In this paper, we establish the existence and stability properties of odd periodic waves related to the Klein–Gordon type equations, which include the well known $\phi^4$ and $\phi^6$ models. Existence of periodic waves is determined by using a general planar theory of ODE. The spectral analysis for the corresponding linearized operator is established using the monotonicity of the period map combined with an improvement of the standard Floquet theory. Orbital stability in the odd sector of the energy space is proved using exclusively the monotonicity of the period map. The orbital instability of explicit solutions for the $\phi^4$ and $\phi^6$ models is presented using the abstract approach of instability for general abstract Hamiltonian systems.

**KEYWORDS**
Klein–Gordon equations, orbital instability, orbital stability, periodic waves

**MSC CLASSIFICATION**
76B25, 35Q51, 35Q70, 35B10, 35B35, 35Q99

1 | INTRODUCTION

The main purpose of this paper is to present orbital stability properties associated to the generalized Klein–Gordon equation

$$\phi_{tt} - \phi_{xx} - \phi + \phi^{2k+1} = 0, \quad (1.1)$$

where $k \geq 1$ is an integer and $\phi : \mathbb{R} \times [0, +\infty) \to \mathbb{R}$ is a periodic function at the spatial variable, that is, it satisfies $\phi(x + L, t) = \phi(x, t)$, where $(x, t) \in \mathbb{R} \times [0, +\infty)$ and $L > 0$ indicates the minimal period of $\phi$. When $k = 1$, we have the well known $\phi^4$ equation, which plays an important role in nuclear and particle physics. For the case $k = 2$, we obtain the $\phi^6$ equation found in the energy transport along the hydrogen-bonded chains. Both models have been studied by several researches along the last years.

Equation (1.1) has an abstract Hamiltonian form,

$$\Phi_t = JE'(\Phi), \quad (1.2)$$

where $J$ is given by

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad (1.3)$$
and \( \Phi = (\phi, \psi) = (\phi, \phi_t) \). \( E' \) in (1.2) indicates the Fréchet derivative of the conserved quantity \( E \) defined by
\[
E(\phi, \psi) = \frac{1}{2} \int_0^L \left( \phi_x^2 + \psi^2 - \phi^2 + \frac{\phi^{2k+2}}{k+1} \right) dx. \tag{1.4}
\]

Moreover, (1.1) has another conserved quantity given by
\[
F(\phi, \psi) = \int_0^L \phi_x \psi dx. \tag{1.5}
\]

Equation (1.1) admits kink and periodic traveling wave solutions of the form \( \phi(x, t) = h(x - ct) \), where \( c \in (-1, 1) \) is the wave speed. In addition, according to the arguments in Pöschel [1], it is possible to obtain, at least for the case \( k = 1 \), the existence of quasi-periodic solutions for (1.1) with homogeneous Dirichlet boundary conditions (see also Berti et al. [2] for an additional reference concerning the existence of quasi-periodic solutions for a class of wave equations containing derivatives in the nonlinear terms).

One of our goals concerns the existence of periodic solutions. Thus, we can substitute the form \( \phi(x, t) = h(x - ct) \) into Equation (1.1) to obtain, for \( \omega = 1 - c^2 > 0 \), the following ODE
\[
-\omega h'' - h + h^{2k+1} = 0. \tag{1.6}
\]

Defining \( G(\phi, \psi) = E(\phi, \psi) - cF(\phi, \psi) \), one has by (1.6) that \( G'(h, ch') = 0 \). In other words, \( (h, ch') \) is a critical point of the Lyapunov functional \( G \).

For the case \( k = 1 \), we have the well known kink solution with hyperbolic tangent profile given by
\[
h(x) = \tanh \left( \frac{x}{\sqrt{2\omega}} \right). \tag{1.7}
\]

Henry et al. [3] determined the orbital stability of (1.7) with respect to small perturbations in the energy space \( X := H^1_{\text{per}}([0, L]) \times L^2_{\text{per}}([0, L]) \). Kowalczyk et al. [4] proved the asymptotic stability of the kink solution (1.7) for odd perturbations in the energy space. The authors based their proof on Virial-type estimates and it has been inspired in some results concerning the asymptotic stability of solitons for the generalized Korteweg–de Vries equations (see previous works [5, 6]).

In periodic context, we can find an explicit solution depending on the Jacobi elliptic function of snoidal type as
\[
h(x) = \sqrt{2\kappa} \frac{\sqrt{2\kappa}}{\sqrt{\kappa^2 + 1}} \text{sn} \left( \frac{4K(\kappa) x}{L}, \kappa \right), \tag{1.8}
\]

where \( \kappa \in (0, 1) \) is called modulus of the elliptic function and \( K(\kappa) \) is the complete elliptic integral of first kind. A result about the orbital stability of periodic waves has been determined in Palacios [7], where the author used the abstract theory in Grillakis et al. [8] adapted to the periodic context (see Definition 4.1 in Section 4 for the precise definition of orbital stability in the periodic context). However, the author performed the stability result by considering the Ginzburg–Landau energy given by
\[
\tilde{E}(\phi, \psi) = \frac{1}{2} \int_0^L \left( \phi_x^2 + \psi^2 + \left( 1 - \frac{\phi^2}{2} \right)^2 \right) dx. \tag{1.9}
\]

As far as we can see, the stability analysis in the periodic context is the same if one considers our energy functional given by (1.4) for the case \( k = 1 \).

Concerning the case \( k = 2 \), kink solutions with hyperbolic tangent profile is given by
\[
h(x) = \sqrt{\frac{2}{3}} \frac{\tanh \left( \frac{x}{\sqrt{\omega}} \right)}{\sqrt{1 - \frac{1}{2} \tanh^2 \left( \frac{x}{\sqrt{\omega}} \right)}}. \tag{1.10}
\]
Unfortunately, we do not know references concerning the orbital/asymptotic stability for this model, but we believe that arguments in Henry et al. [3] can be repeated in order to get the orbital stability in the energy space $X$.

