FIELDS INTERPRETABLE IN $P$-MINIMAL FIELDS

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ABSTRACT. We prove that an infinite field interpretable in a $p$-adically closed field $K$ is definably isomorphic to a finite extension of $K$. The result remains true in any $P$-minimal field where definable functions are generically differentiable.

In [5], interpretable fields in various expansions of real closed valued fields were classified using analysis of one dimensional quotients which are definably embedded in the field. Here we use similar methods to classify interpretable fields in various expansions of $p$-adically closed fields.

In [8] Pillay proved that any field definable in $\mathbb{Q}_p$ is definably isomorphic to a finite extension of $\mathbb{Q}_p$, and asked whether the same result is true of interpretable fields. Our main theorem gives a positive answer to Pillay’s question.

Theorem 1. Let $K$ be a $P$-minimal field. If, additionally, definable functions in $K$ are generically differentiable, then any infinite field interpretable in $K$ is definably isomorphic to a finite extension of $K$. In particular, the result holds for $p$-adically closed fields.

The proof does not use elimination of imaginaries (that is not known, in general, for $P$-minimal fields). It uses implicitly the fact that $P$-minimal fields are uniformities (see [13] for details). The results of Sections 4-5 do not use $P$-minimality beyond those consequences that are treated axiomatically in [13] (and even [2]). For that reason we expect that our methods can also be applied outside of the $P$-minimal context.

Note that recently Walsberg showed, [15], that a positive answer to the above question of Pillay’s would provide an example of an NIP theory not interpreting an infinite field whose Shelah expansion does. We mention also that a positive answer to Pillay’s question on interpretable fields in $\mathbb{Q}_p$ was announced also by E. Alouf, A. Fornasiero and J. de la Nuez Gonzalez.

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1. BACKGROUND AND PRELIMINARIES

The notion of $P$-minimal fields was introduced by Haskell and Macpherson in [4]. Recall that a valued field $K$ is $P$-minimal if it is $p$-valued, its value group is a $\mathbb{Z}$-group and in every structure elementary equivalent to $K$ every definable subset of $K$ is quantifier-free definable in the Macintyre language for valued fields. We denote its value group by $\Gamma$ and its valuation ring by $O$. It is one
of the main results of \cite{A} that \( P \)-minimal fields are henselian, and therefore, as pure valued fields \( p \)-adically closed.

There are various cell decomposition results for \( P \)-minimal fields, but we will only explicitly use the fact, following from the very definition, that definable subsets of \( K \) itself can be partitioned into finite many cells, as in the case of \( p \)-adically closed fields. I.e. every definable set is a disjunction of sets of the form

\[
\{ x \in K : \gamma_1 < v(x) < \gamma_2 \land P_n(\lambda \cdot x) \},
\]

where \( P_n \) is the \( n \)-th power predicate, \( \lambda \in K \) and \( \gamma_1, \gamma_2 \in \Gamma \cup \{ \infty, -\infty \} \) and \( n \in \mathbb{N} \).

In particular, every infinite definable subset of \( K \) has non-empty interior with respect to the valuation topology (this also follows from Simon’s \cite{S}). Since any \( P \)-minimal field is dp-minimal, this shows that \( P \)-minimal fields satisfy the assumptions of Simon and Walsberg’s tameness conditions for topological dp-minimal structures, \cite{SW}. Most of the topological properties of \( P \)-minimal fields we will be using follow from this work of Simon and Walsberg, though – in most cases – specialised to the case of \( P \)-minimal fields they were already known by earlier work.

In particular, this implies that \( P \)-minimal fields eliminate \( \exists^\infty \) (see \cite[Lemma 2.1]{SW}) in the valued field sort. We will use this fact in several places without explicit reference. We note however, that this property does not transfer to imaginary sorts (such as the value group and \( K/O \)). We will also be using the fact that acl (the model theoretic algebraic closure) satisfies the Steiniz Exchange principle in \( P \)-minimal fields (\cite[Corollary 6.2]{A}). As a consequence, in the valued field sort of \( K \), dp-rank, topological dimension and acl-dimension coincide (see \cite[Proposition 2.4]{SW}) and dp-rank is additive. Most of the result we need from \cite{SW} also appear in the the work of Dolich and Goodrick on viscerality, \cite{DG}, and may be relevant for possible generalisations of the present work into structures with IP.

2. Strong internality to \( K \)

Our aim in this section is, given an interpretable field \( F \), to find a dp-minimal \( S \subseteq F \) that is in definable bijection with a subset of \( K \). We proceed in three steps, we first reduce the problem to subsets of \( F \) that are in definable bijection with an infinite subset of \( K/E \), where \( E \) is some definable equivalence relation on \( K \). We then show that for any interpretable set \( S \) of the form \( K/E \), there exists a finite-to-one (partial) function \( f : S \to V \) with infinite domain, where \( V \) is either \( K \), \( \Gamma \) or \( K/O \). Finally we show that if \( S \subseteq K \) then this set can only be \( K \).

We begin with several general results about definable equivalence relations in \( P \)-minimal fields.

2.1. Definable one-dimensional quotients in \( P \)-minimal fields. We assume that \( K \) is a \( P \)-minimal field and \( E \) a definable equivalence relation on \( K \) with infinitely many classes. Our aim is to find a definable function with finite fibres from an infinite subset of \( K/E \) into either \( \Gamma \), \( K/O \) or \( K \) itself. We need some preparation. For \( \gamma \in \Gamma \) and \( a \in K \), we let

\[
B_\gamma(a) = \{ x \in K : v(x - a) > \gamma \},
\]

and call \( \gamma \) the radius of the \( \gamma \)-ball \( B_\gamma(a) \).

By using the fact that any \( P \)-minimal valued field is elementary equivalent to a \( p \)-adic field, it is not hard to verify the following.
Lemma 2.1. (1) For any $\gamma \in \Gamma$ and $n \in \mathbb{N}$, any $\gamma$-ball contains only finitely many $\gamma + n$-balls.

(2) If a $\gamma$-ball is covered by finitely many balls with radii in $\mathbb{Z}$ then $\gamma \in \mathbb{Z}$.

The following is the key lemma for this part of the argument. We thank D. Macpherson for sketching the proof for us. We first recall [4, Lemma 2.3]:

Fact 2.2. Let $K$ be a $p$-adically closed field and let $n \in \mathbb{N}$ with $n > 1$ and $x, y, a \in K$. Suppose that $v(y - x) > 2v(n) + v(y - a)$. Then $x - a, y - a$ are in the same coset of $P_n$.

Lemma 2.3. Assume that $X \subseteq K$ is a definable set, $\gamma_0 \in \Gamma$ and $X$ contains infinitely many $\gamma_0$-balls. Then for every $k \in \mathbb{N}$, $X$ contains a ball of radius $\gamma_0 - k$.

In particular, if $K$ is $\aleph_0$-saturated then $X$ contains a ball of radius $\gamma$ satisfying $\gamma < \gamma_0 - k$ for all $k \in \mathbb{N}$.

Proof. We assume that $X$ contains infinitely many $\gamma_0$-balls.

Partitioning $X$ into cells, and, if needed, translating by an element of $K$, we may assume that $X$ has the form

$$\{ x \in K : \gamma_1 < v(x) < \gamma_2 \land P_n(\lambda \cdot x) \},$$

where $\lambda \in K$, $\gamma_1 < \gamma_2 \in \Gamma \cup \{\infty, -\infty\}$ and $n \in \mathbb{N}$. Multiplying $X$ by a scalar, we may assume that $\gamma_0 = 0$.

