POLISH SPACE PARTITION PRINCIPLES AND THE HALPERN-LÄUCHLI THEOREM

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Abstract. The Halpern-Läuchli theorem, a combinatorial result about trees, admits an elegant proof due to Harrington using ideas from forcing. In an attempt to distill the combinatorial essence of this proof, we isolate various partition principles about products of perfect Polish spaces. These principles yield straightforward proofs of the Halpern-Läuchli theorem, and the same forcing from Harrington’s proof can force their consistency. We also show that these principles are not ZFC theorems by showing that they put lower bounds on the size of the continuum.

1. Introduction

The Halpern-Läuchli theorem, first proven in [5], is a partition principle about products of finitely branching trees. While it is entirely combinatorial in nature, it has deep connections to logic, both in its original motivation — it was a key tool in Halpern and Lévy’s proof [6] that, over ZF, the Boolean Prime Ideal theorem does not imply the Axiom of Choice — and in some methods of proving it. In particular, arguably the most elegant proof of the theorem, due to Harrington, uses some ideas from forcing (see [3] or [16] for a presentation of Harrington’s proof). Here, we investigate some of the combinatorial ideas underlying Harrington’s proof and, building on work from [18], we introduce and study a family of statements about arbitrary (i.e. not necessarily Borel) finite partitions of a product of finitely many perfect Polish spaces. The simplest of these statements are as in the following definition.

Definition 1.1. Recall that a Polish space is perfect if it contains no isolated points. Given $0 < d < \omega$ and a sequence $\langle X_0, \ldots, X_{d-1} \rangle$ of perfect Polish spaces, a somewhere dense grid is a subset of $\prod_{i<d} X_i$ of the form $\prod_{i<d} Y_i$, where each $Y_i \subseteq X_i$ is somewhere dense (in the ordinary topological sense).

The Polish grid principle in dimension $d$, denoted $\text{PG}_d$, is the statement that for any sequence $\langle X_0, \ldots, X_{d-1} \rangle$ of perfect Polish spaces and any coloring $\gamma : \prod_{i<d} X_i \to r$, there is a monochromatic somewhere dense grid. The principle $\text{PG}$ is the statement that $\text{PG}_d$ holds for every $d < \omega$.

More generally, for any cardinal $\kappa$, let $\text{PG}_d(\kappa)$ be the statement that for any sequence $\langle X_0, \ldots, X_{d-1} \rangle$ of perfect Polish spaces and any coloring $\gamma : \prod_{i<d} X_i \to \kappa$,
there is a monochromatic somewhere dense grid, and let $\text{PG}(\kappa)$ denote the statement that $\text{PG}_d(\kappa)$ holds for every $d < \omega$.

We shall see in Section 3 that $\text{PG}_d$ yields a simple, direct proof of the $d$-dimensional Halpern-Läuchli theorem; in fact, it will be immediate from the proof that a natural weakening of $\text{PG}_d$, which we will define in Section 3 and which we will denote by $\text{DDF}_d$, suffices for this derivation. Unlike the Halpern-Läuchli theorem, though, at least for $d > 2$ the principles $\text{PG}_d$ and $\text{DDF}_d$ are independent of ZFC. Much of this paper is concerned with the study of this independence; we obtain, for instance, the following complementary results.

**Theorem A.** Let $2 \leq d < \omega$.

1. If $\text{DDF}_d$ holds, then $2^\aleph_0 \geq \aleph_{d-1}$.
2. If $\text{PG}_d(\aleph_0)$ holds, then $2^\aleph_0 \geq \aleph_d$.
3. $\text{PG}_d(\aleph_0)$ holds after adding at least $(\beth_{d-1})^+$-many Cohen reals to any model of ZFC.

Clause 3 of Theorem A, together with Shoenfield absoluteness, yields a new proof of the Halpern-Läuchli theorem. This proof can be seen as a recasting of Harrington’s proof that seeks to pull apart the forcing machinery and the combinatorial principles underlying the Halpern-Läuchli theorem. Indeed, our proof of Theorem A(3) is essentially an adaptation of Harrington’s proof of the Halpern-Läuchli theorem.

We feel that Theorem A is also of independent interest for isolating a stratified family of natural combinatorial statements that place increasingly strong requirements on the value of the continuum. In this direction, Theorem A yields a sharp result for the principle $\text{PG}_d(\aleph_0)$: for all $2 \leq d < \omega$, if $2^\aleph_0 < \aleph_d$, then $\text{PG}_d(\aleph_0)$ fails, whereas it is consistent that $2^\aleph_0 = \aleph_d$ and $\text{PG}_d(\aleph_0)$ holds (for example, after adding $\aleph_d$-many Cohen reals to a model of GCH). This can be seen as part of a line of investigation into the relationship between questions of dimensionality in the context of the real numbers (or uncountable Polish spaces, more generally), and the cardinals $\{\aleph_d : d < \omega\}$. Other research in this vein includes, e.g., Raghavan and Todorcević’s work on Galvin’s problem ([12], [13]), Komjáth’s work on covering the plane by finite many clouds ([8]) and the work of several authors on additive partition relations for the real numbers ([7], [9], [17]).

The work of Raghavan and Todorcević in particular obtains results with interesting parallels to ours. They consider the problem of finding the $d$-dimensional Ramsey degrees of the topological space $\mathbb{Q}$, i.e. given a coloring of $[\mathbb{R}]^d$ into finitely many colors, find $X \subseteq \mathbb{R}$ homeomorphic to $\mathbb{Q}$ such that $[X]^d$ takes a fixed small number of colors (not depending on the starting number of colors). In [13], they show that $2^\aleph_0 \leq \aleph_{d-2}$ implies that the $d$-dimensional Ramsey degree of the space $\mathbb{Q}$ is infinite, and in [12], they show that in ZFC plus some large cardinal hypotheses, the 2-dimensional Ramsey degree of the space $\mathbb{Q}$ is 2.

Note that Theorem A(1) does not yield any nontrivial information from $\text{DDF}_2$ or $\text{PG}_2$. In fact, we shall see in Section 3 that $\text{DDF}_2$ is a theorem of ZFC. Moreover, $\text{PG}_2$ is a theorem of ZFC when restricted to 2-colorings. It remains open whether the full $\text{PG}_2$ is a theorem of ZFC. It would be especially interesting if similar techniques as in [12] could be used to show that $\text{PG}_2$ is implied by some large cardinal hypotheses.
Finally, the tools developed in our proof of Clauses 1 and 2 of Theorem A yield a new proof of a recent result of Bannister, Bergfalk, Moore, and Todorčević from [1] about provable failures of the Partition Hypothesis introduced in that paper. The original proof of this fact in [1] makes heavy use of ideas from simplicial homology, whereas our proof is purely combinatorial/set theoretic.

The structure of the paper is as follows. In Section 2 we give our background notational conventions and basic definitions regarding trees and the Halpern-Läuchli theorem. In Section 3 we introduce the partition principles that form the main object of study of the paper. We show that they yield immediate derivations of relevant instances of the Halpern-Läuchli theorem. We also show that the principle DDF_{2} is provable in ZFC, as is the principle PG_{2} when restricted to 2-colorings. In Section 4 we prove clauses 1 and 2 of Theorem A. We also give a direct proof of the aforementioned result from [1] about provable failures of the Partition Hypothesis. In Section 5 we prove clause 3 of Theorem A. We conclude in Section 6 with a brief discussion about using the partition principles introduced here to yield variations of the Halpern-Läuchli theorem pertaining to coding trees.

Some related and partially overlapping results were recently independently obtained by Nedeljko Stefanović [14]. In particular, he proves that the principle DDF holds after adding \( \omega \)-many Cohen reals to any model of ZFC, thereby also yielding the proof of the Halpern-Läuchli theorem described above immediately after the statement of Theorem A. Stefanović also investigates these combinatorial principles in certain models of ZF + \( \neg \)AC, in particular proving that PG holds in Cohen’s symmetric model (the same model used by Halpern and Lévy in [6] to show that BPI does not imply AC).

2. Preliminaries

2.1. Notation. If \( a \) is a set of ordinals, then we will sometimes think of \( a \) as the increasing function whose domain is the order type of \( a \) (which we denote by \( \text{otp}(a) \)). In particular, if \( \eta < \text{otp}(a) \), then \( a(\eta) \) is the unique \( \beta \in a \) such that \( \text{otp}(a \cap \beta) = \eta \). Similarly, if \( I \subseteq \text{otp}(a) \), then \( a[I] \) denotes the set \( \{a(\eta) \mid \eta \in I \} \). If \( X \) is any set and \( n < \omega \), then \( |X|^n \) denotes the set of \( n \)-element subsets of \( X \). If \( X \) is a set of ordinals, then we will use the notation \( (\alpha_{0}, \ldots, \alpha_{n-1}) \in |X|^n \) to denote the assertion that \( \{\alpha_{0}, \ldots, \alpha_{n-1}\} \subseteq |X| \) and \( \alpha_{0} < \ldots < \alpha_{n-1} \).

2.2. Trees and the Halpern-Läuchli theorem. A tree is a partially ordered set \( T \) with the property that for every \( t \in T \), the set pred\(_{T}(t) := \{s \in T : s <_{T} t \} \) of predecessors of \( t \in T \) is well-ordered. In particular, any subset \( S \subseteq T \) equipped with the partial order induced from \( T \) is also a tree. We write \( h_{T}(t) := \text{otp}(\text{pred}_{T}(t)) \) and call this the height of \( t \) in \( T \). If \( \alpha \) is an ordinal, we write \( T(\alpha) := \{t \in T : h_{T}(t) = \alpha \} \) for level \( \alpha \) of \( T \). The height of \( T \) is the ordinal \( h(T) := \sup \{h_{T}(t) : t \in T \} \). We write \( \text{succ}_{T}(t) := \{u \in T : t <_{T} u \} \), and given \( \alpha > h_{T}(t) \), we write \( \text{succ}_{T}(t, \alpha) := \{u \in \text{succ}_{T}(t) : h_{T}(u) = \alpha \} \). We write \( \text{IS}_{T}(t) := \text{succ}_{T}(t, h_{T}(t) + 1) \) for the set of immediate successors of \( t \in T \). If \( S \subseteq T \), we can also write \( \text{succ}_{T}(S) := \bigcup_{t \in S} \text{succ}_{T}(t) \) and \( \text{IS}_{T}(S) := \bigcup_{t \in S} \text{IS}_{T}(t) \), and if \( \alpha > h_{T}(t) \) for every \( t \in S \), we can also write \( \text{succ}_{T}(S, \alpha) := \bigcup_{t \in S} \text{succ}_{T}(t, \alpha) \). We say \( T \) is finitely branching if \( \text{IS}_{T}(t) \) is finite for every \( t \in T \), and we say that \( T \) has no terminal nodes if \( \text{IS}_{T}(t) \neq \emptyset \) for every \( t \in T \). We say \( T \) is perfect if for every \( t \in T \), some \( u \in \text{succ}_{T}(t) \) has \( |\text{IS}_{T}(u)| \geq 2 \). We say \( T \) is rooted if \( |T(0)| = 1 \). Unless specified otherwise, all
trees in this paper will be rooted, perfect, finitely branching, height \( \omega \), and without terminal nodes.

