(t, ℓ)-STABILITY AND COHERENT SYSTEMS

L. BRAMBILA-PAZ AND O. MATA-GUTIERREZ

ABSTRACT. Let $X$ be a non-singular irreducible complex projective curve of genus $g \geq 2$. We use $(t, ℓ)$-stability to prove the existence of coherent systems over $X$ that are $\alpha$-stable for all allowed $\alpha > 0$.

1. Introduction

Let $X$ be a non-singular irreducible complex projective curve of genus $g \geq 2$. A coherent system of type $(n, d, k)$ on $X$ is a pair $(E, V)$ where $E$ is a vector bundle on $X$ of rank $n$ and degree $d$ and $V \subset H^0(X, E)$ is a linear subspace of dimension $k$. For any real number $\alpha$ there is a concept of $\alpha$-stability and there exist moduli spaces $G(\alpha; n, d, k)$ (see [14] and [21]). A necessary condition for the non-emptiness of $G(\alpha; n, d, k)$ is that $\alpha > 0$. There are finitely many critical values $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_L$ of $\alpha$; as $\alpha$ varies, the concept of $\alpha$-stability remains constant between two consecutive critical values. We denote by $G_0(n, d, k)$ the moduli spaces corresponding to $0 < \alpha < \alpha_1$ and by $U(n, d, k)$ the scheme $U(n, d, k) := \{(E, V) \in G_0(n, d, k) : (E, V)$ is $\alpha$-stable for all $\alpha > 0$ and $E$ is stable $\}$.

The scheme $U(n, d, k)$ have been studied in extenso for $k \leq n + 1$ (see [6] and [5] for $k \leq n$, and [7], [3] and [4] for $k = n + 1$). We are interested in the non-emptiness of $U(n, d, k)$ for $k \geq n + 2$.

Let $M(n, d)$ be the moduli space of stable vector bundles over $X$ of degree $d$ and rank $n$. M.S. Narasimhan and S. Ramanan in [18] (see also [19]) introduced the concept of $(t, ℓ)$-stability (see Definition 3.1). The aim of this paper is to relate $(t, ℓ)$-stability of the vector bundle $E$ with $\alpha$-stability of the coherent system $(E, V)$.

Write

$$\varepsilon = \begin{cases} 1 & \text{if } d \equiv g - 1 \mod n \\ 0 & \text{otherwise,} \end{cases}$$

For any positive integers $0 \leq a \leq g - 1 - \varepsilon$ denote by $A_a(n, d, k)$ the subscheme

$$A_a(n, d, k) := \{(E, V) \in G_0(n, d, k) : E \text{ is } (0,a) - \text{stable}\},$$

Using the $(t, ℓ)$-stability and the Clifford’s Theorem for coherent systems (see [2.1]), we prove the following theorems (see Theorem 3.5 and 3.7).

Theorem 1.1. Assume $d \geq 2ng + 2b - a$ and $k \geq d + n(1 - g) - b$ with $0 \leq b \leq a < g - 1 - \varepsilon$. If $A_a(n, d, k) \neq \emptyset$ then $U(n, d, k) \neq \emptyset$. Moreover, if $k \leq d + n(1 - g)$ then
\[\emptyset \neq A_a(n, d, k) \subset U(n, d, k)\] and \(U(n, d, k)\) has a component of the expected dimension and birational to a Grassmannian bundle over an open set of \(M(n, d)\).

Given \((n, d, k)\) denote by \(\lambda = d - 2(k - n)\) and by \(A_{a,\lambda}(n, d, k)\) the scheme
\[A_{a,\lambda}(n, d, k) := \{(E, V) \in A_a(n, d, k) : \lambda \leq a}\].

**Theorem 1.2.** If \(A_{a,\lambda}(n, d, k) \neq \emptyset\) then \(U(n, d, k) \neq \emptyset\). Moreover, \(A_{a,\lambda}(n, d, k) \subset U(n, d, k)\).

