A NOTE ON DISCRETENESS OF $F$-JUMPING NUMBERS

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Abstract. Suppose that $R$ is a ring essentially of finite type over a perfect field of characteristic $p > 0$ and that $a \subseteq R$ is an ideal. We prove that the set of $F$-jumping numbers of $\tau_b(R; a^t)$ has no limit points under the assumption that $R$ is normal and $\mathbb{Q}$-Gorenstein – we make no assumption as to whether the $\mathbb{Q}$-Gorenstein index is divisible by $p$. Furthermore, we also show that the $F$-jumping numbers of $\tau_b(R; \Delta, a^t)$ are discrete under the more general assumption that $KR + \Delta$ is $\mathbb{R}$-Cartier.

1. Introduction

The test ideal is an important and subtle object associated to ideals $a$ in positive characteristic rings $R$. It measures the singularities of both the ambient ring and the elements of the ideal; see [HY03]. While the test ideal was initially introduced in the celebrated theory of tight closure of Hochster and Huneke (see [HH90]), more recent interest in the test ideal has been in regards to its connection with the multiplier ideal – a fundamental invariant of higher dimensional algebraic geometry in characteristic zero; see for example [Tak06] or [MY09].

Given a normal ring $R$ essentially of finite type over a perfect field of characteristic $p > 0$, an ideal $a \subseteq R$ and a real number $t \geq 0$, one can form the (big) test ideal $\tau_b(R; a^t)$ – an object which measures both algebraic and arithmetic properties of $R$ and $a$. Inspired by the test ideal’s close relation with the multiplier ideal $J(R, a^t)$, people have studied the numbers $t_i$ where $\tau_b(R; a^t_i)$ changes. That is, people have studied the $F$-jumping numbers (see [MTW05]), real numbers which are by definition the $t_i > 0$ such that for every $\varepsilon > 0$,

$$\tau_b(R; a^{t_i-\varepsilon}) \neq \tau_b(R; a^t_i).$$

One easy to observe fact about multiplier ideals is that their jumping numbers are discrete and rational, at least when $R$ is $\mathbb{Q}$-Gorenstein and normal; see [ELSV04]. Here, by discrete we mean that the set of jumping numbers with respect to a fixed ideal have no limit points. Because of this, various groups have recently worked to show that the $F$-jumping numbers of the test ideal are also discrete and rational; see [Har06], [BMS08], [BMS09], [KLZ09], and [BSTZ10]. In the most recent-mentioned work, the author, along with M. Blickle, S. Takagi, and W. Zhang, showed that

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the $F$-jumping numbers of test ideals formed a discrete set of rational numbers when $R$ is normal and $\mathbb{Q}$-Gorenstein with index not divisible by $p > 0$. Recall that the index of a $\mathbb{Q}$-Gorenstein ring $R$ is the smallest natural number $n$ where $\omega_R^{(n)} = \mathcal{O}_{\text{Spec } R}(nK_R)$ is locally free.

The most fundamental case left open is the case when $R$ is $\mathbb{Q}$-Gorenstein but of arbitrary index; see [BSTZ10 Question 6.1]. We answer this question at least for discreteness.

**Theorem 3.5** Suppose that $R$ is a normal domain essentially of finite type over an $F$-finite field. Further suppose that $a \subseteq R$ is an ideal and $\Delta$ is an $\mathbb{R}$-divisor on $X = \text{Spec } R$ such that $K_X + \Delta$ is $\mathbb{R}$-Cartier (for example, this holds if $\Delta = 0$ and $R$ is $\mathbb{Q}$-Gorenstein). Then, as $t$ varies, the $F$-jumping numbers of $\tau_b(R; \Delta, a^e)$ have no limit points – they are discrete.

We also point out why the existing proofs of rationality do not seem to work in the case that $R$ is $\mathbb{Q}$-Gorenstein with index divisible by $p$.

