On Braided Tensorcategories

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Abstract: We investigate invertible elements and gradings in braided tensor categories. This leads us to the definition of theta-, product-, subgrading and orbitcategories in order to construct new families of BTC’s from given ones. We use the representation theory of Hecke algebras in order to relate the fusionring of a BTC generated by an object $X$ with a two component decomposition of its tensorsquare to the fusionring of quantum groups of type $A$ at roots of unity. We find the condition of ‘local isomorphie’ on a special fusionring morphism implying that a BTC is obtained from the above constructions applied to the semisimplified representation category of a quantum group. This family of BTC’s contains new series of twisted categories that do not stem from known Hopf algebras. Using the language of incidence graphs and the balancing structure on a BTC we also find strong constraints on the fusionring morphism. For Temperley Lieb type categories these are sufficient to show local isomorphie. Thus we obtain a classification for the subclass of Temperley Lieb type categories.

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0. Introduction

An important step in organizing selection rules and defining symmetry principles of Quantum theories in algebraic terms has been the introduction of group theory into physics by Weyl, Yang, Mills and others. Since the works of [DR] and [D] it has become clear that the relevant data can be equivalently and more directly described by a symmetric tensor category (STC). Often in low dimensional physics the axiom that the commutativity constraint squares to one has to be relaxed so that we naturally obtain representation of the braid groups rather than the symmetric groups. The more general braided tensor categories (BTC) are related to quasitriangular quasi-Hopf algebras, but there is no one to one duality correspondence as for STC’s since BTC’s are rarely Tannakian. Interestingly, they appear in many other areas of mathematical physics like the theory of subfactors of von Neumann algebras, two dimensional integrable lattice models, and low dimensional topology.

At generic points in the space of BTC’s many uniqueness statements can be found by using deformation theory. They give some explanation about the relation of affine algebras and quantum groups at generic levels. For rational theories these methods break down. Nevertheless, one has identified equivalent rational BTC’s coming from very different areas. An example of a family of related rational models includes $SU(2)$ and rank=2 WZW-models, the corresponding quantum groups at roots of unity, subfactors with Jones-index $<4$, the Alexander or Jones polynomials, and the $Q$-state Potts model. In order to explain these coincidences in terms of a classification we need to find reasonable constraints on the considered class of BTC’s. Most conveniently they are imposed on the combinatorial part of the $\otimes$-category, i.e., the fusion rules.

In [KW] (see also [FK] for $k=2$) it has been shown that if the entire fusion ring of a BTC is equal to that of $\text{Rep}(U_q(sl(k)))$ then the two categories themselves have to be isomorphic for suitable $q$. In this paper (which is in large parts a summary of results from [FK]) we wish to impose a much weaker condition, namely that the category has a generating object $X$ whose tensor square $X \otimes X$ is the sum of two simple objects. In this situation we face a much larger class of categories including those that are obtained as product-, orbit-, and subgrading-categories from the known ones. The mentioned constructions rely on
the study of gradings and invertible objects of a BTC. Many of the resulting categories are inequivalent to any of the semisimplified representation categories of Hopfalgebras and those occurring in conformal field theory. We find a natural condition in terms of Hecke algebra representations for when this list of categories is complete. We prove it for the case where one of the summands of $X \otimes X$ is invertible, thereby yielding a complete classification.

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1. Braided Tensorcategories

In all our consideration we mean by a braided tensorcategory $\mathcal{C}$ an abelian category (see [M]) for which the morphism sets are finite dimensional vectorspaces over $\mathbb{C}$. In addition we have natural transformations $\epsilon \in Nat(\otimes, P\otimes)$ and $\alpha \in Nat(\otimes(id \times \otimes), \otimes(\otimes \times id))$. They yield the commutativity and associativity isomorphisms $\epsilon(X,Y) : X \otimes Y \to Y \otimes X$ and $\alpha(X,Y,Z) : X \otimes(Y \otimes Z) \to (X \otimes Y) \otimes Z$ which have to obey the pentagonal and two hexagonal equations. For simplicity we shall omit $\alpha$ in the formulas although it can be a non trivial morphism. Also we shall only consider rigid categories. This means that to any object $X \in \mathcal{C}$ we find a conjugate object $X^\vee$ and morphisms $ev : X^\vee \otimes X \to 1$ and $coev : 1 \to X \otimes X^\vee$, with the usual pair of contraction identities. For details see, e.g., [S] for the symmetric and [K] for the braided case.

