GEOMETRIC REALIZATIONS OF KAHLER AND OF PARA-KAHLER CURVATURE MODELS

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Abstract. We show that every Kaehler algebraic curvature tensor is geometrically realizable by a Kaehler manifold of constant scalar curvature. We also show that every para-Kaehler algebraic curvature tensor is geometrically realizable by a para-Kaehler manifold of constant scalar curvature.

MSC: 53B20

1. Introduction

Curvature is a central aspect of modern differential geometry. One can relate curvature to the underlying geometry and topology of a manifold [3, 11, 23], examine analytic properties which are influenced by curvature [20], and study purely algebraic properties of curvature [13, 15, 21]. The study of Hermitian and Kaehler geometry is an active current field of investigation that continues to be important both in pure mathematics and in mathematical physics [4, 8, 25]. Furthermore, the study of para-Kaehler geometry, which is a neutral signature analogue, also is an active research area [9, 10, 12].

In this paper, we extend previous investigations [5, 7, 16, 17, 18] to discuss questions of geometric realizability — when can an algebraic curvature tensor, which is a purely algebraic object, be realized by a Riemannian manifold in suitable contexts and what are the resulting geometric constraints, if any. We will focus our attention on the Kaehler and para-Kaehler settings and give necessary and sufficient linear conditions on the curvature to ensure that a given curvature model is geometrically realizable by a Kaehler manifold or by a para-Kaehler manifold.

Imposing the condition that the manifold in question has constant scalar curvature yields no additional restrictions.

1.1. Hermitian manifolds. We begin with a brief review of previously known results. Let \( M := (M, g) \) be a Riemannian manifold of dimension \( m \); we shall always assume that \( m = 2\bar{m} \) is even and that \( m \geq 4 \). Let \( R \) be the curvature tensor of \( M \); \( R \) satisfies the following identity for all tangent vectors \( x, y, z, w \):

\[
R(x, y, z, w) = -R(y, x, z, w) = R(z, w, x, y),
0 = R(x, y, z, w) + R(y, z, x, w) + R(z, x, y, w).
\] (1.a)

Suppose there exists an almost complex structure \( J \) on \( M \), i.e. an endomorphism \( J \) of the tangent bundle \( TM \) so that \( J^2 = -\mathrm{id} \). We also assume that \( J \) is Hermitian, i.e. that \( J^*g = g \); in this setting, the triple \( C := (M, g, J) \) is said to be an almost Hermitian manifold. An almost Hermitian manifold \( C \) is said to be a Hermitian manifold if \( J \) is integrable; this means that the Nijenhuis tensor

\[
N_J(x, y) := [x, y] + J[x, y] + J[x, Jy] - [Jx, Jy]
\]

vanishes or, equivalently, by the Newlander–Nirenberg Theorem [22] that every point of \( M \) has a neighborhood with local coordinates \( (x_1, y_1, ..., x_{\bar{m}}, y_{\bar{m}}) \) so that

\[
J\partial_{x_i} = \partial_{y_i} \quad \text{and} \quad J\partial_{y_i} = -\partial_{x_i}.
\]
If \( C \) is a Hermitian manifold, then the Riemann curvature tensor satisfies an extra identity discovered by Gray [19]:

**Theorem 1.1.** If \( C \) is a Hermitian manifold, then

\[
0 = R(x, y, z, w) + R(Jx, Jy, Jz, Jw) - R(Jx, Jy, Jz, w) - R(x, Jy, Jz, w) - R(x, y, Jz, Jw)
\]

\( (1.b) \)

