Lower Bound for Envy-Free and Truthful Makespan Approximation on Related Machines *

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Abstract

We study problems of scheduling jobs on related machines so as to minimize the makespan in the setting where machines are strategic agents. In this problem, each job \( j \) has a length \( l_j \) and each machine \( i \) has a private speed \( t_i \). The running time of job \( j \) on machine \( i \) is \( t_i l_j \). We seek a mechanism that obtains speed bids of machines and then assign jobs and payments to machines so that the machines have incentive to report true speeds and the allocation and payments are also envy-free. We show that

1. A deterministic envy-free, truthful, individually rational, and anonymous mechanism cannot approximate the makespan strictly better than \( 2 - 1/m \), where \( m \) is the number of machines. This result contrasts with prior work giving a deterministic PTAS for envy-free anonymous assignment and a distinct deterministic PTAS for truthful anonymous mechanism.

2. For two machines of different speeds, the unique deterministic scalable allocation of any envy-free, truthful, individually rational, and anonymous mechanism is to allocate all jobs to the quickest machine. This allocation is the same as that of the VCG mechanism, yielding a 2-approximation to the minimum makespan.

3. No payments can make any of the prior published monotone and locally efficient allocations that yield better than an \( m \)-approximation for \( Q||C_{max} \ [1, 3, 5, 9, 13] \) a truthful, envy-free, individually rational, and anonymous mechanism.

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1 Introduction

We study problems of scheduling jobs on related machines so as to minimize the makespan (i.e. $Q||C_{\text{max}}$) in a strategic environment. Each job $j$ has a length $l_j$ and each machine $i$ has a private speed $t_i$, which is only known by that machine. The speed $t_i$ is the time it takes machine $i$ to process one unit length of a job — $t_i$ is the inverse of the usual sense of speed. The running time of job $j$ on machine $i$ is $t_il_j$. A single job cannot be performed by more than one machine (indivisible), but multiple jobs can be assigned to a single machine. The workload of a machine is the total length of jobs assigned to that machine and the cost is the running time of its workload. The scheduler would like to schedule jobs to complete in minimum time, but has to pay machines to run jobs. The utility of a machine is the difference between the payment to the machine and its cost. The mechanism used by the scheduler asks the machines for their speeds and then determines an allocation of jobs to machines and payments to machines. Ideally, the mechanism should be fair and efficient. To accomplish this, the following features of mechanism are desirable.

**Individually rational** A mechanism is individually rational (IR), if no agent gets negative utility when reporting his true private information, since a rational agent will refuse the allocation and payment if his utility is negative. In order that each machine accepts its allocation and payment, the payment to a machine should exceed its cost of executing the jobs.

**Truthful** A mechanism is truthful or incentive compatible (IC), if each agent maximizes his utility by reporting his true private information. Under truth-telling, it is easier for the designer to design and analyze mechanisms, since agents’ dominant strategies are known by the designer. In a truthful mechanism, an agent does not need to compute the strategy maximizing his utility, since it is simpler to report his true information.

**Envy-free** A mechanism is envy-free (EF), if no agent can improve his utility by switching his allocation and payment with that of another. Envy-freeness is a strong concept of fairness [10, 11]: each agent is happiest with his allocation and payment.

Prior work on envy-free mechanisms for makespan approximation problems assumes that all machine speeds are public knowledge [6, 15]. We assume that the speed of a machine is private information of that machine. This assumption makes it harder to achieve envy-freeness. Only if the mechanism is also truthful, can the mechanism designer ensure that the allocation is truly envy-free.

In this paper, we prove results about anonymous mechanisms. A mechanism is anonymous, roughly speaking, if when two agents switch their bids, their allocated jobs and payments also switch. This means the allocation and payments depend only on the agents’ bids, not on their names. Anonymous mechanisms are of interest in this problem for two reasons. On the one hand, to the best of our knowledge, all polynomial-time mechanisms for $Q||C_{\text{max}}$ are anonymous [1, 3, 5, 9, 15]. On the other hand, in addition to envy-freeness, anonymity can be viewed as an additional characteristic of fairness [4].

We also study scalable allocations. Scalability means that multiplying the speeds by the same positive constant does not change the allocation. Intuitively, the allocation function should not depend on the “units” in which the speed are measured, and hence scalability is a natural notion. But allocations based on rounded speeds of machines are typically not scalable [1, 5, 13].

The truthful mechanisms and envy-free mechanisms for $Q||C_{\text{max}}$ are both well-understood. There is a payment scheme to make an allocation truthful if and only if the allocation is monotone decreasing [3]. For $Q||C_{\text{max}}$, an allocation is monotone decreasing if no machine gets more workload by bidding a slower speed than its true speed. On the other hand, a mechanism for $Q||C_{\text{max}}$ can be envy-free if and only if its allocation is locally efficient [15]. An allocation is locally efficient if a machine never gets less workload than a slower one.

The complexity of truthful mechanisms and, separately, envy-free mechanisms have been completely settled. $Q||C_{\text{max}}$ is strongly NP-hard, so there is no FPTAS for this problem, assuming $P \neq NP$. On the other hand, there is a deterministic monotone PTAS [5] and a distinct deterministic locally efficient PTAS [15]. This implies the existence of truthful mechanisms and distinct envy-free mechanisms that approximate

the makespan arbitrarily closely. However, neither of these payment functions make the mechanisms both truthful and envy-free.

