EXPLICIT SOLUTION OF TIME FRACTIONAL MODIFIED EQUAL WIDTH WAVE EQUATION BY LIE SYMMETRY ANALYSIS

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Abstract: In present article, the time fractional modified equal width wave equation has been examined by Lie symmetry reduction technique. This schemed methodology with generalized Erdelyi-Kober (E-K) integral and differential operator have been used to transformed the partial differential equations (PDEs) of generalized (non integer) order into ordinary differential equations (ODEs) of fractional order with insertion of some independent variable. At last, explicit solution obtained by power series method.

Keywords: time fractional modified equal width wave (TFMEWW) equation; power series solution; Riemann Liouville (R-L) derivative; Lie symmetry method; Erdelyi-Kober operator.

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1. INTRODUCTION

The generalized calculus literature is as ancient as classical calculus. The concepts of fractional differential equations (FDEs) are utilized in modeling distinct phenomenon of mechanics, dynamics and drug therapy in biological systems. It is also used to study new age advance problems in neurons network, image processing, geology and hydrology. Podlubny [1], Oldham

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[2] and Debnath [3] illustrated the content on fractional order calculus. They provided Grunwald, Caputo and Riemann-Liouville (RL) fractional derivatives and integrals definition along with their physical and geometrical interpretations in real modeling.

The schemed study of Lie symmetry and their application has been derived by Olver [4]. Bakkayaraj and Sehdaven [5] provoked about the group formalism of geometrical transforms in this technique. Biswas et al.[6, 7] suggested the dual dispersion and non-linearity laws with the exclusive use of infinitesimals in symmetry reduction. Invariance criterion of some fractional PDEs, Hirota nonlinear, Hirota-Satsoma systems has been studied by Singla et al. [8]. Sneddon [9] used the concept of E-K operators and remarked that the system of FPDEs can be reduced to FOODEs with the efficient use of these operators. The symmetry properties and exact solution of real time fractional KdV of third, fourth, fifth and generalized order compiled by Zhang [10], Wang et al. [17] and Gandhi [25, 26]. Kaur et al. [11-15] has been implemented Painlike and Lie symmetry to Einstein vacuum field equation, Complex Hirota forms in multiple real and complex solutions. Huang [16] provided the total solutions of time fractional Harry-Dym equation with R-L fractional derivatives approach. Garrido et al. [19] prompted on travelling wave generalized solution of Driffield-Sokolov system and Arora et al. [20] found solitary wave solutions of modified equal width wave equations by Lie infinitesimals. The unremarkable criterion of solitary waves of equal width and regularized long wave equation has been solved by Gardner et al. [21, 22] in late 20th century. The physical phenomenon, scattering of regular long solitary wave has been studied by Morison et al. [23]. Rudin [24] attempted the implicit function theorem in principal of mathematical analysis, which has been used for convergence of power series solution by [10, 17, 25-26].

In recent times, mathematicians have devoted lot of efforts to analyze the explicit and exact solutions of linear and nonlinear PDEs. It’s difficult to obtain the exact solution of nonlinear differential equation as compared to linear differential equation. Therefore, some researchers have used numerical methods. It is always challenging task to find the exact or analytic solutions for nonlinear equations, and finding such solutions is even more difficult for fractional nonlinear equations. Hence, we are accepting the challenge and in this paper, we will
try to find the explicit solution of a nonlinear time fractional equation.

The MEWWE occurring from the nonlinear media with dispersion process has been paid special concentration in the past decades. Our motivation is to generate mathematical formulation of the infinitesimals and investigate the symmetry reduction with power series solution of generalized TFMEWW equation with one parameter ‘γ’

\[ \partial_\gamma u + 3u^2 u_x - \mu u_{xx} = 0 ; 0 < \gamma < 1, \mu > 0 \]  

(1)

and \((x, t)\) is space-time coordinate and \(u(x, t)\) is amplitude of wave for the one dimensional wave-propagation in nonlinear media with dispersion phenomenon. Time fractional MEW model is based on the EW equation. Morrison [23] established this model as modified regular long wave equation and modified KdV equation in fluid mechanics.

