BASIC POLYNOMIAL INVARIANTS, FUNDAMENTAL REPRESENTATIONS AND THE CHERN CLASS MAP

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Introduction

Consider a crystallographic root system $\Phi$ together with its Weyl group $W$ acting on the weight lattice $\Lambda$ of $\Phi$. Let $\mathbb{Z}[\Lambda]^W$ and $S^*(\Lambda)^W$ be the $W$-invariant subrings of the integral group ring $\mathbb{Z}[\Lambda]$ and the symmetric algebra $S^*(\Lambda)$. A celebrated theorem of Chevalley says that $\mathbb{Z}[\Lambda]^W$ is a polynomial ring over $\mathbb{Z}$ in classes of fundamental representations $\rho_1, \ldots, \rho_n$ and $S^*(\Lambda)^W \otimes \mathbb{Q}$ is a polynomial ring over $\mathbb{Q}$ in basic polynomial invariants $q_1, \ldots, q_n$, where $n = \text{rank}(\Phi)$.

In the present paper we establish and investigate the relationship between $\rho_i$'s and $q_i$'s. To do this we introduce an equivariant analogue of the Chern class map $\phi_i$ that provides an isomorphism between the truncated rings $\mathbb{Z}[\Lambda]/I_{m}^j$ and $S^*(\Lambda)/I_{a}^j$ modulo powers of the respective augmentation ideals. This allows us to express basic polynomial invariants in terms of fundamental representations and vice versa, hence relating the geometry of the variety of Borel subgroups $X$ with representation theory of the respective Lie algebra $\mathfrak{g}$.

A multiple of $\phi_i$ restricted to the respective cohomology ($K_0$ and $CH^*$) of $X$ gives the classical Chern class map $c_i: K_0(X) \to CH^i(X)$. This geometric interpretation provides a powerful tool to compute the annihilators of the torsion of the Grothendieck $\gamma$-filtration on $K_0$ of twisted forms of $X$ as well as a tool to estimate the torsion part of its Chow groups in small codimensions.

The paper is organized as follows. In the first section we introduce the $I$-adic filtrations on $\mathbb{Z}[\Lambda]$ and $S^*(\Lambda)$ together with an isomorphism $\phi_i$ on their truncations. Then we study the subrings of invariants and introduce the key notion of an exponent $\tau_i$ of a $W$-action on a free abelian group $\Lambda$. Roughly speaking, the integers $\tau_i$ measure how far is the ring $S^*(\Lambda)^W$ (with integer coefficients) from being a polynomial ring in $q_i$'s. In section 5 we compute all the exponents up to degree 4 and show that they all coincide with the Dynkin index of the Lie algebra $\mathfrak{g}$. In section 6 we apply the obtained results to estimate the torsion in Grothendieck $\gamma$-filtration of some twisted flag varieties. In section 7 we compute the second exponent $\tau_2$ for a non-crystallographic group $H_2$.

Acknowledgments. The first author has been partially supported from the NSERC grants of the other two authors and from the Fields Institute. The second author gratefully acknowledges support through NSERC Discovery grant 8836-20121. The last author has been supported by the NSERC Discovery grant 385795-2010, Accelerator Supplement 396100-2010 and an Early Researcher Award (Ontario).

2000 Mathematics Subject Classification. Primary 13A50; Secondary 14L24.

Key words and phrases. Dynkin index, polynomial invariant, fundamental representation, invariant theory, Chern class map, finite reflection group.
1. Two filtrations

Consider the two covariant functors $S^*(-)$ and $\mathbb{Z}[-]$ from the category of abelian groups to the category of commutative rings

$$S^*(-) : \Lambda \mapsto S^*(\Lambda) \quad \text{and} \quad \mathbb{Z}[-] : \Lambda \mapsto \mathbb{Z}[\Lambda]$$

given by taking the symmetric algebra of an abelian group $\Lambda$ and the integral group ring of $\Lambda$ respectively. The $i$th graded component $S^i(\Lambda)$ is additively generated by monomials $\lambda_1 \lambda_2 \ldots \lambda_i$ with $\lambda_j \in \Lambda$ and the ring $\mathbb{Z}[\Lambda]$ is additively generated by exponents $e^\lambda$, $\lambda \in \Lambda$.

The trivial group homomorphism induces the ring homomorphisms

$$\epsilon_a : S^*(\Lambda) \to \mathbb{Z} \quad \text{and} \quad \epsilon_m : \mathbb{Z}[\Lambda] \to \mathbb{Z}$$
called the augmentation maps. By definition $\epsilon_a$ sends every element of positive degree to 0 and $\epsilon_m$ sends every $e^\lambda$ to 1. Let $I_a$ and $I_m$ denote the kernels of $\epsilon_a$ and $\epsilon_m$ respectively. Observe that $I_a = S^\geq 0(\Lambda)$ consists of elements of positive degree and $I_m$ is generated by differences $(1 - e^{-\lambda})$, $\lambda \in \Lambda$. Consider the respective $I$-adic filtrations:

$$S^*(\Lambda) = I_0^a \supseteq I_a \supseteq I_a^2 \supseteq \ldots \quad \text{and} \quad \mathbb{Z}[\Lambda] = I_0^m \supseteq I_m \supseteq I_m^2 \supseteq \ldots$$

and let

$$gr^*_a(\Lambda) = \bigoplus_{i \geq 0} I_a^i/I_a^{i+1} \quad \text{and} \quad gr^*_m(\Lambda) = \bigoplus_{i \geq 0} I_m^i/I_m^{i+1}$$
denote the associated graded rings. Observe that $gr^*_a(\Lambda) = S^*(\Lambda)$.

1.1. Example. If $\Lambda \cong \mathbb{Z}$, then the ring $S^*(\Lambda)$ can be identified with the polynomial ring in one variable $\mathbb{Z}[\omega]$, where $\omega$ is a generator of $\Lambda$ and the ring $\mathbb{Z}[\Lambda]$ can be identified with the Laurent polynomial ring $\mathbb{Z}[x, x^{-1}]$ where $x = e^{\omega}$. The augmentations $\epsilon_a$ and $\epsilon_m$ are given by

$$\epsilon_a : \omega \mapsto 0 \quad \text{and} \quad \epsilon_m : x \mapsto 1.$$ 

We have $I_a = (\omega)$ and $I_m$ is additively generated by differences $(1 - x^n)$, $n \in \mathbb{Z}$.

