Matrix regularization for tensor fields

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Abstract

We propose a novel matrix regularization for tensor fields. In this regularization, tensor fields are described as rectangular matrices and both of area-preserving diffeomorphisms and local rotations of the orthonormal frame are realized as unitary similarity transformations of matrices in a unified way. We also show that the matrix commutator corresponds to the covariantized Poisson bracket for tensor fields in the large-$N$ limit.
1 Introduction

Tensor fields are important objects in formulating various modern theories in particle physics. For example, they play central roles in the theory of general relativity and its higher-spin generalizations [1]. Superstring theory, which is expected to give a consistent theory of quantum gravity, also yields various tensor fields, including the gravitational field, as various perturbative excitations of strings.

The matrix models [2,3], which are conjectured to be nonperturbative formulations of superstring theory and M-theory, should contain all relevant stringy excitations. It is one of ultimate goals to show that the models can reproduce all perturbative results of the string (field) theory [4]. However, how the tensor fields can be described in terms of matrices has not been fully understood yet, while such description for scalar fields is relatively well-known as the so-called matrix...
In this paper, we generalize the matrix regularization such that it can also be applied to tensor fields. For simplicity, we restrict ourselves to 2-dimensional case but higher dimensional extension will be straightforward. We show that in this regularization, various tensor fields are described as rectangular matrices, where the sizes of the matrices are related to the ranks of the tensor fields and some geometric data of the underlying manifold. We also show that an appropriate combination of the rectangular matrices obtained by the regularization forms a single large matrix. In other words, the single square matrix can be used to regularize a collection of tensor fields with various ranks. In this regularization, the tensor algebra (of the pointwise product with contractions of tensor indices) corresponds to the matrix algebra. Furthermore, area-preserving diffeomorphisms and local rotations of the orthonormal frame are both realized as unitary similarity transformations of matrices.

Let us comment on the earlier work [8], which also gives a method of describing tensor fields on curved spaces in terms of matrices. It was proposed that by interpreting the matrices in the matrix models as covariant derivatives on an infinite dimensional vector space, diffeomorphisms and local Lorentz transformations are realized as unitary transformations on the vector space. Various tensor fields can also be naturally described in this method [8, 9]. Here, the underlying vector space has to be infinite dimensional in order for this formulation to work well. In contrast, the formulation we present in this paper is rather based on the conventional interpretation of the matrix regularization, in which the matrix size is always finite.

Our formulation is based on the so-called Berezin-Toeplitz quantization [10, 11], which gives a practical construction of the matrix regularization. Let us briefly review this quantization for a closed Riemann surface $M$. In this quantization, functions on $M$ are regarded as linear operators on spinor fields on $M$ which couple to a $U(1)$ gauge field with charge $N$. One can then restrict action of the linear operators onto the space of suitable Dirac zero modes, which form an $N$-dimensional vector space. Thus, an $N \times N$ matrix is obtained from a given function on $M$. The matrices constructed in this way enjoy some nice asymptotic behavior in the large-$N$ limit and can be seen as a concrete realization of the matrix regularization.

The generalization discussed in this paper is based on the work [23, 24] (see also [25, 26]), where the quantization of (sections of) vector bundles are proposed. We consider the case that the vector bundle to be quantized is a tensor product of the tangent and the cotangent bundles. In this case, the fields to be quantized are tensor fields. We will derive an asymptotic expansion of the quantized

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\(^1\)The regularization of matrix valued scalar fields is also proposed in [7].

\(^2\)One can also use charged scalar fields on $M$, instead of the spinor fields. The quantization in this case is given by restricting action of functions onto the holomorphic sections [10].

\(^3\)It is notable that the Berezin-Toeplitz quantization naturally appears in the context of the Tachyon condensation in string theory [12, 13]. The inverse procedure of the quantization (i.e. obtaining the classical geometry for a given matrix geometry) is also studied in [10, 18]. See also [7, 19, 22] for related work.
field and show that there is a correspondence between the covariantized Poisson bracket on the
tensor fields and a commutator-like operation for matrices. By using these mapping relations, we
will demonstrate that the matrix regularization of the free Proca field theory on a Riemann surface
gives a rectangular matrix model.

This paper is organized as follows. In section 2, we review the Berezin-Toeplitz quantization
in a general setup for the quantization of vector bundles. In section 3, we apply this quantization
to tensor fields and discuss the asymptotic behavior at large-$N$. In section 4, we consider the
regularization of the free Proca field theory as an example. In section 5, we summarize our results
and discuss future directions.

2 Berezin-Toeplitz quantization for vector bundles

In this section, we review the Berezin-Toeplitz quantization of sections of homomorphism bundles
[23][24][26]. After defining the quantization map, we discuss the large-$N$ asymptotic properties of
Toeplitz operators.

2.1 Quantization of homomorphism bundles

We consider a compact Riemann surface $M$ without boundary. Let $g$ be a metric on $M$ and $\omega$ be a
volume form of $g$. In this case, $\omega$ is a closed and nondegenerate 2-form, which defines a symplectic
structure on $M$.

Let $E$ and $E'$ be some vector bundles on $M$ with Hermitian inner products and Hermitian
connections. Let $\text{Hom}(E, E')$ be the homomorphism bundle on $M$ such that its fiber at $p \in M$
is given by a set of all linear maps from the fiber of $E$ at the point $p$ to that of $E'$. Though
we will review the quantization for general $E$ and $E'$ in this section, we are mainly interested in
the case where $E$ and $E'$ are tensor products of the tangent bundle $TM$ or the cotangent bundle
$T^*M$. In this case, sections of $\text{Hom}(E, E')$ correspond to tensor fields (See [26] for the case that
the vector bundles are not related to $TM$ or $T^*M$ but is associated with gauge groups on $M$). Let
$\Gamma(E)$ be a set of all sections of $E$. The quantization map we will discuss below maps elements of
$\Gamma(\text{Hom}(E, E'))$ to finite size matrices. For two fields $\varphi \in \Gamma(\text{Hom}(E, E'))$ and $\varphi' \in \Gamma(\text{Hom}(E', E''))$, one can consider the pointwise composition $\varphi' \varphi \in \Gamma(\text{Hom}(E, E''))$. We will see that this product
is mapped to the matrix product by the quantization map.

As discussed in the previous section, in order to define the quantization map, we needs spinors
with a $U(1)$ charge. These objects are also mathematically described in terms of vector bundles
as follows. Let $L$ be a complex line bundle with a connection 1-form $A$ satisfying

$$F = dA = \omega/V.$$  

(2.1)
Here, $V$ is the volume defined by $V = \frac{1}{2\pi} \int_M \omega$. In this normalization, the Chern number of $L$ is $\frac{1}{2\pi} \int_M F = 1$. In the physicist’s language, the connection 1-form $A$ can be considered as a background $U(1)$ gauge field with unit homogeneous magnetic flux. The spinors coupling to this gauge field with charge $N$ is said to be sections of the twisted spinor bundle $S \otimes L^\otimes N$, where $S$ is the spinor bundle on $M$ and $N$ is a positive integer. We enlarge the space of spinors in order for the fields $\Gamma(\text{Hom}(E, E'))$ can act on them: We consider spinors $\psi \in \Gamma(S \otimes L^\otimes N \otimes E)$, so that $\varphi \in \Gamma(\text{Hom}(E, E'))$ can be regarded as a linear operator, $\psi \rightarrow \varphi \psi \in \Gamma(S \otimes L^\otimes N \otimes E')$. From the Hermitian inner product of $S, L$ and $E$, an inner product on $\Gamma(S \otimes L^\otimes N \otimes E)$ is induced as

$$
(\psi', \psi) := \int_M \omega (\psi')^\dagger \cdot \psi
$$

(2.2)

for $\psi, \psi' \in \Gamma(S \otimes L^\otimes N \otimes E)$. Here, $\cdot$ represents the contraction of the spinor indices and that for $E$. The norm on $\Gamma(S \otimes L^\otimes N \otimes E)$ is then defined by $|\psi| = \sqrt{(\psi, \psi)}$.

For example, when $E = TM$ and $E' = T^* M$, $\Gamma(\text{Hom}(E, E'))$ is a set of tensors of the form $\varphi_{\alpha\beta}$. The spinors in $\Gamma(S \otimes L^\otimes N \otimes E)$ then corresponds to vector-spinor fields of the form $\psi^\alpha$, where the spinor index is omitted. The action of $\varphi \in \Gamma(\text{Hom}(E, E'))$ gives a spinor, $\varphi_{\alpha\beta} \psi^\beta$, which is an element of $\Gamma(S \otimes L^\otimes N \otimes T^* M)$.

The quantization map is defined by the restriction of $\Gamma(\text{Hom}(E, E'))$ onto suitable Dirac zero modes. The Dirac operator $D^{(E)}$ is defined by

$$
D^{(E)} \psi = i\gamma^\alpha \nabla_\alpha \psi,
$$

(2.3)

where $\psi \in \Gamma(S \otimes L^\otimes N \otimes E)$ and $\{\gamma^\alpha\}$ are the gamma matrices in a local coordinate satisfying $\{\gamma^\alpha, \gamma^\beta\} = 2\delta^{\alpha\beta}$. From the constant gamma matrices $\{\gamma^\alpha\}_{a=1,2}$ in a local orthogonal frame, which satisfy $\{\gamma^\alpha, \gamma^\beta\} = 2\delta^{ab}$, $\gamma^\alpha$ can be constructed as $\gamma^\alpha = e^a_\alpha \gamma^a$ where $e^a_\alpha$ is the inverse of the zweibein for $g$. The covariant derivative $\nabla_\alpha$ acts on $\psi \in \Gamma(S \otimes L^\otimes N \otimes E)$ as

$$
\nabla_\alpha \psi = (\partial_\alpha + \Omega_\alpha - iNA_\alpha - iA^{(E)}_\alpha) \psi,
$$

(2.4)

where $\Omega_\alpha = \frac{1}{4} \Omega_{\alpha a b} \gamma^{a b}$ is the spin connection and $A^{(E)}_\alpha$ is the connection for $E$. We denote by $\ker D^{(E)}$ the set of all normalizable zero modes of $D^{(E)}$ with respect to the inner product (2.2).

