THE PRINCIPAL REPRESENTATIONS OF REDUCTIVE
GROUPS WITH FROBENIUS MAPS

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ABSTRACT. We introduce the principal representation category of reductive
groups with Frobenius maps and show that this category is a highest weight
category when the ground field is complex field $\mathbb{C}$. We also study certain kind
of bound quiver algebras whose representations are related to the principal
representation category.

INTRODUCTION

Let $G$ be a connected reductive group defined over the finite field $\mathbb{F}_q$ with the
standard Forbenius map $F$. Assume that $k$ is a field. According to a result of Borel
and Tits [4, Theorem 10.3 and Corollary 10.4], we know that except the trivial
representation, all other irreducible representations of $kG$ are infinite dimensional
if $G$ is a semisimple algebraic group over $\mathbb{F}_q$ and $k$ is infinite with char $k \neq$ char $\bar{\mathbb{F}}_q$.
So it seems to be difficult to study the abstract representations of $G$. However
in the paper [12], Nanhua Xi studied the abstract representations of $G$ over $k$ by
taking the direct limit of the finite dimensional representations of $G_{q^n}$ and has
got many interesting results. Later, motivated by Xi’s idea, the structure of the
permutation module $k[G/B]$ ($B$ is a Borel subgroup of $G$ ) was studied in [6] for
the cross characteristic case and [7] for the defining characteristic case. The paper
[8] studied the general abstract induced module $M(\theta) = kG \otimes_{kB} k_{\theta}$ for any field $k$
with char $k \neq$ char $\bar{\mathbb{F}}_q$ or $k = \bar{\mathbb{F}}_q$, where $T$ is a maximal splitting torus contained
in a $F$-stable Borel subgroup $B$ and $\theta \in \widehat{T}$ (character group of $T$). The induced
module $M(\theta)$ has a composition series (of finite length) if char $k \neq$ char $\bar{\mathbb{F}}_q$. In the
case $k = \bar{\mathbb{F}}_q$ and $\theta$ is a rational character, $M(\theta)$ has such composition series if and
only if $\theta$ is antidominant (see [8] for details). In both cases, the composition factors
has the form $E(\theta)_J$ with $J \subset I(\theta)$ (see Section 1 for explicit definition).

This paper introduces a category $O(G)$ called principal representation category
to study the abstract representations of $G$. It is the full subcategory of $kG$-Mod
such that any object $M$ in $O(G)$ is of finite length and its composition factors
are $E(\theta)_J$ for some $\theta \in \widehat{T}$ and $J \subset I(\theta)$. When $k = \mathbb{C}$, this category $O(G)$
has enough injectives (see Theorem 2.3) and so it is a highest weight category
(under the definition of [9]). Moreover it has a decomposition $O(G) = \bigoplus \theta \in \widehat{T} O(G)_\theta$,
where each $O(G)_\theta$ is the full subcategory of $O(G)$ containing the objects whose
subquotients are $E(\theta)_J$ for a fixed character $\theta$ of $T$. The weight set of $O(G)_\theta$
is finite and therefore there exists a finite dimensional quasi-hereditary algebra $A_\theta$

Date: July 7, 2019.
2010 Mathematics Subject Classification. 20C07, 20G05.
Key words and phrases. Principal representation, highest weight category, quiver algebra.

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such that $\mathcal{O}(G)_{\theta}$ is equivalent to the right $A_{\theta}$-modules. Thus all the indecomposable projectives in $\mathcal{O}(G)$ are given in Section 3. For each $\theta \in \mathbb{T}$, the algebra $A_{\theta}$ is isomorphic to a bound quiver algebra $\mathcal{A}_{\theta}$ (defined in Section 4) when $|I(\theta)| = n$. When $|I(\theta)| = 1$ or 2, this algebra is of finite representation type which means that the number of the indecomposable modules up to isomorphic is finite. By the equivalence of $\mathcal{O}(G)_{\theta}$ and the the right $A_{\theta}$-modules, we give all the indecomposable modules of $\mathcal{O}(G)$ when the rank of $G$ is 1 or 2. However when $n \geq 3$, the algebra $\mathcal{A}_{\theta}$ is of tame representation type. Moreover, $\mathcal{A}_{\theta}$ is of wild type when $n \geq 4$.

This paper is organized as follows: Section 1 contains some preliminaries and we also introduce the principal representation category $\mathcal{O}(G)$ in this section. In Section 2, we show that this category $\mathcal{O}(G)$ has enough injectives and it is a highest weight category when the ground field is $\mathbb{C}$. The algebra structure of $A_{\theta}$ is also studied in this section. Section 3 is devoted to study the algebras $\mathcal{A}_{\theta}$ which is isomorphic to $A_{\theta}$ when $|I(\theta)| = n$.

Acknowledgements The author is grateful to Prof. Nanhua Xi for his constant encouragement and guidance. The author would also like to thank Prof. Toshiaki Shoji, Prof. Zongzhu Lin and Dr. XiaoYun Chen for their helpful discussions and comments. Part of this work was done during the author’s visit to Institute of Mathematics, Chinese Academy of Sciences. The author is grateful to the institute for hospitality.

