Comments on Chiral $p$-Forms$^*$

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Abstract

Two issues regarding chiral $p$-forms are addressed. First, we investigate the topological conditions on spacetime under which the action for a non-chiral $p$-form can be split as the sum of the actions for two chiral $p$-forms, one of each chirality. When these conditions are not met, we exhibit explicitly the extra topological degrees of freedom and their couplings to the chiral modes. Second, we study the problem of constructing Lorentz-invariant self-couplings of a chiral $p$-form in the light of the Dirac-Schwinger condition on the energy-momentum tensor commutation relations. We show how the Perry-Schwarz condition follows from the Dirac-Schwinger criterion and point out that consistency of the gravitational coupling is automatic.

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I. INTRODUCTION

The talk by M.H. at the Bariloche meeting was devoted to the results obtained in [1–3] on the flip of sign in the quantization condition for $k$-brane dyons ($k$ odd) in $2k + 4$ dimensions. To avoid repetition with what can be found in the literature, the present contribution to the proceedings does not reproduce the actual content of the talk but deals with the Lagrangian formulation of chiral $p$-forms. We refer the reader interested in the quantization condition for $k$-brane dyons to [1–4] for a detailed discussion. See also [5–7] for related information.

Chiral $p$-forms, i.e., $p$-forms, the field strength of which is self-dual, can exist in $(2p + 2)$-Minkowski spacetime for any even $p$. They are notoriously known to suffer from one major difficulty: even though their equations of motion are manifestly Lorentz-invariant, there is no simple (e.g. quadratic in the free case), manifestly Lorentz-invariant Lagrangian that leads to these equations of motion [8].

Although there is no simple manifestly Lorentz-invariant Lagrangian, there is a simple non-manifestly Lorentz-invariant Lagrangian which has been given in [9,10], and which generalizes the Lagrangian of [11] for chiral bosons. This Lagrangian is linear in the first order time derivatives of the spatial components of the $p$-form potential and reads, in the 2-form case that we shall consider for definiteness

$$S[A_{ij}] = \int dx^0 d^5 x B^{ij} \partial_0 A_{ij} - \int dx^0 H \quad (i, j, \ldots = 1, \ldots, 5) \quad (I.1)$$

with

$$B^{ij} = \frac{1}{3!} e^{ijklm} F_{klm}, \quad F_{\mu\nu\lambda} = \partial_\mu A_{\nu\lambda} - \partial_\nu A_{\mu\lambda} - \partial_\lambda A_{\nu\mu} \quad (I.2)$$

and

$$H = \int d^5 x (N\mathcal{H} + N^k \mathcal{H}_k). \quad (I.3)$$

Here, $N$ and $N^k$ are the standard lapse and shift [12]. The magnetic field $B^{ij}$ is a spatial tensor density of weight one. We are considering from the outset the theory in a gravitational background as in [9,10]. In the absence of self-interactions, the energy density $\mathcal{H}$ is given by

$$\mathcal{H} = \frac{1}{\sqrt{g}} B^{ij} B_{ij} \quad (I.4)$$

where the spatial indices are lowered and raised with the spatial metric and its five-dimensional inverse, while $g$ is the determinant of $g_{ij}$. The energy density generates displacements normal to the slices of constant $x^0$. The momentum density $\mathcal{H}_k$, on the other hand, is purely kinematical and generates tangent displacements. It is explicitly given by

$$\mathcal{H}_k = \frac{1}{2} \epsilon_{ijmnk} B^{ij} B_{mn}. \quad (I.5)$$

In order to write the action (I.1), it is necessary to assume that spacetime has the product form “time × space”. This will be done throughout in the sequel.
The action (I.1) is manifestly invariant under the gauge transformations
\[ \delta_\Lambda A_{ij} = \partial_i \Lambda_j - \partial_j \Lambda_i \] (I.6)
since \( B^{ij} \) is gauge-invariant and identically transverse (\( \partial_i B^{ij} \equiv 0 \)). In flat space, it is also invariant under Lorentz transformations, but these do not take the usual form [9,10].

Knowing the action for a chiral form in a gravitational background, one can compute the gravitational anomaly by usual quantum field theoretical methods [13] and compare the result with calculations based on the non-chiral action supplemented by an appropriate projection [14]. As shown in [13], there is agreement.

