Non-orthogonal preferred projectors for modal interpretations of quantum mechanics

R. W. Spekkens and J. E. Sipe

Department of Physics, University of Toronto, 60 St. George Street, Toronto, Ontario, Canada M5S 1A7

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Modal interpretations constitute a particular approach to associating dynamical variables with physical systems in quantum mechanics. Given the ‘quantum logical’ constraints that are typically adopted by such interpretations, only certain sets of variables can be taken to be simultaneously definite-valued, and only certain sets of values can be ascribed to these variables at a given time. Moreover, each allowable set of variables and values can be uniquely specified by a single ‘preferred’ projector in the Hilbert space associated with the system. In general, the preferred projector can be one of several possibilities at a given time. In previous modal interpretations, the different possible preferred projectors have formed an orthogonal set. This paper investigates the consequences of adopting a non-orthogonal set. We present three contributions on this issue: (1) we provide an argument for such non-orthogonality, based on the assumption that perfectly predictable measurements reveal pre-existing values of variables, an assumption which has traditionally constituted a strong motivation for the modal approach; (2) we generalize the existing framework for modal interpretations to accommodate non-orthogonal preferred projectors; (3) we present a novel type of modal interpretation wherein the set of preferred projectors is fixed by a principle of entropy minimization, and we discuss some of the successes and shortcomings of this proposal.

I. INTRODUCTION

In operational quantum mechanics, theoretical predictions take the form ‘if such-and-such a measurement is made after such-and-such a preparation, such-and-such an outcome will be found with such-and-such a probability’. In contrast, a realist interpretation is an attempt to understand quantum mechanics as making stronger claims of the form ‘such-and-such a variable has such-and-such a value with such-and-such a probability’. The ‘elements of reality’ of Einstein, Podolsky and Rosen [1], and Bell’s ‘be-ables’ [2] are two ways of referring to the variables that possess definite values in a realist interpretation. We will simply refer to them as the determinate variables. Since the set of determinate variables in some sense specifies ‘what exists’, we call it the ontology for the system. The specification of the values of the determinate variables will be called the value ascription to the ontology. Within this approach, one assigns a property to a system by assigning a value to a determinate variable. Since the ontology and the value ascription together constitute a complete specification of the properties of a quantum system, they will jointly be referred to as the property ascription.

Within ‘orthodox’ interpretations of this type, a variable is determinate if and only if it is associated with an operator for which the state vector is an eigenstate, and its value is the corresponding eigenvalue. It is also assumed that a variable $V$ defined on a subsystem is determinate only if $V \otimes I$ is determinate on the total system, where $I$ is the identity operator for the part of the total system that is not included in the subsystem. According to these rules, the ontologies and the value ascriptions for all systems are uniquely defined by the state vector. It is widely recognized that this view, together with the assumption that the evolution of the state vector is unitary for all time, leads to the quantum measurement problem, namely, the failure to ensure the determinateness of macroscopic variables such as the pointer reading of an apparatus [3].

One approach to the problem is to introduce a non-unitary dynamics for the state vector into the formalism of the theory (the ‘collapse’ of the state vector). A different approach is to preserve the unitary dynamics, but to reject the notion that a variable is determinate only if it has the state vector as an eigenstate of the associated operator. In the latter type of approach the property ascription need not be fixed at a given time by the state vector. Rather, it may be that the state vector describes only the set of possible property ascriptions, in which case it describes what is possible and what, if anything, is necessary. Since the logic of possibility and necessity is modal logic, realist no-collapse interpretations of this type have been called modal interpretations of quantum mechanics [4].

Modal interpretations typically impose many constraints on the form of the property ascription for a system. Given these constraints, there is always a unique ‘most elementary’ possessed property defined by a property ascription. We call this the preferred property for that property ascription. Since at a given time the property ascription may be one of several possibilities, each of which define a different preferred property, there is in general a set of preferred properties associated with a system. An example may serve to clarify these concepts. Suppose the system is a digital display on an apparatus. The property ascription for the display may include such properties as ‘the digital display shows a number between 1 and 3’, ‘the digital display shows a number smaller than 5’, etc., while the preferred
property may be ‘the digital display shows the number 2’. The set of preferred properties may consist of a list of properties each of the form ‘the digital display shows the number k’, but differing in the value of k. In previous modal interpretations, the preferred properties have been associated with orthogonal projectors.

There are three contributions made in this paper. First, we demonstrate that any modal interpretation which adopts the standard constraints upon the property ascription and which seeks to satisfy a particular criterion of faithful measurement must allow for the set of preferred properties to be associated with non-orthogonal projectors. Second, we introduce a framework for modal interpretations that incorporates such preferred properties. Third, we present a novel proposal within this framework wherein the preferred properties are fixed by a principle of entropy minimization.

The paper is organized as follows. In section 2, we present a review of the constraints upon the property ascription that are standard among modal interpretations, and we provide a rigorous definition of the notion of a preferred property. In section 3, we present the argument that the preferred properties must be associated with non-orthogonal projectors if one hopes to explain the outcomes of perfectly predictable measurements in terms of pre-existing properties of the system under investigation, that is, if perfectly predictable measurements are to be faithful. The argument relies on a particular kind of experiment, involving a sequence of two measurements which have the following critical features: (1) the first measurement disturbs the state of the system differently for different outcomes, resulting in the preparation of non-orthogonal states; and (2) the variable measured by the second device depends on the outcome of the first measurement in such a way that the outcome of the second measurement is always perfectly predictable.

In order to accommodate non-orthogonal preferred properties, we require a new framework for modal interpretations, which is the subject of section 4. We preserve most of the standard constraints on the property ascription, in particular constraints involving the functional relations between the values of variables. However, we show that one must abandon the assumption that different property ascriptions share a common ontology. We assume Healey’s so-called ‘weakening condition’, and adopt Clifton’s rule for relating the properties of composites to the properties of the subsystems of which they are composed. Moreover, we follow previous authors in requiring that the dynamics of the property ascription be Markovian and satisfy certain constraints of analyticity, while also reproducing the standard quantum statistics. Guided by these constraints, we introduce a framework for modal interpretations that incorporates non-orthogonal preferred projectors. This constitutes a generalization of the framework introduced by Bub and Clifton.

With this framework in hand, we proceed in section 5 to present a novel proposal for a modal interpretation. We begin by assuming that there is a distinguished division of the universe into elementary subsystems, or equivalently a distinguished factorization of the total Hilbert space. A preferred decomposition of the state vector is singled out by the minimization of a particular entropic quantity that quantifies the degree of entanglement of the state vector with respect to the distinguished factorization. The actual property ascription is assumed to be fixed by a single element of this decomposition, whose identity evolves by a stochastic process with specified statistical properties. Within this proposal, we demonstrate that the quantum measurement problem is avoided for several models of measurement interactions, and a large class of perfectly predictable measurements are shown to reveal pre-existing properties. In section 6, we present our concluding remarks.

II. THE MODAL APPROACH

A. Review of Constraints

We begin by considering the notion of a property of a physical system. The type of properties in which we are interested are those of the form ‘having a value of the variable V in the range Δ’. For every such property, one can associate an idempotent variable that has value 1 if the value of V is in the range Δ and 0 if the value lies outside this range. Whether a property is possessed or not is given by the value of this idempotent variable; it is possessed if the value is 1, and it is not possessed if the value is 0.

In classical mechanics, a variable is represented by a function on phase space, and the possible values of this variable are just the values in the range of this function. The property of ‘having a value of the variable V in the range Δ’ is associated with the subset of phase space containing all points for which V is in the range Δ. For instance, if the system is a one-dimensional harmonic oscillator, the property ‘having energy between E₁ and E₂’ is associated with an elliptical ring in phase space, while the property ‘having position between x₁ and x₂’ is associated with a vertical band. Suppose the property s is associated with a subset Ω of phase space. The idempotent variable associated with the property s is the function FΩ that takes the value 1 for every point in Ω and 0 for every point outside Ω. If two properties s and s’ are represented by subsets Ω and Ω’ of phase space, then the disjunction of s and s’ is represented by the union of Ω and Ω’, the conjunction of s and s’ is represented by the intersection of Ω and Ω’, and the negation
of \( s \) is represented by the complement of \( \Omega \). The truth tables appropriate for conjunction, disjunction and negation in classical logic place the following constraints on the values of the idempotent variables:

\[
F_{\Omega \cup \Omega'} = F_{\Omega} + F_{\Omega'} - F_{\Omega}F_{\Omega'}
\]
\[
F_{\Omega' \cap \Omega'} = F_{\Omega}F_{\Omega'}
\]
\[
F_{\Omega'} = 1 - F_{\Omega},
\]

where \( \cup, \cap \) and an over-bar respectively denote union, intersection and complementation of subsets of phase space.

We now consider an approach to realist interpretations of quantum mechanics that parallels those features of classical theories described above, except of course that the mathematical structure relevant for the description of a system is no longer phase space but Hilbert space. This approach has its origin in the field of quantum logic \([1]\), and is adopted by most modal interpretations. A variable is represented by a Hermitian operator over the Hilbert space, and its possible values are the eigenvalues of this operator. Properties are associated with subspaces of Hilbert space. The idempotent variable associated with a given property is represented by the projector onto the corresponding subspace. If two properties \( s \) and \( s' \) are represented by subspaces \( S \) and \( S' \), then the disjunction of \( s \) and \( s' \) is represented by the linear span (direct sum) of \( S \) and \( S' \), the conjunction of \( s \) and \( s' \) is represented by the intersection of \( S \) and \( S' \), and the negation of \( s \) is represented by the orthogonal complement of \( S \). These assumptions will be called the constraints on logical connectives.

For simplicity, we denote both the projector onto the subspace \( S \) and the associated idempotent variable by \( P_S \), and denote the value of this idempotent variable by \([P_S]\). Analogously to the classical case, we adopt the following constraint on the values of the idempotent variables:

**Functional relation constraint**

\[
[P_{S \oplus S'}] = [P_S] + [P_{S'}] - [P_S][P_{S'}]
\]
\[
[P_{S \cap S'}] = [P_S][P_{S'}]
\]
\[
[P_{S \perp}] = 1 - [P_S]
\]

where \( \oplus, \cap \) and \( \perp \) denote respectively linear span, intersection, and orthogonal complement.

As it turns out, it is impossible to associate values with all the projectors in a Hilbert space in a way that is consistent with the functional relation constraint and the constraints on logical connectives \([2]\). The response of modal interpretations is to associate definite values with only a subset of all the projectors in the Hilbert space. Thus, in contrast with classical mechanics, only a subset of all idempotent variables correspond to well-defined properties at any given time. The projectors that are associated with definite values are labelled determinate, as are the corresponding idempotent variables. The functional relation constraint is only required to hold among the determinate projectors.

As regards non-idempotent variables, we adopt the convention that a variable is denoted by the same symbol as the associated hermitian operator, for instance \( V \), and that its value is denoted by \([V]\). Moreover, following other modal interpreters \([3]\), we adopt the attitude that if the spectral resolution of a Hermitian operator is \( V = \sum_k v_k P_{S_k} \), where \( v_k \neq 0 \) for all \( k \), and \( v_k \neq v_{k'} \) for \( k \neq k' \), then \( V \) is determinate if and only if all of the projectors in the set \( \{P_{S_k}\}_k \) are determinate, and in this case its value is \([V] = \sum_k v_k[P_{S_k}]\). We call this the spectral constraint.

It follows from this constraint that the set of determinate idempotent variables and their values are sufficient to specify the set of all determinate variables and their values. As noted in the introduction, we will refer to the set of determinate variables as the ontology, and the values of these variables as the value ascription. The ontology and the value ascription together define the property ascription.

We turn now to constraints on the nature of the ontology. It is typically assumed that logical combinations of well-defined properties are also well-defined. Thus, if property \( s \) is well-defined, then so too should be the property ‘not \( s \)’. In other words, if \( P_S \) is determinate, then \( P_{S'} = (I - P_S) \) should be determinate as well. Similarly, if properties \( s \) and \( s' \) are well-defined, so that the associated projectors \( P_S \) and \( P_{S'} \) are determinate, then the properties ‘\( s \) or \( s' \)’ and ‘\( s \) and \( s' \)’ should also be well-defined, and the associated projectors \( P_{S \oplus S'} \) and \( P_{S \cap S'} \) should be determinate. In summary, we require

**Closure constraint:**

If \( P_S \in \text{Ont} \) then \( P_{S'} \in \text{Ont} \)

If \( \{P_S, P_{S'}\} \in \text{Ont} \) then \( \{P_{S \oplus S'}, P_{S \cap S'}\} \in \text{Ont} \).

where \( \text{Ont} \) denotes the ontology. Assuming that for every system there is at least one projector that is determinate, it follows from the closure constraint that the identity operator and the projector onto the null space are determinate for every system.

We refer to all of the constraints on the ontology and value ascription that have been presented thus far as the algebraic constraints.
B. Preferred Projectors

We now derive a few consequences of the algebraic constraints. For this purpose, it is useful to introduce a notational convenience: ‘$P_S > P_{S'}$’ denotes that $S'$ is a subspace of $S$, and ‘$P_S \perp P_{S'}$’ denotes that $S$ and $S'$ are orthogonal subspaces. First, we show that in every property ascription there is a projector $P_S$ that receives the value 1, and for which no projector onto a proper subspace of $S$ receives the value 1; that is, there is a subspace $S$ such that

$$[P_S] = 1 \text{ and there is no subspace } T \text{ such that } P_T < P_S \text{ and } [P_T] = 1.$$  \hspace{1cm} (4)

Such a subspace always exists since at least one projector, namely the identity operator, always receives the value 1. Such a subspace is unique because if $S$ and $S'$ were two distinct subspaces satisfying this definition, then both would receive the value 1, and by the functional relation constraint their intersection would also receive the value 1, which implies that either $S$ or $S'$ has a proper subspace that receives the value 1. We call the unique projector satisfying Eq. (4) the preferred projector for the property ascription. The property associated with this projector is also called preferred.

