Optimal Paired Comparison Experiments for Second-Order Interactions

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Abstract
In real life situations often paired comparisons involving alternatives of either full or partial profiles to mitigate cognitive burden are presented. For this situation the problem of finding optimal designs is considered in the presence of second-order interactions when all attributes have general common number of levels.

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1 Introduction
Paired comparisons are closely related to experiments with choice sets of size two, which are widely used in many fields of applications like health economics, transportation economics and marketing to study people’s or consumer’s preferences towards new products or services where behaviors of interest involve qualitative responses (see e.g., Bradley and Terry, 1952) or quantitative responses (see e.g., Scheffé, 1952). This paper draws on the latter case of quantitative responses (so called conjoint analysis where responses are usually assessed on a rating scale) as frequently encountered in marketing research (Green and Srinivasan, 1990).

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Typically, paired comparisons involve respondents choosing pairs of competing options (alternatives) in a hypothetical (occasionally real) setting which are generated by an experimental design, and are described by a number of attributes. Usually, in applications one may be interested in possible interactions between the attributes. For example, Bradley and El-Helbawy (1976) considered an analysis for a $2^3$-factorial experiment on coffee preferences, where the three attributes were brew strength, roast color and coffee brand, and main effects, two- and three-attribute interactions were of special interest (see also El-Helbawy and Bradley, 1978). This work serves as a motivation for the present paper where any of three of the attributes interact.

Due to the limited cognitive ability to process information in applications a choice task including many attributes may result in respondent decisions that do not reflect their actual preferences. A way to overcome these behaviors is to simplify the choice task by holding the levels of some of the attributes constant in every choice set. The profiles in such a choice set are called partial profiles, and the number of attributes that are allowed with potentially different levels in the partial profiles is called the profile strength (e.g., see Green, 1974; Grasshoff et al., 2003; Chrzan, 2010; Kessels et al., 2011).

In this paper we mainly introduce an appropriate model for the situation of full and partial profiles and derive optimal designs in the presence of three-attribute interactions. We consider the case when the alternatives are specified by general common number of level-attributes. Work on determining the structure of the optimal designs by the two-level situation when the first- and last-levels of the attributes were effects-coded as 1 and $-1$, respectively has been thoroughly investigated by van Berkum (1987) in the case of full profiles in a main-effects and first-order (two-attribute) interactions setup. These designs remain optimal (e.g., see Street et al., 2001) for full profiles and by Schwabe et al. (2003) for partial profiles. Corresponding results when the common number of the attribute levels is larger than two have been obtained by Grasshoff et al. (2003) in a first-order interactions setup for both full and partial profiles. Here, we treat the case of second-order (three-attribute) interactions when the common number of the attribute levels is larger than two and provide a detailed proof. The two-level situation has been investigated by Nyarko and Schwabe (2019) in the case of both full and partial profiles.

It is worthwhile mentioning that the invariant designs considered in this paper possess large number of comparisons. However, these designs can serve as a benchmark to judge the efficiency of competing designs as well as a starting point to construct exact designs or fractions which share the property of optimality and can be realized with a reasonable number of comparisons.
The remainder of the paper is organized as follows. A general model is introduced in Section 2 for linear paired comparisons which is related to a choice set of size two. This is followed by a second-order interactions model for full and partial profiles in Section 3 and optimal designs are characterized in Section 4. The final Section 5 offers some conclusions. All major proofs are deferred to the Appendix.

2 General setting

In any experimental situation the outcome of the experiment depends on some factors (attributes), say, $K$ of influence. In this setting the dependence can be described by a functional relationship $f$ which quantifies the effect of the alternative $i = (i_1, \ldots, i_K)$ for $k = 1, \ldots, K$ of the attributes of influence where $i_k$ is the component of the $k$th attribute. Any observation (utility) $\tilde{Y}_{na}(i)$ of a single alternative $i = (i_1, \ldots, i_K)$ within a pair of alternatives $(a = 1, 2)$ subject to a block effect $\mu_n$ and a random error $\tilde{\varepsilon}_{na}$, which is assumed to be uncorrelated with constant variance and zero mean can be formalized by a general linear model

\[
\tilde{Y}_{na}(i) = \mu_n + f(i)^\top \beta + \tilde{\varepsilon}_{na},
\]

where the index $n$ denotes the $n$th presentation, $n = 1, \ldots, N$, and the alternative $i$ is chosen from a set $\mathcal{I}$ of possible realizations. Here $f$ is a vector of known regression functions which describe the form of the functional relationship between the alternative $i$ and the corresponding mean response $E(\tilde{Y}_{na}(i)) = \mu_n + f(i)^\top \beta$, and $\beta$ is the unknown parameter vector of interest. Usually in order to make statistical inference on the unknown parameters more than one alternative is presented in a choice set to get rid of the influence of the presentation effect $\mu_n$ due to a variety of unobservable influences. Then actually differences of the latent utilities are observed for the alternatives presented in a choice set.

More specifically, unlike in standard experimental designs where there is a possibility of only a single or direct observation, in paired comparison experiments the utilities for the alternatives are usually not directly observed. Only observations $Y_n(i, j) = \tilde{Y}_{n1}(i) - \tilde{Y}_{n2}(j)$ of the amount of preference are available for comparing pairs $(i, j)$ of alternatives $i$ and $j$ which are chosen from the design region $\mathcal{X} = \mathcal{I} \times \mathcal{I}$. In that case the utilities for the alternatives are properly described by the linear paired comparison model

\[
Y_n(i, j) = (f(i) - f(j))^\top \beta + \varepsilon_n,
\]
where $f(i) - f(j)$ is the derived regression function and the random errors 
$\varepsilon_n(i, j) = \hat{\varepsilon}_n^1(i) - \hat{\varepsilon}_n^2(j)$ associated with the different pairs $(i, j)$ are assumed 
to be uncorrelated with constant variance and zero mean. Here, the block 
effects $\mu_n$ are immaterial.

The performance of the statistical analysis based on a paired comparison 
experiment depends on the pairs (alternatives) in the choice sets that are 
presented. The choice of such pairs $(i_1, j_1), \ldots, (i_N, j_N)$ is called a design of 
size $N$. The quality of such a design is measured by its information matrix

$$M((i_1, j_1), \ldots, (i_N, j_N)) = \sum_{n=1}^{N} M((i_n, j_n)), \quad (3)$$

where $M((i, j)) = (f(i) - f(j))(f(i) - f(j))^\top$ is the so-called elemental information 
of a single pair $(i, j)$.

In this paper we restrict our attention to approximate or continuous designs $\xi$ (e.g., see Kiefer, 1959) which are defined as discrete probability measures on the design region $X$ of all pairs $(i, j)$. Moreover, every approximate design $\xi$ which assigns only rational weights $\xi(i, j)$ to all pairs $(i, j)$ in its support points can be realized as an exact design $\xi_N$ of size $N$ consisting of the pairs $(i_1, j_1), \ldots, (i_N, j_N)$.

The information matrix of an approximate design $\xi$ in the linear paired 
comparison model (2) is defined by

$$M(\xi) = \sum_{(i, j) \in X} \xi(i, j)(f(i) - f(j))(f(i) - f(j))^\top, \quad (4)$$

which is proportional to the inverse of the covariance matrix for the best 
linear unbiased estimator of the parameter vector $\beta$. Note that for an exact 
design $\xi_N = ((i_1, j_1), \ldots, (i_N, j_N))$ the normalized information matrix $M(\xi_N)$ 
coincides with the information matrix $M(\xi)$ of the corresponding approximate 
design $\xi$.

Optimality criteria for approximate designs $\xi$ are functionals of $M(\xi)$. 
As a scalar measure of design quality here we consider the criterion of $D$-
optimality. An approximate design $\xi^*$ is $D$-optimal if it maximizes the deter-
minant of the information matrix, that is, if $\det M(\xi^*) \geq \det M(\xi)$ for every approximate design $\xi$ on $X$. 

