Abstract

A generalized algebra of noncommutative coordinates and momenta embracing non-Abelian gauge fields is proposed. Through a two-dimensional realization of this algebra for a gauge field including electromagnetic vector potential and two spin-orbit-like coupling terms, a Dirac-like Hamiltonian in noncommutative coordinates is introduced. We established the corresponding energy spectrum and from that we derived the relation between the energy level quantum number and the magnetic field at the maxima of Shubnikov-de Haas oscillations. By tuning the noncommutativity parameter $\theta$ in terms of the values of magnetic field at the maxima of Shubnikov-de Haas oscillations, we accomplished the experimentally observed Landau plot of the peaks for graphene. Accepting that the experimentally observed behavior is due to the confinement of carriers, we conclude that our method of introducing noncommutative coordinates provides another formulation of the confined massless Dirac fermions in graphene.
1 Introduction

The recent experimental observations of the anomalous quantum Hall effect in monocrystalline graphite films of one atomic layer thickness \[1, 2\] revealed the fact that in this material, called graphene, electrons behave as effectively massless relativistic particles. Theoretically, this unexpected quantization of Hall conductivity can be explained in terms of the massless Dirac-like theory \[3, 4\]. On the other hand magnetic oscillations of electrical conductivity known as Shubnikov-de Haas (SdH) oscillations \[5\] were measured in patterned epitaxial graphene \[6\]. It was shown that its transport properties result from carrier confinement and coherence. Moreover, in \[6\], to explain the observed behavior of the maxima of SdH oscillations, an analytic expression for the energy levels which takes into account the confinement of the charge carriers due to the micrometer-scale of the sample, has been proposed. In fact, it is in accord with the theoretical study of confining massless Dirac fermions by introducing a coordinate dependent mass term \[7\].

Noncommutativity of coordinates naturally appear in Landau problem as well as in string theories where noncommutativity is proportional to the magnetic field in the former and to the background fields in the latter. However, one can also introduce noncommutativity of coordinates by the constant noncommutativity parameter \(\theta\) as an intrinsic property of the space. Then it is legitimate to consider noncommutative version of any physical system employing appropriate formulation. When noncommutativity of space is imposed without attributing a definite physical meaning to the noncommutativity parameter \(\theta\) one should provide its interpretation. Obviously, noncommutativity can be taken as a fundamental property of space by claiming that the noncommutativity parameter \(\theta\) is very small and it is responsible for the errors in the measurements of the related physical quantities. In this case each physical system will provide a different limit for the possible value of the noncommutativity parameter \(\theta\) (see e.g. \[8\] and references therein). However, there exists another interpretation: Noncommutative formulation of a dynamical system can be considered as a tool to link some diverse manifestations of a physical phenomena, e.g. obtaining the fractional Hall effect from the Hall effect in noncommutative coordinates by an appropriate choice of the noncommutativity parameter \(\theta\), as was reported in \[9\]. Because of retaining the terms up to a definite order in \(\theta\) one should show that in the dynamical problem considered the fixed value of \(\theta\) is in accord with this approximation. This interpretation of noncommutable coordinates as a linkage between different phenomena may give same clues of finding an easier method of formulating interacting systems from noninteracting theories \[10\]. Hence, we would like to study massless Dirac theory in two-dimensional noncommutative space to perceive whether a similar interpretation of noncommutativity is possible which can yield a better understanding of some peculiar properties of graphene. However, introducing noncommutative coordinates into spin-dependent Hamiltonian systems defined by constant matrices is not well established. The usual method of introducing noncommutativity is to replace ordinary products with star products which is equivalent to the shift

\[
x_\mu \to x_\mu - \frac{1}{2\hbar} \theta_{\mu\nu} p^\nu,
\]

where \((x_\mu, p_\mu)\) are the quantum phase space variables. Obviously, this method, which does not take into account spin degrees of freedom, is not suitable to deal with matrix valued, constant observables.

Utilizing the semiclassical techniques developed in \[11\], a method of introducing noncommutativity
of space appropriate to deal with non-Abelian spin matrices was presented in \[12\], \[8\]. However, this
semiclassical treatment is not amenable to introduce noncommutativity into Dirac equation of fermions
interacting with external fields. We propose a solution for this issue. First quantum commutation rela-
tions between phase space variables in noncommutative space suitable to consider non-Abelian fields
are presented. We derived them from the semiclassical brackets of classical phase space coordinates
proposed in \[12\]. Then, realizations of this algebra may be employed to introduce noncommutative
coordinates in Hamiltonian systems. It is applicable to systems where the interaction terms are co-
ordinate independent but non-Abelian because of being matrices. Obviously, this formalism can be
applied to Hamiltonian systems whose interaction terms are not matrices, so that it establishes an
alternative to the custom star product approach which is equivalent to the shift \[1\]. We then focus
on the confinement problem of massless Dirac particles in graphene and propose that introducing
noncommutative coordinates within our approach may be used to deal with one of its basic features:
the Landau plot of the maxima of SdH oscillations. To obtain the \(\theta\) deformation we give a realization
of the generalized canonical commutation relations in the presence of spin-orbit-like couplings and a
transverse magnetic field. We obtain the spectrum of the proposed Hamiltonian and show that it can
be used to formulate the SdH effect in graphene. By fixing the noncommutativity parameter \(\theta\) we
actually established a good agreement with the experimental observations which are known to result
from the confinement of massless Dirac fermions. In the last section we discuss the obtained results
and their future applications.