Periodic waves with snoidal profile associated to the model (1.6) are given by

$$h(x) = \frac{\text{asn} \left( \frac{4K(x)\chi}{L}, \kappa \right)}{\sqrt{1 - b\text{sn}^2 \left( \frac{4K(x)\chi}{L}, \kappa \right)}}.$$  \hfill (1.11)

where $a$ and $b$ are complicated functions depending only on the modulus $\kappa \in (0, 1)$ (see the definition of $a$ and $b$ in (4.23) and (4.24), respectively). The value of $\omega$ depends on $\kappa$ and $L$ and it is expressed by

$$\omega = \frac{L^2}{16K(\kappa)^2 \sqrt{\kappa^4 - \kappa^2 + 1}}.$$ \hfill (1.12)

According to Grillakis et al. [8] and since $J$ in (1.3) is invertible with bounded inverse, the first requirement to get the orbital stability/instability of periodic waves for (1.1), is to construct a smooth curve $\omega \in I \subset (0, +\infty) \mapsto h_\omega \in H^2_{\text{per}}([0, L])$ with fixed period solving (1.6). The second claim is to prove that

$$G''(h, ch') \equiv \mathcal{L}_{\mathcal{E}G} = \begin{pmatrix} -\partial_x^2 - 1 + (2k + 1)h^{2k} & c\partial_x \\ -c\partial_x & 1 \end{pmatrix}$$ \hfill (1.13)

has only one negative eigenvalue, which is simple and zero is a simple eigenvalue associated to the eigenfunction $(h', ch'')$. To do so, it makes necessary to combine the min-max characterization of eigenvalues with the study of the first non-positive eigenvalues associated to the single linear operator given by

$$\mathcal{L} = -\omega\partial_x^2 - 1 + (2k + 1)h^{2k}.$$ \hfill (1.14)

Defining $d(c) = E(h, ch') - cF(h, ch') = G(h, ch')$, since $\omega = 1 - c^2$ and $\omega \mapsto h$ is a smooth curve of periodic waves with fixed period, we have from $G'(h, ch') = 0$ that $d'(c) = -cF(h, ch')$. In this setting, if $d''(c) < 0$, we have the orbital instability, and if $d''(c) > 0$, the orbital stability. Here, $d''(c)$ is given by

$$d''(c) = -\int_0^L (h'(x))^2 dx + 2(1 - \omega) \frac{d}{dc} \int_0^L (h'(x))^2 dx.$$ \hfill (1.15)

Palacios [7] uses all requirements above to establish the orbital instability of the periodic wave (1.8) in the energy space $X$. The crucial point to get the second requirement was the exact determination of the instability intervals associated to the Lamé equation

$$\begin{cases} \frac{d^2}{dx^2} \Psi + \left( \rho - 6\kappa^2 \text{sn}^2(x, \kappa) \right) \Psi = 0 \\ \Psi(0) = \Psi(4K(\kappa)), \quad \Psi'(0) = \Psi'(4K(\kappa)). \end{cases}$$ \hfill (1.16)

According to our best knowledge, for $k = 1$, the linearized operator associated to the solution in (1.11) cannot be reduced in a Lamé equation (1.16) and then, the arguments in Palacios [7] can not be used to prove the orbital stability/instability related to the $\phi^6$ model.

To overcome this difficulty, we use a general planar analysis of ODE to deduce the existence of periodic solutions for the general equation (1.1). In addition, let us define the period map $L = L(B)$ in terms of $B \in (0, B_w)$ as

$$L : (0, B_w) \mapsto \int_{\Gamma_B} \frac{dh}{\xi},$$ \hfill (1.17)

where $B$ is the energy level of $\mathcal{E}(h, \xi) = \frac{\xi^2}{2} + \frac{h^2}{2\omega} - \frac{h^{2k+1}}{(2k+2)\omega} = B$, $\xi = h'$, $B_w = \frac{k}{2\omega(k+1)}$, and $\Gamma_B$ stands the orbit in the phase portrait corresponding to the periodic solution $h$ of (1.6). Our analysis establishes that $L_B > 0$ and this fact can be used to determine the required spectral properties associated to the linear operator $\mathcal{L}$ in 1.14 for all $k \geq 1$ integer. This last fact can be determined employing an improvement of the Floquet theory [9–11]. We present our main results. Without
using explicit solutions, we establish for the case $c = 0$, the stability of periodic waves for the general equation for odd periodic waves in the odd sector $X_{\text{odd}} = H^1_{\text{odd}}([0, L]) \times L^2_{\text{odd}}([0, L])$. For the case $k = 1$, our paper improves the arguments in Palacios [7] since it has been used the explicit solution in (1.8) and the Lamé equation in (1.16) to obtain the stability in the odd sector. We use the monotonicity of the period map $L$ in (1.17) to obtain the following basic property:

$$\langle L_{KG}(u, v), (u, v) \rangle_X \geq \gamma \||u, v||^2_X,$$  \hfill (1.18)

where $\gamma > 0$. Inequality in (1.18) is the cornerstone to establish that $(h, 0)$ is minimum point of the energy $E$ in (1.4) for fixed values of the constraint $F$ in (1.5). Thus, the abstract theory in Grillakis et al. [8] can be applied to obtain the stability in $X_{\text{odd}}$. Summarizing, we have

**Theorem 1.1.** Let $h$ be an odd solution for the Equation (1.1). The periodic wave $(h, 0)$ is orbitally stable in $X_{\text{odd}}$ in the sense of Definition 4.1 (without considering the translation symmetry).

Concerning the $\phi^4$ and $\phi^6$ models with explicit solutions given by (1.8) and (1.11), respectively. We see that our planar analysis of ODE gives us that $L_B > 0$ for any $k \geq 1$. Thus, according to the literature [10, 11] and the min-max characterization of eigenvalues, we obtain that $L_{KG}$ has only one negative eigenvalue, which is simple and zero is a simple eigenvalue whose associated eigenfunction is $(h', ch'')$. The orbital instability is then proved by establishing that $d''(c) < 0$. For the $\phi^4$, we calculate $d''(c)$ using the expression in (1.15) and the explicit solution in (1.8). Regarding the $\phi^6$ model, we use the explicit solution in (1.11) combined with simplified formula for $d''(c)$ to avoid heavy expressions containing the complete elliptic integral of third kind (see Byrd and Friedman [12]).

We now present the result concerning the orbital instability of odd periodic waves for the $\phi^4$ model.

**Theorem 1.2.** Let $L_0 > 0$ be fixed. For $\omega \in \left(0, \frac{\delta}{4k+2}\right)$, consider the odd periodic solution $h$ given by (1.8). The periodic wave $(h, ch')$ is orbitally unstable in $X$ in the sense of Definition 4.1.

For the $\phi^6$ model we have:

**Theorem 1.3.** Let $L_0 > 0$ be fixed. For $\omega \in \left(0, \frac{\delta}{4k+2}\right)$, consider the odd periodic solution $h$ given by (1.11). The periodic wave $(h, ch')$ is orbitally unstable in $X$ in the sense of Definition 4.1.

