LOCAL ANALYSIS OF CYCLOTONIC DIFFERENCE SETS

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Abstract. We develop a local analysis method to study cyclotomic difference sets in finite fields. This is by applying a general existence criterion for cyclotomic difference sets via Gauss sums and Gauss periods to various underlying local fields. With this local analysis approach we obtain a series of new necessary conditions. We also extend the previous nonexistence list of $m$-th-cyclotomic difference sets to $m < 34$ except $m = 28$ or 30. As an application, we study the existence of finite non-Desarguesian flag-transitive projective planes, giving a necessary condition in terms of polynomial equations over finite fields of characteristic 3.

Key words: cyclotomic difference sets; flag-transitive projective planes; Gauss sums; $p$-adic gamma function

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1. Introduction

A subset $D = \{a_1, \ldots, a_k\}$ of a finite group $G$ is said to be a $(|G|, k, \lambda)$-difference set (in $G$) or simply a difference set if for each nonidentity $a \in G$ there are exactly $\lambda$ ordered pairs $(a_s, a_t) \in D \times D$ such that $a_s a_t^{-1} = a$. For comprehensive surveys on difference sets, the reader is referred to [7, 18 Part VI] and [17]. In this paper we focus on the case where $G = \mathbb{F}_q^+$ is the additive group of a finite field $\mathbb{F}_q$ and $D \subseteq \mathbb{F}_q^\times$ is a subgroup of the multiplicative group of $\mathbb{F}_q$. More precisely, let $q = m\ell + 1$ be a power of a prime $p$ with positive integers $m$ and $\ell$ such that $1 < m < q - 1$. Then the set $H_{q,m}$ of nonzero $m$th-powers in $\mathbb{F}_q$ form a subgroup of $\mathbb{F}_q^\times$ of order $\ell$. If this is a $(q, \ell, \lambda)$-difference set in $\mathbb{F}_q^+$, then we call it an $m$th-cyclotomic difference set. When not specifying the parameters, we will call it a cyclotomic difference set for simplicity.

Research on cyclotomic difference sets dates back to Paley in 1933 [22], who actually found all the square residue difference sets. Over more than 80 years, cyclotomic difference sets have received attention of wide scope [1, 4, 8, 9, 10, 15, 19, 21, 23, 27, 28, 29]. Recently, the author posed the following conjectural classification of cyclotomic difference sets [29].

Conjecture 1.1. $H_{q,m}$ is a cyclotomic difference set if and only if one of the following appears:

1In some literature, the term $m$th-power residue difference set is used.
(a) $m = 2$ and $q \equiv 3 \pmod{4}$;
(b) $m = 4$ and $q = p = 1 + 4t^2$ for some odd integer $t$;
(c) $m = 8$ and $q = p = 1 + 8u^2 = 9 + 64v^2$ for some odd integers $u$ and $v$.

Note that the “if” parts of the three cases in Conjecture 1.1 are already known to Paly [22], Chowla [4] and Lehmer [19], respectively.

A weaker version of Conjecture 1.1 was made previously for the special case $q = p$ and verified for $q < 10^7$ by Thas and Zagier [26]. Their work roots in finite flag-transitive projective planes as there is a longstanding conjecture stating that every finite flag-transitive projective plane is Desarguesian, which is attributed to the existence problem of related cyclotomic difference sets by Proposition 1.2 below and still “wide open” [14]. For more on finite flag-transitive projective planes including Proposition 1.2, see [25, 26].

**Proposition 1.2.** If there exists a finite non-Desarguesian flag-transitive projective plane of order $n$ with $v$ points, then $v = n^2 + n + 1$ is prime and $H_{v,n}$ is a $(v, n+1, 1)$-difference set in $\mathbb{F}_v^+$ with $n > 8$ even.

In [26], Thas and Zagier gave equivalent conditions for the existence of cyclotomic difference sets in prime fields by Fermat surfaces and Gauss periods, and used the latter to determine cyclotomic difference sets $H_{q,m}$ with $q$ prime and $m < 10$ (although it was already shown by Lehmer in [19]). The same approach was actually taken earlier in the classical textbook of Berndt, Evans and Williams [2, Chapter 5] to determine cyclotomic difference sets $H_{q,m}$ with $q$ prime and $m < 10$ or $m = 12$.

In [29], the author established a criterion for cyclotomic difference sets via a system of polynomial equations on Gauss sums in $\mathbb{C}$. Based on this, it was shown that an $m$th-cyclotomic difference set must have $m$ even and Conjecture 1.1 is true up to $m = 22$. In the present paper, we consider Gauss sums in a general algebraically closed field of characteristic 0, establishing similar criteria for cyclotomic difference sets (Theorem 3.6). Applying this with Gauss sums (and related Gauss periods) in various local fields, we then obtain new necessary conditions for cyclotomic difference sets in Section 3. For instance, Theorems 3.7 and 3.9 are obtained by considering Gauss sums in the local fields $\mathbb{C}$ and $\mathbb{C}_p$, respectively, and Theorems 3.11–3.12 are from the local field $\mathbb{C}_r$ with $r$ coprime to $p$ and $m$. These necessary conditions from various local fields carry “local information” and will provide us a new approach to study the existence of cyclotomic difference sets. We call this approach local analysis of cyclotomic difference sets.

To see how the local analysis approach works, we perform the analysis in Section 4 as applications to small $m$ and finite flag-transitive projective planes. Combing local information from $\mathbb{C}_p$ and $\mathbb{C}_3$ we obtain in Theorem 4.2 a necessary condition for $m$th-cyclotomic difference sets with $m$ not divisible by 3, which immediately yields the nonexistence of 26th and 32nd-cyclotomic difference sets (Lemma 4.4).
Combining local information from $C$, $C_p$, and $C_5$ we can show that there is no 24th cyclotomic difference set (Lemma 4.5). These together with previous known results verify Conjecture 1.1 for $m < 28$ and $m = 32$ (Theorem 4.6). Finally in Theorem 4.8 we use the local information from $C_3$ to deduce a necessary condition for the existence of finite non-Desarguesian flag-transitive projective plane in terms of polynomial equations over finite fields of characteristic 3. We then conjecture that the system of polynomial equations there has no solution, which sheds light on the longstanding conjecture that every finite flag-transitive projective plane is Desarguesian.

The set $H_{q,m} \cup \{0\}$ is called a modified $m$th-cyclotomic difference set or a modified cyclotomic difference set if it is a difference set in $\mathbb{F}_q^+$ (see [29]). We mention that some of the results from the local analysis in this paper does not hold for modified cyclotomic difference sets; see Remark 2 after Theorem 3.9. However, similar ideas may be applied to establish parallel results for modified cyclotomic difference sets, which will need a separate treatment.

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2. Preliminaries

2.1. Notation. Throughout this paper, let $q = m\ell + 1 = p^f$ with prime number $p$ and positive integers $m$, $\ell$ and $f$ such that $1 < m < q - 1$. Let

$$H_{q,m} = \{a^m \mid a \in \mathbb{F}_q^\times\}$$

be the set of nonzero $m$th-powers in $\mathbb{F}_q$, which is the subgroup of $\mathbb{F}_q^\times$ of order $\ell$.

Let $\phi$ be Euler’s totient function. For any positive integer $n$, let $\Phi_n(x)$ be the $n$th cyclotomic polynomial. For any prime number $r$, let $(\cdot \, r)$ be the Legendre symbol defined by

$$\left( \frac{n}{r} \right) = \begin{cases} 
1 & \text{if } n \text{ is a square in } \mathbb{F}_r \\
-1 & \text{if } n \text{ is a non-square in } \mathbb{F}_r
\end{cases}$$

for all integer $n$ coprime to $r$.

For any prime number $r$, let $\mathbb{Z}_r$, $\mathbb{Q}_r$ and $\mathbb{C}_r$ be the ring of $r$-adic integers, the field of $r$-adic numbers and the $r$-adic completion of the algebraic closure of $\mathbb{Q}_r$, respectively. We use $k$ to denote an arbitrary algebraically closed field of characteristic 0, although in our applications $k$ is only taken to be $\mathbb{C}$ or $\mathbb{C}_r$ for some prime number $r$.

2.2. Multiplicative characters. For any integer $n \geq 2$,

$$\{x \in k \mid x^n = 1\}$$
is a cyclic group of order \( n \) under multiplication in \( \mathbb{k} \), and each generator of this group is called a **primitive \( n \)-th root of unity** (in \( \mathbb{k} \)). Under pointwise multiplication, the homomorphisms from \( \mathbb{F}_q^\times \) to \( \mathbb{k}^\times \) form an abelian group, which is denoted by \( \text{Ch}_\mathbb{k}(\mathbb{F}_q^\times) \). Denote the identity of \( \text{Ch}_\mathbb{k}(\mathbb{F}_q^\times) \) by \( \mathbb{1} \). Then \( \mathbb{1} \) is the homomorphism from \( \mathbb{F}_q^\times \) to \( \mathbb{k}^\times \) sending every element of \( \mathbb{F}_q^\times \) to \( 1 \in \mathbb{k}^\times \). For any \( \chi \in \text{Ch}_\mathbb{k}(\mathbb{F}_q^\times) \), extend \( \chi \) to a map from \( \mathbb{F}_q \) to \( \mathbb{C}_r \) by setting

\[
\chi(0) = \begin{cases} 
1 & \text{if } \chi = \mathbb{1} \\
0 & \text{if } \chi \neq \mathbb{1}
\end{cases}
\]

and call this map a (**\( k \)-valued** multiplicative character) on \( \mathbb{F}_q \). The **order** of a \( \mathbb{k} \)-valued multiplicative character \( \chi \) on \( \mathbb{F}_q \) is defined to be the order of \( \chi \) in the group \( \text{Ch}_\mathbb{k}(\mathbb{F}_q^\times) \). If \( r \) is a prime number, then a \( \mathbb{C}_r \)-valued multiplicative character is also called an **\( r \)-adic multiplicative character**.

We remark that for any element \( \chi \) of \( \text{Ch}_\mathbb{k}(\mathbb{F}_q^\times) \) and integer \( s \), the element \( \chi^s \) of \( \text{Ch}_\mathbb{k}(\mathbb{F}_q^\times) \) is also extended to a multiplicative character on \( \mathbb{F}_q \), and by \( \chi^s(\alpha) \) we mean the image of \( \alpha \in \mathbb{F}_q \) under \( \chi^s \) rather than \( (\chi(\alpha))^s \). The equality \( \chi^s(\alpha) = (\chi(\alpha))^s \) may not hold after \( \chi^s \) is extended to a multiplicative character on \( \mathbb{F}_q \). For example, if \( \chi \) has order \( s \geq 2 \) then \( \chi^s(0) = 1 \neq 0 = (\chi(0))^s \).

We need three technical lemmas on multiplicative character sums for later use.