Given a tree \( T \), a branch through \( T \) is a maximal linearly ordered subset of \( T \). We write \([T]\) for the set of branches through \( T \). Every \( x \in [T] \) satisfies \( |x \cap T(m)| = 1 \) for every \( m < \omega \), so we write \( x(m) \in T(m) \) for this unique element. We equip \([T]\) with the topology of pointwise convergence, where \( x_i \to x \) iff for every \( m < \omega \), we eventually have \( x_i(m) = x(m) \). Because of our standing assumptions on trees, we have that \([T]\) is homeomorphic to Cantor space.

Now suppose \( 0 < d < \omega \) and that \( \vec{T} = \langle T_0, \ldots, T_{d-1} \rangle \) is a sequence of trees with no terminal nodes. The level product of these trees is the set \( \prod \vec{T} := \bigcup_{m < \omega} \prod_{i < d} T_i(m) \). We turn \( \prod \vec{T} \) into a tree, where given \( \vec{s} = (s_0, \ldots, s_{d-1}) \) and \( \vec{t} = (t_0, \ldots, t_{d-1}) \) in \( \prod \vec{T} \), we set \( \vec{s} \leq \prod \vec{T} \vec{t} \) iff \( s_i \leq T_i t_i \) for each \( i < d \). To ease notation, we write \( \leq \) in place of \( \leq \prod \vec{T} \). Note that \( \left( \prod \vec{T} \right)(m) = \prod_{i < d} T_i(m) \) and that as spaces, we have \( \prod \vec{T} \cong \prod_{i < d} T_i \). Given \( \vec{x} \in \prod_{i < d} T_i(m) \), if \( i < d \), then we will always denote the \( i \)th entry in \( \vec{x} \) as \( x_i \in [T_i] \). Moreover, if \( m < \omega \), then we let \( \vec{x}(m) := \langle x_0(m), x_1(m), \ldots, x_{d-1}(m) \rangle \in \prod_{i < d} T_i(m) \).

**Definition 2.1.** Let \( T \) be a tree with no terminal nodes, and let \( a \subseteq \omega \) be infinite. An \( a \)-strong subtree of \( T \) is a subset \( S \subseteq T \) built inductively as follows.

- Pick any \( t_0 \in T(a(0)) \) and set \( S(0) = \{t_0\} \).
- Inductively assume for some \( m < \omega \) that \( S(m) \) has been determined and that \( S(m) \subseteq T(a(m)) \). Then for every \( u \in IS_T(S(m)) \), pick some \( t_u \in T(a(m+1)) \), and set \( S(m+1) = \{t_u : u \in IS_T(S(m))\} \).

Notice that if \( S \subseteq T \) is an \( a \)-strong subtree, then the set \( S(m) \) from the inductive construction above is in fact level \( m \) of the tree \( S \). Let us remark that if \( \vec{T} = \langle T_0, \ldots, T_{d-1} \rangle \) is a finite sequence of trees and \( S_i \subseteq T_i \) is an \( a \)-strong subtree for each \( i < d \), then \( \prod S \) is an \( a \)-strong subtree of \( \prod \vec{T} \); however, not every \( a \)-strong subtree of \( \prod \vec{T} \) has this particularly nice form.

We can now state the theorem of Halpern and Läuchli which is the main topic of this paper.

**Theorem 2.2** (Halpern-Läuchli [5]). Let \( 0 < d < \omega \), and let \( \vec{T} = \langle T_0, \ldots, T_{d-1} \rangle \) be a sequence of trees. Let \( r < \omega \), and suppose \( \gamma : \prod \vec{T} \to r \) is a coloring. Then there are an infinite \( a \subseteq \omega \) and \( a \)-strong subtrees \( S_i \subseteq T_i \) so that writing \( \vec{S} = \langle S_0, \ldots, S_{d-1} \rangle \), we have that \( \prod \vec{S} \) is monochromatic for \( \gamma \).

We write \( HL \) for the statement of Theorem 2.2, and \( HL_d \) for its restriction to sequences of trees of length at most \( d \).

### 3. Polish space partition principles

While \( HL_1 \) is trivial, proving \( HL_d \) by induction on \( d \) is quite difficult. Good references for proofs along these lines are [11] and [15]. However, we draw attention to a proof due to Harrington (cf. [16]) using ideas from forcing. Let us begin by giving a very brief, high-level overview of the structure of Harrington’s proof. Given \( \gamma : \prod \vec{T} \to r \) as in Theorem 2.2, one considers the poset \( P \) for adding a large number of Cohen reals, which are viewed as members of \( \prod \vec{T} \). Upon fixing a name \( \vec{U} \) for a non-principal ultrafilter on \( \omega \) and various names \( b \) for \( P \)-generic branches, certain
conditions $q_b \in \mathcal{P}$ force that for $\dot{U}$-many levels, the corresponding node of the branch $b$ is mapped by $\gamma$ to some color $i_b < r$. By using the Erdős-Rado theorem, we can find a rich collection of such $b$ so that various properties of the corresponding $q_b$ and $i_b$ are the same. One then uses the conditions $q_b$ to help build the subtrees $\dot{S}$ with $\otimes \dot{S}$ monochromatic for $\gamma$.

It is natural to attempt to remove some of the forcing formalism from these ideas. Namely, if $U \in \beta \omega \setminus \omega$, is a non-principal ultrafilter and if $\gamma : \otimes \dot{T} \to r$ is a coloring for some $r < \omega$, we can define a coloring $\gamma_U : \prod_{i < \omega} [T_i] \to r$ via $\gamma_U(\vec{x}) = j$ if $\{ m < \omega : \gamma(\vec{x}(m)) = j \} \in U$. Of course, the coloring $\gamma_U$ will typically have horrible definability properties, i.e. fail to have the Baire property. However, we can still attempt to reason about the possible Ramsey-theoretic properties of arbitrary colorings on products of Polish spaces. This line of thought naturally leads to the definition of the partition principles $\text{PG}_d$, which, recalling Definition 3.1, is the assertion that, for every positive $d < \omega$, any sequence $\langle X_0, \ldots, X_{d-1} \rangle$ of perfect Polish spaces, any $r < \omega$, and any coloring $\gamma : \prod_{i < d} X_i \to r$, there is a somewhere dense grid in $\prod_{i < d} X_i$ that is monochromatic for $\gamma$.

**Remark 3.1.** Recall that every perfect Polish space contains a dense $G_{\delta}$ subspace homeomorphic to Baire space $\omega^\omega$. Therefore to show that $\text{PG}_d$ holds, one may assume that each $X_i$ is the Baire space.

**Proposition 3.2.** $\text{PG}_d$ implies $\text{HL}_d$.

**Proof.** Let $\gamma : \otimes \dot{T} \to r$ be a coloring. Fix a non-principal ultrafilter $U \in \beta \omega \setminus \omega$, and form the coloring $\gamma_U : \prod_{i < \omega} [T_i] \to r$ defined before Remark 3.1. Using $\text{PG}_d$, find somewhere dense sets $Y_i \subseteq [T_i]$ so that $\gamma_U[\prod_{i < d} Y_i] = \{ j \}$ for some $j < r$. As each $Y_i$ is somewhere dense, we can find $t_i \in T_i$ so that whenever $u \supseteq_{T_i} t_i$, there is $y \in Y_i$ with $u \supseteq y$. By moving some of the $t_i$ further up if needed to place them all on the same level, we may assume that $(t_0, \ldots, t_{d-1}) := \vec{t} \in \otimes \dot{T}$.

We now inductively construct an infinite $a \subseteq \omega$ and $a$-strong subtrees $S_i \subseteq T_i$ with $\otimes \dot{S}$ monochromatic for $\gamma$. To get started, for each $i < d$, pick some $y_i \in Y_i$ with $t_i \in y_i$. Setting $\vec{y} = (y_0, \ldots, y_{d-1}) \in \prod_{i < d} Y_i$, we have $\gamma_U(\vec{y}) = j$. This means that $W_0 := \{ m < \omega : \gamma(\vec{y}(m)) = j \} \in U$, so in particular is infinite. Pick some $a(0) \in W_0$ above $\ell(\vec{t})$, and for each $i < d$, we set $S_i(0) = \{ y_i(a(0)) \}$.

Now suppose $n > 0$ and that for every $m < n$, both $a(m) < \omega$ and $S_i(m) \subseteq T_i(a(m))$ have been determined so that every $s \in S_i$ satisfies $s \supseteq_{T_i} t_i$. For each $u \in \text{IS}_{T_i}(S_i(n - 1))$, there is some $y_u \in Y_i$ with $u \supseteq y_u$. Set $Z_i := \{ y_u : u \in \text{IS}_{T_i}(S_i(n - 1)) \}$. Because $Z_i \subseteq Y_i$, we have that $W_n := \{ m < \omega : \gamma(\vec{z}(m)) = j \} : \vec{z} \in \prod_{i < d} Z_i \} \in U$. Pick some $a(n) \in W_n$ with $a(n) > a(n - 1)$, and for each $i < d$, set $S_i(n) = \{ z(n) : z \in Z_i \}$.

