Symplectic Sums and Gromov-Witten Invariants

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Abstract

Gromov-Witten invariants of a symplectic manifold are a count of holomorphic curves. We describe a formula expressing the GW invariants of a symplectic sum $X \# Y$ in terms of the relative GW invariants of $X$ and $Y$. This formula has several applications to enumerative geometry. As one application, we obtain new relations in the cohomology ring of the moduli space of complex structures on a genus $g$ Riemann surface with $n$ marked points.

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1. Gromov-Witten invariants

A symplectic structure on a closed smooth manifold $X^{2N}$ consists of a closed, non-degenerate 2-form $\omega$. Gromov’s idea [8] was that one could obtain information about the symplectic structure on $X$ by studying holomorphic curves. For that one needs to introduce an almost complex structure, which is an endomorphism $J \in \text{End}(TX)$ with $J^2 = -\text{Id}$. Such a $J$ is compatible with $\omega$ if the bilinear form $g(v, w) = \omega(v, Jw)$ defines a Riemannian metric on $TX$. For a fixed symplectic structure, the space of compatible almost complex structures is a nonempty, contractible space.

One then considers the moduli space of $J$-holomorphic maps from Riemann surfaces into $X$. Constraints are imposed on the maps, requiring the domain to have a certain form and the image to pass through geometric representatives of fixed homology classes in $X$. When the right number of constraints are chosen there will be finitely many maps satisfying those constraints; the (oriented) count of these maps will give the corresponding Gromov-Witten invariant. In general, there are several technical difficulties one must overcome to get a well-defined Gromov-Witten invariant. The foundations of this theory began with [8], [24], [25] and have been developed since then by the efforts of a large group of mathematicians (see, *Department of Mathematics, University of Wisconsin-Madison, Madison, WI 53706, USA. E-mail: ionel@math.wisc.edu*)
for example, the references in [15] and [22]). Here we present a brief overview of the technical setup.

Consider \((X, \omega)\) a symplectic manifold. For each compatible almost complex structure \(J\) and perturbation \(\nu\) one considers maps \(f : C \rightarrow X\) from a genus \(g\) Riemann surface \(C\) with \(n\) marked points which satisfy the pseudo-holomorphic map equation \(\overline{\partial}f = \nu\) and represent a fixed homology class \(A = [f] \in H_2(X)\). The set of such maps (modulo reparametrizations), together with their limits, forms the compact space of stable maps \(\overline{\mathcal{M}}_{g,n}(X, A)\). For each stable map \(f : C \rightarrow X\), the domain determines a point in the Deligne-Mumford moduli space \(\mathcal{M}_{g,n}\) of genus \(g\) Riemann surfaces with \(n\) marked points (see also §3). The evaluation at each marked point determines a point in \(X\). All together, this gives a natural map

\[
\overline{\mathcal{M}}_{g,n}(X, A) \rightarrow \mathcal{M}_{g,n} \times X^n.
\]

For generic \((J, \nu)\) the image of this map carries a fundamental homology class \([GW_{X,A,g,n}]\) which is defined to be the Gromov-Witten invariant of \((X, \omega)\). The dimension of this homology class, given by an index computation, is

\[
\dim \overline{\mathcal{M}}_{g,n}(X, A) = 2c_1(TX)A + (\dim X - 6)(1 - g) + 2n.
\]

A cobordism argument shows that the homology class \([GW_{X,A,g,n}]\) is independent of generic \((J, \nu)\) and moreover depends only on the isotopy class of the symplectic form \(\omega\). Frequently, the Gromov-Witten invariant is thought of as a collection of numbers obtained by evaluating the homology class \([GW_{X,A,g,n}]\) on a basis of the dual cohomology group. For complex algebraic manifolds these symplectic invariants can also be defined by algebraic geometry, and in important cases the invariants are the same as the counts of curves that are the subject of classical enumerative algebraic geometry.

The next important question is to find effective ways of computing the GW invariants. One useful technique is the method of ‘splitting the domain’. Anytime we have a relation in the cohomology of \(\overline{\mathcal{M}}_{g,n}\) it pulls back to a relation (sometimes trivial) between the GW invariants of a symplectic manifold \(X\). As an example, suppose that the constraints imposed on the domain of the holomorphic curves are boundary classes in \(H^*(\overline{\mathcal{M}}_{g,n})\) (as defined in section 3 below). One then obtains recursive relations which relate such GW invariant to invariants of lower degree or genus. This method was first used by Kontsevich and Ruan-Tian [25] to determine recursively the genus 0 invariants of the projective spaces \(\mathbb{P}^n\). These recursive relations follow from the observation that in the Deligne-Mumford space \(\overline{\mathcal{M}}_{0,4} \cong \mathbb{P}^1\) each boundary class corresponds to a point, and are thus all homologous to each other.