The VCG mechanism for $Q||C_{max}$ is truthful, envy-free, individually rational, and anonymous [5]. However, since the VCG mechanism maximizes the social welfare (i.e. minimizing the total running time), it always allocates all jobs to the quickest machines, yielding a $m$-approximation of makespan for $m$ machines in the worst case. So a question is whether there is a truthful, envy-free, individually rational and anonymous mechanism that approximates the makespan better than the VCG mechanism. Since there already exists many allocation functions that are both monotone and locally efficient, one natural step to answer this question could be checking whether some of these allocation functions admit truthful and envy-free payments.

**Our Results.** We show that

1. A deterministic envy-free, truthful, individually rational, and anonymous mechanism cannot approximate the makespan strictly better than $2 - 1/m$, where $m$ is the number of machines. (Section 3). This result contrasts with prior results [4, 15] discussed above.

2. For two machines of different speeds, the unique deterministic scalable allocation of any envy-free, truthful, individually rational, and anonymous mechanism is to allocate all jobs to the quickest machine. (Section 5). This allocation is the same as that of the VCG mechanism, yielding a 2-approximation of makespan for this case.

3. No payments can make any of the prior published monotone and locally efficient allocations that yield better than an $m$-approximation for $Q||C_{max}$ [1, 3, 5, 9, 13] a truthful, envy-free, individually rational, and anonymous mechanism.

**Related Work.** Hochbaum and Shmoys [12] give a PTAS for $Q||C_{max}$. Andelman, Azar, and Sorani [1] give a 5-approximation deterministic truthful mechanism. Kovács improves the approximation ratio to 3 [13] and then to 2.8 [14]. Randomization has been successfully applied to this problem. Archer and Tardos [3] give a 3-approximate randomized mechanism, which is improved to 2 in [2]. Dhangwatnotai et. al. give a monotone randomized PTAS. All these randomized mechanisms are truthful-in-expectation. However, we can show that no payment function can form a truthful, envy-free, individually rational and anonymous mechanism with any allocation function of these mechanisms. We give a proof for a deterministic allocation [13] in Section 5 and another one for a randomized allocation [3] in Appendix B.

When players have different finite valuation spaces, it is known that a monotone and locally efficient allocation function may not admit prices to form a simultaneously truthful and envy-free mechanism for allocating goods among players [7]. In this paper, we consider mechanisms where all players have identical infinite valuation spaces.

Cohen et. al. [8] study the truthful and envy-free mechanisms on combinatorial auctions with additive valuations where agents have a upper capacity on the number of items they can receive. They seek truthful and envy-free mechanisms that maximize social welfare and show that VCG with Clarke Pivot payments is envy-free if agents’ capacities are all equal. Their result can be interpreted in our setting by viewing that each agent has the same capacity $n$ and the valuation of each agent is the reverse of its cost. So their result implies that the VCG mechanism for $Q||C_{max}$ is truthful and envy-free; but the VCG mechanism does not give a good approximation guarantee for makespan.

## 2 Preliminaries

There are $m$ machines and $n$ jobs. Each agent will report a bid $b_i \in \mathbb{R}$ to the mechanism. Let $t$ denote the vector of true speeds and $b$ the vector of bids.

A mechanism consists of a pair of functions $(w, p)$. An allocation $w$ maps a vector of bids to a vector of allocated workload, where $w_i(b)$ is the workload of agent $i$. For all bid vectors $b$, $w(b)$ must correspond
to a valid job assignment. An allocation $w$ is called scalable if $w_i(b) = w_i(c \cdot b)$ for all bid vectors $b$, all $i \in \{1 \ldots m\}$ and all scalars $c > 0$. A payment $p$ maps a vector of bids to a vector of payments, i.e. $p_i(b)$ is the payment to agent $i$.

The cost machine $i$ incurs by the assigned jobs is $t_iw_i(b)$. Machine $i$’s private value $t_i$ measures its cost per unit work. Each machine $i$ attempts to maximize its utility, $u_i(t_i, b) := p_i(b) - t_iw_i(b)$.

The makespan of allocation $w(b)$ is defined as $\max_i w_i(b) \cdot t_i$. A mechanism $(w, p)$ is $c$-approximate if for all bids $b$ and values $t$, the makespan of the allocation given by $w$ is within $c$ times the makespan of the optimal allocation, i.e.,

$$\max_i w_i(b) \cdot t_i \leq c \cdot OPT(t),$$

where $OPT(t)$ is the minimum makespan for machines with speeds $t$.

A mechanism $(w, p)$ is individually rational, if agents who bid truthfully never incur a negative utility, i.e. $u_i(t_i, (t_i, b_{-i})) \geq 0$ for all agents $i$, true value $t_i$ and other agents’ bids $b_{-i}$.

A mechanism $(w, p)$ is envy-free if no agent wishes to switch his allocation and payment with another. For all $i, j \in \{1, \ldots, m\}$ and all bids $b$,

$$p_i(b) - b_iw_i(b) \geq p_j(b) - b_jw_j(b).$$

Notice that we use bids $b$ instead of the true speeds $t$ in this definition, because a mechanism can determine the envy-free allocation only based on the bids. However, a mechanism can ensure the outcome is envy-free, only if it is also truthful.