The first part of our paper is motivated by this result and aims at understanding better the relationship between the non-chiral action and the chiral ones. We show that when the spatial sections have vanishing Betti numbers \( b_2 \) and \( b_3 \), the action for a non-chiral form is just the sum of the actions for two uncoupled chiral forms of opposite chiralities. Thus the path integral for a non-chiral 2-form supplemented by a projection to one chiral sector, trivially reduces to the path-integral for the corresponding chiral modes. This is no longer true for more general topologies. The non-chiral action and the sum of the chiral actions agree on the local degrees of freedom, but treat differently the harmonic components of the 2-form. However, one can easily keep track of the topological “zero mode” difference. This is explicitly done in section [X] after we have reviewed the necessary background on the dynamics of chiral \( p \)-forms. The importance of global features when dealing with chiral forms has been pointed out and stressed in [13] where the problem of modular invariance has been addressed. Recent developments relevant to the six-torus case are given in [16].

In an interesting series of papers [17], a manifestly covariant formulation of chiral \( p \)-forms has been developed. This formulation is characterized by the presence of an extra field and an extra gauge invariance. This extra field occurs non-polynomially in the action, even for free chiral 2-forms. The manifestly covariant formulation has proved useful for many conceptual developments. It has been shown to be equivalent to the non-manifestly covariant treatment of [1] in Minkowski space [18]. To the extent that the analysis of [17] strongly relies on the Poincaré lemma, it is expected to share also similar global features.

The second question analysed in this paper is that of Lorentz-invariant self-couplings (as well as consistent self-couplings in an external gravitational background) for chiral \( p \)-forms. In view of its relevance to the M-theory five-brane, this question has received a lot of attention, both at the level of the equations of motion [19] and at the level of the action [20–24]. We show that this question can be handled by means of the Dirac-Schwinger condition on the commutation relations of the components of the energy-momentum tensor [25,26]. This condition leads directly to the differential equation obtained in [20] and implies automatically consistency of the gravitational coupling. So, once Lorentz-invariant self-interacting chiral \( p \)-form theories have been found, there is no extra work to be carried out to couple them to gravity. The Dirac-Schwinger criterion, which appears to be quite powerful in

\[ ^1 \text{Since } A_{0i} \text{ does not occur in the action} \text{ -- even if one replaces } \partial_0 A_{ij} \text{ by } \partial_0 A_{ij} - \partial_i A_{0j} - \partial_j A_{i0} \text{ (it drops out because } B^{ij} \text{ is transverse) --, the action is of course invariant under arbitrary shifts of } A_{0i}. \text{ It is also invariant under arbitrary shifts of any other field that does not appear in the action.} \]
the present context, has been used recently in [27], in the investigation of Lorentz-invariance of manifestly duality-invariant theories in the other even (0 mod 4) spacetime dimensions.

II. DYNAMICS OF CHIRAL 2-FORM

As stated above, we assume that spacetime takes the product form $T \times \Sigma$ where $T$ is the manifold of the time variable (usually a line). Furthermore, we also assume that the spatial sections $\Sigma$ are either homeomorphic to $\mathbb{R}^5$ (in which case the theory must be supplemented by fall-off conditions at infinity that insure the vanishing of the relevant surface terms), or compact. Of course, a spatial coordinate could equivalently play the rôle of the time variable, as in [20].

We define the exterior form $B$ to be the (time-dependent) spatial 2-form with components $B_{ij}/\sqrt{g}$. The equations of motion that follow from the action are [9,10]

$$d[N(E - B)] = 0 \quad (\text{II.1})$$

where $E$ is the electric spatial 2-form defined through

$$E_{ij} \equiv \frac{\dot{A}_{ij} - N^s F_{sij}}{N} \quad (\text{II.2})$$

and where $d$ is the spatial exterior derivative operator. In the case where the second Betti number $b_2$ of the spatial sections vanishes, this equation implies $N(E - B) = dm$, where $m$ is an arbitrary spatial 1-form. To bring this equation to a more familiar form, one sets $m_i = A_{0i}$. The equations of motion read then

$$F = *F \quad (\text{II.3})$$

where $F_{bij} = \dot{A}_{ij} - \partial_i A_{0j} + \partial_j A_{0i}$. This is the standard self-duality condition. Alternatively, one may use the gauge freedom to set $m = 0$, which yields the self-duality condition in the temporal gauge.

