Tetrahedron Diagram and Perturbative Calculation in Chern-Simons-Witten Theory

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Abstract

We investigate extended Wilson loop operators, in particular tetrahedron operator in (2 + 1)-dimensional Chern-Simons-Witten theory. This operator emerges naturally from the contribution terms in two-particle scattering amplitude. We evaluate this diagram non-perturbatively in terms of vacuum expectation values of Wilson loop operators, especially for gauge group SU($N$) with specific choices of representations. On the other hand, we also discuss the perturbative calculation of vacuum expectation value in this theory. We show that, up to the third order, this values of unknotted Wilson loop operators are identical to the non-perturbative result.

Keywords: Chern-Simons theory

1 Introduction

The Chern-Simons-Witten (CSW) theory has been providing many interesting topics for both mathematics and physics. Different aspects of the theory have been explored, e.g. abelian and non-abelian theory with the Chern-Simons term, supersymmetric extension, and pure theory. Specifically, after it was pointed out that the pure theory could be relevant for knot theory \cite{1,2}, many

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researchers have contributed to clarify the relation of CSW theory to two-dimensional, conformal field theories and to knot theory. Witten described the exact solution to this theory in non-perturbative case. He and others also showed that there is relation between this theory and the polynomial or quantum group invariants of knot in three dimensions, in particular the Jones polynomial and its generalizations like the HOMFLY and the Kauffman polynomial invariants \[3, 4, 5\]. These knot polynomials can be regarded as vacuum expectation values (VEV) of Wilson loop operators in this theory.

The CSW theory also has interesting features from the perturbative point of view and lead to Goussarov-Vassiliev or finite type invariants. Moreover, Witten’s idea was based on the validity of the path integral formulation of the quantized theory. By examining the existence of a quantum field theory description of the link invariants in perturbative framework, it turned out that the coefficients of the perturbative series correspond to these invariants. The perturbative expansion of the path integral of the theory has been considered for flat \(R^3\) \[6\] and for general three-manifolds \[7\]. Subsequently, there was much attention given to understanding the perturbative series for knots and links in \(R^3\) \[8\].

The perturbative series expansion has been studied for different gauge-fixings which lead to different representations for Vassiliev invariants. The covariant Landau gauge corresponds to the configuration space integrals and the non-covariant light-cone gauge to the Kontsevich integrals. Another studies of the perturbative series expansion in the non-covariant temporal gauge has the important feature that the integrals which are present in the expressions for the coefficients of the perturbative series expansion can be carried out. In this case one obtains combinatorial expressions, instead of integral ones, for Vassiliev invariants and this has been shown to be the case up to order four \[9\]. Perturbative expansion in other gauges might also highlight other aspects of the theory, for example in axial gauge \[10\] and in light-cone gauge \[11\]. The invariants obtained in the perturbative framework with different gauge-fixing are the same since the theory is gauge invariant and Wilson loops are gauge invariant operators.

One main issue in perturbative CSW theory is the calculability of the vacuum expectation value of Wilson loop operator \(\langle W(C) \rangle\) in the three-dimensional field theory framework. It turned out that \(\langle W(C) \rangle\) has a meaningful perturbative expansion in powers of the coupling constant \(k^{-1}\). Although by power counting the theory appears renormalizable, it is in fact UV finite, which means the \(\beta\) function and the anomalous dimensions of the fields vanish to all orders \[12, 13\]. Even, more interestingly, there is no divergences in the computation of \(\langle W(C) \rangle\) in this theory. One loop renormalization constant of the theory has been calculated and the result showed the existence of the famous \(k\) shift \[14\]. Therefore, the framing has nothing to do with divergences of \(\langle W(C) \rangle\), but is
related to the self-linking problem which is topological in origin.

The mathematical establishment of relations between CSW gauge theory, topological theories in 3 manifold and knot theory is still making progress. Some recent studies include the application of Maldacena’s conjecture [15], the use of methods of stochastic analysis [16] and its connection to the Penner models [17].

In this paper we will evaluate the non-perturbative as well as the perturbative aspects of CSW theory. In non-perturbative aspect, it is shown that the VEV of Wilson loop operators are evaluated by using braiding formula [18] which is useful to construct algebraic relations between unknotted Wilson loop operators. It also discussed the emergence of the extended Wilson loop operators, namely baryon type [19] and tetrahedron operators [20]. Especially, we will evaluate the tetrahedron type operator. This diagram emerges by refining the calculation of the gravitational scattering amplitude in previous work [21]. It is important, therefore, to make it clear what contributions will be supplied from these terms.

On the other hand, parallel to the non-perturbative approach, the explicit perturbative calculation of the unknotted Wilson loop operator is presented up to the third order. The coefficients of these expansion is shown to be the same as the non-perturbative results. After that, we also investigate the ghost and auxiliary fields contributions to the Wilson loop operator. A detailed perturbative calculation of the ghost contribution up to the second order is given in the Appendix A and other important integral formulas related to the framing procedure are given in Appendix B.

2 Chern-Simons Theory and Extended Wilson Loop Operators

In this section we will present the rudimentary facts about CSW theory and to discuss the emergence of extended Wilson loop operators. The action for this theory in (2 + 1)-dimension is the Chern-Simons secondary characteristic class defined by

\[ S = \frac{k}{4\pi} \int_M \text{Tr} (A \wedge dA + \frac{2}{3} A \wedge A \wedge A). \] (1)

Here \( k^{-1} \) plays the role of coupling constant whereas \( A \) is a connection on a \( G \)-bundle \( E \) over a space-time three manifold \( M \). Trace is taken over the representation of gauge group \( G \). The partition function \( Z(M,k) \) takes the

\footnote{One of us (FPZ) would like to thank M. Hayashi for collaboration in this part, see ref. [20].}
form
\[ Z(M, k) = \int_M [DA] e^{iS}. \] (2)

Under the gauge transformation, the action will transform as
\[ S \rightarrow S + 2\pi k S_{WZ}, \] (3)

where \( S_{WZ} \in \mathbb{Z} \) is the winding number of the gauge transformation.

The Wilson loop operators \( W_\rho(C) \), which are the basic observables of the theory, are the most important gauge invariant operators in this theory. These operators are related to the link invariants of knot theory and defined by the trace of path-ordered exponential of the connection \( A \) along a closed loop \( C \) which is embedded in \( M \),
\[ W_\rho(C) = \text{Tr}_\rho \left( P \exp \oint_C A \right), \] (4)

where \( \rho \) is a representation of the gauge group \( G \) (\( \bar{\rho} \) is its conjugate representation). Note that a closed loop \( C \) can be knotted in the three manifold.

A normalized VEV of an operator \( O(A) \) is defined as
\[ \langle O(A) \rangle = \frac{Z(M, k, O)}{Z(M, k, 1)}; \quad Z(M, k, O) = \int [DA] O(A) e^{iS}. \] (5)

In the following we use notation for the normalized VEV of unknotted Wilson loop operator
\[ E_0(\rho) = \langle W_\rho(\circ) \rangle, \] (6)

such that the VEV of the baryon-type operator (Figure 1) is denoted as
\[ \langle K_{\rho \bar{\rho}'}(\rho_1, \rho_2, \rho_3) \rangle = \delta_{\rho \bar{\rho}'} \sqrt{E_0(\rho_1)E_0(\rho_2)E_0(\rho_3)}, \] (7)

and if, for example \( \rho_1 = \text{id} \) (identity representation), then from consistency condition \[ \langle K_{\rho \bar{\rho}'}(\text{id}, \bar{\rho}_3, \rho_3) \rangle = \delta_{\rho \bar{\rho}'} E_0(\rho_3). \] (8)

Now, let us imagine the following situation (see Figure 2). A test particle scattered off a massive source particle situated at the spatial origin of the space-time manifold \( M \). Upon quantization of the test particle, the scattering amplitude describing this process becomes a sum of contribution of homotopically inequivalent paths for the test particle owing to the fact that the CSW theory is topological.

Such paths can be labelled by winding numbers between the trajectories of the source and the test particles. If we project the trajectories onto a surface
$\epsilon \equiv K_{ee'}(\rho_1, \rho_2, \rho_3)$

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Figure 1: A baryon-type operator

$\Sigma_t \times \rho_{\sigma} |i\rangle \langle f| \Sigma_f$

Figure 2: A test particle scattered off a massive source particle

$\Sigma$, then the process of winding number $l$ is described by the braiding operator $g^{(2l)}_{\rho\sigma}$. Here we assume that the representations assigned to the source and test particles are $\rho$ and $\sigma$, respectively. Consequently, the scattering amplitude is given by Ref. [20]

$$\langle f | \sum_{l=-\infty}^{\infty} g^{(2l)}_{\rho\sigma} |i\rangle = \sum_{l=-\infty}^{\infty} \left\{ \frac{\langle \hat{g}^{(2l)}_{\rho\sigma} \rangle}{E_0(\rho)E_0(\sigma)} \langle f | g^{(0)}_{\rho\sigma} |i\rangle + \sum_{u=1}^{r-1} b^{(l)}_u \langle f | H_{\rho\sigma}(\phi_u) |i\rangle \right\}. \quad (9)$$

The initial state $|i\rangle$ and the final state $\langle f|$ are connected by the operator $g^{(2l)}_{\rho\sigma}$, which represents trajectories of the source and test particles.

In the above equation, the VEV of the braiding operator $\langle \hat{g}^{(2l)}_{\rho\sigma} \rangle$ has been calculated exactly in the case of compact and simple Lie groups and super Lie group, whereas $b^{(l)}_u$ is given by

$$b^{(l)}_u = \sum_{s=1}^{r-1} (q^{-Q(\rho)-Q(\sigma)+Q(\lambda_s)})^l \sqrt{\frac{E_0(\lambda_s)}{(E_0(\rho)E_0(\sigma))^2}} \langle T(\phi_u, \sigma, \lambda_s, \rho, \bar{\rho}) \rangle, \quad (10)$$
where

\[ q = \exp \left( \frac{2\pi i}{k + Q(\text{Adj})} \right) . \]  

(11)

\( Q(\rho) \) is the quadratic Casimir invariant of the \( \rho \) representation. \( T(\phi_u, \sigma, \bar{\sigma} | \lambda, \rho, \bar{\rho}) \) is a tetrahedron operator (Figure 3).

