Poisson cohomology, Koszul duality, and Batalin-Vilkovisky algebras

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Abstract

We study the noncommutative Poincaré duality between the Poisson homology and cohomology of unimodular Poisson algebras, and show that Kontsevich’s deformation quantization as well as Koszul duality preserve the corresponding Poincaré duality. As a corollary, the Batalin-Vilkovisky algebra structures that naturally arise in these cases are all isomorphic.

Keywords: differential calculus, Koszul duality, deformation quantization, Calabi-Yau

1 Introduction

In this paper we study the noncommutative Poincaré duality between the Poisson homology and cohomology of unimodular Poisson algebras, and show that Kontsevich’s deformation quantization as well as Koszul duality preserve the corresponding Poincaré duality, following the works of Shoikhet [27] and Dolgushev [7] among others.

Let \( A = \mathbb{R}[x_1, \cdots, x_n] \) be the real polynomial algebra in \( n \) variables. A Poisson bivector on \( A \), say \( \pi \), is called quadratic if it is in the form

\[
\pi = \sum_{i_1, i_2, j_1, j_2} c_{i_1 i_2 j_1 j_2}^{j_1 j_2} x_{i_1} x_{i_2} \frac{\partial}{\partial x_{j_1}} \wedge \frac{\partial}{\partial x_{j_2}}, \quad c_{i_1 i_2}^{j_1 j_2} \in \mathbb{R}. \tag{1.1}
\]

Several years ago, Shoikhet [27] observed that if \( \pi \) is quadratic, then the Koszul dual algebra \( A! \) of \( A \), namely, the graded symmetric algebra \( \Lambda(\xi_1, \cdots, \xi_n) \) generated by \( n \) elements of degree \(-1\), has a Poisson structure (let us call it the Koszul dual of \( \pi \)), given by

\[
\pi! = \sum_{i_1, i_2, j_1, j_2} c_{i_1 i_2 j_1 j_2}^{j_1 j_2} \xi_{j_1} \xi_{j_2} \frac{\partial}{\partial \xi_{i_1}} \wedge \frac{\partial}{\partial \xi_{i_2}}, \tag{1.2}
\]

and proved that Kontsevich’s deformation quantization preserves this type of Koszul duality. Shoikhet’s result motivates us to study some other properties of a Poisson algebra under Koszul duality.

First, the following theorem is clear from Shoikhet’s article, once we explicitly write down the corresponding complexes.

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Theorem 1.1. Let \( A = \mathbb{R}[x_1, \cdots, x_n] \) be a quadratic Poisson algebra. Denote by \( A^! \) the Koszul dual Poisson algebra of \( A \). Then there are isomorphisms

\[
\text{HP}_\bullet(A) \cong \text{HP}^{-\bullet}(A^!; A^!) \quad \text{and} \quad \text{HP}^\bullet(A) \cong \text{HP}^\bullet(A^!),
\]

where \( A^! := \text{Hom}_\mathbb{R}(A^!, \mathbb{R}) \) is the linear dual of \( A^! \).

In the above theorem, \( \text{HP}_\bullet(\ - \) \) is the Poisson homology, \( \text{HP}^\bullet(\ - \) \) is the Poisson cohomology, and \( \text{HP}^\bullet(A^!; A^!) \) is the Poisson cohomology of \( A^! \) with values in its dual space.

Historically, the Poisson homology and cohomology were introduced by Koszul [16] and Lichnerowicz [20] respectively. In 1997 Weinstein [33] introduced the notion of unimodular Poisson manifolds, and two years later Xu [36] proved that in this case, there is a Poincaré duality between the Poisson cohomology and homology of \( M \). A purely algebraic version of Weinstein’s notion was later formulated by Dolgushev in [7] (see also [18, 22]), and in this case we also have

\[
\text{HP}^\bullet(A) \cong \text{HP}^n_{\ -\bullet}(A),
\]

for some \( n \) depending on \( A \).

For a finite dimensional algebra such as \( A^! \) above, Zhu, Van Oystaeyen and Zhang introduced in [37] the notion of Frobenius Poisson algebras (in the rest of the paper, we shall use the word symmetric instead of Frobenius, just to be consistent with other references), and proved that if they are unimodular in some sense (to be recalled below), then there also exists a version of Poincaré duality:

\[
\text{HP}^\bullet(A^!) \cong \text{HP}^{n-\bullet}(A^!; A^!).
\]

Combining the above two versions of Poincaré duality (1.4) and (1.5) as well as Theorem 1.1, we have the following:

Theorem 1.2. Let \( A = \mathbb{R}[x_1, \cdots, x_n] \) be a quadratic Poisson algebra. Then \((A, \pi)\) is unimodular if and only if its Koszul dual \((A^!, \pi^!)\) is unimodular symmetric. In this case, we have the following commutative diagram:

\[
\begin{array}{ccc}
\text{HP}^\bullet(A) & \cong & \text{HP}^n_{\ -\bullet}(A) \\
\text{HP}^\bullet(A^!) & \cong & \text{HP}^{n-\bullet}(A^!; A^!).
\end{array}
\]

The main technique to prove the above theorem is the so-called “differential calculus”, a notion introduced by Tamarkin and Tsygan in [28]. Later, Lambre [17] used the terminology “differential calculus with duality” to study the “noncommutative Poincaré duality” in these cases.

In the above-mentioned two references [36, 37], the authors also proved that the Poisson cohomology of a unimodular Poisson algebra (in both cases) has a Batalin-Vilkovisky algebra structure. The Batalin-Vilkovisky structure is a very important algebraic structure that has appeared in, for example, mathematical physics, Calabi-Yau geometry and string topology. For unimodular quadratic Poisson algebras, we have the following:

Theorem 1.3. Suppose \( A = \mathbb{R}[x_1, \cdots, x_n] \) is a unimodular quadratic Poisson algebra. Denote by \( A^! \) its Koszul dual. Then

\[
\text{HP}^\bullet(A) \cong \text{HP}^\bullet(A^!)
\]

is an isomorphism of Batalin-Vilkovisky algebras.
The above three theorems have some analogy to the case of Calabi-Yau algebras, which were introduced by Ginzburg [13] in 2006. Suppose a Calabi-Yau algebra, say $A$, is Koszul, then its Koszul dual is a symmetric algebra. In [13] §5.4 Ginzburg stated a conjecture, which he attributed to R. Rouquier, saying that for a Koszul Calabi-Yau algebra, say $A$, its Hochschild cohomology is isomorphic to the Hochschild cohomology of its Koszul dual $A^!$

\[ \text{HH}^\bullet(A) \cong \text{HH}^\bullet(A^!) \] (1.6)

as Batalin-Vilkovisky algebras. This conjecture is recently proved by two authors of the current paper together with G. Zhou in [4]. In fact, Theorem 1.3 may be viewed as a generalization of Rouquier’s conjecture in Poisson geometry, which has been a folklore for several years.

More than just being an analogy, in [7, Theorem 3], Dolgushev proved that for the coordinate ring $A$ of an affine Calabi-Yau Poisson variety, its deformation quantization in the sense of Kontsevich, say $A_h$, is Calabi-Yau if and only if $A$ is unimodular. Similarly Felder and Shoikhet ([10]) and later Willwacher-Calaque ([35]) proved that, for a symmetric Poisson algebra, its deformation quantization is again symmetric if and only if it is unimodular. Based on these results, Dolgushev asked two questions in [7, §7] (see also [8]). The first question is whether there is any relationship between the roles that the unimodularity plays in these two types of deformation quantizations. The following theorem partially answers his question, although both cases that Dolgushev and Felder-Shoikhet considered are more general (i.e., not necessarily Koszul):

**Theorem 1.4.** Suppose $A = \mathbb{R}[x_1, \ldots, x_n]$ is a quadratic Poisson algebra. Denote by $A^!$ the Koszul dual algebra of $A$, and by $A_h$ and $A^!_h$ the Kontsevich deformation quantization of $A$ and $A^!$ respectively. If $A$ is unimodular (and by Theorem 1.3 $A^!$ is unimodular symmetric), then $A_h$ is Calabi-Yau and $A^!_h$ is symmetric, and the following diagram

\[
\begin{array}{ccc}
\text{HP}^\bullet(A[h]) & \cong & \text{HP}^\bullet(A^![h]) \\
\downarrow & & \downarrow \\
\text{HH}^\bullet(A_h) & \cong & \text{HH}^\bullet(A^!_h).
\end{array}
\] (1.7)

is commutative as Batalin-Vilkovisky algebra isomorphisms, where $A[[h]]$ and $A^![[[h]]$ are equipped with the Poisson bivectors $h\pi$ and $h\pi^!$ respectively.

In other words, the first half of the theorem says that, the unimodularity that appears in the deformation quantization of Poisson Calabi-Yau algebras and the one that appears in the deformation quantization of Poisson symmetric algebras are related by Koszul duality. Note that in the theorem, $A_h$ and $A^!_h$ are Koszul dual to each other by Shoikhet [27].

The second question that Dolgushev asked in [7 §7] is whether there exists a relationship between the Poincaré duality of the Poisson (co)homology of $A$ and the Poincaré duality of the Hochschild (co)homology of $A_h$. The following theorem, on which the proof of the second half of Theorem 1.4 is based, answers this question:

**Theorem 1.5.** (1) Suppose $A = \mathbb{R}[x_1, \ldots, x_n]$ is a unimodular Poisson algebra. Let $A_h$ be its deformation quantization. Then the following diagram

\[
\begin{array}{ccc}
\text{HP}^\bullet(A[h]) & \cong & \text{HP}_{n-\bullet}(A[h]) \\
\downarrow & & \downarrow \\
\text{HH}^\bullet(A_h) & \cong & \text{HH}_{n-\bullet}(A_h)
\end{array}
\]
commutes.

