Abstract. Let $\Omega$ be a star-shaped bounded domain either in the unit $n$-sphere ($S^n, ds^2$) or in paraboloid, $P = \{(x, y, z) \in \mathbb{R}^3 : z = x^2 + y^2\}$, having smooth boundary. In this article, we obtain a sharp lower bound for all Steklov eigenvalues on $\Omega$. This bound is given in terms of the Steklov eigenvalues of the largest geodesic ball contained in $\Omega$ with the same center as $\Omega$. This work is an extension of a result given by Kuttler and Sigillito (SIAM Rev 10:368–370, 1968) on a star-shaped bounded domain in $\mathbb{R}^2$.

1. Introduction

Let $\Omega$ be a bounded domain in a compact connected Riemannian manifold with smooth boundary $\partial \Omega$. The Steklov eigenvalue problem is to find all real numbers $\mu$ for which there exists a nontrivial function $\phi \in C^2(\Omega) \cap C^1(\overline{\Omega})$ such that

$$\begin{align*}
\Delta \phi &= 0 \quad \text{in } \Omega, \\
\frac{\partial \phi}{\partial \nu} &= \mu \phi \quad \text{on } \partial \Omega,
\end{align*}$$

where $\nu$ is the outward unit normal to the boundary $\partial \Omega$. This problem was introduced by Steklov [12] for bounded domains in the plane in 1902. Its importance lies in the fact that the set of eigenvalues of the Steklov problem is same as the set of eigenvalues of the well-known Dirichlet-Neumann map. This map associates to each function defined on $\partial \Omega$, the normal derivative of its harmonic extension on $\Omega$. The eigenvalues of the Steklov problem are discrete and form an increasing sequence $0 = \mu_1 < \mu_2 \leq \mu_3 \leq \cdots \nearrow \infty$. The variational characterization of $\mu_l$, $1 \leq l < \infty$ is given by

$$\mu_l(\Omega) = \sup_{E} \inf_{\phi \neq 0 \in E^\perp} \frac{\int_{\Omega} \|\nabla \phi\|^2 \, dv}{\int_{\partial \Omega} \phi^2 \, ds},$$

where $E$ is a set of $l-1$ functions $\phi_1, \phi_2, \ldots, \phi_{l-1}$ such that $\phi_i \in H^1(\Omega), 1 \leq i \leq l-1$ and $E^\perp = \{\phi \in H^1(\Omega) : \int_{\partial \Omega} \phi \phi_i \, ds = 0, 1 \leq i \leq l-1\}$.

There are several results which estimate first nonzero eigenvalue of the Steklov eigenvalue problem [1, 2, 5, 6]. The first upper bound for $\mu_2$ was given by Weinstock [13] in 1954. He proved that among all simply connected planar domains with analytic boundary of fixed perimeter, the circle maximizes $\mu_2$. Later F. Brock [3] obtained a sharp upper bound for $\mu_2$ by fixing the volume of the domain. He proved that for a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$, $\mu_2(\Omega)$ $(\text{vol}(\Omega))^{\frac{1}{n}} \leq \omega_n^{\frac{1}{n}}$, where $\omega_n$ is the volume of the unit ball in $\mathbb{R}^n$ and equality holds if and only if $\Omega$ is a ball. In several recent papers, bounds for all eigenvalues of the Steklov problem have been studied [4, 8, 10, 14]. In particular, sharp upper bound for specific functions of the Steklov eigenvalues have been derived in [8]. Weyl-type bounds have also been obtained for Steklov eigenvalues in [10, 14].

E-mail address: sheela.verma23@gmail.com.
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Let \( \Omega \subset \mathbb{R}^n \) be a star-shaped domain with smooth boundary \( \partial \Omega \). Let \( p \) be a center of \( \Omega \). Let \( R_m = \min \{ d(p, x) | x \in \partial \Omega \} \), \( R_M = \max \{ d(p, x) | x \in \partial \Omega \} \) and \( h_m = \min \{ \langle x, \nu \rangle | x \in \partial \Omega \} \), where \( \nu \) is the outward unit normal to \( \partial \Omega \). With these notations, Bramble and Payne [2] proved that

\[
\mu_2(\Omega) \geq \frac{R_m^{n-1}}{R_M^{n+1}} h_m.
\]

Equality holds when \( \Omega \) is a ball.

Kuttler and Sigillito [9] proved the following lower bound for a star-shaped bounded domain in \( \mathbb{R}^2 \).

