Randomized Algorithms for Monotone Submodular Function Maximization on the Integer Lattice

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Abstract
Optimization problems with set submodular objective functions have many real-world applications. In discrete scenarios, where the same item can be selected more than once, the domain of the target problem is generalized from a finite set to a bounded integer lattice. In this work, we consider the problem of maximizing a monotone submodular function on the bounded integer lattice subject to a cardinality constraint. In particular, we focus on maximizing DR-submodular functions, i.e., functions defined on the integer lattice that exhibit the diminishing returns property. Given any $\epsilon > 0$, we present a randomized algorithm with probabilistic guarantees of $O\left(1 - \frac{1}{e} - \epsilon\right)$ approximation, using a framework inspired by a stochastic greedy algorithm developed for set submodular functions by Mirzasoleiman et al. We then show that, on synthetic DR-submodular functions, applying our proposed algorithm on the integer lattice is faster than the alternatives, including reducing a target problem to the set domain and then applying the fastest known set submodular maximization algorithm.

1 Introduction

In combinatorial optimization, machine learning, and operations research, Submodular function maximization problems are ubiquitous [1]. Consider, for example, facility location [2] and sensor placement [3] problems in operations research. In machine learning, notable examples include experiment design [4, 5], dictionary learning [6, 7], and sparsity inducing regularizers [8]. In these problems, the goal is to pick a subset of a ground set $\mathcal{V} := \{1, 2, \ldots, n\}$ that maximizes a set function $f : 2^\mathcal{V} \rightarrow \mathbb{R}$ defined on the powerset of $\mathcal{V}$.

Yet, there are practical applications in which one is not only interested in knowing whether an element $e \in \mathcal{V}$ is selected, but is also concerned with the amount of copies to select for a given $e$. One such case is represented by the optimal budget allocation problem [9]. In these scenarios, the ground set can be considered as a multiset, or equivalently as a cube $\{x \in \mathbb{Z}_+^\mathcal{V} \mid x \preceq b\}$ on the integer lattice $\mathbb{Z}_+^\mathcal{V}$, where $b \in \mathbb{Z}_+^\mathcal{V}$ is a known vector such that $b_e$ indicates the quantities available for each element $e \in \mathcal{V}$ and $x \preceq y$ means $x_e \leq y_e$ for every element $e \in \mathcal{V}$. Any set function $f : 2^\mathcal{V} \rightarrow \mathbb{R}$ can be transformed into a pseudo-Boolean function $\phi : \{0, 1\}^\mathcal{V} \rightarrow \mathbb{R}$ defined on the Boolean lattice $\{0, 1\}^\mathcal{V}$, hence submodular optimization performed on the integer lattice $\mathbb{Z}^\mathcal{V}$ can be seen as a natural generalization of optimization on the Boolean lattice. There are, however, some important differences to highlight.

We recall that a set function $f : 2^\mathcal{V} \rightarrow \mathbb{R}$ is called set submodular if and only if

$$f(A) + f(B) \geq f(A \cup B) + f(A \cap B) \quad \text{for all } A, B \subseteq \mathcal{V}$$

(1)

A function $f : \mathbb{Z}^\mathcal{V} \rightarrow \mathbb{R}$ is said to be integer-lattice submodular if

$$f(x) + f(y) \geq f(x \land y) + f(x \lor y)$$

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$$f(x) + f(y) \geq f(x \land y) + f(x \lor y)$$

(2)
for any \( x, y \in \mathbb{Z}_+^n \). On the other hand, \( f : \mathbb{Z}^V \to \mathbb{R} \) is called DR-submodular if

\[
f(x + 1_e) - f(x) \geq f(y + 1_e) - f(y)
\]

for all \( e \in V \), and for all \( x, y \in \mathbb{Z}_+^n \) such that \( x \preceq y \). Properties (2) and (3) are sometimes known as weak and strong DR-submodularity, respectively [5].

