Outer automorphisms of free Burnside groups

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Abstract
In this paper, we study some properties of the outer automorphism group of free Burnside groups of large odd exponent. In particular, we prove that it contains free and free abelian subgroups.

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Introduction
The free Burnside group of rank $r$ and exponent $n$, denoted by $B_r(n)$, is the quotient of the free group $F_r$ by the subgroup $F_r^n$ generated by the $n$-th powers of all its elements. In 1902, W. Burnside asked whether $B_r(n)$ has to be finite or not (see [Bur02]). For a long time, one only knows that the answer was positive for some small exponents (for $n = 2$ see [Bur02], $n = 3$ [Bur02] and [LvdW33], $n = 4$ [San40] and $n = 6$ [Hal57]). In 1968, P.S. Novikov and S.I. Adian achieved a breakthrough (see [NA68a], [NA68b] and [NA68c]). Using the small cancellation theory, developed by V.A. Tartakovskii [Tar49] and M. Greendlinger [Gre60a], [Gre60b], [Gre61], they proved that for large odd exponents, $B_r(n)$ is infinite. Thanks to a diagrammatic formulation of small cancellation, A.Y. Ol’shanskiĭ simplified in 1982 the proof of P.S. Novikov and S.I. Adian [Ol82]. Recently, T. Delzant and M. Gromov, gave a more geometrical
Thus the elements $\phi$ added to the set of relations. We shall now use the fact that the group $F$ is hyperbolic, i.e. the semi-direct product $F \rtimes Z$ defined by $\phi$ is a hyperbolic group. There exists an integer $n_0$ such that for all odd integer $n$ larger than $n_0$, $\phi$ induces an infinite order outer automorphism of $B_r(n)$.

All proofs dealing with free Burnside groups have the same weakness: they involve a presentation of $B_r(n)$ which is not stable under automorphisms. Our work try to regain a little symmetry: we build a sequence of groups $(H_k)$ such that for all $k$, $\phi$ induces an automorphism of $H_k$ and $\lim H_k = B_r(n)$. To that end, we start with $H_0 = F$ and, at each step, we construct $H_{k+1}$ as a small cancellation quotient of $H_k$. Some difficulties appear during this process. Assume that $\rho$ is one of the relations defining the first quotient $F = H_0 \rightarrow H_1$. Since we want $\phi$ to induce an automorphism of $H_1$, the elements $\phi^m(\rho)$, $m \in N$, have to belong to the set of relations. However the small cancellation theory only deals with relations having more or less the same length. In our case, the relations $\phi^m(\rho)$ may have very different lengths. To avoid this problem, we encode the information concerning the automorphism in a larger group: $F \rtimes \phi Z$. Thus the elements $\phi^m(\rho)$ become conjugates of $\rho$ and do not need to be added to the set of relations. We shall now use the fact that the group $F \rtimes Z$ is hyperbolic. In 1991, A.Y. Ol’shanskiĭ provided indeed a generalisation of the Novikov-Adian theorem (see Ol’91). Given a torsion-free, hyperbolic group $G$, he proved that for large odd exponent $n$ the quotient $G/G^n$ is infinite. This result was recovered by T. Delzant and M. Gromov in DG08. We would like to apply the same techniques to $G = F \rtimes Z$. However we must take care not to kill all $n$-th powers of $G$. Indeed, if we did so the automorphism obtained at the end of the construction would have finite order dividing $n$. That is why we propose an extension of the Delzant-Gromov construction where the relations are chosen in a normal subgroup of $F \rtimes Z$. This construction works in a more general situation. It leads to our main theorem

**Theorem 2** (Main Theorem). Let $1 \rightarrow H \rightarrow G \rightarrow F \rightarrow 1$ be a short exact sequence of groups. Assume that $H$ is finitely generated, $G$ is hyperbolic, torsion-free and $F$ is torsion-free. There exists an integer $n_0$ such that for all odd integers $n$ larger than $n_0$, the canonical map $F \rightarrow \text{Out}(H)$ induces an injective homomorphism $F \rightarrow \text{Out}(H/H^n)$.
Theorem 1 is then an application of the main theorem to the short exact sequence $1 \to F_r \to F_r \rtimes \mathbb{Z} \to \mathbb{Z} \to 1$. The work of M. Bestvina, M. Feighn and M. Handel, provides examples of hyperbolic extensions of free groups by free groups. Using this result we obtain our second theorem.

Theorem 3 (see Th. 1.8). Let $r \geq 3$. There exists an integer $n_0$ such that for all odd integers $n$ larger than $n_0$, the group $\text{Out}(B_r(n))$ contains a subgroup which is isomorphic to $F_2$.

The strategy to embed abelian subgroups in $\text{Out}(B_r(n))$ is a little different. We do not apply the main theorem to an appropriate hyperbolic extension of the free group. We construct a family of automorphisms of $F_r$ which already commute in $\text{Aut}(F_r)$ and check “by hand” that they do not satisfy any other relation in $\text{Out}(B_r(n))$. Thus, we obtain the following result.

Theorem 4 (see Th. 1.12). Let $r \geq 1$. There exists an integer $n_0$ such that for all odd integers $n$ larger than $n_0$, the groups $\text{Out}(B_{2r}(n))$ and $\text{Out}(B_{2r+1}(n))$ contain a subgroup which is isomorphic to $\mathbb{Z}$.

Hyperbolic automorphisms induce infinite order automorphisms of free Burnside groups of large exponent. But they are not the only one. For instance, the automorphism $\varphi$ studied by E.A. Cherepanov, characterized by $\varphi(a) = b$ and $\varphi(b) = ab$, is not hyperbolic. Indeed, $\varphi^2$ fixes the commutator $[a^{-1},b^{-1}]$. The semi-direct product $F_r \rtimes \mathbb{Z}$ contains therefore a subgroup which is isomorphic to $\mathbb{Z}^2$. We wonder if there exists a criterion to decide whether an automorphism of $F_r$ induces a infinite order outer automorphism of $B_r(n)$ for some large exponent or not. In particular, is there a link between this property and the growth of the automorphism? Section 1.2 gives a partial answer. We prove that a polynomially growing automorphism always induces a finite order automorphism of $B_r(n)$.

Outline of the paper. In Section 1 we explain the consequences of the main theorem. In particular, we provide examples of infinite order automorphisms of $B_r(n)$. We also construct free and free abelian subgroups of $\text{Out}(B_r(n))$. Section 2 deals with the proof of the main theorem. At first, we recall the geometrical point of view on the small cancellation theory developed by T. Delzant and M. Gromov. We also improve some results of [DG08] which are necessary to control the small cancellation parameters in our situation. Then, we prove an induction lemma (Lemma 2.12) which is the fundamental step of the induction process used in Section 3 to prove the main theorem.

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1 Automorphisms of Burnside groups

Remark: In this paper, we are interested in outer automorphisms of free Burnside groups. One question we look at is the following: given an automorphism of the free group $F_r$, does it induce an infinite order
automorphism of \( B_r(n) \)? Note that any element of the Burnside group has finite order. In particular any inner automorphism of \( B_r(n) \) has finite order. It follows that an element of \( \text{Aut}(B_r(n)) \) has finite order if and only if its image in \( \text{Out}(B_r(n)) \) has finite order.

1.1 Examples of infinite order automorphisms

Using the work of P.S. Novikov and S.I. Adian ([NA68a], [NA68b], [NA68c]), we can produce a first example of an infinite order outer automorphism of Burnside groups. This example was already studied by E.A. Cherepanov in [Che05].

**Proposition 1.1** (see [Che05, Th. 1]). Let \( \{a, b\} \) be a generating set of the free group \( F_2 \). Let \( \varphi \) be the automorphism of \( F_2 \) defined by \( \varphi(a) = ab \) and \( \varphi(b) = a \). There exists an integer \( n_0 \), such that for all odd integers \( n \) larger than \( n_0 \), \( \varphi \) induces an infinite order automorphism of \( B_2(n) \).

