Fluctuation Cumulant Behavior for the Field-Pulse Induced Magnetisation-Reversal Transition in Ising Models

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The universality class of the dynamic magnetisation-reversal transition, induced by a competing field pulse, in an Ising model on a square lattice, below its static ordering temperature, is studied here using Monte Carlo simulations. Fourth order cumulant of the order parameter distribution is studied for different system sizes around the phase boundary region. The crossing point of the cumulant (for different system sizes) gives the transition point and the value of the cumulant at the transition point indicates the universality class of the transition. The cumulant value at the crossing point for low temperature and pulse width range is observed to be significantly less than that for the static transition in the same two-dimensional Ising model. The finite size scaling behaviour in this range also indicates a higher correlation length exponent value. For higher temperature and pulse width range, the transition seems to fall in a mean-field like universality class.

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I. INTRODUCTION

The response of a pure magnetic system to time-dependent external magnetic fields has been of current interest in statistical physics [1, 2, 3, 4]. These studies, having close applications in recording and switching industry, have also got considerable practical importance. These spin systems, driven by time-dependent external magnetic fields, have basically got a competition between two time scales: the time-period of the driving field and the relaxation time of the driven system. This gives rise to interesting non-equilibrium phenomena. Tôme and Oliveira first made a mean-field study [4] of kinetic Ising systems under oscillating field. The existence of the dynamic phase transition for such a system and its nature have been thoroughly studied using extensive Monte-Carlo simulations. Later, investigations were extended to the dynamic response of (ferromagnetic) pure Ising systems under magnetic fields of finite-time duration [5].

All the studies with pulsed field were made below $T_c$, the static critical temperature (without any field), where the system gets ordered. A ‘positive’ pulse is one which is applied along the direction of prevalent order, while the ‘negative’ one is applied opposite to that. The results for the positive pulse case did not involve any new thermodynamic scale [5]. In the negative pulse case, however, interesting features were observed [5]: the negative field pulse competes with the existing order, and the system makes a transition from one ordered state characterised by an equilibrium magnetisation $+m_0$ (say) to the other equivalent ordered state with equilibrium magnetisation $-m_0$, depending on the temperature $T$, field strength $h_p$ and its duration $\Delta t$. This transition is well studied in the limit $\Delta t \to \infty$ for any non-zero value of $h_p$ at any $T < T_c^0$. This transition, for the general cases of finite $\Delta t$, is called here ‘magnetisation-reversal’ transition. Some aspects of this transition has been recently studied extensively [2, 6].

II. MODEL AND THE TRANSITION

The model studied here is the Ising model with nearest-neighbour interaction under a time-dependent external magnetic field. This is described by the Hamiltonian:

$$H = -\frac{1}{2} \sum_{\langle ij \rangle} J_{ij} S_i S_j - h(t) \sum_i S_i,$$

(1)

where $J_{ij}$ is the cooperative interaction between the spins at site $i$ and $j$ respectively, and each nearest-neighbour pair is denoted by $\langle \ldots \rangle$. We consider a square lattice. The static critical temperature is $T_c^0 = 2/\ln(1 + \sqrt{2}) \approx 2.269\ldots$ (in units of $J/K_B$). At $T < T_c^0$, an external field pulse is applied, after the system is brought to equilibrium characterised by an equilibrium magnetisation $m_0(T)$. The spatially uniform field has a time-dependence as follows:

$$h(t) = \begin{cases} -h_p & t_0 \leq t \leq t_0 + \Delta t \\ 0 & \text{otherwise.} \end{cases}$$

(2)

