Conformal (super)gravities with several gravitons

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Abstract

We construct consistent interacting gauge theories for $M$ conformal massless spin-2 fields ("Weyl gravitons") with the following properties: (i) in the free limit, each field fulfills the equation $\mathcal{B}^{\mu\nu} = 0$, where $\mathcal{B}^{\mu\nu}$ is the linearized Bach tensor, (ii) the interactions contain no more than four derivatives, just as the free action and (iii) the internal metric for the Weyl gravitons is not positive definite. The interacting theories are obtained by gauging appropriate non-semi-simple extensions of the conformal algebra $so(4, 2)$ with commutative, associative algebras of dimension $M$. By writing the action in terms of squares of super-curvatures, supersymmetrization is immediate and leads to consistent conformal supergravities with $M$ interacting gravitons.
1 Introduction

Recently, no-go theorems have been proven that prevent interactions between a set of massless symmetric fields $h_{a}^{\mu \nu}$ ($a = 1, \cdots, M$) described in the free limit either by the Pauli-Fierz 1 or the linearized Weyl 2 Lagrangian. These theorems were obtained by using the antifield-based cohomological approach to the problem of consistent interactions 3, 4.

A crucial assumption underlying the no-go theorems is that the metric in the internal space of the fields $h_{a}^{\mu \nu}$ be positive definite. This assumption appears to be quite reasonable in the Pauli-Fierz case, where it guarantees that the free (classical) theory has non-negative energy, leading to a quantum theory free of negative norm states, or ”ghosts”. It is less motivated, however, in the conformal case where the classical energy is unbounded from below even for a positive definite internal metric. For this reason, it is of interest to investigate theories in which positive definiteness of the internal metric is relaxed.

If the assumption on the metric in internal space is dropped, interactions become possible. An explicit example involving two (Weyl) ”gravitational” fields $h_{1}^{\alpha \beta}$ and $h_{2}^{\alpha \beta}$ in interaction was given in 2. The action reads

$$S[h_{a}^{\mu \nu}] = \frac{1}{\alpha} \int d^{4}x \sqrt{-g} h_{\mu \nu}^{2} B_{\mu \nu},$$

(1.1)

where

$$B_{\mu \nu} = 2\nabla^{\alpha} \nabla^{\beta} C_{\mu \nu \alpha \beta} + R^{\lambda \rho} C_{\lambda \mu \nu \rho}$$

(1.2)

is the Bach tensor of the metric $g_{\alpha \beta} = \eta_{\alpha \beta} + \alpha h_{1}^{\alpha \beta}$ and $g$ its determinant. The parameter $\alpha$ is a deformation parameter. In the limit $\alpha \to 0$, one recovers the free theory, with Lagrangian $\sim h_{\mu \nu}^{2} D_{\mu \nu}^{\alpha \beta} h_{\alpha \beta}^{1}$ where $D_{\mu \nu}^{\alpha \beta}$ is the fourth order differential operator appearing in the (linearized) equations of motion. The free equations of motion read $B_{\alpha \beta}^{a \mu \nu} = 0$, where $B_{a}^{a \mu \nu}$ is the linearized Bach tensor, $B_{a}^{a \mu \nu} = D_{\mu \nu}^{\alpha \beta} h_{a}^{\alpha \beta}$ ($a = 1, 2$) (see 2 for more information). The gauge transformations are

$$\delta_{\eta^{\alpha}, \phi^{a}} h_{\mu \nu}^{1} = \eta_{\mu ; \nu}^{1} + \eta_{\nu ; \mu}^{1} + 2\phi^{1} g_{\mu \nu},$$

$$\delta_{\eta^{\alpha}, \phi^{a}} h_{\mu \nu}^{2} = \alpha \mathcal{L}_{\eta^{1}} h_{\mu \nu}^{2} + 2\alpha \phi^{1} h_{\mu \nu}^{1} + \eta_{\mu ; \nu}^{2} + \eta_{\nu ; \mu}^{2} + 2\phi^{2} g_{\mu \nu},$$

(1.3)

(1.4)

where covariant derivatives (;) are computed with the metric $g_{\alpha \beta}$ and where $\mathcal{L}$ denotes the Lie derivative $\mathcal{L}_{\eta^{1}} h_{\mu \nu} = \partial_{\mu} \eta^{1} h_{\rho \nu} + \partial_{\nu} \eta^{1} h_{\rho \mu} - \eta^{1} \partial_{\rho} h_{\mu \nu}$. As one sees from the free action, the metric $k_{ab}$ in internal space is given by $k_{12} = k_{21} = 1$, $k_{11} = k_{22} = 0$ and so has signature (−+). By introducing the linear combinations $\chi_{\alpha \beta} \sim h_{1}^{\alpha \beta} + h_{2}^{\alpha \beta}$ and $\psi_{\alpha \beta} \sim h_{1}^{\alpha \beta} - h_{2}^{\alpha \beta}$, one can diagonalize $k_{ab}$ and get as free Lagrangian $\mathcal{L}^{F} \sim \chi_{\mu \nu} D_{\mu \nu}^{\alpha \beta} \chi_{\alpha \beta} - \psi_{\mu \nu} D_{\mu \nu}^{\alpha \beta} \psi_{\alpha \beta}$.

The purpose of this note is to provide a group-theoretical understanding of (1.1) in terms of the conformal group along the lines of 5, 7. We
show that the above ”2-Weyl-graviton theory” (respectively, the ”M-Weyl-graviton theory”\(^1\)) can be viewed as the gauge theory of the tensor product \(\text{so}(4, 2) \otimes A\) of the conformal algebra \(\text{so}(4, 2)\) with an irreducible, associative and commutative algebra \(A\) of dimension 2 (respectively, \(M\)). The gauging procedure follows the general pattern explained in [7, 8].

The advantage of the group-theoretical viewpoint is that it allows for a direct supersymmetrization. We restrict ourselves to \(N = 1\) (simple) conformal supergravities, i.e., supergravities with one graviton. These we extend to theories with \(M\) gravitons (whereas \(N\)-extended supergravities still always have one graviton). One must simply repeat the steps worked out in \([8, 9, 10]\) for the one-multiplet case. We provide a general existence proof of the \(M\)-multiplet conformal supergravities and give a constructive method for deriving the Lagrangian from the 1-multiplet theory.

