q-Symmetries in DNLS-AL chains and exact solutions of quantum dimers

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Abstract

Dynamical symmetries of Hamiltonians quantized models of discrete non-linear Schrödinger chain (DNLS) and of Ablowitz-Ladik chain (AL) are studied. It is shown that for n-sites the dynamical algebra of DNLS Hamilton operator is given by the su(n) algebra, while the respective symmetry for the AL case is the quantum algebra su_q(n). The q-deformation of the dynamical symmetry in the AL model is due to the non-canonical oscillator-like structure of the raising and lowering operators at each site.

Invariants of motions are found in terms of Casimir central elements of su(n) and su_q(n) algebra generators, for the DNLS and QAL cases respectively. Utilizing the representation theory of the symmetry algebras we specialize to the n = 2 quantum dimer case and formulate the eigenvalue problem of each dimer as a non-linear (q)-spin model. Analytic investigations of the ensuing three-term non-linear recurrence relations are carried out and the respective orthonormal and complete eigenvector bases are determined.

The quantum manifestation of the classical self-trapping in the QDNLS-dimer and its absence in the QAL-dimer, is analysed by studying the asymptotic attraction and repulsion respectively, of the energy levels versus the strength of non-linearity. Our treatment predicts for

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the QDNLS-dimer, a phase-transition like behaviour in the rate of change of the logarithm of eigenenergy differences, for values of the non-linearity parameter near the classical bifurcation point.
1 Introduction

The Discrete Nonlinear Schrödinger and the Ablowitz-Ladik chain models have generated intense interest during the last decade to physicists and mathematicians. Both systems consist of coupled sets of ordinary differential equations, the former has interesting physical motivations while the latter has attractive mathematical properties. The property of DNLS that makes it physically appealing is that of selftrapping [1, 2, 3], viz. the existence of broken symmetry localized states induced by nonlinearity. This property makes DNLS an interesting model for discrete solitons or polarons. On the other hand the AL equation [4], even though not physically as transparent as DNLS has the unique mathematical property of integrability, as a result many of its properties can be investigated analytically [6].

An issue that already has been addressed by various authors (see e.g. [3, 5, 6, 7]), is that of the quantum properties of these two equations. In particular the question of the quantum manifestation of selftrapping and in what precise way the quantum versions of DNLS and AL differ. This is the question we also address in the present article through the use of novel techniques motivated from the area of quantum groups [8]. In particular, we develop a systematic algebraic scheme for the Quantum DNLS (QDNLS) and Quantum AL (QAL) model that uses q-symmetries. This includes the finding of the dynamical symmetry algebra for each model and the determination of the set of independent constants of the motion for the respective Hamiltonians. It is achieved by using a canonical boson realization for the \( su(n) \) algebra, in the case of the \( n \)-site open QDNLS chain. The chain possesses then as invariants the set of \( n - 1 \) independent Casimir operators of the algebra, the eigenvectors of which will determine the linear space of the unitary time evolution of the model. These considerations are further extended to the case of the QAL chain model. In this case we first show that the quantum dynamical variable at each chain site is a particular case of the so called q-oscillator [9] (see [10] for related considerations). Then we invoke a q-oscillator realization of the q-deformed (quantum) su(2) algebra which lead us to determine the invariants of motion of the QAL Hamiltonian that are given in terms of the quantum Casimir elements.

In the second part of the work we turn our attention to the special case of two-site (dimer) DNLS and QAL models where the preceding general algebraic scheme is now applied. In terms of the representation theory of the respective dynamical algebras of the two dimers we formulate the eigenvalue-eigenvector problem and provide analytic solution in each case. Our aim is to use the obtained solutions for a cooperative study of the spectral features of the two dimers from the perspective of the phenomenon of selftrapping. In particular, we investigate the energy level structure, the precise nature of which determines the relative localization-delocalization transition of the quantum excitation for values of the non-linearity parameters near the bifurcation of the classical DNLS dimer. The plan of the paper is as follows: In the following section we discuss the quantum q-symmetries of QDNLS and
QAL, in section 3 we provide the corresponding dimer exact solutions and finally in section 4 we give the numerical results with a physical interpretation of the resulting plots along with conclusions and future prospects of our work.

2 Quantum symmetries

2.1 The QDNLS model

In this chapter we will start the investigation of the dynamical symmetries of the quantum DNLS (QDNLS) and the quantum AL (QAL) chain. The two models are quantized versions of their classical analogues. The quantization procedure is the usual naive method of substituting the classical complex amplitudes by canonical (and non-canonical as in the AL case) boson operators, following the normal ordering rule, and the Poisson brackets by commutators [11].

We consider first the Hamiltonian of the QDNLS $n$-site open chain,

$$H_{DNLS} = \sum_{i=1}^{n} \left\{ -\epsilon(a_{i+1} + a_{i-1})a_i^\dagger - \frac{\gamma}{2}(a_i^\dagger a_i)^2 \right\}$$

(1)

with the canonical commutation relations

$$[a_i, a_j^\dagger] = \delta_{ij},$$

$$[N_i, a_j] = -\delta_{ij}a_i,$$

$$[N_i, a_j^\dagger] = \delta_{ij}a_i^\dagger,$$

(2)

among the boson sites.

