STABILITY AND FINITE ELEMENT ERROR ANALYSIS FOR THE HELMHOLTZ EQUATION WITH VARIABLE COEFFICIENTS

I.G. GRAHAM AND S.A. SAUTER

ABSTRACT. We discuss the stability theory and numerical analysis of the Helmholtz equation with variable and possibly non-smooth or oscillatory coefficients. Using the unique continuation principle and the Fredholm alternative, we first give an existence-uniqueness result for this problem, which holds under rather general conditions on the coefficients and on the domain. Under additional assumptions, we derive estimates for the stability constant (i.e., the norm of the solution operator) in terms of the data (i.e. PDE coefficients and frequency), and we apply these estimates to obtain a new finite element error analysis for the Helmholtz equation which is valid at high frequency and with variable wave speed. The central role played by the stability constant in this theory leads us to investigate its behaviour with respect to coefficient variation in detail. We give, via a 1D analysis, an a priori bound with stability constant growing exponentially in the variance of the coefficients (wave speed and/or diffusion coefficient). Then, by means of a family of analytic examples (supplemented by numerical experiments), we show that this estimate is sharp.

Keywords: Helmholtz equation, high frequency, variable wave speed, variable density, well-posedness, a priori estimates, finite element error analysis

Mathematics Subject Classification (2000): 35J05, 65N12, 65N15, 65N30

1. INTRODUCTION

In this paper we consider the Helmholtz equation

\[(1.1) \quad - \text{div} (a \nabla u) - (\omega/c)^2 u = f \]

on a bounded Lipschitz domain \( \Omega \subset \mathbb{R}^d \), \( d = 1, 2, 3 \), with angular frequency \( \omega \geq \omega_0 > 0 \), given data \( f \in L^2(\Omega) \), and real-valued scalar coefficients \( a, c \) which are allowed to vary spatially, but which will be assumed to be bounded above and below by strictly positive numbers. The problem (1.1) is supplemented with mixed boundary conditions on \( \Gamma := \partial \Omega \) of the form

\[(1.2) \quad a \frac{\partial u}{\partial n} - i\omega\beta u = g \quad \text{on} \quad \Gamma_N, \quad u = 0 \quad \text{on} \quad \Gamma_D, \]

for given \( g \in H^{-1/2}(\Gamma_N) \) and real-valued \( \beta \in L^\infty(\Gamma_N) \). Here \( \Gamma_N, \Gamma_D \) are relatively open pairwise disjoint subsets of \( \Gamma \), with \( \Gamma = \Gamma_N \cup \Gamma_D \) and \( \partial / \partial n \) denotes the outward normal derivative.

For the strong formulation (1.1), a standard, additional requirement is that \( a \) is Lipschitz continuous, so that its gradient is defined almost everywhere. We shall consider rougher coefficients \( a, c \), via the weak form: seek \( u \in \mathcal{H} \) to satisfy

\[(1.3) \quad B_{a,c}(u, v) := \int_\Omega \left( a \nabla u \cdot \nabla v - \left( \frac{\omega}{c} \right)^2 u v \right) - i\omega \int_\Gamma \beta u \overline{v} = \int_\Omega f \overline{v} + \int_\Gamma g \overline{v}, \]

for all \( v \in \mathcal{H} \), where \( \mathcal{H} \) denotes the functions in \( H^1(\Omega) \) with vanishing trace on \( \Gamma_D \). Problem (1.3) arises as a frequency domain scalar approximation of certain elastic or electromagnetic propagation and scattering model problems (see [11]). In this context, the behaviour of \( u \) as frequency \( \omega \) grows is of both physical and numerical importance and occupies considerable contemporary interest.

When \( a \) and \( c \) are constant and (for simplicity) \( \beta = 1 \), it is well-known that the a priori energy bound

\[(1.4) \quad \left( \int_\Omega |a| \nabla u|^2 + \left( \frac{\omega}{c} \right)^2 |u|^2 \right)^{1/2} \leq C_{\text{stab}} \left( \|f\|_{L^2(\Omega)}^2 + \|g\|_{L^2(\Gamma)}^2 \right)^{1/2} \]

holds for some \( C_{\text{stab}} > 0 \) independent of \( f, g \) and \( \omega \), provided \( \Omega \) is Lipschitz and star-shaped with respect to a ball – see [32] Prop 8.1.4 (for \( d = 2 \)), [12] Theorem 1 (\( d = 3 \)) and e.g. [15] for a recent survey.

A standard approach to showing well-posedness of the problem (1.3) is to first establish the a priori bound (1.4) – which provides uniqueness – and then to infer existence via the Fredholm alternative. This approach restricts the range of problems which can be treated to those for which an a priori bound is...
available. A more general well-posedness theory can be obtained via the unique continuation principle (UCP) - and used in [29] Chap. 4.3 (also [32] Rem. 8.1.1). Our first contribution is to present well-posedness results for (1.3), (1.2) which do not require the a priori bound (1.4). This is done by appealing to the literature on the UCP and applying it to the special case of the Helmholtz equation.

Nevertheless, estimates like (1.4), when they are available, play a crucial role in the numerical analysis of (1.3). For the conforming fixed-order Galerkin finite element approximation of (1.3) with constant coefficients, a mesh diameter condition of the form $\omega^2 C_{stab}$ sufficiently small appears as a condition for quasi-optimality [31]. Our second contribution is to extend this finite element convergence theory to the case of variable coefficients, highlighting the role played by the variable-coefficient analogue of (1.4). In order to do this we extend (1.4) to the case of variable wavespeed (requiring that the wavespeed should be essentially non-oscillatory), and then give an analysis of conforming finite element methods for (1.3) under this assumption.

In the final part of the paper we investigate the outcome when the “non-oscillatory” assumption on the coefficients is violated. Here our contribution is a detailed 1D analysis which shows that the estimate (1.3) continues to hold, but the stability constant $C_{stab}$ has an estimate which can blow up exponentially in the variance of $a$ or $c$ (or both) for certain parameter configurations. Investigating this further, we present a class of examples in which this exponential blow-up is realised.

This paper is organized as follows. In §2 we present the existence-uniqueness of (1.1) allowing low regularity of the coefficients $a$ and $c$ via the UCP. The main result in §2 is Theorem 2.1 which requires that $a, c$ are uniformly positive and $a, c^{-2} \in L^\infty(\Omega)$ (for $d \leq 2$). For $d > 3$, the slightly more restrictive assumption that $a$ has a $C^{0,1}$ extension from $\Omega$ to a certain extended domain appears. Moreover the latter condition on $a$ is sharp in 3D, as the “counter-examples” in Remark 2.2 imply.

Turning to the $a$ priori bound (1.3), the standard approach is to use the “Rellich” test function $v = x \cdot \nabla u$ (or linear combinations of $v$ and $u$ - commonly known as the “Morawetz multiplier”) in the weak form (1.3) to obtain stability estimates - see e.g. [15] and a number of authors have recently applied this technique in the variable coefficient case (see discussion of literature below). In §3 we illustrate the outcome applying this analysis in the case of variable $c$ (but restricting to $a = 1$ for illustration) and we explain how this approach yields the bound (1.3), but only under the quite restrictive condition

$$\frac{x \cdot \nabla c}{c} \leq 1 - \theta ,$$

for some $\theta > 0$. Thus, although $c$ is allowed to decrease arbitrarily quickly along the radial directions, there is a severe restriction on any increase in $c$ and hence oscillatory behaviour for $c$ is essentially ruled out.

Section 4 is devoted to the convergence and stability analysis of a conforming Galerkin discretization. As it is common for elliptic problems with compact perturbations, the discretization has to satisfy a “minimal resolution condition” for the adjoint approximability constant (see (4.3)) and then quasi-optimal error estimates follow. In Theorem 4.1 we prove that the quasi-optimality constant for the energy error is independent of the wave number and give the explicit dependence on the coefficients $a$ and $c$. The adjoint approximability is estimated in Theorem 4.3; the estimate is explicit in the wave number $\omega$, reciprocal density $a$, the wave speed $c$, the mesh size $h$ of the finite element mesh, and the stability constant $C_{stab}$. This shows the importance of coefficient-explicit estimates of $C_{stab}$ in order to obtain a coefficient-explicit minimal resolution condition and error estimate. In the setting of §4 such estimates for $C_{stab}$ are available.

In §5 we consider one-dimensional problems with wave speed $c$ and diffusion coefficient $a$ which may both be spatially oscillatory and suffer jumps. We study how this affects $C_{stab}$. We prove the estimate (1.5) with $C_{stab}$ growing exponentially in the variation of the coefficients $a, c$. We conclude §5 with some analytic and numerical calculations on a particular example where the exponential growth of $C_{stab}$ is realised, demonstrating the sharpness of the theoretical estimates. In §6 a refined analysis is presented for perfectly oscillating coefficients $c$. Here, although the variation increases with the oscillation, it is shown that in this case the stability constant is bounded independently of the oscillation.

1.1. Applications. In seismic inversion “the forward model” is the elastic wave equation and the (generally spatially dependent) elastic parameters - density and Lamé parameters - have to be recovered in the inversion process (so-called “full waveform inversion”). The full forward model is thus a 3D system of evolution equations, which can also be converted to a Helmholtz-like system in the frequency domain by Fourier transform. The scalar equation (1.1) can be obtained as an approximation of this system where it is known as “the acoustic approximation”. In this case $a = 1/\rho$ and $c^2 = \rho c_p^2$, where $\rho$ is the density and $c_p$ is the speed of longitudinal waves (also called P-waves) in the medium. (See, e.g. [16].)
In the theory of photonic crystals, the spectra of Maxwell operators on infinite periodic media are analysed by the Floquet transform. The Maxwell system can be decomposed into TM (transverse magnetic) and TE (transverse electric) components and the computation of each case reduces to a scalar problem of form (1.1), with the TM mode having $a$ constant and $c$ variable and the TE mode having $c$ constant and $a$ variable. A key reference is Kuchment [28].

1.2. Literature on this topic. The numerical analysis of heterogeneous Helmholtz problems can be traced back at least to Aziz, Kellogg and Stephens [3]. Here problem (1.1) in 1D with $a = 1$ and $c$ positive (and at least $C^1$), together with mixed Dirichlet-Impedance boundary conditions, was considered. A decomposition of the solution $u$ in terms of explicitly oscillating functions $\exp(\pm i \omega K(x))$ (with $K$ denoting an antiderivative of $1/c$) and with smooth non-oscillatory multiplicative factors was obtained. In this analysis, a test function (of “Morawetz-type” (cf. [36])) of the form $v = au' + ba$, with $a$ and $b$ chosen as solutions of a certain ODE was used.

In [30], the “Rellich-type” test function $v(x) = xu'(x)$ (10) was used to prove stability bounds for a fluid-solid interaction problem, modeled in 1D with piecewise constant material properties having discontinuities at two points. This problem has some resemblance to (1.1) with jumping coefficients $a$ and $c$, but the interface conditions at the jump points are different. Nevertheless [30] provided a frequency-dependent stability result for this problem.

In the very interesting recent PhD thesis [11] (see also [5]) the technique of [30] was adapted to (1.1) in 1D with piecewise constant coefficients, allowing an arbitrary number of jumps of arbitrary magnitude. A frequency-dependent stability estimate is proven with a constant explicit in the number and magnitude of the jumps. This can grow exponentially with respect to the number of jumps. This is a special case of our results obtained in §5.1.

An early paper on the use of multiplier techniques to prove stability for the Helmholtz equation with variable refractive index is [38], where their conditions (1.7), (1.8) are similar to ours. This technique can directly be generalized to variable diffusion (see, e.g. [8]) in general dimension, essentially adding a variable refractive index is [38], where their conditions (1.7), (1.8) are similar to ours. This technique can directly be generalized to variable diffusion (see, e.g. [8]) in general dimension, essentially adding a

Related results are in the recent preprint [37], in which a certain Helmholtz transmission problem is considered, corresponding to scattering from a homogenised multiscale material in 2D. Estimates of the form (1.4) are obtained, but with a stability constant which grows cubically in $\omega$.

An alternative point of view is presented in [11]. This paper is presented in the context of acoustic scattering in random media, but the results can be restated so that they apply to deterministic media modelled by (1.1) with $a = 1$ and $c$ variable. They show that if $1/c = 1 + \varepsilon \eta$ where $\|\eta\|_{L^\infty(\Omega)} \leq 1$ then wavenumber independent stability can be proved provided a frequency-dependent $\varepsilon$ is chosen with $\varepsilon \in O(\omega^{-1})$. Obviously this condition is very restrictive on the range of allowable wave speeds but the result is interesting in that it allows very rough perturbations of a constant speed. Such rough perturbations are forbidden by (1.5). This result is exploited in an uncertainty quantification context in [13]. A condition which is very similar to (1.5) already appeared in [35], generalisations thereof in the case of physically relevant interior and exterior Helmholtz scattering problems in heterogeneous and stochastic media are discussed in detail in [22].

There is considerable current interest in linear algebra problems arising from the discretization of heterogeneous wave problems (e.g. [17], [21]). The stability theory of the underlying PDE turned out to be key to rigorously understanding the performance of iterative methods in the homogeneous case (e.g. [16], [23]), and so analogous results for the heterogeneous case will again be important in the construction and analysis of efficient solvers.

