INFINITESIMAL GENERATORS FOR POLYNOMIAL PROCESSES

WLODEK BRYC AND JACEK WESOŁOWSKI

Abstract. We study the infinitesimal generators of evolutions of linear mappings on the space of polynomials, which correspond to a special class of Markov processes with polynomial regressions. We relate the infinitesimal generator to the unique solution of a certain commutation equation, and we use the commutation equation to find explicit formula for the infinitesimal generators corresponding to the so called free quadratic harnesses.

1. Introduction

The term "polynomial process" as introduced by Cuchiero in [14], see also [16], denotes a time-homogeneous Markov process with infinite state space \( S \subset \mathbb{R}^d \) and polynomial conditional moments of the appropriate degree. For such processes, instead of the semigroup of linear operators on bounded continuous functions on \( S \), one can consider the semigroup of linear transformations \( (P_t)_{t \geq 0} \) that map the linear space \( \mathcal{P} = \mathcal{P}(\mathbb{R}^d) \) of all polynomials in variables \( x_1, \ldots, x_d \) into itself. The crucial property is that transformations \( P_t \) do not increase the degree of a polynomial. Let \( \mathcal{P} = \bigcup_{k=0}^{\infty} \mathcal{P}_{\leq k} \), where \( \mathcal{P}_{\leq k} \) is the linear (finite dimensional) space of all polynomials of degree at most \( k \). Then \( P_t \) maps \( \mathcal{P}_{\leq k} \) into itself for all \( k \). Assuming that \( P_0 \) is the identity and that the semigroup \( (P_t)_{t \geq 0} \) is pointwise continuous at \( t = 0 \), they show in [16] that

\[
P_t = e^{tA},
\]

where the infinitesimal generator

\[
A(f) = \lim_{h \to 0} \frac{1}{h}(P_h(f) - f)
\]

maps \( \mathcal{P}_{\leq k} \) to itself. Furthermore, \( P_t = e^{tA} \) is well defined also for \( t < 0 \), which implies that for \( x = (x_1, \ldots, x_d) \in \mathbb{R}^d \) polynomials \( q_{n_1,\ldots,n_d}(x; t) = P_{-t}(x_1^{n_1} \cdots x_d^{n_d}) \) are martingale polynomials, i.e., \( P_{t-s} q_{n_1,\ldots,n_d}(x; t) = q_{n_1,\ldots,n_d}(x; s) \) (compare [15, Corollary 2.5]). For \( d = 1 \), martingale polynomials arise also for general non-homogeneous Markov processes with polynomial conditional moments, see [30, Theorem 1].

Special cases of Markov processes with polynomial martingales were studied by many authors [3, 4, 10, 12, 17, 18, 22, 23, 24, 25, 26, 27, 28, 29, 31, 32].

We are interested in the study of univariate non-homogeneous Markov processes with polynomial conditional moments, so in our case \( d = 1 \). As in the homogeneous
case, under mild conditions on the support such Markov process generates a unique family \( \{ P_{s,t} : 0 \leq s \leq t \} \) of degree-preserving linear mappings \( \mathcal{P} \to \mathcal{P} \). Some of our examples, in particular those studied in [13], have the property that \( P_{s,t}(f) \) is a polynomial in both \( x \) and \( s \) for all \( f \in \mathcal{P} \). In such cases we can replace the univariate Markov process \( (X_t)_{t \geq 0} \) by the bivariate Markov process \( (t, X_t)_{t \geq 0} \) and apply the theory developed for the time-homogeneous processes on \( \mathbb{R}^2 \). For example, from [2, Corollary 22], we can read out that the infinitesimal generator of the so called \( q \)-Brownian process when considered as \( \mathbb{R}^2 \)-valued homogeneous Markov process acts on a polynomial \( f(x_0, x_1) \) as

\[
A(f)(x_0, x_1) = \frac{\partial}{\partial x_0} f(x_0, x_1) + \int \frac{\partial}{\partial x_1} \left( \frac{f(x_0, y) - f(x_0, x_1)}{y - x_1} \right) P_{q,x_0,x_0}(qx_1, dy),
\]

where \( P_{s,t}(x, dy) \) denotes the transition probability of the \( q \)-Brownian motion. However, in the main example of this paper the conditional moments are rational functions of \( s \), so such a reduction to homogeneous case is not possible.

In this paper we adopt the point of view that the linear mappings \( P_{s,t} \) of \( \mathcal{P} \) can be analyzed “in abstract” without explicit reference to the underlying Markov process and the transition operators. Our main results are algebraic and do not rely on positivity of the mappings \( P_{s,t} \). Of course, a family \( \{ P_{s,t} : 0 \leq s < t \} \) cannot correspond to a Markov process unless it maps positive polynomials to positive polynomials, and sufficient conditions for the existence of a unique Markov process might be difficult to describe in general. On the other hand, there might be some interest in a more algebraic approach where the family \( \{ P_{s,t} : 0 \leq s < t \} \) of degree-preserving mappings of \( \mathcal{P} \) might not correspond to any Markov process, see Concluding Remark 5.1.

With the above in mind, we introduce the following definition. (We would like to emphasize that throughout the paper we assume that the linear mappings preserve the degree of a polynomial.)

**Definition 1.1.** Suppose \( \mathcal{P} \) is the linear space of polynomials in one variable \( x \). A polynomial process on \( \mathbb{R} \) is a family of linear maps \( P_{s,t} : \mathcal{P} \to \mathcal{P} \) with the following properties:

(i) for every \( k = 0, 1, \ldots \) and \( 0 \leq s \leq t \), the linear transformation \( P_{s,t} \) maps \( \mathcal{P}_{\leq k} \) onto itself.

(ii) \( P_{s,t}(1) = 1 \)

(iii) For \( 0 \leq s \leq t \leq u \), we have

\[
P_{s,t} \circ P_{t,u} = P_{s,u}.
\]

Of course, each linear map \( P_{s,t} \) can be identified with an infinite lower-triangular matrix. This is the approach taken in a series of papers [30, 31, 32]. In this paper we follow a more coordinate-free approach in which with each operator we associate a sequence of polynomials. This resembles [1, Chapter 1] who identifies a linear functional on \( \mathcal{P} \) with the sequences of numbers.