Our paper is organized as follows. Section 2 is devoted to present the existence of periodic waves for the general equation (1.1). Spectral analysis for the linearized operator $L_{KG}$ is established in Section 3. The proof of Theorems 1.1,1.2 and 1.3 are presented in Section 4.

## 2 \ EXISTENCE OF ODD PERIODIC WAVES AND THE MONOTONICITY OF THE PERIOD MAP

Our purpose in this section is to present some facts concerning the existence of periodic solutions for the nonlinear ODE written as

$$-\omega h'' - h + h^{2k+1} = 0,$$  \hfill (2.1)

where $\omega > 0$ and $k \geq 1$ is an integer.

It is well known that Equation (2.1) is conservative, and thus, its solutions are contained on the level curves of the energy

$$E(h, \xi) := \frac{\xi^2}{2} + \frac{h^2}{2\omega} - \frac{h^{2k+2}}{(2k+2)\omega},$$  \hfill (2.2)

where $\xi = h'$.

We see that $h$ is a periodic solution of Equation (2.1) if, and only if, $(h, h')$ is a periodic orbit of the planar differential system

$$\begin{align*}
h' &= \xi, \\
\xi' &= -\frac{h}{\omega} + \frac{h^{2k+1}}{\omega}.
\end{align*}$$  \hfill (2.3)

The periodic orbits associated to Equation (2.3) lies inside of convenient energy levels of the energy $E$. This means that the pair $(h, h')$ satisfies the equation $E(h, h') = B$ for all $B \in (0, B_0)$, where $B_0 = \frac{k}{2\omega(k+1)}$. Moreover, the periodic orbits turn round at the center critical point of the system (2.3). In our case, we see that (2.3) has three critical points, being one center
point at \((h, h') = (0, 0)\), and two saddle points at \((h, h') = (\pm 1, 0)\). According to the standard ODE theory, the periodic orbits emanate from the center point to the separatrix curve, which is represented by a smooth solution \(\tilde{h} : \mathbb{R} \to \mathbb{R}\) of \((2.1)\) satisfying \(\lim_{x \to \pm \infty} \tilde{h}(x) = \pm 1\) and \(\tilde{h}'(x) > 0\) for all \(x \in \mathbb{R}\).

From the analysis above, the periodic orbits of the planar system \((2.3)\) correspond to odd periodic solutions \(h\) of \((2.1)\). The period \(L = L(\omega, B)\) of the solution \(h\) can be expressed (formally) by

\[
L = 2 \int_{b_1}^{b_2} \frac{dh}{\sqrt{\frac{h^{2k+2}}{\omega(k+1)} - \frac{h^2}{\omega} + 2B}},
\]

where \(b_1 = \min_{x \in [0, L]} h(x)\) and \(b_2 = \max_{x \in [0, L]} h(x)\).

On the other hand, the energy levels of the first integral \(\mathcal{E}\) in \((2.2)\) parametrize the set of periodic orbits \(\{\Gamma_B\}_{B \in (0, B_0)}\), which emanate from the center point to the separatrix curve. Thus, we can conclude that the set of smooth periodic solutions of \((2.1)\) can be expressed by a smooth family \(h = h_{\omega, B}\), which is parametrized by \(\omega > 0\) and \(B \in (0, B_0)\). Thus, we obtain that the orbit \(\Gamma_B\) is equal to the corresponding periodic solution \(h\). In addition, since the period of the orbit is given by the smooth map

\[
\tilde{L} = \int_{\Gamma_B} \frac{dh}{\xi},
\]

we see from \((2.4)\) and \((2.5)\) that \(L = \tilde{L}\).

Moreover, if \(B \to 0\) we have \(L \to a(\omega)\), and if \(B \to B_0\), one has \(L \to +\infty\), where \(a(\omega) > 0\) is the corresponding period of the equilibrium center point. Next result gives us the monotonicity of the period map \(L\) defined in \((2.5)\).

**Lemma 2.1.** For every \(\omega > 0\) and \(k \geq 1\) integer, the function

\[
L : (0, B_0) \to \mathbb{R}^+, \quad B \mapsto \int_{\Gamma_B} \frac{dh}{\psi}
\]

satisfies \(L_B > 0\).

**Proof.** Consider \(f(h) = h - h^{2k+1}\) and \(F\) satisfying \(F' = f\). According to section 4 of Bonorino et al. [13], it suffices to prove that

\[
I(h) = \frac{F'(h)^2 - 2F(h)F''(h)}{F'(h)^3}
\]

is a strictly increasing function over the interval \((-1, 1)\). Doing the computations, we obtain that

\[
I(h) = -kh^{2k-1} \frac{1 + 2k - h^{2k}}{(1 + k)(h^{2k} - 1)^3},
\]

that is, \(I\) is smooth in the interval \((-1, 1)\). In addition, an exhaustive calculation gives us

\[
I'(h) = -\frac{kh^{2k-2}}{(k + 1)(h^{2k} - 1)^4} \left[(-2k - 8k^2 - 2)h^{2k} + (2k + 1)h^{4k} + 1 - 4k^2\right]
\]

\[
\geq -\frac{kh^{2k-2}}{(k + 1)(h^{2k} - 1)^4}(-4k^2 - 2)h^{2k} > 0.
\]

\[\square\]

### 3 SPECTRAL PROPERTIES

#### 3.1 Floquet theory framework

We need to recall some basic facts concerning Floquet’s theory (see previous works [14, 15]). Let \(Q\) be a smooth \(L\)-periodic function. Consider \(P\) the Hill operator defined in \(L^2(0, L)\), with domain \(\mathcal{D}(P) = H^2_{per}(0, L)\), given by

\[
P = -\partial_x^2 + Q(x).
\]
The spectrum of $P$ is formed by an unbounded sequence of real eigenvalues
\[ \lambda_0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \lambda_4 \leq \cdots \leq \lambda_{2n-1} \leq \lambda_{2n} \cdots , \]
where equality means that $\lambda_{2n-1} = \lambda_{2n}$ is a double eigenvalue. Moreover, according to the oscillation theorem, the spectrum is characterized by the number of zeros of the eigenfunctions as follows: if $p$ is an eigenfunction associated to either $\lambda_{2n-1}$ or $\lambda_{2n}$, then $p$ has exactly $2n$ zeros in the half-open interval $[0, L)$. In particular, the even eigenfunction associated to the first eigenvalue $\lambda_0$ has no zeros in $[0, L]$.

