Kernel formalism applied to Fourier based Wave Front Sensing in presence of residual phases

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Abstract. In this paper, we show how it is possible to understand Fourier-based Wave Front Sensors (WFS) as Linear Integral Operators. Indeed optical laws allow to identify quantities called “Kernels” which are the continuous version of well-known interaction matrices used in Adaptive Optics. The main purpose of this article is to understand the dependence of these Kernels regarding to the optical parameters of the WFSs: the entrance pupil geometry, the filtering mask and the tip/tilt modulation. Kernels are then linked to the classical performance criterion which is the sensitivity. This approach also allows to take into account time-varying phase residuals. As it turns out, they have physically the same status as the modulation: they are both ”pupil plane light shaping”. The only significant difference is their time-dependent behaviour. Modulation is highly regular and set by the experimenter while phase residuals are erratic and inevitable. Fortunately, the latter may be described thanks to their statistics, namely their ”structure function” which naturally arises in the Kernel framework. These results are essential to tackle the Pyramid Wave Front Sensor’s optical gain problem which is at the present time the major factor which constrains the Pyramid WFS performance and operability.

A second part focuses on the special case of ”convolutional Kernels”. They drastically simplify the mathematical formulation since the WFS’s input and output are linked by a convolution product. Such a formalism firstly provides a fast diagnostic tool to study Fourier-based WFSs but also suggests natural algorithms to reconstruct the phase, e.g. Wigner deconvolution. We pay special attention to the assumptions required to be in such a regime (which does not apply in general). Fortunately, we show that a wise choice of the modulation allows to improve its validity.

1 Introduction

Linear integral operators are the continuous version of Matrix. (cf. Appendix A). They linearly transform an input into an output depending on a quantity called Kernel. Mathematically, such an operation may be written:

\[
\text{Output}|_\beta = \int \text{d}\alpha \ K|_{\beta;\alpha} \ \text{Input}|_\alpha
\]

(1)

$\alpha$ (resp. $\beta$) is the variable of the input (resp. output) space. The Kernel $K$ makes the link between these two spaces. Examples of integral operators are many: Fourier transform, Hilbert, Laplace,... are the most famous. In a more general framework, it is absolutely relevant to try to describe a continuous linear system thanks to an integral operator. Advantages of such an approach are many. First of all, since the Kernel completely characterizes the system, it is possible to calculate from it the system’s performance criteria. Moreover, a parallel between integral and matrix transforms may improve the understanding of the discrete description of the system that numerical approach often requires. We finally note that retrieving the input from the output, i.e. inverting (1), depends on our capability to find the ”inverse Kernel” $K^{-1}$ of the system. It has to verify:

\[
\int K^{-1}|_{\gamma;\beta} K|_{\beta;\alpha} \ d\beta = \delta|_{\gamma;\alpha}
\]

(2)
and allows then to get the input from the output:

\[ \text{Input}|_{\gamma} = \int d\beta K^{-1}|_{\gamma,\beta} \text{Output}|_{\beta} \]  

(3)

Depending on the nature of \( K \), many methods exist to calculate this inverse Kernel (if it does exist). Knowing \( K \) is thus critical to build robust and relevant reconstruction algorithms.

The purpose of this article is to use this powerful formalism to study optical systems which probe the wavefront of the light. They are called Fourier-based Wave Front Sensors\(^1\) and are essentially used in Adaptive Optics for Astronomy.

The first part of this paper provides an optical description of Fourier-based WFSensors. We explicit the input and output of such sensors and introduce proper mathematical quantities to characterize the different optical elements. **Kernel of Fourier-based WFSs** is given in a second part. We explicit how it depends on the optical parameters and also how to generalize its expression in presence of **dynamic residual phases** (as it is often the case, in practice, in Adaptive Optics). The third part is dedicated to the particular case of **convolutional Kernel**. Phase reconstruction, in that situation, is greatly facilitated since the inverse Kernel has an explicit formulation. Moreover, it allows to define the WFS’s **Impulse Response** and **Transfer Function** which are compact and meaningful quantities allowing to rapidly visualize the WFS’s sensitivity. Since the convolutional Kernel case is not systematic, we also give the assumptions it relies on. These results are finally applied to the most used Fourier-based WFS which is the Pyramid WFSensor.\(^2\)

### 2 Fourier-based Wave Front Sensing

#### 2.1 Optical system

We consider the optical system shown in Fig. 1. The first plane corresponds to the pupil plane of the telescope. It contains a focusing device which is described as a perfect lens with a focal \( f \), an aperture and another element that we call **modulation**. We describe it in detail in the next paragraph. The second plane is the focal plane which contains a **filtering mask** mathematically described by its transparency function \( m \) and an imaging lens with focal \( f/2 \). The **detector** is finally placed in the next plane which is conjugated to the first pupil plane.

![Fig 1 Schematic view (in 1D) of a Fourier filtering optical system](image)
2.2 Optical propagation

The incoming field is called $\psi_i$, we assume that its light is monochromatic at wavelength $\lambda$ and that weak-turbulence regime applies to phase disturbances undergone during propagation. In the Adaptive Optics Wave Front Sensing context, the pure phase is windowed by an entrance pupil. We call this phase $\phi$ for turbulent phase and describe the pupil geometry thanks to its indicator function $\mathbb{I}_P$. Besides, we assume that the total flux is unitary. Under these assumptions, we have:

$$
\psi_i = \mathbb{I}_P e^{i\phi} 
$$

(4)

Let’s note that turbulent phase has a priori an infinite support. Nevertheless, due to windowing by the finite aperture, only the part which goes through the entrance pupil has a physical interest and may be coded by the WFS. As a consequence, the quantity to be measured, i.e. the sensor’s input is the turbulent phase windowed by the pupil:

WFS’s input: $\mathbb{I}_P \phi$  \hspace{1cm} (5)

The incoming field is then shaped by the “modulation”. Modulation consists in adding a dynamic aberration to the field via a controlled moving device placed in the entry pupil plane. The movement is regular and may be described as a cycle (although it can be generalised). The detector is then synchronized with it in order to have one image per cycle. This system was initially introduced by in order to adjust the Pyramid WFS’s linearity range. It is specific to this sensor yet it should remain general. In this paper, we are interested in a particular case called “tip/tilt modulation”. It consists in adding a tip/tilt aberration in the field. This modulation phase is called $\phi_m$ and defined as:

$$
\phi_m(a_1, a_2)|_{x,y} = \frac{2\pi}{\lambda}(a_1 x + a_2 y) 
$$

(6)

The two variables $x$ and $y$ are the spatial coordinates of the pupil plane. $a_1$ (resp. $a_2$) codes the amplitude of the tip (resp. tilt) aberration. The field after modulation thus becomes:

$$
\psi_i \rightarrow \psi_i e^{i\phi_m(a_1, a_2)} 
$$

(7)