**Theorem 1.1** ([9]). Let \( \Omega \) be a star-shaped bounded domain in \( \mathbb{R}^2 \) with smooth boundary and centered at the origin. Then, for \( 1 \leq k < \infty \),

\[
\mu_{2k+1}(\Omega) \geq \frac{1}{k} \left[ 1 - 2 \left( 1 + \frac{\sin\theta}{R(\theta)} \right) \right]
\]

where \( R(\theta) = \max \{ |x| : x \in \Omega, x = |x|e^{i\theta} \} \) and equality holds for a disc.

Following the idea of Kuttler and Sigillito [9], Garcia and Montano [7] and the author [11] obtained a similar bound for the first nonzero Steklov eigenvalue on a star-shaped domain in \( \mathbb{R}^n \) and \( S^n \), respectively. Let \( \Omega \) be a star-shaped bounded domain with smooth boundary \( \partial \Omega \) centered at a point \( p \) and \( \nu \) be the outward unit normal to \( \partial \Omega \). For any point \( q \in \partial \Omega \), if \( 0 \leq \theta(q) \leq \alpha < \frac{\pi}{2} \), where \( \cos(\theta(q)) = \langle \nu(q), \partial_r(q) \rangle \). Let \( a = \tan^2 \alpha \).

**Theorem 1.2** ([7]). Let \( \Omega \subset \mathbb{R}^n \). Then with the above notations, the first nonzero eigenvalue of the Steklov problem \( \mu_2(\Omega) \) satisfies

\[
\mu_2(\Omega) \geq \left( \frac{R_m}{R_M} \right)^{n-2} \left\{ 2 + a - \sqrt{a^2 + 4a} \right\} \frac{2}{2\sqrt{1+a}}.
\]

**Theorem 1.3** ([11]). Let \( \Omega \) be a star-shaped bounded domain in \( S^n \) such that \( \Omega \subset S^n \setminus \{ -p \} \). Then the first nonzero Steklov eigenvalue \( \mu_2(\Omega) \) satisfies

\[
\mu_2(\Omega) \geq \left( \frac{R_m}{R_M} \right)^{n-1} \left\{ \frac{(2 + a) - \sqrt{a^2 + 4a}}{2\sqrt{1+a}} \right\} \frac{\sin\alpha^{-1}(R_m)}{\sin\alpha^{-1}(R_M)} \mu_2(B'(R_m)).
\]

Here \( R_m \) and \( R_M \) are defined as above.

In Theorem 2.2, we obtain a lower bound similar to [11], for all Steklov eigenvalues in a star-shaped domain \( \Omega \) in \( (S^n, ds^2) \). In Theorem 3.1, we prove a result for a star-shaped domain in a paraboloid in \( \mathbb{R}^3 \) analogous to the above. The main tool used to prove these results is the construction of suitable test function for the variational characterization of the corresponding eigenvalues.

2. Eigenvalues on the unit \( n \)-sphere \( S^n \)

Let \( \Omega \) be a star-shaped bounded domain in \( (S^n, ds^2) \) with respect to a point \( p \in \Omega \) such that \( \Omega \subset S^n \setminus \{ -p \} \). Let \( \partial \Omega \) be the smooth boundary of \( \Omega \) with the outward unit normal \( \nu \). Since \( \Omega \) is star-shaped with respect to the point \( p \), for each point \( q \in \partial \Omega \), there exists unique unit vector \( u \in T_pS^n \) and \( R_u > 0 \) such that \( q = \exp_p(R_u u) \). Then in geodesic polar coordinates, \( \Omega \) and \( \partial \Omega \) can be written as

\[
\partial \Omega = \{ (R_u, u) : u \in T_pS^n, \| u \| = 1 \} \quad \text{and} \quad \Omega \setminus \{ p \} = \{ (r, u) : u \in T_pS^n, \| u \| = 1, 0 < r < R_u \}.
\]

Let \( R_m = \min R_u, R_M = \max R_u, R_u \). Note that \( \partial \Omega \) is the zero set of the function \( F(r, u) = r - R_u \). Thus at any point \( q \in \partial \Omega \), \( \nu(q) = \frac{\nabla F}{\| \nabla F \|} \). Let \( \partial_r \) denote the radial vector field starting at \( p \), the center of \( \Omega \). Then \( \tan^2(\theta(q)) = \frac{|\nabla \nu|^2}{\sin^2(R_u)} \), where \( \cos(\theta(q)) = \langle \nu(q), \partial_r(q) \rangle \). Since \( \Omega \) is a star-shaped
bounded domain, $\theta(q) < \frac{n}{2}$ for all $q \in \partial \Omega$. By compactness of $\partial \Omega$, there exists a constant $\alpha$ such that $0 \leq \theta(q) \leq \alpha < \frac{n}{2}$ for all $q \in \Omega$. Define $U_{\alpha} \Omega = \{ u \in T_{\alpha} \Omega, \| u \| = 1 \}$.