**1.1. Algebra \( \mathcal{Q} \) of sequences of polynomials.** With each linear mapping \( \mathcal{P} : \mathcal{P} \to \mathcal{P} \) we associate a unique sequence of polynomials \( \mathcal{P} = (p_0, p_1, \ldots, p_k, \ldots) \) in variable \( x \) by setting \( p_k = P(x^k) \). Of course, if \( \mathcal{P} \) does not increase the degree, then \( p_k \) is of at most degree \( k \), and if \( \mathcal{P} \) maps \( \mathcal{P}_{\leq n} \) onto \( \mathcal{P}_{\leq n} \) for all \( n \) then \( p_k \) is of degree \( k \). But it will be convenient to allow also the more general \( \mathcal{P} \) where the degree of \( k \)-the element could be higher than \( k \).
Under this correspondence, the composition $R = P \circ Q$ of linear operators $P$ and $Q$ on $\mathcal{P}$ induces the multiplication operation $R = PQ$ for the corresponding sequences of polynomials $\mathbb{P} = (p_0, p_1, \ldots, p_k, \ldots)$ and $\mathbb{Q} = (q_0, q_1, \ldots, q_k, \ldots)$, with $R = (r_0, r_1, \ldots)$ given by

$$r_k(x) = \sum_{j=0}^{\deg(q_k)} [q_k]_j p_j(x).$$

Here, for a polynomial $q$ in variable $x$, by $[q]_k$ we denote the coefficient at $x^k$, so that $q(x) = \sum_{k=0}^\infty [q]_k x^k$ (the sum terminates at the degree of $q$).

The above multiplication can be used to write quickly the inverse correspondence that to each sequence $\mathbb{P}$ of polynomials assigns the linear mapping $P : \mathcal{P} \to \mathcal{P}$ by defining $P(p)$ as the first element of the sequence $\mathbb{P}(p, 0, 0, \ldots)$.

In this paper we study $\mathbb{P}_{s,t}$ through the corresponding elements $\mathbb{P}_{s,t}$ of the algebra $\mathbb{Q}$ of all infinite sequences of polynomials in variable $x$ with the multiplication as defined in (1.3). This algebra has identity $E = (1, x, x^2, \ldots)$.

It is clear that degree-preserving linear mappings of $\mathcal{P}$ are invertible.

**Proposition 1.2.** If for every $n$ polynomial $p_n$ is of degree $n$ then $\mathbb{P} = (p_0, p_1, \ldots)$ has multiplicative inverse $\mathbb{Q} = (q_0, q_1, \ldots)$ and each polynomial $q_n$ is of degree $n$.

**Proof.** Write $p_n(x) = \sum_{k=0}^n a_{n,k} x^k$. The inverse $\mathbb{Q} = (q_0, q_1, \ldots)$ is given by the family of polynomials $q_n$ of degree $n$ which solve the following recursion:

$$q_0 = \frac{1}{a_{0,0}}, \quad q_n = \frac{1}{a_{n,n}} \left( x^n - \sum_{j=0}^{n-1} a_{n,j} q_j(x) \right) \text{ for } n \geq 1.$$

It is then clear that $\mathbb{Q} \mathbb{P} = E$.

Since both $\mathbb{P}$ and $\mathbb{Q}$ consist of polynomials of degree $n$ at the $n$-th position of the sequence, the corresponding linear mappings $P, Q$ on the set of polynomials preserve the degree of a polynomial. Since $Q \circ P$ is the identity on each finite-dimensional space $\mathcal{P}_{\leq n}$, we see that $P \circ Q$ is also an identity, so $PQ = E$.

We rewrite Definition 1.1 into the language of algebra $\mathbb{Q}$.

**Definition 1.3** (Equivalent form of Definition 1.1). A polynomial process is a family $\{\mathbb{P}_{s,t} \in \mathbb{Q} : 0 \leq s \leq t\}$ with the following properties

(i) For $0 < s \leq t$ and $n = 0, 1, \ldots$, the $n$-th component of $\mathbb{P}_{s,t}$ is a polynomial of degree $n$.

(ii) $\mathbb{P}_{s,t}(1,0,0,\ldots) = (1,0,0,\ldots)$

(iii) For $0 < s \leq t \leq u$ we assume

$$(1.4) \quad \mathbb{P}_{s,t} \mathbb{P}_{t,u} = \mathbb{P}_{s,u}.$$

In the sequel we will frequently use two special elements of $\mathbb{Q}$

$$\mathbb{F} = (x, x^2, x^3, \ldots),$$

which represents the operator of multiplication by $x$, and

$$\mathbb{D} = (0, 1, x, x^2, \ldots)$$

is its left-inverse. Since $(1,0,0,\ldots) = E - FD$, property (ii) can be written as $\mathbb{P}_{s,t}(E - FD) = E - FD$. 

\[\]
We note that Proposition 1.2 implies that $P_{s,t}$ is invertible so from (1.4) we see that $P_{t,t} = E$ for all $t \geq 0$. In particular, $P_{0,t}$ is invertible and $M_t = P_{0,t}^{-1}$ consists of polynomials $m_k(x;t)$ in variable $x$ of degree $k$. From (1.4), we get $P_{0,s}P_{s,t}M_t = P_{0,t}M_t = E$. Multiplying this on the left by $M_s = P_{0,s}^{-1}$, we see that

\[ M_s = P_{s,t}M_t, \]

i.e., $M_t$ is a sequence of martingale polynomials for $\{P_{s,t}\}$, see [30, Theorem 1]. Conversely, a sequence $M_t = (m_0(x;t), m_1(x;t), \ldots)$ of martingale polynomials with polynomial $m_k(x;t)$ of degree $k$, determines uniquely $P_{s,t}$ via

\[ P_{s,t} = M_sM_t^{-1}. \]

1.2. Quadratic harnesses. Our main interest is in the special class of Markov processes which have linear conditional means under two-sided conditionings (a harness property, see e.g. [19, 20, 33]) and which have quadratic conditional variances under two-sided conditionings (quadratic harness property, see e.g. [8]).

Since two-sided conditioning is difficult to express in the language of linear mappings of $\mathcal{P}$, in this paper we adopt the following algebraic definition.

**Definition 1.4.** We will say that a polynomial process $\{P_{s,t} : 0 \leq s \leq t\}$ is a quadratic harness with parameters $\eta, \theta, \sigma, \tau, \gamma$ if the following three conditions hold:

(i) (martingale property) $P_{s,t}(0, x, 0, 0, \ldots) = (0, x, 0, 0, \ldots)$.

(ii) (harness property)

\[ P_{0,t}F = (Y + tX)P_{0,t}. \]

(iii) (quadratic harness property)

\[ XY - \gamma YX = E + \eta Y + \theta X + \sigma Y^2 + \tau X^2. \]

As the motivation for the above algebraic properties we offer the following explanations:

(i) The martingale property says that $P_{s,t}(x) = x$.