To describe the modulation, we also need to indicate the time spent for each modulation phase. To do this, we introduce the weighting function $w$. Usually, this function directly depends on the time variable but we prefer here to use the tip/tilt amplitudes $a_1$ and $a_2$. Subsequently, the weighting function may be defined as:

$$
w : \mathbb{R}^2 \rightarrow \mathbb{R}_+ \\
(a_1, a_2) \mapsto w(a_1, a_2) 
$$

(8)

The fact that the weighting function is considered as a 2D function presents several advantages. First, it allows to visualize the tip/tilt modulation in its natural focal plane. Indeed, in this one $(a_1, a_2)$ variables corresponds to spatial shifts. Subsequently, the weighting function directly gives the profile of the modulation. Thus, it becomes possible to envision 2D modulation, as for instance a disk modulation (the classical description only allowed 1D modulation, as for instance

3Brief clarifications about notations: $\phi_m(a_1, a_2)|_{x,y}$ means that the function $\phi_m$ depends on two true spatial coordinates $x, y$ but also on two parameters $a_1$ and $a_2$. They are here scalars but we will see afterwards that it is not systematic.
ring modulation). Moreover, this approach is easily extendable to "non tip/tilt only" modulation: we just have to consider an extended weighting function on the whole phase space. This original approach is described in appendix B. To ensure the energy conservation, we finally enforce that the weighting function has a unitary 1-norm:

$$\int_{\mathbb{R}^2} w(a_1, a_2) \, da_1 da_2 = 1$$

Regarding the optical Fourier filtering stage, it is based on the fact that the focal plane corresponds to the reciprocal space (i.e. the spatial frequencies space) of the phase (or pupil) space. This fact is directly linked to Fraunhofer’s diffraction. Subsequently, a mask placed in this plane acts like a spatial frequencies filter. A relevant way to describe this mask consists in using its transparency function $m$. For a more detailed description of this quantity, the curious reader may refer to 1.

Describing mathematically the filtering process consists in getting the filtered field in the detector’s plane $\psi_d$. Optical propagation laws indicate that this field equates to the convolution product between the pupil plane field and the Fourier transform of the transparency function of the mask:

$$\psi_d = \left[ \mathbb{I}_P \exp \left( i(\phi + \phi_m(a_1, a_2)) \right) \right] \ast \hat{m}$$

where $\hat{m}$ is the Fourier transform of the transparency function of the mask 2.

The last step consists in converting the detector’s field into photo-electrons. Physically, this detection consists in the integration during the modulation cycle of the squared modulus of $\psi_d$. The resulting intensity, which depends on the phase $\phi$, equals to:

$$I(\phi) = \int_{\mathbb{R}^2} \, da_1 da_2 w(a_1, a_2) \left| \left[ \mathbb{I}_P \exp \left( i(\phi + \phi_m(a_1, a_2)) \right) \right] \ast \hat{m} \right|^2$$

In the WFSensing context, it is relevant to give the dependence of this intensity with respect to the phase. A way to efficiently describe this dependence has been shown in detail in 1. It consists in doing a Taylor’s development on the phase term:

$$\exp(i\phi) = \sum_{q=0}^{\infty} \frac{i^q \phi^q}{q!}$$

which allows then to decompose the intensity into phase-constant, linear, quadratic, cubic, etc. terms:

$$I(\phi) = I_{\text{constant}} + I_{\text{linear}}(\phi) + I_{\text{quadratic}}(\phi) + I_{\text{cubic}}(\phi) + \ldots$$

We note that the constant term does not contain any information about phase since it does not depend on $\phi$. Consequently, it is common practice to apply a trivial "return-to-reference" operation on the intensity which consists in numerically retrieving $I_{\text{constant}}$ to $I$. The resulting quantity is usually called meta-intensity:

$$mI(\phi) = I(\phi) - I_{\text{constant}} = I_{\text{linear}}(\phi) + I_{\text{quadratic}}(\phi) + \ldots$$

Actually, it’s not exactly the Fourier transform of the mask but the Fourier transform magnified by a factor depending on the lenses foci and the observation wavelength.
The first phase dependence of \( mI \) is thus the linear one. We note that this numerical operation is easy to do in practice since \( I_{\text{constant}} \) equals to the intensity when there is no turbulent phase, i.e. \( I(0) \). The meta-intensity \( mI \) may be seen as the WFS’s output.

3 Linear Model

A standard approximation consists in assuming that the WFS works in its linearity regime. Mathematically, it means that the meta-intensity only contains the linear term:

\[
mI(\phi) \approx I_{\text{linear}}(\phi)
\]

Calculating the linear regime of a WFS, i.e. the phase subspace where non-linear terms may be neglected, is a challenging task which requires the study of the WFS’s dynamic range but it is not the topic of this paper. Subsequently, we consider that the linear regime of a WFS essentially corresponds to the small phases domain:

\[
\phi \ll 1
\]

3.1 Linear intensity

Within the small phases approximation framework, WFS output equals the linear intensity. We give an explicit expression of it initially developed in:

\[
I_{\text{linear}}(\phi) = 2 \text{Im} \left[ \int_{\mathbb{R}^2} da_1 da_2 \ w(a_1, a_2) \left( \mathbb{I}_{P} e^{i\phi m(a_1, a_2)} \ast \tilde{m} \right) \left( \mathbb{I}_{P} e^{-i\phi m(a_1, a_2)} \ast \tilde{m} \right) \right]
\]

where \( \ast \) means the complex conjugate. The linear intensity is a 2D map corresponding to the perfectly linear response of the WFS regarding to the phase \( \phi \). By using the tip/tilt nature of the modulation (6), it becomes possible to reveal the spatial variations of \( I_{\text{linear}} \):

\[
I_{\text{linear}}(\phi)_{X,Y} = 2 \text{Im} \left[ \int_{\mathbb{R}^2} du dv dx dy \mathbb{I}_{P} \phi \left| x-u, y-v \right| \mathbb{I}_{P} \left| x-u, y-v \right| \tilde{w} \left| u-x, v-y \right| \right]
\]

We realize that, by integrating along \( a_1 \) and \( a_2 \) variables, the 2D Fourier transform of the weighting function appears:

\[
I_{\text{linear}}(\phi)_{X,Y} = 2 \text{Im} \left[ \int_{\mathbb{R}^2} du dv dx dy \mathbb{I}_{P} \phi \left| x-u, y-v \right| \tilde{w} \left| u-x, v-y \right| \right]
\]

Such a result has already been observed for the 1D Pyramid WFS with linear tip/tilt modulation in.\(^3\) Its 2D generalization for any kind of modulations and masks is here allowed thanks to the fact that weighting function is considered as depending on \((a_1, a_2)\) and not only on the time variable.

