A renormalisation group approach to two-body scattering in the presence of long-range forces

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(Dated: March 25, 2022)

We apply renormalisation-group methods to two-body scattering by a combination of known long-range and unknown short-range potentials. We impose a cut-off in the basis of distorted waves of the long-range potential and identify possible fixed points of the short-range potential as this cut-off is lowered to zero. The expansions around these fixed points define the power countings for the corresponding effective field theories. Expansions around nontrivial fixed points are shown to correspond to distorted-wave versions of the effective-range expansion. These methods are applied to scattering in the presence of Coulomb, Yukawa and repulsive inverse-square potentials.

I. INTRODUCTION

Effective field theories (EFT’s) offer the promise of a systematic and model-independent treatment of nuclear and hadronic physics at low energies. In the form of chiral perturbation theory (ChPT), they have been used with some success in both the mesonic and single-nucleon sectors (for reviews see: [1, 2]). Currently there is much interest in extending these applications to few-nucleon systems, as reviewed in Refs. [3, 4, 5, 6].

These theories rely on the existence of a separation of scales between those of interest for low-energy physics, such as momenta, energies or the pion mass, and those of the underlying short-distance physics, such the ρ-meson mass, the nucleon mass or 4πfπ. This makes it possible to expand any observable systematically in powers of the ratio Q/Λ0, where Q denotes a generic low-energy scale and Λ0 a typical scale of the underlying physics. In addition, the effective Lagrangian or Hamiltonian used to calculate these observables can be organised in a similar way. Such an expansion will be useful provided that the separation of scales is wide enough for the expansion to converge rapidly.

An EFT is defined by a Lagrangian or Hamiltonian containing all possible local terms consistent with the symmetries of the underlying theory. Although there is an infinite number of these terms, they can be organised according to a “power counting”, which is related to the number of low-energy scales present in each term. Loop diagrams are of the same or higher order compared with the terms from which they are constructed. This makes systematic calculations possible since, if we want to calculate observables up to some order in Q/Λ0, we need only terms in the effective theory up to a corresponding order. For weakly interacting systems the terms in the expansion can be organised according to naive dimensional analysis, a term proportional to (Q/Λ0)d being counted as of order d [1, 2]. This “Weinberg” power counting is the one familiar from ChPT in the zero- and one-nucleon sectors.

In contrast, for strongly interacting systems there can be new low-energy scales which are generated by nonperturbative dynamics. An important example for nuclear physics is the very large s-wave scattering length in nucleon-nucleon scattering. In such cases we need to resum certain terms in the theory to all orders, and this leads to a new power counting [3, 4, 5, 6], often referred to as Kaplan, Savage and Wise (KSW) power counting.

The theoretical tool that allows us to determine the power counting in these cases is the renormalisation group (RG), which is used to study the scaling behaviour of systems in a wide range of areas of physics. It is an extension of the simple dimensional analysis which leads to Weinberg power counting is weakly-interacting systems.

In our work we use a Wilsonian version of the RG [3], imposing a momentum cut-off, |k| < Λ, on the low-energy EFT. This cut-off should be thought of as separating the low-energy physics, which we wish to treat explicitly, from the underlying high-energy physics.

The cut-off Λ should be chosen to be above all of the low-energy scales of interest, but well below the scales of the underlying physics. This is possible so long as there is a clear separation between these scales. Beyond this, the precise value of Λ is arbitrary and we require that physical observables be independent of Λ. This means that physics on momentum scales higher than Λ has not just been discarded. Rather, it has been “integrated out”, its effects being included implicitly in the couplings of the EFT. As a result, all coupling constants in the EFT become functions of the cut-off Λ. Finally we rescale the theory, expressing all dimensioned quantities in units of Λ. The various coupling constants in the resulting rescaled theory vary with the cut-off Λ, and their flow is described by a first-order differential equation: the RG equation.

For a system with a clear separation of scales, the rescaled coupling constants become independent of Λ as Λ → 0. This is because, for Λ ≪ Λ0, the only scale left is Λ and so the rescaled theory becomes independent of Λ. The theory
is said to flow towards a fixed point of the RG.

Close to a fixed point, deviations from it scale as powers of the cut-off. Since the rescaling means that each low-energy scale appearing in a term of the potential contributes one power of $\Lambda$, we can use this to define the power counting for our EFT. A term that behaves like $\Lambda^\nu$ is assigned an order $d = \nu - 1$, to match with the Weinberg power counting mentioned above.

Perturbations around a fixed point can be classified into three types according to the sign of $\nu$. One with $\nu > 0$ is known as an “irrelevant” perturbation. Its flow is towards the fixed point as $\Lambda \to 0$. A perturbation with $\nu = 0$ is called “marginal”. This is the type of term familiar in conventionally renormalisable field theories. If a marginal perturbation is present, we should expect to find logarithmic flow with $\Lambda$. Finally, a perturbation with $\nu < 0$ is called “relevant”. It leads to flow away from the fixed point as $\Lambda \to 0$.

If all perturbations around a fixed point are irrelevant, then the fixed point is stable: the couplings of any theory close to that point will flow towards it as $\Lambda \to 0$. On the other hand, if there are one or more relevant perturbations then the fixed point is unstable.

In Ref. [13] these ideas were applied to two-body scattering by short-range forces. Two fixed points were found for $s$-wave scattering. One is the trivial fixed point, describing systems with no scattering. The terms in the expansion around this point can be organised according to Weinberg power counting. This defines an EFT which is appropriate for describing a system with weak scattering. The second point is a nontrivial one describing systems with a bound state at zero energy. The flow near it includes one unstable direction and the appropriate power counting for expanding around it is KSW counting. The terms of this expansion are in one-to-one correspondence with the terms of the effective-range expansion [15, 16, 17, 18].

The expansion around this nontrivial fixed point is appropriate for a system with a large scattering length (or equivalently a bound state or resonance close to zero energy). In the case of nucleon-nucleon scattering, it can be used at very low energies, well below the pion mass, where the whole strong interaction between the nucleons can be treated as short-range. It has been applied successfully to the calculation of various deuteron properties and reactions [14]. The corresponding EFT has also been extended to describe three-body systems [9, 20]. (For more examples see Refs. [6, 8].)

At higher energies, where the nucleon momenta are comparable to the pion mass, one would like to treat pion-exchange forces as known long-range interactions, calculable from ChPT [21]. Similarly, in proton-proton scattering the Coulomb interaction provides a known long-range force. In both of these cases, the remaining shorter-range strong force between the nucleons could be described within an EFT in terms of contact interactions.

Kong and Ravndal [22] have studied the example of scattering in the presence of a Coulomb potential. Although they did not establish the corresponding power counting, they did show that the resulting EFT was equivalent to a distorted-wave or “modified” effective-range expansion [15, 16, 17, 18]. Similar ideas have also been applied to scattering with a one-pion exchange potential as the long-range interaction [24, 25].

In the present work, we use the RG to study two-body scattering in a system with a combination of known long-range and unknown short-range forces. To do this, we apply a cut-off to the basis of distorted waves (DW’s) [15] for the known long-range interaction. Applying the RG as outlined above, we identify the possible fixed points of the short-range interaction and establish the corresponding power counting rules for perturbations around these. If a nontrivial fixed point exists, the terms in the resulting EFT can be directly related to those in a DW effective-range expansion.

In order to implement this treatment, it is essential that one has identified all the important low-energy scales for the system. In the case of the Coulomb potential, the Bohr radius provides an additional low-energy scale (apart from the momenta of the particles). So long as this potential is relatively weak, there will be a clear separation of scales between it and the short-range strong interaction. By expressing the Bohr radius in units of the cut-off, we ensure that the Coulomb potential becomes scale independent when we rescale the theory. This means that it can be regarded as part of any fixed point of the RG and so its effects should be treated to all orders.

More generally our approach can be applied whenever a long-range potential can be rescaled to make it scale independent. This potential can then be iterated to all orders to generate the basis of DW’s which forms the starting point for our RG. The fixed points of this RG are then the short-range interactions which, in combination with the long-range one, lead to scale-free behaviour in the limit where all low-energy scales are taken to to zero.

Pion-exchange interactions are of particular interest. These introduce another scale: the pion mass. Clearly this should be treated as a low-energy scale if one wishes to study nucleon-nucleon scattering with momenta comparable to the pion mass. Two schemes for incorporating these forces in EFT’s for nucleon-nucleon scattering can be found in the literature. One, proposed by KSW [11, 26], treats these forces perturbatively. The other, suggested by Weinberg [8] and further developed by van Kolck [3, 21, 27], iterates these forces to all orders.

The KSW scheme is just an extension of the KSW power counting around the effective-range fixed point, in which the pion mass provides the only additional low-energy scale. All of the pion-exchange terms in the rescaled potential vanish as $\Lambda \to 0$ and so can be treated perturbatively. This scheme is consistent with ChPT as implemented in
the zero- and one-nucleon sectors. However, it does seem to be at best slowly convergent in the s-wave channels [24, 25, 28, 29].

The alternative scheme of Weinberg and van Kolck (WvK) uses Weinberg power counting to organise the potential, the leading terms of which are then resummed to all orders. This method has been applied with some success [3, 21, 30], although questions have been raised about its consistency with the usual chiral expansion of ChPT [31, 32]. In our present treatment, we shall see that this WvK scheme corresponds to identifying an additional low-energy scale in the strength of the one-pion-exchange potential.

The outline of this paper is as follows. In Sec. II we summarise the basic results of Ref. [13]. This allows us to set up the basic framework of the RG for two-body scattering, and to establish the associated notation we use. In particular we discuss the two important fixed points: the trivial one and one with a bound state at zero energy. We show how a perturbative expansion around each fixed point can be used to determine the corresponding power counting.

In Sec. III we extend these ideas to describe scattering in presence of a known long-range potential. Our treatment is based on the application of a cut-off to the basis of DW’s for that potential. We derive an RG equations which has a similar form to that obtained in Ref. [13], but which contains an extra term for each low-energy scale associated with the long-range potential. We examine the conditions which that potential must satisfy in order to lead to fixed-point behaviour. In order to handle potentials which are more singular than the long-range potential. We show how a perturbative expansion around each fixed point can be used to determine the corresponding power counting.