(2) Similarly, suppose \( A^i = \Lambda(\xi_1, \cdots, \xi_n) \) is a unimodular symmetric Poisson algebra, and let \( A^i_\hbar \) be its deformation quantization. Then the following diagram

\[
\begin{array}{ccc}
\text{HP}^\bullet(A^i[\hbar]) & \cong & \text{HP}^\bullet - n(A^i[\hbar]; A^i_\hbar) \\
\downarrow & & \downarrow \\
\text{HH}^\bullet(A^i_\hbar) & \cong & \text{HH}^\bullet - n(A^i_\hbar; A^i_\hbar)
\end{array}
\]

commutes.

In other words, the two versions of Poincaré duality, one between the Poisson cohomology and homology, and the other between the Hochschild cohomology and homology, are preserved under Kontsevich’s deformation quantization. Thus as a corollary, one obtains that if \( A = \mathbb{R}[x_1, \cdots, x_n] \) is a unimodular quadratic Poisson algebra, then all the homology and cohomology groups (Poisson and Hochschild) in Theorems 1.4 and 1.5 are isomorphic.

The rest of the paper is devoted to the proof of the above theorems. It is organized as follows: in §2 we collect several facts on Koszul algebras, and their application to quadratic Poisson polynomials; in §3 we first recall the definition of Poisson homology and cohomology, and then prove Theorem 1.1 in §4 we study unimodular quadratic Poisson algebras and their Koszul dual, and prove Theorem 1.2 in §5 we prove Theorem 1.3 by means of the so-called “differential calculus with duality”; in §6 we discuss Calabi-Yau algebras, their Koszul duality and the Batalin-Vilkovisky algebras associated to them; and at last, in §7 we discuss the deformation quantization of Poisson algebras and prove Theorems 1.4 and 1.5.

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Convention. Throughout the paper, \( k \) is a field of characteristic zero, which we may assume to be \( \mathbb{R} \) as in §11. All tensors and morphisms are graded over \( k \) unless otherwise specified. For a chain complex, its homology is denoted by \( H_*(-) \), and its cohomology is \( H^*(-) := H_{-*}(-) \).

2 Preliminaries on Koszul algebras

In this section, we collect some necessary facts about Koszul algebras. The interested reader may refer to Loday-Vallette [21, Chapter 3] for some more details.

Let \( V \) be a finite-dimensional vector space over \( k \). Denote by \( TV \) the free (tensor) algebra generated by \( V \) over \( k \). Suppose \( R \) is a subspace of \( V \otimes V \), and let \( (R) \) be the two-sided ideal generated by \( R \) in \( TV \), then the quotient algebra \( A := TV/(R) \) is called a quadratic algebra.

Consider the subspace

\[
U = \bigoplus_{n=0}^\infty U_n := \bigoplus_{n=0}^\infty \bigcap_{i+j+2=n} V^\otimes i \otimes R \otimes V^\otimes j
\]

of \( TV \), then \( U \) is a coalgebra whose coproduct is induced from the de-concatenation of the tensor products. The Koszul dual coalgebra of \( A \), denoted by \( A^! \), is

\[
A^! = \bigoplus_{n=0}^\infty \Sigma^\otimes n(U_n),
\]
where $\Sigma$ is the degree shifting-up (suspension) functor. $A^i$ has a graded coalgebra structure induced from that of $U$ with

$$(A^i)_0 = k, \quad (A^i)_1 = \Sigma V, \quad (A^i)_2 = (\Sigma \otimes \Sigma)(R), \quad \cdots \quad \cdots$$

The Koszul dual algebra of $A$, denoted by $A^!$, is just the linear dual space of $A^i$, which is then a graded algebra. More precisely, Let $V^* = \text{Hom}(V, k)$ be the linear dual space of $V$, and let $R^\perp$ denote the space of annihilators of $R$ in $V^* \otimes V^*$. Shift the grading of $V^*$ down by one, denoted by $\Sigma^{-1}V^*$, then

$$A^! = \frac{T(\Sigma^{-1}V^*)}{(\Sigma^{-1} \otimes \Sigma^{-1} \circ R^\perp)}$$

Choose a set of basis $\{e_i\}$ for $V$, and let $\{e^*_i\}$ be their duals in $V^*$. There is a chain complex associated to $A$, called the Koszul complex:

$$\cdots \xrightarrow{\delta} A \otimes A_{i+1} \xrightarrow{\delta} A \otimes A_i \xrightarrow{\delta} \cdots \xrightarrow{\delta} A \otimes A_0 \xrightarrow{\delta} k,$$

(2.1)

where for any $r \otimes f \in A \otimes A_i$, $\delta(r \otimes f) = \sum e_i r \otimes \Sigma^{-1} e^*_i f$.

**Definition 2.1** (Koszul algebra). A quadratic algebra $A = TV/(R)$ is called Koszul if the Koszul chain complex (2.1) is acyclic.

**Example 2.2** (Polynomials). Let $A = k[x_1, x_2, \cdots, x_n]$ be the space of polynomials (the symmetric tensor algebra) with $n$ generators. Then $A$ is a Koszul algebra, and its Koszul dual algebra $A^!$ is the graded symmetric algebra $A(\xi_1, \xi_2, \cdots, \xi_n)$, with grading $|\xi_i| = -1$.

**Lemma 2.3** (Shoikhet [27]). Let $A = k[x_1, \cdots, x_n]$ with a bivector $\pi$ in the form (1.1). Then $(A, \pi)$ is quadratic Poisson if and only if $(A^!, \pi^!)$ is quadratic Poisson, where $\pi^!$ is given by (1.2).

So far, we have assumed that $V$ is a $k$-linear space. In [27] we will study the deformed algebras, which are algebras over $k[\hbar]$. In [27], Shoikhet proved that the definitions and results in above subsections remain to hold for algebras over a discrete evaluation ring, such as $k[\hbar]$. For example, $k[x_1, \cdots, x_n][\hbar]$ is Koszul dual to $A(\xi_1, \cdots, \xi_n)[\hbar]$ as graded algebras over $k[\hbar]$ (see [27] Theorem 0.3).

## 3 Poisson homology and cohomology

The notions of Poisson homology and cohomology were introduced by Koszul [16] and Lichnerowicz [20] respectively. Later Huebschmann [14] studied both of them from purely algebraic perspective.

For an commutative algebra $A$, in the following we denote by $\Omega^p(A)$ the set of $p$-th Kähler differential forms of $A$, and by $\mathfrak{X}_A^{-p}(M)$ (or simply $\mathfrak{X}^{-p}(M)$ if $A$ is clear from the context) the space of skew-symmetric multilinear maps $A^{\otimes p} \to M$ that are derivations in each argument. Note that from the universal property of Kähler differentials, there is an identity of left $A$-modules

$$\mathfrak{X}_A^{-p}(M) = \text{Hom}_A(\Omega^p(A), M).$$

\[\text{In the literature such as [21], } A^! \text{ is defined to be } T(V^*)/R^\perp, \text{ or equivalently, } (A^!)_i \cong \Sigma^i \text{Hom}((A^!)_i, k) \text{ but not } \text{Hom}((A^!)_i, k). \text{ This will cause some issues in our later calculations, so in this paper, we take } A^! \text{ as given above, or equivalently } A^! = \text{Hom}(A^!, k).\]
Definition 3.1 (Koszul [16]). Suppose \((A, \pi)\) is a Poisson algebra. Then the Poisson chain complex of \(A\), denoted by \(\text{CP}^\bullet(A)\), is
\[
\cdots \to \Omega^{p+1}(A) \xymatrix{ \ar[r]^-{\partial} & } \Omega^p(A) \xymatrix{ \ar[r]^-{\partial} & } \Omega^{p-1}(A) \xymatrix{ \ar[r]^-{\partial} & } \cdots \to \Omega^0(A) = A,
\]
where \(\partial\) is given by
\[
\partial(f_0 df_1 \wedge \cdots \wedge df_p) = \sum_{i=1}^p (-1)^{i-1}\{f_0, f_i\} df_1 \wedge \cdots \wedge \widehat{df_i} \cdots \wedge df_p + \sum_{1 \leq i < j \leq p} (-1)^{j-i-1} f_0 d\{f_i, f_j\} \wedge df_1 \wedge \cdots \wedge \widehat{df_i} \cdots \wedge \widehat{df_j} \cdots \wedge df_p.
\]
The associated homology is called the Poisson homology of \(A\), and is denoted by \(\text{HP}^\bullet(A)\).

Definition 3.2 (Lichnerowicz [20]). Suppose \((A, \pi)\) is a Poisson algebra and \(M\) is a left Poisson \(A\)-module. The Poisson cochain complex of \(A\) with values in \(M\), denoted by \(\text{CP}^\bullet(A; M)\), is the cochain complex
\[
M = \mathfrak{X}_A^p(M) \xymatrix{ \ar[r]^-{\delta} & } \mathfrak{X}_A^{p+1}(M) \xymatrix{ \ar[r]^-{\delta} & } \mathfrak{X}_A^p(M) \xymatrix{ \ar[r]^-{\delta} & } \cdots
\]
where \(\delta\) is given by
\[
\delta(P)(f_0, f_1, \cdots, f_p) := \sum_{0 \leq i \leq p} (-1)^i\{f_i, P(f_0, \cdots, \widehat{f_i}, \cdots, f_p)\}
\]
\[
+ \sum_{0 \leq i < j \leq p} (-1)^{i+j} P(\{f_i, f_j\}, f_1, \cdots, \widehat{f_i}, \cdots, \widehat{f_j}, \cdots, f_p).
\]
The associated cohomology is called the Poisson cohomology of \(A\) with values in \(M\), and is denoted by \(\text{HP}^\bullet(A; M)\). In particular, if \(M = A\), then the cohomology is just called the Poisson cohomology of \(A\), and is simply denoted by \(\text{HP}^\bullet(A)\).

