BLOW-UP DYNAMICS FOR $L^2$-CRITICAL FRACTIONAL SCHRÖDINGER EQUATIONS

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Abstract. In this paper, we will consider the $L^2$-critical fractional Schrödinger equation $iu_t - |D|^{\beta} u + |u|^{2\beta} u = 0$ with initial data $u_0 \in H^{\beta/2}(\mathbb{R})$ and $\beta$ close to 2. We will show that the solution blows up in finite time if the initial data has negative energy and slightly supercritical-mass. We will also give a specific description for the blow-up dynamics. This is an extension of the work of F. Merle and P. Raphaël for $L^2$-critical Schrödinger equations but the nonlocal structure of this equation and the lack of some symmetries make the analysis more complicated, hence some new strategies are required.

1. Introduction

1.1. Setting of the problem. In this paper, we consider the following fractional Schrödinger equation:

\[
\begin{cases}
iu_t - |D|^{\beta} u + |u|^{2\beta} u = 0, & (t, x) \in \mathbb{R} \times \mathbb{R}, \\
u|_{t=0} = u_0 \in H^{\beta/2}(\mathbb{R}),
\end{cases}
\tag{1.1}
\]

with $1 \leq \beta < 2$. Here $|D|^{\beta}$ is defined as following:

\[
|D|^{\beta} u(\xi) = |\xi|^{\beta} \hat{u}(\xi).
\]

The evolution problems with nonlocal dispersion like (1.1) arise in various physical settings, which include continuum limits of lattice systems [18], models for wave turbulence [4, 22], and gravitational collapse [10, 12, 16]. We also refer to [6, 7, 19, 20, 44] and the references therein for the background of the fractional Schrödinger model in mathematics, numerics and physics.

Let us review some basic properties of this equation. The Cauchy problem (1.1) is an infinite-dimensional Hamiltonian system, which has the following three conservation laws:

- Mass:
  \[M(u(t)) = \int |u(t)|^2 = M(u_0).\tag{1.2}\]

- Energy:
  \[E(u(t)) = \frac{1}{2} \int |D|^{2\beta} u|^2 - \frac{1}{2(\beta + 1)} \int |u(t)|^{2(\beta + 1)} = E(u_0).\tag{1.3}\]

- Momentum:
  \[P(u(t)) = \Im \int u_x(t) \bar{u}(t) = P(u_0).\tag{1.4}\]

The equation (1.1) has the following symmetries:

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• Phase: if \( u(t, x) \) is a solution, then for all \( \theta \in \mathbb{R} \), \( u(t, x)e^{i\theta} \) is also a solution.
• Translation: if \( u(t, x) \) is a solution, then for all \( t_0 \in \mathbb{R}, x_0 \in \mathbb{R} \), \( u(t-t_0, x-x_0) \) is also a solution.
• Scaling: if \( u(t, x) \) is a solution, then for all \( \lambda > 0 \),
\[
u\nu(t, x) = \frac{1}{\lambda^{d/2}}u\left(\frac{t}{\lambda^{2}}, \frac{x}{\lambda}\right)
\]
is also a solution.

**Remark 1.1.** We mention here, unlike the classic nonlinear Schrödinger equations (corresponding to the case \( \beta = 2 \)), the equation (1.1) does not have Galilean transform and pseudo-conformal transform. This fact leads to some additional technical difficulty for obtaining blow-up results for (1.1).

The Cauchy problem (1.1) is \( L^2 \)-critical since the \( L^2 \) norm is invariant under the scaling rule (1.5):
\[
\|u\|_{L^2} = \|u\|_{L^2}, \text{ for all } \lambda > 0.
\]
From [14, 15], we know that the Cauchy problem (1.1) is locally well-posed in the energy space \( H^\frac{d}{2} \). More precisely, for all \( u_0 \in H^\frac{d}{2} \), there exists a unique solution \( u(t) \in C([0, T), H^\frac{d}{2}) \) to (1.1). Moreover, if the maximal lifetime \( T < +\infty \), then we have
\[
\lim_{t \to T^-} \|D\nabla u(t)\|_{L^2} = +\infty.
\]
However, unlike the \( L^2 \)-critical Schrödinger equation which is locally well-posed in \( L^2 \) (see [5]), it is not known whether the Cauchy problem (1.1) is well-posed in the critical space \( L^2 \) when \( 1 \leq \beta < 2 \). Moreover in [8], Choffrut and Pocovnicu proved that in the half-wave case (\( \beta = 1 \)), the Cauchy problem (1.1) is ill-posed in \( H^s \) if \( s < \frac{1}{2} \).

There exists a special class of solutions to (1.1) called the **solitary waves**. It is given by
\[
u(t, x) = e^{it}Q_\beta(x),
\]
where \( Q_\beta \in H^\frac{d}{2} \) is a solution to the following equation:
\[
|D|^\beta Q_\beta + Q_\beta - |Q_\beta|^{2\beta}Q_\beta = 0.
\]
From [12, 13], we know that there exists a unique radial nonnegative \( H^\frac{d}{2} \) solution to (1.8), with
\[
Q_\beta(y) \sim \frac{1}{|y|^{1+\beta}}, \text{ as } |y| \to \infty.
\]
We call this \( Q_\beta \) the **ground state**. It is also the unique optimizer (up to symmetry) for the following Gagliardo-Nirenberg inequality
\[
\|f\|_{L^{2\beta+2}}^{2\beta+2} \leq C^* \|D\nabla f\|_{L^2}^2 \|f\|_{L^2}^{2\beta}, \text{ for all } f \in H^\frac{d}{2}.
\]
Hence, a standard argument shows that if \( \|u_0\|_{L^2} < \|Q_\beta\|_{L^2} \) then the corresponding solution to (1.1) satisfies
\[
\sup_{0 \leq t < T} \|D\nabla u(t)\|_{L^2} \leq C(u_0) < +\infty,
\]
which implies that the solution is global in time and uniformly bounded in \( H^\frac{d}{2} \). However, unlike the \( L^2 \)-critical Schrödinger equation (\( \beta = 2 \)), where Dodson [9] proved that the condition \( \|u_0\|_{L^2} < \|Q_\beta\|_{L^2} \) actually implies scattering at both
time direction, in the fractional case, there exists non-scattering solutions (traveling waves) with arbitrarily small mass, due to [20, 39].

1.2. On the $L^2$-critical NLS problem. Let us give an overview of the results for the $L^2$-critical Schrödinger equations:

\[
\begin{aligned}
    iu_t + \Delta u + |u|^\frac{4}{d}u &= 0, \\
    u|_{t=0} &= u_0 \in H^1(\mathbb{R}^d),
\end{aligned}
\] (1.11)

From Weinstein [43], we know that for all initial data $u_0 \in H^1$ with $\|u_0\|_{L^2} < \|Q\|_{L^2}$, the corresponding solution to (1.11) is global in time and uniformly bounded in $H^1$. Here $Q$ is called the ground state, which is the unique nonnegative radial $H^1$ solution to the following elliptic equations:

\[
\Delta Q - Q + Q^{1+\frac{4}{d}} = 0, \quad Q(y) > 0, \quad Q \in H^1(\mathbb{R}^d).
\] (1.12)

Hence, blow-up for (1.11) can only occur in the case of $\|u_0\|_{L^2} \geq \|Q\|_{L^2}$.

There are several examples of blow-up solutions for (1.11):

1. Using viriel argument: Let the initial data satisfy $u_0 \in H^1$, $xu_0 \in L^2$ and

\[
\tilde{E}(u_0) = \frac{1}{2} \int |\nabla u_0|^2 - \frac{d}{2d+4} \int |u_0|^{\frac{2d+4}{d-4}} < 0,
\]

then the corresponding solution to (1.11) blows up in finite time.

2. Minimal mass blow-up solutions: The pseudo-conformal symmetry of (1.11) yields an explicit minimal blow-up solution:

\[
S(t,x) = \frac{1}{|t|^\frac{4}{d}} Q\left(\frac{x}{t}\right)e^{\frac{4}{d+4} + \frac{4}{d-4} \frac{|x|^2}{|t|^2}},
\] (1.13)

which blows up at $T = 0$ with $\|\nabla S(t)\|_{L^2} \sim 1/|t|$ as $t \to 0$. In [32], Merle proved that the solution given by (1.13) is the unique finite time blow-up solutions in $H^1$ with critical mass $\|u_0\|_{L^2} = \|Q\|_{L^2}$, up to symmetries of the equation.

3. Bourgain-Wang solution: In [3], Bourgain and Wang proved that there exist in dimension $d = 1, 2$, a family of blow-up solutions with blow-up rate: $\|\nabla u(t)\|_{L^2} \sim 1/(T-t)$, other than the minimal mass blow-up solutions given by (1.13). In [38], Merle, Raphaël and Szeftel proved that such solution is unstable in $H^1$.

4. Log-log blow-up solution: Numerical simulations [21], and formal arguments [42], suggest the existence of solutions blowing up like

\[
\|\nabla u(t)\|_{L^2} \sim \sqrt{\log \left(\frac{\log(T-t)}{T-t}\right)},
\]

in dimension $d = 2$. Perelman [40] proved that the existence of a blow-up solution of this type and its stability in some space $E \subset H^1$. More detailed results have been obtained in a series of papers of Merle and Raphaël [34, 35, 36, 37, 41]. They proved the existence of an $H^1$ nonempty open set of initial data leading to finite time blow-up solutions in log-log regime. These solutions behave like a blow-up bubble near the blow-up time:

\[
u(t, x) - \frac{1}{\lambda^\frac{4}{d}(t)} Q\left(\frac{x-x(t)}{\lambda(t)}\right) e^{i\gamma(t)} \to u^* \text{ in } L^2, \quad \text{as } t \to T,
\] (1.14)
for some parameters $\gamma(t) \to +\infty$ and $x(t) \to x(T)$. Here the blow-up speed is given by

$$\|\nabla u(t)\|_{L^2} = \frac{\|Q\|_{L^2}}{\lambda(t)}, \quad \lambda(t) = \sqrt{\frac{2\pi(T-t)}{\log |\log(T-t)|}}(1 + o(1)), \text{ as } t \to T. \quad (1.15)$$

1.3. Blow-up results for $L^2$ critical half wave equations. In the case $\beta = 1$, the Cauchy problem (1.1) is called half wave equation. The existence of blow-up solutions in this case is a long standing open question.

Unlike the $L^2$-critical NLS (1.11), the viriel argument does not work in this case. In deed, we still have:

$$\frac{d}{dt} \left( \Im \int x \cdot \nabla u(t) \bar{u}(t) \right) = 2E(u_0).$$

But the term $\Im \int x \cdot \nabla u(t) \bar{u}(t)$ cannot be written as the derivative of some non-negative term. A suitable generalization of the variance term for the half wave equations should be

$$V(t) := \int |(-\Delta)^{\frac{1}{4}}(xu(t))|^2.$$

However, the appearance of nonlinear terms makes the analysis much more complicate:

$$V'(t) = 2\Im \int x \cdot \nabla u(t) \bar{u}(t) + \text{nonlinear terms}.$$

On the other hand, there is no pseudo-conformal symmetry for half wave equations. We cannot construct minimal mass blow-up solutions directly. However, by a dynamical argument, Krieger, Lenzmann and Raphael [20] constructed a minimal mass blow-up solution to the $L^2$-critical half-wave equation with:

$$\|[D]^{\frac{1}{4}}u(t)\|_{L^2} \sim \frac{C}{T-t}, \text{ as } t \to T,$$

for any given momentum and energy (positive). But unlike the $L^2$ critical Schrödinger case, the uniqueness (up to symmetry) for this minimal mass blow-up solution is still not known.

1.4. Main results. In this paper, we will construct blow-up solutions for (1.1) for $1 \leq \beta < 2$. Similar to the half-wave case, the viriel argument does not work for general $L^2$ critical fractional Schrödinger case either. Besides, we still lack the pseudo-conformal symmetry here to construct minimal mass blow-up solution directly. In [20], the authors claimed that minimal mass blow-up solutions for (1.1) in the case $1 < \beta < 2$ could also be constructed by using the same argument. In [2], the authors proved that under suitable assumptions, there exist solutions to (1.1), which blow up either in finite time or infinite time by using a localized viriel argument. But in this paper, we will focusing on the slightly supercritical mass case:

$$u_0 \in \mathcal{B}_{\alpha_0} := \left\{ u_0 \in H^\frac{\beta}{4} \left| \|Q_\beta\|_{L^2} < \|u_0\|_{L^2} < \|Q_\beta\|_{L^2} + \alpha_0 \right. \right\}. \quad (1.16)$$

First of all, we introduce the following spectral property:
**Definition 1.2** (Spectral Property). Let us consider the following two Schrödinger type operators:

\[ L_1^\beta = |D|^\beta + 2(2\beta + 1)yQ_\beta^2Q_\beta^{2\beta-1}, \quad L_2^\beta = |D|^\beta + 2yQ_\beta^2Q_\beta^{2\beta-1}, \]  

(1.17)

and the following real-valued quadratic form for \( \varepsilon = \varepsilon_1 + i\varepsilon_2 \in H^{\frac{\beta}{2}}(\mathbb{R}) \):

\[ H^\beta(\varepsilon, \bar{\varepsilon}) = (L_1^\beta \varepsilon_1, \varepsilon_1) + (L_2^\beta \varepsilon_2, \varepsilon_2). \]  

(1.18)

For all \( 1 \leq \beta \leq 2 \), we say that the spectral property holds if there exists a universal constant \( \delta > 0 \) such that

\[ H^\beta(\varepsilon, \bar{\varepsilon}) \geq \delta \left( \int |D|^{\frac{\beta}{2}}|\varepsilon|^2 + \int |\varepsilon|^2 e^{-|y|} \right), \]  

(1.19)

if \( (\varepsilon_1, Q_\beta) = (\varepsilon_1, G_1) = (\varepsilon_2, \Lambda Q_\beta) = (\varepsilon_2, \Lambda^2 Q_\beta) = 0 \). Here \( \Lambda \) is the \( L^2 \)-critical scaling operator defined as following:

\[ \Lambda f = \frac{1}{2} f + yf', \]

and \( G_1 \) is some specific odd function in \( H^{\frac{\beta}{2}}(\mathbb{R}) \), which will be defined later.\(^1\)

**Remark 1.3.** In the local case when \( \beta = 2 \), we will see in Section 2 that \( G_1(y) = yQ_\beta \), which coincides with the spectral property introduced in [36, Proposition 2] in this case. Hence, the spectral property (1.19) has already been proved for the local case \( \beta = 2 \).

**Remark 1.4.** The \( L^2 \)-weight \( e^{-|y|} \) appearing on the right hand side of (1.19) is not sharp. By a Hardy’s type estimate\(^2\), it can be replaced by \( (1 + |y|)^{-\beta} \). More precisely, we have

\[ \left( \int |D|^{\frac{\beta}{2}}|\varepsilon|^2 + \int |\varepsilon|^2 e^{-|y|} \right) \sim \left( \int |D|^{\frac{\beta}{2}}|\varepsilon|^2 + \int \frac{|\varepsilon|^2}{(1 + |y|)^\beta} \right). \]

The reason that we choose the weight \( e^{-|y|} \) instead of the more natural one \( (1 + |y|)^{-\beta} \) is mostly for technique reason. Roughly speaking, the weight \( e^{-|y|} \) also us to use a perturbation argument to prove\(^3\)(1.19), when \( \beta \) is close to 2.

**Remark 1.5.** By a standard argument, we know that the spectral property (1.19) has another equivalent statement: there exists a universal constant \( \delta > 0 \) such that for all \( \varepsilon = \varepsilon_1 + i\varepsilon_2 \in H^{\frac{\beta}{2}}(\mathbb{R}) \), there holds

\[ H^\beta(\varepsilon, \bar{\varepsilon}) \geq \delta \left( \int |D|^{\frac{\beta}{2}}|\varepsilon|^2 + \int |\varepsilon|^2 e^{-|y|} \right) \]

\[ - \frac{1}{\delta} \left[ (\varepsilon_1, Q_\beta)^2 + (\varepsilon_1, G_1)^2 + (\varepsilon_2, \Lambda Q_\beta)^2 + (\varepsilon_2, \Lambda^2 Q_\beta)^2 \right]. \]

We will use this equivalence several times in this paper.

Now, we can state the main result of this paper:

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\(^1\)See (2.18) for the detailed definition of \( G_1 \).

\(^2\)See (3.30) for detailed statement.

\(^3\)See Section 4.2 for more details.
Theorem 1.6 (Blow-up dynamics in $B_{\alpha_0}$). Let $1 \leq \beta < 2$, suppose the spectral property (1.19) holds true, then there exists a universal constant $\alpha^* = \alpha^*(\beta) > 0$, such that the following holds true. For all $0 < \alpha_0 < \alpha^*$, if $u_0 \in B_{\alpha_0}$, $E(u_0) < 0$, then the corresponding solution $u(t)$ to the Cauchy problem (1.1) blows up in finite time $T < \infty$, with the following upper bound on the blow-up rate:

$$
\| |D|^\frac{\beta}{2} u(t) |_{L^2} \leq C^* \sqrt{\frac{|\log(T-t)|^{\frac{1}{2}}}{T-t}}, \quad \text{as } t \to T,
$$

(1.20)

for some universal constant $C^* > 0$.

Remark 1.7. The set of initial data satisfying the conditions mentioned in Theorem 1.6 is not empty. Since the ground state has zero energy $E(Q_{\beta}) = 0$, one may consider $u_{0,\delta} = (1 + \delta)Q_{\beta}$ with $0 < \delta \ll 1$. The proof of Theorem 1.6 is similar to that in [34, 35, 36, 41], but due to a lack of Galilean transform and the nonlocal structure, we need some additional strategies.

Remark 1.8. For higher dimension case, similar results can also be proved by using almost the same strategy. But we may need different orthogonality conditions for the spectral property (1.19), since the orthogonality conditions mentioned in Definition 1.2 cannot guarantee that (1.19) holds true in higher dimensions even for the local case $\beta = 2$.

Remark 1.9. Theorem 1.6 gives a first construct of finite time blow-up solutions with supercritical mass for $L^2$ critical fractional NLS (1.1). This solution is also stable in the sense that the set of initial data satisfying the conditions mentioned in Theorem 1.6 is an open subset in $H^\frac{\beta}{2}$, which is completely different from the minimal mass blow-up solution constructed in [20] since the minimal blow-up solution is obviously unstable in $H^\frac{\beta}{2}$. On the other hand, Boulenger, Himmelsbach and Lenzmann proved in [2] that in higher dimension case, radial solutions with negative energy to the $L^2$-critical fractional NLS will blow up in finite time or infinite time. Theorem 1.6 gives an example of finite time blow-up solutions in this case. But existence of infinite time blow-up solutions in this case still remains open.

Remark 1.10. The scaling structure of (1.1) gives the following a priori lower bound for all finite time blow-up solutions:

$$
\| |D|^\frac{\beta}{2} u(t) |_{L^2} \gtrsim (T-t)^{\frac{\beta}{4}}.
$$

However, numerical simulation [19] suggests that the exact blow-up rate for the blow-up solutions introduced in Theorem 1.6 should be

$$
\| |D|^\frac{\beta}{2} u(t) |_{L^2} \sim \sqrt{\frac{\log |\log(T-t)|}{T-t}},
$$

the same as the local case when $\beta = 2$. Sharp description on the blow-up rate in this case is still not known, which if true will lead to a lot more detailed description of the asymptotic dynamics for blow-up solutions introduced in Theorem 1.6. For example, characterization of blow-up behaviors as in [41] and description of the limiting profile as in [37].

Remark 1.11. For initial data with slightly supercritical mass and zero energy, by slightly modifying the proof of Theorem 1.6, we can prove that either the solution
$u(t)$ itself or $v(t) := \bar{u}(-t)$ blows up in finite time with the same estimate on the upper bound of the blow-up rate. In the local case when $\beta = 2$, Merle and Raphael \[35, Theorem 6\] shows that if we additionally assume that $\omega u_0 \in L^2$, then the zero energy solution $u(t)$ will blow up in both time direction. However, this type of result is still not known for the nonlocal case when $1 < \beta < 2$, since we lack here a uniqueness result for the minimal mass blow-up solution constructed in \[20\] as well as a clear characterization of blow-up solutions in the viriel space.

**Remark 1.12.** As a similar model, the $L^2$-critical generalized Benjamin-Ono equation:

$$\partial_t - |D|^\beta \partial_x u + (u|u|^{2\beta})_x = 0,$$

with $1 \leq \beta < 2$, has also received a lot attraction recently. The local case when $\beta = 2$ corresponds to the $L^2$-critical gKdV equation where the theory of formation of singularity has well studied in a series work of Martel and Merle \[23, 24, 25, 26, 27\] and Martel, Merle and Raphael \[28, 29, 30\]. While for the nonlocal case, when $\beta = 1$, this equation becomes the modified Benjamin-Ono equation, where the existence of minimal mass blow-up solution has been proved by Martel and Pilod \[31\], similar as what Krieger, Lenzmann and Raphael \[20\] did for the $L^2$-critical half-wave equation. We expect similar result as Theorem 1.6 can be proved for the $L^2$-critical generalized Benjamin-Ono equation, since the blow-up results in the local case (gKdV) is known and is similar to the local case of (1.1). However, this type of results is still mostly open.

As a natural question, we wonder for which $\beta$ does the spectral property (1.19) hold true. Here we have:

**Theorem 1.13 (Results for $\beta$ close to 2).** There exists $\beta_0 < 2$ such that if $\beta_0 < \beta < 2$, then the spectral property (1.19) holds true. Hence, the blow-up dynamics introduced in Theorem 1.6 also holds true in this case.

**Comments on Theorem 1.13:**

1. From Merle, Raphael \[36\] and Fibich, Merle, Raphael \[11\], we know that the spectral property (1.19) holds true in the case $\beta = 2$. On the other hand, from Frank and Lenzmann \[12\] and Frank, Lenzmann and Silvestre \[13\], we know that the ground state $Q_\beta$ depends continuously on the parameter $\beta$, even at $\beta = 2$:

$$Q_\beta \to Q \text{ in } H^1, \quad \text{as } \beta \to 2,$$

(1.21)

where $Q$ defined as (1.12), is the ground state for $L^2$-critical NLS (1.11). The proof of Theorem 1.13 is based these two facts and a perturbation argument. This is the reason why we need to assume that $\beta$ is close to 2. But due to different asymptotic behaviors at infinity for $Q_\beta$ and $Q$,

$$Q_\beta(y) \sim \frac{1}{|y|^{1+\beta}}, \quad Q(y) \sim e^{-|y|},$$

and different structure of the local operator $-\Delta$ and the nonlocal operator $|D|^\beta$, the perturbation argument is not straightforward.

2. Theorem 1.13 gives an explicit set of $\beta$ where the spectral property holds true, hence the conclusion in Theorem 1.6 also holds true. But for general fractional case,
especially the half-wave case ($\beta = 1$), we still can’t verify the spectral condition (1.19). This is mainly because we lack an explicit expression of the ground state $Q_\beta$, which makes it hard to use numerical methods as in [11, 36] for the local case when $\beta = 2$.

3. In the local case ($\beta = 2$), Merle and Raphaël [34, 35, 37] introduced different spectral property for the higher dimension case, which was proved for $d \leq 5$ by numerical methods [11]. Since the continuity of $Q_\beta$ with respect to $\beta$ still holds true in higher dimension case due to [13], we may similarly prove the corresponding spectral properties for $d \leq 5$ using the same perturbation argument. Hence, the conclusion on Theorem 1.6 also holds true in the case when $d = 2, 3, 4, 5$ and $\beta$ close to $2/d$.

1.5. Outline of the proof. The basic idea of this paper is similar to that of [34, 35, 36, 41]. We first notice that the solution is close to the ground state up to scaling, translation and phase, due to the assumption of slightly supercritical mass and negative energy. Then we can linearize the solution at the ground state so that the Cauchy problem (1.1) can be viewed as an ODE system of the well-chosen parameters and a nonlinear PDE of an error term. We hope to find a suitable control of the error term such that the error does not perturb the ODE system. Therefore the behavior of solutions to the ODE system can fully describe the behavior of the original solution. More precisely, we have the following steps:

1.5.1. The nonlinear blow-up profile. We are seeking for solution of the following form

$$u(t, x) = \frac{1}{\lambda(t)^{\frac{\beta}{2}}} W_{b(t), v(t)} \left( \frac{x - x(t)}{\lambda(t)} \right) e^{i\gamma(t)},$$

$$\frac{ds}{dt} = 1 \frac{\lambda^2}{\lambda^2} \frac{x_s}{\lambda} + b = 0, \quad x_s = v, \quad v_s + bv = 0, \quad \gamma_s + 1 = 0, \quad b_s = 0,$$

which after a direct computation leads to the following equation for $W_{b,v}$:

$$-\Psi_{b,v} := ib\Lambda \lambda^2 W_{b,v} - ivW_{b,v}' - ibc\partial_t W_{b,v} - |D|^{\frac{\beta}{2}} W_{b,v} - W_{b,v} + |W_{b,v}|^{2\beta} W_{b,v} = 0.$$

Using the properties of the linearized operator $L^{\beta}$ at $Q_\beta$, we may find a suitable approximate solution $W_{b,v}$, such that

$$|\Psi_{b,v}| \lesssim |b|^{\beta} + v^2.$$

1.5.2. Geometrical decomposition and modulation theory. Under the assumption of slightly supercritical mass and negative energy, we may use a standard implicit function argument to write the solution in the following form

$$u(t, x) = \frac{1}{\lambda^2(t)} \left[ W_{b(t), v(t)}(\varepsilon(t)) \left( \frac{x - x(t)}{\lambda(t)} \right) e^{i\gamma(t)} \right], \quad (1.22)$$

where the complex valued error term $\varepsilon = \varepsilon_1 + i\varepsilon_2$ satisfies some suitable orthogonality conditions. Here we introduce the velocity parameter $v$ to deal with the lack of Galilean transform. We will see the velocity parameter asymptotically vanishes sufficiently fast so that it does not perturb the system. We also mentioned here that the choice of the orthogonality conditions implies the following relation:

$$b \sim (\varepsilon_2, \Lambda^{2\beta}) \quad (1.23)$$
Differentiating those orthogonality conditions, we have a first control of the parameters:
\[
\begin{align*}
|\frac{\lambda}{\lambda} + b| + |b_s| + |v_s + bv| + |\gamma_s + 1 + \frac{1}{\|\Lambda Q_{\beta}\|_{L^2}}(\epsilon_1, L^0_s \Lambda^2 Q_{\beta})| + |\frac{\gamma_s}{\lambda} - v| \\
\lesssim \delta(\alpha_0) \left( \int |D|\tilde{\theta}_s|\epsilon|^2 + \int |\epsilon|^2 e^{-|y|} \right)^{\frac{1}{2}} + |b|^2 + |v|^2 + \lambda^2 |E(u_0)|.
\end{align*}
\]
(1.24)

Our next task is to find a suitable control of the error term \(\epsilon\).