Typical time-dependent (response) magnetisation $m(t)$ ($= < S_i >$, where $< \ldots >$ denotes the thermodynamic ‘ensemble’ average) of the system under different magnetic field $h(t)$ are indicated in the Fig. 1. The time $t_0$ at which the pulse is applied is chosen such that the system reaches its equilibrium at $T (< T_c^0 )$. As soon as the field is applied, the magnetisation $m(t)$ starts decreasing, continues until time $t + \Delta t$ when the field is withdrawn.
The system relaxes eventually to one of the two equilibrium states (with magnetisation \(-m_0\) or \(+m_0\)). At a particular temperature \(T\), for appropriate combinations of \(h_p\) and \(\Delta t\), a magnetisation-reversal transition occurs, when the magnetisation of the system switches from one state of equilibrium magnetisation \(m_0\) to the other with magnetisation \(-m_0\). This reversal phenomena at \(T < T_c\) is simple and well studied for \(\Delta t \to \infty\) for any non-zero \(h_p\). We study here the dynamics for finite \(\Delta t\) values. It appears that generally \(h_p \to \infty\) as \(\Delta t \to 0\) and \(h_p \to 0\) as \(\Delta t \to \infty\) for any such dynamic magnetisation-reversal transition phase boundary at any temperature \(T\) (< \(T_c\)). In fact, a simple application of the domain nucleation theory gives \(h_p \ln \Delta t = \text{constant}\) along the phase boundary, where the constant changes by a factor \(1/(d+1)\), where \(d\) denotes the lattice dimension, as the boundary changes from single to multi-domain region 

\[
dm(t) = \frac{h_p}{\Delta T} \left( \frac{h_p}{\Delta T} - m_0 \right) \left[ \exp \left( \frac{\Delta T}{T} (t - t_0) \right) \right]^{-1}. 
\]

As a solution of eqn. (3), for \(t_0 \leq t \leq t_0 + \Delta t\). Here \(\Delta T = T_c^{mf} - T\), where \(T_c^{mf} \equiv J(q = 0)\) is the static mean-field critical temperature and \(J(q)\) is the Fourier transform of the interaction \(J_{ij}\). Due to application of the field \(h_p\), \(m(t)\) decreases in magnitude from \(m(t_0) \equiv m_0\) to \(m(t_0 + \Delta t) \equiv m_w\) at the time of withdrawal of the pulse. Due to absence of fluctuation here, magnetisation relaxes back to its original value \(m_0\) if \(m_w\) is positive, or to a value \(-m_0\) if \(m_w\) is negative. In the \(t > t_0 + \Delta t\) regime, where \(h(t) = 0\), the magnetisation (starting from \(m_w\) at \(t = t + \Delta t\)) relaxes back to its final equilibrium value \(\pm m_0\), with a relaxation time 

\[
\tau \sim \frac{1}{(T_c^{mf} - T)} \ln \left( \frac{m_0}{m_w} \right). 
\]

It diverges at the magnetisation-reversal transition boundary, where \(m_w\) vanishes. The prefactor gives the divergence of \(\tau\) at the static mean field transition temperature, and is responsible for critical slowing down phenomena at the static transition point \((h = 0)\). The other factor gives the diverging time scale, at any temperature below the static transition temperature, where magnetisation reversal occurs or \(m_w\) vanishes due to appropriate combination of \(h_p\) and \(\Delta t\). The solution of the susceptibility \(\chi(q)\) gives 

\[
\chi(q) \sim \exp \left(-q^2 \xi^2\right), 
\]

where the correlation length is given by 

\[
\xi \sim \left[ \frac{1}{(T_c^{mf} - T)} \ln \left( \frac{m_0}{m_w} \right) \right]^{\frac{1}{2}}. 
\]

Here, too, the prefactor in \(\chi\) gives the usual divergence at \(T_c^{mf}\), while the other factor gives the divergence at the magnetisation-reversal transition point. Incorporating fluctuations, extensive Monte-Carlo simulation studies have also convincingly demonstrated 
that the fluctuation in the order parameter \(|m_w|\) and in the internal energy of the system grows with the system size and diverges at the magnetisation-reversal transition boundary, where \(m_w\) vanishes.

