Detection of causality in time series using extreme values

Juraj Bodik$^{1,2}$*, Milan Paluš$^1$ and Zbyněk Pawlas$^2$

$^1$Institute of Computer Science, The Czech Academy of Sciences, Prague, Czech Republic.
$^2$Department of Probability and Mathematical Statistics, Charles University, Prague, Czech Republic.

*Email of the corresponding author: Juraj.Bodik@unil.ch;
Contributing authors: mp@cs.cas.cz; pawlas@karlin.mff.cuni.cz;

Abstract

Consider two stationary time series with heavy-tailed marginal distributions. We want to detect whether they have a causal relation, that is, if a change in one of them causes a change in the other. Usual methods for causality detection are not well suited if the causal mechanisms only manifest themselves in extremes. In this article, we propose new insight that can help with causal detection in such a non-traditional case. We define the so-called causal tail coefficient for time series, which, under some assumptions, correctly detects the asymmetrical causal relations between different time series. The advantage is that this method works even if nonlinear relations and common ancestors are present. Moreover, we mention how our method can help detect a time delay between the two time series. We describe some of its properties, and show how it performs on some simulations. Finally, we show on a space-weather and hydro-meteorological data sets how this method works in practice.

Keywords: Granger causality, Causal inference, Nonlinear time series, VAR process, Extremal value theory, Heavy tails
1 Introduction

The ultimate goal of causal inference is to understand relations between random variables and to predict what happens after an intervention on random variables (Peters et al, 2017). Causal inference can be used in almost every scientific field. In climate science, to predict the temperature or the amount of rainfall (Naveau et al, 2020) and to understand what causes the sudden changes in river discharges (Mhalla et al, 2020). In medicine, to understand the spread of epileptic seizures in different regions of a brain (Imbens and Rubin, 2015).

There has been a large effort to develop its mathematical background (Pearl, 2009). A recent theory development consisted in estimating a causal mechanism in structural causal models (Peters and Buhlmann, 2015) and using graphical models (Kalisch et al, 2012). Some articles also dealt with the causal inference in time series (Peters et al, 2009)(Mikosch and Wintenberger, 2015).

Different perspectives can be seen when looking mainly at the tails of the distributions. Many causal mechanisms are present only during extreme events, and interventions often carry information that is likely to be causal (Cox and Wermuth, 1996). Recent work has begun to link extreme value theory and causality. Example is (Kiriliouk and Naveau, 2020), who study probabilities of necessary and sufficient causation as defined in the counterfactual theory using the multivariate generalized Pareto distributions. Other approaches comes from recursive max-linear models on directed acyclic graphs (Gissibl and Klüppelberg, 2018), (Kluppelberg and Krali, 2021). In this paper, we aim to provide a consistent methodology for causal inference in time series using extreme value theory (Coles, 2001).

1.1 Main idea

We give an example of a typical time series, with which we will deal in our paper. Let \((X, Y)^T = (X_t, Y_t)^T, t \in \mathbb{Z}\) be a bivariate strictly stationary time series, defined by the following recurrent relations

\[
X_t = \frac{1}{2}X_{t-1} + \varepsilon^X_t, \\
Y_t = \frac{1}{2}Y_{t-1} + \sqrt{X_{t-5}} + \varepsilon^Y_t,
\]

where \(\varepsilon^X_t, \varepsilon^Y_t \sim \text{iid}\) Pareto(1, 1) \(^1\). A sample realisation of such a model is given in Figure 1. Here, \(X\) causes \(Y\) (in the Granger sense), simply because the knowledge of \(X\) can help in the prediction of the future values of \(Y\). Note that this is not true for the other direction.

Consider that we have data such as in Figure 1; we want to detect a causal relationship between these time series. There is (at least in this realisation) an evident asymmetry between the two time series in the extremes. If the blue

\(^1\)\(\varepsilon^X_t, \varepsilon^Y_t\) are iid (independent and identically distributed), following a Pareto distribution with parameters equal to 1. The distribution function of a Pareto\((a, b)\) random variable is in the form \(F(x) = 1 - \left(\frac{a}{x}\right)^b\) for \(x \geq a\), zero otherwise. When \(a = b = 1\) it is often called the standard Pareto distribution.
**Fig. 1** The figure represents a sample realisation of \((X,Y)\) from Subsection 1.1 (X is the blue line and Y is red the line). We can see that large values of the blue time series cause large values of the red time series, but not the other way. Here, it is easy to see that the blue is causing the red. The lag represents the time delay between the time series, which is equal to 5 in this case.

Bivariate regularly varying time series

We will put this simple idea into mathematical language. The main problem is the *time lag* (or *time delay*). An extreme event of \(X\) does not mean an immediate extreme event of \(Y\) – it takes some time for the information from \(X\) to influence \(Y\) (in this artificial example, it takes exactly 5 time units).

Therefore, we propose the coefficient \(\Gamma_{X,Y}^{\text{time}}(q)\), see Definition 3 below, which mathematically represents how large \(Y\) will be in the next \(q\) steps if \(X\) is extremely large (in their respective scales). In our example, we can consider \(q = 5\). If \(X_0\) is extremely large, then \(Y_5\) will surely be also extremely large (large blue implies large red), but not the other way around. This implies that the following should hold: \(\Gamma_{X,Y}^{\text{time}}(q) = 1\), but \(\Gamma_{Y,X}^{\text{time}}(q) < 1\). The main part of the paper consists of determining the assumptions under which this is true.

A similar idea was used in (Gnecco et al, 2021), which dealt with iid structural causal models instead of time series. The coefficient \(\Gamma_{X,Y}^{\text{time}}(q)\) is a copula-based coefficient, and a similar notion also provides the conditional tail expectation coefficient modified by the expectation.
1.2 Preliminaries

We will consider classical VAR($q$) processes\(^2\) in the form $Z_t = A_1 Z_{t-1} + \cdots + A_q Z_{t-q} + \varepsilon_t$, where $A_i$ are non-random $d \times d$ matrices and $(\varepsilon_t)$ is iid noise. If a stability condition holds, we can rewrite it in so-called causal representation $Z_t = \sum_{i=0}^{\infty} B_i \varepsilon_{t-i}$ for some matrices $B_i$ (see e.g., Helmut (2005)).

A stochastic process $Z$ is strictly stationary if the joint distributions of $n$ consecutive variables are time-invariant. We will not work with other stationarity types. A stationary process refers to strict stationarity.

A more general model is the so-called NAAR model\(^3\), where $Z_t = f_1(Z_{t-1}) + \cdots + f_q(Z_{t-q}) + \varepsilon_t$, for some real functions $f_i$ and iid noise ($\varepsilon_t$). An often used condition $\lim_{|x| \to \infty} \frac{|f_1(x)|}{|x|} < 1$ is “almost” necessary for stationarity (see e.g., Corollary 2.2 in Bhattacharya and Lee, 1995) or Theorem 2.2 in Andel, 1989).

We will use the standard notion of regular variation (e.g., Embrechts et al (1997)). A random variable $X$ is regularly varying with tail index $\theta > 0$, if its distribution function is $F_X(x) = 1 - x^{-\theta} L(x)$ for some slowly varying function $L$ for $x \to \infty$. This fact is denoted by $X \sim RV(\theta)$. We will use notation $f(x) \sim g(x) \iff \lim_{x \to \infty} \frac{f(x)}{g(x)} = 1$, and two random variables $X, Y$ have compatible tails, if $P(X > u) \sim P(Y > u)$.

The main principle that we aim to use is the so-called max-sum equivalence, that is, for two random variables $X, Y$ we have $P(X + Y > x) \sim P(X > x) + P(Y > x) \sim P(\max(X, Y) > x)$ as $x \to \infty$. This holds for example when $X, Y \overset{iid}{\sim} RV(\theta)$ (see Section 1.3.1 in Embrechts et al (1997)). Then also $P(\alpha X > x) \sim \alpha^\theta P(X > x)$ for $\alpha > 0$, as $x \to \infty$. Similar results hold even if we deal with finite or infinite sums of random variables (see e.g., Section 4.5 in Resnick (1987)).

There is a large number of different notions of causality. Generally, the process $X$ causes $Y$ if the knowledge of $X$ can improve the prediction of $Y$. In the notion of Granger causality (Chapter 10 in Peters et al, 2017 or (Palachy, 2019)), this definition is equivalent to the following definition, which will be used in the paper.

**Definition 1.** Let $(X_t, Y_t)^\top = ((X_t, Y_t)^\top, t \in \mathbb{Z})$ follow stable VAR($q$) model, specified by

\[
\begin{align*}
X_t & = \alpha_1 X_{t-1} + \cdots + \alpha_q X_{t-q} + \gamma_1 Y_{t-1} + \cdots + \gamma_q Y_{t-q} + \varepsilon_t^X, \\
Y_t & = \beta_1 Y_{t-1} + \cdots + \beta_q Y_{t-q} + \delta_1 X_{t-1} + \cdots + \delta_q X_{t-q} + \varepsilon_t^Y.
\end{align*}
\]

Then, we say that $X$ (Granger) causes $Y$ if there exists $\delta_i \neq 0$.

Consider its causal representation in the form

\[
\begin{align*}
X_t & = \sum_{i=0}^{\infty} a_i \varepsilon_{t-i}^X + \sum_{i=0}^{\infty} c_i \varepsilon_{t-i}^Y; \\
Y_t & = \sum_{i=0}^{\infty} b_i \varepsilon_{t-i}^Y + \sum_{i=0}^{\infty} d_i \varepsilon_{t-i}^X.
\end{align*}
\]

\(^2\)Vector autoregressive processes of order $q$

\(^3\)Nonlinear additive autoregressive model
Then, $X$ causes $Y$ if and only if $d_i \neq 0$ for some $i$.

**Definition 2.** Let $(X, Y)^\top = ((X_t, Y_t)^\top, t \in \mathbb{Z})$ follow NAAR($q$) model, specified by

$$X_t = f_1(X_{t-1}) + f_2(Y_{t-q}) + \varepsilon_t^X; \quad Y_t = g_1(Y_{t-1}) + g_2(X_{t-q}) + \varepsilon_t^Y.$$ 

Then, we say that $X$ (Granger) causes $Y$ if $g_2$ is a non-constant function on the support of $X_{t-q}$ (a.s.).

### 1.3 Paper organization

The paper is organized as follows.

Section 2 contains the main results, together with a model example of the method.

Section 3 gives some extensions of the proposed method, provides its properties and discusses what will happen under violating the assumptions. Moreover, it also discusses the time lag estimation problem mentioned above.

Section 4 deals with a problem of estimation. It discusses some properties of a proposed estimator and uses simulations on artificial data sets.

In Section 5, the method is applied to a real data set concerning geomagnetic storms; we will confirm results presented by another article, which uses conditional mutual information to determine the cause of this phenomenon.

In order to keep the article short, we moved our proofs to the Appendix. Section A deals with some auxiliary propositions used in the proofs, and the proofs themselves can be found in Section B.

## 2 Causal tail coefficient for time series

Let $(X, Y)^\top = ((X_t, Y_t)^\top, t \in \mathbb{Z})$ be a bivariate time series. The main definition of this paper is the causal tail coefficient for time series $\Gamma_{X,Y}^{\text{time}}$, which gives a numerical value of the causal influence from $X$ to $Y$. The causal tail coefficient for a pair of random variables in SCM was first introduced in (Gnecco et al., 2021). We deal with time series as follows.

**Definition 3.** Let $(X, Y)^\top = ((X_t, Y_t)^\top, t \in \mathbb{Z})$ be a bivariate (strictly) stationary time series. The causal tail coefficient for time series with lag $q$ is defined as the limit (if it exists)

$$\Gamma_{X,Y}^{\text{time}}(q) = \lim_{u \to 1^-} \mathbb{E} \left[ \max \{ F_Y(Y_0), \ldots, F_Y(Y_q) \} \mid F_X(X_0) > u \right],$$

where $F_X, F_Y$ are the distribution functions of $X_0, Y_0$, respectively. The coefficient $\Gamma_{X,Y}^{\text{time}}(q)$ without the zero term $F_Y(Y_0)$ will be denoted by

$$\Gamma_{X,Y}^{\text{time}}(q; -0) = \lim_{u \to 1^-} \mathbb{E} \left[ \max \{ F_Y(Y_1), \ldots, F_Y(Y_q) \} \mid F_X(X_0) > u \right].$$
Lemma 1 (Obvious observations). Always $\Gamma_{X,Y}^{\text{time}}(q) \in [0,1]$, and $\Gamma_{X,Y}^{\text{time}}(q; -0) \leq \Gamma_{X,Y}^{\text{time}}(q) \leq \Gamma_{X,Y}^{\text{time}}(q + 1)$. Moreover, $\Gamma_{X,Y}^{\text{time}}(q)$ is invariant under increasing transformations of the time series.

Remark. The previous definition mathematically expresses very natural questions: How large will $Y$ be if $X$ is large? Does an extreme in $X$ always cause an extreme in $Y$? If $X_0$ is extremely large, will there be any $Y_i$ in the next $q$ steps, which is also extremely large?

We will show that, under some assumptions, $\Gamma_{X,Y}^{\text{time}}(q) = 1$ if and only if $X$ causes $Y$. Hence, if $\Gamma_{X,Y}^{\text{time}}(q) = 1$ and $\Gamma_{Y,X}^{\text{time}}(q) < 1$, we found an asymmetry between time series $X, Y$ and we can detect a causal relationship.

First, we need to establish some assumptions for the time series.

2.1 Models

Definition 4 (Heavy-tailed VAR model). Let $(X, Y)^\top$ follow stable VAR($q$) model specified in Definition 1 and with causal representation given by (1). Assumptions:
- $\varepsilon_t^X, \varepsilon_t^Y \sim RV(\theta),$
- $\alpha_i, \beta_i, \gamma_i, \delta_i \geq 0$ (this assumption is not necessary and will be discussed in Section 3),
- $\exists \hat{\delta} > 0$ such that $\sum_{i=0}^{\infty} a_i^{\theta - \delta} < \infty, \sum_{i=0}^{\infty} b_i^{\theta - \delta} < \infty, \sum_{i=0}^{\infty} c_i^{\theta - \delta} < \infty, \sum_{i=0}^{\infty} d_i^{\theta - \delta} < \infty$.  

Then, we will say that $(X, Y)^\top$ follows the Heavy-tailed VAR model.

Definition 5 (Heavy-tailed NAAR model). Let $(X, Y)^\top$ follow the stationary NAAR($q$) model from Definition 2. We require functions $f_1, f_2, g_1, g_2$ to be either zero functions, or they are continuous non-negative and satisfy $\lim_{x \to \infty} h(x) = \infty$ and $\lim_{x \to \infty} \frac{h(x)}{x} < 1$ for $h = f_1, f_2, g_1, g_2$.

Moreover, let $\varepsilon_t^X, \varepsilon_t^Y \sim RV(\theta)$ be non-negative. Then, we will say that $(X, Y)^\top$ follows the Heavy-tailed NAAR model.

2.2 Causal direction

Theorem 1. Let $(X, Y)^\top$ be a time series which follows either the Heavy-tailed VAR or Heavy-tailed NAAR model. If $X$ causes $Y$, then $\Gamma_{X,Y}^{\text{time}}(q) = 1$.

Remark. We assumed that we know the exact (correct) order $q$. But, for every $p \geq q$ we also have $\Gamma_{X,Y}^{\text{time}}(p) \geq \Gamma_{X,Y}^{\text{time}}(q) = 1$. The choice of appropriate $q$ will be discussed in Section 3.

Remark. Note that we did not use the regular variation condition in the proof.

