APPROXIMATE CATEGORICAL STRUCTURES

ABDELKRIM ALIOUCHE AND CARLOS SIMPSON

ABSTRACT. We consider notions of metrized categories, and then approximate categorical structures defined by a function of three variables generalizing the notion of 2-metric space. We prove an embedding theorem giving sufficient conditions for an approximate categorical structure to come from an inclusion into a metrized category.

1. INTRODUCTION

Gähler \cite{3} introduced the notion of 2-metric space which is a set $X$ together with a function called the 2-metric $d(x, y, z) \in \mathbb{R}$ satisfying some properties generalizing the axioms for a metric space. Notably, the triangle inequality generalizes to the tetrahedral inequality for a 2-metric

$$d(x, y, w) \leq d(x, y, z) + d(y, z, w) + d(x, z, w).$$

One of the main examples of a 2-metric is obtained by setting $d(x, y, z)$ equal to the area of the triangle spanned by $x, y, z$. Here, we consider triangles with straight edges. One might imagine considering more generally triangles with various paths as edges. In this case, in addition to $x, y$ and $z$, we should specify a path $f$ from $x$ to $y$, a path $g$ from $y$ to $z$ and a path $h$ from $x$ to $z$. We could then set $d(f, g, h)$ to be the area of the figure spanned by these paths, more precisely the minimal area of a disk whose boundary consists of the circle formed by these three paths.

This generalization takes us in the direction of category theory: we may think of $d(f, g, h)$ as being some kind of distance between $h$ and a "composition" of $f$ and $g$. We will formalize this notion here and call it an approximate categorical structure.

Generalizing the notion of 2-metric space in this direction may be viewed as directly analogue to the recent paper of Weiss \cite{4} in which he...

\begin{flushleft}
2010 Mathematics Subject Classification. Primary 18A05; Secondary 54E35, 08A72.
\end{flushleft}

\begin{flushleft}
Key words and phrases. Metric, 2-Metric space, Category, Functor, Yoneda embedding, Path, Triangle.
\end{flushleft}
proposed the notion of “metric 1-space” which was a category together with a “distance function” \( d(f) \) for arrows \( f : x \to y \), which would then be required to satisfy the analogues of the usual axioms of a metric space. In his setup, the pair \((x, y)\) is replaced by a pair of objects plus an arrow \( f \) from \( x \) to \( y \).

In our situation, we would like to generalize the notion of 2-metric in a similar way replacing a triple of points \((x, y, z)\) by a triple of objects together with arrows \( f : x \to y, f : y \to z \) and \( h : x \to z \). In this setup, we don’t need to start with a category but only with a graph and the 2-metric itself represents some kind of approximation of the notion of composition.

In an approximate categorical structure, then the underlying set-theoretical object is a graph, consisting of a set of objects \( X \) and sets of arrows \( A(x, y) \) for any \( x, y \in X \). The distance function \( d(f, g, h) \) is required to be defined whenever \( f \in A(x, y), g \in A(y, z) \) and \( h \in A(x, z) \). The main axioms, generalizing the tetrahedral axiom of a 2-metric space, are the left and right associativity properties. These concern the situation of a sequence of objects \( x, y, z, w \) and arrows going in the increasing direction:

The left associativity condition says

\[
d(a, h, c) \leq d(f, g, a) + d(g, h, b) + d(f, b, c).\]

It means that if \( a \) is close to a composition of \( f \) and \( g \), if \( b \) is close to a composition of \( g \) and \( h \) and if \( c \) is close to a composition of \( f \) and \( b \), then \( c \) is also close to a composition of \( a \) and \( h \).

Looking at the same picture but viewed with the arrow \( c \) passing along the top:
the right associativity condition says
\[ d(f, b, c) \leq d(f, g, a) + d(g, h, b) + d(a, h, c). \]

It is natural to add the data of identity elements \(1_x \in A(x, x)\) such that
\[ d(1_x, f, f) = 0 \quad \text{and} \quad d(f, 1_y, f) = 0. \]
Therefore, The theory works out pretty nicely. For example, we obtain a distance function on the arrow sets
\[ \text{dist}_{A(x, y)}(f, g) := d(1_x, f, g). \]
This is a pseudometric, that is to say it satisfies the triangle inequality but there might be distinct pairs \(f, g\) at distance zero apart. We may however identify them together. This is discussed in Section 4.

Perhaps a more direct way to introduce a categorical notion such that the arrow sets are metric spaces, would be just to consider a category enriched in metric spaces. Here, it will be useful for our development to consider the enrichment as being with respect to the product structure where the metric on the product of two metric spaces is the sum of the metrics on the pieces:
\[ d((x, x'), (y, y')) := d(x, y) + d(x', y'). \]
We describe this theory first, in Section 2.

A metrized category then yields an approximate categorical structure, with the tetrahedral inequalities stated in Proposition 2.1.

Approximate categorical structures are weaker objects, in that any subgraph of an AC structure will have an induced AC structure. In particular, if we start with a metrized category and take any subgraph then we get an AC structure. It is natural to ask whether an arbitrary AC structure arises in this way. There is a good notion of functor \((X, A, d) \to (Y, B, d)\) between two AC structures, see Section 10, so we can look at functors from an AC structure to metrized categories. Any such functor induces a distance on the free category \(\text{Free}(X, A)\) generated by the graph \((X, A)\) and we obtain a distance denoted \(d_{\text{max}}\) on \(\text{Free}(X, A)\) as the supremum of these distances. This is discussed in Section 13. The upper bound
\[ d_{\text{max}}(f, g, h) \leq d(f, g, h) \]
is tautological.

Our main result will be to give a strong lower bound under a certain hypothesis. Recall that in [1] and [2], it was useful to introduce a new axiom, called \textit{transitivity}, for 2-metric spaces. This was a metric version of the idea that given four points, if two triples are colinear
then all four are colinear, especially if the two middle points aren’t too close together.

In Section 7, we introduce the analogue of the transitivity axiom for AC structures in Definition 7.1. This axiom turns out to be what is required in order to be able to define the Yoneda functors $Y_u$, for $u \in X$. We would like to set $Y_u(x) := A(u, x)$ together with its distance. This is a metric space and the distance $d(a, f, b)$ allows us to define a metric correspondence from $Y_u(x)$ to $Y_u(y)$, see Section 12. There is a metrized category of (bounded) metric spaces with morphisms the metric correspondences. If $(X, A, d)$ is transitive, then $Y_u$ is a functor from $(X, A, d)$ to this metrized category.

Existence of these functors yields lower bounds on $d_{\text{max}}(f, g, h)$ and somewhat surprisingly the lower bounds are sharp: we have that

$$d_{\text{max}}(f, g, h) = d(f, g, h)$$

whenever $(X, A, d)$ is transitive (also needed are boundedness and a very weak graph transitivity hypothesis [12.1]). We obtain the following embedding theorem saying that an AC structure with these properties is obtained as a subgraph of a metrized category.

**Theorem 1.1.** Suppose $(X, A, d)$ is an approximate categorical structure that is bounded, satisfies the separation property (Definition 4.7), is absolutely transitive (Definition 7.1) and satisfies Hypothesis 12.1. Then there exists a metrized category $C$ with $\text{Ob}(C) = X$ and inclusions $A(x, y) \subset C(x, y)$ such that for any $f \in A(x, y)$, $g \in A(y, z)$ and $h \in A(x, z)$,

$$d(f, g, h) = d_C(g \circ_C f, h).$$

In Section 14 we discuss how the $d_{\text{max}}$ construction, applied to the standard 2-metric space with triangle area, gives rise to the category of piecewise-linear paths. Then, in Section 15 we discuss numerous further questions and directions.

## 2. Metrized categories

A category is a triple $(X, A, \circ)$ where $X$ is the set of objects, $A(x, y)$ is the set of arrows from $x$ to $y$ for each pair of objects and $g \circ f \in A(x, z)$ whenever $f \in A(x, y)$ and $g \in A(y, z)$. These are subject to the existence of an identity arrow $1_x \in A(x, x)$ satisfying $f \circ 1_x = f$ and $1_y \circ f = f$ for all $f \in A(x, y)$ and the associativity axiom for $f \in A(x, y)$, $g \in A(y, z)$ and $h \in A(z, w)$ requiring

$$h \circ (f \circ g) = (h \circ f) \circ g.$$
We can introduce the notion of pseudometric structure on a category as above, which is by definition a pseudometric \(\phi(f, g)\) defined for \(f, g \in A(x, y)\), satisfying the properties of a pseudometric:

\[
\phi(f, f) = 0, \quad \phi(f, g) = \phi(g, f) \quad \text{and} \quad \phi(f, g) \leq \phi(f, h) + \phi(h, g).
\]

If in addition \(\phi(f, g) = 0 \Rightarrow f = g\), then it is a metric. This separation condition will be imposed as appropriate, see also Definition 4.7 below.

We require the following compatibility with the structure of category: for any triple of objects \(x, y, z \in X\), the composition function

\[ A(x, y) \times A(y, z) \to A(x, z), \quad (f, g) \mapsto g \circ f \]

should be nonincreasing, where we provide the product on the left with the metric

\[
(\phi + \phi)((f, g), (f', g')) := \phi(f, f') + \phi(g, g').
\]

In concrete terms this is equivalent to requiring that

\[
(2.1) \quad \phi(g \circ f, g' \circ f') \leq \phi(f, f') + \phi(g, g').
\]

We call such a structure a pseudo-metrized category. If the separation property holds we call it a metrized category.

If there is no confusion, we denote \(\phi(f, f')\) by just \(\text{dist}_{A(x, y)}(f, f')\) or \(d_{A(x, y)}(f, f')\).