In obstacle scattering in homogeneous media, the stability of the solution operator is often associated with the scattering boundary being “non-trapping”, and the introduction of trapping boundaries (e.g. boundaries with cavities) causes the norm of the solution operator to blow up (e.g. [10], [7]). In some sense the imposition of the condition (1.5) can be thought of as a condition which removes the possibility of trapped waves. This correspondence makes more sense in exterior scattering problems and is explored in more detail in [22]. A condition closely related to (1.5) arises in [13]. The role of heterogeneity in the far field is discussed in [39]. Heterogeneity is also discussed in the microlocal analysis literature, where necessarily coefficients are assumed to be smooth. A very general result which shows that the stability constant can grow at worst exponentially in $\omega$ for $C^\infty(\Omega)$ coefficients is given in [14]. For an analogous result for the transmission problem see [6].

Finally we note that wave propagation and scattering in heterogeneous and random media is of considerable interest in the inverse problems and imaging community. There, the Green’s function for the heterogeneous Helmholtz problem - while not known analytically - is an important object of study. For certain coefficient configurations it is possible to analyze its qualitative behavior, derive expansions, as
well as explicit dependencies on certain parameters - see, e.g., [2], [19], although the issues pursued there are somewhat different to the topic of this paper.

1.3. Some notation. For $s \geq 0$, $1 \leq p \leq \infty$, $W^{s,p}(\Omega)$ will denote the classical Sobolev spaces of complex-valued functions with norm $\| \cdot \|_{W^{s,p}(\Omega)}$. Sobolev spaces on the boundary $\Gamma$ of $\Omega$ are defined in the usual manner and are denoted by $W^{s,p}(\Gamma)$ and $H^s(\Gamma)$. As usual we write $L^p(\Omega)$ instead of $W^{0,p}(\Omega)$ and $H^s(\Omega)$ for $W^{s,2}(\Omega)$. The scalar product and norm in $L^2(\Omega)$ and $L^2(\Gamma)$ are denoted respectively by

\[(u,v) := \int_\Omega u \overline{v} \quad \text{and} \quad \|u\| := (u,u)^{1/2} \quad \text{in} \quad L^2(\Omega), \]
\[(u,v)_\Gamma := \int_\Gamma u \overline{v} \quad \text{and} \quad \|u\|_\Gamma := (u,u)_\Gamma^{1/2} \quad \text{in} \quad L^2(\Gamma).\]

For a (topological) vector space $V$, we denote its topological dual space (i.e. all bounded linear functionals on $V$) by $V'$ and we denote its anti-dual (all bounded anti-linear functionals on $V$) by $V^\times$.

For parameters $0 < \alpha_0 < \alpha_1$ and any subspace $V(\Omega) \subseteq L^\infty(\Omega, \mathbb{R})$, we define

\[V(\Omega,[\alpha_0,\alpha_1]) := \left\{ w \in V(\Omega) : \alpha_0 \leq \text{ess inf}_{x \in \Omega} w(x) \leq \text{ess sup}_{x \in \Omega} w(x) \leq \alpha_1 \right\},\]

and in doing so we implicitly assume $-\infty < \alpha_0 < \alpha_1 < \infty$.

In what follows we shall study the Helmholtz problem (1.3) and its adjoint. In abstract form these read:

\[(1.6) \quad \text{Seek } u \in H : \quad B_{a,c}(u,v) = F(v) \quad \text{for all } v \in H,\]
\[(1.7) \quad \text{Seek } z \in H : \quad B_{a,c}(v,z) = G(v) \quad \text{for all } v \in H,\]

where $F \in H^\times$ and $G \in H'$ are given. Throughout we shall make the basic assumptions that $\alpha, c$ and $\beta$ are real-valued and:

\[a \in L^\infty(\Omega,[a_{\min},a_{\max}]）， \quad c \in L^\infty(\Omega,[c_{\min},c_{\max}]）， \quad \beta \in L^\infty(\Gamma,[0,\beta_{\max}]），\]

for some positive $a_{\min}, c_{\min}, \beta_{\max}$. Additional assumptions will be introduced where needed. We also assume that the frequency $\omega$ is bounded away from 0,

\[\omega \geq \omega_0 > 0.\]

For $0 < \lambda < 1$, we denote by $C^{0,\lambda}(\overline{\Omega})$ the space of Hölder continuous functions with Hölder exponent $\lambda$.

2. Well-Posedness via Unique Continuation

In this section we generalize the original ideas in [32] and apply the Unique Continuation Principle (UCP) to obtain uniqueness for problems (1.3), (1.7), under rather general conditions on the coefficients $\alpha, c$ and $\beta$. This, combined with the Fredholm Alternative, proves well-posedness for these problems. In our first result - Theorem 2.1 - we collect what is known about the UCP from various references in a form which should be useful to numerical analysts working on Helmholtz problems. Then, we prove the well-posedness in Theorem 2.4 To our knowledge this procedure provides Helmholtz well-posedness in the most general framework possible. We emphasize that the arguments and reasoning for the proof of well-posedness in this section are original but were made available to the authorship of [22] and anticipated therein.

Another approach is to first obtain a priori bounds which in turn imply uniqueness. However, as we explain below, up to current knowledge, a priori bounds require quite strong conditions on the coefficients.

**Theorem 2.1.** Suppose $a \in L^\infty(\Omega,[a_{\min},a_{\max}]$ is real-valued with $0 < a_{\min}$, and suppose $\kappa \in L^p(\Omega)$, for some $p > 1$. In addition, when $d \geq 3$ we require

\[(2.1) \quad a \in C^{0,1}(\overline{\Omega}) \quad \text{and} \quad \kappa \in L^{d/2}(\Omega).\]

Let $u \in H$ satisfy:

\[(2.2) \quad \int_\Omega \left\{ a \nabla u \cdot \nabla v + \kappa u v \right\} = 0 \quad \text{for all } v \in H.\]

Then, if $u$ vanishes on a ball $B$ of positive radius, with $\overline{B} \subseteq \Omega$, it follows that $u$ vanishes identically on $\Omega$. 


Proof. We start with the case $d \geq 3$. First note that (2.2) implies
\[-\nabla \cdot a \nabla u + \kappa u = 0, \text{ almost everywhere on } \Omega.\]
In the special case when $a = \lambda$, we have $\Delta u = \kappa u$, and the result follows from [27 Theorem 6.3]. In the general case when $a \in C^{0,1}(\Omega)$, Rademacher’s theorem implies $\nabla a \in (L^\infty(\Omega))^d$ and so we can write (cf. [21 Theorem 8.8])
\[-\Delta u = a^{-1}(\nabla a \cdot \nabla u - \kappa u), \text{ almost everywhere on } \Omega.\]
Since $u \in \mathcal{H} \subset H^1(\Omega)$, the Sobolev embedding theorem gives $u \in L^{\frac{2d}{d+2}}(\Omega)$ and using $\kappa \in L^{d/2}(\Omega)$ and Hölder's inequality, we obtain $\kappa u \in L^p(\Omega)$, where $p = 2d/(d+2) = 2 - 4/(d+2) < 2$. Moreover
\[|\nabla a \cdot \nabla u| \leq |\nabla a| |\nabla u|.\]

Then, local ellipticity estimates for the Laplace operator (for example [21 Theorem 9.11]) imply that $u \in W^{1,p}_{\text{loc}}(\Omega)$. The result then follows from [17 Theorem 1], with $A = a^{-1}_{\text{min}}|\kappa|$ and $B = a^{-1}_{\text{min}}|\nabla a|$.

The cases $d = 1, 2$ are somewhat easier. When $d = 2$ the proof can be found in [1 Theorem 1.1]. For $d = 1$ the proof is very similar. If $\kappa = 0$, it is elementary: Let $\sigma \subset \Omega$ be a connected subset on which $u = 0$. Let $\xi, \zeta \in \sigma$ with $\xi \neq \zeta$. Then, the solution of $-\langle au' \rangle + \kappa u = 0$ can be written in the form $u(x) = C \int_{\xi}^{\zeta} \left( \frac{a(s)}{a(\xi)} \right) ds$ for all $x \in \sigma$. Since $u(\zeta) = 0$, we have $C = 0$ and thus $u = 0$. If $\kappa \in L^p(\Omega)$ for $p > 1$ one applies the theory as in [1 Sec. 2] and observes that the Sobolev embedding theorem used in the proofs apply also for $d = 1$. This allows us to transform the equation $-\langle au' \rangle + \kappa u = 0$ in $\Omega$ as explained in [1 Sec. 3] to a local equation in divergence form $-\langle \tilde{a}u' + \tilde{\kappa}u \rangle = 0$, where $\tilde{a} \in L^\infty(\Omega, \{a_0, a_1\})$ for some $0 < a_0 < a_1 < \infty$ and $\tilde{\kappa} \in L^1$ for some $1 < t < p$. The variation of constant formula leads to
\[v(x) = C \left( \int_{t}^{a} \frac{1}{a(t)} \exp \left( \int_{t}^{a} \hat{\kappa}(s) \frac{a(s)}{a(t)} ds \right) \frac{a(s)}{a(t)} ds \right) \exp \left( \int_{t}^{a} \hat{\kappa}(s) \frac{a(s)}{a(t)} ds \right)
\]
and one uses the same argument as in the case $\kappa = 0$ to prove $v = 0$.

\[\Box\]

Remark 2.2. For $d \geq 3$, the condition $a \in C^{0,1}(\Omega, \mathbb{R})$ in Theorem 2.4 is sharp in the following sense. There exists a bounded domain $\Omega \subset \mathbb{R}^d$, a coefficient $a \in \bigcap_{\lambda < 1} C^{0,\lambda}(\Omega)$, a parameter $\omega > 0$, and a function $u \in C^\infty(\Omega)$ with $\text{supp}\ u \subseteq \Omega$ with
\[-\text{div}(au) + \omega^2 u = 0 \quad \text{in } \Omega.\]
A constructive proof is given in [15]. In [27 Rem. 6.5] an example is constructed which shows that also the assumption on $\kappa$ as in (2.1) is sharp.

Before we prove our main result - Theorem 2.4 - we need to state an assumption which will be required.

Assumption 2.3.

1. Either $\beta \in L^\infty(\Gamma_N, [0, \beta_{\text{max}}])$ or $\beta \in L^\infty(\Gamma_N, [-\beta_{\text{max}}, 0])$, with $\beta_{\text{max}} > 0$ and the set $\gamma := \text{int}(\text{supp } \beta) \subset \Gamma_N$ has positive $d - 1$ dimensional measure.

2. There exists a bounded connected domain $\Omega^* \supseteq \Omega$ with the property that $\Gamma \setminus \gamma \subset \partial \Omega^*$ and $\Omega^* \setminus \Omega$ has positive $d$ dimensional measure.

3. The coefficients $a$ and $c$ have extensions $a^*, c^*$ to $\Omega^*$ with $a^* \in L^\infty(\Omega^*, [a_{\text{min}}^*, a_{\text{max}}^*])$ and $c^* \in L^\infty(\Omega^*, [c_{\text{min}}^*, c_{\text{max}}^*])$ with $\omega_{\text{min}} > 0$ and $c_{\text{min}} > 0$.

4. When $d \geq 3$ we require in addition $a^* \in C^{0,1}(\Omega^*)$.

Parts 1. and 2. of this assumption are illustrated in Figure 1.

Theorem 2.4. Suppose Assumption 2.3 is satisfied. Then problems (1.6) and (1.7) have unique solutions in $\mathcal{H}$.

Assume, in addition, that the right-hand sides in (1.6) and (1.7) are given by $F(v) := (f, v)$ and $G(v) := (v, \lambda)$ for some functions $f, \lambda \in L^2(\Omega)$. Then, there exists a constant $C_{\text{stab}}$ independent of $f$ and $\lambda$ such that the corresponding solutions $u$ and $z$ satisfy
\[\|u\|_{\mathcal{H}, a, c} \leq C_{\text{stab}} \|f\| \quad \text{and} \quad \|z\|_{\mathcal{H}, a, c} \leq C_{\text{stab}} \|\lambda\|.
\]
In general this constant depends on $a$, $c$, $\omega$, and $\Omega$.