For any $\otimes-$category $\mathcal{C}$ we can define the fusionring $K_0^+(\mathcal{C})$, which is the ring over $\mathbb{Z}^+$ generated by the equivalence classes $[X]$ of objects subject to the relations $[X] = [Y] + [X/Y]$ whenever $Y$ is included into $X$ and $[X \otimes Y] = [X][Y]$. It is clear that with this definition every object can be written uniquely as the sum of the simple objects that appear in its composition series and the products of the simple objects determine all other products of the fusionring.

A notion that is very useful for our purposes is that of grading. For a BTC the set $\otimes - Nat(id_{\mathcal{C}})$ which consists of natural transformations $\xi(X) \in End(X)$ with $\xi(X \otimes Y) = \xi(X) \otimes \xi(Y)$ is an abelian group. This fact allows us to decompose every object uniquely into

direct sum $X = \bigoplus_{\nu \in Gr(\mathcal{C})} X_{\nu}$. Here $X_{\nu}$ is the maximal subobject
such that the only eigenvalue of $\xi(X_\nu)$ is $\nu(\xi)$ for all $\xi$. $Gr(\mathcal{C})$ is the subgroup of all characters on $\otimes^-\text{Nat}(\text{id}_\mathcal{C})$ of this form. This decomposition has the property that $(X \otimes Y)_\nu = \bigoplus Y_\eta X_{\nu \eta^{-1}} \otimes Y_\eta$ and that to any simple object $X$ we can assign a unique $\nu \in Gr(\mathcal{C})$ with $X = X_\nu$. Thus $Gr(\mathcal{C})$ makes $K_\mathcal{C}^+(\mathcal{C})$ into a graded algebra. We call $\mathcal{C}$ locally rational if every component $K_\mathcal{C}^+(\mathcal{C})_\nu$ is finitely generated, i.e., if there are only finitely many inequivalent, simple objects of a given grading.

A special type of simple objects are the invertible ones, which satisfy $X \otimes X^\vee \cong 1$. They form an abelian group on $K_\mathcal{C}^+(\mathcal{C})$ we shall call $\text{Pic}(\mathcal{C})$. Let us introduce two natural group homomorphisms:

$$\vartheta : \text{Pic}(\mathcal{C}) \longrightarrow Gr(\mathcal{C})$$

$$\mu : \text{Pic}(\mathcal{C}) \longrightarrow \otimes^-\text{Nat}(\text{id}_\mathcal{C})$$

where $\vartheta$ associates a grading to an irreducible element in $\text{Pic}(\mathcal{C})$ and $\mu$ is defined by $1_g \otimes \mu(g)(X) = \epsilon(X, g) \epsilon(g, X)$. A balancing of a tensor-category is a natural transformation of $X \rightarrow X^\vee$ to the identity functor. For a BTC a balancing is equivalently given by a transformation $\theta \in \text{Nat}(\text{id}_\mathcal{C})$ with

$$\epsilon(Y, X)\epsilon(X, Y) = \theta(X) \otimes \theta(Y)\theta(X \otimes Y)^{-1} \text{ and } \theta(X^\vee) = \theta(X)^t.$$ 

If such a balancing exists (there are plenty of examples where it does not) it is unique up to elements of order two in $\otimes^-\text{Nat}(\text{id}_\mathcal{C})$. To a given balancing we can associate a family of traces $tr_X \in \text{End}(X)^*$ by

$$\text{tr}_X(f) : 1 \xrightarrow{\text{coev}} X \otimes X^\vee \xrightarrow{(f \theta(X)) \otimes 1} X \otimes X^\vee \epsilon(X, X^\vee) \xrightarrow{\epsilon(X, X^\vee)} X^\vee \otimes X \xrightarrow{\text{ev}} 1.$$ 

We call a dimension a function $d : K_\mathcal{C}^+(\mathcal{C}) \rightarrow \mathbb{C}$ which respects sums and products and is invariant under conjugation. Since the trace is cyclic, also for pairs of morphisms between different objects, and factorizes w.r.t. tensorproducts, we can define a canonical dimension by $d_{\nu}(X) = tr_X(1)$. Dimension functions can also be constructed in a different way by applying Perron-Frobenius theory to the fusion matrices of $K_\mathcal{C}^+(\mathcal{C})$, representing the action of the ring on itself by multiplication.

**Theorem 1.1** Assume that the fusion ring $K_\mathcal{C}^+(\mathcal{C})$ of a BTC $\mathcal{C}$ is locally rational, then
1. There is exactly one positive dimension \( d_{PF} : K^+_o(C) \to \mathbb{R}^+ \),

2. \( d_{PF} \geq 1 \) and \( d_{PF}(X) = 1 \) if and only if \( X \in \text{Pic}(C) \).

3. If \( X = X_\eta \), then \( X : K^+_o(C)_\nu \to K^+_o(C)_{\nu \eta} \), defined by multiplication has norm \( d_{PF}(X) \) independent of \( \nu \).

