Semi-linear Backward Stochastic Integral Partial Differential Equations driven by a Brownian motion and a Poisson point process

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Abstract: In this paper we investigate classical solution of a semi-linear system of backward stochastic integral partial differential equations driven by a Brownian motion and a Poisson point process. By proving an Itô-Wentzell formula for jump diffusions as well as an abstract result of stochastic evolution equations, we obtain the stochastic integral partial differential equation for the inverse of the stochastic flow generated by a stochastic differential equation driven by a Brownian motion and a Poisson point process. By composing the random field generated by the solution of a backward stochastic differential equation with the inverse of the stochastic flow, we construct the classical solution of the system of backward stochastic integral partial differential equations. As a result, we establish a stochastic Feynman-Kac formula.

Keywords: backward stochastic integral partial differential equation, stochastic differential equation, backward stochastic differential equation, Poisson point process, stochastic flow, Itô-Wentzell formula

AMS Subject Classifications: 60H10, 60H20, 35R09

1 Introduction

Backward stochastic partial differential equations (BSPDEs) are function space-valued backward stochastic differential equations (BSDEs), the theories and applications of which can be found in [2], [4], [7], [9], [30], [31], [42], etc. BSPDEs arise from the study of stochastic control problems. For example, they can serve as the adjoint equations of Duncan-Mortensen-Zakai equation in the optimal control of stochastic differential equations (SDEs) with incomplete information (see [3], [38], [39]). They also appear as the adjoint equations in the stochastic maximum principle of systems governed by stochastic partial differential equations (SPDEs) driven by a Brownian motion (see [44]) or driven by both a Brownian motion and a Poisson random measure (see [27]). A class of fully nonlinear BSPDEs, the so-called backward stochastic Hamilton-Jacobi-Bellman (HJB) equations, were proposed by Peng [32] in the study of the optimal control problems for non-Markovian cases. And Englezos and Karatzas [10] characterized the value function of a utility maximization problem with habit formation as a classical solution of the corresponding stochastic HJB equation, which gives a concrete illustration of BSPDEs in a stochastic control context beyond the classical linear quadratic case. More recently, Meng and Tang [25] have studied the maximum principle of non-Markovian stochastic differential systems driven only by Poisson point processes and obtained a kind of backward stochastic HJB equations of jump type.

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The existence, uniqueness and regularity of the adapted solutions to BSPDEs have been studied by many authors. See Hu and Peng [16], Peng [32], Zhou [43], Ma and Yong [23, 24], Hu, Ma and Yong [15], for non-degenerate and degenerate cases. More recently, Du and Tang [8] explored the Dirichlet problem, rather than the conventional Cauchy problem, of the BSPDEs and established the result of existence and uniqueness in weighted Sobolev spaces. Different from the above literatures where the main instrument is operator semigroup or a prior estimates for differential operators, Tang [40] developed a probabilistic approach to study the properties of the solutions of BSPDEs. To be precise, he constructed the solutions of BSPDEs in terms of the inverse flows of the solutions of SDEs as well as the solutions of BSDEs. As a result, the properties of the solutions of BSPDEs can be obtained by the analysis of the solutions of SDEs and BSDEs with a spatial parameter.

Let \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0 \leq t \leq T}, P)\) be a complete filtered probability space on which is defined a \(d\)-dimensional standard Brownian motion \(\{W_t, 0 \leq t \leq T\}\). Denote by \(\mathcal{P}\) the predictable sub-\(\sigma\)-field of \(\mathcal{B}([0, T]) \otimes \mathcal{F}\). Let \((E, \mathcal{E}, v)\) be a measure space with \(v(E) < \infty\) and \(\mathcal{P} : D_{\mathcal{P}} \subset (0, \infty) \to E\) be an \(\mathcal{F}_t\)-adapted stationary Poisson point process. Then the counting measure induced by \(\mathcal{P}\) is defined by

\[
N((0, t] \times U) := \sharp\{s \in D_{\mathcal{P}}; s \leq t, \mathcal{P}(s) \in U\}, \text{ for } t > 0, U \in \mathcal{E}.
\]

And \(\tilde{N}(dedt) := N(dedt) - v(de)dt\) is a compensated Poisson random measure which is assumed to be independent of the Brownian motion.

This paper is concerned with the probabilistic interpretation of the solution \((p, q, r)\) of the following system of backward stochastic integral partial differential equations (BSIPDEs)

\[
\begin{align*}
dp(t, x) &= -\left[\mathcal{L}(t, x)p(t-, x) + \mathcal{M}(t, x)q(t, x)
+ f(t, x, p(t, x), q(t, x), \partial p(t-, x))\sigma, r(t-, \phi_{t, e}(x)) - p(t-, x) + p(t-, \phi_{t, e}(x))\right]dt
\end{align*}
\]

\[
+ \int_E \left[r(t, e, x) - r(t, e, \phi_{t, e}(x)) + p(t-, x) - p(t-, \phi_{t, e}(x))\right]v(de)dt
\]

\[
+ q(t, x)dW_t + \int_E r(t, e, x)\tilde{N}(dedt), \text{ } (t, x) \in [0, T] \times \mathbb{R}^n,
\]

\[
p(T, x) = \varphi(x),
\]

where we have defined

\[
\begin{align*}
\partial_t &:= \frac{\partial}{\partial t}, \quad \partial^2_t := \frac{\partial^2}{\partial t^2}, \quad \partial p := (\partial_j p^i)_{1 \leq i \leq l, 1 \leq j \leq n}, \\
\mathcal{L}(t, x) &:= \frac{1}{2} \sum_{i,j=1}^{n} \sum_{r=1}^{d} \sigma^{ir}(t, x) \partial^2_t + \sum_{i=1}^{n} [b^i(t, x) - \int_E g^i(t, e, x)v(de)] \partial_i, \\
\mathcal{M}(t, x) &:= \sum_{i=1}^{n} \sigma^{ir}(t, x) \partial_i, \quad 1 \leq r \leq d, \\
\phi_{t, e}(x) &:= x + g(t, e, x), \\
\mathcal{L}p &:= (\mathcal{L}p^1, \cdots, \mathcal{L}p^d), \\
\mathcal{M}q &:= (\sum_{r=1}^{d} \mathcal{M}q^{1r}, \cdots, \sum_{r=1}^{d} \mathcal{M}q^{dr}).
\end{align*}
\]

To be precise, under some suitable conditions on the random coefficients \(b, \sigma, g\) and \(\varphi\), we can
construct the solution \((p, q, r)\) of the above system as follows

\[
p(t, x) = Y_t(X_t^{-1}(x)), \\
q(t, x) = Z_t(X_t^{-1}(x)) - \partial p(t-, x)\sigma(t, x), \\
r(t, e, x) = p(t-, \phi_{t,e}^{-1}(x)) - p(t-, x) + \mathcal{U}_t(e, X_t^{-1}(\phi_{t,e}^{-1}(x)))
\]

where \((X_t(x), Y_t(x), Z_t(x), \mathcal{U}_t(\cdot, x))\in[0, T]\) is the solution of the following non-Markovian forward-backward stochastic differential equation (FBSDE)

\[
\begin{aligned}
\begin{cases}
    dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t + \int_{E} g(t, e, X_{t-})\tilde{N}(d\,e\,dt), \\
    dY_t = -f(t, X_t, Y_t, Z_t)dt + Z_t dW_t + \int_{E} \mathcal{U}_t(e)\tilde{N}(d\,e\,dt), \\
    X_0 = x, \; Y_T = \varphi(X_T), \; t \in [0, T],
\end{cases}
\end{aligned}
\]

and \([X_t^{-1}(x), (t, x) \in [0, T] \times \mathbb{R}^n]\) is the inverse mapping of \([X_t(x), (t, x) \in [0, T] \times \mathbb{R}^n]\) with respect to the spacial variable \(x\). The main contributions of this paper lie in establishing an Itô-Wentzell formula for jump diffusions and deriving the SIPDE satisfied by the inverse flow \([X_t^{-1}(x), (t, x) \in [0, T] \times \mathbb{R}^n]\). As demonstrated in Tang [40], the above two facts play crucial roles in the construction of the classical solutions to BSPDEs. Moreover, by the analysis of solutions of BSDEs driven by a Brownian motion and a Poisson point process, we generalize Tang’s result to the jump case.

The paper is organized as follows. Section 2 contains the notations that are used throughout the paper and some preliminary results. Section 3 is concerned with the derivation of the SIPDE for the inverse flow \([X_t^{-1}(x), (t, x) \in [0, T] \times \mathbb{R}^n]\). Section 4 consists of the estimates of the solutions of BSDEs. Section 5 is constituted the connection between BSIPDEs and non-Markovian FBSDEs.

## 2 Notations and Preliminary results

Let \(\mathbb{E}\) be a Euclidean space. The inner product in \(\mathbb{E}\) is denoted by \(\langle \cdot, \cdot \rangle\) and the norm in \(\mathbb{E}\) is denoted by \(|\cdot|\) or simply by \(|\cdot|\) when there is no confusion. Let \(\gamma := (\gamma_1, \cdots, \gamma_n)\) be a multi-index with nonnegative integers \(\gamma_i, i = 1, \cdots, n\). Denote \(|\gamma| := \gamma_1 + \cdots + \gamma_n\). For a function \(u\) defined on \(\mathbb{R}^n\), \(\partial^\gamma u\) means the derivative of \(u\) of order \(|\gamma|\), of order \(\gamma_i\) with respect to \(x^i\). \(u_i\) means the derivative of \(u\) with respect to \(x^i\) and \(u_{ij}\) means the derivative of \(u\) with respect to \(x^i x^j\), respectively. \(\partial u\) stands for the the gradient of \(u\) and \(\partial^2 u\) stands for the hessian of \(u\), respectively. We use the convention that repeated indices imply summation.

Let \(\mathbb{B}\) be a Banach space with the norm \(\|\cdot\|_\mathbb{B}\). For an integer \(m \geq 0\) and some subset \(K \subseteq \mathbb{R}^n\), we denote by \(C^m(K; \mathbb{B})\) the set of mappings \(f : K \to \mathbb{B}\) which are \(m\)-times continuously differentiable. And denote the norm in this space by

\[
\|f\|_{C^m(K; \mathbb{B})} := \sum_{0 \leq |\gamma| \leq m} \sup_{x \in K} \|\partial^\gamma f(x)\|_\mathbb{B}.
\]

If there is no danger of confusion, \(C^m(K; \mathbb{B})\) will be abbreviated as \(C^m(K)\). Denote by \(D(0, T; \mathbb{B})\) the set of all \(\mathbb{B}\)-valued cadlag functions on the interval \([0, T]\).

For \(p > 1\) and integer \(m \geq 0\), we denote by \(W^m_p\) the Sobolev space of real functions on \(\mathbb{R}^n\) with a finite norm

\[
\|u\|_{m,p} := \left(\sum_{0 \leq |\gamma| \leq m} \int_{\mathbb{R}^n} |\partial^\gamma u|^p \, dx\right)^{1/p}.
\]
The inner product and norm in $W^m_2$ will be denoted by $(\cdot, \cdot)_m$ and $\| \cdot \|_m$, respectively.

We further introduce some other spaces that will be used in the paper. Let $\mathcal{X}$ (resp., $\mathcal{Y}$) be a sub-$\sigma$-algebra of $\mathcal{F}$ (resp., $\mathcal{E}$). $L^p(\mathcal{X}; \mathbb{B})$ (resp., $L^p(\mathcal{Y}; \mathbb{B})$) denotes the set of all $\mathbb{B}$-valued $\mathcal{X}$-measurable (resp., $\mathcal{Y}$-measurable) random variable $\eta$ such that $E\|\eta\|^p_\mathbb{B} < \infty$ (resp., $\int_E \|\eta\|^p_\mathbb{B} v(de) < \infty$). For given two real numbers $1 \leq p, k \leq \infty$, we denote by $\mathcal{L}^p_{\mathcal{F}}(0, T; L^k(\mathcal{F}; \mathbb{B}))$ the set of all adapted $\mathbb{B}$-valued processes $X$ such that

$$\|X\|_{\mathcal{L}^p_{\mathcal{F}}(0, T; L^k(\mathcal{F}; \mathbb{B}))} := \left( \int_0^T [E\|X(t)\|^{k/p/k}_\mathbb{B}] dt \right)^{1/p} < \infty.$$  

When $k = p$, $\mathcal{L}^p_{\mathcal{F}}(0, T; L^k(\mathcal{F}; \mathbb{B}))$ will be abbreviated as $\mathcal{L}^p_{\mathcal{F}}(0, T; \mathbb{B})$. Denote by $\mathcal{L}^{p\otimes E}_{\mathcal{F}}(0, T; \mathbb{B})$ (resp., $\mathcal{L}^{\infty, p}_{\mathcal{F}, \omega}(0, T; \mathbb{B})$) the Banach space of all adapted $\mathbb{B}$-valued strongly (resp., weakly) cadlag processes $X$ for which

$$\|X\|_{\mathcal{L}^{p\otimes E}_{\mathcal{F}}(0, T; \mathbb{B})} := \left( E\sup_{0 \leq t \leq T} \|X(t)\|_\mathbb{B} \right)^{1/p} < \infty.$$  

(resp., $\|X\|_{\mathcal{L}^{\infty, p}_{\mathcal{F}, \omega}(0, T; \mathbb{B})} := \left( E\sup_{0 \leq t \leq T} \|X(t)\|_\mathbb{B} \right)^{1/p} < \infty$).

Denote by $\mathcal{L}^{k,p}_{\mathcal{F}}(0, T; \mathbb{B})$ (resp., $\mathcal{L}^{k,p}_{\mathcal{F}, \omega}(0, T \times \mathcal{F}; \mathbb{B})$) with $k \in [1, \infty)$ the set of all $\mathcal{P}$ measurable (resp., $\mathcal{P} \otimes \mathcal{E}$ measurable) $\mathbb{B}$-valued processes $X$ such that

$$\|X\|_{\mathcal{L}^{k,p}_{\mathcal{F}}(0, T; \mathbb{B})} := \left( E\left( \int_0^T \|X(t)\|^{k/p}_\mathbb{B} dt \right)^{p/k} \right)^{1/p} < \infty.$$  

(resp., $\|X\|_{\mathcal{L}^{k,p}_{\mathcal{F}, \omega}(0, T \times \mathcal{F}; \mathbb{B})} := \left( E\left( \int_0^T \left( \int_E \|X(t, e)\|^{k/p}_\mathbb{B} v(de) dt \right)^{p/k} \right)^{1/p} \right) < \infty$).

Obviously when $k = p$, $\mathcal{L}^p_{\mathcal{F}}(0, T; L^k(\mathcal{F}; \mathbb{B}))$ coincides with $\mathcal{L}^{k,p}_{\mathcal{F}}(0, T; \mathbb{B})$.

We consider the following SDE

$$\begin{cases}
    dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t + \int_E g(t, e, X_{t-})\tilde{N}(dedt), \\
    X_s = x, \quad t \in [s, T],
\end{cases} \quad (2.1)$$

where $b : [0, T] \times \Omega \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $\sigma : [0, T] \times \Omega \times \mathbb{R}^n \rightarrow \mathbb{R}^n \times d$ are $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^n)$ measurable, and $g : [0, T] \times \Omega \times E \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is $\mathcal{P} \otimes \mathcal{E} \otimes \mathcal{B}(\mathbb{R}^n)$ measurable. The functions $b$, $\sigma$, and $g$ are called drift coefficient, diffusion coefficient and jump coefficient, respectively. We denote by \{X^x_t, t \in [s, T] \} the solution of equation (2.1) starting from $x$ at time $s$ and simply denote $X(x) := X^x_0(x)$.

