Approximate Jacobian Elliptic Function Solution of the Modified KdV Equation by Homotopy Perturbation Method

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Abstract: Using He’s homotopy perturbation method (HPM), the modified KdV equation which has not a small parameter is solved. The approximate Jacobi elliptic function solution is obtained. When the modulus of the Jacobian function tends to unit or zero, the corresponding solitary wave solution and trigonometric function solution are obtained. The results reveal that the HPM is very effective, convenient and quite accurate to systems of nonlinear equations.

1. Introduction
In this letter via using the homotopy perturbation method, we consider the Jacobian elliptic function solution of the modified KdV equation

\[ \frac{\partial u}{\partial t} + \alpha u \frac{\partial u}{\partial x} + \beta \frac{\partial^3 u}{\partial x^3} = 0, \quad (1) \]

where \( \alpha \) and \( \beta \) are constants. Fu et al. [1] found some Jacobian elliptic function solutions of Eq. (1) using Jacobian elliptic function expansion method.

There are many methods to find solitary wave solutions of various nonlinear wave equations, but each method has its own shortcoming or limitation, which is valid for a special kind of nonlinear problem.

The discussed mKdV equation was studied by many authors via different approaches with merits on the one hand and disadvantages on the other hand, for example, the Darboux transform method by Chen et al. [2], the inverse scattering method by Huang et al. [3], Jacobian elliptic function method by Fu et al. [1], and Adomian's decomposition method by Yan [4].

The homotopy perturbation method (HPM) was first proposed by He [5,6]. Using homotopy technique in topology, a homotopy is constructed with an embedding parameter \( p \in [0,1] \) which is considered as a “small parameter”. Recently, the homotopy perturbation method have become more and more attractive due to the effective and convenient [7-12]. However, there are few work about the Jacobian elliptic function solutions.

The aim of our Letter is to further extend the method to solve the mKdV equation with the Jacobian elliptic function initial condition. By using the HPM, we get the explicit solutions of the mKdV equation without using any transformation method. The method presented here is also simple to use.

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for obtaining numerical solution of the equations without using any discrete techniques. Furthermore, we will show that considerably better approximations related to the accuracy level would be obtained.

2. Basic idea of He’s homotopy perturbation method

To illustrate the basic ideas of this method, we consider the following nonlinear differential equation [6]:

\[ A(u) - f(r) = 0, \quad r \in \Omega, \]  

with the boundary conditions of

\[ B(u, \partial u / \partial n) = 0, \quad r \in \Gamma, \]  

where \( A \) is a general differential operator, \( B \) a boundary operator, \( f(r) \) a known analytical function and \( \Gamma \) is the boundary of the domain \( \Omega \).

Generally speaking, the operator \( A \) can be decomposed into two operators, \( L \) and \( N \), where \( L \) is linear, and \( N \) is nonlinear operator. Eq. (2) can therefore be rewritten as follows:

\[ L(u) + N(u) - f(r) = 0. \]  

By the homotopy technique, we construct a homotopy \( V(r, p) : \Omega \times [0,1] \rightarrow \mathbb{R} \) which satisfies:

\[ H(V, p) = (1 - p)[L(V) - L(u_0)] + p[A(V) - f(r)] = 0, \quad p \in [0,1], r \in \Omega, \]  

or

\[ H(V, p) = L(u) - L(u_0) + pL(u_0) + p[N(V) - f(r)] = 0, \]  

where \( p \in [0,1] \) is an embedding parameter, \( u_0 \) is an initial approximation of Eq. (2), which satisfies the boundary conditions. Obviously, from Eqs. (5) and (6) we will have:

\[ H(V, 0) = L(V) - L(u_0) = 0, \]  

\[ H(V, 1) = A(V) - f(r) = 0. \]

The changing process of \( p \) from zero to unity is just that of \( V(r, p) \) from \( u_0(r) \) to \( u(r) \). In topology, this is called deformation, \( L(V) - L(u_0) \) and \( A(V) - f(r) \) are called homotopy.

