For many systems in science and engineering, the governing differential equation is either not known or known in an approximate sense. Analyses and design of such systems are governed by data collected from the field and/or laboratory experiments. This challenging scenario is further worsened when data-collection is expensive and time-consuming. To address this issue, this paper presents a novel multi-fidelity physics informed deep neural network (MF-PIDNN). The framework proposed is particularly suitable when the physics of the problem is known in an approximate sense (low-fidelity physics) and only a few high-fidelity data are available. MF-PIDNN blends physics informed and data-driven deep learning techniques by using the concept of transfer learning. The approximate governing equation is first used to train a low-fidelity physics informed deep neural network. This is followed by transfer learning where the low-fidelity model is updated by using the available high-fidelity data. MF-PIDNN is able to encode useful information on the physics of the problem from the approximate governing differential equation and hence, provides accurate prediction even in zones with no data. Additionally, no low-fidelity data is required for training this model. Applicability and utility of MF-PIDNN are illustrated in solving four benchmark reliability analysis problems. Case studies to illustrate interesting features of the proposed approach are also presented.

Keywords multi-fidelity · deep learning · physics-informed · transfer learning · reliability

1 Introduction

The governing equations used in science and engineering are often based on certain assumptions and approximations [1]. For example, heterogeneous material properties are approximated as homogeneous [2], effect of environmental conditions are rarely considered [3] and critical parts such as joints are often ignored [4]. Naturally, results obtained by solving the governing equations only provide an approximation of the true system behavior (i.e., low-fidelity results). An alternative is to perform actual experiments in a laboratory environment. With modern experimental setups and sensors, it is possible to perform highly sophisticated experiments [5, 6]. Results obtained from such experiments are generally accurate (high-fidelity results). However, experiments are expensive and time-consuming, and one can only perform a limited number of experiments (usually in the order of tens). Such small number of experiments is often not sufficient for understanding the system behavior, specifically if dealing with problems such as uncertainty quantification and reliability analysis.

One possible solution to the difficulties raised above resides in multi-fidelity schemes [7, 9] where data fusion techniques are used to combine high-fidelity and low-fidelity data. The most popular multi-fidelity schemes are perhaps the multi-level Monte Carlo (MLMC) methods [10, 13]. The primary idea in MLMC is to accelerate the calculation of the second moments of the quantity of interests. Another popular approach for dealing with multi-fidelity data is co-Kriging [14, 17]. In this method, Kriging [18, 21], aka Gaussian process [22, 26], is coupled with an auto-regressive like information fusion scheme [27, 29]. Methods where the Gaussian process in co-Kriging is replaced by other
machine learning techniques can also be found in the literature \cite{30, 33}. The success of all these methods is already well-established in the literature \cite{34, 35}. Unfortunately, these methods only work for cases where the low-fidelity data is able to capture the trend and the models of different fidelities have a strong linear correlation. Both co-Kriging motivated approaches and MLMC fails when the low-fidelity and high-fidelity data have a space-dependent, complex and nonlinear correlations. To address this issue, researchers have recently proposed methods that are rooted in Bayesian statistics \cite{36} and nonlinear auto-regressive algorithm \cite{37}.

The field of artificial intelligence and machine learning has recently witnessed a huge boom \cite{38} and its influence can also be observed in the multi-fidelity approaches. De et al. \cite{39} developed two multi-fidelity approaches by using deep neural networks. While the first framework uses transfer learning, the second framework utilizes bi-fidelity weighted learning. Meng and Karniadakis \cite{40}, on the other hand, proposed a composite neural network that is trained based on multi-fidelity data. A physics aware component was also added to this network; although, the physics informed component is only used for solving inverse problems. Liu and Wang \cite{41} proposed physics constrained multi-fidelity neural networks for solving partial differential equations.

Based on the discussion above, (at least) two salient conclusions can be drawn about the existing multi-fidelity approaches.

- First, the existing multi-fidelity approaches assume the low-fidelity solver to be computationally efficient so that one can generate sufficient low-fidelity data. This is not always true. For example, compared to wind tunnel test data, a large eddy simulation \cite{42, 43} solver can be treated as a low-fidelity solver. However, computational cost associated with large eddy simulation is significant, even on modern computer clusters.
- Second, the physics informed multi-fidelity approaches proposed in \cite{41} assume that the exact physics corresponding to the high-fidelity data is known. This is not necessarily true. There are problems where the underlying physics is unknown \cite{1}. Also, the apparently known governing equations are often derived based on certain assumptions and hence, only reflect the true scenario in an approximate manner.

The objective of this paper is to present a multi-fidelity physics informed deep learning framework that addresses both the limitations discussed above. Unlike some of the previous studies, it is assumed that the data-generation process for the high-fidelity data is unknown. The low-fidelity model is given by ordinary/partial differential equations. The proposed model needs no low-fidelity data; instead, the initial low fidelity model is directly trained based on the (approximate) physics of the problem. This is achieved by utilizing the recently developed physics informed deep learning algorithm \cite{44, 45}. With this setup, important physical laws such as invariance and symmetries present in the low-fidelity model will be inherently captured by the deep learning framework. Transfer learning \cite{45} and available high-fidelity data is then used to update the trained deep learning framework. Performance of the proposed framework is illustrated on selected reliability analysis problems from the literature.

The rest of the paper is organized as follows. Section 2 provides details on the problem to be solved. Details about the proposed approach are presented in Section 3. Numerical results showcasing the performance of the proposed approach are presented in Section 4. Finally, Section 5 provides the concluding remarks.

2 Problem statement

Consider $\Xi = (\Xi_1, \Xi_2, \ldots, \Xi_N) : \Omega \to \mathbb{R}^N$ to be an $N-$dimensional stochastic vector with cumulative distribution function

$$F_\Xi(\xi) = \mathbb{P}(\Xi \leq \xi) ,$$

where $\xi$ is a realization from the random vector $\Xi$, $\mathbb{P} (\cdot)$ represents the probability measure and $\Omega$ is the input domain. In reliability analysis, one first formulates a limit-state or performance function, $\mathcal{J} (\xi) = 0$ such that $\mathcal{J} (\xi) < 0$ represents the failure domain ($\Omega_f$) and $\mathcal{J} (\xi) \geq 0$ represents the safe domain. Mathematically, this can be represented as

$$\Omega_f \triangleq \{ \xi : \mathcal{J} (\xi) < 0 \} .$$

With this consideration, the probability of failure of the system can be calculated as

$$P_f = \mathbb{P}(\Xi \in \Omega_f) = \int_{\Omega_f} dF_\Xi (\xi) = \int_{\Omega} \mathbb{I}_{\Omega_f} dF_\Xi (\xi) ,$$

where $\mathbb{I}_c$ is an indicator function,

$$\mathbb{I}_c (\xi) = \begin{cases} 1 & \text{if } \xi \in c \\ 0 & \text{if } \xi \notin c \end{cases}$$

where $c$ is the failed state of the system. The objective of this paper is to present a multi-fidelity physics informed deep learning framework that addresses both the limitations discussed above. Unlike some of the previous studies, it is assumed that the data-generation process for the high-fidelity data is unknown. The low-fidelity model is given by ordinary/partial differential equations. The proposed model needs no low-fidelity data; instead, the initial low fidelity model is directly trained based on the (approximate) physics of the problem. This is achieved by utilizing the recently developed physics informed deep learning algorithm \cite{44, 45}. With this setup, important physical laws such as invariance and symmetries present in the low-fidelity model will be inherently captured by the deep learning framework. Transfer learning \cite{45} and available high-fidelity data is then used to update the trained deep learning framework. Performance of the proposed framework is illustrated on selected reliability analysis problems from the literature.

The rest of the paper is organized as follows. Section 2 provides details on the problem to be solved. Details about the proposed approach are presented in Section 3. Numerical results showcasing the performance of the proposed approach are presented in Section 4. Finally, Section 5 provides the concluding remarks.