Note that the rings $\mathbb{Z}[\omega]$ and $\mathbb{Z}[x, x^{-1}]$ are not isomorphic, however they become isomorphic after the truncation. Namely for every $i \geq 0$ there is ring isomorphism

$$\phi_i : \mathbb{Z}[x, x^{-1}]/I_m^{i+1} \cong \mathbb{Z}[\omega]/I_a^{i+1}$$
defined by $\phi_i : x \mapsto (1 - \omega)^{-1} = 1 + \omega + \ldots + \omega^i$ with the inverse defined by $\phi_i^{-1} : \omega \mapsto 1 - x^{-1}$. It is useful to keep the following picture in mind

$$\begin{array}{ccc}
\mathbb{Z}[x, x^{-1}] & \xrightarrow{\phi_i} & \mathbb{Z}[\omega]/I_a^{i+1} \\
\downarrow & & \downarrow \\
\mathbb{Z}[x, x^{-1}]/I_m^{i+1} & \xleftarrow{\phi_i^{-1}} & \mathbb{Z}[\omega]/I_a^{i+1}
\end{array}$$

observing that the inverse $\phi_i^{-1}$ can be lifted to the map $\mathbb{Z}[\omega] \to \mathbb{Z}[x, x^{-1}]$ but $\phi_i$ can’t.

The example can be generalized as follows:
1.2. Lemma. [GaZ 2.1] Assume that $\Lambda$ is a free abelian group of finite rank $n$. The rings $\mathbb{Z}[\Lambda]$ and $S^*(\Lambda)$ become isomorphic after truncation. Namely, if $\{\omega_1, \ldots, \omega_n\}$ is a $\mathbb{Z}$-basis of $\Lambda$, then for every $i \geq 0$ there is a ring isomorphism

$$\phi_i : \mathbb{Z}[\Lambda]/I_{m+1} \cong S^*(\Lambda)/I_{m+1}^i$$

defined by $\phi_i(1) = 1$ and

$$\phi_i(e^{\sum_{j=1}^{n} a_j \omega_j}) = \prod_{j=1}^{n} (1 - \omega_j)^{-a_j}$$

with the inverse defined by $\phi^{-1}_i(\omega_j) = 1 - e^{-\omega_j}$.

Note that the map $\phi_i$ preserves the $I$-adic filtrations. Indeed, by definition $\phi_i(I_m^i) \subseteq I_m^i$ for every $0 \leq j \leq i$. Moreover, we have the following

1.3. Lemma. The isomorphism $\phi_i$ restricted to the subsequent quotients $I_m^i/I_{m+1}^i$ doesn’t depend on the choice of a basis of $\Lambda$. Hence, there is an induced canonical isomorphism of graded rings

$$\phi_* = \oplus_{i \geq 0} \phi_i : gr^*_m(\Lambda) \cong gr^*_m(\Lambda) = S^*(\Lambda).$$

**Proof.** Indeed, in this case we can define the inverse $\phi^{-1}_i : I_m^i/I_{m+1}^i \to I_m^i/I_{m+1}^i$ by

$$\phi^{-1}_i(\lambda_1 \lambda_2 \ldots \lambda_i) = (1 - e^{-\lambda_1})(1 - e^{-\lambda_2}) \ldots (1 - e^{-\lambda_i}).$$

It is well-defined since $(1 - e^{-\lambda - \lambda'}) = (1 - e^{-\lambda}) + (1 - e^{-\lambda'})$ modulo $I_m^2$.

Consider the composite of the map $\phi_i$ with the projections

$$\phi^{(i)} : \mathbb{Z}[\Lambda] \to \mathbb{Z}[\Lambda]/I_{m+1}^i \xrightarrow{\phi_i} S^*(\Lambda)/I_{m+1}^i \to S^i(\Lambda).$$

The map $\phi^{(i)}$, and therefore $\phi_i$, can be computed on generators $e^\lambda, \lambda \in \Lambda$ as follows:

Let $f(z) = \prod_j (1 - \omega_j z)^{-a_j}$, where $\lambda = \sum_j a_j \omega_j$. Then

$$\phi^{(i)}(e^{\sum_j a_j \omega_j}) = \prod_i d^i f(z) \bigg|_{z=0}$$

To compute the derivatives of $f(z)$ we observe that $f'(z) = f(z)g(z)$, where $g(z) = \sum a_j \omega_j (1 - \omega_j z)^{-1}$ and $d^i g(z) = \sum_j \frac{(a_j \omega_j)^{i+1}}{(1 - \omega_j z)^{i+1}}$. Hence, starting with $g_0 = 1$ we obtain the following recursive formulas

$$\frac{d^i f(z)}{dz^i} = f(z) \cdot g_i(z), \text{ where } g_i(z) = g(z)g_{i-1}(z) + g'_{i-1}(z).$$

1.4. Example. For small values of $i$ we obtain

$$\begin{array}{c|c}
 i & d^i \cdot \phi^{(i)}(e^{i}) \\
 1 & 1 \\
 2 & \lambda^2 + \lambda(2) \\
 3 & \lambda^3 + 3\lambda(2)\lambda + 2\lambda(3) \\
 4 & \lambda^4 + 6\lambda(4) + 6\lambda(2)\lambda^2 + 8\lambda(3)\lambda + 3\lambda(2)^2 \\
\end{array}$$

where given a presentation $\lambda = \sum_{j=1}^{n} a_{j, \lambda} \omega_j, a_{j, \lambda} \in \mathbb{Z}$ in terms of the basis $\{\omega_1, \omega_2, \ldots, \omega_n\}$, the character $\lambda(m), m \geq 1$ is defined by

$$\lambda(m) = \sum_{j=1}^{n} a_{j, \lambda} \omega_j^m.$$
2. Invariants and exponents

Let $W$ be a finite group which acts on a free abelian group $\Lambda$ of finite rank by $\mathbb{Z}$-linear automorphisms. Consider the induced action of $W$ on $\mathbb{Z}[\Lambda]$ and $S^*(\Lambda)$. Observe that it is compatible with the $I$-adic filtrations, i.e., $W(I_m) \subseteq I_m$ and $W(I_a^i) \subseteq I_a^i$ for every $i \geq 0$.