We proposed definition and terminology in section 2; in section 3, Lie symmetry approach has been discussed. The application of series solution with its convergence to MEW model described in sections 4 and 5. Finally, the remarks and conclusions established.

2. PRELIMINARIES

In this section, we would like to present the needful definitions and terminology related to fractional calculus.

In the first two definitions, let \(h(t)\) be an integrable function on \((0, t)\) and for \(g > 0\) assume that \(h(t)\) is \(g\)-times differentiable on \((0, t)\) except for a set of measure zero.

2.1 CAPUTO FRACTIONAL DERIVATIVE:

\[ D^\beta_t (h(t)) = \frac{1}{\Gamma(g - \beta)} \int_0^t (t - \xi)^{\beta\gamma - 1} h^\gamma (\xi) d\xi \text{ for } g \leq 1 < \beta \leq g, t > 0, g \in N \]  

(2)

2.2 R-L FRACTIONAL DERIVATIVE:

\[ D^\beta_t (h(t)) = \frac{1}{\Gamma(g - \beta)} \frac{d^g}{dt^g} \int_0^t (t - \xi)^{\gamma - \beta - 1} h(\xi) d\xi \text{ for } g \leq 1 < \beta \leq g, t > 0, g \in N \]  

(3)
2.3 R-L PARTIAL DERIVATIVE OF GENERALIZED ORDER ’β’ FOR THE FUNCTION \( v(x,t) \)

W.R.T. \( t \)

This R-L definition holds for the function of two variables and \( \beta \) is order of fractional derivative.

\[
D^\beta_t(v(x,t)) = \frac{1}{\Gamma(g - \beta)} \frac{\partial^g}{\partial t^g} \int_0^t (t - \xi)^{r - \beta - 1} v(x,\xi) d\xi \text{ for } g - 1 < \beta < g, t > 0, g \in \mathbb{N}
\]

(4)\[\frac{\partial^g v}{\partial t^g} \text{ when } \beta = g\]

2.4 THE LEIBNITZ RULE FOR R-L FRACTIONAL DERIVATIVES:

Leibnitz rule is defined for the product of two functions. Hence, below is the definition of

Leibnitz rule for fractional derivative of the product of two functions. \( v(x,t) \) and \( w(x,t) \) are function of two variable such that they are differentiable and integrable. \( \beta \) is order of fractional derivative.

\[
D^\beta_t(v(x,t),w(x,t)) = \sum_{g=0}^{\infty} \left( \frac{\beta}{g} \right) D^{\beta - g}(v(x,t))D^g(w(x,t)), \beta > 0,
\]

where \( \left( \frac{\beta}{g} \right) = \frac{(-1)^g \beta \Gamma(g - \beta)}{\Gamma(1 - \beta) \Gamma(g + 1)} \)

(5)

2.5 FRACTIONAL DERIVATIVES OF A CONSTANT IS ZERO AND FOR A FUNCTION \( c.w(x,p) \)

\[
D^\beta_x(c.w(x,p)) = c. \frac{\partial^\alpha}{\partial x^\alpha} w(x,p)
\]

also if \( p \in (c,d) \) then \( D^\beta_x(p-c)^\beta = \frac{\Gamma(\beta + 1)}{\Gamma(\beta - \alpha + 1)} (p-c)^{\beta - \alpha} ; \alpha \geq 0, \beta > 0 \)

(6)

2.6 DEFINITION OF A E-K FRACTIONAL DIFFERENTIAL AND INTEGRAL OPERATOR

\( (P^\gamma_x G)(\psi) \) and \( (K^{\gamma + \gamma} H)(\psi) \) are left hand sided Erdelyi-Kober Fractional differential operator and Erdelyi-Kober Fractional integral operator respectively.

