Evolution of cosmological perturbations in a brane-universe

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Abstract

The present article analyses the impact on cosmology, in particular on the evolution of cosmological perturbations, of the existence of extra-dimensions. The model considered here is that of a five-dimensional Anti-de Sitter space-time where ordinary matter is confined to a brane-universe. The homogeneous cosmology is recalled. The equations governing the evolution of cosmological perturbations are presented in the most transparent way: they are rewritten in a form very close to the equations of standard cosmology with two types of corrections: a. corrections due to the unconventional evolution of the homogeneous solution, which change the background-dependent coefficients of the equations; b. corrections due to the curvature along the fifth dimension, which act as source terms in the evolution equations.

I. INTRODUCTION

Although the idea of spatial extra-dimensions is an old one, it has recently been going through a revival with the suggestion that ordinary matter is confined to a three-dimensional subspace, or brane, within a higher dimensional space or bulk. A consequence of this restriction, suggested by recent developments in string theory, is that usual Kaluza-Klein constraints on the size of the extra-dimensions are evaded, i.e. extra dimensions can be large or, in other words, the fundamental gravity mass scale, from which the usual Planck mass would be derived, can be low. An interesting model, due to Randall and Sundrum [1] and which has attracted a lot of attention, takes into account the warping effect on the five-dimensional geometry of the self energy density of the brane and, provided one assumes the existence of a negative cosmological constant in the bulk (which cancels the square of the brane energy density), one can find a static geometry with an infinite fifth dimension but a finite Planck mass, in which usual gravity is recovered at first approximation [2].

All these ideas have aroused a lot of interest in the particle physics community and search for extra-dimensions has been undertaken. However, if extra-dimensions are not so “large”,

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their signature would be visible at energies that are beyond collider reach (or on length scales too small to be probed by gravity experiments). This is why it might be important to consider the effect of extra-dimensions in cosmology, where very high energies have been reached in the early universe and the modifications due to extra-dimensions could have left some imprint that might be observable, e.g. in the Cosmic Microwave Background (CMB) anisotropies.

Many works have been devoted to cosmology with extra-dimensions recently, essentially all in the context of five-dimensional spacetimes, i.e. with only one extra-dimension. In this framework, it has been realized [3] that one cannot recover the usual Friedmann equations in general, the case of an empty bulk, for instance, leading to a Hubble parameter which is proportional to the energy density of the brane. A way out has been found by applying the Randall-Sundrum idea to cosmology ([4] and [5]), i.e. considering an Anti-de Sitter bulk spacetime (with a negative cosmological constant $\Lambda$). One then has to solve, assuming homogeneity in the three ordinary spatial dimensions, the five-dimensional Einstein’s equations $G_{AB} \equiv R_{AB} - R g_{AB}/2 = \kappa^2 T_{AB}$, where the energy-momentum tensor is composed of a cosmological constant contribution $-\Lambda g_{AB}$ and of an energy-momentum tensor confined to the brane. This is equivalent to solve the Einstein’s equations in the bulk, and then apply, at the brane location, junction conditions, which relate the extrinsic curvature jump to the brane matter content. Assuming in addition a planar symmetry, the analog of the Friedmann equation, relating the evolution of the brane-universe scale factor $a(t)$ to its matter content, has been found to be given by ([6,5])

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{\kappa^4}{36} \rho_b^2 + \frac{\Lambda}{6},$$

where $\rho_b$ is the total energy density of the brane (see also [7] for a more general approach). Decomposing it into a constant tension $\sigma$, such that $\kappa^4 \sigma^2 + 6 \Lambda = 0$ (the Randall-Sundrum condition) and an ‘ordinary’ cosmological energy density $\rho$, so that $\rho_b = \sigma + \rho$, one obtains

$$H^2 = \frac{8 \pi G}{3} \left(\rho + \frac{\rho^2}{2\sigma}\right),$$

with the identification $\sigma = 8 \pi G$. This equation gives the usual evolution in the low energy regime $\rho \ll \sigma$ and quadratic corrections in the high energy regime $\rho \gtrsim \sigma$. In general, there is also an extra-term, which behaves effectively like a radiation-component and can be interpreted as a Schwarzschild-type mass (the bulk being then Schwarzschild-Anti de Sitter), but which will be neglected here.

