Stable optical rigidity based on dissipative coupling

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Received 14 March 2019, revised 23 April 2019
Accepted for publication 2 May 2019
Published 16 July 2019

Abstract

We show that stable optical rigidity can be obtained in a Fabry–Perot cavity having pure dissipative optomechanical coupling and detuned pumping under corresponding conditions. Optical detection of a weak classical mechanical force is analyzed under this rigidity. The sensitivity of small force measurement can be better than the standard quantum limit.

Keywords: dissipative coupling, stable optical rigidity, optical spring, Michelson–Sagnac interferometer

(Some figures may appear in colour only in the online journal)

1. Introduction

Resonant optomechanics [1] explores the interaction between an optical cavity and a free mass or mechanical oscillator. The simplest optomechanical interaction is based on the radiation pressure effect, in which a force proportional to the optical power or number of optical quanta circulating in a one-dimensional (1D) optical cavity acts on a test mass so that the size of the optical cavity increases with increasing number of optical quanta localized inside it. This interaction is usually referred to as dispersive coupling. Systems having several degrees of freedom allow more complex optomechanical interactions, including radiation pulling (negative radiation pressure) [2, 3], optomechanical interaction proportional to the quadrature of the electromagnetic field [4–7] and interaction depending on the speed and not on the coordinate of the mechanical system [8, 9].

Optomechanical interaction is important in precise measurements which use an efficient quantum transduction mechanism between the mechanical and optical degrees of freedom, allowing various sensors such as gravitational wave detectors [10–20], torque sensors [21] and magnetometers [22].

The sensitivity of mechanical coordinate measurement in an optomechanical system is usually restricted by the standard quantum limit (SQL) [23, 24] due to quantum backaction. The SQL was investigated in various configurations ranging from macroscopic gravitational wave detectors [7] to microcavities [25, 26]. The sensitivity of other types of measurements that are derivatives of the coordinate detection is also limited by the SQL. An example of such a measurement is detection of a classical force acting on a mechanical degree of freedom in an optomechanical system. However, the SQL of the force measurement is a limit that can be avoided. Several approaches can surpass the SQL, for example variational measurement [4, 7, 27], optomechanical velocity measurement using dispersive coupling [8, 9] and measurements in optomechanical systems having optical rigidity [28, 29]. A quantum speedometer based on dissipative coupling was recently proposed [30].

Dissipative coupling occupies a special place among the variety of optomechanical interactions. Dissipative coupling is characterized by the dependence of optic cavity relaxation on a mechanical coordinate (in the case of a Fabry–Perot cavity the mechanical coordinate is changing transparency of the input mirror), whereas dispersive coupling is characterized by dependence of the cavity frequency on the coordinate. Therefore a system that has dissipative coupling can no longer be considered lossless. But the dissipation here does not lead to decoherence or absorption of light. Instead, it results in lossless coupling between a continuous optical wave and a mode of an optical cavity. The cavity with dissipative coupling can be used as a perfect transducer between the
continuous optical wave and the mechanical degree of freedom, allowing efficient cooling of the mechanical oscillator [21, 31–34], exchange of the quantum states between optical and mechanical degrees of freedom, mechanical squeezing [35–38] and a combination of cooling and squeezing [39, 40]. A combination of conventional, dispersive and dissipative coupling makes the interaction more complex and leads to new effects [41, 42].

Dissipative coupling was proposed theoretically [31] and implemented experimentally about 10 years ago [21, 32, 33, 43]. It was studied in different optomechanical systems including a Fabry–Perot interferometer [21, 32, 33, 43], a Michelson–Sagnac interferometer (MSI) [34, 44, 45] and ring resonators [41, 42].

In this paper we report on the other important feature of a cavity possessing dissipative coupling. If this cavity is non-resonantly pumped it introduces a stable optical rigidity into the mechanical degree of freedom. Recalling that the optical rigidity based on conventional dispersive coupling is unstable, it can only be used with feedback. We formulate the conditions of stability and show that stable optical rigidity based on dissipative coupling allows the SQL restrictions to be avoided.

2. Hamiltonian approach

We consider the 1D optomechanical cavity presented on figure 1. Its optical mode with eigenfrequency \( \omega_0 \) is pumped with detuned light (the pump frequency \( \omega_p = \omega_0 + \delta \)) (this is a generalization of the model presented in [30]). The optical mode is dissipatively coupled with a mechanical system represented by a free test mass \( m \). The relaxation rate \( \kappa \) of the optical mode depends on the displacement \( x \) of the test mass.

The force of interest \( F_x \) acts on the test mass, changing its position.

We use Hamiltonian approach to describe the dissipative coupling following [30, 46]:

\[
H = \hbar \omega_0 \hat{a}_c^\dagger \hat{a}_c + \frac{\hat{p}_c^2}{2m} + H_T + H_s - F_x \hat{x},
\]

(2.1)

where \( \hat{a}_c \) and \( \hat{a}_c^\dagger \) are the annihilation and creation operators describing the intracavity optical field, \( \hat{p}_c \) is the free test mass momentum, \( H_T \) describes electromagnetic continuum [1, 47] and \( H_s \) stands for the attenuation \( \kappa \) of the pump photons and associated quantum noise. From the Hamiltonian (2.1) we obtain [1, 46] a set of corresponding equations describing the time evolution of the optomechanical system

\[
\dot{\hat{a}} + \left( \frac{\kappa}{2} - i\delta \right) \hat{a} = \sqrt{\kappa} \hat{a}_{in},
\]

(2.2a)

\[
\dot{\hat{x}} = i \hbar \frac{\kappa \eta}{2m} (\hat{a}^\dagger \hat{a}_{in} - \hat{a}_{in}^\dagger \hat{a}) + \frac{F_x}{m},
\]

(2.2b)

\[
\kappa = \kappa_0 (1 + \eta \delta), \quad \sqrt{\kappa} \approx \sqrt{\kappa_0} \left( 1 + \frac{\delta^2}{2} \right).
\]

(2.2c)

Here \( \dot{\hat{a}} \) is the slow amplitude (\( \hat{a}_s = \hat{a} e^{-i \omega_p t} \), see (A5c)) of the intracavity wave, \( \kappa \) is the full width at half maximum of the mode (relaxation rate) depending on the position \( x \) of the test mass and \( \eta \) is a constant of dissipative coupling. \( \hat{a}_{in} \) is the slow amplitude of the input wave (for details see appendix A).

Note that equations (2.2), derived from the Hamiltonian (2.1), describe pure dissipative coupling; dispersive coupling is absent.

These equations have to be accompanied by an expression for the output amplitude \( \hat{a}_{out} \), which can be derived in the Hamiltonian approach considering the thermal bath as a transmission line. The details of this derivation are presented in appendix A.1. In case of small transparency \( T_0 \ll 1 \) this expression can be written as

\[
\hat{a}_{out} = -\hat{a}_{in} + \sqrt{\kappa} \hat{a}.
\]

(2.2d)

Below we present amplitudes as a sum of large mean and small addition values

\[
\hat{a} \Rightarrow A + \hat{a}, \quad \hat{a}_{in} \Rightarrow A_{in} + \hat{a}_{in}, \quad \hat{a}_{out} \Rightarrow A_{out} + \hat{a}_{out}
\]

where \( A, A_{in} \) and \( A_{out} \) are expected values of amplitudes of the intracavity, pump and reflected waves, \( \hat{a}_{in} \) and \( \hat{a}_{out} \) are annihilation operators describing vacuum fluctuation in waves falling on the cavity and having the following commutator and correlator:

\[
[\hat{a}_{in}(t), \hat{a}_{in}^\dagger(t')] = \delta(t-t'),
\]

(2.3)

\[
\langle \hat{a}_{in}(t) \hat{a}_{in}^\dagger(t') \rangle = \delta(t-t').
\]

(2.4)

We assume that expected values exceed the fluctuation parts of the operators:

\[
A \gg \hat{a}, \quad A_{in} \gg \hat{a}_{in}, \quad A_{out} \gg \hat{a}_{out}
\]

(2.5)

and apply the method of successive approximations below.