**Lemma 2.1.** Let \( \mathbb{k} \) be an algebraically closed field of characteristic 0. Then for any \( \mathbb{k} \)-valued multiplicative character \( \chi \neq \mathbb{1} \) on \( \mathbb{F}_q \),

\[
\sum_{\alpha \in \mathbb{F}_q} \chi(\alpha) = 0.
\]

**Proof.** As \( \chi \neq \mathbb{1} \), there exists \( \beta \in \mathbb{F}_q^\times \) such that \( \chi(\beta) \neq 1 \). It follows that

\[
\sum_{\alpha \in \mathbb{F}_q} \chi(\alpha) = \sum_{\alpha \in \mathbb{F}_q} \chi(\beta \alpha) = \chi(\beta) \sum_{\alpha \in \mathbb{F}_q} \chi(\alpha)
\]

and thence

\[
\sum_{\alpha \in \mathbb{F}_q} \chi(\alpha) = 0. \quad \Box
\]

**Lemma 2.2.** Let \( \mathbb{k} \) be an algebraically closed field of characteristic 0, and \( \chi \) be a \( \mathbb{k} \)-valued multiplicative character of order \( m \) on \( \mathbb{F}_q \). Then for any integer \( s \),

\[
\sum_{\beta,\gamma \in H_{q,m}} \chi^s(\beta - \gamma) = \ell \sum_{\alpha \in H_{q,m}} \chi^s(1 - \alpha).
\]
Proof. Let $s$ be an integer. Then
\[
\sum_{\beta, \gamma \in H_{q,m}} \chi^s(\beta - \gamma) = \sum_{\beta \in H_{q,m}} \chi^s(\beta) \sum_{\gamma \in H_{q,m}} \chi^s(1 - \beta^{-1} \gamma) = \sum_{\beta \in H_{q,m}} \sum_{\alpha \in H_{q,m}} \chi^s(1 - \alpha) = \ell \sum_{\alpha \in H_{q,m}} \chi^s(1 - \alpha). \quad \Box
\]

Lemma 2.3. Let $k$ be an algebraically closed field of characteristic 0, $\chi$ be a $k$-valued multiplicative character of order $m$ on $F_q$, and
\[
A = \{ \alpha \in H_{q,m} \mid \chi(1 - \alpha) = \chi(\gamma) \}.
\]
Then for any $\gamma \in F_q^\times$,
\[
\sum_{s=0}^{m-1} \chi^{-s}(\gamma) \sum_{\alpha \in H_{q,m}} \chi^s(1 - \alpha) = m|A| + 1.
\]

Proof. Let $\gamma$ be a nonzero element of $F_q$. Then
\[
\sum_{s=0}^{m-1} \chi^{-s}(\gamma) \sum_{\alpha \in H_{q,m}} \chi^s(1 - \alpha)
\]
\[= \sum_{\alpha \in H_{q,m}} \sum_{s=0}^{m-1} \chi^{-s}(\gamma) \chi^s(1 - \alpha)
\]
\[= \sum_{\alpha \in H_{q,m} \setminus \{1\}} \sum_{s=0}^{m-1} \chi^{-s}(\gamma) \chi^s(1 - \alpha) + \sum_{s=0}^{m-1} \chi^{-s}(\gamma) \chi^s(0)
\]
\[= \sum_{\alpha \in H_{q,m} \setminus \{1\}} \sum_{s=0}^{m-1} \left( \frac{\chi(1 - \alpha)}{\chi(\gamma)} \right)^s + \sum_{s=0}^{m-1} \chi^{-s}(\gamma) \chi^s(0)
\]
\[= \sum_{\alpha \in H_{q,m} \setminus \{1\}} m + \sum_{s=0}^{m-1} \chi^{-s}(\gamma) \chi^s(0)
\]
\[= \sum_{\alpha \in H_{q,m} \setminus \{1\}} m + \chi^0(\gamma) \chi^0(0)
\]
\[= \sum_{\alpha \in H_{q,m} \setminus \{1\}} m + 1 = m|A| + 1. \quad \Box
\]

2.3. Gauss and Jacobi sums. For any $k$-valued multiplicative character $\chi$ on $F_q$ and primitive $p$th root of unity $\zeta$ in $k$, the ($k$-valued) Gauss sum $G(\chi, \zeta)$ is defined
by
\[ G(\chi, \zeta) = \sum_{\alpha \in \mathbb{F}_q} \chi(\alpha)\zeta^{\text{tr}(\alpha)}, \]
where tr is the trace map from \( \mathbb{F}_q \) to \( \mathbb{F}_p \). For any \( \mathbb{k} \)-valued multiplicative characters \( \chi \) and \( \psi \) on \( \mathbb{F}_q \), the (\( \mathbb{k} \)-valued) Jacobi sum \( J(\chi, \psi) \) is defined by
\[ J(\chi, \psi) = \sum_{\alpha \in \mathbb{F}_q} \chi(\alpha)\psi(1 - \alpha). \]

If \( r \) is a prime number, then \( \mathbb{C}_r \)-valued Gauss sums and \( \mathbb{C}_r \)-valued Jacobi sums are also called \( r \)-adic Gauss sums and \( r \)-adic Jacobi sums, respectively. It is evident from the above definition that
\[ J(\psi, \chi) = J(\chi, \psi). \]

More facts about Gauss and Jacobi sums are listed in Lemma 2.4 below, whose proof is identical to that of the case \( \mathbb{k} = \mathbb{C} \) in standard textbooks (for example [2, 5]).

**Lemma 2.4.** Let \( \mathbb{k} \) be an algebraically closed field of characteristic 0, \( \chi \) be a \( \mathbb{k} \)-valued multiplicative character of order \( n \) on \( \mathbb{F}_q \), and \( \zeta \) be a primitive \( p \)-th root of unity in \( \mathbb{k} \).

(a) For any integer \( s \) such that \( s \not\equiv 0 \pmod{n} \),
\[ G(\chi^s, \zeta)G(\chi^{-s}, \zeta) = \chi^s(-1)q. \]

(b) For any integers \( s \) and \( t \) such that \( s + t \not\equiv 0 \pmod{n} \),
\[ J(\chi^s, \chi^t)G(\chi^{s+t}, \zeta) = G(\chi^s, \zeta)G(\chi^t, \zeta). \]

(c) For any integer \( s \) such that \( s \not\equiv 0 \pmod{n} \),
\[ J(\chi^s, \chi^{-s}) = -\chi^s(-1). \]

(d) (Davenport-Hasse product formula) For any positive divisor \( d \) of \( n \) and any integer \( s \),
\[ \chi^{ds}(d) \prod_{t=0}^{d-1} G(\chi^{s+nt/d}, \zeta) = G(\chi^{ds}, \zeta) \prod_{t=1}^{d-1} G(\chi^{nt/d}, \zeta). \]

The next lemma relates certain sum of Jacobi sums to that of multiplicative characters.

**Lemma 2.5.** Let \( \mathbb{k} \) be an algebraically closed field of characteristic 0, and \( \chi \) be a \( \mathbb{k} \)-valued multiplicative character of order \( m \) on \( \mathbb{F}_q \). Then for any integer \( s \) such that \( s \not\equiv 0 \pmod{m} \),
\[ \sum_{t=1}^{m-1} J(\chi^s, \chi^t) = 1 + m \sum_{\alpha \in H_{q,m}} \chi^s(1 - \alpha). \]
Proof. Let $s$ be an integer not divisible by $m$. Then $\chi^s \neq 1$, and so
\[
\sum_{t=1}^{m-1} J_q(\chi^s, \chi^t) = \sum_{t=1}^{m-1} \sum_{\beta \in \mathbb{F}_q^\times} \chi^s(\beta) \chi^t(1 - \beta) = \sum_{\beta \in \mathbb{F}_q^\times} \chi^s(\beta) \sum_{t=1}^{m-1} \chi^t(1 - \beta) = \sum_{1-\beta \in H_{q,m}} (m-1) \chi^s(\beta) + \sum_{1-\beta \in \mathbb{F}_q^\times \setminus H_{q,m}} -\chi^s(\beta) = (m-1) \sum_{\alpha \in H_{q,m}} \chi^s(1 - \alpha) - \sum_{\alpha \in \mathbb{F}_q^\times \setminus H_{q,m}} \chi^s(1 - \alpha) = m \sum_{\alpha \in H_{q,m}} \chi^s(1 - \alpha) - \sum_{\alpha \in \mathbb{F}_q^\times} \chi^s(1 - \alpha) = 1 + m \sum_{\alpha \in H_{q,m}} \chi^s(1 - \alpha)
\]
since
\[
\sum_{\alpha \in \mathbb{F}_q} \chi^s(1 - \alpha) = \sum_{\alpha \in \mathbb{F}_q} \chi^s(\alpha) = 0
\]
by Lemma 2.1.

2.4. Discrete Fourier transform. Fix a primitive $n$th root of unity $\mu$ in $\mathbb{k}$ with integer $n \geq 2$. For a $\mathbb{k}$-valued function $X$ on $\mathbb{Z}/n\mathbb{Z}$, the discrete Fourier transform (DFT) of $X$, denoted by $\hat{X}$, is the $\mathbb{k}$-valued function on $\mathbb{Z}/n\mathbb{Z}$ defined by
\[
\hat{X}(s) = \sum_{t=0}^{n-1} \mu^{-st} X(t).
\]
We give in Lemma 2.6 two well-known formulae for DFT, the convolution formula and the inverse formula.

Lemma 2.6. Let $\mathbb{k}$ be an algebraically closed field of characteristic $0$, and $n \geq 2$ be an integer.
(a) If $W$, $X$ and $Y$ are $\mathbb{C}$-valued functions on $\mathbb{Z}/n\mathbb{Z}$ with
\[ W(s) = \sum_{t=0}^{n-1} X(t)Y(s-t) \]
for each $s \in \mathbb{Z}$, then $\hat{W}(s) = \hat{X}(s)\hat{Y}(s)$ for each $s \in \mathbb{Z}/n\mathbb{Z}$.

(b) If $W$ is a $\mathbb{C}$-valued function on $\mathbb{Z}/n\mathbb{Z}$, then $W(s) = \hat{W}(-s)/n$ for each $s \in \mathbb{Z}/n\mathbb{Z}$.

The proof of Lemma 2.6 is identical to that of the case $\mathbb{C} = \mathbb{C}$ in any standard textbook (see for example [24, page 36]).

Let $\chi$ be a $\mathbb{C}$-valued multiplicative character of order $m$ on $\mathbb{F}_q$, $\zeta$ be a primitive $p$th root of unity in $\mathbb{C}$, and $\mu$ be a primitive $m$th root of unity in $\mathbb{C}$. For each $s \in \mathbb{Z}$, let
\[ g_s(\chi, \zeta, \mu) = \sum_{t=1}^{m-1} \mu^{-st}G(\chi^t, \zeta) - 1. \]
The numbers $g_s(\chi, \zeta, \mu)/m$ are the so-called Gauss periods or cyclotomic periods (see [2, § 10.10]). They are essentially the DFT of Gauss sums.