**Proof.** We consider the sequence of iterated images of \( a \) by \( \varphi \).

\[
\begin{align*}
\varphi^0(a) &= a & \varphi^1(a) &= abaaba \\
\varphi^1(a) &= ab & \varphi^2(a) &= abaabaaba \\
\varphi^2(a) &= aba & \varphi^3(a) &= abaabaabaabaaba \\
\varphi^3(a) &= aba & \varphi^4(a) &= abaabaabaabaabaaba \\
& & \vdots
\end{align*}
\]

This sequence converges to a right infinite positive word

\( \varphi^\infty(a) = abaabaabaabaabaaba \ldots \)

which has the following property. For every word \( u \) in \( \{a, b\} \), \( u^4 \) is not a subword of \( \varphi^\infty(a) \) (see [Mos92]). Let \( n \) be an odd integer larger than 10,000. In order to prove that the free Burnside group of large exponent is infinite, P.S. Novikov and S.I. Adian use the following fact: if \( m \) is a reduced word in \( \{a, b\} \) which does not contain a subword equal to a fourth power, then \( m \) defines a non-trivial element of \( B_2(n) \) (see [Adi79, IV. 2.16.] or [AL92, Statement 1]). In particular \( \varphi^\phi(a) \) induces a sequence of pairwise distinct elements of \( B_2(n) \). It follows that \( \varphi \) induces an infinite order automorphism of \( B_2(n) \).

1. We are now interested in a large class of automorphisms of the free group: the hyperbolic automorphisms. We prove that they all induce infinite order automorphisms of the free Burnside group.

**Definition 1.2.** Let \( G \) be a hyperbolic group. An automorphism \( \varphi \) of \( G \) is hyperbolic if the semi-direct product \( G \rtimes \mathbb{Z} \) defined by \( \varphi \) is hyperbolic.

**Example:** Let \( \Sigma \) be the fundamental group of a compact surface \( S \) of genus larger than 2. Thanks to Thurston’s hyperbolisation Theorem, any pseudo-Anosov homeomorphism of \( S \) induces a hyperbolic automorphism of \( \Sigma \) (see [Ota96]).

There exist many characterizations of hyperbolic automorphisms. Assume that \( G \) is endowed with the word metric \( |.| \) relative to a generating set, M. Bestvina and M. Feighn proved in [BF92] that an automorphism \( \varphi \) of \( G \) is hyperbolic if and only if there exist \( \lambda > 1 \) and \( m \in \mathbb{N} \) such that for all \( g \in G \),

\[
\lambda |g| \leq \max \left\{|\varphi^m(g)|, |\varphi^{-m}(g)|\right\}.
\]

On the other hand, an automorphism of the free group is hyperbolic, if and only if it has no non-trivial periodic conjugacy classes (see [BFH97]).
exists an integer $pB$ that $\phi$ under $\Phi$ a power of $\Phi$ an automorphism of rank that is smaller or equal to $r$ for rank one. Let

First case.

we distinguish two cases.

Let Theorem 1.6. Proof. We prove this result by induction on the rank $\Phi$. The outer automorphism group of $B$ denoted by $\phi$ grows polynomially if for every conjugacy class $x$ of $F_r$, the sequence $|\Phi^m(x)|$ grows polynomially.

Definition 1.4. The automorphism $\Phi$ grows polynomially if for every conjugacy class $x$ of $F_r$, the sequence $|\Phi^m(x)|$ grows polynomially.

Proposition 1.5 (see [Lev08]). Let $\Phi$ be a polynomially growing outer automorphism of $F_r$. Up to replace $\Phi$ by a power of $\Phi$, one of the following assertion is true.

(i) There exist $\phi \in \text{Aut}(F_r)$ representing $\Phi$ and a non-trivial free decomposition $F_1 \ast F_2$ of $F_r$ which is invariant under $\phi$.

(ii) There exist $\phi \in \text{Aut}(F_r)$ representing $\Phi$, a non-trivial free decomposition $F_1 \ast \{t\}$ of $F_r$ and an element $f$ of $F_1$ such that $F_1$ is invariant under $\phi$ and $\phi(f) = tf$.

Theorem 1.6. Let $r \geq 1$. Let $\Phi$ be a polynomially growing outer automorphism of $F_r$. For all positive integers $n$, $\Phi$ induces a finite order outer automorphism of $B_r(n)$.

Proof. We prove this result by induction on the rank $r$ of the free group. The outer automorphism group of $Z$ is trivial. Hence the theorem is true for rank one. Let $r \geq 1$. We assume now that the theorem is true for any rank that is smaller or equal to $r$. Let $\Phi$ be a polynomially growing outer automorphism of $F_{r+1}$ and $n$ a positive integer. Due to Proposition 1.5 we distinguish two cases.

First case. There exist an automorphism $\phi \in \text{Aut}(F_{r+1})$ representing a power of $\Phi$ and a non-trivial free decomposition $F_1 \ast F_2$ of $F_{r+1}$ invariant under $\phi$. We denote by $\phi_i$ the restriction of $\phi$ to $F_i$. By induction, there exists an integer $p_i$ such that $\phi_i^{p_i}$ induces the identity of $F_i/F_i^{p_i}$. It follows that $\phi_1^{p_1} \phi_2$ is trivial in $\text{Aut}(B_{r+1}(n))$. Therefore $\Phi$ induces a finite order outer automorphism of $B_{r+1}(n)$. 

1.2 Polynomially growing automorphisms of free groups

We provide now examples of infinite order automorphisms of $F_r$ which induce finite order automorphisms of $B_r(n)$. If $x$ is a conjugacy class of $F_r$, we denote by $[x]$ the length of any cyclically reduced word representing $x$. Given an outer automorphism $\Phi$ of $F_r$, we look at the action of $\Phi$ on the conjugacy classes of $F_r$.
Second case. There exist an automorphism $\varphi \in \text{Aut}(F_{r+1})$ representing a power of $\Phi$, a free decomposition $F_1 * (t)$ of $F_{r+1}$ and an element $f$ of $F_1$ such that $F_1$ is invariant under $\varphi$ and $\varphi(t) = tf$. We denote by $\varphi_1$ the restriction of $\varphi$ to $F_1$. By induction, there exists an integer $p_1$ such that $\varphi_1^{p_1}$ induces the identity of $F_1/F_1^n$. On the other hand, for all integers $q$, $\varphi^q(t)$ is equal to $tf\varphi_1(f)\varphi_1^2(f)\ldots\varphi_1^{p_1-1}(f)$. It follows that the below equality holds in $B_{r+1}(n)$.

$$\varphi^{np_1}(t) = t [f\varphi_1(f)\varphi_1^2(f)\ldots\varphi_1^{p_1-1}(f)]^n = t$$

Hence $\varphi^{np_1}$ is trivial in $\text{Aut}(B_{r+1}(n))$. Therefore $\Phi$ induces a finite order outer automorphism of $B_{r+1}(n)$. $\square$

1.3 Subgroups of $\text{Out}(B_r(n))$

We are now interested in relevant subgroups that can be embedded in $\text{Out}(B_r(n))$. We start with free subgroups. The following result is due to M. Bestvina, M. Feighn and M. Handel

**Theorem 1.7** (see [BFH97b, Th. 5.2]). Let $r \geq 3$. Let $\varphi_1$ and $\varphi_2$ two automorphisms of $F_r$. We assume that the outer automorphisms induced by $\varphi_1$ and $\varphi_2$ are irreducible, do not have common powers and neither have a nontrivial periodic conjugacy class. There exists an integer $m$ such that $\varphi_1^m$ and $\varphi_2^m$ generate a free group. Moreover, the semi-direct product $F_r \rtimes F_2$ defined by $\varphi_1^m$ and $\varphi_2^m$ is hyperbolic.

