ON THE GROWTH OF LOGARITHMIC DIFFERENCE OF MEROMORPHIC FUNCTIONS AND A WIMAN-VALIRON ESTIMATE

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Abstract. This paper gives a precise asymptotic relation between higher order logarithmic difference and logarithmic derivatives for meromorphic functions with order strictly less than one. This allows us to formulate a useful Wiman-Valiron type estimate for logarithmic difference of meromorphic functions of small order. We then apply this estimate to prove a classical analogue of Valiron about entire solutions to linear differential equations with polynomials coefficients for linear difference equations.

1. Introduction

It was shown by Ablowitz, Herbst and Halburd [2] and Halburd and Korhonen [12] (see also [13]) that the integrability of the discrete Painlevé equations in the complex plane can be characterised by the finite Nevanlinna order of growth of meromorphic solutions. A crucial role played in their Nevanlinna theory approach is some difference versions of logarithmic derivative estimate which we re established in [11] and independently by us in [6]. We show in a subsequent paper that given \( \varepsilon > 0 \), there exists a set \( E \subset (1, \infty) \) of finite logarithmic measure, so that

\[
\frac{f(z + \eta)}{f(z)} = e^{\mu(z) + O(r^{\varepsilon + \eta})},
\]

holds for \( r \notin E \cup [0, 1] \), where \( \beta = \max\{\sigma - 2, 2\lambda - 2\} \) if \( \lambda < 1 \) and \( \beta = \max\{\sigma - 2, \lambda - 1\} \) if \( \lambda \geq 1 \) and \( \lambda = \max\{\lambda', \lambda''\} \), from which we deduce when the order \( \sigma(f) < 1 \) and that for each \( \varepsilon > 0 \), there is a set \( E \subset (1, \infty) \) of finite logarithmic measure such that

\[
\frac{\Delta^n f(z)}{f(z)} = \eta^n \frac{f^{(n)}(z)}{f(z)} + O(r^{(n+1)(\sigma - 1) + \varepsilon})
\]

holds for \( |z| = r \notin E \), where \( \Delta f(z) = f(z + \eta) - f(z) \), \( \Delta^n f(z) = \Delta(\Delta^{n-1} f) \). The above results show the different behaviours for meromorphic functions of order less than and greater than unity. Bergweiler and Langley [4, Lemma 4.2] also obtained an asymptotic \( \Delta^n f(z) \sim f^{(n)}(z) \) as \( z \to \infty \) outside an \( \varepsilon \)-set, no precise error bounds where given.

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If one adopts the symbolic operator $D = \frac{d}{dz}$, then it was discussed in \[7, \S 8\] that we could write down the formal expression

$$\Delta^n f = (\eta^n D^n + \frac{n}{2!}\eta^{n+1}D^{n+1} + \cdots)f.$$  

This note has two purposes. We first establish vigorously that the above expression is indeed valid for meromorphic functions of order less than one. This allows us to obtain a more precise difference Wiman-Valiron estimate than established in \[7\]. We state our first main result:

Actually, we have the formal expansion

$$\Delta^n f(z) = n! \sum_{k=n}^{\infty} \frac{S_k(n)}{k!} f^{(k)}(z),$$

where the $S_k(n)$ are the Stirling numbers of the second kind \[1, \S 24.1\]. We recall that the Stirling number of the second kind $S_k(n)$ counts the number of different ways to partitioning a set of $n$ objects into $m$ non-empty subsets. In particular, it has the following generating function \[1, \S 21.1.4\]:

$$x^n = \sum_{m=0}^{n} \frac{S_m(n)}{m!} x(x-1)\cdots(x-m+1).$$

We now give a vigorous justification to the formal expansion (1.4).

**Theorem 1.1.** Let $f$ be a meromorphic function with order $\sigma = \sigma(f) < 1$. Then for any positive integers $n, N$ such that $N \geq n$, we have for each $\varepsilon > 0$, there is a set $E \subset [1, +\infty)$ of finite logarithmic measure so that

$$\Delta^n f(z) = n! \left( \sum_{k=n}^{N} \frac{S_k(n)}{k!} f^{(k)}(z) \right) + O(\eta^n f^{N+1}(\sigma-1)+\varepsilon)$$

for $|z| = r \notin E \cup [0, 1]$.