Recall that the metric $ds^2$ with respect to the geodesic polar coordinates about a point is given by $ds^2 = dr^2 + \sin^2 r g_{S^{n-1}}$, where $g_{S^{n-1}}$ is the standard metric on $S^{n-1}$. For any smooth function $f$ defined on $\Omega$, $\| \nabla f \|^2 = \left( \frac{\partial f}{\partial r} \right)^2 + \frac{1}{\sin r} \| \nabla f \|^2$, where $\nabla f$ represents the component of $\nabla f$ tangential to $S^{n-1}$.

The following lemma is important to prove the main result of this section.

**Lemma 2.1.** [11, Theorem 2.1] Let $\Omega \subset S^n \setminus \{-p\}$, $\nu$, $\alpha$, $R_m$ and $R_M$ be as the above. Let $a = \tan^2(\alpha)$. Then for a continuously differentiable real valued function $f$ defined on $\Omega$, the following holds.

\[
\frac{\int_{\Omega} \| \nabla f \|^2 \, dv}{\int_{\partial \Omega} f^2 \, ds} \geq \left( \frac{R_m}{R_M} \right) \left( \frac{(2 + a) - \sqrt{a^2 + 4a}}{2 \sqrt{1 + a}} \right) \sin^{n-1}(R_m) \left( \frac{\| \nabla f \|^2}{\sin^{n-1}(R_M) \, f^2} \right),
\]

(3)

where $B(R_m) \subset S^n$ is the geodesic ball of radius $R_m$ centered at the point $p$. Further, equality holds if and only if $\Omega$ is a geodesic ball of radius $R_m$.

The following theorem gives a sharp lower bound of Steklov eigenvalues for a star-shaped domain in $(S^n, ds^2)$.

**Theorem 2.2.** Let $\Omega \subset S^n \setminus \{-p\}$, $\nu$, $\alpha$, $R_m$ and $R_M$ be as above. Let $a = \tan^2(\alpha)$. Then $\mu_l(\Omega)$, $1 \leq l < \infty$ satisfies the following inequality.

\[
\mu_l(\Omega) \geq \left( \frac{R_m}{R_M} \right) \left( \frac{(2 + a) - \sqrt{a^2 + 4a}}{2 \sqrt{1 + a}} \right) \sin^{n-1}(R_m) \left( \frac{\| \nabla f \|^2}{\sin^{n-1}(R_M) \, f^2} \right) \mu_l \left( B \left( R_m \right) \right),
\]

(4)

where $B(R_m) \subset S^n$ is the geodesic ball of radius $R_m$ centered at $p$. Furthermore, equality occurs if and only if $\Omega$ is a geodesic ball of radius $R_m$.

**Proof.** We first construct some specific test functions for the variational characterization of $\mu_l(\Omega)$.

We choose the functions $\phi_i$, $1 \leq i < \infty$ such that $\phi_i \sin^{n-2}(R_u) \sqrt{\sin^2(R_u) + \| \nabla R_u \|^2}$ is the $i$th Steklov eigenfunction of $B(R_m)$. Let $\varphi$ be an arbitrary function which satisfies

\[
\int_{\partial B(R_m)} \varphi \phi_i \sin^{n-2}(R_u) \sqrt{\sin^2(R_u) + \| \nabla R_u \|^2} \, ds = 0.
\]

Note that

\[
\int_{\partial \Omega} \varphi \phi_i \, ds = \int_{U_{\alpha} \Omega} \frac{\varphi \phi_i \sqrt{\sin^2(R_u) + \| \nabla R_u \|^2}}{\sin(R_u)} \sin^{n-1}(R_u) \, du.
\]

