Aggregating regular norms

Anatoli Juditsky ∗ Arkadi Nemirovski†

February 13, 2024

Abstract

The subject of this paper is regularity-preserving aggregation of regular norms on finite-dimensional linear spaces. Regular norms were introduced in [5] and are closely related to “type 2” spaces [9, Chapter 9] playing important role in 1) high-dimensional convex geometry and probability in Banach spaces, see [15, 12, 13, 9, 8], and in 2) design of proximal first order algorithms for large-scale convex optimization with dimension-independent, or nearly so, complexity. Regularity, with moderate parameters, of a norm makes applicable, in a dimension-independent fashion, various geometric, probabilistic, and optimization-related results, thus motivating the subject of this paper—aggregations of regular norms resulting in controlled (and moderate) inflation of regularity parameters.

1 Introduction

This paper focuses on regular norms on (finite-dimensional) linear spaces—the notion introduced in [5] in connection with developing dimension-independent bounds on large deviations of sums of i.i.d. random vectors or vector-valued martingales in normed spaces. Postponing definition of regularity till Section 2, right now it suffices to say that this property is quantified by a pair of reals ($\kappa \geq 1, \varsigma \geq 1$), the smaller the better, and is one of numerous existing “measures of deviation” of the norm in question from the standard Euclidean norm (for the latter, $\kappa = \varsigma = 1$).

There are two broad application areas where regularity of norms is of significant interest. The first is high-dimensional convex geometry and probability in Banach spaces, see [15, 12, 13, 9, 8] and references therein. Numerous results in these applications assume that the normed space in question is of “type $p$” (see [9, Chapter 9]), the case of $p = 2$ being, in a sense, the best. The qualitative property of a normed space $X$ to be of type 2 (which is trivially true for finite-dimensional spaces we are working with) has numerical characteristics $T(X)$ which enters the quantitative components of the associated geometric and probabilistic results, the smaller is this constant, the better. It is immediate to see that when the norm on $X$ is ($\kappa, \varsigma$)-regular, one has $T(X) \leq \varsigma^2 \sqrt{\kappa}$. That is, regularity of the norm in question with “moderate” constant makes applicable, in a dimension-independent fashion, numerous powerful geometric and probabilistic results. Another application domain is that of first order proximal type algorithms for large-scale deterministic and stochastic convex optimization (see, e.g., [1, 2, 7, 11]). It turns out that regularity of a norm allows to equip the ball of the conjugate norm with a “good proximal setup” resulting in dimension-independent iteration complexity of associated proximal algorithms for minimizing convex functions over these balls.

Basic examples of regular norms ([5]) are:

1. $\ell^p$ norms, $1 \leq p < \infty$,
2. $\ell^\infty$ norm,
3. $L^p$ norms, $1 \leq p < \infty$,
4. $L^\infty$ norm,
5. Wassezzov-Steinhaus norms.

This work was supported by MIAI @ Grenoble Alpes (ANR-19-P3IA-0003).
1. $\| \cdot \|_2$ on $\mathbb{R}^n$ is $(1,1)$ regular. The norm $\| \cdot \|_p$ on $\mathbb{R}^n$, $2 \leq p \leq \infty$, is $(\kappa, \zeta)$-regular with 
\[ \kappa = O(1)\min[p, \ln(n + 1)] \] and \[ \zeta = O(1) \] (here and in what follows $O(1)$ stands for an absolute constant).

2. When $2 \leq p \leq \infty$, the Schatten norm $\|\sigma(x)\|_p$ on the space $\mathbb{R}^{m \times n}$ of $m \times n$ matrices, $\sigma(x)$ being the singular spectrum of matrix $x$, is $(\kappa, \zeta)$-regular with \[ \kappa = O(1)\min[p, \ln(m, n)] + 1 \] and \[ \zeta = O(1). \]

It should be stressed that parameters of regularity in these examples are either dimension-independent, or deteriorate only logarithmically as the dimension grows.

The potential applications of regular norms outlined above motivate the goal of this paper—investigating regularity-preserving “aggregation” of regular norms.

The main body of the paper is organized as follows: We recall the notion of regular norm in Section 2.1 and then present our principal contributions—Theorems 2.1 and 2.2—in Section 2.2; preliminary versions of these results can be found in our preprint [5]. Then in Section 3 we apply these results to prove the regularity of elliptic and spectratopic norms, as defined in [6, 4]. All proofs are postponed to the appendix.

# 2 Problem statement and main results

## 2.1 Regular norms

Let $\kappa \geq 1$ and $\zeta \geq 1$ be two reals. We refer to the norm $n(\cdot)$ on $\mathbb{R}^n$ as $\kappa$-smooth if the function $\Phi(x) = n^2(x)$ is continuously differentiable and satisfies the relation

\[ \forall x, h : \Phi(x + h) \leq \Phi(x) + [\Phi'(x)]^T h + \kappa \Phi(h), \]

or, which is the same (see Lemma A.4), the gradient of $\Phi$ is Lipschitz continuous with Lipschitz constant $2\kappa$:

\[ n_*(\nabla \Phi(x) - \nabla \Phi(y)) \leq 2\kappa n(x - y) \]

where $n_*(\cdot)$ is the norm conjugate to $n(\cdot)$.

Following [5], we say that a norm $\| \cdot \|$ on $\mathbb{R}^n$ is $(\kappa, \zeta)$-regular if there exists a $\kappa$-smooth norm $n(\cdot)$ on $\mathbb{R}^n$ such that $\| \cdot \|$ is within factor $\zeta$ from $n(\cdot)$:

\[ \zeta^{-1} n(\cdot) \leq \| \cdot \| \leq \zeta n(\cdot). \]

Clearly, if $\| \cdot \|$ is $\kappa$-smooth (or $(\kappa, \zeta)$-regular) and $x \mapsto Ax$ is a linear embedding of $\mathbb{R}^m$ into $\mathbb{R}^n$ (so that $\text{Ker } A = \{0\}$), then the norm $\|y\|_A = \|Ay\|$ on $\mathbb{R}^m$ is $\kappa$-smooth (resp., $(\kappa, \zeta)$-regular). Besides this, if $\| \cdot \|$ is within factor $\alpha \geq 1$ from $(\kappa, \zeta)$-regular norm $\| \cdot \|$, that is,

\[ \alpha^{-1} \| \cdot \| \leq \| \cdot \| \leq \alpha \| \cdot \|, \]

then $\| \cdot \|$ is $(\kappa, \alpha \zeta)$-regular.

The examples of regular norms presented in the introduction stem straightforwardly from the following facts ([5]; in proximal minimization context, these facts are reproduced in [10, 3]) which are important in what follows and in their own right: when $2 \leq p < \infty$,

- The norm $\| \cdot \|_p$ on $\mathbb{R}^n$ is $(p - 1)$-smooth [5, Example 3.2, Section 4.1.1].
- The Schatten norm $\|\sigma(X)\|_p$ on the space $\mathbb{R}^{m \times n}$ of $m \times n$ matrices ($\sigma(x) \in \mathbb{R}^{\min(m,n)}$ stands for the singular spectrum of $X$) is $\max[2, p - 1]$-smooth [5, Example 3.3, Section 4.1.1].
### 2.2 Main results

Our main results on aggregation of regular norms are as follows:

**Theorem 2.1** Let \( \theta(\cdot) : \mathbb{R}^K \to \mathbb{R} \) be a convex continuous homogeneous, of degree 1, function which is monotone on \( \mathbb{R}^K_+ \) and positive outside of the origin. Let \( \| \cdot \|_i \) be \((\varkappa, \varsigma)\)-regular norms on \( \mathbb{R}^n_i \), \( 1 \leq i \leq K \). Then the aggregated norm

\[
\| [x_1; \ldots; x_K] \| = \theta^{1/2}(\| x_1 \|_1^2, \ldots, \| x_K \|_K^2)
\]

(clearly, this indeed is a norm on \( \mathbb{R}^{n_1+\ldots+n_K} \)) is \((c_1 \ln(K+1) + \varkappa, c_2 \varsigma)\)-regular, for properly selected absolute constants \(c_1, c_2\).

Our next result is the refinement of Theorem 2.1 in the situation where the “aggregating function” \( \theta \) is a regular absolute norm:

**Theorem 2.2** Let \( \theta \) be an absolute norm on \( \mathbb{R}^K \) which is \((\varkappa', \varsigma')\)-regular, and let \( \| \cdot \|_i \) be \((\varkappa, \varsigma)\)-regular norms on \( \mathbb{R}^n_i \), \( i \leq K \). Then the norm

\[
\| [x_1; \ldots; x_K] \| = \theta^{1/2}(\| x_1 \|_1^2, \ldots, \| x_K \|_K^2)
\]

is \((\varkappa, \varsigma)\)-regular on \( \mathbb{R}^{n_1+\ldots+n_K} \), with \( \varkappa = 2\varkappa' + \varkappa \) and \( \varsigma = \varsigma' \sqrt{\varsigma} \).

Finally, we have

**Proposition 2.1** Let \( \| \cdot \| \) be a \((\varkappa, \varsigma)\)-regular norm on \( \mathbb{R}^n \) and \( x \mapsto Px : \mathbb{R}^n \to \mathbb{R}^p \) be an onto mapping. Then the factor-norm

\[
\| u' \| = \min_x \{ \| x \| : Px = u \}
\]

induced on \( \mathbb{R}^p \) by \( \| \cdot \|, P \), is \((\varkappa, \varsigma)\)-regular.

**Remark:** Theorems 2.1, 2.2 and Proposition 2.1 are formulated as existence results stating that under such and such assumptions such and such norm \( \| \cdot \| \) is regular with such and such parameters \( \varkappa, \varsigma \), so that there exists a \( \varkappa \)-smooth norm \( \| \cdot \|' \) which is within factor \( \varsigma \) from \( \| \cdot \| \). However, in some applications, e.g., in proximal-type minimization, “simple existence” of \( \| \cdot \|' \) is not enough—we should know what this norm is. In this respect, it should be stressed that, as is seen from the proofs, the above statements are “constructive.” For example, in the context of Theorem 2.1, given \( \varkappa \)-smooth norms which are within the factor \( \varsigma \) from the norms \( \| \cdot \|_i \), we know how to convert these norms and \( \theta(\cdot) \) into a \( (c_1 \ln(K+1) + \varkappa)\)-smooth norm which is within factor \( c_2\varsigma \) of the aggregated norm (1), and similarly for Theorem 2.2 and Proposition 2.1.

### 3 Illustration: regularity of ellitopic and spectratopic norms

#### 3.1 Ellitopes and spectratopes

**Ellitopes.** A basic ellitope [6, Section 4.2.1] in \( \mathbb{R}^n \) is a set \( \mathcal{X} \) given as

\[
\mathcal{X} = \{ x \in \mathbb{R}^n : \exists t \in \mathcal{T} : x^T T_i x \leq t_i, 1 \leq i \leq K \},
\]

where \( T_i \succeq 0, \sum_i T_i \succ 0 \), and \( \mathcal{T} \) is a convex compact set in \( \mathbb{R}^K_+ \) which contains a positive vector and is monotone: whenever \( 0 \leq t' \leq t \in \mathcal{T} \), we have \( t' \in \mathcal{T} \). Here \( A \succeq B \Leftrightarrow B \preceq A \) (\( A \succ B \Leftrightarrow B \prec A \)) mean
that $A, B$ are symmetric matrices of the same size such that $B - A$ is positive semidefinite (resp., positive definite).