This method is applied in Sec. IV to several examples of long-range forces. Sec. IV A revisits the example of scattering by a short-range interaction in the presence of a Coulomb potential. In this case a nontrivial fixed point, if one existed, would have a marginal perturbation leading to a logarithmic evolution with cut-off. The same methods can still be used to construct a potential where the logarithmic terms have been resummed. The power counting for perturbations away from this potential is an extended version of KSW counting and corresponds to the Coulomb DW effective-range expansion.

In Sec. IV B we examine $^1S_0$ NN scattering in the presence of the one-pion exchange. This is a Yukawa potential with the same singularity at $r = 0$ as the Coulomb potential, and so the RG behaviour is very similar. We show how one can obtain either the KSW or WvK schemes for the corresponding EFT, depending on the choice of low-energy scales.

Another important type of long-range interaction is the inverse-square potential. This is relevant to scattering in three-body systems, as shown by Efimov [33]. The corresponding RG could therefore help to clarify the unusual behaviour found in EFT treatments of such systems [24, 25]. This approach can also be applied scattering by a pure short-range potential in higher partial waves, where the centrifugal barrier has an inverse-square form. Scattering in the presence of a repulsive inverse-square potential is examined in Sec. IV C and a nontrivial fixed point is found. Perturbations around from this point scale with noninteger powers of the cut-off and correspond to terms in a DW effective-range expansion. The treatment of the attractive inverse-square potential is more involved and will be treated in a future work.

II. RG FOR SHORT-RANGE FORCES

Before considering the effects of long-range forces, we first set the scene by reviewing the main features of the RG treatment of two-body scattering by short-range forces [13]. For simplicity we consider here only s-wave scattering, although the extension to higher partial waves is straightforward.

The fully off-shell amplitude $T(k', k, p)$ for scattering of two heavy particles of mass $M$ by a potential $V(k', k, p)$ satisfies the Lippmann-Schwinger (LS) equation [18]

$$T(k', k, p) = V(k', k, p) + \frac{M}{2\pi^2} \int q^2 dq \frac{V(k', q, p)T(q, k, p)}{p^2 - q^2 + i\epsilon}.$$  \hspace{1cm} (1)

Here, as throughout this paper, we use $k$ and $k'$ to denote relative momenta and the energy-dependence is expressed in terms of $p \equiv \sqrt{M E}$, the on-shell momentum corresponding to the centre-of-mass energy $E$. The on-shell amplitude

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1 Note that we are careful to distinguish here between Weinberg power counting for the terms in the potential and the scheme based on it for treating pion exchange, which was proposed by Weinberg and developed by van Kolck.
\( T(p, p, p) \) is related to the phase shift \( \delta(p) \) by

\[
T(p, p, p) = -\frac{4\pi}{M} \frac{1}{p \cot \delta(p) - ip}.
\]  (2)

For systems where the scattering is weak at low energies, we can expand the on-shell amplitude as a power series in \( p \). However for nucleon-nucleon scattering, and other systems with bound states or resonances close to threshold, such an expansion is convergent only for a very limited range of energies. In such cases it more useful to expand the inverse of the amplitude in the form of an effective-range expansion (ERE) [15, 16, 17, 18]

\[
-\frac{4\pi}{M} \frac{1}{T(p, p, p)} + ip = p \cot \delta(p) = -\frac{1}{a} + \frac{1}{2} \bar{r} e p^2 + \cdots,
\]  (3)

where \( a \) is the scattering length and \( \bar{r} \) is the effective range.

### A. RG equation

At low energies, where the wavelengths of the particles are large compared to the range of the forces, these forces may be represented by an effective Lagrangian or Hamiltonian that consists of contact interactions only. In coordinate space these interactions can be expressed as \( \delta \)-functions and derivatives of \( \delta \)-functions. In general these will include time derivatives (in a Lagrangian framework) or energy dependence (in a Hamiltonian). In this case, the effective potential has the momentum-space form

\[
V(k', k, p) = c_{00} + c_{20}(k^2 + k'^2) + c_{02} p^2 + \cdots,
\]  (4)

to second order in the momentum expansion. For an EFT to be truly effective, we need to be able to organise the terms in this potential in a systematic way, according to some power counting. The RG provides a framework for doing this.

In order to derive an RG equation for the potential, it is convenient to start from the reactance matrix, \( K \). The off-shell \( K \)-matrix satisfies a Lippmann-Schwinger (LS) equation that is very similar to that for the scattering matrix \( T \), Eq. (1), except that the Green’s function obeys standing-wave boundary conditions. This means that \( K \) is real below all thresholds for particle production. On-shell the \( K \)- and \( T \)-matrices are related by

\[
\frac{1}{K(p, p, p)} = \frac{1}{T(p, p, p)} + \frac{iM p}{4\pi}.
\]  (5)

The LS equation (1) for scattering by contact interactions contains loop integrals that diverge and so need to be regularised. This can be done using a cut-off [13] or dimensional regularisation [11, 32]. It is also possible to renormalise the equation without specifying a particular regularisation by making a subtraction [12]. As the first step in deriving a Wilsonian RG equation, we choose to impose a sharp cut-off at \( q = \Lambda \) on the relative momentum \( q \) in the loop. The LS equation for the \( K \)-matrix is then

\[
K(k', k, p) = V(k', k, p) + \frac{M}{2\pi^2} \mathcal{P} \int_0^\Lambda q^2 dq \frac{V(k', q, p) K(q, k, p)}{p^2 - q^2},
\]  (6)

where \( \mathcal{P} \) denotes the principal value of the integral.

The RG equation for the effective potential is obtained by allowing \( V \) to depend on the cut-off \( \Lambda \) and demanding that the off-shell \( K \)-matrix be independent of \( \Lambda \). This is obviously sufficient to ensure that all scattering observables do not depend on \( \Lambda \). The stronger condition on the off-shell \( K \)-matrix is needed in order to derive the RG equation. Schematically, the LS equation can be written as

\[
K = V(\Lambda) + V(\Lambda) G_0(\Lambda) K,
\]  (7)

where \( G_0(\Lambda) \) is the regulated free Green’s function in Eq. (6). Differentiating this with respect to \( \Lambda \) and demanding that \( K \) be independent of \( \Lambda \) gives

\[
\frac{\partial V}{\partial \Lambda} \left( 1 + G_0(\Lambda) K \right) + V(\Lambda) \frac{\partial G_0}{\partial \Lambda} K = 0.
\]  (8)
Finally, by operating from the right with $(1 + G_0 K)^{-1}$ and using the fact that $K$ satisfies the LS equation (11), we get

$$\frac{\partial V}{\partial \Lambda} = \frac{M}{2\pi^2} V(k', \Lambda, p, \Lambda) \frac{\Lambda^2}{\Lambda^2 - p^2} V(\Lambda, k, p, \Lambda).$$

(9)

The solution to this equation should satisfy boundary conditions that follow from the fact that the effective potential is to describe low-energy scattering by short-ranged interactions. For $s$-wave scattering, it should be an analytic function of $k^2$ and $k'^2$ for small $k$ and $k'$. Rotational invariance would also allow dependence on $k \cdot k'$, but this is needed only for higher partial waves. If the energy lies below all thresholds for production of other particles then the potential should also be an analytic function of the energy, $E = \hat{p}^2 / M$. Under these restrictions, we see that $V$ should have an expansion in non-negative, integer powers of $k^2, k'^2$ and $p^2$.

The second step in obtaining a Wilsonian RG equation is to rescale all dimensioned quantities, expressing them in terms of the cut-off $\Lambda$. Dimensionless momentum variables are defined by $\hat{k} = k / \Lambda$ etc., along with a rescaled potential,

$$\hat{V}(\hat{k}', \hat{k}, \hat{p}, \Lambda) = \frac{M\Lambda}{2\pi^2} V(\Lambda \hat{k}', \Lambda \hat{k}, \Lambda \hat{p}, \Lambda),$$

(10)

where the factor of $M$ corresponds to dividing out $1/M$ from the whole Hamiltonian before rescaling. In terms of these rescaled quantities, Eq. (1) becomes

$$\Lambda \frac{\partial \hat{V}}{\partial \Lambda} = \hat{k}' \frac{\partial \hat{V}}{\partial \hat{k}'} + \hat{k} \frac{\partial \hat{V}}{\partial \hat{k}} + \hat{p} \frac{\partial \hat{V}}{\partial \hat{p}} + \hat{V} + \hat{V}(\hat{k}', 1, \hat{p}, \Lambda) \frac{1}{1 - \hat{p}^2} \hat{V}(1, \hat{k}, \hat{p}, \Lambda).$$

(11)

This is the RG equation for the effective potential.

A systematic expansion of the effective potential can be found if, as the cut-off is taken to zero, the potential tends to a fixed point of the RG. The fixed points are the solutions of Eq. (11) that do not depend on $\Lambda$. Two such points are of particular interest and these are described below.

### B. Trivial fixed point

The simplest example of a fixed-point solution to Eq. (11) is the trivial one:

$$\hat{V}(\hat{k}', \hat{k}, \hat{p}, \Lambda) = 0.$$  

(12)

The $K$-matrix for this potential is also zero, corresponding to no scattering. This obviously describes a scale-free system.

For systems where the scattering at low energies is weak, the rescaled potential tends towards this fixed point as we lower the cut-off towards zero. In such cases we can describe the low-energy behaviour in terms of perturbations around the trivial fixed point. We can find perturbations which scale with definite powers of $\Lambda$ by linearising the RG equation (11) about the fixed point and looking for solutions of the form

$$\hat{V}(\hat{k}', \hat{k}, \hat{p}, \Lambda) = \Lambda^\nu \phi(\hat{k}', \hat{k}, \hat{p}),$$

(13)

where the functions $\phi$ satisfy the eigenvalue equation

$$\hat{k}' \frac{\partial \phi}{\partial \hat{k}'} + \hat{k} \frac{\partial \phi}{\partial \hat{k}} + \hat{p} \frac{\partial \phi}{\partial \hat{p}} + \phi = \nu \phi.$$  

(14)

The solutions to this which are well-behaved as the momenta and energy tend to zero are simply

$$\phi(\hat{k}', \hat{k}, \hat{p}) = \hat{k}'^{2l} \hat{k}^{2m} \hat{p}^{2n},$$

(15)

with RG eigenvalues $\nu = 2(l + m + n) + 1$, where $l$, $m$ and $n$ are non-negative integers. The RG eigenvalues are all positive and so the fixed point is a stable one: starting from any potential in the vicinity of $\hat{V} = 0$ the RG flow will take the potential to the fixed point as $\Lambda \to 0$.

