A TURÁN THEOREM FOR EXTENSIONS VIA AN ERDŐS-KO-RADO THEOREM FOR LAGRANGIANS

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Abstract. The extension of an r-uniform hypergraph G is obtained from it by adding for every pair of vertices of G, which is not covered by an edge in G, an extra edge containing this pair and \((r - 2)\) new vertices. In this paper we determine the Turán number of the extension of an r-graph consisting of two vertex-disjoint edges, settling a conjecture of Hefetz and Keevash, who previously determined this Turán number for \(r = 3\). As the key ingredient of the proof we show that the Lagrangian of intersecting r-graphs is maximized by principally intersecting r-graphs for \(r \geq 4\).

1. Introduction

In this paper we consider r-uniform hypergraphs, which we call r-graphs for brevity. We denote the vertex set of an r-graph \(G\) by \(V(G)\), the number of its vertices by \(v(G)\) and the number of edges by \(e(G)\). (We use \(G\) to denote both the r-graph itself and its edge set.) An r-graph \(G\) is called \(F\)-free if it does not contain \(F\) as a subgraph. We denote the class of all \(F\)-free r-graphs by \(\text{Forb}(F)\). The Turán function \(\text{ex}(n, F)\) is the maximum size of an \(F\)-free r-graph of order \(n\):

\[
\text{ex}(n, F) = \max \{ e(G) : v(G) = n, \ G \in \text{Forb}(F) \}.
\]

The Turán density of an r-graph \(F\) is defined to be the following limit (which was shown to exist by Katona, Nemetz and Simonovits [8]):

\[
\pi(F) = \lim_{n \to \infty} \frac{\text{ex}(n, F)}{\binom{n}{r}}.
\]

The extension of an r-graph \(F\) is an r-graph, denoted by \(\text{Ext}(F)\), obtained from \(F\) by adding an extra edge for every uncovered pair of vertices containing this pair and \((r - 2)\) new vertices. While in general the study of Turán numbers of hypergraphs is a notoriously hard topic, a robust toolkit of stability arguments which can be used to find \(\text{ex}(n, \text{Ext}(F))\), once the maximum Lagrangian of an \(F\)-free r-graph is determined, has been developed in [2, 10, 11, 12]. Using such a stability argument the Turán number of the extension of an edgeless r-graph has been determined by Pikhurko in [12]. Pikhurko’s result has been extended in [2, 10] to determine the Turán number of the extension of all hypergraphs obtained from a fixed r-graph by adding sufficiently many isolated vertices. Our result also relies on stability techniques, including the generic toolkit, which we refer to as the local stability method, developed by two of us in [10, 11].

In [7] Hefetz and Keevash defined the r-graph \(K_{r,r}^{(3)}\) to be the extension of the r-graph consisting of two disjoint edges. In the same paper the authors determined \(\text{ex}(K_{3,3}^{(3)}, n)\) for

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large $n$. To state their result we need to define the balanced blowup $T_3^3(n)$ of $K_5^{(3)}$ on $n$ vertices, where $K_5^{(3)}$ denotes the complete 3-graph on 5 vertices. The 3-graph $T_3^3(n)$ is obtained by partitioning the vertex set of size $n$ into five parts of as equal sizes as possible, and defining the edges of $T_3^3(n)$ to be the triples of vertices belonging to three distinct parts.

**Theorem 1.1** (Hefetz and Keevash, [7]). For sufficiently large $n$, $\text{ex}(n, K_{3,3}^3) = e(T_3^3(n))$ and, moreover, for such $n$ the unique largest $K_{3,3}^3$-free 3-graph on $n$ vertices is $T_3^3(n)$.

In this paper we extend the results of [7] and determine $\text{ex}(K_r^{(r)}, n)$ for all $r \geq 4$ and large $n$. The structure of extremal hypergraphs is different from case $r = 3$. We say that a partition $(A, B)$ of the vertex set of an $r$-graph $H$ is a *star partition* if $|e \cap A| = 1$ for every $e \in H$. We say that $H$ is a *star* if it admits a star partition. We denote by $S(r)[A, B]$ the unique maximal $r$-graph which is a star with a partition $(A, B)$. Finally, we denote by $S(r)(n)$ the star on $n$ vertices with the maximum number of edges. (It is easy to see that $e(S(r)(n)) = (1 - 1/r)^{r-1} \binom{n}{r} + o(n^r)$, and that if $(A, B)$ is a star partition of $S(r)(n)$ then $|A| = n/r + o(n)$.) We are now ready to state our main result, which confirms the aforementioned conjecture of Hefetz and Keevash [7].

**Theorem 1.2.** For every $r \geq 4$, there exists $n_0 := n_0(r)$ such that

$$\text{ex}(n, K_r^{(r)}) = e(S(r)(n))$$

for all $n > n_0(r)$ and, moreover, every $K_r^{(r)}$-free $r$-graph on $n$ vertices with maximum number of edges is a star.

The case $r = 4$ of Theorem 1.2 has been independently established by Wu, Peng and Chen [14]. The proof of Theorem 1.2 as well as the proof of Theorem 1.1 uses the stability method and Lagrangians. The *Lagrangian* $\lambda(F)$ of an $r$-graph $F$ is defined as

$$\lambda(F) = \max_{p} \sum_{e \in F} \prod_{v \in e} p(v),$$

where maximum is taken over all probability distributions on the vertex set $V(F)$, that is, the set of functions $p : V(F) \to [0, 1]$ such that $\sum_{v \in V(F)} p(v) = 1$.

The Lagrangian function for graphs was introduced by Motzkin and Straus [9], who used it to give a new proof of Turán’s Theorem. For hypergraphs, it was introduced independently by Frankl and Rödl [5] and Sidorenko [13], who also established some important properties of the function. In particular, it was shown by them that for any $r$-graph, the Lagrangian is achieved on a subgraph that covers pairs, that is, an $r$-graph in which every pair is contained in an edge (for 2-graphs, this simplifies to maximum sub-clique.) The Lagrangian function is closely related to Turán density of graphs. For any two $r$-graphs $F$ and $G$, any edge-preserving map $\varphi : V(F) \to V(G)$ is called *homomorphism*, that is, for every $f \in F$, $\varphi(F) \in G$. An $r$-graph $G$ is called *$F$-hom-free* if there is no homomorphism from $F$ to $G$. The following lemma was established by Frankl, Füredi in [5] and independently by Sidorenko in [13].

**Lemma 1.3.** For any $r$-graph $F$,

$$\pi(F) = r! \sup_{G \in \text{Forb}_{\text{hom}}(F)} \lambda(G),$$

where $\text{Forb}_{\text{hom}}(F)$ is the family of all $r$-graphs that are $F$-hom-free.
One can further restrict the search of the Turán density of an $r$-graph to the Lagrangian of the family of those $F$-hom-free graphs, which are also dense, where we say an $r$-graph $H$ is dense, if for any proper subgraph $H'$, $\lambda(H') < \lambda(H)$. We say that an $r$-graph $H$ is intersecting if $e \cap f \neq \emptyset$ for all $E, F \in H$. The connection between Turán density of $K_{r,r}$ and the Lagrangians of intersecting $r$-graphs is established in the following lemma, a version of which for $r = 3$ is present in [7], Theorem 4.1.

**Lemma 1.4.** For all $r \geq 3$, $\pi\left(K_{r,r}^{(r)}\right) = r! \sup_{H \in \mathcal{H}} \lambda(H)$, where $\mathcal{H}$ is the family of all intersecting $r$-graphs.