In this work, we focus on monotone submodular function maximization on the integer lattice subject to a cardinality constraint. More precisely, consider the maximization problem:

\[
\max_{x} f(x) \\
\text{s.t.} \quad ||x||_1 \leq r \\
x_e \leq b_e \quad \text{for all } e \in V
\]

where \( f : \mathbb{Z}^V \to \mathbb{R} \) is a monotone submodular function defined on the integer lattice, \( x \in \mathbb{Z}^V \) is defined such that \( x_e \in \mathbb{Z}_+ \) determines how many copies of an element \( e \in V \) should be selected, \( b \in \mathbb{Z}^V \) is a known vector where \( b_e \in \mathbb{Z}_+ \) represents how many copies of an element \( e \in V \) are available in the ground set, and \( r \in \mathbb{Z}_+ \) denotes the maximum cardinality of a feasible solution. We observe that it should also hold that \( r < \|b\|_1 \), otherwise Eq. (4) becomes an unconstrained submodular maximization problem, which is trivially solved in constant time since \( f \) is monotone.

We assume that \( f : \mathbb{Z}^V \to \mathbb{R} \) is given via a value oracle black box, i.e., given some feasible \( x \in \mathbb{Z}^V \), the oracle returns \( f(x) \). We refer to each oracle invocation as an oracle query. This value oracle model is standard in submodular optimization, as it abstracts away the details of the specific problem in a generic algorithm that solves a submodular optimization problem.

We also assume that \( b_e \geq 1 \) for all \( e \in V \) (otherwise, the elements \( e' \in V \) such that \( b_{e'} = 0 \) can be removed from the ground set \( V \) w.l.o.g.). Moreover, notice that when \( b_e = 1 \) for all \( e \in V \), Eq. (4) is equivalent to a set submodular function maximization problem subject to a cardinality constraint, where we one can select a set \( S \subseteq V \) with at most \( |S| = r \) elements.

Considering the wealth of algorithms for the optimization of set submodular functions, which are limited to express binary decisions (i.e., whether to select one element or not), it is natural to consider a reduction from the integer lattice setting to the set submodular setting that enables the use of popular set submodular maximization procedures like the greedy algorithm in the integer lattice domain. The most natural one generates \( b_e \) elements in a new set \( V' \) for each element \( e \in V \) [12]. The drawback is that this reduction yields a pseudo-polynomial-time algorithm in \( n \), which negatively affects the runtime of the set submodular algorithms as the value of \( b_e \) grows for each \( e \). For DR-submodular functions, [13] proposed another reduction to set-submodular optimization, which enacts a bit decomposition argument that yields a ground set \( V' \) of size \( |V'| = O((\log b + \frac{1}{\varepsilon}) \cdot n) \).

The algorithm presented in this work is based on the Stochastic Greedy technique introduced by [14], which is – in turn – based on the Greedy algorithm [15], which is perhaps the single most famous result in submodular function maximization. Indeed, [15] showed that a simple greedy approach that picks an unselected element at each iteration maximizing the local marginal gain provides a tight \((1 - \frac{1}{e})\)-approximation guarantee for maximizing a monotone submodular function subject to cardinality constraints.

[14] devised an algorithm called Stochastic Greedy (SG for short), which greatly improves upon the running time of Greedy while retaining the same approximation ratio in expectation, using only \( O(n \log \frac{1}{\varepsilon}) \) oracle queries. The SG algorithm uses a randomized subsampling technique that, for each of the \( r \) iterations, randomly selects a subset \( Q \subseteq V \) of a fixed size and finds the element \( e \in Q \) that maximizes the marginal gain. The key difference between this approach and Greedy is that \( Q \) changes at each iteration. Thus, in expectation, SG does cover the entire dataset. In contrast,
GREEDY is equivalent to subsampling elements from $\mathcal{V}$ before looking for the most representative elements in it. Here, we extend the idea underlying the STOCHASTIC GREEDY algorithm from a set to an integer lattice domain, proving that our algorithms are both practically and theoretically faster than STOCHASTIC GREEDY consider a reduction of an integer lattice problem to the set domain.

### 1.1 Our Results

Here, we present a randomized algorithm for maximizing a monotone (DR-)submodular function defined on the integer lattice, improving upon the state of the art \cite{16,17} in terms of practical running time, while obtaining a strong approximation guarantee in high probability. We also show that our algorithms are significantly more stable than the previous state of the art in terms of the number of oracle queries required to approximately solve the target problem.

In particular, our contribution comprises:

- a **STOCHASTIC GREEDY LATTICE** (SGL for short) algorithm for maximizing monotone DR-submodular functions defined on the integer lattice subject to cardinality constraints;

- an analysis of the approximation ratio of SGL, which can be made arbitrarily close to $(1 - \frac{\lambda}{\epsilon})$, which is tight \cite{18}, with arbitrary probability greater or equal to one half;

- empirical experiments on a synthetic class of instances, which indicate the scalability of SGL and show the instability of the algorithms of \cite{16} and \cite{17}, which are considered the current state of the art for the considered constrained maximization problem.

