THE LANGUAGE OF PRE-TOPOLOGY IN KNOWLEDGE SPACES

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Abstract We systematically study some basic properties of the theory of pre-topological spaces, such as, pre-base, subspace, axioms of separation, connectedness, etc. Pre-topology is also known as knowledge space in the theory of knowledge structures. We discuss the language of axioms of separation of pre-topology in the theory of knowledge spaces, the relation of Alexandroff spaces and quasi ordinal spaces, and the applications of the density of pre-topological spaces in primary items for knowledge spaces. In particular, we give a characterization of a skill multimap such that the delineate knowledge structure is a knowledge space, which gives an answer to a problem in [14] or [18] whenever each item with finitely many competencies; moreover, we give an algorithm to find the set of atom primary items for any finite knowledge spaces.

Keywords Knowledge space, knowledge structure, learning space, pre-topological space, skill multimap, quasi ordinal space, Alexandroff space, separation of axiom, primary item.

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1. Introduction and preliminaries

This paper is stimulated by a recent paper Danilov (2009), where Danilov discussed the knowledge spaces based on the topological point of view. Indeed, the notion of a knowledge space is a generalization of topological spaces [7]. Doignon and Falmagne (1999)
introduced the theory of knowledge spaces (KST) which is regarded as a mathematical framework for the assessment of knowledge and advices for further learning [8,14]. KST makes a dynamic evaluation process; of course, the accurate dynamic evaluation is based on individuals’ responses to items and the quasi order on domain $Q$ [8].

A field of knowledge is a non-empty set of items or questions, denoted by $Q$. A subset $H$ of $Q$ is called a knowledge state whether an individual is capable of solving it in ideal conditions. A collection $\mathcal{H}$ of knowledge state is called knowledge structure if $\{\emptyset, Q\} \subseteq \mathcal{H}$, denoted by $(Q, \mathcal{H})$. Sometimes we simply say that $\mathcal{H}$ is the knowledge structure if the domain can be omitted without ambiguity. There are two important types of knowledge structures, namely, knowledge spaces and learning spaces. If $H' \subseteq H$ implies $\bigcup H' \in \mathcal{H}$, then the knowledge structure $\mathcal{H}$ is called a knowledge space. Moreover, a special knowledge structure $\mathcal{H}$ is called learning space if $\mathcal{H}$ satisfies learning smoothness and learning consistency, see [6,14]. Moreover, each learning space must be finite.

Let $\mathcal{F} \subseteq 2^Q$, and let $\mathcal{F}_t = \{K \in \mathcal{F} : t \in Q\}$ for each $t \in Q$. For each $t \in Q$, put
\[
t^* = \{r \in Q | H_r = H_t\};
\]
the $t^*$ is called a notion. Therefore, it follows that
\[
t^* = \{r \in Q | H_r = H_t\}.
\]
Moreover, if $t^*$ is single item for each $t \in Q$, then $(Q, \mathcal{H})$ is said to be discriminative. In [18], a knowledge structure $(Q, \mathcal{H})$ is called bi-discriminative if $\mathcal{H}_t \nsubseteq \mathcal{H}_r$ and $\mathcal{H}_r \nsubseteq \mathcal{H}_t$ for any distinct $t, r \in Q$. For each $L \in \mathcal{H}$, put
\[
L^* = \{t^* : t \in L\},
\]
\[
\mathcal{H}^* = \{L^* : L \in \mathcal{H}\},
\]
and
\[
Q^* = \{t^* : t \in Q\}.
\]
Then the knowledge structure $(Q^*, \mathcal{H}^*)$ is discriminative which is produced from $(Q, \mathcal{H})$. We say that $(Q^*, \mathcal{H}^*)$ is the discriminative reduction of $(Q, \mathcal{H})$. A knowledge space is
called a *quasi ordinal space* if it is closed under intersection. A discriminative and quasi ordinal space is called an *ordinal space*. For a detailed description of KST, the reader may refer to Falmagne and Doignon [8–16].

The main motivation of the this paper is systemic to study the theory of pre-topological spaces, and to describe the language of the theory of pre-topology in knowledge spaces.

This paper is organized as follows. Section 1 presents some relevant background about KST. We systematically introduce the theory of pre-topological spaces in section 2. The applications of the theory of pre-topological spaces in knowledge spaces are studied in section 3. Section 4 summarizes the main results of this paper.

Denote the sets of real number, rational number, positive integers, the closed unit interval and all non-negative integers by \( \mathbb{R} \), \( \mathbb{Q} \), \( \mathbb{N} \), \( \mathbb{I} \) and \( \omega \), respectively. For any sets \( A \) and \( B \), we denote \((A \cup B) \setminus (A \cap B)\) by \( A \triangle B \). Readers may refer [14, 17] for terminology and notations not explicitly given here.

2. The theory of pre-topology

It is well known that Császár (2002) in [1] introduced the notions of generalized topological spaces. A *generalized topology* on a set \( Z \) is a subfamily \( \mathcal{T} \) of \( 2^Z \) such that \( \mathcal{T} \) is closed under arbitrary unions. Clearly, \( \emptyset \in \mathcal{T} \). Short introductions to the theory of generalized topology are contained in [1–4]. In this paper, we are interested in a special generalized topological spaces, that is, pre-topological spaces. Indeed, J. Li first discuss the pre-topology (that is, the subbase for the topology) with the applications in rough sets, see [19–21]. Then D. Liu in [22, 23] discuss some properties of pre-topology.

In this section, we are interesting pre-topology in two aspects. On one hand, we will unify the terms of the Császár’s and Li’s such that the theory of pre-topology is well applied in knowledge spaces. On the other hand, in order to give the integrity of the whole theoretical framework of the theory of pre-topology, we shall systematically list and
give some propositions and theorems without proofs except for some part of new results; the reader can find the similar proofs in [17] for some results.

**Definition 1.** [1, 7, 19] A pre-topology on a set $Z$ is a subfamily $\mathcal{T}$ of $2^Z$ such that $\bigcup \mathcal{T} = Z$ and $\bigcup \mathcal{T}' \in \mathcal{T}$ for any $\mathcal{T}' \subseteq \mathcal{T}$. Each element of $\mathcal{T}$ is called an open set of the pre-topology.

Clearly, each topological space is a pre-topological space. Now we provide some examples of pre-topological space such that they are not topological spaces.

**Example 1.** (1) Let $Z$ be a set with the cardinality $|Z| \geq 2$. Put
$$\mathcal{T} = \{\emptyset\} \cup \{U : U \subseteq Z, |U| \geq 2\}.$$ (2) Let $Z$ be a set with the cardinality $|Z| = \omega$. Put
$$\mathcal{T} = \{\emptyset\} \cup \{U : U \subseteq Z, |U| = \omega\}.$$ (3) Let $Z = (-\infty, 1]$ and put
$$\mathcal{T} = \{\emptyset\} \cup \{(-\infty, \frac{1}{n}) : n \in \mathbb{N}\} \cup \{[0, \frac{1}{n}] : n \in \mathbb{N}\} \cup \{(-\infty, 0]\}.$$ (4) Let $Z = \{x, y, s, t\}$. Put $\mathcal{T} = \{\emptyset, \{x, y\}, \{x, s\}, \{x, y, s\}, Z\}$.

Clearly, $(Z, \mathcal{T})$ is a pre-topological space in (1)-(4) respectively which is not a topological space.

**Definition 2.** Let $\mathcal{T}$ and $\mathcal{G}$ be two pre-topologies on a set $X$. We say that the pre-topology $\mathcal{T}$ is coarser than the pre-topology $\mathcal{G}$ when $\mathcal{T} \subseteq \mathcal{G}$; we say that $\mathcal{T}$ is strictly coarser than $\mathcal{G}$ when $\mathcal{T} \subseteq \mathcal{G}$ and $\mathcal{T} \neq \mathcal{G}$. We also say that $\mathcal{G}$ is finer or strictly finer than $\mathcal{T}$ respectively.

The following theorem shows that any relation $\mathcal{R}$ on $2^Z \setminus \{\emptyset\}$ for a nonempty set $Z$ specifies a pre-topology on $Z$. 
Theorem 1. Assume that $Z \neq \emptyset$, and assume that $\mathcal{R}$ is a relation on $2^Z \setminus \{\emptyset\}$. If the family $\tau$ of all subsets of $Z$ satisfies the following condition:

$$U \in \tau \Leftrightarrow (\forall (K, H) \in \mathcal{R} : K \cap U = \emptyset \Rightarrow H \cap U = \emptyset),$$

then $\tau$ is a pre-topology on the set $Z$.

Proof. Clearly, $\{\emptyset, Z\} \subseteq \tau$. Let $\mathcal{F} \subseteq \tau$. For each $(K, H) \in \mathcal{R}$, if $K \cap (\bigcup \mathcal{F}) = \emptyset$, it follows that $K \cap U = \emptyset$ for any $U \in \mathcal{F}$, hence $H \cap U = \emptyset$ for any $U \in \mathcal{F}$ by our assumption, that is, $H \cap (\bigcup \mathcal{F}) = \emptyset$. Therefore, $Z$ is a pre-topological space.

2.1. The pre-base and pre-continuous of pre-topologies.

The following definition of pre-base for pre-topological spaces, which has an important role in our discussion of pre-topological spaces.

Definition 3. If $\mathcal{B}$ is a family of non-empty subsets of the set $Z$, then $\mathcal{B}$ is called a pre-base on $Z$ if $\bigcup \mathcal{B} = Z$.

Remark 1. (1) Clearly, each pre-topological space has a pre-base.

(2) In Example 1, the families $\{W : W \subset Z, |W| = 2\}, \{W \subset Z : |W| = \omega\}, \{[0, \frac{1}{n}] : n \in \mathbb{N}\} \cup \{(-\infty, \frac{1}{n}] : n \in \mathbb{N}\}$ and $\{(x, y), \{x, s\}, Z\}$ are the pre-bases for the pre-topologies of (1), (2), (3) and (4) respectively.

The following three propositions are obvious.

Proposition 1. Let $\mathcal{B}$ be a pre-base on the set $Z$. Put

$$\tau = \left\{ \bigcup \mathcal{B}' : \mathcal{B}' \subseteq \mathcal{B} \right\}.$$

Then $\tau$ is a pre-topology on $Z$ and $\mathcal{B}$ is a pre-base for $\tau$ on $Z$.

Proposition 2. If $(Z, \mathcal{F})$ is a pre-topological space, then, for any $\mathcal{B} \subseteq \mathcal{F}$, the family $\mathcal{B}$ is a pre-base for $\mathcal{F}$ iff for every $W \in \mathcal{F}$ with $z \in W$ there is a $H \in \mathcal{B}$ such that $z \in H \subseteq W$. 
Proposition 3. Suppose that $\tau$ and $\delta$ are two pre-topologies on $Z$. If $\mathcal{B}$ and $\mathcal{P}$ are pre-bases of $\tau$ and $\delta$ on the set $Z$ respectively, then the following (a) and (b) are equivalent:

(a) $\tau$ is coarser than $\delta$, that is, $\tau \subseteq \delta$;

(b) If $B \in \mathcal{B}$ with $z \in B$, then there is $D \in \mathcal{D}$ with $z \in D \subseteq B$.

Definition 4. If $\mathcal{P}$ is a pre-base of a pre-topological space $Z$, then we say that $\mathcal{P}$ is an atom pre-base on the set $Z$ if for each $z \in Z$ and $z \in B \in \mathcal{P}$ there does not exist $P \in \mathcal{P} \setminus \{B\}$ such that $z \in P \subseteq B$.

Remark 2. (1) Any finite pre-topological space has an atom pre-base.

(2) Any Alexandroff space has an atom pre-base, see Section 3.

(3) There exists a pre-topological space which has no atom pre-base.

Indeed, let $Z = [0, 1]$, and put

$$\delta = \{\emptyset\} \cup \{[0, \frac{1}{n}] : n \in \mathbb{N}\}.$$

Then $(Z, \delta)$ is a pre-topological space which has no atom pre-base.

Definition 5. If $\mathcal{P}$ is a pre-base of a pre-topological space $Z$, then we say that $\mathcal{P}$ is a minimal pre-base on $Z$ if for any proper subfamily $\mathcal{B}$ of $\mathcal{P}$ is not a pre-base of $Z$.

Clearly, the following theorem holds.

Theorem 2. Each atom pre-base of a pre-topological space $Z$ is minimal. In particular, if $Z$ is finite, then each minimal pre-base of $Z$ is an atom pre-base.

Remark 3. The pre-topological space (3) in Remark 2 has a minimal pre-base $\{[0, \frac{1}{n}] : n \in \mathbb{N}\}$, but it has no atom pre-bases.

Proposition 4. If $\mathcal{B}$ is a minimal pre-base for a pre-topological space $Z$, then $\mathcal{B} \subseteq \mathcal{F}$ for each pre-base $\mathcal{F}$ of $Z$, which implies that a pre-topological space admits at most one minimal base.
Proof. Take an arbitrary pre-base $F$ of $Z$. Suppose that $B \in B \setminus F$. Hence there exists $F' \subseteq F$ with $B = \bigcup F'$. Then, because $B$ is a pre-base, it follows that $B$ is a union of a subfamily of $B \setminus \{B\}$, a contradiction. Therefore, $Z$ admits at most one minimal base. □

The following example shows that some pre-topological spaces have no minimal pre-base.

Example 2. Let $(\mathbb{R}, \mathcal{O})$ be the usual topology. Put

$$B_1 = \{ (a, b) : a, b \in \mathbb{Q} \}$$

and

$$B_2 = \{ (a, b) : a, b \in \mathbb{P} \}.$$  

Obviously, the families $B_1$ and $B_2$ are pre-bases of $(\mathbb{R}, \mathcal{O})$ respectively. If $(\mathbb{R}, \mathcal{O})$ has a minimal pre-base $B$, then it follows from Proposition 4 that $B \subseteq B_1 \cap B_2 = \emptyset$. Thus, $(\mathbb{R}, \mathcal{O})$ has no minimal pre-base.

Finally, we give some applications of pre-bases in pre-topology.

Definition 6. [21] Let $h : Y \to Z$ be a mapping between two pre-topological spaces $(Y, \tau)$ and $(Z, \upsilon)$. The mapping $h$ is pre-continuous from $Y$ to $Z$ if $h^{-1}(W) \in \tau$ for each $W \in \upsilon$.

Clearly, the following theorem holds.

Theorem 3. Let $G$, $H$ and $L$ be pre-topological spaces. Then

1. The identical mapping $i_L : L \to L$ is pre-continuous;

2. If both $h : G \to H$ and $\tau : H \to L$ are pre-continuous, then $\tau \circ h : G \to L$ is pre-continuous.

Definition 7. Let $h : Y \to Z$ be a mapping and $y \in Y$, where $(Y, \tau)$ and $(Z, \upsilon)$ are two pre-topological spaces. The mapping $h$ is pre-continuous at $z$ if $h^{-1}(W) \in \tau$ with $z \in h^{-1}(W)$ for any $W \in \upsilon$ with $h(z) \in W$. 
By Definition 7, the following two propositions hold.

**Proposition 5.** Assume that \( h : Y \to Z \) is a mapping between two pre-topological spaces \((Y, \tau)\) and \((Z, \upsilon)\). For any \( y \in Y \), \( h \) is pre-continuous at \( y \) iff for any \( U \in \upsilon \) with \( h(y) \in U \) there exists \( V \in \tau \) such that \( h(V) \subseteq U \) and \( y \in V \).

**Proposition 6.** If \( r : Y \to Z \) is a mapping between two pre-topological spaces \( Y \) and \( Z \), then \( r \) is pre-continuous iff \( r \) is pre-continuous at each point of \( Y \).

**Theorem 4.** If \( r : Y \to Z \) is a mapping between two pre-topological spaces \( Y \) and \( Z \), then the following are equivalent:

1. The mapping \( r \) is pre-continuous;
2. There exists a pre-base \( B \) in \( Z \) with \( r^{-1}(B) \) is open in \( Y \) for each \( B \in B \).

For each pre-topological space \( Z \) and \( z \in Z \), denote the set all open neighborhoods of \( z \) by \( \mathcal{U}_z \).