In joint work with Thomas H. Parker, the author established a general formula describing the behavior of GW invariants under the operation of ‘splitting the target’ ([14], [15], [16]). Because we work in the context of symplectic manifolds the natural splitting of the target is the one associated with the symplectic cut operation and its inverse, the symplectic sum. The next section describes the symplectic sum operation and the main ingredients entering the sum formula for GW invariants.
2. Symplectic sums

The operation of symplectic sum is defined by gluing along codimension two submanifolds (see [7], [21]). Specifically, let $X$ be a symplectic manifold with a codimension two symplectic submanifold $V$. Given a similar pair $(Y, V)$ with a symplectic identification between the two copies of $V$ and a complex anti-linear isomorphism between the normal bundles $N_X V$ and $N_Y V$ of $V$ in $X$ and in $Y$ we can form the symplectic sum $X \#_V Y$.

Perhaps it is in more natural to describe the symplectic sum not as a single manifold but as a family $Z \to D$ over the disk depending on a parameter $\lambda \in D$. For $\lambda \neq 0$ the fibers $Z_{\lambda}$ are smooth and symplectically isotopic to $X \#_V Y$ while the central fiber $Z_0$ is the singular manifold $X \cup_Y Y$. In a neighborhood of $V$ the total space $Z$ is $N_X V \oplus N_Y V$ and the fiber $Z_{\lambda}$ is defined by the equation $xy = \lambda$ where $x$ and $y$ are coordinates in the normal bundles $N_X V$ and $N_Y V \cong (N_X V)^*$. The fibration $Z \to D$ extends away from $V$ as the disjoint union of $X \times D$ and $Y \times D$.

Our overall strategy for proving the symplectic sum formula for GW invariants [16] is to relate the pseudo-holomorphic maps into $Z_{\lambda}$ for $\lambda$ small to pseudo-holomorphic maps into $Z_0$. One expects the stable maps into the sum to be pairs of stable maps into the two sides which match in the middle. A sum formula thus requires a count of stable maps in $X$ that keeps track of how the curves intersect $V$.

So the first step is to construct Gromov-Witten invariants for a symplectic manifold $(X, \omega)$ relative to a codimension two symplectic submanifold $V$. These invariants were introduced in a separate paper with Thomas H. Parker [15] and were designed for use in symplectic sum formulas. Of course, before speaking of stable maps one must extend the almost complex structure $J$ and the perturbation $\nu$ to the symplectic sum. To ensure that there is such an extension we require that the pair $(J, \nu)$ be $V$-compatible. The precise definition is given in section §6 of [15], but in particular for such pairs $V$ is a $J$-holomorphic submanifold — something which is not true for generic $J$. The relative invariant gives counts of stable maps for these special $V$-compatible pairs. Such counts are in general different from those associated with the absolute GW invariants described in the first section of this note.

Restricting to $V$-compatible pairs has repercussions. Any pseudo-holomorphic map $f : C \to V$ into $V$ then automatically satisfies the pseudo-holomorphic map equation into $X$. So for $V$-compatible $(J, \nu)$, stable maps may have domain components whose image lies entirely in $V$, so they are far from being transverse to $V$. Worse, the moduli spaces of such maps can have dimension larger than the dimension of $M_{g, n}(X, A)$. We circumvent these difficulties by restricting attention to the stable maps which have no components mapped entirely into $V$. Such ‘$V$-regular’ maps intersect $V$ in a finite set of points with multiplicity. After numbering these points, the space of $V$-regular maps separates into components labeled by vectors $s = (s_1, \ldots, s_\ell)$, where $\ell$ is the number of intersection points and $s_k$ is the multiplicity of the $k^{th}$ intersection point. Each (irreducible) component $M_{g, n, s}^V(X, A)$ of $V$-regular stable maps is an orbifold; its dimension depends of $g, n, A$ and on the vector of multiplicities $s$. 
Next key step is to show that the space of $V$-regular maps carries a fundamental homology class. For this we construct an orbifold compactification $\overline{\mathcal{M}}_{g,n,s}(X, A)$, the space of $V$-stable maps. The relative invariants are then defined in exactly the same way as the GW invariants. We consider the natural map

$$\overline{\mathcal{M}}_{g,n,s}^{V}(X, A) \to \overline{\mathcal{M}}_{g,n+\ell} \times X^n \times V^\ell.$$  