In the last statement we assumed \( K^+_o(C) \) to be equipped with the inner product for which the simple objects are an orthonormal basis.

In order to relate the positivity condition to properties of the categories themselves we introduce \( C^* \)-structures which are known from applications in operator algebras and physics [DR], but are also related to the polarizations in [S]. A \( * \)-structure on a BTC is an antilinear, contravariant, coexact BTC-functor \( * : C \to C \). For simplicity let us assume that \( X^* \otimes Y^* \to (X \otimes Y)^* \) is the identity so that \( \alpha \) and \( \epsilon \) are unitary. We call the category of finite dimensional Hilbert spaces \( \mathcal{H} \) and denote by \( \Omega \) the class of all covariant, exact (not necessarily \( \otimes^- \) ) functors \( \omega : C \to \mathcal{H} \) which commute with \( * \). We say that \( C \) is a \( C^* \)-category if for any morphism \( f \) there is some \( \omega \in \Omega \) with \( \omega(f) \neq 0 \). In this case we can introduce a norm \( \| f \| = \sup_{\omega \in \Omega} \| \omega(f) \| \) which renders the category \( C \) semisimple and equips the algebras \( \text{End}(X) \) with a \( C^* \)-structure in the usual sense. For \( C^* \)-categories we construct a balancing as follows. Define \( \lambda_X \in \text{End}(X) \) by

\[
X \xrightarrow{1 \otimes \text{ev}^*} X \otimes X^\vee \otimes X \xrightarrow{ev(X,X^\vee) \otimes 1} X^\vee \otimes X \otimes X \xrightarrow{\text{ev} \otimes 1} X.
\]

From the positivity of \( < f > = \text{ev}(1 \otimes f)\text{ev}^* = tr_X(\lambda_X^* f) \) and the fact that \( \text{End}(X) \) is a sum of type I factors with trace \( tr_X \) we infer that \( \lambda_X \) is central. Since \( tr_X \) is generally cyclic it follows that the unitary part \( \theta_o(X) = U(\lambda_X) \) gives rise to a natural transformation.

**Theorem 1.2** To any \( C^* \)-BTC \( C \) there exists precisely one balancing such that the associated traces \( tr_X \) are positive \( \forall X \in \text{ob}(C) \). It is given by \( \theta_o \in \text{Nat}(\text{id}_C) \).

Clearly, for this choice, the dimension \( d_o \) associated to the balancing is positive. Thus by Theorem [1.3] we obtain for locally rational \( C^* \)-categories the remarkable identity

\[
d_{PF} = d_o \quad (1.3)
\]

where both quantities are defined in completely independent ways.
2. Hecke - and Temperley Lieb Type Categories

In many examples $K_+^+(\mathcal{C})$ is generated by a single object $\Pi$ (e.g., a fundamental representation) meaning every object is the direct sum of subobjects of tensorpowers of $\Pi \oplus \Pi^v$. It is easy to see that in this situation $Gr(\mathcal{C}) \cong \mathbb{Z}/N$, generated by the character of $\Pi$, and the order $N \geq 1$ is the smallest number such that $Hom(\Pi^n, \Pi^{(n+N)}) \neq 0$ for some $n$.

In order to state a tractable classification problem we confine the class of BTC’s further by restricting the dimension of $End(\Pi \otimes 2)$. The condition $End(\Pi \otimes 2) = \mathbb{C}$ is by rigidity equivalent to $\Pi \in Pic(\mathcal{C})$ whereas $End(\Pi \otimes 2) = \mathbb{C} \oplus \mathbb{C}$ implies that $\Pi \otimes \Pi \cong A \oplus B$ for two inequivalent, simple objects $A$ and $B$. The first is a special case of a $\theta$-category which we classify in the next section. In the second case $\epsilon(\Pi, \Pi)$ has two eigenvalues $\gamma_A$ and $\gamma_B$ so that the rescaled natural representation of $n$-th braidgroup $B_n$ on $E_n = End(\Pi \otimes n)$, defined by $\rho(g_{i+1}) = -\gamma_A 1 \otimes 1 \otimes \epsilon(\Pi, \Pi)$ factors into a representation of the $n$-th Hecke algebra $\rho : H_n(q) \to E_n$ with $q := -\gamma_B \gamma_A^{-1}$. (We choose conventions as in [W1].) This sequence of morphisms is compatible with the inclusions $E_n \hookrightarrow E_{n+1}$: $f \mapsto f \otimes 1_{\Pi}$ and thus extends to $\rho : H_\infty \to E_\infty$. If $\mathcal{C}$ is also a $C^*$-category we have $|q| = 1$ and $\rho$ is a *-representation on every $H_n(q)$. Henceforth we call BTC’s with these properties Hecke type categories. For these $\tau|_{E_n} = d(\Pi)^{-n} tr_{H_n}$ defines a positive, normalized Markov trace on $E_\infty$ with modulus $\eta = \tau(e_A) = d(A)d(\Pi)^{-2}$.