For lower degrees we still have the relation between \((0, a)\)-stability and \(\alpha\)-stability. However, the existence of such bundles depends on the non emptiness of a Brill Noether locus. Nevertheless, for rank 2 and 3 we prove (see Theorem 3.10 and 3.12)

**Theorem 1.3.** Assume \(k = r + 2\) with \(r \geq 1\). If there exists an integer \(0 \leq a \leq g - 1 - \varepsilon\) such that
\[
\max\left\{d - 2g - a, \frac{d - a}{2}\right\} \leq r < d - 2g + g - a + \delta - 3 < \frac{r + 2}{r + 2},
\]
then \(U(2, d, k) \neq \emptyset\). Moreover, \(\emptyset \neq A_a(2, d, k) \subset U(2, d, k)\).

With the notation
\[
\vartheta = \begin{cases} 
1 & \text{if } d - a \equiv 0 \mod 3, \\
-1 & \text{if } d - a \equiv 1 \mod 3 \\
0 & \text{otherwise}, 
\end{cases}
\]
we have the following theorem for rank 3.

**Theorem 1.4.** Assume \(k = 3 + r\). If there exists an integer \(0 \leq a \leq g - 1 - \varepsilon\) such that
\[
\max\left\{d - 3g - a, \frac{d - a}{2}\right\} \leq r < d - 3g + \frac{2g - 2a - 1 - \vartheta}{3 + r},
\]
then \(U(3, d, k) \neq \emptyset\). Moreover, \(\emptyset \neq A_a(3, d, k) \subset U(3, d, k)\).

In Section 2 we give the main results on Brill-Noether Theory and coherent systems that we will use. In Section 3 we recall the main results on \((t, \ell)\)-stability that we will use, and we then prove our main results.

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2. **Brill-Noether theory and Coherent systems**

In this section we recall the main results that we will use on the Brill-Noether Theory and on coherent systems. For a more complete treatment of the subject, see [5] and [20] and [12] and the bibliographies therein.
2.1. Brill-Noether Theory. Let \( M(n, d) \) (resp. \( \widetilde{M}(n, d) \)) denote the moduli space of stable (resp. S-equivalence classes of semistable) bundles of rank \( n \) and degree \( d \) on \( X \). The Brill-Noether loci are defined by

\[
B(n, d, k) := \{ E \in M(n, d) \mid h^0(E) \geq k \},
\]

\[
\widetilde{B}(n, d, k) := \{ [E] \in \widetilde{M}(n, d) \mid h^0(grE) \geq k \},
\]

where \( [E] \) denotes the S-equivalence class of \( E \) and \( grE \) is the graded object associated with \( E \) through a Jordan-Hölder filtration. Since the Brill-Noether loci \( \widetilde{B}(n, d, k) \) are defined as determinantal varieties they are locally closed subschemes of expected dimension

\[
\rho(n, d, k) := n^2(g - 1) - k(k - d + n(g - 1)).
\]

The number \( \rho(n, d, k) \) is often referred to as the Brill-Noether number for \( (g, n, d, k) \). We see at once that:

1. if \( n(g - 1) < d \) and \( 0 \leq k \leq d + n(1 - g) \) then \( B(n, d, k) = M(n, d) \).
2. If \( n(g - 1) < d < 2n(g - 1) \) and \( k > d + n(1 - g) \), \( B(n, d, k) \not\subset M(n, d) \).
3. If \( 0 < d \leq n(g - 1) \) for any \( k \geq 1 \), \( B(n, d, k) \not\subset M(n, d) \).

Recall that a semistable vector bundle \( E \in \widetilde{M}(n, d) \) is called special if \( h^0(E) \cdot h^1(E) \neq 0 \).

Clifford’s Theorem for special bundles (see [3]) gives the bound \( h^0(E) \leq \frac{d}{2} + n \). For special bundles \( E \in \widetilde{M}(n, d) \) with \( d \geq n(g - 1) \) it follows immediately that:

1. \( E^* \otimes K \) is special of degree \( \leq n(g - 1) \);
2. \( h^1(E) \leq ng - \frac{d}{2} \);
3. \( h^0(E) = k_0 + i \) for some \( i = 1, \ldots, ng - \frac{d}{2} \) and \( k_0 = d + n(1 - g) \).

