RELATIVE OUTER AUTOMORPHISMS OF
FREE GROUPS

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Abstract

Let $A_1, \ldots, A_k$ be a system of free factors of $F_n$. The group of relative automorphisms $\text{Aut}(F_n; A_1, \ldots, A_k)$ is the group given by the automorphisms of $F_n$ that restricted to each $A_i$ are conjugations by elements in $F_n$. The group of relative outer automorphisms is defined as $\text{Out}(F_n; A_1, \ldots, A_k) = \text{Aut}(F_n; A_1, \ldots, A_k)/\text{Inn}(F_n)$, where $\text{Inn}(F_n)$ is the normal subgroup of $\text{Aut}(F_n)$ given by all the inner automorphisms. We define a contractible space on which $\text{Out}(F_n; A_1, \ldots, A_k)$ acts with finite stabilizers and we compute the virtual cohomological dimension of this group.

1 Introduction

Let $F_n$ denote the free group on $n$ generators. We consider the group of automorphisms of $F_n$, denoted by $\text{Aut}(F_n)$, and the group of outer automorphisms $\text{Out}(F_n) = \text{Aut}(F_n)/\text{Inn}(F_n)$, where $\text{Inn}(F_n)$ is the normal subgroup of $\text{Aut}(F_n)$ given by all the inner automorphisms.

Culler and Vogtmann introduced a space $\text{CV}_n$ on which the group $\text{Out}(F_n)$ acts with finite stabilizers and proved that $\text{CV}_n$ is contractible. That space $\text{CV}_n$ is called outer space.

Let $A_1, \ldots, A_k$ be a system of free factors of $F_n$, i.e., there exists $B < F_n$ such that $F_n = A_1 * \cdots * A_k * B$. We define the group of relative (to $A_1, \ldots, A_k$) automorphisms $\text{Aut}(F_n; A_1, \ldots, A_k)$ given by the elements $f \in \text{Aut}(F_n)$ such that $f$ restricted to each $A_i$ is a conjugation by an element in $F_n$.

Obviously, $\text{Aut}(F_n) > \text{Aut}(F_n; A_1, \ldots, A_k) \triangleright \text{Inn}(F_n)$. 

We define also the group of relative (to $A_1, \ldots, A_k$) outer automorphisms:

$$\text{Out}(F_n; A_1, \ldots, A_k) = \text{Aut}(F_n; A_1, \ldots, A_k) / \text{Inn}(F_n) < \text{Out}(F_n).$$

We are interested in finding the relative outer space $CV_n(A_1, \ldots, A_k)$ on which $\text{Out}(F_n; A_1, \ldots, A_k)$ acts with finite stabilizers and proving that $CV_n(A_1, \ldots, A_k)$ is contractible. Moreover, we will compute the VCD of $\text{Out}(F_n; A_1, \ldots, A_k)$. Let $s(i)$ be the minimum number of generators for $A_i$.

**Theorem 34.** We have

$$\text{vcd}(\text{Out}(F_n; A_1, \ldots, A_k)) = 2n - 2s(1) - \cdots - 2s(k) + 2k - 2 - m,$$

where $s(i_1) = \cdots = s(i_m) = 1$ and $s(j) > 1$ for $j \neq i_1, \ldots, i_m$.

Our computation of the virtual cohomological dimension of the relative outer space agrees with the computation in [3] when $m = k$. For $k = n$ and $s(1) = \cdots = s(k) = 1$, $\text{Out}(F_n; A_1, \ldots, A_k)$ is called the pure symmetric automorphism group. In [4] Collins showed that the virtual cohomological dimension of the pure symmetric automorphism group is $n - 2$.

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## 2 Out($F_n; A_1, \ldots, A_k$) and CV$_n$(A$_1, \ldots, A_k$)

The goal of this section is to define carefully $\text{Out}(F_n; A_1, \ldots, A_k)$ and $\text{CV}_n(A_1, \ldots, A_k)$.

Consider $A_k = \langle y^1_j, \ldots, y^s(i)_j \rangle$ and $F_n = \langle y^1_1, \ldots, y^k_1, x_1, \ldots, x_{n-\sum_{i=1}^k s(i)} \rangle$.

By a graph, we mean a connected 1-dimensional CW complex.

Let the relative rose $R_n(A_1, \ldots, A_k)$ be a graph obtained by a wedge of $n - \sum_{i=1}^k s(i)$ circles attaching $\sum_{i=1}^k s(i)$ circles $C^1_1, \ldots, C^k_{s(k)}$ on $k$ stems as in Figure 1. The edges are denoted by $C^1_1, \ldots, C^k_{s(k)}, f_1, \ldots, f_k, e_1, \ldots, e_n - \sum_{i=1}^k s(i)$.

Moreover,

$$\pi_1(R_n(A_1, \ldots, A_k), v) \cong F_n = \langle y^1_1, \ldots, y^k_1, x_1, \ldots, x_{n-\sum_{i=1}^k s(i)} \rangle,$$

where $v$ is the central vertex in $R_n(A_1, \ldots, A_k)$ (see Figure 1), by declaring $y^j_i$ to be the homotopy class of $C^j_i$ and $x_i$ to be the homotopy class of the loop $e_i$. 

\[2\]
Let \((R_n(A_1, \ldots, A_k), k)\) be the graph \(R_n(A_1, \ldots, A_k)\) equipped with inclusions \(k_j : \bigvee_{i=1}^{s(j)} S^1 \to R_n(A_1, \ldots, A_k)\) that identifies \(\bigvee_{i=1}^{s(j)} S^1\) with \(\bigvee_{i=1}^{s(j)} C_i^j\), for all \(j = 1, \ldots, k\).

**Definition 1.** Let \(\Gamma\) be a graph of rank \(n\) with vertices of valence at least 3, equipped with embeddings \(l_j : \bigvee_{i=1}^{s(j)} S^1 \to \Gamma\) for \(j = 1, \ldots, k\). We call \(B_j = l_j(\bigvee_{i=1}^{s(j)} S^1)\) wedge cycle. The dual graph of the \(B_j\)'s is the graph with one vertex for each wedge cycle, one vertex \(w\) for each intersection between two or more wedge cycles and edges between \(w\) and vertices corresponding to the wedge cycles meeting in \(w\).

**Definition 2.** An \((A_1, \ldots, A_k, n)\)-graph \((\Gamma, l)\) is a finite graph \(\Gamma\) of rank \(n\) with vertices of valence at least 3, with possible separating edges, equipped with embeddings \(l_j : \bigvee_{i=1}^{s(j)} S^1 \to \Gamma\) for \(j = 1, \ldots, k\), such that any two \(B_j\) intersect in at most a point and the dual graph of the \(B_j\)'s is a forest.

