Cohomology of Oriented Tree Diagram Lie Algebras

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Abstract

Xu introduced a family of root-tree-diagram nilpotent Lie algebras of differential operators, in connection with evolution partial differential equations. We generalized his notion to more general oriented tree diagrams. These algebras are natural analogues of the maximal nilpotent Lie subalgebras of finite-dimensional simple Lie algebras. In this paper, we use Hodge Laplacian to study the cohomology of these Lie algebras. The “total rank conjecture” and “$b_2$-conjecture” for the algebras are proved. Moreover, we find the generating functions of the Betti numbers by means of Young tableaux for the Lie algebras associated with certain tree diagrams of single branch point. By these functions and Euler-Poincaré principle, we obtain analogues of the denominator identity for finite-dimensional simple Lie algebras. The result is a natural generalization of the Bott’s classical result in the case of special linear Lie algebras.

1 Introduction

Cohomology of Lie algebras are important objects in mathematics, which are related to the geometry of the corresponding Lie groups, invariant differential operators, combinatorial identities, integrable systems, Riemannian foliations and cobordism theory [F]. In particular, cohomology of nilpotent Lie algebras with coefficients in the trivial module are more commonly used and has attracted many mathematicians’ attention. However, there are only a few results on the full cohomology of Lie algebras up to this stage.

Santharoubane [S] found the cohomology of Heisenberg Lie algebras. Moreover, Armstrong, Cairns and Jessup [ACJ] studied the cohomology of certain 2-step nilpotent extensions of abelian Lie algebras. Furthermore, Armstrong and Sigg [AS] generalized the latter to the nilpotent Lie algebras which have an abelian ideal with codimension 1. It is also a generalization of Bordemann’s result [B] on the cohomology of filiform Lie algebras. All these algebras are isomorphic to certain tree diagram Lie algebras, and the methods of computation used in all of the above papers are essentially based on the Dixmier’s sequences. These sequences are also used by Fialowski and Millionschikov [FM] in dealing with the cohomology of the graded Lie algebras of maximal class, which are infinite dimensional nilpotent Lie algebras.
Bott [Br] showed that the Betti numbers of the maximal nilpotent subalgebras of 
finite dimensional simple Lie algebras can be expressed by means of the Weyl group. He 
also pointed out that his theorem is equivalent to the Weyl denominator identity. There 
are several distinct proofs on Bott’s result. For example, it was proved in [BBG] by 
representation theory and in [K] by Hodge Laplacian. Both of these two methods are 
very effective. The calculation in [BBG] is generalized by Garland and Lepowsky to the 
case of Kac-Moody algebras and the celebrated Macdonald identities were recovered. The 
main tool in this paper is the Hodge Laplacian introduced by Kostant [K].

Euler-Poincaré Principle says that for a Lie algebra $G=\bigoplus_{\alpha \in \Gamma} G_{\alpha}$ graded by an additive semigroup $\Gamma$,

$$\prod_{\alpha \in \Gamma} (1-e^{\alpha})^{\dim G_{\alpha}} = \sum_{k=0}^{\infty} \sum_{\alpha \in \Gamma} (-1)^k \dim H^k_{\alpha}(G)e^\alpha,$$

where $e^\alpha$ are the base elements of the semigroup algebra $\mathbb{C}[\Gamma]$ (e.g., cf. [KK]). We will use it 
to obtain our combinatorial identities. Let $G$ be a finite dimensional nilpotent Lie algebra. Dixmier [D] proved that all Betti numbers of $G$ are at least two except the zeroth and the highest which are one. So there is a lower bound of total rank, i.e. $\dim H(G) \geq 2 \dim G$. Later, Deninger and Singhof [DS] showed that the length of a polynomial $P(G)$ gives a lower bound for $\dim H(G)$. Moreover, there is a “total rank conjecture” (c.f. [CJP]) which 
has been open for many years:

$$\dim H(G) \geq 2^{\dim C(G)}, \quad \text{where } C(G) \text{ is the centre of } G.$$

Another conjecture that can be found in literatures is the “$b_2$-conjecture” (c.f. [CJP]):

$$b_2 > \frac{b_1^2}{4} \quad \text{if } \dim G > 2, \quad \text{where } b_i = \dim H^i(G) \text{ are the Betti numbers.}$$

In this present paper, we will prove these two conjectures for oriented tree diagram Lie 
algebras.

Oriented tree diagram Lie algebras are introduced by Xu [X] in order to study certain 
evolution partial differential equations. They provide a new realization of some familiar nilpotent Lie algebras such as the ones mentioned in the second paragraph.

An *oriented tree* is a connected oriented graph without cycles. It can be described as 
an ordered pair $\mathcal{T} = (\mathcal{N}, \mathcal{E})$, where

$$\mathcal{N} = \{\iota_1, \iota_2, \ldots, \iota_n\}$$

and

$$\mathcal{E} \subset \{ (\iota_i, \iota_j) \mid 1 \leq i < j \leq n \}$$

are two disjoint sets. The elements of $\mathcal{N}$ are called *nodes* while the elements of $\mathcal{E}$ are called *oriented edges*.
We call $\iota$ the root node if $\{\iota' \mid (\iota', \iota) \in E\} = \emptyset$, and the tip node if $\{\iota' \mid (\iota, \iota') \in E\} = \emptyset$.

Denote

$$\Lambda = \text{the set of root nodes}$$

and

$$\Gamma = \text{the set of tip nodes}.$$

Define an oriented tree diagram

$$\mathcal{T}^d = (\mathcal{N}, \mathcal{E}, d)$$

to be an oriented tree $\mathcal{T} = (\mathcal{N}, \mathcal{E})$ with a weight map $d : \mathcal{E} \to \mathbb{Z}_+$ (the set of positive integers). We identify an oriented tree diagram $\mathcal{T}^d = (\mathcal{N}, \mathcal{E}, d)$ with a graph by depicting a small circle for each node in $\mathcal{N}$ and $d((\iota_i, \iota_j))$ segments connecting $i$th circle to $j$th circle for the edge $(\iota_i, \iota_j) \in \mathcal{E}$, where the orientation is always from left to right. For instance, the following figure

![Figure 1](image1)

represents the oriented tree diagram $\mathcal{T}^d = (\mathcal{N}, \mathcal{E}, d)$ with $\mathcal{N} = \{\iota_1, \iota_2, \ldots, \iota_6\}$, $\mathcal{E} = \{(\iota_1, \iota_3), (\iota_2, \iota_3), (\iota_3, \iota_4), (\iota_4, \iota_5), (\iota_4, \iota_6)\}$, and $d((\iota_3, \iota_4)) = 2$, $d(\mathcal{E}\setminus\{(\iota_3, \iota_4)\}) = 1$.

Given a positive integer $n$, there is an associative algebra of differential operators in $n$ variables:

$$\mathbb{A} = \sum_{m_1, m_2, \ldots, m_n = 0}^{\infty} \mathbb{C}[x_1, x_2, \ldots, x_n] \partial_{x_1}^{m_1} \partial_{x_2}^{m_2} \ldots \partial_{x_n}^{m_n}.$$

We can define a Lie bracket on $\mathbb{A}$ by

$$[A, B] = AB - BA \quad \text{for} \quad \forall A, B \in \mathbb{A}.$$

For any oriented tree diagram $\mathcal{T}^d = (\mathcal{N}, \mathcal{E}, d)$, we define the Lie algebra by

$$L_0(\mathcal{T}^d) = \text{the Lie subalgebra of } \mathbb{A} \text{ generated by } \{\partial_{x_1}, x_j^{d((\iota_j, \iota_k))} \partial_{x_k} \mid \iota_i \in \Lambda, (\iota_j, \iota_k) \in \mathcal{E}\}.$$

Take $\mathcal{T}^d$ to be the following diagrams:

![Figure 2](image2)  ![Figure 3](image3)  ![Figure 4](image4)

It is easy to check that their associated algebras $L_0(\mathcal{T}^d)$ are the algebras in $[S]$([Heisenberg Lie algebras]), $[ACJ]$ and $[B]$, respectively. When we take $\mathcal{T}^d$ to be the following diagram:

![Figure 5](image5)

the Lie algebra $L_0(\mathcal{T}^d)$ is just the maximal nilpotent subalgebras of $sl(i + 1)$. 
A natural generalization of both Figure 3 and Figure 5 is the diagram $\mathcal{A}_n^m$:

\[
\begin{array}{c}
\circ \\
1 \\
\circ \\
2 \\
\circ \\
n - 1 \\
\circ \\
n \\
\circ \\
n + 1 \\
\circ \\
n + 2 \\
\circ \\
n + m \\
\end{array}
\]

(Figure 6)

Without confusion, we also identify $L_0(\mathcal{A}_n^m)$ with $\mathcal{A}_n^m$ for short. In this paper, we will compute $H(\mathcal{A}_n^m)$. The Betti numbers of $\mathcal{A}_n^m$ had been obtained in [ACJ], which is a very special case of ours.

The paper is organized as follows. In Sections 2 and 3, we review all necessary definitions and the known facts concerning oriented tree diagram Lie algebras $L_0(\mathcal{T}^d)$ and their cohomology, especially the Hodge Laplacian introduced by Kostant [K]. We also use the Hodge Laplacian to prove that both of the total rank conjecture and $b_2$-conjecture hold for any oriented tree diagram Lie algebras $L_0(\mathcal{T}^d)$ at the end of Section 3. In Section 4, $H(\mathcal{A}_n^m)$ is computed and then an analogue of the Weyl denominator identity is obtained by Euler-Poincaré principle, where the Vandermonde determinant identity is a special case. The last section is devoted to the calculation of the cohomology the solvable Lie algebra $L_1(\mathcal{T}^d) = \sum_{i=1}^n \mathbb{C}x_i \partial_{x_i} + L_0(\mathcal{T}^d)$.

