Conformal geometrodynamics regained: gravity from duality

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October 8, 2013

Abstract

I propose the following conjecture: the conformal reduction of Hamiltonian general relativity is the sole reduced theory that allows description in the canonical metric phase space by dual spatially covariant theories, each possessing different symmetry content than the other. One of the symmetries is the usual refoliation symmetry of general relativity in 3+1, and the other, its dual, is spatial Weyl symmetry. I prove the conjecture under mild extra assumptions.

“The childhood shows the man,
As morning shows the day.”
John Milton, Paradise Regained.

1 Introduction

In the golden years of the canonical approach to general relativity, one of the most profound thinkers on gravity, John Wheeler, posed the famous question [1]: “If one did not know the Einstein-Hamilton-Jacobi equation, how might one hope to derive it straight off from plausible first principles, without ever going through the formulation of the Einstein field equations themselves?”. In response, Hojman, Kuchař and Teitelboim (HKT), in the aptly titled paper “Geometrodynamics regained” [2], tackled the problem of deriving geometrodynamics directly from first principles rather than by a formal rearrangement of Einstein’s law. They succeeded in obtaining the canonical form of general relativity by imposing meaningful requirements onto a Hamiltonian formulation, ensuring that it represented a foliated space-time.

Here I propose a different answer to Wheeler’s question. Namely, I will look for what I refer to as dual symmetries in the gravitational phase space. Dual symmetries consist of two distinct sets of constraints, which I refer to as the dual partners. Each dual partner should be first of all a (spatially covariant) first class constraint – which by Dirac’s analysis means that each generates a (spatially covariant) symmetry – and secondly, to fix the partnership and establish duality, one partner must gauge-fix the other. In figure 1, we see two first class constraint surfaces intersecting transversally, which illustrates the rather simple principle. These are the central principles that will guide my search, and under mild extra assumptions yield a unique result consisting of the dual symmetries \{ spatial scale \} \times \{ simultaneity \}.

In the case of HKT, they used the fact that the set of vector fields that generate the tangential and normal deformations of an embedded hypersurface in a Riemannian manifold produce a vector commutation algebra.

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Figure 1: First-class constraint surfaces in phase space intersecting transversally. The gauge orbits of one of the constraints is indicated by the dotted lines emanating from the intersection, where the reduced theory lies.
They then sought constraints in the the space of functionals of the spatial metric $g_{ab}$ and momenta $\pi^{ab}$, whose own commutation algebra (Poisson bracket) would mirror the hypersurface deformation algebra. With a few other requirements, they were eventually led to the (super)momentum constraint $H_{\mu}(x) = \nabla_{\mu} \pi^{\mu}(x) = 0$ and the (super) Hamiltonian constraint: \( S(x) = R\sqrt{g} - \frac{\pi^{a} \nabla_{a} \pi^{b}}{\sqrt{g}} = 0 \) (where $\pi = g_{ab} \pi^{ab}$), which are responsible for the entire dynamics of space-time in the Hamiltonian formulation of general relativity. That is, these two functionals have associated symplectic vector fields, which (due to their first class property) generate symmetry transformations in phase space. The Hamiltonian (or scalar) constraint $S(x) = 0$ generates (on-shell) reconfigurations of space-times (i.e. a description of space-time by a different set of surfaces of simultaneity), while the momentum constraint (or diffeomorphism constraint) generates foliation-preserving spatial diffeomorphisms.

It is unknown if the HKT answer satisfied John Wheeler, or if any of the other construction principles that lead more or less uniquely to GR ever did. In the case of gravity there is an almost abundant variety of such principles. Very likely the most well-known was introduced by Lovelock [3], building on earlier work by Weyl. Lovelock showed that the Einstein tensor is the unique generally covariant divergence-free tensor with 2 derivatives of the metric in 4-dimensions. Other construction principles based on the spin-2 character of the graviton have also been introduced as early on as 1962 by Feynman [4], and as late as the thermodynamic–based derivation in 1995 by Tod Jacobson [5]. I point the interested reader to box 17.2 of [6], entitled “6 Routes to Einstein’s geometrodynamical law.”

One way I can justify a shifting in the understanding of a theory to different principles is by just appealing to the intrinsic value of alternative ontologies. Quoting HKT: “The importance of alternate foundations of a basic physical theory cannot be overemphasized. The conceptual reformulation of a theory may open a new path to its development or even lead to its modification. Thus, Feynman’s path-integral approach to quantum field theory led to the implementation of powerful approximation techniques, and Faraday’s reformulation of action-at-a-distance stationary electrodynamics in terms of the field concept developed into Maxwell’s electrodynamics. In this spirit, believing in the potential fruitfulness of the canonical variational-differential approach to the general theory of relativity, I have undertaken the study of a Seventh Route to Einstein’s.”