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Example 1. One-way layout (general levels)
For illustrative purposes we first consider the situation of just a single-attribute \((K = 1)\) (e.g., see Graßhoff et al., 2004) which may vary only over \(v\) levels, and adopt the standard parameterization of effects-coding. In this setting, the effects of each single level \(i = 1, \ldots, v\) has parameters \(\alpha_i\) satisfying the identifiability condition \(\sum_{i=1}^{v} \alpha_i = 0\). Hence,

\[
\tilde{Y}_{na}(i) = \mu_n + \alpha_i + \tilde{\epsilon}_{na},
\]

with \(i \in \mathcal{I} = \{1, \ldots, v\}\), where \(\mu_n\) denotes the block effects, \(n = 1, \ldots, N\), and \(\tilde{\epsilon}_{na}\) the random error is assumed to be uncorrelated with constant variance and zero mean. For effects-coding the regression function \(f = f_1\) is given by \(f_1(i) = e_i\), \(i = 1, \ldots, v - 1\) and \(f_1(v) = -1_{v-1}\), respectively, where \(e_i\) is the \(i\)th unit vector of length \(v - 1\) and \(1_m\) denotes a vector of length \(m\) with all entries equal to 1. This parameterization ensures the usual identifiability condition where the parameter relating to the last level \(v\) can be recovered from the other levels, \(\alpha_v = -\sum_{i=1}^{v-1} \alpha_i\) such that

\[
\tilde{Y}_{na}(i) = \mu_n + f_1(i)\beta + \tilde{\epsilon}_{na},
\]

where the reduced parameter vector \(\beta = (\alpha_1, \ldots, \alpha_{v-1})^\top\).

Then for paired comparisons an observation of the effects \(\alpha_i - \alpha_j\) of level \(i\) compared to level \(j\) can be characterized by the response

\[
Y_n(i, j) = (f_1(i) - f_1(j))\beta + \epsilon_n = \alpha_i - \alpha_j + \epsilon_n,
\]

where \(f_1(i, j) = e_i - e_j\), \(f_1(i, v) = e_i + 1_{v-1}\), \(f_1(v, i) = -f_1(i, v)\), for \(i, j = 1, \ldots, v - 1\) and \(f_1(v, v) = 0\).

Note that \(M((i, j)) = \frac{2}{v-1}(1_{v-1}1_{v-1}^\top)\) for \(i \neq j\) where \(1_m\) denotes the \(m\)-dimensional identity matrix, while \(M((i, i)) = 0\). From this it is obvious that only pairs with different levels should be used and that, in particular, the design \(\bar{\xi}\) which assigns equal weight \(1/(v(v - 1))\) to each of the pair \((i, j), i \neq j\) is optimal with resulting information matrix

\[
M(\bar{\xi}) = \frac{2}{v-1}(1_{v-1}1_{v-1}^\top).
\]

The corresponding information matrix \(M(\bar{\xi})\) has an inverse of the form

\[
M(\bar{\xi})^{-1} = \frac{v-1}{2}(1_{v-1} - \frac{1}{v}1_{v-1}1_{v-1}^\top).
\]

This design \(\bar{\xi}\) will serve as a brick for constructing optimal designs in situations with more than one attribute later on. In particular, this concept
developed for the single-attribute will be extended later on for describing the main-effects as well as the two- and the three-attribute interactions in the case of a couple of, say, \( K \) attributes \( k = 1, \ldots, K \).

It is worthwhile mentioning that under the assumption \( \beta = 0 \) (see e.g. Großmann et al., 2002) the designs considered in this paper carry over to the Bradley and Terry (1952) type choice experiments in which the probability of choosing \( i \) from the pair \((i, i)\) is given by \( \exp[f(i)\top \beta]/(\exp[f(i)\top \beta] + \exp[f(j)\top \beta]) \). Specifically, this assumption simplifies the information matrix of the binary logit model which becomes proportional to the information matrix in the linear paired comparison model (e.g., see Großmann and Schwabe, 2015; Singh et al., 2015).

### 3 Second-order interactions model

In applications one may be interested in the utility estimates of both the main effects and interactions between the levels of the components (attributes). For that setting optimal designs have been derived (van Berkum, 1987; Graßhoff et al., 2003) in a first-order interactions setup. This paper considers a second-order interactions model. Corresponding results for the particular case of binary attributes can be found in Nyarko and Schwabe (2019).

Following Nyarko and Schwabe (2019) we first start with the situation of full profiles. In that case each alternative is represented by level combinations in which all attributes are involved. For such alternatives in a choice set of size two, we denote by \( i = (i_1, \ldots, i_K) \) and \( j = (j_1, \ldots, j_K) \) the first alternative and the second alternative, respectively, which are both elements of the set \( \mathcal{I} = \{1, \ldots, v\}^K \) where \( 1 \) and \( v \) represent the first and last level of each \( k \)th component, \( k = 1, \ldots, K \). Here the choice set \((i, j)\) is an ordered pair of alternatives \( i \) and \( j \) which is chosen from the design region \( \mathcal{X} = \mathcal{I} \times \mathcal{I} \). Note that for each component (attribute) \( k \) the corresponding regression functions \( f_k = f_1 \) as well as the marginal model coincides with that of one attribute as introduced in Example 1.

In the presence of up to second-order interactions direct responses \( \tilde{Y}_{na} \) at alternative \( i = (i_1, \ldots, i_K) \) can be modeled as

\[
\tilde{Y}_{na}(i) = \mu_n + \sum_{k=1}^{K} f_1(i_k)\top \beta_k + \sum_{k<\ell} (f_1(i_k) \otimes f_1(i_\ell))\top \beta_{k\ell} + \sum_{k<\ell<m} (f_1(i_k) \otimes f_1(i_\ell) \otimes f_1(i_m))\top \beta_{k\ell m} + \tilde{\varepsilon}_{na}, \tag{10}
\]

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for full profiles, where \( \otimes \) denotes the Kronecke product of vectors or matrices, \( \beta_k = (\beta_{ik}^{(k)})_{i=1,...,v-1} \) denotes the main effect of the \( k \)th attribute, \( \beta_{k\ell} = (\beta_{ik\ell})_{i=1,...,v-1, \ell=1,...,v-1} \) is the first-order interaction of the \( k \)th and \( \ell \)th attribute, and \( \beta_{k\ell m} = (\beta_{ik\ell m})_{i=1,...,v-1, \ell=1,...,v-1, m=1,...,v-1} \) is the second-order interaction of the \( k \)th, \( \ell \)th and \( m \)th attribute. The vectors \( (\beta_k)_{1 \leq k \leq K} \) of main effects, \( (\beta_{k\ell})_{1 \leq k < \ell \leq K} \) of first-order interactions and \( (\beta_{k\ell m})_{1 \leq k < \ell < m \leq K} \) of second-order interactions have dimensions \( p_1 = K(v-1), p_2 = (1/2)K(K-1)(v-1)^2 \) and \( p_3 = (1/6)K(K-1)(K-2)(v-1)^3 \), respectively. Hence the complete parameter vector

\[
\beta = ((\beta_k)^\top)_{k=1,...,K}, (\beta_{k\ell})^\top_{k<\ell}, (\beta_{k\ell m})^\top_{k<\ell<m}, \tag{11}
\]

has dimension \( p = p_1 + p_2 + p_3 = K(v-1)(1 + (1/6)(K-1)(v-1)(3 + (K-2)(v-1)) \)). The corresponding \( p \)-dimensional vector \( \mathbf{f} \) of regression functions is given by

\[
\begin{align*}
\mathbf{f}(i) &= (\mathbf{f}_1(i_1)^\top, \ldots, \mathbf{f}_1(i_K)^\top, \mathbf{f}_1(i_1)^\top \otimes \mathbf{f}_1(i_2)^\top, \ldots, \mathbf{f}_1(i_{K-1}) \otimes \mathbf{f}_1(i_K)^\top, \\
&\quad \mathbf{f}_1(i_1)^\top \otimes \mathbf{f}_1(i_2)^\top \otimes \mathbf{f}_1(i_3)^\top, \ldots, \mathbf{f}_1(i_{K-2})^\top \otimes \mathbf{f}_1(i_{K-1})^\top \otimes \mathbf{f}_1(i_K)^\top)^\top.
\end{align*}
\]