1. Principal Representation Category

Let $G$ be a connected reductive group defined over $\mathbb{F}_q$ with the standard Frobenius map $F$. Let $B$ be an $F$-stable Borel subgroup, and $T$ be an $F$-stable maximal torus contained in $B$, and $U = R_u(B)$ the $(F$-stable) unipotent radical of $B$. We denote $\Phi = \Phi(G; T)$ the corresponding root system, and $\Phi^+$ (resp. $\Phi^-$) be the set of positive (resp. negative) roots determined by $B$. Let $W = N_G(T)/T$ be the corresponding Weyl group. One denotes $\Delta = \{\alpha_i \mid i \in I\}$ the set of simple roots and $S = \{s_i \mid i \in I\}$ the corresponding simple reflections in $W$. For each $w \in W$, let $\hat{w}$ be one representative in $N_G(T)$. For any $w \in W$, let $U_w$ (resp. $U_w'$) be the subgroup of $U$ generated by all $U_{\alpha}$ (the root subgroup of $\alpha \in \Phi^+$) with $w\alpha \in \Phi^-$ (resp. $w\alpha \in \Phi^+$). The multiplication map $U_w \times U_w' \to U$ is a bijection (see [5]). For any subgroup $H$ of $G$ defined over $\mathbb{F}_q$ and any power of $q^n$ of $q$, denote by $H_q^n$ the set of $F_{q^n}$-points of $H$. Then we have $H = \bigcup_{n=1}^{\infty} H_{q^n}$.

In this paper we fix a field $k$ such that $\text{char } k \neq \text{char } \mathbb{F}_q$. Let $\mathbb{T}$ be the character group of $T$ over $k$. Each $\theta \in \mathbb{T}$ is regarded as a character of $B$ by the homomorphism $B \to T$. Let $k_\theta$ be the corresponding $B$-module. The induced module $M(\theta) = kG \otimes_{kB} k_\theta$ was studied in the paper [3] and all the composition factors of $M(\theta)$ are given in [3]. For convenience, we recall the main results here.

Let $1_\theta$ be a nonzero element in $k_\theta$. We write $x \cdot 1_\theta := x \otimes 1_\theta \in M(\theta)$ for short. It is clear that $M(\theta) = \bigoplus_{w \in W} kU_{-w} \cdot 1_\theta$ and $\{w \cdot 1_\theta \mid w \in W, u \in U_{-w}^{-1}\}$ forms a basis of $M(\theta)$ by the Bruhat decomposition. For each $i \in I$, let $G_i$ be the subgroup of $G$ generated by $U_{\alpha_i}, U_{-\alpha_i}$ and set $T_i = T \cap G_i$. For $\theta \in \mathbb{T}$, define the subset $I(\theta)$ of $I$ by

$$I(\theta) = \{i \in I \mid (\theta|_{T_i}) \text{ is trivial}\}.$$
The Weyl group $W$ acts naturally on $\hat{T}$ by
\[(w \cdot \theta)(t) := \theta^{w}(t) = \theta(w^{-1}t)w\]
for any $\theta \in \hat{T}$.

Let $J \subset I(\theta)$, and $G_J$ be the subgroup of $G$ generated by $G_i$, $i \in J$. We choose a representative $\hat{w} \in G_J$ for each $w \in W_J$. Thus, the element $\hat{w}1_\theta := \hat{w}1_\theta$ ($w \in W_J$) is well defined. For $J \subset I(\theta)$, we set
\[\eta(\theta)_J = \sum_{w \in W_J} (-1)^{\ell(w)}w1_\theta,\]
and let $\Delta(\theta)_J = \sum_{w \in W} kU\hat{w}\eta(\theta)_J$, which is a submodule of $M(\theta)$. In particular, we have $\Delta(\theta)_J = kG\eta(\theta)_J$.

For any $J \subset I(\theta)$, we define
\[E(\theta)_J = \Delta(\theta)_J / \Delta(\theta)'_J,\]
where $\Delta(\theta)'_J$ is the sum of all $\Delta(\theta)_K$ with $J \subset K \subset I(\theta)$. The following theorem (see [3]) gives all the composition factors of $M(\theta)$ explicitly.

**Theorem 1.1.** All modules $E(\theta)_J$ ($J \subset I(\theta)$) are irreducible and pairwise non-isomorphic. In particular, $M(\theta)$ has exactly $2^{\ell(\theta)}\|J\|$ composition factors with each of multiplicity one.

The irreducible $kG$-module $E(\theta)_J$ can also be realized by parabolic induction. Let $\theta \in T$ and $K \subset I(\theta)$. Since $\theta|_{T_i}$ is trivial for all $i \in K$, it induces a character (still denoted by $\theta$) of $T = T/L_K, L_K$. Therefore, $\theta$ is regarded as a character of $L_K$ by the homomorphism $L_K \rightarrow T$ (with the kernel $[L_K, L_K]$), and hence as a character of $P_K$ by letting $U_K$ acts trivially. Set $M(\theta, K) := kG \otimes_{kP_K} \theta$. Let $1_\theta, K$ be a nonzero element in the one dimensional module $k_\theta$ associated to $\theta$. We abbreviate $x1_\theta, K := x \otimes 1_\theta, K \in M(\theta, K)$ as before.