Figure 3: A tetrahedron operator which contribute to scattering amplitude of a test particle and a source particle

Now, we will discussed the detail calculation of the VEV of tetrahedron operator for simple Lie group

Starting from \( u \) channel basis \( H_u \), we can construct the following expansion of \( s \) channel \( I_s \)

\[ \sum_{u'} L_{su'} \]  

(12)

If we attach the Wilson line \( \begin{array}{c} 4 \\ 2 \end{array} \) to the top side of both \( H_u \) and \( I_s \) we will get the following relation

\[ \sum_{u'} L_{su'} \]  

(13)

---

6Here, we also use the following notation \( i \equiv \rho_i \) and \( \bar{i} \equiv \bar{\rho}_i (i = 1, 2, 3, 4) \) as representations and its conjugate of the gauge group \( G \) attach to the Wilson lines.
The loop in the Wilson line can be expressed as a single Wilson line according to the following identity

\[
\bar{4} 2 = \delta_{uu'}(U_{42}^u)^{-1} u' \tag{14}
\]

then, it can be shown easily that a tetrahedron diagram can be constructed from baryon type diagrams as given in the following relation

\[
\includegraphics[width=0.5\textwidth]{tetrahedron_diagram}
\]

Note that \( U_{ij}^s \) is determined by connecting both endpoints in Eq. (14), and therefore can be written as

\[
U_{ij}^s = \sqrt{\frac{E_0(\rho_s)}{E_0(\rho_i) E_0(\rho_j)}}. \tag{16}
\]

Substituting this into the above tetrahedron-baryon relation, we get

\[
\langle 1 2 4 \rangle = L_{su} \left( \frac{E_0(\bar{4}) E_0(2)}{E_0(u)} \right)^{1/2} \langle 3 \rangle, \tag{17}
\]

which, by substituting the definition in equation (7), we finally get the VEV of tetrahedron diagram

\[
\langle s 4 u \rangle = L_{su} \left( \frac{E_0(\bar{4}) E_0(2)}{E_0(u)} \right)^{1/2} (E_0(\bar{3}) E_0(u) E_0(\bar{1}))^{1/2}, \tag{18}
\]

\[
= L_{su}(E_0(\bar{4}) E_0(3) E_0(2) E_0(\bar{1}))^{1/2}, \tag{19}
\]
or
\[ L_{su} = (E_0(\bar{4})E_0(\bar{3})E_0(\bar{1})E_0(2))^{-1/2} \left( \begin{array}{c} s \\ 4 \\ u \\ 2 \\ 1 \end{array} \right). \] (20)

One can also easily see the cyclic symmetry modulo 4 of the tetrahedron diagram as follows

\[ \begin{array}{c}
\begin{array}{c}
1 \\
2 \\
3 \\
\end{array}
\rightarrow
\begin{array}{c}
3 \\
1 \\
2 \\
\end{array}
\rightarrow
\begin{array}{c}
\bar{c} \\
\bar{a} \\
\bar{b} \\
\end{array}
\rightarrow
\begin{array}{c}
\bar{a} \\
\bar{b} \\
\bar{c} \\
\end{array}
\rightarrow
\begin{array}{c}
\begin{array}{c}
1 \\
2 \\
3 \\
\end{array}
\end{array}
\end{array} \]

We can also define similar matrices that relates different channels

\[ J_t = \sum_u \phi_t M_{tu} \phi_u^{-1} H_u H_u = \sum_t \phi_t^{-1} M_{tu} \phi_u J_t \] (21)

and

\[ J_t = \sum_s N_{ts} \phi_s I_s I_s = \sum_t N_{ts} \phi_s^{-1} J_t, \] (22)

where the \( \phi \) factors are defined as

\[ \phi_s = \beta_s q_s^{\frac{1}{4}(Q(3)+Q(4)-Q(s))}, \] (23)

\[ \phi_t = \beta_t q_t^{\frac{1}{4}(Q(t)+Q(4)-Q(1))}, \] (24)

\[ \phi_u = \beta_u q_u^{\frac{1}{4}(Q(2)+Q(4)-Q(u))}, \] (25)

with symmetry factors

\[ \beta_a = \begin{cases} 
+1 & a \text{ is symmetric combination of two representations} \\
-1 & a \text{ is antisymmetric combination of two representations.} 
\end{cases} \] (26)
These $M$ and $N$ matrices can be found by the same manner as the previous derivation of the $L$ matrix. In summary, the matrices $L, M,$ and $N$ are given by

$$L_{su} = \left( E_0(1)E_0(2)E_0(3)E_0(4) \right)^{-\frac{1}{2}} \langle T(4, s, 2|1, u, 3) \rangle,$$  
$$M_{tu} = \left( E_0(1)E_0(2)E_0(3)E_0(4) \right)^{-\frac{1}{2}} \langle T(4, u, 2|3, t, 1) \rangle,$$  
$$N_{ts} = \left( E_0(1)E_0(2)E_0(3)E_0(4) \right)^{-\frac{1}{2}} \langle T(1, t, 4|3, s, 2) \rangle.$$  

From the orthonormality condition of the bases, these matrices will satisfy the following unitary constraints

$$LL^\dagger = 1, MM^\dagger = 1, NN^\dagger = 1,$$  

as well as the following orthogonality conditions, obtained from previous relations

$$LL^T = 1, MM^T = 1, NN^T = 1.$$  

Also, from the consistency condition, one can easily derive the following relation among these matrices

$$N\phi_s L = \phi_t M\phi_u^{-1}.$$  

These constraints and conditions will fix the components of these $L, M,$ and $N$ matrices.

Now, for example, if we choose $\bar{\rho} = \sigma = \bar{\rho}$ and $\bar{\sigma} = \rho = \bar{\rho},$ where $\bar{\rho}$ and $\bar{\rho}$ are the fundamental representation of $SU(N),$ the VEV of tetrahedron operator gives

$$L_{id,S} = \frac{1}{E_0^2(N)} \langle T(S, N, \bar{N}|id, \bar{N}, N) \rangle = \frac{\sqrt{E_0(S)}}{E_0(N)},$$  
$$L_{id,A} = \frac{1}{E_0^2(N)} \langle T(A, N, \bar{N}|id, \bar{N}, N) \rangle = \frac{\sqrt{E_0(A)}}{E_0(N)},$$  
$$N_{Adj,id} = \frac{1}{E_0^2(N)} \langle T(id, N, \bar{N}|Adj, \bar{N}, N) \rangle = \frac{\sqrt{E_0(Adj)}}{E_0(N)},$$  
$$N_{id,id} = \frac{1}{E_0^2(N)} \langle T(id, N, \bar{N}|id, \bar{N}, N) \rangle = \frac{1}{E_0(N)}.$$  

The above relation are consistent with the VEV of baryon type operator in equation (7).

### 3 Perturbative Case

In the previous chapter, we discussed the calculation of vacuum expectation value of extended Wilson loop operators by using unperturbative method. This
The concept has been used for many applications in various physical conditions [10, 22]. Now, we will calculate the vacuum expectation value of unknotted Wilson loop operator \( \langle W^\rho(\bigcirc) \rangle \) within the framework of perturbation theory. The perturbative problem has been discussed in many papers for many different settings [7, 8, 12, 17, 23, 24, 25]. In this section, we use arbitrary gauge group and restrict the order of the calculation of \( \langle W^\rho(\bigcirc) \rangle \) up to order \( \frac{2\pi}{k} \)^3. Particularly, we compare the result with the nonperturbative one for SU(\(N\)) and \(E_6\) gauge groups.

In the perturbative case, the classical CSW action [11] must undergo a modification in order to perform the quantization. We will adopt the standard Faddeev-Popov procedure, and the total action becomes [6]

\[
S_{\text{tot}} = S_{\text{CS}} + S_{\text{gauge-fixing}} + S_{\text{ghost}}
\]

\[
= \frac{k}{4\pi} \int_{M^3} d^3x \epsilon^{\mu\nu\rho} \text{Tr} \left( A_\mu \partial_\nu A_\rho + \frac{i}{3} A_\mu A_\nu A_\rho \right)
\]

\[
+ \frac{k}{4\pi} \int_{M^3} d^3x \sqrt{g} g^{\mu\nu} A_\mu \partial_\nu \phi^a - \int_{M^3} d^3x \sqrt{g} g^{\mu\nu} \partial_\mu c^a (D_\nu c)^a,
\]

where \(c\) and \(\bar{c}^a\) are Faddeev-Popov ghosts, \(\phi^a\) is the Lagrange multiplier (auxiliary field) and

\[
(D_\mu c)^a = \partial_\mu c^a - f^{abc} A_\mu c^c.
\]

Here, \(f^{abc}\) is the structure constant of the group \(G\) and \(g^{\mu\nu}\) is some metric on \(M^3\). The resulting propagators from eq. (37) are

\[
\langle A_\mu^a(x) A_\nu^b(y) \rangle = \frac{i}{k} \delta^{ab} \epsilon_{\mu\nu\sigma} \frac{(x - y)^\sigma}{|x - y|^3},
\]

\[
\langle A_\mu^a(x) \phi^b(y) \rangle = -\frac{i}{k} \delta^{ab} \frac{(x - y)^\mu}{|x - y|^3},
\]

\[
\langle \phi^a(x) \phi^b(y) \rangle = 0,
\]

\[
\langle c^a(x) \bar{c}^b(y) \rangle = -\frac{i}{4\pi} \delta^{ab} \frac{1}{|x - y|}.
\]

The Wilson loop operator in a representation \(\rho\) of \(G\) [5, 6] is defined as

\[
W^\rho(C) = \text{Tr}_\rho(\mathcal{P} \exp \oint_C A)
\]

\[
= \text{Tr}_\rho \left[ 1 + i \oint_C dx^\mu A_\mu(x) - \oint_C dx^\mu \int^x dy^\nu A_\nu(y) A_\mu(x)
\]

\[
- i \oint_C dx^\mu \int^x dy^\nu \int^y dz^\rho A_\rho(z) A_\nu(y) A_\mu(x) \right]
\]
\[ + \oint_C dx^\mu \int^x dy^\nu \int^y dz^\rho \int^z dw^\sigma A_\sigma(w) A_\rho(y) A_\nu(z) A_\mu(x) \]

\[ + i \oint_C dx^\mu \int^x dy^\nu \int^y dz^\rho \int^z dw^\sigma A_\lambda(w) A_\sigma(w) A_\nu(y) A_\mu(x) \]

\[ - \oint_C dx^\mu \int^x dy^\nu \int^y dz^\rho \int^z dw^\sigma A_\tau(u) A_\lambda(w) A_\sigma(w) A_\nu(y) A_\mu(x) + \ldots \] \hspace{1cm} (43)

All line integrals are performed on the same contour \(C\). If an explicit parameterization \{\(x^\mu(t) : 0 \leq t \leq 1\)\} of \(C\) is used, then we will get

\[ \oint_C dx^\mu \int^x dy^\nu = \int_0^1 ds \int_0^s dt \dot{x}^\mu(s) \dot{x}^\nu(t), \] \hspace{1cm} (44)

and so on.