A modal interpretation must account for the fact that a measurement device may have one of several different properties at the end of a measurement despite there being a single state vector for the universe. This is accomplished by assuming that the property ascription is not fixed by the state vector, but may be one of several possibilities at a given time. There are two notions of possibility that are adopted by modal interpreters in this context. In the first, the different possibilities for the final property ascription to the measurement device are attributed to differences in the initial properties (possibly hidden) of the system. In the second, the different possibilities for the final property ascription to the measurement device arise from an objective stochasticity in the evolution of the property state. Although we adopt the latter view in subsequent sections, for the present it suffices to note that in all modal interpretations one associates with a system, at every time, one out of a set of several possible property ascriptions. Since each such property ascription defines a preferred projector, there is in general a set of preferred projectors associated with a system at a given time. We now consider the relation between the elements of this set.

We begin with a definition: two properties are said to be mutually exclusive if their conjunction is a contradiction. This is stronger than simply being distinct, as is illustrated by the properties ‘red’ and ‘red or blue’, which are distinct, but not mutually exclusive. We also use the term ‘mutually exclusive’ to describe two property ascriptions if there is no subspace onto which two different projectors receive the value 1, but such that the intersection of one is mutually exclusive to a property obtaining in the other. We assume that for all systems at all times, the different possible property ascriptions are mutually exclusive.

This assumption can be recast as a constraint upon the set of preferred projectors. Recalling the constraints on logical connectives, for two property ascriptions to be mutually exclusive there must be two projectors $P_R$ and $P_{R'}$, such that $[P_R] = 1$ in the first property ascription, and $[P_{R'}] = 1$ in the second, but such that the intersection of $R$ and $R'$ is the null space. By definition, the preferred projector for the first property ascription, call it $P_S$, must be such that $P_S \leq P_R$, and the preferred projector for the second, call it $P_{S'}$, must be such that $P_{S'} \leq P_{R'}$, from which it follows that the intersection of $S$ and $S'$ must also be the null space. Thus we conclude that the preferred projectors for mutually exclusive properties are orthogonal. Indeed, the possibility of preferred projectors that are non-orthogonal will be the focus of much of this paper.

III. NON-ORTHOGONAL PREFERRED PROJECTORS

A. The Faithfulness Criterion

One can debate the merits of assuming orthogonal preferred projectors in the context of a macroscopic system such as the pointer on a measurement device. On the one hand, the distinguishability of the different physical states of a pointer suggest that they must be associated with orthogonal projectors; on the other hand, the requirement of quantum-classical correspondence suggests that the alternative positions of a pointer should be associated with

\footnote{An example of the first approach is the Bub-Clifton interpretation when the preferred variable has a continuous unbounded spectrum, and the dynamics is given by a guidance equation analogous to the one used in Bohmian mechanics. An example of the second approach is the Bub-Clifton interpretation when one adopts a different guidance equation, or when the preferred variable has a discrete spectrum. See, for instance, section 5.2 of Ref. [4].}
consider a microscopic system that is the object of investigation in a quantum measurement. We do provide an argument for adopting a non-orthogonal set of preferred projectors, but it appeals to the properties that should be assigned to \textit{microscopic} rather than macroscopic systems. In particular, we consider a microscopic system that is the object of investigation in a quantum measurement.

Since operational quantum mechanics only makes reference to the properties of macroscopic preparation and measurement devices, the requirement of agreement with the operational theory does not by itself constrain the properties that are assigned to the microscopic systems under investigation. But if the properties of such microscopic systems are to play any explanatory role in the theory, then one would expect their role to be in determining the outcomes of measurements upon them. In particular, we consider the following criterion for assigning determinate status to a variable:

\textbf{Faithfulness Criterion} If it can be predicted with probability 1 that a measurement of the variable \( V \) will yield the result \( v \), then immediately prior to the measurement the variable \( V \) is determinate with value \( v \).

The motivation for adopting this criterion is presented in the next subsection. It is nonetheless worth emphasizing at this point its importance in this paper: the particular form of the framework for modal interpretations that is presented in section 4 and the particular proposal presented in section 5 are both to a large extent attempts to satisfy the faithfulness criterion.

This criterion is applicable to experiments involving a sequence of measurements, the first of which may be considered a preparation. Within the context of operational quantum mechanics, we consider a sequence of two measurements, associated with distinct Hermitian operators, \( V \) and \( V' \), both belonging to the Hilbert space \( \mathcal{H}^S \). For simplicity, we assume that these operators have non-degenerate eigenvalues, denoted by \( \{ v_k \}_{k=1}^m \) and \( \{ v'_j \}_{j=1}^m \) respectively, and eigenvectors denoted by \( \{ |\varphi_k\rangle \}_{k=1}^m \) and \( \{ |\varphi'_j\rangle \}_{j=1}^m \) respectively. In order to predict the outcome of the second measurement, it is necessary to also specify how, if at all, the first measurement disturbs the state. Suppose then that upon obtaining outcome \( k \) for the first measurement, the state \( |\tilde{\varphi}_k\rangle \) is prepared, and that the set of vectors \( \{ |\tilde{\varphi}_k\rangle \}_{k=1}^m \), although normalized and non-collinear, are non-orthogonal. Since it follows that \( |\tilde{\varphi}_k\rangle \neq |\varphi_k\rangle \) for one or more values of \( k \), we say that the measurement is \textit{disturbing}. For simplicity, the second measurement is assumed to be non-disturbing, and the two measurements are assumed to be immediately consecutive. Finally, we take the preparation procedure that precedes the first measurement to be associated with a state vector \( \sum_k c_k |\varphi_k\rangle \), where \( \sum_k |c_k|^2 = 1 \), and \( c_k \neq 0 \) for all \( k \).

Operational quantum mechanics predicts, via the generalized Born rule, that the probability of the second apparatus indicating outcome \( j \) given that the first apparatus indicates outcome \( k \) is

\[
\text{Prob}(j|k) = |\langle \varphi_k | \varphi'_j \rangle|^2.
\]

It follows that in order for the outcome of the second measurement to be predictable with probability 1 given the outcome of the first measurement, it must be the case that \( \langle \varphi_k | \varphi'_j \rangle = 1 \) for some values of \( k \) and \( j \). Hence, the variable \( V' \) measured by the second apparatus must have at least one of the states in the set \( \{ |\varphi_k\rangle \}_{k=1}^m \) as an eigenstate. We define a set of variables \( \{ V_k \}_{k=1}^m \), such that the variable \( V_k \) has \( |\tilde{\varphi}_k\rangle \) as an eigenstate. In particular, we denote the eigenvalues of \( V_k \) by \( \{ v(k)_j \}_{j=1}^m \), and the associated eigenvectors by \( \{ |\varphi(k)_j\rangle \}_{j=1}^m \), and take \( |\varphi(k)_j\rangle \equiv |\tilde{\varphi}_k\rangle \). It then follows that if the first measurement has outcome \( k \), so that \( |\tilde{\varphi}_k\rangle \) is prepared, and if \( V' = V(k) \), that is, the second apparatus measures \( V(k) \), then with probability 1, the second measurement has the outcome 1. Hence the faithfulness criterion is applicable in this case, and implies that \( V(k) \) is determinate with value \( v(k)_1 \) at time \( t \), immediately prior to the second measurement.

We now introduce a critical assumption about the sequence of measurements: the nature of the second measurement is taken to depend on the outcome of the first. In particular, we imagine a set-up where if the outcome of the first measurement is \( k \), then the second apparatus measures the variable \( V(k) \); we imagine that this is done mechanically by the measurement apparatus, without the intervention of a physicist. In this case, the faithfulness criterion is applicable for all possible outcomes of the first measurement.

We now show that for the faithfulness criterion to be satisfied for such a sequence of measurements, the preferred projectors must be non-orthogonal. Since there is a non-zero probability for the first measurement to have the outcome \( k \) for every \( k \), it follows from the faithfulness criterion that there is a non-zero probability for the system to possess the property \( [V(k) = v(k)_1] \) immediately prior to the second measurement, for every \( k \). If a property has non-zero probability of being possessed, it is a possible property. Since the projector \( P_{\tilde{\varphi}_k} \) associated with the property \( [V(k) = v(k)_1] \) is one-dimensional, it has no non-null proper subspaces, and thus by definition it is the preferred projector for the \( k \)th property ascription. Thus, the set of preferred projectors is \( \{ P_{\tilde{\varphi}_k} \}_{k=1}^m \). Finally, since the set of vectors \( \{ |\varphi_k\rangle \}_{k=1}^m \) is by hypothesis non-orthogonal, the set of preferred projectors must also be non-orthogonal.
For clarity, we briefly repeat this argument in the context of a simple example, illustrated in Fig. 1. Suppose the variables being measured correspond to the components along different spatial axes of the spin operator, $\mathbf{S}$, for a spin 1/2 particle. Denote the component along axis $\hat{\mathbf{n}}$ by $\mathbf{S} \cdot \hat{\mathbf{n}}$, and the eigenstate associated with eigenvalue $\pm \hbar / 2$ by $| \pm \hat{\mathbf{n}} \rangle$. Suppose the first measurement is along $\hat{\mathbf{z}}$, so that $V = \mathbf{S} \cdot \hat{\mathbf{z}}$, $| \varphi_1 \rangle = | + \hat{\mathbf{z}} \rangle$, and $| \varphi_2 \rangle = | - \hat{\mathbf{z}} \rangle$. Suppose moreover that the state $| + \hat{\mathbf{x}} \rangle$ is prepared if the outcome of the first measurement is $- \hat{\mathbf{z}}$, while no disturbance occurs if the outcome is $+ \hat{\mathbf{z}}$, so that $| \tilde{\varphi}_1 \rangle = | + \hat{\mathbf{z}} \rangle$, and $| \tilde{\varphi}_2 \rangle = | + \hat{\mathbf{x}} \rangle$. Now suppose that the manner in which the nature of the second measurement depends on the outcome of the first is the following: if the first measurement has outcome $+ \hat{\mathbf{z}}$, then the second measurement is of $\mathbf{S} \cdot \hat{\mathbf{z}}$, while if it has outcome $- \hat{\mathbf{z}}$, then the second measurement is of $\mathbf{S} \cdot \hat{\mathbf{x}}$. Thus, $V_{(1)} = \mathbf{S} \cdot \hat{\mathbf{z}}$ and $V_{(2)} = \mathbf{S} \cdot \hat{\mathbf{x}}$. Assume the initial state of the spin is $c_1 | + \hat{\mathbf{z}} \rangle + c_2 | - \hat{\mathbf{z}} \rangle$, where $|c_1|^2 + |c_2|^2 = 1$ and $c_1, c_2 \neq 0$.

Using the generalized Born rule, it is straightforward to verify that the result of the second measurement is predictable with probability 1 given the result of the first measurement. It follows from the faithfulness criterion that if the outcome of the first measurement is $+ \hat{\mathbf{z}}$, then $\mathbf{S} \cdot \hat{\mathbf{z}}$ is subsequently determinate with value $+ \hbar / 2$, and if the outcome of the first measurement is $- \hat{\mathbf{z}}$, then $\mathbf{S} \cdot \hat{\mathbf{x}}$ is subsequently determinate with value $+ \hbar / 2$. Since both of these options occur with non-zero probability, the properties $[\mathbf{S} \cdot \hat{\mathbf{z}}] = + \hbar / 2$ and $[\mathbf{S} \cdot \hat{\mathbf{x}}] = + \hbar / 2$ are both possible. It follows that the preferred projectors are $P_{+ \hat{\mathbf{z}}}$ and $P_{+ \hat{\mathbf{x}}}$, which are non-orthogonal.

### B. Motivation

We now consider the reasons for adopting the faithfulness criterion. As is argued by Redhead [14], in seeking a realist interpretation of quantum mechanics one is seeking an explanation of the successes of the operational version of the theory. One way to secure an explanation of a measurement outcome is to demand that the properties of the systems involved ensure this outcome. The faithfulness criterion goes beyond this however, in that it specifies the form that such an explanation must take. Specifically, it is assumed that the reason a variable $V$ is found to have value $v$ in a measurement that is predictable with probability 1 is because immediately prior to the measurement $V$ is determinate and has value $v$. Although this is perhaps the simplest form the explanation could take, it is not the only form, as is evidenced by Bohm’s example and Bell’s be-able interpretation [4], where the outcomes of measurements of certain variables (position in Bohm’s case and lattice fermion number in Bell’s case), are taken to reveal pre-existing values of these variables.

So we see that the faithfulness criterion is not a necessary feature of a realist interpretation. Nonetheless, there have been many attempts to ensure that the outcomes of perfectly predictable measurements do reveal pre-existing values of these variables.

This tradition dates back to von Neumann, who assumed that the determinate variables of a system and their values are fixed by the density operator for the system, $\rho(t)$, by what we shall call the orthodox rule, namely,

$$\text{Ont}(t) = \{V|V \rho(t) \propto \rho(t)\}$$

$$[V]_t = \text{Tr}(V \rho(t)), \quad (5)$$

where $\text{Ont}(t)$ indicates the ontology at time $t$, and $[V]_t$ indicates the value of $V$ at time $t$ (this is simply the rule adopted by the ‘orthodox’ realist interpretations discussed in the introduction). Given that after a non-disturbing measurement of the variable $V$ with outcome $v$, one can predict, with probability 1, that the outcome of an immediately consecutive ideal measurement of $V$ will also be $v$, the faithfulness criterion demands that $V$ be determinate with value $v$ prior to the second measurement. However, given the orthodox rule, this can only occur if the density operator after the first measurement is a projector onto an eigenstate of $V$ associated with eigenvalue $v$. This must be the case regardless of the density operator prior to the first measurement. Thus, in order to satisfy the faithfulness criterion, von Neumann assumed that upon measurement the state vector undergoes a non-unitary evolution (the so-called ‘collapse’) to the eigenvector associated with the outcome of the measurement. As a realist interpretation of quantum mechanics this proposal is at best incomplete since it fails to specify, in terms of the primitives of the theory, the conditions under which a collapse occurs.