In the spirit of HKT, I will implement certain conditions on phase space functionals that will lead more or less directly to my results. Unlike HKT, I will not try to match algebras of constraints. In fact, I will not even assume the existence of space-time (and hypersurface embeddings therein), but only of the gravitational canonical phase space. It is thus even more surprising that any space-time structure emerges. In the interest of full disclosure, departing from my first principles I will not be able to recover the full range of solutions to Einstein gravity, only those that have complete maximal slicing foliations (i.e. those space-times that can be sliced with hypersurfaces with vanishing trace of the extrinsic curvature).

The underlying reason for this restriction can be seen as follows. There are constraints arising in the Hamiltonian formulation of general relativity – also referred to as the Arnowitt-Deser-Misner (ADM) formulation [7] – due to a redundant description of an underlying relational theory. But it is possible to reduce the theory to a physical phase space by the well-known York conformal method [8], which basically exhausts the spatial conformal degrees of freedom of the metric to solve the scalar initial value constraint $S(x) = 0$. This construction gives rise to what is known as conformal superspace [9]. The issue is that a continuous solution curve of the reduced Hamiltonian ADM equations in conformal superspace can only uniquely rebuild space-times with complete maximal slicing or constant mean curvature (CMC) foliations.

The root cause of this property was only uncovered and explored much later [10]: the trace of the momenta, $\pi$, serves a double role in the formalism. It can be seen as the maximal slicing gauge fixing for ADM, $\pi = 0$, but also as a generator of spatial Weyl transformations.

\[
\{ \pi(\rho), g_{ab}(x) \} = \rho(x)g_{ab}(x) =: \delta_{\rho}g_{ab}(x) \\
\{ \pi(\rho), \pi^{ab}(x) \} = -\rho(x)\pi^{ab}(x) =: \delta_{\rho}\pi^{ab}(x)
\]

where I denoted the smearing $\int d^{3}x \rho \pi = \pi(\rho)$ and the canonical Poisson brackets $\{ g_{ab}(x), \pi^{cd}(y) \} = \delta(x, y)\delta^{(cd)}$. This property is the inspiration for the criteria I use in my own construction principle, which can be encapsulated in the question: ‘Are there other sets of symmetries that gauge-fix each other, besides Weyl and reconfigurations?’

It turns out that out of all the possible sets of such dual symmetries, the reconfiguration symmetry generated by the scalar constraint and the Weyl symmetry generated by the Weyl constraint are the only ones that satisfy my requirements. Phrasing it in the vein of the quoted paragraph of HKT’s: “In this spirit, believing in the potential fruitfulness of dual descriptions of gravity, I have undertaken this study, which leads to another Route to Einstein’s.” I thus “regain” the conformally reduced ADM gravity, also known as conformal geometrodynamics.

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1Also as HKT, I have found an appropriate quote from John Milton’s “Paradise Regained”.

2Also called in the mathematical literature conformal transformations, which generates some confusion in the communication between the two fields. I should also note that for maximal slicing on compact manifolds there are restrictions on the Yamabe class of the metrics, restrictions required for the method to work. These technical requirements can be overcome by enlarging the possible foliations to be those of constant mean curvature (CMC). The associated conformal transformations are then those that preserve the total volume of space.
In favor of the 3+1 representation, Dirac wrote [11]: “A few decades ago it seemed quite certain that one had to express the whole of physics in four-dimensional form. But now it seems that four-dimensional symmetry is not of such overriding importance, since the description of nature sometimes gets simplified when one departs from it.” Julian Barbour and collaborators, motivated by relationalist ideas, came to the conclusion that physical theories should have a principle of “relative spatial scale” [12]. It is my hope that providing this new first principles derivation of gravity, I go beyond HKT’s insistence in considering the 3+1 Hamiltonian formalism not only as a technical tool, and give further evidence to Barbour’s relational program. The present results set Hamiltonian 3+1 gravity as an equivalent ontology to, and independent from, space-time, while at the same time elevating spatial scale relationalism to a principle on par with relativity of simultaneity. In the conclusions, I also sketch a dynamical selection principle for my requirements from BRST quantization and arguments coming from renormalization flows in the space of theories.

2 Construction

The results of this section are a shortened version of those contained in [13], which I refer the reader for the more complete treatment and the full calculations, which can become rather involved.

My method here will be to look for certain restrictions on functionals in the phase space of gravity. This phase space is coordinitized by the spatial metric and its conjugate momentum \((g_{ab}(x), \pi_{ab}(x))\) where \(x \in M\), a 3-dimensional manifold which I assume for technical simplicity to be compact without boundary (closed). The functionals should obey the following conditions: i) be scalar, ii) be first class with respect to the spatial diffeomorphism constraint (also referred to as the super-momentum constraint), and iii) gauge-fix each other. Requiring that the constraints be scalar will give the correct number of physical propagating degrees of freedom for gravity, the first class property ensures that they generate symmetries themselves, and the gauge-fixing property with respect to each other (second class) ensures that their symmetries are dual to each other.

