ASYMPTOTIC PLATEAU PROBLEM FOR TWO CONTOURS

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Abstract. Let $\Gamma_1$ and $\Gamma_2$ be two disjoint rectifiable star-shaped Jordan curves in the asymptotic boundary $\partial_{\infty} \mathbb{H}^3$ of the hyperbolic space $\mathbb{H}^3$. If the distance between $\Gamma_1$ and $\Gamma_2$ are bounded above by a constant, then there exists an area minimizing annulus $\Pi \subset \mathbb{H}^3$, which is asymptotic to $\Gamma_1 \cup \Gamma_2$.

The main results of this paper are Theorem 1.7 and Theorem 1.11.

1. Introduction

In this paper we study the asymptotic Plateau problem in hyperbolic 3-space $\mathbb{H}^3$ when the prescribed boundary data consists of two disjoint Jordan curves at infinity. There are several models for $\mathbb{H}^3$, among which we shall use the Poincaré ball model and the upper half-space model.

The Poincaré ball model of $\mathbb{H}^3$ is the open unit ball

$$\mathbb{B}^3 = \{(u, v, w) \in \mathbb{R}^3 \mid u^2 + v^2 + w^2 < 1\}$$

(1.1)
equipped with the hyperbolic metric $ds^2 = 4(du^2 + dv^2 + dw^2)/(1 - r^2)^2$, where $r = \sqrt{u^2 + v^2 + w^2}$. The orientation preserving isometry group of $\mathbb{B}^3$ is denoted by $\text{Möb}(\mathbb{B}^3)$, which consists of Möbius transformations that preserve the unit ball (see [MT98 Theorem 1.7]). The hyperbolic 3-space $\mathbb{B}^3$ has a natural compactification: $\overline{\mathbb{B}^3} = \mathbb{B}^3 \cup S^2_{\infty}$, where $S^2_{\infty} \cong \mathbb{C} \cup \{\infty\}$ is called the asymptotic boundary of $\mathbb{B}^3$ or the idea boundary of $\mathbb{B}^3$ at infinity. Suppose that $X$ is a subset of $\mathbb{B}^3$, we define the asymptotic boundary of $X$ by $\partial_{\infty} X = X \cap S^2_{\infty}$, where $X$ is the closure of $X$ in $\overline{\mathbb{B}^3}$. Obviously we have $\partial_{\infty} \mathbb{B}^3 = S^2_{\infty}$. If $P$ is a geodesic plane in $\mathbb{B}^3$, then $P$ is perpendicular to $S^2_{\infty}$ and $C \overset{\text{def}}{=} \partial_{\infty} P$ is an Euclidean round circle in $S^2_{\infty}$. We also say that $P$ is asymptotic to $C$.

The upper half space model of $\mathbb{H}^3$ is the upper half space

$$\mathbb{U}^3 = \{z + tj \mid z \in \mathbb{C} \text{ and } t > 0\}$$

(1.2)
equipped with the hyperbolic metric $ds^2 = (|dz|^2 + dt^2)/t^2$, where $z = x + iy$ for $x, y \in \mathbb{R}$. The orientation preserving isometry group of $\mathbb{U}^3$ is denoted by $\text{PSL}_2(\mathbb{C})$, which consists of linear fractional transformations. It’s well known that $\text{Möb}(\mathbb{B}^3) \cong \text{PSL}_2(\mathbb{C})$. The asymptotic boundary of $\mathbb{U}^3$ is $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\} \cong S^2_{\infty}$.

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For a collection of disjoint Jordan curves \( \Gamma = \{ \Gamma_1, \ldots, \Gamma_k \} \) in \( S^2_{\infty} \), where \( k \geq 1 \), the asymptotic Plateau problem in \( \mathbb{H}^3 \) asks the existence of an (absolutely) area minimizing surfaces \( \Sigma \subset \mathbb{H}^3 \) asymptotic to \( \Gamma \), that is, \( \partial_{\infty} \Sigma = \Gamma_1 \cup \cdots \cup \Gamma_k \). The asymptotic Plateau problem was first studied by Anderson [And82, And83] in hyperbolic \( n \)-space (\( n \geq 3 \)) for arbitrary codimensions in the case when \( k = 1 \).

In particular, using methods in geometric measure theory, Anderson showed that there exists a complete embedded area minimizing plane asymptotic to a given Jordan curve in \( S^2_{\infty} \).

**Theorem 1.1** ([And83, Theorem 4.1]). For any Jordan curve \( \Gamma \) in \( S^2_{\infty} \), there exists a complete embedded disk-type area minimizing surface \( \Sigma \subset \mathbb{H}^3 \), which is asymptotic to \( \Gamma \).

By the interior regularity results of geometric measure theory (see [Fed69]), the complete area minimizing disk \( \Sigma \) in Theorem 1.1 is smooth. Moreover if \( \Sigma \) is an (absolutely) area minimizing surface asymptotic to a \( C^{1,\alpha} \) Jordan curve \( \Gamma \) in \( S^2_{\infty} \), then \( \Sigma \) is \( C^{1,\alpha} \) at infinity (see [HL87]); the higher boundary regularity of \( \Sigma \) at infinity was studied by Lin in [Lin89b, Lin12]. Moreover, the asymptotic behavior of area-minimizing currents in hyperbolic space with higher codimensions was studied by Lin in [Lin89a]. The reader can read the survey [Cos14] for other topics on asymptotic Plateau problem.

In this paper we shall study the asymptotic Plateau problem in \( \mathbb{H}^3 \) when \( \Gamma = \{ \Gamma_1, \ldots, \Gamma_k \} \subset S^2_{\infty} \) consists of two components.

**Definition 1.2.** If \( C_1 \) and \( C_2 \) are two disjoint round circles in \( \partial_{\infty} \mathbb{H}^3 = S^2_{\infty} \), we define the distance between \( C_1 \) and \( C_2 \) as follows
\[
d(C_1, C_2) = \text{dist}(P_1, P_2),
\]
where \( \text{dist}(\cdot, \cdot) \) is the hyperbolic distance of \( \mathbb{H}^3 \) and \( P_i \) is the totally geodesic plane asymptotic to \( C_i \) for \( i = 1, 2 \).

**Remark 1.3.** In the remaining part of the paper, when we say circles, we mean round circles; otherwise we shall say simple closed curves or Jordan curves.

The following theorem of Gomes (see [Gom87, Proposition 3.2] or [Wan19, Theorem 3.2]) partially solved the asymptotic Plateau problem in \( \mathbb{H}^3 \) when two given disjoint Jordan curves in \( S^2_{\infty} \) are circles.

**Theorem 1.4** (Gomes). Let \( a_c \approx 0.49577 \) be the (unique) critical number of the function \( \varrho \) defined by \( (2.4) \). For two disjoint circles \( C_1, C_2 \subset S^2_{\infty} \), if
\[
d(C_1, C_2) \leq 2\varrho(a_c) \approx 1.00229,
\]
then there exists a minimal surface of revolution, that is, a spherical catenoid in \( \mathbb{H}^3 \), which is asymptotic to \( C_1 \cup C_2 \).
The above theorem of Gomes can’t determine whether the catenoid is area minimizing. Actually we even don’t know if it is globally stable. According to §2.2 any spherical catenoid $C$ in $B^3$ can be determined uniquely up to isometry by the distance from $C$ to its rotation axis. Let $C_a$ denote the spherical catenoid in $B^3$ which has distance $a$ from itself to its rotation axis (see Definition 2.6). The following theorem can determine the stability of spherical catenoids according to the distances from catenoids to their rotation axes (see [BSE10, Proposition 4.10] or [Wan19, Theorem 1.2]).

**Theorem 1.5** (Bérard and Sa Earp). Let $a_c \approx 0.49577$ be the (unique) critical number of the function $\varrho$ defined by (2.4).

1. $C_a$ is unstable if $0 < a < a_c$, and
2. $C_a$ is globally stable if $a \geq a_c$.

The following theorem is an equivalent form of Theorem 1.5.

**Theorem 1.5′.** Let $C_1$ and $C_2$ be disjoint round circles in $S^2_\infty$. 

1. If $d(C_1, C_2) = 2\varrho(a_c)$, there exists exactly one globally stable catenoid in $H^3$ asymptotic to $C_1 \cup C_2$.
2. If $0 < d(C_1, C_2) < 2\varrho(a_c)$, there exist two catenoids in $H^3$ asymptotic to $C_1 \cup C_2$ such that one is unstable and the other one is globally stable.

We shall explain why there are two spherical catenoids in $H^3$ asymptotic to the disjoint round circles $C_1$ and $C_2$ in $S^2_\infty$ if $d(C_1, C_2) < 2\varrho(a_c)$. Suppose that we have the family of catenoids $\{C_a\}_{a>0}$ in $B^3$ such that their rotation axes are the same $u$-axis and they are all symmetric about the $vw$-plane (see (1.1)). According to the arguments in [Wan19, pp. 357–359], the function $\varrho(a)$ is increasing on $(0, a_c)$ and decreasing on $(a_c, \infty)$, and achieves its maximum value at $a = a_c$ (see Figure 3). If $d(C_1, C_2) < 2\varrho(a_c)$, there exist exactly two positive constants $a' < a_c < a''$ such that $d(C_1, C_2) = 2\varrho(a') = 2\varrho(a'')$, that is to say, both $C_{a'}$ and $C_{a''}$ are asymptotic to $C_1$ and $C_2$.

Recall that a stable minimal surface is locally area minimizing, so it could be area minimizing. In the above theorem of Bérard and Sa Earp, we still don’t know whether a stable catenoid is area minimizing. The following theorem (see also [Wan19, Theorem 1.5]) can determine some area minimizing catenoids among the stable ones. As in [Wan19, (4.2) and (4.3)], we define

$$a_l = \cosh^{-1} \left(1/(1 - K)\right) \approx 1.10055 ,$$

where the constant $K$ is defined by

$$K = \int_0^1 \frac{1}{x^2} \left(\frac{1}{\sqrt{1 - x^4}} - 1\right) dx \approx 0.40093 .$$

**Theorem 1.6** (Wang). Each catenoid $C_a$ is area minimizing if $a \geq a_l$. 
According to the proof of above theorem in [Wan19], it seems that there are still some area minimizing catenoids $C_a$ when $a_c \leq a < a_l$. The following theorem is our first main result, which can determine all of the (absolutely) area minimizing spherical catenoids among the globally stable ones.

