COUNTING IDEALS IN \( Z[t]/(f) \)

SARTHAK CHIMNI

Abstract. In this paper we study the growth of ideals in \( Z[t]/(f) \) for a monic cubic polynomial \( f \). We also compute the ideal zeta function of \( Z[t]/(t^n) \) for any \( n \in \mathbb{N} \).

1. Introduction

Given a commutative ring \( R \) with identity whose additive group is isomorphic to \( \mathbb{Z}^n \) for some \( n \in \mathbb{N} \) we define

\[
a^I_R(k) = |\{ I \text{ ideal in } R \mid [R : S] = k \}|
\]

We define the ideal zeta function of \( R \) to be

\[
\zeta^I_R(s) = \sum_{k=0}^{\infty} \frac{a^I_R(k)}{k^s}.
\]

Then we prove the following theorem:

Theorem 1. For any \( n \in \mathbb{N} \)

\[
\zeta^I_{Z[t]/(t^n)}(s) = \zeta(s)\zeta(2s-1)\zeta(3s-2) \cdots \zeta(ns - (n-1)).
\]

It has been brought to my notice by Michael Schein that Theorem 1 is Corollary 4.3 from [6] and can also be proven by counting Hermite Normal Forms of matrices. The proof is left in detail as it is more elementary than that of [6].

It is easy to see that \( \zeta^I_{Z[t]/(t^n)}(s) \) has a pole at \( s = 1 \) of order \( n \).

An application of a standard Tauberian Theorem then gives the following result.

Corollary 2. Let \( c = \frac{1}{n!(n-1)!} \) then

\[
\sum_{k \leq B} a^I_{Z[t]/(t^n)}(k) \sim cB(\log B)^{n-1}.
\]

More generally for any monic polynomial \( f \in \mathbb{Z}[t] \), denote \( \mathbb{Z}[t]/(f) \) by \( \mathbb{Z}_f \). Then the authors in [1] also conjecture that if \( f = g_1^{m_1}g_2^{m_2} \cdots g_k^{m_k} \) where \( g_i \) is irreducible over \( \mathbb{Z}[t] \) then \( \zeta^I_{\mathbb{Z}_f}(s) \) has a pole at \( s = 1 \) of order \( \sum_{i=1}^{k} m_i \). We prove this for the case of cubic polynomials in the following theorem:

Theorem 3. Let \( f \) be a monic cubic polynomial in \( \mathbb{Z}[t] \), then \( \zeta^I_{\mathbb{Z}_f}(s) \) converges for \( R(s) > 1 \) and has a pole at \( s = 1 \). Let \( m_f \) denote the
order of the pole of $\zeta_{\mathbb{Z}[t]}^I$. Then $m_f$ is equal to the number of irreducible factors counted with multiplicity of $f$ in $\mathbb{Z}[t]$. 

In fact the authors of [1] believe that this holds for polynomials of any degree $n$. The ideal zeta function for $\mathbb{Z}[t]$ is computed by Segal in [2] where he proves the following theorem in which the equality is to be interpreted as an identity of formal Dirichlet Series.

**Theorem 4.** Let $S$ be a Dedekind domain, not a field, having only finitely many ideals of each finite index. Let $R = S[t]$. Then $R$ has only finitely many ideals of each finite index, and

$$\zeta_R^I(s) = \prod_{j=1}^{\infty} \zeta_S^I(js - j)$$

The general theory to study these zeta functions using $p$-adic integration techniques was introduced by Grunewald, Segal and Smith in [3]. They show that $\zeta_R(s)$ can be expressed as an Euler product of rational functions of $p^{-s}$ over all primes $p$. In [5] Kaplan, Marcinek and Takloo-Bighash study subring growth of $\mathbb{Z}^n$ and more generally the distribution of orders in number fields by locating the rightmost poles of these zeta functions instead of computing them explicitly. I use these ideas in the proofs in this paper.

The paper is organised as follows. In Section 2 we introduce the $p$-adic integration techniques we use in our proofs. We prove Theorem 1 in Section 3 and Theorem 3 in Section 4.