First and foremost I emphasize the way in which I restrict my search. I demand of my candidate terms the following requirements:

- The constraints must be scalar. That is, they must represent one degree of freedom per space point (so that I obtain a physical theory of two degrees of freedom per space-point).
- Individually, each set must be first class when taken in conjunction with the spatial diffeomorphism constraint (which I take to be a fundamental symmetry of my description).
- I will look for two sets of constraints that are second class with respect to each other. This just means that each will serve as a good gauge-fixing for the other.
- The terms should depend on both the metric and the momenta (so that they include time and are not purely intrinsic to the hypersurface geometry).

To sum up, I are looking for two symmetries that gauge-fix each other and are not “static” (i.e. they must include the momenta). Two further restrictions that I will make, in order to make the explicit calculations more tractable, are i) to consider only constraints with no derivative coupling, i.e. no constraints of the form \(R_{ab}\pi^{ab}\) or others with the momenta contracted with curvature, and ii) to consider only terms up to four derivatives of the metric. However, the fact that I have not included such terms does not mean that I know of a counter-example, only that the equations become more tractable with such restrictions.

The general form of the constraints I will be considering are formed by sums with arbitrary real coefficients of arbitrary contractions between \(\nabla_a\) and \(R_{ab}\) (I remind the reader that in 3 dimensions the Ricci curvature contains all the information of the full Riemann tensor) and among the \(\pi^{ab}\), up to fourth order in derivatives.\(^3\)

\[
A(\alpha, \beta, \gamma, a, b, c) := \alpha R_{ab}R_{ab} + \beta \nabla_a \nabla_b R_{ab} + \gamma \nabla^2 R + \mu_n R^n + \frac{\alpha \pi^{ab}_a \pi^{ab}_b + b \pi^2}{g} + c \frac{\pi}{\sqrt{g}}
\]

(1)

where we use the summation rule for the integer parameter \(n\). It is remarkable that such minimal assumptions generate the strong results I present below.

2.1 The first class property

The first step in the direction of my result is to prove the theorem:

\(^3\)I note that due to the Bianchi identity I could substitute the term \(\nabla_a \nabla_b R_{ab}\) by a term proportional to \(\nabla^2 R\). I have kept both terms here for clarity, and due to the fact that the calculations with \(\nabla_a \nabla_b R_{ab}\) are useful when calculating the 6-th derivative terms.
Theorem 1 Given the constraints $A = 0$ and $\nabla_a \pi^{ab} = 0$, where $A$ is given by (1), the only choice of coefficients which have at least one of $a, b, c \neq 0$ for which $A$ weakly commutes with itself, are $(\alpha, \beta, \gamma, \mu_n, a, b, c) = (0, 0, \mu_1, -2b, 0)$ and $(0, 0, 0, a, b, c)$.

Let us start, for illustration, with the $(\mu_1, a, b, c)$ term:

$$
\{A(\mu_1, a, b, c)(N_1), A(\mu_1, a, b, c)(N_2)\} = \mu_1 \int d^3x N_2 \left(-\nabla^2 N_1 g^{ab} + \nabla^b \nabla^a N_1\right)(2a \pi_{ab} + 2b \pi g_{ab} + cg_{ab}) - (N_1 \leftrightarrow N_2)
$$

This term already presents many of the features needed in the other calculations. First, note that the last term $\int N_2(\nabla^2 \nabla^a N_1 \pi_{ab}) - (N_1 \leftrightarrow N_2)$ can be set proportional to the diffeomorphism constraint and thus vanishes on-shell. The other terms cannot be set proportional to $A(\mu_1, a, b, c)$ nor to some of the higher order terms $\alpha, \beta, \gamma, \mu_n$, nor to the diffeomorphism constraint, unless $a = -2b, c = 0$ (no imposition on $\mu_1$ in this case). Furthermore, it can be checked that they cannot be canceled by including the Poisson bracket of terms of higher order of derivatives [13]. So in order for the constraints to commute for $\mu_1 \not= 0$, it must be true that $a = -2b$ and $c = 0$. Thus we see the main reason to start with the $\mu_1 \not= 0$ term due to there being only one covariant scalar formed from second derivatives of the metric (namely $R$), as opposed to what occurs with fourth derivative onward, one can draw conclusions regarding the coefficients of this term without reference to the higher order derivative terms. In a certain manner this part of the proof is utilizing Lovelock’s result.

