Lax–Oleinik formula on networks

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Abstract

We provide a Lax–Oleinik-type representation formula for solutions of time–dependent Hamilton–Jacobi equations, posed on a network with a rather general geometry, under standard assumptions on the Hamiltonians. It depends on a given initial datum at \( t = 0 \) and a flux limiter at the vertices, which both have to be assigned in order the problem to be uniquely solved. Previous results in the same direction are solely in the frame of junction, namely network with a single vertex. An important step to get the result is to define a suitable action functional and prove existence as well as Lipschitz–continuity of minimizers between two fixed points of the network in a given time, despite the fact that the integrand lacks convexity at the vertices.

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1 Introduction

The aim of the paper is to provide a Lax–Oleinik–type representation formula, obtained via minimization of a suitable action functional, for solutions of time–dependent Hamilton–Jacobi equations posed on a network with a rather general geometry.

This kind of formulas have actually a wider scope since they can enhance the qualitative analysis of Hamilton–Jacobi equations on networks, similarly to what happens for manifolds or Euclidean spaces. In this setting, in fact, they essentially enter into play in a variety of theoretical constructions such as weak KAM theory, see [10], existence of regular subsolutions [2], homogenization problems [19], large time behavior of solutions [6, 9, 13], selection principles in the ergodic approximation [8].

Previous contributions on the same topic we deal with can be found in [12] where, besides some restrictions on the Hamiltonians, the results are given in the case of junctions, namely networks with a single vertex. In [14] the focus is instead on the relationship between Lax–Oleinik formula and weak KAM theory and the connection with time–dependent Hamilton–Jacobi problems are not taken into consideration. Furthermore, the Hamiltonians/Lagrangians are assumed of Tonelli type and required to satisfy a quite stringent condition at the vertices.

We consider a connected network \( \Gamma \) embedded in \( \mathbb{R}^N \) with a finite number of vertices, making up a set denoted by \( V \), linked by regular simple curves \( \gamma \) parametrized in \([0, 1]\), called arcs of \( \Gamma \). A Hamiltonian on \( \Gamma \) is defined as a collection of one–dimensional Hamiltonians \( H_\gamma : [0, 1] \times \mathbb{R} \to \mathbb{R} \), indexed by arcs, depending on state and momentum variable, with the crucial feature that Hamiltonians associated to arcs possessing different support, are totally unrelated.

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We assume the \( H \)'s to be continuous in both arguments plus convex and superlinear in the momentum variable. Namely, not more than the usual conditions to ensure the validity of Lax–Oleinik formula for time–dependent problems posed on manifolds or Euclidean spaces.

The equations we are interested in are

\[
U_t + H_\gamma(s, U') = 0 \quad \text{in} \quad (0, 1) \times (0, +\infty)
\]

on each arc \( \gamma \), and a solution on \( \Gamma \) is a continuous function \( u : \Gamma \times (0, +\infty) \to \mathbb{R} \) such that \( u(\gamma(s), t) \) solves in the viscosity sense (1) for each \( \gamma \), and satisfies suitable additional conditions on the discontinuity interfaces

\[
\{(x, t), t \in [0, +\infty)\} \quad \text{with} \quad x \in V.
\]

It has been established in [11] in the case of junctions and in [17] for general networks that to get existence and uniqueness of solutions, equations (1) must be coupled not only with a continuous initial datum at \( t = 0 \), but also with a flux limiter, that is a choice of appropriate constants \( c_x \) for \( x \) varying in \( V \). We also report the contribution of [15, 16], where the time–dependent problem is studied in junctions, possibly multidimensional, with Kirchoff-type Neumann conditions at vertices.

In [17] flux limiters crucially appear in the conditions a solution must satisfy on the interfaces and, among other things, bond from above the time derivatives of any subsolution on it. Even if an initial datum is fixed, solutions can change according to the choice of flux limiter, so that they must taken into account in representation formulas.

Due to the superlinearity of the \( H \)'s, Lagrangians \( L_\gamma \) can be given, for any arc, via Fenchel transform. An overall Lagrangian \( L \), playing the role of integrand in the Lax–Oleinik formula, is defined on the whole tangent bundle \( TT \) of \( \Gamma \) by changing variables, gluing together the \( L_\gamma \)'s and taking into account the chosen flux limiter \( c_x \) via the formula

\[
L(x, 0) = c_x \quad \text{for any} \quad x \in V.
\]

Setting \( L = +\infty \) outside \( TT \), we get a function which is lower semicontinuous in \( \mathbb{R}^N \times \mathbb{R}^N \), but with \( L(x, \cdot) \) lacking in general convexity when \( x \) is a vertex, see Section 4.1. In fact the tangent space to \( \Gamma \) at such points itself is nonconvex being the union of one–dimensional vector spaces corresponding to different intersecting arcs. This complicates to some extent the proof of the existence of curves minimizing the action functional between two fixed points of \( \Gamma \) in a given time, and of the Lipschitz character of the minimizers as well, see Theorems 5.2 and 5.3.

In this respect condition (2) plays an important role since it allows ruling out Zeno phenomena in the minimization of the action functional, namely the possibility that candidate minimizers wildly oscillate around a discontinuity interface.

This is achieved taking into account the existence of regular solutions to the stationary equation \( H_\gamma = -c_x \) in \( (0, 1) \), for arcs \( \gamma \) ending at a given vertex \( x \), see Proposition 3.4, in this way one can show that to oscillate around the interfaces is more expensive than to stay steady at the vertex for a suitable time interval. This argument seems new, see Propositions 3.6 and 4.2.

In the end we get a formula similar to the one already known in the classical cases, namely the solution \( u(x, t) \) can be expressed as

\[
u(x, t) = \inf \left\{ \int_0^t L(\xi, \dot{\xi}) \, ds + u_0(\xi(0)) \right\}
\]

with the infimum taken over the absolutely continuous curves \( \xi \) from \([0, t]\) to \( \Gamma \) satisfying \( \xi(t) = x \), where \( u_0 \) stands for the initial datum and the flux limiter \( c_x \) is taken into account through the relation (2).
The paper is organized as follows: in Section 2 we fix some notations and conventions. Section 3 provides some basic facts about networks, the definition of Hamiltonians/Lagrangians on networks and of flux limiters, the main assumptions on the model are specified as well. Section 4 is devoted to the definition of the overall Lagrangian $L$ and the corresponding action functional, it is further introduced a class of curves called admissible which is convenient to take in consideration in the minimization of the action functional. Existence of minimizing curves and their Lipschitz–continuity are proved in Section 5.

Finally in Section 6 it is proved the continuity of the function given by Lax–Oleinik formula, it is given the definition of solution for the time–dependent Hamilton–Jacobi equation on $\Gamma$, and written down the main result stating that Lax–Oleinik formula provides the unique solution of the equation with given initial datum and flux limiter.

Two appendices are about some elementary properties of time–dependent equations on a single arc, and a variational result we use to prove the Lipschitz regularity of minimizers.

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2 Preliminaries

We fix a dimension $N$ and $\mathbb{R}^N$ as ambient space. We indicate by $\cdot$ the scalar product in $\mathbb{R}^N$. Given two real numbers $a$ and $b$, we set
\[ a \wedge b = \min\{a, b\} \quad \text{and} \quad a \vee b = \max\{a, b\}. \]

Given any subset $A$ contained in an Euclidean space, we denote by $\overline{A}$ its closure. Given a measurable subset $E \subset \mathbb{R}$, we denote by $|E|$ its Lebesgue measure. By curve contained we mean throughout the paper an absolutely continuous curve with support contained in $\mathbb{R}^N$ or $\mathbb{R}$.

We set
\[ \mathbb{R}^+ = [0, +\infty), \quad Q = (0, 1) \times (0, +\infty), \]
we have $\partial Q = [0, 1] \times \{0\} \cup \{0, 1\} \times \mathbb{R}^+$, we further set
\[ \partial^- Q = [0, 1] \times \{0\} \cup \{0\} \times \mathbb{R}^+. \]

For any $C^1$ function $\Phi : Q \to \mathbb{R}$ and $(s_0, t_0) \in Q$, we denote by $\Phi'(s_0, t_0)$ the space derivative, with respect to $s$, at $(s_0, t_0)$, and by $\psi_t(s_0, t_0)$ or $\frac{d}{dt}\psi(s_0, t_0)$ the time derivative.

Given a continuous function $U : Q \to \mathbb{R}$, we call supertangents (resp. subtangents) to $U$ at $(s_0, t_0) \in Q$ the viscosity test functions from above (resp. below). If needed, we take, without explicitly mentioning, $U$ and test function coinciding at $(s_0, t_0)$ and test function strictly greater (resp. less) than $U$ in a punctured neighborhood of $(s_0, t_0)$.

We say that a subtangent $\Phi$ to $U$ at $(1, t_0)$, $t_0 > 0$, is constrained to $\overline{Q}$ if $(s_0, t_0)$ is a minimizer of $u - \Phi$ in a neighborhood of $(1, t_0)$ intersected with $\overline{Q}$. 
3 Networks

3.1 Basic definitions

An embedded network, is a subset $\Gamma \subset \mathbb{R}^N$ of the form

$$\Gamma = \bigcup_{\gamma \in \mathcal{E}} \gamma([0, 1]) \subset \mathbb{R}^N,$$

where $\mathcal{E}$ is a finite collection of regular (i.e., $C^1$ with non-vanishing derivative) simple oriented curves, called arcs of the network, that we assume, without any loss of generality, parametrized on $[0, 1]$, note that we are also assuming existence of one-sided derivatives at the endpoints 0 and 1. We stress out that a regular change of parameter does not affect our results.

On the support of any arc $\gamma$, we also consider the inverse parametrization defined as

$$\tilde{\gamma}(s) = \gamma(1 - s) \quad \text{for } s \in [0, 1].$$

We call $\tilde{\gamma}$ the inverse arc of $\gamma$. We assume

$$\gamma((0, 1)) \cap \gamma'([0, 1]) = \emptyset \quad \text{whenever } \gamma \neq \gamma', \gamma \neq \tilde{\gamma}'.$$  \hfill (3)

In this case we say that $\gamma$, $\gamma'$ are different arcs. We call vertices the initial and terminal points of the arcs, and denote by $V$ the sets of all such vertices. Note that (3) implies that

$$\gamma((0, 1)) \cap V = \emptyset \quad \text{for any arc } \gamma.$$

We assume that the network is connected, namely given two vertices there is a finite concatenation of arcs linking them. A loop is an arc with initial and final point coinciding. The unique restriction we require on the geometry of the network is

(A1) $\mathcal{E}$ does not contain loops.

See [20] for a comprehensive treatment on graphs and networks.

Given $x \in V$, we define

$$\Gamma_x = \{ \gamma \mid \gamma(1) = x \}.$$

The network $\Gamma$ inherits a geodesic distance, denoted with $d_\Gamma$, from the Euclidean metric of $\mathbb{R}^N$. It is clear that given $x, y$ in $\Gamma$ there is at least a geodesic linking them. The geodesic distance is in addition equivalent to the Euclidean one.