From the index theorem and the vanishing theorem, it follows that $\dim(\ker D^{(E)}) = d^{(E)} N + c^{(E)}$ for sufficiently large $N$, where $d^{(E)}$ and $c^{(E)}$ are the rank and the first Chern number of $E$, respectively \cite{26}.

Now, let us define the Berezin-Toeplitz quantization for homomorphism bundle $\text{Hom}(E, E')$. For any field $\varphi \in \Gamma(\text{Hom}(E, E'))$, the quantization map is defined by

$$
T^{(E', E)}_N(\varphi) = \Pi^\dagger \varphi \Pi.
$$

(2.5)

\footnote{Precisely speaking, $\psi \in \Gamma(S \otimes L^\otimes N \otimes E)$ is expanded by a local smooth frame as $\psi = \psi^M e_M$, where $M$ denotes a collection of all indices of $S \otimes L^\otimes N \otimes E$. The connection 1-form is then represented as a matrix as $\nabla_\alpha e_M = (A_\alpha)_M N e_N$. In \cite{24}, we omitted the indices of $\psi^M$ and the connection 1-forms for simplicity.}
Here, $\Pi : \Gamma(S \otimes L^{\otimes N} \otimes E) \to \text{Ker} D^{(E)}$ is the projection operator onto $\text{Ker} D^{(E)}$ and $\Pi'$ is that for $E'$. The operator $T_N^{(E',E)}(\varphi)$ is called the Toeplitz operator for $\varphi$. By using orthonormal bases of $\text{Ker} D^{(E)}$ and $\text{Ker} D^{(E')}$, $T_N^{(E',E)}(\varphi)$ can be represented as a rectangular matrix with size $(d^{(E')}N + c^{(E')}) \times (d^{(E)}N + c^{(E)})$. Hence, $T_N^{(E',E)}$ is a map from $\Gamma(\text{Hom}(E, E'))$ to matrices with a fixed size and the size is controlled by the parameter $N$.

The quantization map naturally preserves the Hermitian conjugation:

$$T_N^{(E,E')}(\varphi^\dagger) = (T_N^{(E',E)}(\varphi))^\dagger. \quad (2.6)$$

Here, $\varphi^\dagger \in \Gamma(\text{Hom}(E', E))$ is the Hermitian conjugate of $\varphi$ with respect to the inner product (2.2) and the $\dagger$ on the right-hand side is the Hermitian conjugate for rectangular matrices with respect to the Frobenius inner product.

### 2.2 Asymptotic properties of Toeplitz operators

The Toeplitz operator $T_N^{(E',E)}(\varphi)$ for $\varphi \in \Gamma(\text{Hom}(E, E'))$ satisfies a useful asymptotic relation in the large-$N$ limit. Here, we discuss this relation.

Let us consider two Toeplitz operators $T(\varphi) = \Pi' \varphi \Pi$ and $T(\varphi') = \Pi'' \varphi' \Pi'$ for any fields $\varphi \in \Gamma(\text{Hom}(E, E'))$ and $\varphi' \in \Gamma(\text{Hom}(E', E''))$. Here, we omit all the subscripts of the Toeplitz operators for notational simplicity. In [26], it is shown that the product $T(\varphi')T(\varphi)$ has the following asymptotic expansion in $h_N = V/N$:

$$T(\varphi')T(\varphi) = \sum_{i=0}^\infty h_N^i T(C_i(\varphi', \varphi)). \quad (2.7)$$

Here, $C_i : \Gamma(\text{Hom}(E', E'')) \otimes \Gamma(\text{Hom}(E, E')) \to \Gamma(\text{Hom}(E, E''))$ are bilinear differential operators such that, for each $i$, the order of derivatives in $C_i$ is at most $i$ for each argument. The first three $C_i$’s are explicitly given by

$$C_0(\varphi', \varphi) = \varphi' \varphi,$$

$$C_1(\varphi', \varphi) = -\frac{1}{2}(g^{\alpha\beta} + iW^{\alpha\beta})(\nabla_\alpha \varphi')(\nabla_\beta \varphi),$$

$$C_2(\varphi', \varphi) = \frac{1}{8}(g^{\alpha\beta} + iW^{\alpha\beta})(\nabla_\alpha \varphi')(R + 4F_{12}^{(E')})(\nabla_\beta \varphi)$$

$$+ \frac{1}{8}(g^{\alpha\beta} + iW^{\alpha\beta})(\gamma^{\delta \gamma} + iW^{\gamma} \delta)(\nabla_\alpha \nabla_\gamma \varphi')(\nabla_\beta \nabla_\delta \varphi). \quad (2.8)$$

Here, $F_{12}^{(E')} = e_1^{\alpha} e_2^{\beta} F_{\alpha \beta}^{(E')} = e_1^{\alpha} e_2^{\beta}(\partial_\alpha A_\beta^{(E')} - \partial_\beta A_\alpha^{(E')} - i[A_\alpha^{(E')}, A_\beta^{(E')}])$ is the curvature of $E'$ in the orthonormal frame, $R$ is the scalar curvature and $W^{\alpha\beta} := \epsilon^{\alpha\beta}/\sqrt{\det g}$ is the Poisson tensor induced by the symplectic structure. The covariant derivatives in (2.8) are defined by

$$\nabla_\alpha \varphi = \partial_\alpha \varphi - iA_\alpha^{(E')}, \quad \nabla_\alpha \varphi' = \partial_\alpha \varphi' - iA_\alpha^{(E'')} \varphi + i\varphi' A_\alpha^{(E')}. \quad (2.9)$$
Some useful relations can be derived from the expansion \((2.7)\). First, it is easy to see that
\[
\lim_{N \to \infty} |T(\varphi')T(\varphi) - T(\varphi')\varphi)| = 0, \tag{2.10}
\]
where the norm on the left-hand side is a matrix norm. This relation shows that the quantization map approximates the ring structure of fields by using the matrix product and the approximation becomes more and more precise as \(N\) goes to infinity. Second, by using the subleading term \((i = 1)\) in \((2.7)\), one can show that
\[
\lim_{N \to \infty} \left| \frac{1}{\hbar N}[T(f^1), T(\varphi)]^{(E',E)}_N + iT^{(E',E)}_N(\{f, \varphi\}) \right| = 0, \tag{2.11}
\]
where \(\varphi \in \Gamma(\text{Hom}(E, E'))\), \(f \in C^\infty(M)\). Here, we defined
\[
[T(f^1), T(\varphi)]^{(E',E)}_N := T^{(E',E)}_N(f^1_{E'})T^{(E',E)}_N(\varphi) - T^{(E',E)}_N(\varphi)T^{(E,E)}_N(f^1_{E}), \tag{2.12}
\]
and
\[
\{f, \varphi\} := W^{\alpha\beta}(\partial_\alpha f)(\nabla_\beta \varphi). \tag{2.13}
\]
In \((2.12)\), \(1_{E'}\) and \(1_E\) are the identity matrices acting on the fibers of \(E'\) and \(E\), respectively.\(^5\)

The operations \((2.12)\) and \((2.13)\) are generalizations of the matrix commutator and the Poisson bracket, respectively.

If we put both \(E\) and \(E'\) to be an identical trivial line bundle and consider \(\varphi\) as an ordinary function, our quantization map gives the matrix regularization for functions. In this case, \((2.12)\) and \((2.13)\) reduce to the ordinary commutator and Poisson bracket, respectively, and the two relations \((2.10)\) and \((2.11)\) reduce to the main defining axioms of the matrix regularization, showing that the two algebraic structures of the function algebra and the Poisson algebra should be approximately realized in terms of the matrix algebra and the Lie algebra of the matrix commutator, respectively \([6]\). In this sense, the relations \((2.10)\) and \((2.11)\) can be seen as a generalization of those axioms.

Finally, let us consider the trace of the Toeplitz operator. For \(\varphi \in \Gamma(\text{Hom}(E, E))\), the Toeplitz operator \(T(\varphi)\) is a square matrix and we can define the trace operation. In \([26]\), it is shown that the following equation holds:
\[
\lim_{N \to \infty} \hbar N \text{Tr} T(\varphi) = \frac{1}{2\pi} \int_M \omega \text{Tr}_E \varphi. \tag{2.14}
\]
Here, \(\text{Tr}_E\) stands for the trace over the fiber of \(E\). For \(\varphi, \varphi' \in \Gamma(\text{Hom}(E, E'))\), we can define a natural inner product
\[
(\varphi, \varphi') := \frac{1}{2\pi} \int_M \omega \text{Tr}_E (\varphi^\dagger \varphi'). \tag{2.15}
\]
From \((2.6)\), \((2.10)\) and \((2.14)\), one can easily show that
\[
\lim_{N \to \infty} \hbar N \text{Tr}(T(\varphi)^\dagger T(\varphi')) = (\varphi, \varphi'). \tag{2.16}
\]
Thus, the inner product \((2.15)\) is approximated by the Frobenius inner product.

\(^5\)Note that \(\text{Hom}(E, E)\) is a trivial bundle and always has the unit element \(1_E\) as a global section.
3 Quantization of tensor fields

In this section, we apply the generalized Berezin-Toeplitz quantization reviewed in the previous section to tensor fields.

3.1 Toeplitz operators for tensor fields

Let \( TM \) and \( T^*M \) be the tangent bundle and the cotangent bundle of \( M \), respectively. For non-negative integers \( k \) and \( l \), we define a tensor bundle of type \((k,l)\) on \( M \) by \( T^k_M := T^*M^\otimes k \otimes TM^\otimes l \).

We call smooth sections of \( T^k_M \) tensor fields of type \((k,l)\). We set \( \Gamma(T^0_M) = C^\infty(M) \). A tensor field of type \((k,l)\) can be expressed as

\[
f^l_k = (f^l_k)_{\mu_1\cdots\mu_k}^{\nu_1\cdots\nu_l} dx^{\mu_1} \otimes \cdots \otimes dx^{\mu_k} \otimes \partial_{\nu_1} \cdots \otimes \partial_{\nu_l}
\]

in a local coordinate. We introduce a simple multiplication law for tensor fields. For two tensor fields \( f^l_k \) and \( g^m_l \), we define a pointwise product \( \Gamma(T^l_k M) \times \Gamma(T^m_l M) \to \Gamma(T^m_l M) \) by

\[
f^l_k g^m_l := (f^l_k)_{\mu_1\cdots\mu_k}^{\rho_1\cdots\rho_l} (g^m_l)_{\rho_1\cdots\rho_l}^{\nu_1\cdots\nu_m} dx^{\mu_1} \otimes \cdots \otimes dx^{\mu_k} \otimes \partial_{\nu_1} \cdots \otimes \partial_{\nu_m}.
\]

For \( m = 0 \), the above product gives a linear map \( \Gamma(T^l_k M) \times \Gamma(T^0_l M) \to \Gamma(T^0_k M) \) defined by

\[
f^l_k g^0_l := (f^l_k)_{\mu_1\cdots\mu_k}^{\nu_1\cdots\nu_l} (g^0_l)_{\nu_1\cdots\nu_l} dx^{\mu_1} \otimes \cdots \otimes dx^{\mu_k}.
\]

Thus, \( T^k_M \) can also be considered as \( \text{Hom}(T^0_l M, T^0_k M) \).