The commutation of the term $A(\mu_2, a, b, c)$ is exactly the same with the substitution $N_1 \to 2RN_1$ in the first term:

$$
\{A(\mu_2, a, b, c)(N_1), A(\mu_2, a, b, c)(N_2)\} = 2\mu_2 \int d^3x N_2 \left(-\nabla^2 (RN_1)((2a + 4b) \pi + 2c) + 2a(RN_1)^{ab} \pi_{ab}\right) - (N_1 \leftrightarrow N_2)
$$

Although the previous analysis goes through for $\nabla^2 (RN_1)$, the terms $N_2(RN_1)^{ab} \pi_{ab} - (N_1 \leftrightarrow N_2)$ are no longer proportional to any combination of the diffeomorphism constraint. The same analysis follows for all powers of $R$.

The strategy to tackle the more complicated terms $\alpha, \beta, \gamma$ is to first see what happens with the $\alpha$ term. Irrespective of what happens with anything else, one cannot get rid of contractions of the Ricci curvature with the momenta (even after enforcing the momentum constraint), which are not included in my original set (1), and thus cannot be set weakly to zero. A little algebra shows that this sets either $a = 0$ or $\alpha = \beta = \gamma = 0$. If one sets $\alpha = \beta = \gamma = 0$ then the previous analysis is complete for the commutation relations of $A$. If one chooses the $a = 0$ option, this implies that the $\mu_1 R$ term cannot be included, since as I showed above, for $a = 0$ and $\mu_1 \neq 0$, then, irrespective of the higher order terms, $b = 0$ and $c = 0$ (and one of my conditions was exactly that I need to have momentum dependent constraints, which sets one of $a, b, c$ to be non-zero). From this point, a little more algebra shows that the commutation relations of the $\alpha, \beta, \gamma$ terms with either $b$ or $c$ terms necessarily produce terms proportional to $R$, which cannot be weakly zero (as $\mu_1 = 0$ and they don’t factorize). This is proven in [13] and completes the (gist of the) proof of theorem 1.

2.2 The mutual gauge-fixing property

What do I mean by saying that two sets of constraints $A(x)$ and $B(x)$ gauge-fix each other? The usual criterion is that one can find a unique intersection between the orbits of one first class constraint and the surface defined by the other. The infinitesimal criterion is that the bracket $\{A(x), B(y)\} \approx \Delta(x, y)$ be an invertible operator on the intersection $A(x) = B(y) = 0$. The operators $\Delta(x, y)$ can be classified in terms of their principal symbol: they can be elliptic, parabolic or hyperbolic. Parabolic and hyperbolic operators are the ones commonly used to describe heat and wave equations, and will immediately be disallowed by my conditions since in that case

$$
\Delta N(x) := \int d^3x \Delta(x, y)N(x) = 0
$$

has an infinite-dimensional set of non-trivial solutions, for which one cannot find a prescription to invert $\Delta$. This breaks the duality. For elliptic operators on a closed manifold on the other hand, there can be at most a finite-dimensional kernel. Say there is a single generator of the kernel $N_o$ satisfying the appropriate boundary conditions, then the operator $\Delta$ should be inverted by adding the kernel in the following way $\Delta^{-1} \Delta u(x) = u(x) + N_o$. This is equivalent to the usual completion by homogeneous solutions to Green’s functions [15] and related to the inclusion of (a finite number of) poles in Feynman propagators. In any case, I will show that the elliptic operators that I do obtain have trivial kernel on the intersection surface.

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4 Analogous calculations have been performed in [14], obtaining results consistent with ours. There they considered either $R^\mu\nu$ or $b \not= 1$, and never the two at the same time, and they did not consider the pure $(0, a, b, c)$ terms.
The second part of the proof of the present results thus goes as follows. We have the following set of constraints (where the $\alpha, \beta, \gamma$ components have been eliminated from the notation):

$$(\mu_n, a, b, c) = \{(\mu_1 - 2b, b, 0), (0, a, b, c) \mid \mu_1, a, b, c \in \mathbb{R}\}$$

each of which forms a first class system when taken together with the diffeomorphism constraint.

In order to illustrate the power of this second principle on its own rights, I will calculate the restrictions it implements on more general forms of constraints than the ones allowed by theorem 1. Namely, I will look at all the constraints of the following form: $$(\mu_n, a, b, c) = \{(\mu_n - 2b, b, 0), (0, a, b, c) \mid \mu_n, a, b, c \in \mathbb{R}\}$$ thus allowing also $R^n$ terms.

### 2.2.1 Cross-terms: $(\mu_n, -2b, b, 0)$ and $(0, a, b', c)$.

I will here look at the calculation between possible pairs $(\mu_n, -2b, b, 0)$ and $(0, a, b', c)$. The calculation for the pairs $(\mu_n, -2b, b, 0), (\mu_n', -2b', b', 0)$ and $(0, a, b, c), (0, a', b', c')$ (for distinct primed and unprimed variables) follows from the same arguments as the ones used for the cross-terms [13].