The potential near the trivial fixed point can be written in terms of these perturbations as

$$\hat{V}(\hat{k}', \hat{k}, \hat{p}, \Lambda) = \sum_{l,m,n} C_{lmn} \Lambda^\nu \hat{k}'^{2l} \hat{k}^{2m} \hat{p}^{2n}.$$  

(16)
This potential is Hermitian if we take real coefficients with $C_{lmn} = C_{mln}$. Note that terms with forms like $i(k^2 - k'^2)$ need not be included since they vanish after integration by parts in coordinate space. The unscaled form of this potential is

$$V(k', k, p, \Lambda) = \frac{2\pi^2}{M} \sum_{l,n,m} C_{lmn} k'^2 l k^2 m p^2 n. \quad (17)$$

The coefficients in this expansion can be written in a dimensionless form by pulling out a factor of $\Lambda_0^{-\nu}$, where $\Lambda_0$ is some scale associated with the underlying physics. In a “natural” theory it is possible to choose $\Lambda_0$ so that the dimensionless coefficients are all of order unity. This scale $\Lambda_0$ then determines where our expansion of the potential breaks down.

The RG eigenvalues $\nu$ provide a systematic way to classify the terms in this potential, those with the smallest values of $\nu$ dominating for small $\Lambda$. Alternatively, in unscaled units, we see that the terms in the potential can be classified according to the powers of momenta and energy they contain. For the trivial fixed point, this power counting is just the one proposed by Weinberg [8], where an order $d = \nu - 1$ is assigned to each term.

The behaviour near this fixed point can be used to describe systems where the scattering length is small, and so the scattering at low energies can be treated perturbatively. For such systems, this power counting has been shown to provide a systematic treatment in the context of both dimensional regularisation with minimal subtraction [32] and cut-off approaches [34]. The corresponding $K$-matrix is just given by the first Born approximation for the unscaled form of the potential Eq. (16) [13]. This is because higher-order terms from the LS equation are cancelled by higher-order terms in the potential from the full, nonlinear RG equation. On-shell we have

$$K(p, p, p) = \frac{2\pi^2}{M} \left[ C_{000} + (C_{001} + C_{010} + C_{100}) p^2 + \cdots \right], \quad (18)$$

showing that the $C_{lmn}$ are directly related to scattering observables, namely the coefficients in the expansion of the $K$-matrix in powers of the energy.

However, one should note that terms of the same order in the energy, $p^2$, or momenta, $k^2$ or $k'^2$, are equivalent. It is thus possible to swap between energy or momentum dependence in the potential without affecting physical observables. This can be done by a unitary transformation on the wave function or, in field theoretic language, by a transformation of the field variables (see, for example: [35, 36]). Since this involves the combination $p^2 - k'^2$, which vanishes on-shell, it is sometimes referred to as “using the equations of motion” to eliminate either energy or momentum dependence.

C. Effective-range fixed point

The next simplest fixed-point potential $\hat{V} = \hat{V}_0(\hat{p})$ depends on energy, but not on momenta. It satisfies the RG equation

$$\hat{p} \frac{d}{d\hat{p}} \hat{V}_0 + \hat{V}_0 + \hat{V}_0^2 \frac{1}{1 - \hat{p}^2} = 0. \quad (19)$$

A convenient way to solve this equation, as well as other momentum-independent RG equations, is to divide it by $\hat{V}_0^2$ to obtain a linear equation for $1/\hat{V}_0$:

$$\hat{p} \frac{d}{d\hat{p}} \left( \frac{1}{\hat{V}_0} \right) - \frac{1}{\hat{V}_0} - \frac{1}{1 - \hat{p}^2} = 0. \quad (20)$$

This equation is satisfied by the basic loop integral

$$\hat{J}(\hat{p}) = \mathcal{P} \int_0^1 \frac{\hat{q}^2 d\hat{q}}{\hat{p}^2 - \hat{q}^2} = -1 + \frac{\hat{p}}{2} \ln \frac{1 + \hat{p}}{1 - \hat{p}}. \quad (21)$$

This is analytic in $\hat{p}^2$ as $\hat{p}^2 \to 0$ and so it gives us the fixed-point potential

$$\hat{V}_0(\hat{p}) = -\left[ 1 - \frac{\hat{p}}{2} \ln \frac{1 + \hat{p}}{1 - \hat{p}} \right]^{-1}. \quad (22)$$
When this potential used in the LS equation (3), the corresponding \( K \)-matrix is found to give an infinite scattering length,

\[
\frac{1}{K(p)} = 0.
\]  

(23)

Alternatively one can think of it as generating a bound state at zero energy. Since the energy of the bound state is exactly zero, the system has no scale associated with it, which is why it is described by a fixed point of the RG.

This fixed point is the one of most interest for low-energy nuclear physics. Systems such as two nucleons, which have \( s \)-wave bound states or resonances near threshold, can be described by potentials close to this fixed point. Such a potential can be expanded in terms of small perturbations away from \( V_0 \) that scale with definite powers of \( \Lambda \):

\[
\tilde{V}(\hat{k}', \hat{k}, \hat{p}, \Lambda) = \tilde{V}_0(\hat{p}) + CA\nu\phi(\hat{k}', \hat{k}, \hat{p}).
\]  

(24)

These perturbations satisfy the linearised RG equation

\[
\hat{k}' \frac{\partial \phi}{\partial k'} + \hat{k} \frac{\partial \phi}{\partial k} + \hat{p} \frac{\partial \phi}{\partial \hat{p}} + \phi + \frac{\tilde{V}_0(\hat{p})}{1 - p^2} \left[ \phi(\hat{k}', 1, \hat{p}) + \phi(1, \hat{k}, \hat{p}) \right] = \nu \phi.
\]  

(25)

Of particular interest are the eigenfunctions of Eq. (23) that depend only on energy. These have the form

\[
\phi(\hat{p}) = \hat{p}^{\nu+1} \tilde{V}_0(\hat{p})^2.
\]  

(26)

If we demand that they be well-behaved functions of \( p^2 \) as \( p^2 \to 0 \), then we find that the allowed RG eigenvalues are \( \nu = -1, 1, 3, \ldots \). The presence of one negative eigenvalue shows that the fixed point is unstable.

Only potentials which lie on the “critical surface” where the coefficient of the unstable eigenvector is zero flow to the nontrivial fixed point as \( \Lambda \to 0 \). All others go either to the trivial fixed point or to infinity. Potentials which lie close to the cricial surface will flow initially towards the nontrivial fixed point and so their behaviour can be analysed using the expansion around this point. Eventually however, when \( \Lambda \) becomes small enough, their flows are pushed away by the unstable perturbation. This behaviour is illustrated below.

A useful method of examining RG flows is shown in Fig. 1. This illustrates the flow as \( \Lambda \to 0 \) in a two-dimensional slice of the infinite-dimensional space of effective potentials. In particular it shows the behaviour of the first two coefficients in the expansion of the rescaled potential in powers of the energy,

\[
\tilde{V} = b_0(\Lambda) + b_2(\Lambda)p^2 + \cdots.
\]  

(27)

There are two fixed points: the trivial one located at the origin and the nontrivial one at \((-1, -1)\). The flow lines in bold lie along RG eigenvectors close to the fixed points; the dashed lines show more general flow lines. All potentials near the trivial fixed point flow into it, and so that point is stable. In contrast only potentials lying on the critical surface defined by the stable perturbations (the vertical line \( b_0 = -1 \) in the figure) flow to the nontrivial fixed point. We see that flows near the critical surface are governed by the power counting around this fixed point until \( \Lambda \) reaches some critical scale when the unstable perturbation kicks in and takes the flow either into the domain of the trivial fixed point or to infinity. We shall see below that this critical scale is in fact the inverse of the scattering length.

The power counting for perturbations about this fixed point is different from that for the trivial fixed point. The energy-dependent perturbations can be assigned orders \( d = \nu - 1 = -2, 0, 2, \ldots \) in this scheme, which is the one also obtained by Bedaque and van Kolck [9, 10], Kaplan, Savage and Wise [11], and Gegelia [12]. An (unscaled) potential close to the fixed point can be expanded in terms of these perturbations as

\[
V(p, \Lambda) = V_0(p, \Lambda) + \frac{M}{2\pi^2} \sum_{n=0}^{\infty} C_{2n-1} p^{2n} V_0(p, \Lambda)^2.
\]

The connection with scattering observables can be found by solving the LS equation for this potential. The resulting on-shell \( K \)-matrix is given by

\[
\frac{1}{K(p)} = -\frac{M}{2\pi^2} \sum_{n=0}^{\infty} C_{2n-1} p^{2n}.
\]  

(28)

By comparing this with the ERE, Eq. (3), we see that there is a one-to-one correspondence between the perturbations around the nontrivial fixed point and the terms in the effective range expansion. In particular, the first two coefficients are

\[
C_{-1} = -\frac{\pi}{2\alpha}, \quad C_1 = \frac{\pi r_e}{4}.
\]  

(29)
In $s$-wave nucleon-nucleon scattering, the scattering lengths, $a$, are large. This implies that the coefficient $C_{-1}$ can be treated as a small perturbation. In this case, as explained above, the effective potential remains close to the critical surface flowing into the nontrivial fixed point, provided the cut-off $\Lambda$ is kept larger than $1/a$. The power counting about this fixed point is therefore the appropriate one for organising the terms in the nucleon-nucleon potential for momenta between $1/a$ and the pion mass.

Finally we note that there are also momentum-dependent perturbations. The RG analysis of Ref. [13] showed that these have even RG eigenvalues. They are thus of odd orders in the power counting about the nontrivial fixed point and they do not contribute to on-shell observables. For example, the perturbation with eigenvalue $\nu = 2$ has the form (in physical units)

$$C_2 \left\{ \left[ k^2 - p^2 + \frac{1}{3} \frac{M\Lambda^3}{2\pi^2} V_0 \right] V_0(p, \Lambda) + (k \to k') \right\}. \tag{30}$$

This consists of pieces which vanish on-shell, together with a term that cancels a cubic divergence generated by $k^2$ terms. Note that energy and momentum dependence are not equivalent in the vicinity of the nontrivial fixed point. The fact that they appear at different orders in the expansion means that it is not possible to generate a nonzero effective range from a purely momentum-dependent potential without an unnaturally large coefficient. A similar point about the need to include energy dependence has also been noted by Beane and Savage [37].