As motivation for the next section, if \((X, A, \circ, \phi)\) is a pseudo-metrized category, we can define a function of three variables \(d(f, g, h)\) defined whenever \(f \in A(x, y)\), \(g \in A(y, z)\) and \(h \in A(x, z)\) by putting

\[
(2.2) \quad d(f, g, h) := \phi(g \circ f, h).
\]

**Proposition 2.1.** This function satisfies the following properties:

— for all \(f, g \in A(x, y)\) we have

\[
d(f, 1_y, g) = d(1_x, f, g) = \phi(f, g);
\]

— for all \((f, g, h; a, b; c)\) with

\[
x \xrightarrow{f} y \xrightarrow{g} z \xrightarrow{h} w
\]

and

\[
x \xrightarrow{a} z, \quad y \xrightarrow{b} w, \quad x \xrightarrow{c} w
\]

we have:

\[
d(f, b, c) \leq d(f, g, a) + d(g, h, b) + d(a, h, c)
\]

and

\[
d(a, h, c) \leq d(f, g, a) + d(g, h, b) + d(f, b, c).
\]
Proof. For the first statements note that
\[ d(f, 1_y, g) := \phi(1_y \circ f, g) = \phi(f, g) \]
and
\[ d(1_x, f, g) := \phi(f \circ 1_x, g) = \phi(f, g). \]

In the second part, applying the definitions, the first inequality that we would like to show is equivalent to
\[ \phi(b \circ f, c) \leq \phi(g \circ f, a) + \phi(h \circ g, b) + \phi(h \circ a, c). \]

By the triangle inequality in \( A(x, w) \) applied to the sequence \( b \circ f, h \circ g \circ f, h \circ a, c \) we have
\[ \phi(b \circ f, c) \leq \phi(b \circ f, h \circ g \circ f) + \phi(h \circ g \circ f, h \circ a) + \phi(h \circ a, c). \]

The composition axiom (2.1) implies that
\[ \phi(h \circ g \circ f, h \circ a) \leq \phi(h, h) + \phi(g \circ f, a) = \phi(g \circ f, a). \]

Similarly
\[ \phi(b \circ f, h \circ g \circ f) \leq \phi(b, h \circ g) = \phi(h \circ g, b). \]

Therefore we get
\[ \phi(b \circ f, c) \leq \phi(h \circ g, b) + \phi(g \circ f, a) + \phi(h \circ a, c). \]

Putting in the definition \( d(f, g, h) := \phi(g \circ f, h) \) this gives
\[ d(f, b, c) \leq d(g, h, b) + d(f, g, a) + d(a, h, c), \]
which is the first inequality. For the second inequality, we would like to show
\[ \phi(h \circ a, c) \leq \phi(g \circ f, a) + \phi(h \circ g, b) + \phi(b \circ f, c). \]

Apply the triangle inequality in \( A(x, w) \) to the sequence \( h \circ a, h \circ g \circ f, b \circ f, c \). We get
\[ \phi(h \circ a, c) \leq \phi(h \circ a, h \circ g \circ f) + \phi(h \circ g \circ f, b \circ f) + \phi(b \circ f, c). \]

As before, (2.1) implies that
\[ \phi(h \circ a, h \circ g \circ f) \leq \phi(g \circ f, a) \text{ and } \phi(h \circ g \circ f, b \circ f) \leq \phi(h \circ g, b). \]

Hence we obtain
\[ \phi(h \circ a, c) \leq \phi(g \circ f, a) + \phi(h \circ g, b) + \phi(b \circ f, c), \]
in other words
\[ d(a, h, c) \leq d(f, g, a) + d(g, h, b) + d(f, b, c) \]
which is the second inequality. \( \square \)
Suppose \((X, A, \circ, \phi)\) is a category with a pseudometric \(\phi\). Define a new set \(\tilde{A}(x, y)\) to be the quotient of \(A(x, y)\) by the relation that \(x \sim x'\) if \(\phi(x, x') = 0\). This is an equivalence relation. It is compatible with the composition operation by the axiom (2.1). Therefore \(\circ\) induces a composition which we again denote \(\circ\) on \((X, \tilde{A})\). Also the distance \(\phi\) induces a metric \(\tilde{\phi}\) on \(\tilde{A}(x, y)\), and \((X, \tilde{A}, \circ, \tilde{\phi})\) is a metrized category satisfying the separation property.

3. APPROXIMATE CATEGORIES

Abstracting the properties given by Proposition [2.1] we can forget about the composition operation and just look at the function of three variables \(d(f, g, h)\).

Consider a set of objects \(X\) and for each \(x, y \in X\) a set of arrows \(A(x, y)\). Suppose we have isolated an identity arrow \(1_x \in A(x, x)\) for each \(x \in X\). Consider a triangular distance function \(d(f, g, h)\) defined whenever

\[
\begin{array}{c}
\text{that is to say } f \in A(x, y), \quad g \in A(y, z) \quad \text{and} \quad h \in A(x, z).
\end{array}
\]

Assume the following axioms:

**Identity axioms**—
Left identity : for all \(f \in A(x, y)\) we have
\[
d(f, 1_y, f) = 0;
\]
Right identity : for all \(f \in A(x, y)\) we get
\[
d(1_x, f, f) = 0;
\]

**Associativity axioms**—given a “tetrahedron” \((f, g, h; a, b; c)\) with
\[
\begin{array}{c}
x \xrightarrow{f} y \xrightarrow{g} z \xrightarrow{h} w
\end{array}
\]
and
\[
\begin{array}{c}
a \xrightarrow{x} z, \quad b \xrightarrow{y} w, \quad c \xrightarrow{x} w,
\end{array}
\]
Left associativity :
\[
d(a, h, c) \leq d(f, g, a) + d(g, h, b) + d(f, b, c);
\]
Right associativity:
\[
d(f, b, c) \leq d(f, g, a) + d(g, h, b) + d(a, h, c).
\]
Definition 3.1. An approximate categorical structure (AC-structure) is a triple \((X, A, d)\), together with the specified identities \(1_x\), satisfying the above axioms.

An approximate semi-categorical structure is a triple \((X, A, d)\), without specified identities, satisfying just the associativity axioms.

Lemma 3.2. Suppose \((X, A, d)\) is an approximate categorical structure. Then for any \(x, y, z \in X\) with \(f \in A(x, y)\), \(g \in A(y, z)\) and \(h \in A(x, z)\), we have \(d(f, g, h) \geq 0\).

Proof. For notational simplicity we denote the third map by \(a\). Then, use left associativity for \((f, g, 1_z; a, g; a)\). It says
\[
d(a, 1_z, a) \leq d(f, g, a) + d(g, 1_z, g) + d(f, g, a).
\]
Since \(d(a, 1_z, a) = 0\) and \(d(g, 1_z, g) = 0\) we get \(2d(f, g, a) \geq 0\), therefore \(d(f, g, a) \geq 0\) as claimed. \(\Box\)

Lemma 3.3. Suppose \((X, A, d)\) is an approximate semi-categorical structure (resp. categorical structure) and suppose we are given sub-sets \(B(x, y) \subset A(x, y)\) (resp. subsets containing \(1_x\) if \(x = y\)). Then \((X, B, d|_B)\) is an approximate semi-categorical (resp. categorical) structure.

Proof. The conditions for \(d|_B\) follow from the same conditions for \(d\) on \(A\). \(\Box\)

Corollary 3.4. Let \((X, C, \circ, \phi)\) be a pseudo-metrized (resp. metrized) category and suppose \(A(x, y) \subset C(x, y)\) are subsets. Then \((X, A, d|_A)\) is an approximate semi categorical structure and if \(1_x \in A(x, x)\) then we get an approximate categorical structure (resp. separated AC structure).

An approximate categorical structure is clearly also an approximate semi-categorical structure.

Question 3.5. Suppose \((X, A, d)\) is an approximate semi-categorical structure. Let \(A^+(x, y) := A(x, y)\) for \(x \neq y\) and \(A^+(x, y) := A(x, x) \cup \{1_x\}\). Is there a natural way to extend \(d\) to \(A^+\) to obtain an AC structure?

4. Metric on the arrow sets

Lemma 4.1. If \(x, y \in X\) and \(f, g \in A(x, y)\) then
\[
d(f, 1_y, g) = d(1_x, f, g).
\]
Proof. Use the left associativity axiom for \((1_x, f, 1_y; f, f; g)\) which says
\[
d(f, 1_y, g) \leq d(1_x, f, f) + d(f, 1_y, f) + d(1_x, f, g) = d(1_x, f, g).
\]
On the other hand, the right associativity axiom for \((1_x, f, 1_y; f, f; g)\) gives
\[
d(1_x, f, g) \leq d(1_x, f, f) + d(f, 1_y, f) + d(f, 1_y, g) = d(f, 1_y, g).
\]
\[\square\]
By the preceding lemma, we can define a distance on \(A(x, y)\) as follows, for \(f, g \in A(x, y)\) put
\[
\phi(f, g) := d(1_x, f, g).
\]
Note that for any \(x\) we have
\[
d(1_x, 1_x, 1_x) = 0.
\]

**Lemma 4.2.** This distance is reflexive:
\[
\phi(f, f) = 0,
\]
symmetric:
\[
\phi(f, g) = \phi(g, f),
\]
and satisfies the triangle inequality:
\[
\phi(f, g) \leq \phi(f, h) + \phi(h, g).
\]

**Proof.** By definition
\[
\phi(f, f) = d(f, 1_y, f) = 0
\]
by the left identity axiom. Using left associativity we have
\[
\phi(f, g) = d(f, 1_y, g) \leq d(g, 1_y, f) + d(1_y, 1_y, 1_y) + d(g, 1_y, g),
\]
so
\[
\phi(f, g) \leq d(g, 1_y, f) = \phi(g, f),
\]
which by symmetry gives \(\phi(f, g) = \phi(g, f)\). For the triangle inequality suppose \(f, g, h \in A(x, y)\), then applying left associativity we get
\[
\phi(f, g) = d(f, 1_y, g) \leq d(h, 1_y, f) + d(1_y, 1_y, 1_y) + d(h, 1_y, g),
\]
therefore
\[
\phi(f, g) \leq \phi(h, f) + \phi(h, g).
\]
\[\square\]
Lemma 4.3. Given $f, f' \in A(x, y)$, $h \in A(y, z)$ and $c \in A(x, z)$ we have
\[ d(f, h, c) \leq d(f', h, c) + \phi(f, f'). \]

Proof. Applying right associativity with $(f, 1_y, h; f', h; c)$, that is $g := 1_y$, $a := f'$ and $b := h$ we get
\[ d(f, b, c) \leq d(f, g, a) + d(g, h, b) + d(a, h, c), \]
which in our case says
\[ d(f, h, c) \leq d(f, 1_y, f') + d(1_y, h, h) + d(f', h, c). \]
Since $\phi(f, f') = d(f, 1_y, f')$ and $d(1_y, h, h) = 0$ we obtain the desired statement. \hfill $\square$

Similarly,

Lemma 4.4. Given $f \in A(x, y)$, $h, h' \in A(y, z)$ and $c \in A(x, z)$ we have
\[ d(f, h, c) \leq d(f, h', c) + \phi(h, h'). \]

Proof. Applying left associativity with $(f, 1_y, h; f; h'; c)$, that is $g := 1_y$, $a := f$ and $b := h'$ we obtain
\[ d(a, h, c) \leq d(f, g, a) + d(g, h, b) + d(f, b, c), \]
which in our case says
\[ d(f, h, c) \leq d(f, 1_y, f) + d(1_y, h, h') + d(f, h', c). \]
As
\[ d(f, 1_y, f) = 0 \text{ and } d(1_y, h, h') = \phi(h, h'), \]
we get the desired statement. \hfill $\square$