**Definition 8.** Let \( Z \) be a pre-topological space and \( z \in Z \). The family \( \mathcal{V}_z \subseteq \mathcal{U}_z \) is an open neighborhood pre-base at \( z \) if for each \( U \in \mathcal{U}_z \) there exists \( V \in \mathcal{V}_z \) with \( V \subseteq U \).

**Theorem 5.** Let \( h : Y \to Z \) be a mapping between two pre-topological spaces \((Y, \tau)\) and \((Z, \upsilon)\). For any \( y \in Y \), the following (1) \( \iff \) (2).

1. \( h \) is pre-continuous at \( y \in Y \).
2. There exists a pre-base \( \mathcal{B}_{h(y)} \subseteq \upsilon \) at \( h(y) \) such that \( h^{-1}(B) \in \tau \) with \( y \in h^{-1}(B) \) for each \( B \in \mathcal{B}_{h(y)} \).

**Definition 9.** Let \( h : Y \to Z \) be a bijection between two pre-topological spaces \( Y \) and \( Z \). If \( h \) and \( h^{-1} : Z \to Y \) are all pre-continuous, then we say that \( h \) is a pre-homeomorphic mapping. We also say that \( Y \) and \( Z \) are pre-homeomorphic.

**Theorem 6.** If \( G, H \) and \( Z \) be pre-topological spaces, then
(1) the identical mapping $i_Z : Z \to Z$ is a pre-homeomorphic mapping;

(2) if $h : G \to H$ is a pre-homeomorphic mapping, then $h^{-1} : H \to G$ is pre-homeomorphic;

(3) if both $h : G \to H$ and $r : H \to Z$ are pre-homeomorphic, then $r \circ h : G \to Z$ is pre-homeomorphic.

Note that the pre-homeomorphism $h : Y \to Z$ gives us a bijection correspondence between the collections of open sets of $Y$ and of $Z$. Then a property $\mathcal{P}$ of $Y$ is a pre-topological property if $Y$ has the property $\mathcal{P}$ then any pre-homeomorphism image of $Y$ has the the property $\mathcal{P}$.

Assume that $h : Y \to Z$ is a pre-continuous injective mapping between two pre-topological spaces $Y$ and $Z$. Then the mapping $h$ is a pre-topological imbedding of $Y$ in $Z$ if $h : Y \to h(Y)$ is a pre-homeomorphism.

2.2. Subsets of pre-topology.

In this subsection, we discuss subsets of a pre-topological space, and treat the notions of closed set, closure of a set, interior of a set, boundary of a set, and accumulation point. The reader can see the proofs of some results in [20].

Definition 10. A subset $D$ of a pre-topological space $Z$ is closed provided $Z \setminus D$ is open in $Z$.

The following proposition is easily verified.

Proposition 7. Let $(Z, \tau)$ be a pre-topological space, and put $C = \{F : Z \setminus F \in \tau\}$; then

(1) $\emptyset, Z \in C$;

(2) for any non-empty $C_0 \subseteq C$, $\bigcap_{C \in C_0} C \in C$.

Conversely, if a subfamily $C \subseteq 2^Z$ satisfies (1) and (2) above, then

$$\tau = \{U \subseteq Z : Z \setminus U \in C\}$$
is a pre-topology on $Z$.

Take an arbitrary subset $F$ of a pre-topological space $(Z, \tau)$; then it follows from Proposition 7 that

$$\bigcap\{C : F \subseteq C, Z \setminus C \in \tau\}$$

is closed in $Z$, which is called the closure of $F$ and denoted by $\overline{F}$. Clearly, $\overline{F}$ is the smallest closed set containing $F$, and a set $C$ is closed iff $C = \overline{C}$. The proof of the following proposition left to the reader.

**Proposition 8.** Let $H$ be a pre-topological space and $Y \subseteq H$. Then

1. $\emptyset = \emptyset$;
2. $Y \subseteq Y$;
3. $Y = Y$.

For a topological space $Z$, it is well known that for any subsets $R, T \subseteq Z$ we have $\overline{R \cup T} = \overline{R} \cup \overline{T}$. However, the equality of $\overline{R \cup T} = \overline{R} \cup \overline{T}$ does not hold for pre-topological spaces. Indeed, the following example clarifies this point.

**Example 3.** Let $Z = \{z_1, z_2, z_3, z_4\}$ and

$$\tau = \{\emptyset, \{z_1, z_2\}, \{z_1, z_4\}, \{z_1, z_3\}, \{z_1, z_2, z_3\}, \{z_1, z_3, z_4\}, \{z_1, z_2, z_4\}, Z\}.$$ 

Then $(Z, \tau)$ is a pre-topology on $Z$. Let $R = \{z_2, z_3\}$ and $T = \{z_3, z_4\}$. Clearly, $\overline{R} = R$, $\overline{T} = T$ and $\overline{R \cup T} = \overline{z_2, z_3, z_4} = Z$, thus $\overline{R \cup T} \neq \overline{R} \cup \overline{T}$.

**Proposition 9.** If $Y$ is a subset of a pre-topological space $(Z, \tau)$ and $z \in Z$, then $z \in \overline{Y}$ iff $U \cap Y \neq \emptyset$ for any $U \in \tau$ with $z \in U$.

**Corollary 1.** Let $U$ be open in a pre-topological space $Z$ and $A \subseteq Z$. If $U \cap A = \emptyset$, then $U \cap \overline{A} = \emptyset$. 
Definition 11. Let $Z$ be a pre-topological space $(Z, \tau)$, $C \subseteq Z$ and $z \in Z$. We say that $z$ is an accumulation point of $C$ if $U \cap (C \setminus \{z\}) \neq \emptyset$ for any $U \in \tau$ with $z \in U$. The derived set of $C$ is the set of all accumulation points of $C$ and denoted by $C^d$.

Proposition 10. If $B$ is a subset of a pre-topological space $Z$, then $\overline{B} = B \cup B^d$.

Corollary 2. If $G$ is a subset of a pre-topological space $Z$, then $G^d \subseteq G$ iff $G$ is closed in $Z$.

Some equivalent ways of formulating the concept of pre-continuity are provided as follows.

Theorem 7. Let $h : Y \to Z$ be a mapping between two pre-topological spaces $Y$ and $Z$. Then the following are equivalent:

(i) $h$ is pre-continuous;

(ii) $h(B) \subseteq h(B)$ for each $B \subseteq Y$;

(iii) $h^{-1}(H)$ is closed in $Y$ for any closed set $H$ in $Z$.

A closure operator $c$ on $Z$ is an operator that assigns to each subset $B$ of $Z$ a subset $B^c$ of $Z$ so that the following four statements hold.

(a) $\emptyset^c = \emptyset$.

(b) $B \subseteq B^c$ for each $B \subseteq Z$.

(c) $B^{cc} = B^c$ for each $B \subseteq Z$.

(d) If $B \subseteq D$, then $B^c \subseteq D^c$.

The following theorem shows that these four statements are actually characteristic of closure. The pre-topology defined below is the pre-topology associated with a closure operator.

Theorem 8. Let $c$ be a closure operator on $Z$. If

$$\mathcal{F} = \{A : A \subseteq Z, A^c = A\}$$
and
\[ \tau = \{ U : Z \setminus U \in \mathcal{F} \} , \]
to say that \( \tau \) is a pre-topology on \( Z \) and \( A^c \) is the closure of \( A \) for every \( A \subseteq Z \).

**Definition 12.** If \( B \) is a subset of a pre-topological space \((Z, \tau)\), then the set
\[ \bigcup \{ W \subseteq B : W \in \tau \} \]
is called the interior of \( B \) and is denoted by \( \text{int} B \) or \( B^{\circ} \).

Clearly, the following proposition holds.

**Proposition 11.** If \( G \) and \( H \) are subsets of a pre-topological space \((Z, \tau)\), then the following (1)-(3) hold.

1. \( G \subseteq H \Rightarrow G^{\circ} \subseteq H^{\circ} \).
2. \( G \in \tau \Leftrightarrow G = G^{\circ} \).
3. \( G \) is closed in \( Z \Leftrightarrow Z \setminus G = (Z \setminus G)^{\circ} \).

**Theorem 9.** If \( G \) is a subset of a pre-topological space \( H \), then

1. \( G^{\circ} = H \setminus \overline{H \setminus G} \);
2. \( \overline{G} = H \setminus (H \setminus G)^{\circ} \).

**Definition 13.** Let \( Z \) be a pre-topological space and \( B \subseteq Z \). The set \( \partial B = \overline{B} \cap Z \setminus B \) is said to be the boundary of \( B \).

Clearly, \( \partial B = \partial(Z \setminus B) \) is a closed set. From Proposition 9 it follows that \( z \in \partial B \) iff, for each open neighborhood \( O \) of \( z \), we have \( O \cap (Z \setminus B) \neq \emptyset \) and \( O \cap B \neq \emptyset \).

**Theorem 10.** If \( G \) is any subset of pre-topological space \((Z, \tau)\), then the following equalities hold.

1. \( \partial G = \overline{G} \setminus G^{\circ} \);
(2) \( G^\circ = G \setminus \partial G \).

Remark 4. (1) It follows from Theorem 10 that
(a) \( G \in \tau \iff \partial G = \overline{G} \setminus G \);
(b) \( Z \setminus G \in \tau \iff \partial G = G \setminus \overline{G} \);
(c) \( G \) is open and closed \( \iff \partial G = \emptyset \).

(2) For a topological space \( Z \), we have \( \partial (R \cup T) \subseteq \partial R \cup \partial T \). However, this equality does not hold for pre-topological spaces. Indeed, let \( Z \) be the pre-topological space in Example 3, and let \( R \) and \( T \) be the subsets of Example 3. Then \( \partial R = R \), \( \partial T = T \) and \( \partial (R \cup T) = Z \), thus \( \partial (R \cup T) \not\subseteq \partial R \cup \partial T \).

2.3. subspaces of pre-topological spaces.

Let \( Y \) be a subset of pre-topological space \((Z, \tau)\). Put
\[
\tau|_Y = \{ O \cap Z : O \in \tau \}.
\]
Then \((Y, \tau|_Y)\) is a pre-topology too. We say that \((Y, \tau|_Y)\) is the subspace of \((Z, \tau)\).

First, the following proposition and theorem hold.

Proposition 12. If \( Y \) is a subspace of pre-topological space \( Z \), then the family \( \mathcal{B}|_Y = \{ O \cap Z : O \in \mathcal{P} \} \) is a pre-base for \( Y \), where \( \mathcal{P} \) is a pre-base for \( Z \).

Theorem 11. Let \( H \subseteq Z \) and \( F \subseteq H \), where \((Z, \tau)\) is a pre-topological space. Then
(1) \( F \) is closed in \( H \) \( \iff \) there is a closed subset \( F_1 \) in \( Z \) so that \( F = F_1 \cap H \);
(2) \( \text{cl}_H F = H \cap \overline{F} \).

Corollary 3. If \( H \) is closed in the pre-topological space \( Z \) and \( A \subseteq H \), then \( A \) is closed in \( H \) iff \( A \) is closed in \( Z \).

Remark 5. (1) Let \( Z \) be the pre-topological space in Example 3. Then \( W = \{ z_1, z_2 \} \) and \( \{ z_1 \} \) are open in \( Z \) and \( W \) respectively. However, \( \{ z_1 \} \) is not open in \( Z \).
(2) It is well-known that ‘Pasting Lemma’ and ‘Local formulation of continuity’ play important roles in the study of the continuous theory of general topology. However, ‘Pasting Lemma’ and ‘Local formulation of continuity’ do not hold in the class of pre-topological spaces, see the following Example 4.

**Example 4.** Let \( Z = \{z_1, z_2, z_3, z_4\} \), and let
\[
\tau = \{\emptyset, \{z_1, z_2\}, \{z_1, z_4\}, \{z_2, z_3\}, \{z_2, z_4\}, \{z_3, z_4\}, \{z_1, z_2, z_4\}, \{z_1, z_2, z_3\}, \{z_1, z_3, z_4\}, \{z_1, z_2, z_1, z_4\}, \{z_1, z_2, z_3, z_4\}, Z\}
\]
and
\[
\delta = \{\emptyset, \{z_3\}, \{z_1, z_2\}, \{z_1, z_3\}, \{z_1, z_4\}, \{z_1, z_2, z_3\}, \{z_1, z_2, z_4\}, \{z_1, z_3, z_4\}, \{z_1, z_2, z_3, z_4\}, Z\}.
\]
Then \((Z, \tau)\) and \((Z, \delta)\) are two pre-topologies on \( Z \) respectively. Let \( C = \{z_1, z_2\} \) and \( D = \{z_3, z_4\} \). Then \( \tau|_C = \{\emptyset, \{z_1\}, \{z_2\}, C\} \) and \( \tau|_D = \{\emptyset, \{z_3\}, \{z_4\}, D\} \). Clearly, \( Z = C \cup D \) and \( C \) and \( D \) are clopen in \((Z, \tau)\). Let \( h : (Z, \tau) \rightarrow (Z, \delta) \) be the identical mapping from \( Z \) to itself. Then
\[
h|_C : (C, \tau|_C) \rightarrow (Z, \delta) \quad \text{and} \quad h|_D : (D, \tau|_D) \rightarrow (Z, \delta)
\]
are pre-continuous. However, \( h \) is not pre-continuous. Indeed, the set \( \{z_3\} \) is an open set in \((Z, \delta)\); however, the set \( h^{-1}(\{z_3\}) = \{z_3\} \) is not open in \((Z, \tau)\).

**Problem 1.** Assume that \( h : G \rightarrow H \) is a mapping between two pre-topological spaces \( G \) and \( H \) such that there exist closed subsets \( C \) and \( D \) in \( G \) satisfying the following conditions hold.

1. \( G = C \cup D \);
2. the restrict mappings \( h|_C \) and \( h|_D \) are pre-continuous.

Under what conditions must \( h \) be pre-continuous?

By Corollary 3, the following theorem gives a partial answer to Problem 1.
Theorem 12. Assume that $h : G \to H$ is a mapping between two pre-topological spaces $G$ and $H$ such that there exist closed subsets $C$ and $D$ in $G$ satisfying the following conditions hold.

1. $G = C \cup D$;
2. $h^{-1}(F) \subseteq C$ or $f^{-1}(F) \subseteq D$ for each closed subset $F$ of $H$;
3. $h|_C$ and $h|_D$ are pre-continuous.

Then $h$ is pre-continuous.

The following proposition is obvious.

Proposition 13. Let $Y$ and $Z$ be pre-topological spaces.

1. The inclusion mapping $j : B \to Z$ is pre-continuous, where $B$ is a subspace of $Z$.
2. If $r : Y \to Z$ is pre-continuous, then the restricted mapping $r|_H$ is pre-continuous for any subspace $H$ of $Y$.

Let $B$ be a subspace of a pre-topological space $(Z, \tau)$. For any $U \in \tau$, put $[U]_B = \{ O \in \tau : O \cap B = U \cap B \}$; then $[U]_B$ is the set of all elements in $\tau$ such that each element has the same trace on $B$; moreover, $W \cap B \subseteq \bigcap [U]_B$ for each $W \in [U]_B$.

Theorem 13. Let $B$ be a subspace of pre-topological space $(Z, \tau)$. For each $U \in \tau$, the set $\gamma = \{ W : W = T \setminus (\bigcap [U]_B) , T \in [U]_B \} \cup \{ \emptyset \}$ is a pre-topology on $\bigcup \gamma$.

Proof. Clearly, it suffices to prove that $\bigcup \eta \in \gamma$ for any $\eta \subseteq \gamma$. Take any $\eta \subseteq \gamma$. Then, for each $W \in \eta$, there is $V_W \in [U]_B$ with $W = V_W \setminus (\bigcap [U]_B)$. Put $\eta = \{ V_W : W \in \eta \}$. Then $\bigcup \eta = (\bigcup \eta) \setminus (\bigcap [U]_B)$ and $B \cap (\bigcup \eta) = V \cap B$ for any $V \in \eta$. Since $Z$ is a pre-topological space, it follows that $\bigcup \eta$ is open in $Z$, hence $\bigcup \eta \in [U]_B$, which implies that $\bigcup \eta = (\bigcup \eta) \setminus (\bigcap [U]_B) \in \gamma$. \hfill \square

We say that $(\bigcup \gamma, \gamma)$ is an $B$-child pre-topological space of $\tau$, or simply child of $\tau$. Clearly, $\gamma$ is coarser than the pre-topological subspace $B$ of $Z$. 
2.4. The product pre-topology.