### 3.1 The Ghost Fields Contribution

From eq. (37), the total quantized action contains contributions from gauge, ghost and auxiliary fields. This can be rewritten again as

\[ S_{\text{tot}} = \int d^3 x \left( \frac{k}{4\pi} c_2(\rho) e^{\mu\rho} A^a_\mu \partial_\nu A^a_\rho + \frac{k}{4\pi} A^a_\mu \partial^\mu \phi^a + \bar{c}^a \partial^\mu \partial_\mu c^a \right) \]

\[ + \int d^3 x f^{abc} \left( \partial^\mu c^a A^b_\mu A^c_\mu - \frac{k}{12\pi} c_2(\rho) e^{\mu\rho} A^c_\mu A^a_\rho A^b_\rho \right), \] \hspace{1cm} (45)

where \(c_2(\rho)\) is the quadratic Casimir for the fundamental representation and \(\text{dim } \rho\) is the dimension of the gauge group.

The standard VEV of the Wilson loop operator of the action (45) can be written as

\[ \langle W_{\rho}(C) \rangle = \int D A D \phi D c D \bar{c} \text{ Tr}_\rho \left( \mathcal{P} \exp \oint_C A \right) e^{i S_{\text{tot}}}. \] \hspace{1cm} (46)

To solve eq. (46) perturbatively, we insert external source functions \(J\) and \(H\) in the partition function for gauge and ghost fields respectively, and the VEV of the Wilson loop operators is obtained as follows

\[ \langle W_{\rho}(C) \rangle = \left[ \text{dim } \rho + c_2(\rho) \oint_C dx^\tau \int^x dy^\nu \frac{\delta^2}{\delta J^\tau \delta J^\nu} \right. \]

\[ + \frac{i}{2} c_2(\rho) f^{ijk} \oint_C dx^\tau \int^x dy^\nu \int^y dz^\rho \frac{\delta^3}{\delta J^\tau \delta J^\nu \delta J^\rho} + \ldots \]
\[ \times \exp \left[ - f^{def} \int d^3 x \left( \partial^\sigma \frac{1}{\delta H^d} \right) \frac{\delta^3}{\delta J^{e\sigma} \delta \bar{H}^f} \right] \times \exp \left[ \frac{k}{12\pi} c_2(\rho) \int d^3 x f^{abc} \epsilon^{\alpha\beta\gamma} \frac{\delta^3}{\delta J^{a\alpha} \delta J^{b\beta} \delta J^{c\gamma}} \right] Z_0 \bigg|_{J=H=\bar{H}=0} \]  

where \( Z_0 \) is the partition functional that is defined as

\[ Z_0 = \exp \left[ \int d^3 x d^3 y \left( \frac{1}{8 c_2(\rho)} J^s_\mu(x) V_{A A}^{st,\mu\nu}(x-y) J^t_\nu(y) \right. \right. \]

\[ + H^s(x) V_{cc}^{st}(x-y) \bar{H}^t(y) + F(V_{A \phi}) \bigg] \] 

where \( V_{A A}^{ab,\mu\nu}(x-y) = \langle A^a_\mu(x) A^b_\nu(y) \rangle \) and \( V_{cc}^{ab}(x-y) = \langle c^a(x) c^b(y) \rangle \). Note that the contribution of auxiliary fields vanish, because the total action (37) does not contain any vertex of the auxiliary fields.

One would expect that the ghost contribution vanish for all orders due to the unphysical nature of the ghost fields. We show that this contributions vanish up to the second order.

If eq. (47) is expanded as power series of \((1/k)\), we get

(i) For order \((1/k)^0\), \( \langle W^\rho(C) \rangle = \dim \rho \).

(ii) For order \((1/k)^1\), the VEV of the Wilson loop operator for ghost fields is defined as

\[ \langle W^\rho(C) \rangle^{(1)}_{\text{ghost}} = \frac{\dim \rho}{2!} \left[ \int d^3 x f^{def} \left( \partial^\sigma \frac{1}{\delta H^d} \right) \frac{\delta^3}{\delta J^{e\sigma} \delta \bar{H}^f} \right]^2 Z_0 \bigg|_{J=H=\bar{H}=0} \]

\[ - \dim \rho \left[ \int d^3 x f^{def} \left( \partial^\sigma \frac{1}{\delta H^d} \right) \frac{\delta^3}{\delta J^{e\sigma} \delta \bar{H}^f} \right] \]

\[ \times \left[ \frac{k c_2(\rho)}{12\pi} \int d^3 x f^{abc} \epsilon^{\alpha\beta\gamma} \frac{\delta^3}{\delta J^{a\alpha} \delta J^{b\beta} \delta J^{c\gamma}} \right] Z_0 \bigg|_{J=H=\bar{H}=0} \]  

One can see that the value of \( \langle W^\rho(C) \rangle^{(1)}_{\text{ghost}} \) vanish.

(iii) For order \((1/k)^2\), the VEV of the Wilson loop operator for ghost fields contain seven terms that can be written as

\[ \langle W^\rho(C) \rangle^{(2)}_{\text{ghost}} = - \frac{i}{2} f^{ijk} c_2(\rho) \int_C dx^\tau \int^x dy^\lambda \int^y dz^\rho \frac{\delta^3}{\delta J^{i\tau} \delta J^{j\lambda} \delta J^{k\rho}} \]

\[ \times \left[ \int d^3 x f^{def} \left( \partial^\sigma \frac{1}{\delta H^d} \right) \frac{\delta^3}{\delta J^{e\sigma} \delta \bar{H}^f} \right] Z_0 \bigg|_{J=H=\bar{H}=0} \]
- c_2(\rho) \oint_C dx^\tau \int^x dy^\lambda \frac{\delta^2}{\delta J^{i\tau} \delta J^{i\lambda}} \left[ \int d^3 x \ f^{def} \left( \frac{\partial^\sigma}{\delta H^d} \right) \frac{\delta^3}{\delta J^{e\sigma} \delta H^f} \right]

\times \left[ \frac{k}{12\pi} c_2(\rho) \int d^3 x \ f^{abc} \epsilon^{\alpha\beta\gamma} \frac{\delta^3}{\delta J^{a\alpha} \delta J^{b\beta} \delta J^{c\gamma}} \right] Z_0 \bigg|_{J=H=\bar{H}=0}

- \frac{\dim \rho}{3!} \left[ \int d^3 x \ f^{def} \left( \frac{\partial^\sigma}{\delta H^d} \right) \frac{\delta^3}{\delta J^{e\sigma} \delta H^f} \right]

\times \left[ \frac{k}{12\pi} c_2(\rho) \int d^3 x \ f^{abc} \epsilon^{\alpha\beta\gamma} \frac{\delta^3}{\delta J^{a\alpha} \delta J^{b\beta} \delta J^{c\gamma}} \right] Z_0 \bigg|_{J=H=\bar{H}=0}

+ \frac{\dim \rho}{2!} \left[ \int d^3 x \ f^{def} \left( \frac{\partial^\sigma}{\delta H^d} \right) \frac{\delta^3}{\delta J^{e\sigma} \delta H^f} \right]^2

\times \left[ \frac{k}{12\pi} c_2(\rho) \int d^3 x \ f^{abc} \epsilon^{\alpha\beta\gamma} \frac{\delta^3}{\delta J^{a\alpha} \delta J^{b\beta} \delta J^{c\gamma}} \right] Z_0 \bigg|_{J=H=\bar{H}=0}

+ \frac{\dim \rho}{4!} \left[ \int d^3 x \ f^{def} \left( \frac{\partial^\sigma}{\delta H^d} \right) \frac{\delta^3}{\delta J^{e\sigma} \delta H^f} \right]^4

\times \left[ \frac{k}{12\pi} c_2(\rho) \int d^3 x \ f^{abc} \epsilon^{\alpha\beta\gamma} \frac{\delta^3}{\delta J^{a\alpha} \delta J^{b\beta} \delta J^{c\gamma}} \right] Z_0 \bigg|_{J=H=\bar{H}=0}

\frac{1}{2!} c_2(\rho) \oint_C dx^\tau \int^x dy^\lambda \frac{\delta^2}{\delta J^{i\tau} \delta J^{i\lambda}}

\times \left[ \int d^3 x \ f^{def} \left( \frac{\partial^\sigma}{\delta H^d} \right) \frac{\delta^3}{\delta J^{e\sigma} \delta H^f} \right]^2

\times \left[ \frac{k}{12\pi} c_2(\rho) \int d^3 x \ f^{abc} \epsilon^{\alpha\beta\gamma} \frac{\delta^3}{\delta J^{a\alpha} \delta J^{b\beta} \delta J^{c\gamma}} \right] Z_0 \bigg|_{J=H=\bar{H}=0}.