(ii) To decipher the probabilistic meaning of (1.9), let $M_t = (m_0(x;t), m_1(x;t), \ldots)$ be the martingale polynomials as in (1.7). Since $xm_k(x,t)$ is a polynomial of degree $k + 1$, it can be written as a linear combination of $m_0(x,t), \ldots, m_{k+1}(x,t)$. So there exists a sequence $\mathcal{J}_t = (j_0(x;t), j_1(x;t), \ldots)$ of polynomials $j_k(x;t)$ in variable $x$ of degree $k + 1$ such that

\[ FM_t = M_t\mathcal{J}_t \]

(recall (1.5).) In view of (1.8), harness property (1.9) can be written equivalently as

\[ FM_t = M_t(Y + tX). \]

So Definition 1.4(ii) requires that $\mathcal{J}_t = Y + tX$ depends linearly on $t$. For polynomials processes arising from Markov processes with exponential moments, the latter is equivalent to the harness property by [32, Section 4.1]; this equivalence is also implicit in the proof of [8, Theorem 2.3] and is explicitly used in [12, page 1244].
(iii) Suppose \( \{P_{s,t}\} \) arise from a Markov process with polynomial conditional moments which is a harness and has quadratic conditional variances. Then under mild technical assumptions, [8, Theorem 2.3] shows that \( \mathcal{X}, \mathcal{Y} \) from representation (1.11) for the martingale polynomials, satisfy commutation equation (1.10). For a Markov process with exponential moments and polynomial conditional moments, the converse holds too. Namely, (1.10) together with (1.11) imply that conditional expectations are linear under the two-sided conditioning and conditional variances are quadratic under the two-sided conditioning.

Polynomial process \( \{P_{s,t}\} \) with harness property (2.2) is uniquely determined by \( X \) and \( Y \) through formulas (1.11) and (1.8). Since the \( k \)-th element of \( Y + tX \) is a polynomial of degree \( k + 1 \), it follows from (1.11) that \( M_t \) is a rational function of \( t \). Since \( M_t \) is well defined for all \( t \geq 0 \), formula (1.8) shows that the left infinitesimal generator

\[
A_t = \lim_{h \to 0^+} \frac{1}{h}(P_{t,t+h} - E), \quad t > 0
\]

is a well defined element of \( \mathcal{Q} \), and that \( A_t M_t = -\frac{\partial}{\partial t} M_t \).

We note that although the left infinitesimal generator is easier to work with, when \( P_{s,t} \) is continuous in \( t \) the right infinitesimal generator exists and is given by the same expression. To see this, we compute \( \lim_{h \to 0^+} \frac{1}{h}(P_{t,t+h} - E) \) on the polynomials from the sequence \( M_t \), which of course span \( \mathcal{P} \). We have

\[
\lim_{h \to 0^+} \frac{1}{h}(P_{t,t+h}M_t - M_t) = \lim_{h \to 0^+} P_{t,t+h} \frac{1}{h}(M_t - M_{t+h}) = \lim_{h \to 0^+} P_{t,t+h} \lim_{h \to 0^+} \frac{1}{h}(M_t - M_{t+h}) = -\frac{\partial}{\partial t} M_t.
\]

The infinitesimal generator \( A_t \) and its companion operator \( A_t : \mathcal{P} \to \mathcal{P} \) are the main objects of interest in this paper.

**Remark 1.5.** We note that \( A_t = (0, a_1(x;t), a_2(x;t), \ldots) \) always starts with a 0, as from \( P_{s,t}(1) = 1 \) it follows that \( A_t(1) = 0 \). It is clear that \( a_n(x;t) \) is a polynomial in \( x \) of degree at most \( n \). Martingale property implies that \( a_1(x;t) = 0 \), as \( A_t(x) = 0 \). The generator then starts with two zeros, \( A_t = (0, 0, a_2(x;t), a_3(x;t), \ldots) \).

2. Basic properties of infinitesimal generators for quadratic harnesses

Our first result introduces an auxiliary element \( \mathbb{H}_t \) that is related to \( A_t \) by a commutation equation.

**Theorem 2.1.** Suppose that \( \{P_{s,t} \in \mathcal{Q} : 0 \leq s \leq t\} \) is a quadratic harness as in Definition 1.4 with generator \( A_t \). Let

\[
\mathbb{H}_t = A_t F - \mathbb{F} A_t
\]

and denote \( T_t = F - t\mathbb{H}_t \). Then

\[
\mathbb{H}_t T_t - \gamma \tau \mathbb{H}_t \mathbb{H}_t = E + \theta \mathbb{H}_t + \eta T_t + \tau \mathbb{H}_t^2 + \sigma T_t^2.
\]
Proof. Multiplying (1.12) by \( M_t \) from the right we see that \( \frac{\partial}{\partial t} M_t = -A_t M_t \). Therefore, differentiating (1.11) we get

\[
-F A_t M_t = -A_t M_t(Y + tX) + M_t X.
\]

Multiplying this from the right by \( M_t^{-1} \) and using (1.9) to replace \( M_t(Y + tX)M_t^{-1} \) by \( F \) we get

\[
-F A_t = -A_t F + M_t X M_t^{-1}.
\]

Comparing this with (2.1) we see that

\[
\mathbb{H}_t = M_t X M_t^{-1}.
\]

We now use this in the definition of \( T_t \). After replacing \( F \) by \( M_t(Y + tX)M_t^{-1} \) we get

\[
T_t = M_t Y M_t^{-1}.
\]

Equation (2.2) now arises from multiplying (1.10) by \( M_t \) from the left and by \( M_t^{-1} \) from the right.

Remark 2.2. As observed in Remark 1.5 the \( n \)-th element of \( A_t \) is a polynomial of degree at most \( n \). Thus writing \( \mathbb{H}_t = (h_0(x), h_1(x), \ldots) \), from (2.1) we see that \( h_n \) is of degree at most \( n + 1 \). Martingale property implies that \( h_0(x) = 0 \).

We will need several uniqueness results that follow from the more detailed elementwise analysis of the sequences of polynomials. We first show that the generator determines uniquely the polynomial process, at least under the harness (1.9) condition. (For homogeneous polynomial processes (1.1) is a stronger version of this result.)

**Proposition 2.3.** A polynomial process \( \{P_{s,t} : 0 \leq s < t\} \) with property (1.9) is determined uniquely by its generator \( A_t \).