We assume the following conditions

**(C1)** the coefficients $b, \sigma,$ and $g$ are of linear growth with respect to $x$, i.e., there exist positive constant $K$ and deterministic function $K(e)$ such that

$$|b(t, x)| \leq K(1 + |x|), \quad |\sigma(t, x)| \leq K(1 + |x|), \quad |g(t, e, x)| \leq K(e)(1 + |x|)$$

and

$$\int_E K(e)^p v(de) < \infty, \forall p \geq 2;$$

**(C2)** the coefficients $b, \sigma,$ and $g$ are differentiable and their derivatives up to the order $k$ are
bounded, i.e., there exist positive constant $L$ and deterministic function $L(e)$, such that for $1 \leq |\gamma| \leq k$, 
\[
|\partial^\gamma b(t,x)| \leq L, \quad |\partial^\gamma \sigma(t,x)| \leq L, \quad |\partial^\gamma g(t,e,x)| \leq L(e)
\]
and
\[
\int_E L(e)^p v(de) < \infty, \quad \forall p \geq 2;
\]

(C3) the map $\phi_{t,e} : x \to x + g(t,e,x)$ is homeomorphic for any $(t,\omega,e) \in [0,T] \times \Omega \times E$ and the inverse map $\phi_{t,e}^{-1}$ is uniformly Lipschitz continuous and of uniformly linear growth with respect to $x$;

(C4) the Jacobian matrix $I + \partial g(t,e,x)$ of the homeomorphic map $\phi_{t,e}(x)$ is invertible for any $x$, for almost all $(t,\omega,e) \in [0,T] \times \Omega \times E$.

The following two lemmas are due to Kunita [22] or Fujiwara and Kunita [11].

**Lemma 2.1.** Assume the conditions (C1), (C2), and (C3) are satisfied. Then there exists a version of the unique solution of equation (2.1), denoted still by $\{X^i_t(x), \quad (t,x) \in [s,T] \times \mathbb{R}^n\}$, which satisfies

(i) $t \to X^i_s(\cdot)$ is a $C(\mathbb{R}^n)$-valued cadlag process;

(ii) $X^i_t(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ is homeomorphic for any $t \in [s,T]$, a.s.;

(iii) $X^i_t(x) = X^i_t(X^i_s(x)), \quad 0 \leq s \leq t \leq r \leq T$.

**Lemma 2.2.** If conditions (C1), (C2)$_{k+1}$, (C3) and (C4) are satisfied, the unique solution of the equation (2.1) defines a stochastic flow of $C^k$-diffeomorphism.

### 3 SIPDE for the inverse flow $X^{-1}_t(x)$

In this section, under some suitable hypotheses on the coefficients $b$, $\sigma$ and $g$, we will prove that the $i$-th coordinate of the inverse flow $X^{-1}_t(x)$ of the solution $X(x)$ of the SDE (2.1) satisfies a stochastic integral partial differential equation (SIPDE) of the following form

\[
\begin{aligned}
du(t,x) &= (\mathcal{M}^x \mathcal{M}^x - \mathcal{L})(t,x)u(t-,x)dt + \int_E A(t,e)u(t-,x)v(de)dt \\
&\quad - \mathcal{M}^x(t,x)u(t-,x)dW^x_t + \int_E A(t,e)u(t-,x)\tilde{N}(dedt), \quad 0 \leq t \leq T, \\
u(0,x) &= x^i
\end{aligned}
\]  

(3.1)

where the operators $\mathcal{L}(t,x)$ and $\mathcal{M}^x(t,x)$ are defined in (1.2) and

\[
A(t,e)f(x) := -f(x) + f(\phi_{t,e}^{-1}(x)).
\]  

(3.2)

As we will see in Section 5, the above equation plays a crucial role in the construction of the solution of BSIPDE (1.1). In fact, when SDE (2.1) is driven only by a Brownian motion, Krylov and Rozovskii have proved that the inverse flow $X^{-1}_t(x)$ satisfies equation (3.1) with $g = 0$ (see [20] Theorem 3.1, page 89). We generalize their results to the case of jump diffusions.

#### 3.1 An Itô-Wentzell formula for jump diffusions

An Itô-Wentzell formula for forward processes driven by a Poisson point process was established by Øksendal and Zhang in [28] where the integrands in (3.4) are required to be square integrable with respect to $x$ on the whole space $\mathbb{R}^n$, which prevents us from directly applying it to the solution $(Y(x), Z(x), U(x))$ of BSDE (4.10). Recently Krylov [17] considered the case of
Brownian motion-driven semimartingales and proved an Itô-Wentzell formula for distribution-valued processes so that generalized some existing ones (see, for instance, Theorem 3.3.1 of [21] and Theorem 1.4.9 of [35]). Our method is essentially same as that of [17] where the assumptions, except imposed on the jump coefficients \( g \) and \( J \), are weaker than those in Lemma 3.1. However Lemma 3.1 is enough for our subsequent use.

Let \( X \) be an \( \mathbb{R}^n \)-valued stochastic process given by

\[
X_t = X_0 + \int_0^t b(s)ds + \int_0^t \sigma(s)dW_s + \int_0^t \int_E g(s,e)\tilde{N}(deds).
\]

Here \( b(\cdot) \) is predictable \( \mathbb{R}^n \)-valued process, \( \sigma(\cdot) \) is predictable \( \mathbb{R}^{n \times d} \)-valued process and \( g(\cdot,\cdot) \) is \( \mathcal{P} \otimes \mathcal{E} \) measurable \( \mathbb{R}^n \)-valued process such that almost surely

\[
\int_0^T \left[ |b(t)| + \text{tr}(t) \right] dt < \infty,
\]

and

\[
\sup_{(t,e) \in [0,T] \times \mathcal{E}} |g(t,e)| < \infty \tag{3.3}
\]

where \( 2a(t) := \sigma(t)\sigma'(t) \) and \( \text{tr}(t) := \sum_{i=1}^n |a_{ii}(t)| \).

Let \( \{F(t,x), (t,x) \in [0,T] \times \mathbb{R}^n\} \) be a family of \( \mathbb{R} \)-valued semimartingales of the form

\[
F(t,x) = F(0,x) + \int_0^t G(s,x)ds + \int_0^t H(s,x)dW_s + \int_0^t \int_E J(s,e,x)\tilde{N}(deds) \tag{3.4}
\]

where the \( \mathbb{R} \)-valued function \( G(\cdot,\cdot) \) and \( \mathbb{R}^d \)-valued function \( H(\cdot,\cdot) \) are \( \mathcal{P} \otimes \mathcal{B}(\mathbb{R}^n) \) measurable and \( \mathbb{R} \)-valued function \( J(\cdot,\cdot,\cdot) \) is \( \mathcal{P} \otimes \mathcal{E} \otimes \mathcal{B}(\mathbb{R}^n) \) measurable. Assume that

(A1) For any \((\omega,t,e) \in \Omega \times [0,T] \times \mathcal{E},\)

(a) the function \( F(t,x) \) is twice continuously differentiable in \( x, \)

(b) the function \( G(t,x) \) is continuous in \( x, \)

(c) the function \( H(t,x) \) is continuously differentiable in \( x, \)

(d) the function \( J(t,e,x) \) is continuous in \( x; \)

(A2) For any compact subset \( K \subset \mathbb{R}^n \) we have almost surely

\[
\int_0^T \sup_{x \in K} \left[ |F(t,x)| \left[ |b(t)| + \text{tr}(t) \right] + |F(t,x)|^2 \text{tr}(t) \right] dt < \infty,
\]

\[
\int_0^T \sup_{x \in K} \left[ |F(t,x)|^2 + |\partial F(t,x)| + |L(t)F(t,x)| + |M(t)F(t,x)|^2 \right] < \infty,
\]

\[
\int_0^T \sup_{x \in K} \left[ |G(t,x)| + |H(t,x)|^2 + |M^k(t)H^k(t,x)| \right] dt < \infty,
\]

\[
\int_0^T \int_{\mathcal{E}} \sup_{x \in K} |J(t,e,x)|^2 < \infty,
\]

where the differential operators

\[
L(t) := a_{ij}(t)\partial_{ij}^2 + b_i(t)\partial_i,
\]

\[
M^k(t) := \sigma^k(t)\partial_i, \quad k = 1, \ldots, d,
\]

\[
M(t) := (M^1(t), \ldots, M^d(t))^t.
\]
Lemma 3.1. Suppose that the conditions (A1) and (A2) are satisfied. Then we have for each \( t \in [0, T] \) almost surely
\[
F(t, X(t)) = F(0, x) + \int_0^t G(s, X_{s-}) \, ds + \int_0^t H(s, X_{s-}) \, dW_s + \int_0^t \langle \partial F(s- \rightarrow X_{s-}), b(s) \rangle \, ds + \int_0^t \langle \partial F(s- \rightarrow X_{s-}), \sigma(s) \rangle \, dW_s + \int_0^t \frac{1}{2} \langle \sigma \partial^2 F(s- \rightarrow X_{s-}) \sigma^* \rangle \, ds + \int_0^t \langle \sigma \partial H(s- \rightarrow X_{s-}), \sigma \rangle \, ds + \int_0^t \int_E \left[ J(s, e, X_{s-} + g(s, e)) - J(s, e, X_{s-}) \right] v(de) \, ds + \int_0^t \int_E \left[ F(s- \rightarrow X_{s-} + g(s, e)) - F(s- \rightarrow X_{s-}) - \langle \partial F(s- \rightarrow X_{s-}), g(s, e) \rangle \right] v(de) \, ds + \int_0^t \int_E \left[ F(s- \rightarrow X_{s-} + g(s, e)) - F(s- \rightarrow X_{s-}) + J(s, e, X_{s-} + g(s, e)) \right] \tilde{N}(deds)
\]
(3.5)
where \( \ll A, B \gg := \text{tr}(AB^t) \) for \( n \times m \) matrices \( A \) and \( B \).

Proof. Taking nonnegative \( \phi \in C^\infty_c(\mathbb{R}^n, \mathbb{R}) \) with support in the unit ball and \( \int_{\mathbb{R}^n} \phi(x) \, dx = 1 \), define for \( \varepsilon > 0 \), \( \phi_\varepsilon(x) := \varepsilon^{-n} \phi(x/\varepsilon) \). Then for any \( x \in \mathbb{R}^n \), Itô’s formula yields
\[
\phi_\varepsilon(X_{t-} - x)
= \phi_\varepsilon(X_0 - x) + \int_0^t \langle \partial \phi_\varepsilon(X_{s-} - x), b(s) \rangle \, ds + \int_0^t \langle \partial \phi_\varepsilon(X_{s-} - x), \sigma(s) \rangle \, dW_s
+ \frac{1}{2} \int_0^t \langle \partial^2 \phi_\varepsilon(X_{s-} - x), \sigma \sigma^* \rangle \, ds
+ \int_0^t \int_E \left[ \phi_\varepsilon(X_{s-} - x + g(s, e)) - \phi_\varepsilon(X_{s-} - x) \right] \tilde{N}(deds)
+ \int_0^t \int_E \left[ \phi_\varepsilon(X_{s-} - x + g(s, e)) - \phi_\varepsilon(X_{s-} - x) - \langle \partial \phi_\varepsilon(X_{s-} - x), g(s, e) \rangle \right] v(de) \, ds.
\]
Again using Itô’s formula to the product \( F(t, x) \phi_\varepsilon(X_{t-} - x) \), we obtain almost surely for all \( t \in [0, T] \)
\[
F(t, x) \phi_\varepsilon(X_{t-} - x)
= F(0, x) \phi_\varepsilon(X_0 - x) + \int_0^t \phi_\varepsilon(X_{s-} - x) G(s, x) \, ds + \int_0^t \phi_\varepsilon(X_{s-} - x) H(s, x) \, dW_s
+ \frac{1}{2} \int_0^t F(s-, x) \langle \partial^2 \phi_\varepsilon(X_{s-} - x), \sigma \sigma^* \rangle \, ds + \int_0^t H(s, x) \sigma^* \partial \phi_\varepsilon(X_{s-} - x) \, ds
+ \int_0^t \int_E \left[ \phi_\varepsilon(X_{s-} - x + g(s, e)) - \phi_\varepsilon(X_{s-} - x) \right] J(s, e, x) v(de) \, ds
+ \int_0^t \int_E \left[ F(s-, x) \partial \phi_\varepsilon(X_{s-} - x) - \phi_\varepsilon(X_{s-} - x) - \langle \partial \phi_\varepsilon(X_{s-} - x), g(s, e) \rangle \right] v(de) \, ds
+ \int_0^t F(s-, x) \langle \partial \phi_\varepsilon(X_{s-} - x), \sigma \rangle \, dW_s + \int_0^t F(s-, x) \langle \partial \phi_\varepsilon(X_{s-} - x), b(s) \rangle \, ds
+ \int_0^t \int_E \left[ \phi_\varepsilon(X_{s-} - x + g(s, e)) F(s-, x) + \phi_\varepsilon(X_{s-} - x + g(s, e)) J(s, e, x) \right.
- \phi_\varepsilon(X_{s-} - x) F(s-, x)] \tilde{N}(deds).
\]
(3.6)
It is well known that condition [3.3] implies that
\[
\sup_{0 \leq t \leq T} |X_t| < \infty, \ a.s..
\]
In view of assumption (A2), we see that all terms in (3.6) are almost surely finite. For \( r \in \mathbb{N} \), set \( B_r := \{ x \in \mathbb{R}^n : |x| < r \} \). From (3.3) and (A2), we have
\[
\int_0^T \int_{B_r} |\phi(x)(X_s - x)H(s,x)|^2 + |F(s-,x)\sigma^\alpha(s)\partial\phi(x)(X_s - x)|^2 \, dx \, ds \\
\leq \int_0^T \int_{\mathbb{R}^n} |\phi(x)(X_s - x)H(s,x)|^2 + |F(s-,x)|^2 \text{tr}(\sigma(s))\partial\phi(x)(X_s - x)|^2 \, dx \, ds \\
\leq \left( \int_0^T \sup_{x \in K(\omega)} |H(s,x)|^2 \, ds \right) \int_{\mathbb{R}^n} |\phi(x)|^2 \, dx \\
+ \left( \int_0^T \sup_{x \in K(\omega)} |F(s-,x)|^2 \text{tr}(\sigma(s)) \, ds \right) \int_{\mathbb{R}^n} |\partial\phi(x)|^2 \, dx
\]
\(<\infty, \ a.s.,
\]
\[
\int_0^t \int_E \int_{B_r} |\phi(x)(X_s - x + g(s,e)) - \phi(x)(X_s - x)| F(s-,x)|^2 \, dxv \, (de) \, ds \\
\leq \int_0^t \int_E \int_{\mathbb{R}^n} |\phi(x)(X_s - x + g(s,e)) - \phi(x)(X_s - x)| F(s-,x)|^2 \, dxv \, (de) \, ds \\
\leq \left( C\nu(E) \int_0^T \left( \sup_{x \in K'(\omega)} |F(s-,x)|^2 + \sup_{x \in K(\omega)} |F(s-,x)|^2 \right) \, ds \right) \int_{\mathbb{R}^n} |\phi(x)|^2 \, dx
\]
\(<\infty, \ a.s.,
\]
and
\[
\int_0^t \int_E \int_{B_r} |\phi(x)(X_s - x + g(s,e))J(s,e,x)|^2 \, dxv \, (de) \, ds \\
\leq \int_0^t \int_E \int_{\mathbb{R}^n} |\phi(x)(X_s - x + g(s,e))J(s,e,x)|^2 \, dxv \, (de) \, ds \\
\leq \left( C \int_0^T \sup_{E \times K'(\omega)} |J(s,e,x)|^2 v \, (de) \, ds \right) \int_{\mathbb{R}^n} |\phi(x)|^2 \, dx
\]
\(<\infty, \ a.s.,
\]
Here \( K(\omega) \) and \( K'(\omega) \) are two compact subsets of \( \mathbb{R}^n \) depending on \( \omega \). Integrating with respect to \( x \) over the ball \( B_r \) on both sides of (3.6), using Fubini's Theorem to interchange \( dx \) and \( ds \) and the stochastic Fubini's theorem (see [33] Theorem 65, pages 208-209) to interchange \( dx \).
and $dW_s$, and $dx$ and $d\tilde{N}(de)$, and then letting $r \to \infty$, we obtain