**Notation 3.** We will denote by \(A\) the set of free factors \(A_1, \ldots, A_k\).

**Example 4.** Consider the graph in Figure 2. Each loop of the same color is a wedge cycle. That graph is not an \((A_1, A_2, A_3, 4)\)-graph because the dual graph of the \(B_j\)'s is a circle (see Figure 3).

**Definition 5.** A marked \((A, n)\)-graph \((\Gamma, \phi)\) is a graph \(\Gamma\) of rank \(n\) equipped with a homotopy equivalence \(\phi : R_n(A) \to \Gamma\) such that \((\Gamma, \phi \circ \underline{k})\) is an \((A, n)\)-graph.

The map \(\phi\) is called the marking.
Figure 2: Example of a graph that is not an \((A_1, A_2, A_3, 4)\)-graph.

Figure 3: The dual graph of the graph in Figure 2.
The marking induces an isomorphism \( \phi_* : F_n \to \pi_1(\Gamma, \phi(v)) \).

**Definition 6.** A marked metric \((\mathcal{A}, n)-\text{graph}\) \((\Gamma, \phi)\) is a marked graph \((\Gamma, \phi)\) such that each edge \(e\) in \(\Gamma\) has positive real length \(l(e)\).

**Definition 7.** The relative outer space \(CV_n(A_1, \ldots, A_k)\) (or \(CV_n(\mathcal{A})\)) is the space of equivalence classes of marked metric \((\mathcal{A}, n)\)-graphs where

1. the sum of all lengths of the edges in \(\Gamma \setminus \{\phi(C^1_i), \ldots, \phi(C^k_{s(k)})\}\) is 1 (relative volume 1) and
   \[
   \sum_{e \subset \phi(C^1_i)} l(e) = 1 \quad \forall i, j;
   \]

2. \((\Gamma_1, \phi_1) \sim (\Gamma_2, \phi_2)\) if there is an isometry \(h : \Gamma_1 \to \Gamma_2\) with \(h\) such that \(h \circ \phi_1(C^j_i) = \phi_2(C^j_i)\) for all \(i, j\), and \(h \circ \phi_1\) is homotopic to \(\phi_2\) rel. \(\bigcup_{i,j} C^j_i\).

We will usually denote a point in \(CV_n(\mathcal{A})\) by \((\Gamma, \phi)\).

There is a natural right action of \(\text{Out}(F_n; \mathcal{A})\) by homeomorphisms of \(CV_n(\mathcal{A})\): let \(X = (\Gamma, \phi) \in CV_n(\mathcal{A})\), let \(\Psi\) be a relative outer automorphism and consider a map \(\psi : R_n(\mathcal{A}) \to R_n(\mathcal{A})\) such that \([\psi_*] = \Psi\) and which is the identity on \(\bigcup_{i,j} C^j_i\). Define

\[
X \cdot \Psi = (\Gamma, \phi) \cdot \Psi = (\Gamma, \phi \circ \psi).
\]

We can define a topology on \(CV_n(\mathcal{A})\) by varying the lengths of the edges exactly as for outer space (see [7] for the definition in the case of outer space).

Because we suppose that the relative volume is 1 and the sum of the lengths of the edges in each cycle \(\phi(C^j_i)\) is 1, a point \((\Gamma, \phi) \in CV_n(\mathcal{A})\) is in the interior of a polysimplex that is the product of the simplices \(\Delta^j_i\) obtained by varying the lengths of the edges in each cycle \(\phi(C^j_i)\) and the simplex \(\sigma\) given by varying the length of the edges in \(\Gamma \setminus \{\phi(C^1_i), \ldots, \phi(C^k_{s(k)})\}\).

Indeed, if \(\Gamma\) has \(N_{i,j}\) edges in \(\phi(C^j_i)\) of length \(s^1_{i,j}, \ldots, s^{N_{i,j}}_{i,j}\) and \(N\) edges in \(\Gamma \setminus \{\phi(C^1_i), \ldots, \phi(C^k_{s(k)})\}\) of length \(t_1, \ldots, t_N\), then

\[
0 < s^k_{i,j} < 1, \forall i, j, k \quad \text{and} \quad \sum_{k=1}^{N_{i,j}} s^k_{i,j} = 1,
\]

\[
0 < t_m < 1, \forall m \quad \text{and} \quad \sum_{m=1}^{N} t_m = 1.
\]

Let \(\Delta^j_i\) be the open simplex determined by varying the \(s_{i,j}\)’s and \(\sigma\) be the open simplex obtained by varying the \(t\)’s. Define

\[
P \Delta = \Delta^1_i \times \cdots \times \Delta^{k}_{s(k)} \times \sigma.
\]

Changing the length of the edges in \(\Gamma\) gives the open polysimplex \(P \Delta\).
Example 8. Consider $\text{Out}(F_5; A_1, A_2)$, where $F_5 = <a, a', b, b', c>$, $A_1 = <a, a'>$ and $A_2 = <b, b'>$, and consider the point $(\Gamma, \phi) \in CV_5(A_1, A_2)$ in Figure 4.

![Figure 4: The point $(\Gamma, \phi) \in CV_5(A_1, A_2)$ in Example 8.](image)

The open polysimplex given by varying the length of the edges of $\Gamma$ is the open cube $\Delta_1^1 \times \Delta_2^1 \times \Delta_1^2 \times \Delta_2^2 \times \sigma \cong \Delta_1^1 \times \Delta_2^2 \times \sigma$ (see Figure 5).

![Figure 5: The open polysimplex $\Delta_1^1 \times \Delta_2^2 \times \sigma$ in Example 8.](image)

Note that $\text{Out}(F_n; A)$ acts properly and discontinuously on $CV_n(A)$ and that the stabilizer of any marked $(A, n)$-graph $(\Gamma, \phi)$ is isomorphic to the subgroup of isometries of $\Gamma$ which fixes the wedge cycles, hence it is finite.

For an $(A, n)$-graph $(\Gamma, \mathcal{L})$, let $\hat{\Gamma}$ be the graph of rank $n - \sum_{i=1}^{k} s(i)$ obtained from $\Gamma$ by collapsing the wedge cycles $B_1, \ldots, B_k$ to points. Let $e$ be an edge of $\Gamma$ that does not define a loop in $\Gamma$ or in $\hat{\Gamma}$. Then the composition with the edge collapse $\text{col} : \Gamma \to \Gamma/e$ induces embeddings

$$l_j/e : \bigvee_{i=1}^{s(j)} S^1 \to \Gamma/e$$

such that $(\Gamma/e, \mathcal{L}/e)$ is again an $(A, n)$-graph. By an edge collapse in an $(A, n)$-graph, we will always mean the collapse satisfying the above hypothe-
sis. For a marked \((\mathcal{A}, n)\)-graph \((\Gamma, \phi)\), the marking of the collapsed graph is the composition \(\text{col} \circ \phi\).