Proof. Fix \( s < t \). Combining (1.8) with (1.11) we get

\[
P_{s,t} F - F P_{s,t} = M_s M_t^{-1} F - F M_s M_t^{-1} = M_s(Y + tX)M_t^{-1} - M_s(Y + sX)M_t^{-1} = (t-s) M_s M_t^{-1} (M_t M_t^{-1}).
\]

Therefore, from (2.3) we get

\[
P_{s,t} F = F P_{s,t} + (t-s) P_{s,t} \mathbb{H}_t.
\]

We write \( \mathbb{H}_t = (h_0(x), h_1(x), \ldots) \) where \( h_n \) is of degree at most \( n + 1 \), see Remark 2.2. For \( n \geq 0 \), we have

\[
(1 - (t-s)[h_{n+1}]P_{s,t}(x^{n+1}) = x P_{s,t}(x^n) + (t-s) \sum_{k=0}^{n} [h_n]_k P_{s,t}(x^n).
\]

(Recall that \([p]_j\) denotes the coefficient of \( x^j \) of a polynomial \( p \).) We now note that the coefficient \( 1 - (t-s)[h_{n+1}] \) cannot be zero. Indeed, if \( 1 - (t-s)[h_{n+1}] = 0 \) then the left hand side of (2.4) is zero. But \( P_{s,t}(x^n) \) is of degree \( n \), so the right hand side of (2.4) cannot be zero as its degree is \( n + 1 \) due to a single term of degree \( n + 1 \) arising from \( x P_{s,t}(x^n) \).

Since \( P_{s,t}(1) = 1 \), equation (2.4) determines recurrently each polynomial \( P_{s,t}(x^{n+1}) \) in terms of the lower degree polynomials \( P_{s,t}(x), P_{s,t}(x^2), \ldots, P_{s,t}(x^n) \) and the sequence \( \mathbb{H}_t \). Since \( \mathbb{H}_t \) is expressed in terms of \( A_t \) by (2.1), this ends the proof. \( \square \)
Next, we show that under mild technical conditions, $\mathbb{H}_t$ is uniquely determined by the commutation equation (2.2). From the proof of Proposition 2.3 it therefore follows that the entire quadratic harness $\{\mathbb{P}_{s,t}\}$ as well as its generator are also uniquely determined by (2.2).

**Proposition 2.4.** If $\gamma \leq 1$, $\sigma, \tau \geq 0$ and $\sigma \tau \neq 1$ then equation (2.2) has a unique solution among $\mathbb{H}_t \in \mathbb{Q}$ such that $h_0(x) = 0$.

**Proof.** Eliminating $T_t = F - tH_t$ from (2.2) we can rewrite it into the following equivalent form.

\begin{equation}
\mathbb{H}_t F - \gamma \mathbb{H}_t = \mathbb{E} + \theta \mathbb{H}_t + \eta (\mathbb{F} - tH_t) + \tau H_t^2 + \sigma (\mathbb{F} - tH_t)^2 + (1 - \gamma)tH_t^2.
\end{equation}

Write $\mathbb{H}_t = \mathbb{H} = (h_n(x))_{n=0,1,\ldots}$ for a fixed $t > 0$, with $h_0 = 0$. Looking at the $n$-th element of (2.5) for $n \geq 0$, we see that

\begin{equation}
(1 + \sigma t - [h_n]_{n+1}(\sigma t^2 + (1 - \gamma) t + \tau))h_{n+1}(x)
= x^n + \eta x^{n+1} + \sigma x^{n+2} + (\theta - t\eta)h_n(x)
+ (\gamma - \sigma t)xh_n(x) + (\sigma t^2 + (1 - \gamma) t + \tau) \sum_{j=0}^n [h_n]_j h_j(x).
\end{equation}

The highest degree term on the right hand side is $(\sigma + (\gamma - \sigma t)[h_n]_{n+1})x^{n+2}$. We need to consider separately two cases.

**Case $\gamma + \sigma \tau \neq 0$.** Suppose that for some $n \geq 0$ the coefficient at $h_{n+1}(x)$ on the left hand side of (2.6) is 0. Then, noting that for $\gamma \leq 1$ quadratic polynomial $\sigma t^2 + (1 - \gamma) t + \tau$ has no roots $t > 0$, we get

\begin{equation}
[h_n]_{n+1} = \frac{1 + \sigma t}{\sigma t^2 + (1 - \gamma) t + \tau}.
\end{equation}

We now use this value to compute the coefficient at $x^{n+2}$ on the right hand side of (2.6). We get

\begin{equation}
\sigma + (\gamma - \sigma t)[h_n]_{n+1} = \frac{\gamma + \sigma \tau}{\sigma t^2 + (1 - \gamma) t + \tau} \neq 0.
\end{equation}

Since the left hand side of (2.6) is 0, and the degree of the right hand side of (2.6) is $n + 2$, this is a contradiction.

This shows the coefficient of $h_{n+1}(x)$ on the left hand side of (2.6) is non-zero for all $n \geq 0$. So each polynomial $h_{n+1}(x)$ is determined uniquely and has degree at most $n + 2$ for all $n \geq 0$.

**Case $\gamma + \sigma \tau = 0$.** In this case $\sigma t^2 + (1 - \gamma) t + \tau = (1 + \sigma t)(\tau + t)$. Comparing the coefficients at $x^{n+2}$ on both sides of (2.6) we get

\begin{equation}
(1 + \sigma t)(1 - [h_n]_{n+1}(\tau + t))[h_{n+1}]_{n+2} = \sigma(1 - [h_n]_{n+1}(\tau + t)).
\end{equation}

Since $h_0(x) = 0$, this gives $[h_1]_2 = \sigma/(1 + \sigma t)$. So for $n = 1$ we get $1 - [h_n]_{n+1}(\tau + t) = (1 - \sigma \tau)/(1 + \sigma t) \neq 0$. Dividing both sides of (2.7) by this expression we get recursively $[h_n]_{n+1} = \sigma/(1 + \sigma t)$ for all $n \geq 1$. Therefore, for $n \geq 1$, the left hand side of (2.6) simplifies to $(1 - \sigma \tau)h_{n+1}(x)$. Using again $1 - \sigma \tau \neq 0$, this shows that polynomial $h_{n+1}$ is determined uniquely and its degree is at most $n + 2$. Of course, (2.6) determines $h_1(x)$ uniquely, too, as $h_0(x) = 0$. 

$\square$
Our main result is the identification of the infinitesimal generator for quadratic harnesses with \( \gamma = -\sigma \tau \). Such processes were called “free quadratic harnesses” in [8, Section 4.1]. Markov processes with free quadratic harness property were constructed in [9] but the construction required more restrictions on the parameters than what we impose here.

In this section we represent the infinitesimal generators by the series expansion in the left-inverse (1.6) of \( F \). Similar series representations for infinitesimal generators of polynomial processes arising from Lévy processes use the sequence \( E \) for small enough \( z \) solves the quadratic equation
\[
\phi(D) = (c_0, c_0x + c_1, c_0x^2 + c_1x + c_2, \ldots, \sum_{j=0}^{n} c_j x^{n-j}, \ldots)
\]
is a well defined element of \( Q \).