Let $p(x)$ be a nontrivial $L$-periodic solution of the equation
\[ Pf \equiv -f'' + Q(x)f = 0. \tag{3.1} \]

Consider $y(x)$ another solution of (3.1) linearly independent of $p(x)$. There exists a constant $\theta$ (depending on $y$ and $p$) such that (see page 5 of Magnus and Winkler [15])
\[ y(x + L) = y(x) + \theta p(x). \tag{3.2} \]

Consequently, $\theta = 0$ is a necessary and sufficient condition to all solutions of (3.1) be $L$-periodic. This criterion is very useful to establish if the kernel of $P$ is one-dimensional.

Next, for a fixed $\omega > 0$ and $k \geq 1$ integer, let $h = h_{\omega, B}$ be any periodic solution of (2.1). Consider $\mathcal{L}_{\omega, B}$ the linearized operator arising from the linearization of (2.1) at $h = h_{\omega, B}$, that is,
\[ \mathcal{L}_{\omega, B}(y) := -\omega y'' - y + (2k + 1)h^{2k} y. \tag{3.3} \]

We see that $P$ and $\mathcal{L}_{\omega, B}$ can be related as $P = \frac{1}{\omega} \mathcal{L}_{\omega, B}$ and $Q(x) = -\frac{1}{\omega^2} + \frac{2k+1}{\omega} h^{2k}$. By taking the derivative with respect to $x$ in (2.1), we see that $h'$ belongs to the kernel of the operator $\mathcal{L}_{\omega, B}$. In addition, from our construction, $h'$ has exactly two zeros in the half-open interval $[0, L)$, which implies that zero is the second or the third eigenvalue of $\mathcal{L}_{\omega, B}$. The next result gives that it is possible to decide the exact position of the zero eigenvalue by knowing the precise sign of $\theta$ in (3.2).

**Lemma 3.1.** Assume that $\theta$ in (3.2) satisfies $\theta < 0$. The operator $\mathcal{L}_{\omega, B}$, defined in $L^2_{per}([0, L])$, with domain $H^2_{per}([0, L])$, has exactly one negative eigenvalue, a simple eigenvalue at zero and the rest of the spectrum is positive and bounded away from zero.

**Proof.** See theorem 3.1 in Neves [10].

Next, let $\tilde{y}$ be the unique solution of the initial-value problem
\[
\begin{cases}
-\omega \tilde{y}'' - \tilde{y} + (2k + 1)h^{2k} \tilde{y} = 0, \\
\tilde{y}(0) = 0, \\
\tilde{y}'(0) = \frac{1}{h'(0)},
\end{cases}
\tag{3.4}
\]

Since $h'$ is an $L$-periodic solution of the equation in (3.4) and the Wronskian of $\tilde{y}$ and $h'$ is 1, there is a constant $\theta = \theta_{\tilde{y}}$ such that
\[ \tilde{y}(x + L) = \tilde{y}(x) + \theta h'(x). \tag{3.5} \]

By taking the derivative in this last expression and evaluating at $x = 0$, we obtain
\[ \theta = \frac{\tilde{y}(L)}{h'(0)}. \tag{3.6} \]

Now we can give the equality between $L_B$ and $\theta$.

**Lemma 3.2.** We have, $L_B = -\theta$, where $\theta$ is the constant in (3.6).
Proof. Consider \( \check{y} \) and \( h' \) as above. Since \( h \) is odd and periodic one has \( h(0) = h(L) = 0 \). Thus, the smoothness of \( h \) in terms of the parameter \( B \) enables us to take the derivative of \( h(L) = 0 \) with respect to \( B \) to obtain

\[
h'(L)L_B + \frac{\partial h(L)}{\partial B} = 0. \tag{3.7}
\]

Next, we turn back to Equation (2.1) and multiply it by \( h' \) to deduce, after integration, the quadrature form

\[
\frac{\omega h^2(x)}{2} + \frac{h(x)^2}{2} - \frac{h(x)2k+2}{2k+2} = \omega B, \quad \text{for all} \ x \in [0, L]. \tag{3.8}
\]

Deriving Equation (3.8) with respect to \( B \) and taking \( x = 0 \) in the final result, we obtain from (2.1) that \( \frac{\partial h'(0)}{\partial B} = \frac{1}{h(0)} \).

In addition, since \( h \) is odd one has that \( \frac{\partial h}{\partial B} \) is also odd, and thus, \( \frac{\partial h(0)}{\partial B} = 0 \). The existence and uniqueness theorem for ordinary differential equations applied to the problem (3.4) enables us to deduce that \( \check{y} = \frac{\partial h}{\partial B} \). Therefore, we can combine (3.6) with (3.7) to obtain that \( L_B = -\theta \).

Next, under our assumptions, \( \theta \) does not change sign when \( \omega \) and \( B \) vary. Given any periodic solution \( h = h_{\omega,B} \), we denote by \( \theta_{\omega,B} = \theta_{1,0} \) its corresponding constant as in (3.5).

**Definition 3.3.** Let \( Q \) be a smooth \( L \)-periodic function. Let \( P \) be the Hill operator defined in \( L^2_{\text{per}}([0, L]) \) with domain \( D(P) = H^2_{\text{per}}([0, L]) \) given by

\[
P = -\partial_x^2 + Q(x).
\]

The inertial index of \( P \), denoted by \( \text{in}(P) \) is the pair \((n, z)\), where \( n \) denotes the dimension of the negative subspace of \( P \) and \( z \) denotes the dimension of \( \text{ker}(P) \).

**Definition 3.4.** Assume that \( Q(x) = Q_\omega(x) \) is periodic and depends on the parameter \( \omega > 0 \). The family of linear operators \( P_\omega := -\partial_x^2 + Q_\omega(x) \), is said to be isoinertial if \( \text{in}(P_\omega) \) is constant for any \( \omega > 0 \).

**Proposition 3.5.** Let \( h_{\omega,B} \) be the family of solution determined in previous section. Then the family of linear operators \( \mathcal{L}_{\omega,B} = -\omega^2 \partial_x^2 - 1 + (2k + 1)h^{2k} \), is isoinertial. In particular, if \( \theta_{\omega,B} < 0 \) for some \((\omega_0, B_0)\) with \( B_0 \in (0, B_{\omega_0}) \), then \( \theta_{\omega,B} < 0 \) for any \((\omega, B) \in (0, +\infty) \times (0, B_{\omega_0}) \).

**Proof.** See theorem 3.1 in Natali and Neves [9]. \( \Box \)

**Remark 3.6.** Proposition 3.5 establishes that in order to calculate the inertial index of \( \mathcal{L}_{\omega,B} \) it suffices to calculate it for any fixed pair \((\omega_0, B_0)\).

