Young-Capelli bitableaux,

Capelli immanants in $U(gl(n))$

and

the Okounkov quantum immanants

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Abstract

We propose a new approach to unified study of determinants, permanents, immanants, (determinantal) bitableaux and symmetrized bitableaux in the polynomial algebra $\mathbb{C}[M_{n,n}]$ as well as of their “Lie analogues” in the enveloping algebra $U(gl(n))$. This leads to new relevant classes of elements in $U(gl(n))$: Capelli bitableaux, right Young-Capelli bitableaux and Capelli immanants.

The set of standard Capelli bitableaux and the set of standard right Young-Capelli bitableaux are bases of $U(gl(n))$, whose action on the Gordan-Capelli basis of $\mathbb{C}[M_{n,n}]$ have remarkable properties.

Capelli immanants can be efficiently computed and provide a system of generators of $U(gl(n))$. The Okounkov quantum immanants are proved to be simple linear combinations of Capelli immanants.

Several examples are provided throughout the paper.

**Keyword:** Young-Capelli bitableaux; Lie superalgebras; immanants; Capelli determinants; Capelli immanants; quantum immanants; central elements; combinatorial representation theory.

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1 Introduction

In this paper, we propose a new approach to a unified study of determinants, permanents, immanants, determinantal bitableaux and symmetrized bitableaux in the polynomial algebra \( \mathbb{C}[M_{n,n}] \) as well as of their Lie analogues in the enveloping algebra \( \mathfrak{U}(gl(n)) \).

Our method heavily relies upon the “Bitableaux correspondence isomorphism/Koszul map” Theorem (BCK Theorem, for short) [6] that describes a pair of mutually inverse vector space isomorphisms, the Koszul map \( K : \mathfrak{U}(gl(n)) \to \mathbb{C}[M_{n,n}] \), and the bitableaux correspondence isomorphism ([5], [6]) \( B : \mathbb{C}[M_{n,n}] \to \mathfrak{U}(gl(n)) \), that deeply link the enveloping algebra \( \mathfrak{U}(gl(n)) \) of the general linear Lie algebra \( gl(n) \) and the polynomial algebra \( \mathbb{C}[M_{n,n}] \) of polynomials in the entries of a “generic” square matrix of order \( n \). The BCK Theorem has to be regarded as a sharpened version of the PBW Theorem for the enveloping algebra \( \mathfrak{U}(gl(n)) \).

The isomorphism \( B \) maps a (determinantal) bitableau \( (S|T) \) in \( \mathbb{C}[M_{n,n}] \) to the Capelli bitableau \( [S|T] \) in \( \mathfrak{U}(gl(n)) \) ([5], [6], [1]; see section 3.3 below and Theorem 4.1). Since the standard bitableaux are a basis of \( \mathbb{C}[M_{n,n}] \) ([17], [15], [14], [20]; see subsection 2.2.5 below, Theorem 2.1), then the standard Capelli bitableaux are a basis of \( \mathfrak{U}(gl(n)) \) [5].

In the polynomial algebra \( \mathbb{C}[M_{n,n}] \), column bitableaux are, up to a sign, monomials. Their images in \( \mathfrak{U}(gl(n)) \) - under the isomorphism \( B \) - are the column Capelli bitableaux (section 4.3 below).

Although column Capelli bitableaux are far from being “monomials” in the enveloping algebra \( \mathfrak{U}(gl(n)) \), their images under the Koszul isomorphism \( K \) are indeed (commutative) monomials in the polynomial algebra \( \mathbb{C}[M_{n,n}] \). Therefore, column Capelli bitableaux play the same crucial role in \( \mathfrak{U}(gl(n)) \) that monomials play in \( \mathbb{C}[M_{n,n}] \). Capelli bitableaux, right Young-Capelli bitableaux and Capelli immanants (section 5 below) expand - up to a global sign - into column Capelli bitableaux just in the same way as bitableaux, right symmetrized bitableaux and immanants expand into the corresponding monomials in \( \mathbb{C}[M_{n,n}] \).

The expressions of column Capelli bitableaux in \( \mathfrak{U}(gl(n)) \) can be simply computed (Proposition 4.3.1 below). Furthermore, column Capelli bitableaux admit an elegant and meaningful interpretation as polynomial differential operators in the Weyl algebra associated to the polynomial algebra \( \mathbb{C}[M_{n,n}] \) (Proposition 4.12 below).

The isomorphism \( B \) leads to a natural definition of the Capelli immanants

\[
\text{Cimm}_\lambda(i_1 i_2 \cdots i_h; j_1 j_2 \cdots j_h), \quad \lambda \vdash h
\]

in \( \mathfrak{U}(gl(n)) \) as images under \( B \) of the classical immanants

\[
\text{imm}_\lambda(i_1 i_2 \cdots i_h; j_1 j_2 \cdots j_h), \quad (i_1 i_2 \cdots i_h); (j_1 j_2 \cdots j_h) \in \mathbb{R}^h
\]
in the polynomial algebra $\mathbb{C}[M_{n,n}]$ (Littlewood and Richardson [26], see also [27], [37], [19]). Capelli immanants are generalizations of the famous Capelli determinant in $U(gl(n))$, just as immanants are generalizations of the determinant in $\mathbb{C}[M_{n,n}]$. Capelli immanants are linear combinations of suitable column Capelli bitableaux, with coefficients $\chi^\lambda(\sigma)$, $\sigma \in S_h$, $\lambda \vdash h$, $\chi^\lambda$ the character of the irreducible representation of symmetric group $S_h$ associated to the partition $\lambda \vdash h$.

Furthermore, the isomorphism $B$ maps a right symmetrized bitableau $(S|T)$ in $\mathbb{C}[M_{n,n}]$ to the right Young-Capelli bitableau $[S|T]$ in $U(gl(n))$ (section 3.3 below, and Theorem 4.2).

The main results we prove in this paper are the following.

• Since the standard right symmetrized bitableaux $(S|T)$ are the Gordan-Capelli basis of $\mathbb{C}[M_{n,n}]$ ([43], [3], [1]; see subsection 2.3 below, Theorem 2.3), then the standard right Young-Capelli bitableaux $[S|T]$ are a basis of $U(gl(n))$:

**Theorem 4.4.** Let $h \in \mathbb{N}$. The set of right Young-Capelli bitableaux

$$\bigcup_{k=0}^{h} \left\{ [S|T] : S, T \text{ standard, } sh(S) = sh(T) = \lambda \vdash k, \lambda_1 \leq n \right\}$$

is a basis of the filtration element $U(gl(n))^{(h)}$. □

This basis acts on standard right symmetrized bitableaux - and, therefore, on $gl(n)$ - irreducible modules - in a quite notable way (Remark 4.5).

• Right symmetrized bitableaux $(S|T)$ of shape $\lambda \vdash h$ expand into immanants defined by the irreducible character $\chi^\lambda$ of the symmetric group $S_h$ associated the same shape $\lambda \vdash h$, and viceversa (Propositions 2.14 and 2.10). Then, by applying the operator $B$, we obtain:

**Theorem 5.2.** Let $\lambda \vdash h$. Any Capelli immanant

$$C_{\text{imm}}^{\lambda}[i_1i_2\cdots i_h; j_1j_2\cdots j_h]$$

can be written as a linear combination of standard right Young-Capelli bitableaux $[U|V]$ in $U(gl(n))$ of the same shape $\lambda$:

$$C_{\text{imm}}^{\lambda}[i_1i_2\cdots i_h; j_1j_2\cdots j_h] = \sum_{U,V} q_{U,V} [U|V],$$

$q_{U,V} \in \mathbb{C}, \quad sh(U) = sh(V) = \lambda.$

□

**Theorem 5.4.** Let $\lambda \vdash h$. Any right Young-Capelli bitableau $[S|T]$ of shape $sh(S) = sh(T) = \lambda$ can be written as a linear combination of
Capelli immanants $\text{Cimm}_\lambda[i_1 i_2 \cdots i_h; j_1 j_2 \cdots j_h]$ associated to the same shape $\lambda$.

**Theorem 5.5.** The set of Capelli immanants

$$\bigcup_{k=0}^{h} \{ \text{Cimm}_\lambda[i_1 i_2 \cdots i_k; j_1 j_2 \cdots j_k];$$

$$\lambda \vdash k, \lambda_1 \leq n, (i_1 i_2 \cdots i_k), (j_1 j_2 \cdots j_k) \in \mathbb{N}^k \}$$

is a spanning set of the filtration element $\mathbf{U}(\mathfrak{gl}(n))^{(h)}$.

- Amazingly, the Okounkov quantum immanants ([30], [31]), that originate from the study of the center $\zeta(n)$ of $\mathbf{U}(\mathfrak{gl}(n))$ as preimages of the Sahi–Okounkov–Schur shifted symmetric polynomials [38], [32] (with respect to the Harish-Chandra isomorphism in the sense of [32]) turn out to be simple linear combinations of Capelli immanants:

**Theorem 6.1.** Let $\mathbf{T}$ be a multilinear standard Young tableau of shape $s\mathbf{h}(\mathbf{T}) = \lambda \vdash h$, and let $\mathbf{E}_\mathbf{T}$ denote the fusion $(n^h \times n^h)$-matrix with entries in $\mathbf{U}(\mathfrak{gl}(n))$ (see section 6). The quantum immanant

$$\text{Tr}(\mathbf{E}_\mathbf{T}) \in \zeta(n)$$

equals the linear combination of Capelli immanants in $\mathbf{U}(\mathfrak{gl}(n))$:

$$(-1)^{\binom{h}{2}} \sum_{h_1+h_2+\cdots+h_n=h} \frac{H(\lambda)}{h_1!h_2!\cdots h_n!} \text{Cimm}_\lambda[1^{h_1}2^{h_2} \cdots n^{h_n}; 1^{h_1}2^{h_2} \cdots n^{h_n}],$$

where $1^{h_1}2^{h_2} \cdots n^{h_n}$ is a short notation for the non decreasing sequence $i_1 i_2 \cdots i_h$ with

$$h_p = \# \{i_q = p; \ q = 1, 2, \ldots, h\}, \ p = 1, 2, \ldots, n.$$

**Example 1.1.** Let $h = 3, \ \lambda = (2, 1), \ n = 2$. Then $H(\lambda) = 3$. Let $\mathbf{T}$ be a multilinear tableau of shape $\lambda$. The quantum immanant

$$\text{Tr}(\mathbf{E}_\mathbf{T}) \in \zeta(2)$$

equals

$$\text{Tr}(\mathbf{E}_\mathbf{T}) = -\frac{3}{2} \left( \text{Cimm}_{(2, 1)}[112; 112] + \text{Cimm}_{(2, 1)}[122; 122] \right).$$

Theorem 6.1, in combination with Proposition 4.3.1, allows the computation of quantum immanants to be reduced to a fairly simple process (see, e.g. Example 6.4 below).
The quantum immanant $Tr(E_T)$ doesn’t depend from the choice of the multilinear tableau $T$ but only from its shape.

The set of quantum immanants associated to shapes $\lambda \vdash h$, $h \in \mathbb{N}$, $\lambda_1 \leq n$ is a basis of the center $\zeta(n)$ ([30], [31]; for further comments and explanations, see section 6 below).

The paper is organized as follows.

In section 2, we recall the fundamental notions of (determinantal) bitableaux, right symmetrized bitableaux and immanants in the polynomial algebra $\mathbb{C}[M_{n,d}]$ as well as some well known basic results concerning them, such as the straightening laws, the standard basis and the Gordan-Capelli basis Theorems and the connections with the representation theory of the symmetric group.

In section 3, we provide a synthetic presentation of the superalgebraic method of virtual variables for $\mathfrak{gl}(n)$. This method allows us to express remarkable classes of elements in $\mathcal{U}(\mathfrak{gl}(n))$ as images - with respect to the Capelli devirtualization epimorphism - of simple elements in a suitable superalgebraic extension of $\mathcal{U}(\mathfrak{gl}(n))$ and to obtain transparent combinatorial descriptions of their actions on irreducible $\mathfrak{gl}(n)-$modules. Among these classes, we recall and discuss the classes of Capelli bitableaux $[S|T]$ and right Young-Capelli bitableaux $[S \downarrow T]$ (see [4], [5], [1]).

In section 4, we recall the BCK Theorem and discuss some of its main implications. This theorem leads to the notion of column Capelli bitableaux that, in turn, sheds new light on the isomorphisms $\mathcal{B}$ and $\mathcal{K}$. The new notion of Capelli immanant in $\mathcal{U}(\mathfrak{gl}(n))$ arises in a natural way.

In sections 5 and 6, we state and prove the main results.

2 The polynomial algebra $\mathbb{C}[M_{n,d}]$

2.1 Biproducts in $\mathbb{C}[M_{n,d}]$

As usual, the algebra of algebraic forms in $n$ vector variables of dimension $d$ is the polynomial algebra in $n \times d$ (commutative) variables:

$$\mathbb{C}[M_{n,d}] = \mathbb{C}[x_{ij}]_{i=1,...,n; j=1,...,d},$$

and $M_{n,d}$ denotes the matrix with $n$ rows and $d$ columns with “generic” entries $x_{ij}$:

$$M_{n,d} = [x_{ij}]_{i=1,...,n; j=1,...,d} = \begin{bmatrix} x_{11} & \cdots & x_{1d} \\ x_{21} & \cdots & x_{2d} \\ \vdots & \vdots & \vdots \\ x_{n1} & \cdots & x_{nd} \end{bmatrix}.$$

For the sake of readability, we will write $(i|j)$ in place of $x_{ij}$, and call the alphabets $L = \{1, 2, \ldots, n\}$ and $P = \{1, 2, \ldots, d\}$ the letter and the place alphabets, respectively; sometimes, we will consistently write $\mathbb{C}[(i|j)]_{i=1,\ldots,n; j=1,\ldots,d}$ in place of $\mathbb{C}[M_{n,d}]$. 

6
Let $\omega = i_1 i_2 \cdots i_p$ be a word on the alphabet $L = \{1, 2, \ldots, n\}$, and $\varpi = j_1 j_1 \cdots j_q$ a word on the alphabet $P = \{1, 2, \ldots, d\}$.

Following [20] and [3], the biproduct of the two words $\omega$ and $\varpi$

$$ (\omega|\varpi) = (i_1 i_2 \cdots i_p|j_1 j_2 \cdots j_q) $$

is the element of $\mathbb{C}[M_{n,d}]$ defined in the following way:

- If $p = q$, the biproduct $(\omega|\varpi)$ is the signed minor

$$ (\omega|\varpi) = (-1)^{\binom{p}{2}} \det \left( (i_r|j_s) \right)_{r,s=1,2,\ldots,p} \in \mathbb{C}[M_{n,d}]. $$

- If $p \neq q$, the biproduct $(\omega|\varpi)$ is set to be zero.

2.2 Bitableaux in $\mathbb{C}[M_{n,d}]$

2.2.1 Young tableaux

Let $\lambda \vdash h$ be a partition, and label the boxes of its Ferrers diagram with the numbers $1, 2, \ldots, h$ in the following way:

$$
\begin{array}{ccccccc}
1 & 2 & \cdots & \cdots & \lambda_1 \\
\lambda_1 + 1 & \lambda_1 + 2 & \cdots & \lambda_1 + \lambda_2 \\
& \cdots & \cdots & \cdots & \cdots \\
& & & \cdots & \cdots & h
\end{array}
$$

A Young tableau $T$ of shape $\lambda$ over a (finite) alphabet $\mathcal{A}$ is a map $T : \underline{h} = \{1, 2, \ldots, h\} \to \mathcal{A}$; the element $T(i)$ is the symbol in the cell $i$ of the tableau $T$.

The sequences

$$
T(1) T(2) \cdots T(\lambda_1),
T(\lambda_1 + 1) T(\lambda_1 + 2) \cdots T(\lambda_1 + \lambda_2),
\ldots
$$

are called the row words of the Young tableau $T$.

We will also denote a Young tableau by its sequence of rows words, that is $T = (\omega_1, \omega_2, \ldots, \omega_p)$. Furthermore, the word of the tableau $T$ is the concatenation

$$ w(T) = \omega_1 \omega_2 \cdots \omega_p. \quad (2) $$

The content of a tableau $T$ is the function $c_T : \mathcal{A} \to \mathbb{N}$,

$$ c_T(a) = \# \{ h \in \underline{h} : T(h) = a \}. $$

A Young tableau $T$ is said to be multilinear if $\mathcal{A} = \underline{h}$ and the map $T$ is a permutation of $\underline{h}$. In the sequel, multilinear Young tableaux will be always denoted by bold symbols, and $T_0$ will denote the “identity” tableau $T_0(i) = i$, $i = 1, 2, \ldots, h$. 

7
Note that, given any Young tableau $S$ on an alphabet $A$ and any multilinear Young tableau $T$ on the alphabet $h$ of the same shape $\lambda \vdash h$, there exists a unique (specialization) map $I : h \to A$ such that

$$S = I \circ T,$$

that is $S(i) = (I \circ T)(i), \ i = 1, 2, \ldots, h$.