To deal with the case where $b_2$ is not zero, one uses the Hodge decomposition of exterior forms on the spatial sections [28]. Any form - and in particular, any 2-form - can be written as the sum of an exact form, a co-exact form and a harmonic form,

$$A = d\rho + \delta \phi + \sum_A \lambda_A(t) \omega^A. \quad (\text{II.4})$$

Here, the codifferential $\delta$ acting on a $p$-form is equal to $\delta = (-1)^{5p} * d*$, while $\rho$ (respectively $\phi$) is a spatial 1-form (respectively, spatial 3-form) and $\{\omega^A\}$ is, on each spatial slice, a basis of harmonic (= closed and co-closed) 2-forms. These satisfy $\partial_i (\omega^{Aij}\sqrt{g}) = 0$, $\partial_{[i} \omega_{jk]}^{A} = 0$ and are normalized so that $\int d^5x \omega^{Aij} \omega^{B}_{ij} \sqrt{g} = \delta^{AB}$ for each $t$. The harmonic forms are in finite number and thus, the harmonic component of $A$ describes a finite number of global “zero modes”. In the simple case where $b_2 = 0$, there are no zero modes. In the case where $b_2 \neq 0$, $A$ may have a non-trivial harmonic part. The equation of motion (II.1) implies in that case

$$N(E - B) = \sum_A k_A(t) \omega^A + dm. \quad (\text{II.5})$$
Again, one can absorb the exact part of the right-hand side of (II.5) in a redefinition of \( E \) (or set it equal to zero by a gauge transformation), but there is an additional piece which is not determined, namely, the harmonic part. However, this harmonic part turns out to be pure gauge, because the action (I.1) for a chiral 2-form has more gauge invariances than expressed by (I.6). It is actually invariant under addition to \( A \) of an arbitrary closed (and not necessarily exact) 2-form,

\[
\delta_{\lambda, \epsilon} A = d\lambda + \epsilon \omega^A.
\] (II.6)

This follows because \( B \) is co-exact (and not just co-closed), and invariant under (II.6). One can thus gauge away the harmonic part of \( N(E - B) \) and get again the self-duality condition. Therefore, the action (I.1) leads to the correct self-duality condition but is a theory in which the zero modes of \( A \) are pure gauge (no physical component along the harmonic forms). A similar phenomenon was described in [10] for chiral bosons on a circle. To summarize: for a chiral 2-form, both the exact and the harmonic components (i.e., the closed part of \( A \)) are pure gauge and it is the co-exact part only that contains the physical degrees of freedom.

For an anti-chiral 2-form, the action is

\[
S[A_{ij}] = -\int dx^0 d^5 x B^{ij} \partial_0 A_{ij} - \int dx^0 H
\] (II.7)

with \( H = \int d^5 x (N \mathcal{H}' + N^k \mathcal{H}'_k) \). The energy density \( \mathcal{H}' \) is the same as for a chiral form, but the momentum density \( \mathcal{H}'_k \) differs in the sign. The analysis proceeds exactly as above and one finds this time the anti-chiral condition

\[
E + B = 0.
\] (II.8)

An anti-chiral 2-form described by the action (II.7) has no physical harmonic component.

For later purposes, we shall need the brackets of the gauge-invariant magnetic fields \( B^{ij} \). The orthodox way to proceed is to define conjugate momenta and follow the Dirac method for constrained systems [29]. The chirality condition appears as a mixture of second class constraints and of first class constraints, the first class part being related to the gauge invariance of the theory [10]. One may work out the Dirac bracket of the gauge-invariant fields by using the Dirac formula, but one may shortcut the whole procedure and directly read the brackets from the action (I.1), which is already in first-order form. Either way, one finds as Dirac brackets (we consider the chiral case for definiteness, the anti-chiral one differs in the sign)

\[
[B^{ij}(x), B^{mn} (x')] = \frac{1}{4} \epsilon^{ijklm} \delta_{ik}(x - x').
\] (II.9)

We shall also need the brackets of the energy densities \( \mathcal{H}(x) \) at two different space points. A direct calculation using only the form of \( \mathcal{H} \) and the brackets (II.9) yields

\[
[\mathcal{H}(x), \mathcal{H}(x')] = (\mathcal{H}^k(x) + \mathcal{H}^k(x')) \delta_{ik}(x - x').
\] (II.10)

The relation (II.10), derived first on general grounds in [23,24], is deeply connected to Lorentz-invariance and gravitational coupling and we shall return to it below.