Ellitope $\mathcal{X} \subset \mathbb{R}^p$ is a linear image of basic ellitope:

$$\mathcal{X} = P \mathcal{X} \text{ with } \mathcal{X} \text{ given by (2)},$$

where $P \in \mathbb{R}^{p \times n}$.

The simplest examples of basic ellitopes are

- Bounded intersections $\mathcal{X} = \{x : x^T T_k x \leq 1, k \leq K\}$, $T_k \succeq 0$, of centered at the origin ellipsoids/elliptic cylinders.

- $\| \cdot \|_p$-balls, $p \geq 2$: $\{x \in \mathbb{R}^n : \|x\|_p \leq 1\} = \{x \in \mathbb{R}^n : \exists t \in \mathcal{T} : x_i^2 \leq t_i, i \leq n\}, \mathcal{T} = \{t \in \mathbb{R}^p_+ : \|t\|_{p/2} \leq 1\}$.

Spectratopes. A basic spectratope [6, Section 4.3.1] in $\mathbb{R}^n$ is a set $\mathcal{X}$ given as

$$\mathcal{X} = \{x \in \mathbb{R}^n : \exists t \in \mathcal{T} : S^i_2[x] \preceq t I_{d_i}, 1 \leq i \leq K\},$$

where $S_i[x] = \sum_{j=1}^n x_j S^{ij}_i, S^{ij}_i \in S^{d_i}$, are linear mappings taking values in the spaces $S^{d_i}$ of $d_i \times d_i$ symmetric matrices and such that $S_i[x] = 0$ for all $i \leq K$ if and only if $x = 0$, and $\mathcal{T} \subset \mathbb{R}^p_+$ is of the same type as in the definition of a basic ellitope.

Spectratope $\mathcal{X} \subset \mathbb{R}^p$ is a linear image of basic spectratope: $\mathcal{X} = P \mathcal{X}$ with $\mathcal{X}$ given by (4),

where $P \in \mathbb{R}^{p \times n}$.

The simplest examples of basic spectratopes are

- The unit ball of spectral norm $\| \cdot \|_{2,2}$ in the space $S^m$ of symmetric $m \times m$ matrices:

$$\{x \in S^m : \|x\|_{2,2} \leq 1\} = \{x \in S^m : \exists t \in [0,1] : S^2[x] \preceq t I_m\}, S[x] \equiv x.$$

- The unit ball of the spectral norm $\| \cdot \|_{2,2}$ in the space $\mathbb{R}^{m \times n}$ of $m \times n$ matrices:

$$\{x \in \mathbb{R}^{m \times n} : \|x\|_{2,2} \leq 1\} = \{x \in \mathbb{R}^{m \times n} : \exists t \in [0,1] : S^2[x] \preceq t I_{m+n}\}, S[x] = \begin{bmatrix} \frac{x}{x^T x} \end{bmatrix}^T.$$

Calculus of ellitopes/spectratopes. Families of ellitopes and of spectratopes are rather rich: as shown in [6, Section 4.6], each family is closed w.r.t. taking finite intersections, direct products, arithmetic sums, images under linear mappings and inverse images under linear embeddings. Furthermore, ellitopic description (2), (3) and spectratopic description (4), (5) of the results of these operations are readily given by ellitopic and, respectively, spectratopic descriptions of the operands. It should also be mentioned that every ellitope is a spectratope as well.
**Ellitopic and spectratopic norms.** The sets $\mathcal{T} \subset \mathbb{R}^K_+$ participating in the definition of ellitopes and spectratopes are exactly sets of the form

$$\mathcal{T} = \{ t \in \mathbb{R}^K_+ : \theta(t) \leq 1 \} \quad (6)$$

stemming from functions $\theta(\cdot) : \mathbb{R}^K_+ \to \mathbb{R}$ which are

1. convex, continuous, positive outside of the origin, and positively homogeneous, of homogeneity degree $1$,
2. monotone on $\mathbb{R}^K_+$: $0 \leq t \leq t' \Rightarrow \theta(t) \leq \theta(t')$.

Next, basic ellitopes and spectratopes in $\mathbb{R}^n$ are convex compact sets which are symmetric w.r.t. the origin and have the origin in their interior. Consequently, such a set $\mathcal{X}$ is the unit ball of a norm $\| \cdot \|_\mathcal{X}$ on $\mathbb{R}^N$; this norm is

$$\| x \|_\mathcal{X} = \begin{cases} \theta^{1/2} \left( ||T_1^{1/2} x||_2^2 : \ldots : ||T_K^{1/2} x||_2^2 \right), & \text{when } \mathcal{X} \text{ is the ellitope (2)} \\ \theta^{1/2} \left( ||S_1 [x]||_2^2 : \ldots : ||S_K [x]||_2^2 \right), & \text{when } \mathcal{X} \text{ is the spectratope (4)} \end{cases} \quad (7a)$$

where $\theta(\cdot)$ is given by (6).

Finally, if $\overline{\mathcal{X}}$ is an ellitope (a spectratope) given by (3) (resp., by (5)) and the corresponding mapping $x \mapsto Px$ is an onto one, $\overline{\mathcal{X}}$ also is the unit ball of certain norm $\| \cdot \|_{\overline{\mathcal{X}}}$ which is nothing but the factor-norm induced by $\mathcal{X}$ and $P$:

$$\| u \|_{\overline{\mathcal{X}}} = \min \{ \| x \|_{\mathcal{X}} : Px = u \}. \quad (8)$$

**The result** of this section is as follows:

**Theorem 3.1** There exist absolute constants $c_1$, $c_2$ such that

(i) Whenever $\overline{\mathcal{X}}$ is given by (2), (3), and (6) with $\text{Rank}(P) = p$, the ellitopic norm $\| \cdot \|_{\overline{\mathcal{X}}}$ given by (7a) and (8) is $(c_1 \ln(K + 1), c_2)$-regular.

(ii) Whenever $\overline{\mathcal{X}}$ is given by (4), (5), and (6) with $\text{Rank}(P) = p$, the spectratopic norm $\| \cdot \|_{\overline{\mathcal{X}}}$ given by (7b) and (8) is $(c_1 \ln(K + 1) + \max_{i \leq K} \ln(d_i)), c_2)$-regular.

**A Appendix**

**A.1 Proof of Theorem 2.1**

**A.1.1 Preliminary step**

Let us show first that in order to prove Theorem 2.1 “as is,” it suffices to prove it in the case where function $\theta$, in addition to the properties postulated in theorem’s premise (which are convexity and continuity on $\mathbb{R}^K_+$, homogeneity of degree 1, monotonicity, and positivity outside of the origin), possesses two additional properties, namely,

(a) is continuously differentiable outside of the origin, and

(b) satisfies the relation

$$t > 0 \Rightarrow 1/K \leq [\nabla \theta(t)]_i \leq (1 + 1/K), \quad 1 \leq i \leq K. \quad (9)$$

Indeed,
1°. Let \( n(t) = \theta(\text{abs}[t]) \), where \( \text{abs}[t_1; ...; t_K] = [t_1; ...; t_K] \), so that \( n(t) \) is an absolute norm on \( \mathbb{R}^K \). Denoting by \( n_\varepsilon(\cdot) \) the Fenchel conjugate of \( n(\cdot) \), the Fenchel conjugate \( \theta_\varepsilon \) of the function \( \frac{1}{2} n_\varepsilon^2(\cdot) + \frac{t}{2} \| \|_2^2 \) is a function of the form \( \frac{1}{2} \theta_\varepsilon^2(\text{abs}[\cdot]) \) where \( \theta_\varepsilon : \mathbb{R}_+^K \to \mathbb{R} \) is convex homogeneous of degree 1 monotone and positive outside of the origin with Lipschitz continuous gradient; along with \( \theta_\varepsilon \), \( \theta_\varepsilon \) is continuously differentiable outside of the origin. Taking \( \varepsilon > 0 \) small enough, we can make \( \theta_\varepsilon(\cdot) \) to be within the factor, say, 1.1, from \( \theta(\cdot) \). It clearly suffices to prove Theorem 2.1 with \( \theta_\varepsilon \) in the role of \( \theta \), so that from now on let us assume that \( \theta \), in addition to what is stated in the premise of the theorem, possesses properties (a) and (b), which we assume from now on.

2°. Now let \( s_i = \max\{t_i : t \geq 0, \theta(t) \leq 1\} \). By diagonal scaling of \( \mathbb{R}^K \)—passing from \( \theta(t) \) to \( \theta(\mathcal{D}t) \) with positive definite diagonal \( \mathcal{D} \)—we can assume that \( s_i = 1 \), and such scaling does not affect the statement we want to prove. Thus, in addition to what has been already assumed, from now on we assume that \( s_i := \max\{t_i : \theta(t) \leq 1\} = 1 \) for all \( i \). Monotonicity of \( \theta(\cdot) \) implies that the basis orths \( e_i \) belong to the set \( \Theta = \{t \geq 0 : \theta(t) \leq 1\} \), whence \( \theta(t) \leq \sum_i t_i, t \geq 0 \). Besides this, under the assumption that \( s_i = 1, i \leq K \), \( \Theta \) is contained in the box \( \{t : 0 \leq t_i \leq 1, i \leq K\} \), and thus is contained in the set \( \{t \geq 0 : \sum_i t_i \leq K\} \), implying that \( \theta(t) \geq K^{-1} \sum_i t_i \), \( t \geq 0 \). Now let \( \overline{\theta}(t) = \theta(t) + K^{-1} \sum_i t_i \); by the above, \( \theta(\cdot) \leq \overline{\theta}(\cdot) \leq 2\theta(\cdot) \). At the same time, for \( t \geq 0, t \neq 0 \), one has \( |\nabla \overline{\theta}(t)|_i \geq K^{-1} \) for all \( i \). Aside of this, we have \( \overline{\theta}(t) \leq (1 + 1/K) \sum_i t_i \) for all \( t \geq 0 \), implying that for all \( s \geq 0 \)

\[
(1 + 1/K)\sum_j t_j + s \geq \overline{\theta}(t + se_i) \geq \overline{\theta}(t) + s|\nabla \overline{\theta}(t)|_i,
\]

We conclude that \( |\nabla \overline{\theta}(t)|_i \leq 1 + 1/K \) for all \( i \) and all nonzero \( t \geq 0 \). The bottom line is that \( \overline{\theta} \) is shares all properties mentioned in the beginning of Section A.1.1, while \( \theta(\cdot) \) is within factor 2 from \( \overline{\theta}(\cdot) \). As a result, the norms \( \theta(||x||_1^2, ..., ||x_K||_K^2)^{1/2} \) and \( \overline{\theta}(||x||_1^2, ..., ||x_K||_K^2)^{1/2} \) are within factor \( \sqrt{2} \) from each other. We conclude that in order to prove Theorem 2.1 it suffices to prove it in the case when \( \theta \), in addition to what is stated in the premise of the theorem, possesses properties (a) and (b), which we assume from now on.

