Abstract

We study the relation between vertex operators in $AdS_5 \times S^5$ and classical spinning string solutions. In the limit of large quantum numbers the treatment of vertex operators becomes semiclassical. In this regime, a given vertex operator carrying a certain set of quantum numbers defines a singular solution. We show in a number of examples that this solution coincides with the classical string solution with the same quantum numbers but written in a different two-dimensional coordinate system. The marginality condition imposed on an operator yields a relation between the energy and the other quantum numbers which is shown to coincide with that of the corresponding classical string solution. We also argue that in some cases vertex operators in $AdS_5 \times S^5$ cannot be given by expressions similar to the ones in flat space and a more involved consideration is required.
1 Introduction

String theory on the $AdS_5 \times S^5$ background is considered to be integrable. In was shown in [1] that the equations of motion of the classical string sigma model on $AdS_5 \times S^5$ are integrable. However, it is also believed that this theory posses integrability at the quantum level. See the review article [2] and references therein for the recent progress in this direction. A big evidence in favor of integrability comes from the field theory side of the AdS/CFT duality [3, 4, 5]. First, computations of the anomalous dimensions of infinite length single trace operators (corresponding to string states with infinite energy) were summarized in [6, 7] in the form of the asymptotic Bethe ansatz. Furthermore, it was recently proposed in [8, 9, 10, 11] how to take into account the corrections due to the finite length of these operators.

Integrability suggests that it should be possible to identify all quantum states in $AdS_5 \times S^5$ (analogues of particles in flat space) as well as the vertex operators to describe their interaction. In fact, by AdS/CFT correspondence, each vertex operator should be associated with a local gauge-invariant operator in $\mathcal{N} = 4$ gauge theory in the planar limit [12]. Furthermore, integrability also implies that it should also be possible to systematically compute the energy of all these string states at any t’Hooft coupling. At the present moment, it is not known how to perform this program. In particular, it is not much known about construction of the vertex operators in this theory.

This paper is devoted to studying the vertex operators in $AdS_5 \times S^5$ in the limit of large quantum numbers where the analysis becomes semiclassical. Vertex operators in the semiclassical approximation were previously studied in [4, 12, 13]. Our aim is to find the relation between them and various classical spinning string solutions. For a review of string solutions see [14] and references therein. A priory, it is not very clear why such a relation should exist. A vertex operator defines a particle-like quantum state with certain energy, spins and additional quantum numbers. This is not in an obvious way related to any classical solution. However, for large quantum numbers, quantum states often can be approximated with classical field trajectories. Moreover, a semiclassical computation of correlation functions of vertex operators is equivalent to finding a certain classical solution with singularities at the positions of the operators. So in the semiclassical regime both descriptions involve classical solutions. Let us say, we start with a classical string solution with a set of conserved quantities (energy, spins, angular momenta in various directions, ...). If we construct a vertex operators carrying the same quantum numbers, we should expect that in the limit of large charges such an operator corresponds to this classical solution. In fact, we will argue and
show it explicitly in a number of examples that if we compute the two-point function of these vertex operators we obtain the same classical solution we started with but written in a different two-dimensional coordinate system. The two coordinate systems are related by a conformal transformation which is singular at the points where operators are inserted. In this paper, we will present a several examples of this correspondence. We will also show that the relation between the energy and the remaining quantum numbers of the vertex operators coincides with the similar relation of the corresponding classical solutions. Note that since construction of vertex operators in $AdS_5 \times S^5$ is not well understood, one can also view our procedure as a consistency check on the vertex operators themselves.

The paper is organized as follows. In Section 2, we will explain how exactly the singular solution obtained from inserting vertex operators is related to the corresponding classical string solution. For this we will consider the correlation function of two vertex operators in the limit of large quantum numbers so that the description becomes semiclassical. We explain why the semiclassical trajectory in the presence of the operators should coincide with the classical solution carrying the same charges in a different two-dimensional coordinate system. In this framework, the relation between the energy and the remaining quantum numbers comes from requiring that the vertex operator be of the right conformal dimension. As the first demonstration of this approach we consider an example of a string spinning in a two-dimensional plane in flat space [13].

In the rest of the paper, we perform this analysis for various string solutions in $AdS_5 \times S^5$. More precisely, we will consider strings spinning in $AdS_3$ and $S^3$. In these cases the equations of motion are non-linear. We will solve them by starting with a classical solution and performing the appropriate conformal transformation. In Section 3, we consider a folded string spinning in $AdS_3$ [15] and derive the logarithmic correction to the energy. In Section 4, we consider a spinning string in $S^3$. This case is subtle since there are two different solutions with the same energy and spin. So an important question is how to distinguish these two states with vertex operators. One of these solutions has a flat space limit and, thus, its vertex operator can be written by analogy with vertex operators in flat space. However, the second solution does not have such a limit. Furthermore, it carries an additional quantum number which is a topologically trivial winding. We propose that the vertex operator for this state should be written in terms of the T-dual variables. In general, T-duality on $AdS_5 \times S^5$ is rather non-trivial [16, 17, 18]. However, in our case the semiclassical calculations evade this problem since most of the $S^5$ coordinates are constants. We compute the energy-angular momentum trajectory of this operator and show that it coincides with that of the
corresponding classical solution. This can be viewed as a consistency check on the proposed vertex operator.

2 Vertex Operators and Classical String Solution

In this section, we would like to describe the relation between semiclassical vertex operators and classical string solutions. We will be interested in computing the correlation function of two such operators. These operators are inserted on the complex plane whose coordinates we will denote by \((\xi, \bar{\xi})\). On the other hand the classical closed string world-sheet is the cylinder whose coordinates we will denote by \((\tau, \sigma)\) and that is where classical string solutions are defined. We will perform a Wick rotation \(\tau = -i\tau_e\) and consider the Euclidean world-sheet. Now we perform a conformal transformation to map the cylinder to the plane (or, more precisely, to the sphere)

\[ z = e^{\tau_e + i\sigma}. \] (2.1)

In doing this, the points \(\tau_e = \pm\infty\) are mapped to \(z = 0\) and \(z = \infty\). Now we perform one more conformal transformation to map the point \(z = \infty\) to a finite position. From \((z, \bar{z})\) we go to \((\xi, \bar{\xi})\)

\[ z = \frac{\xi}{\xi - \xi_1}. \] (2.2)

Now \(z = 0\) goes to \(\xi = 0\) and \(z = \infty\) goes to \(\xi = \xi_1\). Thus, this sequence of conformal transformations creates two singularities on the \(\xi\)-plane, one at \(\xi = 0\) and the other one at \(\xi = \xi_1\). We will view these two points as the points where two vertex operators are inserted. Of course, we can perform the second conformal transformation to move the singularity at \(\xi = 0\) to an arbitrary point \(\xi_2\),

\[ z = \frac{\xi - \xi_2}{\xi - \xi_1}. \] (2.3)