According to the HPM, we can first use the embedding parameter \( p \) as a “small parameter”, and assume that the solution of Eqs. (5) or (6) can be written as a power series in \( p \):

\[ V = V_0 + pV_1 + p^2V_2 + \cdots. \]  

Setting \( p=1 \) result in the approximate solution of Eq. (2):

\[ u = \lim_{p \to 1} V = V_0 + V_1 + V_2 + \cdots. \]

The combination of the perturbation method and the homotopy method is called the homotopy perturbation method (HPM), which has eliminated the limitations of the traditional perturbation methods. On the other hand, this technique can have full advantage of the traditional perturbation techniques.

The series (10) is convergent for most cases. However, the convergent rate depends on the nonlinear operator \( A(V) \):
(1) The second derivative of $N(V)$ with respect to $V$ must be small because the parameter may be relatively large, i.e., $p \to 1$.

(2) The norm of $L^{-1} \partial N/\partial V$ must be smaller than one so that the series converges.

2. Analysis of the method

To investigate the traveling wave solution of Eq. (1), we first construct a homotopy as follows:

$$(1 - p) \left( \frac{\partial u}{\partial t} + \frac{\partial u_0}{\partial t} \right) + p \left( \alpha u^2 + \beta \frac{\partial^3 u}{\partial x^3} \right) = 0.$$  \hspace{1cm} (11)

Suppose the solution of Eq.(11) and the initial approximations are as follows:

$$u_0(x,t)=u(x,0),$$

$$u(x,t)=U(x,t)=u_0+pu_1+p^2u_2+p^3u_3+\cdots,$$  \hspace{1cm} (13)

where $u_i$ ($i=1, 2, 3, \ldots$) are functions of $(x, t)$ yet to be determined. Substituting Eq. (13) into Eq. (11), and equating the coefficients of the terms with the identical powers of $p$, we have

$$\left( \frac{\partial u_1}{\partial t} + \frac{\partial u_0}{\partial t} + \alpha u_0^2 \frac{\partial u_0}{\partial x} + \beta \frac{\partial^3 u_0}{\partial x^3} \right) p + \left( \frac{\partial u_2}{\partial t} + \beta \frac{\partial^3 u_1}{\partial x^3} + \alpha u_0^2 \frac{\partial u_1}{\partial x} + 2\alpha u_0 u_1 \frac{\partial u_0}{\partial x} \right) p^2$$

$$+ \left( \frac{\partial u_3}{\partial t} + 2\alpha u_0 u_2 \frac{\partial u_0}{\partial x} + 2\alpha u_0 u_1 \frac{\partial u_1}{\partial x} + \alpha u_0^2 \frac{\partial u_2}{\partial x} + \beta \frac{\partial^3 u_2}{\partial x^3} \right) p^3 + \cdots = 0.$$  \hspace{1cm} (14)

In order to obtain the unknowns of $u_i$ ($i=1, 2, 3, \ldots$), we must construct and solve the following system which includes three equations with three unknowns, considering the initial approximations of Eq.(11)

$$\left( \frac{\partial u_1}{\partial t} + \alpha u_0^2 \frac{\partial u_0}{\partial x} + \beta \frac{\partial^3 u_0}{\partial x^3} \right) = 0,$$

$$\left( \frac{\partial u_2}{\partial t} + \beta \frac{\partial^3 u_1}{\partial x^3} + \alpha u_0^2 \frac{\partial u_1}{\partial x} + 2\alpha u_0 u_1 \frac{\partial u_0}{\partial x} \right) = 0,$$  \hspace{1cm} (15)

$$\left( \frac{\partial u_3}{\partial t} + 2\alpha u_0 u_2 \frac{\partial u_0}{\partial x} + 2\alpha u_0 u_1 \frac{\partial u_1}{\partial x} + \alpha u_0^2 \frac{\partial u_2}{\partial x} + \beta \frac{\partial^3 u_2}{\partial x^3} \right) = 0.$$  

If the first three approximations are sufficient, we will obtain:

$$u(x,t) = \lim_{p \to 1} U(x,t) = \sum_{k=0}^{\infty} u_k(x,t),$$  \hspace{1cm} (16)

