Abstract. For the least squares estimator \( \hat{\theta} \) for the drift parameter \( \theta \) of an Ornstein-Uhlenbeck process driven by fractional Brownian motion with Hurst index \( H \in \left[ \frac{1}{2}, \frac{3}{4} \right] \), we show the Berry-Esseen bound of the Kolmogorov distance between Gaussian random variable and \( \sqrt{T}(\hat{\theta}_T - \theta) \) with \( H \in \left( \frac{1}{2}, \frac{3}{4} \right) \) respectively is \( \frac{1}{\sqrt{T}} \sqrt{T - 4H} \), (1.3) respective). The strategy is to exploit Corollary 1 of Kim and Park [Journal of Multivariate Analysis 155, P284-304.(2017)].

Keywords: Berry-Esseen bound; Fourth Moment theorems; fractional Ornstein-Uhlenbeck process; Malliavin calculus.

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1. Introduction

Let \( B^H_t \) be a 1-dimensional fractional Brownian motion with Hurst index \( H \in \left( \frac{1}{2}, \frac{3}{4} \right) \), the least squares estimate of the drift coefficient of 1-dimensional Ornstein-Uhlenbeck process

\[
dX_t = -\theta X_t dt + dB^H_t, \quad X_0 = 0, \quad 0 \leq t \leq T, \tag{1.1}
\]
is given by a ratio of two Gaussian functionals [4]:

\[
\hat{\theta}_T = -\frac{\int_0^T X_t dX_t}{\int_0^T X_t^2 dt} = \theta - \frac{\int_0^T X_t dB^H_t}{\int_0^T X_t^2 dt}. \tag{1.2}
\]

The strong consistency and asymptotic normality of the estimator \( \hat{\theta}_T \) are shown for \( H \in \left( \frac{1}{2}, \frac{3}{4} \right) \) in [4], and recently, this findings is extended to the case of \( H \in \left( 0, \frac{3}{4} \right) \) in [5].

The question naturally arises whether the Berry-Esseen bound of \( \sqrt{T}(\hat{\theta}_T - \theta) \) can be obtained. When \( H = \frac{1}{2} \), it is well known that the Berry-Esseen bound can be shown by means of squeezing techniques, please refer to [2, 3] and the references therein. Recently, two new approaches based on the Malliavin calculus are proposed to show the Berry-Esseen bound [6, 7]. But the case of \( H \neq \frac{1}{2} \) is still unsolved up to now.

In the present paper, we will give an affirmative answer to the case of \( H \in \left[ \frac{1}{2}, \frac{3}{4} \right] \) using one of these two approaches (see also Theorem 2.1 below).

**Theorem 1.1.** Let \( Z \) be a standard Gaussian random variable. When \( H \in \left[ \frac{1}{2}, \frac{3}{4} \right] \), there exists a constant \( C_{\theta, H} \) such that when \( T \) is large enough,

\[
\sup_{z \in \mathbb{R}} \left| P\left( \sqrt{\frac{T}{\theta \sigma^2_H}}(\hat{\theta}_T - \theta) \leq z \right) - P(Z \leq z) \right| \leq \frac{C_{\theta, H}}{\sqrt{T - 4H}}. \tag{1.3}
\]
when $H = \frac{3}{4}$, there exists a constant $C_0$ such that when $T$ is large enough,
\[
\sup_{z \in \mathbb{R}} \left| P \left( \frac{T}{\theta \sigma_H^2 \log T} (\hat{\theta}_T - \theta) \leq z \right) - P(Z \leq z) \right| \leq \frac{C_0}{\log T},
\]
(1.4)
where $\sigma_H^2$ is given in [4, 5] as follows:
\[
\sigma_H^2 = \begin{cases} 
(4H - 1)(1 + \frac{\Gamma(2H)}{(2H)^{2H}}), & H \in [\frac{1}{2}, \frac{2}{3}), \\
\frac{4}{27}, & H = \frac{2}{3}.
\end{cases}
\]
(1.5)

Although the lower bound of Kolmogorov distance between $\sqrt{T}(\hat{\theta}_T - \theta)$ and the Gaussian random variable is known in case of $H = \frac{1}{2}$ [6], we do not give the similar result in case of $H \neq \frac{1}{2}$. Throughout the paper we assume $H \geq \frac{1}{2}$. The case $H < \frac{1}{2}$ will involve much more complex computations and we believe that in this case the upper bound is $\frac{1}{\sqrt{T}}$. We shall investigate this case separately.