Note that in the above definition, the Poisson cochain complex, viewed as a chain complex, is negatively graded, and the coboundary \(\delta\) has degree \(-1\). However, by our convention, the Poisson cohomology are positively graded.

Remark 3.3 (The graded case). The Poisson homology and cohomology can be defined for graded Poisson algebras as well. In this case,
\[
\Omega^p(A) = \bigoplus_{n \in \mathbb{Z}} \{ f_0 df_1 \wedge \cdots \wedge df_n | f_i \in A, |f_0| + |f_1| + \cdots + |f_n| + n = p \}
\]
and \(\mathfrak{X}_A^{-p}(M)\) is again given by \(\text{Hom}_A(\Omega^p(A), M)\). The boundary maps are completely analogous to those of Poisson chain and cochain complexes (with Koszul’s sign convention counted).

Proof of Theorem 1.1. (1) We first show the first isomorphism in (1.3). Since \(A = k[x_1, \cdots, x_n]\), we have an explicit expression for \(\Omega^\bullet(A)\), which is
\[
\Omega^\bullet(A) = \Lambda(x_1, \cdots, x_n, dx_1, \cdots, dx_n),
\]
where \(\Lambda\) means the graded symmetric tensor product, and \(|x_i| = 0\) and \(|dx_i| = 1\) for \(i = 1, \cdots, n\). Similarly,
\[
\Omega^\bullet(A^1) = \Lambda(\xi_1, \cdots, \xi_n, d\xi_1, \cdots, d\xi_n),
\]

where \(|\xi_i| = -1\) and \(|d\xi_i| = 0\) for \(i = 1, \cdots, n\), and therefore

\[
\mathfrak{X}_A^\bullet (A^i) = \text{Hom}_{A^i}(\Omega^\bullet (A^i), A^i)
\]

\[
= \text{Hom}_{A^i(\xi_1, \cdots, \xi_n)}(\Lambda(\xi_1, \cdots, \xi_n, d\xi_1, \cdots, d\xi_n), \text{Hom}(\Lambda(\xi_1, \cdots, \xi_n, k))
\]

\[
= \text{Hom}_{A^i(\xi_1, \cdots, \xi_n)}(\Lambda(\xi_1, \cdots, \xi_n) \otimes \Lambda(d\xi_1, \cdots, d\xi_n), \text{Hom}(\Lambda(\xi_1, \cdots, \xi_n, k))
\]

\[
= \text{Hom}(\Lambda(d\xi_1, \cdots, d\xi_n), \text{Hom}(\Lambda(\xi_1, \cdots, \xi_n, k))
\]

\[
= \text{Hom}(\Lambda(d\xi_1, \cdots, d\xi_n, \xi_1, \cdots, \xi_n), k)
\]

\[
= \text{Hom}(\Lambda(d\xi_1, \cdots, d\xi_n, \xi_1, \cdots, \xi_n, k)
\]

\[
= \Lambda\left(\frac{\partial}{\partial \xi_1}, \cdots, \frac{\partial}{\partial \xi_n}, \xi_1^*, \cdots, \xi_n^*\right). \quad (3.4)
\]

Thus from (3.3) and (3.4) there is a canonical grading preserving isomorphism of vector spaces:

\[
\Phi : \Omega^\bullet (A) \rightarrow \mathfrak{X}_A^\bullet (A^i)
\]

\[
x_i \mapsto \frac{\partial}{\partial x_i}
\]

\[
dx_i \mapsto \xi_i^*, \quad i = 1, \cdots, n.
\]

It is a direct check that \(\Phi\) is a chain map, and thus we obtain an isomorphism of Poisson complexes

\[
\Phi : CP^\bullet (A) \cong CP^\bullet (A^i; A^i), \quad (3.5)
\]

which then induces an isomorphism on the homology.

(2) We now show the second isomorphism in (1.3). Similarly to the above argument, we have

\[
\text{CP}^\bullet (A) = \text{Hom}_{A}(\Omega^\bullet (A), A)
\]

\[
= \text{Hom}_{A(\xi_1, \cdots, \xi_n)}(\Lambda(x_1, \cdots, x_n, dx_1, \cdots, dx_n), \Lambda(x_1, \cdots, x_n))
\]

\[
= \text{Hom}_{A(\xi_1, \cdots, \xi_n)}(\Lambda(x_1, \cdots, x_n) \otimes \Lambda(dx_1, \cdots, dx_n), \Lambda(x_1, \cdots, x_n))
\]

\[
= \text{Hom}(\Lambda(dx_1, \cdots, dx_n), \Lambda(x_1, \cdots, x_n))
\]

\[
= \Lambda\left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}\right) \otimes \Lambda(x_1, \cdots, x_n) \quad (3.6)
\]

and

\[
\text{CP}^\bullet (A^i) = \text{Hom}_{A^i}(\Omega^\bullet (A^i), A^i)
\]

\[
= \text{Hom}_{A^i(\xi_1, \cdots, \xi_n)}(\Lambda(\xi_1, \cdots, \xi_n, d\xi_1, \cdots, d\xi_n), \Lambda(\xi_1, \cdots, \xi_n))
\]

\[
= \text{Hom}_{A^i(\xi_1, \cdots, \xi_n)}(\Lambda(\xi_1, \cdots, \xi_n) \otimes \Lambda(d\xi_1, \cdots, d\xi_n), \Lambda(\xi_1, \cdots, \xi_n))
\]

\[
= \text{Hom}(\Lambda(d\xi_1, \cdots, d\xi_n), \Lambda(\xi_1, \cdots, \xi_n))
\]

\[
= \Lambda\left(\frac{\partial}{\partial \xi_1}, \cdots, \frac{\partial}{\partial \xi_n}\right) \otimes \Lambda(\xi_1, \cdots, \xi_n). \quad (3.7)
\]

Under the identity

\[
x_i \mapsto \frac{\partial}{\partial \xi_i}, \quad \frac{\partial}{\partial x_i} \mapsto \xi_i \quad (3.8)
\]

we again obtain an isomorphism of chain complexes

\[
\Psi : \text{CP}^\bullet (A) \cong \text{CP}^\bullet (A^i).
\]

This completes the proof.

\[\square\]
4 Unimodular Poisson algebras and Koszul duality

In this section, we study unimodular Poisson algebras. We are particularly interested in the algebraic structures on their Poisson cohomology and homology groups, which are summarized by differential calculus, a notion introduced by Tamarkin and Tsygan in [28].

**Definition 4.1** (Differential calculus; Tamarkin-Tsygan [28]). Let $H^*$ and $H_*$ be graded vector spaces. A differential calculus is the sextuple

$$(H^*, H_*, \cup, \iota, [-,-], d)$$

satisfying the following conditions:

1. $(H^*, \cup, [-,-])$ is a Gerstenhaber algebra; that is, $(H^*, \cup)$ is a graded commutative algebra, $(H^*, [-,-])$ is a degree 1 or -1 graded Lie algebra, and the product and Lie bracket are compatible in the following sense

$$[P \cup Q, R] = P \cup [Q, R] + (-1)^{pq}Q \cup [P, R],$$

for homogeneous $P, Q, R \in V$ of degree $p, q, r$, respectively;

2. $H_*$ is a graded (left) module over $(H^*, \cup)$ via the map

$$\iota : H^n \otimes H_m \to H_{m-n}, f \otimes \alpha \mapsto \iota_f \alpha,$$

for any $f \in H^n$ and $\alpha \in H_m$;

3. There is a map $d : H_* \to H_{*+1}$ satisfying $d^2 = 0$, and

$$( -1)^{|f|+1} \iota_{[f,g]} = [L_f, t_g] := L_f t_g - (-1)^{|f|} t_g L_f,$$

where $L_f := [d, \iota_f] = dt_f - (-1)^{|f|} \iota_f d$.

In the following, if $\cup$, $\iota$, $[-,-]$ and $d$ are clear from the context, we will simply write a differential calculus by $(H^*, H_*)$ for short.

4.1 Differential calculus on Poisson (co)homology

Suppose $A$ is a commutative algebra. We have the following operations on $\mathfrak{X}^*(A)$ and $\Omega^*(A)$:

1. Wedge (cup) product: suppose $P \in \mathfrak{X}^{-p}(A)$ and $Q \in \mathfrak{X}^{-q}(A)$, then the wedge product of $P$ and $Q$, denoted by $P \cup Q$, is a polyvector in $\mathfrak{X}^{-p-q}(A)$ defined by

$$(P \cup Q)(f_1, f_2, \cdots, f_{p+q}) := \sum_{\sigma \in S_{p,q}} \text{sgn}(\sigma) P(f_{\sigma(1)}, \cdots, f_{\sigma(p)}) \cdot Q(f_{\sigma(p+1)}, \cdots, f_{\sigma(p+q)}),$$

where $\sigma$ runs over all $(p,q)$-shuffles of $(1, 2, \cdots, p + q)$.