(50)

Because of the anticommutation and the peculiar structure of the indices (see Appendix A), eq. (50) become simpler

\begin{align*}
\langle W_\rho(C) \rangle_{\text{ghost}}^{(2)} &= \frac{\dim \rho}{4!} \left[ \int d^3 x \ f^{def} \left( \frac{\partial^\sigma}{\delta H^d} \right) \frac{\delta^3}{\delta J^{e\sigma} \delta H^f} \right]^4 Z_0 \bigg|_{J=H=\bar{H}=0}

+ \frac{1}{2!} c_2(\rho) \oint_C dx^\tau \int^x dy^\lambda \frac{\delta^2}{\delta J^{i\tau} \delta J^{i\lambda}}

\times \left[ \int d^3 x \ f^{def} \left( \frac{\partial^\sigma}{\delta H^d} \right) \frac{\delta^3}{\delta J^{e\sigma} \delta H^f} \right]^2 Z_0 \bigg|_{J=H=\bar{H}=0}.
\end{align*}
Finally, eq. (51) vanish because of various reasons that will be explained in Appendix A.

3.2 The Zeroth, First and Second Order Contribution of Gauge Fields

In this section, we calculate perturbatively the zeroth, first, and second order contributions of gauge fields following Guadagnini et al. [6]. In the next section we extend the calculation up to the third order contribution. In order to simplify the calculation, the unknotted knot is chosen as a circle

$$U_0 = \{ x(s) = (\cos 2\pi s, \sin 2\pi s, 0); \ 0 \leq s \leq 1 \}. \ (52)$$

For the $\left( \frac{2\pi}{k} \right)^0$ contribution to $\langle W_\rho (C) \rangle$, we will get

$$\langle W_\rho (C) \rangle^{(0)} = \langle W_\rho (\emptyset) \rangle^{(0)} = \dim \rho. \ (53)$$

Then, $\left( \frac{2\pi}{k} \right)$ contribution to $\langle W_\rho (C) \rangle^{(1)}$ is defined as

$$\langle W_\rho (C) \rangle^{(1)} = - \text{Tr} (R^b R^a) \int_C dx^\mu \int dy^\nu \langle A^b_\nu(y) A^a_\mu(x) \rangle$$

$$= -i \left( \frac{2\pi}{k} \right) \dim \rho \ c_2(\rho) \ \varphi(C), \ (54)$$

where the quadratic Casimir for the fundamental representation $c_2(\rho)$ is given by

$$c_2(\rho)1 = R^a R^a, \ (55)$$

and $\varphi(C)$ is defined as

$$\varphi(C) = \frac{1}{2\pi} \int_0^1 ds \int_0^s dt \ \epsilon^{\mu \nu \sigma} \dot{x}^\mu(s) \dot{x}^\nu(t) \frac{(x(s) - x(t))^\sigma}{|x(s) - x(t)|^3}. \ (56)$$

The formula (56) is known as the cotorsion of $C$. The cotorsion is not invariant under the deformation of $C$, because it is metric dependent. This is contrary to the fact that the $\langle W_\rho (C) \rangle$ is a topological invariant. This problem can be solved by inserting a framing contour $C_f$ that is defined as

$$x^\mu \to y^\mu = x^\mu + \epsilon n^\mu(t), \quad (\epsilon > 0, |n(t)| = 1). \ (57)$$
where $n^\mu$ is a vector field orthogonal to $C$. In this paper, we choose the value of $n^\mu$ to be

$$n(s) = [0, 0, e^{\pi i s}]. \tag{58}$$

If the formula (54) is rewritten by inserting a framing contour $C_f$ (57) for the unknot condition (52), we obtain

$$\langle W_\rho(C) \rangle_f^{(1)} = -i \left( \frac{2\pi}{k} \right) \dim \rho c_2(\rho) \varphi_f(U_0) = 0, \tag{59}$$

where $\varphi_f(U_0)$ is the value of $\varphi(C)$ with inserted framing contour $C_f$ (57) for the unknot (52).

Now, we will analyze the $(2\pi)^2$ contribution to $\langle W_\rho(C) \rangle$ which results from the interactions part of the Lagrangian contributed by the $A^3$ and $A^4$ terms of eq. (43).

The first term of the $(2\pi)^2$ part of $\langle W_\rho(C) \rangle$ can be written as

$$\langle W_\rho(C) \rangle_f^{(2a)} = \text{Tr}_\rho \left[ -i \oint_C dx^\mu \int^x dy^\nu \int^y dz^\rho \langle A_\rho(z) A_\nu(y) A_\mu(x) \rangle \right] \tag{60}$$

where the quadratic Casimir for the adjoint representation $c_v$ is obtained through the relation

$$\delta^{ab} c_v = f^{acd} f^{bde}, \tag{61}$$

and

$$H_{\mu\nu\rho}(x, y, z) = e^{\alpha\beta\gamma} \epsilon_{\mu\alpha\sigma} \epsilon_{\nu\beta\lambda} \epsilon_{\rho\gamma\tau} \int d^3 l \frac{(l-x)^\sigma (l-y)^\lambda (l-z)^\tau}{|l-x|^3 |l-y|^3 |l-z|^3}. \tag{62}$$

If we use the unknotted knot (52) in equation (62), we will obtain

$$\zeta_1(U_0) = \frac{1}{32\pi^3} \oint_C dx^\mu \int^x dy^\nu \int^y dz^\rho H_{\mu\nu\rho}(x, y, z) \tag{63}$$

$$= -\frac{1}{16\pi^3} \int_0^{2\pi} d\theta \int_0^\theta d\phi \int_0^{\phi} d\psi \times$$

$$\times \left[ \sin \left( \frac{\theta - \phi}{2} \right) + \sin \left( \frac{\theta - \psi}{2} \right) + \sin \left( \frac{\phi - \psi}{2} \right) \right]$$

$$= -\frac{1}{12},$$
and we get the value of $\langle W_\rho(\zeta) \rangle^{(2a)}$ as

$$\langle W_\rho(\zeta) \rangle^{(2a)} = -\frac{1}{12} \left( \frac{2\pi}{k} \right)^2 \dim \rho \ c_v \ c_2(\rho). \quad (64)$$

The second term of the $\left( \frac{2\pi}{k} \right)^2$ part of $\langle W_\rho(C) \rangle$ can be written as [6]

$$\langle W_\rho(C) \rangle^{(2b)} = \text{Tr}_\rho \left[ \oint_C d\mu \int^x dy \int^y d\rho \int^z d\nu \int^w d\sigma \langle A_\sigma(w) A_\rho(z) A_\nu(y) A_\mu(x) \rangle \right]$$

$$= -\frac{1}{2} \left( \frac{2\pi}{k} \right)^2 \dim \rho \ c^2_2(\rho) \varphi^2(C) + \left( \frac{2\pi}{k} \right)^2 \dim \rho \ c_v \ c_2(\rho) \zeta_2(C) \quad (65)$$

where $\varphi(C)$ is defined in eq. (56) and $\zeta_2(C)$ is defined as

$$\zeta_2(C) = \frac{1}{8\pi^2} \oint_C d\mu \int^x dy \int^y d\nu \int^z d\sigma \int^w d\tau \epsilon_{\sigma\tau\alpha\beta} (w - y)^\alpha (z - x)^\beta \frac{\rho_c}{|w - y|^3 |z - x|^3}. \quad (66)$$

If we use the unknotted knot [52] in equation (65), the value of $\langle W_\rho(\zeta) \rangle^{(2b)}$ is

$$\langle W_\rho(\zeta) \rangle^{(2b)} = 0, \quad (67)$$

where we have applied the framing procedure similar to the first order case [59]. More specific calculations can be found in [6]. Note that the equations (65) and (66) are contributions of order $\left( \frac{2\pi}{k} \right)^2$ to $\langle W_\rho(C) \rangle$.

### 3.3 The $\left( \frac{2\pi}{k} \right)^3$ Contributions

In this section, we discuss the contribution of order $\left( \frac{2\pi}{k} \right)^3$ to $\langle W_\rho(C) \rangle$. It is divided into two parts, $\langle W_\rho(C) \rangle^{(3a)}$ and $\langle W_\rho(C) \rangle^{(3b)}$. $\langle W_\rho(C) \rangle^{(3a)}$ contains the interaction part of the Lagrangian contracted with the $A^3$ term of eq. (43), that is

$$\langle W_\rho(C) \rangle^{(3a)} = \text{Tr} \left[ i \oint_C d\mu \int^x dy \int^y d\rho \int^z d\nu \int^w d\sigma \int^v d\lambda \times \right.$$

$$\times \langle A_\lambda(v) A_\sigma(w) A_\rho(z) A_\nu(y) A_\mu(x) \rangle \left. \right]$$

$$= \frac{i c_v \dim \rho \ c^2_2(\rho)}{8\pi^3 k^3} \oint_C d\mu \int^x dy \int^y d\rho \int^z d\nu \int^w d\sigma \int^v d\lambda \times$$

$$\times \left[ F_{\sigma\rho\nu\mu}(v - w, y, x, z) + F_{\lambda\rho\sigma\nu\mu}(v - z, y, x, w) + F_{\lambda\nu\sigma\rho\mu}(v - x, z, y, w) + F_{\lambda\rho\nu\lambda\mu}(w - z, y, x, v) + F_{\lambda\nu\lambda\rho\mu}(w - y, z, x, v) + F_{\rho\lambda\nu\lambda\sigma\mu}(w - z, y, x, v) + F_{\rho\nu\lambda\lambda\sigma\mu}(w - y, z, x, v) \right]$$

$$+ F_{\lambda\rho\mu\lambda\nu\sigma\mu}(z - x, w, y, v) + F_{\rho\nu\lambda\lambda\sigma\nu}(z - y, w, x, v) + F_{\rho\mu\lambda\lambda\sigma\nu}(z - x, w, y, v) + F_{\nu\rho\lambda\lambda\sigma\nu}(y - x, w, z, v)$$
The contribution is written in equation (70) and includes only terms of order \( \frac{1}{k^2} \) which form combinations of three gauge propagators. The tetrahedron diagram and perturbative calculation in CSW theory involve combinations of two gauge vertices are excluded since they are of order \( \frac{1}{k^3} \). The contribution is related to the \( A^6 \) term of eq. (43). This contribution is written in equation (70) and includes only terms of order \( \frac{1}{k^3} \) which form combinations of three gauge propagators. The \( A^6 \) terms that involve combinations of two gauge vertices are excluded since they are of order \( \frac{1}{k^4} \). The contribution \( \langle W^6 \rangle \) is defined as

\[
\langle W^6 \rangle = \text{Tr} \left[ i \int \frac{d\rho}{k^3} \frac{c^2 C_2(\rho)}{16\pi^2} \int_C x^\mu \int_x^y dy^\nu \int_y^z dz^\rho \int_z^w dw^\sigma \int_w^v dv^\lambda \times \right.
\]
\[
\times \left[ F_{\lambda\rho,\sigma\mu}(w-z, y, x, w) + F_{\nu\rho,\sigma\mu}(v-y, z, x, w) + F_{\tau\nu,\lambda\rho}(w-y, z, x, v) + F_{\mu\nu,\lambda\rho}(w-x, z, y, v) + F_{\rho\nu,\lambda\sigma}(z-x, w, y, v) \right],
\]