For pre-topological spaces $Y$ and $Z$, there is a standard way to define a pre-topology on the cartesian product $Y \times Z$.

**Definition 14.** Let $(Y, \tau)$ and $(Z, \eta)$ be pre-topological spaces, and let $B = \{V \times W : V \in \tau, W \in \eta\}$. The *product pre-topology* on $Y \times Z$ is the pre-topology on the set $Y \times Z$ so that $B$ is a pre-basis on $Y \times Z$.

**Definition 15.** Let $\{Z_\beta\}_{\beta \in J}$ be a family of pre-topological spaces. Let $B = \{\prod_{\beta \in J} W_\beta : W_\beta \text{ is open in } Z_\beta, \beta \in J\}$. We say a pre-topology $\tau$ on the product space $\prod_{\beta \in J} Z_\beta$ is called the *box pre-topology* if $B$ is a pre-basis of $\tau$.

Suppose that $\{Z_\beta\}_{\beta \in J}$ is an indexed family of sets. For any subset $I \subseteq J$, let

$$
\pi_I : \prod_{\alpha \in J} X_\alpha \to \prod_{\beta \in I} Z_\beta
$$

be the mapping which assigns to every element of the product spaces its $\alpha$th coordinate for any $\alpha \in I$,

$$
\pi_I((z_\beta)_{\beta \in J}) = (z_\beta)_{\beta \in I}
$$

it is said to be the *projection mapping* associated with any index $\alpha \in I$.

**Definition 16.** Let $\{(X_\alpha, \tau_\alpha)\}_{\alpha \in J}$ be a family of pre-topological spaces. For any finite subset $I \subseteq J$, let $S_I$ denote the collection

$$
S_I = \{\pi_I^{-1}(\prod_{\alpha \in F} U_\alpha) : U_\alpha \in \tau_\alpha, \alpha \in I\},
$$

and let

$$
S = \bigcup \{S_I : I \text{ is any finite subset of } J\}.
$$

The pre-topology is said to be the *product pre-topology* if it is generated by the collection $S$. In this pre-topology $\prod_{\alpha \in J} X_\alpha$ is called a *product pre-topological space*.

The following three theorems are important.
Theorem 14. Let $B_\beta$ be a subspace of $Z_\beta$ for each $\beta \in J$. Assume that both products $\prod B_\beta$ and $\prod Z_\beta$ are endowed with the box pre-topology or both products are endowed with the product pre-topology, then $\prod B_\beta$ is a subspace of $\prod Z_\beta$.

Theorem 15. Assume that $\{X_\alpha\}$ is a family of pre-topological spaces, and assume that $A_\alpha \subseteq X_\alpha$ for each $\alpha$. If $\prod X_\alpha$ is endowed with either the box or the product pre-topology, then
\[ \prod A_\alpha = \prod A_\alpha. \]

Theorem 16. Assume that $h : B \to \prod_{\beta \in J} Z_\beta$ is a mapping which is defined as follows $h(b) = (h_\beta(b))_{\beta \in J}$, where $b \in B$ and $h_\beta : B \to Z_\beta$ for every $\beta \in J$. If $\prod_{\beta \in J} Z_\beta$ has the product pre-topology, then the mapping $h$ is pre-continuous iff each $h_\beta$ is pre-continuous.

2.5. The quotient pre-topology.

In this subsection, we consider the pre-quotient pre-topology, which is a new way of generating a pre-topology on a set.

Definition 17. Let $p : Y \to Z$ be a surjective mapping between two pre-topological spaces $Y$ and $Z$. The mapping $p$ is called a pre-quotient mapping provided $W$ is open in $Z$ iff $p^{-1}(W)$ is open in $Y$.

Definition 18. Let $r : Y \to Z$ be a mapping between two pre-topological spaces $(Y, \tau)$ and $(Z, \eta)$. If $r(W) \in \eta$ for each $W \in \tau$, then the mapping $r$ is called a pre-open mapping. If the set $r(C)$ is always closed in $Z$ for each closed subset $C$ of $Y$, then the mapping $r$ is called a pre-closed mapping.

Clearly, each pre-quotient mapping is a pre-continuous mapping but not vice-versa. Moreover, since $r^{-1}(Z \setminus B) = Y \setminus r^{-1}(B)$, it follows that $h$ is pre-quotient mapping provided $r^{-1}(C)$ is closed in $Y$ iff $C$ is closed in $Z$. Moreover, if $r : Y \to Z$ is a surjective mapping and $r^{-1}(C)$ is closed in $Y$ for each closed subset $C$ of $Y$, then $r$ is a pre-closed mapping.
pre-continuous mapping which is either pre-closed or pre-open, then from the definition it
follows that $r$ is a pre-quotient mapping. From Example 5, it is easily checked that there
exists a pre-quotient mapping that is neither pre-open nor pre-closed.

Example 5. Let $\mathbb{R}$ endowed with the pre-topology $\tau$ which has a pre-base $\{(x, y) : |y - x| = 1\}$. Suppose that $\pi_1 : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is the projection on the first coordinate. Let
\[ G = \{(x, y) : y = 0\} \cup \{(x, y) : x \geq 0\}, \]
and let $p : G \rightarrow \mathbb{R}$ be obtained by restricting $\pi_1$. Then it is easy to see that $p$ is a
pre-quotient mapping which is neither pre-closed nor pre-open.

Now we can use the concept of pre-quotient mapping to construct a pre-topology on a
set.

Definition 19. Let $h : Y \rightarrow Z$ be a surjective mapping, where $Z$ is a set and $Y$ is a
pre-topological space. Then the following pre-topology
\[ \mathcal{F} = \{O \subseteq Z : h^{-1}(O) \text{ is open in } Y\} \]
on $Z$ is said to be the pre-quotient pre-topology induced by $h$.

Theorem 17. Let $h : Y \rightarrow Z$ be a surjective mapping, where $Z$ is a set and $(Y, \tau)$ is a
pre-topological space. Then the pre-quotient pre-topology on $Z$ is the finest pre-topology on
$Z$ so that $h$ is pre-continuous.

Proof. Let $\delta = \{W \subseteq Z : h^{-1}(W) \text{ is pre-open in } Y\}$ be the pre-quotient pre-topology
on $Z$. For each $W \in \delta$, since $h^{-1}(W) \in \tau$, we conclude that $h : (Y, \tau) \rightarrow (Z, \delta)$ is pre-
continuous. Assume that there is a pre-topology $\delta'$ on $Z$ so that $h : (Y, \tau) \rightarrow (Z, \delta')$ is pre-continuous. Since $h$ is pre-continuous, it follows that for each $O \in \delta'$ we have
$h^{-1}(O) \in \tau$, hence $\delta' \subseteq \delta$. Therefore, $\delta$ is the finest pre-topology on $Z$ which makes $h$
pre-continuous.

Clearly, the following proposition holds.
**Proposition 14.** Let \( h : G \to H \) and \( g : H \to Z \) be pre-quotient mapping, where \( G \), \( H \) and \( Z \) are pre-topological spaces. Then \( g \circ h : G \to Z \) is a pre-quotient mapping too.

From Theorem 3, Proposition 14 and the definition of pre-quotient mapping, it is easily checked the proofs of the following two theorems.

**Theorem 18.** Suppose that \( X_1 \), \( X_2 \) and \( X_3 \) are pre-topological spaces, and suppose that \( h : X_1 \to X_2 \) is pre-quotient and \( g : X_2 \to X_3 \) is a mapping. We conclude that the following statements hold:

1. \( g \circ h : X_1 \to X_3 \) is pre-continuous \( \Leftrightarrow \) \( g \) is pre-continuous.
2. \( g \circ h : X_1 \to X_3 \) is pre-quotient \( \Leftrightarrow \) \( g \) is pre-quotient.

**Theorem 19.** Let \( Z_1 \) and \( Z_2 \) be pre-topological spaces, and let \( h : Z_1 \to Z_2 \) be pre-continuous and bijective. Then the following (1)-(4) are equivalent:

1. \( h \) is pre-homeomorphic;
2. \( h \) is pre-closed;
3. \( h \) is pre-open;
4. \( h \) is pre-quotient.

**Definition 20.** Let \( E \) be an equivalence relation on the set \( Z \), and let \( \tau \) be a pre-topology on \( Z \). Let \( Z/E \) denote the set of all equivalence classes of \( E \) and let \( q \) denote the mapping of \( Z \) to \( Z/E \) assigning to the point \( z \) the equivalence class \( \equiv \) \( z \) \( \in \) \( Z/E \). Then the set \( Z/E \) endowed with the pre-quotient pre-topology is called pre-quotient space.

Assume \( (Z, \tau) \) is a pre-topological space. It can define an equivalence relation \( R \) on the set \( Z \) by \( x \sim y \Leftrightarrow \tau_x = \tau_y \). Put \( Z^* = \{ [z] : z \in Z \} \), and let \( W^* = \{ [z] : z \in W \} \) for each \( W \in \tau \). Put \( \tau^* = \{ W^* : W \in \tau \} \). Then \( \tau^* \) is the pre-quotient pre-topology induced from \( \tau \) on \( Z^* \). Moreover, \( (Z^*, \tau^*) \) is \( T_0 \) (see Definition 21 below). The pre-topological space \( (Z^*, \tau^*) \) is called the \( T_0 \) reduction of \( (Z, \tau) \).
2.6. Axioms of separation.

In this subsection, we consider the axioms of separation in the class of pre-topological spaces, which give the ways of separating points and closed sets in the study of pre-topological spaces.

Definition 21. A pre-topological space \((Z, \tau)\) is called a \(T_0\)-space if for any \(y, z \in Z\) with \(y \neq z\) there exists \(W \in \tau\) such that \(W \cap \{y, z\}\) is exact one-point set.

Definition 22. A pre-topological space \((Z, \tau)\) is called a \(T_1\)-space if for any \(y, z \in Z\) with \(y \neq z\) there are \(V, W \in \tau\) so that \(V \cap \{y, z\} = \{y\}\) and \(W \cap \{y, z\} = \{z\}\).

Definition 23. A pre-topological space \((Z, \tau)\) is called a \(T_2\)-space, or a Hausdorff space, if for any \(y, z \in Z\) with \(y \neq z\) there are \(V, W \in \tau\) so that \(y \in V\), \(z \in W\) and \(V \cap W = \emptyset\).

Remark 6. (1) Suppose that \(Z = \{z_1, z_2, z_3, z_4, z_5, z_6\}\) equipped with the pre-topology as follows:
\[
\tau = \{\emptyset, \{z_4\}, \{z_5, z_6\}, \{z_1, z_3\}, \{z_1, z_3, z_4\}, \{z_4, z_5, z_6\}, \{z_1, z_2, z_3\}, \{z_1, z_2, z_3, z_4\},
\{z_1, z_3, z_5, z_6\}, \{z_1, z_3, z_4, z_5, z_6\}, Z\}.
\]
Hence \((Z, \tau)\) is not a \(T_0\) pre-topological space.

(2) Example 3 is a \(T_0\) pre-topological space.

(3) Let \(Z\) be an infinite set equipped with the following pre-topology
\[
\tau = \{\emptyset\} \cup \{U : |Z \setminus U| \leq 2\}.
\]
Then the pre-topological space \((Z, \tau)\) is \(T_1\).

(4) In Example 4, the pre-topological space \((Z, \tau)\) is \(T_2\).

(5) Each \(T_1\) pre-topological space is \(T_0\), and each \(T_2\) pre-topological space is \(T_1\); but all the inverses are not true. Indeed, Example 3 is a \(T_0\) pre-topological space that is not \(T_1\); \((X, \tau)\) in (3) above is a \(T_1\) pre-topological space that is not \(T_2\).
Each set of cardinal numbers being well-ordered by $\prec$. Let $Z$ be a pre-topological space. Then the infimum of the set
$$\{|B| : B \text{ is a pre-base for } Z \}$$
is said to be the weight of $Z$ and is denoted by $w(Z)$.

**Theorem 20.** For each $T_0$ pre-topological space $(Z, \tau)$ we have $|Z| \leq 2^{w(Z)}$, thus $|Z| \leq |\tau|$.

**Proof.** Let $B$ be a pre-base for $Z$ such that $|B| = w(Z)$. For every $z \in Z$, put $B_z = \{W \in B : z \in W\}$. Because $Z$ is a $T_0$ pre-topological space, it follows that $B_x \neq B_y$ for any $x \neq y$. Because the number of all distinct families $B_z$ is not larger than $2^{|B|}$, we conclude that $|Z| \leq 2^{w(Z)}$. □

**Definition 24.** The locally inner closed points of an open subset $W$ of a $T_0$ pre-topological space $(Z, \tau)$, is the subset of points
$$W^I = \{z \in W : W \setminus \{z\} \in \tau\}.$$

The locally outer closed points of an open subset $W$ of a $T_0$ pre-topological space $(Z, \tau)$, is the subset of points
$$W^O = \{z \in Z \setminus W : W \cup \{z\} \in \tau\}.$$

The locally closed points of an open subset $W$ of a $T_0$ pre-topological space is the set
$$W^{LC} = W^I \cup W^O.$$

**Proposition 15.** Let $Z$ be a $T_0$ pre-topological space, and let $V$ and $W$ be open in $Z$ such that $(V \Delta W) \subseteq V^{LC}$ or $(V \Delta W) \subseteq W^{LC}$. Then $V^I = W^I$ and $V^O = W^O$ iff $V = W$.

**Proof.** Clearly, we only need to prove the necessity. Suppose that $V^I = W^I$ and $V^O = W^O$. Now let $V \neq W$; hence $\emptyset \neq V \Delta W \subseteq V^{LC} = W^{LC}$. Take any $x \in V \Delta W$. Without loss of generality, we assume that $x \in V$ and $x \not\in W$. Clearly, $x \not\in V^O$ because $x \in V$, and $x \not\in V^I$ since $x \not\in W^I$ and $V^I = W^I$, hence $x \not\in V^{LC}$, a contradiction. □
Definition 25. A subset $D$ of a pre-topological space $Z$ is called dense in $Z$ provided $D = Z$.

In a $T_2$ topological space $Z$ with a dense subset $A$, we have $|X| \leq 2^{2^{|A|}}$ and $W \cap A = W$ for each open set $W$ in $Z$. However, from the following example, we conclude that in the class of $T_2$ pre-topological spaces, these situations are different.

Example 6. There exists a $T_2$ pre-topological space $(Z, \tau)$ with a dense subset $C$ such that the following are satisfied.

1. $W \cap C \neq W$ for some $W \in \tau$;
2. $|Z| > 2^{2^{|C|}}$.

Proof. Let $Z$ be a set with $|Z| > 2^8$, and let
$$\mathcal{B} = \{\{x, y\} : x \in \{a, b, c\}, y \in Z, x \neq y\},$$
where $a, b, c$ are three distinct points of $Z$. The pre-topology $\tau$ is endowed on $Z$ such that $\mathcal{B}$ is a pre-base of $\tau$. Obviously, $(Z, \tau)$ is a $T_2$ pre-topological space, and the set $C = \{a, b, c\}$ is a dense subset of $(Z, \tau)$.

1. Take the open set $W = \{c, d\}$. Then we have $\overline{\{c, d\} \cap C} = \{c\} = \{c\}$ and $\overline{\{c, d\}} = \{c, d\}$, which shows $\overline{W \cap C} \neq W$.

2. Since $|Z| > 2^8$ and $2^{2^{|C|}} = 2^8$, it follows that $|Z| > 2^{2^{|C|}}$. \qed

Let $Z$ be a pre-topological space. For each $z \in Z$, the infimum of the set
$$\{|B(z)| : B(z) \text{ is a pre-base at } z\}$$
is called the character of a point $z$ in $Z$, which is denoted by $\chi(z, Z)$. The supremum of the set $\{\chi(z, Z) : z \in Z\}$ is called the character of a pre-topological space $(Z, \tau)$, which is denoted by $\chi(Z)$.

Problem 2. Let $H$ be a $T_2$ pre-topological space. Does the inequality $|H| \leq d(H)^{\chi(H)}$ hold?
Problem 3. For any two pre-continuous mappings $h$, $g$ of a pre-topological space $Y$ into a $T_2$ pre-topological space $Z$, is the set $\{y \in Y : h(y) = g(y)\}$ closed in $Z$?

