Cohomology ring of differential operator rings

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We compute the multiplicative structure in the Hochschild cohomology ring of a differential operators ring and the cap product of Hochschild cohomology on the Hochschild homology.

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Introduction

Let k be a field and A an associative k-algebra with 1. An extension E/A of A is a differential operator ring on A if there exist a Lie k-algebra g and a k-vector space embedding x ↦→ x, of g into E, such that for all x, y ∈ g and a ∈ A, the following conditions hold:

1. \( \tilde{x}a - a\tilde{x} = a^x \), where \( a \mapsto a^x \) is a derivation,
2. \( x y - y x = [x, y]^g + f(x, y) \), where \( [\cdot, \cdot]^g \) is the bracket of g and \( f : g \times g \to A \) is a k-bilinear map,
3. for a given basis \( \{x_i\}_{i \in I} \) of g, the algebra E is a free left A-module with the standard monomials in the \( x_i \)'s as a basis.

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This general construction was introduced in [Ch] and [Mc-R]. Several particular cases of this type of extensions have been considered previously in the literature. For instance:

- when $g$ is a one-dimensional vector space and $f$ is the trivial cocycle, $E$ is the Ore extension $A[x, \delta]$, where $\delta(a) = ax$,
- when $A = k$, we obtain the algebras studied by Sridharan in [S], which are the quasi-commutative algebras $E$, whose associated graded algebra is a symmetric algebra,
- McConnell [Mc, §2] studies this type of extensions under the hypothesis that $A$ is commutative and $(x, a) \mapsto ax$ is an action, and Borho et al. [B-G-R, Theorem 4.2] consider the case in which the cocycle is trivial.

Blattner et al. [B-C-M] and Doi and Takeuchi [D-T] independently begun the study of the crossed products $A \#_H H$ of a $k$-algebra $A$ by a Hopf $k$-algebra $H$, and in [M] it was proved that the differential operator rings on $A$ are the crossed products of $A$ by enveloping algebras of Lie algebras.

In [G-G1] the authors obtained complexes, simpler than the canonical ones, which compute the Hochschild homology and cohomology of a differential operator ring $E$ with coefficients in an $E$-bimodule $M$. In this paper we continue this investigation by studying the Hochschild cohomology ring of $E$ and the cap product

$$H_p(E, M) \times \text{HH}^q(E) \to H_{p+q}(M, E) \quad (q \leq p),$$

in terms of the above mentioned complexes. Moreover we generalize the results of [G-G1] by considering the (co)homology of $E$ relative to a subalgebra $K$ of $A$ which is stable under the action of $g$ (which we also call the Hochschild (co)homology of the $K$-algebra $E$). We also seize the opportunity to fix some minor mistakes and to simplify some proofs in [G-G1].

The paper is organized as follows: In Section 1 we obtain a projective resolution $(X_\bullet, d_\bullet)$ of the $E$-bimodule $E$, relative to the family of all epimorphisms of $E$-bimodules which split as $(E, K)$-bimodule maps. In Section 2 we determine and study comparison maps between $(X_\bullet, d_\bullet)$ and the normalized Hochschild resolution $(E \otimes_K E \otimes_K E, b'_\bullet)$ of $E$, relative to $K$. In Sections 3 and 4 we apply the above results in order to obtain complexes $(\overline{X}^*_K(M, d_\bullet))$ and $(\overline{X}^*_K(M, d^\bullet))$, simpler than the canonical ones, giving the Hochschild homology and cohomology of the $K$-algebra $E$ with coefficients in an $E$-bimodule $M$, respectively. The main results are Theorems 3.4 and 4.4, in which we obtain morphisms

$$\overline{X}^*_{a}(E) \otimes \overline{X}^*_{K}(E) \to \overline{X}^*_{a}(E) \quad \text{and} \quad \overline{X}^*_{a}(M) \otimes \overline{X}^*_{K}(E) \to \overline{X}^*_{a}(M),$$

inducing the cup and cap product, respectively. Finally in Section 5 we obtain further simplifications, assuming that $A$ is a symmetric algebra.

1. Preliminaries

Let $k$ be a field. In this paper all the algebras are over $k$. Let $A$ be an algebra and $H$ a Hopf algebra. We are going to use the Sweedler notation $\Delta(h) = \sum_{(h)} h^{(1)} \otimes_k h^{(2)}$ for the comultiplication $\Delta$ of $H$.

A weak action of $H$ on $A$ is a $k$-bilinear map $(h, a) \mapsto a^h$, from $H \times A$ to $A$, such that

1. $(ab)^h = \sum_{(h)} a^{h^{(1)}} b^{h^{(2)}}$,
2. $1^h = \epsilon(h)1$,
3. $a^1 = a$,

for $h \in H$, $a, b \in A$. By an action of $H$ on $A$ we mean a weak action such that

$$(a^h)^l = a^{hl} \quad \text{for all} \ h, l \in H, \ a \in A.$$
Let \( A \) be an algebra and let \( H \) be a Hopf algebra acting weakly on \( A \). Given a \( k \)-linear map \( f : H \otimes_k H \to A \) we let \( A \#_f H \) denote the algebra (which is not necessarily associative nor with multiplicative unit) whose underlying vector space is \( A \otimes_k H \) and whose multiplication is given by

\[
(a \otimes h)(b \otimes l) = \sum_{(h)(l)} ab^{h^{(1)}} f (h^{(2)}, l^{(1)}) \otimes_k h^{(3)} l^{(2)},
\]

for all \( a, b \in A, h, l \in H \). The element \( a \otimes h \) of \( A \#_f H \) will usually be written \( a \# h \). The algebra \( A \#_f H \) is called a crossed product if it is associative with \( 1 \# 1 \) as identity element. In [B-C-M] it was proved that this happens if and only if the map \( f \) and the weak action of \( H \) on \( A \) satisfy the following conditions:

1. (Normality of \( f \)) for all \( h \in H \) we have \( f(h, 1) = f(1, h) = \epsilon(h)1_A \).
2. (Cocycle condition) for all \( h, l, m \in H \) we have
   \[
   \sum_{(h)(l)(m)} f(l^{(1)}, m^{(1)}) h^{(1)} f(h^{(2)}, l^{(2)} m^{(2)}) = \sum_{(h)(l)} f(h^{(1)}, l^{(1)}) f(h^{(2)} l^{(2)}, m),
   \]
3. (Twisted module condition) for all \( h, l \in H \) and \( a \in A \) we have
   \[
   \sum_{(h)(l)} (a h^{(1)})^{l^{(1)}} f(h^{(2)}, l^{(2)}) = \sum_{(h)(l)} f(h^{(1)}, l^{(1)}) a^{h^{(2)} l^{(2)}}.
   \]

We assume from now on that \( H \) is the enveloping algebra \( U(g) \) of a Lie algebra \( g \). In this case, item (1) of the definition of weak action implies that

\[
(ab)^x = a^x b + ab^x
\]

for each \( x \in g \) and \( a, b \in A \). So, a weak action determines a \( k \)-linear map

\[
\delta : g \to \text{Der}_k(A)
\]

by \( \delta(x)(a) = a^x \). Moreover if \( (h, a) \mapsto a^h \) is an action, then \( \delta \) is a homomorphism of Lie algebras. Conversely, given a \( k \)-linear map \( \delta : g \to \text{Der}_k(A) \), there exists a (generally non-unique) weak action of \( U(g) \) on \( A \) such that \( \delta(x)(a) = a^x \). When \( \delta \) is a homomorphism of Lie algebras, there is a unique action of \( U(g) \) on \( A \) such that \( \delta(x)(a) = a^x \). For a proof of the previous results we refer to [B-C-M]. It is immediate to prove that each normal cocycle

\[
f : U(g) \otimes_k U(g) \to A
\]

is convolution invertible. For a proof see [G-G1, Remark 1.1].

Next we recall some results and notations from [G-G1] that we will need later. Let \( K \) be a subalgebra of \( A \) which is stable under the weak action of \( g \) (that is \( \lambda^x \in K \) for all \( \lambda \in K \) and \( x \in g \)) and let \( E = A \#_f U(g) \) be a crossed product. We are going to modify the sign of some boundary maps in order to obtain simpler expressions for the comparison maps.

To begin, we fix some notations:

1. The unadorned tensor product \( \otimes \) means the tensor product \( \otimes_K \) over \( K \).
(2) For $B = A$ or $B = E$ and each $r \in \mathbb{N}$, we write $\overline{B} = B/K$,

$$B^r = B \otimes \cdots \otimes B \ (r \text{ times}) \quad \text{and} \quad \overline{B}^r = \overline{B} \otimes \cdots \otimes \overline{B} \ (r \text{ times}).$$

Moreover, for $b \in B$ we also let $\overline{b}$ denote the class of $b$ in $\overline{B}$.

(3) For each Lie algebra $g$ and $s \in \mathbb{N}$, we write $g^{\wedge s} = g \wedge \cdots \wedge g \ (s \text{ times})$.

(4) Throughout this paper we will write $\rho_{ij}$ for $a_1 \otimes \cdots \otimes a_r \in A^r$ and $x_{i\alpha}$ for $x_1 \wedge \cdots \wedge x_s \in g^{\wedge s}$.

(5) For $a_{ij}$ and $0 \leq i < j \leq r$, we write $a_{ij} = a_i \otimes \cdots \otimes a_j$.

(6) For $x_{i\alpha}$ and $1 \leq i \leq r$, we write $x_{i\alpha} = x_1 \wedge \cdots \wedge x_i \wedge \cdots \wedge x_s$.

(7) For $x_{i\alpha}$ and $1 \leq i < j \leq s$, we write $x_{i\alpha j\beta} = x_1 \wedge \cdots \wedge x_i \wedge \cdots \wedge x_j \wedge \cdots \wedge x_s$.