**Proof.** By Lemma 1.3 $\pi\left(K_{r,r}^{(r)}\right) = r! \sup_{H} \lambda(H)$, over all dense $K_{r,r}^{(r)}$-hom-free $r$-graphs $H$. It is not hard to see that every intersecting $r$-graph is $K_{r,r}^{(r)}$-hom-free. For the other direction, suppose $H$ is a dense $K_{r,r}^{(r)}$-hom-free $r$-graph. Suppose there are two disjoint edges $f_1$ and $f_2$ in $H$. Since $H$ is dense, for every pair of vertices $v_1 \in f_1$ and $v_2 \in f_2$, there exists an edge of $H$ covering them, thus creating a homomorphic copy of $K_{r,r}^{(r)}$, a contradiction. \[\square\]

Thus to determine $ex(n, K_{r,r})$ asymptotically, as we do it in Theorem 1.2, one is required to find the supremum of Lagrangians of intersecting $r$-graphs. And, indeed, a key ingredient of the proof of Theorem 1.1 Hefetz and Keevash show that the maximum Lagrangian of intersecting 3-graphs is uniquely achieved by $K_{3,3}^{(3)}$. However, as noted in [7], for $r \geq 4$ the analogous result does not hold: The maximum Lagrangian of an intersecting $r$-graph is not obtained by the complete $r$-graph $K_{2r-1}^{(r)}$ on $2r - 1$ vertices. Let $S_{1}^{(r)}(n)$ denote the intersecting $r$-graph on $n$ vertices consisting of all edges containing some fixed vertex $v$. A direct calculation shows that $\lambda(K_{2r-1}^{(r)}) = \frac{1}{r^r}(2r-1)$, while 

$$
\lim_{n \to \infty} \lambda(S_{1}^{(r)}(n)) = \frac{1}{r!}(1 - r)^{r-1},
$$

and the second expression is larger for $r \geq 4$. We show that for $r \geq 4$ the $r$-graphs $S_{1}^{(r)}(n)$ asymptotically achieve the supremum of Lagrangians of intersecting $r$-graphs. In fact, in the proof of Theorem 1.2 we need a slightly stronger result. We say that an $r$-graph $H$ is principal if there exists $v \in V(H)$ such that $v \in e$ for every $e \in H$.

**Theorem 1.5.** For every $r \geq 4$ there exists a constant $c_r$ such that if $H$ is an intersecting, but not principal $r$-graph, then $\lambda(H) < \frac{1}{r^r} \left(1 - \frac{1}{r}\right)^{r-1} - c_r$.

Theorem 1.5 can be considered as a weighted version of the classical Erdős-Ko-Rado theorem which states that for $n \geq 2r + 1$ the intersecting $r$-graph on $n$ vertices with the maximum number of edges is principal. Theorem 1.5 implies that for sufficiently large $n$ the maximum measure of an intersecting $r$-graph under a non-uniform product measure is achieved by a principal $r$-graph. Let us note that Friedgut [6] proved a weighted result for $t$-intersecting set systems using Fourier analytic methods, but, as we consider set systems consisting only of the sets of size $r$, there appears to be no direct way to derive Theorem 1.5 from the results of [3] and vice versa. The proof of Theorem 1.5 relies primarily on the compression techniques, in particular on the tools developed by Ahlswede and Khachatrian in their proof of the Complete Intersection Theorem [1].

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1. A set system $G$ is $t$-intersecting if $|e \cap f| \geq t$ for all $e, f \in G$.
We prove Theorem 1.5 in Section 2. In Section 3 we derive Theorem 1.2 from Theorem 1.5.

As mentioned earlier, our proof relies on the stability method. In Section 3.1 we use the tools developed in [10, 11] to reduce the class of $r$-graphs which need to be considered in the proof of Theorem 1.2 to $K_{r,r}$-free $r$-graphs that are close to $S(r)(n)$ in the edit distance and are nearly regular. In Section 3.2 we prove the upper bound on the number of edges for these $r$-graphs.

1.1. Notation. Our notation is fairly standard. Let $[n] = \{1, 2, \ldots, n\}$. Let $2^X$ denote the set of all subsets of set $X$, and let $X^{(k)}$ denote the set of all $k$-element subsets. For an $r$-graph $F$ and $v \in V(F)$, the link of the vertex $v$ is defined as
\[
L_F(v) := \{I \in (V(F))^{(r-1)} \mid I \cup \{v\} \in F\}.
\]
More generally, for $I \subseteq V(F)$ the link $L_F(I)$ of $I$ is defined as
\[
L_F(I) := \{J \subseteq V(F) \mid J \cap I = \emptyset, I \cup J \in F\}.
\]
We skip the index $F$, whenever $F$ is understood from the context.

For an $r$-graph $F$ and a subset $A \subseteq V(F)$ we define $F[A]$ to be the $r$-graph induced by $A$, that is, an $r$-graph on the vertex set $A$ and all the edges of $F$ which contain only the vertices of $A$.

Given a family $\mathcal{F}$ of $r$-graphs define
\[
\lambda(\mathcal{F}) = \sup_{F \in \mathcal{F}} \lambda(F).
\]

In the next section we will not only consider $r$-graphs, but more general set systems. Extending the hypergraph notation we say that a set system $G$ is intersecting if $e \cap f \neq \emptyset$ for all $e, f \in G$. We say that a set system $G$ is an $(\leq r)$-graph if $|e| \leq r$ for every $e \in G$.

2. Maximum Lagrangian of Intersecting $r$-graphs

In this section we prove Theorem 1.5.

For a positive integer $s$, let $[s]^+ = [s] \cup \{\infty\}$. Our central object of study will be a weighted intersecting set system or w.i.i.s.s. for short, which is a triple $(G, s, p)$, where
\begin{itemize}
\item $s$ is a positive integer,
\item $G \subseteq 2^{[s]}$ is an intersecting $(\leq r)$-graph,
\item $p : [s]^+ \to [0, 1]$ is a probability distribution\footnote{That is $p(\infty) = 1 - \sum_{i=1}^{s} p(i)$.} which is non-increasing on $[s]$.
\end{itemize}

It will be convenient for us to write $p_\infty$ instead of $p(\infty)$. For $e \subseteq 2^s$ such that $|e| \leq r$ and a probability distribution $p : [s]^+ \to [0, 1]$, define the weight $w_p(e)$ of $e$ by
\[
w_p(e) = \frac{r!}{(r - |e|)!} p_\infty^{r-|e|} \prod_{i \in e} p(i).
\]
We frequently use probabilistic intuition to estimate $w_p(e)$. Let $S_p^r$ be a multiset of $r$ elements drawn from $[s]^+$ independently at random according to the probability distribution $p$. Then $w_p(e)$ is the probability that the restriction of $S_p^r$ to $[s]$ is equal to $e$, and, in particular, has no repeated elements. For an $(\leq r)$-graph $G \subseteq 2^s$, we define $w_p(G) = \sum_{e \in G} w_p(e)$. Thus $w_p(G)$ is the probability that the restriction of $S_p^r$ to $[s]$ is equal to an edge of $G$.

Let us further motivate the technical definition of a weighted intersecting set system above. Let $G$ be an intersecting $r$-graph, and let $S \subseteq V(G)$ be such that $e \cap f \cap S \neq \emptyset$ for all $e, f \in G$. 

Let \( G \downarrow S = \{ e \cap S | e \in G \} \). Then \( G \downarrow S \) is an intersecting \((\leq r)\)-graph. Moreover, if \( G \) is a maximal intersecting \( r \)-graph on \( V(G) \) then \( G = \{ e \in V(G)^{[r]} | e \cap S \in G \downarrow S \} \). Thus \( G \downarrow S \) contains all the essential information about \( G \). The \((\leq r)\)-graph \( G \downarrow S \) is referred to as a generating set of \( G \) in \([1]\), and in fact our definition of a weighted intersecting set system is motivated by the definition of generating sets in \([1]\).