Proof. We give the proof for (1.6). The proof for (1.7) is identical. We introduce the parameter-dependent norm on $\mathcal{H}$:
\[\|e\|_{\mathcal{H}, a, c}^2 := \int_{\Omega} \left\{ a|\nabla v|^2 + \left( \frac{\omega}{c} \right)^2 |u|^2 \right\},\]
and we begin by writing
\begin{equation}
B_{a,c} = b_1 + b_2 + b_\Gamma,
\end{equation}
where
\begin{equation}
b_1(u,v) := \int_\Omega \{ a \nabla u \cdot \nabla v + \left( \frac{\omega}{c} \right)^2 u v \}, \quad b_2(u,v) := -2 \int_\Omega \left( \frac{\omega}{c} \right)^2 u v,
\end{equation}
\begin{equation}
and \quad b_\Gamma(u,v) := -i \omega \int_{\Gamma_N} \beta u v.
\end{equation}

We observe that $b_1$ and $b_2$ are Hermitian. Moreover
\begin{equation}
b_1(u,u) = \| u \|_{H_{1/2}}^2,
\end{equation}
and a combination of Hölder and Cauchy-Schwarz inequalities leads to the continuity estimates:
\begin{align}
|b_1(u,v)| &\leq \| u \|_{H,c} \| v \|_{H,c}, \\
|b_2(u,v)| &\leq 2 \left( \frac{\omega}{c} u \right) \left( \frac{\omega}{c} v \right) \leq 2 \| u \|_{H,c} \| v \|_{H,c}, \\
|b_\Gamma(u,v)| &\leq \| \sqrt{\omega} \beta u \|_{\Gamma} \| \sqrt{\omega} \beta v \|_{\Gamma}.
\end{align}

We recall also the multiplicative trace inequality:
\begin{equation}
\| u \|_{\Gamma} \leq C_{\text{trace}} \| u \|_{H_{1/2}}^{1/2} \| u \|_{H_{1/2}}^{1/2}.
\end{equation}
(For $d = 2, 3$, this is the last formula in [25, p.41]. For $d = 1$ it can be obtained by considering the integral of $(|u|^2 Z)'$ where $Z$ is the linear function with values $-1, 1$ at the left- and right-hand boundaries of the domain.) Combining this with Young’s inequality we obtain
\begin{align}
\left( \frac{\omega}{c} u \right) \left( \frac{\omega}{c} v \right) &\leq C_{\text{trace}} \| u \|_{H_{1/2}} \| v \|_{H_{1/2}} \leq C_{\text{trace}} \left( \frac{\omega^2 \beta_{\max}^2}{2} \| u \|_{H_{1/2}}^2 + \frac{1}{2} \| u \|_{H_{1/2}}^2 \right) \\
&\leq C_{\text{trace}} \left( \frac{1 + \omega^2 \beta_{\max}^2}{2} \frac{c_{\max}}{c} \left( \frac{\omega}{c} u \right) \left( \frac{\omega}{c} v \right) \leq \frac{1}{2 \beta_{\max}} \| \nabla u \|_{H_{1/2}}^2 \right) \\
&\leq C_{3} \| u \|_{H,c}^2,
\end{align}
with
\begin{equation}
C_{3} = \frac{C_{\text{trace}}}{\sqrt{2}} \max \left\{ a_{\min}^{-1/2}, c_{\max} \frac{1}{\omega_0^{1/2} + \beta_{\max}} \right\}.
\end{equation}

This proves the continuity of $b_\Gamma$, i.e.
\begin{equation}
|b_\Gamma(u,v)| \leq C_{3} \| u \|_{H,c} \| v \|_{H,c}.
\end{equation}

Now, for $u \in H$, let $K_2 u$ be the unique solution of the problem $b_1(K_2 u, v) = b_2(u,v), v \in H$, which is guaranteed to exist by the Lax-Milgram Lemma. Similarly, let $K_\Gamma u$ denote the unique solution of $b_1(K_\Gamma u, v) = b_\Gamma(u,v), v \in H$, and let $SF \in H$ denote the unique solution of the problem $b_1(SF, v) =$
Moreover we claim that the operators $K_2$ and $K_3$ are compact. (This is verified at the end of the proof.) Hence by the Fredholm Alternative, uniqueness for problem \((2.12)\) implies unique solvability.

To prove uniqueness, suppose $F = 0$ and let $u \in \mathcal{H}$ be a solution of
\[
B(u, v) = 0, \quad \text{for all } v \in \mathcal{H}.
\]
Putting $v = u$ and taking the imaginary part (and noting $\omega \geq \omega_0 > 0$), we obtain
\[
\int_{\Gamma_N} \beta |u|^2 = 0
\]
and then Assumption \((2.3)\) (1) implies that $u = 0$ almost everywhere on $\gamma$. Using Assumption \((2.3)\) (2), (3) we can extend $u$ by zero to a function $u^*$ on $\mathcal{H}^* = H^1(\Omega^*)$. Defining
\[
B^*(u, v) := \int_{\Omega^*} \left\{ a^* \nabla u, \nabla \overline{v} + \left( \frac{\omega}{c} \right)^2 u \overline{v} \right\} \quad \forall u, v \in \mathcal{H}^*,
\]
we have $B^*(u^*, v) = 0$ for all $v \in \mathcal{H}^*$. Now, since $u^*$ vanishes on $\Omega^* \setminus \Omega$, Theorem \(2.1\) tells us that $u$ is identically zero. (The assumptions of Theorem \(2.1\) are satisfied because of Assumptions \((2.3)\) (3) and (4).)

To finish the proof we show the compactness of $K_2$, $K_3$. By \((2.7)\) and \((2.8)\) we have
\[
\|K_2 u\|_{\mathcal{H}, a, c} \leq 2 \|\omega/c\| \leq 2 \frac{\omega}{c_{\min}} \|u\|,
\]
which shows $K_2$ is bounded as an operator from $L_2(\Omega)$ to $\mathcal{H}$, and is thus compact on $\mathcal{H}$. Similarly, using \((2.9)\) and \((2.11)\), we have
\[
\|K_3 u\|_{\mathcal{H}, a, c} \leq C_3 \|\sqrt{\omega/c}\| \leq C\|\sqrt{\omega/c}\| \|u\|_{\mathcal{H}^1(\Omega)},
\]
where we used the continuity of the trace operator from $H^{3/4}(\Omega) \to L^2(\Gamma)$, with continuity constant $C'_{\text{trace}}$. Now since $H^{3/4}(\Omega)$ is compactly embedded in $H^1(\Omega)$ we then have compactness of $K_3$.

In some of our applications in Section \(4\) we will employ the well-posedness of the Helmholtz problem in the following setting.

**Corollary 2.5.** Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain and assume $a \in C^{0,1}(\Omega, [a_{\min}, a_{\max}])$ and $c \in L^\infty(\Omega, [c_{\min}, c_{\max}])$ for some $0 < a_{\min} \leq a_{\max} < \infty$ and $0 < c_{\min} \leq c_{\max} < \infty$. Let $\Gamma_N = \Gamma$ and let $\beta := \sqrt{a/c}$. Assume that the right-hand sides in \((1.3)\) and \((1.7)\) are given by $F(v) := (f, v)$ and $G(v) := (v, \lambda)$ for some functions $f, \lambda \in L^2(\Omega)$. Then, there exists a constant $C_{\text{stab}} = C_{\text{stab}}(\omega, a, c, \Omega, d)$ independent of $f$ and $\lambda$ such that the corresponding solutions $u$ and $z$ satisfy \((2.3)\).

**Proof.** It is easy to verify that the assumptions in this corollary imply Assumption \((2.3)\). In particular, it is well known that the Lipschitz function $a$ can be extended to a Lipschitz function in $\mathbb{R}^d$ (with same Lipschitz constant).

**Remark 2.6.** The condition on $c^*$ in Assumption \((2.3)\) could be relaxed, since Theorem \((2.1)\) only requires that $\kappa = (\omega/c)^2 \in L^p(\Omega^*)$ for some $p > 1$.}

### 3. Frequency Explicit Estimates for $\Omega \subset \mathbb{R}^d$, $d \geq 2$

The first frequency explicit stability estimate for the Helmholtz problem with constant coefficients was given in \[32\] Prop 8.1.4]. There the key idea (used again in many subsequent works) was to use the test function of Rellich type
\[
v := \nabla \cdot \nabla u
\]
in the weak form \((1.3)\) and combine the resulting identity with estimates obtained using the test function $v = u$ and taking real and imaginary parts. An alternative way of thinking about this is to use a parametrized linear combination of these test functions (the so-called Morawetz multiplier) and then to choose the parameters appropriately to obtain the desired result. This method can also be applied to the heterogeneous Helmholtz problem, leading to a stability estimate subject to a strong restriction on the coefficients. Since this is the starting point for our analysis, we provide a sketch of this procedure for the restricted case $A = I$ and $c$ variable, with remarks as to how this can be generalised afterwards.
Assumption 3.1.

(1) $\Omega \subset \mathbb{R}^d$, $d \geq 2$, is a Lipschitz domain which is star-shaped with respect to a ball centred at the origin, i.e., there exists a constant $\gamma > 0$ such that

$$x \cdot n \geq \gamma \quad \text{for all } x \in \Gamma$$

and we set $R := \sup \{ |x| : x \in \Omega \}$.

(2) We restrict to (1.2) with $a = 1$ and $\beta = 1/c$.

(3) $\Gamma = \Gamma_N$, i.e., we consider the pure impedance problem, so $H = H^1(\Omega)$.

(4) The functions on the right-hand side of (1.3) satisfy $f \in L^2(\Omega)$ and $g = 0$.

In this case the equation (1.1) should be understood distributionally and is equivalent to the weak form (1.3).

Theorem 3.2. Let Assumption 3.1 be satisfied; let $\omega \geq \omega_0$ for some $\omega_0 > 0$ and let $c \in L^\infty(\Omega, [c_{\min}, c_{\max}])$ for some $0 < c_{\min} \leq c_{\max} < \infty$. Further we assume $c \in C^{0,1}(\Omega)$ and that there exists some $\theta > 0$ such that

$$\frac{x \cdot \nabla c(x)}{c(x)} \leq 1 - \theta, \quad \text{for all } x \in \Omega.$$ 

Let $u \in H$ denote the solution of (1.3). Then we have the a priori bound:

$$||\nabla u|| + \left| \frac{\omega}{c} u \right| \leq C_{\text{stab}} ||f||,$$

where $C_{\text{stab}}$ depends continuously on the positive real numbers $\omega_0, c_{\min}, c_{\max}, R, \gamma, d, \theta$. Moreover $C_{\text{stab}}$ may become unbounded if one or more of these parameter tends to 0 or $\infty$.

Remark 3.3. The same estimate holds for the adjoint problem where the sign of the boundary integral term in the definition of $B_{u,v}$ in (1.3) is changed from negative to positive.

Proof. First we note that the stated assumptions allow us to apply Corollary 2.5 which implies the existence and uniqueness of the solution $u$. In the following, the parameters $\varepsilon, \varepsilon', \varepsilon_j, \varepsilon'_j, \ldots$ denote positive real numbers, initially arbitrary but eventually fixed. Let $u$ denote the solution of (1.3). First, we choose $v = u$ in equation (1.3) and consider the real part of the equation. This leads to

$$||\nabla u||^2 \leq \left| \frac{\omega}{c} u \right|^2 + \frac{\varepsilon_1}{2} ||u||^2 + \frac{1}{2\varepsilon_1} ||f||^2.$$

The choice $\varepsilon_1 = \varepsilon'_1 \frac{c_{\max}}{2\varepsilon'_2}$ yields

$$||\nabla u||^2 \leq \left( 1 + \frac{\varepsilon'_1}{2} \right) \left| \frac{\omega}{c} u \right|^2 + \frac{c_{\max}}{2\varepsilon'_1 \omega^2} ||f||^2.$$

Recalling $\beta = 1/c$ and $g = 0$ and using the imaginary part of equation (1.3) with $v = -u$ we deduce

$$\left( \frac{\omega}{c} u, u \right)_\Gamma \leq \frac{1}{2} \left( \varepsilon_2 \omega ||u||^2 + \frac{1}{\varepsilon_2 \omega} ||f||^2 \right),$$

from which it follows that

$$||\frac{\omega}{c} u||^2 \leq \frac{\omega}{c_{\min}} \left( \frac{\omega}{c} u, u \right)_\Gamma \leq \frac{1}{2\varepsilon_{\min}} \left( \varepsilon_2 c_{\max} \frac{\omega}{c} u ||u||^2 + \frac{1}{\varepsilon_2} ||f||^2 \right).$$

Next, we choose $v$ as in (3.1). Then it follows by elementary vector calculus that

$$- 2\Re \int_\Omega \left( \nabla u + \left( \frac{\omega}{c} \right)^2 u \right) \nabla \cdot \nabla u + \int_\Omega \nabla \cdot \left( \left( \frac{\omega}{c} \right)^2 x \right) |u|^2$$

$$+ \int_\Gamma (x \cdot n) \left( \nabla |u|^2 - \left( \frac{\omega}{c} \right)^2 |u|^2 \right) - 2\Re \left( i \int_\Gamma \left( \frac{\omega}{c} u \right) \nabla \cdot n \right).$$

(More precisely (3.7) is first proved for arbitrary $u \in C^\infty(\Omega)$, and with $v$ as in (3.1) by elementary vector calculus. When $u$ is the actual solution to (1.1), (1.2) then

$$u \in V(\Omega) := \{ v \in H^1(\Omega) : \Delta u \in L^2(\Omega), \partial u/\partial n \in L^2(\Gamma), u|_\Gamma \in H^1(\Gamma) \}.$$
The proof of (3.7) is completed by observing that \( C^\infty (\Omega) \) is dense in \( V (\Omega) \) (see, e.g. [10] for an analogous argument). Then, rearranging (3.7) and recalling (1.1), leads to

\[
\int_\Omega \nabla \cdot \left( \left( \frac{\omega}{c} \right)^2 x \right) |u|^2 + \int_\Gamma (x.n) |\nabla u|^2 = \int_\Gamma (x.n) \left( \frac{\omega}{c} \right)^2 |u|^2 + 2\Re \left( \frac{1}{c} \left( \left( \frac{\omega}{c} \right) u, v \right)_\Gamma \right) + 2\Re (f, v) + (d - 2) \|\nabla u\|^2 \leq R \left( \frac{\omega}{c} u \right)_\Gamma^2 + 2 \left( \frac{\omega}{c} u \right)_\Gamma^2 \|v\|_\Gamma + \left( \frac{1}{\epsilon} \|f\|^2 + \frac{\varepsilon}{\epsilon} \|v\|^2 \right) + (d - 2) \|\nabla u\|^2.
\]

Since \( \|v\|_\Gamma \leq R \|\nabla u\|_\Gamma \), we obtain

\[
\int_\Omega \nabla \cdot \left( \left( \frac{\omega}{c} \right)^2 x \right) |u|^2 + \int_\Gamma (x.n) |\nabla u|^2 \leq R \left( \frac{\omega}{c} u \right)_\Gamma^2 + R \left( \varepsilon' \left( \frac{\omega}{c} u \right)_\Gamma^2 + \frac{\varepsilon}{\epsilon} \|\nabla u\|^2 \right) + \left( \varepsilon R^2 + d - 2 \right) \|\nabla u\|^2 + \frac{1}{\varepsilon} \|f\|^2.
\]