To stress this relation between $S$ and $T$, we write

$$S = I_T.$$

(3)

Given a linear order on the alphabet $A$, a Young tableau over $A$ is said to be (semi)standard whenever its rows are increasing from left to right and its columns are non-decreasing from top to bottom.

2.2.2 (determinantal) Young bitableau

Let $S = (\omega_1, \omega_2, \ldots, \omega_p)$ and $T = (\varpi_1, \varpi_2, \ldots, \varpi_p)$ be Young tableaux on $L = \{x_1, x_2, \ldots, x_n\}$ and $P = \{1, 2, \ldots, d\}$ of shapes $\lambda$ and $\mu$, respectively.

Following again [20] and [3], the (determinantal) Young bitableau

$$\begin{pmatrix} \omega_1 & \varpi_1 \\ \omega_2 & \varpi_2 \\ \vdots & \vdots \\ \omega_p & \varpi_p \end{pmatrix}$$

(4)

is the element of $\mathbb{C}[M_{n,d}]$ defined in the following way:

- If $\lambda = \mu$, the (determinantal) Young bitableau $(S|T)$ is the signed product of the biproducts of pairs of corresponding rows:

$$\begin{pmatrix} \omega_1 & \varpi_1 \\ \omega_2 & \varpi_2 \\ \vdots & \vdots \\ \omega_p & \varpi_p \end{pmatrix} = \pm (\omega_1|\varpi_1)(\omega_2|\varpi_2)\cdots(\omega_p|\varpi_p),$$

(5)

where

$$\pm = (-1)^{\ell(\omega_2) + \ell(\omega_3) + \ell(\omega_4) + \cdots + \ell(\omega_p) + \ell(\varpi_1) + \ell(\varpi_2) + \cdots + \ell(\varpi_{p-1})},$$

(6)

and the symbol $\ell(w)$ denotes the length of the word $w$.

- If $\lambda \neq \mu$, the Young bitableau $(S|T)$ is set to be zero.

2.2.3 Column bitableaux in $\mathbb{C}[M_{n,d}]$

A column tableau is a Young tableau of shape $\lambda = (1, 1, \ldots, 1) \vdash h$, and the number $h$ of 1’s is called the depth.

A column bitableau in $\mathbb{C}[M_{n,d}]$ is a (determinantal) bitableau $(S|T)$, where $S$ and $T$ are column Young tableaux of the same depth. A column bitableau of
depth \( h \) equals, up to a sign, a monomial in \( \mathbb{C}[M_{n,d}] \):

\[
\begin{pmatrix}
  i_1 & j_1 \\
  i_2 & j_2 \\
  \vdots & \vdots \\
  i_h & j_h \\
\end{pmatrix} = (-1)^{(h)} \binom{h}{2} (i_1|j_1)(i_2|j_2) \cdots (i_h|j_h). \quad (7)
\]

Although the notion of column bitableaux may appear fairly obvious, it will play a crucial role in the passage from the polynomial algebra \( \mathbb{C}[M_{n,d}] \) to the enveloping algebra \( \mathfrak{U}(gl(n)) \) via the bitableaux correspondence isomorphism, Section 4 below.

### 2.2.4 Bitableaux expansion into column bitableaux

Recall that

\[
(i_1i_2 \cdots i_h|j_1j_2 \cdots j_h) = (-1)^{\binom{h}{2}} \det([i_s|j_t])_{s,t=1,2,\ldots,h} \in \mathbb{C}[M_{n,d}],
\]

and, therefore, the biproduct \( (i_1i_2 \cdots i_h|j_1j_2 \cdots j_h) \in \mathbb{C}[M_{n,d}] \) expands into column bitableaux as follows:

\[
(i_1i_2 \cdots i_h|j_1j_2 \cdots j_h) = \sum_{\sigma \in S_h} (-1)^{|\sigma|} \begin{pmatrix}
  i_{\sigma(1)} & j_1 \\
  i_{\sigma(2)} & j_2 \\
  \vdots & \vdots \\
  i_{\sigma(h)} & j_h \\
\end{pmatrix} = \sum_{\sigma \in S_h} (-1)^{|\sigma|} \begin{pmatrix}
  i_1 & j_{\sigma(1)} \\
  i_2 & j_{\sigma(2)} \\
  \vdots & \vdots \\
  i_h & j_{\sigma(h)} \\
\end{pmatrix}.
\]

Notice that, in the passage from monomials to column bitableaux, the sign \((-1)^{\binom{h}{2}}\) disappears, due to eq. (7).

The preceding arguments extend to bitableaux of any shape \( \lambda \), \( \lambda_1 \leq n \).

Given a bitableau \( (S|T) \in \mathbb{C}[M_{n,d}] \) of shape \( \lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m) \vdash h \) with

\[
S = \begin{pmatrix}
  i_{p_1} & \ldots & i_{p_{\lambda_1}} \\
  i_{q_1} & \ldots & i_{q_{\lambda_2}} \\
  \vdots & \ddots & \vdots \\
  i_{r_1} & \ldots & i_{r_{\lambda_m}} \\
\end{pmatrix}, \quad T = \begin{pmatrix}
  j_{s_1} & \ldots & j_{s_{\lambda_1}} \\
  j_{t_1} & \ldots & j_{t_{\lambda_2}} \\
  \vdots & \ddots & \vdots \\
  j_{v_1} & \ldots & j_{v_{\lambda_m}} \\
\end{pmatrix},
\]

we recall that

\[
(S|T) = \pm (i_{p_1} \cdots i_{p_{\lambda_1}}|j_{s_1} \cdots j_{s_{\lambda_1}}) \cdots (i_{r_1} \cdots i_{r_{\lambda_m}}|j_{v_1} \cdots j_{v_{\lambda_m}})
\]

by definition (5), where

\[
\pm = (-1)^{\lambda_2 \lambda_1 + \lambda_3(\lambda_1 + \lambda_2) + \cdots + \lambda_m(\lambda_1 + \lambda_2 + \cdots + \lambda_{m-1})}.
\]

by eq. (6).

Notice that

\[
\lambda_2 \lambda_1 + \lambda_3(\lambda_1 + \lambda_2) + \cdots + \lambda_m(\lambda_1 + \lambda_2 + \cdots + \lambda_{m-1}) + \sum_{k=1}^{m} \binom{\lambda_k}{2} = \binom{h}{2}. \quad (8)
\]
We have

\[(S|T) = \sum_{\sigma_1, \ldots, \sigma_m} (-1)^{\sum_{k=1}^m |\sigma_k|} \left( \begin{array}{c|c}
  i_{p_{\sigma_1(1)}} & j_{s_{\sigma_1(1)}} \\
  \vdots & \vdots \\
  i_{p_{\sigma_1(\lambda_1)}} & j_{s_{\lambda_1}} \\
  \vdots & \vdots \\
  i_{r_{\sigma_m(1)}} & j_{v_{1}} \\
  \vdots & \vdots \\
  i_{r_{\sigma_m(\lambda_m)}} & j_{v_{\lambda_m}} \\
\end{array} \right) \right.

\]

= \sum_{\sigma_1, \ldots, \sigma_m} (-1)^{\sum_{k=1}^m |\sigma_k|} \left( \begin{array}{c|c}
  i_{p_1} & j_{s_{\sigma_1(1)}} \\
  \vdots & \vdots \\
  i_{p_{\lambda_1}} & j_{s_{\lambda_1}} \\
  \vdots & \vdots \\
  i_{r_1} & j_{v_{\sigma_m(1)}} \\
  \vdots & \vdots \\
  i_{r_{\lambda_m}} & j_{v_{\sigma_m(\lambda_m)}} \\
\end{array} \right)

where the multiple sums range over all permutations \(\sigma_1 \in S_{\lambda_1}, \ldots, \sigma_m \in S_{\lambda_m}\).

Notice that only the signs of permutations remain, due to identity (8).

2.2.5 The straightening algorithm and the standard basis of \(\mathbb{C}[M_{n,d}]\)

Given a positive integer \(h \in \mathbb{Z}^+\), let \(\mathbb{C}_h[M_{n,d}]\) denote the \(h\)-th homogeneous component of \(\mathbb{C}[M_{n,d}]\).

Consider the set of all bitableaux \((S|T) \in \mathbb{C}_h[M_{n,d}]\), where \(sh(S) = sh(T) \vdash h\). In the following, let denote by \(\leq\) the linear order on this set defined by the following two steps:

- \((S|T) < (S'|T')\) whenever \(sh(S) \prec_l sh(S')\), where \(\prec_l\) denotes the lexicographic order on partitions \(\lambda \vdash h\).

- \((S|T) < (S'|T')\) whenever \(sh(S) = sh(S')\), \(w(S)w(T) >_l w(S')w(T')\).

where the shapes and the concatenated words \(w(S)w(T), w(S')w(T')\) of the tableaux \(S, T\) and \(S', T'\) (see eq. (2)) are compared in the lexicographic order.

The next Theorem is a well-known result for the polynomial algebra \(\mathbb{C}[M_{n,d}]\) ([17], [15], [14], for the general theory of standard monomials see, e.g. [36], Chapt. 13).

**Theorem 2.1.** (The Standard basis theorem for \(\mathbb{C}_h[M_{n,d}]\))
The set
\[ \{(S\mid T) \text{ standard}; \ sh(S) = sh(T) = \lambda \vdash h, \lambda_1 \leq n, d \} \]
is a basis of \( C_h[M_{n,d}] \).

Furthermore, a Young bitableau \((P\mid Q) \in C_h[M_{n,d}]\) can be uniquely written as a linear combination
\[
(P\mid Q) = \sum_{S,T} a_{S,T} \ (S\mid T),
\]
(9)
of standard bitableaux \((S\mid T)\), where
- the coefficient \( a_{S,T} = 0 \) whenever \((S\mid T) \not\preceq (P\mid Q)\);
- the contents of the tableaux are preserved, that is \( c_S = c_P, c_T = c_Q \).

For a proof, see e.g. [15], [14].

2.3 Right symmetrized bitableaux and the Gordan-Capelli basis of \( C[M_{n,d}] \)

Given a Young tableau \( T \), we say that another tableau \( \overline{T} \) is a column permuted of \( T \) whenever each column of \( \overline{T} \) can obtained by permuting the corresponding column of \( T \).

A right symmetrized bitableau \((S\overline{T})\) is the element of the polynomial algebra \( C[M_{n,d}] \) defined as the following sum of bitableaux:
\[
(S\overline{T}) = \sum_{T} (S\mid T),
\]
where the sum is extended over all \( \overline{T} \) column permuted of \( T \) (hence, repeated entries in a column give rise to multiplicities).

Example 2.2.
\[
\begin{pmatrix} 1 & 3 & 1 & 2 \\ 2 & 4 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 1 & 2 \\ 2 & 4 & 1 & 3 \end{pmatrix} + \begin{pmatrix} 1 & 3 & 1 & 2 \\ 2 & 4 & 1 & 3 \end{pmatrix} + \begin{pmatrix} 1 & 3 & 1 & 2 \\ 2 & 4 & 1 & 3 \end{pmatrix} + \begin{pmatrix} 1 & 3 & 1 & 2 \\ 2 & 4 & 1 & 3 \end{pmatrix} + 2 \begin{pmatrix} 1 & 3 & 1 & 2 \\ 2 & 4 & 1 & 3 \end{pmatrix} + 2 \begin{pmatrix} 1 & 3 & 1 & 2 \\ 2 & 4 & 1 & 3 \end{pmatrix}.
\]

We recall a fundamental result:

Theorem 2.3. (The Gordan-Capelli basis of \( C[M_{n,d}] \)) Let \( h \in \mathbb{N} \).
The set
\[
\{(S \mid T) ; S, T \text{ standard, } sh(S) = sh(T) = \lambda \vdash h, \lambda_1 \leq n, d\}
\]
is a basis of \(\mathbb{C}_h[M_{n,d}]\).

- Any right symmetrized bitableau \((U \mid V)\), \(sh(U) = sh(V) = \lambda \vdash h\), (uniquely) expands into a linear combination of right symmetrized bitableau \((S \mid T)\), \(S, T\) standard of the same shape \(\lambda = sh(S) = sh(T)\).

- Let \((U \mid V)\), \(sh(U) = sh(V) = \lambda \vdash h\), \(\lambda_1 \not\in n, d\). Then
\[
(U \mid V) = 0.
\]

**Corollary 2.4.** The subspace \(\mathbb{C}_h[M_{n,d}]\) decomposes as:
\[
\mathbb{C}_h[M_{n,d}] = \bigoplus_{\lambda \vdash h} \mathbb{C}_h^\lambda[M_{n,d}], \quad \lambda_1 \leq n, d,
\]
where \(\mathbb{C}_h^\lambda[M_{n,d}]\) is the subspace spanned by the right symmetrized bitableaux \((U \mid V)\) of shape \(\lambda = sh(U) = sh(V)\).

Theorem 2.3 was proved, in a different language, by Wallace [43] in the classical commutative case. A superalgebraic version of this result was proved by the present authors in [3]; for a more detailed discussion, see [1].

**2.4 Right symmetrized bitableaux in \(\mathbb{C}_h[M_{n,d}]\), Young symmetrizers and the natural units in the group algebra \(\mathbb{C}[S_h]\).**

In this subsection, we summarize some basic notions from the representation theory of the symmetric group; furthermore, we provide useful descriptions of right symmetrized determinantal bitableaux in terms of Young symmetrizers and of the natural units in the group algebra \(\mathbb{C}[S_h]\).

**2.4.1 The symmetric group \(S_h\)**

Our main reference here is the treatise of James and Kerber [23], Chapter 3, with the proviso that here the role of rows and columns of a Young tableau are interchanged. For the natural basis of the group algebra \(\mathbb{C}[S_h]\), we also refer the reader to Garsia [18]. For item 4) of Proposition 2.5 below, we also refer the reader to Rutherford [37], Chapter V, p. 62.

Given a pair \(S, T\) of multilinear tableaux of the same shape \(sh(S) = sh(T) = \lambda \vdash h\), the Young symmetrizer \(e_{ST}^\lambda \in \mathbb{C}[S_h]\) is the element:
\[
e_{ST}^\lambda = \sum_{\sigma \in R(S), \tau \in C(T)} (-1)^{|\sigma|} \sigma \theta_{ST} \tau,
\]
where $\theta_{ST} = S \circ T^{-1}$, and $R(S)$, $C(T) \subseteq S_h$ are the row subgroup of $S$ and the column subgroup of $T$, respectively.

Clearly,

$$e^\lambda_{ST} = \theta_{ST} e^\lambda_T = e^\lambda_S \theta_{ST}.$$  

We will denote by $n^\lambda_{ST}$ the elements of the natural basis of the group algebra $\mathbb{C}[S_h]$, $\lambda \vdash h$, $S, T$ multilinear standard tableaux of the same shape $sh(S) = sh(T) = \lambda \vdash h$.

Given $\lambda \vdash h$, recall that

$$\mathbb{C}[S_h] = \bigoplus_{\lambda \vdash h} \mathbb{C}^\lambda[S_h],$$

where $\mathbb{C}^\lambda[S_h]$ denotes the isotypic (simple) component of $\mathbb{C}[S_h]$ associated to $\lambda$.

**Proposition 2.5.**

1) The set

$$\{ e^\lambda_{ST}: S, T \text{ multilinear standard tableaux, } sh(S) = sh(T) = \lambda \vdash h \}$$

is a basis of $\mathbb{C}^\lambda[S_h]$.

2) The set

$$\{ n^\lambda_{ST}: S, T \text{ multilinear standard tableaux, } sh(S) = sh(T) = \lambda \vdash h \}$$

is a basis of $\mathbb{C}^\lambda[S_h]$.

3) Let $S, S', T, T'$ be multilinear standard tableaux of shape $\lambda \vdash h$, then

$$n^\lambda_{ST} n^\lambda_{S'T'} = \delta_{T,S'} n^\lambda_{ST'},$$

$$n^\lambda_{ST} e^\lambda_{S'T'} = \delta_{T,S'} e^\lambda_{ST'}.$$  

4) Let $\lambda \vdash h$ be a partition and denote by $\chi^\lambda$ the irreducible character associated to the irreducible representation of shape $\lambda$ of the symmetric group $S_h$.

Let

$$\chi^\lambda = \bigoplus_{\sigma \in S_h} \chi^\lambda(\sigma) \sigma \in \mathbb{C}[S_h].$$

Then

$$\frac{\chi^\lambda(I)}{n!} \chi^\lambda = \sum_T n^\lambda_{TT},$$

where the sum ranges over all multilinear $T$ standard tableaux on $h$ of shape $\lambda$, and

$$\frac{\chi^\lambda(I)}{n!} = \frac{1}{H(\lambda)},$$

$H(\lambda)$ the hook number of the partition $\lambda$.  