\[
(P^\gamma_x G)(\psi) = \prod_{j=0}^{\gamma - 1} \left( r + j - \frac{1}{\partial} \psi \frac{d}{d\psi} \right) (K^{r+\gamma,\gamma + \gamma} G)(\psi) \text{ with } \psi > 0, \partial > 0 \text{ and } \gamma > 0; m = \begin{cases} [\gamma] + 1, \gamma \notin \mathbb{N} \\ \gamma, \gamma \in \mathbb{N} \end{cases}
\]

(7)

\[
(K^{\gamma + \gamma} H)(z) = \begin{cases} \frac{1}{\Gamma(\gamma)} \int_0^z (w-1)^{r-1} w^{-\gamma - \gamma} h(w) d\psi, & \gamma > 0 \\ h(\psi), & \gamma = 0 \end{cases}
\]
3. **Lie Symmetry Analysis**

There are several semi-analytic and analytic techniques to obtain exact and approximate solutions of FPDEs but we imposed Lie approach to address the infinitesimal symmetries of generalized differential equations; as conversion of FPDEs into FODEs is major task and it is feasible after the prolongation technique explained under:

Suppose a general fractional PDE with space-time variables and \( v = v(x, p) \).

\[
\partial^\gamma_v + F(p, x, v, v_x, v_p, v_{xx}, v_{pp}, \ldots) = 0; \text{where } 0 < \gamma < 1,
\]

(8)

Lie group of transformations with parameter \( \epsilon \) is taken as

\[
\begin{align*}
\dot{x}^* &= \dot{x} + \epsilon \zeta(x, p ; v) + O(\epsilon^2), \\
\dot{p}^* &= \dot{p} + \epsilon \tau(x, p ; v) + O(\epsilon^2), \\
\dot{v}^* &= \dot{v} + \epsilon \eta(x, p ; v) + O(\epsilon^2),
\end{align*}
\]

(9)

Associated lie algebra of (9) is generated by vectors fields

\[
X = \zeta \frac{\partial}{\partial x} + \tau \frac{\partial}{\partial p} + \eta \frac{\partial}{\partial v}
\]

(10)

Apply prolongation of 3rd order on (8)

\[
pr^{(r, 3)} X(\Delta)|_{\Delta=0}, \Delta = \partial^\gamma_v F
\]

(11)

By preserving the operator \( pr^{(r, 3)} X \) takes the form below:

\[
pr^{(r, 3)} X(\Delta) = X + \eta \cdot \frac{\partial}{\partial \partial^\gamma_v v} + \eta^\tau \frac{\partial}{\partial v_x} + \eta^v \frac{\partial}{\partial v_{xx}}
\]

(12)

Here, we use essential terms only which are usable in this paper and \( \tau, \zeta \) and \( \eta \) are infinitesimals and \( \eta^\tau, \eta^v \) are extended infinitesimals of order 1 and 3 respectively and \( \eta^{\tau v} \) is an extended time fractional infinitesimal of order \( \gamma \)

\[
\begin{align*}
\eta^\tau &= D_\tau(\eta) - v_p D_\tau(v_p) - D_\tau(v_x) \\
\eta^v &= D_v(\eta) - v_p D_v(v_p) - D_v(v_x) \\
\eta^{\tau v} &= D_v(\eta) + \zeta D^\tau(v_x) - \zeta D^v(v_x) + D^\tau(v_{xx}) - D^v(v_{xx})
\end{align*}
\]

Here \( D_p = \partial_p + v_p \frac{\partial}{\partial v} + v_{pp} \frac{\partial}{\partial v_x} + v_{ppp} \frac{\partial}{\partial v_{xx}} + \ldots \)

\[
D_x = \partial_x + v_x \frac{\partial}{\partial v} + v_{xx} \frac{\partial}{\partial v_x} + v_{xxx} \frac{\partial}{\partial v_{xx}} + \ldots
\]

(13)

(14)

(15)
Now generalized Leibnitz Rule is

$$D^\beta_t (h(t), k(t)) = \sum_{n=0}^{\infty} \binom{\beta}{n} D^n h(t) D^{\beta-n} k(t)$$

where

$$D^\alpha_t (h(t)) = h(t), D^{\alpha+1}_t (h(t)) = D_1 \left( D^\alpha_t (h(t)) \right)$$