(68)

where \( F_{\lambda\rho,\sigma\mu}(v-w, y, x, z) \) is given by

\[
F_{\lambda\rho,\sigma\mu}(v-w, y, x, z) = \epsilon_{\lambda\sigma\rho} \frac{(v-w)^\alpha}{|v-w|} H_{\rho\mu}(y-z, x-z). \tag{69}
\]

The \( \langle W^6 \rangle \) contribution is related to the \( A^6 \) term of eq. (43). This contribution is written in equation (70) and includes only terms of order \( \frac{1}{k^3} \) which form combinations of three gauge propagators. The \( A^6 \) terms that involve combinations of two gauge vertices are excluded since they are of order \( \frac{1}{k^4} \). The contribution \( \langle W^6 \rangle \) is defined as

\[
\langle W^6 \rangle \quad = \quad \text{Tr} \left[ i \int \frac{d\rho}{k^3} \frac{c^2 C_2(\rho)}{16\pi^2} \int_C x^\mu \int_x^y dy^\nu \int_y^z dz^\rho \int_z^w dw^\sigma \int_w^v dv^\lambda \int_v^u du^\tau \times \right.
\]
\[
\times \left[ A_\tau(u)A_\lambda(v)A_\sigma(w)A_\rho(z)A_\nu(y)A_\mu(x) \right] \right]
\]

\[
= \frac{i \dim \rho c^2 C_2(\rho)}{k^3} \int_C x^\mu \int_x^y dy^\nu \int_y^z dz^\rho \int_z^w dw^\sigma \int_w^v dv^\lambda \int_v^u du^\tau \times \right.
\]
\[
\times \left[ \epsilon_{\tau\lambda\sigma} \epsilon_{\rho\sigma\beta} \epsilon_{\nu\mu\gamma} \frac{(u-v)^\alpha (w-z)^\beta (y-x)^\gamma}{|u-v|^3 |w-z|^3 |y-x|^3} + \epsilon_{\tau\lambda\sigma} \epsilon_{\nu\beta\rho} \epsilon_{\rho\mu\gamma} \frac{(u-v)^\alpha (w-y)^\beta (z-x)^\gamma}{|u-v|^3 |w-y|^3 |z-x|^3} + \epsilon_{\tau\lambda\sigma} \epsilon_{\nu\mu\beta} \epsilon_{\rho\nu\gamma} \frac{(u-v)^\alpha (w-x)^\beta (z-y)^\gamma}{|u-v|^3 |w-x|^3 |z-y|^3} + \epsilon_{\tau\rho\alpha} \epsilon_{\sigma\lambda\beta} \epsilon_{\nu\mu\gamma} \frac{(u-z)^\alpha (v-w)^\beta (y-x)^\gamma}{|u-z|^3 |v-w|^3 |y-x|^3} + \right.
\]
\[
1 - \frac{k_1}{2} \epsilon_{\tau\sigma\alpha} \epsilon_{\lambda\rho\beta} \epsilon_{\nu\mu\gamma} \frac{(u-w)^\alpha (v-z)^\beta (y-x)^\gamma}{|u-w|^3 |v-z|^3 |y-x|^3} + \frac{k_1^2}{4} \epsilon_{\tau\sigma\alpha} \epsilon_{\lambda\rho\beta} \epsilon_{\nu\mu\gamma} \frac{(u-w)^\alpha (v-y)^\beta (z-x)^\gamma}{|u-w|^3 |v-y|^3 |z-x|^3}.
\]
\[
\begin{align*}
&+ \left(1 - \frac{k_1}{2}\right) \epsilon_{\tau\sigma\alpha} \epsilon_{\mu\nu\beta} \epsilon_{\rho\gamma} \frac{(u - w)^\alpha (v - x)^\beta (z - y)^\gamma}{|u - w|^3 |v - x|^3 |z - y|^3}, \\
&+ \left(1 - \frac{3k_1}{2} + \frac{k_1^2}{2}\right) \epsilon_{\tau\rho\alpha} \epsilon_{\mu\nu\beta} \epsilon_{\sigma\mu\gamma} \frac{(u - z)^\alpha (v - y)^\beta (w - x)^\gamma}{|u - z|^3 |v - y|^3 |w - x|^3}, \\
&+ \left(1 - k_1 + \frac{k_1^2}{4}\right) \epsilon_{\tau\rho\alpha} \epsilon_{\mu\nu\beta} \epsilon_{\sigma\nu\gamma} \frac{(u - z)^\alpha (v - x)^\beta (w - y)^\gamma}{|u - z|^3 |v - x|^3 |w - y|^3}, \\
&+ \left(1 - \frac{k_1}{2}\right) \epsilon_{\tau\nu\alpha} \epsilon_{\lambda\nu\beta} \epsilon_{\rho\mu\gamma} \frac{(u - y)^\alpha (v - w)^\beta (z - x)^\gamma}{|u - y|^3 |v - w|^3 |z - x|^3}, \\
&+ \left(1 - k_1 + \frac{k_1^2}{4}\right) \epsilon_{\tau\nu\alpha} \epsilon_{\lambda\rho\beta} \epsilon_{\sigma\mu\gamma} \frac{(u - y)^\alpha (v - z)^\beta (w - x)^\gamma}{|u - y|^3 |v - z|^3 |w - x|^3}, \\
&+ \left(1 - \frac{k_1}{2}\right) \epsilon_{\tau\nu\alpha} \epsilon_{\lambda\mu\beta} \epsilon_{\rho\sigma\gamma} \frac{(u - y)^\alpha (v - x)^\beta (w - z)^\gamma}{|u - y|^3 |v - x|^3 |w - z|^3}, \\
&+ \epsilon_{\tau\mu\alpha} \epsilon_{\lambda\sigma\beta} \epsilon_{\rho\nu\gamma} \frac{(u - x)^\alpha (v - w)^\beta (z - y)^\gamma}{|u - x|^3 |v - w|^3 |z - y|^3}, \\
&+ \left(1 - \frac{k_1}{2}\right) \epsilon_{\tau\mu\alpha} \epsilon_{\lambda\rho\beta} \epsilon_{\sigma\nu\gamma} \frac{(u - x)^\alpha (v - z)^\beta (w - y)^\gamma}{|u - x|^3 |v - z|^3 |w - y|^3}, \\
&+ \epsilon_{\tau\mu\alpha} \epsilon_{\lambda\nu\beta} \epsilon_{\rho\sigma\gamma} \frac{(u - x)^\alpha (v - y)^\beta (z - w)^\gamma}{|u - x|^3 |v - y|^3 |z - w|^3},
\end{align*}
\]

(70)

where \( k_1 = c_\nu/c_\rho (\rho) \). If we use the unknot condition (52) in eq. (68), we get the integral

\[
\oint_C dx^\mu \int^x dy^\nu \int^y dz^\rho \int^z dw^\sigma \int^w dv^\lambda \epsilon_{\rho\mu\alpha} \frac{(z - x)^\alpha}{|z - x|^3} H_{\lambda\sigma\nu}(w - v, y - v) = 32i\pi \left(\frac{\pi^2}{6} - 1\right),
\]

(71)

\[
\oint_C dx^\mu \int^x dy^\nu \int^y dz^\rho \int^z dw^\sigma \int^w dv^\lambda \epsilon_{\sigma\mu\alpha} \frac{(w - x)^\alpha}{|w - x|^3} H_{\lambda\rho\nu}(z - v, y - v) = 32i\pi \left(\frac{\pi^2}{6} - 1\right),
\]

(72)
∮_C dx^\mu \int^x dy^\nu \int^y dz^\rho \int^z dw^\sigma \int^w dv^\lambda \left[ \epsilon_{\lambda \rho \alpha} \frac{(v-z)^\alpha}{|v-z|^3} H_{\sigma \nu \mu}(y-w,x-w) + \epsilon_{\sigma \nu \rho} \frac{(w-y)^\alpha}{|w-y|^3} H_{\lambda \rho \mu}(z-v,x-v) \right] = 32i\pi, \quad (73)

∮_C dx^\mu \int^x dy^\nu \int^y dz^\rho \int^z dw^\sigma \int^w dv^\lambda \epsilon_{\lambda \nu \alpha} \frac{(v-y)^\alpha}{|v-y|^3} H_{\sigma \rho \mu}(z-w,x-w) = 32i\pi, \quad (74)

∮_C dx^\mu \int^x dy^\nu \int^y dz^\rho \int^z dw^\sigma \int^w dv^\lambda \epsilon_{\lambda \mu \alpha} \frac{(v-x)^\alpha}{|v-x|^3} H_{\rho \nu \mu}(z-w,y-w) = 32i\pi \frac{\pi^2}{6}, \quad (75)

∮_C dx^\mu \int^x dy^\nu \int^y dz^\rho \int^z dw^\sigma \int^w dv^\lambda \epsilon_{\lambda \sigma \alpha} \frac{(v-w)^\alpha}{|v-w|^3} H_{\rho \nu \mu}(y-z,x-z) + \epsilon_{\sigma \rho \alpha} \frac{(w-z)^\alpha}{|w-z|^3} H_{\lambda \nu \mu}(y-v,x-v) + \epsilon_{\rho \nu \alpha} \frac{(z-y)^\alpha}{|z-y|^3} H_{\lambda \sigma \mu}(w-v,x-v) + \epsilon_{\nu \mu \alpha} \frac{(y-x)^\alpha}{|y-x|^3} H_{\lambda \sigma \rho}(w-v,z-v) \right] = -32i\pi \frac{\pi^2}{6}. \quad (76)