The Brill-Noether loci define a natural filtration

\[
\ldots B(n, d, k) \subset B(n, d, k - 1) \subset \cdots \subset B(n, d, 1) \subset B(n, d, 0) = M(n, d).
\]

\[
\ldots \widetilde{B}(n, d, k) \subset \widetilde{B}(n, d, k - 1) \subset \cdots \subset \widetilde{B}(n, d, 1) \subset \widetilde{B}(n, d, 0) = \widetilde{M}(n, d),
\]

called the Brill-Noether filtration or just the BN-filtration on \( M(n, d) \) (resp. in \( \widetilde{M}(n, d) \)). Note that if \( B(n, d, k) \not\subset M(n, d) \), \( B(n, d, k + 1) \subset \text{Sing}B(n, d, k) \), and for many cases (see [12]) \( B(n, d, k + 1) = \text{Sing}B(n, d, k) \) and \( B(n, d, k) \) has a component of the expected dimension.

Denote by \( Y^{n,d}_k \), or simply by \( Y_k \) when \( (n, d) \) are understood, the scheme given by

\[
Y^{n,d}_k := B(n, d, k) - B(n, d, k + 1).
\]

Note that for any \( E \in Y_k \), \( h^0(E) = k \).

Such schemes \( \{ Y_k \} \) define a schematic stratification (see [13] or [1]) on \( M(n, d) \). Let \( \pi_2 : X \times M(n, d) \to M(n, d) \) be the projection in the second factor. Working locally in the étale topology if necessary, we can assume without loss of generality that exist a universal family \( U \) over \( X \times M(n, d) \). Let \( \mathcal{U}_k \) be the restriction of \( U \) to \( X \times Y_k \). The sheaf \( \pi_{2*}(\mathcal{U}_k) \) is locally free of rank \( k \). Moreover, the Grassmannian bundle \( \text{Grass}(s, \pi_{2*}\mathcal{U}_k) \) of \( s \)-dimensional subspaces has dimension

\[
\dim \text{Grass}(s, \pi_{2*}\mathcal{U}_k) = \dim Y_k + s(k - s).
\]
Remark 2.1. 

(1) Note that if \( d > 2n(g-1) \) then \( \pi_\nu \mathcal{U} \) is locally free of rank \( d+n(1-g) \).

Moreover,
\[
\dim \Grass(k, \pi_\nu \mathcal{U}) = \rho(n, d, k).
\]

(2) If \( d \geq n(g-1) \) and \( k_0 := d+n(1-g) \) then \( \emptyset \neq Y_{k_0} \) is an open set and for \( k \leq k_0 \),
\[
\dim \Grass(k, \pi_\nu \mathcal{U}_{k_0}) = \rho(n, d, k).
\]

(3) For \( k_0 + i \) with \( i = 1, \ldots, ng - \frac{d}{2} \),
\[
\dim \Grass(k, \pi_\nu \mathcal{U}_{k_0+i}) = \dim Y_{k_0+i} + k(k_0 - k) + k_i.
\]

2.2. Coherent systems. Let \((E, V)\) be a coherent system of type \((n, d, k)\) on \(X\). A subsystem of \((E, V)\) is a coherent system \((F, W)\) such that \(F \subset E\) is a subbundle of \(E\) and \(W \subset H^0(F) \cap V\). For a real number \( \alpha > 0 \), the \( \alpha \)-slope of a coherent system \((E, V)\) of type \((n, d, k)\), denoted by \( \mu_\alpha(E, V) \), is the quotient
\[
\mu_\alpha(E, V) := \frac{d + \alpha k}{n}.
\]

A coherent system \((E, V)\) is \( \alpha \)-stable (resp. \( \alpha \)-semistable) if, for all proper subsystems \((F, W)\),
\[
\mu_\alpha(F, W) < \mu_\alpha(E, V) \quad \text{(resp. } \leq \text{)}.
\]