A classical result that entire solutions to linear differential equations with polynomial coefficients have completely regular growths of rational orders (see Valiron \[22\] pp. 106-108}). The second purpose of this paper is to prove analogues of this result that difference equations with polynomial coefficients of order strictly less than one must have completely regular growths of rational orders. Previously, Ishizaki and Yanagihara \[18\] established the same result for order strictly less than $1/2$ via a different method. The authors \[7\] Theorem 7.3 extended the theorem of Ishizaki and Yanagihara to include transcendental entire solutions of order strictly less than unity, but failed to prove the growth of these solutions to have completely regular growth.

This paper is organised as follows. Some preliminary results that are needed for the proof of Theorem 1.1 are given in \[4\] followed by the proof of Theorem 1.1 in \[4\]. We then formulate sharp difference Wiman-Valiron estimates in \[5\] and applying
them to prove a difference version a result as stated in Valiron [22] for difference equations in the next section [6] which is the second main purpose of this paper.

2. Lemmas

**Theorem 2.1.** Let $f$ be a meromorphic function with order $\sigma = \sigma(f) < 1$. Then for each positive integer $k$, we have for any $\varepsilon > 0$, there exists an exceptional set $E^{(n)}$ in $\mathbb{C}$ consisting of union of disks centred at the zeros and poles of $f(z)$ such that when the $z$ lies entirely outside of the $E^{(n)}$

$$\frac{\Delta f}{f} = \frac{f(z + \eta) - f(z)}{f(z)} = \eta f'(z) + \frac{\eta^2}{2!} f''(z) + \cdots + \frac{\eta^k}{k!} f^{(k)}(z) + O(\eta^{k+1} r^{(k+1)(\sigma-1)+\varepsilon}).$$

Moreover, the set $\pi E^{(n)} \cap [1, +\infty)$, where $\pi E^{(n)}$ is obtained from rotating the exceptional disks of $E^{(n)}$ so that their centres all lie on the positive real axis, has finite logarithmic measure.

We recall that a subset $E$ of $\mathbb{R}$ has finite logarithmic measure if

$$\text{Im}(E) = \int_{E \cap (1, \infty)} \frac{dr}{r}$$

is finite. The set $E$ is said to have an infinite logarithmic measure if $\text{Im}(E) = +\infty$.

We need a uniformity estimate for the logarithmic derivatives and logarithmic difference estimates that hold outside an exceptional set of $|z| = r$ of finite logarithmic measure simultaneously. So let us review Gundersen’s important logarithmic derivative estimate [9] Cor. 2 and our earlier difference analogue estimate [6] Thm. 8.2] first.

**Lemma 2.2.** Let $f$ be a transcendental meromorphic function, and let $k$ be a positive integer. Let $\{c_n\}$ be the sequence of zeros and poles of $f$, where $\{c_n\}$ is listed according to multiplicity and ordered by increasing modulus. Let $\gamma > 1$ be a given real constant. Then there exist constants $R_0 > 0$ and $C = C(\gamma, k) > 0$ such that for $|z| \geq R_0$ and $f(z) \neq 0$, $\infty$, we have

$$\left| \frac{f^{(k)}(z)}{f(z)} \right| \leq C \left( \frac{T(\gamma r, f)}{r} + \sum_{|c_n| < \gamma r} \frac{1}{|z - c_n|} \right)^k,$$

where $r = |z|$.

**Lemma 2.3.** Let $f$ be a transcendental meromorphic function, and let $\{c_n\}$ be the sequence of zeros and poles of $f$, where $\{c_n\}$ is listed according to multiplicity and ordered by increasing modulus. Let $\gamma > 1$ and a complex $\eta$ be given, then there are constants $R_1 > \frac{2}{\gamma} |\eta|$ and $\beta = \beta(\gamma)$ such that for $|z| \geq R_1$, we have

$$\left| \log \left| \frac{f(z + \eta)}{f(z)} \right| \right| \leq |\eta| \beta \frac{T(\gamma r, f)}{r} + |\eta| \cdot \sum_{|d_k| < \gamma r} \frac{1}{|z - d_k|},$$

holds, where $|z| = r$ and $(d_k)_{k \in N} = (c_k)_{k \in N} \cup (c_k - \eta)_{k \in N}$. 