With the above generalized law, all our understanding of homogeneous cosmology is safe, provided the universe has been in a low-energy regime since nucleosynthesis. A high-energy regime can be envisaged only in the earlier universe, for example during inflation [8].

The next step is obviously to investigate what will be the influence of extra-dimensions on the cosmological perturbations and their evolution. Several works have developed formalisms to handle the cosmological perturbations for a brane-universe in a five-dimensional spacetime [9–15] but they are rather difficult to manipulate and their connection with the standard theory of cosmological perturbations remains somewhat obscure. The aim of this work is to present, in the most transparent way, how the usual evolution equations for cosmological perturbations are modified in the braneworld scenario.
II. THE MODIFIED PERTURBATION EQUATIONS

Let us start with a perturbed five-dimensional metric, \( g_{AB} = \bar{g}_{AB} + h_{AB} \), of the specific form

\[
d s^2 = -n^2(1 + 2A)dt^2 + 2n^2\partial_iBdt dx^i + a^2 [(1 + 2C)\delta_{ij} + 2\partial_i\partial_j E] dx^i dx^j + dy^2,
\]

where \( t \) and the three \( x^i \) correspond to the ordinary time and space dimensions, whereas \( y \) denotes the fifth dimension; \( n(t, y) \) and \( a(t, y) \) describe the background solution (see [5] for an explicit form), and \( A, B, C \) and \( E \) are four linear perturbations (that depend on all coordinates) of the scalar type (in the Bardeen [16] terminology). Note that restrictions on the coordinate systems have been assumed implicitly in order to get \( h_{A5} = 0 \). Moreover, it will be assumed that the (perturbed) brane is at \( y = 0 \), in other words we consider only Gaussian Normal (GN) coordinate systems. This choice might not be the most convenient to solve for the bulk perturbations but it is very useful in order to make the link with the familiar perturbations of standard cosmology. Indeed the usual cosmological perturbations for the metric will be simply the values taken at \( y = 0 \) of the five-dimensional fields \( A, B, C \) and \( E \).

Let us just recall that the equations for perturbations are obtained in standard cosmology by linearizing Einstein’s equations about a background (homogeneous) solution. The linearized perturbation of the Einstein’s tensor are the basis of the standard theory of cosmological perturbations [17,18]:

\[
\begin{align*}
\delta G_{00}^{st} &= n^2 \left( \frac{\dot{a}}{an^2} \dot{\bar{C}} - \frac{2}{a^2} \Delta C \right) + [B, \dot{E}], \\
\delta G_{ij}^{st} &= \frac{a^2}{n^2} \left\{ -2\ddot{\bar{C}} + \left[ -6 \frac{\ddot{a}}{a} + 2 \frac{\dot{a}}{n} \right] \dot{\bar{C}} + 2 \frac{\ddot{a}}{a} \dot{\bar{A}} + \Delta A + \Delta C + 2 \left( \frac{\dot{a}^2}{a^2} - 2 \frac{\dot{a}\dot{n}}{an} + 2 \frac{\ddot{a}}{a} \right) (A - C) \right\} \delta_{ij} \\
&\quad - \partial_i\partial_j (A + C) + [B, \dot{B}, \dot{E}, \dot{\bar{E}}], \\
\delta G_{0i}^{st} &= \partial_i \left\{ -2\dot{\bar{C}} + 2 \frac{\dot{a}}{a} \dot{\bar{A}} \right\} + [B],
\end{align*}
\]

where we have not written explicitly the linear combinations of \( B \) and \( E \) and their time derivatives, which are summarized by the brackets [in standard cosmology, it is possible to choose a coordinate system such that \( B = E = 0 \)]. The full evolution equations are then obtained by adding on the right hand side the energy-momentum tensor describing cosmological matter.