Consider now a probability distribution \( p \) on \( V(G) \) such that \( \lambda(G) = \sum_{e \in G} \prod_{v \in G} p(v) \). Note that \( r!\lambda(G) \) is the probability that a multiset of \( r \) elements drawn from \( V(G) \) independently at random according to the probability distribution \( p \) produces an edge of \( G \). We assume without loss of generality that \( S = [s] \) for some positive integer, and that \( p \) is non-increasing on \([s]\).\(^3\) Define a probabilistic distribution \( p' : [s]^+ \rightarrow [0,1] \) by setting \( p'(i) = p(i) \) for \( i \in S \), and \( p(\infty) = 1 - \sum_{i=1}^{s} p(i) \). Then \((G \downarrow [s], s, p')\) is a w.i.s.s. and \( w_{p'}(G \downarrow [s]) \) can be interpreted as the probability that a restriction of the multiset of \( r \) elements drawn from \( V(G) \) independently at random according to the probability distribution \( p \) to \([s]\) produces an edge of \( G \downarrow S \). It follows that \( w_{p'}(G \downarrow S) \geq r!\lambda(G) \). If \( G \downarrow [s] \) contains an edge of size \( \leq r - 2 \) then the equality is necessarily strict, but one can construct a sequence of intersecting \( r \)-graphs \( \{G_i\}_{i \in \mathbb{N}} \) such that \( G_i \downarrow [s] = G \downarrow [s] \) and \( \lim_{i \rightarrow \infty} r!\lambda(G_i) = w_{p'}(G \downarrow [s]) \), by increasing the number of vertices of \( G \) and reducing the probability of individual vertices in \( V(G) \setminus [s] \). As we want to upper bound the maximum Lagrangians of intersecting \( r \)-graphs, such sequences of increasingly large \( r \)-graphs with increasing Lagrangians could present a major technical difficulty, as one can not naturally choose an “optimal” object in them. Fortunately restricting our attention to weighted intersecting set systems avoids the issue.

Abusing the notation slightly we will say that the \((\leq r)\)-graph \( G \) is principal if \( 1 \in e \) for all \( e \in G \), and non-principal, otherwise.\(^4\) We say that a w.i.s.s. \((G, s, p)\) is (non)-principal if \( G \) is (non)-principal. Let \( L_r = \left(1 - \frac{1}{r}\right)^{r-1} \). Note that \( L_r \geq 1/e \) for every integer \( r \geq 2 \). For a principal w.i.s.s. \((G, s, p)\) we have
\[
w_{p}(G) \leq rp(1)(1 - p(1))^{r-1} \leq L_r. \tag{1}
\]

The following is the main technical result of this section, which directly implies Theorem 1.3.

**Theorem 2.1.** For every integer \( r \geq 4 \) there exists \( c_r > 0 \) satisfying the following. Let \((G, s, p)\) be a non-principal w.i.s.s. Then
\[
w_{p}(G) \leq L_r - c_r.
\]

First let us derive Theorem 1.3 from Theorem 2.1.

**Proof of Theorem 1.3 assuming Theorem 2.1.** Let \( G \) be an intersecting \( r \)-graph, which is not principal. We may assume \( G \subseteq 2^{|s|} \) for some positive integer \( s \). By definition of the Lagrangian there exists a probability distribution \( p : [s]^+ \rightarrow [0,1] \) with \( p(\infty) = 0 \) such that \( w_{p}(G) = r!\lambda(G) \). By permuting vertices of \( G \) one may further assume that \( p \) is non-increasing on \([s]\), implying that \((G, s, p)\) is a w.i.s.s. By Theorem 2.1 we have
\[
\lambda(G) \leq \frac{1}{r!} w_{p}(G) \leq \frac{1}{r!} \left( \left(1 - \frac{1}{r}\right)^{r-1} - c_r \right).
\]

\(^3\)The condition that \( p \) is non-increasing on \([s]\) might appear artificial at the moment, but is a natural requirement in the compression arguments.

\(^4\)Note that this definition is slightly different from the definition of principal \( r \)-graphs given in the introduction.
as desired.

It remains to prove Theorem 2.1. The proof occupies the remainder of this section.

We say that a non-principal w.i.s.s. $(G, s, p)$ is a target if $w_p(G) \geq w_{p'}(G')$ for every non-principal w.i.s.s. $(G', s', p')$ such that $s' \leq s$. Moreover, if the equality holds then $s = s'$, $p(s) > 0$ and $\sum_{e \in G} \sum_{i \in e} 1 \leq \sum_{e' \in G'} \sum_{i \in e'} i$. Clearly, it suffices to prove Theorem 2.1 for targets.

We use the compression technique to show that targets are very structured. Given a $(\leq r)$-graph $G \subseteq 2^{|s|}$ and $1 \leq i < j \leq s$ we define a compression map $R_{ij} : G \to 2^{|s|}$, by setting

$$R_{ij}(e) = e \setminus \{j\} \cup \{i\},$$

if $j \in e$, $i \notin e$ and $e \setminus \{j\} \cup \{i\} \notin G$, and $R_{ij}(e) = e$, otherwise. Then $R_{ij}$ is an injection. It is well known (see [4 Proposition 2.1]) that if $G$ is intersecting, then so is $R_{ij}(G)$. The next lemma shows that targets are essentially always “compressed”.

**Lemma 2.2.** Let $(G, s, p)$ be a target. Then either

- either $e \setminus \{j\} \cup \{i\} \in G$ for every $e \in G$ and $1 \leq i < j \leq s$ such that $j \in e$, $i \notin e$,
- or $2 \in e$ for every $e \in G$.

**Proof.** Note that if $R_{ij}(G) = G$, for all $1 \leq i < j \leq s$ then the first outcome of the lemma holds. Thus we suppose that $R_{ij}(G) \neq G$ for some $1 \leq i < j \leq s$. Let $G' = R_{ij}(G)$. Then $(G', s, p)$ is a w.i.s.s. Moreover, $w_p(G') \geq w_p(G)$, as $p(i) \geq p(j)$, implying $w_p(R_{ij}(e)) \geq w_p(e)$ for every $e \in G$. Also, $\sum_{e \in G} \sum_{i \in e} i > \sum_{e' \in G'} \sum_{i \in e'} i$. Thus $G'$ is principal, as $(G, s, p)$ is a target. In particular, this implies that $i = 1$, i.e. $R_{i1}(G) = G$ for all $1 < i' < j' \leq s$. Note further that $\emptyset \cap \{1, j\} \neq \emptyset$ for every $e \in G$ as $R_{1j}(G)$ is principal.

As $G$ is non-principal there exists $e \in G$ such that $1 \notin e$, and it follows from the above that $e' = \{2, 3, \ldots, |e| + 1\} \in G$. This in turn implies that $e'' = \{2, 3, \ldots, \min(r + 1, s)\} \in G$, as $e' \subseteq e''$, and if $e'' \notin G$ then it could be added to $G$, violating the condition that $G$ is a target. We further must have $j \in e''$ and $(e'' \setminus \{j\}) \cup \{1\} \notin G$, as otherwise $e'' \in R_{1j}(G)$ violating the assumption that $R_{1j}(G)$ is principal.

Suppose now for a contradiction that $2 \notin f$ for some $f \in G$. If $1 \notin f$, then $j \in f$ and $j \neq 2$. Let $f' = f \setminus \{j\} \cup \{2\}$. Then $f' \in G$, as $R_{2j}(G) = G$, however $f' \cap \{1, j\} = \emptyset$, a contradiction. Thus $1 \notin f$. It follows, as above, that $f'' = \{1, 3, \ldots, \min(r + 1, s)\} \in G$. In this case however, we have $R_{1j}(e'') = e''$, contradicting $R_{1j}(G)$ being principal.

Given a probability distribution $p$ on $[s]^+$, define a probability distribution $\bar{p}$ on $[s-1]^+$ by setting $\bar{p}(i) = p(i)$ for every $i \in [s - 1]$ and $\bar{p}(\infty) = p(\infty) + p(s)$. For a set system $H$ define $H - s = \{e \setminus \{s\} | e \in H\}$. 