By Lemma 2.1, $X$ is not contained in a ball $B_{-m}(0)$ for any $m \in \mathbb{N}$. So for every $k \in \mathbb{N}$, there is some $y_0 \in X$ such that $v(y_0) < -k$. We can thus fix $y_0 \in X$, such that $v(y_0) + 2v(n) < -1$. We claim that $B_{-1}(y_0) \subseteq X$.

Indeed, assume that $v(x - y_0) > -1$. Then, since $v(y_0) < -1$, we have $v(x) = v(y_0)$ and therefore $\gamma_1 < v(x) = v(y_0) < \gamma_2$. Thus it is sufficient to see that $x$ and $y_0$ are in the same $P_n$-coset.

By our choice of $y_0$ we have $v(x - y_0) > -1 > v(y_0) + 2v(n)$, hence by Fact 2.2 (with $a = 0$ there), $x$ and $y_0$ are in the same $P_n$-coset, so $x \in X$. Thus $B_{-1}(y_0) \subseteq X$.

We have thus shown that $X$ contains at least one ball of radius $-1$. After removing from $X$ a single ball of radius $-1$ it still contains infinitely many $0$-balls so we can find in $X$ a second ball of radius $-1$. Continuing in this manner we find infinitely many balls in $X$ of radius $-1$.

It follows that if $X$ contains infinitely many balls of radius $\gamma_0$ then it also contains infinitely many balls of radius $\gamma_0 - 1$. Repeating the process we obtain in $X$ infinitely many balls of radius $\gamma_0 - k$, for every $k \in \mathbb{N}$.

If $K$ is $\aleph_0$-saturated then the existence of a ball of radius $\gamma$ with $\gamma < \gamma_0 - k$ for all $k \in \mathbb{N}$ follows by saturation. \qed

Lemma 2.4. Let $S = \{ S_t : t \in T \}$ be an infinite definable family of pairwise distinct finite sets of 0-balls. Assume that there exists an integer $n \in \mathbb{N}$ with $|S_t| \leq n$ for all $t \in T$. Then there exists a definable infinite subset $T' \subseteq T$ and a definable finite-to-one $f$ from $T'$ into either $\Gamma$ or $K/O$.

Proof. We may assume that $K$ is $\aleph_0$-saturated, by passing to a large enough elementary extension.

By possibly passing to a definable subset of $T$, we may assume that $|S_t| = |S_{t'}| = n$ for all $t, t' \in T$, for some $n \in \mathbb{N}$.

Given two 0-balls, $b_1 \neq b_2$ denote

$$d(b_1, b_2) = \{ v(x_1 - x_2) : \text{ for any } x_i \in b_i \},$$

with $x_i$ a definable function on $b_i$. If $b_1, b_2$ are not 0-balls, we define $d(b_1, b_2)$ as $\infty$. We now prove a technical lemma.

Lemma 2.5. Let $S$ be a definable family of finite sets of 0-balls, with $|S_t| \leq n$ for all $t \in T$. Suppose $S = \bigcup_{i \in \mathbb{N}} S_i$. Then $d(S, S_i)$ is finite for each $i \in \mathbb{N}$.

Proof. We prove the lemma by induction on $i$. The base case $i = 0$ is trivial. Let $i > 0$. Since $S$ is definable, the set $S'$ of all balls $b \in S_i$ with $d(b, S) = i$ is definable. The family $S'$ defines a partition of $S$ into $|S'|$-many (finite) sets of balls at distance $i$. By induction, $d(S', S_i) < i$. Therefore, $d(S, S_i) < 2i$. \qed
and \( d(b_1, b_1) = 1 \) (notice that \( d(b_1, b_2) \) is independent of the choice of \( x_i \in b_i \)). And for \( t \in T \), let \( d(t) = \min\{d(b_1, b_2) : b_i \in S_t\} \) and note that \( d(t) \leq 1 \).

**Case 1** There exists \( d \in \Gamma \) such that \( \{t \in T : d(t) = d\} \) is infinite.

Reducing \( T \), we may assume that for every \( t \in T \), \( d(t) = d \).

**Case 1(a)** \( d \in \mathbb{Z} \).

Given a 0-ball \( b \), and \( \gamma < 0 \) let \( B_\delta(b) \) be the unique ball of radius \( \gamma \) containing \( b \). With this notation, for \( t \in T \) and \( b, b' \in S_t \) we get \( B_{d-2}(b) = B_{d-2}(b') \) (since \( d(b_1, b_2) = d \)). In other words, for \( t \in T \) the ball \( B_t := B_{d-2}(b) \) for some \( b \in S_t \) is independent of the choice of \( b \in S_t \).

Define \( t_1 \sim t_2 \) if \( B_{t_1} = B_{t_2} \). Because \( d - 2 \in \mathbb{Z} \), each \( B_t \) contains only finitely many 0-balls (Lemma 2.1). In fact each \( B_t \) contains at most \( r := p^{d-2} \)-many 0-balls. Thus each \( \sim \)-class contains at most \( r \)-many elements. Therefore, the map \( t \mapsto B_t \) from \( T \) into \( K/p^{d-2} \mathcal{O} \) has fibres of cardinality at most \( r \). Since \( K/r\mathcal{O} \) is definably isomorphic to \( K/\mathcal{O} \), we are done.

**Case 1(b)** \( d + k < 0 \) for all \( k \in \mathbb{N} \).

For each \( t \in T \), let \( S'_t := \{b \in S_t : (\exists b' \in S_t) \ d(b, b') = d\} \). Replacing \( S_t \) with \( S'_t \) we may assume that every \( b \in S_t \) has some \( b' \in S_t \) such that \( d(b, b') = d \).

Let \( B \) be the collection of all balls of radius \( d + 1 \) in \( K \). For any \( B \in B \) there is no \( t \in T \) with \( \bigcup S_t \subseteq B \), for otherwise we could find \( b, b' \in S_t \) with \( d(b, b') = d \) and \( b, b' \subseteq B \), which is absurd. Let \( X = \bigcup_{t \in T} S_t \). By assumption, \( X \) is a union of infinitely many \( \sim \)-balls. By Lemma 2.3, \( X \) contains some ball \( B_\gamma(x_0) \) with \( \gamma + k < 0 \) for all \( k \in \mathbb{N} \). By Lemma 2.1(2) \( B_\gamma(x_0) \) intersects infinitely many of the \( S_t \).

Consider the ball \( B_0 := B_{d+1}(x_0) \in B \). If \( d + 1 \leq \gamma \) then \( B_0 \supseteq B_\gamma(x_0) \) and hence it clearly intersects infinitely many of the \( S_t \). If \( d + 1 > \gamma \) then \( B_0 \subseteq B_\gamma(x_0) \), thus (since \( d < \mathbb{Z} \)) \( B_0 \) contains infinitely many \( \sim \)-balls inside \( B_\gamma(x_0) \). Because every such ball belongs to some \( S_t \) it follows that \( B_0 \) intersects infinitely many of the \( S_t \) as well.

In abuse of notation we write \( B_0 \cap S_t \) for the set of balls in \( S_t \) which are contained in \( B_0 \). If \( \|B_0 \cap S_t\| \leq 1 \) for all \( t \in T \), then by passing to a definable subset of \( T \) we may assume that \( \|B_0 \cap S_t\| = 1 \) for all \( t \in T \) and as a conclusion we get a definable function (with finite fibers) from \( T \) into \( K/\mathcal{O} \). Otherwise, by replacing \( S \) with \( \{S_t \cap B_0 : t \in T\} \), we may finish the proof by induction on \( K \), since as we noted above, \( \|S_t \cap B_0\| < n \).