(2.1)

The new feature is the last factor (the evaluation at the $\ell$ points of contact with $V$) which allows us to constrain how the images of the maps intersect $V$. Thus the relative invariants give counts of $V$-stable maps with constraints on the complex structure of the domain, the images of the marked points, and the geometry of the intersection with $V$. There is one more complication: to be useful for a symplectic sum formula, the relative invariant should record the homology class of the curve in $X \setminus V$ rather than in $X$. This requires keeping track of some additional homology data which is intertwined with the intersection data, as explained in [15].

We now return to the discussion of the symplectic sum formula. As previously mentioned, the overall strategy is to relate the pseudo-holomorphic maps into $Z_0$, which are simply maps into $X$ and $Y$ which match along $V$, with pseudo-holomorphic maps into $Z_\lambda$ for $\lambda$ close to zero. For that we consider sequences of stable maps into the family $Z_\lambda$ of symplectic sums as the ‘neck size’ $\lambda \to 0$. These limit to maps into the singular manifold $Z_0 = X \cup_X Y$. A more careful look reveals several features of the limit maps.

First of all, if the limit map $f_0 : C_0 \to Z_0$ has no components in $V$ then $f_0$ has matching intersection with $V$ on $X$ and $Y$ side. For such a limit map $f_0$ all its intersection points with $V$ are nodes of the domain $C_0$. Ordering this nodes we obtain a sequence of multiplicities $s = (s_1, \ldots, s_\ell)$ along $V$. But it turns out that the squeezing process is not injective in general. For a fixed $\lambda \neq 0$ there are $|s| = s_1 \cdot \ldots \cdot s_\ell$ many stable maps into $Z_\lambda$ close to $f_0$.

Second, connected curves in $Z_\lambda$ can limit to curves whose restrictions to $X$ and $Y$ are not connected. For that reason the GW invariant, which counts stable curves from a connected domain, is not the appropriate invariant for expressing a sum formula. Instead one should work with the ‘Gromov-Taubes’ invariant $GT$, which counts stable maps from domains that need not be connected. Thus we seek a formula of the general form

$$GT_{X \#_V Y} = GT_X^{V} * GT_Y^{V}$$

(2.2)

where $*$ is the operation that adds up the ways curves on the $X$ and $Y$ sides match and are identified with curves in $Z_\lambda$. That necessarily involves keeping track of the multiplicities $s$ and the homology classes. It also involves accounting for the limit maps which have components in $V$; such maps are not counted by the relative invariant and hence do not contribute to the left side of (2.2).

Finally, we need to consider limit maps which have components mapped entirely in $V$. We deal with that possibility by squeezing the neck not in one region, but several regions. As a result, the formula (2.2) in general has an extra term
called the $S$-matrix which keeps track of how the genus, homology class, and intersection points with $V$ change as the images of stable maps pass through the neck region. One sees these quantities changing abruptly as the map passes through the neck — the maps are “scattered” by the neck. The scattering occurs when some of the stable maps contributing to the GT invariant of $Z_\lambda$ have components that lie entirely in $V$ in the limit as $\lambda \to 0$. Those maps are not $V$-regular, so are not counted in the relative invariants of $X$ or $Y$. But this complication can be analyzed and related to the relative invariants of the ruled manifold $\mathbb{P}(N_X V \oplus \mathbb{C})$.

Putting all these ingredients together, we can at last state the main result of [16].

**Theorem 2.1** Let $Z$ be the symplectic sum of $(X,V)$ and $(Y,V)$ and fix a decomposition of the constraints $\alpha$ into $\alpha_X$ on the $X$ side and $\alpha_Y$ on the $Y$ side. Then the GT invariant of $Z$ is given in terms of the relative invariants of $(X,V)$ and $(Y,V)$ by

$$GT_Z(\alpha) = GT^X_V(\alpha_X) * S_V * GT^Y_V(\alpha_Y)$$

where $*$ is the convolution operation and $S_V$ is the $S$-matrix defined in [16].

Several applications of this formula are described in the next two sections (see also [16] for more applications). But the full strength of the symplectic sum theorem has not yet been used.

A.-M. Li and Y. Ruan also have a sum formula [18]. Eliashberg, Givental, and Hofer are developing a general theory for invariants of symplectic manifolds glued along contact boundaries [3]. Jun Li has recently adapted our proof to the algebraic case [19].