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2If the spatial metric depends on \( t \), the 2-form \( \omega^A_{ij} \) will be also time-dependent. The time derivatives \( \dot{\omega}^A_{ij} \) are clearly closed so that \( \int B^{ij} \dot{\omega}^A_{ij} d^5 x = 0 \).
III. ZERO MODES OF A NON-CHIRAL FORM

The action for a non-chiral 2-form is
\[ S[A_{\mu\nu}] = -\frac{1}{2} \cdot \frac{1}{3!} \int d^6x \sqrt{-g} F^{\lambda\mu\nu} F_{\lambda\mu\nu}. \] (III.1)

We keep the same notations for the 2-form, even though \( A_{\mu\nu} \) here \( \neq \) \( A_{\mu\nu} \) before (see relationship (IV.5) below between non-chiral and chiral 2-forms). It is invariant under the gauge transformations
\[ \delta_A A_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \] (III.2)

which enable one to set \( A_{0i} \) equal to zero. The exact part of \( A_{ij} \) can then be also gauged away at any given time, but the harmonic part cannot. Indeed, the action (III.1) is not invariant under shifts of \( A_{ij} \) by an arbitrary closed form, only under shifts of \( A_{ij} \) by an arbitrary exact form. Thus, the harmonic part of a non-chiral 2-form describes true physical degrees of freedom.

It is easy to see that on a flat background, the harmonic part of \( A_{ij} \) behaves like a free particle, i.e., grows linearly with time,
\[ A = d\rho + \delta \phi + \sum A_\lambda(t) \omega^A \] (III.3)

with
\[ \lambda_A(t) = C_A t + D_A \] (III.4)
on-shell. This is because the equation \( \partial_\mu (\sqrt{-g} F^{\mu\nu\sigma}) = 0 \) implies \( \frac{d^2 \lambda}{dt^2} \omega^A = \delta(\text{something}) + d(\text{something'}) \) and thus \( \frac{d^2 \lambda}{dt^2} \omega^A = 0 \). Now, if the integration constant \( C_A \) is different from zero, the form \( A_{\mu\nu} \) cannot be purely chiral or anti-chiral. Indeed, if it is chiral (say), then, the chirality condition implies
\[ C_A \omega^A + d\dot{\rho} + \delta \dot{\phi} = \delta(\text{something}) \] (III.5)

which leads to a contradiction unless \( C_A = 0 \). Accordingly, if one decomposes the field strength into self-dual part and anti-self-dual part, there is no potential neither for the self-dual part, nor for the anti-self-dual part when \( C_A \neq 0 \), although there is a potential for the sum.

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More precisely, the transformation \( \delta_\epsilon A_{ij} = \epsilon_A(t) \omega^A_{ij} \) of the spatial components cannot be supplemented by a transformation of \( A_{0i} \) such that \( \delta_\epsilon F_{0ij} = 0 \) for arbitrary \( \epsilon \)'s. Indeed, this would require \( \epsilon_A \omega^A_{ij} + \epsilon_A \dot{\omega}^A_{ij} = \text{exact form} \), which forces \( \epsilon_A \) to be a solution of the differential equation \( \dot{\epsilon}_A + t_B \epsilon_B = 0 \), where \( \dot{\omega}^A = t_B \omega^B + d(\text{something}) \), showing that \( \epsilon^A \) cannot be an arbitrary function of time. The transformations with \( \epsilon \) solution to this equation should be regarded as rigid symmetries, not gauge symmetries.
The situation is the same as for a chiral boson \( \varphi \) on a circle. The zero mode \( \varphi_0 = at + b \) cannot be written as the sum of single-valued left-movers and right-movers unless \( a = 0 \), even though the sum is single-valued \( (at = (a/2)(t + \sigma) + (a/2)(t - \sigma)) \) but \( t + \sigma \) or \( t - \sigma \) are not single-valued). Of course, the field strength \( F_\mu = \partial_\mu \varphi \) is decomposable into well-defined self-dual and anti-self-dual parts, but these do not derive from a single-valued potential.

We thus see that a non-chiral 2-form contains additional global degrees of freedom besides the local degrees of freedom described by the local chiral actions. It is the presence of the physical zero modes that is responsible for the fact that the sum of a chiral 2-form and an anti-chiral 2-form is not a non-chiral 2-form on a topologically non-trivial background.