**Theorem 1.7.** Let $a_L \approx 0.847486$ be a constant given by Theorem 3.2.

1. If $a_c \leq a < a_L$, then each spherical catenoid $C_a$ is globally stable but not (absolutely) area minimizing.
2. If $a \geq a_L$, then each spherical catenoid $C_a$ is (absolutely) area minimizing among all surfaces asymptotic to $\partial_\infty C_a$.

Moreover, any (absolutely) area minimizing surface asymptotic to two disjoint round circles in $S_2^\infty$ is one of the elements in $\{C_a\}_{a \geq a_L}$ up to isometry.

The following theorem is an equivalent form of Theorem 1.7.

**Theorem 1.7'.** Let $C_1$ and $C_2$ be disjoint round circles in $S_2^\infty$.

1. If $d(C_1, C_2) = 2 \rho(a_c)$, there exists exactly one globally stable spherical catenoid, which is not (absolutely) area minimizing.
2. If $2 \rho(a_L) < d(C_1, C_2) < 2 \rho(a_c)$, there exist two spherical catenoids such that one is unstable and the other one is globally stable. But the stable catenoid is not (absolutely) area minimizing.
3. If $0 < d(C_1, C_2) \leq 2 \rho(a_L)$, there exist two spherical catenoids such that one is unstable and the other one is (absolutely) area minimizing.

Moreover, if there is any (absolutely) area minimizing surface asymptotic to $C_1 \cup C_2$ in $S_2^\infty$, then it’s a spherical catenoid and the distance \[1.3\] between $C_1$ and $C_2$ is $\leq 2 \rho(a_L)$.

Applying the results in Theorem 1.7 or Theorem 1.7', we can say that we have solved the special case of asymptotic Plateau problem when the prescribed boundary data consists of two disjoint round circles.

Next we will generalize the above results to the case when the prescribed boundary data consists of two disjoint rectifiable star-shaped Jordan curves.

**Definition 1.8** (Distance between two Jordan curves). Let $\Gamma_1$ and $\Gamma_2$ be two disjoint Jordan curves in $S_\infty^2$. For $i = 1, 2$, let $\Delta_i$ be the disk component of $S_\infty^2 \setminus (\Gamma_1 \cup \Gamma_2)$ bounded by $\Gamma_i$, and let $C_i$ be any round circle contained in $\Delta_i$. The distance between $\Gamma_1$ and $\Gamma_2$ is defined as follows:

$$d(\Gamma_1, \Gamma_2) = \inf \{d(C_1, C_2) \mid C_i \subset \Delta_i \text{ for } i = 1, 2\},$$

\[1.7\]

where $d(C_1, C_2)$ is given by \[1.3\] for two disjoint circles $C_1$ and $C_2$.

**Remark 1.9.** In [CGT86], p.430, the authors also defined the distance between Jordan curves in $S_\infty^2$, which is different from \[1.7\].
A set $\Omega$ in the plane $\mathbb{R}^2$ is called a star-shaped domain if there exists a point $x_0$ in $\Omega$ such that for each point $x$ in $\Omega$ the line segment from $x_0$ to $x$ is contained in $\Omega$. The point $x_0$ is called a center of the domain $\Omega$. A star-shaped domain in the plane may have more than one center. In particular, any interior point of a convex domain in the plane is its center.

**Definition 1.10** (Star-shaped Jordan curve). A Jordan curve $\Gamma \subset S^2_{\infty}$ is called star-shaped if two components $\Omega_{\pm}$ of $S^2_{\infty} \setminus \Gamma$ are all star-shaped domains. More precisely, there exist points $p_+ \in \Omega_+$ and $p_- \in \Omega_-$, a geodesic $\ell \subset H^3$ connecting $p_+$ and $p_-$ (i.e., $\partial_\infty \ell = \{p_+, p_-\}$), and an isometry $\phi : H^3 \to U^3$ such that the following conditions are satisfied:

1. $\phi(\ell)$ is the $t$-axis of the upper-half space $U^3$ (see (1.2)) with $\phi(p_+) = 0$ and $\phi(p_-) = \infty$.
2. $\phi(\Omega_+) \subset \hat{\mathbb{C}}$ is a star-shaped domain whose center is the origin, and $\phi(\Omega_-) \subset \hat{\mathbb{C}}$ is a star-shaped domain whose center is at infinity, that is, for any point $x \in \phi(\Omega_-)$, the portion of the ray passing through $x$ from the origin that starts at $x$ is contained in the domain $\phi(\Omega_-)$.

The geodesic $\ell$ is called an axis of $\Gamma$.

A Jordan curve $\Gamma \subset S^2_{\infty}$ is rectifiable if its length is finite with respect to the spherical metric on $S^2_{\infty}$. Our second main result can be stated as follows:

**Theorem 1.11.** Let $\Gamma_1$ and $\Gamma_2$ be disjoint rectifiable star-shaped Jordan curves in $S^2_{\infty}$. If the distance between $\Gamma_1$ and $\Gamma_2$ is bounded from above as follows

$$d(\Gamma_1, \Gamma_2) < 2\varrho(a_L) \approx 0.876895,$$

where $\varrho$ is the function defined by (2.4) and $a_L \approx 0.847486$ is the constant given by Theorem 3.2, then there exists an embedded annulus-type area minimizing surface $\Pi \subset H^3$, which is asymptotic to $\Gamma_1 \cup \Gamma_2$.

Moreover, the upper bound in (1.8) is optimal in the following sense: If there is an area minimizing surface in $H^3$ asymptotic to two disjoint round circles in $S^2_{\infty}$, then the distance (1.3) between the circles is $\leq 2\varrho(a_L)$.

**Remark 1.12.** The results of boundary regularity in [HL87, Lin89b, Lin12] also work for Theorem 1.11.

**Remark 1.13.** Coskunuzer proved the following result (see Step 1 in the proof of the Key Lemma in [Cos09]): Let $\Gamma$ be a Jordan curve in $S^2_{\infty}$ with at least one $C^1$-smooth point. If $\Gamma^\pm \subset S^2_{\infty}$ are two Jordan curves in opposite sides of $\Gamma$ and sufficiently close to $\Gamma$, then there exists an area minimizing annulus asymptotic to $\Gamma^+ \cup \Gamma^-$. 




**Organization of the paper.** In §2 we review some definitions on minimal surfaces and catenoids. In §3 we shall prove Theorem 3.2 at first, which is crucial for proving Theorem 1.7 in the same section. In §4 we shall prove several results before we prove Theorem 1.11 in the same section.

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2. **Preliminaries**

2.1. **Review of minimal surfaces.** Suppose that $\Sigma$ is a surface immersed in a complete Riemannian 3-manifold $M^3$. We pick up a local orthonormal frame field $\{e_1, e_2, e_3\}$ for $M^3$ such that, restricted to $\Sigma$, the vectors $\{e_1, e_2\}$ are tangent to $\Sigma$ and the vector $e_3$ is perpendicular to $\Sigma$. Let $A = (h_{ij})_{2 \times 2}$ be the second fundamental form of $\Sigma$, whose entries $h_{ij}$ are defined by $h_{ij} = \langle \nabla e_i e_3, e_j \rangle$ for $i, j = 1, 2$, where $\nabla$ is the covariant derivative in $M^3$, and $\langle \cdot, \cdot \rangle$ is the metric of $M^3$. The immersed surface $\Sigma$ is called a minimal surface in $M^3$ if its mean curvature $H = h_{11} + h_{22}$ is identically equal to zero.

**Definition 2.1** (Area minimizing disk). A compact disk-type minimal surface $\Sigma$ in $\mathbb{H}^3$ is called an area minimizing surface in $\mathbb{H}^3$ if $\Sigma$ has least area among the compact disks in $\mathbb{H}^3$ which are homotopic to $\Sigma$ rel $\partial \Sigma$.

A noncompact complete disk-type surface $\Sigma$ is called an area minimizing surface in $\mathbb{H}^3$ if any compact disk-type sub-domain of $\Sigma$ is an area minimizing surface in $\mathbb{H}^3$.

**Definition 2.2** (Area minimizing annulus). Let $S$ be a compact annulus-type minimal surface immersed in $\mathbb{H}^3$, whose boundary consists of two disjoint Jordan curves $C_1, C_2$, and let $D_1, D_2$ be two area minimizing disks spanning $C_1, C_2$ respectively. The annulus $S$ is called an area minimizing surface in $\mathbb{H}^3$ if

1. $\text{area}(S) < \text{area}(D_1) + \text{area}(D_2)$, and
2. $\text{area}(S) \leq \text{area}(S')$ for each annulus $S'$ homotopic to $S$ rel $\partial S$,

where $\text{area}(\cdot)$ denotes the area of the surfaces in $\mathbb{H}^3$.

A noncompact complete minimal annulus $\Pi \subset \mathbb{H}^3$ whose asymptotic boundary consists of the union of two disjoint Jordan curves $\Gamma_1$ and $\Gamma_2$ in $S_\infty^2$ is called an area minimizing surface if any compact annulus-type subdomain of $\Pi$, which is homotopically equivalent to $\Pi$, is a compact area minimizing annulus.
Remark 2.3. Condition (1) in Definition 2.2 is necessary (see [Dou31, AS79, MY82a]). See also the proof of Theorem 1.7.

Definition 2.4 (Absolutely area minimizing surface). Suppose $S \subset \mathbb{H}^3$ is a compact surface with boundary. The surface $S$ is called absolutely area minimizing if $S$ has least area among all compact surfaces with the same boundary, where these surfaces are not necessarily homotopic to $S$ rel $\partial S$.

A noncompact complete surface $\Sigma$ in $\mathbb{H}^3$ is called an absolutely area minimizing surface provided each compact portion of it is absolutely area minimizing.

Notion Convention 2.5. In the literature, an area minimizing surface is also called a homotopically area minimizing surface, and an absolutely area minimizing surface is also called a homologically area minimizing surface.

2.2. Spherical catenoids in $\mathbb{H}^3$. In this subsection, we shall review some very basic properties of catenoids defined in the Poincaré ball model of $\mathbb{H}^3$. See also [Mor81, Hsi82, dCD83, Gom87, BSE09, BSE10, Wan19].

Let $G \cong SO(2)$ be a subgroup of $\text{M"{o}b}(\mathbb{B}^3)$ that leaves a geodesic $\gamma_0 \subset \mathbb{B}^3$ pointwise fixed. We call $G$ the spherical group of $\mathbb{B}^3$ and $\gamma_0$ the rotation axis of $G$. For two round circles $C_1$ and $C_2$ in $\mathbb{B}^3$, if there is a geodesic $\gamma_0$, such that both $C_1$ and $C_2$ are invariant under the spherical group that fixes $\gamma_0$ pointwise, then $C_1$ and $C_2$ are said to be coaxial, and $\gamma_0$ is called the rotation axis of $C_1$ and $C_2$. It’s well known that any two disjoint round circles $C_1$ and $C_2$ in $S^2_\infty$ are always coaxial (see [Wan19]).