Let $\Lambda(g)$ be the exterior algebra generated by the $k$-vector space $g$ and let $\Lambda(g) \# U(g)$ be the smash product obtained by using the action of $U(g)$ over $\Lambda(g)$, determined by $x^\prime := [x^\prime, x]_g$. We define $Y_*$ as the algebra

$$E \otimes (\Lambda(g) \# U(g)) = (A \# f U(g)) \otimes (\Lambda(g) \# U(g)),$$

endowed with the gradation, obtained giving degree 0 to the elements

$$(a \# 1) \otimes (1 \# 1), \quad y_x := (1 \# x) \otimes (1 \# 1) \quad \text{and} \quad \rho_x := (1 \# 1) \otimes (1 \# x),$$

and degree 1 to the elements $e_x := (1 \# 1) \otimes (x \# 1)$. If we identify each $a \in A$ with $(a \# 1) \otimes (1 \# 1)$, then $Y_*$ is the extension of $A$, generated by the elements $y_x$ and $\rho_x$ of degree 0, and $e_x$, of the degree 1, subject to the relations

$$y_{\lambda x + x^\prime} = \lambda y_x + y_{x^\prime}, \quad y_x y_{x^\prime} = y_x y_{x^\prime} + y_{[y_x, y_{x^\prime}]_g} + f(y^\prime, y) - f(y, y^\prime),$$

$$\rho_{\lambda x + x^\prime} = \lambda \rho_x + \rho_{x^\prime}, \quad \rho_{x^\prime} z_y = y_x \rho_{x^\prime},$$

$$e_{\lambda x + x^\prime} = \lambda e_x + e_{x^\prime}, \quad e_{x^\prime} y_x = y_x e_{x^\prime},$$

$$y_x a = a^x + a y_x, \quad \rho_{x^\prime} \rho_x = \rho_x \rho_{x^\prime} + \rho_{[x^\prime, x]_g},$$

$$\rho_x a = a \rho_x, \quad e_{x^\prime} \rho_x = \rho_x e_{x^\prime} + e_{[x^\prime, x]_g},$$

$$e_x a = a e_x, \quad e^2_x = 0,$$

where $\lambda \in k$, $x^\prime$ and $x$ in $g$ and $[\cdot, \cdot]_g$ denotes the Lie bracket in $g$. Note that $E$ is a subalgebra of $Y_*$ via the embedding that takes $a \in A$ to $a$ and $1 \# x$ to $y_x$ for all $x \in g$. This gives rise to a structure of left $E$-module on $Y_*$. For all $x \in g$, let $z_x = y_x + \rho_x$. Since

$$z_{\lambda x + x^\prime} = \lambda z_x + z_{x^\prime},$$

$$z_x a = a^x + a z_x,$$

$$z_{x^\prime} z_x = z_x z_{x^\prime} + z_{[x^\prime, x]_g} + f(x^\prime, x) - f(x, x^\prime),$$

there is also an algebra map from $E$ to $Y_*$ that takes $a \in A$ to $a$ and $1 \# x$ to $z_x$ for all $x \in g$. This map is also an embedding, since it is a section, with a left inverse given by the algebra map from $Y_*$ to $E$, that takes $a$ to $a$, $y_x$ to $1 \# x$, $\rho_x$ to 0 and $e_x$ to 0.
Remark 1.1. The complex $Y^*$ is slightly different from the similar complex introduced in [G-G1]. However we will obtain in Theorem 1.8 the same projective resolution of $E$ as the one obtained in [G-G1]. We have two reasons to justify the present definition of $Y^*$. On one hand, it allows us to give a very simple proof of the following theorem (corresponding to [G-G1, Theorem 3.1.1]) and, on the other hand, it allows us to obtain a better contracting homotopy of the resolution that appears in Theorem 1.7. For instance the new contracting homotopy will be left $E$-linear.

Remark 1.2. In a first version of this paper we fixed in the following theorem a mistake at the beginning of Section 3.1 of [G-G1]. The error was that the weak action of $g$ on $A \otimes \Lambda(g)$ was poorly defined. Using the notation of that paper it was

$$(a \otimes e)^u = a - a \otimes e^u,$$

but should have been

$$(a \otimes e)^u = \sum_{(u)} a - a \otimes e^u.$$ 

In the current version this weak action does not appear.

Let $(g_i)_{i \in I}$ be a basis of $g$ with indexes running on an ordered set $I$. For each $i \in I$ let us write $y_i := y_{g_i}$, $z_i := z_{g_i}$, $e_i := e_{g_i}$, and $\rho_i := \rho_{g_i}$.

**Theorem 1.3.** Each $Y_s$ is a free left $E$-module with basis

$$\rho_{i_1}^{m_1} e_{i_1}^{\delta_1} \cdots \rho_{i_l}^{m_l} e_{i_l}^{\delta_l} \quad (l \geq 0, \ i_1 < \cdots < i_l \in I, \ m_j \geq 0, \ \delta_j \in \{0, 1\}, \ m_j + \delta_j > 0, \ \delta_1 + \cdots + \delta_l = s).$$

**Proof.** It is sufficient to see that

$$\overline{\rho}_{i_1}^{m_1} \bar{e}_{i_1}^{\delta_1} \cdots \overline{\rho}_{i_l}^{m_l} \bar{e}_{i_l}^{\delta_l} \quad (l \geq 0, \ i_1 < \cdots < i_l \in I, \ m_j \geq 0, \ \delta_j \in \{0, 1\}, \ m_j + \delta_j > 0, \ \delta_1 + \cdots + \delta_l = s),$$

where $\overline{\rho}_i := 1#x_i$ and $\bar{e}_i := x_i#1$, is a basis of $\Lambda(g)#U(g)$ as a $k$-vector space, which follows easily from the fact that

$$x_{j_1} \wedge \cdots \wedge x_{j_s} \quad (j_1 < \cdots < j_l \in I)$$

is a basis of $g^{\wedge s}$ and, by the Poincaré–Birkhoff–Witt theorem,

$$x_{i_1}^{m_1} \cdots x_{i_l}^{m_l} \quad (l \geq 0, \ i_1 < \cdots < i_l \in I, \ m_j \geq 0)$$

is a basis of $U(g)$. □

**Remark 1.4.** A similar, but more involved argument, shows that each $Y_s$ is a free right $E$-module with the same basis. We will not use this result.

**Remark 1.5.** The following result improves [G-G1, Theorem 3.1.3] in the sense that in the current version we obtain that the complex introduced there is contractible as a complex of $(A, E)$-bimodules and not only as a complex of $k$-modules.
Theorem 1.6. Let $\tilde{\mu} : Y_0 \rightarrow E$ be the algebra map defined by $\tilde{\mu}(a) = a$ for $a \in A$ and $\tilde{\mu}(y_i) = \tilde{\mu}(z_i) = 1 \# g_i$ for $i \in I$. There is a unique derivation $\partial : Y_0 \rightarrow Y_{s-1}$ such that $\partial(e_i) = \rho_i$ for $i \in I$. Moreover, the chain complex of $E$-bimodules

$$E \xleftarrow{\tilde{\mu}} Y_0 \xleftarrow{\partial_1} Y_1 \xleftarrow{\partial_2} Y_2 \xleftarrow{\partial_3} Y_3 \xleftarrow{\partial_4} Y_4 \xleftarrow{\partial_5} Y_5 \xleftarrow{\partial_6} \cdots$$

is contractible as a complex of $(E, A)$-bimodules. A chain contracting homotopy

$$\sigma_0^{-1} : E \rightarrow Y_0, \quad \sigma_{s+1}^{-1} : Y_s \rightarrow Y_{s+1} \quad (s \geq 0)$$

is given by

$$\sigma^{-1}(1) = 1,$$

$$\sigma^{-1}(\rho_{i_1}^{m_1} \delta_{i_1} \cdots \rho_{i_l}^{m_l} e_{i_l}) = \begin{cases} (-1)^s \rho_{i_1}^{m_1} e_{i_1} \cdots \rho_{i_{l-1}}^{m_{l-1}} e_{i_{l-1}} \rho_{i_l}^{m_l-1} e_{i_l} & \text{if } \delta_l = 0, \\
0 & \text{if } \delta_l = 1,
\end{cases}$$

where we assume that $i_1 < \cdots < i_l$, $\delta_1 + \cdots + \delta_l = s$ and $m_l + \delta_l > 0$.

Proof. A direct computation shows that

\begin{itemize}
  \item $\mu \circ \sigma^{-1}(1) = \tilde{\mu}(1) = 1$.
  \item $\sigma^{-1} \circ \tilde{\mu}(1) = \sigma^{-1}(1) = 1$ and $\partial \circ \sigma^{-1}(1) = \partial(0) = 0$.
  \item If $x = x' \rho_{i_l}^{m_l} e_{i_l}$, where $m_l > 0$ and $x' = \rho_{i_1}^{m_1} \delta_{i_1} \cdots \rho_{i_{l-1}}^{m_{l-1}} e_{i_{l-1}}$ with $i_1 < \cdots < i_l$, then
    $$\sigma^{-1} \circ \tilde{\mu}(x) = \sigma^{-1}(0) = 0 \quad \text{and} \quad \partial \circ \sigma^{-1}(x) = \partial(x' \rho_{i_l}^{m_l-1} e_{i_l}) = x.$$
  \item If $x = x' \rho_{i_l}^{m_l} e_{i_l}$, where $m_l + \delta_l > 0$ and $x' = \rho_{i_1}^{m_1} \delta_{i_1} \cdots \rho_{i_{l-1}}^{m_{l-1}} e_{i_{l-1}}$ with $i_1 < \cdots < i_l$ and $\delta_1 + \cdots + \delta_l = s > 0$. If $\delta_l = 0$, then
    $$\sigma^{-1} \circ \partial(x) = \sigma^{-1}(\partial(x') \rho_{i_l}^{m_l}) = (-1)^s \partial(x') \rho_{i_l}^{m_l-1} e_{i_l},$$
    $$\partial \circ \sigma^{-1}(x) = \partial((-1)^s x' \rho_{i_l}^{m_l-1} e_{i_l}) = (-1)^s \partial(x') \rho_{i_l}^{m_l-1} e_{i_l} + x,$$
    and if $\delta_l = 1$, then
    $$\sigma^{-1} \circ \partial(x) = \sigma^{-1}(\partial(x') \rho_{i_l}^{m_l} e_{i_l} + (-1)^s x' \rho_{i_l}^{m_l-1}) = x,$$
    $$\partial \circ \sigma^{-1}(x) = \partial(0) = 0.$$ \hspace{1cm} \square
\end{itemize}

The result follows immediately.