We denote by \( G_0(n, d, k) \) the moduli spaces corresponding to small \( \alpha > 0 \) and by \( U(n, d, k) \) the subscheme
\[
U(n, d, k) := \{(E, V) \in G_0(n, d, k) \mid (E, V) \text{ is } \alpha\text{-stable for all } \alpha > 0 \text{ and } E \text{ is stable } \}.\]

For \( d \gg 0 \) and \( k \neq d+n(1-g) \) the non-emptiness of \( U(n, d, k) \) was proved by Ballico in [2] and, using degeneration methods, M. Teixidor i Bigas proves, in [23], non-emptiness of \( U(n, d, k) \), when certain numerical conditions on \( n, d, k \) and \( g \) are satisfied. In [9] the non-emptiness of \( U(n, d, k) \) was related to Butler’s conjecture.

The Clifford’s Theorem for \( \alpha \)-semistable coherent systems (see [16]) states that, for any \( \alpha \)-semistable coherent system \((E, V)\) of type \((n, d, k)\),
\[
\tag{2.1}
k \leq \begin{cases} 
\frac{d + n(1-g)}{n} & \text{if } d \geq 2gn \\
\frac{d}{2} + n & \text{if } d < 2gn.
\end{cases}
\]

There is a forgetful morphism
\[
\Phi : G_0(n, d, k) \longrightarrow \overline{B}(n, d, k) : (E, V) \mapsto [E].
\]

Remark 2.2. An easy computation shows that:

(1) if \( E \) is stable, then, for any linear subspace \( V \subset H^0(E) \) of dimension \( k \), \((E, V) \in G_0(n, d, k)\).

(2) If \( E \in B(n, d, k) \), \( \Phi^{-1}(E) = \Grass(k, H^0(E)) \).

(3) For the stratum \( Y_{k_0+i} \), \( \dim \Phi^{-1}(Y_{k_0+i}) = \dim Y_{k_0+i} + k(k_0 - k) + k_i \).

(4) If \( E \in B(n, d, k) \) then \((E, V) \in U(n, d, k)\) if for all subsystems of type \((n', d', k')\), \( \frac{k'}{n'} \leq \frac{k}{n} \). Moreover, if \((E, V) \in G_0(n, d, k)\) but \((E, V) \notin U(n, d, k)\), then there exists an \( \alpha_i > 0 \) and an \( \alpha_i \)-semistable coherent subsystem \((F, W)\) of type \((n', d', k')\), such that \( \frac{k}{n} \leq \frac{k'}{n'} \).
It is well known that if \((n, d) = (1, 1)\) and \(k \leq d + n(1 - g)\) then \(G_0(n, d, k)\) is birational to the Grassmannian bundle \(\text{Grass}(k, \pi_2 U)\) and \(\dim G_0(n, d, k) = \rho(n, d, k)\).

Set \(k_0 := d + n(1-g)\) and \(t = 0, 1, \ldots, ng - \frac{d}{2} \). Assume that \(d \geq n(g-1)\). The following proposition computes the dimension of \(\Phi^{-1}(Y_{k_0+i}) \subset G_0(n, d, k)\).

**Proposition 2.3.** \(\dim \Phi^{-1}(Y_{k_0}) = \rho(n, d, k)\). Moreover, for any \(0 \leq k \leq k_0 + i\), if \(c = \dim M(n, d) - \dim Y_{k_0+i}\) then

\[
\dim \Phi^{-1}(Y_{k_0+i}) = \rho(n, d, k) + ki - c.
\]

**Proof.** We know that \(\text{Grass}(k, \pi_2 U_{k_0+i})\) is a Grassmannian bundle of rank \(k(k_0 + i - k)\) and

\[
\Phi^{-1}(Y_{k_0+i}) \cong \text{Grass}(k, \pi_2 U_{k_0+i}).
\]