Suppose that $G$ is the spherical group of $\mathbb{B}^3$ associated with the geodesic $\gamma_0 = \{(u, 0, 0) \in \mathbb{B}^3 \mid -1 < u < 1\}$, then we have $\mathbb{B}^3/G \cong \mathbb{B}^2_+$, where $\mathbb{B}^2_+ := \{(u, v) \in \mathbb{B}^2 \subset \mathbb{B}^3 \mid v \geq 0\}$ is still considered as a subset of $\mathbb{B}^3$. For any point $p = (u, v) \in \mathbb{B}^2_+$, there is a unique geodesic segment $\gamma'$ through $p$ which is perpendicular to $\gamma_0$ at $q$. Set $x = \text{dist}(O, q)$ and $y = \text{dist}(p, q) = \text{dist}(p, \gamma_0)$ (see Figure 1). It’s well known that $\mathbb{B}^2_+$ can be equipped with the metric of warped product in terms of the parameters $x$ and $y$, that is, $ds^2 = \cosh^2 y \cdot dx^2 + dy^2$, where $dx$ represents the hyperbolic metric on the geodesic $\gamma_0$ defined in (2.1).

If $\mathcal{C}$ is a minimal surface of revolution in $\mathbb{B}^3$ with respect to the axis $\gamma_0$, where $\gamma_0$ is defined by (2.1), then it is called a catenoid and the curve $\sigma = \mathcal{C} \cap \mathbb{B}^2_+$ is called the generating curve or a catenary of $\mathcal{C}$. Let $\sigma \subset \mathbb{B}^2_+$ be the generating curve of a minimal catenoid $\mathcal{C}$. Suppose that the parametric equations of $\sigma$ are given by: $x = x(s)$ and $y = y(s)$, where $s \in (-\infty, \infty)$ is an arc length parameter of $\sigma$. By the arguments in [Hsi82, pp. 486–488], the curve $\sigma$ satisfies the
following equations
\[
\frac{2\pi \sinh y \cdot \cosh^2 y}{\cosh^2 y + (y')^2} = 2\pi \sinh y \cdot \cosh y \cdot \sin \theta = k \text{ (constant)}, \quad (2.2)
\]
where \(y' = dy/dx\) and \(\theta\) is the angle between the tangent vector of \(\sigma\) and the vector \(e_y = \partial/\partial y\) at the point \((x(s), y(s))\) (see Figure 2).

By the arguments in [Gom87, pp.54–58], up to isometry, we can assume that the curve \(\sigma\) is only symmetric about the \(v\)-axis and intersects the \(v\)-axis orthogonally at \(y_0 = y(0)\), and so \(y'(0) = 0\). Now we solve for \(dx/dy\) in terms of \(y\) in (2.2) and integrate \(dx/dy\) from \(y_0\) to \(y\) for any \(y \geq y_0\) (see [Hsi82] or [Wan19]), then we have the following equality
\[
x(y) = \int_{y_0}^{y} \frac{\sinh(2y_0)}{\cosh t} \frac{dt}{\sqrt{\sinh^2(2t) - \sinh^2(2y_0)}}. \quad (2.3)
\]

The limit of the function \(x(y)\) defined by (2.3) is finite as \(y \to \infty\) for any fixed \(y_0 > 0\). Replacing the initial data \(y_0\) by a parameter \(a \in (0, \infty)\) in (2.3), we can define a function \(\varrho(a)\) of the parameter \(a\) as follows (see Figure 3):
\[
\varrho(a) = \int_{a}^{\infty} \frac{\sinh(2a)}{\cosh t} \frac{dt}{\sqrt{\sinh^2(2t) - \sinh^2(2a)}}. \quad (2.4)
\]
The function (2.4) has a unique critical value \(a_c \approx 0.49577\) so that \(\varrho(a) < \varrho(a_c)\) for all \(a \in (0, a_c) \cup (a_c, \infty)\) by [Wan19, Lemma 3.3].

Let \(\sigma_a \subset \mathbb{B}^2_+\) be the catenary defined by (2.3) for \(y_0 = a\), which is symmetric about the \(v\)-axis and whose initial data is \(a\) (actually the hyperbolic distance between \(\sigma_a\) and the origin of \(\mathbb{B}^2_+\) is equal to \(a\)).

**Definition 2.6.** For \(0 < a < \infty\), the surface of revolution around the axis \(\gamma_0\) in (2.1) generated by the catenary \(\sigma_a\) is called a *catenoid*, which is denoted by \(C_a\).
Notion Convention 2.7. In this paper, let $\mathcal{S}$ denote the set of the spherical catenoids in $\mathbb{B}^3$ that have the same rotation axis $\gamma_0$ (see (2.1)) and the same symmetric plane $P_0 = \{(u, v, w) \in \mathbb{R}^3 \mid v^2 + w^2 < 1 \text{ and } u = 0\}$. Let

$$T_c := \cup \{C_a \in \mathcal{S} : a \geq a_c\} \quad (2.5)$$

be a subregion of $\mathbb{B}^3$. The region $T_c$ is foliated by the spherical catenoids $C_a$ in $S$ for all $a \geq a_c$ according to the three dimensional version of the first statement in [BSE10, Proposition 4.8].

![Figure 3. The graph of the function $\varrho(a)$ defined by (2.4) for $a \in [0, 3]$. This function $\varrho(a)$ has a unique critical number.](image)

3. Existence of Area Minimizing Spherical Catenoids in $\mathbb{H}^3$

In this section we shall prove Theorem 1.7 in §3.3.

3.1. Area difference between catenoids and geodesic planes. Let $C_a \in \mathcal{S}$ be a spherical catenoid in $\mathbb{B}^3$, whose rotation axis is $\gamma_0$. For any $r \geq a$, let

$$\Sigma_{a,r} = C_a \cap \mathcal{N}_r(\gamma_0), \quad (3.1)$$

where $\mathcal{N}_r(\gamma_0)$ denotes the $r$-neighborhood of $\gamma_0$ in $\mathbb{B}^3$. Then $\Sigma_{a,r}$ is a compact surface of revolution whose boundary consists of two round circles $C^\pm$, which are invariant under the spherical group of the rotation axis $\gamma_0$. The area of $\Sigma_{a,r}$ can be calculated by coarea formula (see [Wan19, (4.8)])

$$\text{area}(\Sigma_{a,r}) = \int_a^r \left(4\pi \sinh t \cdot \frac{\sinh(2t)}{\sqrt{\sinh^2(2t) - \sinh^2(2a)}}\right) dt. \quad (3.2)$$

Let $\Delta^\pm_r \subset \mathbb{B}^3$ be the totally geodesic disks bounded by $C^\pm$ respectively, then the area of $\Delta^\pm_r$ is given by $\text{area}(\Delta^+_r) = \text{area}(\Delta^-_r) = 2\pi (\cosh r - 1)$ (see [Bea95, Theorem 7.2.2]) since the radii of $\Delta^\pm_r$ are both equal to $r$.

**Lemma 3.1.** Let $\Phi(a, r) = \text{area}(\Sigma_{a,r}) - (\text{area}(\Delta^+_r) + \text{area}(\Delta^-_r))$ be the area difference, then $\Phi(\cdot, r)$ is increasing and bounded for $a \leq r < \infty$. 

Proof. Actually, the function $\Phi$ is given by

$$
\Phi(a,r) = 4\pi \int_a^r \frac{\sinh t \cdot \sinh(2t)}{\sqrt{\sinh^2(2t) - \sinh^2(2a)}} dt - 4\pi (\cosh r - 1)
$$

$$
= 4\pi \int_a^r \sinh t \cdot \left( \frac{\sinh(2t)}{\sqrt{\sinh^2(2t) - \sinh^2(2a)}} - 1 \right) dt
$$

$$
- 4\pi (\cosh a - 1)
$$

for $r \geq a$. The first integral term in the second equality is positive for all $r \geq a$, so $\Phi(\cdot, r)$ is an increasing function on $r$ for each fixed $a > 0$.

For any fixed $a > 0$, the function $\Phi(a, r)$ is bounded for any $r \geq a$. First of all, $\Phi(a, r)$ is bounded from below, since $\Phi(a, r) \geq -4\pi (\cosh a - 1)$ for $r \geq a$. On the other hand, by Lemma 4.1 and (4.8) in [Wan19], we have

$$
\Phi(a, r) \leq 4\pi \int_a^\infty \sinh t \cdot \left( \frac{\sinh(2t)}{\sqrt{\sinh^2(2t) - \sinh^2(2a)}} - 1 \right) dt
$$

$$
- 4\pi (\cosh a - 1)
$$

$$
\leq 4\pi K \cosh a - 4\pi (\cosh a - 1)
$$

for any $r \geq a$, where $K$ is defined by (1.6). □

![Figure 4](image.png)

**Figure 4.** The graph of the function $\varphi(a)$ defined by (3.3) for $a \in [0, 0.9]$.

Therefore, for any fixed $a \geq 0$, the the limit of $\Phi(a, r)$ as $r \to \infty$ is well defined. Let $\varphi(a) = \lim_{r \to \infty} \Phi(a, r)$, then $\varphi(a)$ is a function of $a$, which can be written as
follows (see Figure 4)

$$\varphi(a) = 4\pi \int_{a}^{\infty} \sinh t \cdot \left( \frac{\sinh(2t)}{\sqrt{\sinh^2(2t) - \sinh^2(2a)}} - 1 \right) dt$$

$$- 4\pi (\cosh a - 1).$$

(3.3)

**Theorem 3.2.** The function $\varphi(a)$ defined by (3.3), $a \geq 0$, has a unique zero $a_L > a_c > 0$ such that the following statements are true.

1. If $0 < a < a_L$, then $\varphi(a) > 0$.
2. If $a > a_L$, then $\varphi(a) < 0$.

**Remark 3.3.** By numerical computation, $a_L \approx 0.847486$. It seems that the critical number of the function $\varphi(a)$ is also equal to $a_c$.

**Lemma 3.4.** Let

$$f(x) = -30 \cosh(3x) - 18 \cosh(5x) + 10 \sinh(7x) + 15(1 - K) \cosh(8x)$$

be a function defined for $x \geq 0$, where $K \approx 0.40093$ is defined by (1.6). Then $f(x)$ is increasing on $[0, \infty)$, and has exactly one zero $x_0 \in (0, a_c)$.