Now set $\mathcal{R}(w) = \{i \in I \mid ws_i < w\}$ and $Z_K = \{w \in W \mid \mathcal{R}(w) \subset I \setminus K\}$. Then by the same argument of [10] Lemma 6.2], we have
\[M(\theta, K) = \sum_{w \in Z_K} kU_{w^{-1}}\hat{w}1_{\theta, K}\]
and $\{u\hat{w}1_{\theta, K} \mid w \in Z_K, u \in U_{w^{-1}}\}$ is a basis of $M(\theta, K)$. Therefore we know that $M(\theta, K)$ a indecomposable $kG$-module. Indeed, we use the basis of $M(\theta, K)$ to consider the endomorphism algebra of $M(\theta, K)$ and have
\[\text{End}_G(M(\theta, K)) \cong \text{Hom}_{P_K}(k_\theta, M(\theta, K)) \cong k\]
which implies that $M(\theta, K)$ is indecomposable.

Using the same proof of [10] Theorem 6.3] and [3] Corollary 3.8], we have the following proposition.

**Proposition 1.2.** For $K \subset I(\theta)$, $M(\theta, K) \cong \bigoplus_{\Delta(\theta)_K} M(\theta)_J$. All the composition factors of $M(\theta, K)$ are $E(\theta)_J$ with $J \subset I(\theta) \setminus K$.

For $J \subset I(\theta)$, set $J' = I(\theta) \setminus J$ and denote $\nabla(\theta)_J = M(\theta, J') = kG \otimes_{kP_J} k_\theta$. Let $E(\theta)'_J$ be the submodule of $\nabla(\theta)_J$ generated by
\[D(\theta)_J := \sum_{w \in W_J} (-1)^{\ell(w)}\hat{w}1_{\theta, J'},\]
Thus we see that \( E(\theta)_J \) is isomorphic to \( E(\theta)_J \) as \( kG \)-modules by [6, Proposition 1.9]. Therefore \( E(\theta)_J \) can be regarded as the scale of \( \nabla(\theta)_J \).

At the end of this section we introduce a category \( \mathcal{O}(G) \) called principal representation category. It is the full subcategory of \( kG\text{-Mod} \) such that any object \( M \) in \( \mathcal{O}(G) \) is of finite length and its composition factors are \( E(\theta)_J \) for some \( \theta \in \hat{T} \) and \( J \subset I(\theta) \). Thus \( \mathcal{O}(G) \) is an abelian category. The category \( \mathcal{O}(G) \) is obviously noetherian and artinian by its construction. By the previous discussion we already have three interesting kinds of modules in \( \mathcal{O}(G) \), the irreducible modules \( E(\theta)_J \), the standard modules \( \nabla(\theta)_J \) and the costandard modules \( \Delta(\theta)_J \). These modules are frequently used in the following sections.

2. Highest Weight Category \( \mathcal{O}(G) \)

Form now on we assume that \( k \) is complex field \( \mathbb{C} \). However we still use \( k \) instead of \( \mathbb{C} \) because I guess the following results are also true for the general field. We will show that the principal representation category \( \mathcal{O}(G) \) is a highest weight category. Firstly we recall the definition of highest weight categories (see [9]).

**Definition 2.1.** Let \( \mathcal{C} \) be a locally artinian, abelian, \( k \)-linear category with enough injectives that satisfies Grothendieck’s condition. Then we call \( \mathcal{C} \) a highest weight category if there exists a locally finite poset \( \Lambda \) (the “weights” of \( \mathcal{C} \)), such that:

(a) There is a complete collection \( \{ S(\lambda)_{\lambda \in \Lambda} \} \) of non-isomorphic simple objects of \( \mathcal{C} \) indexed by the set \( \Lambda \).

(b) There is a collection \( \{ A(\lambda)_{\lambda \in \Lambda} \} \) of objects of \( \mathcal{C} \) and, for each \( \lambda \), an embedding \( S(\lambda) \subset A(\lambda) \) such that all composition factors \( S(\mu)/S(\lambda) \) of \( A(\lambda)/S(\lambda) \) satisfy \( \mu < \lambda \). For \( \lambda, \mu \in \Lambda \), we have that \( \dim_k \Hom_{\mathcal{C}}(A(\lambda), A(\mu)) \) and \( [A(\lambda):S(\mu)] \) are finite.

(c) Each simple object \( S(\lambda) \) has an injective envelope \( I(\lambda) \) in \( \mathcal{C} \). Also, \( I(\lambda) \) has a good filtration \( 0 = F_0(\lambda) \subset F_1(\lambda) \subset \ldots \) such that:

(i) \( F_1(\lambda) \cong A(\lambda) \);

(ii) for \( n > 1 \), \( F_n(\lambda)/F_{n-1}(\lambda) \cong A(\mu) \) for some \( \mu = \mu(n) > \lambda \);

(iii) for a given \( \mu \in \Lambda \), \( \mu = \mu(n) \) for only finitely many \( \lambda \);

(iv) \( \bigcup F_i(\lambda) = I(\lambda) \).

Firstly, we have the following key lemma.

**Lemma 2.2.** For \( \theta \in \hat{T} \), \( K \subset I(\theta) \) and we assume \( k \) is complex field \( \mathbb{C} \), then \( kG \otimes_{kP_k} k_\theta \) is an injective object in \( \mathcal{O}(G) \).