Let us consider the Berezin-Toeplitz quantization of the tensor fields. We consider the tensor field of type \((k,l)\) as operators on \( \Gamma(S \otimes L^\otimes N \otimes T^0_k M) \). We define the inner product for the spinors as

\[
(\psi, \phi) := \int_M \omega g^{\mu_1\nu_1} \cdots g^{\mu_k\nu_k} \psi^\dagger_{\mu_1\cdots\mu_k} \cdot \phi_{\nu_1\cdots\nu_k}
\]

where \( \psi, \phi \in \Gamma(S \otimes L^\otimes N \otimes T^0_k M) \) and \( \cdot \) denotes the contraction of spinor indices. Let \( \Pi_k \) be the orthogonal projection from \( \Gamma(S \otimes L^\otimes N \otimes T^0_k M) \) to the kernel of the Dirac operator \( D(T^k_M) \). Then, the Toeplitz operator of \( f^l_k \in \Gamma(T^l_k M) \) is defined by

\[
T_{kl}(f^l_k) := \Pi_k f^l_k \Pi_l.
\]

Let us again emphasize that, since \( \dim \ker D(T^k_M) = 2^k N \) as shown in appendix \( A \), the Toeplitz operator is a finite rectangular matrix. Thus, the tensor field is regularized as a finite matrix.

The asymptotic expansion \( 2,7 \) implies that for given two tensor fields \( f^l_k \in \Gamma(T^l_k M) \) and \( g^m_l \in \Gamma(T^m_l M) \), the product of their Toeplitz operators satisfies

\[
T_{kl}(f^l_k)T_{lm}(g^m_l) = T_{km}(f^l_k g^m_l) - \frac{\hbar N}{2} T_{km}((g^{\alpha\beta} + iW^{\alpha\beta})(\nabla^l_k f^l_k)(\nabla^m_l g^m_l)) + O(N^{-2}).
\]
Similarly, the trace identity (2.14) implies that
\[ \lim_{N \to \infty} h_N \text{Tr} T_{kk}(f_k^k) = \frac{1}{2\pi} \int_M \omega(f_k^k)_{\mu_1 \cdots \mu_k}^{\mu_k \cdots \mu_1}. \] (3.7)

The relations (3.6) and (3.7) give mapping rules for derivatives and integrals of the tensor fields.
Note that the definition of the Toeplitz operator and the asymptotic relation (3.6) do not depend
on how we set the inner product of spinors. The inner product only affects the form of the equation
(3.7).

### 3.2 Unifying matrix regularization for tensor fields

Here, we show that a single square matrix can be regarded as a regularization of a collection of
tensor fields with various different ranks.

Let us consider an \((r + 1) \times (r + 1)\) matrix \(F\) whose \((k, l)\) element is a tensor field
\[ F_{kl} = f_k^l \in \Gamma(T_k^l M) \] (3.8)
for \(0 \leq k, l \leq r\). Let \(A_r\) be a vector space of all such matrices. We define a multiplication
\(A_r \times A_r \to A_r\) by combining the matrix product with the operation (3.2) as
\[ (FG)_{kl} = \sum_{m=0}^r f_k^m g_{lm}. \] (3.9)

With this multiplication, \(A_r\) forms a large associative algebra. The diagonal elements of \(A_r\) are
\(\Gamma(T_k^k M)\), which form algebras by themselves, while the off-diagonal elements \(\Gamma(T_k^l M)\) with \(l \neq k\)
are bimodules of the \(k\)th and \(l\)th diagonal algebras. In other words, the tensor algebras and the
module structures are embedded into the single large algebra of \(A_r\). In the following, we consider
the matrix regularization of the algebra \(A_r\).

In order to define the Toeplitz operators for \(A_r\), we need an appropriate space of spinor fields
on which \(A_r\) can act. We consider an \((r + 1) \times 1\) column vector \(\Psi\) such that its \(k\)th element is a
spinor field
\[ \Psi_k = \psi_k \in \Gamma(S \otimes L \otimes T_0^k M). \] (3.10)

Let \(V_r\) denote a vector space of all such vectors, which may be identified with \(\Gamma(S \otimes L \otimes (\oplus_{k=0}^r T_0^k M))\), where \(\oplus_{k=0}^r T_0^k M\) is a Whitney sum bundle of \(T_0^k M\), that is the fiber of \(\oplus_{k=0}^r T_0^k M\) is
the direct sum of the fibers of \(T_0^k M\). We define an inner product on \(V_r\) by
\[ (\Psi, \Phi) := \sum_{k=0}^r (\psi_k, \phi_k), \] (3.11)
and the associated norm by \(|\Psi| := \sqrt{(\Psi, \Psi)}\). The algebra \(A_r\) can act on \(V_r\) as
\[ (F\Psi)_k = \sum_{l=0}^r f_k^l \psi_l. \] (3.12)
Let $D$ be the Dirac operator acting on $\mathcal{V}_r$ such that its $(k, l)$ element is given by $(D)_{kl} = \delta_{kl}D_k$. The orthogonal projection $\Pi$ from $\mathcal{V}_r$ onto $\text{Ker} D$ is then an $(r + 1) \times (r + 1)$ diagonal matrix given by $\Pi_{kl} = \delta_{kl}\Pi_k$. Then, we define the Toeplitz operators for $F \in \mathcal{A}_r$ by

$$T(F) = \Pi F \Pi.$$  \hfill (3.13)

The operator $T(F)$ consists of $(r + 1)^2$ blocks and its $(k, l)$ block is given by a matrix $T_{kl}(f^l_k) = \Pi_k f^l_k \Pi_l : \text{Ker} D_l \rightarrow \text{Ker} D_k$. The Toeplitz operator has the following asymptotic relations:

$$T(F)T(G) = T(FG) - \frac{\hbar}{2} T((g^\alpha_\beta + iW^\alpha_\beta)(\nabla_\alpha F)(\nabla_\beta G)) + O(N^{-2}),$$  \hfill (3.14)

$$\lim_{N \to \infty} \hbar N \text{Tr} T(F) = \frac{1}{2\pi} \sum_{k=0}^r \int_M \omega(f^k_k)_{\mu_1 \ldots \mu_k}.$$

(3.15)

Thus, we have obtained an unified quantization map (3.13) for tensor fields with various ranks, which has the asymptotic relations (3.14) and (3.15). The quantization map in the previous section can be obtained by restricting (3.13) to a suitable subalgebra of $\mathcal{A}_r$.

### 3.3 Area preserving diffeomorphism

Here, we consider how area-preserving diffeomorphisms act on Toeplitz operators of tensor fields. A diffeomorphism $\phi : M \rightarrow M$ is area-preserving iff it preserves the area form $\omega$:

$$\phi^*\omega = \omega.$$  \hfill (3.16)

Here, the pullback of a tensor field $X$, denoted by $\phi^*X$, is defined by

$$(\phi^*X)(x) = X(\phi(x)).$$  \hfill (3.17)

In this subsection, we will show that Toeplitz operators transform as

$$T_{kl}(\phi^*f^l_k) = G_k T_{kl}(f^l_k) G_l^{-1} + O(N^{-1}) \quad (f^l_k \in \Gamma(T^l_k M))$$  \hfill (3.18)

for any area preserving diffeomorphism $\phi$ generated by Hamiltonian vector fields. Here, $G_k$ is an element of $GL_{N_k}(\mathbb{C})$ with $N_k = \text{dim} \text{Ker} D_k$ and in particular $G_0$ is an element of $U(N_0)$. The transformation law (3.18) shows that the area-preserving diffeomorphism induces a similarity transformation of the general linear groups, not of the unitary group. Only when $k = l = 0$ and $f$ is a scalar field, it reduces to the unitary similarity transformation.

\footnote{If the first homology group is nontrivial, there exist finite number of area-preserving diffeomorphisms which are not associated with any Hamiltonian vector field.}

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The structure of the transformation law (3.18) can be understood from simple mathematical observations of area-preserving diffeomorphisms. First, the pullback generally satisfies the following properties

\[ \phi^*(X \otimes Y) = (\phi^*X) \otimes (\phi^*Y), \]

\[ \int_M Z = \int_M \phi^*Z, \]

for any tensor fields \(X, Y\) and any 2-form \(Z\). Thus, if \(\phi\) is area-preserving, we obtain

\[ \int_M \omega \text{Tr}(f^k_k) = \int_M \omega \text{Tr}(\phi^* f^k_k) \]

(3.20)

for any \((k, k)\)-type tensor field \(f^k_k\). Here, we used the fact that the pullback operation also commutes with the tensor contraction operation. Now, let us consider what transformation \(\phi_{nm} : M_{nm}(C) \rightarrow M_{nm}(C)\) for Toeplitz operators corresponds to the pullback of area-preserving diffeomorphism. From the general properties of the pullback, \(\phi_{nm}\) should be linear and invertible. The identity map should also be included as a special case of \(\phi_{nm}\). From (3.6) and the first property of (3.19), \(\phi_{nm}\) should also satisfy (up to \(1/N\) corrections)

\[ \phi_{nl}(AB) = \phi_{nm}(A)\phi_{ml}(B) \]

(3.21)

for \(A \in M_{nm}(C)\) and \(B \in M_{ml}(C)\). Furthermore, from (3.7) and (3.20), \(\phi_{nn}\) should preserve the matrix trace:

\[ \text{Tr}[A] = \text{Tr}[\phi_{nn}(A)], \quad (A \in M_n(C)) \]

(3.22)

A general solution of the above requirements is given by the transformation,

\[ \phi_{nm}(A) = M_n AM^{-1}_m \]

(3.23)

for \(M_n \in GL_n(C)\).