**Remark 3.7.** For a fixed \( \omega > 0 \), we have by Lemma 2.1 that \( \theta_{\omega,B} = -L_B < 0 \) for all \( B \in (0, B_{\omega_0}) \). Therefore, \( \text{in}(\mathcal{L}_{\omega,B}) = (1, 1) \), which means that zero is a simple eigenvalue corresponding to the eigenfunction \( h_{\omega,B}' \) and \( n(\mathcal{L}_{\omega,B}) = 1 \), where \( n(A) \) stands the number of negative eigenvalues of a certain linear operator \( A \). From Floquet theory, we see that the eigenfunction associated to the negative eigenvalue is simple, even and it can be assumed positive over \([0, L]\).

### 3.2 Spectral analysis

Let \( h = h_{\omega,B} \) be the periodic solution of (2.1) where \( \omega = 1 - c^2 > 0 \). In this subsection, we are going to analyze the quantity and multiplicity of the non-positive eigenvalues related to the matrix operator given by

\[
\mathcal{L}_{KG} = \begin{pmatrix}
-\partial_x^2 - 1 + (2k + 1)h^{2k} & c \partial_x \\
-c \partial_x & 1
\end{pmatrix} \tag{3.9}
\]

Operator \( \mathcal{L}_{KG} \) in (3.9), is obtained by considering the conserved quantities \( E \) and \( F \) defined in (1.4) and (1.5), respectively. By defining \( G = E - cF \), one has

\[
G'(h, ch') = E'(h, ch') - cF'(h, ch') = 0,
\]

that is, \((h, ch')\) is a critical point of \( G \). In addition, we have \( G''(h, ch') \) and \((h', ch'') \in \text{ker}(\mathcal{L}_{KG})\).
Let us consider the quadratic form associated with matrix operator (3.9),

\[ Q_{KG}(u, v) = \langle L_{KG}(u, v), (u, v) \rangle_X \]

\[ = \int_0^L owu'^2 - u^2 + (2k + 1)h^2u'^2 \, dx + ||cu' - v||_{L^2_{per}}^2 \] \tag{3.10}

where for \( \omega = 1 - c^2 > 0 \), we have that

\[ Q(u) := \int_0^L owu'^2 - u^2 + (2k + 1)h^2u'^2 \, dx, \] \tag{3.11}

represents the quadratic form of the operator \( L_{o,B} \) in (3.3).

We have the following results concerning the linearized operator \( L_{KG} \) defined in (3.9).

**Proposition 3.8.** Let \( \omega > 0 \) be fixed and consider \( k \geq 1 \) an integer. Let \( h = h_{o,B} \) be the periodic wave solution determined in Section 2. The operator \( L_{KG} \) defined in \( L^2_{per}(0, L) \times L^2_{per}(0, L) \), with domain \( H^2_{per}(0, L) \times H^1_{per}(0, L) \) has zero as the second eigenvalue, which is simple. Moreover, the remainder of the spectrum is a discrete set, which is bounded away from zero.

**Proof.** In fact, Remark (3.7) enables us to say that the operator \( L_{o,B} \) in (3.3) has exactly one negative eigenvalue, which is simple and zero is a simple eigenvalue with eigenfunction \( h' \). Let \( d \) be the unique negative eigenvalue of \( L \) with eigenfunction \( v \). Since \( Q(v) = d||v||^2_{L^2_{per}} < 0 \), we see from (3.10) that

\[ Q_{KG}(u, cv') = Q(v) + ||cv' - cu'||^2_{L^2_{per}}, \] \tag{3.12}

Moreover, the smallest eigenvalue \( \sigma_1 \) associated to \( L_{KG} \), is negative. We establish that the next eigenvalue of \( L_{KG} \) is \( \sigma_2 := 0 \) (which is simple) and also, the third eigenvalue \( \sigma_3 \), is strictly positive. To show these facts, we need to use the min-max characterization of eigenvalues. Indeed, in the energy space \( X \), we have

\[ \sigma_2 = \max_{(f_1, f_2) \in X} \min_{(u, v) \in \partial \Omega(0)} \frac{Q_{KG}(u, v)}{||u||_{H^1_{per}}^2 + ||v||_{L^2_{per}}^2}. \] \tag{3.13}

Then, by considering \( f_1 = v \) and \( f_2 = 0 \) we get,

\[ \sigma_2 \geq \min_{(u, v) \in \partial \Omega(0)} \frac{Q_{KG}(u, v)}{||u||_{H^1_{per}}^2 + ||v||_{L^2_{per}}^2} \geq 0 \] \tag{3.13}

and therefore, \( \sigma_2 = 0 \). The proof that \( \sigma_3 > 0 \) is obtained from the same arguments used above when we take the two-dimensional subspace spanned by \( (v, 0) \) and \( (h', 0) \) since in this case \( Q(u) \geq \sigma_3||u||_{L^2_{per}}^2 \), for \( u \perp v, u \perp h' \), where \( \sigma_3 \) is the third eigenvalue related to \( L \), which is obviously positive. Therefore, we conclude that \( L_{KG} \) has one negative eigenvalue, which is simple and zero is a simple eigenvalue with eigenfunction \( (h', ch') \) as desired. \( \square \)

**Corollary 3.9.** In the same framework of Proposition 3.8, the linearized operator \( L_{KG} \) at \( c = 0 \) defined in \( L^2_{per,odd}(0, L) \times L^2_{per,odd}(0, L) \), with domain \( H^2_{per,odd}(0, L) \times H^1_{per,odd}(0, L) \) has no negative eigenvalues and \( \ker(L_{KG}) = \{0, 0\} \). Moreover, the remainder of the spectrum is a discrete set, which is bounded away from zero.

**Proof.** Indeed, the linearized operator \( L_{o,B} \) at \( c = 0 \) in (3.3) does not have negative eigenvalues restricted to the odd sector. In addition, since \( h \) is odd and \( h' \) is even, one has that \( \ker(L_{o,B}) = \{0\} \). Therefore, we obtain the existence of \( \sigma > 0 \) such that \( Q(u) \geq \sigma ||u||_{L^2_{per}}^2 \) for all \( u \in H^1_{per,odd}(0, L) \). The result then follows by a direct application min-max characterization of eigenvalues and the definition of \( Q_{KG} \) given by (3.10). \( \square \)
4 | ORBITAL STABILITY AND INSTABILITY OF ODD PERIODIC WAVES

Since Equation (1.1) is invariant under translations, we define the orbit generated by \((h, ch')\) as

\[
\Omega_{(h, ch')} = \{(h(\cdot + r), ch'(\cdot + r); r \in \mathbb{R})\}.
\]  

(4.1)

In \(X\), we introduce the pseudo metric \(d\) by

\[
d((u_1, v_1), (u_2, v_2)) = \inf \{||u_1 - u_2||_X(\cdot + r); r \in \mathbb{R}\}.
\]

It is to be observed that, by definition, the distance between \((u_1, v_1)\) and \((u_2, v_2)\) is measured by the distance between \((u_1, v_1)\) and the orbit generated by \((u_2, v_2)\).