2. P-adic setting

Let $R$ be a commutative ring with identity whose additive group is isomorphic to $\mathbb{Z}^n$ for some $n \in \mathbb{N}$. Then the following theorem is a summary of results from [3]:

**Theorem 5.** 1. The series $\zeta_R^I(s)$ converges in some right half plane of $\mathbb{C}$. The abscissa of convergence $\alpha_R^I$ of $\zeta_R^I(s)$ is a rational number. There is a $\delta > 0$ such that $\zeta_R^I(s)$ can be meromorphically continued to the domain $\{s \in \mathbb{C} \mid \Re(s) > \alpha_R^I - \delta\}$. Furthermore, the line $\Re(s) = \alpha_R^I$ contains at most one pole of $\zeta_R^I(s)$ at the point $s = \alpha_R^I$.

2. There is an Euler product decomposition

$$\zeta_R^I(s) = \prod_p \zeta_{R,p}^I(s)$$

with the local Euler factor given by

$$\zeta_{R,p}^I(s) = \sum_{l=0}^{\infty} \frac{a_{R,p}(l^s)}{p^l s}.$$
This local factor is a rational function of $p^{-s}$; there are polynomials $P_p, Q_p \in \mathbb{Z}[x]$ such that $\zeta_{R,p}^{I}(s) = \frac{P_p(p^{-s})}{Q_p(p^{-s})}$. The polynomials $P_p, Q_p$ can be chosen to have bounded degree as $p$ varies.

By a theorem of Voll [4], the local Euler factors satisfy functional equations. The paper [3] introduced a $p$-adic formalism to study the local Euler factors $\zeta_{R,p}^{<}(s)$. Fix a $\mathbb{Z}$-basis for $R$ and identify $R$ with $\mathbb{Z}^n$. The multiplication in $R$ is given by a bi-additive map

$$\beta: \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}^n$$

which extends to a bi-additive map

$$\beta_p: \mathbb{Z}_p^n \times \mathbb{Z}_p^n \to \mathbb{Z}_p^n$$

giving $R_p = R \otimes \mathbb{Z} \mathbb{Z}_p$ the structure of a $\mathbb{Z}_p$-algebra.

Let $\mathcal{M}_p(\beta)$ be the subset of the set of $n \times n$ lower triangular matrices $M$ with entries in $\mathbb{Z}_p$ such that if the rows of $M = (x_{ij})$ are denoted by $v_1, \ldots, v_n$, and then for all $j$ satisfying $1 \leq j \leq n$ and for all $v \in \mathbb{Z}_p^n$, there are $p$-adic integers $c_1^v, \ldots, c_n^v$

$$\beta_p(v, v_j) = \sum_{k=1}^n c_k^v v_k. \quad (4)$$

Let $dM$ be the normalized additive Haar measure on $T_n(\mathbb{Z}_p)$, the set of $n \times n$ lower triangular matrices with entries in $\mathbb{Z}_p$. Proposition 3.1 of [3] says:

$$\zeta_{I\mathbb{Z}_p}^{I}(s) = (1 - p^{-1})^{-n} \int_{\mathcal{M}_p(\beta)} |x_{11}|^{s-n+1}|x_{22}|^{s-n+2} \cdots |x_{n,n}|^s dM. \quad (5)$$

We now apply these considerations to the case of $\mathbb{Z}_f = \mathbb{Z}[t]/(f)$ where $f$ is a polynomial of degree $n$ in $\mathbb{Z}[t]$. We fix $B = \{t^{n-1}, \ldots, t, 1\}$ as an ordered basis for $\mathbb{Z}_f$ as a lattice. So that $t^{n-j}$ corresponds to $e_j$ where $e_j$ denotes the $j^{th}$ standard basis vector. Then the product of two vectors $v \cdot w$ is the vector representing the product of the corresponding polynomials in $\mathbb{Z}_f$ in basis $B$. By Theorem 5 there exists an Euler product decomposition

$$\zeta_{\mathbb{Z}_f}^{I}(s) = \prod_{p \text{ prime}} \zeta_p^{I}(\mathbb{Z}_f, s)$$

where

$$\zeta_p^{I}(\mathbb{Z}_f, s) = \sum_{k=1}^{\infty} a_{k,p} \frac{t^k}{p^{ks}}.$$

Here $a_{k,p}$ counts the number of ideals of $\mathbb{Z}_f$ of index $p^k$.

