Estimating Network Link Characteristics using Packet-Pair Dispersion: A Discrete Time Queueing Theoretic View

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Abstract

Packet-dispersion based measurement tools insert pairs of probe packets with a known separation into the network for transmission over a unicast path or a multicast tree. Samples of the separation between the probe pairs at the destination(s) are observed. Heuristic techniques are then used by these tools to estimate the path characteristics from the observations. In this paper we present a queueing theoretic setting for packet-dispersion based probing. Analogous to network tomography, we develop techniques to estimate the parameters of the arrival process to the individual links from the samples of the output separations, i.e., from the end-to-end measurements.

The links are modeled as independent discrete time queues with i.i.d. arrivals. We first obtain an algorithm to obtain the (joint) distribution of the separation between the probes at the destination(s) for a given distribution of the spacing at the input. The parameter estimates of the arrival process are obtained as the minimizer of a cost function between the empirical and calculated distributions. We also carry out extensive simulations and numerical experiments to study the performance of the estimation algorithm under the fairly ‘harsh’ conditions of non stationarity of the arrival process. We find that the estimations work fairly well for two queues in series and for multicast.

Keywords: Network tomography, packet-pair measurements, discrete time queues, packet-pair tomography, multicast queues.
1 Introduction and Background

Network tomography is the estimation of detailed statistics of performance parameters from aggregate or end-to-end measurements of some measurable quantity. The term was first coined in [1], where the problem of estimating flow volumes from link volumes was analyzed. This is also called the traffic matrix estimation problem and has been extensively studied in the transportation literature, for which, an excellent survey is available in [2]. The traffic estimation problem is now being addressed by the networking community (3,4,5,6,7). Another network tomography problem is that of estimating link delay statistics from the samples of end-to-end delays experienced by multicast packets [8,9]. Much of the work in this area has been in estimating the link or segment delay distributions from the end-to-end measurements. See [10] for a fairly comprehensive survey of both delay and traffic estimation tomography problems and solutions related to networking.

The bandwidth on a network path has also been of significant interest and a number of tools have been developed for its estimation. Two kinds of bandwidth related metrics are often estimated. The bottleneck bandwidth is the maximum transfer rate that could be achieved over a path and is essentially the minimum of the transmission rates on the links in the path. The available bandwidth is the portion of the bottleneck bandwidth not used by competing traffic and depends on the traffic load at the inputs to the links.

Many tools have been developed to measure the bottleneck and available bandwidths. To the best of our knowledge, all these tools are based on either ‘packet-pair’ or ‘packet-dispersion’ techniques. The packet-pair technique was first described in [11] to measure the bottleneck bandwidth on a path where the links are rate-servers. In this technique two back-to-back packets of equal length are transmitted by the source. It can be shown that the ratio of the probe packet length to the separation between them at the receiver is the service rate for the packet at the bottleneck link on the path. This idea has been used to develop many tools, e.g., pathchar [12] and clink [13], that measure the bottleneck capacity of a network path. Inspired by the packet-pair technique, the packet-dispersion, also called packet-spacing, technique has been developed to measure the available bandwidth of an Internet path [14,15,16,17,18]. In this technique, a source transmits a number of probe packets with a predetermined separation, say $d$. The samples of the separation at the receiver, say $d_r$, are then used to estimate the available bandwidth on the path. pathload [14] and pathChirp [15] are some examples of tools that use the packet-spacing technique to measure
the available bandwidth. See [16] for an excellent survey of the different packet-spacing techniques used in the measurement tools and [17] for an experimental comparison of the tools. A non cooperative version that does not require a measurement-enabled receiver is described in [18].

Packet-pair and packet-dispersion techniques have an important advantage in that they do not require the sender and the receiver to be synchronized in time. This makes them important in practice. However, the tools that employ these techniques are based on informal arguments. There have been excellent statistical studies based on experiments on packet-pair techniques (e.g., [19, 20]). Recently, there have been a few theoretical models developed for packet dispersion based bandwidth probing techniques [21, 22, 23, 24, 25, 26, 27]. In [21, 22] it is shown that the output dispersion of probe pair samples the sum of three correlated random processes that are derived from the traffic arrival process. The asymptotics of these and the probing process are also analysed. The measurement methods of some bandwidth estimation tools are then related to the analytical models developed. These results are then extended to multihop paths in [23]. In [24], the network is assumed to be a time-invariant, min-plus system that has an unknown service curve. This service curve estimation method is developed for different probing techniques.

While the above works provide interesting theoretical insights into the probing process, our work is more closely related to that of [25, 26, 27]. In [27], a probe train is assumed and the actual delay of each of the probe packets is assumed to be known at the output of the path. The delay sequence is assumed to form a Markov chain whose transition probability matrix is estimated. An inversion is defined to estimate the characteristics of the arrival process from the transition probabilities of the delay Markov chain. In [25], the bottleneck link is modeled as an M/D/1 queue, and the expected dispersion is calculated using the transient analysis. The method of moments is then used to estimate the arrival rate. A similar method is followed in [26] where the more general M/G/1 queue is considered and Takacs’ integro-differential equation is used to obtain the distribution of the dispersion.

In this paper we develop techniques to estimate the parameters of the arrival process distribution on the individual links on a path or a multicast tree from packet-dispersion samples in a queuing theoretic setting. Packet-pair probes with predetermined separation between them are injected at a source. These probe packets traverse discrete-time queues with i.i.d. packet arrivals on a path or in a multicast tree. Samples of the separation between the probe packets at the output nodes of the path or the multicast tree
are used to estimate the parameters of the arrival process distribution at the individual links. The approach is as follows. First, for a single queue and for a given separation between the probes at the input, we derive the conditional distribution of the separation between the probes at the output of the queue in terms of the distribution of the arrival process to the discrete time queue. The key intermediate result here is the joint distribution of the number arrivals to the queue and the number of departures from the queue between the slots in which the probes are injected. Then the output separation is obtained for any given distribution of the input separation. Since the input separation distribution is known, (we assume it to be fixed) this is applied recursively on the path to obtain the separation distribution at the outputs. The parameter estimator is the minimizer of a suitable distance function between the empirical and the theoretical distributions of the output separations.

We first develop our results by assuming that the probe packets are served after all the packets that arrive in their slot, i.e., the probe packets have the least priority among the packets that arrive in their slot. Extension to the case of the probe packet having the same priority (or any arbitrary fixed priority) as the other packets that arrive in the slot will be along the same lines and is also presented. We present numerical results only for the first case, i.e., when the probes have lowest priority.

Our modeling assumption is similar to that of [27] in that we also consider discrete time queues. While [27] considers the delay process of the probe sequence, we consider the dispersion of the packet pair. The inversion methods are also different. While we consider a general i.i.d. arrival process for the discrete time queue, [25] considers an M/D/1 queue and obtains approximations, [26] considers an M/G/1 queue.

The rest of the paper is organized as follows. In the next section, we describe the problem setting and the notation that we will be using in the paper. In Section 3 we consider a single pair of probes that are injected into a stationary discrete-time queue and develop an algorithm for computing the distribution of the separation between a pair of probes at the output of a single queue when the distribution of the separation at the input of the queue and also the arrival process distribution are known. In Section 4 we consider sending the probes over a multi-link path and over a multicast tree and obtain the distribution of the output separation(s) between the probes at their outputs. In Section 5 we describe the method to estimate the parameters of the arrival process at the queues. The numerical evaluation of the estimates for the case of Poisson arrivals are discussed in detail in Section 6. In Section 7 we describe a generalization and an extension.
to the basic theory developed in Section 3. In particular, we outline the method for computing the output separation distribution when the probes have equal priority with the packets arriving in the same slot. We conclude with a discussion in Section 8 where we explore connections with an earlier theoretical study of packet-dispersion techniques. We also contrast our results with delay-tomography results.