**Definition 4.1.** Let \((h, ch')\) be a traveling wave solution for (1.1). We say that \((h, ch')\) is orbitally stable in \(X\) (resp. \(X_{odd}\)) provided that, given \(\varepsilon > 0\), there exists \(\delta > 0\) with the following property: if \((\phi_0, \phi_1) \in X\) (resp. \(X_{odd}\)) satisfies \(||(\phi_0, \phi_1) - (h, ch')||_X < \delta\) and \((\phi, \psi)\) is solution of (1.1) in some local interval \([0, T_0]\) with initial condition \((\phi_0, \phi_1)\), then the solution can be continued to a solution in \(0 \leq t < +\infty\) and satisfies

\[
d((\phi(t), \psi(t)), \Omega_{(h, ch')}) < \varepsilon, \quad \text{for all } t \geq 0.
\]

Otherwise, we say that \((h, ch')\) is orbitally unstable in \(X\) (resp. \(X_{odd}\)).

Definition above establishes that the pair evolution \((\phi, \psi)\), which solves Equation (1.1) must exist in the energy spaces \(X\) (resp. \(X_{odd}\)) for all values of time \(t\) in a local interval of time \([0, T_0]\), where \(T_0 > 0\) is the maximal time where the solution exist. The existence of local solutions in the energy spaces \(X\) (resp. \(X_{odd}\)) for the Cauchy problem

\[
\begin{align*}
\phi_{tt} - \phi_{xx} - \phi + \phi^{2k+1} &= 0, \quad (x, t) \in \mathbb{R} \times \mathbb{R}_+, \\
\phi(x, 0) &= \phi_0(x), \quad \psi(x, 0) = \phi_1(x), \quad x \in \mathbb{R}, \\
\phi(x + L, t) &= \phi(x, t), \quad \forall t \geq 0, x \in \mathbb{R}.
\end{align*}
\]

(4.2)

can be established by the following result:

**Theorem 4.1.** The Cauchy problem (4.2) is locally well-posed for every initial data \((\phi_0, \phi_1) \in X\) (resp. \(X_{odd}\)). More precisely, there exists \(T_0 > 0\) and a unique solution \((\phi, \psi) \in C([0, T_0]; X)\) (resp. \(C([0, T_0]; X_{odd})\)) for the Cauchy problem (4.2) with \(\phi(x, 0) = \phi_0(x)\) and \(\psi(x, 0) = \phi_1(x)\). In addition, for all \(0 < T' < T_0\) the mapping \((\phi_0, \phi_1) \to (\phi, \psi)\) is continuous from \(X \mapsto C([0, T']; X)\) (resp. \(C([0, T']; X_{odd})\)).

**Proof.** The result is determined by using a direct application of the Kato theory for quasilinear equations in Kato [16]. For the better comprehension of the reader, we also refer them to section 6 of Pazy [17].

\[\square\]

4.1 | Orbital stability in the odd sector: Proof of Theorem 1.1

In this subsection, we are going to prove that the periodic wave \((h, ch')\) determined in Section 2 is stable in the energy space \(X_{odd}\) for \(c = 0\). The main tool is to use the approach contained in Grillakis et al. [8]. In fact, according to the Corollary 3.9, we see that

\[
Q(u) = (Lu, u)_{L^2_{per}} \geq \sigma ||u||^2_{L^2_{per}}.
\]

(4.3)

Let \(a, b > 0\) be arbitrary, since \(h\) is a bounded odd periodic wave, we have

\[
a||u_x||^2_{L^2_{per}} + b||u||^2_{L^2_{per}} \leq a||u_x||^2_{L^2_{per}} + \frac{b}{\sigma} (Lu, u)_{L^2_{per}} \\
\leq \left(\frac{a}{\omega} + \frac{a}{\omega \sigma} + \frac{b}{\sigma}\right) (Lu, u).
\]

(4.4)
Choosing \( a > b > 0 \), we see by (4.4) that

\[
Q(u) = (Lu, u)_{L^2} \geq \gamma \|u\|_{H^1}^2,
\]

(4.5)

for some \( \gamma = \gamma(\sigma, k, a, b, \omega) > 0 \). Now, consider \( c = 0 \). From the definition of \( Q_{KG} \) in (3.10), we have

\[
Q(u, v) = \langle L_{KG}(u, v), (u, v) \rangle_X \geq \gamma \|u\|_{H^1}^2 + \|v\|_{L^2}^2 \geq \tilde{\gamma} \|(u, v)\|_X^2,
\]

(4.6)

where \( \tilde{\gamma} = \min\{\gamma, 1\} > 0 \). Inequality (4.6) is the cornerstone to prove our orbital stability result in the restricted subspace \( X_{odd} \) by applying theorem 3.3 in Grillakis et al. [8]. This fact proves Theorem 1.1.

### 4.2 Instability for the \( \phi^4 \) equation: Proof of Theorem 1.2

Let \( L_0 > 0 \) be fixed. In this subsection, we use an explicit form of the wave \( h \) to prove the orbital instability result for the case \( k = 1 \). In fact, using the ansatz

\[
h(x) = \text{asn} \left( \frac{4K(\kappa)x}{L_0}, \kappa \right)
\]

(4.7)

into Equation (2.1), one has that \( h \) is an explicit periodic solution with

\[
a = \frac{\sqrt{2\kappa}}{\sqrt{\kappa^2 + 1}}.
\]

(4.8)

The value of \( \omega \) can be explicitly determined as

\[
\omega = \frac{L_0^2}{16K(\kappa)^2(1 + \kappa^2)}.
\]

(4.9)

Since \( \omega = 1 - c^2 \), we see that \( c > 0 \) is given by

\[
c = \sqrt{\frac{16(1 + \kappa^2)K(\kappa)^2 - L_0^2}{4K(\kappa)\sqrt{1 + \kappa^2}}}.
\]

(4.10)