**Proof.** For convenience we just denote the parabolic subgroup \( P_K \) by \( P \) and show that \( \Hom_{\mathcal{O}(G)}(-, kG \otimes_{kP_k} k_\theta) \) is exact in \( \mathcal{O}(G) \). We introduce the \( kG \)-module \( \Hom_{kP}(kG, k_\theta) \), where \( kG \) is regarded as \( (kP, kG) \)-bimodule. Then it is not difficult to see that we have an embedding

\[
\varphi : kG \otimes_{kP} k_\theta \hookrightarrow \Hom_{kP}(kG, k_\theta) \quad \text{as } kG\text{-modules.}
\]

Indeed, let \( \mathcal{M} \) be the set of all functions \( f : G \rightarrow k_\theta \) satisfying \( f(hg) = \theta(h)f(g) \) for any \( h \in P \) and \( g \in G \). For \( f \in \mathcal{M} \) and \( g \in G \), set \( (g : f)(x) = f(xg) \). This defines a \( kG \)-module structure on \( \mathcal{M} \) which is isomorphic to \( \Hom_{kP}(kG, k_\theta) \). Let \( P \setminus G \) be the right cosets of \( P \) in \( G \). Let \( \mathcal{M}_0 \) be the subset of \( \mathcal{M} \) consisting of all functions \( f \) in \( \mathcal{M} \) with finite support on \( P \setminus G \). Let \( \{x_i\} \) be a set of representatives of right cosets of \( P \) in \( G \). Then the map \( f \rightarrow \sum_i x_i^{-1} \otimes f(x_i) \) defines an isomorphism of \( kG \)-module from \( \mathcal{M}_0 \) to \( kG \otimes_{kP} k_\theta \). It is easy to check that \( kG \otimes_{kP} k_\theta \) is a direct
summand of \( \text{Hom}_P(kG, k_\theta) \) since the complement is just the functions in \( \mathcal{M} \) with infinite support on \( P \setminus G \).

Now given a short exact sequence

\[ 0 \rightarrow M \rightarrow N \rightarrow L \rightarrow 0 \text{ in } \mathcal{O}(G). \]

We claim that there exist \( (M, M_a), (N, N_a) \) and \( (L, L_a) \) such that

\[ 0 \rightarrow M_a \rightarrow N_a \rightarrow L_a \rightarrow 0 \]

is exact as \( kG_q \)-modules for each integer \( a \). Here the notation \( (M, M_a) \) means that each \( M_a \) is a finite dimensional \( kG_q \)-module and for each \( a \mid b \), the injective homomorphism \( M_a \rightarrowtail M_b \) is compatible with the homomorphism \( G_q^a \hookrightarrow G_q^b \) and \( M = \lim M_a \) is a \( kG \)-module. Indeed, suppose \( \{x_1, x_2, \ldots\} \) is a \( k \)-basis of \( N \) (since \( N \) is countable dimensional) and we set \( N_a = kG_q \langle x_1, x_2, \ldots, x_a \rangle \). Denote by \( f : N \rightarrow L \) and let \( L_a = f(N_a) \) which is a \( kG_q \)-module. We denote by \( f_a : N_a \rightarrow L_a \) the restriction of \( f \) and set \( M_a = \text{Ker}(f_a) \). Then

\[ 0 \rightarrow \lim M_a \rightarrow N \rightarrow L \rightarrow 0 \]

is exact as \( kG \)-modules which implies \( M = \lim M_a \). The claim is proved.

When \( k = \mathbb{C} \), we know that

\[ 0 \rightarrow \text{Hom}_{P^c}(L, k_\theta) \rightarrow \text{Hom}_{P^c}(N, k_\theta) \rightarrow \text{Hom}_{P^c}(M, k_\theta) \rightarrow 0 \]

is exact, where \( M_a, N_a, L_a \) are regarded as \( kP^c \)-modules. Since these inverse systems satisfy the Mittag-Leffler condition, then

\[ 0 \rightarrow \lim \text{Hom}_{P^c}(L, k_\theta) \rightarrow \lim \text{Hom}_{P^c}(N, k_\theta) \rightarrow \lim \text{Hom}_{P^c}(M, k_\theta) \rightarrow 0 \]

is a short exact sequence. Therefore we have a short exact sequence

\[ 0 \rightarrow \text{Hom}_P(L, k_\theta) \rightarrow \text{Hom}_P(N, k_\theta) \rightarrow \text{Hom}_P(M, k_\theta) \rightarrow 0. \]

By Frobenius reciprocity, we have

\[ \text{Hom}_{kP}(M, k_\theta) \cong \text{Hom}_{kG}(M, \text{Hom}_{kP}(kG, k_\theta)). \]

Thus we get a short exact sequence

\[ 0 \rightarrow \text{Hom}_{kG}(L, \text{Hom}_{kP}(kG, k_\theta)) \rightarrow \text{Hom}_{kG}(N, \text{Hom}_{kP}(kG, k_\theta)) \rightarrow \text{Hom}_{kG}(M, \text{Hom}_{kP}(kG, k_\theta)) \rightarrow 0 \]

which implies that \( kG \otimes_{kP} k_\theta \) is injective in \( \mathcal{O}(G) \) since \( kG \otimes_{kP} k_\theta \) is a direct summand of \( \text{Hom}_{kP}(kG, k_\theta) \). The lemma is proved.