$$
\int_{\mathbb{R}^n} F(t, x)\phi_\varepsilon(X_t - x)dx
$$

$$
= \int_{\mathbb{R}^n} F(0, x)\phi_\varepsilon(X_0 - x)dx + \int_0^t \int_{\mathbb{R}^n} \phi_\varepsilon(X_s - x)G(s, x)dxdv + \int_0^t \int_{\mathbb{R}^n} \phi_\varepsilon(X_s - x)H(s, x)dxdW_s
$$

$$
+ \frac{1}{2}\int_0^t \int_{\mathbb{R}^n} F(s-, x) \ll \partial^2 \phi_\varepsilon(X_s - x), \sigma \sigma^*(s) \gg dxdv + \int_0^t \int_{\mathbb{R}^n} H(s, x)\sigma^*(s)\partial \phi_\varepsilon(X_s - x)dxdv
$$

$$
+ \int_0^t \int_{\mathbb{E} \times \mathbb{R}^n} F(s-, x)[\phi_\varepsilon(X_s - x + g(s, e)) - \phi_\varepsilon(X_s - x) - \langle \partial \phi_\varepsilon(X_s - x), g(s, e) \rangle]dxdv(de)ds
$$

$$
+ \int_0^t \int_{\mathbb{E} \times \mathbb{R}^n} \left\{ [\phi_\varepsilon(X_s - x + g(s, e)) - \phi_\varepsilon(X_s - x)]F(s-, x)
$$

$$
+ \phi_\varepsilon(X_s - x + g(s, e))J(s, e, x) \right\} dxd\tilde{N}(de)
$$

$$
+ \int_0^t \int_{\mathbb{E} \times \mathbb{R}^n} [\phi_\varepsilon(X_s - x + g(s, e)) - \phi_\varepsilon(X_s - x)]J(s, e, x)dxdv(de)ds.
$$

(3.10)

Indeed, noting the inequality \[ [5.7], \] we see from the dominated convergence theorem that as $r \to \infty$

$$
\int_0^T \left| \int_{B_r} \phi_\varepsilon(X_s - x)H(s, x)dx - \int_{\mathbb{R}^n} \phi_\varepsilon(X_s - x)H(s, x)dx \right|^2 ds \to 0, \text{ in probability}
$$

which implies

$$
\int_0^T \int_{B_r} \phi_\varepsilon(X_s - x)H(s, x)dxdW_s \to \int_0^T \int_{\mathbb{R}^n} \phi_\varepsilon(X_s - x)H(s, x)dxdW_s, \text{ in probability.}
$$

The convergence of other terms can be proved in a similar manner.

Using integration by parts formula, we have

$$
\int_{\mathbb{R}^n} F(s-, x) \ll \partial^2 \phi_\varepsilon(X_s - x), \sigma \sigma^*(s) \gg dx = \int_{\mathbb{R}^n} \phi_\varepsilon(X_s - x) \ll \partial^2 F(s-, x), \sigma \sigma^*(s) \gg dx,
$$

$$
\int_{\mathbb{R}^n} H(s, x)\sigma^*(s)\partial \phi_\varepsilon(X_s - x)dx = \int_{\mathbb{R}^n} \phi_\varepsilon(X_s - x) \ll \partial H(s, x), \sigma(s) \gg dx,
$$

$$
\int_{\mathbb{R}^n} F(s-, x)\partial \phi_\varepsilon(X_s - x)dx = \int_{\mathbb{R}^n} \phi_\varepsilon(X_s - x)\partial F(s-, x)dx.
$$

Finally, letting $\varepsilon \to 0$ in \[ (3.10), \] we can deduce \[ (3.5) \] using arguments analogous to the above. \[ \square \]

### 3.2 An abstract result

In this and the next subsections, following Krylov and Rozovskii [18], [19] and [20], we focus on the derivation of SIPDE \[ (3.1) \] for the inverse flow $X^{-1}(x)$. To this end, we first establish an abstract result.

Let $V$ and $H$ be two separable Hilbert spaces and $V$ is continuously embedded into $H$ such that $V$ is dense in $H$. The space $H$ is identified with its dual space $H^*$, consequently

$$
V \subset H \cong H^* \subset V^*.
$$
where $V^*$ is the dual space of $V$. We denote by $\| \cdot \|_H$ and $\| \cdot \|_V$ the norms in $H$ and $V$, respectively. Denote by $(\cdot, \cdot)$ the inner product in $H$ and $\langle \cdot, \cdot \rangle$ the duality product between $V$ and $V^*$.

We consider the following abstract form of equation (3.1)

$$
\begin{cases}
  du(t) = A(t)u(t)dt + \int_E \tilde{A}(t,e)u(t)v(de)dt + B(t)u(t)dW_t + \int_E \tilde{A}(t,e)u(t-\cdot)\tilde{N}(dedt), \\
  u_0 \in H
\end{cases}
$$

(3.11)

where the three processes

$$
A(\cdot) \in \mathcal{L}_F^\infty(0, T; \mathcal{L}(V, V^*)), \\
B(\cdot) \in \mathcal{L}_F^\infty(0, T; \mathcal{L}(V, H^d)), \\
\tilde{A}(\cdot, \cdot) \in \mathcal{L}_F^\infty(0, T; \mathcal{L}(H, L^2(E, H)))
$$

satisfy the coercive condition

$$
-2\langle A(t)u, u \rangle + \lambda\|u\|_H^2 \geq \alpha\|u\|_V^2 + \|B(t)u\|_H^2 + \int_E \|\tilde{A}(t,e)u\|_H^2 v(de), \quad \forall u \in V,
$$

(3.12)

for some $\alpha > 0$ and $\lambda \in \mathbb{R}$.

Existence and uniqueness of solutions to SPDEs driven by a Poisson random measure or a stable noise are studied by many authors, see e.g. [1, 13, 14, 26, 34, 41], and references therein. Usually the operator $A$ is assumed to be the infinitesimal generator of a strongly continuous semigroup and mild solutions in $H$, rather than weak solutions (in the PDE sense), are considered. In our setting, both $A$ and $B$ are random operators and $B$ is a first-order differential operator, which is a little more complicated than the case in [34] where $B$ is only Lipschitz continuous from $H$ to $H$ in the diffusion term. We have the following theorem

**Theorem 3.2.** Equation (3.11) has a unique solution $u \in \mathcal{L}_F^2(0, T; V) \cap \mathcal{L}_F^{\infty,2}(0, T; H)$. Moreover,

$$
\|u(t)\|_H^2 = \|u_0\|_H^2 + 2\int_0^t \langle A(s)u(s), u(s) \rangle ds + 2\int_0^t \int_E (\tilde{A}(s,e)u(s), u(s))v(de)ds \\
+ 2\int_0^t \langle B(s)u(s), u(s) \rangle ds + \int_0^t \|B(s)u(s)\|_H^2 ds \\
+ \int_0^t \int_E \|\tilde{A}(s,e)u(s-\cdot)\|_H^2 + 2(u(s-\cdot), \tilde{A}(s,e)u(s-\cdot))\tilde{N}(deds) \\
+ \int_0^t \int_E \|\tilde{A}(s,e)u(s-\cdot)\|_H^2 v(de)ds.
$$

(3.13)

**Proof.** Since the weak limit of the Galerkin approximation, as noted in [29] Theorem 1.3], is not necessarily an $H$-valued cadlag adapted process, we split the proof into two steps.

**Step1.** For any given $h \in \mathcal{L}_F^2(0, T; V) \cap \mathcal{L}_F^{\infty,2}(0, T; H)$, we first prove the following equation

$$
\begin{cases}
  du(t) = A(t)u(t)dt + \int_E \tilde{A}(t,e)u(t)v(de)dt + B(t)u(t)dW_t + \int_E \tilde{A}(t,e)h(t-\cdot)\tilde{N}(dedt), \\
  u_0 \in H
\end{cases}
$$

(3.14)
has a unique solution \( u \in L^2_T(0, T; V) \cap L^\infty_T(0, T; H) \). Let \( \{\nu_n\}_{n=1}^\infty \) be a basis of \( V \) and a complete orthonormal basis of \( H \). For \( n \in \mathbb{N} \) and \( 1 \leq i \leq n \), set
\[
V_n := \text{span}\{\nu_1, \nu_2, \ldots, \nu_n\}, \quad u_{0,n} := \sum_{i=1}^{n} g^0_{ni} \nu_i, \quad u_n(t) := \sum_{i=1}^{n} g_n(t) \nu_i
\]
where \( g^0_{ni} := (u_0, \nu_i) \) and \( g_n(t) := (g_{n1}(t), g_{n2}(t), \ldots, g_{nn}(t)) \) is the solution of the following Itô equation:
\[
\begin{cases}
    dg_n(t) = \sum_{j=1}^{n} g_{nj}(t) (A(t)\nu_j, \nu_i) dt + \sum_{j=1}^{n} \int_E g_{nj}(t) (\tilde{A}(t, e)\nu_j, \nu_i) v(de) dt \\
    \quad + \sum_{j=1}^{n} g_{nj}(t) (B(t)\nu_j, \nu_i) dW_t + \int_E (\tilde{A}(t, e) h(t-), \nu_i) \tilde{N}(de dt),
\end{cases} \tag{3.15}
\]
It follows from Itô’s formula and condition (3.12) that
\[
E\left\| u_n(t) \right\|_H^2 \\
\leq \| u_n(0) \|_H^2 + 2E \int_0^t (A(s)u_n(s), u_n(s)) ds + 2E \int_0^t (\tilde{A}(s,e)u_n(s), u_n(s)) (de) ds \\
+ \sum_{i=1}^{n} E \int_0^t [(B(s)u_n(s), \nu_i)]^2 ds + \sum_{i=1}^{n} E \int_0^t \int_E |(\tilde{A}(s,e)h(s-), \nu_i)|^2 (de) ds \\
\leq \| u_n(0) \|_H^2 + E \int_0^t \left[ \| u_n(s) \|_H^2 H - \alpha \| u_n(s) \|_{\tilde{V}}^2 - \| B(s)u_n(s) \|_H^2 - \int_E \| \tilde{A}(s,e)u_n(s) \|_H^2 (de) \right] ds \\
E \int_0^t \int_E \| \tilde{A}(s,e)u_n(s) \|_H^2 v(de) ds + v(E) E \int_0^t \| u_n(s) \|_{\tilde{V}}^2 ds + E \int_0^t \| B(s)u_n(s) \|_H^2 ds \\
+ E \int_0^t \int_E \| \tilde{A}(s,e)h(s-) \|_H^2 v(de) ds.
\tag{3.16}
\]
Gronwall’s inequality yields that
\[
\sup_{0 \leq t \leq T} E\left\| u_n(t) \right\|_H^2 \leq C \left( 1 + E \int_0^T \| h(s) \|_{\tilde{V}}^2 ds \right).
\tag{3.17}
\]
From (3.16), we have
\[
E \int_0^T \| u_n(t) \|_{\tilde{V}}^2 \leq C
\tag{3.18}
\]
which implies that there exist a subsequence \( \{u_{n_k}\} \) and \( u \in L^2_T(0, T; V) \) such that
\[
u_{n_k} \to u, \text{ weakly in } L^2_T(0, T; V).
\tag{3.19}
\]
Let \( \{f(t), t \in [0, T]\} \) be a bounded progressive measurable process on \([0, T]\). It follows from (3.15) that for each \( \nu_i \) and \( k \geq i \),
\[
E \int_0^T f(t)(u_{n_k}(t), \nu_i) dt \\
= E \int_0^T f(t) \left[ (u_0, \nu_i) + \int_0^t (A(s)u_{n_k}(s), \nu_i) ds + \int_0^t \int_E (\tilde{A}(s,e)u_{n_k}(s), \nu_i) v(de) ds \\
+ \int_0^t (B(s)u_{n_k}(s), \nu_i) dW_s + \int_0^t \int_E (\tilde{A}(s,e)h(s-), \nu_i) \tilde{N}(de ds) \right] dt.
\]
Since the operators are bounded, passing to the limit in the last equality we get

\[
E \int_0^T f(t)(u(t), \nu_t)dt = E \int_0^T f(t) \left[ (u_0, \nu_t) + \int_0^t (A(s)u(s), \nu_s)ds + \int_0^t \int_E \tilde{A}(s, e)u(s, \nu_s)v(de)ds + \int_0^t (B(s)u(s), \nu_s)dW_s + \int_0^t \int_E (\tilde{A}(s, e)h(s), \nu_s)\tilde{N}(de)ds \right] dt.
\]

(3.20)

Indeed, it is sufficient to show

\[
E \int_0^T f(t) \left( \int_0^t (B(s)u_{n_k}(s), \nu_s)dW_s \right) dt \to E \int_0^T f(t) \left( \int_0^t (B(s)u(s), \nu_s)dW_s \right) dt,
\]

(3.21)

and the convergence of other terms can be treated in an analogous way. Since

\[
B(\cdot) \in L_\mathcal{F}^2(0, T; \mathcal{L}(V, H^d)),
\]

we can deduce from (3.19) that for any \( t \in [0, T] \),

\[
(B(\cdot)u_{n_k}(\cdot), \nu_t) \to (B(\cdot)u(\cdot), \nu_t), \text{ weakly in } L_\mathcal{F}^2(0, T; \mathbb{R}^d).
\]

Since the stochastic integral with respect to a Brownian motion is a linear and strong continuous mapping from \( L_\mathcal{F}^2(0, t; \mathbb{R}^d) \) to \( L^2(\mathcal{F}_t; \mathbb{R}) \), it is weakly continuous. Therefore,

\[
\int_0^t (B(s)u_{n_k}(s), \nu_s)dW_s \to \int_0^t (B(s)u(s), \nu_s)dW_s, \text{ weakly in } L^2(\mathcal{F}_t; \mathbb{R})
\]

and in particular,

\[
E \left[ f(t) \int_0^t (B(s)u_{n_k}(s), \nu_s)dW_s \right] \to E \left[ f(t) \int_0^t (B(s)u(s), \nu_s)dW_s \right].
\]

Moreover,

\[
\left| E \left[ f(t) \int_0^t (B(s)u_{n_k}(s), \nu_s)dW_s \right] \right| \leq \frac{1}{2} E|f(t)|^2 + CE \int_0^T \|u_{n_k}(s)\|^2 ds \leq C.
\]