2 Problem statement

Consider $\Xi = (\Xi_1, \Xi_2, \ldots, \Xi_N) : \Omega \to \mathbb{R}^N$ to be an $N-$dimensional stochastic vector with cumulative distribution function

$$F_\Xi(\xi) = \mathbb{P}(\Xi \leq \xi) ,$$

where $\xi$ is a realization from the random vector $\Xi$, $\mathbb{P} (\cdot)$ represents the probability measure and $\Omega$ is the input domain. In reliability analysis, one first formulates a limit-state or performance function, $\mathcal{J} (\xi) = 0$ such that $\mathcal{J} (\xi) < 0$ represents the failure domain ($\Omega_f$) and $\mathcal{J} (\xi) \geq 0$ represents the safe domain. Mathematically, this can be represented as

$$\Omega_f \triangleq \{ \xi : \mathcal{J} (\xi) < 0 \} .$$

With this consideration, the probability of failure of the system can be calculated as

$$P_f = \mathbb{P}(\Xi \in \Omega_f) = \int_{\Omega_f} dF_\Xi (\xi) = \int_{\Omega} \mathbb{I}_{\Omega} dF_\Xi (\xi) ,$$

where $\mathbb{I}_c$ is an indicator function,

$$\mathbb{I}_c (\xi) = \begin{cases} 1 & \text{if } \xi \in c \\ 0 & \text{if } \xi \notin c \end{cases}$$

where $c$ is the failed state of the system. The objective of this paper is to present a multi-fidelity physics informed deep learning framework that addresses both the limitations discussed above. Unlike some of the previous studies, it is assumed that the data-generation process for the high-fidelity data is unknown. The low-fidelity model is given by ordinary/partial differential equations. The proposed model needs no low-fidelity data; instead, the initial low fidelity model is directly trained based on the (approximate) physics of the problem. This is achieved by utilizing the recently developed physics informed deep learning algorithm \cite{44, 45}. With this setup, important physical laws such as invariance and symmetries present in the low-fidelity model will be inherently captured by the deep learning framework. Transfer learning \cite{45} and available high-fidelity data is then used to update the trained deep learning framework. Performance of the proposed framework is illustrated on selected reliability analysis problems from the literature.
Although the mathematical formulation of reliability analysis discussed above is relatively simple, the difficulty arises due to the multivariate integral in Eq. (3) Almost all the time, there exists no closed-form solution for the multivariate integral and hence, one has to rely on numerical integration techniques or asymptotic approximations. A detailed account of different reliability analysis methods can be found in [48, 49].

Another important player in reliability analysis is the limit-state function \( J(\xi) \). For computing the probability of failure in Eq. (3) using numerical integration, one needs to evaluate the limit-state function repeatedly; the inherent assumption being, the mathematical model/equation for the limit-state function is known. In this regards, the accuracy of the limit-state function is of utmost importance. However, for many systems in science and engineering, the governing equation is either not available or available in an approximate sense [1]. Under such circumstances, one has no option but to rely on data collected either from the field or from laboratory experiments. Further assuming that the system under consideration is at its design phase, the option of collecting field data becomes invalid and performing laboratory experiments is the only feasible alternative.

Consider, \( \mathcal{D}_h = [\Xi_{hx}, u_h] \) to be the data available from laboratory experiments where

\[
\Xi_{hx} = \Xi_h \otimes x_h \otimes t_h.
\]  

Eq. (5) represents sample/data of the stochastic inputs, \( x_h = [x_1, \ldots, x_s]^T \) are the spatial locations where data is available (sensor locations) and \( t_h = [t_1, \ldots, t_n]^T \) are the time-steps are the times at which observations are available. The operator \( \otimes \) in Eq. (5) indicates Kronecker product and \( u_h = [u_1, \ldots, u_r] \), \( r = N_h \times s \times n \) represents the responses. ‘\( h \)’ in the suffix indicates that the data-collected is high-fidelity. The limit-state function \( J(\xi) \) is generally expressed in terms of the response variable \( u \) and a threshold \( u_t \)

\[
J(\xi) = g(u(\xi, x_i, t_j)) - u_t.
\]  

In case the number of data-points \( N_h \) is significant, it is possible to directly train a surrogate model, \( \mathcal{M} : (\xi, x, t) \rightarrow u \) and then use it to evaluate the probability of failure in Eq. (3). Popular surrogate models available in the literature includes Gaussian process [22, 23], polynomial chaos expansion [50, 51], analysis of variance decomposition [52, 53], support vector machine [54] and hybrid polynomial correlated function expansion [55, 56]. However, in reality, the number of laboratory experiments that can be performed is limited and hence, the number of data-points available is often not sufficient for training a surrogate model. To compensate for the fact that only a limited number of high-fidelity data is available, the approximate (low-fidelity) governing equation of the system is considered,

\[
u_t + h (u, u_x, u_{xx}, \ldots; \xi) = 0.
\]  

\( u_x \) and \( u_{xx} \) in Eq. (7) represent the first and second derivative of \( u \) with respect to \( x \). As already discussed in Section 1 solving Eq. (7) to generate sufficient number of low-fidelity data can also be computationally expensive.

The objective of this paper is to develop a multi-fidelity deep learning framework that can be directly trained by using the low-fidelity model in Eq. (7) (without generating data from it) and the high-fidelity data, \( \mathcal{D}_h \).

# 3 Multi-fidelity physics informed deep neural network

In this section, the proposed multi-fidelity physics informed deep neural network (MF-PIDNN) is presented. However, before proceeding to the proposed framework, details on data-driven and physics-informed deep neural networks are discussed. Data-driven and physics informed deep neural networks form the backbone of the proposed multi-fidelity approach.

## 3.1 Data-driven deep neural networks

One of the primary components of the proposed multi-fidelity approach is a deep neural network (DNN). In this work, a fully connected DNN (FC-DNN) is used and hence, the discussion is limited to FC-DNN only. Having said that, the framework presented is generic and can be used with convolutional [57] and other types of neural networks as well.

An FC-DNN with \( L \)–hidden layers can be represented by using a sequence of activation functions and linear transformations

\[
N(\cdot; \theta) = (\sigma_L \circ W_{L+1}) \circ \cdots \circ (\sigma_0 \circ W_1),
\]  

where \( \sigma_j : \mathbb{R} \rightarrow \mathbb{R} \) and \( W_{j+1} \) respectively represents the activation function and the weight matrix associated with the edges connecting the \( j \)–th and \( (j+1) \)–th layers. The biases of the neural network are absorbed into the weight matrix \( W_j \); the weight matrices \( \{W_{j}\}_{j=1}^{L+1} \) are the parameters of the FC-DNN and are represented using \( \theta \). ‘\( \circ \)’ in Eq. (8) represents operator composition. Note that the \( 0 \)–th layer in Eq. (8) represents the input and \( (L + 1) \)–th layer
represents the output. For using the neural network in practice, the model parameters $\theta$ needs to be estimated. In a data-driven setting, this is achieved by minimizing a loss function. For a detailed account of different loss-functions available in the literature, interested readers may refer \[38, 58\]. In this work, the mean-square loss function ($L_d$) has been used,

$$L_d = \frac{1}{N_d} \sum_{k=1}^{N_d} (u_k - \hat{u}_k)^2.$$ \hspace{1cm} (9)

In Eq. (9), $N_d$ represents the number of data-points, $u_k$ is the observed response corresponding to the $k$−th input, $\xi_k$ and $\hat{u}_k$ represents the neural network predicted response corresponding to $\xi_k$,

$$\hat{u}_k = N(\xi_k; \theta).$$ \hspace{1cm} (10)

The primary challenge behind the application of the DNN for engineering applications is the need for data. It is a well-acknowledged fact that DNNs are data-hungry tools \[36\]. Unfortunately, for the current work, the focus is on problems where one has access to very few high-fidelity data. Therefore, the direct application of data-driven DNN is unlikely to yield satisfactory results.

3.2 Physics-informed deep neural networks

To address the over-reliance of data-driven DNNs on training data, physics informed deep neural networks (PI-DNN) was proposed in \[44\]. The basic idea is to compute the DNN parameters directly from the physics (governing ODE/PDE) of the problem. Since its inception, the PI-DNN has been used for solving a wide range of problems in science and engineering \[40, 45\–47\].

Consider the governing (stochastic) differential equation in Eq. (7). The objective is to solve the stochastic differential equation so as to build a mapping from the input space (stochastic, spatial and temporal inputs) to the response space. In conventional data-driven DNN, this is achieved in three simple steps

- **Generate training data** $D = \{\Xi_{c,i}, u_i\}_{i=1}^{N_r}$, where

$$\Xi_c = \xi_{1:N_s} \otimes x_{1:N_x} \otimes t_{1:N_t},$$ \hspace{1cm} (11)

and

$$N_r = N_s \times N_x \times N_t.$$ \hspace{1cm} (12)

- **Represent the output $y$ using DNN,**

$$u = N(\xi, x, t; \theta).$$ \hspace{1cm} (13)

- **Compute the DNN parameters $\theta$ by minimizing the loss-function in Eq. (9),**

$$\theta^* = \arg \min_{\theta} L_d(\theta).$$ \hspace{1cm} (14)

In PI-DNN, the objective is to remove the data-generation step and compute the DNN parameters $\theta$ directly from the governing differential equation in Eq. (7). The objective is to solve the stochastic differential equation so as to build a mapping from the input space (stochastic, spatial and temporal inputs) to the response space. Following the method presented in \[46\], this is achieved in four simple steps. First, similar to the data-driven case, the response $u$ is represented by using a DNN,

$$u \approx u_{NN}(\xi, x, t; \theta).$$ \hspace{1cm} (15)

Second, the neural network outputs are modified so as to automatically satisfy the initial and Dirichlet boundary conditions.

$$\hat{\xi}(\xi, x, t) = u_b(x_b, t_i) + B \cdot u_{NN}(x, t, \xi),$$ \hspace{1cm} (16)

where the function $B$ is defined in such a way that $B = 0$ at the boundary $(x_b)$ and initial $(t_i)$ points. The function $u_b(x_b, t_i)$ is defined based on the initial and boundary conditions. More details on this can be found in \[45\–46\]. Note that $\hat{\xi}(\xi, x, t)$ can also be viewed as a DNN, $\hat{\xi}(\xi, x, t; \theta)$.