However, by translational invariance we can move the position of one of the operators to \(\xi = 0\). Hence, for simplicity, we will set \(\xi_2 = 0\). If we start with a classical solution on the cylinder and perform the conformal transformations (2.1) and (2.2) we obtain a solution on the \(\xi\)-plane with singularities at \(\xi = 0\) and \(\xi = \xi_1\). The singularities arise due to specifics of these conformal transformations. This singular solution is what we expect to find if we insert the appropriate semiclassical vertex operators at \(\xi = 0\) and \(\xi = \xi_1\). One of our goals in the rest of the paper will be to check this statement in various examples. Once the solution is found, we can calculate semiclassically the two-point function by evaluating the action on this solution. Conformal invariance requires that the dimension of the operator be \((1, 1)\).
On the other hand, in general, the expected behavior of the two-point function is $|\xi_1|^{2\gamma-4}$, where $\gamma$ can be viewed as the anomalous dimension. For operators of dimension $(1,1)$ the anomalous dimension $\gamma$ vanishes. Setting $\gamma = 0$ is supposed to yield the dependence of the energy on the other quantum numbers. Note that in finding the semiclassical solution, all parameters must be fixed in terms of the quantum numbers carried by the vertex operator. Otherwise, if there is an arbitrary parameter which is not fixed, setting $\gamma$ to zero will yield the energy as a function not just of the remaining quantum numbers but also on this unfixed parameter.

Let us finish our general discussion with two simple remarks. The first remark concerns when the semiclassical analysis is reliable. Clearly, to ignore quantum corrections we have to take $\alpha'$ to be small or the 't Hooft coupling $\lambda = 1/\alpha'^2$ to be large. Furthermore, we have to take the energy and other quantum numbers to be large. The second remark is that in our semiclassical computations we can ignore the dependence of the vertex operators on the fermions and take them purely bosonic.

The way we are going to proceed in this paper is as follows. We will start with a specific classical string solution. Then we will write the vertex operator carrying the same quantum numbers as the classical solution. The construction of vertex operators in $AdS_5 \times S^5$ is not well understood and we will have to make certain reasonable guesses. For example, in many cases, the classical string solution admits a flat space limit when the string becomes very small and does not feel the curvature of $AdS_5$ and $S^5$. In these cases, the corresponding vertex operator is expected to have the similar form as in flat space. Once we write the vertex operator, we proceed with the semiclassical evaluation of the two-point function and the computation of the energy-spin relation. In all examples below, this relation will coincide with that of the classical solution we started with.

Let us first illustrate the above procedure in the example of a string spinning in the plane in flat space following [13]. The relevant classical action is

$$A = -\frac{\sqrt{\lambda}}{4\pi} \int d\tau d\sigma (-\partial_a t \partial^a t + \partial_a X \partial^a \bar{X}) \, .$$

(2.4)

The classical solution of interest is

$$t = \kappa \tau, \quad X = \omega \sin \sigma e^{i\tau}, \quad \bar{X} = \omega \sin \sigma e^{-i\tau} \, .$$

(2.5)

Furthermore, the Virasoro constraints give $\omega = \kappa$. This solution has two conserved quantities, the energy $E$ and the spin $S$. They are given by

$$E = \frac{\sqrt{\lambda}}{2\pi} \int_0^{2\pi} d\sigma \dot{t} = \sqrt{\lambda} \kappa$$

(2.6)
and
\[ S = \frac{i\sqrt{\lambda}}{4\pi} \int_{0}^{2\pi} d\sigma (X \dot{X} - \bar{X} \dot{\bar{X}}) = \frac{\sqrt{\lambda} \omega^2}{2}. \] (2.7)

Note that both \( E \) and \( S \) scale as \( \sqrt{\lambda} \). Since \( \omega = \kappa \) we get the following relation between \( E \) and \( S \)
\[ E = \sqrt{2 \sqrt{\lambda} S} = \frac{2}{\alpha'} S. \] (2.8)

Now we will consider the vertex operator carrying the energy and the spin in the \((X, \bar{X})\)-plane. In flat space it is given by
\[ V_{S} = e^{-iE t} (\partial X \bar{\partial} \bar{X})^{S/2}. \] (2.9)

We want to compute the correlation function of \( V_{S} \) inserted at \( \xi = 0 \) and \( V_{-S} \),
\[ V_{-S} = e^{iE t} (\partial \bar{X} \bar{\partial} \bar{X})^{S/2}, \] (2.10)

inserted at \( \xi = \xi_1 \) in the limit of large \( \sqrt{\lambda}, E, S \). We perform a Euclidean rotation
\[ \tau = -i \tau_e, \quad t = -i t_e, \quad A = i A_e \] (2.11)

and consider the action on the plane modified by the vertex operators (2.9) and (2.10) (to simplify our notation we will still denote it by \( A_e \))
\[ A_e = \sqrt{\lambda} \int d^2 \xi (\partial t_e \bar{\partial} t_e + \frac{1}{2} \partial X \bar{\partial} \bar{X} + \frac{1}{2} \partial \bar{X} \bar{\partial} X)
+ E \int d^2 \xi t_e (\delta^2 (\xi) - \delta^2 (\xi - \xi_1))
- \frac{S}{2} \int d^2 \xi \delta^2 (\xi) \ln(\partial X \bar{\partial} X) - \frac{S}{2} \int d^2 \xi \delta^2 (\xi - \xi_1) \ln(\partial \bar{X} \bar{\partial} X), \] (2.12)

where \( d^2 \xi = d Re(\xi) d Im(\xi) \). The two-point function in the semiclassical approximation is then simply given by
\[ \langle V_{S}(0)V_{-S}(\xi_1) \rangle \sim e^{-A_e}. \] (2.13)

Here \( A_e \) is evaluated on the solution to the equations of motion which are as follows
\[ (\partial \bar{\partial} + \bar{\partial} \partial) t_e = \frac{\pi E}{\sqrt{\lambda}} (\delta^2 (\xi) - \delta^2 (\xi - \xi_1)), \] (2.14)
\[ (\partial \bar{\partial} + \bar{\partial} \partial) \bar{X} = \frac{\pi S}{\sqrt{\lambda}} \left[ \partial \left( \frac{\delta^2 (\xi)}{\partial X} \right) + \bar{\partial} \left( \frac{\delta^2 (\xi)}{\partial X} \right) \right], \] (2.15)
\[ (\partial \bar{\partial} + \bar{\partial} \partial) X = \frac{\pi S}{\sqrt{\lambda}} \left[ \partial \left( \frac{\delta^2 (\xi - \xi_1)}{\partial X} \right) + \bar{\partial} \left( \frac{\delta^2 (\xi - \xi_1)}{\partial X} \right) \right]. \] (2.16)
According to our general discussion, the solution to this system is given by the classical solution (2.6) written in the coordinates \((\xi, \bar{\xi})\). More precisely

\[
\begin{align*}
t_e &= \frac{\kappa}{2}[\ln \xi \bar{\xi} - \ln(\xi - \xi_1)(\bar{\xi} - \bar{\xi}_1)], \\
X &= \frac{\omega}{2i} \left( \frac{\xi}{\xi - \xi_1} - \frac{\bar{\xi}}{\bar{\xi} - \bar{\xi}_1} \right), \\
\bar{X} &= \frac{\omega}{2i} \left( \frac{\bar{\xi} - \bar{\xi}_1}{\bar{\xi}} - \frac{\xi - \xi_1}{\xi} \right).
\end{align*}
\]