\(F\) is a forest in a graph \(\Gamma\) if it is a union of edges in \(\Gamma\) that does not contain any loop in \(\Gamma\) or in \(\hat{\Gamma}\). A forest collapse in an \((\mathcal{A}, n)\)-graph \((\Gamma, l)\) is a sequence of edge collapses, where the edges that are collapsed are the edges in the forest. We denote the collapsed graph by \((\Gamma/F, l/F)\).

We define a poset structure on the set of marked \((\mathcal{A}, n)\)-graphs by saying that \((\Gamma_1, \phi_1) \leq (\Gamma_2, \phi_2)\) if there is a forest \(F\) in \(\Gamma_2\) such that \((\Gamma_2/F, \phi_2/F)\) is equivalent to \((\Gamma_1, \phi_1)\).

We denote by \(S_n(\mathcal{A}_1, \ldots, \mathcal{A}_k)\) (or \(S_n(\mathcal{A})\)) the geometric realization of that poset and we call it the relative spine of the relative outer space.

**Example 9.** Consider \(n = 2\), \(F_2 = <a, b>\) and \(A = <a>\).

In that case, \(\text{Out}(F_2; A)\) is isomorphic to the infinite dihedral group \(D_\infty\). Indeed, if \(f \in \text{Aut}(F_2)\) and \(f(a) = a\), then \(f(b) = a^n b^m a^m\), where \(n, m \in \mathbb{Z}\) and \(\varepsilon \in \{\pm 1\}\).

After conjugating by a power of \(a\), we have \(f(b) = a^N b\) or \(f(b) = a^N b^{-1}\).

The map

\[
\begin{align*}
\text{Out}(F_2; A) & \to \mathbb{Z}_2 \\
f & \mapsto \varepsilon
\end{align*}
\]

has kernel \(\mathbb{Z}\), so we get

\[
1 \to \mathbb{Z} \to \text{Out}(F_2; A) \to \mathbb{Z}_2 \to 1,
\]

i.e. \(\text{Out}(F_2; A) \cong \mathbb{Z} \times \mathbb{Z}_2 \cong D_\infty\).

The relative spine \(S_2(\mathcal{A})\) is homeomorphic to the simplicial complex in Figure 6.

**Remark 10.** We can define the reduced relative spine as the subset of the geometric realization of the poset structure described previously, containing only the \((\mathcal{A}, n)\)-graphs with no separating edges. In Example 9 the reduced relative spine is homeomorphic to a line.

Notice that \(CV_n(\mathcal{A})\) is not a polysimplicial complex, because some of the faces are missing, but the relative spine \(S_n(\mathcal{A})\) is a simplicial complex.

Given a polysimplex \(P\Delta\) we define its barycentric subdivision in the following way. Consider the centroid of each face or polyface (i.e. product of faces) of the polysimplex. Those will be the vertices of the barycentric subdivision and we will call them barycentric subdivision vertices (BSV). Now, for each \(n > 0\) and \(n\)-face or \(n\)-polyface \(F\), connect the centroid with each BSV in the \((n-1)\)-faces (or polyfaces) of \(\partial F\) (see Figure 7).
Figure 6: The spine $S_2(A)$ in Example 9 where $(\Gamma, id)$ is the marked graph with identity marking, $(\Gamma_1, \phi_1)$ has the marking given by $\phi_1(a) = a, \phi_1(b) = ab$. In $\Gamma$ and $\Gamma_1$ both the edges have length 1, while in $\Gamma'$ the right edge has length 1 (it corresponds to $\phi'(e_1)$) and the other edges have length $\frac{1}{2}$.

Figure 7: Barycentric subdivision of the polysimplex given by the product of two 1-simplices. The dots are the vertices of the barycentric subdivision.
Note that if $\Delta_j$ is a simplex face in $P\Delta_j$, then the restriction of the barycentric subdivision to $P\Delta_j|\Delta_j$ is the (standard) barycentric subdivision of $\Delta_j$.

There is a natural embedding of $S_n(A)$ into $CV_n(A)$ that sends each vertex to the centroid of the corresponding open polysimplex and each $d$-simplex to the convex hull of the corresponding centroids (see Figure 8 for an example of barycentric subdivision in $CV_5(A_1, A_2)$ of Example 8).

Figure 8: Barycentric subdivision of the polysimplex given in Example 8 where the red dots are the vertices of the simplices in $S_5(A_1, A_2)$.

$CV_n(A)$ deformation retracts onto $S_n(A)$ in the following way. The vertices of $S_n(A)$ correspond to open polysimplices of $CV_n(A)$ and a $d$-simplex is a chain of $d + 1$ open polysimplices, each of which is a face of the next. By pushing within each open polysimplex of $CV_n(A)$ away from the missing faces we have a deformation retraction from $CV_n(A)$ to $S_n(A)$. In other words, $CV_n(A)$ is the union of open polysimplices in a polysimplicial complex $X$. Note that $S_n(A)$ is the maximal full subcomplex of the barycentric subdivision of $X$ that is disjoint from $X \setminus CV_n(A)$. Collapsing every simplex in the barycentric subdivision of $X$ to the face of the simplex contained in $S_n(A)$ gives a deformation retraction of $CV_n(A)$ onto $S_n(A)$.

The action of $\text{Out}(F_n; A)$ extends to a simplicial action on the relative spine $S_n(A)$.

3 Contractibility of $CV_n(A)$

The whole section is dedicated to the proof of the following theorem.

**Theorem 11.** The relative outer space $CV_n(A)$ is contractible.

Because there is a deformation retraction from the relative outer space $CV_n(A)$ to its spine $S_n(A)$, it is enough to prove that the relative spine is contractible. First we prove that if $k = 1$, $S_n(A)$ is contractible.
Recall from [7] that the spine $S_n$ of the outer space $CV_n$ is a poset of marked graphs of rank $n$, where the marking is given by a homotopy equivalence from the rose $R_n = \bigvee^n S^1$. $S_n$ is contractible and admits an action of $\text{Out}(F_n)$. Edge collapses induce a poset structure on $S_n$ with minimal elements the reduced marked graphs, that is roses.

Let $W$ be the set of conjugacy classes of elements in $F_n$. Let $w_1, \ldots, w_m$ be elements in $W$. Following [5], we define the function $f_{w_i}$ from the set of roses to $\mathbb{R}$ by $f((R, \phi)) = nl(w_i)$, where $l$ is the length function on $F_n$ associated to $R$. The minset of $f_{w_i}$ is

$$\text{Minset}(f_{w_i}) = \bigcup_{f_{w_i} \text{ is min at } (R, \phi)} \text{st}((R, \phi)),$$

where $\text{st}(X)$ is the star of the point $X$ in $S_n$.

Now, we consider the function $f$ from the set of roses to $\mathbb{R}^m$, $f = (f_{w_1}, \ldots, f_{w_m})$ and we consider $\mathbb{R}^m$ equipped with the lexicographic order.