Let \( L_0 > 0 \) be fixed such that \( 0 < \omega < \frac{L_0^2}{4\pi^2} \). There exists a smooth curve

\[
\omega \in \left( 0, \frac{L_0^2}{4\pi^2} \right) \mapsto h_\omega
\]

of periodic waves, which solves Equation (2.1). The existence of a smooth curve with fixed period is crucial in our analysis since the function \( d \) in terms of \( c \) as \( d(c) = E(h, c'h') - cF(h, c'h') \) is well defined. Next, since \( d'(c) = -F(h, c'h') = -c \int_0^{L_0} (h'(x))^2 dx \), we see that

\[
d''(c) = -\int_0^{L_0} (h'(x))^2 dx + 2c^2 \frac{d}{d\omega} \int_0^{L_0} (h'(x))^2 dx
\]

(4.11)

\[
= -\int_0^{L_0} (h'(x))^2 dx + 2(1 - \omega) \frac{d}{d\omega} \int_0^{L_0} (h'(x))^2 dx.
\]
On the other hand, using the explicit expression for (4.7), we obtain that

\[
d''(c) = -\int_0^{L_0} (h'(x))^2 \, dx + 2(1 - \omega) \frac{d}{dk} \int_0^{L_0} (h'(x))^2 \, dx \left( \frac{d\omega}{dk} \right)^{-1}
\]

\[
= -\frac{16a^2 K(\kappa)}{L_0} \int_0^{K} \text{cn}^2(x, \kappa) \text{dn}(x, \kappa) \, dx
\]

\[
+ \frac{32(1 - \omega)}{L_0} \frac{d}{dk} \left( a^2 K(\kappa) \int_0^{K} \text{cn}^2(x, \kappa) \text{dn}(x, \kappa) \, dx \right) \left( \frac{d\omega}{dk} \right)^{-1}.
\]

\[
(4.12)
\]

The values of \(a\) and \(\omega\) given by (4.8) and (4.9), respectively, can be used to get combined with formula 361.03 in Byrd and Friedman [12] that

\[
d''(c) = -\frac{16a^2 K(\kappa)}{3k^2 L_0} \left( (1 + k^2)E(\kappa) - (1 - k^2)K(\kappa) \right)
\]

\[
+ \frac{32}{3L_0}(1 - \omega) \frac{d}{dk} \left( \frac{a^2 K(\kappa)}{k^2} \left( (1 + k^2)E(\kappa) - (1 - k^2)K(\kappa) \right) \right) \left( \frac{d\omega}{dk} \right)^{-1}
\]

\[
- \frac{32}{3(1 + k^2)L_0} \left( (1 + k^2)E(\kappa) - (1 - k^2)K(\kappa) \right)
\]

\[
+ \frac{64}{3L_0}(1 - \omega) \frac{d}{dk} \left( \frac{K(\kappa)}{(1 + k^2)} \left( (1 + k^2)E(\kappa) - (1 - k^2)K(\kappa) \right) \right) \left( \frac{d}{dk} \left( \frac{1}{K(\kappa)^2(1 + k^2)} \right) \right)^{-1}.
\]

Since \(p(k) := (1 + k^2)E(\kappa) - (1 - k^2)K(\kappa) > 0\) for all \(k \in (0, 1)\) and \(1 - \omega > 0\), we only need to study the behavior of the last two terms containing derivatives with respect to \(k\). In fact, let us denote

\[
q(k) := \frac{d}{dk} \left( \frac{K(\kappa)}{(1 + k^2)} \left( (1 + k^2)E(\kappa) - (1 - k^2)K(\kappa) \right) \right) \left( \frac{d}{dk} \left( \frac{1}{K(\kappa)^2(1 + k^2)} \right) \right)^{-1}.
\]

Clearly, \(q\) is a negative function in terms of \(k \in (0, 1)\), and thus, \(d''(c) < 0\). To finish the proof, Propositions 3.5 and 3.8 give us the existence of only one negative for \(E_{KO}\) and it results to be simple. In addition, zero is a simple eigenvalue corresponding to the eigenfunction \((h', ch'')\). Since \(d''(c) < 0\), the result in Theorem 1.2 then follows by theorem 4.7 in Grillakis et al. [8].

### 4.3 Instability for the \(\phi^6\) equation: Proof of Theorem 1.3

This subsection is devoted in presenting the orbital instability associated to Equation (1.1) for the case \(k = 2\).

First, let us assume (2.1) with a general \(k \geq 1\) integer. Let \(\omega_0 > 0\) and \(B_0 \in (0, B_{\omega_0})\) be fixed and consider \(h_{\omega_0}\), the corresponding solution for Equation (2.1) with fixed period \(L_0 > 0\). Suppose the existence of a smooth curve \(\omega \in I \mapsto h_{\omega} \in B_{r}(h_{\omega_0})\) of periodic waves with fixed period solving (2.1). Here, \(I \subset (0, +\infty)\) is an open neighborhood around \(\omega_0 = 1 - c^2 > 0\) and \(B_{r}(h_{\omega_0}) \subset H^2_{p_{\omega_0}}([0, L_0])\) is an open neighborhood around \(h_{\omega_0}\). Our intention is to obtain a more convenient expression for \(d''(c)\) in (4.11) to simplify the management of complicated expressions involving elliptic functions, which appears in the case \(k = 2\). Comparing with the results obtained in the previous subsection for the case \(k = 1\), we have a considerable simplification of the calculations since we use few information about the solution \(h\) in comparison with the expression in (4.11) and (4.12).

In fact, let us define \(c \in (-1, 1)\) such that \(c^2 = 1 - \omega\), where \(\omega \in I\) has been determined in the last paragraph. Let \(d\) be the function defined as \(d(c) = E(h, ch') - cf(h, ch)\), where we again consider \(h = h_{\omega}\) to simplify the notation. Since \(d''(c) = -F(h, ch') = -c \int_0^{L_0} (h'(x))^2 \, dx\), we see similarly to (4.11) that

\[
d''(c) = -\int_0^{L_0} (h'(x))^2 \, dx + 2(1 - \omega) \frac{d}{d\omega} \int_0^{L_0} (h'(x))^2 \, dx.
\]

\[
(4.13)
\]