**Theorem 2.5.** Fix \( \eta, \theta \in \mathbb{R} \) and \( \sigma, \tau \geq 0 \) such that \( \sigma \tau \neq 1 \). Then the infinitesimal generator of the quadratic harness with the above parameters and with \( \gamma = -\sigma \tau \) is given by
\[
A_t = \frac{1}{1+\sigma t}(E + \eta F + \sigma F^2)D_1 \phi_t(D)D, \quad t > 0,
\]
where \( \phi_t(z) = \sum_{k=1}^{\infty} c_k(t)z^{k-1} \) for small enough \( z \) solves the quadratic equation
\[
(z^2 + \eta z + \sigma)(t + \tau)\phi_t^2 + ((\theta - \eta)z - 2t\sigma - \sigma \tau - 1)\phi_t + t\sigma + 1 = 0
\]
and the solution is chosen so that \( \phi_t(0) = 1 \).
(For \( \sigma \tau < 1 \) this solution is written explicitly in formula (4.4) below.)

We note that for \( z = 0 \), equation (2.9) is
\[
\sigma(t + \tau)\phi_t^2(0) - (1 + \sigma t + 2t\sigma)\phi_t(0) + 1 + t\sigma = 0
\]
So \( \phi_t(0) = 1 \) indeed is a solution.

### 3. Proof of Theorem 2.5

The plan of proof is to solve equation (2.2) with \( \gamma = -\sigma t \) for \( H_t \), and then to use the solution to determine \( A_t \) from equation (2.1).

#### 3.1. Part I of proof: solution of equation (2.2) when \( \gamma = -\sigma t \).

Equation (2.2) takes the form
\[
H_t F - tH_t^2 + \sigma tFH_t - \sigma tH_t^2 = E + \theta H_t + \eta(F - tH_t) + \tau H_t^2 + \sigma(F^2 - tH_t F - tFH_t - t^2 H_t^2).
\]
So after simplifications, the equation to solve for the unknown \( H_t \) is
\[
(1 + \sigma t)H_t F = E + \eta F + \sigma F^2 + (\theta - \eta t)H_t - \sigma(t + \tau)FH_t + (t + \tau)(1 + \sigma t)H_t^2.
\]
Lemma 3.1. The solution of (3.1) with initial element 0 is

\[ H_t = \frac{1}{1 + \sigma t} (E + \eta F + \sigma F^2) \varphi_t(D) D, \]

where \( \varphi_t \) satisfies equation (2.9) and \( \varphi_t(0) = 1 \).

Proof. Since \( t > 0 \) is fixed, we suppress the dependence on \( t \) and we use Remark 2.2 to write \( H_t = H = (0, h_1(x), \ldots) \). From (3.1) we read out that \( h_1(x) = \frac{1}{1 + \sigma t} (1 + \eta x + \sigma x^2) \).

From Proposition 2.4 we see that (3.1) has a unique solution. In view of uniqueness, we seek the solution in a special form

\[ H = \frac{1}{1 + \sigma t} (E + \eta F + \sigma F^2) \sum_{k=1}^{\infty} c_k D^k \]

with \( c_1 = 1 \) and \( c_k = c_k(t) \in \mathbb{R} \). Note that

\[
H^2 = \frac{1}{(1 + \sigma t)^2} (E + \eta F + \sigma F^2) \sum_{k=1}^{\infty} c_k D^k (E + \eta F + \sigma F^2) \sum_{j=1}^{\infty} c_j D^j \\
= \frac{1}{(1 + \sigma t)^2} (E + \eta F + \sigma F^2) \sum_{k=1}^{\infty} c_k D^k \sum_{j=1}^{\infty} c_j D^j \\
+ \frac{\eta}{(1 + \sigma t)^2} (E + \eta F + \sigma F^2) \sum_{k=1}^{\infty} c_k D^{k-1} \sum_{j=1}^{\infty} c_j D^j \\
+ \frac{\sigma}{(1 + \sigma t)^2} (F + \eta F^2 + \sigma F^3) \sum_{j=1}^{\infty} c_j D^j + \frac{\sigma}{(1 + \sigma t)^2} (E + \eta F + \sigma F^2) \sum_{k=2}^{\infty} c_k D^{k-2} \sum_{j=1}^{\infty} c_j D^j.
\]

Inserting this into (3.1) we get

\[ E + \eta F + \sigma F^2 + (E + \eta F + \sigma F^2) \sum_{k=2}^{\infty} c_k D^{k-1} = E + \eta F + \sigma F^2 \\
+ \frac{\theta - \eta}{1 + \sigma t} (E + \eta F + \sigma F^2) \sum_{k=1}^{\infty} c_k D^k - \frac{\sigma(t + \tau)}{1 + \sigma t} (F + \eta F^2 + \sigma F^3) \sum_{k=1}^{\infty} c_k D^k \\
+ \frac{t + \tau}{1 + \sigma t} [(E + \eta F + \sigma F^2) \sum_{k=1}^{\infty} c_k D^k \sum_{j=1}^{\infty} c_j D^j + \eta(E + \eta F + \sigma F^2) \sum_{k=1}^{\infty} c_k D^{k-1} \sum_{j=1}^{\infty} c_j D^j \\
+ \sigma(F + \eta F^2 + \sigma F^3) \sum_{j=1}^{\infty} c_j D^j + \sigma(E + \eta F + \sigma F^2) \sum_{k=2}^{\infty} c_k D^{k-2} \sum_{j=1}^{\infty} c_j D^j].
\]

The terms with \( (F + \eta F^2 + \sigma F^3) \) cancel out, so \( (E + \eta F + \sigma F^2) \) factors out. We further restrict our search for the solution by requiring that the remaining factors
match, i.e.
\[
\sum_{k=1}^{\infty} c_{k+1} D^k = \frac{\theta - t \eta}{1 + \sigma} \sum_{k=1}^{\infty} c_k D^k
\]
\[+
\frac{t + \tau}{1 + \sigma t} \left( \sum_{k=1}^{\infty} c_k D^k \sum_{j=1}^{\infty} c_j D^j + \eta \sum_{k=1}^{\infty} c_k D^{k-1} \sum_{j=1}^{\infty} c_j D^j + \sigma \sum_{k=2}^{\infty} c_k D^{k-2} \sum_{j=1}^{\infty} c_j D^j \right).\]

Collecting the coefficients at the powers of $D$ we get
\[
\sum_{k=1}^{\infty} c_{k+1} D^k = \frac{\theta - t \eta}{1 + \sigma} \sum_{k=1}^{\infty} c_k D^k
\]
\[+
\frac{t + \tau}{1 + \sigma t} \left( \sum_{k=2}^{\infty} \sum_{j=1}^{k-1} c_j c_{k-j} D^k + \eta \sum_{k=1}^{\infty} \sum_{j=0}^{k-1} c_{j+1} c_{k-j} D^k + \sigma \sum_{k=1}^{\infty} \sum_{j=0}^{k-1} c_j c_{k-j} D^k \right).\]

We now compare the coefficients at the powers of $D$. Since $\sigma \tau \neq 1$, for $k = 1$ we get
\[
c_2 = \left( \theta - \eta \right) / \left( 1 - \sigma \tau \right) = \beta \text{ (say)}.\]

So $c_2 = (\theta + \eta \tau) / (1 - \sigma \tau) = \beta$ (say).