Let $v_f$ be the vector corresponding to $f$ in basis $B$. We define $\mathcal{M}_f(p)$ be the subset of the set of $n \times n$ lower triangular matrices $M$ with entries
in \( \mathbb{Z}_p \) such that if the rows of \( M = (a_{ij}) \) are denoted by \( v_1, \ldots, v_n \), then for all \( j \) satisfying \( 1 \leq j \leq n \), there are \( p \)-adic integers \( \{c_{kj}\}_{k=1}^n \) such that
\[
v_t \cdot v_j = \sum_{k=1}^n c_{kj}v_k \tag{6}\]
Observe that \( M \in \mathcal{M}_f(p) \) if and only if the rows of \( M \) generate an ideal in \( \mathbb{Z}_f \). Using Equation (5) we find that the local factors are given by
\[
\zeta_p^f(\mathbb{Z}_f, s) = (1 - p^{-1})^{-n} \int_{\mathcal{M}_f(p)} |a_{11}|^{s-n+1}|a_{22}|^{s-n+2} \cdots |a_{n,n}|^s dM.
\]
In order to compute these local factors we need the following definition.

**Definition 6.** Let \( (b_1, b_2, \ldots, b_n) \) be a \( n \)-tuple of non negative integers, we set
\[
\mathcal{M}_f(p; b_1, \ldots, b_n) = \left\{ M = \begin{bmatrix} p^{b_1} & 0 & \cdots & 0 \\ a_{21} & p^{b_2} & \cdots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ a_{n1} & \cdots & p^{b_n} \end{bmatrix} \in \mathcal{M}_f(p) \right\}.
\]

Let \( \mu_p^f(b_1, \ldots, b_n) \) be the volume of \( \mathcal{M}_f(p; b_1, \ldots, b_n) \) as a subset of \( \mathbb{Z}_p^{\frac{n(n+1)}{2}} \). It follows from equation (8) that
\[
\zeta_p^f(\mathbb{Z}_f, s) = \sum_{b_i=0}^\infty \frac{p^{(n-1)b_1+(n-2)b_2+\ldots+2b_{n-2}+b_{n-1}}}{p^{b_1+\ldots+b_n}s} \mu_p^f(b_1, \ldots, b_n). \tag{7}\]

3. **Proof of Theorem 1**

We now prove Theorem 1. In this section \( f(t) = t^n \) and \( \mathcal{M}_f(p; b_1, \ldots, b_n) \) is denoted by \( \mathcal{M}_n(p; b_1, \ldots, b_n) \) and \( \mu_p^f(b_1, \ldots, b_n) \) is denoted by \( \mu_p(b_1, \ldots, b_n) \).

We use equation (8) to compute \( \mu_p(b_1, \ldots, b_n) \). Define \( G_M \) be the subgroup of \( \mathbb{Z}_p^n \) generated by the rows of \( M \). Then we use the following standard result to find \( \mu_p(b_1, \ldots, b_n) \).

**Lemma 7.** Let \( M \in \mathcal{M}_n(p; b_1, \ldots, b_n) \), then \( \mu_p(G_M) = p^{-(b_1+\ldots+b_n)} \).

**Proposition 8.**
\[
\mu_p(b_1, \ldots, b_n) = \begin{cases} p^{-(n-2)b_1+(n-3)b_2+\ldots+b_{n-2}} & b_1 \leq \ldots \leq b_n \\ 0 & \text{otherwise} \end{cases}
\]