2. Preliminary

Let $\alpha_H = H(2H - 1)$. Define the Hilbert space
\[
\mathcal{H} = \left\{ f : \mathbb{R}_+ \to \mathbb{R}, \int_0^\infty \int_0^\infty f(t)f(s) |t - s|^{2H-2} dt ds < \infty \right\}.
\]
Then a Gaussian isormal process associated with $\mathcal{H}$ is given by Wiener integrals with respect to a fBm for any deterministic kernel $\in \mathcal{H}$:
\[
B^H(f) = \int_0^\infty f(s) dB^H_s.
\]

Let $H_n$ be the $n$-th Hermite polynomial. The closed linear subspace $\mathcal{H}_n$ of $L^2(\Omega)$ generated by $\{ H_n(B^H(f)) : f \in \mathcal{H}, \|f\|_\mathcal{H} = 1 \}$ is called the $n$-th Wiener-Ito chaos. The linear isometric mapping $I_n : \mathcal{H} \to \mathcal{H}_n$ given by $I_n(\mathcal{H}) = n!H_n(B^H(f))$ is called the $n$-th multiple Wiener-Ito integral. For any $f \in \mathcal{H}$, define $I_n(f) = I_n(f)$ where $\hat{f}$ is the symmetrization of $f$.

Given $f \in \mathcal{H}^{\otimes p}$ and $g \in \mathcal{H}^{\otimes q}$ and $r = 1, \ldots, p \land q$, $r$-th contraction between $f$ and $g$ is the element of $\mathcal{H}^{\otimes (p+q-2r)}$ defined by
\[
f \otimes_r g(t_1, \ldots, t_{p+q-2r}) = \alpha_{2r}^{p+q} \int_{\mathbb{R}_+^{2r}} |u_1 - v_1|^{2H-2} \cdots |u_r - v_r|^{2H-2} f(t_1, \ldots, t_{p-r}, u_1, \ldots, u_r) \times g(t_{p-r+1}, \ldots, t_{p+q-2r}, v_1, \ldots, v_r) du dv,
\]
where $\bar{u} = (u_1, \ldots, u_r), \bar{v} = (v_1, \ldots, v_r)$.

We will make use of the following estimate of the Kolmogorov distance between a nonlinear Gaussian functional and the standard normal (see Corollary 1 of [7]).

**Theorem 2.1** (Kim, Y. T., & Park, H. S). Suppose that $\varphi_T(t, s)$ and $\psi_T(t, s)$ are two functions on $\mathcal{H}^{\otimes 2}$. Let $b_T$ be a positive function of $T$ such that $I_2(\varphi_T) + b_T > 0$ a.s. If $\Psi_i(T) \to 0$, $i = 1, 2, 3$ as $T \to \infty$, then there exist a constant $c$ such that for $T$ large enough,
\[
\sup_{z \in \mathbb{R}} \left| P \left( \frac{I_2(\varphi_T)}{I_2(\psi_T) + b_T} \leq z \right) - P(Z \leq z) \right| \leq c \times \max_{i=1,2,3} \Psi_i(T),
\]
(2.1)
where
\[
\Psi_1(T) = \frac{1}{b_T^2} \sqrt{\left[ \frac{b_T}{2} - 2 \| \varphi_T \|_{\mathcal{H}^2}^2 \right]^2 + 8 \| \varphi_T \otimes_1 \varphi_T \|_{\mathcal{H}^2}^2},
\]
\[
\Psi_2(T) = \frac{2}{b_T^3} \sqrt{2 \| \varphi_T \otimes_1 \psi_T \|_{\mathcal{H}^2}^2 + (\varphi_T, \psi_T)^2_{\mathcal{H}^2}},
\]
\[
\Psi_3(T) = \frac{2}{b_T^2} \sqrt{\| \psi_T \|_{\mathcal{H}^2}^4 + 2 \| \varphi_T \otimes_1 \psi_T \|_{\mathcal{H}^2}^2}.
\]