Now recall the assumption of star-shapedness (3.2) and choose \( \varepsilon' = 2R/\gamma \) to obtain

\[
\int_\Omega \nabla \cdot \left( \left( \frac{\omega}{c} \right)^2 x \right) |u|^2 + \frac{\gamma}{2} \|\nabla u\|^2 \leq R \left( 1 + \frac{2R}{\gamma} \right) \left( \frac{\omega}{c} u \right)_\Gamma^2 + \left( \varepsilon R^2 + d - 2 \right) \|\nabla u\|^2 + \frac{1}{\varepsilon} \|f\|^2.
\]

We now employ (3.9), (3.10) to estimate the first two terms on the right-hand side of (3.8). After a rearrangement this gives

\[
\int_\Omega \nabla \cdot \left( \left( \frac{\omega}{c} \right)^2 x \right) |u|^2 + \frac{\gamma}{2} \|\nabla u\|^2 \leq \frac{\delta}{\varepsilon} \left( \frac{\omega}{c} u \right)_\Gamma^2 + \left( \frac{R}{2\gamma} \left( 1 + \frac{2R}{\gamma} \right) \frac{1}{\epsilon_2} + \varepsilon R^2 + d - 2 \right) \frac{c_2^2}{\varepsilon_2^2} \|\nabla u\|^2 + \frac{1}{\varepsilon} \|f\|^2
\]

with

\[
\delta - (d - 2) = \frac{R}{2\gamma} \left( 1 + \frac{2R}{\gamma} \right) \varepsilon R^2 + \varepsilon R^2 \left( 1 + \frac{\varepsilon}{2} \right) + (d - 2) \varepsilon_1 \frac{c_1}{2}.
\]

Note that, using our assumption (3.3)

\[
\nabla \cdot \left( \left( \frac{\omega}{c} \right)^2 x \right) = \left( \frac{\omega}{c} \right)^2 \left( d - 2 \frac{x}{\varepsilon} \nabla c \right) \geq ((d - 2) + 2\theta) \left( \frac{\omega}{c} \right)^2.
\]

Hence, by making the right-hand side of (3.10) small enough we can “absorb” the term \( \frac{\delta}{\varepsilon} \left( \frac{\omega}{c} u \right)_\Gamma^2 \) on the right-hand side of (3.8) into the left-hand side. We do this by first choosing \( \varepsilon' = \frac{\varepsilon}{\varepsilon_2} \), which ensures \( (d - 2) \varepsilon_1 / 2 \leq \theta / 3 \). Then we choose \( \varepsilon, \varepsilon_2 \) so that

\[
\varepsilon R^2 \left( 1 + \frac{\varepsilon_1}{2} \right) = \theta / 3 = \frac{R}{2\gamma} \left( 1 + \frac{2R}{\gamma} \right) \varepsilon R^2 \frac{c_2^2}{\varepsilon_2^2}
\]

The right hand side of (3.10) is then bounded from above by \( \theta \), and we have derived the estimate

\[
\delta \left( \frac{\omega}{c} u \right)_\Gamma^2 + \frac{\gamma}{2} \|\nabla u\|^2 \leq \left( \frac{R}{2\gamma} \left( 1 + \frac{2R}{\gamma} \right) \frac{1}{\epsilon_2} + \varepsilon R^2 + d - 2 \right) \frac{c_2^2}{\varepsilon_2^2} \|\nabla u\|^2 + \frac{1}{\varepsilon} \|f\|^2.
\]

This leads to the final weighted \( L^2 \) estimate (3.4). The estimate of \( \|\nabla u\| \) follows from this via (3.5). \( \square \)

**Remark 3.4.**

a. The formulation and proof of Theorem 3.2 for \( d = 1 \) is analogous. We discuss it in some detail for a broader class of coefficients in [3].

b. Conditions which ensure the stability estimate (1.4) (with \( C_{\text{stab}} \) independent of \( \omega \)) for the problem (1.1), with the scalar function a generalised to a positive definite matrix \( A \) may be written

\[
(3.11) \quad \frac{x}{c} \nabla c \leq \frac{1}{2} - \theta, \quad (x.n) A \leq A - \theta' I,
\]

with \( \theta, \theta' \) required to be positive. The first condition in (3.11) was introduced in [3] for a Helmholtz equation of the form \( \Delta u + n(x) \omega^2 u = f \) with variable \( n \). By similar multiplier techniques as in [8] these results can be extended to variable \( A \) under relatively restrictive conditions.
on $A$ (see [8]). In [22] the condition on $A$ was formulated in the form (3.7). The second inequality should be understood in the sense of sesquilinear forms with the operator $(\mathbf{x} \cdot \nabla)$ being applied componentwise to the matrix $A$. In [22] it is shown that these conditions imply frequency-independent stability, not only for the interior impedance problem considered here but also for Dirichlet scattering problems on infinite and artificially truncated exterior domains. (Note that when $A$ and $c$ both vary the condition on $c$ in (3.11) is slightly stronger than that in (5.3).)

The main purpose of presenting the proof of Theorem 3.2 here is to emphasis how far one can get by using the “Rellich” test function (3.1). The resulting stability estimates require stronger smoothness requirements on the coefficients than those needed for the well-posedness in Theorem 2.4. Moreover the condition (5.3)2, while allowing $c$ to decay arbitrary quickly in the radial direction, effectively rules out highly oscillatory wave speeds. This is the starting point for [5] which concerns stability of problem (1.1) when (5.3)2 is not satisfied. In this case the Rellich test function (3.1) is not sufficient and we need to use other “coefficient dependent” test functions.

4. Finite element error estimates for heterogeneous problems

In this section we work under the assumptions as stated in Corollary 2.5. We prove estimates for the minimal resolution condition and the Galerkin error for conforming finite element approximation of (1.3) which are explicit in $\omega$, $a$, $c$, $h$, and the stability constant $C_{stab}$. In general the stability constant depends also on the coefficients and the wave number. However, the stronger Assumption 3.1 allows us to apply Theorem 3.2 and Remark 3.3 so that the constant $C_{stab}$ becomes independent of the wavenumber.

Our first results concern the abstract Galerkin method: for a general finite dimensional subspace $S \subset \mathcal{H}$ we seek $u_S \in S$ such that

\begin{equation}
B_{a,c}(u_S, v) = F(v), \quad \text{for all } v \in S.
\end{equation}

For the homogeneous case ($a = c = 1$) existence and uniqueness of the Galerkin solution follow by the “Schatz argument” (see [11]); the notion of adjoint approximability has been introduced in [11], [1], [33] and has been shown to play a fundamental role in the theory of (1.1). Here we generalize this to the heterogeneous case and refer to [13] for a similar reasoning in the context of a posteriori estimates. Let $T_{a,c}^*$ denote the solution operator for the adjoint problem with homogeneous impedance data, that is for $\lambda \in L^2(\Omega)$, $z = T_{a,c}^*\lambda$ is defined to be the solution to the adjoint equation

\begin{equation}
B_{a,c}(v, z) = (v, \lambda) \quad \text{for all } v \in H^1(\Omega).
\end{equation}

The well-posedness of this problem is ensured by Corollary 2.5. Note that (2.3) and (2.11) imply that

\begin{equation}
|B_{a,c}(u, v)| \leq C_{a,c} \|u\|_{H^{1,1}} \|v\|_{H_a,c}.
\end{equation}

Then we define the heterogeneous adjoint approximability constant $\sigma_{a,c}^*(S)$ by

\begin{equation}
\sigma_{a,c}^*(S) := \sup_{\varphi \in L^2(\Omega) \setminus \{0\}} \frac{\inf_{v \in S} \|T_{a,c}^*\left(\frac{\omega}{c}\right)^2 \varphi - v\|_{H_a,c}}{\|\varphi\|_{H_a,c}}.
\end{equation}

Using this we have a result on Galerkin well-posedness and error estimates:

**Theorem 4.1** (Discrete stability and convergence). Suppose the assumptions of Corollary 2.5 hold and suppose

\begin{equation}
\sigma_{a,c}^*(S) \leq \frac{1}{2C_{a,c}},
\end{equation}

with the continuity constant $C_{a,c}$ as given in (4.5). Then the discrete problem (4.4) has a unique solution which satisfies the error estimates:

\begin{equation}
\|u - u_S\|_{H_a,c} \leq 2C_{a,c} \inf_{v \in S} \|u - v\|_{H_a,c},
\end{equation}

\begin{equation}
\left\|\frac{\omega}{c}(u - u_S)\right\|_{L^2(\Omega)} \leq 2C_{a,c}^2 \sigma_{a,c}^*(S) \inf_{v \in S} \|u - v\|_{H_a,c}.
\end{equation}

**Proof.** We first estimate the $L^2$-error in terms of the $H^1$-error via the Aubin-Nitsche technique. Let $e = u - u_S$, set $\psi := T_{a,c}^*\left(\frac{\omega}{c}\right)^2 e$ and let $\psi_S \in S$ denote the best approximation to $\psi$ with respect to $\|\cdot\|_{H_a,c}$. Then, using the definition of $T^*$, we have

\[\left\|\frac{\omega}{c} e\right\|^2 = B_{a,c}(e, \psi).\]
and then using Galerkin orthogonality and continuity, we have
\[
\left\| \frac{\omega}{c} \right\|^2 = B_{a,c}(\varphi, \psi) \leq B_{a,c}(e, \psi - \psi_S) \leq C_{a,c} \left\| e \right\|_{H^{1/2}(\Gamma)} \left\| \psi - \psi_S \right\|_{H^{1/2}(\Gamma)}
\]
(4.8)
\[
\leq C_{a,c} \sigma^*_{a,c}(S) \left\| e \right\|_{H^{1/2}(\Gamma)} \left\| \omega \right\|_{H^{1/2}(\Gamma)}.
\]
To estimate the $H$-norm of the error, note that for any $v_S \in S$, we have, again by Galerkin orthogonality (and using (4.8)),
\[
\left\| e \right\|^2_{H^{1/2}(\Gamma)} = \mathcal{R}(B_{a,c}(e, e)) + 2 \left\| \frac{\omega}{c} \right\|^2 = \mathcal{R}B_{a,c}(e, u - v_S) + 2 \left\| \frac{\omega}{c} \right\|^2
\]
\[
\leq C_{a,c} \left\| e \right\|_{H^{1/2}(\Gamma)} \left\| u - v_S \right\|_{H^{1/2}(\Gamma)} + 2 \left( C_{a,c} \sigma^*_{a,c}(S) \right)^2 \left\| e \right\|^2_{H^{1/2}(\Gamma)}.
\]
Then (4.6) follows on application of (4.5), and (4.7) follows by combination of this with (4.3). \qed

For practical computations the space $S$ is typically chosen to be an $hp$ finite element space. In this case the role of the “resolution condition” (4.5) has been studied in detail for Helmholtz problems with constant coefficients in the sequence of papers [33, 34, 35]. In Theorem 4.3 below we give the first extension of this theory to the heterogeneous case. To reduce technicalities we restrict the argument to lowest order conforming finite elements.

In the argument below we will make use of the following Poisson problem: given $f \in L^2(\Omega)$ and $g \in H^{1/2}(\Gamma)$, find $u \in H^1(\Omega)$ such that
\[
(\nabla u, \nabla v) + (u, v) = (f, v) + (g, v)_{\Gamma}, \quad \forall v \in H^1(\Omega).
\]

Proposition 4.2. Let the assumptions of Corollary 2.2 be satisfied and $\Omega$ be a bounded convex Lipschitz domain. For $g = 0$ in (4.9), the Poisson problem (4.10) is $H^2$ regular i.e. there is a constant $C_{\text{reg}}$ such that
\[
\left\| u \right\|_{H^2(\Omega)} \leq C_{\text{reg}} \left\| f \right\|.
\]
Suppose $d = 2$ and let $\Omega$ be a bounded convex polygon. For $g \in H^{1/2}(\Gamma)$ we have
\[
\left\| u \right\|_{H^2(\Omega)} \leq C_{\text{reg}} \left( \left\| f \right\| + \left\| g \right\|_{H^{1/2}(\Gamma)} \right).
\]
Proof. For $g = 0$ this is [25, Theorem 3.2.1.3]. For inhomogeneous Neumann conditions one can use a lifting for the normal trace to transform the problem to a problem with homogeneous Neumann conditions (see [34, Lemma A.1] or [16, Lemma 2.12] for $d = 2$). \qed

Next we let $T_h$ be a shape regular family of conforming simplicial meshes on $\Omega$ with mesh diameter $h$ and let $S_h$ denote the corresponding space of continuous affine finite element functions. We recall that the nodal interpolant $I_h : C(\overline{\Omega}) \to S_h$ is well-defined for functions in $H^2(\Omega)$ (for $d = 1, 2, 3$) and satisfies, for some constant $C_{\text{int}}$,
\[
\left\| v - I_h v \right\| + h \left\| \nabla (v - I_h v) \right\| \leq C_{\text{int}} h^2 \left\| v \right\|_{H^2(\Omega)} \quad \forall v \in H^2(\Omega).
\]

Theorem 4.3. (i) Let the assumptions of Corollary 2.2 be satisfied and assume in addition that $c \in C^{0,1}(\Omega)$. Assume also that the solution of the Poisson problem (4.1) is $H^2$ regular and satisfies the estimate (4.7).