**Case 2** For every \( d \in \Gamma \) the set \( \{t \in T : d(t) = d\} \) is finite.

In this case the map \( t \mapsto d(t) \) a finite-to-one function into \( \Gamma \). \( \square \)

We are now in position to obtain the second goal of our strategy:

**Proposition 2.5.** Let \( X = \{X_t : t \in T\} \) be a definable family of infinite pairwise disjoint subsets of \( K \). Then,

1. There exists a definable set \( D \subseteq K \) such that for each \( t \in T \), the set \( D \cap X_t \) is the union of finitely many balls of equal radius.
2. There exists a definable infinite subfamily \( T' \subseteq T \) and a definable finite-to-one \( f \) from \( T' \) to either \( \Gamma \) or \( K/\mathcal{O} \).
Proof. Recall that a definable set $D \subseteq K$ is bounded if it is contained in some ball $B_n(0)$. We first reduce to the case where all the $X_t$ are bounded, by replacing each $X_t$ with the set $B_{\gamma(t)}(0) \cap X_t$, for $\gamma(t) \in \Gamma$ maximal satisfying $\text{Int}(B_n(0) \cap X_t) \neq \emptyset$ (since each $X_t$ is infinite and $K$ has no definable infinite discrete sets, each $t$ has some $\gamma \in \Gamma$ such that $B_{\gamma}(0) \cap X_t$ is infinite, so has nonempty interior). The family \{\$B_{\gamma(t)}(0) \cap X_t : t \in T\$\} is definable, so we may assume that each $X_t$ is bounded.

Consequently, for any $t \in T$ the set $\{\gamma \in \Gamma : (\exists a \in X_t)(B_{\gamma}(a) \subseteq X_t)\}$ has a minimum. Let $\gamma_t \in \Gamma$ be this minimum. By minimality of $\gamma_t$ and Lemma 2.3, for any $t \in T X_t$ contains only finitely many balls of radius $\gamma_t$. We may now choose, uniformly in $t$, the set $D_t \subseteq X_t$, consisting of the union of those balls of radius $\gamma_t$ which are contained in $X_t$. Let $D = \bigcup_t D_t$. This ends (1).

Consider the definable map $T \to \Gamma$ sending $t$ to $\gamma_t$.

If the map is finite-to-one, we are done. Otherwise, there exists $\gamma \in \Gamma$ with $\{t \in T : \gamma_t = \gamma\}$ infinite. By passing to a definable subfamily, we may assume that $\gamma_t = \gamma$ for all $t \in T$. After rescaling, we may assume that $\gamma = 0$. We are now in the situation to apply Lemma 2.4. □

2.2. Interpretable fields. We now return to our problem of an interpretable field $\mathcal{F}$. We start with the following general lemma. It uses the well known coding of finite sets using symmetric functions:

Fact 2.6. Let $(L, +, \cdot, \ldots)$ be any field, possibly with additional structure. Then $L$ eliminates bounded finite imaginaries. I.e. if $\{X_t \subseteq L^n : t \in T\}$ is a definable family of finite sets uniformly bounded in size then there exists a definable map $f : T \to L^m$, for some integer $m$, satisfying that $f(t_1) = f(t_2)$ if and only if $X_{t_1} = X_{t_2}$.

Lemma 2.7. Let $\mathcal{F}$ be a field interpretable in an $\aleph_0$-saturated structure $\mathcal{M}$. Then there exists an infinite definable subset $S \subseteq \mathcal{F}$ that is in definable bijection with $M/E$ for some definable equivalence relation $E$ on $M$.

Proof. For any infinite definable subset $S \subseteq \mathcal{F}$ there exist an integer $n$, definable subset $X \subseteq M^n$ and a definable equivalence relation $E$ such that $S$ is in definable bijection with $X/E$.

Choose $S$ so that $n$ is minimal possible and let $X$ and $E$ be the corresponding definable set $X \subseteq K^n$ and definable equivalence relation, respectively. Assume, for simplicity, that $S = X/E$. We claim that $n = 1$. Otherwise, let $\pi : X \to M^{n-1}$ be the projection onto the first $n - 1$ coordinates. Let $W = \pi(X)$. If for some $w \in W$ the set $X_w/E$ is infinite, where $X_w := X \cap \pi^{-1}(w)$ the set $X_w$ contradicts the minimality of $n$ (since we can definably identify $X_w$ with a subset of $M$).

So $|X_w/E|$ is finite for all $w \in W$. By saturation, there exists a uniform bound $k$ on $|X_w/E|$ as $w$ ranges over $W$. Reducing $X$, we may assume that $|X_w/E| = r$ (for some $r \leq k$) for all $w \in W$. This gives a definable correspondence $C : W \to \mathcal{F}$ given by $w \mapsto X_w/E$. Using Fact 2.6 we can replace $C$ with a function $c : W \to \mathcal{F}'$ whose image is infinite. So there is a projection $\tau : \mathcal{F}' \to \mathcal{F}$ such that $\tau \circ c$ has infinite image in $\mathcal{F}$. Define an equivalence relation $E'$ on $W$ by $E'(u, v)$ if $\tau \circ c(u) = \tau \circ c(v)$. This gives us a bijection between an infinite subset of $\mathcal{F}$ and $W/E'$. Since $W \subseteq M^{n-1}$, this contradicts the minimality of $n$.

Hence $S$ is in definable bijection with $X/E$, where $X \subseteq M$ is a definable subset. By possibly enlarging $S$ by one element, we may assume that $X = M$. □
We now add our underlying assumption that $K$ is a $P$-minimal field. We first note that neither $K/O$ nor $\Gamma$ “eliminate $\exists^\infty$” and this remains true for any infinite definable subset.

Lemma 2.8. Let $V$ be either $\Gamma$ or $K/O$ with the induced structure. Let $X$ be an infinite definable subset of $V$. Then there exists a formula $\varphi(x, y)$ satisfying that for every $n < \omega$ there exists $b_n$ such that $\varphi(V, b_n) \subseteq X$, $|\varphi(V, b_n)| < \omega$ and $\sup_n |\varphi(V, b_n)| = \omega$.

Proof. As usual, there is no harm assuming that $K$ is $\omega_0$-saturated. Assume that $V = \Gamma$. Since $X$ is infinite, by quantifier elimination, we may assume without loss of generality that $X$ is of the form $\{x \in \Gamma : \gamma_1 < x < \gamma_2 \land P_m(x)\}$ for some $\gamma_1, \gamma_2 \in \Gamma \cup \{\pm \infty\}$. Fix some $a \in X$. Since $X$ is infinite, at least one of $\gamma_1$ or $\gamma_2$ is not in the same archimedean component as $a$. So the set $\varphi(x, n) := x \in X \cap (a - n, a + n)$ satisfies the requirements.

Assume that $V = K/O$. Let $\pi : K \rightarrow K/O$ be the natural projection and $Y = \pi^{-1}(X)$. Since $X$ is infinite, $Y$ contains infinitely many balls of radius 0. By Lemma 2.3 there exists $a \in Y$ such that $B_{-k}(a) \subseteq Y$ for all $k \in \mathbb{N}$. By Lemma 2.1, for each $k \in \mathbb{N}$, $B_{-k}(a)$ contains only finitely many, say $n_k$, balls of radius 0. In fact, $\sup_{k \in \mathbb{N}} n_k = \omega$ by Lemma 2.1 (2). By choosing elements $b_k \in K$ with $\psi(b_k) = -k$, the definable sets $\pi(B_{\psi(b_k)}(a)) \subseteq X$ satisfy the requirements. \hfill \Box

Proposition 2.9. For $V = \Gamma$ or $V = K/O$, there is no infinite definable subset $I \subseteq F$ and finite-to-one definable function $f : I \rightarrow V$.

