Analytical Newtonian models of finite thin disks in a magnetic field

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Axially symmetric Newtonian thin disks of finite extension in presence of magnetic field are studied based on the well-known Morgan-Morgan solutions. The source of the magnetic field is constructed separating the equation corresponding to the Ampere’s law of electrodinamic in spheroidal oblate coordinates. This produces two associated Legendre equations of first order for the magnetic potential and hence that can be expressed as a series of associated Legendre functions of the same order. The discontinuity of its normal derivate across the disk allows us interpreter the source of the magnetic field as a ringlike current distribution extend on the plane of the disk. We also study the motion of charged test particles around of the disks. In particular we analysis the circular speed curves or rotation curve for equatorial circular orbits of particles both inside and outside the disk. The stability of the orbits is analyzed for radial perturbation using a extension of the Rayleigh criterion.

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I. INTRODUCTION

Axially symmetric solutions of the Poisson’s equation describing the field of a thin disk are important in astrophysics as models of galaxies motivated by the fact that the main part of the mass of the galaxies is concentrated in the galactic disk [1]. Even though a realistic disk have thickness, in first approximation these astrophysical objects can be considered to be very thin, e.g., in our Galaxy the radius of the disk is 10 kpc and its thickness is 1 kpc. On the other hand, there is strong observational evidence that magnetic fields are present in all galaxies and galaxy clusters [2, 3]. These fields are characterized by a strength of the order of $\mu$ G. Observational data also provided a strong support of the existence of extragalactic magnetic fields of at least $B \simeq 10^{-17}$ G on Mpc scales [4, 5]. Thus thin disks in presence of magnetic field are also important as models of flat galaxies.

The simpler model of thin disk is the Kuzmin-Toomre disk [6, 7] that represents a disk-like matter distribution with a concentration of mass in its center and density that decays as $1/r^3$ on the plane of the disc. These models are constructed using the image method that is usually used to solve problems in electrostatics. This structure has no boundary of the mass but as the surface mass density decreases rapidly one can define a cut off radius, of the order of the galactic disk radius, and, in principle, to consider these disks as finite.

Another simple model of disks are the Morgan-Morgan disks [8]. These disks have a mass concentration on their centers and finite radius. The models are constructed using a method developed by Hunter [9] based on obtaining of solutions of Laplace equation in terms of oblate spheroidal coordinates, which are ideally suited to the study of flat disks of finite extension. Several classes of analytical solutions of the Poisson’s equation corresponding thin disks have been obtained by different authors [10, 13].

In this work we considerer analytical Newtonian models of finite thin disk in presence of a magnetic field based on the well-known Morgan-Morgan solution. The method used to construct the source of magnetic field is the Hunter method which involves the use of oblate spheroidal coordinates.

The paper is organized as follows. In Sec. II we we present the formalism to construct models of thin disks with electric current. We also analysis the motion of charged test particles around of the disks and we derivate the stability condition of the system against radial perturbations using an extension of the Rayleigh criterion. In Sec. III we present a simple family of finite thin disks in presence of a magnetic field based on the well-known Morgan-Morgan solutions. We present explicit expression for the main physical quan tities associated to the disks for the two first terms of the series of the solutions. Finally, in Sec. IV we summarize and discuss the results obtained.

II. DISKS AND MAXWELL EQUATIONS

Exact solutions of Poisson’s equation representing the field of a thin disk at $z = 0$ can be constructed assuming the gravitational potential $\Phi$ continuous across the disk, and its first derivative discontinuous in the direction normal to the disk. This can be written as

$$[\Phi, z] = 2 \Phi, z|_{z=0^+}.$$  

Since the disk is thin the Poisson’s equation can be written as

$$\nabla^2 \Phi = 4\pi G\Sigma(R)\delta(z),$$  

where $\delta(z)$ is is the usual Dirac function with support on the disk and $\Sigma(R)$ is the surface mass density. The mass density can be obtained for example using the approach presented in Ref. [14]. Thus, written by the Laplacian operator in cylindrical coordinates and integrating from $z = 0^-$ to $z = 0^+$, we obtain

$$\Sigma(R) = \frac{1}{2\pi G} \frac{\partial \Phi}{\partial z}|_{z=0^+}. $$

Similarly, thin disks with electric current in presence of a pure magnetic field can be obtained assuming the magnetic potential $\mathbf{A}$ continuous and its first derivate discontinuous. That is