(12)

Also here in \( \mathbf{f}(i) \), the first \( K \) components \( \mathbf{f}_1(i_1), \ldots, \mathbf{f}_1(i_K) \) are associated with the main effects and have \( p_1 \), the second components \( \mathbf{f}_1(i_1) \otimes \mathbf{f}_1(i_2), \ldots, \mathbf{f}_1(i_{K-1}) \otimes \mathbf{f}_1(i_K) \) are associated with the first-order interactions and have \( p_2 \), and the remaining components \( \mathbf{f}_1(i_1) \otimes \mathbf{f}_1(i_2) \otimes \mathbf{f}_3(i_3), \ldots, \mathbf{f}_1(i_{K-2}) \otimes \mathbf{f}_1(i_{K-1}) \otimes \mathbf{f}_1(i_K) \) are associated with the second-order interactions and have \( p_3 \).

Due to the limited information processing capacity, in applications a choice task including many attributes may enhance respondent decisions that do not reflect their actual preferences. To overcome these behaviors, only partial profiles are presented within a single paired comparison. Specifically, in a partial profile every choice set consists of alternatives which are described by a predefined number \( S \) of attributes, where the same set of attributes is used throughout both alternatives within a choice set but with potentially different levels, and the remaining \( K - S \) attributes are not shown and remain thus unspecified. The number \( S \) of attributes used in a partial profile is called the profile strength.

For a partial profile a direct observation may be described by model \([10]\) when summation is taken only over those \( S \) attributes contained in the describing subset. This requires that the profile strength \( S \geq 3 \) is needed to ensure identifiability of the second-order interactions. In what follows, we introduce an additional level \( i_k = 0 \), which indicates that the corresponding \( k \)th attribute is not present in the partial profile. The corresponding regression functions are extended to \( \mathbf{f}_k(i_k) = 0 \). With this convention a direct
observation can be described by (10) even for a partial profile \( i \) from the set
\[
\mathcal{I}^{(S)} = \{ i; \ i_k \in \{1, \ldots, v\} \text{ for } S \text{ components and } i_k = 0 \text{ for } K - S \text{ components} \},
\] (13)
of alternatives with profile strength \( S \). In particular, \( \mathcal{I}^{(K)} = \mathcal{I} \) in the case of full profiles \( (S = K) \). For general profile strength \( S \) the vector \( f \) of regression functions and the interpretation of the parameter vector \( \beta \) remain unchanged.

Then for observations in linear paired comparisons the resulting model is given by
\[
Y_n(i, j) = \sum_{k=1}^{K} (f_1(i_k) - f_1(j_k))^\top \beta_k + \sum_{k \lt \ell} ((f_1(i_k) \otimes f_1(i_\ell) - (f_1(j_k) \otimes f_1(j_\ell)))^\top \beta_{k\ell} \\
+ \sum_{k \lt \ell \lt m} ((f_1(i_k) \otimes f_1(i_\ell) \otimes f_1(i_m)) - (f_1(j_k) \otimes f_1(j_\ell) \otimes f_1(j_m)))^\top \beta_{k\ell m} + \varepsilon_n.
\] (14)
However, caution is necessary for the specification of the design region in the case of partial profiles. There it has to be taken into account that the same \( S \) attributes are used in both alternatives. For this situation the design region can be specified as
\[
\mathcal{X}^{(S)} = \{ (i, j); \ i_k, j_k \in \{1, \ldots, v\} \text{ for } S \text{ components and } i_k = j_k = 0 \text{ for exactly } K - S \text{ components} \},
\] (15)
for the set of partial profiles with profile strength \( S \). We note that for full profiles \( (S = K) \) the design region \( \mathcal{X}^{(K)} = \mathcal{I}^{(K)} \times \mathcal{I}^{(K)} \) consists of all pairs of alternatives where all attributes are shown.

4 Optimal designs

In the present setting we consider optimal designs for the second-order interactions paired comparison model (14) with corresponding regression functions \( f(i) \) given by (12). In what follows, we define \( d \) as the comparison depth (e.g., see Graßhoff et al., 2003), which describes the number of attributes in which the two alternatives presented differ, \( 1 \leq d \leq S \leq K \).

For this situation the design region \( \mathcal{X}^{(S)} \) in (15) can be partitioned into disjoint sets
\[
\mathcal{X}_d^{(S)} = \{ (i, j) \in \mathcal{X}^{(S)}; \ i_k \neq j_k \text{ for exactly } d \text{ components} \}.
\] (16)
These sets constitute the orbits with respect to permutations of both the levels \( i_k, j_k = 1, \ldots, v \) within the attributes as well as among attributes \( k = 1, \ldots, K \), themselves. Note that the \( D \)-criterion is invariant with respect to those permutations, which induce a linear reparameterization (see Schwabe, 1996, p. 17). As a result, it is sufficient to look for optimality in the class of invariant designs which are uniform on the orbits of fixed comparison depth \( d \leq S \).

Denote by \( N_d = \binom{K}{d} v^{S_d} (v - 1)^d \) the number of different pairs in \( \mathcal{X}_d^{(S)} \) which vary in exactly \( d \) attributes and by \( \bar{\xi}_d(i, j) = 1/N_d \) to each pair \((i, j)\) in \( \mathcal{X}_d^{(S)} \) and weight zero to all remaining pairs in \( \mathcal{X}_d^{(S)} \). In the following we present the information matrix for the corresponding invariant designs. To begin with, we note that \( M = \frac{2}{v - 1} (\text{Id}_{v - 1} + 1_{v - 1} 1_{v - 1}^\top) \) is the information matrix of the corresponding one-way layout in \( [S] \).

**Lemma 1.** Let \( d \in \{0, \ldots, S\} \). The uniform design \( \bar{\xi}_d \) on the set \( \mathcal{X}_d^{(S)} \) of comparison depth \( d \) has diagonal information matrix

\[
M(\bar{\xi}_d) = \begin{pmatrix}
  h_1(d) \text{Id}_{p_1} \otimes M & 0 & 0 \\
  0 & h_2(d) \text{Id}_{p_2} \otimes M \otimes M & 0 \\
  0 & 0 & h_3(d) \text{Id}_{p_3} \otimes M \otimes M \otimes M
\end{pmatrix}
\]

where

\[
h_1(d) = \frac{d}{K}, \quad h_2(d) = \frac{d}{2vK(K - 1)} (2Sv - 2S - dv - v + 2) \text{ and}
\]

\[
h_3(d) = \frac{d}{4v^2K(K - 1)(K - 2)} (3S^2 + 3S^2v^2 - 6S^2v - 3Sdv^2 + 3Sdv - 6Sv^2 + 15Sv - 9S + d^2v^2 + 3dv^2 - 6dv + 2v^2 - 6v + 6).
\]

Here, \( \text{Id}_m \) is the identity matrix of order \( m \) for every \( m \). The two functions \( h_1(d) \) and \( h_2(d) \) are identical to the corresponding terms for the main-effects and two-attribute interactions in the first-order interaction models considered by Graßhoff et al. (2003).

Note that for \( d = 0 \) all pairs have identical attributes \((i = j)\), \( h_r(0) = 0 \) for \( r = 1, 2, 3 \), and the information is zero. Hence, the comparison depth \( d = 0 \) can be neglected.