2 Generalized algebra

We would like to present a formulation of quantum mechanics in noncommutative coordinates acquired
by quantizing the semiclassical approach of \[11\]. Hence, let us briefly recall this formulation. When one
considers a dynamical system in the classical phase space \((P_I, Q_I), I = 1, \cdots, M\) one can introduce
the star product
\[
\ast = \exp \left[ \frac{i\hbar}{2} \left( \frac{\partial}{\partial Q^I} \frac{\partial}{\partial P_I} - \frac{\partial}{\partial P_I} \frac{\partial}{\partial Q^I} \right) \right],
\]
(2)
to achieve quantization of the system within the Weyl-Wigner-Groenewold-Moyal approach \[13\]. Cor-
responding to the quantum commutators one considers the Moyal bracket of the observables \(f(P, Q)\)
and \(g(P, Q)\) which are some functions as
\[
[f(P, Q), g(P, Q)]_\ast = f(P, Q) \ast g(P, Q) - g(P, Q) \ast f(P, Q).
\]
Poisson brackets follow in the classical limit,
\[
\lim_{\hbar \to 0} -\frac{i}{\hbar} [f(P, Q), g(P, Q)]_\ast = \{f(P, Q), g(P, Q)\} = \frac{\partial f}{\partial Q^I} \frac{\partial g}{\partial P_I} - \frac{\partial f}{\partial P_I} \frac{\partial g}{\partial Q^I}.
\]
(3)
For the matrix observables \(M_{kl}(P, Q)\) and \(N_{kl}(P, Q)\) the Moyal bracket can be defined as
\[
([M(P, Q), N(P, Q)]_\ast)_{kl} = M_{km}(P, Q) \ast N_{ml}(P, Q) - N_{km}(P, Q) \ast M_{ml}(P, Q).
\]
(4)
In contrary to the observables which are functions, matrix observables in general are not Abelian,
so that the classical limit \[3\] of their Moyal bracket \[1\] will yield, in general, a term behaving as
$h^{-1}$ which would be singular. However, if the observables are related to spin they may possess $h$ dependence, then the singularity which we mention is superfluous. To take into account this fact we define the “semiclassical” bracket

$$\{M(P, Q), N(P, Q)\}_C \equiv -\frac{i}{\hbar}[M, N] + \frac{1}{2}\{M(P, Q), N(P, Q)\} - \frac{1}{2}\{N(P, Q), M(P, Q)\}, \quad (5)$$

where one should retain the terms up to $\hbar$. The first term on the right hand side is the ordinary commutator of the matrices and the last two terms are Poisson brackets. The bracket (5) does not satisfy Jacobi identities. This is due to the fact that in its definition one keeps the first two terms of the fully-fledged Moyal bracket (4). However, in our approach the semiclassical limit is taken after multiplying the observables by the star product (2). Hence, we should consider the semiclassical limit of the Jacobi identity which is the first two terms of the Moyal bracket relation

$$-\frac{i}{\hbar}([M, \{N, L\}]_\star + \{N, \{L, M\}\}_\star + \{L, \{M, N\}\}_\star) = \mathcal{O}_1(\frac{1}{\hbar}) + \mathcal{O}_0(h^0) + \mathcal{O}(\hbar) + \cdots.$$  

In fact, one can show that the semiclassical limit of the Jacobi identity is satisfied

$$\mathcal{O}_1(\frac{1}{\hbar}) + \mathcal{O}_0(h^0) = -\frac{i}{\hbar}[M, [N, L]] + [M, \{N, L\}] - [M, \{L, N\}] + \{M, [N, L]\}$$

$$-\{[N, L], M\} + \text{(cyclic permutations of } M, N, L \text{)} = 0.$$  

Nevertheless, once we perform quantization and deal with quantum operators choosing a realization of quantum phase space variables, we should impose that they satisfy Jacobi identities.