### III. RG WITH LONG-RANGE FORCES

Now consider a system of two particles interacting through a potential

$$V = V_L + V_S, \tag{31}$$

where $V_L$ is a known long-range potential and $V_S$ contains the effects of short-range physics, which we wish to parametrise systematically.

To obtain an RG equation for the short-range potential alone, it is convenient to work in terms of the distorted waves (DW’s) of the long-range potential. The $T$-matrix describing scattering by $V_L$ alone is

$$T_L = V_L + V_L G^+_0 T_L, \tag{32}$$

and the corresponding Green's function is

$$\hat{G}_L^+ = G^+_0 + G^+_0 T_L G^+_0. \tag{33}$$
By using the “two-potential trick” to resum the effects of $V_L$ to all orders $\tilde{K}_S = V_S + V_S G_L \tilde{K}_S$. The full $T$-matrix can be written in the form

$$T = T_L + (1 + T_L G_0^+ ) \tilde{T}_S (1 + G_0^+ T_L),$$

where $\tilde{T}_S$ satisfies the LS equation

$$\tilde{T}_S = V_S + V_S G_L^+ \tilde{T}_S.$$  

The expression $T$ is the starting point for the distorted-wave Born approximation since $\Omega = 1 + G_0^+ T_L$ is the Møller wave operator which converts a plane wave into a DW of $V_L$. The operator $\tilde{T}_S$ therefore describes the scattering between the DW’s as a result of the short-range potential.

The effects of the short-range potential can be expressed in terms of

$$\tilde{\delta}_S = \delta - \delta_L,$$

the difference between the full phase shift, $\delta$, and that due to $V_L$ alone, $\delta_L$. By taking on-shell matrix elements of Eq. (34), the matrix elements of $\tilde{T}_S$ between DW’s can be written in the form

$$\langle \psi_L^+(p, r) | \tilde{T}_S(p) | \psi_L^+(p, r) \rangle = -\frac{4\pi}{M} e^{2i\delta_L(p)} e^{2i\delta_S(p)} - \frac{1}{2ip},$$

where $\psi_L^+(p, r)$ is the outgoing DW of $V_L$ with energy $E = p^2 / M$ and $\psi_L^-$ is the corresponding incoming wave (and for simplicity we are considering only $s$-wave scattering). Bethe [17] used this as the starting point for constructing a “modified” or distorted-wave effective-range expansion (DWERE). His result can be written in the form

$$e^{2i\delta_L(p)} |\psi_L^+(p, 0)|^2 \left( \frac{1}{\langle \psi_L^- | \tilde{T}_S(p) | \psi_L^+ \rangle} - ip \right) + \mathcal{M}_L(p)$$

$$= |\psi_L^+(p, 0)|^2 p (\cot \tilde{\delta}_S(p) - i) + \mathcal{M}_L(p) = -\frac{1}{a} + \frac{1}{2} \tilde{r} e p^2 + \cdots.$$  

where the function $\mathcal{M}_L(p)$ is the logarithmic derivative at the origin of the Jost solution to the Schrödinger equation with the potential $V_L$. This expansion has long been used to extract low-energy properties of the strong interaction between two protons [17, 22, 23, 38] and more recently to remove the effects of one-pion exchange from NN scattering [24, 25].

The DWERE is useful because all rapid, and possibly nonanalytic, dependence on the energy is removed in $\mathcal{M}_L(p)$ and $e^{2i\delta_L(p)} |\psi_L^+(p, 0)|^2$. This leaves an amplitude which can be expanded as a power series in the energy, with coefficients whose scale is set by the underlying short-distance physics. Together with the appearance of wave functions at the origin, this strongly suggests that it should be possible to re-express this expansion in the form of an EFT. Indeed Kong and Ravndal [22] have done just this for the case of the Coulomb potential, but without explicitly establishing the form of the corresponding power counting.

### A. RG equation

The power counting which makes possible a systematic expansion of the short-range potential can be obtained from an RG analysis which is similar to that in Sec. I but which incorporates the distortion effects of the long-range potential. Such a DW version of the RG is most conveniently obtained from the DW $K$-matrix which satisfies an LS equation similar to Eq. (35) but with standing-wave boundary conditions,

$$\tilde{K}_S = V_S + V_S G_L \tilde{K}_S.$$  

To regulate this equation, we apply a cut-off to $G_L$ in the DW basis,

$$G_L = \frac{M}{2\pi^2} \int_0^\Lambda dq q^2 \frac{|\psi_L(q)|^2 |\psi_L(q)|}{p^2 - q^2} (+\text{bound states}).$$

If we demand that $\tilde{K}_S$ should be independent of the cut-off $\Lambda$, then we can use the same arguments as above to find that the short-range potential must satisfy the equation

$$\frac{\partial V_S}{\partial \Lambda} = -V_S \frac{\partial G_L}{\partial \Lambda} V_S.$$
If we did not apply the cut-off to the DW basis, $V_S$ would have to change with $\Lambda$ not only to incorporate the physics that has been integrated out but also to correct for the modification of $V_L$ by the cut-off. The resulting equation for $V_S$ would then have taken a much more complicated form.

To simplify the analysis, we assume here that $V_S$ depends only on energy, not on momenta. As shown in Ref. [13], the momentum-dependent solutions to the RG equation have more complicated forms but are not needed to describe on-shell scattering. In Sec. 1 the short-range force was taken to be a contact interaction, $V_S(p, \Lambda; r) = V_S(p, \Lambda) \delta(r)$. This is the simplest form that satisfies the appropriate symmetries for s-wave scattering. However such a choice cannot be used in combination with most of the long-range potentials of interest. These potentials are sufficiently singular that their DW’s $\psi_L(p, r)$ either vanish or diverge as $r \to 0$. As a result the right-hand side of Eq. (41) is either zero or ill-defined for a contact interaction.

Instead of a contact interaction we chose a spherically symmetric potential with a short but nonzero range. By choosing the range $R$ singular that their DW’s $\psi$ cannot be used in combination with most of the long-range potentials of interest. These potentials are sufficiently of the long-range potential. The precise value of this scale is arbitrary and so observables should not depend on it.

A simple and convenient choice for the form of this regulator is the “$\delta$-shell” potential,

$$V_S(p, \kappa, \Lambda; r) = V_S(p, \kappa, \Lambda) \frac{\delta(r - R)}{4\pi R^2}, \quad (42)$$

where $\kappa$ denotes a generic low-energy scale associated with $V_L$. With this choice, the equation (41) for $V_S$ becomes

$$\frac{\partial V_S}{\partial \Lambda} = -\frac{M}{2\pi^2} \psi_L(\Lambda, R)^2 \frac{\Lambda^2}{p^2 - \Lambda^2} V_S^2(p, \kappa, \Lambda), \quad (43)$$

where $\psi_L(k, R)$ is the DW with asymptotic momentum $k$, evaluated at $r = R$. This DW satisfies the Schrödinger equation,

$$\left(\frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr}\right) \psi_L(p, r) - MV_L(r, \kappa) \psi_L(p, r) + p^2 \psi_L(p, r) = 0, \quad (44)$$

and is normalised so that

$$\int_0^\infty dr \, p^2 r^2 \psi_L(p, r) \psi_L(p', r) = \frac{\pi}{2} \delta(p - p'). \quad (45)$$

In order to include nonperturbatively all of the physics associated with the long-range potential we must rescale $\kappa$ and keep $\hat{\kappa} = \kappa/\Lambda$ fixed as we take $\Lambda \to 0$. If, when we rescale $V_L$ as in Eq. (44), the potential is independent of $\Lambda$, we can regard it as part of a fixed point. This ensures that the long-range potential continues to influence the fixed points of the effective potential and the long-range physics is not “integrated out”. If the rescaled $V_L$ vanishes as $\Lambda \to 0$, we can treat it as a small perturbation about one of the fixed points of the pure short-range potential. Finally, if $V_L$ grows as $\Lambda \to 0$, then it is not possible to find any fixed-point behaviour. In such cases we must either give up the RG analysis or try to identify additional low-energy scales in the problem.

The RG equation is obtained in the same way as before, by rescaling $p, \kappa$ and $V_S$ in Eq. (43). The appearance of the factor $|\psi_L(\Lambda, R)|^2$ in this equation means that the effective potential must be rescaled in a way that depends on the long-range potential. We assume that the wave function in the limit $R \ll 1/p$ takes the separable form

$$|\psi_L(p, R)|^2 = |N(\kappa/p)|^2 f(p) F(R). \quad (46)$$

This is general enough to cover all the examples studied here and also the attractive inverse-square potential. Here $N(\kappa/p)$ is a normalisation factor through which information about the long-distance physics is communicated to the short-distance physics.

To remove the dependence on $\Lambda$ and $R$ contained in $|\psi_L(\Lambda, R)|^2$ from the right-hand side of Eq. (43), we define the rescaled short-range potential

$$\hat{V}_S(\hat{p}, \hat{\kappa}, \Lambda) = \frac{M\Lambda}{2\pi^2} F(R) f(\Lambda) V_S(\Lambda \hat{p}, \Lambda \hat{\kappa}, \Lambda). \quad (47)$$

This satisfies the DWRG equation

$$\Lambda \frac{\partial \hat{V}_S}{\partial \Lambda} = \hat{p} \frac{\partial \hat{V}_S}{\partial \hat{p}} + \hat{\kappa} \frac{\partial \hat{V}_S}{\partial \hat{\kappa}} + \left(1 + \Lambda \frac{f'(\Lambda)}{f(\Lambda)}\right) \hat{V}_S + \frac{|N(\hat{\kappa})|^2}{1 - \hat{p}^2} \hat{V}_S^2. \quad (48)$$
Since we have chosen to study potentials which are independent of momentum, as discussed above, the RG equation can be divided by $V_S^2$ to obtain a more practical linear equation for $1/V_S$. This equation has the form

$$\Lambda \frac{\partial}{\partial \Lambda} \left( \frac{1}{V_S} \right) = \hat{p} \frac{\partial}{\partial \hat{p}} \left( \frac{1}{V_0} \right) + \hat{\kappa} \frac{\partial}{\partial \hat{\kappa}} \left( \frac{1}{V_S} \right) - \left( 1 + \Lambda \frac{f'(\Lambda)}{f(\Lambda)} \right) \frac{1}{V_S} - \frac{|N(\hat{\kappa})|^2}{1 - \hat{p}^2}. \quad (49)$$