The answer to Problem 3 is negative. Indeed, let $(Z, \tau)$ be the pre-topological space of Example 4, where $Z = \{z_1, z_2, z_3, z_4\}$. Clearly $(Z, \tau)$ is $T_2$. Moreover, the mappings $h = id_X : Z \to Z$ and $g : Z \to Z$ defined by $g(z_1) = z_1, g(z_2) = z_2, g(z_3) = z_3, g(z_4) = z_1$ are pre-continuous. However, the set $\{z \in Z : h(z) = g(z)\} = \{z_1, z_2, z_3\}$ is not closed in $Z$ since $\{z_4\}$ is not open in $Z$. Moreover, the following proposition gives an another answer to Problem 3.

Proposition 16. If $h$, $g$ are two pre-continuous mappings of a topological space $Y$ into a $T_2$ pre-topological space $Z$, then $\{t \in Y : h(t) = g(t)\}$ is closed in $Z$.

Proof. It need to prove that $G = \{t \in Y : h(t) \neq g(t)\}$ is open in $Y$. Indeed, since $h(t) \neq g(t)$ for each $t \in G$, it follows that there exist open sets $V$ and $W$ in $Z$ such that $h(t) \in V, g(t) \in W$ and $V \cap W = \emptyset$. Hence $y \in h^{-1}(V) \cap g^{-1}(W)$ is open in $Y$ and $h^{-1}(V) \cap g^{-1}(W) \subseteq G$ since $V \cap W = \emptyset$. Thus $\{t \in Y : h(t) = g(t)\}$ is closed in $Y$. □

Proposition 17. For any two pre-continuous mappings $h$, $g$ of a pre-topological space $Y$ into a $T_2$ pre-topological space $Z$, if $h^{-1}(W) = g^{-1}(W)$ for any open set $W$ in $Z$, then $h \equiv g$; thus $\{t \in Y : h(t) = g(t)\}$ is closed in $Y$.

Proof. Assume the set $A = \{t \in Y ; h(t) \neq g(t)\}$ is nonempty. Then take an arbitrary $t \in A$. Hence $h(t) \neq g(t)$; then there are disjoint open sets $V$ and $O$ in $Z$ so that $h(t) \in V, g(t) \in O$. Therefore,

$$t \in h^{-1}(V) \cap g^{-1}(O) = g^{-1}(V) \cap g^{-1}(O) = g^{-1}(V \cap O) = \emptyset,$$

which is a contradiction. Hence $h \equiv g$. □

Definition 26. Let $Z$ be a $T_1$ pre-topological space. We say that $Z$ is a $T_3$ pre-topological space, or a regular space, if for every $z \in Z$ and every closed set $A$ of $Z$ with $z \notin A$ there are open subsets $V$ and $O$ such that $V \cap O = \emptyset$, $z \in V$ and $A \subseteq O$. 

Remark 7. (1) Each regular pre-topological space is $T_2$; however, the inverse is not true. Indeed, let $Z = \{z_1, z_2, z_3, z_4\}$ be endowed with the following pre-topology

$$\tau = \{\emptyset, \{z_1, z_2\}, \{z_1, z_3\}, \{z_1, z_4\}, \{z_2, z_3\}, \{z_2, z_4\}, \{z_3, z_4\}, \{z_1, z_2, z_3\}, \{z_1, z_2, z_4\}, \{z_1, z_3, z_4\}, \{z_2, z_3, z_4\}, Z\}.$$ 

Then $(Z, \tau)$ is a $T_2$ pre-topological space. However, $(Z, \tau)$ is not regular since $\{z_1, z_2\} = Z$.

(2) For a regular topological space $Z$, it follows that $w(Z) \leq 2^{d(Z)}$. However, there is a regular pre-topological space $Z$ with $w(Z) > 2^{d(Z)}$. Indeed, the pre-topological space in Example 6 is regular; but $w(Z) \geq |Z| > 2^8$ and $2^{d(Z)} = 8$.

**Proposition 18.** Let $\mathcal{B}$ be a fixed pre-base for a $T_1$ pre-topological space $Z$. Then $Z$ is regular iff for every $z \in Z$ and each open neighborhood $W$ of $z$ there is an open set $O$ with $z \in O \subseteq \overline{O} \subseteq W$.

**Definition 27.** Let $Z$ be a $T_1$ pre-topological space. Then $Z$ is a $T_{3\frac{1}{2}}$ pre-topological space, or a completely regular pre-topological space, or a Tychonoff pre-topological space, provide for each $z \in Z$ and each closed subset $C \subseteq Z$ with $z \notin C$ there exists a pre-continuous mapping $r : Z \rightarrow I$ so that $r(z) = 0$ and $r(x) = 1$ for each $x \in C$.

Clearly, there exists a regular pre-topological space that is not Tychonoff.

**Proposition 19.** A $T_1$ pre-topological space $Z$ is Tychonoff iff for each $z \in Z$ and each open neighborhood $V$ of $z$ in a fixed pre-base $\mathcal{B}$ there is a pre-continuous function $h_V : Z \rightarrow I$ such that $h_V(z) = 0$ and $h_V(y) = 1$ for any $y \in Z \setminus V$.

**Definition 28.** A $T_1$ pre-topological space $Z$ is called a $T_4$ pre-topological space, or a normal pre-topological space, provided for each any two disjoint closed subsets $C, D \subseteq Z$ there exist disjoint two open sets $U, W$ in $Z$ so that $C \subseteq U$ and $D \subseteq W$.

**Remark 8.** (1) A $T_1$ pre-topological space is normal iff for each closed set $C$ and each open subset $W$ which contains $C$ there exists an open subset $O$ such that $C \subseteq O \subseteq \overline{O} \subseteq W$. 

Each $T_4$ pre-topological space is $T_3$. By the following theorem, we see that each $T_4$ pre-topological space is $T_{3\frac{1}{2}}$, but the inverse is not true.

By using similar methods in [17, Theorem 1.5.11], we can prove the following versions of Urysohn’s lemma and Tietze extension theorem (Theorems 21 and 22, respectively) in the class of pre-topological spaces, thus we do not give out their proofs.

**Theorem 21.** Let $Z$ be a normal pre-topological space. If $C, D$ are two disjoint closed subsets of $Z$, then there exists a pre-continuous function $r : Z \to I$ with $r(C) \subseteq \{0\}$ and $r(D) \subseteq \{1\}$.

By the following lemma, we have the following corollary.

**Lemma 1.** Let $\{h_i\}$ be a sequence of pre-continuous functions from a pre-topological space $Z$ to $\mathbb{R}$ (from $Z$ to $I$) such that $\{h_i\}$ is uniformly convergent to a real-valued function $h$. Then $h$ is a pre-continuous function from $Z$ to $\mathbb{R}$ (from $Z$ to $I$).

A subset $G$ of a pre-topological space $Z$ is called a $G_\delta$-set (resp., a $F_\sigma$-set) if $G$ is the intersections of countably many open subsets (resp., the unions of countably many closed subsets).

**Corollary 4.** Let $C$ be a subset of a normal pre-topological space $Z$. Then $C$ is a closed $G_\delta$-set (resp. an open $F_\sigma$-set) iff there is a pre-continuous function $r : Z \to I$ with $C = r^{-1}(0)$ (resp. $C = r^{-1}((0, 1])$).

Taking complements, we can restate Corollary 4 as follows:

**Corollary 5.** Let $O$ be a subset of a normal pre-topological space $Z$. Then $O$ is an open $F_\sigma$-set iff there is a pre-continuous function $r : Z \to I$ with $O = r^{-1}((0, 1])$.

**Example 7.** There exists a regular, non-normal and finite pre-topological space $Z$. 

Proof. Let \((Z, \tau)\) be a pre-topology such that \(\omega > |Z| > 4\) and the family

\[
P = \left\{ \{y, z\} : y \in \{a_1, a_2, a_3\}, z \in Z, y \neq z, \{y, z\} \neq \{a_1, a_2\} \right\}
\]

is a pre-base of \((Z, \tau)\), where \(a_1, a_2, a_3\) are three distinct points of \(Z\). Obviously, \((Z, \tau)\) is a regular and finite pre-topological space. However, \(Z\) is not normal. Indeed, let \(D = \{a_1, a_2\}\) and \(F = Z \setminus \{a_1, a_2, a_3\}\). Then \(D\) and \(F\) are closed in \(Z\) and \(D \cap F = \emptyset\). Clearly, neither \(D\) nor \(F\) are open in \(Z\). Assume that there exist open sets \(O\) and \(W\) such that \(D \subseteq O, F \subseteq W\) and \(O \cap W = \emptyset\). But since neither \(D\) nor \(F\) are open in \(Z\), we conclude that \(a_3 \in O\) and \(a_3 \in W\), a contradiction. \(\square\)

**Problem 4.** When a regular pre-topological space is normal?

**Theorem 22.** Each pre-continuous function from a closed set \(A\) of a normal pre-topological space \(G\) to \(\mathbb{R}\) or \(I\) can pre-continuously extendable over \(G\).

2.7. **Connected pre-topological spaces.**

In the study of topological spaces, the connectedness is an important concept. By a similar definition of connectedness in topological space, we give the concept of connectedness in the class of pre-topological spaces.

**Definition 29.** Let \(Z\) be a pre-topological space. A pair \(O, W\) of two disjoint nonempty open sets of \(Z\) is a *separation* of \(Z\) whenever \(Z = O \cup W\). The pre-topological space \(Z\) is called *connected* provided there is no separation of \(Z\).

**Remark 9.** (1) If a pre-topological space \(Z\) is connected, then so is any pre-topological space that is pre-homeomorphic to \(Z\).

(2) A pre-topological space \(Z\) is connected iff any clopen subset is either \(\emptyset\) or \(Z\).

We don’t give the proofs of some results in this subsection, the reader can see the proofs in [21].
The following lemma is easily checked.

**Lemma 3.** Let $C$ and $D$ be a separation of pre-topological space $Z$ and $G$ be a connected subspace of $Z$. Then $G \subseteq C$ or $G \subseteq D$.

Then we have the following four theorems.

**Theorem 23.** Let $Z$ be a pre-topological spaces and $\{C_i\}_{i \in I}$ be a family of non-empty connected subspaces. If $\bigcap_{i \in I} C_i \neq \emptyset$, then $\bigcup_{i \in I} C_i$ is connected.

**Theorem 24.** Let $C$ be a connected subspace of a pre-topological space $Z$. If $C \subseteq A \subseteq C$, then $A$ is also connected.

**Theorem 25.** Let $h : Y \rightarrow Z$ be a pre-continuous mapping, where $Y$ and $Z$ are pre-topological spaces. If $Y$ is connected, then $Z$ is connected.

**Theorem 26.** Let $X_1$ and $X_2$ be two connected pre-topological space. Then the product $X_1 \times X_2$ is connected.

**Theorem 27.** The product pre-topology $\prod_{\alpha \in J} X_\alpha$, where each $X_\alpha \neq \emptyset$, is connected iff each $X_\alpha$ is connected.

The family $\{W_s\}_{s \in S}$ of subsets of a set $Z$ is said to be a cover of $Z$ if $\bigcup_{s \in S} W_s = Z$. Further, if $Z$ is a pre-topological space and each the set $W_s$ is open, then the family $\{W_s\}_{s \in S}$ is said to be open. The following theorem is new.

**Theorem 28.** A pre-topological space $Z$ is connected iff for each open cover $\{U_\alpha\}_{\alpha \in J}$ of $Z$ and every pair $x_1, x_2$ of points of $Z$ there exists a finite sequence $\alpha_1, \alpha_2, \ldots, \alpha_k$ of elements of $J$ so that $x_1 \in U_{\alpha_1}$, $x_2 \in U_{\alpha_k}$ and $U_{\alpha_i} \cap U_{\alpha_j} \neq \emptyset$ iff $|i - j| \leq 1$. 
Proof. Sufficiency. Let \( Z \) be not a connected space; hence we can fix a separation \( U \) and \( W \), thus \( \{U, W\} \) is an open cover of \( Z \). Take arbitrary \( x \in U \) and \( y \in W \). However, \( U \) intersects with \( W \) is empty, a contradiction. Therefore, \( Z \) is connected.

Necessity. Let \( Z \) be connected, and let \( O = \{U_\alpha\}_{\alpha \in J} \) be an open cover of \( Z \). Take any pair \( x_1, x_2 \) of \( Z \). Let \( A = \{W : x_1 \in W, W \in \{U_\alpha\}_{\alpha \in J}\} \) and \( B = \{W : x_2 \in W, W \in \{U_\alpha\}_{\alpha \in J}\} \). If \((\bigcup A) \cap (\bigcup B) \neq \emptyset\), then we can choose \( U_{\alpha_1} \in A \) and \( U_{\alpha_2} \in B \) such that \( x_1 \in U_{\alpha_1}, x_2 \in U_{\alpha_2} \) and \( U_{\alpha_1} \cap U_{\alpha_2} \neq \emptyset \). Then the proof complete. Hence we may assume that \((\bigcup A) \cap (\bigcup B) = \emptyset\). Let \( A_1 = \{U \in O : U \cap (\bigcup A) \neq \emptyset\} \setminus A \). If \((\bigcup A_1) \cap (\bigcup B) \neq \emptyset\), then there exist \( U_1 \in A, U_2 \in A_1 \) and \( U_3 \in B \) such that \( U_1 \cap U_2 \neq \emptyset \) and \( U_2 \cap U_3 \neq \emptyset \). Then the proof complete. Otherwise, assume that we can take a finite family \( A_i \subseteq O \) \((i \leq n)\) such that the following conditions hold.

1. For every \( i \leq n \),
\[
A_i = \{W \in O : W \cap (\bigcup A_{i-1}) \neq \emptyset\} \setminus (A \cup \bigcup_{k<i} A_k).
\]
2. For every \( 2 \leq i \leq n \), \((\bigcup A_{i-1}) \cap (\bigcup A_i) \neq \emptyset\).
3. For every \( 2 \leq i \leq n-1 \), \((\bigcup_{j<i} A_j) \cap (\bigcup B) = \emptyset\).

If \((\bigcup A_n) \cap (\bigcup B) \neq \emptyset\), then by our construction of the family \( \{A_i : i \leq n\} \) it is easily verified that theorem holds. Then put \( A_i = \emptyset \) for any \( i > n \). Otherwise, put
\[
A_{n+1} = \{U \in O : U \cap \bigcup A_n \neq \emptyset\} \setminus (A \cup \bigcup_{i \leq n} A_i).
\]
By induction, we have a sequence \( \{A_i : i \in \omega\} \), where \( A_0 = A \), of subsets of \( O \) such that the following conditions hold.

a. For each \( n \in \mathbb{N} \),
\[
A_n = \{U \in O : U \cap \bigcup A_{n-1} \neq \emptyset\} \setminus (A \cup \bigcup_{i \leq n} A_i);
\]
b. For any \( 2 \leq n \), \( \bigcup A_{n-1} \cap \bigcup A_n \neq \emptyset \);
c. For any \( 2 \leq n \), if \((\bigcup_{i \leq n} A_i) \cap (\bigcup B) \neq \emptyset\), then \( A_m = \emptyset \) for \( m > n \).
We claim that there is \( n \in \mathbb{N} \) such that \( \mathcal{A}_n = \emptyset \). Suppose not, \( \mathcal{A}_n \neq \emptyset \) for any \( n \in \omega \), thus \( \bigcup \mathcal{A}_n \cap \bigcup \mathcal{B} = \emptyset \). However, we conclude that \( \bigcup_{n \in \omega} \mathcal{A}_n = Z \), which leads to a contradiction. Indeed, put \( \mathcal{D} = \{ O \in \mathcal{O} : O \cap \bigcup_{n \in \omega} \mathcal{A}_n = \emptyset \} \). Then \( \bigcup \mathcal{D} \) is open in \( Z \) and \( \bigcup \mathcal{D} \cup \bigcup_{n \in \omega} \mathcal{A}_n = Z \). From the connectedness of \( Z \), it follows that \( \bigcup \mathcal{D} = \emptyset \). \( \square \)

**Definition 30.** Let \( C \) be a subset of a pre-topological space \( Z \). The set \( C \) is chain connected in \( Z \), if for every open covering \( \mathcal{U} \) in \( Z \) and any \( x, y \in C \), there can find a finite sequence \( U_1, U_2, \ldots, U_n \) of \( \mathcal{U} \), such that \( U_i \cap U_{i+1} \neq \emptyset \) for any \( i = 1, 2, \ldots, n - 1 \), \( x \in U_1 \) and \( y \in U_n \). If \( C = Z \), we say that \( Z \) is chain-connected.

