New compactifications of 11-dimensional supergravity

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Abstract
Using canonical forms on $S^7$, viewed as an $SU(2)$ bundle over $S^4$, we introduce consistent ansätze for the 4-form field strength of 11-dimensional supergravity and rederive the known squashed, stretched and the Englert solutions. Furthermore, by rewriting the metric of $S^7$ as a $U(1)$ bundle over $CP^3$, we present yet more general ansätze. As a result, we find a new compactifying solution of the type $AdS_5 \times CP^3$, where $CP^3$ is stretched along its $S^2$ fiber. We also find a new solution of $AdS_2 \times H^2 \times S^7$ type in Euclidean space.

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1. Introduction

Eleven-dimensional supergravity solutions have been extensively studied in the 1980s. Among these, the Freund–Rubin solution [1] was the simplest one as it included a 4-form field strength with components only along the AdS direction. Then, attentions were turned to possible solutions with nonvanishing components along the compact directions. Englert was the first to construct such a solution; $AdS_4 \times S^7$ with the round metric on $S^7$ [2]. Later, the so-called squashed solutions with non-standard Einstein metric on $S^7$ were found [3, 4]. Here, $S^7$ is considered as an $SU(2)$ bundle over $S^4$ and squashing corresponds to rescaling the metric along the fiber. For a specific value of the squashing parameter the metric turns out to be Einstein.

In constructing the Englert-type solutions Killing spinors play a significant role. Killing spinors are also required for having supersymmetric solutions [5, 7–10]. Alternatively, on compact manifolds with a bundle structure on a Kähler base, one can use the holomorphic top form and the Kähler form to write consistent ansätze for the 4-form field strength [11]. Algebraic approaches have also been used to study the supergravity solutions [12].

In this work, however, instead of looking for Killing spinors we directly use canonical forms on $S^7$ to write a consistent ansatz for the 4-form field strength. In particular, this allows us to rederive the squashed, stretched and the Englert solutions in a unified scheme. There are some independent earlier works which also use canonical geometric methods [13, 14]. In section 2, we consider $S^7$ as an $S^3$ bundle over $S^4$, and identify a natural basis of such forms.
in terms of the volume forms of the fiber and the base. We will see that a linear combination of these forms provides a suitable ansatz for the Maxwell equation, so that the field equations reduce to algebraic equations for the parameters of the ansatz. In section 3, we rewrite the squashed metric of $S^7$ as a $U(1)$ bundle over $\mathbb{CP}^3$, where it appears as an $S^3$ bundle over $S^7$ with rescaled fibers. Moreover, in this form, we can introduce a different rescaling parameter for the $U(1)$ fibers. This enables us to provide more general ansätze. In section 4, we consider a direct product of a five and six-dimensional spaces and find a new compactifying solution of AdS$_5 \times \mathbb{CP}^3$, in which $\mathbb{CP}^3$ is stretched along its $S^3$ fiber. In section 5, we discuss solutions in which the 11-dimensional space has a Euclidean signature and is a direct product of two two-dimensional spaces and $S^7$. We find a solution of AdS$_2 \times H^2 \times S^7$ type, in which $H^2$ is a hyperbolic surface, and $S^7$ is stretched along its $U(1)$ fiber by a factor of 2.

2. Squashed solution revisited

Let us start our discussion with the Freund–Rubin solution, for which the 4-form field strength has components only along the four dimensions

$$F_i = \frac{1}{8} R^3 e_i,$$  \hspace{1cm}  (1)

and the metric reads

$$dx^2 = R^2 \left( \frac{1}{4} ds^2_{\text{AdS}_5} + dx^2_{S^7} \right).$$  \hspace{1cm}  (2)

The round metric on $S^7$ can be written as an $SU(2)$ bundle over $S^4$ \[3, 15\]

$$dx^2_{S^7} = \frac{1}{4} (d\mu^2 + \frac{1}{2} \sin^2 \mu \Sigma^2_{i} + (\sigma_i - \cos^2 \mu/2 \Sigma_i)^2),$$  \hspace{1cm}  (3)

with $0 \leq \mu \leq \pi$, and $\Sigma_i$ and $\sigma_i$ are two sets of left-invariant 1-forms

$$\Sigma_1 = \cos \gamma d\alpha + \sin \gamma \sin \alpha d\beta,$$

$$\Sigma_2 = -\sin \gamma d\alpha + \cos \gamma \sin \alpha d\beta,$$

$$\Sigma_3 = d\gamma + \cos \alpha d\beta,$$

where $0 \leq \gamma \leq 4\pi$, $0 \leq \alpha \leq \pi$, $0 \leq \beta \leq 2\pi$, and with a similar expression for $\sigma_i$s. They satisfy the $SU(2)$ algebra

$$d\Sigma_i = -\frac{1}{2} \epsilon_{ijk} \Sigma_j \wedge \Sigma_k,$$  \hspace{1cm}  (4)

with $i, j, k, \ldots = 1, 2, 3$.