Many modal interpretations also attempt to satisfy the faithfulness criterion, but unlike von Neumann, they abandon the orthodox rule rather than assuming collapse. For instance, it has occurred to many authors, including Kochen [14], Healey [3] and Dieks [17], that by assigning determinate status to the projectors in the spectral resolution of the density operator one can satisfy the faithfulness criterion for ideal measurements. Modern versions of this approach include the proposals of Vermaas and Dieks [13] and Bacciagaluppi and Dickson [6]. However, it was noted by Bacciagaluppi and Hemmo [14] that the Vermaas and Dieks proposal failed to satisfy the faithfulness criterion for certain non-ideal measurements, specifically, disturbing measurements. The same argument can be applied against the Bacciagaluppi and Dickson proposal.
These results do not rule out the possibility that some new proposal involving a different, but still orthogonal, choice of preferred projectors might satisfy the faithfulness criterion for non-ideal measurements. However, by considering an experiment wherein the nature of the second measurement depends on the outcome of the first, we have shown that the faithfulness criterion fails to be satisfied for any modal interpretation that adopts orthogonal preferred projectors.

Thus any modal proposal seeking to satisfy the faithfulness criterion must allow for non-orthogonal preferred projectors. However, a satisfactory proposal must provide an unambiguous rule for identifying the set of preferred projectors for every system at every time, and it remains to be seen whether there exists any rule that consistently satisfies faithfulness. This rule must also satisfy other constraints, such as predicting properties for macroscopic systems that are in accord with our everyday perceptions of them. In particular, it must yield a solution to the measurement problem. It may be that the preferred set cannot be chosen to satisfy faithfulness for all measurements while also satisfying these other constraints. If this were true, it would certainly remove some of the motivation for pursuing a modal interpretation in the tradition of the authors specified above. We are not able to rule out this possibility here. Nonetheless, the range of measurements for which faithfulness is satisfied can at least be expanded if one assumes non-orthogonal preferred projectors, as we demonstrate in section 5 by a specific proposal.

C. Consequences for the ontology

In a modal interpretation, the property ascription to a system at a given time can be one of several possibilities. A question which we now address is whether or not these possibilities should differ with respect to the ontology they ascribe. Most previous modal interpreters have assumed that they should not. In such interpretations, the possible property ascriptions differ only with respect to the value ascription to a single common ontology. However, as we now prove, such an approach is unable to accommodate non-orthogonal preferred projectors.

**Theorem 1** It is not possible for there to be, at a given time, several possible mutually exclusive property ascriptions which (1) satisfy the algebraic constraints, (2) do not differ with respect to ontology, and (3) are associated with preferred projectors that are non-orthogonal.

**Proof.** The proof is by contradiction. Suppose the ontology and preferred projector for the $k$th property ascription (in the set of possible property ascriptions at a given time) are denoted respectively by $\text{Ont}_k$ and $P_k$. Since by hypothesis the possible property ascriptions do not differ with respect to ontology, there exists a single set of determinate variables, denoted by $\text{Ont}$, such that $\forall k : \text{Ont}_k = \text{Ont}$. Since $\forall k : P_k \in \text{Ont}_k$, it follows that $\{P_k\}_{k=1}^{m} \in \text{Ont}$. In other words, if there is only a single possible ontology at a given time, the preferred projectors for all the different property ascriptions must simultaneously be part of this ontology. Since one of the possible property ascriptions must actually obtain, one of these projectors must receive the value 1. Moreover, since by hypothesis the preferred projectors are non-orthogonal, it follows that the common ontology includes several non-orthogonal projectors, one of which receives the value 1. However, this is in contradiction with the algebraic constraints, as we now demonstrate.

Suppose that $P_S$ and $P_{S'}$ are two non-orthogonal preferred projectors, and that $[P_S] = 1$. Since the two property ascriptions associated with these are by assumption mutually exclusive, the intersection of $S$ and $S'$ is the null space. Moreover, by the definition of a preferred projector, no non-null proper subspace of $S$ or $S'$ can be determinate. By closure and the fact that $P_{S'}$ is determinate, $P_{(S')^\perp}$ is also determinate. By closure and the fact that $P_S$ and $P_{(S')^\perp}$ are determinate, the projector onto the intersection of $(S')^\perp$ and $S$ must also be determinate. Since no non-null proper subspaces of $S$ can be determinate, the intersection of $(S')^\perp$ and $S$ cannot be a non-null proper subspace of $S$. Moreover, this intersection cannot be $S$ itself, since then $S$ and $S'$ would be orthogonal, contradicting our initial assumption. Thus, the intersection of $(S')^\perp$ and $S$ must be the null space. It then follows from the functional relation constraint that $[P_{(S')^\perp}] [P_S] = 0$, and since $[P_S] = 1$, this implies that $[P_{(S')^\perp}] = 0$. It also follows from the functional relation constraint that $[P_{(S')^\perp}] = 1 - [P_{S'}]$, so that $[P_{S'}] = 1$. Thus, both $P_S$ and $P_{S'}$ receive the value 1. But, this is in contradiction with $[P_S][P_{S'}] = 0$ which follows from the fact that the intersection of $S$ and $S'$ is the null space.

QED.

IV. AN INTERPRETIVE FRAMEWORK INCORPORATING NON-ORTHOGONAL PREFERRED PROJECTORS

A. Preliminaries

In the previous section it was established that in order to consider a modal interpretation with non-orthogonal preferred projectors, the different possible property ascriptions to a system must differ with respect to the ontology
they ascribe. The precise form of the ontology associated with a particular property ascription has not yet been specified. It turns out that this form is fixed if an additional constraint on the property ascription is adopted, namely,

**Weakening Condition** If \( P_s \in \text{Ont} \) and \([P_s] = 1\) then for all \( P_R \) such that \( P_R \geq P_S \), \( P_R \in \text{Ont} \) and \([P_R] = 1\).

This condition was introduced by Healey \[3\] and was resurrected recently by Vermaas \[20\]. It is called ‘weakening’ since in Healey’s terminology \( P_R \) is said to be weaker than \( P_S \) if \( P_R \geq P_S \). It is motivated by the same sorts of considerations that lead one to adopt the closure constraint and the functional relation constraint; it is an attempt to preserve the logical structure of classical mechanics. In the language of properties, the weakening condition states that if property \( s \) is well-defined and holds for the system, then any property implied by \( s \), namely any property of the form ‘\( s’ \) or ‘\( s’ \) should also be well-defined and hold for the system. It should be noted that the weakening condition is unlike previous constraints, insofar as the nature of the ontology is made to depend on features of the value ascription.

We now demonstrate the form of property ascription that results from adopting the weakening condition.

**Theorem 2** The algebraic constraints and the weakening condition imply that the set of determinate projectors and the value ascription to these must respectively have the forms

\[
\{P_R|P_R \geq P_S \text{ or } P_R \perp P_S\} \quad (6)
\]

\[
[P_R] = \begin{cases} 1 & \text{if } P_R \geq P_S \\ 0 & \text{if } P_R \perp P_S \end{cases} \quad (7)
\]

where \( P_S \) is the preferred projector for the property ascription.

**Proof.** Recall that the preferred projector \( P_S \) is the unique projector in the property ascription satisfying Eq. \[1\], so that \([P_S] = 1\) and there is no subspace \( T \) such that \( P_T < P_S \) and \([P_T] = 1\). By the weakening condition, \([P_S] = 1\) implies that the set of projectors \( \{P_R|P_R \geq P_S\} \) is determinate. Moreover, for any projector \( P_U \) orthogonal to \( P_S \), \( P_U + P_S \) is determinate, since \( (P_S + P_U) \in \{P_R|P_R \geq P_S\} \). It then follows from the constraint of closure that \( (I - P_S)(P_S + P_U) = P_U \) is determinate. Thus, all projectors orthogonal to \( P_S \), namely the set \( \{P_U|P_U \perp P_S\} \), are also determinate. In summary, all the projectors in the set \( \{P_R|P_R \geq P_S \text{ or } P_R \perp P_S\} \) must be determinate. We now show that the projectors in this set are the only projectors that are determinate.

Suppose the contrary, namely that there exists a determinate projector \( P_V \) such that \( P_V \not\geq P_S \) and \( P_V \not\perp P_S \). If \( P_V \not\geq P_S \) then the intersection of \( V \) and \( S \) is not equal to \( S \), and must therefore be a proper subspace of \( S \). It then follows from the functional relation constraint and the assumption that all proper subspaces of \( S \) receive the value 0 that \([P_S]|P_S = 0\). Since \([P_S] = 1\), we conclude that \([P_V] = 0\). Moreover, since \( P_V \not\perp P_S \) is equivalent to \((I - P_V) \not\geq P_S \), it follows by the same argument that \([I - P_V] = 0\). But \([I - P_V] = 0\) implies \([P_V] = 1\), thereby yielding a contradiction.

Finally, we demonstrate that the value ascription must be of the form of \[7\]. Given that \([P_S] = 1\), it follows trivially from the weakening condition that \([P_R] = 1\) if \( P_R \geq P_S \). Moreover, \([P_R] = 0\) if \( P_R \perp P_S \) since otherwise the intersection of \( R \) and \( S \), which is the null space, would receive the value 1. QED.

**Corollary** If the set of determinate projectors and their values are given by Eqs. \[6\] and \[7\], then the spectral constraint implies that the ontology and its value ascription are given by

\[
\text{Ont} = \{V|VP_S \propto P_S\} \quad (8)
\]

\[
[V] = Tr(V P_S), \quad (9)
\]

where \( P_S \) is the preferred projector for the property ascription.

**Proof.** The spectral constraint states that a non-idempotent variable \( V \) is determinate if and only if all the elements of its spectral resolution are determinate. Thus every variable \( V \) that is determinate has a spectral resolution \( V = \sum_k \lambda_k P_{R_k} \), where \( \forall k : (P_{R_k} \geq P_S \text{ or } P_{R_k} \perp P_S) \). But the latter condition is equivalent to \( \forall k : P_{R_k} P_S \propto P_S \), from which it follows that \( VP_S \propto P_S \). The spectral constraint also states that \([V] = \sum_k \lambda_k [P_{R_k}] \). Since the \( P_{R_k} \) are orthogonal, only one can satisfy \( P_{R_k} \geq P_S \), and thereby receive the value 1 by Eq. \[7\]. Labelling this projector by \( k' \), we have \([V] = \lambda_{k'} \), and \( VP_S = \lambda_{k'} P_S \), from which Eq. \[8\] follows. QED.

The corollary to theorem 2 states that a variable is determinate if it has the subspace associated with the preferred projector as an eigenspace, and the value of this variable is the associated eigenvalue. This has the form of the orthodox rule, defined in Eq. \[1\], but where the role of the density operator is played by the preferred projector.
Note that for systems of dimensionality 3 or greater, theorem 2 implies that the ontologies associated with mutually exclusive property ascriptions are necessarily distinct. This holds true for such systems even if the preferred projectors for the property ascriptions are orthogonal. In this sense, the weakening condition provides another reason, independent of the one provided in section 3.3, for allowing the possible property ascriptions to differ in ontology. Such an argument was in fact made by Vermaas in the context of the Vermaas and Dicks version of the modal interpretation

Thus far, we have focused upon the property ascriptions for individual systems, and nothing has been said concerning the relationship between the property ascriptions to composite systems and the subsystems of which they are formed. Clifton has argued for the following constraint on this relationship, which we call the reductionist rule:

\[ V^A \in \text{Ont}^A \text{ if and only if } V^A \otimes I^B \in \text{Ont}^{AB}, \text{ and } [V^A] = [V^A \otimes I^B], \]

where \( \text{Ont}^A \) is the ontology of system \( A \), and \( AB \) is the composite of systems \( A \) and \( B \) (i.e. \( A(B) \) denotes the system associated with Hilbert space \( \mathcal{H}^A(\mathcal{H}^B) \), and \( AB \) denotes the system associated with \( \mathcal{H}^A \otimes \mathcal{H}^B \)). Denying this constraint leads to what Clifton has called ontological perpectivalism, the view that what exists depends on the level of compositeness of the description. For instance, to deny the ‘only if’ half of the rule amounts to claiming that it is possible for part \( A \) of a composite to have the property \( s \), while the composite itself does not have the property that part \( A \) has property \( s \). Clifton has characterized such a position as ‘metaphysically untenable’.

If one adopts the reductionist rule, the property ascriptions for all subsystems are uniquely fixed by the property ascription for the composite. However, we also wish to assume that the property ascriptions for every system satisfy the algebraic constraints and the weakening condition. It has yet to be demonstrated that these constraints are consistent with the reductionist rule. In fact they are. Specifically, if the property ascription for the composite has the form given in Eqs. (8) and (9), denoted by \( P^A \), then the property ascription for subsystem \( A \) also has the form given in Eqs. (8) and (9), with a preferred projector, denoted by \( P^A_s \), the unique projector satisfying

\[ P^A_s \otimes I^B \leq P^A \otimes I^B \text{ and there is no subspace } T^A \text{ such that } P^A_s \otimes I^B \leq P^A \otimes I^B. \]

(In other words, \( P^A_s \) is the ‘smallest’ projector satisfying \( P^A_s \otimes I^B \geq P^A_{sB} \)) This can be shown to be a limiting case of a result by Dickson and Clifton [21], however for clarity we prove it directly. It suffices to demonstrate the following equivalences:

\[ \{ P^A \mid P^A \otimes I^B \geq P^A_{sB} \} = \{ P^A \mid P^A \geq P^A_s \} \]

and

\[ \{ P^A \mid P^A \otimes I^B \perp P^A_{sB} \} = \{ P^A \mid P^A \perp P^A_s \}, \]

where \( P^A_s \) and \( P^A_{sB} \) are related as above. We first demonstrate that the right hand sides imply the left. \( P^A_s \geq P^A \) trivially implies \( P^A_s \otimes I^B \geq P^A \otimes I^B \), and since by definition, \( P^A_s \otimes I^B \geq P^A_{sB} \), it follows that \( P^A_s \otimes I^B \geq P^A_{sB} \). Similarly, \( P^A_s \perp P^A \) trivially implies \( P^A_s \otimes I^B \perp P^A \otimes I^B \), and together with \( P^A_s \otimes I^B \geq P^A_{sB} \), this implies that \( P^A_s \otimes I^B \leq P^A_{sB} \). To show that the left hand sides imply the right, we make use of the fact that \( P^A_s \otimes I^B \) is the ‘smallest’ projector satisfying \( P^A_s \otimes I^B \geq P^A_{sB} \). This implies that any projector \( P^A_R \) satisfying \( P^A_R \otimes I^B \geq P^A_{sB} \) must also satisfy \( P^A_R \otimes I^B \geq P^A_s \otimes I^B \), and hence \( P^A_R \geq P^A_s \). In addition, any projector \( P^A_R \) satisfying \( P^A_R \otimes I^B \perp P^A_{sB} \) (equivalently \( (I^A - P^A_R) \otimes I^B \geq P^A_{sB} \)) must also satisfy \( (I^A - P^A_R) \otimes I^B \geq P^A_s \otimes I^B \), which implies \( P^A_R \perp P^A_s \). This concludes the proof.