**Remark 1.2.** Theorem 1.1 shows that the integrability of the almost complex structure implies a relation in the curvature. Let \( A \) be a real vector space of dimension \( m \) which is equipped with a positive definite inner product \( \langle \cdot, \cdot \rangle \). We say that \( A \in \otimes^3 V^* \) is an algebraic curvature tensor if \( A \) has the symmetries of Equation (1.a); let \( \mathfrak{A} = \mathfrak{A}(V) \) be the subspace of all such tensors. Fix \( A \in \mathfrak{A} \) and let \( \mathfrak{M} := (V, \langle \cdot, \cdot \rangle, J, A) \) be the associated curvature model. We say \( \mathfrak{C} := (V, \langle \cdot, \cdot \rangle, J, A) \) is an almost Hermitian curvature model if \( J \) is a linear map of \( V \) with \( J^2 = -\mathrm{id} \) and \( J^* \langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle \). Let \( \{e_i\} \) be an orthonormal basis for \( (V, \langle \cdot, \cdot \rangle) \). We adopt the Einstein convention and sum over repeated indices and define the Ricci tensor \( \rho \), the \( * \)-Ricci tensor \( \rho^* \), the scalar curvature \( \tau \), and the \( * \)-scalar curvature \( \tau^* \) by contracting indices:

\[
\rho(x, y) := A(x, e_i, e_i, y), \quad \tau := A(e_i, e_j, e_j, e_i),
\]

\[
\rho^*(x, y) := A(x, e_i, Je_i, Jy), \quad \tau^* := A(e_i, e_j, Je_j, Je_i).
\]

If in addition the Gray identity given in Equation (1.5) is satisfied, then \( \mathfrak{C} \) is said to be a Hermitian curvature model. We say that \( \mathfrak{C} \) is geometrically realizable by an almost Hermitian manifold \( \mathfrak{C} = (M, g, J) \) if there exists a point \( P \) of \( M \) and an isomorphism \( \phi : T_PM \to V \) so that \( \phi^* \langle \cdot, \cdot \rangle = g_P, \phi^* J = J_P \), and \( \phi^* A = R_P \). We refer to [5] for the proof of the following result which provides a converse to Theorem 1.1.

**Theorem 1.3.**

1. If \( \mathfrak{C} \) is an almost Hermitian curvature model, then \( \mathfrak{C} \) is geometrically realizable by an almost Hermitian manifold with constant scalar curvature and with constant \( * \)-scalar curvature.

2. If \( \mathfrak{C} \) is a Hermitian curvature model, then \( \mathfrak{C} \) is geometrically realizable by a Hermitian manifold with constant scalar curvature and with constant \( * \)-scalar curvature.

**Remark 1.4.** Theorem 1.3 shows that the existence of an almost Hermitian structure imposes no additional relations on the curvature other than those generated by Equation (1.a). Furthermore, Equations (1.a) and (1.b) generate the universal symmetries of the curvature of a Hermitian manifold. Finally, assuming that the scalar curvature and \( * \)-scalar curvature are constant imposes no additional relations.
1.3. Kaehler geometry. The Kaehler form of an almost Hermitian manifold \( \mathcal{C} \) is defined by setting \( \Omega(x, y) := g(x, Jy) \). One says that \( \mathcal{C} \) is a Kaehler manifold if \( J \) is integrable and \( d\Omega = 0 \) or, equivalently, if \( \nabla J = 0 \). One has:

**Theorem 1.5.** If \( \mathcal{C} \) is a Kaehler manifold, then:

\[
R(x, y, z, w) = R(Jx, Jy, z, w) \quad \forall \quad x, y, z, w.
\]

An almost Hermitian curvature model \( \mathcal{C} = (V, \langle \cdot, \cdot \rangle, J, A) \) is said to be a Kaehler curvature model if Equation (1.c) is satisfied; this necessarily implies Equation (1.b) is satisfied so any Kaehler curvature model is a Hermitian curvature model. In Section 2, we shall establish the following result which shows that Equations (1.a) and (1.c) generate the universal symmetries of the curvature tensor of a Kaehler manifold and which is a converse to Theorem 1.5. In this setting, necessarily \( \tau = \tau^* \).

**Theorem 1.6.** Let \( \mathcal{C} \) be a Kaehler curvature model. Then \( \mathcal{C} \) is geometrically realizable by a Kaehler manifold of constant scalar curvature.

