Bulk viscosity and trace anomaly in the massive Gross-Neveu model

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Abstract. I present preliminary results concerning the calculation of the bulk viscosity for the Gross-Neveu model in the large $N$ limit with a non-zero bare fermion mass and vanishing chemical potential. This finite mass breaks the integrability of the model and allows for transport of momentum in a non-equilibrium situation. By making this mass arbitrarily small, we can explicitly study the relationship between a peak in the trace anomaly and the bulk viscosity. Since the Gross-Neveu model resembles QCD in many essential aspects, its study allows us to extrapolate some of the conclusions obtained to this physically relevant theory.

1. Introduction
Transport coefficients are essential inputs to describe the space-time evolution of systems not far from equilibrium. During the last years there has been a very active effort to analyze them from both the theoretical and phenomenological points of view in the context of heavy-ion collisions, condensed matter physics, astrophysics and cosmology. The calculation of transport coefficients in quantum field theory at intermediate and strong coupling is still a challenge from both the analytical and the numerical points of view. Due to their intrinsic non-perturbative nature, even in weakly interacting theories a resummation of an infinite number of diagrams is needed in order to obtain the leading-order result. In the strongly coupled regime, the most prominent method available is the AdS/CFT correspondence, although it is only applicable to a limited class of field theories. On the other hand, lattice simulations are still not accurate enough regarding the calculation of spectral densities, specially at finite quark chemical potential because of the sign problem.

It was recently conjectured, based on a sum rule for the spectral density of the trace of the energy-momentum tensor in Yang-Mills theory [1], that a maximum of the trace anomaly near the critical temperature would drive a maximum for the bulk viscosity. The corresponding sum rule was later corrected by Romatschke and Son [2], and the ansatz for the spectral density used to extract the bulk viscosity questioned. Since the trace anomaly measures the breaking of scale invariance in a system, and the bulk viscosity $\zeta$ essentially represents the difficulty for a system to relax back to equilibrium after a scale transformation, it seems in principle reasonable to think that $\zeta$ would be maximum when the breaking of scale invariance is maximum. In what follows I will show an explicit calculation in a model where this can be accurately tested.
2. The massive Gross-Neveu model

We will consider the Gross-Neveu model [3] with an explicit bare mass for the fermion field:

\[ \mathcal{L} = \sum_{i=1}^{N} \bar{\psi}_i i \gamma \psi_i + \frac{g^2}{2} \left( \sum_{i=1}^{N} \bar{\psi}_i \psi_i - Nm \right)^2. \]  

(1)

Since we are interested in studying the large \( N \) limit of the model, in order to properly classify the different Feynman diagrams according to their topologies and power counting in \( 1/N \), it is convenient to introduce an auxiliary field \( \sigma \):

\[ \mathcal{L} = \sum_{i=1}^{N} \bar{\psi}_i i \gamma \psi_i - \frac{1}{2} \sigma^2 - g \sigma \sum_{i=1}^{N} \bar{\psi}_i \psi_i + Nmg \sigma. \]  

(2)

Clearly, the introduction of this field does not affect the dynamics of the system because its equation of motion is \( \sigma = Ngm - g \sum_i \bar{\psi}_i \psi_i \).

In \( 1 + 1 \) space-time dimensions and in the large-\( N \) limit, this model shares many important features with massless QCD in \( 3 + 1 \) dimensions: it is renormalizable, asymptotically free, classically scale invariant (for zero bare fermion mass), it has a dynamically generated mass gap which manifests as a peak in the trace anomaly, and in vacuum undergoes an spontaneous breaking of the discrete “chiral” symmetry \( \psi \rightarrow \gamma_5 \psi \).

The introduction of this bare mass \( m \) is a simple way of allowing the system to relax back to thermodynamic equilibrium after a small perturbation in the distribution of momenta [4]. The Gross-Neveu model (without the bare mass) is an integrable quantum field theory [5, 6], which implies the existence of an infinite number of conserved charges and in \( 1 + 1 \) dimensions the factorization of the \( S \)-matrix in terms of binary collisions, so inelastic processes have vanishing scattering amplitude. Since in \( 1 + 1 \) dimensions binary collisions cannot modify the distribution of momenta, integrability then prevents momentum transport in this system. Consequently, the bulk viscosity\(^1\) of the Gross-Neveu model is infinite. After including the bare mass in the model, this factorization in terms of binary collisions does not longer happen, and it renders the bulk viscosity finite. On the other hand, the bare mass also suppresses the density of kink-anti-kink configurations in the thermodynamic limit and makes the mean-field \( 1/N \) expansion well defined [4, 7, 8].