Now, let us prove (3.18). Firstly, let us consider the infinitesimal form of diffeomorphism \(\phi(x) = x + \epsilon V (|\epsilon| \ll 1)\). The pullback of \(f^l_k \in \Gamma(T_k^l M)\) is given by

\[ \phi^* f^l_k = f^l_k + \epsilon \mathcal{L}_V f^l_k + O(\epsilon^2), \]

(3.24)

where \(\mathcal{L}_V\) is the Lie derivative along a vector field \(V\) defined by

\[ (\mathcal{L}_V f^l_k)_{\mu_1 \cdots \mu_k}^{\nu_1 \cdots \nu_l} = V^\alpha \nabla_\alpha (f^l_k)_{\mu_1 \cdots \mu_k}^{\nu_1 \cdots \nu_l} - (\nabla_\alpha V^\nu_1)(f^l_k)_{\mu_1 \cdots \mu_k}^{\alpha \mu_2 \cdots \nu_l} - (\nabla_\alpha V^\mu_2)(f^l_k)_{\mu_1 \cdots \mu_k}^{\nu_1 \alpha \mu_3 \cdots \nu_l} - \cdots \]

(3.25)

For any area preserving diffeomorphism with \(\mathcal{L}_V \omega = 0\), the vector field \(V\) satisfies \(\nabla_\mu V^\mu = 0\). By the Hodge theorem, \(V\) is decomposed into the harmonic part and the coexact part. For surface
$M$ with genus $g$, there exist $2g$ linearly independent harmonic vector fields and we will ignore transformations generated by such vector fields. The coexact part is simply a Hamiltonian vector field written as
\[ V^\mu = (X_f)^\mu := W^{\mu\nu} \partial_\nu f \] (3.26)
for a real function $f$. Let us consider $\nabla X_f = (\nabla_\alpha (X_f^\beta)) dx^\alpha \otimes \partial_\beta \in \Gamma(T_1^1 M)$. It defines an endomorphism on $\Gamma(T_k^0 M)$ for any $k$ by
\[ (\nabla X_f f_k^0)_{\mu_1\ldots\mu_k} = (\nabla_{\mu_1} (X_f)^\alpha)(f_k^0)_{\alpha\mu_2\ldots\mu_k} + (\nabla_{\mu_2} (X_f)^\alpha)(f_k^0)_{\mu_1\alpha\mu_3\ldots\mu_k} + \cdots \] (3.27)
for $f_k^0 \in \Gamma(T_k^0 M)$. Then, $\mathcal{L}_V f_k^l \in \Gamma(\text{Hom}(T_k^0, T_k^0))$ can be written as
\[ \mathcal{L}_V f_k^l = -\{f, f_k^l\} + (\nabla X_f) f_k^l - f_k^l (\nabla X_f). \] (3.28)
Here, the first and the second $\nabla X_f$ are interpreted as elements of $\Gamma(\text{Hom}(T_k^0, T_k^0))$ and $\Gamma(\text{Hom}(T_k^0, T_k^0))$, respectively, in the sense of (3.27). By applying the quantization map to (3.28), we obtain
\[ T_{kl}(\mathcal{L}_V f_k^l) = [-i\hbar^{-1} T(f 1) + T(\nabla X_f), T_{kl}(f_k^l)] + O(N^{-1}). \] (3.29)
Thus, we obtain the transformation (3.18), where
\[ G_k = \exp \left( -i\hbar^{-1} T_{kk}(f 1) + \epsilon T_{kk}(\nabla X_f) \right). \] (3.30)
By definition, we set $T_{00}(\nabla X_f) = 0$. Note that $T_{kk}(f 1)$ is Hermitian but $T_{kk}(\nabla X_f)$ does not have a definite (anti-)hermiticity in general. This implies that $G_k$ belongs to $GL_{N_k}(\mathbb{C})$ with $N_k = \text{dim Ker} \; D_k$ for $k \in \mathbb{N}$ and $G_0$ belongs to $U(N_0)$. Therefore, we find that for any finite transformation generated solely by a Hamiltonian vector field, the area-preserving diffeomorphism induces the similarity transformation (3.18).

Isometries are special area-preserving diffeomorphisms which preserve not only the area-form but the metric itself. We can see that the transformations (3.18) for the isometries are given by unitary transformations, as follows. Let us decompose
\[ \nabla_\mu X^\nu = \frac{1}{2}(\nabla_\mu X^\nu + \nabla^\nu X_\mu) + \frac{1}{2}(\nabla_\mu X^\nu - \nabla^\nu X_\mu), \] (3.31)
where the indices are raised and lowered by using the metric. If $X^\mu$ generates an isometry, the first term in (3.31) is vanishing and $\nabla_\mu X^\nu$ becomes a real anti-symmetric tensor. In this case, $G_k$ is a unitary matrix because of (2.6).

Note that the general linear transformation (3.18) does not preserve the hermiticity, although diffeomorphisms generally preserve the hermiticity. This is not a contradiction and is understood as follows. The violation of the hermiticity comes from the fact that the Hermitian conjugate defined by the inner product (2.2) depends on the metric. For example, the Hermitian conjugate of a $(1,1)$ tensor field is
\[ (f^\dagger)^{\mu\nu} = g_{\mu\rho} f^*_{\sigma\rho} g^{\sigma\nu}, \] (3.32)
and this obviously depends on the metric $g$. Let us write $f^\dagger g$ to express the $g$-dependence of the conjugation. From the general property of the pullback, we have

$$\phi^*(f^\dagger g) = (\phi^* f)^\dagger (\phi^* g). \quad (3.33)$$

If $f$ is Hermitian, namely, $f^\dagger g = f$, then it also satisfies $\phi^* f = (\phi^* f)^\dagger (\phi^* g)$. Thus, $\phi^* f$ is Hermitian with respect to the metric $\phi^* g$. On the other hand, the Toeplitz operator is defined for a fixed metric and let us write $T_g(f)$ to express the metric dependence. Then, the equation (2.6) is written in this notation as

$$T_g(f)^\dagger = T_g(f^\dagger g). \quad (3.34)$$

This shows that Hermitian tensors with respect to $\dagger g$ are mapped to Hermitian matrices. However, even if $f$ is Hermitian for $\dagger g$, the pullback $\phi^* f$ is not (It is only Hermitian for $\dagger (\phi^* g)$). Therefore, $T_g(\phi^* f)$ is not Hermitian, unless $\phi$ is an isometry with $\phi^* g = g$.

The above violation of the hermiticity is not satisfactory (though it is not a contradiction) as quantization should generally preserve it. However, as we will see below, we can reformulate the quantization map in a way that the hermiticity becomes more transparent. This formulation is given in terms of the local orthogonal frame vector fields and has a great advantage that area-preserving diffeomorphisms and local rotations of the local orthonormal frame are both mapped to unitary similarity transformations in a unified way.

### 3.4 Quantization with local orthonormal frame

Here, we reformulate the Berezin-Toeplitz quantization in terms of the orthonormal frame.

The orthonormal frame vector field is defined up to local rotations. Let us fix orthonormal frame vector fields $\{e_a\}_{a=1,2}$ and their dual 1-form fields $\{\theta^b\}_{b=1,2}$, which satisfy

$$g_{\mu\nu} e^\mu_a e^\nu_b = \delta_{ab}, \quad g^{\mu\nu} \theta^\mu_a \theta^\nu_b = \delta^{ab}, \quad e^\mu_a \theta^\mu_b = \delta^b_a. \quad (3.35)$$

and also

$$\sum_{a=1}^2 e^\mu_a \theta^a_\mu = \delta^\mu, \quad \sum_{a=1}^2 e^\mu_a e^\nu_a = g^{\mu\nu}, \quad \sum_{a=1}^2 \theta^\mu_a \theta^\nu_a = g_{\mu\nu}. \quad (3.36)$$

In the following, we raise and lower the indices $a, b$ of the local orthonormal frame by using the Kronecker delta. From these fields, we define tensor fields

$$E_{a_1a_2\cdots a_k} := e_{a_1} \otimes e_{a_2} \otimes \cdots \otimes e_{a_k} \in \Gamma(T^k_0 M),$$

$$E^{a_1a_2\cdots a_l} := \theta^{a_1} \otimes \theta^{a_2} \otimes \cdots \otimes \theta^{a_l} \in \Gamma(T^l_0 M). \quad (3.37)$$
Any \((k,l)\)-type tensor field \(f^l_k\) can then be expanded as

\[
f^l_k = (f^l_k)_{a_1...a_k} b_1...b_l E^{a_1...a_k} \otimes E_{b_1...b_l}. \tag{3.38}
\]

In terms of the original tensor components in \((3.1)\), the coefficients in \((3.38)\) are given by

\[
(f^l_k)_{a_1...a_k} b_1...b_l = (E_{a_1 a_2...a_k})^{\mu_1\mu_2...\mu_k} (E^{b_1 b_2...b_l})_{\nu_1\nu_2...\nu_l} (f^l_k)_{\mu_1 \mu_2...\mu_k \nu_1 \nu_2...\nu_l}. \tag{3.39}
\]

Similarly, we can expand any spinor \(\psi \in \Gamma(S \otimes L^{\otimes N} \otimes T^0_k M)\) as

\[
\psi = (\psi)_{a_1...a_k} E^{a_1...a_k}, \tag{3.40}
\]

where the spinor index is again omitted. Since \(\{e_a\}\) and \(\{\theta^b\}\) satisfy

\[
\nabla^\mu e^\nu_a = \partial^\mu e^\nu_a + \Omega^\nu_{\mu a} e^\rho_b + \Gamma^\nu_{\mu \rho} e^\rho_a = 0, \\
\nabla^\mu \theta^a_\nu = \partial^\mu \theta^a_\nu + \Omega^a_{\mu b} \theta^b_\nu - \Gamma^a_{\mu \rho} \theta^b_\rho = 0, \tag{3.41}
\]

the fields \((3.37)\) also satisfy

\[
\nabla^\mu (E^{a_1...a_k})_{\nu_1...\nu_k} = 0, \quad \nabla^\mu (E_{a_1...a_k})^{\nu_1...\nu_k} = 0. \tag{3.42}
\]

Because of the property \((3.42)\), if \(\psi\) is a solution to the Dirac equation \((D\psi)_{\mu_1...\mu_k} = 0\) in a local coordinate, it also satisfies \((D\psi)_{a_1...a_k} = 0\). Thus, even when we use the orthonormal frame to expand tensor fields, the definition of the Toeplitz operator \((3.5)\) remains the same. Hence, the asymptotic expansion \((3.6)\) and the unifying formulation in section \(3.2\) also hold in this case.