**Remark 1.7.** The methods used in [7] can be used to extend Theorem 1.6 to the indefinite setting; we omit details in the interests of brevity.

**Remark 1.8.** Theorems 1.1 and 1.3 show that Equation (1.b) provides necessary and sufficient linear identities for a Hermitian curvature model to be geometrically realizable by a Hermitian manifold. Similarly, Theorems 1.5 and 1.6 show that Equation (1.c) provides necessary and sufficient linear identities for a curvature model to be geometrically realizable by a Kaehler manifold. There are examples where one has relations rather than identities. For example, one says that an almost Hermitian manifold is almost Kaehler if \( d\Omega = 0 \). In this setting, we have \( \tau^* - \tau = \frac{1}{2} |\nabla J|^2 \) and thus the curvature lies in the half-space defined by the relation \( \tau^* \geq \tau \). This shows that an almost Hermitian curvature model with \( \tau > \tau^* \) is not geometrically realizable by an almost Kaehler manifold. We refer to [1, 12] for further details concerning almost Kaehler manifolds in both the Riemannian and the higher signature settings.

1.4. Para-Hermitian manifolds. We shall say that the triple \( \tilde{\mathcal{C}} := (M, g, \tilde{J}) \) is an almost para-Hermitian manifold if \( g \) is a pseudo-Riemannian metric on \( M \) of neutral signature \((\bar{m}, \bar{m})\) and if \( \tilde{J} \) is a linear map of \( TM \) satisfying \( \tilde{J}^2 = \text{id} \) and \( \tilde{J}^* g = -g \).

If \( \tilde{\mathcal{C}} \) is an almost para-Hermitian manifold, then one says that \( \tilde{J} \) is integrable if the para-Nijenhuis tensor

\[
N_j(x, y) := [x, y] - \tilde{J}[\tilde{J}x, y] - \tilde{J}[x, \tilde{J}y] + [\tilde{J}x, \tilde{J}y]
\]

vanishes or, equivalently, every point of \( M \) has a neighborhood with local coordinates \((x_1, y_1, ..., x_{\bar{m}}, y_{\bar{m}})\) so that

\[
\tilde{J}\partial_{x_i} = \partial_{y_i} \quad \text{and} \quad \tilde{J}\partial_{y_i} = \partial_{x_i}.
\]

If \( \tilde{J} \) is integrable then \( \tilde{\mathcal{C}} \) is called a para-Hermitian manifold. Theorem 1.1 generalizes to this setting [7] to become the following result – note the changes in sign from Equation (1.b):

**Theorem 1.9.** If \( \tilde{\mathcal{C}} \) is a para-Hermitian manifold, then

\[
0 = R(x, y, z, w) + R(Jx, Jy, Jz, Jw) + R(Jx, Jy, z, w) + R(Jx, Jy, Jz, w) + R(x, Jy, Jz, w) + R(x, Jy, z, w) + R(x, y, Jz, Jw) + R(x, y, Jz, w) \quad \forall \quad x, y, z, w.
\]
1.5. Para-Hermitian curvature models. One defines the notion of an almost para-Hermitian curvature model $\tilde{\mathcal{C}} := (V, \langle \cdot, \cdot \rangle, \tilde{J}, A)$ similarly; if $\tilde{\mathcal{C}}$ satisfies the relations of Equation (1.4), then $\tilde{\mathcal{C}}$ is said to be a para-Hermitian curvature model. Theorem 1.3 extends to this setting [5, 7]:

**Theorem 1.10.**

1. Let $\tilde{\mathcal{C}}$ be an almost para-Hermitian curvature model. Then $\tilde{\mathcal{C}}$ is geometrically realizable by an almost para-Hermitian manifold with constant scalar curvature and with constant $\star$-scalar curvature.

2. Let $\tilde{\mathcal{C}}$ be a para-Hermitian curvature model. Then $\mathcal{C}$ is geometrically realizable by a para-Hermitian manifold with constant scalar curvature and with constant $\star$-scalar curvature.