2 Notation and Preliminaries

We first describe the notation used in this paper. For pedagogical purposes, we consider a packet queuing system. In this section we will consider a single-server queue with infinite buffers and FCFS service discipline. The service time of each packet is equal to the slot length. The convention will be as follows. Packet arrivals to the queue in slot $n$ will occur at the beginning of the slot and the departures will occur at the end of the slot. Thus arrivals in a slot will be available for departure in the slot. We observe the queue at the beginning of a slot, just after the arrival instant. Let $A_n$ denote the number of packets that arrive in slot $n$, $X_n$ the number of departures in slot $n$ and $Q_n$ the number of packets in the queue when the queue is observed at the beginning of the slot. This convention is shown in Figure 1. Of course, $X_n$ can only be 0 or 1. Further, for $m \leq n$, $A_{m,n} := A_m + A_{m+1} + \cdots + A_n$ will denote the cumulative arrivals in the interval $[m, n]$ and $X_{m,n} := X_m + X_{m+1} + \cdots + X_n$ will denote the cumulative departures in the interval $[m, n]$. In particular, $X_{m,m} = X_m$ and $A_{m,m} = A_m$. Note that $X_{m,n}$ is a function of $Q_m$ and $[A_m, \ldots, A_n]$. $X_{m,n}^q$ will denote the number of departures in slots $[m, n]$ that the same arrival sequence $[A_m, \ldots, A_n]$ would cause with $Q_m = q$. Observe that this makes $X_{m,n}^q$ independent of the distribution of $Q_m$ and a function of only $[A_m, \ldots, A_n]$. The work-conserving,
FCFS service discipline will be applied at the granularity of the batch of packets that arrive in a slot and the packets in a batch will be queued in a random order. The queue evolution equation as per the above convention will be

\begin{align*}
Q_{n+1} &= Q_n + A_{n+1} - X_n, \\
Q_{n+m} &= Q_n + A_{n+1,n+m} - X_{n,n+m-1} \quad (1)
\end{align*}

In the rest of this paper we assume that the arrival sequence \( A_n \) is i.i.d. and let \( p_k := \Pr (A_0 = k) \). Recall that if \( \lambda := \mathbb{E}(A_0) < 1 \) then the queue length process is stationary and the moment generating function \( Q(z) \) of the stationary queue length distribution \( \pi_q := \Pr (Q_0 = q) \) is given by

\[ Q(z) = \frac{(1 - \lambda)(z - 1)\mathcal{P}(z)}{z - \mathcal{P}(z)}. \]

where \( \mathcal{P}(z) \) is the moment generating function of \( p_k \). In fact \( \pi_q \) can be obtained using the following recursion.

\begin{align*}
\pi_q &:= \Pr (Q_0 = q) \\
= \begin{cases} 
1 - \mathbb{E}(A_0) = 1 - \lambda & \text{for } q = 0, \\
\frac{1}{p_0} (\pi_0(1 - p_0)) & \text{for } q = 1, \\
\frac{1}{p_0} (\pi_{q-1}(1 - p_1) - \pi_{q-2}p_2 - \cdots \\
- \pi_1p_{q-1} - \pi_0p_{q-1}) & \text{for } q > 1.
\end{cases}
\end{align*} \tag{2}

### 3 Passage of a Probe Pair through a Discrete Time Queue

In this section we consider injecting a single probe pair into a stationary queue and derive the distribution of the output separation, \( d_o \), as a function of the input separation, \( d_i \), and the parameters of the packet arrival process \( \{A_n\} \). For a given distribution of the input separation, the distribution of \( d_o \) is obtained as

\[ \Pr (d_o = s) = \sum_{j=1}^{\infty} \Pr (d_o = s|d_i = j) \Pr (d_i = j). \tag{3} \]

Let us consider a queue in steady state. Consider two probe packets, denoted by \( P_1 \) and \( P_2 \), inserted into the discrete time queue in slots 0 and
$d_i$ respectively. Since we are considering stationary queues, without loss of
generality, we can relabel the slots in this manner. These probe packets are
enqueued along with the other packets that arrive in their respective slots
and are available for transmission in the same slot in which they arrive. In
this section, we assume that the probe packets have the least priority among
the packets that arrive in their slot.

Let $D_1$ and $D_2$ denote the departure slots for the first and the second
probe respectively. Let $d_o := D_2 - D_1$ denote the dispersion of the probe
packets at the output of the queue. Clearly,

\begin{align*}
D_1 &= Q_0, \\
D_2 &= d_i + Q_{d_i} = d_i + Q_0 + 1 + A_{1,d_i} - X_{0,d_i-1}, \\
d_o := D_2 - D_1 &= d_i + 1 + A_{1,d_i} - X_{0,d_i-1} .
\end{align*}

$d_o - d_i$ is the packet-dispersion. In the above, $Q_0$ does not include probe
packet $P_1$ and $Q_{d_i}$ is the number of packets in the queue at the beginning
of slot $d_i$ not including $P_2$. Also, $A_{d_i}$ does not include $P_2$ while $X_{0,d_i-1}$ includes
the possible departure of $P_1$. Note that $X_{0,d_i-1} \geq X_0 = 1$ since the queue
has $Q_0 + 1$ packets (including $P_1$) in slot 0. Observe that $d_o$, the separation
between the probes at the output of the queue, is affected by the arrivals
and departures between the arrivals of $P_1$ and $P_2$, i.e., by $A_{1,d_i}$ and $X_{0,d_i-1}$.

Thus, to obtain the conditional distribution of the output separation, given
the input separation, we need to obtain the joint distribution of the number
of arrivals and departures between the two probe pairs. Computation of
this joint distribution is the key to the computation of the distribution of
the output separation.

From Eqn. 4 for a given $d_i$, say $d_i = m$, the distribution of the output
separation $d_o$ can be expressed as

\[
Pr (d_o = s | d_i = m) = \sum_{l=1}^{m} Pr (X_{0,m-1} = l, A_{1,m} = s - m + l - 1) \tag{5}
\]

for $s = 1, 2, \ldots$. Since we are assuming that we are probing a stationary
queue, the first probe packet $P_1$ will see the queue in steady state. We can
then obtain the joint distribution of $X_{0,m-1}$ and $A_{1,m}$ as

\[
Pr (X_{0,m-1} = l, A_{1,m} = j) \\
= \sum_{q=0}^{\infty} Pr (X_{0,m-1} = l, A_{1,m} = j | Q_0 = q) Pr (Q_0 = q) \\
= \sum_{q=0}^{\infty} Pr (X_{0,m-1} = l, A_{1,m} = j) Pr (Q_0 = q) . \tag{6}
\]
We can see that $Q_0$ and $A_{1,m}$ are independent of each other, but $X_{0,m-1}$ is dependent on both $Q_0$ and on the finite sequence $A_1, \ldots, A_m$. We make the reasonable assumption that the distribution of $Q_0$ is known since it can be derived from the distribution of $A_0$, albeit as a function of its parameters. Hence, to obtain the distribution of $d_o$, we first need to obtain the joint distribution of $X_{0,m-1}$ and $A_{1,m}$ conditioned on $Q_0 = q$, i.e., the joint distribution of $X_{0,m-1}^q$ and $A_{1,m}$. This is derived in the following subsection.

