j \to \kappa \) for \( j \in \{1, 2\} \).

The concluding remark: The time-fractional index \( \beta < 1 \) indicates the sub-diffusivity, and it changes to be the super-diffusivity if we consider \( \beta > 1 \) (see Section 1). In [22], the time-fractional reaction-wave type system with random initial data are studied, in which the first-order time-derivatives of the initial data play the crucial role. To consider
spatial-temporal fractional kinetic systems which is super-diffusive in time and Riesz-Bessel in space will be a task of tremendous analysis. Finally, we mention that, for the classical, i.e. non-fractional, heat-type system with random initial condition, the solution vector-field and the scaling limit are expressed in terms of heat kernels; this more explicit and simpler case is treated in [21].

7 Proofs

In the following proofs, \( \Rightarrow \) denotes the convergence of random variables (or random families) in distributional sense, and \( \overset{d}{=} \) denotes the equality of random variables (or random families) in distributional sense. Moreover, we also denote \( f(t, x; \varepsilon) \sim g(t, x; \varepsilon) \) if there exists a constant \( c := c(t, x) > 0 \) such that \( cg(t, x; \varepsilon) < f(t, x; \varepsilon) < c^{-1}g(t, x; \varepsilon) \) when \( \varepsilon \to 0 \).

Proof of Proposition 1.

For (3.2), we use the solution form (2.5) and Karhunen’s representation (3.1) to get

\[
\begin{align*}
    w(t, x; w_0(\cdot)) &= Q(t, d_1, d_2) \int_{\mathbb{R}^n} G(t, y; \alpha, \gamma) \left( \int_{\mathbb{R}^n} e^{i\langle \lambda, x-y \rangle} Z_{F_1}(d\lambda) \right) dy \\
    &= Q(t, d_1, d_2) \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i\langle \lambda, x-y \rangle} G(t, y; \alpha, \gamma) dy \left( \begin{array}{c} Z_{F_1}(d\lambda) \\ Z_{F_2}(d\lambda) \end{array} \right) \\
    &= Q(t, d_1, d_2) \int_{\mathbb{R}^n} e^{i\langle \lambda, x \rangle} e^{-\mu t |\lambda|^2} Z_{F_1}(d\lambda) \left( \begin{array}{c} Z_{F_1}(d\lambda) \\ Z_{F_2}(d\lambda) \end{array} \right). \tag{7.1}
\end{align*}
\]
For (3.3), it is a consequence by using the independence assumption between the initial data, and we proceed it as follows,

\[
Ew(t, x; w_0(\cdot))w(t', x'; w_0(\cdot)) = Q(t; d_1, d_2)\int_{\mathbb{R}^n} e^{i(\lambda, x - \langle \lambda', x' \rangle)} e^{-\mu(t|\lambda''(1+|\lambda|^2)^]} \lambda''(1+|\lambda'|^2)^] \times \\
E \begin{pmatrix} Z_{F_1}(d\lambda) \\ Z_{F_2}(d\lambda) \end{pmatrix}^T Q(t'; d_1, d_2)^T \\
= Q(t; d_1, d_2)\int_{\mathbb{R}^n} e^{i(\lambda, x - \langle \lambda', x' \rangle)} e^{-\mu(t+|t'|)|\lambda''(1+|\lambda'|^2)^]} \lambda''(1+|\lambda'|^2)^] \times \\
\begin{pmatrix} F_1(d\lambda) \\ F_2(d\lambda) \end{pmatrix} Q(t'; d_1, d_2)^T. \quad \square
\]

Before going to prove our main results we recall the following two arguments, which are powerful to help us to reduce and simplify our problems.

**Slutsky argument** (see, for example, the book of Leonenko [16, p. 6]) Let \( \{\xi\} \) and \( \{\eta\} \) be families of random variables such that \( \{\xi\} \Rightarrow \xi \) and \( \{\eta\} \overset{P}{\Rightarrow} c \), where \( c \in R \). Then (i) \( \xi + \eta \Rightarrow \xi + c \), (ii) \( \xi \eta \Rightarrow c\xi \), and (iii) \( \xi /\eta \Rightarrow \xi /c \).

**Cramer-Wold argument** (see, for example, again [16, p. 6]) A family of \( k \)-dimensional r.v’s \( \xi := (\xi_1(x_1), ..., \xi_k(x_k))^T \) converge in distribution to a \( k \)-dimensional r.v. \( \xi := (\xi(x_1), ..., \xi(x_k))^T \) if and only if, for every \( c := (c_1, ..., c_k)^T \in \mathbb{R}^k \),

\[
< c, \xi > = \sum_{i=1}^{k} c_i \xi_i(x_i) \Rightarrow \sum_{i=1}^{k} c_i \xi(x_i) =< c, \xi >, \quad \text{as } \varepsilon \to 0.
\]

The following lemma, although it is a simple extension of Cramer-Wold argument, is of interest in itself, and will play an important role in the proof of our Theorem 2 and 3.

**Lemma 1.** Let \( X_{\varepsilon} := \{[X_1^{(1)}, X_2^{(2)}]T(t, x), x \in \mathbb{R}^n, t > 0\} \) be a \( \mathbb{R}^2 \)-valued random field which is generated by \( X_\varepsilon(t, x) = Q_\varepsilon(t)[U_\varepsilon, V_\varepsilon]^T(t, x) \), where \( U_\varepsilon(t, x) \) and \( V_\varepsilon(t, x) \) are independent random fields on \( \mathbb{R}^n \times \mathbb{R}^+ \) and \( Q_\varepsilon(t) \) is a non-random \( 2 \times 2 \) matrix. If there exist two random fields \( U_0 \) and \( V_0 \) such that \( U_\varepsilon(t, x) \Rightarrow U_0(t, x) \) and \( V_\varepsilon(t, x) \Rightarrow V_0(t, x) \), respectively, and \( Q_\varepsilon(t) \) converges to \( Q(t) \) in the usual sense when \( \varepsilon \to 0 \), then the finite dimensional distributions of \( X_\varepsilon(t, x), t > 0, x \in \mathbb{R}^n \), converge to the finite dimensional...
Proof of Lemma 1

By Cramer-Wold argument with $k = 2$ there, it suffices to prove: For any given $c_1, c_2 \in \mathbb{R}$ and $x, t$ fixed, we have

$$[c_1, c_2][Q_\varepsilon(t)][U_\varepsilon(x, t), V_\varepsilon(x, t)]_T \Rightarrow [c_1, c_2][Q(t)][U_0(x, t), V_0(x, t)]_T,$$

which is equivalent to

$$(c_1 Q_{11,\varepsilon}(t) + c_2 Q_{21,\varepsilon}(t)) U_\varepsilon(x, t) + (c_1 Q_{12,\varepsilon}(t) + c_2 Q_{22,\varepsilon}(t)) V_\varepsilon(x, t)$$

$$\Rightarrow (c_1 Q_{11}(t) + c_2 Q_{21}(t)) U_0 + (c_1 Q_{12}(t) + c_2 Q_{22}(t)) V_0,$$

where the $i, j$ indicate the $(i, j)$ entry of the matrix. While the above display can be checked by using the characteristic functions, since $U_\varepsilon$ and $V_\varepsilon$ are assumed to be independent (whence so are $U_0, V_0$). □

Proof of Theorem 1.