We will show in our proof of Theorem 3.3 in Section 5 that one can force that $\text{PG}$ is consistent by adding $\omega_2$-many Cohen reals. In particular, as $\text{HL}$ is a $\Pi^1_1$ statement, Shoenfield’s absoluteness theorem then implies that $\text{HL}$ is true in $\text{ZFC}$, yielding a new proof of $\text{HL}$. In a sense, this proof re-interprets Harrington’s forcing proof of $\text{HL}$ by actually passing to the generic extension, whereas Harrington’s proof can be phrased just in terms of combinatorics on the forcing poset.

Upon analyzing the proof of Proposition 3.2, it becomes clear that $\text{PG}$ is actually stronger than what we need to prove $\text{HL}$. We define two weakenings of $\text{PG}$ which
are still strong enough so that the proof of Proposition 3.2 goes through. The first one is the weakest such principle and is in fact true in ZFC.

**Definition 3.3.** If $X$ is a topological space and $\mathcal{U}$ is a collection of open subsets of $X$, a $\mathcal{U}$-set is any $Y \subseteq X$ which meets every member of $\mathcal{U}$.

Given $0 < d < \omega$ and a sequence $\langle X_0, \ldots, X_{d-1} \rangle$ of perfect Polish spaces, a finitary somewhere dense grid is a subset $Y \subseteq \prod_{i<d} X_i$ so that for each $i < d$, there are open $U_i \subseteq X_i$ so that for every sequence $\langle U_0, \ldots, U_{d-1} \rangle$ with $\mathcal{U}_i$ a finite collection of non-empty open subsets of $U_i$, there is for each $i < d$ a $\mathcal{U}_i$-set $Y_i \subseteq X_i$ with $\prod_{i<d} Y_i \subseteq Y$.

The finitary Polish grid principle $\text{FPG}_d$ states that for any sequence $\langle X_0, \ldots, X_{d-1} \rangle$ of perfect Polish spaces, any $r < \omega$, and any coloring $\gamma : \prod_{i<d} X_i \to r$, there is a monochromatic finitary somewhere dense grid.

In fact, one can show in ZFC that for any $d < \omega$, any sequence $\langle X_0, \ldots, X_{d-1} \rangle$ of perfect Polish spaces, any finitary somewhere dense grid $Y \subseteq \prod_{i<d} X_i$, and any finite coloring of $Y$, one of the colors contains a finitary somewhere dense grid. However, we won’t say much more about $\text{FPG}_d$; to us at least, it seems that proofs of $\text{FPG}_d$ either go through forcing the consistency of the much stronger $\text{PG}_d$ or go through repeating many of the steps seen in combinatorial proofs of HL.

In between $\text{FPG}_d$ and $\text{PG}_d$, we have our last principle, which was defined by the second author in [15].

**Definition 3.4.** Given a sequence $\langle X_0, \ldots, X_{d-1} \rangle$ of perfect Polish spaces and $I \subseteq d$, we write $\pi_I : \prod_{i \in I} X_i \to \prod_{i \in I} X_i$ for the natural projection map. Given some $Z \subseteq \prod_{i \in I} X_i$ and some $x \in \prod_{i \in I} X_i$, we write $Z_x := \pi_{d\setminus I}\{z \in Z : \pi_I(z) = x\}$.

We define the notion of a set $Z \subseteq \prod_{i<d} X_i$ being a dense-by-dense filter, DDF for short, by induction on $d > 0$.

1. $Z \subseteq X_0$ is a DDF set if it is dense.

2. Given $d > 0$, we have that $Z \subseteq \prod_{i<d} X_i$ is DDF if $\pi_d[Z] \subseteq \prod_{i<d} X_i$ is DDF and $\{Z_x : x \in \pi_d[Z]\}$ generates a filter of dense subsets of $X_d$.

We say that $Z \subseteq \prod_{i<d} X_i$ is somewhere-DDF if for some non-empty open sets $U_i \subseteq X_i$, $Z$ is DDF as a subset of $\prod_{i<d} U_i$.

The principle $\text{DDF}_d$ states that for any sequence $\langle X_0, \ldots, X_{d-1} \rangle$ of perfect Polish spaces, any $r < \omega$, and any coloring $\gamma : \prod_{i<d} X_i \to r$, there is a monochromatic somewhere-DDF subset.

Since a somewhere dense grid is clearly somewhere-DDF, it is clear that $\text{PG}_d$ implies $\text{DDF}_d$. The next proposition shows that a witness to $\text{DDF}_d$ is also a witness to $\text{FPG}_d$.

**Proposition 3.5.** Given $0 < d < \omega$ and $\langle X_0, \ldots, X_{d-1} \rangle$ a sequence of perfect Polish spaces, if $Z \subseteq \prod_{i<d} X_i$ is somewhere-DDF, then $Z$ is also a finitary somewhere dense grid.

**Proof.** We prove by induction on $d$ that if $U_i \subseteq X_i$ are non-empty open sets with $Z \subseteq \prod_{i<d} U_i$ a DDF subset, then the $U_i$ also witness that $Z$ is a finitary somewhere dense grid. For $d = 1$ this is clear. Now suppose the result is known for dimension $d$. Let $\langle X_0, \ldots, X_d \rangle$ be perfect Polish spaces, and suppose $Z \subseteq \prod_{i<d} U_d$ is DDF. Fix $\langle U_0, \ldots, U_d \rangle$ with each $U_i$ a finite collection of non-empty open subsets of $U_i$.

As $\pi_d[Z] \subseteq \prod_{i<d} X_i$ is DDF, then by induction we can find $U_i$-sets $Y_i \subseteq U_i$ with
\( \prod_{i<d} Y_i \subseteq Z \). Find some finite \( F_i \subseteq Y_i \) which is also a \( \mathcal{U}_i \)-set. As \( \{ Z_x : x \in \pi_d[Z] \} \) generates a filter of dense subsets of \( U_d \), the set \( Y_d = \bigcap \{ Z_y : y \in \prod_{i<d} F_i \} \subseteq U_d \) is dense, so in particular is a \( \mathcal{U}_d \)-set. As \( F_0 \times \cdots \times F_{d-1} \times Y_d \subseteq Z \), we see that \( Z \) is a finitary somewhere dense grid.

\[ \square \]

This principle serves as an interesting middle ground between \( \text{FPG}_d \) and \( \text{PG}_d \); it is rich enough in that one can use non-combinatorial tools to investigate it, yet weak enough that one might hope to prove it in ZFC. Indeed, we have the following.

**Proposition 3.6.** \( \text{DDF}_2 \) is true.

**Proof.** For reasons that will be clear at the end of the proof, we prove something slightly stronger. Let \( X_0 \) be a non-meager subset of some perfect Polish space \( X_0 \), and let \( X_1 \) be a somewhere dense subset of some perfect Polish space \( \tilde{X}_1 \). By zooming in to the relevant open sets, we may assume that \( X_0 \subseteq \tilde{X}_0 \) is nowhere meager and that \( X_1 \subseteq \tilde{X}_1 \) is dense. Let \( r < \omega \), and let \( \gamma : X_0 \times X_1 \rightarrow r \) be a coloring. We will prove that there is a somewhere-\( \text{DDF} \) subset of \( X_0 \times X_1 \) that is monochromatic for \( \gamma \).

The proof is by induction on \( r \). It is trivial if \( r = 1 \), so assume that \( r > 1 \). We attempt to find a \( \text{DDF} \) subset inside \( Z := \gamma^{-1}(\{r - 1\}) \). Let \( \{ U_n : n < \omega \} \) be a basis for \( X_0 \). We inductively attempt to build a decreasing collection \( \{ Y_n : n < \omega \} \) of dense subsets of \( X_1 \) and a dense subset \( \{ x_n : n < \omega \} \) of \( X_0 \) as follows. Set \( Y_0 = X_1 \). If \( n < \omega \) and \( Y_n \) has been determined, pick any \( x_n \in U_n \) such that \( Z_{x_n} \cap Y_n \subseteq X_1 \) is dense, and set \( Y_{n+1} := Z_{x_n} \cap Y_n \). If we can do this for every \( n < \omega \), then \( \bigcup_{n<\omega} \{ x_n \} \times Y_{n+1} \subseteq Z \) is a \( \text{DDF} \) set.

Suppose for some \( k < \omega \) that we fail to construct \( x_k \) and \( Y_{k+1} \). Let \( \{ V_n : n < \omega \} \) be a basis for \( X_1 \). For every \( x \in U_k \), there is some \( n_x < \omega \) so that \( Z_x \cap Y_k \cap V_{n_x} = \emptyset \). For some \( n < \omega \), the set \( W := \{ x \in U_k : n_x = n \} \) is non-meager. Then \( W \subseteq \tilde{X}_0 \) is non-meager, \( Y_k \cap V_n \subseteq \tilde{X}_1 \) is somewhere dense, and \( \gamma \) attains one fewer color on \( W \times (Y_k \cap V_n) \). We can therefore apply the induction hypothesis to \( \gamma \upharpoonright W \times (Y_k \cap V_n) \) to obtain a monochromatic somewhere-\( \text{DDF} \) subset of \( W \times (Y_k \cap V_n) \), and hence also of \( X_0 \times X_1 \).

Using similar ideas, one can also say something about the stronger \( \text{PG}_2 \) principle.

**Proposition 3.7.** \( \text{PG}_2 \) restricted to 2-colorings is true.

**Proof.** Let \( X_0 \) and \( X_1 \) be perfect Polish spaces, and let \( \gamma : X_0 \times X_1 \rightarrow 2 \) be a coloring. We will attempt to construct a dense grid inside \( Z_1 := \gamma^{-1}(\{1\}) \) and will show that if any step of the construction fails, then we can find a somewhere dense grid inside \( Z_0 := \gamma^{-1}(\{0\}) \). Given \( j < 2 \) and \( x \in X_0 \), let \( (Z_j)_x := \{ y \in X_1 : (x, y) \in Z_j \} \). Similarly, given \( y \in X_1 \), let \( (Z_j)^y := \{ x \in X_0 : (x, y) \in Z_j \} \).

For \( i < 2 \), let \( \{ U^i_n : n < \omega \} \) be a basis for \( X_i \). We attempt to build sets \( Y_i = \{ x^i_n : n < \omega \} \) for \( i < 2 \) such that \( Y_0 \times Y_1 \subseteq Z_1 \) and, for all \( n < \omega \) and \( i < 2 \), we have \( x^i_n \in U^i_n \). During the construction, we will also construct \( \subseteq \)-decreasing sequences \( \{ V^i_n : n < \omega \} \) of nowhere meager subsets of \( X_i \) with the property that, for all \( n < \omega \), both \( \{ x^0_n : m < n \} \times V^i_n \) and \( V^i_n \times \{ x^1_m : m < n \} \) are subsets of \( Z_1 \).