The first order matrix Lagrangian adequate to formulate spin dynamics in noncommutative coordinates is

$$L = \dot{r}^\alpha \left[ \frac{p_\alpha}{2} + \rho A_\alpha(r) \right] - \frac{\theta_{\alpha\beta}}{\hbar} p_\beta r^\alpha - H_0(r, p) \quad (6)$$

where $\alpha, \beta = 1, \cdots, d$. We would like to emphasize that $A_\alpha$ is in general matrix valued. $\rho$ denotes the related coupling constant and $\mathbb{1}$ is the unit matrix. The constant, antisymmetric non-commutativity parameter $\theta_{\alpha\beta}$ appears divided by $\hbar$ to set its dimension at $(\text{length})^2$. The definition of canonical momenta

$$\pi_r^\alpha = \frac{\delta L}{\delta \dot{r}_\alpha}, \quad \pi_p^\alpha = \frac{\delta L}{\delta \dot{p}_\alpha}$$

yields the dynamical constraints

$$\psi_1^\alpha \equiv \left( \pi_r^\alpha - \frac{1}{2} p^\alpha \right) \mathbb{1} - \rho A_\alpha,$$

$$\psi_2^\alpha \equiv \left( \pi_p^\alpha + \frac{1}{2} r^\alpha \right) \mathbb{1} + \frac{\theta_{\alpha\beta}}{\hbar} p_\beta. \quad (8)$$

By setting $P_I \equiv (\pi_r^\alpha, \pi_p^\alpha), Q_I \equiv (r^\alpha, p^\alpha)$ in (5) one can show that the constraints (7) and (8) obey the semiclassical brackets

$$\{\psi_1^\alpha, \psi_1^\beta\}_C = \rho F_{\alpha\beta},$$

$$\{\psi_2^\alpha, \psi_2^\beta\}_C = \frac{\theta_{\alpha\beta}}{\hbar},$$

$$\{\psi_1^\alpha, \psi_2^\beta\}_C = -\delta_{\alpha\beta}.$$  

\text{---End---}
\( \delta_{\alpha \beta} \) is the Kronecker delta and \( F_{\alpha \beta} \) is the field strength,

\[
F_{\alpha \beta} = \frac{\partial A_{\beta}}{\partial r^\alpha} - \frac{\partial A_{\alpha}}{\partial r^\beta} - \frac{i\rho}{\hbar} [A_{\alpha}, A_{\beta}],
\]

(9)

where the last term is the ordinary matrix commutator. Thus, we may classify \( \psi^z_{\alpha} ; \ z = 1, 2 \), as second class constraints and the matrix whose elements are

\[
C_{\alpha \beta}^{zz'} = \{ \psi^z_{\alpha}, \psi^{z'}_{\beta} \},
\]

(10)

possesses the inverse \( C^{-1} \):

\[
C_{\alpha \gamma}^{zz''} C^{-1}_{\gamma \beta} = \delta_{\alpha}^{\beta} \delta_{z}^{z''}.
\]

(11)

The inverse matrix elements can be employed to define the “semiclassical Dirac bracket” as

\[
\{ M, N \}_{CD} \equiv \{ M, N \}_{C} - \{ M, \psi^z \}_{C} C_{zz'}^{-1} \{ \psi^{z'}_{N} \},
\]

(12)

so that the constraints (7) and (8) effectively vanish. The basic classical relations between the phase space variables following from (6) can be established, at the first order in \( \theta \) and keeping at most the second order terms in \( \rho \), as

\[
\{ r^\alpha, r^\beta \}_{CD} = \frac{\theta^{\alpha \beta}}{\hbar},
\]

(13)

\[
\{ p^\alpha, p^\beta \}_{CD} = \rho F^\alpha_{\gamma} - \frac{\rho^2}{\hbar} (\theta F)^{\alpha \beta},
\]

(14)

\[
\{ r^\alpha, p^\beta \}_{CD} = \delta^{\alpha \beta} - \frac{\rho}{\hbar} (\theta F)^{\alpha \beta},
\]

(15)

where \( (\theta F)^{\alpha \beta} \equiv F^\gamma_{\alpha \gamma} \), \( (\theta F \theta)^{\alpha \beta} \equiv \theta^{\alpha \gamma} F^\gamma_{\sigma} \theta^{\beta}_{\sigma} \). We omitted the identity matrix \( I \) on the left hand sides.

Indeed, in the sequel we will not write \( I \) explicitly.

The brackets (13)-(15) differ from the Poisson brackets up to commutators of matrices, so that for observables which are not matrices they reduce to the ordinary Dirac brackets. Therefore, we can extend the canonical quantization rules to embrace the matrix observables by substituting the basic brackets with the quantum commutators as \( \{ , \}_{CD} \rightarrow \frac{1}{\hbar} [ , ]_{q} \). To distinguish the matrix commutators and quantum commutation relations we denoted the latter as \( [ , ]_{q} \). This yields the generalized algebra

\[
[r^\alpha, \hat{r}^\beta]_{q} = i\theta^{\alpha \beta},
\]

(16)

\[
[p^\alpha, \hat{p}^\beta]_{q} = i\hbar F^\alpha_{\gamma} - i\rho^2 (\theta F)^{\alpha \beta},
\]