Theorem 2. Let $(X, Y)^\top$ be a time series which follows either the Heavy-tailed VAR or Heavy-tailed NAAR model. If $Y$ does not cause $X$, then $\Gamma_{Y,X}^{\text{time}}(p) < 1$ for all $p \in \mathbb{N}$.

---

4 This condition ensures that the sums $\sum_{i=0}^{\infty} a_i \varepsilon_{t-i}^X$ are a.s. summable, stationarity of $X, Y$ and max-sum equivalence $P(\sum_{i=0}^{\infty} a_i \varepsilon_i > u) \sim (\lim_{n \to \infty} \alpha_i^p) P(\varepsilon_i > u)$ (see e.g., Lemma A.3 in (Mikosch and Samorodnitsky, 2000)). In the proofs, we only use the max-sum equivalence condition, which can be also satisfied if we assume conditions in Lemma A.4 in (Mikosch and Samorodnitsky, 2000).

5 This implies $X_1, Y_1 \sim RV(\theta)$ (see e.g., Theorem 2.3. in Yang and Hongzhi (2005)).
Remark. The primary step of the proof stems from Proposition 2. The idea is that large sums of independent, regularly varying random variables tend to be driven by only a single large value. So if $Y_0$ is large, this can be because some $\varepsilon_i^Y$ is large, which does not affect $X_k$.

Example 1. Let $(X, Y)^T$ follow the following stable VAR model:

$$X_t = 0.5X_{t-1} + \varepsilon_t^X; \quad Y_t = 0.5Y_{t-1} + 0.5X_{t-1} + \varepsilon_t^Y,$$

where $\varepsilon_t^X, \varepsilon_t^Y \overset{iid}{\sim} \text{Pareto}(1, 1)$, that is, with tail index $\theta = 1$. Its causal representation is

$$X_t = \sum_{i=0}^{\infty} \frac{1}{2^i} \varepsilon_{t-i}^X; \quad Y_t = \sum_{i=0}^{\infty} \frac{1}{2^i} \varepsilon_{t-i}^Y + \sum_{i=0}^{\infty} \frac{i}{2^i} \varepsilon_{t-i}^X.$$

In this case the lag is $q = 1$, and it is sufficient to take only $\Gamma_{X,Y}^{\text{time}}(1; -0) = \lim_{u \to 1^-} \mathbb{E}[F_Y(Y_1) \mid F_X(X_0) > u]$

(see also Section 3.3 for discussion). Let us give some vague computation of this coefficient. From Theorem 1 we have $\Gamma_{X,Y}^{\text{time}}(1) = 1$. For the other direction, rewrite

$$\lim_{u \to 1^-} \mathbb{E}[F_X(X_1) \mid F_Y(Y_0) > u] = \lim_{u \to \infty} \mathbb{E}[F_X(X_1) \mid \sum_{i=0}^{\infty} \frac{1}{2^i} \varepsilon_{t-i}^Y + \sum_{i=0}^{\infty} \frac{i}{2^i} \varepsilon_{t-i}^X > u].$$

First, note the following relations (first follows from the independence, second from Lemma 5):

$$\lim_{u \to \infty} \mathbb{E}[F_X(X_1) \mid \sum_{i=0}^{\infty} \frac{1}{2^i} \varepsilon_{t-i}^Y > u] = \mathbb{E}[F_X(X_1)] = 1/2,$$

$$\lim_{u \to \infty} \mathbb{E}[F_X(X_1) \mid \sum_{i=0}^{\infty} \frac{i}{2^i} \varepsilon_{t-i}^X > u] = 1.$$

Next, we know that $P(X_1 < K \mid \sum_{i=0}^{\infty} \frac{1}{2^i} \varepsilon_{t-i}^Y + \sum_{i=0}^{\infty} \frac{i}{2^i} \varepsilon_{t-i}^X > u) = P(X_1 < K)$ for every $K \in \mathbb{R}$, which holds due to Proposition 2. Simply put, with probability $1/2$, $X_1|\{F_Y(Y_0) > u\}$ has the same distribution as non-conditional $X_1$, and with complementary probability it diverges to $\infty$ (as $u \to \infty$). Together,

$$\Gamma_{Y,X}^{\text{time}}(1; -0) = \lim_{u \to 1^-} \mathbb{E}[F_X(X_1) \mid F_Y(Y_0) > u] = \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot \frac{1}{2} = \frac{3}{4}.$$

\[^6\text{Note the identities } \sum_{i=0}^{\infty} \frac{1}{2^i} = 2 = \sum_{i=0}^{\infty} \frac{1}{2^i}.\]
The exact lag $q$ is not always known. If we put for example $q = 2$, we obtain that

$$
\Gamma_{Y,X}^{\text{time}}(2) = \lim_{u \to 1^-} \mathbb{E} \left[ \max\{F_X(X_0), F_X(X_1), F_X(X_2)\} \mid F_Y(Y_0) > u \right]
$$

will be slightly larger than $\frac{3}{4}$. More precisely, it will be equal to $\frac{1}{2} \cdot 1 + \frac{1}{2} \cdot \mathbb{E} \left[ \max\{F_X(X_0), F_X(X_1), F_X(X_2)\} \right]$. Using computer software and many simulations, the true value is somewhere near 0.80.

**Remark.** Note that we keep the “zero term” $F_Y(Y_0)$ in our definition. Our models do not allow instantaneous effects, that is, cases where $X_0$ causes $Y_0$. In real data sets, such situations can happen if the data have considerable time differences between their measurements. Therefore, it is convenient to leave them in our definition.

### 3 Properties and extensions

#### 3.1 Modifications

Up to now, we assumed that large $X$ causes large $Y$. In other words, we assumed that all coefficients in our Heavy-tailed VAR model are non-negative. In this section, we will discuss the extension to possibly negative coefficients and non-direct proportional dependencies.

The most straightforward modification can be used when large $X$ causes small $Y$. In such a case, it is sufficient to consider the maxima of a pair $(X, -Y)$. This simple modification will be used in most of the applications, but can not be done in general.

Consider that our time series are centred around zero (if $\mathbb{E}(X_1), \mathbb{E}(Y_1)$ exist, they are zero) and have full support on $\mathbb{R}$. The idea to extend the causal tail coefficient for time series is to use the absolute values of $|X|, |Y|$ instead of $X, Y$. However, the general VAR series can have very complicated relations.

**Example 2.** Let $(X, Y)^\top$ follow the following VAR model:

$$
X_t = 0.5X_{t-1} + \varepsilon_t^X; \quad Y_t = X_{t-1} - 0.5X_{t-2} + \varepsilon_t^Y.
$$

Then, its causal representation is

$$
X_t = \sum_{i=0}^{\infty} \frac{1}{2^i} \varepsilon_{t-i}^X; \quad Y_t = \varepsilon_t^Y + \varepsilon_{t-1}^X.
$$

Detecting some extremal causal relations can be very difficult, because even though $X$ causes $Y$, the extreme of $X_{t-1}$ does not imply that $Y_t$ will also be extreme (if $X_{t-2}$ was large, then $X_{t-1}$ would also be large, but not $Y_t$). Therefore, we will restrict our time series in such a way that this implication holds.
Definition 6 (Extremal causal condition). Let \((X, Y)\) be a time series such that \(X\) causes \(Y\). Let \((X, Y)\) follow a stable \(\text{VAR}(q)\) model, with its causal representation in the form

\[
X_t = \sum_{i=0}^{\infty} a_i \varepsilon_{t-i}^X + \sum_{i=0}^{\infty} c_i \varepsilon_{t-i}^Y; \quad Y_t = \sum_{i=0}^{\infty} b_i \varepsilon_{t-i}^Y + \sum_{i=0}^{\infty} d_i \varepsilon_{t-i}^X.
\]

We say that it satisfies an extremal causal condition, if there exists \(p \leq q\) such that the following implication holds:

\[
\forall i \in \mathbb{N} \cup \{0\} : a_i \neq 0 \implies d_i + p \neq 0.
\]

Lemma 2. The extremal causal condition holds in the Heavy-tailed \(\text{VAR}\) model (i.e., where the coefficients are non-negative), where \(X\) causes \(Y\).

Proof. In the notion of Definition 4 and Theorem 2, if \(\delta_p > 0\), then

\[
\sum_{i=0}^{\infty} d_i \varepsilon_{p-i}^X + \sum_{i=0}^{\infty} b_i \varepsilon_{p-i}^Y = Y_p = \delta_p X_0 + \cdots = \delta_p \left( \sum_{i=0}^{\infty} a_i \varepsilon_{-i}^X + \sum_{i=0}^{\infty} c_i \varepsilon_{-i}^Y \right) + \cdots
\]

Therefore, if \(a_i > 0\), then \(d_{i+p} \geq \delta_p a_i > 0\).

Remark. The extremal causal condition implies that for every \(k \geq p\), \(|Y_k|\) will be arbitrarily large, provided that \(|X_0|\) is large enough.

Theorem 3. Let \((X, Y)\) be a time series which follows the Heavy-tailed \(\text{VAR}\) model, with possibly negative coefficients, satisfying the extremal causal condition. Moreover, let \(\varepsilon_t^X, \varepsilon_t^Y\) have full support on \(\mathbb{R}\), are iid satisfying tail balance condition. If \(X\) causes \(Y\), but \(Y\) does not cause \(X\), then \(\Gamma_{\text{time}}|X,Y|(q) = 1\), and \(\Gamma_{\text{time}}|Y|,|X|(q) < 1\).

The proof follows similar steps as the proof of Theorem 1.

3.2 Common cause

Reichenbach’s common cause principle states that for every two random variables \(X, Y\) exactly one of the following statements holds: They are independent, \(X\) causes \(Y\), \(Y\) causes \(X\), or there exists \(Z\) causing both \(X\) and \(Y\). The problem is to distinguish between true causality and dependence due to a common cause.

Theorem 4. Let \((X, Y, Z)\) follow a three-dimensional stable \(\text{VAR}(q)\) model, with iid regularly varying noise variables. Let \(Z\) be a common cause of both \(X\) and \(Y\). If \(X\) does not cause \(Y\), then \(\Gamma_{\text{time}}^X|Y,|X|(q) < 1\).

Therefore, we can distinguish between true causality and common cause. We do not observe all relevant data in practice, but Theorem 4 holds even if we do not observe the common cause (see its proof). However, the common cause still needs to fulfill the condition that noise is regularly varying with tail
indexes that are no greater than those of \(X\) and \(Y\). We can not check this assumption if we do not observe all the relevant data.

**Example 3.** Let \((X, Y, Z)^T\) follow a three-dimensional stable VAR\((q)\) model, specified by

\[
\begin{align*}
Z_t &= 0.5Z_{t-1} + \varepsilon^Z_t, \\
X_t &= 0.5X_{t-1} + 0.5Z_{t-1} + \varepsilon^X_t, \\
Y_t &= 0.5Y_{t-1} + 0.5Z_{t-1} + \varepsilon^Y_t,
\end{align*}
\]

where \(\varepsilon^X_t, \varepsilon^Y_t \sim \text{iid Pareto}(2, 2)\) and \(\varepsilon^Z_t \sim \text{iid Pareto}(1, 1)\) (i.e., \(\varepsilon^Z_t\) has a heavier tail than \(\varepsilon^X_t\)). Then \(\Gamma_{X,Y}^{time}(1) = \Gamma_{Y,X}^{time}(1) = 1\), even though \(X\) does not cause \(Y\).

### 3.3 Estimating the lag \(q\)

In all previous sections, we assumed that the exact order \(q\) of our time series is known. What should we do if we do not know this? If \(q\) is too small, we do not obtain correct causal relations (see Lemma 3). On the other hand, choosing very large \(q\) makes distinguishing \(\Gamma_{Y,X}^{time}(q)\) from 1 much harder.

One easy possibility is to look at the extremogram (Davis and Mikosch, 2009) and (such as for the correlogram in the classical case) choose some “reasonable” \(q\) from the plots. Usually, we look at the plot consisting of \(\Gamma_{Y,X}^{time}(q)\) for a variety of values for \(q\). An example is given in Section 4.4.

Consider the following problem: For a time series \((X, Y)^T\), where \(X\) causes \(Y\), we want to estimate how long it takes for information from \(X\) to affect \(Y\). If we undertake an intervention on \(X\), when will it affect \(Y\)? A typical example from the economy can be the following. Let us have two time series representing the price of milk and cheese in time. At one point in time, the government raises taxes for milk prices by 10%. When can we anticipate an increase in cheese prices?

**Definition 7 (Minimal lag).** Let \((X, Y)^T\) follow a stable VAR\((q)\) model specified in Definition 1. We call \(p \in \mathbb{N}\) the minimal lag, if \(\gamma_1 = \cdots = \gamma_{p-1} = \delta_1 = \cdots = \delta_{p-1} = 0\) and either \(\delta_p \neq 0\) or \(\gamma_p \neq 0\). If \(p\) does not exist, we define the minimal lag as \(+\infty\).

We propose a simple method for estimating the minimal lag.

**Lemma 3.** Let \((X, Y)^T\) follow the Heavy-tailed VAR model, where \(X\) causes \(Y\). Let \(p\) be the minimal lag. Then, \(\Gamma_{X,Y}^{time}(r) < 1\) for all \(r < p\), and \(\Gamma_{X,Y}^{time}(r) = 1\) for all \(r \geq p\). **Proof.**

Therefore, the statement that the minimal lag is equal to \(p\) is equivalent to the statement that \(\Gamma_{X,Y}^{time}(p)\) is the first coefficient which is equal to 1.

---

\(^7\)We do not provide a rigorous proof of this equality, but such a proof follows similar steps as the proof of Theorem 1, using the fact that \(P(\varepsilon^X_t + \varepsilon^Y_t > u) \sim P(\varepsilon^X_t > u)\) and causal representation \(X_0 = \sum_{i=0}^{\infty} \frac{1}{2^i} \varepsilon^X_i, \sum_{i=0}^{\infty} \frac{1}{2^i} \varepsilon^Z_i\) and \(Y_1 = \sum_{i=0}^{\infty} \frac{1}{2^i} \varepsilon^Y_1, \sum_{i=0}^{\infty} \frac{1}{2^i} \varepsilon^Z_1\).
4 Estimations and simulations

All methods proposed in this section are programmed in R language (R Core Team, 2020). They can be found in the supplementary package.

4.1 Non-parametric estimator

In this subsection, we discuss a possible estimator of the causal tail coefficient $\Gamma_{X,Y}(q)$ based on a finite sample $(X_1,Y_1)^\top, \ldots, (X_n,Y_n)^\top$.

We propose a very natural estimator, which uses only those values of $Y_i, \ldots, Y_{i+q}$, where $X_i$ is larger than some threshold.

**Definition 8.** We define

$$\hat{\Gamma}_{X,Y}^{\text{time}}(q) := \frac{1}{k} \sum_{i : X_i \geq \tau_k^X} \max\{\hat{F}_Y(Y_i), \ldots, \hat{F}_Y(Y_{i+q})\},$$

where $\tau_k^X = X_{(n-k+1)}$ is the $k$-th largest value of $X_1, \ldots, X_n$, and $\hat{F}_Y(Y_i) = \frac{1}{n} \sum_{j=1}^{n} 1\{Y_j \leq Y_i\}$. \(^8\)

The number $k$ represents the number of extremes which we will take into account. In the following, $k$ will depend on $n$, so to be more precise, we will write $k_n$ instead of $k$. The basic condition in extreme value theory is that

$$k_n \to \infty, \frac{k_n}{n} \to 0, \text{ as } n \to \infty. \quad (2)$$

In the following methods, we take $k_n = \sqrt{n}$. This choice is briefly discussed in Section 4.3.