Recall that from this point \( \hat{a}, \hat{a}_{in}, \hat{a}_{out} \) stand for small slow fluctuation and signal additions.
We select $A_{in} = A_m^\ddagger$ and find steady-state amplitudes
\begin{equation}
A = \frac{\sqrt{\kappa_0}A_{in}}{\frac{\kappa_0}{2} - i\delta}, \quad A_{out} = A_{in}\cdot\frac{\frac{\kappa_0}{2} + i\delta}{\frac{\kappa_0}{2} - i\delta}.
\end{equation}
(2.6)
To the first order of approximation we obtain for small amplitudes and the deviation of the test mass:
\begin{equation}
\hat{a} + \left(\frac{\kappa_0}{2} - i\delta\right)\hat{a} = -\frac{\eta\kappa_0}{2}\hat{A}\xi + \frac{\sqrt{\kappa_0}}{2}\hat{A}_m^\ddagger + \sqrt{\kappa_0}\hat{a}_m^\ddagger.
\end{equation}
(2.7a)
\begin{equation}
\hat{a}_{out} = -\hat{a}_{in} + \sqrt{\kappa_0}\hat{a} + \frac{\sqrt{\kappa_0}}{2}\hat{A}_m^\ddagger,
\end{equation}
(2.7b)
\begin{equation}
\hat{\xi} = \hat{F}_p + F_x,
\end{equation}
(2.7c)
Here $\hat{F}_p$ is a light pressure force.
Below we use the Fourier transform defined as
\begin{equation}
\hat{a}(t) = \int_{-\infty}^{\infty} a(\Omega) e^{-i\Omega t} \frac{d\Omega}{2\pi}
\end{equation}
(2.8)
and in a similar manner we can express other values, denoting the Fourier transform by the same letter but without a hat. For the Fourier transform of the input fluctuation operators one can derive from (2.3) and (2.4):
\begin{equation}
[a_m(\Omega), a_m^\ddagger(\Omega')] = 2\pi \delta(\Omega - \Omega'),
\end{equation}
(2.9)
\begin{equation}
\langle a_m(\Omega)a_m^\ddagger(\Omega')\rangle = 2\pi \delta(\Omega - \Omega').
\end{equation}
(2.10)
We rewrite equations (2.7a) and (2.7c) in the frequency domain:
\begin{equation}
\hat{x} = -\hat{F}_p + F_x
\end{equation}
(2.11)
\begin{equation}
\hat{F}_p = \frac{i\eta\sqrt{\kappa_0}}{2}[\hat{A}^\ddagger\hat{a}_{in} - \hat{A}\hat{a}_{in}^\ddagger - \hat{A}_m(a - a^\ddagger)],
\end{equation}
(2.12)
\begin{equation}
a = \frac{\frac{\kappa_0}{2} - i(\Omega + \delta)}{\frac{\kappa_0}{2} - i(\Omega + \delta)}\frac{\kappa_0}{2} - i(\Omega + \delta)\hat{A}_m^\ddagger
\end{equation}
(2.13)
Here we denote
\begin{equation}
a = a(\Omega), \quad a^\ddagger = a^\ddagger(-\Omega),
\end{equation}
(2.14a)
\begin{equation}
a_{in} = a_{in}(\Omega), \quad a_{in}^\ddagger = a_{in}^\ddagger(-\Omega).
\end{equation}
(2.14b)
Below we express the light pressure force through a sum
\begin{equation}
F_p = F_p - F_x
\end{equation}
(2.15)
of a fluctuation force $F_p$ and a regular rigidity force $F_x$ proportional to displacement $x$, which we calculate in the next section.

3. Optical rigidity
We substitute equation (2.13) into the right part of (2.12) and extract only terms $\sim x$. For the optical rigidity $K = -F_x/x$ one can obtain
\begin{equation}
K = -m\Omega_0^2\delta\left(\frac{\kappa_0}{2} - 2\Omega^\prime - \delta^2\right)\left[\frac{\kappa_0}{2}\left(\frac{\kappa_0}{2} - \Omega^\prime\right) + \delta^2\right].
\end{equation}
(3.1)
\begin{equation}
\Omega_0^2 = \frac{\eta\kappa_0E_0}{m(m\omega_p^2 + \delta^2)} = \frac{\eta^2\kappa_0E_0}{m\omega_p^2} = \frac{\eta^2W_m}{m\omega_p^2}.
\end{equation}
(3.2)
Here $\Omega_0^2$ is the recalculated pump power (the dimension of squared frequency), $E_0 = \hbar\omega_p|A|^2$ is the mean energy stored in the cavity and $W_m = \kappa_0E_0$ is the power of the incident wave.

Recall that in case of dispersive coupling the optical rigidity is always unstable. For example, in the case of detuning on the right slope ($\delta > 0$) of the resonance curve the optical rigidity is positive but the introduced mechanical viscosity is negative. This means instability (when detuning on the left slope the viscosity is positive but the rigidity is negative) [48].

In contrast, the optical rigidity (3.1) is more complicated than the rigidity based on dispersive coupling and one can tune both signs of the rigidity and viscosity by varying the relation between detuning $\delta$ and relaxation rate $\kappa_0$. To demonstrate this we can expand (3.1) into the Taylor series over $(-i\Omega)$ keeping only two first terms:
\begin{equation}
K_m = -m\Omega_0^2\delta\left(\frac{\kappa_0}{2} - \delta^2\right)\left[\frac{\kappa_0}{2}\left(\frac{\kappa_0}{2} - \delta^2\right)\right].
\end{equation}
(3.3a)
\begin{equation}
- m\Omega_0^2\delta\left(\frac{\kappa_0}{2} - \delta^2\right)\left(\frac{\kappa_0}{2} + \delta^2\right)^{-1}(-i\Omega).
\end{equation}
(3.3b)
It is easy to conclude that the rigidity (3.3a) is positive if $\delta < 0$ and $3\left(\frac{\kappa_0}{2}\right)^2 > \delta^2$, whereas the viscosity (3.3b) is positive if additionally $\left(\frac{\kappa_0}{2}\right)^2 < \delta^2$. So we can formulate stable rigidity conditions:
\begin{equation}
\delta < 0, \quad \frac{\kappa_0}{2} < \delta < \sqrt{3}\frac{\kappa_0}{2}.
\end{equation}
(3.4a)

However, conditions (3.4a) are a result of approximation. For accurate consideration we apply the Routh–Hurwitz criterion [49–52] to investigate the stability of the system, described by the susceptibility $\chi = m/(K - m\Omega_0^2)$. We found that the accurate conditions of stability include (3.4a) one more condition applied to the pump
\begin{equation}
0 < \Omega_0^2 < \Omega_{0,\text{max}}^2, \quad \Omega_{0,\text{max}}^2 = \frac{\kappa_0}{|\delta|}\left(\delta^2 - \left(\frac{\kappa_0}{2}\right)^2\right).
\end{equation}
(3.4b)
\begin{equation}
\Omega_0^2 = y\Omega_{0,\text{max}}^2, \quad 0 < y < 1.
\end{equation}
(3.4c)
Here $y$ is a dimensionless power parameter.