Clearly, the preferred projectors for a subsystem can be non-orthogonal if the preferred projectors for the composite are not. What is perhaps more surprising is that the preferred projectors for a subsystem can be non-orthogonal even if the preferred projectors for the composite are not! For example, suppose the preferred projectors for the composite are two orthogonal projectors, \( P^{sB}_{1B} = P^A \otimes P^B \) and \( P^{sB}_{2B} = P^A \otimes P^B \), where \( P^A \) and \( P^B \) are orthogonal projectors, but \( P^A_1 \) and \( P^A_2 \) are not. It then follows from Eq. (11) that the preferred projectors for \( A \) are simply \( P^A_1 \) and \( P^A_2 \), which are non-orthogonal.

In the next subsection, we will introduce a framework for interpretation wherein the possible property ascriptions for the universe are defined first, in accordance with the algebraic constraints and the weakening condition, and the
possible property ascriptions for all subsystems are then inferred using the reductionist rule. The preferred projectors for the universe will be assumed to be orthogonal, but as shown above, this is consistent with the preferred projectors for a subsystem being non-orthogonal and hence does not rule out the possibility of satisfying the faithfulness criterion. A more general approach would be to assume a non-orthogonal preferred set for the universe as well. However, the faithfulness criterion does not necessitate this assumption, and indeed, as we will demonstrate in section 5, one can satisfy this criterion for a wide variety of measurements without it. The case of an orthogonal preferred set for the universe is in any event a natural place to begin such an investigation.

The framework that emerges is similar to the one proposed by Bub and Clifton \[10\]. The most significant difference is in the form of the property ascription, since the latter do not assume the weakening condition. Another difference is in the dynamics of the property ascription. Bub and Clifton defined a dynamics following Vink \[23\] and Bell \[2\]. This approach was subsequently generalized in two respects by Bacciagaluppi and Dickson \[7\], and Dickson \[8\]. First, the preferred projectors were allowed to be time-dependent, and secondly it was shown that there is a plurality of possible dynamics consistent with the quantum statistics. We follow the latter, generalized approach.

Since many of the ingredients of the framework derive from a number of sources, and since we introduce some novel terminology, we have written the rest of this section in such a way that it constitutes a self-contained description of the framework.

B. Details of the framework

It is assumed that the universe is associated with a Hilbert space $\mathcal{H}$ and a vector $|\psi(t)\rangle \in \mathcal{H}$ that evolves deterministically over time in accordance with the Schrödinger equation,

$$\frac{d}{dt} |\psi(t)\rangle = -iH |\psi(t)\rangle,$$

where $H$ is the total Hamiltonian, and where the units are chosen such that $\hbar = 1$. Since, as will be demonstrated shortly, the role of the vector $|\psi(t)\rangle$ in the framework is to determine the probabilities of various different property ascriptions as well as their dynamics, it will be dubbed the dynamical state vector.

Define a decomposition $D$ of a vector $|\psi\rangle$ as a set $\{(c_k, |\phi_k\rangle)\}_{k=1}^m$ of non-zero coefficients $c_k$ and orthonormal vectors $|\phi_k\rangle$ such that $|\psi\rangle = \sum_{k=1}^m c_k |\phi_k\rangle$. It is assumed that every interpretation within the framework selects a preferred decomposition of the dynamical state $|\psi(t)\rangle$ at every time $t$. The projectors onto the elements of the preferred decomposition constitute the preferred projectors for the possible property ascriptions to the universe.

We also introduce a new ‘state vector’ that we denote by $|\Phi(t)\rangle$. It can be any one of the vector elements of the preferred decomposition. The projector onto this vector is the preferred projector for the property ascription to the universe that obtains at time $t$. Assuming the algebraic constraints and the weakening condition, it follows from Theorem 2 that the property ascription for the universe has the form of Eqs. \[8\] and \[10\], which may be rewritten in terms of $|\Phi(t)\rangle$ as

$$Out(t) = \{V|V|\Phi(t)\rangle \propto |\Phi(t)\rangle\},$$

$$[V]_t = \langle \Phi(t) | V | \Phi(t) \rangle.$$ 

The property ascription to any subsystem of the universe is then fixed by the reductionist rule, defined in Eq. \[10\]. Since $|\Phi(t)\rangle$ determines the property ascription to every system, we call it property state vector.

Next, we introduce a restriction on the manner in which the elements of the preferred decomposition can evolve over time. Suppose the set of vectors $\{|\phi_k(t)\rangle\}_{k=1}^d$ at every time $t$ is a complete orthogonal basis for $\mathcal{H}$ that includes as a subset the vector elements of the preferred decomposition. We require that there is an indexing of the basis vectors such that every vector with a given index is an analytic function of time. We call this the constraint of analyticity. It can be satisfied by requiring that the time-dependent vectors in the set $\{|\phi_k(t)\rangle\}_{k=1}^d$ each define a path through Hilbert space obeying the equation

$$\frac{d}{dt} |\phi_k(t)\rangle = -i\tilde{H}(t) |\phi_k(t)\rangle,$$

for some Hermitian operator $\tilde{H}(t)$. It is convenient to refer to these vectors, considered as functions of time, as the preferred paths.

It is assumed that the property state vector evolves according to a Markovian stochastic dynamics that permits hopping among the preferred paths. We require that at every time $t$ the probability $p_k(t)$ that the property state vector lies on the $k$th preferred path is given by
\[ p_k(t) = |\langle \phi_k(t) | \psi(t) \rangle|^2. \] (14)

The latter requirement is called the *Born rule constraint*. Although the basis \( \{ |\phi_k(t)\rangle \}_{k=1}^d \) that is defined at time \( t \) by the preferred paths may include elements that are not part of the preferred decomposition, these elements have no overlap with \( |\psi(t)\rangle \), so that the probability associated with them is zero. It follows therefore that the property state vector always corresponds to one of the vector elements of the preferred decomposition. There are many dynamics that satisfy the Born rule constraint; these will be considered in the next subsection.

We refer to this entire interpretive structure as a ‘framework’ for modal interpretations, since there are a plurality of possible interpretations that have this form. Specifically, there is a different interpretation for every choice of rule for determining the preferred decomposition and every choice of dynamics that satisfies the Born rule constraint.

### C. The general form of the dynamics

We now recall the general form of a Markovian stochastic dynamics that satisfies the Born rule constraint \( \Box \). This constraint, articulated in Eq.(14), can be recast as constraints upon the initial conditions \( \Box \) and the dynamics:

\[ p_k(0) = |\langle \phi_k(0) | \psi(0) \rangle|^2, \]

and

\[ \frac{d}{dt} p_k(t) = \frac{d}{dt} |\langle \phi_k(t) | \psi(t) \rangle|^2. \] (15)

Using Eqs.(12) and (13), the latter becomes

\[ \frac{d}{dt} p_k(t) = 2 \text{Im} \left[ \langle \psi(t) | \phi_k(t) \rangle \langle \phi_k(t) | H - \hat{H}(t) | \psi(t) \rangle \right]. \] (16)

Since we assume Markovian dynamics, it is sufficient to specify the probability \( T_{kj}(t) dt \) of a transition from path \( j \) to path \( k \) during the infinitesimal interval between \( t \) and \( t + dt \), for all \( j \) and \( k \). The evolution of a probability distribution \( p_k(t) \) over the paths is then given by the master equation

\[ \frac{d}{dt} p_k(t) = \sum_j \left[ T_{kj}(t) p_j(t) - T_{jk}(t) p_k(t) \right]. \]

In what follows, we consider the problem of finding a set of functions \( T_{kj}(t) \) that satisfy the master equation given \( p_k(t) \). Following Bell \( \Box \), it is useful to define a new set of functions, namely a set of probability currents, \( J_{kj}(t) \), as follows:

\[ J_{kj}(t) = T_{kj}(t) p_j(t) - T_{jk}(t) p_k(t). \] (17)

The current \( J_{kj}(t) \) describes the net flow of probability from path \( j \) to \( k \) at time \( t \). This definition implies that the current is antisymmetric with respect to an interchange of its indices

\[ J_{kj}(t) = -J_{jk}(t). \] (18a)

In terms of these currents, the master equation becomes a continuity equation:

\[ \frac{d}{dt} p_k(t) = \sum_j J_{kj}(t). \] (19)

Following Bacciagaluppi and Dickson \( \Box \), one can solve for the \( T_{kj}(t) \) in two steps. First, one finds a set of currents \( J_{kj}(t) \) that satisfy Eq.(18a) and that solve Eq.(13) with \( dp_k(t)/dt \) given by Eq.(17). Next, one finds a set of functions \( T_{kj}(t) \) that solve Eq.(17) given a particular solution for \( J_{kj}(t) \). It turns out that there an infinite number of sets of antisymmetric currents which solve the continuity equation. Moreover, for a given set of currents, there are an infinite number of solutions for the \( T_{kj}(t) \), specifically, any set of functions that satisfy

\[ T_{kj} \geq \max \{ 0, \frac{J_{kj}}{p_j} \}, \] (20)
for every pair of indices \( k > j \).

So we see that there is a large number of solutions for the dynamics which satisfy the constraints introduced. It is possible that additional constraints, such as a requirement of quantum-classical correspondence, might eliminate the ambiguity in the choice of dynamics, but this has yet to be demonstrated and some authors argue that it is unlikely.

V. THE MINIMAL ENTROPY PROPOSAL

A. Details of the proposal

We begin by introducing some terminology. A factorization \( F \) of a Hilbert space \( \mathcal{H} \) is defined to be a set of Hilbert spaces each of dimensionality greater than one, the direct product of which is \( \mathcal{H} \), that is, \( F = \{ \mathcal{H}^{(p)} \}_{p=1}^n \), such that \( \mathcal{H} = \mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)} \cdots \otimes \mathcal{H}^{(n)} \), and \( \dim(\mathcal{H}^{(p)}) > 1 \). A more precise definition of this concept is supplied by Bacciagaluppi \(^2\), but this is not required for our purposes. A factorization containing \( n \) elements is called \( n \)-partite, and the elements themselves are called factor spaces. A factorization \( F \) is said to be a coarse-graining of a factorization \( F' \), and \( F' \) a fine-graining of \( F \), if \( F' \) can be generated from \( F \) by factorizing one or more of the elements of \( F \). Finally, a product decomposition of \( |\psi\rangle \) with respect to the factorization \( F = \{ \mathcal{H}^{(p)} \}_{p=1}^n \) is any decomposition \( \{(c_k, \otimes_{p=1}^n |\phi_k^{(p)}\rangle)\}_{k=1}^m \) of \( |\psi\rangle \) every element of which is a product state over \( F \).

The first element of the proposal is to assume that there is a factorization of the Hilbert space of the universe that is more physically relevant than the others; we call it the distinguished factorization. There is a precedent for such an assumption, specifically, in the modal interpretations of Healey \(^4\), Bacciagaluppi and Dickson \(^2\), and Dieks \(^2\). Such interpretations have been called ‘atomic’, since the factor spaces of the distinguished factorization represent the most elementary physical systems. Some restrictions on what the distinguished factorization could be will be discussed briefly in section 5.2.

The first constraint upon the preferred decomposition is that it be a product decomposition with respect to the distinguished factorization. This constraint is not sufficient to uniquely specify a decomposition. Indeed, the number of product decompositions of any state vector with respect to a given factorization is infinite. In order to distinguish between these, we turn our attention towards the coefficients in the decomposition. Since these coefficients define a probability distribution, different decompositions can be ordered with respect to the uniformity of the associated distributions. This uniformity can be quantified by several ‘entropic’ quantities. The most obvious candidate is the Shannon entropy, defined for a probability distribution \( p = (p_1, p_2, \ldots, p_m) \) as

\[
H(p) = -\sum_{k=1}^m p_k \log p_k.
\] (22)

Thus, we can associate with every decomposition \( D = \{(c_k, |\phi_k\rangle)\}_{k=1}^m \) of the state vector \( |\psi\rangle \) the entropy

\[
S_{|\psi\rangle}(D) = -\sum_{k=1}^m |c_k|^2 \log |c_k|^2.
\] (23)

We refer to this quantity as the IU entropy of the state vector \( |\psi\rangle \) for the decomposition \( D \), since it has previously been considered by Ingarden and Urbanik \(^2\), albeit in a very different context.

It is now possible to state our choice of preferred decomposition:

Given a distinguished factorization \( F \), the preferred decomposition of the dynamical state vector \( |\psi\rangle \) is the one that minimizes the IU entropy of \( |\psi\rangle \) from among all product decompositions with respect to \( F \).

By choosing the product decomposition that minimizes the IU entropy, we are choosing the interpretation where the probability distribution over the possible property state vectors is as narrow as possible. Moreover, since the minimum IU entropy (from among IU entropies for product decompositions) is zero if and only if \( |\psi\rangle \) is a product state, it can be thought of as a measure of the entanglement of \( |\psi\rangle \) with respect to the distinguished factorization.
The strongest motivation for such a choice of preferred decomposition is that it appears very promising in securing a solution to the measurement problem and in satisfying the faithfulness criterion, as will be demonstrated in sections 5.2 and 5.3. We do not however offer any a priori justification of the principle.