**Lemma 2.3.** Let $(G, s, p)$ be a target for some $s > 2$. Then there exist $e, f \in G$ such that $e \cap f = \{s\}$, and we have $e \cup f = [s]$ for each such pair $e, f$. In particular, $s \leq 2r - 1$.

**Proof.** Suppose for a contradiction that $(e \cap f) \setminus \{s\} \neq \emptyset$ for all $e, f \in G$. Let $G' = G - s$. Then $G'$ is intersecting, and $w_p(e \setminus \{s\}) \geq w_p(e \setminus \{s\}) + w_p(e)$ for every $e \in G$, implying $w_p(G') \geq w_p(G)$. Thus w.i.s.s. $(G', \bar{p}, s - 1)$ contradicts the assumption that $(G, s, p)$ is a target.

Consider now $e, f \in G$ such that $e \cap f = \{s\}$. Suppose for a contradiction that there exists $i \in [s]$ such that $i \notin e \cup f$. Then $e' = (e \setminus \{s\}) \cup \{i\} \in G$ by Lemma 2.2 as $2 \notin e \cap f$. However, $e' \cap f = \emptyset$, yielding the desired contradiction.
Corollary 2.4. Let \((G, s, p)\) be a target for some \(s > 2\). Then \(e \setminus \{j\} \cup \{i\}\) for every \(e \in G\) and \(1 \leq i < j \leq s\) such that \(j \in e, i \notin e\).

Proof. By Lemma 2.2 either the corollary holds, or \(2 \in e\) for every \(e \in G\). However, if \(2 \in e\) for every \(e \in G\), then \(2 \in e \cap f\) for all \(e, f \in G\), contradicting Lemma 2.3.

The main step in the proof of Theorem 2.1 involves removing \(s\) from every element of \((G, s, p)\), reducing \(p(s)\) to 0, and modifying \(G\) so that the resulting set system is still intersecting, as we did in Lemma 2.3 above. We start by analyzing the change in weight of edges after such a modification.

Lemma 2.5. Let \(s\) be a positive integer, let \(p\) be a probability distribution on \([s]^+\) such that \(p(\infty) \geq p(s) > 0\), and let \(e \subseteq [s], |e| \leq r\) be such that \(s \in e\). Then

\[ w_p(e \setminus \{s\}) \geq 2w_p(e). \tag{2} \]

Proof. Let \(l = |e|\). Recall that \(w_p(e)\) is the probability that the restriction of \(S^*_p\) to \([s]\) is \(e\). Similarly, \(w_p(e \setminus \{s\})\) is the probability that the restriction of \(S^*_p\) to \([s - 1]\) is \(e \setminus \{s\}\). Let \(A\) be the event that the restriction of \(S^*_p\) to \([s - 1]\) is \(e \setminus \{s\}\) and \(s\) occurs in \(S^*_p\) zero times or twice. Clearly, \(Pr[A] \leq w_p(e \setminus \{s\}) - w_p(e)\). We will show that \(Pr[A] \geq w_p(e)\), implying the lemma. Let \(x = \prod_{i \in e} p(i)\). Then

\[ w_p(e) = \frac{r!}{(r - l)!}xp^{r-l}_\infty \tag{3} \]

and

\[ Pr[A] = \frac{r!}{(r - l + 1)!}p^{r-l+1}_\infty \prod_{i \in e \setminus \{s\}} p(i) + \frac{r!}{(r - l - 1)!}p^{r-l-1}_\infty \prod_{i \in e \setminus \{s\}} p(i) \]

\[ = \frac{r!}{(r - l + 1)!}p^{r-l+1}_\infty \frac{x}{p(s)} + \frac{r!}{(r - l - 1)!}p^{r-l-1}_\infty \frac{xp(s)}{2} \tag{4} \]

Combining, (3) and (4), we have

\[ \frac{w_p(e \setminus \{s\}) - w_p(e)}{w_p(e)} \geq \frac{Pr[A]}{w_p(e)} = \frac{p_\infty}{p(s)(r - l + 1)} + \frac{(r - l)p(s)}{2p_\infty}. \]

If \(l = r\), then \((w_p(e \setminus \{s\}) - w_p(e))/w_p(e) \geq 1\), as \(p_\infty \geq p(s)\). Otherwise, \((r - l) \geq (r - l + 1)/2\), and

\[ \frac{w_p(e \setminus \{s\}) - w_p(e)}{w_p(e)} \geq \left( \frac{p_\infty}{p(s)(r - l + 1)} \right) + \frac{1}{4} \left( \frac{p(s)(r - l + 1)}{p_\infty} \right) \geq 1, \]

where the second inequality is AM-GM. The inequality (2) follows.

All the necessary tools in hand, we continue the proof of Theorem 2.1. Let \(c = 1/500\). We prove by induction on \(s\) that if \((G, s, p)\) is a non-principal w.i.s.s. then

\[ w_p(G) \leq L_r - c2^{-\min(2r-2, s-1)}. \tag{5} \]

Theorem 2.1 with \(c_r = c2^{-2r+2}\) is implied by this statement.

The base case \(s = 1\) is trivial as every w.i.s.s. \((G, s, p)\) with \(s = 1\) is principal.

We divide the proof of the induction step into several cases. In the first case, Lemma 2.5 and the argument uses the compression techniques and tools developed above. In the remaining cases, the proof is fairly straightforward for large \(r\), but the small \(r\) cases require brute force computation.
Lemma 2.5 to every element of $H$ for $i$.

Summing (6) and (7) we obtain

$$G_0 = \{ e \in G \mid f \cap e \neq \{s\} \text{ for every } f \in G \},$$

and let $G' = G - G_0$. For $i = 1, 2$. Note further, that as in Lemma 2.3 we have

$$w_p(H_i - s) \geq 2w_p(H_i) \tag{6}$$

for $i = 1, 2$. Indeed, for every $e \in G_0 - s$ such that $\{s\}$ is non-principal. Then $w_p(H_i - s) \geq 2w_p(H_i)$.

Summing (6) and (7) we obtain

$$w_p(G_1) + w_p(G_2) \geq \frac{(w_p(G_0) + 2w_p(H_1)) + (w_p(G_0) + 2w_p(H_2))}{2} = w_p(G). \tag{8}$$

Note that at least one of the w.i.s.s. $(G_1, s - 1, \bar{p})$ and $(G_2, s - 1, \bar{p})$ is non-principal, as $G_1 \cup G_2 = G - e$. By Lemma 2.3 $s \leq 2r - 1$, and we suppose by symmetry, that $(G_1, s - 1, \bar{p})$ is non-principal. Then $w_p(G_1) \leq L_r - c2^{-s+2}$ by the induction hypothesis, and $w_p(G_2) \leq L_r$, using the induction hypothesis if $G_2$ is also non-principal. The inequality (8) now implies

$$w_p(G) \leq \frac{(L_r - c2^{-s+2}) + L_r}{2} = L_r - c2^{-s+1},$$

as desired.

Case 2: $s = 2$.

As $(G, 2, p)$ is non-principal, we have $\{2\} \in G$. Let $x = p(1), y = p(2)$. By the AM-GM inequality we have

$$xy(1 - x - y)^{r-2} = \frac{1}{(r-2)^2}((r-2)x)((r-2)y)(1 - x - y)^{r-2} \leq \frac{1}{(r-2)^2} \left( \frac{r-2}{r} \right)^r, \tag{9}$$

and

$$y(2 - 2y)^{r-1} = \frac{1}{2(r-1)}(2(r-1)y)(2 - 2y)^{r-1} \leq \frac{1}{2(r-1)} \left( \frac{r-1}{r} \right)^r. \tag{10}$$
Using (9) and (10) we obtain

\[ w_p(G) \leq r(r - 1)xy(1 - x - y)^{r-2} + ry(1 - x - y)^{r-1} \]
\[ \leq r(r - 1)xy(1 - x - y)^{r-2} + ry(1 - 2y)^{r-1} \]
\[ \leq \frac{r(r - 1)(r - 2)^{r-2}}{r^r} + \frac{r(r - 1)^{r-1}}{2r^r} \]
\[ = L_r \left( \frac{r - 2}{r - 1} \right)^{r-2} + \frac{1}{2} \]
\[ \leq L_r - \frac{L_r}{18} \leq L_r - c, \]

as desired.