3. Application

Now, we consider the solutions of Eq. (1) with the Jacobian elliptic sine function initial condition [1]:

$$u(x,0) = bsn(kx, m),$$  \hspace{1cm} (17)

where $k$ and $b$ are arbitrary constants, $m$ is the modulus of the Jacobian elliptic function. To calculate the terms of the homotopy series (16) for $u(x, t)$, we substitute the initial condition (17) into the system (15), the solutions of the equation can be obtained as follows:

$$u_0(x,t) = u_0(x,0) = u(x,0) = bsn(kx, m),$$  \hspace{1cm} (18)
In this manner the other components can be easily obtained. Substituting Eqs. (18)–(21) into Eq. (16):

\[ u_i, t(x, t) = -kb_i k^j b_j(1 + m^2)cn(k, m)dt(k, m) - (6\beta k^2 m^2 + \alpha b^2)cn(k, m)\text{dn}(k, m)sn^2(k, m)\], (19)

\[ u_j, t(x, t) = k^3 m^2 k^2(\alpha^2 b^2 + b_\beta m^2 k^2)\text{sn}(k, m) - 5k^2 (1 + m^2)\]

\[ + 84a_\beta b^2 m^2 k^2 + \frac{1}{2} \alpha^2 b^2)\text{sn}(k, m) + 9\beta b^2 m^2 k^2 (1 + m^2) + 15\alpha \beta b^2 k^4 (1 + m^2) + 434b^2 m^2 k^6 + 2\alpha^2 b^2 k^2 \]

\[ + 84a_\beta b^2 m^2 k^2 )\text{sn}(k, m) [1 - k_\beta b^2 k^4 (1 + m^2) + \frac{1}{2} b \beta^2 m^2 k^6 (1 + m^2) + \frac{135}{2} b \beta^2 m^2 k^8 (1 + m^2)\text{sn}(k, m)\] (20)

\[ u_5, t(x, t) = \left[\frac{5962\beta b^2 b_\beta m^4 k^7 (1 + m^2) + 496 \alpha b^3 k^3 (1 + m^2) + 7056\beta b^3 m^6 k^6 (1 + m^2) + 74\beta \alpha b^3 k^5 m^2 (1 + m^2)\right]\]

\[ \text{sn}^3(k, m)\text{cn}(k, m)\text{dn}(k, m) - b m^2 k^5 (1 + m^2) + 6048\beta b^3 m^6 k^6 + 774\beta \alpha b^3 m^6 k^8 + 14292\beta^2 b^4 m^4 k^4\]

\[ \text{sn}^3(k, m)\text{cn}(k, m)\text{dn}(k, m) - d_\beta b^3 m^4 k^3 (1 + m^2) + 2652\beta \alpha b^3 k^5 + 1932\beta^2 m^4 k^6 (1 + m^4)\]

\[ + 625\beta \alpha b^3 m^4 k^3 + 1932\beta^2 b^3 m^4 k^6 (1 + m^4) + 401\beta b^3 m^4 k^7 (1 + m^4)\text{sn}^4(k, m)\text{cn}(k, m)\text{dn}(k, m)\]

\[ + \left[ - 1326\beta b^3 m^4 k^9 + 81 \beta \alpha b^3 k^5 + \frac{5391}{2} \alpha b^3 b^3 m^4 k^7 (1 + m^2) + (\frac{273}{2} + \frac{273}{2} \alpha b^3 k^4 + 82\beta^2 b^3 m^4 k^9 (1 + m^6)\right]\]

\[ \text{sn}^5(k, m)\text{cn}(k, m)\text{dn}(k, m) - 34\alpha b^3 b^3 m^4 k^3 (1 + m^4) + \frac{614}{3} b \beta^2 m^4 k^8 (1 + m^6) + \frac{1}{6} b \beta^2 k^5 (1 + m^6)\]

\[ + 176\alpha b^3 b^3 m^4 k^3 + 913 b^3 m^4 k^6 + 4 \beta \alpha b^3 k^5 \] (21)

With initial conditions (17), the Jacobian elliptic sine function solution of Eq. (1) is in full agreement with the ones constructed by Fu et al. [1].