1.5.3. The local virial estimate and the spectral property. An important feature for (1.1) is that it still has the following virial identity:
\[
\frac{d}{dt} \left( 3 \int x \cdot \nabla u(t) \bar{u}(t) \right) = 2E(u_0).
\]

Although we cannot directly use the virial identity to construct blow-up solutions, we may still use it to get a suitable control of the error term \(\epsilon\). More precisely, if we inject the geometrical decomposition (1.22) into the virial identity using (1.23), we have the following local virial estimate:
\[
b_s \gtrsim H^0(\epsilon, \epsilon) - \lambda^2 E(u_0) + v^2 - |b|^{10} - \delta(\alpha_0) \left( \int |D|\tilde{\theta}_s|\epsilon|^2 + \int |\epsilon|^2 e^{-|y|} \right)^{\frac{1}{2}}.
\]
(1.25)

Thanks to the negative energy assumption and the spectral property (coercivity of \(H^0(\epsilon, \epsilon)\)), we have for all \(s \in [0, +\infty)\),
\[
b_s \geq -C|b|^{10}.
\]

1.5.4. Proof of Theorem 1.6. From (1.24), (1.25) and the spectral property, we can show that
\[
\int |D|\tilde{\theta}_s|\epsilon|^2 + \int |\epsilon|^2 e^{-|y|} \ll |b|,
\]
in a time average sense. Using (1.24) and (1.25) again, we have the following two important results:

1. There exists a \(s_0 > 0\), such that for all \(s \geq s_0\), we have \(b(s) > 0\).
2. For all \(s \geq s_0\), we have \(\lambda_s/\lambda \sim -b\) in a time average sense.

Note that the the local virial estimate implies that \(b_s \geq -Cb^{10}\) for all \(s \geq s_0\). After integration, we have for all \(s\) large enough,
\[
b(s) \geq \frac{C}{s^\beta}, \quad \lambda(s) \geq e^{-Cs^{\beta}},
\]
and for all \(s_2 > s_1\) large enough,
\[
\lambda(s_2) \geq \frac{1}{2} \lambda(s_1).
\]

These estimates remove the possibility that the \(H^\beta\) norm of the solution oscillates in time, which forces the solution to blow up in finite time \(T < +\infty\). Finally, after a change of coordinate, the above estimates also imply that
\[
\lambda(t) \geq \left( \frac{T - t}{|\log(T - t)|^{\frac{1}{\beta}}} \right)^{\frac{1}{\beta}}.
\]

Now the proof of Theorem 1.6 is finished.
1.6. **Notations.** We use $|D|^s$ to denote the fractional order derivatives for $s \geq 0$. That is

$$|D|^s u(\xi) = |\xi|^s \hat{u}(\xi).$$

We use

$$(f, g) = \int \bar{f}g$$

as the inner product on $L^2(\mathbb{R}; \mathbb{C})$.

We denote by $Q_\beta$ the ground state of (1.1), which is the unique radial nonnegative $H^{\frac{2}{\beta}}$ solution of

$$|D|^\beta Q_\beta + Q_\beta - |Q_\beta|^{2\beta} Q_\beta = 0.$$

In the local case when $\beta = 2$, we also denote by $Q = Q_2$, which has an explicit expression in dimension one:

$$Q(x) = \left(\frac{3}{\cosh^2(2x)}\right)^{\frac{1}{4}}.$$

For a regular enough function $f$, we define the $L^2$-critical scaling operator as following:

$$\Lambda f(y) := -\frac{d}{dy}\Big|_{y=1} \frac{1}{\lambda^2} f\left(\frac{y}{\lambda}\right) = \frac{1}{2} f + y f'.$$

We also write $\Lambda^k f$ for $k \in \mathbb{N}$ for the iterates of $\Lambda$.

In some parts of the paper, we will identify any complex valued function $f : \mathbb{R} \to \mathbb{C}$ with $f : \mathbb{R} \to \mathbb{R}^2$ in the following sense:

$$f = \begin{bmatrix} \Re f \\ \Im f \end{bmatrix}.$$ 

We also identify the multiplication by $i$ in $\mathbb{C}$ with the multiplication by the following matrix

$$J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$ 

We denote by $f \cdot g$ the standard multiplication in $\mathbb{C}$,

$$f \cdot g = \begin{bmatrix} \Re f \Re g - \Im f \Im g \\ \Re f \Im g + \Im f \Re g \end{bmatrix},$$

and $\bar{f}$ the complex conjugate of $f$,

$$\bar{f} = \begin{bmatrix} \Re f \\ -\Im f \end{bmatrix}.$$ 

We also denote the linearized operator at the ground state $Q_\beta$ by

$$L^\beta = \begin{bmatrix} L_+^\beta & 0 \\ 0 & L_-^\beta \end{bmatrix},$$

with the following two scalar operators

$$L_+^\beta = |D|^\frac{2}{\beta} + 1 - (2\beta + 1)Q_\beta^{2\beta}, \quad L_-^\beta = |D|^\frac{2}{\beta} + 1 - Q_\beta^{2\beta},$$

acting on $L^2(\mathbb{R}; \mathbb{R})$.

Finally, we denote by $\delta(\alpha) > 0$ a small universal constant such that

$$\lim_{\alpha \to 0^+} \delta(\alpha) = 0.$$
2. Construction of the nonlinear blow-up profile

This section is devoted to construct an approximate blow-up profile \( W_{b,v} \) for the following equation:

\[
ib\Delta W_{b,v} - ivW_{b,v}'' - ibv\partial_\nu W_{b,v} - |D|^\beta W_{b,v} - W_{b,v} + |W_{b,v}|^{2\beta} W_{b,v} = 0.
\]

For convenience, we identify a complex valued function \( f : \mathbb{R} \to \mathbb{C} \) with the vector valued function \( f : \mathbb{R} \to \mathbb{C} \), as we mentioned in Section 1.6.

Now, we can state our result for the approximate blow-up profile.

**Proposition 2.1** (Approximate blow-up profile). For all \( 2 > \beta > 1 \), there exist constants \( b_0 > 0, v_0 > 0 \), such that if \( |b| < b_0, |v| < v_0 \), then there exists a smooth function of the following form

\[
W_{b,v} = Q_\beta + \sum_{j=1}^4 b^j R_{j,0} + v R_{0,1} + v^2 R_{0,2} + bv R_{1,1} + b^2 v R_{2,1}, \quad (2.1)
\]

satisfying the following:

1. The equation of \( W_{b,v} \): the profile \( W_{b,v} \) satisfies

\[
-\Psi_{b,v} = J b \Delta W_{b,v} - J \partial_\nu \Psi_{b,v} - |D|^{\beta} W_{b,v} - W_{b,v} + |W_{b,v}|^{2\beta} W_{b,v}. \quad (2.2)
\]

Here the error term \( \Psi_{b,v} \) satisfies the following estimates:

\[
\|\Psi_{b,v}\|_{H^m} \lesssim_m |b|^5 + v^2(|v| + |b|), \quad (2.3)
\]

\[
\|\langle x \rangle^{(1+\beta)} \Lambda^n \Psi_{b,v}(x)\|_{L^\infty} \lesssim_n |b|^5 + v^2(|v| + |b|). \quad (2.4)
\]

2. Regularity and decay estimate: for \( W_{b,v} \); the functions \( \{R_{k,\ell}\}_{0 \leq k \leq 4, 0 \leq \ell \leq 1} \) satisfy the following estimates:

\[
\|\Lambda^m R_{k,\ell}\|_{H^m} + \|\langle x \rangle^{(1+\beta)} \Lambda^n R_{k,\ell}(x)\|_{L^\infty} \lesssim_{m,n} 1, \quad (2.5)
\]

for all \( m \in \mathbb{N} \) and \( n = 0, 1, 2 \).

3. Degeneracy of the energy:

\[
\beta E(W_{b,v}) = c_0 v^2 + O\left(|b|^5 + v^2(|v| + |b|)\right), \quad (2.6)
\]

where \( c_0 > 0 \) is a positive constant.

4. The scaling invariance: there holds

\[
|D|^{\beta}(\Delta W_{b,v}) + \Delta W_{b,v} - |W_{b,v}|^{2\beta} (\Delta W_{b,v}) - 2\beta W_{b,v} |W_{b,v}|^{2\beta - 2} \Re(W_{b,v} \bar{\Delta} W_{b,v})
\]

\[
= \beta(\Psi_{b,v} + J b \Delta W_{b,v} - J \partial_\nu \Pi_{b,v} - J bv \partial_\nu W_{b,v} - W_{b,v})
\]

\[
- \Lambda^2 W_{b,v} + J v \Lambda(\nabla W_{b,v}) + J bv \Lambda(\partial_\nu W_{b,v}). \quad (2.7)
\]

**Remark 2.2.** For simplicity, we omit the dependence of \( \beta \) here.

**Remark 2.3.** The proof of Proposition 2.1 is similar to [20, Proposition 4.1], but due to the different objects we are dealing with, the nonlinear profile is slightly different.

**Remark 2.4.** In the proof of Proposition 2.1, we actually show that the functions \( R_{k,\ell} \) have the following structure:

\[
R_{1,0} = \begin{cases} 0 & \text{even} \\ 0 & \text{odd} \end{cases}, \quad R_{2,0} = \begin{cases} 0 & \text{even} \\ 0 & \text{odd} \end{cases}, \quad R_{3,0} = \begin{cases} 0 & \text{even} \\ 0 & \text{odd} \end{cases}, \quad R_{4,0} = \begin{cases} 0 & \text{even} \\ 0 & \text{odd} \end{cases},
\]

\[
R_{0,1} = \begin{cases} 0 & \text{even} \\ 0 & \text{odd} \end{cases}, \quad R_{0,2} = \begin{cases} 0 & \text{even} \\ 0 & \text{odd} \end{cases}, \quad R_{1,1} = \begin{cases} 0 & \text{even} \\ 0 & \text{odd} \end{cases}, \quad R_{2,1} = \begin{cases} 0 & \text{even} \\ 0 & \text{odd} \end{cases}. \quad (2.8)
\]
Here the word “even” and “odd” mean that the real or imaginary part of $R_{k,\ell}$ is an even or odd function.

The idea to construct such a blow-up profile is to write down the equation of each $R_{b,v}$. We will see that the solvability of these equations follows from the invertibility of the linearized operator $L^\beta$. We will give a list of properties of $L^\beta$ which will be used in this paper. Some of them are proved by Frank and Lenzmann [12] and Frank, Lenzmann, Silvestre [13].

**Lemma 2.5** (Properties of the linearized operator). The linearized operator

$$L^\beta = \begin{bmatrix} L^\beta_+ & 0 \\ 0 & L^\beta_- \end{bmatrix}$$

has the following properties:

1. **Kernel**: $\ker L^\beta_+ = \text{span}\{\nabla Q_\beta\}$, $\ker L^\beta_- = \text{span}\{Q_\beta\}$, and

$$\ker L^\beta = \text{span}\left\{ \begin{bmatrix} \nabla Q_\beta \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ Q_\beta \end{bmatrix} \right\}. \quad (2.9)$$

Hence, for all $f_1, f_2 \in L^2(\mathbb{R}; \mathbb{R})$, if $(f_1, \nabla Q_\beta) = (f_2, Q_\beta) = 0$, then there exist unique $g_1, g_2 \in H^1(\mathbb{R}; \mathbb{R})$ such that $L^\beta_+ g_1 = f_1$ and $L^\beta_- g_2 = f_2$.

2. **Eigenvalue**: $L^\beta_+$ has exactly two eigenvalues: 0 and a negative one $\kappa_0 < 0$. The negative eigenvalue $\kappa_0$ is associated to an even positive eigenfunction $\chi_0$.

3. **Coercivity**: For all $\varepsilon_1, \varepsilon_2 \in H^{\frac{\beta}{2}}(\mathbb{R}; \mathbb{R})$, if $(\varepsilon_1, \nabla Q_\beta) = (\varepsilon_1, \chi_0) = (\varepsilon_2, Q_\beta) = 0$, then we have

$$(L^\beta_+ \varepsilon_1, \varepsilon_1) \geq (\varepsilon_1, \varepsilon_1), \quad (L^\beta_- \varepsilon_2, \varepsilon_2) \geq (\varepsilon_2, \varepsilon_2). \quad (2.10)$$

4. **Scaling rule**:

$$L^\beta_+(\Lambda Q_\beta) = -\beta Q_\beta, \quad (2.11)$$

5. **Invertibility**: For all $f, g \in H^k(\mathbb{R}; \mathbb{R})$ and $k \in \mathbb{N}$, suppose $(f, \nabla Q_\beta) = (g, Q_\beta) = 0$, then we have

$$\| (L^\beta_+)^{-1} f \|_{H^{k+\beta}} \lesssim_k \| f \|_{H^k}, \quad \| (L^\beta_-)^{-1} g \|_{H^{k+\beta}} \lesssim_k \| g \|_{H^k}, \quad (2.12)$$

and the decay estimates

$$\| (x)^{1+\beta} (L^\beta_+)^{-1} f \|_{L^\infty} \lesssim \| (x)^{1+\beta} f \|_{L^\infty}, \quad (2.13)$$

$$\| (x)^{1+\beta} (L^\beta_-)^{-1} g \|_{L^\infty} \lesssim \| (x)^{1+\beta} g \|_{L^\infty}. \quad (2.14)$$

We leave the proof of Lemma 2.5 to Appendix A.

We now turn to the proof of Lemma 2.5. As mentioned in before, the idea is to solve (2.1) order by order. But here in the nonlocal case when $1 < \beta < 2$, the nonlinear term $|W_{b,v}|^{2\beta} W_{b,v}$ is generally not smooth with respect to $b$ and $v$. But we have

$$\begin{bmatrix} (|W_{b,v}|^{2\beta} W_{b,v}) |_{b=0, v=0} \\ \end{bmatrix} = \begin{bmatrix} Q_\beta^{2\beta+1} \\ \end{bmatrix}. \quad \text{(3.11)}$$

\footnote{Recall that the scalar operators $L^\beta_+$ and $L^\beta_-$ are acting on $L^2(\mathbb{R}; \mathbb{R})$, while the operator $L^\beta$ is acting on $L^2(\mathbb{R}; \mathbb{C})$.}
Here the ground state $Q_\beta$ satisfies

$$Q_\beta(y) \gtrsim \frac{1}{\langle y \rangle^{1+\beta}},$$

for all $y \in \mathbb{R}$. Hence if the decay estimate (2.5) holds true, then for small enough $b$ and $v$, the nonlinear term $|W_{b,v}|^{2\beta}W_{b,v}$ is at least $C^5$-differentiable with respect to $b$ and $v$. Therefore, we will prove Proposition 2.1 in the following steps:

**Step 1:** We first *a priorily* assume that the decay estimate (2.5) holds true, hence the nonlinear term $|W_{b,v}|^{2\beta}W_{b,v}$ is at least $C^5$-differentiable with respect to $b$ and $v$.

**Step 2:** Next we write down the (formal) equation of each $R_{k,\ell}$ order by order. Then we use Lemma 2.5 to solve each equation.

**Step 3:** We then use Lemma 2.5 again to prove the decay and regularity estimate (2.5), which will straightforwardly lead to the error term estimate (2.3).

**Step 4:** The energy estimate and the scaling invariance will follow from direct computation using the estimates obtained above.

**Proof of Proposition 2.1.** As mentioned before, we will prove Proposition 2.1 in the following steps

**Step 1: Solving $R_{k,\ell}$ formally.** We first formally solve each $R_{k,\ell}$ order by order.

*Order $O(1)$:* It is easy to see that $Q_\beta = [Q_\beta, 0]^T$ is what we need.

*Order $O(b)$:* A standard computation shows that $R_{1,0}$ satisfies

$$L^\beta R_{1,0} = J\lambda Q_\beta.$$  

(2.15)

We note that $J\lambda Q_\beta = [0, \lambda Q_\beta]^T$ satisfying $J\lambda Q_\beta \perp \ker L^\beta$, since $(\lambda Q_\beta, Q_\beta) = 0$. Hence we may chose

$$R_{1,0} = \begin{bmatrix} 0 \\ S_1 \end{bmatrix},$$

(2.16)

where $S_1 = (L^\beta)^{-1}\lambda Q_\beta$.

*Order $O(v)$:* For $R_{0,1}$, we have the following equation

$$L^\beta R_{0,1} = -J\nabla Q_\beta.$$  

(2.17)

It is also easy to check that $J\nabla Q_\beta = [0, \nabla Q_\beta]^T$ satisfies $J\nabla Q_\beta \perp \ker L^\beta$, since $Q_\beta$ is an even function while $\nabla Q_\beta$ is an odd function. Hence we may chose

$$R_{0,1} = \begin{bmatrix} 0 \\ G_1 \end{bmatrix},$$

(2.18)

where $G_1 = (-L^\beta)^{-1}\nabla Q_\beta$.

*Order $O(bv)$:* For $R_{1,1}$, we note that $R_{0,1} = R_{0,0,1} = \Im Q_\beta = 0$. Hence, we obtain the following equation for $R_{1,1}$:

$$L^\beta R_{1,1} = -JR_{1,0} + J\lambda R_{0,1} - J\nabla R_{1,0} + 2\beta R_{0,1} \cdot R_{1,0} Q_{\beta}^{2\beta-1}.$$  

(2.19)

To find such $R_{1,1}$, we need to check that the right hand side of (2.19) $\perp \ker L^\beta$, which is equivalent to

$$(G_1 - \lambda G_1 + \nabla S_1 + 2\beta S_1 G_1 Q_{\beta}^{2\beta-1}, \nabla Q_\beta) = 0.$$  

(2.20)
Using the standard commutator formula $[\Lambda, \nabla] = -\nabla$, we have

$$-(\nabla Q_\beta, \Lambda G_1) = (\Lambda \nabla Q_\beta, G_1) = (\nabla \Lambda Q_\beta, G_1) - (\nabla Q_\beta, G_1)$$

$$= (\nabla L^\beta S_1, G_1) - (\nabla Q_\beta, G_1).$$

(2.21)

We also have

$$(\nabla L^\beta S_1, G_1) + (\nabla Q_\beta, \Lambda G_1) = -(S_1, L^\beta \nabla G_1) + (\nabla L^\beta G_1, S_1)$$

$$= (S_1, [\nabla, L^\beta] G_1) = -(S_1, \nabla (Q^2_\beta G_1)) = -2\beta (\nabla Q_\beta, S_1 G_1 Q^{2\beta-1}_\beta)$$

(2.22)

Combining (2.21) and (2.22), we obtain (2.20). Hence, there exist a unique $R_{1,1} \perp \ker L^\beta$ satisfies (2.19). Since, $Q_\beta$ and $S_1$ are even functions and $G_1$ is odd, we have

$$R_{1,1} = \begin{bmatrix} F_1 \\ 0 \end{bmatrix},$$

(2.23)

with some odd function $F_1$.

*Order $\mathcal{O}(\delta^2)$:* We have the following equation for $R_{2,0}$:

$$L^\beta R_{2,0} = J \Lambda R_{1,0} + \beta |R_{1,0}|^2 Q^{2\beta-1}_\beta.$$

(2.24)

Since $R_{1,0} = [0, S_1]^T$, the solvability condition for $R_{2,0}$ is equivalent to

$$-(\nabla Q_\beta, \Lambda S_1) + (\nabla Q_\beta, \beta S_1^2 Q^{2\beta-1}_\beta) = 0$$

(2.25)

Since, $S_1$ and $Q_\beta$ are both even functions, the above condition is obviously true. Hence, we have

$$R_{2,0} = \begin{bmatrix} S_2 \\ 0 \end{bmatrix},$$

(2.26)

with some even function $S_2 = (L^\beta_1)^{-1}(-\Lambda S_1 + \beta S_1^2 Q^{2\beta-1}_\beta)$.

*Order $\mathcal{O}(\nu^2)$:* We have the following equation for $R_{0,2}$

$$L^\beta R_{0,2} = -J \nabla R_{0,1} + \beta |R_{0,1}|^2 Q^{2\beta-1}_\beta.$$

(2.27)

Since $R_{0,1} = [0, G_1]^T$, the solvability condition reduces to

$$(\nabla Q_\beta, \nabla G_1) + (\nabla Q_\beta, G_1^2 Q^{2\beta-1}_\beta) = 0.$$

(2.28)

This condition clearly holds true, since $G_1$ is odd and $Q_\beta$ is even. Hence there exist a unique $R_{0,2}$ satisfying (2.27) with

$$R_{0,2} = \begin{bmatrix} G_2 \\ 0 \end{bmatrix},$$

(2.29)

where $G_2 = (L^\beta_1)^{-1}(-\nabla G_1 + G_1^2 Q^{2\beta-1}_\beta)$ is an even function.

*Order $\mathcal{O}(\delta^3)$:* We notice that $\Re R_{1,0} = \Im R_{2,0} = 0$, so we have the following equation for $R_{3,0}$

$$L^\beta R_{3,0} = J \Lambda R_{2,0} + \beta Q^{2\beta-2}_\beta (2Q_\beta \cdot R_{2,0} + |R_{1,0}|^2) R_{1,0}.$$

(2.30)

The solvability of (2.30) is equivalent to

$$(Q_\beta, \Lambda S_2) + (Q_\beta, 2\beta S_1 S_2 Q^{2\beta-1}_\beta) + (Q_\beta, \beta Q^{2\beta-2}_\beta S_1^3) = 0.$$

(2.31)

Recall that

$$L^\beta_+ S_1 = \Lambda Q_\beta, \quad L^\beta_+ S_2 = -\Lambda S_1 + \beta S_1^2 Q^{2\beta-1}_\beta.$$
We have
\[
(Q_\beta, \Lambda S_2) + (Q_\beta, 2\beta S_1 S_2 Q_\beta^{2\beta-1}) + (Q_\beta, \beta Q_\beta^{2\beta-2} S_1^3)
\]
\[
= - (\Lambda Q_\beta, S_2) + 2\beta (S_2, Q_\beta^{2\beta} S_1) + \beta (Q_\beta^{2\beta-1}, S_1^3)
\]
\[
= - (L_\beta^2 S_1, S_2) + 2\beta (S_2, Q_\beta^{2\beta} S_1) + \beta (Q_\beta^{2\beta-1}, S_1^3)
\]
\[
= - (L_\beta^2 S_1, S_2) + \beta (Q_\beta^{2\beta-1}, S_1^3)
\]
\[
= - (S_1, -\Lambda S_1 + \beta S_1^2 Q_\beta^{2\beta-1}) + \beta (Q_\beta^{2\beta-1}, S_1^3) = 0.
\]
Hence, there exists a unique \( R_{3,0} \) satisfying (2.30) with
\[
R_{3,0} = \begin{bmatrix} 0 \\ S_3 \end{bmatrix},
\]
with some even function \( S_3 = (L_\beta^2)^{-1} (-\Lambda S_2 + 2\beta S_1 S_2 Q_\beta^{2\beta-1} + \beta S_1^2 Q_\beta^{2\beta-2}) \).

**Order \( O(b^3) \):** We notice that \( \Re R_{1,0} = \Im R_{2,0} = \Re R_{3,0} = 0 \), hence
\[
|W_{b,v}|^{2\beta} = |(Q_\beta + b^2 S_2 + b^4 \Re R_{4,0})^2 + b^2 (S_1 + b^2 S_3 + b^4 \Im R_{4,0})^4|^\beta
\]
\[
:= |F(b^2)|^\beta.
\]
Since \( F \in C^3 \) with \( F(0) = Q_\beta^2 > 0 \), we have
\[
|F(b^2)|^\beta = |F(0)|^\beta + \left( F'(0) |F(0)|^\beta \right) b^2
\]
\[
+ \frac{1}{2} \left( F''(0) \right)^2 (\beta - 1) |F(0)|^{2\beta-2} + F'''(0) |F(0)|^\beta \right) b^4 + O(b^6),
\]
which implies
\[
|W_{b,v}|^{2\beta} = Q_\beta^{2\beta} + \beta Q_\beta^{2\beta-2} (2S_2 Q_\beta + S_1^2) b^2 + \left[ \beta (\beta - 1) Q_\beta^{2\beta-4} (2S_2 Q_\beta + S_1^2) (2S_2 Q_\beta + S_1^2) + \beta Q_\beta^{2\beta-2} (S_2^2 + 2Q_\beta \Re R_{4,0} + 2S_1 S_3) \right] b^4 + O(b^6)
\]
Hence, we have the following equation for \( R_{4,0} \),
\[
L^\beta R_{4,0} = J \Lambda R_{3,0} + \beta Q_\beta^{2\beta-2} (2S_2 Q_\beta + S_1^2) R_{2,0} + H Q_\beta,
\]
where \( H = \beta^3 (\beta - 1) Q_\beta^{2\beta-4} (2S_2 Q_\beta + S_1^2) + \beta Q_\beta^{2\beta-2} (S_2^2 + 2S_1 S_3) \) is a real valued even function. The solvability of (2.33) reduces to
\[
\left( \nabla Q_\beta, -\Lambda S_3 + \beta Q_\beta^{2\beta-2} (2S_2 Q_\beta + S_1^2) S_2 + H Q_\beta \right) = 0.
\]
Since \( S_1, S_2, S_3 \) and \( Q_\beta \) are all real valued even functions, the above condition is automatically satisfied. So, there exists a unique \( R_{4,0} \) satisfying (2.33) with
\[
R_{4,0} = \begin{bmatrix} S_4 \\ 0 \end{bmatrix},
\]
where \( S_4 = (L_\beta^2)^{-1} (-\Lambda S_3 + \beta Q_\beta^{2\beta-2} (2S_2 Q_\beta + S_1^2) S_2 + H Q_\beta) \) is an even function.

**Order \( O(b^4) \):** Recall that \( \Re R_{1,0} = \Im R_{2,0} = \Re R_{0,1} = \Im R_{1,1} = 0 \). We then have the following equation for \( R_{2,1} \):
\[
L^\beta R_{2,1} = J (\Lambda R_{1,1} - \nabla R_{2,0}) + \beta Q_\beta^{2\beta-2} \left[ (2Q_\beta S_2 + S_1^2) R_{0,1} + (2Q_\beta F_1 + 2S_1 G_1) R_{1,0} \right].
\]
Using the fact that \( R_{1,0} = R_{2,0} = R_{0,1} = R_{1,1} = 0 \) again, we know that the solvability of (2.36) is equivalent to
\[
\left( Q_{\beta} - \Lambda F_1 + \nabla S_2 + \beta Q_{\beta}^{2^{2-2}} \left[ (2Q_{\beta}S_2 + S_1^2)G_1 + (2Q_{\beta}F_1 + 2S_1G_1)S_1 \right] \right) = 0 \tag{2.37}
\]
Since \( Q_{\beta}, S_1, S_2 \) are even functions while \( F_1, G_1 \) are odd functions, the above condition is automatically satisfied. So there exists a unique \( R_{2,1} \) satisfying (2.33) with
\[
R_{2,1} = \begin{bmatrix} 0 \\ F_2 \end{bmatrix}, \tag{2.38}
\]
where \( F_2 = (L_{\beta}^{-1})^{-1} \left( -\Lambda F_1 + \nabla S_2 + \beta Q_{\beta}^{2^{2-2}} \left[ (2Q_{\beta}S_2 + S_1^2)G_1 + (2Q_{\beta}F_1 + 2S_1G_1)S_1 \right] \right) \) is an odd function.

**Step 2: Regularity and decay estimates for \( R_{k,\ell} \).** Let \( \mathcal{Y} \) be the set of all smooth functions \( f \) satisfying the following property: \( \forall k \in \mathbb{N}, \exists C_k > 0 \) such that
\[
|\partial_k^b f(y)| \leq C_k (1 + |y|)^{-1-\beta-k}.
\]
We can easily show that \( \mathcal{Y} \) has the following properties

1. If \( f \in \mathcal{Y} \), then \( \Lambda f, \nabla f \in \mathcal{Y} \).
2. If \( f, g \in \mathcal{Y} \), then \( fg \in \mathcal{Y} \).
3. \( Q_{\beta} \in \mathcal{Y} \).