**Case 3:** \( r \geq 5 \) and \( p(\infty) \leq p(s) \).

Let \( z = \sum_{i=1}^{s} p(i) \). Then \( z \geq sp(s) \), and so \( z \geq \frac{s}{s+1} \) and therefore \( p(\infty) \leq 1/(s + 1) \). We upper bound \( w_p(G) \) by the probability of the event \( C \) that the restriction of \( S'_p \) to \([s]\) has no repeated elements. Clearly, given \( z \geq \frac{s}{s+1} \), \( \text{Pr}[C] \) is maximized when \( p(1) = p(2) = \ldots = p(s) \) as \( \text{Pr}[C] \) is a symmetric multi-linear function of \( p(1), \ldots, p(s) \), and it is further maximized when \( p(\infty) \) is maximum. Thus we assume that \( p(x) = 1/(s + 1) \) for every \( x \in [s]^+ \). Let \( C_0 \) be the event that \( S'_p \) has no repeated elements at all. Then

\[ \text{Pr}[C_0] = \frac{(s+1)!}{(s+1-r)!(s+1)^r}. \]  

(11)

when \( s + 1 \geq r \), and \( \text{Pr}[C_0] = 0 \), if \( s + 1 < r \). Let \( \overline{C_0} \) denote the negation of the event \( C_0 \). Then \( \text{Pr}[C|\overline{C_0}] \leq \frac{1}{s+1} \), by symmetry. If \( s + 1 < r \) we have

\[ w_p(G) \leq \text{Pr}[C] + \text{Pr}[C|\overline{C_0}](1 - \text{Pr}[C_0]) \]
\[ \leq \frac{1}{s+1} + \frac{(s+1)!}{(s+1-r)!(s+1)^r} \leq \frac{r-1}{r} s + 1 - i \]
\[ \leq \frac{1}{r} + \left( \frac{2s + 3 - r}{2s + 2} \right)^{r} \leq \frac{1}{r} + \left( \frac{3r + 1}{4r} \right)^{r+1}, \]

where the third inequality is by AM-GM inequality. The function \( f(r) = \frac{1}{r} + \left( \frac{2r+1}{4r} \right)^r \) decreases with \( r \), and \( f(7) < 1/3 \). Thus \( w_p(G) \leq L_r - (1/e - 1/3) \leq L_r - c \) for \( r \geq 7 \).

The cases \( r = 5, 6 \) require more care. We use the precise formula

\[ \text{Pr}[C] = \frac{1}{(s+1)^r} \sum_{i=0}^{r} \binom{r}{i} \frac{s!}{(s-i)!}, \]

and verify that \( \text{Pr}[C] \leq L_r - 0.04 \) for \( r \in \{5, 6\} \) and \( r \leq s \leq 2r - 1 \) by computing the corresponding nine values.

**Case 4:** \( r = 4 \) and \( s > 2 \).
Suppose first that \( \{2, 3, 4\} \in G \), then by Corollary 2.4 every three element subset of \([4]\) is an edge of \( G \). Therefore \( |e \cap [4]| \geq 2 \) for every \( e \in G \). It follows \( w_p(G) \) is upper bounded by the probability of the event \( A \) that \( S^r(p) \) contains at least two elements of \([4]\), but no element of \([4]\) appears twice. As in the previous case, the probability of \( A \) is clearly maximized \( p(1) = p(2) = p(3) = p(4) = x \) for some \( 0 \leq x \leq 1/4 \), and so we assume that these equalities hold.

\[
\begin{align*}
\Pr_p(G) &\leq \Pr[A] \\
&\leq \sum_{i=2}^{4} \binom{4}{i} \frac{4!}{(4-i)!} x^i(1-4x)^{4-i} \\
&= 72x^2(1-4x)^2 + 96x^3(1-4x) + 24x^4 \leq 0.41,
\end{align*}
\]

where the last inequality is obtained by explicitly computing the maximum of \( 72x^2(1-4x)^2 + 96x^3(1-4x) + 24x^4 \) on \( 0 \leq x \leq 1/4 \).\(^5\) As \( e_4 = 0.421875 \), it follows that \( \Pr_p(G) \leq e_4 - 0.01 \) in this case.

Thus \( \{2, 3, 4\} \not\in G \). By Corollary 2.4, every edge of \( G \) contains at least two elements of \([s]\), including 1, and no repeated elements of \([s]\), or contains 4 distinct elements of \( \{2, \ldots, s\} \). Let \( B \) be the event that \( S^r(p) \) produces a multiset with the above properties. Thus \( \Pr_p(G) \leq \Pr[B] \). Once again, \( \Pr[B] \) is maximized when \( p(i) = y \) for \( i = 2, 3, \ldots, s \) for some \( y \). Let \( p(1) = x \). Let \( C \) be the event that \( S^r(p) \) contains 1 exactly once. Then

\[
\begin{align*}
\Pr[C] &= 4x(1-x)^3, \quad (12) \\
\Pr[B \setminus C] &= \frac{(s-1)!}{(s-5)!}y^4, \quad (13)
\end{align*}
\]

and

\[
\Pr[C \setminus B] = 4x(1-x-(s-1)y)^3 + 12(s-1)xy^2(1-x-y) + 4(s-1)xy^3. \quad (14)
\]

As \( s \leq 7 \), \( x \geq y \) and \( 1-x-y \geq (s-2)y \), we have

\[
12(s-1)xy^2(1-x-y) \geq \frac{(s-1)!}{(s-5)!}y^4. \quad (15)
\]

Moreover,

\[
\begin{align*}
4x(1-x-(s-1)y)^3 + 4(s-1)xy^3 \\
\geq \frac{4}{(s-1)^2}x((1-x-(s-1)y)^3 + ((s-1)y)^3) \\
\geq \frac{8}{36}x\left(\frac{1-x}{2}\right)^3
\end{align*}
\]

(16)

Combining \((12) - (16)\), we obtain

\[
\begin{align*}
\Pr[B] &= \Pr[C] + \Pr[B \setminus C] - \Pr[C \setminus B] \\
&\leq \left(4 - \frac{1}{36}\right)x(1-x)^3 \leq e_4 - e_4/144 \leq e_4 - e_4,
\end{align*}
\]

\(^5\)The maximum is equal to

\[
24\left(\frac{3(5-\sqrt{3})^4}{21296} - \frac{5(5-\sqrt{3})^3}{2662} + \frac{3}{484}(5-\sqrt{3})^2\right)
\]

and is achieved at \( x = (5-\sqrt{3})/22 \).
as desired.

3. Stability: Proof of Theorem 1.2

3.1. Local Stability. In this section we introduce the result from [10], which builds on the techniques originally presented in [11], and allows us to reduce the proof of Theorem 1.2 to r-graphs which are “close” to the conjectured extremum.

We say that an r-graph G is obtained from an r-graph F by cloning a vertex v to a set W if F ⊆ G, V(G) \ V(F) = W \ {v}, and L_G(w) = L_F(v) for every w ∈ W. We say that G is a blowup of F if G is isomorphic to an r-graph obtained from F by repeatedly cloning and deleting vertices. We denote the set of all blowups of F by B(F). We say that a family of r-graphs is clonable if it is closed under the operation of taking blowups. Note that the family of all stars is clonable.