**IV. DECOMPOSITION OF NON-CHIRAL ACTION**

In order to compare the action for a non-chiral 2-form with the sum of the chiral and anti-chiral actions given above, it is convenient to rewrite the non-chiral action in Hamiltonian form. To that end, one follows the Dirac method. One finds

\[
S[A_{ij}, A_{0i}, \pi^{ij}] = \int dx^0 d^5x [\pi^{ij} \dot{A}_{ij} - \frac{N}{\sqrt{g}} (\pi^{ij} \pi_{ij} + \frac{1}{4} B^{ij} B_{ij}) - N^k \pi^{ij} F_{kij} - 2A_{0i} \pi^{ij}, j] \quad (IV.1)
\]

where \( \pi^{ij} \) is the momentum conjugate to \( A_{ij} \). The component \( A_{0i} \) appears as a Lagrange multiplier for Gauss’law constraint

\[
\partial_i \pi^{ij} = 0. \quad (IV.2)
\]

One can solve Gauss’law for \( \pi^{ij} \) and eliminate the corresponding multiplier from the action. The general solution of (IV.2) is

\[
2\pi^{ij} = \frac{1}{2} \epsilon^{ijklm} \partial_k Z_{lm} + \sqrt{2g} \mu A \omega^{Ai j} \quad (IV.3)
\]

While the 2-form \( A_{ij} \) is determined up to an exact form, the 2-form \( Z_{ij} \) is determined up to a closed form,

\[
\delta_N \chi Z_{ij} = \partial_i A'_j - \partial_j A'_i + \chi A \omega^{Ai j} \quad (IV.4)
\]

Using (IV.3) and making the change of variables of \( Z \),

\[
Z_{ij} = \sqrt{2}(U_{ij} + V_{ij}), \quad A_{ij} = \sqrt{2}(U_{ij} - V_{ij}) \quad (IV.5)
\]

one finds, after straightforward algebra

\[
S[U_{ij}, V_{ij}, \mu_A] = S^{\text{chiral}}[U_{ij}] + S^{\text{anti-chiral}}[V_{ij}]
+ \int d^6x \mu A \sqrt{g} \omega^{Ai j}(U_{ij} - \dot{V}_{ij}) - \frac{1}{2} \int d^6x N^k \mu A \omega^{Ai j} \epsilon^{ijklm} \partial_k(U_{lm} + V_{lm})
- \frac{\sqrt{2}}{2} \int d^6x N^k \sqrt{g} \mu A \omega^{Ai j} F_{kij} - \frac{1}{2} \int dt k^{AB} \mu_A \mu_B \quad (IV.6)
\]

with
The action for the non-chiral form splits thus as the sum of two chiral actions, one of each chirality, plus terms coupling the zero modes $\mu^A$ to the chiral components. The action is invariant under the transformations

$$
\begin{align}
\delta_{\chi,\xi} U_{ij} &= \partial_i X_j + \xi \omega_{ij}^A \\
\delta_{\chi,\xi} V_{ij} &= \partial_i Y_j + \xi \omega_{ij}^A \\
\delta_{\chi,\xi} \mu^A &= 0
\end{align}
$$

with same harmonic component $\xi(t)$ for $\delta_{\chi,\xi} U_{ij}$ and $\delta_{\chi,\xi} V_{ij}$. Consequently, because of the zero mode coupling, the action (IV.6) has less gauge invariances than the sum of two chiral actions. The zero mode of the difference $U_{ij} - V_{ij}$ is also gauge invariant. One easily verifies that it is in fact canonically conjugate to $\mu^A$. For a flat metric, the couplings between the local degrees of freedom and the zero modes simplify because the motion is an isometry so that the time-derivative of a harmonic form is harmonic. One can disentangle the zero modes from the co-exact ones, but this will not be done here.

When $H^2_{DR} \neq 0$, the physical Hilbert space for a non-chiral two-form is bigger than the product of the Hilbert spaces for a chiral two-form and an anti-chiral one. One must also include the states associated with the harmonic modes, 

$$
\mathcal{H}^{non-chiral} = \mathcal{H}^{chiral} \otimes \mathcal{H}^{anti-chiral} \otimes \mathcal{H}^0.
$$

The truncation to the chiral sector is particularly simple when there is no global, topological zero modes, since it simply amounts then to dropping the uncoupled anti-chiral degrees of freedom. How to handle the global modes in the general case depends on the context and will not be addressed here.