(17)

\[
[\hat{r}^\alpha, \hat{p}^\beta]_{q} = i\hbar \delta^{\alpha \beta} - i\rho (\theta F)^{\alpha \beta},
\]

(18)

\[
[p^\alpha, \hat{r}^\beta]_{q} = -i\hbar \delta^{\alpha \beta} + i\rho (\theta F)^{\alpha \beta}.
\]

(19)

Note that, on the right hand side we keep the first order \( \theta \) contributions, so that everything can only depend on \( x_{\alpha} \), defined as \( \hat{r}_{\alpha}|_{\theta=0} = x_{\alpha} \). For Abelian gauge fields this type of algebra has already been considered in [14] and a similar one in noncommutative space for an electromagnetic field was discussed in [15] (see also [16] and references therein).

One can employ realizations of the algebra (16)-(19) to introduce noncommutative coordinates. To illustrate it, let us deal with the commutative case \( \theta = 0 \) and let the gauge field be not a matrix.
but the 2–dimensional electromagnetic one \( a_i = (-Br_2/2, Br_1/2) \), which leads to a constant magnetic field transverse to the \((r_1, r_2)\)-plane \( B \). For these choices the algebra becomes
\[
[r_i, \hat{r}_i)_q = 0, \quad [\hat{p}_i, \hat{p}_j]_q = i\hbar B e_{ij}, \quad [\hat{r}_i, \hat{p}_j]_q = i\hbar \delta_{ij}.
\]

A realization of the algebra (20) is
\[
\hat{p}_i = -i\hbar \frac{\partial}{\partial r_i} + ea_i, \quad \hat{r}_i = r_i.
\]

Through the substitution of classical momenta with the realization (21), in the free Hamiltonian \( H_0 = p^2/2m \), the minimal coupling to the gauge field in quantum mechanics can be achieved as
\[
H_{\text{int}} \equiv H_0(\hat{p}, q) = \frac{1}{2m} \left( -i\hbar \frac{\partial}{\partial r} + e\vec{a} \right)^2.
\]

We will extend this point of view to define quantum mechanics in noncommuting coordinates.

Although we will employ another realization to propose a Hamiltonian adequate to describe graphene on the noncommutative plane, let us present a realization of (16)–(19). In terms of the covariant derivative
\[
D_\alpha = -i\hbar \frac{\partial}{\partial x_\alpha} - \rho A_\alpha \equiv -i\hbar \nabla_\alpha - \rho A_\alpha,
\]
we can realize the algebra (16)–(19) by setting
\[
\hat{p}_\alpha = D_\alpha - \frac{\rho}{2\hbar} F_{\alpha\beta} \theta_{\beta\gamma} D_\gamma,
\quad \hat{r}_\alpha = x_\alpha - \frac{1}{2\hbar} \theta_{\alpha\beta} D_\beta,
\]
as far as \( F_{\alpha\beta} \) are constant, commuting matrices which are equivalent to the conditions
\[
- i\hbar \nabla_\alpha F_{\beta\gamma} - \rho [A_\alpha, F_{\beta\gamma}] = 0, \quad [F_{\alpha\beta}, F_{\gamma\delta}] = 0.
\]

These conditions are also necessary to show that the realization (24)–(25) satisfies the Jacobi identities. We would like to emphasize that this realization is valid for either Abelian or non-Abelian \( A_\alpha \). It can be employed to introduce the related dynamical system in noncommutative coordinates as
\[
H(\theta) \equiv H(0)(\hat{r}, \hat{p}),
\]
where \( H(0)(r, p) \) is the free Hamiltonian appropriate to the considered system. Indeed, this constitutes an alternative method to the star-product approach of introducing noncommutative coordinates in quantum systems.

### 3 Dirac particles in noncommutative space

In graphene, around each Dirac point, which is the point at the corners of Brillouin zone, the free Hamiltonian is written as the massless Dirac-like Hamiltonian
\[
H^{(0)}_{\text{D}}(p, q) = v_F \vec{p} \cdot \vec{\sigma}
\]
for low energies and long wavelengths. Here, \( \vec{p} = (p_x, p_y) \) is the two-dimensional momentum operator and \( \vec{\sigma} = (\sigma_x, \sigma_y) \) where \( \sigma_{x,y,z} \) are the Pauli matrices acting on the states of two sublattices. \( v_F \) is the Fermi velocity playing the role of the speed of light in vacuum.