**Lemma 2.7.** Suppose that $m$ is even. Then with $g_s = g_s(\chi, \zeta, \mu)$, the following hold.

(a) For any integer $s$ such that $s \not\equiv m/2 \pmod{m}$,
\[ \sum_{t=0}^{m-1} g_t g_{s+t} = m(1 - q). \]

(b) If $\chi(2) = \mu^\theta$, then for any integer $s$,
\[ \sum_{t=0}^{m-1} (-1)^t g_t g_{2st-2\theta-t} = \left( g_s + g_{s+m/2} \right) \sum_{t=0}^{m-1} (-1)^t g_t. \]

**Proof.** For each $s \in \mathbb{Z}$, let
\[ Y(s) = \begin{cases} G(\chi^s, \zeta) & \text{if } s \not\equiv 0 \pmod{m} \\ -1 & \text{if } s \equiv 0 \pmod{m}. \end{cases} \]
Then $Y$ is a function on $\mathbb{Z}/m\mathbb{Z}$, and
\[ g_s = \sum_{t=0}^{m-1} \mu^{-st}Y(t) \]
for any integer $s$. By Lemma 2.6(b) we have
\[
Y\left(\frac{m}{2}\right) = \frac{1}{m} \hat{Y}\left(\frac{m}{2}\right) = \frac{1}{m} \sum_{t=0}^{m-1} \left(\mu^{m/2}\right)^t \hat{Y}(t) = \frac{1}{m} \sum_{t=0}^{m-1} (-1)^t g_t.
\]
First suppose that $s$ is an integer with $s \not\equiv m/2 \pmod{m}$. Then $-\mu^{-s} \neq 1$, and by Lemma 2.4(a) we have

\[
\sum_{t=0}^{m-1} g_{t}g_{s+t} = \sum_{t=0}^{m-1} \sum_{u=0}^{m-1} \mu^{-tu}Y(u) \sum_{v=0}^{m-1} \mu^{-(s+t)v}Y(v)
\]

\[
= \sum_{u=0}^{m-1} \sum_{v=0}^{m-1} \sum_{t=0}^{m-1} \mu^{-t(u+v)}\mu^{-sv}Y(u)Y(v)
\]

\[
= \sum_{v=0}^{m-1} m\mu^{-sv}Y(-v)Y(v)
\]

\[
= m + \sum_{v=1}^{m-1} m\mu^{-sv}G(\chi^{-v}, \zeta)G(\chi^{v}, \zeta)
\]

\[
= m + \sum_{v=1}^{m-1} m(-\mu^{-s})^{v} q
\]

\[
= m - mq.
\]

This proves part (a) of the lemma.

Next suppose that $\chi(2) = \mu^{\theta}$ and $s$ is an integer. For any integer $v$ such that $1 \leq v \leq m/2 - 1$ or $m/2 + 1 \leq v \leq m - 1$, we derive from Lemma 2.4(d) that

\[
\mu^{2\theta v}G(\chi^{v}, \zeta)G(\chi^{m/2+v}, \zeta) = G(\chi^{2v}, \zeta)G(\chi^{m/2}, \zeta).
\]

Hence

\[
\mu^{2\theta v}Y \left(\frac{m}{2} + v\right) Y(v) = Y \left(\frac{m}{2}\right) Y(2v)
\]

for all integer $v$. As a consequence,

\[
\sum_{t=0}^{m-1} (-1)^t g_{t}g_{2s-2\theta-t} = \sum_{t=0}^{m-1} \sum_{u=0}^{m-1} \mu^{-tu}Y(u) \sum_{v=0}^{m-1} \mu^{-(2s-2\theta-t)v}Y(v)
\]

\[
= \sum_{u=0}^{m-1} \sum_{v=0}^{m-1} \mu^{t(m/2-u+v)}\mu^{(2\theta-2s)v}Y(u)Y(v)
\]

\[
= \sum_{v=0}^{m-1} m\mu^{(2\theta-2s)v}Y \left(\frac{m}{2} + v\right) Y(v)
\]

\[
= \sum_{v=0}^{m-1} m\mu^{(2\theta-2s)v} \cdot \mu^{-2\theta v}Y \left(\frac{m}{2}\right) Y(2v)
\]

\[
= mY \left(\frac{m}{2}\right) \sum_{v=0}^{m-1} \mu^{-2sv}Y(2v),
\]
and so by (2) we deduce that
\[ \sum_{t=0}^{m-1} (-1)^t g_t g_{2s-2t} = \sum_{t=0}^{m-1} (-1)^t g_t \sum_{v=0}^{m-1} \mu^{-2sv}Y(2v). \]
Since
\[ \sum_{v=0}^{m-1} \mu^{-2sv}Y(2v) = 2 \sum_{t=0}^{m-1} \mu^{-st}Y(t) \]
\[ = \sum_{t=0}^{m-1} (1 + (-1)^t) \mu^{-st}Y(t) \]
\[ = \sum_{t=0}^{m-1} \mu^{-st}Y(t) + \sum_{t=0}^{m-1} \mu^{-(m/2+s)t}Y(t) \]
\[ = g_t + g_{m/2+t}, \]
this leads to
\[ \sum_{t=0}^{m-1} (-1)^t g_t g_{2s-2t} = \sum_{t=0}^{m-1} (-1)^t g_t \left( g_s + g_{m/2+s} \right), \]
which proves part (b) of the lemma.

3. The approach

In this section we illustrate the local analysis approach to cyclotomic difference sets, which leads to various existence conditions. We first extend the criteria in [29] for \( H_{q,m} \) to be a cyclotomic difference set in Subsection 3.1. Then we apply these criteria to derive necessary conditions from \( \mathbb{C}_r \)-valued Gauss sums with \( r = p \) in Subsection 3.2 and \( r \neq p \) in Subsection 3.3.

3.1. Equivalent conditions. Given a \((v,k,\lambda)\)-difference set, we obtain instantly by simple counting that
\[ k(k - 1) = \lambda(v - 1). \]
Therefore, a necessary condition for cyclotomic \((q,\ell,\lambda)\)-difference set is
\[ \ell - 1 = \lambda m. \]

Throughout this subsection, let \( \chi \) be a \( k \)-valued multiplicative character of order \( m \) on \( \mathbb{F}_q \), \( \zeta \) be a primitive \( p \)-th root of unity in \( k \), and \( \mu \) be a primitive \( m \)-th root of unity in \( k \). We shall establish criteria for \( H_{q,m} \) to be an \( m \)-th-cyclotomic difference set in terms of multiplicative character sums, Jacobi sums and Gauss sums, respectively.
Lemma 3.1. Let \( \mathbb{k} \) be an algebraically closed field of characteristic 0, and \( \chi \) be a \( \mathbb{k} \)-valued multiplicative character of order \( m \) on \( \mathbb{F}_q \). Suppose \( \ell - 1 = \lambda m \) for some integer \( \lambda \geq 1 \). Then \( H_{q,m} \) is a \((q, \ell, \lambda)\)-difference set in \( \mathbb{F}_q^+ \) if and only if

\[
\sum_{\alpha \in H_{q,m}} \chi^s(1 - \alpha) = 0 \quad \text{for} \quad s = 1, \ldots, m - 1.
\]

Proof. First assume that \( H_{q,m} \) is a \((q, \ell, \lambda)\)-difference set in \( \mathbb{F}_q^+ \). Then

\[
\sum_{\beta, \gamma \in H_{q,m}} \chi^s(\beta - \gamma) = \lambda \sum_{\alpha \in \mathbb{F}_q^*} \chi^s(\alpha)
\]

for each integer \( s \), and thereby we derive from Lemma 2.1 that

\[
\sum_{\beta, \gamma \in H_{q,m}} \chi^s(\beta - \gamma) = \sum_{\beta, \gamma \in H_{q,m}} \chi^s(\beta - \gamma) = \lambda \sum_{\alpha \in \mathbb{F}_q^*} \chi^s(\alpha) = 0
\]

for \( s = 1, \ldots, m - 1 \). This leads to (4) by Lemma 2.2.

Next assume that (4) holds. Let \( \gamma \) be an element of \( \mathbb{F}_q^* \). Denote

\[
A = \{ \alpha \in H_{q,m} \mid \chi(1 - \alpha) = \chi(\gamma) \}
\]

and

\[
B = \{ (\alpha, \beta) \in H_{q,m} \times H_{q,m} \mid \alpha - \beta = \gamma \}.
\]

It is readily seen that \( (\alpha, \beta) \mapsto \alpha^{-1}\beta \) is a bijection from \( B \) to \( A \) and so \( |B| = |A| \).

By Lemma 2.3 and (4) we have

\[
m|A| + 1 = \sum_{s=0}^{m-1} \chi^{-s}(\gamma) \sum_{\alpha \in H_{q,m}} \chi^s(1 - \alpha) = \chi^0(\gamma) \sum_{\alpha \in H_{q,m}} \chi^0(1 - \alpha) = \ell.
\]

Hence \( |B| = |A| = (\ell - 1)/m = \lambda \). Consequently, \( H_{q,m} \) is a \((q, \ell, \lambda)\)-difference set in \( \mathbb{F}_q^+ \). \( \square \)

Combining Lemma 2.5 and Lemma 3.1 we obtain:

Lemma 3.2. Let \( \mathbb{k} \) be an algebraically closed field of characteristic 0, and \( \chi \) be a \( \mathbb{k} \)-valued multiplicative character of order \( m \) on \( \mathbb{F}_q \). Suppose \( \ell - 1 = \lambda m \) for some integer \( \lambda \geq 1 \). Then \( H_{q,m} \) is a \((q, \ell, \lambda)\)-difference set in \( \mathbb{F}_q^+ \) if and only if

\[
\sum_{t=1}^{m-1} J(\chi^s, \chi^t) = 1 \quad \text{for} \quad s = 1, \ldots, m - 1.
\]

The next lemma gives a criterion for \( H_{q,m} \) to be an \( m \)th-cyclotomic difference set in terms of Gauss sums.
Lemma 3.3. Let $\mathbb{k}$ be an algebraically closed field of characteristic 0, $\chi$ be a $\mathbb{k}$-valued multiplicative character of order $m$ on $\mathbb{F}_q$, and $\zeta$ be a primitive $p$th root of unity in $\mathbb{k}$. Suppose $\ell - 1 = \lambda m$ for some integer $\lambda \geq 1$. Then $H_{q,m}$ is a $(q,\ell,\lambda)$-difference set in $\mathbb{F}_q^+$ if and only if

$$m - 1 \sum_{t=1\atop t \neq s}^{m-1} \chi^{-t}(1)G(\chi^t,\zeta)G(\chi^{s-t},\zeta) = (1 + \chi^s(-1))G(\chi^s,\zeta) \quad \text{for} \quad s = 1, \ldots, m - 1. \quad (6)$$

Proof. Utilizing Lemma 2.4, one can reformulate (5) to

$$\sum_{t=1\atop s+t \neq m}^{m-1} \chi^{s+t}(1)G(\chi^{s+t},\zeta) - \chi^{s}(-1) = 1 \quad \text{for} \quad s = 1, \ldots, m - 1,$$

or equivalently,

$$\sum_{t=1\atop t \neq m-s}^{m-1} \chi^{t}(-1)G(\chi^t,\zeta)G(\chi^{m-s-t},\zeta) = 1 + \chi^{s}(-1) \quad \text{for} \quad s = 1, \ldots, m - 1. \quad (7)$$

After multiplying both sides by $G(\chi^{m-s},\zeta)$ and replacing $s$ by $m-s$, (7) turns out to be (6). Hence Lemma 3.3 follows from Lemma 3.2. □

The criterion in Lemma 3.3 is already applied with $\mathbb{k} = \mathbb{C}$ to deduce necessary conditions for cyclotomic difference sets in [29]. For example, it is shown in [29, Theorem 4.1] that $m$ must be even for any $m$th-cyclotomic difference set.