We obtain expressions

$$\xi D^\gamma_p (v_x) - D^\gamma_p (\xi v_x) = -\sum_{n=0}^{\infty} \binom{\gamma}{n} D^n (\xi) D^{\gamma-n} (v_x)$$

(17)

$$\tau D^\gamma_p (v_p) - D^\gamma_p (\tau v_p) = -\alpha D^\gamma_p (\tau v_p) - \sum_{n=0}^{\infty} \binom{\gamma}{n} D^n (\tau) D^{\gamma-n} (v)$$

(18)

Generalized chain rule for composition of mappings is defined as

$$\frac{d^\gamma}{dp} (f(g(p))) = \sum_{k=0}^{\infty} (-1)^n \binom{n}{k} g^k (p) \frac{\partial^\gamma}{\partial z^n} \left( f \left( g^{n-k} (p) \right) \right)$$

$$\frac{d^n f(z)}{dz^n} \text{ where } z = g(p)$$

(19)

Using generalized Leibnitz rule, we have

$$D^\gamma_p (\eta) = \partial^\gamma_p (\eta) + \eta \partial^\gamma_p (v) - \nu \partial^\gamma_p (\eta) + \sum_{j=0}^{\infty} \binom{\gamma}{j} \partial^j_p (\eta) \partial^{\gamma-j} p (v) + \psi$$

(20)

$$\eta^{x, p} = \partial^\gamma_p (\eta) + \eta \partial^\gamma_p (v) - \nu \partial^\gamma_p (\eta) + \sum_{j=0}^{\infty} \binom{\gamma}{j} \partial^j_p (\eta) \partial^{\gamma-j} p (v) + \psi - \sum_{n=1}^{\infty} \binom{\gamma}{n+1} D^n (\xi) D^{\gamma-n} (v_x)$$

(21)

$$\tau D^\gamma_p (v_p) - \tau D^\gamma_p (\tau v_p) = -\alpha D^\gamma_p (\tau v_p) - \sum_{n=0}^{\infty} \binom{\gamma}{n} D^n (\tau) D^{\gamma-n} (v)$$

After using (12) and (8), substitute (13-21) and comparing the coefficients of v_x, v_xx and solve the set of obtained PDEs and FDEs.

### 3.1 Fractional Equal Width Wave Equation

Application of equation (12) to equation (1), we get

$$\eta^{x, t} + 6\eta u_x + 3\eta^2 u^2 - \mu \eta^{x, x} = 0$$

(22)
Substituting (13-21) in (22)
\[ \frac{\partial}{\partial t} \eta - u \frac{\partial}{\partial x} \eta + 3u^2 \eta_x - \mu \eta_{xx} = 0 \]  
(23)

\[ \left( \frac{\gamma}{n} \right) \frac{\partial}{\partial \tau} \eta - \left( \frac{\gamma}{n+1} \right) D_\tau^{\alpha} (\tau) = 0, n \in N \]  
(24)

\[ 6\rho u + 3u^2 (\eta_u - \xi_u) - \mu (2\eta_{aux} - \xi_{aux}) = 0 \]  
(25)

\[ 3u^2 \tau_u + \mu (\eta_{aux} - \tau_{aux}) = 0 \]  
(26)

\[ 3u^2 \xi_u + \mu (\eta_{aux} - 2\xi_{aux}) = 0 \]  
(27)

\[ 3u^2 \tau_u + \mu (2\eta_{aux} - \xi_{aux} - 2\tau_{aux}) = 0 \]  
(28)

\[ \xi_u = \xi_{aux} = 0 = \tau_u = \tau_{aux}; \eta_u = -\gamma D_\tau (\tau) = 0 \]  
(29)

Solving (23-29), infinitesimals with constants p, q and r found to be

\[ \xi = \frac{q(\gamma - 1)x}{2} + p; \tau = qt + r; \eta = -\frac{q(\gamma + 1)u}{4} \]  
(30)