2 Notations and Facts on $L_0(\mathcal{T}^d)$

Given an oriented tree diagram $\mathcal{T}^d = (\mathcal{N}, \mathcal{E}, d)$, for $\forall t_i, t_j \in \mathcal{N}$, denote

\[ C_{i,j} = \{t_i = t_i, t_{i_2}, \ldots, t_{i_r} = t_j\} \]

to be the sequence of nodes with

\[ (t_{i_1}, t_{i_2}), (t_{i_2}, t_{i_3}), \ldots, (t_{i_{r-1}}, t_{i_r}) \in \mathcal{E}. \]

We remark that $C_{i,j}$ is unique determined by $t_i$ and $t_j$. Of course, sometimes $C_{i,j}$ may be $\emptyset$. We denote $C_{i,i} = \{t_i\}$ for convenience.

Set

\[ C_i = \{t_j \mid C_{j,i} \neq \emptyset\}, \quad D_i = \{t_j \mid C_{i,j} \neq \emptyset\}, \]

and denote

\[ \mathcal{E}_i = \{(t_r, t_s) \in \mathcal{E} \mid t_r, t_s \in C_i\}, \quad \mathcal{E}_{i,j} = \{(t_r, t_s) \in \mathcal{E} \mid t_r, t_s \in C_{i,j}\}. \]

Let

\[ \kappa_i = \prod_{(t_r, t_s) \in \mathcal{E}_i} d[(t_r, t_s)], \quad \kappa_{i,j} = \frac{\kappa_i}{\prod_{(t_r, t_s) \in \mathcal{E}_{i,j}} d[(t_r, t_s)]}. \]

It is obvious that $\mathcal{E}_{i,i} = \emptyset$ and $\kappa_{i,i} = \kappa_j = 1 (\forall t_i \in \mathcal{N}, \forall t_j \in \Lambda)$. Recall that $\Lambda$ is the set of root nodes and $\Gamma$ is the set of tip nodes. We have a basis of $L_0(\mathcal{T}^d)$:

\[ B(\mathcal{T}^d) = \{\partial_{x_i}, (\prod_{t_s \in C_i \setminus \{t_j\}} x_{s}^{m_s}) \partial_{x_j} \mid t_i \in \Lambda, m_s \in \mathbb{N}, \sum_{t_s \in C_i \setminus \{t_j\}} m_s \kappa_{s,j} \leq \kappa_j\}. \]
where $N$ is the set of nonnegative integers. We call it the natural basis of $L_0(\mathcal{T}^d)$.

**Remark 2.1** For a finite dimensional semisimple Lie algebra $G$, its Chevalley basis $B$ possesses a good property: for any $u_1, u_2 \in B$, we always have $[u_1, u_2] = c u_3$, where $u_3 \in B$ and $c \in \mathbb{Z}$. Now we can check easily that the natural basis $B(\mathcal{T}^d)$ also have this property. In the next section, this property will help us introduce the Hodge Laplacian.

The following lemma is obvious and will be used later.

**Lemma 2.2 ([X], [L])** The center of $L_0(\mathcal{T}^d)$ is generated by

$$
\sum_{x_i \in \Gamma} C x_i.
$$

In order to describe the result in latter sections laconically, we add some notations and definitions here.

An oriented tree diagram $\mathcal{T}^{d'} = (N', \mathcal{E}', d')$ is called a subdiagram of the oriented tree diagram $\mathcal{T}^d = (N, \mathcal{E}, d)$ if $N' \subset N, \mathcal{E}' \subset \mathcal{E}, d' = d|\mathcal{E}'$. Further, if $C_i \subset \mathcal{N}'$ for any $i \in N'$, we call $\mathcal{T}^{d'}$ a homo-clan subdiagram of $\mathcal{T}^d$ and call $L_0(\mathcal{T}^{d'})$ a homo-clan subalgebra of $L_0(\mathcal{T}^d)$. For example, Figure 5 is a homo-clan subdiagram of Figure 6 if $i \leq n + 1$.

The following lemma can be got immediately by the definition of homo-clan subdiagram.

**Lemma 2.3** If $L_0(\mathcal{T}^{d'})$ is a homo-clan subalgebra of $L_0(\mathcal{T}^d)$, then $B(L_0(\mathcal{T}^{d'})) \in B(L_0(\mathcal{T}^d))$. Furthermore, for any $u_1, u_2 \in B(L_0(\mathcal{T}^d))$ with $0 \neq [u_1, u_2] \in L_0(\mathcal{T}^{d'})$, we have $u_1, u_2 \in B(L_0(\mathcal{T}^{d'}))$.

Furthermore, there is a graded structure in $L_0(\mathcal{T}^d)$ with $\partial_{x_j} \in L_0(\mathcal{T}^d)_{\epsilon_j - \epsilon_0}$ and

$$
(\prod_{i_s \in C_j \setminus \{j\}} x_{i_s}^{m_{i_s}}) \partial_{x_j} \in L_0(\mathcal{T}^d)_{\epsilon_j - \sum_{i_s \in C_j \setminus \{j\}} m_{i_s} \epsilon_0}.
$$

**Example 1:** Denote $x_0 = 1$ and $y_i = x_{n+i}$ for convenience. The Lie algebra $A_n^m$ (associated with Figure 6) is generated by

$$
\{ x_{i-1} \partial_{x_i}, x_n \partial_{y_j} \mid 1 \leq i \leq n, 1 \leq j \leq m \}.
$$

The natural basis of $A_n^m$ is

$$
B(A_n^m) = \{ x_{i_1} \partial_{x_{i_2}}, x_j \partial_{y_k} \mid 0 \leq i_1 < i_2 \leq n, 0 \leq j \leq n, 1 \leq k \leq m \}.
$$

Obviously,

$$
A_n^m = \bigoplus_{0 \leq i < n; 0 < j \leq m+n} (A_n^m)_{\epsilon_j - \epsilon_i},
$$

where $(A_n^m)_{\epsilon_j - \epsilon_i} = \mathbb{C} x_i \partial x_j$. Hence $\dim(A_n^m)_{\epsilon_j - \epsilon_i} = 1$.

Set

$$
A_1 = \{ \partial_{x_1} \}, A_2 = \{ \partial_{x_2}, x_1 \partial_{x_2} \}, \ldots, A_n = \{ \partial_{x_n}, x_1 \partial_{x_n}, \ldots, x_{n-1} \partial_{x_n} \},
$$
and \(B_{m,n} = \{x_i \partial y_j \mid 0 \leq i \leq n, 1 \leq j \leq m\}\). We have \(B(A^m_n) = (\bigcup_{i=1}^{n} A_i) \cup B_{m,n}\).

Let \(A_0^n\) be the algebra associated with Figure 5. By definition, \(\bigcup_{j=1}^{i} A_j\) is exactly the natural basis of \(A_i^n\), i.e. \(\bigcup_{j=1}^{i} A_j = B(A_0^n)\). Moreover, \(A_0^n\) (0 < \(i \leq n\)) is a homoclasy subalgebra of both \(A^m_n\) and \(A^n_k\) (\(k \geq i\)). It is obvious that \(A_0^n \cong A_{n-1}^1\). Furthermore, one can check easily that \(A_0^n\) is isomorphic to the maximal nilpotent subalgebra of \(sl(n+1)\).

### 3 Lie Algebra Cohomology and Hodge Laplacian

Let \(\mathcal{G}\) be a finite dimensional Lie algebra over \(\mathbb{F}\) and let \(\mathcal{G}^*\) be the vector space dual of \(\mathcal{G}\). The spaces 
\[
\wedge \mathcal{G} = \bigoplus_{i \geq 0} \wedge^i \mathcal{G} \quad \text{and} \quad \wedge \mathcal{G}^* = \bigoplus_{i \geq 0} \wedge^i \mathcal{G}^*
\]
are their exterior algebras. We have a cochain complex:
\[
\mathbb{F} \xrightarrow{D_0} \mathcal{G}^* \xrightarrow{D_1} \wedge \mathcal{G}^* \xrightarrow{D_2} \ldots \xrightarrow{D_{i-1}} \wedge^i \mathcal{G}^* \xrightarrow{D_i} \ldots.
\]
The coboundary operator \(D_p\) is defined by
\[
D_p f(r_0, r_1, \ldots, r_p) = \sum_{0 \leq i < j \leq p} (-1)^{i+j} f([r_i, r_j], r_0, \ldots, \hat{r}_i, \ldots, \hat{r}_j, \ldots, r_p),
\]
where the sign \(^\wedge\) indicates that the argument below it must be omitted.

The cohomology of \((\wedge \mathcal{G}^*, D)\) is called the cohomology (with trivial coefficients) of the Lie algebra \(\mathcal{G}\) and is denoted by \(H(\mathcal{G})\). The gradation from \(\mathcal{G}\) induces a gradation in \(H(\mathcal{G})\). Given a basis \(B = \{u_1, u_2, \ldots, u_n\}\) of \(\mathcal{G}\), denote \(\{u_1^*, u_2^*, \ldots, u_n^*\}\) the basis of \(\mathcal{G}^*\) where \(u_i^*\) is the linear function on \(\mathcal{G}\) with \(u_i^*(u_j) = \delta_{i,j}\). In rest of this paper, we always denote by \(\mathbb{F}(X)\) the polynomial algebra generated by fermionic variables in \(X\) with operation “\(^\wedge\)”. Now we suppose \(\mathcal{G} = L_0(\mathcal{T}^d)\) and take \(B = \{u_1, u_2, \ldots, u_n\}\) to be its natural basis \(B(\mathcal{T}^d)\). Then \(\wedge L_0(\mathcal{T}^d) = \mathbb{F}(B(L_0(\mathcal{T}^d)))\).