Note that the isomorphisms $\phi_i$ and $\phi_i^{-1}$ are not necessarily $W$-equivariant. However, by Lemma 3.3 their restrictions to the subsequent quotients $I_m^i/I_m^{i+1}$ and $I_a^i/I_a^{i+1} = S^i(\Lambda)$ are $W$-equivariant and we have

$$(I_m^i/I_m^{i+1})^W \simeq (I_a^i/I_a^{i+1})^W.$$ 

Let $I_a^W$ denote the ideal of $\mathbb{Z}[\Lambda]$ generated by $W$-invariant elements from the augmentation ideal $I_a$, i.e., from elements of $\mathbb{Z}[\Lambda] \cap I_m$. Similarly, let $I_a^W$ denote the ideal of $S^*(\Lambda)$ generated by $W$-invariant elements from $I_a$, i.e., from elements of $S^*(\Lambda)^W \cap I_a$.

For each $\chi \in \Lambda$ let $\rho(\chi) = \sum_{\lambda \in W(\chi)} e^\lambda$ denote the sum over all elements of the $W$-orbit of $\chi$. Every element in $I_a^W$ can be written as a finite linear combination with integer coefficients of the elements $\hat{\rho}(\chi) = \rho(\chi) - \epsilon_m(\rho(\chi)), \chi \in \Lambda$. Therefore, the ideal $I_a^W$ is generated by the elements $\hat{\rho}(\chi)$, i.e.,

$I_a^W = \langle \hat{\rho}(\chi) \mid \chi \in \Lambda \rangle.$

The image of $I_a^W$ by means of the composite

$$\mathbb{Z}[\Lambda] \rightarrow \mathbb{Z}[\Lambda]/I_a^i \xrightarrow{\phi_i} S^*(\Lambda)/I_a^{i+1}.$$ 

is an ideal in $S^*(\Lambda)/I_a^{i+1}$ generated by the elements $\phi_i(\hat{\rho}(\chi)), \chi \in \Lambda$. Therefore, the image of $I_a^W$ in $S^*(\Lambda)$ is the $i$th homogeneous component of the ideal generated by $\phi^j(\hat{\rho}(\chi))$, where $1 \leq j \leq i, \chi \in \Lambda$, i.e.,

$$\phi^i(I_a^W) = \langle f \cdot \phi^j(\hat{\rho}(\chi)) \mid 1 \leq j \leq i, f \in S^i(\Lambda), \chi \in \Lambda \rangle_{\mathbb{Z}}.$$ 

We are ready now to introduce the central notion of the present paper:

2.1. **Definition.** We say that an action of $W$ on $\Lambda$ has finite exponent in degree $i$ if there exists a non-zero integer $N_i$ such that

$$N_i \cdot (I_a^W)^{(i)} \subseteq \phi^i(I_a^W),$$

where $(I_a^W)^{(i)} = I_a^W \cap S^i(\Lambda)$. In this case the g.c.d. of all such $N_i$s will be called the $i$th exponent of the $W$-action and will be denoted by $\tau_i$.

Observe that if $\phi^i(I_a^W)$ is a subgroup of finite index in $(I_a^W)^{(i)}$, then $\tau_i$ is simply the exponent of $\phi^i(I_a^W)$ in $(I_a^W)^{(i)}$. Note also that by the very definition $\tau_0 = 1$ and $\tau_i \mid \tau_{i+1}$ for every $i \geq 0$.

3. Essential actions

In the present section we study $W$-actions that have no $W$-invariant linear forms, i.e., we assume that $\Lambda^W = 0$. In the theory of reflection groups such actions are called essential (see B 3 V, §3.7 or H). Note that this immediately implies that $\tau_1 = 1$.

3.1. **Lemma.** For every $\chi \in \Lambda$ and $m \in \mathbb{N}_+$ we have $\sum_{\lambda \in W(\chi)} \lambda(m) = 0$. 

Proof. Let $\omega_1, \omega_2, \ldots, \omega_n$ be a $\mathbb{Z}$-basis of $\Lambda$. For $m \in \mathbb{N}_+$ we have

$$\sum_{\lambda \in W(\chi)} \lambda(m) = \sum_{\lambda \in W(\chi)} \left( \sum_{j=1}^n a_{j,\lambda} \omega_j^m \right) = \sum_{j=1}^n \left( \sum_{\lambda \in W(\chi)} a_{j,\lambda} \right) \omega_j^m.$$ 

In particular, for $m = 1$ we obtain

$$\sum_{\lambda \in W(\chi)} \lambda = \sum_{j=1}^n \left( \sum_{\lambda \in W(\chi)} a_{j,\lambda} \right) \omega_i.$$ 

Since $\Lambda^W = 0$, we have $\sum_{\lambda \in W(\chi)} \lambda = 0$. Since $\omega_j$, $1 \leq j \leq n$ are $\mathbb{Z}$-free, we have $\sum_{\lambda \in W(\chi)} a_{j,\lambda} = 0$ for all $1 \leq j \leq n$. \hfill $\square$

3.2. Corollary. For every $\chi \in \Lambda$ we have

$$\phi^{(2)}(\rho(\chi)) = \frac{1}{2} \sum_{\lambda \in W(\chi)} \lambda^2.$$ 

In particular, the quadratic form $\phi^{(2)}(\rho(\chi))$ is $W$-invariant, i.e.

$$\phi^{(2)}(\rho(\chi)) \in S^2(\Lambda)^W.$$ 

Proof. By the formula for $\phi^{(2)}$ in Example 1.4 and by Lemma 3.1 we obtain that

$$\phi^{(2)} \left( \sum_{\lambda \in W(\chi)} e^\lambda \right) = \frac{1}{2} \sum_{\lambda \in W(\chi)} (\lambda^2 + \lambda(2)) = \frac{1}{2} \sum_{\lambda \in W(\chi)} \lambda^2. \hfill \square$$

3.3. Corollary. If $S^2(\Lambda)^W = \langle q \rangle$ for some $q$, then $\phi^{(2)}(I_m^W)$ is a subgroup of finite index in $(I_m^W)^{(2)}$.