In brane cosmology, the way to obtain evolution equations similar to the standard equations is more tortuous and proceeds in several steps. The first step is to write the perturbed five-dimensional Einstein’s tensor, which is more complicated than its four-dimensional counterpart in two respects, on one hand because there are five more components, on the other hand because the metric perturbations now depend on the fifth coordinate \( y \). Considering first the ‘ordinary’ components, they can be written in the form (see [12])

\[
\begin{align*}
\delta^{(5)} G_{00} &= \delta G_{00}^{st} + [h_{\alpha}, h'_{\alpha}, h''_{\alpha}] \\
\delta^{(5)} G_{ij} &= \delta G_{ij}^{st} + [h_{\alpha}, h'_{\alpha}, h''_{\alpha}] \\
\delta^{(5)} G_{0i} &= \delta G_{0i}^{st} + [h_{\alpha}, h'_{\alpha}, h''_{\alpha}],
\end{align*}
\]
where the brackets stand for linear combinations of the four (scalar) metric perturbations $h_\alpha = \{A, B, C, E\}$ ($\alpha = 1, \ldots, 4$), their derivatives with respect to $y$ and their second derivatives with respect to $y$. The coefficients in these linear combinations depend on the background solutions and involve time-derivatives as well as $y$-derivatives of the lapse and scale factor. The first term on the right hand side of each of the equations (5-7) has exactly the same functional form as the corresponding right hand side in (4) but with the difference that the $h_\alpha$ are functions not only of the ordinary coordinates but of $y$ as well [note that, now, we cannot get rid of the terms involving $B$ or $E$ and their derivatives because we do not have the coordinate freedom to set these quantities to zero everywhere; it is however possible to set $B = E = 0$ on one slice, e.g. $y = 0$, but, of course, their $y$-derivatives cannot be eliminated].

Let us now evaluate the relations (5-7) when $y \to 0$, i.e. when one goes in the brane. Second derivatives with respect to $y$ of $a$ and $n$ can be eliminated by resorting to the background Einstein’s equations (which can be found in [5]). First derivatives of $a$ and $n$ can be replaced by the background matter content of the brane, via the junction conditions (see [3]):

$$\left(\frac{a'}{a}\right)_{y=0} = -\frac{\kappa^2}{6} \rho_b, \quad \left(\frac{n'}{n}\right)_{y=0} = \frac{\kappa^2}{6} (3p_b + 2\rho_b),$$

where $p_b = p - \sigma$ represents the total pressure of the brane, $p$ being the ordinary cosmological pressure.

Finally the first derivatives of the metric perturbations can be replaced by the matter perturbations in the brane according, once more, to the junction conditions (established in [12]):

$$A'_{y=0} = \frac{\kappa^2}{6} (2\delta p + 3\delta \rho),$$

$$B'_{y=0} = \kappa^2 (\rho + p) \frac{a_0}{n_0} v,$$

$$\left(C' + \frac{1}{3}\Delta E'\right)_{y=0} = -\frac{\kappa^2}{6} \delta \rho,$$

$$E'_{y=0} = -2\kappa^2 \pi^S,$$

where $v$ is the peculiar velocity potential and $\pi^S$ is related to the scalar part of the anisotropic stress tensor by $\pi^S_{ij} = (\partial_i \partial_j \pi^S - \Delta \pi^S \delta_{ij}/3)$.

After substitution into (5-7) of (8) for the background coefficients and of (9) for the perturbations, one arrives to the modified equations governing the brane cosmological perturbations, which are the main result of this work:

$$\delta G_{00}^{st} - 8\pi G [2\rho A + \delta \rho] = 8\pi G \left(\frac{\rho}{\sigma}\right) [\rho A + \delta \rho] + 3\theta_C + \Delta \theta_E,$$

$$a^{-2} \delta G_{ij}^{st} - 8\pi G \left[2(pC + \delta p) \delta_{ij} + \pi^S_{ij}\right] = 8\pi G \left(\frac{\rho}{\sigma}\right) \left[(2p + \rho) C \delta_{ij} + (\delta p + (1 + w) \delta \rho) \delta_{ij} - (1 + 3w) \pi^S_{ij}/2\right] + \partial_i \partial_j \theta_E - \left(\Delta \theta_E + \theta_A + \frac{3}{2} \theta_C\right) \delta_{ij},$$

