The Mass Spectrum of the 2-dimensional Conformal String

by

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Abstract: We present the mass spectrum of the tensionless string in 2 dimensions where it has been found that the space time conformal symmetry survives quantization. A BRST treatment of the physical states reveals that the string collapses into a massless particle, a result which agrees with the classical treatment.

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1 Introduction

The characteristic energy scale in string theory is set by the level separation in the spectrum, which is of order $\sqrt{T}$, where $T$ is the string tension. The limit $T \to 0$ corresponds to a high energy limit. Understanding this regime of string theory might contribute to clarify the fundamental structure of quantum strings and consequently the short-distance structure of space-time. Null strings were first considered in [1] and they have been found in [2] to correspond to a collection of particles moving on a null geodesic.

The quantum behavior of the tensionless string\(^2\) has been investigated in several articles and it has been found that the Lorentz invariance is preserved for arbitrary space time dimensions [3, 4, 5]. However, it was shown in [6, 7] that the tensionless string is actually space-time conformally invariant. It is this symmetry that replaces the Weyl invariance in the $T \to 0$ limit, something which also occurs for the massless particle. Requiring this symmetry to be a fundamental symmetry of the theory led to the construction of the conformal string, a string with manifest space-time conformal symmetry [8, 9]. Using a Hamiltonian BRST scheme this model was quantized and a critical dimension $d = 2$ was found. It is the mass spectrum of this anomaly free theory that we are investigating in this article.

The content of the paper is as follows: In Section 2 we present the classical theory where using the equations of motion we find that the string collapses to a massless particle. In Section 3 we investigate the quantum case. There using some general arguments for the physical vacuum we find that there are two pairs of genuine physical states. This is examined in the subsequent two subsections where we find that only a state that corresponds to a massless particle survives.

2 The classical case

In this section we discuss the $d = 2$ bosonic string in some detail reviewing known results and providing the necessary background. Its action can be

\(^2\)The tensionless strings currently discussed in the D-brane context have not yet been described in terms of an action. It is therefore unclear how they are related to those treated here.
written [10]

\[ S = \int d^2 \sigma V^\alpha V^\beta \gamma_{\alpha \beta}, \]  

(1)

where \( \gamma_{\alpha \beta} = \partial_\alpha X^\mu \partial_\beta X_\mu \) is the induced metric \( \mu = 0,1 \) is a space-time index, \( \alpha = 0,1 \) is a world sheet index and \( V^\alpha \) is a weight \( w = -\frac{1}{2} \) contravariant 2-dimensional vector density. The action is invariant under world-sheet diffeomorphisms and space-time conformal transformations [6]. Under diffeomorphisms \( X^\mu \) transforms as a scalar field

\[ \delta_\epsilon X^\mu = \epsilon \cdot \partial X^\mu \]

and \( V^\alpha \) as a vector density

\[ \delta_\epsilon V^\alpha = -V \cdot \partial \epsilon + \epsilon \cdot \partial V^\alpha + \frac{1}{2} (\partial \cdot \epsilon) V^\alpha. \]

There are of course many different gauge choices possible for the reparametrization symmetry. It has been found that the following transverse gauge is particularly useful

\[ V^\alpha = (v, 0) \]

(2)

with \( v \) a constant. The transverse gauge corresponds to the conformal gauge \( g_{\alpha \beta} = e^\phi \eta_{\alpha \beta} \) in the tensile theory. Just as in the tensile case there is a residual symmetry that leaves (3) invariant

\[ \delta \tau = f'(\sigma) \tau + g(\sigma), \]

\[ \delta \sigma = f(\sigma), \]

(3) (4)

with \( f(\sigma) \) and \( g(\sigma) \) arbitrary functions. The finite form looks the same with different functions \( f \) and \( g \).

The field equations that follow from the action (1) are

\[ V^\alpha \gamma_{\alpha \beta} = 0, \quad \partial_\alpha (V^\alpha V^\beta \partial_\beta X^\mu) = 0. \]

The first of these equations shows that the induced metric is degenerate. The second group of the field equations is most easily interpreted in the transverse gauge. They become

\[ \ddot{X}^\mu = 0, \]

\[ \dot{X}^2 = 0, \]

\[ \dot{X} \cdot X' = 0. \]
These equations give
\[ \frac{d^2 X^\mu}{d \tau^2} = 0 \quad \Rightarrow \quad X^\mu(\sigma, \tau) = F(\sigma) \tau a_1^\mu + G(\sigma) a_2^\mu + a_3^\mu \]
\[ \dot{X}^2 = 0 \quad \Rightarrow \quad a_1^2 = 0 \quad \Rightarrow \quad a_1^\mu = e^\mu, \]
where \( e^\mu \) is a null vector pointing in either of the two light-like directions. The last equation gives
\[ \dot{X} \cdot X' = 0 \quad \Rightarrow \quad e \cdot a_2 = 0 \quad \Rightarrow \quad a_2^\mu = e^\mu. \]
Thus the solution of the equations of motion is
\[ X^\mu(\sigma, \tau) = \begin{bmatrix} F(\sigma) \tau + G(\sigma) \end{bmatrix} e^\mu + a_3^\mu. \]