The definition of the orthonormal frame has an ambiguity of local rotations. Let us consider two orthonormal frames, which are related by a local \(SO(2)\) rotation as \(e'_a = \Lambda_a^b e_b\). This induces a transformation of tensor fields. For example, a \((1,1)\) tensor field \(f^b_a\) transforms as

\[
f^b_a = \Lambda_a^c f^d_c (\Lambda^T)^d_b, \tag{3.43}
\]

where \(\Lambda^T\) is the transpose of \(\Lambda\). The infinitesimal form of \((3.43)\) is

\[
\delta f^b_a = w^c_a f^b_c - f^c_a w^b_c = [w, f]^b_a, \tag{3.44}
\]

where \(w\) is a real antisymmetric tensor corresponding to the generator of \(\Lambda\). Similarly, any \(f^l_k \in \Gamma(T^l_k)\) transforms as

\[
\delta(f^l_k)_{a_1...a_k} b_1...b_l = (W_k f^l_k - f^l_k W_k)_{a_1...a_k} b_1...b_l, \tag{3.45}
\]

where \(W_k \in \Gamma(T^k_k)\) is defined by

\[
(W_k)_{a_1...a_k} b_1...b_k = \sum_{i=1}^k \delta_{a_i}^{b_i} \cdots \delta_{a_{i-1}}^{b_{i-1}} w_{a_i} b_i \delta_{a_{i+1}}^{b_{i+1}} \cdots \delta_{a_k}^{b_k}. \tag{3.46}
\]
Note that $W_k$ is real and also antisymmetric in exchanging $(a_1, a_2, \cdots, a_k)$ with $(b_1, b_2, \cdots, b_k)$. Let us then consider the transformation law for the Toeplitz operators. From (3.45), we find that

$$T(\delta f^k_i) = T(W_k)T(f^k_i) - T(f^k_i)T(W_i) + O(1/N).$$

(3.47)

Since $W_k$ is real and antisymmetric, $T(W_k)$ is anti-Hermitian. Thus, the finite form of the above transformation is of the form,

$$T(f^l_k) = U_kT(f^l_k)U^\dagger_l + O(1/N),$$

(3.48)

where $U_k \in U(N_k)$ with $N_k = \dim \text{Ker} D_k$. Thus, we find that the local rotation is realized by unitary similarity transformations on the Toeplitz operators.

Note that under diffeomorphisms, the left-hand side of (3.39) transforms as a scalar. Under any infinitesimal area-preserving diffeomorphism generated by a Hamiltonian vector field $X_f$, (3.39) transforms as

$$\delta(\delta f^{l}_k)_{a_1 a_2 \cdots a_k}^{b_1 b_2 \cdots b_l} = -W^{\mu \nu}(\partial_{\mu}f)\partial_{\nu}(f^{l}_k)_{a_1 a_2 \cdots a_k}^{b_1 b_2 \cdots b_l}.$$

(3.49)

This is not covariant under the rotation of the orthonormal indices. However, this can easily be covariantized by applying simultaneously a local rotation with $w_{ab} = -W^{\mu \nu}(\partial_{\mu}f)\Omega_{\nu ab}$. Thus, the transformation law can be written as

$$\delta'(f^{l}_k)_{a_1 a_2 \cdots a_k}^{b_1 b_2 \cdots b_l} = -W^{\mu \nu}(\partial_{\mu}f)\nabla_{\nu}(f^{l}_k)_{a_1 a_2 \cdots a_k}^{b_1 b_2 \cdots b_l} = -\{f, (f^l_k)_{a_1 a_2 \cdots a_k}^{b_1 b_2 \cdots b_l}\},$$

(3.50)

where we used the notation of the generalized Poisson bracket (2.13). Because of (2.11), (3.50) is mapped to the commutator-like operation

$$T(\delta'(f^{l}_k)) = -i\hbar^{-1}N[T(f^1_k), T(f^l_k)] + O(1/N).$$

(3.51)

Then, the finite transformation is unitary and given by the same form as (3.48). Therefore, we find that both area-preserving diffeomorphisms and local rotations are described as the unitary similarity transformation of Toeplitz operators.

Finally, let us come back to the problem of the hermiticity. In the orthonormal frame, the inner product is defined for $\psi, \chi \in \Gamma(S \otimes L^{\otimes N} \otimes T^0_k M)$ as

$$(\psi, \chi) := \int_M \omega \psi^a_{\bar{a}_1 \cdots \bar{a}_k} \cdot \chi_{a_1 \cdots a_k}.$$ 

(3.52)

This is basically equivalent to (3.4). But the hermitian conjugate of tensor fields are now given in this basis as

$$(f^\dagger)_{b_1 \cdots b_l a_1 \cdots a_k} = (f_{a_1 \cdots a_k}^{b_1 \cdots b_l})^*,$$ 

(3.53)
where \( f \in \Gamma(T^\mu_0) \) and \( f^\dagger \in \Gamma(T^\mu_0) \). This does not include the metric unlike (3.32). Since all fields appearing in the definition of the Toeplitz operator are now written in terms of the local orthonormal indices, their transformation laws under diffeomorphisms are trivial. Thus, the hermiticity of Toeplitz operator is preserved by diffeomorphisms. Note that this is also consistent with the fact that area-preserving diffeomorphisms are mapped to unitary transformations in the current formulation.

4 Application to Proca field theory

In this section, we consider the Proca field, which is a massive spin-1 field \( A = A_a E^a \in \Gamma(T^* M) \). By applying our formulation to this theory, we obtain a corresponding matrix model. We then explicitly write down the action for \( M = T^2 \) and show that the matrix action for the massless case has the fuzzy version of the \( U(1) \) gauge symmetry.

4.1 Matrix action for Proca fields

Let us consider the Euclidean action of a Proca field in two dimensions,

\[
S = \alpha \int_M \omega F^{ab} F_{ab} + \beta \int_M \omega A^a A_a,
\]

where \( F^{ab} = \nabla_a A_b - \nabla_b A_a, \nabla_a = \epsilon^\mu_a \nabla_\mu \) and \( \alpha \) and \( \beta \) are constants corresponding to the coupling constant and the mass of the Proca field, respectively. For the massless case \((\beta = 0)\), the action has the \( U(1) \) gauge symmetry \( A \mapsto A' = A + d\lambda \) for \( \lambda \in C^\infty(M) \). The field strength can be written as \( F_{12} = \epsilon^{ab}(\nabla_a A_b) \), where \( \epsilon^{ab} \) is the antisymmetric tensor with \( \epsilon^{12} = 1 \). Employing an isometric embedding \( \{X^A\}_{A=1,\ldots,d} \) which satisfies \( \sum_A \partial_\mu X^A \partial_\nu X^A = g_{\mu\nu} \), we can rewrite \( F_{12} \) as

\[
\epsilon^{ab} \nabla_a A_b = \epsilon^{ab}(\nabla_a A_c) \delta^a_b = \epsilon^{ab}(\nabla_a A_c)(\partial_b X^A)(\partial_\nu X^A) = -((\partial X^A)\{X^A, A_c\},
\]

where we defined \( \partial_a = \epsilon^\mu_a \partial_\mu = \partial^a \). Thus, the action becomes

\[
S = 2\alpha \int_M \omega[(\partial^a X^A)\{X^A, A_a\}]^2 + \beta \int_M \omega A^a A_a.
\]

(4.3)

Let us then consider the matrix regularization of the above action. The vector field \( A = A_a E^a \in \Gamma(T^0_0 M) \) can be seen as a map from \( \Gamma(S \otimes L^{\otimes N}) \) to \( \Gamma(S \otimes L^{\otimes N} \otimes T^0_0 M) \) by the multiplication \( A\psi = A_a \psi E^a \), where \( \psi \in \Gamma(S \otimes L^{\otimes N}) \). Similarly, \( \partial X^A := (\partial^a X^A) E_a \in \Gamma(T^0_0 M) \) gives a map from \( \Gamma(S \otimes L^{\otimes N} \otimes T^0_0 M) \) to \( \Gamma(S \otimes L^{\otimes N}) \). The scalar field \( X^A \in \Gamma(T^0_0 M) \) gives two endomorphisms on \( \Gamma(S \otimes L^{\otimes N}) \) and \( \Gamma(S \otimes L^{\otimes N} \otimes T^0_0 M) \). Hence, we can define the following Toeplitz operators:

\[
\hat{A} = T_{01}(A), \quad \hat{\partial} X^A = T_{01}(\partial X^A), \quad \hat{X}^A = T_{00}(X^A), \quad \hat{X}^A_i = T_{11}(X^A).
\]

(4.4)
The dimensions of the Dirac zero modes are given by \( N_1 = 2N \) and \( N_0 = N \) and the matrix size of \( T_{ij}(f) \) is given by \( N_i \times N_j \) for \( i, j = 0, 1 \) (See appendix \( \text{A} \)). Then, by applying the mapping rules (2.10), (2.11) and (2.14) to the action (4.3), we obtain the regularized action,

\[
S_{MM} = 4\pi \alpha \hbar N \text{Tr}(\hat{F}^2) + 2\pi \beta \hbar N \text{Tr}(\hat{A}^1\hat{A}),
\] (4.5)

where

\[
\hat{F} = i\hbar^{-1} \partial X^A [\hat{X}^A, \hat{A}].
\] (4.6)

Here, the generalized commutator is given by

\[
[\hat{X}^A, \hat{A}] = \hat{X}^A_1 \hat{A} - \hat{A} \hat{X}^A.
\] (4.7)

The dynamical variable in the regularized action (4.5) is \( \hat{A} \), while the other variables are determined by the given geometry of the embedding of \( M \). The latter non-dynamical matrices can be computed case-by-case in principle. In the following, we demonstrate this for the fuzzy torus.

### 4.2 Regularized action on \( T^2 \)

Here, we consider the case of \( T^2 \) and explicitly calculate the Toeplitz operators for non-dynamical variables in (4.5).