A mechanism $(w, p)$ is anonymous if for every bid vector $b = (b_1, \ldots, b_m)$, every $k$ such that $b_k$ is unique and every $l \neq k$,

$$w_l(\ldots, b_{k-1}, b_l, b_{k+1}, \ldots, b_{l-1}, b_k, b_{l+1}, \ldots) = w_k(b)$$

and

$$p_l(\ldots, b_{k-1}, b_l, b_{k+1}, \ldots, b_{l-1}, b_k, b_{l+1}, \ldots) = p_k(b).$$

The condition that $b_k$ is unique is important, because in some case the mechanism may have to allocate jobs of different lengths to agents with the same bids. If mechanism $(w, p)$ is anonymous and the bid of an agent is unique, the workload of that agent stays the same no matter how that agent is indexed. So we can write $w_i(b_i, b_{-i})$ simply as $w(b_i, b_{-i})$ for unique $b_i$ to represent the workload of agent $i$. Similarly, we can write $p_i(b_i, b_{-i})$ simply as $p(b_i, b_{-i})$ for unique $b_i$.

**Characterization of truthful mechanisms**

**Lemma 1** ([3]). The allocation $w(b)$ admits a truthful payment scheme if and only if $w$ is monotone decreasing, i.e., $w_i(b_i', b_{-i}) \leq w_i(b_i, b_{-i})$ for all $i, b_{-i}, b_i' \geq b_i$. In this case, the mechanism is truthful if and only if the payments satisfy

$$p_i(b_i, b_{-i}) = h_i(b_{-i}) + b_iw_i(b_i, b_{-i}) - \int_0^{b_i} w_i(u, b_{-i}) \, du, \quad \forall i$$

where the $h_i$s can be arbitrary functions.

By Lemma 1, the only flexibility in designing the truthful payments for allocation $w$ is to choose the terms $h_i(b_{-i})$. The utility of truth-telling agent $i$ is $h_i(b_{-i}) - \int_0^{b_i} w_i(u, b_{-i})du$, because his cost is $t_iw_i(t_i, b_{-i})$, which cancels out the second term in the payment formula. Thus, in order to make the mechanism individually
rational, the term \( h_i(b_{-i}) \) should be at least \( \int_0^b w_i(u, b_{-i}) \, du \) for any \( b_i \). Since \( b_i \) can be arbitrarily large, \( h_i \) should satisfy
\[
  h_i(b_{-i}) \geq \int_0^\infty w_i(u, b_{-i}) \, du, \quad \forall i, b_{-i}.
\] (2)

Characterization of envy-free mechanisms

An allocation function \( w \) is envy-free implementable if there exists a payment function \( p \) such that the mechanism \( M = (w, p) \) is envy-free. An allocation function \( w \) is locally efficient if for all bids \( b \) and all permutations \( \pi \) of \( \{1, \ldots, m\} \),
\[
  \sum_{i=1}^m b_i \cdot w_i(b) \leq \sum_{i=1}^m b_i \cdot w_{\pi(i)}(b).
\]

Lemma 2 ([15]). Allocation \( w \) is envy-free implementable if and only if \( w \) is locally efficient.

The proof of sufficiency constructs a payment scheme that ensures the envy-freeness for any locally efficient allocation \( w \). Specifically, assuming \( b_1 \geq b_2 \geq \ldots \geq b_m \), the payments for related machines are the following:
\[
p_i(b) = \begin{cases} 
  b_i \cdot w_1(b) & \text{for } i = 1 \\
  p_{i-1}(b) + b_i \cdot (w_i(b) - w_{i-1}(b)) & \text{for } i \in \{2, \ldots, m\}
\end{cases}
\]

These payments are not truthful payments, since \( p_1(b) \) is clearly not in the form of (2). But the set of envy-free payments is a convex polytope for fixed \( w \), since payments satisfying linear constraints \( \forall i, j \ p_i(b) - b_i w_i(b) \geq p_j(b) - b_j w_j(b) \) are envy-free. So there could be other payments that are both envy-free and truthful.

3 Lower Bound on Anonymous Mechanisms

In this section, we will prove an approximation lower bound for truthful, envy-free, individually rational, and anonymous mechanisms.

Theorem 3. Let \( M = (w, p) \) be a deterministic, truthful, envy-free, individually rational, and anonymous mechanism. Then \( M \) is not \( c \)-approximate for \( c < 2 - \frac{1}{m} \).

Since the only flexibility when designing payments in a truthful mechanism is to choose the \( h_i \)'s, we need to know what kind of \( h_i \)'s are required for envy-free anonymous mechanisms. The following two lemmas give necessary conditions on \( h_i \)'s.

**Lemma 4.** If a mechanism \( (w, p) \) is both truthful and anonymous, then there is a function \( h \) such that \( h(v) = h(v) \) in (2) for all bid vector \( v \in \mathbb{R}^{m-1}_+ \) and machines \( i \).

**Proof.** Let \( \beta \) be a real number such that \( \beta < \min_j v_j \). For all \( i \in \{1, \ldots, m-1\} \), define vector \( b^{(i)} = (v_1, \ldots, v_{i-1}, \beta, v_i, \ldots, v_{m-1}) \), \( b^{(i)} = (v_1, \ldots, v_{i+1}, \beta, v_{i+1}, \ldots, v_{m-1}) \). Since \( v = b^{(i)} \) and \( M \) is anonymous, it must be that \( p_i(b^{(i)}) = p_{i+1}(b^{(i)}) \) and \( w_i(b^{(i)}) = w_{i+1}(b^{(i)}) \). Since \( \alpha < v_j \) for any \( 0 < \alpha < \beta, j \in \{1 \ldots m-1\} \), we also have \( w_i(\alpha, v) = w_{i+1}(\alpha, v) \) by anonymity. Thus, for truthful payments, we have
\[
  h_i(v) \geq p_i(b^{(i)}) - \beta w_i(b^{(i)}) + \int_0^\beta w_i(\alpha, v) \, d\alpha = p_{i+1}(b^{(i)}) - \beta w_{i+1}(b^{(i)}) + \int_0^\beta w_{i+1}(\alpha, v) \, d\alpha = h_{i+1}(v).
\]