Combining the above observations with results from [W1] we find the following restrictions:

**Theorem 2.3** For a Hecke type category with $\epsilon(\Pi, \Pi)^2$ non scalar we have

1. $q = -\gamma_B \gamma_A^{-1} = e^{\pm 2\pi i l}$ for some $l = 4, 5, \ldots$

2. $\eta = \frac{d(A)}{d(\Pi)^2} = \frac{(1-q^{-k+1})}{(1+q)(1-q^{-k})}$ for some $k = 1, \ldots, l - 1$.

3. The morphism $\rho$ factors through the semisimple quotient $H_n(q) \to H_n^{(k,l)}$ whose representations are labeled by $(k,l)$-diagrams.

Since $H_n^{(k,l)}$ coincides with the GNS-quotient of the pullback $\rho^*\tau$, the factorized morphism $\tilde{\rho} : H_n^{(k,l)} \to E_\infty$ is an inclusion. It also yields a morphism of (non rigid) fusionrings $K_o(\rho)_n : K_o^+(H_n^{(k,l)}) \to K_o^+(\mathcal{C})_n$ for
positive gradings \( n = 0, 1, \ldots \). Here \( K_0^+(H_{\infty}^{(k,l)}) \) has a unique, smallest extension into a rigid fusionring \( F^{(k,l)} \) with \( \mathbb{Z} \)-grading which is shown in [GW] to be isomorphic to the truncated subfusionring of \( U_q(Gl(k)) \) generated by the usual fundamental representation. If \( C \) is locally rational the norms of \( \|H_n^{(k,l)}\| \) and \( \Pi_{K_0^+(C)} \) and hence of \( \|K_0(\rho)_n\| \) are independent of \( n \) for large \( n \). In this situation we find that \( K_0(\rho)([1^k]) \) has norm one, i.e., it is invertible, and thus can be used to extend \( K_0(\rho) \) to a morphism of rigid fusionrings \( \Psi : F^{(k,l)} \rightarrow K_0^+(C), \) defined also for negative gradings.

The embedding of the Hecke algebras gives us not only information on the fusionring but allows us to compute the balancing phases. In \( H(q) \) the scalar \( \alpha_\lambda \) by which the central braid group element \( \Delta_2^{N} = (g_1 \ldots g_{N-1})^N \) acts in the irreducible representation associated to the diagram \( \lambda \) has been computed in [W2] as a framing anomaly of link invariants. It is possible to factorize the product of \( \epsilon \)'s in \( \mathcal{E}_N \) associated to \( \Delta_2^{N} \) into the expression \( \theta(\Pi)^\otimes N \theta(\Pi^\otimes N)^{-1} \). This observation enters the second part of the following theorem.

**Theorem 2.4** If \( C \) is a locally rational Hecke type category, then

1. there is a unique morphism of rigid fusionrings \( \Psi : F^{(k,l)} \rightarrow K_0^+(C) \) with \( \Psi([1]) = \Pi \)

2. if \( X \in \text{ob}(C) \) is a subobject of \( \Psi(\lambda) \) for some diagramm \( \lambda \) then

\[
\theta(X) = \theta_\lambda 1_X \quad \text{where} \quad \theta_\lambda = \theta(\Pi)^{|\lambda|-|\lambda|^2 \alpha_\lambda^{-1}}
\]

By definition the image of \( \Psi \) generates additively \( \text{ob}(C) \) so that every object is a sum of those considered in b). Hence the balancing of a Hecke type category is completely determined by \( \theta(\Pi), \gamma_A \) and \( \gamma_B \).

In order to explain the constraints on \( \Psi \) resulting from Theorem (2.4) we define the graph of a map of positive lattices \( \Lambda : L_1 \rightarrow L_2 \) as the bicolored graph whose vertices are the generators of \( L_1 \) and \( L_2 \) with respective coloration. The number of edges between them are given by the matrix elements of \( \Lambda \). Denoting by \( \Psi_n, [1]_n \) and \( \Pi_n \) the respective restrictions to the \( n \)-th graded components the relation \( \Psi_{(n+1)}[1]_n = \Pi_n \Psi_n \) means that pairs of neighboring simple objects in the graph of \( [1]_n \) are mapped by \( \Psi \) to sums of pairs of neighboring objects in
the graph of $\Pi_n$. By Theorem 1.1 $\Psi$ is dimension preserving, i.e., $d_{PF}(\Psi(X)) = d_{PF}(X)$. From part $b$ of Theorem 2.4 we see that $\theta$ has to have the same value on every simple object of a connected component of the graph of $\Psi_n$. Knowing the specific values for one coloration namely the $\theta_\lambda$ on $F(k,l)$ this imposes together with the neighborhood condition strong constraints on the structure of $\Psi_n$. In many cases the only remaining possibility is that the components of $\Psi_n$ are pairs of different colorations so that $\Psi_n$ is an isomorphism for every $n$. In this case we say that $\Psi$ is a local isomorphism. A special subclass of such categories are Temperley Lieb type categories which are defined in the next theorem. Its proof is in part a direct consequence of Theorem 1.1 and identity (1.3).