Summing up, the rigidity based on dissipative coupling can be positive on both left and right slopes of the resonance curve; however, it is stable only on the left one, $\delta < 0$. 

Figure 2. The plots of the susceptibilities $|\chi|$ and their approximations $|\chi_m|$ as function of frequency at detuning $\delta = -1.1 \kappa_0/2$ (upper plot) and $(0.1 - (3/\sqrt{3})\kappa_0/2$ as function of frequency for the different power parameters $y$ (3.4c).

It is important that we can control the characteristics of stable rigidity. We can obtain an approximation for eigen-frequency $\Omega_m$, relaxation rate $\delta_m$ and quality factor $Q_m$ of a mechanical oscillator created by the optical rigidity using equations (3.3) to demonstrate this:

$$\Omega_m^2 = \frac{\Omega_0^2[\delta - \left(\frac{\kappa_0}{2}\right)^2 - \delta^2]}{\left[\frac{\kappa_0}{2}\right]^2 + \delta^2}.$$  (3.5)

$$\delta_m = \frac{2\Omega_0^2[\delta - \left(\frac{\kappa_0}{2}\right)^2]}{\left[\frac{\kappa_0}{2}\right]^2 + \delta^2}.$$  (3.6)

$$Q_m = \frac{\Omega_m}{2\delta_m} = \frac{\sqrt{\frac{\kappa_0}{2}} - \delta^2\left(\frac{\kappa_0}{2}\right)^2 + \delta^2)^{1/2}}{4\Omega_0\sqrt{\left[\frac{\kappa_0}{2}\right](\delta^2 - \left[\frac{\kappa_0}{2}\right]^2)}}.$$  (3.7)

Here we assumed for stability that $\delta < 0$.

Despite the fact that this consideration based on the series expansion is convenient, it is only valid for small frequencies and a small pump. This means that the pump parameter $\Omega_0$ should be small. For a large pump $\Omega_0$ we have to use the exact susceptibility $\chi$ instead of its approximation $\chi_m = m/(K_m - m\Omega^2)$. In figure 2 we give plots of $|\chi|$ and $|\chi_m|$, for detunings $\delta$, corresponding to stable rigidity (3.4a) and different $y$ values. We see that this approximation (3.3) gives correct results for a small power parameter $y \leq 0.3$, whereas for $y > 0.3$ the approximation is not valid.

Plots shown in figure 2 also show that by choosing detuning $\delta$ and power parameter $y$ one can obtain an overdamped mechanical oscillator or an oscillator with a high quality factor. So the optical rigidity based on dissipative coupling provides a very promising possibility to create a mechanical system using selected characteristics.

The introduction of stable optical rigidity converts the free mass into an artificially created mechanical oscillator and it is interesting to estimate its noise. It is affected by a fluctuation force (2.12) and its power spectral density $S_{\mathcal{F}_f}$ is equal to

$$S_{\mathcal{F}_f} = 2\hbar m\Omega_0^2\left(\left[\frac{\kappa_0}{2}\right]^2 + \delta^2\right)\times\{|g_+ + j\delta|^2 + |g_+ - j\delta|^2\}.$$  (3.8)

We used the definitions (3.2) and the formula (B5) with notations given in appendix B. This formula can be rewritten through $\Omega_m$ using (3.5).

Due to action of $\mathcal{F}_f$ in the equilibrium state the oscillator has mean fluctuation energy $\mathcal{E}_m = m\Omega_m^2\langle x^2 \rangle$ which is convenient to describe by the mean quantum number $n_{\text{eff}}$:

$$\mathcal{E}_m = \hbar \Omega_m n_{\text{eff}}.$$  (3.9)

The spectral density $S_{\mathcal{F}_f}$ does not practically depend on spectral frequency $\Omega$, and for high quality factor $Q_m$ (3.7) it can be considered as a constant (white noise, not depending on $Q_m$). Consequently, the mean energy $\mathcal{E}_m$ should increase with increasing $Q_m$. For this particular case our estimate gives

$$n_{\text{eff}} \approx 240, \quad \delta = -0.55 \kappa_0, \quad y = 0.01.$$  (3.10a)

For these parameters

$$\Omega_m \approx 0.029 \kappa_0, \quad Q_m \approx 32.37 \Omega_0, \quad Q_m \approx 14.5.$$  (3.10b)

4. Detection of the signal force

Using (2.7b) and (2.13) we obtain for the output amplitude in the frequency domain:

$$a_{\text{out}} = \frac{\frac{\kappa_0}{2} + i[\delta + \Omega]}{\frac{\kappa_0}{2} - i[\delta + \Omega]} a_{\text{in}} + \frac{\eta \kappa_0 A_0}{2} n_{\text{in}} + i\delta \left(\frac{1}{\frac{\kappa_0}{2}} - i[\delta + \Omega]\right) x.$$  (4.1)

We have to substitute the mechanical displacement $x$ in frequency domain into (4.1) accounting for rigidity (3.1):

$$x = \frac{F_f + F_\delta}{-m\Omega^2 Q}, \quad Q = 1 - \frac{K}{m\Omega^2}.$$  (4.2)

and the fluctuation force $F_f$ (for details see appendix B).

We assume that the output wave is registered by a homodyne detector. Hence, we have to calculate the quadratures of the output wave. We define the amplitude quadrature $e_a$ and the phase quadrature $e_p$ in the output wave as...
follows:
\[
e_a = \frac{1}{\sqrt{2}} \left( \frac{\eta_0}{2} - i\delta \right) a_{\text{in}} + \frac{\eta_0}{2} + i\delta a_{\text{out}}^\dagger, \quad (4.3a)
\]
\[
e_p = \frac{1}{i\sqrt{2}} \left( \frac{\eta_0}{2} - i\delta \right) a_{\text{out}} - \frac{\eta_0}{2} + i\delta a_{\text{in}}. \quad (4.3b)
\]
Calculation of the output quadratures as functions of the input amplitude \((a_\text{in})\) and phase \((a_\text{p})\) quadratures
\[
a_{\text{in}} = \frac{a_{\text{in}} + a_{\text{in}}^\dagger}{\sqrt{2}}, \quad a_{\text{p}} = \frac{a_{\text{in}} - a_{\text{in}}^\dagger}{i\sqrt{2}} \quad (4.4)
\]
is given in appendix B; the results are
\[
e_a = E_{\text{in}}a_\text{in} + E_{\text{pp}}a_\text{p} + \Phi_\text{p}f_\text{p}, \quad f_\text{s} = \frac{F_s}{\sqrt{S_{\text{F-SQL}}}}, \quad (4.5a)
\]
\[
e_p = E_{\text{pp}}a_\text{in} + E_{\text{pp}}a_\text{p} + \Phi_\text{p}f_\text{p}, \quad S_{\text{F-SQL}} = 2\hbar m\Omega^2. \quad (4.5b)
\]
Here \(f_\text{s}\) is the Fourier transform of the signal force normalized to the standard quantum limit (SQL). \(S_{\text{F-SQL}}\) is the SQL of power spectral density of the force acting on a free mass \([7–9]\). We follow the tradition of describing the sensitivity of the scheme with optical rigidity in terms of the SQL for a free mass \([28, 29]\). This allows us to compare the sensitivity with the optical rigidity and without it. The expressions for the coefficients in \((4.5)\) are rather cumbersome and we present them using the outcome of notations in appendix B.