For $k \geq 2$, we have the recurrence
\[
c_{k+1} = \frac{\theta - \eta t}{1 + \sigma t} c_k + \frac{t + \tau}{1 + \sigma t} \left( \sum_{j=1}^{k-1} c_j c_{k-j} + \eta \sum_{j=0}^{k-1} c_{j+1} c_{k-j} + \sigma \sum_{j=0}^{k-1} c_j c_{k-j} \right).\]

We solve this recurrence by the method of generating functions. Let $\varphi(z) = \sum_{k=1}^{\infty} c_k z^{k-1}$. Then
\[
\varphi(z) = 1 + \beta z + \sum_{k=2}^{\infty} c_k z^k
\]
\[=
\frac{\theta - \eta t}{1 + \sigma t} \varphi(z) - 1 + \frac{t + \tau}{1 + \sigma t} z^2 \varphi^2(z) + \eta z \frac{t + \tau}{1 + \sigma t} (\varphi^2(z) - 1)
\]
\[+\sigma \frac{t + \tau}{1 + \sigma t} (\varphi(z) (\varphi(z) - 1) - \beta z).\]

This gives quadratic equation (2.9) for $\varphi = \varphi_t$.

\[\square\]

3.2. Part II of proof: solution of equation (2.1). We now use (3.2) to determine $A_t$.

Since $E - \mathbb{F}D = (1, 0, 0, \ldots)$, from Remark 1.5 it follows that $A_t \mathbb{F}D = A_t$. Therefore, multiplying (2.1) by $D$ from the right we get
\[
A_t = FA_t D + H_t D.
\]
Iterating this, we get
\[A_t = \sum_{k=0}^{\infty} F^k H_t D^{k+1},\]
which is well defined as the series consists of finite sums elementwise. Since $H_t$ is given by (3.2), we get

$$A_t = \frac{1}{1+\sigma t} \sum_{k=0}^{\infty} F^k(E + \eta F + \sigma F^2)\varphi_t(D)D^{k+1}.$$  

We now note that

$$\sum_{k=0}^{\infty} F^kD^{k+1} = D_1.$$  

(This can be seen either by examining each element of the sequence, or by solving the equation $D_1 F - FD_1 = E$, which is just a product formula for the derivative, by the previous technique. The latter equation is of course of the same form as (2.1).)

Replacing the series in (3.5) by the right hand side of (3.6) we get (2.8). This ends the proof of Theorem 2.5.

4. Integral representation

In this section we return back to the language of linear operators associated with the sequences of polynomials. We assume that the polynomial process $\{P_{s,t}: 0 \leq s \leq t\}$ corresponds to a quadratic harness from Definition 1.4, and as before we use the parameters $\eta, \theta, \sigma, \tau$ and $\gamma$ to describe the quadratic harness. Infinitesimal generators of several quadratic harnesses, all different than those in Theorem 2.5, have been studied in this language by several authors.

For quadratic harnesses with parameters $\eta = \sigma = 0$ and $\gamma = q \in (-1,1)$, according to [13], the infinitesimal generator $A_t$ acting on a polynomial $f$ is

$$A_t(f)(x) = \int_{\mathbb{R}} \frac{\partial}{\partial x} \left( \frac{f(y) - f(x)}{y - x} \right) \nu_{x,t}(dy),$$

where $\nu_{x,t}(dy)$ is a uniquely determined probability measure. By inspecting the recurrences for the orthogonal polynomials $\{Q_n\}$ and $\{W_n\}$, from [13, Theorem 1.1(ii)] one can read out that for $q^2 t \geq (1 + q)\tau$ probability measure $\nu_{x,t}$ can be expressed in terms of the transition probabilities $P_{s,t}(x, dy)$ of the Markov process by the formula $\nu_{x,t}(dy) = P_{tq^2 -(1+q)\tau,t}(\theta + qx, dy)$. In this form, the formula coincides with Anshelevich [2, Corollary 22] who considered the case $\eta = \theta = \tau = \sigma = 0$ and $\gamma = q \in [-1,1]$. (However, the domain of the generator in [2] is much larger than the polynomials.) Earlier results in Refs. [5, page 392], [6, Example 4.9], and [7] dealt with quadratic harnesses such that $\sigma = \eta = \gamma = 0$.

The following result gives an explicit formula for the infinitesimal generator of the evolution corresponding to the “free quadratic harness” in the integral form similar to (4.1). The main new feature is the presence of an extra quadratic factor in front of the integral in expression (4.3) for the infinitesimal generator. Denote

$$\alpha = \frac{\eta + \theta \sigma}{1 - \sigma \tau}, \quad \beta = \frac{\eta \tau + \theta}{1 - \sigma \tau}.$$  

**Theorem 4.1.** Fix $\sigma, \tau \geq 0$ such that $\sigma \tau < 1$ and $\eta, \theta \in \mathbb{R}$ such that $1 + \alpha \beta > 0$. Let $\gamma = -\sigma \tau$ and let $\varphi_t$ be a continuous solution of (2.9) with $\varphi_t(0) = 1$.  

Then \( \varphi_t \) is a moment generating function of a unique probability measure \( \nu_t \) and the generator of the quadratic harness with the above parameters on \( p \in \mathcal{P} \) is

\[
A_t(p)(x) = \frac{1 + \eta x + \sigma x^2}{1 + \sigma t} \int \frac{\partial}{\partial x} \left( \frac{p(y) - p(x)}{y - x} \right) \nu_t(dy), \quad t > 0.
\]

**Proof.** Since \( \sigma \tau < 1 \), the solution of quadratic equation (2.9) with \( \varphi_t(0) = 1 \) is

\[
\varphi_t(z) = \frac{z(t \eta - \theta) + 2t \sigma + \sigma \tau + 1}{2(t + \tau)(z^2 + z \eta + \sigma)} - \frac{\sqrt{(z(t \eta - \theta) + 2t \sigma + \sigma \tau + 1)^2 - 4(z^2 + z \eta + \sigma)(1 + t \sigma)(t + \tau)}}{2(t + \tau)(z^2 + z \eta + \sigma)}.
\]

We will identify \( \nu_t \) through its Cauchy-Stieltjes transform

\[
G_{\nu_t}(z) = \int \frac{1}{z - x} \nu_t(dx)
\]

which in our case will be well defined for all real \( z \) large enough.