3. Proof of the main theorem

It follows from Eq.(1.2) and the product formula of multiple integrals that
\[
\sqrt{\frac{T}{\theta \sigma_H^2}} (\hat{\theta}_T - \theta) = \frac{I_2(f_T)}{I_2(g_T) + b_T},
\]
where
\[
f_T(t, s) = \frac{1}{2\sqrt{2\theta \sigma_H^4 T}} e^{-\theta |t - s|} 1_{(0 \leq s, t \leq T)},
\]
\[
g_T(t, s) = \sqrt{\frac{\sigma_H}{\theta T}} f_T - \frac{1}{2\theta T} h_T,
\]
\[
h_T(t, s) = e^{-\theta (T - t) - \theta (T - s)} 1_{(0 \leq s, t \leq T)},
\]
\[
b_T(t, s) = \frac{1}{T} \int_0^T \left\| e^{-\theta (t - \cdot)} \right\|_{\mathcal{B}^1}^2 dt.
\]
The reader can also refer to Eq.(17)-(19) of [6] for details.

We need several lemmas before the proof of Theorem 1.1. The following estimate is cited from Proposition 7 or (3.17) of [5].

Lemma 3.1. When \( H \in \left[ \frac{1}{2}, \frac{3}{4} \right] \), there exists a constant \( C_{\theta, H} \) such that
\[
\left\| f_T \otimes_1 f_T \right\|_{\mathcal{S}^2} \leq \frac{C_{\theta, H}}{T^{3 - 4H}}.
\]

We will use \( f_1(T) \sim f_2(T) \) to denote that \( f_1 \) is equal to \( f_2 \) asymptotically, i.e., \( \lim_{T \to \infty} \frac{f_1(T)}{f_2(T)} = 1 \). Since \( H > \frac{1}{2} \), we can write \( b_T \) as
\[
b_T = \frac{\alpha_H}{T} \int_0^T dt \int_{[0,t]^2} e^{-\theta (t-u) - \theta (t-v)} |u - v|^{2H-2} du dv,
\]
\[
= \frac{2\alpha_H}{T} \int_0^T dt \int_{0 \leq u \leq v \leq t} e^{-\theta (t-u) - \theta (t-v)} |u - v|^{2H-2} du dv.
\]

Lemma 3.2. As \( T \to \infty \), \( b_T \) converges to \( H \Gamma(2H) \theta^{-2H} \) with an exponential rate.

Proof. The case of \( H = \frac{1}{2} \) is simple. When \( H > \frac{1}{2} \), by the L’Hospital’s rule, we have that as \( T \to \infty \),
\[
b_T - \frac{H \Gamma(2H) \theta^{-2H}}{2\alpha_H} = \frac{1}{T} \left[ \int_0^T dt \int_{0 \leq u \leq v \leq t} e^{-\theta (t-u) - \theta (t-v)} |u - v|^{2H-2} du dv - \frac{\Gamma(2H) - 1}{2\theta^{2H-1} T} \right].
\]
Similarly, we can show that

\[ \int_{0}^{T} e^{-\theta(T-u)-\theta(T-v)} |u-v|^{2H} \, du \, dv = \frac{\Gamma(2H-1)}{2 \theta^{2H}} \]

(let \( a = T - v, b = v - u \))

\[ = \int_{a+b \leq T, a,b \geq 0} e^{-\theta(2a+b)|2H-2} \, da \, db - \frac{\Gamma(2H-1)}{2 \theta^{2H}} \]

\[ = \int_{a+b \geq T, a,b \geq 0} e^{-\theta(2a+b)|2H-2} \, da \, db \]

\[ = \int_{0}^{T} e^{-\theta b2H-2} \, db \int_{T-b}^{\infty} e^{-2\theta b} \, da + \int_{T}^{\infty} e^{-\theta b2H-2} \, db \int_{0}^{\infty} e^{-2\theta a} \, da \]

\[ = \frac{1}{2 \theta e^{2\theta T}} \int_{0}^{T} e^{\theta b2H-2} \, db + \frac{1}{2 \theta} \int_{T}^{\infty} e^{-\theta b2H-2} \, db, \]

which converges to zero with an exponential rate. \( \Box \)

**Lemma 3.3.** Let \( h_T \) be given as in (3.4). Then as \( T \to \infty \),

\[ \frac{1}{\sqrt{T}} h_T \to 0, \quad \text{in} \quad \mathcal{F}^\otimes_2. \] (3.7)