By the dominated convergence theorem we get (3.21). From (3.20), it follows that for a.e. \( (t, \omega) \in [0, T] \times \Omega \),

\[
u(t) = u_0 + \int_0^t A(s)u(s)ds + \int_0^t \int_E \tilde{A}(s, e)u(s, \nu_s)v(de)ds + \int_0^t B(s)u(s)dW_s + \int_0^t \int_E \tilde{A}(s, e)h(s)\tilde{N}(de)ds.
\]

(3.22)

By [12, Theorem 2, page 156], there exists an \( H \)-valued adapted cadlag process \( \tilde{u} \) which coincides with \( u \) for a.e. \((t, \omega)\) and is equal to the right hand of (3.22) for all \( t \in [0, T] \) a.s.. We identify \( \tilde{u} \) with \( u \). Furthermore, we have

\[
\|u(t)\|^2_H = \|u_0\|^2_H + 2 \int_0^t \langle A(s)u(s), u(s) \rangle ds + 2 \int_0^t \int_E \langle \tilde{A}(s, e)u(s, \nu_s), v(de) \rangle ds + 2 \int_0^t (B(s)u(s), u(s))dW_s + \int_0^t \|B(s)u(s)\|^2_H ds + 2 \int_0^t \int_E \|\tilde{A}(s, e)h(s)\|^2_H + 2(u(s), \tilde{A}(s, e)h(s)\tilde{N}(de)ds + \int_0^t \int_E \|\tilde{A}(s, e)h(s)\|^2_H v(de)ds.
\]

(3.23)
Applying BDG inequality and condition (3.12), we have

\[
E \sup_{0 \leq t \leq T} \|u(t)\|_H^2 + \alpha \int_0^T \|u(s)\|_V^2 ds \\
\leq 2\|u_0\|_H^2 + 2(\lambda + v(E))E \int_0^T \|u(s)\|_H^2 ds + 4E \sup_{0 \leq t \leq T} \left| \int_0^t (B(s)u(s), u(s))dW_s \right| \\
+ 2E \sup_{0 \leq t \leq T} \left| \int_0^t \int_E \left[ \|\tilde{A}(s, e)h(s)\|_H^2 + 2(u(s), \tilde{A}(s, e)h(s)) \right] \tilde{N}(deds) \right| \\
+ 2E \int_0^T \int_E \|\tilde{A}(s, e)h(s)\|_H^2 v(de)ds \\
\leq 2\|u_0\|_H^2 + CE \int_0^T \|u(s)\|_H^2 ds + CE \int_0^T \int_E \|\tilde{A}(s, e)h(s)\|_H^2 v(de)ds \\
+ \frac{1}{2} E \sup_{0 \leq t \leq T} \|u_t\|_H^2
\]

where we have used the conclusion (see [34] page 260-261) for details.

\[
E \sup_{0 \leq t \leq T} \|u_t\|_H^2 \leq C(\|u_0\|_H^2 + E \int_0^T \|u(s)\|_V^2 ds + E \int_0^T \|h(s)\|_H^2 ds) < \infty
\]

which implies

\[
u \in \mathcal{L}_F^2(0, T; V) \cap \mathcal{L}_F^{\infty, 2}(0, T; H).
\]

If \(u^1\) and \(u^2\) are two solutions of the equation (3.14) in \(\mathcal{L}_F^2(0, T; V) \cap \mathcal{L}_F^{\infty, 2}(0, T; H)\). By Itô's formula and condition (3.12), we have

\[
E\|u^1(t) - u^2(t)\|_H^2 + \alpha E \int_0^t \|u^1(s) - u^2(s)\|_V^2 ds \leq (\lambda + v(E))E \int_0^t \|u^1(s) - u^2(s)\|_H^2 ds
\]

which implies

\[
E \int_0^T \|u^1(s) - u^2(s)\|_V^2 ds = 0.
\]

By a similar calculation as (3.24), we have

\[
E \sup_{0 \leq t \leq T} \|u^1(t) - u^2(t)\|_H^2 = 0.
\]

Step 2. We use the contraction mapping principle to prove the existence and uniqueness of the solution of equation (3.11). Let \(h^1, h^2\) be in \(\mathcal{L}_F^2(0, t; V) \cap \mathcal{L}_F^{\infty, 2}(0, t; H)\) where \(t \in [0, T]\) will be determined later. From Step 1 we know there exist \(u^1, u^2 \in \mathcal{L}_F^2(0, t; V) \cap \mathcal{L}_F^{\infty, 2}(0, t; H)\) solving equation (3.14) corresponding to \(h^1\) and \(h^2\) respectively. It follows from Itô's formula
Taking proof is complete. Gronwall’s inequality yields

\[
\|u^1(t) - u^2(t)\|^2_H + \alpha \int_0^s \|u^1(r) - u^2(r)\|^2_H dr \\
\leq (\lambda + v(E)) \int_0^s \|u^1(r) - u^2(r)\|^2_H dr + 2 \int_0^s (B(r)(u^1(r) - u^2(r)), u^1(r) - u^2(r)) dW_r \\
+ \int_0^s \int_E \left[ 2(u^1(r) - u^2(r), \tilde{A}(r,e)(h^1(r) - h^2(r))) + \|\tilde{A}(r,e)(h^1(r) - h^2(r))\|^2_H \right] \tilde{N}(dedr) \\
+ \int_0^s \int_E \|\tilde{A}(r,e)(h^1(r) - h^2(r))\|^2_H v(de) dr, \quad 0 \leq s \leq t.
\]

Gronwall’s inequality yields

\[
E\|u^1(t) - u^2(t)\|^2_H \leq e^{(\lambda + v(E))T} E \int_0^t \int_E \tilde{A}(r,e)(h^1(r) - h^2(r))\|^2_H v(de) dr, \quad 0 \leq s \leq t. \quad (3.26)
\]

Using BDG inequality and (3.26), we can get

\[
E \sup_{0 \leq s \leq t} \|u^1(s) - u^2(s)\|^2_H + E \int_0^t \|u^1(s) - u^2(s)\|^2_V ds \\
\leq CE \int_0^t \|u^1(s) - u^2(s)\|^2_H ds + CE \int_0^t \int_E \|\tilde{A}(s,e)(h^1(s) - h^2(s))\|^2_H v(de) ds \\
\leq CE \int_0^t \|h^1(s) - h^2(s)\|^2_H ds \\
\leq CtE \sup_{0 \leq s \leq t} \|h^1(s) - h^2(s)\|^2_H \\
\leq Ct \left( E \sup_{0 \leq s \leq t} \|h^1(s) - h^2(s)\|^2_H + E \int_0^t \|h^1(s) - h^2(s)\|^2_V ds \right).
\]

Taking \( t \) small enough such that \( Ct < 1 \), by contract mapping theorem we know the equation (3.11) has a unique solution in \( \mathcal{L}_T^2(0,t; V) \cap \mathcal{L}^{2,2}_T(0,t; H) \) on the interval \([0,t]\). We repeat the process on intervals \([t,2t]\), \([2t,3t]\), \ldots, and finally obtain the existence and uniqueness of the solution of equation (3.11) after finite steps. (3.13) follows from [12, Theorem 2, page 156]. The proof is complete. \( \square \)

### 3.3 Degenerate case

In this section, we apply the abstract result proved in the previous section to our equation (3.1) and prove that the inverse flow \( X^{-1}(x) \) is a classical solution to SIPDE (3.1). Consider the following Cauchy problem

\[
\begin{aligned}
du(t, x) &= [a^{ij}(t, x)u_{ij}(t, x) + b^i(t, x)u_i(t, x) + c(t, x)u(t, x)] dt \\
&\quad + \int_E \left[-u(t, x) + \rho(t, e, x)u(t, \phi^{-1}_{i,e}(x))\right] v(de) dt \\
&\quad + \E\left[\tilde{b}^k(t, x)u_k(t, x) + \tilde{c}^k(t, x)u(t, x)\right] dW_t^k \\
&\quad + \int_E \left[-u(t, x) + \rho(t, e, x)u(t, \phi^{-1}_{i,e}(x))\right] \tilde{N}(dedt), \quad (t, x) \in [0, T] \times \mathbb{R}^n, \\
u(0, x) &= \varphi(x).
\end{aligned}
\]
Let $m$ be a nonnegative integer and $K$ be a nonnegative constant. We make the following three assumptions

I) the coefficients $a^{ij}$, $b^i$, $c$, $\tilde{b}^{jk}$, $\tilde{c}^k$ are predictable for each $x$ and $\rho$ is $\mathcal{P} \otimes \mathcal{E}$ measurable for each $x$; the functions $b^i$, $c$, $\tilde{b}^{jk}$, $\tilde{c}^k$, $\rho$ and their derivatives with respect to $x$ up to the order $m$, and the function $a^{ij}$ and its derivatives with respect to $x$ up to the order $m + 1$, are bounded by $K$; $g$ and its derivatives up to the order $m$ are bounded by $K$ and the determinant of the Jacobian matrix $I + \partial g(t,e,x)$ of the homeomorphic map $\phi_{t,e}(x) = x + g(t,e,x)$ is bounded below by a positive constant;

II) the matrix $(a^{ij} - \frac{1}{2}\tilde{b}^{jk}\tilde{b}^{ik}) \geq \delta I$, for some $\delta > 0$;

III) $\varphi \in W^m_2$.

Remark 3.1. Since $\phi_{t,e}^{-1}(x) = x - g(t,e,\phi_{t,e}^{-1}(x))$, our assumptions on the coefficient $g$ imply that the gradient of $\phi_{t,e}$ and the derivatives of $\phi_{t,e}^{-1}$ up to order $m$ with respect to $x$ are bounded.

Definition 3.1. A generalized solution of the problem (3.27) is a function $u \in \mathcal{L}^2_F(0,T;W^m_2) \cap \mathcal{L}^\infty_F(0,T;L^2)$ such that for each $\eta \in C^\infty_0$ and almost all $(t,\omega) \in [0,T] \times \Omega$,

$$
(u(t),\eta)_0 = (\varphi,\eta)_0 + \int_0^t \left[ -2(a^{ij}u_i(s),u_j(s))_m + 2((b^i - a^{ij}_j)u_i(s) + cu(s),u(s))_m \right] ds \\
+ 2 \int_0^t \int_E (-u(s) + \rho(s,e)u(s,\phi_{s,e}^{-1}),u(s))_m v(de) ds \\
+ \int_0^t (\tilde{b}^{jk} u_i(s) + \tilde{c}^k u(s),\eta)_0 dW^k_s \\
+ \int_0^t \int_E (-u(s) + \rho(s,e)u(s,-,\phi_{s,e}^{-1}),\eta)_0 \tilde{N}(de ds).
$$

(3.28)

Theorem 3.3. Under conditions I), II), and III), the Cauchy problem (3.27) has a unique generalized solution $u \in \mathcal{L}^2_F(0,T;W^{m+1}_2) \cap \mathcal{L}^\infty_F(0,T;W^m_2)$ such that the relation (3.28) holds almost surely for any $\eta \in C^\infty_0$ and $t \in [0,T]$. In addition,

$$
\|u(t)\|^2_m = \|\varphi\|^2_m + \int_0^t \left[ -2(a^{ij}u_i(s),u_j(s))_m + 2((b^i - a^{ij}_j)u_i(s) + cu(s),u(s))_m \right] ds \\
+ 2 \int_0^t \int_E (-u(s) + \rho(s,e)u(s,\phi_{s,e}^{-1}),u(s))_m v(de) ds \\
+ \int_0^t (\tilde{b}^{jk} u_i(s) + \tilde{c}^k u(s),u(s))_m dW^k_s + \int_0^t \sum_{k=1}^d \|\tilde{b}^{jk} u_i(s) + \tilde{c}^k u(s)\|^2_m ds \\
+ \int_0^t \int_E \left[ \| -u(s) + \rho(s,e)u(s,-,\phi_{s,e}^{-1})\|^2_m \\
+ 2(u(s), -u(s) + \rho(s,e)u(s,-,\phi_{s,e}^{-1}))_m \tilde{N}(de ds) \\
+ \int_0^t \int_E \| -u(s) + \rho(s,e)u(s,-,\phi_{s,e}^{-1})\|^2_m v(de) ds \right] ds.
$$

(3.29)

Proof. To apply the abstract result, we set

$$
V = W^{m+1}_2, H = W^m_2, V^* = W^{m-1}_2.
$$

(3.30)

For $\zeta \in V$ and $\eta \in V$, it follows from condition I) that

$$
\| - (a^{ij}\zeta_i(s),\eta_j)_m + ((b^i - a^{ij}_j)\zeta_i(s) + c\zeta(s),\eta)_m \| \leq C \|\zeta\|_{m+1} \|\eta\|_{m+1}
$$

for some $C > 0$. This completes the proof.
where \( C \) is independent of \( t, \omega, \zeta \) and \( \eta \). Consequently, the formula
\[
\langle A(t) \zeta, \eta \rangle := -(a^{ij}(t)\zeta_i, \eta_j) + ((b^i(t) - a^{ij}_j(t))\zeta_i + c(t)\zeta, \eta)_m
\]
defines a linear operator \( A(\cdot) \in \mathcal{L}_2^\infty(0, T; \mathcal{L}(V, V^*)) \) and from the elementary inequality \( 2ab \leq \varepsilon a^2 + \frac{1}{\varepsilon}b^2 \),
\[
2\langle A(t) \eta, \eta \rangle = -2(a^{ij}_i \eta_i, \eta_j)_m + 2((b^i - a^{ij}_j(t))\eta_i + c(\eta, \eta)_m
\leq -2 \sum_{|\gamma|=m} (a^{ij}\partial^\gamma \eta_i, \partial^\gamma \eta_j)_0 + \varepsilon_1 \|\eta\|^2_{m+1} + C\|\eta\|^2_m.
\]
(3.31)

In view of Remark 3.31 for \( \eta \in V \) and \( \zeta \in H \) the formulas
\[
B(t) \eta := (\tilde{b}^i(t) \eta_i + \tilde{c}^i(t) \eta_i, \cdots, \tilde{b}^d(t) \eta_i + \tilde{c}^d(t) \eta_i)
\]
and
\[
\tilde{A}(t, e) \zeta := -\zeta + \rho(t, e) \zeta (\phi_{k,e}^{-1})
\]
defines two linear operators \( B(\cdot) \in \mathcal{L}_2^\infty(0, T; \mathcal{L}(V, H^d)) \) and \( \tilde{A}(\cdot, \cdot) \in \mathcal{L}_2^\infty(0, T; \mathcal{L}(H, L^2(\mathcal{E}, H))) \) respectively. Moreover,
\[
\|B(t)\eta\|^2_m = (\tilde{b}^{ik}_i \eta_i + \tilde{c}^{ik}_i \eta_i, \tilde{b}^{ik}_i \eta_i + \tilde{c}^{ik}_i \eta_i)_m
= (\tilde{b}^{ik}_i \eta_i, \tilde{b}^{ik}_i \eta_i)_m + 2(\tilde{b}^{ik}_i \eta_i, \tilde{c}^{ik}_i \eta_i)_m + (\tilde{c}^{ik}_i \eta_i, \tilde{c}^{ik}_i \eta_i)_m
\leq \sum_{|\gamma|=m} (\partial^\gamma (\tilde{b}^{ik}_i \eta_i), \partial^\gamma (\tilde{b}^{ik}_i \eta_i))_0 + \varepsilon \|\eta\|^2_{m+1} + C\|\eta\|^2_m
\leq \sum_{k=1}^d \sum_{|\gamma|=m} \|\tilde{b}^{ik}\partial^\gamma \eta_i\|^2_0 + \varepsilon_2 \|\eta\|^2_{m+1} + C\|\eta\|^2_m
\]
and
\[
\int_E \|\tilde{A}(t, e)\zeta\|^2_m v(de) \leq C\|\zeta\|^2_m.
\]

