On diffusion effects of the perturbed sine-Gordon equation with Neumann boundary conditions

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Abstract

The Neumann boundary problem for the perturbed sine-Gordon equation describing the electrodynamics of Josephson junctions has been considered. The behavior of a viscous term, described by a higher-order derivative with small diffusion coefficient $\varepsilon$, is investigated. The Green function related to the linear third order operator is determined by means of Fourier series, and properties of rapid convergence are established. Furthermore, some classes of solutions of the hyperbolic equation have been determined, proving that there exists at least one solution whose derivatives are bounded. Results prove that diffusion effects are bounded and tend to zero when $\varepsilon$ tends to zero.

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1 Introduction

Let us consider the sine-Gordon equation:

\begin{equation}
    u_{xx} - u_{tt} = \gamma + \sin u.
\end{equation}

This equation models the flux dynamics in the Josephson junction where two superconductors are separated by a thin insulating layer. Indeed, denoting by $\lambda_L$ the London penetration depth of the superconducting electrodes, the spatial coordinate $x$ is normalized to $\lambda_L$, while the time $t$ is normalized to the inverse plasma frequency $\omega_0 = \lambda_L/\tilde{c}$ (where $\tilde{c}$ is the maximum velocity of the electromagnetic waves.

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in the junction). So, function \( u(x,t) \) denotes the phase difference of the electrons between the top and the bottom superconductor, while constant \( \gamma = j/j_0 \) (with \( j_0 \) = maximum Josephson current) represents the normalized current bias.

In this case superconductors are ideal, i.e. there are no quasi-particle currents and all the electrons form Cooper pairs.

Conversely, when a real junction is considered [1], a term \( \alpha u_t \) can denote the dissipative normal electron current flow across the junction, while the flowing of quasi-particles parallel to the junction can be represented by a third term such as \( \varepsilon u_{xxt} \).

So that the following perturbed sine Gordon equation holds:

\[
(1.2) \quad u_{xx} - u_{tt} = \sin u + \gamma + \alpha u_t - \varepsilon u_{xxt}
\]

where the value range for \( \alpha \) and \( \varepsilon \) depends on the material of the real junction. Indeed, denoting by \( C, R, L_p \), respectively, the capacitance, the resistance and the inductance per unit length, it results \( \alpha = 1/\omega_0 RC \) and \( \beta = \omega_0 L_p/R \).

So, there are cases in which \( 0 \leq \alpha, \varepsilon \leq 1 \) [2,3] but, when the resistance of the junction is so low to completely shorten the capacitance, the case \( \alpha \) large with respect to 1 arises [4,5].

Equation (1.2) characterizes rectangular or annular junctions, but other geometries can be considered such as window Josephson junctions (WJJ) (6 and reference therein) or elliptic annular Josephson tunnel junctions (EAJTJs) [7], that reduce to circular annular junctions as soon as eccentricity vanishes. Moreover it is possible to consider also confocal annular Josephson tunnel junctions (CAJTJ) that are subtended by two ellipses with the same foci but do not have a constant annulus width [8]. Besides, if an exponentially shaped Josephson junction (ESJJ) [9]-[11] is examined, the equation achieved is the following:

\[
(1.3) \quad \varepsilon u_{xxt} + u_{xx} - u_{tt} - \varepsilon \lambda u_{xt} - \lambda u_x - \alpha u_t = \sin u - \gamma
\]

where \( \lambda \) is a positive constant and the current due to the tapering is represented by terms \( \lambda u_x \) and \( \varepsilon u_{xxt} \). In particular, \( \lambda u_x \) characterizes the geometrical force driving the fluxons from the wide edge to the narrow edge.

In others cases, such as a semiannular or an S-shaped Josephson junction, indicating by \( L \) the length of the junction and by \( b \) an applied magnetic field parallel to the plane of the dielectric barrier, the term \( b \cos(x\pi/\ell) \), with \( \ell = L/\lambda_L \), has to be considered, too. [2,12].

Moreover, if a harmonically oscillating magnetic field applied parallel to the dielectric barrier, and a dc bias across the superconducting electrodes are considered, one has [13]:

\[
(1.4) \quad u_{tt} - u_{xx} + \sin u = -\gamma - \alpha u_t + \varepsilon u_{xxt} - b \sin(\omega t) \cos(x\pi/\ell)
\]
where $\omega$ is the normalized frequency of the magnetic field normalized to the Josephson plasma frequency $\omega_0$.