Begin by letting \( V^0_0 = X_i \) for \( i < 2 \). Now suppose that \( n < \omega \) and we have chosen \( \langle x^i_m : m < n \rangle \) and \( \langle V^i_n : m \leq n \rangle \). We will choose \( x^0_n \in V^0_n \cap U^0_n \) and a nowhere meager set \( V^1_{n+1} \subseteq V^1_n \), and then we will choose \( x^1_n \in V^1_{n+1} \cap U^1_n \) and a nowhere meager set \( V^0_{n+1} \subseteq V^0_n \).
If we are able to find \( x \in V_{n}^{0} \cap U_{n}^{0} \) such that \( V_{n}^{1} \cap (Z_{1})_{x} \) is nowhere meager, then let \( x_{n}^{0} \) be such an \( x \) and let \( V_{n+1}^{1} := V_{n}^{1} \cap (Z_{1})_{x_{n}} \). Suppose momentarily that we were unable to find such an \( x \). Then, for every \( x \in V_{n}^{0} \cap U_{n}^{0} \), there is \( k_{x} < \omega \) such that \( V_{n}^{1} \cap (Z_{1})_{x} \cap U_{k_{x}}^{1} \) is meager. Then there is a fixed \( k < \omega \) and a non-meager set \( W_{0} \subseteq V_{n}^{0} \cap U_{n}^{0} \) such that \( k_{x} = k \) for all \( x \in W_{0} \). Let \( W_{0}' \) be a countable somewhere dense subset of \( W_{0} \). Then \( V_{n}^{1} \cap U_{k}^{1} \cap \bigcup \{ (Z_{1})_{x} : x \in W_{0}' \} \) is meager, so \( W_{1} := V_{n}^{1} \cap U_{k}^{1} \cap \bigcap \{ (Z_{0})_{x} : x \in W_{0}' \} \) is nonmeager. In particular, \( W_{1} \) is somewhere dense, so \( W_{0}' \times W_{1} \) is a somewhere dense grid contained in \( Z_{0} \).

We can therefore assume that we were able to construct \( x_{0}^{0} \) and \( V_{n+1}^{1} \) and continue to the second half of step \( n \) of the construction, where a symmetric argument shows that we can either find

1. \( y \in V_{n+1}^{1} \) such that \( V_{n}^{0} \cap (Z_{1})^{y} \) is nowhere meager; or
2. a somewhere dense grid contained in \( Z_{0} \).

We can therefore again assume we are in case (1), let \( x_{n}^{1} \) be such a \( y \), and let \( V_{n+1}^{0} := V_{n}^{0} \cap (Z_{1})_{x_{n}}^{1} \).

At the end of the construction, we have produced a dense grid \( Y_{0} \times Y_{1} \). To see that it is a subset of \( Z_{1} \), fix \( m, n \leq \omega \). If \( m \leq n \), then we ensured that \( x_{n}^{1} \in V_{n+1}^{1} \subseteq V_{m+1}^{1} \) and hence \( (x_{m}^{1}, x_{n}^{1}) \in Z_{1} \). Similarly, if \( m > n \), then we ensured that \( x_{m}^{0} \in V_{m}^{0} \subseteq V_{n+1}^{0} \), so again \( (x_{m}^{0}, x_{n}^{1}) \in Z_{1} \). \( \square \)

It remains open whether the full \( \text{PG}_{2} \) is true in ZFC. For \( d > 2 \), we will show in Theorem 4.6 that \( \text{DDF}_{d} \), even restricted to 2-colorings, implies that \( \epsilon \geq \aleph_{d-1} \).

4. Consistent failure of \( \text{DDF}_{d} \)

**Definition 4.1.** Let \( S \) be an infinite set, and suppose that \( \mathcal{H} \) is a collection of subsets of \( S \). We say that \( \mathcal{H} \) is weakly partition regular if for every finite partition \( S = \bigcup_{j} P_{j} \) of \( S \), there is \( j < k \) such that \( P_{j} \in \mathcal{H} \).

Given an infinite cardinal \( \kappa \), we say that \( \mathcal{H} \) is weakly \( \kappa \)-partition regular if for every partition \( S = \bigcup_{\eta < \kappa} P_{\eta} \) of \( S \) into \( \kappa \)-many parts, there is \( \eta < \kappa \) such that \( P_{\eta} \in \mathcal{H} \).

A further weakening of this notion will be useful for us. Given an infinite regular cardinal \( \kappa \), we say that \( \mathcal{H} \) is weakly \( \kappa \)-partition subregular if for every partition \( S = \bigcup_{\eta < \kappa} P_{\eta} \) of \( S \) into \( \kappa \)-many parts, there is \( \xi < \kappa \) such that \( \bigcup_{\eta < \xi} P_{\eta} \in \mathcal{H} \).

**Remark 4.2.** It will often be more convenient to phrase these partition regularity properties in terms of colorings instead of partitions. For instance, a collection \( \mathcal{H} \) of subsets of a set \( S \) is weakly \( \kappa \)-partition subregular if, for every set \( X \) of cardinality \( \kappa \) and every coloring \( c : S \to X \), there is \( Y \in \mathcal{H} \) such that \( |c[Y]| < \kappa \). Throughout, we shall interchangeably use the partition and coloring formulations without explicit comment.

Given \( 0 < d < \omega \) and \( \vec{X} = \langle X_{0}, \ldots, X_{d-1} \rangle \) a sequence of perfect Polish spaces, let \( \text{DDF}(\vec{X}) \) denote the set of subsets of \( \prod_{i < d} X_{i} \) which contain a somewhere-DDF subset. So the principle \( \text{DDF}_{d} \) says that for every such \( \vec{X} \), the collection \( \text{DDF}(\vec{X}) \) is weakly partition regular. Now consider another perfect Polish space \( X_{d} \). We want to consider how partition properties of \( \text{DDF}(\vec{X}) \) affect those of \( \text{DDF}(\vec{X} \setminus X_{d}) \).
Proof. Assume that $\prod DDF$ somewhere-DDF set.

Proposition 4.3. Fix $d \geq 2$ and let $\bar{X}$ and $X_d$ be as above. If $DDF(\bar{X})$ is not weakly $\omega$-partition subregular, then there is a 2-coloring of $\prod_{i \in \omega} X_i$ witnessing that $DDF(\bar{X} \setminus X_d)$ is not weakly partition regular.

Proof. Assume that $DDF(\bar{X})$ is not weakly $\omega$-partition subregular, and fix a partition $\prod_{i \leq d} X_i = \bigcup_{j < \omega} P_j$ such that, for all $k < \omega$, $\bigcup_{j < k} P_j$ does not contain a somewhere-DDF set.

Let $\{S_n : n < \omega\}$ be a sequence of open subsets of $X_d$ with the following property:

- For every non-empty open $U \subseteq X_d$, there is $N < \omega$ such that for all $n \geq N$, we have $S_n \cap U \neq \emptyset$ and $\bigcap_{i < d} P_j$ contains $\emptyset$.

We now describe a partition $\prod_{i \leq d} X_i = P_0^* \cup P_1^*$ that will witness that $DDF(\bar{X})$ is not weakly partition regular. Given $\bar{x} = (x_0, \ldots, x_d) \in \prod_{i \leq d} X_i$, first let $j(\bar{x})$ be the unique natural number $j$ such that $\langle x_0, \ldots, x_{d-1} \rangle \in P_j$. Now put $\bar{x}$ into $P_0^*$ if $x_d \in S_{j(\bar{x})}$ and into $P_1^*$ otherwise.

We claim that this partition is as desired. Suppose for the sake of contradiction that there are non-empty open $U_i \subseteq X_i$ such that $P_0^* (\text{wlog})$ contains a set $Y$ which is DDF in $\prod_{i \leq d} U_i$. However, let $N < \omega$ be such that for all $n \geq N$, we have that $\forall_n U_d$ and $\text{Int}(X_d \setminus S_n) \cap U_d$ are non-empty. Then for every $\bar{x} = (x_0, \ldots, x_d) \in Y$, we must have that $\langle x_0, \ldots, x_{d-1} \rangle \in \bigcup_{j < N} P_j$. This contradicts the assumption that $\bigcup_{j < N} P_j$ does not contain a somewhere-DDF set.

We now show that if the continuum is too small, then for a given $d < \omega$ and any $\bar{X}$ as above, we have that $DDF(\bar{X})$ is not weakly $\omega$-partition subregular. To that end, we now define a sequence of colorings $(c_n : 1 \leq n < \omega)$, where $c_n : [\omega_n]^{n+1} \to \omega$ for all $1 \leq n < \omega$. The definition is by recursion on $n$. First, let $c_1 : [\omega_1]^2 \to \omega$ be any function such that, for all $\beta < \omega_1$, the fiber $c_1(\cdot, \beta) : \beta \to \omega$ is injective. Now suppose that $1 \leq n < \omega$ and we have defined $c_n$. For each $\beta < \omega_{n+1}$, let $e_\beta : \beta \to \omega_n$ be an injective function. Then, for each $\{\alpha_0, \ldots, \alpha_n, \beta\} \in [\omega_{n+1}]^{n+2}$ with $\alpha_0 < \ldots < \alpha_n < \beta$, set

$$c_{n+1}(\alpha_0, \ldots, \alpha_n, \beta) = c_n(e_\beta(\alpha_0), \ldots, e_\beta(\alpha_n)).$$

(Note that the $(n+1)$-tuple $(e_\beta(\alpha_0), \ldots, e_\beta(\alpha_n))$ may not be increasing, but it is certainly injective.)

Now, given $1 \leq n < \omega$ and $a \in [\omega_n]^{n+1}$, we specify a distinguished element $a(\ast) \in a$. We do this by recursion on $n$. First, if $n = 1$, then we simply let $a(\ast) = \text{min}(a)$. Next, if $n > 1$, then let $\beta = \text{max}(a)$, let $a_1 := \{e_\beta(\alpha) : \alpha \in a \setminus \{\beta\}\}$, and let $a(\ast) = e_\beta^{-1}(a_1(\ast))$.

