\textbf{q-ANALOGUES OF THE SUMS OF POWERS OF CONSECUTIVE INTEGERS}

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\textbf{Abstract.} Let $n, k$ be the positive integers ($k > 1$), and let $S_{n,q}(k)$ be the sums of the $n$-th powers of positive $q$-integers up to $k - 1$: $S_{n,q}(k) = \sum_{l=0}^{k-1} q^l l^n$. Following an idea due to J. Bernoulli, we explore a formula for $S_{n,q}(k)$.

1. Introduction

In the early of 17th century Faulhaber [3, 6, 11] computed the sums of powers $1^m + 2^m + \cdots + n^m$ up to $m = 17$ and realized that for odd $m$, it is not just a polynomial in $n$ but a polynomial in the triangular number $N = n(n + 1)/2$. A good account of Faulhaber’s work was given by Knuth [3, 6, 11]. In 1713, J. Bernoulli first discovered the method which one can produce those formulae for the sum $\sum_{l=1}^{n} l^k$ for any natural numbers $k$, cf. [1, 2, 4, 5, 13]. Let $q$ be an indeterminate which can be considered in complex number field, and for any integer $k$ define the $q$-integer as $[k]_q = \frac{q^k - 1}{q - 1}$, cf. [2, 7, 8, 9, 10]. Note that $\lim_{q \to 1} [k]_q = k$. Recently, many authors are studying the problems of $q$-analogues of sums of powers of consecutive integers [3, 4, 5, 6, 7, 8, 11, 12, 13, 14]. In this paper we consider the $q$-analogues of the sums of powers of consecutive integers due to J. Bernoulli. For any positive integers $n, k(> 1)$,
let \( S_{n,q}(k) = \sum_{l=0}^{k-1} q^l l^n \). Following an idea due to J. Bernoulli, we explore a formula for \( S_{n,q}(k) \) as follows:

\[
q^{-k}S_{l-1,q}(k) + (q^{-k} \log q)S_{l,q}(k) = \frac{1}{l} \sum_{i=0}^{l-1} \binom{l}{i} B_{i,q} k^{l-i} + \frac{(1 - q^{-k}) B_{l,q}(0)}{l},
\]

where \( B_{i,q} \) are the \( q \)-analogue of Bernoulli numbers.

2. \( q \)-analogue of the Sums of the \( n \)-th powers of positive integers up to \( k - 1 \)

Let \( j \) be the positive integers. Then we easily see that

\[
q^{j+1}(j + 1) - q^j j = (q - 1)q^j j + q^{j+1}.
\]

From Eq.(1), we note that

\[
(q - 1) \sum_{j=0}^{k-1} q^j j + q[k]_q = q^k k.
\]

Thus, we have the following:

\[
S_{1,q}(k) = \sum_{j=0}^{k-1} q^j j = \frac{q^k k - q[k]_q}{q - 1}.
\]

Note that \( \frac{k(k-1)}{2} = \lim_{q \to 1} \frac{q^k k - q[k]_q}{q - 1} = S_1(k) \). By the same method of Eq.(1), we easily see that

\[
q^{j+1}(j + 1)^2 - q^j j^2 = (q - 1)q^j j^2 + 2q^{j+1} j + q^{j+1}.
\]

Thus, we obtain

\[
S_{2,q}(k) = \sum_{j=0}^{k-1} q^j j^2 = \frac{q^k k^2}{q - 1} - 2q \frac{q^k k - q[k]_q}{(q - 1)^2} - \frac{q[k]_q}{q - 1}.
\]

From the simple calculation, we note that

\[
q^{j+1}(j + 1)^3 - q^j j^3 = (q - 1)q^j j^3 + 3qq^j j^2 + 3qq^j j + q^{j+1},
\]

\[
q^k k^3 = (q - 1)S_{3,q}(k) + 3qS_{2,q}(k) + 3qS_{1,q}(k) + q[k]_q.
\]
Thus, we see that

\begin{equation}
S_{3,q}(k) = \frac{q^k k^3}{q - 1} - \frac{3q}{q - 1} S_{2,q}(k) - \frac{3q}{q - 1} S_{1,q}(k) - \frac{q[k]_q}{q - 1}.
\end{equation}