**Proof.** For any matrix \( M \), let \( M_j \) denote the matrix obtained from \( M \) by deleting the last \( n-j \) rows. Now \( M = (v_1, v_2, \ldots, v_n)^T \in \mathcal{M}_n(p) \) if and only if for all \( 2 \leq j \leq n, v_i \cdot v_j \in G_{M_{n-j}} \). Therefore \( \exists c_{ij} \in \mathbb{Z}_p \) such that
\[
v_t \cdot v_j = (a_{j,2}, \ldots, a_{j,j-1}, p^{b_j}, 0, \ldots, 0) = \sum_{i=1}^{j-1} c_{ij}v_i. \tag{8}\]
This happens only if \( b_{j-1} \leq b_j \) as \( c_{j-1,j} = p^{b_j - b_{j-1}} \). For \( j = 2 \) Equation 8 is satisfied if and only if \( b_1 \leq b_2 \) and this holds on a volume of 1. For \( 3 \leq j \leq n \) we can write \( v_i v_j = u_j + w_j \) where \( u_j \in G_{j-2} \) and \( w_j = p^{b_j} e_{j-1} \) where \( \{e_1, \ldots, e_n\} \) is the standard basis for \( \mathbb{Z}_p^n \). So equation 8 holds on a volume of \( \mu_p(G_{M_{j-2}}) \). Using Lemma 7 we have \( \mu_p(G_{M_{j-2}}) = p^{-(b_1 + \cdots + b_{j-2})} \) for \( 3 \leq j \leq n \). This implies

\[
\mu_p(b_1, \ldots, b_n) = \prod_{j=3}^{n} \mu(G_{M_{j-2}}) = \mu_p(b_1, \ldots, b_n) = p^{-(n-2)b_1 + (n-3)b_2 + \cdots + b_{n-2}}.
\]

We now compute the local factors \( \zeta_{R,p}(s) \). In order to this we need another lemma.

**Lemma 9.** For \( 2 \leq k \leq n \)

\[
\sum_{b_k \geq b_{k-1}} (px)^{b_k} \cdots \sum_{b_{n-2} \geq b_{n-1}} (px)^{b_{n-1}} \sum_{b_n \geq b_{n-1}} x^{b_n} = (p^{n-k} x^{n-k+1})^{b_{k-1}} \prod_{j=1}^{n-k+1} (1-p^{j-1} x^j)^{-1}.
\]

**Proof.** We prove this by induction on the number of summations. If \( k = n \), that is there is only one summation then it is easily seen that the equality holds. Now assume the equation is true for \( n - k \) summations. Then we have

\[
\sum_{b_k \geq b_{k-1}} (px)^{b_k} \cdots \sum_{b_{n-2} \geq b_{n-1}} (px)^{b_{n-1}} \sum_{b_n \geq b_{n-1}} x^{b_n} = \prod_{j=1}^{n-k} (1-p^{j-1} x^j)^{-1} \sum_{b_k \geq b_{k-1}} (px)^{b_k} (p^{n-k-1} x^{n-k})^{b_k}.
\]

So that

\[
\sum_{b_k \geq b_{k-1}} (px)^{b_k} \cdots \sum_{b_{n-2} \geq b_{n-1}} (px)^{b_{n-1}} \sum_{b_n \geq b_{n-1}} x^{b_n} = (p^{n-k} x^{n-k+1})^{b_{k-1}} \prod_{j=1}^{n-k+1} (1-p^{j-1} x^j)^{-1}.
\]

\[\square\]

**Proposition 10.** Let \( x = p^{-s} \) then

\[
\zeta_p^f(\mathbb{Z}[t]/(t^n), s) = \prod_{j=1}^{n} (1-p^{j-1} x^j)^{-1}.
\]  

(9)

**Proof.** Applying Proposition 8 to equation (7) we have

\[
\zeta_p^f(\mathbb{Z}[t]/(t^n), s) = \sum_{0 \leq b_1, \ldots, b_n} \frac{p^{b_1 + \cdots + b_{n-1}}}{p^{(b_1 + \cdots + b_n)s}}.
\]