### 2 Background and Notation

#### Notation

We consider a finite $n$-dimensional set $\mathcal{V} := \{1, 2, \ldots, n\}$, which we refer to as ground set, and its powerset $2^\mathcal{V}$. We denote by $\mathbb{R}_+$ the set of non-negative real numbers, and by $\mathbb{Z}_+$ the set of non-negative integer numbers. We use bold face letters such as $\mathbf{x} \in \mathbb{R}^\mathcal{V}$ and $\mathbf{x} \in \mathbb{R}^n$ interchangeably to denote $n$-dimensional vectors. Similarly, we let 0 and 1 indicate $n$-dimensional vectors whose values are all 0 or 1, respectively. Given a vector $\mathbf{x}$, we denote its $e$-th coordinate by $x_e$ or $\mathbf{x}(e)$. We let $1_e \in \mathbb{Z}_+$ denote the characteristic vector, defined such that $1_e(e) := 1$ and $1_e(e') := 0$ for all $e, e' \in \mathcal{V}$ such that $e \neq e'$. We let $\text{supp}(\mathbf{x}) := \{e \in \mathcal{V} \mid x_e > 0\}$ denote the support of $\mathbf{x} \in \mathbb{Z}_+^\mathcal{V}$. For two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^\mathcal{V}_+$, $\mathbf{x} \preceq \mathbf{y}$ means $x_e \leq y_e$ for every element $e \in \mathcal{V}$. Additionally, given $\mathbf{x}, \mathbf{y} \in \mathbb{Z}^n_+$ we let $\mathbf{x} \land \mathbf{y}$ and $\mathbf{x} \lor \mathbf{y}$ denote the coordinate-wise minimum and maximum, respectively, i.e., $(\mathbf{x} \land \mathbf{y})_e := \min\{x_e, y_e\}$ and $(\mathbf{x} \lor \mathbf{y})_e := \max\{x_e, y_e\}$. We define $||\mathbf{x}||_1 := \sum_{e \in \mathcal{V}} |e_k|$ and $||\mathbf{x}||_{\infty} := \max_{e \in \mathcal{V}} |x_e|$.

#### Definitions

A set function $f : 2^\mathcal{V} \to \mathbb{R}$ is **monotone** if, for every $A \subseteq B \subseteq \mathcal{V}$, $f(A) \leq f(B)$. $f$ is said to be **normalized** if $f(\emptyset) = 0$. The marginal gain obtained by adding an element $e \in \mathcal{V}$ to a set $S \subseteq \mathcal{V}$ is defined as $f(e \mid S) := f(S \cup \{e\}) - f(S)$. We now give the analogous definitions for the integer lattice case.

An integer lattice function $f : \mathbb{Z}_+^\mathcal{V} \to \mathbb{R}$ is **monotone** if $\mathbf{x} \preceq \mathbf{y}$ implies $f(\mathbf{x}) \leq f(\mathbf{y})$ for some $\mathbf{x}, \mathbf{y} \in \mathbb{Z}_+^\mathcal{V}$, and $f$ is **normalized** if $f(\mathbf{0}) = 0$. We only consider monotone and normalized functions. The marginal gain obtained by adding an element $e \in \mathcal{V}$ to a vector $\mathbf{x} \in \mathbb{Z}_+^\mathcal{V}$ is defined as $f(1_e \mid \mathbf{x}) := f(\mathbf{x} + 1_e) - f(\mathbf{x})$, where the sum operation is applied component-wise. Given a known vector $\mathbf{b} \in \mathbb{Z}_+^\mathcal{V}$ that defines the multiplicities of the elements $e \in \mathcal{V}$, the problem of maximizing an integer lattice function $f : \mathbb{Z}_+^\mathcal{V} \to \mathbb{R}$ under a cardinality constraint $r \in \mathbb{Z}_+$ is formalized as selecting a vector $\mathbf{x} \in \mathbb{Z}_+^\mathcal{V}$ whose entries $x_e$ have value at most $b_e$ and such that $||\mathbf{x}||_1 \leq r$.