\(^3\)Such an assumption does make sense in Adaptive Optics context since most of the WFSensing is done in closed loop, i.e. when phase-to-be-measured are residuals of the atmospheric turbulent phase.

\(^4\)Once again, it is not exactly the Fourier transform but the Fourier transform magnificated by a factor depending on \( f \) and \( \lambda \).
At this point, (20) is sufficiently developed to be interpreted as an integral transform performed on the input phase:

\[ I_{\text{linear}}(\phi)|_{X,Y} = \int_{\mathbb{R}^2} dx dy \langle \Pi_P \phi |_{x,y} , K |_{X,Y;x,y} \rangle \]  

(21)

where \( K \) is the Kernel of the Fourier-based WFS in the direct space.

\[ K|_{X,Y;x,y} = 2 \text{Im} \left[ \bar{\hat{m}}|_{X-x,Y-y} \int_{\mathbb{R}^2} du dv \Pi_P |_{u,v} \hat{m}|_{X-u,Y-v} \hat{w}|_{u-x,v-y} \right] \]  

(22)

We observe that such a kernel depends on the WFS optical characteristics. More precisely, we see that it actually depends on "pupil plane" functions: \( \hat{m}, \hat{w} \) and \( \Pi_P \).

\[ K(\hat{m}, \hat{w}, \Pi_P)|_{X,Y;x,y} \]  

(23)

This kernel may also be understood as the continuous interaction matrix with respect to the natural basis of the direct phase space. We call it the "Dirac phase basis"; it is defined as:

\[ \mathcal{B}_\delta = \{ \phi_{x,y}^\delta : (a, b) \rightarrow \delta(a-x)\delta(b-y) ; (x, y) \in \mathbb{R}^2 \} \]  

(24)

The Kernel \( K \) is:

\[ K|_{X,Y;x,y} = I_{\text{linear}}(\phi_{x,y}^\delta)|_{X,Y} \]  

(25)

We emphasize that knowing the response of the WFS with respect to the basis \( \mathcal{B}_\delta \) is fundamental since the decomposition of an arbitrary phase on this basis is absolutely trivial. Such a result means that the Kernel \( K \) is the most general descriptor of a WFS: it allows to compute naturally the WFS’s response to any set of phases.

### 3.2 No modulation case

Since most of the Fourier-based WFSs work without modulation, we give the Kernel when the weighting function is a Dirac function centered at the origin:

\[ w = \delta, \quad \hat{w} = I \]  

(26)

It implies that:

\[ K(\hat{m}, \hat{w} = I, \Pi_P)|_{X,Y;x,y} = 2 \text{Im} \left[ \bar{\hat{m}}|_{X-x,Y-y} (\Pi_P \ast \hat{m})|_{X,Y} \right] \]  

(27)

We conclude that the matrix associated to this kernel may be understood as an Hadamard’s product between a vertical matrix and a circulant one (cf. Appendix A).

### 3.3 WFSensing in presence of residual phases

In this paragraph, we complicate the sensing context by studying a typical Adaptive Optics case: the presence in the pupil plane of residuals phases which are distinct from the phase-to-be-measured. They may be static (as for instance the vast majority of Non Common Path Aberrations) or dynamic (as for instance uncorrected turbulent phase). In both cases, we call these terms \( \phi_r \) for
residuals and assume their statistical behavior to be known. In particular, the notation \( \langle \cdot \rangle \) means "average on the residual phase statistics"\(^5\).

The phase residuals are taken into account by modifying the entrance pupil function:

\[
\mathbb{I}_P \rightarrow \mathbb{I}_P e^{i\phi_r}
\]

Using the linearity of the averaging operation we get the kernel associated to this new situation:

\[
K |_{X,Y;x,y} = 2 \text{Im} \left[ \hat{m} |_{X-x,Y-y} \int_{\mathbb{R}^2} du dv \mathbb{I}_P |_{u,v} \hat{m} |_{X-u,Y-v} \hat{w} |_{u-x,v-y} S |_{x,y,u,v} \right]
\]

where the 4D function \( S \) characterizes the residual phases statistics and is defined as:

\[
S |_{x,y,u,v} \equiv \langle e^{-i\phi_r} |_{x,y} e^{i\phi_r} |_{u,v} \rangle
\]

As a conclusion, in the most general case, the **Kernel in presence of residual phases** may be written:

\[
K(\hat{m}, \hat{w}, \mathbb{I}_P, S) |_{X,Y;x,y}
\]

Unfortunately, without doing any supplementary assumptions, this formula remains difficult to analyze or to numerically compute. Luckily, if we consider Gaussian-distributed, zero-mean phase residuals (which is the case for uncorrected turbulent residuals), the function \( S \) may be simplified to:

\[
S |_{x,y,u,v} = e^{-\frac{1}{2} \langle (\phi_r |_{u,v} - \phi_r |_{x,y})^2 \rangle} = e^{-\frac{1}{2} D |_{x,y,u,v}}
\]

where \( D \) is the **structure function** of the residual phases (see\(^4\)). This quantity is much more easy to compute and well-known concerning the atmospheric turbulence. In that case, the Kernel now depends on:

\[
K(\hat{m}, \hat{w} e^{-\frac{1}{2} D}, \mathbb{I}_P) |_{X,Y;x,y}
\]

In order to go even further, we can use the eventual **stationarity** of the residual phases which says that the structure function only depends on the distance between the points where it is computed:

\[
D |_{x,y,u,v} = D |_{u-x,v-y}
\]

This result is fundamental since it shows that taking into account the residual phases only consists in changing the weighting function by using the structure function of the residual phases:

\[
\hat{w} \rightarrow \hat{w} e^{-\frac{1}{2} D}
\]

Such a fact is, afterwards, quite logical. The modulation itself may be understood as a "residual phases" and its weighting function is a way to characterize its statistics. To conclude, the Kernel under "Gaussian, null average and stationary residuals phases" may be written:

\[
K(\hat{m}, \hat{w} e^{-\frac{1}{2} D}, \mathbb{I}_P) |_{X,Y;x,y}
\]

This equation is crucial in many ways. First, it shows that the Kernel essentially depends in three 2D quantities: the mask, the pupil indicator function and a quantity which is combining the modulation and the residual statistics. It means that "modulation" and "pupil plane statistical behaviors" are fundamentally similar processes. Secondly, it indicates that obtaining the mean Kernel (and subsequently the mean sensitivity) does not require doing a large number of statistical realizations : knowing the structure function is enough (see Fig. 2) ! Such a fact implies a significant acceleration of numerical simulations.