Squashing corresponds to modifying the round metric on $S^7$ as follows:

$$dx^2_{S^7} = \frac{1}{4} (d\mu^2 + \frac{1}{2} \sin^2 \mu \Sigma^2_{i} + \lambda^2 (\sigma_i - \cos^2 \mu/2 \Sigma_i)^2),$$  \hspace{1cm}  (5)

with $\lambda$ the squashing parameter. So, let us take the following ansatz for the 11D metric:

$$dx^2 = \frac{R^2}{4} \left( ds^2_{\text{AdS}_5} + d\mu^2 + \frac{1}{4} \sin^2 \mu \Sigma^2_{i} + \lambda^2 (\sigma_i - \cos^2 \mu/2 \Sigma_i)^2 \right),$$  \hspace{1cm}  (6)

and choose the orthonormal basis of vielbeins as

$$e^0 = d\mu, \quad e^i = \frac{1}{2} \sin \mu \Sigma_i, \quad e^\lambda = \lambda (\sigma_i - \cos^2 \mu/2 \Sigma_i).$$  \hspace{1cm}  (7)

Furthermore, in order to construct our ansatz in the next section we need to evaluate the exterior derivatives of the vielbeins

$$de^i = \cot \mu e^0 \wedge e^i - \frac{1}{\sin \mu} \epsilon_{ijk} e^j \wedge e^k,$$  \hspace{1cm}  (8)

$$d\lambda = \lambda e^0 \wedge e^\lambda + \frac{1}{2} \epsilon_{ijk} \left( \lambda e^i \wedge e^k - \frac{1}{\lambda} e^\lambda \wedge e^j \right) - 2 \left( \frac{1 + \cos \mu}{\sin \mu} \right) e^\lambda \wedge e^\mu,$$  \hspace{1cm}  (9)

where (4) has been used.
2.1. The ansatz

Let us now introduce $\omega_3$, the volume element of the fiber $S^3$:

$$\omega_3 = \hat{e}^1 \wedge \hat{e}^2 \wedge \hat{e}^3,$$

(10)
taking the derivative along with using (9), we obtain

$$d\omega_3 = \lambda_2 (\epsilon_{ijk} e^0 \wedge \hat{e}^j \wedge \hat{e}^k + e^i \wedge \hat{e}^j \wedge \hat{e}^l).$$

(11)

The Hodge dual reads

$$\ast d\omega_3 = \lambda \hat{e}^i \wedge (e^0 \wedge e^i + \frac{1}{2} \epsilon_{ijk} e^j \wedge e^k),$$

(12)

so that using (8) and (9), we derive

$$d \ast d\omega_3 = 6 \lambda^2 \omega_4 - \frac{1}{\lambda} d\omega_3,$$

(13)

where

$$\omega_4 = e^0 \wedge e^1 \wedge e^2 \wedge e^3,$$

(14)
is the volume element of the base. Note that $\omega_4$ is closed; $d\omega_4 = 0$. Furthermore, since $d \ast \omega_4 = d\omega_3$, for a linear combination of these two forms, we have

$$d \ast (\alpha \omega_4 + \beta d\omega_3) = 6 \lambda^2 \beta \omega_4 + (\alpha - \beta / \lambda) d\omega_3$$

(15)
namely, the subspace with a basis of $\omega_4$ and $d\omega_3$ is closed under $d \ast$ operation. This is exactly what we need to construct a consistent ansatz for the 4-form field strength.

The above analysis shows that we can take the following ansatz:

$$F_4 = N \epsilon_4 + \alpha \omega_4 + \beta d\omega_3,$$

(16)

with $N$, $\alpha$ and $\beta$ being constant parameters to be determined by field equations, also note that $dF_4 = 0$. Substituting this into the field equation\(^1\)

$$d \ast_{11} F_4 = - \frac{1}{2} dF_4 \wedge F_4,$$

(17)

we obtain

$$\frac{R^3}{8} d(N \omega_3 \wedge \omega_4 + \alpha \epsilon_4 \wedge \omega_3 + \beta \epsilon_4 \wedge \ast d\omega_3) = -N \epsilon_4 \wedge (\alpha \omega_4 + \beta d\omega_3),$$

(18)

therefore, using (13), we must have

$$6 \lambda^2 \beta = - \frac{8N}{R^3} \alpha, \quad \alpha - \beta / \lambda = - \frac{8N}{R^3} \beta.$$

(19)

A nontrivial solution exists if

$$\lambda \left( \frac{8N}{R^3} \right)^2 - \frac{8N}{R^3} \omega_4 \omega_3 - 6 \lambda^3 = 0.$$

(20)

We will return to this equation after discussing the Einstein equations.