In the third step, collocation points for the inputs, $D_c = \{\xi_k, x_k, t_k\}_{k=1}^{N_c}$ are generated by using some suitable design of experiment scheme \[52\–59\]. Using the collocation points, the physics-informed loss function is formulated as

$$L_p(\theta) = \frac{1}{N_c} \sum_{i=1}^{N_c} R_i^2,$$ \hspace{1cm} (17)
where $N_c$ is the number of collocation points and $R_i$ is the residual of the governing differential equation corresponding to the $i$–th collocation point,

$$R_i = \hat{u}_t(i) + h ((\hat{u})_i, (\hat{u}_x)_i, (\hat{u}_{xx})_i, \ldots ; \xi_i).$$

(\hat{u})_i in Eq. (18) is obtained by substituting the $i$–th collocation point into Eq. (16). (\hat{u}_t)_i, (\hat{u}_x)_i, (\hat{u}_{xx})_i are obtained by using automatic differentiation (AD) [60].

\[
\begin{align*}
\hat{u}_t &= \frac{\partial \hat{u}}{\partial t} = \hat{N}_t(\xi, x, t; \theta), \\
\hat{u}_x &= \frac{\partial \hat{u}}{\partial x} = \hat{N}_x(\xi, x, t; \theta), \\
\hat{u}_{xx} &= \frac{\partial^2 \hat{u}}{\partial x^2} = \hat{N}_{xx}(\xi, x, t; \theta).
\end{align*}
\]

Note that the derivatives in Eq. (19) are also DNN. Since the DNNs in Eq. (19) are obtained by differentiating Eq. (16), they have the same architecture and same parameters; the only difference is in the form of the activation function.

In the fourth and final step, the loss function in Eq. (17) is minimized to compute the parameters of the DNN,

$$\theta^* = \arg \min_\theta \mathcal{L}_p(\theta).$$

For further details on PI-DNN and its application in solving reliability analysis problems, interested readers may refer [46].

PI-DNN has two major advantages. First, unlike other reliability analysis tools including data-driven DNN, PI-DNN needs no simulation data. This is expected to reduce the computational cost significantly. Second, PI-DNN is trained by satisfying the governing differential equation of the system. Therefore, physical properties such as invariance and symmetries are satisfied. However, despite these advantages, the whole idea of PI-DNN is hinged on the fact that the exact governing differential equation for the system under consideration is available. Unfortunately, this is not necessarily true. There exists a number of scenarios in science and engineering where the governing differential equation is not known [1]. Even if the governing equation is known, it is often based on certain assumptions and approximations. In other words, the governing differential equation only represents the reality in an approximate manner. Under such circumstances, results obtained using PI-DNN are bound to be erroneous.

### 3.3 Proposed approach

Neither the data-driven DNN in Section 3.1 nor the PI-DNN presented in Section 3.2 is capable of solving the reliability analysis problem defined in Section 2. The data-driven DNN fails because the number of high-fidelity data available, $N_h$, is very less. On the other hand, the PI-DNN fails as the governing differential equation in Eq. (7) only represents the actual scenario in an approximate manner. To solve the problem defined in Section 3, a multi-fidelity physics informed deep neural network (MF-PIDNN) is presented in this section. MF-PIDNN utilizes the concepts of both data-driven and physics informed DNNs. Unlike available multi-fidelity frameworks, the proposed MF-PIDNN does not assume that generating low-fidelity data is trivial. In fact, no low-fidelity data is needed for the MF-PIDNN presented here.

The key consideration of any multi-fidelity framework is associated with discovering and exploiting the relation between the low-fidelity and high-fidelity model/data. In most of the frameworks available in the literature, this is achieved by using two surrogates; the first surrogate is trained based on the low-fidelity data and the second surrogate is used to find the functional relation between the low-fidelity and the high-fidelity data. This paper takes a separate route; instead of using two DNNs, a single DNN is first trained for the low-fidelity model and then updated based on the high-fidelity data. For updating the DNN, the concept of transfer learning is used in this study. Note that the idea of using transfer learning in a multi-fidelity framework has previously been exploited in [39]. However, unlike the proposed framework, the algorithm presented in [39] is purely data-driven in nature.

MF-PIDNN solves the problem defined in Section 2 in two simple steps. In the first step, PI-DNN is used to solve the low-fidelity model. To that end, the exact procedure as discussed in Section 3.2 is followed. In the second step, the low-fidelity PI-DNN is updated based on the high-fidelity data $D_{h}$. This is achieved by utilizing the concept of data-driven DNN. However, unlike the first step, the second step is not straightforward. More specifically, two specific factors are considered in this step. First, the training algorithm starts by setting the initial value of the neural network parameters to those obtained in step 1. Second, the parameters corresponding to all the layers are not updated. Instead, the concept of transfer learning is used and the parameters corresponding to only the last one or two layers are updated. A schematic representation of MF-PIDNN is shown in Fig. 1. The advantage of transfer learning is three-folds.
Fig. 1: Schematic representation of the proposed MF-PIDNN. During the low-fidelity training phase in (a), the DNN has three building blocks. The physics induced DNN architecture is governed by the low-fidelity governing differential equation of the system. At this stage, the DNN parameters are tuned by using the collocation points $D_c$ and minimizing the residual, $(R_i)$ (physics-informed loss). $\theta_l$ (yellow box) indicates that the DNN parameters obtained at the end of the training phase. During the high-fidelity training phase in (b), the DNN parameters for all but the last one or two layers are fixed at $\theta_l$ (yellow box). The tunable parameters $\theta_t$ (green box) are estimated by minimizing the mean-squared error computed using the high-fidelity data $D_h$.

- First, because of transfer learning, the number of parameters to be updated is reduced. This in turn, accelerates training of the DNN.
- Second, freezing the parameters of the initial layer ensures that the features learned/extracted from the low-fidelity model are retained in the network.
- Thirdly, transfer learning also ensures that the DNN does not overfit the high-fidelity data, $D_h$.

The steps involved in the proposed MF-PIDNN are shown in algorithm 1. For training the MF-PIDNN, RMSProp optimizer [61] followed by L-BFGS algorithm is used. Xavier initialization is used for initializing the DNN parameters. Details on the parameters settings for the optimizers are provided in Section 4. Once the MF-PIDNN is trained, it is possible to predict $u$ corresponding to some unknown inputs by using Eq. (16).

4 Numerical illustration

In this section, four numerical examples are presented to illustrate the performance of the proposed approach. A wide variety of examples involving single and multiple stochastic variables, linear and non-linear problems, ordinary and partial differential equations are selected. For illustrating the performance of the proposed approach, benchmark
Minimize the loss-function to tune the tunable parameters

Formulate data-driven loss function using

Freeze the DNN parameters for initial training the low-fidelity model.

Utilize the low-fidelity physics to formulate a physics-informed loss function;

Minimize the physics-informed loss function to compute the network parameters, \( \theta \);

Freeze the DNN parameters for initial \((L - l_t + 1)\) layers.

Formulate data-driven loss function using \(D_h\);

Minimize the loss-function to tune the tunable parameters

\[
\theta_t = \arg\min_{\theta_t} \mathcal{L}_d(\theta_t),
\]

where \(\theta_t\) represents the tunable parameters.

results using Monte Carlo simulation (MCS)\([62]\) are generated. The software accompanying the proposed approach is developed using TensorFlow\([63]\). For examples 1, 2 and 4, the benchmark results are generated using MATLAB\([64]\). Benchmark results for example 2 are generated using the FeNICS package\([65]\).