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\(^5\)For a static pattern, we just have: \( \langle \cdot \rangle = \text{Identity} \).
Fig 2 Numerical way to compute the effective weighting function from $D$ and $w$. The tip/tilt modulation is, in this example, circular.

3.4 Sensitivity

We are now interested in the sensitivity performance criterion. Sensitivity is a scalar which quantifies the ratio between WFS’s output and input. The sensitivity regarding to a RMS-normalized phase $\phi$, that we call $s(\phi)$ is defined as:

$$s(\phi) = \|I_{\text{linear}}(\mathbb{I}_P \phi)\|_2$$ (37)

We showed in\(^5\) that such a definition was consistent with\(^5\) and\(^6\) approaches. Using the formulation of the linear intensity in terms of integral transform (21) we obtain that:

$$s(\phi)^2 = \int_{\mathbb{R}^4} du_0 dv_0 dx_0 dy_0 (\mathbb{I}_P \phi)\big|_{u_0,v_0} \mathbb{I} K|_{x_0,y_0} (\mathbb{I}_P \phi)|_{x,y}$$ (38)

where

$$\mathbb{I} K|_{u,v;x,y} = \int_{\mathbb{R}^2} dX dY K|_{X,Y;x,y} K|_{X,Y;u,v}$$ (39)

A convenient way to study a WFS in the linear model consists in looking at its sensitivity on a phase basis. For a given basis $\mathcal{B} = \{\phi_i\}$ we thus represent the ”sensitivity with respect to to $\mathcal{B}$” via a function depending on the variable $i$:

$$s_\mathcal{B}|_i = s(\phi_i)$$ (40)

where $i$ is an index allowing to scan $\mathcal{B}$. It may be proven that the ”total sensitivity”:

$$\left( \sum_i s(\phi_i)^2 \right)^{1/2}$$ (41)

\(^6\)It means that $\|\mathbb{I}_P \phi\|_2 = 1$. 

8
does not depend on the chosen basis. However, "locally" the choice of $B$ does have an impact on the sensitivity structure.

A lot of bases may be relevant depending on the studied WFS. The Zernike polynomials are for instance adapted to circular entrance pupils. In order to stay as general as possible, we only consider two sensitivity representations. The first one is the natural basis of the direct space, i.e. the pupil (or phase) plane while the second one describes the reciprocal space, i.e. the focal (of Fourier) plane.

3.4.1 Sensitivity representation in the direct space

The natural basis of the direct space is the Dirac phase basis, see (24). By using the fact that the Kernel $K$ is the response of the WFS with respects to $B_\delta$ (see (25)), we observe that the sensitivity representation in this basis has a convenient expression:

$$s_{B_\delta}|_{x,y} = \mathbb{I}_P|_{x,y} \sqrt{tK|_{x,y;x,y}}$$

This 2D map provides easy visualization of the sensitivity in the pupil plane space. We observe, in particular that the WFS is not able to see phases which are outside the pupil support (that’s completely normal since there is no photon coming from there).

3.4.2 Sensitivity representation in the reciprocal space

The natural phase basis in the focal plane is the Sine and Cosine basis defined as:

$$B_{\sim} = \left\{ \phi_{f_x f_y}^{\cos} : (x, y) \rightarrow \cos \left(2\pi(f_x x + f_y y)\right) \quad \text{and} \quad \phi_{f_x f_y}^{\sin} : (x, y) \rightarrow \sin \left(2\pi(f_x x + f_y y)\right) ; (f_x, f_y) \in \mathbb{R}^2 \right\}$$

In that case, $i$ equals to $(f_x, f_y) \in \mathbb{R}^2$ which codes the spatial frequencies of the oscillating phases. Such sine and cosine phases only contain two spatial frequencies:

$$\hat{\phi}_{f_x f_y}^{\cos} = \frac{\delta_{f_x} \delta_{f_y} + \delta_{-f_x} \delta_{-f_y}}{2} \quad \text{and} \quad \hat{\phi}_{f_x f_y}^{\sin} = \frac{\delta_{f_x} \delta_{f_y} - \delta_{-f_x} \delta_{-f_y}}{2i}$$

A convenient way to represent sensitivity consists in grouping the sine and cosine sensitivities into a unique number associated to $(f_x, f_y)$:

$$s_{B_{\sim}|_{f_x, f_y}} \equiv \sqrt{s\left(\phi_{f_x f_y}^{\cos}\right)^2 + s\left(\phi_{f_x f_y}^{\sin}\right)^2}$$

We may note that such a definition implies that $s(B_{\sim})$ is an even quantity. Such a property comes from the fact that phases have to be real.

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I purposely did not write a scalar coefficient allowing to normalize these phases on the pupil support regarding to RMS norm.
4 Convolutional model

In this part, we try to determine to what extent the Kernel may simplify into a convolutional kernel. In other words, we are looking for the conditions which ensure that a Fourier-based WFS is Linear but also Shift Invariant. In that case, it exists a function, that we call \( \text{IR} \) (for Impulse Response) which allows to write:

\[
K_{|X,Y;x,y} = \text{IR}_{|X-x,Y-y}
\]  

(46)

As a consequence, the output/input relation of the WFS becomes convolutional:

\[
I_{\text{linear}}(\phi) = (\hat{I}_P \phi) \ast \text{IR}
\]  

(47)

where \( \ast \) corresponds to the 2D convolution product. (47) explains why we call such a sensor ”shift invariant”: if \( T \) is a translation operator, the convolution implies that \( I_{\text{linear}}(T[\phi]) = T[I_{\text{linear}}(\phi)] \). In other words, knowing the WFS’s response for a given phase is enough to know the response to any translation of this phase. Moreover, we observe why we use the notation ”Impulse Response” since this quantity corresponds to the WFS response when the phase is the centered Dirac:

\[
\text{IR} = I_{\text{linear}}(\delta)
\]  

(48)

Advantages of (47) are many. First of all, the Kernel may be rapidly computed since the associated matrix is now pure circulant which is a strongly redundant matrix. Moreover it suggests a natural phase reconstruction: the deconvolution. In other words, we know the inverse Kernel \( K^{-1} \) of a convolutional Kernel. Its explicit expression is:

\[
K^{-1}_{|X,Y;x,y} = \left( \hat{1}_{\text{TF}} \right)_{|X-x,Y-y}
\]  

(49)

where \( \hat{1} \) means ”inverse Fourier transform” and \( \text{TF} \) is the Transfer Function of the WFS defined as the Fourier transform of the Impulse response:

\[
\text{TF} = \hat{\text{IR}}
\]  

(50)