Now let us turn to the Einstein equations

$$R_{MN} = \frac{1}{12} F_{MPQR} F_N^{PQR} - \frac{1}{3 \cdot 48} g_{MN} F_{PQRS} F^{PQRS},$$

(21)

where $M, N, P, \ldots = 0, 1, \ldots, 10$. With ansatz (16), we can calculate the right-hand side of the above equations:

$$R_{\mu\nu} = \left( \frac{4}{R^2} \right)^4 \left( - \frac{31}{12} N^2 - \frac{41}{3 \cdot 48} (-N^2 + \alpha^2 + 6 \lambda^2 \beta^2) \right) g_{\mu\nu},$$

(22)

\(^1\) The star in this equation is the 11-dimensional Hodge dual operation. In the rest of equations, it indicates the seven-dimensional Hodge dual operation.
\[ R_{\mu\nu} = \left( \frac{4}{R^2} \right)^4 \left( \frac{31}{12} (\alpha^2 + 3\lambda^2 \beta^2) - \frac{41}{3} \cdot \frac{48}{48} (-N^2 + \alpha^2 + 6\lambda^2 \beta^2) \right) \delta_{\mu\nu}, \quad (23) \]

\[ R_{\hat{a}\hat{b}} = \left( \frac{4}{R^2} \right)^4 \left( \frac{31}{12} (4\lambda^2 \beta^2) - \frac{41}{3} \cdot \frac{48}{48} (-N^2 + \alpha^2 + 6\lambda^2 \beta^2) \right) \delta_{\hat{a}\hat{b}}, \quad (24) \]

with \( \mu, \nu = 0, \ldots, 3, \alpha, \beta = 4, \ldots, 7 \) and \( \hat{a}, \hat{b} = 8, 9, 10 \). Note that different terms in our ansatz (16) do not contract into each other. For the left-hand side, on the other hand, the Ricci tensor of metric (6) becomes

\[ R_{\mu\nu} = \left( \frac{4}{R^2} \right)^4 \left( \frac{3}{2} \right) \delta_{\mu\nu}, \]

\[ R_{\hat{a}\hat{b}} = \left( \frac{4}{R^2} \right)^4 \left( \frac{1}{2} \right) \delta_{\hat{a}\hat{b}}, \quad (25) \]

these are to be substituted on the left-hand side of (23) and (24).

We can now solve (19) and (20) for \( \beta \) and \( N \), and then plug it into (23) and (24). The two resulting equations can be solved for \( \lambda \) and \( \alpha \). We obtain two types of solutions. Those with no internal flux

\[ \alpha = \beta = 0, \quad (26) \]

together with \( \lambda^2 = 1 \), which is the round sphere. Or, we can have \( \lambda^2 = 1/5 \), which corresponds to the squashed sphere solution. We also obtain solutions with fluxes

\[ \alpha^2 = 9/5, \quad \beta^2 = 9, \quad \lambda^2 = 1/5. \quad (27) \]

For \( \lambda = 1/\sqrt{5} \), \( \alpha = -3/\sqrt{5} \), \( \beta = 3 \) and \( N = 3R^3/(4\sqrt{5}) \), we have a non-zero 4-form field strength along \( S^3 \), and it represents the squashed \( S^3 \) with Einstein metric

\[ R_{\mu\nu} = \left( \frac{4}{R^2} \right)^4 \frac{27}{10} \delta_{\mu\nu}, \quad R_{\hat{a}\hat{b}} = \left( \frac{4}{R^2} \right)^4 \frac{27}{10} \delta_{\hat{a}\hat{b}}. \quad (28) \]

The above solution, the so-called squashed solution with torsion, was obtained in the 1980s using the covariantly constant spinors of the squashed sphere without torsion [5, 10]. We can also take \( \lambda = -1/\sqrt{5} \), \( \alpha = -3/\sqrt{5} \), \( \beta = -3 \) and \( N = -3R^3/(4\sqrt{5}) \) instead, this is the skew-whiffed squashed solution.

### 3. CP\(^3\) as an \( S^2 \) bundle over \( S^4 \)

In the previous section the metric of \( S^3 \) was written as an \( S^3 \) bundle over \( S^4 \). It is also possible to write the metric as a \( U(1) \) bundle over \( \text{CP}^3 \). On the other hand, it is observed that \( \text{CP}^3 \) itself can be written as an \( S^3 \) bundle over \( S^4 \). In this form one can construct a family of homogeneous metrics by rescaling the fibers. In fact, we can see that the metric (3) can be rewritten as a \( U(1) \) bundle over such a deformed \( \text{CP}^3 \) [16, 17]. First note that

\[
\begin{align*}
\text{d}x^2_{\text{S}^5} &= \text{d}\mu^2 + \frac{1}{4} \sin^2 \mu \Sigma_i^2 + \lambda^2 (\sigma_i - \cos \Sigma_2 \mu) \Sigma_i^2 \\
&= \text{d}\mu^2 + \frac{1}{4} \sin^2 \mu \Sigma_i^2 + \lambda^2 (\text{d} \tau - A)^2 + \lambda^2 (\text{d} \theta - \sin \phi A_1 + \cos \phi A_2)^2 \\
&+ \lambda^2 \sin^2 \theta (\text{d} \phi - \cot \theta (\cos \phi A_1 + \sin \phi A_2) + A_3)^2, \quad (29)
\end{align*}
\]

where

\[ A_i = \cos^2 \mu / 2 \Sigma_i, \quad (30) \]