### 4.1 An ordinary differential equation

As the first example, a benchmark stochastic ordinary differential equation previously studied in\([66]\) is considered. The low-fidelity model for this problem is given by the following stochastic ordinary differential equation,

\[
\frac{du_t}{dt} = -zu_t,
\]

where \(Z\) is the stochastic variable. The differential equation in \(\text{Eq. (21)}\) is subjected to the following initial condition,

\[
u_t(t = 0) = 1.0. \tag{22}\]

The high-fidelity model, on the other hand, is represented as

\[
u_h = t \sin(t) \left[ \log (u_t^4) \right]^2 + 15t^3 + 1.0 \tag{23}\]

Clearly, the relation between the high-fidelity \(u_h\) and the low-fidelity \(u_t\) is non-linear. The limit-state function for this problem is defined as

\[
\mathcal{J}(Z, t_t) = u_h(Z, t_t) - u_0, \tag{24}\]

where \(u_0\) is the threshold, \(u_h(Z, t_t)\) is the response and \(t_t\) is the time at which the probability of failure is to be estimated. For this example, \(t_t = 1.0\) and \(u_0 = 18.0\) is considered. It is assumed that 15 samples from the high-fidelity model is available, and for each of the 15 high-fidelity samples, the observations are available at \(t = [0.0, 1.0]\). Note that the data-generation process, i.e., \(\text{Eq. (23)}\) is not known. MF-PIDNN only have access to the high-fidelity data and the low-fidelity model in \(\text{Eq. (21)}\). For this particular problem, the stochastic variable \(Z \sim \mathcal{N}(\mu, \sigma^2)\) is considered to follow Gaussian distribution with mean \(\mu = -2.0\) and standard deviation \(\sigma = 1.0\). MCS with \(10^6\) simulations yields a probability of failure of 0.045.

For solving the problem using the proposed approach, the unknown response \(u\) is first represented by using an FC-DNN with 2 inputs, 5 hidden layers and 50 neurons per hidden layer. The 2 inputs to the DNN are time \(t\) and decay parameter \(Z\). Hyperbolic tangent (\(\tanh\)) activation function is considered for all but the last layer. For the last layer, linear activation function is considered. The initial conditions in \(\text{Eq. (22)}\) is automatically satisfied by modifying the DNN output, \(u_{NN}\) using \(\text{Eq. (16)}\) where \(u_0 = 1.0\) and \(B = t\),

\[
\hat{u} = t \cdot u_{NN} + 1.0. \tag{25}\]

The residual for training the low-fidelity DNN is

\[
R_i = \frac{d\hat{u}_i}{dt} + Z_i \hat{u}_i, \tag{26}\]

where \(R_i\) is the residual and \(\hat{u}_i\) is obtained from \(\text{Eq. (25)}\), ‘\(i\)’ in the suffix indicates the \(i\)-th collocation point. For training the low-fidelity model, 8000 collocation points is used and the RMSprop optimizer is run for 15, 000 iterations.
A learning rate of 0.001 is used. The other parameters of RMSprop are kept at their default values. The maximum allowable iterations for L-BFGS optimizer is set to be 10,000.

After training the physics-informed low-fidelity DNN, the next step is to update the model based on the high-fidelity data by using the transfer learning. The parameters corresponding to the last two layers are only updated; parameters corresponding to all other layers are kept fixed. The RMSprop optimizer is run for 10,000 iterations and maximum allowable iterations for the L-BFGS optimizer is set to be 10,000. For RMSprop optimizer, a learning rate of 0.001 is used.

Table 1 shows the results obtained using MCS and MF-PIDNN. Along with the probability of failure \( P_f \), the reliability index \( \beta \) for this problem is also reported.

\[
\beta = \Phi^{-1} (1 - P_f),
\]

where \( \Phi(\cdot) \) represents cumulative distribution function of standard Gaussian distribution. The results obtained using MF-PIDNN matches exactly with the MCS results. To show the utility of the proposed approach, results obtained using only the low-fidelity PI-DNN and the high-fidelity DNN are also presented. Both low-fidelity PI-DNN and high-fidelity DNN are found to yield erroneous results.

| Methods  | \( P_f \)  | \( \beta \)  | \( N_h \)  | \( N_r \)  | \( \epsilon = \frac{|P_f - \beta|}{\beta} \times 100 \) |
|----------|------------|------------|------------|------------|---------------------------------|
| MCS      | 0.045      | 1.6954     | 10\(^9\)   | 10\(^6\) \times 1001 | ~                               |
| LF-DNN   | 0.8133     | -0.8901    | 0          | 0          | 152.5%                          |
| HF-DNN   | 0.0        | \( \infty \) | 15         | 30(15 \times 2) | \( \infty \)                    |
| MF-PIDNN | 0.045      | 1.6954     | 15         | 30(15 \times 2) | 0.0%                            |

To further illustrate the performance of the MF-PIDNN, two additional case studies are performed. In the first case study, the performance of the MF-PIDNN in predicting future reliability is investigated. To that end, it is assumed that for each of the 15 high-fidelity samples, observations are available \( t = [0.0, 0.5, 0.9] \), and the objective is to compute the reliability of the system at \( t = 1.0 \). The difficulty, in this case, arises from the fact that this is an extrapolation problem as no observation is available at or beyond \( t = 1.0 \). The network architecture and other parameters of MF-PIDNN are considered to be same as before; the only difference resides in the fact that the RMSProp optimizer is run for 15,000 iterations (while updating the network using transfer learning). The results obtained are shown in Table 2. Compared to the results presented in Table 1, slight deterioration in the results have been observed; this is expected because this is an extrapolation problem. Nonetheless, the results obtained are still significantly more accurate as compared to HF-DNN and LF-PIDNN.

Table 2: Reliability analysis results for example 1. The results presented illustrate the extrapolation capability of the MF-PIDNN.

| Methods  | \( P_f \)  | \( \beta \)  | \( N_h \)  | \( N_r \)  | \( \epsilon = \frac{|P_f - \beta|}{\beta} \times 100 \) |
|----------|------------|------------|------------|------------|---------------------------------|
| MCS      | 0.045      | 1.6954     | 10\(^9\)   | 10\(^6\) \times 1001 | ~                               |
| LF-DNN   | 0.8133     | -0.8901    | 0          | 0          | 152.5%                          |
| HF-DNN   | 0.014      | 2.9173     | 15         | 45(15 \times 3) | 29.60%                          |
| MF-PIDNN | 0.05       | 1.6449     | 15         | 45(15 \times 3) | 2.98%                           |

Finally, the performance of the MF-PIDNN with variation in the number of high-fidelity data point, \( N_h \) is investigated. For each realization of \( Z \), the responses are observed at \( t = [0.0, 1.0] \) and the probability of failure at \( t=1.0 \) is computed. The variation of the MF-PIDNN predicted probability of failure is shown in Fig. 2. The benchmark result obtained using MCS is also reported. With an increase in \( N_h \), the MF-PIDNN predicted probability of failure converges to the MCS solution. The HF-DNN results, up to \( N_h = 20 \), yields erroneous results (not shown in Fig. 2). This is because, with only observations at two time-instants, the DNN fails to predict the trend of the limit-state function. MF-PIDNN, on the other hand, learns the trend from the physics of the problem and then update itself based on the high-fidelity data.

### 4.2 Burger’s equation

As the second problem, the well-known Burger’s equation is considered. The high-fidelity model for this problem is

\[
\left( u_h \right)_t + u_h \left( u_h \right)_x = \nu \left( u_h \right)_x x,
\]

(28)
with $x \in [-1, 1]$ and $t \in [0, 12]$. $\nu > 0$ in \textit{Eq. (28)} represents the viscosity. The boundary and the initial conditions for this problem are
\begin{equation}
    u_h(t, x = -1) = 1 + \delta 
    \quad \quad u_h(t, x = 1) = -1,
\end{equation}
\begin{equation}
    u_h(t = 0, x) = -1 + (1 + x) \left( 1 + \frac{\delta}{2} \right).
\end{equation}
$\delta$ in \textit{Eq. (29)} is a small perturbation that is applied to the left boundary. The problem as defined has a transition layer at $z$, so that $u_h(z) = 0$. As illustrated in previous studies \cite{67, 68}, the transition layer is super sensitive to $\delta$. Details on different aspects of this problem can be found in \cite{67, 68}.

The low-fidelity model, on the other hand, is considered to be
\begin{equation}
    (u_l)_t = \nu (u_l)_{xx}.
\end{equation}
\textit{Eq. (30)} is obtained by ignoring the nonlinear term in the high-fidelity model. The initial and the boundary conditions are considered to be same as the high-fidelity model. The boundary perturbation $\delta \sim \mathcal{U}(0, 0.1)$ is uniformly distributed between 0.0 and 0.1. It is considered that the high-fidelity model in \textit{Eq. (23)} is not known; instead, data corresponding to five realizations of $\delta$ is available. For each $\delta$, observations at 3 spatial location and 8 temporal locations are available.

$$\Xi_{hx} = [0, 0.025, 0.05, 0.075, 0.1] \otimes [-1, 0, 1] \otimes [1, 2.14, 3.29, 4.43, 5.57, 6.71, 7.86, 9].$$

The limit-state function for this problem is represented as
\begin{equation}
    J(\delta, t_i) = -z(\delta, t) + z_0,
\end{equation}
where $z$ represents the transition layer, $t_i$ is the time at which the reliability is to be computed and $z_0$ is the threshold. For this example, $z_0 = 0.40$ is considered. The objective is to compute the probability of failure at $t_i = 10$. Note that solution of this problem involves extrapolation as no observation at $t = 10$ or beyond is available.