(1) Firstly, for simplification, we set $N_1(\varepsilon) := \varepsilon^{\chi m} L^m(\varepsilon^{-\chi})$. By the Hermite expansion, we can rewrite $s(\varepsilon t, \varepsilon^{\frac{1}{\alpha+\gamma}} x; h(\zeta(\varepsilon^{-\frac{1}{\alpha+\gamma}} \cdot))) - C_0(h)$ as

$$s(\varepsilon t, \varepsilon^{\frac{1}{\alpha+\gamma}} x; h(\zeta(\varepsilon^{-\frac{1}{\alpha+\gamma}} \cdot))) - C_0(h) = \sum_{\rho=m}^{\infty} s(\varepsilon t, \varepsilon^{\frac{1}{\alpha+\gamma}} x ; C_\rho(h) \sqrt{\rho!} H_\rho(\zeta(\varepsilon^{-\frac{1}{\alpha+\gamma}} \cdot))) \quad (7.2)$$

where the summation is in $L^2(\Omega)$ sense. Hence, in accord to the definition [1.2] about the random field $s^\varepsilon(t, x)$, it can be rewritten as

$$s^\varepsilon(t, x) = \sum_{\rho=m}^{\infty} I^\varepsilon_\rho(t, x), \quad (7.3)$$

with $I^\varepsilon_\rho(t, x) := (N_1(\varepsilon))^{-\frac{1}{2}} s(\varepsilon t, \varepsilon^{\frac{1}{\alpha+\gamma}} x ; C_\rho(h) \sqrt{\rho!} H_\rho(\zeta(\varepsilon^{-\frac{1}{\alpha+\gamma}} \cdot))).$
By (2.5) with \( d_1 = d_2 = 0 \), we have

\[
\left( \frac{C_\rho(h)}{\sqrt{\rho!}} \right)^{-1} (N_1(\varepsilon))^{\frac{1}{2}} I^\varepsilon_\rho(t, x)
= \int_{\mathbb{R}^n} G(\varepsilon t, y; \alpha, \gamma) H_\rho(\zeta(\varepsilon^{-\alpha+\gamma} x - y)) dy
= \int_{\mathbb{R}^n} G(\varepsilon t, y; \alpha, \gamma) \int_{\mathbb{R}^n} e^{i<\varepsilon^{-\alpha+\gamma} x - y, \lambda_1 + \ldots + \lambda_\rho>} \prod_{\sigma=1}^\rho \sqrt{f(\lambda_\sigma)} W(d\lambda_\sigma) dy
\]

so (7.4) is equal to

\[
\int_{\mathbb{R}^n} e^{i<\varepsilon^{-\alpha+\gamma} x, \lambda_1 + \ldots + \lambda_\rho>} e^{-\mu \varepsilon^{1-\alpha+\gamma} (\lambda_1 + \ldots + \lambda_\rho)^2} \prod_{\sigma=1}^\rho \sqrt{f(\lambda_\sigma)} W(d\lambda_\sigma)
\]

so (7.4) is equal to

\[
\int_{\mathbb{R}^n} e^{i<\varepsilon^{-\alpha+\gamma} x, \lambda_1 + \ldots + \lambda_\rho>} e^{-\mu \varepsilon^{1-\alpha+\gamma} (\lambda_1 + \ldots + \lambda_\rho)^2} \prod_{\sigma=1}^\rho \sqrt{f(\lambda_\sigma)} W(d\lambda_\sigma)
\]

For the bracket above, by substituting \( t \to \varepsilon t \) and \( \lambda \to \varepsilon^{-\alpha+\gamma} (\lambda_1 + \ldots + \lambda_\rho) \) into (2.3), we have

\[
\int_{\mathbb{R}^n} G(\varepsilon t, y; \alpha, \gamma) e^{-i<\varepsilon^{-\alpha+\gamma} y, \lambda_1 + \ldots + \lambda_\rho>} = e^{-\mu \varepsilon^{1-\alpha+\gamma} (\lambda_1 + \ldots + \lambda_\rho)^2} \prod_{\sigma=1}^\rho \sqrt{f(\lambda_\sigma)} W(d\lambda_\sigma)
\]

so (7.4) is equal to

\[
\int_{\mathbb{R}^n} e^{i<\varepsilon^{-\alpha+\gamma} x, \lambda_1 + \ldots + \lambda_\rho>} e^{-\mu \varepsilon^{1-\alpha+\gamma} (\lambda_1 + \ldots + \lambda_\rho)^2} \prod_{\sigma=1}^\rho \sqrt{f(\lambda_\sigma)} W(d\lambda_\sigma)
\]

where we have used the self-similar property for Gaussian random measure on \( \mathbb{R}^n \) in the

last equality. Therefore, by the orthogonal property for the Gaussian white noise, we can get

\[
(C_\rho(h))^{-2} N_1(\varepsilon) \text{Cov}(I^\varepsilon_\rho(t, x), I^\varepsilon_\rho(t', x'))
= e^{\chi_{pm}} \int_{\mathbb{R}^n} e^{i<x-x', \lambda_1 + \ldots + \lambda_\rho>} e^{-\mu (t+t') \varepsilon^{1-\alpha+\gamma} (\lambda_1 + \ldots + \lambda_\rho)^2} \prod_{\sigma=1}^\rho f(\varepsilon^\chi \lambda_\sigma) d\lambda_\sigma
= e^{\chi_{pm}} \int_{\mathbb{R}^n} e^{i<x-x', \tau_1>} e^{-\mu (t+t') \varepsilon^{1-\alpha+\gamma} (\lambda_1 + \ldots + \lambda_\rho)^2} \frac{f^\rho(\varepsilon^\chi \tau_1)}{\varepsilon^{(\rho-1)m}} d\tau_1
= e^{\chi_{pm}} \int_{\mathbb{R}^n} e^{i<x-x', \tau>} e^{-\mu (t+t') \varepsilon^{1-\alpha+\gamma} (\lambda_1 + \ldots + \lambda_\rho)^2} \frac{f^\rho(\varepsilon^\chi \tau)}{\varepsilon^{(\rho-1)m}} d\tau,
\]
where $f^{\alpha}(\cdot)$ is defined in (3.13).