Define $\Lambda$ as the set of roses $(R, \phi)$ that are minimum in $f = (f_{w_1}, \ldots, f_{w_m})$ with respect to the lexicographic order. Let

$$\text{Minset}(f) = \bigcup_{(R, \phi) \in \Lambda} \text{st}((R, \phi)).$$

**Remark 12.** If $(R_i, \phi_i)$ are roses in $S_n$ for $i = 1, 2$, $f((R_1, \phi_1)) \leq f((R_2, \phi_2))$, $(R_2, \phi_2) \in \Lambda$, then $(R_1, \phi_1) \in \Lambda$.

A useful lemma that we will need in the sequel is the Poset Lemma.

**Theorem 13** (Poset Lemma). Let $X$ be a poset and $f : X \rightarrow X$ be a poset map with the property that $f(x) \leq x$ for all $x \in X$ (or $f(x) \geq x$ for all $x \in X$). Then $f(X)$ is a deformation retract of $X$.

See [11] for a proof of the Poset Lemma.

Let $(\Gamma, \phi)$ be a marked graph in $S_n$ and let $v$ be a vertex of $\Gamma$. Formally, the notion of ideal edges is defined as in [5] in terms of partitions. We can think of an ideal edge $\gamma$ at the vertex $v$ as a partition of the set $E_v$ of half edges of $\Gamma$ terminating at $v$ such that the blow-up $\Gamma^\gamma$ in $v$ is again in $S_n$, where $\Gamma^\gamma$ is the graph obtained by pulling the half edges in $\gamma$ away from $v$ creating a new vertex $v(\gamma)$, a new edge $\gamma$ that goes from $v(\gamma)$ to $v$ and each half edge $e \subset \gamma$ is attached to $v(\gamma)$ instead of $v$. Note that the graph $\Gamma$ can be reobtained by $\Gamma^\gamma$ collapsing $\gamma$.

An ideal forest in a reduced marked graph is a sequence of ideal edges. One can define a poset structure on the set of ideal forests of a rose $(R, \phi)$.
such that the blowing up induces an isomorphism between that poset and the star of \((R, \phi)\) in \(S_n\) (see [5]). As for \(S_n\), we can define ideal edges and ideal forests for \(S_n(A)\).

Let \(A\) be a subset of \(E_v\). We will denote by \(\overline{A}\) the complement of \(A\).

**Definition 14.** Two subsets \(A\) and \(B\) of \(E_v\) are compatible if one of the sets \(A \cap B, \overline{A} \cap B, A \cap \overline{B}, \overline{A} \cap \overline{B}\) is empty.

The upper link of a marked \((A, n)\)-graph \((\Gamma, \phi)\) in \(S_n(A)\) is a set of marked \((A, n)\)-graphs \((\Gamma', \phi')\) that collapse to \((\Gamma, \phi)\). Such marked \((A, n)\)-graphs are said to be obtained by blowing up vertices of \(\Gamma\) into trees. Notice that a set of ideal edges is compatible if it corresponds to a tree.

Let \(B(v)\) be the complex whose vertices are ideal edges at \(v\) and whose \(i\)-simplices are sets of \(i + 1\) compatible ideal edges.

**Definition 15.** We say that an ideal edge \(\gamma\) at a vertex \(v \in \Gamma\) is legal if \(\Gamma_\gamma \in S_n(A)\). We denote the subcomplex of \(B(v)\) spanned by legal ideal edges by \(L(v)\).

**Remark 16.** An ideal edge is legal if and only if it separates at most one pair of half edges contained in a wedge cycle \(B_i\). Indeed, if it separates two pairs of half edges one in \(B_i\) and the other one in \(B_j\), then it blow ups to an edge in \(B_i \cap B_j\) and that contradicts the definition of marked \((A, n)\)-graph.

We have the following remarkable result.

**Theorem 17.** \(\text{Minset}(f)\) is contractible.

The proof of that theorem follows from the following theorem in [5].

**Theorem 18** (Culler-Vogtmann). Let \(W' = \{w_1, \ldots, w_m\}\), where \(w_i \in W\) for all \(i\). Then \(\text{Minset}(f)\) is a contractible subcomplex of \(S_n\), the action is proper and the quotient \(\text{Minset}(f)/\text{Stab}(W')\) is finite.

An alternative proof is given in Section 4 of [9] (in the case of \(G = \{1\}\), \(\|\Lambda\| = \text{Minset}(f)\) and \(E_n^G = S_n\)).

**Lemma 19** ([6]). If \((\Gamma_1, \phi_1)\) and \((\Gamma_2, \phi_2)\) are two marked graphs of rank \(n\) and \(f : \Gamma_1 \to \Gamma_2\) is a map linear on edges such that the following diagram

\[
\begin{array}{ccc}
\Gamma_1 & \xrightarrow{f} & \Gamma_2 \\
\phi_1 \downarrow & & \phi_2 \\
R_n & \xrightarrow{\phi_2} & \\
\end{array}
\]
commutes up to homotopy, then there is a subgraph of \( \Gamma_1 \) where the length of the edges are multiplied by the Lipschitz constant \( \text{Lip}(f) \) and the length of all the edges not in the subgraph are multiplied by a number strictly less than \( \text{Lip}(f) \).

**Definition 20.** Let \( (\Gamma_1, \phi_1) \) and \( (\Gamma_2, \phi_2) \) be two marked graphs of rank \( n \). Given a map \( f \sim \phi_2 \circ \phi_1^{-1} \) linear on edges, we denote by \( \Gamma_f \) the subgraph of \( \Gamma_1 \) whose edges are maximally stretched by \( \text{Lip}(f) \).

**Definition 21.** Let \( (\Gamma_1, \phi_1) \) and \( (\Gamma_2, \phi_2) \) be two marked graphs of rank \( n \). A map \( f \sim \phi_2 \circ \phi_1^{-1} \) linear on edges is *not optimal* if there is some vertex of \( \Gamma_f \) such that all the edges of \( \Gamma_f \) terminating at that vertex have \( f \)-image with a common terminal partial edge. Otherwise, \( f \) is called *optimal*.

A *turn* in \( (\Gamma, \phi) \) is an unordered pair of oriented edges of \( \Gamma \) originating at a common vertex. A turn is *nondegenerate* if it is defined by distinct oriented edges. Otherwise, the turn is called *degenerate*.

A map \( f : \Gamma \to \Gamma \) induces a map \( Df \) from the set of oriented edges of \( \Gamma \) to itself by sending an oriented edge to the first oriented edge in its \( f \)-image as long as no edges are collapsed. We can think of \( Df \) as a sort of derivative.

\( Df \) induces a map \( Tf \) on the set of turns in \( \Gamma \). A turn is *illegal* with respect to \( f \) if its image under some iterate of \( Tf \) is degenerate. Otherwise, the turn is called *legal*. For properties of legal and illegal turns see [2] or [1].