Proof. The image of the ideal $I_m^W$ is generated by $\phi^{(1)}(\rho(\chi))$ and $\phi^{(2)}(\rho(\chi))$. Since $\Lambda^W = 0$, $\phi^{(1)}(\rho(\chi)) = \sum_{\lambda \in W(\chi)} \lambda = 0$ and by Corollary 3.2, $\phi^{(2)}(I_m^W)$ is generated only by the $W$-invariant quadratic forms $\phi^{(2)}(\rho(\chi))$. For every $\chi \in \Lambda$ let

$$\phi^{(2)}(\rho(\chi)) = N_\chi \cdot q, \quad N_\chi \in \mathbb{N}. \quad (1)$$

Then the subgroup $\phi^{(2)}(I_m^W)$ is a subgroup of $(I_m^W)^{(2)}$ of exponent

$$\tau_2 = \gcd_{\chi \in \Lambda} N_\chi. \hfill \square$$

We now investigate the invariants of degree 3 and 4.

3.4. Lemma. For every $\chi \in \Lambda$ we have

$$\phi^{(3)}(\rho(\chi)) = \frac{1}{6} \sum_{\lambda \in W(\chi)} (\lambda^3 + 3\lambda(2)\lambda).$$ 

Proof. By the formula for $\phi^{(3)}$ in Example 1.4 and by Lemma 3.1 we obtain that

$$\phi^{(3)}(\rho(\chi)) = \frac{1}{6} \sum_{\lambda \in W(\chi)} (\lambda^3 + 3\lambda(2)\lambda + 2\lambda(3)) = \frac{1}{6} \sum_{\lambda \in W(\chi)} (\lambda^3 + 3\lambda(2)\lambda). \hfill \square$$

3.5. Lemma. For every $\chi \in \Lambda$ we have

$$\phi^{(4)}(\rho(\chi)) = \frac{1}{24} \sum_{\lambda \in W(\chi)} [\lambda^4 + 6\lambda(2)\lambda^2 + 8\lambda(3)\lambda + 3\lambda(2)^2].$$ 

Proof. It follows from Example 1.4 and Lemma 3.1 \hfill $\square$
4. The Dynkin index

In the present section we show that the action of the Weyl group $W$ of a crystallographic root system $\Phi$ on the weight lattice $\Lambda$ has finite exponent in degree 2 which coincides with the Dynkin index of the respective Lie algebra.

Let $W$ be the Weyl group of a crystallographic root system $\Phi$ and let $\Lambda$ be its weight lattice as defined in [H] §2.9. Let $\{\omega_1, \ldots, \omega_n\}$ be a basis of $\Lambda$ consisting of fundamental weights (here $n$ is the rank of $\Phi$).

The Weyl group $W$ acts on $\lambda \in \Lambda$ by means of simple reflections
\[ s_j(\lambda) = \lambda - \langle \alpha_j^\vee, \lambda \rangle \cdot \alpha_j, \quad j = 1 \ldots n \]
where $\alpha_j^\vee$ is the $j$-th simple coroot and $\langle -, - \rangle$ is the usual pairing. Note that $\langle \alpha_j^\vee, \omega_i \rangle = \delta_{ij}$, where $\delta_{ij}$ is the Kronecker symbol.

The subring of invariants $\mathbb{Z}[\Lambda]^W$ is the representation ring of the respective Lie algebra $g$. By a theorem of Chevalley it is the polynomial ring in fundamental representations $\rho(\omega_i) \in \mathbb{Z}[\Lambda]^W$, i.e.
\[ \mathbb{Z}[\Lambda]^W \simeq \mathbb{Z}[\rho(\omega_1), \ldots, \rho(\omega_n)]. \]
Observe that the dimension of the fundamental representation $\rho(\omega_j)$ equals to the number of elements in the orbit that is $\epsilon_m(\rho(\omega_j))$.

Therefore, the ideal $I_m^W$ is generated by the elements $\hat{\rho}(\omega_j)$, $j = 1 \ldots n$ and its image $\phi^{(i)}(I_m^W)$ is the $i$-th homogeneous component of the ideal generated by $\phi^{(i)}(\rho(\omega_i))$, $1 \leq j \leq i$, $l = 1 \ldots n$.

4.1. Lemma. We have $\Lambda^W = 0$ and hence also
\[ \phi^{(1)}(\mathbb{Z}[\Lambda]^W) = \phi^{(1)}(I_m^W) = 0. \]
Proof. Let $\eta \in \Lambda^W$. Since $\eta = s_{\alpha_j}(\eta) = \eta - \langle \eta, \alpha_j^\vee \rangle \alpha_j$ we have $\langle \eta, \alpha_j^\vee \rangle = \frac{2(\alpha_j, \eta)}{(\alpha_j, \alpha_j)} = 0$ for all simple roots $\alpha_j$ which implies that $\eta = 0$. □

4.2. Lemma. We have $S^2(\Lambda)^W = \langle q \rangle$.
Proof. By [GN] Prop. 4] there exists an integer valued $W$-invariant quadratic form on $\Lambda$ which has value 1 on short coroots. As the group $S^2(\Lambda)^W$ is identical to the group of all integral $W$-invariant quadratic forms on $T_\gamma \otimes \mathbb{R}$, the result follows. □

4.3. Corollary. The image $\phi^{(2)}(I_m^W)$ is a subgroup of $(I_m^W)^{(2)}$ of finite index.
Proof. This follows from Corollary 4.3 and Lemma 4.1. □

We recall briefly the notion of indices of representations introduced by Dynkin [D] §2] (See also [BR]).

Let $f : g \to g'$ be a morphism between Lie algebras. Then there exists a unique number $j_f \in \mathbb{C}$, called the Dynkin index of $f$, satisfying
\[ (f(x), f(y)) = j_f(x, y), \]
for all $x, y \in g$, where $\langle -, - \rangle$ is the Killing form on $g$ and $g'$ normalized such that $(\alpha, \alpha) = 2$ for any long root $\alpha$. In particular, if $f : g \to sl(V)$ is a linear representation, $j_f$ is a positive integer, called the Dynkin index of the linear representation $f$, defined by
\[ \text{tr}(f(x), f(y)) = j_f(x, y). \]
The Dynkin index of $g$ is defined to be the greatest common divisor of all the Dynkin indices of all linear representations of $g$. By [D] (2.24) and (2.25)], the Dynkin index of $g$ is the greatest common divisor of the Dynkin index of its fundamental representations. Moreover, all the Dynkin indices of the fundamental representations were calculated in [D, Table 5].