(i) For $\rho \in \mathbb{N}$ with $m\kappa \leq \rho \kappa < n$ and any $\delta > 0$, by (7.6) and (3.14),

$$\text{Cov}(I^\varepsilon_\rho(t,x)I^\varepsilon_\rho(t',x')) = (C_\rho(h))^2(A_1(\varepsilon) + A_2(\varepsilon)), \quad (7.7)$$

with

$$|A_1(\varepsilon)| = (N_1(\varepsilon))^{-1}\varepsilon^n x| \int_{|\varepsilon^\alpha\tau| > \delta} e^{i<x-x',\tau>} -e^{-\mu(t+t')}|\tau|^\alpha(e^{-\frac{|\tau|^2}{2}} + |\tau|^2)^{\frac{\rho}{2}} f^\rho(\varepsilon^\alpha\tau)d\tau|$$

$$\leq (N_1(\varepsilon))^{-1}\varepsilon^n x \sup\{f^\rho(\tilde{\lambda})| |\tilde{\lambda}| > \delta\} \int_{|\varepsilon^\alpha\tau| > \delta} e^{-\mu(t+t')}|\tau|^n|d\tau$$

$$\leq \varepsilon^n \rho L^{m}(\varepsilon^{-\chi})^{-1} |\varepsilon^\alpha\tau| |\tilde{\lambda}| \sup\{f^\rho(\tilde{\lambda})| |\tilde{\lambda}| > \delta\} \int_{\varepsilon^{-\chi} \delta} e^{-\mu(t+t')}|\tau|^n|d\tau$$

$$\rightarrow 0, \text{ as } \varepsilon \rightarrow 0, \quad (m\kappa \leq \rho \kappa < n)$$

and, by choosing $\delta$ small enough and (3.14),

$$A_2(\varepsilon)$$

$$= (N_1(\varepsilon))^{-1}\varepsilon^n x \int_{|\varepsilon^\alpha\tau| \leq \delta} e^{i<x-x',\tau>} -e^{-\mu(t+t')}|\tau|^\alpha(e^{-\frac{|\tau|^2}{2}} + |\tau|^2)^{\frac{\rho}{2}} f^\rho(\varepsilon^\alpha\tau)d\tau$$

$$= (N_1(\varepsilon))^{-1}\varepsilon^n x \int_{|\varepsilon^\alpha\tau| \leq \delta} e^{i<x-x',\tau>} -e^{-\mu(t+t')}|\tau|^\alpha(e^{-\frac{|\tau|^2}{2}} + |\tau|^2)^{\frac{\rho}{2}} (1 + o(1)) \frac{L^\rho(|\varepsilon^\alpha\tau|^{-1})}{|\varepsilon^\alpha\tau|^{n-\rho\kappa}}|d\tau$$

$$\sim (N_1(\varepsilon))^{-1}\varepsilon^{\rho\kappa} L^{\rho}(\varepsilon^{-\chi}) K(n, \rho\kappa) \int_{\mathbb{R}^n} e^{i<x-x',\tau>} -e^{-\mu(t+t')}|\tau|^\alpha|d\tau$$

as $\varepsilon \rightarrow 0$.

where the asymptotic equivalence is guaranteed by the uniform convergence theorem for the slowly varying function (see, for example, Leonenko [16, Section 1.4]). We remark that the $f$, defined as a spectral density function, is bounded outside of zero (while the singularity at zero is from the LRD assumption, as employed in subsection 3.2); therefore its $\rho$-th convolution remains to have, at most, the singularity only at zero.

The conclusion of (i): Apart from the term $\text{Cov}(I^\varepsilon_\rho(t,x)I^\varepsilon_\rho(t',x'))$,

$$\lim_{\varepsilon \rightarrow 0} \sum_{\rho,m<\rho<n/\kappa} \text{Cov}(I^\varepsilon_\rho(t,x)I^\varepsilon_\rho(t',x')) = 0 \quad (7.8)$$
where we have used the fact that \( \{ l \in \mathbb{N} | m \kappa \leq l \kappa < n \} \) is a finite set and on this set \( \lim_{\varepsilon \to 0} (N_1(\varepsilon))^{-1} \varepsilon^{\rho \kappa} L^\rho (\varepsilon^{-\chi}) = 0 \) except for \( \rho = m \).

(ii) For \( \rho \in \mathbb{N} \) with \( \rho \kappa > n \), by (7.6), (3.15) and \( \sum_{\rho=m}^{\infty} (C_\rho(h))^2 \leq \| h \|_2^2 < \infty \),

\[
\lim_{\varepsilon \to 0} \sum_{\rho, \rho \kappa > n} \text{Cov}(I_\rho^\varepsilon(t, x) I_\rho^{\varepsilon}(t', x')) = 0
\]

Finally, from the expansion (7.3) for the random field \( s^\varepsilon(t, x) \) and combining the observations (7.8) and (7.9) we know that only the component \( I_m^\varepsilon(t, x) \) in (7.3) do contribute to the covariance function of the random field \( s^\varepsilon(t, x) \), that is,

\[
\lim_{\varepsilon \to 0} \text{Cov}(s^\varepsilon(t, x) s^\varepsilon(t', x')) = (C_m(h))^2 K(n, m \kappa) \int_{\mathbb{R}^n} e^{\mu(t+t')|\tau|^{n+\gamma}} d\tau.
\]

(2) From the above discussion, we may apply Chebyshev inequality to obtain that:

\[
\sum_{\rho=m+1}^{\infty} I_\rho^\varepsilon(t, x) \xrightarrow{P} 0.
\]