Using the residual symmetry (3) we can rewrite this as
\[ X^\mu = \tau' e^\mu + a_3^\mu, \tag{5} \]
which is the equation of a massless particle in two space-time dimensions, as observed in [9].

3 The quantum case

Passing to the Hamiltonian formulation we find, [9], that the Hamiltonian is a linear combination of the two constraints
\[ \phi^{-1}(\sigma) = P^\mu P_\mu(\sigma) = 0, \quad \phi^L(\sigma) = P^\mu X'_\mu = 0, \]
which is expected since the theory is reparametrization invariant. In Fourier modes the constraints read
\[ \phi^{-1}_m = \frac{1}{2} \sum_{k=-\infty}^{+\infty} p_k \cdot p_{m-k} = 0, \tag{6} \]
\[ \phi^L_m = -\frac{i}{2} \sum_{k=-\infty}^{+\infty} [k x_k \cdot p_{m-k} + k p_{m-k} \cdot x_k] = 0. \tag{7} \]
and they satisfy the following algebra
\[
\begin{align*}
[\phi^{-1}_m, \phi^L_n] &= (m - n)\phi^{-1}_{m+n}, \\
[\phi^L_m, \phi^L_n] &= (m - n)\phi^L_{m+n},
\end{align*}
\]
where the basic non zero commutators are
\[
[x^\mu_n, p^\nu_m] = i\delta_{m+n}\eta^\mu\nu.
\]
The BRST quantization requires the introduction of new operators, the Faddeev-Popov ghosts. To every constraint \(\phi^A_m, A \in \{-1, L\}\) one introduces a ghost pair \(c^A_m, b^A_m\) that is fermionic. The ghosts satisfy the fundamental anticommutation relations
\[
\{c^A_m, b^B_n\} = \delta_{m+n}\delta^{AB}.
\]
The generator of BRST transformations, the BRST charge is given by
\[
Q = \sum_k (\phi^{-1}_{-k}c^L_k + \phi^L_{-k}c^{-1}_k) - \sum_{k,l}[(k - l)c^{-1}_{-k}c^L_{-k+l}b^{-1}_{-k+l} + \frac{1}{2}(k - l)c^L_{-k}c^L_{-l}b^L_{k+l}].
\]
This BRST operator was found to be nilpotent in [3]. The physical states satisfy the condition
\[
Q|\text{phys}\rangle = 0.
\]
The nilpotency of \(Q\) implies that any state of the form \(Q|\rangle\) is a physical state. These states are called exact states and due to (11) they will decouple from all physical states i.e., have inner product equal to zero. The physics is therefore contained in equivalence classes of physical states (BRST cohomology). In the analysis of the BRST condition (11) it is convenient to introduce the ghost number operator
\[
N_{gh} = \frac{1}{2} \sum_k \left[ c^{-1}_{-k}b^L_{-k} + c^{-1}_{-k}b^{-1}_{-k} - b^{-1}_{-k}c^{-1}_{-k} - b^L_{-k}c^L_{-k} \right].
\]
It satisfies
\[
[\ N_{gh}, c^A_k] = c^A_k, \quad \left[ N_{gh}, b^A_k \right] = -b^A_k.
\]
and also

\[ [N_{gh}, Q] = Q. \]

We can, therefore, classify physical states according to their ghost number

\[ N_{gh}|\text{phys}, n\rangle = n|\text{phys}, n\rangle. \]

Since \( N_{gh} \) is an antihermitian operator with real eigenvalues, the physical states should satisfy

\[ \langle \text{phys}, m|\text{phys}, n\rangle = C_n \delta_{m+n}. \]

In [12] a method of solving the physical state condition (11) was proposed. It is argued that it is sufficient to consider states of the form

\[ |\text{matter}\rangle|\text{ghost}\rangle. \]

A fundamental prescription of this method is the requirement that all inner products have to be finite. As a consequence the genuine physical states will appear in pairs: to every physical state or the form \(|\text{phys}\rangle = |M\rangle|G\rangle\) there is a dual physical bra state of the form \(|\text{phys}'\rangle = \langle -G|\langle M'\rangle\). We choose the normalization of the ghost states to be \(\langle -G|G\rangle = 1\). The ghost state space is built from the vacuum states \(|0\rangle_c\) and \(|0\rangle_b\), which satisfy

\[
\begin{align*}
    c^A_k |0\rangle_c &= 0, \\
    b^A_k |0\rangle_b &= 0, \quad \forall k,
\end{align*}
\]