Proof. Assume towards a contradiction that such $I \subseteq F$ and $f : I \rightarrow V$ existed. Since $I$ is infinite and $f$ is finite-to-one, $f(I) \subseteq V$ is a definable infinite subset. Let $\varphi(x, y)$ and $\{b_n : n \in \mathbb{N}\}$ be as provided by Lemma 2.8. Let $\psi(u, y)$ be the formula $(\exists x)(u \in I \land f(u) = x \land \varphi(x, y))$. Since $f$ is finite-to-one, for every $n \in \mathbb{N}$, $n < |\psi(F, b_n)| < \omega$. On the other hand, $F$, with its induced structure, is a field of finite dp-rank, and thus we get a contradiction to [3 Corollary 2.2]. \hfill \Box

We are ready to show that no interpretable field contains an infinite definable subset almost internal to either $\Gamma$ or $K/O$:

Corollary 2.10. There exists a definable dp-minimal $I \subseteq F$, and a definable injection $I \hookrightarrow K$.

Proof. Let $X \subseteq K^n$ be a definable set $E$ a definable equivalence relation on $X$ such that $X/E$ is the universe of $F$. By Lemma 2.7 there exists a definable infinite $S \subseteq F$ and an equivalence relation $E'$ on $K$ such that $K/E'$ is in definable bijection with $S$.

We first claim that only finitely many $E'$-classes could be infinite. Indeed, if $E'$ had infinitely many infinite classes then by restricting to those classes with nonempty interior, we may assume that all classes are infinite. By applying Proposition 2.9 we obtain a definable injection of some infinite subset of $F$ into either $\Gamma$ or $K/O$, contradicting Proposition 2.9.

We may, therefore, assume that all $E'$-classes are finite, and by uniform finiteness, can restrict to the case where all classes are of size $m$ for some $m \in \mathbb{N}$. The definable quotient map $\pi : K \rightarrow K/E'$ gives rise to a definable family $\{\pi^{-1}(e) : e \in K/E'\}$ of distinct finite sets of cardinality $m$. Hence by Fact 2.6 there exists a definable injection $\tilde{f} : K/E' \rightarrow K^n$ for some integer $n$. Composing, we get a definable injection $f : S \hookrightarrow K^n$.

Since $dp-rk(S) = 1$ the set $f(S) \subseteq K^n$ is one-dimensional and hence, by [13 Proposition 4.6], there exists a definable infinite $Y \subseteq f(S)$ and a projection map $\pi : K^n \rightarrow K$ such that $\pi|Y$ is a bijection with an open subset of $K$. \hfill \Box
3. Subset of $\mathcal{F}$ strongly internal to $K$

Let $K$ be a $P$-minimal valued. The previous section was concluded with the proof that there exists an infinite $I \subseteq \mathcal{F}$ “strongly internal” to $K$, where we borrow the following terminology from [5]:

**Definition 3.1.** A definable set $S \subseteq \mathcal{F}$ is strongly internal to $K$ over $A$ if there exists an $A$-definable injection $f : S \to K^n$ for some $n$. It is called strongly internal to $K$ if it is strongly internal over some $A$.

Since, by [12] Theorem 0.3.$\oplus_0$, dp-rank is additive we may conclude:

**Remark 3.2.** If $S$ is strongly internal to $K$ over $A$ then for any $a, b \in S$

$$dp\text{-}rk(a, b/A) = dp\text{-}rk(a/bA) + dp\text{-}rk(b/A).$$

In the present section we study subsets of $\mathcal{F}$ strongly internal to $K$ of maximal dp-rank. Our aim is to show that such sets, at least on some generic subset, are not too far from being closed under the field operations. The proof is built on the analogous statement from [5]. Specifically, if $Y \subseteq \mathcal{F}$ is of maximal dp-rank among all the subsets of $\mathcal{F}$ strongly internal to $K$. We show, Lemma 3.3, that the function $(x, y, z) \mapsto xy - z$ maps a generic subset of $(I + b) \times Y^2$ (some $b$) into $Y$.

**Lemma 3.3.** Let $X \subseteq Z \subseteq K^n$ be $\emptyset$-definable sets with $dp\text{-}rk(X) = dp\text{-}rk(Z)$. For any $d \in X$ with $dp\text{-}rk(d) = dp\text{-}rk(X)$ there exists an open neighborhood $U \subseteq K^n$ such that $U \cap X = U \cap Z$.

**Proof:** The relative interior of $Z$ in $X$ is the set $D$ of $x \in X$ such that there exists $U \ni x$ an open subset of $K^n$ such that $U \cap Z \subseteq X$. By [13] Corollary 4.4, $dp\text{-}rk(X \setminus D) < dp\text{-}rk(X)$ and hence $d \in D$ so there is an open set $U \ni d$ such that $U \cap Z = U \cap X$.

The following lemma plays an important role in our argument.

**Lemma 3.4.** Let $V \subseteq K^n$ be an open set, $x \in K^n$ any element and $A$ an arbitrary set of parameters. Then there exists $B \supseteq A$ and a $B$-definable open subset $U \subseteq V$ such that $dp\text{-}rk(x/B) = dp\text{-}rk(x/A)$. Moreover, if $x \in V$ then we can find such $U$ with $x \in U$.

**Proof:** Since $V$ is open, and is not assumed to be defined over $A$, we may assume that $V = B_{v(g_1)}(a_1) \times \cdots \times B_{v(g_n)}(a_n)$ for some $a_1, \ldots, a_n \in K$ and $g_1, \ldots, g_n \in \Gamma$.

We prove the result by induction on $n$. For $n = 1$, we use compactness to find $r_1 \in K$ with $v(r_1) > v(g_1)$ such that $dp\text{-}rk(r_1/xA) = 1$ and then find $e_1 \in B_{v(r_1)}(a_1)$, such that $dp\text{-}rk(e_1/r_1xA) = 1$. We have $U := B_{v(r_1)}(e_1) \subseteq B_{v_1}(a_1)$ and by exchange, $dp\text{-}rk(x/e_1r_1A) = dp\text{-}rk(x/A)$, so $B = e_1r_1A$ satisfies the lemma. If $x \in V$ then we may start with $a_1 = x$, and then $x \in B_{v(r_1)}(e_1)$.

As for the general case, we first replace, by induction, $B_{v(g_1)}(a_1) \times \cdots \times B_{v(g_{n-1})}(a_{n-1})$ by an open subset $U_1 \subseteq K^{n-1}$ definable over $A_1 \supseteq A$ such that $dp\text{-}rk(x/A_1) = dp\text{-}rk(x/A)$. Then we apply the case $n = 1$ to the last coordinate with $A_1$ replacing $A$.

We can now prove the main lemma of this section that is the main technical lemma of the paper:
Lemma 3.5. Let $I \subseteq \mathcal{F}$ be as provided by Corollary 2.10 and $Y \subseteq \mathcal{F}$ strongly internal to $K$ of maximal dp-rank. Assume that both are defined over a parameter set $A \subseteq K$.

Then, there exists a definable $J \subseteq I$ and a definable $S \subseteq Y^2$ such that $\text{dp-rk}(J) = 1$ and $\text{dp-rk}(S) = 2\text{dp-rk}(Y)$, and there exists $b \in J$, such that for every $(x,y,z) \in J \times S \subseteq J \times Y^2$, we have $(x-b)y + z \in Y$.

Furthermore, if $(b,c,d) \in I \times Y^2$ are such that $\text{dp-rk}(b,c,d/A) = 2\text{dp-rk}(Y) + 1$ then we can choose $J$ and $S \ni (c,d)$, definable over some set $B \supseteq b$ such that $\text{dp-rk}(c,d/B) = 2\text{dp-rk}(Y)$.