To study the algebraic symmetry of this Hamiltonian we consider the defining relations of the $su(n)$ algebra generators, in the Cartan-Chevalley basis $[12] \{e_i, f_i, h_i\}_{i=1}^{n-1}$, which read

$$[h_i, h_j] = 0$$

(3)

$$[h_i, e_j] = \frac{1}{2}\alpha_{ij}e_j,$$

$$[h_i, f_j] = \frac{1}{2}\alpha_{ij}f_j,$$

$$[e_i, f_i] = 2h_i\delta_{ij},$$

$$\sum_{r,s}(-1)^r \left[ \frac{1}{r} \right] e_i^r e_j^s e_i = 0, \quad i \neq j,$$

$$\sum_{r,s}(-1)^r \left[ \frac{1}{r} \right] f_i^r f_j^s f_i^s = 0, \quad i \neq j$$

(4)

where $r, s$ are non-negative integers constrained by $r+s = 1-\alpha_{ij}$. Notice that the last two equations are the Serre relations which involve cubic products.
of the \( n - 1 \) generators, in terms of these the remaining \( su(n) \) generators are defined, until the total number of the \( n^2 - 1 \) basis elements is completed. Also the Cartan matrix elements which occur as structure constants in eq.\((4)\), are defined as \( \alpha_{ij} = 2\delta_{ij} - \delta_{i,j+1} - \delta_{i,j-1} \). Finally in this algebra basis the raising and lowering generators are conjugate i.e. \( e_i^\dagger = f_i \) and the Cartan subalgebra generators are self-adjoint viz. \( h_i^\dagger = h_i \), \( i = 1, \ldots, n - 1 \).

Then we invoke the bosonic realization of the \( su(n) \) algebra given in terms of the set of \( n \)-bosons of eq.\((2)\),\([13, 12]\),

\[
  e_i = a_i^\dagger a_{i+1}, \\
  f_i = a_{i+1}^\dagger a_i, \\
  h_i = \frac{1}{2} (N_i - N_{i+1}).
\]

Employing the central element \( h = N_1 + N_2 + \ldots + N_n \) (total energy operator), we express the number operator at each site in the form

\[
  N_i = \frac{1}{2} \sum_{j=1}^{n-1} (\Omega^{-1})_{ij} h_i + (\Omega^{-1})_{in} h,
\]

where

\[
  \Omega = \begin{bmatrix}
  1 & -1 & 0 & \ldots & 0 \\
  0 & 1 & -1 & 0 & \ldots & 0 \\
  \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
  1 & 1 & \ldots & 1 & 1
\end{bmatrix}
\]

is an \( n \times n \) invertible matrix.

By virtue of the bosonization of the generators the QDNLS Hamiltonian can now be embedded in the \( su(n) \) algebra

\[
  H_{DNLS} = \sum_{i=1}^{n} \left\{ -\epsilon_i (e_i + f_i) - \frac{\gamma}{2} \left[ \frac{1}{2} \sum_{j=1}^{n-1} (\Omega^{-1})_{ij} h_j \right]^2 - \gamma h \sum_{j=1}^{n-1} (\Omega^{-1})_{ij} h_j - \frac{\gamma}{2} (\Omega^{-1})_{in} h^2 \right\}.
\]

This dynamical symmetry in turn implies that all central (Casimir) elements of the \( su(n) \) algebra commute with \( H_{DNLS} \), and therefore become constants of motion. The number of these independent invariants equals the rank of the \( su(n) \) which is \( n - 1 \). From the various forms available of the Casimir operators in terms of the \( su(n) \) generators\([13]\) we will use one that facilitates the comparison with the corresponding \( q \)-deformed Casimirs that will be discussed next for the case of QAL chain. To this end referring to eq.\((2)\), we introduce the generators

\[
  h_a \equiv \epsilon_a - \epsilon_{a+1}, \quad E_{ab} \equiv \begin{cases} E_{ac}E_{cb} - E_{cb}E_{ac} & a < c < b, \text{no summation} \\
  e_a & a + 1 = b,
\end{cases}
\]

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\[
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\]
and $E_{ab} = 0$, for $a = b, a > b$. Also $F_{ba} \equiv E_{ab}^\dagger$, $e_a^\dagger = e_a$, and the range of values for all indices is from 1 to $n$, while the index $c$ above is arbitrary provided that $a < c < b$. In terms of these generators the desired Casimir invariants are,

$$C_{2p} \equiv (M^p)_{aa},$$

(10)

where the matrix $M$ is defined by $M \equiv E_{ac}F_{cb}$, and recursively the higher powers are given be $(M^{p+1})_{ab} = (M^p)_{ac}M_{cb}$. An alternative set of Casimir operators is defined by

$$C'_{2p} \equiv (N^p)_{aa}$$

(11)

with $N_{ab} = F_{ac}E_{cb}$. The above expressions provide two sets of Casimir operators of any degree for the algebra $su(n)$, i.e. $[H_{DNLS}, C_{2p}] = [H_{DNLS}, C'_{2p}] = 0$. The quantum dynamics generated by the Hamiltonian $H_{DNLS}$ is constrained by the existence of these invariances. The state vector of the system evolves in the vector space spanned by the eigenvectors of the set of Casimir operators that correspond to a given set of eigenvalues. Both the eigenvalues and the set of their respective eigenvectors are labeled by a minimal set of indices the so called Casimir indices, the number of which equals the rank of the dynamical algebra. This is the standard scheme of dynamical symmetries for quantum Hamiltonian systems (see e.g. [13]). It can be utilized to determined the dynamics of the general DNLS chain and shortly it will be used in the DNLS dimer case.

As the spectrum generating algebra of the model, the $su(n)$ algebra, provides means for the solution of the eigenvalue problem. This is particularly so in the case of two sites to which we now turn. Renaming the generators $e_1 = J_+, f_1 = J_-, h_1 = J_0$, in terms of the more usual $su(2)$ spin algebra notation, the quantum dimer Hamiltonian after some rescaling and a shift by a constant term, reads (for the dynamics of the wavefunction zeros of this Hamiltonian see [14]),

$$H_{QD} = J_+ + J_- + \frac{\gamma}{2} J_0^2.$$  

(12)

This form of $H_{QD}$ is particularly suitable for the study of its eigenvalue problem which will be taken up in the next chapter.