Let \( n, k \) be the positive integers \((k > 1)\) and let \( S_{n,q}(k) = \sum_{l=0}^{k-1} q^l l^n \). Then we have

\begin{equation}
q^{l+1}(l+1)^n - l^{n+1} q^l = q^{l+1} \sum_{i=0}^{n} \binom{n+1}{i} l^i + (q-1) l^{n+1} q^l.
\end{equation}

Summing over \( l \) from 0 to \( k-1 \), the left-hand side becomes \( q^{k} k^{n+1} \), but the right-hand side is a linear combination of \( S_{i,q}(k) \) as follows:

\begin{equation}
q^k k^{n+1} = q(n+1) S_{n,q}(k) + q \sum_{i=0}^{n-1} \binom{n+1}{i} S_{i,q}(k) + (q-1) S_{n+1,q}(k).
\end{equation}

Thus, we obtain the following:

**Theorem A.** Let \( n, k(>1) \) be the positive integers. Then we have

\begin{equation}
S_{n,q}(k) = \frac{k^{n+1}}{n + 1} q^{k-1} - \frac{1}{n + 1} \sum_{i=0}^{n-1} \binom{n+1}{i} S_{i,q}(k) + \frac{q - 1}{q(n+1)} S_{n+1,q}(k).
\end{equation}

Remark. Note that

\[
\lim_{q \to 1} S_{n,q}(k) = \lim_{q \to 1} \left( \frac{k^{n+1}}{n + 1} q^{k-1} - \frac{1}{n + 1} \sum_{i=0}^{n-1} \binom{n+1}{i} S_{i,q}(k) + \frac{q - 1}{q(n+1)} S_{n+1,q}(k) \right)
\]

\[=
\frac{k^{n+1}}{n + 1} - \frac{1}{n + 1} \sum_{i=0}^{n-1} \binom{n+1}{i} S_{i}(k) = S_{n}(k), \text{ cf. [13].}
\]

3. A formula for \( S_{n,q}(k) \)

In this section, we assume \( q \in \mathbb{C} \) with \(|q| < 1\). Now, we consider the \( q \)-Bernoulli polynomials as follows:

\begin{equation}
\frac{\log q + t}{q e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_{n,q}(x) \frac{t^n}{n!}, \text{ see [7, 8].}
\end{equation}
In the case \( x = 0 \), \( B_{n,q}(= B_{n,q}(0)) \) will be called the \( q \)-Bernoulli numbers, see [8, 9]. Let \( F_q(t, x) = \frac{\log q + t}{e^{qt-1}} e^{xt} \). Then we see that

\[
F_q(t, x) = -(t + \log q) \sum_{n=0}^{\infty} e^{(n+x)t} q^n, \quad t \in \mathbb{C} \text{ with } |t| < 2\pi, \text{ cf.}[8, 9, 10].
\]

In [8], it was known that

\[
B_{n,q}(x) = \sum_{j=0}^{n} \binom{n}{j} B_{j,q} x^{n-j} = m^{n-1} \sum_{i=0}^{m-1} q^i B_{n,q} m \left( \frac{x+i}{m} \right), \text{ for } n \geq 0.
\]

