COVARIANCE KERNEL OF LINEAR SPECTRAL STATISTICS
FOR HALF-HEAVY TAILED WIGNER MATRICES

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ABSTRACT. In this paper we analyze the covariance kernel of the Gaussian
process that arises as the limit of fluctuations of linear spectral statistics for
Wigner matrices with a few moments. More precisely, the process we study
here corresponds to Hermitian matrices with independent entries that have α
moments for $2 < \alpha < 4$. We obtain a closed form $\alpha$-dependent expression for
the covariance of the limiting process resulting from fluctuations of the Stielt-
jes transform by explicitly integrating the known double Laplace transform
integral formula obtained in [BGM16]. We then express the covariance as an
integral kernel acting on bounded continuous test functions. The resulting for-
mulation allows us to offer a heuristic interpretation of the impact the typical
large eigenvalues of this matrix ensemble have on the covariance structure.

1. INTRODUCTION

The main purpose of this paper is to understand the covariance of the linear
spectral statistics for Wigner matrices $A_N$ whose entries have cumulative distribu-
tion functions decaying like $x^{-\alpha}$ for $2 < \alpha < 4$ (see Definition 1). Linear spectral
statistics are random variables of the type

$$X_N(f) := \sum_{j=1}^{N} f(\lambda_j(A_N))$$

where $\{\lambda_j(A_N)\}_{j=1}^{N}$ are the eigenvalues of $A_N$, $f$ is a test function, and $N$ is the
dimension of $A_N$.

Studying properties of the linear statistic of the eigenvalues allows us to charac-
terize the limiting behavior of the empirical measure,

$$L_N(dx) = \frac{1}{N} \sum_{j=1}^{N} \delta_{\lambda_j}(dx).$$

The celebrated Wigner semicircle law, the Marčenko-Pastur Law and various other
limiting distribution of random matrix eigenvalues are shown by proving the con-
vergence of $L_N(f) = N^{-1}X_N(f)$ for a well-chosen class of functions $f$, for instance,
the resolvent $f(x) = (z - x)^{-1}$ for $z \in \mathbb{C}^+$ or polynomial test functions $f(x) = x^k$
for $k \geq 1$, see [AGZ10] for a detailed introduction.

Once the limiting measure of $L_N$ is understood, a natural way to proceed is
to look at the fluctuations of the linear statistics. This amounts to studying the quantity

$$\tilde{X}_N(f) := \sigma_N \left( X_N(f) - \mathbb{E}[X_N(f)] \right) ,$$

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where \( f \) is suitably smooth and \( \sigma_N \) is an appropriate scaling parameter. For many matrix models, choosing the factor \( \sigma_N = 1 \) causes \( X_N(f) \) (or even \( X_N(f) \) itself) to converge to a centered Gaussian random variable with variance functional \( \Sigma(f) \).

In addition to establishing the Gaussianity of \( X_N(f) \), a thorough analysis of its covariance structure is important both as a theoretical achievement and as a starting point for the development of novel statistical techniques. A deep understanding of such covariance structures can establish which sets of test functions will be asymptotically independent of others. This information underlies many statistical tests arising in applications (see the recent survey [YZB15] for a variety of settings for which Central Limit Theorems for linear spectral statistics can be used for theoretical statistics.)

1.1. Central Limit Theorems for Light-Tailed Ensembles. In the above description, if the matrix \( A_N \) is a Haar distributed Unitary matrix (similar results hold for Haar distributed Orthogonal and Symplectic random matrices), \( L_N(d\,z) \) converges to a uniform measure on the unit circle. The joint fluctuations of the vector
\[
(X_N(f_1), \ldots, X_N(f_k)),
\]
where \( f_\ell(w) = w^\ell \) converge to a centered complex Gaussian random vector whose covariance matrix is
\[
\begin{pmatrix}
1 & 0 & \cdots & \cdots & 0 \\
0 & \sqrt{2} & 0 & \cdots & 0 \\
\vdots & \vdots & \cdots & \cdots & \vdots \\
0 & \cdots & 0 & \sqrt{k-1} & 0 \\
0 & \cdots & \cdots & 0 & \sqrt{k}
\end{pmatrix}
\]
which is to say, \( X_N \) converges to a random Fourier series whose coefficients of \( f_\ell \) are independent Gaussians with variance \( \ell \) [DS94, Theorem 2]. The proof of [DS94] relied on explicit formula of the moments in terms of symmetric polynomials due to their relationship to representation theory of the classical compact groups.

If the matrix \( A_N \) is a Hermitian random matrix whose density is of the form
\[
\mathbb{P}(\{A_N \in S\}) = \frac{1}{Z_N J_S} \int \exp \left(-\frac{N}{2} \text{Tr} V(\Phi)\right) \prod_{1 \leq i < j \leq N} d\Re \Phi_{i,j} d\Im \Phi_{i,j} \prod_{k=1}^N d\Phi_{k,k}
\]
for a polynomial potential \( V \) with even degree and largest coefficient positive then \( L_N \) converges to an “equilibrium measure,” \( \mu_V \). Assuming \( \mu_V \) is supported on an interval \([a, b]\), for Chebyshev polynomials \( T_1, \ldots, T_k \), the multivariate vector
\[
(X_N(T_1) - N\mu_V(T_1), \ldots, X_N(T_k) - N\mu_V(T_k)),
\]
converges to a real multivariate Gaussian with covariance matrix
\[
\frac{1}{2}
\begin{pmatrix}
1 & 0 & \cdots & \cdots & 0 \\
0 & \sqrt{2} & 0 & \cdots & 0 \\
\vdots & \vdots & \cdots & \cdots & \vdots \\
0 & \cdots & 0 & \sqrt{k-1} & 0 \\
0 & \cdots & \cdots & 0 & \sqrt{k}
\end{pmatrix}
\]
again. We may interpret the above result as saying that \( X_N - N\mu_V \) converges as a process to a random Fourier series in the basis \( T_j \) with similar results for real
symmetric and symplectic random matrices [Joh98]. The proof of [Joh98] relied on potential theory arguments and integration by parts formulae for the explicit density of the eigenvalues.

The above Central Limit Theorems hold true for other point processes \( \{ \lambda_j \}_{j=1}^N \subset \mathbb{R} \) even when they do not have an immediate interpretation as the eigenvalues of some random matrix model. For example, when \( \{ \lambda_j \}_{j=1}^N \) are distributed according to a \( \beta \)-ensemble

\[
P(\{ (\lambda_1, \ldots, \lambda_N) \in S \}) = \frac{1}{Z_{N,\beta}} \int_S \prod_{1 \leq i < j \leq N} |\lambda_i - \lambda_j|^\beta \exp \left( -\frac{\beta N}{2} \sum_{i=1}^N V(\lambda_i) \right),
\]

for a wide range of \( V \) and \( \beta \), the Central Limit Theorem proved in [Joh98] still holds.

The interpretation of this point process as a set of random matrix eigenvalues is lost when \( \beta \notin \{1, 2, 4\} \).

When \( \{ \lambda_j \}_{j=1}^N \subset \mathbb{R} \) are distributed according to biorthogonal ensembles [BD17, Theorem 2.5] whose joint distributions are defined by the formula

\[
P(\{ (\lambda_1, \ldots, \lambda_N) \in S \}) = \frac{1}{N!} \det \left[ \psi_j - 1(\lambda_i) \right]_{1 \leq i,j \leq N} \det \left[ \phi_j - 1(\lambda_i) \right]_{1 \leq i,j \leq N} \nu(d\lambda_1) \cdots \nu(d\lambda_N),
\]

where \( \nu \) is a Borel measure and \( \phi_i \) and \( \psi_j \) are a family of functions on \( \mathbb{R}^N \) such that

\[
\int_{\mathbb{R}} \phi_i(x) \psi_j(x) d\nu(x) = \delta_{i,j},
\]

then under certain assumptions on \( \phi_i \) and \( \psi_j \), there exists a measure \( \mu \) supported on a single interval \([a, b]\) such that

\[
L_N(dx) \to \mu
\]

weakly almost surely. Further for any \( f \in C^1(\mathbb{R}) \) it has been shown that the following weak convergence holds

\[
X_N(f) - N\mu(f) \Rightarrow N\left(0, \sum_{k=1}^{\infty} k|\hat{f}_k|^2\right),
\]

where \( \hat{f}_k \) are explicit.

In all of the above cases, when the limit shape of our point process \( L_N \) is not "multi-cut" (supported on several disjoint non-empty intervals), the fluctuations of \( X_N \) is a random Fourier series whose \( k \)-th coefficient is a Gaussian times \( \sqrt{k} \) (up to some model-dependent scaling and centering). Non-Gaussian behavior has arisen for the \( \beta \)-ensemble when the limiting measure \( \mu_V \) is multi-cut [She13], therefore while Gaussianity describes fluctuations of linear spectral statistics for some of the most common random matrix ensembles, this characterization of the fluctuations of linear spectral statistics is not all-encompassing.

### 1.2. Heavy Tailed Matrices

The proof techniques used in the above results for classical compact group random matrices, light-tailed matrices, general \( \beta \)-ensembles and biorthogonal ensembles no longer apply for heavy-tailed random matrix eigenvalues. When the entries \( a_{i,j} \) of \( A_N \) are of the form \( N^{-\frac{2}{\alpha}} x_{i,j} \) with \( 0 < \alpha < 2 \) and \( x_{i,j} \) in the domain of attraction of an \( \alpha \)-stable law, the limit of \( L_N \) is now a
measure $\mu_\alpha$, which satisfies a coupled fixed point equation given in \cite{BAG08} Theorem 1.4. Later, \cite{BCC11} established another characterization of $\mu_\alpha$ in terms of the Poisson Weighted Infinite Tree which provided more information about properties of $\mu_\alpha$, for example, its absolute continuity for $1 < \alpha < 2$ \cite{BCC11} Theorem 1.6. The largest eigenvalues of these matrices were proven to converge to Poisson point processes under suitable scaling conditions \cite{Sos04}. These results were later extend for tail decay in the region $2 < \alpha < 4$ \cite{ABAP09} and a recent generalization of these extreme eigenvalue results can be found in \cite{BCHJ21}. There has also been some extraordinary progress in the study of local eigenvalue statistics, namely, bulk universality has been proven for $0 < \alpha < 2$ \cite{ALY18} and for random matrices with $2 + \epsilon$ moments \cite{Agg19}, in both of these cases the local eigenvalue statistics fall in the GOE universality class. For an overview and a brief survey of how heavy-tailed matrices differ from light-tailed ensembles see \cite{Gui18}. Also, while our focus is symmetric matrix ensembles, there are several results pertaining to other models such as non-Hermitian matrices with heavy-tailed entries, for instance, the recent paper \cite{CO20} analyzes a heavy-tailed elliptic random matrix ensemble and contains a discussion of the known results for the spectra and eigenvector statistics of heavy-tailed random matrix models.