To examine the accuracy and reliability of the HPM for the modified KdV equation, we can also consider the Jacobian elliptic cosine function initial value [1]:

\[ u(x, 0) = bcn(k, m)\], (24)

where \( b \) and \( k \) are arbitrary constant. To calculate the terms of the homotopy series (16) for \( u(x, t) \), we substitute the initial conditions (24) into system (15) and finally using Maple, the solutions of equation can be obtained. Following this procedure as in the first example, if we set

\[ b = m \sqrt{\frac{6c}{\alpha (1 + m^2)}}, k = \sqrt{-\frac{c}{(1 + m^2)\beta}} \] in Eq.(22), we obtain the closed form solutions as follows:

\[ u(x, t) = m \sqrt{\frac{6c}{\alpha (1 + m^2)}} \text{sn}[\sqrt{-\frac{c}{(1 + m^2)\beta}} (x - ct), m]. \] (23)

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\[ b = m \sqrt{\frac{6c}{\alpha (2m^2 - 1)}}, k = \sqrt{-\frac{c}{(2m^2 - 1)\beta}} \] the closed form of solution is:
\[ u(x,t) = m \sqrt{\frac{6c}{\alpha(2m^2 - 1)}} \text{cn}[ \sqrt{\frac{c}{(2m^2 - 1)\beta}}(x - ct), m]. \tag{25} \]

In this case, the Jacobian elliptic cosine function solution of Eq. (1) is in full agreement with the one constructed by Fu et al. [1]. Because the Jacobian elliptic function have special character. When the modulus \( m \to 0 \), \( \text{sn}(\xi) \to \sin(\xi) \), \( \text{cn}(\xi) \to \cos(\xi) \), \( \text{dn}(\xi) \to 1 \), so the trigonometric function solution can be obtained.

\[ u'_{0} (x,t) = u'_{0} (x,0) = u'(x,0) = b \sin(kx), \tag{26} \]

\[ u'_{1} (x,t) = -kbk^{2} \beta(1 + m^{2}) \cos(kx) - (6\beta k^{2} m^{2} + ab^{2}) \cos(kx) \sin^{2}(kx) \]

\[ u'_{2} (x,t) = \{3m^{2}k^{2} \left[ \alpha \beta + b\beta m^{2} \right] (26ab^{2} + 12\beta b^{2} m^{2}) \} \sin^{3}(kx) - 5k^{2}(1 + m^{2})(17\alpha \beta b^{2} m^{2} k^{2} + 84\beta b^{2} m^{4} k^{4} + \frac{1}{2} \alpha \beta b^{2}) \sin^{2}(kx) - 9b \beta^{2} m^{2} k^{2} (1 + m^{4}) + 15\alpha \beta b^{2} k^{4} (1 + m^{4}) + 432 b \beta^{2} m^{6} k^{6} + 2ab^{2} k^{2}, \]

\[ + 84 \alpha \beta b^{2} m^{4} k^{4} \} \sin^{2}(kx) - [1 1/2 \alpha \beta b^{2} k^{4} (1 + m^{2}) + \frac{1}{2} b \beta^{2} k^{6} (1 + m^{2}) + \frac{135}{2} b \beta^{2} m^{4} k^{6} (1 + m^{2})] \sin(kx) \]

\[ u'_{3} (x,t) = \{ -5391 / 2 \alpha \beta b^{2} m^{6} k^{6} (1 + m^{2}) \} \sin^{2}(kx) - (32 b^{2} m^{4} k^{4} + 8 b \beta b^{2} k^{4} + \frac{5391}{2} b \beta^{3} m^{6} k^{6} (1 + m^{2}) + 176 \alpha \beta b^{2} m^{6} k^{6} + 91 b \beta b^{2} k^{4} \} \sin^{2}(kx), \tag{29} \]

\[ u'(x,t) = u'_{0} (x,t) + u'_{1} (x,t) + u'_{2} (x,t) + u'_{3} (x,t) + \cdots. \tag{30} \]

When the modulus \( m \to 1 \), \( \sin(\xi) \to \tanh(\xi) \), \( \sin(\xi) \to \cos(\xi) \), \( \sin(\xi) \to \sech(\xi) \), so we can obtain the solitary wave solution.