\(c\langle 0|0\rangle_c = b\langle 0|0\rangle_b = 0, \quad c\langle 0|0\rangle_b = 1. \)  \hspace{1cm} (13)

The two ghost vacua are related by the equation

\[ |0\rangle_c = N \prod_{k=-\infty}^{+\infty} c^{-1}_k \prod_{l=-\infty}^{+\infty} c^L_l |0\rangle_b, \]  \hspace{1cm} (14)

where \(N\) is a constant chosen to satisfy (13). Applying the operator (12) to the states \(|0\rangle_c\) and \(|0\rangle_b\) we find that their ghost number, \(C\) and \(B\), is \(+\infty\) and \(-\infty\) respectively. An arbitrary ghost state will have the form

\[ |B + k\rangle^{a_1\ldots a_k} = c^{a_1} \ldots c^{a_k} |0\rangle_b \]
or equivalently

\[ |C - k \rangle^{a_1 \ldots a_k} = b^{a_1} \ldots b^{a_k} |0 \rangle_c, \]

where \( a_k = (A, k), A \in \{-1, L\} \) and \( k \) any integer. Assume now the ket state \( |G_s^{-1} \rangle \equiv c_s^{-1} |0 \rangle_b \) which satisfies the condition \( c_s^{-1} |G_s^{-1} \rangle = 0 \). Non-zero inner products can be constructed only with bra states of the form \( c_b^{-1} |b \rangle \) since

\[ c_b^{-1} |b \rangle = \delta_{A, -1} \delta_{m+s}. \] (15)

The state \( \langle -G^{-1}_s | \equiv c_b^{-1} |b \rangle \) satisfies the relation \( \langle -G^{-1}_s | b^{-1} = 0 \). In the same way we can prove for a general ghost state \( |G \rangle \)

\[ c^{A,k} |G \rangle = 0 \Rightarrow \langle -G | b^{A-k} = 0. \] (16)

We find it useful to define also the following states

\[ c_k^{-1} |G_M^{-1} \rangle = c_k^L |G^L_M \rangle = 0, \]
\[ b_k^{-1} |G_M^{-1} \rangle = b_k^L |G^L_M \rangle = 0. \] (17)

According to arguments presented in [1] the vacuum suitable for tensionless strings is not one annihilated by the positive modes of the operators but one annihilated by the momenta

\[ p^\mu_m |0 \rangle_p = 0 \quad \forall m. \] (18)

Following the prescription of [12], we will take the ket states to be built from our vacuum of choice, \( |0 \rangle_p \), and the bra states to be built from \( x \langle 0 | \) satisfying \( x \langle 0 | 0 \rangle_p = 1 \). Asymmetric states for the coordinates and the above restriction may be constructed in different ways. We will make use of the so called wave sector, with the bra states defined by

\[ \langle x | \] or \( x \langle 0 | Y(p) \) (19)

and the ket states defined by

\[ \Phi(x) |0 \rangle_p \] or \( |p \rangle \) (20)

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3States of the form \( c_b^{-1} b^{a_1} \ldots b^{a_k} \) with more than one \( b^a \) operator give zero inner products with \( |G_s^{-1} \rangle \) since we can always move one of the \( b^a \)'s to the right of \( c_s^{-1} \).

4cf. the vacuum for a particle.
respectively. In this relations \(|x\rangle\) and \(|p\rangle\) represent the collective \(|\ldots, x_n, \ldots\rangle\) and \(|\ldots, p_n, \ldots\rangle\) respectively.

For the vacuum \(|18\rangle\) and from the requirement that the BRST charge \(|10\rangle\) should annihilate the vacuum, we obtain further requirements on the ghost part of the vacuum. Defining \(|phys\rangle_0 = |0\rangle_p|G\rangle\) we have

\[
Q|phys\rangle_0 = 0 \Rightarrow [Q, p^\mu_n]|phys\rangle_0 = 0,
\]
while

\[
[Q, p^\mu_n]|phys\rangle_0 = \sum_k (-n)c_L^{\mu_k}p^{\mu_{k+n}}|0\rangle_p|G\rangle = 0,
\]
with the last relation trivially satisfied. In the same way, requiring the bra vacuum \(_0\langle phys| \equiv x\langle 0|\langle -G|\) to be a physical state restricts the ghost part of the state