There exist numerous applications for Josephson junctions. For example, by means of the superconducting quantum interference device (SQUID), it is possible using magnetocardiograms, to diagnose heart and/or blood circulation problems, while, through magnetoencephalography -MEG- magnetic fields generated by electric currents in the brain, can be evaluated [2]. In geophysics, on the other hand, they are used as gradiometers [4] or as gravitational wave detectors ( [14] and reference therein) and they play an important role in the study of the potential virtues of superconducting digital electronics, too [15]. SQUIDs are also used in nondestructive testing as a convenient alternative to ultrasound or x-ray methods ( [2, 4] [16]- [18] and reference therein). Finally, SQUIDs can be used as fast, switchable meta-atoms [19].

1.1 Mathematical considerations

In all the previous equations (1.2)-(1.4), the following linear operator appears:

\begin{equation}
\mathcal{L} u \equiv \partial_{xx} (\varepsilon u_t + u) - \partial_t (u_t + \alpha u).
\end{equation}

This is a third order parabolic operator that, as it is well known, is involved in a vast number of realistic mathematical models concerning superconductivity, neurobiology, and viscoelasticity [20]- [28] where the evolution is often characterized by deep interactions between wave propagation and diffusion. So that, $\mathcal{L}$ can be also considered as a linear hyperbolic operator perturbed by viscous terms described by higher-order derivatives with small diffusion coefficients $\varepsilon$.

When $\varepsilon \equiv 0$, the parabolic operator turns into a hyperbolic one:

\begin{equation}
\mathcal{L}_0 U \equiv U_{xx} - \partial_t (U_t + \alpha U),
\end{equation}

and the influence of the dissipative terms, represented by $\varepsilon \partial_{xxt}$, on the wave behaviour has been estimated.

Similar problems, for Dirichlet conditions, have already been studied in [29,30]. In particular, when $\alpha = 0$, an asymptotic approximation is established by means of the two characteristic times: slow time $\tau = \varepsilon t$ and fast time $\theta = t/\varepsilon$. Moreover, for equation (1.3) in [31], an analytical analysis has proved that the surface damping has little influence on the behaviour of the oscillator, thus confirming numerical results already determined in [32]. Numerical investigations on influence of surface losses can be found in [33], too.

Here the Neumann boundary conditions added with equation (1.5) are considered, and diffusion effects have been evaluated. In order to analyze the influence of the dissipative term on the wave behavior, a rigorous estimate of operator $\mathcal{L}$ has
been achieved by means of the Green function determined by Fourier series, and an evaluation of the following difference:

\[(1.7) \quad d(x, t, \varepsilon) = u(x, t, \varepsilon) - U(x, t)\]

has been done. Hence, the solution of the non-linear problem related to \(d\) is determined. Furthermore, some classes of solutions of the hyperbolic equation have been obtained, proving that there exists at least one solution whose derivatives are bounded. Finally, since the hyperbolic equation admits solutions with limited derivatives, an estimate for the remainder term is achieved by proving that the diffusion effects are of the order of \(\varepsilon^h\) with \(h < 1\) in each interval-time \([0, T_\varepsilon]\), with \(T_\varepsilon = \min\left\{\frac{1}{N} \log\left(\frac{1}{\varepsilon_1 - h}\right); \log\left(\frac{1}{\varepsilon_1 - h}\right)\right\}\) and \(N > 0\) independent from \(\varepsilon\).

The paper is organized as follows: Section II describes the mathematical problem, and attention is fixed on the Green function of the linear operator \(L\) defined in \((1.5)\). In section III properties of the Green Function \(G\) are pointed out in Theorem 1 whose proof can be found in appendix. Moreover, by means of the fixed point theorem, the solution of the problem related to the remainder term is showed. In section IV, some explicit solutions of the non-linear hyperbolic equation \((1.6)\) are determined. Finally, in section V an estimate for the remainder term is given in theorem 3.