Lemma 4.4. Suppose that $1 \leq n < \omega$, $a, b \in [\omega_n]^{n+1}$, $a(\ast) \neq b(\ast)$, and $a \setminus \{a(\ast)\} = b \setminus \{b(\ast)\}$. Then $c_n(a) \neq c_n(b)$.

Proof. We proceed by induction on $n$. If $n = 1$, then there is $\beta > \text{max}\{a(\ast), b(\ast)\}$ such that $a = \{a(\ast), \beta\}$ and $b = \{b(\ast), \beta\}$. Then $c_1(a) \neq c_1(b)$ follows from the fact that $c_1(\cdot, \beta)$ is injective.

Next, suppose that $n > 1$. Let $\beta = \text{max}(a) = \text{max}(b)$, let $a_1 = \{e_\beta(\alpha) : \alpha \in a \setminus \{\beta\}\}$, and let $b_1 = \{e_\beta(\alpha) : \alpha \in b \setminus \{\beta\}\}$. Then

1. $a(\ast) = e_\beta^{-1}(a_1(\ast))$ and $b(\ast) = e_\beta^{-1}(b_1(\ast))$;
2. $a_1 \setminus \{a_1(\ast)\} = b_1 \setminus \{b_1(\ast)\}$;
3. $c_n(a) = c_{n-1}(a_1)$ and $c_n(b) = c_{n-1}(b_1)$.
Items (1) and (2), combined with the induction hypothesis, imply that $c_{n-1}(a_1) \neq c_{n-1}(b_1)$, and then item (3) implies that $c_n(a) \neq c_n(b)$.

**Lemma 4.5.** Fix $1 \leq n < \omega$. There is a coloring $c : (\omega_n)^{n+1} \to (n+2) \times \omega$ and a sequence of natural numbers $(m_k : k < \omega)$ such that, for every $k < \omega$ and every sequence $\langle A_i : i \leq n \rangle$ of elements of $[\omega_n]^{m_k}$, we have

$$c \left( \prod_{i \leq n} A_i \right) > k.$$  

**Proof.** We first define $c : (\omega_n)^{n+1} \to (n+2) \times \omega$. Suppose that $\vec{\alpha} = \langle \alpha_i : i \leq n \rangle \in (\omega_n)^{n+1}$. If there are $i < j \leq n$ such that $\alpha_i = \alpha_j$, then let $c(\vec{\alpha}) := (n+1, 0)$. Otherwise, set $a_{\vec{\alpha}} := \langle \alpha_i : i \leq n \rangle$, and note that $a_{\vec{\alpha}} \in [\omega_n]^{n+1}$. Let $\ell_{\vec{\alpha}}$ be the unique $i \leq n$ such that $a_{\vec{\alpha}}(i) = \alpha_i$, and let $c(\vec{\alpha}) = (\ell_{\vec{\alpha}}, c_n(a_{\vec{\alpha}}))$.

We now define $(m_k : k < \omega)$. First, let $m_0 = 1$. If $k > 0$, then first let $m_k^* < \omega$ be large enough so that

$$m_k^* \to (n+2)k^{n+1},$$

i.e., for every coloring $r : [m_k^*]^{n+1} \to k$, there is $H \in [m_k^*]^{n+2}$ such that $r[H]^{n+1}$ is constant. Then let $m_k = (n+1) \cdot m_k^*$.

We claim that $(m_k : k < \omega)$ is as desired. If $k = 0$, then this is clear, so fix $k > 0$ and a sequence $\langle A_i : i \leq n \rangle$ of elements of $[\omega_n]^{m_k^*}$, and suppose for sake of contradiction that $c \left( \prod_{i \leq n} A_i \right) \leq k$. First, since $m_k = (n+1) \cdot m_k^*$, we can find a sequence of pairwise disjoint sets $\langle A_i^* : i \leq n \rangle$ such that, for all $i \leq n$, $A_i^* \subseteq [A_i]^{m_k^*}$. In particular, for all $\vec{\alpha} \in \prod_{i \leq n} A_i^*$, $\vec{\alpha}$ is injective and hence, if $c(\vec{\alpha}) = (i, \ell)$, then $i \leq n$ and $\ell = c_n(a_{\vec{\alpha}})$.

For each $i \leq n$, enumerate $A_i^*$ in increasing order as $\langle \alpha_i^j \mid j < m_k^* \rangle$, and define a coloring $r$ of $[m_k^*]^{n+1}$ as follows. Given $u \in [m_k^*]^{n+1}$, let $\vec{\alpha}_u := \langle \alpha_{u(i)}^j \mid i \leq n \rangle$, and let $r(u) = c(\vec{\alpha}_u)$. Since $c \left( \prod_{i \leq n} A_i \right) \leq k$, the coloring $r$ takes at most $k$-many colors. Therefore, by our choice of $m_k^*$, we can find $H \in [m_k^*]^{n+2}$ such that $r[H]^{n+1}$ is constant, say with value $(i^*, \ell^*)$. By the last sentence of the previous paragraph, we know that $i^* \leq n$.

Enumerate $H$ in increasing order as $\langle \ell_0, \ldots, \ell_{i^*-1}, \ell_{i^*, 0}, \ell_{i^*, 1}, \ell_{i^*+1}, \ldots, \ell_n \rangle$, i.e.:

- the first $i^*$-many elements of $H$ are $\langle \ell_0, \ldots, \ell_{i^*-1} \rangle$;
- the next 2 elements of $H$ are $\langle \ell_{i^*, 0}, \ell_{i^*, 1} \rangle$;
- the final $(n-i^*)$-many elements of $H$ are $\langle \ell_{i^*+1}, \ldots, \ell_n \rangle$.

Let $u_0 = H \setminus \{ \ell_{i^*, 1} \}$ and $u_1 = H \setminus \{ \ell_{i^*, 0} \}$, let $\vec{\alpha}^0 = \vec{\alpha}_{u_0}$ and $\vec{\alpha}^1 = \vec{\alpha}_{u_1}$, and let $a^0 = a_{\vec{\alpha}^0}$ and $a^1 = a_{\vec{\alpha}^1}$. Since $r(u_0) = r(u_1) = (i^*, \ell^*)$, we have $c(a^0) = c(a^1) = (i^*, \ell^*)$, and hence

- $a_0(*) = a_{\vec{\alpha}^0}^*$ and $a_1(*) = a_{\vec{\alpha}^1}^*$;
- $c_n(a_0) = c_n(a_1) = \ell^*$.

However, we also know that $a_0(*) \neq a_1(*)$ and $a_0 \setminus \{ a_0(*) \} = a_1 \setminus \{ a_1(*) \}$ and hence, by Lemma 4.3, $c_n(a_0) \neq c_n(a_1)$. This is a contradiction.

**Theorem 4.6.** Suppose that $2 \leq d < \omega$ and $2^{\aleph_0} \leq \aleph_{d-1}$. Then, for every $d$-sequence $\vec{X}$ of perfect Polish spaces, $\text{DF}(\vec{X})$ is not weakly $\omega$-partition subregular.

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We are not making any real attempt to optimize the value of $m_k$ here; we just need to know that a natural number $m_k$ with the desired properties exists.
Suppose that \( n < \omega \) and \( \Lambda \) is a directed quasi-order. The Partition Hypothesis \( \text{PH}_n(\Lambda) \) is the following assertion: for all \( c : \Lambda^{n+1} \to \omega \), there is an \((n+1)\)-cofinal function \( F : \Lambda^{\leq n+1} \to \Lambda \) such that \( c \circ F^* : \Lambda^{[n+1]} \to \omega \) is constant.

**Proof.** Proceed by induction on \( d \). Fix a sequence \( \vec{X} = \langle X_0, \ldots, X_{d-1} \rangle \) of perfect Polish spaces. We can assume that we in fact have \( 2^{\aleph_0} = \aleph_{d-1} \), since if \( 2^{\aleph_0} = \aleph_m < \aleph_{d-1} \), then the induction hypothesis will imply that \( \text{DDF}(\vec{X} \langle m + 1 \rangle) \) is not weakly \( \omega \)-partition subregular, which immediately implies that \( \text{DDF}(\vec{X}) \) is not weakly \( \omega \)-partition subregular either. For all \( i < d \), \( X_i \) is a perfect Polish space, so we can injectively enumerate \( X_i \) as \( \langle x_{i,\alpha} : \alpha < \omega_{d-1} \rangle \).

By Lemma 4.5 with \( n = d - 1 \), we can find a coloring \( c : \prod_{i<d} X_i \to (d + 1) \times \omega \) and a sequence of natural numbers \( \langle m_k : k < \omega \rangle \) such that, for every \( k < \omega \) and every sequence \( \langle A_i : i < d \rangle \) such that \( A_i \) is a subset of \( X_i \) of size \( m_k \) for each \( i < d \), we have \( |c(\prod_{i<d} A_i)| > k \). By Proposition 3.5, every \( Y \in \text{DDF}(\vec{X}) \) contains arbitrarily large finite products, which by the discussion above implies that \( c[Y] \) is infinite.

**Corollary 4.7.** Suppose that \( 2 \leq d < \omega \) and \( 2^{\aleph_0} \leq \aleph_{d-1} \). Then, for every \((d + 1)\)-sequence \( \vec{X} \) of perfect Polish spaces, there is a 2-coloring of \( \prod_{i<d} X_i \) witnessing that \( \text{DDF}(\vec{X}) \) is not weakly partition regular.

**Proof.** This is immediate from Proposition 4.3 and Theorem 4.6.

We have now established clauses (1) and (2) of Theorem A.

**Corollary 4.8.** Suppose that \( 2 \leq d < \omega \).

1. If \( \text{DDF}_d \) holds, then \( 2^{\aleph_0} \geq \aleph_{d-1} \).
2. If \( \text{PG}_d(\aleph_0) \) holds, then \( 2^{\aleph_0} \geq \aleph_d \).

**Proof.** Note that \( \text{DDF}_d \) is equivalent to the assertion that, for every \( d \)-sequence \( \vec{X} \) of perfect Polish spaces, \( \text{DDF}(\vec{X}) \) is weakly partition regular, and \( \text{PG}_d(\aleph_0) \) implies that \( \text{DDF}(\vec{X}) \) is weakly \( \omega \)-partition subregular. Clauses (1) and (2) then immediately follow from Corollary 4.7 and Theorem 4.6 respectively.