The next theorem shows that such a statistic is “reasonable”, by showing that it is (under a much more general setting than just assuming the VAR model) asymptotically unbiased.

**Theorem 5.** Let $(X,Y)^\top = ((X_t,Y_t)^\top, t \in \mathbb{Z})$ be a stationary bivariate time series, whose marginal distributions are absolutely continuous with support on some neighbourhood of infinity. Let $\Gamma_{X,Y}^{\text{time}}(q)$ exists. Let $k_n$ satisfy (2) and

$$\frac{n}{k_n} P \left( \frac{n}{k_n} \sup_{x \in \mathbb{R}} |\hat{F}_X(x) - F(x)| > \delta \right) \stackrel{n \to \infty}{\longrightarrow} 0, \forall \delta > 0. \quad (3)$$

Then, $E \hat{\Gamma}_{X,Y}^{\text{time}}(q) \stackrel{n \to \infty}{\longrightarrow} \Gamma_{X,Y}^{\text{time}}(q)$ \(^9\).

**Remark.** The condition (3) is satisfied for iid random variables for example when $\frac{k_n^2}{n} \to \infty$ (this follows from the Dvoretzky–Kiefer–Wolfowitz inequality). For linear time series, the condition differs for different linear coefficients.

\(^8\)This estimator can possibly depend on random variables $Y_{n+1}, \ldots, Y_{n+q}$. If we want to be fully rigorous, we should assume that we observe $n+q$ data, or define that $Y_{n+i} = Y_n$, which is a negligible modification for large $n$.

\(^9\)\hat{\Gamma}_{X,Y}^{\text{time}}(q)$ depends on $n$, although we omitted this index for clarification.
4.2 Some insight using simulations

We will simulate how the estimates \( \hat{\Gamma}_{X,Y}^{time} \) work for a series of models. First, we use the Monte Carlo principle to estimate the distribution of \( \hat{\Gamma}_{X,Y}^{time} \) and \( \hat{\Gamma}_{Y,X}^{time} \) for the following model.

**Definition 9.** Let \((X,Y)^\top\) follow

\[
X_t = 0.5X_{t-1} + \varepsilon_t^X,
\]

\[
Y_t = 0.5Y_{t-1} + \delta X_{t-2} + \varepsilon_t^Y,
\]

where \( \varepsilon_t^X, \varepsilon_t^Y \) are iid and \((X,Y)^\top\) is stable.

Figure 2 shows the histograms of both \( \hat{\Gamma}_{X,Y}^{time} \) and \( \hat{\Gamma}_{Y,X}^{time} \) from 1000 simulated repetitions of time series of length \( n = 10000 \) following the model from Definition 9 with \( \delta = 0.5 \) and \( \varepsilon_t^X, \varepsilon_t^Y \) iid \( \sim \) Cauchy.

![Histograms](image)

**Fig. 2** The histograms represent approximate distributions of \( \hat{\Gamma}_{X,Y}^{time} \) (blue) and \( \hat{\Gamma}_{Y,X}^{time} \) (red) from the model with a correct causal direction \( X \rightarrow Y \) and a number of data \( n = 10000 \).

In the following, we will perform simulations for \((X,Y)^\top\) following the model given by Definition 9. We consider the following three choices for the parameter \( \delta \) and three choices for the sample size \( n \): \( \delta = 0.1, 0.5 \) and \( 0.9 \), \( n = 100, 1000 \) and \( 10000 \). The random variables \( \varepsilon_t^X \) and \( \varepsilon_t^Y \) are generated either from the standard normal\(^{10}\), or the standard Pareto, distribution. For each \( \delta \), each noise distribution and each \( n \), we compute the estimators \( \hat{\Gamma}_{X,Y}^{time} := \hat{\Gamma}_{X,Y}^{time}(2) \). The procedure is repeated 200 times and the means and quantiles of the estimators are computed. Table 1 shows the results; each cell corresponds to the model with given \( \delta \), noise distribution and number \( n \) of data-points.

\(^{10}\)I.e., violating the RV(\(\alpha\)) assumption
Table 1 Consider 200 simulated time series following the model from Definition 9. Each cell represents a different coefficient $\delta$, a different number of data-points $n$ and a different noise distribution. Each value $\hat{\Gamma}_{X,Y}^{time} = \cdot \pm \cdot$ represents the mean of all 200 estimated coefficients $\hat{\Gamma}_{X,Y}^{time}$, and the difference between this mean and the 95% quantile out of all 200 simulations. In every case, $X$ causes $Y$.

### Errors with standard Pareto distributions

| $n$ | $\Gamma_{X,Y}^{time}$ | $\Gamma_{Y,X}^{time}$ | $\Gamma_{X,Y}^{time}$ | $\Gamma_{Y,X}^{time}$ | $\Gamma_{X,Y}^{time}$ | $\Gamma_{Y,X}^{time}$ |
|-----|-------------------|-------------------|-------------------|-------------------|-------------------|-------------------|
| 100 | $0.83 \pm 0.14$  | $0.66 \pm 0.23$  | $0.94 \pm 0.04$  | $0.66 \pm 0.16$  | $0.98 \pm 0.01$  | $0.65 \pm 0.12$  |
| 1000 | $0.68 \pm 0.14$ | $0.66 \pm 0.19$ | $0.68 \pm 0.1$  | $0.66 \pm 0.13$ | $0.69 \pm 0.07$  | $0.62 \pm 0.08$  |
| 10000 | $0.83 \pm 0.11$| $0.64 \pm 0.2$ | $0.86 \pm 0.06$ | $0.65 \pm 0.13$ | $0.90 \pm 0.03$ | $0.66 \pm 0.06$ |
| 10000 | $0.88 \pm 0.07$ | $0.64 \pm 0.19$ | $0.93 \pm 0.03$ | $0.65 \pm 0.13$ | $0.96 \pm 0.01$ | $0.66 \pm 0.09$ |

### Errors with standard Gaussian distributions

| $n$ | $\Gamma_{X,Y}^{time}$ | $\Gamma_{Y,X}^{time}$ | $\Gamma_{X,Y}^{time}$ | $\Gamma_{Y,X}^{time}$ | $\Gamma_{X,Y}^{time}$ | $\Gamma_{Y,X}^{time}$ |
|-----|-------------------|-------------------|-------------------|-------------------|-------------------|-------------------|
| 100 | $0.68 \pm 0.14$  | $0.66 \pm 0.19$  | $0.94 \pm 0.04$  | $0.66 \pm 0.16$  | $0.98 \pm 0.01$  | $0.65 \pm 0.12$  |
| 1000 | $0.68 \pm 0.14$ | $0.66 \pm 0.19$ | $0.68 \pm 0.1$  | $0.66 \pm 0.13$ | $0.69 \pm 0.07$  | $0.62 \pm 0.08$  |
| 10000 | $0.83 \pm 0.11$| $0.64 \pm 0.2$ | $0.86 \pm 0.06$ | $0.65 \pm 0.13$ | $0.90 \pm 0.03$ | $0.66 \pm 0.06$ |
| 10000 | $0.88 \pm 0.07$ | $0.64 \pm 0.19$ | $0.93 \pm 0.03$ | $0.65 \pm 0.13$ | $0.96 \pm 0.01$ | $0.66 \pm 0.09$ |

### X with Pareto error, Y with Gaussian error

| $n$ | $\Gamma_{X,Y}^{time}$ | $\Gamma_{Y,X}^{time}$ | $\Gamma_{X,Y}^{time}$ | $\Gamma_{Y,X}^{time}$ | $\Gamma_{X,Y}^{time}$ | $\Gamma_{Y,X}^{time}$ |
|-----|-------------------|-------------------|-------------------|-------------------|-------------------|-------------------|
| 100 | $0.96 \pm 0.02$  | $0.80 \pm 0.1$  | $0.98 \pm 0.0013$ | $0.92 \pm 0.04$  | $0.997 \pm 0.001$ | $0.98 \pm 0.011$ |
| 1000 | $0.96 \pm 0.02$ | $0.80 \pm 0.1$ | $0.98 \pm 0.0013$ | $0.92 \pm 0.04$ | $0.997 \pm 0.001$ | $0.98 \pm 0.011$ |
| 10000 | $0.65 \pm 0.15$| $0.62 \pm 0.20$ | $0.67 \pm 0.1$ | $0.63 \pm 0.13$ | $0.68 \pm 0.05$ | $0.63 \pm 0.08$ |

### X with Gaussian error, Y with Pareto error

| $n$ | $\Gamma_{X,Y}^{time}$ | $\Gamma_{Y,X}^{time}$ | $\Gamma_{X,Y}^{time}$ | $\Gamma_{Y,X}^{time}$ | $\Gamma_{X,Y}^{time}$ | $\Gamma_{Y,X}^{time}$ |
|-----|-------------------|-------------------|-------------------|-------------------|-------------------|-------------------|
| 100 | $0.96 \pm 0.02$  | $0.80 \pm 0.1$  | $0.98 \pm 0.0013$ | $0.92 \pm 0.04$  | $0.997 \pm 0.001$ | $0.98 \pm 0.011$ |
| 1000 | $0.96 \pm 0.02$ | $0.80 \pm 0.1$ | $0.98 \pm 0.0013$ | $0.92 \pm 0.04$ | $0.997 \pm 0.001$ | $0.98 \pm 0.011$ |
| 10000 | $0.65 \pm 0.15$| $0.62 \pm 0.20$ | $0.67 \pm 0.1$ | $0.63 \pm 0.13$ | $0.68 \pm 0.05$ | $0.63 \pm 0.08$ |

**Example 4.** The notation $\hat{\Gamma}_{X,Y}^{time} = 0.5 \pm 0.1$ means that out of all 200 simulated series from Definition 9, $\hat{\Gamma}_{X,Y}^{time}$ was on average equal to 0.5 and exactly 190 of those simulations were $\hat{\Gamma}_{X,Y}^{time} \leq 0.6$. We write $\pm$, because the 5% quantiles were in all cases symmetrical, cca 190 of those simulations also fulfilled $\hat{\Gamma}_{X,Y}^{time} \geq 0.4$.

**Conclusion.** The method works surprisingly well under violating the assumption of regular variation. If we consider Gaussian noise, the true theoretical value is $\Gamma_{X,Y}^{time} = \Gamma_{Y,X}^{time} = 1$. But if we estimate these values for finite $u < 1$, the value $\mathbb{E} [\max \{F_Y(Y_0), \ldots, F_Y(Y_0)\} \mid F_X(X_0) > u]$ is still larger than in the other direction. This results in seemingly correct causal directions.

On the other hand, if the cause has heavier tails than the effect, our method suggests $\Gamma_{X,Y}^{time} = \Gamma_{Y,X}^{time} = 1$. In this case, for a large $n$, both estimates are very
Detection of causality in time series using extreme values

close to 1, and this results in the wrong causal directions. Therefore, the main problems are caused by a different tail behaviour.

4.3 Choice of a threshold

A common problem in the extreme value theory is the choice of the threshold. In our case, it is the choice of the parameter $k$ from Definition 8. There is a bias-variance trade-off; smaller the $k$, smaller the bias (and the larger the variance) and vice versa. There is no universal method for choosing the threshold. It very much depends on the extremal behavior of our series.

To give an idea of the behavior, consider the time series model given by Definition 9 with the number of data $n = 1000$. Figure 3 represents the estimators $\hat{\Gamma}_{X,Y}^{\text{time}}(2)$ and $\hat{\Gamma}_{Y,X}^{\text{time}}(2)$ using a different $k$. The thick line represents the mean of 100 realisations; the thin lines are 5% and 95% quantiles. The variance of $\hat{\Gamma}_{Y,X}^{\text{time}}(2)$ for small $k$ is very large. On the other hand, the larger $k$, the more negatively biased $\hat{\Gamma}_{X,Y}^{\text{time}}(2)$ will be.

Concluding from this example (and a few others), it seems that $k = \sqrt{n}$ may be a reasonable choice. We want to emphasize that it may not, however, be optimal.

![Fig. 3](image)

The figure represents the estimators $\hat{\Gamma}_{X,Y}^{\text{time}}(2)$ (blue) and $\hat{\Gamma}_{Y,X}^{\text{time}}(2)$ (red) with different choices for parameter $k$ for the specific model of time series with $n = 1000$. The thick line represents the mean of 100 realisations, the thin lines are 5% and 95% quantiles.

4.4 Choice of the lag

To give an example, how $\Gamma_{X,Y}^{\text{time}}(q)$ behaves for different $q$, we consider the model for $(X, Y)^\top$, where

$$X_t = 0.5X_{t-1} + \varepsilon_t^X; \quad Y_t = 0.5Y_{t-1} + 0.5X_{t-6} + \varepsilon_t^Y,$$

and $\varepsilon_t^X, \varepsilon_t^Y \overset{iid}{\sim} \text{Cauchy}$. Notice that the minimal lag is $q = 6$. 
Fig. 4 The figure represents the estimators \( \hat{\Gamma}_{X,Y}^{\text{time}}(q) \) (blue) and \( \hat{\Gamma}_{Y,X}^{\text{time}}(q) \) (red) with different choices for lag \( q \) for the specific model of time series with \( n = 1000 \). The thick line represents the mean of 100 realisations, the thin lines are 5% and 95% quantiles.

Similarly as in Section 4.3, we did many simulations from this model with \( n = 1000 \) and computed \( \hat{\Gamma}_{X,Y}^{\text{time}}(q) \) and \( \hat{\Gamma}_{Y,X}^{\text{time}}(q) \) for different \( q \). The mean, 5% and 95% quantiles of those estimates are drawn in Figure 4.

Usually (at least in the artificial datasets where \( X \) causes \( Y \)), the coefficient \( \hat{\Gamma}_{X,Y}^{\text{time}}(q) \) in the causal direction rises much faster than in the other direction, until it reaches the “correct” lag. Then, this coefficient is very close to 1, and it stays there even for all larger \( q \) (just as the theory suggests). On the other hand, \( \hat{\Gamma}_{Y,X}^{\text{time}}(q) \) rises slower and slowly converges to 1.

4.5 Testing

We want to develop a formal method that tells us the causal direction between two time series. One (quite trivial) option is to put a threshold, for example, we say that \( X \) causes \( Y \) if and only if \( \hat{\Gamma}_{X,Y}^{\text{time}}(q) \geq \tau \), where \( \tau = 0.9 \) or 0.95. The choice of \( \tau \) should depend on the sample size \( n \). If \( n \) is small, we can not expect \( \hat{\Gamma}_{X,Y}^{\text{time}}(q) \) to be large. On the other hand, choosing small \( \tau \) can lead to wrong conclusions.

Ideally, we want to test the hypothesis \( \Gamma_{X,Y}^{\text{time}}(q) = 1 \) against the alternative \( \Gamma_{X,Y}^{\text{time}}(q) < 1 \). To do that, we need to know (at least asymptotically) the distribution of \( \hat{\Gamma}_{X,Y}^{\text{time}}(q) \). This is beyond the scope of this work.

Another method to estimate the confidence intervals is to use the block (sometimes called stationary) bootstrap technique\(^{11}\).