Recently, in [DH09], de Fernex and Hacon gave a definition of the multiplier ideal without the $\mathbb{Q}$-Gorenstein assumption and asked the question of whether discreteness and rationality of the $F$-jumping numbers still holds in this context. Following this, Urbinati showed that rationality need not hold but gave some evidence that discreteness may hold in general; see [Urb10]. This suggests that one should not expect rationality to hold in positive characteristic either.

2. Definition of the test ideal

We only give a very brief description of the big test ideal in this paper. Please see [BSTZ10] for a more detailed description of the test ideal.

First we fix some notation. Given a ring $R$ of characteristic $p > 0$ and $M$ an $R$-module, we set $F^e R$ to be the $R$-module which agrees with $M$ as an additive group but where the $R$-module structure is defined by the rule $r.m = r^e m$. Also recall that $R$ is said to be $F$-finite if $F^e R$ is a finitely generated $R$-module.

**Convention.** Throughout this paper, all rings will be assumed to be $F$-finite.

Recall that an $\mathbb{R}$-divisor on a normal scheme $X$ is a formal linear combination of prime Weil divisors $D_i$ with real coefficients. An $\mathbb{R}$-divisor $D$ is called $\mathbb{R}$-Cartier if it is equal to an $\mathbb{R}$-linear combination of Cartier divisors.

We now define the test ideal $\tau_b(R; \Delta, a^e b_1^{s_1} \cdots b_m^{s_m})$. We work in this greater generality because when proving our main theorem, we perturb our initial triple $(R, \Delta, a^e)$ to a new triple $(R, \Delta', a^e b_1^{s_1} \cdots b_m^{s_m})$ which has the same test ideal.

**Definition 2.1** ([HH90], [Hoc07], [Sch10]). Suppose that $R$ is an $F$-finite normal domain, $\Delta \geq 0$ is an $\mathbb{R}$-divisor on $X = \text{Spec } R$, $a, b_1, \ldots, b_m \subseteq R$ are non-zero ideals and $t, s_1, \ldots, s_m \geq 0$ are real numbers. Then the big test ideal $\tau_b(R; \Delta, a^e b_1^{s_1} \cdots b_m^{s_m})$ is defined to be the unique smallest non-zero ideal $J \subseteq R$ such that

\[
\phi \left( F^e a^{\lceil t(p^{s_1} - 1) \rceil} b_1^{\lceil s_1 (p^{s_1} - 1) \rceil} \cdots b_m^{\lceil s_m (p^{s_m} - 1) \rceil} J \right) \subseteq J
\]

for every $e \geq 0$ and every $\phi \in \text{Hom}_R(F^e R, (\lfloor (p^{s_1} - 1) \Delta \rfloor), R)$. This ideal always exists in the context described.

**Remark 2.2.** In the case that $K_X + \Delta$ is $\mathbb{Q}$-Cartier, the big test ideal is known to equal the (finististic) test ideal (which we will not define here); see [Tak04] and [BSTZ10] for details.
If all $b_i = R$, then we denote the associated big test ideal by $\tau_b(R; \Delta, a^t)$. Likewise if $\Delta = 0$, then we denote the associated big test ideal using the notation $\tau_b(R; a^t b_1^{s_1} \ldots b_m^{s_m})$. Finally, if the $b_i = (f_i)$ are principal, we denote the associated big test ideal by $\tau_b(R; \Delta, a^t f_1^{s_1} \ldots f_m^{s_m})$

**Remark 2.3.** Given a non-zero element $c \in \tau_b(R; \Delta, a^t b_1^{s_1} \ldots b_m^{s_m})$ (such an element is called a big sharp test element), we note that

$$\tau_b(R; \Delta, a^t b_1^{s_1} \ldots b_m^{s_m}) = \sum_{c \geq 0} \phi \left( F^e_x(c \{ p^{e-1} \} b_1^{\lceil s_1(p^{e-1}) \rceil} \ldots b_m^{\lceil s_m(p^{e-1}) \rceil}) \right),$$

where the inner sum is over $\phi \in \text{Hom}_R(F^e_x R([p^{e-1} \{ p^{e-1} \}]), R)$. To see this, simply note that the right side satisfies equation (1), and it is by definition the smallest ideal containing $c$ satisfying equation (1). Designate $a^0 = b^0 = R$.