\[ u'_{0} (x,t) = u''_{0} (x,0) = u''(x,0) = b \tanh(kx), \tag{31} \]

\[ u''_{1} (x,t) = -kbk^{2} \beta(1 + m^{2}) \sec h^{2}(kx) - (6\beta k^{2} m^{2} + ab^{2}) \sec h^{2}(kx) \tan h^{2}(kx) \]. \tag{32} \]
\[ u''_2 (x,t) = (3m^2 k^2 [\alpha^2 \beta^4 + b \phi m^2 k^2 (2 \delta \phi b^2 + 120 \phi m^2 k^2)] \tanh^3 (kx) - 5k^2 (1 + m^2) (17 \alpha^2 \beta^4 m^2 k^2 + 84 \phi \beta^2 m^4 k^4 + \frac{1}{2} \alpha^2 \beta^4 k^4) \tanh^3 (kx) + [9 \phi \beta^2 m^6 k^6 (1 + m^2) + 15 \alpha^2 \beta^4 m^6 k^6 + 434 \phi \beta^2 m^4 k^6 + 2 \alpha^2 \beta^4 k^4] \tanh^3 (kx) - [1.1 \phi \beta^2 m^4 k^4 (1 + m^2) + \frac{1}{2} \beta^2 k^6 (1 + m^2) + \frac{135}{2} \beta^2 m^2 k^8 (1 + m^2)] \tanh (kx), \]  

(33)

\[ u' (x,t) = \left[ \frac{1}{2} \sqrt{\phi \beta m^6 k^6 (1 + m^2)} + \frac{49}{6} \phi \beta^2 m^2 k^2 (1 + m^2) + 7056 \phi \beta^4 m^6 k^6 (1 + m^2) + 74 \phi \alpha \beta^2 m^6 k^6 (1 + m^2) \right] \frac{\tanh (kx)}{\sec (kx)} - b m^2 k (12 \phi \beta^2 + 6048 \phi \beta^4 m^4 k^4 + 774 \phi \alpha \beta^2 m^6 k^4 + 1429 \phi \beta^2 m^4 k^4) \] 

\[ \tan (kx) \sec (kx) - \frac{265}{2} \beta^2 \phi^2 m^2 k^2 (1 + m^2) + 5 \alpha \beta^2 k^2 (1 + m^2) + 1932 \phi \beta^2 m^6 k^6 (1 + m^2) + 62 \phi \alpha \beta^2 m^6 k^6 + 1505 \phi \beta^2 m^2 k^2 (1 + m^2) + 401 \phi \beta^2 m^6 k^6 (1 + m^2) \right] \tan (kx) \sec (kx) \tan (kx) + (132 \phi \beta^2 m^6 k^6 + 8 \sqrt{\phi \alpha \beta^2 m^6 k^6 (1 + m^2) + \frac{5391}{2} \alpha \beta^2 m^2 k^2 (1 + m^2)} + (\frac{273}{2} \alpha \beta^2 m^2 k^2 + 8 \sqrt{\phi \beta^2 m^6 k^6 (1 + m^2) + \frac{1}{6} \phi \beta^2 m^4 k^4 (1 + m^2) + b \beta^2 k^6 (1 + m^2) + 17 \phi \beta^2 m^6 k^6 + 91 \phi \beta^2 m^4 k^4 + 4 \phi \alpha \beta^2 m^4 k^4 \right] \sec (kx) \sec (kx) \]  

(34)

\[ u'' (x,t) = u''_0 (x,t) + u''_1 (x,t) + u''_2 (x,t) + u''_3 (x,t) + \ldots . \]  

(35)

**4. Comparing the results with the exact solutions**

To demonstrate the convergence of the HPM, the results of the numerical example are presented and only few terms are required to obtain accurate solutions. The accuracy of the HPM for the modified KdV equation is controllable, and absolute errors and relatively errors are very small with the present choice of \( t \) and \( x \). These results are listed in Tables 1, Table 2 and Table 3, it is seen that the implemented method achieves a minimum accuracy for the first three approximations for the initial condition (17). It is also evident that when more terms for the HPM are computed the numerical results get much closer to the corresponding exact solutions with the initial conditions (17) of Eq. (1).