**III. MONTE-CARLO STUDY AND THE RESULTS**

Here the Monte-Carlo study has been carried out in two-dimensions (square lattice) with periodic boundary conditions. Spins are updated following Glauber dynamics. The updating rule employed here are both sequential as well as random. In sequential updating rule one

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**FIG. 1.** Typical time variation of the response magnetisation \(m(t)\) for two different field pulses \(h(t)\) with same \(\Delta t\) for an Ising system at a fixed temperature \(T\). The magnetisation-reversal here occurs due to increased pulse strength, keeping their width \(\Delta t\) same. The transition can also be brought about by increasing \(\Delta t\), keeping \(h_p\) fixed. The inset indicates the typical phase boundaries (where the field withdrawal-time magnetisation \(m_w = 0\)) for two different temperatures (sequential updating; note that for random updating the phase boundaries shift upwards).

A mean field study of the problem gives a qualitative understanding of the diverging time and length scales developed near the transition boundary (in the \(h_p - \Delta t\) plane at a fixed \(T < T_c\)). Mean-field approximation for the dynamics gives the equation of motion for the average magnetisation \(m_i\) as 

\[
\frac{dm_i}{dt} = -m_i + \tanh \left( \sum_j J_{ij} m_j + h(t) \right) \frac{T}{T_c}. 
\]

This equation, linearised near the magnetisation-reversal transition point, gives, for uniform magnetisation,
Monte-Carlo step consist of a complete scan of the lattice in a sequential manner; while in random updating a Monte-Carlo step is defined by \( N (= L^2) \) random updates on the lattice, where \( N \) is the total number of spins in a lattice of linear size \( L \). Studies have been carried out at temperatures below the static critical temperature \( (T_c^0 \approx 2.27) \). The system is allowed to evolve from an initial state of perfect order to its equilibrium state at temperature \( T \). The time \( t_0 \) is chosen to be much larger than the static relaxation time at that \( T \), so that the system reaches an equilibrium state with magnetization \( +m_0(T) \) before the external magnetic field is applied at time \( t = t_0 \). The field pulse of strength \( -h_p \) is applied for duration \( \Delta t \) (measured in Monte Carlo steps or MCS).

The magnetisation starts decreasing from its equilibrium value \( m_0 \). The average value of the magnetisation \( m_w \) at the time of withdrawal of pulse is noted. The phase boundary of this dynamic transition is defined by appropriate combination of \( h_p \) and \( \Delta t \) that produces the magnetisation reversal by making \( m(t_0 + \Delta t) \equiv m_w = 0 \) from a value \( m(t_0) = m_0 \), i.e., \( m_w \) changes sign across the phase boundary. The phase boundary changes with \( T \). The behavior of different thermodynamic quantities are studied across the phase boundary. These quantities are averaged over 1000–20000 different initial configurations of the system. The fluctuations over the average value are also noted.

Here we study the behavior of the reduced fourth order cumulant \( U \) near the magnetisation reversal transition. This is defined as

\[
U = 1 - \frac{\langle m_w^4 \rangle}{3 \langle m_w^2 \rangle^2},
\]

where \( \langle m_w^4 \rangle \) is the ensemble average of \( m_w^4 \). \( \langle m_w^2 \rangle \) is similarly defined. The cumulant \( U \) here behaves somewhat differently, compared to that in static and other transitions: Deep inside the ordered phase \( m_w \approx 1 \) and \( U \rightarrow 2/3 \). For other (say, static) transitions the order parameter \( m_w \) goes to zero with a Gaussian fluctuation above the transition point, giving \( U \rightarrow 0 \) there. Here, however, due to the presence of the pulsed field, \( |m_w| \) is non-zero on both sides of the magnetisation-reversal transition. Hence \( U \) drops to zero at a point near the transition and grows again after it.

The universality class of the dynamic transition in Ising model under oscillating field has been studied extensively by investigating \( \Delta t \) the critical point and the cumulant value \( U^* \) at the critical point, where the cumulant curves cross for different system sizes \( L \). In that case, of course, the variation of \( U \) (at any fixed \( L \)) is similar to that in the static Ising transitions \( U = 2/3 \) well inside the ordered phase and \( U \rightarrow 0 \) well within the disordered phase. In fact, \( U^* \) value in this oscillatory field case was found to be the same as that in the static case, indicating the same universality class \( U^* \). We observe different behaviour in the field pulse induced magnetisation-reversal transition case.