At the quantum level, conformal and superconformal theories are, according to the usual perturbative approaches, not unitary. The counting of states is more subtle \([11]\). These drawbacks are of course also present in their multi-multiplet ”daughters”. For this reason, most physicists have used the conformal supergravities as building blocks and backbones of ordinary supergravities without attributing much direct physical meaning to them. However, another point of view, which is increasingly adopted, is to view conformal (super)gravities as effective field theories, in which case standard issues of unitarity would no longer apply \([12]\). From this point of view, conformal supergravities may become of great importance in the future. We should also mention that unconventional possibilities for actually making sense out of theories with ghosts have been explored recently in terms of appropriate averages over the ghost states \([13]\), or through dualities \([14, 15]\) (for some earlier analysis, see also \([16]\)).

Although we concentrate on conformal gravity for reasons explained above, similar considerations apply to standard gravity, provided one replaces the conformal group by the (anti) de Sitter group or its Poincaré contraction \([17]\). In that case also, one can construct interacting multi-graviton theories if the metric in internal space is not positive definite \([18, 19]\). An explicit example, analogous to \((1.4)\), was given in \([19]\). The group-theoretical approach provides considerable insight into the algebraic structures underlying the interactions discussed by Wald \([20]\) by relating them to ordinary Lie algebras and their gaugings. It also yields straightforwardly real-valued (as opposed to algebra-valued) actions. Finally, the approach significantly clarifies the construction of the supersymmetric actions studied in \([21]\).

\section{Algebraic preliminaries}

\(^1\)We shall loosely call hereafter ”\(M\)-Weyl-graviton theories” theories with a set of \(M\) fields \(h_{\mu \nu}^a\) obeying the Bach equations in the free limit.
2.1 Associative, commutative algebras

We first recall the results of [1, 2], obtained by using the cohomological tools developed in [22, 23, 24, 25, 26] for dealing with the local cohomologies $H(s|d)$ ($s$ being the BRST differential and $d$ the spacetime exterior derivative) relevant to the gauge-consistent deformations of a gauge-invariant theory (by gauge-consistent we mean that a $M$-graviton theory will have $M$ times the number of gauge symmetries of the corresponding one-graviton theory). The various fields $h^a_{\mu\nu}$, $a = 1, \ldots M$ of the given set of (Weyl) gravitons are assumed to live in an internal $M$-dimensional algebra $\mathcal{A}$, i.e., a (real) vector space endowed with a bilinear map $\star : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$, i.e. a tensor $a^a_{bc}$ of type $(1,2)$ over $\mathcal{A}$, generally referred to as "multiplication". Upon expansion into a basis $\{E_a\}_{a=1}^M$ the product reads

$$v = v^a E_a, \quad u = u^a E_a,$$

(2.1)

the product reads

$$u \star v = z = z^a E_a = u^b a^a_{bc} v^c E_a.$$  

(2.2)

The $a^a_{bc}$ define the interactions [1, 2]. The algebra $\mathcal{A}$ is also equipped with an internal scalar product, $<u, v> = k_{ab} u^a v^b$, defined by the quadratic part of the action (i.e., the free action).

The tensor $a^a_{bc}$ and the metric $k_{ab}$ were shown to be constrained by the following identities for the deformation to be consistent:

$$a^a_{bc} = a^a_{cb}$$

(2.3)

$$a^a_{b[c} a^b_{d]e} = 0$$

(2.4)

$$k_{ad} a^d_{bc} \equiv a_{abc} = a_{bac}$$

(2.5)

The first equation (2.3) implies that the algebra $\mathcal{A}$ is commutative, (2.4) that it is associative, and (2.3) that the metric is an invariant tensor.

$$u \star v = v \star u$$

(2.6)

$$(u \star v) \star w = u \star (v \star w)$$

(2.7)

$$<u \star v, w> = <u, v \star w>.$$

(2.8)

The same conditions arise in the Einstein case. The conditions (2.3) and (2.4) were first derived in that context in the pioneering work [18], where the emphasis was put on the consistency of the gauge transformations and their algebra. The further symmetry condition (2.8) emerges from the analysis of the action itself. Its importance was particularly stressed in [1, 2].

The structure constants $a^a_{bc}$ characterize actually the only possible consistent deformations of the free action under the condition of Lorentz invariance and the restriction that the complete action should contain no more derivatives (namely 4) than the free action [2]. The conditions (2.6), (2.7) and

\[2\text{the conditions (2.3) and (2.3) together imply that } a_{abc} = a_{(abc)}, \text{ so we will sometimes use the term "symmetric" when referring to the algebras endowed with such a } a_{abc} \text{ tensor.} \]
guarantee consistency of the deformation up to second order included. It follows from the existence of actions explicitly given below that consistency holds in fact to all orders.

Taking the internal metric $k_{ab}$ to be positive-definite ($k_{ab} = \delta_{ab}$), i.e., starting with a free action that is the sum (with only plus signs) of linearized Weyl (or Pauli-Fierz) actions, the algebra $\mathcal{A}$ can be shown to be the direct sum of one-dimensional ideals, which implies the existence of a basis where $a^a_{\ b\ c} = 0$ whenever two indices are different \footnote{The cross-interactions between the various Weyl-gravitons can be removed by redefinitions. If $k_{ab}$ is of mixed signature, however, the algebra $\mathcal{A}$ need not be trivial, and one can construct truly interacting multi-gravitons theories.}: the cross-interactions between the various Weyl-gravitons can be removed by redefinitions. If $k_{ab}$ is of mixed signature, however, the algebra $\mathcal{A}$ need not be trivial, and one can construct truly interacting multi-gravitons theories.

As shown in \footnote{We thank Giulio Bonelli for having pointed this out to us} irreducible, commutative, associative algebras of finite dimension can be divided into three types:

1. $\mathcal{A}$ contains no identity element and every element of $\mathcal{A}$ is nilpotent ($v^m = 0$ for some $m$).

2. $\mathcal{A}$ contains one (and only one) identity element $e$ and no element $j$ such that $j^2 = -e$. In that case, $\mathcal{A}$ contains a $(M - 1)$-dimensional ideal of nilpotent elements and one may choose a basis $\{e, v_k\}$ ($k = 1, \cdots, M - 1$) such that all $v_k$’s are nilpotent.

3. $\mathcal{A}$ contains one identity element $e$ and an element $j$ such that $j^2 = -e$. The algebra $\mathcal{A}$ is then of even dimension $M = 2m$, and there exists a $(2(m - 1))$-dimensional ideal of nilpotent elements. One can choose a basis $\{e, v_k, j, j \cdot v_k\}$ ($k = 1, \cdots, m - 1$) such that all $v_k$’s are nilpotent.

Algebras which are commutative, associative, with a unity and a nondegenerate inner product which makes the algebra symmetric (i.e., a nondegenerate invariant inner product) are called Fröbenius algebras\footnote{We thank Giulio Bonelli for having pointed this out to us}.

