Approximation of plurifinely plurisubharmonic functions

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Abstract

In this paper, we study the approximation of negative plurifinely plurisubharmonic function defined on a plurifinely domain by an increasing sequence of plurisubharmonic functions defined in Euclidean domains.

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1. Introduction

Approximation is one of the most important tools in analysis. Let $\Omega$ be an Euclidean open set of $\mathbb{C}^n$ and let $u$ be a plurisubharmonic function on $\Omega$. The problem of finding characterizations of $u$ and $\Omega$ such that $u$ can be approximated uniformly on $\overline{\Omega}$ by a sequence of smooth plurisubharmonic functions defined on Euclidean neighborhoods of $\Omega$ is classical. The theorem of Fornæss and Wiegerinck \cite{11} asserts that it is always possible if $\Omega$ is bounded domain with $C^1$-boundary and $u$ is continuous on $\overline{\Omega}$. Recently, Avelin, Hed and Persson \cite{3} extended this result to domains with boundaries locally given by graphs of continuous functions. Therefore, it makes sense not only to ask for which domains $\Omega$ such an approximation is possible, but to ask for a characterization of those plurisubharmonic functions $u$ that can be monotonically approximated from outside. According to the results by \cite{5}, \cite{6}, \cite{9}, \cite{15} and the third author, this is possible if the domain $\Omega$ has the $\mathcal{F}$-approximation property and $u$ belongs to the Cegrell’s classes in $\Omega$.

The aim of this paper is to study approximation of $\mathcal{F}$-plurisubharmonic function. More specifically, let $u$ be a negative $\mathcal{F}$-plurisubharmonic function in $\mathcal{F}$-domain $\Omega$. We concern with sufficient conditions on $u$ and $\Omega$ such that $u$ can be approximated by an increasing sequence of plurisubharmonic functions defined on Euclidean neighborhoods of $\Omega$. It is not surprising that we need some kind of $\Omega$ and $u$ in analogy with the set up to make the approximation possible. Namely, we prove the following.

Theorem 1.1. Let $\Omega$ be a bounded $\mathcal{F}$-hyperconvex domain and let $\{\Omega_j\}$ be a decreasing sequence of bounded hyperconvex domains such that $\Omega \subset \Omega_{j+1} \subset \Omega_j$, for all $j \geq 1$. Assume that there exists $p \in \mathcal{E}_0(\Omega)$, $\rho_j \in \text{PSH}^{-}(\Omega_j)$ with $\rho_j \not\rightarrow p$ a.e. in $\Omega$. Then, for every $p > 0$ and for every $u \in \mathcal{F}_p(\Omega)$, there exists an increasing sequence of functions $u_j \in \text{PSH}^{-}(\Omega_j)$ such that $u_j \rightarrow u$ a.e. in $\Omega$.

The paper is organized as follows. In section 2 we recall some notions of (plurifine) pluripotential theory. In Section 3, we give the definition of bounded $\mathcal{F}$-hyperconvex domain $\Omega$ and the class $\mathcal{E}_0(\Omega)$ which is similar as the class introduced in \cite{7} for the case is a bounded hyperconvex domain. In Section 4, we introduce and investigate the class $\mathcal{F}_p(\Omega)$, $p > 0$. Section 5 is devoted to prove the theorem 1.1.

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2. Preliminaries

Some elements of pluripotential theory (plurifine potential theory) that will be used throughout the paper can be found in [1]–[7]. Let Ω be a Euclidean open set of \( \mathbb{C}^n \). We denote by \( PS^*H^*(\Omega) \) the family of negative plurisubharmonic functions in \( \Omega \). The plurifine topology \( \mathcal{F} \) on \( \Omega \) is the smallest topology that makes all plurisubharmonic functions on \( \Omega \) continuous. Notions pertaining to the plurifine topology are indicated with the prefix \( \mathcal{F} \) to distinguish them from notions pertaining to the Euclidean topology on \( \mathbb{C}^n \). Moreover, since \( \Omega \) is complete.

Proposition 2.1. Let \( \Omega \) be an \( \mathcal{F} \)-open set in \( \mathbb{C}^n \) and let \( u \in \mathcal{F} \)-\( PS^*H^*(\Omega) \). Assume that \( \chi : \mathbb{R}^- \to \mathbb{R}^- \) is increasing convex function. Then, \( \chi \circ u \in \mathcal{F} \)-\( PS^*H^*(\Omega) \).

Proof. By Theorem 2.17 in [24], there exists a \( \mathcal{F} \)-closed, pluripolar set \( E \subset \Omega \) such that for every \( z \in \Omega \setminus E \), there is an \( \mathcal{F} \)-open set \( O_z \subset \Omega \) and a decreasing sequence of plurisubharmonic functions \( \{\varphi_j\} \) defined in Euclidean open neighborhoods of \( O_z \) such that \( \varphi_j \searrow u \) on \( O_z \). Since \( \chi \circ \varphi_j \) is plurisubharmonic functions in Euclidean open neighborhoods of \( O_z \) and \( \chi \circ \varphi_j \searrow \chi \circ u \) on \( O_z \), by Theorem 3.9 in [22] we have \( \chi \circ u \in \mathcal{F} \)-\( PS^*H^*(\Omega) \). Therefore, \( \chi \circ u \in \mathcal{F} \)-\( PS^*H^*(\Omega \setminus E) \). Moreover, since \( \chi \circ u \) is \( \mathcal{F} \)-continuous on \( \Omega \), by Theorem 3.7 in [22] it implies that \( \chi \circ u \in \mathcal{F} \)-\( PS^*H^*(\Omega) \). The proof is complete.