We observe two kinds of distinct behavior of the cumulant \( U \). Typically, for low temperature and low pulse-duration region (see the inset in Fig. 1) of the magnetisation-reversal phase boundary, the cumulant crossing for different system sizes \( L \) occur at \( U^* \approx 0.42 \) to 0.46 (see Fig. 2). As mentioned already, we have checked these results for both sequential and random updating. Specifically, for \( T = 0.5 \) and \( \Delta t = 5 \), (see Fig. 2c) we find the transition point value of \( h_p \approx 2.6 \), to be smaller than the value \( (\approx 1.9) \) for sequential updating. However, the value of \( U^* \) at this transition point is again very close to about 0.44. This indicates that updating rule does not affect the universality class \( U^* \) value, as long as the proper region of the phase boundary is considered. For relatively higher temperature and pulse-duration region of the phase boundary, the crossing of \( U \) for different \( L \) values occur for \( U^* \approx 0.4 \). This is true for both sequential (Fig. 3a, b) and random (Fig. 3c) updating. It may be noted that the phase boundary changes with the updating rule, as the system relaxation time (which matches with the pulse width at the phase boundary) is different for sequential and random updating. 

![Graph showing the behavior of cumulant U near the magnetisation reversal transition.](image-url)
FIG. 2. Behavior of $U$ near the transition, driven by (a) $T$ at a fixed value of $h_p (=1.9)$ and $\Delta t (=5)$ with sequential updating, (b) $h_p$ at a fixed value of $T (=0.5)$ and $\Delta t (=5)$ with sequential updating, and (c) $h_p$ at a fixed value of $T (=0.5)$ and $\Delta t (=5)$ with random updating, for different $L$, averaged over 1000 to 20000 initial configurations. The fluctuations are smaller than the symbol size. The insets show the typical behavior of the magnetisation $m_w$ at the time of withdrawal of the field pulse by varying (a) $T$ at a fixed $h_p$ and $\Delta t$, for $L = 100$ and 800, (b) $h_p$ at a fixed $T$ and $\Delta t$, for $L = 100$ and 400, (c) $h_p$ at a fixed $T$ and $\Delta t$, for $L = 50$ and 200; $m_w = 0$ at the effective transition point. (d) Finite size scaling study in this parameter range: the effective $T_c$ or $h^*_c$ values (see the insets), where $m_w = 0$, are plotted against $L^{-1/\nu}$ with $\nu^{-1} = 0.7$. The values of the cumulant crossing points in (a), (b), (c) are taken to correspond the respective transition points for $L \rightarrow \infty$.

It might be noted that in the low temperature and $\Delta t$ regions, there seems to be significant finite size scaling of the transition ($m_w = 0$) point (see the insets of Fig. 2a, b, c). In fact, in Fig. 2d, the finite-size scaling analysis of those data is presented. For the other cases, there seems to be no significant finite size effect on the transition point (cf. insets of Fig. 3a, b, c), indicative of a mean-field nature of the transition in this range. It may be noted that to compare the finite size effects, we normalise the parameters $T$ or $h_p$ by their ranges required for full magnetisation reversal. In fact, this weak finite size effect for high $T$ and $\Delta t$ regions did not lead to any reasonable value for the fitting exponent in the scaling analysis.

FIG. 3. Behavior of $U$ near the transition, driven by (a) $h_p$ at a fixed value of $T (=2.0)$ and $\Delta t (=5)$ with sequential updating, (b) $T$ at a fixed value of $h_p (=0.5)$ and $\Delta t (=10)$ with sequential updating, and (c) $h_p$ at a fixed value of $T (=1.5)$ and $\Delta t (=5)$ with random updating, for different $L$, averaged over 1000 to 6000 initial configurations. The fluctuations are smaller than the symbol size.
The insets show the typical behavior of the magnetisation $m_w$ at the time of withdrawal of the field pulse by varying (a) $h_p$ at a fixed $T$ and $\Delta t$, for $L = 50$ and 400, (b) $T$ at a fixed $h_p$ and $\Delta t$, for $L = 50$ and 200, (c) $h_p$ at a fixed $T$ and $\Delta t$, for $L = 50$ and 200; $m_w = 0$ at the transition point.