The general Poisson bracket for my starting term is:

$$\{A(\mu_n, -2b, b, 0)(N), A(0, a, b', c)(x)\} := \Delta N$$

$$= \left\{ \int d^3x' \sqrt{g} N \left( \mu_n R^n + \frac{-2b_n \pi_{ab} b + b' \pi_{ab}}{g} \right)'(x'), \left( \frac{\sqrt{g}}{\sqrt{g}} N \pi_{ab} + b' \pi_{ab} + c \pi \right)'(x) \right\}$$

where just as a reminder, I am using the summation rule for $n$, and I have defined the phase space dependent second order differential operator $\Delta$ acting on the smearing function $N$.

I would like to make choices for the coefficients $a, b, b', c, \mu_n$ such that the operator is invertible. The terms with the highest order derivatives acting on $N$ can be calculated to be:

$$n \mu_n \left( R^{n-1} \nabla^2 N \left( -\frac{2\pi}{\sqrt{g}} (2b' + a) - 2c \right) + 2a R^{n-1} \nabla^2 N \pi_{ef} \right)$$

Here, for $n > 1$, the principal symbol is phase space dependent even on the intersection of the constraint surfaces. It is not hard to see that it fails to be elliptic on at least some subset of the intersection surface and has there an infinite-dimensional kernel. This is the simple criterion that sets $\mu_n = 0$ for $n > 1$ for the cross terms. One can do a similar calculation for the “diagonal” terms $(\mu_n, -2b, b, 0), (\mu_n', -2b', b', 0)$ and $(0, a, b, c), (0, a', b', c')$.

With these restrictions, $\Delta N$ becomes:

$$\Delta N := c \left( N \left( -3b \frac{G_{abcd} \pi_{ab} \pi_{cd}}{\sqrt{g}} + \mu R \sqrt{g} \right) + \frac{1}{2} \mu R \sqrt{g} \right) + \mu (2 \nabla^2 N)$$

which is elliptic, and generically will have only the trivial kernel but still might have a non-zero finite-dimensional kernel at some singular subsets of phase space. I stop here, since as I have mentioned before, a finite-dimensional kernel does not present an obstruction to constructing an inverse operator by introducing the homogeneous solutions to the Green’s functions.

Thus I obtain the following:

**Theorem 2** Given the two sets of constraints $\{A_i = 0 \text{ and } \nabla_a \pi_{ab} = 0\}_{i=1,2}$, where $A_i(\alpha_i, \beta_i, \gamma_i, \mu_i, a_i, b_i, c_i)$ is given by (1), the only choices of coefficients which: i) have at least one of $a, b, c \neq 0$, ii) for which $A_i$ weakly commutes with itself, and iii) the commutator between $A_1$ and $A_2$ is invertible on the intersection of the constraint surfaces $A_1 = A_2 = 0$ are: $(\alpha, \beta, \gamma, \mu_n, \mu_n - 1, \cdots, \mu_1, a, b, c) = (0, \cdots, 0, \mu_1, -2b, b, 0)$ and $(0, \cdots, 0, c)$.

As a last comment on this section, I note that here I have looked for ellipticity everywhere on the intersection surface. Relaxations of this criterion would allow for extensions of constraints with the second class property. For instance, it is easy to find such a relaxation that allows the inclusion of $\mu_n \neq 0$ if $c = a = b = 0$, for $n$ an odd number. This occurs as follows: for a single $\mu_n$ non-zero and $n$ odd, $\mu_n > 0$, $b > 0 = a = b' = 0$, at the constraint surface $\mu_n R^n g = b \pi_{ab} \geq 0$, which implies that $R \geq 0$ and thus $\mu_n R^{n-1} \geq 0$ at the constraint surface. For the open subset of the intersection surface for which $R[g] > 0$, we see from (5) that although the principal symbol would be phase space dependent, it would be elliptic at this subset of the intersection surface. I have not investigated fully what other terms are allowed by similar restrictions to subsets of the intersection surfaces.

Although the restriction of ellipticity everywhere on the intersection of the constraint surfaces could at first sight appear too strong, it turns out to be a necessary condition for duality to hold around highly symmetric
points of phase space. To be more clear, the resulting operators for the constraints not allowed by theorem 2 have a phase space dependent principal symbol, even when restricted to the intersection of the constraint surfaces $A = 0 = B$. In particular, these will all cease to be elliptic at points of phase space possessing high degree of symmetry (such as $R = 0, \pi^{ab} = 0$). At those points the operators have infinite-dimensional kernels, which means the dual description at such points is no longer valid. Furthermore, the property that such operators have phase space dependent symbol, schematically (i.e. using arbitrary contractions) $\{R^n, \pi^n\} \approx \pi^{n-1}. R^{n-1}. \Delta$ (where $\Delta$ can be elliptic), seems to hold for a wide variety of brackets between higher order terms not considered here (including with derivative coupling). This I also leave for further study.