\[
_0\langle phys| [Q, x^\mu_n] = 0 \Rightarrow \langle x\langle 0|\langle -G| \sum_k \left[c_{-k}^{\mu(-i)p^{\mu_{k+n}} + c_{-k}^{\mu(-k-n)}x^{\mu_{k+n}} \right] = 0
\]

\[
\Rightarrow \langle -G|c_k^{-1} = 0 \Rightarrow b_k^{-1}|G\rangle = 0, \quad \forall k. \quad (21)
\]
This means that any physical state \(|phys\rangle = |M\rangle|G\rangle\) should have a ghost part that satisfies the relations \((21)\). We get further conditions on the physical states by requiring the consistency relations

\[
\langle M'|\langle -G| [Q, c_m^{-1}] = 0, \quad [Q, b_m^{-1}] |M\rangle|G\rangle = 0.
\]
The first one is trivially satisfied. The second restricts the matter part. It requires

\[
[c_{m}^{-1} - \sum_i c_{-i}^{L}b_{m+i}]|M\rangle|G\rangle = 0 \Rightarrow
\]

\[
\phi_m^{-1}|M\rangle = 0, \quad \forall m.
\]
We note that \([Q, b_m^{-1}] \equiv \tilde{\phi}_m^{-1}\) are the extended constraints defined in \([9]\), i.e. the BRST invariant extensions of the original constraints and they satisfy the same algebra. They are very useful in the calculation of the BRST anomaly.

So far, using general arguments on the physical vacuum we found that the physical states have to be of the form

\[
|phys\rangle = |M\rangle|G\rangle, \quad \langle phys'| = \langle M'|\langle -G|.
\]
with the ghost states constrained by the conditions (21). However, there is still a freedom in the ghost part since it can have the form $\prod_k b^L_k \mid -G^{-1}M\rangle$ where $k$ belongs to any set of non repetitive integers. This suggests that there is an infinite number of ghost sectors. That this is not true can be seen as follows: Assume that the ghost part of the physical state satisfies the condition

$$b^L_s \mid G\rangle = 0$$

for a specific $s \neq 0$. Thus, we also have (16)

$$\langle -G \mid c^L_s = 0.$$  

The consistency conditions also require the following relations to hold

$$\langle M' \mid -G \mid Q, c^L_s \rangle = 0 \Rightarrow \langle -G \mid \sum_k (2k + s)c^L_k c^L_{k+s} = 0$$

$$\Rightarrow \langle -G \mid c^L_k = 0 \Rightarrow b^L_k \mid G\rangle = 0, \quad \forall k \neq 0$$

and

$$\left[ Q, b^L_m \right] \mid phys\rangle = 0 \Rightarrow \left[ \phi^L_m - 2mc^L_m b^L_0 \right] \mid M\rangle \mid G\rangle$$

$$\phi^L_m \mid M\rangle = 0, \quad b^L_m \mid G\rangle = 0, \quad \forall m. \quad (22)$$

As a result, by requiring the physical state to be annihilated by just one operator $b^L_s$, consistency conditions force the relation (22) for all $m$. In the same way we can prove that by requiring the ghost part of a physical state to satisfy the condition $b^L_s \mid G\rangle = 0$ just for a specific $s$, the physical state is forced to satisfy the relations

$$\langle M' \mid \phi^L_m = 0, \quad c^L_m \mid G\rangle = 0, \quad \forall m.$$
with the matter part satisfying
\[
\phi^{-1}_m |M\rangle = \phi^L_m |M\rangle = 0, \quad \forall m.
\] (23)

The other pair has ghost number zero and is given by
\[
|M_{\text{phys}, 0}\rangle = |M\rangle |G^{-1}_M, G^L_M\rangle,
\]
\[
\langle M'_{\text{phys}'}, 0| = \langle M'\rangle \langle G^{-1}_M, -G^L_M|,
\]
with the matter part satisfying
\[
\phi^{-1}_m |M\rangle = 0,
\] (24)
\[
\langle M'|\phi^L_m = 0, \quad \forall m.
\] (25)

We will now investigate these two cases in detail.

### 3.1 Case I

The conditions on the physical states in this case are the same as the ones that appear in Dirac quantization of the model. In the wave function sector (19), the solution may be written as
\[
|M\rangle = \Phi(x)|0\rangle_p,
\]
\[
\langle M'| = \langle x|\]
and the inner product gives
\[
\langle \text{phys}', +\infty|\text{phys}, -\infty\rangle = \Phi(x).
\]

Equation (23) then implies the equations
\[
\sum_{k=-\infty}^{+\infty} \frac{\partial}{\partial x^\mu_k} \frac{\partial}{\partial x_{k-m\mu}} \Phi(x) = 0,
\] (26)
\[
\sum_{k=-\infty}^{+\infty} kx^\mu_k \frac{\partial}{\partial x_{k-m\mu}} \Phi(x) = 0.
\] (27)
Before we continue to the solution for general \( m \) let us take a closer look at the conditions (23) for \( m = 0 \).