#### 2.1 Prior Work

The problem of maximizing a set submodular function $f : 2^\mathcal{V} \to \mathbb{R}$, i.e., finding a set $S \subseteq \mathcal{V}$ such that $f(S)$ is maximized, is NP-Hard, as it generalizes the MAXCUT problem. Nemhauser et al. first showed a discrete greedy algorithm (commonly referred to as GREEDY) which yields a constant $(1 - \frac{1}{e})$-approximation to the problem of maximizing a monotone submodular function under a cardinality constraint \cite{15}. The approximation ratio of GREEDY is tight, i.e., it...
is the best possible performance guarantee for the considered problem both in the value oracle model and independently of $P \neq NP$ \cite{18, 19} and \cite{20} pioneered continuous algorithms for constrained submodular maximization based on the multilinear extension \cite{21} and the pipeage rounding method \cite{22, 23} of $f$ \cite{21}. While continuous algorithms are interesting on their own and can model a broad class of constraints, they are generally slower than the fastest greedy counterparts \cite{14}.

Submodular optimization on the lattice domain is almost as old as set-submodular optimization itself \cite{24}, but efficient algorithms for maximizing integer lattice submodular functions are much more recent. \cite{16} proposed two deterministic $(1 - 1/e - \varepsilon)$-approximation algorithms for maximizing monotone integer lattice submodular functions subject to cardinality constraints: one for the case of DR-submodular functions, and the other for the less restrictive case of integer lattice submodular functions (Algorithm 1 and Algorithm 3 of \cite{16}, respectively). We refer to these two algorithms as SOMB-I-DR and SOMB-II. SOMB-I-DR requires $O(\frac{n}{\varepsilon} 1 \log \frac{\log \varepsilon}{\varepsilon})$ running time, whereas SOMB-II has the much worse time complexity of $O(\frac{1}{\varepsilon} \log ||b||_\infty \log \frac{\tau}{\varepsilon} \log \varepsilon)$, where $\tau$ is the ratio of the maximum value of $f$ to the minimum positive increase in the value of $f$. \cite{25} presented a natural “Double Greedy” time algorithm, which is also pseudopolynomial, and with the same approximation ratio of $(1 - 1/e)$ (and which is a $1/3$-approximation when there are no cardinality constraints). Finally, \cite{26} extended the multilinear extension to DR-submodularity, and \cite{27} introduced the generalized multilinear extension of the integer-lattice optimization problems in a similar spirit, but without translating these to algorithms.

**Reduction from Lattice to Set** We are aware of two generic reductions from the integer lattice domain to the set submodular setting, allowing the use of constrained maximization algorithms that target set submodular functions. The most intuitive one \cite{12} generates $b_e$ copies for each element $e \in \mathcal{V}$, letting $\mathcal{V}'$ be the the multiset containing these copies. Thus, problem \cite{14} can then be reformulated into a set function maximization problem with respect to the ground set $\mathcal{V}'$, and one can use the SG algorithm to solve the problem. The drawback is that this reduction yields pseudo-polynomial time algorithm, since $|\mathcal{V}'| = b \cdot 1n$. We denote the SG algorithm simulated in the integer lattice domain by SSG.

Another, more efficient reduction algorithm is proposed by \cite{13}, but it can only be applied to DR-submodular functions. This reduction uses a bit decomposition argument that yields a ground set $\mathcal{V}'$ of size $|\mathcal{V}'| = O(\log b + \frac{1}{\varepsilon}) \cdot n$.

3 Our Algorithm

Consider the drawbacks of the known reduction algorithms to translate an integer lattice submodular function to a set submodular function, explained above, and the unpredictable running time of LA1-DR \cite{28}, which requires solving an integer linear program, whose runtime could be much larger than the time needed to evaluate an oracle query. This motivated us to seek novel algorithms for submodular maximizing of integer lattice submodular functions subject to a cardinality constraint. In particular, we were inspired by the simplicity and performance of the STOCHASTIC GREEDY algorithm \cite{14}, which employs the random subsampling technique, while retaining a tight approximation ratio of $(1 - 1/e)$ on average for the set submodular version of the problem. We thus propose a novel algorithm, which is based on the same random subsampling idea, but combined with the DECREASING THRESHOLD GREEDY framework of \cite{28} and the binary search strategy of \cite{14}.