From Theorem 28, we have the following corollary.

**Corollary 6.** Each connected pre-topological space is chain-connected.

Let \( Z \) be the real line \( \mathbb{R} \), and let \( \mathcal{B} = \{ (a, +\infty) : a \in \mathbb{R} \} \cup \{ (-\infty, a) : a \in \mathbb{R} \} \). Assume \( \mathcal{F} \) is the usual topology on \( Z \). Put

\[
\mathcal{F}' = \{ U \in \mathcal{F} : U = \bigcup \mathcal{B}' \text{ for some } \mathcal{B}' \text{ of } \mathcal{B} \}.
\]

Clearly, \((\mathbb{R}, \mathcal{F}')\) is a connected pre-topological space that is coarser than \((\mathbb{R}, \mathcal{F})\).

**Definition 31.** Let \( Z \) be a pre-topological space. If for any \( y, z \in Z \) there exists a pre-continuous function \( r : I \to Z \) such that \( r(0) = y \) and \( r(1) = z \), then \( Z \) is said to be a pathwise connected space, where \( I \) with the usual topology.

Clearly, we have the following two propositions.

**Proposition 20.** Each pathwise connected pre-topological space is connected.

Note that a connected pre-topological space may not be pathwise connected.

**Proposition 21.** The image of pathwise connected pre-topological space under pre-continuous mapping is pathwise connected as well.
By Theorem 16, the following theorem is easily checked.

**Theorem 29.** The product pre-topology \( \prod_{\alpha \in J} X_\alpha \), where each \( X_\alpha \neq \emptyset \), is pathwise connected iff all spaces \( X_\alpha \) are pathwise connected.

**Definition 32.** Let \((Z, \tau)\) be a finite pre-topological space. We say that \( Z \) is \( n \)-connected provided for any distinct open sets \( U \) and \( W \) there are open subsets \( O_0, O_1, \ldots, O_m \) such that \( O_0 = U, O_1, \ldots, O_m = W \) and \( |O_i \triangle O_{i+1}| = n \) for any \( 0 \leq i \leq m - 1 \).

We say that \( Z \) is tight \( n \)-connected provided for any distinct open sets \( V \) and \( W \) there exist open subsets \( O_0, O_1, \ldots, O_m \) such that \( O_0 = V, O_1, \ldots, O_m = W \) and \( |O_i \triangle O_{i+1}| = n \) for any \( 0 \leq i \leq m - 1 \), where \( m = |V \triangle W| \).

From the definition, the following proposition holds.

**Proposition 22.** If \( Z \) is an \( 1 \)-connected pre-topological space, then \( W^{\mathcal{LC}} \neq \emptyset \) for every open set \( W \) in \( Z \).

**Theorem 30.** If \( Z \) is a finite pre-topological space, then the following are equivalent.

1. \( Z \) is tight \( 1 \)-connected;
2. \( (U \triangle W) \cap U^{\mathcal{LC}} \neq \emptyset \) for any two distinct open sets \( U \) and \( W \);
3. any two open sets \( U \) and \( W \) which satisfy \( U^\mathcal{I} \subseteq W, U^\mathcal{O} \subseteq Z \setminus W \) must be equal.

**Proof.** (1) \( \Rightarrow \) (2). Take any two distinct open sets \( U \) and \( W \). Since \( Z \) is tight \( 1 \)-connected, there exist open subsets \( O_0, O_1, \ldots, O_m \) such that \( O_0 = U, O_1, \ldots, O_m = W \) and \( |O_i \triangle O_{i+1}| = n \) for any \( 0 \leq i \leq m - 1 \), where \( |U \triangle W| = m \). Hence it is easily verified that \( U \cap W \subseteq O_1 \subseteq U \cup W \). Moreover, \( U \) and \( O_1 \) differ by exactly one element \( x \), then \( x \in U \) or \( x \in W \), but not both. Hence, \( x \in (U \triangle W) \cap U^{\mathcal{LC}} \).

(2) \( \Rightarrow \) (3). Suppose not, then there exist distinct open sets \( U \) and \( W \) such that \( U^\mathcal{I} \subseteq W \) and \( U^{\mathcal{O}} \subseteq Z \setminus W \). Take any \( z \in (U \triangle W) \cap U^{\mathcal{LC}} \). If \( z \in U \), then \( z \in U^\mathcal{I} \subseteq W \), which contradicts \( q \in U \triangle W \). Therefore, \( z \notin U \), but then \( z \in W \cap U^{\mathcal{O}} \), hence \( z \in U^{\mathcal{O}} \subseteq Z \setminus W \) and \( z \in W \), a contradiction.
(3) $\Rightarrow$ (1). Let $U$ and $W$ be distinct open subsets in $Z$ with $|U \Delta W| = m > 0$. Since $U \neq W$, from our assumption it follows that $U^c \not\subseteq W$ or $W^c \not\subseteq Z \setminus W$.

Hence there exists an element $z \in Z$ with $z \in (U^c \setminus W) \cup (U^c \setminus (Z \setminus W))$.

If $z \in U^c \setminus W$, we put $W_1 = U \setminus \{z\}$; if $z \in U^c \setminus (Z \setminus W)$, we put $W_1 = U \cup \{z\}$. Then we have $|W_1 \Delta W| = m - 1$. The result follows by induction. $\Box$

**Example 8.** There exists a tight 1-connected and non-connected pre-topological space $Z$.

**Proof.** Let $Z = \{z_1, z_2, z_3, z_4\}$ be endowed with the following pre-topology

$$
\mathcal{H} = \emptyset, \{z_1\}, \{z_4\}, \{z_1, z_4\}, \{z_1, z_2\}, \{z_1, z_2, z_3\}, \{z_1, z_3, z_4\},
$$

$$
\{z_2, z_3, z_4\}, \{z_2, z_3, z_4, Z\}.
$$

It is easily checked that $Z$ is tight 1-connected. However, $Z$ is not connected since $\{z_1, z_2\}$ is open and closed in $Z$. $\Box$

**Example 9.** There exists a connected pre-topological space $Z$ that is not tight 1-connected.

**Proof.** Let $Z = \{z_1, z_2, z_3, z_4, z_5\}$ endowed with the following pre-topology

$$
\mathcal{H} = \emptyset, \{z_1\}, \{z_1, z_2, z_4\}, \{z_1, z_2, z_3\}, \{z_1, z_3, z_4\}, \{z_1, z_2, z_3, z_4\}, Z
$$

Clearly, $Z$ is connected. However, $Z$ is not tight 1-connected since there is not intermediate open set between $\{z_1\}$ and $\{z_1, z_2, z_3\}$. $\Box$

**Remark 10.** It is well known that the cardinality of a $T_1$ connected topological space is infinite. However, there exists a $T_1$ connected pre-topological space which is finite. Indeed, let $Z = \{z_1, z_2, z_3, z_4\}$ endowed with the following pre-topology

$$
\mathcal{H} = \emptyset, \{z_1, z_2, z_4\}, \{z_1, z_2, z_3\}, \{z_1, z_3, z_4\}, \{z_2, z_3, z_4\}, Z
$$

Then $Z$ is a $T_1$ connected pre-topological space.
3. The language of pre-topology in knowledge spaces

In this section, we will discuss the language of pre-topology in the theory of knowledge spaces. From Section 2, we see that knowledge space is just pre-topological space discussed above in the present paper. First, we list some terminology in the theory of pre-topological space and knowledge space respectively that are equivalent.

| pre-topological space | knowledge space | relation |
|-----------------------|-----------------|----------|
| open set              | knowledge state | ⇔        |
| $T_0$-space           | discriminative  | ⇔        |
| $T_1$-space           | bi-discriminative| ⇔        |
| atom pre-base         | base            | ⇒        |
| subspace              | projection      | ⇔        |
| $T_0$-reduction       | discriminative reduction | ⇔ |
| Alexandroff space     | quasi ordinal space | ⇔ |
| $T_0$-Alexandroff space | ordinal space    | ⇔        |
| locally inner closed points | inner fringe | ⇔ |
| locally outer closed points | outer fringe | ⇔ |
| locally closed points | fringe          | ⇔        |
| tight 1-connected     | well-graded     | ⇔        |
| pre-quotient pre-topology | discriminative reduction | ⇔ |

We always say that $(Q, \tau_{\mathcal{H}})$ is a pre-topological space with $\tau_{\mathcal{H}} = \mathcal{H}$ for a knowledge space $(Q, \mathcal{H})$.

3.1. The applications of axioms of separation in knowledge spaces.

In this subsection, we discuss some applications of axioms of separation in knowledge spaces. First, the following two theorems in [18] are provided.

**Theorem 31.** [18] Let $(Q, \mathcal{H})$ be a knowledge space. Then $(Q, \tau_{\mathcal{H}})$ is $T_0$ iff $(Q, \mathcal{H})$ is discriminative.

**Theorem 32.** [18] Let $(Q, \mathcal{H})$ be a knowledge space. Then $(Q, \tau_{\mathcal{H}})$ is $T_1$ iff $(Q, \mathcal{H})$ is bi-discriminative.

**Definition 33.** [14] Assume that $(Q, \mathcal{H})$ is a discriminative knowledge structure. For each $H \in \mathcal{H}$, we say that the set $H^I = \{ t \in H : H \setminus \{ t \} \in \mathcal{H} \}$ is the inner fringe of $H$. 
and that the set $H^O = \{ t \in Q \setminus H : H \cup \{t\} \in \mathcal{H} \}$ is the outer fringe of $H$. Moreover, the set

$$H^F = H^T \cup H^O$$

is said to be the fringe of $H$.

The following theorem shows that we can give a characterization for the bi-discriminative knowledge spaces by inner fringe of $Q$. Since the proof is easy, we do not give out the proof.

**Theorem 33.** For a knowledge space $(Q, \mathcal{H})$, it is bi-discriminative iff $Q \setminus \{t\} \in \mathcal{H}$ for each $t \in Q$, that is, $Q^T = Q$.

Moreover, the following proposition shows that we can give a characterization for knowledge spaces by inner fringe of knowledge states.

**Proposition 23.** Let $(Q, \mathcal{H})$ be a knowledge space. For each $H \in \mathcal{H}$ and $t \in H$, it follows that $t \in H^T$ iff $q \not\in Q \setminus (H \setminus \{t\})$ for any $q \in H \setminus \{t\}$.

**Proof.** Necessity. Assume that $t \in H^T$, then $H \setminus \{t\} \in \mathcal{H}$, hence $(H \setminus \{t\}) \cap (Q \setminus (H \setminus \{t\})) = \emptyset$, that is, $q \not\in Q \setminus (H \setminus \{t\})$ for any $q \in H \setminus \{t\}$.

Sufficiency. Assume that $q \not\in Q \setminus (H \setminus \{t\})$ for any $q \in H \setminus \{t\}$, then for each $q \in H \setminus \{t\}$ there exists $G(q) \in \mathcal{H}$ such that $G(q) \cap (Q \setminus (H \setminus \{t\})) = \emptyset$, hence $G(q) \subseteq H \setminus \{t\}$. Therefore, $H \setminus \{t\} = \bigcup_{q \in H \setminus \{t\}} G(q) \in \mathcal{H}$. □

The following proposition shows that we can give a characterization of outer fringe of a state for knowledge spaces by the derived points.

**Proposition 24.** Let $(Q, \mathcal{H})$ be a knowledge space, $H \in \mathcal{H}$ and $t \in Q \setminus H$. Then $t \in H^O$ iff $t \not\in (Q \setminus H)^d$ in $(Q, \tau_{\mathcal{H}})$.

**Proof.** Necessity. Assume $t \in H^O$, then $H \cup \{t\} \in \mathcal{H}$, hence

$$(H \cup \{t\}) \cap ((Q \setminus H) \setminus \{t\}) = \emptyset,$$
thus $t \notin (Q \setminus H)^d$ in $(Q, \tau, \mathcal{H})$.

Sufficiency. Suppose that $t \notin (Q \setminus H)^d$ in $(Q, \tau, \mathcal{H})$, then there exists $G \in \mathcal{H}$ such that $t \in G$ and $G \cap ((Q \setminus H) \setminus \{t\}) = \emptyset$. Hence $L \subseteq H \cup \{t\}$, hence $H \cup G = H \cup \{t\} \in \mathcal{H}$, that is, $t \in H^G$. □

By Propositions 23 and 24, the following theorem holds.

**Theorem 34.** Let $(Q, \mathcal{H})$ be a knowledge space, $H \in \mathcal{H}$ and $t \in Q$. Then $t \in H^F$ iff either $(H \setminus \{t\}) \cap Q \setminus (Q \setminus H)^d = \emptyset$ and $t \in H$ or $t \notin (Q \setminus H)^d$ in $(Q, \tau, \mathcal{H})$ and $t \notin H$.

In knowledge assessment, the device of the items is very important. Since the time, the manpower and the material resources are limited, ones hope that the process of the assessment is efficient, and that the items are not repeated as far as possible in the process of the assessment. Further, ones hope that the selection of items in the knowledge assessment should not only have depth but also breadth. However, if the device of the items is not appropriate, then it is possible that the process of the knowledge assessment is invalid, see the following example.

**Example 10.** Let $(Q, \mathcal{H})$ be a knowledge space, where $Q = \{a_1, a_2, a_3, a_4, a_5\}$ and

$$\mathcal{H} = \{\emptyset\} \cup \{O : |Q \setminus O| \leq 2, O \subseteq Q\}.$$ 

Then $(Q, \mathcal{H})$ is a bi-discriminative. In a knowledge assessment, a teacher in order to check the level for a course of a student, this teacher designs these five items (that is, knowledge points). Assume this student is capable of solving of items $\{a_1, a_2\}$ (that is, she/his knowledge state indeed). For each $K \in \mathcal{H} \setminus \{\emptyset\}$, the teacher devices an exercise $E_K$. Clearly, the student can not complete any exercise; then it is problem for the teacher to check the level for a course of this student.

In a knowledge assessment devices, we always find that the methods are inadequate, hence ones have to provide further deepening and enriching the methods in the knowledge
assessment. Therefore, we have the following subclass of bi-discriminative knowledge spaces.

**Definition 34.** A knowledge space \((Q, \mathcal{H})\) is completely discriminative provided there exists \(H \in \mathcal{H}_p\) and \(L \in \mathcal{H}_q\) with \(H \cap L = \emptyset\) for any distinct \(p, q \in Q\).

The [18, Example 3] is a completely discriminative knowledge space. Obviously, Example 10 is a bi-discriminative knowledge space which is not completely discriminative. If we enrich the methods of knowledge assessment in Example 10 as follows

\[ \mathcal{H} = \{\emptyset\} \cup \{U : |Q \setminus U| \leq 3, U \subset Q\}, \]

then we will know the level for the course of this student.

**Definition 35.** [11] Let \(\mu : Q \rightarrow 2^S \setminus \{\emptyset\} \setminus \{\emptyset\}\) be a mapping, where \(Q\) and \(S\) are non-empty sets of items and skills respectively. Then a triple \((Q, S, \mu)\) is called a skill multimap. The elements of \(\mu(t)\) are said to be competencies for every \(t \in Q\). Moreover, if the elements of each \(\mu(t)\) are pairwise incomparable, then \((Q, S, \mu)\) is called a skill function.

**Definition 36.** [11] For a skill multimap \((Q, S, \mu)\), we define a mapping \(p : 2^S \rightarrow 2^Q\) as follows: for each \(R \in 2^S\), put

\[ p(R) = \{g \in Q | \text{there exists } R_g \in \mu(g) \text{ with } R_g \subseteq R\}. \]

Then, the mapping \(p\) is said to be the problem function induced by \((Q, S, \mu)\). Put

\[ \mathcal{H} = \{p(R) | R \in 2^S\}. \]

We say that the knowledge structure \((Q, \mathcal{H})\) is delineated by \((Q, S, \mu)\).

In [14, p117, Problem 6], the authors posed the following problem.