Proof. We fix $A$-definable injections $g_1 : I \rightarrow K$ and $g_2 : Y \rightarrow K^I$. For simplicity of notation assume $A = \emptyset$. By assumption $\text{dp-rk}(I) = 1$, and denote $\text{dp-rk}(Y) = n$ for some integer $n > 0$ and fix some $(b,c,d) \in I \times Y \times Y$ such that $\text{dp-rk}(b,c,d) = 2n + 1$. Note that this implies that

$$2n + 1 = \text{dp-rk}(b,c,d) \leq \text{dp-rk}(b,d/c) + \text{dp-rk}(c) \leq \text{dp-rk}(b,d) + \text{dp-rk}(c) \leq 2n + 1$$

and hence $\text{dp-rk}(b,d/c) = n + 1$, $\text{dp-rk}(c) = n$. Similarly, $\text{dp-rk}(b,c) = n + 1$ and $\text{dp-rk}(b/c) = 1$.

For $(x,y,z) \in I \times Y \times Y$ consider the function $f_c(x,z) = xy - z$. Let $e = f_c(b,d)$.

Claim 3.5.1. $b \notin \text{acl}(c,e)$.

Proof. Assume towards a contradiction that there existed an algebraic formula $\varphi(x,e,c)$ isolating $\text{tp}(b/c,e)$, in particular, $\varphi(x,e,c)$ would imply that $x \in I$. By the definition of $f_c$, $d \in \text{dcl}(b,c,e)$, therefore $\varphi(x,e,c) \land f_c(x,y) = e$ is an algebraic formula isolating $\text{tp}(b,d/c,e)$. Hence, there is some integer $m$ with $\exists^m(x,y)(\varphi(x,e,c) \land f_c(x,y) = e)$. By compactness, there is a formula $\psi(z,e) \in \text{tp}(e/c)$ that implies $\exists^m(x,y)(\varphi(x,z,c) \land f_c(x,y) = z)$. In other words, for $D := \psi^{\mathcal{F},c}$, there is a 1-to-$m$ definable partial $c$-definable correspondence $G$ from $D$ into $I \times Y$, sending $e$ to $(b,d)$. Note that if $e_1 \neq e_2 \in D$ then $f_c^{-1}(e_1) \cap f_c^{-1}(e_2) = \emptyset$, thus $G(e_1) \cap G(e_2) = \emptyset$.

The image of $G$ in $I \times Y$ is a $c$-definable set containing $(b,d)$ and since $\text{dp-rk}(b,d/c) = n + 1$, we have $\text{dp-rk}(\text{Im}(G)) = n + 1$.

Recall that $(g_1,g_2) : I \times Y \rightarrow K \times K^I$ is a $\emptyset$-definable injection, so $H = (g_1,g_2) \circ G : \text{Dom}(G) \rightarrow K^{1+1}$ is a one-to-$m$ correspondence, with disjoint images corresponding to distinct $e_1, e_2 \in \text{Dom}(G)$. By Fact 2.6, $H$ induces a definable injection from $\text{Dom}(G)$ into $K^N$, for some integer $N$, as $\text{dp-rk}(\text{Im}(G)) > \text{dp-rk}(Y)$ this is a contradiction to the choice of $Y$. \qed (claim)

We thus conclude that $b \notin \text{acl}(c,f_c(b,d)) = \text{acl}(c,e)$ and in particular $f_c^{-1}(e) \subseteq I \times Y$ is infinite. Notice that by the definition of $f_c$, for every $b' \in I$, and $d_1 \neq d_2 \in Y$, we have $f_c(b_1,e_1) \neq f_c(b_1,e_2)$, thus the projection of $f_c^{-1}(e)$ on the first coordinate, call it $J$, is infinite.

By definition of $J$, for every $x \in J$ there is $z \in Y$ with $xc - z = bc - d = e$ and since $\mathcal{F}$ is a field the map from $J$ to $Y$ mapping $x \in J$ to $z = xc - e = (x-b)c + d$ is injective.

Because $g_1$ is injective, $g_1(J)$ is an infinite subset of $K$ and hence has non-empty interior. By Lemma 3.4 we can find a $B$-definable open subset $U \subseteq g_1(J)$, such that

$$\text{dp-rk}(g_2(c),g_2(d)/Bg_1(b)) = \text{dp-rk}(g_2(c),g_2(d)/g_1(b)) + 2n.$$

Set $J' = g_1^{-1}(U)$. It is an infinite $B$-definable subset of $J$, so for every $x \in J'$, we have $(x-b)c + d \in Y$. Let

$$S = \{(y,z) \in Y^2 : (\forall x \in J')(x-b)y + z \in Y\}.$$
Note that \( (c,d) \in S \) and that \( S \) is \( Bb \)-definable. Because \( (c,d,b) \) and \( (g_1(b),g_2(c),g_2(d)) \) are inter-definable over \( \emptyset \) it follows that \( dp\text{-}rk(c,d/Bb) = 2n \), so the proof is completed. \( \square \)

**Remark 3.6.** Note that in Lemma 3.5 we do not claim that the set \( I \times Y \times Y \) is mapped under \( xy + z \), (or under \( (x - b)y + z \)) onto a subset of \( F \) whose \( dp\text{-}rk \) equals \( dp\text{-}rk(Y) \) (this will turn out to be true once we complete the proof of the main theorem). Instead, at this stage, we only found a subset \( J \times S \subseteq I \times Y \times Y \) of full \( dp\text{-}rk \) whose image has the same rank as \( dp\text{-}rk(Y) \).

### 4. Infinitesimal Neighbourhoods and Topology

In the present section we use methods similar to those in [5] in order to construct a type definable, “infinitesimal” subgroup of \( (F,+) \) which is definably embedded into \( K^n \) for some \( n \). The field \( F \) itself will later be embedded into some \( K^m \), using the subgroup of infinitesimals.

We assume in this section that \( K \) is \( P \)-minimal. We will repeatedly use the fact (already mentioned in the introduction) that in \( P \)-minimal \( acl \) satisfies the Steiniz Exchange principle. Throughout \( +, - , \cdot \) and \( ( )^{-1} \) denote the operations in \( F \).

**Definition 4.1.** For any \( Z \subseteq F \) and a definable injective \( g : Z \to K^m \), let \( \tau_{Z,g} \) be the topology on \( Z \) given by \( \{ g^{-1}(U) : U \subseteq K^m \text{ open} \} \).

We observe that because \( K^n \) has a definable basis for its topology (given, say, by the family of open balls), each \( \tau_{Z,g} \) has a definable basis as well. Also, if \( Z_1 , Z_2 \) are both strongly internal to \( K \) via definable injections \( g_1 \) and \( g_2 \), respectively, then \( \tau_{Z_1 \times Z_2 , g_1 \times g_2} \) is equal to the topology generated by \( \tau_{Z_1,g_1} \times \tau_{Z_2,g_2} \).

We will repeatedly use the following

**Lemma 4.2.** Let \( Z_1 , Z_2 \subseteq F \) be strongly internal to \( K \), witnessed by \( A \)-definable injections \( g : Z_1 \to K^m \) and \( h : Z_2 \to K^m \). Let \( f : Z_1 \to Z_2 \) be an \( A \)-definable function between the topological spaces \( (Z_1,\tau_{Z_1,g}) \) and \( (Z_2,\tau_{Z_2,h}) \). Then the set \( C \subseteq Z_1 \) of \( \tau_{Z_1,g} \)-continuous points of \( f \) is \( A \)-definable, and \( dp\text{-}rk(Z_1 \setminus C) < dp\text{-}rk(Z_1) \).