The topic of interest in this paper is the fluctuation at the global scale for a subclass of such heavy-tailed real symmetric random matrices. Many aspects of the behavior of such fluctuations have been understood for all $\alpha$. In all cases, they form a Gaussian process with a known expression for the covariance see \cite{BGGM14} for $\alpha < 2$, \cite{BGM16} for $2 < \alpha < 4$, and \cite{BS10} for the case $\alpha > 4$. Each of these three cases corresponds to a genuinely different regime, with both the scaling and the covariance changing abruptly at the transition points. For example, when $\alpha \in (0, 2)$ it is known that the fluctuations of linear statistics of its eigenvalues are of order $N^{-\frac{1}{2}}$ rather than the $N^{-1}$ behavior of the Wigner case, and for $2 < \alpha < 4$, the fluctuations are of order $N^{-\frac{2}{\alpha}}$ — note that the exponent linearly interpolates between the $\alpha$-stable regime and the Wigner regime. In the case $0 < \alpha < 2$ the covariance of the resolvent is given by coupled fixed point equations that are formidable to analyze — even more so than the spectral measure since they rely on an understanding of the fixed point equations for the measure $\mu_\alpha$.

We will concentrate on matrices whose entries have tail behavior given by $2 < \alpha < 4$. The limiting spectral measure of such matrices is still the semicircle law, which is a measure supported on a single interval, and therefore is a natural case to compare to the pattern of results discussed in the previous section. This class of matrices provides a natural starting point in understanding the fluctuations of heavy-tailed random matrix eigenvalues.

The main achievement of the present paper is to compute in closed form and interpret the covariance of $\hat{X}_N(f)$ in the context of random matrices whose entries have heavier tails with $2 < \alpha < 4$. As a starting point of our analysis we adopt the double Laplace transform integral formula derived in \cite{BGM16}. We compute the integral to arrive at a simpler expression for the covariance of $\hat{X}_N(f)$ where $f(x) = (z - x)^{-1}$ (Theorem 3) which makes the dependence on $\alpha$ and $m$ (the Stieltjes transform of the semicircle law) easy to see and interpret. In Theorem 5 and Corollary 6 we use our new formula to extract the integral kernel associated with this covariance. The resulting integral kernel demonstrates the impact of large eigenvalues typical of heavy tailed matrix models.
The rest of the paper is organized as follows, Section 4 contains the proof of Theorem 3, Section 5 the proof of Theorem 5 and Section 6 contains the proof of Corollary 6. The Appendix contains elementary integral identities we use in our proofs.

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2. Matrix Model and Past Results

In this paper, $A_N$ is a sequence of $N \times N$ Hermitian random matrix whose entries are i.i.d have first two moments finite, but not necessarily any higher moments, in addition to a power law tail decay condition.

**Definition 1.** Define the sequence of $N \times N$ matrices

$$A_N = [a_{ij}]_{1 \leq i, j \leq N} = \left[ \frac{x_{ij}}{\sqrt{N}} \right]_{1 \leq i, j \leq N},$$

with

- The $x_{i,j}, 1 \leq i \leq j$, are i.i.d real random variables with mean 0 and variance 1 such that for a certain $\alpha \in (2, 4)$ and a certain $c > 0$, as $t \to \infty$,

$$P(|x_{i,j}| > t) \sim \frac{c}{\Gamma(1 - \frac{\alpha}{2})} t^{-\alpha}, \quad (1)$$

or

- $x_{i,j} = x_{ij}^R / \sqrt{2} + i x_{ij}^I / \sqrt{2}$ for $1 < i < j$ and $x_{ii} = x_{ii}^R$ where $x_{ij}^I$ and $x_{ij}^R$ are i.i.d real symmetric random variables with mean 0 and variance 1 that satisfy (1).

For the above matrix model, the semicircle law for the eigenvalues still holds. It was shown in [BGM16] that the spectral statistic

$$\frac{1}{N^{1-\frac{\alpha}{2}}} (\text{Tr } G(z) - \mathbb{E} \text{Tr } G(z)),$$

converges weakly to a centered Gaussian process $X_z$ defined for $z \in \mathbb{C} \setminus \mathbb{R}$ where

$$G(z) = (zI_N - A_N)^{-1}.$$

We restate this result, which is the foundation of our calculations in this paper.

**Theorem 2** ([BGM16]). For

$$G(z) = (zI_N - A_N)^{-1},$$

with $A_N$ as above, the process

$$\frac{1}{N^{1-\frac{\alpha}{2}}} (\text{Tr } G(z) - \mathbb{E} \text{Tr } G(z)),$$
converges to a complex Gaussian centered process \((X_z)_{z \in \mathbb{C} \setminus \mathbb{R}}\) with covariance defined by the fact that \(X_z = \overline{X_z}\) and that for any \(z, w \in \mathbb{C} \setminus \mathbb{R}\), \(\mathbb{E}[X_z X_w] = C(z, w)\), for

\[
C(z, w) := \int_0^\infty \int_0^\infty \partial_z \partial_w \left\{ \left[ (K(z, t) + K(w, s))^{\alpha/2} - (K(z, t)^{\alpha/2} + K(w, s)^{\alpha/2}) \right] \times \exp(\text{sgn}_z itz - K(z, t) + \text{sgn}_w isw - K(w, s)) \right\} \frac{c \, dt \, ds}{2ts}
\]

where \(c\) and \(\alpha\) are as in (1), \(\text{sgn}_z = \text{sgn}(3z)\) and \(K(z, t) := \text{sgn}_z i t m(z)\), \(m(z)\) being the Stieltjes transform of the semicircle law with support \([-2, 2]\).

The branch cut associated to the fractional power is always the principal branch cut. This is in contrast to the choice for the Stieltjes transform, where in the formula:

\[
m(z) = \frac{z - \sqrt{z^2 - 4}}{2},
\]

the branch cut is taken to be on the positive real axis. We will remind the reader of the appropriate branch cut when any branch cut manipulations are performed.

### 3. Main Results

Our first result is a computation of the integral in (2).

**Theorem 3.** Taking \(C(z, w)\) as in Theorem 2, we have that

\[
C(z, w) = \frac{2c \pi m(z)m'(z)m(w)m'(w)}{k_\alpha \sin \left( \frac{\pi \alpha}{2} \right) (m^2(z) - m^2(w))} \left[ (-m(z)^2)^{\frac{\alpha}{2} - 1} - (-m(w)^2)^{\frac{\alpha}{2} - 1} \right]
\]

where \(c\) and \(\alpha\) are as in (1), \(\text{sgn}_z = \text{sgn}(3z)\), \(m(z)\) is the Stieltjes transform of the semicircle law with support \([-2, 2]\), and

\[
k_\alpha := \frac{\Gamma\left(2 - \frac{\alpha}{2}\right)}{\frac{\alpha}{2}\left(\frac{\alpha}{2} - 1\right)}.
\]

**Remark 4.** In a further simplification we can write

\[
C(z, w) = \frac{-2c \pi m(z)m'(z)m(w)m'(w)}{k_\alpha \sin \left( \frac{\pi \alpha}{2} \right) (m^2(z) - m^2(w))} \times \left( \exp \left[ \frac{-\text{sgn}_z \pi \alpha i}{2} \right] m(z)^{\alpha - 2} - \exp \left[ \frac{-\text{sgn}_w \pi \alpha i}{2} \right] m(w)^{\alpha - 2} \right)
\]

The following Theorem expresses the covariance \(C(z, w)\) in an alternative integral form.

**Theorem 5.** Let \(c\) and \(k_\alpha\) be defined as in equations (1) and (4) respectively. Let \(\psi, \phi \in C_b(\mathbb{R})\) be bounded continuous functions and define

\[
\psi \otimes \phi : \mathbb{R}^2 \to \mathbb{R} \quad \psi \otimes \phi := \psi(x)\phi(y) \quad (x, y) \in \mathbb{R}^2.
\]

Then letting, \(z = E + i\eta_1\) and \(w = F + i\eta_2\), the pairing

\[
-\frac{1}{4\pi^2} \int_{\mathbb{R}^2} \left\{ C(z, w) + C(\bar{z}, \bar{w}) - C(\bar{z}, w) - C(z, \bar{w}) \right\} \psi(E)\phi(F) \, dE \, dF
\]
converges as \( \eta_1, \eta_2 \downarrow 0 \) to the pairing
\[
\langle K_\alpha, \phi \otimes \psi \rangle := \frac{c}{k_\alpha} \int_0^\infty \Lambda_u(\psi) \Lambda_u(\phi) \, du
\]
where \( \Lambda_u \) is a measure indexed by \( u \) defined by
\[
\Lambda_u := \frac{1}{\pi} \frac{2u^4 + 2u^2 - u^2 x^2}{\sqrt[4]{\Delta}} \left( \frac{-u^2 x^2 + u^4 + 2u^2 + 1}{2} \right) \delta_{u|\leq 2} + (\delta_u + \delta_{-u-1}) \delta_{u \geq 1}.
\]
and \( \delta_{x_0} \) is the usual point measure \( \delta_{x_0}(\phi) = \phi(x_0) \). In particular for \( \psi(x) = (z-x)^{-1} \) and \( \phi(y) = (w-y)^{-1} \) (where we extend \( K_\alpha \) to complex-valued bounded real continuous functions by linearity) we have
\[
\langle K_\alpha, \phi \otimes \psi \rangle = C(z,w).
\]

The above result suggests that for a finite collection of test functions \( \psi_1, \ldots, \psi_m \in \mathcal{F}_\alpha \) where \( \mathcal{F}_\alpha \) contains at least the subset \( \mathcal{G}_\alpha \) of finite linear combinations of functions of the form \( f(x) = (z-x)^{-1} \) for some \( z \in \mathbb{C} \setminus \mathbb{R} \), the vector
\[
N^{1-\frac{3}{4}} \left( \text{Tr} \psi_1(A_N) - \mathbb{E} \text{Tr} \psi_1(A_N), \ldots, \text{Tr} \psi_m(A_N) - \mathbb{E} \text{Tr} \psi_m(A_N) \right)
\]
converges weakly to a centered Gaussian vector whose covariance matrix has entries \( \langle K_\alpha, \psi_i \otimes \psi_j \rangle \). We conjecture the class \( \mathcal{F}_\alpha \) is the space of all functions \( \psi \) for which \( \langle K_\alpha, \psi \otimes \psi \rangle \) is finite. In order to prove such a Central Limit Theorem, the following approach appearing in [Shc11] seems accessible given the explicit form of \( \psi \).

More qualitatively, our result helps contextualize the fluctuations of the eigenvalues of \( A_N \) at a macroscopic scale with other known results for eigenvalue statistics for this model. Consider, for example, the limiting distribution of the largest eigenvalues of \( A_N \) computed in the paper [ABAP09 Section 4]; the eigenvalues of \( A_N \) were shown to converge to a Poisson process under appropriate scaling
\[
\sum_{j=1}^N \delta_{\gamma_j(A_N)}(d \lambda) 1_{\lambda_j(A_N) > 0} \Rightarrow \mathcal{P}_\alpha,
\]
where
\[
\hat{\lambda}_N := \left( \frac{N(N+1)}{2} \right)^{\frac{1}{2}} \frac{1}{\sqrt{N}} \sim N^{\frac{1}{2}-\frac{1}{2}},
\]
and \( \mathcal{P}_\alpha \) is a Poisson measure on \( (0, \infty) \) with intensity measure
\[
\rho_\alpha(x) := \frac{\alpha}{x^{1+\alpha}}.
\]
Theorem 5 above shows that this intensity measure appears in the covariance formula \( C(z,w) \). In particular, note that the point masses in the measure \( \Lambda_u \) are located at \( u + u^{-1} \) and \( -u - u^{-1} \) for \( u \geq 1 \) taking values on \( [-2,2] \) which are outside the support of the semicircle density. If \( \psi \) and \( \phi \) were test functions supported outside of \( [-2,2] \) then the only contribution in the above covariance would be given by these point masses integrated to a measure proportional to the intensity
In particular, for \( \alpha > 4, 2 < \alpha < 4 \) and the genuinely heavy-tail case of \( \alpha < 2 \). This relationship parallels the changing behavior of the fluctuation of linear statistics of the eigenvalues noted in \([BGM16]\). The original covariance formula presented in Theorem 5 did not provide any insight into this phenomenon and its relationship to other known features about the eigenvalue distribution of this class of matrices. We also provide the following variant of the above formula which may be easier to use for explicit or numerical evaluation.