The thermodynamics of this model has been extensively studied both at finite temperature and fermion chemical potential (see for instance [8, 9, 10] and references therein). In order for the perturbative expansion in powers of \( 1/N \) to be sensible, the bare coupling constant must be re-scaled, \( g \equiv \lambda/\sqrt{N} \), with \( \lambda \) being constant as \( N \rightarrow \infty \). In the mean field approximation (leading order), the renormalized effective potential in terms of the classical auxiliary field \( \sigma_c \) at finite temperature and zero chemical potential is

\[ V_{\text{eff}}^R(\sigma_c; T) = \frac{1}{2g^2} \sigma_c^2 - mN\sigma_c + \frac{N^2 \sigma_c^2}{4\pi} \left[ \ln \left( \frac{\sigma_c}{\sigma_0} \right)^2 - 3 \right] - \frac{2NT}{\pi} \int_0^\infty dk \ln \left( 1 + e^{-k^2 + \sigma_c^2/T} \right), \]

(3)

where \( \sigma_0 \) is the renormalization scale. From the effective potential we can derive useful physical information such as the thermal mass gap \( M(T) \), the pressure, and the rest of thermodynamic quantities. In Fig. 1 I plot the thermal mass gap, speed of sound, and trace anomaly as a function of the temperature for different values of the quotient \( m/M_0 \) (\( M_0 \) is the mass gap at zero temperature). For \( m = 0 \), the thermal mass gap vanishes at the “critical temperature”\(^2\).

\(^1\) The shear viscosity cannot be defined in \( 1 + 1 \) dimensions.
Figure 1. (Left) Thermal mass of the fermion field as a function of the temperature for different values of $m/M_0$. (Center) Speed of sound squared. (Right) Finite-temperature part of the trace anomaly. $P_b \equiv P(T = 0)$ is the bag constant.

$T_c \simeq 0.57M_0$, and above that temperature the leading-order thermodynamics correspond to a free massless gas. As we can also see, the trace anomaly has a pronounced peak at $T_c$. Since we need to introduce a finite bare mass $m$, the lagrangian is no longer scale invariant. Nevertheless, we can make $m$ as small as we wish in order to approach the curve corresponding to $m = 0$ as much as possible and then make sure that the anomalous part dominates $\langle T^\mu_\mu \rangle$. By doing so, naively, the bulk viscosity would increase (see below, though) because the rate of non-binary collisions becomes smaller and the system takes longer to go back to equilibrium. This is perfectly fine for our purpose of testing the possible correlation between the bulk viscosity and the trace anomaly; nonetheless, this model is not suitable to obtain for instance an estimate of the absolute value of the quotient $\zeta/s$ for QCD.

It is important to remark that the quantity which actually controls the breaking of integrability in this model is $m/M(T)$ [4]. Below $T_c$, $m/M(T) \to 0$ for $m \to 0$, hence the bulk viscosity becomes arbitrarily large and integrability is eventually restored. However, for $T > T_c$, $m/M(T)$ goes to a temperature-dependent constant as $m \to 0$, and thus integrability is never restored. In this latter case, the bulk viscosity becomes arbitrarily small as $m \to 0$ because the source term in the Boltzmann equation (see next section) eventually vanishes due to the restoration of scale symmetry.

3. Calculation of the bulk viscosity within kinetic theory

As we have seen in the previous section, the interaction between fermions is suppressed by powers of $1/N$. In addition, the fermion thermal width is $O(1/N)$ [4]. Therefore, in principle it seems reasonable to adopt a kinetic theory treatment to analyze the transport properties of this system in the large $N$ limit. I will use this approach following essentially the previous works [12, 13, 14, 15, 16].

In order to obtain the bulk viscosity, we need to determine the statistical average of the energy-momentum tensor of the system in a cell of fluid for a small departure from equilibrium.