4.1 Required assumptions

We are now interested on the assumptions needed to have a WFS with a convolutional Kernel. First of all, we keep the previous approximations about the turbulence statistics: Gaussian-distributed + zero-mean + stationary phase residuals. Our starting equation is thus:

\[
K(\hat{m}, \hat{w}e^{-\frac{j}{2}D}, \hat{I}_P)|_{X,Y;x,y} = 2\text{Im} \left[ \hat{m}_{|X-x,Y-y} \int_{\mathbb{R}^2} dudv \hat{I}_P|_{u,v} \hat{m}_{|X-u,Y-v}(\hat{w}e^{-\frac{j}{2}D})|_{u-x,v-y} \right]
\]  

(51)

By taking a closer look at this equation, we observe that the main difficulty comes from the \( \hat{I}_P|_{u,v} \) term. It is not such a surprise: the finite size of the pupil is an obvious obstacle to the shift invariance\(^8\). In order to converge towards a convolutional form, we make a Taylor’s development on this function around \( (u - x, v - y) \):

\[
\hat{I}_P|_{u,v} = \hat{I}_P|_{u-x,v-y} + x \partial_x \hat{I}_P|_{u-x,v-y} + y \partial_y \hat{I}_P|_{u-x,v-y} + o(|x| + |y|)
\]  

(52)

\(^8\)For a finite pupil size, a phase Dirac as a definite position with respect to the pupil edge. It is not the case if the pupil is infinite or if this one slides with it...
Such an operation allows to write the Kernel as:

$$K|_{X,Y; x,y} = 2\text{Im} \left[ \tilde{m}(\hat{m} \star (\hat{\omega}e^{-\frac{i}{2}D\hat{I}_P})) \right]|_{X-x,Y-y} + 2x\text{Im} \left[ \tilde{m}(\hat{m} \star (\hat{\omega}e^{-\frac{i}{2}D\partial_x\hat{I}_P})) \right]|_{X-x,Y-y}$$

$$+ 2y\text{Im} \left[ \tilde{m}(\hat{m} \star (\hat{\omega}e^{-\frac{i}{2}D\partial_y\hat{I}_P})) \right]|_{X-x,Y-y} + o(|x| + |y|)$$

(53)

We naturally introduce the following expression of the Impulse Response IR:

$$\text{IR}(\hat{m}, \hat{\omega}) = 2\text{Im} \left[ \tilde{m}(\hat{m} \star \hat{\omega}) \right]$$

(54)

(It is worth noticing that we introduced a new function $\omega$ which may differ from the modulation weighting function $w$. Physical sense of such a quantity is given in the next paragraph.) With such a notation, the Kernel $K$ has a convenient expression:

$$K|_{X,Y; x,y} = \text{IR}(\hat{m}, \hat{\omega}e^{-\frac{i}{2}D\hat{I}_P})|_{X-x,Y-y} + x\text{IR}(m, \hat{\omega}e^{-\frac{i}{2}D\partial_x\hat{I}_P})|_{X-x,Y-y}$$

$$+ y\text{IR}(m, \hat{\omega}e^{-\frac{i}{2}D\partial_y\hat{I}_P})|_{X-x,Y-y} + o(|x| + |y|)$$

(55)

allowing us to understand it as a sum of Hadamard’s product between vertical and circulant matrices:

$$K|_{X,Y; x,y} = \sum \text{Vertical}|_{x,y} \circ \text{Circulant}|_{X-x,Y-y}$$

(56)

4.2 Convolutional model

The convolutional model consists in keeping the first term and neglecting non-convolutional terms. In that case, we indeed get:

$$K(\hat{m}, \hat{\omega}e^{-\frac{i}{2}D\hat{I}_P})|_{X,Y; x,y} \approx \text{IR}(\hat{m}, \hat{\omega}e^{-\frac{i}{2}D\hat{I}_P})|_{X-x,Y-y}$$

(57)

In most cases, such an expression is not exact and constitute an approximation of the linear model. We call this one the sliding pupil approximation since it corresponds to the following assumption:

$$\hat{I}_P|_{u,v} \approx \hat{I}_P|_{u-x,v-y}$$

(58)

Nevertheless, we note that (57) is not an approximation in two cases. First one is a bit unrealistic but worthy of mention. It is the infinite pupil case. Indeed we observe that $\hat{I}_P = \hat{I}$ implies a pure convolutional kernel. The second case is much more interesting since it shows how it is possible to choose the tip/tilt modulation in order to improve the accuracy of the convolutional model. It consists in using a $\omega$ which verifies

$$\hat{\omega}e^{-\frac{i}{2}D\partial\hat{I}_P} = 0$$

(59)

where $\partial\hat{I}_P$ corresponds to the area where the entrance pupil has discontinuities. In other words, a Fourier-based WFS is a convolutional sensor if the Fourier Transform of the weighting function $\hat{\omega}$ is barely null on the edge of the pupil.

Such a condition is actually quite simple to reach. For a circular pupil and a circular modulation, for instance, $\hat{\omega}$ is the first Bessel function of the first kind: $J_0$. In order to ensure (59), we need adapt the modulation radius in such a way that the pupil radius corresponds to a Bessel function’s zero. In other words, it would be possible to efficiently reconstruct the phase with a deconvolution on the condition to properly set the modulation radius.
4.3 Physical interpretation

We discuss in this paragraph the physical meaning of the Impulse Response function. It was defined as:

$$\text{IR}(\hat{m}, \hat{\omega}) = 2\text{Im} \left[ \hat{m} (\hat{m} * \hat{\omega}) \right]$$  \hspace{1cm} (60)

In the infinite pupil case, or when the pupil geometry is not taken into account, we observe that $\omega$ has to verify:

$$\hat{\omega} = \hat{\omega} \Leftrightarrow \omega = w$$  \hspace{1cm} (61)

If we want to tune the model and keep a trace of the pupil geometry we then consider:

$$\hat{\omega} = \hat{\omega} \mathbb{I}_P$$  \hspace{1cm} (62)

which becomes in a ”focal plane” point of view:

$$\omega = w * \mathbb{I}_P$$  \hspace{1cm} (63)

In other words, taking into account the pupil geometry in the convolutional model consists in modifying the modulation weighting function by convolving it with the diffracted field of the pupil. $\omega$ may thus be seen as an effective weighting function of the Fourier-based WFS.

Such a fact has already been noticed in the previous paragraph. Indeed, we saw that considering the residual phases consisted in changing the weighting function:

$$\hat{\omega} = \hat{\omega}_e^{-\frac{1}{2}D}$$  \hspace{1cm} (64)

In conclusion, we emphasize the central role of the Impulse Response $\text{IR}(\hat{m}, \hat{\omega})$. It is the sole function capable of completely characterizing the WFS in the convolutional model. Going through the different WFSensing consists simply in using the proper effective weighting function $\omega$.