1.6. Para-Kaehler geometry. One defines the para-Kaehler form of an almost para-Hermitian manifold $\tilde{\mathcal{C}} = (M, g, \tilde{J})$ by setting $\tilde{\Omega}(x, y) := g(\tilde{J}x, \tilde{J}y)$. We say that $\tilde{\mathcal{C}}$ is para-Kaehler if $\tilde{J}$ is integrable and $d\tilde{\Omega} = 0$ or, equivalently, if $\nabla\tilde{J} = 0$. In this setting one has (note the change in sign from Equation (1.4)):

**Theorem 1.11.** If $\tilde{\mathcal{C}}$ is a para-Kaehler manifold, then:

$$R(x, y, z, w) = -R(\tilde{J}x, \tilde{J}y, z, w) \quad \forall \ x, y, z, w.$$  \hfill (1.e)

We say an almost para-Hermitian curvature model $\tilde{\mathcal{C}}$ is a para-Kaehler curvature model if the relations of Equation (1.4) hold; this implies the relations of Equation (1.3) hold and thus $\tilde{\mathcal{C}}$ is also a para-Hermitian curvature model. Theorem 1.6 generalizes to this setting to become:

**Theorem 1.12.** If $\tilde{\mathcal{C}}$ is a para-Kaehler curvature model, then $\tilde{\mathcal{C}}$ is geometrically realizable by a para-Kaehler manifold of constant scalar curvature.

1.7. Outline of the paper. We show in Section 2 (resp. in Section 3) that any Kaehler (resp. para-Kaehler) curvature model can be geometrically realized by a Kaehler (resp. para-Kaehler) manifold. In Section 4 we show the realizations can be chosen to have constant scalar curvature.

The decomposition of the space of algebraic curvature tensors under the action of the unitary group was given by Tricerri and Vanhecke [24] and is summarized in Theorem 2.1; it plays a central role in the analysis of Section 2 – a similar analysis in the para-Kaehler setting is performed in Section 3. The Cauchy-Kovalevskaya Theorem is used in Section 4 to show that the Kaehler and para-Kaehler realizations in question can be chosen to have constant scalar curvature.

## 2. Kaehler Geometry

### 2.1. Curvature decomposition under the unitary group.**

Let $(V, \langle \cdot, \cdot \rangle, J)$ be a Hermitian structure. Let $\{e_i\}$ be an orthonormal basis for $(V, \langle \cdot, \cdot \rangle)$. Let $U$ be the unitary group:

$$U := \{ T \in \text{GL}(V) : TJ = JT \quad \text{and} \quad T^* \langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle \}.$$  

Define elements $\pi_1$ and $\pi_2$ of $\mathbb{A}$ by setting:

$$\pi_1(x, y, z, w) := \langle x, w \rangle \langle y, z \rangle - \langle x, z \rangle \langle y, w \rangle,$$

$$\pi_2(x, y, z, w) := \langle Jx, w \rangle \langle Jy, z \rangle - \langle Jx, z \rangle \langle Jy, w \rangle - 2 \langle Jx, y \rangle \langle Jz, w \rangle.$$
Let $S^2(V^*)$ denote the set of symmetric 2-tensors and let $\theta \in S^2(V^*)$ with $J^*\theta = \theta$. Define elements $\phi(\theta)$ and $\psi(\theta)$ of $\mathfrak{A}(V)$ by setting:

$$
\phi(\theta)(x, y, z, w) := \langle x, w \rangle \theta(y, z) - \langle x, z \rangle \theta(y, w) + \theta(x, w)\langle y, z \rangle - \theta(x, z)\langle y, w \rangle,
$$

$$
\psi(\theta)(x, y, z, w) := \langle Jx, w \rangle \theta(Jy, z) - \langle Jx, z \rangle \theta(Jy, w) - 2\langle Jx, y \rangle \theta(Jz, w) + \theta(Jx, w)\langle Jy, z \rangle - \theta(Jx, z)\langle Jy, w \rangle.
$$

The following result is due to Tricerri and Vanhecke [24] in the Riemannian setting – the extension to the higher signature context is not difficult [7].