2. Schouten bracket: suppose $P \in \mathfrak{X}^{-p}(A)$ and $Q \in \mathfrak{X}^{-q}(A)$, then their Schouten bracket, denoted by $[P, Q]$, is an element in $\mathfrak{X}^{-p-q+1}(A)$ given by

$$[P, Q](f_1, f_2, \cdots, f_{p+q-1}) := \sum_{\sigma \in S_{q,p-1}} \text{sgn}(\sigma) P(Q(f_{\sigma(1)}, \cdots, f_{\sigma(q)}), f_{\sigma(q+1)}, \cdots, f_{\sigma(q+p-1)})$$

$$- (-1)^{|p-1||q-1|} \sum_{\sigma \in S_{p,q-1}} \text{sgn}(\sigma) Q(P(f_{\sigma(1)}, \cdots, f_{\sigma(p)}), f_{\sigma(p+1)}, \cdots, f_{\sigma(p+q-1)}).$$
(3) Contraction (inner product): suppose $P \in \mathfrak{X}^{-p}(A)$ and $\omega = df_1 \wedge \cdots \wedge df_n \in \Omega^n(A)$, then the contraction of $P$ with $\omega$, denoted by $\iota_P(\omega)$, is an $A$-linear map with values in $\Omega^{n-p}(A)$ given by

$$\iota_P(\omega) = \sum_{\sigma \in S_{p,n-p}} \text{sgn}(\sigma) P(f_{\sigma(1)}, \ldots, f_{\sigma(p)}) df_{\sigma(p+1)} \wedge \cdots \wedge df_{\sigma(n)}, \quad \text{if } n \geq p,$$

$$0, \quad \text{otherwise.}$$

(4) Lie derivative: the Lie derivative is given by the Cartan formula, namely for $P \in \mathfrak{X}^{-p}(A)$ and $\omega \in \Omega^n(A)$, the Lie derivative of $\omega$ with respect to $P$ is given by

$$L_P \omega := [\iota_P, d] = \iota_P(d\omega) - (-1)^p d(\iota_P \omega),$$

where $d$ is the de Rham differential.

**Theorem 4.2.** Suppose $A$ is a Poisson algebra. Then

$$(\mathop{HP^*}(A), \mathop{HP_*(A)}, \cup, \iota, [-,-], d),$$

where $d$ is the de Rham differential, is a differential calculus.

**Proof.** We only have to show the operations listed above respect the Poisson boundary and coboundary. It is a direct check and can be found in [19, Chapter 3].

In the following, we will give another differential calculus structure for a Poisson algebra, which will be used later:

(1) For any $P \in \mathfrak{X}^{-p}(A)$ and $\phi \in \mathfrak{X}^{-q}(A^*)$, let $\iota_p^* (\phi) \in \mathfrak{X}^{-p-q}(A^*)$ be given by

$$\iota_p^* (\phi)(f_1, \cdots, f_{p+q}) := \sum_{\sigma \in S_{p,q}} \text{sgn}(\sigma) P(f_{\sigma(1)}, \cdots, f_{\sigma(p)}) \cdot \phi(f_{\sigma(p+1)}, \cdots, f_{\sigma(p+q)}). \quad (4.1)$$

It is clear that $\iota^*$ is associative, i.e., $\iota_Q^* \circ \iota_P^* = \iota_{P \cup Q}^*$. Also, $\iota^*$ respects the Poisson boundary maps, which is completely analogous to the proof of that $\cup$ commutes with the Poisson coboundary map (cf. [19] §4.3).

(2) Observe that

$$\mathfrak{X}^*(A^*) = \text{Hom}_A(\Omega^*(A), A^*)$$

$$= \text{Hom}_A(\Omega^*(A), \text{Hom}(A, k))$$

$$= \text{Hom}_A(\Omega^*(A) \otimes A, k)$$

$$= \text{Hom}(\Omega^*(A), k).$$

By dualizing the de Rham differential $d$ on $\Omega^*(A)$, we obtain a differential $d^*$ on $\text{Hom}(\Omega^*(A), k)$, i.e., on $\mathfrak{X}^*(A^*)$. It is proved in [37, Theorem 4.10] that $d^*$ commutes with the Poisson boundary.

(3) For any $P \in \mathfrak{X}^*(A)$ and $\omega \in \mathfrak{X}^*(A^*)$, let $L_P \omega := [\iota_P^*, d^*](\omega)$; it is a direct check that

$$[L_P, \iota_Q^*] = \iota_{[P,Q]}^*.$$

By (1)-(3) listed above, we obtain the following.

**Theorem 4.3.** Suppose $A$ is a Poisson algebra, and denote $A^*$ be its dual space. Then

$$(\mathop{HP^*}(A), \mathop{HP_*(A;A^*)}, \cup, \iota^*, [-,-], d^*)$$

is a differential calculus.
4.2 Unimodular Poisson algebras

Suppose $A$ is a commutative algebra, and $\eta \in \Omega^n(A)$. We say $\eta$ is a volume form if $\chi^\bullet(A) \xrightarrow{\iota(-)\eta} \Omega^{n+\bullet}(A)$ is an isomorphism of vector spaces. Now suppose $A$ is Poisson, then we have the following diagram

$$
\begin{array}{ccc}
\chi^\bullet(A) & \xrightarrow{\iota(-)\eta} & \Omega^{n+\bullet}(A) \\
\downarrow \delta & & \downarrow \partial \\
\chi^{\bullet+1}(A) & \xrightarrow{\iota(-)\eta} & \Omega^{n+\bullet+1}(A),
\end{array}
$$

(4.2)

which may not be commutative, i.e., $\eta$ may not be a Poisson cycle. We say $A$ is unimodular if there exists a volume form $\eta$ such that (4.2) commutes. This following is now immediate.

**Theorem 4.4** (Xu). Suppose $A$ is a unimodular symmetric Poisson algebra with the volume form of degree $n$. Then there exists an isomorphism (the Poincaré duality)

$$
\HP^\bullet(A) \cong \HP^{n-\bullet}(A).
$$

4.3 Unimodular symmetric Poisson algebras

Now, we go to unimodular symmetric Poisson algebras, a notion introduced by Zhu, Van Oystaeyen and Zhang in [37].

Suppose $A^!$ is a finite dimensional graded commutative algebra. $A^!$ is called symmetric if it is equipped with a bilinear, non-degenerate symmetric pairing

$$
\langle -, \cdot \rangle : A^! \otimes A^! \to k
$$

of degree $n$ which is cyclically invariant, that is, $\langle a, b \cdot c \rangle = (-1)^{|a||b|}|c|\langle c, a \cdot b \rangle$, for all homogeneous $a, b, c \in A^!$. This is equivalent to saying that there is an $A^!$-bimodule isomorphism

$$
\eta^! : (A^!)^\bullet \to (A^!)_{n+\bullet}, \quad \text{for some } n \in \mathbb{N},
$$

where $A^! = (A^!)^*$. In this case, we may view $\eta^!$ as an element in $\text{Hom}_{A^!}(A^!, A^!) \subset \chi^\bullet_{A^!}(A^!)$. Now assume $A^!$ is Poisson, then we have a diagram

$$
\begin{array}{ccc}
\chi^\bullet_{A^!}(A^!) & \xrightarrow{\iota(-)\eta^!} & \chi^{n+\bullet}_{A^!}(A^!) \\
\downarrow \delta & & \downarrow \delta \\
\chi^{\bullet+1}_{A^!}(A^!) & \xrightarrow{\iota(-)\eta^!} & \chi^{n+\bullet+1}_{A^!}(A^!).
\end{array}
$$

(4.3)

According to Zhu-Van Oystaeyen-Zhang [37], if there exists $\eta^! \in \chi^\bullet_{A^!}(A^!)$ such that $\iota(-)\eta^!$ is an isomorphism, then $\eta^!$ is called a volume form, and if furthermore, the diagram (4.3) commutes, then $A^!$ is called a unimodular symmetric Poisson algebra of degree $n$. From the definition, we immediately have:

**Theorem 4.5** (Zhu-Van Oystaeyen-Zhang [37]). Suppose $A^!$ is a unimodular symmetric Poisson algebra with the volume form of degree $n$. Then there exists an isomorphism

$$
\HP^\bullet(A^!) \cong \HP^{n-\bullet}(A^!; A^!).
$$
In this paper, since we are interested in \( A = k[x_1, \ldots, x_n] \) or \( A^! = \Lambda(\xi_1, \ldots, \xi_n) \), we always assume the volume form is constant.