**Proof.** The definability of \( C \) follows from the definability of a basis for \( \tau_{Z,g} \). The dimension statement follows from the analogous result for definable functions on subsets of \( K^m \), in \( P \)-minimal fields, [4] Theorem 5.1]. \( \square \)

We thus have:

**Lemma 4.3.** Let \( Z \subseteq F \) be definable and \( g : Z \to K^m \), \( h : Z \to K^n \) two \( A \)-definable injections. Then, \( \tau_{Z,g} \) and \( \tau_{Z,h} \) agree at every \( z \in Z \) with \( dp\text{-}rk(z/A) = dp\text{-}rk(Z) \). Namely, there is a common basis for the \( \tau_{Z,g} \)-neighbourhoods and the \( \tau_{Z,h} \)-neighbourhoods of \( z \in Z \).

**Proof.** Apply Lemma 4.2 to \( id : Z \to Z \). \( \square \)

**Definition 4.4.** For \( Z \subseteq F \) a definable set, \( g : Z \to K^m \) a definable injection, and \( d \in Z \), let \( \nu_{Z,g}(d) \) be the partial type given by all definable \( \tau_{Z,g} \)-open sets containing \( d \). We call it the infinitesimal neighborhood of \( d \) with respect to \( \tau_{Z,g} \).
Remark 4.5. If $Z_1, Z_2$ are both strongly internal to $K$ over $A$ via injections $g_1, g_2$, respectively, and $(d_1, d_2) \in Z_1 \times Z_2$ with $\text{dp-rk}(d_1, d_2 / A) = \text{dp-rk}(Z_1) + \text{dp-rk}(Z_2)$ then $\nu_{Z_1,g_1}(d_1) \times \nu_{Z_2,g_2}(d_2) = \nu_{Z_1 \times Z_2,g_1 \times g_2}(d_1, d_2)$ (see above discussion on $\tau_{Z_1,g_1} \times \tau_{Z_2,g_2}$).

By Lemma 4.3 we have:

Corollary 4.6. If $Z \subseteq F$ is strongly internal to $K$ over $A$ and $d \in Z$ is such that $\text{dp-rk}(d / A) = \text{dp-rk}(Z)$ then the infinitesimal neighborhood of $d$ in $Z$ does not depend on any particular $A$-definable injection of $Z$ into some $K^m$. We denote it by $\nu_Z(d)$.

We also have:

Lemma 4.7. Assume that $Z \subseteq F$ is strongly internal to $K$ over $A$ as witnessed by $g$, and $Z_1 \subseteq Z$ is $A$-definable with $\text{dp-rk}(Z_1) = \text{dp-rk}(Z)$. If $d \in Z_1$ is such that $\text{dp-rk}(d / A) = \text{dp-rk}(Z)$ then $\nu_{Z_1}(d) = \nu_Z(d)$.

Proof. By Lemma 4.3 the topologies $\tau_{Z,g}$ and $\tau_{Z_1,g}$ agree on a neighborhood of $d$, thus $\nu_Z(d) = \nu_{Z_1}(d)$. \hfill $\square$

Lemma 4.8. Let $Y_1, Y_2 \subseteq F$ be strongly internal to $K$ over $A$. If $f : Y_1 \to Y_2$ is an $A$-definable partial function, and $a \in \text{dom}(f)$ with $\text{dp-rk}(a / A) = \text{dp-rk}(Y_1)$ then $f$ takes the partial type $\nu_{Y_1}(a)$ into $\nu_{Y_2}(f(a))$. If $f$ is injective then $f(\nu_{Y_1}(a)) = \nu_{Y_2}(f(a))$.

Proof. By Lemma 4.2 $f$ is continuous at $a$ with respect to $\tau_{Y_1,g}$ and $\tau_{Y_2,h}$ (for any $A$-definable $g, h$ witnessing the strong internality of $Y_1, Y_2$, respectively). It is now easy to conclude that $f$ maps $\nu_{Y_1}(a_1)$ into $\nu_{Y_2}(f(a))$. \hfill $\square$

The main result in this section is the following.

Lemma 4.9. Assume that $Y_1, Y_2, Y_3 \subseteq F$ are strongly internal to $K$ over $A$, with $\text{dp-rk}(Y_1) = \text{dp-rk}(Y_2) = \text{dp-rk}(Y_3) = k$, and assume that $Y_1 + Y_2 \subseteq Y_3$. Then

1. For every $c \in Y_1$ and $d \in Y_2$ such that $\text{dp-rk}(c, d / A) = 2k$, we have $\nu_{Y_1}(c) - c = \nu_{Y_2}(d) - d$, and for every $d' \in Y_2$ with $\text{dp-rk}(d' / A) = k$ we have $\nu_{Y_2}(d) - d = \nu_{Y_2}(d') - d'$.
2. For every $d \in Y_2$ such that $\text{dp-rk}(d / A) = k$, the partial type $\nu_{Y_2}(d) - d$ is a type definable subgroup of $(F, +)$.

Proof. It is convenient to work in a $|K|^+$-saturated elementary extension $K$ of $K$ and work with the realizations in $K$ of the partial types $\nu_{Y_1}(c) - c$ and $\nu_{Y_2}(d) - d$ are equal.

Consider the function $F : Y_1 \times Y_2 \to Y_3$, $F(y_1, y_2) = y_1 + y_2$. Applying Lemma 4.8 to $F$ (and using Remark 4.5), we see that $F(\nu_{Y_1}(c) \times \nu_{Y_2}(d)) \subseteq \nu_Y(c + d)$. Consider also the function $H : Y_1 \times Y_3 \to Y_2$, $H(y_1, y_3) = y_3 - y_1$. By Lemma 4.3 (and Remark 4.5), it sends $\nu_{Y_1}(c) \times \nu_{Y_3}(c + d)$ into $\nu_{Y_2}(d)$. It follows that for every $c_1 \in \nu_{Y_1}(c)$, the function $y \mapsto c_1 + y$ is a bijection between $\nu_{Y_2}(d)$ and $\nu_{Y_3}(c + d)$. Indeed, $F(c_1, -)$ is a function from $\nu_{Y_2}(d)$ into $\nu_{Y_3}(c + d)$, whose inverse is $H(c_1, -)$.

By the same observations we show that for every $d_1 \in \nu_{Y_2}(d)$, the function $x \mapsto x + d_1$ is a bijection of $\nu_{Y_1}(c)$ and $\nu_{Y_2}(c + d)$.

In order to prove (1) we show that the sets of realizations in $K$ of the partial types $\nu_{Y_1}(c) - c$ and $\nu_{Y_2}(d) - d$ are equal.
Let \( c_1 \in \nu_{Y_1}(c) \). By our above discussion, there is \( d_1 \in \nu_{Y_2}(d) \) such that \( c_1 + d_1 = c + d \). Thus, \( c_1 - c \in \nu_{Y_2}(d) - d \), so \( \nu_{Y_1}(c) - c \subseteq \nu_{Y_2}(d) - d \). The other inclusion is proved similarly, hence \( \nu_{Y_1}(c) - d = \nu_{Y_2}(d) - d \).

Assume next that \( d' \in Y_2 \) is such that \( \text{dp-rank}(d'/A) = k \). We fix \( c \in Y_1 \) such that \( \text{dp-rank}(c/d, d'/A) = k \). It follows, by Remark 3.2, that \( \text{dp-rank}(c, d/A) = 2k \) and \( \text{dp-rank}(c, d'/A) = 2k \). And then, by what we just saw, \( \nu_{Y_2}(d) - d = \nu_{Y_1}(c) - c = \nu_{Y_2}(d') - d' \).

We obtain \( I \subseteq \nu_{Y_1}(c) \) such that \( c + d_1 = c + d_2 \). It follows that \( d_1 - d_2 = c_1 - c \in \nu_{Y_1}(c) - c \). By what we just showed, \( \nu_{Y_2}(d) - d \), hence \( d_1 - d_2 \in \nu_{Y_2}(d) - d \).

We are now ready to construct the maximal infinitesimal subgroup of \((\mathcal{F}, +)\).