2.3 The resulting reduced system

The resulting reduced system is known by different names: the reduced Einstein Hamiltonian [9], or the 3+1 conformal reduction of ADM [16]. It should be mentioned that I have not included in definition (1) the possible constant density term that would give rise to a cosmological constant and CMC (constant mean curvature) foliations (as opposed to maximal). Nonetheless, I will give here a description of the equations assuming that the present results go through with those additions. I.e. I will describe the CMC reduced Hamiltonian, as derived in appendix D of [17], because otherwise, for maximal slicings, a detour would have to be made into describing the entire boundary formalism. The results obtained in the maximal slicing setting are in [18].

To fully gauge fix the CMC constraint we introduce the following separation of variables:

$$(g_{ab}, \pi^{ab}) \rightarrow ([g]^{-1/3}g_{ab}, [g]^{1/3}(\pi^{ab} - \frac{1}{3} \pi g^{ab})), \frac{2\pi}{3 \sqrt{|g|}} = (\rho_{ab}, \tau, \sigma^{ab}, \tau)$$

(7)

To simplify matters, I choose a reference metric $\gamma_{ij}$ to determine a reference density weight. Then I define a scalar conformal factor $\phi := [g]/[\gamma]$, and replace, in (7) $\varphi \rightarrow \phi$. The variable $\tau$ will give rise to what is commonly known as York time. The inverse transformation from the new variables to the old is given by:

$$\pi^{ab} = \phi^{-1/3}(\sigma^{ab} + \frac{\tau}{3} \rho^{ab}), \quad g_{ab} = \phi^{1/3} \rho_{ab}$$

(8)

I can simultaneously do a phase space reduction by defining the variables $\phi := \phi_o[\tau, \sigma^{ab}, \rho^{ab}; x]$ as a solution to the Lichnerowicz-York equation [8],

$$\nabla^2 \Omega - R\Omega + \frac{1}{8} \frac{\pi^{ab} \pi_{ab}}{g} \Omega^{-7} - \frac{1}{12} \tau^2 \Omega^5 = 0$$

(9)

where $\Omega = e^\phi$, and by setting $\tau$ to be a spatial constant defining York time, i.e. $\dot{\tau} = 1$. Of course, this incorporates the fact that the gauge-fixing $\tau = t = 0$ is second class with respect to $S(x) = 0$, and thus I must symplectically reduce to get rid of these constraints. I am then left with a genuine evolution Hamiltonian:

$$S = \int dt \int d^3x (\sigma^{ab} \dot{\rho}_{ab} - \ln \phi_o - \sigma^{ab} \mathcal{L}_\xi (\rho_{ab}))$$

(10)

note that the last term $\sigma^{ab} \mathcal{L}_\xi (\rho_{ab})$ now only generates diffeomorphisms whose flux is divergenceless (incompressible).

Why introduce shape dynamics?

One of the drawbacks of using a reference density $[\gamma]$ so that my variables have a specific form of coordinate-covariance is that the projected value of any functional $F[g, \pi] \rightarrow F[\rho, \sigma]$ is a priori dependent on the auxiliary (background) metric $\gamma_{ij}$. The only case where it is independent is of course if the functional $F[g, \pi] = F[\rho, \sigma]$ is already conformally invariant. This highlights one of the main advantages of Shape Dynamics is that one has both diffeomorphism invariance and Weyl invariance intact for the full variables $g_{ab}, \pi^{ab}$ without the need to introduce auxiliary quantities.

Another drawback of using the ADM in CMC formalism is that in order to reconstruct the metric (and all the usual physically meaningful quantities calculated with the full metric), one needs to reinsert the non-local York factor. An interesting property of Shape Dynamics is that it is a fully local expression of the kinematics of the theory in terms of the physically meaningful full 3-metric. For instance, the equations of motion can be

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3The metric $\gamma_{ij}$ can be given by a homogeneous metric depending on the topology of the space in question. Having the reference metric as a functional of $g_{ij}$, i.e. $\gamma_{ij}(g)$, would immensely complicate the equations of motion (11), (12) below.