Set \( x = p^{-s} \). Then
\[ \zeta_I^f(\mathbb{Z}[t]/(t^n), s) = \sum_{b_1 \geq 0} (px)^{b_1} \sum_{b_2 \geq b_1} (px)^{b_2} \cdots \sum_{b_{n-1} \geq b_n-2} (px)^{b_{n-1}} \sum_{b_n \geq b_{n-1}} x^{b_n}. \]  

We apply Lemma 9 with \( k = 2 \) to equation \( \text{(10)} \) to obtain

\[ \zeta_I^f(\mathbb{Z}[t]/(t^n), s) = \prod_{j=1}^{n} (1 - p^{j-1}x^j)^{-1} \sum_{b_1 \geq 0} (p^{n-1}x^{n})^{b_1}. \]

Therefore

\[ \zeta_I^f(\mathbb{Z}[t]/(t^n), s) = \prod_{j=1}^{n} (1 - p^{j-1}x^j)^{-1}. \]

\[ \square \]

Multiplying the local factors given in equation \( \text{(9)} \) gives the result.

\[ \zeta_{\mathbb{Z}[t]/(t^n)}(s) = \zeta(s)\zeta(2s-1)\zeta(3s-2)\cdots\zeta(ns-(n-1)). \]

4. Ideals in cubic rings

In this section we prove Theorem 3. Before we prove the theorem we first state the following general result from \[1\]:

**Theorem 11.** Let \( f \in \mathbb{Z}[t] \) be monic and assume that \( f = g_1 \cdots g_k \) with \( g_1, \ldots, g_k \) in \( \mathbb{Z}[t] \) irreducible, monic and pairwise distinct. Then the ideal zeta function \( \zeta_{Z_f}(s) \) converges for \( R(s) > 1 \), has a meromorphic extension to the halfplane \( \{ s \in \mathbb{C} \mid R(s) > 0 \} \) and has a pole of order \( k \) at \( s = 1 \). In particular, \( \sum_{k \leq N} a_{Z_f}(k) \sim cN(\log N)^{k-1} \) for some constant \( c \).

We now prove Theorem 3.

**Proof of Theorem 3.** In the case that the roots of \( f \) are distinct the result follows from Theorem 11. If \( f = (t - \lambda)^3 \) for some \( \lambda \in \mathbb{Z} \) then the result follows from Theorem 1 by a change of variable. Hence we need to show that \( m_f = 3 \) when \( f = (t - \lambda_1)^2(t - \lambda_2) \). Now note that since \( f \) is not separable it is reducible and therefore \( \lambda_1 \) and \( \lambda_2 \in \mathbb{Z} \). Thus by a change of variable we can reduce this to the case \( f = t^2(t - \lambda) \) with \( \lambda \in \mathbb{Z} \setminus \{0\} \). The result now follows from Proposition 12. \( \square \)

**Proposition 12.** Let \( f(t) = t^2(t - \lambda) \in \mathbb{Z}[t] \). Then \( \zeta_{Z_f}^f(s) \) converges for \( R(s) > 1 \) and has a pole of order 3 at \( s = 1 \).

The rest of this section goes into proving Proposition 12. In what follows \( f(t) = t^2(t - \lambda) \). We find the local factors \( \zeta_p(Z_f, s) \) using the \( p \)-adic integration methods of Section 2 and 3 and use them to study the convergence of \( \zeta_{Z_f}(s) \).
Let $M_f(p; b_1, b_2, b_3)$ and $\mu^f_p(b_1, b_2, b_3)$ be as in Definition 6. Then an application of Equation (5) gives the following expression for the local factors of $\zeta_{Z_f}(s)$:

$$\zeta_{Z_f}^f(Z_f, s) = \sum_{b_i=0}^{\infty} \frac{p^{2b_1+b_2}}{p^{b_1+b_2+b_3}s} \mu^f_p(b_1, b_2, b_3).$$ (11)