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2.4.2 Right symmetrized bitableaux and Young symmetrizers: the multilinear case

We consider the algebra \( \mathbb{C}_h[M_{h,h}] \), that is \( n = d = h \), the polynomial algebra generated by the variables \((i|j), i,j = 1,2,\ldots,h\).

We establish the following convention. Given an element

\[
  p = \sum_s c_s \sigma_s \in \mathbb{C}[S_h],
\]

and a column tableau

\[
  \begin{pmatrix}
    I(1) & J(1) \\
    \vdots & \vdots \\
    I(h) & J(h)
  \end{pmatrix},
\]

we set

\[
  \begin{pmatrix}
    I(p(1)) & J(1) \\
    \vdots & \vdots \\
    I(p(h)) & J(h)
  \end{pmatrix} = \sum_s c_s \begin{pmatrix}
    I(\sigma_s(1)) & J(1) \\
    \vdots & \vdots \\
    I(\sigma_s(h)) & J(h)
  \end{pmatrix}.
\] (11)

**Proposition 2.6.** Let \( S, T \) be multilinear tableaux of the same shape \( \lambda \), then

\[
  (S \big| T) = \begin{pmatrix}
    e_{ST}^k(1) & 1 \\
    \vdots & \vdots \\
    e_{ST}^h(h)
  \end{pmatrix}. \] (12)

**Proof.** Since

\[
  (S \big| T) = \sum_{\sigma \in R(S)} (-1)^{\sigma}| \begin{pmatrix}
    \sigma \ S(1) & T(1) \\
    \vdots & \vdots \\
    \sigma \ S(h) & T(h)
  \end{pmatrix},
\]

(see subsection 2.2.4), then

\[
  (S \big| T) = \sum_{\sigma \in R(S), \tau \in C(T)} (-1)^{\sigma}| \begin{pmatrix}
    \sigma \theta_{ST} T(1) & \tau \ T(1) \\
    \vdots & \vdots \\
    \sigma \theta_{ST} T(h) & \tau \ T(h)
  \end{pmatrix}
\]

\[
  = \sum_{\sigma \in R(S), \tau \in C(T)} (-1)^{\sigma}| \begin{pmatrix}
    \sigma \theta_{ST} \tau^{-1} T(1) & T(1) \\
    \vdots & \vdots \\
    \sigma \theta_{ST} \tau^{-1} T(h) & T(h)
  \end{pmatrix}
\]

\[
  = \sum_{\sigma \in R(S), \tau \in C(T)} (-1)^{\sigma}| \begin{pmatrix}
    \sigma \theta_{ST} \tau(1) & 1 \\
    \vdots & \vdots \\
    \sigma \theta_{ST} \tau(h) & h
  \end{pmatrix}.
\]
Since
\[ e_\lambda^{ST} = \sum_{\sigma \in R(S), \tau \in C(T)} (-1)^{|\tau|} \sigma \theta_{ST} \tau, \]
then
\[ (S \blacktriangleright T) = \begin{pmatrix}
  e_\lambda^{ST}(1) & 1 \\
  \vdots & \vdots \\
  e_\lambda^{ST}(h) & h
\end{pmatrix}. \]

2.4.3 Right symmetrized bitableaux and Young symmetrizers: the general case

Let \( U, V \) be Young tableaux on the alphabets \( \underline{n}, \underline{d} \), and let \( S, T \) be multilinear tableaux of the same shape \( \lambda \vdash h \). There exists a unique pair of maps \( I : \underline{n} \to \underline{n} \), \( J : \underline{d} \to \underline{d} \) such that
\[ U = I_S \quad V = J_T \]
(see eq. (3)).

**Proposition 2.7.**
\[ (U \blacktriangleright V) = (I_S \blacktriangleright J_T) = \begin{pmatrix}
  I(e_\lambda^{ST}(1)) & J(1) \\
  \vdots & \vdots \\
  I(e_\lambda^{ST}(h)) & J(h)
\end{pmatrix}. \] (13)

**Proof.** From Proposition 2.6, we get:
\[ (I_S \blacktriangleright J_T) = \sum_{\eta \in R(T), \tau \in C(T)} (-1)^{|\eta|} \begin{pmatrix}
  I(\eta \theta_{ST} \tau^{-1}(1)) & J(1) \\
  \vdots & \vdots \\
  I(\eta \theta_{ST} \tau^{-1}(h)) & J(h)
\end{pmatrix}. \]

Following the notational convention (11), we can compactly write:
\[ (U \blacktriangleright V) = (I_S \blacktriangleright J_T) = \begin{pmatrix}
  I(e_\lambda^{ST}(1)) & J(1) \\
  \vdots & \vdots \\
  I(e_\lambda^{ST}(h)) & J(h)
\end{pmatrix}. \]

2.5 Immanants in \( \mathbb{C}_h[M_{n,d}] \)

The immanant of a matrix was defined by D. E. Littlewood and A. R. Richardson as a generalization of the concepts of determinant and permanent [26] (see also [27], [37], [19]).
Let $\lambda \vdash h$ be a partition and denote by $\chi^\lambda$ the irreducible character associated to the irreducible representation of shape $\lambda$ of the symmetric group $S_h$, and let

$$\chi_\lambda = \sum_{\sigma \in S_h} \chi^\lambda(\sigma) \sigma \in \mathbb{C}[S_h].$$

**Example 2.8.** Let $n = 3$, $\lambda = (2,1) \vdash 3$.

The irreducible character element of the group algebra $\mathbb{C}[S_3]$ associated to the partition $\lambda = (2,1)$ is

$$\chi_{(2,1)} = \sum_{\sigma \in S_3} \chi^\lambda(\sigma) = 2I - (123) - (132) \in \mathbb{C}[S_3].$$

The (generalized) immanant

$$\text{imm}_{\lambda}(i_1 i_2 \cdots i_h; j_1 j_2 \cdots j_h), \quad (i_1 i_2 \cdots i_h) \in \mathbb{Z}_h^h, \quad (j_1 j_2 \cdots j_h) \in \mathbb{Z}_h^h$$

in the polynomial algebra in $\mathbb{C}_h[M_{n,d}]$ is the element:

$$\text{imm}_{\lambda}(i_1 i_2 \cdots i_h; j_1 j_2 \cdots j_h) = \sum_{\sigma \in S_h} \chi^\lambda(\sigma) \begin{pmatrix} i_{\sigma(1)} & j_1 \\ i_{\sigma(2)} & j_2 \\ \vdots & \vdots \\ i_{\sigma(h)} & j_h \end{pmatrix} = \sum_{\sigma \in S_h} \chi^\lambda(\sigma) \begin{pmatrix} i_1 & j_{\sigma(1)} \\ i_2 & j_{\sigma(2)} \\ \vdots & \vdots \\ i_h & j_{\sigma(h)} \end{pmatrix}.$$
that

\[ \text{imm}_\lambda(i_{\tau(1)} j_{\tau(2)} \cdots i_{\tau(h)} j_{\tau(h)}) = \sum_{\sigma \in S_n} \chi^\lambda(\sigma) \begin{pmatrix} i_{\tau(1)} & j_{\tau(\sigma(1))} \\ i_{\tau(2)} & j_{\tau(\sigma(2))} \\ \vdots & \vdots \\ i_{\tau(h)} & j_{\tau(\sigma(h))} \end{pmatrix} = \sum_{\sigma \in S_n} \chi^\lambda(\sigma) \begin{pmatrix} i_1 & j_{\tau(1)} \\ i_2 & j_{\tau(2)} \\ \vdots & \vdots \\ i_h & j_{\tau(h)} \end{pmatrix} = \sum_{\eta \in S_n} \chi^\lambda(\eta) \begin{pmatrix} i_1 & j_{\eta(1)} \\ i_2 & j_{\eta(2)} \\ \vdots & \vdots \\ i_h & j_{\eta(h)} \end{pmatrix} = \text{imm}_\lambda(i_1 i_2 \cdots i_h j_1 j_2 \cdots j_h). \]

Hence,

**Proposition 2.9.** The map

\[ \text{IMM}_\lambda : \begin{pmatrix} i_1 \\ i_2 \\ \vdots \\ i_h \end{pmatrix} \rightarrow \text{imm}_\lambda(i_1 i_2 \cdots i_h j_1 j_2 \cdots j_h) \]

defines a linear map

\[ \text{IMM}_\lambda : \mathbb{C}[M_{n,d}] \rightarrow \mathbb{C}[M_{n,d}]. \]

Clearly, the immanant \( \text{imm}_\lambda(i_1 i_2 \cdots i_h j_1 j_2 \cdots j_h) \in \mathbb{C}[M_{n,d}] \) is the natural generalization of the biproducts (signed minors) \( (i_1 i_2 \cdots i_h j_1 j_2 \cdots j_h) \) in \( \mathbb{C}[M_{n,d}] \).

It is obvious that the immanants \( \text{imm}_\lambda(i_1 i_2 \cdots i_h j_1 j_2 \cdots j_h) \) are homogeneous elements of degree \( h \in \mathbb{N} \) of the polynomial algebra \( \mathbb{C}[M_{n,d}] \); therefore, by Theorem 2.3, the immanants \( \text{imm}_\lambda(i_1 i_2 \cdots i_h j_1 j_2 \cdots j_h) \) expand into linear combination of standard right symmetrized bitableaux of shapes that are partitions of \( h \).

Furthermore, the following stronger result holds.

**Proposition 2.10.** Let \( \lambda \vdash h \). Any immanant \( \text{imm}_\lambda(i_1 i_2 \cdots i_h j_1 j_2 \cdots j_h) \) can be written as a linear combination of standard right symmetrized bitableaux of the same shape \( \lambda \):

\[ \text{imm}_\lambda(i_1 i_2 \cdots i_h j_1 j_2 \cdots j_h) = \sum_{U,V} q_{U,V} \begin{bmatrix} U \\ V \end{bmatrix}, \]

where \( q_{U,V} \in \mathbb{C}, \quad sh(U) = sh(V) = \lambda. \)
Proof. Let \( I : \mathfrak{h} \rightarrow \mathfrak{n}, J : \mathfrak{h} \rightarrow \mathfrak{n}, I(s) = i_s, J(s) = j_s, \ s = 1, 2, \ldots, h. \) Item 4) of Proposition 2.5 implies:

\[
\text{imm}_\lambda(i_1 i_2 \cdots i_h; j_1 j_2 \cdots j_h) = \sum_{\sigma \in \mathcal{S}_h} \chi^\lambda(\sigma) \begin{pmatrix}
I(\sigma(1)) & J(1) \\
I(\sigma(2)) & J(2) \\
\vdots & \vdots \\
I(\sigma(h)) & J(h)
\end{pmatrix}
\]

\[
= \begin{pmatrix}
I(\chi_\lambda(1)) & J(1) \\
I(\chi_\lambda(2)) & J(2) \\
\vdots & \vdots \\
I(\chi_\lambda(h)) & J(h)
\end{pmatrix}
\]

\[
= H(\lambda) \sum_T \begin{pmatrix}
I(\mathbf{n}^\lambda_{TT}(1)) & J(1) \\
I(\mathbf{n}^\lambda_{TT}(2)) & J(2) \\
\vdots & \vdots \\
I(\mathbf{n}^\lambda_{TT}(h)) & J(h)
\end{pmatrix}
\]

Since the natural units \( \mathbf{n}^\lambda_{TT} \) expand into Young symmetrizers of the same shape:

\[
\mathbf{n}^\lambda_{TT} = \sum_{S_1, S_2} C_{TT, S_1 S_2} \cdot \mathbf{e}^\lambda_{S_1 S_2}, \quad C_{TT, S_1 S_2} \in \mathbb{C},
\]

then

\[
\text{imm}_\lambda(i_1 i_2 \cdots i_h; j_1 j_2 \cdots j_h) = H(\lambda) \sum_T \sum_{S_1, S_2} C_{TT, S_1 S_2} \begin{pmatrix}
I(\mathbf{e}^\lambda_{S_1 S_2}(1)) & J(1) \\
I(\mathbf{e}^\lambda_{S_1 S_2}(2)) & J(2) \\
\vdots & \vdots \\
I(\mathbf{e}^\lambda_{S_1 S_2}(h)) & J(h)
\end{pmatrix}
\]

\[
= H(\lambda) \sum_T \sum_{S_1, S_2} C_{TT, S_1 S_2} [I_{S_1} \mathbf{J}_{S_2}].
\]

\( \square \)

From Proposition 2.10 and Theorem 2.3, it follows:

**Corollary 2.11.** Let \( \lambda \vdash h. \) If \( \lambda_1 \not\leq \min\{n, d\} \), then

\[
\text{imm}_\lambda(i_1 i_2 \cdots i_h; j_1 j_2 \cdots j_h) = 0.
\]

**Example 2.12.** Let \( h = 3, \lambda = (2, 1) \vdash 3. \)

Recall that the irreducible character element of the group algebra \( \mathbb{C}[S_3] \) associated to the partition \( \lambda = (2, 1) \) is

\[
\underline{\chi}_\lambda = \sum_{\sigma \in S_3} \chi^\lambda(\sigma) \sigma = 2I - (123) - (132) \in \mathbb{C}[S_3].
\]
By definition, we have:

\[
\text{imm}_\lambda(i_1 i_2 i_3; j_1 j_2 j_3) = 2 \begin{pmatrix} i_1 & j_1 \\ i_2 & j_2 \\ i_3 & j_3 \end{pmatrix} - \begin{pmatrix} i_1 & j_2 \\ i_2 & j_3 \\ i_3 & j_1 \end{pmatrix} - \begin{pmatrix} i_1 & j_3 \\ i_2 & j_1 \\ i_3 & j_2 \end{pmatrix} = -2(i_1 | j_1)(i_2 | j_2)(i_3 | j_3) + (i_1 | j_2)(i_2 | j_3)(i_3 | j_1) + (i_1 | j_3)(i_2 | j_1)(i_3 | j_2);
\]

furthermore,

\[
\begin{pmatrix} i_1 & i_2 \\ i_3 & j_1 & j_2 \\ j_3 & \end{pmatrix} = \begin{pmatrix} i_1 & i_2 \\ i_3 & j_1 \\ j_3 & j_2 \end{pmatrix} + \begin{pmatrix} i_1 & i_2 \\ i_3 & j_3 \\ j_1 & j_2 \end{pmatrix} = -(i_1 | j_1)(i_2 | j_2)(i_3 | j_3) + (i_1 | j_2)(i_2 | j_1)(i_3 | j_3) - (i_1 | j_3)(i_2 | j_2)(i_3 | j_1) + (i_1 | j_2)(i_2 | j_3)(i_3 | j_1),
\]

and

\[
\begin{pmatrix} i_1 & i_3 \\ i_2 & j_1 & j_3 \\ j_2 & \end{pmatrix} = \begin{pmatrix} i_1 & i_3 \\ i_2 & j_1 \\ j_2 & j_3 \end{pmatrix} + \begin{pmatrix} i_1 & i_3 \\ i_2 & j_3 \\ j_1 & j_2 \end{pmatrix} = -(i_1 | j_1)(i_3 | j_3)(i_2 | j_2) + (i_1 | j_3)(i_3 | j_1)(i_2 | j_2) - (i_1 | j_2)(i_3 | j_3)(i_2 | j_1) + (i_1 | j_3)(i_3 | j_2)(i_2 | j_1).
\]

Then:

\[
\text{imm}_\lambda(i_1 i_2 i_3; j_1 j_2 j_3) = \begin{pmatrix} i_1 & i_2 \\ i_3 & j_1 & j_2 \\ j_3 & \end{pmatrix} - \begin{pmatrix} i_1 & i_2 \\ i_3 & j_1 \\ j_3 & j_2 \end{pmatrix} = \begin{pmatrix} i_1 & i_3 \\ i_2 & j_1 & j_3 \\ j_2 & \end{pmatrix} + \begin{pmatrix} i_1 & i_3 \\ i_2 & j_3 \\ j_1 & j_2 \end{pmatrix}.
\]

The scalar multiple $\frac{\chi^\lambda(I)}{n!}$ of the linear operator $\text{IMM}_\lambda$ of Proposition 2.9 acts on $\mathbb{C}_h[M_{n,d}]$ as the projector on the direct summand $\mathbb{C}_h[I][M_{n,d}]$ in the Gordan-Capelli direct sum decomposition (10) of Corollary 2.4.