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**4.1. On the Partition Hypothesis.** In [1], Bannister, Bergfalk, Moore, and Todorčević introduce a partition hypothesis denoted \( \text{PH}_n(\Lambda) \), where \( n < \omega \) is a dimensional parameter and \( \Lambda \) is an arbitrary directed quasi-order. They prove there that, for all \( n < \omega \), \( \text{PH}_n(\omega_n) \) fails, where \( \omega_n \) has the usual ordinal ordering. The proof of this fact presented in [1] makes heavy use of ideas coming from simplicial homology. Here, we show how the results from this section yield a direct, purely combinatorial proof. We first recall the following definitions from [1].

**Definition 4.9.** Suppose that \( 1 \leq n < \omega \) and \( \Lambda \) is a directed quasi-order.

1. If \( \vec{x}, \vec{y} \in \Lambda^{\leq n} \), then we write \( \vec{x} \sqsubseteq \vec{y} \) to indicate that \( \vec{x} \) is a subsequence of \( \vec{y} \) (not necessarily an initial segment). \( \vec{x} \prec \vec{y} \) indicates that \( \vec{x} \) is a proper subsequence of \( \vec{y} \).
2. A function \( F : \Lambda^{\leq n} \to \Lambda \) is \( n \)-cofinal if
   a. \( x \leq F(x) \) for all \( x \in \Lambda \);
   b. \( F(\vec{x}) \leq F(\vec{y}) \) for all \( \vec{x} \sqsubseteq \vec{y} \) in \( \Lambda^{\leq n} \).
3. Let \( \Lambda[n] \subseteq \coprod_{i<n} \Lambda^{i+1} \) consist of all \( \sigma \in \prod_{i<n} \Lambda^{i+1} \) that are \( \leq \)-increasing.
   If \( F : \Lambda^{\leq n} \to \Lambda \) is \( n \)-cofinal, define \( F^* : \Lambda[n] \to \Lambda^n \) by letting \( F^*(\sigma) = F \circ \sigma = \langle F(\sigma(i)) \mid i < n \rangle \) for all \( \sigma \in \Lambda[n] \).

**Definition 4.10.** Suppose that \( n < \omega \) and \( \Lambda \) is a directed quasi-order. The Partition Hypothesis \( \text{PH}_n(\Lambda) \) is the following assertion: for all \( c : \Lambda^{n+1} \to \omega \), there is an \((n+1)\)-cofinal function \( F : \Lambda^{\leq n+1} \to \Lambda \) such that \( c \circ F^* : \Lambda^{[n+1]} \to \omega \) is constant.
We are now ready to give a direct proof of the aforementioned result from [1].

**Theorem 4.11.** For all \( n < \omega \), \( \text{PH}_n(\omega_n) \) fails.

**Proof.** As noted in [1], the identity function \( c : \omega \rightarrow \omega \) readily witnesses the failure of \( \text{PH}_0(\omega) \). Therefore, fix \( 1 \leq n < \omega \), and let \( c : (\omega_n)^{n+1} \rightarrow (n+2) \times \omega \) be the function defined in the proof of Lemma 4.5.

We claim that \( c \) witnesses the failure of \( \text{PH}_n(\omega_n) \). Towards a contradiction, suppose that \( F : (\omega_n)^{n+1} \rightarrow \omega_n \) is an \( (n+1) \)-cofinal function such that \( c \circ F^* \) is constant, taking value \((i^*, k^*) \in (n+2) \times \omega \). As shown in [1] Lemma 7.8, we may assume that \( F \) is strictly increasing, i.e., \( F(\vec{x}) \leq F(\vec{y}) \) for all \( \vec{x} \triangleleft \vec{y} \) in \( (\omega_n)^{\leq n+1} \). In particular, we can assume that \( F^*(\sigma) = F \circ \sigma \) is injective for all \( \sigma \in \omega_n^{[n+1]} \), and hence, recalling the definition of \( c \), we know that \( i^* \neq n + 1 \). In addition, since \( F \circ \sigma \) is strictly increasing for all \( \sigma \in \omega_n^{[n+1]} \), the definition of \( c \) implies that we in fact have \( i^* < n \).

Let \( \alpha^* := F(0,1,\ldots,i^*) + 1 \). Now define \( \sigma_0, \sigma_1 \in \omega_n^{[n+1]} \) as follows:

- for all \( i < i^* \), \( \sigma_0(i) = \sigma_1(i) = \langle 0,1,\ldots,i \rangle \);
- \( \sigma_0(i^*) := \langle 0,1,\ldots,i^* \rangle \);
- \( \sigma_1(i^*) := \langle 0,1,\ldots,i^* - 1, \alpha^* \rangle \);
- for all \( \ell < n - i^* \), \( \sigma_0(i^* + \ell + 1) = \sigma_1(i^* + \ell + 1) := \langle 0,1,\ldots,i^*, \alpha^*, \alpha^* + 1, \ldots, \alpha^* + \ell \rangle \).

Let \( a_0 := F^*(\sigma_0) \) and \( a_1 := F^*(\sigma_1) \). Since \( \sigma_0 \) and \( \sigma_1 \) only differ in their \( i^* \)-th entry and \( F^*(\sigma_0) \) and \( F^*(\sigma_1) \) are strictly increasing, we have \( a_0(i) = a_1(i) \) for all \( i \in (n+1) \setminus \{i^*\} \). By the definition of \( c \) and the fact that \( c \circ F^* \) is constant with value \( (i^*, k^*) \), it follows that \( a_0(*) = a_1(*) = i^* \) (recall the notation from the paragraph preceding Lemma 4.4). Moreover, since \( \alpha^* > F(\sigma_0(i^*)) \) and \( F \) is \( (n+1) \)-cofinal, we know that 

\[
\alpha^* = F(\sigma_0(i^*)) < \alpha^* \leq F(\sigma_1(i^*)) = a_1(i^*).
\]

Then Lemma 4.4 implies that \( c_n(a_0) \neq c_n(a_1) \). However, again by the fact that \( c \circ F^* \) is constant with value \( (i^*, k^*) \), it must be the case that \( c_n(a_0) = c_n(a_1) = k^* \), which is a contradiction. \( \square \)

5. **Forcing PGd**

We now finish the proof of Theorem A by proving part (3). We show that adding \( \beth_{d-1} \)-many Cohen reals to any model of ZFC yields a model of PGd(\( \aleph_0 \)). In particular, by starting with a model of GCH, Theorem 4.6 is consistently sharp. We will need the notion of uniform \( n \)-dimensional \( \Delta \)-system isolated in [10]. We begin by recalling the relevant definitions.

**Definition 5.1.** Suppose that \( a \) and \( b \) are sets of ordinals.

1. We say that \( a \) and \( b \) are **aligned** if \( \text{otp}(a) = \text{otp}(b) \) and \( \text{otp}(a \cap \gamma) = \text{otp}(b \cap \gamma) \) for all \( \gamma \in a \cap b \). In other words, if \( \gamma \) is a common element of two aligned sets \( a \) and \( b \), then it occupies the same relative position in both \( a \) and \( b \).

2. If \( a \) and \( b \) are aligned then we let \( r(a,b) := \{ i < \text{otp}(a) : a(i) = b(i) \} \). Notice that, in this case, \( a \cap b = a[r(a,b)] = b[r(a,b)] \).

**Definition 5.2.** Suppose that \( H \) is a set of ordinals, \( n \) is a positive integer, and \( u_b \) is a set of ordinals for all \( b \in [H]^n \). We call \( \langle u_b : b \in [H]^n \rangle \) a **uniform \( n \)-dimensional \( \Delta \)-system** if there are an ordinal \( \rho \) and, for each \( m \subseteq n \), a set \( r_m \subseteq \rho \) satisfying the following statements.
(1) $\text{otp}(u_b) = \rho$ for all $b \in [H]^n$.
(2) For all $a, b \in [H]^n$ and $m \subseteq n$, if $a$ and $b$ are aligned with $r(a, b) = m$, then $u_a$ and $u_b$ are aligned with $r(u_a, u_b) = r_m$.
(3) For all $m_0, m_1 \subseteq n$, we have $r_{m_0 \cap m_1} = r_{m_0} \cap r_{m_1}$.

The following is a corollary of the main lemma of [10].

**Corollary 5.3.** ([10] Corollary 3.16) Suppose that $1 \leq n < \omega$, and let $\mu = \beth_\omega^{n-1}$. If $\langle u_b : b \in [\mu]^n \rangle$ is a sequence of finite sets of ordinals and $g : [\mu]^n \to \omega$ is a function, then there is $H \in [\mu]^{\aleph_1}$ such that $\langle u_b : b \in [H]^n \rangle$ is a uniform $n$-dimensional $\Delta$-system and $g \upharpoonright [H]^n$ is constant.

We now turn to the proof of Theorem A(3). Recall that every perfect Polish space contains a dense $G_\delta$ subspace homeomorphic to Baire space $\omega^\omega$. Therefore to show that $\text{PG}_d(\mathbb{N}_0)$ holds, one may assume that each $X_i$ is the Baire space. It will be helpful to write $X_i = [T_i]$, where each $T_i$ is a copy of the tree $<^\omega \omega$. Let $\theta := \beth_{d-1}$, and let $\mathbb{P} = \text{Add}(\omega, \beth_{d-1})$ be the forcing to add $\beth_{d-1}$-many Cohen reals. We think of the conditions in $\mathbb{P}$ as being all finite partial functions $p : \theta \to \bigcap_{i<d} T_i$, and $q \leq p$ if and only if $\text{dom}(q) \supseteq \text{dom}(p)$ and, for all $\alpha \in \text{dom}(p)$ and $i < d$, $p(\alpha)(i) \leq q(\alpha)(i)$. For each $\alpha < \theta$ and $i < n$, let $\check{x}_n^\alpha$ be the canonical $\mathbb{P}$-name for $\{p(\alpha)(i) \mid p \in \check{G}\}$, where $\check{G}$ is the canonical $\mathbb{P}$-name for the generic filter.

By standard arguments, $\check{x}_n^\alpha$ is forced to be an element of $[T_i]$ (as defined in $V^\mathbb{P}$).