3.1 Joint Distribution of $A_{1,m}$ and $X_{0,m-1}^q$

In this section we first obtain a recursion on $m$, for the joint distribution of $A_{1,m}$ and $X_{0,m-1}^0$ and then, for arbitrary $q$, express the joint distribution of $A_{1,m}$ and $X_{0,m-1}^q$ in terms of the joint distribution of $A_{1,m}$ and $X_{0,m-1}^0$. We thus begin with the case of $Q_0 = 0$. From our notation, this means that when the first probe packet is injected into the queue in slot 0 it is the only packet in the queue. Since there is the probe packet to be transmitted in slot 0, $X_{0,0}^0 = 1$. Thus we have the following base equations for the joint distribution.

$$
\Pr (X_{0,0}^0 = l, A_{1,1} = j) = \begin{cases} 
\Pr (A_1 = j) & \text{for } l = 1, \\
0 & \text{otherwise.}
\end{cases}
$$

(7)

Now, for arbitrary $m > 1$, we can write the following one-step recursion which also spells out the kind of recursion we seek.

$$
\Pr (X_{0,m-1}^0 = l, A_{1,m} = j) \\
= \sum_i \sum_n \Pr (X_{0,m-1}^0 = l, A_{1,m} = j | X_{0,m-2}^0 = i, A_{1,m-1} = n) \\
\times \Pr (X_{0,m-2}^0 = i, A_{1,m-1} = n).
$$

(8)

For ease of exposition, define

$$
\psi_{i,j}^{i,n} := \Pr (X_{0,m-1}^0 = l, A_{1,m} = j | X_{0,m-2}^0 = i, A_{1,m-1} = n).
$$

We now consider three cases.

1. If $X_{0,m-2}^0 = i < l - 1$ or $X_{0,m-2}^0 = i > l$, then $\psi_{i,j}^{i,n} = 0$. This is because, for there to be $l$ departures in slots $[0, m-1]$, the number of departures in slots $0, \ldots, (m - 2)$, denoted by $i$, should be either $l$ or $l - 1$. 

2. If $X_{0,m-2}^0 = i = l - 1$, then there has to be a departure in slot $m - 1$. Since $Q_0 = 0$, this is possible if, and only if, $A_{1,m-1} = n \geq l - 1$ because only then will the total number of arrivals in slots $[0, m - 1]$, including $P_1$, be greater than or equal to $l$. Hence, for $n < l - 1$, $\psi_{l,j}^{i,n} = 0$ and for $A_{1,m-1} = n \geq l - 1$, $X_{0,m-1}^0 = l$ is a sure event and $A_{1,m} = j$ requires that $j - n$ packets have to arrive in slot $m$ giving us $\psi_{l,j}^{i,n} = \Pr (A_m = j - n)$.

3. For $X_{0,m-2}^0 = i = l$, there should be no departure in slot $m - 1$. Since $Q_0 = 0$, $P_1$ has departed in slot 0. Hence the total number of departures in slots $[1, m - 2]$, i.e., $l - 1$, should be the same as $A_{1,m-1} = n$. Therefore, for $n \neq l - 1$, $\psi_{l,j}^{i,n} = 0$. Thus for $i = l$, only $n = l - 1$ will yield a non zero $\psi_{l,j}^{i,n}$ when the number of arrivals in slot $m$ is $j - n = j - l + 1$, i.e., for $i = l$ and $n = l - 1$, we have $\psi_{l,j}^{i,n} = \Pr (A_m = j - l + 1)$.

Combining the above three cases, we can rewrite the recursion in Eqn. 8 for the joint distribution of $X_{0,m-1}^0$ and $A_{1,m}$ as

$$\Pr (X_{0,m-1}^0 = l, A_{1,m} = j) = \sum_{n=l-1}^{j} \Pr (A_m = j - n) \Pr (X_{0,m-2}^0 = l - 1, A_{1,m-1} = n) + \Pr (A_m = j - l + 1) \Pr (X_{0,m-2}^0 = l, A_{1,m-1} = l - 1).$$

(9)

We obtain the joint distribution $X_{0,m-1}^q$ and $A_{1,m}$ by a suitable transformation of the joint distribution of $X_{0,m-1}^0$ and $A_{1,m}$ derived above. To obtain $\Pr (X_{0,m-1}^q = l, A_{1,m} = j)$ we consider the following cases. Clearly, this probability is zero if $l > m$ (the number of departures is greater than the number of slots) and for $j < 0$.

1. The number of departures in slots $[0, m - 1]$ is less than the number of slots, i.e., $l < m$. We consider two sub cases.

   (a) Clearly, $q > l - 1$ is not possible because in that case, the number of departures including the probe packet $P_1$ would have been at least $l + 1$. Therefore, $\Pr (X_{0,m-1}^q = l, A_{1,m} = j) = 0$ for $q > l - 1$.

   (b) For $q \leq l - 1$, the event

   $$\{(A_1, \ldots, A_m) \mid X_{0,m-1}^q = l\}$$
is the same as the event
\[
\{(A_1, \ldots, A_m) \mid X_{0,m-1}^0 = l - q\}
\]
and so
\[
\{(A_1, \ldots, A_m) \mid X_{0,m-1}^q = l, A_{1,m} = j\} = \\
\{(A_1, \ldots, A_m) \mid X_{0,m-1}^0 = l - q, A_{1,m} = j\}
\]
Thus, for \(q \leq l - 1\), we get
\[
\Pr \left( X_{0,m-1}^q = l, A_{1,m} = j \right) = \\
\Pr \left( X_{0,m-1}^0 = l - q, A_{1,m} = j \right)
\]
2. If \(l = m\), then there is a departure in every slot in \([0, m - 1]\). We once again consider two sub cases.

(a) If \(Q_0 = q < (l - 1)\), then the event
\[
\{(A_1, \ldots, A_m) \mid X_{0,m-1}^q = m, A_{1,m} = j\}
\]
is the same as the union of the disjoint events
\[
\{(A_1, \ldots, A_m) \mid X_{0,m-1}^0 = t, A_{1,m} = j\}
\]
for \(t = m - q, \ldots, m\). Hence, for this case, we can write
\[
\Pr \left( X_{0,m-1}^q = l, A_{1,m} = j \right) = \\
\sum_{t=m-q}^{m} \Pr \left( X_{0,m-1}^0 = t, A_{1,m} = j \right).
\]
(b) If the queue starts with \(Q_0 = q \geq (l - 1)\) packets, then there will be a departure in every slot and \(X_{0,m-1}^q = l\) is a sure event and so for this case \(\Pr \left( X_{0,m-1}^q = l, A_{1,m} = j \right) = \Pr (A_{1,m} = j)\).
Summarizing the above, we have the following transformation.

\[
\Pr \left( X_{0,m-1}^q = l, A_{1,m} = j \right) = \begin{cases} 
\Pr \left( X_{0,m-1}^0 = l - q, A_{1,m} = j \right) & \text{for } l < m \text{ and } q \leq l - 1 \\
\sum_{t=m-q}^{m} \Pr \left( X_{0,m-1}^0 = t, A_{1,m} = j \right) & \text{for } l = m \text{ and } q < l - 1 \\
\Pr (A_{1,m} = j) & \text{for } l = m \text{ and } q \geq l - 1 \\
0 & \text{otherwise.}
\end{cases}
\]

(10)

Once the joint distribution of \( X_{0,m-1}^q \) and \( A_{1,m} \) is found using the joint distribution of \( X_{0,m-1}^0 \) and \( A_{1,m} \), the joint distribution of \( X_{0,m-1}^0 \) and \( A_{1,m} \) can be found from Eqn. 6.

The infinite summations in Eqns. 6 and 3 will need to be truncated in practice. Let \( N_1 \) and \( N_2 \) respectively denote the truncation limits. Also, in Eqn. 5 we need to limit the range of the output separation for which we will compute the probabilities. Let \( N_3 \) be this limit. The algorithm for computing the distribution of the output separation is summarized in Fig. 2.