More details of the calculation of (71)- (76) can be found in Appendix B. By using the values of integrals in eqs. (71)- (76), we can calculate the VEV of an unknotted Wilson loop operator for order \((\frac{2\pi}{k})^3\) in \(\langle W_\rho(C) \rangle^{(3a)}\)

\[ \langle W_\rho(\bigotimes) \rangle^{(3a)} = \frac{c_v^2 \dim \rho \ c_2(\rho)}{k^3} \left( \frac{2\pi^2}{3} \right). \quad (77) \]

The value of \(\langle W_\rho(\bigotimes) \rangle^{(3b)}\) is obtained by using the framing procedure as in eq. (67)

\[ \langle W_\rho(\bigotimes) \rangle^{(3b)} = 0. \quad (78) \]

From the equations (53), (59), (64), (67), (77) and (78), we can conclude that the calculation of VEV of an unknotted Wilson loop operator up to order \((\frac{2\pi}{k})^3\) is given by

\[ \langle W_\rho(\bigotimes) \rangle = \dim \rho \left[ 1 - \frac{1}{12} \left( \frac{2\pi}{k} \right)^2 c_v \ c_2(\rho) + \frac{c_v^2 \ c_2(\rho)}{3} \left( \frac{2\pi^2}{k^3} \right) + \ldots \right]. \quad (79) \]
We use the computation in the previous section for the gauge groups SU\((N)\) and E\(_6\) as examples. For the gauge group SU\((N)\), the values of \(\text{dim} \rho\) and quadratic Casimir are

\[
\text{dim} \rho = N, \quad (80)
\]
\[
c_2(N) = Q(N) = \frac{N^2 - 1}{2N}, \quad (81)
\]
\[
c_v = Q(\text{Adj}) = N. \quad (82)
\]

Then, from non-perturbative case, we get

\[
E_0(N) = [N]_{\sqrt{\pi}} = N \left[ 1 - \frac{\pi^2}{2} \left( \frac{N^2 - 1}{6} \right) + \frac{2N\pi^2}{k^3} \left( \frac{N^2 - 1}{6} \right) + \ldots \right]. \quad (83)
\]

If we calculate eq. (79) by using the values in eqs. (80)-(82), the VEV of an unknotted Wilson loop operator for this gauge group, up to the same order, will have the same values as in the equation (83):

\[
\langle W_N(\odot) \rangle = N \left[ 1 - \frac{\pi^2}{2} \left( \frac{N^2 - 1}{6} \right) + \frac{2N\pi^2}{k^3} \left( \frac{N^2 - 1}{6} \right) + \ldots \right]. \quad (84)
\]

As in SU\((N)\) case, the E\(_6\) group has the following values:

\[
\text{dim} \rho = \text{dim} 27 = 27, \quad (85)
\]
\[
c_2(27) = Q(27) = \frac{26}{3}, \quad (86)
\]
\[
c_v = Q(\text{Adj}) = Q(78) = 12. \quad (87)
\]

From non-perturbative CSW theory, we get

\[
E_0(27) = [3]_{\sqrt{\pi}}[9]_{\sqrt{\pi}} = 27 - 936 \frac{\pi^2}{k^2} + 22464 \frac{\pi^2}{k^3} + \ldots. \quad (88)
\]

Then, as in eq. (83) and (84), the nonperturbative method in eq. (88) will be identical to the perturbative method in eq. (89) up to order \((\frac{2\pi}{k})^3\) of \(\langle W_\rho(\odot) \rangle\):

\[
\langle W_{27}(\odot) \rangle = 27 - 936 \frac{\pi^2}{k^2} + 22464 \frac{\pi^2}{k^3} + \ldots. \quad (89)
\]
4 Conclusions and Discussions

We have discussed the role of Wilson loop operators and extended operators in the CSW theory. We have also discussed a two-particle scattering system, one of which we treat as a test particle scattered off a source. In the calculation in [21], the second term of the equation (9), or the contributions from tetrahedron operator, is missing. We evaluated this term for SU($N$) gauge group.

The calculation of the VEV of the Wilson loop operator in the CSW theory has been discussed by Witten where he has showed that the VEV of the Wilson loop operator in perturbation theory is the same as the polynomial invariants of knot in three dimensions.

Looking at the ($\frac{1}{k}$)$^0$ up to ($\frac{1}{k}$)$^3$ terms of the equation (83), (84), (88) and (89), we summarize that the braiding formula is identical up to the third order of the VEV of an unknotted Wilson loop operator. For example, we have checked this result for the gauge group SU($N$) and E$_6$. In fact, our calculation showed that the symmetry and dynamical terms factorize, so the contribution of the group factor can be computed independently and hence the application of other gauge groups is straightforward. Up to order ($\frac{1}{k}$)$^2$, the VEV of the Wilson loop operator has been computed in [6]. The problem arises in the computation of the VEV of the Wilson loop operator for the unknotted case of order ($\frac{1}{k}$)$^3$. In this case, the use of the framing procedure results in non-simple integral forms.

We hope that the result will help to illuminate more insights of the equivalence between the braiding formula and the VEV of an unknotted Wilson loop operator in perturbation theory and its consistency with the equations (83) and (88) will strengthen this relation.

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A Detailed Calculation of Ghost Contributions

In the last section, we have calculated the contribution of ghost fields to the VEV of the Wilson loop operator. In this appendix, we provide more details of this calculation up to order \(1/k^2\).

The first part of the eq. (51) can be written as

\[
\langle W_\rho(C) \rangle^{(2a)}_{\text{ghost}} = \frac{\dim \rho}{4!} \left[ \int d^3x \ f^{\text{def}} \left( \frac{\delta}{\delta H^i} \right) \frac{\delta^3}{\delta J^\alpha \delta H^i} \right]^4 Z_0 \bigg|_{J=H=H'=0}.
\]

Some terms of the solution of eq. (90) vanish since they contain the determinant form with dependent columns or rows, i.e. \(\epsilon_{\alpha\beta\gamma} \epsilon_{\sigma\delta\lambda} p_4^\alpha p_5^\beta p_2^\delta = 0\). Therefore we can write eq. (90) as

\[
\langle W_\rho(C) \rangle^{(2a)}_{\text{ghost}} = -\dim \rho \left( \frac{\pi}{2k c_2(\rho)} \right)^2 f f_2 e_1 f_1 f f_4 e_3 f_2 f f_1 e_3 f_3 f f_3 e_1 f_4 \epsilon_{\sigma_1 \sigma_4 p_1}
\]

\[
\times \epsilon_{\sigma_2 \sigma_3 p_2} \int d^3x \int \frac{d^3p_1}{(2\pi)^3} \frac{p_1^{\sigma_1} p_4^{\sigma_2}}{p_4^2} \int \frac{d^3p_5}{(2\pi)^3} \frac{p_5^{\sigma_3}}{p_5^2} \left[ \frac{p_5^2}{(p_5 + p_4)^2} - \frac{p_5^2}{p_5^2} \right]
\]

\[
- 4 \cdot \dim \rho \left( \frac{1}{64} \frac{\pi}{2k c_2(\rho)} \right)^2 f f_2 e_3 f_1 f f_4 e_2 f f_1 e_3 f_3 f f_3 e_4 f_4
\]

\[
\times \epsilon_{\sigma_1 \sigma_3 p_1} \epsilon_{\sigma_1 \sigma_4 p_2} \int d^3x \int \frac{d^3p_1}{(2\pi)^3} \frac{p_1^{\sigma_1} p_4^{\sigma_2}}{p_4^5} \int \frac{d^3p_5}{(2\pi)^3} \frac{p_5^{\sigma_3}}{p_5^5} \left[ \frac{p_5^2}{(p_5 + p_4)^2} - \frac{p_5^2}{p_5^2} \right]
\]

\[
= 3 \cdot \dim \rho \left( \frac{\pi}{16k c_2(\rho)} \right)^2 f f_2 e_1 f_1 f f_4 e_3 f_2 f f_1 e_3 f_3 f f_3 e_1 f_4 \int d^3x \epsilon_{\sigma_1 p_2 p_1}
\]

\[
\times \epsilon_{\sigma_2 \sigma_3 p_2} \int d^3x \int \frac{d^3p_1}{(2\pi)^3} \frac{p_1^{\sigma_1} p_4^{\sigma_2}}{p_4^5} \int \frac{d^3p_5}{(2\pi)^3} \frac{p_5^{\sigma_3}}{p_5^5} \left[ \frac{p_5^2}{(p_5 + p_4)^2} - \frac{p_5^2}{p_5^2} \right]
\]

\[
- \dim \rho \left( \frac{\pi}{2k c_2(\rho)} \right)^2 f f_2 e_1 f_1 f f_4 e_3 f_2 f f_1 e_3 f_3 f f_3 e_4 f_4 \epsilon_{\sigma_1 \sigma_4 p_1}
\]

\[
\times \epsilon_{\sigma_2 \sigma_3 p_2} \int d^3x \int \frac{d^3p_1}{(2\pi)^3} \frac{p_1^{\sigma_1} p_4^{\sigma_2}}{p_4^5} \int \frac{d^3p_5}{(2\pi)^3} \frac{p_5^{\sigma_3}}{p_5^5} \left[ \frac{p_5^2}{(p_5 + p_4)^2} - \frac{p_5^2}{p_5^2} \right]
\]

\[
\times \left[(p_4 - p_5)^2 (p_4 - p_5)^\sigma_1 B \left( \frac{7}{2}, 1 \right) - p_5^2 p_5^4 B \left( \frac{7}{2}, 1 \right) - \frac{(p_4 - p_5)^2}{4} \delta^{\sigma_2 \sigma_4} B \left( \frac{5}{2}, 2 \right) + \frac{p_5^2}{4} \delta^{\sigma_2 \sigma_4} B \left( \frac{5}{2}, 2 \right) \right]
\]
\[
+2 \frac{\text{dim } \rho}{3} \left( \frac{\pi}{16k} c_2(\rho) \right)^2 \left( \frac{1}{128} \right) f f_2^e f_1 c f_4^e f_3 c f_2^e f_4 c \\
\times \epsilon_{\sigma_1 \rho_2 \rho_1} \epsilon_{\sigma_1 \rho_2 \sigma_3} \int d^3x \int \frac{d^3p_4}{(2\pi)^3} \frac{p_4^\sigma p_4^\rho}{p_4^2} \\
= 0. \quad (91)
\]