The corresponding generator of lie algebra is described as

\[ T = \left( \frac{q(\gamma - 1)x}{2} + p \right) \frac{\partial}{\partial x} + (qt + r) \frac{\partial}{\partial t} + \left( -\frac{q(\gamma + 1)u}{4} \right) \frac{\partial}{\partial u} \]  
(31)

Taking standard generators of (31)

\[ T_1 = \frac{\partial}{\partial x}, \quad T_2 = \frac{x(\gamma - 1)}{2} \frac{\partial}{\partial x} + t \frac{\partial}{\partial t} - \frac{u(\gamma + 1)}{4} \frac{\partial}{\partial u} \quad \text{and} \quad T_3 = \frac{\partial}{\partial t} \]  
(32)

Corresponding to \( T_1 \), simple characteristic equation is

\[ \frac{dx}{1} = \frac{dt}{0} = \frac{du}{0} \]  
(33)

And, its invariant solution is obtained as \( T = t, u = F(T) \)

By putting in equation (1), it reduced to FODE.

\[ \frac{\partial}{\partial t} F(t) = 0 \]  
(34)

Corresponding to \( T_2 \), characteristic equation found

\[ \frac{2dx}{(\gamma - 1)x} = \frac{dt}{t} = \frac{4du}{-(\gamma + 1)u} \]  
(35)

On solving; explicit solution is obtained as
By putting in equation (1) along with the use of theorem explained below, it reduced to FODE given by (37)

**Theorem 1.** The Similarity transformations form by variables ‘u’ and ‘z’ in (36), reduce time fractional MEWW (1) to nonlinear FODE depends on ‘z’ as

\[
P_{\frac{3-5\gamma}{\gamma-1}} \left( \frac{3-5\gamma}{4} \right) F(z) + 3F^2 F_z - \mu \left[ \frac{3-5\gamma}{4} F_{zz} + \frac{1-\gamma}{2} zF_{zzz} \right] = 0
\]

(37)

Where Erdelyi-Kober fractional differ-integral operators have been explained in (7)

**Proof:**

Let \((g-1) \leq \gamma \leq g, g = 1, 2, \ldots \)

The fractional derivative approach in R-L sense taken as

\[
D^\gamma_{t} u = D^\gamma_{t} \left[ \frac{1}{\Gamma(g-\gamma)} \int_0^t (t-s)^{g-\gamma-1} s^{-(\gamma+1)/4} F\left( xs^{\gamma/2} \right) \, ds \right]
\]

(38)

Putting \( s = \frac{t}{w} \), to obtain

\[
D^\gamma_{t} u = D^\gamma_{t} \left[ \frac{1}{\Gamma(g-\gamma)} \int_0^\infty \left( \frac{t}{w} \right)^{g-\gamma-1} \left( \frac{t}{w} \right)^{-(\gamma+1)/4} F\left( \left( \frac{t}{w} \right)^{\gamma/2} \right) \frac{t}{w^2} \, dw \right]
\]

(39)

\[
= D^\gamma_{t} \left[ t^{\frac{1-\gamma}{4}} \Gamma\left( g-\gamma \right) \int_0^1 \left( w^{-1} \right)^{g-\gamma-1} w^{-(\gamma+1)/4} F\left( z w^{\frac{\gamma}{2}} \right) \, dw \right]
\]

(40)

\[
D^\gamma_{t} u = D^\gamma_{t} \left[ t^{\frac{1-\gamma}{4}} \frac{1}{\Gamma\left( g-\gamma \right)} K_{\frac{2}{\gamma-1}}^{\frac{3-5\gamma}{2}} F\left( z \right) \right]
\]

(41)

If \( z = \frac{1-\gamma}{2} x \), \( F \in C'(0, \infty) \)

\[
tD\gamma_{t} F(z) = x^{\frac{1-\gamma}{2}} t^{\frac{(\gamma+1)}{2}} D\gamma_{x} F(z) = \frac{1-\gamma}{2} z D\gamma_{x} F(z)
\]