Let \(c = \dim M(n, d) - \dim Y_{k_0+i}\), then

\[
\dim \Phi^{-1}(Y_{k_0+i}) = \dim \text{Grass}(k, \pi_2 U_{k_0+i}) = \dim Y_{k_0+i} + k(k_0 + i - k) = \dim M(n, d) - c - k(k - d + n(g - 1)) + ki = \rho(n, d, k) + ki - c.
\]

In particular, if \(i = 0\), \(c = 0\), and this is precisely the assertion of the proposition. \(\square\)

### 3. \((t, \ell)\)-Stability and Main Results

In this section we summarize without proofs the relevant material on \((t, \ell)\)-stability. For a deeper discussion of \((t, \ell)\)-stable bundles we refer the reader to [19] and [17] (see also [18]).

**Definition 3.1.** Let \(t, \ell \in \mathbb{Z}\). A vector bundle \(E\) of rank \(n\) and degree \(d\) is \((t, \ell)\)-stable if, for all proper subbundles \(F \subset E\),

\[
\frac{d_F + t}{n_F} < \frac{d + t - \ell}{n}.
\]

Denote by \(A_{t,\ell}(n, d)\) the set of \((t, \ell)\)-stable bundles of rank \(n\) and degree \(d\). It is known that \((t, \ell)\)-stability is an open condition [19, Proposition 5.3] and that \(A_{t,\ell}(n, d) \neq \emptyset\) if and only if

\[
t(n - r) + r\ell < r(n - r)(g - 1) + \delta_r
\]

for all integers \(r\) with \(1 \leq r \leq n - 1\), where \(\delta_r\) is the unique integer such that \(0 \leq \delta_r \leq n-1\) and \(r(n - r)(g - 1) + \delta_r \equiv rd \mod n\) [17, Proposition 1.9].

Write

\[
\varepsilon = \begin{cases} 
1 & \text{if } d \equiv g - 1 \mod n \\
0 & \text{otherwise,}
\end{cases}
\]

**Proposition 3.2.** For any \(0 \leq a \leq g - 1 - \varepsilon\), \(A_{0,a}(n, d) \neq \emptyset\) and it is an open set of the moduli space \(M(n, d)\). Moreover \(A_{0,g-1}(n, d) \neq \emptyset\) if and only if \(d \not\equiv g - 1 \mod n\).

**Proof.** From the inequalities (3.1) we have that for any \(0 \leq a \leq g - 1 - \varepsilon\), \(A_{0,a}(n, d) \neq \emptyset\). The \((0, a)\)-stability implies that

\[
\mu(F) < \mu(E) - \frac{a}{n} \quad \text{i.e.} \quad \frac{a}{n} < \mu(E) - \mu(F)
\]
for all subbundles of $E$. Therefore, $(0, a)$-stability implies stability. □

We have a filtration of open sets
$$
\emptyset \neq A_{0,g-1-\varepsilon}(n,d) \subset \cdots \subset A_{0,1}(n,d) \subset A_{0,0}(n,d) = M(n,d).
$$

Denote by $A_a(n,d,k)$ the open subscheme
$$
A_a(n,d,k) := \{(E, V) \in G_0(n, d, k) : E \text{ is } (0, a) - \text{stable}\}.
$$

If $\Phi : G_0(n, d, k) \to \tilde{B}(n, d, k)$ is the forgetful map then
$$
\Phi(A_a(n,d,k)) = A_{(0,a)}(n, d) \cap B(n, d, k).
$$

We see at once that $A_a(n,d,k) \neq \emptyset$ in the following cases.

**Proposition 3.3.** If $\dim A_{(0,a)}(n,d)^c < \dim B(n, d, k)$ then $A_a(n,d,k) \neq \emptyset$. Moreover, if $d \geq n(g-1)$ and $k \leq d + n(1-g)$ then for any $0 \leq a \leq g - 1 - \varepsilon$, $A_a(n,d,k) \neq \emptyset$.