For the third edge,

Lemma 4.5. Given $f \in A(x, y)$, $g \in A(y, z)$ and $c \in A(x, z)$ we have
\[ d(f, g, c) \leq d(f, g, c') + \phi(c, c'). \]

Proof. Applying right associativity with $(f, g, 1_z; c'; g; c)$, that is $h := 1_z$, $a := c'$ and $b := g$ we have
\[ d(f, b, c) \leq d(f, g, a) + d(g, h, b) + d(a, h, c), \]
which in our case says
\[ d(f, g, c) \leq d(f, g, c') + d(g, 1_z, g) + d(c', 1_z, c). \]
Since
\[ \phi(c, c') = d(c', 1_z, c) \text{ and } d(g, 1_z, g) = 0, \]
we obtain the desired statement. \hfill $\square$
Putting these together we get:

**Corollary 4.6.** Given \( f, f' \in A(x, y) \), \( g, g' \in A(y, z) \) and \( h, h' \in A(x, z) \) we have

\[
d(f, g, h) \leq d(f', g', h') + \phi(f, f') + \phi(g, g') + \phi(h, h')
\]

**Proof.** By the above lemmas we get successively

\[
d(f, g, h) \leq d(f', g, h) + \phi(f, f') \\
\leq d(f', g', h) + \phi(f, f') + \phi(g, g') \\
\leq d(f', g', h') + \phi(f, f') + \phi(g, g') + \phi(h, h').
\]

□

**Definition 4.7.** We say that an AC structure is separated if

\[
\phi(f, f') = 0 \Rightarrow f = f'.
\]

Equivalently, each \((A(x, y), \phi)\) is a metric space rather than a pseudo-metric space.

The separation property may be ensured by a quotient construction. Given an AC structure in general, define the relation that

\[
f \sim f' \text{ if } \phi(f, f') = 0.
\]

**Lemma 4.8.** This is an equivalence relation on \( A(x, y) \). Let \( \tilde{A}(x, y) := A(x, y)/\sim \). The distance function \( d(f, g, h) \) passes to the quotient to be a function of \( f \in \tilde{A}(x, y) \), \( g \in \tilde{A}(y, z) \) and \( h \in A(x, z) \). Then \((X, \tilde{A}, d)\) is a separated AC structure.

**Proof.** It is an equivalence relation by the triangle inequality of \( \phi \). The above corollary says that \( d \) passes to the quotient. The axioms hold to get an AC structure. □

The lemma shows that an approximate categorical structure can always be replaced by one which satisfies the separation property. We will generally assume that this has been done.

**Lemma 4.9.** Suppose \((X, A, d)\) is a separated AC structure. The function \( d \) is continuous on the topologies associated to the metric spaces \( A(\cdot, \cdot) \). More precisely, for any \( x, y, z \in X \),

\[
d : A(x, y) \times A(y, z) \times A(x, y) \to \mathbb{R}
\]

is a continuous function of its three variables.

**Proof.** This also follows from Corollary 4.6. □
5. The $\epsilon$-categoric condition

We say that $(X, A, d)$ is $\epsilon$-categoric if for any $f \in A(x, y)$ and $g \in A(y, z)$ there exists $h \in A(x, z)$ such that

$$d(f, g, h) \leq \epsilon.$$  

**Lemma 5.1.** Given $f \in A(x, y)$, $g \in A(y, z)$ and $a, a' \in A(x, z)$ we have

$$\phi(a, a') \leq d(f, g, a) + d(f, g, a').$$

*Proof.* Applying left associativity with $(f, g, 1_z; a, g; a')$, that is $h := 1_z$, $b := g$ and $c := a'$ we get

$$d(a, h, c) \leq d(f, g, a) + d(g, h, b) + d(f, b, c),$$

which in our case says

$$d(a, 1_z, a') \leq d(f, g, a) + d(g, 1_z, g) + d(f, g, a').$$

As $d(g, 1_z, g) = 0$ and $d(a, 1_z, a') = \phi(a, a')$ we obtain the desired statement. □

**Corollary 5.2.** If

$$d(f, g, h) = d(f, g, h') = 0,$$

then $\phi(h, h') = 0$. In particular, if the separation property (Definition 4.7) is satisfied then it implies that $h = h'$.

6. Amplitude

We consider now a new function for any $x, y \in X$ denoted

$$\alpha: A(x, y) \to \mathbb{R}.$$  

We would like to consider $\alpha$ as analogous to the metric function considered by Weiss [4], that is $\alpha(f)$ is supposed to represent the “length” of $f$.

We ask first that $\alpha$ satisfy the *reflexivity axiom* $\alpha(1_x) = 0$.

Recall that if $(X, d)$ is a 2-metric space then the distance function $\varphi(x, y)$ satisfies

$$\varphi(x, z) \leq \varphi(x, y) + \varphi(y, z) + d(x, y, z).$$

With this motivation, we ask that $\alpha$ satisfy the *triangle inequality*: for any $x, y, z \in X$ and $f \in A(x, y)$, $g \in A(y, z)$ and $h \in A(x, z)$, we should have

$$\alpha(h) \leq \alpha(f) + \alpha(g) + d(f, g, h).$$
Recall furthermore that Weiss imposed an additional axiom for his metric function, namely (in his notations \([\mathbf{4}]\) that
\[
|\varphi(u) - \varphi(v)| \leq \varphi(v \circ u).
\]
Transposed into our approximately categorical situation, we therefore add the permuted triangle inequalities: for any \(x, y, z \in X\) and \(f \in A(x, y), g \in A(y, z)\) and \(h \in A(x, z)\), we should have
\[
\alpha(f) \leq \alpha(g) + \alpha(h) + d(f, g, h)
\]
and
\[
\alpha(g) \leq \alpha(f) + \alpha(h) + d(f, g, h).
\]

**Lemma 6.1.** Under the above axioms, we have \(\alpha(f) \geq 0\) for all \(f \in A(x, y)\).

**Proof.** Let \(g := 1_y\) and \(h := f\), then
\[
d(f, g, h) = d(f, 1_y, f) = 0
\]
by the left identity property of \(d\), so
\[
0 = \alpha(g) \leq \alpha(f) + \alpha(h) + d(f, g, h) = \alpha(f).
\]

**Lemma 6.2.** Suppose \(f, f' \in A(x, y)\). Then
\[
\alpha(f') \leq \alpha(f) + \phi(f, f'),
\]
implying that
\[
|\alpha(f) - \alpha(f')| \leq \phi(f, f').
\]
In particular, \(\alpha\) is continuous.

**Proof.** Apply the main property of \(\alpha\) with \(g := 1_y\) and \(h := f'\). It says
\[
\alpha(f') \leq \alpha(f) + \alpha(1_y) + d(f, 1_y, f').
\]
Since \(\alpha(1_y) = 0\) by hypothesis and
\[
d(f, 1_y, f') = \phi(f, f'),
\]
we get the first statement. The rest follows.

We might want to assume the *anti-reflexivity property* that
\[
\alpha(f) = 0 \Rightarrow x = y, \ f = 1_x.
\]
However, we might also want to ignore this property for example if we set \(\alpha(f) := 1\) for all \(f\).
7. Transitivity

Suppose \( \alpha \) is a function on the arrow sets. We define a notion of \textit{transitivity}, or rather in fact several related notions. Here by convention an inf over the empty set is \( +\infty \) and its product with 0 is said to be 0.

\textbf{Definition 7.1.} We say that \((X, A, d)\) satisfies left transitivity with respect to \( \alpha \) if for all \( x, y, z, w \in X \) and \( f \in A(x, y) \), \( g \in A(y, z) \), \( h \in A(z, w) \), \( k \in A(y, w) \) and \( l \in A(x, w) \) we have

\[
\alpha(k) \inf_{a \in A(x, z)} (d(f, g, a) + d(a, h, l)) \leq d(g, h, k) + d(f, k, l).
\]

We say that \((X, A, d)\) satisfies right transitivity with respect to \( \alpha \) if for all \( x, y, z, w \in X \) and \( f \in A(x, y) \), \( g \in A(y, z) \), \( h \in A(z, w) \), \( k \in A(x, z) \) and \( l \in A(x, w) \) we have

\[
\alpha(k) \inf_{a \in A(y, w)} (d(g, h, a) + d(f, a, l)) \leq d(g, g, k) + d(k, h, l).
\]

We say that \((X, A, d)\) is \( \alpha \)-transitive if it satisfies both conditions.

We say that \((X, A, d)\) is absolutely (left or right) transitive if it satisfies one or both of the above conditions with \( \alpha(k) = 1 \) for all \( k \).

These notions may be motivated by the transitivity condition introduced in \cite{1} as shown by the following lemma. Another motivation is that these transitivity conditions provide the information to be used in Section 12 below in order to show that the Yoneda constructions give functors.

\textbf{Lemma 7.2.} Suppose \((X, d_X)\) is a set with a function \( d_X(x, y, z) \in \mathbb{R} \). Let \( A^{\text{coarse}}(x, y) := \{*_{x,y}\} \) define the coarse graph structure on \( X \). Define

\[
d(*_{x,y}, *_{y,z}, *_{x,z}) := d_X(x, y, z).
\]

Then \((X, A^{\text{coarse}}, d)\) is an AC structure if and only if \((X, d_X)\) is a 2-metric space. Suppose the 2-metric \( d_X \) is bounded. Put \( \alpha(*_{x,y}) := \varphi(x, y) \) where \( \varphi(x, y) := \sup_{c \in X} d(x, y, c) \) is the distance function \cite{1}. If \((X, d_X)\) satisfies the transitivity axiom of \cite{1}, then \((X, A^{\text{coarse}}, d)\) is an \( \alpha \)/2-transitive AC structure. If \((X, A^{\text{coarse}}, d)\) is an \( \alpha \)-transitive AC structure then \((X, d_X)\) satisfies the transitivity axiom of \cite{1}.

\textbf{Proof.} Suppose \((X, d_X)\) is a 2-metric space, then we obtain an AC structure. The identities are \( 1_x := *_{x,x} \). We have \( d_X(x, x, y) = 0 \) and \( d_X(x, y, y) = 0 \), which show the left and right identity axioms for an AC structure. The left associativity property for an AC structure requires that for any \( x, y, z, w \in X \) we have

\[
d(*_{x,z}, *_{z,w}, *_{x,w}) \leq d(*_{x,y}, *_{y,z}, *_{x,z}) + d(*_{y,z}, *_{z,w}, *_{y,w}) + d(*_{x,y}, *_{y,w}, *_{x,w}).
\]
This translates as
\[ d_X(x, z, w) \leq d_X(x, y, z) + d_X(y, z, w) + d_X(x, y, w) \]
which is the tetrahedral axiom for a 2-metric space with \( y \) as the point in the middle. Similarly, right associativity for the AC structure translates to the same tetrahedral axiom but with \( z \) as the point in the middle. Thus if \((X, d_X)\) is a 2-metric space then \((X, A, d)\) is an AC-structure.