### 2.2 The QAL model

We turn now to the AL discretization of the continues NLS equation [4]. This is a Hamiltonian model with Hamiltonian operator

$$H_{AL} = -\sum_{i=1}^n \left\{ b_i^\dagger (b_{i+1} + b_{i-1}) - 2 \frac{\ln(1 + \frac{\gamma}{2} b_i^\dagger b_i)}{\ln(1 + \frac{\gamma}{2})} \right\} .$$

(13)

The model employs a set of non-canonical oscillators [3] with commutation relations

$$[b_i, b_j^\dagger] = (1 + \frac{\gamma}{2} b_i^\dagger b_i) \delta_{ij} ,$$
\[ [N_i, b_j] = -b_i \delta_{ij}, \]
\[ [N_i, b_j^\dagger] = b_i^\dagger \delta_{ij} . \] (14)

To demonstrate the quantum group symmetry of the QAL Hamiltonian we need first to show the relation of the non-canonical AL-oscillator to the so called \( q \)-oscillator (quantum Heisenberg algebra) and subsequently to consider the \( q \)-bosonization of the quantum group \( su_q(n) \). We start with the abstract \( q \)-oscillator algebra generated by elements \( a, a^\dagger \), and \( N \) with commutation relations

\[ aa^\dagger - qa^\dagger a = q^{-N} , \] (15)

\[ [N, a^\dagger] = a^\dagger , \quad [N, a] = -a . \] (16)

The parameter \( q \) is the so called deformation parameter and in our case is taken to be a non-negative real number. The \( q \)-oscillator is so defined that in the so called classical limit when \( q \to 1 \), it reduces to the standard quantum mechanical oscillator. This \( q \)-deformed algebra possesses a nontrivial central element

\[ C = q^{-N}([N] - a^\dagger a) , \] (17)

where \( [N] = (q^N - q^{-N})/(q - q^{-1}) \), which implies that

\[ a^\dagger a = [N] - q^N C . \] (18)

If we introduce the generators

\[ b^\dagger = a^\dagger q^{-N/2} \quad \text{and} \quad b = q^{-N/2}a , \] (19)

we express the defining relations in terms of commutators i.e.

\[ [b, b^\dagger] = q^{-2N} , \quad [N, b^\dagger] = b^\dagger , \quad [N, b] = b . \] (20)

Then eq.(18) is equivalent to

\[ b^\dagger b = q^{-N+1}[N] - qC , \] (21)

which amounts to

\[ q^{-2N} = 1 - qC - (1 - q^{-2})b^\dagger b . \] (22)

To correctly identify the AL-oscillator we make the choice \( C = 0 \), \( q = \frac{1}{\sqrt{1 + \gamma^2}} \), which imply the relations

\[ [b, b^\dagger] = 1 + \frac{\gamma}{2}b^\dagger b \] (23)

and

\[ N = \frac{\ln(1 + \frac{\gamma}{2}b^\dagger b)}{\ln(1 + \frac{\gamma}{2})} , \] (24)
for the $q$-Heisenberg oscillator. From the representation theory of the abstract $q$-oscillator algebra [13], we know that it possesses a family of irreducible infinite dimensional representations (irrep), with various operator properties assigned to its generators. However the choice $C = 0$, singles out a representation for which the number operator $N$, is positive definite. This Fock-space matrix irrep of the algebra constructed in the Hilbert space $\mathcal{H}_F$, spanned by the basis vector $\{|n\rangle\}_{n=0}^\infty$, makes the $H_{AL}$ a proper energy operator and provides the following matrix realization for its elements:

\[ N|n\rangle = n|n\rangle, \quad (25) \]
\[ b|n\rangle = \sqrt{\{n\}}|n-1\rangle, \quad (26) \]
\[ b^\dagger|n\rangle = \sqrt{\{n+1\}}|n+1\rangle, \quad (27) \]

with $b|0\rangle = 0$ and $\{n\} = \frac{q^{n-1}}{q^{1-1}}$.

We proceed now showing the quantum group symmetry of the QAL-chain. First let us recall the defining relations of the quantum $su_q(n)$ algebra according to the Jimbo-Drin’feld scheme, in the Cartan-Chevalley basis $\{e_i, f_i, k_i \equiv q^{h_i}\}_{i=1}^{n-1}$, they are [8]

\[ k_i^{-1}k_i = 1, \]
\[ k_i^{-1}k_j = k_j^{-1}k_i, \]
\[ k_i^{-1}e_jk_i = q^{\frac{1}{2}\alpha_{ij}}e_j, \]
\[ k_i^{-1}f_jk_i = q^{-\frac{1}{2}\alpha_{ij}}f_j, \]
\[ [e_i,f_i] = [2k_i]\delta_{ij}, \quad (28) \]

(29)

(29)

together with the $q$-Serre relations

\[ \sum_{r,s} (-1)^r \left[ \begin{array}{c} 1 - \alpha_{ij} \\ r \\ q \end{array} \right] e_i^r e_j e_i^s = 0, \quad i \neq j, \]
\[ \sum_{r,s} (-1)^r \left[ \begin{array}{c} 1 - \alpha_{ij} \\ r \\ q \end{array} \right] f_i^r f_j f_i^s = 0, \quad i \neq j. \quad (30) \]

The last ones involve the so called $q$-binomial coefficient [16] defined in terms of $q$-factorial, $[m]! = [1][2]...[m]$ by

\[ \begin{bmatrix} m \\ n \end{bmatrix}_q = \frac{[m]!}{[n]![m-n]!}. \quad (31) \]

