Causal order as a resource for quantum communication

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INTRODUCTION

One of the basic questions in quantum information theory is to characterise the resources necessary for the reliable transmission of quantum information [1]: a sender encodes a quantum state in a system and a receiver has to retrieve it, without a prior knowledge of what the state might be [2]. A typical protocol can involve the physical transfer of the system, allowing it to undergo some time evolution possibly including noise. The essential resource is then how well the evolution preserves the initial state. A different method is teleportation [3], where the resources are entanglement and classical communication. The quality of the communication resource is typically measured as the rate of reliable transmitted qubits per use of the resource, in the limit of many independent uses.

Typical communication protocols presume that the involved parties’ actions take place in a fixed causal order, with the sender’s always preceding the receiver’s. More general situations are possible: The parties might both act on a quantum particle that is exchanged between them, but without knowing to whom the particle goes first. For multiple runs of the protocol, the particle could go one direction or the other randomly, according to some probability. It is natural to ask whether communication is at all possible without a fixed causal order and if causal order should itself be considered as a resource for communication. This can be relevant in scenarios of distributed quantum computation, where separated units have to communicate in order to perform a joint operation, but unknown delays in the network might produce uncertainty in the order in which the units are queried [4]. It is also relevant for foundational questions, such as the informational properties of processes in scenarios where quantum-gravity fluctuations generate uncertainty in causal relations [5–7].

Here we find that, in the asymptotic limit, a communication protocol where the order between two parties is completely unknown allows the transfer of classical information in either direction (although with limited efficiency), but not of quantum information. In particular, we prove that, when the causal order between two parties is completely uncertain (with equal probability for both orders) the asymptotic quantum communication capacity vanishes in both directions.

SINGLE-SHOT PROCESS MATRICES

We consider a general communication protocol where, in an individual run, each party receives a quantum system, which might contain information sent by another party or shared correlations, and then sends away a system in which they encoded the desired information. Each party can perform an arbitrary local operation on their system, namely they can let it interact with a local ancilla in some controlled way. Crucially, the parties have no access to any background causal structure, thus the time of their operations is not set in advance and it might vary probabilistically for different runs of the protocol.

Situations of this type are conveniently modelled within the process matrix framework [8–10], which generalises standard states and channels to scenarios with no background causal structure. We review the framework as formulated in [11] through the notion of higher order maps [12, 13], which turns out to be convenient for the current study of communication protocols.

Bipartite processes defined as higher-order maps are illustrated in the top of fig. 1. Let \( A \) be a completely positive trace preserving (CPTP) map, i.e., a quantum channel, with the input systems \( A_I, A'_I \) and output systems \( A_O, A'_O \), where \( A_I, A_O \) represent the system that Alice receives and respectively send back to the process, while \( A'_I, A'_O \) represent her local ancilla before and after the interaction. Similarly, let \( B \) be a quantum channel with inputs \( B_I, B'_I \) and outputs \( B_O, B'_O \). A processes \( W \) is defined as a linear map acting on \( A_I, A_O, B_I, B_O \), with the requirement that any pair of such channels \( A \) and \( B \) is transformed into a new channel \( N = W(A, B) \), with inputs \( A'_I, B'_I \) and outputs \( A'_O, B'_O \).

Quantum channels and processes can be represented as matrices through the Choi isomorphism [14], which takes...
a completely positive map \( M : \mathcal{L}(\mathcal{H}^a) \to \mathcal{L}(\mathcal{H}^b) \) to its positive semidefinite “Choi matrix”

\[
\sum_{i,j=1}^{d_a} M(\{i\}\{j\}) \otimes \{i\}\{j\} \in \mathcal{L}(\mathcal{H}^b \otimes \mathcal{H}^a),
\]

where \( \{i\} \) is an orthonormal basis of \( \mathcal{H}^a \). In the following, we refer to the Choi matrix of a map \( M \) using the same letter. In this representation, the defining property of a process \( W \), that it maps channels to channels, is captured by the following [9]:

\[
W \geq 0,
\]

\[
\text{Tr}_X W = d_O,
\]

\[
B_1 B_0 W = A_0 B_1 B_0 W,
\]

\[
A_i A_0 W = B_0 A_i A_0 W,
\]

\[
W = A_0 W + B_0 W - A_0 B_0 W,
\]

where \( d_O \) is the product dimension of the output systems and we use the trace-and-replace notation \( \text{Tr}_X M : = \frac{1}{d_X} \otimes \text{Tr}_X M \), where \( d_X \) is the dimension of system \( X \). Any matrix \( W \) obeying conditions (2)–(6) is called a process matrix. It defines a process, whose action on channels \( A \) and \( B \) is represented using Choi matrices as \( N(A, B) = W A B = W (A \otimes B) \). Here we use the “link product” [15]:

\[
M \star N := \text{Tr}_s [M^T \otimes N],
\]

where \( s \) is the system that is the joint support of \( M \) and \( N \), and \( T_s \) is the partial transpose on \( s \). It is understood that the operators act as the identity outside of its original support.