So for any \( \eta \in V \),
\[
2\langle A(t) \eta, \eta \rangle + \|B(t)\eta\|^2_m + \int_E \|\tilde{A}(t, e)\eta\|^2_m v(de)
\leq -2 \sum_{|\gamma|=m} (a^{ij}\partial^\gamma \eta_i, \partial^\gamma \eta_j)_0 + (\varepsilon_1 + \varepsilon_2) \|\eta\|^2_{m+1} + C\|\eta\|^2_m + \sum_{|\gamma|=m} \sum_{k=1}^d \|\tilde{b}^{ik}\partial^\gamma \eta_i\|^2_0
\leq -2\delta \sum_{|\gamma|=m} \sum_{i=1}^n \|\partial^\gamma \eta_i\|^2_0 + (\varepsilon_1 + \varepsilon_2) \|\eta\|^2_{m+1} + C\|\eta\|^2_m
\leq -2\delta \|\eta\|^2_{m+1} + (\varepsilon_1 + \varepsilon_2) \|\eta\|^2_{m+1} + (C + 2\delta)\|\eta\|^2_m.
\]
Taking \( \varepsilon_1 + \varepsilon_2 = \delta \), we have
\[
-2\langle A(t) \eta, \eta \rangle + C\|\eta\|^2_m \geq \delta \|\eta\|^2_{m+1} + \|B(t)\eta\|^2_m + \int_E \|\tilde{A}(t, e)\eta\|^2_m v(de).
\]

So the coercive condition (3.31) is satisfied. According to Theorem 3.32 there exists a unique function \( u \in \mathcal{L}_2^\infty(0, T; W^{m+1}_2) \cap \mathcal{L}_2^\infty(0, T; W^2_2) \) such that almost surely for any \( \zeta \in W^{m+1}_2 \) and
Theorem 3.5. Assume that conditions I), II') and III) hold. Then the equation
\begin{align}
\tag{3.27}
(u(t), \zeta)_m &= (\varphi, \zeta)_m + \int_0^t \left[ -(a^{ij} u_i(s), \zeta_j)_m + ((b^i - a^{ij}) u_i(s) + cu(s), \zeta)_m \right] ds \\
&+ \int_0^t \int_{E} (-u(s) + \rho(s,e)u(s, \phi_{s,e}^{-1}), \zeta)_m v(de) ds \\
&+ \int_0^t (\tilde{b}^k u_i(s) + \tilde{c}^k u(s), \zeta)_m dW_s^k \\
&+ \int_0^t \int_{E} (-u(s) + \rho(s,e)u(s, \phi_{s,e}^{-1}), \zeta)_m \tilde{N}(deds).
\end{align}

Let \( \Delta \) represent the Laplacian on \( \mathbb{R}^n \). It is well known that the operator \( \Lambda := 1 - \Delta \) which maps \( W^2_2 \) into \( L^2 \) has an inverse \( \Lambda^{-1} \) satisfying \( \Lambda^{-1} W^2_2 = W^2_{2+l} \) for any integer \( l \). Moreover, if \( k \) is a nonnegative integer such that \( l + k \geq 0 \), then for \( f \in W^2_{2+k} \), \( g \in W^2_{2+k} \cap W^2_{2+k} \), we have
\begin{align}
\tag{3.33}
(\Lambda^{-1} f, g)_{l+k} = (f, g)_k.
\end{align}

For \( \eta \in C_0^\infty \) in view of (3.33), we can get (3.28) by replacing \( \zeta \) by \( \Lambda^{-m} \eta \) in (3.32). So \( u \) is a generalized solution of the problem (3.27). Suppose \( \hat{u} \in L^2_F(0,T; W^m_{2m+1}) \cap L^\infty_F(0,T; W^m_{2m}) \) is another generalized solution of the problem (3.27). For \( \zeta \in C_0^\infty \), due to (3.33) again and the fact that \( C_0^\infty \) is dense in \( W^m_{2m+1} \), we replace \( \eta \) by \( \Lambda^m \zeta \) in (3.28) and conclude \( \hat{u} \) is also a solution of the abstract equation (3.11). Theorem 3.2 yields \( u = \hat{u} \) so the uniqueness is proved. \( \blacksquare \)

Next we consider the equation (3.27) in the degenerate case, i.e., the assumption II) is replaced by

II') the matrix \( (a^{ij} - \frac{1}{2} \tilde{b}^k \tilde{b}^k) \geq 0 \).

The following lemma is borrowed from [19, Remark 2.1, page 340] with \( p = 2 \).

Lemma 3.4. Under conditions I), II') and III), we have for \( u \in W^m_{2m+1} \),
\begin{align}
-2(a^{ij} u_i, u_j)_m + 2((b^i - a^{ij}) u_i + cu, u)_m + \sum_{k=1}^d \| \tilde{b}^k u_i + \tilde{c}^k u \|^2_m \\
+ 2 \int_E (-u + \rho(s,e)u(\phi_{s,e}^{-1}), u)_m v(de) + \int_E \| -u + \rho(s,e)u(\phi_{s,e}^{-1}) \|^2_m v(de) \\
\leq N \| u \|^2_m,
\end{align}

where the constant \( N \) depends only on \( K, n, d, m \) and \( v(E) \).

Theorem 3.5. Assume that conditions I), II') and III) hold. Then the equation (3.27) has a unique generalized solution
\[ u \in L^\infty_F(0,T; W^m_{2m+1}) \cap L^2_F(0,T; W^m_{2m}) \cap L^\infty_F(0,T; W^m_{2m}). \]

Moreover,
\begin{align}
E \sup_{t \in [0,T]} \| u \|^2_m \leq C \| \varphi \|^2_m
\end{align}

where \( C \) depends on \( v(E), n, d, K, m \) and \( T \).
Proof. Uniqueness. We only need to prove that the equation (6.27) has solution \( u \equiv 0 \) when \( \varphi = 0 \). Using Itô’s formula (see [12, Theorem 2, page 156]) to \( \|u(t)\|_0^2 \) and \( \|u(t)\|_0^2 e^{-Nt} \) where \( N \) is the constant in Lemma 3.4 we get
\[
0 \leq e^{-Nt}\|u(t)\|_0^2 \leq 2 \int_0^t e^{-Ns}(\overline{b}^k u_i^\varepsilon(s) + \overline{c}^k u(s), u(s))_0 dW^k_s \\
+ \int_0^t \int_E e^{-N_s} [2(u(s), -u(s)) + \rho(s, e)u(s, \phi^{-1}_{s,e})]_0 \\
+ \| - u(s) + \rho(s, e)u(s, \phi^{-1}_{s,e})\|_0^2 N(\text{ded}s).
\]

Since a nonnegative local martingale with zero initial value equals to zero, we have \( u(t) = 0 \), \( t \in [0, T] \) almost surely.

Existence. For \( \varepsilon > 0 \), set \( a^{\varepsilon ij} := a^{ij} + \varepsilon \delta^{ij} \). Denote by \( u^\varepsilon \) the unique generalized solution of (3.27) with \( a^{ij} \) replaced by \( a^{\varepsilon ij} \). From Theorem 3.3 we know \( u^\varepsilon \in L^2_{\mathbb{F}}(0, T; W_{0}^{m+1}) \cap L^\infty_{\mathbb{F}}(0, T; W_{0}^{m}) \) and satisfies for any \( \zeta \in W_{0}^{m+1} \),
\[
(u^\varepsilon(t), \zeta)_m = (\varphi, \zeta)_m + \int_0^t \left[-(a^{\varepsilon ij} u_i^\varepsilon(s), \zeta_j)_m + ((b^{ij} - a^{ij})_1 u_i^\varepsilon(s) + c^\varepsilon(s), \zeta)_m \right] ds \\
+ \int_0^t \int_E (-u^\varepsilon(s) + \rho(s, e)u^\varepsilon(s, \phi^{-1}_{s,e}), \zeta)_m v(ds) ds \\
+ \int_0^t \overline{b}^k u_i^\varepsilon(s) + \overline{c}^k u^\varepsilon(s), \zeta)_mdW^k_s \\
+ \int_0^t \int_E (-u^\varepsilon(s) - \rho(s, e)u^\varepsilon(s, \phi^{-1}_{s,e})), \zeta)_m N(\text{ded}s).
\]

We first prove \( u^\varepsilon \) satisfies (3.35) with \( C \) depends only on \( n, d, K, m, T, \) and \( v(E) \). It follows from Lemma 3.4 and (3.29) that
\[
\|u^\varepsilon(t)\|_m^2 \leq \|\varphi\|_m^2 + N \int_0^t \|u^\varepsilon(s)\|_m^2 ds + 2 \int_0^t \overline{b}^k u_i^\varepsilon(s) + \overline{c}^k u^\varepsilon(s), u^\varepsilon(s))_m dW^k_s \\
+ \int_0^t \int_E \| - u^\varepsilon(s) + \rho(s, e)u^\varepsilon(s, \phi^{-1}_{s,e})\|_m^2 \\
+ 2(u^\varepsilon(s) - u^\varepsilon(s) - \rho(s, e)u^\varepsilon(s, \phi^{-1}_{s,e})), \zeta)_m N(\text{ded}s).
\]

From the proof of [19, Lemma 2.1, page 239], we find
\[
\|(\overline{b}^k u_i^\varepsilon(s) + \overline{c}^k u^\varepsilon(s), u^\varepsilon(s))_m \| \leq \sum_{|\gamma|=m} \| (\overline{b}^k \partial^\gamma u_i^\varepsilon(s), \partial^\gamma u^\varepsilon(s))_0 \| + C\|u^\varepsilon(s)\|_m^2 \\
= \sum_{|\gamma|=m} \frac{1}{2} \|(\overline{b}^k, \partial^\gamma [(\partial^\gamma u^\varepsilon(s))^2])_0 \| + C\|u^\varepsilon(s)\|_m^2 \\
= \sum_{|\gamma|=m} \frac{1}{2} \|(-\partial\overline{b}^k, (\partial^\gamma u^\varepsilon(s))^2)_0 \| + C\|u^\varepsilon(s)\|_m^2 \\
\leq C\|u^\varepsilon(s)\|_m^2.
\]

By Gronwall’s inequality and BDG inequality, as well as (3.38), we can deduce from (3.37) that
\[
E \sup_{0 \leq t \leq T} \|u^\varepsilon(t)\|_m^2 \leq C\|\varphi\|_m^2.
\]
It follows from the reflexivity of the process space $L^2_2(0,T;W^m_2)$ that there exist a subsequence, still denoted by $u^\varepsilon$, and $u \in L^2(0,T;W^m_2)$ such that as $\varepsilon \to 0$,

$$u^\varepsilon \rightharpoonup u, \text{ weakly in } L^2_2(0,T;W^m_2).$$

Next we prove $u^\varepsilon$ is also a Cauchy sequence in the space $L^{\infty,2}(0,T;W^{m-1}_2)$. For any $\eta \in W^m_2$, replacing $\zeta$ by $\Lambda^{-1}\eta$ in (3.36) and using the relation (3.33), we have

$$\begin{align*}
(u^\varepsilon(t),\eta)_{m-1} &= \langle \varphi, \eta \rangle_{m-1} + \int_0^t \left[ -\langle a^{ij}\xi_i(s),\eta_j \rangle_{m-1} + \langle (b^i - a^i_j)\xi_i(s) + c\xi(s),\eta \rangle_{m-1} \right] ds \\
&\quad + \int_0^t \int_E (-u^\varepsilon(s) + \rho(s,\varepsilon)u^\varepsilon(s,\phi_{s,e}^{-1}),\eta)_{m-1} v(de)ds \\
&\quad + \int_0^t \int_E (\tilde{b}^k u^\varepsilon(s) + \tilde{c}^k u^\varepsilon(s),\eta)_{m-1} dW_s \\
&\quad + \int_0^t \int_E (-u^\varepsilon(s) + \rho(s,\varepsilon)u^\varepsilon(s,\phi_{s,e}^{-1}),\eta)_{m-1} \tilde{N}(deds).
\end{align*}$$

The above equality implies that $u^\varepsilon$ is also the unique solution of the following evolution equation

$$\begin{align*}
\left\{ 
\begin{array}{l}
\frac{du(t)}{dt} = A'(t)u(t)dt + \int_E \tilde{A}'(t,e)u(t,v(de)dt) + B'(t)u(t)dW_t + \int_E \tilde{A}'(t,e)u(t-)\tilde{N}(dedt), \\
\mbox{$u(0) = \varphi$}
\end{array}
\right.
\end{align*}$$

with triple

$$V = W^m_2, \quad H = W^{m-1}_2, \quad V^* = W^{m-2}_2,$$

and for $\zeta, \eta \in V, \psi \in H$,

$$\langle A'(t)\zeta, \eta \rangle := -\langle a^{ij}\zeta_i,\eta_j \rangle_{m-1} + \langle (b^i - a^i_j)\zeta_i + \varepsilon \zeta, \eta \rangle_{m-1},$$

$$B'(t)\eta := \langle \tilde{b}^1 \eta + \tilde{c}^1 \eta, \cdots, \tilde{b}^d \eta + \tilde{c}^d \eta \rangle,$$

$$\tilde{A}'(t,e)\psi := -\psi + \rho(t,e)\psi(\phi_{t,e}^{-1}).$$

Using Itô’s formula (see again [12, Theorem 2, page 156]) and Lemma 3.4, we have

$$\begin{align*}
E \sup_{0 \leq s \leq t} \| u^\varepsilon(s) - u^\varepsilon'(s) \|_{m-1}^2 \\
&\leq C \int_0^t \| u^\varepsilon(s) - u^\varepsilon'(s) \|^2_{m-1} ds - 2(\varepsilon - \varepsilon') \int_0^t (u^\varepsilon(s), u^\varepsilon(s) - u^\varepsilon'(s))_{m-1} ds \\
&\quad + 2 \int_0^t \int_E \left[ \| (u^\varepsilon(s) - u^\varepsilon'(s)) + \rho(s,\varepsilon)(u^\varepsilon - u^\varepsilon')(s,\phi_{s,e}^{-1}) \|^2_{m-1} \\
&\quad + \rho(s,\varepsilon)(u^\varepsilon - u^\varepsilon')(s,\phi_{s,e}^{-1}) \|^2_{m-1} \right] \tilde{N}(deds).
\end{align*}$$

From (3.38), (3.39) and BDG inequality, we obtain that

$$\begin{align*}
E \sup_{0 \leq t \leq 1} \| u^\varepsilon(s) - u^\varepsilon'(s) \|^2_{m-1} \\
&\leq CE \int_0^t \| u^\varepsilon(r) - u^\varepsilon'(r) \|^2_{m-1} dr + C|\varepsilon - \varepsilon'|E \int_0^t \| u^\varepsilon(r) \|^m \| u^\varepsilon(r) - u^\varepsilon'(r) \|^m dr \\
&\leq C \int_0^t E \sup_{0 \leq s \leq r} \| u^\varepsilon(s) - u^\varepsilon'(s) \|^2_{m-1} dr + CT|\varepsilon - \varepsilon'|\| \varphi \|^2_{m}.
\end{align*}$$
Gronwall’s inequality yields
\[ E \sup_{0 \leq s \leq T} \| u^\varepsilon(s) - u^{\varepsilon'}(s) \|_{m-1}^2 \leq C |\varepsilon - \varepsilon'| \| \varphi \|_m^2 \]
which implies there exists a \( \tilde{u} \in L^\infty_{\mathcal{F}}(0, T; W^{m-1}_2) \) such that
\[ u^\varepsilon \rightarrow \tilde{u}, \text{ strongly in } L^\infty_{\mathcal{F}}(0, T; W^{m-1}_2). \]
Since \( u^\varepsilon \rightharpoonup u \) weakly in \( L^\infty_{\mathcal{F}}(0, T; W^m_2) \), \( u^\varepsilon \rightarrow u \) weakly in \( L^2_{\mathcal{F}}(0, T; W^{m-1}_2) \). So \( u = \tilde{u} \in L^\infty_{\mathcal{F}}(0, T; W^{m-1}_2) \). Using similar arguments to deduce (3.20), we know that both sides of (3.28), corresponding to \( u^\varepsilon \), converge weakly to the corresponding expression to \( u \) in \( L^2_{\mathcal{F}}(0, T; \mathbb{R}) \). So \( u \in L^\infty_{\mathcal{F}}(0, T; W^m_2) \cap L^\infty_{\mathcal{F}}(0, T; W^{m-1}_2) \) is a generalized solution of equation (3.27).