### 3. Relations in $H^\ast(M_{g,n})$

A smooth genus $g$ curve with $n$ marked points is stable if $2g - 2 + n > 0$. The set of such curves, modulo diffeomorphisms, forms the moduli space $M_{g,n}$. The stability condition assures that the group of diffeomorphisms acts with finite stabilizers, and so $M_{g,n}$ has a natural orbifold structure. Its Deligne-Mumford compactification $\overline{M}_{g,n}$ is a projective variety. Elements of $\overline{M}_{g,n}$ are called stable curves; these are connected unions of smooth stable components $C_i$ joined at $d$ double points with a total of $n$ marked points and Euler characteristic $\chi = 2 - 2g + d$. The compactification $\overline{M}_{g,n}$ is also an orbifold, and in fact Looijenga proved that it has a finite degree cover which is a smooth manifold. In any event, the rational cohomology of $\overline{M}_{g,n}$ satisfies Poincaré duality. Throughout this section we work only with rational coefficients.

There are several maps between moduli spaces of stable curves. First, there is a projection $\pi_i : \overline{M}_{g,n+1} \to \overline{M}_{g,n}$ that forgets the marked point $x_i$ (and collapses the components that become unstable). Second, we can consider the attaching maps that build a boundary stratum in $\overline{M}_{g,n}$. For each topological type of a stable curve with $d$ nodes, with components $C_i$ of genus $g_i$ and $n_i$ marked points the attaching map $\xi$ at the $d$ nodes takes $\sqcup_i \overline{M}_{g_i,n_i}$ onto a boundary stratum of $\overline{M}_{g,n}$.
We focus next on three kinds of natural classes in $H^*(\overline{M}_{g,n})$ (or the Chow ring). For each $i$ between 1 and $n$ let $L_i \to \overline{M}_{g,n}$ denote the relative cotangent bundle to the stable curve at the marked point $x_i$. The fiber of $L_i$ over a point $C = (\Sigma, x_1, \ldots, x_n) \in \overline{M}_{g,n}$ is the cotangent space to $\Sigma$ at $x_i$, and its first Chern class $\psi_i$ is called a descendant class. So there are $n$ descendant classes $\psi_1, \ldots, \psi_n$, one for each marked point. Next, there are tautological (or Mumford-Morita-Miller) classes $\kappa_0, \kappa_1, \ldots$ obtained from powers of descendants by the formula $\kappa_a = (\pi_{n+1})_*(\psi_a^{n+1})$ for each $a \geq 0$ (where $\pi_*$ denotes the push forward map in cohomology defined using the Poincaré duality). Finally, the Poincaré dual of a boundary stratum is called a boundary class. These three kinds of natural classes are all algebraic and even dimensional; we define their degree to be their complex dimension.

One natural — and difficult — problem is to describe the structure of the cohomology rings of $\overline{M}_{g,n}$ and $\overline{M}_{0,n}$. This arises from a different perspective as well since $H^*(\overline{M}_{g,n})$ is also the cohomology of the mapping class group (for more details, see Tillman’s I.C.M. talk). In genus zero Keel [17] determined the cohomology ring of $\overline{M}_{0,n}$ in terms of generators (which are boundary classes) and relations. For higher genus far less is known about the cohomology ring.

In this section we will instead focus on finding relations in the cohomology ring. For example, in genus 0 all relations come from the “4-point relation”, essentially that in the cohomology of $\overline{M}_{0,4} \cong \mathbb{P}^1$ the four $\psi_i$ classes as well as the three boundary classes are all cohomologous (all being Poincaré dual to a point). In genus 1 it is also known that $\psi_1$ is equal to $1/12$ of the boundary class in $\overline{M}_{1,1}$. One might wonder whether in higher genus all the $\psi$ classes come from the boundary. That turns out not to be true in genus $g \geq 2$, but in genus 2 Mumford [23] found a relation in $\overline{M}_{2,1}$ expressing $\psi_1^2$ as a combination of boundary classes. Several years ago, Getzler [6] found a similar relation for $\psi_1 \psi_2$ in $\overline{M}_{2,2}$ and he conjectured that this pattern would continue in higher genus. In fact,

**Theorem 3.1** When $g \geq 1$, any product of descendant or tautological classes of degree at least $g$ (or at least $g-1$ when $n=0$) vanishes when restricted to $H^*(\overline{M}_{g,n}, \mathbb{Q})$.