**Theorem 2.5** If $C$ is a Hecke type category with $\epsilon(\Pi, \Pi)^2$ nonscalar, then the following four conditions are equivalent

1.) $k = 2$
2.) $e_A$ and $1 \otimes e_A$ generate $A_\beta(3)$ with $\beta < 4$
3.) $A \in \text{Pic}(C)$
4.) $d(X) < 2$ and $d(A) \leq d(B)$.

Here $A_\beta(n)$ is the Temperley Lieb quotient of the Hecke algebra with modulus $\beta = q + q^{-1} + 2 = d(\Pi)^2$. The elements of $F^{(2,l)}$ are pairs $[\lambda_1, \lambda_2]$ with $\lambda_i \in \mathbb{Z}$ and $0 \leq \lambda_1 - \lambda_2 \leq l - 2$. The graph associated to $[1]_n$ is $A_{l-1}$, where the gradation is $n = \lambda_1 + \lambda_2$ and two simple objects are adjacent if they coincide in one component. Specializing Theorem 2.4 to $k = 2$ we can write the balancing as $\theta_\lambda = c_n t^d$ where $t^4 = q$ is the primitive $l$-th root of unity and $d = \lambda_1 - \lambda_2 + 1$.

The norm of $\Pi_n$ has to be the same as the norm of $[1]$ so that by an old result of Kronecker, see [GHJ], the only possibilities for the graph of $\Pi_n$ are $A_{l-1}$, $D_{l/2+1}$ or $E_{6,7,8}$ ($l = 12, 18, 30$). One readily checks that the neighborhood and component condition discussed above exclude the $D$ and $E$ cases. In summary, we have the following result for $k = 2$:

**Theorem 2.6** Suppose $C$ is a Temperley Lieb type category with $\beta \neq 4$. Then there exists a local isomorphism of rigid, graded fusionrings $\Psi : F^{(2,l)} \rightarrow K_o^+(C)$ with $\Psi([1]) = \Pi$.

### 3. Two Important Examples

A.) A class of braided tensorcategories that can be completely classified are semisimple BTC’s for which all simple objects are invertible.
We call them $\theta$-categories. For a $\theta$-category $C$ we have in particular $K^+_o(C) \cong \mathbb{Z}^+[\text{Pic}(C)]$ and the map $\vartheta$ from $[1,1]$ yields an isomorphism $\text{Pic}(C) \cong \text{Gr}(C)$. To a class of pairs $(\epsilon, \alpha)$ of natural isomorphisms (considered as functions $\alpha \in C(\text{Pic}(C))^3$ and $\epsilon \in C(\text{Pic}(C))^2$ by specialization) that give rise to equivalent BTC’s we can assign a unique class in $H^4(G, 2; C^*)$, the cohomology group of the Eilenberg MacLane space $K_2(G)$. This correspondence results from the fact that the pentagonal and hexagonal equations translate to cocycle conditions and the transformations $g \otimes h \mapsto g' \otimes h'$ of $\otimes$-isomorphisms give rise to coboundaries, see [FK]. The function $\theta : \text{Pic}(C) \to C^* ; g \mapsto \epsilon(g,g)$ is easily shown to be quadratic, only dependent on the cohomology class of $\epsilon$ and a possible balancing of $C$. Combining these observations with results in [EM] we find the following classification:

**Theorem 3.7** To any quadratic form $\theta$ on a finitely generated abelian group $G$ there exists one and up to isomorphism only one $\theta$-category $P(\theta, G)$ such that $\text{Pic}(P(\theta, G)) \cong G$ and $\theta(g) = \epsilon(g,g)$.