Noting \( u \in L^\infty_{\mathcal{F}}(0, T; W^{m-1}_2) \), we can prove \( u \in L^\infty_{\mathcal{F}, w}(0, T; W^m_2) \) and satisfies (3.35) in the same way as [19, Theorem 3.1, page 341].

In order to deduce the SIPDE for the invert flow \( X^{-1}(x) \), we need to introduce weighted Sobolev spaces like in [19] as the initial value \( X_0^{-1}(x) = x \) does not belong to any Sobolev space on the whole \( \mathbb{R}^n \).

For \( p > 1 \) and \( r \in \mathbb{R} \), we denote by \( L^p_{\mathcal{F}}(r) \) the space of real-valued Lebesgue measurable functions on \( \mathbb{R}^n \) with the finite norm
\[ \| f \|_{p, r} := \left( \int_{\mathbb{R}^n} \left( 1 + |x|^2 \right)^{r/2} f(x) |dx| \right)^{1/p}. \]
For \( p = 2 \) we denote the inner product in \( L^2(0) \) by \((\cdot, \cdot)_0\) as before.

Let \( W^m_p(r) \) be the subset of \( L^p_{\mathcal{F}}(r) \) consisting of functions whose generalized derivatives up to the order \( m \) belong to \( L^p_{\mathcal{F}}(r) \). We introduce a norm in this space by
\[ \| f \|_{m, p, r} := \left( \sum_{|\gamma| \leq m} |\gamma|! \int_{\mathbb{R}^n} \left( 1 + |x|^2 \right)^{r/2} |D^{\gamma} f(x)| |dx| \right)^{1/p}, \]
where \( \gamma = (\gamma^1, \ldots, \gamma^n) \) is a multi-index. It is a Banach space and for \( p = 2 \) a Hilbert space.

In the remaining part of this section we assume I), II) and III).

**Definition 3.2.** An \( r \)-generalized solution of the problem (3.27) is a function \( u \in L^\infty_{\mathcal{F}}(0, T; W^1_2(r)) \cap L^\infty_{\mathcal{F}}(0, T; L^2_2(r)) \) such that for each \( \eta \in C_0^\infty \) and almost all \((t, \omega) \in [0, T] \times \Omega \), equation (3.28) holds.

**Remark 3.2.** If an \( r \)-generalized solution satisfies that for almost all \( \omega \in \Omega \), it is cadlag with respect to \( t \) for any \( x \), and twice continuously differentiable with respect to \( x \) for any \( t \), it is a classical solution, i.e., it satisfies the relation (3.27) for all \((t, x) \in [0, T] \times \mathbb{R}^n \), almost surely for \( \omega \in \Omega \).

**Theorem 3.6.** Assume that conditions I), II) and III) are in force. Then the problem (3.27) has a unique \( r \)-generalized solution
\[ u \in L^\infty_{\mathcal{F}}(0, T; W^{m-1}_2(r)) \cap L^2_{\mathcal{F}}(0, T; W^m_2(r)) \cap L^\infty_{\mathcal{F}, w}(0, T; W^m_2(r)). \]
Furthermore,
\[ E \sup_{t \in [0, T]} \| u \|_{m, 2, r} \leq C \| \varphi \|_{m, 2, r} \]
where \( C \) depends on \( v(E) \), \( n \), \( d \), \( K \), \( m \), \( r \), and \( T \).
Proof. The uniqueness can be obtained by analogous arguments to those of Theorem 3.5. Now we prove the existence.

The validity for the case \( r = 0 \) is proved in Theorem 3.5. For the general case we consider the equation for \( \tilde{u}(t, x) := (1 + |x|^2)^{r/2} u(t, x) \) where \( u \) is a solution of the problem (3.27):

\[
\left\{ \begin{array}{l}
   d\tilde{u}(t, x) = [a^{ij}(t, x)\tilde{u}_{ij}(t, x) + \overline{b}^i(t, x)\tilde{u}_i(t, x) + \overline{c}(t, x)\tilde{u}(t, x)]dt \\
   + \int_E (-\tilde{u}(t, x) + \overline{d}(t, e, x)\tilde{u}(t, \phi_t^{-1}(x)))v(de)dt \\
   + \int_E \overline{\phi}^{ik}(t, x)\tilde{u}_i(t, x) + \overline{\phi}^{k}(t, x)\tilde{u}(t, x)\right]dW^k_t \\
   + \int_E (-\tilde{u}(t, x) + \overline{d}(t, e, x)\tilde{u}(t, \phi_t^{-1}(x)))\tilde{N}(dedt), \quad t \in [0, T], \ x \in \mathbb{R}^n, \\
\end{array} \right.
\]

where

\[
\begin{align*}
\overline{b}^i(t, x) &= b^i(t, x) - 2r \frac{a^{ij}(t, x)x_j}{1 + |x|^2}, \\
\overline{c}(t, x) &= r(r + 2) \frac{a^{ij}(t, x)x_i x_j}{(1 + |x|^2)^2} - r \frac{a^{ii}(t, x) + b^i(t, x)x_i}{1 + |x|^2} + c(t, x), \\
\overline{d}(t, e, x) &= \left( \frac{1 + |x|^2}{1 + |\phi_t^{-1}(x)|^2} \right)^{r/2}d(t, e, x), \\
\overline{\phi}^{k}(t, x) &= \phi_k(t, x) - r \frac{\overline{b}^{ik}x_i}{1 + |x|^2}.
\end{align*}
\]

The coefficients and initial data of the above equation satisfy the conditions of the theorem for \( r = 0 \). From the validity for the case \( r = 0 \) we know

\[
\tilde{u} \in L^\infty_r(0, T; W^{m-1}_2) \cap L^2_r(0, T; W^m_2) \cap L^\infty_r(0, T; W^m_2),
\]

\[
E \sup_{t \in [0, T]} \|\tilde{u}\|_{m} \leq C\|\varphi\|_{m,2,r}.
\]

By differentiating the expression \( (1 + |x|^2)^{-r/2}\tilde{u}(t, x) \) we can go back to the equation (3.27) for \( u \). It is a consequence of (3.46) that \( u \) satisfies (3.42) and (3.43). The proof is complete. □

Corollary 3.7. If \((m - j)2 > n\) for some integer \( j \geq 2 \), it follows from the Sobolev theorem on the embedding of \( W^m_2 \) in \( C^2 \) that the \( r \)-generalized solution of (3.27) is classical.

Theorem 3.8. Suppose the integer \( k \) satisfies \( k > 2 + n/2 \). Moreover, assume the coefficients of equation (2.1) satisfy: \( b, \sigma \) and \( g \), as well as their derivatives up to the order \( k + 1 \), are bounded; the determinant of the Jacobian matrix \( I + \nabla g(t, e, x) \) of the homeomorphic map \( \phi_t^{-1}(x) = x + g(t, e, x) \) is bounded below by a positive constant. Then the solution \( X_t(\cdot) : \mathbb{R}^n \rightarrow \mathbb{R}^n \) of equation (2.1) starting at time 0 is a \( C^k \) diffeomorphism for each \( t \in [0, T] \) almost surely. Furthermore, the \( i \)-th coordinate of the inverse flow \( X^{-1} \) is the classical solution of the equation (3.1).

Proof. (2.2) implies the first assertion of the theorem. Since \( \varphi(x) \equiv x^i \in W^m_2(r) \) for any \( r \leq -(n/2 + 1) \), we deduce from Theorem 3.6 and Corollary 3.7 using Sobolev’s embedding theorem that the equation (3.1) has a unique classical solution \( u(t, x) \) which satisfies for any compact subset \( K \subset \mathbb{R}^n \),

\[
E \sup_{0 \leq t \leq T} \|u(t, x)\|_{C^2(K)}^2 \leq C_K \|\varphi\|_{m,2,r}^2.
\]
We apply Itô-Wentzell formula to the expression \( u(t, X_t(x)) \) and obtain that \( du(t, X_t(x)) = 0 \). Hence \( u(t, X_t(x)) = x^t \) for any \( t \in [0, T] \), a.s.. The proof is complete. \( \square \)

**Remark 3.3.**  (see [22], Remark 2, page359) Assume the coefficients \( b, \sigma, \) and \( g \) are deterministic functions and satisfy the conditions in Theorem [3.8] Assume further that the diffusion coefficient \( \sigma \) is \( C^{1,2} \) with respect to \((t, x)\). Then the inverse flow \( \{X_{s,t}^{-1}(x) \colon 0 \leq s \leq t \leq T, x \in \mathbb{R}^n \} \) satisfies the following backward stochastic ordinary differential equation

\[
X_{s,t}^{-1}(x) = x - \int_s^t b(r, X_{r,t}^{-1}(x))dr - \int_s^t \sigma(r, X_{r,t}^{-1}(x))dW_r - \int_s^t \int_E g(r, e, \phi_{r,e}^{-1}(X_{r,t}^{-1}(x)))\tilde{N}(ded\tau) + 2 \int_s^t c(r, X_{r,t}^{-1}(x))dr + \int_s^t \int_E (g(r, e, \phi_{r,e}^{-1}(X_{r,t}^{-1}(x))) - g(r, e, \phi_{r,e}^{-1}(X_{r,t}^{-1}(x))))v(de)dr
\]

where

\[
c(t, x) := \frac{1}{2} \sum_{ij} \partial \sigma^{-1}(t, x)_{ij} \sigma(t, x)
\]

and the superscript “\( j \)” stands for the \( j \)-th column of the underlying matrix; \( d\tilde{W}_r \) and \( \tilde{N}(ded\tau) \) represent the backward Itô integral and backward Poisson integral (see [22] for more details).

## 4 BSDEs with jumps

In this section, let \( p \geq 2 \) is a given constant. We consider the following backward stochastic differential equation

\[
Y_t = \xi + \int_t^T f(s, Y_s, Z_s, U_s)ds - \int_t^T Z_s dW_s - \int_t^T \int_E U_s(e) d\tilde{N}(deds), \quad t \in [0, T]
\]

under the following conditions

(H1) the generator \( f : [0, T] \times \Omega \times \mathbb{R}^d \times \mathbb{R}^{l \times d} \times L^2(\mathcal{F}; \mathbb{R}) \to \mathbb{R}^d \) is \( \mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(L^2(\mathcal{F}; \mathbb{R})) \) measurable and \( f(\cdot, 0, 0, 0) \in L^{2,p}(0, T; \mathbb{R}^d) \).

(H2) there exists a constant \( L > 0 \) such that

\[
|f(t, y, z, u) - f(t, y', z', u')| \leq L(|y - y'| + |z - z'| + |u - u'|),
\]

for all \( t \in [0, T], y, y' \in \mathbb{R}^d, z, z' \in \mathbb{R}^{l \times d}, u, u' \in L^2(\mathcal{F}; \mathbb{R}^d) \).

(H3) \( \xi \in L^p(\mathcal{F}_T; \mathbb{R}^d) \).

**Theorem 4.1.** Under conditions (H1), (H2) and (H3), BSDE (4.1) has a unique solution \((Y, Z, U) \in L^{\infty,p}(0, T; \mathbb{R}^d) \times L^{2,p}(0, T; \mathbb{R}^{l \times d}) \times L^{2,p}(0, T; L^2(\mathcal{F}; \mathbb{R}^d)) \).

Moreover, the solution \((Y, Z, U)\) satisfies

\[
\|Y\|_{L^{\infty,p}(0, T; \mathbb{R}^d)} + \|Z\|_{L^{2,p}(0, T; \mathbb{R}^{l \times d})} + \|U\|_{L^{2,p}(0, T; L^2(\mathcal{F}; \mathbb{R}^d))} \leq C(\|f(\cdot, 0, 0, 0)\|_{L^{2,p}(0, T; \mathbb{R}^d)} + \|\xi\|_{L^p})
\]

for some positive constant \( C \) which depends on \( T, v(E), \) and \( p \).
Proof. The arguments in [9, Theorem 5.1 pages 54-56] can be adopted to get the desired existence and uniqueness of solution to equation (4.3). It remains to prove the estimate (4.4). The techniques are more or less standard now (see [5] and [32]). Using Itô’s formula and the Lipschitz continuity of \( f \), we get

\[
|Y_t|^2 \leq |\xi|^2 + (2L + \frac{L^2}{\varepsilon_1} + \frac{L^2}{\varepsilon_2} + 1) \int_t^T |Y_s|^2 ds + \int_t^T |U_s(e)|^2 v(de)ds + \varepsilon_1 \int_t^T |Z_s|^2 ds
\]

For any \( 0 \leq r \leq t \), letting \( \varepsilon_1 = \varepsilon_2 = 1 \) and taking conditional expectation with respect to \( \mathcal{F}_r \) on both sides of (4.3), we have

\[
E[|Y_r|^2|\mathcal{F}_r] \leq E[|\xi|^2] + \int_0^r |f(s,0,0,0)|^2 ds + C \int_t^T E[|Y_s|^2|\mathcal{F}_r] ds, \quad \forall t \in [r,T].
\]

Using Gronwall’s inequality, we have

\[
E[|Y_t|^2|\mathcal{F}_r] \leq E[|\xi|^2] + \int_0^T |f(s,0,0,0)|^2 ds |\mathcal{F}_r| e^{CT}. \]

In particular, taking \( t = r \), we have

\[
|Y_r|^2 \leq E[|\xi|^2] + \int_0^T |f(s,0,0,0)|^2 ds |\mathcal{F}_r| e^{CT}.
\]

Using Doob’s inequality, we have

\[
E \left( \sup_{0 \leq r \leq T} |Y_r|^2 \right)^{p/2} \leq E \left( \sup_{0 \leq r \leq T} \left( E[|\xi|^2] + \int_0^T |f(s,0,0,0)|^2 ds |\mathcal{F}_r| e^{CT} \right)^{p/2} \right)
\]