Proposition 2.2. Let \( \Omega \) be an \( \mathcal{F} \)-open set in \( \mathbb{C}^n \) and let \( \varphi \) be strictly plurisubharmonic function in \( \mathbb{C}^n \). Assume that \( u, v \in \mathcal{F} \)-\( PS^*H^*(\Omega) \) such that
\[
\int_{\Omega \setminus \{0 < d\varphi \leq 1\}} (dd^c \varphi)^n = 0.
\]
Then, \( u \geq v \) on \( \Omega \).

Proof. Let \( z \in \Omega \cap \{ u > -\infty \} \) and \( \lambda > 0 \) with \( u(z) > -\lambda \). Choose \( r > 0 \) and \( \psi \in PS^*H^*(B(z, r)) \) such that \( \psi(z) > -\frac{1}{r} \) and \( B(z, r) \cap \{ \psi > -1 \} \subset \Omega \). Put
\[
f := \begin{cases} 
\max(-4\lambda, u + 4\lambda \psi) & \text{in } \Omega \\
-4\lambda & \text{in } B(z, r) \setminus \Omega
\end{cases}
\]
and
\[
g := \begin{cases} 
\max(-4\lambda, v + 4\lambda \psi) & \text{in } \Omega \\
-4\lambda & \text{in } B(z, r) \setminus \Omega
\end{cases}
\]
By Proposition 2.3 in [23] and Proposition 2.14 in [22] it follows that \( f, g \in PS^*H^*(B(z, r)) \). From the hypotheses we have
\[
0 \leq \int_{B(z, r) \setminus \{0 < d\psi \leq \epsilon\}} (dd^c \psi)^n \leq \int_{\Omega \setminus \{0 < d\varphi \leq 1\}} (dd^c \varphi)^n \leq 0.
\]
This implies that \( f \geq g \) in \( B(z, r) \). Hence, \( u(z) \geq v(z) \), and therefore, \( u \geq v \) in \( \Omega \cap \{ u > -\infty \} \). Since \( u, v \) are \( \mathcal{F} \)-continuous, by Theorem 3.7 in [22] we obtain that \( u \geq v \) in \( \Omega \). The proof is complete.

Definition 2.3. Let \( \Omega \) be an \( \mathcal{F} \)-open set in \( \mathbb{C}^n \) and let \( QB(\Omega) \) be the trace of \( QB(\mathbb{C}^n) \) on \( \Omega \), where \( QB(\mathbb{C}^n) \) is denoted the measurable space on \( \mathbb{C}^n \) generated by the Borel sets and the pluripolar subsets of \( \mathbb{C}^n \). Assume that \( u_1, \ldots, u_n \in \mathcal{F} \)-\( PS^*H(\Omega) \) be finite. Using the quasi-Lindelöf property of plurifine topology and Theorem 2.17 in [24], there exist a pluripolar set \( E \subset \Omega \), a sequence of \( \mathcal{F} \)-open subsets \( \{O_k\} \) and the plurisubharmonic functions \( f_{j,k}, g_{j,k} \) defined in Euclidean neighborhoods of \( O_k \) such that \( \Omega = E \cup \bigcup_{k=1}^{\infty} O_k \) and \( u_j = f_{j,k} - g_{j,k} \) on \( O_k \). We define \( O_0 := \emptyset \) and
\[
\int_A dd^c u_1 \wedge \ldots \wedge dd^c u_n := \sum_{j=1}^{\infty} \int_{A \cap (O_j \cup \bigcup_{k=0}^{j-1} O_k)} dd^c(f_{j,k} - g_{j,k}) \wedge \ldots \wedge dd^c(f_{j,k} - g_{j,k}), \quad A \in QB(\Omega). \tag{2.1}
\]
By Theorem 3.6 in [24], the measure defined by (2.1) is independent on \( E, \{O_k\}, \{f_{j,k}\} \) and \( \{g_{j,k}\} \). This measure is called the complex Monge-Ampère measure.
Proposition 2.4. Let \( \Omega \) be an \( \mathcal{F} \)-open set in \( \mathbb{C}^n \) and let \( u_1, \ldots, u_n \in \mathcal{F} \)-PS H(\( \Omega \)) be finite. Then, \( dd^c u_1 \wedge \ldots \wedge dd^c u_n \) is non-negative measure on \( QB(\Omega) \).

Proof. The statement follows from [4], Theorem 2.17 in [24] and Lemma 4.1 in [24]. \( \square \)

Proposition 2.5. Let \( \Omega \) be an \( \mathcal{F} \)-open set in \( \mathbb{C}^n \) and let \( \mu \) be non-negative measure on \( QB(\Omega) \). Assume that \( u, v \in \mathcal{F} \)-PS H(\( \Omega \)) are finite such that \( (dd^c u)^n \geq \mu \) and \( (dd^c v)^n \geq \mu \) in \( \Omega \). Then \( (dd^c \max(u, v))^n \geq \mu \) in \( \Omega \).

Proof. Put \( v_j \coloneqq \max(u, v - \frac{1}{j}) \), where \( j \in \mathbb{N}^* \). By Theorem 4.8 in [24] we have

\[
(dd^c v_j)^n \geq 1_{[\mu \geq 0]}(dd^c u)^n + 1_{[\mu \geq \frac{1}{j}]}(dd^c v)^n \geq 1_{[\mu \geq \frac{1}{j}]} \mu.
\]

Since \( v_j \not\to \max(u, v) \) on \( \Omega \), by Theorem 4.5 in [23] we obtain \( (dd^c \max(u, v))^n \geq \mu \) in \( \Omega \). The proof is complete. \( \square \)

Proposition 2.6. Let \( \Omega \) be an \( \mathcal{F} \)-open set in \( \mathbb{C}^n \) and let \( u \in \mathcal{F} \)-PS H(\( \Omega \)) be finite. Assume that \( \{u_j\} \) is a monotone sequence of negative, finite, \( \mathcal{F} \)-plurisubharmonic functions such that \( u_j \to u \) a.e. on \( \Omega \). Then

\[
\int_{\Omega} f(dd^c u)^n \leq \liminf_{j \to +\infty} \int_{\Omega} f(dd^c u_j)^n,
\]

for every non-negative, bounded, \( \mathcal{F} \)-continuous function \( f \) on \( \Omega \).