Recall the property mentioned in Remark 2.1: for \(\forall u_i, u_j \in B\), \([u_i, u_j] = \alpha u_k\), where \(u_k \in B\) and \(\alpha \in \mathbb{Z}\). By this property and under the nature isomorphism \(\wedge^i \mathcal{G}^* \cong (\wedge^i \mathcal{G})^*\), it is easy to check that \(D_p\) is the linear map with
\[
D_p(u_{i_1}^* \wedge u_{i_2}^* \wedge \cdots \wedge u_{i_t}^*) = \sum_{k=1}^{t} (-1)^k u_{i_1}^* \wedge u_{i_2}^* \wedge \cdots \wedge (\Delta u_{i_k})^* \wedge \cdots \wedge u_{i_t}^*,
\]
where \(\Delta : \mathcal{G} \rightarrow \wedge^2 \mathcal{G}\) is the linear map with
\[
\Delta(u_i) = \sum_{[u_j, u_k] = \alpha u_i, \alpha \in \mathbb{Z}^+} \alpha u_j \wedge u_k. \quad (3.1)
\]
Without confusion, we can identify \(\wedge^i \mathcal{G}^*\) with \(\wedge \mathcal{G}\) and redefine
\[
D_p (r_1 \wedge r_2 \wedge \cdots \wedge r_p) = \sum_{i=1}^{p} (-1)^i r_1 \wedge r_2 \wedge \cdots \wedge (\Delta r_i) \wedge \cdots \wedge r_p,
\]
i.e. \(D_p : \wedge^p \mathcal{G} \rightarrow \wedge^{p+1} \mathcal{G}\) is the coboundary operator of complex:
\[
\mathbb{F} \xrightarrow{D_0} \mathcal{G} \xrightarrow{D_1} \wedge \mathcal{G} \xrightarrow{D_2} \ldots \xrightarrow{D_{i-1}} \wedge^i \mathcal{G} \xrightarrow{D_i} \ldots.
\]
On the other hand, there also exists a chain complex:

\[ F \xleftarrow{\delta_1} G \xleftarrow{\delta_2} \wedge^2 G \xleftarrow{\delta_3} \cdots \xleftarrow{\delta_i} \wedge^i G \xrightarrow{\delta_{i+1}} \cdots, \]

where the boundary operator \( \delta \) is defined by

\[
\delta_p(r_1 \wedge r_2 \wedge \cdots \wedge r_p) = \sum_{1 \leq i < j \leq p} (-1)^{i+j} [r_i, r_j] \wedge r_1 \wedge \cdots \wedge \widehat{r_i} \wedge \cdots \wedge \widehat{r_j} \wedge \cdots \wedge r_p.
\]

Now we can define the operator

\[ \mathcal{L} = D\delta + \delta D : \wedge G \to \wedge G \]

and call it the Hodge Laplacian. Precisely,

\[ \mathcal{L}_p = \mathcal{L} \mid_{\wedge^p G} = D_p \delta_{p+1} + \delta_p D_{p-1} : \wedge^p G \to \wedge^p G. \]

**Theorem 3.1 ([K])** One has a direct sum (a “Hodge decomposition”),

\[ \wedge G = \text{Im} \mathcal{L} \oplus \text{Ker} \mathcal{L}, \quad (\text{hence } \wedge^p G = \text{Im} \mathcal{L}_p \oplus \text{Ker} \mathcal{L}_p) \]

and

\[ \text{Ker} \mathcal{L}_p = \text{Ker} D_p \cap \text{Ker} \delta_p ; \quad \text{Im} \mathcal{L}_p = \text{Im} D_{p-1} \oplus \text{Im} \delta_{p+1}. \]

Elements in Ker \( \mathcal{L} \) are called harmonic. By the above theorem, \( c \in \wedge G \) is harmonic if and only if \( D(c) = 0, \delta(c) = 0 \). For convenience, we also denote \( \tilde{H}^p(G) = \text{Ker} \mathcal{L}_p \) and \( \tilde{H}(G) = \text{Ker} \mathcal{L} \).

**Theorem 3.2 ([K],[F])** Every element of the space \( H^p(G) \) can be represented by a unique harmonic cocycle from \( \wedge G \), namely, there is a natural isomorphism

\[ \tilde{H}^p(G) = \text{Ker} \mathcal{L}_p \cong H^p(G). \]

By the above two theorems, we have:

**Lemma 3.3** If \( T^{td} \) is a homo-clan subdiagram of \( T^d \), then \( \tilde{H}(L_0(T^{td})) \subset \tilde{H}(L_0(T^d)) \).

**Proof.** By Lemma 2.3, it follows from the fact that the coboundary operator \( D \) and the boundary operator \( \delta \) of \( L_0(T^{td}) \) are just the ones of \( L_0(T^d) \) restricted to \( L_0(T^{td}) \). \( \square \)

**Theorem 3.4** The total rank conjecture \( \dim H(L_0(T^d)) \geq 2^{\dim C(L_0(T^d))} \) holds.
Proof. By Lemma 2.1, \( \text{dim} C(L_0(\mathcal{T}^d)) = |\Gamma| \) (i.e. the number of elements in \( \Gamma \)). For the trivial case \(|\mathcal{N}| = 1\), it is true that \( \text{dim} H(L_0(\mathcal{T}^d)) = 2 \geq 2^{\text{dim} C(L_0(\mathcal{T}^d))} = 2 \). Thus we assume \(|\mathcal{N}| > 1\), and hence \( \Lambda \cap \Gamma = \emptyset \).

Suppose \( \Gamma = \{\iota_1, \iota_2, \ldots, \iota_t\} \), then \(|\Gamma| = t\). For any \( \iota_i \in \Gamma(i = 1, 2, \ldots, t) \), we can take a \( \iota_p(i) \in \mathcal{N} \) such that \( d[(\iota_p(i), \iota_i)] = d_i \neq 0 \). Take \( A = \{x_{p(i)}^D, \partial_{x_i} \mid i = 1, 2, \ldots, t\} \subset B(L_0(\mathcal{T}^d)) \) and \( \wedge A \) its exterior algebra.

Recall the operator \( \Delta : \mathcal{G} \to \wedge^2 \mathcal{G} \) introduced in (3.1). As \( A \cap [L_0(\mathcal{T}^d), L_0(\mathcal{T}^d)] = \emptyset \), we have \( \Delta(x_{p(i)}^D, \partial_{x_i}) = 0 \). Thus \( D(\wedge A) = 0 \). On the other hand, \( [A, A] = \{0\} \), so \( \delta(\wedge A) = 0 \). Thanks to Theorem 3.1 and 3.2, we have \( \wedge A \subset \text{Ker} \mathcal{L} \cong H(L_0(\mathcal{T}^d)) \). So \( \text{dim} H(L_0(\mathcal{T}^d)) \geq \text{dim}(\wedge A) = 2^t = 2^{\text{dim} C(L_0(\mathcal{T}^d))} \).

Next we turn to the \( b_2 \)-conjecture.

Theorem 3.5 For any \( \mathcal{T}^d \) except the trivial case \(|\mathcal{N}| = 1\), we always have \( b_2 > b_1^2/4 \), where \( b_i = \text{dim} H^i(L_0(\mathcal{T}^d)) \) are the Betti numbers.

Proof. It is obvious that \( \text{Ker} D_1 \cap \text{Ker} \delta_1 = \text{Span}\{\partial_{x_i}, x_j^d[(\iota_j, \iota_k)]\partial x_k \mid \iota_i \in \Lambda, (\iota_j, \iota_k) \in \mathcal{E}\} \). Thus by Theorem 3.1 and 3.2, \( b_1 = \text{dim} H^1(L_0(\mathcal{T}^d)) = |\Lambda| + |\mathcal{E}| \).

On the other hand, one can check that

\[
\{\partial_{x_{i_1}} \wedge \partial_{x_{i_2}}, \partial_{x_{j_1}} \wedge x_j^d[(\iota_j, \iota_k)]\partial x_k, x_j^d[(\iota_j, \iota_k)]\partial x_{k_2} \mid (\iota_i, \iota_{i_1}, \iota_{i_2}) \in \Lambda, (\iota_j, \iota_k), (\iota_j, \iota_{k_2}) \in \mathcal{E}, i \neq j, j_1 \neq k_2, j_2 \neq k_1\} \subset \text{Ker} D_2 \cap \text{Ker} \delta_2 = \text{Ker} \mathcal{L}_2,
\]

and

\[
\{x_j^d[(\iota_j, \iota_k)]^{-1}\partial x_k \wedge x_j^d[(\iota_j, \iota_k)]\partial x_k \mid (\iota_j, \iota_k) \in \mathcal{E}\} \subset \text{Ker} D_2 \cap \text{Ker} \delta_2 = \text{Ker} \mathcal{L}_2.
\]

Hence \( b_2 = \text{dim} H^2(L_0(\mathcal{T}^d)) \geq \left(\frac{|\Lambda| + |\mathcal{E}|}{2}\right) - |\mathcal{E}| + |\mathcal{E}| = \left(\frac{|\Lambda| + |\mathcal{E}|}{2}\right). \)

