Remarks on the horocycle flows for foliations by hyperbolic surfaces

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Abstract. We show that the horocycle flow associated with a foliation on a compact manifold by hyperbolic surfaces is minimal under certain conditions.

1. Introduction

In the 1936 paper \cite{3}, Gustav A. Hedlund showed the minimality of the horocycle flows associated to closed oriented hyperbolic surfaces. In \cite{4}, Matilde Martínez and Alberto Verjovsky consider a generalization of this fact to compact laminations by hyperbolic surfaces. Let us state the question they raised in the case of foliations.

Let $M$ be a closed smooth manifold and $\mathcal{F}$ a foliation by hyperbolic surfaces i.e. a 2-dimensional smooth foliation equipped with a continuous leafwise metric of curvature $-1$. Let $\pi : N \to M$ be the leafwise unit tangent bundle. The total space $N$ admits a locally free action of the universal covering group of $PSL(2,\mathbb{R})$, whose orbit foliation is denoted by $\mathcal{A}$. As a consequence, we have the geodesic flow $g^t$, the stable horocycle flow $h^+_t$ and the unstable horocycle flow $h^-_t$. They satisfy

\begin{equation}
 g^t \circ h^+_s \circ g^{-t} = h^+_s \circ g^{-t}, \quad g^t \circ h^-_s \circ g^{-t} = h^-_s \circ g^{-t}.
\end{equation}

The flows $g^t$ and $h^\pm_t$ jointly define an action of a closed subgroup $B^\pm$, whose orbit foliation is denoted by $B^\pm$. They are subfoliations of $\mathcal{A}$ transverse to each other in a leaf of $\mathcal{A}$ and the intersection is the orbit foliation of $g^t$.

Question 1.1. (Martínez-Verjovsky) Does the minimality of the foliation $B^+_\pm$ imply the minimality of the flow $h^\pm_t$?

Fernando Alcalde-Cuesta and Françoise Dal’bo \cite{1} showed that the answer is positive if $\mathcal{F}$ is a homogeneous Lie foliation. See also F. Alcalde, F. Dal’bo, M. Martín and A. Verjosky \cite{2} for related results. So far no counter-example is reported.

Throughout this paper, $\mathcal{F}$ is to be a codimension $q$ smooth foliation on a closed manifold $M$ by hyperbolic surfaces. We assume that there is a simple closed oriented geodesic $\hat{c}$ in some leaf of $\mathcal{F}$. Let $\hat{D}^i \subset \hat{D}$ be smooth closed $q$-disks in $M$ transverse
to $\mathcal{F}$ such that $\hat{D}' \cap \hat{c} = \hat{D} \cap \hat{c} = \{z_0\}$. Let $\hat{f} : \hat{D}' \to \hat{D}$ be the holonomy map of $\mathcal{F}$ along the curve $\hat{c}$. Thus $\hat{f}(z_0) = z_0$. Let us consider the following condition.

\begin{equation}
\text{There exists an open subset } \hat{U} \subset \hat{D}' \text{ such that } z_0 \in \text{Cl}(\hat{U}), \hat{f}(\hat{U}) \supset \hat{U}, \text{ and for any } z \in \text{Cl}(\hat{U}), d(\hat{f}(z), z_0) \geq d(z, z_0). \tag{\ast}
\end{equation}

The main result of this paper is the following.

**Theorem 1.2.** If there is a leafwise simple closed geodesic which satisfies (\ast), then Question 1.1 has a positive answer.

In Section 3, we show:

**Corollary 1.3.** If $\mathcal{F}$ is a minimal\footnote{i.e. all the leaves are dense} Riemannian foliation and admits a simple closed geodesic on some leaf, then the associated horocycle flow is minimal.

Next consider the case where $\mathcal{F}$ is of codimension 1, that is, $\dim(M) = 3$. As will be shown in Section 4, $\mathcal{F}$ must have a nonplanar leaf. Let $\hat{c}$ be a leafwise simple closed geodesic. Consider the holonomy map $\hat{f}$ along $\hat{c}$ on a transverse open interval $I$ such that $\hat{c} \cap I = \{z_0\}$. If neither $\hat{f}$ nor $\hat{f}^{-1}$ satisfy (\ast) for $z_0$, then there is a fixed point $z_1$ of $\hat{f}$ near $z_0$ for which either $\hat{f}$ or $\hat{f}^{-1}$ satisfies (\ast). See Figure 1.