**Theorem 2.1.** Let $(V, \langle \cdot, \cdot \rangle, J)$ be a Hermitian structure.

1. We have the following orthogonal direct sum decomposition of $\mathfrak{A}$ into irreducible $\mathcal{U}$ modules:
   - (a) If $m = 4$, $\mathfrak{A} = W_1 \oplus W_2 \oplus W_3 \oplus W_4 \oplus W_7 \oplus W_9$.
   - (b) If $m = 6$, $\mathfrak{A} = W_1 \oplus W_2 \oplus W_3 \oplus W_4 \oplus W_5 \oplus W_7 \oplus W_8 \oplus W_9 \oplus W_{10}$.
   - (c) If $m \geq 8$, $\mathfrak{A} = W_1 \oplus W_2 \oplus W_3 \oplus W_4 \oplus W_5 \oplus W_6 \oplus W_7 \oplus W_8 \oplus W_9 \oplus W_{10}$.

   We have $W_1 \approx W_4$ and, if $m \geq 6$, $W_2 \approx W_5$. The other $\mathcal{U}$ modules appear with multiplicity 1.

2. $\mathcal{C} = (V, \langle \cdot, \cdot \rangle, J, A)$ is a Kaehler model if and only if $A \in W_1 \oplus W_2 \oplus W_3$.

3. If $A \in W_1 \oplus W_2 \oplus W_3$, then the projections $p_i$ of $A$ on $W_i$ are given by:
   - (a) $p_1 A = \frac{1}{m(m+1)}(\pi_1 + \pi_2) \tau$.
   - (b) $p_2 A = \frac{1}{m+2}(\phi + \psi)(2p - \frac{1}{m} \tau \langle \cdot, \cdot \rangle)$.
   - (c) $p_3 A = A - p_1 A - p_2 A$.

2.2. Realizability of Kaehler curvature models. Let $\{u_1, \ldots, u_m\}$ be the canonical coordinates on $\mathbb{R}^m$. Set $\partial_i := \frac{\partial}{\partial u_i}$. Let $J$ be the canonical integrable almost complex structure on $\mathbb{R}^m$ given by:

$$
J\partial_{1+2k} = \partial_{2+2k} \quad \text{and} \quad J\partial_{2+2k} = -\partial_{1+2k} \quad \text{for} \quad 0 \leq k < m.
$$

Let $g_0$ be the usual flat Hermitian metric on $\mathbb{R}^m$:

$$
g_{0,ij} := \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise}. \end{cases}
$$

We identify $(V, \langle \cdot, \cdot \rangle, J)$ with $(\mathbb{R}^m, g_0, J)$. Let $S^2_+(V^*)$ be the set of all symmetric 2-tensors $\theta$ such that $J^*\theta = \theta$. If $\Theta \in S^2_+(V^*) \otimes S^2(V^*)$, then set:

$$
g_{\Theta,ij} := g_{0,ij} + \Theta_{ijkl} u^k u^l.
$$

This is positive definite at the origin and hence defines a Riemannian metric which is invariant under $J$ on some neighborhood of the origin. Let $L \Theta \in \mathfrak{A}(V)$ be the curvature of the metric $g_0$ at the origin:

$$
(L\Theta)(x, y, z, w) = \Theta(x, z, y, w) + \Theta(y, w, x, z) - \Theta(x, w, y, z) - \Theta(y, z, x, w).
$$

The map $\Theta \rightarrow d\Omega_{g_0}$ defines a linear map

$$
K_J : S^2_+(V^*) \otimes S^2(V^*) \rightarrow \Lambda^3(V^*) \otimes V^*
$$

which is given by

$$
\{(K_J\Theta)(x, y, z)\}(w) := \Theta(x, Jy, z, w) + \Theta(y, Jz, x, w) + \Theta(z, Jx, y, w).
$$