Therefore, in view of Slutsky argument, we suffice to focus our attention on the term \( I_m^\varepsilon(t, x) \). In the following we will prove \( I_m^\varepsilon(t, x) \) converges in distribution sense to \( s_m(t, x) \), which is defined in (4.4), for each fixed \((t, x) \in \mathbb{R}_+ \times \mathbb{R}^n \). By the definition of \( N_1(\varepsilon) \) and replacing the letter \( \rho \) by \( m \) in (7.5), we can rewrite (7.5) as follows

\[
I_m^\varepsilon(t, x) = \frac{C_m(h)}{\sqrt{m!}} \int_{\mathbb{R}^n \times m} e^{i < x, \lambda_1 + \cdots + \lambda_m >} M_\varepsilon(\lambda) \prod_{\sigma=1}^{m} W(d\lambda_\sigma),
\]

with

\[
M_\varepsilon(\lambda) := \varepsilon^{\frac{\chi(n-n_m)}{2}} L^{-\frac{n}{2}} (\varepsilon^{-\chi}) e^{-\mu(\varepsilon^{-\chi} + |\lambda_1 + \cdots + \lambda_m|^2)^{\frac{1}{2}}} |\lambda_1 + \cdots + \lambda_m|^m \prod_{\sigma=1}^{m} \sqrt{f(\varepsilon^\chi \lambda_\sigma)},
\]

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which, when $\varepsilon \to 0$, satisfies
\[
\lim_{\varepsilon \to 0} M_\varepsilon(\lambda) = \left( K(n, \kappa) \right)^{m} \frac{e^{-\mu|\lambda_1 + \cdots + \lambda_m|^{\alpha + \gamma}}}{(|\lambda_1| \cdots |\lambda_m|)^{\frac{n}{\kappa}}}. \tag{7.12}
\]

Now, applying the isometric property of the multiple Wiener integrals to the difference of (7.10) and (4.4), we have
\[
E[I_m^\varepsilon(t, x) - s_m(t, x)]^2 = \lim_{\varepsilon \to 0} (C_\rho(h))^2 \int_{\mathbb{R}^{n \times m}} |M_\varepsilon(\lambda) - (K(n, \kappa))^{m} \frac{e^{-\mu|\lambda_1 + \cdots + \lambda_m|^{\alpha + \gamma}}}{(|\lambda_1| \cdots |\lambda_m|)^{\frac{n}{\kappa}}}|^2 \prod_{\sigma=1}^{m} d\lambda_\sigma
\to 0, \text{ as } \varepsilon \to 0, \text{ by (7.11) and the assumption } f(\lambda) \text{ is decreasing at infinity in Condition C and }
\int_{\mathbb{R}^{n \times m}} \frac{e^{-2\mu|\lambda_1 + \cdots + \lambda_m|^{\alpha + \gamma}}}{(|\lambda_1| \cdots |\lambda_m|)^{n-k}} \prod_{\sigma=1}^{m} d\lambda_\sigma = r(n, m, \kappa) \int_{\mathbb{R}^n} \frac{e^{-2\mu|\lambda|^{\alpha + \gamma}}}{(|\lambda|)^{n-m\kappa}} d\lambda < \infty, \text{ for } m\kappa < n,
\]
where the constant $r(n, m, \kappa)$ is generated by the Riesz potential. Finally, the assertion (2) of Theorem 1 is followed from Slutsky and Cramer-Wold arguments. \(\square\)

**Proof of Theorem 2 for the case (1):** $m_2 \alpha_2 > m_1 \alpha_1$.

From the solution form (2.5), we have
\[
\begin{pmatrix}
u(t, x; u_0(\cdot)) \\
v(t, x; v_0(\cdot))
\end{pmatrix} = Q(t; d_1, d_2) \begin{pmatrix}
C_0^{(1)} \\
C_0^{(2)}
\end{pmatrix} = Q(t; d_1, d_2) \begin{pmatrix}
U(t, x) \\
V(t, x)
\end{pmatrix} - \begin{pmatrix}
C_0^{(1)} \\
C_0^{(2)}
\end{pmatrix},
\] (7.12)

where $Q(t; d_1, d_2)$, $U(t, x)$ and $V(t, x)$ are defined in (2.6) and (2.7).

By (7.12),
\[
[\varepsilon^{m_1 \kappa_1} L_1^{m_1}(\varepsilon^{-\chi})]^{-\frac{1}{\kappa}} \left\{ \begin{pmatrix}
u(\varepsilon t, \varepsilon^{\frac{1}{\kappa} - \gamma} x; u_0(\varepsilon^{-\frac{1}{\kappa} - \gamma} \cdot)) \\
v(\varepsilon t, \varepsilon^{\frac{1}{\kappa} - \gamma} x; v_0(\varepsilon^{-\frac{1}{\kappa} - \gamma} \cdot))
\end{pmatrix} = Q(\varepsilon t; d_1, d_2) \begin{pmatrix}
C_0^{(1)} \\
C_0^{(2)}
\end{pmatrix} \right\}
\]
\[
= Q(\varepsilon t; d_1, d_2) \left[\varepsilon^{m_1 \kappa_1} L_1^{m_1}(\varepsilon^{-\chi})\right]^{-\frac{1}{\kappa}} \begin{pmatrix}
U(\varepsilon t, \varepsilon^{\frac{1}{\kappa} - \gamma} x; u_0(\varepsilon^{-\frac{1}{\kappa} - \gamma} \cdot)) - C_0^{(1)} \\
V(\varepsilon t, \varepsilon^{\frac{1}{\kappa} - \gamma} x; v_0(\varepsilon^{-\frac{1}{\kappa} - \gamma} \cdot)) - C_0^{(2)}
\end{pmatrix}
\]
\[
:= Q(\varepsilon t; d_1, d_2) \left[\varepsilon^{m_1 \kappa_1} L_1^{m_1}(\varepsilon^{-\chi})\right]^{-\frac{1}{\kappa}} \begin{pmatrix}
U_\varepsilon(t, x) \\
V_\varepsilon(t, x)
\end{pmatrix}.
\] (7.13)

(7.14)
Firstly, by Theorem 1 (2), we have

\[
U_\varepsilon(t, x) \Rightarrow \tilde{X}_{m_1}^{(1)}(t, x),
\]

where \( \tilde{X}_{m_1}^{(1)} \) is defined in (4.5).