(2.17)

Substituting (2.17) into eqs. (2.14)-(2.16) we find that it is indeed a solution\(^1\) with parameters \(\kappa\) and \(\omega\) fixed in terms of the quantum numbers of the operator

\[
\kappa = \frac{E}{\sqrt{\lambda}}, \quad \omega^2 = \frac{S}{\sqrt{\lambda}}.
\]

(2.18)

The two-point function \(\langle V_S(0)V_{-S}(\xi_1) \rangle\) is given by the action evaluated on the solution (2.17). Ignoring the obvious divergence \(\sim \ln(0)\), we obtain

\[
\langle V_S(0)V_{-S}(\xi_1) \rangle \sim |\xi_1|^{E^2/\sqrt{\lambda} - 2S}.
\]

(2.19)

In other words, we get

\[
2\gamma - 4 = \frac{E^2}{\sqrt{\lambda}} - 2S.
\]

(2.20)

Remembering that our analysis is valid only in the regime of large \(E\) and \(S\) (so that \(-4\) can be ignored)\(^2\) we find that the condition \(\gamma = 0\) yields

\[
E = \sqrt{2\sqrt{\lambda}S} = \sqrt{\frac{2}{\alpha'}S}.
\]

(2.21)

This is the same energy-spin relation as for the corresponding classical string solution. Note that eq. (2.21) is the standard Regge trajectory for string states in flat space in the limit of large \(E\) and \(S\).

In the rest of the paper, we apply this consideration for various states in \(AdS_5 \times S^5\). The equations of motion in these cases are very non-linear and, a priori, it is not clear how to solve them. However, as was explained above we can construct the solutions starting with the corresponding classical string solution on the cylinder and performing the conformal transformations (2.1) and (2.2).

\(^1\)To prove that it is a solution we have to use the well-known relations like \(\tilde{\delta}^{12}_\xi = \pi \delta^2(\xi)\).

\(^2\)More precisely, we have to take the limit of large \(E, S, \lambda\) so that \(E, S \sim \sqrt{\lambda}\).
3 Folded Strings in $AdS_3$

As the first example of strings in $AdS_5 \times S^5$ we will consider a folded string spinning in $AdS_3$ [15]. In field theory, this string state corresponds to the twist two operators. We will describe $AdS_5$ in global coordinates in which the metric takes the form

$$ds^2 = d\rho^2 - \cosh^2 \rho dt^2 + \sinh^2 \rho (d\theta^2 + \cos^2 \theta d\phi_1^2 + \sin^2 \theta d\phi_2^2).$$  \hspace{1cm} (3.1)

The relation to the embedding coordinates in $R^{2,4}$ is as follows

$$Y_5 + iY_0 = \cosh \rho e^{it}, \quad Y_1 + iY_2 = \sinh \rho \cos \theta e^{i\phi_1},$$
$$Y_3 + iY_4 = \sinh \rho \sin \theta e^{i\phi_2},$$
$$-Y_0^2 - Y_5^2 + Y_1^2 + Y_2^2 + Y_3^2 + Y_4^2 = -1. \hspace{1cm} (3.2)$$

Since the string is spinning in $AdS_3$ we can set

$$\theta = 0, \quad \phi_2 = 0$$
and denote

$$Y = \cosh \rho e^{it}, \quad \bar{Y} = \cosh \rho e^{-it},$$
$$X = \sinh \rho e^{i\phi_1}, \quad \bar{X} = \sinh \rho e^{-i\phi_1}. \hspace{1cm} (3.4)$$

The action is then given by

$$A = -\frac{\sqrt{\lambda}}{4\pi} \int d^2\sigma (-\cosh^2 \rho \partial_a t \partial^a t + \partial_a \rho \partial^a \rho + \sinh^2 \rho \partial_a \phi_1 \partial^a \phi_1).$$ \hspace{1cm} (3.5)

The folded string solution in the limit of large $E$ and $S$ has the following structure [15]

$$t = \kappa \tau, \quad \phi_1 = \omega \tau, \quad \rho = \mu \sigma. \hspace{1cm} (3.6)$$

The equation of motion for $\rho$ is satisfied only if $^3$

$$\kappa = \omega. \hspace{1cm} (3.7)$$

Furthermore, the equations of motion must be supplemented with the Virasoro constraints. If we denote by $X_M, M = 1, \ldots, 6$ the coordinates on $S^5$ they read

$$\eta^{PQ}(Y_P Y_Q + Y'_P Y'_Q) + \dot{X}_M \dot{X}_M + X'_M X'_M = 0 \hspace{1cm} (3.8)$$

$^3$This relation is valid only in the limit of large $S$. 7
Figure 1: The approximate solution $\rho = \rho(\sigma)$ in the limit of large energy and spin.

and

$$\eta^{PQ} \dot{Y}_P Y_Q' + \dot{X}_M X'_M = 0,$$

where $\eta_{PQ}$ is the metric in $R^{2,4}$,

$$\eta_{PQ} = \text{diag}(-1, -1, 1, 1, 1, 1).$$

In our case the Virasoro constraints yield

$$\kappa = \mu.$$

Note that $\rho$ has to be a periodic function of $\sigma$. Thus, this solution cannot be valid on the whole cylinder $\sigma \in [0, 2\pi]$. In fact, it is valid only on the interval $\sigma \in [0, \pi/2]$. Then it has to be folded and continued by periodicity. The form of the solution in the limit of large energy and spin is depicted in Figure 1. This solution is not smooth and is only approximate in the limit of large $E$ and $S$. The exact solution is also possible to find and it is given in terms of elliptic integrals. We will not need the details of the exact solution in this paper. They can be found in [15, 19, 14].

The energy and the spin of this solution are given by

$$E = \sqrt{\lambda} \int_0^{2\pi} \frac{d\sigma}{2\pi} \cosh^2 \rho \iota = \frac{\sqrt{\lambda}}{2\pi \kappa} \int_0^{2\pi} \frac{d\sigma}{2\pi} \cosh^2 \rho,$$

and

$$S = \sqrt{\lambda} \int_0^{2\pi} \frac{d\sigma}{2\pi} \sinh \rho \phi_1 = \sqrt{\lambda\kappa} \int_0^{2\pi} \frac{d\sigma}{2\pi} \sinh^2 \rho.$$

From these equations one can conclude that $\kappa \sim \ln S$. Therefore, in the limit of large $E$ and $S$ the size of the string $\rho$ becomes very large. In this limit one can eliminate $\kappa$ in eqs. (3.12) and (3.13) to find that $E$ and $S$ are related as follows

$$E = S + \frac{\sqrt{\lambda}}{\pi} \ln \frac{S}{\sqrt{\lambda}} + \ldots,$$

(3.14)
where the ellipsis stands for the subleading correction. The aim of this section is to reproduce this dependence using semiclassical vertex operators. One can find the form of the vertex operator in this case by noticing that in the limit of small $\rho$ the string becomes very small and the curvature of $AdS_3$ can be neglected. Hence, in this limit we have a string spinning in the plane in flat space. In this limit the vertex operator should coincide with (2.9). Thus, the natural expression for the vertex operator for a string spinning in $AdS_3$ is

$$V_S = Y^{-E}(\partial X \bar{\partial} X)^{S/2},$$
$$V_{-S} = \bar{Y}^{-E}(\partial \bar{X} \bar{\partial} \bar{X})^{S/2},$$