### A.1.2 Proving Theorem 2.1 in the case of (a), (b)

When proving the theorem, we lose nothing when assuming that \( \varsigma = 1 \). Indeed, replacing \( \| \cdot \|_i \) with norms which are within some factor of \( \| \cdot \|_i \), the aggregation of the new norms is within the same factor of the aggregation of \( \| \cdot \|_i \). Thus, from now on we assume that the squares

\[
\omega_i(x_i) = ||x_i||_i^2
\]

of the norms \( \| \cdot \|_i \) are continuously differentiable and satisfy the relation

\[
[\omega_i(x) + [\omega_i'(x_i)]^T h_i] \leq \omega_i(x_i + h_i) \leq \omega_i(x) + [\nabla \omega_i(x_i)]^T h_i + \varsigma \omega_i(h_i) \forall x_i, h_i \in \mathbb{R}^{n_i}, 1 \leq i \leq K.
\]

Note that \( ||\nabla \omega(x)||_{i,*} \leq 2 \sqrt{\omega_i(x)} \) where \( || \cdot ||_{i,*} \) is the norm conjugate to \( \cdot \|_i \), thus

\[
||\nabla \omega_i(x_i)]^T h_i || \leq 2 \sqrt{\omega_i(x_i)} \omega_i(h_i).
\] (10)
Let $p$ be a positive integer, and let

$$\mathcal{T} = \{ t \geq 0 : \theta(t) = 1 \},$$

$$f(x,t) = \sum_{i=1}^{K} \omega_i^{p+1} t_i^p : \mathbb{R}^{n_1+\ldots+n_K} \times \mathbb{R}_+^K \to \mathbb{R} \cup \{ +\infty \}$$

$$F(x,t) = f_{\mathcal{T}}(x,t)$$

$$\phi(x) = \min_{t \in \mathcal{T}} f(x,t)$$

$$\Phi(x) = \min_{t \in \mathcal{T}} F(x,t) = \phi_{\mathcal{T}}(x)$$

(here, by convention, $a/0$ is $0$ when $a = 0$ and is $+\infty$ when $a > 0$, making $f(x,t)$ lower semicontinuous on $\mathbb{R}^{n_1+\ldots+n_K} \times \mathbb{R}_+^K$).

$0\text{°}$. Let $q = p/(p+1)$. The function $s^2/\tau^q$ is convex in $s, \tau$ on $\{[s;\tau] : \tau \geq 0 \}$, and $F(x,t) = \|[\omega_1(x_1)/t_1^q; \ldots; \omega_K(x_K)/t_K^q]\|_{p+1}$, that is, $F(x,t)$ is convex in $[x; t] \in \mathbb{R}^{n_1+\ldots+n_K} \times \mathbb{R}_+^K$. Consequently, $\phi$ and $\Phi$ are real-valued convex functions on $\mathbb{R}^{n_1+\ldots+n_K}$. Besides this, the convex and positive outside of the origin function $\Phi$ is homogeneous of degree 2 and as such is the square of a norm. Our goal in what follows is to demonstrate that with properly selected $p$ this norm can be taken as regular approximation of $\|[x_1; \ldots; x_K]\|$ announced in Theorem 2.1.

$1\text{°}$. For evident reasons, minimizers of $f(x,t)$ over $t \in \mathcal{T}$ do exist for every $x$. We claim that given $x \neq 0$, the minimizer $t(x)$ of $f(x,t)$ over $t \in \mathcal{T}$ (or, which clearly is the same, over $t \in \mathcal{T}$) is unique and is continuous in $x \neq 0$. Indeed, assuming w.l.o.g. that the nonzero blocks in $x$ are $x_1, \ldots, x_m$, if $t$ and $t'$ are two distinct minimizers of $f(x,t)$ over $t \in \mathcal{T}$, then $t = t' = 0$ when $i > m$ and the initial blocks $\mathcal{T} = [t_1; \ldots; t_m]$, $\mathcal{T}' = [t_1'; \ldots; t_m']$ of $t$ and $t'$ are two distinct minimizers of the function $\sum_{i=1}^{m} \omega_i^{p+1}(x_i)/t_i^p$ over $\tau$ running through the set $\{ \tau : \tau \in \mathbb{R}^m : [\tau_1; \ldots; \tau_m; 0; \ldots; 0] \in \mathcal{T} \}$, which is impossible, since on the latter set the function in question is strongly convex. To prove that $t(x)$ is continuous on the set $x \neq 0$, let $\bar{x} \neq 0$ and let $x^s \to \bar{x}$ as $s \to \infty$. We should lead to contradiction the assumption that the vectors $t^s = t(x^s)$ do not converge to $t(\bar{x})$. Assuming that the latter is the case and taking into account that $\mathcal{T}$ is compact, we can assume without loss of generality that $t^s$ converge to some $t \neq t(\bar{x})$, $t \in \mathcal{T}$, as $s \to \infty$. Since $f(x,t)$ is lower semicontinuous on $\mathbb{R}^{n_1+\ldots+n_K} \times \mathbb{R}_+^K$ and $\phi(x)$ is convex real valued and thus continuous on $\mathbb{R}^{n_1+\ldots+n_K}$, we have $\phi(x^s) = f(x^s, t^s) \to \phi(\bar{x})$ as $s \to \infty$ and therefore $f(\bar{x}, t) \leq \lim_{s \to \infty} f(x^s, t^s) = \phi(\bar{x})$, implying $f(\bar{x}, t) \leq \phi(\bar{x}) = \min_{\tau \in \mathcal{T}} f(\bar{x}, \tau)$. Thus, $t$ and $t(\bar{x})$ are two distinct minimizers of $f(\bar{x}, \tau)$ over $\tau \in \mathcal{T}$, which, as we already know, is impossible when $\bar{x} \neq 0$.

$2\text{°}$. Observe that when $x \neq 0$, $t = t(x)$ if and only if there exists $\lambda(x) \geq 0$ such that $t$ is the unique solution to the system of equations

$$t \geq 0, \quad \theta(t) = 1,$$

$$x_i = 0 \Rightarrow t_i = 0,$$

$$x_i \neq 0 \Rightarrow p \omega_i^{p+1}(x_i)/t_i^{p+1} = \lambda(x) \frac{[\nabla \theta(t)]}{\theta_i(x)} \quad \text{(11)}$$

in variable $t$. Consequently, when $x \neq 0$, we have $\lambda(x) > 0$. Setting $I(x) = \{ i : x_i \neq 0 \}$, for $i \in I(x)$ we have $\theta_i(x) > 0$ and $t_i(x) = [\lambda(x) \theta_i(x)/p]^{-1/p} \omega_i(x_i)$, whence

$$1 = \theta(t(x)) = \sum_i t_i(x) [\nabla \theta(t(x))]_i = [\lambda(x)/p]^{-1/p} \sum_i \omega_i(x_i) \theta_i^{p+1}(x).$$

7
Thus, when \( x \neq 0 \), we have

\[
\lambda(x) = p \left[ \sum_i \omega_i(x_i) \theta_i^{\frac{x}{\lambda}}(x) \right]^{p+1} > 0
\]

\[
t_i(x) = \begin{cases} 
\frac{\omega_i(x_i) \theta_i^{\frac{x}{\lambda}}(x)}{\sum_j \omega_j(x_j) \theta_j^{\frac{x}{\lambda}}(x)}, & i \in I(x) = \{ i : x_i \neq 0 \} \\
0, & i \notin I(x)
\end{cases}
\]

\[
\phi(x) = \left[ \sum_i \omega_i(x_i) \theta_i^{\frac{x}{\lambda}}(x) \right]^{p+1}
\]

(12a)

(12b)

3°. Let \( x \neq 0 \). For \([x; t] \in \text{Dom } f\) let

\[
[f'_x(x, t)]_i = \begin{cases} 
0, & x_i = 0, \\
(p+1) \omega_i(x_i) \nabla \omega_i(x_i)/t_i^p(x), & x_i \neq 0,
\end{cases}
\]

\[
[f'_t(x, t)]_i = \begin{cases} 
-p \omega_i^{p+1}(x_i)/t_i^{p+1}(x), & x_i \neq 0, \\
0, & x_i = 0,
\end{cases}
\]

\[
f'(x, t) = [f'_x(x, t); f'_t(x, t)],
\]

so that \( f'(x, t) \in \partial f(x, t) \) for all \([x; t] \in \text{Dom } f\); let also \( \phi'(0) = 0 \). We claim that \( \phi'(x) = f'_x(x, t(x)) \) is continuous vector field which is the subgradient (and due to continuity – gradient) vector field of \( \phi(x) \) on \( \mathbf{R}^n \). Indeed, by (12b) for \( x \neq 0 \) we have

\[
[f'_x(x, t)]_i = \begin{cases} 
(p+1) \omega_i^{\frac{x}{\lambda}}(x) \left[ \sum_j \omega_j(x_j) \theta_j^{\frac{x}{\lambda}}(x) \right]^{p} \nabla \omega_i(x_i), & x_i \neq 0, \\
0, & x_i = 0
\end{cases}
\]

and \( \theta_i(x) = [\theta'_i(t(x))] \) is continuous in \( x \) along with \( t(x) \); continuity of \( \phi'(x) \) is evident. To prove that \( \phi(x) \) is the gradient field of \( \phi(x) \), due to continuity of the field it suffices to verify that \( \phi'(x) \in \partial \phi(x) \) whenever \( x \) has no zero blocks. For such an \( x \), setting \( t = t(x) \), we obtain for \( t' \in \overline{T} \)

\[
f(x', t') + \lambda(x)(\theta(t') - 1) \geq \frac{[f'_x(x, t)]^T(x' - x)}{[\phi'(x)]^T(x' - x)} + \frac{[f'_t(x, t)]^T(t' - t) + \lambda(x)[\theta'(t)]^T(t' - t) + f(x, t) + \lambda(x)(\theta(t) - 1)}{= 0 \text{ by (11)}}
\]

whence

\[
f(x', t') \geq \phi(x) + [\phi'(x)]^T(x' - x)
\]

and

\[
\phi(x') = \min_{t' \in T} f(x', t') \geq \phi(x) + [\phi'(x)]^T(x' - x),
\]

as claimed.

4°. We need the following

Lemma A.1 Let \( G \) be an open convex domain in \( \mathbf{R}^n \), \( g(\cdot) : G \rightarrow \mathbf{R} \) be a continuously differentiable convex function, \( \| \cdot \| \) be a norm on \( \mathbf{R}^n \) and \( \gamma \) be a positive real. Assume that for every \( x \in G \) there
exists \( r = r_x > 0 \) such that the ball \( V_x = \{ u : \|u - x\| < r \} \) is contained in \( G \) and for every \( u, v \in V_x \) one has
\[
g(v) \leq g(u) + [g'(u)]^T(v-u) + \frac{\gamma}{2}\|u-v\|^2.
\]
Then \( g'(u) \) is Lipschitz continuous with constant \( \gamma \) on \( G \) w.r.t. \( \|\cdot\| \):
\[
\forall y, z \in G : \|g'(y) - g'(z)\|_* \leq \gamma\|y - z\|, \tag{14}
\]
where \( \|\cdot\|_* \) is the norm conjugate to \( \|\cdot\| \).