Let us mention that here we consider the $su_q(n)$, only at the algebra level, the rest of its Hopf algebra structure is not needed for our present purposes, but they remain worth studying from the physical point of view. By analogy with the bosonization of the $su(n)$ by a set of canonical oscillators, the $q$-bosonization of the quantum $su(n)$ is always possible by using the $q$-deformed
oscillator of eq.(16). Indeed for a set of $n$ such oscillators the bosonized
generators are \[ e_i = a_i^\dagger a_{i+1}, \]
\[ f_i = a_{i+1}^\dagger a_i, \]
\[ k_i = q^{1/2(N_i-N_{i+1})}. \] (32)
We also introduce the generators $C_i = q^{1/2(N_i+N_{i+1})}$, $i = 1, 2, ..., n-1$, which commute with any \{$e_i, f_i, k_i$\} with the same index. Using this commutativity property and eq.(19) we express the $su_q(n)$ generators in terms of the QAL-bosons i.e.
\[ e_i = b_i^\dagger b_{i+1} C_i q^{-1}, \]
\[ f_i = b_{i+1}^\dagger b_i C_i q^{-1}, \]
By means of these equations and the relations (24) we now write the model’s Hamiltonian as
\[ H_{AL} = -\sum_{i=1}^{n}\left\{ qC_i^{-1}(e_i + f_i) - 2N_i\right\}, \] (33)
Due to eq.(4), we can finally embed this Hamiltonian in the $su_q(n)$ algebra as
\[ H_{AL} = -\sum_{i=1}^{n}\left\{ qC_i^{-1}(e_i + f_i) - \sum_{j=1}^{n-1}(\Omega^{-1})_{ij}h_i - (\Omega^{-1})_{in}h\right\} \] (34)
where
\[ C_i^{-1} = q^{-\frac{1}{2}(N_i+N_{i+1})} = q^{-\frac{1}{2}\left\{ \frac{1}{2} \sum_{j=1}^{n-1}(\Omega^{-1})_{ij}h_i + (\Omega^{-1})_{in}h + \frac{1}{2} \sum_{j=1}^{n-1}(\Omega^{-1})_{i+1,j}h_{i+1} + (\Omega^{-1})_{i+1,n}h \right\}}. \] (35)
It is important at this point to emphasize that as our analysis shows that the quantum group symmetry of the QAL model stems from the non-canonical character of the quantum variables defined at each site, (c.f. eq.(14)). This non-canonical character of the degrees of freedom is in fact necessary in order to prove the Hamiltonian structure of the AL equations with respect to the Hamiltonian of eq.(3), \[9\]. We conclude therefore that the found quantum group symmetry is a genuine feature of the model \[18\].

Similarly to the preceding case of the DNLS $su(n)$ invariances, the $su_q(n)$ dynamical symmetry of the QAL chain model implies that all quantum Casimir elements of $su_q(n)$ commute with $H_{QAL}$. In this way they become constants of the motion generated by the QAL Hamiltonian. For the construction of $q$-invariants of $su_q(n)$ there are several approaches \[19, 20, 21\]. For our purpose a construction similar to that outline before for the DNLS case will be employed \[21\]. Let us introduce in terms of the $q$-deformed algebra generators the following elements:
\[ h_a \equiv (\epsilon_a - \epsilon_{a+1} + \epsilon_1 + \epsilon_2 + \cdots + \epsilon_n = 0), \]
\[ E_{ab} \equiv \begin{cases} E_{ac}E_{cb} - q^{-1}E_{cb}E_{ac} & a < c < b, \text{no summation} \\ E_a & a + 1 = b, \\ (q - q^{-1}) & a = b, \end{cases} \] (36)
and $E_{ab} = 0$ for $a > b$. Also $F_{ba} \equiv E_{ab}^\dagger$, $e^\dagger_a = e_a$, and the range of values for all indices is as before. Notice that the $q$-deformed generators $E_{ab}(q)$, are not invariant under $q \to q^{-1}$, so we can define another set of generators $\tilde{E}_{ab} = E_{ab}(q^{-1})$, $\tilde{F}_{ab} = F_{ab}(q^{-1})$. Then we introduce the elements

$$M_{ab} \equiv E_{ab}q^{e_a - 2a}F_{cb}q^{e_c + 2c};$$

which yields the $q$-Casimir elements

$$C_{2p} \equiv q^{2a}(M^p)_{aa}. \quad (37)$$

By the exchange $q \to q^{-1}$, we obtain a new set of invariants

$$\tilde{C}_{2p} \equiv C_{2p}(q^{-1}) = q^{-2a}(\tilde{M}^p)_{aa}, \quad (39)$$

where $\tilde{M}^{p+1}$ as before is obtained recursively from the operator-valued matrix

$$\tilde{M}_{ab} = M_{ab}(q^{-1}) = \tilde{E}_{ac}q^{-e_a + 2a}\tilde{F}_{ab}q^{e_c - 2c}. \quad (40)$$

Finally two Casimir generators invariants under the exchange $q \to q^{-1}$ may be constructed by combining the two previous ones in the form

$$C_{2p} = (C_{2p} + \tilde{C}_{2p})/(q + q^{-1})$$

$$C_{2p+1} = (C_{2p} - \tilde{C}_{2p})/(q - q^{-1}). \quad (41)$$

These last expressions of the invariant Casimir operators modified with appropriate coefficients reduce precisely in the $q \to 1$ limit to the Casimir operators of equal degree for the non-deformed $su(n)$ algebra (see [21], for details and some additional expressions for $q$-Casimir elements).

If we now confine ourselves to a two-site restriction of the QAL-chain then the Hamiltonian

$$H_{AL} = -qC^{-1}_1(a_1^\dagger a_2 + a_2^\dagger a_1) + 2(N_1 + N_2), \quad (42)$$

has a $su_q(2)$ dynamical symmetry. This symmetry will be utilized in the next section in order to solve the eigenvalue problem of the AL-dimer.