So if \(|\Lambda| + |\mathcal{E}| > 2\), it must be \( b_2 > b_1^2/4 \). Otherwise, when \(|\Lambda| + |\mathcal{E}| = 2\), \( \mathcal{T}^d \) must be Figure 4. It is easy to get

\( \text{Ker} \mathcal{L}_1 = \text{Span}\{\partial_{x_1}, x_1^d[(\iota_1, \iota_2)]\partial x_2\} \)

and

\( \{\partial_{x_1} \wedge \partial_{x_2}, x_1^d[(\iota_1, \iota_2)]^{-1}\partial x_2 \wedge x_1^d[(\iota_1, \iota_2)]\partial x_2\} \subset \text{Ker} \mathcal{L}_2, \)

i.e. \( b_1 = 2 \) and \( b_2 \geq 2 \). There also have to be \( b_2 > b_1^2/4 \). \( \square \)

4 Cohomology of \( A_n^m \)

In this section, we will compute the cohomology of \( A_n^m \). In other words, we want to get all the harmonic cocycle of \( \wedge A_n^m \) due to the Theorem 3.2.
With the notations introduced to \( A_n^m \) in Example 1 at the end of Section 2, we define an ordering “\( \prec \)” on \( B(A_n^m) \) by

\[
x_i \partial x_j \prec x_k \partial x_l \quad \text{if} \quad j < l \text{ or } j = l \text{ and } i < k.
\]

Set

\[
\mathcal{B}(A_n^m) = \{ u_1 \wedge u_2 \wedge \cdots \wedge u_p \mid u_1 \prec u_2 \prec \cdots \prec u_p, \ p = 0, 1, 2, \ldots \}.
\]

Obviously, \( \mathcal{B}(A_n^m) \) forms a basis of \( \wedge A_n^m \), which can be regarded as the polynomial algebra \( \mathbb{F}(B(A_n^m)) \) of fermionic variables.

**Lemma 4.1** For any \( a \wedge b \in \wedge A_n^m \) with \( 0 \neq a \in \mathbb{F}(B(A_n^m)) \) and \( 0 \neq b = x_{i_1} \partial y_{j_1} \wedge \cdots \wedge x_{i_k} \partial y_{j_k} \), we have \( \mathcal{L}(a \wedge b) = a \wedge b' \) with \( b' \in \mathbb{F}(B_{m,n}) \). Moreover, each term of \( b' \) must be of the form \( \alpha(x_{i_{s,t}(k)} \partial y_{j_1} \wedge \cdots \wedge x_{i_{s,t}(k)} \partial y_{j_k}) \), where \( \alpha \in \mathbb{F} \) and \( \sigma_{s,t} (s \leq t) \in S_k \) satisfy \( \sigma_{s,t}(s) = t, \sigma_{s,t}(t) = s \) and \( \sigma_{s,t}(r) = r(r \neq s, r \neq t) \). In particular, all of the monomials in \( \mathbb{F}(B(A_n^m)) \) are eigenvectors of the linear map \( \mathcal{L} \). Thus \( \text{Ker } \mathcal{L} \cap \mathcal{B}(A_n^m) \) forms a basis of \( \tilde{H}(A_n^m) \).

**Proof.** By the definitions of \( D \) and \( \delta \), there are at most three distinct factors between any term of \( D \delta(a \wedge b) \) and any term of \( a \wedge b \). So are there between any term of \( \delta D(a \wedge b) \) and any term of \( a \wedge b \).

For any term of \( D \delta(a \wedge b) \) with three distinct factors relative to \( a \wedge b \), it must be produced by

\[
x_i \partial x_j \wedge x_j \partial z_1 \wedge x_s \partial z_2 \wedge \cdots \overset{\delta}{\longrightarrow} -x_i \partial z_1 \wedge x_s \partial z_2 \wedge \cdots \overset{D}{\longrightarrow} -x_i \partial z_1 \wedge x_s \partial x_u \wedge x_u \partial z_2 \wedge \cdots
\]

where \( z_1, z_2 \) may be \( x_k \) or \( y_k \).

(In this paper, the “\( \longrightarrow \)” but not the “\( \rightarrow \)” will be used in the calculation frequently. If we write “\( a \rightarrow f \rightarrow b \)”, \( b \) may be not equal to \( f(a) \) but equal to the terms of \( f(a) \) which we are concerned about.)

At the same time, there must be a term of \( \delta D(a \wedge b) \) produced by

\[
x_i \partial x_j \wedge x_j \partial z_1 \wedge x_s \partial z_2 \wedge \cdots \overset{D}{\longrightarrow} -x_i \partial x_j \wedge x_j \partial z_1 \wedge x_s \partial x_u \wedge x_u \partial z_2 \wedge \cdots \overset{\delta}{\longrightarrow} x_i \partial z_1 \wedge x_s \partial x_u \wedge x_u \partial z_2 \wedge \cdots.
\]

These two terms counteract each other. For any term of \( D \delta(a \wedge b) \) with two distinct factors relative to \( a \wedge b \), it may be produced by

\[
x_i \partial x_j \wedge x_j \partial z \wedge \cdots \overset{\delta}{\longrightarrow} -x_i \partial z \wedge \cdots \overset{D}{\longrightarrow} x_i \partial x_s \wedge x_s \partial z \wedge \cdots
\]

where \( z \) may be \( x_k \) or \( y_k \).

Suppose \( s < j \) (the case of \( s > j \) can be checked similarly). If \( x_s \partial x_j \) is also a factor of \( a \wedge b \), then there must be another term of \( D \delta(a \wedge b) \) produced by

\[
x_i \partial x_j \wedge x_j \partial z \wedge x_s \partial x_j \wedge \cdots \overset{\delta}{\longrightarrow} -x_i \partial x_j \wedge x_s \partial z \wedge \cdots \overset{D}{\longrightarrow} -x_i \partial x_s \wedge x_s \partial z \wedge x_s \partial x_j \wedge \cdots.
\]
If $x_i \partial_{x_j}$ is not a factor of $a \land b$, then there must be a term of $\delta D(a \land b)$ produced by

$$x_i \partial_{x_j} \land x_j \partial_z \land \cdots \rightarrow -x_i \partial_{x_z} \land x_j \partial_z \land \cdots \rightarrow -x_i \partial_{x_z} \land x_j \partial_z \land \cdots .$$

Thus these terms can always be counteracted.

But the following two cannot be counteracted. One is produced by: (when $x_i \partial_{x_j}$ is a factor of $a$)

$$x_i \partial_{x_j} \land x_j \partial_{y_k} \land x_i \partial_{y_l} \land \cdots \rightarrow -x_i \partial_{y_k} \land x_i \partial_{y_l} \land \cdots \rightarrow D \delta -x_i \partial_{x_j} \land x_j \partial_{y_k} \land x_j \partial_{y_l} \land \cdots .$$

The other is produced by: (when $x_i \partial_{x_j}$ is not a factor of $a$)

$$x_j \partial_{y_k} \land x_i \partial_{y_l} \land \cdots \rightarrow -x_i \partial_{y_k} \land x_j \partial_{y_k} \land x_j \partial_{y_l} \land \cdots \rightarrow -D \delta -x_i \partial_{y_k} \land x_j \partial_{y_l} \land \cdots .$$

Moreover, it is obvious that there cannot be a common term of $D \delta (a \land b)$ and $\delta D(a \land b)$ that has only one distinct factor relative to $a \land b$. So $L(a \land b)$ must be of the form described in the lemma.

In particular, if we take $b = 1$, then all of the monomials in $F \langle B(A_n^0) \rangle$ are eigenvectors of $L$. As $B(A_n^0)$ is a basis of $F \langle B(A_n^0) \rangle$, $\ker L \cap B(A_n^0)$ forms a basis of $\tilde{H}(A_n^0)$.

**Lemma 4.2** For any $a \neq a \land b \in \tilde{H}(A_n^m)$ with $a \in B(A_n^0)$ and $0 \neq b \in A_{m,n}$, we have $a \in \tilde{H}(A_n^0)$. In particular, if $0 \neq a \land b \in \tilde{H}(A_n^0)$ with $a \in B(A_{n-1}^0)$ and $0 \neq b \in A_{i}$, then $a \in \tilde{H}(A_{i-1}^0)$.

**Proof.** By Theorem 3.1, we only need to prove that both $D(a)$ and $\delta(a)$ are zero.

$$0 = D(a \land b) = D(a) \land b + (-1)^{\deg(a)} a \land D(b),$$

where $\deg(a)$ is the degree of $a$ as a monomial of $F \langle B(A_{n-1}^0) \rangle$. As $a$ is not contained in any term of $D(a) \land b$, we must have $D(a) \land b = a \land D(b) = 0$. Hence $D(a) = 0$.

$$0 = \delta(a \land b) = \delta(a) \land b + U.$$

Here the $U$ is a sum of terms with form $a' \land b'$, where $a' \in B(A_{n-1}^0)$ and $b' \in A_i$. Furthermore, we can easily observe that each $a'$ loses at most one factor of $a$ but each term of $\delta(a)$ loses two. Thus it must be $\delta(a) \land b = U = 0$. Hence $\delta(a) = 0$.

**Corollary 4.3** (1). If $\sum_{j=1}^{t} a_j \land b_j \in \tilde{H}(A_n^0)$, where $a_j \in B(A_{n-1}^0)$, $b_j \in A_n$ and $a_{j1} \neq a_{j2}$ ($j1 \neq j2$). Then for any $j \in \{1, 2, \ldots, t\}$, we have $a_j \land b_j \in \tilde{H}(A_n^0)$ and $a_j \in \tilde{H}(A_{n-1}^0)$.