Lemma 2.4. Let \( f(z) \) be a meromorphic function of finite order \( \sigma \), \( \eta \) a non-zero complex number, and \( \varepsilon > 0 \) be given real constant. Then there exists an exceptional set \( E^{(3n)} \) in \( \mathbb{C} \) consisting of union of disks centred at the zeros and poles of \( f(z) \), as described in the Theorem 2.1, such that when the \( a \) \( z \) lies entirely outside the \( E^{(3n)} \), then we have

\[
|f^{(k)}(z)| \leq |z|^{k(\sigma-1)+\varepsilon},
\]

and

\[
\exp \left( -|z|^\sigma - \varepsilon \right) \leq \left| \frac{f(z + \eta)}{f(z)} \right| \leq \exp \left( |z|^{\sigma-1} + \varepsilon \right)
\]

hold simultaneously and uniformly in \( t \in [0, 1] \). Moreover, the set \( \pi E^{(3n)} \cap (0, +\infty) \), where \( \pi E^{(3n)} \) is obtained from rotating the exceptional disks of \( E^{(3n)} \) so that their centres all lie on the positive real axis, has finite logarithmic measure.

The above inequality (2.4) can be derived from (2.2) in [9, Lemma 8], while the inequality (2.5) from (2.3) in [6, Eqn. (8.5)]. Both inequalities hold generally under the finite order assumption on \( f \), and both proofs require the Poisson-Jensen formula and the classical Cartan lemma, except that we need to take into the account of the uniformity assumption on \( t \). However, in order to fulfill our later application to establishing the Theorem 2.1 one needs more detailed information concerning the exceptional set that arises from removing the zeros and poles from applying the Cartan lemma (below) than quoting Gundersen [9, §7] directly. So we judge it is appropriate to offer a full proof of the Lemma 2.4 based on that in [9] but tailored to our need here. In fact, our construction of exceptional disks which have larger radii than those constructed in [9] by \( 3\eta \). This is to guarantee the fact that we need below, namely that whenever \( z \) lies outside \( E^{(3n)} \), then the line segment \([z, z + \eta]\) lies outside \( E^{(\eta)} \).

Let us first recall Cartan’s theorem which we adopt from [9].

Lemma 2.5 ([5]). Let \( a_1, \ldots, a_m \) be any finite collection of complex numbers, and let \( d > 0 \) be any given positive number. Then there exists a finite collection of closed disks \( D(a_k, r_k) \) (1 \( \leq k \leq m \)) with corresponding radii that satisfy

\[
\sum_{k=1}^{m} r_k = 2d,
\]

such that for \( z \notin \bigcup_{k=1}^{m} D(a_k, r_k) \), then there is a permutation of the points \( a_1, \ldots, a_m \), say \( b_1, \ldots, b_m \), that satisfies

\[
|z - b_k| > \frac{k}{m} d, \quad 1 \leq k \leq m,
\]

where the permutation may depend on \( z \).

Proof. Let us choose first choose \( \gamma > 1 \) and let us define the annulus

\[
\Gamma_{\nu} := \{ z : \gamma^{\nu} \leq |z| \leq \gamma^{\nu+1} \}, \quad \nu \in \mathbb{N}.
\]
We let \( R = \gamma^{\nu+2} \), \( d_k(t) = (c_k) \cup (c_k-t\eta) \), \((0 \leq t \leq 1)\) so that \((d_k(0)) = (c_k)\) and \((d_k(1)) = (d_k)\). Let \( m = n(R) = n(\gamma^{\nu+2})\) so that \(|d_m(1)| \leq R\) and \(|d_m+1(1)| > R\).

Let us now choose integer \( \nu_0 \) such that
\[
\gamma^{\nu_0+2} \geq |d_1(1)|
\]
and
\[
1 \leq (\gamma - 1) \log n(\gamma^{\nu+2}).
\]

We choose
\[
d_{3\eta} := \frac{\gamma^\nu}{\log(\gamma^\nu)} + 3|\eta|.
\]

The Lemma 2.5 asserts that there is a finite collection of closed disks \( D(a_k, r_k) (1 \leq k \leq m) \) whose radii have a sum of \( 2d_{3\eta} \) and that when \( z \not\in \bigcup_{k=1}^m D(a_k, r_k) \), then there is a permutation of the points \( a_1, \ldots, a_m \), say \( b_1, \ldots, b_m \), that satisfies
\[
|z - b_k| > \frac{k}{m} \left( \frac{\gamma^\nu}{\log(\gamma^\nu)} + 3|\eta| \right), \quad 1 \leq k \leq m.
\]