**Proposition 4.10.** Let \( Y \subseteq \mathcal{F} \) be strongly internal to \( K \) over \( A \), of maximal dp-rank \( n \).

1. Let \( c \in Y \) be with \( \text{dp-rank}(d/A) = n \). Then \( \nu_{Y}(d) - d \) is a type definable subgroup of \((\mathcal{F}, +)\). Moreover, it is independent of the choice of \( d \), and we denote it \( \nu_{Y} \).

2. For any strongly internal \( Y_0 \subseteq \mathcal{F} \), with \( \text{dp-rank}(Y_0) = n \), \( \nu_{Y} = \nu_{Y_0} \).

**Proof.** (1) We return to our one dimensional \( I \subseteq \mathcal{F} \) which is strongly internal to \( K \) (Corollary 2.10). We fix any \( b \in I \) and \( c \in Y \) such that \( \text{dp-rank}(b, c, d/A) = 2n + 1 \) and apply Lemma 3.3.

We obtain \( A \subseteq B \)-definable \( J \subseteq I \) and \( S \subseteq Y \times Y \) containing \((c, d)\), with \( \text{dp-rank}(J) = 1 \), \( \text{dp-rank}(S) = 2n \), and \( B \ni b \), such that the map \((x - b)y + z \) sends \( I \times S \) into \( Y \) and such that \( \text{dp-rank}(c, d/B) = 2n \). Because \((c, d) \in S \subseteq Y \times Y \) has maximal dp-rank, it follows by Lemma 3.4 that there exist \( B \subseteq B_0 \)-definable \( \gamma \)-open definable sets \( Y_1, Y_2 \subseteq Y \), neighborhoods of \( c \) and \( d \), respectively, such that \( Y_1 \times Y_2 \subseteq S \) and \( \text{dp-rank}(c, d/B_0) = \text{dp-rank}(c, d/B) = 2n \).

Since \( \text{dp-rank}(J \times S) = 2n + 1 \), we can find \( b_1 \in J \) with \( \text{dp-rank}(b_1, c, d/B_0) = 2n + 1 \). Let \( Y'_1 = (b_1 - b)Y_1 \) and \( c' = (b - b_1)c \). By our assumptions, \( Y'_1 + Y_2 \subseteq Y \) and \( \text{dp-rank}(Y'_1) = \text{dp-rank}(Y_2) = \text{dp-rank}(Y) \). Therefore, the assumptions of Lemma 4.9 are satisfied, with \( \text{dp-rank}(c', d'/B_0) = 2n \). Thus, \( \nu_{Y_2}(d) - d \) is a subgroup of \((\mathcal{F}, +)\), equal to \( \nu_{Y_2}(c') - c' \). By Lemma 4.7, \( \nu_{Y_2}(d) = \nu_{Y}(d) \) and \( \nu_{Y_2}(c) - c = \nu_{Y_2}(c') - c' \). Thus, \( \nu_{Y}(d) - d = \nu_{Y_2}(c') - c' \) is a subgroup of \((\mathcal{F}, +)\).

If we now have \( d' \in Y \) such that \( \text{dp-rank}(d'/A) = n \), then we choose \( c \in Y \) with \( \text{dp-rank}(c, d/A) = \text{dp-rank}(c, d'/A) = 2n \), and then \( b_1 \in J \) such that \( \text{dp-rank}(b_1, c, d/B_0) = \text{dp-rank}(b_1, c, d'/B_0) = 2n + 1 \). Repeating the above argument we conclude that

\[
\nu_{Y}(d) - d = \nu_{Y}(c') - c' = \nu_{Y}(d') - d'.
\]

(2) Let \( Y_0 \subseteq \mathcal{F} \) be any set strongly internal to \( K \) with \( \text{dp-rank}(Y_0) = n \) and let \( Y' = Y \cup Y_0 \). It follows from Lemma 4.7 that \( \nu_{Y} = \nu_{Y'} = \nu_{Y_0} \).

**Corollary 4.11.** The partial type \( \nu \) is invariant under multiplication by scalars from \( \mathcal{F} \).

**Proof.** Let \( c \in \mathcal{F} \) and let \( Y \subseteq \mathcal{F} \) be any definable subset strongly internal to \( K \) over \( \emptyset \), of maximal dp-rank \( n \). Let \( d \in Y \) be such that \( \text{dp-rank}(d/c) = n \), so also \( \text{dp-rank}(c \cdot d) = n \). The function \( x \mapsto cx \) sends \( Y \) to \( cy \), and by lemma 4.8 it sends \( \nu_{Y}(d) \) onto \( \nu_{cy}(cd) \). Hence,

\[
cv = c(\nu_{Y}(d) - d) = cv_{Y}(d) - cd = \nu_{cy}(cd) - cd = \nu.
\]
5. Embedding the field $\mathcal{F}$ into $M_n(K)$

Assume that $K$ is a P-minimal field. In the present section we prove that $\mathcal{F}$ is in definable isomorphism with a finite extension of $K$. We do so by identifying $\mathcal{F}$ with a subfield of $M_n(K)$. Using dp-minimality of $K$, we show that $\mathcal{F}$ must be a finite extension of the canonical embedding of $K$ into $M_n(K)$.

In order to embed $\mathcal{F}$ into the ring of matrices we need to endow the additive subgroup $\nu$ introduced in the previous section, with a $\mathcal{D}$-structure with respect to $K$. That is, we will see that $\nu$ is an open set, and that group operations are differentiable. Recall,

**Definition 5.1.** Given $U \subseteq K^n$ open, a map $f : U \to K^m$, is *differentiable* at $x_0 \in U$ if there exists a linear map $D_{x_0}f : K^n \to K^m$ such that

$$\lim_{x \to x_0} \frac{f(x) - f(x_0) - D_{x_0}f(x - x_0)}{x - x_0} = 0.$$ 

If $f$ is differentiable at a point $x_0$ we say that $x_0$ is a $\mathcal{D}^1$-point of $f$ and that $f$ is $\mathcal{D}^1$ at $x_0$. It is easy to see that if $f$ is definable then set of $x_0$'s at which $f$ is $\mathcal{D}^1$ is definable as well. For the purposes of the present section we assume:

(A) For every definable open $D \subseteq K^n$, and definable $f : D \to K$, there exists an open subset $U \subseteq K^n$ such that $f \restriction U \cap D$ is differentiable.

**Remark 5.2.** It is standard to see that (A) implies that for any such $\emptyset$-definable $f : D \to K$, any $d \in D$ with $\text{dp-rk}(D) = \text{dp-rk}(d)$ is a $\mathcal{D}^1$-point of $f$.

By [13] Proposition 4.6] if $S \subseteq K^n$ has dimension $d$ then there exists an open set $U \subseteq K^n$ and a projection $\pi : K^n \to K^d$ such that $\pi(U)$ is open and $\pi : (U \cap S) \to K^d$ is a homeomorphism onto its image. Thus, if $Y \subseteq \mathcal{F}$ is strongly internal to $K$ with $\text{dp-rk}(Y) = n$ we can find $Y_0 \subseteq Y$ and a definable injection $g_0 : Y_0 \to K^n$ with open image. This is the setting for the following proof, due to Marikova [6], who proved it for continuity in the o-minimal context. The exact same proof goes through for any property that is generically true for all definable functions.