Let $s := \lfloor \frac{\log 1}{\varepsilon} \rfloor$ indicate the sample size, as in \cite{14}. Our proposed algorithm starts with an empty solution vector $x \in \mathbb{Z}_+^\mathcal{V}$ and with a decreasing threshold value $\Theta$ (which we use to decide whether to add new elements to $x$) initialized with $d$, the maximum value of the function $f$ over every singleton $e$, for $e \in \mathcal{V}$.

Until the cardinality of $x$ reaches $r$, we randomly sample $s$ elements from the available elements in $\mathcal{V}$. Locally, we look for $s$ distinct elements, but we allow sampling with replacement among different iterations. We then look for the maximum integer $k \in \mathbb{Z}_+$ such that the marginal gain $f(k \cdot 1_e | x)$ is greater or equal than $k \cdot \Theta$. The value of $k$ indicates how many copies of an element $e \in \mathcal{V}$ are added to $x$ if any at all), so to make sure that we don’t surpass the cardinality bound $r$, at each iteration $t$ it should hold that $k \leq \min\{b_e - x_e, r - \|x\|_1\}$ for any previous solution $x^{t-1}$ and element $e \in \mathcal{V}$. If such value $k$ exists and satisfies the constraint that $f(x + k \cdot 1_e)$ increase the value $f$, then we add $k$ copies of element $e$ to our solution $x$. We repeat the process until the sum of the entries of $x$ reaches the desired cardinality $r$. 

4
We define the following set $A_t := \text{supp}(\max\{x^* - x_t, 0\})$. We also define
\[
\bar{t} := \frac{\log \left[ 1 - \exp\left\{ -\frac{\log 2}{n} \right\} \right]}{\log \left( 1 - \frac{\epsilon}{n} \right)},
\]
and we prove the following:

**Lemma 1.** Consider $x^t$ the current solution at iteration $t$, and let $k$ be quantity corresponding to a selected element $e \in V$ that should be added to $x^t$ in Algorithm 1. Then, with probability $p > \frac{1}{2}$, it holds that
\[
\frac{f(k^t \cdot 1_e | x^t)}{k^t} \geq \frac{(1 - \bar{t} \epsilon)}{r} \sum_{v \in A_t \backslash \{e\}} f(1_v | x^t).
\]
Proof. We can see that Eq. (7) holds for $\Theta = d$ by the same arguments as \cite{16} Lemma 3, which require DR-submodularity. Let us suppose that it also holds for iteration $t > 0$. Indeed, given $v \in A_t \setminus \{e\}$, if $v$ was picked from the previous sample $Q_{t-1}$ in quantity $k'$, we would have that $f(1, v | x^t) \leq \frac{\Theta}{1 - \epsilon}$. This is by contradiction, since if it was otherwise,

$$f((k' + 1)1_v | x^{t-1}) \geq f(1_v | x^t) + f(k'1_v | x^{t-1}) > \frac{\Theta}{1 - \epsilon} + \frac{k'\Theta}{1 - \epsilon},$$

would hold, contradicting $k'$ being the largest feasible integer found at the previous iteration $t - 1$. Thus, for such a $v$, it holds that $\frac{f(k1_v | x^t)}{k} \geq (1 - \epsilon)f(1_v | x^t)$.

On the other hand, if $v \notin Q_{t-1}$, then

$$\frac{f(k1_v | x^t)}{k} \geq (1 - \epsilon)^{t(\tilde{v}, t)}f(1_v | x^t) \geq (1 - \tilde{t}(v, t)\epsilon)f(1_v | x^t),$$

where $\tilde{t}(v, t)$ denotes the number of iterations which occurred before $t$ since the last time the element $v$ was selected. With a slight abuse of notation, we fix $v$ and $t$ and define $\tilde{t} := \tilde{t}(v, t)$. Clearly, $\tilde{t}$ is a geometric random variable with success probability $\tilde{\rho} := \frac{\epsilon}{2}$ and with cumulative distribution function $1 - (1 - \tilde{\rho})^\tilde{t}$. Thus, for every element $v \in V$ at most $\tilde{t}$ iterations old, this would have a probability of $(1 - (1 - \tilde{\rho})^\tilde{t})^n$. Now,

$$(1 - (1 - \tilde{\rho})^\tilde{t})^n \geq \frac{1}{2} \iff \tilde{t} < \tilde{t} = \frac{\log \left[ 1 - \exp \left\{ - \frac{\log 2}{n} \right\} \right]}{\log (1 - \frac{\epsilon}{2})}$$