B.) It is well known that the category $\text{Rep}(U_t(Sl(k)))$ of quantum group representations, with $q = t^{-2k}$ a primitive $l$-th root of unity, is not semisimple. Nevertheless, it is possible to define a semisimple subquotient category. The morphisms are the quotients of $\text{Hom}_C(X,Y)$ by the nullspaces $\text{Hom}(X,Y)^o$ of the trace pairing

$$\text{Hom}(Y, X) \otimes \text{Hom}(X,Y) \longrightarrow \text{End}(X) \xrightarrow{\text{tr}_X} C^*$$

In this category we also discard objects with $\text{End}(X) = \text{End}(X)^o$ which for indecomposable $X$ is equivalent to $d(X) = 0$. (For details of this construction see [K] and also [A] and [GK].) The full subcategory generated by the image of $\Pi = [1]$ is a semisimple Hecke type category $R(t, k)$ without an apriori $*$-structure. Let us call this an **indefinite Hecke type category**. As for $Sl(k)$ we label the simple objects by Young diagrams with the restrictions $0 \leq \lambda_1 - \lambda_k \leq l - k$ so that $A = [1,1]$,

$B = [2]$ , $\gamma_A = -t^{1+k}$ and $\gamma_B = t^{1-k}$. The group $\text{Pic}(R(t, k)) \cong \mathbb{Z}/k$ is generated by the $\alpha = [l-k]$. The grading group $\text{Gr}(R(t, k))$ is also cyclic of order $k$ and associates to a diagram $\lambda$ the number of boxes $|\lambda| \mod k$. Hence $\vartheta : \mathbb{Z}/k \to \mathbb{Z}/k$ from $[1,1]$ is just multiplication with $l$. A possible balancing of $R(t, k)$ is given by $\vartheta_A = t^{(\lambda)}$ where

$$c(\lambda) = \sum_{i<j}(\lambda_i - \lambda_j)^2 + k(\lambda_i - \lambda_j).$$
The structure of the full subcategory over \( Pic(\mathcal{C}) \) is determined in the sense of Theorem 3.7 by \( \epsilon(\alpha, \alpha) = (-1)^{(l-k)t(l-k)t} \). The map \( \mu \) defined in (1.2) is given by \( \mu(\alpha)([1]) = t^{2l} \). A deformation argument used in \([FK]\) (which should be extendable to general \( k \)) shows that the necessary constraint in Theorem 2.3 for the existence of \(*\)-structures is also sufficient:

**Theorem 3.8** \( R(t, 2) \) is isomorphic to a \( C^* \)-category if and only if \( t^4 = e^{\pm 2\pi i} \).

There is a remarkable uniqueness result on the categories with the same fusion ring as \( R(t, k) \) due to \([KW]\) (for a proof for \( k = 2 \) using structure constants see \([FK]\)).

**Theorem 3.9** Suppose for an indefinite Hecke type category \( \mathcal{C} \) there is an isomorphism of fusionrings \( \psi : K_0^+(\mathcal{C}) \cong K_0^+(R(t, k)) \) mapping generators to each other. If in addition the invariants \( \gamma_A \) and \( \gamma_B \) of \( \mathcal{C} \) coincide with those of \( R(t, k) \) then \( \psi \) extends to an isomorphism of categories \( \mathcal{C} \cong R(t, k) \).

### 4. Product and Orbit Categories

There are a number of natural operations between categories that allow us to produce new categories, e.g., from the examples in the previous section. A special class of \( \otimes^- \) subcategories of a given BTC \( \mathcal{C} \) is obtained by picking a subgroup \( H \subset Gr(\mathcal{C}) \) and defining \( _H\mathcal{C} \hookrightarrow \mathcal{C} \) to be the largest full subcategory for which all objects have grading in \( H \). Of particular interest is the subcategory \( _0\mathcal{C} \) which consists of objects with trivial grading. It is additively generated by the subobjects of all \( j \otimes j' \) with \( j \) simple. Also we denote by \( \mathcal{C}_1 \cap \mathcal{C}_2 \) the largest full subcategory which is contained in two full \( \otimes^- \)subcategories \( \mathcal{C}_i \hookrightarrow \mathcal{C} \).

Dual to the notion of direct products of Hopf algebras we have the notion of a product of categories \( \mathcal{C}_i \) which is a biexact functor \( \otimes : \mathcal{C}_1 \times \mathcal{C}_2 \rightarrow \mathcal{C}_1 \otimes \mathcal{C}_2 \) onto the smallest additive completion of the ordinary product. The precise definition is given in \([D]\). Clearly, this functor induces an isomorphism \( Gr(\mathcal{C}_1) \oplus Gr(\mathcal{C}_2) \cong Gr(\mathcal{C}_1 \otimes \mathcal{C}_2) \).