Lemma 3.4. If $m$ is even and $\ell - 1 = \lambda m$ for some integer $\lambda$, then $\chi(-1) = -1$. In particular, if $H_{q,m}$ is a cyclotomic difference set, then $m$ is even and $\chi(-1) = -1$.

Proof. If $H_{q,m}$ is a $(q,\ell,\lambda)$-difference set, then (3) holds and [29, Theorem 4.1(a)] shows that $m$ is even. Suppose that $m$ is even and $\ell - 1 = \lambda m$ for some integer $\lambda$. Then for any generator $\alpha$ of $\mathbb{F}_q^\times$, $\chi(\alpha)$ is a primitive $m$th root of unity and so $\chi(\alpha)^{m/2} = -1$. It follows that

$$\chi(-1) = \alpha^{(q-1)/2} = \alpha^{m/2} = (-1)^\ell = (-1)^{\lambda m + 1} = -1,$$

which completes the proof. □

Recall the Gauss periods $g_s(\chi,\zeta,\mu)$ defined in (1).

Lemma 3.5. Suppose $\ell - 1 = \lambda m$ for some integer $\lambda$. Then $H_{q,m}$ is a $(q,\ell,\lambda)$-difference set in $\mathbb{F}_q^+$ if and only if $m$ is even and

$$g_s(\chi,\zeta,\mu)g_{m-s}(\chi,\zeta,\mu) = 1 + (m-1)q \quad \text{for all integer} \quad s.$$
Proof. For each \( s \in \mathbb{Z} \), let
\[
Y(s) = \begin{cases} 
G(\chi^s, \zeta) & \text{if } s \not\equiv 0 \pmod{m} \\
-1 & \text{if } s \equiv 0 \pmod{m}
\end{cases}
\]
and
\[
X(s) = (-1)^s Y(s).
\]
Clearly, \( X \) and \( Y \) are both functions on \( \mathbb{Z}/m\mathbb{Z} \). Define a function on \( \mathbb{Z}/m\mathbb{Z} \) by letting
\[
W(s) = \sum_{t=0}^{m-1} X(t)Y(s-t)
\]
for each \( s \in \mathbb{Z} \). Then
\[
W(0) = \sum_{t=0}^{m-1} X(t)Y(-t) = 1 + \sum_{t=1}^{m-1} (-1)^t G(\chi^t, \zeta) G(\chi^{-t}, \zeta)
\]
\[
= 1 + \sum_{t=1}^{m-1} q = 1 + (m-1)q
\]
by Lemma \( \{2,4\}(a) \). Note that \( \hat{Y}(s) = g_s(\chi, \zeta, \mu) \) and \( \hat{X}(s) = g_{m^2+s}(\chi, \zeta, \mu) \). Thus, Lemma \( \{2,6\}(a) \) asserts that
\[
(9) \quad \hat{W}(s) = \hat{X}(s)\hat{Y}(s) = g_s(\chi, \zeta, \mu) g_{m^2+s}(\chi, \zeta, \mu)
\]
for each integer \( s \).

First suppose that \( H_{q,m} \) is a \((q, \ell, \lambda)\)-difference set in \( \mathbb{F}_q^+ \). Then according to Lemma \( \{3,1\} \), \( m \) is even and \( \chi(-1) = -1 \). Hence Theorem \( \{3,6\} \) implies that
\[
\sum_{t=1}^{m-1} (-1)^t G(\chi^t, \zeta) G(\chi^{s-t}, \zeta) = (1 + (-1)^s) G(\chi^s, \zeta) \quad \text{for } s = 1, \ldots, m-1,
\]
which means
\[
(10) \quad \sum_{t=0}^{m-1} X(t)Y(s-t) = 0 \quad \text{for } s = 1, \ldots, m-1.
\]
Consequently,
\[
W(1) = \cdots = W(m-1) = 0
\]
Now for any integer \( s \),
\[
\hat{W}(s) = \sum_{t=0}^{m-1} \mu^{-st} W(t) = W(0) = 1 + (m-1)q.
\]
This leads to \( (8) \) from \( (9) \).
Next suppose that $m$ is even and (8) holds for each integer $s$. It then follows from (9) that \( \hat{W}(s) = 1 + (m - 1)q \) for each integer $s$. Accordingly, Lemma 2.6(b) implies

\[
W(s) = \frac{1}{m} \sum_{t=0}^{m-1} \mu^{-st} \hat{W}(t) = \frac{1 + (m - 1)q}{m} \sum_{t=0}^{m-1} \mu^{-st} = 0 \quad \text{for} \quad s = 1, \ldots, m - 1,
\]

which leads to (10). Since $m$ is even, we see as in the proof of Lemma 3.4 that $\chi(-1) = -1$. Hence (10) is equivalent to (6). This implies that $H_{q,m}$ is a \((q,\ell,\lambda)\)-difference set in $\mathbb{F}_q^+$ by Theorem 3.6.

We summarize results of this section so far in the following theorem.

**Theorem 3.6.** Let $\mathbb{k}$ be an algebraically closed field of characteristic 0, $\chi$ be a $\mathbb{k}$-valued multiplicative character of order $m$ on $\mathbb{F}_q$, $\zeta$ be a primitive $p$th root of unity in $\mathbb{k}$, and $\mu$ be a primitive $m$th root of unity in $\mathbb{k}$. Suppose $\ell - 1 = \lambda m$ for some integer $\lambda \geq 1$. Then the following are equivalent:

(i) $H_{q,m}$ is a \((q,\ell,\lambda)\)-difference set in $\mathbb{F}_q^+$;

(ii) \( \sum_{\alpha \in H_{q,m}} \chi^s(1 - \alpha) = 0 \) for $s = 1, \ldots, m - 1$;

(iii) \( \sum_{t=1}^{m-1} J(\chi^s, \chi^t) = 1 \) for $s = 1, \ldots, m - 1$;

(iv) \( \sum_{\substack{t=1 \atop t \neq s}}^{m-1} \chi^t(-1)G(\chi^t, \zeta)G(\chi^{s-t}, \zeta) = (1 + \chi^s(-1))G(\chi^s, \zeta) \) for $s = 1, \ldots, m - 1$;

(v) $m$ is even and \( g_s(\chi, \zeta, \mu)g_{m+s}(\chi, \zeta, \mu) = 1 + (m - 1)q \) for all integer $s$.

As mentioned above, some of the criteria in Theorem 3.6 is already applied with $\mathbb{k} = \mathbb{C}$ in [29, Theorem 4.1] to obtain results of cyclotomic difference sets. We give one more such result below, which follows from the proof of [29, Theorem 4.2].

**Theorem 3.7.** Let $\chi$ be a $\mathbb{C}$-valued multiplicative character of order $m$ on $\mathbb{F}_q$. Suppose that $H_{q,m}$ is a cyclotomic difference set. Then $m$ is even, and there exist $x_1, \ldots, x_{m-1} \in \mathbb{C}$ such that

\[
\begin{align*}
\sum_{t=0}^{m-1} (-1)^t x_t x_{2s-t} &= 0, \quad s = 1, \ldots, \frac{m}{2} - 1, \\
x_s x_{m-s} &= (-1)^s, \quad s = 1, \ldots, \frac{m}{2}, \\
\chi(4)^s x_s x_{\frac{m}{2}+s} &= x_{2s} x_{\frac{m}{2}}, \quad s = 1, \ldots, \frac{m}{2} - 1,
\end{align*}
\]

where subscripts of $x$’s are counted modulo $m$ and $x_0 = -1/\sqrt{q}$.\[\square\]
In the rest of the section we apply Theorem 3.6 with \( k = \mathbb{C}_r \) for various prime numbers \( r \). The results turn to be quite different depending on whether \( r \) equals the characteristic \( p \) of \( \mathbb{F}_q \) or not.

3.2. Defining characteristic. In this subsection we apply Theorem 3.6 with \( k = \mathbb{C}_p \) to obtain new necessary conditions for cyclotomic difference sets. In view of Lemma 3.4, we assume that \( m \) is even henceforth. It follows that \( q = m\ell + 1 \) is odd and so \( p \) is odd.

For any integer \( n \geq 0 \), denote by \( s_p(n) \) the sum of the base-\( p \) digits of \( n \). If

\[
n = \sum_{j=0}^{f-1} n_j p^j
\]

with \( n_j \in \{0, 1, \ldots, p-1\} \) and the subscripts of \( n_0, n_1, \ldots, n_{f-1} \) counted modulo \( f \), then for any integer \( t \), denote

\[
n(t) = \sum_{j=0}^{f-1} n_j + tp^j.
\]

The next lemma is elementary and fairly easy to prove.