By the above lemma, \( \nabla(\theta)_J \) is the injective envelope of \( E(\theta)_J \). The lemma also suggests the existence of enough injectives:

**Theorem 2.3.** \( \mathcal{O}(G) \) has enough injectives.

**Proof.** We do induction on the length of \( M \in \mathcal{O}(G) \). By Lemma 2.2, \( \nabla(\theta)_J = kG \otimes_{kP} k_\theta \) is the injective envelope of \( E(\theta)_J \). Assuming \( M \) has length \( > 1 \), it has a simple quotient \( E(\theta)_J \), giving rising to a short exact sequence \( 0 \rightarrow N \xrightarrow{i} M \xrightarrow{\partial} E(\theta)_J \rightarrow 0 \). So by induction there exists a monomorphism \( N \xrightarrow{j} Q \) for some injective \( Q \in \mathcal{O}(G) \). Then there exists a morphism \( M \xrightarrow{h} Q \) such that \( i = hf \). Therefore either \( M \xrightarrow{\delta} Q \) is monomorphism or \( M \cong N \oplus E(\theta)_J \). In the latter case, \( M \rightarrow Q \oplus \nabla(\theta)_J \) is monomorphism and the theorem is proved.
Now we show that \( \mathcal{O}(G) \) is a highest weight category. In Definition 2.1, the set of weights is \( \Lambda = \{ (\theta, J) \mid \theta \in \hat{T}, J \subset I(\theta) \} \) and we define the order by
\[(\theta_1, J_1) \leq (\theta_2, J_2), \text{ if } \theta_1 = \theta_2 \text{ and } J_1 \supseteq J_2.\]
Set \( S(\lambda) = A(\lambda) = E(\theta)_J \text{ and } I(\lambda) = \nabla(\theta)_J, \) then the condition Definition 2.1 is easy to check. So \( \mathcal{O}(G) \) is a highest weight category.

The highest weight category has many good properties (see [9]). We list some interesting propositions in the category \( \mathcal{O}(G) \).

**Proposition 2.4.** (1) For \( n \geq 0, \) Ext\(^n\)\(_{\mathcal{O}(G)}(M, N) \) is finite dimensional for all \( M, N \in \mathcal{O}(G) \).

(2) If Ext\(^n\)\(_{\mathcal{O}(G)}(E(\lambda)_J, E(\mu)_K) = 0 \) then \( \lambda = \mu \) and \( J \subseteq K \). Moreover, if \( n > 0 \), we have \( \lambda = \mu \) and \( J \subseteq K \).

According to [8, Corollary 5.6], we know that any finite dimensional irreducible representations of \( G \) are one dimensional when \( k \) is algebraically closed with char \( k \neq \text{char } \overline{F}_q \). Moreover, these irreducible representations are isomorphic to \( E(\theta)\_\omega \) for some \( \theta \in \hat{T} \) with \( I(\theta) = I \). By Proposition 2.4 we have the following corollary immediately.

**Corollary 2.5.** The finite dimensional complex irreducible representations of the group \( G \) is one dimensional and all the finite dimensional complex representations of \( G \) are semisimple.

For \( \theta \in \hat{T} \), let \( \mathcal{O}(G)\_\theta \) be the subcategory of \( \mathcal{O}(G) \) containing the objects whose subquotients are \( E(\theta)_{J} \) for some \( J \subset I(\theta) \). Then by Proposition 2.4, we have \( \mathcal{O}(G) = \bigoplus_{\theta \in \hat{T}} \mathcal{O}(G)\_\theta \). For each \( \theta \in \hat{T} \), \( \mathcal{O}(G)\_\theta \) is a highest weight category and then there exists a finite dimensional quasi-hereditary algebra \( A_{\theta} \) such that \( \mathcal{O}(G)\_\theta \) is equivalent to the right \( A_{\theta} \)-modules. Indeed, if we set \( \mathcal{S}_{\theta} = \bigoplus_{J \subset I(\theta)} \nabla(\theta)_{J} \), then
\[ A_{\theta} \cong \text{End}_{\mathcal{O}(G)}(\mathcal{S}_{\theta}) \]. The functor \( \text{Hom}_{G}(-, \mathcal{S}_{\theta})^* \) form \( \mathcal{O}(G)\_\theta \) to the right \( A_{\theta} \)-modules is an equivalence of categories. Therefore we also see that the category \( \mathcal{O}(G) \) is a Krull-Schmidt category.

Now we want to understand the structure of the algebra \( A_{\theta} \), we have
\[ A_{\theta} \cong \text{End}_{\mathcal{O}(G)}(\mathcal{S}_{\theta}) \cong \bigoplus_{J \subset I(\theta)} \text{Hom}_{\mathcal{O}(G)}(\nabla(\theta)_{J}, \mathcal{S}_{\theta}). \]