**Proof.** It’s easy to verify that the following functions

$$f_1(x) = -14 \cosh(5x) + 10 \sinh(7x)$$

$$f_2(x) = -30 \cosh(3x) - 4 \cosh(5x) + 15(1 - K) \cosh(8x)$$

are increasing for $x \geq 0$, so is $f(x) = f_1(x) + f_2(x)$ for $x \geq 0$. Actually direct computation shows that $f_1'(x) = -70 \sinh(5x) + 70 \cosh(7x) > 0$ for all $x \geq 0$, where we use the inequalities $\cosh(7x) > \cosh(5x) > \sinh(5x)$ for all $x \geq 0$, so $f_1(x)$ is increasing on $[0, \infty)$. Direct computation shows that

$$f_2''(x) = -270 \cosh(3x) - 100 \cosh(5x) + 960(1 - K) \cosh(8x)$$

$$> -270 \cosh(3x) - 100 \cosh(5x) + 480 \cosh(8x)$$

is positive for all $x \geq 0$, since $\cosh(8x) > \cosh(5x) > \cosh(3x)$ for all $x > 0$. It’s easy to verify that $f_2'(0) = 0$, so $f_2'(x) > 0$ for all $x > 0$, which means that $f_2(x)$ is increasing on $[0, \infty)$.

Since $f(0) = -48 + 15(1 - K) < 0$ and

$$f(\log(3/2)) = \frac{171374697 - 215561285 \cdot K}{1119744} > 0,$$

there exists a unique zero $0 < a_0 < \log(3/2) \approx 0.405465 < a_c$ of $f(x)$. □

**Proof of Theorem 3.2.** We shall prove the theorem in four steps. Suppose that the catenoids in the family $\mathcal{S} = \{C_a\}_{a > 0}$ have the same rotation axis $\gamma_0$ and the same symmetric plane $P_0$ (see Notion Convention 2.7).

**Step 1.** $\varphi(a) > 0$ for $0 < a < a_c$. In particular, we have $\varphi(a_c) \geq 0$. 


Proof of Step 1. Otherwise, if \( \varphi(a) \leq 0 \) for any fixed \( a \in (0, a_c) \), then we have \( \Phi(a, r) < 0 \) for any \( r > a \), that is,

\[
\text{area}(\Sigma_{a, r}) < \text{area}(\Delta^+_r) + \text{area}(\Delta^-_r), \quad r > a,
\]

where \( \Sigma_{a, r} = C_a \cap \mathcal{N}_r(\gamma_0) \) and \( \Delta^+_r \) are the totally geodesic disks bounded by the round circles \( \partial \Sigma_{a, r} \). We may choose \( r \gg a \) such that \( \partial \Sigma_{a, r} \subset T_c \), since \( \partial_{\infty} C_a \) is contained in \( \partial_{\infty} T_c \) by Theorem 1.4 where \( T_c \) is defined by (2.5). By Theorem 1.5 \( C_a \) is unstable since \( a < a_c \). According to the three dimensional version of the second statement of Proposition 4.8 in [BSE10], \( \Sigma_{a, r} \) is also unstable.

Let \( \Omega \subset \mathbb{B}^3 \) be the region bounded by \( \Delta^+_r \), \( \Delta^-_r \) and \( \Sigma_{a, r} \). Since \( \Omega \) is a simply connected region whose boundary \( \partial \Omega = \Delta^+_r \cup \Delta^-_r \cup \Sigma_{a, r} \) is mean convex with respect to the inward normal vector field, together with the condition (3.4), there exists an annulus-type area minimizing surface \( \Sigma' \subset \Omega \) with \( \partial \Sigma' = \partial \Sigma_{a, r} \) according to [AS79, MY82b].

But the existence of \( \Sigma' \) is impossible. The argument is as follows. Since \( \Sigma_{a, r} \) is unstable, \( \Sigma' \) is not identical to \( \Sigma_{a, r} \). On the other hand, similar to the arguments in [Wan19, §4], we know that \( \Sigma' \) is a minimal surface of revolution about \( \gamma_0 \), and it’s also symmetric about the same plane \( P_0 \), therefore it’s a portion of some catenoid \( C_{a'} \in \mathcal{I} \), where \( a' = \text{dist}(\Sigma', \gamma_0) \). Since \( \Sigma' \subset \Omega \), we have

\[
a' = \text{dist}(\Sigma', \gamma_0) < \text{dist}(\Sigma_{a, r}, \gamma_0) = a < a_c.
\]

This implies that \( C_{a'} \) is unstable. Recall that \( \partial \Sigma' = \partial \Sigma_{a, r} \subset T_c \), the compact minimal annulus \( \Sigma' \) is unstable according to the second statement of Proposition 4.8 in [BSE10], therefore it can not be area minimizing either.

This is a contradiction to the above assumption, which implies \( \varphi(a) \) must be positive for all \( 0 < a < a_c \). In particular, this implies that \( \varphi(a_c) \geq 0 \). □

Step 2. \( \varphi(a) < 0 \) if \( a \geq a_l > a_c \), where \( a_l \approx 1.10055 \) is defined by (1.5). In particular, \( \varphi(a) \rightarrow -\infty \) as \( a \rightarrow \infty \).

Proof of Step 2. Using the substitution \( t \mapsto t + a \), we have the following estimate according to the arguments in the proof of Lemma 4.1 in [Wan19]

\[
\varphi(a) < 4\pi K \cosh a - 4\pi(\cosh a - 1) = -4\pi(1 - K) \cosh a + 4\pi
\]

for all \( a \geq 0 \), where \( K < 1 \) is defined by (1.6). By Lemma 4.1 and (4.8) in [Wan19], we have \( \varphi(a) < 0 \) if \( a \geq a_l \). Furthermore, \( \varphi(a) > -4\pi(\cosh a - 1) \) for all \( a > 0 \), therefore we have \( \varphi(a) \rightarrow -\infty \) as \( a \rightarrow \infty \) by the squeeze theorem. □

Step 3. \( \varphi(a) \) is concave downward if \( a > a_0 \), where \( a_0 < a_c \) is the constant determined in Lemma 3.4.
Proof of Step 3. Direct computation shows that the second derivative of \( \varphi(a) \)
can be written as \( \varphi''(a) = I_1(a) + I_2(a) \), where
\[
I_1(a) = 4\pi \int_0^\infty \sinh(a + t) \cdot \left( \frac{\sinh(2a + 2t)}{\sinh^2(2a + 2t) - \sinh^2(2a)} - 1 \right) dt
- 4\pi K \cosh a
\]
and
\[
I_2(a) = -4\pi \int_0^\infty \frac{5\cosh(a + t) - 3\cosh(3a + 3t) - 3\cosh(5a + t) + \cosh(7a + 3t)}{\sqrt{\sinh^2(2a + 2t) - \sinh^2(2a) \cdot \sinh^2(4a + 2t)}} dt
- 4\pi (1 - K) \cosh a.
\]
Similar to the argument in Step 2, we have \( I_1(a) < 0 \) for any \( a \geq 0 \). Next we need show that \( I_2(a) < 0 \) for \( a > a_0 \). We shall estimate both the numerator and denominator of the integrand of \( I_2(a) \): For the numerator, we have
\[
\text{numerator} = 5\cosh(a + t) - 3\cosh(3a + 3t) - 3\cosh(5a + t) + \cosh(7a + 3t)
\geq -3\cosh(3a)e^{3t} - 3\cosh(5a)e^{5t} + \cosh(7a)e^{3t}.
\]
For the denominator, we have
\[
\text{denominator} = \sqrt{\sinh^2(2a + 2t) - \sinh^2(2a) \cdot \sinh^2(4a + 2t)}
\leq \sinh(2a + 2t) \cdot (\sinh(4a + 2t))^2
\leq \cosh(2a) \cosh^2(4a)e^{6t}.
\]
So we have the following estimates
\[
I_2(a) \leq -4\pi \int_0^\infty \frac{-3\cosh(3a)e^{3t} - 3\cosh(5a)e^{5t} + \sinh(7a)e^{3t}}{\cosh(2a) \cosh^2(4a)} dt
- 4\pi (1 - K) \cosh a
\]
\[
= -4\pi \left\{ \frac{- \cosh(3a) - \frac{3}{2} \cosh(5a) + \frac{1}{3} \sinh(7a)}{\cosh(2a) \cosh^2(4a)} \right\}
- 4\pi (1 - K) \cosh a
\]
\[
\leq -4\pi \cdot \frac{-30 \cosh(3a) - 18 \cosh(5a) + 10 \sinh(7a) + 15(1 - K) \cosh(8a)}{\cosh(2a) \cosh^2(4a)},
\]
where we use the facts that \( \cosh(2a) \geq \cosh(a) \geq 1 \) and \( \cosh^2(4a) \geq \cosh(8a)/2 \)
for the last inequality. According to Lemma 3.4, \( I_2(a) \leq 0 \) if \( a > a_0 \).

Therefore \( \varphi''(a) < 0 \) as long as \( a > a_0 \). In other words, \( \varphi(a) \) is concave downward on \( (a_0, \infty) \). \( \square \)

Step 4. \( \varphi(a_c) > 0 \).
Proof of Step 4. By Lemma 3.3 in [Wan19], we know that \( 0 < a_c < A_3 \), where the constant \( A_3 \approx 0.530638 \) is defined by [Wan19] (6.1). According to Step 3, \( \varphi(a) \) is concave downward on \((a_0, A_3) \ni a_c\), since \( \varphi(a_0) > 0 \) by Step 1 and \( \varphi(A_3) \approx 0.781314 > 0 \) by direct numerical computation using softwares, we have \( \varphi(a_c) > 0 \).

According to Steps 1–3 there exists a (unique) zero \( a_L > a_0 > 0 \) of \( \varphi(a) \) such that the conditions in Theorem 3.2 are satisfied. Moreover, we know that \( a_L > a_c \) since \( \varphi(a) > 0 \) if \( a \in (0, a_c] \) according to Step 4. \( \square \)

3.2. Area minimizing annuli asymptotic to circles. In order to prove the last statement of Theorem 1.7, we have to determine whether a connected minimal surface \( \Sigma \subset H^3 \) asymptotic to two disjoint round circles in \( S^2_{\infty} \) is a spherical catenoid. The following theorem of Levitt and Rosenberg shows that \( \Sigma \) is a spherical catenoid if it is regular at infinity (see [LR85, Theorem 3.2] or [JCGT86, Theorem 3]). Without the regularity at infinity, \( \Sigma \) still could be a spherical catenoid as long as it is (absolutely) area minimizing (see Corollary 3.7).