Every invariant design \( \bar{\xi} \) can be written as a convex combination \( \bar{\xi} = \sum_{d=1}^{S} w_d \bar{\xi}_d \) of uniform designs on the comparison depths \( d \) with corresponding weights \( w_d \geq 0 \), \( \sum_{d=1}^{S} w_d = 1 \). Consequently, for every invariant design the information matrix can be obtained as a convex combination of the information matrices for the uniform designs on fixed comparison depths.
Lemma 2. Let $\bar{\xi}$ be an invariant design on $X^{(S)}$. Then $\bar{\xi}$ has diagonal information matrix

$$M(\bar{\xi}) = \begin{pmatrix} h_1(\bar{\xi}) \text{Id}_{p_1} \otimes M & 0 & 0 \\ 0 & h_2(\bar{\xi}) \text{Id}_{p_2} \otimes M & 0 \\ 0 & 0 & h_3(\bar{\xi}) \text{Id}_{p_3} \otimes M \otimes M \otimes M \end{pmatrix}$$

where $h_r(\bar{\xi}) = \sum_{d=1}^{S} w_d h_r(d)$, $r = 1, 2, 3$.

First we consider optimal designs for the main effects, the first-order interaction and the second-order interaction terms separately by maximizing the corresponding entries $h_r(d)$ for $r = 1, 2, 3$ in the corresponding information matrix $M(\bar{\xi}_d)$. The resulting designs may optimize every design criterion which is invariant with respect to both permutations of the levels and permutations of the attributes if one considers the full parameter vector, satisfying the aforementioned identifiability conditions. Hence, the reduced parameter vector $\beta = ((\beta_k)_{k=1,\ldots,K}, (\beta_{k\ell})_{k<\ell}, (\beta_{k\ell m})_{k<\ell<m})^{\top}$ in (11) is also invariant in particular, with the $D$-criterion. To begin with, we mention that the following Result 1 and Result 2 paraphrase theorems given in Graßhoff et al. (2003) for first-order interaction models and translate them to the present setting of second-order interaction models.

Result 1. The uniform design $\bar{\xi}_S$ on the largest possible comparison depth $S$ is optimal for the vector of main effects $(\beta_1, \ldots, \beta_K)^{\top}$.

This means that for the main effects only those pairs of alternatives should be used which differ in all attributes subject to the profile strength.

For first-order interactions the number of the attributes subject to the profile strength with distinct levels does not provide enough information. As a consequence, only those pairs of alternatives should be used which differ in approximately half of the profile strength $S$. In particular, one has to consider the intermediate comparison depth $d^* = S - 1 - \left\lfloor \frac{S-2}{2} \right\rfloor$ where $[q]$ denotes the integer part of the decimal expansion for $q$, satisfying $[q] \leq q < [q] + 1$.

Result 2. The uniform designs $\bar{\xi}_{d^*}$ is optimal for the vector of first-order interaction effects $(\beta_{k\ell})_{k<\ell}^{\top}$.

Obviously the optimal designs of Results 1 and 2 are the same as in the first-order interactions model. However, for the second-order interactions we obtain the following result.

Theorem 1. There exists a single comparison depth $d^*$ subject to the profile strength $S$ such that the uniform design $\bar{\xi}_{d^*}$ is $D$-optimal for the second-order interaction effects $(\beta_{k\ell m})_{k<\ell<m}^{\top}$. 
This means that also for the second-order interactions those pairs of alternatives should be used which differ in the comparison depth \( d^* \) subject to the profile strength \( S \). Specifically, in the following Table 1 we note that the corresponding values of \( d^* \) in Theorem 1 were obtained by first calculating the values of \( h_3(d) \) and determining the maximum. It is worthwhile mentioning that generally for moderate values of \( v \) the optimal comparison depth \( d^* = S \) but this is not true for the case when \( S = 3 \) and \( K = 4 \). Moreover, the optimal comparison depth \( d^* = S - 2 \) for sufficiently large values of \( v \). We further note that the corresponding results for the case \( v = 2 \) can be found in Nyarko and Schwabe (2019).

Table 1: Values of the optimal comparison depths \( d^* \) of the \( D \)-optimal uniform designs \( \bar{\xi}_{d^*} \) for the second-order interactions with \( S = K - 1 \) attributes and \( v \)-levels

| \( K \) | \( S \) | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 20 |
|-------|------|---|---|---|---|---|---|---|---|----|----|
| 4     | 3    | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1   | 1   |
| 5     | 4    | 1 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2   | 2   |
| 6     | 5    | 2 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3   | 3   |
| 7     | 6    | 3 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4   | 4   |
| 8     | 7    | 4 | 4 | 4 | 5 | 5 | 5 | 5 | 5 | 5   | 5   |
| 9     | 8    | 5 | 5 | 5 | 5 | 5 | 5 | 6 | 6 | 6   | 6   |
| 10    | 9    | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 7 | 7   | 7   |

It is worthwhile mentioning that a single comparison depth \( d \) may be sufficient for non-singularity of the information matrix \( M(\xi_d) \), i.e. for the identifiability of all parameters. This can be easily seen by observing \( h_r(1) > 0 \), \( r = 1, 2, 3 \), for \( d = 1 \). But this is not true for all comparison depths as for example \( h_2(S) = 0 \). Moreover, in view of Results 1 and Theorem 1, no design exists which simultaneously optimizes the information of the whole parameter vector. Therefore we focus on the \( D \)-criterion to derive optimal designs for the whole parameter vector.

Define by \( V((i,j), \xi) = (f(i) - f(j))^\top M(\xi)^{-1}(f(i) - f(j)) \) the variance function for a design \( \xi \) with nonsingular information matrix \( M(\xi) \). This variance function plays an important role for the \( D \)-criterion. According to the Kiefer-Wolfowitz equivalence theorem (Kiefer and Wolfowitz, 1960) a design \( \xi^* \) is \( D \)-optimal if the associated variance function is bounded by the number of parameters, \( V((i,j), \xi^*) \leq p \).

Now, for invariant designs \( \bar{\xi} \) the variance function \( V((i,j), \bar{\xi}) \) is also invariant with respect to permutations of active levels and attributes and is,
hence, constant on the orbits $\mathcal{X}_d^{(S)}$ of fixed comparison depth $d$. Hence, the value of the variance function for an invariant design $\check{\xi}$ evaluated at comparison depth $d$ may be denoted by $V(d, \check{\xi})$, say, where $V(d, \check{\xi}) = V((i, j), \check{\xi})$ on $\mathcal{X}_d^{(S)}$. The following result provides a formula for calculating the variance function.

**Theorem 2.** For every invariant design $\check{\xi}$ the variance function $V(d, \check{\xi})$ is given by

$$V(d, \check{\xi}) = d(v - 1) \left( \frac{1}{h_1(\check{\xi})} + \frac{v-1}{4v h_2(\check{\xi})} (2Sv - 2S - dv - v + 2) + \frac{(v-1)^2}{24v^2 h_3(\check{\xi})} \lambda(d) \right),$$

where

$$\lambda(d) = 3S^2 + 3S^2 v^2 - 6S^2 v - 3Sdv^2 + 3Sdv - 6Sv^2 + 15Sv - 9S + d^2 v^2 + 3dv^2 - 6dv + 2v^2 - 6v + 6.$$

If the invariant design $\check{\xi}$ is concentrated on a single comparison depth, then this representation simplifies.