Remember that

\[
A_i = \langle y_1^i, \ldots, y_{s(i)}^i \rangle \quad \text{and} \quad F_n = \langle y_1^1, \ldots, y_{s(k)}^k, x_1, \ldots, x_n - \sum_{i=1}^{k} s(i) \rangle.
\]

Consider the function \( f \) given by \( f = (f_1, f_2, \ldots, f_k) \), where

\[
f_j = (f_{w_1^j}, \ldots, f_{w_{s(j)}^j}, f_{w_{1,2}^j}, \ldots, f_{w_{1,i}^j}, \ldots, f_{w_{1,\ell}^j}, \ldots)
\]

and

\[
w_i^j = y_i^j, \quad \text{for } i = 1, \ldots, s(k),
w_{i,l}^j = y_i^j y_l^j, \quad \text{for all } i < l, i, l = 1, \ldots, s(k),
w_{i,l}^j = y_i^j y_l^j, \quad \text{for all } i < l, i, l = 1, \ldots, s(k).
\]

**Lemma 22.** Consider \( F_{s(j)} = \langle y_1^j, \ldots, y_{s(j)}^j \rangle \). Then \( \text{Minset}(f_j) \) consists of the star of a single rose in \( S_{s(j)} \), and hence is contractible.
Proof. Suppose that \((R, \phi)\) is a rose as in Figure 9 with
\[
l(y_j^i) = 1, \forall i = 1, \ldots, s(j); \quad l(y_j^i y_l^j) = l(y_j^i y_l^j) = 2, \forall i < l, i, l = 1, \ldots, s(j).
\]
(1)

Let \((R_1, \phi_1)\) be another rose in \(\text{Minset}(f_j)\). We will prove that the optimal homotopy map \(f\) such that the following diagram commutes

\[
\begin{array}{ccc}
R & \xrightarrow{f} & R_1 \\
\downarrow{\phi} & & \downarrow{\phi_1} \\
R_{s(j)} & \xleftarrow{f} & R_{s(j)}
\end{array}
\]

is an isometry up to homotopy. First of all, we need to show that \(\Gamma_f = R\).

Notice that by Proposition 3.15 in [6], the Lipschitz constant is determined by a cycle or a figure eight graph. By (1), \(\text{Lip}(f) = 1\).

By contradiction, if \(\Gamma_f\) is not the whole graph, then there is a loop not in \(\Gamma_f\) that has length less than one by Theorem 19 and that gives a contradiction with our assumption (1). Hence, \(\Gamma_f = R\).

In order to prove that \(f\) is an isometry, we need to show that we do not have any illegal turn.

If a loop contains an illegal turn, then its length is stretched by a number less than \(\text{Lip}(f)\) (see [3]). Therefore, if we have a loop with an illegal turn, then the length of that loop would be less than the Lipschitz constant, 1, but that is a contradiction.

If \(s(j) = 2\), then we have other two possibilities for illegal turns (see Figure 10). In both cases the length of the figure eight graph would be less than 2 and that leads to a contradiction. In general, if we have an illegal turn in a path contained in a subrose of \(m\) petals, then the sum of the lengths of the edges in the subrose would be less than \(m\), and that contradicts (1).

In conclusion, \(f\) is an isometry and \(\text{Minset}(f_j)\) consists of the star of a single rose in \(S_{s(j)}\).
A different proof of that lemma can be found in [5].

Applying Lemma 22 to Minset($f_j$) $\hookrightarrow S_n$, for each $(\Gamma, \psi)$ in Minset($f_j$), the marking $\psi$ is an embedding on the wedge cycle $B_j$, and we have the following corollary.

**Corollary 23.** Minset($f_j$) = $S_n(A_j)$.

Therefore, by Corollary 23 and Lemma 22 if $k = 1$ (i.e., we have only one wedge cycle), then $S_n(A_1)$ is contractible.

Note that Minset($f$) $\subseteq$ Minset($f_j$), for all $j = 1, \ldots, k$.

Now it remains to prove that $S_n(A)$ is contractible for $k > 1$. We will follow the approach described in [3].

**Definition 24.** A forest $F$ in $(\Gamma, \phi) \in$ Minset($f$) is called admissible if the marked graph $(\Gamma', \phi')$ obtained by collapsing each tree in $F$ to a point is also in Minset($f$).

**Lemma 25.** Let $(\Gamma, \phi) \in$ Minset($f$) and $\phi(C^j_i)$ be the reduced path representing $\phi(w^j_i)$, for $1 \leq i \leq s(j)$, $1 \leq j \leq k$. Then

- $B_j = \bigvee_{i=1}^{s(j)} \phi(C^j_i)$ is a wedge cycle in $\Gamma$ for all $1 \leq j \leq k$;
- $B_j \cap B_{j'}$ ($j' \neq j$) is either empty, a point or a tree;
- $\bigcup(B_j \cap B_{j'})$ is a forest in $\Gamma$;
- If $F$ is an admissible forest in $\Gamma \setminus \{\phi(C^1_1), \ldots, \phi(C^k_{s(k)})\}$, then $F \cup \bigcup(B_j \cap B_{j'})$ is an admissible forest in $\Gamma$. 

Figure 10: Two possible illegal turns.
Proof. Let \((R, \psi)\) be any marked rose in Minset\((f)\) with \((\Gamma, \phi)\) in its star. By Lemma 22, for each \((\Gamma, \phi)\) in Minset\((f)\), the marking \(\phi\) is an embedding on \(\bigvee_{i=1}^{s(j)} \phi(C_j^i)\). Thus, \(B_j = \bigvee_{i=1}^{s(j)} \phi(C_j^i)\) is a wedge cycle in \(\Gamma\) for all \(1 \leq j \leq k\).

Because \((\Gamma, \phi)\) is obtained by blowing up the vertex in \(R\) into a tree \(T\), the intersection \(B_j \cap B_j'\) is contained in \(T\) and it is connected (see Figure 11), so the union of all such intersections is a forest in \(T\). The last statement of the lemma follows from the previous observation.

![Diagram](image)

Figure 11: A point \((\Gamma, \phi)\) in Minset\((f)\), where \(A_1 = y_1^1, y_1^2\) and \(A_2 = y_2^1\). The union of the edges \(e_1\) and \(e_2\) is the intersection \(B_1 \cap B_2\).

Hence, because \(B_j \cap B_j'\) can be a tree, Minset\((f)\) is not contained in \(S_n(A)\). However, we have the following theorem.

**Theorem 26.** Minset\((f)\) deformation retracts onto \(S_n(A)\).

**Proof.** First of all, notice that we have \(S_n(A) \hookrightarrow \text{Minset}(f)\).

Let \((\Gamma, \phi) \in \text{Minset}(f)\). Collapsing each component of \(\bigcup (B_j \cap B_j')\) to a point we obtain a map \(g\) from Minset\((f)\) to \(S_n(A)\) (see Figure 12).