**Lemma 5.** Let \( L = \sum_k l_k \). If mechanism \( M = (w, p) \) is truthful, envy-free, and anonymous, then
\[
  h(t_{-i}) - h(t_{-j}) \leq L \cdot t_i + (t_j - t_i) w_i(t),
\] (3)
for all \( t \in \mathbb{R}^m_+ \) and \( i, j \in \{1, \ldots, m\} \).
Claim 6. If machine $j$ does not envy machine $i$, then $p_j(t) - t_j w_j(t) \geq p_i(t) - t_j w_i(t)$. Using (1) to substitute in for $p_i$ and $p_j$, and Lemma [4] this yields

$$
(h(t_j) + t_j w_j(t) - \int_0^{t_j} w(x, t_j) \, dx) - t_j w_j(t) \geq \left( h(t_i) + t_i w_i(t) - \int_0^{t_i} w(x, t_i) \, dx \right) - t_j w_i(t).
$$

Rearranging terms gives

$$
h(t_i) - h(t_j) \leq \int_0^{t_i} w(x, t_i) \, dx - \int_0^{t_j} w(x, t_j) \, dx + (t_j - t_i) w_i(t)
$$

$$
\leq \int_0^{t_i} L \, dx - \int_0^{t_j} 0 \, dx + (t_j - t_i) w_i(t)
$$

$$
= L \cdot t_i + (t_j - t_i) w_i(t).
$$

Proof of Theorem \[3\] Consider $n = m$ jobs of length $l = (1, \ldots, 1, m)$. Let $L := 2m - 1$ denote the total length of the jobs. Define speed vector $t = (m \alpha, \ldots, m \alpha, \alpha)$, where $\alpha$ is a real number that only depends on $m$ and $c$ and will be determined at the end of this section. We will show that if $M$ is deterministic, truthful, envy-free, individually rational and anonymous, it should allocate all jobs to the quickest machine in this instance.

Claim 6. For speed vector $t = (m \alpha, \ldots, m \alpha, \alpha)$ and jobs $l = (1, \ldots, 1, m)$, if $M$ is $c$-approximate and $w_i(t) \geq 1$ for some $i \in \{1, \ldots, m - 1\}$, then

$$
h(t_{-1}) \geq (L + \frac{m - 1}{Lc}) \cdot \alpha.
$$

Proof. Since $M$ is truthful and individually rational, inequality (2) applies, and

$$
h(t_{-1}) \geq \int_0^{\infty} w(x, t_{-1}) \, dx \geq \int_0^{\frac{\alpha}{m}} w(x, t_{-1}) \, dx + \int_{\frac{\alpha}{m}}^{\alpha} w(x, t_{-1}) \, dx + \int_{\alpha}^{\infty} w(x, t_{-1}) \, dx.
$$

Apply $M$ to vector $(x, t_{-1})$. By the local efficiency of $w$, job $m$ should be assigned to the quickest machine. So for $x < \alpha$, $w(x, t_{-1}) \geq l_m$. When $x < \frac{\alpha}{Lc}$, all the jobs should be assigned to the machine with speed $x$ for a makespan less than $\alpha/c$. Otherwise the makespan is at least $\alpha$, contradicting $M$ is $c$-approximate. Since $w_i(m \alpha, t_{-1}) \geq 1$ for some $i \in \{1, \ldots, m - 1\}$, monotonicity implies $w_i(x, t_{-1}) \geq 1$ for all $x \in (\alpha, m \alpha)$. Since $x \in (\alpha, m \alpha)$ is unique in vector $(x, t_{-1})$, we get $w(x, t_{-1}) \geq 1$ by anonymity. Thus

$$
h(t_{-1}) \geq \int_0^{\frac{\alpha}{m}} L \, dx + \int_{\frac{\alpha}{m}}^{\alpha} m \, dx + \int_{\alpha}^{\infty} w(x, t_{-1}) \, dx = \frac{1}{c} \alpha + m \alpha - \frac{m}{Lc} \alpha + m \alpha - \alpha = (L + \frac{m - 1}{Lc}) \alpha.
$$

Let $t' = (1, m \alpha, \ldots, m \alpha, \alpha)$. Applying $M$ to $t'$, Lemma [5] implies

$$
h(t'_{-1}) - h(t'_{-m}) \leq L \cdot (\alpha - 1) w_1(t') \leq L \cdot \alpha.
$$

Since $t'_{-1} = t_{-1}$, this implies

$$
h(t_{-1}) \leq L \cdot \alpha + h(t'_{-m}).
$$

Claim 7. If $M$ is $c$-approximate, then $h(t_{-1}) < L \cdot \alpha + f(m, c)$, where $f(m, c) = \gamma^{m-1} L + h(\gamma^{m-2}, \gamma^{m-1}, \ldots, \gamma, 1)$ and $\gamma = cL + \epsilon$ for some $0 < \epsilon < 1$. 

6
Proof. Define speed vector \( t^{(i)} = (\gamma^{i-1}, \gamma^{i-2}, \ldots, \gamma, 1, m_\alpha, \ldots, m_\alpha) \) of length \( m \) for \( i \geq 2 \), where \( \gamma = cL + \epsilon \) for some \( 0 < \epsilon < 1 \).