Here the first two properties follow from direct computation, and the third one is proved in [12].

Combining these properties with the equation of each \( R_{k,\ell} \), we have for all \( k, \ell \)
\[
L^\beta R_{k,\ell} \in \mathcal{Y}. \tag{2.39}
\]
Since \( [L^\beta, \nabla] = (2\beta+1)\nabla(Q_{\beta}^{2^{2}}) \) and \( [L^\beta, \nabla] = \nabla(Q_{\beta}^{2^{2}}) \). From (2.12), (2.13), (2.14), we obtain that \( R_{k,\ell} \in \mathcal{Y} \), which implies (2.5) immediately.

**Step 3: Error term estimate.** As we mentioned before the regularity and decay condition of \( R_{k,\ell} \) implies that for \( |b|, |v| \) small enough, \( \Psi_{b,v} \) is at least \( C^{10} \)-differentiable with respect to \( b \) and \( v \). Hence, by Taylor’s extension formula and the construction of \( R_{k,\ell} \), we have
\[
\Psi_{b,v} = b^5 T_{5,0} + vb^3 T_{3,1} + v^2 b T_{1,2} + v^3 T_{0,3}, \tag{2.40}
\]
where
\[
T_{5,0} = \frac{1}{5!} \int_0^1 \partial_b^5 \Psi_{tb,0} \, dt, \quad T_{3,1} = \frac{1}{3!} \int_0^1 \partial_b^3 \partial_v \Psi_{tb,0} \, dt,
\]
\[
T_{1,2} = \frac{1}{2!} \int_0^1 \partial_b \partial_v^2 \Psi_{tb,0} \, dt, \quad T_{0,3} = \frac{1}{3!} \int_0^1 \partial_v^3 \Psi_{b,te} \, dt.
\]
Combining the regularity and decay estimate (2.5) for \( R_{k,\ell} \), we obtain (2.3) and (2.4) immediately.
Step 4: Energy estimate. The energy estimate (2.6) follows from direct computation of the equation of $W_{b,v}$. More precisely, we claim that the following identity holds:

\[
\beta E(W_{b,v}) = \mathbb{R} \int (|D|^\beta W_{b,v} + W_{b,v} - |W_{b,v}|^{2\beta}W_{b,v} - J_\beta AW_{b,v})\Lambda W_{b,v}. \tag{2.41}
\]

Indeed, it is easy to see that

\[
\mathbb{R} \int (W_{b,v} - J_\beta AW_{b,v})\Lambda W_{b,v} = 0, \tag{2.42}
\]

and

\[
\mathbb{R} \int |W_{b,v}|^{2\beta}W_{b,v}(\Lambda W_{b,v}) = \mathbb{R} \int |W_{b,v}|^{2\beta} (\mathbb{R} W_{b,v}(\Lambda W_{b,v}) + \mathbb{I} W_{b,v}(\Lambda W_{b,v}))
= \frac{1}{2} \int |W_{b,v}|^{2\beta+2} + \int |W_{b,v}|^{2\beta} (y \nabla (\mathbb{R} W_{b,v}) W_{b,v} + y \nabla (\mathbb{I} W_{b,v}) \mathbb{I} W_{b,v})
= \frac{1}{2} \int |W_{b,v}|^{2\beta+2} + \frac{1}{2\beta+2} \int y \cdot \nabla (|W_{b,v}|^{2\beta+2})
= \frac{\beta}{2\beta+2} \int |W_{b,v}|^{2\beta+2}. \tag{2.43}
\]

Moreover, let $\mathcal{F}$ be the standard Fourier transform given by:

\[
\mathcal{F} f(\xi) = \frac{1}{\sqrt{2\pi}} \int f(x) e^{-ix\xi} \, dx.
\]

Then we have:

\[
\mathbb{R} \int |D|^\beta W_{b,v}(\Lambda W_{b,v}) = \mathbb{R} \int \mathcal{F}(|D|^\beta W_{b,v}) \mathcal{F}(\Lambda W_{b,v})
= \mathbb{R} \int |\xi|^\beta \mathcal{F}(W_{b,v}) \left( \frac{1}{2} \mathcal{F}(W_{b,v}) + J_\beta \xi (J_\beta \mathcal{F}(W_{b,v})) \right)
= \frac{1}{2} \int ||D|^\beta W_{b,v}|^2 + \mathbb{R} \int \xi \partial_\xi (|\xi|^\beta \mathcal{F}(W_{b,v})) \mathcal{F}(W_{b,v})
= \frac{1}{2} \int ||D|^\beta W_{b,v}|^2 + \beta \mathbb{R} \int |\xi|^\beta \mathcal{F}(W_{b,v}) \mathcal{F}(W_{b,v})
+ \frac{1}{2} \mathbb{R} \int (\xi |\xi|^\beta) \partial_\xi (\mathcal{F}(W_{b,v}) \mathcal{F}(W_{b,v}))
= \frac{\beta}{2} \int ||D|^\beta W_{b,v}|^2. \tag{2.44}
\]

From (2.42)–(2.44), we obtain (2.41) immediately.

Combining (2.2) and (2.41), we have

\[
\beta E(W_{b,v}) = \mathbb{R} \int \mathfrak{W}_{b,v}(\Lambda W_{b,v}) - v \Im \int \nabla W_{b,v}(\Lambda W_{b,v})
- bv \Im \int \partial_v(W_{b,v})\Lambda W_{b,v} := I + II + III. \tag{2.45}
\]

For $I$, from the estimate (2.3) we obtain

\[
|I| \lesssim |b|^5 + v^2(|v| + |b|). \tag{2.46}
\]
For II and III, from the construction of $W_{b,v}$, we know that $W_{b,v}$ has the following form

$$W_{b,v} = [\text{even}][0] + b[0][\text{even}] + b^2[0][\text{even}] + b^3[0][\text{even}] + b^4[0][\text{even}] + v[0][\text{odd}] + v^2[0][\text{even}] + bv[0][\text{odd}] + vb^2[0][\text{odd}].$$

Hence, we have

$$II = -v\Im\int \nabla W_{b,v}(\Lambda W_{b,v}) = -v^2\left(\int \nabla Q_\beta \Lambda G_1 - \nabla G_1 \Lambda Q_\beta\right) + O(|b| + |v|)v^2. \quad (2.47)$$

Using the commutator formula $[\nabla, \Lambda] = \nabla$ and (2.10), we have

$$\int \nabla Q_\beta \Lambda G_1 - \nabla G_1 \Lambda Q_\beta = (G_1, [\nabla, \Lambda]Q_\beta) = (G_1, \nabla Q_\beta) = -(L_\beta^2 G_1, G_1) < 0.$$ Similarily, we have

$$III = -bv\Im\int \partial_v(W_{b,v})\Lambda W_{b,v} = O(|b| + |v|)v^2. \quad (2.48)$$

Injecting (2.46)–(2.48) into (2.45), we obtain (2.6) immediately.

**Step 5: Scaling invariance.** Finally, for the scaling invariance, we use a similar argument as the proof of $L_+^\beta \Lambda Q_\beta = -\beta Q_\beta$. More precisely, for any regular enough functions $\omega$ and $\Omega$ satisfying

$$D^\beta |\omega - \omega|^2\omega = \Omega,$$

we consider $\omega_\lambda(y) = \lambda^{1/2}\omega(\lambda y), \Omega_\lambda(y) = \lambda^{1/2}\Omega(\lambda y)$ for $\lambda > 0$. Then we have

$$D^\beta \omega_\lambda + \lambda^2 \omega_\lambda - |\omega_\lambda|^2\omega_\lambda = \lambda^2 \Omega_\lambda. \quad (2.49)$$

Differentiating (2.49) with respect to $\lambda$ and taking $\lambda = 1$, we obtain

$$D^\beta(\Lambda \omega) + \Lambda \omega - \Lambda |\omega|^{2\beta} - 2\beta |\omega|^{2\beta-2}\Re(\omega \overline{\omega}) = \beta(\Omega - \omega) + \Lambda \Omega, \quad (2.50)$$

where we use the following facts for the above estimate:

$$\left.\left(\frac{d}{d\lambda} \omega_\lambda\right)\right|_{\lambda=1} = \Lambda \omega, \quad \left.\left(\frac{d}{d\lambda} |\omega_\lambda|^{2\beta}\right)\right|_{\lambda=1} = 2\beta |\omega|^{2\beta-2}\Re(\omega \overline{\omega}).$$

Now, we apply (2.50) to $\omega = W_{b,v}$ and

$$\Omega = \Psi_{b,v} + Jb\Lambda W_{b,v} - Jv\nabla W_{b,v} - Jbv\partial_v W_{b,v},$$

then we obtain (2.7), which concludes the proof of Proposition 2.1. \hfill \Box

### 3. Modulation theory

In this section, we build a general setting of solution with negative energy and slightly super critical mass to (1.1). We use the variation properties of the ground state $Q_\beta$ and conservation laws of (1.1) to establish a sharp geometrical decomposition for such solution and study its basic properties.
3.1. **Geometrical decomposition.** We start with the variation structure of $Q_\beta$.

**Lemma 3.1** (Variation characterization). There exists $\alpha_1 > 0$ such that for all $0 < \alpha' < \alpha_1$ and $u_0 \in H^{\frac{\alpha}{2}}(\mathbb{R})$ satisfying

$$\left| \int |u_0|^2 - \int Q_\beta \right| \leq \alpha', \quad E(u_0) \leq \alpha' \int \| D^{|\frac{\alpha}{2}} u_0 \|^2,$$

then there exist constants $\lambda_0 > 0$, $x_0 \in \mathbb{R}$, $\gamma_0 \in \mathbb{R}$ such that

$$\| Q_\beta(\cdot) - e^{i\gamma_0} \lambda_0^{1/2} u_0 (\lambda_0 \cdot + x_0) \|_{H^{\frac{\alpha}{2}}} \leq \delta(\alpha').$$

**Proof.** The proof is based on the variational properties of the ground state and a standard concentration compactness argument. We refer to [17, Lemma 9], [33, Lemma 1], [36, Lemma 1] and the references therein for detailed proof. □

We now turn to the Cauchy problem (1.1). Let $u(t) \in C([0, T), H^{\frac{\alpha}{2}})$ be a solution of (1.1) with maximal lifespan $T > 0$. Suppose the initial data $u_0 \in B_{\alpha_0}$ with $E(u_0) < 0$ for some small enough $\alpha_0 > 0$. Then from the mass conservation law (1.2) and energy conservation law (1.3), we have for all $0 \leq t < T$, there exists $\lambda_1(t) > 0$, $x_1(t) \in \mathbb{R}$, $\gamma_1(t) \in \mathbb{R}$ such that

$$\left\| Q_\beta(\cdot) - e^{-i\gamma_1(t)} \lambda_1(t)^{1/2} u(t, \lambda_1(t) \cdot + x_1(t)) \right\|_{H^{\frac{\alpha}{2}}} \leq \delta(\alpha_0),$$

provided that $\alpha_0 > 0$ is small enough.

Now we can establish the **geometrical decomposition** for solutions to (1.1) with negative energy and slightly supercritical mass.

**Proposition 3.2** (Geometrical decomposition). Let $u(t)$ be a solution to (1.1) satisfying the conditions mentioned in Theorem 1.6. Then there exist five $C^1$ functions on $[0, T)$: $\lambda(t)$, $x(t)$, $b(t)$, $v(t)$, $\gamma(t)$ such that the following holds true:

1. **Geometrical decomposition:** for all $t \in [0, T)$
   $$u(t, x) = \frac{1}{\lambda_1^2(t)} [W_{b(t), v(t)} + \varepsilon(t)] \left( \frac{x - x(t)}{\lambda(t)} \right) e^{i\gamma(t)},$$

   where $W_{b, v}$ is the nonlinear blow-up profile\(^6\) constructed in Proposition 2.1.

2. **Orthogonality conditions:** for all $t \in [0, T)$
   $$\begin{align*}
   (\varepsilon_1, \Lambda \Theta) - (\varepsilon_2, \Lambda \Sigma) &= 0, \\
   (\varepsilon_1, \partial_b \Theta) - (\varepsilon_2, \partial_b \Sigma) &= 0, \\
   (\varepsilon_1, \partial_\sigma \Theta) - (\varepsilon_2, \partial_\sigma \Sigma) &= 0, \\
   (\varepsilon_1, \nabla \Theta) - (\varepsilon_2, \nabla \Sigma) &= 0, \\
   (\varepsilon_1, \Lambda^2 \Theta) - (\varepsilon_2, \Lambda^2 \Sigma) &= 0.
   \end{align*}$$

Here we use the following notation:

$$W_{b, v} = \Sigma + i \Theta, \quad \varepsilon = \varepsilon_1 + i \varepsilon_2.$$

3. **A priori estimates on the parameters:** for all $t \in [0, T)$
   $$\| b(t) \| + \| v(t) \| + \| \varepsilon(t) \|_{H^{\frac{\alpha}{2}}} \leq \delta(\alpha_0).$$

---

\(^6\)Here we use the normal notation of complex valued functions instead of the vector form introduced in Section 1.6.
Proof. The proof of Proposition 3.2 follows from a standard argument of implicit function theory. We leave the detailed proof in Appendix B. □

3.2. Modulation estimates. With the geometrical decomposition obtained in Proposition 3.2, we are now able to derive some crucial estimates for the parameters \((\lambda(t), b(t), x(t), v(t), \gamma(t))\).

We first introduce the following scaling invariant coordinate:

\[
s = \int_0^t \frac{1}{\lambda^\beta(\tau)} \, d\tau, \quad y = \frac{x - x(t)}{\lambda(t)},
\]

(3.12)

Remark 3.3. We mention here that the maximal lifespan of the solution in the rescaled setting is always \(+\infty\):

\[s(T) = +\infty.\]

This is a consequence of the scaling structure of the equation. If the solution does not blow up in finite time, then it is obviously that \(s(T) = +\infty\), since the negative energy condition removes the possibility that \(\lambda(s) \to +\infty\). If the solution blows up in finite time \(T < +\infty\), then the scaling structure ensures that \(\lambda(t) \lesssim (T - t)^{1/\beta}\), which also implies that \(s(T) = +\infty\).

Under this new coordinate, we have the following a priori estimates for the parameters \((\lambda, b, v, x, \gamma)\) and the error term \(\varepsilon\):

**Proposition 3.4.** For all \(s \in [0, +\infty)\), the following estimates hold true:

1. Equation of \(\varepsilon\):

\[
\begin{aligned}
b_s(\partial_b \Sigma) + (v_s + bv)\partial_v \Sigma + \partial_s \varepsilon_1 - M_- (\varepsilon) + b\lambda \varepsilon_1 - v \cdot \nabla \varepsilon_1 \\
= \left( \frac{\lambda s}{\lambda} + b \right) (\Lambda \Sigma + \Lambda \varepsilon_1) + \left( \frac{x_s}{\lambda} - v \right) \cdot \left( \nabla \Sigma + \nabla \varepsilon_1 \right) \\
+ \tilde{\gamma}_s (\Theta + \varepsilon_2) + \mathbb{R}(\Psi_{b,v}) - R_2(\varepsilon),
\end{aligned}
\]

(3.13)

\[
\begin{aligned}
b_s(\partial_b \Theta) + (v_s + bv)\partial_v \Theta + \partial_s \varepsilon_2 + M_+ (\varepsilon) + b\lambda \varepsilon_2 - v \cdot \nabla \varepsilon_2 \\
= \left( \frac{\lambda s}{\lambda} + b \right) (\Lambda \Theta + \Lambda \varepsilon_2) + \left( \frac{x_s}{\lambda} - v \right) \cdot \left( \nabla \Theta + \nabla \varepsilon_2 \right) \\
- \tilde{\gamma}_s (\Sigma + \varepsilon_1) - \mathbb{R}(\Psi_{b,v}) + R_1(\varepsilon).
\end{aligned}
\]

(3.14)

Here \(\tilde{\gamma}(s) = -s - \gamma(s)\) and \(M = (M_+, M_-)\) are small perturbations of the linearized operator \(L^\beta = (L_+^\beta, L_-^\beta)\) given by

\[
\begin{aligned}
M_+(\varepsilon) &= |D|^\beta \varepsilon_1 + \varepsilon_1 - |W_{b,v}|^{2\beta} \varepsilon_1 \\
&\quad - 2\beta |W_{b,v}|^{2(\beta - 1)} (\Sigma^2 \varepsilon_1 + \Sigma \Theta \varepsilon_2),
\end{aligned}
\]

(3.15)

\[
\begin{aligned}
M_- (\varepsilon) &= |D|^\beta \varepsilon_2 + \varepsilon_2 - |W_{b,v}|^{2\beta} \varepsilon_2 \\
&\quad - 2\beta |W_{b,v}|^{2(\beta - 1)} (\Theta^2 \varepsilon_2 + \Sigma \Theta \varepsilon_1).
\end{aligned}
\]

(3.16)

\footnote{See [20, Appendix C] and [36, Lemma 2] for similar discussion.}
The nonlinear terms $R_1(\varepsilon)$, $R_2(\varepsilon)$ are given by

\[
R_1(\varepsilon) = |W_{b,v} + \varepsilon|^{2\beta} (\Sigma + \varepsilon_1) - \Sigma|W_{b,v}|^{2\beta} - |W_{b,v}|^{2\beta} \varepsilon_1 - 2\beta|W_{b,v}|^{2(\beta-1)}(\Sigma^2 \varepsilon_1 + \Sigma \Theta \varepsilon_2),
\]

\[
R_2(\varepsilon) = |W_{b,v} + \varepsilon|^{2\beta}(\Theta + \varepsilon_2) - \Theta|W_{b,v}|^{2\beta} - |W_{b,v}|^{2\beta} \varepsilon_2 - 2\beta|W_{b,v}|^{2(\beta-1)}(\Theta \varepsilon_2 + \Sigma \Theta \varepsilon_1).
\]  

(3.17) \hspace{1cm} (3.18)

(2) Estimates induced by the conservation laws:

\[
|\lambda^\beta E_0| - \langle \varepsilon_1, \Sigma + b\Lambda \Theta - v\nabla \Theta \rangle - \langle \varepsilon_2, \Theta - b\Lambda \Sigma + v\nabla \Sigma \rangle \leq \left( \int |D|^\beta \varepsilon \right)^2 + \int |\varepsilon|^2 e^{-|y|} + |b|^5 + v^2. \]

(3.19)

(3) Estimates for the modulation parameters:

\[
\left| \frac{\lambda}{\lambda_s} + b \right| + |v_s + bv| + \left| \frac{x_s}{\lambda} - v \right| + |b_s| + \left| \tilde{\eta}_s - \frac{1}{\|\Lambda Q_\beta\|_{L^2}}(\varepsilon_1, L_\beta(\Lambda^2 Q_\beta)) \right| \leq \delta(\alpha_0) \left( \int |D|^\beta \varepsilon \right)^\frac{1}{2} + |b|^5 + v^2 + \lambda^\beta |E_0|. \]

(3.20)

**Proof.** Proof of (1): The equations of (3.13) and (3.14) follows from direct computation.

**Proof of (2):** For the estimate (3.19), we expand the energy conservation law as the following:

\[
2\langle \varepsilon_1, \Sigma + b\Lambda \Theta - v\nabla \Theta \rangle + 2\langle \varepsilon_2, \Theta - b\Lambda \Sigma + v\nabla \Sigma \rangle - \frac{1}{\beta + 1} \int F(\varepsilon) + \int |D|^\beta \varepsilon - \int (|W_{b,v}|^{2\beta} + 2\beta \Sigma^2 |W_{b,v}|^{2\beta-2}) \varepsilon_1^2
\]

\[
- \int (|W_{b,v}|^{2\beta} + 2\beta \Theta^2 |W_{b,v}|^{2\beta-2}) \varepsilon_2^2 - 4\beta \int \Sigma \Theta |W_{b,v}|^{2\beta-2} \varepsilon_1 \varepsilon_2,
\]

(3.21)

where

\[
F(\varepsilon) = |W_{b,v} + \varepsilon|^{2\beta + 2} - |W_{b,v}|^{2\beta + 2} - (2\beta + 2)|W_{b,v}|^{2\beta}(\Sigma \varepsilon_1 + \Theta \varepsilon_2)
\]

\[
- (\beta + 1) \left( |W_{b,v}|^{2\beta} + 2\beta \Sigma^2 |W_{b,v}|^{2\beta-2} \right) \varepsilon_1^2
\]

\[
- (\beta + 1) \left( |W_{b,v}|^{2\beta} + 2\beta \Theta^2 |W_{b,v}|^{2\beta-2} \right) \varepsilon_2^2
\]

\[
- 4\beta(\beta + 1) \Sigma \Theta |W_{b,v}|^{2\beta-2} \varepsilon_1 \varepsilon_2.
\]

(3.22)

Recalling from the construction of $W_{b,v}$, we have for all $y \in \mathbb{R}$,

\[
|W_{b,v}(y)| \geq Q_\beta(y) - O(|b| + |v|)(1 + |y|)^{-1-\beta} \gtrsim \frac{1}{(1 + |y|)^{\frac{1}{\beta}}} > 0.
\]

(3.23)

Applying the following estimate\(^8\)

\[
\left| (1 + z)^{2\beta} - 1 - (2 + \beta)\Re z - (\beta + 1)(2\beta + 1)(\Re z)^2 - (\beta + 1)(\Im z)^2 \right| \leq C(|z|^3 + |z|^{2+2\beta}), \quad \forall z \in \mathbb{C}
\]

(3.24)

\(^8\)We use the fact that $2\beta + 2 > 3$ for this estimate.
to \( z = \varepsilon / W_{b,v} \), we have:

\[
|F(\varepsilon)| = |W_{b,v}|^{2\beta} \times \left| 1 + \frac{\varepsilon}{W_{b,v}} \right|^{2\beta^2} - 1 - (2 + \beta) \Re \left( \frac{\varepsilon}{W_{b,v}} \right) - (\beta + 1)(2\beta + 1) \left| \Re(\varepsilon / W_{b,v}) \right|^2 - (\beta + 1) \left| \Im(\varepsilon / W_{b,v}) \right|^2
\]

\[
\leq C(|\varepsilon|^{2\beta^2} + |\varepsilon|^3|W_{b,v}|^{2\beta^2 - 1}),
\]

Hence

\[
\int |F(\varepsilon)| \lesssim \int |\varepsilon|^{2\beta^2} + \int |\varepsilon|^3|W_{b,v}|^{2\beta^2 - 1}
\]

\[
\lesssim \| \varepsilon \|_{L^2}^{2\beta} \left( \int |D|^{\frac{\beta}{2}} \varepsilon |^2 \right) + \left( \int |\varepsilon|^{\beta + 3} \right) \frac{1}{\beta + 3} \left( \int |\varepsilon|^2 |W_{b,v}|^{\frac{\beta + 1}{\beta + 3}} \right) \frac{\beta^2 + 1}{\beta + 3}
\]

\[
\lesssim \| \varepsilon \|_{L^2}^{2\beta} \left( \int |D|^{\frac{\beta}{2}} \varepsilon |^2 \right) + \| \varepsilon \|_{L^2} \left( \int |D|^{\frac{\beta}{2}} \varepsilon |^2 \right) \frac{1}{\beta + 3} \left( \int |\varepsilon|^2 |W_{b,v}|^{\frac{\beta + 1}{\beta + 3}} \right) \frac{\beta^2 + 1}{\beta + 3}
\]

\[
\lesssim \| \varepsilon \|_{H^2} \left( \int |D|^{\frac{\beta}{2}} \varepsilon |^2 \right) + \int \frac{|\varepsilon|^2}{(1 + |y|)^\beta}, \quad (3.25)
\]

where we use the Gagliardo-Nirenberg’s inequality:

\[
\| \varepsilon \|_{L^p}^p \leq C \| |D|^{\frac{\beta}{2}} \varepsilon |^2 \|_{L^2} \| \varepsilon \|_{L^2} \frac{p\beta + 1}{p\beta - 1 + 2}, \quad \forall p \geq 2
\]

and the decay estimate (2.5) for the inequality (3.25).

Similarly, we have

\[
\int \left( |W_{b,v}|^{2\beta} + 2\beta \Sigma^2 |W_{b,v}|^{2\beta - 2} \right) \varepsilon_1^2 + \int \left( |W_{b,v}|^{2\beta} + 2\beta \Theta^2 |W_{b,v}|^{2\beta - 2} \right) \varepsilon_2^2
\]

\[
- 4\beta \int \Sigma \Theta |W_{b,v}|^{2\beta - 2} \varepsilon_1 \varepsilon_2 \lesssim \int \frac{|\varepsilon|^2}{(1 + |y|)^\beta}, \quad (3.26)
\]

and

\[
2(\varepsilon_1, bv \partial_v \Theta + \Re(\Psi_{b,v})) - 2(\varepsilon_2, bv \partial_v \Sigma - \Im(\Psi_{b,v}))
\]

\[
\lesssim (|b|^{5} + v^{2}(|v| + |b|) + |bv|) \left( \int \frac{|\varepsilon|}{(1 + |y|)^{1+\beta}} \right)
\]

\[
\lesssim |b|^{5} + v^{2} + \int \frac{|\varepsilon|^2}{(1 + |y|)^{\beta}}, \quad (3.27)
\]

Recalling from (2.6), we have:

\[
|E(W_{b,v})| \lesssim v^2 + |b|^5. \quad (3.28)
\]

Injecting (3.25)–(3.28) into (3.22), using the a priori smallness estimate (3.11) and the energy condition \( E_0 < 0 \), we obtain

\[
|\lambda^2| E_0 | - (\varepsilon_1, \Sigma + b\Lambda \Theta - v \nabla \Theta) - (\varepsilon_2, \Theta - b\Lambda \Sigma + v \nabla \Sigma)
\]

\[
\lesssim \left( \int |D|^{\frac{\beta}{2}} \varepsilon |^2 \right) + \frac{|\varepsilon|^2}{(1 + |y|)^\beta} + |b|^5 + v^2, \quad (3.29)
\]

provided that \( \alpha_0 \) is small enough.
Now, it is easy to see that the estimate (3.19) follows from the following Hardy’s type estimate:

**Lemma 3.5 (Hardy’s type estimate).** Suppose $1 \leq \beta < 2$, then we have

$$
\int \frac{|f|^2}{(1 + |y|)^\beta} \leq C \left( \int |D\tilde{s}f|^2 + \int |f|^2 e^{-|y|} \right),
$$

(3.30)

for all $f \in H^{\frac{s}{2}}(\mathbb{R})$.

**Remark 3.6.** The proof of (3.30) follows from the fractional Hardy’s inequality\(^9\) in dimension 1 as well as a localization argument. We leave the detailed proof in Appendix C.