Let us first introduce the following fundamental functions on \( T^2 \):

\[
u(x^1, x^2) = e^{i x^1}, \quad v(x^1, x^2) = e^{i x^2}.
\] (4.8)

Here, \((x^1, x^2)\) is the orthonormal coordinate of \( T^2 \) with the identification \((B.1)\). We also define an isometric embedding of \( T^2 \) in \( \mathbb{R}^4 \) by

\[
X^1(x^1, x^2) = \cos(x^1) = \frac{1}{2}(u + u^*), \quad X^2(x^1, x^2) = \sin(x^1) = \frac{1}{2i}(u - u^*),
\]
\[
X^3(x^1, x^2) = \cos(x^2) = \frac{1}{2}(v + v^*), \quad X^4(x^1, x^2) = \sin(x^2) = \frac{1}{2i}(v - v^*).
\] (4.9)

They satisfy the following relations:

\[
\partial_1 X^1 = -X^2, \quad \partial_1 X^2 = X^1,
\]
\[
\partial_2 X^3 = -X^4, \quad \partial_2 X^4 = X^3,
\]
\[
\partial_1 X^3 = \partial^1 X^4 = \partial_2 X^1 = \partial_2 X^2 = 0.
\] (4.10)

In order to compute Toeplitz operators of the above functions, we first need to compute Dirac zero modes. Recall that the action (4.5) depends on two kinds of the Dirac operators, \( D_0 \) and \( D_1 \). The operator \( D_0 \) is the usual Dirac operator on charged spinor fields, while \( D_1 \) is the so-called twisted Dirac operator on charged 1-form spinor fields. See appendix \( \text{B} \) where we obtain
an orthonormal basis of \( \text{Ker} \, D_0 \). Then, by using this basis, we can also construct an orthonormal basis for \( \text{Ker} \, D_1 \) as follows. Because of the flatness of \( T^2 \), there exists a decomposition \( \text{Ker} \, D_1 = \text{Ker} \, D_0 \oplus \text{Ker} \, D_0 \) such that

\[
\Psi = \psi E^1 + \phi E^2 \quad (4.11)
\]

for \( \Psi \in \text{Ker} \, D_1 \) and \( \psi, \phi \in \text{Ker} \, D_0 \). Here, the orthonormal frame fields are defined simply as \( E^a = dx^a \) and \( E_a = \partial_a \). Let \( \{ \psi_I \}_{I=1}^N \) be the orthonormal basis of \( \text{Ker} \, D_0 \) defined in \( (B.23) \). Then, we can construct an orthonormal basis of \( \text{Ker} \, D_1 \), which we denote by \( \{ \Psi_I \}_{I=1}^N \), as

\[
\Psi_I = \begin{cases} 
\psi_I E^1 & (I = 1, 2, \ldots, N), \\
\psi_{I-N} E^2 & (I = N + 1, N + 2, \ldots, 2N). 
\end{cases} \quad (4.12)
\]

Now, let us compute the Toeplitz operators, \( \hat{X}^A = T_{00}(X^A), \hat{X}_1^A = T_{11}(X^A) \) and \( \partial \hat{X}^A = T_{01}(\partial X^A) \) defined in \( (4.4) \). From the form of the zero modes \( (4.12) \), we have the following block structure:

\[
\partial \hat{X}^A = (\partial_1 \hat{X}^A, \partial_2 \hat{X}^A), \quad \hat{X}_1^A = \begin{pmatrix} \hat{X}^A & 0 \\ 0 & \hat{X}^A \end{pmatrix}, \quad (4.13)
\]

where each block is an \( N \times N \) matrix and \( \partial_a \hat{X}^A := T_{00}(\partial_a X^A) \). Since \( \partial_a X^A \) are written in terms of \( X^A \) as shown in \( (4.10) \), the matrices \( \partial \hat{X}^A \) and \( \hat{X}_1^A \) are both made of \( \hat{X}^A \). Thus, we find that calculations of those matrices reduce to computing \( \hat{X}^A \). Furthermore, since \( X^A \) are linear combinations of \( u, \, v \) and their complex conjugates, it suffices to compute the Toeplitz operators \( \hat{u} = T_{00}(u) \) and \( \hat{v} = T_{00}(v) \). As shown in appendix \( \Box \) those Toeplitz operators are given by the clock-shift matrices as

\[
\hat{u} = e^{-\frac{a}{2N}} \begin{pmatrix} 1 & & 1 \\ & \ddots & \\ 1 & & 1 \end{pmatrix}, \quad \hat{v} = e^{-\frac{a}{2N}} \begin{pmatrix} q^{-1} & & q^{-2} \\ & \ddots & \\ q^{-N} & & \end{pmatrix} \quad (4.14)
\]

for \( q = e^{i2\pi/N} \). These matrices satisfy the algebra of non-commutative torus \( \hat{u} \hat{v} = q \hat{v} \hat{u} \) \( [27] \). By using the mapping rule \( (2.6) \) for the complex conjugate, we obtain \( \hat{X}^A \) as

\[
\begin{align*}
\hat{X}^1 &= \frac{1}{2}(\hat{u} + \hat{u}^\dagger), \\
\hat{X}^2 &= \frac{1}{2t}(\hat{u} - \hat{u}^\dagger), \\
\hat{X}^3 &= \frac{1}{2}(\hat{v} + \hat{v}^\dagger), \\
\hat{X}^4 &= \frac{1}{2t}(\hat{v} - \hat{v}^\dagger).
\end{align*} \quad (4.15)
\]

\(^7\)Here, \( \partial_a \hat{X}^A \) should not be confused with \( \partial \hat{X}^A \). The former is \( N \times N \) matrix for a scalar field \( \partial_a X^A \), while the latter is \( N \times 2N \) matrix for a vector field \( \partial^a X^A E_a \).
From the above matrices, we can easily construct \( \hat{X}^A_1 \) and \( \hat{\partial}X^A \) by using the equation (4.13).

Finally, we compute the matrix action (4.5). We also write \( \hat{A} \) as

\[
\hat{A} = \begin{pmatrix} \hat{A}_1 \\ \hat{A}_2 \end{pmatrix}, \tag{4.16}
\]

where \( \hat{A}_1 \) and \( \hat{A}_2 \) are \( N \times N \) matrices. Then, the matrix \( \hat{F} \) defined in (4.6) is given by

\[
\hat{F} = \frac{\hbar}{2N} \left( \partial_1 X^A_1 [X^A_1, \hat{A}_1] + \partial_2 X^A_2 [X^A_2, \hat{A}_2] \right)
= \frac{\hbar}{2N} \left( X^1 [X^2, \hat{A}_1] - X^2 [X^1, \hat{A}_1] + X^3 [X^4, \hat{A}_2] - X^4 [X^3, \hat{A}_2] \right). \tag{4.17}
\]

Substituting (4.16), we can rewrite \( \hat{F} \) in a simple form as

\[
\hat{F} = \partial_1 \hat{A}_2 - \partial_2 \hat{A}_1, \tag{4.18}
\]

where, we defined linear operations \( \hat{\partial}_a \) \( (a = 1, 2) \) on \( N \times N \) matrices by

\[
\hat{\partial}_1 X := \frac{\hbar}{2N} \left( \hat{\partial}_1 [\hat{\partial}_1, X] \right), \quad \hat{\partial}_2 X := \frac{-\hbar}{2N} \left( \hat{\partial}_2 [\hat{\partial}_2, X] \right). \tag{4.19}
\]

The above operators are the regularized versions of the partial derivatives \( \partial_1 \) and \( \partial_2 \). Note that these operators commute with each other, \( [\hat{\partial}_1, \hat{\partial}_2, X] = 0 \). The matrix action (4.5) for the Proca fields on \( T^2 \) is finally given by

\[
S_{MM} = 4\pi \alpha \hbar N \text{Tr}(\hat{\partial}_1 \hat{A}_2 - \hat{\partial}_2 \hat{A}_1)^2 + 2\pi \beta \hbar N \text{Tr}(\hat{A}_a \hat{A}_a). \tag{4.20}
\]

For the massless case with \( \beta = 0 \), the action (4.20) has an extra symmetry, which corresponds to the \( U(1) \) gauge symmetry in the original commutative theory. The action indeed has the symmetry under the transformation,

\[
\hat{A}_a \mapsto \hat{A}_a + \hat{\partial}_a \hat{\lambda} \tag{4.21}
\]

for any \( N \times N \) matrix \( \hat{\lambda} \). The invariance of the action follows from the linearity and the commutativity of \( \hat{\partial}_a \).

5 Summary and discussion

In this paper, we proposed the matrix regularization for tensor fields. We defined the matrix regularization in terms of the so-called the Berezin-Toeplitz quantization. We considered tensor fields as operators acting on tensor-spinor fields and the quantization was essentially given by restricting those operators onto finite-dimensional subspaces of the Dirac zero modes. We saw that after the quantization, tensor fields are mapped to finite-size rectangular matrices. Those
matrices satisfy an asymptotic expansion in the large-$N$ limit, which at the leading order gives a mapping rule for the pointwise product of tensor fields and the matrix product. At the next-leading order, it also gives a mapping rule for the covariantized Poisson bracket and the commutator-like operation of matrices.

For the conventional matrix regularization, it is well-known that area-preserving diffeomorphisms are mapped to unitary similarity transformations. We showed that the regularization proposed in this paper also possesses this property. Namely, area-preserving diffeomorphism for tensor fields are mapped to unitary similarity transformations for their Toeplitz operators. Furthermore, we showed that local rotations of the orthonormal frame are also realized as unitary similarity transformations.

We then applied our formulation to the Proca field theory. We obtained the matrix-regularized action of this theory. For the case of the fuzzy torus, we explicitly wrote down the action and showed that the regularized theory has the matrix gauge symmetry in the massless limit.