**Proof.** The case of \( H = \frac{1}{2} \) is simple. When \( H > \frac{1}{2} \), by the symmetrical property and the L’Hospital’s rule, we have that as \( T \to \infty \),

\[ \frac{1}{\alpha_H^2 T} \|h_T\|^2_{\mathcal{F}^\otimes_2} = \frac{1}{T} \int_{[0,T]^4} e^{-\theta(T-t_1)+(T-s_1)+(T-t_2)+(T-s_2)} |t_1 - t_2|^{2H-2} |s_1 - s_2|^{2H-2} \, d\vec{s} \]

\[ = \frac{8}{T e^{4\theta T}} \int_{0 \leq t_2 \leq t_1 \leq T, 0 \leq s_1 \leq s_2 \leq T, s_1 \leq t_1} e^{\theta (t_1 + t_2 + s_1 + s_2)} |t_1 - t_2|^{2H-2} |s_1 - s_2|^{2H-2} \, d\vec{s} \]

\[ \sim \frac{8}{(1 + 4\theta T) e^{4\theta T}} \int_{0 \leq t_2 \leq t_1 \leq T, 0 \leq s_1 \leq s_2 \leq T} e^{\theta (t_1 + s_1 + s_2)} (T - t_2)^{2H-2} (s_1 - s_2)^{2H-2} \, dt_2 \, ds_2. \]

We divide the domain \( \{0 \leq t_2 \leq T, 0 \leq s_2 \leq s_1 \leq T, s_1 \leq T \} \) into three disjoint regions according to the distinct orders of \( s_1, s_2, t_2 \):

\[ \Delta_1 = \{0 \leq s_2 \leq s_1 \leq t_2 \leq T \}, \quad \Delta_2 = \{0 \leq s_2 \leq t_2 \leq s_1 \leq T \}, \quad \Delta_3 = \{0 \leq t_2 \leq s_2 \leq s_1 \leq T \}. \]

We also denote \( I_i = \int_{\Delta_i} e^{\theta (t_1 + s_1 + s_2 - 3T)} (T - t_2)^{2H-2} (s_1 - s_2)^{2H-2} \, dt_2 \, ds_2. \) Thus, we have that as \( T \to \infty \),

\[ \frac{1}{\alpha_H T} \|h_T\|^2_{\mathcal{F}^\otimes_2} \sim \frac{8}{1 + 4\theta T} (I_1 + I_2 + I_3). \] (3.8)

Firstly, we consider \( I_1 \). By making the change of variables \( T - t_2 = x, t_2 - s_1 = y, s_1 - s_2 = z \), we have that

\[ I_1 = \int_{\mathbb{R}^3_+, x+y+z \leq T} e^{-\theta(3x+2y+z)} x^{2H-2} z^{2H-2} \, dx dy dz \]

\[ < \int_{\mathbb{R}^3_+} e^{-\theta(3x+2y+z)} x^{2H-2} z^{2H-2} \, dx dy dz < \infty. \]

Similarly, we can show that \( I_2, I_3 < \infty \), which implies that \( \frac{1}{\sqrt{T}} \|h_T\|^2_{\mathcal{F}^\otimes_2} \to 0 \) as \( T \to \infty \). \( \Box \)
Lemma 3.4. Let $g_T$ be given as in (3.3). When $H \in \left[\frac{1}{2}, \frac{3}{4}\right)$, we have that as $T \to \infty$,

$$T \|g_T\|^2_{\mathcal{BH}^3} \to \frac{\delta_H}{2\theta^{1+4H}}, \quad T\langle f_T, g_T \rangle_{\mathcal{BH}^3} \to \frac{\delta_H^2}{4\theta^{1+8H} \sigma_H^2},$$

$$T \|f_T \otimes_1 g_T\|^2_{\mathcal{BH}^3} \to 0, \quad T \|g_T \otimes_1 g_T\|^2_{\mathcal{BH}^3} \to 0; \quad (3.9)$$

when $H = \frac{3}{4}$, we have that

$$\frac{T}{\log T} \|g_T\|^2_{\mathcal{BH}^3} \to \frac{\delta_H}{2\theta^{1+4H}}, \quad \frac{T}{\log^2 T} \langle f_T, g_T \rangle_{\mathcal{BH}^3} \to \frac{\delta_H^2}{4\theta^{1+8H} \sigma_H^2},$$