The composition factors of \( \nabla(\theta)_{J} = M(\theta, J') \) with \( J' = I(\theta) \setminus J \) are given in Proposition 1.2 therefore
\[ \text{Hom}_{\mathcal{O}(G)}(\nabla(\theta)_{J}, \mathcal{S}_{\theta}) = \text{Hom}_{\mathcal{O}(G)}(\nabla(\theta)_{J}, \bigoplus_{K \subset I(\theta)} \nabla(\theta)_{K}) \]
\[ = \bigoplus_{K \subset J} \text{Hom}_{\mathcal{O}(G)}(\nabla(\theta)_{J}, \nabla(\theta)_{K}) \]
\[ \cong \bigoplus_{K \subset J} \text{Hom}_{\mathcal{O}(J')}\( (1_{\theta}, J'), M(\theta, K') \)) \]
So if let $\varphi_{K \in J}$ be a $kG$-module morphism such that $\varphi_{K \in J}(1_{\theta,J}) = 1_{\theta,K'}$, then $\text{Hom}_{G}(\nabla(\theta)_{J}, \nabla(\theta)_{K}) \cong k\varphi_{K \in J}$. Now the $k$-algebra $A_{\theta}$ has a $k$-basis $\{\varphi_{K \in J} \mid K \subset J \subset I(\theta)\}$. The multiplications are given by

$$\varphi_{K \subset M} \varphi_{L \subset J} = \begin{cases} \varphi_{K \subset J} & \text{if } L = M, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, we get the following corollary.

**Corollary 2.6.** If $|I(\lambda)| = |I(\mu)|$, then $A_{\lambda} \cong A_{\mu}$ as $k$-algebras and thus $\mathcal{O}(G)_{\lambda}$ is equivalent to $\mathcal{O}(G)_{\mu}$.

For each $\theta \in \mathcal{T}$, the right $A_{\theta}$-modules has enough projectives. Then $\mathcal{O}(G)_{\theta}$ and hence $\mathcal{O}(G)$ also have enough projectives. Moreover we have the following proposition.

**Proposition 2.7.** For $\theta \in \mathcal{T}$ and $J \subset I(\theta)$, $\Delta(\theta)_{J}$ is the projective cover of $E(\theta)_{J}$.

**Proof.** The functor $\text{Hom}_{G}(\cdot, \mathcal{F}_{\theta})^{*}$ form $\mathcal{O}(G)_{\theta}$ to the right $A_{\theta}$-modules keeps the projectives. So it is enough to show that $\text{Hom}_{G}(\Delta(\theta)_{J}, \mathcal{F}_{\theta})^{*}$ is a projective right $A_{\theta}$-module.

We denote $\varphi_{J} := \varphi_{K \in J}$ when $J = K$. As a right $A_{\theta}$-module, $A_{\theta}$ has a decomposition $A_{\theta} = \bigoplus_{J \subset I(\theta)} \varphi_{J}A_{\theta}$ and each $\varphi_{J}A_{\theta}$ is indecomposable projective. In the following we will show that $\text{Hom}_{G}(\Delta(\theta)_{J}, \mathcal{F}_{\theta})^{*} \cong \varphi_{J}A_{\theta}$.

All the composition factors of $\Delta(\theta)_{J}$ are $E(\theta)_{K}$ with $J \subset K \subset I(\theta)$. Thus

$$\text{Hom}_{G}(\Delta(\theta)_{J}, \mathcal{F}_{\theta}) = \bigoplus_{K \subset I(\theta)} \text{Hom}_{G}(\Delta(\theta)_{J}, \nabla(\theta)_{K}) = \bigoplus_{J \subset K \subset I(\theta)} \text{Hom}_{G}(\Delta(\theta)_{J}, \nabla(\theta)_{K})$$

Let $f_{K \supset J}$ be the $kG$-module morphism such that $f_{K \supset J}(\eta(\theta)_{J}) = 1_{\theta,K'}$, where $K' = I(\theta) \setminus K$. Then $\text{Hom}_{G}(\Delta(\theta)_{J}, \nabla(\theta)_{K}) \cong k f_{K \supset J}$. Thus $\{f_{K \supset J} \mid J \subset K \subset I(\theta)\}$ is a basis of $\text{Hom}_{G}(\Delta(\theta)_{J}, \mathcal{F}_{\theta})$. The operation of $A_{\theta}$ on this basis is:

$$\varphi_{L \subset M} f_{K \supset J} = \begin{cases} f_{L \supset J} & \text{if } K = M \text{ and } L \supset J, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\tilde{f}_{K \supset J}$ be the dual basis of $f_{K \supset J}$ and set $\tilde{f}_{J} = \tilde{f}_{K \supset J}$ when $J = K$. It is easy to check that the homomorphism send $\tilde{f}_{J}$ to $\varphi_{J}$ induce an isomorphism $\text{Hom}_{G}(\Delta(\theta)_{J}, \mathcal{F}_{\theta})^{*} \cong \varphi_{J}A_{\theta}$ as right $A_{\theta}$-modules. Therefore $\Delta(\theta)_{J}$ is the projective cover of $E(\theta)_{J}$ and the proposition is proved.

**Remark 2.8.** Since the proof in Lemma 2.2 using the fact that each complex representation of a finite group is semisimple, so we need to assume $k$ is the complex field $\mathbb{C}$ or an algebraic closed field of characteristic 0. However, I guess all results in this section are true for general algebraic closed field $k$ with $\text{char} k \neq \text{char} \mathbb{F}_{q}$.