The second part of eq. (51) vanishes

\[
\langle W_\rho(C) \rangle_{\text{ghost}}^{(2b)} = \frac{1}{2!} c_2(\rho) \int_C dx^\tau \int x^\lambda \frac{\partial^2}{\delta J^\sigma \delta J^\lambda} \\
\times \left[ \int d^3x f^{\text{def}} \left( \partial^\sigma \frac{1}{\delta H^4} \right) \frac{\delta^3}{\delta J^\sigma \delta J^4} \right]^2 Z_0 \bigg|_{J=H=\bar{H}=0} \\
= - \left( \frac{\pi}{8k} \right)^2 \left( \frac{c_v \text{ dim } \rho}{c_2(\rho)} \right) \int_C dx^\tau \int x^\lambda \epsilon_{\sigma_\lambda \alpha} \epsilon_{\sigma_\beta} \\
\times \int \frac{d^3p}{(2\pi)^3} \frac{p_\alpha p_\beta}{p^3} \cos[p_\gamma(x-y)] \\
= 0, \quad (92)
\]

because of the following relations

\[
\cos[p_\gamma(x-y)] = 1 - \frac{[p_\gamma(x-y)]^2}{2!} + \frac{[p_\gamma(x-y)]^4}{4!} \ldots, \quad (93)
\]

and

\[
\int \frac{d^3p}{(2\pi)^2} \frac{p_{\mu_1} p_{\mu_2}}{p^3} = \int \frac{d^3p}{(2\pi)^2} \frac{p_{\mu_1} p_{\mu_2} p_{\mu_3}}{p^3} = \ldots = 0. \quad (94)
\]

Finally, the third part of the eq. (51) is

\[
\langle W_\rho(C) \rangle_{\text{ghost}}^{(2c)} = - \frac{\text{dim } \rho}{3!} \left[ \int d^3x f^{\text{def}} \left( \partial^\sigma \frac{1}{\delta H^4} \right) \frac{\delta^3}{\delta J^\sigma \delta J^4} \right]^3 \\
\times \left[ \frac{k}{12\pi} c_2(\rho) \int d^3x f^{abc} \epsilon^\alpha \epsilon^\beta \epsilon^\gamma \frac{\delta^3}{\delta J^\alpha \delta J^\beta \delta J^\gamma} \right] Z_0 \bigg|_{J=H=\bar{H}=0} \quad (95)
\]

which, after eliminating the zero terms can be written as

\[
\langle W_\rho(C) \rangle_{\text{ghost}}^{(2c)} = \frac{4 \text{ dim } \rho}{6\pi} \left( \frac{\pi}{k} \right)^3 \left( \frac{1}{2} c_2(\rho) \right)^2 f f_b^e f_1 c f_2^a f_3 e f_4 c \epsilon_{\sigma_2 \rho_1 \rho_2}
\]
\begin{align*}
\times \epsilon_{\sigma_2,\sigma_1,\rho_1} & \int d^3 x \left[ \frac{d^3 p_1}{(2\pi)^3} \frac{\rho_1^\sigma}{64} \frac{d^3 p_2}{(2\pi)^3} \frac{\rho_2^\rho}{p_2^2 (p_2 - p_1)^6} \right. \\
& \left. + \frac{d^3 p_2}{(2\pi)^3} \frac{\rho_2^\sigma}{64} \frac{d^3 p_1}{(2\pi)^3} \frac{\rho_1^\rho}{p_1^2 (p_1 - p_2)^6} \right] \\
- & \frac{\dim \rho k}{18\pi} \left( \frac{\pi}{k} \right)^3 \left( -\frac{1}{2 \c_2(\rho)} \right)^2 f_{f_1 b f_1} f_{f_1 c f_2} f_{f_2 a f_3} f^{abc} \epsilon_{\sigma_2,\sigma_1,\rho_1} \\
& \times \epsilon_{\sigma_3,\sigma_1,\rho_3} \int d^3 x \int \frac{d^3 p_1}{(2\pi)^3} \frac{\rho_1^\sigma}{p_1^2} \int \frac{d^3 p_2}{(2\pi)^3} \frac{\rho_2^\rho}{p_2^2 (p_2 - p_1)^2} \\
& \times \frac{1}{4\pi^2 (p_1 - p_2)^3} \left[ p_1^\sigma p_1^\rho B \left( \frac{7}{2}, 1 \right) - \frac{p_1^2}{4} \delta_{\rho^2 \rho^3} B \left( \frac{5}{2}, 2 \right) \\
& - p_2^\rho p_2^\rho B \left( \frac{7}{2}, 1 \right) + \frac{p_2^2}{4} \delta_{\rho^2 \rho^3} B \left( \frac{5}{2}, 2 \right) \right] \\
- & \frac{\dim \rho k}{2\pi} \left( \frac{\pi}{k} \right)^3 \left( \frac{1}{64 \c_2(\rho)} \right)^2 f_{f_2 b f_1} f_{f_3 c f_2} f_{f_1 a f_3} f^{abc} \epsilon_{\sigma_3,\sigma_1,\rho_1} \\
& \times \epsilon_{\sigma_1,\rho_1,\rho_1} \epsilon_{\sigma_2,\sigma_3,\rho_1} \int d^3 x \int \frac{d^3 p_1}{(2\pi)^3} \frac{\rho_1^\sigma}{p_1^2} \frac{\rho_1^\rho}{p_1^2} \\
& \times \left[ \int \frac{d^3 p_2}{(2\pi)^3} \frac{\rho_2^\rho}{p_2^2} - \int \frac{d^3 p_2}{(2\pi)^3} \frac{\rho_2^\rho}{(p_1 + p_2)^2} \right] \\
+ & \frac{\dim \rho k}{6\pi} \left( \frac{\pi}{k} \right)^3 \left( \frac{1}{2 \c_2(\rho)} \right)^2 f_{f_2 b f_1} f_{f_3 c f_2} f_{f_1 a f_3} f^{abc} \epsilon_{\sigma_1,\rho_1,\rho_1} \\
& \times \epsilon_{\sigma_2,\sigma_3,\rho_2} \int d^3 x \int \frac{d^3 p_2}{(2\pi)^3} \frac{\rho_2^\rho}{p_2^2} \int \frac{d^3 p_1}{(2\pi)^3} \frac{\rho_1^\rho}{p_1^2 (p_1 + p_2)^2} \\
& \times \left[ (p_1 + p_2)^\rho (p_1 + p_2)^\rho B \left( \frac{7}{2}, 1 \right) - p_2^\rho p_2^\rho B \left( \frac{7}{2}, 1 \right) \\
& - \frac{(p_1 + p_2)^2}{4} \delta_{\rho_1 \rho_2} B \left( \frac{5}{2}, 2 \right) + \frac{p_2^2}{4} \delta_{\rho_1 \rho_2} B \left( \frac{5}{2}, 2 \right) \right]
= 0. \quad (96)
\end{align*}

From the results of the above calculations, we can conclude that the contributions of ghost fields to the VEV of the Wilson loop operator up to order \((1/k^2)\) vanish.
Detailed Calculation of The Integrals Using Framing

In this paper, we use circle for the unknotted knot which parametrization is given in eq. (52) and the vector field orthogonal to $C$ is $n^\mu$ which is given in eq. (58). Therefore we get

$$
\epsilon_{\mu\nu} \dot{x}^\mu(s) (\dot{x}^\nu(t) + \epsilon n^\nu(t)) (x(s) - x(t) - \epsilon n(t))^\sigma \\
= \det[\dot{x}(s)|\dot{x}(t)|x(s) - x(t)] + \epsilon \det[\dot{x}(s)|\dot{n}(t)|x(s) - x(t)] \\
- \epsilon \det[\dot{x}(s)|\dot{n}(t)|n(t)] - \epsilon^2 \det[\dot{x}(s)|\dot{n}(t)|n(t)] \\
= \epsilon e^{i\pi t} \left[ 4\pi^4 i(s - t)^2 - (2\pi)^2 \left( t - s - \frac{(t - s)^3}{3!} \right) \right].
$$

(97)

By using the integral in the eq. (57), the imaginary part of the integral is

$$
\int_0^t du \epsilon_{\mu\nu} \dot{x}^\mu(s) (\dot{x}^\nu(u) + \epsilon n^\nu(u)) \frac{(x(s) - x(u) - \epsilon n(u))^\sigma}{|x(s) - x(u) - \epsilon n(u)|^3} \\
= \epsilon e^{i\pi s} \int_{s - \delta}^t du \left[ \frac{32\pi^4 i(u - s)^2 - 32\pi^2(u + s)}{16\pi^2(u - s)^2 + 4\epsilon^2} \right]^{3/2} = -\pi \epsilon e^{i\pi s} \\
= -i\pi \sin \pi s,
$$

(98)

$$
\int_0^t du \dot{x}^\mu(s)\dot{x}^\nu(t)\dot{x}^\rho(u) H_{\mu\nu\rho} (x(t) - x(s), x(u) - x(s)) \\
= (2\pi)^4 \int_0^t du [\sin \pi |s - t| + \sin \pi |s - u| + \sin \pi |u - t|]^{-1} \\
= (2\pi)^3 \sec^2 \left[ \frac{\pi(s - t)}{2} \right] \ln \left( 1 - \cot \frac{\pi s}{2} \tan \frac{\pi t}{2} \right).
$$

(99)

$$
\int_0^u dg \int_0^g dh \dot{x}^\sigma(g)\dot{x}^\nu(t)\dot{x}^\lambda(h) H_{\lambda\sigma\nu}(x(g) - x(h), x(t) - x(h)) \\
= 16\pi^2 \int_{g=0}^u \ln \left( 1 - \cot \frac{\pi t}{2} \tan \frac{\pi g}{2} \right) d \tan \left( \frac{\pi(g - t)}{2} \right) \\
= 16\pi^2 \ln \left( 1 - \cot \frac{\pi t}{2} \tan \frac{\pi u}{2} \right) \tan \left( \frac{\pi(u - t)}{2} \right) \\
- \cot \frac{\pi t}{2} \ln \left( 1 + \tan \frac{\pi t}{2} \tan \frac{\pi u}{2} \right).
$$

(100)
Next, we will compute the integral in eq. (71)- (76) by framing the unknotted knot.