By substituting $r = \frac{\varphi R_u}{R_m}$, the above integral becomes

\[
\int_{\partial \Omega} \varphi \phi_i \, ds = \frac{1}{\sin^{n-1}(R_m)} \int_{\partial B(R_m)} \varphi \phi_i \sqrt{\sin^2(R_u) + \| \nabla R_u \|^2} \sin^{n-2}(R_u) \, ds
\]

\[
= 0.
\]
Fix $E = \{\phi_1, \phi_2, \ldots, \phi_{l-1}\}$ in (2). Then it follows from (2) and (3) that

$$
\mu_l(\Omega) \geq \inf_{\phi \neq 0} \frac{\int_\Omega \|\nabla \phi\|^2 \, dv}{\int_{\partial \Omega} \phi^2 \, ds}
\geq \left( \frac{R_m}{R_M} \right) \left( \frac{(2 + a) - \sqrt{a^2 + 4a}}{2\sqrt{1 + a}} \right) \frac{\sin^{n-1}(R_m)}{\sin^{n-1}(R_M)}
\frac{\int_{\partial B(R_m)} \|\nabla \phi\|^2 \, dv}{\int_{\partial B(R_m)} \phi^2 \, ds}.
$$

(5)

Since $\phi_i \sin^{n-2}(R_u) \sqrt{\sin^2(R_u) + \|\nabla R_u\|^2}$ is the $i$th Steklov eigenfunction of $B(R_m)$, we have

$$
\inf_{0 \neq \phi} \frac{\int_{\partial B(R_m)} \|\nabla \phi\|^2 \, dv}{\int_{\partial B(R_m)} \phi^2 \, ds} = \mu_l(B(R_m)).
$$

By substituting the above value in (5), we get (4). Equality case follows from Lemma 2.1. \(\square\)

**Remark 2.3.** In [7], authors obtained a lower bound for the first nonzero Steklov eigenvalue on a star-shaped bounded domain in $\mathbb{R}^n$. Using the above idea, a similar bound can be obtained for all nonzero Steklov eigenvalues on a star-shaped bounded domain in $\mathbb{R}^n$.

### 3. Eigenvalues on a Paraboloid in $\mathbb{R}^3$

In this section, we state and prove the result for a star-shaped domain in a paraboloid $P = \{(x, y, z) \in \mathbb{R}^3 : z = x^2 + y^2 \}$. We first fix some notations which will be used to state the main result.

We use the parametrization $(r \cos \theta, r \sin \theta, r^2)$ for paraboloid $P$, where $\theta \in [0, 2\pi)$ and $r \geq 0$. Then the line element $ds^2$ and the area element $dA$ on $P$ is given by $ds^2 = (1 + 4r^2) \, dr^2 + r^2 \, d\theta^2$ and $dA = r \, \sqrt{1 + 4r^2} \, dr \, d\theta$, respectively.

Let $\Omega \subset P$ be a star-shaped bounded domain with respect to the origin and with smooth boundary $\partial \Omega$. Then there exists a function $R : [0, 2\pi) \rightarrow \mathbb{R}^+$ such that

$$
\partial \Omega = \{(R(\theta), \theta) : \theta \in [0, 2\pi)\}
$$

and

$$
\Omega \setminus \{0\} = \{(r, \theta) : \theta \in [0, 2\pi), 0 < r < R(\theta)\}.
$$

Hereafter, we denote $R(\theta)$ by $R_\theta$. Let $R_m = \min\{R_\theta : \theta \in [0, 2\pi)\}$ and $R_M = \max\{R_\theta : \theta \in [0, 2\pi)\}$. Define $B(R_m) = \{(R_m, \theta) : \theta \in [0, 2\pi)\}$. Let $\nu$ be the outward unit normal to $\partial \Omega$. Let $a = \max\left\{\left(1 + 4R_\theta^2\right) \left(\frac{R_m}{R_\theta}\right)^2 : \theta \in [0, 2\pi)\right\}$. With these notations, we prove the following theorem.

**Theorem 3.1.** Let $\Omega, \nu, a, R_m$ and $R_M$ be as the above. Then $\mu_l(\Omega)$, $1 \leq l < \infty$ satisfies

$$
\mu_l(\Omega) \geq \left( \frac{R_m}{R_M} \right)^3 \left( \frac{(2 + a) - \sqrt{a^2 + 4a}}{2\sqrt{1 + a}} \right) \mu_l(B(R_m)).
$$

(6)

Furthermore, equality holds if and only if $\Omega$ is a geodesic ball of radius $R_m$.