For each $s \geq 0$ we consider $E \otimes_k g^{\wedge s}$ as a right $K$-module via $(c \otimes_k x) \lambda = c \lambda \otimes_k x$. For $r, s \geq 0$, let

$$X_{rs} = (E \otimes_k g^{\wedge s}) \otimes \bar{A}^r \otimes E.\) The groups $X_{rs}$ are $E$-bimodules in an obvious way. Let us consider the diagram of $E$-bimodules and $E$-bimodule maps.
where $\mu_s : X_{0s} \to Y_s$ and $d^0_{ss} : X_{ss} \to X_{s-1,ss}$ are defined by:

$$
\mu(1 \otimes_k x_{1s} \otimes 1) = e_{x_1} \cdots e_{x_s},
$$

$$
d^0(1 \otimes_k x_{1s} \otimes a_{1r} \otimes 1) = (-1)^s a_1 \otimes_k x_{1s} \otimes a_{2r} \otimes 1
$$

$$
- \sum_{i=1}^{r-1} (-1)^{i+s} \otimes_k x_{i-1} \otimes a_{1i-1} \otimes a_{i+1} \otimes a_{i+1,r} \otimes 1
$$

$$
+ (-1)^{r+s} \otimes_k x_{1s} \otimes a_{1,r-1} \otimes a_r.
$$

Each horizontal complex in this diagram is contractible as a complex of $(E, K)$-bimodules. A chain contracting homotopy is the family $\sigma_{0s} : Y_s \to X_{0s}$, $\sigma_{r+1,s} : X_{rs} \to X_{r+1,s}$ for $r \geq 0$, of $(E, K)$-bimodule maps, defined by

$$
\sigma^0(1 \otimes_k x_{1s} \otimes a_1r \otimes a_{r+1} \# w) = (1 \# x_{s+1}) \cdots (1 \# x_0),
$$

where $\sum_j a_j \# w_j = (1 \# x_{s+1}) \cdots (1 \# x_0)$, and

$$
\sigma^0(1 \otimes_k x_{1s} \otimes a_1r \otimes a_{r+1} \# w) = (-1)^{r+s+1} \otimes_k x_{1s} \otimes a_{1, r+1} \otimes 1 \# w
$$

for $r \geq 0$. (In order to prove that the $\sigma^0$s are right $K$-linear it is necessary to use that $K$ is stable under the action of $g$.) Moreover, each $X_{rs}$ is a projective $E$-bimodule relative to the family of all epis of $E$-bimodules which split as $(E, K)$-bimodule maps. We define $E$-bimodule maps

$$
d^l_{rs} : X_{rs} \to X_{r+l-1,s-l} \quad (r \geq 0 \text{ and } 1 \leq l \leq s)
$$

recursively by:

$$
d^l(y) = \begin{cases} 
-\sigma^0 \circ d \circ \mu(y) & \text{if } l = 1 \text{ and } r = 0, \\
-\sigma^0 \circ d^1 \circ d^0(y) & \text{if } l = 1 \text{ and } r > 0, \\
-\sum_{j=1}^{l-1} \sigma^0 \circ d^{-j} \circ d^j(y) & \text{if } l > 1 \text{ and } r = 0, \\
-\sum_{j=0}^{l-1} \sigma^0 \circ d^{-j} \circ d^j(y) & \text{if } l > 1 \text{ and } r > 0,
\end{cases}
$$

where $y = 1 \otimes_k x_{1s} \otimes a_{1r} \otimes 1$. 

Theorem 1.7. The complex

\[
\begin{array}{cccccccc}
E & \xrightarrow{\bar{\pi}} & X_0 & \xleftarrow{d_1} & X_1 & \xleftarrow{d_2} & X_2 & \xleftarrow{d_3} & X_3 & \xleftarrow{d_4} & X_4 & \xleftarrow{d_5} & X_5 & \xleftarrow{d_6} & \ldots
\end{array}
\]

(1)

where

\[
\bar{\pi}(1 \otimes 1) = 1, \quad X_n = \bigoplus_{r+s=n} X_{rs} \quad \text{and} \quad d_n = \sum_{r+s=n} \sum_{l=0}^{s} d_{rs}.
\]

is a projective resolution of the \(E\)-bimodule \(E\), relative to the family of all epimorphisms of \(E\)-bimodules which split as \((E, K)\)-bimodule maps. Moreover an explicit contracting homotopy

\[
\bar{\sigma}_0 : E \to X_0, \quad \bar{\sigma}_{n+1} : X_n \to X_{n+1} \quad (n \geq 0)
\]

of (1), as a complex of \((E, K)\)-bimodules, is given by

\[
\bar{\sigma}_0 = \sigma^0 \circ \sigma_0^{-1} \quad \text{and} \quad \bar{\sigma}_{n+1} = - \sum_{l=0}^{n+1} \sigma^l_{n-l+1} \circ \sigma_{n+1}^{-l} + \sum_{r=0}^{n} \sum_{l=0}^{n-r} \sigma^l_{r+l+1, n-l-r},
\]

where

\[
\sigma^l_{i,s-i} : Y_s \to X_{i,s-i} \quad \text{and} \quad \sigma^l_{i+l+1, s-l} : X_{rs} \to X_{i+l+1, s-l} \quad (0 \leq l \leq s, \ r \geq 0)
\]

are recursively defined by

\[
\sigma^l = - \sum_{j=0}^{l-1} \sigma^0 \circ d^{-j} \circ \sigma^j.
\]

Proof. It follows from [G-G2, Corollary A.2]. \(\Box\)

The boundary maps of the projective resolution of \(E\) that we just found are defined recursively. Next we give closed formulas for them.

Theorem 1.8. For \(x_i, x_j \in g\), we put \(\hat{f}_{ij} = f(x_i, x_j) - f(x_j, x_i)\). We have:

\[
d^1(1 \otimes_k x_{1s} \otimes a_{1r} \otimes 1) = \sum_{i=1}^{s} (-1)^{i+1} x_i \otimes_k x_{1s} \otimes a_{1r} \otimes 1
\]

\[
+ \sum_{i=1}^{s} (-1)^i \otimes_k x_{1s} \otimes a_{1r} \otimes 1 \# x_i
\]

\[
+ \sum_{i=1}^{s} (-1)^i \otimes_k x_{1s} \otimes a_{1r} \otimes 1 \# x_i
\]

\[
+ \sum_{i=1}^{s} (-1)^{i+j} \otimes_k [x_i, x_j] \otimes x_{1s} \otimes a_{1r} \otimes 1
\]

\[
+ \sum_{1 \leq i < j \leq s} (-1)^{i+j} \otimes_k \left[ x_i, x_j \right] \otimes x_{1s} \otimes a_{1r} \otimes 1.
\]
\[ d^2(1 \otimes_k x_{1s} \otimes a_{1r} \otimes 1) = \sum_{1 \leq i < j \leq s} (-1)^{i+j+s} \otimes_k x_{1ij} \otimes a_{1h} \otimes \hat{f}_{ij} \otimes a_{h+1,r} \otimes 1 \]

and \(d^l = 0\) for all \(l \geq 3\).

**Proof.** The proof of [G-G1, Theorem 3.3] works in our more general context. \(\square\)

2. The comparison maps

In this section we introduce and study comparison maps between \((X_s, d_s)\) and the canonical normalized Hochschild resolution \((E \otimes \tilde{E}^s \otimes E, b'_s)\) of the \(K\)-algebra \(E\). It is well known that there are morphisms of \(E\)-bimodule complexes

\[
\theta_s : (X_s, d_s) \to (E \otimes \tilde{E}^s \otimes E, b'_s) \quad \text{and} \quad \vartheta_s : (E \otimes \tilde{E}^s \otimes E, b'_s) \to (X_s, d_s),
\]

such that \(\theta_0 = \vartheta_0 = \text{id}_{E \otimes E}\) and that these morphisms are inverse of each other up to homotopy. They can be recursively defined by \(\theta_0 = \vartheta_0 = \text{id}_{E \otimes E}\) and

\[
\theta(1 \otimes_k x_{1s} \otimes a_{1r} \otimes 1) = (-1)^n \theta \circ d(1 \otimes_k x_{1s} \otimes a_{1r} \otimes 1) \otimes 1
\]

and

\[
\vartheta(1 \otimes c_{1n} \otimes 1) = \vartheta \circ b'(1 \otimes c_{1n} \otimes 1),
\]

for \(n \geq 1\), where \(r + s = n\) and \(c_{1n} = c_1 \otimes \cdots \otimes c_n \in \tilde{E}^n\). The following result was established without proof in [G-G1].

**Proposition 2.1.** We have:

\[
\theta(1 \otimes_k x_{1s} \otimes a_{1r} \otimes 1) = \sum_{\tau \in \mathfrak{S}_s} \text{sg}(\tau) \otimes (1 \# x_{\tau(1)} \otimes \cdots \otimes 1 \# x_{\tau(s)}) \ast a_{1r} \otimes 1,
\]

where \(\mathfrak{S}_s\) is the symmetric group in \(s\) elements and \(\ast\) denotes the shuffle product, which is defined by

\[
(\beta_1 \otimes \cdots \otimes \beta_s) \ast (\beta_{s+1} \otimes \cdots \otimes \beta_n) = \sum_{\sigma \in \{s, n-s\}-\text{shuffles}} \text{sg}(\sigma) \beta_{\sigma(1)} \otimes \cdots \otimes \beta_{\sigma(n)}.
\]

**Proof.** We proceed by induction on \(n = r + s\). The case \(n = 0\) is obvious. Suppose that \(r + s = n\) and the result is valid for \(\theta_{n-1}\). By the recursive definition of \(\theta\), Theorem 1.8, and the inductive hypothesis we obtain that:

\[
\theta(1 \otimes_k x_{1s} \otimes a_{1r} \otimes 1) = (-1)^n \theta \circ d(1 \otimes_k x_{1s} \otimes a_{1r} \otimes 1) \otimes 1
\]

\[
= (-1)^n \theta \circ (d^0 + d^1 + d^2)(1 \otimes_k x_{1s} \otimes a_{1r} \otimes 1) \otimes 1
\]

\[
= \theta(1 \otimes_k x_{1s} \otimes a_{1r-1} \otimes a_r) \otimes 1
\]

\[
+ \theta \left( \sum_{i=1}^s (-1)^{i+n} \otimes_k x_{1is} \otimes a_{1r} \otimes 1 \# x_i \right) \otimes 1.
\]

The desired result follows now using again the inductive hypothesis. \(\square\)
Lemma 2.2. Let \((g_i)_{i \in I}\) be the basis of \(g\) considered in Theorem 1.3. As in that theorem, let us write \(e_i = e_{g_i}\) for each \(i \in I\). The following facts hold:

1. \(\sigma_{n+1} \circ \sigma_n = 0\) for all \(n \geq 0\).
2. \(\sigma^l((E \otimes_k g^{I^L}) \otimes K^\perp g) = 0\) for all \(0 \leq l \leq s\).
3. \(\sigma^l(e_i \cdots e_{i_n}) = 0\) for all \(0 < l \leq n\).
4. \(\sigma^l((E \otimes_k g^{I^L}) \otimes K^\perp A) = 0\) for all \(0 < l \leq s\).
5. \(\sigma^{-1} \circ \mu(A \otimes_k g^{I^L}) = 0\).
6. Assume that \(i_1 < \cdots < i_n\). Then,

\[
\sigma^{-1} \circ \mu(1 \otimes_k g_{i_1} \wedge \cdots \wedge g_{i_n} \otimes 1#g_{i_{n+1}}) = \begin{cases} (-1)^n e_{i_1} \cdots e_{i_{n+1}} & \text{if } i_n < i_{n+1}, \\ 0 & \text{otherwise}. \end{cases}
\]

Proof. (1) An inductive argument shows that there are maps (which are left \(E\)-linear and right \(K\)-linear)

\[
y_{r,s}^l : X_{r+1,s} \rightarrow X_{r+l,s-l},
\]

such that \(\sigma_{r+l+1,s-l}^l = \sigma_{r+l+1,s-l}^0 \circ \gamma_{r,s}^l \circ \sigma_{r,s}^0\). Because of \(\sigma^0 \circ \sigma^0 = 0\), this implies that \(\sigma^l \circ \sigma^l = 0\), for all \(l, l' \geq 0\). Thus,

\[
\sigma_{n+1} \circ \sigma_n = \sum_{l=0}^{n+1} \sigma^l \circ \sigma^{-1} \circ \mu \circ \sigma^0 \circ \sigma^{-1} \circ \mu = 0,
\]

where the last equality holds because \(\mu \circ \sigma^0 = \text{id} \) and \(\sigma^{-1} \circ \sigma^{-1} = 0\).