The recursion in step 3b of Fig. 2 is computationally the most intensive part of the algorithm. It can be shown that this step requires

\[
\left( \frac{N_3(N_3 + 1)}{2} + \frac{5}{2} \right) \frac{N_2(N_2 + 1)}{2} - \frac{N_2(N_2 + 1)(2N_2 + 1)}{12}
\]

additions and multiplications.

4 Passage of a Probe Pair through a Network of Independent Queues

We now characterize the output separation between the probes as they pass through multiple queues. Since we are able to express the distribution of the output separation of the probe pair in terms of the distributions of the input separation and the arrival process distribution, we can obtain the distribution of the output separation at the output of a path of independent queues fairly easily.

We first consider the case when the probe pair passes through a path consisting of \( K \) independent queues as shown in Figure 3. The probe packets leaving queue \( k \) are placed in queue \( (k + 1) \) after an arbitrary but fixed delay. Since the delays are fixed, the separation between the probe pairs at the input to queue \( (k + 1) \) is
1. Input: Arrival distribution \( \{p_k\}_{k \geq 0} \)
   Input separation distribution \( \{\Pr(d_i = k)\}_{k \geq 1} \).

2. Obtain \( \pi_q \) using the following equations.

   \[
   \pi_q := \Pr(Q_0 = q) = \begin{cases} 
   1 - \mathbb{E}(A_0) = 1 - \lambda & \text{for } q = 0, \\
   \frac{1}{p_0} (\pi_0(1 - p_0)) & \text{for } q = 1, \\
   \frac{1}{p_0} (\pi_q^{-1}(1 - p_1) - \pi_{q-2}p_2 - \cdots - \pi_1p_{q-1} - \pi_0p_{q-1}) & \text{for } q > 1.
   \end{cases}
   \]

3. Obtain the joint distribution of \( \{X_{0,m-1}^q, A_1, m\} \) for each \( q \) using the following equations.

   (a) Base Equations

   \[
   \Pr(X_{0,0}^0 = l, A_{1,1} = j) = \begin{cases} 
   \Pr(A_1 = j) & \text{for } l = 1 \text{ and } j \geq 0 \\
   0 & \text{otherwise}
   \end{cases}
   \]

   (b) Recursions

   \[
   \Pr(X_{0,m-1}^0 = l, A_{1,m} = j) = \sum_{n=l-1}^{j} \Pr(A_m = j - n) \Pr(X_{0,m-2}^0 = l-1, A_{1,m-1} = n) \\
   + \Pr(A_m = j - l + 1) \Pr(X_{0,m-2}^0 = l, A_{1,m-1} = l-1).
   \]

   (c) Transformations

   \[
   \Pr(X_{0,m-1}^q = l, A_{1,m} = j) = \begin{cases} 
   \Pr(X_{0,m-1}^0 = l - q, A_{1,m} = j) & l < m \text{ and } q \leq l - 1 \\
   \sum_{t=m-q}^{l} \Pr(X_{0,m-1}^0 = t, A_{1,m} = j) & l = m \text{ and } q < l - 1 \\
   \Pr(A_{1,m} = j) & l = m \text{ and } q \geq l - 1 \\
   0 & \text{otherwise}
   \end{cases}
   \]

4. Obtain the joint distribution of \( \{X_{0,m-1}, A_1, m\} \) by using the following equations.

   \[
   \Pr(X_{0,m-1} = l, A_{1,m} = j) = \sum_{q=0}^{N_1} \Pr(X_{0,m-1}^q = l, A_{1,m} = j) \pi_q
   \]

5. Obtain the output separation distribution for a given input separation, \( d_i \), say \( d_i = m \).

   \[
   \Pr(d_o = s|d_i = m) = \sum_{l=1}^{m} \Pr(X_{0,m-1} = l, A_{1,m} = s - m + l - 1) \quad \text{for } s = 1, 2, \ldots, N_3.
   \]

6. Obtain the distribution of the output separation for a given input separation distribution.

   \[
   \Pr(d_o = s) = \sum_{j=1}^{N_2} \Pr(d_o = s|d_i = j) \Pr(d_i = j).
   \]
the same as that at the output of queue \( k \). (Generalization to variable delay is presented in Section \( \PageIndex{7} \).) Let \( d_0 \) be the separation at the input to queue 1 and \( d_k \) the separation between the probe pairs at the output of queue \( k \) with \( d_{\text{out}} := d_K \).

For \( k = 1, 2, \ldots, K \), let \( A^{(k)}_n \) denote the i.i.d. arrival sequence into queue \( k, p^{(k)}_i \), the distribution of \( A^{(k)}_n \) and \( \pi^{(k)}_i \) the stationary distribution of the queue occupancy at the beginning of a slot. Probe packets \( P_1 \) and \( P_2 \) are injected into queue 1 in slots 0 and \( d_0 \) respectively with a known distribution of \( d_0 \). The distribution of \( d_k \) is obtained from the recursive relation

\[
\Pr (d_k = s) = \sum_{j=1}^{\infty} \Pr (d_k = s | d_{k-1} = j) \Pr (d_{k-1} = j).
\]

The distribution of \( d_K \) can be obtained by repeated use of the algorithm in the previous section from the distribution of \( d_0 \). Notice that, if \( d_0 = 1 \), then \( d_1 \) has the same distribution as \( A^{(1)}_0 \).

The second case that is of interest is when the probe pairs traverse a multicast tree of independent queues. Figure \( \PageIndex{4} \) shows an example of a three-node tree. Here
copies of the probe packet are made at the output of Queue 1 and the copies are simultaneously placed in queues 2 and 3 after arbitrary but fixed delays on each of the links. As with the case of a path, since the delays are fixed, the separation between the probe pairs at the input to queues 2 and 3 is the same as that at the output of queue 1. We are interested in the joint distribution of the separation at the output of both queues 2 and 3, \(d_2\) and \(d_3\) respectively. This joint distribution is given by

\[
\Pr(d_2 = r, d_3 = s) = \sum_{t=1}^{\infty} \Pr(d_2 = r, d_3 = s | d_1 = t) \Pr(d_1 = t)
\]

\[
= \sum_{t=1}^{\infty} \Pr(d_2 = r | d_1 = t) \Pr(d_3 = s | d_1 = t) \Pr(d_1 = t)
\]

(12)

The last equality follows because conditional on \(d_1\), \(d_2\) and \(d_3\) are independent. The individual conditional probabilities can be obtained as before.

The above ideas can be extended to compute the joint distribution of the separation at the leaves of an arbitrary tree rooted at the node which introduces the probe pairs into the network.

5 Estimating the Arrival Process Parameters

The algorithm developed for computing the distribution of the output separation between the probes can, in principle, be used to estimate the traffic parameters in the network provided the parameters are identifiable from the distribution. However, often the parameters may not be identifiable from the distribution, i.e., the same distribution may be resulted from two different sets of traffic parameters. For instance, consider a series of two queues with independent Poisson traffic with means \(\lambda_1\) and \(\lambda_2\). The output separation has the same distribution for \((\lambda_1, \lambda_2) = (0, \lambda)\) and \((\lambda_1, \lambda_2) = (\lambda, 0)\). We performed some simulation experiments to judge the viability of using this distribution for traffic parameter estimation. However, these simulations use ideal settings like independent Poisson traffic arrivals to different queues which are usually not strictly satisfied by practical networks, and as a result, should be taken only as proof of concept. They do not guarantee that the method will work in practical networks.

Consider a rooted tree of \(K\) independent queues. Probe pairs are introduced into this network at the root of the tree with unit separation. While the theory allows for an arbitrary but known distribution of the input separation, we choose a fixed input separation of one slot. This minimizes the computational requirements.