In the homodyne detector we measure a quadrature \(e_\theta = e_\text{in}\cos\theta + e_\text{in}\sin\theta\) in the output wave, where \(\theta\) is a homodyne angle. It is convenient to describe the sensitivity as a quadrature \(e_\theta\) recalculated to the SQL:
\[
\delta = \frac{e_\text{in}\cos\theta + e_\text{in}\sin\theta}{\Phi_\text{p}\cos\theta + \Phi_\text{p}\sin\theta} \quad (4.6)
\]
with the power spectral density (PSD)
\[
S_\text{f}(\Omega) = S_a + S_p,
\]
\[
S_a = \frac{|E_{\text{in}} + E_{\text{pp}}\tan\theta|^2}{|\Phi_\text{a} + \Phi_\text{p}\tan\theta|^2}, \quad S_p = \frac{|E_{\text{pp}} + E_{\text{pp}}\tan\theta|^2}{|\Phi_\text{a} + \Phi_\text{p}\tan\theta|^2}. \quad (4.7)
\]
Below we only analyze sensitivity for stable rigidity (i.e. conditions \((3.4)\) are valid).

4.1. Amplitude detection

We start our consideration from amplitude detection. Formally it corresponds to \(\sin\theta = 0\) in the formulae \((4.7)\):
\[
S_\text{f}(\Omega) = S_a^4 + S_p^4, \quad S_a^4 = \frac{|E_{\text{in}}|^2}{|\Phi_\text{a}|^2}, S_p^4 = \frac{|E_{\text{pp}}|^2}{|\Phi_\text{p}|^2}. \quad (4.8)
\]
We obtain that even amplitude detection allows the SQL (i.e. \(S_\text{f} < 1\)) to be surpassed by more than 100 times. Choosing the pump parameter \(y\) one can vary both spectral frequency and range of the surpassing of the SQL (demonstrated by plots in figure 3). Recall that amplitude detection in the scheme with optical rigidity based on dispersive coupling does not provide a sensitivity much better than the SQL \([28, 29]\).

4.2. Homodyne detection

In this case we have a homodyne angle \(\theta\) featuring an additional degree of freedom which provides the possibility of controlling the sensitivity. Indeed, even for constant pumping we can change PSD by tuning the homodyne angle. As shown in figure 5, PSD has a minimum at frequency \(\Omega_{\text{min}}\) and its
width \( \Delta \Omega \) (where the SQL is surpassed, i.e. \( S_1 < 1 \)) can be shifted and changed.

The plots in figure 5 demonstrate that frequency \( \Omega_{\text{min}} \) grows with increasing homodyne angle \( \theta \), whereas the bandwidth \( \Delta \Omega \) initially decreases until \( \Omega_{\text{min}} < \Omega_m \) and increases when \( \Omega_{\text{min}} > \Omega_m \). Note that if \( \Omega_{\text{min}} \approx \Omega_m \) we have the strongest minimum in PSD but at a very narrow bandwidth.

Detailed analysis shows that for the particular plots at the top of figure 5 the amplitude part \( S_a \) makes the main contribution to PSD (4.7). The minimum of PSD (practically the minimum of \( S_a \)) occurs when the shot noise term \( \sim (\beta_x - \beta_e^* \tan \theta) \) and the backaction noise term \( \sim (E_{\text{in}} + E_{\text{out}} \tan \theta) \) compensate each other. For details see equations (B7) in appendix B. The plot of susceptibility is also presented (it has a different dimension) in order to show that the above compensation occurs at frequencies close to the frequency \( \Omega_m \) of the mechanical resonance.

5. Model of dissipative coupling based on a Michelson–Sagnac interferometer

For realization of dissipative coupling without dispersive coupling we consider a MSI as first suggested in [44]. In the Fabry–Perot cavity shown in figure 6, the MSI plays the role of the input generalized mirror (GM). Here we present a generalized model with the non-balanced beam splitter having amplitude transmittance \( T_{\text{bs}} \) and reflectivity \( R_{\text{bs}} \) and the partially reflecting mirror \( M \) having transmittance \( T \) and reflectivity \( R \). We assume that the size of the GM is smaller than the distance \( L \) between the non-movable beam splitter and the end mirror so both amplitude transmittance \( T \) and reflectivity \( R \) of the GM depend on the position \( X \) of the movable mirror \( M \) having mass \( m \) and do not depend on spectral frequency.

We start from the boundary conditions on the beam splitter:

\[
A_e = T_{\text{bs}} B_e - R_{\text{bs}} B_m, \quad A_n = R_{\text{bs}} B_e + T_{\text{bs}} B_m, \tag{5.1a}
\]

\[
B_{\text{out}} = T_{\text{bs}} B_n - R_{\text{bs}} B_e, \quad D_1 = T_{\text{bs}} B_e + R_{\text{bs}} B_m, \tag{5.1b}
\]

where \( B_n, B_{\text{out}}, B_e, B_m, A_n, A_e, \) are complex amplitudes of the incident and reflected waves on the beam splitter (see the notations in figure 6). The boundary conditions on mirror \( M \) give

\[
B_e = -R A_e e^{i k d_m} + T A_n e^{i (k d_m + t_c)}, \tag{5.2a}
\]

\[
B_n = R A_n e^{i k d_e} + T A_e e^{i (k d_e + t_c)} \tag{5.2b}
\]

where \( k = \omega_j / c \) is the wave vector, \( c \) is the speed of light and \( k d_e \) (\( k d_m \)) is the accumulated phase of the light traveling between the beam splitter and mirror \( M \) through the east (north) arm.
We define the reflectivity and transmittance of the GM as
\[ D_i = T B_{in} + R \triangleright P B_c, \quad B_{out} = T B_k + R \triangleleft B_{in}. \] (5.3)

Using (5.1) and (5.2) one can derive
\[ T = e^{i \phi_0} \{ 2 R R_h B_k \cos \phi_\perp - T \Delta_b \}, \] (5.4a)
\[ \triangleright = R e^{i \phi_0} \times \left\{ \Delta_b \cos \phi_\perp - i \sin \phi_\perp + \frac{2 T T_k R_h}{R} \right\}, \] (5.4b)
\[ \triangleleft = - R e^{i \phi_0} \times \left\{ \Delta_b \cos \phi_\perp + i \sin \phi_\perp + \frac{2 T T_k R_h}{R} \right\}, \] (5.4c)
\[ \phi_\perp = k (\ell_c \pm t_c), \quad \Delta_b = R^2 - T^2. \] (5.4d)

It is obvious that the sum phase \( \phi_\perp \) does not depend on the displacement \( X \) of mirror \( M \), but the phase difference \( \phi_\perp \) does. Below we express the displacement \( X = x_0 + x \) as a sum of the constant mean value \( x_0 \) and a small addition \( x \) so that \( \phi_\perp = \phi_0 + 2 k x \) and expand the reflectivity and transmittance of the GM in a series over \( x \).