**Lemma 3.8.** For any nonnegative integers \( a \) and \( b \), if \( c \) is the number of carries required to add \( a \) and \( b \) in base-\( p \), then

\[
s_p(a) + s_p(b) = s_p(a + b) + (p-1)c.
\]

Let \( \zeta_p \) and \( \zeta_{q-1} \) be a primitive \( p \)th and \( (q-1) \)th root of unity in \( \mathbb{C}_p \) respectively, \( K = \mathbb{Q}_p(\zeta_{q-1}) \), and \( \mathcal{O}_{K(\zeta_p)} \) be the ring of integers of \( K(\zeta_p) = \mathbb{Q}_p(\zeta_{q-1}, \zeta_p) \). Then the residue field \( \mathcal{O}_{K(\zeta_p)}/(\zeta_p - 1)\mathcal{O}_{K(\zeta_p)} \) of \( \mathcal{O}_{K(\zeta_p)} \) is isomorphic to \( \mathbb{F}_q \), and thus the map

\[
(11) \quad \omega : \mathcal{O}_{K(\zeta_p)}/(\zeta_p - 1)\mathcal{O}_{K(\zeta_p)} \to K(\zeta_p), \quad x + (\zeta_p - 1)\mathcal{O}_{K(\zeta_p)} \mapsto \lim_{n\to\infty} x^p^n
\]

induces a map from \( \mathbb{F}_q \) to \( K(\zeta_p) \), which is called the *Teichmüller character* of \( \mathbb{F}_q \). For any \( s \in \mathbb{Z}_p \), let

\[
\Gamma_p(s) = \lim_{n \to s} (-1)^n \prod_{j=1 \atop p|j}^{n-1} j.
\]

The function \( \Gamma_p \) is called the *\( p \)-adic Gamma function* [20]. According to the Gross-Koblitz formula [13],

\[
(12) \quad G(\omega^n, \zeta_p) = -\pi^{s_p(n)} \prod_{i=0}^{f-1} \Gamma_p \left( \frac{n(i)}{q-1} \right) \quad \text{for} \quad n = 0, 1, \ldots, q - 2,
\]

where \( \pi \in \mathbb{Q}_p(\zeta_p) \) such that \( \pi^{p-1} = -p \) and \( \zeta_p \) is congruent to \( 1 + \pi \) modulo \( \pi^2 \).

We are now in the position to give the main result of this section.
Theorem 3.9. Suppose that $H_{q,m}$ is a cyclotomic difference set. Then $m$ is an even divisor of $p - 1$, $q = p^f$ with odd $f$, and

$$
\sum_{t=1}^{2s-1} (-1)^t \left( \Gamma_p \left( \frac{t}{m} \right) \right)^f \left( \Gamma_p \left( \frac{2s-t}{m} \right) \right)^f

+ (-p)^f \sum_{t=2s+1}^{m-1} (-1)^t \left( \Gamma_p \left( \frac{t}{m} \right) \right)^f \left( \Gamma_p \left( \frac{m+2s-t}{m} \right) \right)^f

= -2 \left( \Gamma_p \left( \frac{2s}{m} \right) \right)^f \quad \text{for} \quad s = 1, \ldots, \frac{m}{2} - 1.
$$

Proof. We have already seen that $m$ is even as Lemma 3.4 implies. When $m = 2$, (13) says nothing and the theorem follows from the condition that $q \equiv 3 \pmod{4}$. Thus we assume $m \geq 4$ in the remaining of the proof.

First we prove that $m$ divides $p - 1$. According to Theorem 3.6,

$$
\sum_{t=1}^{m-1} \chi^t (-1) \chi^t (\zeta_p) G(\chi^{m-2-t}, \zeta_p) = (1 + \chi^{m-2}(-1)) G(\chi^{m-2}, \zeta_p)
$$

for any multiplicative character $\chi$ of order $m$ on $\mathbb{F}_q$. Let $\omega$ be defined in (11). Taking $\chi = \omega^f$ and substituting (12) into the above equality we derive that

$$
\sum_{t=1}^{m-3} (-1)^{t+s_p(\ell t)+s_p(m\ell-2\ell-t\ell)} \prod_{j=0}^{f-1} \left( \Gamma_p \left( \frac{(t\ell)^{(j)}}{q-1} \right) \Gamma_p \left( \frac{(m\ell-2\ell-t\ell)^{(j)}}{q-1} \right) \right)

- \pi^{2s_p(m\ell-\ell)} \prod_{j=0}^{f-1} \left( \Gamma_p \left( \frac{(m\ell-\ell)^{(j)}}{q-1} \right) \right)^2 = -2 \pi^{s_p(m\ell-2\ell)} \Gamma_p \left( \frac{(m\ell-2\ell)^{(j)}}{q-1} \right).
$$

Note that $1 \leq \ell \leq q - 1 = p^f - 1$. Let

$$
\ell = \sum_{j=0}^{f-1} \ell_j p^j
$$

be the base-$p$ expansion of $\ell$, where $\ell_j \in \{0, 1, \ldots, p - 1\}$ for each $j$ such that $0 \leq j \leq f - 1$.

Assume that $(m - 2)\ell_0 \geq p$. Then for any integer $t$ such that $1 \leq t \leq m - 3$, there are at least one carry required to add $t\ell$ and $m\ell - 2\ell - t\ell$. Thus Lemma 3.8 implies that

$$
s_p(t\ell) + s_p(m\ell - 2\ell - t\ell) > s_p(m\ell - 2\ell) \quad \text{for} \quad t = 1, \ldots, m - 3.
$$

For the same reason,

$$
2s_p(m\ell - \ell) = s_p(m\ell - \ell) + s_p(m\ell - \ell) > s_p(2m\ell - 2\ell) = s_p(m\ell - 2\ell).
$$

This violates (14).
Now we know that $(m - 2)\ell_0 \leq p - 1$. If $m\ell_0 + 1 \geq 2p$, then

$$m\ell_0 + 1 \geq 2(m - 2)\ell_0 + 2,$$

which turns out to be $m \leq (4\ell_0 - 1)/\ell_0$, contrary to our assumption that $m \geq 4$. Consequently, $m\ell_0 + 1 < 2p$. Moreover, the equality

$$\sum_{j=1}^{f-1} m\ell_j p^j + m\ell_0 + 1 = m\ell + 1 = q$$

implies that $p$ divides $m\ell_0 + 1$. Therefore, $m\ell_0 + 1 = p$ and so $m$ divides $p - 1$.

Next we show that $f$ is odd. Suppose for a contradiction that $f$ is even. Then $m\ell = q - 1 = p^f - 1$ is divisible by $p^2 - 1$. As $m$ divides $p - 1$, it follows that $\ell$ is divisible by $(p^2 - 1)/m = (p + 1)(p - 1)/m$ and thus divisible by $p + 1$. Then since $p + 1$ is even, we deduce that $\ell$ is even. However, \(3\) together with the fact that $m$ is even implies that $\ell$ is odd. This contradiction implies that $f$ is odd.

Finally we prove \(13\). Viewing

$$\ell = \frac{q - 1}{m} = (p^f - 1 + \cdots + p + 1) \cdot \frac{p - 1}{m}$$

we see that for any $t \in \{1, \ldots, m - 1\}$,

$$t\ell = \sum_{j=0}^{f-1} \frac{t(p - 1)}{m} p^j$$

is the base-$p$ expansion of $t\ell$ and so \(12\) implies

\begin{equation}
G(\omega^{t\ell}, \zeta_p) = -\pi^{ft(p-1)/m} \prod_{j=0}^{f-1} \Gamma_p \left( \frac{t\ell}{q - 1} \right) = -\pi^{ft(p-1)/m} \left( \Gamma_p \left( \frac{t}{m} \right) \right)^f.
\end{equation}

Since $m$ is even, we deduce from \(6\) that $\ell$ is odd and hence $\omega^{\ell}(-1) = -1$. Then together with \(15\), Theorem 3.6 gives

$$\sum_{t=1}^{s-1} (-1)^t \left( \Gamma_p \left( \frac{t}{m} \right) \right)^f \left( \Gamma_p \left( \frac{s - t}{m} \right) \right)^f + (-p)^f \sum_{t=s+1}^{m-1} (-1)^t \left( \Gamma_p \left( \frac{t}{m} \right) \right)^f \left( \Gamma_p \left( \frac{m + s - t}{m} \right) \right)^f = -2 \left( \Gamma_p \left( \frac{s}{m} \right) \right)^f \text{ for } s = 2, 4, \ldots, m - 2,$$

which is indeed \(13\).

**Remark 1.** If we only prove that $m$ divides $p - 1$ in Theorem 3.9 then we do not need the Gross-Koblitz formula and instead Stickelberger’s theorem (see for example [6])
Propostion 11.7.11]) would suffice. In fact, Stickelberger’s theorem can be deduced immediately from the Gross-Koblitz formula.

**Remark 2.** Although (13) seems to be strong restriction on the existence of cyclotomic difference sets, we do not know how to use it properly at this stage. On the other hand, the assertion that $m$ divides $p-1$ in Theorem 3.9 will play an important role in the subsequent cross characteristic analysis. It is also worth mentioning that the assertions in Theorem 3.9 does not hold for modified cyclotomic difference set; for example, $H_{16,3} \cup \{0\}$ is a difference set in $\mathbb{F}_{16}^+$, but $m = 3$ is neither even not a divisor of $p-1 = 1$ and $q = 16$ is an even power of $p = 2$.

### 3.3. Cross characteristic.

Let $r$ be a prime number other than $p$, $\chi$ be an $r$-adic multiplicative character of order $m$ on $\mathbb{F}_q$, $\zeta$ be a primitive $p$th root of unity in $\mathbb{C}_r$, and $\mu$ be a primitive $m$th root of unity in $\mathbb{C}_r$. It is clear from the definition of Gauss sums that

$$G(\chi^s, \zeta) \in \mathbb{Z}[\mu, \zeta]$$

for each integer $s$. Let $|\cdot|_r$ be an $r$-adic absolute value on $\mathbb{Q}_r(\mu, \zeta)$,

$$\mathcal{O} = \{ z \in \mathbb{Q}_r(\mu, \zeta) \mid |z|_r \leq 1 \}$$

be the ring of integers of $\mathbb{Q}_r(\mu, \zeta)$, and

$$\mathcal{M} = \{ z \in \mathbb{Q}_r(\mu, \zeta) \mid |z|_r < 1 \}$$

be the unique maximal ideal of $\mathcal{O}$. Then the residue field $\mathcal{O}/\mathcal{M}$ is a finite field of characteristic $r$. For any $z \in \mathcal{O}$, let

$$(16) \quad \overline{z} = z + \mathcal{M} \in \mathcal{O}/\mathcal{M}.$$  

Since $\mathbb{Z}[\mu, \zeta] \subseteq \mathcal{O}$, we may consider $G(\chi^s, \zeta)$ for $s \in \mathbb{Z}$.

**Lemma 3.10.** Let $r$ be a prime number other than $p$ such that $\gcd(r, m) = 1$. Then in the above notation, the following hold.