Definition 3.5. For an integer \( k \geq 1 \), a complete minimal surface \( \Sigma \subset H^3 \) is C^k-regular at infinity if \( \partial_{\infty} \Sigma \) is a C^k-submanifold of \( S^2_{\infty} \) and \( \Sigma = \Sigma \cup \partial_{\infty} \Sigma \) is a C^k-submanifold (with boundary) of \( H^3 \).

Theorem 3.6 (Levitt and Rosenberg). Let \( \mathcal{C} \) be a connected minimal surface immersed in \( H^3 \) whose asymptotic boundary consists of two disjoint round circles \( C_1 \) and \( C_2 \) in \( S^2_{\infty} \). If \( \mathcal{C} \) is C^2-regular at infinity, then \( \mathcal{C} \) is a spherical catenoid asymptotic to \( C_1 \cup C_2 \).

The following corollary is a direct application of Theorem 3.6 and boundary regularity, which will be applied to prove the last statement of Theorem 1.7.

Corollary 3.7. Let \( \mathcal{C} \) be a connected minimal surface immersed in \( H^3 \) whose asymptotic boundary consists of two disjoint round circles \( C_1 \) and \( C_2 \) in \( S^2_{\infty} \). If \( \mathcal{C} \) is an (absolutely) area minimizing surface, then \( \mathcal{C} \) is a spherical catenoid asymptotic to \( C_1 \cup C_2 \).

Proof. Since round circles in \( S^2_{\infty} \) are always smooth, the (absolutely) area minimizing surface \( \mathcal{C} \) is C^2-regular at infinity by the results of boundary regularity in [HL87, Lin89b, Lin12] \(^1\) Therefore \( \mathcal{C} \) must be a minimal surface of revolution by the theorem of Levitt and Rosenberg, that is, \( \Sigma \) is a spherical catenoid, whose asymptotic boundary is \( C_1 \cup C_2 \). \( \square \)

\(^1\)The key result of Theorem 2.2 in [HL87] works for both homotopically and homologically area minimizing surfaces in \( U^3 \).
3.3. Existence of area minimizing catenoids. Now we are able to prove the first main result of this paper.

**Theorem 1.7.** There exists a constant \( a_L \approx 0.847486 \) given by Theorem 3.2

1. If \( a_c \leq a < a_L \), then each spherical catenoid \( C_a \) is globally stable but not (absolutely) area minimizing.

2. If \( a \geq a_L \), then each spherical catenoid \( C_a \) is (absolutely) area minimizing among all surfaces asymptotic to \( \partial \infty C_a \).

Moreover, any (absolutely) area minimizing surface asymptotic to two disjoint round circles in \( S^2_\infty \) must be one element in \( \{ C_a \}_{a \geq a_L} \) up to isometry.

**Proof.** Suppose that all the spherical catenoids in the family \( \mathcal{S} = \{ C_a \}_{a \geq 0} \) have the same axis \( \gamma_0 \) and the same symmetric plane \( P_0 \) (see Notion Convention 2.7).

1. If \( a_c \leq a < a_L \), then \( \varphi(a) > 0 \) by Theorem 3.2; hence there exists some \( r \gg a \) such that \( \Phi(a, r) > 0 \), this is equivalent to the following inequality

\[
\text{area}(\Sigma_{a, r}) > \text{area}(\Delta^+_{a, r}) + \text{area}(\Delta^-_{a, r}) = 4\pi(\cosh r - 1),
\]

where \( \Sigma_{a, r} \) is defined by \([3.1]\), and \( \Delta^\pm_{a, r} \) are the totally geodesic disks bounded by the two boundary components of \( \Sigma_{a, r} \) respectively.

Next we shall construct annuli \( \Pi_{a, r}(s) \) such that \( \partial \Pi_{a, r}(s) = \partial \Sigma_{a, r} \) for all \( 0 < s \ll r \) and the area of \( \Pi_{a, r}(s) \) is less than that of \( \Sigma_{a, r} \) when \( s > 0 \) is sufficiently small. Therefore \( C_a \) is not an area minimizing surface if \( a_c \leq a < a_L \), and it is not absolutely area minimizing either.

The distance between \( \Delta^+_{a, r} \) and \( \Delta^-_{a, r} \) is given by (see \([5, 19] \))

\[
L = \text{dist}(\Delta^+_{a, r}, \Delta^-_{a, r}) = 2 \int_a^r \frac{\sinh(2a)}{\cosh t} \frac{dt}{\sqrt{\sinh^2(2t) - \sinh^2(2a)}}.
\]

Let \( 0 < s \ll r \) be sufficiently small, and let \( Y_{a, r}(s) \) be the annulus-type region of \( \partial N_s(\gamma_0) \) between \( \Delta^+_{a, r} \) and \( \Delta^-_{a, r} \), where \( N_s(\gamma_0) \) is the \( s \)-neighborhood of \( \gamma_0 \), then \( Y_{a, r}(s) \) is an equidistant cylinder with radius \( s \) and height \( L \), so its area is easy to obtain

\[
\text{area}(Y_{a, r}(s)) = 2\pi L \sinh s \cosh s.
\]

Let \( D^\pm_{a, r}(s) = \Delta^\pm_{a, r} \cap N_s(\gamma_0) \), then \( D^\pm_{a, r}(s) \) are totally geodesic disks with radii \( s \). We can define a new annulus \( \Pi_{a, r}(s) \) as follows:

\[
\Pi_{a, r}(s) = Y_{a, r}(s) \cup (\Delta^+_{a, r} \setminus D^+_{a, r}(s)) \cup (\Delta^-_{a, r} \setminus D^-_{a, r}(s)).
\]

Obviously \( \partial \Pi_{a, r}(s) = \partial \Sigma_{a, r} \) and \( \Pi_{a, r}(s) \) is homotopic to \( \Sigma_{a, r} \) rel \( \partial \Sigma_{a, r} \). Recall that \( \text{area}(D^+_{a, r}(s)) + \text{area}(D^-_{a, r}(s)) = 4\pi(\cosh s - 1) \), so the area of \( \Pi_{a, r}(s) \) is

\[
\text{area}(\Pi_{a, r}(s)) = 2\pi L \sinh s \cosh s + 4\pi(\cosh r - 1) - 4\pi(\cosh s - 1).
\]
Since $2\pi L \sinh s \cosh s - 4\pi (\cosh s - 1) \to 0$ as $s \to 0$, we can make $s$ sufficiently small so that $2\pi L \sinh s \cosh s - 4\pi (\cosh s - 1) < \text{area}(\Sigma_{a,r}) - 4\pi (\cosh r - 1)$, and then $\text{area}(\Pi_{a,r}(s)) < \text{area}(\Sigma_{a,r})$.

Therefore, if $a_c \leq a < a_L$, then each stable catenoid $C_a$ is not (absolutely) area minimizing.

(2) On the other hand, $\varphi(a) \leq 0$ for $a \geq a_L$ according to Theorem 3.2, hence for any $r \geq a$, we have $\Phi(a,r) < 0$, which implies the following inequality

$$\text{area}(\Sigma_{a,r}) < \text{area}(\Delta_a^+) + \text{area}(\Delta_a^-),$$

for all $r > a$.

Recall that $\Sigma_{a,r} \subset C_a \subset \mathbf{T}_c$ for all $r \geq a$, since $a \geq a_L > a_c$, where $\mathbf{T}_c$ is defined by (2.3). We claim that $\Sigma_{a,r}$ is a compact area minimizing annulus for all $r > a$. Otherwise, similar to the arguments in the first step of the proof of Theorem 3.2, there exists a compact area minimizing annulus $\Sigma'$ such that $\partial \Sigma' = \partial \Sigma_{a,r}$ and it’s a portion of some spherical catenoid $C_{a'} \in \mathcal{S}$, where $a'$ is the distance from $C_{a'}$ to its rotation axis. Since the elements in $\{C_a\}_{a \geq a_c}$ are disjoint to each other, we have $a' < a_c$. Similar to the arguments in the proof of Theorem 3.2, $\Sigma'$ can’t be area minimizing. This is a contradiction. So $\Sigma_{a,r}$ is area minimizing for all $r > a$, which implies that $C_a$ is an area minimizing surface as long as $a \geq a_L$.

Next we will show that $C_a$ is actually an absolutely area minimizing surface if $a \geq a_L$. Let $\Sigma$ be an absolutely area minimizing surface which is asymptotic to $C_1 \cup C_2 =: \partial_{\infty} C_a$, where $a \geq a_L$. According to Corollary 3.7, $\Sigma$ must be a spherical catenoid asymptotic to $C_1 \cup C_2$. Recall that there are exactly two spherical catenoids asymptotic to $C_1 \cup C_2$ (see [BSE10] or [Wan19]), one is $C_a$ whereas the other one is unstable. Since $\Sigma$ is also area minimizing, $\Sigma$ must be identical to $C_a$, where $a \geq a_L$.

For the last statement, let $\Pi \subset \mathbb{H}^3$ be an (absolutely) area minimizing surface asymptotic to two round circles $C_1$ and $C_2$ in $S^2_{\infty}$. Then $\Pi$ must be a spherical catenoid by Corollary 3.7. Let $a$ be the distance from $\Pi$ to its rotation axis, then $a \geq a_L$ by the above arguments. \(\square\)

4. Existence of complete area minimizing annuli in $\mathbb{H}^3$

In this section we shall prove Theorem 1.11 (see §4.4). At first, we shall prove three important results: Proposition 4.3, Theorem 4.5 and Proposition 4.7. To finish the proof of Theorem 1.11 we also need the help of geometric measure theory, in particular the theory of varifolds, see [Fed69, All75, Sim83, LY02, KP08, Mor16] for details.

Let $\Lambda$ be a set in $S^2_{\infty}$, the convex hull of $\Lambda$, which is denoted by $\text{CH}(\Lambda)$, is the intersection of all the closed half spaces in $\mathbb{H}^3$ whose asymptotic boundary contains $\Lambda$. Suppose that $\Gamma_1$ and $\Gamma_2$ are two disjoint (star-shaped) Jordan curves
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\[ \partial_{\infty} \text{CH}(\Gamma_1 \cup \Gamma_2) = \Gamma_1 \cup \Gamma_2. \]

The boundary of \( \text{CH}(\Gamma_1 \cup \Gamma_2) \) consists of three surfaces in \( B_3 \):

- \( D_1 \) and \( D_2 \) are disk-type surfaces asymptotic to \( \Gamma_1 \) and \( \Gamma_2 \) respectively,
- \( A \) is an annulus-type surface asymptotic to \( \Gamma_1 \cup \Gamma_2 \).