(3.15)

where $E$ is the energy and $S$ is the spin of the corresponding state and $X$ and $Y$ are now given by eq. (3.4). We want to compute the two-point function of the operator $V_S$ inserted on the complex plane at $\xi = 0$ and the operator $V_{-S}$ inserted at $\xi = \xi_1$. The Euclidean action on the plane (including the operator insertions) is as follows

$$A_e = \frac{\sqrt{\lambda}}{\pi} \int d^2 \xi (\partial \rho \bar{\partial} \rho + \cosh^2 \rho \partial t_e \bar{\partial} t_e + \sinh^2 \rho \bar{\partial} \phi \bar{\partial} \phi)$$
$$+ E \int d^2 \xi \ln Y(\delta^2(\xi) - \delta^2(\xi - \xi_1))$$
$$- \frac{S}{2} \int d^2 \xi \delta^2(\xi) \ln(\partial X \bar{\partial} X) - \frac{S}{2} \int d^2 \xi \delta^2(\xi - \xi_1) \ln(\partial \bar{X} \bar{\partial} \bar{X}),$$

(3.16)

where we have also Wick rotated the $AdS$ time $t$. As we mentioned above, in the regime of large $E$ and $S$ (in which our semiclassical analysis is valid) $\rho$ becomes very large. In this limit $\cosh \rho \sim \sinh \rho \sim e^\rho$. Then the action becomes

$$A_e = \frac{\sqrt{\lambda}}{\pi} \int d^2 \xi (\partial \rho \bar{\partial} \rho + \cosh^2 \rho \partial t_e \bar{\partial} t_e + \sinh^2 \rho \bar{\partial} \phi \bar{\partial} \phi)$$
$$+ E \int d^2 \xi (\delta^2(\xi) - \delta^2(\xi - \xi_1))t_e - S \int d^2 \xi (\delta^2(\xi) - \delta^2(\xi - \xi_1))i\phi_1$$
$$+ (E - S) \int d^2 \xi (\delta^2(\xi) + \delta^2(\xi - \xi_1))\rho$$
$$- \frac{S}{2} \int d^2 \xi \delta^2(\xi) \ln[\partial(\rho + i\phi_1)] - \frac{S}{2} \int d^2 \xi \delta^2(\xi) \ln[\bar{\partial}(\rho + i\phi_1)]$$
$$- \frac{S}{2} \int d^2 \xi \delta^2(\xi - \xi_1) \ln[\partial(\rho - i\phi_1)] - \frac{S}{2} \int d^2 \xi \delta^2(\xi - \xi_1) \ln[\bar{\partial}(\rho - i\phi_1)].$$

(3.17)

We will start with the equation of motion for $\rho$,

$$\partial \bar{\partial} \rho - \sinh \rho \cosh \rho (\partial t_e \bar{\partial} t_e + \bar{\partial} \phi \bar{\partial} \phi_1) = \frac{\pi(E - S)}{2\sqrt{\lambda}}(\delta^2(\xi) + \delta^2(\xi - \xi_1))$$
$$+ \frac{\pi S}{4\sqrt{\lambda}} \left[ \partial \left( \frac{\delta^2(\xi)}{\partial(\rho + i\phi_1)} \right) + \bar{\partial} \left( \frac{\delta^2(\xi - \xi_1)}{\bar{\partial}(\rho - i\phi_1)} \right) + \partial \rightarrow \bar{\partial} \right].$$

(3.18)
According to our prescription, we expect the solution to be of the form (3.6) written in the coordinates \((\xi, \bar{\xi})\). In particular, we expect \(\phi_1\) to behave as

\[
i\phi_1 = \omega \tau_e = \frac{\omega}{2} [\ln \xi \bar{\xi} - \ln(\xi - \xi_1)(\bar{\xi} - \bar{\xi}_1)].
\]  

(3.19)

So

\[
\partial(i\phi_1) \sim \frac{1}{\xi(\xi - \xi_1)}.
\]  

(3.20)

This implies that

\[
\frac{\delta^2(\xi)}{\partial(\rho + i\phi_1)} \sim \delta(\xi - \xi_1)\delta^2(\xi) = 0, \quad \frac{\delta^2(\xi - \xi_1)}{\partial(\rho + i\phi_1)} \sim \delta(\xi - \xi_1)\delta^2(\xi - \xi_1) = 0.
\]  

(3.21)

Thus, all terms in the second line vanish. Furthermore, we similarly expect that

\[
t_e = \frac{\kappa}{2} [\ln \xi \bar{\xi} - \ln(\xi - \xi_1)(\bar{\xi} - \bar{\xi}_1)].
\]  

(3.22)

Therefore,

\[
\partial_t e \bar{\partial}_t e + \partial \phi_1 \bar{\partial} \phi_1 \sim \frac{(\kappa^2 - \omega^2)}{|\xi||\xi - \xi_1|}.
\]  

(3.23)

If we multiply both sides of (3.18) by \(|\xi||\xi - \xi_1|\) we find that the delta-functional term vanishes and the equation for \(\rho\) becomes non-singular as is expected since the classical solution for \(\rho\) does not develop singularities under the conformal transformation to the \(\xi\)-plane. Since the equation is the same as in the absence of the operators, we find the same solution as before. In the limit of large parameters we get

\[
t_e \approx i\phi_1 = \frac{\kappa}{2} [\ln \xi \bar{\xi} - \ln(\xi - \xi_1)(\bar{\xi} - \bar{\xi}_1)],
\]

\[
\rho = \mu \sigma = \frac{\mu}{2} \left( \ln \frac{\xi}{\xi_1} - \ln \frac{\xi - \xi_1}{\bar{\xi} - \bar{\xi}_1} \right), \quad \sigma \in [0, \pi/2]
\]  

(3.24)

and \(\rho\) is folded as is Figure 1. Note that so far \(\kappa\) and \(\mu\) are unrelated. Now let us consider equations of motion for \(t_e\) and \(\phi_1\). They read

\[
\partial(\cosh^2 \rho \, \bar{\partial} t_e) + \bar{\partial}(\cosh^2 \rho \, \partial t_e) = \frac{\pi E}{\sqrt{\lambda}} (\delta^2(\xi) - \delta^2(\xi - \xi_1))
\]  

(3.25)

and

\[
\partial(\sinh^2 \rho \, \bar{\partial} \phi_1) + \bar{\partial}(\sinh^2 \rho \, \partial \phi_1) = \frac{i\pi S}{\sqrt{\lambda}} (\delta^2(\xi) - \delta^2(\xi - \xi_1)).
\]  

(3.26)