We would like to deal with the dynamics of the massless Dirac particle on the noncommutative \((x, y)\)-plane whose free Hamiltonian is \((28)\) within the method presented in the previous section. For this purpose let the gauge field be

\[
A_i = -\frac{eB}{2} \epsilon_{ij} x_j + ik \epsilon_{ij} \sigma_j + l \sigma_i, \quad i, j = 1, 2, \tag{29}
\]

which is non–Abelian. The first term corresponds to the transversal, constant magnetic field \( B \) and the others are spin-orbit-like coupling terms. However one should keep in mind that for graphene \( \vec{\sigma} \) act on the states of sublattices, so that though \( k \) and \( l \), respectively, like the coupling constants related to the Rashba and Dresslhauss spin-orbit interaction terms for electrons, their effect is to give rise to terms proportional to \( \sigma_z \) and unity in the non–deformed Hamiltonian. In fact, shifting momenta in \((28)\) with the gauge field \((29)\) yields the following Dirac–like Hamiltonian

\[
H_D = v_F (\vec{p} - \vec{A}) \cdot \vec{\sigma} - \frac{2}{h} eB^2 \theta + \frac{4i}{h} eB \theta (l^2 - k^2) \sigma_z.
\]

We would like to get a \( \theta \)-deformation of this Hamiltonian employing the procedure outlined in the previous section. Hence, we set \( \rho = 1 \) and by using the definition \((27)\) we obtain the field strength corresponding to \((29)\) as

\[
F_{ij} = \left( eB + \frac{2}{h} (l^2 - k^2) \sigma_z \right) \epsilon_{ij}. \tag{31}
\]

The algebra \((16) - (18)\) now becomes

\[
[\hat{r}_i, \hat{r}_j] = i \epsilon_{ij} \theta, \tag{32}
\]

\[
[\hat{p}_i, \hat{p}_j] = \left[ eB + \frac{2}{h} (l^2 - k^2) \sigma_z \right] \epsilon_{ij} + \left[ i eB^2 \theta + \frac{4i}{h} eB \theta (l^2 - k^2) \sigma_z \right] \epsilon_{ij}, \tag{33}
\]

\[
[\hat{p}_i, \hat{r}_j] = -ih \delta_{ij} \left[ 1 + \frac{\theta}{2l_B^2} + (l^2 - k^2) \frac{2\theta}{h^2} \sigma_z \right] \tag{34}
\]

where \( l_B^2 = \frac{\hbar}{eB} \). We deal with small \( l, k \), so that we neglect the terms at the order \( ln^k m \) for \( n + m \geq 4 \). Obviously, \((29)\) and \((31)\) do not satisfy the conditions \((26)\), so that one cannot make use of the realization \((24), (25)\). Nevertheless, we accomplish a realization of \((32) - (34)\) as follows:

\[
\hat{p}_i = \left[ 1 + \frac{\theta}{2l_B^2} + (l^2 - k^2) \frac{\theta}{h^2} \sigma_z \right] \left( -ih \nabla_i + \frac{eB}{2} \epsilon_{ij} x_j - ik \epsilon_{ij} \sigma_j + l \sigma_i \right) + (l^2 - k^2) \frac{\theta}{h^2} \epsilon_{nm} x_n \left( -ih \nabla_m \right) (ik \epsilon_{ij} \sigma_j + l \sigma_i), \tag{35}
\]

\[
\hat{r}_i = \left[ 1 + \frac{\theta}{2l_B^2} + (l^2 - k^2) \frac{\theta}{h^2} \sigma_z \right] x_i - \frac{\theta}{2\hbar} \epsilon_{ij} \left( -ih \nabla_j - \frac{eB}{2\hbar} \epsilon_{jn} x_n \right) \tag{36}
\]

\[
-\frac{\theta}{h^2} (l^2 - k^2) \epsilon_{ij} \left[ (ik \epsilon_{jn} \sigma_n + l \sigma_j) x_m - 2 (ik \epsilon_{nm} \sigma_m + l \sigma_n) x_n x_j \right].
\]

One can demonstrate that \((35)\) and \((36)\) satisfy the Jacobi identities at the first order in \( \theta \) and ignoring the terms at the order of \( ln^k m \) for \( n + m \geq 4 \).
Through the procedure outlined with (27) θ-deformation of the Hamiltonian (30) can be achieved by substituting momenta with the realization (35) in the Hamiltonian (28) as

\[ H_D^{(θ)} = \frac{v_F}{2} \left[ \hat{\mathbf{p}} \cdot \hat{\mathbf{σ}} + (\hat{\mathbf{p}} \cdot \hat{\mathbf{σ}})^\dagger \right]. \] (37)

Indeed, plugging (35) into (37) yields

\[
H_D^{(θ)} = v_F \left( 1 + \frac{θ}{2l_B^2} \right) \left( -i\hbar \nabla_i + \frac{eB}{2} \epsilon_{ij} x_j \right) \sigma_i - 2v_F \left[ 1 + \frac{θ}{2l_B^2} + (l^2 - k^2) \frac{θ}{\hbar^2} \sigma_z \right. \\
\left. + 2(l^2 - k^2) \frac{θ}{\hbar^2} \epsilon_{nm} x_n (-i\hbar \nabla_m) \right] (\sigma_z + 1).
\] (38)

One can observe that for \( θ = 0 \), it yields the Hamiltonian given in (30). In [19] noncommutative structure emerging in graphene was studied where the noncommutativity parameter considered is due to a lattice distortion term present in the Hamiltonian. In fact, it is similar to the mass term, so that it is related the our constant parameter \( k \). However, it is like a coordinate dependent mass term because they also consider a nonconstant distortion.