$$\delta G_{0i}^{st} + 8\pi G \rho (1 + w) a \partial_i v = 4\pi G \left(\frac{\rho}{\sigma}\right) \rho (1 + w) (1 + 3w) a \partial_i v + \frac{1}{2} \partial_i \theta_B,$$
in a coordinate system where $B(y = 0) = E(y = 0) = 0$ (i.e. in the longitudinal gauge) and such that $n(y = 0) = 1$. All five-dimensional quantities in (10-12) are evaluated at $y = 0$. We have defined $w = p/\rho$ and

$$
\theta_A = \frac{1}{n^2} \left( n^2 A' \right)', \quad \theta_B = \frac{1}{n^2} \left( n^2 B' \right)', \quad \theta_C = \frac{1}{a^2} \left( a^2 C' \right)', \quad \theta_E = \frac{1}{a^2} \left( a^2 E' \right)'.
$$

(13)

In (10-12), we have isolated the corrective terms on the right hand side, i.e. the standard equations correspond to the system (10-12) with zero on the right hand sides.

It is then clear that there are two types of corrective terms. First, new terms arise because of the non-conventional nature of the generalized Friedmann law (2). One can see that these terms become negligible in the low energy density regime, i.e. $\rho \ll \sigma$. Second, there is another category of terms ($\theta_A, \theta_B, \theta_C, \theta_E$), which clearly come from the dependence of the metric perturbations on the fifth dimension. They represent the gradients of the metric perturbations along the fifth dimension, and, therefore, all (scalar) information about the outside of the brane is embodied in these four four-dimensional fields, which in fact are not independent as shown just below.

At this stage, we have not yet taken into account the other components of the five-dimensional Einstein’s equations, the $(5-5)$ component, and the $(5-0)$ and $(5-i)$ components. As far as the latter are concerned, it is not very difficult to show that, via the same procedure of substitution using (8) and (9), the component $(5-0)$ is equivalent to the linearized energy conservation equation, whereas the components $(5-i)$ yield the linearized cosmological Euler equation. These components therefore just correspond to the usual conservation of the energy-momentum tensor. Finally, the $(55)$ component, or rather its version in terms of the Ricci tensor, $\delta R_{55} = 0$, is expressible in the very simple form (see [12])

$$
\theta_A + 3\theta_C + \Delta \theta_E = 0.
$$

(14)

In view of (10-12), it is then natural to reexpress these four quantities in terms of the following three quantities,

$$
8\pi G \rho_5 = 3\theta_C + \Delta \theta_E = -\theta_A, \quad 8\pi G \pi_5 = \theta_E, \quad 8\pi G (\rho + p)a v_5 = -\theta_B/2,
$$

(15)

which can be interpreted respectively as an effective energy density, an effective anisotropic stress and an effective peculiar velocity potential. The label ‘5’ for these quantities refers to their ‘extra-dimensional’ origin. One can also define an effective pressure, but which is not independent of $\rho_5$ because of (14),

$$
P_5 = -(8\pi G)^{-1} \left( \theta_A + 2\theta_C + \frac{2}{3}\Delta \theta_E \right) = \frac{1}{3} \rho_5.
$$

(16)

It should not be surprising to find this equation of state, when one remembers the existence of a radiation-like term in the homogeneous cosmology. All Einstein’s equations have now been exhausted. Before discussing influence of the corrective terms, let us just mention that the same procedure applies to vector and tensor perturbations. Moreover, it is straightforward to generalize the equations (10-12) to a system of several fluids, by simply summing on all fluids in the right hand sides of (8) and (9). These extensions as well as more details on the derivation of the evolution equations will be given elsewhere [13].
III. INFLUENCE OF THE UNCONVENTIONAL BACKGROUND CORRECTIONS

Let us first concentrate on the corrective terms that just change the coefficients of the familiar evolution equations, ignoring in this section the corrections due to the curvature along the fifth dimension. If $\rho$ is negligible with respect to the tension $\sigma$, one recovers exactly the standard evolution for the perturbations. Deviations from the standard evolution will appear in the high energy regime, which is possible only before nucleosynthesis. However, cosmological perturbations of observational interest have entered the Hubble radius long after nucleosynthesis, which implies that possible modifications due to a high energy radiation era have necessarily taken place while their lengthscale was much bigger than the Hubble radius.