**Proof of (3):** Now, we turn to the proof of (3.20). We first differentiate (3.5) to obtain

$$
(\partial_s \epsilon_1, \Lambda \Theta) - (\partial_s \epsilon_2, \Lambda \Sigma) + b_s \left[ (\epsilon_1, \Lambda (\partial_b \Theta)) - (\epsilon_2, \Lambda (\partial_b \Sigma)) \right]
$$

$$
= -v_s \left[ (\epsilon_1, \Lambda (\partial_b \Theta)) - (\epsilon_2, \Lambda (\partial_b \Sigma)) \right].
$$

(3.31)

Then we project (3.13) and (3.14) onto $-\Lambda \Theta$ and $\Lambda \Sigma$ using (3.31) to obtain

$$
b_s \left[ (\partial_b \Theta, \Lambda \Sigma) - (\partial_b \Sigma, \Lambda \Theta) - (\epsilon_2, \Lambda (\partial_b \Sigma)) + (\epsilon_1, \Lambda (\partial_b \Theta)) \right]
$$

$$
+ (v_s + bv) \left[ (\partial_s \Phi, \Lambda \Sigma) - (\partial_s \Sigma, \Lambda \Theta) - (\epsilon_2, \Lambda (\partial_s \Sigma)) + (\epsilon_1, \Lambda (\partial_s \Theta)) \right]
$$

$$
+ \left( \frac{x_s}{\Lambda} - v \right) \left\{ (\Theta, (\Lambda \Sigma)_y) - (\Sigma, (\Lambda \Theta)_y) + (\epsilon_2, (\Lambda \Sigma)_y) - (\epsilon_1, (\Lambda \Theta)_y) \right\}
$$

$$
+ \left( \frac{\lambda_s}{\Lambda} + b \right) \left\{ - (\Lambda \Theta, \Lambda \Sigma) + (\Lambda \Sigma, \Lambda \Theta) + (\epsilon_2, \Lambda \Sigma) - (\epsilon_1, \Lambda \Sigma) \right\}
$$

$$
- \beta \left\{ (\Sigma, \Lambda \Sigma) + (\Sigma, \Lambda \Sigma) + (\epsilon_1, \Lambda \Sigma) + (\xi_2, \Lambda \Theta) \right\}
$$

$$
= -bv \left\{ (\epsilon_2, \Lambda (\partial_s \Sigma)) - (\epsilon_1, \Lambda (\partial_s \Theta)) \right\} - M_\pm (\epsilon, \Lambda \Sigma) - M_\pm (\epsilon, \Lambda \Theta)
$$

$$
- b \left\{ (\Lambda \Sigma, \Phi) - (\Lambda \Sigma, \Lambda \Theta) \right\} + v \left\{ (\nabla \xi_1, \Lambda \Theta) - (\nabla \xi_1, \Lambda \Theta) \right\}
$$

$$
- (\Lambda \Sigma, \Re(\Psi_{b,v})) - (\Lambda \Phi, \Im(\Psi_{b,v})) + (R_1(\epsilon), \Lambda \Sigma) + (R_2(\epsilon), \Lambda \Theta). \quad (3.32)
$$

By applying the same argument to (3.6)–(3.9), we can obtain the other four equations similar as (3.32). For simplicity, we do not write down the other four equations explicitly.

Now, we view these five equations as a linear system of

$$
\begin{pmatrix}
    b_s \frac{\lambda_s}{\Lambda} + b \frac{x_s}{\Lambda} - v, v_s + bv, \tilde{\gamma}_s
\end{pmatrix}.
$$

Hence, (3.32) and the other four similar equations can be written in the following form

$$
A \begin{bmatrix}
    b_s \\
    \frac{\lambda_s}{\Lambda} + b \\
    \frac{x_s}{\Lambda} - v \\
    v_s + bv \\
    \tilde{\gamma}_s
\end{bmatrix} = 
B \begin{bmatrix}
    B_1 \\
    B_2 \\
    B_3 \\
    B_4 \\
    B_5
\end{bmatrix}, \quad (3.33)
$$

\(^9\)See [1] for more details.
where \( A = A(b, v, \varepsilon) \) is a \( 5 \times 5 \) matrix and
\[
B_1 = -b \left\{ \varepsilon_2, \partial_v(\Lambda \Sigma) - (\varepsilon_1, \partial_v(\Lambda \Theta)) \right\} - (M_+(\varepsilon), \Lambda \Sigma) - (M_-(\varepsilon), \Lambda \Theta) \\
- b \left\{ \Lambda \varepsilon_2, \Lambda \Sigma \right\} - (\Lambda \varepsilon_1, \Lambda \Theta) + v \left\{ (\nabla \varepsilon_2, \Lambda \Sigma) - (\nabla \varepsilon_1, \Lambda \Theta) \right\} \\
- (\Lambda \Sigma, \Re(\Psi_{b,v})) - (\Lambda \Theta, \Im(\Psi_{b,v})) + (R_1(\varepsilon), \Lambda \Sigma) + (R_2(\varepsilon), \Lambda \Theta). \tag{3.34}
\]

While for \( B_i \), \( i = 2, 3, 4, 5 \), they are defined similarly as \( B_1 \) with \( \Lambda \Sigma \) replaced by \( \partial_v \Sigma, \partial_v \Sigma, \nabla \Sigma, \Lambda^2 \Sigma \) and \( \Lambda \Theta \) replaced by \( \partial_v \Theta, \partial_v \Theta, \nabla \Theta, \Lambda^2 \Theta \) respectively.

We claim that
\[
|B_1| + |B_2| + |B_3| + |B_4| + |B_5| - (\varepsilon_1, L^5_\varepsilon(\Lambda^2 Q_\beta)) \leq \delta(\alpha_0) \left( \int |D|^2 |\varepsilon|^2 \right) + \frac{1}{2} |\varepsilon|^2 + |\varepsilon|^2 + \lambda^\beta |E_0|. \tag{3.35}
\]
Indeed, from Proposition 2.1 and the \textit{a priori} estimate (3.11), we can easily obtain that
\[
|B_1 + (M_+(\varepsilon), \Lambda \Sigma) + (M_-(\varepsilon), \Lambda \Theta) - (R_1(\varepsilon), \Lambda \Sigma) - (R_2(\varepsilon), \Lambda \Theta)| \leq \delta(\alpha_0) \left( \int |D|^2 |\varepsilon|^2 \right) + \frac{1}{2} |\varepsilon|^2 + |\varepsilon|^2 + \lambda^\beta |E_0|. \tag{3.36}
\]
where we use the Hardy’s type estimate (3.30) for the last inequality. Then we apply the following inequality:

\[
\forall \varepsilon \in \mathbb{C}, \quad |(1 + z)|1 + \varepsilon|^2 \beta - 1 - (2 \beta + 1) \Re \varepsilon - i \Im \varepsilon|^2 \leq (|\varepsilon|^2 + |\varepsilon|^2 + 1)
\]

to \( z = \frac{x}{W_{b,v}} \), using Hölder’s inequality to obtain
\[
|\varepsilon|^{2 \beta+2} + \frac{|\varepsilon|^2}{(1 + |\varepsilon|)^2} \leq \int |D|^2 |\varepsilon|^2 + \int |\varepsilon|^2 e^{-|\varepsilon|}, \tag{3.37}
\]
where we use Gagliardo-Nirenberg’s inequality and the Hardy’s type estimate (3.30) for the last inequality. We also have
\[
(M_+(\varepsilon), \Lambda \Sigma) + (M_-(\varepsilon), \Lambda \Theta) = (L^5_\varepsilon \varepsilon_1, \Lambda Q_\beta) + O\left( \delta(\alpha_0) \left( \int \frac{|\varepsilon|^2}{(1 + |\varepsilon|)^2} \right)^{\frac{1}{2}} \right) \\
= -\beta(\varepsilon_1, Q_\beta) + O\left( \delta(\alpha_0) \left( \int |\varepsilon|^2 e^{-|\varepsilon|} \right)^{\frac{1}{2}} \right). \tag{3.38}
\]
From (2.11), (3.19) and the Hardy’s type estimate (3.30), we know that
\[
|(M_+(\varepsilon), \Lambda \Sigma) + (M_-(\varepsilon), \Lambda \Theta)| \leq \delta(\alpha_0) \left( \int |D|^2 |\varepsilon|^2 \right) + \int |\varepsilon|^2 e^{-|\varepsilon|} + |\varepsilon|^2 + \lambda^\beta |E_0|. \tag{3.39}
\]

\[\text{Recall from (3.23), we have } |W_{b,v}(y)| \neq 0 \text{ for all } y \in \mathbb{R}.\]
Combining (3.36), (3.37) and (3.38), we obtain
\[ |B_1| \lesssim \delta(\alpha_0) \left( \int |D|^2 |\varepsilon|^2 + \int |\varepsilon|^2 e^{-|y|} \right)^{\frac{1}{2}} + |b|^5 + \nu^2 + \lambda^\beta |E_0|. \] (3.40)

Similarly, we have
\[ |B_2 + (M_+(\varepsilon), \partial_b \Sigma) + (M_-(\varepsilon), \partial_b \Theta)| + |B_3 + (M_+(\varepsilon), \partial_v \Sigma) + (M_-(\varepsilon), \partial_v \Theta)| \]
\[ + |B_4 + (M_+(\varepsilon), \nabla \Sigma) + (M_-(\varepsilon), \nabla \Theta)| + |B_5 + (M_+(\varepsilon), \Lambda^2 \Sigma) + (M_-(\varepsilon), \Lambda^2 \Theta)| \]
\[ \lesssim \delta(\alpha_0) \left( \int |D|^2 |\varepsilon|^2 + \int |\varepsilon|^2 e^{-|y|} \right)^{\frac{1}{2}} + |b|^5 + \nu^2. \] (3.41)

Using the same argument as (3.38), we have:
\[ \begin{align*}
(M_+(\varepsilon), \partial_b \Sigma) + (M_-(\varepsilon), \partial_b \Theta) \\
= (L_+^{\beta} \varepsilon_2, S_1) + O\left( \delta(\alpha_0) \left( \int \frac{|\varepsilon|^2}{(1 + |y|)^\beta} \right)^{\frac{1}{2}} \right), \quad (3.42)
\end{align*} \]
\[ \begin{align*}
(M_+(\varepsilon), \partial_v \Sigma) + (M_-(\varepsilon), \partial_v \Theta) \\
= (L_+^{\beta} \varepsilon_2, G_1) + O\left( \delta(\alpha_0) \left( \int \frac{|\varepsilon|^2}{(1 + |y|)^\beta} \right)^{\frac{1}{2}} \right), \quad (3.43)
\end{align*} \]
and
\[ \begin{align*}
(M_+(\varepsilon), \nabla \Sigma) + (M_-(\varepsilon), \nabla \Theta) \\
= (L_+^{\beta} \varepsilon_1, \nabla Q_\beta) + O\left( \delta(\alpha_0) \left( \int \frac{|\varepsilon|^2}{(1 + |y|)^\beta} \right)^{\frac{1}{2}} \right), \quad (3.44)
\end{align*} \]
\[ \begin{align*}
(M_+(\varepsilon), \Lambda^2 \Sigma) + (M_-(\varepsilon), \Lambda^2 \Theta) \\
= (L_+^{\beta} \varepsilon_1, \Lambda^2 Q_\beta) + O\left( \delta(\alpha_0) \left( \int \frac{|\varepsilon|^2}{(1 + |y|)^\beta} \right)^{\frac{1}{2}} \right). \quad (3.45)
\end{align*} \]

From the orthogonality condition (3.5) and (3.8) as well as the translation invariance (2.9), we have:
\[ \begin{align*}
(L_+^{\beta} \varepsilon_2, S_1) &= (\varepsilon_2, \Lambda Q_\beta) = O\left( \delta(\alpha_0) \left( \int \frac{|\varepsilon|^2}{(1 + |y|)^\beta} \right)^{\frac{1}{2}} \right), \quad (3.46)
(L_+^{\beta} \varepsilon_2, G_1) &= (\varepsilon_2, -\nabla Q_\beta) = O\left( \delta(\alpha_0) \left( \int \frac{|\varepsilon|^2}{(1 + |y|)^\beta} \right)^{\frac{1}{2}} \right), \quad (3.47)
(L_+^{\beta} \varepsilon_1, \nabla Q_\beta) &= 0. \quad (3.48)
\end{align*} \]

Injecting (3.46)–(3.48) into (3.42)–(3.45), together with (3.40) and (3.41), we conclude the proof of (3.35).
Finally, for the matrix $A$, we denote by $A_0 = A(0,0,0)$. From the proof\footnote{See Appendix B for more details.} of Proposition 3.2, we have

$$A_0 = \begin{bmatrix}
(-S_1, \Lambda Q_\beta) & 0 & 0 & 0 & 0 \\
0 & (-S_1, \Lambda Q_\beta) & 0 & 0 & 0 \\
0 & 0 & (G_1, \nabla Q_\beta) & 0 & 0 \\
0 & 0 & 0 & (G_1, \nabla Q_\beta) & 0 \\
(Q_\beta^2, S_1) & 0 & 0 & 0 & (Q_\beta, \Lambda^2 Q_\beta)
\end{bmatrix}.
$$

(3.49)

This follows from the construction of the parameters. Since they are found by using the implicit function theory, the matrix $A_0$ has to be the Jacobian matrix at $(b, \lambda, x, v, \gamma, u) = (0, 1, 0, 0, 0, Q_\beta)$. We also have

$$A(b, v, \varepsilon) = A_0 + O(|b|, |v|, \|\varepsilon\|_{H^\beta}).$$

Combining with (3.49), we have

$$\det A \sim 1.
$$

(3.50)

Injecting (3.49) and (3.50) into (3.33), using (3.35), we obtain (3.20) immediately. $\square$

4. The local viriel argument

This section is devoted to exhibit the dispersion structure of the Cauchy problem (1.1) by using the viriel identity.

Similar argument has been introduced by Merle and Raphaël [34, 35, 36, 37, 41] for the local case when $\beta = 2$, where they exhibit a local dispersive relation which can be roughly written as the following:

$$(\varepsilon_2, Q) \geq \bar{H}(\varepsilon, \varepsilon) + \lambda^2|E_0| - Ce^{-\frac{C}{\varepsilon^2}}.$$  

(4.1)

Here $\bar{H}(\varepsilon, \varepsilon)$ is some $H^1$ quadratic form of $\varepsilon$, which is positive except on a three dimensional vector space. However, all these three negative direction can be controlled by the conservation laws and modulation theory up to an exponentially small correction in $(\varepsilon_2, Q)$. Hence, the estimate (4.1) implies:

$$(\varepsilon_2, Q) \geq \delta_0 \left( \int |\varepsilon_y|^2 + \int |\varepsilon|^2e^{-|\varepsilon|} \right) - Ce^{-\frac{C}{\varepsilon^2}},$$

(4.2)

which will lead to the log-log blow-up dynamics.

While for (1.1) in the nonlocal case when $\beta < 2$ the following viriel identity

$$\frac{d}{dt}\left( 3 \int x \cdot \nabla u(t) \bar{u}(t) \right) = 2E(u_0)$$

(4.3)

still holds true, we can prove the following estimate\footnote{We mention here that the parameter $b$ plays a similar role as $(\varepsilon_2, Q)$, since it is governed by the orthogonality condition $\langle \varepsilon_1, \Theta \rangle - (\varepsilon_2, \Sigma) = 0$.}

$$b_s \geq \mu_0 \left( \int |D|^2 |\varepsilon|^2 + \int |\varepsilon|^2e^{-|\varepsilon|} \right) - C|b|^{10}.$$  

(4.4)
Here the error term $|b|^{10}$ comes from the estimate (2.3) for the self-similar equation\footnote{We mention here that if the estimate of the error term $\Psi_{b,v}$ can be improved to $e^{-C/|b|}$, we can improve the upper bound on the blow-up rate to “log-log” rate as the numerical simulation suggested.}, which will lead to the upper bound on the blow-up rate introduced in Theorem 1.6.

4.1. The viriel identity in $\varepsilon$ variable. In this section, we will derive the viriel identity in $\varepsilon$ variable. First, we denote by

$$\Phi(u)(t) = \Im \int x \cdot \nabla u(t) \bar{u}(t), \quad \Phi(\varepsilon)(t) = \Im \int y \varepsilon y(s) \bar{\varepsilon}(s).$$

We note that

$$\Phi(u)(t) = \Im \left( \int y(W_{b,v} + \varepsilon)(W_{b,v} + \varepsilon) \right) (s)$$

The viriel identity (4.3) implies

$$\Phi(\varepsilon)(s) - 2(\varepsilon, \Lambda \Sigma) + 2(\varepsilon, \Theta) + \Phi(W_{b,v}) = -2|E_0|t + C(u_0).$$

From the orthogonality condition (3.5), we know that

$$(\varepsilon, \Lambda \Sigma) - (\varepsilon, \Theta) = 0.$$  

Moreover, from the construction\footnote{See Remark 2.4 for more details.} of $W_{b,v}$, we obtain

$$\Phi(W_{b,v}) = -bC_{b,v},$$

where $C_{b,v} = 2(L^2S_1, S_1) + O(|v| + |b|) > 0$.

Combining all the above, we get

$$[\Phi(\varepsilon)](s) = (bC_{b,v})_s - 2\lambda^2(\varepsilon)|E_0| \sim C_{b,v}b_s - 2\lambda^2(s)|E_0|. \quad (4.6)$$

We will see that in the blow-up region, the term $\lambda^2|E_0|$ is a relatively “small” term when $t$ is close to the blow-up time. Hence, we expect that the viriel type relation in $\varepsilon$ on the term $\Phi(\varepsilon)$ is formally the same as the parameter $b$.

According to the above argument, we are led to compute $b_s$. From Proposition 3.4 and the orthogonality condition (3.5)–(3.9), we have the following lemma:

Lemma 4.1 (Viriel identity in $\varepsilon$). Under the assumption of Theorem 1.6, we have for all $s \in [0, +\infty)$

$$b_s \left[ (\partial_b \Theta, \Lambda \Sigma) - (\partial_b \Sigma, \Lambda \Theta) - (\varepsilon, \Lambda(\partial_b \Theta)) \right]$$

$$= - (v_s + bv) \left[ (\partial_b \Theta, \Lambda \Sigma) - (\partial_b \Sigma, \Lambda \Theta) - (\varepsilon, \Lambda(\partial_b \Theta)) \right]$$

$$- \left( \frac{x_s}{\lambda} - v \right) \left\{ (\Theta, (\Lambda \Sigma)_y) - (\Sigma, (\Lambda \Theta)_y) + (\varepsilon, \Lambda(\partial_b \Theta)) \right\}$$

$$- \left( \frac{\lambda s}{\lambda} + b \right) \left\{ (\varepsilon, \Lambda^2 \Sigma) - (\varepsilon, \Lambda^2 \Theta) \right\} - \tilde{\gamma}_s \left\{ (\varepsilon, \Lambda \Sigma) + (\varepsilon, \Lambda \Theta) \right\}$$

$$- v \left\{ (\Theta, (\Lambda \Sigma)_y) - (\Sigma, (\Lambda \Theta)_y) \right\} - b v \left\{ (\Theta, \Lambda(\partial_b \Sigma)) - (\Sigma, \Lambda(\partial_b \Theta)) \right\}$$

$$+ \beta \lambda^2|E_0| + H_{b,v}(\varepsilon, \varepsilon) - (\varepsilon, \mathfrak{H}(\Lambda \Psi_{b,v})) - (\varepsilon, \Im(\Lambda \Psi_{b,v})) + G(\varepsilon). \quad (4.7)$$
Here the nonlinear term $G(\varepsilon)$ is given by

$$G(\varepsilon) = -\frac{1}{1 + \beta} \int F(\varepsilon) + (\tilde{R}_1(\varepsilon), \Lambda \Sigma) + (\tilde{R}_2(\varepsilon), \Lambda \Theta)$$

(4.8)

with

$$\tilde{R}_1(\varepsilon) = R_1(\varepsilon) - \varepsilon_1^2 |W_{b,v}|^{(2\beta - 4) \{ \beta(2 + 1)\Sigma^2 + 3\beta \Sigma \Theta^2 \}} - \varepsilon_2^2 |W_{b,v}|^{(2\beta - 4) \{ \beta \Sigma^2 + \beta(2 - 1) \Sigma \Theta^2 \}} - 2\beta \varepsilon_1 \varepsilon_2 |W_{b,v}|^{(2\beta - 4) \{ (2 - 1) \Sigma \Theta + \Theta^3 \}},$$

(4.9)

and

$$\tilde{R}_2(\varepsilon) = R_2(\varepsilon) - \varepsilon_1^2 |W_{b,v}|^{(2\beta - 4) \{ \beta(2 + 1) \Theta^2 + 3\beta \Sigma^2 \Theta \}} - \varepsilon_2^2 |W_{b,v}|^{(2\beta - 4) \{ \beta \Theta^3 + \beta(2 - 1) \Sigma \Theta^2 \}} - 2\beta \varepsilon_1 \varepsilon_2 |W_{b,v}|^{(2\beta - 4) \{ (2 - 1) \Sigma \Theta^2 + \Theta^3 \}},$$

(4.10)

and the quadratic form $H_{b,v}(\varepsilon, \varepsilon)$ can be written explicitly in the following form:

$$H_{b,v}(\varepsilon) = \frac{\beta}{2} H^\beta(\varepsilon, \varepsilon) + \tilde{H}_{b,v}(\varepsilon, \varepsilon)$$

(4.11)

with $H^\beta(\varepsilon, \varepsilon) = (L_1^\beta \varepsilon_1, \varepsilon_1) + (L_2 \varepsilon_2, \varepsilon_2)$,

$$L_1^\beta = |D|^\beta + 2(2 + 1) y Q^\beta_\beta Q^{\beta - 1}_\beta, \quad L_2^\beta = |D|^\beta + 2y Q^\beta_\beta Q^{\beta - 1}_\beta,$$

(4.12)

and

$$|\tilde{H}_{b,v}(\varepsilon, \varepsilon)| \lesssim \delta(\alpha_0) \left( \int ||D|^\frac{2\beta}{2} \varepsilon|^2 + \int |\varepsilon|^2 e^{-|y|} \right).$$

(4.13)

**Proof.** The proof of Lemma 4.1 is mostly algebraic computation. The following three things are crucial for the proof:

- The scaling invariance (2.7).
- The choice of orthogonality condition (3.5).
- The energy conservation law (3.21).

Now, we turn to the proof of (4.7). From (3.32), we have

$$b_s \left[ (\partial_\Theta, \Lambda \Sigma) - (\partial_\Sigma, \Lambda \Theta) - (\varepsilon_2, \Lambda (\partial_\Sigma) + (\varepsilon_1, \Lambda (\partial_\Theta)) \right]$$

$$= -(v_s + bv) \left[ (\partial_\Theta, \Lambda \Sigma) - (\partial_\Sigma, \Lambda \Theta) - (\varepsilon_2, \Lambda (\partial_\Sigma) + (\varepsilon_1, \Lambda (\partial_\Theta)) \right]$$

$$- \left( \frac{x_s}{\lambda} - v \right) \left[ \left( \Theta, (\Lambda \Sigma)_y \right) - \left( \Sigma, (\Lambda \Theta)_y \right) + (\varepsilon_2, (\Lambda \Sigma)_y) - (\varepsilon_1, (\Lambda \Theta)_y) \right]$$

$$- \left( \frac{\lambda}{\lambda} + b \right) \left[ (\varepsilon_2, \Lambda^2 \Sigma) - (\varepsilon_1, \Lambda^2 \Theta) \right] - \gamma_s \left( (\varepsilon_1, \Lambda \Sigma) + (\varepsilon_2, \Lambda \Theta) \right)$$

$$- bv \left( (\varepsilon_2, \Lambda (\partial_\Sigma)) - (\varepsilon_1, \Lambda (\partial_\Theta)) \right) - (M_+ (\varepsilon), \Lambda \Sigma) - (M_- (\varepsilon), \Lambda \Theta)$$

$$+ b (\varepsilon_2, \Lambda^2 \Sigma) - (\varepsilon_1, \Lambda^2 \Theta) \right) - v \left( (\varepsilon_2, \nabla \Lambda \Sigma) - (\varepsilon_1, \nabla \Lambda \Theta) \right)$$

$$- (\Lambda \Sigma, \Re (\Psi_{b,v}))) - (\Lambda \Theta, \Im (\Psi_{b,v})) + (R_1 (\varepsilon), \Lambda \Sigma) + (R_2 (\varepsilon), \Lambda \Theta).$$

(4.14)

---

15Recall that $R_1(\varepsilon)$ and $R_2(\varepsilon)$ are defined by (3.17) and (3.18), while $F(\varepsilon)$ is given by (3.22).
Then we use (2.7) to compute:

\[(M_+ (\varepsilon), \Lambda \Sigma) + (M_- (\varepsilon), \Lambda \Theta)\]

\[= (|D|^\beta \varepsilon_1 + \varepsilon_1 - |W_{b,v}|^{2\beta} \varepsilon_1 - 2\beta |W_{b,v}|^{2\beta-2} (\Sigma^2 \varepsilon_1 + \Theta \varepsilon_2), \Lambda \Sigma)\]

\[+ (|D|^\beta \varepsilon_2 + \varepsilon_2 - |W_{b,v}|^{2\beta} \varepsilon_2 - 2\beta |W_{b,v}|^{2\beta-2} (\Theta^2 \varepsilon_2 + \Sigma \varepsilon_1), \Lambda \Theta)\]

\[= (\varepsilon_1, |D|((\Lambda \Sigma) + \Lambda \Sigma - |W_{b,v}|^{2\beta} \Lambda \Sigma - 2\beta \Sigma |W_{b,v}|^{(2\beta-2)} (\Sigma \Lambda \Sigma + \Theta \Lambda \Theta))\]

\[+ (\varepsilon_2, |D|((\Lambda \Theta) + \Lambda \Theta - |W_{b,v}|^{2\beta} \Lambda \Theta - 2\beta \Theta |W_{b,v}|^{(2\beta-2)} (\Sigma \Lambda \Sigma + \Theta \Lambda \Theta))\]

\[= - \beta \left[ (\varepsilon_1, \Sigma - b \Lambda \Theta + v \nabla \Theta - \Re(\Psi_{b,v})) + (\varepsilon_2, \Theta + b \Lambda \Sigma - v \nabla \Sigma - \Im(\Psi_{b,v})) \right]\]

\[+ b \left[ (\varepsilon_2, \Lambda^2 \Sigma) - (\varepsilon_1, \Lambda^2 \Theta) \right] - v \left[ (\varepsilon_2, \Lambda \nabla \Sigma) - (\varepsilon_1, \Lambda \nabla \Theta) \right]\]

\[= (\varepsilon_2, \Lambda \nabla \Sigma) - (\varepsilon_1, \Lambda \nabla \Theta).\]

Then we use the following identity\(^\text{16}\)

\[(\varepsilon_2, \Lambda \nabla \Sigma) - (\varepsilon_1, \Lambda \nabla \Theta)\]

\[= (\varepsilon_2, \Lambda \nabla \Sigma) - (\varepsilon_1, \nabla \Lambda \Theta) - (\varepsilon_2, \nabla \Lambda \Sigma) + (\varepsilon_1, [\nabla, \Lambda] \Theta)\]

\[= (\varepsilon_2, \Lambda \nabla \Sigma) - (\varepsilon_1, \nabla \Lambda \Theta) - (\varepsilon_2, \nabla \Sigma) + (\varepsilon_1, \nabla \Theta)\]

\[= (\varepsilon_2, \Lambda \nabla \Sigma) - (\varepsilon_1, \nabla \Lambda \Theta).\]