In the other direction, if \((X, A, d)\) is an AC structure then we have seen that \( d(*_{x,y}, *_{y,z}, *_{x,z}) \geq 0 \), so \( d_X(x, y, z) \geq 0 \). The axioms for a 2-metric space now translate from the axioms for an AC structure as above.

We now relate the transitivity conditions. This is not a perfect correspondence, because we have modified our definition of transitivity slightly in order that it work better with the discussion to come later in the paper. Suppose \((X, d_X)\) is a bounded 2-metric space and put
\[ \alpha(*_{x,y}) := \varphi(x, y) = \sup_{c \in X} d_X(x, y, c). \]

The transitivity axiom of \([1]\) says that given 4 points \( x, y, z, w \) we have
\[ d_X(x, y, w) \varphi(y, z) \leq d_X(x, y, z) + d_X(y, z, w). \]
It implies by permutation that
\[ d_X(x, z, w) \varphi(y, z) \leq d_X(x, y, z) + d_X(y, z, w). \]

On the other hand, for an amplitude \( \alpha \) our left and right transitivity axioms in Definition \([7,1]\) both translate in terms of \( d_X \) to
\[ \alpha(*_{y,z})(d_X(x, y, w) + d_X(x, z, w)) \leq d_X(x, y, z) + d_X(y, z, w). \]
In the case of left transitivity we should apply Definition \([7,1]\) to the points in order \( y, x, z, w \) which is the same as right transitivity of Definition \([7,1]\) for the points in order \( x, y, w, z \).

If we assume the transitivity of \([1]\) then by adding the two previous equations and dividing by 2 we get Definition \([13,1]\) for the function \( \alpha/2 \). On the other hand, by positivity of the distances, if we know the condition of Definition \([7,1]\) for \( \alpha \) then we get the transitivity property of \([1]\). □

Notice that the function \( \alpha \) used for the definition of absolutely transitive, is not an amplitude since it doesn’t satisfy the property \( \alpha(1_x) = 0 \). It will be interesting to see if Theorems \([13,1]\) and \([13,6]\) which use the absolute transitivity condition in an essential way, can be extended to relative transitivity with respect to an amplitude function \( \alpha \).
8. The 0-categoric situation

If \((X, A, d)\) is 0-categoric, then it corresponds to an actual category, and the composition is a non expansive function \(A(x, y) \times A(y, z) \rightarrow A(x, z)\) with respect to the sum distance on the product.

**Theorem 8.1.** Suppose \((X, A, d)\) is a 0-categoric approximate category, and suppose that it is separated (Definition 4.7). Then for any \(f \in A(x, y)\) and \(g \in A(y, z)\) there is a unique element denoted \(g \circ f \in A(x, z)\) such that \(d(f, g, g \circ f) = 0\). This defines a composition operation making \((X, A, \circ, \phi)\) into a metrized category. If we let \(d_\phi(f, g, h) := \phi(g \circ f, h)\) then we have \(d_\phi(f, g, h) \leq d(f, g, h)\) whenever these are defined. The composition maps

\[ A(x, y) \times A(y, z) \rightarrow A(x, z) \]

are continuous, and indeed they are distance nonincreasing if the left hand side is provided with the sum metric.

**Proof.** By Corollary 5.2, if \(h\) and \(h'\) are any elements such that

\[ d(f, g, h) = d(f, g, h') = 0, \]

then \(\phi(h, h') = 0\). Since \((X, A, d)\) is separated, this implies that \(h = h'\). Therefore, the composition \(h = g \circ f\) is unique. The associativity (resp. unit) properties imply that the composition is associative (resp. has units).

We would now like to bound the norm of the composition operation. Suppose \(f, f' \in A(x, y)\), \(g, g' \in A(y, z)\) and let \(h := g \circ f\) and \(h' := g' \circ f'\). Apply Corollary 4.6 to \(f', f, g', g\), and two times \(h\). As \(d(f', g', h') = 0\) and \(\phi(h', h') = 0\) we get

\[ d(f, g, h') \leq \phi(f, f') + \phi(g, g'). \]

On the other hand,

\[ \phi(h, h') = d(h, 1_z, h'). \]

Applying the the associativity tetrahedral property to \(f, g, 1_z; h, g; h'\) we get

\[ d(h, 1_z, h') \leq d(f, g, h) + d(g, 1_z, g) + d(f, g, h'). \]

This gives

\[ \phi(h, h') \leq \phi(f, f') + \phi(g, g'). \]

It says that the composition map is non increasing from the sum distance on \(A(x, y) \times A(y, z)\) to \(A(x, z)\). \(\Box\)

**Proposition 8.2.** Suppose that \((X, A, d)\) is \(\epsilon\)-categoric for any \(\epsilon > 0\) and each metric space \((A(x, y), \phi)\) is complete. Then it is 0-categoric.
Proof. Given \( f \in A(x, y) \) and \( g \in A(y, z) \), for every positive integer \( m \), choose an \( h_m \) such that
\[
d(f, g, h_m) \leq 1/m.
\]
By left associativity for \( f, g, 1_z; h_m, g; h_n \) we have
\[
\phi(h_m, h_n) = d(h_m, 1_z, h_n) 
\leq d(f, g, h_m) + d(g, 1_z, g) + d(f, g, h_n) 
\leq \frac{1}{m} + \frac{1}{n}.
\]
It follows that \((h_m)\) is a Cauchy sequence. By the completeness hypothesis, it has a limit which we denote \( g \circ f \). By left associativity for \( 1_x, f, g; f, h_m; g \circ f \) we get
\[
d(f, g, g \circ f) \leq d(1_x, f, f) + d(f, g, h_m) + d(1_x, h_m, g \circ f) 
\leq \frac{1}{m} + \phi(h_m, g \circ f).
\]
The right side \( \to 0 \) as \( m \to \infty \) so we obtain \( d(f, g, g \circ f) = 0 \). This is the 0-categoric property. \( \square \)

**Lemma 8.3.** If \((X, A, d, \alpha)\) is \( \epsilon \)-categoric for any \( \epsilon > 0 \), then it is absolutely transitive.

**Proof.** We show absolute left transitivity. Suppose given \( f, g, h, k, l \) as in the definition. For any \( \epsilon > 0 \) there exists \( a \in A(x, z) \) such that \( d(f, g, a) < \epsilon \). By the tetrahedral axiom,
\[
d(a, h, l) \leq d(f, g, a) + d(g, h, k) + d(f, k, l) = \epsilon + d(g, h, k) + d(f, k, l).
\]
Therefore
\[
d(f, g, a) + d(a, h, l) \leq 2\epsilon + d(g, h, k) + d(f, k, l).
\]
Such an \( a \) exists for any \( \epsilon > 0 \), thus
\[
\inf_{a \in A(x, z)} (d(f, g, a) + d(a, h, l)) \leq d(g, h, k) + d(f, k, l).
\]
This is the absolute left transitivity condition. The proof for absolute right transitivity is similar. \( \square \)

9. **Examples**

**9.1. Example from a 2-metric space.** Suppose \((X, d)\) is a bounded 2-metric space. That is, we suppose first that \( d \) satisfies the axioms denoted \((\text{Sym}), (\text{Tetr}), (Z), (N), \) and \((B)\) of \( \mathbb{I} \). As there, define \( \varphi(x, y) \) to be the supremum of \( d(x, y, c) \) for \( c \in X \). Then we put \( A(x, y) := \)
\{f_{x,y}\} \text{ (the set with a single element which is denoted as } f_{x,y}). \text{ Set } 1_x := f_{x,x}. \text{ Define }
\[ d(f_{x,y}, f_{y,z}, f_{x,z}) := d(x, y, z) \text{ and } \alpha(f_{x,y}) := \varphi(x, y). \]

**Lemma 9.1.** With the above notations and hypotheses, the structure \((X, A, d, \alpha)\) is an approximate categorical structure with amplitude \(\alpha\) and it satisfies the small transitivity condition. If \((X, d)\) satisfies transitivity \([\text{Trans}]\), then the approximate categorical structure is transitive with respect to the amplitude \(\alpha = \varphi\) in the sense of Definition 7.1.

**Proof.** The associativity conditions come from (Tetr) using symmetry (Sym). The identity conditions with \(1_x := f_{x,x}\) come from (Z). The axioms for \(\alpha(f_{x,y}) := \varphi(x, y)\) come from the property
\[ \varphi(x, z) \leq \varphi(x, y) + \varphi(y, z) + d(x, y, z), \]
which in turn came from the tetrahedral condition in view of the definition \(\varphi(x, y) := \sup_c d(x, y, c)\). Small transitivity comes about because
\[ d(x, y, z) \leq \min\{\alpha(f_{x,y}), \alpha(f_{y,z})\} \]
in view of the definition of \(\varphi\). Similarly, transitivity of the 2-metric implies transitivity for the approximate categorical structure. \(\square\)

**Remark:** If \(L \subset X\) is a line, then it corresponds to a 0-categoric substructure of \((X, A, d)\).

**Remark:** The AC structure \((X, A, d)\) defined from a 2-metric space as above, is generally not absolutely transitive, because we need to use the amplitude and in particular \(\alpha(1_x) = 0\).

9.2. **A finite example.** Consider a very first case. Let \(X = \{x\}\) have a single object and \(A(x, x) = \{1, e\}\) with \(1 = 1_x\). Put
\[ \phi := \phi(1, e) = d(1, e, 1) = d(e, 1, 1) = d(1, 1, e). \]

The remaining quantities to consider are \(d(e, e, e)\) and \(d(e, e, 1)\). Recall that there are two categorical structures, with \(e^2 = e\) or \(e^2 = 1\) and these two numbers represent the distances to these two cases.

From the various associativity laws we get the following inequalities:
\[ d(e, e, e) \leq \phi, \]
\[ d(e, e, 1) \leq 2\phi, \]
\[ |d(e, e, e) - \phi| \leq d(e, e, 1), \]
and
\[ |d(e, e, 1) - \phi| \leq d(e, e, e). \]
Since everything is invariant under scaling (and trivial if $\phi = 0$) we may assume $\phi = 1$ and set

$$u := d(e, e, e), \quad v := d(e, e, 1).$$

Note that $u, v \geq 0$. The inequalities become

$$u \leq 1, \quad v \leq 2, \quad |u - 1| \leq v \quad \text{and} \quad |v - 1| \leq u$$

which reduce to

$$u \leq 1, \quad u + v \geq 1 \quad \text{and} \quad v \leq u + 1.$$

Hence, the graph of the allowed region in the $(u, v)$-plane looks like:

The categorical structures are $(u, v) = (1, 0)$ for $e^2 = 1$ and $(u, v) = (0, 1)$ for $e^2 = e$. The third vertex $(1, 2)$ is an extremal case where no categorical relations hold.