**Proof of Theorem 1.2.** First, we show that a quadratic Poisson algebra \((A = k[x_1, \ldots, x_n], \pi)\) is unimodular if and only if \((A^!, \pi^!)\) is unimodular symmetric. In fact, recall that for \( A = k[x_1, \ldots, x_n] \),

\[
\begin{align*}
\mathfrak{X}_A^*(A) &= \Lambda \left( x_1, \ldots, x_n, \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n} \right), \\
\Omega^*(A) &= \Lambda(x_1, \ldots, x_n, dx_1, \ldots, dx_n), \\
\mathfrak{X}_{A^!}^*(A^!) &= \Lambda \left( \xi_1, \ldots, \xi_n, \frac{\partial}{\partial \xi_1}, \ldots, \frac{\partial}{\partial \xi_n} \right), \\
\mathfrak{X}^*_A(A^!) &= \Lambda \left( \xi_1^*, \ldots, \xi_n^*, \frac{\partial}{\partial \xi_1^*}, \ldots, \frac{\partial}{\partial \xi_n^*} \right).
\end{align*}
\]

Let

\[
\eta = dx_1 dx_2 \cdots dx_n \quad \text{and} \quad \eta^! = \xi_1^* \xi_2^* \cdots \xi_n^*,
\]

where \( \eta^! \) is understood as contraction, namely,

\[
\eta^!(\xi_1 \cdots \xi_{i_p}) := \sum_{\sigma \in S_{p,n-p}} \langle \xi_{\sigma(1)} \cdots \xi_{\sigma(p)}, \xi_{\sigma(p+1)}^* \cdots \xi_{\sigma(n)}^* \rangle \cdot \xi_{\sigma(1)}^* \cdots \xi_{\sigma(n)}^*.
\]

then under the identification

\[
x_i \mapsto \frac{\partial}{\partial \xi_i}, \quad dx_i \mapsto \xi_i^*, \quad \frac{\partial}{\partial x_i} \mapsto \xi_i,
\]

the diagram

\[
\begin{array}{c}
\mathfrak{X}_A^*(A) = \Lambda \left( x_1, \ldots, x_n, \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n} \right) \\
\Omega^*(A) = \Lambda(x_1, \ldots, x_n, dx_1, \ldots, dx_n)
\end{array}
\]

commutes. This means \( \eta \) is a Poisson cycle for \( A \) if and only if \( \eta^! \) is a Poisson cocycle for \( A^! \), which proves the claim.

Second, for \( A \) as above, we show the following diagram

\[
\begin{array}{c}
\mathfrak{X}_{A^!}^*(A^!) = \Lambda \left( \xi_1, \ldots, \xi_n, \frac{\partial}{\partial \xi_1}, \ldots, \frac{\partial}{\partial \xi_n} \right)
\end{array}
\]

commutes. In fact, the two vertical isomorphisms are given by Theorem 1.1 and the two horizontal isomorphisms are given by Theorems 4.4 and 4.5 respectively. The commutativity of the diagram (4.5) follows from the chain level commutative diagram (4.4). \( \square \)

### 5 Poisson cohomology and the Batalin-Vilkovisky algebra

The purpose of this section is to show that for unimodular quadratic Poisson polynomial algebras, the horizontal isomorphisms in (4.5) naturally induce on \( \text{HP}^*(A) \) and \( \text{HP}^*(A^!) \) a Batalin-Vilkovisky algebra structure, and the vertical isomorphisms in (4.5) are isomorphisms of Batalin-Vilkovisky algebras. We start with the notion of differential calculus with duality.
Definition 5.1 (Lambre [17]). A differential calculus \((H^*, H_*, \cup, \iota, [-,-], d)\) is called a differential calculus with duality if there exists an integer \(n\) and an element \(\eta \in H_n\) such that

(a) \(\iota_1 \eta = \eta\), where \(1 \in H^0\) is the unit, \(d(\eta) = 0\), and

(b) for any \(i \in \mathbb{Z}\),

\[
PD(-) := \iota_{(-)} \eta : H^i \to H_{n-i}
\]  

(5.1)

is an isomorphism.

Such isomorphism \(PD\) is called the Van den Bergh duality (also called the noncommutative Poincaré duality), and \(\eta\) is called the volume form.

Definition 5.2 (Batalin-Vilkovisky algebra). Suppose \((V, \bullet)\) is an graded commutative algebra. A Batalin-Vilkovisky algebra structure on \(V\) is the triple \((V, \bullet, \Delta)\) such that

(1) \(\Delta : V^i \to V^{i-1}\) is a differential, that is, \(\Delta^2 = 0\); and

(2) \(\Delta\) is second order operator, that is,

\[
\Delta(a \bullet b \bullet c) = \Delta(a \bullet b) \bullet c + (-1)^{|a|} a \bullet \Delta(b \bullet c) + (-1)^{|a|-1} |b| b \bullet \Delta(a \bullet c) - (\Delta a) \bullet b \bullet c - (-1)^{|a|} |a| a \bullet (\Delta b) \bullet c - (-1)^{|a|+|b|} a \bullet b \bullet (\Delta c).
\]

Equivalently, if we define the bracket

\[
[a, b] := (-1)^{|a|+1}(\Delta(a \bullet b) - \Delta(a) \bullet b - (-1)^{|a|} a \bullet \Delta(b)),
\]

then \([-,-]\) is a derivation with respect to \(\bullet\) for each component. In other words, a Batalin-Vilkovisky algebra is a Gerstenhaber algebra \((V, \bullet, [-,-])\) with a differential \(\Delta : V^i \to V^{i-1}\) such that

\[
[a, b] = (-1)^{|a|+1}(\Delta(a \bullet b) - \Delta(a) \bullet b - (-1)^{|a|} a \bullet \Delta(b)),
\]  

(5.2)

for any \(a, b \in V\) (cf. [12, Proposition 1.2]). \(\Delta\) is also called the Batalin-Vilkovisky operator, or the generator (of the Gerstenhaber bracket).

Now suppose \((H^*, H_*, \cup, \iota, [-,-], d, \eta)\) is a differential calculus with duality. Let \(\Delta : H^* \to H^{*-1}\) be the linear operator such that

\[
\begin{array}{ccc}
H^* & \xrightarrow{\Delta} & H^{*-1} \\
\downarrow{PD} & & \downarrow{PD} \\
H_{n-*} & \xrightarrow{d} & H_{n-*+1}
\end{array}
\]  

(5.3)

commutes. Then we have the following theorem:

Theorem 5.3 (Lambre [17]). Let \((H^*, H_*, \cup, \iota, [-,-], d, \eta)\) be a differential calculus with duality. Then the triple \((H^*, \cup, \Delta)\) is a Batalin-Vilkovisky algebra.

The proof can be found in Lambre ([17, Théorème 1.6]); however, since some details in loc. cit. are omitted, we give a proof here for completeness.
Proof. Since \((\mathbb{H}^*, \cup, [-,-])\) is a Gerstenhaber algebra, we only need to show that the Gerstenhaber bracket is compatible with the operator \(\Delta\) in \([5,3]\); that is, equation \([5.2]\) holds. For any homogeneous elements \(f, g \in \mathbb{H}^*\), by the definition of Poincaré duality \([5.1]\) and the Cartan formulae (Lemma \([6,3]\), we have

\[
(-1)^{|f|+1} PD([f, g]) = (-1)^{|f|+1} t_{[[f,g]]}(\eta) = L_f t_g(\eta) - (-1)^{|g|(|f|+1)} t_g L_f(\eta) = d t_f t_g(\eta) - (-1)^{|f|+1} t_f d t_g(\eta) + (-1)^{|g|(|f|+1)+|f|} t_g t_f d(\eta) = d \circ PD(f \cup g) - (-1)^{|f|(|f|+1)} t_g d \circ PD(f) - (-1)^{|f|} t_f d \circ PD(g) = PD(\Delta(f \cup g)) - (-1)^{|g|(|f|+1)} t_g PD(\Delta(f)) - (-1)^{|f|} t_f PD(\Delta(g)) = \iota_{\Delta(f \cup g)}(\eta) - (-1)^{|g|(|f|+1)} t_g \iota_{\Delta(f)}(\eta) - (-1)^{|f|} t_f \iota_{\Delta(g)}(\eta) = (\iota_{\Delta(f \cup g)} - (-1)^{|g|(|f|+1)} t_g \iota_{\Delta(f)} - (-1)^{|f|} t_f \iota_{\Delta(g)})(\eta) = PD(\Delta(f \cup g) - \Delta(f) \cup g - (-1)^{|f|} f \cup \Delta(g)).
\]

Since PD is an isomorphism, we thus have

\[
[f, g] = (-1)^{|f|+1}(\Delta(f \cup g) - \Delta(f) \cup g - (-1)^{|f|} f \cup \Delta(g)).
\]

Corollary 5.4 (see also Xu \([36]\) and Zhu-Van Oystaeyen-Zhang \([37]\)). Suppose \(A\) is a unimodular Poisson or unimodular symmetric Poisson algebra. Then \(\mathbb{H}^*(A)\) admits a Batalin-Vilkovisky algebra structure.

Proof. If \(A\) is unimodular Poisson, then Theorems \([1,2]\) and \([4,4]\) imply the pair \((\mathbb{H}^*(A), \mathbb{H}_*(A))\) is in fact a differential calculus with duality; similarly, if \(A\) is unimodular symmetric Poisson, Theorem \([4,3]\) and \([4,5]\) \((\mathbb{H}^*(A), \mathbb{H}^*(A; A^*))\) is a differential calculus with duality. The theorem then follows from Theorem \([5,3]\).

Proof of Theorem \([7,3]\). Note that in Theorem \([1,2]\) the right vertical isomorphism preserves the Kähler differential as well as the volume form, that is, the two differential calculus with duality \((\mathbb{H}^*(A), \mathbb{H}_*(A))\) and \((\mathbb{H}^*(A'), \mathbb{H}^*(A'; A^{'})\)

are isomorphic. Combining with Corollary \([5,4]\) the theorem follows.