In order to compute $\mu^f_p(b_1, b_2, b_3)$ we use the following lemma

**Lemma 13.** $M = \begin{bmatrix} p^{b_1} & 0 & 0 \\ a_{21} & p^{b_2} & 0 \\ a_{31} & a_{32} & p^{b_3} \end{bmatrix} \in M_f(p; b_1, b_2, b_3)$ if and only if the entries of $M$ satisfy the following inequalities.

\begin{align*}
    b_2 & \leq v(p^{b_1} + \lambda a_{31}) \quad (12) \\
    b_1 + b_2 & \leq v(p^{b_2} a_{32} - (p^{b_3} + \lambda a_{31}) a_{21}) \quad (13) \\
    b_2 & \leq v(a_{21}) \quad (14) \\
    b_1 + b_2 & \leq v(p^{2b_2} - \lambda a_{21}^2) \quad (15) \\
    b_2 & \leq b_1. \quad (16)
\end{align*}

**Proof.** These inequalities follow from an application of Equation (6).

**Lemma 14.** If $(\lambda, p) = 1$ then

$$\mu^f_p(b_1, b_2, b_3) = p^{-b_1-2b_2-\lceil \frac{b_1-b_2}{2} \rceil}$$

**Proof.** Suppose $(\lambda, p) = 1$. Using Inequality (14) we can write $a_{21} = p^{b_2} z$ for some $z \in \mathbb{Z}_p$. Therefore Inequality (15) can be rewritten as $b_1 - b_2 \leq v(1 - \lambda z^2)$. Since Inequality (16) holds, we have that these inequalities hold on a volume of $p^{-b_2 - \lceil \frac{b_1-b_2}{2} \rceil}$ for $a_{21}$. Now Inequality (12) holds on a volume of $p^{-b_2}$ while Inequality (13) holds on a volume of $p^{-b_1}$. Multiplying these volumes we get $\mu^f_p(b_1, b_2, b_3) = p^{-b_1-2b_2-\lceil \frac{b_1-b_2}{2} \rceil}$ in this case.

**Lemma 15.** If $(\lambda, p) > 1$ then $\mu^f_p(b_1, b_2, b_3) \leq p^{-2b_2} \text{ and } b_1 = b_2$.

**Proof.** Now suppose $\lambda = p^a$ for some $a \in \mathbb{N}$. As before we can write $a_{21} = p^{b_2} z$ and therefore we have $b_1 - b_2 \leq v(1 - p^a z^2)$. This inequality only holds if $b_1 = b_2$. This implies that Inequality (13) holds on a volume of $p^{-b_2}$. Therefore all the inequalities hold on a volume of at most $p^{-2b_2}$. 

\[ \square \]
We now evaluate the local factors using Equation (11).

**Lemma 16.** Let \( x = p^{-s} \). If \((\lambda, p) = 1\) then

\[
\zeta_p' \left( \mathbb{Z}_f, s \right) = \frac{1 - x^2 + p^{-1}x - p^{-1}x^2 + x^2 - x^3}{(1 - x)^2(1 - px^2)(1 - x^4)}
\]

**Proof.**

\[
\zeta_p' \left( \mathbb{Z}_f, s \right) = \sum_{b_2 \leq b_1} \sum_{b_3=0}^\infty \frac{p^{2b_1+b_2}}{p^{b_1+b_2+b_3}s} \mu_p' \left( b_1, b_2, b_3 \right)
\]

Using Lemma [4]

\[
= \sum_{b_2 \leq b_1} \sum_{b_3=0}^\infty \frac{p^{b_1-b_2-[\frac{b_1+b_2}{2}]} - [\frac{b_1+b_2}{2}]}{p^{b_1+b_2+b_3}s}
\]

\[= A_{00} + A_{01} + A_{10} + A_{11}.
\]

Where

\[A_{ij} = \sum_{k_2 \leq k_1} \sum_{b_3=0}^\infty \frac{p^{2k_1+i-(2k_2+j)-[\frac{2k_1+i-2k_2-j}{2}]-[\frac{2k_1+i-2k_2-j}{2}]} - [\frac{2k_1+i-2k_2-j}{2}]}{p^{2k_1+i+2k_2+j+b_3}s}
\]

Setting \( x = p^{-s} \), \( b_1 = 2k_1 + i \) and \( b_2 = 2k_2 + j \).