By (10) and (11), we easily see that the \( q \)-Bernoulli numbers can be rewritten as

\[
B_{0,q} = \frac{q-1}{\log q}, \quad q(B_q + 1)^k - B_{k,q} = \delta_{k,1}, \text{ for } k \geq 1,
\]

where \( \delta_{k,1} \) is Kronecker symbol and we use the usual convention about replacing \( B^i_q \) by \( B_{i,q} \), cf. [8, 9, 10]. From the simple calculation, we note that

\[
-\sum_{n=0}^{\infty} e^{(n+k)t} q^n + \sum_{n=0}^{\infty} e^{nt} q^{n-k} = \sum_{l=0}^{\infty} (q^{-k} \sum_{n=0}^{l} n^n q^n) \frac{t^l}{l!} = \sum_{l=0}^{\infty} (q^{-k} \sum_{n=0}^{l-1} n^{l-1} q^n) \frac{t^{l-1}}{l!}.
\]

Thus, we have

\[
\sum_{l=0}^{\infty} (B_{l,q}(k) - q^{-k} B_{k,q}(0)) \frac{t^l}{l!} = -(t + \log q) \sum_{n=0}^{\infty} e^{(n+k)t} q^n + (t + \log q) \sum_{n=0}^{\infty} e^{nt} q^{n-k}
\]

\[
= \sum_{l=0}^{\infty} (q^{-k} l S_{l-1,q}(k) + (q^{-k} \log q) S_{l,q}(k)) \frac{t^l}{l!}.
\]

By comparing the coefficient on both side in Eq.(14), we obtain the following:

**Theorem B.** Let \( l, k \) be positive integers \( (k > 1) \). Then we see that

\[
q^{-k} S_{l-1,q}(k) + (q^{-k} \log q) S_{l,q}(k) = \frac{B_{l,q}(k) - q^{-k} B_{l,q}(0)}{l}.
\]

Remark. Note that

\[
S_{l-1}(k) = \lim_{q \to 1} \left( q^{-k} S_{l-1,q}(k) + (q^{-k} \log q) S_{l,q}(k) \right) = \frac{B_l(k) - B_l(0)}{l}, \text{ cf. [13]},
\]

where \( B_l(k) \) are called ordinary Bernoulli polynomials.
References

1. T. Apstol, *Introduction to analytic number theory*, Undergraduate Texts in Math., Springer-Verlag, New York, 1986.
2. L. Carlitz, *q-Bernoulli numbers and polynomials*, Duke Math. J. 15 (1948), 987-1000.
3. J. Faulhaber, *Academia Algebræ, Darinnen die miraculosische inventiones zu den höchsten Cossen weithers continuirt und profitiert werden*, Augspurg, bey Johann Ulrich Schönigs, 1631.
4. K. C. Garrett and K. Hummel, *A combinatorial proof of the sum of q-cubes*, Electronic J. Combinatorics 11 (2004), R 9.
5. G. Gasper, M. Rahman, *Basic hypergeometric series*, Encyclopedia of Mathematics and Its Applications, Vol 96, Secod Edition, Cambridge Univ. Press, 2004.
6. V. J. W. Guo, J. Zeng, *A q-analogue of Faulhaber’s formula for sums of powers*, arXiv:math.CO/0501441 (2005).
7. T. Kim, *Sums of powers of consecutive q-integers*, Advan. Stud. Contemp. Math. 9 (2004), 15-18.
8. T. Kim, *A new approach to q-zeta function*, arXiv:math.NT/0502005 (2005).
9. T. Kim, *q-Volkenborn Integration*, Russian J. Math. Phys. 9 (2002), 288-299.
10. T. Kim, *Analytic continuation of multiple q-zeta functions and their values at negative integers*, Russian J. Math. Phys. 11 (2004), 71-76.
11. D. E. Knuth, *Johann Faulhaber and sums of powers*, Math. Comput. 61 (1993), 277-294.
12. M. Schlosser, *q-analogues of the sums of consecutive integers, squares, cubes, quarts and quints*, Electronic J. Combinatorics 11 (2004), R 71.
13. Y.-Y. Shen, *A note on the sums of powers of consecutive integers*, Tunghai Science 5 (2003), 101-106.
14. S. O. Warnaar, *On the q-analogues of the sums of cubes*, Electronic J. Combinatorics 11 (2004), N 13.