Remark 5.5. Not all quadratic Poisson algebras are unimodular. For example, for \(A = \mathbb{R}[x_1, x_2, x_3]\), Etingof-Ginzburg \([9]\) Lemma 4.2.3 and Corollary 4.3.2 showed that any unimodular Poisson structure is of the form

\[
\{x, y\} = \frac{\partial \phi}{\partial z}, \quad \{y, z\} = \frac{\partial \phi}{\partial x}, \quad \{z, x\} = \frac{\partial \phi}{\partial y},
\]

for some \(\phi \in A\) (taking \(\phi\) to be cubic then the Poisson structure is quadratic); for \(A = \mathbb{C}[x_1, x_2, x_3, x_4]\), Pym \([23]\) §3 showed that any unimodular quadratic Poisson bracket on \(A\) may be written uniquely in the following form

\[
\{f, g\} := \frac{df \wedge dg \wedge d\alpha}{dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4}, \quad f, g \in A,
\]

where \(\alpha = \sum_{i=1}^4 \alpha_i dx_i \in \Omega^1(A)\) such that \(\alpha \wedge d\alpha = 0\), and \(\alpha_i\)'s are homogeneous cubic polynomials satisfying \(\sum_{i=1}^4 x_i \alpha_i = 0\).
6 Calabi-Yau algebras

At the end of §1 we sketched some analogy between unimodular Poisson algebras and Calabi-Yau algebras. In the following sections, we study their relationships in more detail.

6.1 Calabi-Yau algebras and the Batalin-Vilkovisky algebra structure

Definition 6.1 (Calabi-Yau algebra; Ginzburg [13]). Let $A$ be an associative algebra over $k$. $A$ is called a Calabi-Yau algebra of dimension $n$ if

1. $A$ is homologically smooth, that is, $A$, viewed as an $A^e$-module, has a finite-long resolution of finitely generated projective $A^e$-modules, and

2. there is an isomorphism
   \[ \text{RHom}_{A^e}(A, A \otimes A) \cong \Sigma^{-n}A \]  
   in the derived category $D(A^e)$ of $A^e$-modules.

In the above definition, $A^e$ is the enveloping algebra of $A$, namely $A^e := A \otimes A^{op}$. There are a lot of examples of Calabi-Yau algebras, such as the universal enveloping algebra of semi-simple Lie algebras, the skew-product of complex polynomials with a finite subgroup of $\text{SL}(n, \mathbb{R})$, the Yang-Mills algebras, etc.

We next study Van den Bergh’s noncommutative Poincaré duality for Calabi-Yau algebras ([32]). To this end, we first recall the differential calculus structure for associative algebras.

For an associative algebra $A$, denote by $(\bar{C}^*(A; A), \delta)$ and $(\bar{C}_*(A; A), b)$ the reduced Hochschild cochain and chain complexes of $A$. Recall that the Gerstenhaber cup product and the Gerstenhaber bracket on $\bar{C}^*(A; A)$ are given as follows: for any $f \in \bar{C}^n(A; A)$ and $g \in \bar{C}^m(A; A)$,

\[ f \cup g(\bar{a}_1, \ldots, \bar{a}_{n+m}) := (-1)^{nm}f(\bar{a}_1, \ldots, \bar{a}_n)g(\bar{a}_{n+1}, \ldots, \bar{a}_{n+m}), \]

and

\[ \{f, g\} := f \circ g - (-1)^{|f||g|+1}g \circ f, \]

where

\[ f \circ g(\bar{a}_1, \ldots, \bar{a}_{n+m-1}) := \sum_{i=0}^{n-1} (-1)^{|g|+i}f(\bar{a}_1, \ldots, \bar{a}_i, g(\bar{a}_{i+1}, \ldots, \bar{a}_{i+m}), \bar{a}_{i+m+1}, \ldots, \bar{a}_{n+m-1}). \]

Gerstenhaber proved in [11, Theorems 3-5] $\cup$ and $\{-, -\}$ are well-defined on the cohomology level, and moreover, $\cup$ is graded commutative. Therefore we obtain on the Hochschild cohomology $\text{HH}^*(A)$ a Gerstenhaber algebra structure.

Next, we consider the action of the Hochschild cochain complex on the Hochschild chain complex. Given any homogeneous elements $f \in \bar{C}^n(A; A)$ and $\alpha = (a_0, a_1, \ldots, a_m) \in \bar{C}_m(A; A)$,

1. the cap product $\cap : \bar{C}^n(A; A) \times \bar{C}_m(A; A) \to \bar{C}_{m-n}(A; A)$ is given by

   \[ f \cap \alpha := \begin{cases} 
   (a_0f(\bar{a}_1, \ldots, \bar{a}_n), \bar{a}_{n+1}, \ldots, \bar{a}_m), & \text{if } m \geq n \\
   0, & \text{otherwise.} \end{cases} \]

   If we denote by $\iota_f(-) := f \cap -$ the contraction operator, then $\iota_f \iota_g = (-1)^{|f||g|} \iota_{g \cup f} = \iota_{f \cup g}$.
(2) the Lie derivative $L : \bar{C}^n(A; A) \times \bar{C}_m(A; A) \to \bar{C}_{m-n}(A; A)$ is given as follows: for any $\alpha = (a_0, a_1, \ldots, a_m) \in \bar{C}_m(A; A)$, if $n \leq m + 1$, then
\[ L_f(\alpha) := \sum_{i=0}^{m-n} (-1)^{i(n+1)} (a_0, a_1, \ldots, \hat{a}_i, f(\bar{a}_{i+1}, \ldots, \bar{a}_{i+n}), \ldots, a_m) \]
\[ + \sum_{i=m-n+1}^{m} (-1)^{m(i+1)+n+1} (f(\bar{a}_{i+1}, \ldots, \bar{a}_m, \bar{a}_0, \ldots, \bar{a}_{n-m+i-1}, \bar{a}_{n-m+i}, \ldots, \bar{a}_i), \]
where the second sum is taken over all cyclic permutations such that $a_0$ is inside of $f$, and otherwise if $n > m + 1$, $L_f(\alpha) = 0$;

(3) the Connes operator $B : \bar{C}_\bullet(A; A) \to \bar{C}_{\bullet+1}(A; A)$ is given by
\[ B(\alpha) := \sum_{i=0}^{m} (-1)^{mi} (1, a_1, \ldots, a_m, a_0, \ldots, a_{i-1}). \]

The following two lemmas first appeared in Daletskii-Gelfand-Tsygan [5], which we learned from Tamarkin-Tsygan in [28].

**Lemma 6.2.** Keep the notations as in the above definition. Then

1. $(\bar{C}_\bullet(A; A), b, \cap)$ is a DG module over $(\bar{C}^\bullet(A; A), \delta, \cup)$, that is,
\[ \iota_{sf} = (-1)^{|f|+1} [b, \iota_f], \quad \iota_{fg} = \iota_{f \cup g}, \]
for any homogeneous elements $f, g \in \bar{C}^\bullet(A; A)$;

2. for any homogeneous elements $f, g \in \bar{C}^\bullet(A; A)$,
\[ [L_f, L_g] = L_{\iota_{f,g}}, \]
and in particular $(-1)^{|f|+1} [b, L_f] + L_{\delta f} = 0$.

**Lemma 6.3** (Homotopy Cartan formulae). Suppose $i, L, B$ are given as above and $f, g \in \bar{C}^\bullet(A; A)$ are any homogeneous elements.

1. Define an operation (cf. [28] Equ. (3.5))
\[ S_f(\alpha) := \sum_{i=0}^{n} \sum_{j=i+n}^{m} (-1)^{ij} (1, \bar{a}_{j+1}, \ldots, \bar{a}_m, \bar{a}_0, \ldots, \hat{a}_i, f(\bar{a}_{i+1}, \ldots, \bar{a}_{i+n}), \bar{a}_{i+n+1}, \ldots, \bar{a}_j) \]
for any $\alpha = (a_0, a_1, \ldots, a_m) \in \bar{C}_m(A; A)$ (the sum is taken over all cyclic permutations and $a_0$ always appears on the left of $f$), where $\eta_{ij} := (n+1)m + (m-j)m + (n+1)(j-i)$. Then we have
\[ L_f = [B, \iota_f] + [b, S_f] - S_{\delta f}. \]  
(6.3)

2. Define
\[ T(f, g)(\alpha) := \sum_{i=l-n+2}^{l} \sum_{j=0}^{n+i-l-2} (-1)^{\theta_{ij}} (f(\bar{a}_{i+1}, \ldots, \bar{a}_l, \bar{a}_0, \ldots, \hat{a}_j, g(\bar{a}_{j+1}, \ldots, \bar{a}_{j+m}), \ldots, \bar{a}_{n+m+1-l-2}, \ldots, \bar{a}_i) \]
for any $\alpha = (a_0, a_1, \ldots, a_l) \in \bar{C}_l(A; A)$, where $\theta_{ij} = (m+1)(i+j+l) + l(i+1)$. Then we have
\[ [L_f, \iota_g] - (-1)^{|f|+1} \iota_{(f,g)} = [b, T(f, g)] - T(\delta f, g) - T(f, \delta g). \]  
(6.4)
The above two lemmas say that Definition 4.1 (2) (3) hold up homotopy on the chain level. Together with Gerstenhaber’s theorem, we have the following.

**Theorem 6.4** (Daletskii-Gelfand-Tsygan [5]). Let $A$ be an associative algebra. Then the following sextuple

$$(\text{HH}^*(A), \text{HH}_*(A), \cup, \iota, \{-,-\}, B)$$

is a differential calculus.

In [6, Proposition 5.5], de Thanhoffer de Völcsey and Van den Bergh proved that, for a Calabi-Yau algebra $A$ of dimension $n$, there exists a class $\eta \in \text{HH}_n(A)$ such that the contraction $\text{HH}^*(A) \rightarrow \text{HH}_n(A)$ is an isomorphism. This immediately implies the following:

**Theorem 6.5** ([13, 17]). Suppose $A$ is a Calabi-Yau algebra $A$ of dimension $n$. Then

$$(\text{HH}^*(A), \text{HH}_*(A), \cup, \iota, \{-,-\}, B)$$

is a differential calculus with duality, and in particular, $(\text{HH}^*(A), \cup, \Delta)$ is a Batalin-Vilkovisky algebra.