The first, from the eq. (71), there is integral upper limit \( u \to t \) for integration variable \( dg \):

\[
\oint_C dx^\mu \int_x^y dy^\nu \int_y^z dz^\rho \int_z^w dw^\sigma \int_w^\infty dv^\lambda \epsilon_{\mu\sigma\alpha} (x-z)\alpha \frac{(x-z)\alpha}{|x-z|^3} H_{\lambda\sigma\nu}(w-v, y-v) \]

\[
= \int_0^1 ds \int_0^s dt \int_0^u du \int_0^{u-t} dg \int_0^g dh \epsilon_{\mu\sigma\alpha} \hat{x}^\mu(s) (\hat{x}^\rho(u) + \epsilon \hat{n}^\rho(u)) \\
\times \frac{(x(s)-x(u) - \epsilon n(u))\alpha}{|x(s)-x(u) - \epsilon n(u)|^3} \hat{x}^\sigma (t) \hat{x}^\lambda (h) H_{\lambda\sigma\nu}(x(g) - x(h), x(t) - x(h)) \\
= 16\pi^3 i \int_0^1 ds \sin \pi s \int_0^s dt \cot \frac{\pi t}{2} \ln \left( 1 + \tan^2 \frac{\pi t}{2} \right) \\
= 32\pi i \left( \frac{\pi^2}{6} - 1 \right) \quad (101)
\]

and from the eq. (72), there is integral limit \( g \to u \) for integration variable \( dh \):

\[
\oint_C dx^\mu \int_x^y dy^\nu \int_y^z dz^\rho \int_z^w dw^\sigma \int_w^\infty dv^\lambda \epsilon_{\mu\sigma\alpha} (x-w)\alpha \frac{(x-w)\alpha}{|x-w|^3} H_{\lambda\rho\nu}(z-v, y-v) \]

\[
= \int_0^1 ds \int_0^s dt \int_0^t du \int_0^{u-t} dg \int_0^g dh \epsilon_{\mu\sigma\alpha} \hat{x}^\mu(s) (\hat{x}^\sigma (g) + \epsilon \hat{n}^\sigma (g)) \\
\times \frac{(x(s)-x(g) - \epsilon n(g))\alpha}{|x(s)-x(g) - \epsilon n(g)|^3} \hat{x}^\rho (u) \hat{x}^\lambda (h) H_{\lambda\rho\sigma\nu}(x(u) - x(h), x(t) - x(h)) \\
= -8\pi^4 i \int_0^1 ds \sin \pi s \int_0^s dt \int_0^t du \sec^2 \left( \frac{\pi(u-t)}{2} \right) \ln \left( 1 - \cot \frac{\pi t}{2} \tan \frac{\pi u}{2} \right) \\
= 32\pi i \left( \frac{\pi^2}{6} - 1 \right) . \quad (102)
\]

Then, from the eq. (73), we take the limit \( u \to t \) and \( g \to u \) for integration variables \( dg \) and \( dh \) respectively:

\[
\oint_C dx^\mu \int_x^y dy^\nu \int_y^z dz^\rho \int_z^w dw^\sigma \int_w^\infty dv^\lambda \left[ \epsilon_{\rho\lambda\alpha} \frac{(z-v)\alpha}{|z-v|^3} H_{\sigma\nu\mu}(y-w, x-w) \\
+ \epsilon_{\mu\sigma\alpha} \frac{(y-w)\alpha}{|y-w|^3} H_{\lambda\rho\mu\nu}(z-v, x-v) \right] 
\]
and by using the limit \( t \to s \) in the eq. (74), we get

\[
\frac{1}{1 + \tan \frac{\pi t}{2} \tan \frac{\pi s}{2}} + \sin \pi t \cot \left( \frac{\pi s}{2} \right) \ln \left( 1 + \tan \frac{\pi t}{2} \tan \frac{\pi s}{2} \right) \]

(103)

Next, from the eq. (75), we take the limit \( s \to 1 \) for integration variable \( dt \):

\[
\frac{1}{1 + \tan \frac{\pi t}{2} \tan \frac{\pi s}{2}} + \sin \pi t \cot \left( \frac{\pi s}{2} \right) \ln \left( 1 + \tan \frac{\pi t}{2} \tan \frac{\pi s}{2} \right)
\]

(104)

\[
\frac{1}{1 + \tan \frac{\pi t}{2} \tan \frac{\pi s}{2}} + \sin \pi t \cot \left( \frac{\pi s}{2} \right) \ln \left( 1 + \tan \frac{\pi t}{2} \tan \frac{\pi s}{2} \right)
\]

(105)
and finally, the four integrals in equation (76) can be evaluated as

\[
\oint C \ d\xi^\mu \int^x_0 dy^\nu \int^y_0 dz^\rho \int^z_0 dw^\sigma \int^w_0 dv^\lambda
\times \left[ \epsilon_{\sigma\lambda\alpha} \frac{(w-v)^\alpha}{|w-v|} H_{\rho\mu}(y-z, x-z) + \epsilon_{\rho\sigma\alpha} \frac{(z-w)^\alpha}{|z-w|} H_{\lambda\nu}(y-v, x-v) \\
+ \epsilon_{\nu\rho\alpha} \frac{(y-z)^\alpha}{|y-z|} H_{\lambda\sigma}(w-v, x-v) + \epsilon_{\mu\nu\alpha} \frac{(x-y)^\alpha}{|x-y|} H_{\lambda\sigma\rho}(w-v, z-v) \right]
\]

\[
= \int_0^1 ds \int_0^s dt \int_0^t du \int_0^{u-1} dg \int_0^{s-t} dh \dot{x}^\mu(s) \dot{x}^\nu(t) \dot{x}^\rho(u) \dot{x}^\lambda(h)
\times \left[ \epsilon_{\alpha\lambda\sigma} \frac{(x(g) - x(h))^\alpha}{|x(g) - x(h)|^3} H_{\rho\mu}(x(t) - x(u), x(s) - x(u)) \\
+ \epsilon_{\rho\alpha\lambda} \frac{(x(u) - x(g))^\alpha}{|x(u) - x(g)|^3} H_{\lambda\nu}(x(t) - x(h), x(s) - x(h)) \\
+ \epsilon_{\nu\rho\alpha} \frac{(x(t) - x(u))^\alpha}{|x(t) - x(u)|^3} H_{\lambda\sigma}(x(g) - x(h), x(s) - x(h)) \\
+ \epsilon_{\mu\nu\alpha} \frac{(x(s) - x(t))^\alpha}{|x(s) - x(t)|^3} H_{\lambda\sigma\rho}(x(g) - x(h), x(u) - x(h)) \right]
\]

\[
= -\pi \left[ \int_0^1 ds \int_0^s dt \int_0^t du \int_0^{u-1} dg e^{i\pi g} \dot{x}^\mu(s) \dot{x}^\nu(t) \dot{x}^\rho(u) H_{\rho\mu}(x(t) - x(u), x(s) - x(u)) \\
+ \int_0^1 ds \int_0^s dt \int_0^t du \int_0^{s-t} dh e^{i\pi u} \dot{x}^\mu(s) \dot{x}^\nu(t) \dot{x}^\lambda(h) H_{\lambda\rho}(x(t) - x(h), x(s) - x(h)) \\
+ \int_0^1 ds \int_0^s dt \int_0^t du \int_0^{u-s} dg e^{i\pi t} \dot{x}^\mu(s) \dot{x}^\nu(u) \dot{x}^\lambda(g) H_{\lambda\sigma}(x(g) - x(h), x(s) - x(h)) \\
+ \int_0^1 ds \int_0^t dt \int_0^u du \int_0^{g} dh e^{i\pi s} \dot{x}^\rho(u) \dot{x}^\sigma(g) \dot{x}^\lambda(h) H_{\lambda\sigma\rho}(x(g) - x(h), x(u) - x(h)) \right]
\]

\[
= -\pi \left[ \int_0^1 ds \int_0^s dt \int_0^t du \int_0^{u-1} dg e^{i\pi g} \dot{x}^\mu(s) \dot{x}^\nu(t) \dot{x}^\rho(u) H_{\rho\mu}(x(t) - x(u), x(s) - x(u)) \\
+ \int_0^1 ds \int_0^s dt \int_0^t du \int_0^{s-t} dg e^{i\pi u} \dot{x}^\mu(s) \dot{x}^\nu(t) \dot{x}^\sigma(g) H_{\sigma\nu}(x(t) - x(g), x(s) - x(g)) \\
+ \int_0^1 ds \int_0^s dt \int_0^t du \int_0^{u} dg e^{i\pi t} \dot{x}^\mu(s) \dot{x}^\nu(u) \dot{x}^\sigma(g) H_{\sigma\rho}(x(u) - x(g), x(s) - x(g)) \\
+ \int_0^1 ds \int_0^t dt \int_0^u du \int_0^{g} dg e^{i\pi s} \dot{x}^\rho(t) \dot{x}^\nu(u) \dot{x}^\sigma(g) H_{\sigma\rho}(x(u) - x(g), x(t) - x(g)) \right]
\]

\[
= -\frac{\pi}{4} \times 4 \int_0^1 ds \int_0^s dt \int_0^t du \int_0^{u-1} dg e^{i\pi g} \dot{x}^\mu(s) \dot{x}^\nu(t) \dot{x}^\rho(u) H_{\rho\mu}(x(t) - x(u), x(s) - x(u))
= -\pi i \frac{(1)}{6\pi} \frac{(2)}{\pi} = -32\pi i \left( \frac{\pi^2}{6} \right).
\]
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