$$[\mathbf{A}, z] = 2 \mathbf{A}, z|_{z=0^+}.$$  

The magnetic field is governed by the Maxwell’s equations

$$\nabla \cdot \mathbf{B} = 0,$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}. $$
where $J$ is the electric current density vector. For axially symmetric fields the magnetic potential is $\mathbf{A} = A \hat{\varphi}$ where $A$ is function of $R$ and $z$ only and $\hat{\varphi}$ a unit vector in the azimuthal direction. Since $\mathbf{B} = \nabla \times \mathbf{A}$, the Eq. 55 can be cast as

$$\nabla^2 A - \frac{1}{R^2} A = -\mu_0 j_\varphi \delta(z). \tag{6}$$

where $j_\varphi$ is the surface azimuthal electric current density. Again written by the Laplacian operator in cylindrical coordinates and integrating from $z = 0^-$ to $z = 0^+$, we obtain

$$j_\varphi = -\frac{2}{\mu_0} \frac{\partial A}{\partial z} \bigg|_{z=0^+}. \tag{7}$$

Now we analyze the motion of charged test particles around of the disks. For equatorial circular orbits the motion equation of test particles reads

$$\frac{\partial \Phi}{\partial R} - \hat{c} v \frac{1}{R} \frac{\partial (RA)}{\partial R} = \frac{v^2}{R}, \tag{8}$$

where $\hat{c}$ is the specific electric charge of the particles and $v$ is the speed of the particles which is given by

$$v_c = \frac{\hat{c}}{2} \frac{\partial}{\partial R} (RA) \pm \sqrt{\left(\frac{\hat{c}}{2} \frac{\partial}{\partial R} (RA)\right)^2 + R \frac{\partial \Phi}{\partial R}}. \tag{9}$$

The positive sign corresponds to the direct orbits or co-rotating and the negative sign to the retrograde orbits or counter-rotating.

To analyze the stability of the particles of the disks in the case of circular orbits in the equatorial plane we use an extension of Rayleigh criteria of stability of a fluid at rest in a gravitational field [15–17]. The method works as follows. Any small element of the matter distribution analyzed (in our case a test particle of the disks) is displaced slightly from its path. As a result of this displacement, forces appear which act on the displaced matter element. If the matter distribution is stable, these forces must tend to return the element to its original position.

The term first on the left-hand side of the motion equation (5) is the gravitational force $F_\theta$ (per unit of mass), the term second the magnetic force $F_m$, and the term on the right-hand side the centrifugal force $F_c$ acting on the test particle. So we have a balance between the total force $F = F_\theta + F_c$ and the centrifugal force. We consider the particle to be initially in a circular orbit with radius $R = R_0$. In terms of specific angular momentum $L = Rv + \hat{c}AR$, $F_c(R_0) = (L_0 - \hat{c}AR_0)^2/R_0^3$. We slightly displace the particles to a higher orbit $R > R_0$. Since the angular momentum of particle remains equal to its initial value, the centrifugal force in its new position is $\hat{F}_c(R) = (L_0 - \hat{c}AR)^2/R^3$. In order that the particle returns to its initial position must be met that $F(R) > \hat{F}_c(R)$, but according to the balance equation (5) $F(R) = F_c(R)$ so that $F_c(R) > \hat{F}_c(R)$, and hence $(L - \hat{c}AR)^2 > (L_0 - \hat{c}AR)^2$. It follows that the condition of stability is $L^2 > L_0^2$. By doing a Taylor expansion of $L$ around $R = R_0$, we can write this condition in the form

$$L \frac{dL}{dR} > 0, \tag{10}$$

or, in other words, $dL^2/dR > 0$.

From the relation $\mathbf{B} = \nabla \times \mathbf{A}$, we find

$$\frac{\partial}{\partial R} (RA)dR + \frac{\partial}{\partial z} (RA)dz = 0. \tag{11}$$

Thus the equation $RA = C$, with $C$ constant, gives the lines of force of the magnetic field.