![Figure 1](image)

**Figure 1.** $z_0$ does not satisfy (\ast), but an endpoint $z_1$ of a component of $I \setminus \text{Fix}(\hat{f})$ satisfies (\ast).

Therefore we have:

**Corollary 1.4.** If $\mathcal{F}$ is of codimention one, then Question 1.1 has an affirmative answer.

2. **Proof of Theorem 1.2**

Let $\hat{c}$ be a leafwise simple closed geodesic in $M$ which satisfies (\ast). Associated to $\hat{c}$, there is a periodic orbit of the leafwise geodesic flow $g^t$ denoted by $c = (\hat{c}, \hat{c}')$. For a point $z_0$ in (\ast), let $\pi^{-1}(z_0) \cap c = \{\zeta_0\}$, that is, $\zeta_0$ is the tangent vector of $\hat{c}$ at $z_0$. Let $E$ be a smooth closed $(q + 2)$-disk in the $(q + 3)$-dimensional manifold $N$ transverse to $g^t$ centered at $\zeta_0$. Let $D$ be a $q$-disk centered at $\zeta_0$ contained in $\text{Int}(E)$ and transverse to the foliation $\mathcal{A}$. The projection $\pi$ yields a diffeomorphism from $D$ to its image $\hat{D}$, a $q$-disk transverse to $\mathcal{F}$. The disk $E$, being transverse to the flow $g^t$, is transverse to the foliation $B_\pm$ and $\mathcal{A}$. Let $\beta_\pm$ and $\alpha$ be the restriction of the foliation $B_\pm$ and $\mathcal{A}$ to $E$. See Figure 2.
The 1-dimensional foliations \( \beta_+ \) and \( \beta_- \) are subfoliations of the 2-dimensional foliation \( \alpha \), transverse to each other in a leaf of \( \alpha \). Given a point \( x \in E \), let us denote by \( \beta_\pm(x) \) and \( \alpha(x) \) the leaves of the corresponding foliations which pass through \( x \). Given \( \zeta \in D \) and small \( r > 0 \), let \( \iota_\pm, \zeta : [−r, r] \to \beta_\pm(\zeta) \) be the isometric embedding such that \( \iota_\pm(0) = \zeta \). Let us denote by \( [\xi, \eta, \zeta] \) the unique point of the intersection of \( \beta_+(\iota_-, \zeta(\xi)) \) and \( \beta_-(\iota_+, \zeta(\eta)) \). See Figure 3.

Recall the open subset \( \hat{U} \) of \( \hat{D}' \) in condition (\( * \)). For small \( r > 0 \), define

\[
\hat{U}_r = \{ z \in \hat{U} \mid d(z, z_0) \leq r \}.
\]

We have \( \hat{f}(\hat{U}_r) \supset \hat{U}_r \). Let \( U_r = \pi^{-1}(\hat{U}_r) \cap E \) and define

\[
V_r = \{ [\xi, \eta, \zeta] \mid |\xi| \leq r, |\eta| \leq r, \zeta \in \text{Cl}(U_r) \}.
\]

The first return map \( F : V_r \to E \) of the flow \( g^t \) can be written as

\[
F[\xi, \eta, \zeta] = [\phi_{\zeta}(\xi), \psi_{\zeta}(\eta), f(\zeta)].
\]

The map \( f \) is a conjugate of the map \( \hat{f} \) in (\( * \)) and it satisfies \( f(\zeta_0) = \zeta_0 \), \( f(U_r) \supset U_r \) and

\[
d(f(\zeta), f(\zeta_0)) \geq d(\zeta, \zeta_0), \quad \forall \zeta \in \text{Cl}(U_r),
\]

for an appropriate metric \( d \). By the hyperbolicity (1.1) of the leafwise geodesic flow \( g^t \), there is \( \lambda \in (0, 1) \) such that

\[
d(\phi_{\zeta}(\xi), \phi_{\zeta}(\xi')) \geq \lambda^{-1}d(\xi, \xi'), \quad \forall \xi, \xi' \in [−r, r],
\]
(2.1) \[ d(\psi(\xi(\eta)), \psi(\xi'(\eta'))) \leq \lambda d(\eta, \eta'), \quad \forall \eta, \eta' \in [-r, r]. \]