(a) For any integer $s$ such that $s \not\equiv 0 \pmod{m}$,

$$G(\chi^{rs}, \zeta) = (\chi^r(r))^s \left( G(\chi^s, \zeta) \right)^r$$

(b) If $\chi(r) = \mu^s$, then for any integer $s$,

$$\left( g_s(\chi, \zeta, \mu) \right)^r = g_{s+\sigma}(\chi, \zeta, \mu).$$
Proof. Let \( s \) be an integer such that \( s \not\equiv 0 \pmod{m} \). As \( \gcd(r,m) = 1 \), we have \( \chi^{rs}(0) = 0 \). Then since \( \mathcal{O}/\mathcal{M} \) is a field of characteristic \( r \), we derive that
\[
G(\chi^{rs}, \zeta) = \sum_{\alpha \in \mathbb{F}_q^*} \chi^{rs}(\alpha) \zeta^{tr(\alpha)}
= \sum_{\alpha \in \mathbb{F}_q^*} \chi^{rs}(\alpha) \zeta^{tr(\alpha)}
= \sum_{\alpha \in \mathbb{F}_q^*} \chi^{rs}(\alpha) \chi^{rs}(\alpha) (\zeta^{tr(\alpha)})^r
= (\chi^r(1))^{s} \sum_{\alpha \in \mathbb{F}_q^*} (\chi^{s}(\alpha) \zeta^{tr(\alpha)})^r
= (\chi^r(1))^{s} G(\chi^{s}, \zeta)^r.
\]
Hence part (a) holds. Now suppose \( \chi(r) = \mu^\sigma \). It follows that
\[
\left( G(\chi, \zeta) \right)^r = \left( \chi(1) \right)^{-rt} G(\chi^t, \zeta) = \mu^{-\sigma rt} G(\chi^t, \zeta) \quad \text{for} \quad t = 1, \ldots, m - 1.
\]
Then for any integer \( s \) we have
\[
\left( g_s(\chi, \zeta, \mu) \right)^r = \left( \sum_{t=1}^{m-1} \mu^{-st} G(\chi^t, \zeta) - 1 \right)^r
= \sum_{t=1}^{m-1} \mu^{-rst} \left( G(\chi^t, \zeta) \right)^r - 1
= \sum_{t=1}^{m-1} \mu^{-(s+\sigma)rt} G(\chi^t, \zeta) - 1
= \sum_{t=1}^{m-1} \mu^{-(s+\sigma)rt} G(\chi^t, \zeta) - 1
= g_{s+\sigma}(\chi, \zeta, \mu).
\]
This proves part (b). \[\square\]

Now we are in the position to give the main results of this subsection in the following theorems, which are necessary conditions for cyclotomic difference sets from local fields \( \mathbb{C}_p \).

**Theorem 3.11.** Let \( r \neq p \) be a prime number with \( \gcd(r,m) = 1 \), \( \chi \) be an \( r \)-adic multiplicative character of order \( m \) on \( \mathbb{F}_q \), and \( \mu \) be a primitive \( m \)th root of unity in \( \mathbb{C}_p \). Suppose that \( H_{q,m} \) is a cyclotomic difference set, \( \chi(r) = \mu^\sigma \) and \( \chi(2) = \mu^\theta \).
Then \( m \) is even, and there exist \( w_d \in \mathbb{F}_{\mu \phi(m)} \) for each odd prime divisor \( d \) of \( m \) and \( u, w, x_1, \ldots, x_{m-1} \in \mathbb{F}_{\mu \phi(m)} \) such that

\[
\begin{align*}
\Phi_{\mu \phi(m)}(u) &= 0 \\
\Phi_{\mu \phi(2\phi(m))}(w) &= 0 \\
x_{rs} &= u^s x_{s}^r, \quad s = 1, \ldots, m-1 \\
\sum_{t=1}^{m-1} (-1)^t x_t x_{2s-t} &= 2x_{2s}, \quad s = 1, \ldots, \frac{m}{2} - 1 \\
x_s x_{m-s} &= (-1)^s q, \quad s = 1, \ldots, \frac{m}{2} \\
w^s x_s x_{\frac{m}{2}+s} &= x_{2s} x_{\frac{m}{2}}, \quad s = 1, \ldots, \frac{m}{2} - 1
\end{align*}
\]

for each odd prime divisor \( d \) of \( m \), where subscripts of \( x \)'s are counted modulo \( m \).

\begin{proof}
Let \( \zeta \) be a primitive \( p \)th root of unity in \( \mathbb{C}_r \). With the notation in (16), take \( w_d = \chi^{d(r)} \) for each odd prime divisor \( d \) of \( m \), \( u = \chi^{(r)^2} = \overline{\mu}^{\sigma r} \), \( w = \chi^{2(2)} = \overline{\mu}^{2\phi} \) and \( x_s = \overline{G}(\chi^s, \zeta) \) for \( s = 1, \ldots, m-1 \). Let \( d \) be an odd prime divisor of \( m \) (if there is any). Then we have \( w_d^{m/d} = 1 \) since \( (\chi^{d(r)})^{m/d} = \chi^{m(d)} = 1 \). Note that \( \mu^{\sigma r} \) is a root of \( x^m \mod \gcd(\sigma, m) - 1 \) but not a root of \( x^j - 1 \) for any positive integer \( j < m \mod \gcd(\sigma, m) \). We deduce \( \Phi_{m \mod \gcd(\sigma, m)}(\mu^{\sigma r}) = 0 \) and so \( \Phi_{m \mod \gcd(\sigma, m)}(u) = 0 \).

In the same vein we obtain \( \Phi_{m \mod \gcd(2\phi(m))}(w) = 0 \). These verify the first two lines of (17). As \( w_d^m = w^m = w^{m} = 1 \), we have \( w_d, u, w \in \mathbb{F}_{\phi(m)} \subseteq \mathbb{F}_{\mu \phi(m)} \).

The third line of (17) follows immediately from Lemma 3.10(a). According to Lemma 3.3, \( m \) is even and \( \chi(-1) = -1 \). Then Theorem 3.6(iv) and Lemma 2.4(a) give

\[
\sum_{t=1}^{m-1} (-1)^t G(\chi^t, \zeta) G(\chi^{2s-t}, \zeta) = 2G(\chi^{2s}, \zeta), \quad s = 1, \ldots, \frac{m}{2} - 1
\]

and

\[
G(\chi^s, \zeta) G(\chi^{-s}, \zeta) = (-1)^s q, \quad s = 1, \ldots, m - 1,
\]

respectively. These imply the fourth and the fifth line of (17). For \( s = 1, \ldots, m/2-1 \), we derive from Lemma 2.4(d) that

\[
\chi^{2s}(2) G(\chi^s, \zeta) G(\chi^{m/2+s}, \zeta) = G(\chi^{2s}, \zeta) G(\chi^{m/2}, \zeta),
\]

\end{proof}
which yields the sixth line of (17). For \( s = 1, \ldots, m/d - 1 \), as \( d \) is odd, we derive from Lemma 2.4 that

\[
\chi^{ds}(d) \prod_{t=0}^{d-1} G(\chi^{s+mt/d}, \zeta) = G(\chi^{ds}, \zeta) \prod_{t=1}^{d-1} G(\chi^{mt/d}, \zeta)
\]

\[
= G(\chi^{ds}, \zeta) \prod_{t=1}^{(d-1)/2} \left( G(\chi^{mt/d}, \zeta) G(\chi^{m(d-t)/d}, \zeta) \right)
\]

\[
= G(\chi^{ds}, \zeta) \prod_{t=1}^{(d-1)/2} q = q^{(d-1)/2} G(\chi^{ds}, \zeta),
\]

which leads to the second line of (18).

Finally, from the third line of (17) we deduce inductively that \( x_{rjs} = u^{p^r-1}s x_{js} \) for any positive integer \( j \). Taking \( j = m\phi(m) \) and viewing that \( r^{m\phi(m)} \equiv 1 \pmod{m} \), we then obtain

\[
x_s = x_{r,m\phi(m)s} = u^{m\phi(m)r^{m\phi(m)-1}s} x^{r^{m\phi(m)}} = x_s^{r^{m\phi(m)}}.
\]

This implies \( x_s \in F_{r,m\phi(m)} \) for \( s = 1, \ldots, m - 1 \), completing the proof.

\[ \square \]

**Theorem 3.12.** Let \( r \neq p \) be a prime number with \( \gcd(r, m) = 1 \), \( \chi \) be an \( r \)-adic multiplicative character of order \( m \) on \( F_q \), and \( \mu \) be a primitive \( m \)-th root of unity in \( \mathbb{C}_r \). Suppose that \( H_{q,m} \) is a cyclotomic difference set, \( \chi(r) = \mu^r \) and \( \chi(2) = \mu^2 \).

Then \( m \) is even, and there exist \( y_0, y_1, \ldots, y_{m-1} \in F_{r, \gcd(s,m)} \) such that

\[
y_{s+\sigma} = y_s^r, \quad s = 0, 1, \ldots, m - 1
\]

\[
y_s y_{m+s} = 1 + (m-1)q, \quad s = 0, 1, \ldots, m - 1
\]

\[
\sum_{t=0}^{m-1} y_t y_{s+t} = m(1-q), \quad s = 0, 1, \ldots, m - 1
\]

\[
\sum_{t=0}^{m-1} (-1)^t y_t y_{2s-2} y_{-t} = \left( y_s + y_{m+s} \right) \sum_{t=0}^{m-1} (-1)^t y_t, \quad s = 0, 1, \ldots, m - 1, -1,
\]

where subscripts of \( y \)'s are counted modulo \( m \).

**Proof.** From Lemma 3.4 we see that \( m \) is even. Let \( \zeta \) be a primitive \( p \)-th root of unity in \( \mathbb{C}_r \). With the notation in (16), take \( F = \mathcal{O}/\mathcal{M} \) and \( y_s = \bar{g}(\zeta^s, \zeta, \mu) \) for \( s = 0, 1, \ldots, m - 1 \). Then the first two lines of (19) follow from Lemma 3.10(b) and Theorem 3.6(v), respectively, and the last two lines of (19) follow from Lemma 2.7.

Moreover, the first line of (19) implies \( y_{s+\sigma j} = y_{js}^r \) for any nonnegative integer \( j \). Hence

\[
y_s = y_{s+\sigma} m / \gcd(s,m) = y_{s}^{m/\gcd(s,m)}
\]

for any integer \( s \), which implies \( y_0, y_1, \ldots, y_{m-1} \in F_{r,m/\gcd(s,m)} \). Thus Theorem 3.12 is true. \[ \square \]
Remark 3. For \( r \) not congruent to 1 modulo \( m \), the assumption \( r \neq p \) appearing in both Theorem 3.11 and Theorem 3.12 is satisfied by the conclusion of Theorem 3.9 that \( m \) divides \( p - 1 \).

4. Applications

4.1. Characteristic 3. Applying Theorem 3.12 with a feasible prime number \( r \) gives a necessary condition for \( m \)-th-cyclotomic difference sets in terms of polynomial equations over \( \mathbb{F}_{r^m} \). We illustrate this by giving the details for \( r = 3 \). First we present a lemma for general \( r \).