**5. Conclusions**

In this paper, the homotopy perturbation method (HPM) was used for finding soliton solutions of a modified KdV equation with initial conditions. It can be concluded that the HPM is a very powerful and efficient technique in finding exact solutions for wide classes of problems. It is worth pointing out that the HPM presents a rapid convergence for the solutions. The obtained solutions are compared with the Adomian’s decomposition method[4]. All the examples show that the results of the present method are in excellent agreement with those obtained by the Adomian’s decomposition method. The HPM has got many merits and much more advantages than the Adomian’s decomposition method. This method is to overcome the difficulties arising in calculation of Adomian polynomials. Also the HPM does not require small parameters in the equation, so that the limitations of the traditional perturbation methods can be eliminated, and also the calculations in the HPM are simple and straightforward. The reliability of the method and the reduction in the size of computational domain give this method a wider applicability. The results show
that the HPM is a powerful mathematical tool for solving systems of nonlinear partial differential equations having wide applications in physics and engineering.

Table 1: The HPM result for $u(x,t)$ for the first three approximation in comparison with the analytical solutions when $b = \sqrt[17]{\frac{16}{17}}$, $k = \sqrt[17]{\frac{16}{17}}$, $m=1/4$, of Eq.(22), for the solitary wave solutions with the initial conditions (17) of Eq.(1), respectively

| $(x,t)$   | $|u_{exact} - u_{homotopy}|$ | $|\frac{u_{exact} - u_{homotopy}}{u_{exact}}|$ |
|-----------|-------------------------------|-----------------------------------------------|
| (15, 0.1) | 5.957E-07                     | 2.465271682E-06                               |
| (15, 0.2) | 9.04745E-06                   | 3.90655571E-05                               |
| (15, 0.3) | 4.77444E-05                   | 1.978788485E-07                              |
| (13, 0.1) | 1.504136E-04                  | 6.322475797E-04                              |
| (13, 0.2) | 3.665645E-04                  | 1.5771878E-07                                |
| (13, 0.3) | 2.916E-07                     | 4.76768133E-06                               |
| (13, 0.4) | 5.10712E-06                   | 6.112989886E-05                              |
| (13, 0.5) | 2.81039E-05                   | 2.673857402E-04                              |
| (10, 0.1) | 9.55916E-05                   | 7.608344971E-04                              |
| (10, 0.2) | 1.75028E-07                   | 2.677377202E-05                              |
| (10, 0.3) | 8.88581E-06                   | 2.2203773109E-04                             |
| (10, 0.4) | 1.981474E-05                  | 3.132076006E-04                              |
| (10, 0.5) | 2.832933E-05                  | 3.310266730E-04                              |

Table 2: The HPM result for $u(x,t)$ for the first three approximation in comparison with the analytical solutions when $b = \sqrt[17]{\frac{16}{17}}$, $k = \sqrt[17]{\frac{16}{17}}$, $m=1/2$, of Eq.(35), for the solitary wave solutions with the initial conditions (31) of Eq.(1), respectively

| $(x,t)$   | $|u_{exact} - u_{homotopy}|$ | $|\frac{u_{exact} - u_{homotopy}}{u_{exact}}|$ |
|-----------|-------------------------------|-----------------------------------------------|
| (15, 0.1) | 4.7161E-07                     | 6.66579391E-07                               |
| (15, 0.2) | 8.9785E-06                     | 8.183824576E-05                              |
| (15, 0.3) | 5.25179E-05                     | 3.558084049E-04                              |
| (15, 0.4) | 1.871696E-04                   | 1.016760288E-03                              |
| (15, 0.5) | 5.05449E-04                     | 2.309522254E-03                              |
| (13, 0.1) | 1.153E-07                      | 2.646476025E-07                              |
| (13, 0.2) | 1.4784E-06                      | 3.466876189E-06                              |
| (13, 0.3) | 5.3995E-06                      | 1.302701969E-05                              |
| (13, 0.4) | 9.7619E-06                      | 2.441613085E-05                              |
| (13, 0.5) | 4.3519E-06                      | 1.137978855E-05                              |
| (10, 0.1) | 4.732E-07                      | 1.153667027E-06                              |
| (10, 0.2) | 6.864E-06                      | 1.603107536E-05                              |
| (10, 0.3) | 3.13117E-05                     | 7.165905863E-05                              |
| (10, 0.4) | 8.886821E-05                    | 2.001680817E-04                              |
| (10, 0.5) | 1.929858E-04                    | 4.322857978E-04                              |