### 3 Exact solutions

#### 3.1 The DNLS dimer

We now turn to the analytic determination of the eigenvalues and eigenvectors of the quantum DNLS chain in the simplest case of two sites. As was already mentioned there is a number of previous investigations in the case of classical dimer, which predict the well known phenomenon of selftrapping [1]. But also for the quantum case there are numerical and perturbative efforts [3], and quasiclassical treatments [7] along with formal techniques based on
the inverse scattering method \[5\], aiming to study the quantum manifestations of selftrapping.

Our approach will be entirely analytic and will be based in the \(su(2)\) pseudo-spin expression given to \(H_{QD}\) Hamiltonian \((12)\). According to previously given general prescription of the group symmetry of the chain, the two-boson realization of the \(su(2)\) generators reads

\[
J_+ = a_1^\dagger a_2^\dagger, \quad J_- = a_1^\dagger a_2, \quad J_0 = \frac{1}{2}(N_1 - N_2).
\]

The generators obey the standard relations of the quantum angular momentum

\[
[J_0, J_\pm] = \pm J_\pm, \quad [J_+, J_-] = 2J_0.
\]

With the Casimir operator adapted from the preceding general formulae

\[
C = J_0(J_0 - 1) + J_+ J_- = J_0(J_0 + 1) + J_- J_+ ,
\]

the matrix form of the Hamiltonian is induced by the \((2j + 1)\)-dimensional matrix representation of the generators in the spin basis \(|jm\rangle\) where \(j = \frac{1}{2}, 1, \frac{3}{2}, ...\), reads:

\[
C|jm\rangle = j(j + 1)|jm\rangle, \quad J_0|jm\rangle = m|jm\rangle, \quad J_\pm|jm\rangle = \sqrt{(j \mp m)(j \pm m + 1)}|jm\pm\rangle.
\]

In the two boson labeling of the spin state vector

\[
|j\rangle = \frac{1}{2}(n_1 + n_2); \quad m = \frac{1}{2}(n_1 - n_2) \equiv |n_1, n_2\rangle, \quad n_{1,2} \geq 0,
\]

we see that the total boson excitation number \(n_1 + n_2\), determines the dimensionality of the Hamiltonian matrix, and also the total energy allocated to the dimer. Then the eigenvalue equation

\[
H_{QD}|\phi\rangle = \lambda|\phi\rangle
\]

for a Hamiltonian eigenvector which is expressed by

\[
|\phi\rangle = \sum_{n=-j}^j c_n|n\rangle,
\]

leads to the difference equation

\[
\gamma n^2 c_n + \epsilon \left( \sqrt{(j+n)(j-n+1)}c_{n-1} + \sqrt{(j-n)(j+n+1)}c_{n+1} \right) = \lambda c_n.
\]
with boundary conditions \( c_{-j-1} = c_{j+1} = 0 \). We introduce new variables \( \psi_n = c_n \gamma_n \) and impose upon \( \gamma_n \)'s the relation

\[
\gamma_n \sqrt{(j + n)(j - n + 1)} = \gamma_{n-1}
\]

with solution

\[
\gamma_{-j+n} = \sqrt{n! \prod_{i=0}^{n-1} (2j - i) \gamma_{-j}}, \quad n = 1, 2, \ldots, 2j.
\]

In this way we obtain a normalized form of the iteration

\[
\psi_{i-j+1} = (\lambda + \mu_i^{(1)}) \psi_{i-j} + \mu_i^{(2)} \psi_{i-j-1}, \quad i = 0, 1, \ldots, j,
\]

where

\[
\mu_i^{(1)} = \gamma (i - j)^2
\]

and

\[
\mu_i^{(2)} = -i(2j + 1 - i).
\]

(also we set \( \mu_i^{(0)} = 1 \) for later use). Then from the boundary conditions \( \psi_{-j-1} = \psi_{j+1} = 0 \) and the seed of the recurrence relation \( \psi_{-j} = 1 \), we deduce that the polynomial \( p(\lambda) \equiv \psi_{j+1}(\lambda) \), is the characteristic polynomial of the Hamiltonian matrix \( H_{QD} \). Let us write

\[
p(\lambda) = \sum_{i=0}^{2j+1} \delta_i^{(2j)} \lambda^2,
\]

then the coefficients of the characteristic polynomial are determined by induction. Indeed if we introduce the concept of the weight of a coefficient \( \delta_i^{(k)} \) defined as

\[
\text{wt}(\delta_i^{(k)}) \equiv w_i = \text{deg}p(\lambda) - i = 2j + 1 - i,
\]

then we shall obtain the following expression for the coefficients

\[
\delta_i^{(\nu)} = \sum_{i_1, i_2, \ldots, i_w = 0}^{2} \sum_{t_w = t_1 + \ldots + t_{i_w} - 1}^{\nu} \sum_{t_w - t_i = 1}^{t_w \ldots t_{i_w}} \ldots \sum_{t_{w-1} = t_{w-2} + \ldots + t_{w-1} - 1}^{t_{w-1}} \sum_{t_{w-1} = t_{w-2} + \ldots + t_{w-1} - 1}^{t_{w-1}} \ldots \mu_{t_{w_1}}^{(i_{w_1})} \ldots \mu_{t_{l_2}}^{(i_1)} \mu_{t_1}^{(i_2)},
\]

where the summation indices satisfy the constrains

\[
i_1 \neq 0, \quad i_{w_i} \neq 0, \quad i_1 + i_2 + \ldots + i_{w_i} = w_i,
\]

and if \( i_k = 2 \) then \( i_{k+1} = 0 \).