Let us confine our \( z \) within the annulus \((2.7)\). Because of the choice of \( \nu_0 \) as defined in \((2.9)\) we deduce
\[
\sum_{|d_k| < \gamma r} \frac{1}{|z - d_k(1)|} \leq \sum_{k=1}^m \frac{1}{|z - b_k|} \leq \frac{m}{\gamma^\nu + 3|\eta|} \sum_{k=1}^m \frac{1}{k} \leq \frac{n(R)}{\gamma^\nu + 3|\eta|} (1 + \log m)
\]
\[
\leq \frac{n(2\gamma^2 r) \log^2 r}{\gamma^\nu + 3|\eta|} \left( 1 + \log n(\gamma^2 r) \right)
\]
\[
< \frac{n(2\gamma^2 r) \log^2 r}{\gamma^\nu} \left( 1 + \log n(\gamma^2 r) \right)
\]
\[
\leq \gamma \frac{n(2\gamma^2 r)}{r} \log^2 r \left( 1 + \log n(\gamma^2 r) \right)
\]
\[
\leq \gamma \frac{n(2\gamma^2 r)}{r} \log^2 r \log n(\gamma^2 r)
\]

So given an \( \varepsilon > 0 \), we deduce from the inequalities \((2.2)\) and \((2.3)\) and \((2.12)\) that there is an exceptional set \( E^{(3\eta)} = \bigcup_{\nu=1}^\infty E^{(3\eta)}_{\nu} \) where each \( E^{(3\eta)}_{\nu} \) consists of the union of closed disks \( D(a_k, r_k) (1 \leq k \leq m) \) whose centres lie in \( \Gamma_{\nu} \) defined above such that for all \( z \not\in E^{(3\eta)}_{\nu} \), the \((2.4)\) and \((2.5)\) hold simultaneously and uniformly for all \( t \in [0, 1] \). So we can choose an \( R \) so large such that, when \(|z| > R\) and \( z \not\in E^{(3\eta)} \), the inequalities \((2.4), (2.6)\) and \((2.12)\) hold simultaneously.

It remains to compute the size of the exceptional sets. According to the Lemma 2.6 that the sum of the diameters of the disks \( \bigcup_{k=1}^m D(a_k, r_k) \) is
We argue in the spirit of [9] that we revolve each of the disks $D(a_k, r_k)$ about the origin to form annuli centered about the origin. We then consider the logarithmic measure of the union of the intersection of these annuli with the positive real axis in $\Gamma_\nu$ which we denote by $\pi E^{(3\eta)}$. Here we need to clarify an exceptional situation, namely when one of those disks already contains the origin, then we simply count the line segment $[0, |a_k|]$ part of the exceptional set in the computation of the logarithmic measure. That is, we have

$$E_\nu := [\gamma^\nu, \gamma^{\nu+1}] \cap \pi E^{(3\eta)}$$

and

$$E := \bigcup_{\nu=\nu_0}^\infty E_\nu.$$

It follows from (2.13) that the linear measure of $E_\nu$ does not exceed $4 d_{3\eta}$. So

$$\int_E \frac{dx}{x} = \sum_{\nu=\nu_0}^\infty \int_{E_\nu} \frac{dx}{x} \leq \sum_{\nu=\nu_0}^\infty \left\{ \log\left(\gamma^\nu + \frac{\gamma^\nu}{\log(\gamma^\nu)} + 3|\eta|\right) - \log(\gamma^\nu) \right\}$$

$$= \sum_{\nu=\nu_0}^\infty \log\left(1 + \frac{1}{\log(\gamma^\nu)\gamma} + \frac{1}{3|\eta|\gamma^\nu}\right) < \infty$$

thus proving that the exceptional set (2.15) has finite logarithmic measure. \[ \square \]

**Remark 2.6.** We would like to emphasis that we shall take the exceptional set to be $E^{(\eta)}$ in our application below. Since the $E^{(\eta)}$ is a subset of $E^{(3\eta)}$. So the set $E^{(\eta)}$ also has finite logarithmic measure.

We will also require a complex form of Lagrange’s version of Taylor’s theorem. Although we cannot find an exact reference for the result, one can easily modify the argument in [9 p. 242]:

**Lemma 2.7.** Let $f$ be an analytic function in a domain $D$. Let $c \in D$, then

$$f(z) = f(c) + f'(c)(z - c) + \frac{f''(c)}{2!}(z - c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(z - c)^n + R_n(z)$$

where

$$R_n(z) = \frac{1}{n!} \int_c^z (z - t)^n f^{(n+1)}(t) dt$$

for each $z \in D$. 