4.4 Transfer Function

The Impulse Response is a way to characterize a Fourier-based WFS in the direct phase space, i.e. in a pupil plane. To understand it in the phase spatial frequencies space, i.e. in a focal plane, we have to consider the Transfer Function, previously defined as the Fourier transform of the impulse response:

$$\text{TF} \equiv \hat{\text{IR}}$$  \hspace{1cm} (65)

By Fourier transforming (54), we get a concise expression of the $\text{TF}$ which depends in a very simple way on the mask and the effective weighting function:

$$\text{TF}(m, \omega) = i(m * \overline{m\omega} - \overline{m} * m\omega)$$  \hspace{1cm} (66)

Indeed, getting the $\text{TF}$ only requires basic mathematical operations: conjugation, convolution and double Fourier transform, i.e. symmetry operation\(^9\).

(66) is even clearer if the optical system is centro-symmetric. (It is the case for the 4-faces Pyramid WFS with a circular modulation and and isotropic residual phases). As a matter of fact, we get:

$$\text{TF}(m, \omega) = 2\text{Im}[m * \overline{m\omega}]$$  \hspace{1cm} (67)

\(^9\)For a given 2D function $f$, we have $\hat{f}|_{x,y} = f|_{-x,-y}$. 

12
4.5 Sensitivity

This paragraph is dedicated to the sensitivity criterion in the convolutional model. Its expression may be simplified due to the convolutional nature of the WFS’s input/output relationship. Under that framework, the key quantity $\mathcal{K}\mathcal{K}$ given in (39) which allows to compute the sensitivity with respect to a phase $\phi$ reveals the auto-correlation of the Impulse Response:

$$\mathcal{K}\mathcal{K}_{u,v,x,y} = \mathcal{I}\mathcal{R} \ast \mathcal{I}\mathcal{R}_{u-x,v-y}$$

(68)

$$Autocorrelation(\mathcal{I}\mathcal{R})_{u-x,v-y}$$

(69)

This property and (42) allow to calculate the total sensitivity regarding to the Dirac basis $B_{\delta}$:

$$||\mathcal{I}\mathcal{R}||_{2} \mathcal{I}\mathcal{P}$$

(70)

Therefore, it means that in the convolutional model, the sensitivity in the direct space is null outside the pupil support and ”flat” inside; the associated constant level equals to the 2-norm of the Impulse Response. Moreover, (70) allows to easily get the ”total sensitivity”

$$||s_{B_{\delta}}||_{2} = ||\mathcal{I}\mathcal{R}||_{2} ||\mathcal{I}\mathcal{P}||_{2}$$

(71)

In the reciprocal space, i.e. in the spatial frequencies space, we calculate the sensitivity regarding to the Sine and Cosine basis, see (45). After some calculations which use the Plancherel theorem we get:

$$s_{B_{\sim}}(m,\omega,\mathcal{I}\mathcal{P}) = \sqrt{|\mathcal{I}\mathcal{F}(m,\omega)|^{2} \ast ||\mathcal{I}\mathcal{P}||^{2}}$$

(72)

It is worth noticing that $||\mathcal{I}\mathcal{P}||^{2}$ is the Point Spread Function (PSF) of the imaging system without incident phase.

The fundamental property of (72) is its great simplicity. Its dependence on the WFS’s optical parameters (the filtering mask, the effective weighting function and the PSF) is explicit and comprehensive. Moreover, getting $s_{B_{\sim}}$ does not require complex or time-demanding computations. It was not the case in non-convolutional models since it requires the construction of interaction matrices.

Such a result is very promising since $s_{B_{\sim}}$ equals to a quantity called Fourier’s Filter of the WFS which is the cornerstone of many phase reconstruction algorithms (see, when applied to the Pyramid WFS). This quantity is also a relevant and meaningful quantity in the AO WFSensing: it gives a direct overview of the sensitivity for any kind of Fourier-based WFSs. It is thus a perfect tool to compare and optimize these sensors.

It is especially adapted to the study of Pyramid WFSensors: n-faces Pyramids, Axicon (see, and Flattened Pyramid (cf.); masks with manufacturing errors, etc. Moreover, because of the computing time gain, it becomes possible to study in an exhaustive way the modulation stage. All the 1D modulations (circular, square, etc.) are easily taken into account in the convolution model but 2D modulations are as well: disc, ring or Gaussian modulation are indeed easily described.

---

10It is another way to understand the shift-invariance.

11We remind the reader that this quantity does not depend on the phase basis.

12$s(\phi) = ||(\mathcal{I}\mathcal{P}\phi) \ast \mathcal{I}\mathcal{R}||_{2} = ||\mathcal{I}\mathcal{P}\phi \ast \mathcal{I}\mathcal{F}||_{2}$
thanks to the 2D variables weighting function. Other tests about, for instance, erratic modulation become also possible.

Finally, it is a perfect tool to study WFSensing in presence of residual phases. Since the previous mathematical developments allows to compute the optical gain (see\textsuperscript{16} and\textsuperscript{17}) which is a quantity defined as the ratio between sensitivities in different turbulence regime.

5 Application to Pyramid WFS

This section is dedicated to the application of the previous developments to the most famous and used (see,\textsuperscript{18},\textsuperscript{19},\textsuperscript{20} and\textsuperscript{21}) Fourier-based WFS which is the Pyramid Wave Front Sensor (cf.\textsuperscript{2}). We study its historical configuration: 4-faces, large angle and circular modulation. Moreover, we assume that there is no residual phases. Such a case, which tackles the fundamental problem of "optical gain tracking for the Pyramid WFS", will have a dedicated paper (see\textsuperscript{22}).

5.1 Optical description

The entrance pupil is assumed circular (left insert of Fig. 3) with a diameter $D$. Its indicator function thus equals to:

$$\Pi_P(x, y) = \Theta \left( D - \sqrt{x^2 + y^2} \right)$$

where $\Theta$ is the Heaviside function. The filtering mask (right insert of Fig. 3) is a transparent 4-faces pyramid with an apex angle $\theta$ which is large enough to completely separate the pupil images on the detector’s (Fig. 5).

Fig 3 Pupil indicator function (left) and optical shape of the pyramid mask (right).

The transparency function of the mask is:

$$m_\Delta(x, y) = \exp \left( \frac{2\pi}{\lambda} \theta (|x| + |y|) \right)$$

We are interested in the specific case of "circular" tip/tilt modulation. The weighting function thus equals to:

$$w_\circ(a_1, a_2) = \frac{1}{2\pi r_m} \delta \left( r_m - \sqrt{a_1^2 + a_2^2} \right)$$

where $r_m$ is the modulation radius. Fig. 4 shows this weighting function for four modulation radius: 0, 2, 5 and 10 $\lambda/D$. It is worth noticing that $\Pi_P$, $m_\Delta$ and $w_\circ$ both present the same symmetry\textsuperscript{13}:

$$\hat{\Pi}_P = \Pi_P, \quad \hat{m}_\Delta = m_\Delta \quad \text{and} \quad \hat{w}_\circ = w_\circ$$

\textsuperscript{13}It allows to use the simplified expression (67) of the transfer function.
Let’s finally note that the output of this sensor is the meta-intensity defined in (14). Concretely, it means that we consider that the whole detector—and not only the pupil images—contains phase information. We give on Fig. 5 the intensities when the incoming phase is null which are required to perform the return-to-reference operation of (14).