Then,
\[
\sigma^*_{a,c}(S) \leq K \left( \frac{\alpha_{\max}}{\alpha_{\min}} \right)^{1/2} \left( \frac{\omega}{\alpha_{\min}^2} \right) \left( \frac{c_{\min}}{\omega_0 h} + C_{\text{stab}} \right)^2 \left( \frac{\omega}{\alpha_{\min}^2} \right)
\]
(4.12)

with
\[
K := K (a,c,\omega_0,\Omega) := C_{\text{reg}} C_{\text{int}} \sqrt{\alpha_{\min}} \left( C_0 + C_0 \right) \frac{K_a}{\omega_0} \sqrt{\alpha_{\min} c_{\min}}.
\]

The constants $C_0$ and $C_0'$ are defined in (4.18) and (4.19).

(ii) If, in addition, the assumptions of Theorem 5.3 are satisfied, then $C_{\text{stab}}$ is independent of $\omega$ and the estimate (4.12) is explicit with respect to the coefficients $a, c, \omega, h$.

Proof. For $\varphi \in L^2(\Omega)$, let $z := T_{a,c}(\varphi)$. Then (4.11) leads to
\[
\inf_{v \in S} \left\| z - v \right\|_{H^{1/2}(\Gamma)} \leq \left\| z - I_h z \right\|_{H^{1/2}(\Gamma)} \leq C_{\text{int}} h \left( \frac{a_{\max}^{1/2}}{c_{\min}} + \frac{\omega h}{c_{\min}} \right) \left\| z \right\|_{H^2(\Omega)}.
\]
(4.14)
To get a bound for $\sigma^*_{a,c}(S)$ it remains to estimate $\|z\|_{H^2(\Omega)}$ in terms of $\|\varphi\|$. To do this we write the adjoint Helmholtz equation (4.12) for $\lambda := (\frac{\omega}{c})^2 \varphi$ which defines $z$ as the solution to a Poisson-type problem

$$-\Delta z + z = \left(\frac{\omega}{\sqrt{ac}}\right)^2 \varphi + \nabla \cdot \nabla z + \left(1 + \left(\frac{\omega}{\sqrt{ac}}\right)^2\right) z \quad \text{in } \Omega \quad \text{a.e.,}
$$

$$\frac{\partial z}{\partial n} = -i \frac{\omega}{\sqrt{ac}} z \quad \text{on } \Gamma.$$

Then, the $H^2$ regularity (4.10) implies

$$\|z\|_{H^2(\Omega)} \leq C_{\text{reg}} \left(\left\|\nabla \cdot \nabla \left(\frac{\omega}{\sqrt{ac}}\right)^2 \varphi\right\| + \kappa_a \frac{\sqrt{a_{\min}c_{\min}}}{c} \|\sqrt{a} \nabla z\| + \left\|1 + \left(\frac{\omega}{\sqrt{ac}}\right)^2\right\| z\right\|_{H^{1/2}(\Gamma)},$$

where $\kappa_a = \|\nabla a / a\|_{\infty}$. Utilising the pointwise estimate

$$1 + \left(\frac{\omega}{\sqrt{ac}}\right)^2 \leq C_0 \left(\frac{\omega}{\sqrt{ac}}\right)^2 \quad \text{with } C_0 = \left(1 + \left(\frac{1/2}{\min} c_{\max}\right)^2\right),$$

we obtain

$$\|z\|_{H^2(\Omega)} \leq C_{\text{reg}} \left(\frac{\omega}{a_{\min}c_{\min}} \left\|\varphi\right\| + \frac{\kappa_a}{\sqrt{a_{\min}}} \|\sqrt{a} \nabla z\| + C_0 \frac{\omega}{a_{\min}c_{\min}} \|z\|_{H^{1/2}(\Gamma)} + \left(\frac{\omega}{c}\right)\right),$$

To estimate the last term, we employ a trace inequality and then some elementary differentiation to obtain

$$\left\|\frac{\omega}{\sqrt{ac}} z \left\|_{H^{1/2}(\Gamma)} \leq C_{\text{trace}} \left\|\varphi \left\|_{H^1(\Omega)} \leq C_0 \left(\frac{\omega}{\sqrt{a_{\min}c_{\min}}} \right) \|z\|_{H_{a,c}},$$

with

$$C_0 := C_{\text{trace}} \left(\frac{1}{\sqrt{a_{\min}}} + \frac{c_{\min}}{\omega_0} \left(1 + \kappa_c + \kappa_a / 2\right)\right),$$

where $\kappa_c = \|\nabla c / c\|_{\infty}$. Hence, combining (4.18) and (4.19) with (4.17), and using the definition of $C_{\text{stab}}$ as in Corollary 2.24 for the adjoint problem (4.12) with $\lambda := (\frac{\omega}{c})^2 \varphi$, we obtain

$$\|z\|_{H^2(\Omega)} \leq C_{\text{reg}} \left(\frac{\omega}{a_{\min}c_{\min}} \left\|\varphi\right\| + \left(C_0 + C_0' \sqrt{a_{\min}} + \frac{\kappa_a}{\omega_0} \sqrt{a_{\min}c_{\min}}\right) \|z\|_{H_{a,c}}\right)$$

$$\leq C_{\text{reg}} \left(1 + \left(C_0 + C_0' \sqrt{a_{\min}} + \frac{\kappa_a}{\omega_0} \sqrt{a_{\min}c_{\min}}\right) \|z\|_{H_{a,c}}\right),$$

The combination of (4.14) with (4.20) leads to

$$\inf_{v \in S} \left\|z - v\right\|_{H_{a,c}} \leq h \left(\frac{\sqrt{a_{\max}}}{\sqrt{a_{\min}c_{\min}}} \right) \times$$

$$\left[C_{\text{reg}} C_{\text{int}} \sqrt{a_{\min}} \left(1 + \left(C_0 + C_0' \sqrt{a_{\min}} + \frac{\kappa_a}{\omega_0} \sqrt{a_{\min}c_{\min}}\right) C_{\text{stab}} \frac{\omega}{c_{\min}}\right) \right] \frac{\omega}{a_{\min}c_{\min}}.$$

With $K$ as defined in (4.13), the right hand side of (4.21) can be written

$$h \left(\frac{\sqrt{a_{\max}}}{\sqrt{a_{\min}c_{\min}}} \right) \left[C_{\text{reg}} C_{\text{int}} \sqrt{a_{\min}} + KC_{\text{stab}} \frac{\omega}{c_{\min}}\right] \frac{\omega}{a_{\min}c_{\min}}.$$

Now, using

$$K \geq C_{\text{reg}} C_{\text{int}} \sqrt{a_{\min}} \quad (\text{since } C_0 \geq 1),$$

we obtain (4.12).

**Remark 4.4.** a. The right-hand side in the estimate (4.12) blows up if $a_{\min}^{-1}, a_{\max}, c_{\min}^{-1}, c_{\max}, \omega_0^{-1}$, $\kappa_a, \kappa_c$ tend to infinity but remains bounded otherwise.
b. The combination of Theorems 4.1 and 4.3 gives a complete theory for the lowest order Galerkin discretisation of the Helmholtz problem with variable a and c in the case when \( C_{\text{stab}} < \infty \): If \( \omega^2 h \) is chosen sufficiently small (with respect to the values of \( a_{\min}, a_{\max}, c_{\min}, c_{\max}, \kappa_a, \kappa_c \)), then the Galerkin method is well-posed and enjoys quasi-optimal error estimates in the weighted norm \( \| \cdot \|_{H,a,c} \). The condition on \( \omega^2 h \) becomes more stringent if \( C_{\text{stab}} \), \( a_{\min}^{-1}, a_{\max}^{-1}, c_{\min}^{-1}, c_{\max}^{-1}, \kappa_a \) or \( \kappa_c \) increase. This result shows the key role played by the stability constant \( C_{\text{stab}} \) in the Galerkin theory. We also know from [3] that a sufficient condition for a frequency-independent bound \( C_{\text{stab}} < \infty \) (under the assumptions of Theorem 3.3) involves upper bounds on \( c_{\min}^{-1}, c_{\max}^{-1} \) and \( \kappa \) and so the stability and discretisation theories are intimately linked.

c. For constant coefficients \( a = c = 1 \) it is shown in [33], [34] that higher order methods perform much better in the pre-asymptotic range than low order methods: the condition \( \omega^2 h \lesssim 1 \) can then be replaced by \( \omega h / p \lesssim 1 \) if the polynomial degree is chosen according to \( p \gtrsim \log k \). For heterogeneous Helmholtz problems this question is open; first numerical results are reported in [13].

5. The 1-dimensional Case

Without loss of generality we assume \( \Omega = (-L, L) \); a problem on any other interval can be transferred to \( \Omega \) via an affine change of variable. The boundary consists of the two endpoints \( \Gamma = \{-L, L\} \) and we consider the following choices of the Dirichlet- and impedance parts of the boundary:

\[
\begin{align*}
\Gamma_D & = \emptyset \quad \text{and} \quad \Gamma_N = \Gamma \quad \text{pure impedance,} \quad (5.1) \\
\Gamma_D & = \{L\} \quad \text{and} \quad \Gamma_N = \{-L\} \quad \text{Dirichlet-impedance,} \\
\Gamma_D & = \{-L\} \quad \text{and} \quad \Gamma_N = \{L\} \quad \text{impedance-Dirichlet.}
\end{align*}
\]

To reduce technicalities we assume throughout this section that \( \beta \) given in (4.2) is

\[
\beta(x) = \frac{\sqrt{a(x)}}{c(x)} \quad \text{for} \quad x \in \Gamma_N.
\]

The weak form is defined using the sesquilinear form on \( \mathcal{H} \):

\[
B_{a,c}(u, v) := \int_{-L}^{L} \left( a u' \overline{v}' - \left( \frac{\omega}{c} \right)^2 u \overline{v} \right) - i \sum_{x \in \Gamma_N} \frac{\sqrt{a(x)}}{c(x)} u(x) \overline{v}(x).
\]

We recall that the norm is

\[
\|v\|_{\mathcal{H}, a, c}^2 = \|\sqrt{a} v\|^2 + \left\| \frac{\omega}{c} v \right\|^2.
\]

Then the problem is to seek a solution \( u \in \mathcal{H} \), such that

\[
B_{a,c}(u, v) = F(v) + G(v) \quad \text{for all} \quad v \in \mathcal{H},
\]

\[
(5.5)
\]

where \( G(v) = \sum_{x \in \Gamma_N} g_x \overline{v}(x) \) and \( F(v) = \int_{-L}^{L} f \overline{v} \), for given data \( f \in L^2(\Omega) \), and \( g_x \in \mathbb{C}, x \in \Gamma_N \).

To describe the properties of \( a, c \) we need the following definition of functions with a finite number of jumps and changes of sign.

**Definition 5.1.** We denote by \( C^1_{\text{pw}} [-L, L] \) the space of functions \( g : [-L, L] \to \mathbb{R} \) such that there exists a finite partition

\[
(5.6)
\]

\[-L = z_0 < \ldots < z_N = L,\]

with \( g \in C^1 [z_{j-1}, z_j] \) for each \( j = 1, \ldots, N \), and

\[
(5.7)
\]

either \( g'(x) > 0 \) or \( g'(x) \leq 0 \), when \( x \in (z_{j-1}, z_j) \).

The partition depends on \( g \) and is not unique; once a partition for \( g \) is identified, any refinement of it is also a partition. For each \( z_j \), we define the one-sided limits

\[
g^- (z_j) := \lim_{x \to z_j^-} g(x) \quad \text{and} \quad g^+ (z_j) = \lim_{x \to z_j^+} g(x).
\]

and the jumps of \( g \) at each \( z_j \) (here taken from left to right) are defined by

\[
[g]_{z_j} := \begin{cases} 
g^-(z_j) - g^+(z_j) & 1 \leq j \leq N - 1, \\
g^-(L) & j = 0, \\
g^+(L) & j = N. \end{cases}
\]
(When \( g \) is one-signed and without any discontinuities, we have \( z_0 = -L, z_1 = L \) and no interior points in the partition.) Then we define the regular part of the derivative of \( g \in C^1_{pw}(\Omega) \) by
\[
\partial_{pw} g(x) = g'(x), \quad x \in (z_{j-1}, z_j), \quad j = 1, \ldots, N,
\]
and the variation of \( g \) on \([-L, L]\) is defined by
\[
\text{Var}(g) = \sum_{\ell = -M+1}^{N-1} |g|_{z_\ell} + \int_{-L}^{L} |(\partial_{pw} g)(s)| \, ds.
\]
For later notational convenience, we denote the subintervals of \([5.6]\) as:
\[
\tau_j = (z_{j-1}, z_j), \quad j = 1, \ldots, N.
\]
In the following we shall make the following assumption on the coefficients \( a, c \).

**Assumption 5.2.** We assume that \( a, c \in C^1_{pw} \) and that
\[
a_{\text{min}} \leq a(x) \leq a_{\text{max}} \quad \text{and} \quad c_{\text{min}} \leq c(x) \leq c_{\text{max}}, \quad x \in [-L, L],
\]
for some positive \( a_{\text{min}}, c_{\text{min}} \). Then, without loss of generality, there is a partition (which we again write as \([5.6]\)) so that, for each \( \tau_j, j = 1, \ldots, N \), \( a' \) and \( c' \) are both one-signed.