**Proposition 2.13.** Let $U, V$ be Young tableaux of the same shape $sh(U) = sh(V) = \mu + h$. We have:

1. if $\mu = \lambda$, then

\[
\frac{\chi^\lambda(I)}{n!} \text{IMM}_\lambda \left( \begin{array}{c|c} U & V \end{array} \right) = \begin{array}{c|c} U & V \end{array};
\]

2. if $\mu \neq \lambda$, then

\[
\frac{\chi^\lambda(I)}{n!} \text{IMM}_\lambda \left( \begin{array}{c|c} U & V \end{array} \right) = 0.
\]

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Proof. Set 

\[ U = I_{T_0}, \quad V = J_{T_0}, \quad sh(U) = sh(V) = sh(T_0) = \mu. \]

Equation (13) implies 

\[ (U | V) = (I_{T_0} | J_{T_0}) = \begin{pmatrix} I(e^\mu_{T_0T_0}(1)) & J(1) \\ \vdots & \vdots \\ I(e^\mu_{T_0T_0}(h)) & J(h) \end{pmatrix}. \]

Item 4) of Proposition 2.5 implies 

\[ \chi^\lambda(I) \left( \frac{1}{n!} IMM_\lambda \left( \begin{pmatrix} U \\ V \end{pmatrix} \right) \right) = \left( \sum_T n^\lambda_{TT} \begin{pmatrix} I(e^\mu_{T_0T_0}(1)) & J(1) \\ \vdots & \vdots \\ I(e^\mu_{T_0T_0}(h)) & J(h) \end{pmatrix} \right). \]

If \( \lambda \neq \mu \), the natural units \( n^\lambda_{TT} \) and the Young symmetrizer \( e^\mu_{T_0T_0} \) belong to different simple components of the semisimple algebra 

\[ \mathbb{C}[S_h] = \bigoplus_{\nu \vdash h} \mathbb{C}^\nu[S_h], \]

and are therefore orthogonal. This proves the second assertion.

If \( \lambda = \mu \), since (Proposition 2.5, item 3) 

\[ n^\lambda_{TT} e^\lambda_{T_0T_0} = \delta_{T,T_0} e^\lambda_{T_0T_0}, \]

we get 

\[ \frac{\chi^\lambda(I)}{n!} IMM_\lambda \left( \begin{pmatrix} U \\ V \end{pmatrix} \right) = \left( \begin{pmatrix} I(e^\lambda_{T_0T_0}(1)) & J(1) \\ \vdots & \vdots \\ I(e^\lambda_{T_0T_0}(h)) & J(h) \end{pmatrix} \right) = (U | V), \]

and the first assertion is proved. \( \square \)

**Proposition 2.14.** Let \( \lambda \vdash h \). Any right symmetrized bitableau \( (U | V) \) of shape \( sh(U) = sh(V) = \lambda \) can be written as a linear combination of immanants \( imm^\lambda(i_1i_2\ldots i_h; j_1j_2\ldots j_h) \) associated to the same shape \( \lambda \).

**Proof.** Expand the right symmetrized bitableau \( (U | V) \) into monomials and apply to each summand the linear operator \( IMM_\lambda \). \( \square \)
Example 2.15. From the definitions we have:

\[
\begin{pmatrix}
  i_1 & i_2 & j_1 & j_2 \\
  i_3 & & j_3 & \\
\end{pmatrix}
= -(i_1|j_1)(i_2|j_2)(i_3|j_3) + (i_1|j_2)(i_2|j_1)(i_3|j_3)
- (i_1|j_3)(i_2|j_2)(j_3|j_1) + (i_1|j_2)(i_2|j_3)(j_3|j_1),
\]

\[
\text{IMM}_{(2,1)}(i_1|j_1)(i_2|j_2)(i_3|j_3) =
= (i_1|j_1)(i_2|j_2)(i_3|j_3) - (i_1|j_2)(i_2|j_3)(i_3|j_1) - (i_1|j_3)(i_2|j_1)(i_3|j_2),
\]

\[
\text{IMM}_{(2,1)}(i_1|j_2)(i_2|j_1)(i_3|j_3) =
= 2(i_1|j_2)(i_2|j_1)(i_3|j_3) - (i_1|j_2)(i_2|j_3)(i_3|j_1) - (i_1|j_1)(i_2|j_3)(i_3|j_2),
\]

\[
\text{IMM}_{(2,1)}(i_1|j_3)(i_2|j_2)(i_3|j_1) =
= 2(i_1|j_3)(i_2|j_2)(i_3|j_1) - (i_1|j_1)(i_2|j_3)(i_3|j_2) - (i_1|j_2)(i_2|j_1)(i_3|j_3).
\]

Then

\[
\begin{pmatrix}
  i_1 & i_2 & j_1 & j_2 \\
  i_3 & & j_3 & \\
\end{pmatrix}
= \frac{1}{3}
\left[- \text{imm}_{(2,1)}(i_1i_2i_3;j_1j_2j_3) + \text{imm}_{(2,1)}(i_1i_2i_3;j_2j_3j_1)
- \text{imm}_{(2,1)}(i_1i_2i_3;j_3j_2j_1) + \text{imm}_{(2,1)}(i_1i_2i_3;j_2j_3j_1)\right].
\]

By combining Theorem 2.3 and Proposition 2.14, we get

Proposition 2.16. The set of immanants

\[
\text{imm}_\lambda(i_1i_2\cdots i_h; j_1j_2\cdots j_h),
\]

with

\[
\lambda \vdash h, \lambda_1 \not\prec \min\{n, d\}, (i_1i_2\cdots i_h) \in \mathcal{A}^h, (j_1j_2\cdots j_h) \in \mathcal{A}^h,
\]

is a spanning set of \( \mathbb{C}_h[M_{n,d}] \).
3 The superalgebraic approach to the enveloping algebra $U(gl(n))$

In this section, we provide a synthetic presentation of the superalgebraic method of virtual variables for $gl(n)$.

This method was developed by the present authors for the general linear Lie superalgebras $gl(m|n)$, in the series of notes [1], [2], [3], [4], [5], [6], [7].

The technique of virtual variables is an extension of Capelli’s method of variabili ausiliarie (Capelli [13], see also Weyl [45]).

Capelli introduced the technique of variabili ausiliarie in order to manage symmetrizer operators in terms of polarization operators and to simplify the study of some skew-symmetrizer operators (namely, the famous central Capelli operator).

Capelli’s idea was well suited to treat symmetrization, but it did not work in the same efficient way while dealing with skew-symmetrization.

One had to wait the introduction of the notion of superalgebras (see e.g. [39], [24], [44]) to have the right conceptual framework to treat symmetry and skew-symmetry in one and the same way. To the best of our knowledge, the first mathematician who intuited the connection between Capelli’s idea and superalgebras was Koszul in 1981 [25]; Koszul proved that the classical determinantal Capelli operator can be rewritten - in a much simpler way - by adding to the symbols to be dealt with an extra auxiliary symbol that obeys to different commutation relations.

The superalgebraic method of virtual variables allows us to express remarkable classes of elements in $U(gl(n))$ as images - with respect to the Capelli de-virtualization epimorphism (Subsection 3.2.1 below) - of simple monomials and to obtain transparent combinatorial descriptions of their actions on irreducible $gl(n)$-modules.

Among these classes, here we recall the classes of Capelli bitableaux $[S|T]$ and right Young-Capelli bitableaux $\left[\begin{array}{c|c} S & T \end{array}\right]$ (see [4], [5], [1], and subsection 3.3.2 below), and introduce the new class of Capelli immanants

$$Cim\lambda[i_1i_2\cdots i_h; j_1j_2\cdots j_h]$$

(see Section 5 below).

Moreover, this method throws a bridge between the theory of $U(gl(n))$ and the (super)straightening techniques in (super)symmetric algebras (see e.g. [20], [5], [6], [1]).

3.1 The superalgebras $\mathbb{C}[M_{m_0|m_1+n, d}]$ and $gl(m_0|m_1+n)$

3.1.1 The general linear Lie super algebra $gl(m_0|m_1+n)$

Given a vector space $V_n$ of dimension $n$, we will regard it as a subspace of a $\mathbb{Z}_2$-graded vector space $W = W_0 \oplus W_1$, where

$$W_0 = V_{m_0}, \quad W_1 = V_{m_1} \oplus V_n.$$
The vector spaces $V_{m_0}$ and $V_{m_1}$ (informally, we assume that $\dim(V_{m_0}) = m_0$ and $\dim(V_{m_1}) = m_1$ are “sufficiently large”) are called the spaces of even virtual (auxiliary) vectors and of odd virtual (auxiliary) vectors, respectively, and $V_n$ is called the space of (odd) proper vectors.

The inclusion $V_n \subset W$ induces a natural embedding of the ordinary general linear Lie algebra $gl(n)$ of $V_n$ into the auxiliary general linear Lie superalgebra $gl(m_0|m_1 + n)$ of $W = W_0 \oplus W_1$ (see, e.g. [24], [39]).

Let $A_0 = \{\alpha_1, \ldots, \alpha_{m_0}\}$, $A_1 = \{\beta_1, \ldots, \beta_{m_1}\}$, $L = \{x_1, \ldots, x_n\}$ denote fixed bases of $V_{m_0}$, $V_{m_1}$ and $V_n$, respectively; therefore $|\alpha_s| = 0 \in \mathbb{Z}_2$, and $|\beta_t| = |x_i| = 1 \in \mathbb{Z}_2$.

Let
\[
\{e_{a,b}; a, b \in A_0 \cup A_1 \cup L\}, \quad |e_{a,b}| = |a| + |b| \in \mathbb{Z}_2
\]
be the standard $\mathbb{Z}_2$–homogeneous basis of the Lie superalgebra $gl(m_0|m_1 + n)$ provided by the elementary matrices. The elements $e_{a,b} \in gl(m_0|m_1 + n)$ are $\mathbb{Z}_2$–homogeneous of $\mathbb{Z}_2$–degree $|e_{a,b}| = |a| + |b|$.

The superbracket of the Lie superalgebra $gl(m_0|m_1 + n)$ has the following explicit form:
\[
[e_{a,b}, e_{c,d}] = \delta_{bc} e_{a,d} - (-1)^{|(a|+|b)|(|c|+|d|)}\delta_{ad} e_{c,b},
\]
\[a, b, c, d \in A_0 \cup A_1 \cup L.\]

### 3.1.2 The supersymmetric algebra $\mathbb{C}[M_{m_0|m_1+n,d}]$

As already said, we will write $(i|j)$ in place of $x_{ij}$, and regard the (commutative) algebra $\mathbb{C}[M_{n,d}]$ as a subalgebra of the “auxiliary” supersymmetric algebra
\[
\mathbb{C}[M_{m_0|m_1+n,d}] = \mathbb{C}[\alpha_s|j], (\beta_t|j), (i|j)]
\]
generated by the ($\mathbb{Z}_2$-graded) variables $(\alpha_s|j), (\beta_t|j), (i|j), j = 1, 2, \ldots, d$, where
\[
|\alpha_s| = 1 \in \mathbb{Z}_2 \quad \text{and} \quad |\beta_t| = |(i|j)| = 0 \in \mathbb{Z}_2,
\]
subject to the commutation relations:
\[
(a|h)(b|k) = (-1)^{|(a|h)||b|k|} (b|k)(a|h),
\]
for $a, b \in \{\alpha_1, \ldots, \alpha_{m_0}\} \cup \{\beta_1, \ldots, \beta_{m_1}\} \cup \{1, 2, \ldots, n\}$.

In plain words, all the variables commute each other, with the exception of pairs of variables $(\alpha_s|j), (\alpha_t|j)$ that skew-commute:
\[
(\alpha_s|j)(\alpha_t|j) = -(\alpha_t|j)(\alpha_s|j).
\]

In the standard notation of multilinear algebra, we have:
\[
\mathbb{C}[M_{m_0|m_1+n,d}] \cong \Lambda [W_0 \otimes P_d] \otimes \text{Sym}[W_1 \otimes P_d] = \Lambda [V_{m_0} \otimes P_d] \otimes \text{Sym}[(V_{m_1} \otimes V_n) \otimes P_d]
\]

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where $P_d = (P_d)_1$ denotes the trivially (odd) $\mathbb{Z}_2$–graded vector space with distinguished basis \( \{ j; j = 1, 2, \ldots, d \} \).

The algebra \( \mathbb{C}[M_{n_0}[m_1+n,d]] \) is a supersymmetric $\mathbb{Z}_2$–graded algebra (superalgebra), whose $\mathbb{Z}_2$–gradation is inherited by the natural one in the exterior algebra.

### 3.1.3 Left superderivations and left superpolarizations

A **left superderivation** \( D \) (\( \mathbb{Z}_2 \)–homogeneous of degree \( |D| \)) (see, e.g. [39], [24]) on \( \mathbb{C}[M_{n_0}[m_1+n,d]] \) is an element of the superalgebra \( \text{End}_\mathbb{C}[\mathbb{C}[M_{n_0}[m_1+n,d]] \) that satisfies "Leibniz rule"

\[
D(p \cdot q) = D(p) \cdot q + (-1)^{|D||p|} p \cdot D(q),
\]

for every \( \mathbb{Z}_2 \)–homogeneous of degree \( |p| \) element \( p \in \mathbb{C}[M_{n_0}[m_1+n,d]] \).

Given two symbols \( a, b \in A_0 \cup A_1 \cup L \), the **superpolarization** \( D_{a,b} \) of \( b \) to \( a \) is the unique left superderivation of \( \mathbb{C}[M_{n_0}[m_1+n,d]] \) of parity \( |D_{a,b}| = |a|+|b| \in \mathbb{Z}_2 \) such that

\[
D_{a,b}((c|j)) = \delta_{bc} (a|j), \ c \in A_0 \cup A_1 \cup L, \ j = 1, \ldots, d. \quad (16)
\]

Informally, we say that the operator \( D_{a,b} \) **annihilates** the symbol \( b \) and **creates** the symbol \( a \).

### 3.1.4 The superalgebra \( \mathbb{C}[M_{n_0}[m_1+n,d]] \) as a \( \textbf{U}(\text{gl}(m_0|m_1+n)) \)-module

Since

\[
D_{a,b}D_{c,d} - (-1)^{|a|+|b|(|c|+|d|)} D_{c,d}D_{a,b} = \delta_{b,c}D_{a,d} - (-1)^{|a|+|b|(|c|+|d|)} \delta_{a,d}D_{c,b},
\]

the map

\[
e_{a,b} \rightarrow D_{a,b}, \quad a, b \in A_0 \cup A_1 \cup L.
\]

(that send the elementary matrices to the corresponding superpolarizations) is an (even) Lie superalgebra morphism from \( \text{gl}(m_0[m_1+n]) \) to \( \text{End}_\mathbb{C}[\mathbb{C}[M_{n_0}[m_1+n,d]] \) and, hence, it uniquely defines a morphism (i.e. a representation):

\[
\varrho : \textbf{U}(\text{gl}(m_0[m_1+n])) \rightarrow \text{End}_\mathbb{C}[\mathbb{C}[M_{n_0}[m_1+n,d]] \).
\]

In the following, we always regard the superalgebra \( \mathbb{C}[M_{n_0}[m_1+n,d]] \) as a \( \textbf{U}(\text{gl}(m_0[m_1+n])) \)–supersubmodule, with respect to the action induced by the representation \( \varrho \):

\[
e_{a,b} : p = D_{a,b}(p),
\]

for every \( p \in \mathbb{C}[M_{n_0}[m_1+n,d]] \).

We recall that \( \mathbb{C}[M_{n_0}[m_1+n,d]] \) is a **semisimple** \( \textbf{U}(\text{gl}(m_0[m_1+n])) \)–supersubmodule, whose irreducible (simple) submodules are - up to isomorphism - **Schur supermodules** (see, e.g. [3], [4], [1]).

Clearly, \( \textbf{U}(\text{gl}(0|n)) = \textbf{U}(\text{gl}(n)) \) is a subalgebra of \( \textbf{U}(\text{gl}(m_0[m_1+n])) \) and the subalgebra \( \mathbb{C}[n,d] \) is a \( \textbf{U}(\text{gl}(n)) \)–submodule of \( \mathbb{C}[M_{n_0}[m_1+n,d]] \).
3.2 The virtual algebra $\text{Virt}(m_0 + m_1, n)$ and the virtual presentations of elements in $U(gl(n))$

3.2.1 The virtual algebra $\text{Virt}(m_0 + m_1, n)$ as a subalgebra of $U(gl(m_0|m_1 + n))$ and the Capelli devirtualization epimorphism $p : \text{Virt}(m_0 + m_1, n) \twoheadrightarrow U(gl(n))$

We say that a product

$$e_{a_m,b_m} \cdots e_{a_1,b_1} \in U(gl(m_0|m_1 + n))$$

is an irregular expression whenever there exists a right subword

$$e_{a_i,b_i} \cdots e_{a_2,b_2} e_{a_1,b_1},$$

$i \leq m$ and a virtual symbol $\gamma \in A_0 \cup A_1$ such that

$$\# \{ j; b_j = \gamma, j \leq i \} > \# \{ j; a_j = \gamma, j < i \}. \quad (17)$$

The meaning of an irregular expression in terms of the action of $U(gl(m_0|m_1 + n))$ on the algebra $\mathbb{C}[M_{m_0|m_1+n,d}]$ is that there exists a virtual symbol $\gamma$ and a right subsequence in which the symbol $\gamma$ is annihilated more times than it was already created.

Example 3.1. Let $\gamma \in A_0 \cup A_1$ and $x_i, x_j \in L$. The product

$$e_{\gamma,x_j} e_{x_i,\gamma} e_{x_i,\gamma} e_{\gamma,x_i}$$

is an irregular expression.