**Fig 5** Reference intensity $I(\phi = 0)$ for four modulation radius. From top-left to bottom-right: $r_m = 0, 2, 5, 10\lambda/D$. These intensities come from end-to-end code.

### 5.2 Convolutional model

We decide, from now, to use the convolutional model to study the Pyramid WFS. This choice may seem risky: this WFS has no reason to be shift-invariant. Nevertheless, we want to know if results given by the convolutional model are consistent with previous studies. A further article will be dedicated to a rigorous comparison between the different models (exact, linear and convolutional).

The image of the Impulse Responses (6) allows to understand the shape of WFS’s output depending on the input phase. Indeed, $mI$ equals to convolution product between IR and $\phi$. We identify, for instance, 4 bright spots with lines between them. Each bright spot will thus correspond to the center of a pupil image whereas the lines will correspond to phase information which is ejected outside pupil images.

Moreover, we observe that contrast between the bright spots of the IRs and the lines is improving with the modulation radius. It means that phase information lies more and more into the 4 pupil images with an increasing modulation radius. Such a fact is a well-know effect of the modulation: it allows to condense signal into the geometrical pupil images.
Fig 6 Absolute value of the Impulse response $|\text{IR}_s|$ for four modulation radius. From top-left to bottom-right: $r_m = 0, 2, 5, 10\lambda/D$.

Fig 7 Linear intensity (left) and its convolutional approximation (right). Injected phase has the typical atmosphere profile. Considered WFS is the 4-faces Pyramid with circular modulation.

We give an example on Fig. 7 of the WFS’s output for the linear and convolutional models when the input phase is a typical atmospheric turbulent phase with an amplitude of 2 rad RMS. We may observe that the main differences are on the pupil’s edge. Such a fact confirms the predictions of (59) which says that the sliding pupil approximation is not valid on the pupil discontinuities.

5.3 Sensitivity in the Spatial Frequency Domain

We now provide the 2D sensitivity with respect to the sine and cosine basis (Fig. 8). To do so, we use (72) which gives the Fourier Filters of the circularly modulated Pyramid WFS.

In terms of structures, we observe that without modulation, the sensitivity is approximately flat: the PWFS codes all the spatial frequencies in the same way. When the modulation radius increases, a depletion zone appears in the midst of the image, i.e. for the lower spatial frequencies. By looking at Fig. 4, it appears that this area has exactly the same radius as the weighting function. Physically, it shows that this quantity directly sets the transition frequency between flat response (high frequencies) and degraded response (low frequencies). Such a result is similar to previous studies (3 and 5) showing the link between sensitivity and modulation. It is also worth noticing that sensitivity is maximum on the neighborhood of the edges of the pyramid mask. Such a fact is not surprising since we know from Foucault and his knife-edge test (see23) that WFSensing efficiently works where there are discontinuities. We note that this area of maximum signal grows with the
modulation radius. It comes from the fact that the modulated Point Spread Function enlarges with the modulation and thus communicates more with the edges of the pyramid mask.

![Fourier Filters](image)

**Fig 8 Fourier Filters** (or Sensitivity with respects to regarding to sine and cosine basis) for four modulation radius. From left to right: \( r_m = 0, 2, 5, 10\lambda/D \).

Another way to visualize the sensitivity consists in converting Fourier Filter (which is 2D quantity, see Fig. 8) into 1D sensitivity via an orthoradial average. The resulting quantity (dashed lines of Fig. 9) may then be compared with 1D usual spatial frequencies sensitivity used in the literature\(^{14}\) (solid lines).

![Spatial frequencies sensitivity](image)

**Fig 9** Spatial frequencies sensitivity derived from the orhoradial average of the linear (solid line) and convolutional sensitivity (dashed line) maps. Four modulation radii: \( r_m = 0, 2, 5, 10\lambda/D \)

Once again, we observe a constant sensitivity for the non modulated PWFS. Such a ”phase sensor” behavior is well known.\(^{24,5}\) As the modulation radius rises, the PWFS becomes a ”slope sensor” on an increasing low frequencies range. Transition frequency simply equals to the modulation radius. Such a behavior exactly corresponds to previous results coming from theory and simulations.\(^{3,25}\) Finally, we notice that convolution model implies an overestimation of the sensitivity for the low spatial frequencies.\(^{14}\)

\(^{14}\)i.e. coming from the linear model.
5.4 Phase reconstruction

We insist on the fact that the following results are preliminary: phase reconstruction is a separate field. The deconvolution process is not that simple; especially when it is applied to WFSensors which may be, in practice, very different from the physical model (photon, read-out noises, non-linearities, mis-registrations, mis-alignments, etc.) Unfortunately, studying the impact of all these model deviations on the quality reconstruction and providing technical solutions (as for instance, proper regularization) is beyond the scope of this paper. Nevertheless, we decided to show in this short paragraph that a basic deconvolution provides interesting phase estimation, even if we recognize that numerical optimization investigations have to be done.

We consider a random phase which follows typical atmosphere statistics (see left insert of Fig. 10). Its RMS norm is chosen at 2 radians. It means that this phase does not respect the small phases approximation; in other words, WFS will not work in its linear regime. This phase is propagated via an end-to-end algorithm in a modulated Pyramid WFS with a modulation radius which equals to $2\lambda/D$. We get from this simulation a meta-intensity given on the right insert of Fig 10.

![Fig 10](image)

**Fig 10** Left: Random phase following atmosphere statistics. Its norm RMS equals to 2 radians. Middle: associated Point Spread Function. Right: resulting meta-intensity.

We then apply a deconvolution on this meta-intensity by using the inverse Kernel (49). Reconstructed phase (and its residual PSF) are given on Fig. 11. The RMS norm of the residual phases equals 1.30 radians.

![Fig 11](image)

**Fig 11** Reconstructed phases and residual PSF obtained after deconvolution under infinite (top) and sliding (bottom) approximations.

We observe that deconvolution algorithm does work: it calculates a phase which has the same structure as the incoming one even if the reconstruction is far from perfect. Such a fact comes from the conjugated effects of non-linearities (the phase to be measured is not small at all) and model deviations (the real WFS is not convolutional).