\[
\phi_0^{-1}|M\rangle = \frac{1}{2} \sum_k p_k \cdot p_{-k} |M\rangle = \frac{1}{2} p_0^2 |M\rangle + \sum_{k>0} p_k \cdot p_{-k} |M\rangle = 0
\]

\[
\Rightarrow -p_0^2 |M\rangle = 2 \sum_{k>0} p_k \cdot p_{-k} |M\rangle \Rightarrow M^2 |M\rangle = 2 \sum_{k>0} M_k^2 |M\rangle,
\]

(28)

where we have used the definition of the mass operator \( M^2 = -p_0^2 \) and defined \( p_k \cdot p_{-k} \equiv M_k^2 \). We also have

\[
\phi_L^0 = -i \sum_{k>0} k (x_k \cdot p_{-k} - x_{-k} \cdot p_k) \equiv \sum_{k>0} k S_k.
\]

It turns out that the commutators \([M^2, \phi_L^0], [M^2, S_k], [M^2, M_k^2], [\phi_L^0, S_k], [\phi_L^0, M_k^2], [M_k^2, S_l], [S_k, S_l], [M_k^2, M_l^2] \) vanish. The problem of finding mass eigenstates can, therefore, be reduced to that of simultaneously diagonalizing \( S_k \) and \( M_k^2 \)

\[
S_k \Phi_k = s_k \Phi_k \text{ and } M_k^2 \Phi_k = m_k^2 \Phi_k.
\]

The wave function of a generic state will be written as a product of functions, a factor for each \( k \leq 0 \)

\[
\Phi(x) = \prod_{k=0}^{+\infty} \Phi_k(x_k, x_{-k}).
\]

(29)

The zero-modes \( p_0^\mu, x_0^\mu \) are not contained in \( \phi_L^0 \) and \( M^2 \). To them corresponds a plane wave describing the motion of the string’s center of mass

\[
\Phi_{k=0} = e^{i l_0 \cdot x_0}.
\]

We now look for the other functions \((n \neq 0)\). They have to satisfy the equations

\[
- \frac{\partial}{\partial x_k^\mu} \frac{\partial}{\partial x_{-k} \mu} \Phi_k(x_k, x_{-k}) = m_k^2 \Phi_k(x_k, x_{-k}), \quad (30)
\]

\[
\left(x_{-k}^\mu \frac{\partial}{\partial x_{-k} \mu} - x_k^\mu \frac{\partial}{\partial x_k \mu}\right) \Phi_k(x_k, x_{-k}) = s_k \Phi_k(x_k, x_{-k}). \quad (31)
\]
In order to solve the equations (30) and (31) we notice that the operator \( X(\sigma) \) is Hermitian. This means that the modes \( x_k \) and \( x_{-k} \) are related by the equation \( x_k^\dagger = x_{-k} \) and correspondingly, they are represented in the wave sector by

\[
x_k \rightarrow x_k, \quad x_{-k} \rightarrow x_k^*, \quad k > 0.
\]

where \( x_k^* \) is the complex conjugate of \( x_k \). For a real wave function the equation (31) is trivially satisfied. Hence we can distinguish between two cases. In the first we take the wave function to be real i.e. we assume that \( x_k = x_{-k} \). In the second, which will be investigated later, we allow \( x_{-k} \neq x_k \). Applying (30) to the first case we get

\[
\frac{\partial}{\partial x_\mu} \frac{\partial}{\partial x_{k\mu}} \Phi_k(x_k) = -m_k^2 \Phi_k(x_k)
\]

\[
\Rightarrow \Phi_k(x_k) = e^{i l_k \cdot x_k},
\]

with the condition \( l_k \cdot l_k = m_k^2 \). The total wave function (29) should then have the form

\[
\Phi(x) = \exp \left( i \sum_{k=0}^{s} l_k \cdot x_k \right),
\]

where we have taken \( s \) to be a very large positive integer. This wave function has to satisfy the equations (26) and (27) for any integer \( m \). Those equations are trivially satisfied for \( m > 2s \). For \( m = 2s \) (26) gives

\[
\frac{\partial}{\partial x_{s\mu}} \frac{\partial}{\partial x_{s\mu}} \exp \left( i \sum_{k=0}^{s} l_k \cdot x_k \right) = 0 \Rightarrow l_s^2 = 0.
\]