Combining (3.22), (4.14) and (4.15) we obtain

\[b_s \left[ (\partial_\theta \Theta, \Lambda \Sigma) - (\partial_\Sigma \Lambda \Theta) - (\varepsilon_2, \Lambda (\partial_\theta \Sigma)) + (\varepsilon_1, \Lambda (\partial_\theta \Theta)) \right]\]

\[= -(v_s + bv) \left[ (\partial_\theta \Theta, \Lambda \Sigma) - (\partial_\Sigma \Lambda \Theta) - (\varepsilon_2, \Lambda (\partial_\theta \Sigma)) + (\varepsilon_1, \Lambda (\partial_\theta \Theta)) \right]\]

\[- \left( \frac{x_s}{\lambda} - v \right) \left\{ (\Theta, (\Lambda \Sigma)_y) - (\Sigma, (\Lambda \Theta)_y) + (\varepsilon_2, (\Lambda \Sigma)_y) - (\varepsilon_1, (\Lambda \Theta)_y) \right\}\]

\[- \left( \frac{x_s}{\lambda} + b \right) \left\{ (\varepsilon_2, \Lambda^2 \Sigma) - (\varepsilon_1, \Lambda^2 \Theta) \right\} - \tilde{\gamma}_s \left\{ (\varepsilon_2, \Lambda \Sigma) + (\varepsilon_2, \Lambda \Theta) \right\}\]

\[+ \beta \lambda^2 |E_0| + H_{b,v}(\varepsilon, \varepsilon) - (\varepsilon_1, \Re(\Lambda \Psi_{b,v})) - (\varepsilon_2, \Im(\Lambda \Psi_{b,v}))) + G(\varepsilon),\]

\[+ \beta E(W_{b,v}) - (\Lambda \Sigma, \Re(\Psi_{b,v})) - (\Lambda \Theta, \Im(\Psi_{b,v})).\]

\(^\text{16}\)Here we the commutator relation $[\nabla, \Lambda] = \nabla$ and the orthogonality condition (3.8) for this equality.
where

\[
H_{b,v}(\varepsilon, \varepsilon) = \frac{\beta}{2} \left\{ \int ||D|^2 \varepsilon||^2 - \int (||W_{b,v}|^{2\beta} + 2\beta \Sigma^2 ||W_{b,v}|^{2\beta-2}) \varepsilon_1^2 \right. \\
- \int (||W_{b,v}|^{2\beta} + 2\beta \Theta^2 ||W_{b,v}|^{2\beta-2}) \varepsilon_2^2 - 4\beta \int \Sigma \Theta |W_{b,v}|^{2\beta-2} \varepsilon_1 \varepsilon_2 \left. \right\} \\
+ \int \Lambda \Sigma \left\{ \varepsilon_1^2 ||W_{b,v}|^{2\beta-4} [\beta(2\beta + 1) \Sigma^3 + 3\beta \Sigma \Theta^2] \\
+ \varepsilon_2^2 ||W_{b,v}|^{2\beta-4} [\beta \Sigma^3 + \beta(2\beta - 1) \Sigma \Theta^2] \\
+ 2\beta \varepsilon_1 \varepsilon_2 |W_{b,v}|^{2\beta-4} [(2\beta - 1) \Sigma \Theta + \Theta^3] \right\} \\
+ \int \Lambda \Theta \left\{ \varepsilon_1^2 ||W_{b,v}|^{2\beta-4} [\beta(2\beta + 1) \Theta^3 + 3\beta \Sigma^2 \Theta] \\
+ \varepsilon_2^2 ||W_{b,v}|^{2\beta-4} [\beta \Theta^3 + \beta(2\beta - 1) \Sigma \Theta^2] \\
+ 2\beta \varepsilon_1 \varepsilon_2 |W_{b,v}|^{2\beta-4} [(2\beta - 1) \Sigma \Theta^2 + \Theta^3] \right\},
\]

(4.17)

From the construction of \(W_{b,v}\), we know that

\[
\Sigma(y) = Q_\beta(y) + O(|b| + |v|)(1 + |y|)^{-1-\beta} \\
\Theta(y) = O(|b| + |v|)(1 + |y|)^{-1-\beta}.
\]

Hence, we can rewrite the quadratic form \(H_{b,v}(\varepsilon, \varepsilon)\) as following

\[
H_{b,v}(\varepsilon, \varepsilon) = \frac{\beta}{2} \{ (\mathcal{L}_1 \varepsilon_1, \varepsilon_1) + (\mathcal{L}_2 \varepsilon_2, \varepsilon_2) \} + \mathcal{H}_{b,v}(\varepsilon, \varepsilon)
\]

(4.18)

where

\[
\mathcal{L}_1^\beta = |D|^\beta + 2(2\beta + 1)yQ_\beta Q_\beta^{-1}, \quad \mathcal{L}_2^\beta = |D|^\beta + 2yQ_\beta Q_\beta^{-1},
\]

(4.19)

and

\[
|\mathcal{H}_{b,v}(\varepsilon, \varepsilon)| \lesssim (|b| + |v|) \left( \int ||D|^2 \varepsilon||^2 \right) + \int |\varepsilon|^2 (1 + |y|)^{1-\beta} \right) \\
\lesssim \delta(\alpha) \left( \int ||D|^2 \varepsilon||^2 \right) + \int |\varepsilon|^2 |y| \right).
\]

(4.20)

Here we use the \textit{a priori} estimate (3.11) and the Hardy’s estimate (3.30) for the last inequality.

Finally, we need to deal with the term \(E(W_{b,v})\). From (2.41), we have

\[
\beta E(W_{b,v}) = (\mathcal{R}(\Psi_{b,v}), \Lambda \Sigma) + (\mathcal{R}(\Psi_{b,v}), \Lambda \Theta) - v \{ (\nabla \Sigma, \Lambda \Theta) - (\nabla \Theta, \Lambda \Sigma) \} \\
- b v \{ (\Lambda \Sigma, \partial_v \Theta) - (\Lambda \Theta, \partial_v \Sigma) \} \\
= (\mathcal{R}(\Psi_{b,v}), \Lambda \Sigma) + (\mathcal{R}(\Psi_{b,v}), \Lambda \Theta) - v \{ (\Theta, (\Lambda \Sigma)_y) - (\Sigma, (\Lambda \Theta)_y) \} \\
- b v \{ (\Theta, \Lambda (\partial_v \Sigma)) - (\Sigma, \Lambda (\partial_v \Theta)) \}.
\]

(4.21)

Combining (4.16)-(4.21) and the energy condition \(E_0 < 0\), we conclude the proof of (4.7).
4.2. Spectral properties. This subsection is devoted to show that the quadratic form $H^2(\varepsilon, \varepsilon)$ is positive except on a dimension three subspace provided that $\beta$ is close to 2. Recall that we say the spectral property holds true for $1 \leq \beta \leq 2$, if there exists a universal constant $\delta > 0$ such that

$$H^2(\varepsilon, \varepsilon) \geq \delta \left( \int |D[1/2]_\varepsilon|^2 + \int |\varepsilon|^2 e^{-|y|} \right),$$

(4.22)

for all $\varepsilon \in H^2(\mathbb{R})$ with $(\varepsilon_1, Q_\beta) = (\varepsilon_1, G_1) = (\varepsilon_2, \Lambda Q_\beta) = (\varepsilon_2, \Lambda^2 Q_\beta) = 0$.

We mention here that for general $\beta \in [1, 2]$ it is still not known whether the spectral property holds true. But for the local case when $\beta = 2$, the spectral property has been proved by Merle and Raphaël [36] with the help of some numeric tools. As mentioned before the ground state $Q_\beta$ is continuous with respect to $\beta$ up to $\beta = 2$. More precisely, we have

**Lemma 4.2** (Continuity of $Q_\beta$ with respect to $\beta$). We denote by $Q = Q_2$ for the ground state in the local case when $\beta = 2$, and

$$L_+ = -\Delta + 1 - 5Q^4, \quad L_- = -\Delta + 1 - Q^4$$

the linearized operator at $Q$. Then we have

(1) **Continuity of $Q_\beta$:**

$$Q_\beta \to Q, \text{ in } H^1,$$

as $\beta \to 2^-$.

(2) **Uniform boundedness from the above:** there exist a constant $C$ independent of $\beta$ such that

$$|Q_\beta(y)| \leq \frac{C}{(1 + |y|)^{1+\varepsilon}}, \quad |Q(y)| \leq Ce^{-\frac{|y|}{2}}.$$ 

(4.24)

(3) **Convergence of $L^2_\beta$ in the norm-resolvent sense:**

$$\left\| \frac{1}{L^2_\pm + z} \right\|_{L^2 \to L^2} \to 0,$$

(4.25)

as $\beta \to 2^-$ for all $z \in \mathbb{C}$ with $\Im z \neq 0$.

Hence, we may use a perturbation argument to show that the spectral property (1.19) also holds true for $\beta$ close to 2.

4.2.1. Notations. We start with some basic notations. For all $\varepsilon = \varepsilon_1 + i\varepsilon_2 \in H^2$, we denote by

$$\mathcal{H}^2_1(\varepsilon_1, \varepsilon_1) = \int |D[1/2]_{\varepsilon_1}|^2 + \frac{10}{9} \left( 2(2\beta + 1) \int y\nabla Q_\beta Q_\beta^{2\beta - 1} \varepsilon_1^2 - \frac{1}{10} \int \frac{\varepsilon_1^2}{\cosh^2(\frac{10}{9}y)} \right),$$

$$\mathcal{H}^2_2(\varepsilon_2, \varepsilon_2) = \int |D[1/2]_{\varepsilon_2}|^2 + \frac{10}{9} \left( 2 \int y\nabla Q_\beta Q_\beta^{2\beta - 1} \varepsilon_2^2 - \frac{1}{10} \int \frac{\varepsilon_2^2}{\cosh^2(\frac{10}{9}y)} \right),$$

and

$$\mathcal{L}^2_1 = |D|^2 + \frac{20(2\beta + 1)}{9} y\nabla Q_\beta Q_\beta^{2\beta - 1} - \frac{1}{9 \cosh^2(\frac{10}{9}y)},$$

$$\mathcal{L}^2_2 = |D|^2 + \frac{20}{9} y\nabla Q_\beta Q_\beta^{2\beta - 1} - \frac{1}{9 \cosh^2(\frac{10}{9}y)}.$$
Hence, we have
\[ \overline{\mathcal{H}}_1^\beta(\varepsilon_1,\varepsilon_1) = (\mathcal{L}_1^\beta \varepsilon_1,\varepsilon_1), \quad \overline{\mathcal{H}}_2^\beta(\varepsilon_2,\varepsilon_2) = (\mathcal{L}_2^\beta \varepsilon_2,\varepsilon_2) \]
and
\[ H^\beta(\varepsilon,\varepsilon) = \frac{1}{10} \left( \int ||D|^{\frac{3}{2}}|\varepsilon|^2 + \int \frac{|\varepsilon|^2}{\cosh^2(\frac{10}{9}y)} \right) + \frac{9}{10} \left( \overline{\mathcal{H}}_1^\beta(\varepsilon_1,\varepsilon_1) + \overline{\mathcal{H}}_2^\beta(\varepsilon_2,\varepsilon_2) \right) \] (4.26)
for all \( \varepsilon = \varepsilon_1 + i\varepsilon_2 \in H^\beta_1 \).

For simplicity, we denote by
\[ H(\varepsilon,\varepsilon) = (\mathcal{L}_1^\beta \varepsilon_1,\varepsilon_1) + (\mathcal{L}_2^\beta \varepsilon_2,\varepsilon_2), \quad \forall \varepsilon = \varepsilon_1 + i\varepsilon_2 \in H^1 \]
with \( \mathcal{L}_1 = -\Delta + 10yQ'Q^3 \), \( \mathcal{L}_2 = -\Delta + 2yQ^2Q^{2\beta-1} \).

We also denote by
\[ \overline{H}_1(\varepsilon_1,\varepsilon_1) = \int |\nabla \varepsilon_1|^2 + \frac{10}{9} \left( 10 \int y\nabla QQ^3 \varepsilon_1^2 - \frac{1}{10} \int \varepsilon_1^2 \cosh^2(\frac{10}{9}y) \right) \]
\[ \overline{H}_2(\varepsilon_2,\varepsilon_2) = \int |\nabla \varepsilon_2|^2 + \frac{10}{9} \left( 2 \int y\nabla QQ^3 \varepsilon_2^2 - \frac{1}{10} \int \varepsilon_2^2 \cosh^2(\frac{10}{9}y) \right) \]
and
\[ \overline{L}_1 = -\Delta + \frac{100}{9} yQQ^3 - \frac{1}{9 \cosh^2(\frac{10}{9}y)} \]
\[ \overline{L}_2 = -\Delta + \frac{20}{9} yQQ^3 - \frac{1}{9 \cosh^2(\frac{10}{9}y)} \]
for all \( \varepsilon = \varepsilon_1 + i\varepsilon_2 \in H^1_1(\mathbb{R}) \).

Finally, we introduce the index of a bilinear form on a vector space \( V \):
\[ \text{ind}_V(B) := \max\{ k \in \mathbb{N} | \text{there exists a subspace } P \text{ of codimension } k \text{ such that } B|_P \text{ is positive} \} \].

Let \( H^1 \) (respectively \( H^1_1 \)) be the subspace of all even (respectively odd) \( H^1 \) functions. Assume that \( H^1 \) is \( B \)-orthogonal to \( H^1_1 \). We say that \( B \) defined on \( H^1 \) has index \( k + j \), if \( \text{ind}_{H^1}(B) = k \) and \( \text{ind}_{H^1_1}(B) = j \).

4.2.2. Proof of the spectral property for \( \beta \) close to 2. First, we claim that the following almost coercivity holds true for \( \overline{\mathcal{H}}_1^\beta \) and \( \overline{\mathcal{H}}_2^\beta \):

**Lemma 4.3.** For all \( \kappa > 0 \), there exists \( \beta_\kappa < 2 \) such that if \( \beta_\kappa < \beta < 2 \), then for all \( \varepsilon = \varepsilon_1 + i\varepsilon_2 \in H^\beta_1 \) with \( (\varepsilon_1, G_1) = (\varepsilon_1, Q_\beta) = 0 \) and \( (\varepsilon_2, \Lambda Q_\beta + \frac{1}{2} \Lambda^2 Q_\beta) = 0 \), there holds
\[ \overline{\mathcal{H}}_1^\beta(\varepsilon_1,\varepsilon_1) + \overline{\mathcal{H}}_2^\beta(\varepsilon_2,\varepsilon_2) \geq -\kappa \left( \int ||D|^{\frac{3}{2}}|\varepsilon|^2 + \int |\varepsilon|^2 e^{-|\varepsilon|} \right) \] (4.27)

We may easily see that (4.26) and (4.27) imply the spectral property (1.19) immediately, when \( \beta \) is close enough to 2. On the other hand, by a standard density argument, we only need to show that (4.27) holds true for \( \varepsilon = \varepsilon_1 + i\varepsilon_2 \in H^1(\mathbb{R}) \).
Proof of Lemma 4.3. First, we claim that $\Pi_1^\beta$ has index $1 + 1$ and $\Pi_2^\beta$ has index $1 + 0$, if $\beta$ is close enough to $2$.

From [36, Lemma 10], we know that $\Pi_1$ has index $1 + 1$, while $\Pi_2$ has index $1 + 0$. Since we have $Q_\beta \to Q$ in $H^1$ as $\beta \to 2^-$. For $\beta$ close enough to $2$, we know that the number of the eigenvalues for $\Pi_1^\beta$ ($\Pi_2^\beta$ respectively) is the same as $\Pi_1$ ($\Pi_2$ respectively). Thus $\Pi_1^\beta$ must have index $1 + 1$ while $\Pi_2^\beta$ has index $1 + 0$, provided that $\beta$ is close enough to $2$.

Next, we state some numerical result obtained in [36, Lemma 11].

**Lemma 4.4** (Numerical estimates). For the operator $\Pi_1$ and $\Pi_2$, we have

1. There exists a unique even function $\phi_1 \in L^\infty \cap \hat{H}^1$ such that $\Pi_1 \phi_1 = Q$ and
   $$- \langle \phi_1, Q \rangle \left( 1 - \Pi_1(Q, Q) \frac{\langle \phi_1, Q \rangle}{(Q, Q)^2} \right) > 0. \quad (4.28)$$
2. There exists a unique odd function $\phi_2 \in L^\infty \cap \hat{H}^1$ such that $\Pi_1 \phi_2 = yQ$ and
   $$- \langle \phi_2, yQ \rangle \left( 1 - \Pi_1(Q, Q) \frac{\langle \phi_2, yQ \rangle}{(Q, Q)^2} \right) > 0. \quad (4.29)$$
3. Let $\tilde{Q} = \Lambda Q + \frac{1}{9} \Lambda^2 Q$, then there exists a unique odd function $\phi_3 \in L^\infty \cap \hat{H}^1$ such that $\Pi_2 \phi_3 = \tilde{Q}$ and
   $$- \langle \phi_3, \tilde{Q} \rangle \left( 1 - \Pi_2(Q, Q) \frac{\langle \phi_3, \tilde{Q} \rangle}{(Q, Q)^2} \right) > 0. \quad (4.30)$$

**Remark 4.5.** The estimates $(4.28)$–$(4.30)$ are verified by numerical methods.

**Remark 4.6.** As mentioned in [36], these three functions $\phi_j$ are not given by the Lax-Milgram theory, hence may not be in $L^2$.

Now, we turn back to the proof of $(4.27)$. Let $\chi$ be a smooth cut-off function such that $\chi(y) = 1$ if $|y| < 1$, $\chi(y) = 0$ if $|y| > 2$. We denote by
$$\langle \phi_1 \rangle_A(y) = \phi_1(y) \chi(y/A)$$
for some $A > 1$ to be chosen later. One may easily show that
$$||\nabla(\phi_1)\rangle_A||_{L^2} \leq C$$
for some constant $C$ independent of $A$. We also have
$$\Delta \phi_1 = \frac{100}{9} y \nabla Q \nabla \phi_1 - \frac{1}{9 \cosh^2\left(\frac{Q}{3y}\right)} \phi_1 - Q \in L^2,$$
which implies that $||\Delta(\phi_1)\rangle_A||_{L^2} \leq C$ for some constant $C$ independent of $A$. Hence we have for all $s \in [1, 2]$, there holds
$$||D^s(\phi_1)\rangle_A||_{L^2} \leq C, \text{ for some constant } C \text{ independent of } A, \quad (4.31)$$
$$||D^s((\phi_1)\rangle_A - \phi_1)||_{L^2} \to 0, \text{ as } A \to +\infty, \text{ uniformly in } s. \quad (4.32)$$

We first claim that for all $\kappa > 0$, there exists $\beta_\kappa < 2$ such that if $\beta_\kappa < \beta < 2$, then for all real valued function $\varepsilon_1 \in H^1_\varepsilon$ with $\langle \varepsilon_1, Q_\beta \rangle = 0$, we have
$$\Pi_1^\beta(\varepsilon_1, \varepsilon_1) \geq -\kappa \left( \int ||D\tilde{\varepsilon_1}|^2 + \int |\varepsilon_1|^2 e^{-|y|} \right). \quad (4.33)$$

The proof of $(4.33)$ can be divided into the following steps:
(1) We consider the subspace $(P_1)_A \subset H^1_\ell$ spanned by $Q$ and $(\phi_1)_A$. We show that for $A$ large enough, there exists a $\beta(A) < 2$ such that if $\beta \in (\beta(A), 2)$, then $\overline{H}_1^{\phi}$ restricted to $(P_1)_A$ is not degenerate.

(2) Let $(P_1)_A^\perp$ be the orthogonal of $(P_1)_A$ in $H^1_\ell$ with respect to the quadratic form $\overline{H}_1^{\phi}$. Then by an index argument, we can show that $\overline{H}_1^{\phi}$ is nonnegative on $(P_1)_A^\perp$ under the same assumption of the previous step.

(3) For all $\varepsilon_1 \in (P_1)_A$ with $(\varepsilon_1, Q_\beta) = 0$, we show that $\overline{H}_1^{\phi}(\varepsilon_1, \varepsilon_1) > 0$ provided that $A \geq A_0$ large enough and $\beta > \beta(A)$.

(4) Using the fact that $Q_\beta$ is continuous in $H^1$ with respect to $\beta$, we are able to prove (4.33).

Now, let $(P_1)_A = \text{span}\{Q_\beta, (\phi_1)_A\}$. We need to show that if $A$ is large enough and $2 > \beta > \beta(A)$ for some constant $\beta(A) < 2$, then we have

$$\det \begin{bmatrix} \overline{H}_1^{\phi}(Q_\beta, Q_\beta) & \overline{H}_1^{\phi}(Q_\beta, (\phi_1)_A) \\ \overline{H}_1^{\phi}(Q_\beta, (\phi_1)_A) & \overline{H}_1^{\phi}((\phi_1)_A, (\phi_1)_A) \end{bmatrix} \neq 0 \quad (4.34)$$

Indeed, from the fact that $Q_\beta$ is continuous in $H^1$ with respect to $\beta$, we have

$$\overline{H}_1^{\phi}(Q_\beta, Q_\beta) = \overline{H}_1(Q, Q) + \delta([2 - \beta]). \quad (4.35)$$

Since, $(\phi_1)_A \in H^1$, we have $\overline{H}_1^{\phi}(\phi_1)_A \rightarrow \overline{H}_1(\phi_1)_A$ in $L^2$, as $\beta \rightarrow 2^-$. But this convergence may not be uniform with respect to $A$, since we do not know whether $\phi_1 \in H^\perp_\ell$ for $\beta < 2$. However, we still have for all fixed $A$ and $\kappa_0 > 0$, there exits a $\beta(A, \kappa_0) < 2$ such that if $2 > \beta > \beta(A, \kappa_0)$, then

$$\overline{H}_1^{\phi}(Q_\beta, (\phi_1)_A) = \overline{H}_1(Q, (\phi_1)_A) + O(\kappa_0), \quad (4.36)$$

$$\overline{H}_1^{\phi}((\phi_1)_A, (\phi_1)_A) = \overline{H}_1((\phi_1)_A, (\phi_1)_A) + O(\kappa_0). \quad (4.37)$$

On the other hand, we have

$$\det \begin{bmatrix} \overline{H}_1(Q, Q) & \overline{H}_1(Q, (\phi_1)_A) \\ \overline{H}_1(Q, (\phi_1)_A) & \overline{H}_1((\phi_1)_A, (\phi_1)_A) \end{bmatrix}$$

$$= -(Q, Q)^2 \left(1 - \overline{H}_1(Q, Q)\left(\frac{(\phi_1)_A, Q}{(Q, Q)^2}\right) + O\left(\frac{1}{A}\right)\right) \neq 0, \quad (4.38)$$

provided that $A \geq A_0$ is large enough. Combining (4.35)-(4.38), we obtain (4.34).

Then, it follows that $H^1_\ell = (P_1)_A \oplus (P_1)_A^\perp$ where $(P_1)_A^\perp$ is the orthogonal of $(P_1)_A$ in $H^1_\ell$ with respect to the quadratic form $\overline{H}_1^{\phi}$ in the sense that for all $f \in (P_1)_A$ and $g \in (P_1)^\perp_\ell$, we have $\overline{H}_1^{\phi}(f, g) = 0$. Since the index of $\overline{H}_1^{\phi}$ on $H^1_\ell$ is 1, and

$$\overline{H}_1^{\phi}(Q_\beta, Q_\beta) = \overline{H}_1(Q, Q) + \delta([2 - \beta]) < 0,$$

we conclude that $\overline{H}_1^{\phi}$ is nonnegative on $(P_1)_A^\perp$ for $A$ large and $\beta \in (\beta(A), 2)$.

Next, let $A \geq A_0$ large enough, and $\beta \in (\beta(A), 2)$ such that the above statement holds true. For all real valued function $\varepsilon_1 \in (P_1)_A$ with $(\varepsilon_1, Q_\beta) = 0$, we have $\overline{H}_1^{\phi}(\varepsilon_1, \varepsilon_1) > 0$. Indeed, let $\varepsilon_1 = \alpha Q_\beta + \gamma (\phi_1)_A$. Since $(\varepsilon_1, Q_\beta) = 0$, one has $\gamma \neq 0$.

\textsuperscript{17}Here we use the fact that $\overline{H}_1(Q, Q) < 0$, which was proved in [36, Appendix A].
Indeed, let $\alpha = \frac{((\phi_1),Q_\beta)}{(Q_\beta,\kappa_\beta)}$. Then we have:

$$\frac{\overline{H}_1^\beta(\varepsilon,\varepsilon)(\varepsilon_1,\varepsilon_1)}{\gamma^2} = \frac{\alpha^2}{\gamma^2} \overline{H}_1^\beta(Q_\beta,\varepsilon_1,\varepsilon_1) + 2\left(\frac{\alpha}{\gamma}\right) \overline{H}_1(Q_\beta,\varepsilon_1,\varepsilon_1) + \overline{H}_1^\beta((\phi_1),\varepsilon_1,\varepsilon_1)$$

$$= \overline{H}_1^\beta(Q_\beta,\varepsilon_1,\varepsilon_1)\frac{((\phi_1),Q_\beta)(\varepsilon_1,\varepsilon_1)^2}{(Q_\beta,\varepsilon_1,\varepsilon_1)^2} - 2\left(\frac{((\phi_1),Q_\beta)(\varepsilon_1,\varepsilon_1)}{(Q_\beta,\varepsilon_1,\varepsilon_1)}\right) + \left(\frac{((\phi_1),\varepsilon_1,\varepsilon_1)}{(Q_\beta,\varepsilon_1,\varepsilon_1)}\right)$$

By similar argument as we used for (4.36) and (4.37), we have for all $\kappa_0 > 0$ small enough and $A \geq A_0$ large enough, there exists $\beta(A,\kappa_0) < 2$ such that if $2 > \beta > \beta(A,\kappa_0)$, then

$$\frac{\overline{H}_1^\beta(\varepsilon_1,\varepsilon_1)}{\gamma^2} = -(\phi_1, Q)\left(1 - \overline{H}_1(Q_\beta,\varepsilon_1,\varepsilon_1)\frac{(\phi_1, Q)}{(Q_\beta,\varepsilon_1,\varepsilon_1)}\right) + O\left(\frac{1}{A} + \kappa_0\right) > 0. \quad (4.39)$$

Finally, for all nonzero $\varepsilon_1 \in H_1^\beta$ with $(\varepsilon_1, Q_\beta) = 0$, we assume that $A \geq A_0$ is large enough and let

$$\varepsilon_1 = \varepsilon_A^{(1)} + \varepsilon_A^{(2)},$$

with $\varepsilon_A^{(1)} \in (P_1)_A$ and $\varepsilon_A^{(2)} \in (P_1)_A^\perp$.

Since $(P_1)_A$ is orthogonal to $(P_1)_A$ with respect to the quadratic form $\overline{H}_1^\beta$, we have

$$\overline{H}_1^\beta(\varepsilon_1,\varepsilon_1) = \overline{H}_1^\beta(\varepsilon_A^{(1)},\varepsilon_A^{(1)}) + \overline{H}_1^\beta(\varepsilon_A^{(2)},\varepsilon_A^{(2)}) \geq \overline{H}_1^\beta(\varepsilon_A^{(1)},\varepsilon_A^{(1)}), \quad (4.40)$$

since $\overline{H}_1^\beta$ is nonnegative on $(P_1)_A$.

Now let

$$\varepsilon_A^{(1)} = \alpha_A Q_\beta + \varepsilon_A^{(3)},$$

such that $(\varepsilon_A^{(3)}, Q_\beta) = 0$ and $\alpha_A = \frac{((\phi_1), Q_\beta)}{(Q_\beta,\kappa_\beta)}$.