4Here we have taken the choice $\mu_1/b = 2$ which is the usual ADM Hamiltonian.
calculated by using an alternative method involving Dirac brackets and extended phase space [18], and yield (for shape dynamics):

\[ \dot{g}_{ab} = 4\rho g_{ab} + 2e^{-6\phi_{_o}}N_o \pi^{ab} + \mathcal{L}_\xi g_{ab} \]  
(11)

\[ \dot{\pi}^{ab} = N_o e^{2\phi_{_o}} \sqrt{g} \left( \mathcal{R}^{ab} - 2\nabla^a \nabla^b \phi_{_o} + 4\phi_{_{_o}}^a \phi_{_{_o}}^b - \frac{1}{2} \mathcal{R}g^{ab} + 2\nabla^2 \phi_{_o} g^{ab} \right) 
- \frac{N_o}{\sqrt{g}} e^{-6\phi_{_o}} \left( 2(\pi^{ac} \pi_{_c}^b) - \frac{1}{2} (\pi^{cd} \pi_{_d}^b) g^{ab} \right) 
- e^{2\phi_{_o}} \sqrt{g} \left( \nabla^b \nabla^a N_o - 4\phi_{_{_o}}^a (\pi_{_o}^b) - \nabla^2 N_o g^{ab} \right) + \mathcal{L}_\xi \pi^{ab} - 4\rho \pi^{ab} \]  
(12)

The conformally reduced equations of motion themselves are given by (11), (12) by imposing the restrictions \( \left\{ \pi \approx \pi, \rho = 0, N_o \right\} \approx 0 \) and the lapse \( N_o \) to satisfy the appropriate foliation condition (which is introduced via use of the Dirac brackets in the construction of shape dynamics), and are given in [16], equations 7.89 and 7.91.

### 3 Conclusions

In their paper, HKT gave their own response to Wheeler’s question “If Einstein’s law is inevitable, such a modified potential \([as R^{ab} R_{ab}]\) must be excluded. But what natural requirement, formulated directly in the geometrodynamical language, does exclude it?” My own answer is: the principle of duality. I have shown in this paper that there is basically one set of dual symmetries in the phase space of canonical gravity. To obtain the result I had to require some extra conditions on my trial terms.

Some of these requirements, such as that of no derivative coupling, seem solely technical, in order to reduce the number of possible terms. On the other hand, the requirement that each constraint include the momenta is more fundamental, and it avoids constraints that do not possess a “time” component at all. If this requirement is dropped the number of dual symmetries is enlarged.

Regarding the technical requirements, it is quite likely that the principles utilized here can be used for a more general class of constraints, making such requirements obsolete. In particular, the mutually gauge-fixing property – manifested by invertibility of the Poisson bracket – seems to have been considerably underused in the present study. The fundamental property that makes this requirement so restrictive is that Poisson brackets in the present study. The fundamental property – manifested by invertibility of the Poisson bracket – seems to have been considerably underused more general class of constraints, making such requirements obsolete. In particular, the mutually gauge-fixing is dropped the number of dual symmetries is enlarged.

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### Final remarks

What really drove the efforts of HKT was a mixture of aesthetic requirements and a hope that a purely geometrodynamical formulation would be better suited for quantization. In their words: “A desire to have geometrodynamics derived from purely geometrodynamical principles is aesthetic in its origin. There is, however, yet another motivation for undertaking such an enterprise. The language of geometrodynamics is much closer to the language of quantum dynamics than the original language of Einstein’s law ever was. ... Thus, one may hope that the first principles of geometrodynamics can be adapted more readily to the quantum theory and lead to a deeper understanding of how the macroscopic spacetime theory grows from the quantum geometrodynamical roots.” As my final remarks, let me take the issues of aesthetics and a quantum mechanical appropriate representation into consideration.

In my view, requiring a dual description of gravitational phenomena is already a powerful enough principle in its own right. I see the fact that it is not based on a space-time perspective at all as a strength, not a weakness. But I would like to go beyond the preference for one aesthetic principle over another. For this, I would like to propose a mechanism for self-selection of the principles introduced here. This would imply
not only that the construction here may require “less” or “better” principles than other such constructions of general relativistic gravity, but that the principles presented here are in fact in some sense “preferred” by Nature.

I thus conclude the paper with a foray into the realm of BRST formulations of Hamiltonian field theories. In the case of pure constraints theories (fully covariant) - such as ADM - gauge fixing terms that are also symmetry generators possess a special role in the classical BRST formulation of the gauge-fixed theory. More specifically, the gauge-fixed BRST Hamiltonian in this instance has the BRST symmetries related to both the original symmetry and to that of the gauge-fixing term. To see this simple result, termed “symmetry doubling” [10], see the appendix 3. Symmetry doubling completely fulfills, in the BRST setting, the desire expressed in the introduction, that spatial scale relationalism be considered as a principle on par with relativity of simultaneity. A sufficient condition to obtain symmetry doubling is exactly to find dual symmetries, as explained in this paper.

Thus symmetry doubling could possibly provide a BRST-symmetry based dynamical selection principle for the dual condition. It implies that systems that have a dual description have also a preferred status upon BRST quantization. Thus the duality criterion would not only be aesthetically appealing, but might provide a preferred set of theories in theory space. One could argue for example that the augmented symmetry content would make the set of theories more stable under renormalization group flows [20, 21]. In the case of the gravitational phase space, I have shown that under certain assumptions, the conformal reduction of the canonical formulation of gravity (conformally reduced ADM) is the sole theory that emerges from this criterion.