Note that the additional delta-functional terms of the type \(\frac{\delta^2(\xi)}{\partial(\rho + i\phi_1)}\) which also arise in eq. (3.26) vanish by the same arguments as given below eq. (3.19). We have to show that the
expressions in (3.24) indeed solve these two equations. First, we subtract (3.26) from (3.25). Since \( t_e \approx i \phi_1 \) we get

\[
(\partial \bar{\partial} + \bar{\partial} \partial)t_e = \frac{E - S}{\sqrt{\lambda}} \pi (\delta^2(\xi) - \delta^2(\xi - \xi_1)).
\]  

(3.27)

We see that (3.24) is indeed a solution if we set

\[
\kappa = \frac{E - S}{\sqrt{\lambda}}.
\]  

(3.28)

Now we consider eq. (3.26). At this point we can approximate \( \sinh \rho \sim e^\rho \)

\[
\partial(e^{2\rho} \bar{\partial} \phi_1) + \bar{\partial}(e^{2\rho} \partial \phi_1) = -\frac{i\pi S}{\sqrt{\lambda}} (\delta^2(\xi) - \delta^2(\xi - \xi_1)).
\]  

(3.29)

At the first glance it looks problematic that (3.24) can be a solution if \( \rho \) is large and behaves as \( \mu \sigma \). However, our conformal transformations (2.1), (2.2) are well-defined only for function on the cylinder, that is periodic in \( \sigma \). Thus, we have to use the fact that \( \rho \) behaves as \( \mu \sigma \) only for \( \sigma \in [0, \pi/2] \) and really is a periodic function. Periodicity of \( \rho \) means that we can expand it in the Fourier series

\[
e^{2\rho} = (e^{2\rho})_0 + \sum_{n \neq 0} (e^{2\rho})_n e^{i n \sigma}.
\]  

(3.30)

Now we substitute (3.30) into (3.29). We know that away from the singularities \( \xi = 0, \xi = \xi_1 \), eq. (3.29) is the classical equation which is satisfied by (3.24). So all we have to understand is where the singular terms in the left hand side come from. It is straightforward to realize that singular contributions on the left hand side can come only from the zero Fourier component \( (e^{2\rho})_0 \). All the higher Fourier modes do not contribute to the delta-functions. The zero Fourier mode is easy to find:

\[
(e^{2\rho})_0 = \int_0^{2\pi} \frac{d\sigma}{2\pi} e^{2\rho} = 4 \int_0^{\pi/2} \frac{d\sigma}{2\pi} e^{2\mu \sigma} = \frac{1}{\pi \mu} (e^{\pi \mu} - 1).
\]  

(3.31)

If we now substitute (3.31) into (3.29) we find that the logarithmic function \( \phi \) in eq. (3.24) solves (3.29) provided \( \mu \) is related to the spin \( S \) as follows

\[
\mu = \frac{1}{\pi} \ln \frac{S}{\sqrt{\lambda}} + \mathcal{O}(1).
\]  

(3.32)

The \( t_e \) and \( \phi_1 \)- dependent terms in the action are straightforward to compute. Up to the irrelevant divergence \( \sim \ln(0) \) they give

\[
A_e(t_e, \phi) = -\sqrt{\lambda} \kappa^2 \ln |\xi_1|.
\]  

(3.33)
The $\rho$-dependent part of the action,

$$A_e(\rho) = \frac{\sqrt{\lambda}}{\pi} \int d^2\xi \partial \rho \bar{\partial} \rho,$$

(3.34)
is easier to compute if we go back to the $(\tau, \sigma)$ coordinates. In these coordinates $\rho$ is just $\mu \sigma$ and, hence, we obtain

$$A_e(\rho) = \frac{\sqrt{\lambda}}{4\pi} 2\pi \mu^2 \int_{\tau_1}^{\tau_2} d\tau = \frac{\sqrt{\lambda} \mu^2}{2} (\tau_2 - \tau_1),$$

(3.35)

where $\tau_2 \to \infty$ and $\tau_1 \to -\infty$. If we now go to the $\xi, \bar{\xi}$ coordinates we have

$$\tau_2 = \frac{1}{2} \ln \left( \frac{\xi \bar{\xi}}{(\xi - \xi_1)(\bar{\xi} - \bar{\xi}_1)} \right) \big|_{\xi \to \xi_1}$$

(3.36)

and

$$\tau_1 = \frac{1}{2} \ln \left( \frac{\xi \bar{\xi}}{(\xi - \xi_1)(\bar{\xi} - \bar{\xi}_1)} \right) \big|_{\xi \to 0}.$$  

(3.37)

Substituting it into (3.35) and ignoring the obvious divergence $\sim \ln(0)$ we find that

$$A_e(\rho) = \sqrt{\lambda} \mu^2 \ln |\xi_1|.$$  

(3.38)

Combining it with $A_e(t_e, \phi)$ we obtain

$$\langle V_S(0)V_{-S}(\xi_1) \rangle \sim e^{-A_e} \sim |\xi_1|^\sqrt{\lambda(\kappa^2 - \mu^2)}.$$  

(3.39)

Thus, in the limit of large $\lambda$ the marginality condition implies that

$$\kappa = \mu.$$  

(3.40)

Recalling that $\kappa = (E - S)/\sqrt{\lambda}$ and $\mu = \frac{1}{\pi} \ln \frac{\bar{\lambda}}{\sqrt{\lambda}}$ we obtain the following relation between the energy and the spin

$$E - S = \frac{\sqrt{\lambda}}{\pi} \ln \frac{S}{\sqrt{\lambda}} + \text{subleading terms}.$$  

(3.41)

Thus, we have obtained the same energy-spin dependence as for the corresponding classical solution. As was already stated above, our analysis is valid only in the regime of large $E$ and $S$ and, thus, it cannot be used for computing the subleading terms in (3.41). To compute the subleading corrections one has to go beyond the semiclassical approximation and, in particular, include the dependence on the fermions in the vertex operators. This is beyond the scope of the present paper.
4 Strings Spinning in $S^3$

In the last section, we will consider solutions describing strings spinning in $S^3 \subset S^5$. For given energy $E$ and angular momentum $J$ there are two different solutions of this type [20]. Thus, there is the question whether we can write vertex operators distinguishing these two solutions. Before we discuss the vertex operators, let us first set up our notation. We will perform our analysis in global coordinates in which the metric on $S^5$ looks as follows

$$ds^2 = d\gamma^2 + \cos^2 \gamma d\varphi_3^2 + \sin^2 \gamma (d\psi^2 + \cos^2 \psi d\varphi_1^2 + \sin^2 \psi d\varphi_2^2).$$

(4.1)

The embedding coordinates in $R^6$ can then be written as

$$X_1 + iX_2 = \sin \gamma \cos \psi e^{i\varphi_1}, \quad X_3 + iX_4 = \sin \gamma \sin \psi e^{i\varphi_2},$$

$$X_5 + iX_6 = \cos \gamma e^{i\varphi_3},$$

$$X_1^2 + X_2^2 + X_3^2 + X_4^2 + X_5^2 + X_6^2 = 1.$$  

(4.2)

Since string is spinning in $S^3$ we will set for the rest of the section

$$\varphi_3 = 0.$$  

(4.3)

Both $S^3$-solutions are located at $\rho = 0$ and the only non-trivial dependence on the $AdS_5$ coordinates is

$$t = \kappa \tau$$  

(4.4)

which yields the energy

$$E = \sqrt{\lambda \kappa}.$$  

(4.5)

The two solutions are distinguished by the radius of $S^3$. In the first case, the radius of $S^3$ is given by an arbitrary constant $a < 1$. It is possible to take the limit of small $a$ in which this solution reduces to a flat space one. We will refer to this $S^3$ as to the small $S^3$. In the second case, string is spinning in the big $S^3$ of unit radius. In this case the flat space limit does not exist. We will refer to this $S^3$ as to the big $S^3$. Now we will discuss both solutions in turns.