For solving the problem using the proposed MF-PIDNN, $u$ is first represented by using a FC-DNN with 6 hidden layers. Each of the 6 hidden layers has 50 neurons. The DNN has 3 inputs, $x$, $t$ and $\delta$ and one output $u_{NN}$. $\tanh$ activation function is considered for all but the last layer. For the last layer, linear activation function is used. To automatically satisfy the boundary and initial conditions, the DNN output is modified as
\begin{equation}
    \hat{u} = u_h(t = 0, x) + t(1 - x)(1 + x)u_{NN},
\end{equation}
where $u_h(t = 0, x)$ is obtained from \textit{Eq. (29b)} Using $\hat{u}$ and its derivatives, the residual of the low-fidelity model is formulated as
\begin{equation}
    R_i = (\tilde{u}_i)(t) - \nu(\tilde{u}_i)_{xx},
\end{equation}
where $R_i$ is the residual. $i$ in the suffix indicates that the quantities are evaluated corresponding to the $i$–th collocation point. The physics-informed loss-function for training the low-fidelity model is formulated by using 30,000
Because of the simplicity of the low-fidelity model, the RMSProp optimizer is run for 500 iterations and the maximum allowable iterations for the L-BFGS optimizer is set to be 1000. The learning rate in RMSProp optimizer is set to be 0.001. Once the low-fidelity physics informed DNN is trained, the next step is to update the DNN model by using transfer learning. To retain information gained from the low-fidelity model and avoid over-fitting, parameters corresponding to only the last two layers of the DNN are allowed to update; all the other parameters are frozen. The RMSprop optimizer is run for 6000 iterations with a learning rate of 0.003. The maximum allowed iterations for the L-BFGS algorithm is set to 10,000. The L-BFGS optimizer is only allowed to update the DNN parameters corresponding to the last layer. For this problem, the MF-PIDNN is found to be highly sensitive to the initial point of the parameters and varies from run to run. Therefore, the MF-PIDNN results presented are mean predictions after running the model for 20 times.

For the purpose of validation, benchmark results using MCS with $10^4$ simulations are generated. To that end, finite element package FeNICS [65] is used. The same-solver is used for generating the high-fidelity data as well.

The reliability analysis results are shown in Table 3. Along with MCS and MF-PIDNN results, LF-PIDNN and HF-DNN predicted results are also presented. Similar to the previous example, both probability of failure and reliability index are reported. It is observed that MF-PIDNN predicted results are extremely close to the MCS results. HF-DNN and LF-PIDNN, on the other hand, yields erroneous results. Fig. 3 shows the performance of MF-PIDNN with increase in $N_t$ (i.e, number of time-steps at which high-fidelity data is available). It is observed that with an increase in $N_t$, the MF-PIDNN predicted result moves closer to the MCS results. However, at $N_t = 6$ and 8, the probability of failure obtained is found to be similar, indicating convergence of the proposed approach.

| Methods    | $P_f$  | $\beta$ | $N_{th}$ | $N_r$       | $\epsilon = |\beta - \beta_e| \times 100$ |
|------------|--------|---------|----------|-------------|---------------------------------|
| MCS        | 0.2036 | 0.8288  | $10^4$   | $10^4 \times 33 \times 10^3$ | -                               |
| LF-PIDNN   | 0      | $\infty$| 0        | 0           | $\infty$                        |
| HF-DNN     | 0.932  | -1.4909 | 5        | 120($5 \times 3 \times 8$)     | 280%                            |
| MF-PIDNN   | 0.2242 | 0.7581  | 5        | 120($5 \times 3 \times 8$)     | 8.5304%                         |

Fig. 3: Variation of MF-PIDNN predicted results with $N_t$.

4.3 Nonlinear oscillator

As the third example, a nonlinear oscillator, previously studied in [69] has been considered. The high-fidelity model for this problem is given as

$$\frac{d(x_h)_1}{dt} = (x_h)_2,$$
$$\frac{d(x_h)_2}{dt} = -\alpha_1(x_h)_2 - \alpha_2 \sin((x_h)_1),$$
where $\alpha_1$ and $\alpha_2$ are the stochastic parameter. The initial conditions for the problem are

$$
(x_h)_1(t = 0) = -1.193, \quad (x_h)_2(t = 0) = -3.876.
$$

The low-fidelity model, on the other hand, is given as

$$
\frac{d(x_l)_1}{dt} = (x_l)_2, \\
\frac{d(x_l)_2}{dt} = -\alpha_1(x_l)_1 - \alpha_2(x_h)_1.
$$

The initial condition for the low-fidelity model is considered to be the same as the high-fidelity model. Similar to the previous examples, the high-fidelity equation is assumed to be unknown and only has access to the low-fidelity model and high-fidelity data. Specifically, data corresponding to five realizations of the stochastic parameters $\alpha_1$, and $\alpha_2$ are considered to be uniformly distributed. The limit-state function for this problem is defined as

$$
\mathcal{J}(\alpha_1, \alpha_2, t_i) = -|x_2(\alpha_1, \alpha_2, t_i)| + x_0,
$$

where $x_0$ is the threshold and $t_i$ is the time at which the reliability is to be evaluated. For this problem, $t_i = 5.0$ and $x_0 = 4.0$ is considered.

To solve the problem using MF-PIDNN, $x_i$, $i = 1,2$ is first represented using a FC-DNN having 4 hidden layers. Each hidden layer has 50 neurons. The DNN has three inputs, $t$, $\alpha_1$ and $\alpha_2$ and two outputs $x_1$ and $x_2$. The activation function for all but the last layer. Linear activation function is used for the last layer. To automatically generate the low-fidelity model is formulated as

$$
\hat{x}_1 = t \cdot x_{NN,1} - 1.193, \\
\hat{x}_2 = t \cdot x_{NN,2} - 3.876,
$$

where $x_{NN,1}$ and $x_{NN,2}$ are the DNN outputs. Using $\hat{x}_1$ and $\hat{x}_2$, the residuals are computed,

$$
R_{1,i} = ((\hat{x}_1)_i - (\hat{x}_2)_i), \\
R_{2,i} = ((\hat{x}_2)_i + (\alpha_1)_i(\hat{x}_2)_i + (\alpha_2)_i(\hat{x}_1)_i).
$$

$i$ in Eq. (40) indicates the $i$-th collocation point. Using the residuals, the physics-informed loss function for training the low-fidelity model is formulated as

$$
L_p(\theta_i) = \frac{1}{N_c} \sum_{i=1}^{N_c} \left( R_{1,i}^2 + R_{2,i}^2 \right),
$$

where $N_c$ is the number of collocation points. For this problem, 10000 collocation points have been used. The RMSProp optimizer is run for 15000 iterations with a learning-rate of 0.001. For L-BFGS optimizer, the maximum allowable iterations is set to 10000. The trained low-fidelity model is then updated using transfer learning and high-fidelity data. Only the parameters corresponding to the last two layers of DNN are allowed to be updated. A learning rate of 0.001 is used and the RMSProp optimizer is run for 10000 iterations. Maximum allowable iterations for the L-BFGS optimizer is set to 10000.

The benchmark results for validation are generated using MCS with $10^4$ simulations. To that end, the differential equations are solved using the ODE45 routine in MATLAB [64]. The high-fidelity data-set discussed earlier was also generated by using the same solver.

Table 4 shows the reliability analysis results for the nonlinear oscillator problem. Similar to the previous examples, results obtained using HF-DNN and LF-PIDNN are also presented. The MF-PIDNN is found to yield highly accurate results, matching closely with the MCS results. The variation of probability of failure with threshold $x_0$ is shown in Fig. 4. Corresponding to all the thresholds, the MF-PIDNN predicted results matches closely with the MCS results. This indicates that the proposed MF-PIDNN is able to capture the response over the whole domain.

Lastly, to illustrate the robustness of the proposed MF-PIDNN, the same model is used to compute the probability of failure at a $t_i = 3.0$ and $x_0 = 2.0$. The results obtained are shown in Table 5. In this case also, MF-PIDNN is found to yield accurate results outperforming both HF-DNN and LF-PIDNN.
Table 4: Reliability analysis results for nonlinear oscillator.

| Methods     | $P_f$  | $\beta$ | $N_h$ | $N_r$  | $\epsilon = \frac{|\beta - \beta_e|}{\beta_e} \times 100$ |
|-------------|--------|---------|-------|-------|--------------------------------------------------|
| MCS         | 0.1599 | 0.9949  | 10000 | $10^4 \times 10^3$ | - |
| LF-PIDNN    | 0.27   | 0.6128  | 0     | 0     | 38.41%                                           |
| HF-DNN      | 0.19   | 0.8779  | 5     | $5 \times 5$ | 11.76%                                           |
| MF-PIDNN    | 0.1576 | 1.0044  | 5     | $5 \times 5$ | 0.95%                                            |

Fig. 4: Variation of probability of failure with threshold $x_0$.