This shows that $\ker(K_J)$ is invariant under the action of $\mathcal{U}$. Clearly $\Theta \in \ker(K_J)$ if and only if $g_0$ is a Kaehler metric;

$$
L : \ker(K_J) \rightarrow W_1 \oplus W_2 \oplus W_3.
$$
is a linear map which is equivariant with respect to the action $\mathcal{U}$. To show every Kaehler curvature model is geometrically realizable by a Kaehler metric, it suffices to show that $\mathcal{L}$ is surjective. Take 

$$\Theta = \frac{1}{2}(e^1 \otimes e^1 + e^2 \otimes e^2) \otimes (e^1 \otimes e^1 + e^2 \otimes e^2)$$

so that the metric has the form

$$g_{\Theta} = (du_1^2 + ... + du_m^2) + \frac{1}{2}(u_1^2 + u_2^2)(du_1^2 + du_2^2).$$

The metric $g_{\Theta}$ is Kaehler since it takes the form $M_2 \times \mathbb{C}^{m-1}$ where $M_2$ is a Riemann surface. Thus $\Theta \in \ker(K_J)$. Furthermore, the only non-zero curvature components of the curvature tensor at the origin, up to the usual $\mathbb{Z}_2$ symmetries, are

$$R(e_1, e_2, e_2, e_1) = 1.$$ 

The non-zero components of $\rho$ are $\rho(e_1, e_1) = \rho(e_2, e_2) = 1$. We compute:

$$\pi_1(e_3, e_4, e_4, e_3) = 1, \quad \pi_2(e_3, e_4, e_4, e_3) = 3, \quad \tau = 2, \quad \rho_1(A(e_3, e_4, e_4, e_3)) = \frac{2}{m(m+1)}.$$ 

Thus the component of $A$ in $\mathcal{W}_1$ is non-zero. Similarly, we compute:

$$(\phi(\rho))(e_3, e_4, e_4, e_3) = 0, \quad (\psi(\rho))(e_3, e_4, e_4, e_3) = 0,$$

$$(\phi(\cdot, \cdot))(e_3, e_4, e_4, e_3) = 2, \quad (\psi(\cdot, \cdot))(e_3, e_4, e_4, e_3) = 6,$$

$$(p_2A)(e_3, e_4, e_4, e_3) = -\frac{1}{m(m+2)}.$$ 

This shows the component of $A$ in $\mathcal{W}_2$ is non-zero. We have

$$((\text{id} - p_1 - p_2)A)(e_3, e_4, e_4, e_3) = -\frac{2(m+2)-4(m+1)}{m(m+1)(m+2)} = -\frac{2}{(m+1)(m+2)}$$

and thus the component of $A$ in $\mathcal{W}_3$ is non-zero. Thus $A$ has non-zero components in all 3 factors. Since these 3 factors are not isomorphic unitary modules, we may conclude that $\mathcal{L}$ is in fact a surjective map from $\ker(K_J)$ to $\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3$ as desired. 

\[ \square \]

3. Para-Kaehler geometry

The proof that every para-Kaehler curvature model is geometrically realizable by a para-Kaehler manifold is essentially the same as the proof in the Kaehler setting given above in Section 2. Let $\tilde{J}$ be the canonical integrable almost para-complex structure on $\mathbb{R}^m$ given by 

$$\tilde{J} \partial_{1+2k} = \partial_{2+2k} \quad \text{and} \quad \tilde{J} \partial_{2+2k} = \partial_{1+2k} \text{ for } 0 \leq k < \tilde{m}.$$ 

Let $g_0$ be the canonical flat para-Hermitian metric on $\mathbb{R}^m$:

$$\tilde{g}_{0,ij} = \begin{cases} 
1 & \text{if } i = j \equiv 0 \mod 2, \\
-1 & \text{if } i = j \equiv 1 \mod 2, \\
0 & \text{otherwise}.
\end{cases}$$