Implementing the proposal requires solving the minimization problem for a given dynamical state vector and a given choice of distinguished factorization. If the distinguished factorization is bi-partite, the solution is given by the following theorem.

Theorem 3 Suppose $|\psi\rangle$ is any vector in a Hilbert space with a bi-partite distinguished factorization $F_{bi}$. Any decomposition of $|\psi\rangle$ that is bi-orthogonal with respect to $F_{bi}$ minimizes the IU entropy from among all product decompositions of $|\psi\rangle$ with respect to $F_{bi}$.

The proof of this theorem is relegated to appendix A. In the case of an n-partite distinguished factorization, with $n > 2$, we have not yet found a solution to the minimization problem for all state vectors. However, the bi-partite result can be used to identify the preferred decomposition for some state vectors, as follows.

Theorem 4 Suppose $|\psi\rangle$ is a vector in a Hilbert space with an n-partite distinguished factorization, $F_n$, where $n > 2$. If there exists a decomposition of $|\psi\rangle$ that is a product decomposition with respect to $F_n$ and that is a bi-orthogonal decomposition with respect to the distinguished factorization of $F_n$, then this decomposition minimizes the IU entropy from among all product decompositions with respect to $F_n$.

Proof. Suppose $F_{bi}$ is a bi-partite coarse-graining of $F_n$. The set $S_n$ of decompositions of $|\psi\rangle$ that are product decompositions with respect to $F_n$ is a subset of the set $S_{bi}$ that are product decompositions with respect to $F_{bi}$. We can denote this by $S_{n} \subseteq S_{bi}$. Moreover, suppose $D_{bi}^{\text{min}}(D_{bi})$ is the decomposition that minimizes the IU entropy from among all the elements of $S_{bi}$. Theorem 3 shows that for every state vector $|\psi\rangle$, $D_{bi}^{\text{min}}$ is the bi-orthogonal decomposition of $|\psi\rangle$. For certain state vectors, it may happen that $D_{bi}^{\text{min}}$ lies among the elements of $S_n$. Since we know that $D_{bi}^{\text{min}}$ minimizes the IU entropy from among all the elements of $S_{bi}$, and $S_n \subseteq S_{bi}$, it follows that in this case $D_{bi}^{\text{min}}$ also minimizes the IU entropy from among all the elements of $S_n$. Thus, in this case $D_n^{\text{min}} = D_{bi}^{\text{min}}$. QED.

Theorem 4 is not a complete solution to the minimization problem because there exist state vectors for which $D_{bi}^{\text{min}}$ does not lie among the elements of $S_n$. Further work is required to determine the decomposition that minimizes the IU entropy in such cases.

We note that in the proof of theorem 3, presented in appendix A, the only relevant feature of the IU entropy is that it has the form $\sum_{k=1}^{m} f(|c_k|^2)$ for some concave function $f$. It follows that one would obtain the same results if, instead of minimizing the IU entropy, one minimized any other entropic quantity having this form. However, there is no guarantee that this insensitivity to the choice of entropic quantity persists in the more general case of state vectors for which theorem 4 does not apply.

A possible difficulty with the minimal entropy proposal as it stands has to do with the uniqueness of the preferred decomposition. It is well known that the bi-orthogonal decomposition of a state vector is not unique when the eigenvalues of the reduced density operator for one of the subsystems are degenerate. It follows from theorem 3 that if the distinguished factorization is bi-partite, then the decomposition that minimizes the IU entropy may not be unique, and the minimal entropy proposal may fail to uniquely specify a preferred decomposition. For instance, this occurs if the dynamical state vector is the EPR-Bell state for two spins $|\psi\rangle = 2^{-1/2}(|+a\rangle |+a\rangle + |a\rangle |-a\rangle)$. This difficulty persists in the case of an n-partite distinguished factorization, $F_n$, where $n > 2$, since there are dynamical state vectors for which theorem 4 applies and the decomposition that minimizes the IU entropy is non-unique; an example being a tensor product of EPR-Bell states. It should be noted however that a degeneracy among the eigenvalues of the reduced density operator for one of the factor spaces of $F_n$ does not always lead to a non-unique preferred decomposition. For instance, if the dynamical state vector has a decomposition that is n-orthogonal with respect to the factorization $F_n$, then it follows from theorem 4 that this decomposition minimizes the IU entropy, and since the n-orthogonal decomposition is unique for $n > 2$, so is the preferred decomposition. It is an open question whether the minimization of the IU entropy leads to a unique preferred decomposition when the dynamical state vector is such that theorem 4 does not apply.

It is useful to distinguish two cases of non-uniqueness of the preferred decomposition: an instantaneous non-uniqueness, occurring at an isolated moment in time, and an extended non-uniqueness, occurring over a finite interval of time. If the constraint of analyticity (defined in Eq. (13)) holds for the minimal entropy proposal, then the instantaneous non-uniqueness problem can be solved easily: the preferred paths at the moment of non-uniqueness are simply taken to be the limit of the preferred paths at adjoining times. This is the same solution as was proposed in the context of the atomic modal interpretation by Bacciagaluppi and Dickson [1]. The extended non-uniqueness problem is not so easily solved. One possible approach to the problem is to argue that cases wherein there is an extended non-uniqueness have negligible probability. Since such an argument has been made for the occurrence of a non-unique
The dynamics is assumed to be such that operational quantum mechanics predicts, via the Born rule, that the measurement will have outcome \( \{ |\phi_k(t)\rangle \}_{k=1}^2 \). We follow Bacciagaluppi and Dickson \[7\] in choosing:

\[
J_{kj}(t) = 2 \text{Im} \left[ \langle \psi(t)|\phi_k(t)\rangle \langle \phi_k(t)|H - \tilde{H}(t)|\phi_j(t)\rangle \langle \phi_j(t)|\psi(t)\rangle \right],
\]

and

\[
T_{kj}(t) = \max \{0, \frac{J_{kj}(t)}{p_j(t)} \}.
\]

This is a generalization to time-dependent preferred decompositions of the choice made by Bell \[2\], Vink \[23\] and Bub \[2\]. Since the inequality in Eq.\[23\] is saturated, this choice of \( T_{kj}(t) \) minimizes the degree of stochasticity for a given form of the current. Such a choice is motivated by the fact that classical mechanics, which is deterministic, must be obtained as a limit of quantum mechanics.

B. The quantum measurement problem

We now consider whether the minimal entropy proposal solves the quantum measurement problem. Although this term is often taken to refer to the whole cluster of conceptual difficulties surrounding measurement, we shall use it to refer to the particular problem of deriving operational quantum mechanics from a realist no-collapse interpretation. To consider the problem, we must introduce a quantum mechanical model of the measurement procedure, that is, a model of the interaction between the degrees of freedom of the system under investigation, the apparatus, and the environment. We discuss both single measurements and sequences of measurements.

1. Single measurements

Following the notation introduced in section 3.1, we consider the measurement of a Hermitian operator \( V \), belonging to a Hilbert space \( \mathcal{H}^S \), the eigenvalues of which are non-degenerate and the eigenvectors of which are denoted by \( \{|\phi_k\rangle\}_{k=1}^2 \). Assuming the preparation procedure is associated with a state vector \( \sum_k c_k |\phi_k\rangle \), where \( \sum_k |c_k|^2 = 1 \), operational quantum mechanics predicts, via the Born rule, that the measurement will have outcome \( k \) with probability \( |c_k|^2 \).

We now consider a quantum mechanical model of the measurement process. The system under investigation is called the object system and is assumed to be microscopic. This is made to interact with a macroscopic apparatus, associated with a Hilbert space \( \mathcal{H}^A \), which in turn interacts with a macroscopic environment, associated with a Hilbert space \( \mathcal{H}^E \). Given an initial state vector in \( \mathcal{H}^S \otimes \mathcal{H}^A \otimes \mathcal{H}^E \), one could in principle determine the evolution of the total system using the full microscopic Hamiltonian.

In practice of course the problem is far too complex to be solved exactly. Nonetheless, there is a set of standard toy models of measurement that are commonly used to investigate realist interpretations. These models adopt some simplifying assumptions about the initial state and the form of the evolution. Specifically, it is assumed that the object system, apparatus and environment are all initially uncorrelated, so that the initial dynamical state vector has the form \( |\varphi_k\rangle \otimes |A_0\rangle \otimes |E_0\rangle \), a product state with respect to the factorization \( \{ \mathcal{H}^S, \mathcal{H}^A, \mathcal{H}^E \} \) of the Hilbert space. The dynamics is assumed to be such that

\[
|\varphi_k\rangle \otimes |A_0\rangle \otimes |E_0\rangle \rightarrow |\tilde{\varphi}_k\rangle \otimes |A_k\rangle \otimes |E_k\rangle ,
\]
where \( \{|A_k\rangle\}_{k=1}^m \) is a set of orthonormal vectors for the apparatus, \( \{|E_k\rangle\}_{k=1}^m \) is a set of orthonormal vectors for the environment, and \( \{|\tilde{\varphi}_k\rangle\}_{k=1}^m \) is a set of normalized but possibly non-orthogonal vectors for the object system, and where ‘\( \rightarrow \)’ denotes the mapping corresponding to the unitary evolution.

If the initial state for the object system is \( \sum_k c_k \langle \tilde{\varphi}_k \rangle \), the final dynamical state vector for the total system, given Eq. (26) and the assumption that the evolution is linear, is

\[
|\psi_{\text{final}}\rangle = \sum_{k=1}^m c_k |\tilde{\varphi}_k\rangle \otimes |A_k\rangle \otimes |E_k\rangle.
\]  

We are now in a position to ask whether a given realist no-collapse interpretation falls prey to the quantum measurement problem within this model. We begin by illustrating the problem in the traditional manner, specifically, in the context of the simplest realist no-collapse interpretation one can imagine: one where the property ascriptions for systems are fixed by the orthodox rule, defined in Eq. (5). Such an interpretation has been called the ‘bare theory’ by Albert [30]. Within the framework of section 4, it corresponds to adopting the trivial decomposition of the dynamical state vector as preferred (the trivial decomposition of \( |\psi(t)\rangle \) is simply \( \{(1, |\psi(t)\rangle)\} \)).

Consider first a case where \( c_k \neq 0 \) for only a single value of \( k \), that is, where the initial state vector of the object system is an eigenstate of \( V \). The final state vector is then of the form \( |\varphi_k\rangle \otimes |A_k\rangle \otimes |E_k\rangle \). By the orthodox rule and the reductionist rule, the preferred projector for the property ascription to the apparatus is \( P_{A_k} \). If the bare theory is to reproduce the predictions of operational quantum mechanics in this case, then the property associated with the projector \( P_{A_k} \) must be such that the apparatus can be accurately described as 'indicating outcome \( k \)' (for instance, if the apparatus indicates the outcome by a digital display, \( P_{A_k} \) could correspond to the property of displaying the number \( k \)). We refer to this as the assumption of ontological correspondence.

If, on the other hand, the initial state is such that \( c_k \neq 0 \) for more than one value of \( k \), then the final state vector is of the form \( \sum_k c_k |\varphi_k\rangle \otimes |A_k\rangle \otimes |E_k\rangle \), where \( \sum_k \) indicates a sum over values of \( k \) for which \( c_k \neq 0 \). In this case, the preferred projector for the property ascription to the apparatus is \( \sum_k P_{A_k} \), while no projector of the form \( P_{A_k} \) receives the value 1. Thus, even given the assumption of ontological correspondence, the bare theory does not predict that the apparatus indicates the outcome \( k \) for any value of \( k \) for which \( c_k \neq 0 \). Hence the bare theory does not reproduce the predictions of operational quantum mechanics. This is the quantum measurement problem.

We now specify the assumptions under which the minimal entropy proposal solves this problem. These involve the nature of the distinguished factorization, which we have not yet specified. Whatever it might be, the distinguished factorization should be defined in terms of primitives of the theory and selected by physical principles, for instance, from considerations of symmetry. We do not here present an argument for the identity of the distinguished factorization, however a discussion of the issue can be found in Dieks [26], where it is argued that a necessary condition on this choice is that the factor spaces carry an irreducible representation of the space-time group (the Galilei group in nonrelativistic quantum mechanics). For the present, we insist on only that the distinguished factorization, which we denote by \( F \), has the factorization \( \{H^S, H^A, H^E\} \) as a coarse-graining, and that its elements correspond to microscopic degrees of freedom (for instance, they could correspond to degrees of freedom of elementary particles).

Now, suppose that \( F \) is such that all the vectors in the sets \( \{|\varphi_k\rangle\}_{k=1}^m, \{|A_k\rangle\}_{k=1}^m, \) and \( \{|E_k\rangle\}_{k=1}^m \) are product states with respect to it. One can then determine that the preferred decomposition of \( |\psi_{\text{final}}\rangle \) is

\[
D_{\text{final}} = \{(c_k, |\varphi_k\rangle \otimes |A_k\rangle \otimes |E_k\rangle)\}_{k=1}^m.
\]

This follows from theorem 4 and the fact that \( D_{\text{final}} \) is a product decomposition with respect to \( F \) that is bi-orthogonal with respect to the coarse-graining \( F_{\text{bi}} = \{\{H^S \otimes H^A, H^E\} \} \) of \( F \). It then follows from the Born rule constraint that, with probability \( |c_k|^2 \), the property state vector is

\[
|\Phi_{\text{final}}\rangle = |\tilde{\varphi}_k\rangle \otimes |A_k\rangle \otimes |E_k\rangle.
\]

Using the reductionist rule we find that the projector \( P_{A_k} \) is determinate and receives the value 1 with probability \( |c_k|^2 \). Finally, by the assumption of ontological correspondence, the apparatus has the property of indicating outcome \( k \) with probability \( |c_k|^2 \). This is in agreement with the predictions of operational quantum mechanics.

Thus, we have obtained a solution to the measurement problem within the standard model of measurement. In so doing, we have had to assume that when the initial state vector of the object system is an eigenstate of \( V \), the final dynamical state vector for the total system is unentangled with respect to the distinguished factorization.