(42)

Then \( D^\gamma_{t} \left[ t^{\frac{1-\gamma}{4}} \frac{1}{\Gamma\left( g-\gamma \right)} K_{\frac{2}{\gamma-1}}^{\frac{3-5\gamma}{2}} F\left( z \right) \right] = D^\gamma_{t} \left[ D^\gamma_{t} \left[ t^{\frac{1-\gamma}{4}} \frac{1}{\Gamma\left( g-\gamma \right)} K_{\frac{2}{\gamma-1}}^{\frac{3-5\gamma}{2}} F\left( z \right) \right] \right] \)
TIME FRACTIONAL MODIFIED EQUAL WIDTH WAVE EQUATION

\begin{equation}
D_t^{g_{-1}} \left[ t^{g_{-1} - 1} \frac{5y}{4} \left( g - \frac{1}{4} - \frac{5y}{4} \frac{(1 - \gamma)}{2} zD_z \right) K^\frac{3 - \gamma}{y - 1} F \right](z) \right]
\end{equation}

(43)

Repeating process \((g-1)\) times, we find

\begin{equation}
D_t^{g_{-1}} \left[ t^{g_{-1} - 1} \frac{5y}{4} \left( K^\frac{3 - \gamma}{y - 1} F \right) \right] = t^{g_{-1} - 1} \prod_{j=0}^{g_{-1}-1} \left[ 1 + j - \frac{5y}{4} + \frac{(1 - \gamma)}{2} zD_z \right] \left( K^\frac{3 - \gamma}{y - 1} F \right)(z)
\end{equation}

(44)

\begin{equation}
\Rightarrow D_t^{g_{-1}} u = t^{g_{-1} - 1} \left( K^\frac{3 - \gamma}{y - 1} F \right)(z)
\end{equation}

(45)

At last, we obtain

\begin{equation}
\left( K^\frac{3 - \gamma}{y - 1} F \right)(z) + 3F^2F_z - \mu \left[ \left( \frac{3 - \gamma}{4} \right) F_{zz} + \frac{1 - \gamma}{2} zF_{zzz} \right] = 0
\end{equation}

(46)

4. EXPLICIT POWER SERIES SOLUTION

To execute the solution of obtained FODE (46), we would like to pursue power series solution to obtain explicit solution of generalized MEWW equation

Let \( F(z) = \sum_{g=0}^{\infty} a_g z^g \) be the solution of (46), where coefficients \( a_g \) are to be determined. Putting the value of \( F(z) \) in (46), we get

\begin{equation}
\sum_{g=0}^{\infty} \frac{\Gamma \left( \frac{3 - \gamma}{4} - \frac{(g - 1)}{2} \right)}{\Gamma \left( \frac{3 - \gamma}{4} \right)} a_g z^g + 3 \sum_{g=0}^{\infty} a_g z^g \left( \sum_{g=0}^{\infty} (g + 1) a_{g+1} z^g \right) - \mu \left[ \left( \frac{3 - \gamma}{4} \right) \sum_{g=0}^{\infty} (2 + g)(1 + g) a_{g+2} z^g + \frac{1 - \gamma}{2} \sum_{g=0}^{\infty} (1 + g)(2 + g)(3 + g) a_{g+3} z^g \right] = 0
\end{equation}

(47)

\begin{equation}
\sum_{g=0}^{\infty} \frac{\Gamma \left( \frac{3 - \gamma}{4} - \frac{(g - 1)}{2} \right)}{\Gamma \left( \frac{3 - \gamma}{4} \right)} a_g z^g + 3 \sum_{g=0}^{\infty} a_g z^g \left( \sum_{g=0}^{\infty} (g + 1) a_{g+1} z^g \right) - \mu \left[ \left( \frac{3 - \gamma}{4} \right) \sum_{g=0}^{\infty} (2 + g)(1 + g) a_{g+2} z^g + \frac{1 - \gamma}{2} \sum_{g=0}^{\infty} (1 + g)(2 + g)(3 + g) a_{g+3} z^g \right] = 0
\end{equation}