(2). If $0 \neq a_1 \land a_2 \land \cdots \land a_n \in \tilde{H}(A_n^0)$ where $a_i \in A_i$ ($i = 1, 2, \ldots, n$), then $a_1 \land a_2 \land \cdots \land a_j \in \tilde{H}(A_j^0)$ ($j = 1, 2, \ldots, n$).

**Proof.** Both of these two statements can be obtained by Lemmas 4.1 and 4.2 directly.

Thanks to the first claim of the above corollary, we only need to consider the monomials in $F \langle B(A^0) \rangle$ in order to obtain a basis of $\tilde{H}(A_n^0)$. Precisely, $B(A_n^0) \cap \tilde{H}(A_n^0)$ is a basis of $\tilde{H}(A_n^0)$. 

For each $a \in B(A^0_i) \cap \tilde{H}(A^0_i)$, we introduce a total ordering “$\preceq_a$” into $\{0, 1, 2, \ldots, i\}$:

$$
\begin{aligned}
\text{for any } 0 \leq j < k \leq i, & \quad \begin{cases} 
\ k \preceq_a j, & \text{if } x_j \partial_{x_k} \text{ is a factor of } a; \\
\ j \preceq_a k, & \text{otherwise.}
\end{cases} \\
\end{aligned}
$$

Although we have not checked that “$\preceq_a$” is well defined, it will be indicated in the next theorem.

**Theorem 4.4** The total orderings defined in (4.2) are well defined. For any $a \wedge b \in B(A^0_i) \cap \tilde{H}(A^0_i)$ with $0 \neq a \in \wedge A^0_i$ and $0 \neq b \in \wedge A_{i+1}$, if $x_j \partial_{x_k}$ is a factor of $b$, then all $x_k \partial_{x_{i+1}}$ ($j \preceq_a k$) are factors of $b$. Conversely, each element of this form must be in $B(A^0_i) \cap \tilde{H}(A^0_i)$.

**Proof.** We will prove it by induction on $i$. For $i = 1$, there are only two elements 1 and $\partial_{x_1}$ in $B(A^0_1) \cap \tilde{H}(A^0_1)$. We have

$$
0 \preceq_1 1; \quad 1 \preceq_{\partial_{x_1}} 0.
$$

Thus “$\preceq_1$” and “$\preceq_{\partial_{x_1}}$” are indeed total orderings. For $a \wedge b \in B(A^0_2) \cap \tilde{H}(A^0_2)$ with $0 \neq a \in \wedge A^0_1$ and $0 \neq b \in \wedge A_2$, an easy calculation indicates that $a \wedge b = x_1 \partial_{x_2}$, $\partial_{x_2} \wedge x_1 \partial_{x_2}$, $\partial_{x_1} \wedge \partial_{x_2}$, or $\partial_{x_1} \wedge \partial_{x_2} \wedge x_1 \partial_{x_2}$. One can check that the statement is true for $i = 1$.

Suppose that the statement is true for $i - 1$. Take any $a \wedge b \in B(A^0_{i+1}) \cap \tilde{H}(A^0_{i+1})$ with $0 \neq a \in \wedge A^0_i$ and $0 \neq b \in \wedge A_{i+1}$. By Lemma 4.2, one has $a \in B(A^0_i) \cap \tilde{H}(A^0_i)$. Assume $a = a_1 \wedge a_2$ with $0 \neq a_1 \in \wedge A^0_{i-1}$ and $0 \neq a_2 \in \wedge A_i$. By definition, we know that “$\preceq_a$” restricted to $\{1, 2, \ldots, i \} - 1 \}$ is the unique element such that $x_j \partial_{x_i}$ is a factor of $a_2$ but $x_k \partial_{x_i}$ ($\forall k \preceq_{a_1} j$) are not. Hence we get that $j$ must be the next number of $i$ under the total ordering “$\preceq_a$”. Precisely,

$$
i \preceq_a k \text{ if and only if } k = j \text{ or } j \preceq_{a_1} k \text{ (hence } k \preceq_a i \text{ if and only if } k \preceq_{a_1} j).$$

So we have proved that “$\preceq_a$” is well defined.

Taking any pair $(k, l)$ such that $x_k \partial_{x_{i+1}}$ is a factor of $b$ and $k \preceq_a l$, we need to prove that $x_l \partial_{x_{i+1}}$ is a factor of $b$. Suppose not.

If $k < l$, then $x_k \partial_{x_l}$ is not a factor of $a$. Thus there must be a nonzero term of $D(a \wedge b)$:

$$
x_k \partial_{x_{i+1}} \wedge \cdots \xrightarrow{D} -x_k \partial_{x_l} \wedge x_l \partial_{x_{i+1}} \wedge \cdots.
$$

This contradicts Theorem 3.1. If $l < k$, then $x_l \partial_{x_k}$ is a factor of $a$. Thus there must be a nonzero term of $\delta(a \wedge b)$:

$$
x_l \partial_{x_k} \wedge x_k \partial_{x_{i+1}} \wedge \cdots \xrightarrow{\delta} -x_l \partial_{x_{i+1}} \wedge \cdots.
$$

This again leads a contradiction to Theorem 3.1. Therefore $x_l \partial_{x_{i+1}}$ is a factor of $b$. 

At last, if \( a \land b \) is an element of the form we mentioned. For any pair \((k, l)\) such that \( x_k \partial_{x_{k+1}} \) is a factor of \( b \), we only need to check the following two cases to prove \( D(a \land b) = 0 \) and \( \delta(a \land b) = 0 \) (other cases are so trivial that the total ordering \( \prec_a \) is needless). If \( l < k \) and \( x_l \partial_{x_k} \) is a factor of \( a \) (hence \( k \prec_a l \)), then \( x_l \partial_{x_{l+1}} \) is a factor of \( b \). Thus
\[
x_l \partial_{x_k} \land x_k \partial_{x_{k+1}} \land x_l \partial_{x_{l+1}} \land \cdots \rightarrow x_l \partial_{x_{l+1}} \land x_l \partial_{x_{l+1}} \land \cdots (= 0).
\]
If \( l > k \) and \( x_k \partial_{x_l} \) is not a factor of \( a \) (hence \( k \prec_a l \)). Then \( x_l \partial_{x_{l+1}} \) is a factor of \( b \). Thus
\[
x_k \partial_{x_{l+1}} \land x_l \partial_{x_{l+1}} \land \cdots \rightarrow x_k \partial_{x_l} \land x_l \partial_{x_{l+1}} \land x_l \partial_{x_{l+1}} \land \cdots (= 0).
\]
Hence \( a \land b \in \mathcal{B}(A^0_{i+1}) \cap \widetilde{H}(A^0_{i+1}) \).

**Corollary 4.5** The generating function of the Betti numbers of \( A^0_n \) is
\[
\sum_{i=0}^{\infty} b_it^i = (1 + t)(1 + t + t^2)(1 + t + t^2 + t^3) \cdots (1 + t + t^2 + \cdots + t^n) = \prod_{i=1}^{n}(1 - t^{i+1})/(1 - t)^{n}.
\]

**Proof.** For \( n = 1 \), the statement holds trivially. Suppose the statement is true for \( n - 1 \). Then for any \( a \in \mathcal{B}(A^0_{n-1}) \cap \widetilde{H}(A^0_{n-1}) \), there are \( n \) distinct \( b \in \land A_n \) such that \( a \land b \in \mathcal{B}(A^0_n) \cap \widetilde{H}(A^0_n) \).

Furthermore, if \( \{i_1, i_2, \ldots, i_n\} \) is a permutation of \( \{0, 1, \ldots, n - 1\} \) such that \( i_n \prec_a i_{n-1} \prec_a \cdots \prec_a i_1 \), then the \( n \) distinct \( b \) are \( 1, x_1 \partial_{x_n}, x_1 \partial_{x_n} \land x_2 \partial_{x_n}, \ldots, x_1 \partial_{x_n} \land x_2 \partial_{x_n} \land \cdots \land x_{i_n} \partial_{x_n} \), respectively. Hence the statement holds for \( n \). \( \square \)

**Remark 4.6** Bott’s theorem has indicated that \( b_i = \dim H_i(A^0_n) = |S^{(i)}_{n+1}| \) (\( \lvert \cdot \rvert \) means the number of elements in \( S^{(i)}_{n+1} \) where \( S^{(i)}_{n+1} \) is the set of elements in \( S_{n+1} \) (the \( n + 1 \)th symmetric group which can be regarded as a Weyl group) with length \( i \)). So our generating function is just the Poincaré polynomial which gives the description of the length about the elements of Weyl group \( S_{n+1} \). (The definition of Poincaré polynomial can be found in \([H]\).)

Moreover, we can get the explicit relationship about Bott’s theorem between the cohomology and the Weyl group in the case of type \( A \). That is, for each \( a \in \mathcal{B}(A^0_n) \cap \widetilde{H}(A^0_n) \), the total ordering \( \prec_a \) defined above is relative to an element \( \sigma_a \in S_{n+1} \) with \( i \prec_a j \iff \sigma_a(i) < \sigma_a(j) \).

In order to describe the result about \( A^0_n \) laconically, we first introduce a notation. Given a total ordering \( \prec_a \) on \( \{0, 1, \ldots, n\} \), we assume \( i_{n+1} \prec_a i_n \prec_a \cdots \prec_a i_1 \), where \( \{i_1, i_2, \ldots, i_{n+1}\} \) is a permutation of \( \{0, 1, \ldots, n\} \).