3. The Algebras $\mathcal{A}_{n}$

Assume that $X$ is a finite set with cardinality $|X| = n \geq 1$. We let

$$\mathcal{A}_{n} = \{ (a_{Y,Z}) \in M_{2^{n}}(k) \mid a_{Y,Z} = 0 \text{ if } Y \text{ is not a subset of } Z \}$$
which is a subalgebra of the matrix algebra $M_{2^n}(k)$ whose rows and columns are indexed by the subsets of $X$. The algebra $\mathcal{A_n}$ just depend on the cardinality of $X$. Let $\{e_{Y,Z} \mid Y \subset Z \subset X\}$ be a basis of $\mathcal{A_n}$. Without lost of generality, we can assume each element in $\mathcal{A_n}$ has the form of a upper triangular matrix. The Jacobson radical of $\mathcal{A_n}$ is $\text{Rad}(\mathcal{A_n}) = \sum_{Y \subsetneq Z} k e_{Y,Z}$. As a right $\mathcal{A_n}$-module, $\mathcal{A_n}$ has a decomposition

$$\mathcal{A_n} = \bigoplus_{Y \subset X} e_{Y,Y} \mathcal{A_n}$$

and for each $Y \subset X$, $e_{Y,Y} \mathcal{A_n}$ is a indecomposable projective module. For each integer $n$, it is not difficult to see that $\mathcal{A_n}$ is a basic and connected algebra. Therefore there exists a bound quiver $(\Omega, \mathcal{J})$ such that $k\Omega/\mathcal{J} \cong \mathcal{A_n}$ (see [1, Chapter II]).

As a matter of fact, we can construct the bound quiver $(\Omega, \mathcal{J})$ associate to $X$ with $|X| = n$. The vertices $Q_0$ of the quiver $\Omega = (Q_0, Q_1, s, t)$ is indexed by the subsets of $X$ and we denote it by $Q_0 = \{i_Y \mid Y \subset X\}$. For two vertices $i_Y, i_Z \in Q_0$, there exists an edge $\alpha \in Q_1$ between them if $Y \subset Z$ with $|Z \setminus Y| = 1$ and the orientation is given by $s(\alpha) = i_Y, t(\alpha) = i_Z$. We denote such arrow by $\alpha_{Y,Z}$. The admissible ideal $\mathcal{I}$ in $k\Omega$ is generated by all elements $\omega_1 - \omega_2$ given by the pairs $\{\omega_1, \omega_2\}$ of paths in $\Omega$ having the same starting and ending vertices. Then we have $k\Omega/\mathcal{I} \cong \mathcal{A_n}$. It is not difficult to see that the algebra $A_\theta$ is isomorphic to the algebra $A_m$ with $m = |I(\theta)|$.

For $n = 1$, the algebra $\mathcal{A}_1$ is the path algebra of the Dynkin quiver of type $A_2$. The number of isomorphism classes of indecomposable representations of $\mathcal{A}_1$ is 3. By the correspondence of $O(G)_\theta$ and the right $A_\theta$-modules under the assumption $k = \mathbb{C}$, then for $G = SL_2$, the indecomposable modules in $O(G)$ is $M(tr), St, k_{tr}$ and $\{M(\theta) \mid \theta \in \hat{T} \text{ nontrivial}\}$. Therefore each module in $O(G)$ is a direct sum of these modules.

For $n = 2$, the incidence algebra $\mathcal{A}_2$ is isomorphic the algebra given by the following quiver

$$\begin{array}{cccc}
  a & \xleftarrow{\alpha} & b & \xrightarrow{\beta} \\
  \gamma & \xrightarrow{c} & \delta & \xrightarrow{d} \\
\end{array}$$

bound by the relation $\beta \alpha = \delta \gamma$. Therefore the representations of the algebra $\mathcal{A}_2$ is given by the following diagram

$$\begin{array}{cccc}
  V_a & \xrightarrow{f_{ac}} & V_c & \xrightarrow{f_{cd}} V_d \\
  f_{ab} & & f_{bd} \uparrow & \downarrow f_{bd} \\
  V_b & & V_d \downarrow & \uparrow f_{bd} \\
\end{array}$$

such that $V_a, V_b, V_c, V_d$ are vector spaces and $f_{ab}, f_{bd}, f_{ac}, f_{cd}$ are linear morphisms which satisfy $f_{bd}f_{ab} = f_{cd}f_{ac}$. The Auslander-Reiten quiver of the algebra $\mathcal{A}_2$ was known in the example of Chap VII.2 of the book [2]. Thus the algebra $\mathcal{A}_2$ is of finite type and there are 11 indecomposable $\mathcal{A}_2$-modules up to isomorphism. For
an given representation $V$ of $A_2$, the dimension vector of $V$ is denoted by
\[ \dim V = (\dim V_a, \dim V_b, \dim V_c, \dim V_d). \]
Hence all the indecomposable $A_2$-modules (up to isomorphism) are replaced by their dimension vectors which are the following:

\[ (1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 0), (1, 1, 0, 0), (1, 0, 1, 0), (0, 1, 0, 1), (0, 0, 1, 1), (1, 1, 1, 0), (1, 1, 1, 1). \]

Now we consider the category $O$ of irreducible modules of $A$. Therefore the Tits form of $A$ is known when the rank of $G$ is 2.

Before we consider the algebra $A_n$ for $n \geq 3$, we recall some facts and results about the Tits form of an algebra. For the algebra $A \cong kQ/\langle R \rangle$, let $R$ be the minimal set of relations which generate the ideal $J$. Then the Tits form of $A$ is the integral quadratic form $q_A : Z^m \to Z$ defined by the formula
\[ q_A(x) = \sum_{i \in Q_0} x_i^2 - \sum_{\alpha \in Q_1} x_{\alpha}x_{t(\alpha)} + \sum_{i,j \in Q_0} r_{ij} x_i x_j, \]
where $r_{ij}$ is the cardinality of $R \cap kQ(i, j)$ and $kQ(i, j)$ is the vector space spanned by the paths from $i$ to $j$ (see [3]). It is well known: if $A$ is a tame algebra then the Tits form $q_A$ is weakly positive, that is $q_A(x) \geq 0$ for any $x \in Z^m$ with nonnegative coordinates (see [11]).