Given a continuous function $u : \Gamma \times \mathbb{R}^+ \to \mathbb{R}$ and an arc $\gamma$, we define $u \circ \gamma : [0, 1] \times [0, +\infty) \to \mathbb{R}$ as

$$u \circ \gamma(s, t) = u(\gamma(s), t) \quad \text{for any } (s, t) \in \mathcal{Q}. \hfill (4)$$

The tangent bundle of $\Gamma$, $TT$ in symbols, is made up by elements $(x, q) \in \Gamma \times \mathbb{R}^N$ with $q$ of the form

$$q = \lambda \dot{\gamma}(s) \quad \text{if } x = \gamma(s), s \in [0, 1], \text{ with } \lambda \in \mathbb{R},$$

note that $\dot{\gamma}(s)$ is univocally determined, up to a sign, if $x \in \Gamma \setminus V$ or in other words if $s \neq 0, 1$. 

4
3.2 Curves on $\Gamma$

We define

$$m = \min\{|\dot{\gamma}(s)| \mid s \in [0,1], \gamma \in \mathcal{E}\} > 0. \quad (5)$$

**Lemma 3.1.** Given an arc $\gamma$, the function

$$\gamma^{-1} : \gamma[0,1] \to [0,1]$$

is Lipschitz continuous with respect to the distance $d_\Gamma$ on $\Gamma$.

**Proof.** It is enough to consider $x, y$ in $\gamma(0,1)$ with $\gamma^{-1}(x) < \gamma^{-1}(y)$, if the geodesic between $x$ and $y$ is given by the portion of $\gamma$ between $\gamma^{-1}(x)$ and $\gamma^{-1}(y)$, we have

$$d_\Gamma(x,y) = \int_{\gamma^{-1}(x)}^{\gamma^{-1}(y)} |\dot{\gamma}| \, ds \geq m (\gamma^{-1}(y) - \gamma^{-1}(x)).$$

If instead such a geodesic is given by the concatenation of different arcs, then taking into account that the geodesic distance between vertices of $\Gamma$ is bounded by below by $m$, we get

$$d_\Gamma(x,y) \geq m (1 - 0) \geq m (\gamma^{-1}(y) - \gamma^{-1}(x)).$$

We then derive in both cases

$$\gamma^{-1}(y) - \gamma^{-1}(x) \leq \frac{1}{m} d_\Gamma(x,y).$$

$\square$

**Lemma 3.2.** For any given arc $\gamma$ and curve $\xi : [a,b] \to \gamma([0,1])$, the function

$$\gamma^{-1} \circ \xi : [a,b] \to [0,1]$$

is absolutely continuous, and

$$\frac{d}{dt} \gamma^{-1} \circ \xi(t) = \frac{\dot{\xi}(t) \cdot \dot{\gamma}(\gamma^{-1} \circ \xi(t))}{|\dot{\gamma}(\gamma^{-1} \circ \xi(t))|^2} \text{ for a.e. } t. \quad (6)$$

**Proof.** The function $\gamma^{-1} \circ \xi$ is absolutely continuous as composition of an absolutely continuous and a Lipschitz continuous function. Let $t_0$ be a time where $\xi$ is differentiable, to ease notations we put $s_0 = \gamma^{-1} \circ \xi(t_0)$, $\lambda_0 = \frac{\dot{\xi}(t_0) \cdot \dot{\gamma}(s_0)}{|\dot{\gamma}(s_0)|^2}$ so that

$$\xi(t_0) = \gamma(s_0) \quad \text{ and } \quad \dot{\xi}(t_0) = \lambda_0 \dot{\gamma}(s_0),$$

and formula (6) boils down to

$$\frac{d}{dt} \gamma^{-1} \circ \xi(t) = \lambda_0.$$
where $o(\cdot)$ is the Landau symbol. Exploiting the Lipschitz continuity of $\gamma^{-1}$, we have
\[ |\gamma^{-1}(\gamma(s_0 + \lambda_0 h) + o(h)) - \gamma^{-1}(\gamma(s_0 + \lambda_0 h))| \leq o(h) \]
or in other words
\[ \gamma^{-1}(\gamma(s_0 + \lambda_0 h) + o(h)) = \gamma^{-1}(\gamma(s_0 + \lambda_0 h)) + o(h). \]
Continuing the computation in (7) we therefore get
\[
\frac{\gamma^{-1}(\gamma(s_0 + \lambda_0 h) + o(h)) - s_0}{h} = \frac{\gamma^{-1}(\gamma(s_0 + \lambda_0 h)) + o(h) - s_0}{h} = \frac{s_0 + \lambda_0 h - s_0 + o(h)}{h} = \frac{\lambda_0 h + o(h)}{h}.
\]
Sending $h$ to 0, we obtain in the end
\[
\lim_{h\to 0} \frac{\gamma^{-1} \circ \xi(t_0 + h) - \gamma^{-1} \circ \xi(t_0)}{h} = \lambda_0
\]
which shows that $\gamma^{-1} \circ \xi$ is differentiable at $t_0$ and (6) holds true.

\[\square\]

3.3 Hamiltonians and Lagrangians on the arcs

A Hamiltonian on $\Gamma$ is a collection of Hamiltonians $H_\gamma : [0, 1] \times \mathbb{R} \to \mathbb{R}$, indexed by the arcs satisfying
\[ H_\tilde{\gamma}(s, \mu) = H_\gamma(1 - s, -\mu) \quad \text{for any arc } \gamma. \]
We emphasize that, apart the above compatibility condition, the Hamiltonians $H_\gamma$ are unrelated.

We require any $H_\gamma$ to be:

**(H1)** continuous in both arguments;

**(H2)** convex in $\mu$;

**(H3)** \[ \lim_{|\mu| \to \infty} \inf_{s \in [0,1]} \frac{H_\gamma(s, \mu)}{|\mu|} = +\infty \quad \text{for any } \gamma \in \mathcal{E}. \]

Note that by the Corollary of Proposition 2.2.6 in [5] the above assumptions imply that the $H_\gamma$'s are locally Lipschitz continuous in $\mu$ uniformly with respect to $s \in [0, 1]$. Namely, given $M > 0$, there exists $C_M$ such that
\[
H_\gamma(s, \mu_1) - H_\gamma(s, \mu_2) \leq C_M |\mu_1 - \mu_2| \quad \text{for any } s \in [0,1], \lambda_1, \lambda_2 \text{ in } (-M, M). \tag{8}
\]

Assumptions (A1), (H1), (H2), (H3) are in force, without further mentioning, throughout the paper.

Thanks to the superlinearity condition (H3), we can define for any $\gamma \in \mathcal{E}$, the Lagrangian corresponding to $H_\gamma$ as
\[
L_\gamma(s, \lambda) := \max_{\lambda \in \mathbb{R}} (\lambda \mu - H_\gamma(s, \mu)).
\]
where the supremum is actually achieved thanks to (H3). We have for each \( \lambda \in \mathbb{R} \) and \( s \in [0, 1] \),
\[
L_\gamma(s, \lambda) = L_\gamma(1 - s, -\lambda).
\]

From (H3) we also derive that the Lagrangians \( L_\gamma \) are superlinear.

We set
\[
c_\gamma = -\max_s \min_\mu H_\gamma(s, \mu) \quad \text{for any arc } \gamma.
\]

**Lemma 3.3.** We have
\[
c_\gamma = \min_s L_\gamma(s, 0).
\]

**Proof.** Given \( s \in [0, 1] \), we have
\[
L_\gamma(s, 0) = \max_\mu -H_\gamma(s, \mu) = -\min_\mu H_\gamma(s, \mu)
\]
and consequently
\[
c_\gamma = -\max_s \min_\mu H_\gamma(s, \mu) = \min_s [-\min_\mu H_\gamma(s, \mu)] = \min_s L_\gamma(s, 0).
\]

The following characterization will play a relevant role in the sequel, see [18] for a comprehensive analysis of Eikonal equations on networks and graphs.

**Proposition 3.4.** The stationary equation
\[
H_\gamma(s, U') = a \quad \text{in } (0, 1)
\]
(9)

admits a \( C^1 \) solution if and only if \( a \geq -c_\gamma \).

**Proof.** If \( a \) satisfies the inequality in the statement, we define
\[
\sigma^+_a(s) = \max\{\mu \mid H_\gamma(s, \mu) = a\},
\]
which is apparently a continuous function as \( s \) varies in \([0, 1]\). We further define
\[
U(s) = \int_0^s \sigma^+_a(\tau) d\tau,
\]
which is the sought \( C^1 \) solution to (9). The converse implication is immediate.

Following [11], we call \textit{flux limiter} any function \( x \mapsto c_x \) from \( V \) to \( \mathbb{R} \) satisfying
\[
c_x \leq \min_{\gamma \in \Gamma_x} c_\gamma = \min_\gamma L_\gamma(s, 0) \quad \text{for any } x \in V.
\]

It is convenient for future use to introduce a modification of \( L_\gamma \). For any arc \( \gamma \) we define
\[
\bar{L}_\gamma(s, \lambda) = \begin{cases} 
L_\gamma(s, \lambda) & \text{for } s \neq 0, 1 \\
L_\gamma(s, \lambda) + (c_{\gamma(0)} \wedge c_{\gamma(1)}) - L_\gamma(s, 0) & \text{for } s = 0, 1
\end{cases}
\]

It is clear from the definition of \( \bar{L}_\gamma \) and Lemma 3.3 that

**Lemma 3.5.** For any arc \( \gamma \) the function \( (s, \lambda) \mapsto \bar{L}_\gamma(s, t) \) satisfies the following properties:
(i) is lower semicontinuous in \((s, \lambda)\);
(ii) is convex in \(\lambda\) for any fixed \(s \in [0, 1]\);
(iii) \(\bar{L}_\gamma(0, 0) = \bar{L}_\gamma(1, 0) = c_{\gamma(0)} \wedge c_{\gamma(1)}\);
(iv) is superlinear in \(\lambda\) for any fixed \(s\).