Table 5: Reliability analysis results for nonlinear oscillator at $t = 3.0$ and $x_0 = 2.0$

| Methods     | $P_f$  | $\beta$ | $N_h$ | $N_r$  | $\epsilon = \frac{|\beta - \beta_e|}{\beta_e} \times 100$ |
|-------------|--------|---------|-------|-------|--------------------------------------------------|
| MCS         | 0.0651 | 1.5133  | 10000 | $10^4 \times 10^3$ | - |
| LF-PIDNN    | 0.98   | -2.0537 | 0     | 0     | 235.7%                                           |
| HF-DNN      | 0.5955 | -0.2417 | 5     | $5 \times 5$ | 115.97%                                           |
| MF-PIDNN    | 0.0729 | 1.4545  | 5     | $5 \times 5$ | 3.88%                                            |

4.4 Cell signaling cascade

As the last example, a mathematical model of autocrine cell-signaling cascade is considered.

\[
\begin{align*}
\frac{de_{1p}}{dt} &= \frac{I}{1 + G_4 e_{3p}} \frac{V_{max,1}(1 - e_{1p})}{K_{m,1} + (1 - e_{1p})} - \frac{V_{max,2} e_{1p}}{K_{m,2} + e_{1p}}, \\
\frac{de_{2p}}{dt} &= \frac{V_{max,3} e_{1p}(1 - e_{2p})}{K_{m,3} + (1 - e_{2p})} - \frac{V_{max,4} e_{2p}}{K_{m,4} + e_{2p}}, \\
\frac{de_{3p}}{dt} &= \frac{V_{max,5} e_{2p}(1 - e_{3p})}{K_{m,5} + (1 - e_{3p})} - \frac{V_{max,6} e_{3p}}{K_{m,6} + e_{3p}}, \quad t \in [0, 10],
\end{align*}
\]

where $e_{1p}$, $e_{2p}$, and $e_{3p}$ are the state variables and denotes concentrations of the active form of enzymes. $I$ in Eq. (42) is the tuning parameter. The initial conditions for this problem are

\begin{align*}
e_{1p}(t = 0) &= 0, \quad e_{2p}(t = 0) = 1.0, \quad e_{3p}(t = 0) = 0.
\end{align*}

This model was first developed in [71]. Overall the model has 13 parameters, $K_{m,1:6}$, $V_{max,1:6}$ and $G_4$. For biological meaning and other details on the model parameters, interested readers may refer [71].

For reliability analysis, all the 13 parameters defined above are considered to be stochastic. The mean of the parameters are adopted from [71] and a 10% relative noise is added. For clarity of readers, the mean of the 13 parameters is presented in Table 6. The same parameter settings have previously been used in [69].

A low-fidelity model for this problem is set up by considering $I = 0$. With this, the coupled differential equations in Eq. (42) are decoupled and it becomes possible to solve the equations sequentially. Moreover, the stochastic variables
\[ G_4, V_{\text{max},1} \text{ and } K_{m,1} \text{ become inactive. This further complicates the problem. It is further assumed that the governing differential equation in Eq. (42) is not available; instead, responses corresponding to 10 realizations of the stochastic variables are available. For each of the 10 realizations, observations are available at 5 time-steps. The observation time-instants are equally spaced in [4, 7].}

The limit-state function for this problem is
\[ \mathcal{J}(\xi) = e_{3p}(\xi, t_i) - e_{3,0}, \] (44)

where \( \xi \in \mathbb{R}^{13} \) represents the stochastic variables, \( e_{3,0} \) is the threshold parameter and \( t_i \) is the time-instants at which the reliability is to be estimated. For this problem, \( t_i = 3.0 \) is considered. Since all the high-fidelity observation are available in [4, 7], this is an extrapolation problem.

For reliability analysis using MF-PIDNN, the output responses are first represented by using a FC-DNN. The DNN has 14 inputs (13 stochastic variables and time), 3 outputs and 4 hidden layers. Each of the hidden layers has 100 neurons. All but the last layer of the DNN have \text{tanh} \ activation function. For the last layer, linear activation function is used. To automatically satisfy the initial conditions, the DNN outputs are modified as

\[
\begin{align*}
\hat{e}_{1p} &= t \cdot e_{1p,NN} \\
\hat{e}_{2p} &= t \cdot e_{2p,NN} + 1.0, \\
\hat{e}_{3p} &= t \cdot e_{3p,NN},
\end{align*}
\] (45)

where \( e_{1p,NN}, e_{2p,NN} \text{ and } e_{3p,NN} \) are the DNN outputs. The residuals for formulating the physics-informed loss function are given as

\[
\begin{align*}
R_{1,i} &= ((K_{m,2})_i + (\hat{e}_{1p})_i) ((\hat{e}_{1p})_i + (V_{\text{max},2})_i (\hat{e}_{1p})_i, \\
R_{2,i} &= ((K_{m,4})_i + (\hat{e}_{2p})_i) ((K_{m,3})_i + (1 - (\hat{e}_{2p})_i)) - (V_{\text{max},3})_i (\hat{e}_{1p})_i (1 - (\hat{e}_{2p})_i), \\
R_{3,i} &= ((K_{m,6})_i + (\hat{e}_{3p})_i) ((K_{m,5})_i + (1 - (\hat{e}_{3p})_i)) - (V_{\text{max},5})_i (\hat{e}_{2p})_i (1 - (\hat{e}_{3p})_i),
\end{align*}
\] (46)

where ‘i’ in suffix represents the \( i \)-th collocation point. The residuals in Eq. (46) corresponds to the low-fidelity model and hence, \( I, G_4 \text{ and } V_{\text{max},1} \) are not present. Using the residuals, the physics-informed loss function for the low-fidelity model is computed as

\[ L_p(\Theta_i) = \frac{1}{N_c} \sum_{i=1}^{N_c} \sum_{k=1}^{3} R_{k,i}^2, \] (47)

where \( N_c \) represents the number of collocation points. For minimizing \( L_p(\Theta) \), the RMSProp optimizer is run for 5000 iterations. A learning rate of 0.001 is used. As for the L-BFGS optimizer, the maximum allowed iterations is set to 10000. The trained low-fidelity model is then updated by using the high-fidelity data and transfer learning. At this stage, only the parameters corresponding to the last layer is allowed to be tuned. All the other parameters are fixed at \( \Theta_i \). A learning rate of 0.001 is used and the RMSProp optimizer is run for 5000 iterations. As for the L-BFGS optimizer, the maximum allowed iterations is set to 10000.

The benchmark results for this problem are generated by using MCS with \( 10^4 \) simulations. To that end, the \texttt{ode45} routine available in \texttt{MATLAB} is used. The high-fidelity data discussed before were also generated by using the same procedure.

Table 7 shows the reliability analysis results for the cell signaling cascade problem. Along with MCS and MF-PIDNN, results obtained using LF-PIDNN and HF-DNN are also presented. The proposed MF-PIDNN is found to yield highly accurate results with a prediction error of 1.52%. Results obtained using LF-PIDNN and HF-DNN respectively have an error of 64.35% and 98.04%. The variation of probability of failure with the change in threshold \( e_{3,0} \) is shown in Fig. 5. For all the thresholds, MF-PIDNN predicted results are found to closely match with the MCS results. This indicates that MF-PIDNN is able to capture the response over the whole domain.

Finally, the trained MF-PIDNN is used to compute the probability of failure at different time instants. The corresponding results are illustrated in Fig. 6. To be specific, probability of failures around \( t = 3, 5, 7 \text{ and } 9 \) are presented. The threshold \( e_{3,0} \) for the four cases are set to be 0.40, 0.575, 0.70 and 0.78. MF-PIDNN for all the four cases is found to yield reasonably accurate results. Do note that high-fidelity data was only available at five equidistant time-instants.
Table 7: Reliability analysis results for cell signaling cascade problem

| Methods     | $P_f$  | $\beta$ | $N_h$   | $N_r$   | $\epsilon = \frac{|\beta - \beta_e|}{\beta_e} \times 100$ |
|-------------|--------|---------|---------|---------|-------------------------------------------------|
| MCS         | 0.1663 | 0.9689  | 10000   | $10^4 \times 10^3$ | 64.35%                                          |
| LF-PIDNN    | 0.3649 | 0.3454  | 0       | 0       | 64.35%                                          |
| HF-DNN      | 0.0275 | 1.9189  | 10      | $10^5 \times 5$ | 98.04%                                          |
| MF-PIDNN    | 0.17   | 0.9542  | 10      | $10^5 \times 5$ | 1.52%                                           |

Fig. 5: Variation of the probability of failure with threshold $e_{3,0}$.

between $t = 4.0$ and $t = 7.0$. The fact that the proposed approach yields reasonable results outside this domain illustrates the extrapolability of the proposed approach. This capability of the MF-PIDNN is because of the fact that some physics is learnt (and retained) from the low-fidelity data.