One can easily deduce that realizing pure dissipative coupling (but not a combination of dissipative and dispersive coupling) we should obtain the relative derivatives of \( T, \triangleright \) and \( \triangleleft \) over \( \phi_\perp \) as real. Calculations give:
\[ \frac{\partial_x T}{T} = - \frac{2 R R_h T_k \sin \phi_\perp}{2 R R_h T_k \cos \phi_\perp - T \Delta_b}, \] (5.5a)
\[ \frac{\partial_x \triangleright}{\triangleright} = - \frac{R \Delta_b \sin \phi_\perp - i R \cos \phi_\perp}{R \Delta_b \cos \phi_\perp - i R \sin \phi_\perp + 2 T T_k R_h}. \] (5.5b)

We see that the relative derivative (5.5a) is real for any combination of parameters. In order to have the real derivative (5.5b) we have the following two possibilities:

(a) a balanced beam splitter (\( \Delta_b = 0 \)) and a perfectly reflective mirror \( M (T = 0) \); this case was analyzed in [30, 44];

(b) a non-balanced beam splitter and a partially transparent mirror \( M (T \neq 0) \); in this case we have to choose \( \phi_\perp = \phi_0 \), where \( \phi_0 \) is the solution of the equation
\[ \cos \phi_0 = - \frac{R \Delta_b}{2 T T_k R_h}, \] (5.5c)
\[ T_0 = T |_{\phi_\perp = \phi_0} = - e^{i \phi_0} \frac{\Delta_b}{T}, \] (5.5d)
\[ |R_0| = |R| |_{\phi_\perp = \phi_0} = \sqrt{\left(2 T T_k R_h\right)^2 - T^2}, \] (5.5e)
\[ \frac{\partial_\phi T}{T} \bigg|_{\phi_\perp = \phi_0} = \frac{2 R R_h T_k \sin \phi_\perp}{\Delta_b} = \frac{R |R_0|}{|T_0|}. \] (5.5f)

The cavity should have high finesse. Hence, for \( |T_0| \ll 1 \) one has to have \( |\Delta_b| \ll T \). So, assuming
\[ e^{i \phi_0} = - 1, \] we obtain to first-order approximation over \( x \)
\[ T = T_0 \left( 1 + \frac{R |R_0|}{|T_0|} 2 k x + \ldots \right), \] (5.6)
\[ \triangleright = R_0 \left( 1 - \frac{R |T_0|}{|R_0|} 2 k x + \ldots \right). \] (5.7)

We should point out that at these conditions pure dissipative coupling is realized and dispersive coupling is absent as the effective length of the cavity does not change.

We see that realization of dissipative coupling with a partially transparent mirror \( M \) requires the choice of the correct angle \( \phi_0 \) (i.e. constant displacement \( x_0 \)). It is important that we can choose GM parameters on demand by varying the beam splitter parameters \( (R_{\triangleright}, T_{\triangleright}) \).

A small displacement \( x \) of mirror \( M \) from the mean position \( x_0 \) provides modulation of the relaxation rate of Fabry–Perot interferometer:
\[ \kappa = \kappa_0 (1 + \eta x), \quad \kappa_0 = \frac{|T|^2}{\tau}, \quad \tau = \frac{2 L}{c} \] (5.8a)
\[ \eta = 4 k R |R_0| \frac{|R_0|}{|T_0|}. \] (5.8b)

It is easy to demonstrate that all equations for this optomechanical system are the same as those derived in section 2.

It is important that the example of an interferometer considered as a GM allows us to demonstrate the peculiar property of the light pressure force in an optomechanical system having dissipative coupling. Indeed, using the notation in figure 6 we can write a ponderomotive force acting on mirror \( M \):
\[ F = 2 h k R^2 (|A|^2 - |A|^2) \] (5.9)
\[ = 4 h k R^2 (B_c B_m^* + B_c^* B_m). \] (5.10)

In the last equation we used the input–output relation (5.1), putting \( R_{\triangleright} = T_{\triangleright} \). Recall that for dispersive coupling the ponderomotive force is proportional to the square of the amplitude of the intracavity wave. In contrast, for an optomechanical system having dissipative coupling the force is proportional to the cross product of incident amplitude \( B_m \) and amplitude \( B_c \), as follows from (5.9). The light pressure force depends on the phase difference between \( B_m \) and \( B_c \), so it can be also referred to as the interferometric pressure. It is this property that provides the additional possibility of realizing stable rigidity. We would like to draw attention to the similarity of formulae (5.9) and (2.12), obtained through the Hamiltonian approach.

Note that the realization of stable optical rigidity was proposed in [45] and elegantly demonstrated in [34, 53] for a similar scheme with an MSI alone as shown in figure 6, without any cavity and without paying attention to dissipative or dispersive coupling. In contrast, in the scheme analyzed in this paper stable optical rigidity is a property of a cavity having dissipative coupling. So we formulated conditions when a MSI acts as a generalized mirror with dissipative coupling (but not a combination of dissipative and dispersive coupling).
6. Conclusion

We analyzed optical rigidity based on dissipative coupling and formulated conditions (3.4) for stable optical rigidity. (Recall that by using dispersive coupling one can get only unstable rigidity [28, 29].)

We underline that in this paper we restrict our considerations by using a linear approximation of $\kappa$ over $x$. In the model of dissipative coupling presented in section 5 this means that displacement $x$ must be small (see (5.8)):

$$\eta x \ll 1, \quad \eta x \approx \frac{4kx}{|T_0|}. \quad \text{(6.1)}$$

Physically this means that displacement $x$ produces a small deformation of the resonance curve. Usually this condition is valid in optomechanical experiments. Note that a similar deformation of the resonance curve. Usually this condition is formulated for dispersive coupling.

One can get an estimation of displacement using (3.9), (3.10) and (6.4):

$$\sqrt{\langle x^2 \rangle} = \frac{m\Omega_0}{|n_{\text{eff}}|} \approx 5.4 \cdot 10^{-15} \text{ m}, \quad \text{(6.2)}$$

$$\frac{4k \sqrt{\langle x^2 \rangle}}{|T_0|} \approx 1.4 \cdot 10^{-5} \ll 1. \quad \text{(6.3)}$$

The last estimation demonstrates the validity of approximation (6.1).

The rigidity based on dissipative coupling can be positive on both the left and right slopes of the resonance curve (but it is stable only on the left one, $\delta < 0$), whereas positive (unstable) rigidity for dispersive coupling occurs only on the right slope.

We show that the physical meaning of stability of the rigidity based on dissipative coupling concerns interference between the intracavity and extracavity waves, giving a more complicated dependence of the light pressure force than with dispersive coupling.

We have shown that pure dissipative coupling can be realized in a MSI having a partially transparent mirror $M$; this is a generalization of previous results [30, 44] for a perfectly reflecting mirror $M$. It provides the possibility of using a thin membrane [54–56] with extremely small mass $m$ as the mirror $M$ for experimental realization.

For the estimation we assume:

$$m = 10^{-8} \text{ g}, \quad k = \frac{2\pi}{\lambda}, \quad \lambda = 10^{-6} \text{ m}, \quad \text{(6.4a)}$$

$$W_m = 10^{-4} W, \quad R^2 = 0.7, \quad \frac{|T|^2}{W} = 10^{-4}. \quad \text{(6.4b)}$$

Using (3.2) and (3.5) we obtain estimations of the power parameter $\Omega_0$ and mechanical eigenfrequency $\Omega_m$:

$$\Omega_0 = \sqrt{\frac{4kW_m}{mcT_0^2}} \approx 92 \cdot 10^3 \text{ rad/s}, \quad \text{(6.5a)}$$

$$\Omega_m \approx 86 \cdot 10^3 \text{ rad/s}. \quad \text{(6.5b)}$$

In the last estimation we put $\delta = -0.55 \kappa_0$.