**Problem 5.** Under which condition on a skill multimap is the delineated structure a knowledge space?
Indeed, in [18] the authors also asked which kind of skill multimaps delineate knowledge spaces. The following theorem gives an answer to this problem when each item with finitely many competencies. For a skill multimap \((Q, S, \mu)\) and each \(t \in Q\), we denote \(\mu_M(t)\) by the set of minimum elements in \(\mu(t)\).

**Theorem 35.** Let \((Q, S, \mu)\) be a skill multimap, where each \(\mu(t)\) is a finite set. Then the delineate knowledge structure \((Q, H)\) is a knowledge space iff, for any \(H \subseteq Q\), \(H \in H\) iff there is \(P_H \subseteq \bigcup_{t \in Q} \mu_M(t)\) such that \(H = \bigcup_{D \in P_H} p(D)\).

**Proof.** Sufficiency. Let \(p\) be the problem function induced by \(\mu\), and let \((Q, H)\) be the knowledge structure which is delineated by \((Q, S, \mu)\). Take any a subfamily \(H' \subseteq H\). We claim that \(\bigcup_{H \in H'} \bigcup_{D \in P_H} p(D) = \bigcup_{D \in P} p(D)\). Clearly, we have

\[
\bigcup_{H \in H'} \bigcup_{D \in P_H} p(D) \subseteq \bigcup_{D \in P} p(D).
\]

From our assumption, it follows that \(\bigcup_{H \in H'} \bigcup_{D \in P_H} p(D) \subseteq H\).

Necessity. Let \((Q, H)\) be the knowledge space which is delineated by \((Q, S, \mu)\). Take any \(H \subseteq Q\). Assume that \(H \in H\). Hence we can take a subset \(R \subseteq S\) such that \(p(R) = H\). For each \(t \in H\), there exists a \(C_t \in \mu_M(t)\) such that \(C_t \subseteq R\). We claim that \(H = \bigcup_{t \in H} p(C_t)\). Clearly, \(H \subseteq \bigcup_{t \in H} p(C_t)\); moreover, since each \(C_t \subseteq R\) and \(p(R) = H\), it follows that \(\bigcup_{t \in H} p(C_t) \subseteq H\). Therefore, \(H = \bigcup_{t \in H} p(C_t)\). Now assume that there exists \(P_H \subseteq \bigcup_{t \in Q} \mu_M(t)\) such that \(H = \bigcup_{D \in P_H} p(D)\). Since \((Q, H)\) is a knowledge space and each \(p(D) \in H\), it follows that \(\bigcup_{D \in P_H} p(D) \in H\), hence \(H \in H\). \(\square\)

**Corollary 7.** Let \((Q, S, \mu)\) be a skill multimap such that each \(\mu(t)\) is a finite set. If, for any subset \(Q' \subseteq Q\) and \(g \in Q\), the following condition \((\star)\) holds, then \((Q, H)\) is a knowledge space.

\[(\star)\] For any \(M \subseteq \bigcup_{t \in Q'} \mu(t)\), if \(C \setminus D \neq \emptyset\) for any \(C \in \mu_M(g)\) and \(D \in M\), then \(C \setminus \bigcup M \neq \emptyset\) for each \(C \in \mu_M(g)\).
Proof. By Theorem 35, we need to prove that for any \( H \subseteq Q, H \in \mathcal{H} \) iff there exists \( \mathcal{M} \subseteq \bigcup_{t \in Q} \mu_M(t) \) such that \( H = \bigcup_{D \in \mathcal{M}} p(D) \).

Suppose that \( H \in \mathcal{H} \), then there exists a subfamily \( \mathcal{M} \) of \( \bigcup_{t \in H} \mu_M(t) \) such that \( H = p(\bigcup \mathcal{M}) \). We claim that \( H = p(\bigcup \mathcal{M}) = \bigcup_{D \in \mathcal{M}} p(D) \). Clearly, it follows that \( \bigcup_{D \in \mathcal{M}} p(D) \subseteq p(\bigcup \mathcal{M}) \). Assume that \( p(\bigcup \mathcal{M}) \setminus \bigcup_{D \in \mathcal{M}} p(D) \neq \emptyset \). Take any \( g \in p(\bigcup \mathcal{M}) \setminus \bigcup_{D \in \mathcal{M}} p(D) \). Hence \( C \setminus D \neq \emptyset \) for each \( C \in \mu_M(g) \) and \( D \in \mathcal{M} \), then from the condition (\( \ast \)) it follows that \( C \setminus \bigcup \mathcal{M} \neq \emptyset \) for any \( C \in \mu_M(g) \), which leads to a contradiction with \( g \in p(\bigcup \mathcal{M}) \setminus \bigcup_{D \in \mathcal{M}} p(D) \).

Pick any \( H \subseteq Q \), and assume that there is \( \mathcal{M} \subseteq \bigcup_{t \in Q} \mu_M(t) \) such that \( H = \bigcup_{D \in \mathcal{M}} p(D) \).

We claim that \( H \in \mathcal{H} \). Indeed, \( H = p(\bigcup \mathcal{M}) \) by the condition (\( \ast \)), hence \( H \in \mathcal{H} \). \( \square \)

**Corollary 8.** Let \((Q, S, \mu)\) be a skill multimap. If, for each \( t \in Q \) and \( C \in \mu(t) \), there exists \( s_C \in C \) such that \( \{s_C\} \in \mu(t) \), then the delineate knowledge structure is a knowledge space.

**Proof.** Indeed, it is easily verified that the condition (\( \ast \)) holds in Corollary 7; moreover, each \( \mu_M(q) \) consists of elements of singleton. Therefore, the delineate knowledge structure is a knowledge space by Corollary 7. \( \square \)

It follows from the following example that the condition in Corollary 8 is sufficient and non-necessary condition.

**Example 11.** Assume that \((Q, S, \mu)\) is a skill multimap such that \( \mu(t) = \{S\} \) for each \( t \in Q \). It is obvious that the delineate knowledge structure \( \mathcal{H} = \{\emptyset, Q\} \) is a knowledge space. However, the condition in Corollary 8 does not hold.

By Corollary 7, the following corollary holds.

**Corollary 9.** Let \((Q, S, \mu)\) be a skill function. If, for any subset \( Q' \subseteq Q \) and \( g \in Q \), the following condition (\( \ast \)) holds, then \((Q, \mathcal{H})\) is a knowledge space.
For each $M \subseteq \bigcup_{t \in Q'} \mu(t)$, if $C \setminus D \neq \emptyset$ for any $C \in \mu(g)$ and $D \in M$, then $C \setminus \bigcup M \neq \emptyset$ for each $C \in \mu(g)$.

Definition 37. [18] Suppose that $Z$ is a non-empty set, $O$ and $W$ be two family of subsets on $Z$. If for any $O \in O$ there exists $W \in W$ with $W \subseteq O$, then we say that $O$ is refined by $W$ which is denoted by $O \sqsubseteq W$. Otherwise, $O$ is not refined by $W$ which is denoted by $O \not\sqsubseteq W$. Further, if $O = \{O\}$, then we say that $O$ is refined by $W$, which is denoted by $O \sqsubseteq W$; otherwise, we say that $O$ is not refined by $W$, which is denoted by $O \not\sqsubseteq W$.

Theorem 36. For a skill multimap $(Q, S, \mu)$ with each $\mu(r)$ being finite, then the delineate knowledge structure $(Q, \mathcal{H})$ is completely discriminative iff for any distinct $h, q$ in $Q$, there exist $C_h \in \mu_M(h)$ and $C_q \in \mu_M(q)$ such that, for any $g \in Q$, at most one of $C_h \in \mu_M(g)$ and $C_q \in \mu_M(g)$ holds.

Proof. Necessity. Assume that $(Q, \mathcal{H})$ is completely discriminative. Then for any distinct $h, q$ in $Q$, there exist $C_h \in \mu_M(h)$ and $C_q \in \mu_M(q)$ such that $p(C_h) \cap p(C_q) = \emptyset$. For any $g \in Q$, without loss of generality, suppose that $C_h \in \mu_M(g)$, then $g \in p(C_h)$. Since $p(C_h) \cap p(C_q) = \emptyset$, it follows that $g \not\in p(C_q)$, then $C \setminus C_q \neq \emptyset$ for any $C \in \mu_M(g)$. Hence $C_q \not\in \mu_M(g)$.

Sufficiency. For any distinct $h, q$ in $Q$, it follows from the assumption that there exist $C_h \in \mu_M(h)$ and $C_q \in \mu_M(q)$ such that, for any $g \in Q$, at most one of $C_h \in \mu_M(g)$ and $C_q \in \mu_M(g)$ holds. Then it is easily verified that $p(C_h) \cap p(C_q) = \emptyset$. Therefore, $(Q, \mathcal{H})$ is completely discriminative. \hfill \Box

The following corollary is easily checked by Theorems 35 and 36.

Corollary 10. For a skill multimap $(Q, S, \mu)$ with each $\mu(r)$ being finite, then the delineate knowledge structure $(Q, \mathcal{H})$ is a completely discriminative knowledge space iff the following two conditions hold:
(1) For any \( H \subseteq Q \), \( H \in \mathcal{H} \) iff there exists \( P_H \subseteq \bigcup_{t \in Q} \mu_M(t) \) such that \( H = \bigcup_{D \in P_H} \mu_M(D) \);

(2) For any distinct \( h, q \) in \( Q \), there exist \( C_h \in \mu_M(h) \) and \( C_q \in \mu_M(q) \) such that, for any \( g \in Q \), at most one of \( C_h \cup g \) and \( C_q \cup g \) holds.

The following Theorems 37 and 38 show that the language of the regularity of pre-topology in knowledge spaces.

**Theorem 37.** Let \((Q, \mathcal{H})\) be a knowledge space, and let \((Q, \tau_\mathcal{H})\) be regular. For each \( t \in Q \) and \( H \in \mathcal{H}_t \), we have \( Q \setminus H \in \mathcal{H} \) or \( H \) is not an atom at \( t \).

**Proof.** Take an arbitrary \( t \in Q \), and any \( H \in \mathcal{H}_t \). Assume that \( Q \setminus H \notin \mathcal{H} \), then \( H \) is not closed in \((Q, \tau_\mathcal{H})\). Since \((Q, \tau_\mathcal{H})\) is regular and \( t \notin Q \setminus H \), there are \( L \in \mathcal{H} \) and \( K \in \mathcal{H} \) with \( t \in L \), \( Q \setminus H \subseteq K \) and \( L \cap K = \emptyset \). Because \( Q \setminus K \notin \mathcal{H} \), it follows that \( K \cap H \neq \emptyset \); moreover, it is obvious that \( L \subseteq Q \setminus K \subseteq H \), then \( L \neq H \) since \( Q \setminus H \notin \mathcal{H} \), hence \( H \) is not an atom at \( t \). \( \Box \)

**Corollary 11.** Let \((Q, \mathcal{H})\) be a knowledge space with \((Q, \tau_\mathcal{H})\) being regular. If there exists an atom \( K \) at some \( t \in Q \), then \((Q, \tau_\mathcal{H})\) is not a connected space. In particular, if \( Q \) is finite and \((Q, \tau_\mathcal{H})\) is regular, then \((Q, \tau_\mathcal{H})\) is not a connected space.

By a similar proof of Theorem 37, the following theorem holds.

**Theorem 38.** Let \((Q, \mathcal{H})\) be a knowledge space with \((Q, \tau_\mathcal{H})\) being regular. For each \( K \in \mathcal{H} \) and \( t \in Q \), if \( t \in K^\circ \), then \( Q \setminus (K \cup \{t\}) \in \mathcal{H} \) or \( K \cup \{t\} \) is not an atom at \( t \).

**Remark 11.** It is an interesting topic to find the applications of axioms of separation of Tychonoff and normality in knowledge spaces.

### 3.2. Alexandroff spaces and quasi ordinal spaces.

It is well known that a topological space is called *Alexandroff spaces* provided the
intersection of arbitrary many open sets is open. From Theorem 31, we see that ordinal spaces is equivalent to $T_0$-Alexandroff spaces.

**Theorem 39.** The knowledge space $(Q, \mathcal{K})$ is an ordinal space iff $(Q, \tau_\mathcal{K})$ is a $T_0$-Alexandroff space.

**Proof.** Since each Alexandroff space is equivalent to quasi ordinal space, it following Theorem 31 that each ordinal space is equivalent to a $T_0$-Alexandroff space. □

The following theorem holds by [14, Theorem 3.8.3] and Theorem 39.

**Theorem 40.** There exists a bijection between the collection of all quasi orders $Q$ on $Q$ and the collection of all Alexandroff spaces $\mathcal{F}$ on $Q$. Under this bijection, $T_0$-Alexandroff spaces are mapped onto partial orders.

If a relation on a set $Z$ is reflexive and transitive, then we say this relation is a quasi order $Z$, and if $Z$ is equipped with a quasi order is called quasi ordered. Given a quasi-order $Q$ construct the Alexandroff space $Q\langle Q \rangle$ as the set $Q$ with the topology generated by $\{ \leftarrow, q : q \in Q \}$. Conversely, given an Alexandroff space $(Q, \tau)$, we construct a quasi-order $Q\langle Q \rangle$ as the set with the order $x \preceq y$ iff $x \in \bigcap \tau(y)$.

**Proposition 25.** Let $(Q, \mathcal{F}_1)$ and $(Q, \mathcal{F}_2)$ be two pre-topologies on $Q$ such that $(Q, \mathcal{F}_1)$ and $(Q, \mathcal{F}_2)$ are two Alexandroff space. Then $\mathcal{F}_1 = \mathcal{F}_2$ iff, for any $p, q \in Q$, $\mathcal{F}_1(p) \subseteq \mathcal{F}_1(q) \Leftrightarrow \mathcal{F}_2(p) \subseteq \mathcal{F}_2(q)$.

**Proof.** The necessity is obvious. In order to prove the converse implication, suppose that $O \in \mathcal{F}_1$. Hence $O \subseteq \bigcup_{p \in O}(\bigcap \mathcal{F}_2(p)) \in \mathcal{F}_2$. Put $O' = \bigcup_{p \in O}(\bigcap \mathcal{F}_2(p))$. We conclude that $O = O'$. Indeed, take any $u \in O'$. Then there exists $v \in O$ such that $u \in \bigcap \mathcal{F}_2(v)$. Hence $\mathcal{F}_2(v) \subseteq \mathcal{F}_2(u)$, then $\mathcal{F}_1(v) \subseteq \mathcal{F}_1(u)$. We obtain $u \in \bigcap \mathcal{F}_1(v)$, which implies that $u \in O$. This gives $O' \subseteq O$, hence $O = O'$. We conclude that $\mathcal{F}_1 \subseteq \mathcal{F}_2$, and by symmetry, $\mathcal{F}_1 = \mathcal{F}_2$. □
Proposition 26. If \((Q, \mathcal{H})\) is a knowledge structure, then the following are equivalent:

1. \((Q, \mathcal{H})\) is a finite ordinal space;
2. \((Q, \mathcal{H})\) is a learning space and \(\bigcap \mathcal{H}_p \in \mathcal{H}\) for each \(p \in Q\);
3. \((Q, \tau_{\mathcal{H}})\) is a finite \(T_0\)-Alexandroff space.

Proof. By Theorem 39, we have (1) \(\iff\) (3). It follows from [14, Theorem 3.8.7] that (1) \(\implies\) (2). Hence it suffices to prove (2) \(\implies\) (3). Since \((Q, \mathcal{H})\) is a learning space, it follows that \((Q, \mathcal{H})\) is finite and discriminative, hence \((Q, \tau_{\mathcal{H}})\) is \(T_0\). Suppose that \(H, L \in \mathcal{H}\).

Let \(\mathcal{H}\) be a family of subsets of a finite set \(Q\) such that \(\mathcal{H}\) is closed under union. The family \(\mathcal{H}\) is called an antimatroid provided \(Q = \bigcup \mathcal{H}\) and \(\mathcal{H}\) satisfies the condition: for each non-empty element \(H\) of \(\mathcal{H}\), there is \(t \in H\) such that \(H \setminus \{t\} \in \mathcal{H}\).

Lemma 4. Any finite \(T_0\)-Alexandroff space \(X\) is antimatroid.