5 Conclusions

In this paper, a multi-fidelity physics informed deep neural network (MF-PIDNN) is presented. The proposed approach is ideally suited for problems where the physics of the problem is known in an approximate sense (low-fidelity physics) and only a few high-fidelity data is available. MF-PIDNN blends the concepts of physics-informed and data-driven deep learning; the primary idea is to first train a low-fidelity deep learning model based on the available approximate physics and then use transfer learning to update the model based on the high-fidelity data. With this, MF-PIDNN is able to extract useful information from both the low-fidelity physics and high-fidelity data. There are two distinct advantages of MF-PIDNN. First, the low-fidelity model is directly trained from the physics of the problem and hence, no low-fidelity data is needed in this framework. Second, because of the physics-informed framework within MF-PIDNN, the proposed approach is able to capture some of the physical laws that are present in the approximate model. As a result, it provides reasonable predictions even in zones with no-data.

The proposed approach is used for solving benchmark reliability analysis problems from the literature. For all the problems, the proposed approach is able to correctly predict the probability of failure and the reliability index of the system. To illustrate the advantage of the proposed approach, results obtained are compared with those obtained from only the high-fidelity data-driven model and low-fidelity physics-driven model. The proposed approach is found to outperform both these approaches. Case studies are also presented to illustrate different features of MF-PIDNN.

Despite the several advantages of the MF-PIDNN, certain aspects can be further enhanced. For example, during updating the model using transfer learning, mean-squared loss-function with no regularization has been used. This can lead to over-fitting. One future direction is to study the effect of regularization on the results. Second, the number of tunable parameters during transfer learning are selected manually in this study. Automating the transfer learning step will be hugely beneficial. Third, the network architecture and the activation functions in this study are manually provided. Automating this will also be beneficial. In future, some of these aspects will be investigated.
Fig. 6: MF-PIDNN and MCS predicted results at different time-instants. The threshold \( \epsilon_{3,0} \) for these four cases are set at (a) 0.40, (b) 0.575, (c) 0.70 and (d) 0.78.

**Acknowledgements**

The author would like to thank Soumya Chakraborty for proof-reading this article and Somdatta Goswami, Tanmoy Chatterjee and Rajdip Nayek for the useful discussions during the preparation of this paper. The TensorFlow codes were run on Google Colab service.

**References**

[1] Steven L Brunton, Joshua L Proctor, and J Nathan Kutz. Discovering governing equations from data by sparse identification of nonlinear dynamical systems. *Proceedings of the national academy of sciences*, 113(15):3932–3937, 2016.

[2] DES Rodrigues, J Belinha, FMA Pires, LMJS Dinis, and RM Natal Jorge. Material homogenization technique for composites: A meshless formulation. *Science and Technology of Materials*, 30(1):50–59, 2018.

[3] Appu Haapio and Pertti Viitaniemi. Environmental effect of structural solutions and building materials to a building. *Environmental impact assessment review*, 28(8):587–600, 2008.

[4] Bibekananda Mandal and Anupam Chakrabarti. A simple homogenization scheme for 3d finite element analysis of composite bolted joints. *Composite Structures*, 120:1–9, 2015.

[5] K Ramesh and Vivek Ramakrishnan. Digital photoelasticity of glass: A comprehensive review. *Optics and Lasers in Engineering*, 87:59–74, 2016.

[6] Lukas Mennel, Joanna Symonowicz, Stefan Wachter, Dmitry K Polyushkin, Aday J Molina-Mendoza, and Thomas Mueller. Ultrafast machine vision with 2d material neural network image sensors. *Nature*, 579(7797):62–66, 2020.
[7] Benjamin Peherstorfer, Karen Willcox, and Max Gunzburger. Survey of multifidelity methods in uncertainty propagation, inference, and optimization. *Siam Review*, 60(3):550–591, 2018.

[8] M Giselle Fernández-Godino, Chanyoung Park, Nam H Kim, and Raphael T Haftka. Issues in deciding whether to use multifidelity surrogates. *AIAA Journal*, 57(5):2039–2054, 2019.

[9] Souvik Chakraborty, Tanmoy Chatterjee, Rajib Chowdhury, and Sondipon Adhikari. A surrogate based multifidelity approach for robust design optimization. *Applied Mathematical Modelling*, 47:726–744, 2017.

[10] Claudio Bierig and Alexey Chernov. Approximation of probability density functions by the multilevel monte carlo maximum entropy method. *Journal of Computational Physics*, 314:661–681, 2016.

[11] Michael B Giles. Multilevel monte carlo path simulation. *Operations research*, 56(3):607–617, 2008.

[12] Mike B Giles, Tigran Nagapetyan, and Klaus Ritter. Adaptive multilevel monte carlo approximation of distribution functions. *arXiv preprint arXiv:1706.06869*, 2017.

[13] Stefan Heinrich. Multilevel monte carlo methods. In *International Conference on Large-Scale Scientific Computing*, pages 58–67. Springer, 2001.

[14] Loic Le Gratiet and Josselin Garnier. Recursive co-kriging model for design of computer experiments with multiple levels of fidelity. *International Journal for Uncertainty Quantification*, 4(5), 2014.

[15] Paris Perdikaris, Daniele Venturi, Johannes O Royset, and George Em Karniadakis. Multi-fidelity modelling via recursive co-kriging and gaussian–markov random fields. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 471(2179):20150018, 2015.

[16] Slawomir Koziel, Stanislav Ogurtsov, Ivo Couckuyt, and Tom Dhaene. Variable-fidelity electromagnetic simulations and co-kriging for accurate modeling of antennas. *IEEE transactions on antennas and propagation*, 61(3):1301–1308, 2012.

[17] Loic Le Gratiet. *Multi-fidelity Gaussian process regression for computer experiments*. PhD thesis, Universite Paris-Diderot, 2013.

[18] Subhadip Biswas, Souvik Chakraborty, Satish Chandra, and Indrajit Ghosh. Kriging-based approach for estimation of vehicular speed and passenger car units on an urban arterial. *Journal of Transportation Engineering, Part A: Systems*, 143(3):04016013, 2017.

[19] Tanmoy Mukhopadhyay, S Chakraborty, S Dey, S Adhikari, and R Chowdhury. A critical assessment of kriging model variants for high-fidelity uncertainty quantification in dynamics of composite shells. *Archives of Computational Methods in Engineering*, 24(3):495–518, 2017.

[20] Arpita Saha, Souvik Chakraborty, Satish Chandra, and Indrajit Ghosh. Kriging based saturation flow models for traffic conditions in indian cities. *Transportation Research Part A: Policy and Practice*, 118:38–51, 2018.

[21] Irfan Kaymaz. Application of kriging method to structural reliability problems. *Structural Safety*, 27(2):133–151, 2005.

[22] Ilias Bilionis and Nicholas Zabaras. Multi-output local Gaussian process regression: Applications to uncertainty quantification. *Journal of Computational Physics*, 231(17):5718–5746, 2012.

[23] Ilias Bilionis, Nicholas Zabaras, Bledar A. Konomi, and Guang Lin. Multi-output separable Gaussian process: Towards an efficient, fully Bayesian paradigm for uncertainty quantification. *Journal of Computational Physics*, 241:212–239, 2013.

[24] Rajdip Nayek, Souvik Chakraborty, and Sriram Narasimhan. A Gaussian process latent force model for joint input-state estimation in linear structural systems. *Mechanical Systems and Signal Processing*, 128:497–530, 2019.

[25] Souvik Chakraborty and Rajib Chowdhury. Graph-Theoretic-Approach-Assisted Gaussian Process for Non-linear Stochastic Dynamic Analysis under Generalized Loading. *Journal of Engineering Mechanics*, 145(12):04019105, 2019.

[26] Souvik Chakraborty, Sondipon Adhikari, and Ranjan Ganguli. The role of surrogate models in the development of digital twins of dynamic systems. *arXiv preprint arXiv:2001.09292*, 2020.

[27] H Babaei, P Perdikaris, C Chryssostomidis, and GE Karniadakis. Multi-fidelity modelling of mixed convection based on experimental correlations and numerical simulations. *Journal of Fluid Mechanics*, 809:895–917, 2016.

[28] Rohit Batra, Ghanshyam Pilania, Blas P Uberuaga, and Rampi Ramprasad. Multifidelity information fusion with machine learning: A case study of dopant formation energies in hafnia. *ACS applied materials & interfaces*, 11(28):24906–24918, 2019.
[29] Paris Perdikaris, Daniele Venturi, and George Em Karniadakis. Multifidelity information fusion algorithms for high-dimensional systems and massive data sets. *SIAM Journal on Scientific Computing*, 38(4):B521–B538, 2016.