For each $p \in \mathbb{P}$, we define a “collapsed” version of $p$, denoted $\check{p}$, as follows. Let $\ell := |\text{dom}(p)|$, and enumerate $\text{dom}(p)$ in increasing order as $\langle \alpha_k : k < \ell \rangle$. Then define $\check{p} : \ell \to \bigcap_{i<d} T_i$ by letting $\check{p}(k) = p(\alpha_k)$ for all $k < \ell$. Note that each collapsed condition is a function from some natural number to a countable set, so there are only countably many such collapsed conditions.

Let $\check{c}_0$ be a $\mathbb{P}$-name for a function from $\prod_{i<d}[T_i] \to \omega$. We will really only be interested in the values of $\check{c}_0$ on $(n+1)$-tuples of the generic branches $\check{x}_n^\alpha$, so let $\check{c}$ be a $\mathbb{P}$-name for a function from $[\theta]^d$ to $\omega$ defined in the following way: for all $(\alpha_0, \ldots, \alpha_{d-1}) \in [\theta]^d$, let $\check{c}(\alpha_0, \ldots, \alpha_{d-1}) = \check{c}_0(\check{x}_0, \ldots, \check{x}_n^{\alpha_{d-1}})$ (recall our convention that the notation $(\alpha_0, \ldots, \alpha_{d-1}) \in [\theta]^d$ implies that $\alpha_0 < \alpha_1 < \ldots < \alpha_{d-1}$).

Fix an arbitrary $p \in \mathbb{P}$. We will find $q \leq p$ and $j < \omega$ such that $q$ forces the existence of a sequence $\langle Y_i \mid i < d \rangle$ such that each $Y_i$ is a somewhere dense subset of $[T_i]$ and $c' \upharpoonright \prod_{i<d} Y_i$ is constant, taking value $j$.

For each $a \in [\theta]^d$, find a condition $q_a \leq p$ and $j_a < \omega$ such that $q_a \forces \check{c}_a = j_a$. Let $u_a := \text{dom}(q_a)$. Without loss of generality, assume that $a \subseteq u_a$ for every $a \in [\theta]^d$.

By Corollary 5.3 we can find $H \in [\theta]^{\aleph_1}$, a “collapsed” condition $\check{q}^*$, natural numbers $j^*$ and $\rho$, and a set $r^* \in [\rho]^d$ such that

- for all $a \in [H]^d$, we have $q_a = \check{q}^*$ and $j_a = j^*$;
- $(u_a : a \in [H]^d)$ forms a uniform $d$-dimensional $\Delta$-system; and
- for all $a \in [H]^d$, we have $|u_a| = \rho$ and $a = u_a[r^*]$.

By taking an initial segment if necessary, assume that $\text{otp}(H) = \omega_1$. Let $(r_m : m \subseteq d)$ witness the fact that $\langle u_a : a \in [H]^d \rangle$ is a uniform $d$-dimensional $\Delta$-system. For each $m < d$ and each $a \in [H]^m$, define $u_a$ and $q_a$ by choosing any $b \in [H]^d$ for which $b[m] = a$ and setting $u_a := u_b[r_m]$ and $q_a := q_b \upharpoonright u_a$. By our uniformization of $H$ (cf. [2] Lemma 2.3), these definitions are independent of our choice of $b$. 


Let \( q := q_0 \). Since \( q_a \leq_p \emptyset \) for every \( a \in [H]^d \), it follows that \( \text{dom}(p) \subseteq u_0 \) and hence \( q_0 \leq p \). Also, for each \( i < d \), let \( s_i := \overline{\eta}(r^*(i))(i) \). In other words, \( s_i \in T_i \) is such that, for all \( a = (\alpha_0, \ldots, \alpha_{d-1}) \in [H]^d \), we have \( q_a(\alpha_i)(i) = s_i \). We claim that \( q \) is as desired; in particular, \( q \) forces the existence of a sequence of sets \( \langle Y_i : i < d \rangle \) such that

- for all \( i < d \), \( Y_i \) is \( s_i \)-dense in \([T_i]\), i.e., for all \( t \geq T_i \), \( s_i \), there is \( y \in Y_i \) such that \( t \in y \);
- \( \bar{c}_0 \restriction \prod_{i < d} Y_i \) is constant, taking value \( j^* \).

**Claim 5.4.** Suppose that \( m < d \), \( a \in [H]^m \), and \( \gamma \in H \setminus (\max(a) + 1) \). Then the set \( D_{a, \gamma} := \{ q_{a^{-}\langle \beta \rangle} : \beta \in H \setminus \gamma \} \) is pre-dense below \( q_a \) in \( \mathbb{P} \).

**Proof.** Let \( r \leq_p q_a \) be arbitrary. The set \( \{ u_{a^{-}\langle \beta \rangle} : \beta \in H \setminus \gamma \} \) forms an uncountable \( \Delta \)-system with root \( u_a \). We can therefore find \( \beta \in H \setminus \gamma \) for which \( \text{dom}(r) \cap (u_a \setminus \langle u_a \setminus \langle \beta \rangle \rangle) = \emptyset \). We also know that \( q_{a^{-}\langle \beta \rangle} \restriction u_a = q_a \), and \( r \leq_p q_a \). It follows that \( r \) and \( q_{a^{-}\langle \beta \rangle} \) are compatible in \( \mathbb{P} \), so \( D_{a, \gamma} \) is indeed pre-dense below \( q_a \). \( \square \)

Now let \( G \) be \( \mathbb{P} \)-generic over \( V \) with \( q \in G \), and let \( c \) be the realization of \( \bar{c} \) in \( V[G] \). By recursively applying Claim 5.4 \( d \)-many times, we can find a set \( \delta \in [H]^d \), enumerated in increasing order as \( \langle \delta_0, \ldots, \delta_{d-1} \rangle \) such that

- for all \( i < d - 1 \), \( H \cap (\delta_i, \delta_{i+1}) \) is infinite;
- \( H \cap \delta_0 \) is infinite;
- \( q_\delta \in G \).

Let \( H_0 \) denote the set of the first \( \omega \)-many elements of \( H \), and for \( i < d - 1 \), let \( H_{i+1} \) denote the set of the first \( \omega \)-many elements of \( H \cap (\delta_i, \delta_{i+1}) \). Note that each \( H_i \) is an element of \( V \). Now, working in \( V[G] \), we will recursively construct a matrix of ordinals \( \langle \alpha_{i,k} : i < d, k < \omega \rangle \) such that, setting \( A_i := \{ \alpha_{i,k} : k < \omega \} \) and \( Y_i := \{ x_{i}^{\alpha} : \alpha \in A_i \} \) for all \( i < d \), we have the following:

- for all \( i < d \), \( A_i \subseteq H_i \cup \{ \delta_i \} \);
- for all \( i < d \), \( Y_i \) is \( s_i \)-dense in \([T_i]\);
- for all \( a \in \prod_{i < d} A_i \), we have \( q_a \in G \), and hence \( c \restriction \prod_{i < d} Y_i \) is constant, taking value \( j^* \).

The construction of the matrix of ordinals is by recursion on \( k < \omega \) and, for fixed \( k \), by recursion on \( i < d \); in other words, the construction is by recursion on the anti-lexicographic ordering of \( d \times \omega \).

For each pair \( (i, k) \in d \times \omega \) and each \( j < d \), let \( A_j \restriction (i, k) \) be the portion of \( A_j \) constructed before stage \( (i, k) \) of the process, i.e., \( A_j \restriction (i, k) = \{ \alpha_{j,\ell} : \ell \leq k \} \) if \( j < i \) and \( A_j \restriction (i, k) = \{ \alpha_{j,\ell} : \ell < k \} \) if \( j \geq i \). Our recursion hypothesis will be the assumption that \( q_a \in G \) for all \( a \in \prod_{j \leq i} A_j \restriction (i, k) \) by the time we have reached stage \( (i, k) \) of the construction. Enumerate \( <^\omega \omega \) as \( \langle t_k : k < \omega \rangle \), with \( t_0 = \emptyset \). We will also maintain the requirement that, for all \( (i, k) \in d \times \omega \), \( x_{i,k}^{\alpha} \) extends \( s_i^{-}t_k \); this is what will ensure that \( Y_i \) is \( s_i \)-dense in \([T_i]\).

Begin by setting \( \alpha_{i,0} := \delta_i \) for all \( i \leq n \). The fact that \( q_\delta \in G \) ensures that this satisfies the recursion hypotheses. Now suppose that \( (i, k) \in d \times \omega \), with \( i \geq 1 \), and we have reached stage \( (i, k) \) of the construction. Let

\[ r_{i,k} = \bigcup \{ q_a : a \in \prod_{j < d} A_j \restriction (i, k) \} \].
By our recursion hypothesis, $r_{i,k}$ is a condition in $\mathbb{P}$ and is in fact in $G$. Let $B_0 := \prod_{i < j} A_j \upharpoonright (i,k)$ and $B_1 := \prod_{i < j < d} A_j \upharpoonright (i,k)$. Note that both $B_0$ and $B_1$ are in $V$, as they are finite sets of finite sequences of ordinals. For each $\alpha \in H_i$, let

$$q_\alpha^* := \bigcup \{ q_{b_0}^\alpha : b_0 \in B_0, b_1 \in B_1 \}.$$ 

Notice that, for all $b_0, b_1, b_0', b_1' \in B_0$ and $b_0^\alpha(b_1) - b_1^\alpha(b_1)$ are aligned; it follows that $q_\alpha^*$ is a condition in $\mathbb{P}$. Moreover, for all $(b_0, b_1) \in B_0 \times B_1$, we have $q_{b_0}^\alpha(b_1) = s_i$, and hence $q_\alpha^*(b_1(i)) = s_i$. We can therefore extend $q_\alpha^*$ to a condition $q_{\alpha^*}$ with the same domain by letting $q_{\alpha^*}(b_1(i)) = q_{\alpha^*}(b_1(i)) = q_{\alpha^*}(b_1(i))$ for all $(\eta, j) \in (\text{dom}(q_\alpha^*) \times d) \setminus \{ (\alpha, i) \}$. 

**Claim 5.5.** The set $E := \{ q_{\alpha^*} : \alpha \in H_i \setminus \{ \alpha_{i,\ell} \mid \ell < k \} \}$ is predense in $\mathbb{P}$ below $r_{i,k}$. 

