Supplemental Document

Generation of quantum states with nonlinear squeezing by Kerr nonlinearity: supplement

Šimon Bräuer and Petr Marek¹,*

Optics Department, Faculty of Science, Palacký University, 17. Listopadu 12, 77146 Olomouc, Czech Republic

*marek@optics.upol.cz

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This supplemental document presents details on calculation, numerical optimization, and statistical evaluation of results presented in the paper.

1. OPTIMAL GAUSSIAN STATES FOR EVALUATING NONLINEAR SQUEEZING.

To properly evaluate the nonlinear squeezing of quantum states given by $\xi_{3,4}$ we require the knowledge of the optimal variance of the ideal Gaussian state that is used in the denominator. Due to the symmetry of the operator $\hat{O}_3$, the ideal Gaussian state for the cubic nonlinearity is the vacuum state squeezed along the $\hat{x}$ axis. For this state, the optimal squeezing parameter and the minimal variance can be analytically calculated in the Heisenberg’s picture, in which the operators $\hat{x}$ and $\hat{p}$ evolve as

$$\hat{x} \rightarrow g \hat{x},$$  \hspace{1cm} (S1)
$$\hat{p} \rightarrow \frac{1}{g} \hat{p}.$$  \hspace{1cm} (S2)

We can then substitute the new $\hat{x}$ and $\hat{p}$ into the variance

$$\text{var}(\hat{O}_3) = \langle \hat{O}_3^2 \rangle - \langle \hat{O}_3 \rangle^2; \quad \hat{O}_3 = \hat{x} - \hat{p}^2,$$

$$\langle \hat{O}_3 \rangle = g \langle 0 | \hat{x} | 0 \rangle - \frac{1}{g^2} \langle 0 | \hat{p}^2 | 0 \rangle,$$

$$\langle \hat{O}_3^2 \rangle = g^2 \langle 0 | \hat{x}^2 | 0 \rangle - \frac{1}{g^2} \langle 0 | \hat{p}^2 + \hat{x}^2 | 0 \rangle + \frac{1}{g^4} \langle 0 | \hat{p}^4 | 0 \rangle,$$

$$\text{var}(\hat{O}_3) = \frac{g^2}{2} + \frac{1}{g^4} \frac{3}{4} - \frac{1}{g^4} \frac{1}{4} = \frac{g^2}{2} + \frac{1}{2g^4}.$$  \hspace{1cm} (S3)

The next step is to perform a partial derivative and to set it equal to the zero value for finding the value of the minimum:

$$\frac{\partial \text{var}(\hat{O}_3)}{\partial g} = 0,$$

$$g - \frac{2}{g^3} = 0 \rightarrow g = \sqrt[3]{2},$$  \hspace{1cm} (S4)

$$\text{var}_{\text{min}}(\hat{O}_3) = 3 \times 2^{\frac{2}{3}}.$$  \hspace{1cm} (S5)

For the case of quartic squeezing, the optimal state is the vacuum state, this time however squeezed in an arbitrary direction. By applying the same approach as in the previous case, the quadrature operators of the squeezed and rotated vacuum state can be expressed as:

$$\hat{x} \rightarrow g \cos(\phi) \hat{x} + \frac{\sin(\phi)}{g} \hat{p},$$  \hspace{1cm} (S6)
$$\hat{p} \rightarrow \frac{\cos(\phi)}{g} \hat{p} - g \sin(\phi) \hat{x}.$$  \hspace{1cm} (S7)
Subsequently, we substitute them into the nonlinear variance and obtain
\[
\text{var}(\hat{O}_4) = \langle \hat{O}_4^2 \rangle - \langle \hat{O}_4 \rangle^2; \quad \hat{O}_4 = \hat{x} - \hat{p}^3, \quad \text{(S16)}
\]
where
\[
\hat{x}' = g (\cos(\phi)x + \sin(\phi)p), \quad \text{(S17)}
\]
\[
\hat{p}' = \frac{1}{g} \left( (\sin(\phi)x - \cos(\phi)p) + \beta \right), \quad \text{(S18)}
\]
\[
\hat{O}'_3 = \hat{x}' - \hat{p}'^3. \quad \text{(S19)}
\]

The optimal parameters for the Gaussian state minimizing this formula can be numerically found to be
\[
g = -0.637, \quad \text{(S14)}
\]
\[
\phi = -1.949, \quad \text{(S15)}
\]
and the minimal variance of the Gaussian state is \( \text{var}_{\text{min}}(\hat{O}_4) = 0.971. \)