### 6.2 Symmetric algebras and the Batalin-Vilkovisky algebra structure

We now recall a differential calculus structure on the Hochschild complexes of symmetric algebras.

First, for an associative algebra $A$, denote $A^* := \text{Hom}(A, k)$, which is an $A$-bimodule. Denote by $\tilde{C}^*(A; A^*)$ the reduced Hochschild cochain complex of $A$ with values in $A^*$. Then under the identity

$$\tilde{C}^*(A; A^*) = \bigoplus_{n \geq 0} \text{Hom}(\overline{A}^\otimes n, A^*) = \bigoplus_{n \geq 0} \text{Hom}(A \otimes \overline{A}^\otimes n, k),$$

one may equip on $\tilde{C}^*(A; A^*)$ the dual Connes differential, which is denoted by $B^*$, i.e., $B^*(g) := (-1)^{|g|}g \circ B$ for homogeneous $g \in \tilde{C}^*(A; A^*)$. $B^*$ commutes with the Hochschild coboundary map $\delta$, and thus is well-defined on the homology level.

Second, let

$$\tilde{C}^*(A; A^*) \times \tilde{C}^*(A; A^*) \xrightarrow{cr} \tilde{C}^*(A; A^*)$$

$$\begin{array}{ccc}
(f, \alpha) & \mapsto & t_f^\ast(\alpha) := (-1)^{|f||\alpha|} \alpha \circ t_f,
\end{array}$$

for any homogeneous $f \in \tilde{C}^*(A; A)$ and $\alpha \in \tilde{C}^*(A; A^*)$. We have the following.

**Theorem 6.6.** Let $A$ be an associative algebra. Then

$$(\text{HH}^*(A), \text{HH}^*(A; A^*), \cup, \iota^*, \{-,-\}, B^*)$$

is a differential calculus.

**Proof.** By the definition of differential calculus, we only need to show the last two equalities given in Definition 4.1.

(1) By the definition of $\iota^*$ and Lemma 6.2 (1), one has

$$t_{f^\ast}^\ast(\iota^\ast) = (1)^{|g||\alpha|}t_{f^\ast}^\ast(\alpha \circ \iota_g) = (-1)^{|\alpha|+|f|(|\alpha|)}(\alpha \circ t_f) \circ \iota_f.$$
Theorem 6.7. Thus we have the following.

For any homogenous elements \( f, g \in \HH^\bullet(A) \) and \( \alpha \in \HH^\bullet(A; A^*) \). This means that the cap product is a left module action.

(2) Given any homogenous elements \( f \in \HH^\bullet(A) \) and \( \alpha \in \HH^\bullet(A; A^*) \), we define \( L_f^\bullet(\alpha) := (-1)^{|f||\alpha|+|\alpha|+1}\circ L_f^{\bullet}([B^\bullet, t_f^\bullet](\alpha)) \), and by Lemma 6.3 one has

\[
[L_f^\bullet, t_g^\bullet](\alpha) = (L_f^\bullet t_g^\bullet - (-1)^{|f|+|g|+1}\circ t_g^\bullet L_f^\bullet)(\alpha) \\
= (-1)^{|f|+|\alpha|+|\alpha|+1}\circ (t_g L_f) - (-1)^{|g|+|\alpha|+1}\circ (L_f t_g) \\
= (-1)^{|f|+|\alpha|+1}\circ [(L_f, t_g)] \\
= (-1)^{|f|+|\alpha|+1}\circ ((-1)^{1+1}t_{f,g}) \\
= (-1)^{|f|+1}t_{f,g}(\alpha).
\]

This completes the proof.

Now suppose \( A^1 \) is symmetric. Recall that the existence of the degree \( n \) cyclic pairing is equivalent to an isomorphism

\[ \eta : A^1 \cong \Sigma^{-n} A^1 \]

as \( A^1 \)-bimodules. Such \( \eta \) may be viewed as an element in \( \bar{C}^{-n}(A^1; A^1) \), which is a cocycle, and hence represents a cohomology class. By abuse of notation, this class is also denoted by \( \eta \). The following map

\[
- \cap \eta : \bar{C}^\bullet(A^1; A^1) = \bigoplus_{q \geq 0} \text{Hom}((\bar{A}^1)^{\otimes q}, A^1)
\]

is symmetric of degree \( n \), and only if \( A^1 \) is symmetric of degree \( n \).

Theorem 6.7 (17, 29). Suppose \( A^1 \) is a symmetric algebra of degree \( n \).

\[
(\HH^\bullet(A), \HH^\bullet(A; A^*), \cup, t^\bullet, \{-,-\}, B^*)
\]

is a differential calculus with duality, and in particular, \( \HH^\bullet(A^1) \) is a Batalin-Vilkovisky algebra.

6.3 Koszul Calabi-Yau algebras and Rouquier’s conjecture

Analogously to the quadratic Poisson algebra case, the Koszul dual of a Koszul Calabi-Yau algebra is symmetric ( chronologically the latter is discovered first), and we have the following theorem due to Van den Bergh (see [31, Theorem 9.2] or [1, Proposition 28] for a proof): Suppose \( A \) is a Koszul algebra and let \( A^1 \) be its Koszul dual algebra. Then \( A \) is Calabi-Yau of dimension \( n \) if and only if \( A^1 \) is symmetric of degree \( n \).

It has been well-known that for a Koszul algebra, say \( A \),

\[ \HH^\bullet(A) \cong \HH^\bullet(A^1), \]

as Gerstenhaber algebras, and Rouquier conjectured (it is stated in Ginzburg [13]) that, for a Koszul Calabi-Yau algebra, the above two Batalin-Vilkovisky are isomorphic, which turns out to be true (see [21, Theorem A] for a proof):
Theorem 6.8 (Rouquier’s conjecture). Suppose \( A \) is a Koszul Calabi-Yau algebra. Denote by \( A^! \) and \( A^\cdot \) the Koszul dual algebra and coalgebra of \( A \) respectively. Then

\[
\left( \operatorname{HH}^*(A), \operatorname{HH}_*(A) \right) \quad \text{and} \quad \left( \operatorname{HH}^*(A^!), \operatorname{HH}_*(A^!; A^\cdot) \right)
\]

are isomorphic as differential calculus with duality. In particular, \( \operatorname{HH}^*(A) \) and \( \operatorname{HH}^*(A^!) \) are isomorphic as Batalin-Vilkovisky algebras.

The key point of the proof is that, with the differentials properly assigned on \( A \otimes A^! \) and \( A \otimes A^\cdot \) respectively, then

\[
\overline{C}_\bullet(A; A) \simeq A \otimes A^\cdot \simeq \overline{C}_\bullet(A^!; A^\cdot)
\]

and via these quasi-isomorphisms, the volume forms as well as the contractions given by (6.2) and (6.5) are identical on the above middle terms (compare with the proof of Theorem 1.2).

Example 6.9 (The polynomial case). Let \( A = \mathbb{R}[x_1, x_2, \cdots, x_n] \), which is \( n \)-Calabi-Yau. Its Koszul dual algebra \( A^! = \Lambda(\xi_1, \xi_2, \cdots, \xi_n) \) is symmetric. As in the Poisson case, the volume forms on \( \operatorname{HH}_*(A) \) and \( \operatorname{HH}^*(A^!; A^\cdot) \) are, via the above quasiisomorphisms, represented by \( 1 \otimes \xi_1^* \cdots \xi_n^* \) in \( A \otimes A^\cdot \).

7 Calabi-Yau/symmetric algebras and their deformations

In this section, we take \( k \) to be \( \mathbb{R} \). Dolgushev [7, Theorem 3] (respectively Felder and Shoikhet [10] and Willwacher-Calaque [35, Theorem 37]) proved that for a Calabi-Yau algebra (respectively symmetric algebra), if it is unimodular Poisson (respectively unimodular symmetric Poisson), then its deformation quantization is again Calabi-Yau (respectively symmetric). We use their results to prove Theorems 1.4 and 1.5.

7.1 Deformation quantization of Calabi-Yau Poisson algebras

Recall that for a Poisson algebra \( A \) with bracket \( \{-, -\} \), its deformation quantization, denoted by \( A[h] \), is a \( k[h] \)-linear associative product (called the star-product) on \( A[h] \)

\[
a \star b = a \cdot b + \mu_1(a, b)h + \mu_2(a, b)h^2 + \cdots,
\]

where \( h \) is the formal parameter and \( \mu_i \) are bilinear operators, satisfying

\[
\lim_{h \to 0} \frac{1}{h} \{ a \star b - b \star a \} = \{ a, b \}, \quad \text{for all} \ a, b \in A.
\]

In [15], Kontsevich showed that there is a one-to-one correspondence between the equivalence classes of the star-products and the equivalence classes of Poisson algebra structures on \( A[h] \), where \( A \) is the algebra of smooth functions on the manifold. He also constructed an explicit \( L_\infty \)-quasiisomorphism from the space of polyvector fields to the Hochschild cochain complex; via this map, the Poisson bivector \( h \pi \) on \( A[h] \) gives rise to a star-product on \( A[h] \), which is called Kontsevich’s deformation quantization.