There is one additional comment that I would like to make regarding this emergence: no space-time was assumed anywhere. Einstein space-times uniquely re-emerge from the picture, but not all space-times, only those that have complete maximal slicings. Furthermore, one can always rebuild a line-element from the phase space solution curves of the reduced theory, but it sometimes happens that this line-element does not form a bona-fide geometrically 4-dimensional space-time, as the rebuilt line element might be degenerate [22, 23].

To close the paper, I find a citation from York very appropriate for my results [8]: “An increasing amount of evidence shows that the true dynamical degrees of freedom of the gravitational field can be identified directly with the conformally invariant geometry of three-dimensional spacelike hypersurfaces embedded in space-time.” I would scratch ‘hypersurfaces embedded in space-time’, but otherwise keep York’s sentence, with what I believe to be an yet increased amount of evidence.

Appendix

For a rank one BRST charge, related to the constraints \( \chi_\alpha \) with structure functions \( U_{ab}^c \), I have

\[
\Omega = \eta^a \chi_a - \frac{1}{2} \eta^b \eta^c U_{ab}^c P_c \tag{13}
\]

where \( \eta^a \) are the ghosts associated to the constraint transformations, and \( P_b \) the canonically conjugate ghost momenta.

The gauge-fixed Hamiltonian is constructed by choosing a ghost number –1 fermion, called the gauge-fixing fermion, \( \tilde{\Psi} = \tilde{\sigma}^a P_a + ... \), where \( \{ \tilde{\sigma}^a \}_{\alpha \in \mathcal{A}} \) is a set of proper gauge fixing conditions [24]. Denoting the BRST invariant extension of the on-shell Hamiltonian (where all constraints are set to vanish) by \( H_o \), the general gauge fixed BRST-Hamiltonian is written as

\[
H_{\tilde{\Phi}} = H_o + \eta^a V^\beta_{\alpha} P_\beta + \{ \Omega, \tilde{\Psi} \}, \tag{14}
\]

where \( \{ H_o, \chi_\alpha \} = V^\beta_{\alpha} \chi_\beta \) and the bracket is extended to include the conjugate ghost variables. The gauge fixing changes the dynamics of ghosts and other non-BRST invariant functions, but maintains evolution of all BRST-invariant functions. The crux of the BRST-formalism is that the gauge-fixed Hamiltonian \( H_{\tilde{\Phi}} \) commutes strongly with the BRST generator \( \Omega \). Although gauge symmetry is completely encoded in the BRST transformation \( s := \{ \Omega, . \} \), and the gauge has been fixed, the system retains a notion of gauge-invariance through BRST symmetry.

Applying this to a generally covariant theory, i.e. a system with vanishing on-shell Hamiltonian \( H_o = 0 \), I find that the gauge-fixed BRST-Hamiltonian takes the form

\[
H_{\tilde{\Phi}} = \{ \Omega, \tilde{\Psi} \}. \tag{15}
\]

Now comes the rather simple point: suppose that \( \{ \sigma^\alpha \}_{\alpha \in \mathcal{A}} \) is both a classical gauge fixing for \( \chi_\alpha \), and also a first class set of constraints: \( \{ \sigma^\alpha, \sigma^\beta \} = C^\gamma_{\alpha \beta} \sigma^\gamma \). One then can construct a nilpotent gauge-fixing \( \Psi \) with the same form as the BRST charge related to the system \( \{ \sigma^\alpha \}_{\alpha \in \mathcal{A}} \), the only difference being that ghosts and antighosts are swapped. For this I only need a gauge-fixing fermion of the form:

\[
\Psi = \sigma^\alpha P_\alpha - \frac{1}{2} P_\alpha P_\beta C_{\alpha \beta} U_{ab}^c \eta^c \tag{16}
\]
Using this gauge-fixing fermion would mean that the BRST extended gauge-fixed Hamiltonian would be invariant under two BRST transformations

\[
\begin{align*}
{s}_1 &= \{\Omega, \cdot\} \\
{s}_2 &= \{\cdot, \Psi\},
\end{align*}
\] (17)

which follows directly from the super-Jacobi identity and nilpotency of both \(\Omega\) and \(\Psi\). This implies that the BRST gauge-fixed Hamiltonian has an additional, dual symmetry. In this formalism it does not distinguish between the original Hamiltonian generating evolution and the gauge-fixing fermion, which makes them bona-fide symmetry doubling pairs. As we can see, a sufficient condition for having the symmetry doubling pairs is to find dual symmetries, as defined in the main text.

Acknowledgements

HG was supported in part by the U.S. Department of Energy under grant DE-FG02-91ER40674. I would like to thank Steve Carlip and Lee Smolin for reading this manuscript and giving valuable feedback.

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