**Proof.** Let us fix \( x \in G \). Applying Lemma A.4 to \( V = V_x \) and \( f \)—restriction of \( g \) on \( V \), we conclude that (14) holds true when \( y, z \in V_x \); since \( x \in G \) is arbitrary, (14) holds true for all \( y, z \in G \). \( \square \)

Lemma A.1 admits the following immediate

**Corollary A.1** Let \( g(x) \) be a continuously differentiable convex function on \( \mathbb{R}^n \), and let \( \mathbb{R}_0^n \) be the set of all vectors from \( \mathbb{R}^n \) with all entries different from 0. Let \( \|\cdot\| \) be a norm on \( \mathbb{R}^n \) and \( \gamma > 0 \) be a real. Assume that for every \( x \in \mathbb{R}_0^n \) there exists \( r_x > 0 \) such that for the centered at \( x \) \( \|\cdot\| \)-ball \( V_x \) of radius \( r_x \), \( V_x \subset \mathbb{R}_0^n \); it holds
\[
\forall u, v \in V_x : g(v) \leq g(u) + [g'(u)]^T(v-u) + \frac{\gamma}{2}\|u-v\|^2.
\]
Then
\[
\forall x, y \in \mathbb{R}^n : \|g'(x) - g'(y)\|_* \leq \gamma\|x - y\|. \tag{15}
\]
Indeed, since \( g \) is continuously differentiable and \( \mathbb{R}_0^n \) is dense in \( \mathbb{R}^n \), it suffices to verify (15) for \( x, y \in \mathbb{R}_0^n \). In this case, setting \( x_t = x + t(y-x), 0 \leq t \leq 1 \), we can find points \( t_1 < t_2 < \ldots < t_k \) in (0, 1) such that setting \( t_0 = 0, t_{k+1} = 1 \), the intervals \( \Delta_s = \{ x_t : t_s < t < t_{s+1} \} \), \( s = 0, 1, \ldots, k \), belong to \( \mathbb{R}_0^n \), that is, \( \text{sign}([x_t]_i) = \epsilon_i(s) \in \{-1, 1\} \), \( t_s < t < t_{s+1} \). Then \( G_s = \{ x \in \mathbb{R}^n : \text{sign}(x_i) = \epsilon_i(s), i \leq n \} \) is an open convex domain containing \( \Delta_s \). By Lemma A.1, when \( t_s < t' < t'' < t_{s+1} \) we have
\[
\|g'(x_{t''}) - g'(x_{t'})\|_* \leq \gamma\|x_{t''} - x_{t'}\| = \gamma[t'' - t']\|y - x\|.
\]
Passing to limit as \( t' \to t_s + 0 \) and \( t'' \to t_{s+1} - 0 \) and taking into account that \( g \) is \( C^1 \) on \( \mathbb{R}^n \), we get
\[
\|g'(x_{t_s}) - g'(x_{t_{s+1}})\|_* \leq \gamma\|x_{t_{s+1}} - x_{t_s}\|y - x\|,
\]
whence \( \|g'(x_{t_{s+1}}) - g'(x_{t_s})\|_* \leq \gamma\|y - x\| \). \( \square \)

5. Let now \( x \in \mathbb{R}^{n+\ldots+nK} \) be a vector with all entries different from 0 and \( h \in \mathbb{R}^{n+\ldots+nK} \) be a vector with
\[
\omega_i^{1/2}(h_i) < \omega_i^{1/2}(x_i)/[(2p + 2)^2(1 + \kappa)], \quad i \leq K.
\]
One has
\[
\omega_i(x_i) + [\nabla \omega_i(x_i)]^T h_i \leq \omega_i(x_i + h_i) \leq \omega_i(x_i) + [\nabla \omega_i(x_i)]^T h_i + \kappa \omega_i(h_i).
\]
Thus, when setting
\[
\Delta_i = [\omega_i(x_i + h_i) - \omega_i(x_i)]/\omega_i(x_i)
\]
we have
\[
\Delta_i = [\nabla \omega_i(x_i)]^T h_i/\omega_i(x_i) + \delta_i, \quad |\delta_i| \leq \kappa \omega_i(h_i)/\omega_i(x_i),
\]
and taking into account (10) and (16),
\[
|\Delta_i| \leq 2\sqrt{\omega_i(h_i)/\omega_i(x_i)} + \kappa \omega_i(h_i)/\omega_i(x_i) \leq 3\sqrt{\omega_i(h_i)/\omega_i(x_i)} \leq 3/(2p + 2)^2.
\]
We have
\[
\phi(x) + [\phi'(x)]^T h \leq \phi(x + h) \leq \sum_i \left[ \sum_i \omega_i^{p+1}(x_i + h_i) \right] f_i^p(x) = \sum_i \omega_i^{p+1}(x_i)(1 + \Delta_i)^{p+1}
\]
\[
\leq \sum_i \omega_i^{p+1}(x_i) \left[ 1 + (p+1)\Delta_i \right] + \frac{p(p+1)}{2} \left[ \frac{\Delta_i}{\sqrt{\Delta_i}} \right] t_i^{-p}(x)
\]
\[
\leq \sum_i \omega_i^{p+1}(x_i) \left[ 1 + (p+1)\Delta_i \right] \frac{p(p+1)}{2} \exp \{ (p-1)|\Delta_i| \Delta_i^2 \} t_i^{-p}(x)
\]
\[
= \phi(x) + [\phi'(x)]^T h + (p+1)(5p + \infty) \sum_i \omega_i(h_i)\omega_i^p(x_i) t_i^{-p}(x)
\]
\[
= \phi(x) + [\phi'(x)]^T h + (p+1)(5p + \infty) \left[ \sum_i \omega_i(h_i)\theta_i^{p+1}(x) \right] \left[ \sum_i \omega_i(x_i)\theta_i^{p+1}(x) \right]^p
\]
where the concluding equality is due to (12a).

Note that for
\[
\psi(s) = \begin{cases} 
\frac{s^{p+1}}{p+1} & s \geq 0, \\
-\infty & s < 0,
\end{cases}
\]
\[\bar{s} > 0 \text{ and } \delta \in \mathbb{R} \text{ one has}
\]
\[
\psi(\bar{s} + \delta) \leq \psi(\bar{s}) + \frac{1}{p+1} \bar{s}^{p+1} \delta = \psi(\bar{s}) + \frac{1}{p+1} \frac{\delta}{\psi^p(\bar{s})}.
\]
Hence,
\[
\Phi(x) + [\Phi'(x)]^T h \leq \Phi(x + h) = \psi(\phi(x + h)) \leq \psi(\phi(x) + \delta) \leq \psi(\phi(x)) + \frac{1}{p+1} \delta / \psi^p(\phi(x))
\]
\[
= \Phi(x) + \frac{1}{p+1} [\phi'(x)]^T h / \psi^p(\phi(x))
\]
\[
+ (5p + \infty) \left[ \sum_i \omega_i(h_i)\theta_i^{p+1}(x) \right] \left[ \sum_i \omega_i(x_i)\theta_i^{p+1}(x) \right]^p \psi^{-p}(\phi(x))
\]
\[
= \Phi(x) + [\Phi'(x)]^T h + (5p + \infty) \left[ \sum_i \omega_i(h_i)\theta_i^{p+1}(x) \right] \left[ \sum_i \omega_i(x_i)\theta_i^{p+1}(x) \right]^p \psi^{-p}(\phi(x))
\]
\[
= \Phi(x) + [\Phi'(x)]^T h + (5p + \infty) \left[ \sum_i \omega_i(h_i)\theta_i^{p+1}(x) \right], \quad (17)
\]
due to \(\psi^p(\phi(x)) = \rho_{p+1}(x) = \left[ \sum_i \omega_i(x_i)\theta_i^{p+1}(x) \right]^p\) by (12b).

6. Now, for \(x \in \mathbb{R}^{n_1 + \cdots + n_K}\), let us set \(\omega[x] = [\omega_1(x); \ldots; \omega_K(x)]\) and \(\|x\| = \theta^{1/2}(\omega[x])\). Clearly, \(\| \cdot \|\) indeed is a norm, and the set \(\{ x : \|x\| = 1 \}\) is nothing but the set \(\{ x : \omega[x] \in \mathcal{F} \}\).

Given \(\beta \geq 1\), we say that two nonnegative reals \(a, b\) are within factor \(\beta\), if \(\beta^{-1}a \leq b \leq \beta a\), or, which is the same, \(\beta^{-1}b \leq a \leq \beta b\). Let us say that two block vectors \(u, v \in \mathbb{R}^{n_1 + \cdots + n_K}\) are within factor \(\beta\), if \(\sqrt[\theta]{\omega_i(u_i)}\) and \(\sqrt[\theta]{\omega_i(v_i)}\) are within factor \(\beta\) for every \(i\).

Let us make the following observation
Let $x \in \mathbb{R}^{n_1+\ldots+n_k}$ be a vector with nonzero blocks such that $\|x\| = 1$. Then there exists vector $z$ with nonzero blocks such that $\phi(z) = 1/p$ and $z$ is within factor

$$\kappa = \left[pK\right]^{1/(p+1)}$$

from $x$. Vice versa, if $z$ is a vector with nonzero blocks such that $\phi(z) = 1/p$, then there exists vector $x$ with nonzero blocks such that $\|x\| = 1$ and $x$ is within factor $\kappa$ from $z$.

Indeed, let $x$ be vector with nonzero blocks such that $\|x\| = 1$, and let $t = \omega [x]$, so that $t \in \mathcal{T}$ and $t > 0$. Let now $z$ be with nonzero blocks $z_i \in \mathbb{R}^{n_i}$ such that $\sqrt{\omega_i(z_i)} = \left[p^{-1}t_i^{p+1} |\nabla \theta(t)|_i\right]^{2/(p+1)}$, so that

$$\omega_i(z_i) = t_i [p^{-1} |\nabla \theta(t)|_i]^{2/(p+1)} = \omega_i(x_i)[p^{-1} |\nabla \theta(t)|_i]^{2/(p+1)}.$$

Note that $z$ and $x$ are within the factor $\kappa$. We have $t \in \mathcal{T}$ and $p\omega_i^{p+1}(z_i)/t_i^{p+1} = |\nabla \theta(t)|_i$, implying that $t = t(z)$ by (11). Consequently,

$$\phi(z) = \sum_i \omega_i^{p+1}(z_i)/t_i^p = p^{-1}\sum_i t_i |\nabla \theta(t)|_i = p^{-1}\theta(t) = p^{-1},$$

as claimed. Vice versa, let $z$ be a vector with nonzero blocks such that $\phi(z) = 1/p$, so that for $t = t(z) \in \mathcal{T}$ and $\lambda = \lambda(z) \geq 0$ we have

$$\omega_i^{p+1}(z_i) = (\lambda/p)t_i^{p+1}|\nabla \theta(t)|_i,$$

$$1/p = \phi(z) = \sum \omega_i^{p+1}(z_i)/t_i^p = (\lambda/p)\sum_i t_i |\nabla \theta(t)|_i = (\lambda/p)|\theta(t)| = \lambda/p.$$

We see that $\lambda = 1$, and therefore $\omega_i(z_i)$ are within factor $\kappa^2$ from $t_i$. Selecting $x_i \in \mathbb{R}^{n_i}$ such that $\omega_i(x_i) = t_i$, we get vector with nonzero blocks such that $\|x\| = \theta(t) = 1$ and $z$ is within factor $\kappa$ from $x$.