Proof. From Theorem 3.9 in [22], there exists a \( \mathcal{F} \)-closed, pluripolar set \( E \subset \Omega \) such that \( u_j \to u \) on \( \Omega \backslash E \). By Theorem 4.5 in [23] we have the sequence of measures \( (dd^c u_j)^n \) converges \( \mathcal{F} \)-locally vaguely to \( (dd^c u)^n \) on \( \Omega \backslash E \). Using the quasi-Lindelöf property of plurisubharmonic topology, there exist a pluripolar set \( F \subset \Omega \backslash E \), a sequence of \( \mathcal{F} \)-open subsets \( \{O_k\} \) and non-negative \( \mathcal{F} \)-continuous functions \( \chi_k \) in \( \mathbb{C}^n \) with compact support on \( O_k \) such that \( \Omega \backslash E = F \cup \bigcup_{k=1}^{\infty} O_k \), \( 0 \leq \chi_k \leq 1 \), \( \sum_{k=1}^{\infty} \chi_k = 1 \) on \( \Omega \backslash (E \cup F) \) and

\[
\int_{O_k} f\chi_k(dd^c u)^n = \lim_{j \to +\infty} \int_{O_k} f\chi_k(dd^c u_j)^n, \text{ for all } k \geq 1.
\]

It follows that

\[
\int_{\Omega} f(dd^c u)^n = \int_{\bigcup_{k=1}^{\infty} O_k} f(dd^c u)^n = \sup_{k \geq 1} \sum_{j=1}^{k} \int_{O_k} f\chi_k(dd^c u)^n
\]

\[
= \sup_{k \geq 1} \lim_{j \to +\infty} \int_{\Omega} f\left( \sum_{k=1}^{j} \chi_k \right)(dd^c u)^n \leq \liminf_{j \to +\infty} \int_{\Omega} f(dd^c u_j)^n.
\]

The proof is complete. \( \square \)

3. The class \( E_0(\Omega) \)

Definition 3.1. Let \( \Omega \) be bounded \( \mathcal{F} \)-domain in \( \mathbb{C}^n \). Then, \( \Omega \) is called \( \mathcal{F} \)-hyperconvex if there exist a negative bounded plurisubharmonic function \( \gamma_\Omega \) defined in a bounded hyperconvex domain \( \Omega' \) such that \( \Omega = \Omega' \cap \{\gamma_\Omega > -1\} \) and \( -\gamma_\Omega \) is \( \mathcal{F} \)-plurisubharmonic in \( \Omega \).

We say that a bounded negative \( \mathcal{F} \)-plurisubharmonic function \( u \) defined on bounded \( \mathcal{F} \)-hyperconvex domain \( \Omega \) belongs to \( E_0(\Omega) \) if \( \int_{\Omega} (dd^c u)^n < +\infty \) and satisfy for every \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that \( \Omega \cap \{u < -\varepsilon\} \subset \Omega \cap \{\gamma_\Omega > -1 + \delta\} \).

Remark 3.2. If \( \Omega \) is bounded hyperconvex domain then it is \( \mathcal{F} \)-hyperconvex. Moreover, there exists a bounded \( \mathcal{F} \)-hyperconvex domain that has no Euclidean interior point.

Proposition 3.3. Let \( \Omega \) be a bounded \( \mathcal{F} \)-hyperconvex domain in \( \mathbb{C}^n \). Then \( E_0(\Omega) \neq \emptyset \).
Proof. Let $\Omega'$ be a bounded hyperconvex domain in $\mathbb{C}^n$ and let $\gamma_\Omega \in PS H^-(\Omega') \cap L^\infty(\Omega')$ such that $\Omega = \Omega' \cap \{\gamma_\Omega > -1\}$ and $-\gamma_\Omega \in \mathcal{F}^{-}PS H(\Omega)$. Let $\psi \in E_0(\Omega') \cap C(\Omega')$ such that $-1 \leq \psi < 0$ in $\Omega'$. Choose $\varepsilon_0 > 0$ such that

$$G := (\psi < -2\varepsilon_0) \cap \{\gamma_\Omega > -1 + 2\varepsilon_0\} \neq \emptyset.$$

We define

$$\rho := \sup \{\psi \in \mathcal{F}^{-}PS H^-(\Omega) : \psi \leq \max(-1 - \gamma_\Omega, \psi) \text{ on } G\}.$$

Since $\max(-1 - \gamma_\Omega, \psi) \in \mathcal{F}^{-}PS H^-(\Omega)$ and $G$ is $\mathcal{F}^{-}$-open set, we have $\rho \in \mathcal{F}^{-}PS H^-(\Omega)$. Let $\varepsilon > 0$. Choose $\delta \in (0, \varepsilon)$. Because

$$-1 \leq \max(-1 - \gamma_\Omega, \psi) \leq \rho < 0 \text{ in } \Omega,$$

and $\gamma_\Omega$ is upper semi-continuous on $\Omega'$, it follows that

$$\frac{\rho}{\varepsilon} \in \Omega' \cap \{\gamma_\Omega > -1 + \delta\}.$$

It remains to prove that $\int_\Omega (dd^c \rho)^n < +\infty$. Put

$$u := \begin{cases} \max\left(-\frac{1}{\varepsilon_0}, \rho + \frac{\rho}{\varepsilon_0} \gamma_\Omega\right) & \text{in } \Omega; \\ -\frac{1}{\varepsilon_0} & \text{in } \Omega' \setminus \Omega. \end{cases}$$