This result was proved by the author in [11]. It extends an earlier result of Looijenga [20], who proved that a product of descendant classes of degree at least $g+n-1$ vanishes in the Chow ring $A^*(C_g^n)$ of the moduli space $C_g^n$ of smooth genus $g$ curves with $n$ not necessarily distinct points.

The idea of proof of Theorem 3.1 is simple. We start with the moduli space $\overline{Y}_{d,g,n}$ of degree $d$ holomorphic maps from smooth genus $g$ curves with $n$ marked points to $S^2$ which have a fixed ramification pattern over $r$ marked points in the target. We then consider its relative stable map compactification $\overline{Y}_{d,g,n}$ (closely related to the space of admissible covers [9]). The space $\overline{Y}_{d,g,n}$ has an orbispace structure and it comes with two natural maps $st$ and $q$ that record respectively the domain and the target of the cover.

\[
\begin{array}{c}
\overline{M}_{g,n} \xleftarrow{st} \overline{Y}_{d,g,n} \\
\overline{M}_{0,r} \xrightarrow{q}
\end{array}
\]
A simple way to get relations in the cohomology of $\overline{M}_{g,n}$ is to pull back by $q$ known relations in the cohomology of $\overline{M}_{0,r}$, and then push them forward by $st$.

To begin with, note that the diagram above provides several other natural classes in $\overline{M}_{g,n}$: for each choice of ramification pattern, $st_*Y_{d,g,n}$ defines a cycle in $\overline{M}_{g,n}$. The most useful ones turn out to be the “2-point ramification cycles”, for which all but at most two of the branch points are simple. Pushing forward such cycles by the attaching map of a boundary stratum gives a generalized 2-point cycle.

To prove Theorem 3.1, we choose a degree $d$ of the cover and a 2-point ramification cycle $Y_{d,g,n}$ in such a way that the stabilization map $st: Y_{d,g,n} \to \overline{M}_{g,n}$ has finite, nonzero degree. The key step is the following proposition.

**Proposition 3.2** The Poincaré dual of any degree $m$ product of descendant and tautological classes can be written as a linear combination of generalized 2-point ramification cycles of codimension $m$.

But the codimension of a 2-point ramification cycle is at most $g$. A simple degeneration argument proves that the cycles of codimension exactly $g$ vanish on $\overline{M}_{g,n}$, thus implying Theorem 3.1.

There are three main ingredients in the proof of Proposition 3.2. First, the relative cotangent bundle to the domain is related to the pullback of the relative cotangent bundle to the target, so we can express the descendant classes in the domain via descendant classes in the target. Second, the target has genus zero and (nontrivial) products of descendants in $\overline{M}_{0,r}$ are Poincaré dual to boundary cycles $D$. This means that we can relate a product of descendants on the domain to cycles of type $st_*q^*D$. Finally, a degeneration formula, which is essentially a consequence of the symplectic sum Theorem 2.1, expresses cycles of type $st_*q^*D$ in terms of 2-point ramification cycles.

The degree $g$ in Theorem 3.1 is the lowest degree in which some monomial in descendants would vanish on $\overline{M}_{g,n}$ (see the discussion in [10]). However, there are lower degree polynomial relations in descendent and tautological classes. For example, if we restrict our attention to the moduli space $\mathcal{M}_g$ of smooth genus $g$ curves then the subring generated by the tautological classes is called the tautological ring $R^*_g$. Looijenga’s result [20] implies that $R^*_g = 0$ for $* \geq g - 1$ and Faber [4] made the following

**Conjecture 3.3** The classes $\kappa_1, \ldots, \kappa_{[g/3]}$ generate the tautological ring $R^*_g$.

We refer the reader to [4] for the full conjecture.

It turns out that techniques similar to those of Theorem 3.1 produce several other sets of relations between tautological classes. One such set of relations implies that, for each $a > [g/3]$, the class $\kappa_a$ can be written as polynomial in lower degree tautological classes, as required by Faber’s conjecture. A detailed proof will appear in [11].

4. Further applications
There are other applications of the sum formula (2.3). One such application considered in [16] begins with the following simple observation. Given any symplectic manifold \( X \) with a codimension 2 symplectic submanifold \( V \), we can write \( X \) as a (trivial) symplectic sum \( X \#_VP \) where \( P \) is the ruled manifold \( \mathbb{P}(N_XV \oplus \mathbb{C}) \) and \( V \) is identified with its infinity section. We can then obtain recursive formulas for the GW invariants of \( X \) by moving constraints from one side to the other and applying the symplectic sum formula.