Hence, with probability $p = \frac{1}{2}$ and for all iterations $t$ and elements $v \in V$, it holds that

$$\frac{f(k1_v | x^t)}{k} \geq (1 - \epsilon)^{\tilde{t}}f(1_v | x^t) \geq (1 - \tilde{t})(1 - \epsilon)f(1_v | x^t).$$

If we average over the elements, we obtain that

$$\frac{f(k1_v | x^t)}{k} \geq \frac{1 - \tilde{t}}{r} \sum_{v \in A_t \setminus \{e\}} f(1_v | x^t).$$

Lemma \cite{16} allows us to prove the following approximation bound.

**Theorem 1.** Algorithm \cite{16} achieves an approximation ratio of $(1 - \frac{1}{r} - \epsilon)$ with probability $p > \frac{1}{2}$.

Proof. By Lemma \cite{16} with probability $p > \frac{1}{2}$ we have that

$$\frac{f(k1_v | x^t)}{k} \geq (1 - \epsilon)^{\tilde{t}}f(x^* - x^t | x^t),$$

where $x^t$ is the solution of Algorithm \cite{16} at iteration $t$, and $x^*$ is the optimal solution. Given $k_{sum}$ the sum of all $k$ selected for a fixed element $e \in V$ at iteration $t$, it follows that

$$f(x^{t+1}) - f(x^t) \geq \frac{1 - \epsilon}{r} k_{sum} f(x^*) - f(x^t).$$

Let $x^{t'}$ be the result of Algorithm \cite{16} By induction,

$$f(x^{t'}) \geq \left( 1 - \prod_t \left( 1 - \frac{(1 - \epsilon)k_{sum}}{r} \right) \right) f(x^*) \geq \left( 1 - \prod_t \exp(-\frac{(1 - \epsilon)k_{sum}}{r}) \right) f(x^*) = \left( 1 - \exp(-\frac{(1 - \epsilon)\sum_t k_{sum}}{r}) \right) f(x^*) \geq \left( 1 - \frac{1}{e} - \epsilon \right) f(x^*).$$
Notice that the probability \( p \) in Lemma 1 can be adjusted by probability amplification [29, Section 6.8]. In effect, Algorithm SGL-III is likely to achieve an arbitrarily close performance in terms of the approximation ratio as the state of the art [16], but with a lower overall computational cost.

4.2 Bounding the Number of Oracle Calls

Proposition 1. The largest possible value of \( k_{\text{max}}^t \) is \( \min \{ ||b||_\infty, r \} \).

Proof. In Algorithm 1, \( k_{\text{max}}^t \) is computed as \( \min \{ b_e - x_e^{t-1} - r - ||x^{t-1}||_1 \} \) for a given \( e \in Q \). We recall that \( b \) and \( r \) are two given constants, and that \( ||x||_1 \) is non-decreasing over time (starting from \( x = 0 \)). Thus, in the worst case, \( k_{\text{max}}^t = \min \{ b_e - 0, r - 0 \} = \min \{ ||b||_\infty, r \} \), which can only happen before the first element is selected to be part of the solution.

Proposition 2. The binary search procedure requires \( \lceil \log_2 (k_{\text{max}}^t + 1) \rceil \) oracle queries for a given \( k_{\text{max}}^t \). The number of oracle queries performed by one iteration of Algorithm 1 is \( \mathcal{O}(\log(\min \{ ||b||_\infty, r \})) \).

Proof. At each iteration, we consider at most \( s := \lfloor n r \cdot \log_2(1/\epsilon) \rfloor \) distinct elements. From Proposition 1 we have at most \( \mathcal{O}(\log(\min \{ ||b||_\infty, r \})) \) oracle queries per iteration.

The number of iterations is a function of a number of parameters, including the function \( f \) and its \( n := |V| \) and the resulting \( \Theta^0 = d \), as well as \( b, r, \epsilon \), and the associated \( \Theta_{\text{stop}} = \frac{\epsilon}{r} \cdot d \). We present details of the runtime of several variants of the algorithm in an extended version on-line. Let us illustrate the behaviour computationally.