Proof. For each point \(x \in X\), let \(V_x\) be a minimal open set containing the point \(x\). Let \(U\) be any non-empty open set in \(X\). Take an arbitrary point \(y_0 \in U\). If \(U\) is a single set, then it is obviously true. Thus we assume that \(|U| \geq 2\). If \(\bigcup_{y \in U \setminus \{y_0\}} V_y = U \setminus \{y_0\}\), then \(U \setminus \{y_0\}\) is obviously open in \(X\), hence the assertion of the Lemma is true. Therefore, now we assume that \(\bigcup_{y \in U \setminus \{y_0\}} V_y \neq U \setminus \{y_0\}\), then there is \(y_1 \in U \setminus \{y_0\}\) with \(y_1 \notin V_{y_0}\) since \(X\) is \(T_0\). If \(U \setminus V_{y_0} = \{y_1\}\), then the assertion of the Lemma is true. Otherwise, \(|U \setminus V_{y_0}| \geq 2\).

If \(V_{y_0} \cup \bigcup_{y \in U \setminus (V_{y_0} \cup \{y_1\})} V_y = U \setminus \{y_1\}\), then \(U \setminus \{y_1\}\) is obviously open in \(X\), hence the assertion of the Lemma is true. Therefore, we may assume that \(V_{y_0} \cup \bigcup_{y \in U \setminus (V_{y_0} \cup \{y_1\})} V_y \neq U \setminus \{y_1\}\), then there exists \(y_2 \in U \setminus (V_{y_0} \cup \{y_1\})\) with \(y_2 \notin V_{y_1} \cup V_{y_0}\) since \(X\) is \(T_0\).
U is finite, by induction, it follows that there exists a point \( t \in U \) so that \( U \setminus \{t\} \) is open in \( X \).

\[ \square \]

**Theorem 41.** Any finite \( T_0 \)-Alexandroff space \( X \) is tight 1-connected.

**Proof.** Since \( X \) is a finite \( T_0 \)-Alexandroff space, we denote the minimal open neighborhood of \( x \) by \( W_x \). By Theorem 30, it only need to prove that \( (O \triangle W) \cap O^{cc} \neq \emptyset \) for any open sets \( O \) and \( W \) with \( O \neq W \). We divide the rest proof into the following two cases.

**Case 1:** \( O \setminus W \neq \emptyset \).

It follow from Lemma 4 and [7, Lemma 6] that we can take a point \( z \in O \setminus W \) so that \( O \setminus \{z\} \) is open in \( X \). Therefore, \( z \in (O \triangle W) \cap O^{cc} \).

**Case 2:** \( O \setminus W = \emptyset \).

Indeed, we shall use an induction on the size of the set \( W \setminus O \). If \( W \setminus O \) is a single set \( \{x\} \), then \( x \in (O \triangle W) \cap O^{cc} \). Suppose that there is a point \( x \in W \setminus O \) such that \( O \cup \{x\} \) is open if \( |W \setminus O| = n \) for each \( n \geq 2 \). Now assume that \( |W \setminus O| = n + 1 \). Then it follow from Lemma 4 and [7, Lemma 6] that there exists a point \( z \in W \setminus O \) such that \( W(z) = W \setminus \{z\} \) is open in \( X \). Then \( |W(z) \setminus O| = n \). By our assumption, there exists \( y \in W(z) \setminus O \) such that \( O \cup \{y\} \) is open in \( X \).

\[ \square \]

Let \( \mathcal{F} \) be a family of subsets of a finite set \( Z \). We say that \( \mathcal{F} \) is well-graded provided, for any \( K, H \in \mathcal{F} \) with \( L \neq H \), there exists a finite sequence of states \( L = H_0, H_1, \ldots, H_m = H \) so that \( d(H_{i-1}, H_i) = 1 \) for \( 1 \leq i \leq m \), where \( m = d(L, H) \).

**Corollary 12.** Each finite ordinal space \((Q, \mathcal{H})\) is well-graded.

**Remark 12.** There is a finite ordinal space \((Q, \mathcal{H})\) such that \((Q, \tau_{\mathcal{H}})\) is not connected.

Indeed, let \( Q = \{z_1, z_2, z_3, z_4, z_5, z_6\} \) endowed with the following knowledge structure

\[ \mathcal{H} = \{\emptyset, \{z_4\}, \{z_1, z_3\}, \{z_5, z_6\}, \{z_1, z_3, z_4\}, \{z_1, z_2, z_4\}, \{z_4, z_5, z_6\}, \{z_1, z_3, z_5, z_6\}, \{z_1, z_2, z_4, z_6\}, Q\}. \]
Then \((Q, \mathcal{H})\) is a finite ordinal space, which is not connected since \(\{z_1, z_2, z_3\}\) is open and closed in \((Q, \tau_H)\). However, it follow from [5, Theorem 2.7] that we have the following Theorem 42. First, we give the following proposition.

**Proposition 27.** Assume that \((Q, \mathcal{H})\) is a knowledge space. Then \((Q, \mathcal{H})\) is a quasi ordinal space iff each point in \(Q\) has a unique minimal state \(M(t)\) containing the point \(t\).

**Proof.** Necessity. Let \((Q, \mathcal{H})\) be a quasi ordinal space with \(t \in Q\). Put

\[ M(t) = \{ G : t \in G \in \mathcal{H} \} \]

and \(M(t) = \bigcap M(t)\). Since \((Q, \mathcal{H})\) is a quasi ordinal space, it follows that \(M(t) \in \mathcal{H}\). From the definition of \(M(t)\), it is obvious that \(M(t)\) is a unique minimal state containing the point \(t\).

Sufficiency. Assume each \(t \in Q\) has a unique minimal state \(M(t)\) containing the point \(t\). Consider an arbitrary intersection of states, \(H = \bigcap_{\beta \in A} H_\beta\), where every \(H_\beta \in \mathcal{H}\). If \(H = \emptyset\), then \(H \in \mathcal{H}\) and we are done. If \(H \neq \emptyset\), then take any \(t \in H\) and we have \(t \in H_\beta\) for each \(\beta \in A\), hence \(M(t) \subseteq H_\beta\) for each \(\beta \in A\) since \(M(t)\) is the unique state containing the point \(t\). Therefore, \(M(t) \subseteq H\) for each \(\beta \in A\), thus \(H = \bigcup_{t \in H} M(t) \in \mathcal{H}\) since \((Q, \mathcal{H})\) is a knowledge space. Therefore, \((Q, \mathcal{H})\) is a quasi ordinal space. \(\square\)

**Theorem 42.** If \((Q, \mathcal{H})\) is an ordinal space, then the following conditions are equivalent:

(a) \((Q, \tau_{\mathcal{H}})\) is connected;

(b) \((Q, \tau_{\mathcal{H}})\) is chain connected;

(c) \((Q, \tau_{\mathcal{H}})\) is pathwise connected;

(d) for any \(r, t \in Q\), there exists a finite sequence \(q_0, \ldots, q_{n+1} \in Q\) such that \(r = q_0, q_{n+1} = t\) and \(M(q_i) \cap M(q_j) \neq \emptyset\) if \(|i - j| \leq 1\).

**Theorem 43.** If \((Q, \mathcal{H})\) is a bi-discriminative quasi ordinal space, then \(\mathcal{H} = 2^Q\).
Proof. It only need to prove that \( \{ t \} \in \mathcal{H} \) for each \( t \in Q \). Indeed, take any \( t \in Q \). Obviously, \((Q, \mathcal{H})\) is a \( T_1 \) Alexandroff space. Therefore, it follows that

\[
\{ t \} = \bigcap \{ H \in \mathcal{H} : t \in H \} \in \mathcal{H}.
\]

\( \square \)

From [5, Theorem 2.9], the following two theorems hold.

**Theorem 44.** If \((Q, \mathcal{H})\) is a quasi ordinal space, then \( Q \setminus M(t) \in \mathcal{H} \) for each \( t \in Q \) iff the pre-topological space \((Q, \tau_{\mathcal{H}})\) is regular, where \( M(t) \) is the unique minimal state containing the point \( t \).

**Theorem 45.** Let \((Q, \mathcal{H})\) be a quasi ordinal space. If \((Q, \tau_{\mathcal{H}})\) is a regular pre-topological space with a countable dense subset, then \((Q, \tau_{\mathcal{H}})\) is normal.

### 3.3. Primary items in knowledge spaces.

In this section, we discuss the density of pre-topological spaces with the application in knowledge spaces. By Theorem 20, it is easily verified the following theorem.

**Theorem 46.** The inequality \(|Q| \leq |\mathcal{H}|\) holds for each discriminative knowledge space \((Q, \mathcal{H})\).

We say that a pairwise disjoint family consisting of non-empty open subsets of a pre-topological space \((Z, \tau)\) is called a *cellular family*. The *cellularity* of \( Z \) is defined as follows:

\[
c(Z) = \sup \{|\mathcal{V}| : \mathcal{V} \text{ is a cellular family in } Z \}.
\]

Here, the cellularity of a pre-topological space maybe finite.

**Lemma 5.** Let \( Z \) be a pre-topological space such that \( c(Z) = \kappa \). If \( \mathcal{V} \) is an open cover of \( Z \), then there is a subcollection \( \mathcal{W} \) of \( \mathcal{V} \) with \(|\mathcal{W}| \leq \kappa \) and \( \bigcup \mathcal{W} = Z \).

**Proof.** Let \( \mathcal{C} \) be the family of all non-empty open sets in \( Z \) that are subsets of some element of \( \mathcal{V} \). It follows from Zorn’s lemma that we can take a maximal cellular family
$O' \subset O$. Hence $|O'| \leq c(X) = \kappa$, and $Z = \bigcup O'$ by the maximality of $O'$. For each $O \in O'$, fix a $W_O \in \mathcal{V}$ such that $O \subset W_O$. Put $\mathcal{W} = \{W_O : O \in O'\}$. Then $\bigcup \mathcal{W} = Z$. \hfill \Box

The following corollary holds by Lemma 5.

**Corollary 13.** Let $\mathcal{B}$ be a atom pre-base of a finite pre-topological space $Z$. Then there is a maximally disjoint subfamily $\mathcal{B}'$ of $\mathcal{B}$ such that $Z = \overline{\mathcal{B}}$.

Let $(Q, \mathcal{H})$ be a knowledge space. If $D$ is a dense subset of $(Q, \tau_{\mathcal{H}})$, we say that $D$ is the primary items of $Q$. In other work, one person in order to master an item, she or he must master some items of $D$.

**Definition 38.** Let $Z$ be a pre-topological space. The infimum of the set

$$\{ |D| : D \subseteq Z, \overline{D} = Z \}$$

is called the density $d(Z)$ of $Z$.

For a knowledge space $(Q, \mathcal{H})$, we say that $A$ is an atom primary items of $(Q, \mathcal{H})$ if $A$ is dense and $|A| = d(Q)$. Moreover, the atom primary items of a knowledge space is not unique. Indeed, the sets $\{z_1, z_2, z_3\}$, $\{z_1, z_2, z_4\}$ and $\{z_2, z_3, z_4\}$ are all atom primary items of the pre-topological space $(Z, \tau)$ in Example 4. Clearly, we have the following proposition.

**Proposition 28.** For any knowledge space $(Q, \mathcal{H})$, if $A$ is an atom primary items of $(Q, \mathcal{H})$, then $|A| \leq \min\{ |Q|, w(Q, \tau_{\mathcal{H}}) \}$.

From Lemma 5, the following proposition holds.

**Proposition 29.** For any knowledge space $(Q, \mathcal{H})$, if $A$ is an atom primary items of $(Q, \mathcal{H})$, then $|A| \geq c(Q)$.

The following theorem gives a complement for Example 6.
**Theorem 47.** Let \((Q, \mathcal{H})\) be a knowledge space, and let \(D\) be a dense subset of \((Q, \tau_{\mathcal{H}})\). If \((Q, \tau_{\mathcal{H}})\) is Hausdorff and \(\overline{H} \cap D = \overline{H}\) for each \(H \in \mathcal{H}\), then \(|Q| \leq 2^{2^{\|D\|}}\).

**Proof.** For every \(t \in Q\), the family \(\mathcal{N}(t) = \{U \cap A : U \in \mathcal{H}_t\}\) of subsets of \(D\). Since \((Q, \tau_{\mathcal{H}})\) is Hausdorff and \(\overline{L} \cap D = \overline{L}\) for any \(L \in \mathcal{H}\), we conclude that the intersection of the closures of all members of \(\mathcal{N}(t)\) equals \(\{t\}\); thus \(\mathcal{N}(t) \neq \mathcal{N}(g)\) for \(g \neq q\). Clearly, the number of all distinct families \(\mathcal{N}(t)\) is not larger than \(2^{2^{\|D\|}}\), hence \(|Q| \leq 2^{2^{\|D\|}}\). \(\square\)

**Remark 13.** The Hausdorff pre-topological space \((Z, \tau)\) in Example 4, which has an atom primary items \(D = \{z_1, z_2, z_3\}\) such that \(|Z| \leq 3^2\) and \(|D| = 3 < 4\). However, \[
\{z_2\} = \{z_2\} = \{z_2, z_4\} \neq \{z_2, z_3\} = \{z_2, z_4\}.
\]
Hence the condition \(\overline{\overline{H}} \cap D = \overline{H}\) for each \(H \in \mathcal{H}\) is sufficient and not necessary.

**Lemma 6.** Let \((Q, \mathcal{H})\) be a finite knowledge space, and let \(\mathcal{H}'\) be a subfamily of \(\mathcal{H}\). Put \(m = \max\{|\mathcal{H}'_q| : q \in Q\}\). For any distinct \(r, t \in Q\), if \(|\mathcal{H}'_r| = |\mathcal{H}'_t| = m\), then \(\bigcap \mathcal{H}'_r = \bigcap \mathcal{H}'_t\) or \((\bigcap \mathcal{H}'_r) \cap (\bigcap \mathcal{H}'_t) = \emptyset\).

**Proof.** Assume that \(\bigcap \mathcal{H}'_r \neq \bigcap \mathcal{H}'_t\). Then, without loss of generality, we assume that there is \(H \in \mathcal{H}'_r\) with \(r \notin H\). Now assume that \((\bigcap \mathcal{H}'_r) \cap (\bigcap \mathcal{H}'_t) \neq \emptyset\). Then there is \(q \in (\bigcap \mathcal{H}'_r) \cap (\bigcap \mathcal{H}'_t)\), hence \(q \in H\) and \(q \in L\) for any \(L \in \mathcal{H}'_r\), which implies that \(\mathcal{H}'_q = m + 1\), this is a contradiction. Therefore, \((\bigcap \mathcal{H}'_r) \cap (\bigcap \mathcal{H}'_t) = \emptyset\). \(\square\)

Now we give a method to construct a primary items for a finite pre-topological space as follows.

Let \(\mathcal{B}\) be an atom pre-base for a finite pre-topological space \((Q, \mathcal{H})\). Now we define a dense subset \(A\) of \((Q, \tau_{\mathcal{H}})\), by induction, as follows. First, let \(\mathcal{B}(1) = \mathcal{B}\), and take an arbitrary \(p_1 \in Q\) such that \(\mathcal{B}(1)_{p_1} = \max\{|\mathcal{B}(1)_q| : q \in Q\}\). Put \(A_1 = \{p_1\}\). If \(\mathcal{B} \setminus \mathcal{B}(1)_{p_1} = \emptyset\), then we stop our induction, and put \(A = A_1\). If \(\mathcal{B} \setminus \mathcal{B}(1)_{p_1} \neq \emptyset\), then put \(\mathcal{B}(2) = \mathcal{B} \setminus \mathcal{B}(1)_{p_1}\). Then take an arbitrary \(p_2 \in Q\) such that \(\mathcal{B}(2)_{p_2} = \max\{|\mathcal{B}(2)_q| : q \in Q\}\). Put \(A_2 = A_1 \cup \{p_2\}\). Assume that we have define a subset
$A_k = \{p_i : i \leq k\} (k \geq 2)$ such that $|\mathcal{R}(i)_{p_i}| = \max\{|\mathcal{R}(i)_{q_i}| : q \in Q\}$ for any $1 \leq i \leq k$, where $\mathcal{R}(i) = \mathcal{R} \setminus \bigcup_{j<i} \mathcal{R}(j)_{p_j}$. If $\mathcal{R} \setminus \bigcup_{i \leq k} \mathcal{R}(i)_{p_i} = \emptyset$, then we stop our induction, and put $A = A_k$; otherwise, let $\mathcal{R}(k+1) = \mathcal{R} \setminus \bigcup_{i \leq k} \mathcal{R}(i)_{p_i}$. Then take an arbitrary $p_{k+1} \in Q$ such that $|\mathcal{R}(k+1)_{p_{k+1}}| = \max\{|\mathcal{R}(k)_{q_i}| : q \in Q\}$. Then put $A_{k+1} = A_k \cup \{p_{k+1}\}$. Since $Q$ is finite, there exists a minimum $n \in \mathbb{N}$ such that $\mathcal{R} \setminus \bigcup_{i \leq n} \mathcal{R}(i)_{p_i} = \emptyset$. Moreover, from Lemma 6, it follows that, for any $j \leq n$, if $|\mathcal{R}(j)| = 1$ we may assume that $p_j \notin \bigcup(\bigcup_{i<j} \mathcal{R}(i)_{p_i})$. Then we put $A = A_n$. Clearly, $A$ is dense in $(Q, \mathcal{H})$.