From Proposition 2.3 in [23] and Proposition 2.14 in [22] we get $u \in PS H(\Omega')$. By Proposition 3.2 in [23] we have $\rho$ is $\mathcal{F}^{-}$-maximal in $\{\psi > -2\varepsilon_0\} \cup \{-1 < \gamma_\Omega < -1 + 2\varepsilon_0\}$. Moreover, since $\rho = u - \frac{1}{\varepsilon_0} \gamma_\Omega$ in $\{\gamma_\Omega > -1 + \varepsilon_0\}$ and $\{\psi < -\varepsilon_0\} \subseteq \Omega'$, by Theorem 4.8 in [23] it follows that

$$\int_\Omega (dd^c \rho)^n = \int_{\Omega' \cap \{\psi < -\varepsilon_0\} \cup \{\gamma_\Omega > -1 + \varepsilon_0\}} (dd^c \rho)^n$$

$$\leq \int_{\Omega' \cap \{\psi < -\varepsilon_0\}} (dd^c (u - \frac{1}{\varepsilon_0} \gamma_\Omega))^n < +\infty$$

(because $\Omega' \cap \{\psi < -\varepsilon_0\} \subseteq \Omega'$). Therefore, $\rho \in E_0(\Omega)$, and hence, $E_0(\Omega) \neq \emptyset$. The proof is complete.

**Proposition 3.4.** Let $\Omega$ be a bounded $\mathcal{F}^{-}$-hyperconvex domain in $\mathbb{C}^n$. Assume that $u \in E_0(\Omega)$ and $v \in \mathcal{F}^{-}PS H(\Omega)$ such that $u \leq v < 0$ in $\Omega$. Then, $v \in E_0(\Omega)$ and

$$\int_\Omega (-\rho)(dd^c v)^n \leq \int_\Omega (-\rho)(dd^c u)^n,$$

for every $\rho \in \mathcal{F}^{-}PS H^-(\Omega) \cap L^\infty(\Omega)$. Moreover, if $u = v$ in $[u > -\varepsilon_0]$ for some $\varepsilon_0 > 0$ then

$$\int_\Omega (dd^c v)^n = \int_\Omega (dd^c u)^n.$$

**Proof.** Let $\Omega'$ be a bounded hyperconvex domain in $\mathbb{C}^n$ and let $\gamma_\Omega \in PS H^-(\Omega') \cap L^\infty(\Omega')$ such that $\Omega = \Omega' \cap \{\gamma_\Omega > -1\}$ and $-\gamma_\Omega \in \mathcal{F}^{-}PS H(\Omega)$. Fix $\varepsilon > 0$. Choose $\delta > 0$ such that

$$\Omega \cap \{u < -\varepsilon\} \subseteq \Omega' \cap \{\gamma_\Omega > -1 + \delta\}.$$

Since $u \leq v < 0$ in $\Omega$, we get

$$\Omega \cap \{v < -\varepsilon\} \subseteq \Omega \cap \{u < -\varepsilon\} \subseteq \Omega' \cap \{\gamma_\Omega > -1 + \delta\},$$

and hence,

$$\int_\Omega (dd^c v)^n \leq \int_\Omega (dd^c u)^n.$$
It remains to prove that
\[ \int_{\Omega} (-\rho)(dd^c v)^n \leq \int_{\Omega} (-\rho)(dd^c u)^n, \]
for every \( \rho \in F^{PS H^c}(\Omega) \cap L^\infty(\Omega)\). We consider two cases follows.

Case 1. \( u = v \) in \( \Omega \cap \{ u > -\varepsilon_0 \} \) for some \( \varepsilon_0 > 0 \). Let \( \psi \in E_0(\Omega') \cap C(\Omega') \). Choose \( \delta_0 > 0 \) such that
\[ \Omega \cap \{ \psi > -2\delta_0 \} \cup \{ \gamma_\Omega < -1 + 2\delta_0 \} \subset \Omega \cap \{ u > -\varepsilon_0 \}. \]
Without loss of generality we can assume that \(-1 \leq u \leq v < 0 \) and \(-1 \leq \rho \leq 0 \) in \( \Omega \). Put
\[ f := \begin{cases} \max(-\frac{1}{\varepsilon_0}, u + \frac{1}{\varepsilon_0}\gamma_\Omega) & \text{in } \Omega \\ \max(-\frac{1}{\varepsilon_0}, v + \frac{1}{\varepsilon_0}\gamma_\Omega) & \text{in } \Omega' \setminus \Omega \end{cases}, \]
and
\[ \varphi := \begin{cases} \max(-\frac{1}{\varepsilon_0}, \rho + \frac{1}{\varepsilon_0}\gamma_\Omega) & \text{in } \Omega \\ \max(-\frac{1}{\varepsilon_0}, \rho) & \text{in } \Omega' \setminus \Omega \end{cases}. \]
From Proposition 2.3 in [23] and Proposition 2.14 in [22] we get \( f, g, \varphi \in PS H(\Omega') \). By Theorem 4.8 in [24] we have
\[ (dd^c u)^n = (dd^c v)^n \text{ in } \Omega \cap \{ \gamma_\Omega < -1 + 2\delta_0 \}. \]
Since \( \rho = h - \frac{1}{\varepsilon_0}\gamma_\Omega \), \( u = f - \frac{1}{\varepsilon_0}\gamma_\Omega \), \( v = g - \frac{1}{\varepsilon_0}\gamma_\Omega \) in \( \{ \gamma_\Omega > -1 + \delta_0 \} \) and \( f = g \) in \( \{ \psi > -2\delta_0 \} \cup \{ \gamma_\Omega < -1 + 2\delta_0 \} \), by integration by parts yields
\[ \int_{\Omega} \rho((dd^c v)^n - (dd^c u)^n) = \int_{\{ \gamma_\Omega > -1 + \delta_0 \}} \rho((dd^c v)^n - (dd^c u)^n) \]
\[ = \int_{\{ \gamma_\Omega > -1 + \delta_0 \}} (\varphi - \frac{1}{\delta_0}\gamma_\Omega)(dd^c(g - \frac{1}{\delta_0}\gamma_\Omega))^n - (dd^c(f - \frac{1}{\delta_0}\gamma_\Omega))^n \]
\[ = \int_{\Omega} (\varphi - \frac{1}{\delta_0}\gamma_\Omega)(dd^c(g - \frac{1}{\delta_0}\gamma_\Omega))^n - (dd^c(f - \frac{1}{\delta_0}\gamma_\Omega))^n \]
\[ = \int_{\Omega} (\varphi - \frac{1}{\delta_0}\gamma_\Omega)dd^c(g - f) \wedge \sum_{j=0}^{n-1} (dd^c(f - \frac{1}{\delta_0}\gamma_\Omega))^j \wedge (dd^c(g - \frac{1}{\delta_0}\gamma_\Omega))^{n-1-j} \]
\[ = \int_{\Omega} (\varphi - \frac{1}{\delta_0}\gamma_\Omega)dd^c(g - f) \wedge \sum_{j=0}^{n-1} (dd^c(f - \frac{1}{\delta_0}\gamma_\Omega))^j \wedge (dd^c(g - \frac{1}{\delta_0}\gamma_\Omega))^{n-1-j} \]
\[ = \int_{\Omega} (v - u)dd^c\rho \wedge \sum_{j=0}^{n-1} (dd^c u)^j \wedge (dd^c v)^{n-1-j} \geq 0. \]
This follows that
\[ \int_{\Omega} (dd^c u)^n = \int_{\Omega} (dd^c v)^n \]
and
\[ \int_{\Omega} (-\rho)(dd^c v)^n \leq \int_{\Omega} (-\rho)(dd^c u)^n. \]