### III. FINITE DISKS WITH MAGNETIC FIELD

Solutions representing the field of a finite thin disk can be obtained resolving the Laplace equation in oblate spheroidal coordinates $(u,v)$, which are defined in terms of the cylindrical coordinates $(R, z)$ by

$$R^2 = a^2(1 + u^2)(1 - v^2), \tag{12a}$$

$$z = auv. \tag{12b}$$
and explicitly
\begin{align}
\sqrt{2}u &= \sqrt{[(\tilde{R}^2 + \tilde{z}^2 - 1)^2 + 4\tilde{z}^2]^{1/2} + \tilde{R}^2 + \tilde{z}^2 - 1}, \\
\sqrt{2}v &= \sqrt{[(\tilde{R}^2 + \tilde{z}^2 - 1)^2 + 4\tilde{z}^2]^{1/2} - (\tilde{R}^2 + \tilde{z}^2 - 1),}
\end{align}
(13a)
(13b)
where \( u \geq 0 \), \(-1 < v < 1 \), \( \tilde{R} = R/a \), \( \tilde{z} = z/a \), and \( a \) the radius of the disk. The disk is located in \( u = 0 \), \(-1 < v < 1 \), and when is crossed the coordinate \( v \) changes of sign but not its value absolute, whereas \( u \) is continuous. This implies that an even function of \( v \) is a continuous function everywhere but has a discontinuous \( v \) derivative at the disk.

In this coordinate system the Laplacian operator has the form
\[ \nabla^2 = \frac{1}{a^2(u^2 + v^2)} \left[ \frac{\partial}{\partial u}(1 + u^2) \frac{\partial}{\partial u} + \frac{\partial}{\partial v}(1 - v^2) \frac{\partial}{\partial v} \right]. \]
(14)

The general solution of Laplace's equation can be written as
\[ \Phi = -\sum_{n=0}^{\infty} c_{2n}q_{2n}(u)P_{2n}(v), \]
(15)
where \( c_{2n} \) are constants, \( P_{2n} \) are the Legendre polynomials of order \( 2n \) and
\[ q_{2n}(u) = i^{2n+1}Q_{2n}(iu), \]
(16)
being \( Q_{2n}(iu) \) the Legendre functions of the second kind.

The mass surface density \( \Sigma \) takes the form
\[ \Sigma(R) = \frac{1}{2\pi aG} \sum_{n=0}^{\infty} c_{2n}(2n + 1)q_{2n+1}(0)P_{2n}(v) \]
(17)
with \( v = \sqrt{1 - R^2/a^2} \).

For \( n = 0 \) we have the zeroth order Morgan-Morgan disk and for the two first terms of the series the first Morgan-Morgan disk \( \Sigma \). In this case, the gravitational potential is
\[ \Phi = -\frac{MG}{a} \left\{ \cot^{-1}(u) + \frac{1}{4} \left[ (3u^2 + 1) \cot^{-1}(u) - 3u \right] (3v^2 - 1) \right\}, \]
(18)
being \( M \) is the mass of the disks, and the surface mass density
\[ \Sigma(R) = \frac{3M}{2\pi a^2} \sqrt{1 - \frac{R^2}{a^2}}. \]
(19)

Now we consider a thin disk with electric current in presence of a pure magnetic field. With the Laplace operator in the form \( \nabla^2 \), the change of variable \( u = ix \), the algebraic relation
\[ \frac{x^2 - v^2}{(1 - x^2)(1 - v^2)} = \frac{1}{1 - x^2} - \frac{1}{1 - v^2}, \]
(20)
and letting \( A = X(x)V(v) \), the expression for the magnetic potential \( \Phi \) yields two associated Legendre equations of first order
\begin{align}
(1 - x^2) \frac{d^2X}{dx^2} - 2x \frac{dX}{dx} + l(l + 1)X - \frac{1}{1 - x^2} X &= 0, \quad (21a) \\
(1 - v^2) \frac{d^2V}{dv^2} - 2v \frac{dV}{dv} + l(l + 1)V - \frac{1}{1 - v^2} V &= 0. \quad (21b)
\end{align}
The solution which vanishes at infinity is \( A_n = b_n^1 Q_n^1(-iu)P_n^1(v) \), where \( b_n \) is a constant, \( P_n^1(v) \) are the associated Legendre functions of the first kind of order \( 1 \) and \( Q_n^1(-iu) \) the associated Legendre functions of the second kind of order \( 1 \). But \( Q_n^m(-z) = (-1)^{n + m} Q_n^m(z) \), hence \( A_n = b_n(-1)^{n+1} Q_n^1(iu)P_n^1(v) \). Moreover, since \( A \) must be invariant to reflection in the equatorial plane by symmetry of the problem, \( A(u, v) = A(u, -v) \), the parity property
\( P^m_n(-v) = (-1)^{n+m} P^m_n(v) \) shows that \( n \) is an odd integer. Thus the most general solutions for the magnetic potential can be written as

\[
A = \sum_{n=0}^{\infty} b_{2n+1} (-1)^{2n+3} Q^1_{2n+1}(iu) P^1_{2n+1}(v).
\]