For a family of r-graphs \( \mathcal{F} \), let

\[
m(\mathcal{F}, n) := \max_{F \in \mathcal{F}, v(F) = n} |F|
\]

denote the maximum number of edges in an r-graph in \( \mathcal{F} \) on n vertices.

Let \( \mathcal{F} \) and \( \mathcal{H} \) be two families of r-graphs. We define the distance \( d_\mathcal{F}(F) \) from an r-graph F to a family \( \mathcal{F} \) as

\[
d_\mathcal{F}(F) := \min_{F' \in \mathcal{F}, v(F') = v(F)} |F \triangle F'|.
\]

For \( \varepsilon, \alpha > 0 \), we say that \( \mathcal{F} \) is \((\mathcal{H}, \varepsilon, \alpha)\)-locally stable if there exists \( n_0 \in \mathbb{N} \) such that for all \( F \in \mathcal{F} \) with \( v(F) = n \geq n_0 \) and \( d_\mathcal{H}(F) \leq \varepsilon n' \) we have

\[
|F| \leq m(\mathcal{H}, n) - \alpha d_\mathcal{H}(F). \tag{17}
\]

We say that \( \mathcal{F} \) is \( \mathcal{H} \)-locally stable if \( \mathcal{F} \) is \((\mathcal{H}, \varepsilon, \alpha)\)-locally stable for some choice of \( \varepsilon \) and \( \alpha \). We say that \( \mathcal{F} \) is \((\mathcal{H}, \alpha)\)-stable if it is \((\mathcal{H}, 1, \alpha)\)-locally stable, that is the inequality (17) holds for all \( F \in \mathcal{F} \) with \( v(F) = n \geq n_0 \). We say that \( \mathcal{F} \) is \( \mathcal{H} \)-stable, if \( \mathcal{F} \) is \((\mathcal{H}, \alpha)\)-stable for some choice of \( \alpha \). We refer the reader to [11] for the detailed discussion of this notion of stability and its differences from the classical definition.

For \( \varepsilon, \alpha > 0 \), we say that a family \( \mathcal{F} \) of r-graphs is \((\mathcal{H}, \varepsilon, \alpha)\)-vertex locally stable if there exists \( n_0 \in \mathbb{N} \) such that for all \( F \in \mathcal{F} \) with \( v(F) = n \geq n_0 \), \( d_\mathcal{H}(F) \leq \varepsilon n' \), and \( |L_F(v)| \geq r(1 - \varepsilon)m(\mathcal{H}, n)/n \) for every \( v \in V(F) \), we have

\[
|F| \leq m(\mathcal{H}, n) - \alpha d_\mathcal{H}(F).
\]

We say that \( \mathcal{F} \) is \( \mathcal{H} \)-vertex locally stable if \( \mathcal{F} \) is \((\mathcal{H}, \varepsilon, \alpha)\)-vertex locally stable for some \( \varepsilon, \alpha \). It is shown in [11] that vertex local stability implies local stability under mild conditions.

Let \( M_2^{(r)} \) denote the r-graph consisting of two vertex disjoint edges. Note that \( K_{r,r}^{(r)} = \text{Ext}(M_2^{(r)}) \) and that \( \text{Forb}(M_2^{(r)}) \) is exactly the class of intersecting r-graphs.

We are now ready to state the main result from [10] that we will be using this paper. The result in full generality requires one to extend the notions of distance and stability to weighted r-graphs. In the interest of brevity, we do not present the corresponding definitions and instead state a direct corollary of [10, Corollary 2.8], which is necessary for our purposes. An interested reader can verify that the next theorem is indeed a direct weakening of [10, Corollary 2.8].
Theorem 3.1. Let $G$ be an $r$-graph, let $\mathcal{F} = \text{Forb}(\text{Ext}(G))$, let $\mathcal{F}^* = \text{Forb}(G)$, and let $\mathcal{H} \subseteq \mathcal{F}$ be a clonable family of $r$-graphs. If the following conditions hold

(T1): $\mathcal{F}$ is $\mathcal{H}$-vertex locally stable,

(T2): there exists a constant $c > 0$ such that $\lambda(F) \leq \lambda(\mathcal{H}) - c$ for every $F \in \mathcal{F}^* - \mathcal{H}$.

then $\mathcal{F}$ is $\mathcal{H}$-stable. In particular, there exists $n_0 \in \mathbb{N}$ such that if $F \in \mathcal{F}$ satisfies $v(F) = n$ and $|F| = m(\mathcal{F}, n)$ for some $n \geq n_0$ then $F \in \mathcal{H}$.

Let $S$ denote the family of $r$-graphs which are stars. To derive Theorem 1.2 from Theorem 3.1 it suffices to show that (T1) and (T2) hold when $\mathcal{F} = \text{Forb}(K_r^{(r)})$, $\mathcal{F}^*$ is the family of all intersecting $r$-graphs, and $\mathcal{H} = S$. The validity of condition (T2) in this case follows directly from Theorem 1.3. Thus it remains to verify condition (T1). It will follow from the theorem we introduce next.

Let

$$d_r := \frac{(1 - \frac{1}{r})^{r-1}}{(r-1)!},$$

$$e_r := \frac{(1 - \frac{1}{r})^{r-1}}{r!}.$$ 

Then $m(S, n) = e_r n^r + o(n^r)$.

Theorem 3.2. For every $r \geq 2$ there exist $n_0 \in \mathbb{N}$ and $\delta > 0$ such that the following holds. Let $F$ be a $K_r^{(r)}$-free $r$-graph with $v(F) = n \geq n_0$. If

(S1): $|L_F(v)| \geq (1 - \delta)d_r n^{r-1}$ for every $v \in V(F)$, and

(S2): there exists a star $S$ such that $|F \triangle S| \leq \delta n^r$,

then $F$ is a star.

It is easy to see that Theorem 3.2 implies that $\text{Forb}(K_r^{(r)})$ is $\mathcal{F}$-vertex locally stable. Thus it remains to prove Theorem 3.2. The proof occupies the rest of the section.

3.2. Vertex Local Stability: Proof of Theorem 3.2

Before we start the main proof, let us state and prove two auxiliary lemmas which will need later. Our first lemma ensures that if a large star $S$ has edge density close to the maximum possible (i.e. $e_r n^r$), then the star-partition of $S$ is almost as balanced as the one for $S^{(r)}(n)$. More precisely, given a star $S$, with star-partition $(A, B)$ we call $S$, $\varepsilon$-balanced, if $|A - \frac{n}{r}| \leq \varepsilon$.

Lemma 3.3. Let $r \geq 3$. For every $\varepsilon > 0$ there exists $\delta > 0$ and $n_0$ such that if $S$ is a star on $n \geq n_0$ vertices such that $e(S) \geq (e_r - \delta)n^r$ then $S$ is $\varepsilon$-balanced.

Proof. Let $(A, B)$ be the star-partition of $S$, then $|S| \leq |A|(|n - |A|)^{r-1}/(r-1)!$. Setting $|A|/n = x$ and $f(x) = x(1 - x)^{r-1}/(r-1)!$ we rewrite the above inequality as $|S| \leq f(x)n^r$. Note that $f : [0, 1] \to [0, 1]$ is a continuous function with the unique maximum $e_r$ achieved at $x^* = 1/r$. It follows that for every $\varepsilon > 0$ there exists $\delta > 0$ such that $f(x) \geq e_r - \delta$ implies $|x - x^*| \leq \varepsilon$, implying the desired inequality.

\[\text{In fact, Theorem 3.2 is stronger as it implies that }\text{Forb}(K_r^{(r)}) \text{ is } (\mathcal{F}, \delta, \alpha)\text{-vertex locally stable for every } \alpha > 0.\]
Let the constants $n_0, k, \varepsilon_1, \varepsilon_2, \delta$ be chosen implicitly to satisfy the inequalities appearing throughout the proof. It will be clear that these inequalities are satisfied as long as

$$\frac{1}{n_0} \ll \delta \ll \varepsilon_1 \ll \varepsilon_2 \ll \frac{1}{k} \ll \frac{1}{r}.$$  

Let $S$ be the star satisfying (S2), let $(A, B)$ be the star partition of $S$, and let $S_e$ be the complete star with the partition $A$ and $B$. It follows from (S1) that $|F| \geq (1 - \delta)e_n r^n$, and thus $|S| \geq (e_r - 2\delta)n^r$. Therefore, $|S_e - S| \leq 2\delta n^r$, as $|S_e| \leq e_r n^r$. Finally, it follows that $|F \Delta S| \leq 3\delta n^r$. By replacing $\delta$ with $3\delta$, we may assume that $S = S_e$.