**Corollary 6.** Let \( c \) and \( k_\alpha \) be defined as in equations (1) and (4) respectively and let \( m_\pm(E) = \lim_{\delta \to 0} m(E \pm i \delta) \). We may write the action of \((K_\alpha, \psi \otimes \phi)\) as the integral of \( \psi(E)\phi(F) \) against the following distribution

\[
K_\alpha(E, F) = \frac{c}{k_\alpha} \frac{1}{\pi k_\alpha \sin \left( \frac{\pi}{2} \right) \times}
\begin{cases}
\frac{|m_+(E)|^\alpha - 1 (2 + m_+(E)^2) m_+(E)}{\pi(-m_+(E)^2 F^2 + (m_+(E)^2 + 1)^2) \sqrt{4 - F^2}} m_+(E) & \text{if } |E| < 2 \text{ and } |F| > 2 \\
\frac{|m_-(E)|^\alpha - 1 (2 + m_-(E)^2) m_-(E)}{\pi(-m_-(E)^2 F^2 + (m_-(E)^2 + 1)^2) \sqrt{4 - F^2}} m_-(E) & \text{if } |E| > 2 \text{ and } |F| < 2 \\
\frac{|m_+(E)|^\alpha - 1 (2 + m_+(E)^2) m_+(E)}{\pi(-m_+(F)^2 E^2 + (m_+(E)^2 + 1)^2) \sqrt{4 - E^2}} (\delta_{E_- F} + \delta_{E_+ F}) & \text{if } |E| > 2 \text{ and } |F| > 2,
\end{cases}
\]

and for \( |E|, |F| < 2 \),

\[
K_\alpha(E, F) = \frac{-c}{\pi k_\alpha \sin \left( \frac{\pi}{2} \right) \times}
\begin{align*}
&\left\{ \frac{m_+(E) m_+(E) m_+(E) m_+(E) m_+(E)}{m_+(E) - m_+(E)} \right\} \left[ \left( -m_+(E)^2 \right)^\frac{\alpha}{2} - \left( -m_+(E)^2 \right)^\frac{\alpha}{2} \right] \\
&\frac{m_-(E) m_-(E) m_-(E) m_-(E) m_-(E)}{m_-(E) - m_-(E)} \left[ \left( -m_-(E)^2 \right)^\frac{\alpha}{2} - \left( -m_-(E)^2 \right)^\frac{\alpha}{2} \right] \\
&\frac{m_+(E) m_+(E) m_-(E) m_+(E) m_-(E)}{m_-(E) - m_+(E)} \left[ \left( -m_+(E)^2 \right)^\frac{\alpha}{2} - \left( -m_+(E)^2 \right)^\frac{\alpha}{2} \right] \\
&\frac{m_-(E) m_-(E) m_+(E) m_-(E) m_-(E)}{m_-(E) - m_+(E)} \left[ \left( -m_-(E)^2 \right)^\frac{\alpha}{2} - \left( -m_-(E)^2 \right)^\frac{\alpha}{2} \right].
\end{align*}
\]

In particular, for \( \alpha = 3 \) we have

\[
K_3(E, F) = \frac{15c}{8 \sqrt{\pi}} \frac{1}{\sqrt{4 - F^2 \sqrt{4 - E^2}}} \sqrt{4 - E^2} \sqrt{4 - F^2},
\]

\begin{align*}
&\frac{|m_+(E)|^3 - 1 (2 + m_+(E)^2) m_+(E)}{\pi(-m_+(E)^2 F^2 + (m_+(E)^2 + 1)^2) \sqrt{4 - F^2}} m_+(E) & \text{if } |E| < 2 \text{ and } |F| > 2, \\
&\frac{|m_-(E)|^3 - 1 (2 + m_-(E)^2) m_-(E)}{\pi(-m_-(E)^2 F^2 + (m_-(E)^2 + 1)^2) \sqrt{4 - F^2}} m_-(E) & \text{if } |E| > 2 \text{ and } |F| < 2, \\
&\frac{|m_+(E)|^3 - 1 (2 + m_+(E)^2) m_+(E)}{\pi(-m_+(F)^2 E^2 + (m_+(E)^2 + 1)^2) \sqrt{4 - E^2}} (\delta_{E_- F} + \delta_{E_+ F}) & \text{if } |E| > 2 \text{ and } |F| > 2.
\end{align*}

### 4. Proof of Theorem 3: Rewriting the Integral

**Proof.** By the fixed point equation representation of \( m(z) \):

\[
m(z) = \frac{1}{z - m(z)},
\]

we can write

\[
C(z, w) = \frac{c}{2} \int_0^\infty \int_0^\infty \partial_z \partial_w L(z, w; t, s) \, dt \, ds,
\]

(6)
Figure 1. Plot of $K_3$ with $|E|, |F| < 2$ and $c = \frac{8\sqrt{\pi}}{15}$.

where

$$L(z, w; t, s) := \frac{1}{ts} \left\{ \left( it \text{sgn}_z m(z) + i{s} \text{sgn}_w m(w) \right)^{\frac{3}{2}} - \left( it \text{sgn}_z m(z) \right)^{\frac{3}{2}} 
- \left( is \text{sgn}_w m(w) \right)^{\frac{3}{2}} \right\} \exp \left( \frac{it \text{sgn}_z m(z)}{m(z)} + \frac{is \text{sgn}_w m(w)}{m(w)} \right).$$

we have that

$$\Re \left( \frac{i \text{sgn}_z}{m(z)} \right) < 0 \quad \text{and} \quad \Re \left( \frac{i \text{sgn}_w}{m(w)} \right) < 0,$$

so for each $z, w \in \mathbb{C} \setminus \mathbb{R}$, $L(z, w; t, s)$ decays in $t$ and $s$ exponentially. We wish to pull out the derivatives of $z$ and $w$ out of the integral in (6). To do so, observe that by the Cauchy integral formula,

$$\partial_z \partial_w L(z, w; t, s) = -\frac{1}{4\pi^2} \oint_{C_z} \oint_{C_w} \frac{L(\rho, \gamma; t, s)}{(\rho - z)^2(\gamma - w)^2} \, d\rho \, d\gamma,$$

where $C_z$ and $C_w$ are contours containing $z$ and $w$ respectively — we take them to be balls of a small radius so they avoid the branch cut. Now we have

$$\int_0^\infty \int_0^\infty \partial_z \partial_w L(z, w; t, s) \, dt \, ds =
- \int_0^\infty \int_0^\infty \frac{1}{4\pi^2} \oint_{C_z} \oint_{C_w} \frac{L(\rho, \gamma; t, s)}{(\rho - z)^2(\gamma - w)^2} \, d\rho \, d\gamma \, dt \, ds,$$
now by choosing contours to be balls of radius $\delta$ we have

$$
\int_0^\infty \int_0^\infty \oint_{C_z} \oint_{C_w} \frac{|L(\rho, \gamma; t, s)|}{(\rho - z)^2(\gamma - w)^2} \, d\rho \, d\gamma = \frac{1}{\delta^2} \int_0^\infty \int_0^\infty \oint_{C_z} \oint_{C_w} |L(\rho, \gamma; t, s)| \, d\rho \, d\gamma \, dt \, ds,
$$

we claim that the integral in $t$ and $s$ of $|L(\rho, \gamma; t, s)|$ is uniformly bounded for all $\rho$ and $\gamma$ in the chosen contours. This allows us to rearrange the original integrals using Fubini’s Theorem,

$$
C(z, w) = \frac{c}{2} \partial_z \partial_w \int_0^\infty \int_0^\infty L(z, w; t, s) \, dt \, ds.
$$

Next, observe that

$$
\Re(-it \text{sgn}_z m(z)) < 0, \quad \text{and} \quad \Re(-is \text{sgn}_w m(w)) < 0,
$$

so we may use Lemma [11] to write

$$
(t \text{sgn}_z m(z) + is \text{sgn}_w m(w))^\frac{\alpha}{2} - (it \text{sgn}_z m(z))^\frac{\alpha}{2} - (is \text{sgn}_w m(w))^\frac{\alpha}{2} = \int_0^\infty \frac{1}{k_\alpha r^{1 + \frac{1}{2}} t^s} \left\{ \exp \left\{ -ir \{ t \text{sgn}_z m(z) + s \text{sgn}_w m(w) \} \right\} - \exp(-irt \text{sgn}_z m(z)) - \exp(-irs \text{sgn}_w m(w)) + 1 \right\} \, dr,
$$

obtaining the equality

$$
L(z, w; t, s) = \int_0^\infty \frac{1}{k_\alpha r^{1 + \frac{1}{2}} t^s} \left\{ \exp \left[ it \text{sgn}_z \left( -r m(z) + \frac{1}{m(z)} \right) \right] - \exp \left[ it \text{sgn}_z m(z) \right] \right\} \left\{ \exp \left[ is \text{sgn}_w \left( -r m(w) + \frac{1}{m(w)} \right) \right] - \exp \left[ is \text{sgn}_w m(w) \right] \right\} \, dr.
$$

Using the above representation, we can integrate $t$ and $s$ first by applying Fubini’s Theorem again. Observe that

$$
\exp \left[ it \text{sgn}_z \left( -r m(z) + \frac{1}{m(z)} \right) \right] - \exp \left[ it \text{sgn}_z m(z) \right] \frac{t}{r} = -ir \text{sgn}_z m(z) \int_0^1 \exp \left[ it \text{sgn}_z \left( -r \nu m(z) + \frac{1}{m(z)} \right) \right] \, d\nu,
$$

so

$$
\int_0^\infty \exp \left[ it \text{sgn}_z \left( -r m(z) + \frac{1}{m(z)} \right) \right] - \exp \left[ it \text{sgn}_z m(z) \right] \frac{d\nu}{r} = -ir \text{sgn}_z m(z) \int_0^1 \int_0^\infty \exp \left[ it \text{sgn}_z \left( -r \nu m(z) + \frac{1}{m(z)} \right) \right] \, dt \, d\nu,
$$

$$
= rm(z) \int_0^1 \frac{d\nu}{\left( -r \nu m(z) + \frac{1}{m(z)} \right)}
$$

$$
= - \left\{ \log \left( -r m(z) + \frac{1}{m(z)} \right) - \log \left( \frac{1}{m(z)} \right) \right\},
$$