The separation between the probe pairs at the output of the leaves is observed. We assume that there is no loss of probe pairs in the network. We also assume that the link delays, i.e., the delay between the output of a queue and the input to the next queue on a path is arbitrary but fixed.
For \( k = 1, \ldots, K \), let \( A_n^{(k)} \) be the arrival sequence of packets into queue \( k \). For queue \( k \), \( A_n^{(k)} \) is an i.i.d. sequence and further, the sequences for the different queues are independent. Let \( \lambda_k \) be the vector of parameters of the distribution of \( A_0^{(k)} \) and let us define \( \Lambda := [\lambda_1, \ldots, \lambda_K] \). Our goal is to estimate \( \Lambda \) by observing only the separation of the probes at the output of the leaves of the paths. Let \( j_1, j_2, \ldots, j_\kappa \) denote these leaf nodes and \( d_{j_1}, d_{j_2}, \ldots, d_{j_\kappa} \) the separation of the probe pairs at the output of these queues.

We send a sequence of probe pairs and corresponding to every probe pair we obtain an observation of the vector \( d := [d_{j_1}, d_{j_2}, \ldots, d_{j_\kappa}] \). From these observations we can obtain the empirical distribution of \( d \), denoted by \( \hat{\Psi}_d \). We probe with a very low rate, i.e., the interval between the probe pairs is made very large so that the probing load is negligible. This allows us to assume that the stationary distribution of the occupancy of queue \( k \) is governed only by the distribution of \( A_0^{(k)} \) and is a function of only \( \lambda_k \). Further, we assume that the first packet of every probe pair, packet \( P_1 \), ‘sees’ each of the queues in steady state. This allows us to express the distribution of \( d \) as a function of only \( \Lambda \). We then use the results of the previous sections to compute the analytical distribution \( \Psi_d(\Lambda) \) for a given value of \( \Lambda \). Let \( D \left( \hat{\Psi}_d, \Psi_d(\Lambda) \right) \) denote a suitably defined distance between \( \hat{\Psi}_d \) and \( \Psi_d(\Lambda) \). The estimate \( \hat{\Lambda} \) of the parameter vector can be obtained as

\[
\hat{\Lambda} = \arg \min_{\Lambda} D \left( \hat{\Psi}_d, \Psi_d(\Lambda) \right)
\]

i.e., \( \hat{\Lambda} \) is obtained as the minimizer of the distance between the empirical and the analytical distributions of the separation vector.

A well known measure of the distance between two distributions is the Kullback-Leibler (KL) distance. The KL distance between the empirical and the analytical distributions for a given \( \Lambda \), \( D_{KL} \left( \hat{\Psi}_d, \Psi_d(\Lambda) \right) \), is defined as

\[
D_{KL} \left( \hat{\Psi}_d, \Psi_d(\Lambda) \right) := \sum_x \hat{\Psi}_{d=x} \log \frac{\hat{\Psi}_{d=x}}{\Psi_{d=x}(\Lambda)}.
\]

It can be shown that the minimizer of \( D_{KL} \left( \hat{\Psi}_d, \Psi_d(\Lambda) \right) \) is the maximum-likelihood estimate of \( \Lambda \). Alternatively, we could use the squared Euclidean distance, as the measure of the distance between the two distributions. The squared Euclidean distance \( D_E \left( \hat{\Psi}_d, \Psi_d(\Lambda), \right) \) between the empirical and the analytical distributions for a given \( \Lambda \) is given by

\[
D_E \left( \hat{\Psi}_d, \Psi_d(\Lambda) \right) := \sum_x \left( \hat{\Psi}_{d=x} - \Psi_{d=x}(\Lambda) \right)^2.
\]

A simplistic way to obtain \( \hat{\Lambda} \) is as follows. The analytical distribution is computed and stored for all the values of the parameter vector \( \Lambda \) over a suitably discretized grid on the feasible range of \( \Lambda \). We obtain the empirical distribution \( \hat{\Psi}_d \)
from a sufficiently large number of probes. The minimizer of \( D(\hat{\Psi}_d, \Psi_d(\Lambda)) \) is then obtained by an exhaustive search over the grid. To find a reasonably accurate estimate, the exhaustive search will require a very fine grid and consequently a large number of distribution calculations. Rather than an exhaustive search, we can also use an optimization method to reach the minimum of \( D(\hat{\Psi}_d, \Psi_d(\Lambda)) \).

As we had mentioned earlier, for the case of one queue, if we fix \( d_0 = 1 \), then \( d_1 \) will have the same distribution as \( A_1^{(0)} \) and estimating the parameters of \( A_1^{(0)} \) is straightforward. We will not pursue that any more. The case of a two-queue path has interesting numerical properties and we explore them in detail in the next section. This will give us an insight into the difficulties of probing paths that have multiple bottlenecks. We expand upon this in Section 7. We will also see that the estimation quality when probing a multicast tree is significantly better.

6 Numerical Results

In this section we present some numerical results obtained as follows. Different paths and multicast trees of independent discrete-time queues are simulated. The probe packets are injected into the system at very low probing rate. The passage of the probes through the different queues are simulated and the output separation obtained for each probe. We do not simulate lost probes although it is easy to see that they do not affect the algorithm. Although our results are applicable for all distributions of the arrival process, we report our results for the case when the packet arrivals form an i.i.d. Poisson sequence. Since Poisson is a one-parameter distribution, it is easier to discuss the issues in the estimation of \( \Lambda \).

We first consider a two-queue path. Let \( \lambda_1 \) and \( \lambda_2 \) be the arrival rates of packets to queues 1 and 2 respectively. Probe pairs are generated according to a Poisson process. We obtain the empirical distribution from \( 2 \times 10^8 \) slots of simulation time. Table 1 shows the estimates that minimize the squared Euclidean distance and KL distance between the empirical distribution and the calculated distribution for different true values of \( \lambda_1 \) and \( \lambda_2 \) and for two probing rates. Notice that both the distance measures give comparable precision in almost all cases. Since the squared Euclidean distance is computationally simpler than the KL distance, the rest of the numerical results are based on the squared Euclidean distance.

The exhaustive search method has the drawback that it is primarily a one time estimation and not an adaptive one. An alternative is to reach the optimum value by a suitable optimization algorithm, for example, a steepest descent algorithm. This may not however give the best estimate if the distance function has a local minima. We performed an extensive numerical investigation of the nature of the cost function for the two-queue case. However, the cost function was seen to be non-convex. The level sets of the Euclidan cost function as shown in Fig. 6 are clearly not convex. This may present some problem in convergence of an optimization technique. However, the limited numerical simulations performed by us gave reasonably good performance under ideal setup.
Table 1: Exhaustive search based estimates for the cascade of two queues

| True values $(\lambda_1, \lambda_2)$ | inter probe mean | Estimates in 3 different executions | Euclidean cost based | KL distance based |
|--------------------------------------|------------------|-------------------------------------|----------------------|------------------|
|                                      |                  | $(\lambda_1, \lambda_2)$           | $(\lambda_1, \lambda_2)$ | $(\lambda_1, \lambda_2)$ |
| $(0.9, 0.8)$                         | 1000             | $(0.91, 0.79)$                      | $(0.91, 0.79)$        | $(0.91, 0.79)$ |
| $(0.5, 0.6)$                         | 200              | $(0.54, 0.57)$                      | $(0.51, 0.59)$        | $(0.54, 0.57)$ |
| $(0.7, 0.4)$                         | 1000             | $(0.70, 0.40)$                      | $(0.69, 0.41)$        | $(0.70, 0.40)$ |
| $(0.2, 0.3)$                         | 200              | $(0.20, 0.30)$                      | $(0.19, 0.31)$        | $(0.20, 0.30)$ |

Figure 5: A typical Euclidean cost as a function of $(\lambda_1, \lambda_2)$. This plot was obtained with $\lambda_1 = 0.3$ and $\lambda_2 = 0.8$. 