2.2 Tensor product of a Lie algebra and an associative, commutative algebra

We have just seen that multi-graviton theories require associative, commutative algebras. On the other hand, the theory of a single type of Weyl gravitons requires the Lie algebra of the conformal group \footnote{We thank Giulio Bonelli for having pointed this out to us}. How can the two be put together?

Let $L$ be a Lie algebra (e.g., the Lie algebra of the conformal group) whose generators we denote by $T_A$. One has

$$[T_A, T_B] = f_{AC}^B T_C$$  \hspace{1cm} (2.9)

where the $f_{AB}^C$ are the structure constants. Consider the direct product $L' = L \otimes \mathcal{A}$ of the Lie algebra $L$ with the associative commutative symmetric
M-dimensional algebra \( \mathcal{A} \). It is easy to see that \( L' \) has a natural Lie algebra structure. An arbitrary element of \( L' \) is a sum of terms of the form \( X \otimes x \), where \( X = X^A T_A \in \mathcal{L} \) and \( x = x^a E_a \in \mathcal{A} \). The Lie bracket in \( L' \) is defined by

\[
[X \otimes x, Y \otimes y]_{L'} = [X, Y] \otimes x \ast y.
\] (2.10)

The bilinearity of \([\cdot, \cdot]_{L'}\) follows from the bilinearity of \([\cdot, \cdot]_L\) and \(\ast\). The \(L'\)-bracket is antisymmetric because the \(L\)-bracket is antisymmetric and the \(\ast\)-product is symmetric. Similarly, the Jacobi identity holds in \(L'\) because it holds in \(L\) and because \(\mathcal{A}\) is commutative and associative. In general, the algebra \(L'\) is not semi-simple, even if \(L\) is, because \(\mathcal{A}\) contains nilpotent elements.

If \(L\) is equipped with a quadratic form, \((X, Y) = g_{AB} X^A Y^B\), one can extend it to \(L'\) through the formula

\((X \otimes x, Y \otimes y)_{L'} \equiv (X, Y) \ast x \ast y\). (2.11)

Because of (2.8), \((\cdot, \cdot)_{L'}\) is invariant in \(L'\),

\(([X \otimes x, Y \otimes y]_{L'}, Z \otimes z)_{L'} = ([X \otimes x, Y \otimes y, Z \otimes z]_{L'})_{L'}\)

whenever \((\cdot, \cdot)_{L}\) is invariant in \(L\), \(([[X, Y], Z) = (X, [Y, Z])\).

### 2.3 Examples

From now on, \(L\) is the conformal algebra \(so(4,2)\)

\[
[M_{ab}, M_{cd}] = \eta_{bc} M_{ad} + \eta_{ad} M_{bc} - \eta_{ac} M_{bd} - \eta_{bd} M_{ac},
\]
\[
[P_a, M_{bc}] = \eta_{ab} P_c - \eta_{ac} P_b, \quad [K_a, M_{bc}] = \eta_{ab} K_c - \eta_{ac} K_b,
\]
\[
[P_a, K_b] = 2(\eta_{ab} D - M_{ab}),
\]
\[
[P_a, \mathcal{D}] = P_a, \quad [K_a, \mathcal{D}] = -K_a.
\] (2.13)

with bilinear form

\((X, Y) = \frac{1}{2} \varepsilon_{abcd} X^{ab} Y^{cd}\) (2.14)

considered in [3] in order to construct the action. Here, \(X^{ab}\) and \(Y^{ab}\) are the components of \(X\) and \(Y\) along the Lorentz generators. We follow the

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4For \(L'\) to be a Lie algebra, \(\mathcal{A}\) need not be finite-dimensional. In fact, the same construction forms the basis of "loop algebras", where \(L\) is taken to be a finite-dimensional Lie algebra and \(\mathcal{A}\) the (infinite-dimensional) algebra of functions on the circle. Setting \(T_A^n \equiv T_A \exp(2\pi i n \varphi)\), one gets

\[T^m_A \cdot T^n_A = f_{ABC} T^{n+m}_C, \quad n, m \in \mathbb{Z}.\]
conventions of [3, 4]. The scalar product (2.14) is not invariant under the full conformal algebra, although it is invariant under the Lorentz subalgebra.

When the associative algebra \( \mathcal{A} \) contains an identity element \( e \), the Lie algebra \( L' \) contains a subalgebra isomorphic to the conformal algebra, namely, \( \text{so}(4, 2) \otimes \{ e \} \). As we shall see, the gauging leads then to Riemannian geometry (+ extra fields). We shall restrict the present discussion to those cases. It is clear that case 3 in the above classification of associative algebras is the complexification of case 2, so we just consider this latter case. Theories of complex gravity, i.e. theories with a complex metric have been extensively studied in the past. A more recent study is in [20].

The simplest non-trivial algebra \( \mathcal{A} \) is spanned by two elements, a unit one \( e \) and a nilpotent one of order two \( n (e^2 = e, en = ne = n, n^2 = 0) \). The most general (invertible) internal metric \( k_{ab} \) compatible with (2.5) is, up to a multiplicative factor, given by

\[
k_{ab} = \begin{pmatrix} k & 1 \\ 1 & 0 \end{pmatrix}.
\]

(2.15)

where \( k \) is an arbitrary number. Explicitly,

\[
k(e, e) = k; \quad k(n, n) = 0;
\]

\[
k(e, n) = k(n, e) = 1.
\]

(2.16)

We shall adopt the following notation : the elements \( T_A \otimes e \) of \( L' \) are written as \( T_A^e \), while the \( T_A \otimes n \) are written as \( T_A^n \). Since \( L \equiv \text{so}(4, 2) \) is the 15-parameter conformal algebra, the Lie algebra \( L' \) has dimension 30. In fact, since \( L \) is isomorphic to the subalgebra \( L \otimes \{ e \} \), we will simply write \( T_A \) instead of \( T_A \otimes e \). Then the algebraic structure of \( L' \) is given by

\[
[T_A, T_B] = f_{AB}^C T_C \\
[T_A, T_B^n] = f_{AB}^C T_C^n \\
[T_A^n, T_B^n] = 0,
\]

(2.17)

with \( f_{AB}^C \) given in (2.13). The subspace generated by the \( T_A^n \) is an abelian ideal.

According to (2.11), the scalar product (2.14) extends to \( L' \) as

\[
(X, Y) = \frac{1}{2} \varepsilon_{abcd}(X^{ab}Y^{cd} + X^{ab}Y^{cd} + kX^{ab}Y^{cd})
\]

(2.18)

(using (2.10) and an obvious notations). It is invariant under the subalgebra \( \mathcal{L} \otimes \mathcal{A} \), where \( \mathcal{L} \) is the Lorentz subalgebra of \( \text{so}(4, 2) \).