The role of the flux limiters is highlighted by the following result:

**Proposition 3.6.** Given an arc \(\gamma\) and a curve \(\eta : [a, b] \rightarrow [0, 1]\) with \(\eta(a) = \eta(b)\), one has
\[
\int_a^b L_\gamma(\eta, \dot{\eta}) \, d\tau \geq \int_a^b \bar{L}_\gamma(\eta, \dot{\eta}) \, d\tau \geq (c_{\gamma(0)} \wedge c_{\gamma(1)}) (b - a).
\]

**Proof.** We set
\[
E = \{ t \in [a, b] \mid \eta(t) \notin \{0, 1\} \}, \\
F = \{ t \in [a, b] \mid \eta(t) \in \{0, 1\} \}.
\]

If \(E\) is empty then \(\eta\) is constant, equal either to 0 or to 1, then the assertion is immediate in view of Lemma 3.3. If \(E\) is instead nonempty, it is open in \([a, b]\), and it is then the union, up to a set of vanishing measure, of a countable family of intervals \([a_k, b_k]\) with disjoint interiors. According to Proposition 3.4, there exists a \(C^1\) subsolution \(U(s)\) of \(H_\gamma = -c\), where \(c = c_{\gamma(0)} \wedge c_{\gamma(1)}\), in \((0, 1)\) so that we have
\[
\bar{L}_\gamma(\eta(t), \dot{\eta}(t)) = L_\gamma(\eta(t), \dot{\eta}(t)) \geq H_\gamma(\eta(t), U'(\eta(t))) + \eta(t)(U'(\eta(t)) - H_\gamma(\eta(t), U'(\eta(t))))
\]
for a.e. \(t \in E\), and consequently
\[
\int_{a_k}^{b_k} \bar{L}_\gamma(\eta, \dot{\eta}) \, d\tau \geq \int_{a_k}^{b_k} U'(\eta) \dot{\eta} \, d\tau - \int_{a_k}^{b_k} H_\gamma(\eta, U'(\eta)) \, d\tau = U(\eta(b_k)) - U(\eta(a_k)) + c (b_k - a_k)
\]
for any \(k\). The indices \(k\) for which the curve \(\eta\) restricted to \([a_k, b_k]\) is noncyclic are finitely many because the length of \(\eta\) is finite, when we sum the contributions \(U(\eta(b_k)) - U(\eta(a_k))\) over this family of indices we get \(U(\eta(t_1)) - U(\eta(t_0))\), where
\[
t_0 = \inf\{ t \in [a, b] \mid \eta(t) \in E \}, \\
t_1 = \sup\{ t \in [a, b] \mid \eta(t) \in E \}.
\]

If instead \(\eta\) restricted to \([a_k, b_k]\) is cyclic then clearly \(U(\eta(b_k)) - U(\eta(a_k)) = 0\). Altogether we have
\[
\int_E \bar{L}_\gamma(\eta, \dot{\eta}) \, d\tau = \sum_k \int_{a_k}^{b_k} \bar{L}_\gamma(\eta, \dot{\eta}) \, d\tau \geq \sum_k (U(\eta(b_k)) - U(\eta(a_k))) + c (b_k - a_k)
\]
(10)
\[
= U(\eta(t_1)) - U(\eta(t_0)) + c |E|.
\]

Taking into account that
\[
\dot{\eta}(t) = 0 \quad \text{for a.e. } t \in F
\]
we further get
\[
\int_F \bar{L}_\gamma(\eta, \dot{\eta}) \, d\tau = c |F|.
\]
(11)
If \(a, b \in E\), then \(t_0 = a, t_1 = b\) and by the assumption
\[
U(\eta(t_0)) = U(\eta(a)) = U(\eta(b)) = U(\eta(t_1)).
\]
(12)

If instead \(a, b \in F\) then \(U(\eta(t_0)) = U(\eta(a))\) and \(U(\eta(t_1)) = U(\eta(b))\) and we again obtain (12).
By (10), (11), (12) we conclude that
\[
\int_a^b \bar{L}_\gamma(\eta, \dot{\eta}) \, d\tau \geq c(b - a),
\]
as was asserted.

Remark 3.7. If we have in the statement of Proposition 3.6 the additional information that \(\eta([a, b]) \cap \{0\} = \emptyset\) and set
\[
F(t) = \begin{cases} 
L_\gamma(\eta(t), \dot{\eta}(t)) & \text{for } \eta(t) \neq 1 \\
c_\gamma(1) & \text{for } \eta(t) = 1
\end{cases}
\]
(13)
the same argument in the proof of the proposition allows showing
\[
\int_a^b F(\tau) \, d\tau \geq c_\gamma(1)(b - a).
\]

4 The action functional
4.1 The Lagrangian on \(\Gamma\)
We assume that it is given a flux limiter \(c_x\) for any \(x \in V\). We set for \((x, q) \in V \times \mathbb{R}\)
\[
E(x, q) = \{ \gamma \in \Gamma_x \mid q \text{ parallel to } \dot{\gamma}(1) \},
\]
note that \(E(x, q) \neq \emptyset\) if and only if \((x, q) \in T \Gamma\). In the definition of the Lagrangian \(L : T \Gamma \to \mathbb{R}\)
we distinguish three cases:

\(- x \in \Gamma \setminus V, \ x \in \gamma((0, 1)), \) then
\[
L(x, q) = L_\gamma \left( \gamma^{-1}(x), \frac{q \cdot \dot{\gamma}(\gamma^{-1}(x))}{|\dot{\gamma}(\gamma^{-1}(x))|^2} \right);
\]
(14)

\(- x \in V, \ q \neq 0, \) then
\[
L(x, q) = \min_{\gamma \in E(x, q)} L_\gamma \left( 1, \frac{q \cdot \dot{\gamma}(1)}{|\dot{\gamma}(1)|^2} \right);
\]
(15)

\(- x \in V, \ q = 0, \) then
\[
L(x, 0) = c_x.
\]
(16)

Note that formula (15) is more involved because there is a problem to take into account, namely different arcs ending at \(x\) could have parallel tangent vectors, in this case we should have
\[
q = \lambda_1 \dot{\gamma}_1(1) = \lambda_2 \dot{\gamma}_2(1) \quad \text{for arcs } \gamma_1 \neq \gamma_2, \text{ scalars } \lambda_1, \lambda_2.
\]

As already pointed out in the Introduction, (16) provides the link between flux limiter and Lagrangian, and consequently between flux limiter and representation formula.
Proposition 4.1. The Lagrangian $L : T \Gamma \to \mathbb{R}$ is lower semicontinuous.

Proof. It is clearly continuous in $T T \cap ((\Gamma \setminus V) \times \mathbb{R}^N)$. Now consider $x \in V$, $q \neq 0$, and $(x_n, q_n) \to (x, q)$. Taking into account that the arcs are finitely many, we can assume, up to extracting a subsequence, that $x_n = \gamma(s_n)$ for an arc $\gamma \in \Gamma_x$ with $s_n \to 1$, $q_n = \mu_n \hat{\gamma}(s_n)$, for some scalar $\mu_n$, and $L(x_n, q_n) = L_\gamma(s_n, \mu_n)$. Hence $\mu_n \to \mu$, $q = \mu \hat{\gamma}(1)$ and

$$L(x_n, q_n) = L_\gamma(s_n, \mu_n) \to L_\gamma(1, \mu) \geq L(x, q).$$

Now assume $q = 0$ and consider $(x_n, q_n) \to (x, 0)$, if $q_n \neq 0$ or $x_n \neq x$, we can argue as above, bearing in mind Lemma 3.3. It is left the case $(x_n, q_n) = (x_0, 0)$, which is trivial. \hfill \square

It is convenient to extend $L$ on the whole of $\mathbb{R}^N \times \mathbb{R}^N$, keeping it lower semicontinuous, setting

$$L(x, q) = +\infty \quad \text{if } (x, q) \notin T \Gamma.$$

It is apparent that $L$ is bounded from below. We set

$$m_L = \min_{(x,q) \in T \Gamma} L(x, q).$$

(17)

From the superlinearity of the $L_\gamma$’s we finally derive that there exists a function $\theta : \mathbb{R}^+ \to \mathbb{R}$ with $\lim_{t \to +\infty} \frac{\theta(t)}{t} = +\infty$ such that

$$L(x, q) \geq \theta(|q|) \quad \text{for any } (x, q) \in T \Gamma.$$  

(18)

4.2 Admissible curves

Given a curve $\xi : [0, T] \to \mathbb{R}^N$, we define the corresponding action functional as

$$\int_0^T L(\xi, \dot{\xi}) \, d\tau.$$

Note that for any pair of points $x$ and $y$ of $\Gamma$ and $T > 0$ there are curves linking $x$ to $y$ in the time $T$ with finite action functional. It is enough for that to take a geodesic linking $x$ to $y$, which does exist since the network is connected, and to change the parametrization in order to define it in $[0, T]$.

The following result is a consequence of Proposition 3.6 and Remark 3.7.

Proposition 4.2. Let $\xi$ be a curve defined in $[a, b]$, $x \in V$. Assume that $\xi(a) = \xi(b)$, and $\xi([a, b]) \subset \gamma([0, 1])$ for some $\gamma \in \Gamma_x$. Assume further that $\xi([a, b]) \cap (V \setminus \{x\}) = \emptyset$, then

$$\int_a^b L(\xi, \dot{\xi}) \, d\tau \geq c_x(b - a).$$

Proof. We set

$$E = \{t \in [a, b] \mid \xi(t) \neq x\},$$

$$F = \{t \in [a, b] \mid \xi(t) = x\}$$

and define

$$\eta(t) = \gamma^{-1} \circ \xi(t) \quad \text{for } t \in [a, b].$$

10
Note that $\eta$ satisfies the assumptions of Proposition 3.6 plus $\eta([a, b]) \cap \{0\} = \emptyset$. Taking into account the definition of the Lagrangian $L$ and Lemma 3.2, we have that

$$L(\xi(t), \dot{\xi}(t)) = \mathcal{L}_\gamma(\eta(t), \dot{\eta}(t)) \quad \text{for a.e. } t \in E,$$

in addition

$$\dot{\xi}(t) = 0 \quad \text{for a.e. } t \in F,$$

$$L(x, 0) = c_x. \quad \text{(21)}$$

If we define $F(t)$, for $t \in [a, b]$, as in (13), we get by (19), (20), (21)

$$F(t) = L(\xi(t), \dot{\xi}(t)) \quad \text{for a.e. } t$$

and the assertion is then a consequence of Proposition 3.6, Remark 3.7. \hfill \Box

**Definition 4.3.** We say that a curve $\xi : [0, T] \to \Gamma$ is admissible if there exists a finite partition $\{t_1, \ldots, t_m\}$ of the interval $[0, T]$ with

- $t_1 = 0$, $t_m = T$, $\xi(t_i) \in V$ for $i = 2, \ldots, m - 1$ if $m > 2$;
- for any $i$, either $\xi([t_i, t_{i+1}]) \cap V = \emptyset$ and $\xi(t_i) \neq \xi(t_{i+1})$ or $\xi(t) \equiv x \in V$ for $t \in (t_i, t_{i+1})$.

**Lemma 4.4.** Let $\xi : [0, T] \to \Gamma$ be a curve. Assume that for any nondegenerate interval $[a, b]$ with $\xi(a), \xi(b) \in V$, $\xi([a, b]) \cap V = \emptyset$, one has $\xi(a) \neq \xi(b)$ then $\xi$ is admissible.