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Figure 6: Contours of the Euclidean cost function between the analytical and the empirical distributions for $\lambda_1 = 0.3$ and $\lambda_2 = 0.8$. The contours were plotted by taking logarithm of the cost function so that more details can be seen near the valley (that almost touches zero).
Fig. 5 shows the plot of a typical Euclidean cost function for an empirical distribution obtained from simulation with $(\lambda_1, \lambda_2) = (0.3, 0.8)$. The minima is at $(0.31, 0.79)$. However, notice that the function shows a distinct valley along a line with an extremely low slope along it. The existence of the valley is more clear from the contour plot of the cost function shown in Fig. 6. The steepest descent method will result in a very slow convergence due to zig-zagging along the valley.

Our main simulation involves an iterative stochastic gradient based online estimation. We divide the probe sequence into equal sized blocks and compute the ‘instantaneous’ empirical distribution of $d$ for every block. In our simulations we have used a block size of 500. Let $\hat{\Psi}_{d,n}^{\text{inst}}$ be the instantaneous empirical distribution obtained from the $n$-th block of probes. The overall empirical distribution after the $n$-th iteration, $\hat{\Psi}_{d,n}$, is updated using the exponential update rule

$$\hat{\Psi}_{d,n} = \begin{cases} \hat{\Psi}_{d,1}^{\text{inst}} & \text{if } n = 1 \\ (1 - a)\hat{\Psi}_{d,n-1} + a\hat{\Psi}_{d,n}^{\text{inst}} & \text{if } n > 1. \end{cases} \quad (13)$$

Here $a$ is a constant between 0 and 1. We used $a = 0.05$ in all our simulations. For each $n$, we then update our estimate of $\Lambda$ based on the current empirical distribution $\hat{\Psi}_{d,n}$. We use the steepest descent algorithm for one iteration for each $n$ and update the estimate of $\Lambda$ as

$$\hat{\Lambda}_n = \hat{\Lambda}_{n-1} - \alpha_n \frac{\nabla D \left( \hat{\Psi}_{d,n}, \Psi_d(\Lambda) \right)}{\| \nabla D \left( \hat{\Psi}_{d,n}, \Psi_d(\Lambda) \right) \|}$$

where $\| \cdot \|$ denotes the length of a vector and $\alpha_n$ is the step size. We used an adaptive step size so that whenever a particular step increases the distance, the step size is decreased and whenever the gradient is more than a threshold, the step size is increased. The increase and decrease of the step size is not done if it is respectively above or below some thresholds. If the arrival process is non stationary, this scheme allows us to track slowly varying $\Lambda$. Our simulations will study the effectiveness of such tracking.

We first consider the two-queue path. $\lambda_1$ and $\lambda_2$ are varied linearly with time and a step change is also simultaneously introduced. Fig. 7 plots the true values of $\lambda_1$ and $\lambda_2$ and the estimates as functions of time. For the estimation, the initial estimates are taken as $\hat{\lambda}_1 = \hat{\lambda}_2 = 0.5$. From these plots we see that the steepest descent algorithm converges and tracks the true values reasonably well even when there is step change. It is instructive to study the evolution of the estimate. Fig. 8 shows the evolution of the estimate pair $(\hat{\lambda}_1, \hat{\lambda}_2)$. A close look at the figure reveals that initially as well as when the parameters change suddenly, the estimates go to the nearest point on the new valley and then zig-zags along the valley towards the true values.

We next consider the three-queue multicast tree of Fig. 4. As with the two-queue case, the true values of $\lambda_i$ are varied linearly and also a step change is introduced. In Fig. 9 we plot $\lambda_1$, $\lambda_2$ and $\lambda_3$ as functions of time along with the true
Figure 7: The estimates of $\lambda_1$ and $\lambda_2$ and their true values for two-queue path. The straight lines represent the true values of the parameters.
values. Observe that these estimates are far superior to those for the two-queue path. This is because here we have the empirical joint distribution of the output separations at two leaf nodes of the network and this gives much more information about the arrivals to the queues.

The slow convergence of the adaptive estimation due to the valley in the Euclidean cost function that we saw for the two-queue path is expected to be even more pronounced for a three-queue path. In fact, in our simulations, the estimates did not converge to the true values even after $2 \times 10^8$ slots of simulation time. We conjecture that there will be a valley along a two dimensional manifold.

We have seen that in a multicast network, having multiple outputs through different paths helps improve the quality of the estimation. However two or more queues in series makes for slow convergence of the estimates. To investigate the interplay of these two we carried out the estimation for a four-queue multicast tree shown in Fig. [10]. The plots of the estimates and the true values of the arrival means are shown in Fig. [11]. As expected, we observed faster convergence for this network. However, the queues 2 and 4 are in series in this network. The estimates of $\lambda_2$ and $\lambda_4$ can be seen to be worse than the estimates for the two-queue path. This is possibly because of the error in the estimate of $\lambda_1$ (this misleads the estimation of $\lambda_2$ and $\lambda_4$ with an incorrect distribution of the probe separation at the input of queue 2) and also because the probe pairs entering queue 2 are not separated by a fixed one slot. Hence, the convergence of the estimates of $\lambda_2$ and $\lambda_4$ is not as fast.
Figure 9: The estimates of $\lambda_1$, $\lambda_2$ and $\lambda_3$ and their true values for the multicast network.

Figure 10: The four-queue multicast tree used in some of the simulation results.
In the adaptive estimation, we need to consider two performance criteria. If the arrivals are stationary, we need to minimize the error in the estimate. However, if the arrivals are non-stationary, we would like the estimates to track the true values closely. Achieving these objectives is governed by the choices of the various parameters of the estimation algorithm. In the rest of this section we will discuss the role of these parameters. The performance of the estimation was seen to be quite sensitive to the selection of the parameters, and quite a bit of tuning was necessary to get the performance shown in the figures.

Choice of $a$, block length and $\alpha_n$: The parameter $a$ (in Eqn. 13) and the number of probe pairs in a block dictate how fast the empirical distribution tracks the true distribution of the output separations. However, there is a trade-off between tracking the true distribution for non-stationary arrivals and the variance of the empirical distribution under stationary arrivals. Increasing $a$ or decreasing the block size enables the empirical distribution to track the actual distribution faster but increases the variance of the empirical distribution under stationary arrivals. This reflects in $\Lambda_n$ tracking the true value faster but in the components of $\Lambda_n$
having larger variance under stationary arrivals.

The step size $\alpha_n$ also plays a role in the convergence rate and the tracking ability of the estimation. A larger $\alpha_n$ enables faster tracking but results in larger variance of the estimates under stationary arrivals.

*How well to track the minima of $D(\hat{\Psi}_{d,n}, \Psi_d(\Lambda))$*: In our simulations, we have used one iteration of steepest descent for each $n$. Alternatively we could use an exhaustive search to obtain the minima of $D(\hat{\Psi}_{d,n}, \Psi_d(\Lambda))$. As a second alternative, we may execute many iterations of steepest descent for each $n$. This may involve a fixed number of iterations or termination based on a specified condition. A larger number of iterations for each $n$ will enable $\hat{\Lambda}_n$ to be nearer the minima of the current cost function $D(\hat{\Psi}_{d,n}, \Psi_d(\Lambda))$. Hence, a larger number of iterations for each $n$ will result in the estimate $\hat{\Lambda}_n$ tracking the minima of $D(\hat{\Psi}_{d,n}, \Psi_d(\Lambda))$ more closely. Although this results in better tracking, it makes the estimate more tuned to the current empirical distribution and as a result more sensitive to the variation of the current empirical distribution under stationary arrivals.