We give a convenient expression for \(D = \frac{d}{d\omega} \int_0^{L_0} (h'(x))^2 \, dx\). The reason for that is to avoid heavy calculations as determined for the case \(k = 1\), where we used the explicit form in (4.7) to calculate \(D\) is terms of the complete elliptic integrals.
of the first and second kinds. In fact, deriving Equation (2.1) with respect to \( \omega \in I \) and multiplying the resulting equation by \( h \), we obtain after integration over \([0, L_0]\) that

\[
-\omega \int_0^{L_0} \eta'' h dx + \int_0^{L_0} h''^2 dx - \int_0^{L_0} \eta h dx + (2k + 1) \int_0^{L_0} h^{2k+1} \eta dx = 0, \quad (4.14)
\]

where \( \eta = \frac{dh}{d\omega} \) and we omit the variable \( x \) in (4.14) to simplify the notation. An integration by parts applied to the first integral in (4.14) establishes using the ODE (2.1) that

\[
\int_0^{L_0} h''^2 dx + \frac{k}{k + 1} \frac{d}{d\omega} \int_0^{L_0} h^{2k+2} dx = 0. \quad (4.15)
\]

The same Equation (4.14) also gives

\[
\omega \frac{d}{d\omega} \int_0^{L_0} h''^2 dx + \frac{1}{2} \int_0^{L_0} h''^2 dx - \frac{1}{2k + 2} \int_0^{L_0} h^{2k+2} dx = 0. \quad (4.16)
\]

On the other hand, multiplying Equation (2.1) by \( h' \), we obtain after integration over \([0, L_0]\)

\[
-\frac{\omega}{2} \int_0^{L_0} h''^2 dx - \frac{1}{2} \int_0^{L_0} h''^2 dx + \frac{1}{2k + 2} \int_0^{L_0} h^{2k+2} dx + AL = 0. \quad (4.17)
\]

where \( A \) is constant of integration depending smoothly on \( \omega \). Now, deriving Equation (4.17) with respect to \( \omega \in I \) and combining the final result with (4.16), we obtain from (4.15) that

\[
\omega \frac{d}{d\omega} \int_0^{L_0} h''^2 dx - \frac{1}{2} \int_0^{L_0} h''^2 dx - L_0 \frac{dA}{d\omega} = 0. \quad (4.18)
\]

By (4.14) and (4.18), we deduce a simple expression for \( d''(c) \) as

\[
d''(c) = -\frac{1}{\omega} \int_0^{L_0} h''^2 dx + \frac{2(1 - \omega)L_0}{\omega} \frac{dA}{d\omega}. \quad (4.19)
\]

It is possible to obtain a convenient expression for \( \frac{dA}{d\omega} \). Indeed, multiplying Equation (2.1) by \( h' \), integrating over \([0, x]\), using the fact that \( h \) is odd, and deriving the final result with respect to \( \omega \), we have

\[
\frac{dA}{d\omega} = \omega h'(0)\eta'(0) + \frac{h'(0)^2}{2}. \quad (4.20)
\]

Gathering (4.19) and (4.20), we get the more convenient equality

\[
d''(c) = -\frac{1}{\omega} \int_0^{L_0} h''^2 dx + 2L_0(1 - \omega)h'(0)\eta'(0) + \frac{1 - \omega}{\omega} h'(0)^2 L_0. \quad (4.21)
\]

Let \( L_0 > 0 \) be fixed. The explicit solution for Equation (2.1) with \( k = 2 \) is given by

\[
h(x) = \frac{\text{asn} \left( \frac{4K(\kappa)}{L_0}, \kappa \right)}{\sqrt{1 - b \sin^2 \left( \frac{4K(\kappa)}{L_0}, \kappa \right)}}, \quad (4.22)
\]
where \( a, b, \) and \( \omega \) are complicated functions in terms of \( \kappa \) given by

\[
a = \frac{\sqrt{1458} \left( (-1 - \kappa^6 + \frac{3}{2} \kappa^4 + \frac{3}{2} \kappa^2) \sqrt{s(\kappa) + (s(k))^2} \right)}{s(k)}.
\]  
(4.23)

\[
b = \frac{1}{3} \kappa^2 + \frac{1}{3} - \frac{1}{3} \sqrt{\kappa^4 - \kappa^2 + 1},
\]  
(4.24)

and

\[
\omega = \frac{L_0^2}{16K(\kappa)^2 \sqrt{\kappa^4 - \kappa^2 + 1}},
\]  
(4.25)

where the notation \( s(\kappa) = \kappa^4 - \kappa^2 + 1 \) in (4.23) was employed to simplify the notation. Let \( L_0 > 0 \) be fixed. Equality in (4.25) enables us to deduce that \( \omega > 0 \) must be considered over the interval \( (0, \frac{L_0}{4\pi}) \).

By Propositions 3.5 and 3.8, we see that \( \mathcal{L}_{KG} \) has only one negative eigenvalue, which is simple, zero is a simple eigenvalue associated to the eigenfunction \((h', ch'')\) and the remainder of the spectrum consists in a discrete set, which is bounded away from zero. It remains to calculate \( d''(c) \) using the simplified formula in (4.21). In fact, we see from the chain rule that

\[
d''(c) = -\frac{1}{\omega} \int_0^{L_0} h'^2 \, dx + 2L_0(1 - \omega)h'(0) \frac{d\eta'(0)}{d\kappa} \left( \frac{d\omega}{d\kappa} \right)^{-1} + \frac{1 - \omega}{\omega} h'^2(0)L_0.
\]  
(4.26)

Consider \( \beta(\kappa) := 2L_0(1 - \omega)h'(0) \frac{d\eta'(0)}{d\kappa} \left( \frac{d\omega}{d\kappa} \right)^{-1} + \frac{1 - \omega}{\omega} h'^2(0)L_0 \). We are going to prove that \( \beta(\kappa) < 0 \) for all \( \kappa \in (0, 1) \). In fact, an exhaustive calculation gives us that

\[
\beta(\kappa) = \frac{72}{L_0^3} \left( \kappa + 1 \right) \kappa^2 \left( -16K(\kappa)^2 \sqrt{\kappa^4 - \kappa^2 + 1} + L_0 \right) \tau(\kappa),
\]  
(4.27)

where \( \tau(\kappa) \) is complicated smooth function depending only on \( \kappa \in (0, 1) \). Figure 1 shows that \( \tau(\kappa) > 0 \) for all \( \kappa \in (0, 1) \).
On the other hand, for a fixed $L_0 > 0$ such that $\omega \left( 0, \frac{L_0}{4\kappa^2} \right)$, one has
\[-16K(\kappa)^2 \sqrt{\kappa^4 - \kappa^2 + 1} + L_0 < 0, \quad \forall \kappa \in (0, 1).\]

Therefore, $d''(c) < 0$ and Theorem 1.3 is established.

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CONFLICT OF INTEREST STATEMENT
The authors declare that they have no conflict of interest.

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