Let us consider the allocation \( M \) makes to machine 1 for bid vector \( t^{(i)} \). The speed of machine 1 is \( \gamma^{i-1} \geq \gamma \) for \( i \geq 2 \). The speed of machine \( i \) is 1. The makespan of allocating all jobs to machine \( i \) is \( L \) while the makespan of allocating at least one job to machine 1 is at least \( \gamma = cL + \epsilon \). Since \( M \) is \( c \)-approximate, this means \( w_1(t^{(i)}) = 0 \). Using Lemma 5 we have

\[
h(t^{(i)}_1) - h(t^{(i)}_{m-1}) \leq t^{(i)}_1 L + (t^{(i)}_m - t^{(i)}_1)w_1(t^{(i)}) = \gamma^{i-1}L.
\]

Since \( t^{(i)}_m = t^{(i+1)}_{-1} \), this implies \( h(t^{(i)}_1) - h(t^{(i+1)}_{-1}) \leq \gamma^{i-1}L \) for \( i \in \{2, \ldots, m-1\} \). Summing up these inequalities on all \( i \), we have

\[
h(t^{(2)}_1) - h(t^{(m)}_{-1}) \leq L \sum_{i=2}^{m-1} \gamma^{i-1} < \gamma^{m-1}L.
\]

Since \( t^{(2)}_{-1} = t^{(2)}_{-1} \), we get \( h(t^{(m)}_{-1}) < \gamma^{m-1}L + h(t^{(m)}_{-1}) = f(m, c) \). Plugging this into \( \text{(4)} \) yields

\[
h(t^{(1)}_{-1}) < L \cdot \alpha + f(m, c).
\]

To complete the proof of Theorem 3 consider speed vector \( t \) with \( \alpha = \frac{L}{m-1}f(m, c) \). If mechanism \( M \) does not allocate all jobs to machine \( m \), then \( w_1(t) \geq 1 \) for some \( i \in \{1, \ldots, m-1\} \). Then Claim 6 implies that \( h(t_{-1}) \geq \alpha \cdot L + f(m, c) \), contradicting \( \text{(5)} \). So \( M \) must allocate all jobs to machine \( m \) in this case, yielding a makespan of \( (2m-1)\alpha \) while the makespan of the schedule that assigns job \( j \) to machine \( j \) for all \( j \) is \( m_\alpha \). Thus, \( M \) is \( c \)-approximate for some \( c \geq 2 - 1/m \).

4 Characterizing Scalable Mechanisms on Two Machines

We show that known monotone and locally efficient allocations do not have payments to form truthful, envy-free, individually rational, and anonymous mechanisms. (See Section 5 and Appendix B.)

In this section, we will show that for two machines, there is just one deterministic scalable allocation that can be made truthful, envy-free, individually rational, and anonymous. This allocation turns out to be the same allocation as the VCG mechanism.

Lemma 8. Let \( w \) be a deterministic and scalable allocation function for 2 machines. For some \( k > 1 \), if \( w(x, a) > 0 \) for all \( a > 0 \) and \( x < ka \), then there is some \( g(k) > 0 \) such that

\[
\int_a^{ka} w(x, a) \, dx \geq \int_a^{ka} w(a, x) \, dx + g(k) \cdot a.
\]

Proof. For \( a < x < ka \), let \( x = \frac{a^2}{t} \).

\[
\int_a^{ka} w(x, a) \, dx = \int_a^{\frac{a^2}{t}} w(\frac{a^2}{t}, a)(-\frac{a^2}{t^2}) \, dt \quad \text{(integrate by substitution)}
\]

\[
= \int_a^{ka} \frac{a^2}{t^2} w(a, t) \, dt \quad \text{(\( w \) is scalable)}
\]

\[
= \int_{\frac{a}{k}}^{a} \frac{a^2}{t^2} w(a, t) \, dt + \int_{\frac{a}{k}+\frac{a}{2a}}^{a} \frac{a^2}{t^2} w(a, t) \, dt
\]

\[
= \int_{\frac{a}{k}}^{a} \frac{a^2}{t^2} w(a, t) \, dt + \int_{\frac{a}{k}+\frac{a}{2a}}^{a} \frac{a^2}{t^2} w(a, t) \, dt
\]
For \( \frac{a}{k} < t < \frac{k+1}{2k} a \) and \( k > 1 \), we have \( \frac{a^2}{k^2} \geq \frac{a^2}{(k+1)^2} = \frac{4k^2}{(k+1)^2} > k > 1 \). For \( \frac{k+1}{2k} a < t < a \), we have \( \frac{a^2}{k^2} \geq 1 \). Therefore,

\[
\int_{a}^{k a} w(x, a) \, dx \geq \frac{4k^2}{(k+1)^2} \int_{\frac{a}{k}}^{k a} w(a, t) \, dt + \int_{a}^{k a} w(a, t) \, dt
\]

\[
= \left( \frac{4k^2}{(k+1)^2} - 1 \right) \int_{\frac{a}{k}}^{k a} w(a, t) \, dt + \int_{a}^{k a} w(a, t) \, dt
\]

We also have

\[
\int_{\frac{a}{k}}^{k a} w(a, t) \, dt = \int_{\frac{a}{k}}^{a} w(1, \frac{t}{a}) \, dt = a \int_{\frac{a}{k}}^{a} w(1, y) \, dy.
\]

The first equality follows the scalability of \( w \) and we get the second equality by substituting \( t \) with \( ay \).