In our setting, the minimal set of relations $R$ which generate the ideal $J$ contains all the relations
\[ \alpha_{Y,U} \alpha_{U,Z} = \alpha_{Y,V} \alpha_{V,Z}, \]
with $\alpha_{Y,U}, \alpha_{U,Z}, \alpha_{Y,V}, \alpha_{V,Z}$ are arrows in $Q_1$. Thus for $i_Y, i_Z \in Q_0$, we have
\[ r_{i_Y, i_Z} = \begin{cases} 1 & \text{if } Y \subset Z \text{ with } |Z \setminus Y| = 2, \\ 0 & \text{otherwise.} \end{cases} \]
Therefore the Tits form of $A_n$ is
\[ q_n(x) = \sum_Y x_Y^2 - \sum_{Y \subset Z, |Z \setminus Y| = 1} x_Y x_Z + \sum_{Y \subset Z, |Z \setminus Y| = 2} x_Y x_Z, \]
where $Y, Z$ are subsets of a fixed set $X$ such that $|X| = n$.

We consider the Tits form of the incidence algebra $B_5$ which is given by the following quiver

```
1 -- 2 -- 5
|   |   |
|   |   |
3 -- 6 -- 8

4 --- 7
```
bounded by six relations, where each one is given by \( \omega_1 = \omega_2 \) of paths having the same starting and ending vertices of a parallelogram like the case of \( \mathcal{A}_2 \). Now the Tits form is given by

\[
q(x) = x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2 + x_7^2 + x_8^2 + x_1x_5 + x_1x_6 + x_1x_7 + x_2x_8 + x_3x_8 + x_4x_8 - x_1x_2 - x_1x_3 - x_1x_4 - x_2x_5 - x_3x_6 - x_4x_7 - x_5x_8 - x_6x_8 - x_7x_8.
\]

Do variable substitution with \( x_1 = x + z, x_2 = z, y_1 = x_2 - x_5, y_2 = x_2 - x_6, y_3 = x_3 - x_5, y_4 = x_3 - x_7, y_5 = x_4 - x_6, y_6 = x_4 - x_7 \), then we have

\[
q(x) = 2(x + z)z + x^2 - \frac{1}{2}x(y_1 + y_2 + y_3 + y_4 + y_5 + y_6) + \frac{1}{2}(y_1^2 + y_2^2 + y_3^2 + y_4^2 + y_5^2 + y_6^2)
\]

which is nonnegative for any \( x, z, y_i \in \mathbb{Z} \). Thus the Tits form of \( \mathcal{A}_3 \) is weakly positive.

In the following we use a basic method to show that \( \mathcal{A}_3 \) is a tame algebra (I do not know whether there is a criterion to get this property by the weakly positive Tits form). We classify any indecomposable representation \( \mathbb{V} = (V_i, f_i) \) of \( \mathcal{A}_3 \) to the following three cases by considering the morphism \( f : V_1 \rightarrow V_5 \): (a) When \( V_1 = V_5 = 0 \), then \( \mathbb{V} \) can be regarded as a representation of the Euclidean quiver of type \( \hat{A}_5 \). (b) When \( V_1 \neq 0 \) and \( V_5 \neq 0 \), in this case \( \dim V_1 = \dim V_5 = 1 \) and the only indecomposable module is the projective module \( P(1) \). (c) When \( V_1 = 0 \) or \( V_5 = 0 \), these two situations are symmetric and we consider the case \( V_1 = 0 \) and \( V_5 \neq 0 \). Therefore the linear morphisms \( f_{25}, f_{26}, f_{35}, f_{36}, f_{47}, f_{48}, f_{58}, f_{68}, f_{78} \) are all injective. Since \( \mathbb{V} \) is indecomposable, we also have \( \dim V_i \leq 1 \) for \( 2 \leq i \leq 4 \) and \( \dim V_j \leq 2 \) for \( 2 \leq j \leq 4 \). Therefore there are finitely many indecomposable representations \( \mathbb{V} = (V_i, f_i) \) such that \( V_1 = 0 \) and \( V_5 \neq 0 \).

Now consider the incidence algebra \( \mathcal{A}_4 \), for the vertex \( i_Y \in Q_0 \), we set

\[
x_Y = \begin{cases} 
0 & \text{when } |Y| = 0 \text{ or } 4, \\
1 & \text{when } |Y| = 1 \text{ or } 3, \\
2 & \text{when } |Y| = 2,
\end{cases}
\]

for any vertex \( i_Y \in Q_0 \) in the Tits form of \( \mathcal{A}_4 \). Then by a direct calculation we have \( q(x) = -4 \). This example also implies that the Tits form of \( \mathcal{A}_n \) is not weakly positive when \( n \geq 4 \). Then we have the following proposition.

**Proposition 3.1.** The incidence algebra \( \mathcal{A}_n \) is of wild type when \( n \geq 4 \).

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