(48)

After rearranging the terms, we obtain:
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\[
\sum_{g=0}^{\infty} \frac{\Gamma\left(3 - \frac{5}{2} \gamma \right)}{4} \frac{(\gamma - 1) g}{2} a_g z^g + 3 \mu \sum_{g=0}^{\infty} \left[ \sum_{m=0}^{g} \left( g + 1 - m \right) a_{g+m} \right] (g + 1 - m) a_{g+m} z^g = 0
\]

(49)

\[
\mu \left[ \frac{3 - 5}{4} g - (\gamma - 1) \right] a_0 + 3a_0 a_i - 2 \mu \left( \frac{3 - 5}{4} \right) a_2 = 0
\]

(50)

\[
3 \sum_{g=0}^{\infty} \left[ \sum_{m=0}^{g} \left( g + 1 - m \right) a_{g+m} \right] (g + 1 - m) a_{g+m} z^g = 0
\]

(51)

Comparing the powers of ‘\(z\)’ on both sides,

\[
\frac{\Gamma\left(3 - \frac{5}{2} \gamma \right)}{4} \frac{(\gamma - 1) g}{2} a_0 + 3a_0 a_i - 2 \mu \left( \frac{3 - 5}{4} \right) a_2 = 0
\]

\[
\Rightarrow a_2 = \frac{2}{\mu(3 - 5\gamma)} \left[ \frac{\Gamma\left(3 - \frac{5}{2} \gamma \right)}{4} \frac{(\gamma - 1) g}{2} a_0 + 3a_0 a_i \right] a_0
\]

(52)

\[
a_{g+2} = \frac{4}{\mu(3 + 2g - (2g + 5)\gamma)(g + 1)(g + 2)} \left[ \frac{\Gamma\left(3 - \frac{5}{2} \gamma \right)}{4} \frac{(\gamma - 1) g}{2} a_g + \sum_{m=0}^{g} \left( g + 1 - m \right) a_{g+m} \right]
\]

(53)

Hence, power series solution of the equation (46) is given below:

\[
F(z) = a_0 + a_1 \left( xt \right)^{1/2} + a_2 \left( xt \right)^{3/2} + \sum_{g=0}^{\infty} a_{g+2} \left( xt \right)^{g+2}
\]

(54)

Where, the explicit power series solution for time fractional modified equal width wave equation
is provided below:

\[
 u(x,t) = \frac{1}{(y+1)} \left[ a_0 + a_1 \left( \frac{1-y}{2} \right) + a_2 \left( \frac{1-y}{2} \right)^2 + \sum_{g=1}^{\infty} a_{g+2} \left( \frac{1-y}{2} \right)^{g+2} \right]
\]  \hspace{2cm} (55)

4.1 CONVERGENCE OF THE SERIES SOLUTION OF TFM EWW EQUATION

We can take \( a_0 \) and \( a_1 \) as two arbitrary constants from equation (52). So let us assume \( a_0 = \rho \neq 0 \)

and \( a_1 = \sigma \neq 0 \) therefore, by equation (52) \( a_2 = \frac{2}{\mu(3-5\gamma)} \left[ \Gamma \left( \frac{3-5\gamma - (\gamma - 1)}{4} \right) \right] + 3\sigma \rho \).

Similarly, we can find other coefficient from (53). From (53), we observe that

\[
 |a_{g+2}| \leq N \left[ |a_g| + \sum_{m=0}^{g} \sum_{i=0}^{m} |a_i| |a_{m-i}| g^{g+1-m} \right]
\]  \hspace{2cm} (56)

Where \( N = \max \left[ \frac{4}{\mu(3+2g-(2g+5)\gamma)}, \frac{12}{\mu(3+2g-(2g+5)\gamma)} \right] \quad \text{and} \quad \Gamma \left( \frac{3-5\gamma - g(\gamma - 1)}{4} \right) < 1 \quad \text{for all} \quad g. \)