\(^2\) For the sake of clarity, we will set \( R^2 = 4 \) from now on.
and
\[ A = \cos \theta d\phi + \sin \theta (\cos \phi A_1 + \sin \phi A_2) + \cos \theta A_3. \] (31)

\(\sigma_i\)s are left-invariant 1-forms that are chosen as follows:
\[ \sigma_1 = \sin \phi \, d\theta + \sin \theta \cos \phi \, d\tau, \]
\[ \sigma_2 = -\cos \phi \, d\theta + \sin \theta \sin \phi \, d\tau, \]
\[ \sigma_3 = -d\phi + \cos \theta \, d\tau. \]

In the new form of the metric (29), we can further rescale the \(U(1)\) fibers so that the Ricci tensor (on a basis we introduce shortly) is still diagonal. Hence, we take the metric to be
\[ ds^2_{\tilde{\gamma}} = d\mu^2 + \frac{1}{4} \sin^2 \mu \Sigma_i^2 + \lambda^2 (d\theta - \sin \phi A_1 + \cos \phi A_2)^2 + \lambda^2 \sin^2 \theta (d\phi - \cot \theta (\cos \phi A_1 + \sin \phi A_2) + A_3)^2 + \tilde{\lambda}^2 (d\tau - A)^2, \] (32)
and choose the following basis:
\[ e^0 = d\mu, \quad e^i = \frac{1}{2} \sin \mu \Sigma_i, \]
\[ e^5 = \lambda (d\theta - \sin \phi A_1 + \cos \phi A_2), \]
\[ e^6 = \lambda \sin \theta (d\phi - \cot \theta (\cos \phi A_1 + \sin \phi A_2) + A_3), \]
\[ e^7 = \tilde{\lambda} (d\tau - A). \] (33)

In this basis, the Ricci tensor is diagonal and reads
\[ R_{00} = R_{11} = R_{22} = R_{33} = 3 - \lambda^2 - \tilde{\lambda}^2/2, \]
\[ R_{55} = R_{66} = \lambda^2 + 1/\lambda^2 - \tilde{\lambda}^2/2\lambda^4, \quad R_{77} = \tilde{\lambda}^2 + \lambda^2/2\lambda^4. \] (34)

3.1. The ansatz

As in the previous section, a natural 3-form to begin with is \(\omega_3 = e^{567}\). To proceed, however, it proves useful to define the following forms:
\[ R_1 = \sin \phi (e^{01} + e^{23}) - \cos \phi (e^{02} + e^{31}), \]
\[ R_2 = \cos \theta \cos \phi (e^{01} + e^{23}) + \cos \theta \sin \phi (e^{02} + e^{31}) - \sin \theta (e^{03} + e^{12}), \]
\[ K = \sin \theta \cos \phi (e^{01} + e^{23}) + \sin \theta \sin \phi (e^{02} + e^{31}) + \cos \theta (e^{03} + e^{12}). \] (35)

The key feature of this definition, that we will use frequently in this paper, is that these three forms are orthogonal to each other, i.e.
\[ R_1 \wedge R_2 = K \wedge R_1 = K \wedge R_2 = 0. \] (36)

Let us also define
\[ \text{Re } \Omega = R_1 \wedge e^3 + R_2 \wedge e^6, \quad \text{Im } \Omega = R_1 \wedge e^6 - R_2 \wedge e^3. \] (37)

We will further need to work out the exterior derivatives of the above forms
\[ d\text{Re } \Omega = 4\lambda \omega_4 - \frac{2}{\lambda} e^{56} \wedge K, \quad d\text{Im } \Omega = 0, \] (38)
for \(d\omega_3\) in the new basis we obtain
\[ d\omega_3 = \lambda \text{Im } \Omega \wedge e^7 - \tilde{\lambda} e^{56} \wedge F, \] (39)
with
\[ F = d\Lambda = -K - e^{56}/\lambda^2. \] (40)
Note that since
\[ \text{d} \text{Im} \Omega = 0, \quad \text{Im} \Omega \wedge F = - \text{Im} \Omega \wedge K = 0, \] (41)
we have three independent 4-forms \( \omega_4, e^7 \wedge \text{Im} \Omega \) and \( e^{56} \wedge K \), which are closed and do not contract into each other. Furthermore, the set of these 4-forms is closed under \( d^* \) operation, and hence a suitable ansatz for \( F_4 \) is as follows:
\[ F_4 = N e_4 + \alpha \omega_4 + \beta e^7 \wedge \text{Im} \Omega + \gamma K \wedge e^{56}, \] (42)
for \( \alpha, \beta \) and \( \gamma \) being three real constants. Taking the Hodge dual, we have
\[ *_{11} F_4 = N \omega_3 \wedge \omega_4 + \epsilon_4 \wedge (\alpha \omega_3 - \beta \text{Re} \Omega + \gamma K \wedge e^7). \] (43)
Using \( \text{d} e^{56} = \lambda \text{Im} \Omega, \quad \text{d} K = - \frac{1}{\lambda} \text{Im} \Omega \), (38), we see that Maxwell equations (17) reduce to
\[ - \alpha \lambda^2 + N \lambda \beta + \gamma = 0, \]
\[ \alpha \tilde{\lambda} + 2 \beta / \lambda + (\tilde{\lambda} / \lambda^2 + N) \gamma = 0, \]
\[ N \alpha - 4 \beta \beta + 2 \lambda \gamma = 0. \] (45)