Set
\[
Y = \{(j_1, j_2, \ldots, j_k) \mid 1 \leq j_1 \leq j_2 \leq \cdots \leq j_k \leq m, 0 \leq k \leq n\}.
\]
In which \( (j_1, j_2, \ldots, j_k) = 0 \) if \( k = 0 \). For any \((j_1, j_2, \ldots, j_k) \in Y\), denote
\[
\varphi^{(i)}_{(j_1, j_2, \ldots, j_k)} = \begin{cases} 1, & \text{if } k = 0; \\ \sum_{\sigma \in S_k} x_{i_1} \partial_{y_{\sigma(1)}} \land \cdots \land x_{i_k} \partial_{y_{\sigma(k)}}, & \text{if } k > 0. \end{cases}
\]
With this notation, we can easily find numbers of elements which belong to \( \tilde{H}(A_n^m) \). Denote

\[
Q = \{a \land \varphi_j^a \land \varphi_j^a \land \cdots \land \varphi_j^a \mid a \in \mathcal{B}(A_n^0) \cap \tilde{H}(A_n^0), J_1, J_2, \ldots, J_t \in Y, t \in \mathbb{Z}_+ \}.
\]

We have

**Lemma 4.7** \( Q \subset \tilde{H}(A_n^m) \).

**Proof.** Given any \( \varphi_j^a = \sum_{\sigma \in S_k} x_{i_1} \partial y_{i_1} + \cdots + x_{i_k} \partial y_{i_k} \), we take the pair \((i_1, i_2)\) (other pairs can be discussed similarly). Suppose \( i_1 < i_2 \), then \( x_{i_1} \partial y_{i_2} \) must be a factor of \( a \). Hence

\[
x_{i_1} \partial y_{i_2} \land (x_{i_1} \partial y_{i_1} + x_{i_2} \partial y_{i_2} + x_{i_3} \partial y_{i_3} + \cdots) \land \cdots \land \partial y_{i_1} \land x_{i_2} \partial y_{i_2} + x_{i_1} \partial y_{i_1} \land x_{i_2} \partial y_{i_2} + x_{i_3} \partial y_{i_3} + \cdots \land \cdots (= 0).
\]

Suppose \( i_1 > i_2 \), then \( x_{i_2} \partial y_{i_1} \) can not be a factor of \( a \). Thus

\[
(x_{i_1} \partial y_{i_1} + x_{i_2} \partial y_{i_2} + x_{i_3} \partial y_{i_3} + \cdots) \land \cdots \land \partial y_{i_1} \land x_{i_2} \partial y_{i_2} + x_{i_1} \partial y_{i_1} \land x_{i_2} \partial y_{i_2} + x_{i_3} \partial y_{i_3} + \cdots \land \cdots (= 0).
\]

So we always have \( D(a \land \varphi_j^a \land \varphi_j^a \land \cdots \land \varphi_j^a) = 0 \) and \( \delta(a \land \varphi_j^a \land \varphi_j^a \land \cdots \land \varphi_j^a) = 0 \). That is \( a \land \varphi_j^a \land \varphi_j^a \land \cdots \land \varphi_j^a \in \tilde{H}(A_n^m) \).

By the above lemma, we know

\[
\text{Span}Q \subset \tilde{H}(A_n^m).
\]

The next theorem will show that the “\( \subset \)” in the above formula can be changed to “\( = \)”.

In fact, we are even able to take a proper subset \( P \subset Q \) such that \( \tilde{H}(A_n^m) = \text{Span}P \). Now we first define a such subset \( P \) and then give the main theorem.

We call \( a \land \varphi_j^a \land \varphi_j^a \land \cdots \land \varphi_j^a \in Q \) a basic element if \( J_s = (j_{1,s}, j_{2,s}, \ldots, j_{p_s,s}) \in Y \) \((s = 1, 2, \ldots, t)\) satisfy that \( p_1 \geq p_2 \geq \cdots \geq p_t \) and \( j_{q,s_1} < j_{q,s_2} \) (with \( s_1 < s_2 \)). The set of all basic elements in \( Q \) is denoted by \( P \).

**Theorem 4.8** The set \( P \) of all basic elements in \( Q \) is a basis of \( \tilde{H}(A_n^m) \).
We shall divide the proof of Theorem 4.8 into several lemmas.

By Lemma 4.1 and 4.2, we know \( \tilde{H}(A^m_n) \) can be spanned by the elements of the form \( a \wedge b \) with \( a \in \mathcal{B}(A^m_n) \cap \tilde{H}(A^0_n) \) and \( b \in \wedge B_{m,n} \). Lemma 4.1 also allows us to fetch \( b \) better. Precisely, we can take \( b \) to satisfy the following condition:

**Condition (\( \ast \))** If \( x_{s_1} \partial_{y_1} \wedge \cdots \wedge x_{s_k} \partial_{y_k} \) is a term of \( b \), then other terms of \( b \) should be of the form \( x_{s_{(1)}} \partial_{y_1} \wedge \cdots \wedge x_{s_{(k)}} \partial_{y_k} \), (\( \sigma \in S_k \))

Assume \( i_{n+1} < a_i < a \cdots < a_{i-1} \) where \( \{ i_1, i_2, \ldots, i_{n+1} \} \) is a permutation of \( \{ 0, 1, \ldots, n \} \) all the time. And denote \( C_i = \{ x_i \partial_{y_j} \mid 1 \leq j \leq m \}, (i = 0, 1, \ldots, n) \).

**Lemma 4.9** If \( j < k \), then the number of elements in \( C_{ij} \) which are factors of a term of \( b \) must be equal to or greater than the number of elements in \( C_{ik} \) which are also factors of the same term of \( b \).

**Proof.** Suppose not. We see that one term of \( b \) is of the form

\[
c = x_{i_j} \partial_{y_1} \wedge x_{i_j} \partial_{y_2} \wedge \cdots \wedge x_{i_j} \partial_{y_\alpha} \wedge x_{i_k} \partial_{y_1} \wedge x_{i_j} \partial_{y_2} \wedge \cdots \wedge x_{i_j} \partial_{y_\beta} \wedge U
\]

where \( \alpha < \beta \) and \( U \) has no factor in \( C_{ij} \) or \( C_{ik} \). If there exist \( s_l = t_{l'} \), then we can omit the two factors \( x_{i_j} \partial_{y_1} \) and \( x_{i_k} \partial_{y_{l'}} \) that will not influence our discuss because of

\[
\begin{align*}
&\text{(if } i_j > i_k) \quad x_{i_j} \partial_{y_1} \wedge x_{i_k} \partial_{y_1} \wedge \cdots \wedge D x_{i_j} \partial_{y_1} \wedge x_{i_k} \partial_{x_{ij}} \wedge x_{i_j} \partial_{y_2} \wedge \cdots (= 0); \\
&\text{(if } i_j < i_k) \quad x_{i_j} \partial_{x_{ik}} \wedge x_{i_j} \partial_{y_1} \wedge x_{i_k} \partial_{y_1} \wedge \cdots \wedge D x_{i_j} \partial_{x_{ij}} \wedge x_{i_j} \partial_{y_2} \wedge \cdots (= 0).
\end{align*}
\]

So we also assume that \( s_l \neq t_{l'} \) for any \( 1 \leq l \leq \alpha, 1 \leq l' \leq \beta \). If \( i_j > i_k \), then \( x_{i_k} \partial_{x_{ij}} \) is not a factor of \( a \). Since

\[
\begin{align*}
x_{i_j} \partial_{y_1} \wedge x_{i_j} \partial_{y_2} \wedge \cdots \wedge x_{i_j} \partial_{y_\alpha} \wedge x_{i_k} \partial_{y_1} \wedge x_{i_k} \partial_{y_2} \wedge \cdots \wedge x_{i_k} \partial_{y_\beta} \wedge \cdots D(a \wedge b) = 0,
\end{align*}
\]

and \( D(a \wedge b) = 0 \), there should be a term of \( b \) of the form

\[
x_{i_j} \partial_{y_1} \wedge \cdots \wedge x_{i_j} \partial_{y_{l'}} \wedge \cdots \wedge x_{i_j} \partial_{y_\alpha} \wedge x_{i_k} \partial_{y_1} \wedge \cdots \wedge x_{i_k} \partial_{y_2} \wedge \cdots \wedge x_{i_k} \partial_{y_\beta} \wedge U \quad (4.3)
\]

in which \( x_{i_j} \partial_{y_{l'}} \) and \( x_{i_k} \partial_{y_\alpha} \) are at the places where \( x_{i_j} \partial_{y_1} \) and \( x_{i_k} \partial_{y_{l'}} \) used to be in \( c \), respectively. However, since \( \beta > \alpha \), there are not enough \( l \) to match all \( l' \in \{ 1, 2, \ldots, \beta \} \). That is impossible.