The notion of quotients of BTC’s related to branching of representations to sub-Hopf algebras needs more explanation: To this end assume that \( P \) is a full \( \otimes^- \)subcategory with a \( \otimes^- \)-fibre functor \( \nu : P \rightarrow Vect(\mathcal{C}) \) (or \( \mathcal{H} \)) of strict, symmetric categories. To any object
$X \in \text{ob}(\mathcal{C})$ we have - up to isomorphism - a unique maximal subobject $X_P \hookrightarrow X$ with $X_P \in \text{ob}(P)$. We define a category $\mathcal{C}/P$ with $\text{ob}(\mathcal{C}/P) = \text{ob}(\mathcal{C})$ and morphisms $\widetilde{\text{Hom}}(X,Y) = \nu((Y \otimes X^\vee)_P)$. (see [D],[DM] for Tannakian categories.) The canonical morphism in $P$,
$(Z \otimes Y^\vee)_P \otimes (Y \otimes X^\vee)_P \rightarrow (Z \otimes X^\vee)$, obtained from $ev$, determines the composition of morphisms in $\mathcal{C}/P$. Using the natural braid isomorphisms we find two canonical isomorphisms in $P \otimes \mathcal{P}$:

\begin{equation}
\otimes^\pm : (Y_1 \otimes X_1^\vee)_P \otimes (Y_2 \otimes X_2^\vee)_P \rightarrow ((Y_1 \otimes Y_2) \otimes (X_1 \otimes X_2)^\vee)_P \tag{4.4}
\end{equation}

both of which define tensor products of morphisms in $\mathcal{C}/P$.

Viewing the invariances as subobjects $Z_1 \hookrightarrow Z_P$ the map $\text{Hom}(X,Y) \rightarrow \text{Hom}(1,(Y \otimes X^\vee)_1)$ \(\nu\) $\rightarrow \text{Hom}_\mathcal{C}(1,\nu((Y \otimes X^\vee)_1))$ $\rightarrow \nu((Y \otimes X^\vee)_1) \rightarrow \nu((Y \otimes X^\vee)_P)$ gives then rise to a \(\otimes\)-functor $p : \mathcal{C} \rightarrow \mathcal{C}/P$ such that the following diagram commutes:

\begin{equation}
\begin{array}{ccc}
\mathcal{C} & \overset{p}{\longrightarrow} & \mathcal{C}/P \\
\downarrow & & \downarrow \otimes 1_{\mathcal{C}} \\
\mathcal{P} & \overset{\nu}{\longrightarrow} & \text{Vect}(\mathcal{C})
\end{array}
\end{equation}

Clearly, the images of the natural isomorphisms $\tilde{\epsilon} = p(\epsilon)$ and $\tilde{\alpha} = p(\alpha)$ satisfy the pentagonal and hexagonal equations and are natural with respect to morphisms in the image of $p$. But since the functor $p$ is by definition not full for $P \neq \text{Vect}(\mathcal{C})$ there is a priori no reason for $\tilde{\epsilon}$ and $\tilde{\alpha}$ to be natural in $\mathcal{C}/P$. It turns out that naturality is equivalent to demanding that $P$ \textit{decouples}, i.e.,

\[ \epsilon(Q,X)\epsilon(X,Q) = 1 \quad \text{for all} \quad Q \in \text{ob}(P), \; X \in \text{ob}(\mathcal{C}) \, . \]

In this case the two morphisms $\otimes^\pm$ from (4.4) coincide. Suppose $j \in \text{ob}(\mathcal{C})$ is simple and $j \otimes j^\vee$ contains nontrivial subobjects from $P$. Then $\widetilde{\text{End}}(j) \neq \mathcal{C}$, and since kernels and cokernels have to stem from $\mathcal{C}$, $\mathcal{C}/P$ fails to be abelian. Also na"ive abelian completions usually spoil naturality of $\tilde{\epsilon}$. In order to avoid this situation we have to impose the condition $\mathcal{C}_0 \cap P = \text{Vect}(\mathcal{C}) \otimes 1$. It is easily seen that the only subcategories with this property are $\theta$-categories over subgroups $R \subset \text{Pic}(\mathcal{C})$ on which the grading $\vartheta|_R$ from (1.1) is injective. In this case the fusion ring morphism associated to $p$ is locally isomorphic and the
inequivalent objects of $\mathcal{C}/P$ are identical with orbits of $R$. Hence we call $\mathcal{C}/R$ an orbit category. (In [FK] the term induced category was used.) Conversely, any local isomorphism $\psi$ is of the form that it sends simple objects to their orbits under the action of $\psi^{-1}(1) \subset \text{Pic}(\mathcal{C})$. Moreover, we can pullback every category along such $\psi$ by setting

$$\text{Hom}_\mathcal{C}(X,Y) := \bigoplus_{\nu \in \text{Gr}(\mathcal{C})} \text{Hom}_{\mathcal{C}/P}(\psi(X_\nu),\psi(Y_\nu)).$$

Note that the decoupling condition for $R$ is that $R$ lies in the kernel of the map $\mu$ from (1.2). We conclude with a survey of properties of orbit categories. For more details see [FK].