2 Statement of the problem

Let \(T\) be a prefixed positive value constant and

\[\Omega = \{(x, t) : 0 \leq x \leq \ell, \quad 0 < t \leq T\}.\]

The Neumann boundary value problem for equation \((1.2)\) refers to the phase gradient value and is proportional to the magnetic field \([34, 35]\). So that one has:

\[
\begin{cases}
\begin{align*}
\partial_{xx}(\varepsilon u_t + u) - \partial_t (u_t + \alpha) &= \sin u + \gamma, & (x, t) \in \Omega, \\
u(x, 0) &= h_0(x), & x \in \Omega, \\
u_t(x, 0) &= h_1(x), & x \in \Omega, \\
u_x(0, t) &= \varphi_0(t), & 0 < t \leq T, \\
u_x(\ell, t) &= \varphi_1(t), & 0 < t \leq T.
\end{align*}
\end{cases}
\]

When \(\varepsilon \equiv 0\), problem \((2.1)\) turns into the Neumann problem related to parabolic operator \(L_0\), which has, of course, the same initial boundary conditions:

\[
\begin{cases}
\begin{align*}
U_{xx} - \partial_t (U_t + \alpha U) &= \sin U + \gamma, & (x, t) \in \Omega, \\
U(x, 0) &= h_0(x), & x \in \Omega, \\
U_t(x, 0) &= h_1(x), & x \in \Omega, \\
U_x(0, t) &= \varphi_0(t), & 0 < t \leq T, \\
U_x(\ell, t) &= \varphi_1(t), & 0 < t \leq T.
\end{align*}
\end{cases}
\]
The influence of the dissipative term on the wave behavior of $U$ can be estimated when the difference $d$, defined in (1.7), is evaluated.

So, let us consider the following problem related to the remainder term $d$:

$$
\begin{align*}
\partial_{xx}(\varepsilon \partial_t + 1)d - \partial_t(\partial + \alpha)d &= F(x, t, d), \quad (x, t) \in \Omega, \\
F(x, t, d) &= \sin(d + U) - \sin U - \varepsilon U_{xxt}.
\end{align*}
$$

(2.3)

The operator $L$ has already been examined by means of convolutions of Bessel functions and a short review can be found in [36], while for Dirichlet problem it has already been studied in [37, 38] and a recent approach can be found in [40]. Here, in order to deduce an exhaustive asymptotic analysis, it has been analyzed by means of the Green function determined by Fourier series. So, assuming

$$
\begin{align*}
\gamma_n &= \frac{n\pi}{\ell}, \\
h_n &= \frac{1}{2}(\alpha + \varepsilon \gamma_n^2), \\
\omega_n &= \sqrt{h_n^2 - \gamma_n^2}, \\
H_n(t) &= \frac{1}{\omega_n} e^{-h_n t} \sinh(\omega_n t),
\end{align*}
$$

(2.5)

by means of standard techniques, the Green function $G = G(x, t)$ of problem (2.3) is given by

$$
G(x, t, \xi) = \frac{1}{\ell} \frac{1 - e^{-\alpha t}}{\alpha} + \frac{2}{\ell} \sum_{n=1}^{\infty} H_n(t) \cos \gamma_n \xi \cos \gamma_n x.
$$

(2.6)

3 Properties of the Green function and solution related to the remainder term

In order to achieve the explicit solution of problem (2.3), attention should be paid to function $G$. Some properties of the Green function have already been determined in [37–39] proving among other things, that function $G$ is exponentially vanishing as $t \to \infty$. Moreover, since (2.6), it results:
(3.1) \[ G(x, t, \xi) \leq \frac{2}{\ell} \sum_{n=0}^{\infty} H_n(t) \cos \gamma_n \xi \cos \gamma_n x, \]

and denoting by 

\[ \beta \equiv \min \left\{ \frac{1}{\varepsilon + \alpha (\ell/\pi)^2}, \frac{\alpha + \varepsilon (\ell/\pi)^2}{2}, \alpha/2 \right\}, \]

(3.2)

the following theorem, whose proof can be found in appendix, holds:

**Theorem 1** The function \( G(x, \xi, t) \) defined in (2.6), and all its time derivatives are continuous functions and there exist some positive constants \( A_j \) \((j \in \mathbb{N})\) depending on \( \alpha, \varepsilon \) and positive constants \( M, N \) depending on \( \alpha \) and independent from \( \varepsilon \) such that:

(3.3) \[ |G(x, \xi, t)| \leq 1/2 \left( M \varepsilon^r + N \, e^{-\beta t} \right) \]

(3.4) \[ \left| \frac{\partial^j G}{\partial \theta^j} \right| \leq A_j e^{-\beta t}, \quad j \in \mathbb{N} \]

Furthermore, one has:

(3.5) \[ \mathcal{L} G = \partial_{xx}(\varepsilon G_t + G) - \partial_t(G_t + \alpha G) = 0. \]

Now, let us consider the nonlinear source (2.4) \( F(x, t, d) = \sin(d + U) - \sin U - \varepsilon U_{xx} \). By means of standard methods related to integral equations and thanks to the fixed point theorem, owing to basic properties of the Green function \( G \) and the source \( F \), it is possible to prove that problem (2.3)-(2.4) admits a unique regular solution in \( \Omega \) and it results: \([41]-[43]\)

(3.6) \[ d(x, t) = -\frac{1}{\ell} \int_0^t \int_0^\ell \left[ e^{-\alpha (t-\tau)} \right] F(\xi, \tau, d(\xi, \tau)) d\xi \]

\[ - \frac{2}{\ell} \int_0^t \int_0^\ell H(x, \xi, t-\tau) F(\xi, \tau, d(\xi, \tau)) d\xi. \]

where \( H = \sum_{n=1}^{\infty} H_n(t) \cos \gamma_n \xi \cos \gamma_n x. \)
4 Explicit solutions of the hyperbolic equation

Let us consider the semilinear second order equation:

\[ U_{xx} - U_{tt} - \alpha U_t = \sin U + \gamma \]  

When \( \alpha = \gamma = 0 \), (4.1) represents the sine - Gordon equation and there is plenty of literature about its classes of solution. [44]- [47]

Now, let \( f \) be an arbitrary function, and let us consider the following function \( \Pi(f) \):

\[ \Pi(f) = 2 \arctan e^f \]

so that

\[ \sin \Pi(f) = \frac{1}{\cosh(f)}, \quad \cos \Pi(f) = -\tanh(f). \]

By means of function (4.2) it is possible to find a class of solutions of equation (4.1).

Indeed, it is possible to verify that the following function:

\[ U = 2 \Pi[f(\xi)] \quad \text{with} \quad \xi = \frac{x - t}{\alpha} \]

is a solution of (4.1) provided that one has:

\[ -\alpha U_t = \sin U + \gamma. \]

Moreover, since (4.3) and being \( \Pi = \frac{1}{\cosh f} \), it results:

\[ -\alpha U_t = 2 \frac{f'}{\cosh f}; \quad \sin U = 2 \sin \Pi \cos \Pi = -2 \frac{\tanh f}{\cosh f}. \]

So, from (4.5), one deduces that function \( f \) must satisfy the following equation:

\[ \frac{df}{-\tanh f + \gamma/2 \cosh f} = d\xi \]

When \( 0 \leq \gamma \leq 1 \), we point our attention to those cases in which it results:
(4.7) \[ U = 4 \arctan(y + \sqrt{y^2 + 1}). \]

So, let \( h \) be an arbitrary constant of integration, one obtains:

(4.8) \[ y = h e^{-\xi} \quad \text{when} \quad \gamma = 0, \]

(4.9) \[ y = \frac{1 - (\xi - h)}{1 + \xi - h} \quad \text{when} \quad \gamma = 1. \]

Moreover, assuming \( \gamma^2 < 1 \), let

\[ A = \mp \sqrt{1 - \gamma^2} \quad \text{and} \quad \delta = -1 + A. \]

For \( \gamma^2 \neq \delta^2 \), it results:

(4.10) \[ y = \frac{h \gamma}{\delta} e^{\xi A} - \frac{\delta}{\gamma}. \]

When \( \gamma > 1 \), it is possible to prove that

(4.11) \[ U = 2 \arctan\left\{ \frac{1}{\gamma} \left[ \sqrt{\gamma^2 - 1} \tan\left( \frac{\sqrt{\gamma^2 - 1}}{2} \xi + h \right) - 1 \right] \right\} \]