(2) Since \(\sigma^l = \sigma^0 \circ \gamma^l \circ \sigma^0\) for \(l > 0\), we can assume that \(l = 0\). In this case the assertion follows immediately from the definition of \(\sigma^0\).

(3) By the definition of \(\sigma^0\) and Theorem 1.8,

\[
\sigma^0 \circ d^1 \circ \sigma^0(e_{i_1} \cdots e_{i_n}) = \sigma^0 \circ d^1(1 \otimes_k g_{i_1} \wedge \cdots \wedge g_{i_n} \otimes 1) = 0
\]

and

\[
\sigma^0 \circ d^2 \circ \sigma^0(e_{i_1} \cdots e_{i_n}) = \sigma^0 \circ d^2(1 \otimes_k g_{i_1} \wedge \cdots \wedge g_{i_n} \otimes 1) = 0.
\]

Item (3) follows now easily by induction on \(l\), since, by the recursive definition of \(\sigma^l\) and Theorem 1.8,

\[
\sigma^1 = -\sigma^0 \circ d^1 \circ \sigma^0 \quad \text{and} \quad \sigma^l = -\sigma^0 \circ d^1 \circ \sigma^{l-1} - \sigma^0 \circ d^2 \circ \sigma^{l-2} \quad \text{for } l \geq 2.
\]

(4) It is similar to the proof of item (3).

(5) Since \(e_i a = ae_i\) for all \(i \in I\) and \(a \in A\),

\[
\sigma^{-1} \circ \mu(a \otimes_k g_{i_1} \wedge \cdots \wedge g_{i_n} \otimes a') = \sigma^{-1}(ae_{i_1} \cdots e_{i_n} a') = \sigma^{-1}(aa'e_{i_1} \cdots e_{i_n}) = 0.
\]

where the last equality follows from the definition of \(\sigma^{-1}\).
(6) We have

$$\sigma^{-1} \circ \mu(1 \otimes_k g_{i_1} \wedge \ldots \wedge g_{i_n} \otimes 1 \# g_{i_{n+1}}) = \sigma^{-1}(e_{i_1} \ldots e_{i_n} z_{i_{n+1}})$$

$$= \sigma^{-1}(e_{i_1} \ldots e_{i_n} (y_{i_{n+1}} + \rho_{i_{n+1}}))$$

$$= \sigma^{-1}(y_{i_{n+1}} e_{i_1} \ldots e_{i_n}) + \sigma^{-1}(e_{i_1} \ldots e_{i_n} \rho_{i_{n+1}}),$$

where $z_{i_{n+1}}, y_{i_{n+1}}$ and $\rho_{i_{n+1}}$ are as in Theorem 1.3. So, in order to finish the proof it suffices to note that $\sigma^{-1}(y_{i_{n+1}} e_{i_1} \ldots e_{i_n}) = 0$ and

$$\sigma^{-1}(e_{i_1} \ldots e_{i_n} \rho_{i_{n+1}}) = \begin{cases} (-1)^n e_{i_1} \ldots e_{i_{n+1}} & \text{if } i_n < i_{n+1}, \\ 0 & \text{otherwise,} \end{cases}$$

which follows immediately from

$$e_{i_j} \rho_{i_{n+1}} = \rho_{i_{n+1}} e_{i_j} + e_{[x_{i_j}, x_{i_{n+1}}]} 1_g$$

for all $j$ such that $i_j > i_{n+1},$

and the definition of $\sigma^{-1}$. □

**Theorem 2.3.** Let $(g_i)_{i \in I}$ be the basis of $\mathfrak{g}$ considered in Theorem 1.3. Assume that $c_{i_1} = c_1 \otimes \cdots \otimes c_n \in \mathbb{F}^n$ is a simple tensor with $c_j \in A \cup \{1 \# g_i: i \in I\}$ for all $j \in \{1, \ldots, n\}$. If there exist $0 \leq s \leq n$ and $i_1 < \cdots < i_s$ in $I$, such that $c_j = 1 \# g_{i_j}$ for $1 \leq j \leq s$ and $c_j \in A$ for $s < j \leq n$, then

$$\vartheta(1 \otimes c_{i_1} \otimes 1) = 1 \otimes_k g_{i_1} \wedge \ldots \wedge g_{i_s} \otimes c_{s+1,n} \otimes 1.$$  

Otherwise, $\vartheta(1 \otimes c_{i_1} \otimes 1) = 0$.

**Proof.** For all $n \geq 0$ we define $P_n$ by $c_{i_1} \in P_n$ if there are $i_1 < \cdots < i_n$ in $I$ such that $c_j = 1 \# g_{i_j}$ for $j \leq n$ and $c_j \in A$ for $j > n$. We now proceed by induction on $n$. The case $n = 0$ is immediate. Assume that the result is valid for $\vartheta_n$. By item (1) of Lemma 2.2 and the recursive definition of $\vartheta_n$, we have

$$\overline{\sigma} \circ \vartheta_n = \overline{\sigma} \circ \vartheta \circ b'(c_{i_1} \otimes 1) = 0,$$

and so

$$\vartheta(1 \otimes c_{1,n+1} \otimes 1) = (-1)^n \overline{\sigma} \circ \vartheta(1 \otimes c_{1,n+1}).$$

Assume that $c_j \in A \cup \{1 \# g_i: i \in I\}$ for all $j \in \{1, \ldots, n + 1\}$. In order to finish the proof it suffices to show that:

- If $c_{1,n+1} \notin P_{n+1}$, then $\overline{\sigma} \circ \vartheta(1 \otimes c_{1,n+1}) = 0$,
- If $c_{1,n+1} = 1 \# g_{i_1} \otimes \cdots \otimes 1 \# g_{i_s} \otimes a_{s+1,n+1} \in P_{n+1}$, then

$$\overline{\sigma} \circ \vartheta(1 \otimes c_{1,n+1}) = (-1)^n \otimes_k g_{i_1} \wedge \cdots \wedge g_{i_s} \otimes a_{s+1,n+1} \otimes 1.$$

If $c_{i_1} \notin P_n$, then $\vartheta(1 \otimes c_{1,n+1}) = 0$ by the inductive hypothesis. It remains to consider the case $c_{i_1} \in P_n$. We divide this into three subcases.
(1) If \( c_{1n} = 1 \# g_{l_1} \otimes \cdots \otimes 1 \# g_{l_s} \otimes a_{s+1,n} \) and \( c_{n+1} = a_{n+1} \in A \), then

\[
\overline{\sigma} \circ \vartheta (1 \otimes c_{1,n+1}) = \sigma (1 \otimes_k (g_{l_1} \wedge \cdots \wedge g_{l_s} \otimes a_{s+1,n+1}))
\]

\[
= \sigma^0 (1 \otimes_k (g_{l_1} \wedge \cdots \wedge g_{l_s} \otimes a_{s+1,n+1}))
\]

\[
= (-1)^{n+1} \otimes_k (g_{l_1} \wedge \cdots \wedge g_{l_s} \otimes a_{s+1,n+1} \otimes 1),
\]

by the inductive hypothesis, items (4) and (5) of Lemma 2.2, and the definitions of \( \overline{\sigma} \) and \( \sigma^0 \).

(2) If \( c_{1n} = 1 \# g_{l_1} \otimes \cdots \otimes 1 \# g_{l_s} \otimes a_{s+1,n} \) with \( s < n \) and \( c_{n+1} = 1 \# g_{l_{n+1}} \), then

\[
\overline{\sigma} \circ \vartheta (1 \otimes c_{1,n+1}) = \sigma (1 \otimes_k (g_{l_1} \wedge \cdots \wedge g_{l_s} \otimes a_{s+1,n} \otimes 1 \# g_{l_{n+1}})) = 0,
\]

by the inductive hypothesis, the definition of \( \overline{\sigma} \) and item (2) of Lemma 2.2.

(3) If \( c_{1n} = 1 \# g_{l_1} \otimes \cdots \otimes 1 \# g_{l_s} \) and \( c_{n+1} = 1 \# g_{l_{n+1}} \), then

\[
\overline{\sigma} \circ \vartheta (1 \otimes c_{1,n+1}) = \sigma (1 \otimes_k (g_{l_1} \wedge \cdots \wedge g_{l_m} \otimes 1 \# g_{l_{n+1}}))
\]

\[
= \sigma^0 (1 \otimes_k (g_{l_1} \wedge \cdots \wedge g_{l_m} \otimes 1 \# g_{l_{n+1}}))
\]

\[
= \left\{ \begin{array}{ll}
(-1)^{n+1} \otimes_k (g_{l_1} \wedge \cdots \wedge g_{l_{m+1}} \otimes 1) & \text{if } c_{1,n+1} \in P_{n+1}, \\
0 & \text{otherwise},
\end{array} \right.
\]

by the inductive hypothesis, items (2), (3) and (6) of Lemma 2.2, and the definitions of \( \overline{\sigma} \) and \( \sigma^0 \). \( \square \)

3. The Hochschild cohomology

Let \( E = A \# U(g) \) and let \( M \) be an \( E \)-bimodule. In this section we obtain a cochain complex \((\mathcal{X}_K^*(M), \overline{a}^s)\), simpler than the canonical one, giving the Hochschild cohomology of the \( K \)-algebra \( E \) with coefficients in \( M \). When \( K = k \) our result reduces to the one obtained in [G-G1, Section 5]. Then, we obtain an expression that gives the cup product of the Hochschild cohomology of \( E \) in terms of \((\mathcal{X}_K^*(E), \overline{a}^s)\). As usual, given \( c \in E \) and \( m \in M \), we let \([m, c] \) denote the commutator \( mc - cm \).