[30] Bo Liu, Slawomir Koziel, and Qingfu Zhang. A multi-fidelity surrogate-model-assisted evolutionary algorithm for computationally expensive optimization problems. *Journal of computational science*, 12:28–37, 2016.

[31] Chanyoung Park, Raphael T Haftka, and Nam H Kim. Remarks on multi-fidelity surrogates. *Structural and Multidisciplinary Optimization*, 55(3):1029–1050, 2017.

[32] Liang Yan and Tao Zhou. Adaptive multi-fidelity polynomial chaos approach to bayesian inference in inverse problems. *Journal of Computational Physics*, 381:110–128, 2019.

[33] Pramudita Satria Palar, Takeshi Tsuchiya, and Geoffrey Thomas Parks. Multi-fidelity non-intrusive polynomial chaos based on regression. *Computer Methods in Applied Mechanics and Engineering*, 305:579–606, 2016.

[34] Han Gao, Xueyu Zhu, and Jian-Xun Wang. A bi-fidelity surrogate modeling approach for uncertainty propagation in three-dimensional hemodynamic simulations. *Computer Methods in Applied Mechanics and Engineering*, 366:113047, 2020.

[35] Alexander Forrester, Andras Sobester, and Andy Keane. *Engineering design via surrogate modelling: a practical guide*. John Wiley & Sons, 2008.

[36] Jonas Nitzler, Jonas Biehler, Niklas Fehn, Phaedon-Stelios Koutsourelakis, and Wolfgang A Wall. A generalized probabilistic learning approach for multi-fidelity uncertainty propagation in complex physical simulations. *arXiv preprint arXiv:2001.02892*, 2020.

[37] Paris Perdikaris, Maziar Raissi, Andreas Damianou, ND Lawrence, and George Em Karniadakis. Nonlinear information fusion algorithms for data-efficient multi-fidelity modelling. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 473(2198):20160751, 2017.

[38] Ian Goodfellow, Yoshua Bengio, and Aaron Courine. *Deep learning*. MIT press, 2016.

[39] Subhayan De, Jolene Britton, Matthew Reynolds, Ryan Skinner, Kenneth Jansen, and Alireza Doostan. On transfer learning of neural networks using bi-fidelity data for uncertainty propagation. *arXiv preprint arXiv:2002.04495*, 2020.

[40] Xuhui Meng and George Em Karniadakis. A composite neural network that learns from multi-fidelity data: Application to function approximation and inverse pde problems. *Journal of Computational Physics*, 401:109020, 2020.

[41] Dehao Liu and Yan Wang. Multi-fidelity physics-constrained neural network and its application in materials modeling. *Journal of Mechanical Design*, 141(12), 2019.

[42] Yang Zhiyin. Large-eddy simulation: Past, present and the future. *Chinese journal of Aeronautics*, 28(1):11–24, 2015.

[43] Matthew F Barone, Julia Ling, Kenny Chowdhary, Warren Davis, and Jeffrey Fike. Machine learning models of errors in large eddy simulation predictions of surface pressure fluctuations. In *47th AIAA Fluid Dynamics Conference*, page 3979, 2017.

[44] Maziar Raissi, Paris Perdikaris, and George E Karniadakis. Physics-informed neural networks: A deep learning framework for solving forward and inverse problems involving nonlinear partial differential equations. *Journal of Computational Physics*, 378:686–707, 2019.

[45] Somdatta Goswami, Cosmin Anitescu, Souvik Chakraborty, and Timon Rabczuk. Transfer learning enhanced physics informed neural network for phase-field modeling of fracture. *Theoretical and Applied Fracture Mechanics*, 106:102447, 2020.

[46] Souvik Chakraborty. Simulation free reliability analysis: A physics-informed deep learning based approach. *arXiv preprint arXiv:2005.01302*, 2020.

[47] Yinhao Zhu, Nicholas Zabaras, Phaedon-Stelios Koutsourelakis, and Paris Perdikaris. Physics-constrained deep learning for high-dimensional surrogate modeling and uncertainty quantification without labeled data. *Journal of Computational Physics*, 394:56–81, 2019.

[48] Achintya Haldar and Sankaran Mahadevan. *Probability, reliability, and statistical methods in engineering design*. John Wiley, 2000.

[49] Achintya Haldar and Sankaran Mahadevan. *Reliability assessment using stochastic finite element analysis*. John Wiley & Sons, 2000.
[50] Dongbin Xiu and George Em Karniadakis. The Wiener-Askey polynomial chaos for stochastic differential equations. *SIAM Journal on Scientific Computing*, 24(2):619–644, 2002.

[51] Bruno Sudret. Global sensitivity analysis using polynomial chaos expansions. *Reliability Engineering & System Safety*, 93(7):964–979, 2008.

[52] Souvik Chakraborty and Rajib Chowdhury. Sequential experimental design based generalised ANOVA. *Journal of Computational Physics*, 317:15–32, 2016.

[53] Souvik Chakraborty and Rajib Chowdhury. Polynomial Correlated Function Expansion. In *Modeling and Simulation Techniques in Structural Engineering*, pages 348–373. IGI Global, 2017.

[54] Atin Roy and Subrata Chakraborty. Support vector regression based metamodel by sequential adaptive sampling for reliability analysis of structures. *Reliability Engineering & System Safety*, page 106948, 2020.

[55] Souvik Chakraborty and Rajib Chowdhury. An efficient algorithm for building locally refined hp–adaptive h-pcfe: Application to uncertainty quantification. *Journal of Computational Physics*, 351:59–79, 2017.

[56] Souvik Chakraborty and Rajib Chowdhury. Hybrid framework for the estimation of rare failure event probability. *Journal of Engineering Mechanics*, 143(5):04017010, 2017.

[57] Yinhao Zhu and Nicholas Zabaras. Bayesian deep convolutional encoder–decoder networks for surrogate modeling and uncertainty quantification. *Journal of Computational Physics*, 366:415–447, 2018.

[58] Hagan Demuth Beale, Howard B Demuth, and MT Hagan. Neural network design. *Pws, Boston*, 1996.

[59] Biswarup Bhattacharyya. A Critical Appraisal of Design of Experiments for Uncertainty Quantification. *Archives of Computational Methods in Engineering*, 25(3):727–751, 2018.

[60] Atılım Günes Baydin, Barak A Pearlmutter, Alexey Andreyevich Radul, and Jeffrey Mark Siskind. Automatic differentiation in machine learning: a survey. *The Journal of Machine Learning Research*, 18(1):5595–5637, 2017.

[61] Tijmen Tieleman and Geoffrey Hinton. Lecture 6.5-rmsprop: Divide the gradient by a running average of its recent magnitude. *COURSERA: Neural networks for machine learning*, 4(2):26–31, 2012.

[62] RY. Rubinstein. *Simulation and the Monte Carlo method*. Wiley, New York, U.S.A., 1981.

[63] Martín Abadi, Paul Barham, Jianmin Chen, Zhifeng Chen, Andy Davis, Jeffrey Dean, Matthieu Devin, Sanjay Ghemawat, Geoffrey Irving, Michael Isard, et al. Tensorflow: A system for large-scale machine learning. In *12th {USENIX} Symposium on Operating Systems Design and Implementation (OSDI'16)*, pages 265–283, 2016.

[64] The Mathworks Inc., Natick, Massachusetts, US. MATLAB and Statistics Toolbox Release 2019b, 2019.

[65] Martin Alnæs, Jan Blechta, Johan Hake, August Johansson, Benjamin Kehlet, Anders Logg, Chris Richardson, Johannes Ring, Marie E Rognes, and Garth N Wells. The fenics project version 1.5. *Archive of Numerical Software*, 3(100), 2015.

[66] Jing Li and Dongbin Xiu. Evaluation of failure probability via surrogate models. *Journal of Computational Physics*, 229(23):8966–8980, 2010.

[67] Dongbin Xiu and George Em Karniadakis. Supersensitivity due to uncertain boundary conditions. *International journal for numerical methods in engineering*, 61(12):2114–2138, 2004.

[68] Jens Lorenz. *Nonlinear singular perturbation problems and the Engquist-Osher difference scheme*. Katholieke Universiteit Nijmegen. Mathematisch Instituut, 1981.

[69] Tong Qin, Zhen Chen, John Jakeman, and Dongbin Xiu. A neural network approach for uncertainty quantification for time-dependent problems with random parameters. arXiv preprint [arXiv:1910.07096], 2019.

[70] R. L. Iman, J. M. Davenport, and D. K. Zeigler. Latin hypercube sampling (program user’s guide). Technical report, Sandia laboratories, 1980.

[71] Stanislav Yefimovic Shvartsman, Michael P Hagan, A Yacoub, Paul Dent, HS Wiley, and Douglas A Lauffenburger. Autocrine loops with positive feedback enable context-dependent cell signaling. *American Journal of Physiology-Cell Physiology*, 282(3):C545–C559, 2002.