Next, $\Phi(x)$ is convex positive outside of the origin homogeneous, of degree 2, function and thus $n(x) := \sqrt{\Phi(x)} = \phi^{(p+1)}(x)$ is a norm. By construction $n(\cdot)$ is an “absolute block-norm,” meaning that $n(x)$ depends solely and monotonically on the norms $|x_i| = \omega_i(x_i)$ of the blocks $x_i$. As an immediate corollary, if $u, v$ are two vectors with nonzero blocks which are within some factor from each other, then $n(u)$ and $n(v)$ are within the same factor from each other as well. Given vector $x$ with nonzero blocks and such that $\|x\| = 1$, there exists vector $z$ within factor $\kappa$ from $x$ with $\phi(z) = 1/p$ and thus with $n(z) = \pi := p^{-(p+1)}$, so that $n(x)$ is within factor $\kappa$ from $\pi$. Thus, $\|\cdot\|$ is within factor $\pi = \kappa/\pi$ from $n(\cdot)$.

$\mathcal{T}^*$. Our next observation is as follows.

Let $t \in \mathcal{T}$ and $h \in \mathbb{R}^n$. Then

$$\sum_i \omega_i(h_i) |\nabla \theta(t)|_i^{p+1} \leq K^{1/p+1} \|h\|^2.$$

Indeed, by (9) we have $|$\nabla \theta(t)|_i^{p+1} \leq K^{1/p+1} |\nabla \theta(t)|_i$, whence

$$\sum_i \omega_i(h_i) |\nabla \theta(t)|_i^{p+1} \leq K^{1/p+1} |\nabla \theta(t)|^T \omega[h] \leq K^{1/p+1} [\theta(t + \omega[h]) - \theta(t)]$$

$$\leq K^{1/p+1} [\theta(t) + \theta(\omega[h]) - \theta(t)] = K^{1/p+1} \|h\|^2.$$
As a consequence, we have
\[
\sum_i \omega_i(h_i)[\nabla \theta(t)]_{i}^{\pi} \leq K^{\frac{1}{p+1}}(\kappa/\pi)^2 n^2(h)
\]
Finally, invoking (17), we conclude that for every vector \(x\) with nonzero blocks and all \(h\) such that
\[
\omega_i^{1/2}(h_i) \leq \omega_i^{1/2}(x_i)/[(2p + 2)^2(1 + \varepsilon)], \quad i \leq K,
\]
one has
\[
n^2(x + h) \leq n^2(x) + [\nabla n^2(x)]^T h + \frac{1}{2}n^2(h), \quad \gamma = 2K^{\frac{1}{p+1}}(\kappa/\pi)^2(5p + \varepsilon).
\]
We arrive at the following conclusion:

In the situation in question, for every positive integer \(p\) there exists a norm \(n(\cdot)\) on \(R^{n_1+\ldots+n_K}\) such that

- \(\Phi(x) = n^2(x)\) is continuously differentiable with Lipschitz continuous gradient and satisfies
  \[
  \forall x, h \in R^n : \Phi(x + h) \leq \Phi(x) + [\nabla \Phi(x)]^T h + [pK]^{\frac{1}{p+1}}(5p + \varepsilon)\Phi(h)
  \]
- \(n(\cdot)\) is within the factor \(\kappa/\pi = K^{\frac{1}{2p+1}}p^{\frac{1}{p+1}}\) from the norm \(\|x\| = \sqrt{\theta(\omega[x]^2)}\).

Setting \(p = \lceil \ln(K + 1) \rceil^1\) we complete the proof of Theorem 2.1. \(\square\)

A.2 Proof of Proposition 2.1

As is immediately seen, we lose nothing when assuming that the mapping \(z \rightarrow Pz\) in (3) is just the natural projection \(R^n = R^{p+q} \ni [u; y] \mapsto u \in R^p\). In this case, factor-norm induced by this projection and the norm \(\| \cdot \|\) on \(R^{p+q}\):
\[
\|u\|' = \min_y \|u; y\|.
\]
Observe that when two norms on \(R^{p+q}\) are within some factor from each other, their factor-norms induced by \(P\) are within the same factor as well. Thus, it suffices to prove Theorem 2.1 in the case when the function \(\Phi(x) = \|x\|^2\) is continuously differentiable and such that
\[
\|\nabla \Phi(x) - \nabla \Phi(x')\|_* \leq 2\varepsilon\|x - x'\| \quad \forall x, x';
\]
where, as always, \(\| \cdot \|_*\) is the norm conjugate to \(\| \cdot \|\). In this situation the statement of Proposition 2.1 is readily given by the following Lemma A.2

Lemma A.2 Let \(\Psi(u, y) : R^{p+q}\) be a convex function, \(\| \cdot \|\) be a norm on \(R^{p+q}\), and let \(\| \cdot \|'\) be the factor-norm:
\[
||h||' = \min_{d \in R^q} ||h; d||.
\]
induced by the natural projection of \(R^{p+q}\) onto \(R^p\). Assume that \(\Psi\) is smooth with Lipschitz continuous with constant \(L\) w.r.t. \(\| \cdot \|\) gradient, i.e.,
\[
\|\nabla \Psi(z) - \nabla \Psi(z')\|_* \leq L\|z - z'\| \quad \forall z, z',
\]
\(^1\)Here \([a]\) stands for the upper integer part of \(a\)—the smallest integer greater or equal to \(a\).
and let the set
\[ Y(u) = \text{Argmin}_{y \in \mathbb{R}^p} \Psi(u, y) \]
be nonempty for every \( u \in \mathbb{R}^p \). Then the function
\[ \overline{\Psi}(u) = \min_y \Psi(u, y) \]
on \( \mathbb{R}^p \) is convex and smooth with Lipschitz continuous with constant \( L \) w.r.t. \( \| \cdot \|' \) gradient.

**Proof of the lemma.** Convexity of \( \overline{\Psi} \) is evident. Let \( u \in \mathbb{R}^p \), and let \( y \in Y(u) \). Then for all \((u', y')\),
\[ \Psi(u', y') \geq \Psi(u, y) + [\Psi'_u(u, y)]^T (u' - u) + [\Psi'_y(u, y)]^T (y' - y) = \Psi(u, y) + [\Psi'_u(u, y)]^T (u' - u), \]
that is \( \Psi(u', y') \geq \overline{\Psi}(u) + [\Psi'_u(u, y)]^T (u' - u) \) for all \( u', y' \); hence,
\[ \overline{\Psi}(u') \geq \overline{\Psi}(u) + [\Psi'_u(u, y)]^T (u' - u) \quad \forall u' \in \mathbb{R}^p. \quad (18) \]
Denoting for brevity \( g = \Psi'_u(u, y) \), for every \( h \in \mathbb{R}^p \), selecting \( d \in \mathbb{R}^q \) such that \( \|h; d\| = \|h\|' \), we get
\[ \overline{\Psi}(u + h) \leq \Psi(u + h, y + d) \leq \Psi(u, y) + g^T h + \frac{L}{2} \|h\|'^2 = \Psi(u, y) + g^T h + \frac{L}{2} \|h\|^2, \]
implying that
\[ \overline{\Psi}(u) + g^T h \leq \overline{\Psi}(u + h) \leq \overline{\Psi}(u) + g^T h + \frac{L}{2} \|h\|^2 \quad (19) \]
with the first inequality given by (18). Consequently, \( g \) is the Frechet derivative of \( \overline{\Psi} \) at \( u \) and is independent of how \( y \in Y(u) \) is selected. Next, \( \overline{\Psi} \) is convex on \( \mathbb{R}^p \) and thus is locally Lipschitz continuous. We claim that \( \overline{\Psi}(\cdot) \) is continuous. Indeed, from local Lipschitz continuity of \( \overline{\Psi} \) it follows that \( \overline{\Psi}(\cdot) \) is locally bounded. Thus, to prove the continuity of \( \overline{\Psi}(\cdot) \), it suffices to lead to the contradiction the assumption that for some sequence \( u_i \to u, i \to \infty \), such that \( g_i = \overline{\Psi}(u_i) \) converge to some \( e \), one has \( e \neq g := \overline{\Psi}(u) \). But this is evident: \( e \) clearly is a subgradient of \( \overline{\Psi} \) at \( u \), and since \( \overline{\Psi} \) is differentiable, we get \( e = g \), which is a contradiction.

Thus, \( \overline{\Psi} \) is continuously differentiable convex function satisfying (19), implying by Lemma A.1 that the gradient of \( \overline{\Psi} \) is Lipschitz continuous, with constant \( L \), w.r.t. \( \| \cdot \|' \). \( \square \)

**A.3 Proof of Theorem 2.2**

Since \( \theta \) is \( (\mathfrak{F}, \mathfrak{S}) \)-regular and \( \| \cdot \|_i \) are \( (\mathfrak{S}', \mathfrak{S}) \)-regular, there exist \( \mathfrak{F} \)-smooth norm \( \vartheta(\cdot) \) and \( \mathfrak{S}' \)-smooth norms \( \vartheta_i(\cdot) \) such that
\[ \vartheta^{-1}(\vartheta(\cdot)) \leq \theta(\cdot) \leq \mathfrak{F}(\vartheta(\cdot)), \quad (\mathfrak{S}')^{-1}(\vartheta_i(\cdot)) \leq \| \cdot \|_i \leq \mathfrak{S}'(\vartheta_i(\cdot)), \quad i \leq K. \quad (20) \]
Since \( \theta \) is an absolute norm, these relations hold true when replacing \( \vartheta(\cdot) \) with the absolute norm
\[ \overline{\vartheta}(y) = 2^{-K} \sum_{E \in \mathcal{E}_K} \vartheta(Ey) \]
where \( \mathcal{E}_K \) is the multiplicative group of diagonal \( K \times K \) matrices with diagonal entries \( \pm 1 \), and the norm \( \overline{\vartheta} \) clearly is \( \mathfrak{F} \)-smooth along with \( \vartheta \). Thus, we can assume, on the top of what was just stated, that \( \vartheta \) is an absolute norm.