Secondly, by Theorem 1 (1), we can obtain

\[
V_\varepsilon(t, x) \xrightarrow{P} 0
\]

since we can apply Chebyshev inequality to observe that for any \( c > 0 \), as \( \varepsilon \to 0 \),

\[
P(|V_\varepsilon(t, x)| > c) \leq c^{-2} \text{Var}(V_\varepsilon(t, x)) \approx c^{-2} \left[ \varepsilon^{-m_1 \alpha_1} L^{-m_1} \left( \varepsilon^{-\chi} \right) \right] \cdot \left[ \varepsilon^{-m_2 \alpha_2} L^{-m_2} \left( \varepsilon^{-\chi} \right) \right] \to 0.
\]

Meanwhile, since

\[
\lim_{\varepsilon \to 0} Q_\varepsilon(t) := \lim_{\varepsilon \to 0} Q(\varepsilon t; d_1, d_2) = P \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} P^{-1} = I_{2 \times 2}.
\]

Therefore, we may apply Lemma 1 to those \( U_\varepsilon(t, x) \), \( V_\varepsilon(t, x) \) and \( Q_\varepsilon(t) \) on the above to obtain that

\[
\left[ \varepsilon^{m_1 \alpha_1} L^{m_1} \left( \varepsilon^{-\chi} \right) \right]^{-\frac{1}{2}} \left\{ u(\varepsilon t, \varepsilon^{-\alpha_1} x; u_0(\varepsilon^{-\alpha_1} \chi \cdot x)) - Q(\varepsilon t; d_1, d_2) \begin{pmatrix} C_0^{(1)} \\ C_0^{(2)} \end{pmatrix} \right\} \Rightarrow I_{2 \times 2} \begin{pmatrix} \tilde{X}_{m_1}^{(1)}(t, x) \\ 0 \end{pmatrix}, \quad t > 0, \ x \in \mathbb{R}^n.
\]

**Proof of Theorem 2 for the case (2): \( m_1 \alpha_1 > m_2 \alpha_2 \).**

The proof is proceeded as the case (1), yet under the new assumption and the different renormalization \( \left[ \varepsilon^{m_2 \alpha_2} L^{m_2} \left( \varepsilon^{-\chi} \right) \right]^{-\frac{1}{2}} \). Now (7.15) becomes as

\[
U_\varepsilon(t, x) := \left[ \varepsilon^{m_2 \alpha_2} L^{m_2} \left( \varepsilon^{-\chi} \right) \right]^{-\frac{1}{2}} \left\{ U(\varepsilon t, \varepsilon^{-\alpha_2} x; u_0(\varepsilon^{-\alpha_2} \chi \cdot x)) - C_0^{(1)} \right\} \xrightarrow{P} 0,
\]

and (7.16) becomes as

\[
V_\varepsilon(t, x) := \left[ \varepsilon^{m_2 \alpha_2} L^{m_2} \left( \varepsilon^{-\chi} \right) \right]^{-\frac{1}{2}} \left\{ V(\varepsilon t, \varepsilon^{-\alpha_2} x; v_0(\varepsilon^{-\alpha_2} \chi \cdot x)) - C_0^{(2)} \right\} \Rightarrow \tilde{X}_{m_2}^{(2)}(t, x),
\]
Because the representation for the limiting fields $\tilde{X}_{m_2}^{(2)}$ is kept unchanged. Therefore, we again apply Lemma 1 to get

$$\epsilon^{m_2kx^2}L^{m_2}(e^{-x})^{-\frac{1}{2}} \left\{ \left( \begin{array}{c} u(\epsilon t, \epsilon \frac{1}{\alpha + \gamma} x; u_0(\epsilon \frac{1}{\alpha + \gamma} x)) \\ v(\epsilon t, \epsilon \frac{1}{\alpha + \gamma} x; v_0(\epsilon \frac{1}{\alpha + \gamma} x)) \end{array} \right) - Q(\epsilon t; d_1, d_2) \left( \begin{array}{c} C_0^{(1)} \\ C_0^{(2)} \end{array} \right) \right\} \Rightarrow I_{2\times 2} \left( \begin{array}{c} 0 \\ \tilde{X}_{m_2}^{(2)}(t, x) \end{array} \right), \; t > 0, \; x \in \mathbb{R}^n. \quad \square$$

**Proof of Theorem 2 for the case (3):** $m_1 = m_2 = m$, $\alpha_1 = \alpha_2 = \alpha$.

By Theorem 1 (2), we have

$$U_\epsilon(t, x) := \epsilon^{m_2kx^2}L^m(e^{-x})^{-\frac{1}{2}} \left\{ U(\epsilon t, \epsilon \frac{1}{\alpha + \gamma} x; u_0(\epsilon \frac{1}{\alpha + \gamma} x)) - C_0^{(1)} \right\} \Rightarrow \tilde{X}_{m}^{(1)}(t, x),$$

$$V_\epsilon(t, x) := \epsilon^{m_2kx^2}L^m(e^{-x})^{-\frac{1}{2}} \left\{ V(\epsilon t, \epsilon \frac{1}{\alpha + \gamma} x; v_0(\epsilon \frac{1}{\alpha + \gamma} x)) - C_0^{(2)} \right\} \Rightarrow \tilde{X}_{m}^{(2)}(t, x),$$

where $\tilde{X}_{m}^{(j)}$, $j \in \{1, 2\}$, are defined in (4.5) and (4.6) with $m_1 = m_2 = m$.

Because in this case the equality $\lim_{\epsilon \to 0} Q_\epsilon(t) = I$ is still unchange, in the same way, we obtained

$$\epsilon^{m_2kx^2}L^m(e^{-x})^{-\frac{1}{2}} \left\{ \left( \begin{array}{c} u(\epsilon t, \epsilon \frac{1}{\alpha + \gamma} x; u_0(\epsilon \frac{1}{\alpha + \gamma} x)) \\ v(\epsilon t, \epsilon \frac{1}{\alpha + \gamma} x; v_0(\epsilon \frac{1}{\alpha + \gamma} x)) \end{array} \right) - Q(\epsilon t; d_1, d_2) \left( \begin{array}{c} C_0^{(1)} \\ C_0^{(2)} \end{array} \right) \right\} \Rightarrow I_{2\times 2} \left( \begin{array}{c} \tilde{X}_{m}^{(1)}(t, x) \\ \tilde{X}_{m}^{(2)}(t, x) \end{array} \right), \; t > 0, \; x \in \mathbb{R}^n. \quad \square$$