Lemma 4.1. Let \( r \neq p \) be a prime number with \( \gcd(r, m) = 1 \), \( \chi \) be an \( r \)-adic multiplicative character of order \( m \) on \( \mathbb{F}_q \), and \( \mu \) be a primitive \( m \)th root of unity in \( \mathbb{C}_r \). Suppose that \( \chi(r) = \mu^\sigma \) and \( m \) is even. Then

\[
(-1)^\sigma = \left( \frac{(-1)^{m/2}q}{r} \right).
\]

Proof. Taking \( s = m/2 \) in Lemma 3.10(a) we obtain

\[
\left( G(\chi^{m/2}, \zeta) \right)^r = \left( \chi(r) \right)^{-rs} G(\chi^{rm/2}, \zeta) = \mu^{-\sigma rm/2} G(\chi^{m/2}, \zeta)
\]

in the notation of (16), so

\[
\left( G(\chi^{m/2}, \zeta) \right)^{r-1} = \mu^{-\sigma rm/2} = (-1)^\sigma.
\]

Moreover, we deduce from Lemma 2.4(a) that

\[
\left( G(\chi^{m/2}, \zeta) \right)^2 = (-1)^{m/2}q.
\]

Thus,

\[
(-1)^\sigma = \left( G(\chi^{m/2}, \zeta) \right)^{r-1} = \left( \left( G(\chi^{m/2}, \zeta) \right)^2 \right)^{(r-1)/2} = \left( (-1)^{m/2}q \right)^{(r-1)/2} = \left( \frac{(-1)^{m/2}q}{r} \right).
\]

Now we deduce the local information from Theorem 3.12 with \( r = 3 \). This will play a crucial role in proving the nonexistence of \( m \)-th-cyclotomic difference sets for some \( m \) coprime to 3.

Theorem 4.2. Suppose that \( H_{q,m} \) is a cyclotomic difference set with \( m > 2 \) and \( m \equiv \pm 2 \pmod{6} \). Then there exist a positive integer \( \sigma \) dividing \( m \), an integer
Let \( \theta \in \{1, 2, \ldots, m/2\} \) and \( y_0, y_1, \ldots, y_{m-1} \in \mathbb{F}_{3^{m/2}} \) such that

\[
\begin{align*}
  y_{s+\sigma} = y_s^3, & \quad s = 0, 1, \ldots, m-1 \\
  y_s y_{m/2+s} = 1 + (m-1)(-1)^{\sigma+m/2}, & \quad s = 0, 1, \ldots, m/2 - 1 \\
  \sum_{t=0}^{m-1} y_t y_{s+t} = m(1 - (-1)^{\sigma+m/2}), & \quad s = 0, 1, \ldots, m/2 - 1 \\
  \sum_{t=0}^{m-1} (-1)^t y_t y_{2s-2t} = \left( y_s + y_{m/2+s} \right)^{m-1} \sum_{t=0}^{m-1} (-1)^t y_t, & \quad s = 0, 1, \ldots, m/2 - 1,
\end{align*}
\]

where subscripts of \( y \)'s are counted modulo \( m \).

**Proof.** Let \( \chi \) be a 3-adic multiplicative character of order \( m \) on \( \mathbb{F}_q \), and take \( \mu \) to be a primitive \( m \)-th root of unity in \( \mathbb{C}_3 \) such that \( \chi(3) = \mu^\sigma \) for some positive integer \( \sigma \) dividing \( m \). If \( p = 3 \), then Theorem 3.9 would imply that \( m \) divides 2, which is not possible as \( m > 2 \). Hence we have \( p \neq 3 \) and \( \gcd(3, m) = 1 \). It then follows from Theorem 3.12 that there exist an integer \( \theta \) and \( y_0, y_1, \ldots, y_{m-1} \in \mathbb{F}_{3^{m/2}} \) with

\[
\begin{align*}
  y_{s+\sigma} = y_s^3, & \quad s = 0, 1, \ldots, m-1 \\
  y_s y_{m/2+s} = 1 + (m-1)q, & \quad s = 0, 1, \ldots, m/2 - 1 \\
  \sum_{t=0}^{m-1} y_t y_{s+t} = m(1 - q), & \quad s = 0, 1, \ldots, m/2 - 1 \\
  \sum_{t=0}^{m-1} (-1)^t y_t y_{2s-2t} = \left( y_s + y_{m/2+s} \right)^{m-1} \sum_{t=0}^{m-1} (-1)^t y_t, & \quad s = 0, 1, \ldots, m/2 - 1,
\end{align*}
\]

where subscripts of \( y \)'s are counted modulo \( m \). Note that (21) does not change if we replace \( \theta \) with \( \theta + jm/2 \) for any integer \( j \). Thus we may assume \( \theta \in \{1, 2, \ldots, m/2\} \).

By Lemma 4.1,

\[
(-1)^\sigma = \frac{(-1)^{m/2} q}{3}.
\]

Accordingly, \((-1)^{m/2} q \equiv (-1)^\sigma \pmod{3}\), which means \( q = (-1)^{\sigma+m/2} \in \mathbb{F}_{3^{m/2}} \). Substituting this into (21) we obtain (20), as desired. \(\square\)

In the case when \( \sigma = 1 \), the first line of (20) implies that \( y_s = y_0^{s^2} \) for all integer \( s \geq 0 \) and thus (20) is essentially a univariate system on \( y_0 \in \mathbb{F}_{3^m} \). For example, if \( \sigma = 1 \) and \( m/2 \) is odd, then the first three lines of (20) gives

\[
\begin{align*}
  y_0^{1+3^{m/2}} - m = 0 \\
  \sum_{t=0}^{m-1} y_0^{(1+3^t)3^t} = 0, & \quad s = 0, 1, \ldots, m/2 - 1,
\end{align*}
\]

which seems enough to conclude that there is no solution \( y_0 \in \mathbb{F}_{3^m} \). In fact, computation results of the greatest common divisor of the polynomials suggests:
Conjecture 4.3. For an odd prime \( r \) and integer \( n \geq 2 \), the system of equations
\[
\begin{align*}
&y^{1+r^n} - 2n = 0 \\
&\sum_{t=0}^{2n-1} y^{(1+r^s)r^t} = 0, \quad s = 0, 1, \ldots, n - 1
\end{align*}
\]
has a solution \( y \in \mathbb{F}_{r^{2n}} \) if and only if \( r \) divides \( n \).

4.2. Small \( m \). Computing the Gröbner basis for the system (21) of polynomial equations in Magma\textsuperscript{3} we find that, for \( m = 26 \) and 32 respectively, (21) has no solution in any field of characteristic 3 for any positive divisor \( \sigma \) of \( m \) and \( \theta \in \{1, 2, \ldots, m/2\} \). This yields the following result by Theorem 4.2.

Lemma 4.4. There is neither 26th nor 32nd-cyclotomic difference set.

We can show in the same way as above that there is no 14th-cyclotomic difference set. It is hopeful that for the family \( m \equiv 2 \pmod{6} \) we are able to obtain more nonexistence results from Theorem 4.2, but the author’s PC fail to compute the Gröbner basis of (21) for the next value \( m = 38 \) in this family.

Next we combine the local information from \( \mathbb{C}, \mathbb{C}_p \) and \( \mathbb{C}_5 \) to prove that there is no 24th-cyclotomic difference set.

Lemma 4.5. There is no 24th-cyclotomic difference set.

Proof. Suppose on the contrary that \( H_{q,24} \) is a cyclotomic difference set. If \( 2 \in H_{q,6} \) then \( \chi^2(4) = \chi^4(2) = 1 \) for any \( \mathbb{C} \)-valued multiplicative character \( \chi \) of order 24 on \( \mathbb{F}_q \), and so by Theorem 3.7 there exist \( x_1, \ldots, x_{23} \in \mathbb{C} \) such that
\[
\begin{align*}
&\sum_{t=0}^{23} (-1)^t x_t x_{28-t} = 0, \quad s = 1, \ldots, 11 \\
&x_s x_{24-s} = (-1)^s, \quad s = 1, \ldots, 12 \\
&x_s x_{12+s} = x_{2s} x_{12}, \quad s = 1, \ldots, 11
\end{align*}
\]

or
\[
\begin{align*}
&\sum_{t=0}^{23} (-1)^t x_t x_{28-t} = 0, \quad s = 1, \ldots, 11 \\
&x_s x_{24-s} = (-1)^s, \quad s = 1, \ldots, 12 \\
&(-1)^s x_s x_{12+s} = x_{2s} x_{12}, \quad s = 1, \ldots, 11,
\end{align*}
\]

where subscripts of \( x \)'s are counted modulo 24 and \( x_0 = -1/\sqrt{q} \). Using FGb library \textsuperscript{11} \textsuperscript{12} to compute the Gröbner basis\textsuperscript{3} of (22) and (23) we find that (22) and (23) imply
\[
(x_0 - 11)(x_0 + 11)(5x_0 - 1)(5x_0 + 1) = 0
\]

\textsuperscript{2}by the command \texttt{GroebnerBasis}
\textsuperscript{3}The computation is performed by Dr. Xiaoxian Tang.
and
\[ x_0^2(x_0 - 11)(x_0 + 11)(11x_0 + 49)(11x_0 - 49)(x_0^2 + 23) = 0, \]
respectively, both contradicting the condition
\[ 0 < x_0^2 < \frac{1}{25} \]
as \( x_0^2 = 1/q \). Therefore, \( 2 \not\in H_{q,6} \).

Now let \( \chi \) be a 5-adic multiplicative character of order 24. We have \( p \neq 5 \) since 24 divides \( p - 1 \) by Theorem 3.9. Write \( \chi(5) = \mu^\sigma \) and \( \chi(2) = \mu^\theta \). Then since the polynomial \( \Phi_{24/\gcd(\sigma,24)}(x) \) divides \( x^{24} - 1 \), we deduce from Theorem 3.11 that there exist \( w_3, u, w, x_1, \ldots, x_{23} \in \mathbb{F}_{5^{192}} \) such that
\[
\begin{align*}
&u^{24} - 1 = 0 \\
&\Phi_{\frac{12}{\gcd(\theta,12)}}(w) = 0 \\
x_{5s} = w^s x_s, \quad s = 1, \ldots, 23 \\
\sum_{t=1}^{23} (-1)^t x_t x_{2s-t} = 2x_{2s}, \quad s = 1, \ldots, 11 \\
x_s x_{24-s} = (-1)^s q, \quad s = 1, \ldots, 12 \\
w^s x_{s} x_{12+s} = x_{2s} x_{12}, \quad s = 1, \ldots, 11 \\
w_3^8 = 1 \\
w_3^s x_s x_{8+s} x_{16+s} = q x_{3s}, \quad s = 1, \ldots, 7,
\end{align*}
\]
where subscripts of \( x \)'s are counted modulo 24. As \( 2 \not\in H_{q,6} \) we have \( \mu^{4\theta} = \chi^4(2) \neq 1 \). This indicates \( \gcd(\theta,12) \neq 6 \) or 12, and so \( \gcd(\theta,12) \in \{1, 2, 3, 4\} \). Moreover, \( q \neq 0 \) (mod 5) since \( p \neq 5 \), whence \( q \equiv 1, 2, 3, 4 \) (mod 5). However, by computing the Gröbner basis in Magma [3] we find that the system (24) of polynomial equations has solution for none of the above possibilities for \( \gcd(\theta,12) \) and \( q \). This contradiction completes the proof.