To this end we compute

\[
\varphi_t(1/z)/z = \frac{(1 + \sigma \tau + 2t \sigma)z + t \eta - \theta}{2(t + \tau)(z^2 + z \eta + \sigma)} - \frac{\sqrt{[(1 - \sigma \tau)z - (\alpha + \sigma \beta)t - \beta - \alpha \tau]^2 - 4(1 + \sigma t)(t + \tau)(1 + \alpha \beta)}}{2(t + \tau)(z^2 + z \eta + \sigma)}.
\]

Under our assumptions, the above expression is well defined for real large enough \( z \in \mathbb{R} \).

Expression (4.5) coincides with the Cauchy-Stieltjes transform in [21, Proposition 2.3], with their parameters

\[
c_{SY} = \frac{1 - \sigma \tau}{1 + \sigma t}, \quad \alpha_{SY} = \frac{\eta \tau + \theta}{1 - \sigma \tau}, \quad a_{SY} = \frac{2 \eta \tau + \theta \sigma \tau + \theta + t(\eta \sigma \tau + \eta + 2 \theta \sigma)}{(\sigma \tau - 1)^2}
\]

and

\[
b_{SY} = \frac{(\sigma t + 1)(t + \tau)(\eta^2 \tau + \eta \theta (\sigma \tau + 1) + \theta^2 \sigma + (1 - \sigma \tau)^2)}{(1 - \sigma \tau)^3}.
\]

(We added subscript ”SY” to avoid confusion with our use of \( \alpha \) in (4.2).)

This shows that \( \varphi_t(1/z)/z \) is a Cauchy-Stieltjes transform of a unique compactly-supported probability measure \( \nu_t \).

It is well known that a Cauchy-Stieltjes transform is an analytic function in the upper complex plane, determines measure uniquely, and if it extends to real \( z \) with \( |z| \) large enough then the corresponding moment generating function is well defined for all \( |z| \) small enough and is given by \( G_{\nu_t}(1/z)/z = \varphi_t(z) \). This shows that \( \varphi_t(z) \) is the moment generating function of the probability measure \( \nu_t \).

For a more detailed description of measure \( \nu_t \) and explicit formulas for its discrete and absolutely continuous components we refer to [21, Theorem 2.1]; see also Remark 4.2 below.

Next we observe that (3.3) gives

\[
H_t(x^n) = \frac{1 + \eta x + \sigma x^2}{1 + \sigma t} \sum_{k=1}^{n} c_k(t)x^{n-k}
\]
Writing \( c_k(t) = \int y^{k-1} \nu_t(dy) \), we therefore get
\[
H_t(f)(x) = \frac{1 + \eta x + \sigma x^2}{1 + \sigma} \int \frac{f(y) - f(x)}{y - x} \nu_t(dy).
\]
(4.6)

Since the operator version of relation (2.1) is \( A_n(x^{n+1}) = H_t(x^n) + xA(x^n) \), we derive (4.3) from (4.6) by induction on \( n \); for a similar reasoning see [13, Lemma 2.4].

\[\Box\]

Remark 4.2. Denote by \( \pi_{t,\eta,\theta,\sigma,\tau}(dx) \) the univariate law of \( X_t \) for the free quadratic harness \( (X_t) \) with parameters \( \eta, \theta, \sigma, \tau \) in [9, Section 3]. Then \( \nu_t \) is given by
\[
(4.7) \quad \nu_t(dx) = \frac{1}{t(t + \tau)} (t^2 + \theta tx + \tau x^2) \pi_{t,\eta,\theta,\sigma,\tau}(dx).
\]
We read out this answer from [9, Eqtn. (3.4)] using the following elementary relation between the Cauchy-Stieltjes transforms:

If \( \nu(dx) = (ax^2 + bx + c)\pi(dx) \) and \( m = \int x\pi(dx) \) then the Cauchy-Stieltjes transforms of \( \pi \) and \( \nu \) are related by the formula
\[
(4.8) \quad G_{\pi}(z) = (az^2 + bz + c)G_{\pi}(z) - \alpha m - az - b.
\]

In our setting, \( m = 0, a = \frac{\tau}{t(t + \tau)}, b = \theta/(t + \tau), c = t/(t + \tau) \), and [9, Eqtn. (3.4)] gives
\[
(4.9) \quad G_{\pi}(z) = \frac{\tau z + \theta t}{\tau z^2 + \theta tz + t^2} + \frac{t \left[ t(1 + \sigma z + 2\sigma t)z + \tau \eta - \theta \right]}{2(\sigma z^2 + \eta z + 1)(\tau z^2 + \theta tz + t^2)}
\]
\[\quad - \frac{t \sqrt{[(1 - \sigma \tau)z - (\alpha + \sigma \beta)t - \beta - \alpha \tau]^2 - 4(1 + \sigma \tau)(t + \tau)(1 + \alpha \beta)}}{2(\sigma z^2 + \eta z + 1)(\tau z^2 + \theta tz + t^2)}.
\]

Inserting this expression into the right hand side of (4.8) we get the right hand side of (4.5). Uniqueness of Cauchy-Stieltjes transform implies (4.7).

5. Concluding remarks

5.1. Here we show how one can use harness property (1.9) to construct polynomial processes that do not correspond to a genuine Markov process. From [8, Example 4.8] it follows that for any \( \eta, \theta \in \mathbb{R} \) sequences \( X = D + \eta FD + \eta \theta FD^2 \) and \( Y = F + \theta FD \) satisfy equation (1.10) with \( \sigma = \tau = \gamma = 0 \). Therefore (1.11) and (1.8) define a unique polynomial process \( \{P_{s,t}: 0 \leq s \leq t \} \). However, from [11, Eqtn (15)] it follows that for \( 1 + \eta \theta < 0 \) there are no probability measures that could serve as the univariate laws for a Markov process with the same martingale polynomials \( P_{0,t}^{-1} \).

5.2. It would be interesting to determine which families \( \{A_t: t \geq 0\} \) of operators on \( \mathcal{P} \) are the infinitesimal generators polynomial processes, as well as under what conditions on \( \{A_t\} \) the corresponding polynomial process \( \{P_{s,t}\} \) maps positive polynomials into positive polynomials.