9.3. Paths. Consider $X := \mathbb{R}^2$, and let $A(x, y)$ be the set of continuous paths $f : [0, 1] \to X$ with $f(0) = x$ and $f(1) = y$. Let $d(f, g, h)$ denote the infimum of the areas of disks mapping to $X$ such that the boundary maps to the circle defined by joining the paths $f$, $g$ and $h$. Let $1_x$ denote the constant path at $x$.

**Lemma 9.2.** The resulting triple $(X, A, d)$ is an approximate categorical structure.

**Proof.** Suppose $f : [0, 1] \to X$ is a path from $x$ to $y$. If we define $\varphi(s, t) := f(s)$ and restrict to the triangle whose vertices are $(0, 0)$, $(1, 0)$ and $(1, 1)$. The triangle is homeomorphic to a disk and we obtain a disk mapping to $X$ whose boundary consists of the paths $f$, $1_y$ and $f$ again with total area zero. This shows $d(f, 1_y, f) = 0$. The other identity axiom holds similarly. For the tetrahedral axioms, given three disks corresponding to triangles in the interior of the tetrahedron.
we can paste them together to get a disk whose boundary consists of the three outer edges and whose area is the sum of the three areas. This shows the required tetrahedral property for either left or right associativity. □

10. Functors

Given \((X,A)\) and \((Y,B)\), a prefunctorial map \(F : (X,A) \rightarrow (Y,B)\) consists of a map \(F : X \rightarrow Y\) and, for all \(x,y \in X\), a map \(F : A(x,y) \rightarrow A(Fx,Fy)\). We generally assume that it is unital, that is \(F(1_x) = 1_{Fx}\).

Given approximate categorical structures denoted \(d\) on \((X,A)\) and \((Y,B)\) and \(k \geq 0\), we say that a prefunctorial map \(F\) is \(k\)-functorial if it is unital and whenever \(x,y,z \in X\) and \(f \in A(x,y)\), \(g \in A(y,z)\) and \(h \in A(x,z)\) we have

\[
d(F(f), F(g), F(h)) \leq kd(f, g, h).
\]

If \((X,A)\) and \((Y,B)\) are 0-categorical and separated, then this implies that \(F\) respects composition, hence it defines a functor between categories.

Lemma 10.1. Suppose \(F : (X,A) \rightarrow (Y,B)\) is a \(k\)-functorial map. Then for any \(x, y \in X\) and \(f, f' \in A(x,y)\) we have

\[
\phi(F(f), F(f')) \leq k\phi(f, f').
\]

Proof. It follows from the definition of \(\phi\) and the condition that \(F\) is unital. □

11. Metric correspondences

Suppose \((X,d_X)\) and \((Y,d_Y)\) are bounded metric spaces. We define a set of metric correspondences denoted \(\mathcal{M}(X,Y)\) as follows. An element of \(\mathcal{M}(X,Y)\) is a bounded function \(f : X \times Y \rightarrow \mathbb{R}\) satisfying the following axioms:

(MC0)—if \(X\) is nonempty then \(Y\) is nonempty;

(MC1)—for any \(x, x' \in X\) and \(y \in Y\) we have

\[
f(x, y) \leq d_X(x, x') + f(x', y);
\]

(MC2)—for any \(x \in X\) and \(y, y' \in Y\) we get

\[
f(x, y) \leq f(x, y') + d_Y(y, y').
\]

A functional metric correspondence is a metric correspondence which also satisfies the axiom

(MF)—for any \(x \in X\) and \(y, y' \in Y\) we obtain

\[
d_Y(y, y') \leq f(x, y) + f(x, y').
\]
Let $\mathcal{F}(X, Y) \subset \mathcal{M}(X, Y)$ be the subset of functional metric correspondences.

**Definition 11.1.** If $X, Y, Z$ are metric spaces, and $f \in \mathcal{M}(X, Y)$ and $g \in \mathcal{M}(Y, Z)$ define their composition by

$$(g \circ f)(x, z) := \inf_{y \in Y} (f(x, y) + g(y, z)).$$

If $X \neq \emptyset$ then by (MC0) also $Y \neq \emptyset$ so we can form the inf. If $X = \emptyset$ then nothing needs to be given to define $(g \circ f)$.

Define the identity $i_X \in \mathcal{M}(X, X)$ by

$$i_X(x, x) := d_X(x, x).$$

Define a distance on $\mathcal{M}(X, Y)$ by

$$d_M(X, Y)(f, f') := \sup_{x \in X, y \in Y} |f(x, y) - f'(x, y)|.$$  

The supremum exists since we have assumed that our correspondence function $f$ is bounded.

**Proposition 11.2.** The composition operation

$$\circ : \mathcal{M}(Y, Z) \times \mathcal{M}(X, Y) \rightarrow \mathcal{M}(X, Z)$$

defined in the previous definition, with the identities $i_X$, provides a structure of metrized category denoted $\textbf{MetCor}$ whose objects are the metric spaces and whose morphism spaces are the metric spaces $(\mathcal{M}(X, Y), d_M(X, Y))$.

Proof. First, suppose $f \in \mathcal{M}(X, Y)$, and consider the composition $g := f \circ i_X$. We have

$$g(x, y) = \inf_{u \in X} (i_X(x, u) + f(u, z)).$$

Taking $u := x$ we get $g(x, y) \leq f(x, y)$, but on the other hand, by hypothesis

$$f(x, y) \leq i_X(x, u) + f(u, y)$$

for any $u$, so

$$f(x, y) \leq g(x, y).$$

This shows the right identity axiom $f \circ i_X = f$ and the proof for left identity is the same.

Suppose $f \in \mathcal{M}(X, Y)$, $g \in \mathcal{M}(Y, Z)$ and $h \in \mathcal{M}(Z, W)$. Put $a := g \circ f$. Then

$$a(x, z) = \inf_{y \in Y} (f(x, y) + g(y, z)).$$
and
\[(h \circ a)(x, w) = \inf_{z \in Z} (a(x, z) + h(z, w)) \]
\[= \inf_{y \in Y, z \in Z} (f(x, y) + g(y, z) + h(z, w)).\]

If we put \(b := h \circ g\) then the expression for \((b \circ f)(x, w)\) is the same, showing associativity. The composition operation therefore defines a category.

To show that the metric gives a metrized structure, we need to show that
\[d_M(X, Z)(g \circ f, g \circ f') \leq d_M(X, Y)(f, f') + d_M(Y, Z)(g, g').\]

Suppose
\[d_M(X, Y)(f, f') < \epsilon \quad \text{and} \quad d_M(Y, Z)(g, g') < \epsilon.\]

It means that
\[f(x, y) < f'(x, y) + \epsilon, \quad f'(x, y) < f(x, y) + \epsilon\]
and
\[g(y, z) < g'(y, z) + \epsilon, \quad g'(y, z) < g(y, z) + \epsilon.\]

Then
\[(g' \circ f')(x, z) = \inf_{y \in Y} (f'(x, y) + g'(y, z)) \]
\[\leq \inf_{y \in Y} (f(x, y) + \epsilon + g(y, z) + \epsilon) \]
\[= (g \circ f)(x, z) + \epsilon + \epsilon.\]

Similarly
\[(g \circ f)(x, z) \leq (g' \circ f')(x, z) + \epsilon + \epsilon.\]

It follows from this statement that
\[d_M(X, Z)(g \circ f, g \circ f') \leq d_M(X, Y)(f, f') + d_M(Y, Z)(g, g')\]
as required. \(\square\)

We can also provide the collection of sets \(\mathcal{M}(X, Y)\) with an AC structure. If \(X, Y, Z\) are three metric spaces, define for \(f \in \mathcal{M}(X, Y)\), \(g \in \mathcal{M}(Y, Z)\) and \(h \in \mathcal{M}(X, Z)\) the distance
\[d(f, g, h) := \sup_{x \in X, z \in Z} \left| h(x, z) - \inf_{y \in Y} (f(x, y) + g(y, z)) \right|.\]

Write \(X \xrightarrow{f} Y\) if \(f \in \mathcal{M}(X, Y)\).
Lemma 11.3. The condition $d(f, g, h) < \epsilon$ is equivalent to the conjunction of the following two conditions:

$(d1)$—for any $x, y, z$ we have
\[ h(x, z) \leq f(x, y) + g(y, z) + \epsilon \]
and

$(d2)$—for any $x, z$ there exists $y \in Y$ with
\[ f(x, y) + g(y, z) \leq h(x, z) + \epsilon. \]

Proof. This is similar to the technique used in the previous proof. \qed

Corollary 11.4. The above distance satisfies the axioms for an AC structure. Furthermore, it is absolutely transitive. It is the AC structure associated to the metrized category $\text{MetCor}$.

Proof. This follows from Propositions 11.2 and 2.1. Absolute transitivity follows from Lemma 8.3. \qed

We can more generally define, for any $k > 0$, the $k$-contractive metric correspondences $M(X, Y; k)$. For this, we keep the second condition the same but modify the first condition so it says

$(MC1)$—for any $x, x' \in X$ and $y \in Y$ we have
\[ f(x, y) \leq kd_X(x, x') + f(x', y); \]

$(MC2)$—for any $x \in X$ and $y, y' \in Y$ we get
\[ f(x, y) \leq f(x, y') + d_Y(y, y'). \]

Again, the functionality condition (MF) is the same as before. Notice that the identity $i_X$ will be in here only if $k \geq 1$ and furthermore if $f$ is $k$-contractive then we would need $k \leq 1$ in order to get $d(i_X, f, f) = 0$.

It will undoubtedly be interesting to try to iterate the composition (to be defined in the proof of transitivity above) of $k$-contractive correspondences and to study convergence of the iterates.

12. The Yoneda Functors

Suppose $(X, A, d)$ is an AC structure, such that each $A(x, y)$ is a bounded metric space. Choose $u \in X$. Then we would like to define a “Yoneda functor” $x \mapsto A(u, x)$ from $(X, A, d)$ to the AC structure of metric correspondences. Put
\[ Y_u(x) := (A(u, x), \text{dist}_{A(u, x)}) \]
where the distance $\text{dist}_{A(u, x)}$ is the distance $\phi$ coming from $d$ as in Section 4. We assume the separation axiom of Definition 4.7, so $Y_u(x)$ is a metric space and it is bounded by assumption.
Make the following hypothesis.