On the other hand, since \( \zeta_0 = [0, 0, \zeta_0] \) is a fixed point of \( F \), we have for small \( r > 0 \),
\[
\phi_{f^{-1}(\zeta)}(-r) < -r < \phi_{f^{-1}(\zeta)}(r)
\]
and
(2.2) \[-r < \psi_{f^{-1}(\zeta)}(-r) < \psi_{f^{-1}(\zeta)}(r) < r.\]

Therefore
\[ F(V_r) \cap V_r = \{ [\xi, \eta, \zeta] \mid |\xi| \leq r, \psi_{f^{-1}(\zeta)}(-r) \leq \eta \leq \psi_{f^{-1}(\zeta)}(r), \zeta \in U_r \}. \]

Now we have by (2.2)
\[ -r < \psi_{f^{-1}(\zeta)}(-r) < \psi_{f^{-1}(\zeta)}^2(-r) < \cdots < \psi_{f^{-1}(\zeta)}^{2^n}(r) < \psi_{f^{-1}(\zeta)}(r) < r \]
and by (2)
\[ \lim_{n \to \infty} \psi_{f^{-1}(\zeta)}^n(-r) = \lim_{n \to \infty} \psi_{f^{-1}(\zeta)}^n(r) =: \Psi(\zeta). \]

Therefore for any \( n > 0 \),
\[ \bigcap_{0 \leq i \leq n} F^i(V_r) = \{ [\xi, \eta, \zeta] \mid |\xi| \leq r, \psi_{f^{-1}(\zeta)}^n(-r) \leq \eta \leq \psi_{f^{-1}(\zeta)}^n(r), \zeta \in U_r \}, \]
and
\[ K_r := \bigcap_{n \geq 0} F^n(V_r) = \{ [\xi, \Psi(\zeta), \zeta] \mid |\xi| \leq r, \zeta \in U_r \}. \]

The function \( \Psi \) is continuous since \( K_r \) is closed. Notice that if \( x \in K_r \) and \( n > 0 \), then \( F^{-n}(x) \in V_r \).

Let \( \tau : V_r \to (0, \infty) \) be the first return time to \( E \) of the flow \( g^t \) and let
\[ \tilde{V}_r = \{ g^t(x) \mid x \in V_r, t \in [0, \tau(x)] \} \quad \text{and} \quad \tilde{K}_r = \{ g^t(x) \mid x \in K_r, t \in [0, \tau(x)] \}. \]

Then we have:

(\text{**}) If \( x \in \tilde{K}_r \) and \( t > 0 \), then \( g^{-t}(x) \in \tilde{V}_r \).

Now let us finish the proof of Theorem 1.2. We assume that the foliation \( B_+ \) is minimal. Let \( M \) be a minimal set of the flow \( h^t_+ \). Then we have
\[ \bigcap_{t_0 \in \mathbb{R}} \bigcup_{t \geq t_0} g^t(M) = N, \]
since the LHS is \( B_+ \)-invariant, closed and nonempty. Since \( \tilde{V}_r \) has nonempty interior, there is \( x \in M \) and \( t > 0 \) such that \( y = g^t(x) \in \tilde{V}_r \). Then an orbit segment of \( h^t_+ \) through \( y \) intersects \( \tilde{K}_r \), say at a point \( y' \). See Figure 4.

\[ \text{Figure 4.} \]
By (1.1), the point \( x' = g^{-t}(y') \) can be written as \( x' = h^t_\pm(x) \), and hence \( x' \in \mathcal{M} \). On the other hand by (**), \( x' \in \hat{V}_r \). That is, \( \mathcal{M} \cap \hat{V}_r \neq \emptyset \). Since \( r \) is arbitrary, we get

(***) \( \mathcal{M} \cap c \neq \emptyset \).