While the problem is properly defined by \([5.4]\), we also wish to derive estimates using test functions with discontinuities at the points \( z_j \). To allow this we rewrite \([5.4]\) as an interface problem. By adapting the argument in \([11, \text{Theorem 1]} \) one can show that \([5.4]\) is equivalent to the problem
\[
(a u')' - \left(\frac{a'}{c'}\right)^2 u = f \quad \text{in} \quad \tau_j, \quad \text{for all} \quad j = 1, \ldots, N
\]
together with the interface conditions at interior points
\[
[u]_{z_j} = 0 \quad \text{and} \quad [au']_{z_j} = 0, \quad j = 1, \ldots, N - 1
\]
and the boundary conditions (cf. \([12]\) with \([5.2]\))
\[
\left(\frac{\partial u}{\partial n} - i \omega \frac{\sqrt{a}}{c} u\right)(x) = g_x \quad \text{for all} \quad x \in \Gamma_N \quad \text{and} \quad u(x) = 0 \quad \text{for} \quad x \in \Gamma_D
\]
with the “normal” derivative defined as \( \partial u/\partial n (\pm L) := \pm u' (\pm L) \). Note that Assumption 5.2 allows \( a \) to be discontinuous at some of the \( z_j \) (but does not imply discontinuity at any particular \( z_j \)). If \( a \) is continuous at \( z_j \), the second interface condition in \([5.10]\) simplifies to \([u']_{z_j} = 0 \).

### 5.1. Stability estimate for oscillatory and jumping coefficients

**Lemma 5.3.** Suppose Assumption 5.2 is satisfied and let \( u \) solve \([5.4]\). Then for any real-valued \( q \) which satisfies \( q|_{\tau_j} \in C^1(\tau_j) \), for each \( j = 1, \ldots, N \) and \( q(x) = 0 \) for \( x \in \Gamma_D \), we have
\[
\frac{1}{2} \int_{-L}^{L} \left( \frac{\partial}{\partial n} \left( \frac{q}{a} \right) |au'|^2 + \omega^2 \partial_{pw} \left( \frac{q}{c^2} \right) |u|^2 \right) \, dx
\]
\[
- \frac{1}{2} \sum_{j=1}^{N-1} \left( \frac{q}{a} |(au')(z_j)|^2 + \omega^2 \frac{q}{c^2} |u(z_j)|^2 \right)
\]
\[
\leq \frac{3}{2} \sum_{x \in \Gamma_N} |q(x)| \omega \frac{a}{c} |u(x)| + \frac{1}{a_{\text{min}}} \sum_{x \in \Gamma_N} |q(x)||g_x|^2 + |(f, qu')|.
\]

**Proof.** Suppose \( q|_{\tau_j} \in C^1(\tau_j) \) for each \( j = 1, \ldots, N \). Then,
\[
- \Re \int_{z_{j-1}}^{z_j} (au')' qu' = - \frac{1}{2} \int_{z_{j-1}}^{z_j} \frac{a}{q} |au'|^2 = \frac{1}{2} \int_{z_{j-1}}^{z_j} \frac{q}{a} |(au')'|^2 = \frac{1}{2} \int_{z_{j-1}}^{z_j} \frac{q}{c^2} |u|^2 - \frac{1}{2} \frac{q}{c^2} |u|^2_{z_{j-1}}
\]
and
\[
- \Re \int_{z_{j-1}}^{z_j} u \left( \frac{q}{c^2} \right) u' = - \frac{1}{2} \int_{z_{j-1}}^{z_j} \left( |u|^2 \right)' \left( \frac{q}{c^2} \right) = \frac{1}{2} \int_{z_{j-1}}^{z_j} \left( \frac{q}{c^2} \right)' |u|^2 - \frac{1}{2} \frac{q}{c^2} |u|^2_{z_{j-1}}.
\]
Now add (5.13) to (5.14) (multiplied by \( \omega^2 \)) and sum over \( j = 1, \ldots, N \), to obtain (recalling that \( u \) satisfies the interface conditions (5.10))

\[
\begin{align*}
\Re \int_{-L}^{L} \left( \partial_{pw} (au') + \left( \frac{\omega}{c} \right)^2 a \right) qu' & = \frac{1}{2} \int_{-L}^{L} \left( \partial_{pw} \left( \frac{q}{a} \right) \right) |au'|^2 + \omega^2 \left( \partial_{pw} \left( \frac{q}{c^2} \right) \right) |u|^2 \\
- \frac{1}{2} \sum_{j=1}^{N} \left( \frac{q}{a} |(au'(z_j)|^2 \right)_{z_{j-1}} - \frac{1}{2} \sum_{j=1}^{N} \omega^2 \left( \frac{q}{c^2} \right) |u|^2 \right)_{z_{j-1}} = \frac{1}{2} \int_{-L}^{L} \left( \partial_{pw} \left( \frac{q}{a} \right) \right) |au'|^2 + \omega^2 \left( \partial_{pw} \left( \frac{q}{c^2} \right) \right) |u|^2 \\
- \frac{1}{2} \sum_{j=1}^{N-1} \left( \frac{q}{a} |(au'(z_j)|^2 + \omega^2 \left( \frac{q}{c^2} \right) |u(z_j)|^2 \right) \right)_{z_{j}} = \frac{1}{2} \left( |(au'|^2 + \omega^2 \left( \frac{q}{c^2} |u|^2 \right) \right) q \right)_{z_{j}} - L + \Re (f, qu'). 
\end{align*}
\]

On the other hand, the left-hand side in (5.15) equals \( \Re (f, qu') \) so that

\[
\begin{align*}
\frac{1}{2} \int_{-L}^{L} \left( \partial_{pw} \left( \frac{q}{a} \right) \right) |u'|^2 + \omega^2 \left( \partial_{pw} \left( \frac{q}{c^2} \right) \right) |u|^2 \\
- \frac{1}{2} \sum_{j=M+1}^{N-1} \left( \frac{q}{a} |(au'(z_j)|^2 + \omega^2 \left( \frac{q}{c^2} \right) |u(z_j)|^2 \right)_{z_{j}} = \frac{1}{2} \left( |(au'|^2 + \omega^2 \left( \frac{q}{c^2} |u|^2 \right) \right) q \right)_{z_{j}} - L + \Re (f, qu'). 
\end{align*}
\]

The required result is then obtained by estimating the first term on the right-hand side of (5.16). To do this we use the boundary conditions (5.11) to obtain, for \( x \in \Gamma_N \),

\[
a (x) |u'(x)|^2 \leq 2 \left( \frac{\omega}{\epsilon(x)} \right)^2 |u(x)|^2 + \frac{q(x)^2}{a(x)}.
\]

Combining this with (5.10) yields the result. \( \square \)

This lemma leads to the following theorem, which identifies suitable properties of \( q \) which will lead to an \( a \) priori bound for \( u \). Following this we will describe how to construct \( q \) satisfying these properties.

**Theorem 5.4.** Suppose Assumption (5.2) is satisfied. Let \( u \) solve (5.4) and suppose \( q \) can be chosen as in Lemma (5.3) with the two additional properties:

1. For any \( j = 1, \ldots, N \),

\[
\partial_{pw} \left( \frac{q}{a} \right) (x) \geq \frac{1}{a(x)}, \quad \text{and} \quad \partial_{pw} \left( \frac{q}{c^2} \right) (x) \geq \frac{1}{c^2(x)}, \quad x \in \tau_j.
\]

2. We have the negative interior jumps:

\[
\left( \frac{q}{a} \right)_{z_j} \leq 0 \quad \text{and} \quad \left( \frac{q}{c^2} \right)_{z_j} \leq 0, \quad j = 1, \ldots, N - 1.
\]

Then for \( \omega_0 > 0 \), \( f \in L^2(\Omega) \) and \( g : \Gamma_N \rightarrow C \) the a priori bound

\[
\|u\|_{H^{1,\alpha,\epsilon}} \leq C_{stab}^{t} ||f|| + C_{stab}^{t} \sqrt{Q} ||g||_{\Gamma_N}
\]

holds, for all \( \omega \geq \omega_0 \), with \( Q = ||q||_{L^\infty([-L,L])} \).

\[
C_{stab}^{t} := \frac{2}{\sqrt{\alpha_{\min}}} \left( 1 + 3 \frac{c_{\max}}{c_{\min}} \right) \quad \text{and} \quad C_{stab}^{t} := \frac{2}{\sqrt{\alpha_{\min}}} \left( \frac{3}{2} \frac{c_{\max}}{c_{\min}} \right) + 1.
\]

**Proof.** In the following, \( \epsilon, \epsilon_1, \epsilon_2 \ldots \) denote positive real numbers. Also, we will make frequent use of the following elementary estimate. For two functions \( \mu, \nu \in L^2(\Omega) \) and any positive function \( \delta \in L^\infty(\Omega, [\delta_0, \delta_1]) \) for some \( 0 < \delta_0 \leq \delta_1 < \infty \), it follows

\[
|\langle \mu, \nu \rangle| \leq \frac{1}{2} ||\delta \mu||^2 + \frac{1}{2} ||\nu||^2.
\]
Using Lemma 5.3 and making use of (5.17) and (5.18), we obtain

\[ (5.22) \quad \frac{1}{2} \|u\|_{H_{a,c}}^2 \leq \frac{3}{2} \sum_{x \in \Gamma_N} |q(x)| \left| \frac{\omega}{c(x)} u(x) \right|^2 + \frac{1}{a_{\text{min}}} \sum_{x \in \Gamma_N} |q(x)||g_x|^2 + |(f, qv')|. \]

For convenience we introduce the notation (for suitable functions \(f, g\)),

\[ (f, g)v_N = \sum_{x \in \Gamma_N} f(x)g(x) \quad \text{and} \quad \|f\|_{L^2}^2 = (f, f)v_N. \]

Then (5.22) yields

\[ (5.23) \quad \frac{1}{2Q} \|u\|_{H_{a,c}}^2 \leq \frac{3}{2} \left\| \frac{\omega}{c} u \right\|_{\Gamma_N}^2 + \int_{-L}^L |f| |u'| + \frac{1}{a_{\text{min}}} \|g\|_{L^2}^2. \]

To estimate the first term on the right-hand side of (5.23), we insert \(v = u\) into (5.4) and take the imaginary part of each side to obtain

\[ (5.24) \quad \left\| \sqrt{\frac{\omega}{c}} u \right\|_{\Gamma_N}^2 \leq |(f, u)| + |(g, u)_{\Gamma_N}|. \]

We then use (5.21) with \(\mu = u|_{\Gamma_N}, \nu = g, \text{ and } \delta = \sqrt{\frac{\omega}{c}}\) to obtain

\[ |(g, u)_{\Gamma_N}| \leq \frac{1}{2} \left\| \sqrt{\frac{\omega}{c}} u \right\|_{\Gamma_N}^2 + \frac{1}{2} \left\| \sqrt{\frac{\omega}{c}} g \right\|_{\Gamma_N}^2 \]

so that (5.24) yields

\[ \left\| \sqrt{\frac{\omega}{c}} u \right\|_{\Gamma_N}^2 \leq \frac{\omega}{\delta_2^2} + \delta_2 u^2 + \left\| \sqrt{\frac{\omega}{c}} g \right\|_{\Gamma_N}^2, \quad \text{for any } \delta_2 > 0. \]

Hence

\[ \left\| \frac{\omega}{c} u \right\|_{\Gamma_N}^2 \leq \frac{\omega}{\delta_2} \left( \frac{\omega}{\delta_2} \right)^2 + \frac{c_{\text{max}}}{\sqrt{\delta_2}} \left( \frac{\omega}{\delta_2} \right)^2 u^2 + \frac{c_{\text{max}}}{\sqrt{\delta_2}} \left( \frac{\omega}{\delta_2} \right)^2 g^2 \]

We choose \(\delta_2 = \sqrt{\frac{1}{\delta_2}}\) to finally obtain

\[ \left\| \frac{\omega}{c} u \right\|_{\Gamma_N}^2 \leq \frac{c_{\text{max}}}{\epsilon_1 \sqrt{\delta_2}} \left( \frac{1}{\epsilon_1} \|f\|_{L^2} + \frac{\omega}{\epsilon_1} \|u\|_{L^2}^2 + \frac{1}{\sqrt{\delta_2}} \|g\|_{L^2}^2 \right). \]

Now, substituting this for the first term on the right-hand side of (5.23) and estimating the second term similarly, we obtain

\[ \frac{1}{2Q} \|u\|_{H_{a,c}}^2 \leq \frac{3}{2} \frac{c_{\text{max}}}{\epsilon_1 \sqrt{\delta_2}} \left( \frac{1}{\epsilon_1} \|f\|_{L^2}^2 + \frac{\omega}{\epsilon_1} \|u\|_{L^2}^2 + \frac{1}{\sqrt{\delta_2}} \|g\|_{L^2}^2 \right) + \frac{1}{2\epsilon_2} \|f\|_{L^2}^2 + \frac{\epsilon_2}{2\epsilon_2} \|au\|_{L^2}^2 + \frac{\epsilon_2}{\sqrt{\delta_2}} \|g\|_{L^2}^2. \]

The choices of \(\epsilon_1\) and \(\epsilon_2\) given by

\[ \epsilon_1 \frac{3}{2} \frac{c_{\text{max}}}{\epsilon_1 \sqrt{\delta_2}} = \frac{1}{4Q} \quad \text{and} \quad \frac{\epsilon_2}{2\epsilon_2} = \frac{1}{4Q} \]

lead to

\[ \frac{1}{4Q} \|u\|_{H_{a,c}}^2 \leq Q \left( \frac{1}{\delta_2} + 9 \frac{c_{\text{max}}^2}{\epsilon_1 \sqrt{\delta_2}} \|f\|_{L^2}^2 + \frac{1}{\sqrt{\delta_2}} \|g\|_{L^2}^2 \right), \]

which yields the result after straightforward algebraic manipulations. \(\square\)
Recall that the coefficients \( a, c \) are required to satisfy Assumption 5.2. In order to construct an appropriate function \( q \) in Theorem [5.3] we introduce the following definition. From the function \( a \) defined above, and for each \( j \), we define, for \( x \in \tau_j = (z_{j-1}, z_j) \),

\[
\tilde{a}(x) = \begin{cases} 
    a(x) & \text{when } a'(x) > 0, \\
    a^+(z_{j-1}) & \text{when } a'(x) \leq 0.
\end{cases}
\]

(5.25)

The values of \( \tilde{a} \) at the breakpoints \( z_j \) are unimportant in what follows, but for definiteness we shall require \( \tilde{a} \) to be right continuous at each \( z_j \), \( j < N \) and left continuous at \( z_N \).