Since \( w(x, a) > 0 \) for all \( a > 0 \) and \( x < ka \), we have \( w(1, y) > 0 \) for \( y > 1/k \). In sum, \( \int_{a}^{k a} w(x, a) \, dx \geq \int_{\frac{a}{k}}^{a} w(x, a) \, dx + g(k) \cdot a \), where \( g(k) = \left( \frac{4k^2}{(k+1)^2} - 1 \right) \int_{\frac{a}{k}}^{a} w(1, y) \, dy > 0. \)

**Theorem 9.** Let \( M = (w, p) \) be deterministic, truthful, envy-free, individually rational, and anonymous. If \( w \) is scalable, then for two machines of different speeds, \( w \) allocates all jobs to the quickest machine.

**Proof.** Let \( L \) denote the total length of jobs. First, consider two machines of speed \( t_1 = 1 \) and \( t_2 = a (a > 1) \).

Since \( M \) is truthful, envy-free, and anonymous, by Lemma 8\( a \) we have

\[
h(a) - h(1) \leq L + (a - 1)L = L \cdot a
\]

(6)

Since \( w \) is individually rational, \( h(1) \geq \int_{0}^{\infty} w(x, 1) \geq 0 \). We will show that \( w(ka, a) = 0 \) for any \( k > 1 \). For a contradiction, assume \( w(ra, a) > 0 \) for some \( r > 1 \). Let \( k \) be such that \( w(x, a) > 0 \) for \( x < ka \) and \( w(x, a) = 0 \) for \( x > ka \). By monotonicity, such \( a \) exists. By the assumption that \( w(ra, a) > 0 \) for some \( r > 1 \), we know that \( k > 1 \). Since \( w \) is scalable, we have for any \( x > ka \), \( w(y, a) = w(a, x) = L \) if \( y/a = a/x \), i.e. \( y = a^2/x < a/k \). Therefore,

\[
h(a) \geq \int_{0}^{\infty} w(x, a) \, dx
\]

\[
= \int_{0}^{\frac{a}{k}} L \, dx + \int_{\frac{a}{k}}^{a} w(x, a) \, dx + \int_{a}^{ka} w(x, a) \, dx
\]

\[
\geq \frac{L}{k} a + \int_{\frac{a}{k}}^{a} w(x, a) \, dx + \int_{a}^{ka} w(a, x) \, dx + g(k) \cdot a
\]

(Lemma 8\( a \))

\[
\geq \frac{L}{k} a + \int_{\frac{a}{k}}^{a} (w(x, a) + w(a, x)) \, dx + g(k) \cdot a
\]

\[
= \frac{L}{k} a + \int_{\frac{a}{k}}^{a} L \, dx + g(k) \cdot a
\]

(7)

\[
= (L + g(k))a
\]

Take \( a > h(1)/g(k) \). We have \( h(a) > aL + h(1) \) from (7). This contradicts (6).

5 Payments for Known Allocation Rules

Although the VCG mechanism is truthful, envy-free, individually rational, and anonymous, it does not have a good approximation guarantee for makespan. In [13], the LPT* algorithm is described and shown to
be monotone decreasing. In this section, we will show that \(LPT^*\) is locally efficient and no payment function can form an envy-free, truthful, individually rational, and anonymous mechanism with the \(LPT^*\) algorithm. We also prove a similar result for randomized mechanisms in Appendix B: the randomized 2-approximation algorithm in [2 3], whose expected allocation is monotone decreasing and locally efficient, admits no payment function that can make it simultaneously truthful-in-expectation, envy-free-in-expectation, individually rational, and anonymous. We can show similar results with similar proofs for the allocations in [1 5 9].

The \(LPT^*\) algorithm is the following: Let \(w_i^j\) be the workload of machine \(i\) before job \(j\) is assigned. Assume the jobs are indexed so that \(l_1 \geq l_2 \geq \ldots \geq l_m\). Note that this algorithm rounds the speeds and hence is not scalable.

**Algorithm 1 \(LPT^*\) Algorithm**

1. Define rounded speed of machine \(i\) to be \(s_i := 2^{\lceil \log b_i \rceil}\).
2. for \(j = 1\) to \(m\) do
3. Assign job \(j\) to machine \(i\) that minimizes \((w_i^j + l_j) \cdot s_i\).
4. end for
5. Among machines of same rounded speed, reorder bundles on these machines so that a machine with smaller bid gets more jobs.

**Lemma 10.** \(LPT^*\) is locally efficient.

**Proof.** We need to show that \(w_i(b) \leq w_k(b)\) for any \(b_i > b_k\). If \(s_i = s_k\), then step 5 ensures \(w_i(b) \leq w_k(b)\). So suppose \(s_i > s_k\). Consider the last job, \(j\), assigned to machine \(i\). Since job \(j\) is assigned to machine \(i\) rather than machine \(k\), it should be that

\[
(w_i^j + l_j)s_i \leq (w_k^j + l_j)s_k,
\]

where \(w_i^j\) is the workload of machine \(i\) before job \(j\) is assigned. Thus,

\[
w_i^j + l_j \geq \frac{s_i}{s_k}(w_k^j + l_j) \geq 2(w_i^j + l_j).
\]

That is \(w_k^j \geq 2w_i^j + l_j\). Since \(w_k(b) \geq w_i^j\) and \(w_i(b) = w_i^j + l_j\), we get \(w_k(b) \geq w_i(b)\). □

**Theorem 11.** There is no payment function that will make \(LPT^*\) simultaneously truthful, envy-free, individually rational, and anonymous.