Definition 10. Let \((X, Y, Z)^\top\) follow

\[
\begin{align*}
Z_t &= 0.5Z_{t-1} + \varepsilon^Z_t,
X_t &= 0.5X_{t-1} + 0.5Z_{t-2} + \varepsilon^X_t,
Y_t &= 0.5Y_{t-1} + 0.5Z_{t-1} + (X_{t-3})^3 + 5\varepsilon^Y_t,
\end{align*}
\]

\(^{11}\)More precisely so-called Reverse Bootstrap Percentile Interval (Section 5.4 in (Hesterberg, 2014)). Although in each resample, we always have \( \hat{\Gamma}_{X,Y}^{\text{time}}(q) < 1 \), this bootstrap modification can overcome this problem.
where $\varepsilon^X_t, \varepsilon^Y_t, \varepsilon^Z_t \overset{iid}{\sim} \text{Pareto}(1,1)$. A sample realisation can be found in Figure 5.

We understand that the process $Z$ from Definition 10 represents unobserved common cause. Granger testing fails for such a case because $Z$ creates spurious $Y \to X$ direction, although the truth is the opposite. However, we obtain correct causal directions using $\hat{\Gamma}^{\text{time}}_{X,Y}(q)$. For example, for $n = 1000$ we obtain very similar values of $\hat{\Gamma}^{\text{time}}_{X,Y}(q)$ and $\hat{\Gamma}^{\text{time}}_{Y,X}(q)$ such as in Figure 2. We will try to use the bootstrap method in the rest of the section to make this more formal.

![Figure 5](image)

**Table 2** We consider two time series models, one is a simple VAR model, the second is a more complex nonlinear model with a common cause. We used three methods for the causal inference. The resulting percentage shows how many times the result was correct (in each direction, sensitivity corresponds to the $X \to Y$ direction, specificity to the opposite one). For the first and second tests, the results represent the percentage of cases when the corresponding p-value was less than $\alpha = 0.05$. For the third method (it is not a formal test), sensitivity 99% represents that $\hat{\Gamma}_{X,Y} > \tau$ held in 99% of the cases. Specificity 79% represents that in 79% of cases was $\hat{\Gamma}_{Y,X} \leq \tau$.

We perform small simulations. Consider two models of time series. The first is a simple model given by Definition 9 with $\delta = 0.5$ and Pareto(1,1) noise. The second model follows Definition 10. We will discuss three different methods for detecting the causal directions. The first is the bootstrap method...
mentioned above. The second method is the classical Granger test. We consider the significance level $\alpha = 0.05$. We will also use the method (not a formal test), when we estimate $\hat{\Gamma}_{X,Y}^{time}(3)$ and conclude that $X$ causes $Y$ if and only if $\hat{\Gamma}_{X,Y}^{time}(3) \geq \tau$ with the choice $\tau = 0.9$.

We perform 100 simulations of the time series mentioned above, with the number of data $n = 500, 5000$. Finally, we compute the number that correctly infer the causal directions $X \rightarrow Y$ and $Y \not\rightarrow X$ using these three methods. Table 2 shows the results in percentage. Here, the specificity represents the percentage of correct conclusions “$X$ causes $Y$”. The sensitivity represents the percentage of correct conclusions “$Y$ does not cause $X$”.

The results suggest that the Granger test behaves well under a simple model, but that it can not handle a more complex model. The bootstrap method does not behave properly either. It is a common bootstrap problem that we obtain smaller confidence intervals. Even for $n = 5000$, we still rejected the hypothesis $\Gamma_{X,Y}^{time}(3) = 1$ in 40\% of cases even though the hypothesis holds. Therefore, using the bootstrap method may not be an appropriate approach here.

5 Applications

Again, all data together with a detailed R code are available in the supplementary package.

5.1 Space weather

In the following, we deal with a problem from space weather studies. The term “space weather” refers to the variable conditions on the Sun and in space that can influence the performance of technology we use on Earth. Extreme space weather could potentially cause damage to critical infrastructures – especially the electric grid. In order to protect people and systems that might be at risk from space weather effects, we need to understand the causes of space weather.

Geomagnetic storms and substorms are indicators of geomagnetic activity. Visually, a substorm is seen as a sudden brightening and increased movement of auroral arcs. It can cause magnetic field disturbances in the auroral zones up to a magnitude of 1000 nT (Tesla units). The basis of this geomagnetic activity begins in the Sun. Specifically, there is a significant correlation with the solar wind (stream of negatively charged particles from the Sun) and also with an interplanetary magnetic field (a component of a solar magnetic field dragged away from the Sun by the solar wind).

One of the fundamental problems in this area is determining and predicting some specific characteristics – magnetic storm index (SYM) and a substorm index (AE). It may seem that AE is a driving factor (cause) of SYM because, usually, the accumulation of successive substorms precedes the occurrence of

---

12 Using “grangertest” function from “lmtest” package (Zeileis and Hothorn, 2002).
13 Text taken from a webpage https://www.ready.gov/space-weather, accessed 18.5.2021.
Detection of causality in time series using extreme values

Fig. 6 Space weather. The first plot represents SYM (magnetic storm index), the second one AE (substorm index) and the last one BZ (vertical component of an interplanetary magnetic field). Data were measured in 5 minutes intervals for the year 2000 by NASA.

magnetic storms. However, a recent article (Manshour et al, 2021) induces that this is not the case. A vertical component of an interplanetary magnetic field (BZ) seems to be a common cause of both of these indexes. We will apply our method to check this result and determine if the causal influence manifests itself in the extremes.

Our data consist of three time series (SYM, AE, BZ) with about 100000 measurements (every 5 minutes for the entire year 2000). Data are available
Detection of causality in time series using extreme values

(besides in the supplementary package) online on the NASA webpage\textsuperscript{14}. Their plot can be found in Figure 6. From the nature of the data, we compare extremes when SYM is extremely small, AE is extremely large, BZ is extremely small (i.e. taking $-\text{SYM}$, $+\text{AE}$, $-\text{BZ}$ and comparing only maximums). We also know that an appropriate lag will be smaller than $q = 24$ (2 hours).

First things first, we discuss whether the assumptions for our method are fulfilled. We estimate the tail indexes of our data\textsuperscript{15}. Resulting numbers are the following: SYM has the estimated tail index $0.25 (0.015, 0.5)$, AE has $0.18 (0.08, 0.28)$ and BZ has $0.30 (0.12, 0.46)$. Therefore, the assumption of the regular variation with the same tail index seems reasonable. Moreover, neither confidence interval includes the zero value (although SYM is quite close), and therefore our time series can be considered regularly varying. The time series also seem stationary. Therefore, all our assumptions seem reasonable for this application.

Finally, we compute the causal tail coefficient with different lags (with $k = \sqrt{n}$, but the results are similar if we consider $k$ other than $\sqrt{n}$). The resulting numbers can be found in Figure 7. They suggest that there is a strong asymmetry between BZ and SYM ($\hat{\Gamma}_{\text{time \, BZ,SYM}} \approx 1$, and $\hat{\Gamma}_{\text{time \, SYM,BZ}} \ll 1$), the asymmetry between BZ and AE ($\hat{\Gamma}_{\text{time \, BZ,AE}} \approx 1$, and $\hat{\Gamma}_{\text{time \, SYM,AE}} < 1$) and no asymmetry between SYM and AE ($\hat{\Gamma}_{\text{time \, AE,SYM}} < 1$ and both $\hat{\Gamma}_{\text{time \, SYM,AE}}$, $\hat{\Gamma}_{\text{time \, AE,SYM}}$ are very similar).

\textbf{Fig. 7} The figure refers to the real data set from Section 5.1. It represents all values of $\hat{\Gamma}_{\cdot \cdot \cdot}(q)$ for a range of lag $q \in [1, 24]$, with all pairs of time series SYM (magnetic storm index), AE (substorm index) and BZ (interplanetary magnetic field). We can see an asymmetry in the causal influence between the time series BZ-SYM and BZ-AE.

Such results correspond to the hypothesis about BZ being a common cause of SYM and AE, with no causal relation between them. Note that our method can deal with a common cause (at least in theory). Although classical methods

\textsuperscript{14}NASA webpage https://cdaweb.gsfc.nasa.gov, accessed 18.05.2021.
\textsuperscript{15}Using “HTailIndex” function in “ExtremeRisks” package in R with variable $k=500$, reference (Padoan and Stupfler, 2020).
suggest that AE causes SYM, there are some extreme events where AE is extreme, but SYM is not.

5.2 Hydrometeorology

There are many important phenomena that affect weather, although their significance and effects are not yet well known. In this application, we take a few of the most interesting ones to find some causal relations between them.

- **El Niño** is a climate pattern that describes the unusual warming of surface waters in the eastern tropical Pacific Ocean. It has an impact on ocean temperatures, the speed and strength of ocean currents, the health of coastal fisheries, and local weather from Australia to South America and beyond.

- **North Atlantic Oscillation (NAO)** is a weather phenomenon over the North Atlantic Ocean of fluctuations in the difference of atmospheric pressure.

- **Indian Dipole Index (DMI)** (sometimes referred to as the Dipole Mode Index) is commonly measured by an index describing the difference between sea surface temperature anomalies in two specific regions of the tropical Indian Ocean.

- **Pacific Decadal Oscillation (PDO)** is a recurring pattern of ocean-atmosphere climate variability centred over the mid-latitude Pacific basin.

- **East Asian Summer Monsoon Index (EASMI)** is defined as an area-averaged seasonally dynamical normalized seasonality at 850 hPa within the East Asian monsoon domain.

- **Amount of rainfall in a region in India (AoR)** is a data set consisting of the amount of rainfall in the region which is in the centre of India.

Of all of these variables, we have monthly measurements since 1.1.1953. After some basic data handling (erasing seasonality and ensuring the stationarity of our data, etc., all details are included in the supplementary package) we look at the assumption of heavy-tailness. Again, using the same method as in the previous subsection, we obtain the estimated tail indexes (0.43, 0.28, 0.28, 0.19, 0.20, 0.34, 0.43), respectively. It is true that, for example, El Niño has a significantly larger tail index than PDO. Still, we will proceed keeping in mind that these relations with different tail indexes may be misleading.

Finally, we compute the Causal tail coefficient for time series for each pair. Only in this application, we conclude that $X$ causes $Y$ if and only if $\hat{\Gamma}_{X,Y}^{\text{time}} > 0$ and $\hat{\Gamma}_{Y,X}^{\text{time}} < 0$. This corresponds to a quite clear asymmetry in extremes, although we do not provide a formal p-value. We choose for all pairs the lag $q = 12$, that is, one year. Resulting relations are drawn in Table 3. We can see some extremal relations after drawing a corresponding graph (see Figure 8). Here, a directed arrows indicate pairs where there is a clear asymmetry in corresponding \(\hat{\Gamma}\) coefficients (bold numbers in Table 3).

\[16\] In words, we answer the following question: When an extreme event happens in one time series, will an extreme event happen in the next 12 months in the other time series? And is this not valid for the opposite direction? We answer yes if $\hat{\Gamma}_{X,Y}^{\text{time}} > 0.9$ and $\hat{\Gamma}_{Y,X}^{\text{time}} < 0.8$. 
Table 3  Estimated Causal tail coefficient for time series for each pair of variables. Concerning the order of subscripts, the column comes first, then the row. For example, $\hat{\Gamma}_{\text{El Niño}, \text{NAO}}^{\text{time}} = 0.88$ and $\hat{\Gamma}_{\text{NAO}, \text{El Niño}}^{\text{time}} = 0.74$.

|       | El Niño | NAO | DMI | PDO | EASMI | AoR |
|-------|---------|-----|-----|-----|-------|-----|
| El Niño  | x      | 0.88| 0.94| 0.87| 0.92  | 0.95|
| NAO     | 0.74   | x   | 0.83| 0.77| 0.90  | 0.94|
| DMI     | 0.86   | 0.89| x   | 0.77| 0.94  | 0.95|
| PDO     | 0.92   | 0.88| 0.92| x   | 0.94  | 0.94|
| EASMI   | 0.65   | 0.92| 0.74| 0.72| x     | 0.93|
| AoR     | 0.69   | 0.92| 0.77| 0.72| 0.90  | x   |

Fig. 8  Estimated causal directions using Causal tail coefficient for the set of variables AoR (Amount of rainfall in India), DMI (Indian Dipole Index), PDO (Pacific Decadal Oscillation), NAO (North Atlantic Oscillation), EASMI (East Asian Summer Monsoon Index) and El Niño pattern. Arrows represent the asymmetry in the extreme events, more specifically the arrow between two variables $X, Y$ is present if and only if $\hat{\Gamma}_{X,Y} > 0.9$ and $\hat{\Gamma}_{Y,X} < 0.8$.

Concluding that these causal relations are surely true would be a very bold statement. We can not say for sure that these highly complex data sets follow a simple time series model. Their tail indexes are also a bit different. However, we do not say that other causal relations are not present (for example, there can be a causal relation without a significant presence in extremes). Nevertheless, this simple method points out asymmetries in extremes and gives us some idea of which directions can be interesting to focus on further.

6 Conclusion

Causal inference in time series is a common problem in the literature. In many real-world scenarios, the causal mechanisms are much more visible during some extreme periods, and not in the bulk of the distribution. We propose a method that uses mostly extreme events to detect some causal relations. This method is put into a mathematical framework and some of its properties are rigorously proved under certain model assumptions.
This work can potentially have a broad impact on the causal inference theory. It sheds light on some connections between causality and extremes. Many scientific disciplines use causal inference as a baseline of their work. A method that can detect a causal direction in complex heavy-tailed datasets can be very useful in some domains.

This topic provides a wide range of possibilities for future research. For example, what is the distribution of $\hat{\Gamma}_{X,Y}^{\text{time}}(q)$? Can we create some better (consistent) statistic where we cancel the negative bias of $\hat{\Gamma}_{X,Y}^{\text{time}}(q)$ and provide better testing than by a bootstrap (which does not work very well)? Can we also use some observed covariates for a causal inference between $X,Y$? Does this method work even for some light-tailed time series? Can a similar method be used in a more general setup? These questions can lead to potential future research and important results.

Acknowledgements

This project was supported by the Czech Science Foundation (Project No. GA19-16066S) and by the Czech Academy of Sciences (Praemium Academiae awarded to M. Paluš). We wish to thank Valérie Chavez and Katerina Schindler for helpful discussions.

A Auxiliary propositions

Proposition 1. Let $X,Y,(\varepsilon_i, i \in \mathbb{N})$ be independent continuous random variables with support on some neighbourhood of infinity. Let $a_i, b_i \geq 0, i \in \mathbb{N}$ and $M_1, M_2 \in \mathbb{R}$ be constants. Then

$$P(X+Y > M_1 \mid a_1 X + a_2 Y > M_2) \geq P(X+Y > M_1),$$

or more generally,

$$P\left(\sum_{i=1}^{\infty} a_i \varepsilon_i > M_1 \mid \sum_{i=1}^{\infty} b_i \varepsilon_i > M_2\right) \geq P\left(\sum_{i=1}^{\infty} a_i \varepsilon_i > M_1\right),$$

provided that the sums are a.s. summable, non-trivial.