Again, we identify $(V, \langle \cdot, \cdot \rangle, \tilde{J})$ with $(\mathbb{R}^m, \tilde{g}_0, \tilde{J})$. Let $S^2_2(V^*)$ be the space of symmetric 2-tensors $\theta$ so $\tilde{J}^* \theta = -\theta$. Given $\Theta$ in $S^2_2(V^*) \otimes S^2_2(V^*)$, we construct the para-Hermitian metric:

$$\tilde{g}_{\Theta,ij} := \tilde{g}_{0,ij} + \Theta_{ijkl} u^k \bar{u}^l.$$ 

We use Equation 2.2 to define $K_J : S^2_2(V^*) \otimes S^2_2(V^*) \rightarrow \Lambda^3(V^*) \otimes V^*$ by

$$\{(K_J \Theta)(x, y, z)\}(w) := \Theta(x, \tilde{J}y, z, w) + \Theta(y, \tilde{J}z, x, w) + \Theta(z, \tilde{J}x, y, w).$$ 

The curvature of $\tilde{g}_\Theta$ at the origin is given by

$$(\mathcal{L}\Theta)(x, y, z, w) := \Theta(x, z, y, w) + \Theta(y, w, x, z) - \Theta(x, w, y, z) - \Theta(y, z, x, w).$$
We may decompose $\mathfrak{g}$ as a direct sum of irreducible factors under the action of the para-unitary group. This decomposes the para-Kaehler tensors as a direct sum $\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3$. We show $\mathcal{L}$ is a surjective map from $\ker(K_J)$ to $\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3$ by taking
\[
\Theta = \frac{1}{2}(e^1 \otimes e^1 - e^2 \otimes e^2) \otimes (e^1 \otimes e^1 - e^2 \otimes e^2),
\]
\[
\tilde{g}_\Theta = \tilde{g}_0 + \frac{1}{2}(u^2_1 - u^2_2)(du^2_1 - du^2_2).
\]
One now shows exactly as before that $A$ has non-zero components in each of these 3 factors; we omit details as the analysis is exactly the same as in the Kaehler setting. This shows every para-Kaehler curvature model is geometrically realizable by a para-Kaehler manifold.

4. The Cauchy-Kovalevskaya Theorem

In this section, we show the realization of Theorem 1.6 can be chosen to have constant scalar curvature; the corresponding argument in the para-Kaehler case is similar and is therefore omitted. We shall adapt an argument given in [7] and refer to that paper for additional details. The major difference is that we consider a $4^{th}$ order quasi-linear partial differential equation rather than a $2^{nd}$ order equation.

We begin by recalling the classical Cauchy-Kovalevskaya Theorem as formulated by Evans [13]. Set $u = (y, u_m)$ for $y = (u_1, ..., u_{m-1}) \in \mathbb{R}^{m-1}$. Let $\Phi = \Phi(u)$ be real analytic. We consider the $3^{rd}$ order jets of $\Phi$:
\[
\xi := \{\Phi, \partial_j \Phi, \partial_j \partial_j \Phi, \partial_j \partial_k \partial_j \Phi\}.
\]
We consider a $4^{th}$ order quasi-linear equation in $\Phi$:
\[
\psi^{j_1 j_2 j_3 j_4}(\xi) \partial_{j_1} \partial_{j_2} \partial_{j_3} \partial_{j_4} \Phi + \psi(\xi) = 0 \quad (4.a)
\]
where the coefficients $\psi^{j_1 j_2 j_3 j_4}$ and $\psi$ are real analytic functions of the variables $\xi$. Impose the Cauchy data $\xi = 0$ on the initial hypersurface $u_m = 0$, i.e.
\[
\Phi(y, 0) = 0, \quad \partial_m \Phi(y, 0) = 0, \quad \partial_m \partial_m \Phi(0, y) = 0, \quad \partial_m \partial_m \partial_m \Phi(y, 0) = 0. \quad (4.b)
\]

Theorem 4.1. If $\psi^{m_m_m_m_m}(0) \neq 0$, there is $\epsilon > 0$ and a unique real analytic $\Phi$ defined for $|u| < \epsilon$ which satisfies Equations (4.a) and (4.b).