The function \( \tilde{c} \) is defined analogously and it is easily verified that \( \tilde{c}^2 = c^2 \). From this definition we have the following two propositions. The proof of the first is very elementary and so omitted.

**Proposition 5.5.** Under Assumption 5.2 for \( j = 1, \ldots, N \),

\[
\tilde{a}(x) \geq a_{\min} > 0, \quad \tilde{a}'(x) \geq 0, \quad (\tilde{a}/a)'(x) \geq 0, \quad \text{for all } x \in \text{int}(\tau_j),
\]

with the analogous result for \( \tilde{c} \).

**Proposition 5.6.** Under Assumption 5.2

\[
\text{Var}(\tilde{a}) \leq \text{Var}(a),
\]

with the analogous result for \( \tilde{c} \).

**Proof.** Let \( \tilde{\Omega} := \cup\{\tau_j : a'|_{\tau_j} > 0\} \). Then, by definition of \( \tilde{a} \),

\[
\int_{-L}^L |\partial_{pw}\tilde{a}| = \sum_{\tau_j \subset \tilde{\Omega}} \int_{\tau_j} |a'|.
\]

Next, we note that if \( a \) is increasing in \( \tau_j \), then \([\tilde{a}]_{z_j} = [a]_{z_j}\). On the other hand if \( a \) is non-increasing in \( \tau_j \), then

\[
[\tilde{a}]_{z_j} = a^+(z_{j-1}) - a^+(z_j) = (a^+(z_{j-1}) - a^-(z_j)) + (a^-(z_j) - a^+(z_j)) = \left( \int_{\tau_j} |a'| \right) + [a]_{z_j}.
\]

The result follows on combination of these relations. \( \square \)

**Notation 5.7.** Noting that, by Assumption 5.2 and definition (5.25), \( a, c, \tilde{a} \) and \( \tilde{c} \) are all \( C^1_{pw} \) functions with respect to the partition (5.16), we introduce, for \( j = 1, \ldots, N - 1 \), the quantities

\[
\alpha_j = \max \left\{ \frac{\tilde{a}^-(z_j)}{a^+(z_j)}, 1 \right\},
\]

\[
\sigma_j = \max \left\{ \frac{(\tilde{c}^2)^-(z_j)}{(\tilde{c}^2)^+(z_j)}, 1 \right\},
\]

and

\[
\gamma_j = \max \left\{ \frac{a^+(z_j)}{a^-(z_j)}, \frac{(c^2)^+(z_j)}{(c^2)^-(z_j)}, 1 \right\}.
\]

This leads us to the definition of the function \( q \) which we shall use in conjunction with Theorem 5.3.

**Definition 5.8** (the function \( q \)). We define the increasing sequence of positive numbers \( \{A_j : j = 1, \ldots, N\} \) inductively by

\[
A_1 = 0 \quad \text{and} \quad A_{j+1} = \alpha_j \sigma_j \gamma_j \left( \int_{\tau_j} \frac{1}{\tilde{a} c^2} + A_j \right), \quad j = 1, \ldots, N,
\]

Then we define the function \( q \in C^1_{pw} [-L, L] \) by

\[
q(x) = \tilde{a}(x)c^2(x) \left( \int_{z_{j-1}}^x \frac{1}{\tilde{a}(s)c^2(s)} ds + A_j \right), \quad x \in \tau_j, \quad 1 \leq j \leq N.
\]

**Lemma 5.9.** Under Assumption 5.2 the function \( q \) defined in (5.27) is increasing on \([-L, L]\) with \( q(-L) = 0 \) and satisfies the requirements of Theorem 5.3.
Proof. First note that for \( x \in \tau_j \), and using Proposition 5.7, we have
\[
\left( \frac{q}{a} \right)'(x) = \left( \frac{\tilde{a}}{a} \right)'(x) (\tilde{c}^2(x) + \frac{\tilde{a}}{a} (\tilde{c}^2(x)) \left( \int_{z_{j-1}}^{x} \frac{1}{a(s)c^2(s)} \, ds + A_j \right) + \frac{1}{a(x)} \geq \frac{1}{a(x)}.
\]
Similarly \( (q/c')'(x) \geq c^2 (x) \). Moreover,
\[
\frac{q}{a} \bigg|_{z_j} = \frac{\tilde{a}^-(z_j)}{a^-(z_j)} \left( \int_{\tau_j} \frac{1}{a(s)c^2(s)} \, ds + A_j \right) = \frac{\tilde{a}^+(z_j)}{a^+(z_j)} \left( \int_{\tau_j} \frac{1}{a(s)c^2(s)} \, ds + A_j \right)
\]
and the result follows on combining (5.31), (5.32) and (5.33).

We now have the main result of this section.

**Theorem 5.10.** Suppose Assumption 5.3 holds and let \( u \) solve (5.4). Then \( u \) satisfies the a priori bound (5.19), (5.20), with
\[
Q \leq 2L a_{\text{max}} c_{\text{max}}^2 \exp \left( \frac{2}{a_{\text{min}}} \text{Var}(a) + \frac{2}{c_{\text{min}}} \text{Var}(c^2) \right).
\]

In the pure impedance case (see (5.21)), the multiplicative factor \( 2L \) on the right-hand side can be replaced by \( L \).

**Proof.** We begin by considering the “Dirichlet-Impedance” case \( \Gamma_D = \{ -L \} \). With \( q \) as defined above (and since \( q(-L) = 0 \), Theorem 5.4.1 and Lemma 5.9 then imply that (5.19) and (5.20) hold, with \( Q = q(L) \). By induction on (5.20), and using the crucial fact that \( \alpha \ell \geq 1, \sigma \ell \geq 1, \gamma \ell \geq 1 \), we obtain
\[
A_{j+1} \leq \left( \prod_{\ell=1}^{j} \alpha_\ell \sigma_\ell \gamma_\ell \right) \left( \int_{-L}^{z_j} \frac{1}{a c^2} \right), \quad j = 1, \ldots N - 1,
\]
from which it follows that
\[
A_N \leq \left( \prod_{\ell=1}^{N-1} \alpha_\ell \right) \left( \prod_{\ell=1}^{N-1} \sigma_\ell \right) \left( \prod_{\ell=1}^{N-1} \gamma_\ell \right) \left( \int_{-L}^{z_{N-1}} \frac{1}{a c^2} \right).
\]
Thus, from (5.27),
\[
q(L) \leq a_{\text{max}} c_{\text{max}}^2 \left( \prod_{\ell=1}^{N-1} \alpha_\ell \right) \left( \prod_{\ell=1}^{N-1} \sigma_\ell \right) \left( \prod_{\ell=1}^{N-1} \gamma_\ell \right) \left( \int_{-L}^{z_{N-1}} \frac{1}{a c^2} \right)
\]
\[
\leq 2L a_{\text{max}} c_{\text{max}}^2 \exp \left( \frac{2}{a_{\text{min}}} \text{Var}(a) + \frac{2}{c_{\text{min}}} \text{Var}(c^2) \right).
\]
To bound the products in (5.31), we appeal to Lemma 5.1. This, combined with Proposition 5.6 gives immediately
\[
\prod_{\ell=1}^{N-1} \alpha_\ell \leq \exp \left( \frac{1}{a_{\text{min}}} \text{Var}(a) \right), \quad \text{and} \quad \prod_{\ell=1}^{N-1} \sigma_\ell \leq \exp \left( \frac{1}{c_{\text{min}}} \text{Var}(c^2) \right).
\]
Also
\[
\prod_{\ell=1}^{N-1} \gamma_\ell \leq \left( \prod_{\ell=1}^{N-1} \max \left( \frac{a^+(z_\ell)}{a^-(z_\ell)} \right) \right) \left( \prod_{\ell=1}^{N-1} \max \left( \frac{c^2(z_\ell)}{(c^2)^+(z_\ell)} \right) \right)
\]
\[
\leq \exp \left( \frac{1}{a_{\text{min}}} \text{Var}(a) \right) \exp \left( \frac{1}{c_{\text{min}}} \text{Var}(c^2) \right)
\]
and the result follows on combining (5.31), (5.32) and (5.33).

For the Impedance-Dirichlet case we replace \( q \) by \( q - q(L) \). This function has the same derivative as \( q \), vanishes at \( x = L \) and has maximum modulus \( q(L) \) occurring at \( x = -L \), so the proof is the same as
before. For the Impedance-Impedance case it is natural to replace $q$ by $q - q(L)/2$, which has maximum modulus $q(L)/2$ giving an extra factor of 1/2 in the estimate. □

**Discussion 5.11.** We finish this subsection with a short illustration of how Theorem 5.10 handles both oscillations and jumps in the coefficients $a$, $c$. These examples also show that the use of Lemma 7.1 to bound the right-hand side of (5.31) is sharp in terms of the order of its dependence on the variance when $a$ or $c$ is oscillatory, but can be pessimistic in terms of its dependence on $a_{\text{max}}/a_{\text{min}}$ and $c_{\text{max}}^2/c_{\text{min}}^2$, when $a$, $c$ are not oscillatory.

**Example 1.** Consider the case when $a = 1$ and $c$ is piecewise constant with respect to the partition (5.6). Suppose $N$ is odd and set

$$c(x) = c_{\text{max}}, \quad x \in [z_{j-1}, z_j), \quad j \text{ odd, } j < N$$

$$c(x) = c_{\text{max}}, \quad x \in [z_{N-1}, z_N], \quad j = N$$

$$c(x) = c_{\text{min}}, \quad x \in [z_{j-1}, z_j), \quad j \text{ even},$$

where $c_{\text{max}} > c_{\text{min}} > 0$. Then it is easy to see that (with the definitions as in Notation 5.7), for each $j = 1, \ldots, N - 1$, $\alpha_j = 1$ and

$$\sigma_j = \left( \frac{c_{\text{max}}}{c_{\text{min}}} \right)^2 \quad \text{and} \quad \gamma_j = 1, \quad \text{when } j \text{ is odd}$$

$$\sigma_j = 1 \quad \text{and} \quad \gamma_j = \left( \frac{c_{\text{max}}}{c_{\text{min}}} \right)^2, \quad \text{when } j \text{ is even} .$$

Hence the estimate (5.31) yields

$$Q \leq 2L \left( \frac{c_{\text{max}}}{c_{\text{min}}} \right)^{N} .$$

In this case $\text{Var}(c^2) = (N - 1)(c_{\text{max}}^2 - c_{\text{min}}^2)$. Thus, $\text{Var}(c^2)$ grows linearly in $N$ which implies that the bound on $Q$ grows exponentially in $\text{Var}(c^2)$. Similar results are implied by the estimates in [11] and [5].

**Example 2.** Consider the case when $c = 1$ and $a$ is oscillatory, given by

$$a(x) = 2 + \sin(m \pi x / L), \quad x \in [-L, L],$$

where $m$ is chosen to be an even positive integer. Then $a'$ changes sign at the points

$$z_j := (j - m - 1/2)L/m, \quad j = 1, \ldots, 2m .$$

These form the interior points of the partition (5.6), so that $N = 2m + 1$, and we set $z_0 := -L$, and $z_{2m+1} := L$. Recall the definition of $\alpha_j, \sigma_j$ and $\gamma_j$ from Notation 5.7. It is easily seen that $\sigma_j = \gamma_j = 1$, for $j = 1, \ldots, 2m$. Moreover $a(z_0) = a(z_{2m+1}) = 2$ and

$$a(z_j) = \begin{cases} 1, & j \text{ is even} \\ 3, & j \text{ is odd} \end{cases}$$

Since $a$ switches from decreasing to increasing at $z_j$ with $j$ even, we have

$$\alpha_j = \frac{a(z_{j+1})}{a(z_j)} = 3, \quad \text{when } j \text{ is even} \quad \text{and} \quad \alpha_j = 1 \quad \text{when } j \text{ is odd} .$$

Hence the estimate (5.31) yields

$$Q \leq 2L \frac{a_{\text{max}}}{a_{\text{min}}} \left( \prod_{j=1}^{2m} \alpha_j \right) = 6L 3^m .$$

noting that $\text{Var}(a)$ grows linearly with $m$, we see that again the above estimate grows exponentially with the order of the variance.

**Example 3.** Consider the non-oscillatory case when both $a$ and $c$ are monotonic (decreasing or increasing) functions on $[-L, L]$. Then there are no interior points in the partition (5.6), and $z_0 = -L$, $z_1 = L$.