**Proof.** Let \(w\) denote the allocation of the \(LPT^*\) algorithm. For a contradiction, assume there exists a payment function \(p\) such that mechanism \(M = (w, p)\) is truthful, envy-free, individually rational, and anonymous.

Apply \(M\) to the problem of two jobs with lengths \(l_1 = 2\) and \(l_2 = 1\), and two machines with speeds \(t_1 = 1, t_2 = a\) where \(a > 1\) and \(a\) is a power of 2. By Lemma 5 we have

\[
h(a) - h(1) \leq 3 + (a - 1) \cdot 3 = 3a.
\]  

(8)

Since \(M\) is individually rational, we also have

\[
h(a) \geq \int_0^\infty w(x, a) \, dx
\]

\[
\geq \int_0^{\frac{a}{4}} w(x, a) \, dx + \int_{\frac{a}{4}}^a w(x, a) \, dx + \int_a^{2a} w(x, a) \, dx
\]

\[
\geq \frac{a}{4} w\left(\frac{a}{4}, a\right) + \int_{\frac{a}{4}}^a w(x, a) \, dx + a \cdot w(2a, a),
\]

where the last inequality follows the monotonicity of \(w\). Since \(a\) is a power of 2, the \(LPT^*\) algorithm ensures \(w\left(\frac{a}{4}, a\right) = 3\) and \(w(2a, a) = 1\). Since \(w\) is locally efficient, for any \(\frac{a}{4} < x < a\), a machine with speed \(x\) gets at least job one, i.e., \(w(x, a) \geq 2\). Therefore, \(h(a) \geq \frac{a}{4} \cdot 3 + \frac{a}{4} \cdot 2 + a \cdot 1 = \frac{13}{4} a\). Now take \(a = 8h(1)\), we have \(h(a) \geq \frac{13}{4} a = 26h(1)\) and \(h(a) \leq h(1) + 3a = 25h(1)\) from (8), a contradiction. □
Theorem 11 implies that local efficiency and monotonicity of an allocation are not sufficient for the existence of a payment function to form an envy-free, truthful, individually rational, and anonymous mechanism. This insufficiency of monotonicity and local efficiency still holds, even if the allocation function is also scalable. See Proposition 13 in Appendix C for more details.

6 Open Questions

In this paper, we establish an approximation lower bound $2 - 1/m$ for any deterministic, envy-free, truthful, individually rational, and anonymous mechanism while the upper bound is $m$ given by the VCG mechanism. So one open question is whether the VCG mechanism is the best among all truthful and envy-free mechanisms.

The proof of Lemma 5 implicitly gives a characterization of mechanisms that are truthful, envy-free, and anonymous. Ideally, there would be a characterization of allocations for which there exist prices that make the resulting mechanism truthful and envy-free. Another interesting question is whether there is a characterization of truthful and envy-free mechanisms for $Q||C_{max}$.

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Appendix

A Randomized mechanisms

In this section, we define the randomized mechanisms for $Q||C_{\text{max}}$ in our setting. A randomized mechanism $(w, p)$ is truthful-in-expectation, if each agent $i$ maximizes his expected utility by bidding his true value $t_i$, i.e., for all agent $i$, all possible $t_i$, $b_i$ and $b_{-i}$,

$$
\mathbb{E}[p_i(t_i, b_{-i}) - t_i w_i(t_i, b_{-i})] \geq \mathbb{E}[p_i(b_i, b_{-i}) - t_i w_i(b_i, b_{-i})].
$$

A randomized mechanism $(w, p)$ is envy-free-in-expectation if no agent wishes to switch his expected allocation and expected payment with another, i.e., for all $i, j \in \{1, \ldots, m\}$ and all bids $b$,

$$
\mathbb{E}[p_i(b)] - b_i \mathbb{E}[w_i(b)] \geq \mathbb{E}[p_j(b)] - b_i \mathbb{E}[w_j(b)].
$$

A randomized mechanism $(w, p)$ is anonymous if for every bid vector $b = (b_1, \ldots, b_m)$, every $k$ such that $b_k$ is unique and every $l \neq k$,

$$
\mathbb{E}[w_l(b_{k-1}, b_k b_{k+1}, \ldots, b_{l-1}, b_k, b_{l+1}, \ldots)] = \mathbb{E}[w_k(b)]
$$

and

$$
\mathbb{E}[p_l(b_{k-1}, b_k b_{k+1}, \ldots, b_{l-1}, b_k, b_{l+1}, \ldots)] = \mathbb{E}[p_k(b)].
$$

So we seek randomized mechanisms that are truthful-in-expectation, envy-free-in-expectation, individually rational, and anonymous. For a randomized mechanism that is truthful-in-expectation, Lemma 1 still holds after replacing all $w_i(b)$ with $\mathbb{E}[w_i(b)]$ [3]. By Lemma 1, the payments are deterministic of a randomized mechanism that is truthful-in-expectation. It is easy to check that [2], Lemma 4 and Lemma 5 still hold after replacing the allocation of a machine with its expected allocation for all machines.

B

In [3], a randomized 3-approximation algorithm (see Algorithm 2) is described and its expected allocation is shown to be monotone decreasing. In this section, we will show that its expected allocation is locally efficient and there is no payment that makes it a truthful-in-expectation, envy-free-in-expectation, individually rational and anonymous mechanism.