**Proof.** Let $f$ be a continuously differentiable real valued function defined on $\bar{\Omega}$. We find a lower bound for $\int_{\Omega} \|\nabla f\|^2 \, dA$ and upper bound for $\int_{\partial \Omega} f^2 \, ds$ to find a lower bound for the Rayleigh
quotient $\frac{\int_{\Omega} \| \nabla f \|^2 \, dA}{\int_{\partial \Omega} f^2 \, ds}$. We first obtain a lower bound for $\int_{\Omega} \| \nabla f \|^2 \, dA$.

$$
\int_{\Omega} \| \nabla f \|^2 \, dA = \int_0^{2\pi} \int_0^{R_m} \left[ \frac{1}{1 + 4\rho^2} \left( \frac{\partial f}{\partial r} \right)^2 + \frac{1}{r^2} \left( \frac{\partial f}{\partial \theta} \right)^2 \right] r \sqrt{1 + 4\rho^2} \, dr \, d\theta 
$$

$$
= \int_0^{2\pi} \int_0^{R_m} \left[ \frac{r}{\sqrt{1 + 4\rho^2}} \left( \frac{\partial f}{\partial r} \right)^2 + \frac{\sqrt{1 + 4\rho^2}}{r} \left( \frac{\partial f}{\partial \theta} \right)^2 \right] \, dr \, d\theta
$$

Let $\phi = \theta$, $\rho = \frac{r R_m}{R_\phi}$. Since $\rho = \frac{r R_m}{R_\phi} \leq r$, we have $\sqrt{1 + 4\rho^2} \geq \sqrt{1 + 4\rho^2}$ and $\frac{\sqrt{1 + 4\rho^2}}{\rho} \geq \frac{\rho}{\sqrt{1 + 4\rho^2}}$. Thus the above integral can be written as

$$
\int_{\Omega} \| \nabla f \|^2 \, dA \geq \int_0^{2\pi} \int_0^{R_m} \left[ \frac{\rho}{\sqrt{1 + 4\rho^2}} \left( \frac{R_m \partial f}{R_\phi \partial \rho} \right)^2 + \frac{R_m \sqrt{1 + 4\rho^2}}{\rho R_\phi} \left( \frac{\partial f}{\partial \phi} - \frac{\rho R'_\phi \partial f}{R_\phi \partial \rho} \right)^2 \right] \frac{R_\phi}{R_m} d\rho \, d\phi
$$

For any function $\beta^2$ on $\Omega$, Cauchy-Schwarz inequality gives

$$
-2 \frac{\rho R'_\phi \partial f}{R_\phi \partial \rho} \frac{\partial f}{\partial \phi} \geq - \frac{1}{\beta^2} \left( \frac{\rho R'_\phi}{R_\phi} \right)^2 \left( \frac{\partial f}{\partial \rho} \right)^2 - \beta^2 \left( \frac{\partial f}{\partial \phi} \right)^2.
$$

As a consequence, we have

$$
\int_{\Omega} \| \nabla f \|^2 \, dA \geq \int_0^{2\pi} \int_0^{R_m} \left[ \frac{\rho}{\sqrt{1 + 4\rho^2}} \left( \frac{\partial f}{\partial \rho} \right)^2 + \frac{\sqrt{1 + 4\rho^2}}{\rho} \left( 1 - \beta^2 \right) \left( \frac{\partial f}{\partial \phi} \right)^2 \right] \frac{R_\phi}{R_m} d\rho \, d\phi
$$

$$
= \int_0^{2\pi} \int_0^{R_m} \left\{ \left( 1 - (1 + 4\rho^2) \left( \frac{R'_\phi}{R_\phi} \right)^2 \right) \frac{\rho}{\sqrt{1 + 4\rho^2}} \left( \frac{\partial f}{\partial \rho} \right)^2 \right\} \frac{R_\phi}{R_m} d\rho \, d\phi
$$

$$
+ \left( 1 - \beta^2 \right) \frac{\sqrt{1 + 4\rho^2}}{\rho} \left( \frac{\partial f}{\partial \phi} \right)^2 \frac{R_\phi}{R_m} \, d\rho \, d\phi
$$