**Proof.** We will show that, under the assumption in the statement, there is a finite partition of $[0, T]$ satisfying Definition 4.3. If $\xi(0, T) \cap V = \emptyset$, then $\{0, T\}$ is such a partition. Otherwise let us denote by $r > 0$ the minimum of the geodesic distance $d_F(x, y)$ between distinct vertices of the network. Being $\xi$ absolutely continuous, there exists a $\delta > 0$ such that

$$\int_E |\dot{\xi}(\tau)| \, d\tau < r, \quad \text{if } |E| \leq \delta.$$ 

Let us partition $[0, T]$ into a finite number of closed intervals $I_1, \ldots, I_n$ of length less than $\delta$. Then $\xi(I_k)$ can contain at most one vertex $x_k$, for each $k \in \{1, \ldots, n\}$. Let us set

$$E_k := \{t \in I_k : \xi(t) \neq x_k\}$$

and pick $k$ such that $E_k \neq I_k$. Then $E_k$ is open in $I_k = [a_k, b_k]$. Let us assume there exists a connected component $(a, b)$ of $E_k$ with $a_k < a$ and $b < b_k$. Then $\xi(a) = \xi(b) = x_k$, hence by assumption $\xi((a, b)) = \{x_k\}$, in contradiction with the definition of $E_k$. This shows that, if not empty,

$$E_k = [a_k, \alpha_k) \cup (\beta_k, b_k], \quad \text{for some } a_k \leq \alpha_k \leq \beta_k \leq b_k. \quad \text{(22)}$$

Finally (22) implies that

$$\{t \in [0, T] : \xi(t) \notin V\} = \bigcup_{k=1}^n E_k$$

is the disjoint union of a finite number of intervals whose extreme points, possibly adding 0 and $T$, form a partition of $[0, T]$ satisfying Definition 4.3. \hfill \Box
Proposition 4.5. Given $T > 0$, and a curve $\xi$ defined in $[0, T]$, we can find an admissible curve $\zeta$ defined in the same interval with $\xi(0) = \zeta(0)$, $\xi(T) = \zeta(T)$ and
\[
\int_0^T L(\xi, \xi') \, d\tau \geq \int_0^T L(\zeta, \zeta') \, d\tau.
\]

Proof. We can assume that $\xi$ is supported in $\Gamma$, otherwise the action is infinite and there is nothing to prove. If $\xi$ is admissible then, again, we are done. If not, there is, according to Lemma 4.4, a collection of nondegenerate intervals $[a_i, b_i] \subset [0, T]$ with disjoint interiors such that $\xi(a_i) = \xi(b_i) \in V$ and $\xi((a_i, b_i)) \cap V = \emptyset$. Note that they are not more than countably many because $\mathbb{R}$ is separable. We fix $i = 1$ and find an arc $\gamma \in \Gamma_x$ with $\xi(a_1, b_1) \subset \gamma((0, 1))$. We define
\[
\zeta(t) = \begin{cases} x & t \in [a_1, b_1] \\ \xi(t) & t \in [0, T] \setminus [a_1, b_1] \end{cases}
\]
and get from Proposition 4.2 and the very definition of $L$
\[
\int_0^T L(\zeta, \zeta') \, d\tau = \int_0^{a_1} L(\xi, \xi') \, d\tau + c_x (b_1 - a_1) + \int_{b_1}^T L(\xi, \xi') \, d\tau \leq \int_0^T L(\xi, \xi') \, d\tau.
\]

We can repeat the above procedure for any other $[a_i, b_i]$ for $i > 1$. Note that $[0, T] \setminus \bigcup_i [a_i, b_i]$ is made up by intervals which have disjoint interiors and endpoints at two different vertices, except possibly the interval starting at $\xi(0)$ and the one ending at $\xi(T)$. These intervals must be finitely many because $\xi$ is absolutely continuous and so possess finite length. Consequently the $[a_i, b_i]$’s can be glued together in a finite number of disjoint intervals corresponding to steady states at a given vertex. We have therefore constructed in this way an admissible curve satisfying the statement.

Given $\xi : [0, T] \to \Gamma$, we will denote in what follows by $\xi_{ad}$ the admissible curve obtained from $\xi$ through the procedure detailed in the above proof.

Remark 4.6. Looking back at the construction performed in the proof of the Proposition 4.5, it is apparent that
\begin{enumerate}
  \item \(\xi_{ad}(t) \in \Gamma \setminus V \Rightarrow \xi_{ad}(t) = \xi(t)\);
  \item \(\xi(t) \in V \Rightarrow \xi_{ad}(t) = \xi(t)\);
  \item if \(\{t_i\}\) is an admissible partition of $[0, T]$ associated to $\xi_{ad}$, then $\xi_{ad}(t_i) = \xi(t_i)$.
\end{enumerate}

Proposition 4.7. Let $\xi : [a, b] \to \Gamma$ be an admissible curve and assume that there is an $x \in V$ such that
\[
\xi([a, b]) \cap V = \{x\}
\]
or
\[
\xi([a, b]) \subseteq \gamma((0, 1]), \quad \text{for some } \gamma \in \Gamma_x.
\]

Then there exists a positive constant $\ell$ such that
\[
\int_a^b L(\xi, \xi') \, d\tau \geq c_x (b - a) - \ell (d_G(x, \xi(a)) \vee d_G(x, \xi(b))).
\]
Proof. We recall from Lemma 3.1 that \( \frac{1}{m} \), with \( m \) defined as in (5), is a Lipschitz constants of \( \gamma^{-1} : \gamma([0,1]) \to [0,1] \) for all arcs \( \gamma \). We further set

\[
M = \max_{\gamma \in \mathcal{E}} \max_{s \in [0,1]} L(\gamma(s), \dot{\gamma}(s)) \lor 1.
\]

We will show that the inequality in the statement holds true with interval \( \gamma \) where \( \gamma \) goes back to \( \xi \) for a positive time, and finally enters the second arc, where it dwells until the time \( \Gamma \).

We start looking into the case (25). We assume, to fix ideas, that \( \alpha := \gamma^{-1} \circ \xi(a) \leq \gamma^{-1} \circ \xi(b) =: \beta \). We then consider the curve \( \zeta \) obtained via concatenation of \( \xi \) and \( \tilde{\gamma} \) restricted to \([1 - \beta, 1 - \alpha] \), we have

\[
1 - \alpha \leq \frac{d}{m} \quad \text{and} \quad 1 - \beta \leq \frac{d}{m},
\]

where \( d = d_{\mathcal{E}}(x, \xi(a)) \lor d_{\mathcal{E}}(x, \xi(b)) \). We can apply to \( \zeta \) Proposition 4.2 to get

\[
\int_{a}^{b+\beta-\alpha} L(\zeta, \dot{\zeta}) \, d\tau \geq c_x (b - a + \beta - \alpha).
\]

We further have

\[
\int_{b}^{b+\beta-\alpha} L(\zeta, \dot{\zeta}) \, d\tau = \int_{1-\beta}^{1-\alpha} L(\tilde{\gamma}, \dot{\tilde{\gamma}}) \, d\tau \leq M (\beta - \alpha) \leq M (1 - \alpha) \leq \frac{M}{m} d
\]

and accordingly, if \( c_x \geq 0 \),

\[
\int_{a}^{b} L(\xi, \dot{\xi}) \, d\tau \geq c_x (b - a) + c_x (\beta - \alpha) - \frac{M}{m} d \geq c_x (b - a) - \frac{M}{m} d = c_x (b - a) - \frac{\ell}{2} d
\]

or, if \( c_x < 0 \),

\[
\int_{a}^{b} L(\xi, \dot{\xi}) \, d\tau \geq c_x (b - a) + c_x (1 - \alpha) - \frac{M}{m} d \geq c_x (b - a) - \left( \frac{M - c_x}{m} \right) d \geq c_x (b - a) - \frac{\ell}{2} d.
\]

In both cases we then get

\[
\int_{a}^{b} L(\xi, \dot{\xi}) \, d\tau \geq c_x (b - a) - \frac{\ell}{2} d.
\]  \hspace{1cm} (26)

The case where \( \gamma^{-1} \circ \xi(a) \geq \gamma^{-1} \circ \xi(b) \) can be handled similarly.

Now let consider the case (24). By the assumption, \( \xi \) visits exactly two different arcs in the time interval \([a, b]\). It starts in one arc at time \( a \), then intersects the vertex \( x \), and possibly stops on it for a positive time, and finally enters the second arc, where it dwells until the time \( b \). It cannot go back to \( x \), otherwise the admissibility condition should be violated. To fix ideas, assume that

\[
\xi([a, b]) \cap \gamma_1((0, 1)) \neq \emptyset, \quad \xi([a, b]) \cap \gamma_2((0, 1)) \neq \emptyset,
\]

where \( \gamma_1 \) and \( \gamma_2 \) are different arcs, both belonging to \( \Gamma_x \), and that \( \xi(a) \in \gamma_1((0, 1)), \xi(b) \in \gamma_2((0, 1)) \). We thus have three disjoint intervals \( I_1, I_2 \) and \( I_3 \) covering \([a, b]\) and such that

\[
\xi(I_1) \subseteq \gamma_1((0, 1)), \quad \xi(I_2) \subseteq \gamma_2((0, 1)), \quad \xi(I_3) \equiv x.
\]
By definition
\[ \int_{I_3} L(\xi, \dot{\xi}) \, d\tau = c_x |I_3| \]
and it follows from (26) that
\[ \int_{I_1} L(\xi, \dot{\xi}) \, d\tau + \int_{I_2} L(\xi, \dot{\xi}) \, d\tau \geq c_x (|I_1| + |I_2|) - \ell(d_\Gamma(x, \xi(a)) \cup d_\Gamma(x, \xi(b))). \]
This concludes the proof. \(\square\)

5 Minimal action functional

We proceed proving a semicontinuity property for the action functional with respect to the weak topology of \(H^{1,1}([0, T])\), the Sobolev space of the absolutely continuous curves with the norm
\[ \|\xi\| := \int_0^T (|\xi| + |\dot{\xi}|) \, d\tau. \]
The difficulty here is that the Lagrangian \(L\) is not convex in the velocity variable at the vertices. A deeper analysis using Proposition 4.7 is therefore needed.

**Proposition 5.1.** Let \(\xi_n\) be a sequence of admissible curves, defined in \([0, T]\), weakly converging to \(\xi\) in \(H^{1,1}\), then
\[ \liminf_n \int_0^T L(\xi_n, \dot{\xi}_n) \, d\tau \geq \int_0^T L(\xi_{\text{ad}}, \dot{\xi}_{\text{ad}}) \, d\tau. \]