Taking \( \varepsilon_1 = \frac{1}{2} \) and \( t = 0 \) in (4.3), we have

\[
E \left( \int_0^T |Z_s|^2 ds \right)^{p/2} + E \left( \int_0^T \int_E |U_s(e)|^2 N(ded) \right)^{p/2}
\]

\[
\leq C \left\{ E[|\xi|^p] + E \sup_{0 \leq r \leq T} |Y_r|^p + \varepsilon_2^{p/2} E \left( \int_0^T \int_E |U_s(e)|^2 v(de)ds \right)^{p/2} + E \left( \int_0^T |f(s,0,0,0)|^2 ds \right)^{p/2} + E \left( \int_0^T \langle Y_s, Z_s dW_s \rangle \right)^{p/2} \right\}^{p/2}.
\]
Using BDG inequality, we have

\[
E \left| \int_0^T (Y_s, Z_s \, dW_s) \right|^{p/2} \leq CE \left( \int_0^T |Y_s|^2 |Z_s|^2 \, ds \right)^{p/4}
\]

\[
\leq CE \left( \sup_{0 \leq t \leq T} |Y_t|^{p/2} \left( \int_0^T |Z_s|^2 \, ds \right)^{p/4} \right)
\]

\[
\leq \frac{C^2}{2\varepsilon} E \sup_{0 \leq t \leq T} |Y_t|^p + \frac{\varepsilon}{2} E \left( \int_0^T |Z_s|^2 \, ds \right)^{p/2}
\]

and

\[
E \left| \int_0^T \int_E (Y_{s-}, U_s(e)) \, \tilde{N}(de) \, ds \right|^{p/2}
\]

\[
\leq CE \left( \int_0^T \int_E |Y_{s-}|^2 |U_s(e)|^2 \, N(de) \, ds \right)^{p/4}
\]

\[
= CE \left( \sum_{0 \leq s \leq T} |Y_{s-}|^2 |U_s(\mathbb{P}(s))|^2 \right)^{p/4}
\]

\[
\leq CE \left( \sup_{0 \leq s \leq T} |Y_s|^{p/2} \left( \int_0^T \int_E |U_s(e)|^2 \, N(de) \, ds \right)^{p/4} \right)
\]

\[
\leq \frac{C^2}{2\varepsilon} E \sup_{0 \leq s \leq T} |Y_s|^p + \frac{\varepsilon}{2} E \left( \int_0^T \int_E |U_s(e)|^2 \, N(de) \, ds \right)^{p/2}.
\]

Moreover,

\[
E \left( \int_0^T \int_E |U_s(e)|^2 v(de) \, ds \right)^{p/2}
\]

\[
\leq (v(E)T)^{p/2-1} E \int_0^T \int_E |U_s(e)|^p v(de) \, ds
\]

\[
\leq (v(E)T)^{p/2-1} E \int_0^T \int_E |U_s(e)|^p N(de) \, ds
\]

\[
\leq (v(E)T)^{p/2-1} E \sum_{0 < s \leq T} |U_s(\mathbb{P}(s))|^p
\]

\[
= (v(E)T)^{p/2-1} E \sum_{0 < s \leq T} (|U_s(\mathbb{P}(s))|^2)^{p/2}
\]

\[
\leq (v(E)T)^{p/2-1} E \left( \sum_{0 < s \leq T} |U_s(\mathbb{P}(s))|^2 \right)^{p/2}
\]

\[
= (v(E)T)^{p/2-1} E \left( \int_0^T \int_E |U_s(e)|^2 \, N(de) \, ds \right)^{p/2}.
\]

Taking \(\varepsilon_2\) in (4.6) and \(\varepsilon\) in (4.7) and (4.8) small enough, we can deduce from (4.6) that

\[
E \left( \int_0^T |Z_s|^2 \, ds \right)^{p/2} + E \left( \int_0^T \int_E |U_s(e)|^2 \, N(de) \, ds \right)^{p/2}
\]

\[
\leq C \left\{ E|\xi|^p + E \sup_{0 \leq t \leq T} |Y_t|^p + E \left( \int_0^T |f(s, 0, 0, 0)|^2 \, ds \right)^{p/2} \right\}.
\]

Combining the estimates (4.5) and (4.9), we can get the desired result. \(\square\)
Now we consider the following BSDE
\[
\begin{aligned}
dY_t &= -f(t, X_t(x), Y_t, Z_t, U_t)dt + Z_t dW_t + \int_E U_t(e) \tilde{N}(de dt), \quad t \in [0, T], \\
Y_T &= \varphi(X_T(x))
\end{aligned}
\]  
(4.10)
where \( X \) is the solution of equation (2.1) starting at time 0 and coefficients \( f \) and \( \varphi \) satisfy the following conditions

\( (C5)_k \)
\[
f \in L^\infty_F(0, T; C^k(\mathbb{R}^n \times \mathbb{R}^l \times \mathbb{R}^{l \times d} \times L^2(\mathcal{E}; \mathbb{R}^l); \mathbb{R}^l)),
\]
\[
\varphi \in L^\infty(\mathcal{F}_T; C^k(\mathbb{R}^n; \mathbb{R}^l)).
\]
In what follows the derivatives of Lemma 4.2.

The adapted solution to BSDE (4.10) will be denoted by \((Y(x), Z(x), U(x))\). We have the following lemma.

**Lemma 4.2.** ([7] Proposition 3.3]) Assume \((C1)\), \((C2)_k\) and \((C5)_k\) are satisfied for \( k = 1 \) or \( k = 2 \). Then the unique adapted solution \((Y(x), Z(x), U(x))\) to BSDE (4.10) satisfies almost surely

\[
Y \in C^{k-1}(\mathbb{R}^n; D(0, T; \mathbb{R}^l)),
\]
\[
Z \in C^{k-1}(\mathbb{R}^n; L^2(0, T; \mathbb{R}^{l \times d})),
\]
\[
U \in C^{k-1}(\mathbb{R}^n; L^2(0, T; L^2(\mathcal{E}; \mathbb{R}^l))).
\]
And for any \( p \geq 2 \) and any multi-index \( \gamma \) with \( |\gamma| \leq k - 1 \), we have

\[
\|\partial^\gamma Y(x)\|_{L^\infty_F(0, T; \mathbb{R}^l)}^p + \|\partial^\gamma Z(x)\|_{L^2_F(0, T; \mathbb{R}^{l \times d})}^p + \|\partial^\gamma U(x)\|_{L^2_F(0, T; L^2(\mathcal{E}; \mathbb{R}^l))}^p \leq C_p, \quad \forall x \in \mathbb{R}^n,
\]
and

\[
\|\partial^\gamma Y(x_1) - \partial^\gamma Y(x_2)\|_{L^\infty_F(0, T; \mathbb{R}^l)}^p + \|\partial^\gamma Z(x_1) - \partial^\gamma Z(x_2)\|_{L^2_F(0, T; \mathbb{R}^{l \times d})}^p + \|\partial^\gamma U(x_1) - \partial^\gamma U(x_2)\|_{L^2_F(0, T; L^2(\mathcal{E}; \mathbb{R}^l))}^p
\]
\[
\leq C_p |x_1 - x_2|^p, \quad \forall x_1, x_2 \in \mathbb{R}^n.
\]
Moreover, the gradient \((\partial Y, \partial Z, \partial U)\) satisfies the following BSDE
\[
\partial Y_t(y) = \partial \varphi(X_T(y)) \partial X_T(y) + \int_t^T \left[ f_x(\Xi_t^y) \partial X_s(y) + f_y(\Xi_t^y) \partial Y_s(y) + f_z(\Xi_t^y) \partial Z_s(y) \\
+ f_u(\Xi_t^y) \partial U_s(y) \right] ds - \int_t^T \partial Z_s(y) dW_s - \int_t^T \int_E \partial U_s(e, x) \tilde{N}(de dt)
\]
\[
(4.11)
\]
where \( \Xi_t^y := (s, X_s(y), Y_s(y), Z_s(y), U_s(y)) \) and \( \partial Z_t(y) dW_t := \partial Z_t^y(y) dW_t^y \).

To get higher regularity of \((Y(x), Z(x), U(x))\) with respect to \( x \), we introduce the following assumption

\( (C6) \) the function \( f(t, x, y, z, u) \) is linear in \( z \) and \( u \) with the derivatives \( f_z \) and \( f_u \) being bounded and being independent of \((x, y, z, u)\).

**Theorem 4.3.** Suppose that for some positive integer \( k \), \((C1)\), \((C2)_k\), \((C5)_k\) and \((C6)\) are all satisfied. Then we have almost surely

\[
Y \in C^{k-1}(\mathbb{R}^n; D(0, T; \mathbb{R}^l)),
\]
\[
Z \in C^{k-1}(\mathbb{R}^n; L^2(0, T; \mathbb{R}^{l \times d})),
\]
\[
U \in C^{k-1}(\mathbb{R}^n; L^2(0, T; L^2(\mathcal{E}; \mathbb{R}^l))).
\]
And for any \( p \geq 2 \) and any multi-index \( \gamma \) with \( |\gamma| \leq k - 1 \), we have
\[
\| \partial^\gamma Y(x) \|_{L^p_f(0,T;\mathbb{R}^l)}^p + \| \partial^\gamma Z(x) \|_{L^{p,1}_f(0,T;\mathbb{R}^l \times d)}^p + \| \partial^\gamma U(x) \|_{L^{p,2}_f(0,T;L^2(\mathcal{E};\mathbb{R}^l))}^p \\
\leq C_p, \quad \forall x \in \mathbb{R}^n,
\]
and
\[
\| \partial^\gamma Y(x_1) - \partial^\gamma Y(x_2) \|_{L^p_f(0,T;\mathbb{R}^l)}^p + \| \partial^\gamma Z(x_1) - \partial^\gamma Z(x_2) \|_{L^{p,1}_f(0,T;\mathbb{R}^l \times d)}^p + \| U(x_1) - U(x_2) \|_{L^{p,2}_f(0,T;L^2(\mathcal{E};\mathbb{R}^l))}^p \\
\leq C_p|x_1 - x_2|^p, \quad \forall x_1, x_2 \in \mathbb{R}^n.
\]
Moreover, the triple \((\partial^\gamma Y(x), \partial^\gamma Z(x), \partial^\gamma U(x))\) satisfies the following BSDE
\[
\begin{aligned}
\partial^\gamma Y_t(x) &= -\{f_p(t, X_t(x), Y_t(x))\partial^\gamma Y_t(x) + f_z(t)\partial^\gamma Z_t(x) + f_u(t)\partial^\gamma U_t(x)\}dt \\
&\quad - \{f_p(t, X_t(x), Y_t(x)) + f_z(t)X_t(x) + P_\gamma(t, x)\}dt + \partial^\gamma Z_t(x)dW_t \\
\partial^\gamma Y_T(x) &= \partial^\gamma \varphi(X_T(x))
\end{aligned}
\]
where \( P_\gamma(t, x) \) is a \( n \)-dimensional vector whose components are polynomials of the partial derivatives up to order \( |\gamma| - 1 \) of the components of \( X_t(x) \) and \( Y_t(x) \), with the partial derivatives of order \( |\gamma| \) of the components of \( f \) as coefficients.

**Proof.** We use the principle of induction. For the case \( k = 2 \), Theorem 4.3 holds in view of Lemma 4.2. Suppose that Theorem 4.3 is true for \( k > 2 \). Now we prove Theorem 4.3 is true for the case \( k + 1 \). By Theorem 4.1 and the induction assumptions, we can proceed as [6, Proposition 3.3] to deduce from the equation (4.14) that for any \( p \geq 2 \) and multi-index \( \gamma \) with \( |\gamma| = k - 1 \), there exists a positive constant \( C_p \) such that for any \( x, x' \in \mathbb{R}^n, h, h' \in \mathbb{R}, 1 \leq i \leq n, \)
\[
\begin{aligned}
\| \Delta_i^h \partial^\gamma Y(x) \|_{L^p_f(0,T;\mathbb{R}^l)}^p + \| \Delta_i^h \partial^\gamma Z(x) \|_{L^{p,1}_f(0,T;\mathbb{R}^l \times d)}^p + \| \Delta_i^h \partial^\gamma U(x) \|_{L^{p,2}_f(0,T;L^2(\mathcal{E};\mathbb{R}^l))}^p \\
\leq C_p,
\end{aligned}
\]
where \( \Delta_i^h \Phi(x) := \frac{1}{h} (\Phi(x + he_i) - \Phi(x)) \) and \((e_1, \cdots, e_n)\) is the orthogonal basis of \( \mathbb{R}^n \). From Kolmogorov’s theorem (see, for instance, [21, Theorem 1.4.1, page 31]), there exists a modification, still denoted by \((Y(x), Z(x), U(x))\), of the solution of equation (4.10) such that almost surely
\[
Y \in C^k(\mathbb{R}^n; D(0, T; \mathbb{R}^l)), \\
Z \in C^k(\mathbb{R}^n; L^2(0, T; \mathbb{R}^{l \times d})), \\
U \in C^k(\mathbb{R}^n; L^2(0, T; L^2(\mathcal{E}; \mathbb{R}^l))).
\]
And the estimates (4.12) and (4.13) hold for \( |\gamma| = k \). In view of the equations corresponding to \((\partial^\gamma Y(x + he_i), \partial^\gamma Z(x + he_i), \partial^\gamma U(x + he_i))\) and \((\partial^\gamma Y(x), \partial^\gamma Z(x), \partial^\gamma U(x))\) and passing to the limit as \( h \to 0 \) in the expression
\[
\frac{1}{h} (\partial^\gamma Y(x + he_i) - \partial^\gamma Y(x)),
\]

we can obtain that for $|\gamma| = k, (\partial^n Y(x), \partial^n Z(x), \partial^n U(x))$ satisfies some BSDE of the form (4.14).

The proof is complete. □

To get the differentiability of $Z_t(x)$ with respect to $x$ for $(t, \omega) \in [0, T] \times \Omega$ and the differentiability of $U_t(e, x)$ with respect to $x$ for $(t, e, \omega) \in [0, T] \times E \times \Omega$, we need the following version of Kolmogorov’s continuity criterion.

Lemma 4.4. ([30, Lemma 1, pages 46-47]) Let $\mathbb{B}$ be a Banach space and $X \in C^k, a(R^n; L^p(F; \mathbb{B}))$ with $k + a > \frac{n}{p} + j$ ($j$ is a nonnegative integer). Then there is unique mapping $\tilde{X} : R^n \times \Omega \to \mathbb{B}$ such that

(i) for each $\omega \in \Omega$, $\tilde{X}(\cdot, \omega) \in C^j(R^n; \mathbb{B})$;

(ii) for each $x \in R^n$, $\tilde{X}(x, \cdot)$ is $\mathcal{F}$ measurable, and the derivatives $\partial^\gamma \tilde{X}(x, \omega)$ for all multi-index $\gamma$ such that $0 \leq |\gamma| \leq j$ are indistinguishable with the derivatives $\partial^\gamma X$ in $L^p(F; \mathbb{B})$.

Moreover, if $B := B(x_0, R)$ is a disk of $R^n$, there is a nonnegative constant $C_{k,a,p,B}$ such that for all $X \in C^{k, a}(R^n; L^p(F; \mathbb{B}))$ we have

$$|||\tilde{X}(\cdot, \cdot)|||_{C^j(B, \mathbb{B})}||_{L^p} \leq C_{k,a,p,B} |||X|||_{C^{k, a}(B, L^p)}$$

where $\tilde{X}$ is a regular version of $X$ satisfying (i) and (ii).