If \((\Gamma', \phi') \in \text{Minset}(f)\) is obtained from \((\Gamma, \phi)\) by collapsing a forest \(F\), then \(F \cup \bigcup (B_j \cap B_j')\) is also a forest in \(\Gamma\) by Lemma 25. Hence, \(g\) is a poset map.

By the Poset Lemma, \(g\) is a deformation retraction from Minset\((f)\) onto \(S_n(A)\).

\(\square\)

We are now able to prove Theorem 11.

**Proof.** By Theorem 26, \(S_n(A)\) is a deformation retraction of Minset\((f)\). Because Minset\((f)\) is contractible by Theorem 17, \(S_n(A)\) is contractible.

\(\square\)
We conclude that $\text{CV}_n(\mathcal{A})$ is contractible because $\text{CV}_n(\mathcal{A})$ deformation retracts onto $S_n(\mathcal{A})$.

4 Relative Spine vs. Small Spine

We introduce a new spine, called small spine, that is a simplicial complex smaller than the relative spine, but which carries all the data coming from the relative outer space.

Consider the relative outer space $\text{CV}_n(\mathcal{A})$.

**Definition 27.** Let $D_n(\mathcal{A})$ be the subcomplex of $S_n(\mathcal{A})$ spanned by vertices $(\Gamma, \phi)$ in which all the wedge cycles are disjoint. $D_n(\mathcal{A})$ is called small spine.

Thus the small spine $D_n(\mathcal{A})$ is a simplicial complex. The definition of small spine shall be more clear after few examples.

**Example 28.** Suppose $s(i) > 1$ for $i = 1, \ldots, k$. Let $(\Gamma, \phi)$ be a maximal graph in the centroid of a maximal dimensional open polysimplicex in the relative outer space $\text{CV}_n(\mathcal{A})$ with vertices on the wedge cycles. So the basepoints of the wedge cycles have valence $2s(1), \ldots, 2s(k)$ and the other vertices have valence 3. The maximal simplex of the small spine that contains $\Gamma$ is given by the barycentric subdivision of the polysimplicex obtained by varying the length of the edges in $\phi(C_1), \ldots, \phi(C_{s(k)})$ and leaving the length of the edges in $\Gamma \setminus \{\phi(C_1), \ldots, \phi(C_{s(k)})\}$ equal to $\frac{1}{N}$, where $N$ is the number of edges in $\Gamma \setminus \{\phi(C_1), \ldots, \phi(C_{s(k)})\}$. 


Example 29. Consider the group $\text{Out}(F_4; A_1, A_2)$, where $F_4 = \langle a, a', b, b' \rangle$ and $A_1 = \langle a, a' \rangle$, $A_2 = \langle b, b' \rangle$. Notice that, using Stalling’s method (see [12]), an element in $\text{Out}(F_4; A_1, A_2)$ is of the form:

$$
\begin{align*}
 a & \mapsto \omega(a, a') a^{\varepsilon_1} \, \varpi(a, a') \\
 a' & \mapsto \omega(a, a') a^{\varepsilon_2} \, \varpi(a, a') \\
 b & \mapsto \omega(b, b') b^{\varepsilon_3} \, \varpi(b, b') \\
 b' & \mapsto \omega(b, b') b^{\varepsilon_4} \, \varpi(b, b')
\end{align*}
$$

where $\varepsilon_i \in \{\pm 1\}$, $1 \leq i \leq 4$, $\omega(a, a') \in A_1$ and $\omega(b, b') \in A_2$.

Modulo separating edges, a point in $\text{CV}_4(A_1, A_2)$ is given by two wedge cycles, with two cycles each, attached in one point (see Figure 13).

The relative spine and the small spine $D_4(A_1, A_2)$ are both equal to the product of two trees $T_1$ and $T_2$ that correspond to the universal coverings of the wedge cycles associated to $A_1$ and $A_2$.

![Figure 13: A point $(\Gamma, \phi)$ in $\text{CV}_4(A_1, A_2)$.

Example 30. Consider the group $\text{Out}(F_5; A_1, A_2)$, where $F_5 = \langle a, a', b, b', c \rangle$ and $A_1 = \langle a, a' \rangle$, $A_2 = \langle b, b' \rangle$ (see Example 8).

By Example 29 and Stalling’s method, an element in $\text{Out}(F_5; A_1, A_2)$ is of the form:

$$
\begin{align*}
 a & \mapsto \omega(a, a') a^{\varepsilon_1} \, \varpi(a, a') \\
 a' & \mapsto \omega(a, a') a^{\varepsilon_2} \, \varpi(a, a') \\
 b & \mapsto \omega(b, b') b^{\varepsilon_3} \, \varpi(b, b') \\
 b' & \mapsto \omega(b, b') b^{\varepsilon_4} \, \varpi(b, b') \\
 c & \mapsto u_1(a, a', b, b') c^{\varepsilon_5} u_2(a, a', b, b')
\end{align*}
$$

where $\varepsilon_i \in \{\pm 1\}$, $1 \leq i \leq 5$, $\omega(a, a') \in A_1$, $\omega(b, b') \in A_2$ and $u_i(a, a', b, b')$’s are elements in $F_4 = \langle a, a', b, b' \rangle$.

The relative outer space, the relative spine and the small spine are more complicated than in Example 29, but let us understand what is happening in this case.
Consider the point \((\Gamma, \phi)\) in \(CV_5(A_1, A_2)\) described in Figure 14. Varying the length of the edges in \(\Gamma\) we can move in an open (maximal) 5-polysimplex of \(CV_5(A_1, A_2)\). When we shrink an edge \(e\) that is not in a wedge cycle, we end up in an open 4-polysimplex. Because the only way to move away from that open 4-polysimplex is to blow up the vertex given by collapsing \(e\), that open 4-polysimplex is a free face (i.e., it is a face of a unique polysimplex) in the relative outer space.

Hence, it is possible to deformation retract the free face onto the interior of the (maximal) polysimplex.

First collapsing \(e\) and then four edges in the wedge cycles we get a 5-simplex \(\sigma\) in the relative spine. By Definition 27 of small spine, the edge \(e\) cannot be collapsed and so \(\sigma\) is not in the small spine.

Repeating the same argument for the graph in Figure 4 of Example 8 we can compare the simplices in \(S_5(A_1, A_2)\) (see Figure 8) with the simplices in \(D_5(A_1, A_2)\) in Figure 15.

Therefore, the small spine is strictly smaller than the relative spine, but
what are missing are vertices in the relative outer space corresponding to free faces.

Following [3] we will prove that there is a deformation retraction from the relative spine to the small spine.