It is useful to consider causally-ordered processes, i.e., processes that cannot transmit information in certain directions. We use \( W^{A \rightarrow B} \) (respectively \( W^{A \leftarrow B} \)) to denote a process that cannot be used to signal from \( B \) to \( A \) (respectively \( A \) to \( B \)). It holds that [15, 16]

\[
W^{A \rightarrow B} = B_0 W^{A \leftarrow B}, \quad W^{A \leftarrow B} = A_0 W^{A \rightarrow B}.
\]

Incidentally, the processes generalize quantum channels. For instance, one can verify that a quantum channel from \( A \) to \( B \) is a special case of \( W^{A \leftarrow B} \) with the systems \( A_I \) and \( B_0 \) set to be the one-dimensional trivial system. General causally-ordered processes represent channels with memory [17].

Here we are interested in more general processes, \( W^{AB} = p W^{A \leftarrow B} + (1 - p) W^{A \rightarrow B} \), where Alice might come before Bob with probability \( p \) and Bob before Alice with probability \( 1 - p \). Such processes are called causally separable, and it is known that more general situations, where the causal order is indefinite, are possible too [8, 18–23]. In this work, however, we are mostly concerned with definite, albeit possibly unknown, causal order.

**THE ASYMPTOTIC SETTING**

Measures of communication capacity are typically defined as the optimal rate of transmitted information per use of the resource, in the limit of infinite uses [1, 24]. Recall that, in the standard asymptotic setting for channel communication, \( n \to \infty \) copies \( N^{\otimes n} \) of the channel \( N \) are sandwiched between a joint encoding channel \( E \) and a joint decoding channel \( D \).

To generalise this notion to processes where the causal order is not fixed, we need to clarify in what ways the parties can use multiple copies of a process. Each copy \( W_{ni} \) of the process is associated with input-output spaces \( A_i, A_D \), which can be accessed by letting them interact with ancillary systems \( A_i', A_D' \) (and similarly for Bob). As the parties have no access to a background causal structure, they do not know in which order different copies will be instantiatted. Therefore, they can communicate in no other way than through the process. In other words, \( n \) uses of a bipartite process \( W \) are described by the the \( 2n \)-partite process \( W^{\otimes n} \) where, for \( i = 1, 2, \cdots, n \), each party can only apply independent channels \( A_i, B_i \). (We can equivalently say that the process \( W^{\otimes n} \) can be composed with arbitrary product channels \( \otimes_{i=1}^{n} (A_i \otimes B_i) \).) We can still think that all the \( A_i \) channels are controlled by a single agent, Alice, and the \( B_i \) ones by Bob, who are restricted to product channels because of the unknown causal order. We will refer to Alice and Bob as “agents”, to distinguish them from “parties”, which we reserve to the individual access points to each copy of the process.
Formally, at each iteration the parties convert the process into a channel for the corresponding ancillary systems, $N_i := W_i \ast (A_i \otimes B_i)$. The $N_i$’s thus obtained can then be used according to the ordinary asymptotic settings for channels, with the difference that now each agent can both receive and send information. Therefore, the channel $N_i^\otimes n$ can be preceded by encoding channels $E_A, E_B$, which prepare joint states in the spaces $A_i^\otimes n_i, B_i^\otimes n_i$, and followed by decoding channels $D_A, D_B$, which transform respectively $A_i^\otimes n_i, B_i^\otimes n_i$ into the final output state. Note that no joint encoding on $A_i^\otimes n_i \otimes B_i^\otimes n_i$ should be allowed, as this could introduce additional entanglement, not modelled in the process, and thus an additional communication resource. Similarly, common decodings on $A_i^\otimes n_i \otimes B_i^\otimes n_i$ are excluded, as they would allow the parties to exchange information beyond what is enabled by the process. As illustrated in the bottom of fig. 1, the setting described generates a shared channel $M$, which can be used to communicate information.

We observe that there are also other ways in which agents could use multiple copies $W^\otimes n$ of a process, for example by arranging the access to the different parties in some given order or adding extra entanglement. Note, however, that there are constraints on process composition: For example, $W^\otimes n$ cannot be used as a bipartite process, where all the $A$’s and the $B$’s each act ‘simultaneously’ as a collective party [25, 26]. The setting introduced above is appropriate for the study of causal order as a communication resource in the asymptotic setting, as it precludes the agents from using additional communication resources, such as entanglement or causal order. A more general study can be based on the one-shot setting, which is beyond the scope of this work.