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3. Proof of Theorem 2.1

We apply the Lemma 2.4 with when $z \not\in E^{(3\eta)}$. Then the inequalities (2.4) and (2.5) both hold. In fact, it is not difficult to see that the whole line segment $[z, z + \eta]$ lies outside $E^{(\eta)}$ which also has finite logarithmic measure. So let us choose a path in the complex plane that connects $z$ and $z + \eta$ that does not intersect with the $z$ in the $C$ that forms the exceptional set $E^{(\eta)}$. We replace $z$ by $z + \eta$ and $c$ by $z$ in (2.17), and divide through both sides of the Taylor expansion (2.7) by $f(z)$ to yields

$$\frac{f(z + \eta) - f(z)}{f(z)} = \frac{\eta f'(z)}{f(z)} + \frac{\eta^2 f''(z)}{2! f(z)} + \cdots + \frac{\eta^n f^{(n)}(z)}{n! f(z)} + \frac{R_n(z + \eta)}{f(z)}$$

where

$$R_n(z + \eta) = \frac{1}{n!} \int_z^{z+\eta} (z + \eta - t)^n \frac{f^{(n+1)}(t)}{f(z)} \, dt$$

Lemma 2.4 asserts that for each $\varepsilon > 0$, one can have an exceptional set $\pi E^{(\eta)}$ of real numbers such that (2.4) and (2.5) hold simultaneous and uniformly for $t \in [0, 1]$ for $|z|$ outside of $\pi E^{(\eta)}$. Thus (3.2) becomes

$$|\frac{R_n(z + \eta)}{f(z)}| = \left| \frac{\eta^{n+1}}{n!} \int_0^1 (1 - T)^n \frac{f^{(n+1)}(z + T\eta)}{f(z)} \, dT \right|$$

\begin{align*}
\leq & \frac{\eta^{n+1}}{n!} 2^n \cdot O(e^{\sigma-1+\varepsilon}) \cdot O(|z + T\eta|^{(n+1)(\sigma-1)+\varepsilon}) \\
\leq & O(|z|^{n+1}|z|^{(n+1)(\sigma-1)+\varepsilon})
\end{align*}

as $|z| \to \infty$ and outside of $\pi E^{(\eta)}$ which is a subset of $E$ from (2.15). This proves (2.1).

4. Proof of Theorem 1.1

Proof. We shall also make use of the recurrence formula [1, p. 825]

$$\left( \begin{array}{c} m \\ r \end{array} \right) \mathcal{S}^{(m)}_n = \sum_{k=m-r}^{n-r} \left( \begin{array}{c} n \\ k \end{array} \right) \mathcal{S}^{(r)}_{n-k} \mathcal{S}^{(m-r)}_k, \quad n \geq m \geq r.$$
\[
\frac{\Delta^{n+1} f(z)}{f(z)} = \frac{\Delta F(z)}{F(z)} \frac{F(z)}{f(z)} = \frac{F(z)}{f(z)} \left( n! \left( \sum_{k=n}^{\infty} \frac{\eta^k F^{(k)}(z)}{k! F(z)} \right) + O(\eta^{n+N+1} \epsilon) \right)
\]
\[
= n! \sum_{k=1}^{\infty} F^{(k)}(z) \frac{\eta^k \Delta^n (f(z)) [\Delta^n (f(z))]^{(k)}}{\Delta^n (f(z))} f(z) + O(\eta^{n+N+1} \epsilon)
\]
\[
= n! \sum_{k=1}^{\infty} \frac{\eta^k \Delta^n (f^{(k)}(z))}{k!} f(z) + O(\eta^{n+N+1} \epsilon)
\]
\[
= n! \sum_{k=1}^{\infty} \frac{\eta^k f^{(k)}(z) \Delta^n (f^{(k)}(z))}{k!} f(z) + O(\eta^{n+N+1} \epsilon)
\]
\[
= n! \sum_{k=1}^{\infty} \frac{\eta^k f^{(k)}(z) \Delta^n (f^{(k)}(z))}{k!} f(z) + O(\eta^{n+N+1} \epsilon)
\]
\[
= n! \sum_{k=1}^{\infty} \frac{\eta^k f^{(k)}(z)}{k!} \left( \sum_{s=n}^{\infty} \eta^s \frac{\Theta_s^{(n)}(z)}{s!} \right) f(z) + O(\eta^{n+N+1} \epsilon)
\]
\[
= n! \sum_{k=1}^{\infty} \frac{\eta^k f^{(k)}(z)}{k!} \left( \sum_{s=n}^{\infty} \eta^s \frac{\Theta_s^{(n)}(z)}{s!} \right) f(z) + O(\eta^{n+N+1} \epsilon)
\]
\[
= n! \sum_{k=1}^{\infty} \frac{\eta^k f^{(k)}(z)}{k!} \left( \sum_{s=n}^{\infty} \eta^s \frac{\Theta_s^{(n)}(z)}{s!} \right) f(z) + O(\eta^{n+N+1} \epsilon)
\]
\[
(4.2)
\]