**Theorem 1.8.** Let $r \geq 3$. There exists an integer $n_0$, such that for all odd integers $n$ larger than $n_0$, $\text{Out}(B_r(n))$ contains a subgroup which is isomorphic to $F_2$.\[\square\]

Proof. Theorem 1.7 provides a hyperbolic extension of $F_r$ by $F_2$. In other words, $1 \rightarrow F_r \rightarrow F_r \rtimes F_2 \rightarrow F_2 \rightarrow 1$ is a short exact sequence such that $F_r \rtimes F_2$ is hyperbolic. The result follows from the main theorem (see Theorem 2). $\square$

We are now looking for free abelian subgroups of $\text{Out}(B_r(n))$. Let $G_1$ and $G_2$ be two torsion-free groups. We denote by $G$ the free product $G_1 * G_2$. Since $G_1$ and $G_2$ are torsion-free, so is $G$ (see [Sc77]).

**Lemma 1.9.** Let $g$ be a non-trivial element of $G$. If there exists a positive integer $k$ such that $g^k$ belongs to $G_1$, then $g$ belongs to $G_1$.

Proof. We use the Bass-Serre theory of groups acting on trees (see [Sc77]). There exist a simplicial tree $T$ and a simplicial action without inversion of $G$ on $T$ satisfying the following properties. The stabilizers of the vertices are the conjugates of $G_1$ and $G_2$. The stabilizers of the edges are trivial. Since $G$ is torsion-free, $g^k$ is a non-trivial element of $G_1$. In particular it fixes a unique point $x$ of $T$: the one whose stabilizer is $G_1$. Therefore $g$ is an elliptic isometry and fixes a point of $T$ (see [CDP90] Chap. 9, Cor. 3.2). This last point has to be $x$, otherwise $g^k$ would fix two distinct points of $T$. Consequently $g$ belongs to the stabilizer of $x$, i.e. $G_1$. $\square$

**Lemma 1.10.** Let $n$ be an integer. The canonical map $G_1 \hookrightarrow G$ induces an injective homomorphism $j : G_1/G_1^n \hookrightarrow G/G^n$.\[\square\]

Proof. By the previous lemma $G_1 \cap G^n = G_1^n$. Thus the kernel of the map $G_1 \rightarrow G \rightarrow G/G^n$ is exactly $G_1^n$. $\square$
Lemma 1.11. Let $n$ be an integer. Let $\varphi$ be an automorphism of $G$ which stabilizes the factor $G_1$. We denote by $\varphi_1$ the restriction of $\varphi$ to $G_1$. If $\varphi$ induces a finite order automorphism of $G/G^n$ then $\varphi_1$ induces a finite order automorphism of $G_1/G_1^n$.

Proof. We respectively denote by $\bar{\varphi}_1$ and $\bar{\varphi}$ the automorphisms of $G_1/G_1^n$ and $G/G^n$ induced by $\varphi_1$ and $\varphi$. By assumption, there exists an integer $k$ such that $\bar{\varphi}^k = \text{id}$. However, the following diagram is commutative.

$$
\begin{array}{ccc}
G_1/G_1^n & \xrightarrow{j} & G/G^n \\
\bar{\varphi}_1 & \downarrow & \bar{\varphi} \\
G_1/G_1^n & \xrightarrow{j} & G/G^n
\end{array}
$$

Thus $j \circ \bar{\varphi}_1^k = \bar{\varphi}^k \circ j = j$. Since $j$ is injective (see Lemma 1.11), $\bar{\varphi}_1^k = \text{id}$. In particular, $\bar{\varphi}_1$ has finite order.

Theorem 1.12. Let $r \geq 2$. There exists an integer $n_0$ such that for all odd integers $n$ larger than $n_0$, $\text{Out}(B_2(n))$ and $\text{Out}(B_{2r+1}(n))$ contain a subgroup which is isomorphic to $\mathbb{Z}^r$.

Proof. We denote by $\varphi$ the automorphism of $F_2$ studied in Proposition 1.1. There exists an integer $n_0$ such that for all odd integers $n$ larger than $n_0$, $\varphi$ induces an infinite order automorphism of $B_2(n)$. We consider $F_{2r}$ as a free product $F_1 \ast \cdots \ast F_r$ of $r$ copies of $F_2$. For all $i \in \{1, \ldots, r\}$, we define an automorphism $\bar{\varphi}_i$ of $F_{2r}$ as follows.

(i) The restriction of $\varphi_i$ to $F_i$ is $\varphi$.

(ii) The restriction of $\varphi_i$ to any other factor is the identity.

We respectively denote by $\bar{\varphi}_i$ and $\bar{\varphi}$ the automorphisms of $B_{2r}(n)$ and $B_2(n)$ induced by $\varphi_i$ and $\varphi$. By construction, the $\bar{\varphi}_i$’s generate an abelian subgroup of $\text{Out}(B_2(n))$. We now study the relations between the $\bar{\varphi}_i$’s in $\text{Out}(B_{2r}(n))$. Consider $r$ integers $k_1, \ldots, k_r$ such that $\psi = \varphi_1^{k_1} \cdots \varphi_r^{k_r}$ induces an inner automorphism $\psi$ of $B_{2r}(n)$. In particular, $\psi$ has finite order. By Lemma 1.11, the restriction of $\psi$ to $F_i$ induces a finite order automorphism of $F_i/F_i^n$. In other words, $\varphi_i^{k_i}$ has finite order. By construction, $\varphi$ has finite order. This forces $k_i$ to be zero. There is hence no relation between the $\bar{\varphi}_i$’s in $\text{Out}(B_{2r}(n))$. Thus the $\bar{\varphi}_i$’s generate a subgroup of $\text{Out}(B_{2r}(n))$ which is isomorphic to $\mathbb{Z}^r$. For $\text{Out}(B_{2r+1}(n))$ we apply the same argument with the following free factorization: $F_{2r+1} = F_1 \ast \cdots \ast F_r \ast \mathbb{Z}$. □

2 Small cancellation theory

In this section, we expose the geometrical point of view on small cancellation developed by T. Delzant and M. Gromov in [DG06] and used in Section 3 to prove the main theorem.

2.1 Hyperbolic spaces

Let $X$ be a proper, geodesic, $\delta$-hyperbolic (in the sense of Gromov) space. The distance between two points $x$ and $x'$ of $X$ is denoted by $|x - x'|_X$ (or simply $|x - x'|$). Although it may not be unique, denote by $[x, x']$ a geodesic joining $x$ and $x'$. The boundary at infinity of $X$ is
denoted by \( \partial X \) (see [CDP90, Chap. 2]). A part \( Y \) of \( X \) is \( \alpha \)-quasi-convex if every geodesic of \( X \) joining two points of \( Y \) lies in the \( \alpha \)-neighbourhood of \( Y \), denoted by \( Y^+ \).

**Lemma 2.1** (see [DG08, Lemma 2.1.5] or [Cou09, Cor. 1.2.2]). Let \( x, x', y \) and \( y' \) be four points of \( X \). Let \( u \) be a point of \( [x, x'] \) such that \(|u - x| > |x - y| + 8\delta\) and \(|u - x'| > |x' - y'| + 8\delta\). Then \( u \) belongs to the \( 8\delta \)-neighbourhood of \([y, y']\).