For \( m = 2s - 1 \) we have

\[
\left( \frac{\partial}{\partial x_{s\mu}} \frac{\partial}{\partial x_{s-1\mu}} + \frac{\partial}{\partial x_{s-1\mu}} \frac{\partial}{\partial x_{s\mu}} \right) \exp \left( i \sum_{k=0}^{s} l_k \cdot x_k \right) = 0
\]

\[
\Rightarrow l_s \cdot l_{s-1} = 0.
\]

But in two space-time dimensions

\[
\begin{align*}
l_s^2 &= 0 \\
l_s \cdot l_{s-1} &= 0
\end{align*}
\]

\[
\Rightarrow l_{s-1}^2 = 0,
\]

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both pointing in the same light-like direction. In the same way we find that \( l_i \cdot l_j = 0, \forall i, j \in [t, s] \), where \( t \) is any integer lesser than \( s \). Then for \( m = 2s - t - 1 \)

\[
\langle x | \phi^{-1}_{2s-t-1} | M \rangle = 0 \Rightarrow l_s \cdot l_{s-t-1} + l_{s-1} \cdot l_{s-t} + \ldots + l_{s-t-1} \cdot l_s = 0
\]

\[
\Rightarrow l_s \cdot l_{s-1} = 0 \Rightarrow l_s^2 = 0
\]

so \( l_n \cdot l_m = 0, \forall n, m < s \) and accordingly the states are massless since (28) gives

\[
M^2 \Phi(x) = 2 \sum_{k=1}^{s} l_k \cdot l_k \Phi(x) = 0.
\]

A physical state should also satisfy the relations (27). For \( m = 2s \)

\[
x_s \mu \frac{\partial}{\partial x_s \mu} \exp \left( i \sum_{k=0}^{s} l_k \cdot x_k \right) = 0 \Rightarrow x_s \cdot l_s = 0
\]

\[
\Rightarrow x^2 = 0, \quad x_s \cdot l_k = 0, \quad \forall k < s.
\]

In the same way we can prove

\[
l_k \cdot x_m = 0, \quad \forall m \neq 0.
\]

Thus, a physical state can only have the form

\[
\Phi(x) = Ce^{il_0 x_0},
\]

where \( C \) is a constant. This is the wave function of a massless particle.

We now investigate the second case where we take \( x_k \neq x_{-k} \). To construct the eigenstates of \( S_k \), we start from a general Lorentz invariant wave function of the form

\[
\Phi(x_k, x_{-k}) \equiv Z_k(x_k, x_{-k}) = F_k(l^k \cdot x_k)G_k(l^k \cdot x_{-k})H_k(x^2_k)I_k(x^2_{-k})\Psi_k(\chi_k),
\]

where \( \chi_k \equiv x_k \cdot x_{-k} \). This function should satisfy eq. (31)

\[
S_k Z_k(x_k, x_{-k}) = s_k Z_k(x_k, x_{-k})
\]

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\[ (x^\mu_k \frac{\partial}{\partial x_k^\mu} - x^\mu_k \frac{\partial}{\partial x_k^\mu}) F_k G_k H_k I_k \Psi_k = s_k F_k G_k H_k I_k \Psi_k \]

\[ - \frac{(l^k \cdot x_k)}{F_k} \frac{d F_k}{d(l^k \cdot x_k)} + \frac{(l^q_k \cdot x_{-k})}{G_k} \frac{d G_k}{d(l^q_k \cdot x_{-k})} \]

\[ - \frac{2(x^2_{-k})}{H_k} \frac{d H_k}{d(x^2_{-k})} + \frac{2(x^2_{-k})}{I_k} \frac{d I_k}{d(x^2_{-k})} = s_k \]

Thus the eigenfunctions of \( S_k \) should have the form

\[ Z_k = (l^k_1 \cdot x_k) \ldots (l^k_q \cdot x_k)(l^k_{q_k+1} \cdot x_{-k}) \ldots (l^k_{q_k+r_k} \cdot x_{-k})(x^2_{-k})^{q_k}(x^2_k)^{q_k} \Psi_k(\chi_k). \]

The corresponding eigenvalue is \( s_k = r_k - q_k - 2n_k + 2g_k \). To simplify the notation we define \( A_k \equiv (l^k_1 \cdot x_k) \ldots (l^k_q \cdot x_k), B_k \equiv (l^k_{q_k+1} \cdot x_{-k}) \ldots (l^k_{q_k+r_k} \cdot x_{-k}), \)