The next simplest case is three-dimensional. There are actually two three-dimensional algebras. The first one is generated by the identity and two nilpotent elements of order 2, \( n^2 = 0, n'^2 = 0 \), such that \( nn' = 0 \). The internal scalar product is however degenerate. [The scalar products \( <n, n'> \),
< n', n' > and < n, n' > vanish because \( n^2 = 0 \), \( n'^2 = 0 \) and \( nn' = 0 \) (e.g., < \( n, n' \) >= < \( n, n'e \) >= < \( nn', e \) > (symmetry) = 0) so there is a linear combination of \( n \) and \( n' \) that is orthogonal to all other elements.

For this reason, it is the other three-dimensional irreducible, commutative, associative algebra, which is of most interest. This algebra is generated by an identity \( e \) and a nilpotent element \( u \) of order 3, \( u^3 = 0 \). The algebra is spanned by the three basis elements \( \{e, u, v = u^2\} \) and the multiplication table reads

\[
\begin{array}{c|ccc}
* & e & u & v \\
\hline
 e & e & u & v \\
 u & u & v & 0 \\
v & v & 0 & 0 \\
\end{array}
\] (2.19)

The Lie algebra formed by the tensorial product of \( L \) and \( A \) possesses, schematically, the following commutation rules:

\[
\begin{align*}
[T^e, T^e] = & T^e, & [T^e, T^u] = & T^u, \\
[T^e, T^v] = & T^v, & [T^u, T^u] = & T^v, \\
[T^u, T^v] = & [T^v, T^v] = & 0 \\
\end{align*}
\] (2.20)

where \([T^e, T^e] = T^e\) really stands for \([T^e_B, T^e_C] = f_{BC}^A T^e_A\), etc.

The (non-positive definite) internal metric \( k_{ab} \) (where \( a, b \) run over \( e, u, v \)) that makes the algebra symmetric is, up to an overall multiplicative factor,

\[
k_{ab} = \begin{pmatrix} k & l & 1 \\ l & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}
\] (2.21)

where \( k \) and \( l \) are arbitrary.

### 3 Two-Weyl-graviton theory

In this section we first show that the gauging of the extension \( L' \equiv so(4,2) \otimes \{e, n\} \) of the conformal algebra leads to the 2-Weyl-graviton theory with action (1.1). Then we construct the three-Weyl-graviton theory.

#### 3.1 Curvatures

We assign gauge fields to each of the generators of \( L' \) and form a "big" gauge field

\[
h_{\mu} = h_{\mu}^A(T_A \otimes e) + h_{\mu}^n(T_A \otimes n) = h_{\mu}^A T_A + h_{\mu}^n T_A.
\] (3.1)
The components are explicitly denoted

\[ e_a^\mu, \omega_{ab}^\mu, b^\mu, f_a^\mu, e_{na}^\mu, \omega_{nab}^\mu, b_n^\mu, f_{na}^\mu. \] (3.2)

The gauge parameters can be similarly expanded, \( \epsilon = e^AT_A + e^nAT^n_A \), and yield, when inserted into the gauge transformation formula \( \delta_\epsilon h_\mu = \partial_\mu \epsilon + [h_\mu, \epsilon] \equiv D_\mu \epsilon \)

\[
\delta_\epsilon h_\mu^A &= D_\mu^L \epsilon^A, \\
\delta_\epsilon h_n^A &= D_\mu^L \epsilon^{nA} + f_{BC}^A h_\mu^n B^C. 
\] (3.4)

where \( D^L \) is the \( L \)-covariant derivative

\[ D^L \chi^{nA} = \partial_\mu \chi^{nA} + f_{BC}^A h_\mu^B \chi^{nC}. \] (3.5)

The second term in the right-hand side of (3.4) indicates that \( h_n^A \) transform in the adjoint representation of the original group.

The \( L' \)-curvatures are given by:

\[ R_{\mu\nu} = R_{\mu\nu}^A T_A + R_{\mu\nu}^{nA} T^n_A = 2 \partial_{[\mu} h_{\nu]} + [h_\mu, h_\nu] \] (3.6)

(antisymmetrization is defined with the factor \( 1/2 \) so that it is a projection operator). Explicitly, one gets

\[ R_{\mu\nu}^A = 2 \partial_{[\mu} h_{\nu]}^A + f_{BC}^A h_\mu^B h_\nu^C, \] (3.7)

for the components along \( L \otimes \{ e \} \) and

\[ R_{\mu\nu}^{nA} = 2D_\mu^L h_n^{nA}. \] (3.8)

for the components along \( L \otimes \{ n \} \). Note in particular that the components along \( L \otimes \{ e \} \) are unchanged. With the help of the formula \( \delta_\epsilon R_{\mu\nu} = [R_{\mu\nu}, \epsilon] \) for the variation of the curvatures, we find

\[
\delta_\epsilon R_{\mu\nu}^A &= f_{BC}^A R_{\mu\nu}^{Bn} \epsilon^C \\
\delta_\epsilon R_{\mu\nu}^{nA} &= f_{BC}^A (R_{\mu\nu}^{Bn} \epsilon^C + R_{\mu\nu}^{nB} \epsilon^C). 
\] (3.9)

Finally, the symmetric fields of the gravitons are obtained from the "tetrads" \( e_\mu^a \) and \( e_{na}^\mu \) through

\[ g_{\mu\nu} = e_\mu^a e_{\nu}^a \] (3.11)

and

\[ h_{\mu\nu}^{nA} = 2e_{(\mu\nu)}^n = e_\mu^{na} e_\nu^{n} + e_\nu^{na} e_\mu^{n}. \] (3.12)

The tetrads \( e_\mu^a \) are assumed to be invertible. World indices are lowered and raised with the metric \( g_{\mu\nu} \) and its inverse \( g^{\mu\nu} \).
3.2 Action and constraints

We take the action to be the natural extension of the 1-graviton action of [5], namely, the action is quadratic in the curvatures with bilinear form given by (2.18). Taking \( k = 0 \) for simplicity (see end of subsection 3.3 for the general case), this leads to

\[
(2) \quad S[e, e^n, \omega, \omega^n, b, b^n, f, f^n] = 2 \int d^4x \varepsilon^{\mu
u\rho\sigma} \varepsilon_{abcd} R_{\mu\nu}^{ab}(M) R_{\rho\sigma}^{cd}(M^n). \quad (3.13)
\]