**Proposition 2.2** (see [DG08 Lemma 2.2.2] or [Cou09, Prop. 1.2.4]). Let \( Y \) and \( Z \) be two \( \alpha \)-quasi-convex parts of \( X \). For all \( A > 0 \), we have

\[
\text{diam} \left( Y^+ \cap Z^+ \right) \leq \text{diam} \left( Y^{+\alpha + 10\delta} \cap Z^{+\alpha + 10\delta} \right) + 2A + 20\delta.
\]

Let \( G \) be a group acting properly, co-compactly, by isometries on \( X \). An element \( g \) of \( G \) is either elliptic (in particular it has finite order) or hyperbolic (see [CDP90, Chap. 9]). In the second case, \( g \) fixes exactly two points of \( \partial X \) denoted by \( g^- \) and \( g^+ \). In order to measure the action of an isometry \( g \) of \( X \), we define two translation lengths. The translation length \( [g]_X \) (or simply \([g]\)) is given by

\[
[g] = \inf_{x \in X} |gx - x|.
\]

The asymptotic translation length \([g]^\infty_X \) (or simply \([g]^\infty\)) is defined by

\[
[g]^\infty = \lim_{n \to \infty} \frac{1}{n} |g^n x - x|.
\]

These two lengths satisfy the following inequality (see [CDP90, Chap. 10, Prop 6.4]):

\[
\forall g \in G, \ [g]^\infty \leq [g] \leq [g]^\infty + 32\delta.
\]

An isometry of \( X \) is hyperbolic if and only if its asymptotic translation length is positive (see [CDP90, Chap. 10, Prop. 6.3]). The axis \( A_g \) of an isometry \( g \), defined as follows, is a \( 40\delta \)-subset of \( X \) (cf. [DG08 Prop. 2.3.3]).

\[
A_g = \{ x \in X | |gx - x| \leq \max \{ \|g\|, 40\delta \} \}
\]

**Proposition 2.3.** Let \( g \) be a hyperbolic element of \( G \). We denote by \( \sigma \) a geodesic joining \( g^- \) and \( g^+ \), the points of \( \partial X \) fixed by \( g \). Let \( Y \) be a \( \alpha \)-quasi-convex part of \( X \). If \( Y \) is \( g \)-invariant, then \( \sigma \) is contained in the \((\alpha + 8\delta)\)-neighbourhood of \( Y \). In particular \( \sigma \) is contained in the \( 50\delta \)-neighbourhood of \( A_g \).

**Proof.** Let \( x \) be a point of \( \sigma \). We denote by \( d \) the distance from \( x \) to \( Y \) and by \( y \) a point of \( Y \) such that \(|x - y| \leq d + \delta \). Since \( g \) is hyperbolic, there exists an integer \( m \) such that \(|g^m x - g^{-m} x| \geq 2d + 100\delta\) (see [CDP90, Chap. 10, Lemme 6.5]). We respectively denote by \( p_- \) and \( p_+ \) the projections of \( g^{-m} x \) and \( g^m x \) on \( \sigma \). The geodesics \( \sigma \) and \( g^m \sigma \) have the same extremities. It follows that they are \( 8\delta \)-closed (see [CDP90, Chap. 2, Prop 2.2]). In particular \(|g^m x - p_+| \leq 8\delta \). In the same way, we have \(|g^{-m} x - p_-| \leq 8\delta \). Note that \( x \) lies on the subgeodesic of \( \sigma \) delimited by \( p_- \) and \( p_+ \). Indeed, if it was not the case, we should have

\[
|g^m x - g^{-m} x| \leq |p_- - p_+| + 16\delta \leq \|x - p_- - |x - p_+| + 16\delta \leq |x - g^{-m} x| - |x - g^m x| + 32\delta \leq 32\delta.
\]
Contradiction. On the other hand, we have
\[ |x - p_+| \geq |x - g^m x| - 8\delta \geq \frac{1}{2} |g^{-m} x - g^m x| - 8\delta \geq d + 30\delta. \]

In the same way, we have \( |x - p_-| \geq d + 30\delta \). By lemma 2.1, \( x \) belong to the \( 8\delta \)-neighbourhood of \( [g^m y, g^m y] \). However \( g^{-m} y \) and \( g^m y \) belongs to \( Y \) which is \( \alpha \)-quasi-convex. Therefore the distance between \( x \) and \( Y \) is smaller than \( \alpha + 8\delta \).

The injectivity radius of a part \( P \) of \( G \) on \( X \) is defined by
\[ r_{\text{inj}}(P, X) = \inf \{|g|/g \text{ is a hyperbolic element of } P\}. \]

A subgroup of \( G \) is called elementary if it is virtually cyclic. Since \( G \) is a hyperbolic group, any non-elementary subgroup of \( G \) contains a copy of \( \mathbb{F}_2 \), the free group of rank 2 (see [GdlH90, Chap. 8, Theo. 37]). Given a hyperbolic isometry \( g \) of \( X \), the normalizer of \( \langle g \rangle \) is elementary (see [CDP90, Chap. 10, Cor. 7.2]).

The group \( G \) satisfies the small centralizers hypothesis if \( G \) is non-elementary and any elementary subgroup of \( G \) is cyclic. In order to study such a group we define the invariant \( \Delta(G, X) \). It is the upper bound of \( \text{diam} \left( \bigcup_{\rho \neq \tau} Y_{\rho}^{+20\delta} \cap Y_{\tau}^{-20\delta} \right) \), where \( g \) and \( g' \) are two elements of \( G \) which generate a non-elementary subgroup and whose translation lengths are smaller than \( 100\delta \) (see also [DG08]).

### 2.2 Small cancellation theorem

For the rest of Section 2, we assume that \( X \) is simply-connected, and \( G \) satisfies the small centralizers hypothesis. Let \( H \) be a normal subgroup of \( G \) and \( P \) a set of hyperbolic elements of \( H \) stable by conjugation. We also assume that \( P \) only contains a finite number of conjugacy classes. Let \( N \) be the subgroup of \( G \) generated by \( P \). The goal is to study the quotient \( \bar{G} = G/N \). To that end, we use a small cancellation assumption whose statement requires the following objects. Let \( \rho \) be an element of \( P \). We denote by \( Y_{\rho} \) the set of points of \( X \) which are \( 10\delta \)-closed to a geodesic joining \( \rho^- \) and \( \rho^+ \). The set \( Y_{\rho} \) is \( 10\delta \)-quasi-convex (see [Cou09, Lemma 1.2.8]). The subgroup of \( G \) which stabilizes \( Y_{\rho} \) is denoted by \( E_{\rho} \). It is an elementary subgroup of \( G \) (see [CDP90, Chap. 10, Prop. 7.1]). The parameters \( \Delta(P) \) and \( r_{\text{inj}}(P) \), defined below, respectively play the role of the length of the largest piece and the length of the smallest relation in the usual small cancellation theory.

\[
\Delta(P) = \sup_{\rho \neq \tau} \text{diam} \left( \bigcup_{\rho \neq \tau} Y_{\rho}^{+20\delta} \cap Y_{\tau}^{-20\delta} \right)
\]
\[
r_{\text{inj}}(P) = \inf_{\rho \in P} |\rho|^{\infty}
\]

We are interested in situations where the ratios \( \frac{\Delta(P)}{r_{\text{inj}}(P)} \) and \( \frac{\Delta(P)}{r_{\text{inj}}(P)} \) are very small (see Theorem 2.4 below). We construct now a space \( \bar{X} \) on which \( \bar{G} \) acts properly, co-compactly by isometries. Let \( r_0 \) be a positive number. Its value will be made precise in the small cancellation theorem (see Theorem 2.4). Let \( \rho \in P \). We endow \( Y_{\rho} \) with the length metric \( |\cdot|_{\rho} \) induced by the restriction of \( |\cdot|_X \) to \( Y_{\rho} \). The cone over \( Y_{\rho} \) denoted by
construction is a kind of Margulis decomposition. The cones play the role of the thick part: the translation length of a hyperbolic element of \(\nu\) in a Margulis decomposition, we use the fact that the map \(\bar{\nu}\) (see [DG08, Lemme 5.9.3]):

\[
\text{ch} \left( \|y, r\| - \|y', r'\| \right) = \text{ch} r \text{ch} r' - \text{sh} r \text{sh} r' \cos \left( \min \left\{ \pi, \frac{|y - y'|_r}{\text{sh} r_0} \right\} \right).
\]

The cone-off over \(X\) relatively to \(P\), denoted by \(\bar{X}_P(r_0)\) (or simply \(\bar{X}\)) is obtained by attaching, for all \(\rho \in P\), the cone \(C_{\rho}\) on \(\bar{X}\) along \(Y_{\rho}\). The distances \(|\cdot|_X\) and \(|\cdot|_{C_{\rho}}\) induce a metric on \(\bar{X}\) (see [Cou09, Prop. 3.1.7]). We extend by homogeneity the action of \(G\) on \(X\) in an action of \(G\) on \(\bar{X}\): if \(x = (y, r)\) is a point of the cone \(C_{\rho}\) and \(g\) an element of \(G\), then \(\bar{gx}\) is the point of the cone \(C_{\rho+g}\) defined by \(\langle gy, r \rangle\). Thus, \(G\) acts by isometries on \(\bar{X}\) (see [Cou09] Lemma 4.3.1). The metric space \(\bar{X}_P(r_0)\) (or simply \(\bar{X}\)) is the quotient of \(X\) by \(N\). It is a proper, geodesic, simply-connected metric space. Moreover \(G\) acts properly, co-compactly, by isometries on it (see [Cou10 Prop II.3.12]).