**Proof.** We only need to make the following observation. If $A_{(0,a)}(n,d) \cap B(n, d, k) \neq \emptyset$ then $A_a(n,d,k) \neq \emptyset$. The hypotheses in the proposition give $A_{(0,a)}(n,d) \cap B(n, d, k) \neq \emptyset$.

**Remark 3.4.** We have proved more, namely that if $\dim A_{(0,a)}(n,d)^c < \dim Y_r$ then $A_a(n,d,k) \neq \emptyset$ and for $r \geq k$,
$$
Grass(k, \pi_2, U_r) \cap A_{(0,a)}(n,d) \subset A_a(n,d,k) \subset G_0(n, d, k).
$$

Moreover, $\dim Y_r + k(r-k) \leq \dim G_0(n, d, k)$.

The following theorems establish a relation between $(0, a)$-stable bundles and $\alpha$-stable coherent systems with $\alpha > 0$.

**Theorem 3.5.** Assume $d \geq 2ng + 2b - a$ and $k \geq d + n(1-g) - b$ with $0 \leq b \leq a < g - 1 - \varepsilon$. If $A_a(n,d,k) \neq \emptyset$ then $U(n, d, k) \neq \emptyset$. Moreover, if $k \leq d + n(1-g)$ then $\emptyset \neq A_a(n,d,k) \subset U(n, d, k)$ and $U(n, d, k)$ has a component of the expected dimension and birational to a Grassmannian bundle over an open set of $M(n,d)$.

**Proof.** Let $(E, V) \in A_a(n,d,k)$. We shall prove that $(E, V)$ is $\alpha$-stable for all $\alpha > 0$.

Suppose for a contradiction that $(E, V) \notin U(n, d, k)$. From Remark 2.2 there exists an $\alpha_i$-semistable coherent subsystem $(F, W)$ of type $(n', d', k')$, such that $\frac{k}{n} \leq \frac{k'}{n'}$.

By hypothesis, one has
$$
\frac{d + n(1-g) - b}{n} \leq \frac{k}{n} \leq \frac{k'}{n'}.
$$

If $\mu(F) \geq 2g$, the Clifford bound (2.1) for coherent systems gives $\frac{k'}{n'} \leq \mu(F) + 1 - g$. Using this, together with the previous inequality, we obtain
$$
\mu(E) + 1 - g - \frac{b}{n} \leq \frac{k'}{n'} \leq \mu(F) + 1 - g,
$$

which implies
$$
\mu(E) \leq \mu(F) + \frac{b}{n} \leq \mu(F) + \frac{a}{n}.
$$

This contradicts the $(0, a)$-stability of $E$ (see (3.2)).
If $\mu(F) < 2g$, the Clifford bound for $(F, W)$ gives $k' \leq \frac{\mu(F)}{2} + 1$. Hence

$$\mu(E) - g - \frac{b}{n} \leq \frac{k}{n} \leq \frac{k'}{n} \leq \frac{\mu(F)}{2} + 1.$$ 

So, since $E$ is $(0, a)$-stable,

$$\mu(E) - g - \frac{b}{n} \leq \frac{k}{n} \leq \frac{\mu(F)}{2} + 1$$

which implies

$$\mu(E) < 2g + \frac{b}{n} - \frac{a}{n}.$$ 

This contradicts the assumption that $d \geq 2ng + 2b - a$. Hence, $(E, V) \in U(n, d, k)$ as required.

If $k \leq d + n(1 - g)$, from Proposition 3.3, $A_a(n, d, k) \neq \emptyset$. Therefore the theorem follows from the observation that $\Phi(A_a(n, d, k))$ is an open set of $M(n, d)$.

**Remark 3.6.** A slight change in the proof of Theorem 3.5 actually shows that if $(E, V) \in A_a(n, d, k)$ with $E$ special then $(E, V) \in U(n, d, k)$ if $d \geq 2ng + 2(b - \nu) - a$ and

$$d + n(1 - g) - b \leq k \leq h^0(E) = d + n(1 - g) + \nu$$

when $0 \leq b - \nu \leq a$.