To establish energy eigenvalues it is convenient to write (38) in terms of the complex variables \( z = x + iy, \ \bar{z} = x - iy \) as

\[
H_D^{(θ)} = \begin{pmatrix}
g_+ - h_+ L_z & iK \left( -2h \nabla_z + \frac{eB}{2} \bar{z} \right) \\
iK \left( 2h \nabla_z + \frac{eB}{2} z \right) & g_- - h_- L_z
\end{pmatrix}
\] (39)

where \( L_z = -i\epsilon_{ij} x_i \nabla_j = h(z \nabla_z - \bar{z} \nabla_{\bar{z}}) \) is the angular momentum operator. The involved constants are defined as

\[
g_{±} = -2v_F (l ± k) \left[ 1 + \frac{θ}{2l_B^2} ± \frac{θ}{\hbar^2} (l^2 - k^2) \right], \quad h_{±} = 4v_F \frac{θ}{l_B^2} (l ± k)(l^2 - k^2), \quad K = v_F \left( 1 + \frac{θ}{2l_B^2} \right).
\]

To derive the eigenvalues of (39) algebraically, we introduce two pairs of annihilation and creation operators:

\[
a = -\frac{d}{\sqrt{2n}} \left( 2h \nabla_z + \frac{eB}{2} z \right), \quad a^\dagger = \frac{d}{\sqrt{2n}} \left( -2h \nabla_z + \frac{eB}{2} \bar{z} \right), \\
b = -\frac{d}{\sqrt{2n}} \left( 2h \nabla_z + \frac{eB}{2} \bar{z} \right), \quad b^\dagger = \frac{d}{\sqrt{2n}} \left( -2h \nabla_z + \frac{eB}{2} z \right),
\]

which are mutually commuting and satisfy the commutation relations

\[
[a, a^\dagger] = [b, b^\dagger] = 1.
\]

Hence, the Hamiltonian (39) acquires the form

\[
H_D^{(θ)} = \begin{pmatrix}
g_+ - h_+(b^\dagger b - a^\dagger a) & \tilde{K} a^\dagger \\
\tilde{K} a & g_- - h_- (b^\dagger b - a^\dagger a)
\end{pmatrix}
\]

where \( \tilde{K} = 2v_F eB \left( 1 + \frac{θ}{2l_B^2} \right) \). The eigenvalue equation for the two component spinor

\[
H_D^{(θ)} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = E \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}.
\]
leads to two coupled equations

\[
\begin{align*}
[g_+ - h_+ (b^\dagger b - a^\dagger a) - E] \psi_1 &= -\tilde{K} a^\dagger \psi_2, \\
[g_- - h_- (b^\dagger b - a^\dagger a) - E] \psi_2 &= -\tilde{K} a \psi_1.
\end{align*}
\]

(40) (41)

After some calculation, one can show that the equation satisfied by the spinor component \(\psi_1\) takes the form

\[
\left[ E^2 + 4E \left( Kl + \frac{\theta v_F}{\hbar^2} l (l^2 - k^2)(1 + 2b^\dagger b + 2a^\dagger a) \right) - \tilde{K}^2 a^\dagger a + 4\tilde{K}^2 (l^2 - k^2) \right] \psi_1 = 0.
\]

(42)

To draw the energy eigenvalues, let us write the state corresponding to the spinor component \(\psi_1\) as

\[
|\psi_1\rangle = \frac{1}{\sqrt{n!(m + n)!}} (b^\dagger)^m (a^\dagger)^n |0\rangle;
\]

(43)

where \(m, n = 0, 1, 2 \cdots\), and by definition \(a|0\rangle = b|0\rangle = 0\). In the complex plane (43) yields

\[
\langle z, \bar{z} | n, m \rangle = N_{mn} z^m L^m_n \left( \frac{z\bar{z}}{2} \right) e^{-\frac{1}{4} z\bar{z}}
\]

where \(L^m_n\) are the Laguerre polynomials and \(N_{mn}\) are the normalization constants whose explicit forms are not needed.

Obviously, (43) satisfies the relations

\[
(b^\dagger b - a^\dagger a) | n, m \rangle = m | n, m \rangle,
\]

\[
a^\dagger a | n, m \rangle = n | n, m \rangle,
\]

where \(m\) and \(n\) are the quantum numbers corresponding, respectively, to the angular momentum eigenvalues and the Landau levels. Now, (42) can be solved to deduce the energy spectrum as

\[
E_{n,m}(k, l, \theta, B) = \pm 2v_F \left( 1 + \frac{\theta}{2l_B^2} \right) \sqrt{\frac{\hbar^2}{2l_B^2} n + k^2} - 2v_F l \left[ 1 + \frac{\theta}{2l_B^2} + \frac{\theta}{\hbar^2} (l^2 - k^2)(2m + 1) \right].
\]

(44)

Moreover, one can show that the corresponding spinor components are given by

\[
\Psi_{n,m} = \left( \begin{array}{c} n, m \rangle \\ s' | n - 1, m + 1 \rangle \end{array} \right)
\]

with the convention \(\psi_{-1,m} \equiv 0\). Here \(s'\) is a constant, which can be read from (41).