**Theorem 2.4** (Small cancellation theorem, see [DG08 Th. 5.5.2] or [Cou09 Th. 4.2.2]). There exist positive numbers \(\delta_0, \delta_1, \Delta_0\) and \(r_0 \geq 10^3 \delta_1\), that do not depend on \(X, G\) or \(P\) such that, if \(\delta \leq \delta_0, \Delta(P) \leq \Delta_0\) and \(r_{\text{maj}}(P) \geq 3 \text{sh} r_0\), then the space \(\bar{X}_P(r_0)\) is \(\delta\)-hyperbolic. In particular, \(G\) is a hyperbolic group.

**Remark**: The fact that the constants \(r_0, \delta_0, \delta_1\) and \(\Delta_0\) do not depend on \(X, P\) or \(G\) is very important in order to iterate the small cancellation construction.

### 2.3 Estimation of the injectivity radius of \(\bar{H}\)

We suppose now that the assumptions of the small cancellation theorem are fulfilled. In order to iterate the construction, we need an estimation of the small cancellation parameters for \(G\). This can be achieved by controlling the constants \(\Delta(\bar{G}, \bar{X})\) and \(r_{\text{maj}}(\bar{H}, \bar{X})\), where \(\bar{H}\) is the image of \(H\) by the projection \(\pi : G \to \bar{G}\). Let \(\nu\) be the canonical map \(\nu : \bar{X} \to \bar{H}\). The space \(\bar{X}\) is obtained by gluing cones of large radius on \(\nu(X)\). This construction is a kind of Margulis decomposition. The cones play the role of the thick part: the translation length of a hyperbolic element of \(G\) on a cone is very large. In particular we have the following lemma.

**Lemma 2.5** (see [DG08 Lemme 5.9.3]). Let \(\bar{y}\) be an element of \(\bar{G}\) such that \(|\bar{y}| \leq 200 \delta_1\). Assume that, for all \(\rho \in P\), \(\bar{y}\) does not belong to \(E_\rho = \pi(E_\rho)\). Then \(A_{\bar{y}}\) is contained in \(\nu(X) + 100 \delta_1\), and \(A_{\bar{y}} \cap \nu(X)\) is non-empty.

To study \(\nu(X) + 100 \delta_1\), which is an analogue of the thin part of the Margulis decomposition, we use the fact that the map \(\nu(X) \to \bar{X}\) is a local quasi-isometry:

**Lemma 2.6** (see [Cou09 Prop. 3.1.8]). Let \(x, x'\) be two points of \(X\). First, \(|x - x'|_X \leq |x - x'|_X\). Moreover, if \(|x - x'|_X \leq \frac{2}{\text{sh} r_0}\), then \(|x - x'|_X \leq \frac{2 \text{sh} r_0}{\text{sh} r_0} |x - x'|_X\).

Using this point of view, T. Delzant and M. Gromov proved the following result.

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Proposition 2.7 (see [DG08, Lemme. 5.10.1]). Let $\tilde{C}$ be a $50\delta_1$-quasi-convex part of $\nu(X)^{+100\delta_1}$. There exists a part $C$ of $X$ having the following properties:

(i) the map $\nu : \tilde{X} \to \tilde{X}$ induces an isometry from $C$ onto $\tilde{C}$,

(ii) the projection $\pi : G \to \tilde{G}$ induces an isomorphism from $\text{Stab}(C)$ onto $\text{Stab}(\tilde{C})$ which are respectively the stabilizers of $C$ and $\tilde{C}$.

Proposition 2.8. The injectivity radius of $\tilde{H}$ on $\tilde{X}$ is bounded below by $\min \{ \kappa l, \delta_1 \}$ where $l$ is the smallest asymptotic translation length of a hyperbolic element of $H$ that does not belong to any $E_\rho$ and $\kappa$ is equal to $\frac{2 \pi r_0}{\rho}$.

Remark : This lemma improves Lemma 5.11.1 proved by T. Delzant and M. Gromov in [DG08]. They gave indeed a lower bound for $r_{\nu_{\rho \epsilon}}(\tilde{G}, \tilde{X})$. For our purpose, we need a more accurate result. We propose here an estimation of the injectivity radius of a normal subgroup of $\hat{G}$.

Proof. Let $m$ be the largest integer such that $m \min \{ \kappa l, \delta_1 \} \leq 40\delta_1$. Let $h$ be a hyperbolic element of $H$. We assume that $[\tilde{h}^m] \leq m \min \{ \kappa l, \delta_1 \} + 40\delta_1$. We denote by $\tilde{C}$, the axis of $\tilde{h}^m$ in $\tilde{X}$, which is $50\delta_1$-quasi-convex (see [DG08, Prop. 2.3.3]). Since $[\tilde{h}^m] \leq 80\delta_1$, the axis $C$ is contained in the $100\delta_1$-neighbourhood of $\nu(X)$ (see Lemma 2.5). By Proposition 2.7 there exists a part $C$ of $X$ such that

(i) the map $\nu : \tilde{X} \to \tilde{X}$ induces an isometry from $C$ onto $\tilde{C}$,

(ii) the map $\pi : G \to G$ induces an isomorphism from $\text{Stab}(C)$ onto $\text{Stab}(\tilde{C})$.

However $\tilde{h}$ belongs to $\text{Stab}(\tilde{C})$. We denote by $h$ its preimage in $\text{Stab}(C)$. Since $h$ is hyperbolic, $\tilde{h}$ is necessarily hyperbolic and does not belong to any $E_\rho$, $\rho \in P$. Note that the relations $P$ are contained in $H$. Thus $N$ lies in $\tilde{H}$. It follows that $h$ is an element of $H$. Hence, by assumption, $[h]^\infty \geq l$.

On the other hand, by Lemma 2.6 the intersection $\tilde{C} \cap \nu(X)$ is non-empty. We chose a point $\tilde{x}$ in $\tilde{C} \cap \nu(X)$ and denote by $x$ its preimage in $C$. The map $\nu : C \to \tilde{C}$ is an equivariant isometry, thus we have

$|h^m x - x|_X = |\tilde{h}^m \tilde{x} - \tilde{x}|_{\tilde{X}} \leq \max \{ [\tilde{h}^m]_1, 40\delta_1 \} \leq 80\delta_1$.

By Lemma 2.6 $|h^m x - x|_X$ is smaller than $\frac{2 \pi r_0}{\rho} |h^m x - x|_X$. Consequently,

$ml \leq m[h]^\infty_X \leq [h^m]_X \leq |h^m x - x|_X \leq \frac{160\pi \delta_1 \sh r_0}{r_0} = \frac{20\delta_1}{\kappa}$.