**Theorem 4.10**

1. If $R \subset \text{Pic}(\mathcal{C})$ is a subgroup on which $\mu$ is trivial, $\vartheta$ is injective and the associated $\theta$-subcategory $P$ is trivial then there exist a unique, abelian BTC $\mathcal{C}/P$, and functors $\nu$ and $p$ such that (4.5) commutes.

2. For any local isomorphism $\psi : F \to K^+_o(\bar{\mathcal{C}})$ of rigid fusion rings there is a unique BTC $\mathcal{C}$ with $K^+_o(\mathcal{C}) \cong F$, a functor $p : \mathcal{C} \to \bar{\mathcal{C}}$ and a fibre functor on the subcategory associated to $\psi^{-1}(1)$ extending $\psi$ such that (4.4) commutes.

5. **A New Family of Hecke Categories and a Classification of Temperley Lieb Categories**

Combining the constructions and examples given in the previous sections we can define a class of indefinite Hecke type categories with fusionring $F^{(k,l)}$ by

$$D'(\theta,t,k) := \Delta\left(\mathcal{P}(\theta,\mathbb{Z}) \odot D(t,k)\right)$$

where $\Delta \subset \mathbb{Z} \oplus \mathbb{Z}/k = \text{Gr}(\mathcal{P} \odot D)$ is the diagonal subgroup. The basic invariants with respect to the canonical generator $\Pi' = (1) \odot \Pi$ are $\gamma_A = -\theta(1)t^{1+k}$ and $\gamma_B = \theta(1)t^{1-k}$, where (1) is the generator of $\mathcal{P}$. In fact Theorem 3.9 and Theorem 4.10 show that $D'$ is the only category with this fusionring and these invariants. We have an isomorphism

$$\varphi : \mathbb{Z} \oplus \mathbb{Z}/(k,l) \cong \text{Pic}(D') ; (i,j) \mapsto ((k,l)i) \odot \alpha^{jk'+ii''},$$

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where \( k' = k/(k, l) \) and \( ll'' = (k, l) \mod k \). The grading \( \vartheta \) is the projection onto the first factor \( i(k, l) \). The \( \theta \)-category \( P_{ij} \) associated to the infinite cyclic subgroup generated by an object \( \varphi(i, j) = (n) \odot \alpha^m \) with \( n \neq 0 \) is trivial and decouples iff \( \epsilon(\alpha, \alpha)^{m^2} = \theta(1)^{-n^2} \) and \( t^{2m} = \theta(1)^{2n} \). For these values we denote by

\[
D''(\theta, t, k, i, j) := D'(\theta, t, k)/P_{ij}
\]  

(5.6)

the orbit category as defined in Theorem 4.10. In the list of the categories of the form (5.6) we recover the ones obtained from \( \hat{\ell}(l-k) \) and \( \hat{\ell}(l-k) \) and products of these with level one theories. Using that the group extension

\[
0 \rightarrow \text{Pic}(\mathcal{C}) \rightarrow \text{Pic}(\mathcal{C}) \rightarrow \text{Gr}(\mathcal{C})
\]

is an invariant of \( \mathcal{C} \) we can easily check that the orbit construction yields categories inequivalent to any subcategories of the known representation categories of Hopf algebras. The easiest such case is found for \( l = 6, k = 2 \), if we divide by the \( \theta \)-subcategory generated by \( \varphi(1, 1) = (2) \odot [4] \). The set of simple objects \( \{[1], \ldots, [4]\} \) is the same as for the \( U_t(sl_2) \) category, but we have modifies products \( [1][1] = [3][3] = [2] + [4] \) and \( [1][3] = [1] + [2] \). In general the requirement 2.) of local isomorphie from Theorem 4.10 is difficult to verify. However for \( k = 2 \) we can use Theorem 2.6 and the uniqueness of the \( D' \)-categories to prove the following classification.

**Theorem 5.11** Every Temperley Lieb type category with \( \epsilon(\Pi, \Pi)^2 \) non-scalar is of the form \( D''(\theta, t, 2, i, j) \) for \( t^4 = e^{\pm \frac{2\pi i}{l}} \) and admissible \( \theta, i \) and \( j \).

In the case where \( \epsilon^2 \) is scalar we can consider products with suitable \( \theta \)-categories and reduce the problem to the case where \( \epsilon(\Pi, \Pi)^2 = 1 \). Since \( \Pi \) is a generator this implies that the category is symmetric and we can apply the result of [DR] to find a classification in terms of \( U(2) \)-subgroups.

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