3.1. The complex \((\mathcal{X}_K^*(M), \overline{a}^s)\)

For \( r, s \geq 0 \), let

\[
\mathcal{X}_K^{rs}(M) = \text{Hom}_K(\overline{A}^r \otimes_k g^\otimes s, M).
\]

where \( \overline{A}^r \otimes_k g^\otimes s \) is considered as a \( K \)-bimodule via the canonical actions on \( \overline{A}^r \). We define the morphism

\[
\overline{a}_l^s : \mathcal{X}_K^{r+1,s-l}(M) \rightarrow \mathcal{X}_K^{rs}(M) \quad \text{(with } 0 \leq l \leq \min(2, s) \text{ and } r + l > 0 \text{)}
\]

by:

\[
\overline{a}_0(\varphi)(a_{1_r} \otimes_k x_{1_s}) = a_1 \varphi(a_{2_r} \otimes_k x_{1_s})
\]

\[
+ \sum_{i=1}^{r-1} (-1)^i \varphi(a_{1,r-1} \otimes a_{i+1} \otimes a_{i+2,r} \otimes_k x_{1_s})
\]

\[
+ (-1)^r \varphi(a_{1,r-1} \otimes_k x_{1_s}) a_r,
\]
Theorem 3.1. The Hochschild cohomology $H^n_K(E, M)$ of the $K$-algebra $E$ with coefficients in $M$, is the cohomology of $(\bar{X}_K^s(M), d^s)$. 

Proof. It is an immediate consequence of the above discussion. □

3.2. The comparison maps

The maps $\theta_s$ and $\varphi_s$, introduced in Section 2, induce quasi-isomorphisms

$$\vartheta^* : (\text{Hom}_K^\ast(\bar{E}^s, M), b^s) \to (\bar{X}_K^s(M), d^s)$$

and

$$\overline{\vartheta}^* : (\bar{X}_K^s(M), d^s) \to (\text{Hom}_K^\ast(E^s, M), b^s)$$

which are inverse of each other up to homotopy.
Proposition 3.2. We have
\[
\overline{\theta}(\psi)(a_{1r} \otimes_k x_{1s}) = \sum_{\tau \in \mathcal{S}_s} (-1)^{rs} \text{sg}(\tau) \psi \left((1\#x_{\tau(1)} \otimes \cdots \otimes 1\#x_{\tau(s)}) * a_{1r}\right).
\]

Proof. This follows immediately from Proposition 2.1. \( \square \)

In the sequel we consider immediately that \( \overline{X}^r_K \subseteq \overline{X}^{r+s}_K \) in the canonical way.

Theorem 3.3. Let \((g_i)_{i \in I}\) be the basis of \( g \) considered in Theorem 1.3 and let \( \varphi \in \overline{X}^{r+s}_K \). Assume that \( c_{1,r+s} = c_1 \otimes \cdots \otimes c_{r+s} \in E^{r+s} \) is a simple tensor with \( c_j \in A \cup \{1\#g_i: i \in I\} \) for all \( j \in \{1, \ldots, r+s\} \). If \( c_j = 1\#g_{i_j} \) with \( i_1 < \cdots < i_s \) in \( I \) for \( 1 \leq j \leq s \) and \( c_j \in A \) for \( s < j \leq r+s \), then
\[
\overline{\theta}(\varphi)(c_{1,r+s}) = (-1)^{rs} \varphi(c_{s+1,r+s} \otimes_k g_{i_1} \wedge \cdots \wedge g_{i_s}).
\]

Otherwise, \( \overline{\theta}(\varphi)(c_{1,r+s}) = 0 \).

Proof. This follows immediately from Theorem 2.3. \( \square \)

As usual, in the following subsection we will write \( \text{HH}^*_K(E) \) instead of \( \text{HH}^*_K(E,E) \).

3.3. The cup product

Recall that the cup product of \( \text{HH}^*_K(E) \) is given in terms of \( (\text{Hom}_{K^*}(E^*, E), b^*) \), by
\[
\left( \psi \smile \psi' \right)(c_{1,m+n}) = \psi(c_{1m}) \psi'(c_{m+1,m+n}).
\]

where \( \psi \in \text{Hom}_{K^*}(\widetilde{E}^m, E) \) and \( \psi' \in \text{Hom}_{K^*}(\widetilde{E}^0, E) \). In this subsection we compute the cup product of \( \text{HH}^*_K(E) \) in terms of the small complex \((\overline{X}^*_K(E), \overline{d}^*)\). Given
\[
\varphi \in \overline{X}^{r+s}_K(E) \quad \text{and} \quad \varphi' \in \overline{X}^{r'+s'}_K(E)
\]
we define \( \varphi \bullet \varphi' \in \overline{X}^{r+r'+s+s'}_K(E) \) by
\[
\left( \varphi \bullet \varphi' \right)(a_{1r'} \otimes_k x_{1s'}) = \sum_{1 \leq j_1 < \cdots < j_s \leq s'} \text{sg}(j_{1s}) \varphi(a_{1r} \otimes_k x_{j_{1s}}) \varphi'(a_{r+1,r''} \otimes_k x_{j_{s'}}),
\]
where
\begin{itemize}
  \item \( \text{sg}(j_{1s}) = (-1)^{r's + \sum_{u=1}^s (j_u - b)} \),
  \item \( r'' = r + r' \) and \( s'' = s + s' \),
  \item \( 1 \leq l_1 < \cdots < l_{s'} \leq s'' \) denote the set defined by
  \[
  \{j_1, \ldots, j_s\} \cup \{l_1, \ldots, l_{s'}\} = \{1, \ldots, s''\},
  \]
  \item \( x_{j_{1s}} = x_{j_1} \wedge \cdots \wedge x_{j_s} \) and \( x_{l_{1s'}} = x_{l_1} \wedge \cdots \wedge x_{l_{s'}} \).
\end{itemize}

Theorem 3.4. The cup product of \( \text{HH}^*_K(E) \) is induced by the operation \( \bullet \) in the complex \((\overline{X}^*_K(E), \overline{d}^*)\).
Proof. Let \( \varphi \in \mathbb{X}^s_c(E) \) and \( \varphi' \in \mathbb{X}^{s'}_c(E) \). Let \( r'' \) and \( s'' \) be natural numbers satisfying \( r'' + s'' = r + r' + s + s' \) and let \( \mathbf{a}_{1^{r''}} \otimes_k \mathbf{x}_{1^{s''}} \in \mathbb{X}^K_{r+s} \). Let \( (g_i)_{i \in I} \) be the basis of \( g \) considered in Theorem 1.3. Clearly we can assume that there exist \( i_1 < \ldots < i_{s'} \) in \( I \) such that \( x_j = g_{i_j} \) for all \( 1 \leq j \leq s' \). By Proposition 3.2,

\[
\bar{\theta}(\varphi - \varphi')(\mathbf{a}_{1^{r''}} \otimes_k \mathbf{x}_{1^{s''}}) = (\bar{\theta}(\varphi) - \bar{\theta}(\varphi'))(T)
\]

where

\[
T = \sum_{\tau \in \Sigma_{s''}} (-1)^{r'' s''} s g(\tau) ((1^#x_1) \otimes \cdots \otimes (1^#x_{s''})) \ast \mathbf{a}_{1^{r''}}.
\]

In order to finish the proof it suffices to note that by Theorem 3.3, this is zero if \( r'' \neq r + r' \) and this is \( (\varphi \ast \varphi')(\mathbf{a}_{1^{r''}} \otimes_k \mathbf{x}_{1^{s''}}) \) if \( r'' = r + r' \).

4. The Hochschild homology

Let \( E = A \# U(g) \) and let \( M \) be an \( E \)-bimodule. In this section we obtain a chain complex \((\mathbb{X}^K_c(M), \bar{d}_a)\), simpler than the canonical one, giving the Hochschild homology of the \( K \)-algebra \( E \) with coefficients in \( M \). When \( K = k \) our result reduces to the one obtained in [G-G1, Section 4]. Then, we obtain an expression that gives the cap product of \( H^i_c(M) \) in terms of \((\mathbb{X}^K_c(E), \bar{d}^a)\) and \((\mathbb{X}^K_c(E, M), \bar{d}_a)\). As in the previous section \([m, c] \) denotes the commutator \( mc - cm \) of \( m \in M \) and \( c \in E \).

4.1. The complex \((\mathbb{X}^K_c(M), \bar{d}_a)\)

For \( r, s \geq 0 \), let

\[
\mathbb{X}^K_{r+s}(M) = \frac{M \otimes \bar{A}^r}{[M \otimes \bar{A}^r, K]} \otimes g^{s},
\]

where \([M \otimes \bar{A}^r, K]\) is the \( k \)-vector space generated by the commutators \([m \otimes \mathbf{a}_r, \lambda]\), with \( \lambda \in K \) and \( m \otimes \mathbf{a}_r \in M \otimes \bar{A}^r \). We let \( \bar{m} \otimes \mathbf{a}_r \) denote the class of \( m \otimes \mathbf{a}_r \) in \( M \otimes \bar{A}^r/\{M \otimes \bar{A}^r, K\} \). We define the morphism

\[
\bar{d}_{r,s}: \mathbb{X}^K_{r+s}(M) \to \mathbb{X}^K_{r+s+1}(M) \quad (\text{with } 0 \leq l \leq \min(2, s) \text{ and } r + l > 0)
\]

by:

\[
\bar{d}^0 (\bar{m} \otimes \mathbf{a}_r \otimes_k \mathbf{x}_s) = \frac{\bar{m} \mathbf{a}_1 \otimes \mathbf{a}_{2^s} \otimes_k \mathbf{x}_s}{\mathbf{a}_r} + \sum_{i=1}^{r-1} (-1)^i \bar{m} \otimes \mathbf{a}_1 \otimes_k \mathbf{x}_{i+1} \otimes_k \mathbf{x}_s + \cdots + \sum_{i=1}^{s} (-1)^{i+s} \mathbf{a}_r \otimes_k \mathbf{x}_s,
\]