Case 2. The general case. Fix \( \lambda \in (0, 1) \) and define
\[ v_j = \max(u, \lambda v - \frac{1}{j}), \text{ where } j \in \mathbb{N}^+. \]
Since \( u = v_j \) in \( \{ u > -\frac{1}{j} \} \), by the case 1 and Theorem 4.8 in [24] we get
\[ \int_{\Omega} (-\rho)(dd^c u)^n \geq \int_{\Omega} (-\rho)(dd^c v_j)^n \]
\[
\geq \int_{\{u < v\}} (-p)(dd^c v)^{p} = \mathcal{A}^{p} \int_{\{u < \frac{1}{4}\}} (-p)(dd^c v)^{p}.
\]

It follows that
\[
\int_{\Omega} (-p)(dd^c u)^{p} \geq \sup_{\lambda \in (0,1)} \left[ \mathcal{A}^{p} \sup_{j \geq 1} \int_{\{u < \lambda\}} (-p)(dd^c v)^{p} \right]
\]
\[
= \sup_{\lambda \in (0,1)} \left[ \mathcal{A}^{p} \int_{\Omega} (-p)(dd^c v)^{p} \right] = \int_{\Omega} (-p)(dd^c v)^{p}.
\]

The proof is complete. \(\square\)

4. The class \(\mathcal{F}_p(\Omega)\)

**Definition 4.1.** Let \(\Omega\) be a bounded \(\mathcal{F}\)-hyperconvex domain in \(\mathbb{C}^n\) and let \(p > 0\). Denote by \(\mathcal{F}_p(\Omega)\) is the family of negative \(\mathcal{F}\)-plurisubharmonic functions \(u\) defined on \(\Omega\) such that there exist a decreasing sequence \(\{u_j\} \subset \mathcal{E}_0(\Omega)\) that converges pointwise to \(u\) on \(\Omega\) and
\[
\sup_{j \geq 1} \int_{\Omega} (1 + (-u_j)^p)(dd^c u_j)^p < +\infty.
\]

**Remark 4.2.** If \(u \in \mathcal{F}_p(\Omega)\) then \(u \in \mathcal{F}_q(\Omega)\) for all \(q \in (0, p)\).

**Proposition 4.3.** Let \(\Omega\) be a bounded \(\mathcal{F}\)-hyperconvex domain in \(\mathbb{C}^n\) and let \(p > 0\). Assume that \(u \in \mathcal{F}_p(\Omega)\) and \(\{u_j\} \subset \mathcal{E}_0(\Omega)\) such that \(u_j \searrow u\) on \(\Omega\) and
\[
\sup_{j \geq 1} \int_{\Omega} (1 + (-u_j)^p)(dd^c u_j)^p < +\infty.
\]

Then,
\[
\int_{\{u > -\infty\}} (dd^c u)^p = \sup_{j \geq 1} \int_{\Omega} (dd^c u_j)^p.
\]

Moreover, if \(u\) is bounded then
\[
\int_{\Omega} (-v)(dd^c u)^p = \sup_{j \geq 1} \int_{\Omega} (-v)(dd^c u_j)^p,
\]
for every \(v \in \mathcal{F}\cdot PS H^*(\Omega) \cap L^\infty(\Omega)\).