(22)

As \( A \) is an even function of \( v \), it is continuous across the disk. Consequently its normal derivative is a function odd of \( v \) and hence discontinuous across the disks. This implies that the source of the magnetic field also is planar. Using the relation

\[
\frac{d}{dz} Q^m_n(z) \bigg|_{z=0} = -(n - m + 1) Q^m_{n+1}(0),
\]

(23)

one finds that the surface current density \( j_\varphi \) is

\[
\dot{j}_\varphi = \frac{2}{\mu_0 a v} \sum_{n=0}^{\infty} b_{2n+1} (2n+1)(-1)^{2n+3} i Q^1_{2n+2}(0) P^1_{2n+1}(v),
\]

(24)

where \( v = \sqrt{1 - R^2/a^2} \).

Since the surface current density diverges at the disc edge, when \( v = 0 \), must we impose the condition

\[
\sum_{n=0}^{\infty} b_{2n+1} (2n+1)(-1)^{2n+3} i Q^1_{2n+2}(0) P^1_{2n+1}(0) = 0.
\]

(25)

Using the identities

\[
P^1_{2n+1}(0) = (-1)^n \frac{(2n+1)!}{(2n)!},
\]

(26a)

\[
iQ^1_{2n+2}(0) = (-1)^{n+2} (2)^{n+1} \frac{(n+1)!}{(2n+1)!}.
\]

(26b)

one finds

\[
b_1 = -\sum_{n=1}^{\infty} b_{2n+1} (2n+1) \frac{(n+1)!}{n!},
\]

(27)

which allows us to express the value of the constants \( b_{2n+1} > 1 \) in terms of \( b_1 \). Thus \( b_1 \) is the parameter that controls the magnetic field and can be determined by knowing the magnetic field at some point of galaxy, through the relation \( B = |\nabla \times \vec{A}| \).

In oblate coordinate the circular velocity \( v_c \) takes the form

\[
v_c = -\frac{\dot{E}}{2} \left( A + \frac{(v^2 - 1)}{v} \frac{\partial A}{\partial v} \right) \pm \sqrt{\frac{\dot{E}^2}{4} \left( A + \frac{(v^2 - 1)}{v} \frac{\partial A}{\partial v} \right)^2 + \frac{(v^2 - 1)}{v} \frac{\partial \Phi}{\partial v}}.
\]

(28)

We consider the two first terms of the series \( (22) \). In this case \( b_1 = -6b_3 \) and the magnetic potential is given by

\[
A = \frac{b_1}{8} \sqrt{1 - v^2} \left\{ \frac{u[8 + (13 + 15u^2)(5v^2 - 1)]}{\sqrt{1 + u^2}} - \left[ 8 + 3(1 + 5u^2)(5v^2 - 1) \right] \sqrt{1 + u^2} \cot^{-1}(u) \right\}.
\]

(29)

The surface electric charge density is

\[
\dot{j}_\varphi = -\frac{20b_1}{\mu_0 a} (R/a) \sqrt{1 - \frac{R^2}{a^2}}.
\]

(30)

Inside the disk the circular speed of particles is given by

\[
v_c = -\frac{5}{8} b \pi R(3R^2 - 2) \pm \sqrt{\frac{25}{64} b^2 \pi^2 R^2 (3R^2 - 2)^2 + 3\pi R^2},
\]

(31)
whereas outside the disk is

\[ v_c = -\frac{5}{4}ebR[-3u + (3R^2 - 2) \cot^{-1}(u)] \pm \frac{25}{16}e^2b^2R^2[-3u + (3R^2 - 2) \cot^{-1}(u)]^2 + \frac{3}{4}[-2u + \pi R^2 - 2R^2 \cot^{-1}(u)], \]

with \( u = \sqrt{R^2 - 1} \).