Let $\text{Irr}$ be the left ideal of $U(gl(m_0|m_1 + n))$ generated by the set of irregular expressions.

Remark 3.2. The action of any element of $\text{Irr}$ on the subalgebra $\mathbb{C}[M_{n,d}] \subset \mathbb{C}[M_{m_0|m_1+n,d}]$ - via the representation $\varrho$ - is identically zero.

Proposition 3.3. ([3], [2]) The sum $U(gl(0|n)) + \text{Irr}$ is a direct sum of vector subspaces of $U(gl(m_0|m_1 + n))$.

We come now to one of the main notions of the virtual method. The virtual algebra $\text{Virt}(m_0 + m_1, n)$ is the subalgebra

$$\text{Virt}(m_0 + m_1, n) = U(gl(0|n)) \oplus \text{Irr} \subset U(gl(m_0|m_1 + n)).$$

The proof of the following proposition is immediate from the definitions.

Proposition 3.4. The left ideal $\text{Irr}$ of $U(gl(m_0|m_1 + n))$ is a two sided ideal of $\text{Virt}(m_0 + m_1, n)$.
The Capelli devirtualization epimorphism is the projection
\[ p : \text{Virt}(m_0 + m_1, n) = U(gl(0|n)) \oplus \text{Irr} \to U(gl(0|n)) = U(gl(n)) \]
with \( \text{Ker}(p) = \text{Irr} \).

**Example 3.5.** Let \( x \in L, \alpha \in A_0 \). The element
\[ e_{x,\alpha}e_{\alpha,x} = -e_{\alpha,x}e_{x,\alpha} + e_{x,x} + e_{\alpha,\alpha} \]
belongs to the virtual algebra \( \text{Virt}(m_0 + m_1, n) \) and
\[ p(e_{x,\alpha}e_{\alpha,x}) = e_{x,x} \in U(gl(n)). \]

**Example 3.6.** Let \( x, y \in L, \alpha \in A_0 \). Then
\[ e_{y\alpha}e_{x\alpha}e_{x\alpha}e_{y\alpha} = -e_{y\alpha}e_{x\alpha}e_{x\alpha}e_{y\alpha} + e_{y\alpha}e_{x\alpha}e_{x\alpha}e_{y\alpha} + e_{y\alpha}e_{x\alpha}e_{x\alpha}e_{y\alpha} \]
\[ = +e_{y\alpha}e_{x\alpha}e_{y\alpha}e_{x\alpha} - e_{y\alpha}e_{x\alpha}e_{x\alpha}e_{y\alpha} \]
\[ - e_{x\alpha}e_{y\alpha}e_{x\alpha} + e_{x\alpha}e_{y\alpha}e_{x\alpha} 
+ e_{y\alpha}e_{x\alpha}e_{y\alpha} + e_{y\alpha}e_{x\alpha}e_{y\alpha} \]
\[ = +e_{y\alpha}e_{x\alpha}e_{y\alpha}e_{x\alpha} - e_{x\alpha}e_{y\alpha}e_{x\alpha} - e_{y\alpha}e_{x\alpha}e_{y\alpha} \]
\[ - e_{x\alpha}e_{y\alpha}e_{y\alpha} + e_{x\alpha}e_{y\alpha}e_{y\alpha} 
+ e_{y\alpha}e_{x\alpha}e_{y\alpha} + e_{y\alpha}e_{x\alpha}e_{y\alpha} \]
\[ + e_{y\alpha}e_{x\alpha}e_{y\alpha} + e_{y\alpha}e_{x\alpha}e_{y\alpha} + e_{y\alpha}e_{x\alpha}e_{y\alpha} + e_{y\alpha}e_{x\alpha}e_{y\alpha} \in U(gl(m_0|m_1 + n)). \]

Therefore
\[ e_{y\alpha}e_{x\alpha}e_{x\alpha}e_{y\alpha} \in \text{Virt}(m_0 + m_1, n) \]
and
\[ p(e_{y\alpha}e_{x\alpha}e_{x\alpha}e_{y\alpha}) = -e_{y\alpha}e_{x\alpha}e_{x\alpha}e_{y\alpha} + e_{y\alpha}e_{x\alpha}e_{x\alpha}e_{y\alpha} \in U(gl(n)). \]

Any element in \( M \in \text{Virt}(m_0 + m_1, n) \) defines an element in \( m \in U(gl(n)) \)
- via the map \( p \) - and \( M \) is called a virtual presentation of \( m \).

Since the map \( p \) is a surjection, any element \( m \in U(gl(n)) \) admits several virtual presentations. In the sequel, we even take virtual presentations as the true definition of special elements in \( U(gl(n)) \), and this method will turn out to be quite effective.

**Example 3.7. (A virtual presentation of the Capelli determinant)** As a generalization of Example 3.6, we describe a “monomial” virtual presentation in \( \text{Virt}(m_0 + m_1, n) \) of the classical Capelli determinant in \( U(gl(n)) \).

Let \( \alpha \in A_0 \). The monomial element
\[ C = e_{x,\alpha} \cdots e_{x,\alpha}e_{x,\alpha} \cdots e_{x,\alpha}e_{x,\alpha} \in U(gl(m_0|m_1 + n)) \]
(18)
belongs to the virtual algebra $\text{Virt}(m_0|m_1+n)$. The image of the element $C$ under the Capelli devirtualization epimorphism $p$ equals the column determinant

$$H_n(n) = \text{cdet} \begin{pmatrix}
e_{x_1,x_1} + (n-1) & e_{x_1,x_2} & \cdots & e_{x_1,x_n} \\
e_{x_2,x_1} & e_{x_2,x_2} + (n-2) & \cdots & e_{x_2,x_n} \\
\vdots & \vdots & \ddots & \vdots \\
e_{x_n,x_1} & e_{x_n,x_2} & \cdots & e_{x_n,x_n}
\end{pmatrix} \in U(gl(n)).$$

This result is a special case of the result that we called the “Laplace expansion for Capelli rows” ([7] Theorem 2, [1] Theorem 6.3). A sketchy proof of it can also be found in Koszul [25].

The next results will play a crucial role in the study of central elements of $U(gl(n))$.

**Proposition 3.8.** For every $e_{x_i,x_j} \in gl(n) \subset gl(m_0|m_1+n)$, let $\text{ad}(e_{x_i,x_j})$ denote its adjoint action on $\text{Virt}(m_0+m_1,n)$; the ideal $\text{Irr}$ is $\text{ad}(e_{x_i,x_j})$-invariant. Then

$$p(\text{ad}(e_{x_i,x_j})(m)) = \text{ad}(e_{x_i,x_j})(p(m)), \quad m \in \text{Virt}(m_0+m_1,n). \quad (19)$$

**Corollary 3.9.** The Capelli epimorphism image of an element of $\text{Virt}(m_0|m_1+n)$ that is an invariant for the adjoint action of $gl(n)$ is in the center $\zeta(n)$ of $U(gl(n))$.

**Example 3.10.** Recall that

$$\text{ad}(e_{x_i,x_j})(e_{x_h,\alpha}) = \delta_{jh}e_{x_i,\alpha},$$

$$\text{ad}(e_{x_i,x_j})(e_{\alpha,x_k}) = -\delta_{ki}e_{\alpha,x_j},$$

for every virtual symbol $\alpha$, and that $\text{ad}(e_{x_i,x_j})$ acts as a derivation, for every $i,j = 1,2,\ldots,n$.

The monomial $C$ of Example 3.7, eq.(18) is annihilated by $\text{ad}(e_{x_i,x_j})$, $i \neq j$, by skew-symmetry. Furthermore, $\text{ad}(e_{x_i,x_j})(C) = C - C = 0$, $i = 1,2,\ldots,n$; hence, $C$ is an invariant for the adjoint action of $gl(n)$.

Since $p(C) = H_n(n)$, the Capelli determinant $H_n(n)$ is central in $U(gl(n))$, by Corollary 3.9.

---

$^1$The symbol $\text{cdet}$ denotes the column determinat of a matrix $A = [a_{ij}]$ with noncommutative entries: $\text{cdet}(A) = \sum_{\sigma} (-1)^{||\sigma||} a_{\sigma(1),1}a_{\sigma(2),2}\cdots a_{\sigma(n),n}$. 

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3.2.2 The action of $\text{Virt}(m_0 + m_1, n)$ on the subalgebra $\mathbb{C}[M_{n,d}]$

From the representation-theoretic point of view, the core of the method of virtual variables lies in the following result.

**Theorem 3.11.** The action of $\text{Virt}(m_0 + m_1, n)$ leaves invariant the subalgebra $\mathbb{C}[M_{n,d}] \subseteq \mathbb{C}[M_{m_0 + n,d}]$, and, therefore, the action of $\text{Virt}(m_0 + m_1, n)$ on $\mathbb{C}[M_{n,d}]$ is well defined. Furthermore, for every $v \in \text{Virt}(m_0 + m_1, n)$, its action on $\mathbb{C}[M_{n,d}]$ equals the action of $p(v) \in \mathbb{U}(gl(n))$.

Therefore, instead of studying the action of an element in $\mathbb{U}(gl(n))$, one can study the action of a virtual presentation of it in $\text{Virt}(m_0 + m_1, n)$. The advantage of virtual presentations is that they are frequently of monomial form, admit quite transparent interpretations and are much easier to be dealt with (see, e.g. [3], [4], [7], [1], [2]).

A prototypical instance of this method is provided by the celebrated Capelli identity [9], [45], [21], [22], [41]. From Example 3.7, it follows that the action of the Capelli determinant $H_n(n)$ on a form $f \in \mathbb{C}[M_{n,d}]$ is the same as the action of its monomial virtual presentation, and this leads to a few lines proof of the identity [7], [2].

3.2.3 Balanced monomials as elements of the virtual algebra $\text{Virt}(m_0 + m_1, n)$

In order to make the virtual variables method effective, we need to exhibit a class of nontrivial elements that belong to $\text{Virt}(m_0 + m_1, n)$.

A quite relevant class of such elements is provided by balanced monomials.

In plain words, a balanced monomial is product of two or more factors where the rightmost one annihilates the $k$ proper symbols $x_{j_1}, \ldots, x_{j_k}$ and creates some virtual symbols; the leftmost one annihilates all the virtual symbols and creates the $k$ proper symbols $x_{i_1}, \ldots, x_{i_k}$; between these two factors, there might be further factors that annihilate and create virtual symbols only.

In a formal way, balanced monomials are elements of the algebra $\mathbb{U}(gl(m_0|m_1 + n))$ of the forms:

- $e_{x_{i_1}, \gamma_{p_1}} \cdots e_{x_{i_k}, \gamma_{p_k}} \cdot e_{\gamma_{p_1}, x_{j_1}} \cdots e_{\gamma_{p_k}, x_{j_k}}$,
- $e_{x_{i_1}, \theta_{q_1}} \cdots e_{x_{i_k}, \theta_{q_k}} \cdot e_{\theta_{q_1}, \gamma_{p_1}} \cdots e_{\theta_{q_k}, \gamma_{p_k}} \cdot e_{\gamma_{p_1}, x_{j_1}} \cdots e_{\gamma_{p_k}, x_{j_k}}$,
- and so on,

where $x_{i_1}, \ldots, x_{i_k}, x_{j_1}, \ldots, x_{j_k} \in L$, i.e., the $x_{i_1}, \ldots, x_{i_k}, x_{j_1}, \ldots, x_{j_k}$ are $k$ proper symbols.

The next result is the (superalgebraic) formalization of the argument developed by Capelli in [13], CAPITOLO I, §X.Metodo delle variabili ausiliarie, page 55 ff.

**Proposition 3.12.** ([3], [4], [7], [1], [2]) Every balanced monomial belongs to $\text{Virt}(m_0 + m_1, n)$. Hence its image under the Capelli epimorphism $p$ belongs to $\mathbb{U}(gl(n))$. 

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In plain words, the action of a balanced monomial on the subalgebra $\mathbb{C}[M_{n,d}]$ equals the action of a suitable element of $U(gl(n))$.

The following result lies deeper and is a major tool in the proof of identities involving monomial virtual presentation of elements of $U(gl(n))$. Since the adjoint representation acts by superderivation, it may be regarded as a version of the Laplace expansion for the images of balanced monomials.

**Proposition 3.13.** *(Monomial virtual presentation and adjoint actions)* In $U(gl(n))$, the element

$$p \left[ e_{x_1,1} \ldots e_{x_n,n} \right]$$

equals

$$p \left[ \text{ad}(e_{x_1,1}) \ldots \text{ad}(e_{x_n,n}) \right].$$

**Example 3.14.** Let $\alpha \in A_0$. Then

$$[x_1, x_2 \ldots x_k | x_1, x_2 \ldots x_k] = p[x_{1,\alpha} x_{2,\alpha} \cdots x_{k,\alpha}] =$$

$$= p \left[ \text{ad}(e_{x_1,1}) \text{ad}(e_{x_2,2}) \ldots \text{ad}(e_{x_k,k}) \right] =$$

$$= (-1)^{\binom{k}{2}} \text{cdet} \begin{pmatrix}
  e_{x_{1,1},x_{1,1}} + (k-1) & e_{x_{1,1},x_{1,2}} & \cdots & e_{x_{1,1},x_{1,k}} \\
  e_{x_{1,2},x_{1,1}} & e_{x_{1,2},x_{1,2}} + (k-2) & \cdots & e_{x_{1,2},x_{1,k}} \\
  \vdots & \vdots & \ddots & \vdots \\
  e_{x_{1,k},x_{1,1}} & e_{x_{1,k},x_{1,2}} & \cdots & e_{x_{1,k},x_{1,k}}
\end{pmatrix} \in U(gl(n)).$$

**Example 3.15.** Let $\alpha \in A_1$. The element

$$p = p \left[ e_{x_3,\alpha} e_{x_2,\alpha} e_{x_1,\alpha} e_{\alpha,x_1} e_{\alpha,x_2} e_{\alpha,x_3} \right] =$$

$$= p \left[ \text{ad}(e_{x_3,\alpha}) \text{ad}(e_{x_2,\alpha}) \text{ad}(e_{x_1,\alpha}) \right] \left[ e_{\alpha,x_1} e_{\alpha,x_2} e_{\alpha,x_3} \right]$$

equals the column permanent$^2$

$$\text{cper} \left( \begin{array}{ccc}
  e_{x_1,x_1} - 2 & e_{x_1,x_2} & e_{x_1,x_3} \\
  e_{x_2,x_1} & e_{x_2,x_2} - 1 & e_{x_2,x_3} \\
  e_{x_3,x_1} & e_{x_3,x_2} & e_{x_3,x_3}
\end{array} \right) \in U(gl(3)).$$

**Example 3.16.** Let $\alpha, \beta \in A_0$. Then

$$\begin{pmatrix}
  x_1 \\
  x_2
\end{pmatrix} \begin{pmatrix}
  x_2 \\
  x_1
\end{pmatrix} = p \left( e_{x_1,\alpha} e_{x_2,\beta} e_{\alpha,x_2} e_{\beta,x_1} \right) =$$

$$= -e_{x_1,x_2} e_{x_2,x_1} + e_{x_1,x_1} \in U(gl(2)).$$

---

$^2$The symbol $\text{cper}$ denotes the column permanent of a matrix $A = [a_{ij}]$ with noncommutative entries: $\text{cper}(A) = \sum_{\sigma} a_{\sigma(1),1} a_{\sigma(2),2} \cdots a_{\sigma(n),n}$. 

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3.3 Two special classes of elements in $\text{Virt}(m_0 + m_1, n)$ and their images in $U(gl(n))$

We will introduce two classes of remarkable elements of the enveloping algebra $U(gl(n))$, that we call Capelli bitableaux and right Young-Capelli bitableaux, respectively.

Capelli bitableaux are the analogues in $U(gl(n))$ of bitableaux in the polynomial algebra $\mathbb{C}[M_{n,d}]$, as well as right Young-Capelli bitableaux are the analogues in $U(gl(n))$ of right symmetrized bitableaux. Besides this analogy, their meaning lies deeper, as we shall see in section 4.

3.3.1 Bitableaux monomials in $U(gl(m_0 + m_1, n))$

Let $S$ and $T$ be two Young tableaux of same shape $\lambda \vdash h$ on the alphabet $A_0 \cup A_1 \cup L$:

$$S = \begin{pmatrix} z_1, \ldots, z_{\lambda_1} \\ z_j, \ldots, z_{\lambda_2} \\ \vdots \\ z_{s_1}, \ldots, z_{s_{\lambda_p}} \end{pmatrix}, \quad T = \begin{pmatrix} z_{h_1}, \ldots, z_{h_{\lambda_1}} \\ z_{k_1}, \ldots, z_{k_{\lambda_2}} \\ \vdots \\ z_{t_1}, \ldots, z_{t_{\lambda_p}} \end{pmatrix}$$

(20)

To the pair $(S, T)$, we associate the bitableau monomial:

$$e_{S,T} = e_{z_1, z_{h_1}} e_{z_j, z_{k_1}} e_{z_s, z_{t_1}} \cdots e_{z_{\lambda_1}, z_{\lambda_{\lambda_1}}}$$

in $U(gl(m_0|m_1 + n))$.