2. OPTIMIZATION OF THE STATE PREPARATION

In the following we would like to describe the numerical optimization tools employed to achieve the results in the paper. The optimized parameter is always the relative squeezing parameter \( \xi_{3,4} \), but since the denominator is always fixed, as shown in Section 1 of this supplemental document, it is sufficient to minimize the variance of the quantum operators \( \hat{O}_3 \) and \( \hat{O}_4 \) with \( \hat{O}_n = \hat{x} - \hat{p}^{n-1} \). These variances, which we will denote \( V_3(\alpha; \chi, \phi, \beta, r) \) and \( V_4(r; \chi, \phi_1, \omega, \phi_2) \), respectively, are the functions of the input parameters of the preparation circuit.

The variances for each combination of parameters were calculated in two steps. In the first step we numerically applied the Kerr nonlinearity by expressing the input state in the Fock basis of the dimension \( N \) and by multiplying it by matrix form of the unitary operator for the Kerr operation to produce the approximate representation of the states \( |\xi_3^3\rangle = \hat{K}(\chi)\hat{D}(\alpha)|0\rangle \) and \( |\xi_4^4\rangle = \hat{K}(\chi)\hat{S}(r)|0\rangle \) for the preparation of the cubic and the quartic squeezing, respectively (see Section 4 of this supplemental document for more details about numerical representation of the operators and states used). The application of the Kerr operator could be done perfectly because the operator is diagonal in the Fock basis. In our simulations we have used dimension \( N = 300 \) which was sufficient to faithfully represent the selected input states. We then used these quantum states to evaluate the moments of the quadrature operators.

For generation of the cubic squeezing, the relevant moments can be obtained by the Gaussian transformation of the quadrature operators. That leads to the polynomial formula for the variance
\[
V_3 = \langle \xi_3^3|\hat{O}_3^2|\xi_3^3\rangle - \langle \xi_3^3|\hat{O}_3'\xi_3^3\rangle^2, \quad \text{(S16)}
\]
where
\[
\hat{x}' = g (\cos(\phi)x + \sin(\phi)p), \quad \text{(S17)}
\]
\[
\hat{p}' = \frac{1}{g} \left( (\sin(\phi)x + \cos(\phi)p) + \beta \right), \quad \text{(S18)}
\]
\[
\hat{O}'_3 = \hat{x}' - \hat{p}'^3. \quad \text{(S19)}
\]
Similarly for the quartic squeezing we obtain

\[ V_4 = \langle \zeta_4 | (\hat{O}_4')^2 | \zeta_4 \rangle - \langle \zeta_4 | \hat{O}_4' | \zeta_4 \rangle^2, \tag{S20} \]

where

\[ \hat{x}_1 = \omega \sin(\phi_2) (\sin(\phi_1) \hat{x} + \cos(\phi_1) \hat{p}), \tag{S21} \]
\[ \hat{x}_2 = \frac{1}{\omega} \cos(\phi_2) (-\cos(\phi_1) \hat{x} + \sin(\phi_1) \hat{p}), \tag{S22} \]
\[ \hat{p}_1 = \frac{1}{\omega} \sin(\phi_2) (-\cos(\phi_1) \hat{x} + \sin(\phi_1) \hat{p}), \tag{S23} \]
\[ \hat{p}_2 = \omega \cos(\phi_2) (\sin(\phi_1) \hat{x} + \cos(\phi_1) \hat{p}), \tag{S24} \]
\[ \hat{x}'' = \hat{x}_1 + \hat{x}_2, \tag{S25} \]
\[ \hat{p}'' = \hat{p}_1 - \hat{p}_2, \tag{S26} \]
\[ \hat{O}_4 = \hat{x}'' - \hat{p}''. \tag{S27} \]

**Fig. S1.** Left column - the optimal parameters of obtained by the numerical optimization for the given input displacement for the case of the Cubic nonlinear squeezing. (a) The parameters of Kerr nonlinearity \( \chi \), and phase shift parameter \( \phi \). (b) The displacement parameter \( \beta \), and the squeezing parameter, \( g \). Right column - the optimal parameters obtained by the numerical optimization for a given input displacement for the case of the Quartic nonlinear squeezing. (c) The parameters of Kerr nonlinearity \( \chi \), and rotation \( \phi_1 \). (d) The phase shift parameter \( \phi_2 \), and the squeezing parameter \( \omega \).