4 Shubnikov-de Haas effect

The Shubnikov-de Haas (SdH) effect is a magnetotransport phenomena that occurs in materials in a strong magnetic field of about 1 Tesla and for low temperature about few kelvins [5]. It is an oscillatory dependence of the electrical resistivity of a metal or a semiconductor as a function of the applied constant magnetic field. More precisely, the SdH effect is produced by the oscillations of the density of states at the Fermi level. The mechanism can be understood for metals considering Landau levels [20] which are the energy levels of electrons in the presence of a magnetic field. If the electrons
fill the energy levels up to the level \( n + 1 \), the Fermi energy which is equal to the chemical potential at absolute zero, will lie in this level. As the magnetic field increases the degeneracy of the Landau level increases. Thus the electrons move to the level \( n \), depopulating the level \( n + 1 \), so that the Fermi energy is decreased. Now, increasing magnetic field leads to less populated Fermi level until all electrons migrate to the lower energy level. Hence, the conductance or the resistivity will oscillate as a function of the external magnetic field. The maxima of the SdH effect occur at the magnetic fields \( B_N \) which can be calculated by equating the energy level corresponding to the index \( N \) with the chemical potential \( \mu \) (Fermi energy). Hence, the relation between \( N \) and \( B_N \) predicted by our approach is established as

\[
N = \frac{1}{2e\hbar B_N} \left[ \frac{\mu^2}{v_F^2} + 4\frac{\mu}{v_F} l + 4(l^2 - k^2) + \frac{\tilde{\theta}}{\hbar^2} \left( \frac{2}{e\hbar B_N} l(k^2 - l^2)(2m + 1) + \frac{\mu}{2v_F} + l \right) \right],
\]

by solving the equation \( E_{N,m}(k,l,\theta,B_N) = \mu \) obtained from (44). For convenience, we rescaled the noncommutativity parameter as

\[
\theta = -\frac{v_F}{\mu} \tilde{\theta}.
\]

To analyze the SdH effect in graphene within our formulation we shall choose the involved parameters adequately. To start with, we require that the spin-orbit-like coupling constants obey

\[
l = -\frac{\mu}{2v_F} + k.
\]

With this choice (45) is simplified and takes the form

\[
N = \frac{\tilde{\theta}}{\hbar^2} \left( B(m,k) + k \right)
\]

where we defined

\[
B(m,k) = \frac{\mu}{v_F e\hbar} \left( \frac{\mu}{2v_F} - 2k \right) \left( \frac{\mu}{2v_F} - k \right) (1 + 2m).
\]

Since the noncommutativity parameter \( \theta \) is a free parameter, it can be fixed in diverse fashions. However, one should keep in mind that its value should be consistent with the approximation of retaining the terms up to the first order in \( \theta \). In particular, for the limiting values of \( B_N \), we propose to choose \( \tilde{\theta} \) as

\[
\tilde{\theta}(B) = \begin{cases} 
\beta/B_N, & B_N > B(m,k)/k \\
\gamma B_N, & B_N \ll B(m,k)/k
\end{cases}
\]

where \( \gamma, \beta \) are two constants and we assume that \( k \neq 0 \). We can analyze separately for each case given in (47). For \( B_N > B(m,k)/k \) we deduce the behavior

\[
N_> = \frac{\beta k}{\hbar^2} \frac{1}{B_N},
\]

by neglecting a term behaving as \( 1/B_N^2 \). Thus, for large \( B_N \), \( N \) changes linearly with respect to \( 1/B_N \). However, in the second case, \( B_N \ll B(m,k)/k \), \( N \) leads to the constant value

\[
N_< = \frac{\gamma B(m,k)}{\hbar^2}.
\]
Let us link these considerations to the experimental observations of [6]. They obtained the limiting values

\[ N_{\text{exp}} = \begin{cases} \frac{B_0}{B_N}, & B_N > 2.5 \text{ T} \\ 25, & B_N \ll 2.5 \text{ T} \end{cases} \]  

(50)

where the constant is given by

\[ B_0 = \frac{\mu^2}{2e\hbar v_F} \approx 35 \text{ T}. \]

This fixes the ratio

\[ \frac{\mu}{v_F} \approx 34 \times 10^{-27} \text{ kg.m/s}. \]