From the renormalization group equation (48) we can see that there is always a trivial fixed-point solution, $\hat{V}_S = 0$. However, nontrivial fixed points can only exist if the right-hand side is independent of $\Lambda$. This occurs if the wave function has a power-law form close to the origin,

$$|\psi_L(p, R)|^2 = |N(\kappa/p)|^2 (pR)^{\sigma - 1}, \quad (50)$$

where $\sigma$ is a real number. This is the case for all the examples considered here, but not for the attractive inverse-square potential. Taking $f(\Lambda) = \Lambda^{\sigma - 1}$, the DWRG equation can be written

$$\Lambda \frac{\partial}{\partial \Lambda} \left( \frac{1}{V_S} \right) = \hat{p} \frac{\partial}{\partial \hat{p}} \left( \frac{1}{V_0} \right) + \hat{\kappa} \frac{\partial}{\partial \hat{\kappa}} \left( \frac{1}{V_S} \right) - \sigma \frac{1}{V_S} - \frac{|N(\hat{\kappa})|^2}{1 - \hat{p}^2}. \quad (51)$$

As in the pure short-range case, the effective potential must satisfy the boundary condition of being analytic in $p^2$ at low energies. This arises from the assumption that all the nonanalytic energy dependence generated by long-range physics has been factored out into the DW’s. The corresponding condition for the dependence on $\kappa$ depends on the nature of that scale. In most cases we need to demand only that the effective potential is analytic in $\kappa$ depends on the current quark mass in the underlying theory, QCD and hence, as in ChPT, the effective potential should be analytic in $m_{\pi}^2$.

### B. Trivial fixed point

As already noted, a trivial fixed point, $\hat{V}_0 = 0$, always exists for Eq. (51). The interpretation of this fixed point is similar to that in the pure short-range case. It describes a system with no scattering of the distorted waves of the long-range potential. Perturbations about this point can be used to describe systems where the short-range interactions provide small corrections to the scattering by $V_L$. They correspond to an expansion of the DW $K$-matrix $K_S$ of Eq. (33) in powers of energy and $\kappa$. They have the forms

$$\hat{V}_S = C_{mn} \Lambda^{m+2n+\sigma} \kappa^m \hat{\rho}^{2n}. \quad (52)$$

The boundary conditions demand that $m$ and $n$ are non-negative integers. In addition, if $\kappa$ is a scale like the pion mass, $m$ must be even.

The stability of the fixed point depends upon the value of $\sigma$, which describes the power-law behaviour of the DW’s near the origin. For positive $\sigma$ the fixed point is stable, while for negative $\sigma$ there is one or more unstable perturbation, with $m + 2n + \sigma < 0$. If $\sigma = 0$ then the perturbation with $m = n = 0$ is marginal. A marginal eigenfunction may also exist when $\sigma$ is a negative integer, if $m + 2n = -\sigma$ is allowed. These marginal eigenfunctions have no power-law evolution with $\Lambda$. Instead we expect to find a logarithmic dependence which will allow us to classify such perturbations as marginally stable or unstable.

### C. Nontrivial fixed point

A nontrivial fixed point of Eq. (51), if one exists, satisfies the equation,

$$\hat{p} \frac{\partial}{\partial \hat{p}} \left( \frac{1}{V_0} \right) + \hat{\kappa} \frac{\partial}{\partial \hat{\kappa}} \left( \frac{1}{V_0} \right) = \sigma \frac{1}{V_0} + \frac{|N(\hat{\kappa})|^2}{1 - \hat{p}^2}. \quad (53)$$

It is useful to note that this equation for $1/V_0$ is satisfied by the loop integral

$$\hat{J}_0(\hat{p}, \hat{\kappa}) = \mathcal{P} \int_0^1 q^{\sigma + 1} dq \frac{|N(\hat{\kappa}/q)|^2}{\hat{p}^2 - q^2}. \quad (54)$$
Unlike the pure short-range case, we cannot directly identify \( 1/\hat{V}_0 = \hat{J}_0 \) as a fixed-point solution of the RG equation since it is not in general analytic as \( \hat{p}, \hat{\kappa} \to 0 \). Nonetheless we can use \( \hat{J}_0 \) as a starting point for finding such solutions. The method will be described in more detail in the examples below, but the basic idea is to subtract from \( \hat{J}_0 \) a solution to the homogeneous version of Eq. (53) to cancel all its nonanalytic behaviour. We can then write

\[
\frac{1}{\hat{V}_0} = \hat{J}_0(\hat{p}, \hat{\kappa}) - \hat{M}(\hat{p}, \hat{\kappa}),
\]

where \( \hat{M}(\hat{p}, \hat{\kappa}) \) is a homogenous function of order \( \sigma \) in \( \hat{p} \) and \( \hat{\kappa} \).

Perturbations around a fixed-point solution to Eq. (53) have the form

\[
\frac{1}{V_S} = \frac{1}{V_0} - C_{mn} \Lambda^{m+2n-\sigma} \hat{\kappa}^m \hat{p}^{2n},
\]

where \( n \) and \( m \) satisfy the conditions described above in the case of the trivial fixed point. If \( \sigma \) is positive then the fixed point is unstable, the eigenfunctions with \( m + 2n - \sigma < 0 \) corresponding to the unstable directions. If \( \sigma \) is negative then this fixed point is stable. Marginal eigenvectors may exist for integer \( \sigma \) if \( m + 2n = \sigma \) is allowed. In particular, for \( \sigma = 0 \) the fixed point is marginal with the \( m = n = 0 \) perturbation having zero eigenvalue.

### D. Regaining the DW expansion

Having obtained a solution to the full RG equation near the nontrivial fixed point, we can use it in the DW Lippmann-Schwinger equation for \( \hat{K}_S \) in order to connect the potential to scattering observables. As with a pure short-range interaction, we find a direct connection to an effective range expansion, in this case the DW version of \( L \) equation for \( \hat{p}, \hat{\kappa} \).

Note that the integral in the denominator is just \( \Lambda^\sigma R^{\sigma-1} \hat{J}_0(\hat{p}, \hat{\kappa}) \) where \( \hat{J}_0 \) is defined in Eq. (54).

This equation can be rewritten in terms of the cotangent of the additional phase shift as

\[
|\mathcal{N}(\hat{p}/p)|^2 \frac{\pi p^\sigma}{2} \cot \tilde{\delta}_S = \Lambda^\sigma \left( \hat{J}_0(\hat{p}, \hat{\kappa}) - \frac{1}{V_S(\hat{p}, \hat{\kappa})} \right).
\]

This result is independent of \( R \), as anticipated. Despite initial appearances it is also independent of \( \Lambda \). The difference between \( 1/V_0 \) and \( \hat{J}_0 \) is just the homogeneous function \( \hat{M}(\hat{p}, \hat{\kappa}) \) introduced in Eq. (55). The corresponding piece of the right-hand side is a homogeneous function of order \( \sigma \) in the physical variables \( p \) and \( \kappa \). Including perturbations of the form given in Eq. (56), we can therefore rewrite Eq. (59) as

\[
|\mathcal{N}(\hat{p}/p)|^2 \frac{\pi p^\sigma}{2} \cot \tilde{\delta}_S = -\hat{M}(p, \kappa) + \sum_{n,m} C_{nm} p^n \kappa^m,
\]

where \( \hat{M}(p, \kappa) = \Lambda^\sigma \hat{M}(\hat{p}, \hat{\kappa}) \).

This resulting equation (60) has exactly the form of the distorted wave expansion, Eq. (88). The only difference is that, for sufficiently singular potentials, the wave functions must be evaluated close to, but not exactly at, the origin. However, provided the wave functions have a power-law form in this region, the \( R \) dependence cancels leaving Eq. (60). All nonanalytic effects of the long-range force have been factored or subtracted out in the functions \( \mathcal{N} \) and \( \hat{M} \). The remainder can be written as a power-series expansion in \( p \) and \( \kappa \), which corresponds directly to the expansion of the short-range effective potential.
IV. EXAMPLES

A. Coulomb potential

Scattering from a combination of Coulomb and short-range potentials has already been studied from an EFT viewpoint by Kong and Ravndal \[22\]. Here we examine it using the RG and show how a power counting can be established which matches exactly with the DWERE.

Since the fine structure constant $\alpha$ is small, a low-energy scale for this potential is provided by the inverse of the Bohr radius,

$$\kappa = \frac{\alpha M}{2}. \quad (61)$$

By rescaling this quantity as described above, we ensure that the long-range potential remains independent of $\Lambda$ as $\Lambda \to 0$. In contrast, if we were to treat $\kappa$ as a high-energy scale, then the Coulomb potential would act like a relevant perturbation and would destroy any possible infrared fixed points.

The correctly normalised distorted $s$-wave for this potential is, in terms of a hypergeometric function,

$$\psi_L(p, r) = e^{-\pi \kappa/2p} \Gamma(1 - i\kappa/p) F_1(1 + i\kappa/p, 2, -2ipr/\kappa) e^{ipr}. \quad (62)$$

This tends to a finite, nonzero value at the origin and so it has the form assumed in the previous section with $\sigma = 1$. The long-range phase-shift is $\delta_L = \text{Arg}\Gamma(1 + i\kappa/p)$ and the square of the wave function at the origin is given by the well-known Sommerfeld factor

$$|N(\kappa/p)|^2 = \lim_{R \to 0} |\psi_L(p, R)|^2 = C(\kappa/p) = \frac{2\pi\kappa/p}{e^{2\pi\kappa/p} - 1}. \quad (63)$$

A trivial fixed point always exists and the power counting for the expansion around it can be found using the general analysis in Sec. \[11\]. Of more interest are possible nontrivial solutions to the RG and the expansions around them. We take the basic loop integral of Eq. \[54\] as our starting point for the construction of these solutions. This satisfies Eq. \[53\], the fixed-point version of the DW RG equation \[51\] but it contains nonanalytic terms which should not be present. To identify these, we follow Kong and Ravndal \[22\] and break the integral up as

$$\hat{J}_0(\hat{p}, \hat{\kappa}) = -\int_0^1 dq C(\hat{\kappa}/q) + \hat{p}^2 P \int_0^\infty dq C(\hat{\kappa}/q) q^2 - \hat{p}^2 \int_1^\infty dq C(\hat{\kappa}/q) q^2. \quad (64)$$

When written in terms of physical, rather than rescaled, variables, the first term contains linear and logarithmic divergences. The second term is finite but depends nonanalytically on $\hat{p}$ and $\hat{\kappa}$, whereas the final term is analytic.