By summarizing the previous results we are able to show that Conjecture 1.1 holds for \( m < 28 \) and \( m = 32 \).

**Theorem 4.6.** If \( m < 28 \) or \( m = 32 \) then Conjecture 1.1 is true.

**Proof.** If \( H_{q,m} \) is a cyclotomic difference set then \( m \) is even, as Theorem 3.6 states. Hence the result for \( m \leq 8 \) is due to Storer [23]. If \( 10 \leq m \leq 22 \), then there is no \( m \)th cyclotomic difference set by [29, Theorem 5.1]. For \( m = 26 \) or 32, \( H_{q,m} \) is not a cyclotomic difference set by Lemma 4.3. Finally, Lemma 4.5 asserts that there is no 24th-cyclotomic difference set. Thus Conjecture 1.1 is true if \( m < 28 \) or \( m = 32 \). □
4.3. Finite flag-transitive projective planes. We have seen in Proposition 1.2 that the existence problem of finite non-Desarguesian flag-transitive projective plane is related to the existence problem of certain cyclotomic difference set. Suppose that $H_{v,n}$ is a $(v, n+1, 1)$-difference set in $\mathbb{F}_v^+$ with $v = n^2 + n + 1$ prime, as in Proposition 1.2. Then clearly $n$ is not congruent to 1 modulo $v$, and we have

$$n^3 = (n^2 + n + 1)(n-1) + 1 = v(n-1) + 1 \equiv 1 \pmod{v}.$$ 

Thereby we conclude that $n$ has order 3 in $\mathbb{F}_v^\times$. Now applying the First Multiplier Theorem (see [17, Theorem 2.1]) and [19, Theorem IV] we deduce that every prime divisor of $n = (n+1) - 1$ lies in $H_{v,n}$. This has two consequences. Firstly, $n$ lies in $H_{v,n}$ so that $n^3 = 1 \in \mathbb{F}_v$, which implies that $n + 1$ is divisible by 3. Secondly, 2 is a prime divisor of $n$ as $n$ is even, so we have $2 \in H_{v,n}$. To sum up, we have the following lemma.

Lemma 4.7. If $H_{v,n}$ is a $(v, n+1, 1)$-difference set in $\mathbb{F}_v^+$ with $v = n^2 + n + 1$ prime, then $n \equiv 2 \pmod{3}$ and $2 \in H_{v,n}$.

In light of the necessary condition $n \equiv 2 \pmod{3}$ in Lemma 4.7 we make use of the local information from $\mathbb{C}_3$ to study the existence of finite non-Desarguesian flag-transitive projective plane.

Theorem 4.8. If there exists a finite non-Desarguesian flag-transitive projective plane of order $n$ with $v$ points, then $v = n^2 + n + 1$ is prime with $n \equiv 8 \pmod{24}$ and $n > 8$, and there exist a positive even integer $\sigma$ dividing $n$ and $y_0, y_1, \ldots, y_{n-1} \in \mathbb{F}_{3v/\sigma}$ such that

$$y_{s+\sigma} = y_s^3, \quad s = 0, 1, \ldots, n-1$$

$$y_s y_{2+s} = -1, \quad s = 0, 1, \ldots, \frac{n}{2} - 1$$

$$\sum_{t=0}^{n-1} y_t y_{s+t} = 0, \quad s = 0, 1, \ldots, \frac{n}{2} - 1$$

$$\sum_{t=0}^{n-1} (-1)^t y_t y_{2s-t} = \left( y_s + y_{\frac{n}{2}+s} \right) \sum_{t=0}^{n-1} (-1)^t y_t, \quad s = 0, 1, \ldots, \frac{n}{2} - 1,$$

where subscripts of $y$’s are counted modulo $n$.

Proof. Suppose that there exists a finite non-Desarguesian flag-transitive projective plane of order $n$ with $v$ points. Then Proposition 1.2 shows that $v = n^2 + n + 1$ is prime and $H_{v,n}$ is a $(v, n+1, 1)$-difference set in $\mathbb{F}_v^+$ with $n > 8$ even. By Lemma 4.7 we have $n \equiv 2 \pmod{3}$ and $2 \in H_{v,n}$. Moreover, we derive from [18, Theorem 3.5] that $n \equiv 0 \pmod{8}$. Hence $n \equiv 8 \pmod{24}$.

Let $\chi$ be a 3-adic multiplicative character of order $n$ on $\mathbb{F}_v$, and take $\mu$ to be a primitive $n$th root of unity in $\mathbb{C}_3$ such that $\chi(3) = \mu^\sigma$ for some positive integer $\sigma$.
dividing \( n \). It follows that \( \chi(2) = 1 \) since \( 2 \in H_{v,n} \). Then by virtue of Theorem 3.12 there exist \( y_0, y_1, \ldots, y_{n-1} \in \mathbb{F}_{3n/\sigma} \) such that

\[
\left\{
\begin{array}{l}
y_{s+\sigma} = y_s, \\ y_s y_{2s} + 1 = (n - 1)v, \\ \sum_{t=0}^{n-1} y_t y_{s+t} = n(1 - v), \\ \sum_{t=0}^{n-1} (-1)^t y_t y_{2s-t} = \left(y_s + y_{\frac{n}{2}+s}\right) \sum_{t=0}^{n-1} (-1)^t y_t, 
\end{array}
\right.
\]

where subscripts of \( y \)'s are counted modulo \( n \). Since \( n \equiv 2 \pmod{3} \) and \( v = n^2 + n + 1 \equiv 2^2 + 2 + 1 \equiv 1 \pmod{3} \), the above system of equations turns out to be (25). Finally, as Lemma 4.1 asserts

\[
(-1)^\sigma = \left(-\frac{1}{3}\right)^{n/2} = \left(-\frac{1}{3}\right)^{\frac{n}{3}} = \left(-\frac{1}{3}\right) = 1,
\]

we deduce that \( \sigma \) is even. This completes the proof. \( \square \)

We finish the paper with the following conjecture whose affirmative answer will imply the nonexistence of finite non-Desarguesian flag-transitive projective plane by Theorem 4.8.

**Conjecture 4.9.** For any integer \( n > 8 \) with \( n \equiv 8 \pmod{24} \) and any positive even integer \( \sigma \) dividing \( n \), the system (25) of equations on \( y_0, y_1, \ldots, y_{n-1} \in \mathbb{F}_{3n/\sigma} \) has no solution.

**References**

[1] L. D. Baumert and H. Fredricksen, The cyclotomic numbers of order eighteen with applications to difference sets, *Math. Comp.*, 21 (1967), 204–219.
[2] B. C. Berndt, R. J. Evans, and K. S. Williams, *Gauss and Jacobi sums*, A Wiley-Interscience Publication, New York, 1998.
[3] W. Bosma, J. Cannon and C. Playoust, The magma algebra system I: The user language, *J. Symbolic Comput.*, 24 (1997), no. 3-4, 235–265.
[4] S. Chowla, A property of biquadratic residues, *Proc. Nat. Acad. Sci. India, Sect. A*, 14 (1944), 45–46.
[5] H. Cohen, *Number theory Vol. I Tools and Diophantine equations*, Springer, New York, 2007.
[6] H. Cohen, *Number theory Vol. II Analytic and modern tools*, Springer, New York, 2007.
[7] C. J. Colbourn and J. H. Dinitz, *Handbook of combinatorial designs*, Second edition, CRC Press, Boca Raton, FL, 2007.
[8] R. J. Evans, Biotic Gauss sums and sixteenth power residue difference sets, *Acta Arith.*, 38 (1980), 37–46.
[9] R. J. Evans, Twenty-fourth power residue difference sets, *Math. Comp.*, 40 (1983), 677–683.
[10] R. J. Evans, Nonexistence of twentieth power residue difference sets, *Acta Arith.*, 84 (1999), 397–402.
[11] J.-C. Faugère, FGb: a library for computing Gröbner bases, in Mathematical Software - ICMS 2010, vol. 6327 of Lecture Notes in Computer Science, Springer, Berlin, Heidelberg, 2010, pp. 84–87.
[12] J.-C. Faugère and S. Lachartre, Parallel Gaussian elimination for Gröbner bases computations in finite fields, in Proceedings of the 4th International Workshop on Parallel and Symbolic Computation, ACM, 2010, pp. 89–97.
[13] B. H. Gross and N. Koblitz, Gauss sums and the $p$-adic $Γ$-function, Ann. of Math. (2), 109 (1979), no. 3, 569–581.
[14] N. Gill, Transitive projective planes and insoluble groups, Trans. Amer. Math. Soc., 368 (2016), 3017–3057.
[15] M. Hall Jr., Characters and cyclotomy, in Proc. Symp. Pure Math. Vol. 8, Amer. Math. Soc., Providence, R.I., 1965, pp. 31–43.
[16] D. R. Hughes and F. C. Piper, Projective planes, Springer-Verlag, New York-Berlin, 1973.
[17] D. Jungnickel, Difference sets, in Contemporary design theory, Wiley, New York, 1992, pp. 241–324.
[18] D. Jungnickel and K. Vedder, On the geometry of planar difference sets, European J. Combin., 5 (1984), no. 2, 143–148.
[19] E. Lehmer, On residue difference sets, Canadian J. Math., 5 (1953), 425–432.
[20] Y. Morita, A $p$-adic analogue of the $Γ$-function, J. Fac. Sci. Univ. Tokyo Sect. IA Math., 22 (1975), no. 2, 255–266.
[21] J. B. Muskat, The cyclotomic numbers of order fourteen, Acta Arith., 11 (1966), 263–279.
[22] R. E. A. C. Paley, On orthogonal matrices, J. Math. Phys., 12 (1933), 311–320.
[23] T. Storer, Cyclotomy and difference sets, Markham Publishing Company Chicago, 1967.
[24] A. Terras, Fourier analysis on finite groups and applications, Cambridge University Press, Cambridge, 1999.
[25] K. Thas, Finite flag-transitive projective planes: a survey and some remarks, Discrete Math., 266 (2003), no. 1-3, 417–429.
[26] K. Thas and D. Zagier, Finite projective planes, Fermat curves, and Gaussian periods, J. Eur. Math. Soc., 10 (2008), no. 1, 173–190.
[27] A. L. Whiteman, The cyclotomic numbers of order ten, in Proc. Sympos. Appl. Math., Vol. 10, Amer. Math. Soc., Providence, R.I., 1960, pp. 95–111.
[28] A. L. Whiteman, The cyclotomic numbers of order twelve, Acta Arith., 6 (1960), 53–76.
[29] B. Xia, Cyclotomic difference sets in finite fields, to appear in Math. Comp., https://arxiv.org/abs/1501.03275.

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