Next we determine the associated eigenvectors by specifying their expansion coefficients in the spin basis (c.f. eq.(49)). Since the \( n \)-th of these coefficients is identified with the \( n \)-th order iteration of the recurrence relation issued in eq.(53) we write

\[
\psi_{-j+n}(\lambda_a) = q(\lambda_a).
\]
The polynomial \( q(\lambda_a) \) for each of the eigenvalues \( \{\lambda_a\}_{a=0}^{2j} \), has \( \deg q(\lambda_a) = n \), then we can write

\[
q(\lambda_a) = \sum_{i=0}^{n} \delta_i^{(n-1)} \lambda_a^i. \tag{61}
\]

The finding of the eigenvector amounts then to the determination of the coefficients \( \delta_i^{(n-1)} \). This is done by use of eq.\((53)\) and the solution is again given by eq.\((58)\), with \( \delta_i^{(-1)} = 1 \). The variable \( w_i \), this time stands for the weight of the coefficients of the \( q(\lambda) \) polynomial and it is defined by

\[
\text{wt}(\delta_i^{(n-1)}) = w_i = n - i, \quad i = 0, 1, ..., n. \tag{62}
\]

Finally, we shall show that the found operator spectrum possesses proper mathematical properties, namely that the energy eigenvalues are real numbers with zero multiplicity (non-degenerate) and that the corresponding set of eigenvectors form an orthonormal and complete basis. To this end we first recall some generalities \[22\]. Let us assume that we have a set of single variable polynomials \( \{q_n(x)\}_{n=0}^{N-1} \), obeying the three term recurrence relation

\[
\alpha_n q_{n+1}(x) + \beta_n q_n(x) + \gamma_n(x) = x q_n(x), \tag{63}
\]

with \( q_{-1}(x) = 0 \), which have roots \( x_i \), \textit{i.e.} \( p_n(x_i) = 0 \), if we introduce the numbers \( d_n = \sqrt{\frac{\alpha_n}{\alpha_{n-1}}} d_{n-1} \), with \( d_0 \) arbitrary positive number and the variables \( a_n = \alpha_n d_{n+1} \), then the so-called Darboux-Christoffel (DF) formula given by

\[
\sum_{n=0}^{N-1} \frac{q_n(x) q_n(y)}{d_n^2} = \frac{a_{N-1}}{a_N} \frac{1}{d_{N-1}^2} \frac{q_N(x) q_{N-1}(y) - q_{N-1}(x) q_N(y)}{x - y}, \tag{64}
\]

is valid for the polynomials. For two roots \( x = x_i, y = x_j \) the DF formula becomes

\[
\sum_{n=0}^{N-1} \frac{q_n(x) q_n(y)}{d_n^2} = \delta_{ij} \mathcal{N}_i^2 \tag{65}
\]

with

\[
\mathcal{N}_i^2 = \sum_{n=0}^{N-1} \frac{q_n^2(x_i)}{d_n^2} = \frac{a_{N-1}}{a_N} \frac{1}{d_{N-1}^2} q_N(x_i) q_{N-1}(x_i). \tag{66}
\]

The non-degeneracy of the roots is now obtained from eq.\((65)\), which implies that for \( x_i = x_j \), we get \( q_N(x_i) \neq 0 \). Turning now to our case we can write for the \( H_{DNLS} \) eigenvectors the expression

\[
|\psi(\lambda_a)\rangle = \mathcal{N}_a \sum_{k=0}^{2j} \frac{\psi_{j+k}(\lambda_a)}{\epsilon_k} |j + k\rangle, \tag{67}
\]

where the normalization constant (which determines the value of \( \gamma_{-j} \))

\[
\frac{1}{\mathcal{N}_a} = \sum_{k=0}^{2j} \frac{\psi_{j+k}(\lambda_a)}{\epsilon_k^2}, \tag{68}
\]
involves the factor
\[ \epsilon_k = \delta_{o,k} + k! \sqrt{\binom{2j}{k}}, \]  
with \( a = 0,1,\ldots,2j \).

To verify the orthonormality of the eigenvectors we resort to the quoted DF formula and write
\[ \langle \psi(\lambda_a) | \psi(\lambda_b) \rangle = N_a N_b \sum_{k=0}^{2j} \frac{\psi_{-j+k}(\lambda_a)\psi_{-j+k}(\lambda_b)}{\epsilon_k^2} \delta_{ab} \].

We note that the orthonormality of the eigenvectors is equivalent to the orthonormality of the coefficients of the eigenvectors in the spin basis taken as polynomials with discrete variable, which is the corresponding eigenvalue. Also the completeness of the eigenvalue basis can be based on the DF formula; indeed it is straightforward to write:
\[ \sum_{a=0}^{2j} |\psi(\lambda_a)\rangle \langle \psi(\lambda_a)| = \sum_{k,l=0}^{2j} \sum_{a=0}^{2j} N_a^2 \frac{\psi_{-j+k}(\lambda_a)\psi_{-j+k}(\lambda_a)}{\epsilon_k \epsilon_l} | -j+k\rangle \langle -j+l| = 1. \]

Having constructed the eigenvector basis of the Hamiltonian, we can now investigate dynamical questions, by expanding a suitable initial state vector on this basis, here however we refrain from doing so and shortly we will turn to the study of the quantum manifestations of the classical selftrapping.

### 3.2 The QAL dimer.

In this section we put forward an analysis of the eigenvalue problem for the QAL-dimer based, as in the DNLS-dimer, on the symmetry algebra of the \( H_{QAL} \) Hamiltonian. The new feature in this case is that we deal with the quantum, or \( q \)-deformed algebra \( su_q(2) \). As the value of \( q \)-deformed parameter is a non-negative real number, the representation theory (which as we have shown in the preceding case is an indispensable element for the concise formulation of the eigenvalue problem) of \( su_q(2) \) is very similar to its \( q = 1 \) limiting case.