More generally, the gauge symmetry can be incorporated in our regularization as follows, based on the result in [26]. First, we enlarge the vector bundle $E$, which was a tensor product of $TM$ or $T^*M$ in this paper, as $E \otimes E_G$, where $E_G$ is a vector bundle of a representation space of the given gauge group $G$. By enlarging $E'$ in the similar way, we can consider fields in $\Gamma(\text{Hom}(E \otimes E_G, E' \otimes E'_G))$, which correspond to tensor fields coupled to the gauge field of $G$. The results in [26], in particular, the asymptotic expansion (2.7) is valid in this case as well, and the covariant derivatives in (2.7) now contain the gauge field of $G$. Thus, the gauge covariant derivative is regularized in the same way as in [26]. We can also regularize the action of the gauge field in terms of the Wilson line operators. Let us consider Wilson lines of all the connections of $S \otimes L \otimes N \otimes E \otimes E_G$. Since such operators induce linear maps on $\Gamma(S \otimes L \otimes N \otimes E \otimes E_G)$, they are regularized by using the projection onto the Dirac zero modes. As is well known, infinitesimally small Wilson loops produce the standard gauge field action, their matrix regularization will give the regularization of the gauge field action. For example, on the two-dimensional torus, let $U$ and $V$ be the straight Wilson lines from $(x^1, x^2)$ to $(x^1 + \epsilon, x^2)$ and $(x^1, x^2 + \epsilon)$, respectively. The operator $\hat{U}^\dagger \hat{V}^\dagger \hat{U} \hat{V} + h.c.$, which transports spinors along the small squares, is mapped to the product of the corresponding matrices. The matrix action is then given by $\text{Tr}(\hat{U}^\dagger \hat{V}^\dagger \hat{U} \hat{V}) + c.c.$, which is just the Eguchi-Kawai model [29]. Here, the matrices contain not only the connection of $G$, but also the other connections in $S \otimes L \otimes N \otimes E \otimes E_G$. However, the other connections are fixed by the geometry and the only dynamical field is the gauge field of $G$. The gauge symmetry is realized as the left and right actions onto the matrices $\hat{U}$ and $\hat{V}$. We will elaborate more on this correspondence in the future work.

Although we here considered the two-dimensional case to present our ideas, it is straightforward to extend this work to higher (even) dimensional cases. In contrast to the two-dimensional case, there are many interesting tensor field theories in higher dimensions. It is interesting to apply our matrix regularization to those theories to see how the fuzziness affects the structures of the
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A Dimension of $\text{Ker } D^{(T^0_kM)}$

In this appendix, we show that the dimension of $\text{Ker } D^{(T^0_kM)}$ is $2^kN$ for sufficiently large $N$.

For the elliptic differential operator $D^{(T^0_kM)}$ acting on $\Gamma(S \otimes L^\otimes N \otimes T^0_kM)$, the analytical index is defined by $\text{Ind } D^{(T^0_kM)} = \dim \text{Ker } D^{(T^0_kM)+} - \dim \text{Ker } D^{(T^0_kM)-}$ as usual, where $D^{(T^0_kM)\pm}$ are the restrictions of $D^{(T^0_kM)}$ onto the positive and the negative chirality modes, respectively. Then, the Atiyah-Singer index theorem for $D^{(T^0_kM)}$ states that the analytical index can be computed by

$$\text{Ind } D^{(T^0_kM)} = \int_M \hat{A}(M) \wedge \text{ch}(L) \wedge \text{ch}(T^*M)^\wedge k.$$  \hspace{1cm} (A.1)

Here, $\hat{A}(M)$ is the $\hat{A}$-genus of $M$ and $\text{ch}(E)$ is the Chern character of $E$. We have

$$\hat{A}(M) = 1 - \frac{1}{24} p_1 + \left( \frac{7}{5760} p_1^2 - \frac{1}{1440} p_2 \right) + \cdots,$$

$$\text{ch}(L) = \exp \left( \frac{F}{2\pi} \right) = 1 + \frac{F}{2\pi} + \frac{1}{8\pi^2} F^\wedge 2 + \cdots,$$

$$\text{ch}(T^*M) = 2 + p_1 + \frac{1}{12} (p_1^\wedge 2 - 2p_2) + \cdots,$$  \hspace{1cm} (A.2)

where $p_k$ is a $4k$-form called the $k$-th Pontryagin class. Since $M$ is assumed to be two-dimensional in this paper, only the 2-form part of the integrand contributes to the index of $D^{(T^0_kM)}$:

$$\text{Ind } D^{(T^0_kM)} = \frac{2^k N}{2\pi} \int_M F = 2^kN.$$  \hspace{1cm} (A.3)

Moreover, the vanishing theorem states that $\dim \text{Ker } D^{(T^0_kM)-} = 0$ for large enough $N$ \cite{26}. This implies that

$$\dim \text{Ker } D^{(T^0_kM)} = \text{Ind } D^{(T^0_kM)} = 2^kN$$  \hspace{1cm} (A.4)

for sufficiently large $N$.

It is sometimes useful to consider the decomposition $T^*M = T^*M^{(1,0)} \otimes T^*M^{(0,1)}$ for complex manifolds, where $(1,0)$ and $(0,1)$ represent subspaces spanned by holomorphic and anti-holomorphic 1-forms, respectively. For example, if we consider the Dirac operator $D_1$ on 1-form spinor fields, its zero modes can be decomposed to holomorphic and anti-holomorphic forms. The
number of the holomorphic zero modes can be counted by replacing $T^*M$ with $T^*M^{(1,0)}$ in (A.1). By using the formulas, $\text{ch}(T^*M^{(1,0)}) = 1 + c_1(T^*M^{(1,0)}) + \cdots$ and $\int c_1(T^*M^{(1,0)}) = -\chi_M$, where $c_1$ is the first Chern class and $\chi_M$ is the Euler number of $M$, we find that the number of zero modes in $\Gamma(S \otimes L \otimes T^0_1 M^{(1,0)})$ is equal to $N$.

Similarly, because $\int c_1(T^*M^{(0,1)}) = \chi_M$, the number of zero modes in $\Gamma(S \otimes L \otimes T^0_1 M^{(0,1)})$ is equal to $N + \chi_M$. The total number of zero modes is given by $(N - \chi_M) + (N + \chi_M) = 2N$ and this is consistent with (A.4).

**B Dirac zero modes on torus**

In this appendix, we construct Dirac zero modes on two-dimensional torus \[25\].

Let us start with a flat plane $\mathbb{R}^2$ equipped with the flat metric $ds^2 = (dx^1)^2 + (dx^2)^2$. With the identifications

\[ x^a \sim x^a + 2\pi \quad (a = 1, 2), \tag{B.1} \]

we define 2-torus $T^2$ as the quotient space

\[ T^2 = \mathbb{R}^2 / \sim. \tag{B.2} \]

$T^2$ inherits the flat metric and the corresponding volume form is given by

\[ \omega = dx^1 \wedge dx^2. \tag{B.3} \]

The volume of $T^2$ is then given by

\[ V := \frac{1}{2\pi} \int_{T^2} \omega = \frac{1}{2\pi} \int_{T^2} dx^1 dx^2 = 2\pi \tag{B.4} \]

and the Chern number density $\hbar^{-1}$ is

\[ \hbar^{-1} = \frac{N}{V} = \frac{N}{2\pi}. \tag{B.5} \]

We define the $U(1)$ gauge field $A$ for the line bundle $L$ by

\[ A = \frac{1}{4\pi}(-x^2 dx^1 + x^1 dx^2), \tag{B.6} \]

so that it satisfies the requirement (2.1). Note that $A$ is not periodic under the identification $\sim$:

\[ A(x^1 + 2\pi, x^2) = A(x^1, x^2) + \frac{1}{2} dx^2, \]
\[ A(x^1, x^2 + 2\pi) = A(x^1, x^2) - \frac{1}{2} dx^1. \tag{B.7} \]

However, this variation can be considered as a gauge transformation:

\[ A(x^1 + 2\pi, x^2) = A(x^1, x^2) + d\lambda_1, \]
\[ A(x^1, x^2 + 2\pi) = A(x^1, x^2) + d\lambda_2, \tag{B.8} \]
where the parameters are given by
\[ \lambda_1 = \frac{1}{2} x^2, \quad \lambda_2 = -\frac{1}{2} x^1. \] (B.9)

Note that the parameters (B.9) are not single valued. However, the multivaluedness does not affect (B.8).

Let us then consider spinor fields coupling to \( A \). Because of (B.8), any spinor \( \psi \in \Gamma(S \otimes L \otimes N) \) satisfies the twisted boundary condition,
\[
\psi(x^1 + 2\pi, x^2) = e^{iN\lambda_1} \psi(x^1, x^2) = \exp\left(i\frac{N}{2} x^2\right) \psi(x^1, x^2), \\
\psi(x^1, x^2 + 2\pi) = e^{iN\lambda_2} \psi(x^1, x^2) = \exp\left(-i\frac{N}{2} x^1\right) \psi(x^1, x^2).
\] (B.10)

Let us rewrite \( \psi \) as
\[
\psi(x^1, x^2) = \exp\left(i\frac{N}{4\pi} x^1 x^2\right) \chi(x^1, x^2),
\] (B.11)
introducing another spinor field \( \chi \). The boundary condition for \( \chi \) is
\[
\chi(x^1 + 2\pi, x^2) = \chi(x^1, x^2), \\
\chi(x^1, x^2 + 2\pi) = \exp\left(-iN x^1\right) \chi(x^1, x^2).
\] (B.12)

The first condition of (B.12) is solved by
\[
\chi(x^1, x^2) = \sum_{n \in \mathbb{Z}} c_n(x^2) e^{in x^1},
\] (B.13)
where \( c_n(x^2) \) are arbitrary complex functions. The second condition of (B.12) then leads to
\[
c_n(x^2 + 2\pi) = c_{n+N}(x^2)
\] (B.14)
for all \( n \in \mathbb{Z} \).