Physical cases show that generally \( \gamma \) is less than 1, and in this case, indicating by \( \eta = \sqrt{1 - \gamma^2} \), it also results:

(4.12) \[ U(x, t) = 2 \arctan \left[ \frac{\eta}{\gamma} \left( \frac{1 + h e^{\xi}}{1 - h e^{\xi}} - \frac{1}{\eta} \right) \right] \]

So, indicating by

(4.13) \[ z = \frac{\eta}{\gamma} \left( \frac{1 + h e^{\xi}}{1 - h e^{\xi}} - \frac{1}{\eta} \right), \]

one has:

(4.14) \[ U_{xxt}(x, t) = \frac{2 z_{xxt}}{1 + z^2} + \frac{12 z z_x z_{xx} + 4 z_x^3}{(1 + z^2)^2} - \frac{16 z^2 z_x^3}{(1 + z^2)^3} \]

8
which is bounded for all \((x, t) \in \Omega_T\), being

\[
\begin{align*}
z_x &= \frac{\eta}{\gamma \alpha} \left[ \frac{he^\xi}{1-he^\xi} + \frac{h(1+he^\xi)}{(1-he^\xi)^2} \right]; \\
z_{xx} &= \frac{\eta}{\gamma \alpha^2} \left[ \frac{he^\xi}{1-he^\xi} + \frac{2\xi^2(1+he^\xi)}{(1-he^\xi)^2} + \frac{2h^2e^\xi(1-h^2e^{2\xi})}{(1-he^\xi)^4} \right] \\
z_{xxt} &= -\frac{\eta}{\gamma \alpha^3} \left[ \frac{he^\xi}{1-he^\xi} + \frac{3h^2e^{2\xi} + h^2e^\xi}{(1-he^\xi)^2} + \frac{2h^2e^\xi - 6h^4e^{3\xi}}{(1-he^\xi)^4} + \frac{8h^3e^{2\xi}(1+h^2e^{2\xi})}{(1-he^\xi)^5} \right].
\end{align*}
\]

5 Estimates for the remainder term

Let us assume \(\varepsilon = 0\) and let \(U\) be a solution of the reduced problem (2.2).

In the following we will have to refer to a known inequality of Gronwall type due to S.M. Sardar’ly (\cite{48} p 359):

**Theorem 2** Let \(x\) and \(a_2\) be continuous and \(a, a_1, \int_0^t b(t,s)ds\) Riemann integrable functions on \(J = [0, \beta]\), with \(a_1\) and \(a_2\) nonnegative on \(J\).

If

\[
(5.1) \quad x(t) \leq a(t) + \int_0^t b(t,s)ds + a_1(t) \int_0^t a_2(s)x(s)ds \quad t \in J
\]

then

\[
(5.2) \quad x(t) \leq a(t) + \int_0^t b(t,s)ds + a_1(t) \int_0^t a(s)a_2(s) \exp\left( \int_s^t a_1(z)a_2(z)dz \right) ds + a_1(t) \int_0^t a_2(s) \int_0^s b(t,z)dz \exp\left( \int_0^t a_1(z)a_2(z)dz \right) ds \quad t \in J.
\]

According to this, it is possible to state:

**Theorem 3** Let us assume

\[
(5.3) \quad S(t) = \sup_{0 \leq x \leq \ell} |d(x,t)|.
\]
If there exists a positive constant $k$ such that

$$
|U_{xxt}(x,t)| \leq k,
$$

then there exist two positive constants $\Gamma$ and $h$ with $h < \frac{q-1}{q} (1 < q < \infty)$ such that, indicated by $N$ the positive constant defined in (3.3) and

$$
T_\varepsilon := \min \left\{ \frac{1}{N} \log \left( \frac{1}{\varepsilon^{1-h}} \right); \log \left( \frac{1}{\varepsilon^{1-h}} \right) \right\},
$$

it results:

$$
0 \leq S(t) \leq \Gamma \varepsilon^h
$$

for every $t \leq T_\varepsilon$.