Next, the function \( \Psi(x_1, ..., x_K) = \theta^{1/2}(\|x_1\|^2, ..., \|x_K\|^2_K) \) is positively homogeneous of degree 1, and since \( \theta \) is an absolute norm, function \( \Psi^2(y) \) is monotone on \( \mathbb{R}^K_+ \). Consequently, \( \Psi^2 \) is convex, and

13
therefore the Lebesgue sets of $\Psi$ are convex; together with positive homogeneity of degree 1 of the function this implies that $\Psi(x_1, \ldots, x_K)$ is a norm. By the same argument, function $\vartheta^{1/2}(\vartheta_1^2(x_1), \ldots, \vartheta_K^2(x_K))$ also is a norm, so that monotonicity and homogeneity of degree 1 of $\vartheta$ and $\vartheta$ on $\mathbb{R}_+^K$ combines with (20) to imply that for $\varsigma = \varsigma' \sqrt{x}$ and all $[x_1; \ldots; x_K] \in \mathbb{R}^{n_1+\cdots+n_K}$

$$\varsigma^{-1} \vartheta^{1/2}(\vartheta_1^2(x_1), \ldots, \vartheta_K^2(x_K)) \leq \vartheta^{1/2}(\|x_1\|_1^2, \ldots, \|x_K\|_K^2) \leq \varsigma \vartheta^{1/2}(\vartheta_1^2(x_1), \ldots, \vartheta_K^2(x_K)).$$

To complete the proof of the theorem, all we need is the following statement.

**Lemma A.3** Let $F(y) = \|y\|_o^2$ be the square of an absolute norm on $\mathbb{R}^K$ such that $\nabla F$ is Lipschitz continuous w.r.t. $\| \cdot \|_o$ with Lipschitz constant $L$. Let also $\phi_i(x_i) = \|x_i\|_i^2$ be squares of norms on $\mathbb{R}^{n_i}$ such that $\nabla \phi_i$ is Lipschitz continuous with constant $\ell$ with respect to $\| \cdot \|_i$, $1 \leq i \leq K$. For $x = [x_1; \ldots; x_K] \in \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_K}$, we set $\phi(x) = [\phi_1(x_1); \ldots; \phi_K(x_K)]$ and

$$f(x) = F^{1/2}(\phi(x)).$$

Then $f$ is the square of a norm $\| \cdot \|$ on $\mathbb{R}^{n_1+\cdots+n_K}$ with gradient $\nabla f$ which is Lipschitz continuous with constant $2L + \ell$ w.r.t. $\| \cdot \|$.

**Proof of Lemma A.3.** The fact that $f$ is square of a norm is evident—since $F$ is square of an absolute norm the function $F(\phi(x))$ is convex. Besides this, the function is continuously differentiable, and is of homogeneity degree 4; as a result, $F^{1/4}(\phi(x))$ is continuous positively homogeneous, of degree 1, even function with convex level sets and as such is convex, and therefore is a norm. Note also that $f$ is continuously differentiable. All we need to prove is that

$$\forall (x, h) : f(x + h) \leq f(x) + [f'(x)]^T h + \frac{1}{2}[2L + \ell]f(h).$$

Assume for the sake of simplicity that $F$ and $\phi_i$ are twice continuously differentiable outside of the respective origins (we can arrive at this situation by suitable approximation, see Lemma A.5 below). Invoking Lemma A.4, it suffices to show that for every $x$ with nonzero $x_i$’s and every $h$ it holds

$$D^2 f(x)[h, h] \leq [2L + \ell]f(h).$$

(21)

Let us put

$$D = \{x \in \mathbb{R}^{n_1+\cdots+n_K} : x_i \neq 0, i \leq K\}, \quad D_o = \{y \in \mathbb{R}^K : y \neq 0\}.$$  

$F$ is twice continuously differentiable on $D_o$ and

$$0 \leq D^2 F(y)[d, d] \leq LF(d) \quad \forall (y \in D_o, d \in \mathbb{R}^K), \quad (a)$$

$$0 \leq D^2 \phi_i(x)[h, h] \leq \ell \phi(h) \quad \forall (x \in D, h \in \mathbb{R}^{n_1+\cdots+n_K}). \quad (b)$$

Besides this, taking into account that $F$ and $\phi_i$ are squares of norms continuously differentiable outside of the origin, we have

$$\text{abs}[D \phi(x)[h]] \leq 2[\phi(x)]^{1/2} \cdot [\phi(h)]^{1/2}, \quad (c)$$

$$|DF(y)[d]| \leq 2\sqrt{F(y)} \sqrt{F(d)}, \quad (d)$$

$$y \geq 0 \Rightarrow DF(y)[h] \text{ is monotone in } h \quad (e)$$

where for a vector $z$, abs$[z]$ is the vector of modulae of entries in $z$, for a nonnegative vector $z$ and $\alpha > 0$, $[z]^\alpha$ stands for the vector comprised of $\alpha$-th degrees of the entries of $z$, and $u \cdot v$ is coordinatewise product of two vectors; (e) stems from the fact that $F$ is the square of an absolute norm.
Now, let \( x \in \mathcal{D} \) and let \( y = \phi(x) \), so that \( y \in \mathcal{D}_0 \). We have \( f(x) = F^{1/2}(\phi(x)) \) with

\[
Df(x)[h] = \frac{1}{2F^{1/2}(\phi(x))} DF(\phi(x))[D\phi(x)[h]]
\]

and

\[
D^2 f(x)[h, h] = \frac{1}{2F^{1/2}(\phi(x))} \left[ D^2 F(\phi(x))[D\phi(x)[h], D\phi(x)[h]] + DF(\phi(x))[D^2 \phi(x)[h, h]] \right]
\]

whence

\[
\leq \frac{1}{2F^{1/2}(y)} \left[ D^2 F(y)[D\phi(x)[h, D\phi(x)[h]] + DF(y)[D^2 \phi(x)[h, h]] \right]. \tag{22}
\]

By homogeneity, it suffices to verify (21) for \( x \in \mathcal{X} \) such that \( F(\phi(x)) = 1 \). In this case we have

\[
Df(x)[h] = \frac{1}{2} DF(y)[D\phi(x)[h]]
\]

\[
D^2 f(x)[h, h] \leq \frac{1}{2} \left[ D^2 F(y)[D\phi(x)[h], D\phi(x)[h]] + DF(y)[D^2 \phi(x)[h, h]] \right]
\]

Putting \( d = D^2 \phi(x)[h, h] \), we get by (b),

\[
DF(y)[D^2 \phi(x)[h, h]] \leq 2 \sqrt{F(D^2 \phi(x)[h, h])} \leq 2 \ell F(\phi(x)) = 2 \ell f(h)
\]

by (b) and due to homogeneity of degree 2 of \( F \) and monotonicity of \( F(\cdot) \) on \( \mathbb{R}^K_+ \). Besides this,

\[
D^2 F(y)[D\phi(x)[h], D\phi(x)[h]] \leq LF(D\phi(x)[h]) \quad \text{[by (a)]}
\]

\[
\leq LF[|\phi(x)|^{1/2} \cdot \phi(h)]^{1/2} \quad \text{[since \( F(\cdot) \) is the square of an absolute norm]}
\]

\[
\leq LF(\lambda \phi(x) + \lambda^{-1} \phi(h)) \forall \lambda > 0 \quad \text{[since \( F \) is monotone on \( \mathbb{R}^K_+ \)]}
\]

\[
\leq L[\lambda F^{1/2}(\phi(x)) + \lambda^{-1} F^{1/2}(\phi(h))] \forall \lambda > 0 \quad \text{[since \( F^{1/2}(\cdot) \) is a norm on \( \mathbb{R}^K \)],}
\]

whence

\[
D^2 F(y)[D\phi(x)[h], D\phi(x)[h]] \leq L \left[ \inf_{\lambda > 0} [\lambda F^{1/2}(\phi(x)) + \lambda^{-1} F^{1/2}(\phi(h))] \right]^2
\]

\[
= 4LF^{1/2}(\phi(x))F^{1/2}(\phi(h)) = 4L f(h).
\]

The bottom line is that when \( x \in \mathcal{D} \) and \( F(\phi(x)) = 1 \), we have by (22)

\[
D^2 f(x)[h, h] \leq \frac{1}{2} \left[ D^2 F(y)[D\phi(x)[h], D\phi(x)[h]] + DF(y)[D^2 \phi(x)[h, h]] \right]
\]

\[
\leq \frac{1}{2} [4Lf(h) + 2 \ell f(h)] = [2L + \ell] f(h),
\]

as required in (21). \( \square \)

### A.4 Proof of Theorem 3.1

(i): By Theorem 2.1 as applied to the (1,1)-regular norms \( \| \cdot \|_i = \| \cdot \|_2 \) on \( \mathbb{R}^n \), the norm

\[
\|[x_1; \ldots; x_K]\| := \theta^{1/2}(\|x_1\|_2^2, \ldots, \|x_K\|_2^2)
\]

on \( \mathbb{R}^{nK} \) is \( (O(1) \ln(K + 1), O(1)) \)-regular. With \( \mathcal{X} \) given by (2), the norm

\[
\|x\| := \theta^{1/2}(\|T_1^{1/2} x\|_2^2, \ldots, \|T_k^{1/2} x\|_2^2)
\]

on \( \mathbb{R}^{nK} \) is \( (O(1) \ln(K + 1), O(1)) \)-regular.
is the superposition of \( \| \cdot \| \) and linear embedding \( x \mapsto [T^{1/2}_1 x; \ldots; T^{1/2}_k x] \) and therefore is regular with the same parameters as \( \| \cdot \| \), see Section 2.1. Finally, \( \| \cdot \|_V \) is the factor-norm of \( \| \cdot \|_+ \), and passing from a norm to its factor-norm preserves regularity parameters by Proposition 2.1.

(ii): The norms \( \| \cdot \| = \| \cdot \|_2 \) on \( S^{d_i} \) are \((O(1) \ln(d_i + 1), O(1))-regular, see “basic facts” at the end of in Section 2.1. By Theorem 2.1\) as applied to these norms, the norm

\[
\| y_1; \ldots; y_K \| := \theta^{1/2}(\| y_1 \|_2^2, \ldots, \| y_K \|_2^2)
\]
on \( S^{d_1} \times \ldots \times S^{d_K} \) is \((O(1) [\ln(K + 1) + \max_{i \leq K} \ln(d_i)], O(1))-regular. Similarly to the case of (i), with \( X \) given by (5), the norm \( \| \cdot \|_+ \) which is the superposition of \( \| \cdot \| \) and linear embedding

\[
x \mapsto (S_1(x), \ldots, S_K(x))
\]
is regular with the same parameters as \( \| \cdot \| \), and these parameters are inherited by the factor-norm \( \| \cdot \|_V \).

\[\square\]

A.5 Technical lemmas

Lemma A.4 Let \( \| \cdot \| \) be a norm on \( \mathbb{R}^n \) with conjugate norm \( \| \cdot \|_* \). Let also \( \alpha \geq 0 \), let \( V \) be a nonempty open convex domain in \( \mathbb{R}^n \) and \( f : V \to \mathbb{R} \) be a continuously differentiable convex function. The following 3 properties of \( f \) are equivalent:

1. \( h^T \nabla f(x + h) − \nabla f(x) \leq \alpha \| h \|^2 \quad \forall x, x + h \in V; \)
2. \( \| \nabla f(x + h) − \nabla f(x) \|_* \leq \alpha \| h \| \quad \forall x, x + h \in V; \)
3. \( f(x + h) \leq f(x) + h^T \nabla f(x) + \frac{\alpha}{2} \| h \|^2 \quad \forall x, x + h \in V. \)