**Proof.** We fix \(\varepsilon > 0\). Taking into account that the \(\dot{\xi}_n\)'s are uniformly integrable by the convergence assumption on \(\xi_n\), we can choose \(\delta \in (0, \varepsilon)\) such that
\[ |E| < 2\delta \Rightarrow \begin{cases} \int_E |\dot{\xi}_n| \, d\tau < \varepsilon & \forall n \\smallint_E |L(\xi_{\text{ad}}, \dot{\xi}_{\text{ad}})| \, d\tau < \varepsilon \end{cases} \tag{27} \]
Since \(\xi_{\text{ad}}\) is admissible, \(\xi_{\text{ad}}^{-1}(V)\) is made up by finitely many compact intervals, some of them reduced to a singleton, and some other nondegenerate, where \(\xi_{\text{ad}}\) is constant, say
\[ \xi_{\text{ad}}^{-1}(V) = \bigcup_{k=1}^m J_k. \]
We denote by \(x_k, k = 1, \ldots, m\), the element of \(V\) univocally determined by the condition
\[ x_k = \xi_{\text{ad}}(t) \quad \text{for some } t \in J_k. \]
Accordingly
\[ A_\delta = \{ t \in [0, T] \mid d(t, \xi_{\text{ad}}^{-1}(V)) \leq \delta \} \]
is a finite union of disjoint compact intervals denoted by \(J'_1, \ldots, J'_m\), with \([a_k, b_k] =: J'_k \supset J_k\), provided that \(\varepsilon\), and so \(\delta\), is sufficiently small. Since \(\xi_n\) uniformly converges to \(\xi\), we deduce, taking into account Remark 4.6 (ii), that the \(\xi_n\)'s restricted to any \(J'_k\) satisfy the assumption of Proposition 4.7 for \(n\) large enough, with \(x = x_k\). In addition
\[ d_\Gamma(\xi_n(a_k), x_k) \leq \int_{a_k}^{a_k+\delta} |\dot{\xi}_n| \, d\tau \leq \varepsilon \]
\[ d_\Gamma(\xi_n(b_k), x_k) \leq \int_{b_k-\delta}^{b_k} |\dot{\xi}_n| \, d\tau \leq \varepsilon \]
for any such \( n \), and any \( k \), by (27). We consequently have by Proposition 4.7
\[
\int_{J_k'} L(\xi_n, \dot{\xi}_n) \, d\tau \geq c_{x_k} |J_k'|-\ell \varepsilon. \tag{28}
\]
On the other side, by the very definition of \( J_k \) and (27), we have
\[
\int_{J_k'} L(\xi_{ad}, \dot{\xi}_{ad}) \, d\tau = \int_{J_k} L(\xi_{ad}, \dot{\xi}_{ad}) \, d\tau + \int_{J_k' \setminus J_k} L(\xi_{ad}, \dot{\xi}_{ad}) \, d\tau \leq c_{x_k} |J_k| + \varepsilon \leq c_{x_k} |J_k'| + \varepsilon.
\]
By combining (28), (29) we get
\[
\int_{J_k'} L(\xi_n, \dot{\xi}_n) \, d\tau \geq \int_{J_k'} L(\xi_{ad}, \dot{\xi}_{ad}) \, d\tau - (\ell + 1) \varepsilon, \tag{29}
\]
for any \( k = 1, \ldots, m, n \) sufficiently large. The complement in \([0, T]\) of \( \overline{A_\delta} \) is a finite union of open (in \([0, T]\)) intervals denoted by \( I_1, \ldots, I_l \), where \( \xi_{ad} \) and \( \xi \) coincide by Remark 4.6. Note that on the intervals \( I_r \), the Lagrangian satisfies the usual conditions of lower semicontinuity and convexity which imply that the action functional is sequentially lower semicontinuous in the weak topology of \( H^{1,1} \), see Theorem 3.6 in [4], so that we get
\[
\liminf_{n \to +\infty} \int_{I_r} L(\xi_n, \dot{\xi}_n) \, d\tau \geq \int_{I_r} L(\xi_{ad}, \dot{\xi}_{ad}) \, d\tau \quad \text{for any } r = 1, \ldots, l. \tag{30}
\]
By combining (29), (30), we further get
\[
\liminf_{n \to +\infty} \int_0^T L(\xi_n, \dot{\xi}_n) \, d\tau \geq \int_0^T L(\xi_{ad}, \dot{\xi}_{ad}) \, d\tau - m (\ell + 1) \varepsilon,
\]
which gives the assertion since \( \varepsilon \) has been arbitrarily chosen.

**Theorem 5.2.** Given \( x \) and \( y \) in \( \Gamma \) and \( T > 0 \), there is an admissible curve defined in \([0, T]\) and linking \( x \) to \( y \) which minimizes the action.

**Proof.** Let \( \xi_n \) be a minimizing sequence that we can assume, according to Proposition 4.5, made up by admissible curves. Since the minimal action in \([0, T]\) between \( x, y \) is finite, say equal to \( M \), we find
\[
M + 1 \geq \int_0^T L(\xi_n, \dot{\xi}_n) \, dt \geq \int_0^T \theta(|\dot{\xi}_n|) \, dt,
\]
for \( n \) large, where \( \theta \) has been introduced in (18). Since the \( \dot{\xi}_n \)'s are bounded in \( L^1 \), we can use the Dunford–Pettis compactness criterion, see Theorem 4.7.18 in [3], to get that the \( \xi_n \) converges weakly in \( H^{1,1} \), up to subsequences, to a limit curve \( \xi \). By Proposition 5.1
\[
\liminf_n \int_0^T L(\xi_n, \dot{\xi}_n) \, d\tau \geq \int_0^T L(\xi_{ad}, \dot{\xi}_{ad}) \, d\tau.
\]
This shows that \( \xi_{ad} \) is the sought minimizer.

Next result is relevant by itself.

**Theorem 5.3.** Any admissible curve minimizing the action between two elements \( z_1, z_2 \) of \( \Gamma \) in a given time \( T \) is Lipschitz continuous.
Proof. Let $\xi : [0, T] \to \Gamma$ be an admissible minimizing curve and $\{t_1, \ldots, t_m\}$ the corresponding finite partition of $[0, T]$. We will show that the restriction of $\xi$ to any interval $[t_i, t_{i+1}]$, $i = 1, \ldots, m-1$ is Lipschitz continuous. If such a restriction is constant there is clearly nothing to prove. We therefore focus on an interval $[t_i, t_{i+1}]$ with

$$\xi((t_i, t_{i+1})) \subset \gamma((0,1))$$

for a suitable arc $\gamma$.

We consequently have

$$\int_{t_i}^{t_{i+1}} L(\xi, \dot{\xi}) \frac{dt}{t} = \int_{t_i}^{t_{i+1}} L_\gamma(\eta, \dot{\eta}) \frac{dt}{t},$$

where $\eta = \gamma^{-1} \circ \xi$, see Lemma 3.2. To fit the setting of Section B we can extend $L_\gamma$ in the whole of $\mathbb{R} \times \mathbb{R}$ defining for instance

$$L_\gamma(s, \lambda) = \begin{cases} L_\gamma(0, \lambda) & \text{if } s < 0 \\ L_\gamma(1, \lambda) & \text{if } s > 1 \end{cases}$$

By the optimality properties of $\xi$, it is clear that $\eta$ minimizes the action induced by $L_\gamma$ among the curves $\zeta$ with $\zeta(t_i) = \eta(t_i)$, $\zeta(t_{i+1}) = \eta(t_{i+1})$, spt $\zeta \subset \text{spt} \eta$. We can therefore invoke Theorem B.3 to deduce that $\eta$, and consequently $\xi$, is Lipschitz continuous in $[t_1, t_2]$. \hfill \Box

We proceed proving:

**Theorem 5.4.** *The minimal action functional*

$$(x, t, y, r) \mapsto \min \left\{ \int_0^{r-t} L(\xi, \dot{\xi}) \, d\tau \mid \xi \text{ AC with } \xi(0) = x, \xi(r - t) = y \right\}$$

is continuous for $x, y$ varying in $\Gamma$, $t, r$ in $\mathbb{R}^+$ with $r > t$.

Proof. We denote by $S(x, y, \cdot, \cdot)$ the minimal action functional. We first show that it is lower semicontinuous. We fix $(x_0, t_0, y_0, r_0)$ with $r_0 > t_0$, and consider $(x_n, t_n, y_n, r_n) \to (x_0, t_0, y_0, r_0)$ with

$$S(x_n, t_n, y_n, r_n) \to a \quad \text{for some } a \in \mathbb{R} \cup \{+\infty\}.$$ 

If $a = +\infty$ there is nothing to prove, we can therefore assume that

$$\int_0^{r_n-t_n} L(\xi_n, \dot{\xi}_n) \, d\tau \quad \text{is bounded from above},$$

(31)

where $\xi_n$ denotes a sequence of admissible curves defined in $[0, r_n-t_n]$ and realizing the minimal action between $x_n$ and $y_n$. Given $\varepsilon > 0$, we set $T_\varepsilon = r_0 - t_0 + \varepsilon$, so that $r_n - t_n < T_\varepsilon$ for $n$ large, and we extend all such $\xi_n$’s in $[0, T_\varepsilon]$ defining

$$\xi_n(t) = y_n \quad \text{for } t \in (r_n - t_n, T_\varepsilon].$$

Arguing as in the proof of Theorem 5.2 and taking into account (31), we see that the $\xi_n$’s weakly converge in $H^{1,1}((0, T_\varepsilon))$ to a curve $\xi$ with $\xi(0) = x, \xi(r - s) = y$. By Proposition 5.1, we have

$$\liminf_n \int_0^{T_\varepsilon} L(\xi_n, \dot{\xi}_n) \, d\tau \geq \int_0^{T_\varepsilon} L(\xi_{ad}, \dot{\xi}_{ad}) \, d\tau.$$ 

We further have

$$\int_0^{T_\varepsilon} L(\xi_n, \dot{\xi}_n) \, d\tau = S(x_n, t_n, y_n, r_n) + \int_{r_n-t_n}^{T_\varepsilon} L(y_n, 0) \, d\tau$$

$$\leq S(x_n, t_n, y_n, r_n) + M (r_0 - t_0 + \varepsilon - r_n + t_n),$$

16
where \( M = \max_{x \in \Gamma} |L(x, 0)| \), and consequently
\[
\int_{0}^{T_{e}} L(\xi_{n}, \dot{\xi}_{n}) \, d\tau \leq S(x_{n}, t_{n}, y_{n}, r_{n}) + 2 \bar{M} \varepsilon
\]
for \( n \) suitably large. We deduce that
\[
a \geq \lim \inf_{n} \int_{0}^{T_{e}} L(\xi_{n}, \dot{\xi}_{n}) \, d\tau - 2 \bar{M} \varepsilon \geq \int_{0}^{T_{e}} L(\xi_{ad}, \dot{\xi}_{ad}) \, d\tau - 2 \bar{M} \varepsilon
\]
where \( m_{L} \) is defined as in (17). This shows the claimed lower semicontinuity.