Using the detailed expression of Kong and Ravndal \[22\] for the second term and expanding the first in powers of $\hat{\kappa}$, we can write $\hat{J}_0$ in the form

$$\hat{J}_0(\hat{p}, \hat{\kappa}) = -1 - \pi \hat{\kappa} \left[ \ln \hat{\kappa} + \gamma \right] - \pi \hat{\kappa} \text{Re}[H(\hat{\kappa}/\hat{p})] + \text{terms analytic in } \hat{p}, \hat{\kappa}, \quad (65)$$

where $\gamma$ is Euler’s constant and the function $H$ is given by

$$H(x) = \psi(ix) + \frac{1}{2ix} - \ln(ix), \quad (66)$$

in terms of the logarithmic derivative of the $\Gamma$-function, denoted by $\psi$.

A potential which is analytic as $\hat{p}^2, \hat{\kappa} \to 0$ can be built from $\hat{J}_0$ by cancelling the terms proportional to $\hat{\kappa} \ln \hat{\kappa}$ and $\hat{\kappa} \text{Re}[H(\hat{\kappa}/\hat{p})]$. The second of these satisfies the homogeneous version of Eq. \[53\],

$$\hat{p} \frac{\partial}{\partial \hat{p}} \left( \frac{1}{V_0} \right) + \hat{\kappa} \frac{\partial}{\partial \hat{\kappa}} \left( \frac{1}{V_0} \right) = \frac{1}{V_0}, \quad (67)$$

and so it can be subtracted from $\hat{J}_0$ to leave a solution to the full equation.

In contrast, the term with logarithmic dependence on $\hat{\kappa}$ can not be removed in this way. Hence no true nontrivial fixed point exists in this case. Nonetheless it is possible to cancel the logarithmic term, leaving a potential which satisfies the RG equation and obeys the correct boundary conditions. This can be done with the aid of the function

$$\hat{\kappa} \left( \ln \hat{\kappa} + \ln \Lambda/\mu \right), \quad (68)$$
which satisfies the homogeneous version of the full RG equation (51),

$$\Lambda \frac{\partial}{\partial \Lambda} \left( \frac{1}{V_S} \right) = \tilde{p} \frac{\partial}{\partial \tilde{p}} \left( \frac{1}{V_S} \right) + \kappa \frac{\partial}{\partial \kappa} \left( \frac{1}{V_S} \right) - \frac{1}{V_S},$$

(69)

and so can be used to cancel the term with the logarithmic dependence on $\tilde{\kappa}$.

The resulting potential is given by

$$\frac{1}{V_0(\tilde{p}, \tilde{\kappa}, \Lambda)} = \tilde{J}_0(\tilde{p}, \tilde{\kappa}) + \pi \tilde{\kappa} \left( \text{Re}[H(\tilde{\kappa}/\tilde{p})] + \ln \tilde{\kappa} \Lambda / \mu \right).$$

(70)

It satisfies the DW RG equation (51), is analytic in both $\tilde{p}^2$ and $\tilde{\kappa}$, but is not a fixed point since it does depend on $\Lambda$, albeit only logarithmically. This should not be too surprising since the general analysis in Sec. III C showed that we would get a potential with a marginal perturbation. Although such perturbations have no power-law dependence on $\Lambda$, in general they give rise to logarithmic evolution with $\Lambda$. If we invert Eq. (70) we see that the potential we have obtained resums the leading logarithms ($\kappa \ln \Lambda$) to all orders.

Although this potential does evolve slowly with $\Lambda$, we can still expand around it and use the DW RG equation (51) to determine the forms of the perturbations around it. These are the same as the ones given above in Eq. (56) and so the full short-range potential is given by

$$\frac{1}{V_S} = \frac{1}{V_0} + \sum_{n,m=0}^{\infty} C_{nm} \Lambda^{m+2n-1}\kappa^m \tilde{p}^{2n}.$$ 

(71)

This provides the power-counting for a strong short-range potential in the presence of the Coulomb potential. Note that for the $\tilde{\kappa}$-independent terms this is just KSW counting, which corresponds to the ERE.

Another consequence of the marginal perturbation is the appearance in the potential (70) of an arbitrary scale $\mu$. This perturbation is independent of $\Lambda$ and so cannot be separated unambiguously from a fixed-point potential. Its coefficient $C_{01}$ can be chosen to depend on $\mu$ in such a way that the term

$$\left( \pi \ln \tilde{\kappa} \Lambda / \mu + C_{01}(\mu) \right) \tilde{\kappa}$$

(72)

and hence the full potential (71) are independent of $\mu$. (An alternative prescription would be to set $C_{01} = 0$ and to take $\mu$ as the corresponding parameter in the potential.)

To illustrate this RG flow, we plot the behaviour of the leading coefficients in the expansion:

$$\tilde{V}(\tilde{p}, \tilde{\kappa}, \Lambda) = b_{00}(\Lambda) + b_{01}(\Lambda) \tilde{\kappa} + b_{20}(\Lambda) \tilde{p}^2 + b_{21}(\Lambda) \tilde{p}^2 \tilde{\kappa} + \cdots.$$ 

(73)

In the $(b_{00}, b_{20})$ plane the flow is identical to that in Fig. 1 above. It may be interpreted in a similar way to the pure short-range case. However the point $(-1, -1)$ is not a fixed point in the $b_{01}$ direction. This can be seen from the flow in the $(b_{01}, b_{21})$ plane shown in Fig. 2. The only fixed point in this plane is the trivial one.

As above, the bold lines show the flows as $\Lambda \to 0$ corresponding to the RG eigenvectors. The flow associated with the marginal perturbation carries the potential up the vertical flow line $b_{00} = -1$ at a logarithmic rate. Although the coefficient $b_{01}$ appears to be tending to infinity along this flow line, it will eventually become so large that the expansion in Eq. (73) breaks down. A general potential, for which $b_{00}$ is not exactly $-1$, will flow either into the trivial point or to infinity. However, provided the coefficient of the unstable perturbation is small, it is still possible to expand around the logarithmic flow line, at least until some critical value of $\Lambda$.

Substituting the effective potential into equation (59), we get the following expansion for DW scattering observables:

$$C(\kappa/p) \cot \hat{\delta}_S = -2\kappa \left( \text{Re}[H(\kappa/p)] + \ln \kappa / \mu \right) + \frac{2}{\pi} \sum_{n,m=0}^{\infty} C_{nm} \kappa^m p^{2n},$$

(74)

where $\kappa = \alpha M/2$. This result can now be compared with the DWERE for the Coulomb potential [14, 17, 22].

$$C(\kappa/p)(\cot \hat{\delta}_S - i) + \alpha M H(\kappa/p) = -\frac{1}{a} + \frac{1}{2} \tilde{\kappa} \tilde{p}^2 + \cdots,$$

(75)

where the imaginary parts of the two terms on the left-hand side of this equation cancel exactly. Apart from the logarithmic term, the nonanalytic effects of the long-range Coulomb potential are contained in the functions $C(\kappa/p)$ and $H(\kappa/p)$. 
As in the pure short-range case, there is a direct correspondence between the expansion of the potential in powers of the energy and the terms in the ERE. The full effective potential gives a further expansion of this in powers of $\kappa$, or equivalently $\alpha$. This is organised according to the power counting around the logarithmically evolving potential of Eq. (70). Equating the two expansions, we get the following expressions for the Coulomb-modified scattering length and effective range,

$$\frac{1}{\tilde{a}} = \alpha M \ln \left( \frac{\alpha M}{2\mu} \right) - \frac{2}{\pi} \sum_{m=0}^{\infty} C_{0m} \left( \frac{\alpha M}{2} \right)^m,$$

$$\tilde{r}_e = \frac{4}{\pi} \sum_{m=0}^{\infty} C_{1m} \left( \frac{\alpha M}{2} \right)^m,$$

We see that for $\alpha = 0$ these reduce to the forms given in Eq. (29) for the coefficients in the ordinary ERE.

The scattering length defined in Eq. (75) is independent of $\mu$, provided $C_{01}(\mu)$ is chosen as discussed above (Eq. (72)). However, it still contains a logarithmic dependence on $\alpha$ and so cannot be a pure short-range effect. One could define a strong scattering length by

$$\frac{1}{\bar{a}} = \frac{1}{\tilde{a}} - \alpha M \ln \left( \frac{\alpha M}{2\mu} \right) = -\frac{2}{\pi} \left( C_{00} + C_{01}(\mu) + \frac{\alpha M}{2} + \cdots \right),$$

but this would depend on the arbitrary scale $\mu$. This expression may be compared with that of Jackson and Blatt [38],

$$\frac{1}{a_{pn}} = \frac{1}{a_{pp}} - \alpha M \left( \ln \alpha M r_0 + 0.33 \right),$$

relating the proton-proton DW scattering length and the neutron-proton scattering length. Their relation assumes that the strong interaction is charge independent and so one might naively interpret any deviation from it as a signal of isospin violation. However from Eq. (75), we see that Jackson and Blatt’s form corresponds to a particular choice of the scale $\mu$. A different choice of $\mu$ would lead to a different estimate of the charge-independence breaking and so Eq. (79) must be treated with caution.

B. Yukawa potential

The renormalisation of scattering in the presence of a Yukawa potential is very similar to the Coulomb case since they both have a $1/r$ singularity at the origin and so their distorted waves have the same short-distance behaviour.
We consider in particular the potential for one-pion exchange (OPE) between nucleons in the $^1S_0$ channel,

$$V_L(r) = -\alpha_\pi \frac{e^{-m_\pi r}}{r},$$  \hfill (80)

where the strength of the potential is

$$\alpha_\pi = \frac{g_A^2 m_\pi^2}{16\pi f_\pi^2}$$  \hfill (81)

in terms of the pion mass, $m_\pi = 140$ MeV, the pion decay constant, $f_\pi = 93$ MeV, and the axial coupling of the nucleon, $g_A = 1.26$.

In this case, the scaling behaviour depends crucially on the quantities we choose to identify as low-energy scales. As in ChPT, the pion mass clearly provides one such scale. Treating this scale explicitly will allow us to make a chiral expansion of an EFT which we might hope would be valid for momenta below the $\rho$-meson mass.