We also impose the constraints that the translation curvatures be zero

\[
\begin{aligned}
R^a_{\mu\nu}(P) &= 0 \\
R^n_{\mu\nu}(P^n) &= 0
\end{aligned}
\]

since, as we shall see in a moment, this is necessary for invariance under conformal boosts. The first constraint, being the same as the translation constraint of the 1-graviton theory, can be solved in the same way for \( \omega^{ab}_\mu \) to yield

\[
\omega^{ab}_\mu = -2g^{\rho\nu} e_{\mu}^{[a} \partial_{\nu} e_{\nu]} + e^{ap} e^{bq} e_{ap} \partial_{[q} e_{p]} + 2b^{[a} e_{\mu]} b^{b]}, \quad b^a \equiv e^{\mu a} b_{\mu}. \quad (3.15)
\]

Similarly, the second constraint can be solved for \( \omega^{nab}_\mu \) and gives

\[
\omega^{nab}_\mu = D^{[a} e_{\mu}^{\nu]} - D_{\mu} e^{n[a} + 2b^{[a} e^{\nu]b]} - 2e_{\mu}^{[a} e_{\nu]} b^{b]} + 2e_{\mu}^{[a} e_{c]b]} b^{c}. \quad (3.16)
\]

In the action (3.13), the \( \omega \)'s are not varied independently, but must be regarded as functions of the other fields, given by (3.15) and (3.16).

The constraints (3.14) are preserved under \( M-, K- \) and \( D- \) gauge transformations. It follows that the \( \omega \)'s transform exactly as connections under these transformations, which implies, in turn, that the curvatures transform as in (3.9) and (3.10) (under those same gauge transformations). Therefore, the action (3.13) is invariant under the transformations generated by \( M_{ab} \otimes e, M_{ab} \otimes n, D \otimes e, D \otimes n, K_a \otimes e \) and \( K_a \otimes n \) (using Lorentz invariance of the quadratic form and the constraints). The action is also trivially diffeomorphism invariant (because it is a 4-form), i.e., invariant under

\[
\delta_{\eta} h^A_\mu = \mathcal{L}_{\eta} h^A_\mu, \quad \delta_{\eta} h^{nA}_\mu = \mathcal{L}_{\eta} h^{nA}_\mu \quad (3.17)
\]

as well as invariant under the transformations

\[
\delta_{\eta_2} h^A_\mu = 0, \quad \delta_{\eta_2} h^{nA}_\mu = \mathcal{L}_{\eta_2} h^{nA}_\mu \quad (3.18)
\]

under which only the components of the gauge field along the generators \( T_A \otimes n \) transform (one has \( \delta_{\eta_2} R_{\mu\nu}^{ab}(M) = 0 \) and \( \delta_{\eta_2} R_{\mu\nu}^{ab}(M^n) = \mathcal{L}_{\eta_2} R_{\mu\nu}^{ab}(M) \)). Finally, the constraints are not preserved under \( P- \) gauge transformations. This means that the \( \omega \)'s acquire an extra variation term under translations. Following the mechanism explained in [3, 4], this has the net effect of trading \( P \)-translations for the ”coordinate” transformations (3.17) and (3.18).
3.3 Comparison with original action (1.1)

To go from (3.13) to (1.1), one proceeds as follows:

1. One first observes that $\omega_{ab}^\mu$ and $\omega_{na}^\mu$ have been eliminated by means of the constraints;

2. One observes next that the fields $f^a_\mu$ and $f_{na}^\mu$ taken together are auxiliary, in the sense that one can eliminate them by using their own equations of motion. More precisely, the equation of motion for $f_{na}^\mu$ can be solved for $f^a_\mu$ and yield the same expression as in the 1-graviton case, whereas the equation of motion for $f^a_\mu$ can be solved for $f_{na}^\mu$. One eliminates both $f^a_\mu$ and $f_{na}^\mu$ using their equations of motion. One can in fact view these equations as other constraints involving the Ricci tensor [8, 10]. [The explicit on-shell expression of $f_{na}^\mu$ is not needed because $f_{na}^\mu$ drops out once $f^a_\mu$ is on-shell. The action has schematically the form $a_{ij}(f^i - F^i)(f^{nj} - F^{nj}) + S_0$ where $f^i$ (respectively $f^{ni}$) stands for $f^a_\mu$ (respectively $f_{na}^\mu$) and where $F^i$, $F^{nj}$ and $S_0$ do not depend on the $f^i$'s. The matrix $a_{ij}$ is invertible and the equations of motion for $f^i$ and $f^{ni}$ imply $f^i - F^i = 0$ and $f^{nj} - F^{nj}$. Once $f^i$ is put on-shell, $f^{ni}$ disappears.]

3. Once this is done, one finds that the fields $b_\mu$ and $b_{na}^\mu$ drop out from the action. This is not surprising, because these are the only fields left that transform under conformal boosts $K_a \otimes e$ and $K_a \otimes n$.

4. At this stage, the remaining fields in the action are the tetrads $e^a_\mu$ and $e_{na}^\mu$. They enter only through the symmetric combinations (3.11) and (3.12) because of Lorentz invariance. The resulting action is in fact (1.1) with gauge symmetries (1.3) and (1.4). This shows that indeed, the action (1.1) describes the gauging of the extension $so(4,2) \otimes \{e,n\}$ of the conformal algebra. Note that the fields along the identity $e$ and the fields along the nilpotent element $n$ play quite different roles. The former are related to Riemannian geometry through $g_{\mu\nu}$, which is a standard Riemannian metric, while the latter appear in the end as fields propagating on that Riemannian spacetime.

To summarize: one can construct an action with two Weyl fields $h^a_{\mu\nu}$ which are interacting, by gauging the extension (2.17) of the conformal group. The action is (3.13). There are two constraints $R^a_{\mu\nu}(P) = 0$; $R_{\mu\nu}^a(P^n) = 0$, and a pair of auxiliary fields $\{f^a_\mu, f_{na}^\mu\}$. If we had taken the most general metric (2.13) in the commutative algebra $A$ ($k \neq 0$), we would have obtained as action the sum of the 1-graviton action of [8] and of (3.13). The discussion for this case proceeds straightforwardly, along exactly the same lines as above and simply amounts to adding to (1.1) the Weyl action $\frac{1}{\alpha} \int \sqrt{-g} C_{\alpha\beta\mu\nu} C_{\alpha\beta\mu\nu}$ for the metric $g_{\mu\nu}$. 