**Proof of Proposition 3.**

$$E \left( \begin{array}{c} Y_{1}^{***}(t, x) \\ Y_{2}^{***}(t, x) \end{array} \right) \left( \begin{array}{c} Y_{1}^{***}(t', x') \\ Y_{2}^{***}(t', x') \end{array} \right) = E \left( \begin{array}{c} \tilde{X}_{m}^{(1)}(t, x) \\ \tilde{X}_{m}^{(2)}(t, x) \end{array} \right) \left( \begin{array}{c} \tilde{X}_{m}^{(1)}(t', x') \\ \tilde{X}_{m}^{(2)}(t', x') \end{array} \right) = \left( \begin{array}{cc} E\tilde{X}_{m}^{(1)}(t, x)\tilde{X}_{m}^{(1)}(t', x') & 0 \\ 0 & E\tilde{X}_{m}^{(2)}(t, x)\tilde{X}_{m}^{(2)}(t', x') \end{array} \right). \quad (7.18)$$

Because the representation for the limiting fields $\tilde{X}_{m}^{(1)}(t, x)$ and $\tilde{X}_{m}^{(2)}(t, x)$ is the same as the limiting field $s_{m}(t, x)$, defined in (4.4), we can apply the result (4.3) to get

$$E\tilde{X}_{m}^{(j)}(t, x)\tilde{X}_{m}^{(j)}(t', x') = (C^{(j)}^2K(n, \kappa)m) \int_{\mathbb{R}^n} e^{i < \tau - x, \tau >} e^{-\mu(\tau + t')|\tau|^{\alpha+\gamma}} \frac{e^{-\mu(\tau + t')|\tau|^{\alpha+\gamma}}}{|\tau|^{n-\kappa}} d\tau.$$

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Therefore, the covariance structure (7.18) is equal to
\[
\int_{\mathbb{R}^n} e^{i<x-x',\tau>} K(n, km) \frac{e^{-\mu|t|^\alpha\gamma}}{|\tau|^{n-m\epsilon}} \left( \begin{array}{cc} (C_m^{(1)})^2 & 0 \\ 0 & (C_m^{(2)})^2 \end{array} \right) d\tau. \quad \Box
\]

**Proof of Proposition 4**

(1) Here, for simplification, we set \(G(t, x) := G(t, x; \alpha, \gamma)\) and \(N(\varepsilon) := \varepsilon^{\frac{m}{\alpha}} L^m(e^{-\frac{t}{\varepsilon}})\), then by the solution form (2.7) for the differential equation \(\frac{\partial}{\partial t}s = -\mu(I - \Delta) \frac{2}{\gamma} (\Delta)^{\frac{\gamma}{2}} s\) we have

\[
(N(\varepsilon))^{-\frac{1}{2}} \left\{ \int_{\mathbb{R}^n} G(t, x; \varepsilon) h(\zeta(\cdot)) - C_0(h) \right\}
\]

\[
= (N(\varepsilon))^{-\frac{1}{2}} \left\{ \int_{\mathbb{R}^n} G(t, x; \varepsilon) - (t, x) \int_{\mathbb{R}^n} G(t, x; \varepsilon) h(\zeta(\cdot)) dy - C_0(h) \right\}
\]

\[
\int_{\mathbb{R}^n} G(t, x; \varepsilon) h(\zeta(\cdot)) dy := \sum_{k=m}^{\infty} \zeta_k(t, x). \quad (7.19)
\]

From (3.9), the cross terms of the left hand side blow have zero covariance, thus we have

\[
\text{Cov}(\sum_{k=m}^{\infty} \zeta_k(t, x), \sum_{k=m}^{\infty} \zeta_k(t', x')) = \sum_{k=m}^{\infty} \text{Cov}(\zeta_k(t, x), \zeta_k(t', x')). \quad (7.20)
\]

For each \(k \in \{m, m + 1, \ldots\}\), by (3.9)

\[
\text{Cov}(\zeta_k(t, x), \zeta_k(t', x'))
\]

\[
= (C_k(h))^{(N(\varepsilon))^{-\frac{1}{2}}} \int_{\mathbb{R}^n} G(t, x; \varepsilon) G(t', x'; \varepsilon) - (t', x') R^k(y - y') dy dy' \]

\[
= (C_k(h))^{(N(\varepsilon))^{-\frac{1}{2}}} \int_{\mathbb{R}^n} G(t, x; \varepsilon) G(t', x'; \varepsilon) - (t', x') \left\{ \int_{\mathbb{R}^n} e^{i<y-y',\lambda>} f^{\ast k}(\lambda) d\lambda \right\} dy dy' \]

\[
= (C_k(h))^{(N(\varepsilon))^{-\frac{1}{2}}} \int_{\mathbb{R}^n} \left\{ \int_{\mathbb{R}^n} e^{i<y-y',\lambda>} G(t, x; \varepsilon) - (t', x') dy \int_{\mathbb{R}^n} e^{i<y-y',\lambda>} G(t', x'; \varepsilon) - (t, x) dy' \right\} f^{\ast k}(\lambda) d\lambda \]

\[
= (C_k(h))^{(N(\varepsilon))^{-\frac{1}{2}}} \int_{\mathbb{R}^n} e^{i<y-y',\lambda>} e^{-\mu|t+t'||\lambda|^2(1+|\lambda|^2)^{\frac{2}{\gamma}}} f^{\ast k}(\lambda) d\lambda \]

\[
= (C_k(h))^{(N(\varepsilon))^{-\frac{1}{2}}} \int_{\mathbb{R}^n} e^{i<y-y',\lambda>} e^{-\mu|t+t'||\lambda|^2(1+|\lambda|^2)^{\frac{2}{\gamma}}} f^{\ast k}(\frac{1}{\varepsilon \lambda}) d\lambda, \quad (7.21)
\]

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by rescaling \( \lambda \) into \( \varepsilon^{\frac{1}{n}} \lambda \).