A transition between a high energy regime and a low energy regime will thus affect all the relevant perturbations in the same way. Let us compute this ‘transfer coefficient’. Assuming, for simplicity, that the anisotropic stress vanishes, the traceless part of (6) yields $A = -C \equiv \Phi$, $\Phi$ being the usual Bardeen potential. Combining then the trace of (5) with (6), one obtains the following evolution equation for the gravitational potential $\Phi$:

$$\ddot{\Phi} + \left[4 + 3c_s^2 - \frac{\dot{H}}{\lambda^2 \sigma}\right] \dot{H} \dot{\Phi} + \left[2 \dot{H} + 3H^2 \left(1 + c_s^2 - 2 \frac{\dot{H}}{\lambda^2 \sigma}\right)\right] \Phi - \left(c_s^2 - 2 \frac{\dot{H}}{\lambda^2 \sigma}\right) \frac{\Delta \Phi}{a^2} = 0,$$

assuming a single barotropic fluid with sound speed $c_s^2 = \dot{p}/\dot{\rho}$. The only difference between this equation and its analog in standard cosmology comes from the presence of terms proportional to $\dot{H}/\lambda^2 \sigma$, terms which are negligible with respect to the others when one considers the low-energy regime.

Taking the long wavelength limit of the above equation, and reexpressing $c_s^2$ in terms of the scale factor only, one recovers, even in the high energy regime, exactly the same equation as in standard cosmology (which is not surprising if one adopts a purely geometric description of the perturbation), and its solution is given by

$$\Phi = C_1 \left(1 - \frac{H}{a} \int dt \frac{a}{H}\right) + C_2 \frac{H}{a},$$

where $C_1$ and $C_2$ are two constants ($C_2$ corresponds to a decaying mode). During the radiation era, $a(t) \sim t^{1/4}$ in the high energy regime and $a(t) \sim t^{1/2}$ in the low energy regime so that the relation between the gravitational potentials in the two regimes is $\Phi$ (low energy) = $(5/6)\Phi$ (high energy). There is therefore a small attenuation of the perturbation amplitude during the high/low energy transition.

IV. IMPACT OF THE FIFTH DIMENSION CURVATURE CORRECTIONS

Let us turn now to the five dimensional curvature corrections. It must be clear that the three four-dimensional fields $\rho_5$, $\pi_5^S$ and $v_5$ are arbitrary fields from the four-dimensional point of view in the brane, and that they can be determined only from a five-dimensional analysis of the bulk perturbations. If it is not difficult to compute the perturbations in the
Anti-de Sitter bulk, which are five-dimensional gravitational waves, it is more delicate in general to extract from all possible bulk perturbations those which are compatible with the brane and its perturbations.

As far as we are interested here, perturbations from the bulk appear in the four-dimensional-like equations for cosmological perturbations like source terms, and formally their impact on cosmological perturbations is similar to that of the ‘active seeds’, which have been studied in the context of topological defects [20]. Part of the methods developed in order to investigate the effect of topological defects on CMB anisotropies could thus be converted to the study of brane-universe perturbations. In order to compute the CMB anisotropy power spectrum, one would need correlators between the various quantities $\rho_5$, $v_5$, etc, which once more requires a specific description of the perturbations in the bulk, far beyond the scope of the present work.

What can be inferred, however, from the various studies of topological defects, is that it seems unlikely that the ‘extra-dimension seeds’ may give the dominant contribution in the CMB anisotropies. Indeed, active seeds were shown to fail to reproduce the gross features of the CMB anisotropies, in particular the acoustic peaks, because they produce new metric perturbations at all times, thus blurring any specific features by cumulative effect. One can expect the same to happen with extra-dimension seeds. However, they could constitute a subdominant contribution, that might be observable by high precision CMB experiments in the near future. Therefore, an interesting question left for future investigation is whether it would be possible to discriminate between topological defects and extra-dimension seeds.
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