5 Numerical Results

We have implemented our algorithm SGL, as well as the algorithms of:

- **SOMA-DR-I** by Soma and Yoshida [16];
- **LAI-DR** by Lai [17];
- **SSG**, i.e., the Simulated **STOCHASTIC GREEDY** algorithm due to Mirzasoleiman et al. [14] lifted from the set to the integer lattice domain.

in an open-source package available on-line. As far as we know, the code we release is the first open-source package for submodular function maximization on the integer lattice subject to cardinality constraints.

We have validated our proposed algorithm on some synthetic problem instances of increasing size \( n \), cardinality constraint \( r \), and for various uniformly random configurations of the vector of available quantities \( b \). We measure the performance of the algorithms both in terms of the number of oracle queries and in terms of the total running time.

**Synthetic Monotone Function** We define a simple monotone DR-submodular function \( f : \mathbb{Z}_+^V \rightarrow \mathbb{R}_+ \) where \( f(x) := w^T \cdot x \), wherein \( w \in \mathbb{Z}_+^V \) is a vector sorted in ascending order whose components \( w_e \) are such that \( 1 \leq w_e \leq 100 \), for every \( e \in V \). One can observe that \( f \) is normalized. We consider the following parameter settings:

- \( n \in \{ 100, 200, 500, 750 \} \);
- \( r \in \{ 0.25 \cdot n, 0.5 \cdot n, 1 \cdot n, 2 \cdot n \} \);
- Six equidistant \( b \in \mathbb{Z}_+ \cap [\lfloor \frac{r}{20} \rfloor, \lfloor \frac{r}{2} \rfloor] \)

\( b \) is uniformly sampled in range \( \{ b, b \cdot 4 \} \). We discard every unfeasible combination of \( (n, b, r) \) such that \( r > ||b||_\infty \)

and \( r \geq n \cdot ||b||_\infty \).

[https://github.com/jkomyno/lattice-submodular-maximization](https://github.com/jkomyno/lattice-submodular-maximization)
Table 1: Average number of oracle queries required by each considered algorithm as \( n \) increases on the synthetic monotone DR-submodular instances. Smaller numbers are better. We see that SGL takes the least amount of time on average, and that the reduction from integer lattice to set domain approach does not scale well.

Experimental Setting  
We remark that every algorithm under consideration uses randomization, except Soma-DR-I. To reduce the bias of our benchmark, for each parameter setting, we repeated every experiment on a randomized algorithm 5 times. Moreover, we seeded our random generator to ensure that every algorithm consumes exactly the same problem instances, even though those are generated randomly “on the fly”. We fixed the timeout for the experiments at 6 hours and 30 minutes. While SGL and SSG managed to complete their computations in time, both Soma-DR-I and Lai-DR (currently considered the state of the art for our target problem) timed out on the most challenging parameter settings, with \( n \geq 500 \). The code for the experiments is written in PYTHON 3.8. For all algorithms that require an error threshold \( \varepsilon > 0 \) in input (our included), we fix \( \varepsilon := \frac{1}{4n} \), where \( n \) is the size of the ground set.

Figure 1: An illustration of the growth of the number of oracle calls as a function of the average value of \( b \) on a synthetic DR-submodular monotone function, for all the considered algorithms. On the left, \( n = 200 \) and \( r = 100 \). On the right, \( n = 500 \) and \( r = 125 \). It is clear that SSG depends positively on the \( b \), whereas Soma-DR-I seem to present an inverse relationship (but not always). SGL and Lai-DR do not seem to be affected by \( b \). SGL is the only algorithm that consistently scales as both \( n \) and \( b \) grow.

6 Conclusion and Future Work

In this extended abstract, we considered the problem of maximizing a monotone integer lattice and DR-submodular functions subject to cardinality constraints and proposed one randomized algorithm for this problem, inspired by the random sampling technique that, until now, has only been applied to set submodular functions [14]. We showed that the algorithm achieves state-of-the-art approximation ratio for monotone DR-submodular functions. Experimentally,
Table 2: Average value returned by each considered algorithm as $n$ increases on the synthetic monotone DR-submodular instances. Higher numbers are preferred. SSG tops the ranking in terms of value precision, with SGL coming close, less than 3% far. Surprisingly, LAI-DR’s approximation performs much worse than expected compared to the other algorithms.

we have shown that the algorithm requires considerably fewer value-oracle queries than state-of-the-art deterministic algorithms for the same problem, as well as the naive baseline lifting the integer lattice to a set domain.

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