**Theorem 31.** There is an Out($F_n; A$)-equivariant deformation retraction of $S_n(A)$ onto $D_n(A)$.

*Proof.* By the definition of $D_n(A)$, we can build $S_n(A)$ from the small spine $D_n(A)$ by adding marked $(A, n)$-graphs $(\Gamma, \phi)$ in order of decreasing number of vertices in $\Gamma$. Thus at each stage, we are attaching $(\Gamma, \phi)$ along its entire upper link in $S_n(A)$. Hence, it is suffices to show that the upper link is contractible.

Note that a marked $(A, n)$-graph in $S_n(A)$ with $k$ vertices (the basepoints of the wedge cycles) of valence $2s(1), \ldots, 2s(k)$ and the remaining vertices of valence 3 is in $D_n(A)$.

Let $(\Gamma, \phi) \in S_n(A) \setminus D_n(A)$. Then $\Gamma$ contains at least one vertex that is in at least two wedge cycles. Let $v$ be one of those vertices. In order to prove that the upper link of $(\Gamma, \phi)$ in $S_n(A)$ is contractible, it is suffices to prove the following lemma.

**Lemma 32.** If $v$ is contained in at least two wedge cycles, then $L(v)$ is contractible.

That lemma can be proved as the Claim in the proof of Proposition 17 in [3]. We will briefly sketch the argument of the proof.

The set of half edges $E_v$ at $v$ is the union of half edges $A = \{a_1, \overline{a}_1, \ldots, a_r, \overline{a}_r\}$ contained in some wedge cycle $B_i$ and $B = \{b_1, \ldots, b_s\}$ not contained in any wedge cycle. Fix an element $a \in A$ and define the *inside* of an ideal edge to be the side containing $a$, and the *size* to be the number of half edges on the inside. By hypothesis, $r \geq 2$. The lemma is proved by induction on $s$. If $s = 0$, consider the ideal edge $\alpha$ that separates $a$ and $\overline{a}$ from all the other half edges. Let $st(\alpha)$ denote the star of $\alpha$ in $L(v)$. By adding vertices of $L(v) \setminus st(\alpha)$ in order of increasing size, $L(v)$ deformation retracts onto $st(\alpha)$. Therefore, $L(v)$ is contractible. The inductive step can be proved in a similar way.

That concludes the proof of the theorem. \qed

See Figure 16 for an example of the deformation retraction of $S_n(A)$ onto $D_n(A)$. 19
Corollary 33. The small spine $D_n(A)$ is contractible.

5 Virtual Cohomological Dimension of Out$(F_n; A)$

In this section we obtain a corollary of the fact that the relative outer space is contractible computing the virtual cohomological dimension of Out$(F_n; A)$.

Theorem 34. We have

$$\text{vcd}(\text{Out}(F_n; A_1, \ldots, A_k)) = 2n - 2s(1) - \cdots - 2s(k) + 2k - 2 - m,$$

where $s(i_1) = \cdots = s(i_m) = 1$ and $s(j) > 1$ for $j \neq i_1, \ldots, i_m$.

Proof. Suppose $s(i_1) = \cdots = s(i_m) = 1$ and $s(j) > 1$ for $j \neq i_1, \ldots, i_m$.

Recall that we consider $F_n = \langle y^i_1, \ldots, y^i_k, x_1, \ldots, x_{n - \sum_{i=1}^k s(i)}, \bar{y}^i_1 \rangle$.

We denote $\theta^i = (y^j_1, \ldots, y^j_{s(i)})$. Reordering the $y$’s if necessary, we can suppose $i_1 = k - m + 1, \ldots, i_m = k$. Consider the quotient map Aut$(F_n) \to$ Out$(F_n)$.

In order to compute the lower bound, we notice that the image of the Abelian subgroup of Aut$(F_n)$

$$A = \langle \alpha_i, \beta_i, \gamma_j, \delta_r | 1 \leq i \leq n - \sum_{i=1}^k s(i), 1 < j \leq k, 1 < r \leq k - m \rangle,$$

where $\alpha_i$ fixes all the elements of the basis except $x_i \mapsto y^i_1x_i$, $\beta_i$ fixes all the elements of the basis except $x_i \mapsto x_i\bar{y}^i_1$, $\gamma_j$ fixes all the elements of the
basis except $\theta^j \mapsto y_1^i \theta^j y_1^i$ and $\delta_r$ fixes all the elements of the basis except $\theta^r \mapsto y_1^i \theta^r y_1^i$, is in $\text{Out}(F_n; A)$.

Indeed, obviously $\{\alpha_i\}_{i=1,...,n-\sum_{i=1}^k s(i)}$ and $\{\beta_i\}_{i=1,...,n-\sum_{i=1}^k s(i)}$ commute with all the generators of the subgroup. Because we have

$$
\gamma_i \circ \delta_i(y_j^i) = \gamma_i(y_1^i y_j^i y_1^i) = y_1^i y_j^i y_1^i y_1^i y_1^i y_1^i = y_1^i y_j^i y_j^i y_1^i y_1^i = \delta_i(y_1^i y_j^i) = \delta_i \circ \gamma_i(y_j^i),
$$

$\gamma_i$ commutes with $\delta_j$ for all $i$ and $j$. Therefore, $A$ is Abelian.

It remains to check that all the basis elements are independent. Notice that $\{\alpha_i, \beta_j\}_{i,j=1,...,n-\sum_{i=1}^k s(i)}$ are independent (the proof is analogous to the one for $\text{Out}(F_n)$, see [5]) and that $y_1^i \gamma_j y_1^i$ is the conjugation by $y_1^i$ of all the elements except $\theta^j$. Moreover, any compositions of conjugates of elements in the free Abelian subgroup $D$ generated by $\alpha_i$ and $\beta_i$ for $1 \leq i \leq n-\sum_{i=1}^k s(i)$ cannot be equal to $\gamma_j$ or $\delta_j$, for all $1 < j \leq k$, $1 < r \leq k-m$. Indeed, for all $1 < j \leq k$, $1 < r \leq k-m$ we have

1. $\gamma_j(x_i) = x_i$, $\delta_r(x_i) = x_i$, $\forall i \in \{1,...,n-\sum_{i=1}^k s(i)\}$;

2. $\gamma_j(y_p^1) = y_p^1$, $\delta_r(y_p^1) = y_p^1$, $\forall p \in \{1,...,s(1)\},$

and a composition of conjugates of elements in $D$ contradicts (1) or (2). For example, let $F$ be the conjugate by $y_1^i$ of

$$
\alpha_1^{-1} \circ \cdots \circ \alpha_{n-\sum_{i=1}^k s(i)}^{-1} \circ \beta_1^{-1} \circ \cdots \circ \beta_{n-\sum_{i=1}^k s(i)}^{-1}.
$$

We have $F(x_i) = x_i$, but $F(y_p^1) = y_1^i y_p^1 y_1^i$ for $p \in \{1,...,s(1)\}$.