For each \( t \in \mathbb{R} \), we have either \( \mathcal{M} \cap g^t(\mathcal{M}) = \emptyset \) or \( \mathcal{M} = g^t(\mathcal{M}) \) since the both sets are minimal sets of \( h^t_\pm \). Let

\[ T = \{ t \in \mathbb{R} \mid g^t(\mathcal{M}) = \mathcal{M} \}. \]

Then \( T \) is a closed subgroup of \( \mathbb{R} \). The statement (***') shows that \( T \) is nontrivial. If \( T = \mathbb{R} \), then \( \mathcal{M} \) is \( B_+ \)-invariant and we have \( \mathcal{M} = \mathcal{N} \), as is required. Consider the remaining case where \( T \) is isomorphic to \( \mathbb{Z} \). In this case, the minimal set \( \mathcal{M} \) is a global cross section of \( g^t \). It is easy to show that \( \mathcal{M} \) must be a tamely embedded topological submanifold of codimension one. Thus the manifold \( N \) must be a bundle over \( S^1 \) and admit a closed 1-form \( \omega \) which takes positive value at \( \frac{dg^t}{dt}(x) \) for any \( x \in N \). The closed geodesic \( c \) which we started with and the closed geodesic with the reverse orientation correspond to two periodic orbits \( c \) and \( c' \) of \( g^t \) and we must have \( \int_c \omega > 0 \) and \( \int_{c'} \omega > 0 \). However \( c' \) is homotopic to \( -c \). A contradiction completes the proof of Theorem 1.2.

3. Foliations with transverse invariant measures

In this section, we shall prove Corollary 1.3. Since Theorem 1.2 is applicable for any leafwise simple closed geodesic of Riemannian foliations and since Riemannian foliations admit transverse invariant measures, the following theorem is sufficient for Corollary 1.3.

**Theorem 3.1.** Let \( \mathcal{F} \) be a minimal foliation on \( M \) with a transverse invariant measure \( \mu \). Then the foliation \( B_+ \) on \( N \) is minimal.

**Proof.** Assume for contradiction that there is a proper minimal set \( \mathcal{M} \) of the foliation \( B_+ \). Since \( \mathcal{F} \) is minimal, we have \( \pi(\mathcal{M}) = M \). Consider a foliation chart \( D \times T \) of \( \mathcal{F} \), where \( T \) is a transverse disk. Its inverse image \( \pi^{-1}(D \times T) \) can be written as

\[ \pi^{-1}(D \times T) = D \times S^1 \times T, \]

where \( D \times S^1 \times \{t\} \) is a plaque of \( \mathcal{A} \) and \( D \times \{(\theta, t)\} \) is a plaque of \( B_+ \). The intersection \( \mathcal{M} \cap (D \times S^1 \times T) \) has the form

\[ \mathcal{M} \cap (D \times S^1 \times T) = D \times \mathcal{N} \]

for some closed subset \( \mathcal{N} \) of \( S^1 \times T \). Since \( \mathcal{N} \) and \( T \) are standard Borel spaces and the inverse image of any point in \( T \) by the projection \( \mathcal{N} \to T \) is compact, there is a Borel cross section \( \sigma : T \to \mathcal{N} \). This can also be shown in an elementary way as follows. For each \( t \in T \), the set

\[ \mathcal{N}_t = \{ \theta \in S^1 \mid (\theta, t) \in \mathcal{N} \} \]

is a proper nonempty closed subset of \( S^1 \). In fact, if it is \( S^1 \) for some \( t \), then \( \mathcal{M} \) would contain at least one leaf of \( \mathcal{A} \). But the foliation \( \mathcal{A} \) is minimal since \( \mathcal{F} \) is minimal, contradicting the properness of \( \mathcal{M} \). Choosing \( T \) smaller if necessary, one may assume that there is an open interval \( I \) of \( S^1 \) such that \( \mathcal{N}_t \) is contained in
Then there is an upper semi-continuous cross section $\sigma : T \to S^1$ defined by $\sigma(t) = \sup N_t$.

Let $\mu$ be an ergodic transverse invariant measure. Together with the leafwise hyperbolic volume, it defines an ergodic completely invariant harmonic measure $\lambda$ on $M$. Identifying $S^1$ with the circle at infinity of the leaves of $\mathcal{F}$ and considering the leafwise Brownian motion associated to $\lambda$, M. Martínez and A. Verjovsky showed \cite{MartinezVerjovsky} that either $N_t = S^1$ for $\mu$-almost all $t \in T$ or $N_t$ is a singleton for $\mu$-almost all $t$. In our case the former cannot happen and we have the following lemma.