$$\frac{T}{\log T} \|f_T \otimes_1 g_T\|^2_{\mathcal{BH}^3} \to 0, \quad \frac{T}{\log T} \|g_T \otimes_1 g_T\|^2_{\mathcal{BH}^3} \to 0,$$

where $\delta_H$ is given in [4]:

$$\delta_H = \left\{ \begin{array}{ll}
H^2(4H - 1)(\Gamma^2(2H) + \frac{\Gamma(2H)(3 - 4H)(4H - 1)}{\Gamma(2 - 2H)}), & H \in \left[\frac{1}{2}, \frac{3}{4}\right),
\frac{9}{16}, & H = \frac{3}{4}.
\end{array} \right.$$

Proof. We only show the case of $H \in \left[\frac{1}{2}, \frac{3}{4}\right)$. The case of $H = \frac{3}{4}$ is similar.

It follows from (3.3) that

$$T \|g_T\|^2_{\mathcal{BH}^3} = \frac{\sigma_H^2}{\theta} \|f_T\|^2_{\mathcal{BH}^3} + \frac{1}{4\theta^2 T} \|h_T\|^2_{\mathcal{BH}^3} - \frac{\sqrt{T}}{\theta} \langle f_T, h_T \rangle_{\mathcal{BH}^3}.$$

The Cauchy-Schwarz inequality implies that the third term is bounded by $\frac{\sqrt{T}}{\theta} \|f_T\| \cdot \|h_T\|$. By Lemma 3.3 and Eq.(3.12)-(3.14) of [4], we have that

$$\lim_{T \to \infty} T \|g_T\|^2_{\mathcal{BH}^3} = \frac{\sigma_H^2}{\theta} \lim_{T \to \infty} \|f_T\|^2_{\mathcal{BH}^3} = \frac{\delta_H}{2\theta^{1+4H}}.$$

Similarly, we have that

$$\lim_{T \to \infty} \sqrt{T} \langle f_T, g_T \rangle_{\mathcal{BH}^3} = \sqrt{\frac{\sigma_H^2}{\theta}} \lim_{T \to \infty} \|f_T\|^2_{\mathcal{BH}^3} = \sqrt{\frac{\theta}{\sigma_H^2}} \frac{\delta_H}{2\theta^{1+4H}}.$$

Next, it is clear that

$$\sqrt{T} f_T \otimes_1 g_T = \sqrt{\frac{\sigma_H^2}{\theta}} f_T \otimes_1 f_T - \frac{1}{2\theta} f_T \otimes_1 \left( \frac{1}{\sqrt{T}} h_T \right).$$

The fourth moment theorem implies that $f_T \otimes_1 f_T \to 0$ in $\mathcal{BH}^3$ as $T \to \infty$, please refer to [4, 5] for details. The Cauchy-Schwarz inequality (or Lemma 4.2 of [1]) and Lemma 3.3 imply that as $T \to \infty$,

$$\left\| f_T \otimes_1 \left( \frac{1}{\sqrt{T}} h_T \right) \right\|_{\mathcal{BH}^3} \leq \|f_T\|_{\mathcal{BH}^3} \cdot \frac{1}{\sqrt{T}} \|h_T\|_{\mathcal{BH}^3} \to 0,$$

which implies that $\sqrt{T} f_T \otimes_1 g_T \to 0$ in $\mathcal{BH}^3$.

Finally, the Cauchy-Schwarz inequality or Lemma 4.2 of [1] implies that

$$\sqrt{T} \|g_T \otimes_1 g_T\|^2_{\mathcal{BH}^3} \leq \sqrt{T} \|g_T\|^2_{\mathcal{BH}^3} = \frac{1}{\sqrt{T}} \cdot T \|g_T\|^2_{\mathcal{BH}^3} \to 0.$$

\[\square\]
Lemma 3.5. When \( H \in \left[ \frac{3}{4}, \frac{1}{2} \right) \), the convergence speed of \( 2 \| f_T \|^2_{\ell^2} \rightarrow \left[ H \Gamma(2H) \theta^{-2H} \right]^2 \) is at least \( \frac{1}{2^3 T^2} \) as \( T \rightarrow \infty \). When \( H = \frac{3}{4} \), the convergence speed of \( \frac{2 \| f_T \|^2_{\ell^2}}{\log T} \rightarrow \frac{9\pi}{64T^2} \) is at least \( \frac{1}{T} \) as \( T \rightarrow \infty \).