We proceed considering again \((x_{n}, t_{n}, y_{n}, r_{n}) \to (x_{0}, t_{0}, y_{0}, r_{0})\) with
\[
S(x_{n}, t_{n}, y_{n}, r_{n}) \to a \quad \text{for some } a,
\]
and denote by \( \xi : [0, r_{0} - t_{0}] \to \Gamma \) an admissible curve realizing the minimal action between \( x_{0} \) and \( y_{0} \). We further set \( \tau_{k} = \frac{1}{k}, \quad \tau'_{k} = (r_{0} - t_{0}) - \frac{1}{k} \), the velocity \(|\dot{\xi}|\) being bounded by Theorem 5.3, we have
\[
\begin{align*}
\int_{0}^{\tau_{k}} L(\xi, \dot{\xi}) \, d\tau &= O(1/k) \quad (32) \\
\int_{\tau'_{k}}^{r_{0} - t_{0}} L(\xi, \dot{\xi}) \, d\tau &= O(1/k), \quad (33)
\end{align*}
\]
where \( O(\cdot) \) stands for the Landau symbol, and
\[
d_{\Gamma}(\xi(\tau_{k}), x_{0}) = O(1/k), \quad d_{\Gamma}(\xi(\tau'_{k}), y_{0}) = O(1/k).
\]

We select \( n_{k} \) such that
\[
|r_{n_{k}} - t_{n_{k}} - r + s| = O(1/k^{2}), \quad d_{\Gamma}(x_{n_{k}}, x_{0}) = O(1/k^{2}), \quad d_{\Gamma}(y_{n_{k}}, y_{0}) = O(1/k^{2}).
\]

We therefore have
\[
d_{\Gamma}(x_{n_{k}}, \xi(\tau_{k})) = O(1/k) \quad \text{and} \quad d_{\Gamma}(y_{n_{k}}, \xi(\tau'_{k})) = O(1/k).
\]

Let \( \zeta_{k} \) be a geodetic in \( \Gamma \) linking \( \xi(\tau_{k}) \) and \( x_{n_{k}} \) reparametrized in order to be defined in the interval \([0, \tau_{k}]\). We have
\[
\int_{0}^{\tau_{k}} L(\zeta_{k}, \dot{\zeta}_{k}) \, d\tau = O(1/k). \quad (34)
\]

Let \( \zeta'_{k} \) be a geodetic linking \( \xi(\tau'_{k}) \) and \( y_{n_{k}} \) reparametrized in order to be defined in the interval \([\tau'_{k}, r_{n_{k}} - t_{n_{k}}]\). We have
\[
\int_{\tau'_{k}}^{r_{n_{k}} - t_{n_{k}}} L(\zeta'_{k}, \dot{\zeta}'_{k}) \, d\tau = O(1/k). \quad (35)
\]

The curve
\[
\xi_{k}(\tau) = \begin{cases} 
\zeta_{k}(\tau) & \tau \in [0, \tau_{k}] \\
\xi(\tau) & \tau \in (\tau_{k}, \tau'_{k}) \\
\zeta'_{k}(\tau) & \tau \in [\tau'_{k}, r_{n_{k}} - t_{n_{k}}]
\end{cases}
\]

17
links \( x_{nk} \) to \( y_{nk} \) in the time interval \([0, r_{nk} - t_{nk}]\) for \( k \) large. We have by (34), (35), (32), (33)

\[
\int_{0}^{r_{nk} - t_{nk}} L(\xi_k, \dot{\xi}_k) \, d\tau = \int_{\tau_k}^{\tau_k'} L(\xi, \dot{\xi}) \, d\tau + O(1/k) \\
= \int_{0}^{r - s} L(\xi, \dot{\xi}) \, d\tau + O(1/k).
\]

This implies

\[
a = \lim_k S(x_{nk}, t_{nk}, y_{nk}, r_{nk}) \\
\leq \lim_k \int_{0}^{r_{nk} - t_{nk}} L(\xi_k, \dot{\xi}_k) \, d\tau = S(x_0, t_0, y_0, r_0),
\]

which shows the upper semicontinuity of the minimal action.

\[\square\]

6 The time–dependent equation on \( \Gamma \)

For any given arc \( \gamma \), we consider the time–dependent equation

\[
U_t + H_\gamma(s, U') = 0 \quad \text{in } Q. \tag{HJ_\gamma}
\]

We are interested in finding a continuous function \( v : \Gamma \times \mathbb{R}^+ \to \mathbb{R} \) such that \( v \circ \gamma \) solves \( \text{(HJ}_\gamma) \) in \( Q \), for any \( \gamma \), taking into account, in the sense we are going to specify, a flux limiter \( c_x \) at any vertex. We denote by \( \text{(HJ}_\Gamma \) the problem as a whole.

The definition of (sub / super) solution to \( \text{(HJ}_\Gamma \) is as follows:

**Definition 6.1.** We say that a continuous function \( v(x, t), \ v : \Gamma \times \mathbb{R}^+ \to \mathbb{R}, \) is a *supersolution* if

(i) \( v \circ \gamma \) is a viscosity supersolution of \( \text{(HJ}_\gamma \) in \( Q \) for any arc \( \gamma \);

(ii) for any vertex \( x \) and time \( t_0 > 0 \), if

\[
\frac{d}{dt} \phi(t_0) < c_x
\]

for some \( C^1 \) sub tangent \( \phi \) to \( v(x, \cdot) \) at \( t_0 \), then there is an arc \( \gamma \in \Gamma_x \) such that all the \( C^1 \) sub tangents \( \Phi \), constrained to \( \overline{Q} \), to \( v \circ \gamma \) at \( (1, t_0) \) satisfy

\[
\Phi_t(1, t_0) + H_\gamma(1, \Phi'(1, t_0)) \geq 0.
\]

**Definition 6.2.** We say that a continuous function \( v(x, t), \ v : \Gamma \times \mathbb{R}^+ \to \mathbb{R}, \) is a *subsolution* if

(i) \( v \circ \gamma \) is a viscosity subsolution of \( \text{(HJ}_\gamma \) in \( Q \) for any arc \( \gamma \);

(ii) for any vertex \( x \) and time \( t_0 > 0 \), all supertangents \( \psi(t) \) to \( v(x, \cdot) \) at \( t_0 \) satisfy

\[
\frac{d}{dt} \psi(t_0) \leq c_x.
\]
We finally say that a continuous function $v$ is solution to $(HJ\Gamma)$ if it subsolution and supersolution at the same time.

We recall the following result of [17]:

**Theorem 6.3.** For any continuous initial datum $u_0$ and flux limiter $c_x$, there exists one and only one continuous solution to $(HJ\Gamma)$ continuously attaining the datum $u_0$ at $t = 0$.

From now on we fix a continuous initial datum $u_0$ and a flux limiter $c_x$.

### 6.1 The equation on a single arc

We fix an arc $\gamma$, Given a continuous boundary datum $g : \partial Q \to \mathbb{R}$,

we consider the equation $(HJ\gamma)$ coupled with the boundary conditions

$$U(s,t) = g(s,t) \quad \text{for } (s,t) \in \partial Q.$$  \hspace{1cm} (36)

or

$$U(s,t) = g(s,t) \quad \text{for } (s,t) \in \partial^- Q.$$  \hspace{1cm} (37)

We say that the boundary data in (36) (resp. (37)) are admissible if there exists a continuous function $V : Q \to \mathbb{R}$ solution of $(HJ\gamma)$ in $Q$ and equal to $g$ on $\partial Q$ (resp. $\partial^- Q$). The following results are well known, a proof is provided in Appendix A for reader’s convenience.

**Theorem 6.4.** If the datum $g$ in (36) is admissible then the formula

$$V(s,t) = \inf \left\{ g(s_0,t_0) + \int_{t_0}^t L_\gamma(\eta,\dot{\eta}) \, d\tau \right\},$$  \hspace{1cm} (38)

where the infimum is over the elements $(s_0,t_0) \in \partial Q$ with $t_0 < t$, and the curves $\eta : [t_0,t] \to \overline{Q}$ with $\eta(t_0) = s_0$, $\eta(t) = s$, is a solution to $(HJ\gamma)$, (36).

**Theorem 6.5.** If the datum $g$ in (37) is admissible then the function

$$W(s,t) = \inf \left\{ g(s_0,t_0) + \int_{t_0}^t L_\gamma(\eta,\dot{\eta}) \, d\tau \right\},$$  \hspace{1cm} (39)

where the infimum is over the elements $(s_0,t_0) \in \partial^- Q$ with $t_0 < t$, and the curves $\eta : [t_0,t] \to \overline{Q}$ with $\eta(t_0) = s_0$, $\eta(t) = s$, is solution to $(HJ\gamma)$, (37) and satisfies

$$\Psi_t(1,t) + H_\gamma(1,\Psi_t(1,t)) \geq 0$$

for any $t > 0$, any subtangent $\Psi$ to $W$, constrained to $\overline{Q}$, at $(1,t)$.  

19
6.2 The main result

We define for \((x,t) \in \Gamma \times \mathbb{R}^+\)

\[
u(x,t) = \inf_{\xi} \left\{ \int_0^t L(\xi, \dot{\xi}) \, ds + u_0(\xi(0)) \right\}, \tag{40}\]

where \(\xi\) is any curve from \([0, t]\) to \(\Gamma\) with \(\xi(t) = x\).

**Proposition 6.6.** The function \(\nu\) defined in (40) is continuous in \(\Gamma \times \mathbb{R}^+\).

**Proof.** The continuity at any point \((x_0, t_0)\) with \(t_0 > 0\) comes straightforwardly from the continuity of the minimal action functional and of \(u_0\). Assume now that \((x_n, t_n)\) converges to \((x_0, 0)\) and \(\lim u(x_n, t_n) =: a\). We have

\[
u(x_n, t_n) \leq u_0(x_n) + \int_0^{t_n} L(x_n, 0) \, d\tau
\]

which, sending \(n\) to infinity, shows that \(a \leq u_0(x_0)\). Assume now that \(u(x_n, t_n) = u_0(y_n) + \int_0^{t_n} L(\xi_n, \dot{\xi}_n) \, d\tau\), (41)

we claim that \(y_n \to x_0\). If not \(y_n \to y_0 \neq x_0\), up to subsequences, and we find by the super-linearity of \(L\) two sequences of real numbers \(\alpha_k, \beta_k\) with \(\beta_k \to +\infty\) such that

\[L(x, q) \geq \alpha_k + \beta_k |q|,\]

We derive that

\[
\int_0^{t_n} L(\xi_n, \dot{\xi}_n) \, d\tau \geq \alpha_k t_n + \beta_k d_\Gamma(x_n, y_n) \to +\infty \quad \text{as} \quad k \to +\infty, \quad \text{for any} \ n
\]

and

\[
\liminf_{n \to +\infty} \int_0^{t_n} L(\xi_n, \dot{\xi}_n) \, d\tau \geq \beta_k d_\Gamma(x_0, y_0) \quad \text{for any} \ k.
\]

This contradicts (41). Finally, we have

\[
u(x_n, t_n) \geq u_0(y_n) + t_n m_L,
\]

with \(m_L\) defined as in (17), which implies \(a \geq u_0(x_0)\), and concludes the proof. \(\square\)

We keep in mind existence and uniqueness Theorem 6.3. Our main result is:

**Theorem 6.7.** The function \(\nu\) defined in (40) is the unique solution to (\(HJ_\Gamma\)) with initial datum \(u_0\) and flux limiter \(c_2\) for \(x \in \mathbf{V}\).

We start by a preliminary result. We fix an arc \(\gamma\).