\[
\bar{d}^1 (\bar{m} \otimes \mathbf{a}_r \otimes_k \mathbf{x}_s) = \sum_{i=1}^{s} (-1)^{i+s} \left[ \mathbf{a}_r \otimes_k \mathbf{x}_{s+1} \right] - \sum_{i=1}^{s} (-1)^{i+s} \mathbf{a}_r \otimes_k \mathbf{x}_s + \sum_{1 \leq h \leq r} (-1)^{i+s} \mathbf{a}_r \otimes_k \mathbf{x}_s.
\]
and
\[
d^2(m \otimes a_1r \otimes_k x_{1s}) = \sum_{1 \leq i < j \leq s} (-1)^{i+j+h} m \otimes a_{1h} \otimes \hat{f}_{ij} \otimes a_{h+1, r} \otimes_k x_{ij,s},
\]
where \( \hat{f}_{ij} = f(x_i, x_j) - f(x_j, x_i) \). Recall that \( X_{rs} = (E \otimes_k g^{s}) \otimes \tilde{A}^r \otimes E \) and let \( E^e \) be enveloping algebra of \( E \). By tensoring on the left \( X_{rs} \) over \( E^e \) with \( M \), and using Theorem 1.8 and the identifications \( \gamma_{rs} : X_{K}(M) \to M \otimes_{E^e} X_{rs} \), given by
\[
\gamma(m \otimes a_1r \otimes_k x_{1s}) = (-1)^{rs} m \otimes_{E^e} (1 \otimes_k x_{1s} \otimes a_1r \otimes 1),
\]
we obtain the complex
\[
X^K_0(M) \xleftarrow{\partial_1} X^K_1(M) \xleftarrow{\partial_2} X^K_2(M) \xleftarrow{\partial_3} X^K_3(M) \xleftarrow{\partial_4} X^K_4(M) \xleftarrow{\partial_5} \cdots,
\]
where
\[
X^K_n(M) = \bigoplus_{r+s=n} X_{rs}^K(M) \quad \text{and} \quad \partial_n = \sum_{r+s=n} \sum_{r+l>0} \partial_{rs}^l.
\]
Note that if \( f(g \otimes_k g) \subseteq K \), then the chain complex \( (\bigoplus X^K_{rs}(M), \partial_n) \) is the total complex of the double complex \( (\bigoplus X_{rs}^K(M), \partial^1_{rs}, \partial^2_{rs}) \).

**Theorem 4.1.** The Hochschild homology \( H^K_e(E, M) \), of the \( K \)-algebra \( E \) with coefficients in \( M \), is the homology of \( (\bigoplus X^K_{rs}(M), \partial_n) \).

**Proof.** It is an immediate consequence of the above discussion. □

### 4.2. The comparison maps

The maps \( \vartheta_* \) and \( \vartheta_* \), introduced in Section 2, induce quasi-isomorphisms
\[
\vartheta_* : (\bigoplus X^K_{rs}(M), \partial_n) \to \left( \frac{M \otimes E^*}{[M \otimes E^*, K]}, b_* \right)
\]
and
\[
\overline{\vartheta}_* : \left( \frac{M \otimes E^*}{[M \otimes E^*, K]}, b_* \right) \to (\bigoplus X^K_{rs}(M), \partial_n)
\]
which are inverse one of each other up to homotopy.

**Proposition 4.2.** We have
\[
\vartheta(m \otimes a_1r \otimes_k x_{1s}) = \sum_{\tau \in \mathcal{S}_x} (-1)^{rs} \text{sg}(\tau)m \otimes (1 \# x_{r(1)} \otimes \cdots \otimes 1 \# x_{r(s)}) \ast a_{1r}.
\]

**Proof.** This follows immediately from Proposition 2.1. □
Theorem 4.3. Let \((g_i)_{i \in I}\) be the basis of \(q\) considered in Theorem 1.3. Assume that \(c_{1n} = c_1 \otimes \cdots \otimes c_n \in E^n\) is a simple tensor with \(c_j \in A \cup \{1 \# g_i; \ i \in I\}\) for all \(j \in \{1, \ldots, n\}\). If there exist \(0 \leq s \leq n\) and \(i_1 < \cdots < i_s\) in \(I\), such that \(c_j = 1 \# g_{i_j}\) for \(1 \leq j \leq s\) and \(c_j \in A\) for \(s < j \leq n\), then

\[
\varphi(m \otimes c_{1n}) = (-1)^{s(n-s)}m \otimes c_{s+1,n} \otimes_{k} g_{i_1} \wedge \cdots \wedge g_{i_s}.
\]

Otherwise, \(\varphi(m \otimes c_{1n}) = 0\).

Proof. This follows immediately from Theorem 2.3. \(\square\)

4.3. The cap product

Recall that the cap product

\[
H^K_p(E, M) \times HH^q_K(E) \to H^K_{p-q}(E, M) \quad (q \leq p)
\]

is defined in terms of \((\overline{M \otimes E^p}, b_s)\) and \((\text{Hom}_{K^*}(E^s, E), b^s)\), by

\[
\overline{m} \otimes c^{1p} \prec \psi = \overline{m}\psi(c_{1q}) \otimes c_{q+1,p},
\]

where \(\psi \in \text{Hom}_{K^*}(E^q, E)\). In this subsection we compute the cap product in terms of the small complexes \((\overline{X}^K_a(M), \overline{a}_a)\) and \((\overline{X}^K_K(E), \overline{a}^a)\). Given

\[
\overline{m} \otimes \overline{a}^{1r} \otimes_k x_{1s} \in \overline{X}_K^{K}(M) \quad \text{and} \quad \varphi' \in \overline{X}_K^{r's'}(E) \quad \text{with} \quad r \geq r' \quad \text{and} \quad s \geq s',
\]

we define \((\overline{m} \otimes \overline{a}^{1r} \otimes_k x_{1s}) \cdot \varphi' \in \overline{X}_K^{r''s''}(M)\) by

\[
(\overline{m} \otimes \overline{a}^{1r} \otimes_k x_{1s}) \cdot \varphi' = \sum_{1 \leq j_1 < \cdots < j_{s'} \leq s} s g(j_{1s'}) \overline{m} \varphi'(\overline{a}^{1r} \otimes_k x_{j_{1s'}}) \otimes \overline{a}^{r+1,r} \otimes_k x_{1_{s''} r''},
\]

where

- \(s g(j_{1s'}) = (-1)^{r's' + r's' + \sum_{u=1}^s (j_u - s')},\)
- \(1 \leq l_1 < \cdots < l_{s-s'} \leq s\) denote the set defined by
  \[
  \{j_1, \ldots, j_{s'}\} \cup \{l_1, \ldots, l_{s-s'}\} = \{1, \ldots, s\},
  \]
- \(x_{j_{1s'}} = x_{j_1} \wedge \cdots \wedge x_{j_{s'}}\) and \(x_{l_{1s-s'}} = x_{l_1} \wedge \cdots \wedge x_{l_{s-s'}}\).

Theorem 4.4. In terms of the complexes \((\overline{X}^K_a(M), \overline{a}_a)\) and \((\overline{X}^K_K(E), \overline{a}^a)\), the cap product

\[
H^K_p(E, M) \times HH^q_K(E) \to H^K_{p-q}(E, M)
\]

is induced by \(\bullet\).
Proof. Let \( \overline{m} \otimes a_{1r} \otimes_k x_{1s} \in \overline{X}_k^s(M) \) and \( \varphi' \in \overline{X}_k^{s'}(E) \). Let \((g_i)_{i \in I}\) be the basis of \( g \) considered in Theorem 1.3. Clearly we can assume that there exist \( i_1 < \cdots < i_s \) in \( I \) such that \( x_j = g_{ij} \) for all \( 1 \leq j \leq s \). By Proposition 4.2,

\[
\overline{\delta}(\overline{\delta}(m \otimes a_{1r} \otimes_k x_{1s}) \wedge \overline{\delta}(\varphi')) = \overline{\delta}(T \wedge \overline{\delta}(\varphi')),
\]

where

\[
T = \sum_{\sigma \in S_s} (-1)^{rs} s \sigma((1#x_{\sigma(1)}) \otimes \cdots \otimes (1#x_{\sigma(s)})) * a_{1r}.
\]

Hence, by Theorem 3.3, if \( r' > r \) or \( s' > s \), then

\[
\overline{\delta}(\overline{\delta}(m \otimes a_{1r} \otimes_k x_{1s}) \wedge \overline{\delta}(\varphi')) = 0,
\]

and, if \( r' \leq r \) and \( s' \leq s \), then

\[
\overline{\delta}(\overline{\delta}(m \otimes a_{1r} \otimes_k x_{1s}) \wedge \overline{\delta}(\varphi')) = \sum_{1 \leq j_1 < \cdots < j_{s'} \leq s} \overline{\delta}(m\varphi'(a_{1r} \otimes_k x_{j_{s'}}) \otimes T'_{l_{j_1, \cdots, j_{s'}}}),
\]

where

\[
T'_{l_{j_1, \cdots, j_{s'}}} = \sum_{\tau \in S_{s'-s'}} (-1)^{rs+s'} s \tau((1#x_{\tau(1)}) \otimes \cdots \otimes (1#x_{\tau(\tau(s')-1)})) * a_{r'+1,r}.
\]

In order to finish the proof it suffices to apply Theorem 4.3. \( \square \)

5. The (co)homology of \( S(V) \#_f U(g) \)

In this section we obtain complexes \((\overline{Z}_*(M), \overline{\delta}_*)\) and \((\overline{Z}^*(M), \overline{\delta}^*)\), simpler than \((\overline{X}_k^*(M), \overline{\delta}^*)\) and \((\overline{X}_k^*(M), \overline{\delta}^*)\) respectively, giving the Hochschild homology of the \( K \)-algebra \( E := A \#_f U(g) \) with coefficients in an \( E \)-bimodule \( M \), when

- \( K = k \) and \( A \) is a symmetric algebra \( S(V) \),
- \( v^x \in k \oplus V \) for all \( v \in V \) and \( x \in g \),
- \( f(x_1, x_2) \in k \oplus V \) for all \( x_1, x_2 \in g \).

Then, we obtain an expression that gives the cup product of \( HH_k^s(E) \) in terms of \( (\overline{Z}_*(M), \overline{\delta}_*) \), and we obtain an expression that gives the cap product of \( H_k^s(E, M) \) in terms of \( (Z_*(M), \delta_*) \) and \( (Z^*(E), \delta^*) \).