Proof. The case of \( H = \frac{1}{2} \) is easy.

Next, suppose that \( H \in \left( \frac{1}{2}, \frac{3}{4} \right) \). By the symmetrical property, the L'Hospital's rule and Lemma 5.3 in the web-only Appendix of \cite{4}, we have that as \( T \rightarrow \infty \),

\[
\left\{ 2 \| f_T \|^2_{\ell^2} \left[ H \Gamma(2H) \theta^{-2H} \right]^2 \right\} \times \frac{\theta \sigma_H^2}{2a_H^2} = \frac{1}{4T} \int_{[0,T]^3} \frac{e^{-2T\varepsilon_1 - \theta |s_2|} |t_1 - T|^2 2H - 2|s_1 - s_2|^{2H - 2} d\bar{s}}{\theta \sigma_H^2} - \frac{\theta \sigma_H^2}{2a_H^2}.
\]

\[
\sim \int_{[0,T]^3} e^{-\theta \varepsilon_1 - \theta |T - s_2|} |t_1 - T|^2 2H - 2|s_1 - s_2|^{2H - 2} d\bar{s} - \frac{\theta \sigma_H^2 \delta_H}{2a_H^2}
\]

(1)

\[
\text{(let } x = T - s_2, y = T - s_1, z = T - t_1 \})
\]

\[
\int_{[0,T]^3} e^{-\theta (x + y - z)} z^{2H - 2} |x - y|^{2H - 2} dxdydz - \frac{\theta \sigma_H^2 \delta_H}{2a_H^2}
\]

\[
\int_{[0,T]^3} e^{-\theta (x + y - z)} z^{2H - 2} |x - y|^{2H - 2} dxdydz
\]

\[
:= -\sum_{i=1}^{6} I_i,
\]

where for \( i = 1, \ldots, 6 \),

\[
I_i = \int_{\Delta_i} e^{-\theta (x+y-z)} z^{2H-2} |x-y|^{2H-2} dxdydz,
\]

\[
\Delta_i = \lim_{T \to \infty} \Delta_i(T) = \Delta_i(T),
\]

\[
\Delta_1(T) = \{ 0 \leq x \leq y \leq z \leq T \}, \quad \Delta_2(T) = \{ 0 \leq x \leq z \leq y \leq T \}, \quad \Delta_3(T) = \{ 0 \leq z \leq x \leq y \leq T \},
\]

\[
\Delta_4(T) = \{ 0 \leq y \leq x \leq z \leq T \}, \quad \Delta_5(T) = \{ 0 \leq y \leq z \leq x \leq T \}, \quad \Delta_6(T) = \{ 0 \leq z \leq y \leq x \leq T \}.
\]

By making the change of variables \( a = x, b = y - x, c = z - y \), we have that

\[
I_1 = \int_{\mathbb{R}_+^3, a + b + c > T} e^{-\theta(a+c)} z^{2H-2} (a+b+c)^{2H-2} \text{dadbdc}.
\]

Since on \( \{ (a,b,c) \in \mathbb{R}_+^3, a + b + c > T \} \), we have that

\[
\{ a + b + c > T, b \geq 1 \} = \{ 1 \leq b \leq T, a + c > T - b \} \cup \{ b > T \},
\]

\[
\{ a + b + c > T, 0 < b < 1 \} \subset \{ 0 < b < 1, a + c > T - 1 \},
\]

\[
(a + b + c)b \geq b^2 1_{\{b \geq 1\}} + (a + c)b 1_{\{0 < b < 1\}}.
\]

Hence

\[
I_1 \leq I_{11} + I_{12} + I_{13},
\]
Hence, the convergence speed of $I$ is dominated by Thus, we obtain that the speed of exponential. In addition, by the L'Hospital's rule, we have that as 

Finally, suppose that $H$ is at least $\frac{1}{4}$. Similarly, we have that as $T \to \infty$, 

Thus, we obtain that the speed of $I_1 \to 0$ is also at least $\frac{1}{T^{1-\delta_H}}$. Clearly, all $I_i$, $i = 2, 4, 6$ can be dominated by $I_1$. And it is easy to check that $I_3 = I_5$ converges to zero with exponential rate. Hence, the convergence speed of $2 \| f_T \|_{\delta_H=2}^2 \to \| H \Gamma (2H) \theta^{-2H} \|_{\delta_H=2}^2$ is at least $\frac{1}{T^{1-\delta_H}}$ as $T \to \infty$.