4.1 Strings Spinning in the Small $S^3$

This solution is characterized by

$$\sin \gamma = a < 1.$$  

(4.6)
Equations for $\gamma$, $\psi$, $\varphi_1$ and $\varphi_2$ give
\begin{equation}
\psi = n\sigma, \quad \varphi_1 = \varphi_2 = n\tau. \tag{4.7}
\end{equation}

For simplicity, we will set $n = 1$ and comment on the case $n \neq 1$ later. One can perform an $SO(4)$ rotation and rewrite (4.6), (4.7) in the “chiral” form
\begin{equation}
\sin \gamma = a, \quad \psi = \pi/4, \quad \varphi_1 = \tau + \sigma, \quad \varphi_2 = \tau - \sigma. \tag{4.8}
\end{equation}

It is easy to check that in both cases $\sum_{i=1}^{4} x_i^2 = a^2$ and, thus, these two solutions are indeed related by an $SO(4)$ rotation. Let us make a comment on the solution (4.8). If we substitute $\sin \gamma = a$, $\psi = \pi/4$ into the equations of motion we will find that they are satisfied if
\begin{equation}
\partial_a \varphi_1 \partial^a \varphi_1 = 0, \quad \partial_a \varphi_2 \partial^a \varphi_2 = 0. \tag{4.9}
\end{equation}

This means that we could take both $\varphi_1$ and $\varphi_2$ to be a function of $\tau + \sigma$ or of $\tau - \sigma$. However, the second Virasoro condition (3.9) requires that one of the angles, say $\varphi_1$, depend on $\tau + \sigma$ and the other one on $\tau - \sigma$. This solution carries the equal angular momentum in both $\varphi_1$ and $\varphi_2$ directions given by
\begin{equation}
J_1 = J_2 = J = \frac{\sqrt{\lambda}a^2}{2}. \tag{4.10}
\end{equation}

The Virasoro constraint (3.8) relates the $AdS_5$ and $S^5$ parts of the solution. More precisely, it sets
\begin{equation}
\kappa^2 = 2a^2. \tag{4.11}
\end{equation}

Using eqs. (4.5), (4.10) and (4.11) we obtain the following relation between the energy and the angular momentum
\begin{equation}
E = \sqrt{4J\sqrt{\lambda}}. \tag{4.12}
\end{equation}

This solution has a limit of small $a$ in which the angular momentum and the energy are very small and we get a string spinning in two complex planes in flat space. On the other hand, since $a$ is bounded from above the limit of infinite angular momentum does not exist. This means that this state is dual to a short operator in field theory. In particular, it was argued in [21, 22, 23] that some of these states are dual to members of the Konishi multiplet. See also [24] for a worldsheet approach to studying spectra of short strings.

Since this solution has a flat space limit, it is natural to expect that the corresponding vertex operator has a structure similar to that in flat space. Let us work in the coordinate system in which the classical solution has the “chiral” form (4.8). Since $\varphi_1$ carries only
the "left moving" modes and $\varphi_2$ carries only the "right moving" modes we write the vertex operator in the form

$$V_J = e^{-E_t} (\partial X)^J (\bar{\partial} Y)^J, \quad (4.13)$$

where

$$X = \sin \gamma \cos \psi e^{i\varphi_1}, \quad Y = \sin \gamma \sin \psi e^{i\varphi_2}. \quad (4.14)$$

In (4.13), for simplicity, we have set $\rho = 0$. We want to compute the correlation function

$$\langle V_J(0)V_{-J}(\xi_1) \rangle \sim e^{-A_e} \quad (4.15)$$

and to show that it produces the relation (4.12).

The Euclidean action $A_e$ is given by

$$A_e = \frac{\sqrt{\lambda}}{\pi} \int d^2 \xi (\partial t_e \bar{\partial} t_e + \partial \gamma \bar{\partial} \gamma + \sin^2 \gamma \partial \psi \bar{\partial} \psi + \sin^2 \gamma \cos^2 \psi \partial \varphi_1 \bar{\partial} \varphi_1 + \sin^2 \gamma \sin^2 \psi \partial \varphi_2 \bar{\partial} \varphi_2) + E \int d^2 \xi (\delta^2(\xi) - \delta^2(\xi - \xi_1)) t_e$$

$$- J \int d^2 \xi \delta^2(\xi) \ln(\partial X \bar{\partial} Y) - J \int d^2 \xi \delta^2(\xi - \xi_1) \ln(\partial \bar{X} \bar{\partial} \bar{Y}). \quad (4.16)$$

As in the previous case, the expected semiclassical solution is given by (4.8) rewritten in the coordinates $(\xi, \bar{\xi})$. That is,

$$t_e = \frac{\kappa}{2} [\ln \xi \bar{\xi} - \ln(\xi - \xi_1)(\bar{\xi} - \bar{\xi}_1)],$$

$$\sin \gamma = a, \quad \psi = \pi/4, \quad i\varphi_1 = \ln \xi - \ln(\xi - \xi_1), \quad i\varphi_2 = \ln \bar{\xi} - \ln(\bar{\xi} - \bar{\xi}_1). \quad (4.17)$$

Note that the holomorphic properties of $\varphi_1$ and $\varphi_2$ are encoded in the form of the vertex operator (4.13). Let us first consider the equation of motion for $t_e$. It is the same as in the previous section. We find that it is solved by (4.17) if $\kappa$ is related to energy as follows

$$\kappa = \frac{E}{\sqrt{\lambda}}. \quad (4.18)$$

Computing the corresponding part of the action gives

$$A_e(t_e) = -\frac{E^2}{\sqrt{\lambda}} \ln |\xi|. \quad (4.19)$$

Now we will consider the $S^3$ part of the action. It is straightforward but tedious to show that with our ansatz (4.17) the equations of motion for $\gamma$ and $\psi$ are satisfied provided

$$a^2 = \frac{2J}{\sqrt{\lambda}}. \quad (4.20)$$

\footnote{It is easy to check that the equation of motion for $\rho$ is satisfied even when the vertex operators are inserted.}
In proving this, we used the facts like
\[
\frac{\delta^2(\xi)X}{\partial X} = \frac{\delta^2(\xi)}{\partial \ln X} = \delta^2(\xi)(\xi - \xi_1) = 0 \tag{4.21}
\]
and similarly for \(X\) replaced with \(Y\) and \(\delta^2(\xi)\) replaced with \(\delta^2(\xi - \xi_1)\). Now we consider the equation of motion for \(\varphi_1\). Using eq. (4.21) it can be written as
\[
(\partial \bar{\partial} + \bar{\partial} \partial) i \varphi_1 = \frac{2\pi J}{a^2 \sqrt{\lambda}} (\delta^2(\xi) - \delta^2(\xi - \xi_1)). \tag{4.22}
\]
From here we find the logarithmic solution for \(\varphi_1\) provided eq. (4.20) is satisfied. One can check that the equation of motion for \(\varphi_2\) is analogously satisfied if eq. (4.20) is fulfilled.