Moreover, \( \partial \text{CH}(\Gamma_1 \cup \Gamma_2) = D_1 \cup D_2 \cup A \) is mean convex with respect to the inward normal vector field. For any minimal surface \( \Sigma_i \) asymptotic to \( \Gamma_i \), \( i = 1, 2 \), and any minimal surface \( \Pi \) asymptotic to \( \Gamma_1 \cup \Gamma_2 \), it's well known that \( \Sigma_1, \Sigma_2, \Pi \subset \text{CH}(\Gamma_1 \cup \Gamma_2) \) according to [Sim83, Theorem 19.2].

4.1. Minimal surfaces asymptotic to star-shaped curves. Let \( \Gamma \subset S^2_\infty \) be a star-shaped Jordan curve. According to [And83, Theorem 4.1] and [HL87, Theorem 4.1] (see also [Has86, Example on pp.14–15]), there exists a unique complete embedded disk-type minimal surface \( \Sigma \subset B_3 \) asymptotic to \( \Gamma \), which minimizes area in the category of immersed surfaces (no topological restriction) asymptotic to \( \Gamma \). In other words, \( \Sigma \) is actually an absolutely area minimizing surface asymptotic to \( \Gamma \).

**Definition 4.1.** Let \( \Gamma \subset S^2_\infty \) be a star-shaped Jordan curve with an axis \( \ell \) and let \( \Sigma \subset B_3 \) be the minimal disk asymptotic to \( \Gamma \). Let \( \phi : B^3 \to \mathbb{R}^3 \) be the isometry that maps the axis \( \ell \) of \( \Gamma \) to the \( t \)-axis of \( \mathbb{R}^3 \) (see (1.2)). For any positive real number \( \lambda \), we define

\[
\lambda(z, t) = (\lambda z, \lambda t) \text{ for any } (z, t) \in \mathbb{R}^3.
\]

Then each \( \lambda(z, t) \) is an isometry of \( B^3 \) that translates a point in the geodesic \( \ell \) at distance \( \log \lambda \) along \( \ell \).

For any \( \lambda > 0 \), the surface \( \Sigma_\lambda = \lambda(z, t) \) is an area minimizing disk asymptotic to the Jordan curve \( \Gamma_\lambda = \lambda(z, t) \). In particular \( \Sigma_1 = \Sigma \) when \( \lambda = 1 \).

By [HL87, Theorem 4.1] and [Lin89b, Corollary 2.4] (see also [Has86, Example on pp.14–15]), we have the following corollary (still using the above notations and settings in Definition 4.1).

**Proposition 4.2.** The area minimizing disk \( \Sigma \subset B^3 \) asymptotic to a star-shaped Jordan curve \( \Gamma \subset S^2_\infty \) is a Killing graph (see Definition 10.4.1 in [Lóp13]). Moreover the family of complete area minimizing disks \( \{\Sigma_\lambda\}_{\lambda>0} \) foliates \( B^3 \).

Next we try to understand the intersection of a minimal surface asymptotic to a star-shaped Jordan with a 3-ball. We expect that this intersection just consists of exactly one component when the radius of the ball is sufficiently large. More precisely we have the following proposition.

**Proposition 4.3.** Let \( \Sigma \subset H^3 \) be a minimal surface asymptotic to a star-shaped Jordan curve \( \Gamma \subset S^2_\infty \). Let \( B^3(p, r) \) be any 3-ball in \( \mathbb{R}^3 \) with the center \( p \) and the radius \( r \), where \( p \in \mathbb{H}^3 \) is an arbitrary point. If \( r \) is sufficiently large, then
$B^3(p, r) \cap \Sigma$ consists of exactly one disk, whose boundary is simple closed curve. Moreover, $B^3(p, r) \cap \Sigma$ converges to $\Gamma$ as $r \to \infty$.

Before we prove Proposition 4.3, we need prove the following lemma.

**Lemma 4.4.** Let $\Sigma \subset U^3$ be a minimal surface asymptotic to a star-shaped Jordan curve $\Gamma \subset \overline{\mathbb{C}}$. Let $P(t)$ be the horizontal plane through the point $(0, 0, t)$ for $t > 0$. There exists a positive number $r_\Gamma$ such that $P(t) \cap \Sigma$ consists of exactly one simple closed curve for all $t \in [0, r_\Gamma)$. Moreover $P(t) \cap \Sigma$ converges to $\Gamma$ as $t \to 0$.

**Proof.** According to [HL87] or [Lin89a], there exists a constant $r_\Gamma$ depending on $\Gamma$ such that

$$\Sigma' := (\Sigma \cup \Gamma) \cap \{(x, y, t) \in \mathbb{R}^3 \mid t < r_\Gamma\}$$

(4.2)

is a finite union of surfaces with boundary which can be viewed as a graph over $\Gamma \times [0, r_\Gamma)$. We assume that $\Gamma$ is a star shaped Jordan curve, therefore $\Gamma \times [0, r_\Gamma)$ is an annulus, so is $\Sigma'$. This means that $P(t) \cap \Sigma$ consists of exactly one simple closed curve if $t < r_\Gamma$. As $t \to 0$, $P(t) \cap \Sigma$ converges to $\Gamma$.

**Proof of Proposition 4.3.** Consider the upper half space model $U^3$. When $r$ is sufficiently large (which might depend on the choice of $p$), $B^3(p, r)$ is sufficiently close to a horizontal horosphere in $U^3$. In particular, when $r$ is sufficiently large, we have $(\Sigma \cup \Gamma) \cap \left(U^3 \setminus B^3(p, r)\right) \subset \Sigma'$, where $\Sigma'$ is given by (4.2). Applying Lemma 4.4, we prove the statement of the proposition.

**4.2. Density at infinity.** Let $\Sigma \subset H^3$ be a minimal surface asymptotic to a Jordan curve $\Gamma$ in $S^2_{\infty}$. Fix a point $p \in H^3$, for any $r > 0$, let

$$\Theta(\Sigma, p, r) = \frac{\text{area}(\Sigma \cap B^3(p, r))}{4\pi \sinh^2(r/2)} = \frac{\text{area}(\Sigma \cap B^3(p, r))}{2\pi (\cosh r - 1)},$$

(4.3)

where $B^3(p, r) \subset H^3$ is an open three ball with (hyperbolic) radius $r$ centered at $p$. According the hyperbolic version of monotonicity formula (see [And82, Theorem 1]), $\Theta(\Sigma, p, r)$ is a nondecreasing function of $r > 0$, so the limit of $\Theta(\Sigma, p, r)$ exists as $r \to \infty$. Note that $B^3(p, r) \subset B^3(q, r + \text{dist}(p, q))$ for any point $q \in H^3$, from which it easily follows that $\lim_{r \to \infty} \Theta(\Sigma, p, r)$ is independent of the choice of $p$, and therefore we call

$$\Theta_\infty(\Sigma) = \lim_{r \to \infty} \Theta(\Sigma, p, r)$$

(4.4)

the density of $\Sigma$ at infinity (see [Whi16]).

The following result belongs to Gromov [Gro83, Theorem 8.3.A] (see also [EWW02]), which is crucial to prove Theorem 1.11.

**Theorem 4.5 (Gromov).** If $\Sigma \subset H^3$ is a minimal surface asymptotic to a rectifiable star-shaped Jordan curve $\Gamma \subset S^2_{\infty}$, then $\Theta_\infty(\Sigma)$ is finite.
**Proof.** Let $\text{CH}(\Gamma) \subset \mathbb{H}^3$ be the convex hull of $\Gamma$, then $\Sigma$ is contained in $\text{CH}(\Gamma)$. Choose a point $p$ in $\Sigma$, and consider the geodesic cone $\mathcal{C}$ over $\Gamma$ with vertex $p$. Then $\mathcal{C}$ is also contained in $\text{CH}(\Gamma)$. Since $\Gamma$ is rectifiable, $\Theta(\mathcal{C}, p, r)$ is a finite constant for any $r > 0$, so is $\Theta_{\infty}(\mathcal{C})$.

Let $B^3(p, r) \subset \mathbb{H}^3$ be the 3-ball with the radius $r$ and the center $p$. According to Proposition 4.3, there exists a constant $r_0 > 0$ such that $\Sigma(r) := B^3(p, r) \cap \Sigma$ consists of exactly one disktype component and $A(r) := \partial B^3(p, r) \cap \text{CH}(\Gamma)$ is an annulus for all $r > r_0$. Also set $\mathcal{C}(r) := B^3(p, r) \cap \mathcal{C}$.

Obviously both $\partial \Sigma(r) = \partial B^3(p, r) \cap \Sigma(r)$ and $\partial \mathcal{C}(r) = \partial B^3(p, r) \cap \mathcal{C}$ are contained in $A(r)$ for all $r > r_0$. Let $E(r) \subset A(r)$ be the domain bounded by $\partial \Sigma(r)$ and $\partial \mathcal{C}(r)$. Then $\Sigma(r)$ and $\mathcal{C}(r) \cup E(r)$ are the surfaces in $\mathbb{H}^3$ with the same boundary. Since $\Sigma(r)$ is an absolutely area minimizing surface for any $r > r_0$, we must have

$$\text{area}(\Sigma(r)) < \text{area}(\mathcal{C}(r) \cup E(r)) < \text{area}(\mathcal{C}(r)) + \text{area}(A(r))$$

for all $r > r_0$. The area of $A(r)$ can be estimated as the product of an exponentially small factor and the length of $\partial \mathcal{C}(r)$, where the former term is obtained from the argument that is similar to [Thu80, p.40] by the definition of convex hull (see also [Gro83, p.111]). More precisely, we have

$$\text{area}(A(r)) = O\left(e^{-(r-r_0)} \text{length}(\partial \mathcal{C}(r))\right) = O\left(e^{-(r-r_0)} \sinh r\right)$$

for all $r > r_0$, where the second equality comes from the fact that $\Theta(\mathcal{C}, p, r)$ is a finite constant for any $r > 0$, which can imply that $\text{length}(\partial \mathcal{C}(r))/(2\pi \sinh r)$ is the same constant for all $r > 0$.