Proof. Let \( V_0 \) be an open convex domain such that for some \( \varepsilon[V_0] > 0 \) the \( \| \cdot \|_2 \)-norm \( \varepsilon \)-neighbourhood of \( V_0 \) is contained in \( V \). It suffices to demonstrate that the properties (1\(_o\))–(3\(_o\)) obtained from respective properties (1)–(3) by replacing the requirement \( x, x + h \in V \) with \( x, x + h \in V_0 \) are equivalent to each other, whatever be \( V_0 \) of the indicated type. Let us fix such a \( V_0 \) and note that when \( f \) satisfies property \( i \), \( i = 1, 2, 3 \), then property \( i_o \) holds true for the restrictions onto \( V_0 \) of all shifts \( f(\cdot − u) \) of \( f(\cdot) \) with \( \| u \|_2 < \varepsilon[V_0] \), same as for convex combinations of these shifts. Selecting a \( C^\infty \) nonnegative kernel \( \delta(\cdot) \) with unit integral which vanishes outside of the \( \varepsilon[V_0]/2 \)-neighbourhood of the origin and setting

\[
f_\delta(x) = \int f(x − u)\delta(u)du : V_0 \to \mathbb{R},
\]
we obtain a \( C^\infty \) convex function on \( V_0 \) which possesses properties \( i_o \), \( i = 1, 2, 3 \), provided that \( f \) possesses property \( i \). When \( \delta \to +0 \), functions \( f_\delta \) converge to \( f \) uniformly on compact subsets of \( V_0 \) along with their gradients. Therefore, when \( f_\delta \) possesses one of the properties \( i_o \), so does \( f \), and vice versa. The bottom line is that all we need is to show the equivalence (1\(_o\)) – (3\(_o\)) in the case when \( f \) is a convex \( C^\infty \) function on \( V_0 \). This is immediate: when \( f \) is convex and smooth, every one of the three properties in question is equivalent to

\[\forall (x \in V_o, u, v \in \mathbb{R}^n). \]

Indeed,

\[(1_o) \iff (s_o): \text{When } f \text{ obeys } (1_o), x \in V_o \text{ and } h \in \mathbb{R}^n, \text{ we clearly have}
\]

\[
h^T \nabla^2 f(x)h = \lim_{t \to +0} t^{-1} h^T [\nabla f(x + th) − \nabla f(x)] = \lim_{t \to +0} t^{-2} [th]^T [\nabla f(x + th) − \nabla f(x)] \leq \alpha \| th \|^2,
\]

\[\leq \alpha \| th \|^2 \]

16
whence, due to the fact that $\nabla^2 f(x)$ is symmetric positive semidefinite,
\[
    u^T \nabla^2 f(x)v \leq |u^T \nabla^2 f(x)u|^{1/2}|v^T \nabla^2 f(x)v|^{1/2} \leq \alpha \|u\|\|v\|,
\]
so that $(\ast_o)$ holds. Vice versa, assuming that $f$ obeys $(\ast_o)$, for $x, x + h \in V_o$ we have
\[
    h^T [\nabla f(x + h) - \nabla f(x)] = \int_0^1 h^T \nabla^2 f(x + th)hdt \leq \int_0^1 \alpha \|h\|^2 dt = \alpha \|h\|^2.
\]
as required in $(i_o)$.

$(2_o) \iff (\ast_o)$: Assuming that $(2_o)$ holds, for $x \in V_o$ and $u, v \in \mathbb{R}^n$ we have
\[
    u^T \nabla^2 f(x)v = \lim_{t \to +0} t^{-1} u^T [\nabla f(x + tv) - \nabla f(x)] \leq \lim_{t \to +0} t^{-1} \|u\| \|\nabla f(x + tv) - \nabla f(x)\|_o,
\]
and the latter limit is $\leq \alpha \|u\|\|v\|$ by $(2_o)$. Vice versa, if $f$ obeys $(\ast_o)$, and $x \in V_o$, $x + h \in V_o$, we have
\[
    \|\nabla f(x + h) - \nabla f(x)\|_o = \max_{u: \|u\| \leq 1} u^T [\nabla f(x + h) - \nabla f(x)] = \max_{u: \|u\| \leq 1} \int_0^1 u^T \nabla^2 f(x + th)hdt
\]
\[
    \leq \max_{u: \|u\| \leq 1} \int_0^1 \alpha \|u\| \|h\| dt = \alpha \|h\|,
\]
as required in $(2_o)$.

$(3_o) \iff (\ast_o)$: $(\ast_o)$ clearly implies $(3_o)$. On the other hand, when $x \in V_o$ and $h \in \mathbb{R}^n$, for properly selected $C = C(x, h) < \infty$ and $\overline{T} = \overline{T}(x, h) > 0$ we have
\[
    0 \leq t \leq \overline{T} \Rightarrow f(x + th) \geq f(x) + th^T \nabla f(x) + \frac{t^2}{2} h^T \nabla^2 f(x)h - Ct^3.
\]
It follows that when $f$ obeys $(3_o)$, we have
\[
    f(x) + th^T \nabla f(x) + \frac{t^2}{2} h^T \nabla^2 f(x)h - Ct^3 \leq f(x) + th^T \nabla f(x) + \frac{\alpha}{2} \|h\|^2
\]
for $0 \leq t \leq \overline{T}$, implying that $h^T \nabla^2 f(x)h \leq \alpha \|h\|^2$, whence, $(\ast_o)$ holds true. \hfill \Box

**Lemma A.5** Let $f(\cdot): \mathbb{R}^n \to \mathbb{R}_+$ be a norm such that the function $F(x) = f^2(x)$ is continuously differentiable and for certain $\alpha > 0$ satisfies the relation
\[
    h^T [\nabla F(x + h) - \nabla F(x)] \leq \alpha F(h) \ \forall x, h \in \mathbb{R}^n.
\]

Then

(i) For every $\epsilon \in (0, 1)$ there exists a norm $f^\epsilon(\cdot)$ on $\mathbb{R}^n$ such that
\[
    (1 - \epsilon)f(\cdot) \leq f^\epsilon(\cdot) \leq (1 + \epsilon)f(\cdot)
\]
and the function $F^\epsilon(x) := [f^\epsilon(x)]^2$ is $C^\infty$ outside of the origin and satisfies the relations
\[
    h^T [\nabla F^\epsilon(x + h) - \nabla F^\epsilon(x)] \leq (1 + \epsilon)\alpha F^\epsilon(h) \ \forall (x, h \in \mathbb{R}^n),
\]
\[
    0 \leq D^2 F^\epsilon(x)[h, h] \leq \alpha F^\epsilon(h) \ \forall (x \neq 0, h).
\]
Moreover, as $\epsilon \to +0$, functions $F^\epsilon$ converge to $F$ uniformly on compact subsets of $\mathbb{R}^n$ along with their gradients.
(ii) If, in addition, \( f \) is an absolute norm, \( f^* \) in (i) can be made absolute norm as well.

Proof. (i) Let \( S = \{ x \in \mathbb{R}^n : \|x\| = 1 \} \) and let \( SO_n \) be the group of orthogonal \( n \times n \) matrices with determinant 1, both sets equipped with the natural structures of \( C^\infty \) manifolds. Let also \( \mu(\cdot) \) be the invariant measure on \( SO_n \). In the sequel, for \( g \in SO_n \) and function \( \phi(\cdot) \) on \( \mathbb{R}^n \), \( \phi_g(\cdot) \) is given by \( \phi_g(x) = \phi(gx) \).

1°. Observe that for every \( \epsilon \in (0,1) \) there exists \( \delta(\epsilon) > 0 \) such that

\[
\|f\|_{2,2} \leq \delta < \delta(\epsilon) \Rightarrow (1 - \epsilon) f(x) \leq f_g(x) \leq (1 + \epsilon) f(x) \quad \forall x
\]

where, same as above, \( \|\cdot\|_{2,2} \) is the spectral norm. Indeed, for properly selected \( C < \infty \) and any linear mapping \( x \mapsto Hx : \mathbb{R}^n \to \mathbb{R}^n \) it holds

\[
\|H\|_f \leq C \|H\|_{2,2}
\]

where \( \cdot \|f \) is the norm induced by the norm \( f(\cdot) \) on \( \mathbb{R}^n \). Consequently, for \( g \in SO_n \), by the triangle inequality, one has for all \( x \):

\[
\begin{align*}
\sum_{\mathbb{R}^n} f(gx) &= f(\int [g - I_n]x + x) \leq f(x) + \|g - I_n\|_f f(x) \leq (1 + C \|g - I_n\|_{2,2}) f(x), \\
\sum_{\mathbb{R}^n} f(gx) &= f(\int [g - I_n]x + x) \geq f(x) - \|g - I_n\|_f f(x) \geq (1 - C \|g - I_n\|_{2,2}) f(x),
\end{align*}
\]

so that it suffices to put \( \delta(\epsilon) = \epsilon/C \).

2°. Given \( \delta > 0 \), let \( \theta(\cdot) \) be a nonnegative \( C^\infty \) function on \( SO_n \) such that \( \theta(g) = 0 \) for \( \|g - I_n\|_{2,2} \geq \delta \) and \( \int_{SO_n} \theta_\delta(g) \mu(dg) = 1 \), and let

\[
F^{(\delta)}(x) = \int_{SO_n} F(gx) \theta_\delta(g) \mu(dg).
\]

Observe that function \( F^{(\delta)} \) is continuous, homogeneous of degree 2 and positive outside of the origin, so that function \( f(\delta)(x) = \sqrt{F^{(\delta)}(x)} \) is positive outside of the origin and satisfies \( f(\delta)(\lambda x) = |\lambda| f(x) \). Besides this,

\[
f^{(\delta)}(x) = \sqrt{\int_{SO_n} f_\delta^2(gx) \theta_\delta(g) \mu(dg)},
\]

so that \( f^{(\delta)} \) is convex, since the functional \( \Phi[\phi] = \sqrt{\int_{SO_n} \phi^2(g) \theta_\delta(g) \mu(dg)} \) on the space of continuous real-valued functions on \( SO_n \) is convex and satisfies \( \Phi[\psi] \geq \Phi[\phi] \) whenever \( \psi(\cdot) \geq \phi(\cdot) \geq 0 \). The bottom line is that \( f^{(\delta)} \) is a norm on \( \mathbb{R}^n \). In addition, taking into account that \( \theta_\delta(g) \geq 0 \), \( \int_{SO_n} \theta_\delta(g) \mu(dg) = 1 \), and \( \theta_\delta(g) = 0 \) when \( \|g - I_n\|_{2,2} \geq \delta \), by 1° we conclude that

\[
\forall \epsilon \in (0,1), \quad 0 < \delta < \delta(\epsilon) : \quad (1 - \epsilon) f(\cdot) \leq f^{(\delta)}(\cdot) \leq (1 + \epsilon) f(\cdot).
\]

3°. Since \( F \) is continuously differentiable, we have

\[
\forall (x, \mu) : \nabla F^{(\delta)}(x) = \int_{SO_n} gT \nabla F(gx) \theta_\delta(g) \mu(dg),
\]

whence

\[
\int_{SO_n} [g \nabla F(gx + gh) - \nabla F(gx)] \theta_\delta(g) \mu(dg)
\]

\[
\leq \int_{SO_n} \alpha F(gh) \theta_\delta(g) \mu(dg),
\]

18
that is,
\[ \forall x, h : h^T [\nabla F^{(\delta)}(x + h) - \nabla F^{(\delta)}(x)] \leq \alpha F^{(\delta)}(h). \]

4\textsuperscript{o}. Our goal is to prove that \( F^{(\delta)} \) is \( C^\infty \) outside of the origin.