**Hypothesis 12.1.** Whenever \( A(x, y) \) and \( A(y, z) \) are nonempty, then \( A(x, z) \) is nonempty too.

For any \( f \in A(x, y) \) define \( Y_u(f) \in \mathcal{M}(Y_u(x), Y_u(y)) \) by

\[
Y_u(f)(a, b) := d(a, f, b).
\]

**Lemma 12.2.** Assuming Hypothesis 12.1, if \( f \in A(x, y) \) then

\[
Y_u(f) \in \mathcal{M}(Y_u(x), Y_u(y)).
\]

**Proof.** We need to show (MC0), (MC1) and (MC2). Suppose \( Y_u(x) = A(u, x) \) is nonempty. Then by Hypothesis 12.1, \( Y_u(y) = A(u, y) \) is also nonempty, giving (MC0).

Suppose \( a, a' \in Y_u(x) = A(u, x) \) and \( b \in Y_u(y) = A(u, y) \). We have

\[
Y_u(f)(a, b) = d(a, f, b) \leq d_{A(u,x)}(a, a') + d(a', f, b) = d_{A(u,x)}(a, a') + Y_u(f)(a', b)
\]

by Lemma 4.3, giving (MC1).

Suppose \( a \in Y_u(x) = A(u, x) \) and \( b, b' \in Y_u(y) = A(u, y) \), then

\[
Y_u(f)(a, b) = d(a, f, b) \leq d_{A(u,y)}(b, b') + d(a, f, b') = d_{A(u,y)}(b, b') + Y_u(f)(a, b')
\]

by Lemma 4.3, giving (MC2).

**Proposition 12.3.** Suppose \((X, A, d)\) satisfies absolute left transitivity (Definition 7.7). Then the Yoneda map \( Y_u \) defined above is a functor (with contractivity constant 1).

**Proof.** Suppose \( x, y, z \in X \) and \( f \in A(x, y), \ g \in A(y, z) \) and \( h \in A(x, z) \). We would like to show

\[
d_{\mathcal{M}(x,z)}(Y_u(g) \circ Y_u(f), Y_u(h)) \leq d(f, g, h).
\]

We have for \( a \in Y_u(x) = A(u, x) \) and \( c \in Y_u(z) = A(u, z) \),

\[
Y_u(g) \circ Y_u(f)(a, c) = \inf_{b \in A(u,y)} (Y_u(g)(b, c) + Y_u(f)(a, b)).
\]

Note that by Hypothesis 12.1, given \( a \) and \( f \) it follows that \( A(u, y) \) is nonempty so the inf exists. Now

\[
d_{\mathcal{M}(x,z)}(Y_u(g) \circ Y_u(f), Y_u(h)) = \\
\sup_{a \in A(u,x), c \in A(u,z)} |Y_u(h)(a, c) - Y_u(g) \circ Y_u(f)(a, c)|
\]

\[
= \sup_{a \in A(u,x), c \in A(u,z)} \left| Y_u(h)(a, c) - \inf_{b \in A(u,y)} (Y_u(g)(b, c) + Y_u(f)(a, b)) \right|
\]
We would like to show that this is \( \leq d(f, g, h) \). This is equivalent to asking that for all \( a \in A(u, x) \) and \( c \in A(u, z) \) we should have

\[
(12.1) \quad d(a, h, c) - \inf_{b \in A(u, y)} (d(b, g, c) + d(a, f, b)) \leq d(f, g, h)
\]

and

\[
(12.2) \quad \inf_{b \in A(u, y)} ((d(b, g, c) + d(a, f, b)) - d(a, h, c) \leq d(f, g, h).
\]

In turn, the first one (12.1) is equivalent to

\[
d(a, h, c) \leq d(f, g, h) + \inf_{b \in A(u, y)} (d(b, g, c) + d(a, f, b))
\]

and this is true by the tetrahedral inequality

\[
d(a, h, c) \leq d(f, g, h) + d(b, g, c) + d(a, f, b)
\]

for any \( b \). The second one (12.2) is equivalent to

\[
\inf_{b \in A(u, y)} (d(b, g, c) + d(a, f, b)) \leq d(f, g, h) + d(a, h, c),
\]

but that is exactly the statement of the absolute left transitivity condition of Definition 7.1. Thus under our hypothesis, (12.2) is true. We obtain the required inequality.

For the identities, we need to know that \( Y_u(1_x) = i_{Y_u(x)} \). Recall that the identity \( i_{Y_u(x)} \) in \( M(Y_u(x), Y_u(x)) \) is just the distance function \( d_{Y_u(x)} \), and \( Y_u(x) = A(u, x) \). Its distance function is

\[
d_{Y_u(x)}(f, f') = d(f, 1_x, f')
\]

by the discussion of Section 11 and in turn this is exactly \( Y_u(1_x) \). This shows that \( Y_u \) preserves identities, and completes the proof that \( Y_u \) is a functor with contractivity constant 1. \( \square \)

We can similarly define Yoneda functors in the other direction

\[
Y^u(x) := A(x, u)
\]

with the same properties. The opposed statement of the previous proposition says

**Proposition 12.4.** Suppose \((X, A, d)\) satisfies absolute right transitivity (Definition 7.1). Then the Yoneda map \( Y^u \) is a functor (with contractivity constant 1).

The proof is similar.
13. Functors to metrized categories

Suppose \((X, A, d)\) is an AC structure. We would like to look at functors \(F : (X, A, d) \to (C, d_C)\) to metrized categories.

First, we consider the free category on \((X, A)\). Let \(\text{Free}(X, A)\) denote the free category on the graph \((X, A)\). Thus, the set of objects of \(\text{Free}(X, A)\) is equal to \(X\) and
\[
\text{Free}(X, A)(x, y) := \{(x_0, \ldots, x_k; a_1, \ldots, a_k) : x_i \in X, \ x_0 = x, x_k = y, a_i \in A(x_{i-1}, x_i)\}.
\]
Arrows will be denoted just \((a_1, \ldots, a_k)\) if there is no confusion. Composition is by concatenation and the identity of \(x\) in \(\text{Free}(X, A)\) is the sequence \((\cdot)_x\) of length \(k = 0\) based at \(x_0 = x\).

Suppose given a functor \(F : (X, A, d) \to (C, d_C)\). In particular, \(F : X \to \text{Ob}(C)\) and for any \(x, y \in X\) we have \(F : A(x, y) \to C(Fx, Fy)\). This induces a functor of usual categories
\[
\text{Free}(F) : \text{Free}(X, A) \to C
\]
defined by sending a sequence \((a_1, \ldots, a_k)\) to the composition \(F(a_k) \circ \cdots \circ F(a_1)\) and sending \((\cdot)_x\) to \(1_{Fx}\). Pulling back the distance on \(C\) we obtain a pseudometric \(d_F\) on \(\text{Free}(X, A)\), which is to say for each \(x, y\) we have a distance \(d_F\) on \(\text{Free}(X, A)(x, y)\). These distances form a structure of pseudometrized category on \(\text{Free}(X, A)\), although the separation axiom doesn’t hold.

Now \(F\) is a functor of AC structures, if and only if
\[
d_C(F(g) \circ F(f), F(h)) \leq d(f, g, h)
\]
for any \(f \in A(x, y), g \in A(y, z)\) and \(h \in A(x, z)\) and also if \(d_C(F(1_x), 1_{Fx}) = 0\). These conditions are equivalent to the following conditions on the pullback distance \(d_F\):
\[
d_F((f, g), (h)) \leq d(f, g, h) \text{ and } d_F((1_x), (\cdot)_x) = 0.
\]
Let \(\text{dfunct}(X, A, d)\) denote the set of pseudo-metric structures \(d_F\) on \(\text{Free}(X, A)\) satisfying these conditions, we call these elements functorial distances.

**Proposition 13.1.** Assume Hypothesis [12.1]. For any two arrows \(a\) and \(b\) in \(\text{Free}(X, A)(x, y)\), the set of values of \(d_F(a, b)\) over all \(d_F \in \text{dfunct}(X, A, d)\) is bounded. Therefore we may set
\[
d^{\text{max}}(a, b) := \sup_{d_F \in \text{dfunct}(X, A, d)} d_F(a, b),
\]
and \(d^{\text{max}} \in \text{dfunct}(X, A, d)\) is the unique maximal functorial distance.
Proof. Given that \( \text{Free}(X, A)(x, y) \) is non empty, Hypothesis \([12.1]\) implies that \( A(x, y) \) is nonempty, so we may fix some \( h \in A(x, y) \). We have

\[
d_F(a, b) \leq d_F(a, h) + d_F(h, b),
\]

so it suffices to show that \( d_F(a, h) \) is bounded (the case of \( d_F(h, b) \) being the same by symmetry). Suppose \( a \) is the composition of a sequence of arrows \( a_i \in A(x_{i-1}, x_i) \) with \( x_0 = x \) and \( x_n = y \). Again by our hypothesis, we may choose \( h_i \in A(x_0, x_i) \) with \( h_n = h \). We show by induction on \( i \) that \( d_F((a_i, \cdots, a_0), h_i) \) is bounded as \( F \) ranges over all functorial distances. This is true for \( i = 1 \) since

\[
d_F(a_1, h_1) = d_F((1_x, a_1), h_1) \leq d(1_x, a_1, h_1) = \phi(a_1, h_1).
\]

Suppose it is known for \( i - 1 \). Then

\[
d_F((a_i, \cdots, a_0), h_i) = d_F((a_i, (a_{i-1}, \cdots, a_0), h_i) \leq d_F((a_i, h_{i-1}), h_i) + d_F((a_{i-1}, \cdots, a_0), h_{i-1})
\]

and

\[
d_F((a_i, h_{i-1}), h_i) \leq d(a_i, h_{i-1}, h_i)
\]

whereas \( d_F((a_{i-1}, \cdots, a_0), h_{i-1}) \) is bounded by hypothesis. This completes the induction step and we conclude for \( i = n \) that \( d_F(a, h) \) is bounded.

In view of the form of the axioms for a pseudometric structure on the category \( \text{Free}(X, A)(x, y) \), a supremum of a family of pseudometric structures, individually bounded on any pair of arrows, is again a pseudometric structure. Also the supremum will satisfy the conditions for a functorial distance. Therefore \( d^\text{max} \in \text{dfunct}(X, A, d) \). \( \square \)

We also denote by \( d^\text{max}(f, g, h) \) the value \( d^\text{max}((f, g), (h)) \).

**Remark 13.2.** We have the upper bound

\[
d^\text{max}(f, g, h) \leq d(f, g, h).
\]

We would like to get a lower bound.