Suppose that $X = \text{Spec } R$ is normal. Then given $\phi \in \text{Hom}_R(F^e_x R, R) \cong F^e_x \mathcal{O}_X((1-p^e)K_X)$, we may view $\phi$ as determining an effective Weil divisor linearly equivalent to $(1-p^e)K_X$.

**Definition 2.4.** We use $D_\phi$ to denote the Weil divisor associated to $\phi$ in this way.

Given an $\mathbb{R}$-divisor $\Delta \geq 0$ on $X$, one has an inclusion

$$\text{Hom}_R(F^e_x R([p^{e-1} \Delta]), R) \subseteq \text{Hom}_R(F^e_x R, R).$$

The following lemma gives a nice interpretation of this submodule.

**Lemma 2.5.** An element $\phi \in \text{Hom}_R(F^e_x R, R)$ is contained inside the submodule $\text{Hom}_R(F^e_x R([p^{e-1} \Delta]), R)$ if and only if $D_\phi \geq (p^{e-1})\Delta$.

**Proof.** Because all the modules are reflexive, the statement can be reduced to the case where $R$ is a discrete valuation ring and $\Delta = s \text{div}(x)$, where $x$ is the parameter for the DVR $R$ and $s \geq 0$ is a real number. In this case, the inclusion from equation (3) can be identified with the multiplication map $R \to R$ which sends 1 to $x^{\lfloor s(p^{e-1}) \rfloor}$. Thus, $\phi \in \text{Hom}_R(F^e_x R, R)$ is contained inside $\text{Hom}_R(F^e_x R([p^{e-1} \Delta]), R) \cong x^{\lfloor s(p^{e-1}) \rfloor} R$ if and only if $D_\phi \geq [s(p^{e-1}) \text{div}(x) = [p^{e-1} \Delta]$. However, since $D_\phi$ is integral, it is harmless to remove the round-up $\lceil \cdot \rceil$.

### 3. Discreteness of $F$-jumping numbers

In this section we prove our main result. We accomplish this by perturbing our triples $(R, \Delta, a^t)$ in order to reduce the discreteness statement to the case where the (log) $\mathbb{Q}$-Gorenstein index is not divisible by $p > 0$. First we need a lemma.

**Lemma 3.1.** Suppose that $(X = \text{Spec } R, \Delta, a^t b_1^{s_1} \ldots b_m^{s_m})$ is a triple and that $\Delta = \Gamma + b \text{div}(f)$ for some $f \in R \setminus \{0\}$ and non-negative number $b \in \mathbb{R}$. Then

$$\tau_b(R; \Delta, a^t b_1^{s_1} \ldots b_m^{s_m}) = \tau_b(R; \Gamma, f^b a^t b_1^{s_1} \ldots b_m^{s_m}).$$

This type of statement is essentially obvious for multiplier ideals, but because of certain issues surrounding the construction of test ideals we have thus-far presented, it is somewhat less obvious in this context. However, it is still quite straightforward, especially from the definition of the generalized test ideal by Hara-Yoshida-Takagi (the proof in that case uses the theory tight closure); see [HY03] and [Tak04]. We provide a short proof here, certainly acknowledging that this statement is obvious to experts.
Thus, since $t \geq 0$, 

\[ \phi \left( \sum_{\phi, D_\phi \geq (p^s-1)\Gamma} F_\phi X f [t(p^s-1)] b_1^{[x_1]} \ldots b_m^{[x_m]} [p^s-1] \right) \]