In particular $m l \leq 20\delta_1$. It follows that $m$ is not the largest integer such that $m \min \{ \kappa l, \delta_1 \} \leq 40\delta_1$. Contradiction. Therefore, $[\tilde{h}^m]$ is larger than $m \min \{ \kappa l, \delta_1 \} + 40\delta_1$. We now use the inequality linking asymptotic translation lengths and translation lengths (see [CDP90, Chap. 10, Prop. 6.4]):

$m[h]^\infty = [\tilde{h}^m]^\infty \geq [\tilde{h}^m] - 32\delta_1 \geq m \min \{ \kappa l, \delta_1 \}$.

This last inequality holds for any hyperbolic element $\tilde{h}$ in $\tilde{H}$. Thus $r_{\nu_{\rho \epsilon}}(\tilde{H}, \tilde{X})$ is larger than $\min \{ \kappa l, \delta_1 \}$.

\hfill $\Box$
2.4 Other properties of $\bar{G}$ and $\bar{X}$

We recall here some results obtained by T. Delzant and M. Gromov in [DG08].

**Proposition 2.9** (see [DG08, Lemme 5.9.5]). The constant $\Delta(\bar{G},\bar{X})$ satisfies the following inequality:

$$\Delta(\bar{G},\bar{X}) \leq \Delta(G,X) + 1000\delta_1 e^{350\delta_1}.$$

**Proposition 2.10** (see [DG08, Lemme 5.10.2 and Lemme 5.10.3]). Assume that any element of $P$ is an odd power of an element of $G$ which is not a proper power. Then $\bar{G}$ satisfies the following properties:

(i) every elementary subgroup of $\bar{G}$ is cyclic,

(ii) let $\bar{F}$ be a finite subgroup of $\bar{G}$. Either $\bar{F}$ is the image of a finite subgroup of $G$, or there exists $\rho \in P$ such that $\bar{F}$ is a subgroup of $\pi(E_{\rho})$.

**Proposition 2.11** (see [DG08, Th. 5.7.1]). The Euler characteristic of $\bar{G}$ satisfies $\chi(\bar{G},Q) = \chi(G,Q) + |P/G|$, where $|P/G|$ denotes the number of conjugacy classes of $P$.

2.5 An induction lemma

The following lemma must be seen as the fundamental step of the induction which will be used to prove the main theorem. We recall that the invariant $\Delta(G,X)$ represents the maximal overlap between the axis of two small hyperbolic elements (see p.9). The injectivity radius $r_{\text{inj}}(H,X)$ denotes the smallest asymptotic translation length of a hyperbolic element of $H$ (see p.9).

**Lemma 2.12** (Induction lemma). There exist positive numbers $\delta_1$, $\Delta_1$, $l_1$, $l_2$, $l_3$ and an integer $n_0$ satisfying the following properties. Let $n$ be an odd integer larger than $n_0$.

(i) $G$ satisfies the small centralizers hypothesis and the order of every finite subgroup of $G$ divides $n$,

(ii) $\Delta(G,X) \leq \Delta_1$ and $r_{\text{inj}}(H,X) \geq \frac{l_2}{\sqrt{n}}$.

We denote by $R$ the set of hyperbolic elements of $H$, which are not a proper powers in $G$ and whose asymptotic translation lengths are smaller than $l_1$. Let $N$ be the normal subgroup of $G$ generated by $\{h^n/h \in R\}$. $G/N$ is connected, simply-connected, $\delta_1$-hyperbolic. Let $G$ be a group acting properly, co-compactly, by isometries on $X$ and $H$ a normal subgroup of $G$ such that

(i) $\delta_1$-hyperbolic space $X$ on which $G$ acts properly, co-compactly, by isometries. Moreover $G$, $H$ and $X$ satisfy the points (i) and (ii). Furthermore, $\Delta(G,X) > 0$ and

$$\forall g \in G, \quad |\pi(g)|_X^N \leq \frac{l_3}{\sqrt{n}} |g|_X^N.$$
Remark: If $G$, $H$, $X$ and $n$ satisfy the hypothesis of the previous lemma we will say that $(G,H,X)$ satisfies the induction assumptions for exponent $n$. The Induction lemma says in particular that if $(G,H,X)$ satisfies the induction assumptions for exponent $n$, so does $(\bar{G},\bar{H},\bar{X})$.

Proof. The positive constants $r_0$, $\delta_0$, $\delta_1$, and $\Delta_0$ are given by the small cancellation theorem (see Theorem 2.4). The constant $\kappa = \frac{r_0^{\log 2}}{\pi \sh r_0}$ is the one that appears in Proposition 2.13. We define a renormalization parameter $L_n = \sqrt{\frac{n\delta_0}{\pi \sh r_0}}$. The sequence $(L_n)$ is increasing and tends to infinity. Up to choose $n_0$ large enough, we may assume that for all $n \geq n_0$,

$$\frac{2000\delta_1 e^{3\Delta_0}}{L_n} + 300\delta_1 \leq \min \left\{ \Delta_0, 1000\delta_1 e^{3\Delta_0} \right\},$$

$$\frac{\delta_1}{L_n} \leq \delta_0,$$

$$\frac{3n\delta_1}{L_n} \leq \delta_1.$$

Note that $n_0$ only depend on $\delta_1$ and $r_0$. We now define the following constants:

$$\Delta_1 = 2000\delta_1 e^{3\Delta_0}, \quad l_1 = 3\delta_1, \quad l_2 = 3\sqrt{n\delta_1} \pi \sh r_0 \quad \text{and} \quad l_3 = \frac{\pi \sh r_0}{n\delta_1}.$$

Let $n$ be an odd integer larger than $n_0$. We assume that $(G,H,X)$ satisfies the induction assumptions for exponent $n$. In particular, $R$ is the set of hyperbolic elements of $H$, which are not a proper powers in $G$ and whose asymptotic translation lengths are smaller than $l_1$.

Lemma 2.13. There exists a subset $R_0$ of $R$, stable by conjugation, such that for all $h \in R$, exactly one of the elements $h$ or $h^{-1}$ belongs to $R_0$.

Proof. It is sufficient to prove that an element $h$ of $R$ cannot be conjugate to its inverse. Assume that this fact is false. There exist $h \in R$ and $g \in G$ such that $ghg^{-1} = h^{-1}$. Therefore, $g$ belongs to the normalizer of $h$, which is elementary (see [CDP90, Chap. 10, Prop. 7.1]). In particular $g$ and $h$ generate an elementary subgroup of $G$. Thus $g$ and $h$ commute. It follows that $h = h^{-1}$. In particular, $h$ is not hyperbolic. Contradiction.

We now study the set of relations $P = \{h^n, h \in R_0\}$. To that end, we consider the action of $G$ on the renormalized space $\frac{1}{L_n} X$, which is $\delta_0$-hyperbolic.

Lemma 2.14. The set $P$ satisfies the small cancellation assumptions of Theorem 2.4.

Proof. First note that $\frac{1}{L_n} X$ is $\delta_0$-hyperbolic. Let $h_1$ and $h_2$ be two elements of $R_0$ such that $h_1^4 \neq h_2^2$. By Proposition 2.3 $Y_{h_1}^{*+20\delta_0}$ is contained in $A_{h_1}^{*+100\delta_0}$. Hence Proposition 2.2 gives

$$\text{diam} \left( Y_{h_1}^{*+20\delta_0} \cap Y_{h_2}^{*+20\delta_0} \right) \leq \text{diam} \left( A_{h_1}^{*+120\delta_0} \cap A_{h_2}^{*+120\delta_0} \right) \leq \text{diam} \left( A_{h_1}^{*+50\delta_0} \cap A_{h_2}^{*+50\delta_0} \right) + 300\delta_0.$$

Assume that $h_1$ and $h_2$ generate an elementary subgroup. Since $G$ satisfies the small centralizers hypothesis, this subgroup should be cyclic. However $h_1$ and $h_2$ are not proper powers. Thus they are either equal or inverse. By construction of $R_0$, they cannot be inverse. Thus $h_1 = h_2$, and $a$
fortxi h^n ∼ h^n. Contraction. Consequently, h_1 and h_2 generate a non-
elementary subgroup. In the other hand, [h_1]_∞ and [h_2]_∞ are smaller
than 3δ_0. By definition of Δ(G, X), we have
\[ \text{diam} \left( Y_{h_1}^{+2000_0} \cap \overline{H^{+2000_0}} \right) \leq \Delta \left( G, \frac{1}{L_n} X \right) + 300 \delta_0 \]
\[ \leq \Delta + 300\delta_1 \frac{e^{350\delta_1}}{L_n} + 300\delta_1. \]
Hence Δ(P) is smaller than Δ₀.