For \( r, s \geq 0 \), let \( Z_r = E \otimes g^{rs} \otimes V^{\vee} \otimes E \). The groups \( Z_r \) are \( E \)-bimodules in an obvious way. Let

\[
\delta^l_r : Z_r \rightarrow Z_{r+l-1,s-l} \quad (0 \leq l \leq \min(2,s) \text{ and } r+l > 0)
\]

be the \( E \)-bimodule morphisms defined by

\[
\delta^0(1 \otimes x_{1s} \otimes v_{1r} \otimes 1) = \sum_{i=1}^r (-1)^{r+s}(v_{ij} \otimes x_{1s} \otimes v_{1r} \otimes 1 - 1 \otimes x_{1s} \otimes v_{1r} \otimes v_{ij}),
\]

\[
\delta^1(1 \otimes x_{1s} \otimes v_{1r} \otimes 1) = \sum_{i=1}^r (-1)^{r+s+1} x_{ij} \otimes x_{1s} \otimes v_{1r} \otimes 1
\]

\[
+ \sum_{i=1}^s (-1)^i \otimes x_{1s} \otimes v_{1r} \otimes 1 \otimes x_{ij}
\]
\[ + \sum_{1 \leq i \leq r} (-1)^i \otimes x_{1, s} \otimes v_{1, h-1} \wedge v_{h}^{s_i} \otimes v_{h+1, r} \otimes 1 \]
\[ + \sum_{1 \leq i < j \leq s} (-1)^{i+j} \otimes [x_i, x_j]_B \otimes x_{1, j} \otimes v_{1, r} \otimes 1, \]

and

\[ \delta^2(1 \otimes x_{1, s} \otimes v_{1, r} \otimes 1) = \sum_{1 \leq i < j \leq s} (-1)^{i+j+s} \otimes x_{1, j} \otimes \tilde{f}_{ij} \wedge v_{1, r} \otimes 1, \]

where

- \( v_{h} = v_h \wedge \cdots \wedge v_1 \),
- \( v_{h}^{s_i} \) is the \( V \)-component of \( v_{h}^{s_i} \) (that is \( v_{h}^{s_i} \in V \) and \( v_{h}^{s_i} - v_{h}^{s_i} \in k \)),
- \( \tilde{f}_{ij} = f_V(x_i, x_j) - f_V(x_j, x_i) \) in which \( f_V(x_i, x_j) \) and \( f_V(x_j, x_i) \) are the \( V \)-components of \( f(x_i, x_j) \) and \( f(x_j, x_i) \), respectively.

**Theorem 5.1.** The complex

\[ E \xleftarrow{\delta_1} Z_0 \xrightarrow{\delta_2} Z_1 \xrightarrow{\delta_3} Z_2 \xrightarrow{\delta_4} Z_3 \xrightarrow{\delta_5} Z_4 \xrightarrow{\delta_6} \cdots, \]

where

\[ \mu(1 \otimes 1) = 1, \quad Z_n = \bigoplus_{r+s=n} Z_{rs} \quad \text{and} \quad \delta_n = \sum_{r+s=n} \sum_{l=0}^{\min(s,2)} \delta_{l,rs}, \]

is a projective resolution of the \( E \)-bimodule \( E \). Moreover, the family of maps

\[ \Gamma: Z_n \rightarrow X_n, \]

given by

\[ \Gamma(1 \otimes x_{1, s} \otimes v_{1, r} \otimes 1) = \sum_{\sigma \in S} s(g(\sigma)) \otimes x_{1, s} \otimes v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(r)} \otimes 1, \]

defines a morphism of \( E \)-bimodule complexes from \( (Z_n, \delta_n) \) to \( (X_n, d_n) \).

**Proof.** It is clear that each \( Z_n \) is a projective \( E \)-bimodule and a direct computation shows that \( \Gamma \) is a morphism of complexes. Let

\[ G_n^0 \subseteq G_n^1 \subseteq G_n^2 \subseteq \cdots \quad \text{and} \quad F_n^0 \subseteq F_n^1 \subseteq F_n^2 \subseteq F_n^3 \subseteq \cdots \]

be the filtrations of \( (Z_n, \delta_n) \) and \( (X_n, d_n) \), defined by

\[ G_n^i = \bigoplus_{r+s=n, s \leq i} Z_{rs} \quad \text{and} \quad F_n^i = \bigoplus_{r+s=n, s \leq i} X_{rs}, \]
respectively. In order to see that $\Gamma_*$ is a quasi-isomorphism it is sufficient to show that it induces a quasi-isomorphism between the graded complexes associated with the filtrations introduced above. In other words, the maps

$$\Gamma_{as} : (Z_{as}, \delta^0_{as}) \to (X_{as}, d^0_{as}) \quad (s \geq 0),$$

defined by

$$\Gamma(1 \otimes x_{1s} \otimes v_{1r} \otimes 1) = \sum_{\sigma \in \mathcal{S}_r} s_g(\sigma) \otimes x_{1s} \otimes v_1 \otimes \cdots \otimes v_{\sigma(r)} \otimes 1,$$

are quasi-isomorphisms, which follows easily from Proposition 2.1. □

5.1. Hochschild cohomology

Let $M$ be an $E$-bimodule. For $r, s \geq 0$, let

$$\overline{Z}^{rs}(M) = \text{Hom}_k(V^r \otimes g^{s}, M).$$

We define the morphism

$$\overline{\delta}^r_l : \overline{Z}^{r+l-1,s-l}(M) \to \overline{Z}^{rs}(M) \quad (\text{with } 0 \leq l \leq \min(2, s) \text{ and } r+l > 0)$$

by:

$$\overline{\delta}_0(\varphi)(v_{1r} \otimes x_{1s}) = \sum_{i=1}^{r} (-1)^i [v_{i}, \varphi(v_{1r} \otimes x_{1s})],$$

$$\overline{\delta}_1(\varphi)(v_{1r} \otimes x_{1s}) = \sum_{i=1}^{s} (-1)^{i+r} [\varphi(v_{1r} \otimes x_{1s}), 1 \# x_i]$$

$$+ \sum_{1 \leq h \leq r} (-1)^{i+r} \varphi(v_{1r-1} \wedge v_{h} \otimes x_{1s})$$

$$+ \sum_{1 \leq i < j \leq s} (-1)^{i+j+r} \varphi(v_{1r} \otimes [x_i, x_j]_0 \otimes x_{1s})$$

and

$$\overline{\delta}_2(\varphi)(v_{1r} \otimes x_{1s}) = \sum_{1 \leq i < j \leq s} (-1)^{i+j} \varphi(f_{ij} \wedge v_{1r} \otimes x_{1s}).$$

Applying the functor $\text{Hom}_{E^r}(-, M)$ to the complex $(Z_*, \delta_*)$, and using Theorem 5.1 and the identifications $\xi^{rs} : \overline{Z}^{rs}(M) \to \text{Hom}_{E^r}(Z_{rs}, M)$, given by

$$\xi(\varphi)(1 \otimes x_{1s} \otimes v_{1r} \otimes 1) = (-1)^{rs} \varphi(v_{1r} \otimes x_{1s}),$$

we obtain the complex
\[ Z^0(M) \overset{\delta^1}{\longrightarrow} Z^1(M) \overset{\delta^2}{\longrightarrow} Z^2(M) \overset{\delta^3}{\longrightarrow} Z^3(M) \overset{\delta^4}{\longrightarrow} Z^4(M) \overset{\delta^5}{\longrightarrow} \cdots, \]

where

\[ Z^n(M) = \bigoplus_{r + s = n} Z^{rs}(M) \quad \text{and} \quad \delta^n = \sum_{r + s = n} \delta^{rs}_i. \]

Note that if \( f(g \otimes g) \subseteq k \), then the cochain complex \((\overline{Z}^*(M), \overline{\delta}^*)\) is the total complex of the double complex \((Z^{rs}(M), \delta^{rs}, \overline{\delta}^{rs})\).

**Theorem 5.2.** The Hochschild cohomology \( H^*(E, M) \), of \( E \) with coefficients in \( M \), is the cohomology of \((\overline{Z}^*(M), \overline{\delta}^*)\).

The map \( \Gamma_* : (Z_*, \delta_*) \rightarrow (X_*, d_*) \) induces a quasi-isomorphism

\[ \overline{\mathcal{F}}^* : (\overline{X}_k^*, \overline{\delta}_*) \rightarrow (\overline{Z}^*(M), \overline{\delta}^*). \]

**Proposition 5.3.** We have

\[ \overline{F}(\varphi)(v_{1r} \otimes x_{1s}) = \sum_{\sigma \in \mathcal{O}_r} \text{sg}(\sigma) \varphi(v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(r)} \otimes x_{1s}). \]

**Proof.** This follows immediately from Theorem 5.1. \( \square \)

5.2. The cup product

In this subsection we compute the cup product of \( \text{HH}^*(E) \) in terms of the complex \((\overline{Z}^*(E), \overline{\delta}^*)\). Given \( \phi \in \overline{Z}^{rs}(E) \) and \( \phi' \in \overline{Z}^{r''s'}(E) \), we define \( \phi \ast \phi' \in \overline{Z}^{r + r', s + s'}(E) \) by

\[ (\phi \ast \phi')(v_{1r} \otimes x_{1s}) = \sum_{1 \leq i_1 < \cdots < i_r \leq r''} \text{sg}(i_{1r}, j_{1s}) \phi(v_{i_{1r}} \otimes x_{j_{1s}}) \phi'(v_{h_{1r}} \otimes x_{l_{1s}}), \]

where

- \( \text{sg}(i_{1r}, j_{1s}) = (-1)^{r's' + \sum_{u=1}^{r'}(i_u - u) + \sum_{u=1}^{s'}(j_u - u)}, \)
- \( r'' = r + r' \) and \( s'' = s + s', \)
- \( 1 \leq h_1 < \cdots < h_{r'} \leq r'' \) denote the set defined by

\[ \{i_1, \ldots, i_r\} \cup \{h_1, \ldots, h_{r'}\} = \{1, \ldots, r''\}, \]

- \( 1 \leq l_1 < \cdots < l_{s'} \leq s'' \) denote the set defined by

\[ \{j_1, \ldots, j_s\} \cup \{l_1, \ldots, l_{s'}\} = \{1, \ldots, s''\}, \]

- \( v_{ih} = v_{i1} \wedge \cdots \wedge v_{ir} \) and \( v_{ih_{1r}} = v_{h1} \wedge \cdots \wedge v_{h_{1r}} \),
- \( x_{j_{1s}} = x_{j1} \wedge \cdots \wedge x_{js} \) and \( x_{l_{1s'}} = x_{l1} \wedge \cdots \wedge x_{l_{s'}} \).
Theorem 5.4. The cup product of $HH^n(E)$ is induced by the operation $\star$ in the complex $(\mathcal{Z}^*(E), \delta^*)$.