Assume now that $0 < d \leq 2gn$. From Clifford’s Theorem for $\alpha$-semistable coherent systems of type $(n, d, k)$, $k \leq \frac{d}{2} + n$. Denote by $\lambda$ the difference

$$\lambda := d - 2(k - n)$$

and by $A_{a, \lambda}(n, d, k)$ the subscheme

$$A_{a, \lambda}(n, d, k) = \{(E, V) \in G_0(n, d, k) : E \in A_{0,a}(n, d) \text{ and } \lambda \leq \alpha\}.$$

**Theorem 3.7.** If $A_{a, \lambda}(n, d, k) \neq \emptyset$ then $U(n, d, k) \neq \emptyset$. Moreover, $A_{a, \lambda}(n, d, k) \subset U(n, d, k)$.

**Proof.** Let $(E, V) \in A_{a, \lambda}(n, d, k)$. Analysis similar to that in the proof of Theorem 3.5 shows that if $(E, V) \notin U(n, d, k)$ we get a contradiction. Indeed, suppose that there exists an $\alpha$-semistable coherent subsystem $(F, W)$ of type $(n', d', k')$, such that $\frac{k}{n} \leq \frac{k'}{n'}$. Since $E$ is $(0, a)$-stable, and hence stable, $\mu(F) < 2g$. Thus, from Clifford’s Theorem for coherent systems we have that $\frac{k'}{n'} \leq \frac{\mu(F)}{2} + 1$. Hence,

$$\frac{\mu(E)}{2} - \frac{\lambda}{2n} + 1 = \frac{k}{n} \leq \frac{k'}{n'} \leq \frac{\mu(F)}{2} + 1.$$ 

The assumption $\lambda \leq a$ implies that

$$\mu(E) \leq \mu(F) + \frac{\lambda}{n} \leq \mu(F) + \frac{a}{n}$$

which contradicts the $(0, a)$-stability of $E$. This gives $U(n, d, k) \neq \emptyset$, and the theorem follows.

$\square$
For rank 2 and 3, we can prove that \( U(n, d, k) \neq \emptyset \) for a wider range of values of \( d \) and \( k \) by computing the dimension of \( A_{0,a}(n, d)^c := M(n, d) \setminus A_{0,a}(n, d) \). An estimate for this was given in [17, Theorem 1.10], but it is possible to compute it precisely using the Segre invariants. Recall (see [10]) that the \( m \)-Segre invariant \( s_m(E) \) of a bundle of rank \( n \) and degree \( d \) is defined by

\[
 s_m(E) := \min_{F \subset E} \{ md - nd_F \mid F \text{ a subbundle of rank } m \text{ of } E \},
\]

that is,

\[
(3.3) \quad \frac{s_m(E)}{mn} = \min_{F \subset E} \{ \mu(E) - \mu(F) \}
\]

Let \( M(n, d, m, s) \) be the set of stable vector bundles of rank \( n \) and degree \( d \) such that the \( m \)-Segre invariant is \( s \), that is

\[
 M(n, d, m, s) := \{ E \in M(n, d) \mid s_m(E) = s \}.
\]

In [10] (see also [22]) it was proved that for an integer \( 0 < s \leq m(n - m)(g - 1) \) such that \( s \equiv md \mod n \), \( M(n, d, m, s) \) is non empty and irreducible and

\[
 \dim M(n, d, m, s) = n^2(g - 1) + 1 + s - m(n - m)(g - 1).
\]

In the following result we describe the \( A_{0,a}(n, d) \) in terms of Segre invariants. First, we introduce the following notation

\[
(3.4) \quad \tilde{s}_m := \max\{ s \mid s \leq ma, \ s \equiv md \mod n \},
\]

and

\[
(3.5) \quad s_\Delta := \min_m \{ m(n - m)(g - 1) - \tilde{s}_m \}.
\]

**Theorem 3.8.** For any \( 0 \leq a \leq g - 1 - \varepsilon \),

\[
 A_{0,a}(n, d) = \bigcap_{m=1}^{n-1} \left( \bigcup_{s>ma} M(n, d, m, s) \right).
\]

Moreover, \( \dim A_{0,a}(n, d) = n^2(g - 1) + 1 - s_\Delta \).