Starting with the Hamiltonian of eq.(42) and after appropriate shifting and scaling with constant factors we get
\[ H_{AL} = \sum_{i=1}^{2} a_i^\dagger (a_{i+1} + a_{i-1}). \]

This can be expressed in terms of the effective quantum \( su(2) \) generator (\( \text{viz.} \)
\[ H_{AL} = J_+^q + J_-^q. \]

(We note here that the \( H_{AL} \) Hamiltonian is exactly that of Azbel-Hofstadter model which describes Bloch electrons in magnetic field [23], however in that case the \( q \)-deformation parameter is determined by the value of the magnetic
flux and is root of unity; given that the latter is solvable by Bethe ansatz techniques the relation between the two models is worth studying.) These quantum generators bosonized in an angular momentum fashion become

\[ J_q^+ = a_1^\dagger a_2, \]
\[ J_q^- = a_1^\dagger a_2, \]
\[ J_q^0 = \frac{1}{2}(N_1 - N_2). \]  

(74)

They satisfy the quantum \( su_q(2) \) algebra commutation relations i.e. \((q \equiv e^s)\),

\[ [J_q^0, J_q^\pm] = \pm J_q^\pm \]
\[ [J_q^+, J_q^-] = [2J_q^0] = \frac{\sinh(2sJ_q^0)}{\sinh s} \]  

(75)

With the \( q \)-Casimir operator adapted from the preceding general formulae for \( q \)-deformed central elements,

\[ C_q = [J_q^0][J_q^0 - 1] + J_q^+J_q^- = [J_q^0][J_q^0 + 1] + J_q^-J_q^+, \]  

(76)

the matrix representation of the \( q \)-deformed generators

\[ C_q|jm> = [j][j + 1]|jm>, \]
\[ J_q^+|j,m> = \sqrt{[j + m][j + m + 1]}|j,m + 1>, \]
\[ J_q^-|j,m> = m|j,m>. \]  

(77)

is very similar to their \( q = 1 \) counterparts. We notice that the non-linearity of the model resides in the fact that the off-diagonal matrix elements of the Hamiltonian matrix are expressed in terms of \( q \)-numbers (c.f. eq.(77)). Simple inspection shows that in the limit \( \gamma \to 0, q \to 1, \) where for any \( q \)-number the limit \([x] \to x, \) is valid, the generator matrices \( J_q^\pm \) become the \( su(2) \) generators i.e. \( J_q^\pm \to J_\pm. \) In this limit the QAL-dimer becomes a linear angular momentum problem that is easily solved.

To address the eigenvalue problem we first note that a similar relation as in eq.(17) for \( su(2), \) holds between the state vectors of \( su_q(2) \) and their parametrization in terms of the excitation numbers of two \( q \)-bosons. If we now assume an expansion \( |\phi> = \sum_{n=-j}^j c_n|n>, \) for the eigenvector \( |\phi>, \) associated to the eigenvalues \( \lambda \) i.e. \( H_{QAL}|\phi> = \lambda|\phi>, \) then this last relation by means of eqs.(13),(77) yields

\[ \sqrt{[j - n + 1][j + n]}c_{n-1} + \sqrt{[j + n + 1][j - n]}c_{n+1} = \lambda c_n. \]  

(78)

Again we normalize this difference equation with a change of variables \( c_n = \frac{\psi_n}{\gamma_n}, \) where the \( \gamma_n \)'s satisfy the relation

\[ \gamma_n = \gamma_{n-1}\sqrt{[j + n][j - n + 1]}; \]  

(79)
and explicitly are given by
\[\gamma_{-j+n} = \sqrt{[n]! \prod_{i=0}^{n-1} \gamma_{-j}} \quad n = 1, 2, \ldots, 2j.\] (80)

Then the normalized iteration with vanishing boundaries i.e. \(\psi_{j+1} = \psi_{-j-1} = 0\) and seed \(\psi_{-j} = 1\) reads,
\[\psi_{n+1} = \lambda \psi_n - [j + n][j - n + 1] \psi_{n-1}.\] (81)

As was explained in the similar situation of the DNLS-dimer, the eigenvalues are the roots of the real polynomial \(\psi_{j+1}(\lambda) = 0\). By induction we find that one needs to distinguish between integer and half-integer values for the index \(j\). In other words we need to distinguish between even total number of excitation quanta and odd total number of excitation quanta that are available in the two \(q\)-boson sites of the model, (c.f. eq.(47)). Explicitly we find that for \(j\)-integer
\[\psi_{j+1}(\lambda) = \lambda q(\lambda^2) = 0\] (82)
where \(\deg q(\lambda) = j\) and \(\deg \psi_{j+1}(\lambda) = 2j + 1\). This implies that the energy spectrum contains the zero eigenvalue and that the remaining energy levels are arranged in doublets for each pair \(\pm \lambda\). If we now define \(\chi = \lambda^2\), and assume that
\[q(\chi) = \sum_{m=0}^{j} \beta_m^{(2J)} \chi^m,\] (83)
then we seek to determine the coefficients \(\beta_m^{(2J)}\).
Similarly for \(j\)-half-integer we obtain that
\[\psi_{j+1}(\lambda) = p(\lambda^2) = 0,\] (84)
where \(\deg p(\lambda) = \frac{2j+1}{2}\), and
\[p(x) = \sum_{m=0}^{\frac{j}{2}(2j+1)} \beta_m^{(2J)} \chi^m.\] (85)

Then by induction of the recurrence relation of eq.(81), we specify the \((j+1)\)-order of iteration \(\psi_{j+1}(\lambda)\), which provides the coefficients
\[\beta_m^{(\nu)} = \sum_{i_j = p_j \cdots p_1} \sum_{i_j = p_j \cdots p_1} \mu_{i_j \cdots i_1}.\] (86)