**Proof.** We consider two cases.

Case 1. \(u\) is bounded. First, we claim that if \(v \in \mathcal{E}_0(\Omega)\) then
\[
\int_{\Omega} (-v)(dd^c u)^p = \sup_{j \geq 1} \int_{\Omega} (-v)(dd^c u_j)^p. \tag{4.1}
\]

Indeed, without loss of generality we can assume that \(-1 \leq u \leq u_j < 0\) in \(\Omega\). Let \(\Omega'\) be a bounded hyperconvex domain in \(\mathbb{C}^n\) and let \(\gamma \in PS H^*(\Omega') \cap L^\infty(\Omega')\) such that \(\Omega = \Omega' \cap \{\gamma > -1\}\) and \(-\gamma \in \mathcal{F}\cdot PS H(\Omega)\). Let \(\{\delta_k\}\) be a decreasing sequence of positive real numbers such that \(\delta_k \searrow 0\) and
\[
\Omega \cap \{v < -\frac{2}{k}\} \subset \Omega' \cap \{\gamma > -1 + 2\delta_k\} \text{ for all } k \geq 1.
\]

Define
\[
\chi_k := \begin{cases} 
\max(-k(v - 1), \frac{1}{\delta_k}(1 + \gamma), 0) & \text{in } \Omega; \\
0 & \text{in } \mathbb{C}^n \setminus \Omega.
\end{cases}
\]

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It is clear that $\chi_k$ is $\mathcal{F}$-continuous function with compact support on $\Omega'$. Fix $k \geq 1$. Put

$$f := \begin{cases} 
\max(-\frac{1}{\delta_k} u + \frac{1}{\delta_k} \gamma \Omega) & \text{in } \Omega \\
-\frac{1}{\delta_k} u & \text{in } \Omega' \setminus \Omega
\end{cases}$$

and

$$f_j := \begin{cases} 
\max(-\frac{1}{\delta_k} u_j + \frac{1}{\delta_k} \gamma \Omega) & \text{in } \Omega \\
-\frac{1}{\delta_k} u_j & \text{in } \Omega' \setminus \Omega
\end{cases}$$

By Proposition 2.3 in [23] and Proposition 2.14 in [22] it follows that $f, f_j \in PSH^-(\Omega') \cap L^\infty(\Omega')$. Since $u = f - \frac{1}{\delta_k} \gamma \Omega$, $u_j = f_j - \frac{1}{\delta_k} \gamma \Omega$ in $\{\gamma \Omega > -1 + \delta_k\}$ and $\{\gamma \Omega \neq 0\} \subset \{\gamma \Omega > -1 + \delta_k\}$, by [4] we get

$$\int_{\Omega} \chi_k(-v)(dd^c u)^n = \int_{\Omega'} \chi_k(-v)(dd^c(f - \frac{1}{\delta_k} \gamma \Omega))^n$$

$$= \lim_{j \to +\infty} \int_{\Omega} \chi_k(-v)(dd^c(f_j - \frac{1}{\delta_k} \gamma \Omega))^n$$

$$= \lim_{j \to +\infty} \int_{\Omega} \chi_k(-v)(dd^c u_j)^n.$$

Moreover, since $\{\gamma \Omega \neq 1\} \subset \{v \geq -\frac{1}{2}\}$, we get

$$\limsup_{j \to +\infty} \int_{\Omega} (-v)(dd^c u_j)^n \geq \int_{\Omega} \chi_k(-v)(dd^c u)^n$$

$$\geq \liminf_{j \to +\infty} \int_{\Omega} (-v)(dd^c u_j)^n - \limsup_{j \to +\infty} \int_{\Omega} (1 - \chi_k)(-v)(dd^c u_j)^n$$

$$\geq \liminf_{j \to +\infty} \int_{\Omega} (-v)(dd^c u_j)^n - \limsup_{j \to +\infty} \int_{\{\gamma \Omega \geq -\frac{1}{2}\}} (-v)(dd^c u_j)^n$$

$$\geq \liminf_{j \to +\infty} \int_{\Omega} (-v)(dd^c u_j)^n - \frac{2}{k} \sup_{j \geq 1} \int_{\Omega} (dd^c u_j)^n.$$

Let $k \not\to +\infty$, by Proposition 3.4 we obtain that

$$\int_{\Omega} (-v)(dd^c u)^n = \lim_{j \to +\infty} \int_{\Omega} (-v)(dd^c u_j)^n = \sup_{j \geq 1} \int_{\Omega} (-v)(dd^c u_j)^n. $$

This proves the claim. Now, fix $\rho \in \mathcal{E}_0(\Omega)$ and define $v_k := \max(v, k \rho)$, where $k \in \mathbb{N}^+$. By Proposition 3.4 it implies that $v_k \in \mathcal{E}_0(\Omega)$. Hence, by [4.1] and Proposition 3.4 we get

$$\int_{\Omega} (-v)(dd^c u)^n = \sup_{k \geq 1} \int_{\Omega} (\chi_k(dd^c u)^n)$$

$$= \sup_{k \geq 1} \left[ \sup_{j \geq 1} \int_{\Omega} (v_k(dd^c u_j)^n) \right] = \sup_{j \geq 1} \int_{\Omega} (-v)(dd^c u_j)^n. $$

**Case 2.** The general case. Let $k \in \mathbb{N}^+$. Since $u_j \leq \max(u_j, -k) < 0$ in $\Omega$, by Proposition 3.4 we have $\max(u_j, -k) \in \mathcal{E}_0(\Omega)$ and