Therefore we have the following estimates

$$\Theta(\Sigma, p, r) = \frac{\text{area}(\Sigma \cap B^3(p, r))}{4\pi \sinh^2(r/2)} < \frac{\text{area}(\mathcal{C} \cap B^3(p, r))}{4\pi \sinh^2(r/2)} + \frac{\text{area}(A(r))}{4\pi \sinh^2(r/2)} = \Theta_{\infty}(\mathcal{C}) + O\left(e^{-({r-r_0})} \frac{\sinh r}{\sinh^2(r/2)}\right) = \Theta_{\infty}(\mathcal{C}) + O\left(e^{-({r-r_0})} \frac{\sinh r}{\cosh r - 1}\right)$$

for all $r > r_0$. As $r \to \infty$, we have $\Theta_{\infty}(\Sigma) \leq \Theta_{\infty}(\mathcal{C}) < \infty$. □

### 4.3. Intersection of minimal surfaces.

In this subsection, we study the intersection of a spherical catenoid with a minimal disk asymptotic to a star-shaped Jordan curve. At first let’s fix some notations in the following definition.

**Definition 4.6.** A catenoid $\mathcal{C}$ with $\partial_{\infty}\mathcal{C} = C_1 \cup C_2$ divides the hyperbolic space $\mathbb{H}^3$ into two regions, the one containing the rotation axis of $\mathcal{C}$ is homeomorphic to...
a solid cylinder, and the other one is homeomorphic to a solid torus. The former is called the interior region of $C$, denoted by $X$; the latter is called the exterior region of $C$, denoted by $T$. One can verify that \( \partial T = C = \partial X, \partial X = B_1 \cup B_2 \) and \( \partial \infty T = S_\infty \setminus (B_1 \cup B_2) \), where $B_i$ is the disk-type component of $S_\infty \setminus (C_1 \cup C_2)$ bounded by $C_i$ for $i = 1, 2$. Note that both $T$ and $X$ are the subregions of $B_3$ with mean convex boundary.

**Proposition 4.7.** Let $C \subset B_3$ be an area minimizing catenoid asymptotic to $C_1 \cup C_2$, whose interior region is denoted by $X$. Let $\Gamma \subset S_\infty$ be a star-shaped Jordan curve with the axis $\ell$ which separates $C_1$ and $C_2$ (that is, two components of $S_\infty \setminus \Gamma$ contain $C_1$ and $C_2$ respectively). Suppose that $\Sigma \subset B_3$ is the area minimizing disk asymptotic to $\Gamma$, then $\Sigma$ must intersect $C$ transversely, and $\Sigma \cap X$ consists of a single disk-type subdomain of $\Sigma$, denoted by $\Delta$, such that

1. $\partial \Delta$ is a Jordan curve which is essential in $C$, and
2. $\left(\Sigma \setminus \Delta\right) \cap (X \cup C) = \emptyset$.

**Proof.** According to [And83, Theorem 4.1] and [HL87, Theorem 4.1], $\Sigma$ is the unique embedded minimal disk asymptotic to $\Gamma$, which has the least area among all surfaces asymptotic to $\Gamma$.

It’s easy to see that $\Gamma$ is essential in $T = B_3 \setminus X$, which implies that the complete minimal disk $\Sigma$ must intersect $C$.

**Claim 1:** If $\Sigma$ and $C$ intersect transversely at some simple closed curve $\alpha$, then $\alpha$ must be essential in $C$.

**Proof of Claim 1.** Otherwise, there exist two compact minimal disks $D \subset C$ and $\Delta \subset \Sigma$ such that $\partial D = \alpha = \partial \Delta$. For any $\lambda > 0$, let $\Sigma_\lambda = h_\lambda(\Sigma)$, where $h_\lambda$ be the isometry of $B_3$ defined by (4.1), then \{\Sigma_\lambda\}_0<\lambda<\infty foliates $B_3$ by Proposition 4.2. Since $\partial D = \alpha \subset \Sigma$ by assumption, there exists some $\lambda_0$ such that $\Sigma_{\lambda_0}$ is tangent to $D$ from one side, which is impossible because of the maximum principle. Therefore any curve of $\Sigma \cap C$ must be essential in $C$. \(\square\)

**Claim 2:** $\Sigma$ and $C$ intersect transversely.

**Proof of Claim 2.** Otherwise we may assume that $\Sigma$ and $C$ intersect at a point $p$ non-transversely. By [MY82a, Lemm 2] there exist neighborhoods $U \subset C$ and $V \subset \Sigma$ of $p \in \Sigma \cap C$ such that $U$ and $V$ intersect along a finite number of curves passing through $p$ and the intersection is transversal at points other than $p$ (see also [FHS83, Figure 1.2 and Lemma 1.4]).

By applying an isometry of $B_3$, we may assume that $p$ is the origin of $B_3$ and the unit normal vector to both $C_a$ and $\Sigma$ at $p$ is parallel to the $w$-axis (see the second paragraph in [4] for the definition of $B_3$). Let $g_t$ be a translation along the $w$-axis about distance $t$ for $t \in (-\infty, \infty)$, then $g_t$ is an isometry of $B_3$. Let $\varepsilon > 0$
be a sufficiently small number such that \( g_t(\partial_\infty C_a) \cap \Gamma = \emptyset \) if \(|t| < \varepsilon \). According to [FHS83] Lemma 1.5, we may slightly translate \( C_a \) along the \( w \)-axis via \( g_t \) for \(|t| \ll \varepsilon \) so that the minimal disk \( \Sigma \) intersects the catenoid \( g_t(C_a) \) transversely at a finite number of simple closed curves (see [FHS83] Figure 1.2]), which are all null homotopic in \( g_t(C_a) \) by topological arguments. But this is impossible according to Claim 1, so \( \Sigma \) must intersect \( C_a \) transversely, and Claim 2 is proved. \( \square \)

**Claim 3**: \( \Sigma \cap C \) consists of exactly one simple closed curve that is essential in the spherical catenoid \( C \).

**Proof of Claim 3.** If \( \Sigma \) intersects \( C \) more than once, then there is a compact portion \( C' \) of \( C \) such that \( \partial C' \subset \Sigma \) consists of two components in \( \Sigma \cap C \) and \( C' \) is totally contained in one component of \( \mathbb{B}^3 \setminus \Sigma \). Since \( \{ \Sigma_\lambda \}_0 \leq \lambda < \infty \) foliates \( \mathbb{B}^3 \) by Proposition [1.2] there exists some \( 0 < \lambda_0 < 1 \) or \( \lambda_0 > 1 \) such that \( \Sigma_\lambda \) is tangent to \( C' \) from one side. As usual, this is impossible because of the maximum principle. This implies that \( \Sigma \) intersects \( X \) exactly once, and so \( \Sigma \) intersects \( C \) exactly once.

Now each component of \( \Sigma \cap C \) is a simple closed curve, which is essential in \( C \).

Let \( \alpha \) denote one of the components of \( \Sigma \cap C \). Since \( X \) is a subregion of \( \mathbb{H}^3 \) with mean convex boundary (actually \( \partial X = C \)) and \( \alpha \) is null homotopic in \( X \), the curve \( \alpha \) must bound an area minimizing disk \( \Delta \) in \( X \) by [AS79, MY82a, MY82b], which is also a subdomain of \( \Sigma \). \( \square \)

### 4.4. Existence of area minimizing annuli.

Let \( C \) be any area minimizing catenoid in \( \mathbb{B}^3 \) (i.e., the distance from \( C \) to its rotation axis is \( \geq a_L \)) asymptotic to disjoint round circles \( C_1 \) and \( C_2 \) in \( S_\infty^2 \). Suppose that \( \Gamma_1 \) and \( \Gamma_2 \) are disjoint star-shaped Jordan curves contained in the annulus-type component of \( S_\infty^2 \setminus (C_1 \cup C_2) \) such that \( d(\Gamma_1, \Gamma_2) < 2\varrho(a_L) \), and that \( \Sigma_1 \) and \( \Sigma_2 \) are area minimizing disks asymptotic to \( \Gamma_1 \) and \( \Gamma_2 \) respectively. By Proposition [1.7] \( \alpha_i = C \cap \Sigma_i \) is a Jordan curve in \( \mathbb{B}^3 \) for \( i = 1, 2 \).

Since \( \Gamma_1 \) and \( \Gamma_2 \) are disjoint, it’s well known that the disk-type area minimizing surfaces \( \Sigma_1 \) and \( \Sigma_2 \) are also disjoint (see for example [FHS83] Lemma 1.2]). We need some notations:

- Let \( B \) be the subregion of \( \mathbb{B}^3 \) such that \( \partial B = \Sigma_1 \cup \Sigma_2 \) and \( \partial_\infty B \) is the annulus-type component of \( S_\infty^2 \setminus (\Gamma_1 \cup \Gamma_2) \), then \( B \) is a subregion of \( \mathbb{B}^3 \) with mean convex boundary.
- Let \( C' = C \cap B \), then \( C' \) is a compact annulus-type minimal surface with \( \partial C' = \alpha_1 \cup \alpha_2 \), where \( \alpha_i = C \cap \Sigma_i \) for \( i = 1, 2 \).
- Suppose that \( \Delta_i \) is the disk-type subdomain of \( \Sigma_i \) such that \( \partial \Delta_i = \alpha_i \) for \( i = 1, 2 \).
Lemma 4.8. Using the above settings. Suppose \( \gamma_i \subset \Sigma_i \setminus \Delta_i \) is a rectifiable Jordan curve for \( i = 1, 2 \), then there exists an embedded compact annulus-type area minimizing surface \( \Pi \subset \mathbb{B}^3 \) such that \( \partial \Pi = \gamma_1 \cup \gamma_2 \).

Proof. Let \( \Sigma'_i \) be the compact disk-type subdomain of \( \Sigma_i \) such that \( \partial \Sigma'_i = \gamma_i \) for \( i = 1, 2 \). Obviously \( \Sigma'_i \) is the area minimizing disk spanning \( \gamma_i \) for \( i = 1, 2 \).