Proof. Let $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$, then for any non-decreasing functions $f, g : \mathbb{R}^n \to \mathbb{R}$ we have

$$\text{cov}(f(\varepsilon), g(\varepsilon)) \geq 0.$$

This is a well-known result from the theory of associated random variables, see e.g. Theorem 2.1 in (Esary et al, 1967). The functions $f(x_1, \ldots, x_n) =$
$1\{\sum_{i=1}^{n} a_ix_i > M_1\}$ and $g(x_1,\ldots,x_n) = 1\{\sum_{i=1}^{n} b_ix_i > M_2\}$ are non-decreasing because $a_i, b_i \geq 0$. Therefore, we obtain

$$0 \leq \text{cov}(f(\varepsilon), g(\varepsilon))$$

$$= P(\sum_{i=1}^{n} a_i\varepsilon_i > M_1, \sum_{i=1}^{n} b_i\varepsilon_i > M_2) - P(\sum_{i=1}^{n} a_i\varepsilon_i > M_1)P(\sum_{i=1}^{n} b_i\varepsilon_i > M_2).$$

Dividing both sides by $P(\sum_{i=1}^{n} b_i\varepsilon_i > M_2)$ (which is positive), we obtain the inequality

$$P(\sum_{i=1}^{n} a_i\varepsilon_i > M_1 \mid \sum_{i=1}^{n} b_i\varepsilon_i > M_2) \geq P(\sum_{i=1}^{n} a_i\varepsilon_i > M_1)$$

for arbitrary $n \in \mathbb{N}$. The assertion of the proposition follows by taking the limits as $n \to \infty$. 

\begin{prop}

\begin{itemize}
\item Let $\varepsilon^X_i, \varepsilon^Y_i \overset{iid}{\sim} \text{RV}(\theta)$ be continuous.
\item Let $a_i, b_i, c_i \geq 0$ be constants, such that for some $\delta > 0$ we have $
\sum_{i=0}^{\infty} a_i^{\theta-\delta} < \infty, \sum_{i=0}^{\infty} b_i^{\theta-\delta} < \infty, \sum_{i=0}^{\infty} c_i^{\theta-\delta} < \infty$ (it implies that all
$\sum_{i=0}^{\infty} a_i X_i, \sum_{i=0}^{\infty} b_i X_i, \sum_{i=0}^{\infty} c_i Y_i$ are a.s. summable).
\item Denote $A = \sum_{i=0}^{\infty} a_i^\theta, B = \sum_{i=0}^{\infty} b_i^\theta, C = \sum_{i=0}^{\infty} c_i^\theta$, for which it holds that
$A, B, C \in (0, \infty)$.
\item Let $\Phi = \{i \in \mathbb{N} \cup \{0\} : b_i > 0 = a_i\}$.
\end{itemize}

Then

$$\lim_{u \to \infty} P\left(\sum_{i=0}^{\infty} a_i \varepsilon^X_i < M \mid \sum_{i=0}^{\infty} b_i \varepsilon^X_i + \sum_{i=0}^{\infty} c_i \varepsilon^Y_i > u\right) = P\left(\sum_{i=0}^{\infty} a_i \varepsilon^X_i < M\right) \frac{C + \sum_{i \in \Phi} b_i^\theta}{C + B}$$

for all $M \in \mathbb{R}$.

We consider only those $M \in \mathbb{R}$ such that $P(\sum_{i=0}^{\infty} a_i \varepsilon^X_i < M) > 0$, otherwise the statement is trivial. We prove this proposition using the following series of lemmas.

\begin{lem}

Let $X, Y \sim \text{RV}(\theta)$ be independent. Then

$$\lim_{u \to \infty} P(X < M \mid X + Y > u) = P(X < M) \lim_{u \to \infty} \frac{P(Y > u)}{P(Y > u) + P(X > u)},$$

for every $M \in \mathbb{R}$.
\end{lem}

\begin{lem}

Under the conditions from Proposition 2,

$$\lim_{u \to \infty} P\left(\sum_{i=0}^{n} a_i \varepsilon^X_i < M \mid \sum_{i=0; i \notin \Phi}^{n} b_i \varepsilon^X_i > u\right) = 0$$

for all $n \in \mathbb{N}$.
\end{lem}
Lemma 6. Let $Z \sim RV(\theta)$ be independent of $(\varepsilon_i^X, i \in \mathbb{Z})$. Under the conditions from Proposition 2,

$$
\lim_{u \to \infty} P\left(\sum_{i=0}^{n} a_i \varepsilon_i^X < M \mid \sum_{i=0; i \notin \Phi}^{n} b_i \varepsilon_i^X + Z > u\right) = P\left(\sum_{i=0}^{n} a_i \varepsilon_i^X < M \mid Z > u\right) \lim_{u \to \infty} P\left(Z > u\right) + P\left(\sum_{i=0; i \notin \Phi}^{n} b_i \varepsilon_i^X > u\right)
$$

for all $n \in \mathbb{N}$.

Proof of Lemma 4. Using the Bayes theorem, we obtain

$$
P(X < M \mid X + Y > u) = P(X + Y > u \mid X < M) \frac{P(X < M)}{P(X + Y > u)}.
$$

For the denominator we use the sum-equivalence $P(X + Y > u) \sim P(X > u) + P(Y > u)$. Therefore, it is sufficient to show that $P(X + Y > u \mid X < M) \sim P(Y > u)$.

Now, let $W$ be a random variable independent of $Y$ with a distribution satisfying $P(W \leq t) = P(X \leq t \mid X < M)$ for all $t \in \mathbb{R}$. Then, $P(X + Y > u \mid X < M) = P(W + Y > u)$. We obviously have $\lim_{u \to \infty} \frac{P(W > u)}{P(W > u)} = 0$ and we can use e.g., Theorem 2.1. from (Bingham et al, 2006) to obtain $\lim_{u \to \infty} \frac{P(X + Y > u \mid X < M)}{P(Y > u)} = 1$. Therefore $\lim_{u \to \infty} P(X + Y > u \mid X < M) = \lim_{u \to \infty} P(Y > u)$, which we wanted to prove. \hfill \square

Proof of Lemma 5. Without loss of generality $\Phi = \emptyset$, otherwise we have only lower $n$. Denote $\omega = \min_{i \leq n} a_i$, it holds that $\omega > 0$. In this proof only, we denote $B = \sum_{i=0}^{n} b_i$, and $A = \sum_{i=0}^{n} a_i$. The following events relation are valid:

$$
\begin{align*}
\left\{ \sum_{i=0}^{n} a_i \varepsilon_i^X < M; \sum_{i=0}^{n} b_i \varepsilon_i^X > u \right\} & \subseteq \left\{ \exists j \leq n : \varepsilon_j^X > \frac{u}{B} \sum_{i=0}^{n} a_i \varepsilon_i^X < M \right\} \\
& \subseteq \left\{ \exists i, j \leq n : \frac{u}{B} \varepsilon_i^X < M - \frac{\omega u}{A} \right\}.
\end{align*}
$$
(Simply put, there needs to be one large and one small $\varepsilon^X_i$). Therefore, we can rewrite

$$
\lim_{u \to \infty} P\left( \sum_{i=0}^{n} a_i \varepsilon_i^X < M \mid \sum_{i=0, i \notin \Phi}^{n} b_i \varepsilon_i^X > u \right)
\leq \lim_{u \to \infty} \frac{P\left( \exists i, j \leq n : \varepsilon_i^X < \frac{M - \frac{\omega u}{A}, \varepsilon_j^X > \frac{u}{B} \right)}{P\left( \sum_{i=0}^{n} b_i \varepsilon_i^X > u \right)}
= \lim_{u \to \infty} \frac{n(n + 1) P\left( \varepsilon_1^X < \frac{M - \frac{\omega u}{A}, \varepsilon_2^X > \frac{u}{B} \right)}{P\left( \sum_{i=0}^{n} b_i \varepsilon_i^X > u \right)}
= \lim_{u \to \infty} \frac{n(n + 1)}{P\left( \varepsilon_1^X < \frac{M - \frac{\omega u}{A}, \varepsilon_2^X > \frac{u}{B} \right)} \sum_{i=0}^{n} b_i^\theta = 0.
$$

\[\square\]

**Proof of Lemma 6.** Without loss of generality $\Phi = \emptyset$, otherwise we have only lower $n$. The proof is very similar to that of Proposition 1. In this proof only, we denote $B = \sum_{i=0}^{n} b_i$, and $A = \sum_{i=0}^{n} a_i$.

Let $W$ be a random variable independent of $Z$ with a distribution satisfying $P(W \leq t) = P\left( \sum_{i=0}^{n} b_i \varepsilon_i^X \leq t \mid \sum_{i=0}^{n} a_i \varepsilon_i^X < M \right)$ for all $t \in \mathbb{R}$. Then, $P(X + Y > u \mid X < M) = P(W + Y > u)$. Using the Bayes theorem, we have

$$
\lim_{u \to \infty} P\left( \sum_{i=0}^{n} a_i \varepsilon_i^X < M \mid \sum_{i=0}^{n} b_i \varepsilon_i^X + Z > u \right)
= \lim_{u \to \infty} P\left( \sum_{i=0}^{n} b_i \varepsilon_i^X + Z > u \mid \sum_{i=0}^{n} a_i \varepsilon_i^X < M \right) \frac{P\left( \sum_{i=0}^{n} a_i \varepsilon_i^X < M \right)}{P\left( \sum_{i=0}^{n} b_i \varepsilon_i^X + Z > u \right)}
= \lim_{u \to \infty} \frac{P(W + Z > u)}{P\left( \sum_{i=0}^{n} b_i \varepsilon_i^X + Z > u \right)}
= \lim_{u \to \infty} \frac{P(W > u) + P(Z > u)}{P\left( \sum_{i=0}^{n} b_i \varepsilon_i^X > u \right) + P(Z > u)}.
$$

In the last equality, we used the fact that $W$ does not have a heavier tail than $Z$ and therefore we can use the sum-equivalence.
All we need to prove is that \( \lim_{u \to \infty} \frac{P(W > u)}{P(\sum_{i=0}^{n} b_{i}^{X} > u)} = 0. \) Again, using the Bayes theorem, we obtain

\[
\lim_{u \to \infty} \frac{P(W > u)}{P(\sum_{i=0}^{n} b_{i}^{X} > u)} = \lim_{u \to \infty} \frac{P(\sum_{i=0}^{n} b_{i}^{X} > u | \sum_{i=0}^{n} a_{i}^{X} < M)}{P(\sum_{i=0}^{n} b_{i}^{X} > u)} = \lim_{u \to \infty} \frac{1}{P(\sum_{i=0}^{n} a_{i}^{X} < M)} \lim_{u \to \infty} P(\sum_{i=0}^{n} a_{i}^{X} < M | \sum_{i=0}^{n} b_{i}^{X} > u). \]

The rest follows from Lemma 5. \( \square \)

**Proof of Proposition 2.** Let \( \delta > 0, \) define \( \zeta = 1 - \sqrt{1 - \delta} > 0^{17} \) and choose large \( n_0 \in \mathbb{N} \) such that the following conditions hold:

- \( P(| \sum_{i=n_0+1}^{\infty} a_{i}^{X} | > \delta) < \delta, \)
- \( \sum_{i=0}^{n_0} b_{i}^{X} + C > 1 - \zeta, \)
- \( P(| \sum_{i=n_0+1}^{\infty} b_{i}^{X} | < \delta) > 1 - \zeta. \)

Denote

- \( E = \sum_{i=0}^{n_0} a_{i}^{X}, \) \( F = \sum_{i=n_0+1}^{\infty} a_{i}^{X}, \)
- \( G = \sum_{i=0}^{n_0} b_{i}^{X}, \) \( H = \sum_{i=n_0+1}^{\infty} b_{i}^{X}, \)
- \( Z = \sum_{i=0}^{\infty} c_{i}^{Y} + \sum_{i \in \Phi} b_{i}^{X}. \)

Then, \( E, F, Z \) and also \( G, H, Z \) are pair-wise independent. Since due to the sum-equivalence, \( P(Z > u) \sim \left( \sum_{i=0}^{\infty} c_{i}^{Y} + \sum_{i \in \Phi} b_{i}^{Y} \right) P(\varepsilon_{1}^{X} > u) \) and \( P(G + H + Z > u) \sim \left( \sum_{i=0}^{\infty} c_{i}^{Y} + \sum_{i \in \Phi} b_{i}^{Y} \right) P(\varepsilon_{1}^{X} > u) \), with our notation, we want to prove that

\[
\lim_{u \to \infty} \frac{P(E + F < M | G + H + Z > u)}{P(E + F < M) \lim_{u \to \infty} \frac{P(Z > u)}{P(G + H > u) + P(Z > u)}}.
\]

First, due to Lemma 4,

\[
\lim_{u \to \infty} \frac{P(H > \delta | H + (G + Z) > u)}{1 - \lim_{u \to \infty} P(H \leq \delta | H + (G + Z) > u)} = 1 - \lim_{u \to \infty} \frac{P(G + Z > u)}{P(G + Z > u) + P(H > \delta) P(\sum_{i=0}^{n_0} b_{i}^{Y} + C \sum_{i=0}^{\infty} b_{i}^{Y} + C)} < 1 - (1 - \zeta)(1 - \zeta) = \delta.
\]

\(^{17}\) \( 1 - \sqrt{1 - \delta} \) is a solution of \( 1 - (1 - \zeta)(1 - \zeta) = \delta. \) When \( \delta \to 0 \) then also \( \zeta \to 0. \)
Second, using previous results and independence of \( F \) and \((G, Z)\), we obtain

\[
\lim_{u \to \infty} P(F > \delta \mid G + H + Z > u) = \lim_{u \to \infty} \frac{P(F > \delta, G + H + Z > u, H > \delta)}{P(G + H + Z > u)} + \frac{P(F > \delta, G + H + Z > u, H \leq \delta)}{P(G + H + Z > u)} \\
\leq \lim_{u \to \infty} \frac{P(G + H + Z > u, H > \delta)}{P(G + H + Z > u)} + \frac{P(F > \delta, G + Z > u - \delta)}{P(G + H + Z > u)}
\]

\[
= \lim_{u \to \infty} P(H > \delta \mid G + H + Z > u) + \frac{P(F > \delta \mid G + Z > u - \delta)}{P(G + H + Z > u)} = \delta + \frac{P(G + Z > u)}{P(G + Z > u) + P(H > u)} \leq \delta + P(F > \delta) < 2\delta.
\]

Finally, we obtain (the first inequality is trivial; the second uses the identity \( P(A \cap B) \geq P(A) - P(B^c) \); the third uses the previous result; the equality follows from Lemma 6; the next two inequalities follow from the sum-equivalence and trivial \( P(H > u) \geq 0 \); the next is trivial and the last inequality follows from \( P(F + \delta > 0) > 1 - \delta \) and independence of \( E, F \)):

\[
\lim_{u \to \infty} P(E + F < M \mid G + H + Z > u) \geq \lim_{u \to \infty} P(E + \delta < M; F \leq \delta \mid G + H + Z > u) \geq \lim_{u \to \infty} P(E + \delta < M \mid G + H + Z > u) - P(F > \delta \mid G + H + Z > u) \geq \lim_{u \to \infty} P(E < M - \delta \mid G + (H + Z) > u) - 2\delta
\]

\[
= P(E < M - \delta) \lim_{u \to \infty} \frac{P(H + Z > u)}{P(H > u) + P(G > u)} - 2\delta
\]

\[
\geq P(E < M - \delta) \lim_{u \to \infty} \frac{P(H > u) + P(Z > u)}{P(Z > u) + P(H > u) + P(G > u)} - 2\delta
\]

\[
\geq P(E < M - \delta) \lim_{u \to \infty} \frac{P(Z > u)}{P(Z > u) + P(G + H > u)} - 2\delta
\]

\[
\geq P(E + (F + \delta) < M - \delta; (F + \delta) > 0) \lim_{u \to \infty} \frac{P(Z > u)}{P(Z > u) + P(G + H > u)} - 2\delta
\]

\[
\geq (1 - \delta)P(E + F + \delta < M - \delta) \lim_{u \to \infty} \frac{P(Z > u)}{P(Z > u) + P(G + H > u)} - 2\delta.
\]