Note that $(1 + 4\rho^2) \left( \frac{R'_\phi}{R_\phi} \right)^2 \leq (1 + 4R^2_a \left( \frac{R'_\phi}{R_\phi} \right)^2 \leq a$ and $\frac{R'_\phi}{R_\phi} \geq \frac{R'_\phi}{R_m}$. Let’s assume $\beta^2 < 1$, then the above integral becomes

$$
\int_{\Omega} \| \nabla f \|^2 \, dA \geq \left( \frac{R_m}{R_M} \right) \int_0^{2\pi} \int_0^{R_m} \left[ \left( 1 - (1 + 4\rho^2) \left( \frac{R'_\phi}{R_\phi} \right)^2 \right) \frac{\rho}{\sqrt{1 + 4\rho^2}} \left( \frac{\partial f}{\partial \rho} \right)^2 \right] \frac{R_\phi}{R_m} d\rho \, d\phi
$$

$$
+ \left( 1 - \beta^2 \right) \frac{\sqrt{1 + 4\rho^2}}{\rho} \left( \frac{\partial f}{\partial \phi} \right)^2 \frac{R_\phi}{R_m} \, d\rho \, d\phi
$$

\[\]
Solving the equation \( 1 - \left( \frac{1}{\beta^2} - 1 \right) a = 1 - \beta^2 \) for \( \beta^2 \), we obtain
\[
1 - \left( \frac{1}{\beta^2} - 1 \right) a = 1 - \beta^2 = \frac{(2 + a) - \sqrt{a^2 + 4a}}{2} > 0.
\]
By substituting these values, we have
\[
\int_{\Omega} \|\nabla f\|^2 dA \geq \left( \frac{R_m}{R_M} \right)^2 \frac{(2 + a) - \sqrt{a^2 + 4a}}{2} \int_{0}^{2\pi} \int_{0}^{R_m} \left[ \frac{\rho}{\sqrt{1 + 4\rho^2}} \left( \frac{\partial f}{\partial \rho} \right)^2 \right] d\rho d\phi
+ \frac{\sqrt{1 + 4\rho^2}}{\rho} \left( \frac{\partial f}{\partial \phi} \right)^2 d\rho d\phi
= \left( \frac{R_m}{R_M} \right)^2 \frac{(2 + a) - \sqrt{a^2 + 4a}}{2} \int_{0}^{2\pi} \int_{0}^{R_m} \left[ \frac{1}{1 + 4\rho^2} \left( \frac{\partial f}{\partial \rho} \right)^2 + \frac{1}{\rho^2} \left( \frac{\partial f}{\partial \phi} \right)^2 \rho \sqrt{1 + 4\rho^2} d\rho d\phi \right]
= \left( \frac{R_m}{R_M} \right)^2 \frac{(2 + a) - \sqrt{a^2 + 4a}}{2} \int_{B(R_m)} \|\nabla f\|^2 dA.
\]
Now we give a lower bound for \( \int_{\partial \Omega} f^2 ds \).
\[
\int_{\partial \Omega} f^2 ds = \int_{0}^{2\pi} f^2 \sqrt{1 + (1 + 4R_{\phi}^2) \left( \frac{R_{\theta}}{R_{\phi}} \right)^2} R_{\theta} d\theta \\
\leq \sqrt{1 + a} \int_{0}^{2\pi} f^2 R_{\theta} d\theta.
\]
By substituting \( \phi = \theta \), \( \rho = \frac{r R_m}{R_{\phi}} \) and using the fact that \( R_{\theta} \leq R_M \), we get
\[
\int_{\partial \Omega} f^2 ds \leq \frac{R_M \sqrt{1 + a}}{R_m} \int_{0}^{2\pi} f^2 R_m d\phi = \frac{R_M \sqrt{1 + a}}{R_m} \int_{\partial B(R_m)} f^2 ds.
\]
Hence for a continuously differential real valued function \( f \) defined on \( \overline{\Omega} \), it follows from (7) and (8) that
\[
\frac{\int_{\Omega} \|\nabla f\|^2 dA}{\int_{\partial \Omega} f^2 ds} \geq \left( \frac{R_m}{R_M} \right)^2 \frac{(2 + a) - \sqrt{a^2 + 4a}}{2\sqrt{1 + a}} \frac{\int_{B(R_m)} \|\nabla f\|^2 dA}{\int_{\partial B(R_m)} f^2 ds}.
\]
Now using the same argument as in Theorem 2.2, we get the desired result. Further, assume that the equality holds in (6). This is true if and only if \( R_{\phi} = R_m = R_M \). Hence \( \Omega \) is a geodesic ball.

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