By Proposition 4.7, we can define an embedded compact annulus \( S \subset \mathbb{B} \) whose boundary is \( \gamma_1 \cup \gamma_2 \):

\[
S = C' \cup (\Sigma'_1 \setminus \Delta_1) \cup (\Sigma'_2 \setminus \Delta_2) .
\]

(4.5)

Since \( C \) is assumed to be an area minimizing catenoid, we have the inequality \( \text{area}(C') < \text{area}(\Delta_1) + \text{area}(\Delta_2) \). Therefore we have

\[
\text{area}(S) = \text{area}(C') + \text{area}(\Sigma'_1 \setminus \Delta_1) + \text{area}(\Sigma'_2 \setminus \Delta_2) \\
< \text{area}(\Delta_1) + \text{area}(\Delta_2) + \text{area}(\Sigma'_1 \setminus \Delta_1) + \text{area}(\Sigma'_2 \setminus \Delta_2) \\
= \text{area}(\Sigma'_1) + \text{area}(\Sigma'_2) .
\]

By [AS79] Theorem 7 or [MY82a] Theorem 1], there exists an area minimizing annulus \( \Pi \subset \mathbb{B} \) such that \( \partial \Pi = \gamma_1 \cup \gamma_2 \).

Next we need show that \( \Pi \) is also the area minimizing annulus in \( \mathbb{B}^3 \). Otherwise, assume that \( \Pi' \) is an area minimizing annulus in \( \mathbb{B}^3 \) with boundary components \( \gamma_1 \) and \( \gamma_2 \) such that \( \text{area}(\Pi') < \text{area}(\Pi) \). We shall prove that \( \Pi' \) is actually contained in \( \mathbb{B} \). In fact, \( \Pi' \) can’t intersect the component of \( \mathbb{B}^3 \setminus \mathbb{B} \) bounded by the minimal disk \( \Sigma_1 \) since the family of the minimal disks \( \{h_\lambda(\Sigma_1)\}_{0 < \lambda < 1} \) foliates this subregion by Proposition 4.2 where each \( h_\lambda \) is defined by (4.1). Similarly, \( \Pi' \) can’t intersect the component of \( \mathbb{B}^3 \setminus \mathbb{B} \) bounded by \( \Sigma_2 \).

Therefore \( \Pi' \subset \mathbb{B} \). But we have proved that \( \Pi \) is the area minimizing annulus in \( \mathbb{B} \). This is a contradiction. So \( \Pi \) is the area minimizing annulus in \( \mathbb{B}^3 \). \( \square \)

Now we are able to prove Theorem 1.11

Theorem 1.11. Let \( \Gamma_1 \) and \( \Gamma_2 \) be disjoint rectifiable star-shaped Jordan curves in \( S^2_\infty \). If the distance between \( \Gamma_1 \) and \( \Gamma_2 \) is bounded from above as follows

\[
d(\Gamma_1, \Gamma_2) < 2\varrho(a_L) \approx 0.876895 ,
\]

(1.8)

where \( \varrho \) is the function defined by (2.4) and \( a_L \approx 0.847486 \) is the constant given by Theorem 3.2, then there exists an embedded annulus-type area minimizing surface \( \Pi \subset \mathbb{H}^3 \), which is asymptotic to \( \Gamma_1 \cup \Gamma_2 \).

Moreover the upper bound (1.8) is optimal in the following sense: If there is an area minimizing surface in \( \mathbb{H}^3 \) asymptotic to two disjoint round circles in \( S^2_\infty \), then the distance (1.3) between the circles is \( \leq 2\varrho(a_L) \).
Because of Proposition 4.3, there exists an area minimizing catenoid \( C \) such that two components of \( \partial_\infty C \) are contained in the disk-type components of \( S^2_\infty \setminus (\Gamma_1 \cup \Gamma_2) \) respectively and \( \partial_\infty C \cap (\Gamma_1 \cup \Gamma_2) = \emptyset \).

Let \( O \) be a fixed point contained in the region \( B \) of \( \mathbb{B}^3 \setminus (\Sigma_1 \cup \Sigma_2) \) such that \( \partial B = \Sigma_1 \cup \Sigma_2 \) and \( \partial_\infty B \) is the annulus component of \( S^2_\infty \setminus (\Gamma_1 \cup \Gamma_2) \). Let \( B^3(s) \subset \mathbb{B}^3 \) denote the open 3-ball of radius \( s \) (centered at the origin \( O \)), that is,

\[
B^3(s) = B^3(O,s) = \{ x \in \mathbb{H}^3 \mid \text{dist}(x,O) < s \}.
\]

There exists a sufficiently large positive number \( r_0 \) such that the following two conditions are satisfied for any \( s > r_0 \) (see Proposition 4.3):

1. \( B^3(s) \cap \Sigma_i \) consists of exactly one disk, denoted by \( \Sigma_i(s) \) for \( i = 1, 2 \).
2. \( B^3(s) \cap \mathcal{A} \) is homeomorphic to \( \mathcal{A} \), where \( \mathcal{A} \) is the annulus-type part of the boundary of \( \text{CH}(\Gamma_1 \cup \Gamma_2) \), which is asymptotic to \( \Gamma_1 \cup \Gamma_2 \).

For \( i = 1, 2 \), we define a Jordan curve \( \gamma_i(s) \) as follows

\[
\gamma_i(s) = \partial \Sigma_i(s) = \Sigma_i \cap \partial B^3(s).
\]

Because of Proposition 4.3, \( \gamma_i(s) \to \Gamma_i \) as \( s \to \infty \) for \( i = 1, 2 \).

According to Lemma 4.8 there exists an embedded compact annulus-type area minimizing surface \( \Pi(s) \subset B^3(s) \) with \( \partial \Pi(s) = \gamma_1(s) \cup \gamma_2(s) \subset \partial B^3(s) \) for each \( s \geq r_0 \) so that

1. \( \text{area}(\Pi(s)) < \text{area}(\Sigma_1(s)) + \text{area}(\Sigma_2(s)) \), and
2. \( \Pi(s) \) is contained in \( \text{CH}(\Gamma_1 \cup \Gamma_2), B^3(s) \) and \( \overline{B^3(s)} \), where \( \overline{B^3(s)} \) is the subregion of \( \mathbb{B}^3 \) bounded by \( \Sigma_1 \) and \( \Sigma_2 \).

**Claim.** For any \( 0 < r < s \), there exists a constant \( C_r \), depending only on \( r, \Sigma_1 \) and \( \Sigma_2 \) such that \( \text{area}(\Pi(s) \cap B^3(r)) \leq C_r \).

**Proof.** Let \( \theta = \theta_1 + \theta_2 \), where \( \theta_i \) is the density at infinity of \( \Sigma_i \) for \( i = 1, 2 \). According to Theorem 4.5 both \( \theta_1 \) and \( \theta_2 \) are finite, so is \( \theta \).

For any \( 0 < r \leq s \), we have

\[
\frac{\text{area}(\Pi(s) \cap B^3(r))}{4\pi \sinh^2(r/2)} \leq \frac{\text{area}(\Pi(s))}{4\pi \sinh^2(s/2)} \leq \frac{\text{area}(\Sigma_1(s)) + \text{area}(\Sigma_2(s))}{4\pi \sinh^2(s/2)} = \frac{\text{area}(\Sigma_1(s))}{4\pi \sinh^2(s/2)} + \frac{\text{area}(\Sigma_2(s))}{4\pi \sinh^2(s/2)} \leq \theta_1 + \theta_2 = \theta,
\]

where we use the facts \( \Pi(s) \cap B^3(s) = \Pi(s) \) and \( \Sigma_i(s) = \Sigma_i \cap B^3(s) \) for \( i = 1, 2 \), therefore

\[
\text{area}(\Pi(s) \cap B^3(r)) \leq \theta \cdot 4\pi \sinh^2(r/2) =: C_r \quad (4.6)
\]
for all $0 < r \leq s$. The proof of the Claim is complete. □

Pick up a sequence of increasing positive real numbers $r_0 < r_1 < r_2 < \cdots$ such that $r_k \to \infty$ as $k \to \infty$. According to Lemma 4.8 there exists an area minimizing annulus $\Pi(r_k) \subset \mathbb{H}^3$ spanning $\gamma_1(r_k)$ and $\gamma_2(r_k)$. Now we have a sequence of compact annulus-type area minimizing surfaces $\Pi(r_1), \Pi(r_2), \ldots, \Pi(r_k), \ldots$ so that $\text{area}(\Pi(r_k)) < \text{area}(\Sigma_1(r_k)) + \text{area}(\Sigma_2(r_k))$ for $k = 1, 2, \ldots$

We shall prove that $\{\Pi_k := \Pi(r_k)\}_{k \geq 1}$ converges smoothly to a complete area minimizing annulus $\Pi \subset \mathbb{H}^3$ which is asymptotic to $\Gamma_1 \cup \Gamma_2$ as $k \to \infty$. This can be done via geometric measure theory, the reader can check [AS79, And82] for details. Here we just sketch the whole process:

1) For $k = 1, 2, \ldots,$ we can associate each (compact) area minimizing annulus $\Pi_k$ with a varifold $\nu(\Pi_k) \in V_2(\mathbb{H}^3)$ (see [All75, §3.5]).

2) According to the estimate (4.6) in the above claim, we have

$$V = \lim_{k \to \infty} \nu(\Pi_k) \in V_2(\mathbb{H}^3)$$

by the weak convergence of Randon measures (see [CM11, Theorem 3.2]).

3) By Theorem 2 in [AS79] or Proposition 3.5 in [CM11], the varifold $V$ is stationary.

4) According to Allard’s regularity theorem in [All75, §8] and the arguments in sections 4, 5, and 6 of [AS79], for each point $x_0 \in \text{spt} \|V\|$ there is a positive integer $n_{x_0}$, a $\rho_{x_0} > 0$, and an analytic minimal surface $\Sigma_{x_0}$ such that

$$V \cap B^3(x_0, \rho_{x_0}) \times G(3, 2) = n_{x_0} \nu(\Sigma_{x_0}).$$

This implies that there exists a smooth minimal surface $\Pi \subset \mathbb{H}^3$ with $\partial_\infty \Pi = \Gamma_1 \cup \Gamma_2$ such that $V = \nu(\Pi)$. By the arguments in section 9 of [AS79], this minimal surface $\Pi$ is of annulus-type.

5) It’s well known that a limit of area minimizing surfaces is itself area minimizing. The proof can be found in the last paragraph of the proof of Theorem 3.1 in [MY92]. Therefore the minimal surface $\Pi$ is (homotopically) area minimizing, that is, any compact subdomain of $\Pi$ is an area minimizing surface.

To show that the upper bound (1.8) is optimal, let’s consider the special case when $\Gamma_1$ and $\Gamma_2$ are two round circles in $S^2_\infty$. If $\Sigma \subset \mathbb{H}^3$ is an area minimizing surface asymptotic to $\Gamma_1 \cup \Gamma_2$, then $d(\Gamma_1, \Gamma_2) \leq 2\varrho(a_L)$ by Theorem 1.7.

Now the proof of Theorem 1.11 is done. □

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