We first claim that there exists a constant $C_0$ independent of $A$ such that

$$\int |D|^{\frac{2}{\beta}} |\varepsilon_A^{(1)}|^2 + \int |\varepsilon_A^{(1)}|^2 e^{-|y|} \leq C_0 \left(\int |D|^{\frac{2}{\beta}} |\varepsilon_1|^2 + \int |\varepsilon_1|^2 e^{-|y|}\right). \quad (4.41)$$

Indeed, let

$$\varepsilon_A^{(1)} = d_A Q_\beta + \varepsilon_A(\phi_1)_A.$$

Then we have

$$\left[\begin{array}{c} d_A \\ e_A \end{array}\right] = \left[\begin{array}{cc} \overline{H}_1^\beta(Q_\beta, Q_\beta) & \overline{H}_1^\beta(\phi_1)_A \\ \overline{H}_1^\beta((\phi_1)_A, Q_\beta) & \overline{H}_1^\beta((\phi_1)_A, (\phi_1)_A) \end{array}\right]^{-1} \left[\begin{array}{c} \overline{H}_1^\beta(\varepsilon_1^{(1)}, Q_\beta) \\ \overline{H}_1^\beta((\phi_1)_A, (\phi_1)_A) \end{array}\right]. \quad (4.42)$$

where we use the fact that $\overline{H}_1^\beta$ is not degenerate on $(P_1)_A$ and $\overline{H}_1^\beta(f, g) = 0$ for all $f \in (P_1)_A$, $g \in (P_1)_A^\perp$. It is easy to obtain that there exists some constant $C$ independent of $A$ such that

$$\overline{H}_1^\beta(\varepsilon_1, Q_\beta) \leq C \left(\int |D|^{\frac{2}{\beta}} |\varepsilon_1|^2 + \int |\varepsilon_1|^2 e^{-|y|}\right)^{\frac{1}{2}}. \quad (4.43)$$
and
\[
|\mathcal{H}_1^g(\varepsilon_1, (\phi_1)_A)| \leq C \|D\tilde{\varphi}(\phi_1)_A\|_{L^2} \|D\tilde{\varphi}_1\|_{L^2} + C \left( \int |\varepsilon_1|^2 (1 + |y|)^{-1 - \beta} \right)^{\frac{1}{2}},
\]

\[
\leq C \left( \int |D\tilde{\varphi}_1|^2 + \int |\varepsilon_1|^2 e^{-|y|} \right)^{\frac{1}{2}}, \tag{4.44}
\]

for \( A \geq A_0 \) large enough and \( 2 > \beta > \beta(A) \). Here we use the following estimate for the last inequality of (4.44): if \( A \geq A_0 \) is large enough and \( 2 > \beta > \beta(A) \) for some \( \beta(A) < 2 \), then we have

\[
\|D\tilde{\varphi}(\phi_1)_A\|_{L^2} \leq \|\nabla(\phi_1)_A\|_{L^2} + \|(|D|\tilde{\varphi} - |D|)(\phi_1)_A\|_{L^2} \leq C \|\nabla \phi_1\|_{L^2},
\]

with some constant \( C \) independent of \( A \). Combining (4.34), (4.42)–(4.44), we obtain (4.41).

Now, using (4.39), Cauchy-Schwarz inequality and the Hardy’s type estimate (3.30), we have for all \( \kappa > 0 \), there exists a \( C_\kappa > 0 \) such that

\[
\mathcal{H}_1^g(\varepsilon_A^{(1)}, \varepsilon_A^{(1)}) = \mathcal{H}_1^g(\alpha_A Q_\beta + \varepsilon_A^{(3)}, \alpha_A Q_\beta + \varepsilon_A^{(3)})
\]

\[
\geq \alpha_A^2 \mathcal{H}_1^g(Q_\beta, Q_\beta) + 2\alpha_A \mathcal{H}_1^g(Q_\beta, \varepsilon_A^{(3)}) + \mathcal{H}_1^g(\varepsilon_A^{(3)}, \varepsilon_A^{(3)})
\]

\[
\geq \alpha_A^2 \mathcal{H}_1^g(Q_\beta, Q_\beta) - 2\alpha_A \mathcal{H}_1^g(Q_\beta, \varepsilon_A^{(3)}) - \varepsilon_A^{(3)}
\]

\[
\geq -C_\kappa \alpha_A^2 - \frac{\kappa}{2C_0} \left( \int |D\tilde{\varphi}_1|^2 + \int |\varepsilon_1|^2 e^{-|y|} \right), \tag{4.45}
\]

where \( C_0 \) is the constant introduced in (4.41).

Our goal is to show that for all \( \kappa > 0 \), there exists \( A(\kappa) \gg 1 \) and \( \beta_\kappa < 2 \) such that if \( A \geq A(\kappa) \) and \( \beta_\kappa \in (\beta_\kappa, 2) \), then we have

\[
|\alpha_A| \leq \sqrt{\frac{\kappa}{2C_\kappa}} \left( \int \|D\tilde{\varphi}_1|^2 + \int |\varepsilon_1|^2 e^{-|y|} \right)^{\frac{1}{2}}, \tag{4.46}
\]

which together with (4.40) and (4.45), implies (4.33) immediately.

Indeed, since \((\varepsilon_1, Q_\beta) = 0\), we have \( \alpha_A = \frac{(\varepsilon_A^{(2)}, Q_\beta)}{(Q_\beta, Q_\beta)} \). From the definition of \((P_1)_A^\perp\), we have:

\[
|\varepsilon_A^{(2)}, Q_\beta| = |(\varepsilon_A^{(2)}, Q_\beta - \mathcal{L}_1^g(\phi_1)_A)|
\]

\[
\leq |\varepsilon_A^{(2)}, Q_\beta| + |(\varepsilon_A^{(2)}, \mathcal{L}_1^g - \mathcal{L}_1)(\phi_1)_A| + |(\varepsilon_A^{(2)}, \mathcal{L}_1((\phi_1)_A - \phi_1))|, \tag{4.47}
\]

where we use the fact that

\[
0 = \mathcal{H}_1^g(\varepsilon_A^{(2)}, (\phi_1)_A) = (\varepsilon_A^{(2)}, \mathcal{L}_1(\phi_1)_A)
\]

and \( \mathcal{L}_1 \phi_1 = Q \) for (4.47). We estimate each term in (4.47) separately.
where we also use the Hardy’s type estimate for the last inequality.

Then using a similar argument as above, we have
\[
\left| (\varepsilon_A^{(2)}, Q \beta - Q) \right| \leq C\|Q \beta - Q\|_{L^2} \left( \int |\varepsilon_A^{(2)}|^2 (1 + |y|)^{-1-\beta} \right)^{\frac{1}{2}}
\]
\[
\leq \delta(2 - \beta) \left( \int ||D|\frac{\partial}{\partial y} \varepsilon_A^{(2)}|^2 + \int |\varepsilon_A^{(2)}|^2 (1 + |y|)^{-1-\beta} \right)^{\frac{1}{2}}
\]
\[
\leq \delta(2 - \beta) \left( \int ||D|\frac{\partial}{\partial y} \varepsilon_A^{(2)}|^2 + \int |\varepsilon_A^{(2)}|^2 e^{-|y|} \right)^{\frac{1}{2}},
\]
(4.48)

where we also use the Hardy’s type estimate for the last inequality.

Next, using a similar argument as above, we have
\[
\left| (\varepsilon_A^{(2)}, (\mathcal{L}^\beta_1 - \mathcal{L}_1) \phi_1, A) \right| 
\leq \left| \left( \left| D \right|\frac{\partial}{\partial y} \varepsilon_A^{(2)}, \left| D \right|\frac{\partial}{\partial y} - \left| D \right|^{2-2/\beta} (\phi_1, A) \right) \right|
+ \left| \left( \varepsilon_A^{(2)}(\phi_1, A), \left( \frac{20(2\beta + 1)}{9} y\nabla Q \beta Q^{2-1} - \frac{100}{9} y\nabla QQ^3 \right) \right) \right|
\leq \tilde{\delta}_A \left( \int \left| D \right|\frac{\partial}{\partial y} \varepsilon_A^{(2)}|^2 + \int |\varepsilon_A^{(2)}|^2 (1 + |y|)^{-1-\beta} \right)^{\frac{1}{2}}
\]
\[
\leq \tilde{\delta}_A \left( \int \left| D \right|\frac{\partial}{\partial y} \varepsilon_A^{(2)}|^2 + \int |\varepsilon_A^{(2)}|^2 e^{-|y|} \right)^{\frac{1}{2}},
\]
(4.49)

for \( A \geq A_0 \) large enough and \( 2 > \beta > \beta(A) \) for some \( \beta(A) < 2 \). Here \( \tilde{\delta}_A \) is a small constant depending only on \( A \), with \( \tilde{\delta}_A \to 0 \) as \( A \to +\infty \).

Next, using (4.32) and a similar argument as above, we have
\[
\left| (\varepsilon_A^{(2)}, \mathcal{L}_1((\phi_1, A) - \phi_1)) \right| 
\leq \left| \left( \left| D \right|\frac{\partial}{\partial y} \varepsilon_A^{(2)}, \left| D \right|^{2-2/\beta} ((\phi_1, A) - \phi_1) \right) \right|
+ \left| \left( \varepsilon_A^{(2)}, \left( \frac{100}{9} y\nabla QQ^3 - \frac{1}{9 \cosh^2(\frac{10}{9} y)} \right) ((\phi_1, A) - \phi_1) \right) \right|
\leq \tilde{\delta}_A \left( \int \left| D \right|\frac{\partial}{\partial y} \varepsilon_A^{(2)}|^2 + \int |\varepsilon_A^{(2)}|^2 e^{-|y|} \right)^{\frac{1}{2}},
\]
(4.50)

for \( A \geq A_0 \) large enough and \( 2 > \beta > \beta(A) \) with some \( \beta(A) < 2 \).

Combining (4.41) and (4.47)–(4.50), we have for \( A \geq A_0 \) large enough and \( 2 > \beta > \beta(A) \), there exists a small constant \( \delta_A \), with \( \delta_A \to 0 \) as \( A \to +\infty \), such that
\[
|\alpha_A| \leq C\delta_A \left( \int \left| D \right|\frac{\partial}{\partial y} \varepsilon_A^{(2)}|^2 + \int |\varepsilon_A^{(2)}|^2 e^{-|y|} \right)^{\frac{1}{2}}
\]
\[
\leq C\delta_A \left( \int \left| D \right|\frac{\partial}{\partial y} \varepsilon_A^{(1)}|^2 + \int |\varepsilon_A^{(1)}|^2 e^{-|y|} + \int \left| D \right|\frac{\partial}{\partial y} \varepsilon_1|^2 + \int |\varepsilon_1|^2 e^{-|y|} \right)^{\frac{1}{2}}
\]
\[
\leq \delta_A \left( \int \left| D \right|\frac{\partial}{\partial y} \varepsilon_1|^2 + \int |\varepsilon_1|^2 e^{-|y|} \right)^{\frac{1}{2}},
\]
(4.51)
Now, for all $\kappa > 0$, we fix an $A = A(\kappa)$ large enough such that

$$\delta_A \leq \sqrt{\frac{\kappa}{2C_\kappa}},$$

where $C_\kappa$ is the constant introduced in (4.45). We also let $\beta_\kappa = \beta(A(\kappa)) < 2$, then from (4.51), we conclude the proof of (4.46), hence the proof of (4.33).

Similar as (4.33), we have for all $\kappa > 0$, there exists $\beta_\kappa < 2$ such that if $\beta_\kappa < \beta < 2$, then for all real valued function $\varepsilon_1 \in H^1_\kappa$ and $\varepsilon_2 \in H^1_\kappa$, with $(\varepsilon_1, G_1) = 0$ and $(\varepsilon_2, \Lambda Q_\beta + \frac{1}{2} \Lambda^2 Q_\beta) = 0$, there holds

$$\overrightarrow{\mathcal{P}}_1^{\beta}(\varepsilon_1, \varepsilon_1) \geq -\kappa \left( \int |D|^{\frac{\beta}{2}} |\varepsilon_1|^2 \right) + \int |\varepsilon_1|^2 e^{-|y|},$$

(4.52)

$$\overrightarrow{\mathcal{P}}_2^{\beta}(\varepsilon_2, \varepsilon_2) \geq -\kappa \left( \int |D|^{\frac{\beta}{2}} |\varepsilon_2|^2 \right) + \int |\varepsilon_2|^2 e^{-|y|}.$$  

(4.53)

The proof of (4.52) and (4.53) is similar to (4.33). We give a strategy of the proof as following and omit the details:

1. Let $(\phi_2)_A(y) = \phi_2(y) \chi(y/A)$ and $(\phi_3)_A(y) = \phi_3(y) \chi(y/A)$, where $\chi$ is the cut-off function introduced in the proof of (4.33).
2. We consider the subspace $(P_2)_A \subset H^1_\kappa$ spanned by $\{Q, (\phi_2)_A\}$ and $(P_3)_A \subset H^1_\kappa$ spanned by $\{Q, (\phi_2)_A\}$. We show that for $A$ large enough, there exists a $\beta(A) < 2$ such that if $\beta \in (\beta(A), 2)$, then $\overrightarrow{\mathcal{P}}_1^{\beta}$ restricted to $(P_2)_A$ is not degenerate and $\overrightarrow{\mathcal{P}}_2^{\beta}$ restricted to $(P_3)_A$ is not degenerate either.
3. Let $(P_2)_A^{\perp}$ be the orthogonal of $(P_2)_A$ in $H^1_\kappa$ with respect to the quadratic form $\overrightarrow{\mathcal{P}}_1^{\beta}$ and $(P_3)_A^{\perp}$ be the orthogonal of $(P_3)_A$ in $H^1_\kappa$ with respect to the quadratic form $\overrightarrow{\mathcal{P}}_2^{\beta}$. Then by an index argument, we can show that $\overrightarrow{\mathcal{P}}_1^{\beta}$ is nonnegative on $(P_2)_A^{\perp}$ and $\overrightarrow{\mathcal{P}}_2^{\beta}$ is also nonnegative on $(P_3)_A^{\perp}$, under the same assumption of the previous step.
4. For all $\varepsilon_1 \in (P_2)_A$ with $(\varepsilon_1, G_1) = 0$ and $\varepsilon_2 \in (P_3)_A$ with $(\varepsilon_2, \Lambda Q_\beta + \frac{1}{2} \Lambda^2 Q_\beta) = 0$, we show that $\overrightarrow{\mathcal{P}}_1^{\beta}(\varepsilon_1, \varepsilon_1) > 0$ and $\overrightarrow{\mathcal{P}}_2^{\beta}(\varepsilon_2, \varepsilon_2) > 0$ provided that $A \geq A_0$ large enough and $\beta > \beta(A)$.
5. Using the fact that $Q_\beta$ is continuous in $H^1$ with respect to $\beta$, we can prove (4.52) and (4.53).

Remark 4.7. We mention here that in the proof of (4.52) and (4.53) we use the fact that

$$(G_1, \Lambda Q_\beta, \Lambda^2 Q_\beta) \rightarrow (yQ, \Lambda Q, \Lambda^2 Q), \quad H^1 \times H^1 \times H^1, \quad \text{as } \beta \rightarrow 2^-.$$  

This is a direct consequence of the continuity of $Q_\beta$ with respect to $\beta$. Since, we have

$$L^\beta G_1 = \Lambda Q_\beta, \quad L_- (yQ) = -\nabla Q,$$

$$L^\beta_+ (\Lambda Q_\beta) = -\beta Q_\beta, \quad L_+ (\Lambda Q) = -2Q,$$

and $L^\beta_\pm \rightarrow L_\pm$ as $\beta \rightarrow 2^-$ in the norm -resolvent sense\textsuperscript{18}.

Combining (4.33), (4.52) and (4.53) we conclude the proof of (4.27), hence the proof of Theorem 1.13.
4.3. Local viriel estimate. In this section, we will derive the crucial local viriel estimate under the assumption that the spectral property (1.19) holds true.

**Proposition 4.8** (Local viriel estimate). Assume the spectral property (1.19) holds true for some $\delta > 0$, and $\alpha_0$ is small enough, then there exist universal constants $C > 0$, $\mu_0 > 0$ such that for all $s \in [0, +\infty)$, there holds

$$b_s \geq \mu_0 \left( \lambda^2 |E_0| + v^2 + \int ||D|\tilde{x}_\varepsilon|^2 + \int |\varepsilon|^2 e^{-|y|} \right) - Cb^{10}. \quad (4.54)$$

Moreover, for all $s_1 < s_2$, there holds

$$\int_{s_1}^{s_2} \left( \lambda^2 |E_0| + v^2 + \int ||D|\tilde{x}_\varepsilon|^2 + \int |\varepsilon|^2 e^{-|y|} \right) \leq \delta(\alpha_0) + \int_{s_1}^{s_2} b^{10}. \quad (4.55)$$

**Proof.** The proof of (4.54) follows from estimating every term in (4.7). More precisely, we proceed in several steps.

**Step 1:** Estimate for the terms generated by the velocity parameter $v$.

Recall from the construction of $W_{b,v}$, we have

$$W_{b,v} = 
\begin{bmatrix}
  \text{even} \\
  \text{even} \\
  \text{even} \\
  \text{even} \\
  \text{even} \\
\end{bmatrix}
+b
\begin{bmatrix}
  0 \\
  0 \\
  0 \\
  0 \\
  0 \\
\end{bmatrix}
+b^2
\begin{bmatrix}
  \text{even} \\
  \text{even} \\
  \text{even} \\
  \text{even} \\
  \text{even} \\
\end{bmatrix}
+b^3
\begin{bmatrix}
  \text{even} \\
  \text{even} \\
  \text{even} \\
  \text{even} \\
  \text{even} \\
\end{bmatrix}
+b^4
\begin{bmatrix}
  \text{even} \\
  \text{even} \\
  \text{even} \\
  \text{even} \\
  \text{even} \\
\end{bmatrix}
+bv
\begin{bmatrix}
  0 \\
  0 \\
  \text{odd} \\
  \text{odd} \\
  \text{odd} \\
\end{bmatrix}
+vb
\begin{bmatrix}
  0 \\
  0 \\
  \text{odd} \\
  \text{odd} \\
  \text{odd} \\
\end{bmatrix}, \quad (4.56)
$$

It is easy to see that

$$\left| bv \left\{ \left( \Theta, \Lambda (\partial_c \Sigma) \right) - \left( \Sigma, \Lambda (\partial_c \Theta) \right) \right\} \right| \lesssim (|v| + |b|) v^2 \leq \frac{c_0}{1000} v^2, \quad (4.57)$$

and

$$v \left\{ \left( \Theta, (\Lambda \Sigma)_y \right) - \left( \Sigma, (\Lambda \Theta)_y \right) \right\} = v^2 \left( (G_1, \nabla \Lambda Q_\beta) - (G_1, \Lambda \nabla Q_\beta) \right) + O(|b| + |v|) v^2. \quad (4.58)$$

where $c_0 > 0$ is the constant introduced in (2.6).

From (2.10) and the commutator relation $[\nabla, \Lambda] = \nabla$, we have

$$([G_1, \nabla \Lambda \Sigma] - (G_1, \Lambda \nabla \Sigma)) = (G_1, [\nabla, \Lambda] Q_\beta) = (G_1, \Lambda \nabla Q_\beta) = -\langle L_\beta \nabla Q_\beta, \nabla Q_\beta \rangle < 0.$$ 

Recall from the proof of (2.6), we know that $c_0 = \langle L_\beta \nabla Q_\beta, \nabla Q_\beta \rangle > 0$. Hence, we have

$$-v \left\{ \left( \Theta, (\Lambda \Sigma)_y \right) - \left( \Sigma, (\Lambda \Theta)_y \right) \right\} \geq \frac{c_0}{2} v^2, \quad (4.59)$$

provided that $\alpha_0$ is small enough.

**Step 2:** Elliptic estimate for the quadratic terms.

By assuming the spectral property (1.19) holds true, we claim that there exist universal constants $\delta_0 > 0$ and $C > 0$ such that

$$H_{b,v}(\varepsilon, \varepsilon) - \frac{1}{\|\Lambda Q_\beta\|_{L^2}} (\varepsilon_1, L_+^\beta (\Lambda^2 Q_\beta))(\varepsilon_1, \Lambda Q_\beta) \geq \delta_0 \left( \int ||D|\tilde{x}_\varepsilon|^2 + \int |\varepsilon|^2 e^{-|y|} \right) - \frac{c_0}{1000} v^2 - C(b^{10} + \lambda^2 |E_0|), \quad (4.60)$$

where $c_0 > 0$ is the positive constant introduced in (2.6).

To prove (4.60), we first introduce the following elliptic estimate:
Lemma 4.9. Suppose that the spectral property (1.19) holds true for some \( \delta > 0 \). Then there exists some constant \( \delta_0 \in (0, \delta) \) depending only on \( \delta \), such that for all \( \varepsilon_1 \in H^2(\mathbb{R}; \mathbb{R}) \),

\[
\tilde{H}^\beta(\varepsilon_1, \varepsilon_1) \geq \delta_0 \left( \int |D|^{2\beta} \varepsilon_1^2 + \int |\varepsilon_1|^2 e^{-|y|} \right) - \frac{1}{\delta_0} \left( (\varepsilon_1, Q_\beta)^2 + (\varepsilon_1, G_1)^2 + (\varepsilon_1, S_1)^2 \right),
\]

(4.61)

where

\[
\tilde{H}^\beta(\varepsilon_1, \varepsilon_1) = \frac{\beta}{2} (L_1 \varepsilon_1, \varepsilon_1) - \frac{1}{\|Q_\beta\|_{L^2}^2} (\varepsilon_1, L_+^\beta \Lambda^2 Q_\beta)(\varepsilon_1, \Lambda Q_\beta)
\]

and \( \delta > 0 \) is the constant on the right hand side of (1.19).

Proof of Lemma 4.9. We denote by

\[
\tilde{B}(f, g) = \frac{\beta}{2} ([L_1 f, g]) - \frac{1}{2\|Q_\beta\|_{L^2}^2} (f, L_+^\beta \Lambda^2 Q_\beta)(g, \Lambda Q_\beta)
\]

the bilinear form underlying \( \tilde{H}^\beta \) for all \( f, g \in H^2(\mathbb{R}; \mathbb{R}) \).

We claim that for all \( \varepsilon_1 \in H^2(\mathbb{R}; \mathbb{R}) \), there holds

\[
\tilde{B}(\varepsilon_1, \Lambda Q_\beta) = 0.
\]

(4.62)

Indeed, for the linear operator \( L_1 \) we claim that

\[
L_1(\Lambda Q_\beta) = \frac{1}{\beta} [L_+^\beta(\Lambda^2 Q_\beta) - \Lambda (L_+^\beta \Lambda Q_\beta)] = \frac{1}{\beta} L_+^\beta(\Lambda^2 Q_\beta) + \Lambda Q_\beta.
\]

(4.63)

Here for the first equality of (4.63), we have

\[
[L_+^\beta, \Lambda] = [D]^{\beta, y \cdot \nabla} - (2\beta + 1)[Q_\beta^{2\beta}, y \cdot \nabla] = [D]^{\beta, y \cdot \nabla} + (2\beta + 1) y \nabla (Q_\beta^{2\beta}) = [D]^{\beta, y \cdot \nabla} + 2\beta(2\beta + 1) y (\nabla Q_\beta) Q_\beta^{2\beta-1}.
\]

(4.64)

and

\[
([D]^{\beta, y \cdot \nabla} f, g) = ([D]^{\beta}(y \nabla f), g) - (y \nabla [D]^{\beta} f, g)
\]

\[
= \int |\xi|^{\beta} i \partial_x (i\xi \hat{f}) \hat{g} - \int i \partial_x (i\xi |\xi|^{\beta} \hat{f}) \hat{g}
\]

\[
= - \int |\xi|^{\beta} \hat{f} \hat{g} - \int \xi |\xi|^{\beta} (\partial_x \hat{f}) \hat{g} + \int \xi |\xi|^{\beta} (\partial_x \hat{f}) \hat{g} + (\beta + 1) \int |\xi|^{\beta} \hat{f} \hat{g}
\]

\[
= \beta \int |\xi|^{\beta} \hat{f} \hat{g} = \beta ([D]^{\beta} f, g).
\]

(4.65)

for all regular enough functions \( f \) and \( g \). Combining (4.64) and (4.65), we obtain (4.63).
Now using (4.63), we have
\[
\tilde{B}(\tilde{\varepsilon}_1, \Lambda Q_\beta) \leq \frac{\beta}{2} (\tilde{\varepsilon}_1, L_1 \Lambda Q_\beta) - \frac{1}{2\|\Lambda Q_\beta\|_{L^2}} \left\{ (\tilde{\varepsilon}_1, L^\beta_+ (\Lambda^2 Q_\beta)) (\Lambda Q_\beta, \Lambda Q_\beta) + ((\tilde{\varepsilon}_1, \Lambda Q_\beta) (\Lambda Q_\beta, L^\beta_+ (\Lambda^2 Q_\beta))) \right\} \\
= \left( \tilde{\varepsilon}_1, \frac{1}{2} L^\beta_+ (\Lambda^2 Q_\beta) + \frac{\beta}{2} \Lambda Q_\beta - \frac{1}{2} L^\beta_+ (\Lambda^2 Q_\beta) - \frac{(-\beta Q_\beta, \Lambda^2 Q_\beta)}{2\|\Lambda Q_\beta\|_{L^2}^2} \Lambda Q_\beta \right) \\
= 0.
\]

Next, we let \( \tilde{\varepsilon}_1 \in H^\beta_+ (\mathbb{R}; \mathbb{R}) \) such that \( (\tilde{\varepsilon}_1, G_1) = (\tilde{\varepsilon}_1, S_1) = 0 \). We also set \( \tilde{\varepsilon}_1 = \tilde{\varepsilon}_1 + \nu \Lambda Q_\beta \) with \( \nu = -(\tilde{\varepsilon}_1, \Lambda Q_\beta) / \|\Lambda Q_\beta\|_{L^2}^2 \), so that \( (\tilde{\varepsilon}_1, G_1) = (\tilde{\varepsilon}_1, \Lambda Q_\beta) = 0 \).

From the Hardy’s type estimate (3.30), we have
\[
\nu^2 \leq \frac{(\tilde{\varepsilon}_1, \Lambda Q_\beta)^2}{\|\Lambda Q_\beta\|_{L^2}^2} \lesssim \int \frac{|\tilde{\varepsilon}_1|^2}{(1 + |y|)^2} \lesssim \int |D^\frac{\beta}{2} \tilde{\varepsilon}_1|^2 + \int |\tilde{\varepsilon}_1|^2 e^{-|y|},
\]
which implies that
\[
\frac{1}{C} \left( \int |D^\frac{\beta}{2} \tilde{\varepsilon}_1|^2 + \int |\tilde{\varepsilon}_1|^2 e^{-|y|} \right) \lesssim \int |D^\frac{\beta}{2} \tilde{\varepsilon}_1|^2 + \int |\tilde{\varepsilon}_1|^2 e^{-|y|}
\]
\[
\lesssim C \left( \int |D^\frac{\beta}{2} \tilde{\varepsilon}_1|^2 + \int |\tilde{\varepsilon}_1|^2 e^{-|y|} \right),
\]
for some universal constant \( C > 0 \).