##
where we have used that \(-rm(z) + \frac{1}{m(z)}\) does not cross the principal branch cut of the logarithm. We conclude
\[
\int_0^\infty \int_0^\infty \mathcal{L}(z, w; t, s) \, dt \, ds = \\
\int_0^\infty \frac{1}{k_\alpha r^{1+\frac{\alpha}{2}}} \left( \log \left[ -rm(z) + \frac{1}{m(z)} \right] - \log \left[ \frac{1}{m(z)} \right] \right) \\
\times \left( \log \left[ -rm(w) + \frac{1}{m(w)} \right] - \log \left[ \frac{1}{m(w)} \right] \right) \, dr.
\]
Insert the partial derivatives with respect to \(z\) and \(w\) back into the integrand above to obtain
\[
C(z, w) = \frac{c}{2} \int_0^\infty 4r^{\frac{\alpha}{2} - 1} \frac{\alpha m(z)m'(z)m(w)m'(w)}{k_\alpha} \frac{1}{(rm^2(z) - 1)(rm^2(w) - 1)} \, dr,
\]
\[
= \frac{c}{2} \int_0^\infty 4r^{\frac{\alpha}{2} - 1} \frac{m(z)m'(z)m(w)m'(w)}{k_\alpha} \frac{1}{(r - m^2(z))(r - m^2(w))} \, dr. \quad (7)
\]
We apply Lemma 12 to obtain
\[
C(z, w) = \frac{2c \pi m(z)m'(z)m(w)m'(w)}{k_\alpha \sin \left( \frac{\pi \alpha}{2} \right)} \left[ \left( -m^2(z) \right)^{\frac{\alpha}{2} - 1} - \left( -m^2(w) \right)^{\frac{\alpha}{2} - 1} \right]. \quad (8)
\]
Simplifying this further, we see that if \(\Im z > 0\) then \(\arg[m(z)] \in (-\pi, 0)\) yielding that \(\arg[-m^2(z)] = \arg[\exp(i\pi m^2(z))] \in (-\pi, \pi)\) which implies
\[
\left( -m^2(z) \right)^{\frac{\alpha}{2} - 1} = \exp \left[ i\pi \left( 1 - \frac{\alpha}{2} \right) \right] m(z)^{\alpha - 2}.
\]
On the other hand, if \(\Im z < 0\) then \(\arg m(z) \in (0, \pi)\) yielding that \(\arg[-m^2(z)] = \arg[\exp(-i\pi m^2(z))] \in (-\pi, \pi)\) which implies
\[
\left( -m^2(z) \right)^{\frac{\alpha}{2} - 1} = \exp \left[ -i\pi \left( 1 - \frac{\alpha}{2} \right) \right] m(z)^{\alpha - 2}.
\]
Applying these observations in equation (8) proves Remark 4. Observe that \(C(z, w)\) is analytic in both \(z\) and \(w\) so long as \(z\) and \(w\) are not on the real line. \(\square\)

5. Proof of Theorem 5

Recall from Section 4 the formula
\[
C(z, w) = \frac{c}{2} \int_0^\infty 4r^{\frac{\alpha}{2} - 1} \frac{m(z)m'(z)m(w)m'(w)}{k_\alpha} \frac{1}{(r - m^2(z))(r - m^2(w))} \, dr,
\]
\[
= \frac{c}{k_\alpha} \int_0^\infty \frac{1}{u^{1+\alpha}} \left( \frac{2u^2 m(z)m'(z)}{1 - u^2 m^2(z)} \right) \left( \frac{2u^2 m(w)m'(w)}{1 - u^2 m^2(w)} \right) \, du.
\]
where we have mapped \(r\) to \(u^{-2}\) so that the intensity measure of limiting point process of the largest eigenvalues of \(A_N\) appears. From this representation
\[
C(z, w) - C(\bar{z}, \bar{w}) = \frac{2ic}{k_\alpha} \int_0^\infty \frac{1}{u^{1+\alpha}} 3 \left( \frac{2u^2 m(z)m'(z)}{1 - u^2 m^2(z)} \right) \left( \frac{2u^2 m(w)m'(w)}{1 - u^2 m^2(w)} \right) \, du,
\]
\[
C(z, \bar{w}) - C(\bar{z}, w) = \frac{2ic}{k_\alpha} \int_0^\infty \frac{1}{u^{1+\alpha}} 3 \left( \frac{2u^2 m(z)m'(z)}{1 - u^2 m^2(z)} \right) \left( \frac{2u^2 m(w)m'(w)}{1 - u^2 m^2(w)} \right) \, du,
\]
hence
\[- \frac{1}{4\pi^2} \left\{ C(z, w) + C(\bar{z}, \bar{w}) - C(\bar{z}, w) - C(z, \bar{w}) \right\} \]
\[= \frac{c}{\pi^2k\alpha} \int_0^\infty \frac{1}{u^{1+\alpha}} \left\{ \frac{2u^2m(z)m'(z)}{1-u^2m^2(z)} \right\} \Im \left\{ \frac{2u^2m(w)m'(w)}{1-u^2m^2(w)} \right\} du. \]

This representation will be the basis for our proof of Theorem 5. We need the following Lemmas in order to proceed.

**Lemma 7.** Let \( \psi \in C_b(\mathbb{R}) \) and let \( \zeta = x + i\eta \) with \( \eta > 0 \). For fixed \( \eta \), define the functions
\[ r_\eta(\theta) := \frac{\eta - \sqrt{\eta^2 + 4\sin^2(\theta)}}{2\sin \theta} \quad \theta \in (-\pi, 0) \]
\[ E_\eta(\theta) := \left( r_\eta(\theta) + \frac{1}{r_\eta(\theta)} \right) \cos \theta \quad \theta \in (-\pi, 0) \]
then for every \( \eta > 0 \),
\[ \int_{\mathbb{R}} \frac{1}{\pi} \left\{ \frac{2u^2m(x + i\eta)m'(x + i\eta)}{1-u^2m^2(x + i\eta)} \right\} \psi(x) \, dx \]
\[= -\frac{1}{\pi} \int_{-\pi}^{\pi} \psi(E_\eta(\theta)) \frac{d}{d\theta} \arg \left\{ 1-u^2r_\eta^2(\theta) \exp(i2\theta) \right\} \, d\theta. \]

**Proof.** For all \( x \in \mathbb{R} \), \( -m^2(\zeta) \) is in the complex disk with negative reals removed, \( D \setminus [-1, 0] \), therefore the function \( 1-u^2m^2(\zeta) \) never crosses the principal branch cut of the logarithm and
\[ \frac{d}{d\zeta} \log \left\{ 1-u^2m^2(\zeta) \right\} = -\frac{2u^2m(\zeta)m'(\zeta)}{1-u^2m^2(\zeta)}. \]
By the Cauchy-Riemann equations,
\[ \Im \left[ \frac{d}{d\zeta} \log \left\{ 1-u^2m^2(\zeta) \right\} \right] = \frac{\partial}{\partial x} \Im \log \left\{ 1-u^2m^2(\zeta) \right\}, \] (9)

note that the imaginary part of the logarithm is the principal branch of the argument
\[ \Im \log \left\{ 1-u^2m^2(x + i\eta) \right\} = \arg \left\{ 1-u^2m^2(x + i\eta) \right\}. \]
Let \( \psi \in C_b(\mathbb{R}) \), by equation (9)
\[ \int_{\mathbb{R}} \frac{1}{\pi} \left\{ \frac{2u^2m(x + i\eta)m'(x + i\eta)}{1-u^2m^2(x + i\eta)} \right\} \psi(x) \, dx \]
\[= -\frac{1}{\pi} \int_{-\pi}^{\pi} \psi(x) \frac{\partial}{\partial x} \arg \left\{ 1-u^2m^2(x + i\eta) \right\} \, dx. \] (10)
Let the polar form of \( m(\zeta) = R \exp(i\theta) \) note that \( \theta = \arg \{m(\zeta)\} \in (-\pi, 0) \). The rewritten fixed point equation
\[ \zeta = m(\zeta) + \frac{1}{m(\zeta)} \]
implies
\[ x + i\eta = m(\zeta) + \frac{\overline{m(\zeta)}}{|m(\zeta)|^2} = \Re \{m(\zeta)\} \left( 1 + \frac{1}{|m(\zeta)|^2} \right) + i\Im \{m(\zeta)\} \left( 1 - \frac{1}{|m(\zeta)|^2} \right), \]
which is equivalent to
\[ x = \left( R + \frac{1}{R} \right) \cos \theta \quad \text{and} \quad \eta = \left( R - \frac{1}{R} \right) \sin \theta. \]  
(11)

We use equation (11) to express the radius \( r \) solely as a function of \( \theta \) and \( \eta \)
\[ R^2 \sin \theta - R \eta - \sin \theta = 0 \implies R = r_\eta(\theta) = \frac{\eta - \sqrt{\eta^2 + 4 \sin^2 \theta}}{2 \sin \theta}. \]  
(12)

then using the function \( r_\eta(\theta) \) we write \( x \) as a function of \( \theta \) and \( \eta \)
\[ x = E_\eta(\theta) = \left( r_\eta(\theta) + \frac{1}{r_\eta(\theta)} \right) \cos \theta. \]

The function \( E_\eta(\theta) \) is a bijection from \((-\pi, 0)\) to \(\mathbb{R}\) and is differentiable. We may therefore change variables in equation (10)
\[- \frac{1}{\pi} \int_{\mathbb{R}} \psi(x) \frac{\partial}{\partial x} \arg \left\{ 1 - u^2 m^2(x + i \eta) \right\} \, dx = - \frac{1}{\pi} \int_{-\pi}^{\pi} \psi(E_\eta(\theta)) \frac{d}{d\theta} \arg \left\{ 1 - u^2 r^2_\eta(\theta) \exp(i \theta) \right\} \, d\theta, \]
which is the required result. \( \square \)

We wish to bound the expression in Lemma 7 for arbitrary test functions \( \psi \) and arbitrary \( u \geq 0 \) and \( \eta > 0 \).

**Lemma 8.** Let \( \eta > 0 \) and let \( r_\eta(\theta) \) be as defined in Lemma 7. The interval \((-\pi, 0)\) can be partitioned into at most 9 intervals for which the function
\[ \arg \left\{ 1 - u^2 r^2_\eta(\theta) \exp(i \theta) \right\}, \]  
(13)
is piecewise monotonic. When \( u \leq 1 \) in particular, there exists a point \( \tilde{\theta}_{\eta,u} \in (-\frac{\pi}{2}, 0) \) such that the function (13) is decreasing on \((-\pi, -\tilde{\theta}_{\eta,u} - \pi)\), increasing on \((-\tilde{\theta}_{\eta,u} - \pi, \tilde{\theta}_{\eta,u})\) and decreasing once more on \((\tilde{\theta}_{\eta,u}, 0)\). Further the point \( \tilde{\theta}_{\eta,u} \) is defined to be the unique solution to
\[- u^2 r^6_\eta(\theta) - (1 - u^2) r^6_\eta(\theta) + (5 + u^2) r^4_\eta(\theta) - (u^2 + 4 \eta^2 + 7) r^2_\eta(\theta) + 3 = 0 \]
on \( \theta \in (-\frac{\pi}{2}, 0) \).