For the static transition of the pure two-dimensional Ising system, $U^* \simeq 0.6107$ [2, 3, 4]. For low temperature (and low $\Delta t$) regions of the magnetisation-reversal phase boundary, the observed values of $U^*$ (in the range $0.42 - 0.46$) are considerably lower than the above mentioned value for the static transition. There is not enough indication of finite-size effect in the $U^*$ value either (cf. [2]). This suggests a new universality class in this range. Also, the finite-size scaling study for the effective transition points here (see Fig. 2d) gives a correlation length exponent value ($\nu \simeq 1.4$) larger than that of the static transition. For comparatively higher temperatures (and high $\Delta t$), the $U^* \simeq 0.4_+$ at the crossing point. Such small value of the cumulant at the crossing point can hardly be imagined to be a finite-size effect; it seems unlikely that one would get here also the same universality class and $U^*$ value will eventually shoot up to $U^* \simeq 0.44$ (for larger system sizes), as for the other range of the transition. On the other hand, such low value of $U^*$ might indicate a very weak singularity, as indicated by the mean field calculations [2] mentioned in the introduction. In fact, even for the static transition, as the dimensionality increases, and the singularity becomes weaker (converging to mean field exponents) with increasing lattice dimension, the cumulant crossing point $U^*$ decreases ($U^* \simeq 0.61$ in $d = 2$ to $U^* \simeq 0.44$ in $d = 4$) [2]. We believe the mean field transition behavior here, as mentioned earlier, is even weaker in this dynamic case as reflected by the value $U^* \simeq 0.4_+$, corresponding to a logarithmic singularity (as in eqns. [5] and [7]).

IV. SUMMARY AND CONCLUSIONS

The universality class of the dynamic magnetisation-reversal transition, induced by a competing pulse, in an Ising model on a square lattice, below its static ordering temperature, is studied here using Monte Carlo simulations. Both sequential and random updating have been used. The phase boundary at any $T (< T^*_p)$ is obtained first in the $h_p - \Delta t$ plane. They of course depend on the updating rule. The phase boundaries obtained compare well with the nucleation theory estimate $h_p \ln \Delta t = \text{constant}$ along the boundary [2]. The mean-field theory applications [2, 3] indicated time and length (eqns. [6] and [7]) respectively scale divergences at these phase boundaries. Extensive Monte-Carlo studies for the fluctuations in the order parameter $|m_w|$ and internal energies etc. showed prominent divergences along the phase boundaries [6]. Fourth order cumulant ($U$) of the order parameter distribution is studied here for different system sizes (upto $L = 800$) around the phase boundary region. The crossing point of the cumulant (for different system sizes) gives the transition point and the value $U^*$ of the cumulant at the transition point indicates the universality class of the transition. In the low temperature and low pulse range, the $U^*$ value is found to be around 0.44 (see Figs. 2a, b, c). The prominent discrepancy with the $U^*$ value ($\simeq 0.61$) for the static transition in the same model in two dimensions indicates a new universality class for this dynamic transition. Indeed, the finite-size scaling analysis (Fig. 2d) suggests a different (larger) value of the correlation length exponent also. For comparatively higher temperatures and higher pulse widths, the $U^*$ values are very close to zero (see Fig. 3a, b, c), and the transitions here seem to fall in a mean-field-like weak-singularity universality class similar to that obtained earlier [3], and indicated by eqns. [6] and [7]. Here, the finite size effects in the order parameter and the transition point are also observed to be comparatively weaker (see insets of Fig. 3).

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