Now we proceed by induction. We start considering the subgroup $D_1$ generated by $\gamma_2, \alpha_i, \beta_i$ for $1 \leq i \leq n-\sum_{i=1}^k s(i)$. An argument similar to the previous one shows that a composition of conjugates of elements in $D_1$ cannot be equal to $\gamma_3$. Hence, $\{\alpha_i, \beta_i, \gamma_2, \gamma_3\}$ are independent. By induction, $\{\alpha_i, \beta_l, \gamma_j\}$ are independent for $1 \leq i, l \leq n-\sum_{i=1}^k s(i)$, $1 < j \leq k$.

Again using a similar argument, it is easy to prove (by induction) that $\{\alpha_i, \beta_l, \gamma_j, \delta_r\}$ are independent for $1 \leq i, l \leq n-\sum_{i=1}^k s(i)$, $1 < j \leq k$, $1 < r \leq k-m$.

Hence, there is an Abelian free group of rank $2n-2\sum_{i=1}^k s(i)+2k-2-m$ contained in our group.

For an upper bound, we compute the dimension of the small spine. Suppose that the wedge cycles lie in a maximally blown up graph in the small
spine and that the graph has $V$ vertices and $E$ edges. The vertices corresponding to the basepoints of the wedge cycles have valence $2s(1), \ldots, 2s(k-m)$ and the remaining vertices have valence 3.

Therefore,

$$E = \frac{3(V - k + m)}{2} + s(1) + \cdots + s(k - m) = \frac{3V - 3k + 3m + 2s(1) + \cdots + 2s(k - m)}{2}.$$

Because $V - E = 1 - n$, we get

$$V = 2n + 3k - 2s(1) - \cdots - 2s(k - m) - 3m - 2.$$

Because the wedge cycles must stay disjoint, we can collapse $V$ vertices to $k$ vertices (which is the number of wedge cycles). Then,

$$\dim(D_n(A)) = 2n + 2k - 2s(1) - \cdots - 2s(k - m) - 3m - 2.$$

Because $s(k - m + 1) = \cdots = s(k) = 1$,

$$\vcd(\text{Out}(F_n; A)) \leq 2n + 2k - 2s(1) - \cdots - 2s(k - m) - 3m - 2 = 2n - 2\sum_{i=1}^{k} s(i) + 2k - 2 - m.$$

The result follows from Theorem 11.

\[ \Box \]

**Corollary 35.** If $m = 0$, then

$$\vcd(\text{Out}(F_n; A_1, \ldots, A_k)) = 2n - 2s(1) - \cdots - 2s(k) + 2k - 2.$$

Our computation of the virtual cohomological dimension of the relative outer space agrees with the computation in \[3\] when $m = k$. For $k = n$ and $s(1) = \cdots = s(k) = 1$, Out$(F_n; A_1, \ldots, A_k)$ is called the *pure symmetric automorphism group*. In \[4\] Collins showed that the virtual cohomological dimension of the pure symmetric automorphism group is $n - 2$.

**Remark 36.** Suppose that the wedge cycles lie in a maximally blown up graph in $c$ connected components in the relative spine $S_n(A)$ and that the graph has $V$ vertices and $E$ edges. Because two wedge cycles must meet in a valence 4 vertex and the dual graph of the wedge cycles is a forest, there are $k - c$ vertices of valence 4 in the graph. The vertices corresponding to the basepoints of the wedge cycles have valence $2s(1), \ldots, 2s(k - m)$. The remaining vertices have valence 3.
Therefore,

\[ E = \frac{3(V - 2k + c + m)}{2} + s(1) + \cdots + s(k - m) + 2(k - c) = \]

\[ = \frac{3V - 2k - c + 3m + 2s(1) + \cdots + 2s(k - m)}{2}. \]

Because \( V - E = 1 - n \), we get

\[ V = 2n + 2k + c - 2s(1) - \cdots - 2s(k - m) - 3m - 2. \]

We can collapse \( V \) vertices to 1 vertex by first collapsing all except one edge in each cycle \( \phi(C_i^j) \) and then collapsing some remaining tree. That gives a simplex of dimension \( 2n + 2k + c - 2s(1) - \cdots - 2s(k) - 3 - m \) \((c \leq k)\).

Collapse all the separating edges and note that the maximum \( c \) is \( k \) if \( n - \sum_{i=1}^{k} s(i) \geq 1 \) and 1 if \( n = \sum_{i=1}^{k} s(i) \).

If \( n - \sum_{i=1}^{k} s(i) \geq 1 \), then the maximum \( c = k \) and the dimension of a maximal simplex is \( 2n + 3k - 2s(1) - \cdots - 2s(k) - 3 - m \). Notice that if \( k = 0 \), then \( m = 0 \) and we get the classical result \( \text{vcd}(\text{Out}(F_n)) \leq 2n - 3 \).

If \( n = \sum_{i=1}^{k} s(i) \), then the maximum \( c = 1 \) and the dimension of a maximal simplex is \( 2n + 2k - 2s(1) - \cdots - 2s(k) - 2 - m \).

Hence,

\[ \dim(S_n(A)) = \begin{cases} 
2n + 3k - 2 \sum_{i=1}^{k} s(i) - 3 - m, & \text{if } n - \sum_{i=1}^{k} s(i) \geq 1 \\
2n + 2k - 2 \sum_{i=1}^{k} s(i) - 2 - m, & \text{if } n = \sum_{i=1}^{k} s(i). 
\end{cases} \]

Notice that if \( n = \sum_{i=1}^{k} s(i) \), then \( \dim(S_n(A)) = \dim(D_n(A)) \) (see Example 23).

Because a maximal graph has \( 3n + 3k - 2 \sum_{i=1}^{k} s(i) - 3 - m \) edges if \( n - \sum_{i=1}^{k} s(i) \geq 1 \) and \( 3n + 2k - 2 \sum_{i=1}^{k} s(i) - 2 - m \) edges if \( n = \sum_{i=1}^{k} s(i) \) and we impose the conditions that the relative volume is 1 (if \( n - \sum_{i=1}^{k} s(i) \geq 1 \)) and that the sum of the length of the edges in each cycle is 1, we have the following result.

**Corollary 37.**

\[ \dim(CV_n(A)) = \begin{cases} 
3n + 3k - 3 \sum_{i=1}^{k} s(i) - 4 - m, & \text{if } n - \sum_{i=1}^{k} s(i) \geq 1 \\
3n + 2k - 3 \sum_{i=1}^{k} s(i) - 2 - m, & \text{if } n = \sum_{i=1}^{k} s(i). 
\end{cases} \]

Note that if \( k = 0 \) and \( n > 1 \), then \( m = 0 \) and we have \( \dim(CV_n) = 3n - 4 \).
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