As for the Einstein equations, we use (34) and the ansatz (42) to obtain
\[ 3 - \lambda^2 - \frac{\tilde{\lambda}^2}{2} = \frac{1}{3} \left( \alpha^2 + \beta^2 + \frac{1}{2} \gamma^2 + \frac{1}{2} N^2 \right), \]
\[ \lambda^2 + \frac{1}{\lambda^2} - \frac{\tilde{\lambda}^2}{2 \lambda^4} = \frac{1}{3} \left( - \alpha^2 + \beta^2 + 2 \gamma^2 + \frac{1}{2} N^2 \right), \]
\[ \tilde{\lambda}^2 + \frac{\tilde{\lambda}^2}{2 \lambda^4} = \frac{1}{3} \left( - \alpha^2 + 4 \beta^2 - \gamma^2 + \frac{1}{2} N^2 \right). \] (46)

In general, it is not easy to solve set of coupled equations (45) and (46). In fact, apart from the known solutions, we have found no real (i.e. real coefficients for \( F_4 \)) solutions. In special cases, though, we can reduce the equations further and find solutions. Let us start by assuming
\[ \lambda = \tilde{\lambda}; \]
then by the Einstein equations we must have \( \beta^2 = \gamma^2 \). Taking \( \beta = - \gamma \) yields \( \lambda = \tilde{\lambda} = 1 / \sqrt{5}, N = - 6 / \sqrt{5} \) and \( \alpha^2 = \beta^2 = \gamma^2 = 9 / 5 \), which is the squashed solution (with torsion) of the previous section with \( R_{\mu \nu} = - 45 / 10 g_{\mu \nu} \).
For \( \beta = \gamma \), we obtain \( \lambda = \tilde{\lambda} = 1, N = - 2 \) and \( \alpha^2 = \beta^2 = \gamma^2 = 1 \); this is an Englert-type solution with \( R_{\mu \nu} = - 5 / 2 g_{\mu \nu} \). This has the same four-dimensional Ricci tensor as the original solution found by Englert in [2] using parallelizing torsions on the 7-sphere, and later by [6] and [11] using Killing spinors.

### 3.2. Pope–Warner solution

In this section, we rederive the Pope–Warner ansatz and the solution [11] using the canonical forms language. Let us then begin by defining
\[ \text{Re} L = - R_1 \wedge e^5 + R_2 \wedge e^6, \quad \text{Im} L = R_1 \wedge e^6 + R_2 \wedge e^5. \] (47)
We note that in the vielbein basis (33), \( A \) in (31) can be written as
\[
A = \cot \theta \frac{e^6}{\lambda} + \cot \frac{\mu}{2} \frac{e^6}{\sin \theta} (\cos \phi e^1 + \sin \phi e^2),
\] (48)
which, together with (35), allows us to write \( de^5 \) and \( de^6 \) more compactly as
\[
de^5 = -e^6 \wedge A + \lambda R_1, \quad de^6 = e^5 \wedge A + \lambda R_2.
\] (49)

Taking the exterior derivative once more yields
\[
\lambda dR_1 = \lambda R_2 \wedge A + e^6 \wedge K, \quad \lambda dR_2 = -\lambda R_1 \wedge A - e^5 \wedge K.
\] (50)

Having derived (49) and (50), it is now easy to prove that
\[
dRe L = -2A \wedge {\text{Im}} L, \quad d{\text{Im}} L = 2A \wedge {\text{Re}} L.
\] (51)

To absorb \( A \) into \( e^7 \) in the above equations, we define
\[
P = e^{-2i \tau} L,
\] (52)
by using equations (51), we see that
\[
dP = -\frac{2i}{\tilde{\lambda}} \tilde{\lambda} e^7 \wedge P.
\] (53)

On the other hand, note that
\[
* L = i L \wedge e^7,
\] (54)
so we can write (53) as
\[
dP = \frac{2}{\lambda} \tilde{\lambda} * P.
\] (55)

This implies that for the 4-form field strength we can take
\[
F_4 = N e_4 + \eta e^7 \wedge (\sin 2 \tau {\text{Re}} L - \cos 2 \tau {\text{Im}} L),
\] (56)
with \( \eta \) a real constant. Maxwell equation (17) then requires \( N = -2/\tilde{\lambda} \), whereas, the Einstein equations imply \( \lambda^2 = 1 \), and \( \tilde{\lambda}^2 = 2 \), together with \( \eta^2 = 2 \). Note that in this solution the \( U(1) \) fibers of \( S^7 \) are stretched by a factor of 2.