Using the $\mathfrak{sl}_2$-representation theory, the Dynkin index of a linear representation $f : g \rightarrow \mathfrak{sl}_2$ can be described as follows. Let $\alpha$ be a long root. For the formal character $\chi(V) = \sum_\lambda n_\lambda e^\lambda$, one has (see [LS, Lemma 2.4] or [KLR] 5.1 and Lemma 5.2)

$$j_f = \frac{1}{2} \sum_\lambda (\lambda, \alpha^\vee)^2.$$  \hfill (2)

4.4. Theorem. The integers $N(\omega_j)$ for the $j$-th fundamental weight as defined in (1) coincide with the Dynkin index of the fundamental representation with highest weight $\omega_j$. In particular, the second exponent $\tau_2$ coincides with the Dynkin index of $g$.

Proof. To find the precise value of $\tau_2$ we use the explicit formula for $\phi^{(2)}$, that is

$$\phi^{(2)}(\rho(\chi)) = \frac{1}{2} \sum_{\lambda \in W(\chi)} \lambda^2.$$  \hfill (2)

We know that $\tau_2$ is the greatest common divisor of the integers $N_j = N_{\omega_j}$, using the notation of the proof of Corollary 3.3, where $\omega_j$ is the $j$-th fundamental weight of $g$. As the Dynkin index is the greatest common divisor of the Dynkin indices of the fundamental representations $\omega_j$, it suffices to show that $N_j$ coincides with the Dynkin index of the representation $V_j$ corresponding to $\omega_j$. We can view $\phi^{(2)}(\rho(\chi))$ for $\chi = \omega_j$ as a function on the lattice $\mathbb{Z}_\alpha = \text{Span}\{\alpha^\vee | \alpha \in \Phi \text{ long}\}$. Since $V_j$ has character $\chi(V_j) = \sum_\lambda \lambda e^\lambda$, by (2) the Dynkin index of the representation $V_j$ is $\frac{1}{2} \sum_{\lambda \in W(\omega_j)} (\lambda, \alpha^\vee)^2$, where $\alpha$ is any long root in $\Phi$. Thus, $\phi^{(2)}(\rho(\omega_j))$ is the constant function with value $N_j$. \hfill $\Box$

We note that a different proof of Theorem 4.4 was given in [GaZ, §2].

5. Exponents of degrees 3 and 4

In the present section we show that $\tau_2 = \tau_3 = \tau_4$ for all crystallographic root systems.

Let $S = \{\lambda_1, \ldots, \lambda_r\}$ be a finite set of weights. We denote by $-S$ the set of opposite weights $\{-\lambda_1, \ldots, -\lambda_r\}$, by $S_+$ the set of sums $\{\lambda_i + \lambda_j\}_{i<j}$, by $S_-$ the set of differences $\{\lambda_i - \lambda_j\}_{i<j}$ and by $S_{\pm}$ the disjoint union $S_+ \cup S_-$. By definition we have $|S_+| = |S_-| = \binom{r}{2}$.

Using the fact that $(\lambda + \lambda')(m) = \lambda(m) + \lambda'(m)$ for every $\lambda, \lambda' \in \Lambda$ and $m \geq 0$ we obtain the following lemma which will be extensively used in the computations.

5.1. Lemma. (i) For every integer $m_1, m_2, x, y \geq 0$ and a finite subset $S \subset \Lambda$ we have

$$\sum_{\lambda \in S_{\pm}} \lambda(m_1)^x \lambda(m_2)^y = (1 + (-1)^{x+y}) \sum_{\lambda \in S} \lambda(m_1)^x \lambda(m_2)^y.$$

In particular, $\sum_{\lambda \in S_{\pm}} \lambda(2) \lambda^2 = 0$.  

(ii) For every subset $S \subset \Lambda$ with $|S| = r$ and for every $m_1, m_2 \geq 0$ we have
\[
\sum_{\lambda \in S_{+}} \lambda(m_1)\lambda(m_2) = (r - 1) \sum_{\lambda \in S} \lambda(m_1)\lambda(m_2) + \sum_{i \neq j} \lambda_i(m_1)\lambda_j(m_2)
\]
and
\[
\sum_{\lambda \in S_{-}} \lambda(m_1)\lambda(m_2) = (r - 1) \sum_{\lambda \in S} \lambda(m_1)\lambda(m_2) - \sum_{i \neq j} \lambda_i(m_1)\lambda_j(m_2).
\]
In particular, this implies that $\sum_{\lambda \in S_{\pm}} \lambda(m_1)\lambda(m_2) = 2(r - 1) \sum_{\lambda \in S} \lambda(m_1)\lambda(m_2)$.

**$A_n$-case.** Let $\Phi$ be of type $A_n$ for $n \geq 3$. We denote the canonical basis of $\mathbb{R}^{n+1}$ by $e_i$ with $1 \leq i \leq n + 1$. According to [11, §3.5 and §3.12] the basic polynomial invariants of the $W$-action on $\Lambda$ (algebraically independent homogeneous generators of $S^*(\Lambda)^W$ as a $\mathbb{Q}$-algebra) are given by the symmetric power sums
\[
q_i := e_1^i + \cdots + e_{n+1}^i, \quad 2 \leq i \leq n + 1.
\]
Let $s_i$ denote the $i$th elementary symmetric function in $e_1, \ldots, e_{n+1}$. Using the classical identities
\[
q_1 = s_1, \quad q_i = s_1 q_{i-1} - s_2 q_{i-2} + \cdots + (-1)^{i-1} s_{i-1} q_1 + (-1)^{i+1} i \cdot s_i, \quad 1 < i < n + 1
\]
and the fact that $s_1 = 0$, we obtain that
\[
q_2/2 = -s_2, \quad q_3/3 = s_3, \quad \text{and} \quad q_4/2 = s_2^2 - 2s_4.
\]
generate (with integral coefficients) the ideal $I^W$ up to degree 4.

The fundamental weights of $\Phi$ can be expressed as follows
\[
\omega_1 = e_1, \quad \omega_2 = e_1 + e_2, \quad \ldots, \quad \omega_{n-1} = e_1 + \cdots + e_{n-1}, \quad \omega_n = -e_{n+1},
\]
where $e_1 + e_2 + \cdots + e_{n+1} = 0$. The orbits of $\omega_1, \omega_1 + \omega_n, \omega_n$ and $\omega_2, \omega_{n-1}$ under the action of the Weyl group $W = S_{n+1}$ are given by
\[
W(\omega_1) = \{e_1, \ldots, e_{n+1}\} = -W(\omega_n), \quad W(\omega_1 + \omega_n) = \{e_1 - e_j\}_{i \neq j}
\]
and
\[
W(\omega_2) = \{e_i + e_j\}_{i < j} = -W(\omega_{n-1}).
\]
Therefore, $W(\omega_1 + \omega_n) = S_{-} \Pi S_{-}$ and $W(\omega_2) = S_{+}$, where $W = W(\omega_1)$.