**Proposition 6.8.** Let \(u\) be the function defined in (40), then the boundary datum

\[
g(s, t) = \begin{cases} u_0 \circ \gamma(s) & \text{in } [0, 1] \times \{0\} \\ u \circ \gamma(0, t) & \text{in } \{0\} \times \mathbb{R}^+ \\ u \circ \gamma(1, t) & \text{in } \{1\} \times \mathbb{R}^+, \end{cases} \tag{42}\]

where \(u \circ \gamma\) is defined in (4), is admissible for (\(HJ_\gamma\)) with boundary conditions (36), and \(u \circ \gamma\) is the solution given by formula (38).
Proof. We know that \( u \circ \gamma \) is continuous in \( \overline{Q} \) by Proposition 6.6. It is then enough by Theorem 6.4 to show that \( u \circ \gamma \) is given by (38). Let \((s, t) \in Q\), we denote by \( \xi : [0, t] \to \Gamma \) an optimal curve for \( u(\gamma(s), t) \), and set

\[
A = \{ \tau < t \mid \xi(\tau) \in \{ \gamma(0), \gamma(1) \} \}
\]

\[
t_0 = \begin{cases} \max A & \text{if } A \neq \emptyset \\ 0 & \text{if } A = \emptyset \end{cases}
\]

It is clear that \( \xi((t_0, t]) \subset \gamma((0, 1)) \). If \( t_0 > 0 \), it is also clear that the curve \( \xi \) restricted to \([0, t_0]\) is optimal for \( u(\xi(t_0), t_0) \), hence we have by (14), Lemma 3.2 and (6)

\[
u(\gamma(s), t) = u_0(\xi(0)) + \int_0^{t_0} L(\xi, \dot{\xi}) \, d\tau + \int_{t_0}^t L(\xi, \dot{\xi}) \, d\tau
\]

\[
= u(\xi(t_0), t_0) + \int_{t_0}^t L(\gamma^{-1} \circ \xi(\tau), d/d\tau(\gamma^{-1} \circ \xi(\tau))) \, d\tau
\]

\[
= g(\gamma^{-1} \circ \xi(t_0), t_0) + \int_{t_0}^t L(\gamma^{-1} \circ \xi(\tau), d/d\tau(\gamma^{-1} \circ \xi(\tau))) \, d\tau,
\]

and the same formula holds true if \( t_0 = 0 \). Taking into account that \( \gamma^{-1} \circ \xi \) is a curve by Lemma 3.2, we derive that

\[
u \circ \gamma \geq V \quad \text{in } Q,
\]

where \( V \) is the function defined in (38). Assume for purposes of contradiction that there is \((s_1, t_1) \in \partial Q\) with \( t_1 < t \), and a curve \( \eta : [t_1, t] \to \overline{Q} \) with \( \eta(t_1) = s_1, \eta(t) = s \) such that

\[
u(\gamma(s), t) > g(s_1, t_1) + \int_{t_1}^t L(\eta(\tau), \dot{\eta}(\tau)) \, d\tau.
\]

We denote by \( \bar{\xi} : [0, t_1] \to \Gamma \) an optimal curve for \( u(\gamma(s_1), t_1) \), we further denote by \( \zeta : [0, t] \to \Gamma \) the curve obtained by concatenating \( \bar{\xi} \) and \( \gamma \circ \eta \), namely

\[
z(\tau) = \begin{cases} \bar{\xi}(\tau) & \tau \in [0, t_1) \\ \gamma \circ \eta(\tau) & \tau \in [t_1, t) \end{cases}
\]

We then deduce from (43)

\[
u(\gamma(s), t) > u(\gamma(s_1), t_1) + \int_{t_1}^t L(\eta(\tau), \dot{\eta}(\tau)) \, d\tau
\]

\[
= u_0(\bar{\xi}(0)) + \int_0^{t_1} L(\bar{\xi}, \dot{\bar{\xi}}) \, d\tau + \int_{t_1}^t L(\eta(\tau), \dot{\eta}(\tau)) \, d\tau
\]

\[
= u_0(\zeta(0)) + \int_0^t L(\zeta, \dot{\zeta}) \, d\tau,
\]

which cannot be. The proof is therefore concluded. \( \square \)

Proof of Theorem 6.7. We know, thanks to Proposition 6.8 that \( u \circ \gamma \) is viscosity solution to (HJ\_s) for any \( \gamma \). It is then enough to check the solution conditions at the vertices. Assume by contradiction that there is \( x \in V, t_0 > 0 \) and a supertangent \( \psi \) to \( u(x, \cdot) \) at \( t_0 \) with

\[
\frac{d}{dt}\psi(t_0) > c_x.
\]
We derive that there exists $\delta > 0$ with
\[ u(x, t_0 - \delta) + h c_x \leq \psi(t_0 - \delta) + \delta c_x < \psi(t_0) = u(x, t_0), \]
therefore
\[ u(x, t_0) > u(x, t_0 - \delta) + \int_{t_0-\delta}^{t_0} L(x, 0) \, d\tau \]
which is in contrast with the very definition of $u$. We have so proved the subsolution condition for $u$ at the vertices. We proceed considering an admissible curve $\xi : [0, t_0] \to \Gamma$ realizing $u(x, t_0)$, we further assume the existence of a subtangent $\phi$ to $u(x, \cdot)$ at $t_0$ with
\[ \frac{d}{dt} \phi(t_0) < c_x. \]
This implies that for any $\delta > 0$ sufficiently small we have
\[ u(x, t_0 - \delta) + \delta c_x \geq \psi(t_0 - \delta) + \delta c_x > \psi(t_0) = u(x, t_0). \]
We in turn derive that
\[ \xi(t) \neq x \quad \text{for } \delta > 0 \text{ small, and } t \in (t_0 - \delta, t_0). \]
Taking into account the definition of admissible curve, there thus exists $0 \leq t_1 < t_0$, $\gamma \in \Gamma_x$ such that $\xi(t_1, t_0)) \subset \gamma(Q)$ and
\[
u(x, t_0) = \begin{cases} \nu(\gamma(0), t_1) + \int_{t_1}^{t_0} L(\xi, \xi') \, d\tau & \text{if } t_1 > 0, \\ \nu_0(\gamma(s)) + \int_0^{t_1} L(\xi, \xi') \, d\tau & \text{for some } s \in [0, 1) \text{ if } t_1 = 0 \end{cases} \]
If we define the boundary datum $g$ as in (42) and set $\eta = \gamma^{-1} \circ \xi$, we can also write
\[
u(\gamma(1), t) = \begin{cases} g(0, t_1) + \int_{t_1}^{t_0} L(\eta, \eta') \, d\tau & \text{if } t_1 > 0, \\ g(s, 0) + \int_0^{t_1} L(\eta, \eta') \, d\tau & \text{for some } s \in [0, 1) \text{ if } t_1 = 0 \end{cases} \]
Since by Proposition 6.8 $u \circ \gamma$ is given by formula (38), with datum $g$ taken continuously on $\partial Q$, we therefore see that
\[ u \circ \gamma = W \quad \text{at } (1, t_0), \quad u \circ \gamma \leq W \quad \text{in } \overline{Q}, \]
where $W$ is defined as in Theorem 6.5. If $\varphi$ is subtangent, constrained to $\overline{Q}$, to $u \circ \gamma$ at $(1, t_0)$, then $\varphi$ is also subtangent to $W$ at $(1, t_0)$ and by Theorem 6.5 we get
\[ \varphi_t(1, t_0) + H_\gamma(1, \varphi'(1, t_0)) \geq 0. \]
The proof is then complete. \qed

## A Proof of Theorems 6.4, 6.5

The proof is based on the following lemmata:

**Lemma A.1.** Let $U : \overline{Q} \to \mathbb{R}$ be a continuous function. If for any $(s, t) \in Q$, one has
\[ U(s, t) - U(\eta(t - \delta), t - \delta) \leq \int_{t-\delta}^{t} L(\eta, \eta') \, d\tau \]
for any $C^1$ curve $\eta$ taking values in $[0, 1]$ with $\eta(t) = s$, $\delta > 0$ small enough, then $U$ is a subsolution to $\text{(HJ)}_\gamma$. 

22
Proof. We fix $\lambda \in \mathbb{R}$. Let $\Phi$ be a supertangent to $U$ at $(s,t)$. If $\eta$ is a $C^1$ curve with $\eta(t) = s$ and $\dot{\eta}(t) = \lambda$, we have for any $\delta > 0$ small enough
\[
\frac{\Phi(s,t) - \Phi(\eta(t-\delta), t-\delta)}{\delta} \leq \frac{U(s,t) - U(\eta(t-\delta), t-\delta)}{\delta} \leq \frac{1}{\delta} \int_{t-\delta}^{t} L_\gamma(\eta, \dot{\eta}) \, d\tau.
\]
Sending $\delta$ to 0 we get
\[
\Phi_t(s,t) + \lambda \Phi'(s,t) - L_\gamma(s, \lambda) \leq 0.
\]
By the very definition of $L_\gamma$ and the fact that $\lambda$ has been taken arbitrarily, we further get
\[
\Phi_t(s,t) + H_\gamma(s, \Phi'(s,t)) \leq 0,
\]
which gives the assertion. \qed

**Lemma A.2.** Let $U : \mathbb{Q} \to \mathbb{R}$ be a continuous function. If for any $(s,t) \in \mathbb{Q}$ there is a curve $\eta$ taking values in $[0,1]$ with $\eta(t) = s$ such that
\[
U(s,t) - U(\eta(t-\delta), t-\delta) \geq \int_{t-\delta}^{t} L_\gamma(\eta, \dot{\eta}) \, d\tau,
\]
when $\delta > 0$ is small enough, then $U$ is a supersolution to (HJ$_\gamma$) in $\mathbb{Q}$.

**Proof.** Let $\Phi$ be a subtangent to $U$ at $(s,t)$, then we have for any $\mu \in \mathbb{R}$, $\delta$ small
\[
\Phi(s,t) - \Phi(\eta(t-\delta), t-\delta) \geq \frac{U(s,t) - U(\eta(t-\delta), t-\delta)}{\delta} \geq \frac{1}{\delta} \int_{t-\delta}^{t} L_\gamma(\eta, \dot{\eta}) \, d\tau \geq \frac{1}{\delta} \int_{t-\delta}^{t} (\mu \dot{\eta}(\tau) - H_\gamma(\eta(\tau), \mu)) \, d\tau \geq \frac{1}{\delta} \int_{t-\delta}^{t} H_\gamma(\eta(\tau), \mu) \, d\tau.
\]

Given any infinitesimal sequence $\delta_n$, we find, by the mean value theorem applied to the function $\Phi$, points $(s_n, t_n)$ in the segment joining $(\eta(t - \delta_n), t - \delta_n)$ to $(s,t)$ with
\[
\Phi(s,t) - \Phi(\eta(t-\delta_n), t-\delta_n) = \Phi_t(s_n, t_n) \delta_n + \Phi'(s_n, t_n)(\eta(t) - \eta(t-\delta_n)),
\]
therefore (44) with $\mu = \Phi'(t_n, s_n)$ yields, for any $n \in \mathbb{N},$
\[
\Phi_t(t_n, s_n) + \Phi'(t_n, s_n) \frac{\eta(t) - \eta(t-\delta_n)}{\delta_n} \geq \frac{\eta(t) - \eta(t-\delta_n)}{\delta_n} - \frac{1}{\delta_n} \int_{t-\delta_n}^{t} H_\gamma(\eta(\tau), \Phi'(s_n, t_n)) \, d\tau.
\]
We thus have
\[
\Phi_t(s,t) + H_\gamma(s, \Phi'(s,t)) = \lim_{n \to \infty} \Phi_t(t_n, s_n) + \frac{1}{\delta_n} \int_{t-\delta_n}^{t} H_\gamma(\eta, \Phi'(t_n, s_n)) \, d\tau \geq 0,
\]
This shows the assertion, since $(s,t)$ and $\Phi$ are arbitrary. \qed