The assumption of no entanglement between the distinguished factor spaces of the apparatus and the environment is not particularly realistic, given that these factor spaces are taken to correspond to microscopic degrees of freedom, and typical interactions between the apparatus and the environment are likely to entangle these degrees of freedom. However, this assumption can be relaxed somewhat without changing any of our conclusions, as we now demonstrate.
indicating outcome \( k \)

\[ P_{\mu} \]

probability

await further progress on the problem of the minimization of the IU entropy. The assumption of ontological correspondence in this case becomes the assumption that for every value of \( \mu \), the projector \( P_{A_{k,\mu}} \) corresponds to the apparatus indicating outcome \( k \). That there can be more than one projector corresponding to indicating a particular outcome is not unreasonable since there can be many different microscopic configurations of the apparatus leading to the same overall macroscopic appearance.

An arbitrary initial state vector for the object system, \( \sum_k c_k |\varphi_k\rangle \), leads, via Eq. (28), to the following final dynamical state vector for the total system

\[ |\psi_{\text{final}}\rangle = \sum_{k=1}^m c_k |\tilde{\varphi}_k\rangle \otimes \sum_{\mu=1}^M f_{k,\mu} \langle A_{k,\mu} \otimes |E_{k,\mu}\rangle \].

The preferred decomposition of this state vector is

\[ D_{\text{final}}' = \left\{ \left\{ (c_k f_{k,\mu}, |\tilde{\varphi}_k\rangle \otimes |A_{k,\mu}\rangle \otimes |E_{k,\mu}\rangle) \right\}_{\mu=1}^M \right\}_{k=1}^m \],

since this is a product decomposition with respect to \( F \) that is bi-orthogonal with respect to \( F_{\text{bi}} \). It follows that the property state vector is

\[ |\Phi_{\text{final}}\rangle = |\tilde{\varphi}_k\rangle \otimes |A_{k,\mu}\rangle \otimes |E_{k,\mu}\rangle \]

with probability \( |c_k f_{k,\mu}|^2 \). By the reductionist rule, the projector \( P_{A_{k,\mu}} \) is determinate and receives value 1 with probability \( |c_k f_{k,\mu}|^2 \). Finally, by the assumption of ontological correspondence, the apparatus has the property of indicating outcome \( k \) with probability \( \sum_{\mu} |c_k f_{k,\mu}|^2 = |c_k|^2 \) in agreement with operational quantum mechanics.

Note that the model of measurement provided by Eq. (28) can also describe error-prone measurements. This occurs if for some values of \( \mu \), \( P_{A_{k,\mu}} \) corresponds to the property of indicating an outcome \( k' \neq k \), or to the property of indicating a malfunction. Furthermore, this model can incorporate measurements described by positive operator-valued measures (POVMs) [3]. This follows from the fact that such measurements are implemented by adjoining an ancilla to the system under investigation and measuring a projector-valued measure (PVM) on the composite. By including the ancilla in our definition of the object system, the model presented above can describe these measurements. Note however that we are restricted to PVMs whose eigenvectors are product states with respect to the distinguished factorization.

Despite the possibility of incorporating some error-prone and POVM measurements, the model of measurement provided by Eq. (28) is still not the most general or realistic. Although it is true that an arbitrary state vector has many decompositions into product states with respect to the distinguished factorization, it is not necessarily the case that any of these decompositions are bi-orthogonal with respect to a coarse-graining of the distinguished factorization. For instance, if any of the vectors in the set \( \{|\tilde{\varphi}_k\rangle\}_{k=1}^m \) are entangled with respect to the distinguished factor spaces of \( \mathcal{H}^S \), then theorem 4 fails to apply if the final dynamical state vector is of the form of \( |\psi_{\text{final}}\rangle \). Since the problem of minimizing the IU entropy for arbitrary state vectors has not yet been solved, it is not clear what the preferred decomposition will be in this case and whether the measurement problem is resolved or not.

It is nonetheless interesting to consider one particular type of modification of the evolution where the only change from the model considered above is that the set of vectors \( \{|E_{k,\mu}\rangle\}_{\mu=1}^M \) (describing the states of the environment that are relative to the apparatus states \( \{|A_{k,\mu}\rangle\}_{\mu=1}^M \) is only approximately orthogonal. This is an instance where theorem 4 may fail to apply. However, the difference between \( |\psi_{\text{final}}\rangle \) when the elements of \( \{|E_{k,\mu}\rangle\}_{\mu=1}^M \) are orthogonal and when they are very nearly orthogonal, is not significant. Thus, if the preferred decomposition does not depend sensitively on small variations in the dynamical state vector, the preferred decomposition in the nearly orthogonal case should be ‘close to’ \( D_{\text{final}}' \), and it is then likely that the apparatus will be assigned an ontology that is ‘close to’ the one it receives for the orthogonal case. We see therefore that whether or not there is a measurement problem in this case depends on whether or not there is such sensitive dependence. The answer to this question must await further progress on the problem of the minimization of the IU entropy.3

3The analogous question in the Vermaas-Dieks version of the modal interpretation is whether the spectral resolution of a
Finally, we note that the assumption that the apparatus and environment are initially unentangled is also an unrealistic feature of the standard model of measurement. For that matter, the assumption that the composite of system, apparatus and environment is unentangled with the rest of the universe may not be realistic either. However, this difficulty is not unique to the minimal entropy proposal. Every realist no-collapse interpretation must contend with the fact that the dynamical state vector for the universe is in general not factorizable with respect to subsystems that have interacted in the past, even if this interaction is quite weak. Further work is required to determine whether the predictions of the minimal entropy proposal remain satisfactory when these assumptions are relaxed.

2. Sequences of measurements

We now demonstrate the extent to which the minimal entropy proposal is in agreement with operational quantum mechanics for sequences of measurements. Consider in particular the sequence of two measurements described in section 3.1. Recall that the first measurement is of a variable $V$ with eigenstates $\{|\varphi_k\rangle\}_{k=1}^m$, the second measurement is of a variable $V'$ with eigenstates $\{|\varphi'_k\rangle\}_{k=1}^m$, and the state prepared by the first apparatus given outcome $k$ is denoted by $|\tilde{\varphi}_k\rangle$.

In the last subsection we considered two distinct models of measurement which differed in the extent to which the apparatus and the environment became entangled due to their interaction. In this subsection, we consider only the simpler of the two models. The reader can verify that the more realistic model leads to the same conclusions.

There are now two apparatuses, and an environment for each. We denote their Hilbert spaces by $\mathcal{H}^{A1}, \mathcal{H}^{A2}, \mathcal{H}^{E1}$, and $\mathcal{H}^{E2}$ respectively, and we distinguish state vectors for the two apparatuses(environments) by a superscript. It is again assumed that the object system, the two apparatuses and the two environments are all initially uncorrelated.

The distinguished factorization $F$ is assumed to have $\{\mathcal{H}^S, \mathcal{H}^{A1}, \mathcal{H}^{A2}, \mathcal{H}^{E1}, \mathcal{H}^{E2}\}$ as a coarse-graining.

We assume that the first measurement is well described by Eq.(26) with the exception of a change of notation: $|A_0\rangle, |E_0\rangle, |A_k\rangle$ and $|E_k\rangle$ become $|A^1_k\rangle, |E^1_k\rangle, |A^2_k\rangle$ and $|E^2_k\rangle$ in order to specify that the object system interacts with the first rather than the second apparatus. We assume that the second apparatus and its environment remain uncorrelated with the rest of the system and each other during this first measurement. It follows that the dynamical state vector for the total system after the first measurement is

$$|\psi_{\text{final} 1}\rangle = \left( \sum_{k=1}^m c_k |\tilde{\varphi}_k\rangle \otimes |A^1_k\rangle \otimes |E^1_k\rangle \right) \otimes |A^2_0\rangle \otimes |E^2_0\rangle .$$  \hspace{1cm} (29)

Suppose that $|A^1_0\rangle$ and $|E^1_0\rangle$ are product states with respect to $F$. If we make all the same assumptions about $|A^1_k\rangle$ and $|E^1_k\rangle$ as were made for $|A_k\rangle$ and $|E_k\rangle$ in the previous subsection, and if we use the bi-partite factorization $\{\mathcal{H}^S \otimes \mathcal{H}^{A1} \otimes \mathcal{H}^{A2}, \mathcal{H}^{E1} \otimes \mathcal{H}^{E2}\}$ in place of the bi-partite factorization $\{\mathcal{H}^S \otimes \mathcal{H}^{A}, \mathcal{H}^{E}\}$ in the arguments found therein, then it is straightforward to show that the preferred decomposition of $|\psi_{\text{final} 1}\rangle$ is

$$D_{\text{final} 1} = \left\{ (c_k, |\tilde{\varphi}_k\rangle \otimes |A^1_k\rangle \otimes |E^1_k\rangle \otimes |A^2_0\rangle \otimes |E^2_0\rangle ) \right\}_{k=1}^m .$$  \hspace{1cm} (30)

We conclude that with probability $|c_k|^2$, the first apparatus indicates outcome $k$, while the second apparatus remains ready to measure.

Now assume that the second measurement is also well described by Eq.(26) with the notational change that $|A_0\rangle, |E_0\rangle, |A_k\rangle$ and $|E_k\rangle$ become $|A^2_0\rangle, |E^2_0\rangle, |A^2_k\rangle$ and $|E^2_k\rangle$ since the object system is now interacting with the second apparatus, and where the vectors for the object system acquire a prime since the second measurement is of $V'$ rather than $V$. For simplicity, we take this second measurement to be non-disturbing, so that $|\tilde{\varphi}'_k\rangle = |\varphi'_k\rangle$. Assume also that the first apparatus and its environment have no interactions during this measurement. It then follows that the dynamical state vector after the second measurement is

$$|\psi_{\text{final} 2}\rangle = \sum_{k=1}^m \sum_{j=1}^m d_{kj}^2 |\varphi'_j\rangle \otimes |A^2_j\rangle \otimes |E^2_j\rangle \otimes |A^2_k\rangle \otimes |E^2_k\rangle .$$  \hspace{1cm} (31)

Density operator is sensitive to small changes in the density operator. Bacciagaluppi, Donald and Vermaas [28] have shown that this does in fact occur when the density operator has nearly degenerate eigenvalues.
where the coefficients \( \{ d_j^k \}_{k=1}^m \) are defined by \( | \tilde{\varphi}_k \rangle = \sum_{j=1}^m d_j^k | \varphi'_j \rangle \). Again, if we make all the same assumptions about \( | A_j^2 \rangle \) and \( | E_j^2 \rangle \) as were made for \( | A_j \rangle \) and \( | E_j \rangle \) in the previous subsection, and if we use the bi-partite factorization \( \{ \mathcal{H}^S \otimes \mathcal{H}^{A1} \otimes \mathcal{H}^{A2} \otimes \mathcal{H}^{E1} \otimes \mathcal{H}^{E2} \} \) in place of the bi-partite factorization \( \{ \mathcal{H}^S \otimes \mathcal{H}^A, \mathcal{H}^E \} \) in all the arguments found therein, the preferred decomposition of \( | \psi_{\text{final}} \rangle_2 \) is found to be

\[
D_{\text{final} \ 2} = \left\{ \{ c_k d_j^k, | \varphi'_j \rangle \otimes | A_k^1 \rangle \otimes | E_k^1 \rangle \otimes | A_k^2 \rangle \otimes | E_k^2 \rangle \} \right\}_{k=1}^m \left\{ \right\}_{j=1}^m .
\] (32)

We can therefore conclude that there is a probability \( | c_k d_j^k |^2 \) that the first apparatus indicates outcome \( k \) and the second apparatus indicates outcome \( j \) after the second measurement. It follows that the probability of the second apparatus indicating outcome \( j \) given that the first apparatus indicates outcome \( k \) after the second measurement is \( | d_j^k |^2 = | \langle \varphi_j | \tilde{\varphi}_k \rangle |^2 \).

However, we have still not determined the probability for the second apparatus to indicate outcome \( j \) after the second measurement given that the first apparatus indicates outcome \( k \) after the first measurement, which is the quantity specified by the generalized Born rule. The problem is that it has not been shown that the outcome indicated by the first apparatus is stable over time. Whether it is or not depends on the dynamics of the property state vector, which is determined by Eqs. \( (24) \) and \( (25) \). Now although it may be reasonable to assume that the dynamical state vector after a measurement is such that theorem 4 applies, it is unlikely that this theorem applies during the entire interaction leading up to this outcome. Given this, we cannot at present determine the time-sequence of preferred decompositions nor the preferred paths through Hilbert space defined by this sequence. Since Eqs. \( (24) \) and \( (25) \) depend on the identity of these preferred paths, we cannot at present determine the dynamics of the property state vector.

Thus, for the moment we simply assume that within the minimal entropy proposal, the apparatus is never described as ‘jumping’ between macroscopically different readings. We call this assumption stability. Given stability, the minimal entropy proposal reproduces the predictions of the generalized Born rule.

C. The Faithfulness criterion revisited

We now reconsider the experiment of section 3.1 in the context of the minimal entropy proposal. Since this experiment involves a sequence of two measurements, we can make use of the model of measurement presented in the previous section. Consider the property ascription at the time \( t \), after the first measurement. The dynamical state vector is \( | \psi_{\text{final}} \rangle_1 \), defined in Eq. \( (23) \), and its preferred decomposition is \( D_{\text{final} \ 1} \), defined in Eq. \( (30) \). If the first apparatus indicates the outcome \( k \) at time \( t \), the property state vector must be the \( k \)th element of \( D_{\text{final} \ 1} \), that is, \( | \Phi_{\text{final} \ 1} \rangle = | \tilde{\varphi}_k \rangle \otimes | A_k^1 \rangle \otimes | E_k^1 \rangle \otimes | A_k^2 \rangle \otimes | E_k^2 \rangle \). The critical feature of the experiment of section 3.1 is that the vector \( | \tilde{\varphi}_k \rangle \) that is prepared when the first apparatus indicates outcome \( k \) is an eigenstate of \( V(k) \), the variable measured by the second apparatus. It follows that the variable \( V(k) \otimes I \) (where \( I \) is the identity operator for \( \mathcal{H}^{A1} \otimes \mathcal{H}^{A2} \otimes \mathcal{H}^{E1} \otimes \mathcal{H}^{E2} \)) is determinate and has value \( v(k) \) at time \( t \). Finally, it follows from the reductionist rule that \( V(k) \) is determinate and has value \( v(k) \) at time \( t \). This is precisely what is required in order for the faithfulness criterion to be satisfied.