**Proof.** Fix an arbitrary condition $r \leq_{\mathbb{P}} r_{i,k}$; we will find a condition in $E$ that is compatible with $r$. Let $m := d \setminus \{ i \}$, and let $H^* := H_i \setminus \{ \alpha_{i,\ell} \mid \ell < k \}$. For each $(b_0, b_1) \in B_0 \times B_1$, the set $\{ u_{b_0}^\alpha \setminus b_0^\alpha : \alpha \in H^* \}$ forms a $\Delta$-system whose root is equal to $v_{b_0} = u_{b_0}^\alpha \setminus b_0^\alpha[r_m]$ for some (and therefore all) $\alpha \in H^*$. Since there are only finitely many such pairs $(b_0, b_1)$ and since $H^*$ is infinite, we can therefore fix $\alpha \in H^*$ such that, for all $(b_0, b_1) \in B_0 \times B_1$, we have

$$(u_{b_0}^\alpha \setminus b_0^\alpha) \setminus v_{b_0} \cap \text{dom}(r) = \emptyset.$$ 

In particular, we have $\alpha \notin \text{dom}(r)$. 

We claim that $q_{\alpha^*}^*$ and $r$ are compatible. Since $\alpha \notin \text{dom}(r)$, it suffices to show that $q_{b_0}^\alpha \setminus b_0^\alpha$ and $r$ are compatible for all $(b_0, b_1) \in B_0 \times B_1$. Thus, fix such a $(b_0, b_1)$. Note that $b_0^\alpha \setminus b_0^\alpha$ and $b_0^\alpha \setminus b_0^\alpha$ are aligned and that $q_{b_0}^\alpha \setminus b_0^\alpha \upharpoonright v_{b_0} = q_{b_0}^\alpha \setminus b_0^\alpha \upharpoonright v_{b_0}$. Since $r_{i,k} \leq_{\mathbb{P}} q_{b_0}^\alpha \setminus b_0^\alpha$ and $r \leq_{\mathbb{P}} r_{i,k}$, we know that $r$ is compatible with $q_{b_0}^\alpha \setminus b_0^\alpha \upharpoonright v_{b_0}$, and since $u_{b_0}^\alpha \setminus b_0^\alpha \setminus v_{b_0}$ is disjoint from $\text{dom}(r)$, it follows that $r$ is compatible with $q_{b_0}^\alpha \setminus b_0^\alpha$ and therefore with $q_{\alpha^*}$.

By the claim and the fact that $r_{i,k} \in G$, we can fix an $\alpha_{i,k} \in H_i \setminus \{ \alpha_{i,\ell} : \ell < k \}$ such that $q_{\alpha^*}^* \in G$ and proceed to the next stage of the recursive construction. At the end of the construction, our recursion hypothesis ensures that, for all $a \in \prod_{i < d} A_i$, we have $q_a \in G$ and hence $c \upharpoonright \prod_{i < d} A_i$ is constant, taking value $j^*$. It follows that $c_0 \upharpoonright \prod_{i < d} Y_i$ is constant, also taking value $j^*$. Finally, our construction ensures that for every $i < d$ and every $t \in <\omega$, there is $x \in Y_i$ extending $s_i \upharpoonright t$, and hence $Y_i$ is $s_i$-dense in $[T_i]$, as desired. This completes the proof of Theorem A(3).

**Corollary 5.6.** For every $1 \leq d < \omega$, it is consistent that $2^{\aleph_0} = \aleph_d$ and, for every $d$-sequence $\bar{X}$ of perfect Polish spaces, $\text{DDF}(\bar{X})$ is weakly $\omega$-partition regular.

6. **PG$_d$ AND CODING TREES**

We conclude with a brief discussion of using PG to prove more detailed versions of the Halpern-Läuchli theorem pertaining to coding trees. In recent years, coding trees in various forms have been developed in [4] and [19] to code countable structures in a finite binary language, such as the Rado graph or Henson’s triangle-free graph. Halpern-Läuchli theorems for the strict similarity types of [4] or the aged embeddings of [19], proven using a Harrington-style forcing argument, form the pigeon-hole principle used to show that certain Fraïssé classes have finite big Ramsey degrees. Indeed, this is a major motivation for trying to develop new proofs of the Halpern-Läuchli theorem, as these new proofs might generalize to previously unknown settings.
With this in mind, we show how PG can be used to prove a version of HL for coding trees of a simple form. Fix $0 < d, k < \omega$, and for each $i < d$, let $T_i$ be a copy of the tree $\omega^k$. We assume that $T_i \cap T_j = \emptyset$ for $i \neq j < d$. We now enrich each $T_i$ to a structure $T_i$ by declaring that for each $m < \omega$, at most one node of $\bigcup_{i<d} T_i(m)$ is a coding node. Write $T = (T_0, ..., T_{d-1})$ for this sequence of structures. If $\bigcup_{i<d} T_i(m)$ contains a coding node, we write $c_T(m)$ for this node. We assume that for every $i < d$ and every $t \in T_i$, there is some $n < \omega$ so that $c_T(n) \in T_i$ and $t \subseteq c_T(n)$.

An embedding of $T$ into itself is an injection $f: \bigcup_{i<d} T_i \to \bigcup_{i<d} T_i$ satisfying the following properties:

1. $f[T_i] \subseteq T_i$ for each $i < d$.
2. $f$ preserves tree order, meets, relative levels, and lexicographic order. Write $\tilde{f}: \omega \to \omega$ for the induced function on levels.
3. If $c_T(m)$ exists, then so does $c_T(\tilde{f}(m))$, and we have $f(c_T(m)) = c_T(\tilde{f}(m))$.

Write $\text{Emb}(T, T)$ for the set of embeddings of $T$ into itself.

Note that if we remove the extra coding node structure and item (3) from the above, then items (1) and (2) describe strong subtrees of the form appearing in the ordinary Halpern-Läuchli theorem. To state the version for these coding trees, first observe that the level product $\bigotimes T_i$ now contains $d + 1$ different types of elements; given $(t_0, ..., t_{d-1}) \in \bigotimes T_i$, either none of the $t_i$ is a coding node, or exactly one of the $t_i$ is a coding node. We refer to the former case as type $-1$ and the latter as type $i$ for a given $i < d$. Given $p \in \{-1, ..., d - 1\}$, let $T(p) \subseteq \bigotimes T_i$ denote those tuples of type $p$.

**Theorem 6.1.** Given $p \in \{-1, 0, ..., d - 1\}$, $r < \omega$, and a coloring $\gamma: T(p) \to r$, there is $f \in \text{Emb}(T, T)$ such that $f[T(p)] \subseteq T(p)$ is monochromatic for $\gamma$.

We opt to give a relatively straightforward proof using PG$_d$. Then by Shoenfield absoluteness, this yields a ZFC proof.

**Proof.** For each $i < d$, let $X_i \subseteq [T_i]$ consist of those branches which contain infinitely many coding nodes. Then $X_i$ is a dense $G_\delta$ subspace of $[T_i]$, so is itself a perfect Polish space. In the case where $p = -1$, the proof is almost identical to that of Proposition 5.2, the key difference being that one works inside the space $\prod_{i<d} X_i$ so that when building $f$, we can ensure there are coding nodes where we need them.

So now assume that without loss of generality $p = 0$. For each $y \in X_0$, let $U: X_0 \to \beta \omega$ be a function such that for every $y \in X_0$, we have that $C(y) := \{n < \omega : c_T(n) \in y\} \in U(y)$. Form a coloring $\tilde{\gamma}: \prod_{i<d} X_i$ by setting $\tilde{\gamma}(y_0, ..., y_{d-1}) = j < r$ if $\{n \in C(y_0): \gamma(y_0(n), ..., y_{d-1}(n)) = j\} \in U(y_0)$. Using PG$_d$, find somewhere dense sets $Y_i \subseteq X_i$ for each $i < d$ such that $\prod_{i<d} Y_i$ is monochromatic, say with color $j < r$. Say that $(s_0, ..., s_{d-1}) \in \bigotimes T_i$ is such that each $Y_i$ is dense above $s_i$. We now proceed to define $f \in \text{Emb}(T, T)$. Suppose $m < \omega$ and that $f$ has been defined on $\bigcup_{i<d} T_i(t)$. If $m = 0$, then letting $\emptyset_t \in T_t(0)$ denote the root, define $f'(\emptyset_t) = s_t$. If $m > 0$, we define the map $f': \bigcup_{i<d} T_i(m)$, where given $t \in \bigcup_{i<d} T_i(m)$ with $t = s^t b$ for some $s \in \bigcup_{i<d} T_i(m-1)$ and some $b < k$, then we set $f'(t) = f(s)^t b$. If $T_0(m)$ does not contain a coding node, then we can define $f$ on $\bigcup_{i<d} T_i(m)$ as follows. If $c_T(m)$ exists, pick any $n < \omega$ such that $c_T(n) \subseteq f'(c_T(m))$, set $f(c_T(m)) = c_T(n)$, and for every other $t \in \bigcup_{i<d} T_i(m)$, let $f(t) \in \text{succ}(f'(t), n)$. If $c_T(m) \in T_0(m)$, first for
each \(i < d\) and each \(t \in T_i(m)\), fix a branch \(y_t \in Y_i\) with \(f'(t) \in y_0\). Writing \(y = y_{CT(m)}\), then for any choice of \(t_i \in T_i(m)\) for \(0 < i < d\), we have \(\{n \in C(y) : \gamma(y(n), y_{i_1}(n), ..., y_{d-1}(n)) = j\} \in U(y)\). Hence we can find \(n \in C(y)\) so that for every \(0 < i < d\) and every \(t_i \in T_i(m)\), we have that \(\gamma(y(n), y_{i_1}(n), ..., y_{d-1}(n)) = j\). We then set \(f(t) = y_i(n)\) for every \(t \in \bigcup_{i < d} T_i(m)\).

One major difficulty in generalizing the above argument to the more general coding trees and aged embeddings of [19] is that in general, the forcing one needs to use is not Cohen forcing. This suggests that rather than the principle PG, one would ask for a combinatorial principle corresponding to each specific type of forcing used. Nonetheless, the following seems like a worthwhile question to ask.

**Question 6.2.** Is there a family of consistent combinatorial principles about partitions of structures on products of Polish spaces which implies all of the variants of the Halpern-Läuchli theorem appearing in [14] and [19]?