\( \hat{A}_k \equiv (l^k_1 \cdot x_k) \ldots (l^k_{q_k+1} \cdot x_{-k}) \ldots (l^k_{q_k+t} \cdot x_{-k}) \) and \( \hat{B}_k \equiv (l^k_{q_k+1} \cdot x_{-k}) \ldots (l^k_{q_k+t} \cdot x_{-k}) \). Demanding the function \( Z_k \) to be an eigenfunction of the operator \( \{34\} \) we find that \( n_k = g_k = 0 \). We also have

\[ \frac{\partial}{\partial x_k^\mu} \frac{\partial}{\partial x_{-k}^\mu} Z_k(x_k, x_{-k}) = -m_k^2 Z_k(x_k, x_{-k}) \]

\[ \Rightarrow \sum_{s=1}^{q_k} \sum_{t=1}^{r_k} (l^k_s \cdot x_k l^k_{q_k+t} \hat{A}_k \hat{B}_k \Psi_k(\chi_k) + A_k B_k \sum_{s=1}^{q_k} \frac{\partial \Psi_k}{\partial \chi_k} \]

\[ + A_k B_k \sum_{s=1}^{r_k} \frac{\partial \Psi_k}{\partial \chi_k} + A_k B_k \frac{\partial}{\partial x_k} \left[ x^\mu_k \frac{\partial \Psi_k}{\partial x_k} \right] = -m_k^2 A_k B_k \Psi_k(\chi_k). \]

Thus \( l^k_s \cdot l^k_{q_k+t} = 0, \forall s, t \) and \( \Psi_k \) has to satisfy the equation

\[ \chi_k \Psi''_k + (q_k + r_k + d) \Psi'_k + m_k^2 \Psi_k = 0, \]

where \( d \) is the space-time dimension. The solution of this equation has the form \([4, 13]\)

\[ \Psi_k^m(\chi_k) = N_m \chi_k^{-(1+q_k+r_k)/2} J_{1+q_k+r_k}(2m_k \sqrt{\chi_k}), \]

where \( J_{1+q_k+r_k} \) is a Bessel function of rank \( 1 + q_k + r_k \). Thus an eigenfunction that simultaneously diagonalizes \([30]\) and \([31]\) has the form

\[ Z_k^m(x_k, x_{-k}) = N(l^k_1 \cdot x_k) \ldots (l^k_{q_k+r_k} \cdot x_{-k}) \chi_k^{-(1+q_k+r_k)/2} J_{1+q_k+r_k}(2m_k \chi_k^{1/2}). \]
Since $m_k^2 = p_k \cdot p_{-k}$ is a Hermitian operator its eigenvalues $m_k^2$ should be real numbers. If $m_k^2 < 0 \Rightarrow m_k$ is purely imaginary. Take $m_k = i\mu_k$ with $\mu_k$ real. Then

$$\Psi_k^{m}(\chi_k) = N_m(i\mu_k)^{1+q_k+r_k} \sum_{n=0}^{+\infty} \frac{(\mu_k \chi_k^{1/2})^{2n}}{n! \Gamma(q_k + r_k + n + 2)}.$$  

According to the latter, for the tachyonic solutions every term in the same factor is positive and hence the corresponding wave functions cannot be normalizable. This means that $m_k^2$ should be real and positive and so the mass spectrum is continuous ranging from zero to infinity.

Special consideration is needed for the case $m_k^2 = 0$. The corresponding equation is then

$$\chi_k \Psi_k'' + (q_k + r_k + 2) \Psi_k' = 0 \Rightarrow Z_k^0(\chi_k) = D_k \chi_k^{-(1+q_k+r_k)} + C_k,$$

where $D_k$ and $C_k$ are constants.

We will assume again the wave function to depend on a finite number of modes i.e

$$\Phi(x) = \prod_{k=0}^{K} Z_k(x_k, x_{-k}),$$

where $K$ is a large but finite integer. Since (33) is the wave function of a physical state it should also satisfy the equations (26) for any $m$. The first non-trivially satisfied constraint is again $\phi_{2K}^{-1}$. This gives

$$\frac{\partial}{\partial x_k^{\mu}} \frac{\partial}{\partial x_{-K}^{\nu}} \prod_{k=1}^{K} A_k B_k e^{i l_0 \cdot x_0} \Psi_k(\chi_k) = 0$$

$$\Rightarrow \sum_{s=1}^{r_K} \sum_{t \neq s}^{r_K} l^{K}_{q_k+s} \cdot l^{K}_{q_k+t} \hat{B}_k^{st} \Psi_K(\chi_K) + 2 \sum_{s=1}^{r_K} l^{K}_{q_k+s} \cdot x_K \hat{B}_K^{s} \Psi_K'(\chi_K)$$