**Proof.** For \( \theta \in (-\frac{\pi}{2}, 0) \) we have \( \sin(-\pi - \theta) = \sin(\theta) \) further \( \exp(i2(-\pi - \theta)) = -\exp(-i2\theta) \) from which we conclude
\[ \arg \left\{ 1 - u^2 r^2_\eta(-\pi - \theta) \exp \left\{ i2(-\pi - \theta) \right\} \right\} = - \arg \left\{ 1 - u^2 r^2_\eta(\theta) \exp(i2\theta) \right\}, \]
therefore it suffices to prove the existence of a \( \theta^*_{\eta,u} \) such that the function (13) is increasing on \((-\frac{\pi}{2}, \theta^*_{\eta,u})\) and decreasing on \((\theta^*_{\eta,u}, 0)\). For what proceeds assume \( \theta \in (-\frac{\pi}{2}, 0) \). Let \( \arctan \) be the principal branch of the inverse tangent taking values in \((-\frac{\pi}{2}, \frac{\pi}{2})\). Observe that
\[ \arg \left\{ 1 - u^2 r^2_\eta(\theta) \exp(i2\theta) \right\} = 2 \arctan \left( \frac{- u^2 r^2_\eta(\theta) \sin(2\theta)}{1 - u^2 r^2_\eta(\theta) \cos(2\theta) + |1 - u^2 r^2_\eta(\theta) \exp(i2\theta)|} \right). \]
We define
\[ g(\theta) := \frac{1 - u^2 r_n^2(\theta) \cos(2\theta)}{-u^2 r_n^2(\theta) \sin(2\theta)} = \frac{1 - u^2 r_n^2(\theta)\{1 - 2\sin^2(\theta)\}}{-2u^2 r_n^2(\theta) \sin(\theta)\sqrt{1 - \sin^2(\theta)}}, \]

\[ = \frac{1 - u^2 r_n^2(\theta)\{1 - 2\theta^2(\theta)\}}{-u^2 r_n^2(\theta) \sin(\theta)\sqrt{1 - \sin^2(\theta)}}, \]

\[ = \frac{1 - u^2 r_n^2(\theta)\{1 - 2\theta^2(\theta)\}}{2u^2 r_n^2(\theta) \sin(\theta)\sqrt{1 - \sin^2(\theta)}}, \]

\[ = \frac{(1 - r_n^2(\theta))^2 - u^2 r_n^2(\theta)\{1 - 2\theta^2(\theta)\}}{2\eta u^2 r_n^2(\theta) \sin(\theta)\sqrt{1 - \sin^2(\theta)}}, \]

\[ = \frac{-u^2 \theta^2(\theta) + \{1 + 2u^2(1 + \eta^2)\}r_n^4(\theta) - (2 + u^2)r_n^2(\theta) + 1}{2\eta u^2 r_n^4(\theta) \sqrt{r_n^4(\theta) - (2 + \eta^2)r_n^2(\theta) + 1}}, \]

where we have used the equation (12) to express \( g(\theta) \) as a function of \( r_n(\theta) \) only.

Observe that
\[
\frac{d}{d\theta} \arg \{1 - u^2 r_n^2(\theta) \exp(i2\theta)\} = -g'(\theta) \times \frac{2}{g(\theta) + \sqrt{1 + g^2(\theta)}} (1 + \frac{g(\theta)}{\sqrt{1 + g^2(\theta)}} \frac{1}{\sqrt{g^2(\theta) + 1}}),
\]

therefore \( \frac{d}{d\theta} \arg \{1 - u^2 r_n^2(\theta) \exp(i2\theta)\} \) has the same sign as \( -g'(\theta) \). Next,
\[
-g'(\theta) = -r_n'(\theta) \times \left\{ \frac{-u^2 r_n^{10} - (1 - 2u^2)r_n^8 + 6u^6 - 2(u^2 + 2\eta^2 + 6)r_n^4 + (u^2 + 4\eta^2 + 10)r_n^2 - 3}{2\eta u^2 r_n^4(r_n^4 - (\eta^2 + 2)r_n^2 + 1)^2} \right\},
\]

(14)

Note that \( -r_n'(\theta) > 0 \) for all \( \theta \in (-\frac{\pi}{2}, 0) \), so the sign of \( -g'(\theta) \) is the same as the sign of the numerator of the above expression. Therefore, it suffices to study the behavior of the polynomial in the numerator in the variables \( v = r_n^2 \)
\[
p(v) := -u^2 v^3 - (1 - 2u^2)v^4 + 6v^5 - 2(u^2 + 2\eta^2 + 6)v^2 + (u^2 + 4\eta^2 + 10)v - 3.
\]

Note that \( p(1) = 0 \) so we may write
\[
p(v) = (v - 1)q(v),
\]
\[
q(v) := -u^2 v^3 - (1 - u^2)v^5 + (5 + u^2)v^2 - (u^2 + 4\eta^2 + 7)v + 3.
\]

For general \( u \), the polynomial \( q(v) \) admits at most 4 real roots in the interval \( (0, 1) \), in between these roots \( q \) is either positive or negative (with the opposite sign for \( p(v) \)) which proves the first part of the Theorem. We now study the roots of \( q(v) \) for \( u^2 \leq 1 \). Observe that
\[
q'(v) = -4u^2 v^3 - 3(1 - u^2)v^2 + 2(5 + u^2)v - u^2 - 4\eta^2 - 7,
\]

further note that
\[
q'(v + 1) = -4u^2 v^3 - 3(3u^2 + 1)v^2 - 4(u^2 - 1)v - 4\eta^2,
\]
by Budan’s Theorem [AkS2, Theorem 3], if \( s_{q'(v)} \) is the number of sign changes in the non-zero coefficients of \( q'(v) \) and \( s_{q'(v+1)} \) is the number of sign changes in the non-zero coefficients of \( q'(v+1) \) then

\[
s_{q'(v)} - s_{q'(v+1)} - \#\{ v \in (0, 1) : q'(v) = 0 \}
\]

is a non-negative even integer. When \( u^2 \leq 1, s_{q'(v)} = s_{q'(v+1)} = 2 \) so that \( q'(v) \) has no roots in \( (0, 1] \). Because \( q'(0) = -u^2 - 4\eta^2 - 7 < 0 \), it follows that \( q'(v) < 0 \) for all \( v \in (0, 1] \). Further, \( q(0) = 3 > 0 \) and \( q(1) = -4\eta^2 < 0 \), which implies \( q(v) \) is monotonically decreasing and has exactly one root in \( [0, 1] \). Since the polynomial is in the variable \( r^2_\eta \), we must verify this root is attained in the range of \( r^2_\eta \); to this end note

\[
v^* = \sup_{\theta \in (-\tfrac{\pi}{2}, 0)} r^2_\eta(\theta) = \left( \sqrt{1 + \frac{\eta^2}{4} - \frac{\eta}{2}} \right)^2 \Rightarrow (1 - v^*)^2 = \eta^2 v^*,
\]

and

\[
q(1) - q(v^*) = \int_{0}^{1-v^*} q'(1 - t) \, dt,
\]

\[
= \int_{0}^{1-v^*} t \left( 4u^2 t^2 - 3(3u^2 + 1) t + 4(1 - u^2) \right) \, dt - 4\eta^2(1 - v^*),
\]

\[
= u^2 \eta^4 (v^*)^2 - (3u^2 + 1)(1 - v^*)\eta^2 v^* + 2(1 - u^2)\eta^2 v^* - 4\eta^2(1 - v^*),
\]

\[
= (\eta^4 u^2 + \eta^2 (3u^2 + 1))(v^*)^2 + 5\eta^2(1 - u^2)v^* - 4\eta^2 > q(1),
\]

so \( q(v^*) < 0 \) meaning the unique root of \( q \) in \( [0, 1] \) is achieved for \( v \) of the form \( v = r^2_\eta(\theta) \) and \( \theta \in (-\tfrac{\pi}{2}, 0) \). The second statement of the Theorem now holds because if \( v = r^2_\eta \) is the root of \( q(v) \), the sign of \(-q'(\theta)\) is positive for \( r^2_\eta < r^2_\eta \), and negative for \( r^2_\eta > r^2_\eta \), defining \( \tilde{\theta}_{\eta,u} \) so that \( r^2_\eta = r^2_\eta(\tilde{\theta}_{\eta,u}) \), gives the required result. \( \square \)

Using the above Lemma, we are able to obtain the following bound.

**Lemma 9.** Let \( \psi \in C_b(\mathbb{R}) \) be a bounded continuous test function, then the following estimate holds

\[
\frac{1}{\pi} \int_{\mathbb{R}} \left| \log \left( \frac{2\pi^2 m(x + i\eta)m'(x + i\eta)}{1 - u^2 m^2(x + i\eta)} \right) \psi(x) \right| \, dx
\]

\[
\leq \left\{ \begin{array}{ll}
18 \| \psi \|_{\infty} & \text{for all } u, \\
\frac{4}{\pi} \| \psi \|_{\infty} \text{arg} \{ 1 - u^2 r^2_\eta(\tilde{\theta}_{\eta,u}) \text{exp}(i\tilde{\theta}_{\eta,u}) \} & \text{for } u \leq 1,
\end{array} \right.
\]

where \( \tilde{\theta}_{\eta,u} \) is defined in Lemma 8.

**Proof.** Using Lemma 7, the absolute value of the above integral is the same as

\[
\frac{1}{\pi} \int_{-\pi}^{\pi} \left| \psi(E_\eta(\theta)) \frac{d}{d\theta} \text{arg} \{ 1 - u^2 r^2_\eta(\theta) \text{exp}(i2\theta) \} \right| \, d\theta,
\]

now using Lemma 8, we partition \( (-\pi, 0) \) into subintervals \( P_1, \ldots, P_k \) with \( k \leq 9 \), such that \( \text{arg} \{ 1 - u^2 r^2_\eta(\theta) \text{exp}(i2\theta) \} \) is monotonically increasing or decreasing on
each $P_i$. Using linearity and triangle inequality equation (16) is bounded by
\[
\|\psi\|_{\infty} \sum_{i=1}^{k} \frac{1}{\pi} \int_{P_i} \left| \frac{d}{d\theta} \arg \left\{ 1 - u^2 r_2^2(\theta) \exp(i2\theta) \right\} \right| d\theta \leq 2k \|\psi\|_{\infty} \leq 18 \|\psi\|_{\infty}.
\]
For $u \leq 1$, we use the same style of argument, except now equation (16) is bounded by
\[
\frac{\|\psi\|_{\infty}}{\pi} \left( \int_{-\pi}^{-\theta_{\eta, \nu} - \pi} - \frac{d}{d\theta} \arg \left\{ 1 - u^2 r_2^2(\theta) \exp(i2\theta) \right\} d\theta + \int_{\theta_{\eta, \nu} + \pi}^{\theta_{\eta, \nu}} - \frac{d}{d\theta} \arg \left\{ 1 - u^2 r_2^2(\theta) \exp(i2\theta) \right\} d\theta + \int_{\theta_{\eta, \nu}}^{0} - \frac{d}{d\theta} \arg \left\{ 1 - u^2 r_2^2(\theta) \exp(i2\theta) \right\} d\theta \right),
\]
which, by $r_2(-\pi - \theta) = r_2(\theta)$ for all $\theta \in (-\frac{\pi}{2}, 0)$, equals
\[
\frac{4\|\psi\|_{\infty}}{\pi} \arg \left\{ 1 - u^2 r_2^2(\tilde{\theta}_{\eta, \nu}) \exp(i2\tilde{\theta}_{\eta, \nu}) \right\},
\]
as required. \qed

With these estimates in place, we may now argue that the limit as $\eta_1$ and $\eta_2 \downarrow 0$ can be taken under the integral with respect to $u$.