By expressing the Young tableaux $S, T$ in the functional form (see subsection 2.2.1):

$$S : h \to A_0 \cup A_1 \cup L, \quad T : h \to A_0 \cup A_1 \cup L,$$

the bitableau monomial $e_{S,T}$ of eq. (21) becomes:

$$e_{S,T} = e_{S(1),T(1)} e_{S(2),T(2)} \cdots e_{S(h),T(h)}.$$

Example 3.17. Let $S$ and $T$ be tableaux of shape $\lambda = (3, 2, 2)$ on the alphabet $A_0 \cup A_1 \cup L$:

$$S = \begin{pmatrix} z & x & y \\ z & u & v \end{pmatrix}, \quad T = \begin{pmatrix} z & s & w \\ x & t & y \end{pmatrix}.$$  

(22)

To the pair $(S, T)$, we associate the monomial:

$$e_{S,T} = e_{z,z} e_{x,s} e_{y,w} e_{z,x} e_{u,t} e_{x,y} e_{v,w}$$

in $U(gl(m_0|m_1 + n))$. 

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Let $\alpha_1, \ldots, \alpha_p \in A_0$, $\beta_1, \ldots, \beta_{\lambda_1} \in A_1$ families of mutually distinct positive and negative virtual symbols, respectively. Set

$$D_\lambda^* = \begin{pmatrix} \beta_1 & \ldots & \ldots & \beta_{\lambda_1} \\ \beta_1 & \ldots & \beta_{\lambda_2} \\ \vdots \\ \beta_1 & \ldots & \beta_{\lambda_p} \end{pmatrix}, \quad C_\lambda^* = \begin{pmatrix} \alpha_1 & \ldots & \alpha_1 \\ \alpha_2 & \ldots & \alpha_2 \\ \vdots \\ \alpha_p & \ldots & \alpha_p \end{pmatrix} \quad (23)$$

The tableaux $D_\lambda^*$ and $C_\lambda^*$ are called the virtual Deruyts and Coderuyts tableaux of shape $\lambda$, respectively.

### 3.3.2 Capelli bitableaux and right Young-Capelli bitableaux

Given a pair of Young tableaux $S, T$ of the same shape $\lambda$ on the proper alphabet $L$, consider the elements

$$e_{S,C_\lambda^*} e_{C_\lambda^*,T} \in U(gl(m_0|m_1 + n)), \quad (24)$$

$$e_{S,C_\lambda^*} e_{C_\lambda^*,D_\lambda^*} e_{D_\lambda^*,T} \in U(gl(m_0|m_1 + n)). \quad (25)$$

Since elements (24) and (25) are balanced monomials in $U(gl(m_0|m_1 + n))$, then they belong to the subalgebra $Virt(m_0 + m_1, n)$ (section 3.2.3).

Hence, we can consider their images in $U(gl(n))$ with respect to the Capelli epimorphism $p$.

We set

$$p(e_{S,C_\lambda^*} e_{C_\lambda^*,T}) = [S|T] \in U(gl(n)), \quad (26)$$

and call the element $[S|T]$ a Capelli bitableau.

We set

$$p(e_{S,C_\lambda^*} e_{C_\lambda^*,D_\lambda^*} e_{D_\lambda^*,T}) = [S \overline{T}] \in U(gl(n)). \quad (27)$$

and call the element $[S \overline{T}]$ a right Young-Capelli bitableau.

**Example 3.18.** Let $\lambda = (3, 2, 2)$, then

$$C_\lambda^* = \begin{pmatrix} \alpha_1 & \alpha_1 & \alpha_1 \\ \alpha_2 & \alpha_2 & \alpha_2 \\ \alpha_3 & \alpha_3 & \alpha_3 \end{pmatrix}, \quad \alpha_1, \alpha_2, \alpha_3 \in A_0. \quad (28)$$

Let $S$ and $T$ be the tableaux of shape $\lambda = (3, 2, 2)$ on the alphabet $A_0 \cup A_1 \cup L$, of Example 3.17. Then

$$[S|T] = p(e_{S,C_\lambda^*} e_{C_\lambda^*,T}) = p(e_{\alpha_1} e_{\alpha_1} e_{\alpha_1} e_{\alpha_2} e_{\alpha_2} e_{\alpha_2} e_{\alpha_3} e_{\alpha_3} e_{\alpha_1} e_{\alpha_2} e_{\alpha_2} e_{\alpha_3} e_{\alpha_3} e_{\alpha_1} e_{\alpha_2} e_{\alpha_1} e_{\alpha_2} e_{\alpha_2} e_{\alpha_3} e_{\alpha_3} e_{\alpha_1} e_{\alpha_2} e_{\alpha_2} e_{\alpha_3} e_{\alpha_3}) =$$

$$= p(e_{\alpha_1} e_{\alpha_1} e_{\alpha_1} e_{\alpha_2} e_{\alpha_2} e_{\alpha_2} e_{\alpha_3} e_{\alpha_3} e_{\alpha_1} e_{\alpha_2} e_{\alpha_2} e_{\alpha_3} e_{\alpha_3} e_{\alpha_1} e_{\alpha_2} e_{\alpha_1} e_{\alpha_2} e_{\alpha_2} e_{\alpha_3} e_{\alpha_3} e_{\alpha_1} e_{\alpha_2} e_{\alpha_2} e_{\alpha_3} e_{\alpha_3}).$$

\[\square\]
Remark 3.19. In the present notation, the element described in Example 3.14 is the element

\((-1)^\binom{k}{2} [S|S], \quad \lambda = (k), \quad S = (x_1, x_2 \ldots x_k)\).

The element described in Example 3.16 is the element

\([S|T], \quad \lambda = (1, 1), \quad S = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad T = \begin{pmatrix} x_2 \\ x_1 \end{pmatrix}\)

The next result will play a crucial role subsection 4.2 below. In plain words, it states that right Young-Capelli bitableaux expand into Capelli bitableaux in the enveloping algebra \(U(gl(n))\) just in the same formal way as right symmetrized bitableaux expand into bitableaux in the polynomial algebra \(C[M_{n,d}]\) (subsection 2.3).

Proposition 3.20. Let \(S, T\) be Young tableaux, \(sh(S) = sh(T)\). The following identity holds in the enveloping algebra \(U(gl(n))\):

\([S|T]) = \sum_T [S|T],\)

where the sum is extended over all \(T\) column permuted of \(T\) (hence, repeated entries in a column give rise to multiplicities).

The proof easily follows from the definitions, by applying the commutator identities in the superalgebra \(U(gl(m_0|m_1+n))\).

Example 3.21. (cfr. Example 2.2)

\[
\begin{bmatrix}
  x_1 & x_3 \\
  x_2 & x_4
\end{bmatrix}
\begin{bmatrix}
  x_1 & x_2 \\
  x_1 & x_3
\end{bmatrix} = \begin{bmatrix}
  x_1 & x_3 & x_1 & x_2 \\
  x_2 & x_4 & x_1 & x_3
\end{bmatrix} + \begin{bmatrix}
  x_1 & x_3 & x_1 & x_2 \\
  x_2 & x_4 & x_1 & x_3
\end{bmatrix} + \begin{bmatrix}
  x_1 & x_3 & x_1 & x_2 \\
  x_2 & x_4 & x_1 & x_3
\end{bmatrix} + \begin{bmatrix}
  x_1 & x_3 & x_1 & x_2 \\
  x_2 & x_4 & x_1 & x_3
\end{bmatrix} + \begin{bmatrix}
  x_1 & x_3 & x_1 & x_2 \\
  x_2 & x_4 & x_1 & x_3
\end{bmatrix} = 2 \begin{bmatrix}
  x_1 & x_3 & x_1 & x_2 \\
  x_2 & x_4 & x_1 & x_3
\end{bmatrix} + 2 \begin{bmatrix}
  x_1 & x_3 & x_1 & x_2 \\
  x_2 & x_4 & x_1 & x_3
\end{bmatrix}.
\]

4 The bitableaux correspondence isomorphism \(B\) and the Koszul map \(K\)

4.1 The BCK theorem

Our next aim is to describe an extremely relevant pair of (mutually inverse) vector space isomorphisms between the polynomial algebra of forms \(C[M_{n,n}]\) and the universal enveloping algebra \(U(gl(n))\).

In order to do this, it is worth to simplify the notation in the following way:
• we will write $i$ in place of $x_i$ and $e_{ij}$ in place of $e_{x_i,x_j}$;
• consistently, we set $L = P = \mathbf{n} = \{1, 2, \ldots, n\}$.

The main advantage of this convention is that it allows us to write bitableaux in $\mathbb{C}[M_{n,n}]$ and Capelli bitableaux in $U(gl(n))$ as elements associated to pairs of Young tableaux on the same alphabet.

More specifically, given a shape (partition) $\lambda$ with $\lambda_1 \leq n$, to any pair of Young tableaux $S, T$ on the alphabet $\mathbf{n} = \{1, 2, \ldots, n\}$ and of the same shape $sh(S) = sh(T) = \lambda$, one associates the (determinantal) bitableau $(S|T) \in \mathbb{C}[M_{n,n}]$, and the Capelli bitableau $[S|T] \in U(gl(n))$.

**Theorem 4.1. (The BCK theorem)** The “bitableaux correspondence” map

$$B : (S|T) \mapsto [S|T]$$

uniquely defines a linear isomorphism

$$B : \mathbb{C}[M_{n,n}] \rightarrow U(gl(n)).$$

Furthermore, this isomorphism is the inverse of the Koszul map

$$K : U(gl(n)) \rightarrow \mathbb{C}[M_{n,n}]$$

introduced by J.-L. Koszul in [25].

Eq. (29) indeed defines a linear operator since bitableaux in $\mathbb{C}[M_{n,n}]$ and Capelli bitableaux in $U(gl(n))$ are ruled by the same straightening laws (see [5], Proposition 7).

The linear isomorphism $B$ was introduced in [6], Theorem 1. The fact that $B$ and $K$ are inverse of each other was proved in [6], Theorem 2 (see also, [2]).

### 4.2 Right symmetrized bitableaux and right Young-Capelli bitableaux

Amazingly, the "bitableaux correspondence" and the Koszul isomorphisms well-behave with respect to right symmetrized bitableaux

$$(S\left[ T \right]) \in \mathbb{C}[M_{n,n}]$$

and right Young-Capelli bitableaux

$$[S\left[ T \right] = p(e_{SC\lambda}e_{C\lambda}D_T e_{D_T\lambda}e_{\lambda}) \in U(gl(n)).$$

In plain words, any right Young-Capelli bitableaux $[S\left[ T \right]$ is the image - with respect to the linear operator $B$ - of the right symmetrized bitableaux $(S\left[ T \right]$.

**Theorem 4.2. We have:**

$$B : (S\left[ T \right] \mapsto [S\left[ T \right],$$

$$K : [S\left[ T \right] \mapsto (S\left[ T \right].$$
Proof. Indeed, we have:

\[ B(S \square T) = B \left( \sum_T (S|T) \right) = \sum_T [S|T], \]

where the sum is extended over all column permuted of \( T \).

By Proposition 3.20, the last summation equals the right Young-Capelli bitableaux \([S|T]\).

Example 4.3. Let \( \lambda = (2, 1) \). Let

\[ S = \begin{pmatrix} 1 & 3 \\ 2 & 3 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} \]

Consider the Young-Capelli bitableau

\[ \begin{array}{c|c} 1 & 3 \\ \hline 2 & 3 \end{array} \begin{array}{c|c} 1 & 2 \\ \hline 3 & 3 \end{array} = [S \square T] = p(e_{SC} e_{C^*} D_{\lambda} e_{D^* T}) \in U(gl(3)). \]

We have

\[ p(e_{SC} e_{C^*} D_{\lambda} e_{D^* T}) = p(e_{1\alpha_1, e_{3\alpha_1}, e_{2\alpha_2}} \cdot e_{\alpha_1, \beta_1} e_{\alpha_1, \beta_2} e_{\alpha_2, \beta_1} \cdot e_{\beta_1, 1} e_{\beta_2, 2} e_{\beta_3, 3}) \]

that can be proved (for example, by using the commutation rules in a suitable \( U(gl(m_0|m_1 + 3), m_0, m_1 \geq 2) \) to be equal to

\[ p(e_{1\alpha_1, e_{3\alpha_1}, e_{2\alpha_2}} \cdot e_{\alpha_1, 1} e_{\alpha_2, 2} e_{\alpha_3}) + p(e_{1\alpha_1, e_{3\alpha_1}, e_{2\alpha_2}} \cdot e_{\alpha_1, 3} e_{\alpha_2, 1}) = \]

\[ = \begin{array}{c|c} 1 & 3 \\ \hline 2 & 3 \end{array} + \begin{array}{c|c} 1 & 3 \\ \hline 2 & 1 \end{array} = \]

\[ = B \left( \begin{pmatrix} 1 & 3 \\ 2 & 3 \end{pmatrix} \right) + B \left( \begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix} \right) = \]

\[ = B \left( \begin{array}{c|c} 1 & 2 \\ \hline 3 & 3 \end{array} \right) \in U(gl(3)). \]

By Theorem 4.2 and Theorem 2.3, we have:

Theorem 4.4. Let \( h \in \mathbb{N} \). The set of right Young-Capelli bitableaux

\[ \bigcup_{k=0}^{h} \left\{ [S \square T] : S, T \text{ standard, } sh(S) = sh(T) = \lambda \vdash k, \lambda_1 \leq n \right\} \]

is a basis of the filtration element \( U(gl(n))^{(h)} \).

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Remark 4.5. The basis elements
\[
\{ [S|T] : S, T \text{ standard, } sh(S) = sh(T) = \lambda \vdash k, \lambda_1 \leq n \}
\]
act in a quite remarkable way on Gordan-Capelli basis elements
\[
\{ (U|V) : U, V \text{ standard, } sh(U) = sh(V) = \mu \vdash h, \lambda_1 \leq n \}.
\]
Indeed, we have:
- If \( h < k \), the action is zero.
- If \( h = k \), the action is nondegenerate triangular.

See [4] and [1], Theorem 10.1.

\[\square\]

4.3 Column Capelli bitableaux in \( \mathbf{U}(gl(n)) \)

A column Capelli bitableau in \( \mathbf{U}(gl(n)) \) is a Capelli bitableau \([S|T]\), where \( S \) and \( T \) are column Young tableaux of the same depth.

Although column Capelli bitableaux are far from being “monomials” in \( \mathbf{U}(gl(n)) \), they play the same role that column bitableaux – signed monomials – play in the polynomial algebra \( \mathbb{C}[M_{n,n}] \). Specifically, Capelli bitableaux and right Young-Capelli bitableaux expand into column Capelli bitableaux just in the same way as bitableaux and right symmetrized bitableaux expand into column bitableaux in the polynomial algebra \( \mathbb{C}[M_{n,n}] \).

Remark 4.6. The column Capelli bitableau \([i|j]\) of depth \( h = 1 \) equals the generator \( e_{i,j} \) of the algebra \( \mathbf{U}(gl(n)) \), \( i, j = 1, 2, \ldots, n \). Indeed
\[
[i|j] = p [e_{i,\alpha} e_{\alpha,j} - e_{\alpha,j} e_{i,\alpha} + \delta_{i,j} e_{\alpha,\alpha}] = e_{ij}.
\]

From Theorem 4.1, it immediately follows:

Corollary 4.7. We have:

\[\begin{align*}
\mathcal{B} : \begin{pmatrix} i_1 & j_1 \\ i_2 & j_2 \\ \vdots & \vdots \\ i_h & j_h \end{pmatrix} & \mapsto \begin{pmatrix} i_1 & j_1 \\ i_2 & j_2 \\ \vdots & \vdots \\ i_h & j_h \end{pmatrix}, \\
\mathcal{K} : \begin{pmatrix} i_1 & j_1 \\ i_2 & j_2 \\ \vdots & \vdots \\ i_h & j_h \end{pmatrix} & \mapsto \begin{pmatrix} i_1 & j_1 \\ i_2 & j_2 \\ \vdots & \vdots \\ i_h & j_h \end{pmatrix}.
\end{align*}\]
Since column bitableaux in the polynomial algebra $C[M_{n,n}]$ are signed commutative monomials, then column Capelli bitableaux are invariant with respect to permutations of their rows, that is

$$\begin{bmatrix} i_1 & j_1 \\ i_2 & j_2 \\ \vdots & \vdots \\ i_h & j_h \end{bmatrix} = \begin{bmatrix} i_{\sigma(1)} & j_{\sigma(1)} \\ i_{\sigma(2)} & j_{\sigma(2)} \\ \vdots & \vdots \\ i_{\sigma(h)} & j_{\sigma(h)} \end{bmatrix}$$

for every $\sigma \in S_h$.