Setting \( x = p^{-s} \), \( b_1 = 2k_1 \) and \( b_2 = 2k_2 \) we have

\[A_{00} = \sum_{k_2=0}^\infty \left( p^{-1}x^2 \right)^{k_2} \sum_{k_1=k_2}^\infty \left( px^2 \right)^{k_1} \sum_{b_3=0}^\infty \left( x \right)^{b_3}
\]

\[= \frac{1}{1-x} \sum_{k_2=0}^\infty \left( p^{-1}x^2 \right)^{k_2} \sum_{k_1=k_2}^\infty \left( px^2 \right)^{k_1}
\]

\[= \frac{1}{(1-x)(1-px^2)} \sum_{k_2=0}^\infty x^{4k_2}
\]

\[A_{00} = \frac{1}{(1-x)(1-px^2)(1-x^4)}.
\]

We compute \( A_{01}, A_{10} \) and \( A_{11} \) similarly to get

\[\zeta_p \left( \mathbb{Z}_f, s \right) = \frac{1 + x + p^{-1}x + x^2}{(1-x)(1-px^2)(1-x^4)}
\]

\[= \frac{1 - x^2 + p^{-1}x - p^{-1}x^2 + x^2 - x^3}{(1-x)^2(1-px^2)(1-x^4)}.
\]

\[\square
\]

**Lemma 17.** If \((\lambda, p) > 1\) then \( \zeta_p(\mathbb{Z}_f, s) \) converges for \( R(s) > 1/2 \).
Proof. Let $R(s) = \sigma$, an application of Lemma 15 gives

$$|\zeta_p^I(\mathbb{Z}_f, s)| \leq \sum_{b_2, b_3} \frac{p^{b_2}}{p^{(2b_2 + b_3)\sigma}}$$

Therefore

$$|\zeta_p^I(\mathbb{Z}_f, s)| \leq \frac{1}{(1 - p^{-\sigma})(1 - p^{1 - 2\sigma})}$$

Since the right hand side converges for $\sigma > 1/2$ so does $\zeta_p(\mathbb{Z}_f, s)$. $\square$

Now let

$$F(s) = \prod_{p: (\lambda, p) > 1} \zeta_p(\mathbb{Z}_f, s)$$

and

$$G(s) = \prod_{p: (\lambda, p) = 1} \zeta_p(\mathbb{Z}_f, s).$$

Then it is easy to see that $F(s)$ is a finite product and therefore converges for $R(s) > 1/2$ by Lemma 17. On the other hand it follows from Lemma 16 that $G(s)$ has a pole at $s = 1$ of order 3. Hence $\zeta_{\mathbb{Z}_f}(s) = F(s)G(s)$ converges for $R(s) > 1$ and has a pole of order 3 at $s = 1$, proving Proposition 12.

References

[1] Fukshansky, L; Kuhnlein, S; Schwerdt, R Counting Ideals in Polynomial Rings ArXiv e-prints, January 2017, arXiv:1701.04633
[2] Segal, Dan. Ideals of finite index in a polynomial ring. Quart. J. Math. Oxford (2), 48 (1997), 83-92
[3] Grunewald, F. J.; Segal, D.; Smith, G. C. Subgroups of finite index in nilpotent groups. Invent. Math. 93 (1988), no. 1, 185–223.
[4] Voll, Christopher. Functional equations for zeta functions of groups and rings. Ann. of Math. (2) 172 (2010), no. 2, 1181–1218.
[5] Kaplan, Nathan; Marcinek, Jake; Takloo-Bighash, Ramin. Distribution of orders in number fields. Res. Math. Sci. 2 (2015), Art. 6, 57 pp.
[6] Rossmann, Tóibás. Enumerating submodules invariant under an endomorphism Math. Ann. 368 (2017), pp. 391-417