**Proposition 5.3.** Let $Y \subseteq \mathcal{F}$ be a definable subset with dp-rank $n$, $g : Y \to K^n$ an injective $\mathcal{A}$-definable map with open image, and let $\mathcal{K}$ be a $|K|^+$-saturated elementary extension of $K$. Then there exists $a \in Y$, with $\text{dp-rk}(a/A) = \text{dp-rk}(Y)$, such that,

1. The map $x \mapsto g(x + a)$ induces on $\nu(\mathcal{K})$ a structure of a $\mathcal{D}^1$-group, namely, the group operations are $\mathcal{D}^1$ when read via $g$.
2. Any $K$-definable endomorphism $\alpha : (\mathcal{F}(\mathcal{K}),+) \to (\mathcal{F}(\mathcal{K}),+)$ sending $\nu(\mathcal{K})$ to itself is a $\mathcal{D}^1$-map with respect to the above differential structure on $\nu$.

**Proof.** All group operations appearing in the proof are the restriction to $Y$ of $\mathcal{F}$ addition (and subtraction).

Let $d \in Y$ be such that $\text{dp-rk}(d/A) = \text{dp-rk}(Y)$. By replacing $Y$ with $Y - d$, we may assume that $\nu \vdash Y$. Now absorb $A$ and $d$ into the language. In order to keep notation simple we identify $Y$...
with its image under $g$ (so $g = \text{id}$).

(1) Note that $\nu = \{ U \subseteq K^n : U \text{ is } K\text{-open and } 0 \in U \}$ and thus $\nu(K)$ is open in $K^n$.

Since $\nu$ is a group, type definable over $K$, there are, by compactness, $K$-definable open sets $V_1, V_0$, such that $\nu \vdash V_1 \subseteq V_0 \subseteq K^n$ and

$$\varphi_4 : \langle x, y \rangle \mapsto x + y : V_1^2 \to V_0.$$ 

Similarly, we find $V_2 \subseteq V_1$ such that

$$\varphi_3 : \langle x, y, z \rangle \mapsto \langle -z, x + y \rangle : V_2^2 \to V_1^2.$$ 

We also find $V_3 \subseteq V_2$ such that

$$\varphi_2 : \langle x, y, z \rangle \mapsto \langle x + y, z \rangle : V_3^2 \to V_2^2,$$

and $V_4 \subseteq V_3$ such that

$$\varphi_1 : \langle x, y, z, w \rangle \mapsto \langle w + x, -y, z \rangle : V_4^3 \to V_3^3.$$ 

We may assume that all the above are $\emptyset$-definable. Let $a, b \in V_1$ with $\text{dp-rk}(a, b) = 2\text{dp-rk}(V_4)$. By assumption (A), $\varphi_1(x, y, z, b) \in D^1$ at $(a, a, a)$, $\varphi_2$ is $D^1$ at $(b + a, -a, a)$, $\varphi_3(x, y, b) \in (b, a)$ and $\varphi_4$ at $(b^{-1}, b + a)$. Composing, we obtain $\varphi_1 \circ \varphi_3 \circ \varphi_2 \circ \varphi_1(x, y, z) = x - y + z$, so we have shown that the map $\langle x, y, z \rangle \mapsto x - y + z$ is $D^1$ at $(a, a, a)$. In fact, the proof actually provides an open set, $U \subseteq V_4$ which, by compactness, we may take to be $K$-definable, with $a \in U$, such that $\langle x, y, z \rangle \mapsto x - y + z : U^3 \to V_0$ is $D^1$.

Our goal is to show that the push-forward of $+ \downarrow \nu^2$ and $-() \downarrow \nu$ under the map $x \mapsto x + a$ are $D^1$ in the sense of $K$. Namely, we need to prove that the functions $(x - a) + (y - a) + a$ and $-(x - a) + a$ on $\nu_Y(a)^2$ and $\nu_Y(a)$, respectively, are $D^1$. Both follow immediately from our choice of $U$.

(2) Let $e \in \nu(K)$ with $\text{dp-rk}(e/K) = \text{dp-rk}(\nu) = n$. By assumption (A), $e$ is a $D^1$-point of $\alpha$. Since $\alpha$ is a homomorphism and $\nu$ is a $D^1$-group, $\alpha$ is a $D^1$-function on $\nu$. \qed

We need the following easy and well known fact:

**Remark 5.4.** If $K$ is a dp-minimal field, then $K$ has no definable infinite subfields. Indeed, if $L$ were such a field then $L$ itself is dp-minimal. And if we had some $v \in K \setminus L$ we could define $T : L^2 \to K$ by $(a, b) \mapsto a + bv$. Since $v$ is $L$-linearly independent of 1, we get that $T$ is a linear injection, so $\text{dp-rk}(T(L^2)) = 2$ which is impossible.

**Proposition 5.5.** The field $F$ is definably isomorphic to a finite extension of $K$.

**Proof.** Let $g$ be as before, let $Y \subseteq F$ be strongly internal of maximal dp-rank $n$ and assume that $g : Y \to K^n$ is the corresponding injective map.

By Corollary 4.11, for any $z \in F$ the function $\lambda_z : x \mapsto zx$ leaves the type $\nu$ invariant. By Proposition 5.3, $\lambda_z$ is a $D^1$ map on $\nu(K)$. Thus, for every $z \in F$ there exists a definable neighborhood $V \ni 0_F$ such that $\lambda_z$ is $D^1$ on $V$. This is a first order property which holds in $K$ as well. Consequently, to each $z \in F$ may associate, definably, the Jacobian matrix $J_z \in M_n(K)$ of $\lambda_z$ at 0.
As in the proof of [7, Lemma 4.3], an application of the chain rule [9, Remark 4.1.ii], implies that the map \( z \mapsto J_z \) is a ring homomorphism sending \( 1 \in F \) to the identity matrix \( I_n \). Since \( F \) is a field, the map is injective so we may embed \( F \) into a definable subring of \( M_n(K) \).

We may now view \( F \) as a definable subfield of \( M_n(K) \). Let \( K_0 = \{ xI_n : x \in K \} \), where now we take the usual scalar multiplication in the algebra of matrices. Thus \( K_0 \cap F \) is an infinite definable subfield of \( K \) (as they both contain \( I_n \) and are of characteristic 0). Since \( K_0 \cong K \) is dp-minimal, it has no infinite definable subfield, so \( K_0 \cap F = K_0 \) i.e. \( K_0 \subseteq F \). Thus \( F \) is a finite extension of \( K_0 \).

Summing all the above we get:

**Theorem 5.6.** Let \( K \) be a \( P \)-minimal field \( K \). If for every definable \( f : D \to K \), \( D \subseteq K^n \) with non-empty interior, there exists an open subset \( U \subseteq K^n \) such that \( f \upharpoonright U \cap D \) is differentiable then every infinite interpretable field is definably isomorphic to a finite extension of \( K \).

Let \( L^{an} \) be the subanalytic language for the \( p \)-adics, see [1]. For any prime \( p \), let \( \mathbb{Q}_p^{an} \) be the \( p \)-adic field in the subanalytic language.

**Corollary 5.7.** Let \( (K,v) \) be a valued field. Assume that either

1. \( K \) is elementary equivalent to \( \mathbb{Q}_p^{an} \) in the subanalytic language \( L^{an} \) or
2. \( K \) is \( p \)-adically closed.

Then every infinite field interpretable in \( K \) is definably isomorphic to a finite extension of \( K \).

**Proof.** In case of (1), \( K \) is \( P \)-minimal by [14, Theorem B]). As a result it is sufficient to verify assumption (A). Since having non-empty interior is definable in families and being differentiable is definable (using the parameters needed to define the function), it is enough to check assumption (A) for \( \emptyset \)-definable functions whose domain has non-empty interior and assume that either \( f \) is a definable function in \( \mathbb{Q}_p^{an} \) or in a finite extension of \( \mathbb{Q}_p \) (in the valued field language). Since every analytic map is differentiable [9, Proposition 6.1], the result follows readily from [11, Proposition 3.29] in case (1) and [10, Theorem 1.1] in case (2).

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1See, also, the discussion in [10, Section 5]
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