Let \( e_1, \ldots, e_n \) be the canonic basis orts of \( \mathbb{R}^n \). For \( u \in S \), let
\[ S(u) = \{ v \in S : v^T u > 0 \} = \{ v \in S : \|v - u\| < \sqrt{2} \}. \]

We claim that there exists mapping \( \chi : S(e_1) \to SO_n \) such that \( \chi(e_1) = I_n \) and \( \chi(x)e_1 = x \) for all \( x \in S(e_1) \). Indeed, when \( x \in S(e_1) \), vectors \( x, e_2, e_3, \ldots, e_n \) are linearly independent. When applying to this sequence the Gram-Schmidt orthogonalization process, we obtain an orthonormal system \( e_1(x) \equiv x, e_2(x), \ldots, e_n(x) \) with vector-functions \( e_i(x) \) which are \( C^\infty \) on \( S(e_1) \), and we can set \( \chi(x) = [e_1(x), e_2(x), \ldots, e_n(x)] \).

We are now ready to show that \( F^{(\delta)} \) is \( C^\infty \) outside of the origin when \( \epsilon \in (0, 1) \) and \( 0 < \delta < \delta(\epsilon) \). Since \( F^{(\delta)} \) is homogeneous of degree 2 and positive outside of the origin, it suffices to verify that the restriction of \( F^{(\delta)} \) onto the sphere \( S \) is \( C^\infty \). To justify the latter claim, let \( \delta > 0 \), \( \bar{x} \in S \), and let \( \bar{g} \in SO_n \) be such that \( \bar{g}e_1 = \bar{x} \). Then for all \( x \in S(\bar{x}) \)
\[ \bar{g}^{-1}x \in S(e_1) \Rightarrow \bar{g}^{-1}x = \chi(\bar{g}^{-1}x)e_1 \Rightarrow x = \bar{g}\chi(\bar{g}^{-1}x)e_1, \]
whence for such \( x \) and \( h = g\bar{g}\chi(\bar{g}^{-1}x) \) we get
\[ F^{(\delta)}(x) = \int_{SO_n} F(g\bar{g}\chi(\bar{g}^{-1}x)e_1)\theta_\delta(g)\mu(dg) = \int_{SO_n} F(he_1)\theta_\delta(h\chi^{-1}(\bar{g}^{-1}x)\bar{g}^{-1})\mu(dh) \]
due to invariance of \( \mu(\cdot) \). Because \( \theta(\cdot) \) is \( C^\infty \) on \( SO_n \) and \( \chi(z) \) is \( C^\infty \) in \( S(e_1) \), function \( \theta_\delta(\cdot, h) \) is \( C^\infty \) in the first argument in the neighbourhood \( S(\bar{x}) \) with derivatives which are continuous in \( x \) and \( h \), implying that \( F^{(\delta)}(x) \) is \( C^\infty \) on \( S \).

5\textsuperscript{o}. The bottom line of the argument is that for every \( \epsilon \in (0, 1) \), for properly selected \( \delta(\epsilon) > 0 \) and all \( \delta \in (0, \delta(\epsilon)) \) function \( F^{(\delta)}(\cdot) \) is the square of some norm \( f^{(\delta)}(\cdot) \) satisfying
\[ (1 - \epsilon)f(\cdot) \leq f^{(\delta)}(\cdot) \leq (1 + \epsilon)f(\cdot), \]

\( F^{(\delta)} \) is \( C^\infty \) outside of the origin, and
\[ \forall x, h : h^T [\nabla F^{(\delta)}(x + h) - \nabla F^{(\delta)}(x)] = \alpha F^{(\delta)}(h). \quad (24) \]

In particular, when \( x \neq 0 \), we have
\[ 0 \leq h^T [\nabla F^{(\delta)}(x + th) - \nabla F^{(\delta)}(x)] \leq t^{-1}F^{(\delta)}(th) = t\alpha F^{(\delta)}(h), \]

implying that
\[ D^2 F^{(\delta)}(x)[h, h] = \lim_{t \to 0^+} t^{-1}h^T [\nabla F^{(\delta)}(x + th) - \nabla F^{(\delta)}(x)] \leq \alpha F^{(\delta)}(h) \forall x \neq 0, h. \quad (25) \]

Thus, given \( \epsilon \in (0, 1) \), the function \( F^\epsilon := F^{(\delta(\epsilon)/2)} \) satisfies all requirements of (i). (i) is proved.

6\textsuperscript{o}. It remains to prove (ii). To this end, assume that \( f \) is an absolute norm. Let \( E_n \) be the multiplicative group comprised by the \( 2^n \) diagonal matrices with diagonal entries \( \pm 1 \). Given \( \epsilon \in (0, 1) \) and assuming that \( \delta \in (0, \delta(\epsilon)) \), let us set
\[ F^{\delta, E}(x) := F^{(\delta)}(Ex) = [f^{(\delta)}(Ex)]^2, E \in E_n. \]
Functions $F^{\delta,E}(x)$ are $C^\infty$ outside of the origin functions, while due to (24) and (25) we have

$$\forall x, h : \quad h^T \left[ \nabla F^{\delta,E}(x + h) - \nabla F^{\delta,E}(x) \right] \leq \alpha F^{\delta,E}(h),$$

$$\forall x \neq 0, h : \quad D^2 F^{\delta,E}(x)[h,h] \leq \alpha F^{\delta,E}(h).$$

We conclude that the function

$$F^{\delta}(x) = 2^{-n} \sum_{E \in \mathcal{E}_n} F^{\delta,E}(x)$$

is $C^\infty$ outside of the origin and satisfies

$$\forall x, h : \quad h^T \left[ \nabla F^{\delta}(x + h) - \nabla F^{\delta}(x) \right] \leq \alpha F^{\delta}(h),$$

$$\forall x \neq 0, h : \quad D^2 F^{\delta}(x)[h,h] \leq \alpha F^{\delta}(h).$$

Besides this,

$$(F^{\delta}(x))^{1/2} = \left( 2^{-n} \sum_{E \in \mathcal{E}_n} [f^{(\delta)}(Ex)]^2 \right)^{1/2},$$

so that $(F^{\delta}(x))^{1/2}$ is a norm along with the functions $f^{(\delta)}(Ex)$. Furthermore, by construction, $F^{\delta}(Ex) = F^{\delta}(x)$ for all $x$, so that $(F^{\delta}(x))^{1/2}$ is an absolute norm. Finally, by (23) and due to $0 < \delta < \delta(\epsilon)$ we have

$$(1 - \epsilon)^2 F(\cdot) \leq [f^{(\delta)}(\cdot)]^2 \leq (1 + \epsilon)^2 F(\cdot),$$

and because $f(\cdot)$ is an absolute norm, it follows that

$$(1 - \epsilon)^2 F(x) \leq [f^{(\delta)}(Ex)]^2 \leq (1 + \epsilon)^2 F(x) \quad \forall x, \forall E \in \mathcal{E}_n.$$ 

Hence,

$$(1 - \epsilon)^2 F(\cdot) \leq F^{\delta}(\cdot) \leq (1 + \epsilon)^2 F(\cdot),$$

and thus

$$(1 - \epsilon) f(\cdot) \leq \left( F^{\delta}(\cdot) \right)^{1/2} \leq (1 + \epsilon) f(\cdot).$$

The bottom line is that to meet all requirements of (ii), it suffices to set $F^\epsilon = F^{\delta(\epsilon)/2}$. Finally, we can ensure that $\delta(\epsilon) \to +0$ as $\epsilon \to +0$, and in this case, as is immediately seen from the above, functions $F^\epsilon$ converge, along with their gradients, to $F$ uniformly on bounded subsets of $\mathbb{R}^n$ as $\epsilon \to +0$. □

References

[1] Beck, A. (2017) First-Order Methods in Optimization. – MOS-SIAM Series on Optimization, SIAM.

[2] Ben-Tal, A., Nemirovski, A. Lectures on Modern Convex Optimization. https://www2.isye.gatech.edu/~nemirovs/LMCOLN2022Fall.pdf

[3] Ben-Tal, A. Nemirovski, A. On solving large-scale polynomial convex problems by randomized first-order algorithms. - Mathematics of Operations Research 40:2 (2015), 474–494.
[4] Juditsky, A., Kotsalis, G., Nemirovski, A. (2021). Tight Computationally Efficient Approximation of Matrix Norms with Applications. arXiv:2110.04389 https://arxiv.org/pdf/2110.04389.pdf

[5] Juditsky, A., Nemirovski, A. (2008) Large Deviations of Vector-Valued Martingales in 2-Smooth Normed Spaces. arXiv:0809.0813 https://arxiv.org/pdf/0809.0813.pdf

[6] Juditsky, A., Nemirovski, A. (2020). Statistical Inference via Convex Optimization. In Statistical Inference via Convex Optimization. Princeton University Press.

[7] Lan, G. (2020) First-order and stochastic optimization methods for machine learning, Springer.

[8] Ledoux, M., Talagrand, M. (2002) Probabilities in Banach Spaces. Series of Modern Surveys in Mathematics, v. 23, Springer.

[9] Milman, V., Schechtman, G. (2001) Asymptotic Theory of Finite Dimensional Normed Spaces, With an Appendix by M. Gromov “Isoperimetric Inequalities in Riemannian Manifolds”—Lecture Notes in Mathematics v. 1200, Springer.

[10] Nesterov, Yu., Nemirovski, A. On first-order algorithms for $\ell_1$/nuclear norm minimization. - Acta Numerica 22 (2013), 509-575.

[11] Nesterov, Yu. (2018) Lectures on Convex Optimization. Springer Optimization and Its Applications, v. 137, Springer.

[12] Pinelis, I. (1992). An approach to inequalities for the distributions of infinite-dimensional martingales. In Probability in Banach Spaces, 8: Proceedings of the Eighth International Conference, 128–134. Birkhäuser, Boston, MA.

[13] Pinelis, I. (1994). Optimum bounds for the distributions of martingales in Banach spaces. The Annals of Probability, 1679-1706.

[14] Pinelis, I. (2015). Rosenthal-type inequalities for martingales in 2-smooth Banach spaces. Theory of Probability & Its Applications, 59(4), 699-706.

[15] Pisier, G. (1975). Martingales with values in uniformly convex spaces. Israel Journal of Mathematics, 20(3), 326-350.