*Alternative Optimization Methods*: The valley in the cost function makes the convergence of steepest descent algorithm very slow due to the typical zig-zag path followed by this algorithm. Other optimization algorithms like conjugate gradient or Newton’s algorithm may also be tried to overcome the problem.

*Method of Moments Based Estimation*: A common method in parameter estimation is the method of moments. A similar approach can be considered here. We can compute a few moments of the calculated and the empirical distribution and try to match them by varying the parameters. In the absence of closed form expressions for the moments, we may again need to estimate by minimizing a suitable cost function between the calculated moments and the moments from the empirical distributions. However, since we do not have a way of calculating the moments directly without calculating the distributions first, this method presents little attraction, especially since we can expect to lose some information in working only with some moments instead of the distributions themselves. Indeed, the results obtained by using the squared Euclidean distance between the vectors of the first ten moments of the calculated and the empirical distributions gave no promising results. It is also to be noted that computation of higher order moments is computationally intensive and needs numerical caution in dealing with large numbers.

*Probing rate*: Finally, we remark on the probing rate. A higher probing rate gives us more samples and hence a quicker convergence for stationary arrivals and a better ability to track fast changing arrival processes. However, the probing process makes the aggregate arrival process, including the probe packets, non Poisson and disturbs the estimate because the calculated distribution will not be correct. Further, the samples may not be independent if the probes are closely spaced. To minimize the effect, the mean probe arrivals should be small. In our simulations, the times at which probes $P_1$ are inserted were generated according to a Poisson process of rate 0.005.
7 An Extension and a Generalization

In this section we consider an extension and a generalization to the algorithms presented in Sections 3 and 4. First, we relax the assumption that the probe packet has the least priority within its batch. We will see that this essentially involves an extension of the joint distribution of $A_{1,m}$ and $X_{n,m-1}$ of Section 6.1 to include packets that arrive in the same slot as the probes but may be queued after the probe packet. We then present a generalization in which we allow the transfer delays in the network to be random rather than fixed but arbitrary.

7.1 Generalization to Equal Priority for the probes

In our discussions earlier, we assumed that the probe packets have the lowest priority among the packets that arrive in the slot. We now consider the case when all the packets that arrive in a slot have the same priority and the position of the probe packets within the batch may have a distribution.

For convenience, we redefine the notation and let $Q_n$ be the number of packets in the queue at the beginning of slot $n$ but before the arrival of the packets in slot $n$. This new notation is shown in Figure 12. Under this notation the queue evolution described by Eqn. 1 changes to

\[
Q_{n+1} = Q_n + A_n - X_n ,
\]

\[
Q_{n+m} = Q_n + A_{n,n+m-1} - X_{n,n+m-1} .
\]  

Note that according to this definition the stationary distribution of $Q_n$ would be different from that given in Eqn. 2. The probability generating function for the stationary distribution would be

\[
Q(z) = \frac{(1-\lambda)(z-1)}{z - \hat{P}(z)}
\]
and the recursion for the probabilities will be

\[
\pi_q = \begin{cases} 
\frac{1-\lambda}{p_0} & \text{for } q = 0 \\
\frac{p_0}{p_0} (1 - p_0 - p_1) & \text{for } q = 1 \\
\frac{1}{p_0} (\pi_{q-1}(1 - p_1) - \pi_{q-2}p_2) & \text{for } q > 1.
\end{cases}
\]

For a probe packet arriving into the queue in slot \(k\), define \(A'_k\) (resp. \(\tilde{A}_k\)) to be the number of packets from those that arrived in slot \(k\) that are queued before (resp. after) the probe packet. Thus \(A'_k + \tilde{A}_k = A_k\). Also, for probe packets arriving in slots \(i\) and \(j\), define, \(A'_{i,j} := A_i + A_{i+1} + \cdots + A_{j-1} + A'_j\), \(\tilde{A}_{i,j} := \tilde{A}_i + A_{i+1} + \cdots + A_{j-1} + A_j\), \(\bar{A}_{i,j} := \tilde{A}_i + A_{i+1} + \cdots + A_{j-1} + A'_j\).

As before, we assume that the probe packets \(P_1\) and \(P_2\) enter the queue in slots 0 and \(d_i\) respectively. They depart in slots \(D_1\) and \(D_2\). Along the lines of Eqn. 4 we obtain the output separation of the probe packets as follows.

\[
D_1 = Q_0 + A'_0, \\
D_2 = d_i + Q_{d_i} + A'_{d_i}, \\
d_{\text{out}} := D_2 - D_1 = d_i + 1 + \tilde{A}_{0,d_i} - X_{0,d_i-1}.
\]

(15)

For any \(d_i = m+1 > 1\), to obtain the distribution of the output separation, we thus need to know the joint distribution of \(X_{0,m}, \tilde{A}_{0,m+1}\).

The distribution of \(A'_{m}\) and \(\tilde{A}_m\) is derived from the distribution of \(A_0\) as

\[
\Pr(A'_{m} = k) = \Pr(\tilde{A}_m = k) = \sum_{j=k}^{\infty} \frac{1}{j+1} \Pr(A_0 = j).
\]

Following the procedure in obtaining the recursion in Section 4 we begin by defining \(X_{0,m}^{q,a}\) to be the number of departures in slots 0, \ldots, \(m\) when \(Q_0 = q\) and \(A_0 = a\).

As before, we begin by assuming \(Q_0 = A_0 = 0\) and compute the joint distribution of \(X_{0,0}^{0,0}\) and \(A_{1,m}\). The base equation will be

\[
\Pr\left(X_{0,0}^{0,0} = l, A_{1,0} = j\right) = \begin{cases} 
1 & \text{for } l = 1 \text{ and } j = 0, \\
0 & \text{otherwise}.
\end{cases}
\]

(16)

Here \(A_{1,0}\) is an empty summation and we define it to be 0. We first obtain the recursion for the joint distribution of \(X_{0,m}^{0,0}\) and \(A_{1,m}\) and then the joint distribution
of $X_{0,m}^{0,0}$ and $A_{1,m+1}'$. Following the arguments of Section 3 we obtain the following recursions.

$$
\Pr \left( X_{0,m}^{0,0} = l, A_{1,m} = j \right)
= \begin{cases} 
\Pr \left( X_{0,m-1}^{0,0} = l, A_{1,m-1} = l - 1 \right) \\
\times \Pr \left( A_m = 0 \right) \delta(j - l + 1) \\
+ \sum_{n=l-2}^{j} \Pr \left( X_{0,m-1}^{0,0} = l - 1, A_{1,m-1} = n \right) \\
\times \Pr \left( A_m = j - n \right), \\
& \text{for } l \leq m + 1, \text{ and } j > l - 2, \\
0 & \text{otherwise}
\end{cases}
$$

(17)

$$
\Pr \left( X_{0,m}^{0,0} = l, A_{1,m+1}' = j \right) = \\
\sum_{k=0}^{j} \Pr \left( X_{0,m}^{0,0} = l, A_{1,m} = j - k \right) \Pr \left( A_{1,m+1}' = k \right),
$$