$$+ B_K x_K^2 \Psi_K''(\chi_K) = 0.$$  

This equation holds if all of its terms are equal to zero. In particular it requires that $l^{K}_{q_k+s} \cdot l^{K}_{q_k+t} = 0$, $\forall s \neq t \in [1, r_K]$ and $l^{K}_{q_k+s} \cdot x_K = 0$, $\forall s \in [1, r_K]$. Working in the same way for all $m$, we find that the constraints $\phi_{m}^{-1}$ imply the conditions $l^{K}_{s} \cdot l^{m}_{t} = l^{K}_{s} \cdot x_n = 0$, $\forall k, n \neq 0$ and $\forall s, t$, all being null vectors pointing in the same light like direction. Thus the wave
function should have the form \( e^{i\theta_0 \cdot x_0} \prod \Psi_k \). The last term in (34) requires \( \Psi''_K(\chi_K) = 0 \). This is satisfied if \( \Psi_K \) is given by the massless solution (32) with \( D_K = 0 \). Notice that eq. (34) is also satisfied if we choose \( x^2_K = 0 \) instead of \( \Psi_K = C_K \). Requiring the funcion \( \Phi \) to satisfy the constraints (26) for all the other values of \( m \) as well, we find that \( \Psi_k = C_k, \forall k \in \mathcal{A} \), \( \mathcal{A} \) being a subset of \( \{-K, \ldots, -1, 1, \ldots, K\} \) and that \( x_i \cdot x_j = 0, \forall i, j \in \mathcal{B} \), where \( \mathcal{B} \) is the compliment of \( \mathcal{A} \). The second conditions give in particular that \( \chi_n = 0, \forall n \in \mathcal{B} \). So again the only solution which survives is the massless particle.

\[
\Phi(x) = \left( \prod_k C_k \right) e^{i\theta_0 \cdot x_0},
\]

with \( k \in \mathcal{A} \).

### 3.2 Case II

We are now going to solve the equations (24) and (25). In the wave function sector (19), the solution may be written as

\[
|M\rangle = |p\rangle, \quad \langle M'| = x \langle 0|Y(p),
\]

and the inner product gives

\[
\langle \text{phys}', 0|\text{phys}, 0 \rangle = Y(p).
\]

Equations (24) and (25) then imply the relations

\[
\begin{align*}
\sum_{k=-\infty}^{+\infty} p_k \mu p_{m-k} Y(p) &= 0, \\
\sum_{k=-\infty}^{+\infty} k p_{m-k} \mu \frac{\partial Y(p)}{\partial p_{-k} \mu} &= 0.
\end{align*}
\]

(35)

(36)

We take a finite number of modes \( s \) again. For \( m = 2s \), equation (33) gives

\[
\begin{align*}
\sum_{k=-s}^{s} p_k \cdot p_{2s-k} Y(p) &= p_s^2 Y(p) = 0 \\
\Rightarrow \quad p_s^2 &= 0.
\end{align*}
\]
For \( m = 2s - 1 \) we find that \( p_i \cdot p_j = 0, \forall i, j \in [s, s - 1] \). We can work in the same way to prove by induction that
\[
p_i \cdot p_j = 0, \quad \forall i, j \in [-s, s]
\]
all pointing in the same light-like direction. Therefore, the function \( Y(p) \) should have the form
\[
Y(p) = Y(a_s \cdot p_{-s}, a_{s-1} \cdot p_{-s+1}, \ldots, a_{-s} \cdot p_s).
\]
The spectrum is massless again since from (28) we deduce that \( M^2 Y(p) = 0 \).

A physical state should also satisfy the relations (36). For \( m = 2s \) we get
\[
\sum_{k=-s}^{+s} kp_{2s-k} \frac{\partial Y(p)}{\partial p_{-k}^\mu} = 0 \Rightarrow sp_s^\mu \frac{\partial Y(p)}{\partial p_{-s}^\mu} = 0
\]
\[
\Rightarrow \quad p_s \cdot a_s = 0 \Rightarrow p_{-s} \cdot a_s = 0.
\]
Working in the same way as before we can prove that \( a_k \cdot p_{-k} = 0, \forall k \neq 0 \).
So we get
\[
Y(p) = f(a_0 \cdot p_0),
\]
which is the equation of a massless particle. Thus we find once again that the string collapses into a massless particle as in the classical case.

Since our theory does not have any parameter to start with, it can be argued that the spectrum should consist of massless states only or that it should be continuous. From the point of view of the limit \( \alpha' = 1/(\sqrt{2\pi T}) \to \infty \) of usual string theories, it is clear that all the states in the leading Regge trajectory collapse to zero mass. The above reasoning shows that all daughter trajectories should also collapse to zero mass.

One might ask how the spectrum would look like if supersymmetry was also included. In the case of world-sheet supersymmetry it has been shown in [14] that the critical dimension is negative and hence the theory is anomalous. In the case of space-time supersymmetry on the other hand one cannot apply the same formalism because of difficulties that have to do with the covariant quantization of the superparticle.

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