We say that a pair $(A', B')$ with $A' \subseteq A, B' \subseteq B$ is perfect if $F[A' \cup B'] = S^{(r)}[A', B']$, i.e. the restriction of $F$ to $A' \cup B'$ coincides with the restriction of $S$. Let $P$ denote the set of all perfect pairs.

For a positive integer $k$, let the random variables $X^k$ and $Y^k$ be subsets of size $k$ of $X \subseteq A$ and $Y \subseteq B$, respectively, chosen uniformly and independently at random. We say that a vertex $v \in V(F)$ is $(A, k, \varepsilon)$-regular if

$$\Pr[(A^{k-1} \cup \{v\}, B^k) \in P] \geq 1 - \varepsilon.$$  

Similarly, a vertex $v \in V(F)$ is $(B, k, \varepsilon)$-regular if

$$\Pr[(A^k, B^{k-1} \cup \{v\}) \in P] \geq 1 - \varepsilon.$$  

The next claim motivates the above definitions. (It will be applied with $\varepsilon = \varepsilon_1$ and $\varepsilon = \varepsilon_2$ later in the proof.)

**Claim 1.** If $\varepsilon \ll 1/r$ then

(C11): if $v_1, v_2$ are distinct $(A, k, \varepsilon)$-regular vertices then there exists no $e \in F$ such that $\{v_1, v_2\} \subseteq e$,

(C12): if $v_1, v_2, \ldots, v_r$ are $(B, k, \varepsilon)$-regular then $\{v_1, v_2, \ldots, v_r\} \notin F$.

*Proof. (C11):* Suppose for a contradiction that there exists $e \in F$ such that $\{v_1, v_2\} \subseteq e$. By $(A, k, \varepsilon)$-regularity of $v_1$ and $v_2$ we have

$$\Pr[(A^{k-1} \cup \{v_1\}, B^k), (A^{k-1} \cup \{v_2\}, B^k) \in P] \geq 1 - 2\varepsilon.$$  

If $\varepsilon < 1/2$ and $k > r^3 + r$ there exist $(A' \cup \{v_i\}, B') \in P$ for some $A', B'$ with $|A'| \geq (r - 1)^2 + r$ and $|B'| \geq 2r + (r - 2)(r^2 + 1)$ and $i = 1, 2$. Let $f_i \subseteq B' \cup \{v_i\}$ for $i = 1, 2$ be chosen so that $e \cap f_i = \{v_i\}$ and $f_1 \cap f_2 = \emptyset$. One can then straightforwardly find an extension of $\{f_1, f_2\}$ in $F[A' \cup B' \cup \{v_1, v_2\}] \cup \{e\}$, by using $e$ to extend the pair $\{v_1, v_2\}$, and selecting the edges to extend other pairs of vertices greedily in $F[A' \cup B']$. Thus we obtain the desired contradiction as $F$ is $K_{r-r}^{(r)}$-free.

(C12): The proof is similar to (C11). Suppose that there exists $e = \{v_1, \ldots, v_r\} \in F$, such that $v_1, v_2, \ldots, v_r$ are $(B, k, \varepsilon)$-regular. As in (C11), if $\varepsilon < 1/r$ we can find sufficiently large $A' \subseteq A, B' \subseteq B$ such that $(A', B' \cup \{v_i\}) \in P$ for $i \in [r]$. Choose $f \subseteq A' \cup B'$, such that $|f \cap A'| = 1$ and $f \cap e = \emptyset$. Then one can find an extension of $\{e, f\}$ in $F[A' \cup B' \cup \{v_1, \ldots, v_r\}]$, choosing edges to extend pairs of vertices in $e$ and $f$ greedily. \(\square\)

Note that Claim [11] implies that, if every vertex of $F$ is either $(A, k, \varepsilon)$-regular or $(B, k, \varepsilon)$-regular for some $k, \varepsilon$ satisfying the conditions of Claim [11] then $F$ is a star, as desired. Thus our goal is to show that all vertices are “sufficiently regular”. We start by showing that almost all vertices are.
Claim 2. There exist $A_0 \subseteq A, B_0 \subseteq B$ such that

(C21): $|A_0| \geq (1/r - \varepsilon_1)n, |B_0| \geq ((r - 1)/r - \varepsilon_1)n,$

(C22): every $v \in A_0$ is $(A, k, \varepsilon_1)$-regular, and

(C23): every $v \in B_0$ is $(B, k, \varepsilon_1)$-regular.

Proof. Let $e$ be an edge of $S$ chosen uniformly at random.

$$\Pr[e \notin F] \leq \frac{|F \triangle S|}{|S|} \leq \frac{\delta}{e_r - 2\delta} \leq \frac{2}{e_r}\delta.$$  

Therefore,

$$\Pr[(A^k, B^k) \notin P] \leq k \left(\frac{k}{r-1}\right) \Pr[e \notin F] \leq k \left(\frac{k}{r-1}\right) \frac{2}{e_r}\delta,$$  

(18)

where the first inequality holds, as one can choose $e$ by first choosing the pair $(A^k, B^k)$, and then choosing an $r$-element $e \subseteq A^k \cup B^k$ such that $|e \cap A^k| = 1$ uniformly at random.

Let $A_0$ be the set of all vertices in $A$ which are $(A, k, \varepsilon_1)$-regular. Then

$$\Pr[(A^k, B^k) \notin P] \geq \varepsilon_1 \frac{|A| - |A_0|}{|A|},$$  

(19)

as $A^k$ can be chosen by first choosing a single element of $v \in A$ uniformly at random to be in it, and if such $v$ is not in $A_0$, then the resulting pair $(A^k, B^k)$ is not perfect with probability at least $\varepsilon_1$.

Combining, (18) and (19) we obtain

$$|A_0| \geq \left(1 - k \left(\frac{k}{r-1}\right) \frac{2}{\varepsilon_1 e_r}\right) |A| \geq \left(1 - \frac{\varepsilon_1}{2}\right) |A|.$$  

Moreover, by Lemma 3.3 we have $|A| \geq (1/r - \varepsilon_1/2)n$, as $\delta \ll \varepsilon_1$. It follows that $|A_0| \geq (1/r - \varepsilon_1)n$, as desired.

Analogously, we have $|B_0| \geq ((r - 1)/r - \varepsilon_1)n$, where $B_0$ is the set of all $(B, k, \varepsilon_1)$-regular vertices in $B$, finishing the proof of the claim. □

Our final step is to show that all vertices of $F$ are $(A, k, \varepsilon_2)$-regular or $(A, k, \varepsilon_2)$-regular for some $\varepsilon_1 \ll \varepsilon_2 \ll 1/k$.

Claim 3. Let $v \in V$.

(C31): if there exists $f_0 \in L(v)$ such that $f_0 \cap A_0 \neq \emptyset$ then $v$ is $(B, k, \varepsilon_2)$-regular,

(C32): otherwise, $v$ is $(A, k, \varepsilon_2)$-regular.

Proof. We start by proving (C31). Let $e \in F$, $u \in A_0$ be such that $\{v, u\} \in e$.