When rescaled the OPE potential is, in momentum space,

$$\hat{V}_L(\hat{k}', \hat{k}, \hat{m}_\pi, \Lambda) = -\Lambda \frac{M g_A^2}{8\pi^2 f_\pi^2} \hat{m}_\pi^2 |\hat{k}' - \hat{k}|^2 + \hat{m}_\pi^2.$$  \hfill (82)

This is proportional to the cut-off $\Lambda$ and so its effect on the RG flow vanishes as $\lambda \rightarrow 0$. This means that the fixed points will be the same as in the pure short-range case. The DW Green’s function $G_L$ can be expanded in powers of the long-ranged potential and pion exchange treated as perturbative corrections to the ordinary ERE. This is the KSW scheme for including pions explicitly in an EFT for nucleon-nucleon scattering [11, 26].

Although the KSW scheme allows ChPT to be extended to the two-nucleon sector, the resulting expansion turns out to be, at best, slowly convergent [24, 25, 28, 29]. The problem is that the pion-nucleon coupling is large, or equivalently that the scale which sets the the strength of the potential,

$$\lambda_{NN} = \frac{16\pi f_\pi^2}{M g_A^2} \approx 300 \text{ MeV},$$  \hfill (83)

is small. There is thus no good separation of scales and so it is unsurprising that the corresponding EFT shows poor convergence.

An alternative approach is to treat $\lambda_{NN}$ or equivalently the inverse “pionic Bohr radius”,

$$\kappa_\pi = \frac{\alpha_\pi M}{2},$$  \hfill (84)

as an additional low-energy scale. The rescaled OPE potential is then independent of $\Lambda$ and so has been promoted to form part of any fixed point.

The effects of the distortion show up in the normalisation of the DW’s near the origin. Although it is not possible to write this normalisation in analytic form, from dimensional analysis, it must be of the form

$$|\psi_L(p, R \rightarrow 0)|^2 = C_\pi(\kappa_\pi/p, m_\pi/p).$$  \hfill (85)

If we rescale both $\kappa_\pi$ and $m_\pi$, as just discussed, the normalisation of the DW with $p = \Lambda$ is independent of $\Lambda$. The resulting RG equation is

$$\frac{\Lambda}{\partial \Lambda} \frac{\partial \hat{V}_S}{\partial \hat{p}} = \hat{p} \frac{\partial \hat{V}_S}{\partial \hat{p}} + \kappa_\pi \frac{\partial \hat{V}_S}{\partial \kappa_\pi} + \hat{m}_\pi \frac{\partial \hat{V}_S}{\partial \hat{m}_\pi} + \hat{V}_S + \frac{C_\pi(\kappa_\pi, \hat{m}_\pi)}{1 - \hat{p}^2} \hat{V}_S^2.$$  \hfill (86)

The expansion around the trivial fixed point is

$$\hat{V}_S = \sum_{l,m,n} C_{lmn} \Lambda^\nu \hat{m}_\pi^{2l} \kappa_\pi^{2m} \hat{p}^{2n},$$  \hfill (87)

where the eigenvalues are $\nu = 2l + m + 2n + 1 = 1, 3, 5, \cdots$, as in the Coulomb case.

The more interesting expansion around a nontrivial fixed point takes the form

$$\frac{1}{V_S} = \frac{1}{V_0} + \sum_{l,m,n} C_{klm} \Lambda^\nu \hat{m}_\pi^{2l} \kappa_\pi^{2m} \hat{p}^{2n},$$  \hfill (88)
where \( \nu = 2l + m + 2n - 1 = -1, 0, +1, \cdots \). This includes a marginal perturbation which is linear in the inverse Bohr radius and leads to logarithmic evolution of the potential \( V_0 \) with \( \Lambda \). We can resum this using the RG method described above in Sec. IV A. The presence of one unstable perturbation, as well as the marginal one, means that the RG flows are very similar to those shown in Figs. 1 and 2.

The expansion of the potential corresponds to a DWERE of the form
\[
C_\pi(\kappa_\pi/p, m_\pi/p) \cot \delta_S = 2\kappa_\pi (H_\pi(\kappa_\pi/p, m_\pi/p) + \ln \kappa_\pi/\mu) + \frac{2}{\pi} \sum_{l,m,n} C_{lmn} m_\pi^{2l} \kappa_\pi^m p^{2n}, \tag{89}
\]
where the dependence on the arbitrary scale cancels between the first term and \( C_{010}(\mu) \). All nonanalytic behaviour is contained in the functions \( C_\pi(\kappa_\pi/p, m_\pi/p) \) and \( H_\pi(\kappa_\pi/p, m_\pi/p) \). For the Yukawa potential these must be calculated using numerical methods. This DWERE has been applied to nucleon-nucleon scattering in Refs. 24, 25.

The EFT corresponding to this DWERE is one in which the effects of OPE and an energy-independent short-range force are iterated to all orders. The resummation of these two terms, which would be of leading order in Weinberg power counting about a trivial fixed point, is precisely the scheme suggested by Weinberg in his original paper 8 on EFT’s for nucleon-nucleon scattering. This WvK scheme was then developed and applied by van Kolck and others 3, 21, 27, 30, with some success. The RG analysis shows that the expansion around the logarithmically evolving potential defines the power counting for this scheme, a term \( m_\pi^d \alpha_\pi^p p^{2n} \) being of order \( d = 2l + m + 2n - 2 \). For the terms which depend on energy only, this counting is similar to that of the KSW scheme.

Although the original motivation for developing EFT’s for nucleon-nucleon scattering was to extend ChPT to few-nucleon systems, it is not obvious that the WvK scheme is consistent with the chiral expansion. For example the term proportional to \( \alpha_\pi \) is of order \( d = -1 \) in this scheme, whereas a term proportional to \( m_\pi^2 \) is of order \( d = 0 \). Since \( \alpha_\pi \) is proportional to \( m_\pi^2 \), both terms would be of the same order in a pure chiral expansion. Within the chiral expansion the quantity \( \lambda_{NN} \) would be a high-energy scale, but in the WvK scheme it has been treated as a low-energy scale. Hence in contrast to the the KSW scheme, the direct link to ChPT has been lost.

C. Repulsive inverse-square potential

As a final example we consider the inverse-square potential, which is of interest because of its relevance to the three-body problem. Efimov 33 has shown that this potential can describe the scattering of a particle off a two-particle bound state in the limit where the two-body potential has zero range and infinite scattering length. The renormalisation of the short-ranged three-body forces which appear in EFT treatments of such systems is currently the object of keen study 20, 29.

In coordinate space the potential has the form
\[
V_L(r) = \frac{\beta}{M r^2}. \tag{90}
\]
When an overall factor of \( 1/M \) is taken out, as here, we see that the remaining coupling \( \beta \) is dimensionless. The potential is thus scale-free and so it can naturally be thought of as part of a fixed point. This is why it appears in the three-body problem with a scale-free-two-body interaction.

The inverse-square potential acts like the centrifugal term in the free Schrödinger equation and so the results we obtain here can also be applied to scattering in higher partial waves by a short-range potential. In general, the distorted waves are given in terms of Bessel functions of noninteger order and have the form
\[
\psi_L(p, r) = A_1 r^{-1/2} J_\nu(pr) + A_2 r^{-1/2} J_{-\nu}(pr), \tag{91}
\]
where the order \( \nu = \sqrt{1/4 + \beta} \) depends on the strength of the coupling. This makes it clear that two cases need to be considered separately, real \( \nu (\beta > 1/4) \) and imaginary \( \nu (\beta < 1/4) \). In this paper we consider only the repulsive inverse-square potential. The more complicated case of a strongly attractive attractive potential, where \( \nu \) is imaginary, will be discussed in a future work.

In the repulsive case, the relevant solution, which is regular at the origin and correctly normalised, is
\[
\psi_L(p, r) = \sqrt{\frac{\pi}{2pr}} J_\nu(pr). \tag{92}
\]
The corresponding s-wave phase-shift is easily determined to be \( \delta_L = \pi/4 - \nu \pi/2 \). To construct the RG, we need the square of the DW at small \( R \),
\[
|\psi_L(p, R)|^2 = \frac{1}{\Gamma(1+\nu)^2} \left(\frac{pr}{2}\right)^{2\nu-1}. \tag{93}
\]
This is of the form in Eq. (10) with \( \sigma = 2\nu \) and \( \mathcal{N}^2 = 1/[2^{2\nu-1}(1+\nu)^2] \). For scattering in a partial wave with \( l > 0 \), the centrifugal barrier provides a \( 1/r^2 \) potential with \( \beta = l(l+1) \) and hence \( \nu = l + \frac{1}{2} \).

Since \( \mathcal{N}^2 \) is simply a constant in this case, it is convenient to absorb it into the rescaling of the potential. The resulting RG equation is then

\[
\Lambda \frac{\partial \hat{V}_S}{\partial \Lambda} = \hat{p} \frac{\partial \hat{V}_S}{\partial \hat{p}} + 2\nu \hat{V}_S + \frac{1}{1 - \hat{p}^2} \hat{V}_S^3.
\]

From the general analysis in Sec. [11] we see that perturbations around the trivial fixed are of the form,

\[
\hat{V}_S = C_{2n} \Lambda^{2n+2\nu} \hat{p}^{2n}.
\]

Since \( \nu > 0 \) all of the eigenvalues are positive and the fixed point is stable as \( \Lambda \to 0 \). The term proportional to \( \hat{p}^{2n} \) is of order \( d = 2n + 2\nu - 1 \) in the corresponding power counting. For nucleon-nucleon scattering in partial waves with \( l > 0 \) there are no bound states or resonances close to threshold and so this fixed point is the appropriate one. The power counting for this case is given by \( d = 2(n+1) \).

Also of interest is the nontrivial fixed point, which corresponds to a DWERE. To construct it, we start from the basic loop integral of Eq. (53). The cases of integer and noninteger \( \nu \) behave differently and need to be considered separately. The loop integral can be evaluated to give

\[
\hat{J}_0(\hat{p}) = \mathcal{P} \int_0^1 \hat{q}^{2\nu+1} d\hat{q} \frac{1}{\hat{p}^2 - \hat{q}^2} = \frac{1}{2} \sum_{n=0}^{\infty} \frac{\hat{p}^{2n}}{n - \nu} + \frac{\pi}{2} \hat{p}^{2\nu} G(\hat{p}, \nu),
\]

where,

\[
G(\hat{p}, \nu) = \begin{cases} \cot \pi \nu, & \nu \notin \mathbb{N} \\ 2 \ln \hat{p}, & \nu \in \mathbb{N}. \end{cases}
\]

The prime on the sum here indicates that the term with \( n = \nu \) term must be omitted when \( \nu \) is an integer.