**Lemma 10.** Let $z = E + i\eta_1$ and $w = F + i\eta_2$. For any $\psi, \phi \in C_b(\mathbb{R})$ we have the following inequalities
\[
\sup_{\eta_1, \eta_2 \leq 1} \int_{\mathbb{R}^2} \frac{\left| \psi(E) \right| \left| \phi(F) \right|}{\pi^2} \left| \begin{array}{c} 2u^2 m(z) m'(z) \\ 1 - u^2 m^2(z) \\ 2u^2 m(w) m'(w) \\ 1 - u^2 m^2(w) \end{array} \right| dE dF \leq 18^2 \|\psi\|_{\infty} \|\phi\|_{\infty} (u^2 1_{u < \frac{1}{2}} + 1_{|u| \geq \frac{1}{2}})
\]
and
\[
\int_{0}^{\infty} \frac{1}{u^{1+\alpha}} \sup_{\eta_1, \eta_2 \leq 1} \left\{ \int_{\mathbb{R}^2} \left( \frac{\left| \psi(E) \right| \left| \phi(F) \right|}{\pi^2} \right) \left( \begin{array}{c} 2u^2 m(z) m'(z) \\ 1 - u^2 m^2(z) \\ 2u^2 m(w) m'(w) \\ 1 - u^2 m^2(w) \end{array} \right) \right\} dE dF \right\} d\alpha \leq \frac{18^2 (2^\alpha)}{4 - \alpha} \|\psi\|_{\infty} \|\phi\|_{\infty}.
\]

**Proof.** Using the notation of Lemma 8
\[
\sup_{\theta \in (-\pi, 0)} \sup_{\eta_1 \leq 1} \left| r_2^2(\theta) \right| \leq 1,
\]
therefore $1 - u^2 r_2^2(\tilde{\theta}_{\eta, \nu}) \exp(i2\tilde{\theta}_{\eta, \nu})$ for any $u < 1$ is away from the branch cut of the argument function and its distance to 1 is bounded by $u^2$. Due to the bound
\[
|\zeta| < \frac{1}{2} \implies |\arg\{1 - \zeta\}| \leq |\zeta|,
\]
we have for $|u| < \frac{1}{2}$

$$\arg \left\{ 1 - u^2 \overline{r}_{\eta_j}(\tilde{\theta}_{\eta_j,u}) \exp(i2\tilde{\theta}_{\eta_j,u}) \right\} \leq u^2.$$  

Next, applying Lemma 9 into the integrand we obtain the upper bound

$$\int_{R^2} |\psi(E)||\phi(F)| \left\{ 3 \left\{ \frac{2u^2m(z)m'(z)}{1 - u^2m^2(z)} \right\} \right\} \left\{ 3 \left\{ \frac{2u^2m(w)m'(w)}{1 - u^2m^2(w)} \right\} \right\} \ dE \ dF$$

$$\leq ||\psi||_{\infty} ||\phi||_{\infty} \left( 16 \prod_{j=1}^{2} \arg \left\{ 1 - u^2 r_{\eta_j}(\tilde{\theta}_{\eta_j,u}) \exp(i2\tilde{\theta}_{\eta_j,u}) \right\} \right) \left( u^4 1_{u < \frac{1}{2}} + 18^2 1_{u \geq \frac{1}{2}} \right),$$

uniformly in $\eta_j \leq 1$, proving the first part of the Lemma. The second part holds by applying the first bound and integrating over $u$. □

With the results in Lemmas 7, 8, 9 and 10 in hand, we will be able to prove Theorem 5. Our sketch is as follows. Given test functions $\psi, \phi \in C_b(\mathbb{R})$ we must compute

$$\lim_{\eta_1, \eta_2 \downarrow 0} \frac{c}{\pi^2 \hbar \alpha} \int_0^{\infty} \frac{1}{u^{1+\alpha}} \int_{R^2} \psi(E)\phi(F) \left[ 3 \left\{ \frac{2u^2m(E + i\eta_1)m'(E + i\eta_1)}{1 - u^2m^2(E + i\eta_1)} \right\} \right]$$

$$\times \left\{ \frac{2u^2m(F + i\eta_2)m'(F + i\eta_2)}{1 - u^2m^2(F + i\eta_2)} \right\} \ dE \ dF \ d\mu,$$

because of the bound in Lemma 10 we may not only interchange orders of integration freely by Fubini Theorem, but we may take the limit in $\eta_1, \eta_2 \downarrow 0$ inside the integral in $u$ by dominated convergence Theorem. It suffices then to compute the limit

$$\lim_{\eta_1, \eta_2 \downarrow 0} \int_{R^2} \psi(E)\phi(F) \left[ 3 \left\{ \frac{2u^2m(E + i\eta_1)m'(E + i\eta_1)}{1 - u^2m^2(E + i\eta_1)} \right\} \right]$$

$$\times \left\{ \frac{2u^2m(F + i\eta_2)m'(F + i\eta_2)}{1 - u^2m^2(F + i\eta_2)} \right\} \ dE \ dF,$$

this is the product of two one-dimensional integrals, for which we will apply dominated convergence once more by starting with the useful representation of Lemma 7.

To facilitate computation, we work with a mollified version of $\psi_t$ of $\psi$ and $\phi_{s,\alpha}$ of $\phi$ so that we may integrate by parts. This mollification poses no issues in the analysis since Lemma 10 implies that uniformly in $\eta_1, \eta_2 \leq 1$ the difference between these limits is bounded by $18^2(2^\alpha)(4 - \alpha)^{-1} ||\psi_t - \psi||_{\infty} ||\phi_{s,\alpha} - \phi||_{\infty}$ which will go to zero when we take the appropriate limits in $s$ and $t$.

**Proof of Theorem 5.** Following the sketch above, let $t > 0$ and consider $\psi_t = P_t * \psi$ where $P_t$ is the Poisson kernel. Recall that $||\psi_t - \psi||_{\infty}$ goes to 0 in the limit that $t \to 0$ and $\psi_t$ is infinitely differentiable, further $||\psi_t'||_{\infty} \leq ||P_t'||_1 ||\psi||_{\infty}$ for every
\[ t > 0. \text{ Applying the representation in Lemma 7 to } \psi_t \text{ we have} \]
\[
\int_{\mathbb{R}} \psi_t(E) \cdot \left\{ \frac{2u^2 m(E + i\eta)m'(E + i\eta)}{1 - u^2 m^2(E + i\eta)} \right\} \, dE
\]
\[
= - \int_{\mathbb{R}} \psi_t(x) \frac{\partial}{\partial x} \arg \left\{ 1 - u^2 m^2(x + i\eta) \right\} \, dx
\]
\[
= \int_{\mathbb{R}} \psi_t'(x) \arg \left\{ 1 - u^2 m^2(x + i\eta) \right\} \, dx.
\]

For each \( u \) we will apply Dominated Convergence Theorem to take the limit \( \eta \downarrow 0 \) inside the integral. To this end note that it suffices to show
\[
\| \psi \|_{\infty} \| P' \|_1 \sup_{\eta \leq 1} \| \arg \left\{ 1 - u^2 m^2(x + i\eta) \right\} \|
\]
is integrable on \( \mathbb{R} \) for almost every \( u > 0 \). Observe that
\[
|\zeta| < \frac{1}{2} \implies |\arg\{1 - \zeta\}| \leq |\zeta|.
\](17)

Recall from Lemma 7 that \( |m^2(x + i\eta)| = r^2_\eta(\theta) \) where \( \theta \) is the argument of \( m(z) \). Further, we have \( x = E_\eta(\theta) \). Let
\[
\epsilon_u := 2^{-\frac{1}{2}} \min(1, u^{-1}) \quad \text{and} \quad \theta_u^* := \arcsin \left( \frac{-\eta \epsilon_u}{1 - \epsilon_u^2} \right),
\]
where \( \arcsin \) is taken to be the principal value. Note that \( r_\eta(\theta_u^*) = \epsilon_u \) and since \( \theta_u^* \in \left( -\frac{\pi}{2}, 0 \right) \) we have
\[
E_\eta(\theta_u^*) = \left( \frac{\epsilon_u^2 + 1}{\epsilon_u (1 - \epsilon_u^2)} \right) \sqrt{\epsilon_u^4 - (2 + \eta^2)\epsilon_u^2 + 1}
\]
\[
\leq \left( \frac{\epsilon_u^2 + 1}{\epsilon_u (1 - \epsilon_u^2)} \right) \sqrt{\epsilon_u^4 - 2\epsilon_u^2 + 1} =: x_u^*,
\]
moved since \( E_\eta(\theta) \) is strictly increasing and is a bijection from \( (-\pi, 0) \) to \( \mathbb{R} \) it follows that
\[
|x| \geq x_u^* \implies |m^2(x + i\eta)| \leq \epsilon_u^2,
\]
since \( x_u^* \) is a constant only depending on \( u \), it follows that uniformly in \( \eta > 0 \) we have
\[
|x| \geq \max(x_u^*, 3) \implies |\arg\{1 - u^2 m^2(x + i\eta)\}| \leq u^2|m^2(x + i\eta)|,
\]
\[
|x| \leq \max(x_u^*, 3) \implies |\arg\{1 - u^2 m^2(x + i\eta)\}| \leq \pi,
\]
but observe that \( \sup_{\eta \leq 1} |m^2(x + i\eta)| \) is integrable over \( |x| \geq \max(x_u^*, 3) \). Hence Dominated Convergence Theorem applies if we prove that the pointwise limit
\[
\lim_{\eta \downarrow 0} \arg\{1 - u^2 m^2(x + i\eta)\},
\]
exists almost everywhere. To compute this limit, we first consider the set of \( x \) with \( |x| < 2 \). In this region, for each point \( x \) there is a small enough \( \eta \) such that \( 1 - u^2 m^2(x + i\eta) \), is not on the branch cut of \( \arg \). Since \( \arg \) is continuous in this region we may take the limit as \( \eta \downarrow 0 \) under the argument. We have the limits
\[
m^2(x + i\eta) \to \begin{cases} 
\frac{x^2 - 2}{2} - i \frac{x \sqrt{4 - x^2}}{2} & |x| < 2 \\
\frac{x^2}{2} - 1 - \frac{x \sgn(x) \sqrt{x^2 - 4}}{2} & |x| \geq 2
\end{cases}
\]
so that

$$|x| < 2 \implies \lim_{\eta \to 0} \arg \left\{ 1 - u^2 m^2(x + i\eta) \right\} = \arg \left\{ 1 + u^2 - \frac{u^2 x^2}{2} + i \frac{u^2 x \sqrt{4 - x^2}}{2} \right\}$$