This makes it feasible to create a mechanical nanooptical oscillator with an eigenfrequency in the range of hundreds of kHz using a free mass and stable optical rigidity. The fluctuating light pressure force gives rise to excitation of an oscillator; we show that in equilibrium the mean quantum number $n_{\text{eff}}$ of such an oscillator can be about 200. This estimate corresponds to a coherent pump; however, for a specially tuned squeezed pump the mean quantum number $n_{\text{eff}}$ can be smaller.

Acknowledgments

The authors acknowledge support from the Russian Science Foundation (grant no. 17-12-01095).

Appendix A. Description of dissipation

In this appendix we present a detailed description of the Hamiltonian (2.1) and the derivation of equations (2.2) for the field $\hat{a}_c$ inside the cavity and the mechanical coordinate $\hat{x}$.

We write the Hamiltonians $H_f, H_e$ in the form

$$H_e = i\hbar \sqrt{\kappa \Delta \omega \over 2\pi} \sum_{q=1}^{\infty} (\hat{a}_q^\dagger \hat{b}_q - \hat{b}_q^\dagger \hat{a}_q), \quad \text{(A1)}$$

$$H_f = \sum_{q=1}^{\infty} \hbar \omega_q \hat{b}_q \hat{b}_q^\dagger. \quad \text{(A2)}$$

Here we present a thermal bath as an infinite number of oscillators with annihilation and creation operators $\hat{b}_q$, $\hat{b}_q^\dagger$, $q$ is an integer number and the frequencies $\omega_q$ of these oscillators are separated by $\Delta \omega = \omega_q - \omega_{q-1}$, recalling that below we put

$$\Delta \omega \to 0. \quad \text{(A3)}$$

The commutators and correlators are

$$[\hat{b}_q, \hat{b}_q^\dagger] = \delta_{qq'}, \quad \langle \hat{b}_q^\dagger \hat{b}_q \rangle = \delta_{qq'}. \quad \text{(A4)}$$

(The temperature of the bath is assumed to be zero.) We write down the movement equations:

$$\hat{\dot{a}}_c = -i\hbar [\hat{a}_c, H] = -i\omega_0 \hat{a}_c + \sqrt{\kappa \Delta \omega \over 2\pi} \sum_{q=1}^{\infty} \hat{b}_q, \quad \text{(A5a)}$$

$$\hat{\dot{b}}_q = -i\omega_q \hat{b}_q - \sqrt{\kappa \Delta \omega \over 2\pi} \hat{a}_c. \quad \text{(A5b)}$$

Introducing slow amplitudes

$$\hat{a}_c = \hat{a} e^{-i\omega_0 t}, \quad \hat{b}_q \approx \hat{b}_q e^{-i\omega_q t} \quad \text{(A5c)}$$

we get

$$\hat{\dot{a}}(t) - i\delta \hat{a} = \sqrt{\kappa \Delta \omega \over 2\pi} \sum_{q=1}^{\infty} \hat{b}_q e^{i(\omega_q - \omega_0) t}, \quad \text{(A5d)}$$

$$\hat{\dot{b}}_q = -\sqrt{\kappa \Delta \omega \over 2\pi} \hat{a} e^{-(\omega_q - \omega_0) t}. \quad \text{(A5e)}$$
We substitute the formal solution of (A5c) for \( \hat{b}_q \) into (A5d) using the method of successive approximations based on (A3):

\[
\hat{b}_q = \hat{b}_q(0) - \frac{\kappa \Delta \omega}{2\pi} \int_0^t \hat{a}_i(t') e^{-i(\omega_q - \omega) t'} dt',
\]
(A6a)

\[
\hat{a}_i = i \delta \hat{a}_i = \frac{1}{\kappa \Delta \omega} \int_{-\infty}^t \hat{b}_q(0) e^{i(\omega_q - \omega) t'} dt'
\]
(A6b)

\[
= \frac{1}{\kappa \Delta \omega} \int_0^t \hat{a}_c(t') e^{i(\omega_q - \omega) t'} dt' = \sqrt{\kappa} \hat{a}_m - \frac{\kappa}{\sqrt{2}} \hat{a}.
\]
(A6d)

Below we give details of the derivation of (A6d).

In subsequent calculations in the limit (A3) we replace the sum by the integral using the rule

\[
\int_{-\infty}^\infty d\omega_q \Rightarrow \int_0^\infty d\omega_q.
\]
(A7)

Using (A4) we calculate the commutator (2.3) for \( a_m \) defined in (A6b)

\[
\hat{a}_m(t) \equiv \frac{1}{\kappa \Delta \omega} \int_{-\infty}^t \hat{b}_q(0) e^{i(\omega_q - \omega) t'} dt',
\]
(A8a)

\[
[\hat{a}_m(t), \hat{a}_m^\dagger(t') ] \equiv \frac{1}{\kappa \Delta \omega} \int_{-\infty}^t \int_{-\infty}^{t'} e^{i(\omega_q - \omega) t''} dt''
\]
(A8b)

\[
= \delta(t - t').
\]
(A8c)

Here \( \delta(t) \) is the Dirac delta function. A similar calculation gives the correlator (2.4) assuming that thermostat oscillators are in the main state:

\[
\langle b_q b_{q'} \rangle = \delta_{kk'}, \quad \langle b_q b_q^\dagger \rangle = 0
\]
(A8d)

where \( \delta_{kk'} \) is the Kronecker delta.

We calculate the term (A6c) using the rule (A7):

\[
\int_0^t \hat{a}_c(t') \left[ \sum_{q=1}^\infty \frac{\kappa \Delta \omega}{2\pi} e^{i(\omega_q - \omega) t'} \right] dt'
\]
(A8e)

\[
\int_0^t \hat{a}_c(t') \frac{\kappa}{2\pi} \int_0^\infty e^{i(\omega_q - \omega) t'} \delta(t - t') dt' = \frac{\kappa}{2} \hat{a}_c(t).
\]
(A8f)

From (A1) we get for the mechanical coordinate, using the definition (2.2c),

\[
\xi = -\frac{i\hbar n}{2m} \sqrt{\kappa \Delta \omega} \sum_{q=1}^\infty (\hat{a}_q \hat{b}_q - \hat{b}_q^\dagger \hat{a}_q) + \frac{F_i}{m}.
\]
(A9)

Using the definition (A8a) we obtain (2.2b).

In order to derive relation (2.2d), below we consider the transmission line as a model of the thermal bath.

### A.1. The transmission line model

Consider the model of the transmission line coupled with the LC circuit as a model of the thermal bath (see figure A1). The transmission line has wave resistance \( \rho = \sqrt{L/C} \), linear capacitance \( C \) and linear inductance \( L \). The LC circuit consists of inductance \( L_0 \) and capacitance \( C_0 \), its eigenfrequency is \( \omega_0 = 1/\sqrt{L_0 C_0} \) and its characteristic resistance is \( \rho_0 = \sqrt{L_0/C_0} \).

The length of the transmission line is \( L \) (below we put \( X \to \infty \)). We expand fields in line into field oscillators corresponding to waves propagating from left to right, \( U_i \) (input wave), and backwards \( U_r \) (reflected wave): \n
\[
U_i(x,t) = \sum_{n=0}^\infty \frac{\hbar \omega_n \rho}{2} \frac{\Delta \omega}{2\pi} (\beta_n(t) e^{i\kappa x} + \beta_n^\dagger(t) e^{-i\kappa x}),
\]
\( U_r(x,t) = \sum_{n=0}^\infty \frac{\hbar \omega_n \rho}{2} \frac{\Delta \omega}{2\pi} (\delta_n(t) e^{i\kappa x} + \delta_n^\dagger(t) e^{-i\kappa x}).
\]

Here \( \beta_n, \beta_n^\dagger \) are the annihilation and creation operators of the field oscillators, their commutators are \( [\beta_n, \beta_n^\dagger] = \delta_{nk} \). In order to fulfill translation boundary conditions \( U_i(x,t) = U_i(x + X, t) \), \( U_r(x,t) = U_r(x + X, t) \) we choose frequencies \( \omega_n = n \Delta \omega \) where the spectral range \( \Delta \omega = 2\pi v / \sqrt{LC} \) is the wave speed.