Nevertheless this approach which, for the first time\textsuperscript{15}, takes into account the geometrical shape of the pupil seems promising and suggests that advances may be done in Fourier phase reconstruction (it is the purpose of the future paper\textsuperscript{27}) by adapting algorithms to sliding pupil approximation.

\textsuperscript{15}The infinite pupil approximation (rather the sliding one) is usually assumed to justify the convolutional model (,\textsuperscript{267} and\textsuperscript{10}).
6 Conclusion

This paper is dedicated to a natural extension of the theoretical work done in¹ about Fourier-based WFSensors. Using the mathematical framework describing these sensors, we showed that they may be understood as 2D integral operators which are completely characterized by their Kernel. This quantity may be understood as the continuous version of interaction matrix.

In the first part, we provided an explicit expression of the Kernel depending on the WFS’s optical parameters: pupil geometry, weighting function of the tip/tilt modulation and filtering mask. It is thus possible to calculate it for any kind of Fourier-based WFSs. We then extended it to the sensing in presence of phase residuals: under realistic assumptions, we simply had to modify the modulation weighting function by using the structure function of the residual phases. This result is an important step in the theoretical understanding WFSensing in presence of uncorrected turbulent residuals since it indicates how to adapt the phase reconstruction to the phase statistics context: problem which is known as the optical gain tracking.

We studied in a second part the particular case of convolutional kernels. In this case, the WFS’s input/output relation is convolutional. Advantages are manifold: a concise characterization of the WFS thanks to the Impulse Response and Transfer Function and a natural phase reconstruction, the deconvolution. Main results relate to the assumptions required to be in that interesting case: we showed in particular that a wise choice of the weighting function ensures a convolutional sensor. We then give the analytic expression of the Fourier’s Filter for any kind of Fourier-based WFS’s which is the cornerstone of many numerical reconstructors. These results are finally applied to the 4-faces Pyramid WFS. They allow to retrieve, via fast numerical simulations, most of the Pyramid’s behaviors.

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Appendix A: From Integral transform to Matrix transform

We give in this appendix some elements allowing to understand the link between kernels and matrices. First, we use the lexicographic order for all the 2D images¹⁶ handled in this article. As a consequence, 2D variables become 1D variable:

\[(X, Y) \rightarrow \beta \quad (77)\]
\[(x, y) \rightarrow \alpha \quad (78)\]

¹⁶Phase, mask, weighting function, meta-intensity, etc.
With such notations, an integral transform may be written:

$$\text{Output}|_\beta = \int d\beta \ K|_{\beta;\alpha} \ \text{Input}|_\alpha$$

(79)

In order to make appear a matrix transform, we discretize the spatial variables. Previous integral thus becomes a sum:

$$\text{Output}_i = \sum_j K_{i,j} \ \text{Input}_j$$

(80)

We obviously identify a matrix operation:

$$\text{Output} = K \ \text{Input}$$

(81)

where $K$ may be understood as:

$$
\begin{pmatrix}
\text{Output}_1 \\
\text{Output}_2 \\
\vdots \\
\text{Output}_n
\end{pmatrix}
= \begin{pmatrix}
K_{1,1} & K_{1,2} & \cdots & K_{1,n} \\
K_{2,1} & K_{2,2} & \cdots & K_{2,n} \\
\vdots & \vdots & \ddots & \vdots \\
K_{n,1} & K_{n,2} & \cdots & K_{n,n}
\end{pmatrix}
\begin{pmatrix}
\text{Input}_1 \\
\text{Input}_2 \\
\vdots \\
\text{Input}_n
\end{pmatrix}
$$

(82)

$\alpha$ is thus the input space variable while $\beta$ is the output space variable.

Thanks to this parallel, we may interpret Kernels in the matrix formalism. A typical decomposition of Kernel is for instance:

$$K|_{\beta;\alpha} = V|_\beta \times H|_\alpha \times C|_{\beta-\alpha} \times U|_{\beta;\alpha}$$

(83)

$V$, $H$ and $C$ are 1-variable functions and $K$ and $U$ have 2-variables. $\times$ is the usual scalar multiplication. In the matrix formalism, previous equation may be written:

$$K_{i,j} = V_i \circ H_j \circ C_{i-j} \circ M_{i,j}$$

(84)

where $\circ$ is the Hadamard product which corresponds to a "coefficient by coefficient" multiplication:

$$(A \circ B)_{i,j} = A_{i,j} \times B_{i,j}$$

(85)

Thanks to this interpretation we may detail the structure of $V$, $H$ and $C$ matrices. $H$ is a matrix only depending on the input variable $\alpha$, it is thus an horizontal matrix:

$$H = \begin{pmatrix}
h_1 & h_2 & \cdots & h_n \\
h_1 & h_2 & \cdots & h_n \\
\vdots & \vdots & \ddots & \vdots \\
h_1 & h_2 & \cdots & h_n
\end{pmatrix}$$

(86)

$\text{There is nothing to say about the Unspecified matrix U.}$
In the same way, \( V \) only depends on the output variable \( \beta \). Is is a vertical matrix:

\[
V = \begin{pmatrix}
v_1 & v_1 & \ldots & v_1 \\
v_2 & v_2 & \ldots & v_2 \\
\vdots & \vdots & \ddots & \vdots \\
v_n & v_n & \ldots & v_n \\
\end{pmatrix}
\]

\( C \) depends on \( \beta - \alpha \). In the matrix formalism, such a type of matrix is called \textbf{circulant}. It has the following form:

\[
C = \begin{pmatrix}
c_0 & c_1 & \ldots & c_{n-1} \\
c_{n-1} & c_0 & \ldots & c_{n-2} \\
\vdots & \vdots & \ddots & \vdots \\
c_1 & c_2 & \ldots & c_0 \\
\end{pmatrix}
\]

**Appendix B: More general definition of the modulation**

We generalize in this appendix the tip/tilt modulation. We call \( \phi_m \) a "modulation phase" which is not necessarily a tip/tilt. The modulated incoming field becomes:

\[
\psi_i \rightarrow \psi_i e^{i \phi_m}
\]  
(86)

We introduce the "generalized weighting function" defined on the whole "Phase space" \( \mathcal{E}_\phi \):

\[
w : \mathcal{E}_\phi \rightarrow \mathbb{R}_+ \\
\phi_m \mapsto w(\phi_m)
\]  
(87)

To ensure the energy conservation, we enforce that the weighting function has a unitary norm:

\[
\int_{\mathcal{E}_\phi} w(\phi_m) \, d\phi_m = 1
\]  
(88)

With these notations, the intensity on the detector equals to:

\[
I(\phi) = \int_{\mathcal{E}_\phi} \delta \phi_m \, w(\phi_m) \left| \mathbb{I}_P \exp \left(i(\phi + \phi_m)\right) * \tilde{m} \right|^2
\]  
(89)

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