For $|x| \geq 2$, let $\sigma(x)$ be the semi-circular density, recall

$$m(x + i\eta) = \int_{\mathbb{R}} \frac{(x - y)\sigma(dy)}{(x - y)^2 + \eta^2} - i \int_{\mathbb{R}} \frac{\eta \sigma(dy)}{(x - y)^2 + \eta^2}$$

by squaring this integral representation above, we see for $x \geq 2$, $1 - u^2 m^2(x + i\eta)$ will be in the upper-half plane and for $x \leq -2$, $1 - u^2 m^2(x + i\eta)$ will be in the lower-half plane. The branch cut of arg is approached for $|x| > 2$ only for those $x$

for which

$$\lim_{\eta \to 0} \Re \{1 - u^2 m^2(x + i\eta)\} \leq 0,$$

the limiting region for which this holds is

$$1 \leq u^2 m^2_+(x),$$

corresponding to

$$2 \leq |x| \leq u + \frac{1}{u},$$

which is non-empty only for $u \geq 1$. Hence we have the following limit

$$|x| \geq 2 \implies \lim_{\eta \to 0} \arg \left\{ 1 - u^2 m^2(x + i\eta) \right\} = \pi \left( 1_{2 \leq x \leq u+1} - 1_{-u-1 \leq x \leq -2} \right) 1_{u \geq 1},$$

holding for almost every $u > 0$. Combining these results

$$\lim_{\eta \to 0} \int_{\mathbb{R}} \psi_t'(x) \arg \left\{ 1 - u^2 m^2(x + i\eta) \right\} \, dx =$$

$$\int_{\mathbb{R}} \psi_t'(x) \arg \left\{ 1 + u^2 - \frac{u^2 x^2}{2} + i \frac{u^2 x \sqrt{4 - x^2}}{2} \right\} 1_{|x| \leq 2} \, dx +$$

$$\pi \left( \psi_t(u + u^{-1}) - \psi_t(2) \right) 1_{u \geq 1} - \pi \left( \psi_t(-2) - \psi_t(-u - u^{-1}) \right) 1_{u \geq 1},$$

for almost every $u > 0$. Integrating the first term by parts yields

$$\int_{\mathbb{R}} \psi_t'(x) \arg \left\{ 1 + u^2 - \frac{u^2 x^2}{2} + i \frac{u^2 x \sqrt{4 - x^2}}{2} \right\} 1_{|x| \leq 2} \, dx =$$

$$\int_{\mathbb{R}} \psi_t(x) \frac{d}{dx} \arg \left\{ 1 + u^2 - \frac{u^2 x^2}{2} + i \frac{u^2 x \sqrt{4 - x^2}}{2} \right\} 1_{|x| \leq 2} \, dx$$

$$+ \pi \left( \psi_t(2) + \psi_t(-2) \right) 1_{u \geq 1},$$

holding for almost every point $u > 0$. Note that for $|x| < 2$,

$$\frac{d}{dx} \arg \left\{ 1 + u^2 - \frac{u^2 x^2}{2} + i \frac{u^2 x \sqrt{4 - x^2}}{2} \right\} = \frac{2u^4 + 2u^2 - u^2 x^2}{\sqrt{4 - x^2} \left( -u^2 x^2 + u^4 + 2u^2 + 1 \right)}$$
which is absolutely integrable in \([-2, 2]\) so that
\[
\lim_{\eta \downarrow 0} \int_{\mathbb{R}} \psi(t) \Im \left\{ \frac{2u^2m(E + i\eta)m'(E + i\eta)}{1 - u^2m^2(E + i\eta)} \right\} \, dE
= \int_{\mathbb{R}} \psi(x) \frac{2u^2 + 2u^2 - u^2x^2}{\sqrt{4 - x^2}} \left(-u^2x^2 + u^4 + 2u^2 + 1\right) 1_{|x| \leq 2} \, dx
+ \pi \left\{ \psi(u + u^{-1}) + \psi(-u + u^{-1}) \right\} 1_{u \geq 1}.
\]

Using this argument with the original integral (with \(\phi_s = P_s \ast \phi\)) along with the previous Lemmas allowing us to pull the limit as \(\eta_1, \eta_2 \downarrow 0\) under the integral over \(u\) which implies the limit
\[
\lim_{\eta_1, \eta_2 \downarrow 0} \frac{c}{\pi^2k_0} \int_0^\infty \frac{1}{u^{1+\alpha}} \int_{\mathbb{R}^2} \psi(t) \phi_s(F) \left[ \Im \left\{ \frac{2u^2m(E + i\eta)m'(E + i\eta)}{1 - u^2m^2(E + i\eta)} \right\} \right] \, dE \, dF \, du = c \int_0^\infty \Lambda_u(\psi) \Lambda_u(\phi_s) \, du.
\]

Using the bound in Lemma 10 allows us to take the limit as \(t\) and \(s\) go down to 0 so that the above formula holds for \(\psi\) and \(\phi\) as well (we omit the argument that \(u^{-1-\alpha}\Lambda_u(\psi)\Lambda_u(\phi_s)\) converges in \(L^1(\mathbb{R})\) as \(t, s \downarrow 0\) since it is straightforward). To prove the final statement of the Theorem it suffices to show, for \(z \in \mathbb{C}\backslash\mathbb{R}\),
\[
\Lambda_u \left( \frac{1}{x - z} \right) = \frac{2u^2m(z)m'(z)}{1 - u^2m^2(z)}, \tag{19}
\]

note that the right hand side, for \(u < 1\) is analytic on \(\mathbb{C}\backslash[-2, 2]\) while for \(u \geq 1\) is meromorphic with simple poles at \(\pm(u + u^{-1})\), that is, the function
\[
\Phi(z) := \frac{2u^2m(z)m'(z)}{1 - u^2m^2(z)} + \left( \frac{1}{z - u - u^{-1}} + \frac{1}{z + u + u^{-1}} \right) 1_{u \geq 1}, \tag{20}
\]
is analytic on \(\mathbb{C}\backslash[-2, 2]\). Let
\[
\Phi_{\pm}(x) = \lim_{\eta \downarrow 0} \Phi(x \pm i\eta),
\]
then by the well-known Sokhotski-Plemelj relation \([\text{Ga66} \text{ Section 34.2}]\)
\[
\frac{1}{2\pi i} \int_{\mathbb{R}} \frac{\Phi_{+}(x) - \Phi_{-}(x)}{x - z} \, dx = \Phi(z), \tag{21}
\]
given our computation above,
\[
\frac{\Phi_{+}(x) - \Phi_{-}(x)}{2\pi i} = \frac{2u^4 + 2u^2 - u^2x^2}{\pi \sqrt{4 - x^2} \left(-u^2x^2 + u^4 + 2u^2 + 1\right)} 1_{|x| \leq 2}, \tag{22}
\]
inserting equation (22) and equation (20) into equation (21) gives the desired equation 19 upon rearranging terms. \(\square\)
6. Proof of Corollary

Proof. Using the representation of Theorem, we may write the covariance kernel as the sum

\[ \langle K_\alpha, \psi \otimes \phi \rangle = T_1 + T_2 + T_3 + T_4, \]  

(23)

where we have defined

\[ T_1 := \frac{c}{k_\alpha} \int_{-2}^{2} \int_{-2}^{2} \int_{0}^{\infty} \frac{u^{1-\alpha}(2u^2 - 2 - E^2)(2u^2 + 2 - F^2)}{\pi^2(-u^2E^2 + (u^2 + 1)^2)(-u^2F^2 + (u^2 + 1)^2) \sqrt{4 - E^2 \sqrt{4 - F^2}}} \, \psi(E) \phi(F) \, dE \, dF \, du, \]

\[ T_2 := \frac{c}{k_\alpha} \int_{-2}^{2} \psi(E) \int_{0}^{\infty} \frac{1}{\pi(-u^2E^2 + (u^2 + 1)^2)} \, \phi(u + u^{-1}) + \phi(-u - u^{-1}) \, du, \]

\[ T_3 := \frac{c}{k_\alpha} \int_{-2}^{2} \phi(F) \int_{0}^{\infty} \frac{1}{\pi(-u^2F^2 + (u^2 + 1)^2)} \, \psi(u + u^{-1}) + \psi(-u - u^{-1}) \, du, \]

\[ T_4 := \frac{c}{k_\alpha} \int_{-2}^{2} \int_{-2}^{2} \frac{1}{u^{1-\alpha}(2u^2 - 2 - E^2)(2u^2 + 2 - F^2)} \, \psi(E) \phi(F) \, dE \, dF \, du. \]

(24)

Note that we have freely interchanged orders of integration to obtain the equality \([23]\). We will rewrite each of the integrals in \([24]\) starting with the term \(T_4\) in \([24]\). Consider the change of variables

\[ u = \frac{1}{m_+(s)} \quad s \in [2, \infty), \]

which is well-defined since \(s \in [2, \infty)\), \(m_+\) is a decreasing function and takes values in \((0, 1]\). Using the fixed point equation satisfied by \(m_+\), we change variables for \(T_4\) and obtain

\[ T_4 = \frac{c}{k_\alpha} \int_{2}^{\infty} \frac{\psi(E) + \psi(-E)}{\phi(E) + \phi(-E)} \, \phi(E) \, dE \]

\[ = \frac{c}{k_\alpha} \int_{(\mathbb{R} \setminus [-1, 2])^2} (\delta_{E-F} + \delta_{E+F}) \, m_+(E) \psi(E) \phi(F) \, dE \, dF. \]

(25)

Next, consider the term \(T_3\) in \([24]\) (with the term \(T_2\) being identical). We once more change variables to \(u = \frac{1}{m_+(E)}\) to rewrite this integral as

\[ T_3 = \frac{c}{\pi k_\alpha} \int_{-2}^{2} \int_{-2}^{2} \frac{|m_+(E)|^{1-\alpha}(2 + m_+(E)^2(2 - F^2))m_+'(E)}{(-m_+(E)^2F^2 + (m_+(E)^2 + 1)^2) \sqrt{4 - F^2}} \psi(E) \phi(F) \, dF \, dE \]

\[ + \int_{-2}^{2} \int_{-\infty}^{2} \frac{|m_+(E)|^{1-\alpha}(2 + m_+(E)^2(2 - F^2))m_+'(E)}{(-m_+(E)^2F^2 + (m_+(E)^2 + 1)^2) \sqrt{4 - F^2}} \psi(E) \phi(F) \, dF \, dE. \]

(26)

For the remaining term, \(T_1\), in \([23]\) we will compute for almost every \(E, F \in (-2, 2)\) the integral

\[ \frac{c}{k_\alpha \sqrt{4 - E^2 \sqrt{4 - F^2}}} \int_{0}^{\infty} \frac{u^{3-\alpha}(2u^2 - 2 - E^2)(2u^2 + 2 - F^2)}{\pi^2(-u^2E^2 + (u^2 + 1)^2)(-u^2F^2 + (u^2 + 1)^2)} \, du. \]