In this expression we have used the abbreviation
\[\mu_n = -[j + n][j - n + 1],\] (87)
and the \(p_r\)’s are indices determined by the iteration
\[p_r = p_{r-1} + 2 \quad r = 2, 3, \ldots, \quad p_1 = 1.\] (88)
Next we construct the set of \((2j+1)\)-eigenvectors of \(H_{QAL}\) by specifying their expansion coefficients \(\psi_{-j+n}, n = 0, 1, \ldots, 2j\), in the \(q\)-spin basis. We find that for \(n\)-even
\[
\psi_{-j+n}(\lambda) = u(\lambda^2),
\]
and
\[
u(\lambda) = \sum_{m=0}^{n/2} \beta_m^{(n-1)} \chi^m.
\]
Similarly for \(n\)-odd
\[
\psi_{-j+n}(\lambda) = w(\lambda^2),
\]
with
\[
w(\lambda) = \sum_{m=0}^{1/2(n-1)} \beta_m^{(n-1)} \chi^m.
\]
where the coefficients \(\delta_m^{(n-1)}\) are determined again by the relation (86).

Concerning normalization of the eigenvectors we note that since we have set \(\psi_{-j} = 1\), the remaining unspecified variable \(\gamma_{-j}\), is determined from the normalization condition of the eigenvectors as in the preceding case. The latter are given explicitly by
\[
|\psi(\lambda_a) > = \mathcal{N}_a \sum_{k=0}^{2j} \frac{\psi_{-j+k}(\lambda_a)}{\epsilon_k} | - j + k >,
\]
where the normalization constant
\[
\frac{1}{\mathcal{N}_a} = \sum_{k=0}^{2j} \frac{\psi_{-j+k}^2(\lambda_a)}{\epsilon_k^2},
\]
involves the factor
\[
\epsilon_k = \delta_{o,k} + [k]! \sqrt{\left[ \begin{array}{c} 2j \\ k \\
\end{array} \right]_q},
\]
for \(a = 0, 1, \ldots, 2j\).

4 Numerical Results and Discussion

We use now the previously derived exact results to study the spectral behavior of QDNLS and QAL models. These results are shown in Figure 1 for QAL and in Figure 2 for QDNLS respectively. In Figure 1a we present the spectrum of QAL as a function of nonlinearity for \(j = 4\). We note the distinct ”repulsion” of the levels: the larger the value of nonlinearity the greater the repulsion between the levels. This tendency for repulsion of the energy levels is not related to similar repulsive and avoiding crossing behavior one encounters in quantum non-integrable systems (see e.g. [24]), since our QAL-dimer is an integrable model. It should rather be attributed to the non-linear dependence of the eigenvalues on the \(q\)-deformation parameter, that is to say that it should be an effect of the quantum group symmetry itself.
In Figure 1b we plot the lowest QAL energy levels for a given nonlinearity value ($\gamma = 2$) but as a function of the number of quanta in the system. The classical AL is obtained in the limit of very large $j$-values. We note here the repulsive aspect of the levels as well.

In Figure 2a the spectrum of QDNLS is shown for $j = 3$ as a function of the DNLS nonlinearity parameter. We note a great difference from the corresponding spectrum of QAL; here pairs of adjacent levels are grouped together and they merge as nonlinearity increases. This property is evident for all pairs of levels and especially in the lowest ones. We note that the rate of convergence is $\gamma$-dependent. In order to investigate this property we plot in Figure 2b the energy difference of the lowest two pairs of levels as a function of the nonlinearity parameter in a log-log plot. The labels (1) and (2) signify the lowest and next to lowest pairs respectively. There is a drastic change in the convergence plot as the amount of nonlinearity exceeds a certain value. While at small $\gamma$ the energy difference shrinks very slowly, the rate changes very rapidly for large $\gamma$. The convergence is done seemingly in an algebraic fashion with energy dependent exponents that can be easily obtained.

The numerical results obtained for the energy splittings demonstrate the role of classical selftrapping in the quantum mechanical regime. In its absence, as in the AL equation, the system energy spectrum shows repulsion, while in its presence the opposite effect is manifested, viz. level clustering. This energy clustering in the quantum regime is a signature of long-lived excitations. Indeed, if an excitation is created on one site of the QDNLS dimer, while the latter is in a large nonlinearity regime, the tunneling time will be very large, leading thus to a very long lived and localized excitation. This excitation can be thought of as a quantum breather induced by the nonlinearity of the corresponding classical problem. Such a breather does not exist in the QAL case, as becomes evident from its spectrum. Both classical counterparts of the two problems studied in this work are integrable; it would be interesting to investigate in detail and compare cases with and without selftrapping but while the system is also non integrable. In such problems the simultaneous presence of classical localization and chaos will have interesting reprecautions in the quantum mechanical spectrum.

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Figure captions

Fig. 1a. The spectrum of QAL as a function of nonlinearity for $j = 4$, i.e. nine energy quanta.
Fig. 1b. The lowest QAL energy levels for a given nonlinearity value ($\gamma = 2$) as function of the number of quanta in the system.
Fig. 2a. The spectrum of QDNLS is shown for $j = 3$ as a function of the DNLS nonlinearity parameter.
Fig. 2b. The difference of the lowest two pairs of energy levels as a function of the nonlinearity parameter in a log-log plot. The labels (1) and (2) signify the lowest and next to lowest pairs respectively.
Fig. 1a
Fig. 2a

J=3

$E$ vs $\gamma$ for different values of $J$. The graph shows a clear trend of decreasing energy ($E$) with increasing $\gamma$. The axes are labeled with values ranging from $-50.0$ to $10.0$ for $E$, and from $0.0$ to $6.0$ for $\gamma$. The curves indicate a linear relationship between $E$ and $\gamma$. 
Fig. 2b

J = 3

Graph showing the relationship between \( \ln(\gamma) \) and \( \ln(dE) \) with two curves labeled (1) and (2).