If \( i_j < i_k \), then \( x_{i_j} \partial_{x_{ik}} \) is a factor of \( a \). As

\[
\begin{align*}
x_{i_j} \partial_{x_{ik}} \wedge x_{i_j} \partial_{y_1} \wedge x_{i_j} \partial_{y_2} \wedge \cdots \wedge x_{i_j} \partial_{y_\alpha} \wedge x_{i_k} \partial_{y_1} \wedge x_{i_k} \partial_{y_2} \wedge \cdots \wedge x_{i_k} \partial_{y_\beta} \wedge \cdots \\
\delta
\end{align*}
\]

and \( \delta(a \wedge b) = 0 \), there also should be a term of \( b \) of the form

\[
x_{i_j} \partial_{y_1} \wedge \cdots \wedge x_{i_j} \partial_{y_{l'}} \wedge \cdots \wedge x_{i_j} \partial_{y_\alpha} \wedge x_{i_k} \partial_{y_1} \wedge \cdots \wedge x_{i_k} \partial_{y_2} \wedge \cdots \wedge x_{i_k} \partial_{y_\beta} \wedge U, \quad (4.4)
\]
which is the same as we mentioned before. There are not enough $l$ to match all $l' \in \{1, 2, \ldots, \beta\}$, either. That is impossible, too. \hfill \square

Indeed the proof of the above lemma (i.e. (4.3) and (4.4)) also indicated other information:

Lemma 4.10 If $x_{i_k} \partial_{y_l}$ is a factor of a term (denote by $c$) of $b$, then there should be other terms $c_s$ of $b$ and integers $l_s$ ($s = 1, 2, \ldots, k - 1$) such that $c_s$ comes from $c$ by replacing $x_{i_k} \partial_{y_l}$ and $x_{i_s} \partial_{y_{t_s}}$ by $x_{i_k} \partial_{y_{t_s}}$ and $x_{i_s} \partial_{y_l}$, respectively. \hfill \square

Now we can begin our proof of the main theorem.

Proof of Theorem 4.8. We should prove two things. One is that the elements $a \wedge b \in \hat{H}(\mathcal{A}_m^n)$ can be represented as a linear combination of the elements in $\mathcal{P}$. The other is that the elements in $\mathcal{P}$ are linear independent.

We have discussed before that it is enough to consider the elements satisfying condition $(\ast)$. Obviously, each $a \wedge \varphi_{j_1}^a \wedge \varphi_{j_2}^a \wedge \cdots \wedge \varphi_{j_t}^a$ satisfies $(\ast)$.

Each term of $b$ can be adjusted to the “standard” form:

$$(x_{i_1} \partial_{y_{t_1}} \wedge x_{i_1} \partial_{y_{t_2}} \wedge \cdots \wedge x_{i_1} \partial_{y_{t_1}}) \wedge \cdots \wedge (x_{i_k} \partial_{y_{t_k}} \wedge x_{i_k} \partial_{y_{t_2}} \wedge \cdots \wedge x_{i_k} \partial_{y_{t_k}}),$$

where $t_1 \geq t_2 \geq \cdots \geq t_k$ (because of Lemma 4.9) and $l_{s,1} < l_{s,2} < \cdots < l_{s,t_s}$ ($\forall s = 1, 2, \ldots, k$). For any term $c$ of $b$ of the above “standard” form, we define a map

$$\omega : \{\text{terms of } b\} \rightarrow \mathbb{Z}_+,$$

$$c \mapsto \overline{l_{1,1}l_{1,2} \cdots l_{1,t_1} \cdots l_{k,1}l_{k,2} \cdots l_{k,t_k}},$$

where the overline means not to multiply the elements under it but just to represent a number of base-$(m + 1)$ number system. (For example, if $l_1 = 1, l_2 = 12, l_3 = 2$ and $m = 19$, then $\overline{l_{1}l_{2}l_{3}}$ means the number $1 \times 20^2 + 12 \times 20 + 2$.)

Since we have assumed $a \wedge b$ satisfies condition $(\ast)$, each term of $b$ should be with different values under the map $\omega$. We call $c$ the leading term of $b$ if $\omega(c) < \omega(c')$ where $c'$ is any other term of $b$. Assume $c = (x_{i_1} \partial_{y_{t_1}} \wedge x_{i_1} \partial_{y_{t_2}} \wedge \cdots \wedge x_{i_1} \partial_{y_{t_1}}) \wedge \cdots \wedge (x_{i_k} \partial_{y_{t_k}} \wedge x_{i_k} \partial_{y_{t_2}} \wedge \cdots \wedge x_{i_k} \partial_{y_{t_k}})$ is the leading term of $b$. We have $l_{1,p} \leq l_{2,p} \leq \cdots \leq l_{s,p}$, ($\forall s = 1, 2, \ldots, k, p = 1, 2, \ldots, t_s$). If not, thanks to Lemma 4.10, we may exchange certain $l_{s,p}, l_{s,p+1}$ to get another term $c'$ of $b$ such that $\omega(c') < \omega(c)$ which leads a contradiction to our fetching way of $c$. Hence we have $t(= t_1)$ chains:

$$l_{1,p} \leq l_{2,p} \leq \cdots \leq l_{s,p} \ (\forall s = 1, 2, \ldots, k; \\ t_{s+1} < p \leq t_s).$$

Denote $J_p = (l_{1,p}, l_{2,p}, \ldots, l_{s,p}), (\forall s = 1, 2, \ldots, k; \\ t_{s+1} < p \leq t_s)$. We can observe that $a \wedge \varphi_{j_1}^a \wedge \varphi_{j_2}^a \wedge \cdots \wedge \varphi_{j_t}^a \in \mathcal{P}$ and $c$ is also the leading term of $\varphi_{j_1}^a \wedge \varphi_{j_2}^a \wedge \cdots \wedge \varphi_{j_t}^a$. Then $a \wedge (b - \varphi_{j_1}^a \wedge \varphi_{j_2}^a \wedge \cdots \wedge \varphi_{j_t}^a)$ is also an element in $\hat{H}(\mathcal{A}_m^n)$ and satisfies the condition $(\ast)$. Moreover, for any term $c'$ of $(b - \varphi_{j_1}^a \wedge \varphi_{j_2}^a \wedge \cdots \wedge \varphi_{j_t}^a)$, we have $\omega(c') > \omega(c)$. Hence we can replace $b$ by $(b - \varphi_{j_1}^a \wedge \varphi_{j_2}^a \wedge \cdots \wedge \varphi_{j_t}^a)$ and use induction on $\omega(c)$. As the $\omega(c)$ becomes
larger and larger, and the $\omega(c)$ has an upper bound (because the sum $t_1 + t_2 + \cdots + t_k$ is fixed), there should be an end of our inductive process. Thus we know $a \wedge b$ can be presented as a linear combination of elements in $P$.

Now we turn to prove that $P$ is a linear independent set. Since the only change among the all terms of $b = \varphi_{t_1}^a \wedge \varphi_{t_2}^a \wedge \cdots \wedge \varphi_{t_k}^a$ is the permutation of $y_i$'s, we only need to show the linear independence of the elements in the set

$$P^a_{(t; k_1, k_2, \ldots, k_t)} = \{ \varphi_{J_1}^a \wedge \varphi_{J_2}^a \wedge \cdots \wedge \varphi_{J_t}^a \mid a \wedge \varphi_{J_1}^a \wedge \varphi_{J_2}^a \wedge \cdots \wedge \varphi_{J_t}^a \in P, |J_s| = k_s, s = 1, 2, \ldots, t \},$$

where $a \in \mathcal{B}(A^0_n) \cap \tilde{H}(A^0_n)$ and $t, k_1, k_2, \ldots, k_t \in \mathbb{Z}_+$ with $k_1 \geq k_2 \geq \cdots \geq k_t$ are fixed. ($|J_s|$ means the length of $J_s$. For example, if $J_s = (j_1, j_2, \ldots, j_k)$, then $|J_s| = k$.)

For any two elements of $P^a_t$, their leading terms are different from each other. So are the values of these two leading terms under the map $\omega$.

If $\sum_{s=1}^p \alpha_s b_s = 0$ where $0 \neq \alpha_s \in \mathbb{F}$ and $0 \neq b_s \in P^a_t$ with leading term $c_s$. Suppose $\omega(c_{s_1}) < \omega(c_{s_2}) < \cdots < \omega(c_{s_p})$, then the leading term of $\sum_{s=1}^p \alpha_s b_s$ is $c_{s_1} \neq 0$. But the leading term of 0 is of course 0. Contradiction! Therefore the elements in $P^a_{(t; k_1, k_2, \ldots, k_t)}$ are linear independent. So are the elements in $P$. □

By Theorem 4.8, the calculation of the Betti numbers can be transformed to a combinatorial problem. Precisely, we should to computer the number of the elements in a such set:

$$S^i_{m,n} = \{(l_1,1, l_1,2, \ldots, l_{1,t_1}), (l_2,1, l_2,2, \ldots, l_{2,t_2}), \ldots, (l_k,1, l_k,2, \ldots, l_{k,t_k}) \mid 1 \leq k \leq n + 1, t_k \leq t_{k-1} \leq \cdots \leq t_1, t_1 + t_2 + \cdots + t_k = i, 1 \leq l_{s,1} < l_{s,2} < \cdots < l_{s,t_s} \leq m, l_{1,p} \leq l_{2,p} \leq \cdots \leq l_{s,p}, 1 \leq s \leq k, p \leq t_s \},$$

in which $0 \leq i \leq mn + m$.

Using the definitions and notations of Young tableaux in [Fw], we say $S^i_{m,n}$ is the set of all Young tableaux whose entries are taken from $\{1, 2, \ldots, m\}$ and whose shape is $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_s) \vdash i$ with $\lambda_1 \leq n + 1$ and $s \in \mathbb{Z}_+$.

Denote by $d_\lambda(m)$ the number of Young tableaux on the shape $\lambda$ with entries in $\{1, 2, \ldots, m\}$. One has

$$|S^i_{m,n}| = \sum_{\lambda=(\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_s) \vdash i; \lambda_1 \leq n + 1} d_\lambda(m).$$

There is a hook length formula for the number $d_\lambda(m)$ due to Stanley (c.f. [Fw]):

$$d_\lambda(m) = \prod_{(i,j) \in \lambda} \frac{m + j - i}{h_\lambda(i,j)},$$

where $h_\lambda(i,j)$ is the hook length in the $i$-th row and $j$-th column of shape $\lambda$.