(18)

where $\delta(\cdot)$ is the Kronecker delta function. If $Q_0 = q$ and $A_0 = a$, then, following the arguments in the development of Eqn. 10 the joint distribution of $X_{0,m}^{q,a}$ and $A_{1,m+1}'$ would be obtained in exactly the same manner as Eqn. 10 except that $q$ would be replaced by $q + a$. Thus we will have

$$
\Pr \left( X_{0,m}^{q,a} = l, A_{1,m+1}' = j \right)
= \begin{cases} 
\Pr \left( X_{0,m}^{0,0} = l - q - a, A_{1,m+1}' = j \right) \\
& \text{for } l < m + 1 \text{ and } q + a \leq l - 1, \\
\sum_{t=m+1-q-a}^{m+1} \Pr \left( X_{0,m}^{0,0} = t, A_{1,m+1}' = j \right) \\
& \text{for } l = m + 1 \text{ and } q + a < l - 1, \\
\Pr \left( A_{1,m}' = j \right) \\
& \text{for } l = m + 1 \text{ and } q + a \geq l - 1, \\
0 & \text{otherwise}.
\end{cases}
$$

(19)

Of the $a$ packets that arrive in slot 0, if $\tilde{a}$ are queued after $P_1$, then we need that many fewer arrivals in 1, 2, ..., $m$. Hence, we have

$$
\Pr \left( X_{0,m}^{q,a} = l, \tilde{A}_{0,m+1} = j \right)
= \sum_{a=0}^{a} \frac{1}{a + 1} \Pr \left( X_{0,m}^{q,a} = l, A_{1,m+1}' = j - \tilde{a} \right).
$$

(20)

Finally, we can uncondition on $q$ and $a$ as

$$
\Pr \left( X_{0,m} = l, \tilde{A}_{0,m+1} = j \right) = \sum_{q=0}^{\infty} \sum_{a=0}^{\infty} \Pr \left( A_0 = a \right) \\
\Pr (Q_0 = q) \Pr \left( X_{0,m}^{q,a} = l, \tilde{A}_{0,m+1} = j \right).
$$

(21)
The output separation of the probes can now be obtained following the same method as in Section 3.

### 7.2 Random Delays on the Links

In deriving the distribution of the output separation, in Eqns. 11 and 12, we assumed that the transfer delays from the output of a queue to the input of the next queue on the path is arbitrary but fixed. We can relax this requirement as follows.

Consider the series of queues of Figure 3 first. In going from the output of queue $k$ to the input of queue $k+1$, let $P_1$ be delayed by $t_1$ slots and $P_2$ by $t_2$ slots. Let $d_k$ be the probe separation at the output of queue $k$. The probe separation at the input to queue $k+1$ is $d_k = d_{k-1} + (t_2 - t_1)$. It is reasonable to assume that the order of the probes is maintained in the transfer from $k$ to $k+1$. Hence $t_1$ and $t_2$ cannot be independent. If the conditional distribution of $d_k'$ given $d_k$ is known then Eqn. 11 can be generalized as

$$
Pr(d_k = s) = \sum_{j=1}^{\infty} Pr(d_k = s | d_{k-1} = j) Pr(d_{k-1} = j)
$$

Along similar lines, we can generalize for random transfer delays on the multicast tree. The schematic is shown in Fig. 13. Eqn. 12 can be generalized as

$$
Pr(d_2 = r, d_3 = s) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sum_{l=1}^{\infty} Pr(d_2 = r | d_{1,2} = l) Pr(d_3 = s | d_{1,2} = m) \times Pr(d_{1,2} = l; d_{1,3} = m; d_1 = n) Pr(d_1 = n)
$$

There are many uses for this generalization. If the transmission rates on the links are different, then the separation is compressed or expanded depending on
whether the downstream link has a higher or lower data rate respectively. The conditional distribution of the separation can be obtained by knowing the two transmission rates.

A second use is in tomography of paths with many links with one or two of them being ‘bottleneck links.’ Consider a $K$-queue path. Let links $i$ and $j$, $1 \leq i < j \leq K$, be the bottleneck links. The delays from the source to the input of link $i$, from the output of $i$ to input of $j$ and from the output of $j$ to the destination can have simple models. For example, let $K = 3$, $i = 1$ and $j = 3$ with link 2 being lightly loaded. We can assume that over link 2, each of the probes $P_1$ and $P_2$ will see at most one cross traffic packet independently. Let $\alpha$ be the probability of this event. Then we can see that, if the input separation is greater than 1, the separation increases by 1 if only $P_2$ sees a cross traffic packet, while it shrinks by 1 if only $P_1$ sees such a packet. If the input separation is 1, then the output separation will also be 1 if neither $P_2$ does not see a cross traffic packet. We can then use

$$\Pr (d_2 = n|d_1 = m) = \begin{cases} 
\alpha & \text{if } m = 1 \text{ and } n = 2 \\
1 - \alpha & \text{if } m = n = 1 \\
\alpha(1 - \alpha) & \text{if } n = m + 1 \text{ or } n = m - 1 \text{ and } m > 1 \\
1 - 2\alpha(1 - \alpha) & \text{if } n = m \text{ and } m > 1 \\
0 & \text{otherwise.}
\end{cases}$$

Here link 2 is playing the role of the random delay element.

## 8 Discussion and Conclusion

We conclude with a discussion on the connection with some earlier results on the analysis of packet-dispersion based probing and also on network delay tomography.

We first explore the connection of this work with that in [21][22]. A theoretical analysis of packet pair probing of a single bottleneck path is carried out in [21]. A continuous time model is used and the key result is Theorem 2. This theorem states that for a probe pair inserted into the queue at time $a_1$ with input separation $\delta$ and probe packet length $s$, the output separation $\delta'$ is given by

$$\delta' = \frac{Y_\delta(a_1)}{C} + \frac{s}{C} + \max\left(\frac{B_\delta(a_1) - s}{C}, 0\right) \quad (22)$$

where

- $C$ is the capacity of the bottleneck link,
- $Y_\delta(t)$ is the work (sum of the service times of the packets) that enters the queue in $(t, t + \delta)$ and
- $B_\delta(t)$ is the unused capacity of the server in the interval $(t, t + \delta)$. 


Thus the output separation is a sample of a linear combination of three interdependent random processes. This theorem essentially implies that obtaining the joint law of these processes is the key to the use of packet pair probing.

We can interpret Eqn. 22 for the discrete time queues that we consider as follows. Note that $s = C = 1$. The first term is the number of packets entering the queue between the enqueuing of the two probe packets and the third term is the number of idle slots between the enqueuing of the two probe packets. Thus we can rewrite Eqn. 22 as follows.

$$d_{\text{out}} = A_{1,d_i} + 1 + \max\{d_i - X_{0,d_i-1}, 0\}$$

$$= A_{1,d_i} + 1 + d_i - X_{0,d_i-1}.$$ 

The last equality is true because the number of departures in $d_i$ slots can never be greater than $d_i$. This is nothing but Eqn. 4.

Eqn. 22 essentially says that the output separation of a packet pair is a function of the work entering the queue between the probe arrivals and the amount of work done in this interval. These two quantities are dependent and one needs their joint distribution for finding the distribution of $d_{\text{out}}$. The key difficulty in developing the analytical model for packet pair probing is to obtain this joint distribution, and we believe, that is a key contribution of this paper.

It is interesting to contrast our problem setting with delay tomography of, for example, [8, 9], since both estimate how the individual links will affect the traffic flowing, albeit, in different forms. An important assumption in the development of network-delay tomography is that the end-to-end delays of the individual packets can be measured. This requires that the sources and the destinations be suitably time synchronized. This is clearly not an easy task. In fact, the packet-pair techniques were developed to avoid this problem. Furthermore, much of the delay-tomography results were developed assuming that the probes are multicast packets. This means that when there are links in series, the estimation is for the delay across the series and not on the individual links.

Finally, we remark that the primary objective in this paper is to develop an analytical framework for ‘capacity tomography’ and not develop a practical tool.

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