Then from (4.62) and the fact that \( (\tilde{\varepsilon}_1, \Lambda Q_\beta) = 0 \), we have
\[
\tilde{H}^\beta (\varepsilon_1, \tilde{\varepsilon}_1) = \tilde{H}^\beta (\tilde{\varepsilon}_1 - \nu \Lambda Q_\beta, \tilde{\varepsilon}_1 - \nu \Lambda Q_\beta) = \tilde{H}^\beta (\tilde{\varepsilon}_1, \tilde{\varepsilon}_1) = \frac{\beta}{2} (L_1 \tilde{\varepsilon}_1, \tilde{\varepsilon}_1).
\]

Finally, from the fact that \( (Q_\beta, \Lambda Q_\beta) = 0, (\tilde{\varepsilon}_1, Q_\beta) = (\tilde{\varepsilon}_1, \tilde{\varepsilon}_1) \) and the spectral property (1.19), we know that there exists some constant \( \delta > \delta_0 > 0 \) such that\(^{19}\)
\[
\tilde{H}^\beta (\tilde{\varepsilon}_1, \tilde{\varepsilon}_1) = \frac{\beta}{2} (L_1 \tilde{\varepsilon}_1, \tilde{\varepsilon}_1) \gtrsim \frac{\beta}{2} \left( \int |D^\frac{\beta}{2} \tilde{\varepsilon}_1|^2 + \int |\tilde{\varepsilon}_1|^2 e^{-|y|} \right) - C_\delta (\tilde{\varepsilon}_1, Q_\beta)^2
\]
\[
\gtrsim \delta_0 \left( \int |D^\frac{\beta}{2} \tilde{\varepsilon}_1|^2 + \int |\tilde{\varepsilon}_1|^2 e^{-|y|} \right) - \frac{1}{\delta_0} (\tilde{\varepsilon}_1, Q_\beta)^2,
\]
which implies (4.61) immediately. Hence we conclude the proof of Lemma 4.9. \( \square \)

Now we turn back to the proof of (4.60). From (4.61) and the spectral property (1.19), we know that
\[
\frac{\beta}{2} [(L_1 \varepsilon_1, \varepsilon_1) + (L_2 \varepsilon_2, \varepsilon_2)] - \frac{1}{\|\Lambda Q_\beta\|_{L^2}^2} (\varepsilon_1, L^\beta_+ (\Lambda^2 Q_\beta)) (\varepsilon_1, \Lambda Q_\beta)
\]
\[
g \geq \delta_0 \left( \int |D^\frac{\beta}{2} \varepsilon_1|^2 + \int |\varepsilon_1|^2 e^{-|y|} \right) - \frac{1}{\delta_0} [(\varepsilon_1, Q_\beta)^2 + (\varepsilon_1, S_1)^2 + (\varepsilon_1, G_1)^2
\]
\[
+ (\varepsilon_2, \Lambda Q_\beta)^2 + (\varepsilon_2, \Lambda^2 Q_\beta)^2]. \tag{4.66}
\]
\(^{19}\)Here we use the fact that \( \beta \geq 1 \) for the first inequality.
From the orthogonality condition (3.5)–(3.9) and the Hardy’s type estimate (3.30), we know that
\[
(\varepsilon_1, S_1)^2 + (\varepsilon_1, G_1)^2 + (\varepsilon_2, \Lambda Q_\beta)^2 + (\varepsilon_2, \Lambda^2 Q_\beta)^2 = O(|b| + |v|) \left( \int ||D|^2 \varepsilon ||^2 + \int |\varepsilon|^2 (\Lambda^2 + 1 + |y|^{1+\beta}) \right)
\lesssim \delta(\alpha_0) \left( \int ||D|^2 \varepsilon ||^2 + \int |\varepsilon|^2 e^{-|y|} \right). \tag{4.67}
\]

From (3.19), we know that
\[
(\varepsilon_1, Q_\beta)^2 \lesssim b^{10} + \nu^2 (|b| + |v|) + (\lambda^2 |E_0|)^2 + \left( \int ||D|^2 \varepsilon ||^2 + \int |\varepsilon|^2 e^{-|y|} \right)^2 \lesssim \delta(\alpha_0) \left( v^2 + \int ||D|^2 \varepsilon ||^2 + \int |\varepsilon|^2 e^{-|y|} \right) + b^{10} + (\lambda^2 |E_0|)^2. \tag{4.68}
\]

From the definition of the quadratic form \( H_{b,v}(\varepsilon, \varepsilon) \) (4.11)–(4.13), we know that
\[
H_{b,v}(\varepsilon, \varepsilon) - \frac{1}{||LQ_\beta||^2_{L^2}} (\varepsilon_1, L^\beta_+ (\Lambda^2 Q_\beta)) (\varepsilon_1, Q_\beta)
= \frac{\beta}{2} \left[ (L_1 \varepsilon_1, L_1 \varepsilon_1) + (L_2 \varepsilon_2, L_2 \varepsilon_2) \right] - \frac{1}{||LQ_\beta||^2_{L^2}} (\varepsilon_1, L^\beta_+ (\Lambda^2 Q_\beta)) (\varepsilon_1, Q_\beta)
+ O \left( \delta(\alpha_0) \left( \int ||D|^2 \varepsilon ||^2 + \int |\varepsilon|^2 e^{-|y|} \right) \right). \tag{4.69}
\]

Combining (4.66)–(4.69), we obtain (4.60) immediately provided that \( \alpha_0 \) is small enough.

**Step 3:** Estimates for the geometrical parameters using modulation theory.

From the orthogonality condition (3.9), we know that
\[
\left( \frac{\lambda_\Theta}{\lambda} + b \right) \left\{ (\varepsilon_2, \Lambda^2 \Sigma) - (\varepsilon_1, \Lambda^2 \Theta) \right\} = 0. \tag{4.70}
\]

Recalling from (4.56), we have
\[
| (\Theta, (\Lambda \Sigma)_y) - (\Sigma, (\Lambda \Theta)_y) | + | (\partial_x \Theta, \Lambda \Sigma) - (\partial_x, \Lambda \Sigma, \Lambda \Theta) | = O(|v|). \tag{4.71}
\]

Hence using the modulation estimate (3.20), we have
\[
\left| \left( \frac{x}{\lambda} - b \right) \left\{ (\Theta, (\Lambda \Sigma)_y) - (\Sigma, (\Lambda \Theta)_y) + (\varepsilon_2, (\Lambda \Sigma)_y) - (\varepsilon_1, (\Lambda \Theta)_y) \right\} \right|
\lesssim \frac{\delta_0}{1000} \left( \int ||D|^2 \varepsilon ||^2 + \int |\varepsilon|^2 e^{-|y|} \right) + \frac{\alpha_0}{1000} v^2 + C \left( b^{10} + (\lambda^2 |E_0|)^2 \right), \tag{4.72}
\]
provided that \( \alpha_0 \) is small enough. Similarly, we have
\[
\left| \left( \frac{x}{\lambda} - v \right) \left\{ (\Theta, (\Lambda \Sigma)_y) + (\varepsilon_2, (\Lambda \Sigma)_y) - (\varepsilon_1, (\Lambda \Theta)_y) \right\} \right|
\lesssim \frac{\delta_0}{1000} \left( \int ||D|^2 \varepsilon ||^2 + \int |\varepsilon|^2 e^{-|y|} \right) + \frac{\alpha_0}{1000} v^2 + C \left( b^{10} + (\lambda^2 |E_0|)^2 \right), \tag{4.73}
\]
and
\[
\left| \dot{\gamma}_t \left\{ (\varepsilon_1, \Lambda \Sigma) + (\varepsilon_2, \Lambda \Theta) \right\} \right| - \frac{1}{\| \Lambda Q \|_{L^2}}(\varepsilon_1, L^\beta_x \Lambda^2 Q_\beta)(\varepsilon_1, \Lambda Q_\beta) \\
\leq \frac{\delta_0}{1000} \left( \int ||D|^\beta \varepsilon |^2 + \int |\varepsilon|^2 e^{-|y|} \right) + \frac{c_0}{1000} \mu^2 + C(\mu^{10} + (\lambda^\beta |E_0|)^2). \tag{4.74}
\]
Here \( \delta_0 \) is the constant introduced in (4.60), while \( c_0 \) is the constant introduced in (2.6).

**Step 4:** Estimate for the nonlinear terms and the interaction terms.

From Proposition 2.1 and the Hardy’s type estimate (3.30), it is easy to see that
\[
\left| \{ (\varepsilon_1, \Re(\Lambda \Psi b, v)) - (\varepsilon_1, \Im(\Lambda \Psi b, v)) \} \right| \\
\lesssim (b^5 + (|v| + |\theta|)|^2 \left( \int ||D|^\beta \varepsilon |^2 + \int |\varepsilon|^2 e^{-|y|} \right)^{\frac{1}{2}} \\
\leq \frac{\delta_0}{1000} \left( \int ||D|^\beta \varepsilon |^2 + \int |\varepsilon|^2 e^{-|y|} \right) + \frac{c_0}{1000} \mu^2 + Cb^{10}, \tag{4.75}
\]
where we use Cauchy-Schwarz’s inequality for the last estimate.

While for the nonlinear term \( G(\varepsilon) \), we first use (3.25) and the Hardy’s type estimate (3.30) to obtain
\[
\left| \int F(\varepsilon) \right| \lesssim \delta \left( \int ||D|^\beta \varepsilon |^2 + \int |\varepsilon|^2 e^{-|y|} \right). \tag{4.76}
\]
Next, we apply the following inequality
\[
\left| (1 + \Re z)(1 + z)^{2\beta - 1} - (2\beta + 1)\Re z - \beta(2\beta + 1)(\Re z)^2 - \beta(\Im z)^2 \right| \\
\lesssim (|z|^3 + |z|^{2\beta + 1}), \quad \forall z \in \mathbb{C},
\]
to \( z = \frac{W_{\mu, v}}{W_{\mu, v}} \) using Hölder’s inequality to obtain
\[
\left| (\widetilde{R}_1(\varepsilon), \Lambda \Sigma) \right| \lesssim \int (|\varepsilon|^{2\beta + 1} + |\varepsilon|^3)|\Lambda \Sigma| \\
\leq C_{\mu_1} \int |\varepsilon|^{2\beta + 2} + \mu_1 \int |\varepsilon|^2(1 + |y|)^{-1 - \beta}, \tag{4.77}
\]
for all \( \mu_1 > 0 \). Here we use the following Hölder’s inequality
\[
\forall \mu_1 > 0, \quad |\varepsilon|^{2\beta} \leq C_{\mu_1} |\varepsilon|^{2\beta + 2} + \mu_1 |\varepsilon|^2, \quad |\varepsilon|^3 \leq C_{\mu_1} |\varepsilon|^{2\beta + 2} + \mu_1 |\varepsilon|^2,
\]
for the last inequality of (4.77). Now, taking \( \mu_1 \) small enough, using the Gagliardo-Nirenberg’s inequality (1.10) and the Hardy’s type estimate (3.30) to obtain
\[
\left| (\widetilde{R}_1(\varepsilon), \Lambda \Sigma) \right| \leq \frac{\delta_0}{1000} \left( \int ||D|^\beta \varepsilon |^2 + \int |\varepsilon|^2 e^{-|y|} \right), \tag{4.78}
\]
as long as \( \alpha_0 \) is small enough.

Then we apply the following inequality
\[
\forall z \in \mathbb{C}, \quad \left| (1 + z)(1 + z)^{2\beta - 1} - (2\beta + 1)\Re z - i\Im z \right| \lesssim (|z|^2 + |z|^{2\beta + 1})
\]
\[\text{Remark from (3.23), we have } |W_{\mu, v}(y)| \neq 0 \text{ for all } y \in \mathbb{R}.\]
to $z = \frac{\varepsilon}{W_{b,v}}$, using the same argument as above, to obtain that

$$|(R_2(\varepsilon), \Lambda \Theta)| \lesssim \int (|\varepsilon|^{2\beta+1} + |\varepsilon|^2)|\Lambda \Theta|$$

$$\lesssim \int |\varepsilon|^{2\beta+2} + O(|b| + |v|) \int |\varepsilon|^2(1 + |y|)^{-1-\beta}$$

$$\leq \frac{\delta_0}{1000} \left( \int ||D|^2 \varepsilon|^2 + \int |\varepsilon|^2 e^{-|y|} \right),$$

(4.79)

where we use the fact that $\Lambda \Theta = O(|b| + |v|)(1 + |y|)^{-1-\beta}$, for the second inequality of (4.79).

**Step 5:** Conclusion.

Combining (4.57), (4.59), (4.60), (4.72)-(4.76), (4.78) and (4.79), we have

$$b_s \left[ (\partial_b \Theta, \Lambda \Sigma) - (\partial_b \Sigma, \Lambda \Theta) - (\varepsilon_2, \Lambda (\partial_b \Sigma)) + (\varepsilon_1, \Lambda (\partial_b \Theta)) \right]$$

$$\geq \frac{\delta_0}{2} \left( \int ||D|^2 \varepsilon|^2 + \int |\varepsilon|^2 e^{-|y|} \right) + \frac{c_0}{2} v^2 + \beta \lambda^\beta |E_0|$$

$$- \frac{\delta_0}{50} \left( \int ||D|^2 \varepsilon|^2 + \int |\varepsilon|^2 e^{-|y|} \right) - \frac{c_0}{50} v^2 - C(b^{10} + (\lambda^\beta |E_0|)^2)$$

$$\geq \delta_1 \left( \lambda^\beta |E_0| + v^2 + \int ||D|^2 \varepsilon|^2 + \int |\varepsilon|^2 e^{-|y|} \right) - Cb^{10},$$

(4.80)

for some universal constant $\delta_1 > 0$, provided that $\alpha_0$ is small. Here we use the fact that

$$\lambda^\beta |E_0| \lesssim \delta(\alpha_0) \ll 1$$

for the last inequality of (4.80).

Now, we observe from $W_{b,v}|_{b=0,v=0} = Q_b$ and $\partial_b W_{b,v}|_{b=0,v=0} = iS_1$ that

$$(\partial_b \Theta, \Lambda \Sigma) - (\partial_b \Sigma, \Lambda \Theta) - (\varepsilon_2, \Lambda (\partial_b \Sigma)) + (\varepsilon_1, \Lambda (\partial_b \Theta))$$

$$= (S_1, \lambda \Sigma_b) + O(|b| + |v| + ||\varepsilon||_H^2)$$

$$= (L_b^\Lambda \lambda \Sigma_b, \lambda \Sigma_b) + O(\delta(\alpha_0)) > 0,$$

(4.81)

where we use (2.10) and (3.11) for the last inequality. Injecting (4.81) into (4.80), we obtain (4.54) immediately. Finally, integrating (4.54) from $s_1$ to $s_2$ we obtain (4.55), which concludes the proof of Proposition 4.8.

5. **Proof of Theorem 1.6**

In this section, we will finish the proof of Theorem 1.6. We consider initial data $u_0 \in \mathcal{B}_{\alpha_0}$ with $E_0 = E(u_0) < 0$. Assume that $\alpha_0$ is small enough so that all the estimates obtained in the previous sections hold true. Then we can prove Theorem 1.6 in the following steps.

---

21This is a direct consequence of (3.11) and (3.19).
5.1. Monotony properties. In this subsection, we will derive from the local viriel estimate (4.54) some monotony properties of the geometrical parameters. Roughly speaking, we will show that the parameter \( b \) is always positive for large time. Since the scaling parameter \( \lambda \) satisfies
\[
\frac{\lambda_s}{\lambda} \sim -b,
\]
the sign structure of \( b \) implies that \( \lambda \) is almost monotony in time. This fact removes the possibility that the \( H^s \) norm of the original solution \( u(t) \) oscillates in time, which forces the solution to blow up in finite time.

More precisely, we have

**Proposition 5.1.** Assume that \( \alpha_0 \) is small enough and the spectral property (1.19) holds true. Then there exists a unique \( s_0 \in [0, +\infty) \) such that

1. **Sign structure of \( b \):**
   \[
   \forall s < s_0, \ b(s) < 0; \quad \forall s > s_0, \ b(s) > 0. \tag{5.1}
   \]

2. **Monotonicity of \( \lambda \):** for all \( s_2 > s_1 \geq s_0 \), we have
   \[
   \frac{1}{2} \int_{s_1}^{s_2} b(s) \, ds - \delta(\alpha_0) \leq -\log \left( \frac{\lambda(s_2)}{\lambda(s_1)} \right) \leq \frac{3}{2} \int_{s_1}^{s_2} b(s) \, ds + \delta(\alpha_0), \tag{5.2}
   \]
   and
   \[
   \lambda(s_1) > \frac{1}{2} \lambda(s_2). \tag{5.3}
   \]

**Proof.** We proceed the proof in the following steps.

**Step 1:** Equation of the scaling parameters.

We claim that for all \( s_2 > s_1 \geq 0 \), there holds
\[
\left| \log \left( \frac{\lambda(s_2)}{\lambda(s_1)} \right) + \int_{s_1}^{s_2} b(s) \, ds \right| \leq \delta(\alpha_0) + \frac{1}{2} \int_{s_1}^{s_2} |b| \, ds. \tag{5.4}
\]

We first differentiate (3.6) to obtain
\[
(\partial_x \varepsilon_1, \partial_y \Theta) - (\partial_x \varepsilon_2, \partial_y \Sigma) + b_s \left[ (\varepsilon_1, \partial_y \Theta) - (\varepsilon_2, \partial_y \Sigma) \right] - v_s \left[ \varepsilon_2, \partial_y (\partial_v \Sigma) \right] = b_s \left[ (\varepsilon_1, \partial_y \Theta) - (\varepsilon_2, \partial_y \Sigma) \right] \tag{5.5}
\]

Then we project (3.13) and (3.14) onto \( -\partial_y \Theta \) and \( \partial_y \Sigma \) using (3.31) to obtain
\[
\left( \frac{\lambda_s}{\lambda} + b \right) \left\{ - (\Lambda \Theta, \partial_y \Sigma) + (\Lambda \Sigma, \partial_y \Theta) + (\varepsilon_2, \Lambda (\partial_y \Sigma)) - (\varepsilon_1, \Lambda (\partial_y \Theta)) \right\}
+ b_s \left[ (\partial_y \Theta, \partial_y \Sigma) - (\partial_y \Sigma, \partial_y \Theta) - (\varepsilon_2, \partial_y ^2 \Sigma) + (\varepsilon_1, \partial_y ^2 \Theta) \right]
+ (v_s + bv) \left[ (\partial_v \Theta, \partial_y \Sigma) - (\partial_v \Sigma, \partial_y \Theta) - (\varepsilon_2, \partial_y (\partial_v \Sigma)) + (\varepsilon_1, \partial_y (\partial_v \Theta)) \right]
+ \left( \frac{x_s}{\lambda} - v \right) \left\{ (\Theta, (\partial_y \Sigma)_y) - (\Sigma, (\partial_y \Theta)_y) + (\varepsilon_2, (\partial_y \Sigma)_y) - (\varepsilon_1, (\partial_y \Theta)_y) \right\}
- \gamma_s \left\{ (\partial_y \Sigma, \partial_y \Theta) + (\partial_y \Sigma, \partial_y (\partial_v \Theta)) + (\varepsilon_2, \partial_y (\partial_v \Theta)) \right\}
= -bv \left\{ (\varepsilon_2, \partial_y (\partial_v \Sigma)) - (\varepsilon_1, \partial_y (\partial_v \Theta)) \right\} - (M_+ (\varepsilon), \partial_y \Sigma) - (M_- (\varepsilon), \partial_y \Theta)
- b \left\{ (\Lambda \varepsilon_2, \partial_y \Sigma) - (\Lambda \varepsilon_1, \partial_y \Theta) \right\} + v \left\{ (\nabla \varepsilon_2, \partial_y \Sigma) - (\nabla \varepsilon_1, \partial_y \Theta) \right\}
- (\partial_y \Sigma, \Re \langle \Psi_{\nu,v} \rangle) - (\partial_y \Theta, \Im \langle \Psi_{\nu,v} \rangle) + (R_1 (\varepsilon), \partial_y \Sigma) + (R_2 (\varepsilon), \partial_y \Theta). \tag{5.6}
\]
From (2.3), (3.11) and (3.20), we can easily obtain the following rough estimate:

\[
\left( \frac{\lambda s}{\lambda} + b \right) \{ (\Lambda \Sigma, \partial_b \Theta) - (\Lambda \Theta, \partial_b \Sigma) \} = O \left( \lambda^2 |E_0| + v^2 + \int \|D|^2 |\frac{\partial}{\partial x} \| \| + \int |\varepsilon^2 e^{-|y|} + \delta(\alpha_0) |b| \right).
\]

(5.7)

Since

\[
(\Lambda \Sigma, \partial_b \Theta) - (\Lambda \Theta, \partial_b \Sigma) = (\Lambda Q, S_1) + O(|b| + |v|),
\]

using (3.20) again, we have

\[
\left( \frac{\lambda s}{\lambda} + b \right) \{ (\Lambda \Sigma, \partial_b \Theta) - (\Lambda \Theta, \partial_b \Sigma) \} = \left( \frac{\lambda s}{\lambda} + b \right) (\Lambda Q, S_1)

+ O \left( \lambda^2 |E_0| + v^2 + \int \|D|^2 |\frac{\partial}{\partial x} \| \| + \int |\varepsilon^2 e^{-|y|} + \delta(\alpha_0) |b| \right).
\]

(5.8)

Combining (5.7), (5.8) and the fact that \((\Lambda Q, S_1) \neq 0\), we have

\[
\left( \frac{\lambda s}{\lambda} + b \right) = O \left( \lambda^2 |E_0| + v^2 + \int \|D|^2 |\frac{\partial}{\partial x} \| \| + \int |\varepsilon^2 e^{-|y|} + \delta(\alpha_0) |b| \right).
\]

(5.9)

Integrating (5.9) from \(s_1\) to \(s_2\), using (4.55), we obtain (5.4) immediately.

**Step 2:** Proof of (5.1).

Suppose that (5.1) does not hold. Then there are two possibilities:

1. There exists \( s_0 \in [0, +\infty) \) such that \( b(s_0) \leq 0 \) and \( b(s_0) = 0 \)

2. For all \( s \in [0, +\infty) \), we have \( b(s) < 0 \).

If the first one holds true, then from (4.54), we have \( \lambda^3 (s_0) |E_0| \leq 0 \). Since \( E_0 < 0 \), we have \( \lambda(s_0) = 0 \), which leads to a contradiction. Hence, we have for all \( s \in [0, +\infty) \), \( b(s) < 0 \).

If \( \int_{s_1}^{+\infty} b = -\infty \), then from (5.4), we have \( \lambda(s_2) \rightarrow +\infty \) as \( s_2 \rightarrow +\infty \), which contradicts with (3.11), (3.19) and the energy condition \( E_0 < 0 \). Now, we have

\[
\int_{s_1}^{+\infty} b < +\infty.
\]

(5.10)

Using (5.4) again, we can show that there exists \( 0 < \lambda_- < \lambda_+ \) such that for all \( s \geq s_0 \), we have

\[
\lambda_- < \lambda(s) < \lambda_+.
\]

(5.11)

From (3.20), (4.55) and (5.10), we have:

\[
\int_{s_1}^{+\infty} |bb_s| \lessapprox \int_{s_1}^{+\infty} \left( |b| + \lambda^3 |E_0| + v^2 + \int \|D|^2 |\frac{\partial}{\partial x} \| \| \right) \) ds < +\infty.
\]

Using (5.10) again, we have

\[
b(s) \rightarrow 0, \quad \text{as } s \rightarrow +\infty.
\]

(5.12)

We also have

\[
b_s(s_n) \rightarrow 0, \quad \text{as } n \rightarrow +\infty,
\]

(5.13)

for some sequence \( \{s_n\}_{n=1}^{\infty} \) with \( s_n \rightarrow +\infty \), as \( n \rightarrow +\infty \).
Injecting (5.12) and (5.13) into (4.54), we obtain that
\[ \lambda^2(s_n)|E_0| \to 0, \text{ as } n \to +\infty, \]
which contradicts with (5.11) and the energy condition \( E_0 < 0 \).

Hence, we conclude the proof of (5.1).

**Step 3:** Proof of (5.2) and (5.3).

Since for all \( s \geq s_0 \), we have \( b(s) > 0 \), we know from (5.4) that
\[ \int_{s_1}^{s_2} |b| = \int_{s_1}^{s_2} b. \]
Then (5.2) follows immediately.

Now, using (5.1) and (5.2), we have
\[ \log \left( \frac{\lambda(s_2)}{\lambda(s_1)} \right) \leq \delta(\alpha_0) - \int_{s_1}^{s_2} b(s) \, ds \leq \delta(\alpha_0) \leq e^{\frac{1}{2}}, \]
which implies (5.3).

Now we conclude the proof of Proposition 5.1. \( \square \)

5.2. **Finite time or infinite time blow-up.** This subsection is devoted to show that the solution \( u(t) \) must blow up either in finite time or in infinite time. More precisely, we claim that
\[ \lambda(s) \to 0, \text{ as } s \to +\infty. \] (5.14)

Indeed, from (4.54), we have for all \( s > s_0 \)
\[ b_s \geq -Cb^{10}. \]
Since \( b(s) > 0 \) for all \( s > s_0 \), we have for all \( s > s_0 \)
\[ \frac{b_s}{b^{10}} \geq -C. \]
After integration, we can show that for some \( \tilde{s}_1 > s_0 \), there holds
\[ b(s) \geq \frac{C}{s^{\tilde{s}}} \] (5.15)
Hence from (5.2), we have for all \( s > \tilde{s}_1 \)
\[ -\log \left( \frac{\lambda(s)}{\lambda(\tilde{s}_1)} \right) + \delta(\alpha_0) \geq \frac{1}{2} \int_{s_1}^{s_2} b(s) \, ds \to +\infty, \text{ as } s \to +\infty, \] (5.16)
which implies (5.14) immediately.

5.3. **Finite time blow-up and upper bound on the blow-up rate.** In this subsection, we will finish the proof of Theorem 1.6.

Combining (5.2) and (5.15), we have for all \( s > \tilde{s}_1 \),
\[ \log \left( \frac{\lambda(s)}{\lambda(\tilde{s}_1)} \right) \leq -\frac{1}{2} \int_{s_1}^{s} b(s) \, ds + \delta(\alpha_0) \leq C(s^{\tilde{s}} - s^{\tilde{s}_1}) + \delta(\alpha_0). \] (5.17)
Hence, there exists some \( \tilde{s}_2 \gg \tilde{s}_1 \), such that for all \( s > \tilde{s}_2 \)
\[ \log(\lambda(s)) \leq -Cs^{\tilde{s}} \leq -Cb^{-8}, \] (5.18)
for some universal constant \( C \geq 0 \).

From (5.14), we may chose a sequence of time \( \{t_n\} \) such that \( \lambda(t_n) = 2^{-n} \) and \( t_n \to T \), the maximal lifespan \( T \), as \( n \to +\infty \). Denote by \( \hat{t}_2 \) such that \( s(\hat{t}_2) = \tilde{s}_2 \).
and $s_n$ such that $s(t_n) = s_n$. We also assume that for all $n \geq n_0$, there holds $t_n \geq \tilde{t}_2$. Recall from (5.3), for all $n \geq n_0$ and $s \in [s_n, s_{n+1}]$, we have
\[ 2^{-n-2} \leq \lambda(s) \leq 2^{-n+1}. \] (5.19)

From (5.2) and (5.18), we have
\[ \delta(\alpha_0) - \log \left( \frac{\lambda(s_{n+1})}{\lambda(s_n)} \right) \geq \int_{s_n}^{s_{n+1}} b(s) \, ds \geq C \int_{s_n}^{s_{n+1}} \frac{ds}{|\log(\lambda(s))|^{1+\frac{1}{k}}} \]
Combining the above two estimates, we have
\[ \int_{t_n}^{t_{n+1}} \frac{dt}{\lambda^\beta(t)|\log(\lambda(t))|^{1+\frac{1}{k}}} \leq C. \] (5.20)

It is easy to see from (5.19) that for all $n \geq n_0$, $t_n \leq t \leq t_{n+1}$ there holds
\[ \frac{1}{C} 2^{-\beta n} |\log(2^{-n})|^{1+\frac{1}{k}} \leq \lambda^\beta(t)|\log(\lambda(t))|^{1+\frac{1}{k}} \leq C 2^{-\beta n} |\log(2^{-n})|^{1+\frac{1}{k}}, \] (5.21)
for some universal constant $C > 0$.