The following corollary can be obtained by Theorem 4.8 and Corollary 4.5.
Corollary 4.11  The generating function of the Betti numbers of $A^m_n$ is

$$
\sum_{i=0}^{\infty} b_i t^i = \left( \sum_{i=0}^{m+n} |S^i_{m,n}| t^i \right) \prod_{j=1}^{n} \frac{(1 - t^{j+1})}{(1 - t)^n},
$$

where $|S^i_{m,n}| = \sum_{\lambda=(\lambda_1\geq \lambda_2\geq \cdots \geq \lambda_s)=0; \lambda_1 \leq n+1} \prod_{(i,j) \in \lambda} m + j - i \prod_{k=1}^{s} \eta_{\lambda_{k}(i,j)}$. \hfill \square

By the total order (4.1) and Theorem 4.4, there is a one-to-one correspondence between $B(A^0_n) \cap \tilde{H}(A^0_n)$ and the $(n+1)$-th symmetric group $S_{n+1}$. Precisely, the element which corresponds to $\sigma \in S_{n+1}$ belongs to $H^k(A^0_n)$ with $(-1)^k = \text{sign}(\sigma)$ and $\tau = \sum_{0 \leq i<j \leq n; \sigma(i) > \sigma(j)} \epsilon_{\sigma(i)} - \epsilon_{\sigma(j)}$.

Apply Euler-Poincaré Principle to $A^0_n$. We get an identity:

$$
\prod_{0 \leq i<j \leq n} (1 - e^{\epsilon_{i} - \epsilon_{j}}) = \sum_{\sigma \in S_{n+1}} \text{sign}(\sigma) \prod_{0 \leq i<j \leq n; \sigma(i) > \sigma(j)} e^{\epsilon_{\sigma(i)} - \epsilon_{\sigma(j)}}. \tag{4.5}
$$

Multiply the both sides of (4.5) by $\prod_{i=0}^{n} e^{(n-i)\epsilon_i}$. We get

$$
\prod_{0 \leq i<j \leq n} (e^{\epsilon_{i}} - e^{\epsilon_{j}}) = \sum_{\sigma \in S_{n+1}} \text{sign}(\sigma) \prod_{i=0}^{n} e^{(n-i)\epsilon_{\sigma(i)}},
$$

which is just the Vandermonde Determinant!

For $A^m_n$, there is a one-to-one correspondence between $P$ and $\{(\sigma, T) \mid \sigma \in S_{n+1}, T \in S_{m,n}^t, 0 \leq t \leq mn + m\}$. Now the Euler-Poincaré Principle induces an identity as follows:

$$
\prod_{0 \leq i<j \leq n; i<j \leq m+n} (1 - e^{\epsilon_{i} - \epsilon_{j}}) = \sum_{\sigma \in S_{n+1}} \text{sign}(\sigma) \prod_{0 \leq i<j \leq n; \sigma(i) > \sigma(j)} e^{\epsilon_{\sigma(i)} - \epsilon_{\sigma(j)}} \left( \sum_{t=0}^{m+n} \prod_{k=1}^{m} e^{c_{k}t} \right) \left( \prod_{k=1}^{s} e^{-\lambda_{k}t} \right),
$$

where $(\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_s) \vdash t$ and $(c_1, c_2, \cdots, c_m)$ are the shape and content of $T$, respectively.

5 Final Remarks about $H(L_1(T^d))$

In [L], we also introduced a class of solvable Lie algebras $L_1(T^d)$, which is an extension of $L_0(T^d)$ with $H = \{ x_i \partial_{x_i} \mid i \in \mathcal{N} \}$. The cohomology of $L_1(T^d)$ is much simpler than that of $L_0(T^d)$. In fact, using the following lemma, we can obtain a theorem about $H(L_1(T^d))$ immediately.

To describe the Lemma, we have to introduce some notations and definitions firstly.

Given a finite dimensional Lie algebra $\mathcal{G}$, suppose $g_1, g_2, \ldots, g_t \in \mathcal{G}$ are pairwise commuting elements such that $\mathcal{G}$ possesses a basis consisting of the vectors which are eigenvectors for all the operators $ad \ g_i : g \mapsto [g, g]$. Denote

$$
\mathcal{G}(\lambda_1, \lambda_2, \ldots, \lambda_t) = \{ g \in \mathcal{G} \mid [g, g] = \lambda_i g, i = 1, 2, \ldots, t \}.
$$
It is obvious that
\[ [\mathcal{G}(\lambda_1, \ldots, \lambda_t), \mathcal{G}(\mu_1, \ldots, \mu_t)] \subset \mathcal{G}(\lambda_1 + \mu_1, \ldots, \lambda_t + \mu_t). \]

Denote
\[ \bigwedge^k (\lambda_1, \lambda_2, \ldots, \lambda_t) \mathcal{G} = \text{Span}\{ r_1 \wedge r_2 \wedge \cdots \wedge r_k \mid r_i \in \mathcal{G}(\lambda_1, \ldots, \lambda_i), (1 \leq i \leq k), \sum_{i=1}^k \lambda_{i_j} = \lambda_j (1 \leq j \leq t) \} \]
and
\[ \bigwedge (\lambda_1, \lambda_2, \ldots, \lambda_t) \mathcal{G} = \bigoplus_{k \geq 0} \bigwedge^k (\lambda_1, \lambda_2, \ldots, \lambda_t) \mathcal{G}. \]

**Lemma 5.1** ([F]) *The inclusion \( \bigwedge_{(0,0,\ldots,0)} \mathcal{G} \to \bigwedge \mathcal{G} \) induces an isomorphism in cohomology.*

Hence there comes a theorem:

**Theorem 5.2** *The cohomology group of \( L_1(\mathcal{T}^d) \) is isomorphic to \( \bigwedge \mathcal{H} = \bigoplus_{i \geq 0} \bigwedge^i \mathcal{H}, \) where \( \mathcal{H} = \{ x_i \partial_{x_i} \mid i \in \mathcal{N} \}. \)

**Proof.** Take the natural basis of \( L_0(\mathcal{T}^d) \), i.e.
\[ B(\mathcal{T}^d) = \{ \partial_{x_i}, (\prod_{\kappa \in \mathcal{C}_j \setminus \{i_j\}} x_i^{m_s}) \partial_{x_j} \mid i \in \Lambda, m_s \in \mathbb{N}, \sum_{\kappa \in \mathcal{C}_j \setminus \{i_j\}} m_s \kappa_s \leq \kappa_j \}. \]

Then \( B(\mathcal{T}^d) \cup \mathcal{H} \) is a basis of \( L_1(\mathcal{T}^d) \). Furthermore, the elements in \( \mathcal{H} \) are pairwise commutative and the elements in \( B(\mathcal{T}^d) \cup \mathcal{H} \) are the eigenvectors for all the operators \( \alpha d(x_i \partial_{x_i}), (\forall x_i \partial_{x_i} \in \mathcal{H}) \). Hence we can define the \( \mathcal{G}(\lambda_1, \lambda_2, \ldots, \lambda_t) \) and \( \bigwedge (\lambda_1, \lambda_2, \ldots, \lambda_t) \mathcal{G} \) for \( \mathcal{G} = L_1(\mathcal{T}^d) \).

Take any element \( r_1 \wedge r_2 \wedge \cdots \wedge r_p \in \bigwedge_{(0,0,\ldots,0)} \mathcal{G}, (r_i \in B(\mathcal{T}^d) \cup \mathcal{H}) \). If \( r_i = f_i \partial_{x_{s_i}} \in B(\mathcal{T}^d), \) then we have \( [x_{s_{i}}, \partial_{x_{s_{i}}}, r_{i_{1}}] = -r_{i_{1}} \). So there must be an \( r_{i_2} \) and \( \alpha > 0 \) such that \( [x_{s_{i}}, \partial_{x_{s_{i}}}, r_{i_2}] = \alpha r_{i_2} \). Thus we can assume \( r_{i_2} = f_2 \partial_{x_{s_2}} \in B(\mathcal{T}^d) \) satisfying that \( x_{s_1} \) is a factor of \( f_2 \) and \( \iota_{s_2} \in \mathcal{D}_{s_1} \setminus \{\iota_{s_1}\} \). Analogously, there must be \( r_{i_k} = f_k \partial_{x_{s_k}} \) such that \( x_{s_{k-1}} \) is a factor of \( f_k \) and \( \iota_{s_k} \in \mathcal{D}_{s_{k-1}} \setminus \{\iota_{s_{k-1}}\} \). So \( k \) can be taken as arbitrary positive integer, but there exists no infinite chain \( \iota_{s_1}, \iota_{s_2}, \ldots, \iota_{s_l}, \ldots \) such that \( \iota_{s_k} \in \mathcal{D}_{s_{k-1}} \setminus \{\iota_{s_{k-1}}\} \). Thus \( r_1, r_2, \ldots, r_p \) must be all in \( \mathcal{H} \), i.e. \( \bigwedge_{(0,0,\ldots,0)} \mathcal{G} \subset \bigwedge \mathcal{H} \). As it is obvious that \( \bigwedge \mathcal{H} \subset \bigwedge_{(0,0,\ldots,0)} \mathcal{G}; \) we get \( \bigwedge \mathcal{H} = \bigwedge_{(0,0,\ldots,0)} \mathcal{G} \).

Thanks to Lemma 5.1, \( \bigwedge \mathcal{H} \) is isomorphic to the cohomology group of \( L_1(\mathcal{T}^d) \). \( \square \)

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