Thus, we have found that (4.20) is indeed a solution and the parameters \(\kappa\) and \(a\) are fixed in terms of the quantum numbers of the vertex operator \(E\) and \(J\). Now it is straightforward to compute the \(S^3\) part of the action \(A_e\). Since \(\gamma\) and \(\psi\) are constants and since \(\varphi_1\) and \(\varphi_2\) are (anti)holomorphic the classical action vanishes. The delta-functional terms give
\[
A_e(S^3) = 4J \ln |\xi_1|, \tag{4.23}
\]
where we ignored the obvious divergence \(\sim \ln(0)\) and used the relation between \(a\) and \(J\) (4.20). Combining \(A_e(S^3)\) and \(A_e(t_e)\) we obtain
\[
\langle V_J(0)V_{-J}(\xi_1) \rangle \sim |\xi_1|^{\frac{1}{\sqrt{\lambda}}(E^2 - 4J \sqrt{\lambda})} \tag{4.24}
\]
In the limit of large \(E\) and \(J\) we get the following energy-angular momentum trajectory
\[
E = \sqrt{4J \sqrt{\lambda}} \tag{4.25}
\]
which coincides with (4.12). Note that eq. (4.25) coincides with the flat space leading Regge trajectory. The reason is that the angles \(\gamma\) and \(\psi\) are constants and the consideration is essentially the same as in flat space.

We finish this subsection by noticing that one can generalize the above analysis for the case \(n \neq 1\). The corresponding vertex operator in this case is given by
\[
V_J = e^{-E_t} (\partial^n X)^J (\bar{\partial}^n Y)^J \tag{4.26}
\]
and the energy-angular momentum trajectory is
\[
E = \sqrt{4J n \sqrt{\lambda}}. \tag{4.27}
\]
4.2 Strings Spinning in the Big $S^3$

The solution describing a string spinning in the big $S^3$ is characterized by

$$\sin \gamma = 1, \quad \gamma = \pi/2.$$  \hspace{1cm} (4.28)

Then the equations of motion give

$$\psi = \pi/4, \quad \varphi_1 = \omega \tau + m \sigma, \quad \varphi_2 = \omega \tau - m \sigma.$$  \hspace{1cm} (4.29)

Furthermore, from the Virasoro constraints it follows that

$$\kappa^2 = \omega^2 + m^2.$$  \hspace{1cm} (4.30)

One can check that $\sum_{i=1}^4 X_i = 1$ so the string is spinning in $S^3$ or unit radius. Similarly to the case discussed in the previous subsection, the angular momenta in the $\varphi_1$ and $\varphi_2$ directions are still equal and are given by

$$J_1 = J_2 = J = \frac{\sqrt{\lambda} \omega}{2}.$$  \hspace{1cm} (4.31)

From eqs. (4.5), (4.30) and (4.31) we get the following relation between the energy and the angular momentum

$$E = \sqrt{4J^2 + m^2 \lambda}.$$  \hspace{1cm} (4.32)

Note that since the radius of $S^3$ is fixed to be unity this solution does not have a flat space limit. On the other hand it admits the large $J$ limit as well as the BMN [25] expansion in powers of $\lambda/J^2$. Also note that, in addition to $E$ and $J$, this solution has one more quantum number. This is the winding number $m$. This winding number is topologically trivial and the solution is unstable [20, 26]. Similarly, we expect the corresponding quantum state to be unstable under decay to the BPS state with $m = 0$ and $E = 2J$.

Now we would like to discuss the vertex operator corresponding to this solution. Since this solution does not have a flat space limit the vertex operator must be more subtle than in the previous examples. Moreover, it has to depend not just on $E$ and $J$ but also on $m$. First, let us note that when $m = 0$ we obtain a BPS state with $E = 2J$. The corresponding vertex operator then must be

$$V_{J,m=0} = e^{-Et} X^J Y^J,$$  \hspace{1cm} (4.33)

where $X$ and $Y$ are given by eq. (4.14) and just like in the previous subsection, for simplicity, we set $\rho = 0$. Now we want to add a non-trivial $m$. Though it is topologically trivial it
still has a meaning of winding number. In flat space, to describe states with a non-trivial winding number one has to introduce the T-dual coordinates $\tilde{X}$ or write the vertex operators in terms of the left- and right-moving fields and treat them as the independent variables. We would like to propose that a similar procedure should be performed in this case. One more reason in favor of using the T-dual variables is the following simple observation. The formula for the energy (4.32) is invariant under the T-duality looking transformation

$$J \leftrightarrow \pm \frac{m\sqrt{\lambda}}{2}.$$  \hspace{1cm} (4.34)

Thus, on general grounds, one can expect that it should be possible to perform calculations so that this symmetry is manifest. In particular, if this is the case, it should be possible to write the vertex operators in the form which respects this symmetry. T-duality in $AdS_5 \times S^5$ was studied in [16, 17, 18]. However, in our case, as far as the semiclassical analysis is concerned, the discussion will be much simpler. First, we do not have to worry about the fermions. Second, since $\gamma$ and $\psi$ are constants, we have only the two decoupled field $\varphi_1$ and $\varphi_2$ whose actions are free. Thus, we can perform T-duality as in flat space. Requiring that the vertex operator respects the symmetry (4.34) we write it in the form

$$V_{J,m} = e^{-Et} X^J Y^J e^{\pm \alpha \sqrt{\lambda} \tilde{\varphi}_1} e^{-\pm \alpha \sqrt{\lambda} \tilde{\varphi}_2},$$  \hspace{1cm} (4.35)

where $\tilde{\varphi}_1$ and $\tilde{\varphi}_2$ are the T-dual field. We will see below that this expression is consistent with the classical solution (4.29). In particular, the coefficient in front of $\sigma$ will be exactly $m$ which is not obvious at this stage. We would like to point out that our discussion in this subsection is rather heuristic and that eq. (4.35) is not the full vertex operator but its simplified version. In principle, it can also depend on other $S^5$ coordinates. Fortunately, this is not relevant for our purposes since all the remaining coordinates are constants.