We now consider the new power series

\[
 M = R(Y) = \sum_{g=0}^{\infty} r_g Y^g \quad \text{as} \quad r_0 = |a_0| |\rho|, \quad r_1 = |a_1| |\sigma|,
\]

\[
r_2 = |a_2|, \quad \text{And} \quad r_{g+2} = N \left[ r_g + \sum_{m=0}^{g} \sum_{i=0}^{m} r_i r_{m-i} r_{g+1-m} \right], \quad g = 1,2,3, \ldots
\]

We can easily show that \( |a_g| \leq r_g, \quad g = 1,2,3, \ldots \)

Now, we claim for the positive radius of convergence of the series \( M = R(Y) = \sum_{g=0}^{\infty} r_g Y^g \)
\[ R(Y) = r_0 + r_1 Y + r_2 Y^2 + \sum_{g=1}^{\infty} r_g Y^{g+2} \]
\[
= r_0 + r_1 Y + r_2 Y^2 + N[\sum_{g=1}^{\infty} r_g Y^{g+2} + \sum_{g=1}^{\infty} \sum_{m=0}^{g} r_m r_{g+1-m} Y^{g+2}] 
\]
\[
= r_0 + r_1 Y + r_2 Y^2 + N[(M - r_0 + 2Mr_0r_1 - M^2r_1 - r_1r_0^2)Y^2 + (M^3 + 3Mr_0(r_0 - M) - r_0^3)Y] \quad (57)
\]

Now, consider the implicit functional equation,
\[
H(Y, M) = M - R(Y)
\]
\[
= M - r_0 - r_1 Y - r_2 Y^2 - N[(M - r_0 + 2Mr_0r_1 - M^2r_1 - r_1r_0^2)Y^2 + (M^3 + 3Mr_0(r_0 - M) - r_0^3)Y] \quad (58)
\]

Here, H is analytic function in \((Y, M)\) plane and \(H(0, r_0) = 0, H'_m(0, r_0) = 1 \neq 0\)

Therefore, \(M = R(Y)\) has positive radius of convergence and is analytic in the vicinity of the \((0, r_0)\).

Hence, series solution (54) of equation (1) is convergent in the vicinity of the point \((0, r_0)\) and can be verified by implicit function theorem [24].

Thus, solution of equation (46) in power series is as:
\[
F(z) = \rho + \sigma Z + \frac{2}{\mu(3-5\gamma)} \left[ \frac{\Gamma\left(\frac{3-5\gamma - (\gamma-1)}{4}\right)}{\Gamma\left(\frac{3-\gamma - (\gamma-1)}{4}\right)} + 3\sigma \rho \right] Z^2 + ... \quad (59)
\]

Therefore, power series solution of equation (1) is
\[
u(x,t) = \frac{1}{t^{\frac{4}{\gamma+3}}}
\]
\[
+ \frac{2}{\mu(3-5\gamma)} \left[ \frac{\Gamma\left(\frac{3-5\gamma - (\gamma-1)}{4}\right)}{\Gamma\left(\frac{3-\gamma - (\gamma-1)}{4}\right)} + 3\sigma \rho \right] \left(\frac{1}{t^{\frac{4}{\gamma+3}}}\right)^2 + ... \quad (60)
\]

\(\rho\) and \(\sigma\) are the arbitrary constants and \(\mu\) is a positive parameter.

**CONCLUSIONS**

In present work, the time fractional MEWW equation with one parameter analyzed by means of Lie symmetry analysis in the sense of RL derivative. We have obtained the infinitesimal generators along with explicit similarity transformations by invariance analysis of the problem.
In addition, with the help of the similarity variables, FPDE are converted into nonlinear FODE by making use of EK differ-integral operators. Power series method is implemented to the FODE. In last, convergence of the power series solution is proved. Overall, we conclude that the mathematical solution with use of dual techniques Lie symmetry analysis along with power series solution is reliable and efficient tool in obtaining the solution of such kind of linear and nonlinear time fractional wave models.

CONFLICT OF INTERESTS
The authors declare that there is no conflict of interests.

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