Let $B_1 = \{b \in B||L(v, b)| \geq n^{r-5/2}\}$. Suppose that there exists $f_1 \in L(v)$ such that $f_1 \subseteq B_1$, $f_1 \cap e = \emptyset$ and $(A', B') \in P$ such that $u \in A'$, $|A'|, |B'| \gg r$ and $f_1 \subseteq B'$. Choosing $f_2 \in B'$ with $|f_2| = r - 1, f_2 \cap (f_1 \cup e) = \emptyset$ one can find an extension of $\{\{v\} \cup f_1, \{u\} \cup f_2\}$ in $F$, using $e$ to extend $\{v, u\}$, extending the pairs of vertices not containing $v$ in $F[A' \cup B']$ greedily, which can be done as $(A', B') \in P$, and extending the remaining pairs containing $v$ greedily using the fact $f_2 \subseteq B_1$. Thus no such choice of $f_1$ and $(A', B')$ is possible.

Let $b$ be an element of $B - B_1$, and let $f_b$ be a random subset of $B$ of size $r - 1$ containing $b$. Then

$$\Pr[f_b \in L(v)] = \frac{|L(v, b)|}{(|B| - 1)^{r-2}} \leq \frac{n^{r-5/2}}{(n/2)^{r-2}} = \frac{2^{r-2}}{\sqrt{n}}.$$  

(20)
Let \( f \) now be a random subset of \( B \) of size \( r - 1 \) then it follows from (20) that
\[
\Pr[\{ f \in L(v) | f \not\subseteq B_1 \}] \leq \frac{(r - 1)2^{r-2}}{\sqrt{n}}.
\]
(21)

Note next that \( f \) can be chosen by choosing a random pair \((A^k, B^k)\) and randomly choosing \( f \subseteq B^k \). Using (21) and the discussion above, we have
\[
\Pr[\{ f \in L(v) \}= \Pr[\{ f \in L(v) \land f \subseteq B_1 \} + \Pr[\{ f \cap e = \emptyset \} + \Pr[\{ f \not\subseteq B_1 \}]
\leq \Pr[\{ (A^k) \cup \{ u \} \not\subseteq P \} + \Pr[\{ f \cap e \not\subseteq \emptyset \} + \Pr[\{ f \in L(v) | f \not\subseteq B_1 \}]
\leq \epsilon_1 + \frac{(r - 1)r}{|B|} + \frac{(r - 1)2^{r-2}}{\sqrt{n}} \leq 2\epsilon_1.
\]

Thus \(|L(v) \cap B^{(r-1)}| \leq 2\epsilon_1 n^{r-1} \). By (C11) no element of \( L(v) \) contains two elements of \( A_0 \).

Let \( T = \{ f \in V(F)^{(r-1)} | f \cap A = \emptyset \} \), and let \( L_0(v) = L(v) \cap T \). By the above we have, that every element of \( L(v) - L_0(v) \) is contained in \( B^{(r-1)} \) or contains an element of \( A - A_0 \). Therefore
\[
|L(v) - L_0(v)| \leq |A - A_0| n^{r-2} + 2\epsilon_1 n^{r-1} \leq 3\epsilon_1 n^{r-1}.
\]
Thus \(|L_0(v)| \geq (d_r - \delta - 3\epsilon_1)n^{r-1} \). Now let \( f \subseteq T \) be chosen uniformly at random. Note that
\[
\Pr[\{ f \not\subseteq L_0(v) \}] \leq 1 - \frac{|L_0(v)|}{|T|} \leq 1 - \frac{d_r - \delta - 3\epsilon_1}{d_r + r\epsilon_1} \leq \frac{(r + 4)\epsilon_1}{d_r}.
\]
Thus
\[
\Pr[\{ (A^k, B^{k-1}) \not\subseteq P \}]
\leq \Pr[\{ (A^k, B^{k-1}) \not\subseteq P \} + \Pr[\exists f \subseteq A^k \cup B^{k-1} : f \in T - L_0(v)]
\leq \frac{k(k - 1)}{r - 1} \frac{2}{d_r} \delta + k \frac{(k - 2)}{r - 2} \frac{(r + 4)\epsilon_1}{d_r}
\leq \epsilon_2,
\]
as desired.

\[\Box\]

As mentioned earlier, the proof of Claim 3 concludes the proof of Theorem 3.2. Indeed, Claim 3 implies that \( V(F) \) can be partitioned into the set \( A' \) of \( (A, k, \epsilon_2) \)-regular vertices and a set \( B' \) of \( (B, k, \epsilon_2) \)-regular vertices. It follows from Claim 1 that \( (A', B') \) is a star partition of \( F \), and so \( F \) is a star.

4. Concluding Remarks

Towards a Complete Intersection Theorem for Lagrangians. As mentioned in the introduction our proof of Theorem 1.5 relies significantly on the techniques introduced by Ahlswede and Khachatrian in their proof of the Complete Intersection Theorem [1], which determines the maximum number of edges in a \( t \)-intersecting \( r \)-graph on \( n \) vertices. It seems therefore natural to ask whether Theorem 1.5 can be extended to determine the supremum of Lagrangians of \( t \)-intersecting \( r \)-graphs. To continue the discussion we need to recall the statement of the Complete Intersection Theorem. For integers \( t \geq 1, i \geq 0 \), and \( r \geq t + i \),
let $F(r, t, i)$ denote the $(\leq r)$-graph with $F(r, t, i) \subseteq 2^{[t+2]}$ such that $t + i \leq |e| \leq r$ for every $e \in F(r, t, i)$. Then it is easy to see that $F(r, t, i)$ is $t$-intersecting. Consider now $n \geq 2t + i$ and an $r$-graph $G(r, t, i, n)$ consisting of all edges $e \subseteq [n]$, $|e| = r$ such that $e \cap [2t + i] \in F(r, t, i)$. Then $G(r, t, i, n)$ is $t$-intersecting $r$-graph.

**Theorem 4.1 (The Complete Intersection Theorem [1]).** Let $G$ be a $t$-intersecting $r$-graph with $\nu(G) = n$ then

$$|G| \leq \max_{0 \leq i \leq r-t} |G(r, t, i, n)|.$$ 

The analogous statement for Lagrangians is best stated in terms of weighted $t$-intersecting set systems. Define the weighted $t$-intersecting set system $(G, s, p)$ analogously to our definition of a weighted intersecting set system in Section 2 except that the $(\leq r)$-graph $G$ is required to be $t$-intersecting.

**Conjecture 4.1.** Let $G$ be a $t$-intersecting $r$-graph then

$$r!\lambda(G) \leq \max_{0 \leq i \leq r-t} w_p(F(r, t, i)),$$

where the maximum is implicitly taken over all $p : [t+2]^{\rightarrow} [0, 1]$ such that $(F(r, t, i), t + 2i, p)$ is a weighted $t$-intersecting set system.

The validity of Conjecture [4.1] for $t = 1$ follows from Theorem [1.5] for $r \geq 4$. For $r = 3$, it follows from the results in [7] mentioned in the introduction. It is also easy to verify for $r = 2$.

Theorem [4.1] is relatively easy to establish for $n \gg r, t$. In this regime the maximum $t$-intersecting $r$-graphs are principal, i.e. consist of all set of size $r$ containing a fixed set of size $t$. One would expect the same phenomenon to hold in our setting, i.e. for $r \gg t$, the weighted $t$-intersecting set system of maximum weight consists of single edge of size $t$. The validity of this intuition for $t = 1$ is supported by Theorem [1.5]. Surprisingly, the first author have shown that the above statement is false for large $t$.

**Vertex Local Stability.** We have taken slightly non-standard route in the proof of Theorem [3.2] by considering a vertex to be well-behaved if probability that a sample containing this vertex matches the expected structure. A different proof of Theorem [3.2] is given in the third author’s PhD thesis [15, Theorem 6.2.1].

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