**Theorem 12.** There is no payment function that will make Algorithm 2 simultaneously truthful-in-expectation, envy-free-in-expectation, individually rational, and anonymous.

**Proof.** Let $w$ denote the allocation of Algorithm 2. For a contradiction, assume there exists a payment function $p$ such that mechanism $M = (w, p)$ is truthful-in-expectation, envy-free-in-expectation, individually rational and anonymous.

Apply $M$ to the problem of two jobs with lengths $t_1 = 2$ and $l_2 = 1$, and two machines with speeds $t_1 = 1, t_2 = a$ where $a > 1$. By Lemma 5 we have

$$
h(a) - h(1) \leq 3 + (a - 1) \cdot 3 = 3a,
$$

(9)
Algorithm 2 Randomized 3-approximation Algorithm by Archer and Tardos

Require: Jobs and machines are indexed so that \( b_1 \geq b_2 \geq \ldots \geq b_m \) and \( b_1 \leq b_2, \ldots, \leq b_m \).

1: Compute a lower bound of makespan

\[
T_{LB}(b) = \max_j \min_i \max \left\{ \sum_{k=1}^j \frac{l_k}{b_k} \right\}
\]

2: For each machine \( i \), create a bin \( b \) of size \( s_i(b) = T_{LB}(b)/b_i \).

3: Assign jobs 1, 2, \ldots, \((k - 1)\) to bin 1, where \( k \) is the first job that would cause the bin to overflow. Then assign to bin 1 a piece of job \( k \) exactly as large as the remaining capacity of bin 1. Continue by assigning jobs to bin 2, starting with the rest of job \( k \) and so on, until all jobs are assigned.

4: For each job \( j \), assign job \( j \) to machine \( i \) with probability equal to the proportion of job \( j \) that is fractionally assigned to bin \( i \).

where \( h(1) \geq \int_0^\infty w(x, 1) \, dx \geq 0 \). The expected workload of machine \( i \) is equal to the size of bin \( i \) by step 4, i.e., \( \mathbb{E}[w_i(b)] = s_i(b) \). Algorithm 2 also ensures that for all \( c > 0 \),

\[
s_i(c \cdot b) = T_{LB}(c \cdot b)/c \cdot b_i = s_i(b).
\]

So we get \( \mathbb{E}[w_i(c \cdot b)] = \mathbb{E}[w_i(b)] \) for all \( c > 0 \). This implies, for two machines,

\[
\int_0^\infty \mathbb{E}[w(x, a)] \, dx = \int_0^\infty \mathbb{E}[w(x/a, 1)] \, dx = a \int_0^\infty \mathbb{E}[w(t, 1)] \, dt,
\]

where the last equality is obtained by substituting \( x \) with \( at \). Since \( M \) is individually rational, we have

\[
T_{LB}(b) = \frac{\mathbb{E}[w_i(b)]}{b_i} \geq \frac{\mathbb{E}[w_j(b)]}{b_j} = \frac{T_{LB}(b)}{b_j} = \mathbb{E}[w_j(b)]
\]

for \( b_i \leq b_j \). So the expected allocation of Algorithm 2 is locally efficient.

C

**Proposition 13.** Local efficiency, monotonicity and scalability of an allocation are not sufficient for the existence of a payment function to form an envy-free, truthful, individually rational, and anonymous mechanism.

**Proof.** Define allocation \( w \) for 2 machines to be the allocation that minimizes the makespan. If there are more than one such allocations, let \( w \) be the one that also minimizes the total completion time. It is easy to verify that \( w \) is unique. Hence \( w \) is well-defined and anonymous. \( w \) is also locally efficient and scalable, since it minimizes the makespan. Now, we show that \( w \) is monotone. Let the allocation of \( w \) for 2 machines with bids \( (b_1, b_2) \) be \( \mathcal{O} = (L_1, L_2) \). Assume w.l.o.g. that \( b_1 L_1 \geq b_2 L_2 \). If machine 1 increases its bid, its allocation can only go down. Consider the allocation of \( w \): \( \mathcal{O}' = (L_1', L_2') \) for 2 machines with bids \( (b_1, b_2') \), where \( b_2 > b_2' \). For a contradiction, assume \( L_2' > L_2 \). So \( L_2' < b_2 < b_2' \). If the makespan of \( \mathcal{O}' \) is \( b_1 L_1' \), i.e. \( b_1 L_1' \geq b_2 L_2' \), then we have \( b_2 L_2' < b_2' L_2' < b_1 L_1' < b_1 L_1 \). So \( \max\{b_2 L_2', b_1 L_1'\} < b_1 L_1 \). It contradicts the optimality of \( \mathcal{O}' \). If the makespan of \( \mathcal{O}' \) is \( b_2' L_2' \), i.e. \( b_2' L_2' \geq b_1 L_1' \), then \( b_2' L_2' \leq \max\{b_1 L_1, b_2 L_2\} \) by the optimality of \( \mathcal{O}' \). Since \( b_2 L_2' > b_2 L_2 \), we have \( b_2' L_2' \leq b_1 L_1 \). Since \( b_1 L_1' < b_1 L_1 \), we have \( \max\{b_2 L_2', b_1 L_1'\} < b_1 L_1 \). It contradicts the optimality of \( \mathcal{O} \). Therefore, \( L_2' \leq L_2 \) and \( w \) is monotone. Since \( w \) is different from the VCG allocation for 2 machines, this theorem follows from Theorem 9. 

\( \square \)