where the (4.2) follows from applying \( r = 1 \) from the recurrence formula (4.1). This completes the proof. \( \square \)
5. Sharp Difference Wiman-Valiron Estimates

We recall that if \( f(z) = \sum_{n=0}^{\infty} \) is an entire function in the complex plane. Let \( M(r, f) = \max_{|z|=r} |f(z)| \) denote the maximum modulus of \( f \) on \( r > 0 \), and the maximal term \( \mu(r, f) \) of \( f \) by \( \mu(r, f) = \max_{n \geq 0} |a_n| r^n \). The central index \( \nu(r, f) \) is the greatest exponent \( m \) such that
\[
|a_m| r^m = \mu(r, f).
\]
We note that \( \nu(r, f) \) is a real, non-decreasing function of \( r \).

It is well-known that for finite order \( \sigma \) function \( f \) that its central index satisfies
\[
\limsup_{r \to \infty} \frac{\log \nu(r, f)}{\log r} = \sigma \quad \text{(see \cite{wiman}, and \textit{also} [15] for the latest development).}
\]

We next quote the classical result of Wiman-Valiron (see also [15] in the form

**Lemma 5.1.** \cite{WV} pp. 28–30 Let \( f \) be a transcendental entire function. Let \( 0 < \varepsilon < \frac{1}{2} \) and \( z \) be such that \( |z| = r \) and that
\[
|f(z)| > M(r, f)(\nu(r, f))^{-\frac{1}{2} + \varepsilon}
\]
holds. Then there exists a set \( E \subset (1, \infty) \) of finite logarithmic measure, such that
\[
\frac{f^{(k)}(z)}{f(z)} = \left( \frac{\nu(r, f)}{z} \right)^k (1 + R_k(z)),
\]
\[
R_k(z) = O((\nu(r, f))^{-\frac{1}{2} + \varepsilon})
\]
holds for all \( k \in \mathbb{N} \) and all \( r \notin E \cup [0, 1] \).

We note that the error term \( R_k \) in (5.3) is independent of \( k \). In [7], We deduce from (1.2), (5.2) and (5.3) in Lemma 5.1, if \( \sigma(f) = \sigma < 1 \), and \( 0 < \varepsilon < \frac{1}{2} \), and \( |z| = r \) satisfies (5.1). Then for each positive integer \( k \), there exists a set \( E \subset (1, \infty) \) that has finite logarithmic measure, such that for all \( r \notin E \cup [0, 1] \),
\[
\frac{\Delta^k f(z)}{f(z)} = \left( \frac{\nu(r, f)}{z} \right)^k (1 + O((\nu(r, f))^{-\frac{1}{2} + \varepsilon})), \quad \text{if} \quad \sigma = 0,
\]
\[
\frac{\Delta^k f(z)}{f(z)} = \left( \frac{\nu(r, f)}{z} \right)^k + O(r^{k\sigma-k-\gamma+\varepsilon}), \quad \text{if} \quad 0 < \sigma < 1,
\]
where \( \gamma = \min\{\frac{1}{2} \sigma, 1 - \sigma\} \).

The reminder in (5.5) is not sharp and in particular it depends on the integer \( k \). The Theorem \cite{11} allows us to remove this restriction and establish a sharp error bound on (5.5).