The numerical optimization of the functions was performed in Python, with help of the `scipy.optimize.minimize` library and the L-BFGS-B function. This is a quasi-Newtonian optimization method that allows to set the intervals of parameters in which the optimization will take place and thus reduce the computational time. This method uses the Broyden-Fletcher-Goldfarb-Shanno algorithm [1]. The optimization searches for the local minima and always starts from the pre-selected entry points. In our analysis, in which we searched for the minimal values for the different fixed values of \( \alpha \) (for the cubic squeezing) and \( \tau \) (for the quartic squeezing), we always chose one of the sets of the parameters as those that were optimal for the previously calculated value of \( \alpha \) or \( \tau \), and the other 299 sets were chosen randomly. Together there were 300 different starting sets of the parameters for each individual instance of the numerical optimization.

The parameters for which the optimal nonlinear squeezing was found are plotted for the cubic squeezing in Fig. S1(a),(b) and for the quartic squeezing in Fig. S1(c),(d).
3. STATISTICAL EVALUATION OF ERRORS

In the following we would like describe, in detail, the error analysis employed to obtain the results presented in Section V of the paper. When simulating the errors, we started from the optimal set of the parameters and then simulated the random deviations. This was done by running the Monte Carlo simulation, in which \( N_{\text{runs}} = 10000 \) runs of the quantum state preparation were simulated with parameters that were randomly chosen from Gaussian distributions with mean values \( \mu \), that were corresponding to the parameters’ optimal value, and standard deviations \( \sigma = \gamma \mu \), that were considered to be a certain fraction of the mean value. The obtained nonlinear variances from each run were then statistically evaluated.

In each simulated run of the experiment, the obtained nonlinear variance can be expressed as \( \xi_{3,4}(k) \), where \( k = 1, \cdots, N_{\text{runs}} \) denotes the particular run. The fundamental information is provided by the statistical moments. The most important one is the mean value, \( \xi_{3,4} = \sum_{k} \xi_{3,4}(k) / N_{\text{runs}} \), but important insight is also given by the upper and the lower standard deviations

\[
\sigma^+ = \frac{1}{N_+} \sum_{k} (\max(\xi_{3,4}(k) - \bar{\xi}_{3,4}, 0))^2, \tag{S28}
\]

\[
\sigma^- = \frac{1}{N_-} \sum_{k} (\min(\xi_{3,4}(k) - \bar{\xi}_{3,4}, 0))^2, \tag{S29}
\]

where \( N_+ \) and \( N_- \) represent the number of runs in which the measured nonlinear variance \( \xi_{3,4}(k) \) is larger or lower, respectively, than the mean variance \( \bar{\xi}_{3,4} \).

4. NUMERICAL REPRESENTATION OF QUANTUM STATES AND OPERATIONS

To implement the numerical simulation of the quantum states and operations, we chose the dimension of the Hilbert space to be \( N = 300 \) to avoid nonphysical results by ensuring the states we work with are faithfully represented. On this dimension we can represent the truncated annihilation operator by matrix \( A \) with elements

\[
a_{m,n} = \delta_{m,n+1} \sqrt{m} \quad \forall \ m = 1, 2, \ldots, N - 1,.
\]

The quadrature operators \( \hat{x} \) and \( \hat{p} \) can then be represented by matrices

\[
X = \left( A^\dagger + A \right) / \sqrt{2},
\]

\[
\hat{p} = i \left( A^\dagger - A \right) / \sqrt{2}.
\]

Subsequently, we use these matrices to define transformation matrices of displacement \( D(\alpha) \), squeezing \( S(r) \), phase rotation \( R(\phi) \) and Kerr nonlinearity \( K(\chi) \),

\[
D(\alpha) = \exp \left[ -i \alpha X \right],
\]

\[
S(r) = \exp \left[ \frac{ir}{2} \left( XP + PX \right) \right],
\]

\[
R(\phi) = \exp \left[ \frac{i\phi}{2} \left( X^2 + X^2 \right) \right],
\]

\[
K(\chi) = \exp \left[ -iX \left( X^2 + P^2 \right) \right],
\]

and use them to calculate the vector representation of the quantum state obtained by applying Kerr operation to a coherent state for the case of preparation of the cubic squeezing,

\[
|\xi_3\rangle = \hat{K}(\chi) \hat{D}(\alpha) |0\rangle,
\]
or to a squeezed state for the case of preparation of the quartic squeezing,

$$|\zeta_4\rangle = \hat{K}(\chi)\hat{S}(r)|0\rangle.$$ 

In both cases, the initial vacuum state is represented by numerical vector

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$ 

REFERENCES

1. R. Fletcher, *Practical methods of optimization* (Wiley, Chichester New York, 1987).