As a consequence of Theorem 4.3 and Lemma 4.4, we have

Theorem 4.5. Suppose (C1), (C2)$_{k+1}$ and (C5)$_{k+1}$ are satisfied with $k > \frac{n}{p} + j$ for some positive integer $j \geq 2$, and (C6) is satisfied. Let $(Y(x), Z(x), U(x))$ be the solution of BSDE (1.10). We have for any compact subset $K \subseteq R^n$,

$$E \sup_{0 \leq t \leq T} ||Y_t(\cdot)||_{C^k(K; R)} < \infty,$$

$$E \int_0^T ||Z_t(\cdot)||_{C^j(K; R^{n \times d})}^2 dt < \infty,$$

$$E \int_0^T \int_E ||U_t(e, \cdot)||_{C^j(K; R^{n \times d})}^2 v(de) dt < \infty.$$

5 Classical solutions to BSIPDEs

In this section, we consider classical solutions to BSIPDEs driven by a Brownian motion and a Poisson point process. First of all, we establish the relationship between BSIPDEs and non-Markovian FBSDEs.

Denote by $(X_t^y(x), Y_t^y(x), Z_t^y(x), U_t^y(\cdot, x))_{t \in [s, T]}$ the solution of the following non-Markovian FBSDE

$$\begin{cases}
    dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t + \int_E g(t, e, X_0)\tilde{N}(dedt), \\
    dY_t = -f(t, X_t, Y_t, Z_t, U_t)dt + Z_t dW_t + \int_E U_t(e)\tilde{N}(dedt), \\
    X_s = x, Y_T = \varphi(X_T), \ t \in [s, T].
\end{cases}$$

Theorem 5.1. Suppose $p(s, x) := Y_s^x(x)$ can be written as a semimartingale of the following form

$$p(s, x) = \varphi(x) - \int_s^T \Phi(l, x)dl - \int_s^T q(l, x)dW_l - \int_s^T \int_E r(l, e, x)\tilde{N}(dedl), \ s \in [0, T].$$

Then $(p, q, r)$ formally satisfies the BSIPDE (1.1).
Proof. Applying Itô-Wentzell formula to \( p(t, X^s_t(x)) \), we have

\[
p(t, X^s_t(x)) = \Phi(1, X^s_{t^-}(x)) + \int_s^t \Phi(l, X^s_{t^-}(x))dl + \int_s^t q(l, X^s_{t^-}(x))dW_l + \int_s^t (\partial p(l-, X^s_{t^-}(x))b(l, X^s_t(x)))dl
\]

\[
+ \int_s^t (\partial p(l-, X^s_{t^-}(x))\sigma(l, X^s_t(x))dW_l) + \frac{1}{2} \int_s^t \sigma\sigma^*(l, X^s_{t^-}(x)) \gg dl
\]

\[
+ \int_s^t \ll q(l, X^s_{t^-}(x))\sigma(l, X^s_t(x)) dl
\]

\[
- \int_s^t \int_E \left[ r(l, e, \phi_{l,e}(X^s_{t^-}(x))) - r(l, e, X^s_{t^-}(x)) \right] v(de)dl
\]

\[
+ \int_s^t \int_E \left[ p(l-, \phi_{l,e}(X^s_{t^-}(x))) - p(l-, X^s_{t^-}(x)) - (\partial p(l-, X^s_{t^-}(x)), g(l, e, X^s_{t^-}(x))) \right] v(de)dl
\]

\[
+ \int_s^t \int_E \left[ p(l-, \phi_{l,e}(X^s_{t^-}(x))) - p(l-, X^s_{t^-}(x)) + r(l, e, \phi_{l,e}(X^s_{t^-}(x))) \right] \tilde{N}(dedl).
\]

(5.3)

On the other hand,

\[
p(t, X^s_t(x)) = Y^s_t(X^s_t(x)) = Y^s_t(x)
\]

\[
= p(s, x) - \int_s^t f(l, X^s_{t^-}(x), Y^s_{t^-}(x), Z^s_{t^-}(x), U^s_{t^-}(e, x))dl
\]

\[
+ \int_s^t Z^s_{t^-}(x) dW_l + \int_s^t \int_E U^s_{t^-}(e, x) \tilde{N}(dedl).
\]

(5.4)

Comparing (5.3) with (5.4), we obtain

\[
\Phi(s, x)
\]

\[
= - f(s, x, p(s, x), q(s, x) + \partial p(s-, x)\sigma(s, x), p(s-, \phi_{s,-}(x)) - p(s-, x) + r(s, \cdot, \phi_{s,-}(x)))
\]

\[
- (\partial p(s-, x), b(s, x)) - \frac{1}{2} \ll \sigma\sigma^*(s, x) \gg
\]

\[
- \ll \partial q(s, x), \sigma(s, x) \gg - \int_E \left[ r(s, e, \phi_{s,e}(x)) - r(s, e, x) \right] v(de)
\]

\[
- \int_E \left[ p(s-, \phi_{s,e}(x)) - p(s-, x) - (\partial p(s-, x), g(s, e, x)) \right] v(de).
\]

(5.5)

Taking (5.5) into (5.2), we can get the desired result.

Next we concern the construction of the classical solution of BSIPDE (1.1) via the solution of FBSDE (5.1).

Definition 5.1. A triple of random fields \( \{(p(t, x), q(t, x), r(t, e, x)), (t, x, e) \in [0, T] \times \mathbb{R}^n \times E\} \) is called an adapted classical solution of BSIPDE (1.1), if

(i) \( p(\cdot, x) \) is an adapted cadlag process for any \( x \) and is twice continuously differentiable with respect to \( x \) for any \( t \) almost surely;

(ii) \( q(\cdot, x) \) is a predictable process for any \( x \) and is continuously differentiable with respect to \( x \) for almost any \( (t, \omega) \in [0, T] \times \Omega \);
Assume further
At the jump time
Remark 5.1.
Theorem 3.8, we have
Assume the same conditions for coefficients
So
\[
\sup_{0 \leq t \leq T, x \in K} |\partial^\beta p(t, x)| < \infty,
\]
\[
\int_0^T \sup_{x \in K} |\partial^\gamma q(t, x)|^2 dt < \infty,
\]
\[
\int_0^T \int_E \sup_{x \in K} |r(t, e, x)|^2 v(de) dt < \infty;
\]
(v) the following holds almost surely
\[
p(t, x) = \varphi(x) + \int_t^T [\mathcal{L}(s, x)p(s-, x) + \mathcal{M}(s, x)q(s, x)
+ f(s, x, p(s, x), q(s, x)) + \partial p(s-, x)\sigma(r(s, \cdot, \phi_{s,\cdot}(x)) - p(s-, x) + p(s-, \phi_{s,\cdot}(x)))] ds
- \int_t^T \int_E [r(s, e, x) - r(s, e, \phi_{s,e}(x)) + p(s-, x) - p(s-, \phi_{s,e}(x))] v(de) ds
- \int_t^T q_k(s, x)dW^k_s - \int_t^T \int_E r(s, e, x)\tilde{N}(deds), \forall (t, x) \in [0, T] \times \mathbb{R}^n.
\]
(5.6)
(5.7)
Denote \((X_t(x), Y_t(x), Z_t(x), U_t(\cdot, x))_{t \in [0, T]} := (X_t^0(x), Y_t^0(x), Z_t^0(x), U_t^0(\cdot, x))_{t \in [0, T]}\). We define
\[
\begin{align*}
p(t, x) &:= Y_t(X_t^{-1}(x)), \\
q(t, x) &:= Z_t(X_t^{-1}(x)) - \partial p(t-, x)\sigma(t, x), \\
r(t, e, x) &:= p(t-, \phi_{t,e}^{-1}(x)) - p(t-, x) + U_t(e, X_t^{-1}(\phi_{t,e}^{-1}(x))).
\end{align*}
\]
(5.8)
Remark 5.1. At the jump time \(\tau\) of the point process \(\mathbb{P}\), we have the relation
\[
X_\tau(x) = X_{\tau-}(x) + g(\tau, p(\tau), X_{\tau-}(x)) = \phi_{\tau\cdot, p(\tau)}(X_{\tau-}(x))
\]
which implies
\[
X_{\tau-}(x) = \phi_{\tau\cdot, p(\tau)}^{-1}(X_\tau(x)).
\]
So \(X_{\tau-}(\cdot) : \mathbb{R}^n \to \mathbb{R}^n\) is also a homeomorphic mapping for any \(t \in [0, T]\) almost surely.

Theorem 5.2. Assume the same conditions for coefficients \(b, \sigma\) and \(g\) as in Theorem 3.8.
Assume further (C5)_{k+1} and (C6) are satisfied with \(k > 2 + \frac{n}{2}\). Then the triple \((p, q, r)\) defined in (5.8) is a classical solution of BSIPDE (1.1).

Proof. From Theorem 4.5 and (3.47), we know that the triple \((p, q, r)\) satisfies (5.6). From Theorem 3.8 we have
\[
\begin{align*}
dx_t^{-1}(x) &= (\mathcal{M}^T \mathcal{M}^T - \mathcal{L})(t, x)X_{t-}^{-1}(x) dt + \int_E \mathcal{A}(t, e)X_{t-}^{-1}(x)v(de) dt \\
&\quad - \mathcal{M}^T(t, x)X_{t-}^{-1}(x) dW^r_t + \int_E \mathcal{A}(t, e)X_{t-}^{-1}(x)\tilde{N}(dedt), \quad 0 \leq t \leq T,
\end{align*}
\]
(5.9)
In view of Theorem 4.5 again, we apply the Itō-Wentzell formula to calculate $Y_t(X_t^{-1}(x))$ and obtain

$$dp(t, x) = -f(t, x, p(t, x), Z_t(X_t^{-1}(x)), U_t(X_t^{-1}(x)))dt + Z_t(X_t^{-1}(x))dW_t$$

$$+ \partial_i Y_t(\bar{x})|_{\bar{x}=X_t^{-1}(x)}[(M^{r}\mathcal{M}^r - \mathcal{L})(t, x)X_t^{-1,i}(x) + \int_E A(t, e)X_t^{-1,i}(x)v(de)]dt$$

$$- \partial_i Y_t(\bar{x})|_{\bar{x}=X_t^{-1}(x)}M^r(t, x)X_t^{-1,i}(x)dW_t^r$$

$$+ \frac{1}{2} \delta_{ij} Y_t(\bar{x})|_{\bar{x}=X_t^{-1}(x)}[\mathcal{M}^r(t, x)X_t^{-1,i}(x)][\mathcal{M}^r(t, x)X_t^{-1,j}(x)]dt$$

$$- \partial_i Z^r(\bar{x})|_{\bar{x}=X_t^{-1}(x)}M^r(t, x)X_t^{-1,i}(x)dt$$

$$+ \int_E [Y_t(X_t^{-1}(\phi_{t,e}^{-1}(x))) - Y_t(X_t^{-1}(x)) - \partial_i Y_t(\bar{x})|_{\bar{x}=X_t^{-1}(x)}(A(t, e)X_t^{-1,i}(x))]v(de)dt$$

$$+ \int_E [Y_t(X_t^{-1}(\phi_{t,e}^{-1}(x))) + U_t(e, X_t^{-1}(\phi_{t,e}^{-1}(x))) - Y_t(X_t^{-1}(x))]\tilde{N}(dedt)$$

$$+ \int_E [U_t(e, X_t^{-1}(\phi_{t,e}^{-1}(x))) - U_t(e, X_t^{-1}(x))]v(de)dt, \quad t \in [0, T],$$

$$p(T, x) = \varphi(x)$$

(5.10)

where $X_t^{-1,i}(x)$ is the $i$-th component of $X_t^{-1}(x)$. By computation, we have

$$\mathcal{M}^r(t, x)Z_t^{r'}(X_t^{-1}(x)) = \partial_i Y_t(\bar{x})|_{\bar{x}=X_t^{-1}(x)}\mathcal{M}^r(t, x)X_t^{-1,i}(x),$$

$$\mathcal{M}^r(t, x)p(t, x) = \partial_i Y_t(\bar{x})|_{\bar{x}=X_t^{-1}(x)}\mathcal{M}^r(t, x)X_t^{-1,i}(x),$$

$$\mathcal{M}^r\mathcal{M}^r p(t, x) = \mathcal{M}^r(t, x)\partial_i Y_t(\bar{x})|_{\bar{x}=X_t^{-1}(x)}\mathcal{M}^r(t, x)X_t^{-1,i}(x)$$

$$= \mathcal{M}^r(t, x)(\partial_i Y_t(\bar{x})|_{\bar{x}=X_t^{-1}(x)})\mathcal{M}^r(t, x)X_t^{-1,i}(x)$$

$$+ \partial_i Y_t(\bar{x})|_{\bar{x}=X_t^{-1}(x)}\mathcal{M}^r(t, x)X_t^{-1,i}(x)$$

$$= \partial_{ij} Y_t(\bar{x})|_{\bar{x}=X_t^{-1}(x)}[\mathcal{M}^r(t, x)X_t^{-1,i}(x)][\mathcal{M}^r(t, x)X_t^{-1,j}(x)]$$

$$+ \partial_i Y_t(\bar{x})|_{\bar{x}=X_t^{-1}(x)}(\mathcal{M}^r\mathcal{M}^r(t, x)X_t^{-1,i}(x),$$

$$\mathcal{L}(t, x)p(t, x) = \partial_i Y_t(\bar{x})|_{\bar{x}=X_t^{-1}(x)}\mathcal{L}(t, x)X_t^{-1,i}(x)$$

$$+ \frac{1}{2} \partial_{ij} Y_t(\bar{x})|_{\bar{x}=X_t^{-1}(x)}[\mathcal{M}^r(t, x)X_t^{-1,i}(x)][\mathcal{M}^r(t, x)X_t^{-1,j}(x)].$$

So the triple $(p, q, r)$ is a classical solution to BSIPDE (111).

The following theorem is concerned with the uniqueness of the classical solution to the system (111).

**Theorem 5.3.** Let the assumptions of Theorem 5.2 be satisfied. Let $(\tilde{p}, \tilde{q}, \tilde{r})$ be a classical solution to BSIPDE (111). Then we have

$$\tilde{p}(t, X_t(x)) = Y_t(x),$$

$$\tilde{q}(t, X_t^{-1}(x)) = Z_t(x) - \partial \tilde{p}(t, X_t^{-1}(x))\sigma(t, X_t^{-1}(x)),$$

$$\tilde{r}(t, e, \phi_{t,e}(X_t^{-1}(x))) = U_t(e, x) + \tilde{p}(t, X_t^{-1}(x)) - \tilde{p}(t, \phi_{t,e}(X_t^{-1}(x)))$$

or equivalently,

$$\tilde{p}(t, x) = Y_t(X_t^{-1}(x)).$$
\[ \tilde{q}(t, x) = Z_t(X_t^{-1}(x)) - \partial[Y_t(X_t^{-1}(x))]\sigma(t, x), \]
\[ \tilde{r}(t, e, x) = Y_t(X_t^{-1}(\phi_{t,e}^{-1}(x))) - Y_t(X_t^{-1}(x)) + U_t(e, X_t^{-1}(\phi_{t,e}^{-1}(x))). \]

Proof. Using the Itô-Wentzell formula to calculate \( \tilde{p}(t, X_t(x)) \), we see that
\[ (\tilde{p}(t, X_t(x)), \tilde{q}(t, X_t(x)) + \partial \tilde{p}(t, X_t(x))\sigma(t, X_t(x)), \]
\[ \tilde{r}(t, e, \phi_{t,e}(X_t(x))) - \tilde{p}(t, X_t(x)) + \tilde{p}(t, \phi_{t,e}(X_t(x))) \]
is an adapted solution of BSDE (4.10). We have the desired result. \( \square \)

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