We can construct another consistent ansatz by taking a linear combination of Pope–Warner ansatz and the one introduced in the previous section. However, by this we obtain non-zero off diagonal components of energy–momentum tensor, i.e. \( T_{56} \neq 0 \), unless we set \( \beta = 0 \). Let us then set
\[
F_4 = N e_4 + \alpha \omega_4 + \gamma K \wedge e^{56} + \eta e^7 \wedge (\sin 2 \tau {\text{Re}} L - \cos 2 \tau {\text{Im}} L),
\] (57)
Maxwell equations (17) and (45) then require
\[
N = -2/\tilde{\lambda}, \quad \lambda^2 = \tilde{\lambda}^2 = 1, \quad \alpha = \gamma,
\] (58)
while, the Einstein equations imply
\[
\alpha^2 = \gamma^2 = \eta^2 = 1,
\] (59)
which is the Englert solution with \( R_{\mu \nu} = -5/2 g_{\mu \nu} \). Note that here we have \( \alpha = \gamma = 1 \), hence the second and the third terms in (57) combine to
\[
\omega_4 + K \wedge e^{56} = \frac{1}{2} F \wedge F,
\] (60)
with \( F \) being the Kähler form defined in (40). We can now recognize (57) as exactly the Englert solution of [11]. The \( F \wedge F \) term and the term proportional to \( \eta \) are each invariant under an \( SU(4) \) symmetry, but with the given values of the constant coefficients, \( \alpha, \beta \) and \( \gamma \), the symmetry enhances to \( SO(7) \).
4. New AdS$_5 \times$ CP$^3$ compactification

With the ansatz introduced in section 3.1, we can think of 11-dimensional metrics which are direct product of five- and six-dimensional spaces with $F_4$ given by (42) setting $N$ and $\beta$ equal to zero. By this, apart from the result of [18] we derive a new solution of AdS$_5 \times$ CP$^3$ so that the CP$^3$ factor is stretched along its $S^2$ fiber by a factor of 2.

Let us then take the 11-dimensional spacetime to be the direct product of five- and six-dimensional spaces

$$ds^2_{11} = ds^2_5 + ds^2_6.$$  
(61)

For the six-dimensional space, we take the same metric that appeared in $S^7$ description in (29):

$$ds^2_6 = d\mu^2 + \frac{1}{4} \sin^2 \mu \Sigma^2 + \lambda^2 (d\theta - \sin \phi A_1 + \cos \phi A_2)^2$$

$$+ \lambda^2 \sin^2 \theta (d\phi - \cot \theta (\cos \phi A_1 + \sin \phi A_2) + A_3)^2,$$  
(62)

as mentioned before, this is an $S^2$ bundle over $S^4$, and for $\lambda^2 = 1$ we obtain the Fubini–Study metric on CP$^3$. By taking the basis $e^0, \ldots, e^6$ as in (33) the Ricci tensor reads

$$R_{00} = R_{11} = R_{22} = R_{33} = 3 - \lambda^2,$$

$$R_{55} = R_{66} = \lambda^2 + 1/\lambda^2.$$  
(63)

As for $F_4$, we choose the following ansatz:

$$F_4 = \alpha \omega_4 + \gamma K \wedge e^5,$$  
(64)

which is closed. Taking the Hodge dual, we have

$$*_{11} F_4 = \epsilon_5 \wedge (\alpha e^5 + \gamma K).$$  
(65)

As $F_4 \wedge F_4 = 0$, in this case the Maxwell equation reads

$$d *_{11} F_4 = - (\alpha \lambda - \gamma / \lambda) \epsilon_5 \wedge \text{Im} \Omega = 0,$$  
(66)

where use has been made of (44). So, we must have

$$\alpha \lambda^2 = \gamma.$$  
(67)

The Einstein equations along compact six dimensions, on the other hand, imply

$$3 - \lambda^2 = \frac{1}{3} \left( \alpha^2 + \frac{1}{2} \gamma^2 \right) = \frac{1}{3} \left( 1 + \frac{\lambda^4}{2} \right) \alpha^2,$$

$$\lambda^2 + \frac{1}{\lambda^2} = \frac{1}{3} \left( - \alpha^2 + 2 \gamma^2 \right) = \frac{1}{3} \left( - \frac{1}{2} + 2 \lambda^4 \right) \alpha^2,$$  
(68)

where we used (67) in the last equalities. From the above equations, we obtain two solutions:

$$\lambda^2 = 1, \quad \alpha^2 = \gamma^2 = 4,$$  
(69)

for which the metric is the standard Fubini–Study metric of CP$^3$. The five- dimensional Ricci tensor becomes

$$R_{\mu \nu} = -2g_{\mu \nu},$$  
(70)

with $\mu, \nu = 0, \ldots, 4$. Therefore, the five-dimensional spacetime is anti-de Sitter. This solution was first derived in [18].