\[ = \sum_{\phi, D_\phi \geq (p^s-1)\Gamma + \text{div } f[b(p^s-1)]} \phi \left( F_\phi X f [t(p^s-1)] b_1^{[x_1]} \ldots b_m^{[x_m]} [p^s-1] \right) \]

\[ \subseteq \sum_{\phi, D_\phi \geq (p^s-1)\Gamma + [b(p^s-1) \text{div}(f)]} \phi \left( F_\phi X f [t(p^s-1)] b_1^{[x_1]} \ldots b_m^{[x_m]} [p^s-1] \right) \]

\[ \subseteq \sum_{\phi, D_\phi \geq (p^s-1)\Gamma + [b(p^s-1) \text{div}(f)]} \phi \left( F_\phi X f [t(p^s-1)] b_1^{[x_1]} \ldots b_m^{[x_m]} [p^s-1] \right) \]

\[ = \tau_0 (R; \Delta, a^s b_1^{s_1} \ldots b_m^{s_m}), \]

and so $\tau_0 (R; \Gamma, f^s a^s b_1^{s_1} \ldots b_m^{s_m}) \subseteq \tau_0 (R; \Delta, a^s b_1^{s_1} \ldots b_m^{s_m})$. For the converse inclusion, observe first that 

\[ \text{div}(f[b(p^s-1)]) - b(p^s-1) \text{div}(f) \leq \text{div}(f). \]

Thus, since $cf$ is also a test element, 

\[ \tau_0 (R; \Delta, a^s b_1^{s_1} \ldots b_m^{s_m}) = \sum_{\phi, D_\phi \geq (p^s-1)\Delta} \phi \left( F_\phi X f [t(p^s-1)] b_1^{[x_1]} \ldots b_m^{[x_m]} [p^s-1] \right) \]

\[ = \sum_{\phi, D_\phi \geq (p^s-1)\Gamma + \text{div } f[b(p^s-1)]} \phi \left( F_\phi X f [t(p^s-1)] b_1^{[x_1]} \ldots b_m^{[x_m]} [p^s-1] \right) \]

\[ \subseteq \sum_{\phi, D_\phi \geq (p^s-1)\Gamma + \text{div } f[b(p^s-1)]} \phi \left( F_\phi X f [t(p^s-1)] b_1^{[x_1]} \ldots b_m^{[x_m]} [p^s-1] \right) \]

\[ = \sum_{\phi, D_\phi \geq (p^s-1)\Gamma} \phi \left( F_\phi X f [b(p^s-1)] a^s b_1^{[x_1]} \ldots b_m^{[x_m]} [p^s-1] \right) \]

\[ = \tau_0 (R; \Gamma, f^s a^s b_1^{s_1} \ldots b_m^{s_m}), \]

and so $\tau_0 (R; \Delta, a^s b_1^{s_1} \ldots b_m^{s_m}) \subseteq \tau_0 (R; \Gamma, f^s a^s b_1^{s_1} \ldots b_m^{s_m})$, as desired. \[ \square \]

We also need a very special case of Skoda’s theorem.

**Lemma 3.2** ([HT04 Theorem 4.1]). Suppose that $X = \text{Spec } R$, $\Delta > 0$, $a \subseteq R$ and $t \geq 0$ is as above. Further suppose that $f \in R$ is a non-zero element. Then 

\[ \tau_0 (X; \Delta + \text{div } f, a^s b_1^{s_1} \ldots b_m^{s_m}) = f \tau_0 (X; \Delta, a^s b_1^{s_1} \ldots b_m^{s_m}). \]

**Proof.** We leave the proof to reader; see [HT04 Theorem 4.2] and [BSTZ10 Lemma 3.26]. \[ \square \]

Now we can prove the following result.