Let denote by $C_h[M_{n,n}]$ the homogeneous component of degree $h \in \mathbb{N}$ of the polynomial algebra $C[M_{n,n}]$ and denote $U(gl(n))^{(h)}$ the $h$–th filtration element of the enveloping algebra $U(gl(n))$.

**Corollary 4.8.** The bitableaux correspondence isomorphism $B$ and the Koszul isomorphism $K$ induce, by restriction, a pair of mutually inverse isomorphism

$$B : \bigoplus_{k=0}^{h} C_k[M_{n,n}] \rightarrow U(gl(n))^{(h)}$$

and

$$K : U(gl(n))^{(h)} \rightarrow \bigoplus_{k=0}^{h} C_k[M_{n,n}].$$

The preceding assertion can be regarded as a sharpened version of the PBW Theorem for $U(gl(n))$.

### 4.3.1 Devirtualization of column Capelli bitableaux in $U(gl(n))$

Given any column Capelli bitableau, *devirtualized expressions* of it as an element of $U(gl(n))$ can be easily obtained by means of iterations of the following identities.

**Proposition 4.9.** In the enveloping algebra $U(gl(n))$, we have:

$$(-1)^{h-1} e_{i_{11},j_{11}} \begin{bmatrix} i_2 & j_2 \\ \vdots & \vdots \\ i_{h-1} & j_{h-1} \\ i_h & j_h \end{bmatrix} + (-1)^{h-2} \sum_{k=2}^{h} \delta_{i_k, j_1} \begin{bmatrix} i_1 & j_k \\ \vdots & \vdots \\ i_{k-1} & j_{k-1} \\ i_{k+1} & j_{k+1} \\ \vdots & \vdots \\ i_h & j_h \end{bmatrix} =$$
Proof. By definition,

\[
(-1)^{h-1} \begin{bmatrix} i_1 & j_1 \\ i_2 & j_2 \\ \vdots & \vdots \\ i_h & j_h \end{bmatrix} e_{i_h,j_h} + (-1)^{h-2} \sum_{k=1}^{h-1} \delta_{i_k,j_k} e_{i_k,j_k} = \begin{bmatrix} i_1 & j_1 \\ \vdots & \vdots \\ i_k & j_k \end{bmatrix}.
\]

Notice that as elements of the algebra \( U \),

\[
\delta_{i_h,j_1} e_{i_1,\alpha_1} e_{i_2,\alpha_2} \cdots e_{i_{h-1},\alpha_{h-1}} e_{i_h,\alpha_h} \cdot e_{\alpha_1,j_1} e_{\alpha_2,j_2} \cdots e_{\alpha_{h-1},j_{h-1}} e_{\alpha_h,j_h} = \delta_{i_h,j_2} (-1)^{h-2} e_{i_1,\alpha_1} e_{i_2,\alpha_2} \cdots e_{i_{h-1},\alpha_{h-1}} e_{\alpha_1,j_1} e_{\alpha_2,j_2} \cdots e_{\alpha_{h-1},j_{h-1}} e_{\alpha_h,j_h}
\]

as elements of the algebra \( gl(m_0|n_1 + n) \).

Therefore, the summand

\[
p e_{i_1,\alpha_1} e_{i_2,\alpha_2} \cdots e_{i_{h-1},\alpha_{h-1}} \cdot \delta_{i_h,j_1} e_{\alpha_2,j_2} \cdots e_{\alpha_{h-1},j_{h-1}} e_{\alpha_h,j_h}
\]

equals

\[
(-1)^{h-2} \delta_{i_h,j_1} \begin{bmatrix} i_1 & j_h \\ i_2 & j_2 \\ \vdots & \vdots \\ i_h & j_h \end{bmatrix}.
\]

By repeating the above procedure of moving left the element \( e_{\alpha_1,j_1} \) - using the commutator identities in \( U(gl(m_0|n_1 + n)) \) - we finally get

\[
\begin{bmatrix} i_1 & j_1 \\ i_2 & j_2 \\ \vdots & \vdots \\ i_h & j_h \end{bmatrix} = \begin{bmatrix} i_1 & j_1 \\ \vdots & \vdots \\ i_k & j_k \end{bmatrix}.
\]

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Example 4.10.

equals

\[ U \]

Furthermore

\[ \notag = \begin{pmatrix} 1 & 2 \\ 2 & 1 \\ 3 & 1 \end{pmatrix} = [1]2 \begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 1 \\ 3 & 1 \end{pmatrix} = -e_{12} e_{21} e_{31} + e_{11} e_{31} \in U(gl(n)). \]
Notice that
\[
\begin{bmatrix}
1 & 2 \\
2 & 1 \\
3 & 1
\end{bmatrix}
= \begin{bmatrix}
3 & 1 \\
2 & 1 \\
1 & 2
\end{bmatrix}
= [3|1] \begin{bmatrix}
2 & 1 \\
1 & 2
\end{bmatrix}
- \begin{bmatrix}
3 & 1 \\
2 & 1
\end{bmatrix}
= (-[3|1]|2|1|1|2 - [2|2]) + [2|1]|3|2
= -e_{31}e_{21} + e_{31}e_{22} + e_{21}e_{32}
= \begin{bmatrix}
1 & 2 \\
3 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 2 \\
3 & 1
\end{bmatrix}
\begin{bmatrix}
3 & 1 \\
2 & 1
\end{bmatrix}
= (-[1|2]|2|1 + [1|1]|3|1
= -e_{12}e_{21}e_{31} + e_{11}e_{31} \in U(gl(n)).
\]

\[\square\]

Remark 4.11. Theorems 4.1 and 4.2, in combination with Proposition 4.9, allows the explicit devirtualized forms in $U(gl(n))$ of Capelli bitableaux and of right Young-Capelli bitableaux to be easily computed. The process can be illustrated by an example. Let $n \geq 2$, $h = 3$, $\lambda = (2,1)$. Consider the Capelli bitableaux
\[
\begin{bmatrix}
1 & 2 \\
1 & 2
\end{bmatrix}
in U(gl(n)).
\]

By Theorem 4.1:
\[
\begin{bmatrix}
1 & 2 \\
1 & 2
\end{bmatrix}
= B\left(\begin{bmatrix}
1 & 2 \\
1 & 2
\end{bmatrix}\right)
= B\left(\begin{bmatrix}
1 & 1 \\
2 & 1
\end{bmatrix}\right) - \begin{bmatrix}
1 & 2 \\
1 & 2
\end{bmatrix}
= \begin{bmatrix}
1 & 1 \\
2 & 2
\end{bmatrix}
- \begin{bmatrix}
1 & 2 \\
2 & 1
\end{bmatrix}
= \begin{bmatrix}
1 & 1 \\
2 & 2
\end{bmatrix}
- \begin{bmatrix}
1 & 2 \\
2 & 1
\end{bmatrix}.
\]

By Proposition 4.9,
\[
\begin{bmatrix}
1 & 1 \\
2 & 2 \\
1 & 2
\end{bmatrix}
= -e_{11}e_{22}e_{12} + e_{12}e_{21} - e_{12} \in U(gl(n)),
\begin{bmatrix}
1 & 2 \\
2 & 1 \\
1 & 2
\end{bmatrix}
= -e_{12}e_{21}e_{12} + e_{12}e_{22} + e_{11}e_{12} - e_{12} \in U(gl(n)).
\]

\[\square\]
4.3.2 Column Capelli bitableaux as polynomial differential operators on $\mathbb{C}[M_{n,d}]$

The next result will play a crucial role in section 6. In the language of Procesi ([36], chapter 3), it describes the action of column Capelli bitableaux as elements of the Weyl algebra associated to the polynomial algebra $\mathbb{C}[M_{n,d}]$.

**Proposition 4.12.** The action of the column Capelli bitableau

\[
\begin{bmatrix}
  i_1 & j_1 \\
  i_2 & j_2 \\
  \vdots & \vdots \\
  i_h & j_h \\
\end{bmatrix}
\in U(gl(n))
\]

on the algebra $\mathbb{C}[M_{n,d}]$ equals the action of the polynomial differential operator

\[
(-1)^{\binom{h}{2}} \sum_{(\varphi_1,\varphi_2,\ldots,\varphi_h) \in \mathbb{N}^h} (i_1|\varphi_1)(i_2|\varphi_2) \cdots (i_h|\varphi_h) \partial_{(j_1|\varphi_1)} \partial_{(j_2|\varphi_2)} \cdots \partial_{(j_h|\varphi_h)}.
\]

**Proof.** Consider a monomial $M \in \mathbb{C}[M_{n,d}]$,

\[
M = \prod_{i=1}^n (i|1)^{s_{i1}}(i|2)^{s_{i2}} \cdots (i|d)^{s_{id}}
\]

and let $\alpha \in A_0$ be a positive virtual symbol. Given $j_h = 1, 2, \ldots, n$, consider the action of the superpolarization $D_{\alpha,j_h}$ on the supersymmetric algebra $\mathbb{C}[M_{m_0|m_1+n,d}] \supseteq \mathbb{C}[M_{n,d}]$. A straightforward computation shows that

\[
D_{\alpha,j_h}(M) = \sum_{\varphi=1}^d \partial_{(j_1|\varphi)}(M)(\alpha|\varphi).
\] (30)

Furthermore, notice that:

\[
D_{\alpha,s,j_h}D_{\alpha,t,j_h}(M) = \sum_{\varphi=1}^d \left(D_{\alpha,s,j_h}(\partial_{(j_1|\varphi)}(M))\right)(\alpha_t|\varphi),
\]

that equals

\[
\sum_{\varphi_1,\varphi_2=1,2,\ldots,d} \left(\partial_{(j_h|\varphi_1)}\partial_{(j_h|\varphi_2)}(M)\right)(\alpha_s|\varphi_1)(\alpha_t|\varphi_2).
\] (31)

Recall that the action of the column Capelli bitableau

\[
\begin{bmatrix}
  i_1 & j_1 \\
  i_2 & j_2 \\
  \vdots & \vdots \\
  i_h & j_h \\
\end{bmatrix}
\in U(gl(n))
\]
on the algebra $\mathbb{C}[M_{n,d}]$ is implemented by the product of superpolarizations

$$D_{i_1, \alpha_1} \cdots D_{i_{h-1}, \alpha_{h-1}} D_{i_h, \alpha_h} D_{\alpha_1,j_1} \cdots D_{\alpha_{h-1},j_{h-1}} D_{\alpha_h,j_h},$$

where $\alpha_1, \ldots, \alpha_{h-1}, \alpha_h$ are distinct arbitrary positive virtual symbols. Note that $|D_{i_r, \alpha_r}| = |D_{\alpha_r,j_r}| = 1 \in \mathbb{Z}_2$, for every $r = 1, 2, \ldots, h$.

From eqs. (30) and (31), it immediately follows:

$$D_{\alpha_1,j_1} \cdots D_{\alpha_h,j_h} (\mathbb{M}) = \sum_{(\varphi_1, \ldots, \varphi_h) \in d^h} \partial_{(j_1|\varphi_1)} \cdots \partial_{(j_h|\varphi_h)} (\mathbb{M}) (\alpha_1|\varphi_1) \cdots (\alpha_h|\varphi_h)$$

(32)

Since $|(\alpha_r|\varphi_r)| = 1 \in \mathbb{Z}_2$, for every $r = 1, 2, \ldots, h$, from eq. (32), we infer:

$$D_{i_1, \alpha_1} \cdots D_{i_{h-1}, \alpha_{h-1}} D_{i_h, \alpha_h} \left(D_{\alpha_1,j_1} \cdots D_{\alpha_{h-1},j_{h-1}} D_{\alpha_h,j_h} (\mathbb{M})\right)$$

equals

$$(1)^{(2)} \sum_{(\varphi_1, \varphi_2, \ldots, \varphi_h) \in d^h} (i_1|\varphi_1)(i_2|\varphi_2) \cdots (i_h|\varphi_h) \partial_{(j_1|\varphi_1)} \partial_{(j_2|\varphi_2)} \cdots \partial_{(j_h|\varphi_h)} (\mathbb{M}).$$

$\Box$

5 Capelli immanants and Young-Capelli bitableaux in $U(gl(n))$

The bitableaux correspondence (linear) isomorphism

$$\mathcal{B} : \mathbb{C}[M_{n,n}] \rightarrow U(gl(n)),$$

leads to the following natural definition of Capelli immanant

$$\text{Cimm}_\lambda[i_1i_2 \cdots i_h; j_1j_2 \cdots j_h]$$

in the enveloping algebra in $U(gl(n))$:

$$\text{Cimm}_\lambda[i_1i_2 \cdots i_h; j_1j_2 \cdots j_h] = \mathcal{B}(\text{imm}_\lambda[i_1i_2 \cdots i_h; j_1j_2 \cdots j_h]).$$

By linearity of the operator $\mathcal{B}$, we get

$$\text{Cimm}_\lambda[i_1i_2 \cdots i_h; j_1j_2 \cdots j_h] = \sum_{\sigma \in S_h} \chi^\lambda(\sigma) \left[ \begin{array}{cccc} i_{\sigma(1)} & j_1 \\ i_{\sigma(2)} & j_2 \\ \vdots & \vdots \\ i_{\sigma(h)} & j_h \end{array} \right]$$

$$= \sum_{\sigma \in S_h} \chi^\lambda(\sigma) \left[ \begin{array}{cccc} i_1 & j_{\sigma(1)} \\ i_2 & j_{\sigma(2)} \\ \vdots & \vdots \\ i_h & j_{\sigma(h)} \end{array} \right].$$

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Clearly, the notion of Capelli immanants provides a natural generalization of the notion of Capelli determinant (see Example 3.14).

**Example 5.1.** Consider the symmetric group $S_3$, the irreducible character $\chi^\lambda$, $\lambda = (2, 1) \vdash h = 3$, and the irreducible character element

$$\chi_\lambda = \sum_{\sigma \in S_3} \chi^\lambda(\sigma)\sigma = 2I - (123) - (132) \in \mathbb{C}[S_3].$$

We have:

$$\text{Cimm}_\lambda[123; 123] = \sum_{\sigma \in S_3} \chi^\lambda(\sigma) \cdot \begin{bmatrix} 1 & \sigma(1) \\ 2 & \sigma(2) \\ 3 & \sigma(3) \end{bmatrix} = 2 \begin{bmatrix} 1 & 1 \\ 2 & 2 \\ 3 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 3 \\ 2 & 1 \\ 3 & 2 \end{bmatrix}.$$

We have

$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \\ 3 & 3 \end{bmatrix} = -e_{11}e_{22}e_{33},$$

$$\begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 1 \end{bmatrix} e_{31} - \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} = -e_{12}e_{23}e_{31} + e_{13}e_{31} + e_{12}e_{21} - e_{11},$$

$$\begin{bmatrix} 1 & 3 \\ 2 & 1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & 1 \\ 3 & 2 \end{bmatrix} e_{32} - \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} = -e_{13}e_{21}e_{32} + e_{12}e_{21} - e_{11}.$$

Therefore, for $n \geq 3$,

$$\text{Cimm}_\lambda[123; 123]$$

is equal to

$$-2e_{11}e_{22}e_{33} + e_{12}e_{23}e_{31} + e_{13}e_{31} + e_{12}e_{21} - 2e_{12}e_{21} + 2e_{11} \in U(gl(n)).$$

Since a right Young-Capelli bitableau $[U \begin{bmatrix} V \end{bmatrix}] \in U(gl(n))$ is the image of the right symmetrized bitableau $(U \begin{bmatrix} V \end{bmatrix}) \in \mathbb{C}[M_{n,n}]$ with respect to the isomorphism $\mathcal{B}$, Proposition 2.10 implies

**Theorem 5.2.** Let $\lambda \vdash h$. Any Capelli immanant $\text{Cimm}_\lambda[i_1i_2\cdots i_h; j_1j_2\cdots j_h]$ can be written as a linear combination of standard right Young-Capelli bitableaux $[U \begin{bmatrix} V \end{bmatrix}]$ in $U(gl(n))$ of the same shape $\lambda$:

$$\text{Cimm}_\lambda[i_1i_2\cdots i_h; j_1j_2\cdots j_h] = \sum_{U,V} g_{U,V} [U \begin{bmatrix} V \end{bmatrix}],$$

where $g_{U,V} \in \mathbb{C}$, $sh(U) = sh(V) = \lambda$.

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From Corollary 2.11, it follows:

**Corollary 5.3.** Let $\lambda \vdash h$. If $\lambda_1 \not\leq n$, then

$$C_{\text{imm}}(i_1 i_2 \cdots i_k; j_1 j_2 \cdots j_k) = 0.$$ 

Furthermore, Proposition 2.14 implies

**Theorem 5.4.** Let $\lambda \vdash h$. Any right Young-Capelli bitableau \([U] [V]\) in $U(gl(n))$ of shape $sh(U) = sh(V) = \lambda$ can be written as a linear combination of Capelli immanants $C_{\text{imm}}(i_1 i_2 \cdots i_k; j_1 j_2 \cdots j_k)$ associated to the same shape $\lambda$.