5.3. The generators of polynomial processes arising from Lévy processes are of the form \( \varphi(D) \). Theorem 2.5 identifies generators of the form \( (1 + \eta x + \sigma x^2)D_1 \varphi(D)D \). For \( \gamma < 1 \) we were unable to use Theorem 2.1 to determine the generators for \( \gamma \neq \sigma \tau \) even in the simplest case of \( \eta = \theta = \sigma = \tau = 0 \). In the integral form, this generator is given in [2, Corollary 22], see also [13, Theorem 1.1] which is presented in (4.1) here.
Acknowledgement

WB research was supported in part by NSF grant #DMS-0904720. JW research was supported in part by NCN grant 2012/05/B/ST1/00554.

References

[1] N. Akhiezer, The classical moment problem, Oliver & Boyd Edinburgh, 1965.
[2] M. Anshelevich, Generators of some non-commutative stochastic processes, Probability Theory Related Fields, 157 (2013), pp. 777–815. Arxiv preprint arXiv:1104.1381.
[3] D. Bakry and O. Mazet, Characterization of Markov semigroups on $\mathbb{R}$ associated to some families of orthogonal polynomials, in Séminaire de Probabilités XXXVII, Springer, 2003, pp. 69–80.
[4] P. Barrieu and W. Schoutens, Iterates of the infinitesimal generator and space–time harmonic polynomials of a Markov process, Journal of Computational and Applied Mathematics, 186 (2006), pp. 300–323.
[5] P. Biane and R. Speicher, Stochastic calculus with respect to free Brownian motion and analysis on Wigner space, Probab. Theory Related Fields, 112 (1998), pp. 373–409.
[6] M. Bożežko, B. Kümmerer, and R. Speicher, $q$-Gaussian processes: non-commutative and classical aspects, Comm. Math. Phys., 185 (1997), pp. 129–154.
[7] W. Bryc, Markov processes with free Meixner laws, Stoch. Processes Appl., 120 (2010), pp. 1383–1403.
[8] W. Bryc, W. Matysiak, and J. Wesołowski, Quadratic harnesses, $q$-commutations, and orthogonal martingale polynomials, Trans. Amer. Math. Soc., 359 (2007), pp. 5449–5483. arxiv.org/abs/math.PR/0504194.
[9] ———, Free quadratic harness, Stoch. Proc. Appl., 121 (2011), pp. 657–671. arxiv.org/abs/1003.4771.
[10] W. Bryc and J. Wesołowski, Conditional moments of $q$-Meixner processes, Probab. Theory Related Fields, 131 (2005), pp. 415–441. arxiv.org/abs/math.PR/0403016.
[11] ———, Bi-Poisson process, Infin. Dimens. Anal. Quantum Probab. Relat. Top., 10 (2007), pp. 277–291. arxiv.org/abs/math.PR/0404241.
[12] W. Bryc and J. Wesołowski, Askey–Wilson polynomials, quadratic harnesses and martingales, Ann. Probab., 38 (2010), pp. 1221–1262.
[13] W. Bryc and J. Wesołowski, Infinitesimal generators of $q$-Meixner processes, Stoch. Proc. Appl., 124 (2014), pp. 914–936.
[14] C. Cuchiero, Affine and polynomial processes, PhD thesis, ETH ZURICH, 2011.
[15] C. Cuchiero, M. Keller-Ressel, and J. Teichmann, Polynomial processes and their applications to mathematical finance, arXiv, arXiv:0812.4740 (2008).
[16] ———, Polynomial processes and their applications to mathematical finance, Finance and Stochastics, 16 (2012), pp. 711–740.
[17] E. Di Nardo, An application of symbolic $\mathcal{D}$-harmonic polynomials to the Dirichlet problem, J. Math. Anal. Appl., 112 (1984), pp. 482–490.
[18] E. Di Nardo, A new family of $\mathcal{D}$-harmonic polynomials via symbolic Lévy processes, J. Math. Anal. Appl., 200 (1996), pp. 126–139.
[19] J. M. Hammersley, Harnesses, in Proc. Fifth Berkeley Sympos. Mathematical Statistics and Probability (Berkeley, Calif., 1965/66), Vol. III: Physical Sciences, Univ. California Press, Berkeley, Calif., 1967, pp. 89–117.
[20] R. Mansuy and M. Yor, Harnesses, Lévy bridges and Monsieur Jourdain, Stochastic Process. Appl., 115 (2005), pp. 329–338.
[21] N. Saitoh and H. Yoshida, The infinite divisibility and orthogonal polynomials with a constant recursion formula in free probability theory, Probab. Math. Statist., 21 (2001), pp. 159–170.
[22] W. Schoutens, Stochastic processes and orthogonal polynomials, Springer Verlag, 2000.
[23] W. Schoutens and J. L. Teugels, Lévy processes, polynomials and martingales, Stochastic Models, 14 (1998), pp. 335–349.
[24] A. Sengupta, Time-space harmonic polynomials for continuous-time processes and an extension, Journal of Theoretical Probability, 13 (2000), pp. 951–976.
[25] ——, Markov processes, time–space harmonic functions and polynomials, Statistics & Probability Letters, 78 (2008), pp. 3277–3280.
[26] A. Sengupta and A. Sarkar, Finitely polynomially determined Lévy processes, Electron. J. Probab, 6 (2001), pp. 1–22.
[27] J. L. Solé and F. Utzet, On the orthogonal polynomials associated with a Lévy process, The Annals of Probability, 36 (2008), pp. 765–795.
[28] ———, Time–space harmonic polynomials relative to a Lévy process, Bernoulli, 14 (2008), pp. 1–13.
[29] P. J. Szablowski, Lévy processes, martingales, reversed martingales and orthogonal polynomials, arXiv preprint arXiv:1212.3121, (2012).
[30] ———, On Markov processes with polynomials conditional moments, Trans. AMS (to appear), (2012). arXiv:1210.6055.
[31] ———, On stationary Markov processes with polynomial conditional moments, arXiv preprint arXiv:1312.4887, (2013).
[32] ———, Markov processes, polynomial martingales and orthogonal polynomials, arXiv preprint arXiv:1410.6731, (2014).
[33] D. Williams, Some basic theorems on harnesses, in Stochastic analysis (a tribute to the memory of Rollo Davidson), Wiley, London, 1973, pp. 349–363.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CINCINNATI, PO BOX 210025, CINCINNATI, OH 45221–0025, USA
E-mail address: Wlodzimierz.Bryc@UC.edu

FACULTY OF MATHEMATICS AND INFORMATION SCIENCE WARSAW UNIVERSITY OF TECHNOLOGY PL. POLITECHNIKI 1 00-661 WARSZAWA, POLAND
E-mail address: wesolo@alpha.mini.pw.edu.pl