**Theorem 3.3.** Suppose that $R$ is an $F$-finite normal domain and further suppose that $(X, \Delta, a^s)$ is a triple where $K_X + \Delta$ is $R$-Cartier. Then for each point $x \in X$, there exists an open set $U = \text{Spec } R' = \text{Spec } R[h^{-1}]$ containing $x \in X$
with the following properties: There exists an effective \( \mathbb{Q} \)-divisor \( \Gamma \) on \( U \), elements \( f_1, \ldots, f_m \in R' \setminus \{0\} \) and non-negative real numbers \( b_1, \ldots, b_m \) such that

1. \( K_U + \Gamma \) is \( \mathbb{Q} \)-Cartier with index not divisible by \( p > 0 \), and, furthermore, 
   \( (p^e - 1)(K_U + \Gamma) \sim 0 \) for some integer \( e > 0 \).
2. The \( F \)-jumping numbers of \( \tau_b(U, \Delta_{|U}, (aR')^t) \) are the same as the \( F \)-jumping numbers of \( \tilde{\tau}_b(U, (aR')^t) \) (both sets of jumping numbers are with respect to \( t \)).

Proof. Choose a non-zero section \( \phi \) of \( \text{Hom}_R(F^*_R R, R) \) and set \( \Gamma := \frac{1}{p^e - 1} D_\phi \). It follows that \( K_X + \Gamma \) satisfies condition (1) on \( X \). Therefore, \( (K_X + \Delta) - (K_X + \Gamma) = \Delta - \Gamma \) is \( \mathbb{R} \)-Cartier, and so we may write \( \Delta - \Gamma = d_1 D_1 + \cdots + d_m D_m \) for some integral effective Cartier divisors \( D_i \) and real numbers \( d_i \in \mathbb{R} \). We choose our open set \( U = \text{Spec} R[h^{-1}] = \text{Spec} R' \) to be any such set containing \( x \in X \) where all of the \( D_i|_U \) are principal divisors.

Now write \( D_i|_U = \text{div}(f_i) \) for some \( f_i \in R' \setminus 0 \) and also by abuse of notation denote \( \Gamma := \Gamma|_U \). Choose natural numbers \( l_i \) such that \( b_i := l_i + d_i > 0 \) for all \( i \) and set \( g := f_1^{l_1} \cdots f_m^{l_m} \in R' \). Notice that \( (\Delta|_U + \text{div}(g)) - \Gamma = b_1 \text{div}(f_1) + \cdots + b_m \text{div}(f_m) \).

By Lemma 3.2 the \( F \)-jumping numbers of \( \tau_b(U, \Delta_{|U}, (aR')^t) \) and the \( F \)-jumping numbers of \( \tau_b(U, \Delta_{|U} + \text{div}(g), (aR')^t) \) coincide. Now using Lemma 3.1 we have

\[
\tau_b(U, \Delta_{|U} + \text{div}(g), (aR')^t) = \tau_b(U, \Gamma + b_1 \text{div}(f_1) + \cdots + b_m \text{div}(f_m), (aR')^t) = \tau_b(U, \Gamma, f_1^{b_1} \cdots f_m^{b_m} (aR')^t),
\]

which proves the theorem. \( \square \)

Remark 3.4. If, in Theorem 3.3 \( K_X + \Delta \) is \( \mathbb{Q} \)-Cartier, then one needs only a single \( f_i^{b_i} \) (and no other \( f_i^{b_i} \)). However, if the index of \( K_X + \Delta \) is divisible by \( p > 0 \), then it follows by construction that \( b_1 \) will be a rational number with its denominator divisible by \( p > 0 \).

We are now in a position to prove the discreteness of the \( F \)-jumping numbers in the case that \( X \) is essentially of finite type over a field. The proof idea follows the usual lines.

**Theorem 3.5.** Suppose that \( R \) is a normal domain essentially of finite type over an \( F \)-finite field. Further suppose that \( a \subseteq R \) is an ideal and \( \Delta \) is an \( \mathbb{R} \)-divisor on \( X = \text{Spec} R \) such that \( K_X + \Delta \) is \( \mathbb{R} \)-Cartier (for example, this holds if \( \Delta = 0 \) and \( R \) is \( \mathbb{Q} \)-Gorenstein). Then, as \( t \) varies, the \( F \)-jumping numbers of \( \tau_b(R; \Delta, a^t) \) have no limit points – they are discrete.