For issues that depend on the local (high-energy) behaviour of the theory, such as anomalies in local symmetries, the topological modes should not be relevant. In the absence of such modes, the change of variables (IV.5) can be implemented easily in the path integral and yields

$$
\begin{align}
Z &= \int DAD\pi \exp i(S[A, \pi]) \\
&= \int DUDV \exp i(S^{chiral}[U] + S^{anti-chiral}[V])
\end{align}
$$

where the measures $DAD\pi$ and $DUDV$ involve of course the ghost modes and gauge conditions. Note that neither the change of variables (IV.3) nor the parametrization (IV.3) (when there is no $\omega^A$) involves the metric. Projecting out to the chiral sector by interting a delta-function $\delta(\pi_{ij} - B^{ij})$ of the chirality condition is equivalent to setting the anti-chiral component $V_{ij}$ to zero, leaving one with the path-integral for a chiral 2-form. Thus, implementing the chiral condition by a projection or dealing with the non-manifestly invariant chiral action are clearly equivalent in the absence of harmonic modes.
When one can use the tensor calculus, it is rather easy to construct interactions that preserve Lorentz invariance. These interactions should also preserve the number of (possibly deformed) gauge symmetries (if any), but this aspect is rather immediate for p-form gauge symmetries – although it is less obvious for the extra gauge symmetry of \([17]\).

There is an alternative way to control Lorentz invariance. It is through the commutation relations of the energy-momentum tensor components. Because the energy-momentum tensor is the source of the gravitational field, the method gives at little extra price a complete grasp on the gravitational interactions. As shown by Dirac and Schwinger \([25,26]\), a sufficient condition for a manifestly rotation and translation invariant theory (in space) to be also Lorentz-invariant is that its energy density fulfills the commutation relations (II.10). The condition is necessary when one turns to gravitation. The method is more cumbersome than the tensor calculus when one can use the tensor calculus, but has the advantage of being still available even when manifestly invariant methods do not exist.

In the Dirac-Schwinger approach, the question is to find the most general \(H\) fulfilling (II.10). The energy-density \(H\) must be a spatial scalar density in order to fulfill the kinematical commutation relations \([H(x), H(x')] \sim H(x')\delta_{ik} (x-x')\) and depends on \(A_{ij}\) through \(B_{ij}\) in order to be gauge-invariant. In five dimensions, there are only two independent invariants that can be made out of \(B_{ij}\),

\[
y_1 = -\frac{1}{2g}B_{ij}B^{ij}, \quad y_2 = \frac{1}{4g^2}B_{ij}B^{jk}B_{km}B^{mi},
\]

as can easily be seen by bringing \(B_{ij}\) to canonical form by a rotation (only \(B_{12}\) and \(B_{34}\) non-zero; note that in this local frame the only non-vanishing component of \(H^k\) is \(H^5\)). Set

\[
H = f(y_1, y_2)\sqrt{g}, \quad f_1 = \partial_1 f, \quad f_2 = \partial_2 f.
\]

Then, a calculation following the standard pattern and paralleling the free case calculation yields

\[
[H(x), H(x')] = (\Lambda(x)H^k(x) + \Lambda(x')H^k(x'))\delta_{ik} (x-x')
\]

with

\[
4\Lambda = f_1^2 + y_1 f_1 f_2 + (\frac{1}{2}y_1^2 - y_2) f_2^2
\]

Requiring that (II.10) be fulfilled gives

\[
f_1^2 + y_1 f_1 f_2 + (\frac{1}{2}y_1^2 - y_2) f_2^2 = 4
\]

which is precisely the equation (31) of Perry & Schwarz with \(f\) replaced by \(2f\). The Dirac-Schwinger criterion yields thus directly the Perry-Schwarz equation, whose solutions are investigated in \([20]\).

In the flat space context \((g_{ij} = \delta_{ij}, N = 1, N^k = 0)\), the equation (II.10) guarantees that the interactions are Lorentz-invariant and no further work is required \([25,24]\). It also guarantees complete consistency in a gravitational background because of locality of \(H\) in the metric \(g_{ij}\) \([10,11]\).
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