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3.4 Three-graviton conformal gravity

The 3-graviton case - more generally, the $M$-graviton case - is a direct extension of what we just found. The underlying algebra to be gauged is in this case the extension of the conformal group, so we have now three families of generators, fields, parameters and curvatures

$$T^{e}_{A}, T^{u}_{A}, T^{v}_{A}, h^{A}_{\mu}, h^{u}_{\mu}, h^{v}_{\mu}, e^{A}_{\mu}, e^{u}_{\mu}, e^{v}_{\mu}, R^{A}_{\mu\nu}, R^{u}_{\mu\nu}, R^{v}_{\mu\nu}. \quad (3.19)$$

The action reads

$$I[h^{u}_{\mu}, h^{v}_{\mu}, h^{u}_{\mu}] = \int d^{4}x \varepsilon^{\mu\nu\rho\sigma} \varepsilon_{abcd}[2R^{ab}_{\mu\nu}(M)R^{cd}_{\rho\sigma}(M) + R^{ab}_{\mu\nu}(M)R^{cd}_{\rho\sigma}(M)]. \quad (3.20)$$

where we have set $k = l = 0$ in the quadratic form (2.21). Keeping $k$ and $l \neq 0$ would simply amount to adding to the action given below the 1-graviton action – directly from the 1-graviton case – as well as the $M$-graviton action – directly from the 1-graviton action. The approach explains also the rationale behind the above construction and somehow "demystifies" the occurrence of nilpotent elements in the underlying algebras.

The constraints are again that all the components of the curvatures along the translation generators, here $P_{a} \otimes e_{1}, P_{a} \otimes u_{1}, P_{a} \otimes u^{2}$, are equal to zero. One can solve these constraints for the $\omega$’s. Furthermore, the $f$’s are again found to be auxiliary fields. Upon their elimination using their own equations of motion, the action becomes a function of the symmetric combinations

$$g_{\mu\nu} = e^{a}_{\mu} e^{b}_{\nu} \eta_{ab} \quad (3.21)$$
$$h^{u}_{\mu\nu} = (e^{a}_{\mu} e^{b}_{\nu} + e^{a}_{\mu} e^{b}_{\nu}) \eta_{ab} \quad (3.22)$$
$$h^{v}_{\mu\nu} = (e^{a}_{\mu} e^{b}_{\nu} + 2 e^{a}_{\mu} e^{b}_{\nu}) \eta_{ab} \quad (3.23)$$

of the tetrads only (the $b$’s drop out), invariant under

$$\delta^{e, c}_{e, \xi, \xi^{u}, \xi^{v}} g_{\mu\nu} = \mathcal{L}_{\xi^{u}} g_{\mu\nu},$$
$$\delta^{e, c}_{e, \xi, \xi^{u}, \xi^{v}} h^{u}_{\mu\nu} = \mathcal{L}_{\xi^{u}} h^{u}_{\mu\nu} + \mathcal{L}_{\xi^{v}} g_{\mu\nu},$$
$$\delta^{e, c}_{e, \xi, \xi^{u}, \xi^{v}} h^{v}_{\mu\nu} = \mathcal{L}_{\xi^{u}} h^{v}_{\mu\nu} + \mathcal{L}_{\xi^{v}} h^{u}_{\mu\nu} + \mathcal{L}_{\xi^{v}} g_{\mu\nu}. \quad (3.24)$$

The dilatation part of the gauge transformations gives

$$\delta^{e, c}_{e, \phi, \phi^{u}, \phi^{v}} g_{\mu\nu} = \phi^{e} g_{\mu\nu},$$
$$\delta^{e, c}_{e, \phi, \phi^{u}, \phi^{v}} h^{u}_{\mu\nu} = \phi^{e} h^{u}_{\mu\nu} + \phi^{u} g_{\mu\nu},$$
$$\delta^{e, c}_{e, \phi, \phi^{u}, \phi^{v}} h^{v}_{\mu\nu} = \phi^{e} h^{v}_{\mu\nu} + \phi^{v} h^{u}_{\mu\nu} + \phi^{v} g_{\mu\nu}. \quad (3.25)$$

We shall not write explicitly the final action but rather provide a method for deriving it – as well as the $M$-graviton action – directly from the 1-graviton action. The approach explains also the rationale behind the above construction and somehow "demystifies" the occurrence of nilpotent elements in the underlying algebras.
4 Truncation of Taylor expansions

4.1 General theory

The existence of the M-graviton action is in fact a consequence of the following elementary observation. Let \( S^{(0)}[y^i] \) be an action invariant under the gauge transformations

\[
\delta_\epsilon y^i = R^i_\alpha \epsilon^\alpha
\]  

(4.1)

(we adopt DeWitt’s condensed notations). The Noether identities read

\[
\frac{\delta S^{(0)}}{\delta y^i} R^i_\alpha = 0.
\]  

(4.2)

By functional differentiation, one derives further identities

\[
\frac{\delta^2 S^{(0)}}{\delta y^i \delta y^j} R^i_\alpha + \frac{\delta S^{(0)}}{\delta y^i} \frac{\delta R^i_\alpha}{\delta y^j} = 0.
\]  

(4.3)

It follows that the action

\[
S[y^i, Y^i] = k S^{(0)}[y^i] + S^{(1)}[y^i, Y^i], \quad S^{(1)} \equiv \frac{\delta S^{(0)}}{\delta y^i} Y^i
\]  

(4.4)

is invariant under the gauge transformations

\[
\delta_{\epsilon, \eta} y^i = R^i_\alpha \epsilon^\alpha
\]  

(4.5)

\[
\delta_{\epsilon, \eta} Y^i = \frac{\delta R^i_\alpha}{\delta y^j} Y^j \epsilon^\alpha + R^i_\alpha \eta^\alpha.
\]  

(4.6)

If the field \( y^i \) obeys the linear equation \( D_{ij} y^j = 0 \) in the free limit \((S^{(0)}_{\text{Free}}[y^i]) \sim y^i D_{ij} y^j\) where \( D_{ij} \) is a symmetric differential operator), then the linearized equations of motion following from (4.4) will also be \( D_{ij} y^j = 0, D_{ij} Y^j = 0 \). One can view both the action (4.4) and its gauge symmetries as obtained from the original action (times \( k \)) by a limited Taylor expansion

\[
y^i \rightarrow y^i + x Y^i, \quad \epsilon^\alpha \rightarrow \epsilon^\alpha + x \eta^\alpha
\]  

(4.7)

in which one truncates to terms of degree \( \leq 1 \), which formally amounts to assuming that \( x \) is nilpotent of order 2, \( x^2 = 0 \). Invariance is automatic because it holds order by order in \( x \). The form (4.4) follows from a rescaling of \( y^i \) in which one absorbs \( k \) and \( x \) (a similar rescaling must be performed on \( \eta^\alpha \)). If one starts from the 1-Weyl-graviton action of [5] and performs this limited Taylor expansion, one gets the 2-Weyl-graviton action (3.13).