When we send \( \delta \to 0 \), we finally obtain

\[
P(E + F < M \mid G + H + Z > u) \geq P(E + F < M) \lim_{u \to \infty} \frac{P(Z > u)}{P(Z > u) + P(G + H > u)},
\]

which we wanted to show. The inequality in the opposite direction can be done analogously.
Consequence. Under the conditions from Proposition 2,
\[
\lim_{u \to \infty} P\left(\sum_{i=0}^{\infty} a_i |\varepsilon_i^X| < M \mid \sum_{i=0}^{\infty} b_i \varepsilon_i^X + \sum_{i=0}^{\infty} c_i \varepsilon_i^Y > u\right) = P\left(\sum_{i=0}^{\infty} a_i |\varepsilon_i^X| < M\right) \frac{C + \sum_{i \in \Phi} b_i^\theta}{C + B}.
\]

Proof. The proof is analogous as that of Proposition 2. Modified Lemma 4 and Lemma 6 are still valid, just with $|\varepsilon_i^X|$ instead of $\varepsilon_i^X$ in the equations. Modification for Lemma 5 is trivial, because
\[
\lim_{u \to \infty} P\left(\sum_{i=0}^{n} a_i |\varepsilon_i^X| < M \mid \sum_{i=0;}^{n} b_i \varepsilon_i^X > u\right) \leq \lim_{u \to \infty} P\left(\sum_{i=0}^{n} a_i |\varepsilon_i^X| < M \mid \sum_{i=0;}^{n} b_i \varepsilon_i^X > u\right) = 0.
\]

The limiting argument for $n \to \infty$ remains the same. \hfill \Box

Proposition 3. Let $(X,Y)^\top$ follow NAAR model, specified by
\[
X_t = f(X_{t-1}) + \varepsilon_t^X,
Y_t = g_1(Y_{t-1}) + g_2(X_{t-q}) + \varepsilon_t^Y,
\]
where $f,g_1,g_2$ are continuous and satisfy $\lim_{x \to \infty} h(x) = \infty$ and $\lim_{x \to \infty} \frac{h(x)}{x} < 1$, $h = f,g_1,g_2$. Moreover, let $\varepsilon,\varepsilon_t^X,\varepsilon_t^Y \iid RV(\theta)$ be non-negative. If $(X,Y)^\top$ is stationary, then
\[
\lim_{u \to \infty} \frac{P(Y_t > u)}{P(\varepsilon > u)} < \infty.
\]

Lemma 7. Under assumptions from Proposition 3,
\[
\lim_{u \to \infty} \frac{P(X_t > u)}{P(\varepsilon > u)} < \infty.
\]

Proof of Lemma 7. Let $c = \lim_{x \to \infty} \frac{f(x)}{x} \in [0,1)$. First, notice that
\[
\lim_{u \to \infty} \frac{P(f(X_t) > u)}{P(X_t > u)} = c^\theta.
\]
Compute
\[
\lim_{u \to \infty} \frac{P(X_t > u)}{P(\varepsilon > u)} = \lim_{u \to \infty} \frac{P(f(X_{t-1}) + \varepsilon^Y_t > u)}{P(\varepsilon > u)} \\
= 1 + \lim_{u \to \infty} \frac{P(f(X_{t-1}) > u)}{P(\varepsilon > u)} \leq 1 + c^\theta \lim_{u \to \infty} \frac{P(X_{t-1} > u)}{P(\varepsilon > u)} \\
= 1 + c^\theta \lim_{u \to \infty} \frac{P(X_t > u)}{P(\varepsilon > u)}.
\]

We have used the sum-equivalence, independence and the previous equation. Therefore, we have \( \lim_{u \to \infty} \frac{P(X_t > u)}{P(\varepsilon > u)} \leq \frac{1}{1-c^\theta} < \infty. \)

**Proof of Proposition 3.** Find \( c < 1, K \in \mathbb{R} \) such that for all \( x > 0 \) we have
\[
f(x) < K + cx, g_1(x) < K + cx, g_2(x) < K + cx.
\]

Note that \( f(x+y) \leq (K + cx) + (K + cy) \). Then, the following holds a.s.
\[
Y_0 = \varepsilon^Y_0 + g_2(X_{-q}) + g_1(Y_{-1}) \leq \varepsilon^Y_0 + g_2(X_{-q}) + K + cY_{-1} \\
\leq \varepsilon^Y_0 + g_2(X_{-q}) + K + c(\varepsilon^Y_{-1} + g_2(X_{-q-1}) + K + cY_{-2}) \\
\leq (\varepsilon^Y_0 + c\varepsilon^Y_{-1} + c^2 \varepsilon^Y_{-2} + \ldots) \\
+ (g_2(X_{-q}) + cg_2(X_{-q-1}) + c^2 g_2(X_{-q-2}) + \ldots) + (K + cK + c^2 K + \ldots) \\
= \sum_{i=0}^{\infty} c^i \varepsilon^Y_{-i} + \sum_{i=0}^{\infty} c^i K + \sum_{i=0}^{\infty} c^i g_2(X_{q-i}) \leq \sum_{i=0}^{\infty} c^i \varepsilon^Y_{-i} + \frac{2K}{1-c} + \sum_{i=0}^{\infty} c^{i+1} X_{q-i}.
\]

Finally, because \( X_i \) and \( \varepsilon^Y_i \) are all independent,
\[
\lim_{u \to \infty} \frac{P(Y_t > u)}{P(\varepsilon > u)} \leq \lim_{u \to \infty} \frac{P(\sum_{i=0}^{\infty} c^i \varepsilon^Y_{-i} + \frac{2K}{1-c} + \sum_{i=0}^{\infty} c^{i+1} X_{q-i} > u)}{P(\varepsilon > u)} \\
= \lim_{u \to \infty} \frac{P(\sum_{i=0}^{\infty} c^i \varepsilon^Y_{-i} > u) + P(\sum_{i=0}^{\infty} c^{i+1} X_{q-i} > u)}{P(\varepsilon > u)} \\
= \sum_{i=0}^{\infty} c^i \theta + \lim_{u \to \infty} \frac{P(\sum_{i=0}^{\infty} c^{i+1} X_{q-i} > u)}{P(\varepsilon > u)} < \infty,
\]

where we used regular variation property, sum-equivalence, and Lemma 7. \( \square \)

**Remark.** We proved a stronger claim. We showed that for every Heavy-tailed NAAR model, there exist a stable VAR\((q)\) sequence which is a.s. larger. Note that the VAR\((q)\) process defined by
\[
X_t = aX_{t-1} + \varepsilon^X_t; \quad Y_t = bY_{t-1} + dX_{t-q} + \varepsilon^Y_t,
\]
with \( 0 < a, b, d < 1 \), is stable.
B Proofs

**Observation:** Let $X, Y$ be continuous random variables with support on some neighbourhood of infinity, and $F_X, F_Y$ their distribution functions. Then,

$$\lim_{u \to 1^-} \mathbb{E}[F_Y(Y) \mid F_X(X) > u] = 1$$

if and only if $\lim_{u \to \infty} P(Y > M \mid X > u) = 1$ for every $M \in \mathbb{R}$.

**Theorem 1 (Heavy-tailed VAR model).** Let $(X, Y)^\top$ be a time series which follows the Heavy-tailed VAR model. If $X$ causes $Y$, then $\Gamma_{X,Y}^{time}(q) = 1$.

**Proof.** Since $X$ causes $Y$, we get $\delta_p > 0$ for some $p \leq q$. Then

$$\Gamma_{X,Y}^{time}(q) = \lim_{u \to 1^-} \mathbb{E}[\max\{F_Y(Y_0), \ldots, F_Y(Y_q)\} \mid F_X(X_0) > u]$$

$$\geq \lim_{u \to 1^-} \mathbb{E}[F_Y(Y_p) \mid F_X(X_0) > u] = \lim_{u \to \infty} \mathbb{E}[F_Y(Y_p) \mid X_0 > u].$$

Now, if we prove that $\lim_{u \to \infty} P(Y_p > M \mid X_0 > u) = 1$ for all $M \in \mathbb{R}$, it will imply that $\lim_{u \to \infty} \mathbb{E}[F_Y(Y_p) \mid X_0 > u] = 1$ (see **Observation**). Rewrite

$$P(Y_p > M \mid X_0 > u)$$

$$= P(\delta_p X_0 + \sum_{i=1}^q \beta_i Y_{-i} + \sum_{i=1; i \neq p}^q \delta_i X_{-i} + \varepsilon_p^Y > M \mid X_0 > u)$$

$$\geq P(\delta_p u + \sum_{i=1}^q \beta_i Y_{-i} + \sum_{i=1; i \neq p}^q \delta_i X_{-i} + \varepsilon_p^Y > M \mid X_0 > u).$$

Now, using causal representation, we can write

$$X_0 = \sum_{i=0}^\infty a_i \varepsilon_{-i}^X + \sum_{i=0}^\infty c_i \varepsilon_{-i}^Y,$$

$$\sum_{i=1}^q \beta_i Y_{-i} + \sum_{i=1; i \neq p}^q \delta_i X_{-i} + \varepsilon_p^Y = \sum_{i=0}^\infty \phi_i \varepsilon_{-i}^X + \sum_{i=0}^\infty \psi_i \varepsilon_{-i}^Y$$

for some $\phi_i, \psi_i \geq 0$. 
We obtain
\[
\lim_{u \to \infty} P(\delta_p u + \sum_{i=1}^q \beta_i \epsilon_{p-i} + \sum_{i \neq p} \delta_i \epsilon_{p-i} > M \mid X_0 > u)
\]
\[
= \lim_{u \to \infty} P(\sum_{i=0}^\infty \phi_i \epsilon_{q-i} + \sum_{i=0}^\infty \psi_i \epsilon_{q-i} > M - \delta_p u \mid \sum_{i=0}^\infty \alpha_i \epsilon_{-i} + \sum_{i=0}^\infty c_i \epsilon_{-i} > u)\]
\[
\geq \lim_{u \to \infty} P(\sum_{i=0}^\infty \phi_i \epsilon_{q-i} + \sum_{i=0}^\infty \psi_i \epsilon_{q-i} > M - \delta_p u) = 1,
\]
where we used Proposition 1 in the last step. Therefore, \(\lim_{u \to \infty} P(Y_p > M \mid X_0 > u) \geq 1\), which proves the theorem.

**Theorem 1 (Heavy-tailed NAAR model).** Let \((X, Y)\) be a time series which follows Heavy-tailed NAAR model. If \(X\) causes \(Y\), then \(\Gamma_{X,Y}^{\text{time}}(q) = 1\).

**Proof.** We proceed very similarly as in the proof for Heavy-tailed VAR model. We rewrite \(\Gamma_{X,Y}^{\text{time}}(q) \geq \lim_{u \to \infty} E[F_Y(Y_q) \mid X_0 > u]\), which is equal to 1 if \(\lim_{u \to \infty} P(Y_q > M \mid X_0 > u) = 1\) for all \(M \in \mathbb{R}\). We rewrite
\[
\lim_{u \to \infty} P(Y_q > M \mid X_0 > u) = \lim_{u \to \infty} P(g_1(Y_{q-1}) + g_2(X_0) + \epsilon_Y > M \mid X_0 > u).
\]
Since \(X\) causes \(Y\), it holds that \(g_2\) is not constant and \(\lim_{x \to \infty} g_2(x) = \infty\). This implies that there exists \(x_0 \in \mathbb{R}\) such that \(g_2(x) > M\) for all \(M \in \mathbb{R}\). Therefore, for all \(u > x_0\),
\[
P(g_2(X_0) > M \mid X_0 > u) = 1.
\]
Finally, we only use the fact that \(\epsilon_Y\) and \(g_1\) are non-negative. Hence,
\[
\lim_{u \to \infty} P(g_1(Y_{q-1}) + g_2(X_0) + \epsilon_Y > M \mid X_0 > u)
\]
\[
\geq \lim_{u \to \infty} P(g_2(X_0) > M \mid X_0 > u) = 1,
\]
which we wanted to prove.

**Theorem 2 (Heavy-tailed VAR model).** Let \((X, Y)\) be a time series which follows the Heavy-tailed VAR model. If \(Y\) does not cause \(X\), then \(\Gamma_{Y,X}^{\text{time}}(p) < 1\) for all \(p \in \mathbb{N}\).

**Proof.** Let \(M \in \mathbb{R}\) be such that \(P(X_0 < M) > 0\). We show that
\[
\lim_{u \to \infty} P(\max(X_0, \ldots, X_p) < M \mid Y_0 > u) > 0,
\]
from which it follows that \(\lim_{u \to \infty} E[\max(F_X(X_0), \ldots, F_X(X_p)) \mid Y_0 > u] < 1\).
Rewrite

\[ P(\max(X_0, \ldots, X_p) < M \mid Y_0 > u) = P(\max(X_0, \ldots, X_p < M \mid Y_0 > u) \geq P(\max(|X_0| + |X_1| + \cdots + |X_p| < M \mid Y_0 > u). \]

Now, we use the causal representation of the time series, which, because we know that \( Y \) does not cause \( X \), can be written in the form

\[ X_t = \sum_{i=0}^{\infty} a_i \varepsilon_{L-i}^X; \quad Y_t = \sum_{i=0}^{\infty} b_i \varepsilon_{L-i}^Y + \sum_{i=0}^{\infty} d_i \varepsilon_{L-i}^X. \]

We obtain

\[ P(\sum_{t=0}^{p} |X_t| < M \mid Y_0 > u) = P(\sum_{t=0}^{\infty} \sum_{i=0}^{\infty} a_i \varepsilon_{L-i}^X < M \mid Y_0 > u) \geq P(\sum_{t=0}^{\infty} \sum_{i=0}^{\infty} a_i \varepsilon_{L-i}^X < M \mid Y_0 > u) \]

\[ = P(\sum_{i=0}^{\infty} \phi_i \varepsilon_{p-i}^X < M \mid \sum_{i=0}^{\infty} b_i \varepsilon_{-i}^Y + \sum_{i=0}^{\infty} d_i \varepsilon_{-i}^X > u), \]

for \( \phi_i = a_i + \cdots + a_{i-p} \) (we define \( a_j = 0 \) for \( j < 0 \)). Finally, it follows from the consequence of Proposition 2 that

\[ \lim_{u \to \infty} P(\sum_{i=0}^{\infty} \phi_i \varepsilon_{p-i}^X < M \mid \sum_{i=0}^{\infty} b_i \varepsilon_{-i}^Y + \sum_{i=0}^{\infty} d_i \varepsilon_{-i}^X > u) > 0, \]

which we wanted to prove (Theorem 2 requires non-trivial sums, but if \( \forall i : d_i = 0 \) then the series are independent and this inequality holds trivially).

**Theorem 2 (Heavy-tailed NAAR model).** Let \((X, Y)^\top \) be a time series which follows the Heavy-tailed NAAR model. If \( Y \) does not cause \( X \), then \( \Gamma_{Y,X}^{\text{time}}(p) < 1 \) for all \( p \in \mathbb{N} \).