It should be noted that since the variable measured by the second apparatus depends upon the outcome of the first measurement, the initial state of the second apparatus may well be different for different outcomes of the first measurement. Thus, rather than the first measurement interaction being described by Eq. \( (26) \), it may be described by

\[
| \varphi_k \rangle \otimes | A_0^1 \rangle \otimes | E_0^1 \rangle \otimes | A_0^2 \rangle \otimes | E_0^2 \rangle \mapsto | \tilde{\varphi}_k \rangle \otimes | A_k^1 \rangle \otimes | E_k^1 \rangle \otimes | A_k^2 \rangle \otimes | E_k^2 \rangle ,
\] (33)

where \( \{ | A_k^1 \rangle \}_{k=1}^m \) and \( \{ | E_k^2 \rangle \}_{k=1}^m \) are orthonormal sets of vectors, and \( | A_k^2 \rangle \) corresponds to the apparatus being ready to measure the variable \( V(k) \). In any event, by making the same assumptions for \( | A_k^1 \rangle \) and \( | E_k^2 \rangle \) as were made for \( | A_0^1 \rangle \) and \( | E_0^2 \rangle \) in the last subsection, one can show that the minimal entropy proposal is in agreement with the predictions of operational quantum mechanics even when the nature of the second measurement depends on the outcome of the first.

We end this section with a discussion of the case wherein the second measurement is of a variable whose eigenstates are not all product states with respect to the distinguished factorization \( F \). For such measurements, the faithfulness criterion cannot be satisfied within the minimal entropy proposal. The reason is as follows. Suppose \( | \varphi \rangle \) is an
eigenstate of the measured variable that is entangled with respect to $F$. If $|\varphi\rangle$ is prepared by the first measurement and measured by the second, then the faithfulness criterion requires that the projector $P_\varphi$ be determinate with value 1 immediately prior to the second measurement. However, for this to occur the property state vector must be an eigenstate of $P_\varphi$, and hence must be entangled with respect to $F$. But the property state vector is always a product state with respect to $F$ in the minimal entropy proposal.

The failure of the faithfulness criterion for such measurements in the context of the atomic modal interpretation of Bacciagaluppi and Dickson \[7\] and Dicks \[28\] has been discussed by Dieks, and also by Vermaas \[32\]. These authors have suggested that an explanation of the outcomes of these measurements might be provided by dispositional properties or collective effects of the composite. This explanation can also be invoked in the context of the minimal entropy proposal.

VI. CONCLUSIONS

In modal interpretations, the properties of a system are given by a specification of the set of determinate variables (the ontology) and the value ascription to these variables, jointly referred to as the property ascription. Such interpretations also assume that the property ascription which obtains at a given time is just one of several possibilities. There is always a unique ‘smallest’ projector which receives the value 1 in each of these possible property ascriptions, which we call the preferred projector for that property ascription.

We have shown that these preferred projectors must be non-orthogonal if one seeks to satisfy the faithfulness criterion, that is, if one seeks to explain the outcomes of certain perfectly predictable measurements in terms of pre-existing properties of the system under investigation. The possibility of such an explanation has historically been a strong motivation for the modal approach.

We have also shown that non-orthogonal preferred projectors are inconsistent with the assumption, common among previous modal interpretations, that at a given time there is only a single possible ontology. In order to consider non-orthogonal preferred projectors, we have developed a framework for modal interpretations wherein at a given time, the possible property ascriptions may differ with respect to ontology. As is required for any modal interpretation, the state vector appearing in the Schrödinger equation, which we call the dynamical state vector, does not uniquely fix the property ascription. Rather, a preferred decomposition of the dynamical state vector into a sum of orthogonal vectors must be specified at every time, and a single element of this decomposition, dubbed the property state vector, fixes the property ascription. The property state vector evolves stochastically according to a Markovian dynamics. Finally, subsystems receive only those properties they inherit from the total system by the reductionist rule.

It is of course possible to generalize this framework in many ways. One could consider non-Markovian dynamics, alternatives to the reductionist rule, and even non-orthogonal decompositions of the dynamical state vector. Nonetheless, we feel that the framework presented is a natural starting place for the interpretive program at hand.

Within the context of this framework, we have presented a novel proposal for the preferred decomposition. The proposal assumes that there is a distinguished set of subsystems of the universe, that is, a distinguished factorization of the total Hilbert space into a tensor product of Hilbert spaces. It is also assumed that the preferred decomposition is a product decomposition with respect to this factorization. In the case of a distinguished factorization that is bi-partite, it is then natural to follow previous authors in identifying the bi-orthogonal decomposition as preferred. However, the obvious generalization of the bi-orthogonal decomposition to an $n$-partite distinguished factorization, namely the $n$-orthogonal decomposition, does not exist for all state vectors, as shown by Peres \[28\]. The preferred decomposition in our proposal is the one that minimizes the IU entropy from among all product decompositions with respect to the distinguished factorization. This decomposition always exists and turns out to be equal to the $n$-orthogonal decomposition when the latter exists. It therefore can be thought of as a natural generalization of the bi-orthogonal decomposition to $n$-partite systems.

At present the strongest justification for the minimal entropy proposal is its success in dealing with the quantum measurement problem and in satisfying the faithfulness criterion. The measurement problem is resolved for a wide variety of measurements including certain types of non-ideal measurements, in particular, disturbing measurements, assuming particular microscopic models of the apparatus and environment. Within the same microscopic models, the faithfulness criterion is satisfied for sequences of disturbing measurements. It is this feature of the minimal entropy proposal that sets it apart from previous modal interpretations.

\footnote{Other modal interpreters \[7\] \[32\] \[28\] have been led to this assumption by considerations of the correlations between the properties of a system and its subsystems.}
The solution of the measurement problem relies on the assumption of ontological correspondence, that the ontology of macroscopic systems corresponds to our everyday perceptions of them, and the assumption of stability, that the dynamics of the properties assigned to macroscopic systems are consistent with our stable perceptions of them. Ideally, these features would be demonstrated rather than assumed. However, the demonstration of ontological correspondence is likely to require a better specification of the distinguished factorization than has been provided in the present work, while the demonstration of stability must await progress in solving the entropy minimization problem in cases where theorem 4 does not apply. Progress on the minimization problem will also help to determine whether one can solve the quantum measurement problem for more general types of measurements than the ones considered here, for example, measurements of variables whose eigenstates are entangled with respect to the distinguished factorization. In addition, such progress is required to determine what the proposal has to say about more realistic models of measurements. Finally, it may indicate whether the IU entropy is the correct quantity to minimize in the rule for determining the preferred decomposition, or whether some other entropic quantity might be a better choice.

So we see that there remain many unanswered questions. In addition to these, there are difficulties with the minimal entropy proposal. For one, the product decomposition that minimizes the IU entropy may fail to be unique for certain dynamical state vectors. It may be that further technical work will show that this is not a problem after all. For instance, dynamical state vectors for which the preferred decomposition is non-unique for a finite interval of time may constitute a set of measure zero. Another difficulty is that the faithfulness criterion explicitly fails to be satisfied in measurements of variables whose eigenstates are entangled with respect to the distinguished factorization. Given that not all Hermitian operators necessarily correspond to variables that can be measured, it may happen that with a suitable choice of distinguished factorization, the measurements for which the faithfulness criterion fails to be satisfied are precisely those which are impossible to implement. On the other hand, it may be that this problem cannot be avoided within the minimal entropy proposal, but can be avoided if some other choice of preferred decomposition is made. As a third possibility, one might find that the faithfulness criterion for variables with entangled eigenstates, cannot be satisfied by any interpretation within the framework we have set out. Justifying any one of these answers would certainly be an interesting result, and motivates further investigation of these issues.

The use of a preferred decomposition, sometimes called an ‘interpretation basis’, has been viewed by some as necessary within interpretive strategies distinct from modal interpretations. This has been suggested by Deutsch in the context of the many-worlds interpretation and by Kent and McElwaine in the context of consistent histories. A preferred decomposition might also be useful in nonlinear modifications of quantum mechanics. Thus, the preferred decomposition of the minimal entropy proposal may well be of relevance to such interpretive strategies as well. In any event, a mathematically precise proposal, even though not without problems, can be useful in stimulating progress on interpretive issues, as is evidenced by the recent profusion of work on modal interpretations. We hope that the minimal entropy proposal will not be an exception in this respect.

VII. ACKNOWLEDGMENTS

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[1] A. Einstein, B. Podolsky, N. Rosen, *Phys. Rev.* **47**, 777 (1935).
[2] J. S. Bell, *Speakable and Unspeakable in Quantum Mechanics*, (Cambridge University Press, Cambridge, 1993), chap. 19.
[3] J. Bub, *Interpreting the Quantum World*, (Cambridge University Press, Cambridge, 1997).
[4] This is consistent with the characterization found in Ref. [6].
[5] R. Healey, *The Philosophy of Quantum Mechanics* (Cambridge University Press, Cambridge, 1989).
[6] R. Clifton, *Brit. J. Phil. Sci.*, **47**, 371 (1996).
[7] G. Bacciagaluppi and M. Dickson, quant-ph/9711048 (1997).
[8] M. Dickson, in Healey and Hellman, Ref. [9], p. 160.
[9] R. Healey and G. Hellman (eds.), *Quantum Measurement: Beyond Paradox* (University of Minnesota Press, Minneapolis, 1997).
[10] J. Bub and R. Clifton, *Stud. Hist. Phil. Mod. Phys.* **27**, 181 (1996).
[11] The classic article is G. Birkhoff and J. von Neumann, *Ann. Math.* **37**, 823 (1936). For an overview, see K. Svozil, *Quantum Logic* (Springer-Verlag, Singapore, 1998).
Lemma A.2
For any vector $|\psi\rangle$, if $D$ is an arbitrary product decomposition of $|\psi\rangle$, then there always exists an $A$-orthogonal decomposition of $|\psi\rangle$, $D_{A-orth}$, such that

$$S_{|\psi\rangle}(D_{A-orth}) \leq S_{|\psi\rangle}(D).$$

Lemma A.1
For any vector $|\psi\rangle$, if $D_{A-orth}$ is any $A$-orthogonal decomposition of $|\psi\rangle$, and $D_{bi-orth}$ is any bi-orthogonal decomposition of $|\psi\rangle$, then

$$S_{|\psi\rangle}(D_{bi-orth}) \leq S_{|\psi\rangle}(D_{A-orth}).$$

VIII. APPENDIX: PROOF OF THEOREM 3.

It will be assumed throughout that the distinguished factorization is bi-partite, and the two factor spaces are denoted by $\mathcal{H}^A$ and $\mathcal{H}^B$. All references to product decompositions are to be understood as product decompositions with respect to this factorization. We say that a product decomposition $\{\rho_k, \{|A_k\rangle \otimes |B_k\rangle\}_{k=1}^m\}$ is $A$-orthogonal ($B$-orthogonal) if the set of vectors $\{|A_k\rangle\}_{k=1}^m$ ($\{|B_k\rangle\}_{k=1}^m$) is orthogonal. A bi-orthogonal decomposition (also called a Schmidt decomposition) is one that is both $A$-orthogonal and $B$-orthogonal. We shall make use of several well-known properties of bi-orthogonal decompositions, an exposition of which can be found in Ref. [38]. Finally, we remind the reader that $S_{|\psi\rangle}(D)$ denotes the IU entropy of $|\psi\rangle$ for the decomposition $D$, which is defined by Eq.(23).

Theorem 3 follows from two lemmas:

Lemma A.1 For any vector $|\psi\rangle$, if $D$ is an arbitrary product decomposition of $|\psi\rangle$, then there always exists an $A$-orthogonal decomposition of $|\psi\rangle$, $D_{A-orth}$, such that

$$S_{|\psi\rangle}(D_{A-orth}) \leq S_{|\psi\rangle}(D).$$

Lemma A.2 For any vector $|\psi\rangle$, if $D_{A-orth}$ is any $A$-orthogonal decomposition of $|\psi\rangle$, and $D_{bi-orth}$ is any bi-orthogonal decomposition of $|\psi\rangle$, then

$$S_{|\psi\rangle}(D_{bi-orth}) \leq S_{|\psi\rangle}(D_{A-orth}).$$
Together these imply that for any vector $|\psi\rangle$, if $D$ is an arbitrary product decomposition of $|\psi\rangle$, and $D_{\text{bi-orth}}$ is any bi-orthogonal decomposition of $|\psi\rangle$, then

$$S_{|\psi\rangle}(D_{\text{bi-orth}}) \leq S_{|\psi\rangle}(D),$$

which is simply theorem 3.

The task at hand is therefore to prove lemmas A.1 and A.2. We begin by reviewing a partial ordering relation among probability distributions, namely that of majorization [39], which has recently seen application in the study of entanglement purification [40]. Suppose $p \equiv (p_1, p_2, ..., p_m)$ and $q \equiv (q_1, q_2, ..., q_m)$ are two $m$-element probability distributions. By definition, $p$ majorizes $q$ if for every $l$ in the range $\{1, ..., m\},$

$$\sum_{k=1}^{l} p_k^l \geq \sum_{k=1}^{l} q_k^l,$$

where $p_k^l$ indicates the $k$th largest element of $p$, so that $p_1^l \geq p_2^l \geq ... \geq p_m^l$.

The notion of majorization is important in the present investigation because of the following well-known result (Theorem II.3.1 of Ref. [39]): The following two conditions are equivalent

(i) $p$ majorizes $q$.

(ii) $\sum_{k=1}^{m} f(p_k) \leq \sum_{k=1}^{m} f(q_k)$ for all concave functions $f$.