The injectivity radius of H on \( \frac{1}{L_n} X \) is larger than
\[ \frac{1}{L_n} \frac{l_2}{\sqrt{n}} = \frac{\pi \text{sh} r_0}{n \delta_1} \frac{3 \sqrt{n} \delta_1 \pi \text{sh} r_0}{\sqrt{n}} = \frac{3 \pi \text{sh} r_0}{n}. \]
In particular for all \( h \in R_0 \), \([h^n]_∞ \geq 3 \pi \text{sh} r_0 \). Therefore, \( r_{\text{inj}} (P) \geq 3 \pi \text{sh} r_0 \).

Applying the small cancellation theorem, the space \( \tilde{X} = \tilde{X}_P (r_0) \) is
proper, geodesic, simply-connected, δ_1-hyperbolic and \( \tilde{G} = G/\ll P \gg \)
acts properly, co-compactly, by isometries on it.

**Lemma 2.15.** Every elementary subgroup of \( \tilde{G} \) is cyclic, either infinite
or finite with order dividing \( n \).

**Proof.** All elements of \( P \) are odd powers of elements of \( G \) which are not
proper powers. By Proposition 2.10, all elementary subgroups of \( \tilde{G} \) are cyclic. Assume now, that \( \tilde{F} \) is a finite subgroup of \( \tilde{G} \). Applying the same
proposition, we distinguish two cases.

(i) \( \tilde{F} \) is the image of a finite subgroup of \( G \). However, the order of every
finite subgroup of \( G \) divides \( n \). Thus the order of \( \tilde{F} \) divides \( n \).

(ii) There exists \( h \in R_0 \) such that \( \tilde{F} \) is a subgroup of \( \tilde{E}_h^n = \pi (E_h^n) = \langle \pi (h) \rangle \), whose order divides \( n \).

**Lemma 2.16.** The constant \( \Delta (\tilde{G}, \tilde{X}) \) is bounded above by \( \Delta_1 \). The
injectivity radius \( r_{\text{inj}} (H, \tilde{X}) \) is bounded below by \( \frac{l_2}{\sqrt{n}} \).

**Proof.** By Proposition 2.10, \( \Delta (\tilde{G}, \tilde{X}) \leq \Delta \left( G, \frac{1}{L_n} X \right) + 1000 \delta_1 e^{350\delta_1} \). However, we assumed that
\[ \Delta \left( G, \frac{1}{L_n} X \right) = \frac{1}{L_n} \Delta (G, X) \leq \frac{\Delta_1}{L_n} = \frac{2000 \delta_1 e^{350\delta_1}}{L_n} \leq 1000 \delta_1 e^{350\delta_1}. \]
Hence \( \Delta (\tilde{G}, \tilde{X}) \leq 2000 \delta_1 e^{350\delta_1} = \Delta_1 \).

Let \( g \) be a hyperbolic element of \( H \), which does not belong to any
subgroup \( E_h^n = \langle h \rangle, h \in R_0 \). Its asymptotic translation length in \( \frac{1}{L_n} X \)
is larger than \( \frac{l_2}{L_n} = \frac{3 \delta_1}{L_n} \). By Proposition 2.8, we have
\[ r_{\text{inj}} (H, \tilde{X}) \geq \min \left\{ \frac{3 \delta_1}{L_n}, \frac{l_2}{\sqrt{n}} \right\} = \frac{3 \delta_1}{L_n} = \frac{3 \sqrt{n} \delta_1 \pi \text{sh} r_0}{\sqrt{n}} = \frac{l_2}{\sqrt{n}}. \]
Lemma 2.17. The Euler characteristic of \( \bar{G} \) satisfies \( \chi(\bar{G}, \mathbb{Q}) = \chi(G, \mathbb{Q}) + \frac{1}{2} |R/G| \), where \( |R/G| \) is the number of conjugacy classes of \( R \). In particular \( \bar{G} \) is non-elementary.

Proof. By construction, there are twice more conjugacy classes in \( R \) than in \( R_0 \). The result follows from Proposition 2.11.

Lemma 2.18. For all \( g \in G \), we have \( \| \pi(g) \|^\infty_X \leq \sqrt{\frac{\pi \text{sh} r_0 n}{n \delta_1}} \| g \|^\infty_X \).

Proof. By Lemma 2.12, the map \( \frac{1}{L_n} X \rightarrow \bar{X} \) contracts the distances. Thus for all \( g \in G \),

\[
\| \pi(g) \|^\infty_X \leq \frac{1}{L_n} \| g \|^\infty_X = \sqrt{\frac{\pi \text{sh} r_0 n}{n \delta_1}} \| g \|^\infty_X.
\]

This last lemma ends the proof of the induction lemma.

3 Proof of the main theorem

Recall the statement of the main theorem.

Theorem 3.1. Let \( 1 \rightarrow H \rightarrow G \rightarrow F \rightarrow 1 \) be a short exact sequence of groups. Assume that \( H \) is finitely generated, \( G \) is torsion-free, hyperbolic and \( F \) is torsion-free. Then there exists an integer \( n_0 \) such that for all odd integers \( n \) larger than \( n_0 \), the canonical map \( F \rightarrow \text{Out}(H) \) induces an injective homomorphism \( F \hookrightarrow \text{Out}(H/H_n) \).

Proof. The constants \( \delta_1, \Delta_1, l_1, l_2, l_3 \) and \( n_0 \) are given by the Induction lemma (see Lemma 2.12). Up to increase \( n_0 \), we may also assume that \( \frac{l_3}{\sqrt{n_0}} < 1 \). Let \( 1 \rightarrow H \rightarrow G \rightarrow F \rightarrow 1 \) be a short exact sequence of groups, which satisfies the hypotheses of the theorem. The strategy is to build by induction a family of short exact sequences \( 1 \rightarrow H_k \rightarrow G_k \rightarrow F \rightarrow 1 \) with an action of \( G_k \) on a hyperbolic space \( X_k \), such that the direct limit \( H_\infty ) \) is the Burnside group \( H/H^n \).

Initialization. We put \( H_0 = H \), and \( G_0 = G \). Let \( X_0 \) be a proper, geodesic, simply-connected, hyperbolic space on which \( G \) acts properly, co-compactly, by isometries. Take for instance the Rips’ polyhedron of \( G \) (see [CDP90 Chap 5]). Up to renormalize \( X_0 \), we may assume that:

- \( X_0 \) is \( \delta_1 \)-hyperbolic,
- \( \Delta(G_0, X_0) \leq \Delta_1 \),
- \( \{ h \in H/[h]^\infty \leq l_1, h \) is not a proper power \( \} \) contains a number of conjugacy classes bounded below by \( 2\chi(G, \mathbb{Q}) \).

Since \( G \) is a hyperbolic group, the injectivity radius of \( H \) is positive (see [Del96]). Thus, up to increase one more time \( n_0 \), we may assume that \( r_{n_0}(H_0, X_0) \geq \frac{l_2}{\sqrt{n_0}} \). It follows that \( (G_0, H_0, X_0) \) satisfies the induction assumptions for exponent \( n_0 \).