The Hamiltonian of the transmission line consists of two parts:

\[
H = \int_0^X \left( \frac{CU_i^2}{2} + \frac{LF_i^2}{2} \right) dx,
\]
(A10)

\[
= \sum_{n=0}^\infty \hbar \omega_n \left( \beta_n^\dagger \beta_n + \frac{1}{2} \right),
\]
(A11)

\[
H = \int_0^X \left( \frac{CU_r^2}{2} + \frac{LF_r^2}{2} \right) dx
\]
(A12)

\[
= \sum_{n=0}^\infty \hbar \omega_n \left( \delta_n^\dagger \delta_n + \frac{1}{2} \right)
\]
(A13)

Below we use slow varying amplitudes (operators) \( b_n, d_n \) defined as:

\[
\beta_n = b_n e^{-i\omega_n t}, \quad \beta_n^\dagger = b_n^\dagger e^{i\omega_n t},
\]
(A14)

\[
\delta_n = d_n e^{-i\omega_n t}, \quad \delta_n^\dagger = d_n^\dagger e^{i\omega_n t}.
\]
(A15)
In the limit $X \to \infty$ we get

$$U_\circ = \int_0^\infty \sqrt{\hbar \omega / 2} \left( b(\omega) e^{-i\omega t + ikx} + b^\dagger(\omega) e^{i\omega t - ikx} \right) d\omega / 2\pi,$$

$$[b(\omega), b^\dagger(\omega')] = 2\pi\delta(\omega - \omega')$$

$$U_- = \int_0^\infty \sqrt{\hbar \omega / 2} \left( d(\omega) e^{-i\omega t + ikx} + d^\dagger(\omega) e^{i\omega t + ikx} \right) d\omega / 2\pi.$$ (A16)

The charge $q$ of capacitance in the LC circuit and the current $i$ flowing in inductance are equal to

$$q = \frac{\hbar}{2\rho_0} (\alpha + \alpha')^2, \quad i = \omega_0 \frac{\hbar}{2\rho_0} \left( \alpha - \alpha'^2 \right).$$ (A17)

where $\alpha, \alpha'$ are the annihilation and creation operators of the LC circuit. Below we use slow amplitudes:

$$\alpha = a e^{-\omega_0 t}, \quad \alpha' = a^* e^{\omega_0 t}.$$ (A18)

We have to equate voltages and currents in the circuit and the line at $x = x_0$:

$$L_0 \dot{q} + \frac{q}{C_0} = U_{in} + U_{out},$$ (A19)

$$\rho \dot{i} = U_{in} - U_{out},$$ (A20)

where we denote $U_{in} = U_i(x_0, t)$, $U_{out} = U_o(x_0, t)$. Summing equations (A19) and (A20) we obtain

$$\dot{q} + \kappa \dot{q} + \omega_0^2 q = \frac{2U_{in}}{L_0}, \quad \kappa = \frac{\rho}{\omega_0} \rho_0.$$ (A21)

Substituting (A17) and (A18) into (A21) we obtain a set of equations for slowly varying amplitudes $a, a^*$:

$$\frac{\hbar}{2\rho_0} \left( \dot{a} e^{-\omega_0 t} + \dot{a}^* e^{\omega_0 t} \right) = 0,$$ (A22)

$$-i\omega_0 \frac{\hbar}{2\rho_0} \left( \dot{a} e^{-\omega_0 t} - \dot{a}^* e^{\omega_0 t} \right) = \frac{2U_{in}}{L_0} - \kappa q.$$ (A23)

One can solve this set and after omitting fast oscillating terms obtain

$$\dot{a} + \frac{\kappa}{2} a = \frac{e^{ikx_0}}{-i} \int_0^\infty \sqrt{\kappa} b(\Omega) e^{-i(\omega - \omega_0)t} d\Omega / 2\pi,$$ (A24)

where $\Omega = \omega - \omega_0$. In the frequency domain we get:

$$a(t) = \int_0^\infty a(\Omega) \frac{d\Omega}{2\pi} \Rightarrow a(\Omega) \left( \frac{\kappa}{2} - i\Omega \right) = \frac{e^{ikx_0}}{-i} \sqrt{\kappa} b(\Omega).$$ (A25)

Putting $e^{ikx_0} = -i$ one obtains equation (A6b). Using the boundary condition (A20), we get the input–output relation:

$$d(\Omega) = -b(\Omega) + \sqrt{\kappa} a(\Omega)$$ (A27)

which coincides with (2.2d), taking into account that in the transmission line model $d(\Omega) = a_{out}(\Omega)$ and $b(\Omega) = a_{in}(\Omega)$.

**Appendix B. Calculations of the output quadratures**

Here we present details of the calculation of output quadratures. In order to make the formulæ below less cumbersome we use the following notations:

$$\psi = \frac{\kappa_0}{2} - i\delta, \quad \psi^* = \frac{\kappa_0}{2} + i\delta,$$ (B1a)

$$\Psi = \frac{\kappa_0}{2} - i\delta - i\Omega, \quad \Psi^* = \frac{\kappa_0}{2} + i\delta + i\Omega,$$ (B1b)

$$\Psi_- = \frac{\kappa_0}{2} - i\delta + i\Omega, \quad \Psi_+ = \frac{\kappa_0}{2} + i\delta - i\Omega.$$ (B1c)

For the fluctuation part of the light pressure force $F_\beta$ in the frequency domain we get, using (2.12) and the notations (2.14):

$$F_\beta = i\hbar A_0 \eta \sqrt{\kappa_0} \left\{ a_{in} \psi^* + a_{out} \psi^* \right\}$$ (B2)

$$\eta = \frac{\kappa_0}{4} \left( \frac{1}{\psi^*} + \frac{1}{\psi} \right), \quad \eta = \frac{\kappa_0}{4} \left( \frac{1}{\psi^*} - \frac{1}{\psi} \right).$$ (B3)

Substituting this into (4.2) and then into (4.1) using (4.4) we obtain:

$$a_{out} = \frac{\psi^*}{\psi} a_{in} + \frac{\eta A_0 \hbar_0}{2}, \quad \psi^* \left( \frac{1}{\psi^*} - \frac{1}{\psi} \right)$$

$$\times \sqrt{\kappa_0} \eta \left( -a_{in} (g_- + j_-) + a_{out} (g_+ + j_+) \right).$$ (B5)

Further, we substitute it into the definitions (4.3) and after simple but tedious calculations we finally obtain the coefficients in (4.5):

$$E_{in} = \beta_\circ + E_{out}$$ (B7a)

$$E_{out} = \frac{2\Omega_0^2}{Q} (g_+ + j_+)(g_- + j_-),$$ (B7b)

$$E_{up} = -\beta_\circ + E_{up},$$ (B7c)

$$E_{up} = \frac{2\Omega_0^2}{Q} (g_+ + j_+)(g_- + j_-),$$ (B7d)

$$\Phi_\circ = \frac{4\Omega_0^2}{Q} \left( -g_+ + j_+ \right).$$ (B7e)

$$E_{pa} = \beta_\circ - E_{pa},$$ (B7f)

$$E_{pa} = \frac{2\Omega_0^2}{Q} (g_+ + j_+)(g_- + j_-),$$ (B7g)

$$E_{pp} = \beta_\circ + E_{pp},$$ (B7h)

$$E_{pp} = \frac{2\Omega_0^2}{Q} (g_+ + j_+)(g_- + j_-).$$ (B7i)
where
\[ \beta_s = \frac{1}{2} \left( \psi^\dagger \psi^s + \psi^s \psi^\dagger \right), \quad (B8a) \]
\[ \beta_n = \frac{1}{2i} \left( \psi^\dagger \psi^n - \psi^n \psi^\dagger \right), \quad (B8b) \]

and \( \Omega_0^2 \) is the normalized pump (3.2).