Applying Lemma 5.5 and Lemma 5.1, we obtain that
\[
\phi(\rho(\omega_1) + \rho(\omega_n)) = \frac{1}{24} \sum_{\lambda \in S} (\lambda^4 + 8\lambda(3)\lambda + 3\lambda(2)^2)
\]
and
\[
\phi(\rho(\omega_1 + \omega_n) + \rho(\omega_2) + \rho(\omega_{n-1})) = \frac{1}{24} \sum_{\lambda \in S_{-} \Pi S_{-}} (\lambda^4 + 8\lambda(3)\lambda + 3\lambda(2)^2) = \frac{1}{24} \sum_{\lambda \in S_{-} \Pi S_{-}} \lambda^4 + \frac{n}{6} \sum_{\lambda \in S} (8\lambda(3)\lambda + 3\lambda(2)^2).
\]
Then the difference
\[
\phi(\rho(\omega_1 + \omega_n) + \rho(\omega_2) + \rho(\omega_{n-1})) - 2n \cdot \phi(\rho(\omega_1) + \rho(\omega_n))
\]
is a symmetric function in $e_1, \ldots, e_{n+1}$ and, therefore, it can be always written as a polynomial in $q_i$s. Indeed, since
\[
\sum_{\lambda \in S_{-} \Pi S_{-}} \lambda^4 = 2 \sum_{i < j} ((e_i + e_j)^4 + (e_i - e_j)^4) = 4n \sum_{\lambda \in S} \lambda^4 + 24 \sum_{i < j} e_i^2 e_j^2,
\]
the difference equals
\[ \sum_{i<j} c_i^2 c_j^2 = (q_2^2 - q_4)/2. \]

5.2. Lemma. For a root system of type \( A_n, n \geq 2 \), we have \( \tau_2 = \tau_3 = \tau_4 = 1 \).

Proof. It is enough to show that the generators \( q_2/2, q_3/3 \) and \( q_4/2 \) are in the ideal generated by the image of \( \phi^{(4)} \), \( i \leq 4 \).

By Corollary 3.2 we have \( \phi^{(2)}(\rho(\omega_1)) = \frac{1}{2} \sum_{\lambda \in S} \lambda^2 = q_2/2 \). By Lemma 3.4 we have \( q_3/3 = \phi^{(3)}(\rho(\omega_1)) - \phi^{(3)}(\rho(\omega_n)) \) (see also [GaZ, §1C]). If \( \Phi \) is of type \( A_2 \), then \( s_4 = 0 \) and, hence, \( q_4 = q_2^3/2 \). If \( \Phi \) is of type \( A_n, n \geq 3 \), then by 3 the generator \( q_4/2 \) belongs to the ideal generated by the images of \( \phi^{(2)} \) and \( \phi^{(4)} \). \( \Box \)

5.3. Lemma. For any crystallographic root system \( \Phi \) the third exponent \( \tau_3 \) of the \( W \)-action coincides with \( \tau_2 \) (the Dynkin index).

Proof. If \( \Phi \) is of type \( A_n, n \geq 2 \), this follows from Lemma 3.2 for the other types there are no basic polynomial invariants of degree 3 [H, §3.7 Table 1]. Therefore, \( \tau_3 = \tau_2 \). \( \Box \)

\( B_n, C_n \) and \( D_n \) cases. Let \( \Phi \) be of type \( B_n \) or \( C_n \) for \( n \geq 2 \) or of type \( D_n \) for \( n \geq 4 \). We denote the canonical basis of \( \mathbb{R}^n \) by \( e_i \) with \( 1 \leq i \leq n \). By [H, §3.5 and §3.12] the basic polynomial invariants of the \( W \)-action on \( \Lambda \) are given by even power sums
\[ q_{2i} := e_1^{2i} + \cdots + e_n^{2i}, \quad 1 \leq i \leq n. \]

The first two fundamental weights of \( \Phi \) are given by \( \omega_1 = e_1, \omega_2 = e_1 + e_2 \) and their \( W \)-orbits are
\[ W(\omega_1) = \{ \pm e_1, \ldots, \pm e_n \} \] and \( W(\omega_2) = \{ \pm e_i \pm e_j \}_{i<j} \).

Hence \( W(\omega_1) = S \oplus S \) and \( W(\omega_2) = S_{\pm} \oplus S_{\pm} \), where \( S = \{ e_1, \ldots, e_n \} \).

Applying Lemma 3.5 and Lemma 5.1 we obtain that
\[ \phi^{(4)}(\rho(\omega_1)) = \frac{1}{17} \sum_{\lambda \in S} \lambda^4 + \frac{1}{17} \sum_{\lambda \in S} (8\lambda(3)\lambda + 3\lambda(2)^2) \] and
\[ \phi^{(4)}(\rho(\omega_2)) = \frac{1}{17} \sum_{\lambda \in S_{\pm} \oplus S_{\pm}} \lambda^4 + \frac{1}{17} \sum_{\lambda \in S} (8\lambda(3)\lambda + 3\lambda(2)^2). \]

Then similar to the \( A_n \)-case we obtain
\[ \phi^{(4)}(\rho(\omega_2)) - 2(n - 1)\phi^{(4)}(\rho(\omega_1)) = (q_2^2 - q_4)/2, \tag{4} \]

where \( q_i = e_1^{i} + \ldots + e_n^{i} \).

5.4. Lemma. For a root system of type \( B_n \) or \( C_n, n \geq 2 \) or \( D_n, n \geq 4 \) we have \( \tau_4 = \tau_2 \).

Proof. It is enough to show that \( q_4/2 \) is in the ideal generated by the image of \( \phi^{(2)} \) and \( \phi^{(4)} \).