**Proposition 13.3.** Suppose \( F : (X, A, d) \rightarrow (\mathcal{C}, d_\mathcal{C}) \) is an AC-functor from \( (X, A, d) \) to a metrized category \( (\mathcal{C}, d_\mathcal{C}) \). Assume Hypothesis \([12.1]\). Then we have

\[
d_\mathcal{C}(F(g) \circ F(f), F(h)) \leq d^\text{max}(f, g, h).
\]

There exists a functor on which this inequality is an equality for all \( f, g, h \).
Proof. Let $d_F$ be the pullback distance induced by $F$. Then $d_F \in \text{dfunct}(X, A, d)$. By the construction of $d^\text{max}$ we have $d_F \leq d^\text{max}$, so

$$d_C(F(g) \circ F(f), F(h)) = d_F((f, g), (h)) \leq d^\text{max}(f, g, h).$$

This shows the inequality.

Let $C^\text{max}$ be the metrized category obtained from $\text{Free}(X, A)(x, y)$ by identifying arrows $a$ and $b$ whenever $d^\text{max}(a, b) = 0$, as described in the paragraph at the end of Section 2. This is a metrized category with distance induced by $d^\text{max}$, the distance is a metric and the map $f \mapsto (f)$ defines an AC functor $F^\text{max}$ from $(X, A, d)$ to $C^\text{max}$. Tautologically, the inequality is an equality for this functor. □

We now apply the previous corollary to the Yoneda functors. Assume that $(X, A, d)$ satisfies absolute transitivity, i.e. transitivity for $\alpha = 1$. Then we have seen that the Yoneda constructions define AC functors $Y_u$ from $(X, A, d)$ to the metrized category $\text{MetCor}$ of metric correspondences with

$$Y_u(x) = (A(u, x), \varphi_{u,x})$$

and for $a \in A(x, y)$,

$$Y_u(a) \in \mathcal{M}(Y_u(x), Y_u(y)) \text{ with } Y_u(f)(a, b) := d(a, f, b).$$

If $(X, A, d)$ is absolutely transitive, then $Y_u$ is an AC functor (Proposition 12.3).

**Corollary 13.4.** Suppose $(X, A, d)$ is absolutely transitive and satisfies Hypothesis 12.1. Then for any $u, x, y, z$ and $a \in A(u, x)$, $b \in A(u, y)$, $c \in A(u, z)$ and $f \in A(x, y)$, $g \in A(y, z)$ and $h \in A(x, z)$ we have

$$d(a, h, c) \leq d^\text{max}(f, g, h) + d(a, f, b) + d(b, g, c).$$

For any $a, c, f, g, h$ as above,

$$\inf_{b \in A(u, y)} (d(a, f, b) + d(b, g, c)) \leq d^\text{max}(f, g, h) + d(a, h, c).$$

**Proof.** For any $f \in A(x, y)$, $g \in A(y, z)$ and $h \in A(x, z)$ we have

$$\sup_{a \in A(u, x), c \in A(u, z)} \left| d(a, h, c) - \inf_{b \in A(u, y)} (d(a, f, b) + d(b, g, c)) \right| \leq d^\text{max}(f, g, h),$$

by the corollary using the fact that $Y_u$ is an AC-functor. We get, for any $a$ and $c$, the two inequalities

$$d(a, h, c) - \inf_{b \in A(u, y)} (d(a, f, b) + d(b, g, c)) \leq d^\text{max}(f, g, h)$$
APPROXIMATE CATEGORIES

and
\[
\inf_{b \in A(u, y)} (d(a, f, b) + d(b, g, c) - d(a, h, c) \leq d^{\max}(f, g, h).
\]
The first one implies the same inequality for any \(b\), giving the first statement of the corollary. The second one gives the second statement of the corollary. □

**Corollary 13.5.** Suppose \((X, A, d)\) is absolutely transitive and satisfies Hypothesis \([12.1]\). Then for any \(f, g, h\) we have
\[
\inf_{b \in A(x, y)} \text{dist}_{A(x, y)}(f, b) + d(b, g, h) \leq d^{\max}(f, g, h).
\]

**Proof.** Apply the previous corollary to \(u = x, a = 1_x\) and \(c = h\). Then
\[
d(a, c, h) = d(1_x, h, h) = 0
\]
and
\[
d(a, f, b) = d(1_x, f, b) = \text{dist}_{A(x, y)}(f, b).
\]
□

We obtain the following embedding theorem.

**Theorem 13.6.** Suppose \((X, A, d)\) is bounded, absolutely transitive and satisfies Hypothesis \([12.1]\). Then for any \(f, g, h\) we have
\[
d^{\max}(f, g, h) = d(f, g, h).
\]
Hence, there exists an AC functor \(F : (X, A, d) \rightarrow (\mathbb{C}, d_{\mathbb{C}})\) to a metrized category, such that for any \(f, g, h\) we have
\[
d(f, g, h) = d_{\mathbb{C}}(F(g) \circ F(f), F(h)).
\]

**Proof.** We have \(d^{\max}(f, g, h) \leq d(f, g, h)\) by Remark \([13.2]\). On the other hand, for any \(b \in A(x, y)\) we have
\[
d(f, g, h) \leq d(b, g, h) + \text{dist}_{A(x, y)}(f, b),
\]
by Lemma \([4.3]\). Applying the result of the previous corollary, this gives
\[
d(f, g, h) \leq d^{\max}(f, g, h).
\]
Now the last statement of Proposition \([13.3]\) gives existence of an AC functor \(F\) satisfying \((13.1)\). □

**Proof of Theorem 1.1:** Notice that the functor constructed in Theorem \([13.6]\) is the identity on the set of objects. The target metrized category \(\mathbb{C}^{\max}\) is the quotient of \(\text{Free}(X, A)\) by the equivalence relation induced by the pseudo-metric \(d^{\max}\). Assume that \((X, A, d)\) satisfies the separation property (Definition \([4.7]\)). The maps on arrow sets \(F : A(x, y) \rightarrow \mathbb{C}(x, y)\) preserve distances, as may be seen by taking \(y = x\) and \(f := 1_x\) in the property. It follows that they are injective,
so we may consider that $A(x, y) \subset C(x, y)$. This gives the inclusion
required to finish the proof of Theorem 1.1.

The following corollary may be viewed as an improvement of Theorem 8.1.

**Corollary 13.7.** Suppose $(X, A, d)$ is bounded and $\epsilon$-categoric for all $\epsilon > 0$. Then there exists an AC functor $F : (X, A, d) \to (C, d_C)$ to a
metrized category, such that for any $f, g, h$ we have

$$d(f, g, h) = d_C(F(g) \circ F(f), F(h)).$$

**Proof.** By Lemma 8.3, $(X, A, d)$ is absolutely transitive. Furthermore,
the $\epsilon$-categoric condition for any one $\epsilon$ implies Hypothesis 12.1. Apply
Theorem 13.6. \qed

14. Paths in $\mathbb{R}^n$

As we have noted above, the absolute transitivity condition doesn’t
apply to the AC structure coming from a 2-metric space. We look at
how to calculate $d^{\text{max}}$, but for simplicity we restrict to the standard
example of euclidean space as a 2-metric space. In what follows, let
$X := \mathbb{R}^n$ with the standard 2-metric: $d_X(x, y, z)$ is the area of the
triangle spanned by $x, y, z$. Let $(X, A, d)$ be the associated AC structure.
Recall that $A(x, y) = \{x, y\}$.

Arrows in $\text{Free}(X, A)$ have the following geometric interpretation.
An arrow from $x$ to $y$ corresponds to a composable sequence in $(X, A)$
going from $x$ to $y$, which in this case just means a sequence of points
$(x_0, \ldots, x_k)$ with $x_0 = x$ and $x_k = y$. We may picture this sequence
as being a piecewise linear path composed of the line segments $x_ix{i+1}$.
Thus, we may consider elements $a \in \text{Free}(X, A)(x, y)$ as being piecewise
linear paths from $x$ to $y$.

Suppose $D \subset \mathbb{R}^2$ is a compact convex polygonal region. Its boundary
$\partial D$ is a closed piecewise linear path. If $s : D \to \mathbb{R}^n$ is a piecewise linear
map, its boundary $\partial s$ is a closed piecewise linear path in $\mathbb{R}^n$. We can
divide $D$ up into triangles on which $s$ is linear, and define $\text{Area}(D, s)$
to be the sum of the areas of the images of these triangles in $\mathbb{R}^n$. This
could also be written as

$$\text{Area}(D, s) = \int_D |ds|.$$

Suppose $a$ is a closed piecewise linear path in $X = \mathbb{R}^n$. Put

$$\text{MinArea}(a) := \inf_{\partial s = a} \text{Area}(D, s)$$

be the minimum of the area of piecewise linear maps from compact
convex polyhedral regions to $X$ with boundary $a$. 
Remark: MinArea\((a)\) is also the minimum of areas of piecewise \(C^1\) maps from the disk, with boundary \(a\).

If \(f, g\) are piecewise linear paths from \(x\) to \(y\), let \(g^{-1}f\) denote the closed path based at \(x\) obtained by following \(f\) by the inverse of \(g\) (the path \(g\) run backwards). Define a metrized structure on the category \textbf{Free}(\(X, A\)) of piecewise linear paths, by
\[
d_{\text{Area}}(f, g) := \text{MinArea}(g^{-1}f).
\]
The associated AC structure also denoted by \(d_{\text{Area}}\) is given by
\[
d_{\text{Area}}(f, g, h) = d_{\text{Area}}(g \circ f, h) = \text{MinArea}(h^{-1}gf).
\]

\textbf{Lemma 14.1.} Suppose two paths \(f, g\) differ by an elementary move, in the sense that one is \(f = (x_0, \ldots, x_i, z, x_{i+1}, \ldots, x_k)\) and the other is \(g = (x_0, \ldots, x_i, z, x_{i+1}, \ldots, x_k)\). Then
\[
d_{\text{max}}(f, g) \leq d_{\text{Area}}(f, g) = d_X(x_i, z, x_{i+1}).
\]

\textbf{Proof.} By the axiom (2.1) for a metrized category applied to the compositions on either side with the paths \((x_0, \ldots, x_i)\) and \((x_{i+1}, \ldots, x_k)\), we get
\[
d_{\text{max}}(f, g) \leq d_{\text{max}}((x_i, z, x_{i+1}), (x_i, x_{i+1})).
\]
By the definition of \(d_{\text{max}}\),
\[
d_{\text{max}}((x_i, z, x_{i+1}), (x_i, x_{i+1})) \leq d(*_{x_i, z}, *_{x_i, x_{i+1}}, *_{x_i, x_{i+1}}) = d_X(x_i, z, x_{i+1}).
\]
Note that the minimal area of a disk whose boundary is a triangle, is the area of the triangle, so \(d_X(x_i, z, x_{i+1})\) is also equal to \(d_{\text{Area}}(f, g)\). \(\Box\)

\textbf{Theorem 14.2.} For \((X, A)\) the AC structure associated to the 2-metric space \(X = \mathbb{R}^n\) with its standard area metric, on the category of piecewise linear paths \textbf{Free}(\(X, A\)) we have \(d_{\text{max}} = d_{\text{Area}}\). In particular, for \(f \in A(x, y), g \in A(y, z)\) and \(h \in A(x, z)\) we have
\[
d(f, g, h) = d_X(x, y, z) = d_{\text{max}}(g \circ f, h).
\]

\textbf{Sketch of proof:} The pseudo-metric \(d_{\text{Area}}\) gives a structure of pseudo-metrized category on \textbf{Free}(\(X, A\)), which agrees with \(d\) on \((X, A)\). It follows from the definition of \(d_{\text{max}}\) that \(d_{\text{max}} \geq d_{\text{Area}}\).