$$\sup_{j \geq 1} \int_{\Omega} [1 + \max(u_j, -k)](dd^c \max(u_j, -k))^n$$

$$\leq (1 + k\rho) \sup_{j \geq 1} \int_{\Omega} (dd^c u_j)^n < +\infty.$$
Therefore, \( \max(u, -k) \in \mathcal{F}_p(\Omega) \). Hence, by (4.2) and Proposition 3.4 we get
\[
\int_{\Omega} (dd^c \max(u, -k))^n = \sup_{j \geq 1} \int_{\Omega} (dd^c \max(u_j, -k))^n = \sup_{j \geq 1} \int_{\Omega} (dd^c u_j)^n.
\]
Moreover, by Proposition 2.1 we have \(-(u_m)^{\min(p,1)} \in \mathcal{F}^{-PS H}(-\Omega)\) for all \( m \geq 1 \). Hence, again by Proposition 3.4 it implies that
\[
\int_{\Omega} (-u)^{\min(p,1)} (dd^c \max(u, -k))^n = \sup_{m \geq 1} \int_{\Omega} (-u_m)^{\min(p,1)} (dd^c \max(u, -k))^n
\[
= \sup_{m \geq 1} \left[ \sup_{j \geq 1} \int_{\Omega} (-u_m)^{\min(p,1)} (dd^c u_j)^n \right]
\[
\leq \sup_{m \geq 1} \int_{\Omega} (-u)^{\min(p,1)} (dd^c u_j)^n
\[
\leq \sup_{j \geq 1} \int_{\Omega} (1 + (-u_j)^p) (dd^c u_j)^n.
\]
It follows that
\[
\int_{\{u \leq 0\}} (dd^c \max(u, -k))^n \leq \frac{1}{k^{\min(p,1)}} \sup_{j \geq 1} \int_{\Omega} (1 + (-u_j)^p) (dd^c u_j)^n.
\]
Therefore, by Theorem 4.8 in [24] we get
\[
\int_{\{u \geq 0\}} (dd^c u)^n = \lim_{k \to +\infty} \int_{\{u \geq 0\}} (dd^c \max(u, -k))^n
\[
= \lim_{k \to +\infty} \int_{\Omega} (dd^c \max(u, -k))^n = \sup_{j \geq 1} \int_{\Omega} (dd^c u_j)^n.
\]
The proof is complete.  

\[\square\]

**Proposition 4.4.** Let \( \Omega \) be a bounded \( \mathcal{F} \)-hyperconvex domain in \( \mathbb{C}^n \) and let \( p > 0 \). Assume that \( u \in \mathcal{F}_p(\Omega) \) and \( v \in \mathcal{F}^{-PS H}(\Omega) \) with \( u \leq v < 0 \) then \( v \in \mathcal{F}^{-min(p,1)}(\Omega) \) and
\[
\int_{\{u \geq -\infty\}} (dd^c v)^n \leq \int_{\{u \geq -\infty\}} (dd^c u)^n.
\]

**Proof.** Let \( \{u_j\} \subset \mathcal{E}_0(\Omega) \) such that \( u_j \searrow u \) in \( \Omega \) and
\[
\sup_{j \geq 1} \int_{\Omega} (1 + (-u_j)^p) (dd^c u_j)^n < +\infty.
\]
Put \( v_j := \max(u_j, v) \). By Proposition 3.4 we have \( v_j \in \mathcal{E}_0(\Omega) \). Moreover, by Proposition 2.1 we have \(-(v_j)^{\min(p,1)} \in \mathcal{F}^{-PS H}(\Omega)\). Hence, again by Proposition 3.4 it implies that
\[
\sup_{j \geq 1} \int_{\Omega} (1 + (-v_j)^{\min(p,1)}) (dd^c v_j)^n \leq \sup_{j \geq 1} \int_{\Omega} [1 + (-v_j)^{\min(p,1)}] (dd^c u_j)^n
\[
\leq \sup_{j \geq 1} \int_{\Omega} [2 + (-u_j)^p] (dd^c u_j)^n < +\infty.
\]
Since \( v_j \downarrow v \) in \( \Omega \), it implies that \( v \in \mathcal{F}_{\min(p,1)}(\Omega) \). Therefore, by Proposition 3.3 and Proposition 4.3, we obtain

\[
\int_{[0,\infty)} (dd^c v)^n = \sup_{j \geq 1} \int_{\Omega} (dd^c v_j)^n \\
\leq \sup_{j \geq 1} \int_{\Omega} (dd^c u_j)^n = \int_{[0,\infty)} (dd^c u)^n.
\]

The proof is complete.

**Proposition 4.5.** Let \( \Omega \) be a bounded \( \mathcal{F} \)-hyperconvex domain in \( \mathbb{C}^n \) and let \( p > 0 \). Assume that \( u \in \mathcal{F}_{\min(p,1)}(\Omega) \) and \( v \in \mathcal{F}^{-PS H^+}(\Omega) \) such that \( (1 + (-v)^p)(dd^c u)^n \leq (1 + (-v)^p)(dd^c v)^n \) in \( \Omega \cap [u > -\infty] \cap [v > -\infty] \). Then \( u \geq v \) in \( \Omega \).

**Proof.** Let \( \varphi \) be smooth strictly plurisubharmonic function in \( \mathbb{C}^n \) such that \( \Omega \subset \{ \varphi < 0 \} \). Put \( v_j := \max(u, v + \frac{1}{j} \varphi) \) on \( \Omega \), where \( j \in \mathbb{N}^* \). First, we claim that

\[
(dd^c v_j)^n \geq (dd^c u)^n \text{ in } \Omega \cap [u > -\infty] \cap [v > -\infty].
\]

Indeed, by the hypotheses it implies that

\[
(dd^c (v + \frac{1}{j} \varphi))^n \geq (dd^c v_j)^n \geq \frac{1 + (-v)^p}{1 + (-v)^p} (dd^c u)^n \geq 1_{[u \leq \varphi + \frac{1}{j} \varphi]} (dd^c u)^n
\]

in \( \Omega \cap [u > -\infty] \cap [v > -\infty] \). Hence, by Proposition 2.5 we get

\[
1_{[u \leq \varphi + \frac{1}{j} \varphi]} (dd^c v_j)^n \geq 1_{[u \leq \varphi + \frac{1}{j} \varphi]} (dd^c u)^n \text{ in } \Omega \cap [u > -\infty] \cap [v > -\infty].
\]

Moreover, by Theorem 4.8 in [24] we have \( (dd^c v_j)^n = (dd^c u)^n \) in \( \Omega \cap [u > \varphi + \frac{1}{j} \varphi] \cap [u > -\infty] \cap [v > -\infty] \). Therefore,

\[
(dd^c v_j)^n \geq (dd^c u)^n \text{ in } \Omega \cap [u > -\infty] \cap [v > -\infty].
\]

This proves the claim. Since \( v_j \geq u \) in \( \Omega \), by Proposition 4.3 we have

\[
\int_{[0,\infty)} (dd^c u)^n \leq \int_{[0,\infty)} (dd^c v_j)^n \leq \int_{[0,\infty)} (dd^c u)^n < +\infty.
\]