**Lemma 3.2.** The measure $\sigma_* \mu$ is independent of the choice of the Borel cross section $\sigma$. \hfill \qedsymbol

Now the family formed by $\sigma_* \mu$ for each foliation chart yields a transverse invariant measure of the lamination $B_+|_M$, and hence a completely invariant harmonic measure on $M$. But the geodesic flow on $M$ preserves the transverse measure on one hand, and contracts the leafwise hyperbolic measure on the other hand. A contradiction shows Theorem 3.1.

### 4. Codimension one foliations

Here we shall prove the following.

**Theorem 4.1.** There is no foliation by hyperbolic disks on any closed 3-manifold $M$.

**Proof.** Assume on the contrary that there is a smooth foliation $\mathcal{F}$ by hyperbolic disks on closed 3-manifold $M$. Harold Rosenberg \cite{Rosenberg} showed that the 3-torus $T^3$ is the only 3-manifold which admits a smooth foliation by planes. So we have $M = T^3$. According to William Thurston \cite{Thurston}, $\mathcal{F}$ can be isotoped to be transverse to a fibration $S^1 \to T^3 \to T^2$. Let $f, g \in \text{Diff}^\infty_+(S^1)$ be the holonomy associated to generators of $\pi_1(T^2)$. Then there is an orientation preserving homeomorphism $h$ of $S^1$ such that $h \circ f \circ h^{-1}$ and $h \circ g \circ h^{-1}$ are rigid rotations. This implies that there is an $\mathcal{F}$-preserving topological $S^1$-action whose orbit foliation is the above smooth fibration. Consider the covering space $S^1 \times \mathbb{R}^2$ of $T^3$, where the lifted foliation is $\{\{t\} \times \mathbb{R}^2\}$, and let $\{\phi^t\}_{t \in S^1}$ be the lifted $S^1$-action. Each leaf $\{t\} \times \mathbb{R}^2$ is equipped with a hyperbolic metric. Fix one leaf $\{0\} \times \mathbb{R}^2$ which has a hyperbolic metric $g_0$ and replace the metric of the other leaf $\{t\} \times \mathbb{R}^2$ by $(\phi^{-1})^* g_0$. Then the new metric is $K$-quasiconformally equivalent to the old one with fixed constant $K$. The quotient space of $S^1 \times \mathbb{R}^2$ by the $S^1$-action is identified with the Poincaré upper half plane $\mathbb{H}$. The covering transformation induces a $K$-quasiconformal action of $\mathbb{Z}^2$ on $\mathbb{H}$. Now a theorem of Dennis Sullivan \cite{Sullivan} shows that such an action is topologically conjugate to an action of a subgroup of $PSL(2, \mathbb{R})$. Being a quotient action of a covering transformation, this action must be cocompact, that is, there is a compact subset of $\mathbb{H}$ which intersects each orbit. But this is impossible since the group is $\mathbb{Z}^2$, showing Theorem 4.1. \hfill \qedsymbol

### References

[1] F. Alcalde Cuesta and F. Dal’bo, *Remarks on the dynamics of the horocycle flow for homogeneous foliations by hyperbolic surfaces*, Preprint, arXiv:1410.7181.

[2] F. Alcalde, F. Dal’bo, M. Martínez and A. Verjovsky, *Minimality of the horocycle flow on foliations by hyperbolic surfaces with non-trivial topology*, Preprint, arXiv:1412.3259.
[3] G. A. Hedlund, *Fuchsian groups and transitive horocycles*, Duke Math. J. 2(1936), 530-542.

[4] M. Martínez and A. Verjovsky, *Horocycle flows for laminations by hyperbolic Riemann surfaces and Hedlund’s theorem*, Preprint arXiv:0711.2907.

[5] H. Rosenberg, *Foliations by planes*, Topology, 7(1968), 131-138.

[6] D. Sullivan, *On the ergodic theory at infinity of an arbitrary discrete group of hyperbolic motions*, in “Riemann Surfaces and Related Topics,” 465-496, Ann. Math. Studies 97(1981), Princeton Univ. Press.

[7] W. Thurston, *Foliations of 3-manifolds which are circle bundles*, Thesis Berkeley 1971.

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