Proposition 2.16 implies

**Theorem 5.5.** The set of Capelli immanants

$$\bigcup_{k=0}^{h} \{C_{\text{imm}}(i_1 i_2 \cdots i_k; j_1 j_2 \cdots j_k); \lambda \vdash k, \lambda_1 \leq n, (i_1 i_2 \cdots i_k), (j_1 j_2 \cdots j_k) \in \mathbb{P}^k\}$$

is a spanning set of $U(gl(n))^{(h)}$.

## 6 The Okounkov quantum immanants

In this section, we follow along the lines of previous work of Okounkov [31], [30]. Our main result is a description of quantum immanants as simple linear combinations of Capelli immanants. This result – in combination with Proposition 4.9 – allows the computation of quantum immanants as elements of $U(gl(n))$ to be reduced to a fairly simple process (see, e.g. Example 6.4 below).

Quantum immanants are central elements associated to shapes $\lambda$, $\lambda_1 \leq n$, and equal (up to the scalar factor $\frac{1}{H(\lambda)}$) the elements of the Schur basis, that is the preimage, in the center $\zeta(n)$ of $U(gl(n))$, of the Sahi–Okounkov–Schur basis of the algebra of shifted symmetric polynomials, with respect to the Harish-Chandra isomorphism [38], [32].

We adopt the following notational conventions:

- Given a positive integer $h \in \mathbb{Z}^+$, and a partition $\lambda \vdash h$, $V^\lambda$ denotes the irreducible representation of the symmetric group $S_h$, associated to the shape $\lambda$. Recall that $\dim(V^\lambda) = \frac{h!}{H(\lambda)}$.

- $T$ denotes a multilinear standard Young tableau of shape $sh(T) = \lambda \vdash h$.

- For every $s = 1, 2, \ldots, h$, let $(i, j)$ be the pair of row and column indices of the cell of $T$ that contains $s$. Set $c_T(s) = j - i$ (the “Frobenius content” of the cell $(i, j)$).

- $v_T$ denotes the element of the seminormal Young basis of $V^\lambda$ associated to the multilinear standard tableau $T$. Since each basis vector is defined only up to a scalar factor, we assume that $(v_T, v_T) = 1$ (see Okounkov and Vershik [33]; for a more traditional approach, see James and Kerber [23]).
- given the element
\[ \Psi_T = \sum_{\sigma \in S_h} (\sigma \cdot v_T, v_T) \sigma^{-1} \in \mathbb{C}[S_h], \]

\(\Psi_T^h\) denotes the matrix that represents the element \(\Psi_T\) as a linear operator on the tensor space \((\mathbb{C}^n)^{\otimes h}\).

- \(\chi^h_T\) denotes the matrix that represents the element
\[ \chi^h_T = \sum_{\sigma \in S_h} \chi^h(\sigma) \sigma \in \mathbb{C}[S_h] \]
as a linear operator on the tensor space \((\mathbb{C}^n)^{\otimes h}\).

- Let \(E = [e_{ij}]_{i,j=1,2,...,n}\) be the matrix whose entries are the elements of the standard basis of \(gl(n)\).

- Let
\[ E_T = (E - c_T(1)) \otimes (E - c_T(2)) \otimes \cdots \otimes (E - c_T(h)) \times \Psi_T^h \]
be the fusion matrix; the fusion matrix \(E_T\) is a \((n^h \times n^h)\)-matrix with entries in \(U(gl(n))\).

Following Okounkov ([31], [30]), the element
\[ Tr(E_T) \in U(gl(n))^{(h)} \]
is the quantum immanant associated to the multilinear standard tableau \(T\).

The higher Capelli identities ([31], [30]), imply ([31], eq. (5.1)) that the action of the quantum immanant
\[ Tr(E_T) \]
on the algebra \(\mathbb{C}[M_{n,d}]\) equals the action of the polynomial differential operator
\[ \frac{1}{\dim(V^\lambda)} Tr(X^{\otimes h} \times (D')^{\otimes h} \times \chi^h_T), \] (33)
where
- \(X\) denotes the matrix \([i|\varphi]_{i=1,...,n; \varphi=1,...,d}\)
- \(D\) denotes the matrix \([\partial_{i|\varphi}]_{i=1,...,n; \varphi=1,...,d}\) of partial derivatives on the algebra \(\mathbb{C}[M_{n,d}]\), and the prime stands for transposition.

Since the action of \(U(gl(n))\) on the algebra \(\mathbb{C}[M_{n,d}]\) is a faithful action whenever \(n \leq d\), and the differential operator of eq. (33) is independent from the choice of the multilinear standard tableau \(T\), the quantum immanant \(Tr(E_T)\) depends only from the shape \(\lambda\).
Theorem 6.1. The quantum immanant

\[ Tr(\mathbb{E}_\tau) \]

equals the linear combination of Capelli immanants:

\[ (-1)^{\binom{n}{2}} \sum_{h_1 + h_2 + \cdots + h_n = h} \frac{H(\lambda)}{h_1! h_2! \cdots h_n!} \text{Cimm}_\lambda[i_1^{h_1} 2^{h_2} \cdots n_i^{h_n}; 1^{h_1} 2^{h_2} \cdots n_i^{h_n}], \]

where \( 1^{h_1} 2^{h_2} \cdots n_i^{h_n} \) is a short notation for the non decreasing sequence \( i_1 i_2 \cdots i_h \) with

\[ h_p = \#\{i_q = p; \ q = 1, 2, \ldots, h\}, \quad p = 1, 2, \ldots, n. \]

**Proof.** For every \( \sigma \in S_h \), \( \overline{i} = (i_1, \ldots, i_h) \in n^h \), \( \overline{\varphi} = (\varphi_1, \ldots, \varphi_h) \in d^h \), we set

\[ P_\sigma[\overline{i}; \overline{\varphi}] = (i_1|\varphi_1) \cdots (i_h|\varphi_h) \partial_{(i_\tau(1)|\varphi_1)} \cdots \partial_{(i_\tau(h)|\varphi_h)}. \]

By straightforward computation, the right-hand side of eq. (33) equals

\[ \frac{1}{\dim(V^\lambda)} \sum_{\overline{i} = (i_1, \ldots, i_h) \in n^h} \left( \sum_{\sigma \in S_h} \chi^\lambda(\sigma) \left( \sum_{\overline{\varphi} = (\varphi_1, \ldots, \varphi_h) \in d^h} P_\sigma[\overline{i}; \overline{\varphi}] \right) \right). \quad (34) \]

By Proposition 4.12, the action of the Capelli immanant

\[ \text{Cimm}_\lambda[i_1 i_2 \cdots i_h; i_1 i_2 \cdots i_h] \in U(gl(n)) \]

on the algebra \( \mathbb{C}[M_{n,d}] \) equals the action of the polynomial differential operator

\[ (-1)^{\binom{n}{2}} \sum_{\sigma \in S_h} \chi^\lambda(\sigma) \left( \sum_{\overline{\varphi} = (\varphi_1, \ldots, \varphi_h) \in d^h} P_\sigma[\overline{i}; \overline{\varphi}] \right), \]

for every \( \overline{i} = (i_1, \ldots, i_h) \in n^h \).

Since the action of \( U(gl(n)) \) on the algebra \( \mathbb{C}[M_{n,d}] \) is a faithful action whenever \( n \leq d \), it immediately follows that any quantum immanant equals - up to a scalar factor - a linear combination of Capelli immanants. Indeed, we have:

\[ Tr(\mathbb{E}_\tau) = (-1)^{\binom{n}{2}} \frac{1}{\dim(V^\lambda)} \sum_{(i_1, \ldots, i_h) \in n^h} \text{Cimm}_\lambda[i_1 i_2 \cdots i_h; i_1 i_2 \cdots i_h] \in U(gl(n)). \quad (35) \]

Since

\[ \text{Cimm}_\lambda[i_1 i_2 \cdots i_h; i_1 i_2 \cdots i_h] = \text{Cimm}_\lambda[i_{\tau(1)} i_{\tau(2)} \cdots i_{\tau(h)}; i_{\tau(1)} i_{\tau(2)} \cdots i_{\tau(h)}], \]

for every \( \tau \in C[S_h] \), the right-hand side of eq. (35) equals

\[ (-1)^{\binom{n}{2}} \sum_{h_1 + h_2 + \cdots + h_n = h} \frac{H(\lambda)}{h_1! h_2! \cdots h_n!} \text{Cimm}_\lambda[i_1^{h_1} 2^{h_2} \cdots n_i^{h_n}; 1^{h_1} 2^{h_2} \cdots n_i^{h_n}]. \]

\[ \square \]
From Theorem 6.1 and Corollary 5.3, it follows:

**Corollary 6.2.** Let $T$ be a multilinear standard tableau, $sh(T) = \lambda$. If $\lambda_1 \not\leq n$, then

$$Tr(E_T) = 0.$$ 

\[ \square \]

Let $\lambda, \lambda_1 \leq n$, and let recall that $\zeta(n)$ is the center of $U(gl(n))$. According with Okoukov [31], [30], the Schur element $S_\lambda(n) \in \zeta(n)$ is defined by setting

$$S_\lambda(n) = \frac{dim(V^\lambda)}{h!} Tr(E_T).$$

Since $dim(V^\lambda) = \prod_{i=1}^{h} H(\lambda)$, Theorem 6.1 implies:

**Corollary 6.3.**

$$S_\lambda(n) = (-1)^{\frac{\lambda_1}{2}} \sum_{h_1 + \cdots + h_n = h} \frac{1}{h_1! \cdots h_n!} Cimm_{\lambda}[1^{h_1} \cdots n^{h_n}; 1^{h_1} \cdots n^{h_n}].$$ (36)

**Example 6.4.** Let $h = 3, \lambda = (2,1), n = 2$. Recall that $H(\lambda) = 3$. Then

$$S_{(2,1)}(2) = -\frac{1}{2} \left( Cimm_{(2,1)}[11;112] + Cimm_{(2,1)}[122;12] \right).$$ (37)

Indeed,

$$Cimm_{(2,1)}[11;112] = 2 \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 2 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 2 & 2 \end{bmatrix} = 2 \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 2 & 2 \end{bmatrix} - 2 \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 1 \end{bmatrix}.$$ 

$$Cimm_{(2,1)}[122;12] = 2 \begin{bmatrix} 1 & 1 \\ 2 & 2 \\ 2 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 2 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 2 & 2 \end{bmatrix} = 2 \begin{bmatrix} 1 & 1 \\ 2 & 2 \\ 2 & 2 \end{bmatrix} - 2 \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 2 & 2 \end{bmatrix}.$$ 

Hence

$$S_{(2,1)}(2) = - \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 2 & 2 \\ 2 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 2 & 2 \\ 2 & 2 \end{bmatrix}.$$ 

By Proposition 4.9, we obtain:
\[ \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = -e_{11}^2 e_{22} + e_{11} e_{22}, \]

\[ \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} = -e_{11} e_{12} e_{21} + e_{11}^2 + e_{12} e_{21} - e_{11}, \]

\[ \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} = -e_{11} e_{22}^2 + e_{11} e_{22}, \]

\[ \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} = -e_{12} e_{21} e_{22} + e_{11} e_{22} + e_{12} e_{21} - e_{11}. \]

According with Theorem 5.2, the central element \( S_{(2,1)}(2) \) also equals, in turn, the linear combination of Young-Capelli bitableaux:

\[ S_{(2,1)}(2) = -\frac{1}{2} \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}. \] (38)

Identity (38) is an instance of an alternative presentation (see our preprint [8], subsection 4.5.1) of the Schur element \( S_\lambda(n) \in \zeta(n) \), \( \lambda_1 \leq n \) of Corollary 6.3.

In the notation of subsection 3.3.1, we have:

\[ S_\lambda(n) = \frac{1}{H(\lambda)} \sum_S p(e_{S,C_\lambda^*} \cdot e_{C_\lambda^*} \cdot e_{D_\lambda^*} \cdot e_{D_\lambda^*} \cdot e_{S,C_\lambda^*} \cdot e_{C_\lambda^*} \cdot e_{C_\lambda^*} \cdot e_{S,C_\lambda^*}) \] (39)

where the sum is extended to all row (strictly) increasing tableaux \( S \) on the proper alphabet \( L = \{1, 2, \ldots, n\} \).

**Example 6.5.** We show how identity (39) specializes to identity (38), and hence to identity (37).

By eq. (39), we have:

\[ S_{(2,1)}(2) = \frac{1}{3} p \begin{pmatrix} 1 & 2 & a_1 a_1 & \cdot e_{a_1 a_1} & \cdot e_{a_1 a_1} & \cdot e_{a_2 a_2} & \cdot e_{a_1 a_1} & \cdot e_{a_1 a_1} & \cdot e_{a_1 a_1} & \cdot e_{a_1 a_1} & \cdot e_{a_2 a_2} & \cdot e_{a_1 a_1} & \cdot e_{a_1 a_1} & \cdot e_{a_1 a_1} \\ 1 & 2 & a_1 a_1 & \cdot e_{a_1 a_1} & \cdot e_{a_1 a_1} & \cdot e_{a_1 a_1} & \cdot e_{a_1 a_1} & \cdot e_{a_1 a_1} & \cdot e_{a_1 a_1} & \cdot e_{a_1 a_1} & \cdot e_{a_1 a_1} & \cdot e_{a_1 a_1} & \cdot e_{a_1 a_1} & \cdot e_{a_1 a_1} & \cdot e_{a_1 a_1} & \cdot e_{a_1 a_1} \end{pmatrix} \text{.} \]
Now,  
\[
p\left(\ell_1 2 \frac{\alpha_1}{\alpha_2}, \ell_1 2 1 \frac{\beta_1}{\beta_2} 1 \frac{\alpha_1}{\alpha_2}, \ell_1 2 1 \frac{\beta_1}{\beta_2} 1 \frac{\alpha_1}{\alpha_2}, \ell_1 2 1 \frac{\alpha_1}{\alpha_2} \right) =
\]
\[
= p\left(-\ell_1 2 \frac{\alpha_1}{\alpha_2}, \ell_1 2 1 \frac{\beta_1}{\beta_2} 1 \frac{\alpha_1}{\alpha_2}, \ell_1 2 1 \frac{\beta_1}{\beta_2} 1 \frac{\alpha_1}{\alpha_2} + \ell_1 2 \frac{\alpha_1}{\alpha_2} \right) =
\]
\[
= \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} = 2 \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix},
\]
and
\[
p\left(\ell_1 2 \frac{\alpha_1}{\alpha_2}, \ell_1 2 1 \frac{\beta_1}{\beta_2} 1 \frac{\alpha_1}{\alpha_2}, \ell_1 2 1 \frac{\beta_1}{\beta_2} 1 \frac{\alpha_1}{\alpha_2}, \ell_1 2 1 \frac{\alpha_1}{\alpha_2} \right) =
\]
\[
= p\left(-\ell_1 2 \frac{\alpha_1}{\alpha_2}, \ell_1 2 1 \frac{\beta_1}{\beta_2} 1 \frac{\alpha_1}{\alpha_2}, \ell_1 2 1 \frac{\beta_1}{\beta_2} 1 \frac{\alpha_1}{\alpha_2} + \ell_1 2 \frac{\alpha_1}{\alpha_2} \right) =
\]
\[
= \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix} = 2 \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix}.
\]

Therefore, we get again identity (38):
\[
S_{(2,1)}(2) = -\frac{1}{2} \left( \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \right) - \left( \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix} \right) =
\]
\[
= -\frac{1}{2} \left( Cimm_{(2,1)}[112;112] + Cimm_{(2,1)}[122;122] \right).
\]

Presentation (39) is more supple than presentation (36). Indeed:

- The centrality of the elements \( S_\lambda(n) \) is fairly obvious, as well as the fact that the set
  \[
  \{ S_\lambda(n); \lambda_1 \leq n \}
  \]
  is a basis of the center \( \zeta(n) \).

- If \( \lambda = (k) \) is the row shape of length \( k \), then \( S_{(k)}(n) \) is immediately recognized as the \( k \)-th determinantal Capelli generator \( H_k \) (see, e.g. [6], [2]; see also Capelli [9], [11], [12] and [13], Howe and Umeda [22]).

- If \( \lambda = (1^k) \) is the column shape of length \( k \), then \( S_{(1^k)}(n) \) is immediately recognized as the \( k \)-th permanental Nazarov-Umeda generator \( I_k \) (see, e.g. [2], Nazarov [29], Umeda [42]).
Presentation (39) is better suited to the study of the eigenvalues on irreducible $gl(n)-$modules, and of the duality in the algebra $\zeta(n)$ (see our preprint [8], section 4).

Presentation (39) is better suited to the study of the limit $n \to \infty$, via the Olshanski decomposition, see our preprint [8], section 5, Olshanski [34], [35] and Molev [28], pp. 928 ff.

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