Finally, suppose that $H = \frac{1}{4}$. Similarly, we have that as $T \to \infty$, 

where 

The L'Hospital's rule implies that as $T \to \infty$, 

$$J_1 \sim \frac{1}{e^{\theta T} T^{-\frac{1}{2}}} \int_{0 \leq x \leq y \leq T} e^{(y-x)\theta} (y-x)^{-\frac{1}{2}} \, dx \, dy$$

$$\sim \frac{1}{(\theta - \frac{1}{2} T^{-1}) e^{\theta T} T^{-\frac{1}{2}}} \int_{0}^{T} e^{(T-x)\theta} (T-x)^{-\frac{1}{2}} \, dx$$

$$= \frac{1}{(\theta - \frac{1}{2} T^{-1}) e^{\theta T} T^{-\frac{1}{2}}} \int_{0}^{T} e^{u\theta} u^{-\frac{1}{2}} \, du,$$
which converges to $\frac{1}{a}$ with the rate as $\frac{1}{T}$. Similarly, as $T \to \infty$,
\[
J_2 \sim \frac{T}{e^{zT}} \int_{0 \leq z \leq T} e^{(z-x)\theta} z^{-\frac{1}{2}} (T-x)^{-\frac{1}{2}} dx dz
\]
(\text{let } a = T-x)
\[
= \frac{T}{e^{zT}} \int_{0 \geq z \geq T} e^{(z+a)\theta} z^{-\frac{1}{2}} a^{-\frac{1}{2}} da dz
\]
\[
= \frac{T}{e^{zT}} \left[ \int_{[0,T]^2} e^{(z+a)\theta} z^{-\frac{1}{2}} a^{-\frac{1}{2}} da dz - \int_{[0,T]^2, a+z \leq T} e^{(z+a)\theta} z^{-\frac{1}{2}} a^{-\frac{1}{2}} da dz \right],
\]
where the first term converges to $\frac{1}{a}$ with rate as $\frac{1}{T}$, and the second term converges to zero with exponential rate. It is easy to check that each $J_i$, $i = 3, \ldots , 6$, converges to zero with exponential rate. Thus, we have that the speed of $\frac{2||f_T||^2_{\sigma\theta}}{\log T}$ converges to $\frac{a}{T}$ as $T \to \infty$. \hfill \Box

Proof of Theorem 1.1. We only show the case of $H \in \left[ \frac{1}{4}, \frac{1}{2} \right)$. The case of $H = \frac{1}{2}$ is similar.

It follows from Theorem 2.1, Lemma 3.2 and Eq. (3.1)-(3.5) that there exists a constant $C_{\theta,H}$ such that for $T$ large enough,
\[
\sup_{z \in \mathbb{R}} \left| P \left( \frac{T}{16\sigma_H^2} (\theta_T - \theta) \leq z \right) - P(Z \leq z) \right| \leq C_{\theta,H} \times \max \left\{ |b_T^2| - 2 \|f_T\|^2, \|f_T \otimes_1 f_T\|, \|f_T \otimes_1 g_T\|, \|g_T\|^2, \|g_T \otimes_1 g_T\| \right\}. \quad (3.11)
\]
Denote $a = H^2(2H)\theta^{-2H}$. Lemma 3.2 and Lemma 3.5 imply that there exists a constant $c$ such that for $T$ large enough,
\[
|b_T^2 - 2 \|f_T\|^2| \leq |b_T^2 - a^2| + 2 \|f_T\|^2 - a^2 | \leq c \times \frac{1}{T^{3-4H}}.
\]
Lemma 3.4 imply that there exists a constant $c$ such that for $T$ large enough,
\[
\|f_T \otimes_1 g_T\|, \|f_T, g_T\|, \|g_T \otimes_1 g_T\| \leq c \times \frac{1}{\sqrt{T}}, \quad \|g_T\|^2 \leq c \times \frac{1}{T}.
\]
Substituting (3.6) and the above inequalities into (3.11), we obtain that the Berry-Esseen bound (1.3) holds. \hfill \Box

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