**Corollary 1.** For a uniform design $\check{\xi}_{d'}$ on a single comparison depth $d'$ the variance function is given by

$$V(d, \check{\xi}_{d'}) = \frac{d}{d'} \left( p_1 + p_2 \frac{2Sv - 2S - dv - v + 2}{2Sv - dv - v + 2} + p_3 \frac{\lambda(d)}{\lambda(d')} \right),$$

where

$$\lambda(d) = 3S^2 + 3S^2 v^2 - 6S^2 v - 3Sdv^2 + 3Sdv - 6Sv^2 + 15Sv - 9S + d^2 v^2 + 3dv^2 - 6dv + 2v^2 - 6v + 6.$$

For $d = d'$ we obtain $V(d, \check{\xi}_{d}) = p_1 + p_2 + p_3 = p$ which shows the $D$-optimality of $\check{\xi}_{d}$ on $\mathcal{X}_d^{(S)}$ in view of the Kiefer-Wolfowitz equivalence theorem.

The following result gives an upper bound on the number of comparison depths required for a $D$-optimal design.

**Theorem 3.** In the second-order interactions model the $D$-optimal design $\xi^*$ is supported on, at most, three different comparison depths $S$, $d^*$ and $d^* + 1$, say.

In contrast to the results in Results 1, 2 and Theorem 1 on parts of the parameter vector the $D$-optimal design for the full parameter vector may depend on both the profile strength $S$ and the number $K$ of attributes as can be seen by the numerical examples for the case of arbitrary levels, $v \geq 2$ presented in Table 2. We note that for the case $v = 2$, $S = K = 3$ of full
profiles and complete interactions the $D$-optimal design (see Graßhoff et al., 2003) indicate that all three comparison depths are needed for $D$-optimality. Corresponding results for the case $S = 3 < K$ indicate that for $S = 3$ and $K = 4$ two comparison depths $d^* = 1$ and $S = 3$ are needed for $D$-optimality, while for $S = 3$ and $K > 4$ one comparison depth is sufficient (see Nyarko and Schwabe, 2019).

For $S \geq 4$ numerical computations indicate that at most two different comparison depths $S$ and $d^*$ may be required for $D$-optimality. The following Table 2 shows the corresponding optimal designs with their optimal comparison depths $d^*$ in boldface and their corresponding weights $w^*_d$, for various choices of attributes $K$ between 4 and 10, profile strengths $S \leq K$ and levels $v = 2, \ldots, 8$. Entries of the form $(d^*, w^*_d)$ indicate that invariant designs $\xi^*_d = w^*_d \xi_d + (1 - w^*_d) \xi_S$ have to be considered, while for single entries $d^*$ the optimal design $\xi^* = \xi_{d^*}$ has to be considered which is uniform on the optimal comparison depth $d^*$. Moreover, the corresponding values of the normalized variance function $V(d, \xi^*)/p$ which shows $D$-optimality of the design $\xi^*$ in view of the Kiefer and Wolfowitz (1960) equivalence theorem is exhibited in Table 3. We note that for the situation when $S = K$, $K = 4, \ldots, 10$ and $v = 2$ the corresponding results can be found (see Nyarko and Schwabe, 2019).
Table 2: Optimal designs with intermediate comparison depths $d^*$ in boldface and optimal weights $w_{d^*}$ of the form $(d^*, w_{d^*})$ for $S \leq K$ attributes and $v$-levels

| $K$ | $S$ | $v$ | 2  | 3  | 4  | 5  | 6  | 7  | 8  |
|-----|-----|-----|----|----|----|----|----|----|----|
| 4   | 3   | (1, 0.900) | (1, 0.937) | 1  | 1  | 1  | 1  | 1  | 1  |
|     | (2, 0.857) | 2  | 2  | 2  | 2  | 2  | 2  | 2  | 2  |
| 5   | 3   | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  |
|     | (2, 0.800) | 2  | 2  | 2  | 2  | 2  | 2  | 2  | 2  |
|     | (2, 0.833) | 3  | 3  | 3  | 3  | 3  | 3  | 3  | 3  |
| 6   | 3   | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  |
|     | (2, 0.732) | 2  | 2  | 2  | 2  | 2  | 2  | 2  | 2  |
|     | (2, 0.802) | 2  | 2  | 2  | 2  | 2  | 2  | 2  | 2  |
|     | (3, 0.732) | 3  | 3  | 3  | 3  | 3  | 3  | 3  | 3  |
| 7   | 3   | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  |
|     | (1, 0.836) | 2  | 2  | 2  | 2  | 2  | 2  | 2  | 2  |
|     | (2, 0.756) | 2  | 2  | 2  | 2  | 2  | 2  | 2  | 2  |
|     | (2, 0.728) | 3  | 3  | 3  | 3  | 3  | 3  | 3  | 3  |
|     | (3, 0.697) | 4  | 4  | 4  | 4  | 4  | 4  | 4  | 4  |
| 8   | 3   | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  |
|     | (1, 0.832) | 2  | 2  | 2  | 2  | 2  | 2  | 2  | 2  |
|     | (2, 0.707) | 2  | 2  | 2  | 2  | 2  | 2  | 2  | 2  |
|     | (2, 0.687) | 3  | 3  | 3  | 3  | 3  | 3  | 3  | 3  |
|     | (3, 0.643) | 4  | 4  | 4  | 4  | 4  | 4  | 4  | 4  |
|     | (3, 0.644) | 5  | 5  | 5  | 5  | 5  | 5  | 5  | 5  |
| 9   | 3   | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  |
|     | (1, 0.819) | 2  | 2  | 2  | 2  | 2  | 2  | 2  | 2  |
|     | (2, 0.659) | 2  | 2  | 2  | 2  | 2  | 2  | 2  | 2  |
|     | (2, 0.645) | 3  | 3  | 3  | 3  | 3  | 3  | 3  | 3  |
|     | (3, 0.594) | 4  | 4  | 4  | 4  | 4  | 4  | 4  | 4  |
|     | (3, 0.598) | 5  | 5  | 5  | 5  | 5  | 5  | 5  | 5  |
|     | (4, 0.577) | 6  | 6  | 6  | 6  | 6  | 6  | 6  | 6  |
| 10  | 3   | 1  | 1  | 1  | 1  | 1  | 1  | 1  | 1  |
|     | (1, 0.800) | 2  | 2  | 2  | 2  | 2  | 2  | 2  | 2  |
|     | (2, 0.615) | 2  | 2  | 2  | 2  | 2  | 2  | 2  | 2  |
|     | (2, 0.604) | 3  | 3  | 3  | 3  | 3  | 3  | 3  | 3  |
|     | (3, 0.551) | 4  | 4  | 4  | 4  | 4  | 4  | 4  | 4  |
|     | (3, 0.556) | 5  | 5  | 5  | 5  | 5  | 5  | 5  | 5  |
|     | (4, 0.533) | 6  | 6  | 6  | 6  | 6  | 6  | 6  | 6  |
|     | (4, 0.538) | 7  | 7  | 7  | 7  | 7  | 7  | 7  | 7  |
The optimality of the so obtained designs has been checked numerically by virtue of the Kiefer-Wolfowitz equivalence theorem. For full profiles ($S = K$) the corresponding values of the normalized variance function $V(d, \xi^*)/p$ are recorded in Table 3 in the Appendix, where maximal values less than or equal to 1 establish optimality.

5 Discussion

In paired comparison experiments with many attributes impose cognitive burden due to the limited information processing capacity. The use of partial profiles are the means to thwart this potential problem. For paired comparisons where the alternatives are described by an analysis of variance model with main effects only optimal designs require that the components of the alternatives in the choice sets show distinct levels in all attributes subject to the profile strength (see Graßhoff et al., 2004). In a first-order interactions model pairs have to be used for an optimal design in which approximately $(v - 1)/v$ of the attributes are distinct and $1/v$ of the attributes coincide subject to the profile strength (see Graßhoff et al., 2003). In this paper it is shown that in a second-order interactions model one has to consider both types of pairs in which either all attributes have distinct levels or approximately $(v - 1)/v$ of the attributes are distinct and $1/v$ of the attributes coincide to obtain a $D$-optimal design for the whole parameter vector. The resulting optimal designs for the particular situation of $v = 2$ levels for each attribute (component) have been obtained by Nyarko and Schwabe (2019). Optimal designs may be concentrated on one, two or three different comparison depths depending on the number of levels, attributes and the profile strength. The so obtained designs can serve as a benchmark to judge the efficiency of competing designs as well as a starting point to construct exact designs or fractions which share the property of optimality and can be realized with a reasonable number of comparisons.