For \( k : mk \leq k\kappa < n \) and any \( \delta > 0 \), by (7.21)

\[
\text{Cov}(s_k^s(t, x), s_k^s(t', x')) = (C_k(h))^2(A_1(\varepsilon) + A_2(\varepsilon)), \tag{7.22}
\]

with

\[
|A_1(\varepsilon)| = (N(\varepsilon))^{-1} \left| \int_{|\varepsilon^{\frac{1}{n}} \lambda| > \delta} e^{i \langle x - x', \lambda \rangle - \mu (t + t') |\lambda|^\alpha (1 + |\varepsilon^{\frac{1}{n}} \lambda|^2)^{\frac{1}{2}}} \varepsilon^{\frac{n}{n}} f^{s_k}(\varepsilon^{\frac{1}{n}} \lambda) d\lambda \right|
\leq (N(\varepsilon))^{-1} \varepsilon^{\frac{n}{n}} \sup \{ f^{s_k}(\tilde{\lambda}) : |\tilde{\lambda}| > \delta \} \left( e^{-\mu (t + t') |\lambda|^\alpha} \int_{|\varepsilon^{\frac{1}{n}} \lambda| > \delta} e^{-\mu (t + t') |\lambda|^\alpha} d\lambda \right)
\leq C(\delta) \varepsilon^{\frac{n-k\kappa}{n}} L^m(\varepsilon^{\frac{1}{n}}) \int_{|\varepsilon^{\frac{1}{n}} \lambda| > \delta} e^{-\mu (t + t') |\lambda|^\alpha} r^{n-1} dr \to 0, \text{ as } \varepsilon \to 0,
\]

and by choosing \( \delta \) small enough

\[
A_2(\varepsilon) = (N(\varepsilon))^{-1} \varepsilon^{\frac{n}{n}} \int_{|\varepsilon^{\frac{1}{n}} \lambda| \leq \delta} e^{i \langle x - x', \lambda \rangle - \mu (t + t') |\lambda|^\alpha (1 + |\varepsilon^{\frac{1}{n}} \lambda|^2)^{\frac{1}{2}}} f^{s_k}(\varepsilon^{\frac{1}{n}} \lambda) d\lambda.
\]

By (7.23)

\[
\lim_{\varepsilon \to 0} \sum_{k : mk \leq k\kappa < n} \text{Cov}(s_k^s(t, x), s_k^s(t', x')) = (C_m(h))^2 K(n, m\kappa) \int_{\mathbb{R}^n} e^{i \langle x - x', \lambda \rangle - \mu (t + t') |\lambda|^\alpha} |\lambda|^{n-m\kappa} d\lambda, \tag{7.23}
\]

since \( \{ l \in \mathbb{N} | m\kappa \leq l\kappa < n \} \) is a finite set and \( \lim_{\varepsilon \to 0} (N(\varepsilon))^{-1} \varepsilon^{\frac{k\kappa}{n}} L^k(\varepsilon^{\frac{1}{n}}) = 0 \) for \( k \in \{ l \in \mathbb{N} | m\kappa \leq l\kappa < n \} \) except for the term \( k = m \).
For $k : k\kappa > n$, by (7.21) and (3.15)

$$
\lim_{\varepsilon \to 0} \sum_{k : k\kappa > n} \text{Cov}(s^\varepsilon_k(t, x), s^\varepsilon_k(t', x')) = (7.24)
$$

since by (3.15) $f^{\varepsilon k}(0)$ is bounded by $f^{\varepsilon \tilde{k}}(0)$ with

$$
\tilde{k} = \inf\{ l \in \mathbb{N} | l\kappa > n \}
$$

so we set $M := f^{\varepsilon \tilde{k}}(0)$.

The proof of Proposition 4 (1) is completed by combining (7.20), (7.23) and (7.24) to obtain

$$
\lim_{\varepsilon \to 0} \text{Cov}(s^\varepsilon(t, x), s^\varepsilon(t', x')) = (C_m(h))^2 K(n, m\kappa) \int_{\mathbb{R}^n} e^{i<x-x', \lambda>-\mu(t+t')|\lambda|^\alpha} d\lambda.
$$

(2) This is derived from (1) by the same way as that in the proof of [3, Theorems 2.2 and 2.3], and thus we omit it.

**Proof of Theorem 3 for the case (1): $m_2\kappa_2 > m_1\kappa_1$ and $d_1 > d_2$.**

From (7.12), we have

$$
]\left[ e^{\frac{m_1\kappa_1}{\alpha}L_{m_1}(\varepsilon^{-\frac{1}{\alpha}})} \right]^{-\frac{1}{2}} e^{-d_1\frac{t}{\varepsilon}} \left\{ \left( \frac{u(t, \varepsilon, x)}{v(t, \varepsilon, x)} \right) - 2^{C_0^{(1)}} \right\} = e^{-d_1\frac{t}{\varepsilon}Q(t ; d_1, d_2)[e^{\frac{m_1\kappa_1}{\alpha}L_{m_1}(\varepsilon^{-\frac{1}{\alpha}})}]^{-\frac{1}{2}} \left( \frac{U(t, \varepsilon, x)}{V(t, \varepsilon, x)} - C_0^{(1)} \right)} \right.
$$

(7.25)

Firstly, by Proposition 4 (2), we have

$$
U_\varepsilon(t, x) := \left[ e^{\frac{m_1\kappa_1}{\alpha}L_{m_1}(\varepsilon^{-\frac{1}{\alpha}})} \right]^{-\frac{1}{2}} \left\{ U(t, \varepsilon, x) - C_0^{(1)} \right\} \implies X^{(1)}_{m_1}(t, x),
$$

(7.26)

where $X^{(1)}_{m_1}$ is defined in (5.3).

Secondly, by Proposition 4 (1), we have

$$
V_\varepsilon(t, x) := \left[ e^{\frac{m_1\kappa_1}{\alpha}L_{m_1}(\varepsilon^{-\frac{1}{\alpha}})} \right]^{-\frac{1}{2}} \left\{ V(t, \varepsilon, x) - C_0^{(2)} \right\} \overset{P}{\to} 0,
$$

(7.27)
since we can apply Chebyshev inequality to observe that, for any $c > 0$, as $\varepsilon \to 0$,

$$P(|V_\varepsilon(t,x)| > c) \leq c^{-2}\text{Var}(V_\varepsilon(t,x)) \times c^{-2}[\varepsilon^{-\frac{m_1}{\kappa_1}}L^{-m_1}(\varepsilon^{-\frac{1}{\alpha}})] \cdot [\varepsilon^{\frac{m_2}{\kappa_2}}L^{m_2}(\varepsilon^{-\frac{1}{\alpha}})] \to 0.$$  

Meanwhile, since $d_1 > d_2$,

$$\lim_{\varepsilon \to 0} Q_\varepsilon(t) = \lim_{\varepsilon \to 0} e^{-d_1 \varepsilon^\frac{1}{\alpha}} P\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} p_{11}p_{22} - p_{11}p_{12} \\ p_{21}p_{22} - p_{12}p_{21} \end{pmatrix}.$$  