Now, let us construct Dirac zero modes for \( D_0 \). For sufficiently large \( N \), any zero-mode only has a positive chirality component as a consequence of the vanishing theorem. Then, if we denote by \( \psi^+ \) and \( \psi^- \) the positive and negative chirality components of a zero mode \( \psi \), respectively, we always have \( \psi^- = 0 \). The positive mode \( \psi^+ \) is determined by
\[
\left[ \partial_1 + i\partial_2 + \frac{N}{4\pi} (x^1 + ix^2) \right] \psi^+ = 0,
\] (B.15)
which is the positive chirality mode of the massless Dirac equation. By plugging the general form (B.11) into the above equation, we obtain
\[
\left( \partial_1 + i\partial_2 + i\frac{N}{2\pi} x^2 \right) \chi^+ = 0.
\] (B.16)
With the mode expansion \([B.13]\), we have
\[
\forall n \in \mathbb{Z} : \quad \partial_2 c_n(x^2) + (n + \frac{N}{2\pi} x^2) c_n(x^2) = 0. \tag{B.17}
\]

It is easy to see that this equation is solved by
\[
c_n(x^2) = d_n \exp\left(-\frac{N}{4\pi} (x^2)^2 - n x^2\right), \tag{B.18}
\]
where, \(d_n\) are integration constants. The equation \([B.14]\) imposes the recursion relation,
\[
d_{n+N} = d_n e^{-\pi(N+2n)} \tag{B.19}
\]
for any integer \(n\). By writing the integer \(n\) as
\[
n = Nk + I \quad k \in \mathbb{Z}, \ I \in \{1, 2, \cdots, N\}, \tag{B.20}
\]
we can solve the recursion relation \([B.19]\) as
\[
d_{Nk+I} = d_I e^{-\pi(Nk^2+2Ik)}. \tag{B.21}
\]

Hence, the only unknown coefficients are \(\{d_I| I = 1, 2, \cdots, N\}\) and they can also be determined by normalizing the zero modes as we will see shortly. We thus obtain the general zero-mode solution as
\[
\psi^+(x^1, x^2) = e^{-\frac{N}{4\pi} (x^2)^2} e^{i \frac{N}{4\pi} x^1 x^2} \sum_{I=1}^{N} d_I \sum_{k \in \mathbb{Z}} e^{-\pi(Nk^2+2Ik)} e^{i(Nk+I)(x^1+i x^2)}. \tag{B.22}
\]

Here, notice that the above general solution is a superposition of
\[
\psi^+_I(x^1, x^2) = d_I e^{-\frac{N}{4\pi} (x^2)^2} e^{i \frac{N}{4\pi} x^1 x^2} \sum_{k \in \mathbb{Z}} e^{-\pi(Nk^2+2Ik)} e^{i(Nk+I)(x^1+i x^2)} \tag{B.23}
\]
for \(I = 1, 2, \cdots, N\). It is easy to see that \(\psi^+_I\) are orthogonal to each other. Furthermore, by putting
\[
d_I = \left(\frac{N}{8\pi^4}\right)^{1/4} e^{-\frac{\pi}{4\pi^2}}, \tag{B.24}
\]
they become orthonormal (see also appendix C for this calculation.). Because the index theorem shows that the number of the zero modes is exactly equal to \(N\), we can take \(\{\psi^+_I\}\) as an orthonormal basis of the Dirac zero modes.
C Computation of $\hat{u}^n v^m$

In this appendix, we explicitly calculate $\hat{u}^n v^m = T_{00}(u^n v^m)$ for general integers $n$ and $m$, where $u$ and $v$ are the fundamental functions on $T^2$ defined in (4.13).

We will evaluate the following integral:

$$\left(\hat{u}^n v^m\right)_{IJ} = \int_{T^2} \omega \psi_I^* \cdot u^n v^m \psi_J$$

$$= d_I d_J \sum_{k,k' \in \mathbb{Z}} e^{-\pi(Nk^2 + 2Ik)} e^{-\pi(Nk'^2 + 2Ik')} \int_{S^1} dx^2 e^{-\frac{N}{2\pi}(x^2)^2} e^{-\frac{N}{2\pi}(Nk + I + J)x^2} e^{inx^1}$$

$$\times \int_{S^1} dx^1 e^{-i[N(k-k') + I - J]x^1} e^{inx^1}, \quad (C.1)$$

where $\psi_I$ are the Dirac zero modes [B.23] on $T^2$. We first write the integers $n, m$ in the following forms:

$$n = Nk_n + \tilde{n}, \quad m = Nk_m + \tilde{m}, \quad (k_n, k_m \in \mathbb{Z}, \quad \tilde{n}, \tilde{m} \in \{1, 2, \cdots, N\}) \quad (C.2)$$

where $k_n$ and $\tilde{n}$ are the quotient and the remainder for $n$ divided by $N$ and similarly $k_m$ and $\tilde{m}$ are those for $m$. Note that we have $-2N + 1 \leq I - J - \tilde{n} \leq N - 2$. The integral over $x^1$ in (C.1) is then given by

$$\int_{S^1} dx^1 e^{-i[N(k-k') + I - J]x^1} e^{inx^1}$$

$$= \begin{cases} 
2\pi \delta_{k-k'-k_n,0} \delta_{I-J-\tilde{n},0} & ((0) : 0 \leq I - J - \tilde{n} \leq N - 2), \\
2\pi \delta_{k-k'-k_n-1,0} \delta_{I-J-\tilde{n}+N,0} & ((1) : -N \leq I - J - \tilde{n} \leq -1), \\
2\pi \delta_{k-k'-k_n-2,0} \delta_{I-J-\tilde{n}+2N,0} & ((2) : -2N + 1 \leq I - J - \tilde{n} \leq -N - 1),
\end{cases} \quad (C.3)$$

where we labeled the above three cases by $(a) = (0), (1), (2)$. By using the index $a = 0, 1, 2$ which label the above cases, we can write (C.1) as

$$\left(\hat{u}^n v^m\right)_{IJ} = 2\pi d_I d_J \delta_{I-J-\tilde{n}+aN,0} \sum_{k \in \mathbb{Z}} e^{-\pi(2Nk^2 - 2kn + 4Ik + Nk_n^2 - Na^2 - 2Ik_n + 2\tilde{n}k_n - 2Ia + 2\tilde{n}a)}$$

$$\times \int_{S^1} dx^2 e^{-\frac{N}{2\pi}(x^2)^2} e^{-\frac{N}{2\pi}(2Nk + I - J)x^2} e^{inx^1}$$

$$= 2\pi d_I d_J \delta_{I-J-\tilde{n}+aN,0} \sum_{k \in \mathbb{Z}} e^{-\pi(2Nk^2 - 2kn + 4Ik + Nk_n^2 - Na^2 - 2Ik_n + 2\tilde{n}k_n - 2Ia + 2\tilde{n}a)}$$

$$\times \int_{S^1} dx^2 e^{-\frac{N}{2\pi}[x^2 + \frac{2}{N}(2Nk + I - J)x^2]} e^{\frac{N}{2\pi}[(2Nk + I - J)x^2]}$$

$$= 2\pi d_I d_J \delta_{I-J-\tilde{n}+aN,0} e^{-N\pi(k_n^2 - a^2)} e^{2\pi a \tilde{n} k_n (I - \tilde{n})} e^{-\frac{N}{2\pi}(2I - n - im)^2}$$

$$\times \sum_{k \in \mathbb{Z}} \int_{S^1} dx^2 e^{-\frac{N}{2\pi}(x^2 + 2\pi k + \frac{2\pi}{N}(I + n - im))^2}.$$
Let us choose the integration range of $x^2$ as $[0, 2\pi)$. By shifting the integration variable, we obtain

\[ \hat{u}^n \hat{v}^m_{IJ} = 2\pi d_I d_J \delta_{I-J-n+a} e^{-\frac{N\pi}{2}(k_n^2-a^2)} e^{2\pi k_n (I-n)} e^{\frac{\pi}{2N}(2I-n-im)^2} \]

\[ \times \sum_{k\in\mathbb{Z}} \int_{2\pi k}^{2\pi(k+1)} dx^2 e^{-\frac{K}{2}(x^2+\frac{2\pi l}{N} + \frac{1}{2}(n+im))^2} \]

\[ = 2\pi d_I d_J \delta_{I-J-n+a} e^{-\frac{N\pi}{2}(k_n^2-a^2)} e^{2\pi k_n (I-n)} e^{\frac{\pi}{2N}(2I-n-im)^2} \]

\[ \times \int_{\mathbb{R}} dx^2 e^{-\frac{K}{2}(x^2+\frac{2\pi l}{N} + \frac{1}{2}(n+im))^2} \]

\[ = 2\pi \sqrt{\frac{2\pi^2}{N}} d_I d_J \delta_{I-J-n+a} e^{-\frac{N\pi}{2}(k_n^2-a^2)} e^{2\pi k_n (I-n)} e^{\frac{\pi}{2N}(2I-n-im)^2}. \]

When $n = m = 0$, the equation (C.5) is just equal to the inner product $(\psi_I, \psi_J)$. In this case, only nontrivial matrix entries arise for $a = 0$ in (C.5). We then find that

\[ (\psi_I, \psi_J) = \sqrt{\frac{8\pi^4}{N}} d_I d_J \delta_{I,J}. \]

Hence, we can check that if $d_I$ are chosen as in (B.24), the basis $\{\psi_I\}_{I=1,2,\ldots,N}$ is indeed orthonormal satisfying $(\psi_I, \psi_J) = \delta_{I,J}$.

For general $m$ and $n$, we substitute (B.24) into (C.5) and obtain

\[ \hat{u}^n \hat{v}^m_{IJ} = \delta_{I-J-n+a} e^{-\frac{N\pi}{2}(n^2-2im+m^2)} e^{2\pi a (I-n)} q^{-mI}, \]

where $q := e^{\frac{2\pi i}{N}}$. When $(n, m) = (1, 0)$, (C.7) reduces to

\[ \hat{u}_{IJ} = \delta_{I-J-1+a} e^{-\frac{\pi}{2N} e^{2\pi a (1-I)}}. \]

The only non-vanishing cases are $a = 0, 1$ and we obtain

\[ \hat{u}_{IJ} = \begin{cases} 
\delta_{I-J-1,0} e^{-\frac{\pi}{2N}} & (1 \leq I - J \leq N - 1), \\
\delta_{I-J-1+N,0} e^{2\pi (1-I)} & (-N + 1 \leq I - J \leq 0).
\end{cases} \]

Similarly, when $(n, m) = (0, 1)$, (C.7) becomes

\[ \hat{v}_{IJ} = \delta_{I-J,0} e^{-\frac{\pi}{2N} e^{2\pi (1-I)} q^{-J}}. \]

Thus, we obtain the Toeplitz operators (4.14). Finally, it is easy to show that (C.7) with general $(n, m)$ can be written in terms of $\hat{u}$ and $\hat{v}$ as

\[ \hat{u}^n \hat{v}^m = e^{-\frac{\pi}{2N}(n^2-a^2-m^2)} q^{-\frac{nm}{2}} \hat{u}^n \hat{v}^m. \]
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