**References**

1. Aspelmeyer M, Kippenber T and Marquardt F 2014 Rev. Mod. Phys. 86 1391–452
2. Povinelli M L, Lončar M, Ibanescu M, Smythe E J, Johnson S G, Capasso F and Joannopoulos J D 2005 Opt. Lett. 30 3042–4
3. Maslov A V, Astratov V N and Bakunov M I 2013 Phys. Rev. A 87 053848
4. Vyatchanin S P and Matsko A B 1993 Sov. Phys. – JETP 77 218–21
5. Vyatchanin S P and Matsko A B 1996 Sov. Phys. – JETP 82 107
6. Matsko A B and Vyatchanin S P 1997 Appl. Phys. B 64 167–71
7. Kimble H J, Levin Y, Matsko A B, Thorne K S and Vyatchanin S P 2001 Phys. Rev. D 65 022002
8. Braginsky V B and Khalili F Y 1990 Phys. Lett. A 147 251–6
9. Braginsky V B, Gorodetsky M L, Khalili F Y and Thorne K S 2000 Phys. Rev. D 61 044002
10. Abbott B P et al 2018 Living Rev. Relativ. 21 3
11. Ayo S, Michimura Y, Somiya K, Ando M, Miyakawa O, Sekiguchi T, Tatsumi D and Yamamoto H 2013 Phys. Rev. D 88 043007
12. Dooley K, Akutsu T, Dwyer S and Puppo P 2015 J. Phys.: Conf. Ser. 610 012012
13. Affeld C, Danzmann K, Dooley K, Grote H, Hewitson M, Hild S, Hough J, Leong J, Luck H and Prijateli M 2014 Class. Quantum Grav. 32 224002
14. Aasi J et al (LIGO Scientific Collaboration) 2015 Class. Quantum Grav. 32 074001
15. Acernese F et al (Virgo Collaboration) 2015 Class. Quantum Grav. 32 024001
16. Abbott B P et al (LIGO Scientific Collaboration, Virgo Collaboration and KAGRA Collaboration) 2018 Living Rev. Relativ. 21 3
17. Abbott B P et al (LIGO Scientific Collaboration and Virgo Collaboration) 2016 Phys. Rev. Lett. 116 241103
18. Abbott B P et al (LIGO Scientific Collaboration and Virgo Collaboration) 2017 Phys. Rev. Lett. 119 161101
19. Abbott B P et al (LIGO Scientific Collaboration and Virgo Collaboration) 2017 Phys. Rev. Lett. 119 141101
20. Acernese F et al (Virgo Collaboration) 2018 Class. Quantum Grav. 35 205004
21. Wu M, Hryciw A C, Healey C, Lake D P, Jayakumar H, Freeman M R, Davis J P and Barclay P E 2014 Phys. Rev. X 4 021052
22. Forstner S, Prams S, Knittel J, van Ooijen E D, Swaim J D, Harris G I, Szorkovszky A, Bowen W P and Rubinsztein-Dunlop H 2012 Phys. Rev. Lett. 108 120801
23. Braginsky V B 1968 Sov. Phys. – JETP 26 831–4
24. Braginsky V B and Khalili F Y 1992 Quantum Measurement (Cambridge: Cambridge University Press)
25. Kippenberg T and Vahala K 2008 Science 321 1172–6
26. Dobrindt J M and Kippenberg T J 2010 Phys. Rev. Lett. 104 033901
27. Vyatchanin S and Zubova E 1995 Phys. Lett. A 201 269–74
28. Braginsky V B and Khalili F Y 1999 Phys. Lett. A 257 241
29. Khalili F Y 2001 Phys. Lett. A 288 251–6
30. Vyatchanin S P and Matsko A B 2016 Phys. Rev. A 93 063817
31. Elste F, Girvin S M and Clerk A A 2009 Phys. Rev. Lett. 102 207209
32. Li M, Pernice W H P and Tang H X 2009 Phys. Rev. Lett. 103 223901
33. Hryciw A, Wu M, Kanaliloo B and Barclay P 2015 Optica 2 491
34. Sawadsky A, Kaufer H, Nia R M, Tarabrin S P, Khalili F Y, Hammerner K and Schnabel R 2015 Phys. Rev. Lett. 114 043601
35. Kronwald A, Marquardt F and Clerk A A 2013 Phys. Rev. A 88 063833
36. Tan H, Li G and Meyestro P 2013 Phys. Rev. A 87 033829
37. Zhu J, Huang H and Li G 2014 J. Appl. Phys. 115 033102
38. Qu K and Agarwal G S 2015 Phys. Rev. A 91 063815
39. Gu W, Li G X and Yang Y P 2013 Phys. Rev. A 88 013835
40. Gu W and Li G 2013 Opt. Express 21 20423
41. Huang S and Agarwal G S 2010 Phys. Rev. A 81 053810
42. Huang S and Agarwal G S 2010 Phys. Rev. A 82 033811
43. Weiss T, Bruder C and Nunnemark A 2013 New J. Phys. 15 045017
44. Xuereb A, Schnabel R and Hammerner K 2011 Phys. Rev. Lett. 107 213604
45. Tarabrin S P, Kaufer H, Khalili F Y, Schnabel R and Hammerner K 2013 Phys. Rev. A 88 023809
46. Clerk A A, Devoret M H, Girvin S M, Marquardt F and Schoelkopf R J 2010 Rev. Mod. Phys. 82 1155
47. Marquardt F and Girvin S 2009 Physics 2 40
48. Braginsky V B and Minakova I I 1964 Vestn. Mosk. Univ. 36 69 (in Russian)
49. Routh E and Treatise A 1877 On the Stability of a Given State of Motion: Particularly Steady Motion (London: Macmillan)
50. Hurwitz A 1895 Math. Ann. 46 273–84
51. Rabinovich M and Trubetskov D 1984 Introduction into Theory of Oscillations and Waves (in Russian) (Moscow: Nauka)
52. Gopal M 2002 Control Systems: Principles and Design 2nd edn (New Delhi: Tata McGraw-Hill Education)
53. Crive D, Danz B, Lane B, Lorio M, Falcone J, Cole G and Corbitt T 2018 Opt. Lett. 43 2193
54. Arcieto O, Cohadon P-F, Briant T, Pinard M, Heidmann A, Mackowski J-M, Michel C, Pinard L, Francois O and Rousseau L 2006 Phys. Rev. Lett. 97 133601
55. Zwicki B M, Shank W E, Jayich A M, Yang C, Jayich A C B, Thompson J D and Harris J G E 2008 Appl. Phys. Lett. A 92 103125
56. Borkje K, Nunnemark A, Zwicki B M, Yang C, Harris J G E and Girvin S M 2010 Phys. Rev. A 82 013818

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