By Corollary 3.2 we have \( \phi^{(2)}(\rho(\omega_1)) = \sum_{\lambda \in S} \lambda^2 = q_2 \). Therefore, by (4)
\[ q_4/2 = (q_2^2/2) \cdot \phi^{(2)}(\rho(\omega_1)) - \phi^{(4)}(\rho(\omega_2)) + 2(n - 1)\phi^{(4)}(\rho(\omega_1)) \]
and the proof is finished. \( \Box \)

5.5. Theorem. For any crystallographic root system \( \Phi \) we have \( \tau_2 = \tau_3 = \tau_4 \).
6. TORSION IN THE GROTHENDIECK $\gamma$-FILTRATION

The goal of the present section is to provide geometric interpretation (see [8]) of the map $\phi_i$ and the exponents $\tau_i$.

Let $G$ be a simple simply-connected Chevalley group over a field $k$. We fix a maximal split torus $T$ of $G$ and a Borel subgroup $B \supset T$. Let $\Lambda$ be the group of characters of $T$. Since $G$ is simply-connected, $\Lambda$ coincides with the weight lattice of $G$.

Let $X$ denote the variety of Borel subgroups of $G$ (conjugate to $B$). Consider the Chow ring $CH^* (X)$ of algebraic cycles modulo rational equivalence and the Grothendieck ring $K_0 (X)$. Following [De74, §1] to every character $\lambda \in \Lambda$ we may associate the line bundle $\mathcal{L}(\lambda)$ over $X$. It induces the ring homomorphisms (called the characteristic maps)

$$c_\lambda : S^\lambda (\Lambda) \to CH^*(X)$$

by sending $\lambda \mapsto c_1 (\mathcal{L}(\lambda))$ and $e^\lambda \mapsto [\mathcal{L}(\lambda)]$ respectively. Note that the map $c_\lambda$ is an isomorphism in codimension one, hence, giving

$$c_\lambda : S^\lambda (\Lambda) = \Lambda \overset{\sim}{\to} Pic(X) = CH^1 (X)$$

and the map $c_m$ is surjective. Let $W$ be the Weyl group and let $I^W_m$ and $I^W_n$ denote the respective $W$-invariant ideals. Then according to [De73, §4 Cor.2,§9] and [CPZ, §6]

$$\ker c_m = I^W_m$$

and their multiples are in $I^W_n$.

Consider the Grothendieck $\gamma$-filtration on $K_0 (X)$ (see [GaZ, §1]). Its $i$th term is an ideal generated by products

$$\gamma_i (X) := \{ (1 - [\mathcal{L}_1^\gamma]) (1 - [\mathcal{L}_2^\gamma]) \cdots (1 - [\mathcal{L}_i^\gamma]) \},$$

where $\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_i$ are line bundles over $X$. Consider the $i$th subsequent quotient $\gamma_i (X)/\gamma_i+1 (X)$. The usual Chern class $c_i$ induces a group homomorphism $c_i : \gamma_i (X)/\gamma_i+1 (X) \to CH^i (X)$.

6.1. Proposition. For every $i \geq 0$ there is a commutative diagram of group homomorphisms

$$\begin{array}{ccc}
I^i_m / I^{i+1}_m & \xrightarrow{(-1)^{i-1} (i-1) \cdot \phi_i} & S^i (\Lambda) \\
\downarrow{c_m} & & \downarrow{c_\lambda} \\
\gamma_i (X)/\gamma_i+1 (X) & \xrightarrow{c_i} & CH^i (X)
\end{array}$$

Proof. Indeed, the $\gamma$-filtration on $K_0 (X)$ is the image of the $I_m$-adic filtration on $\mathbb{Z}[\Lambda]$, i.e. $\gamma_i (X) = \mathfrak{c}_m (I^i_m)$ for every $i \geq 0$. The Proposition then follows from the identity

$c_i \left( (1 - [\mathcal{L}_1^\gamma]) (1 - [\mathcal{L}_2^\gamma]) \cdots (1 - [\mathcal{L}_i^\gamma]) \right) = (-1)^{i-1} (i-1)! \cdot c_1 (\mathcal{L}_1) c_1 (\mathcal{L}_2) \cdots c_1 (\mathcal{L}_i),$
where \( \mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_i \) are line bundles over \( X \) and \( \mathcal{L}_i^\vee \) denotes the dual of \( \mathcal{L}_i \).

\[ \square \]

6.2. Remark. Note that \( \mathbb{Z}[A] \) can be identified with the \( T \)-equivariant \( K_0 \) of a point \( pt = \text{Spec} \ k \) and \( S(A) \) with the \( T \)-equivariant \( CH \) of a point (see [GiZ]). The maps \( c_a \) and \( c_m \) then can be identified with the pull-backs \( K_T(pt) \to K_T(G) \) and \( CH_T(pt) \to CH_T(G) \) induced by the structure map \( G \to pt \).

In view of these identifications the map \( \phi_i \) can be viewed as an equivariant analogue of the Chern class map \( c_i \).

Consider the diagram \( \square \) with \( \mathbb{Q} \)-coefficients. In this case the Chern class map \( c_i \) will become an isomorphism (by the Riemann-Roch theorem), the characteristic map \( c_a \) will turn into a surjection and the map \((-1)^{i-1}(i - 1)! \cdot \phi_i \) will be an isomorphism as well. In view of \( \square \) we obtain an isomorphism

\[ \phi(i) \otimes \mathbb{Q} : I_m^W \cap I_m^i / I_m^i \otimes I_m^1 \otimes \mathbb{Q} \longrightarrow (I_a^W)^{(i)} \otimes \mathbb{Q} \]

on the kernels of \( c_m \) and \( c_a \). By the very definition of the exponents \( \tau_i \) this implies that

6.3. Corollary. The action of the Weyl group of a crystallographic root system has finite exponent \( \tau_i \) for every \( i \).

We are now ready to prove the main result of this section

6.4. Theorem. The integer \( \tau_i \cdot (i - 1)! \) annihilates the torsion of the \( i \)th subsequent quotient \( \gamma^i(X)/\gamma^{i+1}(X) \) of the \( \gamma \)-filtration on \( K_0(X) \) for \( i = 3, 4 \).

6.5. Remark. Note that by [SGA] Exposé XIV, 4.5] for groups of types \( A_n \) and \( C_n \) the quotients \( \gamma^i(X)/\gamma^{i+1}(X) \) have no torsion.