The 3-field action (respectively, M-field action) is obtained by Taylor expanding and keeping terms up to order 2 in \( x \) (respectively, \( M - 1 \))

\[
y^i \rightarrow y^i + x Y^i + x^2 Z^i, \quad \epsilon^\alpha \rightarrow \epsilon^\alpha + x \eta^\alpha + x^2 \lambda^\alpha.
\]  

(4.8)
For $M = 3$, the action and gauge symmetries read

$$S[y^i, Y^i, Z^i] = k S^{(0)}[y^i] + l S^{(1)}[y^i, Y^i] + S^{(2)}[y^i, Y^i, Z^i]$$  \hspace{1cm} (4.9)

with

$$S^{(2)}[y^i, Y^i, Z^i] = \frac{\delta S^{(0)}}{\delta y^i} Z^i + \frac{1}{2} \frac{\delta^2 S^{(0)}}{\delta y^i \delta y^j} Y^i Y^j$$  \hspace{1cm} (4.10)

and

$$\delta_{\epsilon, \eta \lambda} y^i = R^i_{\alpha} \epsilon^\alpha \hspace{1cm} (4.11)$$
$$\delta_{\epsilon, \eta \lambda} Y^i = \frac{\delta R^i_{\alpha}}{\delta y^j} Y^j \epsilon^\alpha + R^i_{\alpha} \eta^\alpha \hspace{1cm} (4.12)$$
$$\delta_{\epsilon, \eta \lambda} Z^i = \left( \frac{1}{2} \frac{\delta^2 R^i_{\alpha \beta}}{\delta y^j \delta y^j} Y^j Y^j + \frac{\delta R^i_{\alpha}}{\delta y^j} Z^j \right) \epsilon^\alpha + \frac{\delta R^i_{\alpha}}{\delta y^j} Y^j \eta^\alpha + R^i_{\alpha} \lambda^\alpha \hspace{1cm} (4.13)$$

Invariance of the action follows from the Noether identities (4.2) and the identities that follow by differentiating them once (see (4.3)) and twice.

### 4.2 Extension of Lie algebras

The above formulas hold for any given theory. If the original theory is associated with Lie groups and gaugings, then the multi-field theory has also an underlying Lie algebra structure, which one easily identifies by expanding the fields and the gauge transformations and truncating as above. This yields explicitly, for the $M$-field theory

$$[T^i_A, T^j_B] = f^C_{AB} T^C = f^C_{AB} T^C_t \hspace{1cm} \text{for } 0 \leq i + j < M$$
$$= 0 \hspace{1cm} \text{otherwise}, \hspace{1cm} (4.14)$$

This is in fact $L \otimes A$ where $A$ is the associative, commutative algebra generated by a unit and an element which is nilpotent of order $M$. $L \otimes A$ is a truncation of the loop algebra

$$[T^i_A, T^j_B] = f^C_{AB} T^C \hspace{1cm} i, j \in Z \hspace{1cm} (4.15)$$

in which one retains only non-negative-moded elements and drops furthermore the $T^i_A$ with $i \geq M$. Truncating is equivalent to assuming that the Taylor expansion parameter fulfills $x^M = 0$, which clearly explains the emergence of nilpotent elements in the associative algebra appearing in the tensor product $L \otimes A$. 

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5 Supersymmetrization

After the discovery of ordinary supergravity [27, 28] based on the super Poincaré algebra (which was later extended to the anti-de Sitter case [29, 30, 31]), a theory of supergravity was constructed, based on the conformal group instead of the Poincaré group. Rather than using the Noether method, the starting point was to consider curvatures of the form $dh^A + f^A_{BC}h^Ch^B$ where $f^A_{BC}$ are the (constant) structure constants of the $N = 1$ superconformal algebra [5]. The notion that group theory can be used to construct gauge theories was used to construct ordinary (not conformal) supergravity in several papers both for the $N = 1$ anti-de Sitter case [17, 27] and for the $N = 2$ anti-de Sitter case [32, 33]. As a warming-up exercise, first the bosonic case was considered, the 15-parameter conformal group was gauged, and the action that followed from this approach turned out to be the square of the Weyl tensor. This was a bit of a surprise because one might have expected that also the Maxwell action for the dilatation gauge field $b_\mu$ (Weyl’s invention) should be present. The local conformal boost symmetry acted only on $b_\mu$ and this explained why $b_\mu$ was absent from the action.

An extension of this approach to $N$-extended supergravities with $U(N)$ internal symmetry group was begun in [34]. To every generator of the superalgebra a gauge field $h^A_\mu$ was associated, but many of the $h^A_\mu$ were non-propagating. To eliminate them, two approaches were followed: initially the equations of motion with non-propagating fields were used to eliminate such auxiliary gauge fields, but later constraints on the curvatures were used. The former method is dynamical in origin, whereas the latter method is free from dynamics and emphasizes the geometrical aspects. Clearly the latter method is more fundamental.

The complete theory of $N = 1$ conformal supergravity was announced in [5], and a detailed derivation was given in [8]. The relevant superalgebra is $su(2,2|1)$. The gauge fields and corresponding generators are $(e^a_\mu, P_a)$ for the translations, $((\psi^\alpha_\mu, Q_\alpha)$ for ordinary supersymmetry, $((\omega^{ab}_\mu, M_{ab})$ for the Lorentz transformations, $(b_\mu, D)$ for the dilations, $(A_\mu, A)$ for the $U(1)$ axial symmetry, $((f^a_\mu, K_a)$ for the conformal boosts and $(\varphi^\alpha_\mu, S_\alpha)$ for conformal supersymmetry. Of these fields, only $e^a_\mu, \psi^\alpha_\mu$ and $A_\mu$ remain in the final action. Due to local conformal boost gauge symmetry, the dilatation gauge field $b_\mu$ drops out of the action (just like for example the longitudinal part of the electromagnetic field drops out of the Maxwell action). The spin connection $\omega^{ab}_\mu$ was expressed in terms of $e^a_\mu, \psi^\alpha_\mu, b_\mu$ by the constraint that the torsion (the curvature of the translation generator $P_a$) vanishes, $R^a_\mu(P) = 0$. The conformal gauge fields $f^a_\mu$ and $\varphi^\alpha_\mu$ were eliminated as follows. The action was assumed to be quadratic in curvatures, i.e., of the form $e^{\mu\nu\rho\sigma}R^A_\mu R^B_\rho Q_{AB}$, where $Q_{AB}$ were constant Lorentz invariant tensors ($\epsilon_{abcd}, \gamma_5$ and $\eta^{ab}$). It preserved parity. In addition there was a term $R^a_\mu(A)R_{p\rho}(A)g^{\mu\rho}g^{\alpha\beta}\sqrt{-g}$ which explicitly depended on the metric; the rest of the terms was affine. Invariance of the action led to further constraints on the curvatures from which
\[ \varphi^\alpha \] could be eliminated in terms of other fields. Finally, \( f^a_\mu \) was eliminated from its own nonpropagating field equations.