In Fig. 4 we plot the electric current density \( j_\varphi \) for \( \tilde{e} = 1 \) different values of the parameter of magnetic fields \( b_1 = 0 \) (dashed curves), 0.1, 0.3, and 0.5 (dash-dotted curves), as functions of \( R \). We see that \( j_\varphi \) is zero in the center of the disk where the surface mass density is greater (Eq. (19)), reaches a maximum and then falls to zero at rim of the disk where there is no matter.

In Fig. 2 we plot the curves of the circular speed \( v_+ \) and \( v_- \) for the motion of charged test particles inside the the disk (Figs. (a) and (b)) and for particles outside the disk (Figs. (c) and (d)), for \( \tilde{e} = 1 \) and the same value of the other parameters. Inside the disk, we find that for direct orbits the magnetic field increases the speed of particles to a certain values of \( R \), and then decreases it, whereas for retrograde orbits the contrary occurs. Outside the disk, we observer that for direct orbits the magnetic field decreases the speed of particles everywhere of the disk whereas for retrograde orbits the contrary occurs.

In Fig. 3 we plot the specific angular momentum \( L_+^2 \) and \( L_-^2 \) for the motion of test particles inside the the disk (Figs. (a) and (b)) and for particles outside the disk (Figs. (c) an (d)), with \( \tilde{e} = 1 \) and the same value of the other parameters. Inside the disk, we find that for direct orbits the magnetic field can make less stable the motion of the particles against radial perturbations, whereas for retrograde orbits the particles are always stable. Outside the disk, we observer that for direct orbits the magnetic field can stabilize the particles, whereas for retrograde orbits the magnetic field enhances the zone of instability .

In Fig. 5 we plot the surfaces and level curves of the function \( RA \) that represent the magnetic field lines as function of \( R \) and \( z \) with parameter \( b_1 = 0.5 \). We find that the lines are closed curves surrounding the disk which suggests that the source of the magnetic field is a ringlike current distribution.

### IV. DISCUSSION

Finite thin disk models in presence of magnetic field were presented based on the Morgan-Morgan solutions. The source of the magnetic field was constructed separating the equation corresponding to the Ampere’s law of electrodynamic in oblate spheroidal coordinates. We found that lines of magnetic field are closed curves surrounding the disk which suggests that the field is generated for a ringlike current distribution. We obtained general expressions for the magnetic potential and explicit expressions for the the two first terms of series of solutions for the main physical quantities associated to the disks. In particular, we found that the surface electric current density is is zero in the center of the disk where the surface mass density is greater, reaches a maximum and then falls to zero at rim of the disk where there is no matter.

We also analyzed the equatorial circular motion of charged test particles for the two first terms of series of solutions and we discuss their stability against radial perturbations. Inside the disk, we found regions where the magnetic field increases the speed of particles and regions where the opposite occurs. For direct orbits, we observed that the field can make less stable the motion of the particles. Outside the disk, we found that the magnetic field decreases the speed of particles for direct orbits and that can stabilize them, whereas for retrograde orbits the field increases the speed of the particles and enhances the zone of instability.

Finally, the relativistic version of these models is being considered.

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FIG. 1. The electric current density $j_\varphi$ for $\bar{c} = 1$, and different values of the parameter of magnetic fields $b_1 = 0$ (dashed curves), 0.1, 0.3, and 0.5 (dash-dotted curves), as functions of $R$. 
FIG. 2. The circular speed $v_+$ and $v_-$ for the motion of test particles inside of the disk (Figs. (a) and (b)) and for particles outside of the disk (Figs. (c) and (d)) for $\tilde{e} = 1$ and different values of the parameter of magnetic fields $b_1 = 0$ (dashed curves), 0.1, 0.3, and 0.5 (dash-dotted curves), as functions of $R$. 
FIG. 3. The specific angular momentum $L^2_+$ and $L^2_-$ for the motion of test particles inside of the disk (Figs. (a) an (b)) and for particles outside of the disk (Figs. (c) an (d)) for $\vec{e} = 1$, and different values of the parameter of magnetic fields $b_1 = 0$ (dashed curves), 0.1, 0.3, and 0.5 (dash-dotted curves), as functions of $R$.

FIG. 4. Surfaces and level curves of the functions $RA$ as functions of $R$ and $z$ for $b_1 = 0.5$. 