For the second solution, we have

$$\lambda^2 = 2, \quad \alpha^2 = 1, \quad \gamma^2 = 4,$$  
(71)
with the five-dimensional Ricci tensor;
\[ R_{\mu\nu} = -\frac{3}{2} g_{\mu\nu}. \] (72)
This new solution corresponds to an stretched CP^3, in which the S^2 fibers are stretched by a factor of 2. Note that, for this solution the six-dimensional metric is no longer Einstein. Also, note that according to our discussion at the end of the previous section the first solution, (69), has an SU(4) symmetry, whereas in the new solution, (71), this symmetry is reduced to \( SO(3) \times SO(5) \), i.e. to the direct product of the symmetry subgroups of the fiber and the base.

5. \( \text{AdS}_2 \times H^2 \times S^7 \) compactification

With metric (32) for the \( S^7 \), we can take yet another ansatz for the metric and \( F_4 \) and come up with a new compactification. In fact, in this section we obtain a new solution of type \( \text{AdS}_2 \times H^2 \times S^7 \), with \( H^2 \) being a hyperbolic surface. As we will see, this solution exists only in 11-dimensional space with Euclidean signature, and like the Pope–Warner solution the \( S^7 \) metric gets stretched along its \( U(1) \) fibers by a factor of 2.

Let the 11-dimensional spacetime be the direct product of two two-dimensional spaces and \( S^7 \),
\[ ds_{11}^2 = ds_2^2 + ds_2^2 + ds_7^2, \] (73)
where \( ds_7^2 \) is the same as (32). For \( F_4 \), we take
\[ F_4 = N \epsilon_2^2 \wedge \epsilon_2 + \alpha \omega_4 + \beta e^7 \wedge \Im \Omega + \gamma K \wedge e^{e^7} + \epsilon_2 \wedge (\xi_1 K + \eta_1 e^{e^7}) + 2A \wedge (\xi_2 K + \eta_2 e^{e^7}), \] (74)
note that the first four terms are the same as those appeared in (42). \( \xi_1, \xi_2, \eta_1 \) and \( \eta_2 \) are constant parameters. We take the four-dimensional space to be the direct product of two Euclidean subspaces with \( \epsilon_2^1 \) and \( \epsilon_2 \) as their two-dimensional volume elements.

The Bianchi identity requires that
\[ \tilde{\lambda}(2 \xi_1 + \eta_1/\lambda^2) = -i(2 \xi_2 \gamma + \alpha \eta_2), \]
\[ \tilde{\lambda}(2 \xi_2 + \eta_2/\lambda^2) = -i(2 \xi_1 \gamma + \alpha \eta_1), \]
\[ \alpha \tilde{\lambda} + 2\beta/\lambda + \gamma (\tilde{\lambda}/\lambda^2 + iN) = -i(\xi_1 \eta_2 + \eta_1 \xi_2), \]
\[ iN \alpha - 4\lambda \beta + 2\lambda \gamma = -2i \xi_1 \xi_2, \]
\[ -\alpha \lambda^2 + iN \lambda \beta + \gamma = 0. \] (75)

For the Maxwell equation, first note that in Euclidean 11-dimensional space we need to account for an extra \( i \) factor coming from the Chern–Simons term so that (17) is replaced by
\[ d \ast_{11} F_4 = -\frac{i}{2} F_4 \wedge F_4, \] (76)
therefore, with our ansätze (32) and (74) the Maxwell equations reduce to the following algebraic equations:
\[ \tilde{\lambda}(2 \xi_1 + \eta_1/\lambda^2) = -i(2 \xi_2 \gamma + \alpha \eta_2), \]
\[ \tilde{\lambda}(2 \xi_2 + \eta_2/\lambda^2) = -i(2 \xi_1 \gamma + \alpha \eta_1), \]
\[ \alpha \tilde{\lambda} + 2\beta/\lambda + \gamma (\tilde{\lambda}/\lambda^2 + iN) = -i(\xi_1 \eta_2 + \eta_1 \xi_2), \]
\[ iN \alpha - 4\lambda \beta + 2\lambda \gamma = -2i \xi_1 \xi_2, \]
\[ -\alpha \lambda^2 + iN \lambda \beta + \gamma = 0. \] (77)

Using (75), the first two equations above imply
\[ \xi_1^2 = \xi_2^2, \quad \eta_1^2 = \eta_2^2. \] (78)

Had we chosen a Lorentzian signature metric for the four-dimensional space, since \( \ast \epsilon_2^2 = -\epsilon_2 \) and \( \ast \epsilon_2 = \epsilon_2^1 \), we would have obtained \( \xi_1^2 = -\xi_2^2 \), with no real solution. On the other hand, in Euclidean 11-dimensional space equation (76) implies that whenever the RHS is nonvanishing
$F_4$ is necessarily complex valued, and so there is no restriction on the coefficients of $F_4$ to be real. However, for having a well-defined metric we still require that $\lambda$ and $\tilde{\lambda}$ to be real.