Proof. By Theorem 3.4 it suffices to prove that

$$\overline{T}(\varphi \bullet \varphi') = \overline{T}(\varphi) \star \overline{T}(\varphi')$$

for all $\varphi \in \mathcal{X}^{rs}_k(E)$ and $\varphi' \in \mathcal{X}^{r's'}_k(E)$. Let $\varphi = \overline{T}(\varphi)$ and $\varphi' = \overline{T}(\varphi')$. On one hand

$$\left( \varphi \star \varphi' \right)(v_{1r''} \otimes x_{1s''}) = \sum_{1 \leq i_1 < \cdots < i_r \leq r''} \sum_{1 \leq j_1 < \cdots < j_s \leq s''} sg(i_1, \ldots, i_r) \varphi(v_{i_1} \otimes x_{j_1}) \varphi'(v_{i_1'} \otimes x_{i_1'})$$

On the other hand

$$\overline{T}(\varphi \bullet \varphi')(v_{1r''} \otimes x_{1s''}) = \sum_{\sigma \in \mathcal{S}_{r''}} sg(\sigma)(\varphi \bullet \varphi')(v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(r''-1)} \otimes x_{1s''})$$

Now, formula (2) follows immediately from these facts. □

5.3. Hochschild homology

Let $M$ be an $E$-bimodule. For $r, s \geq 0$, let

$$\overline{Z}_{r,s}(M) = M \otimes V^{\wedge r} \otimes g^{\wedge s}.$$ 

We define the morphisms

$$\delta^l_{rs} : \overline{Z}_{r,s}(M) \to \overline{Z}_{r+l-1,s-l}(M) \quad (0 \leq l \leq \min(2, s) \text{ and } r + l > 0)$$

by:
\[
\bar{\delta}^0(m \otimes v_{1r} \otimes x_{1s}) = \sum_{i=1}^{r} (-1)^i [m, v_i] \otimes v_{1r} \otimes x_{1s},
\]

\[
\bar{\delta}^1(m \otimes v_{1r} \otimes x_{1s}) = \sum_{i=1}^{s} (-1)^{i+r} [1 \otimes x_i, m] \otimes v_{1r} \otimes x_{1s}
\]

\[
+ \sum_{1 \leq h \leq r} (-1)^{i} m \otimes v_{1h-1} \wedge v_{h} \wedge v_{h+1} \otimes x_{1s}
\]

\[
+ \sum_{1 \leq i < j \leq s} (-1)^{i+j+r} m \otimes v_{1r} \otimes [x_i, x_j] \wedge x_{1i} \wedge x_{1j}.
\]

and

\[
\bar{\delta}^2(m \otimes v_{1r} \otimes x_{1s}) = \sum_{1 \leq i < j \leq s} (-1)^{i+j} m \otimes \hat{f}_{ij} \wedge v_{1r} \otimes x_{1i} \wedge x_{1j}.
\]

By tensoring on the left the complex \((Z^\ast, \delta^\ast)\) over \(Ee\) with \(M\), and using Theorem 5.1 and the identifications \(\xi: Z_{rs}(M) \to M \otimes_{Ee} Z_{rs}\), given by

\[
\xi(m \otimes v_{1r} \otimes x_{1s}) = (-1)^{rs} m \otimes_{Ee} (1 \otimes x_{1s} \otimes v_{1r} \otimes 1),
\]

we obtain the complex

\[
\bar{Z}_0(M) \xleftarrow{\bar{\delta}_1} \bar{Z}_1(M) \xleftarrow{\bar{\delta}_2} \bar{Z}_2(M) \xleftarrow{\bar{\delta}_3} \bar{Z}_3(M) \xleftarrow{\bar{\delta}_4} \bar{Z}_4(M) \xleftarrow{\bar{\delta}_5} \cdots,
\]

where

\[
\bar{Z}_n(M) = \bigoplus_{r+s=n} Z_{rs}(M) \quad \text{and} \quad \bar{\delta}_n = \sum_{r+s=n} \sum_{l=0}^{\min(s,2)} \bar{\delta}_{rs}^l.
\]

Note that if \(f(g \otimes g) \subseteq k\), then the chain complex \((Z^\ast, \delta^\ast)\) over \(E^s\) with \(M\), is the homology of \((\bar{Z}^\ast, \bar{\delta}^\ast)\).

**Theorem 5.5.** The Hochschild homology \(H^\ast(E, M)\), of \(E\) with coefficients in \(M\), is the homology of \((\bar{Z}^\ast, \bar{\delta}^\ast)\).

**Proof.** It is an immediate consequence of the above discussion. \(\square\)

The map \(\Gamma^\ast: (Z^\ast, \delta^\ast) \to (X^\ast, d^\ast)\) induces a quasi-isomorphism

\[
\bar{\Gamma}^\ast: (\bar{Z}^\ast, \bar{\delta}^\ast) \to (\bar{X}^\ast, \bar{d}^\ast).
\]

**Proposition 5.6.** We have

\[
\bar{\Gamma}(m \otimes v_{1r} \otimes x_{1s}) = \sum_{\sigma \in \Sigma_r} \text{sg}((\sigma)m \otimes v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(r)} \otimes x_{1s}).
\]

**Proof.** This follows immediately from Theorem 5.1. \(\square\)
5.4. The cap product

In this subsection we compute the cap product

\[ H_p(E, M) \times HH^q(E) \rightarrow H_{p-q}(E, M) \quad (q \leq p), \]

in terms of the complexes \((Z^*(E), \delta *)\) and \((\overline{Z}^*(E), \overline{\delta} *)\). Given

\[ m \otimes v_{1r} \otimes x_{1s} \in \overline{Z}_{rs}(M) \quad \text{and} \quad \phi' \in \overline{Z}^{r's'}(E) \quad \text{with} \ r \geq r' \quad \text{and} \ s \geq s', \]

we define \((m \otimes v_{1r} \otimes x_{1s}) \ast \phi' \in \overline{Z}_{r-r', s-s'}(M)\) by

\[
(m \otimes v_{1r} \otimes x_{1s}) \ast \phi' = \sum_{1 \leq j_1 < \cdots < j_r \leq r} \sum_{1 \leq j_1 < \cdots < j_{s'} \leq s'} 
sg(i_{1r'}, j_{1s'}) m\phi'(v_{1r'} \otimes x_{1s'}) \otimes v_{h_{1-r'}t} \otimes x_{i_{1-s'}},
\]

where

- \(sg(i_{1r'}, j_{1s'}) = (-1)^{rs' + r's' + \sum_{u=1}^{s'} (j_u - u) + \sum_{u=1}^{r'} (j_u - u)}\),
- \(1 \leq h_1 < \cdots < h_{r-r'} \leq r\) denote the set defined by
  \[\{i_1, \ldots, i_r\} \cup \{h_1, \ldots, h_{r-r'}\} = \{1, \ldots, r\},\]
- \(1 \leq l_1 < \cdots < l_{s-s'} \leq s\) denote the set defined by
  \[\{j_1, \ldots, j_{s'}\} \cup \{l_1, \ldots, l_{s-s'}\} = \{1, \ldots, s\},\]
- \(v_{i_{1r'}} = v_{i_1} \wedge \cdots \wedge v_{i_{r'}}\) and \(v_{h_{1-r't}} = v_{h_1} \wedge \cdots \wedge v_{h_{r-r'}}\),
- \(x_{j_{1s'}} = x_{j_1} \wedge \cdots \wedge x_{j_{s'}}\) and \(x_{l_{1-s'}} = x_{l_1} \wedge \cdots \wedge x_{l_{s-s'}}\).

**Theorem 5.7.** The cap product

\[ H_p(E, M) \times HH^q(E) \rightarrow H_{p-q}(E, M) \quad (q \leq p) \]

is induced by \(\ast\), in terms of the complexes \((Z^*(E), \delta *)\) and \((\overline{Z}^*(E), \overline{\delta} *)\).

**Proof.** By Theorem 4.4 it suffices to prove that

\[
\overline{T}(m \otimes v_{1r} \otimes x_{1s}) \ast \phi' = \overline{T}((m \otimes v_{1r} \otimes x_{1s}) \ast \overline{T}(\phi'))
\]

for all \(m \otimes v_{1r} \otimes x_{1s} \in \overline{Z}_{rs}(M)\) and \(\phi' \in \overline{X}_{k}^{r's'}(E)\). Let \(\phi' = \overline{T}(\phi')\). On one hand

\[
\overline{T}(m \otimes v_{1r} \otimes x_{1s}) \ast \phi' = \sum_{1 \leq j_1 < \cdots < j_{s'} \leq s'} 
sg(\sigma) sg(j_{1s'}) m\phi'(v_{\sigma(1r')} \otimes x_{j_{1s'}}) \otimes v_{\sigma(r'+1,r)} \otimes x_{i_{1-s'}},
\]

where

\[v_{\sigma(1r')} = v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(r')} \quad \text{and} \quad v_{\sigma(r'+1,r)} = v_{\sigma(r'+1)} \otimes \cdots \otimes v_{\sigma(r)}.\]
On the other hand

\[
(m \otimes v_{1r} \otimes x_{1s}) \star \phi' = \sum_{1 \leq i_1 < \cdots < i_{r'} \leq r} \sum_{1 \leq j_1 < \cdots < j_{s'} \leq s} \sg(i_{1r'}, j_{1s'}) \cdot \phi'(v_{i_1r'} \otimes x_{j_1s'}) \otimes v_{t_{1s'}-r} \otimes x_{l_{1s'}-s}
\]

\[
= \sum_{1 \leq i_1 < \cdots < i_{r'} \leq r} \sum_{1 \leq j_1 < \cdots < j_{s'} \leq s} \sg(\tau) \cdot \sg(i_{1r'}, j_{1s'}) \cdot \phi'(v_{\tau(i_{1r'})} \otimes x_{j_{1s'}}) \otimes v_{t_{1s'}-r} \otimes x_{l_{1s'}-s},
\]

where \( v_{\tau(i_{1r'})} = v_{\tau(i_1)} \otimes \cdots \otimes v_{\tau(i_{r'})} \). Formula (3) follows immediately. □

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