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Appendix

Proof of Lemma 7. The quantities $h_1(d)$ and $h_2(d)$ can be obtained as in Graßhoff et al. (2003). For $h_3(d)$ we proceed similarly by first noting that $\sum_{i=1}^n f_1(i)f_1(i)^\top = \frac{n-1}{2} M$ and $\sum_{i\neq j} f_1(i)f_1(j)^\top = -\frac{n-1}{2} M$. 

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For the second-order interactions, we consider attributes $k$, $\ell$ and $m$, say, and distinguish between pairs in which all three attributes are distinct, pairs in which two of these attributes $k$ and $\ell$, say, have distinct levels in the alternatives while the same level is presented in both alternatives for the remaining attribute and, finally, pairs in which only one of the attributes, say, $k$ has distinct levels in the alternatives while the same level is presented in both alternatives for the two remaining attributes:

$$
\sum_{i_k \neq j_k} \sum_{i_\ell \neq j_\ell} \sum_{i_m \neq j_m} (f_1(i_k) \otimes f_1(i_\ell) \otimes f_1(i_m) - f_1(j_k) \otimes f_1(j_\ell) \otimes f_1(j_m))
$$

$$
\cdot (f_1(i_k) \otimes f_1(i_\ell) \otimes f_1(i_m) - f_1(j_k) \otimes f_1(j_\ell) \otimes f_1(j_m))^T
$$

$$
= \sum_{i_k = 1}^{v} \sum_{j_k \neq 1}^{v} \sum_{i_\ell = 1}^{v} \sum_{j_\ell \neq 1}^{v} \sum_{i_m = 1}^{v} \sum_{j_m \neq 1}^{v} (f_1(i_k)f_1(i_\ell)f_1(i_m)^T - f_1(j_k)f_1(j_\ell)f_1(j_m)^T)
$$

$$
+ f_1(j_k)f_1(j_\ell)f_1(j_m)^T \otimes f_1(i_k)f_1(i_\ell)f_1(i_m)^T
$$

$$
- f_1(i_k)f_1(i_\ell)f_1(i_m)^T \otimes f_1(j_k)f_1(j_\ell)f_1(j_m)^T
$$

$$
- f_1(j_k)f_1(j_\ell)f_1(j_m)^T \otimes f_1(i_k)f_1(i_\ell)f_1(i_m)^T
$$

$$
= 2(v - 1)^3 \sum_{i_k = 1}^{v} f_1(i_k)f_1(i_k)^T \otimes \sum_{i_\ell = 1}^{v} f_1(i_\ell)f_1(i_\ell)^T \otimes \sum_{i_m = 1}^{v} f_1(i_m)f_1(i_m)^T
$$

$$
- 2 \sum_{i_k \neq j_k} f_1(i_k)f_1(j_k)^T \otimes \sum_{i_\ell \neq j_\ell} f_1(i_\ell)f_1(j_\ell)^T \otimes \sum_{i_m \neq j_m} f_1(i_m)f_1(j_m)^T
$$

$$
= \frac{1}{4} v(v - 1)^3(v^2 - 3v + 3)M \otimes M \otimes M, \quad (17)
$$
also
\[
\sum_{i_k \neq j_k} \sum_{i_\ell \neq j_\ell} \sum_{i_m = j_m} (f_i(i_k) \otimes f_i(i_\ell) \otimes f_i(i_m) - f_i(j_k) \otimes f_i(j_\ell) \otimes f_i(j_m)) \cdot \left( f_i(i_k) \otimes f_i(i_\ell) \otimes f_i(i_m) - f_i(j_k) \otimes f_i(j_\ell) \otimes f_i(j_m) \right)^T
\]
\[
= \sum_{i_k = 1}^v \sum_{i_\ell = 1}^v \sum_{i_m = 1}^v \sum_{j_k = 1}^v \sum_{j_\ell = 1}^v \sum_{j_m = 1}^v (f_i(i_k)f_i(i_\ell)f_i(i_m)^T - f_i(j_k)f_i(j_\ell)f_i(j_m)^T)
\]
\[
= 2(v - 1)^2 \sum_{i_k = 1}^v f_i(i_k)f_i(i_k)^T \otimes \sum_{i_\ell = 1}^v f_i(i_\ell)f_i(i_\ell)^T \otimes \sum_{i_m = 1}^v f_i(i_m)f_i(i_m)^T
\]
\[
= 1/4 v(v - 1)^3 M \otimes M \otimes M, \tag{18}
\]

and
\[
\sum_{i_k \neq j_k} \sum_{i_\ell \neq j_\ell} \sum_{i_m = j_m} (f_i(i_k) \otimes f_i(i_\ell) \otimes f_i(i_m) - f_i(j_k) \otimes f_i(j_\ell) \otimes g(j_m)) \cdot \left( f_i(i_k) \otimes f_i(i_\ell) \otimes f_i(i_m) - f_i(j_k) \otimes f_i(j_\ell) \otimes f_i(j_m) \right)^T
\]
\[
= \sum_{i_k = 1}^v \sum_{i_\ell = 1}^v \sum_{i_m = 1}^v \sum_{j_k = 1}^v \sum_{j_\ell = 1}^v \sum_{j_m = 1}^v (f_i(i_k)f_i(i_\ell)f_i(i_m)^T - f_i(j_k)f_i(j_\ell)f_i(j_m)^T)
\]
\[
= 2(v - 1) \sum_{i_k = 1}^v f_i(i_k)f_i(i_k)^T \otimes \sum_{i_\ell = 1}^v f_i(i_\ell)f_i(i_\ell)^T \otimes \sum_{i_m = 1}^v f_i(i_m)f_i(i_m)^T
\]
\[
= 1/4 v(v - 1)^3 M \otimes M \otimes M, \tag{19}
\]

respectively.

Now for the given attributes \(k\), \(\ell\) and \(m\) the pairs with distinct levels in the three attributes occur \(\binom{K-3}{3} \binom{S-3}{d-3} v^{S-3}(v - 1)^{d-3}\) times in \(\lambda_d^{(S)}\),
while those which differ in two attributes occur \((\frac{3}{2})\left(\frac{K}{S} - 3\right)\left(\frac{S}{d-2}\right) v^{S-3}(v - 1)^{d-2}\) times in \(X_d^{(S)}\). Finally, those which differ only in one attribute occur \((\frac{3}{1})\left(\frac{K}{S} - 3\right)\left(\frac{S}{d-1}\right) v^{S-3}(v - 1)^{d-1}\) times in \(X_d^{(S)}\). As a consequence, the diagonal elements \(h_3(d)\) for the second-order interactions are given by

\[
h_3(d) = \frac{1}{N_d} \left[ \frac{3}{4} \left(\frac{K}{S} - 3\right)\left(\frac{S}{d-3}\right) v(v - 1)^3(v^2 - 3v + 3) v^{S-3}(v - 1)^{d-3} M \otimes M \otimes M \right] \\
+ \frac{3}{4} \left(\frac{K}{S} - 3\right)\left(\frac{S}{d-2}\right) v(v - 1)^3(v - 2) v^{S-3}(v - 1)^{d-2} M \otimes M \otimes M \\
+ \frac{3}{4} \left(\frac{K}{S} - 3\right)\left(\frac{S}{d-1}\right) v(v - 1)^3 v^{S-3}(v - 1)^{d-1} M \otimes M \otimes M
\]

\[
= \left(\frac{d(d-1)(d-2)}{4v^2K(K-1)(K-2)}(v^2 - 3v + 3) \\
+ \frac{3(S-d)d(d-1)}{4v^2K(K-1)(K-2)}(v-1)(v-2) \\
+ \frac{3(S-d)(S-d-1)d}{4v^2K(K-1)(K-2)}(v-1)^2\right) M \otimes M \otimes M
\]

\[
= \frac{d}{4v^2K(K-1)(K-2)}(3S^2 + 3S^2v^2 - 6S^2v - 3Sdv^2 + 3Sdv - 6Sv^2 \\
+ 15Sv - 9S + d^2v^2 + 3dv^2 - 6dv + 2v^2 - 6v + 6) M \otimes M \otimes M,
\]

in the information matrix.