In [15] we used this method to obtain both (a) the Caporaso-Harris formula for the number of nodal curves in \( \mathbb{P}^2 \) [2], and (b) the “quasimodular form” expression for the rational enumerative invariants of the rational elliptic surface [1]. In hindsight, our proof of (a) is essentially the same as that in [2]; using the symplectic sum formula makes the proof considerably shorter and more transparent, but the key ideas are the same. Our proof of (b), however, is completely different from that of Bryan and Leung in [1].

We end with another interesting application of the Symplectic Sum Theorem 2.1. For each symplectomorphism \( f \) of a symplectic manifold \( X \), one can form the symplectic mapping cylinder

\[
X_f = X \times \mathbb{R} \times S^1/\mathbb{Z}
\]  

(4.1)

where the \( \mathbb{Z} \) action is generated by \((x,s,\theta) \mapsto (f(x),s+1,\theta)\). In a joint paper [13] with T. H. Parker we regarded \( X_f \) as a symplectic sum and computed the Gromov invariants of the manifolds \( X_f \) and of fiber sums of the \( X_f \) with other symplectic manifolds. The result is a large set of interesting non-Kähler symplectic manifolds with computational ways of distinguishing them. In dimension four this gives a symplectic construction of the ‘exotic’ elliptic surfaces of Fintushel and Stern [5]. In higher dimensions it gives many examples of manifolds which are diffeomorphic but not ‘equivalent’ as symplectic manifolds.

More precisely, fix a symplectomorphism \( f \) of a closed symplectic manifold \( X \), and let \( f^*k \) denote the induced map on \( H_k(X;\mathbb{Q}) \). Note that \( X_f \) fibers over the torus \( T^2 \) with fiber \( X \). If \( \det (I - f^*_1) = \pm 1 \) then there is a well-defined section class \( T \). Our main result of [13] computes the genus one Gromov invariants of the multiples of this section class. These are the particular GW invariants that, in dimension four, C.H. Taubes related to the Seiberg-Witten invariants (see [27] and [12]).

**Theorem 4.1** If \( \det (I - f^*_1) = \pm 1 \), the partial Gromov series of \( X_f \) for the section class \( T \) is given by the Lefschetz zeta function of \( f \) in the variable \( t = t_T \):

\[
Gr^T(X_f) = \zeta_f(t) = \frac{\prod_{k \text{ odd}}^{\infty} \det(I - tf^*_k)}{\prod_{k \text{ even}}^{\infty} \det(I - tf^*_k)}.
\]

When \( X_f \) is a four-manifold, a wealth of examples arise from knots. Associated to each fibered knot \( K \) in \( S^3 \) is a Riemann surface \( \Sigma \) and a monodromy diffeomorphism \( f_K \) of \( \Sigma \). Taking \( f = f_K \) gives symplectic 4-manifolds \( X_K \) of the homology type of \( S^2 \times T^2 \) with

\[
Gr(X_K) = \frac{A_K(t_T)}{(1 - t_T)^2}
\]
where $A_K(t) = \det(I - tf_*)$ is the Alexander polynomial of $K$ and $T$ is the section class.

We can elaborate on this construction by fiber summing $X_f$ with other 4-manifolds. For example, let $E(n)$ be the simply-connected minimal elliptic surface with fiber $F$ and holomorphic Euler characteristic $n$. Then $E(1)$ is the rational elliptic surface and $K3 = E(2)$. Forming the fiber sum of $X_K$ with $E(n)$ along the tori $T = F$, we obtain a symplectic manifold

$$E(n, K) = E(n)\#_{F=T} X_K.$$  

homeomorphic to $E(n)$. In fact, for fibered knots $K$, $K'$ of the same genus there is a homeomorphism between $E(n, K)$ and $E(n, K')$ preserving the periods of $\omega$ and the canonical class $\kappa$. For $n > 1$ we can compute the full (not just partial) Gromov series.

**Proposition 4.2** For $n \geq 2$, the Gromov and Seiberg-Witten series of $E(n, K)$ are

$$Gr(E(n, K)) = SW(E(n, K)) = A_K(t_F) (1 - t_F)^{n-2}. \quad (4.2)$$

Thus fibered knots with distinct Alexander polynomials give rise to symplectic manifolds $E(n, K)$ which are homeomorphic but not diffeomorphic. In particular, there are infinitely many distinct symplectic 4-manifolds homeomorphic to $E(n)$. Fintushel and Stern [5] have independently shown how (4.2) follows from knot theory and results in Seiberg-Witten theory.

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