It follows that \( 1_{[v > -\infty]}(dd^c v_j)^n = 1_{[u > -\infty]}(dd^c u)^n \) in \( \Omega \). Therefore, by Theorem 4.8 in [24] we get

\[
\int_{[u < \varphi < v]} (dd^c \varphi)^n \leq \int_{[u < \varphi < v]} [(dd^c (v + \frac{1}{j} \varphi))^n - (dd^c v_j)^n] \leq \int_{[u < \varphi < v]} [(dd^c v)^n - (dd^c u)^n] = 0.
\]

Thus,

\[
\int_{[u < \varphi < v]} (dd^c \varphi)^n = \sup_{j \geq 1} \int_{[u < \varphi < v]} (dd^c \varphi)^n = 0.
\]

From Proposition 2.3 we have \( u \geq v \) on \( \Omega \). The proof is complete.

**5. Proof of theorem 1.1**

**Proof.** Let \( \{ \varphi_j \} \subset \mathcal{E}_0(\Omega) \) such that \( \varphi_j \downarrow u \) on \( \Omega \) and

\[
\sup_{j \geq 1} \int_{\Omega} (1 + (-\varphi_j)^p)(dd^c \varphi_j)^n < +\infty.
\]
By Proposition 2.6 we have
\[
\int_{\{u > -\infty\}} (1 + (-u)^p)(dd^c u)^n \leq \sup_{k \geq 1} \int_{\Omega \cap \{u > -\infty\}} (1 + (-\varphi_k)^p)(dd^c u)^n
\]
\[
\leq \sup_{k \geq 1} \liminf_{j \to +\infty} \int_{\Omega} (1 + (-\varphi_k)^p)(dd^c \varphi_j)^n
\]
\[
\leq \sup_{j \geq 1} \int_{\Omega} (1 + (-\varphi_k)^p)(dd^c \varphi_j)^n < +\infty.
\]
Moreover, since the measure \(1_{\Omega \cap \{u > -\infty\}}(1 + (-u)^p)(dd^c u)^n\) vanishes on all pluripolar subsets of \(\Omega_j\), by Theorem 4.10 in \([12]\) there exists \(u_j \in \mathcal{F}_p(\Omega_j)\) such that
\[
(1 + (-u_j)^p)(dd^c u_j)^n = 1_{\Omega \cap \{u > -\infty\}}(1 + (-u)^p)(dd^c u)^n \quad \text{in } \Omega_j.
\]
By Theorem 4.8 in \([12]\) we have \(u_j \geq u_{j+1}\) in \(\Omega_{j+1}\). Moreover, since \(u \in \mathcal{F}_{\min(p,1)}(\Omega)\), by Proposition 4.5 it implies that \(u \geq u_j\) in \(\Omega\) for all \(j \geq 1\). Let \(v\) be the least \(\mathcal{F}\)-upper semi-continuous regularization of \(\lim_{j \to +\infty} u_j\) on \(\Omega\). By Theorem 3.9 in \([23]\) we get \(u_j \to v\) a.e. in \(\Omega\).

We claim that \(v \in \mathcal{F}_{\min(p,1)}(\Omega)\). Indeed, put \(v_k := \max(v, k\rho)\), where \(k \in \mathbb{N}^*\). By Proposition 3.4 we have \(v_k \in \mathcal{E}_0(\Omega)\). Since \(\max(u_j, k\rho) \not\nearrow v_k\) a.e. in \(\Omega\), by Proposition 2.6 and Lemma 3.3 in \([1]\) we get
\[
\int_{\Omega} [1 + (-\varphi_k)^\min(p,1)](dd^c v_k)^n \leq \liminf_{j \to +\infty} \int_{\Omega} [1 + (-\varphi_k)^\min(p,1)](dd^c \max(u_j, k\rho))^n
\]
\[
\leq \liminf_{j \to +\infty} \int_{\Omega} [1 + (-\max(u_j, k\rho))^\min(p,1)](dd^c \max(u_j, k\rho))^n
\]
\[
\leq \liminf_{j \to +\infty} \int_{\Omega} [1 + (-u_j)^\min(p,1)](dd^c u_j)^n
\]
\[
\leq 2 \liminf_{j \to +\infty} \int_{\Omega} [1 + (-u_j)^p](dd^c u_j)^n
\]
\[
= 2 \int_{\Omega \cap \{u > -\infty\}} (1 + (-u)^p)(dd^c u)^n.
\]
Hence,
\[
\sup_{k \geq 1} \int_{\Omega \cap \{u > -\infty\}} [1 + (-\varphi_k)^\min(p,1)](dd^c v_k)^n \leq 2 \int_{\Omega \cap \{u > -\infty\}} (1 + (-u)^p)(dd^c u)^n < +\infty.
\]
It follows that \(v \in \mathcal{F}_{\min(p,1)}(\Omega)\). This proves the claim. Since \(v \geq u_j\) in \(\Omega\), so
\[
(1 + (-v)^p)(dd^c v)^n \leq (1 + (-u_j)^p)(dd^c u_j)^n \quad \text{in } \Omega \cap \{u > -\infty\}.
\]
Moreover, since \(u_j \not\nearrow v\) a.e. in \(\Omega\), by Theorem 4.5 in \([23]\) we have
\[
(1 + (-v)^p)(dd^c v)^n \leq (1 + (-u)^p)(dd^c u)^n \quad \text{in } \Omega \cap \{u > -\infty\} \cap \{v > -\infty\}.
\]
Hence, by Proposition 4.5 it implies that \(v \geq u\) in \(\Omega\), and therefore, \(u = v\) in \(\Omega\). Thus, \(u_j \to u\) a.e. in \(\Omega\). The proof is complete.

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