To carry on, we set $\xi_1 = \xi_2 = \xi$ without loss of generality, and (77) becomes

$$\begin{align*}
\alpha \lambda^2 + 2\lambda^4 \gamma - i\tilde{\lambda}(1 + 2\lambda^4) &= 0, \\
(\tilde{\lambda} \lambda^2 - iN)\alpha + 6\lambda \beta + (iN\lambda^2 - \tilde{\lambda})\gamma &= 0, \\
\tilde{\lambda} \lambda^2 - 2\lambda \beta + iN\alpha/2 + i\xi^2 &= 0, \\
-\alpha \lambda^2 + iN\lambda \beta + \gamma &= 0,
\end{align*}$$

(79)

where the second equation is obtained by dividing the third and fourth equations in (77). For $\beta \neq 0$, we have found no solution of (79) for which $\lambda$ and $\tilde{\lambda}$ are both real. Let us then discuss the case with $\beta = 0$. In this case, the second and the fourth equations above imply

$$\begin{align*}
\lambda^2 &= 1, \\
\alpha &= \gamma.
\end{align*}$$

(80)

Plugging this into the first equation, we have

$$\begin{align*}
\alpha &= \gamma = i\tilde{\lambda},
\end{align*}$$

(81)

and finally the third equation gives

$$\begin{align*}
\xi^2 &= -\tilde{\lambda}^2 - \frac{1}{2}iN.
\end{align*}$$

(82)

Let us now look at the Einstein equations along $S^7$. Taking into account $\lambda^2 = 1$ and $\alpha = \gamma$, they read

$$\begin{align*}
2 - \frac{\tilde{\lambda}^2}{2} &= \frac{1}{2} \alpha^2 - \frac{1}{6} N^2 - \frac{1}{2} \xi^2, \\
\frac{3}{2} \lambda^2 &= -\frac{1}{2} \alpha^2 - \frac{1}{6} N^2 - \frac{1}{2} \xi^2,
\end{align*}$$

(83)

note the sign change of $N^2$ as a result of using the Riemannian signature (compare with (46)).

Using (81), we can solve for $\tilde{\lambda}$:

$$\begin{align*}
\tilde{\lambda}^2 &= 2, \\
N^2 + 3\xi^2 + 12 &= 0.
\end{align*}$$

(84, 85)

this last equation together with (82) can be solved to give $N$ and $\xi$.

The Ricci tensor along two two-dimensional spaces reads

$$\begin{align*}
R_{ab} &= (\xi^2 \frac{1}{2} \eta_2^2 + \frac{1}{4} N^2 - \frac{1}{6} (N^2 + \alpha^2 + 2\gamma^2) - \frac{1}{2} \xi^2) g_{ab}, \\
R_{a'b'} &= (\xi^2 \frac{1}{2} \eta_1^2 + \frac{1}{4} N^2 - \frac{1}{6} (N^2 + \alpha^2 + 2\gamma^2) - \frac{1}{2} \xi^2) g_{a'b'},
\end{align*}$$

(86)

with $a, b = 0, 1$, and $a', b' = 2, 3$. Now, using (81), (84) and (85), we obtain

$$\begin{align*}
R_{ab} &= -3g_{ab}, \\
R_{a'b'} &= -3g_{a'b'}.
\end{align*}$$

(87)

Therefore, the four-dimensional space is a direct product of a Euclidean AdS$_2$ and a two-dimensional hyperbolic surface. Interestingly, this solution has some common features with the Pope–Warner and the Freund–Rubin solutions. As in the Pope–Warner solution, here the metric of $S^7$ is stretched by a factor of 2 along its $U(1)$ fiber with $SU(4)$ isometry group. And, on the other hand, the four-dimensional Ricci tensor is equal to that of the Freund–Rubin solution.
6. Conclusions

In this paper, we provided a unified approach to study the squashed, stretched and the Englert-type solutions of 11-dimensional supergravity, especially when there are fluxes in the compact direction. With the special form of the metric (32), we were able to construct more general ansätze by bringing together the earlier known ones and those constructed in section 3. We then used the ansatz to reduce the field equations to algebraic ones and rederive the known solutions. Furthermore, using these ansätze we were able to find new compactifying solutions to five and four dimensions. In compactifying to five dimensions, we derived a solution of AdS$_5$ \times CP$^1$ type with the CP$^3$ factor stretched. We also derived a solution of AdS$_2$ \times H$^2$ \times S$^7$ type compactifying to Euclidean four dimensions. In this solution, the compact space was a stretched S$^7$.

Having derived the above solutions, the next important issue to address is that of stability. It is also worth studying the new solutions in the context of holographic superconductivity in M-theory [19].

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