**THE QUANTUM COMMUNICATION TASK**

The standard definition of the quantum communication task for channels through entanglement or subspace transmission [1] can be generalized to processes [1], and their capacities agree [2]. Without loss of generality we present the communication task and define the quantum communication capacity for processes through entanglement transmission.

In an entanglement transmission task from Alice to Bob, Alice, in addition to sharing copies of the process $W^{AB}$ with Bob, also shares a preexisting state $\tau$ with a third party Charlie. The goal is for Alice to “transmit” her share of the preexisting state to Bob so that in the end Bob and Charlie share a state $\rho$ that is as close to $\tau$ as possible. In the above asymptotic setting, the protocol takes the form

$$\rho^{CB} = M \ast \tau^{CA} = (N_i^\otimes n \ast E_A \ast E_B \ast D_B) \ast \tau^{CA},$$

where $N_i^\otimes n = \otimes_{i=1}^n N_i$ and $N_i = W_i \ast (A_i \otimes B_i)$. Without loss of generality the system $A_i'$ and the operation $D_A$ have been taken to be trivial, since they are not accessible to Bob and will eventually be traced out. In addition, $E_B$ is taken to be a state rather than a channel, since even if it were a channel in the beginning, a state needs to be fed into its input to turn the channel into a state by the end of the protocol.

We say there is a $(R, n, \epsilon)$ code for entanglement transmission if for $R = (1/n) \log m$ there exists a protocol with $(A_i, B_i, E_A, E_B, D_B)$ such that for any input state $\sigma^{CA}$ with $\dim A = \dim C = m$, the fidelity $F(\sigma^{CA}, \rho^{CB}) \geq 1 - \epsilon$. A rate $R$ is said to be achievable if there is a sequence of $(R, n, \epsilon_n)$ codes with $\epsilon_n \rightarrow 0$. The quantum communication capacity of $W$, $Q(W)$, is the supremum of the achievable rates.

**RESULTS**

Suppose two agents Alice and Bob can interact with multiple uses of the process $W$ in the above asymptotic setting. How does the lack of a definite causal order affect their quantum communication capacity? We know that if $W = W^{A\rightarrow B}$, Alice cannot communicate any information to Bob, but what about the general case of a causally separable $W = pW^{A\rightarrow B} + (1-p)W^{A\rightarrow B}$, which becomes $W^{A\rightarrow B}$ only at $p = 0$? Here we prove that the quantum capacity actually starts to vanish at the much higher value $p = 1/2$, which implies that, for any causally separable $W$ of this form, there is quantum capacity in at most one direction.

**Theorem 1.** $W^{AB} = pW^{A\rightarrow B} + (1-p)W^{A\rightarrow B}$ can have positive quantum communication capacity in the Alice to Bob direction if and only if $p > 1/2$.

**Proof.** In a general protocol, Alice applies a channel $A_i$ on the $i$-th copy $W'_i$ of $W$, and as explained above, $A_i'$.

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1 The definition through entanglement generation generalized to processes, however, does not capture “communication”. States are special cases of processes. Entanglement can be generated from them even when they do not allow signalling at all from one party to the other.

2 This can be seen as a consequence of the theorem in the appendix of [27] and the fact that the capacities agree for channels [25].
can be taken to be trivial.

\[ W_i \ast A_i = pW_i^{A \times B} \ast A_i + (1-p)W_i^{A \to B} \ast A_i \]  

(11)

\[ = pW_i^{A \times B} \ast A_i + (1-p)[A_i \otimes W_i^{A \times B} \ast A_i] \]  

(12)

\[ = pW_i^{A \times B} \ast A_i + (1-p)\mathbb{I}^{A'_i} \otimes \operatorname{Tr}_{A'_i}\sigma^{B_i} \]  

(13)

\[ = pW_i^{A \times B} \ast A_i + (1-p)\mathbb{I}^{A'_i} \otimes \sigma^{B_i} \]  

(14)

\[ = pW_i^{A \times B} \ast A_i + (1-p)A_i \otimes \sigma^{B_i} \]  

(15)

\[ = [p\operatorname{Tr}_{B_i}W_i^{A \times B} \ast A_i + (1-p)\mathbb{I}^{A'_i} \otimes \sigma^{B_i}] \otimes \mathbb{I}^{B_i}, \]  

(16)

where is the density operator defined by \( \sigma^{B_i} := \operatorname{Tr}_{A'_i}\sigma^{B_i} W_i^{A \times B} \). Equation (13) holds by eq. (9). Equation (14) holds because \( \mathbb{I}^{A'_i} \ast A_i = \mathbb{I}^{A'_i}A'_i \), which is true for any channel \( A_i \). Equation (15) holds by eq. (5). Equation (16) holds by eq. (8). \( \mathbb{I}^{B_i} \) in (16) means that whatever that is sent into \( B_i \) is traced out. This implies that \( E_B \) and \( B_i \) can be omitted. Their non-trivial part is obtained after tracing out \( B_i \), which can be absorbed into \( D_B \).