It should be noted that the off-diagonal elements all vanish because the terms in the corresponding entries sum up to zero due to the effects-type coding.

\[\checkmark\]

**Proof of Theorem 2.** First we note that

\[
M(\xi)^{-1} = \begin{pmatrix}
\frac{1}{h_1(\xi)} \text{Id}_{p_1} & 0 & 0 \\
0 & \frac{1}{h_2(\xi)} \text{Id}_{p_2} & 0 \\
0 & 0 & \frac{1}{h_3(\xi)} \text{Id}_{p_3}
\end{pmatrix},
\]

for the inverse of the information matrix of the design \(\bar{x}\).

Now, by Lemma 2 of [Graßhoff et al. (2003)] it is sufficient to note that for the \(k\)-th main effects the variance function is given by

\[
(f_i(i_k) - f_i(j_k))^\top M^{-1}(f_i(i_k) - f_i(j_k)) = v - 1.
\]
Further for the regression function associated with the first-order interactions of the attributes \( k \) and \( \ell \), say, we obtain

\[
\begin{align*}
(f_1(i_k) \otimes f_1(i_\ell) - f_1(j_k) \otimes f_1(j_\ell))\trans M^{-1} \otimes M^{-1} & (f_1(i_k) \otimes f_1(i_\ell) - f_1(j_k) \otimes f_1(j_\ell)) \\
& = f_1(i_k)\trans M^{-1} f_1(i_\ell) \cdot f_1(i_k)\trans M^{-1} f_1(i_\ell) + f_1(j_k)\trans M^{-1} f_1(j_\ell) \cdot f_1(j_k)\trans M^{-1} f_1(j_\ell) \\
& \quad - f_1(i_k)\trans M^{-1} f_1(j_k) \cdot f_1(i_\ell)\trans M^{-1} f_1(j_\ell) - f_1(j_k)\trans M^{-1} f_1(i_k) \cdot f_1(j_\ell)\trans M^{-1} f_1(i_\ell) \\
& = \begin{cases} 
\frac{(v-1)^2(v-2)}{2v} & \text{for } i_k \neq j_k, i_\ell \neq j_\ell \\
\frac{(v-1)^3}{2v} & \text{for } i_k \neq j_k, i_\ell = j_\ell \text{ or } i_k = j_k, i_\ell \neq j_\ell.
\end{cases} 
\tag{21}
\end{align*}
\]

Accordingly, for the regression function associated with the interaction of the attributes \( k, \ell \) and \( m \), say, we obtain

\[
\begin{align*}
(f_1(i_k) \otimes f_1(i_\ell) \otimes f_1(i_m) - f_1(j_k) \otimes f_1(j_\ell) \otimes f_1(j_m))\trans M^{-1} \otimes M^{-1} \otimes M^{-1} & \cdot (f_1(i_k) \otimes f_1(i_\ell) \otimes f_1(i_m) - f_1(j_k) \otimes f_1(j_\ell) \otimes f_1(j_m)) \\
& = f_1(i_k)\trans M^{-1} f_1(i_\ell) \cdot f_1(i_m)\trans M^{-1} f_1(i_\ell) \cdot f_1(i_m)\trans M^{-1} f_1(i_\ell) \\
& \quad + f_1(j_k)\trans M^{-1} f_1(j_\ell) \cdot f_1(j_m)\trans M^{-1} f_1(j_\ell) \cdot f_1(j_m)\trans M^{-1} f_1(j_\ell) \\
& \quad - f_1(i_k)\trans M^{-1} f_1(j_k) \cdot f_1(i_\ell)\trans M^{-1} f_1(j_\ell) \cdot f_1(i_m)\trans M^{-1} f_1(j_\ell) \\
& \quad - f_1(j_k)\trans M^{-1} f_1(i_k) \cdot f_1(j_\ell)\trans M^{-1} f_1(j_\ell) \cdot f_1(i_m)\trans M^{-1} f_1(i_\ell) \\
& = \begin{cases} 
\frac{(v-1)^3(v^2 - 3v + 3)}{4v^2} & \text{for } i_k \neq j_k, i_\ell \neq j_\ell, i_m \neq j_m \\
\frac{(v-1)^2(v-2)}{4v^2} & \text{for } i_k \neq j_k, i_\ell \neq j_\ell, i_m = j_m \\
\frac{(v-1)^5}{4v^2} & \text{for } i_k \neq j_k, i_\ell = j_\ell, i_m = j_m.
\end{cases} 
\tag{22}
\end{align*}
\]

Now for a pair of alternatives \((i,j) \in \mathcal{X}_d^{(S)}\) of comparison depth \(d\): there are exactly \(d\) attributes of the main effects for which \(i_k\) and \(j_k\) differ, there are \(\frac{1}{2}d(d-1)\) first-order interaction terms for which \((i_k,i_\ell)\) and \((j_k,j_\ell)\) differ in all two attributes \(k \) and \(\ell\), there are \(d(S-d)\) first-order interaction terms for which \((i_k,i_\ell)\) and \((j_k,j_\ell)\) differ in exactly one attribute \(k \) or \(\ell\), there are \(\frac{1}{6}d(d-1)(d-2)\) second-order interaction terms for which \((i_k,i_\ell,i_m)\) and \((j_k,j_\ell,j_m)\) differ in all three attributes \(k, \ell \) and \(m\), there are \(\frac{1}{2}d(S-d)\) second-order interaction terms for which \((i_k,i_\ell,i_m)\) and \((j_k,j_\ell,j_m)\) differ in exactly two of the associated three attributes and finally, there are \(\frac{1}{24}(S-d)(S-d-1)d\) second-order interaction terms for which \((i_k,i_\ell,i_m)\) and \((j_k,j_\ell,j_m)\) differ in exactly one
of the associated three attributes. As a consequence, we obtain

\[
V(d, \xi) = (f(i) - f(j))^\top M(\xi)^{-1} (f(i) - f(j))
\]

\[
= \frac{d(v-1)}{h_1(\xi)} + \frac{d(d-1)(v-1)^2(v-2)}{2vh_2(\xi)} + \frac{d(S-d)(v-1)^3}{2vh_2(\xi)}
\]

\[
+ \frac{(S-d)(d-1)(v-1)^3(v^2 - 3v + 3)}{6 4v^2h_3(\xi)}
\]

\[
= \frac{d(v-1)}{h_1(\xi)} + \frac{d(v-1)^2}{4vh_2(\xi)} ((d-1)(v-2) + 2(S-d)(v-1))
\]

\[
+ \frac{d(v-1)^3}{24v^2h_3(\xi)} ((d-1)(d-2)(v^2 - 3v + 3)
\]

\[
+ 3(S-d)(d-1)(v-1)(v-2)
\]

\[
+ 3(S-d)(S-d-1)(v-1)^2)
\]

\[
= \frac{d(v-1)}{h_1(\xi)} + \frac{d(v-1)^2}{4vh_2(\xi)} (2Sv - 2S - dv - v + 2)
\]

\[
+ \frac{d(v-1)^3}{24v^2h_3(\xi)} (3S^2v^2 - 6S^2v - 6Sv^2 + 3S^2 - 3Sdv^2 + 3Sdv
\]

\[
+ 3dv^2 + 15Sv - 9S + d^2v^2 - 6dv + 2v^2 - 6v + 6),
\]

for \((i, j) \in \mathcal{X}_d^{(S)}\) which proves the proposed formula.