Therefore, by the independence between $U_\varepsilon(t,x)$ and $V_\varepsilon(t,x)$ we may apply Lemma 1 to obtain

$$\begin{align*}
&\left[\varepsilon^{\frac{m_1}{\kappa_1}}L^{m_1}(\varepsilon^{-\frac{1}{\alpha}})\right]^{-\frac{1}{2}} e^{-d_1 \varepsilon^\frac{1}{\alpha}} \begin{pmatrix} u(t, x) \\ v(t, x) \end{pmatrix} - Q\left(\varepsilon^{\frac{1}{\alpha}}; d_1, d_2\right) \begin{pmatrix} C_0^{(1)} \\ C_0^{(2)} \end{pmatrix} \\
&\Rightarrow \begin{pmatrix} p_{11}p_{22} - p_{11}p_{12} \\ p_{21}p_{22} - p_{12}p_{21} \end{pmatrix} \begin{pmatrix} X_{m_1}^{(1)}(t, x) \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} p_{11}p_{22}X_{m_1}^{(1)}(t, x) \\ p_{21}p_{22}X_{m_1}^{(1)}(t, x) \end{pmatrix}, \quad t > 0, \ x \in \mathbb{R}^n.
\end{align*}$$

**Proof of Theorem 3 for the case (2):** $m_2 \kappa_2 < m_1 \kappa_1$ and $d_1 > d_2$.

We use the same scheme as in the proof of the case (1) with the roles of $U_\varepsilon(t,x)$ and $V_\varepsilon(t,x)$ being replaced as follows

$$U_\varepsilon(t,x) := \left[\varepsilon^{\frac{m_2}{\kappa_2}}L^{m_2}(\varepsilon^{-\frac{1}{\alpha}})\right]^{-\frac{1}{2}} \begin{pmatrix} u(t, x) \\ v(t, x) \end{pmatrix} - C_0^{(1)} \to 0.$$  

and

$$V_\varepsilon(t,x) := \left[\varepsilon^{\frac{m_2}{\kappa_2}}L^{m_2}(\varepsilon^{-\frac{1}{\alpha}})\right]^{-\frac{1}{2}} \begin{pmatrix} v(t, x) \\ -u(t, x) \end{pmatrix} - C_0^{(2)} \Rightarrow X_{m_2}^{(2)}(t, x),$$

where $X_{m_2}^{(2)}$ is defined in (5.4). Additionally, in this case the limit of the matrix $Q_\varepsilon(t)$
coincides with (7.28) so

\[
\begin{align*}
\varepsilon_{m}^{2} \kappa_{2} \alpha L_{m}^{2} \left( \varepsilon - 1 \right) - 2 e^{d_{1} t} \varepsilon \left\{ u\left( t^{\varepsilon}, x^{\varepsilon} \right), v\left( t^{\varepsilon}, x^{\varepsilon} \right) \right\} - Q\left( t^{\varepsilon}; d_{1}, d_{2} \right) \left( C_{0}^{(1)} C_{0}^{(2)} \right) \\
= \begin{pmatrix}
  p_{11} p_{22} & -p_{11} p_{12} \\
  p_{21} p_{22} & -p_{12} p_{21}
\end{pmatrix}
\begin{pmatrix}
  X_{m_{2}}^{(1)}(t, x) \\
  X_{m_{2}}^{(2)}(t, x)
\end{pmatrix}, \quad t > 0, \ x \in \mathbb{R}^{n}.
\end{align*}
\]

Proof of Theorem 3 for the case (3): \( m_{1} = m_{2} = m, \ \kappa_{1} = \kappa_{2} = \kappa \) and \( d_{1} > d_{2} \).

By Proposition 4 (2), we have

\[
U_{\varepsilon}(t, x) := \left[ \varepsilon^{\frac{m_{1}}{m_{2}}} L_{m}^{2}(\varepsilon^{-\frac{1}{m}}) \right]^{-\frac{1}{2}} \left\{ U\left( t^{\varepsilon}, x^{\varepsilon} \right) - C_{0}^{(1)} \right\} \implies X_{m_{2}}^{(1)}(t, x), \quad (7.31)
\]

and

\[
V_{\varepsilon}(t, x) := \left[ \varepsilon^{\frac{m_{1}}{m_{2}}} L_{m}^{2}(\varepsilon^{-\frac{1}{m}}) \right]^{-\frac{1}{2}} \left\{ V\left( t^{\varepsilon}, x^{\varepsilon} \right) - C_{0}^{(2)} \right\} \implies X_{m_{2}}^{(2)}(t, x), \quad (7.32)
\]

where \( X_{m_{2}}^{(1)} \) and \( X_{m_{2}}^{(2)} \) is defined in (5.3) and (5.4) with \( m_{1} = m_{2} = m \).

Additionally, in this case the matrix \( Q_{\varepsilon}(t) \) is also unchanged so by applying Lemma 1 to \( U_{\varepsilon}(t, x), V_{\varepsilon}(t, x) \) and \( Q_{\varepsilon}(t) \) which are given in (7.31), (7.32) and (7.28), respectively, we see that the finite dimensional distributions of the rescaled random field

\[
\begin{align*}
\left[ \varepsilon^{\frac{m_{1}}{m_{2}}} L_{m}^{2}(\varepsilon^{-\frac{1}{m}}) \right]^{-\frac{1}{2}} e^{-d_{1} t} \left\{ u\left( t^{\varepsilon}, x^{\varepsilon} \right), v\left( t^{\varepsilon}, x^{\varepsilon} \right) \right\} - Q\left( t^{\varepsilon}; d_{1}, d_{2} \right) \left( C_{0}^{(1)} C_{0}^{(2)} \right) \\
= \begin{pmatrix}
  p_{11} p_{22} - p_{11} p_{12} \\
  p_{21} p_{22} - p_{12} p_{21}
\end{pmatrix}
\begin{pmatrix}
  X_{m_{2}}^{(1)}(t, x) \\
  X_{m_{2}}^{(2)}(t, x)
\end{pmatrix}, \quad t > 0, \ x \in \mathbb{R}^{n}.
\end{align*}
\]

Proofs of Theorems 4 and 5.

The proofs can be proceeded parallel to the proofs of Theorem 3 and 2, respectively, and thus we leave them to the reader.
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