Proof: Let us consider function $F$ defined in (2.4):

$$
F = \sin (d + U) - \sin U - \varepsilon U_{xxt}.
$$

Since (3.1) and (3.6), it results:

$$
|d(x,t,\varepsilon)| \leq \frac{2}{\ell} \int_0^t d\tau \int_0^l |G(x,t-\tau,\xi)| |F(x,\tau,\xi)| d\xi
$$

where, function $F(x,t,u)$, according to (5.4) and (5.7), satisfies the following inequalities:

$$
|F(x,t)| \leq |d(x,t)| + \varepsilon k
$$

$$
|F(x,t)| \leq 2 + \varepsilon k.
$$

So that, by means of properties of the Green function $G$, and in particular since (3.3), one obtains:

$$
|d(x,t,\varepsilon)| \leq \frac{1}{\ell} \int_0^t d\tau \int_0^l M e^{\tau}(2 + \varepsilon k)d\xi + \frac{1}{\ell} \int_0^t d\tau \int_0^l N e^{-\beta(t-\tau)} [||d(\tau,\xi)|| + \varepsilon k] d\xi
$$
where $\beta$ and $r$ are defined in (A.1).

Hence, it follows:

$$ S(t) \leq (2 + \varepsilon k) M \varepsilon^r t + k \varepsilon N \int_0^t e^{-\beta (t-z)} \, dz + N e^{-\beta t} \int_0^t e^{\beta z} S(z) \, dz. $$  

(5.11)

Applying theorem 2 it results:

$$ S(t) \leq M \varepsilon^r (2 + \varepsilon k) t + N k \frac{1 - e^{-\beta t}}{\beta} \varepsilon + N e^{-\beta t} \int_0^t [N k \frac{1 - e^{-\beta s}}{\beta} + M \varepsilon^r (2 + \varepsilon k) s] e^{\beta s} e^{N(t-s)} \, ds. $$

(5.12)

So, one has:

$$ S(t) \leq \left[ M (2 + \varepsilon k) t + N \frac{1 - e^{-\beta s}}{\beta} \left( \frac{t}{\beta - N} + \frac{e^{-t(\beta-N)} - 1}{(\beta - N)^2} \right) \right] \varepsilon^r + $$

$$ + \left[ N k \frac{1 - e^{-\beta t}}{\beta} + \frac{N^2}{\beta} \left( \frac{1}{\beta - N} + \frac{e^{-t(\beta-N)}}{N - \beta} + \frac{e^{-t\beta}}{N} \right) \right] \varepsilon $$

(5.13)

Now, since for all $t \in \mathbb{R}$ one has $t + 1 \leq e^t$, it results

$$ t \varepsilon^r \leq \varepsilon^h; \quad \varepsilon^r e^{Nt} \leq \varepsilon^h $$

as soon as $h < r = \frac{q-1}{q} (q > 1)$ and $t \leq T_\varepsilon$.

**Remark 6.1** Estimate (5.6) specifies the infinite time-intervals where the effects of diffusion are of the order $\varepsilon^h$ with $0 < h < 1$. Indeed, the evolution of the superconductive model is characterized by diffusion effects which are of the order of $\varepsilon^h$ in each time-interval $[0, T_\varepsilon]$ with $T_\varepsilon$ defined in (5.5).

**Remark 6.2** Formula (4.14) shows that the class of functions satisfying hypotheses of Theorem 3 is not empty.

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A Theorem 1

Let us assume:

\[ \beta \equiv \min \left\{ \frac{1}{\varepsilon + \alpha (\ell / \pi)^2}, \frac{\alpha + \varepsilon (\ell / \pi)^2}{2}, \frac{\alpha}{2} \right\}, \]

(A.1)

\[ r = \frac{q - 1}{q} \quad (q > 1) \]

The function \( G(x, \xi, t) \) defined in (2.6) and all its time derivatives are continuous functions and there exist some positive constants \( A_j \) \((j \in \mathbb{N})\) depending on \( \alpha, \varepsilon \) and positive constants \( M, N \) depending on \( \alpha \) and independent from \( \varepsilon \) such that:

(A.2) \[ |G(x, \xi, t)| \leq \frac{1}{2} \left( M \varepsilon^r + N e^{-\beta t} \right) \]

(A.3) \[ \left| \frac{\partial^j G}{\partial \theta^j} \right| \leq A_j e^{-\beta t}, \quad j \in \mathbb{N} \]

Furthermore, one has:

(A.4) \[ \mathcal{L} G = \partial_{xx}(\varepsilon G_t + G) - \partial_t(G_t + \alpha G) = 0. \]

Proof:

Since \( \alpha \varepsilon < 1 \), indicating by \( N_1, N_2 \) the integer part of \( \ell / (\pi \varepsilon) (1 \mp \sqrt{1 - \alpha \varepsilon}) \), respectively, circular functions have to be distinguished from hyperbolic terms. In this case, since Taylor formula, it results \( \sqrt{1 - b_n^2 h_n^2} < 1 - \frac{b_n^2}{2 h_n^2} \), and for all \( n \geq 1 \) one has:

(5) \[ e^{-t(h_n - \omega_n)} \leq e^{-h_n t} e^{h_n \left( \frac{b_n^2}{2 h_n^2} \right)^t} \leq e^{\frac{1}{\alpha + \varepsilon (\ell / \pi)^2}} \]

Moreover, indicating by \( c \) an arbitrary constant less than 1, denoting by \( N_c \) the integer part of \( \ell / (\pi \varepsilon \sqrt{c})(1 + \sqrt{1 - \alpha \varepsilon c}) \), for all \( n \geq N_c \), it results \( h_n > b_n \) and \( \frac{b_n}{\sqrt{c}} < h_n \). So that one has:

(6) \[ \omega_n = h_n \sqrt{1 - \frac{b_n^2}{h_n^2}} \geq h_n \sqrt{1 - c} \]
and hence
\[
\sum_{n=N_\varepsilon}^{\infty} \frac{e^{-t(h_n-\omega_n)}}{\omega_n} \leq \frac{2\ell^2 \varepsilon}{\pi^2 \sqrt{1-\alpha \varepsilon}} \xi(2) e^{\frac{1}{1+\alpha \varepsilon/(2\pi)\varepsilon}}.
\] (7)

Besides, if \( \ell \geq 2\pi/a(1+\sqrt{1-\alpha \varepsilon}) \), terms \( \sum_{n=0}^{N_1-1} \frac{e^{-t(h_n-\omega_n)}}{\omega_n} \) have to be considered.

Since \( \sqrt{1-\alpha \varepsilon} = 1 - \alpha \varepsilon/2 - (\alpha \varepsilon)^2/8 (0 < \theta < 1) \) there exists a positive constant \( B \) such that:

\[
\sum_{n=0}^{N_1-1} \frac{e^{-t(h_n-\omega_n)}}{\omega_n} \leq B m \left( 1 + e^{\varepsilon/(1+\alpha \varepsilon)^2} \right) \varepsilon
\] (8)

being \( m \) the minimum value of \( \frac{\omega_n}{\varepsilon^2} \).

Otherwise, when \( N_1 < 1 \) attention must be paid to \( \sum_{n=0}^{N_2-1} \frac{e^{-h_n t \sinh(\omega_n t)}}{\omega_n} \). If \( 1 < q < \infty \) and \( 1/p + 1/q = 1 \) Holder inequality can be considered:

\[
\sum_{n=0}^{N_2-1} \frac{e^{-h_n t \sinh(\omega_n t)}}{\omega_n} \leq \left( \sum_{n=0}^{N_2-1} \left| e^{-h_n t \sinh(\omega_n t)} \right|^p \right)^{1/p} \left( \sum_{n=0}^{N_2-1} \left| \frac{1}{\omega_n} \right|^q \right)^{1/q}.
\] (9)

So, having
\[
\left| \frac{1}{\omega_n} \right|^q \leq \frac{\varepsilon^q}{\sqrt{1-\alpha \varepsilon}}
\]
it is possible to find a positive constant \( D \) such that the following inequality holds:

\[
\sum_{n=0}^{N_2-1} \frac{e^{-h_n t \sinh(\omega_n t)}}{\omega_n} \leq D \varepsilon^{\frac{2+1}{q}} \frac{1}{\sqrt{1-\alpha \varepsilon}}.
\] (10)

In this way (A.2) holds.

As for the \( x \)-differentiation of Fourier series like (A.4), attention must be given to convergence problems. Therefore, we consider \( x \)-derivatives of the operator \( (\varepsilon \partial_t + 1)G \) instead of \( G \) and \( G_t \). Following [37] theorem can be completely proven.
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