Since $-x \log x$ is a concave function of $x$, it follows that $H(p) \leq H(q)$ if and only if $p$ majorizes $q$, where $H(p)$ is the Shannon entropy of a probability distribution $p$, defined in Eq. (24). Now consider two decompositions of a state vector, $D = \{(c_k, |\phi_k\rangle)\}_{k=1}^{m}$ and $D' = \{(c'_k, |\phi'_k\rangle)\}_{k=1}^{m'}$. Although these may have different cardinalities, they can be associated with probability distributions of equal cardinality by simply adding zeroes. Specifically, if $m \geq m'$, then $D$ is associated with the distribution $p_k = |c_k|^2$ for $k \in \{1, ..., m\}$ and $D'$ is associated with the distribution $q_k = |c'_k|^2$ for $k \in \{1, ..., m'\}$ and $q_k = 0$ for $k \in \{m' + 1, ..., m\}$. Since the IU entropy of $|\psi\rangle$ for the decomposition $D(D')$ is simply the Shannon entropy of $p(q)$, it follows that $S_{|\psi\rangle}(D) \leq S_{|\psi\rangle}(D')$ if $p$ majorizes $q$.

In order to facilitate the proof of lemma A.1, we set out two minor lemmas.

**Lemma A.3** Consider two probability distributions $p \equiv (p_1, p_2, ..., p_m)$ and $q \equiv (q_1, q_2, ..., q_m)$. If for every $l$ in the range $\{1, ..., m\},$

$$\sum_{k=1}^{l} p_k \geq \sum_{k=1}^{l} q_k,$$

then $p$ majorizes $q$.

**Proof.** This result follows from the definition of majorization and the fact that

$$\sum_{k=1}^{l} p_k^l \geq \sum_{k=1}^{l} p_k$$

for every $l$ in the range $\{1, ..., m\}$. This inequality is obviously true since the $l$-element subset of $p$ with the largest sum must be the subset containing the $l$ largest elements of $p$. QED.

For the second minor lemma, we make use of some notational conventions introduced in the text: $P_S$ denotes the projector onto the subspace $S$, and $P_{S'} < P_{S''}$ denotes that $S$ is a proper subspace of $S'$.

**Lemma A.4** If $P_S \leq P_{S'}$, then $\langle \psi | P_{S'} | \psi \rangle \leq \langle \psi | P_{S'} | \psi \rangle$.

**Proof.** If $P_S = P_{S'}$, then the inequality is saturated. Otherwise, $P_S < P_{S'}$, and there exists a projector $P_T$ such that $P_S + P_T = P_{S'}$. The desired inequality follows from the positivity of $\langle \psi | P_{S'} | \psi \rangle$. QED.

We are now in a position to prove lemma A.1.

**Proof of lemma A.1.** An arbitrary product decomposition has the form $D = \{(d_k, |\phi_k^A\rangle \otimes |\chi_k^B\rangle)\}_{k=1}^{m}$, where the lists of vectors $\{|\phi_k^A\rangle\}_{k=1}^{m}$ and $\{|\chi_k^A\rangle\}_{k=1}^{m}$ are not necessarily orthogonal nor even linearly independent (although the
list of vectors \( \{ |\phi^A_k \rangle \otimes |\chi^B_k \rangle \}_{k=1}^m \) is orthogonal. The decomposition \( D \) defines an \( m \)-element probability distribution \( q = \{ q_1, q_2, ..., q_m \} \), where \( q_k \equiv |d_k|^2 \). As before, let \( q_k^i \) denote the \( k \)th largest element of \( q \), and let \( |\phi^A_k \rangle \) and \( |\chi^A_k \rangle \) denote the vectors associated with \( q_k^i \).

Now, identify every vector in the list \( \{ |\phi^A_k \rangle \}_{k=1}^m \) that cannot be obtained as a linear combination of vectors with lower indices from this list. Suppose there a number \( m' \) of such vectors, corresponding to a particular subset \( S \) of the indices \( \{1, 2, ..., m\} \), so that the set of vectors is denoted by \( \{ |\phi^A_k \rangle \}_{k \in S} \). By definition, this is a linearly independent set. The remaining vectors are denoted by \( \{ |\phi^A_k \rangle \}_{k \in \bar{S}} \), where \( \bar{S} \) is the set of indices that remain after removing the elements of \( S \) from \( \{1, 2, ..., m\} \). Obviously the elements of \( \{ |\phi^A_k \rangle \}_{k \in \bar{S}} \) can all be written as linear combinations of the elements of \( \{ |\phi^A_k \rangle \}_{k \in S} \).

Finally, for future reference, we define \( g(k) \) as the number of indices \( k' \) in \( S \) such that \( k' \leq k \). It is clear from the definition of \( S \) that \( g(k) \leq k \).

Let \( \{ |\mu^A_j \rangle \}_{j=1}^m \) be the ordered set of orthogonal vectors that are obtained by applying the Gram-Schmidt orthogonalization procedure to \( \{ |\phi^A_k \rangle \}_{k \in \bar{S}} \), in order of ascending \( k \). This new set yields an \( A \)-orthogonal decomposition of \( |\psi \rangle \), \( D_{A\text{-orth}} = \{ (c_j, |\mu^A_j \rangle \otimes |\nu^B_j \rangle) \}_{j=1}^m \), where \( |\nu^B_j \rangle = \langle \mu^A_j |\psi \rangle / c_j \) and \( c_j = |\langle \mu^A_j |\psi \rangle | \). It also defines an \( m \)-element probability distribution \( p = \{ p_1, p_2, ..., p_m \} \) where \( p_j \equiv |c_j|^2 \) for \( j \) in the range \( \{1, ..., m'\} \), and \( p_j \equiv 0 \) for \( j \) in the range \( \{m'+1, ..., m\} \).

Let \( P_{\phi^A} \) denote the projector onto the ray spanned by \( |\phi^A \rangle \) and for convenience define \( P_{\mu^A} \equiv P_{null} \) for \( j \) in the range \( \{m'+1, ..., m\} \). The nature of the Gram-Schmidt orthogonalization procedure ensures that for every \( k \in S \), \( |\phi^A_k \rangle = \sum_{j=1}^{g(k)} f_{j,k} |\mu^A_j \rangle \) for some set of complex amplitudes \( \{f_{j,k}\}_{j=1}^{g(k)} \). Thus, \( (\sum_{j=1}^{g(k)} P_{\mu^A_j}) |\phi^A_k \rangle = |\phi^A_k \rangle \), or equivalently, \( \sum_{j=1}^{g(k)} P_{\mu^A_j} \geq P_{\phi^A_k} \). Since \( g(k) \leq k \), this is trivially extended to \( \sum_{j=1}^{\tilde{k}} P_{\mu^A_j} \geq P_{\phi^A_k} \). Moreover, if \( k \in \bar{S} \), then \( |\phi^A_k \rangle \) can be written as a linear combination of the elements of \( \{ |\phi^A_{k'} \rangle \}_{k' \in \bar{S}, k' \leq k} \), so that \( |\phi^A_k \rangle = \sum_{j=1}^{g(h(k))} f_{j,k} |\mu^A_j \rangle \) for some set of complex amplitudes \( \{f_{j,k}\}_{j=1}^{g(h(k))} \), where \( h(k) = \max_{k' \in \bar{S}, k' \leq k} k' \). It follows that \( \sum_{j=1}^{g(h(k))} P_{\mu^A_j} \geq P_{\phi^A_k} \) for all \( k \in \bar{S} \). Since \( g(h(k)) < k \), this is trivially extended to \( \sum_{j=1}^{\tilde{k}} P_{\mu^A_j} \geq P_{\phi^A_k} \). It follows therefore that for every \( k \) in the range \( \{1, ..., m\} \) we have \( \sum_{j=1}^{\tilde{k}} P_{\mu^A_j} \geq P_{\phi^A_k} \). Now, since \( IB \geq P_{\chi^B} \), we can infer that \( \sum_{j=1}^{\tilde{k}} P_{\mu^A_j} \otimes IB \geq P_{\phi^A_k} \otimes P_{\chi^B} \), and by the orthogonality of the projectors in the set \( \{ P_{\phi^A_k} \otimes P_{\chi^B_k} \}_{k=1}^l \), we conclude that \( \sum_{j=1}^{\tilde{k}} P_{\mu^A_j} \otimes IB \geq \sum_{k=1}^{l} P_{\phi^A_k} \otimes P_{\chi^B_k} \) for every \( k \) in the range \( \{1, ..., m\} \).

Now we note that the probability distributions \( q \) and \( p \) are related to the projectors by \( q_k^i = \langle \psi | P_{\phi^A_k} \otimes P_{\chi^B_k} |\psi \rangle \) and \( p_j = \langle \psi | P_{\mu^A_j} \otimes IB |\psi \rangle \). From the inequality derived above together with lemma A.4, we find that \( \sum_{j=1}^{\tilde{k}} p_j \geq \sum_{l=1}^{\tilde{k}} q_l \) for every \( k \) in the range \( \{1, ..., m\} \). By lemma A.3, it follows that \( p \) majorizes \( q \). QED.

Finally, we prove lemma A.2.

Proof of lemma A.2. An arbitrary \( A \)-orthogonal decomposition of \( |\psi \rangle \) has the form \( D_{A\text{-orth}} = \{ (c_j, |\mu^A_j \rangle \otimes |\nu^B_j \rangle) \}_{j=1}^m \), where the vectors \( \{ |\mu^A_j \rangle \}_{j=1}^m \) are orthogonal, but \( \{ |\nu^B_j \rangle \}_{j=1}^m \) need not be orthogonal nor even linearly independent. A bi-orthogonal decomposition of \( |\psi \rangle \) has the form \( D_{\text{bi-orth}} = \{ (c_j, |\tilde{\mu}^A_j \rangle \otimes |\tilde{\nu}^B_j \rangle) \}_{j=1}^m \), where both the vectors \( \{ |\tilde{\mu}^A_j \rangle \}_{j=1}^m \) and \( \{ |\tilde{\nu}^B_j \rangle \}_{j=1}^m \) form orthogonal sets. The probability distributions associated with each decomposition are \( \{ |c_1|^2, |c_2|^2, ..., |c_m|^2 \} \) and \( \{ |\tilde{c}_1|^2, |\tilde{c}_2|^2, ..., |\tilde{c}_m|^2 \} \) respectively (even if there is more than one bi-orthogonal decomposition for a particular state vector, these do not differ in their coefficients). For ease of comparison of these distributions, we add zeroes until the number of elements in each is equal to the dimensionality, \( d \), of the Hilbert space \( \mathcal{H}^A \).

Denote the resulting distributions by \( p \) and \( \tilde{p} \) respectively. We establish that \( S_{|\psi \rangle} (D_{\text{bi-orth}}) \leq S_{|\psi \rangle} (D_{A\text{-orth}}) \) by showing that \( \tilde{p} \) majorizes \( p \).

To begin, we express the probabilities as expectation values of projectors. We introduce an arbitrary orthogonal set of vectors \( \{ |\mu^A_j \rangle \}_{j=m+1}^d \) which together with \( \{ |\mu^A_j \rangle \}_{j=1}^m \) form an orthogonal basis for the Hilbert space \( \mathcal{H}^A \), and similarly for \( \{ |\tilde{\mu}^A_j \rangle \}_{j=1}^d \). Then, we have for all \( j \) in the range \( \{1, ..., d\} \),

\[
p_j = Tr_A (\rho^A P_{\mu^A_j}),
\]

\[
\tilde{p}_j = Tr_A (\rho^A P_{\tilde{\mu}^A_j}),
\]

Let the unitary operator that transforms the elements of \( \{ |\mu^A_j \rangle \}_{j=1}^d \) to the elements of \( \{ |\tilde{\mu}^A_j \rangle \}_{j=1}^d \) be denoted by \( U^A \), so that
\[ |\mu_j^A \rangle = U^A |\tilde{\mu}_j^A \rangle. \]

It follows that
\[
p_j = Tr_A (\rho^A U^A |\tilde{\mu}_j^A \rangle \langle \tilde{\mu}_j^A | U^A \rho^A U^A) = Tr_A (U^A \rho^A U^A |\tilde{\mu}_j^A\rangle \langle \tilde{\mu}_j^A |),
\]
where in the last step we have used the cyclic property of the trace. What distinguishes the bi-orthogonal decomposition from other A-orthogonal decompositions is that the projectors \( \{ \tilde{P}_{\tilde{\mu}_k^A} \}_{k=1}^d \) diagonalize \( \rho^A \),
\[
\rho^A = \sum_{k=1}^d \tilde{p}_k \tilde{P}_{\tilde{\mu}_k^A}.
\]

Plugging this form of \( \rho^A \) into the expression for \( p_j \), we obtain
\[
p_j = \sum_{k=1}^d |U^A_{jk}|^2 \tilde{p}_k,
\]
where \( U^A_{jk} = \langle \tilde{\mu}_j | U^A |\tilde{\mu}_k \rangle \). By the unitarity of \( U^A \), we find that \( \sum_j |U^A_{jk}|^2 = \langle \tilde{\mu}_k | U^A |U^A |\tilde{\mu}_k \rangle = 1 \), and \( \sum_k |U^A_{jk}|^2 = \langle \tilde{\mu}_j | U^A |U^A |\tilde{\mu}_j \rangle = 1 \). Thus, the transition matrix between the probability distributions \( \tilde{p} \) and \( p \) is doubly stochastic, from which it follows by a well-known result (theorem II.1.9 of Ref. [39]) that \( \tilde{p} \) majorizes \( p \). QED.
Fig. 1 An example of a sequence of measurements upon a spin 1/2 particle for which the faithfulness criterion can only be satisfied if one adopts non-orthogonal preferred projectors. In the diagram, \(SG(n)\) denotes a Stern-Gerlach magnet oriented along the \(n\) axis. D1 and D2 are detectors. A and B denote the two possible locations at which the particle might be detected by D1. The magnetic field after D1 is designed to map the state \(|-z\rangle\) onto \(|+x\rangle\). The axis \(n\) of the second Stern-Gerlach magnet is assumed to depend on the measurement outcome at D1. If the particle is found at A, then \(n\) is set to \(z\), while if it is found at B, then \(n\) is set to \(x\). Although the steering mechanism for the beams is not indicated, an appropriate mechanism can be devised (particularly since it can be taken to depend on the measurement outcome at D1).