Then using (5.31) to estimate $Q$ directly we would obtain

$$Q \leq 2L \frac{a_{\text{max}}}{a_{\text{min}}} \left( \frac{c_{\text{max}}}{c_{\text{min}}} \right)^2 ,$$

whereas the estimate (5.30) (which made use of Lemma 7.1) would be somewhat worse:

$$Q \leq 2L \frac{a_{\text{max}}}{a_{\text{min}}} \left( \frac{c_{\text{max}}}{c_{\text{min}}} \right)^2 \exp \left( 2 \frac{a_{\text{max}}}{a_{\text{min}}} + 2 \frac{c_{\text{max}}^2}{c_{\text{min}}^2} - 4 \right) .$$
5.2. On the sharpness of the estimates. In this section we will show by means of a family of examples that the bound in Theorem 5.11 is sharp in the sense that the exponential growth of the stability constant in $\text{Var}(c)$ can be realised in numerical simulations. The derivation of these “nearly unstable” examples can be found in [22] and further classes of examples which may also serve as benchmark problems are derived in [32]. Each member of our family has the general (strong) form:

\[
\begin{align*}
-u'' - \left(\frac{\pi}{\ell} \right)^2 u &= 0 \quad \text{in } \Omega = (-1,1), \\
\text{with } \left(-u' - \frac{i}{c}u\right)(-1) &= g_1 \\
\text{and } \left(u' - \frac{i}{c}u\right)(1) &= g_2.
\end{align*}
\]

(5.34)

The family will be specified by a countably infinite sequence of frequencies $\omega_m$, where $m$ ranges over the positive even integers, and a corresponding sequence of piecewise constant wave speeds $c_m$ with increasing numbers of jumps. For this problem it is possible to write the analytic solution in each subinterval on which $c_m$ is constant as a linear combination of left- and right-travelling waves, with coefficients determined by the boundary data $g_1$, $g_2$, the frequency $\omega_m$ and the wave-speed $c_m$. The analytic solution is then equivalent to solving a system of linear equations, the properties of which can be studied by symbolic manipulation. Using this approach, and looking for situations in which this system becomes ill-conditioned, the following unstable case has been derived.

Let $r \in (0,1)$ and let $m$, be an even positive integer. For each $m$ we specify the frequency

\[
\omega_m = \frac{\pi}{2}(1 - r + m), \quad m = 2, 4, 6, \ldots.
\]

(5.35)

To specify the wave speed $c_m$ we choose a partition of $[-1,1]$ with $2m + 1$ subintervals of the form:

\[
-1 = x_0^m < x_1^m < \ldots < x_{2m+1}^m = 1.
\]

(5.36)

For each $\ell = 1, \ldots, 2m + 1$ we define the wave speed on $\tau_{m,\ell}^m = (x_{\ell-1}^m, x_1^m)$ to be

\[
c_{m,\ell} = \begin{cases}
1 - r & \text{when } \ell \text{ is odd}, \\
1 + r & \text{when } \ell \text{ is even}.
\end{cases}
\]

(5.37)

The partition $(x_{\ell}^m)_{\ell=0}^{2m+1}$ is fixed by setting

\[
(x_{\ell}^m - x_{\ell-1}^m) = \begin{cases}
\frac{c_{m,\ell}}{1 - r + m} & \text{when } \ell \neq m + 1, \\
2\frac{c_{m,\ell}}{1 + r + m} & \text{when } \ell = m + 1.
\end{cases}
\]

Combining this prescription with $x_0 := -1$, it is easy to see that the resulting partition is of the form (5.36).

Since the right-hand side in the first equation in (5.34) vanishes, by writing down (5.4) with $v = u$ and taking the real part, we have

\[
\int_{\Omega} |u'|^2 = \int_{\Omega} \left(\frac{\omega}{c}\right)^2 |u|^2,
\]

which implies $||u'|| = ||(\omega/c)u||$ and $||u||_{H^2(a,c)} = \sqrt{2} ||u'||$. In the tables below we present computed values of $||u'||$ corresponding to varying choices of $r$ and $m$. The computations are done by applying the standard continuous linear finite element method to (5.4). For given $m$, we construct an initial piecewise uniform grid with 800 equal elements on each subinterval $\tau_{m,\ell}^m$, $\ell = 1, \ldots, 2m + 1$. On this grid we compute the finite element solution $u_h$ and then $||u_h'||$. All integrations in the implementation are done exactly. Then we repeat this calculation using a sequence of six additional uniform refinements, the finest one having $(800 \times 2^6) \times (2m + 1) = 51200 \times (2m + 1)$ elements in each $\tau_{m,\ell}^m$. Provided the computed value of $||u_h'||$ does not change (in its first four significant figures) in the final three of these seven successive refinements, then that value is recorded as the true value of $||u'||$ (to four figures).

Computations are done in MATLAB and the required linear systems are solved using the standard sparse backslash. Overall, the linear systems being solved are quite ill-conditioned and the computation of $u_h$ can become unstable on the finest meshes for the largest values of $m$, especially when $r$ is relatively close to 1. In some of our experiments we failed to achieve convergence to four significant figures. Such results are labelled with $^*$ in the tables.

Table I illustrates the properties of $||u'||$ as $m$ and $r$ vary. In the column labelled $\kappa$ we give the estimated condition number of the system matrix on the finest grid (computed using the MATLAB function condest).

From these computations we see clearly the blow up of the Helmholtz energy $2||u'||$ as $m$ increases, with the rate of blow-up increasing as $r$ increases. The results can be seen to reflect the theoretical worst case bound as follows.
Since \( f = 0 \) and \( \|g\|_N = 1 \), Theorem 5.10 gives the bound

\[
\|u'\| \leq \frac{1}{\sqrt{2}} C_{II}^{stab} \sqrt{Q_*}
\]

with

\[
C_{II}^{stab} = 2 \sqrt{\frac{3(1 + r)}{2(1 - r)}} + 1, \quad Q_* = 2 \left( \frac{1 + r}{1 - r} \right)^2 \exp \left( 4m \frac{(1 + r)^2}{(1 - r)^4} \right).
\]

Thus, with \( u^m \) denoting the solution of problem (5.34) for each \( m \), we have

\[
(5.38) \quad \log \|u'\| \leq 2m \left( \frac{(1 + r)^2}{1 - r} \right) + \log \left( C_{II}^{stab} \frac{1 + r}{1 - r} \right).
\]

Extrapolation on the data in Table 1 indicates that \( \log(\|u^m'\|) \) grows approximately linearly with \( m \). The gradient of the linear least squares fit to the data \( (m, \log(\|u^m'\|)) \) is indicated in the last line of Table 1 and is seen to grow weakly as \( r \) increases, numerically supporting the estimate (5.38).

The instability illustrated above is sensitive to the boundary data \( (g_1, g_2) \). In Table 2 we illustrate two different cases, one stable and one not. In Table 3 we study the sensitivity of the instability to small changes in the data. We consider the same data as for the problem in Table 1 except that we perturb the mesh point \( x_{k+1} \) by a small parameter \( \varepsilon \), i.e. the mesh is as in (5.36) except that

\[
(5.39) \quad x_{m+1} = x_{m+1} + \varepsilon.
\]

Computations for various \( m \) and \( \varepsilon \) are given in Table 3. Computations with \( \varepsilon \leq 10^{-7} \) and \( m \geq 14 \) are not sufficiently convergent and hence are not reported.

In order to illustrate the substantial effect a small perturbation can have on the solution near an instability we give in Figure 2 graphs of the solution to the problem studied in Table 3 for the cases \( \varepsilon = 0 \) and \( \varepsilon = 10^{-5} \). Note the substantial difference in the vertical scales in these two graphs, while the data is only different \( 10^{-5} \).

| \( m \) | \( g_1 = 1 = g_2 \) | \( g_1 = 2, \ g_2 = 0.5 \) |
|---|---|---|
| 2 | 4.677(-1) | 1.520 |
| 4 | 3.480(-1) | 1.918 |
| 6 | 2.887(-1) | 1.386(+1) |
| 8 | 2.520(-1) | 4.838(+1) |
| 10 | 2.26(-1)*+ | 1.70(+2)* |
| 12 | 2.1(-1)* | 6.30(+2)* |

Table 2. Values of \( \|u'\| \) for problem (5.34), with \( \omega^m \) given in (5.35) and \( c^m \) given in (5.37), with \( r = 0.6 \).
Table 3. Values of $\|u^\prime\|$ for problem (5.34), with $\omega^m$ given in (5.35) and $c^m$ given in (5.37) with $r = 0.5$. Mesh is as in (5.36) with perturbation (5.39).

| $m$, $\varepsilon$ | $10^{-7}$ | $10^{-6}$ | $10^{-5}$ | $10^{-7}$ | $10^{-6}$ | $10^{-5}$ | $10^{-4}$ | $10^{-5}$ |
|---------------------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|
| 6                   | 4.59      | 4.59      | 4.59      | 4.59      | 4.59      | 3.829     | 0.7256    |
| 8                   | 12.03     | 12.03     | 12.03     | 12.03     | 11.99     | 9.49      | 1.547     |
| 10                  | 32.47     | 32.47     | 32.47     | 32.47     | 32.47*    | 3.73*     | 0.7256    |
| 12                  | 89.35     | 89.28     | 88.9*     | 65*       | 9.57*     | 0.978*    | 0.1424    |
| 14                  |           |           |           |           | 2.57*     | 0.3030    | 0.1644    |
| 16                  |           |           |           |           | 0.72*     | 0.1644    | 0.1509    |
| 18                  |           |           |           |           | 0.244*    | 0.1437    | 0.1424    |
| 20                  |           |           |           |           | 0.1466    | 0.1354    | 0.1353*   |

Figure 2. Graphs of the real and imaginary parts of the solution $u$ to the problem computed in Table 3 with $m = 12$ and $\varepsilon = 0$ (left) and $\varepsilon = 10^{-5}$ (right).

6. Appendix

Lemma 6.1. Suppose $f \in C^1_{pc} [-L, L]$ with break points as in (5.3), so that, on each $\tau_j$, either $f'(x) \leq 0$ or $f'(x) > 0$. Suppose also $f(x) \geq f_{\min} > 0$ for all $x \in [-L, L]$. Then

$$
\left(\prod_{\ell=1}^{N-1} \max \left\{ f^+ (z_\ell), f^- (z_\ell) \right\} \right) \leq \exp \left( \frac{1}{f_{\min}} \text{Var}(f) \right).
$$

Proof. We restrict the proof to the case of $f^+$ in the numerator and $f^-$ in the denominator on the left-hand side of (6.1). The other case is analogous. Let the left hand side of this inequality be denoted $C$. Then

$$
\log(C) = \sum_{\ell=1}^{N-1} \max \left\{ f^+ (z_\ell), f^- (z_\ell) \right\}
= \sum_{\ell=1}^{N-1} \log \left( 1 + \frac{f^+ (z_\ell) - f^- (z_\ell)}{f^- (z_\ell)} \right)
\leq \frac{1}{f_{\min}} \sum_{\ell=1}^{N-1} ||f||_{z_\ell} \leq \frac{1}{f_{\min}} \text{Var}_{[z_0, L]}(f).
$$

Acknowledgement. We are grateful to the Hausdorff Research Institute for Mathematics in Bonn for Visiting Fellowships in their 2017 Trimester Programme on Multiscale Methods, during which part of this work was carried out. The first author also thanks the Institut für Mathematik at the University of Zürich for financial support. We would like to thank Euan Spence for very useful discussions.
[36] C. S. Morawetz and D. Ludwig. An inequality for the reduced wave operator and the justification of geometrical optics. *Comm. Pure Appl. Math.*, 21:187–203, 1968.

[37] M. Ohlberger and B. Verfürth. A new heterogeneous multiscale method for the Helmholtz equation with high contrast. *arXiv:1605.03400v2*, 2016. *Multiscale Model. Simul.*, to appear.

[38] B. Perthame and L. Vega. Morrey-Campanato estimates for Helmholtz equations. *J. Funct. Anal.*, 164:340–355, 1999.

[39] B. Perthame and L. Vega. Energy concentration and Sommerfeld condition for Helmholtz equation with variable index at infinity. *Geom. Funct. Anal.*, 17(5):1685–1707, 2008.

[40] F. Rellich. Darstellung der Eigenwerte von $\Delta u + \lambda u = 0$ durch ein Randintegral. *Math. Z.*, 46:635–636, 1940.

[41] S. Sauter. A Refined Finite Element Convergence Theory for Highly Indefinite Helmholtz Problems. *Computing*, 78(2):101–115, 2006.

[42] S. A. Sauter and C. Torres. Stability estimate for the Helmholtz equation with rapidly jumping coefficients. Technical Report *arXiv:1711.05439* [math.NA], arXiv, 2017.

[43] S. A. Sauter and C. Torres. Stability and instabilities for the heterogeneous Helmholtz equation in 1D. Technical Report; in preparation, Universität Zürich, 2018.

[44] A. Schatz. An observation concerning Ritz-Galerkin methods with indefinite bilinear forms. *Math. Comp.*, 28:959–962, 1974.

[45] E. A. Spence. Wavenumber-explicit bounds in time-harmonic acoustic scattering. *SIAM J. Math. Anal.*, 46(4):2987–3024, 2014.

[46] A. Tarantola. Inversion of seismic reflection data in the acoustic approximation. *Geophysics*, 49(8):1259–1266, 1984.

[47] T. Wolff. A property of measures in $\mathbb{R}^n$ and an application to unique continuation. *Geom. Funct. Anal.*, 2(2):225–284, 1992.

---

**Department of Mathematical Sciences, University of Bath, Bath BA2 7AY, United Kingdom**

E-mail address: i.g.graham@bath.ac.uk

**Institut für Mathematik, Universität Zürich, Winterthurerstrasse 190, CH-8057 Zürich, Switzerland**

E-mail address: stas@math.uzh.ch