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2 The chiral symmetry is actually immediately restored at $T = 0^+$ due to kink-anti-kink configurations (satisfying so the Mermin-Wagner theorem), and thus this critical temperature turns out to be an artifact of the mean-field approximation. However, as mentioned above, the introduction of a finite bare mass suppresses these kink-anti-kink configurations in the thermodynamic limit, and therefore we can approach the curve $m = 0$ as much as we wish provided we keep $m$ finite.

3 Strictly, due to infrared divergencies characteristic of low-dimensional systems, this calculation is valid in $1 + 1$ dimensions only in the limit $N \to \infty$, where the long-time tail in the energy-momentum tensor correlator $\sim t^{-1/2}$ becomes negligible [11]. Otherwise, for $N$ finite, the bulk viscosity of the massive Gross-Neveu model would be infinite.
In kinetic theory, this average is [17]

\[ T^{\mu\nu}(t, x) = \sum_A \int_{-\infty}^{\infty} \frac{dk}{(2\pi)^3 E_k} k_\mu k_\nu f^A(t, x, k), \quad (4) \]

where \( f^A = f^A(t, x, k) \) is the non-equilibrium distribution function, \( A \) is a collective index which denotes its fermionic or anti-fermionic character as well as the flavor (the sum runs over all the possible types of fermions), \( E_k \equiv \sqrt{M(T)^2 + k^2} \), and \( k = (E_k, k) \) is the canonical momentum (the underline emphasizes that it is on-shell).

The Boltzmann-Uehling-Uhlenbeck equation determines the space-time evolution of distribution functions for dilute systems due to the change in the number of particles of type \( a \) produced by collisions in the fluid. In 1+1 dimensions it reads

\[ \left( \frac{\partial}{\partial t} + \frac{k}{E_k} \frac{\partial}{\partial x} \right) f^A = \left. \frac{\partial f^A}{\partial t} \right|_{\text{gain}} - \left. \frac{\partial f^A}{\partial t} \right|_{\text{loss}} = \frac{1}{E_k} C_k^A[f], \quad (5) \]

with \( f = \{\{f^A\}\} \) a vector containing the distribution functions for every fermion state.

To derive an expression for the bulk viscosity we need to solve (5) for small departures from equilibrium. To do it, we first write \( f^A(t, x, k) = f^A_{eq}(t, x, k) + \delta f^A(t, x, k) \), where \( \delta f^A \) is linear in spatial gradients of the fluid velocity and small. The fermion distribution function at equilibrium for zero chemical potential is \( f^A_{eq}(t, x, k) = \exp(\beta k \cdot u) + 1)^{-1} \), with \( \beta^{-1} \equiv T(t, x) the local temperature, and \( u^\alpha(t, x) \) the velocity of the corresponding fluid cell. Linearizing both sides of (5) with respect to spatial gradients in the local rest frame (\( u^1 = 0 \)), one finds that the deviation from equilibrium must be of the form

\[ \delta f^A_k = -\beta n_F(E_k)|1 - n_F(E_k)|B_k^A \frac{\partial u^1}{\partial x}, \quad (6) \]

with \( B_k^A = B^A(|k|) \) some dimensionless function to be determined by solving the integral equation obtained after the previous linearization and \( n_F(E_k) \equiv (e^{\beta E_k} + 1)^{-1} \) is the Fermi-Dirac distribution function. This new integral equation can be represented as the action of a linear operator over the departure from equilibrium:

\[ |S\rangle = \mathcal{E} |B\rangle, \quad (7) \]

where the source term on the left-hand side is given by \( S(p) = p^2 - c_s^2 (E_p^2 - MTdM/dT). \) An arbitrary matrix element of the collision operator is, in the large \( N \) limit,\(^4\)

\[ \langle \psi | \mathcal{E} | \chi \rangle = N^3 \beta \int_{-\infty}^{\infty} \prod_{i=1}^{6} \frac{dp_i}{(2\pi)^2 E_i} \left\{ (2\pi)^2 \delta^{(2)}(p_1 + p_2 + p_3 - p_4 - p_5 - p_6) n_{F,1} n_{F,2} n_{F,3} \right. \]