In [15 Theorem 4.10], Kontsevich showed that there is an \( L_\infty \)-quasiisomorphism

\[
\mathfrak{X}^*(A)[h] \longrightarrow \overline{C}_\bullet(A_h; A_h),
\]
and in loc. cit. §8, he sketched that this \( L_\infty \)-morphism also respects the cup product on both sides, which implies that on the cohomology level

\[
\text{HP}^* (A[h]) \longrightarrow \text{HH}^* (A_h)
\]

is an isomorphism of Gerstenhaber algebras; see also Manchon and Torossian in \[23\] Théorèm 1.1 and by Mochizuki \[24\] Theorem 1.1 for more details about the signs, etc.

Note that \( \Omega^* (A) \) and \( \bar{C}_\bullet (A; A) \) are modules over \( \mathbb{R}^* (A) \) and \( \mathcal{C}^* (A; A) \) respectively, and in \[30\] Conjecture 5.3.2, Tsygan conjectured that Kontsevich’s deformation quantization also gives an \( L_\infty \)-quasiisomorphism of \( L_\infty \)-modules between \( \bar{C}_\bullet (A; A) \) and \( \Omega^* (A) \). This is known as Tsygan’s Formality Conjecture for chains, and is proved by Shoikhet in \[20\] Theorem 1.3.1. Shoikhet also conjectured that such \( L_\infty \)-morphism is also compatible with the cup product, which was later proved by Calaque and Rossi in \[22\] Theorem A.

Recall that \( \Omega^* (A) \) and \( \bar{C}_\bullet (A; A) \), we have the de Rham differential operator and the Connes boundary operator respectively. One naturally expects the \( L_\infty \)-quasiisomorphism constructed above respects these two operators. This is known as the Cyclic Formality Conjecture for chains, and is proved by Willwacher in \[31\] Theorem 1.3 and Corollary 1.4.

Proof of Theorem \[1.5\] (1). The above cited references, in summary, show that the two pairs

\[
(\text{HP}^* (A[h]), \text{HP}_\bullet (A[h])) \quad \text{and} \quad (\text{HH}^* (A_h), \text{HH}_\bullet (A_h))
\]

are isomorphic as differential calculus (see also Dolgushev, Tamarkin and Tsygan \[8\] Theorem 9). Restricting to the Calabi-Yau case, once we know the two versions of Poincaré duality between the pairs (7.1) are given by capping with the volume form (note that in our case this volume form is an isomorphism of \( \mathcal{B}_\bullet (\mathcal{B}^1; \mathcal{B}) \) and \( \Omega^* (A) \)), that they are further isomorphic as differential calculus with duality. Thus by Theorem \[5.3\]

\[
\text{HP}^* (A[h]) \cong \text{HH}^* (A_h)
\]

as Batalin-Vilkovisky algebras.

\[ \square \]

### 7.2 Deformation quantization of symmetric Poisson algebras

We first rephrase Kontsevich’s Cyclic Formality Conjecture for cochains, published in Felder-Shoikhet \[10\] §1, in the case \( \mathbb{R}^{0|n} \). Note that in this case, \( A^1 = \mathcal{O}(\mathbb{R}^{0|n}) \), the space of functions on \( \mathbb{R}^{0|n} \).

Fix a constant volume form \( \Omega \) on \( \mathbb{R}^{0|n} \) (where in previous sections we use the notation \( \eta^1 = \xi_1^1 \cdots \xi_n^1 \) ), then via the pairing

\[
\langle f, g \rangle := \int_{\mathbb{R}^{0|n}} f \cdot g \cdot \Omega
\]

one can identify \( \bar{C}^* (A^1; A^1) \) with \( \bar{C}_\bullet^* (A^1; A^1) \), which is the same as capping with the “volume form” as given in (0.6). Denote by \( \mathcal{D}_{\text{poly}} (\mathbb{R}^{0|n}) \) the polydifferential Hochschild cochain subspace of \( \bar{C}^* (A^1; A^1) \), and denote the image of \( \psi \in \mathcal{D}_{\text{poly}} (\mathbb{R}^{0|n}) \) in \( \bar{C}^* (A^1; A^1) \) by \( \tilde{\psi} \). Let

\[
[\tilde{\mathcal{D}}_{\text{poly}} (\mathbb{R}^{0|n})]_{\text{cyc}} := \left\{ \psi \in \mathcal{D}_{\text{poly}} (\mathbb{R}^{0|n}) \mid \tilde{\psi} (f_1, \cdots, f_{n+1}) = (-1)^{|f_{n+1}|+|f_1|+\cdots+|f_n|+n} \tilde{\psi} (f_{n+1}, f_1, \cdots, f_n) \right\},
\]

where the latter is a subspace of the Connes cyclic invariant subspace of \( \bar{C}^* (A^1; A^1) \).
Given Kontsevich’s $L_\infty$-quasiisomorphism $U : T^\bullet_{\text{poly}}(\mathbb{R}^{0/n}) \to \mathcal{D}_{\text{poly}}(\mathbb{R}^{0/n})$, the Cyclic Formality Conjecture for cochains can be rephrased as the existence of an $L_\infty$-quasiisomorphism of Lie modules

$$\left(\mathfrak{X}^\bullet(A^i) \otimes \mathbb{R}[u], u \cdot d^\ast\right) \simeq \left(\left[\mathcal{D}_{\text{poly}}(\mathbb{R}^{0/n})\right]_{\text{cycl}}, \delta\right).$$

(7.2)

**Theorem 7.1** (Felder-Shoikhet [10]; Willwacher-Calaque [35]). For $A^i = \Lambda(\xi_1, \cdots, \xi_n)$, Kontsevich’s $L_\infty$-quasiisomorphism $U : T^\bullet_{\text{poly}}(\mathbb{R}^{0/n}) \to \mathcal{D}_{\text{poly}}(\mathbb{R}^{0/n})$ induces an $L_\infty$-quasiisomorphism (7.2). In this case, Kontsevich’s deformation quantization of $A^i$, say $A^i_{\hbar}$, is symmetric if and only if $A^i$ is unimodular symmetric Poisson.

**Proof.** Kontsevich’s $L_\infty$-quasiisomorphism holds for the supermanifold case, as has been shown in Cattaneo and Felder [3, Appendix] (it has also been used by Shoikhet [27]). The theorem follows verbatim from [35], in particular, Theorem 37 therein. \hfill \square

According to the fact that the set of Poisson structures on $A^i$ and the set of star products on $A^i[\hbar]$ are one-to-one correspondent, (7.2) also induces a quasiisomorphism of mixed complexes (called the tangent homomorphism)

$$\left(\text{CP}^\bullet(A^i[\hbar]; A^i[\hbar]), \delta_{h^{\pi^i}}, d^\ast\right) \simeq \left(\tilde{C}^\bullet(A^i_{\hbar}; A^i_{\hbar}), \delta, B^\ast\right),$$

where $\delta_{h^{\pi^i}}$ is the Poisson coboundary operator with respect to $h^{\pi^i}$. In particular,

$$\left(\text{HP}^\bullet(A^i[\hbar]), \text{HP}^{\bullet-n}(A^i[\hbar]; A^i[\hbar])\right) \text{ and } \left(\text{HH}^\bullet(A^i_{\hbar}), \text{HH}^{\bullet-n}(A^i_{\hbar}; A^i_{\hbar})\right)$$

are isomorphic as differential calculus.

**Proof of Theorem 7.1 (2).** According to the above argument, together with Theorems 4.5, 5.7 and 7.1, we have that

$$\left(\text{HP}^\bullet(A^i[\hbar]), \text{HP}^{\bullet-n}(A^i[\hbar]; A^i[\hbar])\right) \text{ and } \left(\text{HH}^\bullet(A^i_{\hbar}), \text{HH}^{\bullet-n}(A^i_{\hbar}; A^i_{\hbar})\right)$$

are isomorphic as differential calculus with duality, and hence by Theorem 5.3 the conclusion follows. \hfill \square

**Proof of Theorem 7.4.** By Shoikhet [27, Theorem 0.3], $A_h$ and $A^i_{\hbar}$ are Koszul dual algebras over $k[\hbar]$, and hence the theorem follows from a combination of Theorems 1.3, 1.5 and Rouquier’s conjecture. \hfill \square

### 7.3 Twisted Poincaré duality for Poisson algebras

For a general associative algebra, say $A$, it may not be Calabi-Yau, and therefore there may not exist any Poincaré duality between $\text{HH}^\bullet(A)$ and $\text{HH}_\bullet(A)$. In [1], Brown and Zhang introduced the so-called “twisted Poincaré duality” for associative algebras. That is, for such $A$, keeping its left $A$-module structure (the multiplication) as usual, the right $A$-module structure of $A$ is the multiplication composed with an automorphism $\sigma : A \to A$. Denote such $A$-bimodule by $A_\sigma$, then Brown and Zhang showed that for a lot of algebras, there exists a twisted Poincaré duality $\text{HH}^\bullet(A) \cong \text{HH}_{n-\sigma}(A; A_\sigma)$ for some $n \in \mathbb{N}$ (cf. [1 Corollary 5.2]). In this case $A$ is called a twisted Calabi-Yau algebra of dimension $n$.

Such phenomenon also occurs for Poisson algebras. Namely, not all Poisson algebras are unimodular, and hence there may not exist an isomorphism between $\text{HP}^\bullet(A)$ and $\text{HP}_\bullet(A)$. In
the authors studied the so-called twisted Poincaré duality for Poisson algebras, similarly to that of associative algebras. They also studied some comparisons with twisted Calabi-Yau algebras. However, it would be very interesting to study the relationships between the deformation quantization of twisted unimodular Poisson algebras and twisted Calabi-Yau algebras, and obtain a theorem similar to Theorem 1.4 in this twisted case.

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