Proof. Assume that \( \alpha \) is a torsion element in \( \gamma^i(X)/\gamma^{i+1}(X) \). Then \( c_i(\alpha) = 0 \) since \( CH^i(G/B) \) has no torsion. Let \( \hat{\alpha} \) be a preimage of \( \alpha \) via \( c_a \) in \( I_m^i / I_m^{i+1} \subseteq \mathbb{Z}[A] / I_m^{i+1} \). By the same analysis as in [GaZ §1B, §1C] one can show that \( \ker(c_a(i)) = (I_a^W)^{(i)} \) for \( i \leq 4 \). By \( \square \) we obtain that

\[ (i - 1)! \phi_i(\hat{\alpha}) \in (I_a^W)^{(i)} \]

By definition of the index \( \tau_i \) we have

\[ \tau_i \cdot (i - 1)! \phi_i(\hat{\alpha}) = \phi_i(\beta), \quad \text{where } \beta \in I_m^W / I_m^{i+1} \cap I_m^W. \]

Applying \( \phi_i^{-1} \) to both sides we obtain

\[ \tau_i \cdot (i - 1)! \cdot \hat{\alpha} = \beta \in I_m^W / I_m^{i+1} \cap I_m^W \]

Applying \( c_m \) to both sides and observing that \( I_m^W = \ker c_m \) we obtain that

\[ \tau_i \cdot (i - 1)! \cdot \alpha = 0. \]

Let \( \xi X \) be a twisted form of the variety \( X \) by means of a cocycle \( \xi \in Z^1(k, G) \). By [P] Thm. 2.2.(2)] the restriction map \( K_0(\xi X) \to K_0(X) \) (here we identify \( K_0(X) \) with the \( K_0(X \times_k \bar{k}) \) over the algebraic closure \( \bar{k} \)) is an isomorphism. Since the characteristic classes commute with restrictions, this induces an isomorphism between the \( \gamma \)-filtrations, i.e. \( \gamma^i(\xi X) \simeq \gamma^i(X) \) for every \( i \geq 0 \), and between the respective quotients

\[ \gamma^i(\xi X)/\gamma^{i+1}(\xi X) \simeq \gamma^i(X)/\gamma^{i+1}(X) \quad \text{for every } i \geq 0. \]

In view of this fact Theorem 6.4 imply that
6.6. Corollary. Let $G$ be a split simple simply connected group of type $B_n$ ($n \geq 3$) or $D_n$ ($n \geq 4$). Then for every $\xi \in Z^1(k, G)$ the torsion in $\gamma^4(\xi X)/\gamma^5(\xi X)$ is annihilated by 12.

Consider the topological filtration on $K_0(Y)$ given by the ideals

$$\tau^i(Y) := \langle [O_Y] \mid V \hookrightarrow Y, codim_V Y \geq i \rangle.$$ It is known (see [GaZ, §2]) that $\gamma^i(Y) \subseteq \tau^i(Y)$ for every $i \geq 0$.

6.7. Corollary. In the notation of Corollary 6.6 assume in addition that the induced map

$$\gamma^4(\xi X)/\gamma^5(\xi X) \to \tau^4(\xi X)/\tau^5(\xi X)$$

is surjective. Then the 2-torsion of $CH^4(\xi X)$ is annihilated by 8.

Proof. By the Riemann-Roch theorem [F, Ex.15.3.6], the composition

$$CH^4(\xi X) \to \tau^4(\xi X)/\tau^5(\xi X) \to CH^4(\xi X)$$

is the multiplication by $(-1)^{4-1}(4-1)! = -6$, where the first map is surjective. Hence, the torsion subgroup of $CH^4(\xi X)$ is annihilated by 72 and so the result follows.

7. ‘THE DYNKIN INDEX’ IN THE $H_2$ CASE

Note that the notion of an exponent $\tau_i$ can be defined over a unique factorisation domain in the same way. As an example we compute the second exponent $\tau_2$ for the action of the Weyl group of of the non-crystallographic root system $H_2$ over the base ring $\mathbb{Z}[\frac{1+\sqrt{5}}{2}]$, hence, giving rise to an interesting question about its geometric/Lie algebra interpretation.

7.1. Theorem. For the non-crystallographic root system $H_2 := I_2(5)$, the second exponent $\tau_2$ is $\sqrt{5}$.

Proof. We follow the notations in [CMP]. In the root system $H_2$, the Weyl group $W$ is the dihedral group of order 10 and $M$ is the $\mathbb{Z}[\tau]$-lattice generated by two simple roots $\alpha_1$ and $\alpha_2$, where $\tau = (1 + \sqrt{5})/2$. Observe that $\mathbb{Z}[\tau]$ is an Euclidean domain.

The dual basis $\{\omega_1, \omega_2\}$ is defined by

$$\begin{cases} \omega_1 = \frac{1}{1+\tau}(2\alpha_1 + \tau\alpha_2) \\ \omega_2 = \frac{1}{3-\tau}(\tau\alpha_1 + 2\alpha_2) \end{cases} \quad \text{or} \quad \begin{cases} \alpha_1 = 2\omega_1 - \tau\omega_2 \\ \alpha_2 = -\tau\omega_1 + 2\omega_2 \end{cases}$$

One computes the orbits of $\omega_1$ and $\omega_2$ as follows:

$$W(\omega_1) = \{\omega_1, -\omega_2, -\omega_1 + \tau\omega_2, -\tau\omega_1 + \omega_2, \tau\omega_1 - \tau\omega_2\},$$

$$W(\omega_2) = -W(\omega_1).$$

As the action of $W$ on $M$ is essential, by Corollary 3.2, we have

$$\phi^{(2)}(\rho(\omega_2)) = \phi^{(2)}(\rho(\omega_1)) = \frac{1}{2}(\omega_1^2 + \omega_2^2 + (\omega_1 - \tau\omega_2)^2 + (\tau\omega_1 - \omega_2)^2 + (\tau\omega_1 - \tau\omega_2)^2)$$

$$= (1 + \tau^2)\omega_1^2 + (1 + \tau^2)\omega_2^2 - (2\tau + \tau^2)\omega_1\omega_2. \quad (7)$$

Since $\phi^{(2)}(\rho(\omega_2))$ is $W$-invariant by Corollary 3.2, we have

$$\tau_2 = gcd(1 + \tau^2, 2\tau + \tau^2) = gcd(2 + \tau, 2\tau - 1).$$
But $2\tau - 1 = \sqrt{5}$ is a prime in $\mathbb{Z}[\tau]$, and we have $2 + \tau = (2\tau - 1)\tau$ proving that $\tau^2 = \sqrt{5}$. \hfill \Box

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