\[ \times \left. (1 - n_{F,4})(1 - n_{F,5})(1 - n_{F,6}) \left[ \frac{1}{6} |M_{123\rightarrow456}|^2 + \frac{3}{2} |M_{123\rightarrow456}|^2 \right] \right. \]

\[ \times \left. (\psi_1 + \psi_2 + \psi_3 - \psi_4 - \psi_5 - \psi_6)(\chi_1 + \chi_2 + \chi_3 - \chi_4 - \chi_5 - \chi_6) + (2\pi)^2 \delta^{(2)}(p_1 + p_2 - p_3 - p_4 - p_5 - p_6) n_{F,1} n_{F,2} \right. \]

\[ \times \left. (1 - n_{F,3})(1 - n_{F,4})(1 - n_{F,5})(1 - n_{F,6}) \left[ |M_{12\rightarrow3456}|^2 + \frac{3}{2} |M_{12\rightarrow3456}|^2 \right] \right. \]

\[ \times \left. (\psi_1 + \psi_2 - \psi_3 - \psi_4 - \psi_5 - \psi_6)(\chi_1 + \chi_2 - \chi_3 - \chi_4 - \chi_5 - \chi_6) \right\}, \quad (8) \]

\(^4\) We here use that the deviation from equilibrium \( B^A \) is flavor-independent (the \( O(N) \) flavor symmetry is not broken in \( 1+1 \) dimensions), and it is also the same for fermions and anti-fermions by invariance of the source term under charge conjugation.
thus the collision integral is dominated by the processes $3 \to 3$ and $2 \leftrightarrow 4$ in this limit. Then the bulk viscosity is given by \[ \zeta = \langle S|B \rangle = \langle S|\hat{c}^{-1}|S \rangle. \] As shown in \cite{12, 13, 14}, in order to calculate numerically this expectation value for the inverse of the collision operator, it is most convenient to do it variationally. If we define the functional

\[ Q[\chi] = \langle \chi|S \rangle - \frac{1}{2}\langle \chi|\hat{c}|\chi \rangle, \] \hspace{1cm} (10)

then the solution of (7) corresponds to a maximum of this functional. Hence, the bulk viscosity is proportional to this maximum, $\zeta = 2 Q_{\text{max}}$. Expanding the solution of the Boltzmann equation in terms of a set of $D$ linearly independent functions, $B_i(k) = \sum_{i=1}^{D} b_i^A \phi_i(k)$, and maximizing (10) with respect to the coefficients $b_i^A$ then, when the set constitutes a basis, the bulk viscosity can be expressed as

\[ \zeta = S^\dagger b = S^\dagger \tilde{C}^{-1} S, \] \hspace{1cm} (11)

with $S_i = \langle \phi_i|S \rangle$, $C_{ij} = \langle \phi_i|\tilde{c}|\phi_j \rangle$. Note that $S_i = \mathcal{O}(N)$, and since $C_{ij} = \mathcal{O}(1/N)$ \cite{4}, hence $\zeta = \mathcal{O}(N^3)$. If the set of vectors used is not a basis then the result obtained is a lower bound for the bulk viscosity, which converges to the actual value as we increase $D$. A particularly convenient set of functions, which becomes a basis when $D \to \infty$, is \cite{14}

\[ \phi_i(k) = \frac{\langle |k| \rangle^{i-1}}{(1 + \langle |k| \rangle)^{D-3}}, \hspace{1cm} i = 1, \ldots, D, \] \hspace{1cm} (12)

with the thermal average $\langle |k| \rangle \sim \sqrt{M_0 T}$ for $T \to 0$, and $\langle |k| \rangle \sim T$ for $T \to \infty$. This set of functions automatically incorporates the required asymptotic behavior for the solution of the Boltzmann equation in the bulk channel: $B(|k|) \sim 1$ for $|k| \to 0$, and $B(|k|) \sim k^2$ for $|k| \to \infty$. At this point it is important to notice that the collision operator has a zero-mode associated to energy conservation, $B(|k|) \propto E_k$ (it is easy to see from (8)). Therefore, in order to be able to invert the collision matrix, it is necessary to calculate it in the vector space orthogonal to this zero-mode.