Proof. By [BSTZ10] Proposition 3.28, it is sufficient to answer this question on a finite affine cover of \( X \). Therefore, we reduce to the case that \( X \) is one of the charts from Theorem 3.3. In particular, it is sufficient to prove our result for triples of the form \( \tau_b(R; \Gamma, f_1^{b_1} \cdots f_m^{b_m} a^t) \), where \( (p^e - 1)(K_X + \Gamma) \sim 0 \) for some \( e > 0 \). Using [BSTZ10] Lemma 4.2, Proposition 3.28, one can further assume that \( R \) is of finite type over an \( F \)-finite field of characteristic \( p > 0 \). One then has two options:

(a) Mimic the proof of the main result of [BSTZ10] Section 4. In other words, use the methods of \( F \)-adjunction (as worked out in [Sch09a] and [BSTZ10]) to reduce to the case where \( R \) is a polynomial ring and then use...
degree bounding methods similar to those found in [BMS08]. Note that in [BSTZ10] one worked with triples \((R, \Delta, a^t)\) and not with the more complicated objects \((R, \Gamma, f_1^{b_1} \ldots f_m^{b_m} a^t)\), but the methods are easily generalized to our setting.

(b) Use the new language of [Bli09, Section 4]. We claim that the algebra of \(p^{-e}\)-linear maps associated to the triple \((R, \Gamma, f_1^{b_1} \ldots f_m^{b_m})\), as in [Sch09b, Remark 3.10], is “gauge bounded” (see [Bli09, Definition 4.7]). To see this claim, note that by [Sch09a, Lemma 3.9] or [Sch09b, Remark 4.4] the Cartier-algebra associated to \((R, \Gamma)\) is finitely generated and thus gauge bounded by [Bli09, Proposition 4.8]. It then follows from [Bli09, Proposition 4.13] that the Cartier-algebra associated to \((R, \Gamma, f_1^{b_1} \ldots f_m^{b_m})\) is also gauge bounded, as claimed. To finish the proof, apply [Bli09, Theorem 4.14]. In either case, the result follows easily from the theories previously developed. □

4. On the question of rationality

Note that the usual way to prove the rationality of the \(F\)-jumping numbers employs the following theorem. First recall that a pair \((X, \Delta)\) is called \(\log \mathbb{Q}\)-Gorenstein with index \(n\) if \(n(K_X + \Delta)\) is Cartier and \(n > 0\) is the smallest integer with this property.

**Theorem 4.1** ([BMS08, BSTZ10]). Suppose \((X, \Delta)\) is \(\log \mathbb{Q}\)-Gorenstein with index \(n\) such that \(n\) divides \((p^e - 1)\) for some fixed \(e > 0\). Further suppose that \(a\) is an ideal sheaf of \(X\). Then if \(t_0\) is an \(F\)-jumping number of \(\tau(X; \Delta, a^t)\), then \(p^e t_0\) is also an \(F\)-jumping number.

However, without the “index not divisible by \(p\)” assumption, this theorem is false. Consider the following example (which in some sense is typical by Remark 3.4).

**Example 4.2.** Set \(X = \mathbb{A}^1_k = \text{Spec } k[x]\), \(\Delta = \frac{1}{p} \text{div}(x)\) and \(a = (x)\). Then the \(F\)-jumping numbers of \((X, \Delta, a^t) = (X, (x)^{1/p} a^t)\) with respect to \(t\) are 
\[
p - 1, \quad \frac{2p - 1}{p}, \quad \frac{3p - 1}{p}, \ldots
\]
In particular, \(p\) (or \(p^e\)) times any of them is not an \(F\)-jumping number.

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