**Theorem 5.2.** Let \( f \) be a transcendental entire function of order \( \sigma(f) = \sigma < 1 \), let \( 0 < \varepsilon < \frac{1}{2} \) and \( z \) satisfies (5.1). Then for each positive integer \( k \), there exists a set \( E \subset (1, \infty) \) that has finite logarithmic measure, such that for all \( r \notin E \cup [0, 1] \),
\[
\frac{\Delta^k f(z)}{f(z)} = \left( \frac{\nu(r, f)}{z} \right)^k (1 + R_k(z)),
\]
where the \( R_k(z) \) is given by (5.3).
Proof. Let \( \varepsilon > 0 \) be given. We choose \( N \) so large such that

\[
\left( \frac{1}{8} - \varepsilon \right) (\sigma + \varepsilon) < (n + N + 1)(1 - \varepsilon) + \varepsilon.
\]

Substituting (6.2) into (1.3) yields

\[
\frac{\Delta^n f}{f} = n! \left( \sum_{k=n}^{N} \eta^{k} \Theta_k^{(n)} f^{(k)}(z) \right) + O\left( \eta^{n+N+1} r^{(n+N+1)(\sigma-1)+\varepsilon} \right)
\]

\[
= n! \sum_{k=n}^{N} \eta^{k} \Theta_k^{(n)} \left( \frac{\nu(r, f)}{z} \right)^k (1 + R_k(z)) + O\left( \eta^{n+N+1} r^{(n+N+1)(\sigma-1)+\varepsilon} \right)
\]

\[
= \left( \frac{\nu(r, f)}{z} \right)^n \left[ 1 + \sum_{k=n+1}^{N} n! \eta^{k} \Theta_k^{(n)} \left( \frac{\nu(r, f)}{z} \right)^k (1 + R_k(z)) \right] + O\left( r^{(n+N+1)(\sigma-1)+\varepsilon} \right)
\]

\[
= \left( \frac{\nu(r, f)}{z} \right)^n \left( 1 + O((\nu(r, f))^{-\frac{1}{2}+\varepsilon}) \right) + O\left( r^{(n+N+1)(\sigma-1)+\varepsilon} \right)
\]

\[
= \left( \frac{\nu(r, f)}{z} \right)^n \left( 1 + O((\nu(r, f))^{-\frac{1}{2}+\varepsilon}) \right) \cdot (1 + o((\nu(r, f))^{-\frac{1}{2}+\varepsilon}))
\]

\[
= \left( \frac{\nu(r, f)}{z} \right)^n \left( 1 + O((\nu(r, f))^{-\frac{1}{2}+\varepsilon}) \right)
\]

as required. \( \square \)

6. Applications to Difference Equations

We considered linear difference equations

\[
a_n(z) \Delta^n f(z) + \cdots + a_1(z) \Delta f(z) + a_0(z) f(z) = 0,
\]

in [7] where \( a_0(z), \cdots, a_n(z) \) are polynomials. We have shown that any entire solution \( f \) to (6.1) with order of growth less than one has a positive rational order of growth and the rational order \( \chi \) can be calculated explicitly from the gradients of the corresponding Newton-Puiseux diagram of the equation. This falls short of showing that the growth of these entire solutions have a completely regular growth as is well-known for entire solutions to linear differential equations (i.e., replacing the \( \Delta^k f \) in (6.1) by \( f^{(k)} \) for \( k = 1, \cdots, n \)). We shall strengthen our earlier result that the growth order \( \chi \) of entire solutions to (6.1) with \( \chi < 1 \) is indeed completely regular. This also improves an earlier result of Ishizaki and Yanagihara [15] where they have proved the following result for solutions of order < 1/2 by developing a Wiman-Valiron theory based on binomial basis. Ramis considered the corresponding problems for \( q \)-difference equations [21].

**Theorem 6.1.** Let \( a_0(z), \cdots, a_n(z) \) be polynomial coefficients of the difference equation (6.1), and let \( f \) be an entire solution with order \( \sigma(f) = \chi < 1 \). Then \( \chi \) is a rational number which can be determined from a gradient of the corresponding Newton-Puiseux diagram equation (6.7). In particular,

\[
\log M(r, f) = L r^\chi (1 + o(1))
\]

where \( L > 0, \chi > 0 \) and \( M(r, f) = \max_{|z|=r} |f(z)| \). That is, the solution has completely regular growth.
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The original proof of the corresponding result for linear differential equations is based on application of (5.2). Thus our estimate (5.6) implies that we may simply replace the differential operators by difference operators and the proof then follows exactly the same pattern. So we omit the proof. Wittich [23, pp. 65–68] and [10] discussed if the possible rational orders for the entire solutions of linear differential equations with polynomial coefficients obtained by the Newton-Puiseux diagram ([13, §22]) method are realised. It would be interesting to know if this also holds for the difference equation (6.1).

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