(27)
the above integral is simply the pointwise limit
\[
- \lim_{\eta_1, \eta_2 \to 0} \frac{1}{4\pi^2} \left\{ C(E + i\eta_1, F + i\eta_2) + C(E - i\eta_1, F - i\eta_2) - C(E - i\eta_1, F + i\eta_2) - C(E + i\eta_1, F - i\eta_2) \right\},
\]
which is easy to evaluate given the explicit formula in Remark \[\text{as}\] as
\[
\frac{-c}{\pi k_3 \sin \left( \frac{\pi \alpha}{2} \right)} \times \\
\left\{ \frac{m_+(E)m'_+(E)m_+(F)m'_+(F)}{m^2_+(E) - m^2_+(F)} \left[ \left( -m_+(E)^2 \right)^{\frac{\alpha}{2}} - \left( -m_+(F)^2 \right)^{\frac{\alpha}{2}} \right] \right. \\
+ \frac{m_-(E)m'_-(E)m_-(F)m'_-(F)}{m^2_-(E) - m^2_-(F)} \left[ \left( -m_-(E)^2 \right)^{\frac{\alpha}{2}} - \left( -m_-(F)^2 \right)^{\frac{\alpha}{2}} \right] \\
- \frac{m_+(E)m'_+(E)m_-(F)m'_-(F)}{m^2_+(E) - m^2_+(F)} \left[ \left( -m_+(E)^2 \right)^{\frac{\alpha}{2}} - \left( -m_+(F)^2 \right)^{\frac{\alpha}{2}} \right] \\
- \frac{m_-(E)m'_-(E)m_+(F)m'_+(F)}{m^2_-(E) - m^2_+(F)} \left[ \left( -m_-(E)^2 \right)^{\frac{\alpha}{2}} - \left( -m_+(F)^2 \right)^{\frac{\alpha}{2}} \right] \right\}. \tag{28}
\]
Using the formulas \[25, 26\] and \[28\] for \( T_1, T_2, T_3 \) and \( T_4 \) in the original equations \[23\] and \[24\], proves the first part of the Corollary. We now describe the simplifications yielding the second part of the Corollary pertaining to the case \( \alpha = 3 \). When \( \alpha = 3 \), equation \[28\] further simplifies to
\[
\frac{c}{\pi k_3} \left\{ \frac{im_+(E)m'_+(E)m_+(F)m'_+(F)}{m_+(E) + m_+(F)} - \frac{im_-(E)m'_-(E)m_-(F)m'_-(F)}{m_-(E) + m_-(F)} \\
- \frac{im_+(E)m'_+(E)m_-(F)m'_-(F)}{m_+(E) - m_-(F)} + \frac{im_-(E)m'_-(E)m_+(F)m'_+(F)}{m_-(E) - m_+(F)} \right\} \\
= \frac{2c}{\pi k_3} \left\{ \frac{m_+(E)m'_+(E)m_+(F)m'_+(F)}{m_+(E) + m_+(F)} + \frac{m_-(E)m'_-(E)m_+(F)m'_+(F)}{m_-(E) - m_+(F)} \right\}, \tag{29}
\]
to compute this imaginary part we combine the terms inside the brackets
\[
m_+(F)m'_+(F) \times \\
\frac{m_+(E)m'_+(E)(m_-(E) - m_+(F)) + m_-(E)m'_-(E)(m_+(E) + m_+(F))}{(m_+(E) + m_+(F))(m_-(E) - m_+(F))},
\]
note that \(|m_\pm(E)|^2 = 1\) so that
\[
m_+(F)m'_+(F) \frac{2\Re\{m'_+(E)\} + 2im_+(F)\Im\{m_-(E)m'_-(E)\}}{1 + im_+(F)sqrt{4 - E^2 - m_+(F)^2}}, \tag{30}
\]
note that
\[
2\Re\{m'_+(E)\} + 2im_+(F)\Im\{m_-(E)m'_-(E)\} = 1 + im_+(F)\frac{2 - E^2}{\sqrt{4 - E^2}}.
\]
multiplying this by the conjugate of the denominator of equation (30) gives

\[
\left(1 + im_+(F)\frac{2 - E^2}{\sqrt{4 - E^2}}\right)\left(1 - im_-(F)\sqrt{4 - E^2} - m_-(F)^2\right)
\]

\[
= 1 - im_-(F)\sqrt{4 - E^2} - m_-(F)^2 - i(m_+(F) - m_-(F))\frac{2 - E^2}{\sqrt{4 - E^2}} + (2 - E^2),
\]

multiplying this by the remaining factor of \(m_+(F)m'_+(F)\) in equation (30) gives

\[
-m'_+(F)(\sqrt{4 - F^2} + \sqrt{4 - E^2}) + m_+(F)m'_+(F)(\sqrt{4 - F^2} + \sqrt{4 - E^2})\frac{2 - E^2}{\sqrt{4 - E^2}}
\]

the imaginary part of this expression is

\[
-\left(\frac{\sqrt{4 - F^2} + \sqrt{4 - E^2}}{2}\right)(1 + \frac{(2 - F^2) (2 - E^2)}{\sqrt{4 - F^2}\sqrt{4 - E^2}})
\]

the modulus square of the denominator of equation (30) can similarly be verified to equal

\[
(\sqrt{4 - E^2} + \sqrt{4 - F^2})^2,
\]

hence the imaginary part of equation (30) inserted into equation (29) yields

\[
\frac{15c}{8\sqrt{\pi}n} \left(\frac{1}{\sqrt{4 - F^2} + \sqrt{4 - E^2}}\right)\left(1 + \frac{(2 - F^2) (2 - E^2)}{\sqrt{4 - F^2}\sqrt{4 - E^2}}\right)
\]

as claimed. \(\square\)

**Appendix A. Integral Identities**

**Lemma 11.** Let \(\alpha \in (2, 4)\) and let \(\sigma \in \mathbb{C}\) satisfy \(\text{Re}(\sigma) < 0\). Then

\[
\int_0^\infty \frac{\exp(r\sigma) - r\sigma - 1}{r^{\frac{\alpha}{2} + 1}} \, dr = k_\alpha(-\sigma)^{\frac{\alpha}{2}}
\]

where

\[
k_\alpha = \frac{\Gamma\left(2 - \frac{\alpha}{2}\right)}{\frac{\alpha}{2}(\frac{\alpha}{2} - 1)},
\]

and the principal branch of the power function is taken.

**Proof.** Starting from,

\[
\int_0^1 (1 - \nu) \frac{d^2}{d\nu^2} \exp(r\nu\sigma) \, d\nu = \exp(r\sigma) - r\sigma - 1,
\]

we have

\[
\int_0^\infty \frac{\exp(r\sigma) - r\sigma - 1}{r^{\frac{\alpha}{2} + 1}} \, dr = \int_0^\infty \int_0^1 r^{1 - \frac{\alpha}{2}} \sigma^2 (1 - \nu) \exp(r\nu\sigma) \, d\nu \, dr,
\]

interchanging orders of integration yields

\[
\int_0^1 \int_0^\infty (1 - \nu)\sigma^2 r^{1 - \frac{\alpha}{2}} \exp(r\nu\sigma) \, dr \, d\nu,
\]

changing variables in the integrand for \(r\) yields

\[
\sigma^2 |\sigma|^{\frac{\alpha}{2} - 2} \int_0^1 (1 - \nu)\nu^{\frac{\alpha}{2} - 2} \, d\nu \int_0^\infty r^{1 - \frac{\alpha}{2}} \exp(\omega r) \, dr \quad \text{where} \quad \omega := \frac{\sigma}{|\sigma|},
\]
The integral over \( \nu \) is the Beta integral:

\[
\int_0^1 \nu^{\frac{\alpha}{2} - 1}(1 - \nu) \, d\nu = B\left(\frac{\alpha}{2} - 1, 2\right) = \frac{\Gamma\left(\frac{\alpha}{2} - 1\right)\Gamma(2)}{\Gamma\left(\frac{\alpha}{2} + 1\right)}
\]

while the integral

\[
\int_0^\infty r^{1 - \frac{\alpha}{2}} \exp(\omega r) \, dr,
\]

will be computed using a Cauchy Integral argument. Since \( \Re(\sigma) < 0 \), this implies \(-\omega\) lies in the right-half plane \( \{ z \in \mathbb{C} : \Re(z) > 0 \} \), in the sector defined by the positive real line and the ray along \(-\omega\), the function

\[
f(z) = z^{1 - \frac{\alpha}{2}} \exp(\omega z),
\]

with principal branch cut selected for the power \( z^{1 - \frac{\alpha}{2}} \), is holomorphic with an integrable singularity at the origin, so

\[
\int_0^\infty r^{1 - \frac{\alpha}{2}} \exp(\omega r) \, dr = -\omega(1 - \omega)^{1 - \frac{\alpha}{2}} \int_0^\infty t^{1 - \frac{\alpha}{2}} \exp(-t) \, dt,
\]

\[
= -\omega(1 - \omega)^{1 - \frac{\alpha}{2}} \Gamma\left(2 - \frac{\alpha}{2}\right),
\]

note that since \( \omega \) is on the unit circle, away from the principal branch cut, and \( \frac{\alpha}{2} - 1 \in (0, 1) \),

\[
(1 - \omega)^{1 - \frac{\alpha}{2}} = (-\omega)^{\frac{\alpha}{2} - 1},
\]

which implies

\[
\sigma^2|\sigma|^{\frac{\alpha}{2} - 2}(1 - \omega)(1 - \omega)^{1 - \frac{\alpha}{2}} = -\sigma(-\sigma)^{\frac{\alpha}{2} - 1} = (-\sigma)^{\frac{\alpha}{2}}
\]

where in the last equality, we used that \( \Re(\sigma) < 0, \frac{\alpha}{2} - 1 \in (0, 1) \) (so that we do not cross branch cuts while combining powers). To obtain the formula for the constant note

\[
\frac{\Gamma\left(\frac{\alpha}{2} - 1\right)\Gamma(2)\Gamma\left(2 - \frac{\alpha}{2}\right)}{\Gamma\left(\frac{\alpha}{2} + 1\right)} = \frac{\Gamma\left(2 - \frac{\alpha}{2}\right)}{\frac{\alpha}{2}(\frac{\alpha}{2} - 1)} = k_\alpha.
\]

\[\square\]

**Lemma 12.** Let \( \alpha \in (2, 4) \) and let \( \sigma_1, \sigma_2 \in \mathbb{C}\backslash \mathbb{R}_- \). Then

\[
\int_0^\infty \frac{r^{\frac{\alpha}{2} - 1}}{(r - \sigma_1)(r - \sigma_2)} \, dr = \frac{\pi}{\sin\left(\frac{\alpha\pi}{2}\right)} (\sigma_1 - \sigma_2)^{\frac{\alpha}{2} - 1} \left(\sigma_1^{\frac{\alpha}{2} - 1} - \sigma_2^{\frac{\alpha}{2} - 1}\right)
\]

where the principal branch cut is taken for the power function. When \( \sigma_1 = \sigma_2 = \sigma \in \mathbb{C}\backslash \mathbb{R}_- \), then the integral equals

\[
\int_0^\infty \frac{r^{\frac{\alpha}{2} - 1}}{(r - \sigma)^2} \, dr = \frac{\pi(\frac{\alpha}{2} - 1)(\sigma)^{\frac{\alpha}{2} - 2}}{\sin\left(\frac{\alpha\pi}{2}\right)}.
\]

**Proof.** Let

\[
f(z) := \frac{z^{\frac{\alpha}{2} - 1}}{(z + \sigma_1)(z + \sigma_2)},
\]
if we integrate this function on the counterclockwise keyhole contour that avoids the branch cut on the negative reals then the residue theorem yields

\[
\left( \exp \left[ i\pi \left( \frac{\alpha}{2} - 1 \right) \right] - \exp \left[ -i\pi \left( \frac{\alpha}{2} - 1 \right) \right] \right) \int_0^\infty \frac{r^{\frac{\alpha}{2} - 1}}{(r - \sigma_1)(r - \sigma_2)} \, dr = \frac{2\pi i}{\sigma_2 - \sigma_1} \left( \left(-\sigma_1\right)^{\frac{\alpha}{2} - 1} - \left(-\sigma_2\right)^{\frac{\alpha}{2} - 1} \right)
\]

which when rearranged yields

\[
\int_0^\infty \frac{r^{\frac{\alpha}{2} - 1}}{(r - \sigma_1)(r - \sigma_2)} \, dr = \frac{\pi}{\sin \left( \frac{\pi}{\alpha} \right)} \frac{\pi}{(\sigma_1 - \sigma_2)} \left( \left(-\sigma_1\right)^{\frac{\alpha}{2} - 1} - \left(-\sigma_2\right)^{\frac{\alpha}{2} - 1} \right)
\]

as required. Setting \( \sigma_2 = \sigma \) and letting \( \sigma_1 \to \sigma_2 \) yields the second integral. \( \square \)

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