As before, we consider $\langle V_{J,m}(0) V_{-J,-m}(\xi) \rangle \sim e^{-A_e}$, where the Euclidean action $A_e$ is given by

$$A_e = \frac{\sqrt{\lambda}}{\pi} \int d^2 \xi (\partial t_e \bar{t}_e + \partial \gamma \bar{\partial} \gamma + \sin^2 \gamma \partial \psi \bar{\partial} \psi + \sin^2 \gamma \cos^2 \gamma \bar{\partial} \varphi_1 \bar{\partial} \varphi_1 + \sin^2 \gamma \sin^2 \psi \partial \varphi_1 \bar{\partial} \varphi_2) + E \int d^2 \xi (\delta^2(\xi - \xi_1)) t_e - J \int d^2 \xi \delta(\xi) \ln(XY) - J \int d^2 \xi \delta(\xi - \xi_1) \ln(\bar{X}\bar{Y}) - \frac{i\sqrt{\lambda} m}{2} \int d^2 \xi (\delta^2(\xi - \xi_1)(\tilde{\varphi}_1 - \tilde{\varphi}_2)).$$  \hspace{1cm} (4.36)

The part of the action depending on $t_e$ is identical to the previous cases. It leads to the solution

$$t_e = \frac{E}{2\sqrt{\lambda}} [\ln \xi - \ln(\xi - \xi_1)(\xi - \xi_1)]$$  \hspace{1cm} (4.37)
and to the contribution to the action given by

$$A_e(t_e) = -\frac{E^2}{\sqrt{\lambda}} \ln |\xi_1|.$$  (4.38)

Furthermore, the equations for $\gamma$ and $\psi$ are trivially satisfied since $\cos \gamma = 0$ and $\sin \psi = \cos \psi$. Thus, we can concentrate on the action and the operators depending on $\varphi_1$ and $\varphi_2$ only. The relevant classical action is

$$A_e(\varphi_1, \varphi_2) = \frac{\sqrt{\lambda}}{2\pi} \int d^2\xi \left( \partial_1 \bar{\partial}_1 \varphi_1 + \partial_2 \bar{\partial}_2 \varphi_2 \right)$$  (4.39)

and the relevant vertex operator is

$$V_{J,m} = e^{iJ\varphi_1} e^{i\sqrt{m\lambda} \bar{\varphi}_1} e^{iJ\varphi_2} e^{-i\sqrt{m\lambda} \bar{\varphi}_2}.$$  (4.40)

Clearly, it is enough to consider only one field, say $\varphi_1$, since the contribution from the second one is obtained by replacing $m \to -m$. The consideration will be relatively standard. Nevertheless, for completeness we will perform it below. First, we will write the action using both $\varphi_1$ and $\bar{\varphi}_1$. It reads

$$A_e(\varphi_1, \bar{\varphi}_1) = \frac{\sqrt{\lambda}}{8\pi} \int d^2\xi \left[ -(\partial - \bar{\partial})\varphi_1(\partial - \bar{\partial})\varphi_1 - (\partial - \bar{\partial})\bar{\varphi}_1(\partial - \bar{\partial})\bar{\varphi}_1 + 2(\partial + \bar{\partial})\varphi_1(\partial - \bar{\partial})\bar{\varphi}_1 \right].$$  (4.41)

If we find the equation of motion for $\bar{\varphi}_1$ and substitute it back in (4.41) we obtain the action for $\varphi_1$. The equations of motion for $\varphi_1$ and $\bar{\varphi}_1$ are as follows

$$(\partial - \bar{\partial})^2 \varphi_1 - (\partial + \bar{\partial})(\partial - \bar{\partial})\varphi_1 = \frac{4\pi i J}{\sqrt{\lambda}}(\delta^2(\xi) - \delta^2(\xi - \xi_1)),$$  (4.42)

$$(\partial - \bar{\partial})^2 \bar{\varphi}_1 - (\partial + \bar{\partial})(\partial - \bar{\partial})\varphi_1 = 2\pi i m(\delta^2(\xi) - \delta^2(\xi - \xi_1)).$$  (4.43)

It is straightforward to check that these equations are solved by

$$i\varphi_1 = \frac{J}{\sqrt{\lambda}} \left[ \ln \xi \xi - \ln(\xi - \xi_1)(\xi - \xi_1) \right] + \frac{m}{2} \left[ \ln \xi / \xi - \ln(\xi - \xi_1)/(\xi - \xi_1) \right]$$  (4.44)

and

$$i\bar{\varphi}_1 = \frac{m}{2} \left[ \ln \xi \xi - \ln(\xi - \xi_1)(\xi - \xi_1) \right] + \frac{J}{\sqrt{\lambda}} \left[ \ln \xi / \xi - \ln(\xi - \xi_1)/(\xi - \xi_1) \right].$$  (4.45)

\[5\] If the winding part of the vertex operators depends on $\gamma$ and $\psi$ there is an additional contribution to the singular part of the equations of motion. Here we are assuming that they are still satisfied if $\gamma = \pi/2$, $\psi = \pi/4$.

\[6\] Note that the equations of motion by construction must be invariant under $\varphi_1 \leftrightarrow \bar{\varphi}_1$, $J \leftrightarrow m\sqrt{\lambda}/2$. 
If we recall the conformal transformations (2.1) and (2.2) and rewrite $\varphi_1$ in terms of $\tau$ and $\sigma$ we will get precisely the classical solution

$$\varphi_1 = -i\omega \tau + m \sigma,$$

where

$$\omega = \frac{2J}{\sqrt{\lambda}}$$

and the coefficient in front of sigma is precisely $m$. Note that we wrote our vertex operator imposing the symmetry (4.34). However, it turned out to be what is needed to reproduce the classical solution. This can be viewed as a consistency check on our proposed vertex operator.

Evaluating the action on our solution we obtain

$$A_e(\varphi_1) = \frac{1}{\sqrt{\lambda}} \left( J + \frac{\sqrt{\lambda}m}{2} \right)^2 \ln \xi_1 + \frac{1}{\sqrt{\lambda}} \left( J - \frac{\sqrt{\lambda}m}{2} \right)^2 \ln \bar{\xi}_1.$$  

(4.48)

Note that if $m \neq 0$ the coefficients at $\ln \xi_1$ and $\ln \bar{\xi}_1$ are different. This means that $e^{-A_e(\varphi_1)}$ does not behave as $|\xi_1|$ to some power which is not consistent with conformal symmetry. To achieve conformal symmetry we need to add the contribution from the second field $\varphi_2$. Its action is the same as above but with $m \to -m$. Then the action of these two fields is

$$A_e(\varphi_1, \varphi_2) = A_e(\varphi_1) + A_e(\varphi_2) = \frac{4J^2 + m^2 \lambda}{\sqrt{\lambda}} \ln |\xi_1|. $$  

(4.49)

Combining this with $A_e(t_e)$ we get

$$\langle V_{J,m}(0)V_{-J,-m}(\xi_1) \rangle \sim |\xi_1|^\frac{1}{\sqrt{\lambda}(E^2-4J^2-m^2\lambda)},$$

(4.50)

from which we obtain the dependence of the energy on $J$ and $m$ of the form

$$E = \sqrt{4J^2 + m^2 \lambda}.$$  

(4.51)

This coincides with eq. (4.32) and we have an agreement between the proposed semiclassical vertex operator (4.35) and the classical string solution (4.28), (4.29).

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