**Proof of Corollary 3.** In view of Theorem 2 it is sufficient to note that the representation of the variance function follows immediately by inserting the values of \(h_r(\xi_d)\) from Lemma 1 and \(p_r = \binom{k}{r}(v-1)^r, r = 1, 2, 3.\)

**Proof of Theorem 3.** According to a corollary of the Kiefer-Wolfowitz equivalence theorem for the \(D\)-optimal design \(\xi^*\) the variance function \(V(d, \xi^*)\) is equal to the number of parameters \(p\) for all \(d\) such that \(\xi^* = \sum_{d=1}^{S} w_d^* \xi_d\) for \(w_d^* > 0\). By Theorem 2 the variance function is a cubic polynomial in the comparison depth \(d\) with positive leading coefficient. According to the fundamental theorem of algebra the variance function \(V(d, \xi^*)\) may thus be equal to \(p\) for, at most, three different values \(d_1 < d_2 < d_3\) of \(d\), say. Now, by the Kiefer-Wolfowitz equivalence theorem itself \(V(d, \xi^*) \leq p\) for all \(d = 0, 1, \ldots, S\). Hence, by the shape of the variance function we obtain that
in the case of three different comparison depths $d^* = d_1$, $d_2 = d^* + 1$ and $d_3 = S$ must hold. For two comparison depths either $d_3 = S$ or two adjacent comparison depths $d^* = d_1$ and $d_2 = d^* + 1$ are included.
Table 3: Values of the variance function \( V(d, \xi^*) \) for \( \xi^* \) from Table 2 in the case of full profiles \((S = K)\) and \(v\)-levels (boldface 1 corresponds to the optimal comparison depths \(d^*\))

| \(K\) | \(v\) | 1   | 2   | 3   | 4   | 5   | 6   | 7   | 8   | 9   | 10  |
|------|------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 4    | 2    | 0.875 | 1   | 0.875 | 1   |
|      | 3    | 0.813 | 1   | 0.938 | 1   |
|      | 4    | 0.793 | 1   | 0.953 | 0.983 |
|      | 5    | 0.783 | 1   | 0.962 | 0.980 |
|      | 6    | 0.777 | 1   | 0.968 | 0.980 |
|      | 7    | 0.773 | 1   | 0.973 | 0.981 |
|      | 8    | 0.770 | 1   | 0.976 | 0.982 |
| 5    | 2    | 0.760 | 1   | 0.960 | 0.880 | 1   |
|      | 3    | 0.723 | 1   | 1     | 0.954 | 1   |
|      | 4    | 0.689 | 0.967 | 1     | 0.952 | 0.987 |
|      | 5    | 0.666 | 0.951 | 1     | 0.961 | 0.981 |
|      | 6    | 0.653 | 0.941 | 1     | 0.968 | 0.980 |
|      | 7    | 0.644 | 0.934 | 1     | 0.972 | 0.981 |
|      | 8    | 0.638 | 0.929 | 1     | 0.976 | 0.982 |
| 6    | 2    | 0.701 | 0.983 | 1     | 0.906 | 0.855 | 1   |
|      | 3    | 0.624 | 0.921 | 1     | 0.968 | 0.932 | 1   |
|      | 4    | 0.591 | 0.895 | 1     | 0.993 | 0.963 | 0.997 |
|      | 5    | 0.576 | 0.882 | 0.997 | 1     | 0.972 | 0.992 |
|      | 6    | 0.560 | 0.865 | 0.987 | 1     | 0.976 | 0.989 |
|      | 7    | 0.550 | 0.854 | 0.981 | 1     | 0.979 | 0.988 |
|      | 8    | 0.543 | 0.846 | 0.977 | 1     | 0.982 | 0.988 |
| 7    | 2    | 0.615 | 0.917 | 1     | 0.956 | 0.879 | 0.863 | 1   |
|      | 3    | 0.553 | 0.860 | 0.988 | 1     | 0.963 | 0.941 | 1   |
|      | 4    | 0.519 | 0.822 | 0.965 | 1     | 0.981 | 0.962 | 0.997 |
|      | 5    | 0.498 | 0.800 | 0.952 | 1     | 0.992 | 0.974 | 0.993 |
|      | 6    | 0.487 | 0.787 | 0.944 | 1     | 0.999 | 0.983 | 0.995 |
|      | 7    | 0.479 | 0.777 | 0.937 | 0.997 | 1     | 0.985 | 0.994 |
|      | 8    | 0.471 | 0.768 | 0.929 | 0.994 | 1     | 0.987 | 0.993 |

(To be continued)
Table 3 (continued)

| $K$ | $v$ | 1    | 2    | 3    | 4    | 5    | 6    | 7    | 8    | 9    | 10   |
|-----|-----|------|------|------|------|------|------|------|------|------|------|
| 8   | 2   | 0.559| 0.872| 1    | 1    | 0.945| 0.884| 0.884| 1    |      |      |
|     | 3   | 0.490| 0.792| 0.948| 1    | 1    | 0.948| 1    |      |      |      |
|     | 4   | 0.462| 0.759| 0.924| 0.993| 1    | 0.980| 0.969| 1    |      |      |
|     | 5   | 0.442| 0.732| 0.902| 0.981| 1    | 0.988| 0.977| 0.995|      |      |
|     | 6   | 0.429| 0.716| 0.889| 0.974| 1    | 0.994| 0.982| 0.994|      |      |
|     | 7   | 0.421| 0.706| 0.880| 0.970| 1    | 0.997| 0.987| 0.995|      |      |
|     | 8   | 0.415| 0.698| 0.874| 0.960| 1    | 1    | 0.991| 0.996|      |      |
| 9   | 2   | 0.504| 0.811| 0.962| 1    | 0.969| 0.910| 0.868| 0.883| 1    |      |
|     | 3   | 0.437| 0.726| 0.894| 0.972| 1    | 0.969| 0.946| 0.946| 1    |      |
|     | 4   | 0.414| 0.696| 0.872| 0.965| 1    | 0.994| 0.977| 0.971| 1    |      |
|     | 5   | 0.397| 0.674| 0.853| 0.953| 0.995| 1    | 0.989| 0.981| 1    |      |
|     | 6   | 0.384| 0.657| 0.836| 0.940| 0.989| 1    | 0.992| 0.985| 0.996|      |
|     | 7   | 0.376| 0.645| 0.825| 0.932| 0.985| 1    | 0.995| 0.988| 0.995|      |
|     | 8   | 0.370| 0.637| 0.817| 0.927| 0.982| 1    | 0.997| 0.990| 0.996|      |
| 10  | 2   | 0.462| 0.763| 0.932| 1    | 0.997| 0.956| 0.905| 0.874| 0.896| 1    |
|     | 3   | 0.395| 0.669| 0.843| 0.938| 1    | 0.972| 0.953| 0.938| 0.947| 1    |
|     | 4   | 0.374| 0.642| 0.822| 0.929| 0.981| 1    | 0.987| 0.974| 0.972| 1    |
|     | 5   | 0.359| 0.622| 0.803| 0.917| 0.977| 1    | 1    | 0.989| 0.985| 1    |
|     | 6   | 0.348| 0.606| 0.786| 0.903| 0.968| 0.996| 1    | 0.993| 0.988| 0.998|
|     | 7   | 0.340| 0.594| 0.774| 0.892| 0.961| 0.993| 1    | 0.995| 0.990| 0.997|
|     | 8   | 0.335| 0.586| 0.765| 0.885| 0.956| 0.990| 1    | 0.996| 0.991| 0.996|