To obtain a well-behaved short-range potential, we apply the boundary condition of analyticity by cancelling the final, nonanalytic term from \( \hat{J}_0 \). When \( \nu \) is noninteger we can do this using the fact that \( \hat{p}^{2\nu} \) satisfies the homogeneous version of the RG equation (94). In the case of integer \( \nu \), the logarithm of \( \hat{p} \) can be cancelled in similar manner to the logarithm of \( \hat{k} \) which appears for the Coulomb potential. This leads to a potential with a logarithmic dependence on \( \Lambda \), in which the leading logarithms are resummed to all orders. The result in either case can be expressed in the form

\[
\frac{1}{\hat{V}_0(\hat{p})} = \hat{J}_0(\hat{p}) - \frac{1}{2} \hat{p}^{2\nu} G(\hat{p}/\mu, \nu).
\]

Once again the full solution to the RG equation is obtained by adding perturbations around this fixed point,

\[
\frac{1}{\hat{V}_S} = \frac{1}{\hat{V}_0} + \sum_{n=0}^{\infty} C_{2n} \Lambda^{2n-2\nu} \hat{p}^{2n}.
\]

This fixed point is unstable, with the number of negative eigenvalues being governed by \( \nu \). If \( \nu \) lies between the integers \( N - 1 \) and \( N \), the first \( N \) perturbations are unstable. If \( \nu = N \) then there is also a marginal eigenvector, \( \hat{p}^{2N} \), which is the origin of the logarithmic behaviour. The corresponding coefficient \( C_{2N}(\mu) \) depends on the arbitrary scale \( \mu \) so that the full potential is \( \mu \)-independent.

Figs. 3 and 4 show this flow for the cases \( \nu = 0.85 \) and \( \nu = 1.15 \) respectively. We again expand the potential in powers of energy, as in Eq. (27), and plot the RG flow in the \((b_0, b_2)\) plane. For \( \nu = 0.85 \) there is only one unstable perturbation. However the flow in the direction of the lowest stable perturbation is quite weak, \( \propto \Lambda^{0.3} \), and so the flow lines peel away from the critical surface much more rapidly than in Fig. 1. This flow weakens as \( \nu \to 1 \) until at \( \nu = 1 \) it becomes the logarithmic flow of a marginal perturbation. For \( \nu > 1 \), the lowest two perturbations are unstable, as shown in Fig. 4.

The power counting around the nontrivial fixed point is \( d = 2n - 2\nu - 1 \) for a term proportional to \( \hat{p}^{2n} \). This is quite different from the counting found for scattering in the presence of the Coulomb potential. Since the inverse-square potential is scale-free, its strength does not provide an expansion parameter in the low-energy EFT. Instead it appears in the energy power-counting itself, determining the number of relevant (unstable) perturbations. In the limit where this strength vanishes, and \( \nu \to \frac{1}{2} \), we have precisely the power-counting established earlier for a pure short-range potential.
FIG. 3: The flow as $\Lambda \to 0$ of the first two coefficients in the expansion Eq. (27) of the short-range potential in the presence of an inverse-square potential with $\nu = 0.85$.

FIG. 4: As Fig. 3 but for $\nu = 1.15$.

The scattering amplitude for this potential can be calculated as in Sec. III. The result can then be expanded the form of a DWERE as

$$p^{2\nu}\left(\cot \delta_S - \frac{1}{\pi}G(p/\mu, \nu)\right) = \frac{2}{\pi} \sum_{n=0}^{\infty} C_{2n} p^{2n}.$$  (100)

This is an expansion in powers of the energy, which is the only scale in this system. In general the coefficients have unusual, noninteger dimensions, as a result of the noninteger power of the energy on the left-hand side. For example the leading coefficient, which is the analogue of a modified scattering length, has a dimension of $2\nu$.

In the case of scattering of a particle with angular momentum $l$ by a short-range potential, we have $\nu = l + \frac{1}{2}$ and
there is no nonanalytic energy dependence in \( \cot \delta_S = \cot(\delta + l\pi/2) \). The ERE becomes

\[
p^{2l+1} \cot \left( \frac{\delta + l\pi}{2} \right) = \frac{2}{\pi} \sum_{n=0}^{\infty} C_{2n} p^{2n}.
\]

(101)

For \( l = 0 \) this is the familiar \( s \)-wave ERE given above in Eq. (3). For \( l = 1 \) we get the \( p \)-wave expansion, whose leading term is a scattering volume rather than a length.

V. DISCUSSION

Over the last few years, EFT’s have been developed for low-energy scattering of two heavy particles by short-range interactions. In particular these have been successfully applied to nucleon-nucleon scattering. The techniques described here extend the RG ideas which underly these theories to systems where the particles interact by a combination of known long-range and unknown short-range forces. These provide a framework for constructing EFT’s for such systems.

By expanding the short-range potential about a fixed point of the RG, we can systematically organise the terms in the potential according a power counting defined by their RG eigenvalues. These eigenvalues also determine whether a perturbation is stable, unstable or marginal.

A crucial feature of our approach is that we regulate the loop integrals by cutting them off in the basis of DW’s for the long-range potential. This ensures that the long-range potential is not modified by the cut-off. As a result the RG equation has a simple form, which is very similar to that found for a pure short-range potential.

Another important feature is the need to identify all low-energy scales associated with the long-range potential. Our method can be applied when the resulting rescaled potential is independent of the cut-off and so it can be treated as part of a fixed point, its effects being resummed to all orders in the DW’s. If the rescaled long-range potential diverges as the cut-off is lowered, then no fixed point can be found. On the other hand, if the rescaled potential vanishes in this limit, then it can be treated as a perturbation about a fixed point of the RG for a pure short-range potential.

As in the case of a pure short-range potential, we always find a trivial fixed point. The expansion around this point can be used to describe systems where the additional scattering by the short-range potential forms a small correction to the effect of the long-range potential. The terms in this potential correspond to an expansion of the DW \( K \)-matrix in powers of the energy and any other low-energy scales.

In some cases we also find an energy-dependent potential which forms a nontrivial fixed point of the RG. In other cases, such a fixed point would have a marginal perturbation and instead we find a potential which evolves logarithmically with the cut-off. These potentials describe systems which have bound states lying exactly at threshold and the expansions around them correspond to DW versions of the effective-range expansion.

We have applied this method to several examples. An effective field theory for the strong interaction in proton-proton scattering can be constructed by considering the Coulomb potential as the long-range distorting potential. The inverse Bohr radius then provides an additional low-energy scale. This is a case where a nontrivial fixed point would have a marginal perturbation and the RG can be used to resume the resulting logarithmic corrections. The expansion around this potential corresponds to the DWERE for the Coulomb potential, where the coefficients in the expansion in powers of the energy have themselves been expanded in powers of the inverse Bohr radius. The logarithmic evolution of the potential shows up in the fact that the purely short-range scattering is not uniquely defined, as it depends logarithmically on an arbitrary scale.

The RG analysis of the scattering in the presence of a Yukawa potential is similar to that for the Coulomb one. In the specific example of nucleon-nucleon scattering in the \( 1^S_0 \) channel, the Yukawa potential is provided by OPE. Here we clearly want to identify the pion mass as a low-energy scale if we wish to make contact with ChPT. However we have a choice about how we treat the scale

\[
\kappa_\pi = \frac{g_\pi^2 M m_\pi^2}{32\pi f_\pi^2}.
\]

(102)

In strict chiral counting this inverse “Bohr radius” is of second order in \( m_\pi \). Treating it in this way, the rescaled OPE potential vanishes linearly with the cut-off and so can be treated a perturbation in an EFT based on a fixed point of a pure short-range potential. This is is the basis for the KSW scheme for including pion-exchange forces in EFT’s for nucleon-nucleon scattering.

The alternative is to regard \( \kappa_\pi \) as a low-energy scale, analogous to the treatment of the inverse Bohr radius in the Coulomb case. The rescaled Yukawa potential is then independent of cut-off and should be resummed. This is the WvK scheme for including OPE. The EFT based on the resulting nontrivial (logarithmically evolving) potential
is equivalent to a DWERE. However, although this scheme results in a consistent EFT, the treatment of $\kappa_\pi$ as a quantity of first-order in low-energy scales means that the connection with the chiral expansion has been lost.

Our final example is the inverse-square potential which is of particular interest because of its relevance to three-body scattering. We have considered here the case of repulsive potentials, which are relevant to three-body systems such as neutron-deuteron scattering with $J = \frac{3}{2}$, and also to two-body scattering in higher partial waves. This potential is scale free and so its strength shows up in the RG eigenvalues, and hence the power counting. Although it is possible to find a nontrivial fixed point corresponding to a DWERE, the number of unstable perturbations increases with the strength of the long-range potential. This implies that an extremely delicate fine-tuning would be required to generate a bound state at threshold. In general one would expect scattering in such systems to be weak, as it is for neutron-proton scattering in high partial waves. The appropriate EFT’s for them are based on the (stable) trivial fixed point.

We are currently extending this approach to describe scattering in the presence of an attractive inverse-square potential, which is more complicated than the repulsive case since the DW’s oscillate rapidly near the origin. The resulting EFT’s will be relevant to three-body systems such as neutron-deuteron scattering with $J = \frac{1}{2}$. It will also be interesting to explore more singular potentials, such as OPE in the $^3S_1$ channel. Finally the real power of the EFT’s is their ability to form direct connections between observables for different processes. Doing this will require enlarging the present treatment to included couplings to external electromagnetic and weak currents.

Acknowledgments

We wish to thank J. McGovern for helpful discussions and a critical reading of this paper. MCB is grateful to the Institute for Nuclear Theory, Seattle for hospitality during the programme on Effective Field Theories and Effective Interactions where these ideas first took shape, to S. Beane, T. Cohen, S. Coon, H. Griesshammer, D. Phillips, U. van Kolck and others for useful discussions, and to K. Richardson for an introduction to DW effective-range expansions. This work was supported by the EPSRC.

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