Later a new constraint, this time involving the Ricci tensor, was shown to achieve the same result for the elimination of \( f^a_\mu \) \([10]\). A final constraint on the supercurvatures was found in \([8, 35]\) : \( R_{\mu \nu}(D) + \frac{i}{3} \epsilon_{\mu \nu \rho \sigma} R_{\rho \sigma}(A) = 0 \) was derived from requiring closure of the gauge algebra. This constraint allowed one to replace the term \( R_{\mu \nu}(A) R_{\rho \sigma}(A) g^{\mu \nu} g^{\rho \sigma} \sqrt{-g} \) by \( R_{\mu \nu}(D) R_{\rho \sigma}(A) \epsilon^{\mu \nu \rho \sigma} \), so that now the whole action was affine. The final action read

\[
S = \int d^4x \epsilon^{\mu \nu \rho \sigma} \left[ R_{\mu \nu}^{ab}(M) R_{\rho \sigma}^{cd}(M) \epsilon_{abcd} + \alpha R_{\mu \nu}^{\alpha}(Q) (\gamma_5)_{\alpha \beta} R_{\rho \sigma}^\beta(S) + \beta R_{\mu \nu}(A) R_{\rho \sigma}(D) \right]
\]

where \( \alpha \) and \( \beta \) are constants which were fixed by requiring invariance of the action under all 24 local symmetries. The final set of constraints is given by

\[
\begin{align*}
R_{\mu \nu}^a(P) &= 0, \\
\gamma^{\mu} R_{\mu \nu}(Q) &= 0, \\
R_{\mu \nu}(M) + R_{\mu \nu}(D) + \bar{\psi} \gamma^5 R_{\nu \lambda}(Q) &= 0, \\
R_{\mu \nu}(D) + \frac{i}{3} \epsilon_{\mu \nu \rho \sigma} R_{\rho \sigma}(A) &= 0,
\end{align*}
\]

where \( R_{\mu \nu}(M) \) is the Ricci tensor. The first three constraints are field equations in Poincaré theories, but here they define the geometry. The last constraint is related to the chirality-dualities which later became important.

Of course, conformal gravities and supergravities were also studied in other dimensions. In \( d = 3 \) for \( N = 0 \) \([36, 37]\), \( N = 1 \) \([38, 39]\) and \( N \geq 2 \) \([40]\). The actions were not squares of curvatures but rather Chern-Simons actions \((d = 3)\) multi-graviton theories which break PT invariance have been considered recently in \([41]\). Also in \( d = 5 \) incomplete results were obtained \([12]\). A review of conformal supergravities may be found in \([13]\). Our general analysis clearly shows how to construct the theory of \( M \) Weyl supermultiplets in interaction. One simply replaces the algebra \( su(2, 2|1) \) by its extension \( su(2, 2|1) \otimes A \) (where \( A \) is generated by a nilpotent element of order \( M \)).

In the simplest case where \( A \) is spanned by a unity \( e \) and a nilpotent element \( n \) of order two with \( e \epsilon n = n, n \epsilon n = 0 \) and with a non-positive-definite internal metric \((2.16)\), the action for two superconformal Weyl multiplets in interaction reads

\[
S = \int d^4x \epsilon^{\mu \nu \rho \sigma} \left[ 2 R_{\mu \nu}^{ab}(M^e) \epsilon_{abcd} R_{\rho \sigma}^{cd}(M^n) + \alpha R_{\mu \nu}^{\alpha}(Q^e)(\gamma_5)_{\alpha \beta} R_{\rho \sigma}^\beta(S^n) + \alpha R_{\mu \nu}^{\alpha}(Q^a)(\gamma_5)_{\alpha \beta} R_{\rho \sigma}^\beta(S^n) + \beta R_{\mu \nu}(A^e) R_{\rho \sigma}(D^n) + \beta R_{\mu \nu}(A^n) R_{\rho \sigma}(D^e) \right].
\]

The action \((5.18)\) is defined on a set a constraints which are, on the one hand, the constraints \((6.17)\) on the generators \( T_\Lambda \otimes e \) of \( su(2, 2|1) \otimes A \), and on the other hand, the constraints obtained from the set \((5.17)\) in a way similar to the way the action \( S[y^i, Y^i] \) \((1.4)\) was obtained from the action \( S^{(0)}[y^i] \) in section \( 4 \). That is, if we denote by \( C^a[y^i] = 0, a = 1, ..., K \) the set of
constraints for the gauge theory of $su(2, 2|1)$, the set of constraints for the
gauge theory of $su(2, 2|1) \otimes \mathcal{A}$ is given by the constraints $C^a[y^i] = 0$
together with the constraints $D^a[y^i, Y^i] = 0$, $D^a[y^i, Y^i] \equiv \frac{\delta C^a}{\delta y^i} Y^i$. The fields $y^i$
are the gauge fields associated to the generators $T_A \otimes e$, while the fields $Y^i$
are associated to the generators $T_A \otimes n$.

6 Conclusions

In this paper, we have constructed interacting theories in $3 + 1$ dimensions,
involving a set of $M$ (super)-conformal Weyl multiplets. These theories are
obtained by replacing in a theory with one graviton all fields $y^i$ by $y^i + x Y^i$
and all parameters $e^a$ by $e^a + x \eta^a$, and then truncating the action and trans-
formation laws to terms at most linear in $x$. Because of this truncation, these
theories are truly interacting multi-graviton theories. In principle our con-
struction can be extended to lower dimensions and higher dimensions, and
to $\mathcal{N}$-extended conformal supergravities (which are Chern-Simons theories
in odd dimensions). These theories a priori suffer from the same physical
drawbacks as ordinary conformal (super)gravities since their linearized ver-
sions have ghosts. However, they exhibit interesting algebraic structures that
make them gauge-consistent (in the sense that the interacting theory has the
same number of gauge symmetries as the free theory). This is a feature
that could make them of interest. We have also explained why algebras with
nilpotent elements arise in this context, shedding thereby light on previous
work concerning the purely gravitational case.

The structures that we have encountered are reminiscent of loop exten-
sions of $so(4, 2)$. However, if one performs the same analysis for such exten-
sions, one loses the ”triangular form” which enabled one to explicitly elimi-
nate the $\omega$’s and the $f$’s from the action. For this reason, our construction
does not appear to be immediately generalizable to that case.

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