Let \( n \) be an odd integer larger than \( n_0 \). \( (G_0, H_0, X_0) \) satisfies a fortiori the induction assumptions for exponent \( n \).
**Induction.** Let \((G_k, H_k, X_k)\) satisfying the induction assumptions for exponent \(n\). We denote by \(R_k\) the set of hyperbolic elements of \(H_k\) which are not proper powers in \(G_k\) and whose asymptotic translation lengths are smaller than \(l_1\). Let \(N_k\) be the normal subgroup of \(G_k\) generated by \(\{h^n/h \in R_k\}\), \(G_{k+1}\) the quotient \(G_k/N_k\) and \(H_{k+1}\) the image of \(H_k\) by the canonical map \(\pi_k : G_k \to G_{k+1}\). By the Induction lemma, there exists a metric space \(X_{k+1}\) such that \((G_{k+1}, H_{k+1}, X_{k+1})\) satisfies the induction assumptions for the exponent \(n\). In this way, we obtain two sequences of groups \((H_k)\) and \((G_k)\) whose properties we want to study now.

**Properties of \(H_k\) and \(G_k\).**

**Lemma 3.2.** For all integers \(k\), there exists a map \(G_k \to F\) such that the following diagram is commutative. Moreover its rows are short exact sequences.

\[
\begin{array}{cccc}
1 & \longrightarrow & H & \longrightarrow & G & \longrightarrow & F & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 & \longrightarrow & H_k & \longrightarrow & G_k & \longrightarrow & F & \longrightarrow & 1
\end{array}
\]

**Proof.** Following the construction of the groups \(H_k\) and \(G_k\), we prove this lemma by induction on \(k\). The result is obvious for \(k = 0\). Consider now an integer \(k\) for which the lemma holds. The subgroup \(N_k\) is generated by elements of \(H_k\). It follows that \(N_k\) is contained in \(H_k\), which is also the kernel of the map \(G_k \to F\). Hence \(G_k \to F\) induces a map from \(G_{k+1} = G_k/N_k\) to \(F\) such that the following diagram is commutative.

\[
\begin{array}{ccc}
G_k & \longrightarrow & F \\
\downarrow & & \downarrow \\
G_{k+1} & \longrightarrow & F
\end{array}
\]

By definition, \(H_{k+1}\) is the image of \(H_k\) by the projection \(\pi_k\). Since \(\pi_k\) is onto, \(H_{k+1}\) is the kernel of the map \(G_{k+1} \to F\). Consequently, the following diagram commutes and its rows are short exact sequences.

\[
\begin{array}{cccc}
1 & \longrightarrow & H_k & \longrightarrow & G_k & \longrightarrow & F & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 & \longrightarrow & H_{k+1} & \longrightarrow & G_{k+1} & \longrightarrow & F & \longrightarrow & 1
\end{array}
\]

Thus the lemma holds for \(k + 1\). \(\Box\)

We would like now to compare the groups \(H/H^n\) and \(\lim \limits_{\to} H_k\). For notational convenience, we will denote by \(h\) an element of \(H\) as well as its images in \(H_k\), \(\lim \limits_{\to} H_k\) or \(H/H^n\).

**Lemma 3.3.** The kernel of the canonical map \(H \to \lim \limits_{\to} H_k\) is exactly \(H^n\), the subgroup of \(H\) generated by all \(n\)-th powers.

**Proof.** Let \(h\) be an element of \(H \setminus \{1\}\). For all integers \(k\), we have \(|h|_{X_k}^\infty \leq \left(\frac{l_1}{\sqrt[3]{l_0}}\right)^k |h|_{X_0}^\infty \leq \left(\frac{l_1}{\sqrt[3]{l_0}}\right)^k |h|_{X_0}^\infty\) (see Lemma 2.12). However, we chose \(n_0\) in such a way that \(\frac{l_1}{\sqrt[3]{l_0}} < 1\). It follows that there exists an integer \(k\) such
that $[h]_{X_k}^\infty < \frac{1}{\sqrt{r}}$. The group $G_k$ satisfies the small centralizers hypothesis (i.e. $G_k$ is non-elementary and its elementary subgroups are cyclic). Thus there exists an element $r$ of $G_k$, which is not a proper power, and a positive integer $m$ such that $h = r^m$. In particular $r^m$ belongs to the kernel of the map $G_k \to F$. Since $F$ is torsion-free, $r$ also belongs to $H_k$. Note that its asymptotic length in $X_k$ is smaller than $\frac{1}{\sqrt{r}}$. By construction, the injectivity radius of $H_k$ on $X_k$ is larger than $\frac{1}{\sqrt{r}}$ (point (ii) of Lemma 3.4). Therefore $r$ is an elliptic isometry. In particular, it has finite order dividing $n$ (point (i) of Lemma 2.12). It follows that the image of $h^n$ in $H_k$ is trivial. Hence $H^n$ is contained in the kernel of $H \to \lim_{\to} H_k$.

On the other hand, at each step of the construction, the kernel of the map $H_k \to H_{k+1}$ is generated by $n$-th powers of elements of $H_k$. It follows that the kernel of the morphism $H \to \lim_{\to} H_k$ is contained in $H^n$.

Lemma 3.4. The groups $H/H^n$ and $\lim_{\to} H_k$ are isomorphic.

Proof. The map $H \to \lim_{\to} H_k$ is onto. Thanks to Lemma 3.3 its kernel is $H^n$. It follows that it induces an isomorphism between $H/H^n$ and $\lim_{\to} H_k$.

Lemma 3.5. Let $f$ be a non trivial element of $F$. Let $g$ be a preimage of $f$ by the map $G \to F$. The conjugation by $g$ defines an automorphism of $H$ which induces a non trivial outer automorphism of $H/H^n$.

Proof. Let $S$ be a finite generating set of $H$. We denote by $\varphi$ the automorphism of $H$ defined as follows: for all $h \in H$, $\varphi(h) = g h g^{-1}$. Assume that $\varphi$ induces an inner automorphism of $H/H^n$. There exists $l \in H$ such that for all $h \in H$, $\varphi(h)$ and $l h l^{-1}$ have the same image in $H/H^n$. We proved previously that $H/H^n$ and $\lim_{\to} H_k$ are isomorphic (see Lemma 3.4). Since $S$ is finite, there exists an integer $k$ such that, for all $s \in S$, $\varphi(s)$ and $l s l^{-1}$ are equal in $H_k$. However $S$ is a generating set of $H$. Thus for all $h \in H$, $\varphi(h) = g h g^{-1}$ and $l h l^{-1}$ are equal in $H_k$. We use now the following commutative diagram (see Lemma 2.2).

$$
\begin{array}{c}
1 & \longrightarrow & H & \longrightarrow & G & \longrightarrow & F & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \\
1 & \longrightarrow & H_k & \longrightarrow & G_k & \longrightarrow & F & \longrightarrow & 1
\end{array}
$$

The image of $l^{-1} g$ in $G_k$ commutes with every element of $H_k$. The subgroup $H_k$ is a normal subgroup of $G_k$ which satisfies the small centralizers hypothesis. Therefore, $H_k$ is non-elementary. In particular, it contains a hyperbolic element $h$. Hence $h$ and $l^{-1} g$ generates an abelian subgroup of $G_k$ which has to be cyclic. There exists $(p, q) \in \mathbb{Z}^* \times \mathbb{Z}$ such that $(l^{-1} g)^p = h^q$ in $G_k$. Using one more time the commutative diagram, we push this identity in $F$ and obtain $f^p = 1$. Since $F$ is torsion-free, $f$ is trivial. Contradiction.

End of the proof of the main theorem. The map $F \to \text{Out}(H)$ can be constructed as follows. Let $f$ be an element of $F$ and $g$ a preimage of $f$ by $G \to F$. The image of $f$ by the map $F \to \text{Out}(H)$ is exactly the outer automorphism of $H$ induced by the conjugation by $g$ in $G$. The previous lemma is hence a reformulation of the following fact: the map
$F \to \text{Out}(H)$ induces an injective homomorphism $F \hookrightarrow \text{Out}(H/H^n)$. This remark ends the proof of the main theorem.

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