**Proof.** We have

\[ X_t = f(X_{t-1}) + \varepsilon_t^X; \quad Y_t = g_1(Y_{t-1}) + g_2(X_{t-q}) + \varepsilon_t^Y. \]

Choose large \( M \in \mathbb{R} \), such that \( \sup_{x<M} f(x) < M \) and such that\(^{18} \)

\[ P(\varepsilon_0^X < M - \sup_{x<M} f(x)) > 0. \]

\(^{18}\)This is possible from the assumptions on continuity and the limit behaviour of \( f \).
Denote $M^* = \sup_{x < M} f(x)$. Rewrite

$$P(\max(X_0, \ldots, X_q) < M \mid Y_0 > u) = P(X_0 < M, \ldots, X_q < M \mid Y_0 > u)$$

$$= \prod_{i=0}^{q} P(X_i < M \mid X_0 < M, \ldots, X_{i-1} < M, Y_0 > u).$$

Then, as in the proof for the Heavy-tailed VAR model case, if we show that this is strictly larger than 0, it will imply that $\Gamma_{Y,X}^{time}(q) < 1$. We know that for every $i \geq 1$ the following holds

$$\lim_{u \to \infty} P(X_i < M \mid X_0 < M, \ldots, X_{i-1} < M, Y_0 > u) \leq \lim_{u \to \infty} P(f(X_{i-1}) + \varepsilon_i^X < M \mid X_0 < M, \ldots, X_{i-1} < M, Y_0 > u)$$

$$\leq \lim_{u \to \infty} P(M^* + \varepsilon_i^X < M \mid X_0 < M, \ldots, X_{i-1} < M, Y_0 > u)$$

$$= P(M^* + \varepsilon_i^X < M) > 0.$$

We only need to show for the case when $i = 0$ that $\lim_{u \to \infty} P(X_0 > M \mid Y_0 > u) < 1$. Let $Z = g_1(Y_{-1}) + g_2(X_{-q})$, $Z$ is independent with $\varepsilon_0^Y$. After rewriting, we obtain

$$P(X_0 > M \mid Y_0 > u) = P(X_0 > M \mid \varepsilon_0^Y + Z > u) = \frac{P(X_0 > M; \varepsilon_0^Y + Z > u)}{P(\varepsilon_0^Y + Z > u)}.$$

Let $\frac{1}{2} < \delta < 1$ (we will later send $\delta \to 1$). The following events relation are valid:

$$\{X_0 > M; \varepsilon_0^Y + Z > u\}$$

$$\subseteq \{X_0 > M; \varepsilon_0^Y > \delta u\} \cup \{Z > \delta u\} \cup \{Z > (1 - \delta)u; \varepsilon_0^Y > (1 - \delta)u\}.$$

Applying it to the previous equation, we obtain

$$\lim_{u \to \infty} \frac{P(X_0 > M; \varepsilon_0^Y + Z > u)}{P(\varepsilon_0^Y + Z > u)}$$

$$\leq \lim_{u \to \infty} \frac{P(X_0 > M; \varepsilon_0^Y > \delta u) + P(Z > \delta u) + P(Z > (1 - \delta)u; \varepsilon_0^Y > (1 - \delta)u)}{P(\varepsilon_0^Y + Z > u)}$$

$$= \lim_{u \to \infty} \frac{P(X_0 > M)\varepsilon_0^Y > \delta u)}{P(\varepsilon_0^Y + Z > u)} + \frac{P(Z > \delta u)}{P(\varepsilon_0^Y + Z > u)}$$

$$+ \lim_{u \to \infty} P(Z > (1 - \delta)u) \frac{(1 - \delta - \delta)^\theta P(\varepsilon_0^Y > u)}{P(\varepsilon_0^Y + Z > u)}$$

$$= \frac{1}{\delta^\theta} \lim_{u \to \infty} \frac{P(X_0 > M)P(\varepsilon_0^Y > u)}{P(\varepsilon_0^Y + Z > u)} + \frac{P(Z > \delta u)}{P(\varepsilon_0^Y + Z > u)} + 0.$$
The last summand is 0 because \( \lim_{u \to \infty} P(Z > (1 - \delta)u) = 0 \) and \( \frac{P(\varepsilon_0^Y > u)}{P(\varepsilon_0^Y + Z > u)} \leq 1 \) (simply because \( Z \) is a non-negative random variable).

Now, we will use the result from Proposition 3. In the case when \( \lim_{u \to \infty} \frac{P(Z > u)}{P(\varepsilon_0^Y > u)} = 0 \), we obtain (see e.g., Lemma 1.3.2 in (Kulik and Soulier, 2020)) \( \lim_{u \to \infty} \frac{P(\varepsilon_0^Y > u)}{P(\varepsilon_0^Y + Z > u)} = 1 \) and \( \lim_{u \to \infty} \frac{P(Z > u)}{P(\varepsilon_0^Y + Z > u)} = 0 \). Therefore,

\[
\lim_{u \to \infty} \frac{\frac{1}{\delta^\theta} P(X_0 > M) P(\varepsilon_0^Y > u) + P(Z > \delta u)}{P(\varepsilon_0^Y + Z > u)} = \frac{1}{\delta^\theta} P(X_0 > M) < 1,
\]

for \( \delta \) close enough to 1.

On the other hand, if \( \lim_{u \to \infty} \frac{P(Z > u)}{P(\varepsilon_0^Y > u)} = c \in \mathbb{R}^+ \), we also have that \( Z \sim RV(\theta) \) (this follows trivially from the definition of regular variation, tails behaviour is the same up to a constant). Therefore, we can apply the sum-equivalence and we obtain

\[
\lim_{u \to \infty} \frac{P(X_0 > M) P(\varepsilon_0^Y > u) + P(Z > \delta u)}{P(\varepsilon_0^Y + Z > u)} = \frac{1}{\delta^\theta} P(X_0 > M) + c,
\]

which is less than 1 for \( \delta \) close enough to 1. Therefore, we obtain

\[
\lim_{u \to \infty} P(X_0 > M \mid Y_0 > u) < 1,
\]

which we wanted to prove. \( \square \)

**Theorem 3.** Let \((X, Y)^T\) be a time series which follows Heavy-tailed VAR model, with possibly negative coefficients, satisfying the extremal causal condition. Moreover, let \( \varepsilon_i^X, \varepsilon_i^Y \) have full support on \( \mathbb{R} \), iid and satisfy tail balance condition. If \( X \) causes \( Y \), but \( Y \) does not cause \( X \), then \( \Gamma_{|X|,|Y|}^{time}(q) = 1 \), and \( \Gamma_{|Y|,|X|}^{time}(q) < 1 \).

**Proof.** First, we show that if \( Y \) does not cause \( X \), then \( \Gamma_{|Y|,|X|}^{time}(q) < 1 \). This holds even without the extremal causal condition. Similarly as in the proof of Theorem B, it is sufficient to show that for some \( M > 0 \) we have \( \lim_{u \to \infty} P(\sum_{i=0}^\infty a_i \varepsilon_{i-1}^X > M \mid \sum_{i=0}^\infty b_i \varepsilon_{i-1}^Y + \sum_{i=0}^\infty d_i \varepsilon_{i-1}^X > u) < 1 \) for \( t \leq q \).

We use the following fact. Since we assumed that the \( \varepsilon_i^X \) are \( RV(\theta) \) and satisfy the tail balance condition, the following holds:

\[
P(\sum_{i=0}^\infty a_i \varepsilon_{i-1}^X > u) \sim \sum_{i=0}^\infty |a_i|^\theta P(|\varepsilon_0^X| > u) \sim P(\sum_{i=0}^\infty |a_i| \varepsilon_{i-1}^X > u),
\]
(see e.g., page 6 in (Jessen and Mikosch, 2006)). The second step follows simply from the max-sum equivalence. Finally, we use this fact and the triangle inequality to obtain the following relations

\[
P(\left| \sum_{i=0}^{\infty} a_i \varepsilon_{t_{i-1}}^X \right| > M \mid \sum_{i=0}^{\infty} b_i \varepsilon_{t_{i-1}}^Y + \sum_{i=0}^{\infty} d_i \varepsilon_{t_{i-1}}^X > u) \\
\leq \frac{P \left( \sum_{i=0}^{\infty} \left| a_i \right| \varepsilon_{t_{i-1}}^X > M ; \sum_{i=0}^{\infty} \left| b_i \right| \varepsilon_{t_{i-1}}^Y + \sum_{i=0}^{\infty} \left| d_i \right| \varepsilon_{t_{i-1}}^X > u \right)}{P \left( \left| \sum_{i=0}^{\infty} b_i \varepsilon_{t_{i-1}}^Y + \sum_{i=0}^{\infty} d_i \varepsilon_{t_{i-1}}^X \right| > u \right)} \\
\sim \frac{P \left( \sum_{i=0}^{\infty} \left| a_i \right| \varepsilon_{t_{i-1}}^X > M ; \sum_{i=0}^{\infty} \left| b_i \right| \varepsilon_{t_{i-1}}^Y + \sum_{i=0}^{\infty} \left| d_i \right| \varepsilon_{t_{i-1}}^X > u \right)}{P \left( \sum_{i=0}^{\infty} \left| b_i \right| \varepsilon_{t_{i-1}}^Y + \sum_{i=0}^{\infty} \left| d_i \right| \varepsilon_{t_{i-1}}^X > u \right)} \\
= P \left( \sum_{i=0}^{\infty} \left| a_i \right| \varepsilon_{t_{i-1}}^X > M \mid \sum_{i=0}^{\infty} \left| b_i \right| \varepsilon_{t_{i-1}}^Y + \sum_{i=0}^{\infty} \left| d_i \right| \varepsilon_{t_{i-1}}^X > u \right).
\]

This is for \( u \to \infty \) less than 1 due to the classical non-negative case from Proposition 2 (for any \( M \in \mathbb{R} \) such that \( P(\left| \sum_{i=0}^{\infty} a_i \varepsilon_{t_{i-1}}^X > M \right| < 1) \).

Second, we show that if \( X \) causes \( Y \), then \( \mathbb{I}_{\text{time}} \mathbb{I}_{|X|,|Y|}(q) = 1 \). Similarly, as in the proof of Theorem 1, it is sufficient to show that

\[
\lim_{u \to \infty} P(\left| Y_p \right| < M \mid |X_0| > u) = 0
\]

for every \( M \in \mathbb{R} \). Here, \( p \leq q \) is some index with \( \delta_p \neq 0 \). Using the causal representation with the same notation as in the proof of Theorem 1,

\[
\lim_{u \to \infty} P(\left| \sum_{i=0}^{\infty} b_i \varepsilon_{p_{i-1}}^Y + \sum_{i=0}^{\infty} d_i \varepsilon_{p_{i-1}}^X \right| < M \mid \sum_{i=0}^{\infty} a_i \varepsilon_{p_{i-1}}^X > u) \\
\leq \lim_{u \to \infty} P \left( \sum_{i=0}^{\infty} \left| b_i \right| \varepsilon_{p_{i-1}}^Y + \sum_{i=0}^{\infty} \left| d_i \right| \varepsilon_{p_{i-1}}^X < M \mid \sum_{i=0}^{\infty} \left| a_i \right| \varepsilon_{p_{i-1}}^X > u \right),
\]

where we used the same argument as in the first part of the proof. Therefore, we simplified our model and obtained the classical non-negative case. The result follows from the previous theory. Using Lemma 5 we obtain the result for finite \( n \),

\[
\lim_{u \to \infty} P(\sum_{i=0}^{n} \left| b_i \right| \varepsilon_{p_{i-1}}^Y + \sum_{i=0}^{n} \left| d_i \right| \varepsilon_{p_{i-1}}^X < M \mid \sum_{i=0}^{n} \left| a_i \right| \varepsilon_{p_{i-1}}^X > u) = 0,
\]

because \( \Phi = \emptyset \) due to the extremal causal condition. The argument for limiting case \( n \to \infty \) follows the same steps as those in the proof of Proposition 2. \( \square \)

**Theorem 4.** Let \( (X,Y,Z)^T \) follow the three-dimensional stable \( VAR(q) \) model, with non-negative coefficients, where independent noise variables have
the RV(θ) distribution. Let Z be a common cause of both X and Y, and neither X nor Y cause Z. If Y does not cause X, then Γ_{Y,X}^{time}(q) < 1.

Proof. Let our series have the following representation:

\[
\begin{align*}
Z_t &= \sum_{i=0}^{\infty} a_i \varepsilon_{t-i}^Z, \\
X_t &= \sum_{i=0}^{\infty} b_i \varepsilon_{t-i}^X + \sum_{i=0}^{\infty} c_i \varepsilon_{t-i}^Z, \\
Y_t &= \sum_{i=0}^{\infty} d_i \varepsilon_{t-i}^X + \sum_{i=0}^{\infty} e_i \varepsilon_{t-i}^Y + \sum_{i=0}^{\infty} f_i \varepsilon_{t-i}^Z.
\end{align*}
\]

Just as in the proof of Theorem 2, it is sufficient to show that \(\lim_{u \to \infty} P(X_t > M|Y_0 > u) < 1\) for some \(M > 0\). After rewriting,

\[
\lim_{u \to \infty} P(\sum_{i=0}^{\infty} b_i \varepsilon_{t-i}^X + \sum_{i=0}^{\infty} c_i \varepsilon_{t-i}^Z > M | \sum_{i=0}^{\infty} d_i \varepsilon_{t-i}^X + \sum_{i=0}^{\infty} e_i \varepsilon_{t-i}^Y + \sum_{i=0}^{\infty} f_i \varepsilon_{t-i}^Z > u) < 1,
\]

which follows from Proposition 2 (two countable sums can be written as one countable sum).

Lemma 3. Let \((X, Y)\top\) follow the Heavy-tailed VAR model, where X causes Y. Let \(p\) be the minimal lag. Then, \(\Gamma_{X,Y}^{time}(r) < 1\) for all \(r < p\), and \(\Gamma_{X,Y}^{time}(r) = 1\) for all \(r \geq p\).

Proof. Proving that \(\Gamma_{X,Y}^{time}(r) = 1\) for all \(r \geq p\), is an obvious consequence of the proof of Theorem 1 (in the first row of the proof, instead of choosing some \(p \leq q : \delta_p > 0\), we choose \(p\) to be the minimal lag).

Concerning the first part, we only need to prove that \(\Gamma_{X,Y}^{time}(p-1) < 1\), because then also \(\Gamma_{X,Y}^{time}(p-i) \leq \Gamma_{X,Y}^{time}(p-1) < 1\). As in the proof of Theorem 2, we only need to show that \(\lim_{u \to \infty} P(Y_{p-1} > M|X_0 > u) > 0\) for some \(M > 0\).

By rewriting to its causal representation, we obtain the following relation

\[
\lim_{u \to \infty} P(\sum_{i=0}^{\infty} b_i \varepsilon_{p-1-i}^Y + \sum_{i=0}^{\infty} d_i \varepsilon_{p-1-i}^X < M | \sum_{i=0}^{\infty} a_i \varepsilon_{-i}^X + \sum_{i=0}^{\infty} c_i \varepsilon_{-i}^Y > u) > 0.
\]

We only need to realize that \(d_i = 0\) for \(i \in \{1, \ldots, p-1\}\) because \(p\) is the minimal lag. Therefore, \(\varepsilon_0^X\) is independent of \(Y_{p-1}\) and the rest follows from Proposition 2 (where we deal with the two sums as one, and single \(\varepsilon_0^X\) is the second “sum”).

Theorem 5. Let \((X, Y)\top = ((X_t, Y_t)\top, t \in \mathbb{Z})\) be a stationary bivariate time series, whose marginal distributions are absolutely continuous with support on
some neighbourhood of infinity. Let $\Gamma^{time}_{X,Y}(q)$ exists. Let $k_n$ satisfy (2) and
\[
\frac{n}{k_n} P\left( \sup_{x \in \mathbb{R}} |\hat{F}_X(x) - F(x)| > \delta \right) \xrightarrow{n \to \infty} 0, \ \forall \delta > 0. \tag{3}
\]
Then, $\mathbb{E} \hat{\Gamma}^{time}_{X,Y}(q) \xrightarrow{n \to \infty} \Gamma^{time}_{X,Y}(q)^{19}$.