**Lemma A.3.** The value functions defined in (38) and (39) are continuous.
Proof. We will only prove that (38) is continuous, since the proof for (39) is identical. We consider the set
\[ E = \{(s_0, t_0, s, t) \in \overline{Q}^2 \mid t > t_0 \text{ or } t = t_0, s = s_0 \} \]
and the function
\[ S_\gamma(s_0, t_0, s, t) := \inf \left\{ \int_{t_0}^t L_\gamma(\eta, \dot{\eta}) d\tau \mid \eta \text{ curve with } \eta(t_0) = s_0, \eta(t) = s \right\} \]
in \( E \), and \( S_\gamma = +\infty \) in \( \overline{Q} \setminus E \). It is continuous when \( t > t_0 \) and lower semicontinuous otherwise, see [10] and the proof of Proposition 6.6. This implies that if \( \{(s_n, t_n)\}_{n \in \mathbb{N}} \) is a sequence in \( \overline{Q} \) converging to an element \( (\bar{s}, \bar{t}) \), there is \( (s_0, t_0) \in \partial Q \) with \( t_0 \leq \bar{t} \) and a sequence \( \{(s'_n, t'_n)\}_{n \in \mathbb{N}} \subset \partial Q \) with \( t'_n \leq t_n \) such that
\[ V(\bar{s}, \bar{t}) = g(s_0, t_0) + S_\gamma(s_0, t_0, \bar{s}, \bar{t}), \quad V(s_n, t_n) = g(s'_n, t'_n) + S_\gamma(s'_n, t'_n, s_n, t_n). \]
When \( \bar{t} > t_0 \) the continuity is easily checked, we therefore focus on the \( \bar{t} = t_0, \bar{s} = s_0 \) case. In other term, we assume \( (\bar{s}, \bar{t}) \in \partial Q \) and, taking into account that \( g \) is admissible, \( V(\bar{s}, t) = g(\bar{s}, \bar{t}) \).
We further assume that \( V(s_n, t_n) \) converges to a quantity \( a \). The lower semicontinuity of \( S_\gamma \) and the admissibility of \( g \) yield
\[ a = \lim_n g(s'_n, t'_n) + S_\gamma(s'_n, t'_n, s_n, t_n) \geq g(\bar{s}, \bar{t}) = V(\bar{s}, \bar{t}). \tag{45} \]
If \( t_n = t'_n \), up to a subsequence, then \( (s_n, t_n) \in \partial Q \) and \( V(s_n, t_n) = g(s_n, t_n) \) so that
\[ a = \lim_n g(s_n, t_n) = g(\bar{s}, \bar{t}) = V(\bar{s}, \bar{t}). \tag{46} \]
If \( \bar{t} = 0 \) then
\[ a \leq \lim_n g(s_n, 0) + t_n \max_{s \in [0,1]} L_\gamma(s, 0) = g(\bar{s}, 0) = V(\bar{s}, \bar{t}). \tag{47} \]
If \( \bar{t} > 0 \) and \( t'_n < t_n \), for any \( n \), then we can choose \( t''_n > 0 \) such that
\[ t''_n < t_n, \quad |s_n - s| \leq t_n - t''_n, \quad t''_n \to \bar{t} \]
so that we have
\[ a \leq \lim_n g(\bar{s}, t''_n) + \max_{(s, \lambda) \in [0,1] \times [-1,1]} L_\gamma(s, \lambda) (t_n - t''_n) = g(\bar{s}, \bar{t}) = V(\bar{s}, \bar{t}). \tag{48} \]
By combining (46), (47), (48) with (45), we get the assertion. \( \square \)

Proof of Theorem 6.4. We take for granted that for any \((s, t) \in Q\) there exists a curve \( \eta \) realizing the infimum in formula (38). Then we have for any \( \delta > 0 \) small enough
\[ V(s, t) = V(\eta(t - \delta), t - \delta) + \int_{t-\delta}^t L_\gamma(\eta, \dot{\eta}) d\tau \]
which implies by Lemma A.2 that \( V \) is a subsolution. Since \( V \) is defined by (38) and is continuous by Lemma A.3, it also satisfies the assumption of Lemma A.1, which shows that it is subsolution as well. \( \square \)
Proof of Theorem 6.5. Arguing as in the proof of Theorem 6.4, we see that the function defined in (39) is solution to (HJγ). If we take a point (1, t) for some t > 0, then there exists an optimal curve, say η, realizing W(1, t), if we consider a subtangent Φ, constrained to Q, to W at (1, t) then arguing as in Lemma A.2 we derive that

\[ \Phi_t(1, t) + H_γ(1, \Phi'(1, t)) \geq 0. \]

This concludes the proof.

B Lipschitz regularity of the minimizers

We present the results of this appendix in a rather general frame. They are applied in Theorem 5.3 to the one–dimensional Hamiltonians/Lagrangians appearing in the equations on networks.

We consider an Hamiltonian \( \tilde{H} : \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R} \) enjoying the usual properties of continuity in both arguments plus convexity and superlinearity in the second one, and denote by \( \tilde{L} : \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R} \) the corresponding Lagrangian defined through Fenchel transform. We provide the notion of \( a \)-Lagrangian parametrization of a curve.

**Definition B.1.** Given an absolutely continuous curve \( \eta : [0, T] \rightarrow \mathbb{R}^N \), we say that \( a \in \mathbb{R} \) is admissible if

\[ a \geq \max_{x \in \text{spt} \eta} \min_{p \in \mathbb{R}^N} \tilde{H}(x, p). \]

We say that \( \eta \) has an \( a \)-Lagrangian parametrization, for some admissible \( a \), if

\[ \tilde{L}(\eta(t), \dot{\eta}(t)) = \sigma_a(\eta(t), \dot{\eta}(t)) - a \quad \text{for a.e. } t \in [0, T], \]

where

\[ \sigma_a(x, q) = \max\{p \cdot q \mid p \text{ with } \tilde{H}(x, p) \leq a\}. \]

Note that the sublevels in the above formula are nonempty because of the admissibility of \( a \).

**Proposition B.2.** Any curve \( \eta : [0, T] \rightarrow \mathbb{R}^N \) with an \( a \)-Lagrangian parametrization is Lipschitz continuous.

**Proof.** By assumption we have

\[ \tilde{L}(\eta(t), \dot{\eta}(t)) = p(t) \dot{\eta}(t) - \tilde{H}(\eta(t), p(t)) \quad \text{for a.e. } t, \]

where \( p(t) \) satisfies \( \dot{\eta}(t), p(t) = a \). This implies that \( \dot{\eta}(t) \in \partial_p \tilde{H}(\eta(t), p(t)) \), and by the uniform coercivity of \( \tilde{H} \) in \( p \), we get that \( |p(t)| < M \), for a.e. \( t \) and some \( M > 0 \). Since \( \tilde{H} \) is locally Lipschitz continuous in \( p \) uniformly in \( \text{spt} \eta \), see (8), we find a constant \( C_M \) with \( |\dot{\eta}(t)| < C_M \) for a.e. \( t \), concluding the proof.

The main result we aim at showing is:

**Theorem B.3.** Given an absolutely continuous curve \( \xi : [0, T] \rightarrow \mathbb{R}^N \), the problem

\[ \inf \left\{ \int_0^T \tilde{L}(\zeta, \dot{\zeta}) \, dt \mid \zeta : [0, T] \rightarrow \mathbb{R}^N \text{ A.C. with } \xi(0) = \zeta(0), \xi(T) = \zeta(T), \text{spt } \zeta \subset \text{spt } \xi \right\} \]

admits minimizers. All of them possess an \( a \)-Lagrangian representation, for some admissible \( a \), and consequently are Lipschitz continuous.
We need a further notion.

**Definition B.4.** Given an absolutely continuous curve \( \xi : [0, T] \rightarrow \mathbb{R}^N \), a curve \( \zeta : [0, T'] \rightarrow \mathbb{R}^N \) is said a reparametrization of \( \xi \) if there exists a nondecreasing absolutely continuous function \( \varphi \) from \( [0, T'] \) onto \( [0, T] \) with

\[
\zeta(t) = \xi \circ \varphi(t) \quad \text{for any } t \in [0, T'].
\]

Since the composition of two absolutely continuous functions, with the second one monotonic, is still absolutely continuous, we see that any reparametrization \( \zeta \) of \( \xi \) keeps absolutely continuity. It is further immediate that the support of a curve and its length are invariant under reparametrization. Similarly, exploiting the positive homogeneity of \( \sigma_a(x, \cdot) \) we see that

\[
\int_0^T \sigma_a(\zeta, \dot{\zeta}) dt = \int_0^T \sigma_a(\xi, \dot{\xi}) dt \quad (49)
\]

for any admissible \( a \). Note finally that if \( \zeta \) is a reparametrization of \( \xi \), the converse property in general is not true for \( \varphi \) could not have strictly positive derivative for a.e. \( t \), see Zarecki criterion for an absolutely continuous inverse in [1]. We have:

**Proposition B.5.** Any absolutely continuous curve in \([0, T]\) is a reparametrization of a curve \( \eta \) with constant speed, namely absolutely continuous with \( |\dot{\eta}(t)| \equiv \text{constant} \) for a.e. \( t \), defined in the same interval.

**Proof.** See [1, 7].

The following crucial result of [7] is in the spirit of Erdmann condition.

**Theorem B.6.** Given a curve \( \xi : [0, T] \rightarrow \mathbb{R}^N \) with constant speed, there is a reparametrization in the same interval with an \( a \)-Lagrangian parametrization, for some admissible \( a \).

**Proof.** See Theorem 3.16 and Remark 3.17 in [7].

**Proof of Theorem B.3.** By the usual existence argument, see Theorem 5.2, and the fact that spt \( \xi \) is compact, we find a minimizer \( \zeta_0 : [0, T] \rightarrow \mathbb{R}^N \) of the optimization problem in the statement. By Proposition B.5, \( \zeta_0 \) is the reparametrization of curve \( \eta_0 : [0, T] \rightarrow \mathbb{R}^N \) with constant speed. Taking into account (49), we have

\[
\int_0^T \tilde{L}(\zeta_0, \dot{\zeta}_0) dt \geq \int_0^T \sigma_a(\zeta_0, \dot{\zeta}_0) dt - a T = \int_0^T \sigma_a(\eta_0, \dot{\eta}_0) dt - a T \quad (50)
\]

for any admissible \( a \). By Theorem B.6, there is an admissible \( a \) and a reparametrization \( \zeta \) of \( \eta_0 \) with an \( a \)-Lagrangian parametrization. We therefore get

\[
\int_0^T \tilde{L}(\zeta, \dot{\zeta}) dt = \int_0^T \sigma_a(\zeta, \dot{\zeta}) dt - a T = \int_0^T \sigma_a(\eta_0, \dot{\eta}_0) dt - a T \leq \int_0^T \tilde{L}(\zeta_0, \dot{\zeta}_0) dt.
\]

We deduce from the optimality of \( \zeta_0 \) that equality must prevail in the above formula, as well as in (50). This implies the assertion.
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