In the other direction, suppose \(f, g \in \textbf{Free}(X, A)(x, y)\) and suppose we are given a piecewise linear map \(s\) from a polyhedron \(D\) to \(\mathbb{R}^n\) with boundary \(\partial s = g^{-1}f\). Dividing the polyhedron up into triangles, we obtain a sequence of paths \(f_0 = f, f_1, \ldots, f_m = g\) such that \(f_i\) and \(f_{i+1}\) differ by an elementary move as in the previous lemma, and the triangles occurring in these elementary moves together make up the polyhedron \(D\). Applying the previous lemma, and in view of the fact
that \( \text{Area}(D, s) \) is the sum of the areas of the triangles in \( \mathbb{R}^n \) in the image of \( D \), we have

\[
\sum_{i=1}^{k} d_{\text{max}}(f_{i-1}, f_i) \leq \text{Area}(D, s).
\]

By the triangle inequality for \( d_{\text{max}} \) we get

\[
d_{\text{max}}(f, g) \leq \text{Area}(D, s),
\]

and since \( d_{\text{Area}}(f, g) \) is the minimum of \( \text{Area}(D, s) \) over all choices of \( D, s \), this shows that \( d_{\text{max}} \leq d_{\text{Area}} \). Therefore \( d_{\text{max}} = d_{\text{Area}} \). \( \square \)

15. FURTHER QUESTIONS

15.1. More transitivity conditions. Various other transitivity conditions may be considered. For example, a weaker \( \leq \epsilon \) transitivity axiom would require existence of \( m \) only when the right hand side is \( \leq \epsilon \).

The small transitivity condition is that if \( f \in A(x, y) \) and \( g \in A(y, z) \) then there exists \( m \in A(x, z) \) such that

\[
d(f, g, m) \leq \min\{\alpha(f), \alpha(g)\}.
\]

We could also weaken the basic transitivity conditions of Definition 7.1 for example by allowing an error of some \( \epsilon \). It would be interesting to see what kinds of lower bounds for \( d_{\text{max}} \) could be obtained with these conditions.

15.2. Lower bound for \( \alpha \)-transitive AC structures. In Theorem 13.6 we used the absolute transitivity condition. However, in examples such as the AC-structure coming from a transitive 2-metric space, only transitivity relative to an amplitude \( \alpha \) holds. Therefore, it is an interesting question to see what kind of lower bound for \( d_{\text{max}} \) could be obtained using \( \alpha \)-transitivity.

15.3. Paths in a 2-metric space. In the example of a 2-metric space \((X, d_X)\), we put

\[
A(x, y) := \{*, x, y\} \quad \text{and} \quad d(*, x, y, *, y, z, *, x, z) := d_X(x, y, z).
\]

We can then construct the induced pseudometric \( d_{\text{max}} \) on \( \text{Free}(X, A) \). Morphisms in \( \text{Free}(X, A) \) may be viewed as paths in \( X \). If \( X = \mathbb{R}^n \) then we view such a path as a piecewise linear path in \( \mathbb{R}^n \), replacing \(*, x, y\) by the straight line segment joining \( x \) to \( y \), and we have seen in Section 14 that \( d_{\text{max}}(f, g) \) is the minimal area of a disk whose boundary consists of the paths \( f \) and \( g \), as in the example of 9.3. It will be interesting to generalize this to arbitrary 2-metric spaces.
15.4. **Natural transformations.** Classically the next step after functors is to consider natural transformations. It is an interesting question to understand the appropriate generalization of this notion to the approximate categorical context.

Suppose \( F, G : (X,A) \rightarrow (Y,B) \) are two prefunctorial maps. A **pre-natural transformation** \( \eta : F \rightarrow G \) is a function which for any \( x, y \in X \) and \( f \in A(x,y) \) associates \( \eta(f) \in B(Fx,Gy) \). We say that \( \eta \) is a **\( k \)-natural transformation** if, for any \( x, y, z \in X \) and \( f \in A(x,y), g \in A(y,z) \) and \( h \in A(x,z) \) we have

\[
d(F(f), \eta(g), \eta(h)) \leq kd(f,g,h) \quad \text{and} \quad d(\eta(f), G(g), \eta(h)) \leq kd(f,g,h).
\]

If \( F \) is \( k \)-functorial then defining \( \eta(f) := F(f) \) gives an “identity prenatural transformation” from \( F \) to itself, and it is also \( k \)-natural.

Suppose \( F, G, H : (X,A) \rightarrow (Y,B) \) are three prefunctorial maps. Suppose \( \eta : F \rightarrow G \), \( \zeta : G \rightarrow H \) and \( \omega : F \rightarrow H \) are prenatural transformations.

For any \( x, y, z \in X \) and \( f \in A(x,y), g \in A(y,z) \) and \( h \in A(x,z) \), consider

\[
Fx \xrightarrow{\eta(f)} Gy \xrightarrow{\zeta(g)} Fz
\]

and ask that

\[
d(\eta(f), \zeta(g), \omega(h)) \leq kd(f,g,h) + \delta_0(\eta, \zeta, \omega).
\]

Let \( \delta(\eta, \zeta, \omega) \) be the inf of the \( \delta_0(\eta, \zeta, \omega) \) which work here. We hope that this will allow to define an approximate categorical structure on the functors and natural transformations. This is left open as a question for the future.

15.5. **Correspondence functors.** It was useful to introduce a notion of metric correspondence between metric spaces. One may ask whether this notion can be extended naturally to metrized categories and AC structures.

Let \((X, A, d_A)\) and \((Y, B, d_B)\) be AC structures. We would like to define a notion of functor from \( A \) to \( B \) using the idea of metric correspondences on the morphism sets. Let us try as follows. Consider a function \( F : X \rightarrow Y \) and functions \( f(x, x', a, b) \in \mathbb{R} \) for any \( x, x' \in X \) and \( a \in A(x, x') \) and \( b \in B(Fx, Fx') \). Roughly speaking we would like to have

\[
\inf_{a''} f(x, x'', a'', b'') + d_A(a, a', a'')
\]

\[
\leq \inf_{b,b'} f(x, x', a, b) + f(x', x'', a', b') + d_B(b, b', b'').
\]
This translates also into: for any \(x, x', x'' \in X\), any \(a \in A(x, x')\), any \(a' \in A(x', x'')\) and any \(a'' \in A(x, x'')\), and any \(b'' \in B(Fx, Fx'')\) we have

\[
\inf_{b \in B(Fx, Fx')} \left( f(x, x', a, b) + f(x', x'', a', b') - f(x, x'', a'', b'') + d_B(b, b', b'') \right) \leq d_A(a, a', a'').
\]

15.6. **Reconstruction questions.** Given a category \((X, C, \circ)\) with a metric structure \(\psi\), we get an approximate categorical structure by Proposition 2.1.

We could then consider a collection of subsets \(A(x, y) \subset C(x, y)\) such that \(A(x, x)\) contains \(1_x\). This will again give an AC structure.

One question will be, to what extent can we recover the structure of \(C\) just by knowing \((X, A, d)\)? Our main theorem provides an existence result in the absolutely transitive case. One may then ask, to what extent is the enveloping metrized category unique?

Another somewhat similar question: suppose \(A(x, y) = C(x, y)\). However, suppose \(d'\) is a different AC-structure obtained by perturbing the original one. For example, note that \(d' = d + \epsilon d_1\) is again an AC-structure for any AC-structure \(d_1\). Question: can we recover the categorical structure, i.e. the composition \(\circ\), from the perturbed \(d'\)?

The AC-structures on \((X, A)\) form a cone, because if \(d_1\) and \(d_2\) are AC-structures and \(c_1, c_2\) are positive constants then \(c_1 d_1 + c_2 d_2\) is again an AC-structure. So, another question is, what properties does this cone have? What do the boundary points or extremal rays look like?

The picture in subsection 9.2 suggests that we should look for the position of the categorical structures within this cone. To recast the questions of two paragraphs ago, given a finite graph, are there small values of \(\epsilon\) such that any \(\epsilon\)-categorical structure is near to a 0-categorical structure?

15.7. **Category theory.** To what extent can we generalize the classical structures and constructions of category theory to the situation of metrized categories and AC structures?

15.8. **Fixed points and optimization.** One of our main original motivations for looking at 2-metric spaces in [1, 2] was to consider the theory of fixed points and other fixed subsets such as lines. In the more general setting of AC structures, we can envision several different kinds of fixed point and iteration problems, for example fixed points of a functor, fixed points of a metric correspondence between metric
spaces, as well as fixed arrows within a metrized category or AC structure. It will be interesting to see what applications these might have to optimization problems.

REFERENCES

[1] A. Aliouche, C. Simpson. Fixed points and lines in 2-metric spaces. *Adv. Math.* **229** (2012), 668-690.

[2] A. Aliouche, C. Simpson. Common fixtures of several maps on 2-metric spaces. *Indian J. Math.* **56** (2014), 229-262.

[3] S. Gähler. 2-metrische Räume und ihre topologische Struktur. *Math. Nachr.* **26** (1963), 115-148.

[4] I. Weiss. Metric 1-spaces. Preprint [arXiv:1201.3960](http://arxiv.org/abs/1201.3960) (2012).

Laboratoire des systèmes dynamiques et contrôle, Université Larbi Ben M’Hidi, Oum-El-Bouaghi, 04000, Algérie

E-mail address: alioumath@gmail.com

CNRS, Laboratoire J. A. Dieudonné, UMR 6621, Université de Nice-Sophia Antipolis, 06108 Nice, Cedex 2, France

E-mail address: carlos@unice.fr