In Fig. 2, I plot the numerical result of a variational computation of the bulk viscosity in the massive Gross-Neveu model with $m = (0 + \epsilon) M_0$ using $D = 3$ basis functions. The numerical error corresponding to considering only $D = 3$ basis functions (including the error from the numerical evaluation of integrals) is estimated to be of order $0.1\%$ for temperatures above $T_c$, $\sim 1\%$ for temperatures close to $T_c$ from below, and $\sim 60\%$ for temperatures smaller than $0.1 M_0$. By considering for instance $D = 9$, this latter error can be decreased down to $\sim 20\%$. However, since the result shown corresponds to a lower bound and because of the exponential growth, I do not expect that the qualitative behavior with temperature will change significantly. From the numerical result we clearly see that there is no maximum in the bulk viscosity near $T_c$, it is a monotonously decreasing function of the temperature. The behavior at very low temperatures is exponentially increasing $\sim e^{2M_0/T}$ as $T \to 0$, similarly to the case of $\lambda \phi^4$ \cite{15}. By reducing further the value of $m$, we would eventually reconstruct (continuously) a discontinuity for $\zeta$ at $T_c$. For infinitesimally small $m$, above $T_c$ the bulk viscosity would be arbitrarily small, then going down in temperatures it would increase very sharply right at $T_c$ (with an arbitrarily large value), and it would continue increasing exponentially at very low temperatures.

5 We define the following scalar product in the space of solutions of the transport equation:

\[ \langle \chi|\psi \rangle = \beta \sum_A \int_{-\infty}^{\infty} \frac{dk}{2\pi} E_k n_F(E_k)[1 - n_F(E_k)]\chi^A(k)\psi^A(k). \] \hspace{1cm} (9)

6 The tilde emphasizes the matrix or vector character.
4. Conclusions
We have seen that there is not direct correlation between the bulk viscosity and the trace anomaly in the sense that a peak in the latter does not necessarily imply a peak in the former. This result can be extrapolated qualitatively to the case of pure Yang-Mills theory in 3 + 1 dimensions in the large $N_c$ limit, because below $T_c$ the interaction between gluons is suppressed by powers of $N_c$. Since the speed of sound in this case has a minimum near $T_c$, a moderately sharp increase of the bulk viscosity is expected when crossing the phase transition from high temperatures. At very low temperatures the bulk viscosity would behave $\sim \exp(m_g/T)$ with $m_g$ the mass of the lightest glueball. In massless two-flavor QCD, the picture would change due to the dominance of Goldstone modes at low temperatures and the critical behavior of the bulk viscosity near $T_c$. I plan to carry out further study of these different cases in the near future.

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References
[1] Kharzeev D and Tuchin K 2008 JHEP 0809 093
[2] Romatschke P and Son D T 2009 Phys. Rev. D 80 065021
[3] Gross D J and Neveu A 1974 Phys. Rev. D 10 3235
[4] Fernandez-Fraile D, arXiv:1009.2741
[5] Zamolodchikov A B and Zamolodchikov A B 1979 Annals Phys. 120 253
[6] Witten E 1978 Nucl. Phys. B 142 285
[7] Dashen R F, Ma S -k and Rajaraman R 1975 Phys. Rev. D 11 1499
[8] Barducci A, Casalbuoni R, Modugno M and Pettini G 1995 Phys. Rev. D 51 3042
[9] Schnetz O, Thies M and Urlichs K (2006) Annals Phys. 321 2604
[10] Blaizot J -P, Mendez-Galain R and Wschebor N 2003 Annals Phys. 307 209
[11] Kovtun P and Yaffe L G 2003 Phys. Rev. D 68 025007
[12] Arnold P B, Moore G D and Yaffe L G 2000 JHEP 0011 001
[13] Arnold P B, Moore G D and Yaffe L G 2003 JHEP 0305 051
[14] Arnold P B, Dogan C and Moore G D 2006 Phys. Rev. D 74 085021
[15] Jeon S 1995 Phys. Rev. D 52 3591
[16] Jeon S and Yaffe L G 1996 Phys. Rev. D 53 5799
[17] de Groot S R, van Leeuwen W A and van Weert C G 1980 Relativistic kinetic theory: principles and applications (North-Holland Pub. Co.)