Generally, $A$ is not an atom primary items of $(Q, \mathcal{H})$. We must delete some surplus points of $A$ such that it is an atom primary items of $(Q, \mathcal{H})$.

If $B \cap (A \{p_1\}) \neq \emptyset$ for any $B \in \mathcal{R}(1)_{p_1}$, then put $B_1 = A \{p_1\}$; otherwise, put $B_1 = A$. If $p_1 \in B_1$ and $B \cap (B_1 \{p_2\}) \neq \emptyset$ for any $B \in \mathcal{R}(2)_{p_2}$, then put $B_2 = B_1 \{p_2\}$; if $p_1 \notin B_1$, $B \cap (B_1 \{p_2\}) = \emptyset$ for some $B \in \mathcal{R}(2)_{p_2}$, then put $B_2 = B_1$; if $p_1 \notin B_1$, $B \cap (B_1 \{p_2\}) \neq \emptyset$ for any $B \in \mathcal{R}(2)_{p_2}$, and $O \cap (B_1 \{p_2\}) \neq \emptyset$ whenever $p_2 \in O \in \mathcal{R}(1)_{p_1}$, then put $B_2 = B_1 \{p_2\}$; otherwise, put $B_2 = B_1$. By induction, we can obtain the subsets $B_1, \ldots, B_n$ of $A$ such that $B_1 \supset \ldots \supset B_n$ and $B_n$ is dense in $(Q, \mathcal{H})$. Clearly, we have $p_n \in B_n$. We claim that $D = B_n$ is an atom primary items of $(Q, \mathcal{H})$, see the following theorem.

**Theorem 48.** The set $D$ is dense in $(Q, \tau_{\mathcal{H}})$ and $|D| = d(Q)$, that is, $D$ is an atom primary items of $(Q, \mathcal{H})$.

**Proof.** From our construction of $D$ above, it follows that $D$ is dense in $(Q, \tau_{\mathcal{H}})$. Hence it only need to prove that $\overline{C} \neq Q$ for each subset $C$ of $Q$ with $|C| < |D|$. Indeed, we shall prove by induction on the cardinality of atom pre-base $\mathcal{R}$.

Clearly, if $|\mathcal{R}| = 1$, then $|D| = 1$, hence $\overline{C} = \emptyset$. If $|\mathcal{R}| = 2$, it suffice to verify the case of $D = \{p_1, p_2\}$. Let $C = \{q_0\}$. By our construction, if $C$ is dense in $(Q, \tau_{\mathcal{H}})$, then $|\mathcal{R}_{q_0}| > |\mathcal{R}_{p_1}|$, which is a contradiction. Assume that $\overline{C} \neq Q$ for any subset $C$ of $Q$ with $|C| < |D|$ when $|\mathcal{R}| \leq k$. Now assume that $|\mathcal{R}| = k + 1$. From our construction,
we conclude that $|A \cap B| = 1$ for each $B \in \mathcal{B}(n)_{p_n}$. We divide the rest proof into the following two cases.

**Case 1:** $|\mathcal{B}(n)_{p_n}| \geq 2$.

Let $B' = \bigcup \mathcal{B}(n)_{p_n}$, and put $\mathcal{B} = (\mathcal{B} \setminus \mathcal{B}(n)_{p_n}) \cup \{B'\}$. Let $(Q, \mathcal{F})$ be the knowledge space generated by $\mathcal{B}$. Clearly, $\tau_{\mathcal{F}}$ is coarser than $\tau_{\mathcal{H}}$, hence $D$ is dense in $\tau_{\mathcal{F}}$. Since $|\mathcal{B}'| \leq k$, from our assumption it follows that $\mathcal{F}_{\mathcal{F}} \neq Q$ for each subset $F$ of $Q$ with $|F| < |D|$, hence $\mathcal{F}_{\mathcal{F}} \neq Q$ for each subset $F$ of $Q$ with $|F| < |D|$.

**Case 2:** $|\mathcal{B}(n)_{p_n}| = 1$.

If $|\mathcal{B}(i)_{p_i}| = 1$ for each $i \leq n$, then it is obvious. Otherwise, there exists a maximal $m \leq n$ such that $|\mathcal{B}(i)_{p_i}| = 1$ for any $i > m$ and $|\mathcal{B}(m)_{p_m}| \geq 2$. Then put $B'' = \bigcup \mathcal{B}(m)_{p_m}$. Let $(Q, \mathcal{L})$ be the knowledge space which is generated by the family $\mathcal{B}'' = (\mathcal{B} \setminus \mathcal{B}(m)_{p_m}) \cup \{B''\}$. Then $\tau_{\mathcal{L}}$ is coarser than $\tau_{\mathcal{H}}$, hence $D$ is dense in $(Q, \tau_{\mathcal{L}})$. Since $|\mathcal{B}''| \leq k$, from our assumption it follows that $\mathcal{F}_{\mathcal{L}} \neq Q$ for each subset $F$ of $Q$ with $|F| < |D|$, hence $\mathcal{F}_{\mathcal{L}} \neq Q$ for each subset $F$ of $Q$ with $|F| < |D|$.

Finally we give an algorithm for constructing the set of atom primary items for a finite knowledge spaces.

**Sketch of Algorithm.** Let $Q = \{q_1, \ldots, q_m\}$, and list the atom pre-base $B$ as $B_1, \ldots, B_n$. Form an $m \times n$ array $T = (T_{ij})$ with the rows and columns representing the atom pre-base $B$ and the elements of $Q$ respectively; thus, both the rows and the columns are indexed from 1 to $m$ and 1 to $n$ respectively. Initially, set $T_{ij}$ to 1 if $q_i \in B_j$; otherwise, set $T_{ij}$ to 0.

1. First, take an arbitrary $i_1 \in \{1, \ldots, m\}$ such that $\sum_{j=1}^{n} T_{i_1j} = \max\{\sum_{j=1}^{n} T_{ij} : i \in \{1, \ldots, m\}\}$, then swap the $i_1$-th row of the matrix with the first row, and swap all the columns $N_1 = \{j : T_{i_1j} = 1, j = 1, \ldots, n\}$ with the first $n_1$ columns, where $n_1 = |N_1|$. If $n_1 = n$, then we terminates this step, and put $A = \{p_{i_1}\}$; otherwise, denote the new matrix by $(T^{(1)}_{ij})_{m \times n}$.
(2) For any \( k \geq 2 \), take an arbitrary \( i_k \in \{ k, \ldots, m \} \) such that
\[
\sum_{j=(\sum_{j=1}^{k-1} n_j)+1}^{n} T_{ikj}^{(k-1)} = \max \left\{ \sum_{j=(\sum_{j=1}^{k-1} n_j)+1}^{n} T_{ij}^{(k-1)} : i \in \{ k, \ldots, m \} \right\},
\]
then swap the \( i_k \)-th row of the matrix with the \( k \)-th row, and swap all the columns
\( N_k = \{ j : T_{ikj} = 1, j = (\sum_{j=1}^{k-1} n_j) + 1, \ldots, n \} \) with the \( j = (\sum_{j=1}^{k-1} n_j) + 1 \)-th column to
\( j = (\sum_{j=1}^{k} n_j) \)-th column, where \( n_k = |N_k| \). If \( n_k = 1 \), we may require the \( i_k \) satisfying
that \( T_{ikj}^{(k)} = 0 \) for any \( j \leq \sum_{j=1}^{k-1} n_j \). If \( \sum_{j=1}^{k} n_j = n \), then we terminates this step, and
put \( A = \{ p_{i_1}, \ldots, p_{i_k} \} \); otherwise, denote the new matrix by \( (T_{ij}^{(k)})_{m \times n} \).

(3) The set \( A = \{ p_{i_1}, \ldots, p_{i_N} \} \) and the sub-matrix \( (T_{ij}^{(N)})_{N \times M} \) of \( (T_{ij}^{(N)})_{m \times n} \) obtained
after step \( N \) are the desired set and the desired matrix respectively, where \( M = \sum_{i=1}^{N} n_i \).

(4) Then initialize \( D \) to \( A \) again.

(5) For each \( l \leq N \), check whether, there exists \( \sum_{k=1}^{l-1} n_k < j \leq \sum_{k=1}^{l} n_k \) such that
\( T_{ij}^{(N)} = 0 \) for any \( i > l \); if the condition holds, the set \( D \) is invariant. If the condition does
not hold, check whether there exists \( t < l \) such that \( p_{i_t} \not\in D \) and \( \sum_{j=a_{t-1}+1}^{a_t} T_{ij}^{(N)} = 1 \),
where \( a_t = \sum_{i=1}^{t} n_i \) and \( n_0 = 0 \); otherwise, delete \( p_{i_t} \) from \( D \). (This terminates step \( N \.)

The set \( D \) obtained after step \( N \) is the set of atom primary items for a finite knowledge
spaces \((Q, \mathcal{K})\).

**Example 12.** Let a knowledge space \((Q, \mathcal{K})\) have an atom pre-base
\[
\{\{ z_1 \}, \{ z_2 \}, \{ z_1, z_3 \}, \{ z_2, z_3, z_4 \}, \{ z_1, z_3, z_4, z_5 \}\},
\]
where \( Q = \{ z_1, z_2, z_3, z_4, z_5 \} \). Then

\[
\begin{array}{cccccc}
\{ \} & \{ z_1 \} & \{ z_2 \} & \{ z_1, z_3 \} & \{ z_2, z_3, z_4 \} & \{ z_1, z_3, z_4, z_5 \} \\
\hline
z_1 & 1 & 0 & 1 & 0 & 1 \\
z_2 & 0 & 1 & 0 & 1 & 0 \\
z_3 & 0 & 0 & 1 & 1 & 0 \\
z_4 & 0 & 0 & 0 & 1 & 0 \\
z_5 & 0 & 0 & 0 & 0 & 1 \\
\end{array}
\]

\[
\begin{array}{cccccc}
\{ \} & \{ z_1, z_3 \} & \{ z_2, z_3, z_4 \} & \{ z_1, z_3, z_4, z_5 \} & \{ z_1 \} & \{ z_2 \} \\
\hline
z_3 & 1 & 1 & 1 & 0 & 0 \\
z_2 & 1 & 0 & 1 & 1 & 0 \\
z_1 & 0 & 1 & 0 & 0 & 1 \\
z_4 & 0 & 1 & 1 & 0 & 0 \\
z_5 & 0 & 0 & 1 & 0 & 0 \\
\end{array}
\]
Since $T_{2j} + T_{3j} = 1$ for each $j = 1, 2, 3$, it follows that the set of atom primary items of $(Q, \mathcal{H})$ is $D = \{z_1, z_2\}$.

3.4. The applications of connectedness and pre-quotient mapping in knowledge spaces.

In this subsection, we discuss some applications of connectedness and pre-quotient mapping in knowledge spaces. First, we have the following proposition.

Proposition 30. If $(Q, \mathcal{H})$ is a knowledge space, then $(Q, \tau_{\mathcal{H}})$ is connected iff $\mathcal{H} \cap \overline{\mathcal{H}} = \{\emptyset, X\}$, where $\overline{\mathcal{H}} = \{K : Q \setminus K \in \mathcal{H}\}$.

A knowledge space $(Q, \mathcal{H})$ is called granular if for each state $L \in \mathcal{H}$ and $t \in Q$, there exists an atom $H$ at $t$ with $H \subset L$. By Theorem 37, the following proposition holds.

Proposition 31. If $(Q, \mathcal{H})$ is a granular knowledge space and $(Q, \tau_{\mathcal{H}})$ is regular, then both $H$ and $Q \setminus H$ belong to $\mathcal{H}$ for each atom $H$. Thus $(Q, \tau_{\mathcal{H}})$ is not connected.

Assume that $\sigma$ is a function mapping from a non-empty set $Q$ into $2^{2^Q}$. We say that $\sigma$ is an attribution if $\sigma(t) \neq \emptyset$ for every $t \in Q$. The following theorem is obvious.

Theorem 49. Let $(Q, \mathcal{H})$ be a knowledge space which is derived from the attribution $\sigma$ on $Q$. If there exist a proper subset $Q'$ of $Q$ and $C_t \in \sigma(t)$ for each $t \in Q$ such that $\bigcup_{t \in Q'} C_t \subset Q'$ and $\bigcup_{t \in Q \setminus Q'} C_t \subset Q \setminus Q'$, then $(Q, \tau_{\mathcal{H}})$ is not connected.

Theorem 50. Let $(Q, \mathcal{H})$ be a knowledge space. Then the mapping $h : (Q, \mathcal{H}) \rightarrow (Q', \mathcal{H}')$ defined by $h(t) = t'$ for each $t \in Q$ is a pre-quotient mapping, that is, the discriminative reduction $(Q', \mathcal{H}')$ is a pre-quotient pre-topology of $(Q, \mathcal{H})$. 
Proof. For each $K^* \in \mathcal{H}^*$, we have $h^{-1}(K^*) = K$. Moreover, let $h^{-1}(A)$ is open $(Q, \mathcal{H})$. Then there is a subfamily $\mathcal{H} \subseteq \mathcal{H}$ with $h^{-1}(A) = \bigcup \mathcal{H}$, hence $A = \bigcup_{H \in \mathcal{H}} h(H) = \bigcup_{H \in \mathcal{H}} H^*$ is open in $(Q^*, \mathcal{H}^*)$. Therefore, $h$ a pre-quotient mapping. \qed

4. Conclusions

The theory of knowledge spaces (KST) was originally regarded as a mathematical framework for knowledge training and assessment. In fact, the concept of a knowledge spaces is a generalization of topological spaces. Indeed, it is equivalent to the notion of pre-topological spaces in the present paper. Therefore, the language of topology and its established generalizations are very useful for studying in the theory of knowledge spaces.

For example, Falmagne’s discriminative resemble the axiom of separation $T_0$ for topologies. His notion of quasi ordinal space is just the well-known Alexandroff space in topology.

In order to study the theory of knowledge spaces, we systematically introduce the theory of pre-topological spaces, such as, pre-base, subspace, axioms of separation, connectedness, etc. Then we discuss the applications of axioms of separation for knowledge spaces. For a bi-discriminative knowledge space, we prove that the necessary and sufficient condition of an item belongs to some inner fringe or outer fringe of some knowledge state. Moreover, we give a characterization of a skill multimap such that answer to a problem in $[14]$ or $[18]$ whenever each item with finitely many competencies. Further, we discuss the relation of Alexandroff spaces and quasi ordinal spaces. Indeed, the concept of Alexandroff spaces in topology is equivalent to the quasi ordinal spaces in knowledge spaces. We prove that each bi-discriminative quasi ordinal space $(Q, \mathcal{H})$ has $\mathcal{H} = 2^Q$, and give a characterization of quasi ordinal spaces that are connected. Finally, we study the roles of the density of pre-topological spaces in primary items for knowledge spaces. We define the concept of primary items in knowledge spaces, that is, one person in order to master an item, she or he must master items. We give an algorithm to find the set of atom primary items for any finite knowledge space.
Our paper only takes a first step to study the language of pre-topology in the theory of knowledge spaces. The theoretical framework introduced here and, in particular, the method of studying the theory of knowledge spaces from the topological views are expected to prove useful for future work in this direction.

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