The communication resource above can be simulated by a quantum erasure channel 3. In a communication protocol, the \( \operatorname{Tr}_{B_i}W_i^{A \times B} \ast A_i \otimes \mathbb{I}^{B_i} \) part of Equation (16) is equivalent to the channel \( L_i := \operatorname{Tr}_{B_i}W_i^{A \times B} \ast A_i \), while the \( \mathbb{I}^{A'_i} \otimes \sigma^{B_i} \) part is equivalent to a fixed state \( \sigma^{B_i} \). \( W_i \ast A_i \) can be simulated by a quantum erasure channel \( \rho \rightarrow pp+(1-p)|e\rangle\langle e| \): Bob applies the local channel \( L_i \) when no erasure occurs, and locally sends \( |e\rangle \) to \( \sigma \) when the erasure occurs. By doing this for all \( i \), the whole protocol of communicating using \( W \) can be simulated by one using the erasure channel. Consequently the quantum capacity of \( W^{AB} \) is upper-bounded by that of the quantum erasure channel, which is known to be \( Q = \max \{0, 2p-1\} \) [30]. Therefore the capacity of the process to communicate from \( A \) to \( B \) can be positive only if \( p > 1/2 \).

To see that when \( p > 1/2 \) there can indeed be positive capacity, simply let \( W^{A \times B} \) describe the identity channel on a subspace, and let \( W^{A \to B} \) induce a state \( \sigma^{B_j} \) on the orthogonal subspace. Then Equation (16) is effectively a quantum erasure channel, which has capacity \( Q = \max \{0, 2p-1\} \) that is positive for \( p > 1/2 \).

**Corollary 1.1.** \( W^{AB} = pW^{A \times B} + (1-p)W^{A \to B} \) can have positive quantum communication capacity in at most one direction (either Alice to Bob or Bob to Alice).

**Proof.** By the previous theorem, to have positive capacity in either direction \( p \) or \( 1-p \) has to be greater than \( 1/2 \). Yet this can only hold for at most one of them. \( \square \)

**Corollary 1.2.** \( W^{AB} = \frac{1}{2}W^{A \times B} + \frac{1}{2}W^{A \to B} \) has no quantum communication capacity in either direction.

This is a simple consequence of the previous theorem. When the uncertainty in the causal order is maximal \( (p = 1-p = 1/2) \), there is no quantum communication capacity in either direction.

**DISCUSSION**

We have seen that some bias in the causal order is necessary to have any quantum communication, with the consequence that quantum information can only be exchanged in one direction when the same system is used for read-out and encoding. Interestingly, this is not the case for classical communication: agents can transmit perfect classical bits asymptotically, as long as in each run there is a non-zero probability of having a channel in the right direction.

The impossibility to communicate quantum information bidirectionally does not contradict recent results proving two-way quantum communication with a single particle, exchanged in a superposition of directions [31, 32]. Indeed, in those scenarios each agent performs a preparation first and a measurement afterwards, with each agent’s measurement always after the other’s preparation. This corresponds to a four-partite process with fixed causal order, although with the interesting constraint that only one particle per run is exchanged.

Finally, there are several promising directions to extend the analysis presented here. We have only considered classical uncertainty of causal order, modelled by causally separable processes. It remains to be established whether processes with *indefinite* causal order [8, 18] can outperform separable ones in this respect, for example if they permit bidirectional quantum communication. Since indefinite causal order can provide advantages in certain communication tasks [33–35], it is an interesting open question whether it also constitutes a quantum communication resource in the asymptotic scenario treated here, in particular in view of the recent experimental interest [36–42]. Furthermore, there are other communication settings such as different asymptotic settings or the one-shot setting where the results in this work do not apply. These and the general topic of quantifying causal order as a resource for communication are left for further investigations.

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3 A quantum erasure channel is defined to send an input state \( \rho \) to the output state \( p\rho + (1-p)|e\rangle \langle e| \), where the erasure flag state \( |e\rangle \) is orthogonal to any possible input state [29]. The idea is that when the erasure occurs the receiver can detect it from the flag state in an orthogonal subspace.
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