We use Theorem 4.1 to show the metric of Theorem 1.6 can be chosen to be real analytic and with constant scalar curvature. We adopt the notation of Section 2. For $0 \leq k < \tilde{m}$, set:
\[
\begin{align*}
z_k & := u_{1+2k} + \sqrt{-1}u_{2+2k}, & \bar{z}_k & := u_{1+2k} - \sqrt{-1}u_{2+2k}, \\
dz_k & := du_{1+2k} + \sqrt{-1}du_{2+2k}, & \bar{dz}_k & := du_{1+2k} - \sqrt{-1}du_{2+2k}, \\
\partial_{z_k} & := \frac{1}{2}\{\partial_{1+2k} - \sqrt{-1}\partial_{2+2k}\}, & \partial_{\bar{z}_k} & := \frac{1}{2}\{\partial_{1+2k} + \sqrt{-1}\partial_{2+2k}\}.
\end{align*}
\]
Let $\circ$ denote the symmetric product. If $\Phi$ is a real analytic function (which is called the Kaehler potential), form
\[
\kappa_{\Phi} := \{\partial_{z_k} \partial_{\bar{z}_k} \Phi\} dz^j \circ d\bar{z}^k \in S_2^2(V^*).
\]
This is a $J$-invariant symmetric real 2-tensor with $d\Omega_{\kappa_{\Phi}} = 0$.

Suppose given a Kaehler curvature model $\mathfrak{c} = (V, \langle \cdot, \cdot \rangle, J, A)$ with scalar curvature $c$. Use the methods of Section 2 to choose $\Theta$ so that the curvature tensor of the metric $g_{\Theta}$ is given by $A$ at the origin where $g_{\Theta}$ is the real analytic metric defined by Equation (2.a) and where $J$ is the integrable complex structure given by Equation (2.a). Consider the metric $h_{\Theta, \Phi} := g_{\Theta} + \kappa_{\Phi}$. We impose the Cauchy initial data given by Equation (4.a) so $h_{\Theta, \Phi}(0) = \delta$ and thus $h_{\Theta, \Phi}$ is a Riemannian metric in
some neighborhood of the origin. The scalar curvature $\tau_{\Phi}$ of $h_{\Phi \Phi}$ is given by a quasi-linear 4th-order equation of the form given by Equation (4.a).

We suppress terms which do not involve maximal derivatives in $\partial_m$ to write:

\[ h_{\bar{m} \bar{m}} = h_{mm} = \frac{1}{4} \partial^2_2 m \Phi + \ldots, \quad R_{\bar{m} \bar{m} \bar{m} \bar{m}} = -\frac{1}{8} \partial^4_2 m \Phi + \ldots, \quad \tau_{\Phi} = -\frac{1}{4} \partial^4_2 m \Phi + \ldots \]

Thus the non-degeneracy condition of Theorem 4.1 is satisfied and we can solve the equation $\tau_{\Phi} - c = 0$ with vanishing Cauchy initial data. All the 4th order derivatives of $\Phi$ vanish except possibly for $\partial^4_2 m \Phi$ – the equation $\tau_{\Phi} - c = 0$ implies this vanishes as well. Thus $\Phi = O(|u|^5)$ so $\Phi$ makes no contribution to the curvature tensor at the origin. This shows the Kaehler manifold in question can be chosen to have constant scalar curvature; a similar calculation using a para-Kaehler potential pertains in the para-Kaehler setting as well. This completes the proof of all the assertions of this paper. \(\square\)

ACKNOWLEDGMENTS

Research of M. Brozos-Vázquez partially supported by Project MTM2006-01432 (Spain). Research of P. Gilkey partially supported by DFG PI 158/4-6 (Germany) and by Project MTM2006-01432 (Spain). Research of E. Merino partially supported by FEDER and Project MTM2008-05861 MICINN (Spain).

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