Proof. Throughout the proof, we use the copula fact that $P(F_X(X_1) \leq t) = t$ for $t \in [0, 1]$ and the fact that follows from the stationarity $P(\hat{F}_X(X_1) \leq \frac{k}{n}) = P(X_1 \leq X(_k)) = \frac{k}{n}$, for $k \leq n, k \in \mathbb{N}$. Please note that $X(_k)$ is always meant with respect to (not written) index $n$.

First, notice the following (the third equation follows from the linearity of expectation and stationarity of our series; the fourth equation follows from the definition of conditional expectation; the fifth is quite trivial):
\[
\mathbb{E} \hat{\Gamma}^{time}_{X,Y}(q) = \mathbb{E} \frac{1}{k_n} \sum_{i: X_i \geq \tau_{kn}} \max\{\hat{F}_Y(Y_i), \ldots, \hat{F}_Y(Y_{i+q})\}
= \mathbb{E} \frac{1}{n} \sum_{i=1}^{n} \frac{k_n}{n} \max\{\hat{F}_Y(Y_i), \ldots, \hat{F}_Y(Y_{i+q})\}1[\hat{F}_X(X_i) > 1 - \frac{k_n}{n}]
= \frac{n}{k_n} \mathbb{E}[\hat{F}_Y(\max\{Y_1, \ldots, Y_{q+1}\})1[\hat{F}_X(X_1) > 1 - \frac{k_n}{n}]]
= \frac{n}{k_n} P(\hat{F}_X(X_1) > 1 - \frac{k_n}{n})
\cdot \mathbb{E}[\hat{F}_Y(\max\{Y_1, \ldots, Y_{q+1}\}) | \hat{F}_X(X_1) > 1 - \frac{k_n}{n}]
= \mathbb{E}[\hat{F}_Y(\max\{Y_1, \ldots, Y_{q+1}\}) | \hat{F}_X(X_1) > 1 - \frac{k_n}{n}].
\]

Now, use $\hat{F} = F + \hat{F} - F$ to obtain
\[
\mathbb{E}[\hat{F}_Y(\max\{Y_1, \ldots, Y_{q+1}\}) | \hat{F}_X(X_1) > 1 - \frac{k_n}{n}]
= \mathbb{E}[F_Y(\max\{Y_1, \ldots, Y_{q+1}\}) | \hat{F}_X(X_1) > 1 - \frac{k_n}{n}]
+ \mathbb{E}[(\hat{F}_Y - F_Y)(\max\{Y_1, \ldots, Y_{q+1}\}) | \hat{F}_X(X_1) > 1 - \frac{k_n}{n}].
\]

The second term is less than $\mathbb{E}[\sup_{x \in \mathbb{R}} |\hat{F}_Y(x) - F_Y(x)|] \to 0$ as $n \to \infty$ from the assumptions. All we need to show is that the first term converges to

\[19\text{Do not forget that } \hat{\Gamma}^{time}_{X,Y}(q) \text{ depends on } n.\]
Therefore, all we need to show is the following

\[
\mathbb{E}[F_Y(\max\{Y_1,\ldots,Y_{q+1}\}) \mid \hat{F}_X(X_1) > 1 - \frac{k_n}{n}] = \mathbb{E}[F_Y(\max\{Y_1,\ldots,Y_{q+1}\}) \mid X_1 > X_{n-k_n}].
\]

Therefore, all we need to show is the following

\[
\Gamma^{\text{time}}_{X,Y}(q) = \lim_{u \to \infty} \mathbb{E}[F_Y(\max\{Y_1,\ldots,Y_{q+1}\}) \mid X_1 > u] = \lim_{n \to \infty} \mathbb{E}[F_Y(\max\{Y_1,\ldots,Y_{q+1}\}) \mid X_1 > X_{n-k_n}].
\]

Denote \(Z = F_Y(\max\{Y_1,\ldots,Y_{q+1}\})\). Choose \(u_n \in \mathbb{R}\) such as \(1 - \frac{k_n}{n}\) quantiles of \(X_1\), that is, numbers fulfilling \(P(X_1 > u_n) = \frac{k_n}{n}\). Because \(u_n \to \infty\) it is sufficient to show that

\[
\lim_{n \to \infty} \mathbb{E}[Z \mid X_1 > u_n] \overset{?}{=} \lim_{n \to \infty} \mathbb{E}[Z \mid X_1 > X_{n-k_n}].
\]

Rewrite (using identity \(1[a > b] = 1[c > a > b] + 1[a > c > b] + 1[a > b > c]\) when no ties are present):

\[
\mathbb{E}[Z \mid X_1 > u_n] = \frac{1}{P(X_1 > u_n)} \int_{\Omega} Z \cdot 1[X_1 > u_n]dP = \frac{n}{k_n} \int_{\Omega} Z \cdot 1[X_1 > u_n]dP = \frac{n}{k_n} \int_{\Omega} Z \cdot 1[X_1 > X_{n-k_n} > u_n]dP + \frac{n}{k_n} \int_{\Omega} Z \cdot 1[X_1 > u_n > X_{n-k_n}]dP.
\]

On the other hand, rewrite also

\[
\mathbb{E}[Z \mid X_1 > X_{n-k_n}] = \frac{1}{P(X_1 > X_{n-k_n})} \int_{\Omega} Z \cdot 1[X_1 > X_{n-k_n}]dP = \frac{n}{k_n} \int_{\Omega} Z \cdot 1[u_n > X_1 > X_{n-k_n}]dP + \frac{n}{k_n} \int_{\Omega} Z \cdot 1[X_1 > u_n > X_{n-k_n}]dP.
\]

Note that these two equations differ only in the first term. Therefore, to show the equality, we only need to show that

\[
\frac{n}{k_n} \int_{\Omega} Z \cdot 1[X_{n-k_n} > X_1 > u_n]dP - \frac{n}{k_n} \int_{\Omega} Z \cdot 1[u_n > X_1 > X_{n-k_n}]dP \overset{n \to \infty}{\to} 0.
\]

We show that the first term goes to 0. Analogously, the second term can be shown to converge to 0.
We know that $0 \leq Z \leq 1$ and we have for the first term:

\[
\frac{n}{k_n} \int_{\Omega} Z \cdot 1_{\{X_{(n-k_n)} > X_1 > u_n\}} dP \leq \frac{n}{k_n} P(X_{(n-k_n)} > X_1 > u_n)
\]

\[
= P(X_{(n-k_n)} > X_1 \mid X_1 > u_n) = P(X_{(n-k_n)} > X_1 \mid F_X(X_1) > 1 - \frac{k_n}{n})
\]

\[
= 1 - P(X_1 \geq X_{(n-k_n)} \mid F_X(X_1) > 1 - \frac{k_n}{n})
\]

\[
= 1 - P(\hat{F}_X(X_1) \geq 1 - \frac{k_n}{n} \mid F_X(X_1) > 1 - \frac{k_n}{n})
\]

\[
= 1 - P(F_X(X_1) + (\hat{F}_X(X_1) - F_X(X_1)) \geq 1 - \frac{k_n}{n} \mid F_X(X_1) > 1 - \frac{k_n}{n})
\]

\[
\leq 1 - P(F_X(X_1) - \sup_{x \in \mathbb{R}} |\hat{F}_X(x) - F_X(x)| \geq 1 - \frac{k_n}{n} \mid F_X(X_1) > 1 - \frac{k_n}{n}).
\]

Denote $S_n := \sup_{x \in \mathbb{R}} |\hat{F}_X(x) - F_X(x)|$. It is sufficient for our proof to show that

\[
P(F_X(X_1) - S_n \geq 1 - \frac{k_n}{n} \mid F_X(X_1) > 1 - \frac{k_n}{n}) \xrightarrow{n \to \infty} 1.
\]

Choose $\varepsilon > 1$, define $\delta = 1 - \frac{1}{\varepsilon}$. Rewrite

\[
P(F_X(X_1) - S_n \geq 1 - \frac{k_n}{n} \mid F_X(X_1) > 1 - \frac{k_n}{n})
\]

\[
= \frac{n}{k_n} P(F_X(X_1) - S_n \geq 1 - \frac{k_n}{n} ; F_X(X_1) > 1 - \frac{k_n}{n})
\]

\[
\geq \frac{n}{k_n} P(F_X(X_1) - S_n \geq 1 - \frac{k_n}{n} ; F_X(X_1) > 1 - \frac{k_n}{\varepsilon n})
\]

\[
\geq \frac{n}{k_n} P(S_n \leq \frac{k_n - k_n/\varepsilon}{n} ; F_X(X_1) > 1 - \frac{k_n/\varepsilon}{n})
\]

\[
= \frac{n}{k_n} P(\frac{n}{k_n} S_n \leq \delta ; F_X(X_1) > 1 - \frac{k_n/\varepsilon}{n}).
\]
Use the identity $P(A \cap B) = 1 - P(A^c) - P(B^c) + P(A^c \cap B^c)$ and continue

$$\frac{n}{k_n}P\left(\frac{n}{k_n}S_n \leq \delta; F_X(X_1) > 1 - \frac{k_n/\varepsilon}{n}\right)$$

$$= \frac{n}{k_n}\left[1 - P\left(\frac{n}{k_n}S_n > \delta\right) - P\left(F_X(X_1) \leq 1 - \frac{k_n/\varepsilon}{n}\right)\right]$$

$$+ P\left(\frac{n}{k_n}S_n > \delta; F_X(X_1) \leq 1 - \frac{k_n/\varepsilon}{n}\right)$$

$$\geq \frac{n}{k_n}\left[1 - P\left(\frac{n}{k_n}S_n > \delta\right) - \left(1 - \frac{k_n/\varepsilon}{n}\right) + 0\right]$$

$$= \frac{n}{k_n}\left[\frac{k_n/\varepsilon}{n} - P\left(\frac{n}{k_n}S_n > \delta\right)\right] = \frac{1}{\varepsilon} - \frac{n}{k_n}P\left(\frac{n}{k_n}S_n > \delta\right) \xrightarrow{n \to \infty} 1.$$

Altogether, we proved that $\lim_{n \to \infty} \mathbb{E}[Z \mid X_1 > u_n] = \lim_{n \to \infty} \mathbb{E}[Z \mid X_1 > X_{(n-k_n)}]$, from which the theorem follows. \qed

References

Andel J (1989) On nonlinear models for time series. Statistics https://doi.org/10.1080/02331888908802217

Bhattacharya RN, Lee C (1995) Ergodicity of nonlinear first order autoregressive models. J Theor Probab https://doi.org/10.1007/BF02213462

Bingham N, Goldie C, Omey E (2006) Regularly varying probability densities. Publications de L’Institut mathematique https://doi.org/10.2298/PIM0694047B

Coles S (2001) An Introduction to Statistical Modeling of Extreme Values. Springer, New York, USA, URL https://link.springer.com/book/10.1007/978-1-4471-3675-0

Cox DR, Wermuth N (1996) Multivariate Dependencies: Models, Analysis and Interpretation. Chapman and Hall, URL https://www.routledge.com/Multivariate-Dependencies-Models-Analysis-and-Interpretation/Cox-Wermuth/p/book/9780367401375

Davis RA, Mikosch T (2009) The extremogram: A correlogram for extreme events. Bernoulli https://doi.org/10.3150/09-bej213

Embrechts P, Klüppelberg C, Mikosch T (1997) Modelling Extremal Events for Insurance and Finance. Cambridge University Press

Esary JD, Proschan F, Walkup DW (1967) Association of random variables, with applications. The Annals of Mathematical Statistics https://doi.org/10.1214/aoms/11776987012
Gissibl N, Klüppelberg C (2018) Max-linear models on directed acyclic graphs. URL https://arxiv.org/abs/1512.07522

Gnecco N, Meinshausen N, Peters J, et al (2021) Causal discovery in heavy-tailed models. The Annals of Statistics https://doi.org/10.1214/20-AOS2021

Helmut L (2005) The New Introduction to Multiple Time Series Analysis. Springer, https://doi.org/10.1007/978-3-540-27752-1

Hesterberg T (2014) What teachers should know about the bootstrap: Resampling in the undergraduate statistics curriculum. The American Statistician https://doi.org/10.1080/00031305.2015.1089789

Imbens G, Rubin D (2015) Causal Inference for Statistics, Social, and Biomedical Sciences: An Introduction. Cambridge University Press, https://doi.org/10.1017/CBO9781139025751

Jessen A, Mikosch T (2006) Regularly varying functions. Publications De L’institut Mathematique https://doi.org/10.2298/PIM0694171J

Kalisch M, Mächler M, Colombo D, et al (2012) Causal inference using graphical models with the r package pcalg. Journal of Statistical Software https://doi.org/10.18637/jss.v047.i11

Kiriliouk A, Naveau P (2020) Climate extreme event attribution using multivariate peaks-over-thresholds modeling and counterfactual theory. The Annals of Applied Statistics https://doi.org/10.1214/20-AOAS1355

Kluppelberg C, Krali M (2021) Estimating an extreme bayesian network via scalings. Journal of Multivariate Analysis https://doi.org/10.1016/j.jmva.2020.104672

Kulik R, Soulier P (2020) Heavy-Tailed Time Series. Springer, https://doi.org/10.1007/978-1-0716-0737-4

Manshour P, Balasis G, Paluš M, et al (2021) Causality and information transfer between the solar wind and the magnetosphere–ionosphere system. Entropy https://doi.org/10.3390/e23040390

Mhalla L, Chavez-Demoulin V, Dupuis DJ (2020) Causal mechanism of extreme river discharges in the upper danube basin network. Journal of the Royal Statistical Society https://doi.org/10.1111/rssc.12415

Mikosch T, Samorodnitsky G (2000) The supremum of a negative drift random walk with dependent heavy-tailed steps. The Annals of Applied Probability
Detection of causality in time series using extreme values

Mikosch T, Wintenberger O (2015) A large deviations approach to limit theory for heavy-tailed time series. Probability Theory and Related Fields https://doi.org/10.1007/s00440-015-0654-4

Naveau P, Hannart A, Ribes A (2020) Statistical methods for extreme event attribution in climate science. Annual Review of Statistics and Its Application https://doi.org/10.1146/annurev-statistics-031219-041314

Padoan S, Stupfler G (2020) ExtremeRisks: Extreme Risk Measures. URL https://CRAN.R-project.org/package=ExtremeRisks, r package version 0.0.4

Palachy S (2019) Inferring causality in time series data. URL https://towardsdatascience.com/inferring-causality-in-time-series-data-b8b75fe52c46

Pearl J (2009) Causality: Models, Reasoning and Inference, 2nd edn. Cambridge University Press, USA

Peters J, Buhlmann P (2015) Structural intervention distance for evaluating causal graphs. Neural computation https://doi.org/10.1162/NECO_a_00708

Peters J, Janzing D, Gretton A, et al (2009) Detecting the direction of causal time series. Association for Computing Machinery, https://doi.org/10.1145/1553374.1553477

Peters J, Janzing D, Schölkopf B (2017) Elements of Causal Inference: Foundations and Learning Algorithms. Springer, URL http://library.oapen.org/handle/20.500.12657/26040

R Core Team (2020) R: A Language and Environment for Statistical Computing. R Foundation for Statistical Computing, Vienna, Austria, URL https://www.R-project.org/

Resnick SI (1987) Extreme Values, Regular Variation and Point Processes. Springer, https://doi.org/10.1007/978-0-387-75953-1

Yang J, Hongzhi A (2005) Nonlinear autoregressive models with heavy-tailed innovation. Science China-mathematics https://doi.org/10.1360/03za00321

Zeileis A, Hothorn T (2002) Diagnostic Checking in Regression Relationships. URL https://cran.r-project.org/web/packages/lmtest/vignettes/lmtest-intro.pdf