Table 3: The HPM result for $u(x,t)$ for the first three approximation in comparison with the analytical solutions when $b = \sqrt[17]{\frac{16}{17}}$, $k = \sqrt[17]{\frac{16}{17}}$, $m=1$, of Eq.(36), for the solitary wave solutions with the initial conditions (32) of Eq.(1), respectively

| $(x,t)$   | $|u_{exact} - u_{homotopy}|$ | $|\frac{u_{exact} - u_{homotopy}}{u_{exact}}|$ |
|-----------|-------------------------------|-----------------------------------------------|
| (15, 0.1) | 7.0571E-07                     | 6.66579391E-07                               |
| (15, 0.2) | 8.9785E-06                     | 8.183824576E-05                              |
| (15, 0.3) | 5.25179E-05                     | 3.558084049E-04                              |
| (15, 0.4) | 1.871696E-04                   | 1.016760288E-03                              |
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| (13, 0.5) | 4.3519E-06                      | 1.137978855E-05                              |
| (10, 0.1) | 4.732E-07                      | 1.153667027E-06                              |
| (10, 0.2) | 6.864E-06                      | 1.603107536E-05                              |
| (10, 0.3) | 3.13117E-05                     | 7.165905863E-05                              |
| (10, 0.4) | 8.886821E-05                    | 2.001680817E-04                              |
| (10, 0.5) | 1.929858E-04                    | 4.322857978E-04                              |
\begin{tabular}{|c|c|c|}
\hline
(x,t) & \( |u_{exact} - u_{homotopy}| \) & \( |u_{exact} - u_{homotopy}| / |u_{exact}| \) \\
\hline
(15, 0.1) & 1.10E-10 & 1.414213564E-10 \\
(15, 0.2) & 1.10E-10 & 1.414213565E-10 \\
(15, 0.3) & 0 & 0 \\
(15, 0.4) & 1.10E-10 & 1.414213566E-10 \\
(15, 0.5) & 0 & 0 \\
(13, 0.1) & 0 & 0 \\
(13, 0.2) & 1.10E-10 & 1.414213601E-10 \\
(13, 0.3) & 1.10E-10 & 1.414213607E-10 \\
(13, 0.4) & 0 & 0 \\
(13, 0.5) & 1.10E-10 & 1.414213622E-10 \\
(10, 0.1) & 0 & 0 \\
(10, 0.2) & 2.1E-10 & 2.828432539E-10 \\
(10, 0.3) & 1.41E-09 & 1.979903353E-09 \\
(10, 0.4) & 4.81E-09 & 6.788242342E-09 \\
(10, 0.5) & 1.231E-08 & 1.739487771E-08 \\
\hline
\end{tabular}

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\textbf{References}

[1] Fu Z T, Lou S K and Liu S D 2001 \textit{Phys. Lett. A} 290 72
[2] Chen Z Y, Liu Z Z, and Xiao Y 1993 \textit{J. Phys. A} 26 1365
[3] Huang N N, Chen Z Y and Yue H 1996 \textit{Phys. Lett. A} 221 167
[4] Yan Z Y 2005 \textit{Appl. Math. Comput.} 166 571
[5] He J H 2005 \textit{Int. J. Nonlinear Sci.} 6 207
[6] He J H 2005 \textit{Chaos. Soliton. Fract.} 26 695
[7] Tari H, Ganji D D, and Rostamian M 2007 \textit{Int. J. Nonlinear Sci.} 8 203
[8] Biazar J, Eslami M, and Ghazvini H 2007 \textit{Int. J. Nonlinear Sci.} 8 413
[9] Ghorbani A and Saberi-Nadjafi J 2007 \textit{Int. J. Nonlinear Sci.} 8 229
[10] Ariel P D, Hayat T, and Asghar S 2006 \textit{Int. J. Nonlinear Sci.} 7 399
[11] Rafei M and Ganji D D 2006 \textit{Int. J. Nonlinear Sci.} 7 321
[12] Beléndez A, Hernández T, and Beléndez et al 2007 \textit{Int. J. Nonlinear Sci.} 8 79