Injecting (5.21) into (5.20), we have for all $n \geq n_0$,
\[ t_{n+1} - t_n \leq C 2^{-\beta n} |\log(2^{-n})|^{\frac{1}{k}} \] (5.22)
Summing (5.22) with respect to $n$ for all $n \geq n_0$, we obtain that $T < +\infty$ or equivalently, the solution blows up in finite time. On the other hand, we also obtain that for all $n \geq n_0$
\[ T - t_n \leq C \sum_{k=n}^{+\infty} 2^{-\beta k} k^{\frac{1}{k}} \leq C 2^{-\beta n} n^{\frac{1}{k}} \leq C \lambda^\beta(t_n)|\log(\lambda(t_n))|^{\frac{1}{k}}, \] (5.23)
where we use the fact that $\lambda(t_n) = 2^{-n}$ for the last inequality.

Next, for all $t$ close to $T$, there exists $n \geq n_0$ such that $t \in [t_n, t_{n+1}]$. From (5.19) and (5.23), we have
\[ \lambda^\beta(t)|\log(\lambda(t))|^{\frac{1}{k}} \geq C \lambda^\beta(t_n)|\log(\lambda(t_n))|^{\frac{1}{k}} \geq C(T - t_n) \geq C(T - t). \] (5.24)
Let $f(x) = x^\beta |\log x|^{\frac{1}{k}}$ for $x \in (0, x_0)$. It is easy to verify that for $x_0$ small enough, $f(x)$ is increasing in $x$. Since, for all $t$ close to $T$, there holds
\[ f \left( \left( \frac{T - t}{|\log(T - t)|^{\frac{1}{k}}} \right)^{\frac{1}{\beta}} \right) \leq \frac{T - t}{|\log(T - t)|^{\frac{1}{k}}} \log \left( \frac{T - t}{|\log(T - t)|^{\frac{1}{k}}} \right)^{\frac{1}{\beta}} \leq C(T - t) \leq C \lambda^\beta(t)|\log(\lambda(t))|^{\frac{1}{k}} = C f(\lambda(t)). \] (5.25)

For $t$ close to $T$, we have $|\log(\lambda(t))| \gg 1$, which implies
\[ C f(\lambda(t)) \leq f \left( (2C)^{\frac{1}{\beta}} \lambda(t) \right). \]
Injecting the above inequality into (5.25), we have
\[ \lambda(t) \geq C \left( \frac{T - t}{|\log(T - t)|^{\frac{1}{k}}} \right)^{\frac{1}{\beta}}, \] (5.26)
for all $t$ close to $T$, which implies that
\[ \| |D|^{\frac{1}{k}} u(t) \|_{L^2} \leq C \sqrt{\frac{|\log(T - t)|^{\frac{1}{k}}}{T - t}}, \] (5.27)
for all $t$ close to $T$. Now we conclude the proof of Theorem 1.6.
Appendix A. Estimates for the linearized operator $L^\beta$

This section is devoted to prove Lemma 2.5. First of all, we refer to [12, 13] for the proof of (1)–(3) of Lemma 2.5. While for (4), it is a consequence of the scaling rule (1.5). Let

$$Q_{\beta,\lambda}(x) = \frac{1}{\lambda^2} Q_{\beta}\left(\frac{x}{\lambda}\right).$$

Then $Q_{\beta,\lambda}$ satisfies the following equation:

$$|D|^\beta Q_{\beta,\lambda} + \lambda^\beta Q_{\beta,\lambda} - |Q_{\beta,\lambda}|^{2\beta} Q_{\beta,\lambda} = 0. \quad (A.1)$$

We note that

$$\Lambda Q_{\beta} = -\frac{\partial Q_{\beta,\lambda}}{\partial \lambda} \bigg|_{\lambda=1}$$

Hence, differentiating (A.1) with respect to $\lambda$ and taking $\lambda = 1$, we obtain (2.11).

We now turn to the proof of (2.11)–(2.14). It suffices to prove the estimate for $L_+^\beta$. The proof for $L_-^\beta$ is similar. Let $h = (L_+^\beta)^{-1} f$. Then we have

$$|D|^\beta h + h = (2\beta + 1)Q_{\beta}^{2\beta} h + f.$$ We use the fact that $Q_{\beta} \in W^{k,\infty}$ for all $k \in \mathbb{N}$ (since $Q_{\beta} \in H^s$ for all $s \geq 0$) and apply $\nabla^k + 1$ to the above equation to obtain:

$$\|h\|_{H^{s+k}} \sim \|(\nabla^k + 1)(|D|^\beta h + h)\|_{L^2} \lesssim \|f\|_{H^k} + \|Q_{\beta}\|_{W^{k,\infty}}^{2\beta} \|h\|_{H^k}.$$ From the kernel property of $L_+^\beta$, we know that (2.11) hold true for $k = 0$. Then, (2.11) follows from the above estimate and a standard induction argument (note that $\beta \geq 1$) on $k$.

Finally, for the decay estimates (2.13) and (2.14), we argue as the following. Let $h = (L_+^\beta)^{-1} f$. Then we have

$$h = \frac{1}{|D|^\beta + 1} \left( (2\beta + 1)Q_{\beta}^{2\beta} h \right) + \frac{1}{|D|^\beta + 1} f.$$ We denote by $K$ the kernel of $1/(|D|^\beta + 1)$. Namely, $K(\xi) = 1/(|\xi|^\beta + 1)$. Since $\beta \geq 1$, it is easy to see that $K \in L^p$ for all $1 < p < \infty$. Since $f \in L^2$, $Q_{\beta}^{2\beta} h \in L^2$, we know that

$$h(x) = K \ast \left( (2\beta + 1)Q_{\beta}^{2\beta} h + f \right)(x)$$

is continuous and vanishes as $|x| \to \infty$. From [12, Lemma A.4], we know that for $|x| \geq 1$,

$$|K(x)| \lesssim \frac{1}{|x|^{1+\beta}}.$$ We claim that the decay assumption on $f$ implies the following:

$$|K \ast f(x)| \lesssim \frac{1}{\langle x \rangle^{1+\beta}} \|\langle \cdot \rangle^{1+\beta} f(\cdot)\|_{L^\infty}. \quad (A.2)$$ Indeed, for $|x| \leq 1$, we have

$$|\langle x \rangle^{1+\beta} K \ast f(x)| \lesssim \|K\|_{L^2} \|f\|_{L^2} \lesssim \|\langle \cdot \rangle^{1+\beta} f(\cdot)\|_{L^\infty}.$$
While for $|x| > 1$, we have
\[
|\langle x \rangle^{1+\beta} K * f(x)| \lesssim \|\langle \cdot \rangle^{1+\beta} f(\cdot)\|_{L^\infty} \times \langle x \rangle^{1+\beta} \int \frac{1}{\langle y \rangle^{1+\beta}} |K(x-y)| \, dy
\]
\[
\lesssim \|\langle \cdot \rangle^{1+\beta} f(\cdot)\|_{L^\infty} \int_{|x-y| \leq \frac{|x|}{2}} \frac{\langle x \rangle^{1+\beta}}{\langle y \rangle^{1+\beta}} |K(x-y)| \, dy
\]
\[
+ \|\langle \cdot \rangle^{1+\beta} f(\cdot)\|_{L^\infty} \int_{|x-y| \geq \frac{|x|}{2}} \frac{\langle x \rangle^{1+\beta}}{\langle y \rangle^{1+\beta}} |K(x-y)| \, dy
\]
\[
= I + II.
\]
Since $|x| > 1$, so in the region $|x-y| \leq |x|/2$, we have $\langle \frac{x}{y} \rangle \leq \langle y \rangle \leq \frac{3 \langle x \rangle}{2}$. Hence,
\[
I \leq \|\langle \cdot \rangle^{1+\beta} f(\cdot)\|_{L^\infty} \left( \int_{|y| \leq \frac{|x|}{2}} |K(y)| \, dy + \int_{|y| \geq \frac{|x|}{2}} |K(y)| \, dy \right)
\]
\[
\lesssim \|\langle \cdot \rangle^{1+\beta} f(\cdot)\|_{L^\infty} \left[ \left( \int_{|y| \leq \frac{|x|}{2}} |K(y)|^2 \, dy \right)^{\frac{1}{2}} + \int_{|y| \geq \frac{|x|}{2}} \frac{1}{|y|^{1+\beta}} \, dy \right]
\]
\[
\lesssim \|\langle \cdot \rangle^{1+\beta} f(\cdot)\|_{L^\infty}.
\]
While for $II$, we have
\[
II \leq \|\langle \cdot \rangle^{1+\beta} f(\cdot)\|_{L^\infty} \langle x \rangle^{1+\beta} \int_{|x-y| \geq \frac{|x|}{2}} \frac{1}{|x-y|^{1+\beta}} \frac{1}{\langle y \rangle^{1+\beta}} \, dy
\]
\[
\lesssim \|\langle \cdot \rangle^{1+\beta} f(\cdot)\|_{L^\infty} \int \frac{1}{\langle y \rangle^{1+\beta}} \, dy \lesssim \|\langle \cdot \rangle^{1+\beta} f(\cdot)\|_{L^\infty},
\]
which concludes the proof of $(A.2)$.

Now we turn to the proof of the decay estimates $(2.13)$ and $(2.14)$. Recall that $h = (L^\beta_+)^{-1} f$ satisfies
\[
h(x) = K * [(2\beta + 1)Q^2_\beta h + f](x).
\]
Hence from the fact that $\|\langle \cdot \rangle^{1+\beta} Q^2_\beta h(\cdot)\|_{L^\infty} < +\infty$ and $(A.2)$, we have
\[
\|\langle \cdot \rangle^{1+\beta} h(\cdot)\|_{L^\infty} \lesssim \|\langle \cdot \rangle^{1+\beta} Q^2_\beta h(\cdot)\|_{L^\infty} + \|\langle \cdot \rangle^{1+\beta} f(\cdot)\|_{L^\infty}
\]
\[
\lesssim \left\| \langle \cdot \rangle^{1+\beta} Q^2_\beta h(\cdot) K * [(2\beta + 1)Q^2_\beta h + f](\cdot) \right\|_{L^\infty} + \|\langle \cdot \rangle^{1+\beta} f(\cdot)\|_{L^\infty}
\]
\[
\lesssim \left\| K * [(2\beta + 1)Q^2_\beta h](\cdot) \right\|_{L^\infty} + \|\langle \cdot \rangle^{1+\beta} f(\cdot)\|_{L^\infty}
\]
\[
\lesssim \|K\|_{L^2} \|h\|_{L^2} + \|\langle \cdot \rangle^{1+\beta} f(\cdot)\|_{L^\infty}
\]
\[
\lesssim \|\langle \cdot \rangle^{1+\beta} f(\cdot)\|_{L^\infty},
\]
where we use the following estimate for the last inequality
\[
\|h\|_{L^2} \leq \|h\|_{H^s} = \|L^\beta_+\|^{-1} f\|_{H^s} \lesssim \|f\|_{L^2} \lesssim \|\langle \cdot \rangle^{1+\beta} f(\cdot)\|_{L^\infty}.
\]
Now we conclude the proof of $(2.13)$ and $(2.14)$. 
APPENDIX B. PROOF OF THE GEOMETRICAL DECOMPOSITION

This section is devoted to prove the geometrical decomposition (3.4)–(3.11), using a standard argument of the implicit function theory. We first define

\[ U_a = \{ u \in H^\frac{1}{2} \| u - Q_\beta \|_{H^\frac{1}{2}} < a \}, \]

for some small enough constant \( a > 0 \). We then define a map \( \varepsilon = \varepsilon(b, \lambda, x, v, \gamma, u) \):

\[ (-a, a) \times (1 - a, 1 + a) \times (a, a)^3 \times U_a \mapsto H^\frac{1}{2} \]

as following:

\[
\varepsilon(y) = \lambda^\frac{1}{4} u(\lambda y + x) e^{-i\gamma} - W_{b,v}(y). \tag{B.1}
\]

We also define \( \sigma_i = \sigma_i(b, \lambda, x, v, \gamma, u) \) as

\[
\sigma_1 = (\varepsilon_1, \Lambda \Theta) - (\varepsilon_2, \Lambda \Sigma),
\sigma_2 = (\varepsilon_1, \partial_b \Theta) - (\varepsilon_2, \partial_b \Sigma),
\sigma_3 = (\varepsilon_1, \partial_\lambda \Theta) - (\varepsilon_2, \partial_\lambda \Sigma),
\sigma_4 = (\varepsilon_1, \nabla \Theta) - (\varepsilon_2, \nabla \Sigma),
\sigma_5 = (\varepsilon_1, \Lambda^2 \Theta) - (\varepsilon_2, \Lambda^2 \Sigma),
\]

where we still use the notation \( W_{b,v} = \Sigma + i\Theta \) and \( \varepsilon = \varepsilon_1 + i\varepsilon_2 \).

We claim that the relations \( \sigma_i = 0 \) implies an implicit map from \( U_a \) to \((-a, a) \times (1 - a, 1 + a) \times (a, a)^3 \). First, it is easy to verify that when \((b, \lambda, x, v, \gamma, u) = (0, 1, 0, 0, 0, Q_\beta)\) we have \( \sigma_i = 0 \) for all \( i = 1, 2, 3, 4, 5 \). By the implicit function theory, we only need to show that the Jacobian matrix at \((0, 1, 0, 0, 0, Q_\beta)\) is non-degenerate. Indeed, at \((0, 1, 0, 0, 0, Q_\beta)\) we have

\[
\frac{\partial \varepsilon}{\partial b} = -\partial_b W_{b,v}\big|_{(b,v)=(0,0)} = -iS_1, \quad \frac{\partial \varepsilon}{\partial v} = -\partial_v W_{b,v}\big|_{(b,v)=(0,0)} = -iG_1,
\]

\[
\frac{\partial \varepsilon}{\partial \lambda} = \Lambda Q_\beta, \quad \frac{\partial \varepsilon}{\partial x} = \nabla Q_\beta, \quad \frac{\partial \varepsilon}{\partial \gamma} = -iQ_\beta,
\]

where we recall that \( S_1 \) is an even function with \( L_- S_1 = \Lambda Q_\beta \) and \( G_1 \) is an odd function with \( L_- G_1 = -\nabla Q_\beta \). Hence we can compute the Jacobian matrix at \((0, 1, 0, 0, 0, Q_\beta)\) as following:

\[
\frac{\partial \sigma_1}{\partial b} = -(S_1, \Lambda Q_\beta), \quad \frac{\partial \sigma_1}{\partial \lambda} = 0, \quad \frac{\partial \sigma_1}{\partial x} = 0, \quad \frac{\partial \sigma_1}{\partial v} = 0, \quad \frac{\partial \sigma_1}{\partial \gamma} = 0, \tag{B.2}
\]

\[
\frac{\partial \sigma_2}{\partial b} = 0, \quad \frac{\partial \sigma_2}{\partial \lambda} = -(S_1, \Lambda Q_\beta), \quad \frac{\partial \sigma_2}{\partial x} = 0, \quad \frac{\partial \sigma_2}{\partial v} = 0, \quad \frac{\partial \sigma_2}{\partial \gamma} = 0, \tag{B.3}
\]

\[
\frac{\partial \sigma_3}{\partial b} = 0, \quad \frac{\partial \sigma_3}{\partial \lambda} = 0, \quad \frac{\partial \sigma_3}{\partial x} = (G_1, \nabla Q_\beta), \quad \frac{\partial \sigma_3}{\partial v} = 0, \quad \frac{\partial \sigma_3}{\partial \gamma} = 0, \tag{B.4}
\]

\[
\frac{\partial \sigma_4}{\partial b} = 0, \quad \frac{\partial \sigma_4}{\partial \lambda} = 0, \quad \frac{\partial \sigma_4}{\partial x} = 0, \quad \frac{\partial \sigma_4}{\partial v} = (\nabla Q_\beta, G_1), \quad \frac{\partial \sigma_4}{\partial \gamma} = 0, \tag{B.5}
\]

\[
\frac{\partial \sigma_5}{\partial b} = (\Lambda^2 Q_\beta, S_1), \quad \frac{\partial \sigma_5}{\partial \lambda} = 0, \quad \frac{\partial \sigma_5}{\partial x} = 0, \quad \frac{\partial \sigma_5}{\partial v} = 0, \quad \frac{\partial \sigma_5}{\partial \gamma} = -(Q_\beta, \Lambda^2 Q_\beta). \tag{B.6}
\]

From (2.10), we know that

\[
(S_1, \Lambda Q_\beta) = (S_1, L_-^3 S_1) \neq 0, \quad (G_1, \nabla Q_\beta) = -(G_1, L_-^3 G_1) \neq 0.
\]
We also have \((Q_β, \Lambda^2 Q_β) = (-\Lambda Q_β, \Lambda Q_β) \neq 0\). Injecting these estimate into (B.2)–(B.6), we obtain that the determinant of the Jacobian matrix is not 0 at \((0, 1, 0, 0, 0, Q_β)\). Now using implicit function theory we have: for all \(u \in U_a\), there exist constant \(\delta = \delta(a) > 0\) and \((b_u, \lambda_u, x_u, v_u, \gamma_u) \in (-\delta, \delta) \times (1-\delta, 1+\delta) \times (\delta, \delta)^3\), such that

\[
\varepsilon_u(y) := \lambda_u^2 u(\lambda_u y + x_u) e^{-i\gamma_u} - W_{b_u,v_u}(y) \tag{B.7}
\]
satisfies the orthogonality condition (3.5)–(3.9).

Now, from (3.3), we know that for all \(t \in [0, T)\),

\[
u_1(t, \cdot) := e^{-i\gamma_1(t)} [\lambda_1(t)]^{1/2} u(t, \lambda_1(t) \cdot + x_1(t)) \in U_a,
\]
provided that \(\alpha_0\) is small enough. Applying (B.7) to the above function \(u_1(t, x)\), we can find \((\tilde{b}(t), \tilde{\lambda}(t), \tilde{x}(t), \tilde{v}(t), \tilde{\gamma}(t)) \in (-\delta, \delta) \times (1-\delta, 1+\delta) \times (\delta, \delta)^3\), such that

\[
\bar{\varepsilon}(t, y) := e^{-i\gamma_1(t)} [\tilde{\lambda}(t)]^{1/2} u_1(t, \tilde{\lambda}(t) y + \tilde{x}(t)) - W_{\tilde{b}(t),\tilde{v}(t)}(y)
\]
satisfies the orthogonality condition (3.5)–(3.9).

Next, we let

\[
\begin{align*}
b(t) &= \tilde{b}(t), \quad \lambda(t) = \lambda_1(t) \tilde{\lambda}(t), \quad x(t) = \lambda_1(t) \tilde{x}(t) + x_1(t), \\
v(t) &= \tilde{v}(t), \quad \gamma(t) = \gamma_1(t) + \tilde{\gamma}(t).
\end{align*}
\]

It is to see that the parameters chosen above satisfies the geometrical condition (3.4) and the orthogonality condition (3.5)–(3.9). Finally, we can easily see that the a priori smallness estimate (3.11) follows from the fact that

\[
(\tilde{b}(t), \tilde{\lambda}(t), \tilde{x}(t), \tilde{v}(t), \tilde{\gamma}(t)) \in (-\delta, \delta) \times (1-\delta, 1+\delta) \times (\delta, \delta)^3,
\]

which concludes the proof of (3.2).

**APPENDIX C. PROOF OF THE HARDY’S TYPE ESTIMATE**

This section is devoted to prove the Hardy’s type estimate (3.30). We first choose a cut-off function \(\chi\) such that \(\chi(y) = 1\), if \(|y| < 1\); \(\chi(y) = 0\), if \(|y| > 2\). Then we have

\[
\int \frac{|f(y)|^2}{(1 + |y|)^\beta} \, dy \lesssim \int \frac{|f(y)\chi(y)|^2}{(1 + |y|)^\beta} \, dy + \int \frac{|f(y)(1 - \chi(y))|^2}{(1 + |y|)^\beta} \, dy := I + II. \tag{C.1}
\]

For \(I\) it is to obtain:

\[
I \lesssim \int_{|y| \leq 2} |f|^2 \lesssim \int |f|^2 e^{-|y|}. \tag{C.2}
\]

While for \(II\), we first introduce the fractional Hardy’s inequality in dimension 1 introduced in [1],

\[
\frac{1}{2} \int_D \int_D \frac{|u(x) - u(y)|^2}{|x - y|^{d+\alpha}} \, dx \, dy \geq \kappa_{d,\alpha} \int_D \frac{|u(x)|^2}{x_d^{2\alpha}} \, dx, \tag{C.3}
\]

for all \(u \in C_b(D)\). Here \(D = \{(x_1, \ldots, x_d)|x_d > 0\}\) is the upper half space and \(\alpha \in (0, 2)\), \(\kappa_{d,\alpha} > 0\) are some constants.

Applying (C.3) for \(\alpha = \beta\) and \(d = 1\), together with the following characterization of the fractional Sobolev norm:

\[
\int |D|^{\|eta/2\|} f^2 \sim \int \int \frac{|f(x) - f(y)|^2}{|x - y|^{1+\beta}} \, dx \, dy, \tag{C.4}
\]

\(^{22}\) Here \(\delta(a) \to 0\) as \(a \to 0\).
we obtain
\[ \int \frac{|f(x)|^2}{|x|^{2\beta}} \, dx \lesssim \int \int \frac{|f(x) - f(y)|^2}{|x-y|^{1+\beta}} \, dxdy \sim \int ||D|^{\frac{\beta}{2}} f|^2, \] (C.5)
for all \( f \in H^{\frac{2\beta}{\beta}} \cap C_0(\mathbb{R}\setminus\{0\}).

Hence, we have
\[ II \lesssim \int \int \frac{|f(x)(1 - \chi(x)) - f(y)(1 - \chi(y))|^2}{|x-y|^{1+\beta}} \, dxdy \]
\[ \lesssim \int \int \frac{|f(x) - f(y)|^2(1 - \chi(x))^2}{|x-y|^{1+\beta}} \, dxdy + \int \int \frac{|f(y)|^2|\chi(x) - \chi(y)|^2}{|x-y|^{1+\beta}} \, dxdy \]
\[ \lesssim \int ||D|^{\frac{\beta}{2}} f|^2 + \int |f(y)|^2 \, dy \int \frac{|\chi(x) - \chi(y)|^2}{|x-y|^{1+\beta}} \, dx. \] (C.6)

We denote by
\[ N_\prec = \int |f(y)|^2 \, dy \int_{|t| \leq 1} \frac{|\chi(y) - \chi(y - t)|^2}{|t|^{1+\beta}} \, dt, \] (C.7)
\[ N_\succ = \int |f(y)|^2 \, dy \int_{|t| \geq 1} \frac{|\chi(y) - \chi(y - t)|^2}{|t|^{1+\beta}} \, dt. \] (C.8)

For \( N_\prec \), using the Leibniz’s rule we have
\[ N_\prec \lesssim \int |f(y)|^2 \, dy \int_{|t| \leq 1} |t|^{-1-\beta} |t|^{1+\beta} \, dt \int_0^1 \chi'(y - t + st) \, ds \, dt. \] (C.9)

Since, \( \chi'(y) = 0 \), if \( |y| > 2 \), we have for all \( |t| \leq 1, s \in [0,1] \) and \( |y| \geq 3 \),
\[ \chi'(y - t + st) = 0, \]
which implies that for all \( |t| \leq 1, s \in [0,1] \) and \( y \in \mathbb{R}, \)
\[ \chi'(y - t + st) \lesssim e^{-|y|}. \] (C.10)

Injecting (C.10) into (C.9), we obtain\(^{23}\)
\[ N_\prec \lesssim \int |f(y)|^2 e^{-|y|} \, dy \int_{|t| \leq 1} |t|^{-1-\beta} \, dt \int_0^1 ds \lesssim \int |f(y)|^2 e^{-|y|} \, dy. \] (C.11)

For \( N_\succ \), we have
\[ N_\succ \lesssim \int |f(y)|^2 \, dy \left( \int_{|t| \geq 1} \frac{|\chi(y)|^2}{|t|^{1+\beta}} \, dt + \int_{|t| \geq 1} \frac{|\chi(y - t)|^2}{|t|^{1+\beta}} \, dt \right) \]
\[ \lesssim \int |f(y)| \chi(y) |^2 \, dy + \int_{|y| \leq A} |f(y)|^2 \, dy \int_{|t| \geq 1} \frac{|\chi(y - t)|^2}{|t|^{1+\beta}} \, dt \]
\[ + \int_{|y| \geq A} |f(y)|^2 \, dy \int_{|t| \geq 1} \frac{|\chi(y - t)|^2}{|t|^{1+\beta}} \, dt \]
\[ := \tilde{I} + \tilde{II} + \tilde{III}, \] (C.12)
with some large constant \( A > 10 \) to be chosen later.

\(^{23}\)Here, we use the fact that \( \beta < 2 \).
It is easy to obtain that
\[ \dot{I} \lesssim \int |f(y)|^2 e^{-|y|} \, dy, \tag{C.13} \]
and
\[ \dot{II} \lesssim \int_{|y| \leq A} |f(y)|^2 \, dy \leq e^A \int |f(y)|^2 e^{-|y|} \, dy. \tag{C.14} \]

While for \( \dot{III} \), we have
\[ \dot{III} \lesssim \int_{|y| \geq A} |f(y)|^2 \, dy \int_{\{t||t|\geq 1,|y-t|\leq 2\}} |t|^{-1-\beta} \, dt. \tag{C.15} \]

Since \( |y| \geq A > 10 \), we have for all \( t \) with \( |t| \geq 1 \) and \( |y-t| \leq 2 \), there holds \( |t| \geq \frac{1}{2} |y| \) and
\[ \int_{\{t||t|\geq 1,|y-t|\leq 2\}} |t|^{-1-\beta} \, dt \sim \frac{1}{|y|^{1+\beta}}. \]

Thus, we have
\[ \dot{III} \lesssim \int_{|y| \geq A} |f(y)|^2 \, dy \leq \frac{C}{A} \int \frac{|f(y)|^2}{(1+|y|)^{\beta}} \, dy. \tag{C.16} \]

Combining (C.13), (C.14) and (C.16), we have
\[ N_> \leq C_A \int |f(y)|^2 e^{-|y|} \, dy + \frac{C}{A} \int \frac{|f(y)|^2}{(1+|y|)^{\beta}} \, dy, \tag{C.17} \]
with some constant \( C_A > 0 \) depending only on \( A \).

Finally, by collection all the estimate above, we have
\[ \int \frac{|f(y)|^2}{(1+|y|)^{\beta}} \, dy \leq C_A \left( \int ||D|^2 f|^2 + \int |f(y)|^2 e^{-|y|} \, dy \right) + \frac{C}{A} \int \frac{|f(y)|^2}{(1+|y|)^{\beta}} \, dy. \]

Choosing \( A \) large enough, we obtain (3.30) immediately.

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