Now, we would like to determine the value of the noncommutativity parameter \( \theta \) comparing (50) with (48) and (49) for \( m = 0 \). The other values of \( m \) can be treated similarly. First of all observe that we may impose

\[ \frac{B(0, k)}{k} = 2.5 \text{ T}. \]  

(51)

To simplify let \( k = (\mu/2v_F)\delta \), so that (51) yields the equation

\[ 2\delta^2 - \left( 3 + \frac{2.5}{B_0} \right) \delta + 1 = 0 \]

whose solutions are

\[ \delta \approx 0.77 \pm 0.3. \]

Hence, we may set

\[ k = 1 \times 10^{-26} \text{ kg.m/s} \]

which implies to choose

\[ \beta \approx 4 \times 10^{-41} \text{ JmT}, \quad \gamma \approx 1 \times 10^{-41} \text{ JmT}^{-1}. \]

It worths to observe that the magnitude of the noncommutativity parameter for the limiting cases \([47]\) reads

\[ |\theta(B)| = \begin{cases} B_N^{-1} \times 10^{-15} \text{ m}^2, & B_N > 2.5 \text{ T} \\ B_N \times 10^{-16} \text{ m}^2, & B_N \ll 2.5 \text{ T} \end{cases} \]

Therefore, there is no conflict with keeping the terms up to the first order in \( \theta \). Until now we dealt with the values of \( \theta \) for the limiting values of the magnetic field \( B_N \). However, we can also choose it appropriately for all values of \( B_N \). Inserting the choice (51) into (46) yields

\[ N = \frac{\tilde{k}}{\hbar^2} \left( \frac{2.5 \text{ T}}{B_N} + 1 \right). \]

(52)

To write the full expression for \( \tilde{\theta} \), let us introduce the Heaviside step function

\[ H(x) = \begin{cases} 0, & x < 0 \\ 1/2, & x = 0 \\ 1, & x > 0 \end{cases} \]
which can be given analytically as \[21\]

\[
H(x) = \lim_{\ell \to 0} \left[ \frac{1}{2} + \frac{1}{\pi} \tan^{-1} \frac{x}{t} \right].
\]

We choose the noncommutativity parameter to be

\[
\tilde{\theta} = \frac{\hbar^2 / k}{1 + 2.5B_N^{-1}} \left\{ 35B_N^{-1} \left[ \frac{1}{2} + \frac{1}{\pi} \tan^{-1} \left( \frac{0.4 - B_N^{-1}}{0.01} \right) \right] + \frac{53}{0.9 + B_N} \left[ \frac{1}{2} + \frac{1}{\pi} \tan^{-1} \left( \frac{B_N^{-1} - 0.4}{0.01} \right) \right] \right\}
\times \left[ \frac{1}{2} + \frac{1}{\pi} \tan^{-1} \left( \frac{0.83 - B_N^{-1}}{0.01} \right) \right] + 24.8 \left[ \frac{1}{2} + \frac{1}{\pi} \tan^{-1} \left( \frac{B_N^{-1} - 0.83}{0.01} \right) \right]
\]

which produces the limiting values correctly and with this choice (52) yields Figure 1 (using MATHEMATICA). Indeed, we have chosen (53) appropriately so that the Landau plot of the peaks given in Figure 1 matches well with the experimental one obtained in [6]. Moreover, one can check that the order of magnitude of the noncommutativity parameter is \( \theta \approx 10^{-16} \) m\(^2\), so that it is in accord with the approximation of ignoring the second order terms in \( \theta \).

5 Discussions

The results which we obtained are twofold:

- A new method of introducing noncommutative coordinates into quantum mechanics is established.

- An analytic method of obtaining confinement of massless Dirac particles in graphene is proposed.

We introduced a generalized algebra of quantum phase space operators in noncommuting space on general grounds with momenta involving non-Abelian gauge fields. This constitutes an alternative to the custom method of introducing noncommuting coordinates by star products. It may lead to some new features of quantum mechanics in noncommutative coordinates. Moreover, it should be possible to extend it to field theory formulations. These are currently under inspection.

We considered a two-dimensional space by a particular choice of gauge fields. A realization of the associated algebra is presented and employed to obtain a massless Dirac-like Hamiltonian on the
noncommutative plane. Its energy eigenvalues are established. Through an appropriate choice of the noncommutativity parameter $\theta$ we showed that this energy spectrum is adequate to accomplish the experimentally observed behavior of the SdH oscillations in graphene, which are known to result due to the confinement of its charge carriers which are massless Dirac particles. Obviously, our main objective is to employ this noncommutative theory to understand those features of graphene which are not well understood within other formalisms. This work should be considered as the first step in this direction. We obtained a satisfactory noncommutative version of Dirac-like theory of graphene which led to some predictions. One of the next steps would be to obtain a field theory in terms of the Hamiltonian (38), which can be used to introduce other interactions like the spin of electron as in [22] into the noncommutative theory.

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