**Proof.** The first part follows immediately from (3.2) and (3.3).
The dimension of \((A_0,a(n,d))^c\) follows from the next equalities:

\[
\dim(A_0,a(n,d))^c = \dim \bigcap_{m=1}^{n-1} \left[ \bigcup_{s>ma} M(n,d,m,s) \right]^c \\
= \dim \bigcup_{m=1}^{n-1} \left[ \bigcup_{s>ma} M(n,d,m,s) \right]^c \\
= \dim \bigcup_{m=1}^{n-1} \left[ \bigcup_{s\leq ma} M(n,d,m,s) \right]^c \\
= \max_m \left\{ \max_s \{\dim(M(n,d,m,s))\} \right\} \\
= \max_m \left\{ \max_s \{n^2(g-1) + 1 + s - m(n-m)(g-1)\} \right\} \\
= \max_s \{n^2(g-1) + \tilde{s}_m - m(n-m)(g-1)\} \\
= n^2(g-1) + 1 - s_\Delta. \\
\]

□

The following results are an application of Theorem 3.8 for vector bundles of rank 2 and 3.

**Corollary 3.9.** \(\dim A_0,a(2,d)^c = 3g + a - \delta\), where

\[
\delta = \begin{cases} 
2 & \text{if } a \equiv d \mod 2 \\
3 & \text{otherwise}, 
\end{cases}
\]

**Proof.** From Theorem 3.8

\[
A_0,a(2,d)^c = \bigcup_{0 \leq s \leq a} M(2,d,s)
\]

and

\[
\dim M(2,d,s) = 3g + s - 2
\]

for \(s \leq g - 1\) (see also [15, Proposition 3.1]). Since \(s \equiv d \mod 2\), it follows that \(\dim M(2,d,s)\) attains its maximum for \(s \leq a\) when \(s = a\) if \(a \equiv d \mod 2\) or when \(s = a - 1\) otherwise. The result follows. □

**Theorem 3.10.** Assume \(k = 2 + r\). If there exists an integer \(0 \leq a \leq g - 1 - \varepsilon\) such that

\[
\max \left\{ d - 2g - a, \frac{d - a}{2} \right\} \leq r < d - 2g + \frac{g - a + \delta - 3}{r + 2},
\]

then \(U(2,d,k) \neq \emptyset\). Moreover, \(\emptyset \neq A_a(2,d,k) \subset U(2,d,k)\).

**Proof.** We begin by proving that \(A_a(2,d,k) \neq \emptyset\). Since \(\dim B(2,d,k) \geq \beta(2,d,k)\), it is sufficient by Proposition 3.3 to prove that \(\dim A_0,a(2,d)^c < \beta(2,d,k)\). According to Corollary 3.9 this means we need to prove that

\[
3g + a - \delta < 4(g - 1) + 1 - (k)(r - d + 2g).
\]

This follows from the second inequality in (3.6).
It remains to show that \( A_\alpha(2, d, k) \subset U(2, d, k) \). For this, we argue as in the proof of Theorem 3.7 and 3.5. Let \((E, V) \in A_\alpha(2, d, k)\) and suppose \((E, V) \notin U(2, d, k)\). Let \((F, W)\) be a subsystem of \((E, V)\) of type \((1, d', k')\) such that \(\frac{k}{2} \leq k'\). From (3.6), we have
\[
\mu(E) + 1 - g - \frac{a}{n} \leq k' \leq \mu(F) + 1 - g,
\]
which implies
\[
\mu(E) \leq \mu(F) + \frac{a}{n}.
\]
This contradicts the \((0, a)\)-stability of \(E\). If \(d' < 2g\),
\[
\frac{k}{2} \leq d' + 1 \leq \frac{d - a}{4} + 1.
\]
This contradicts the first inequality in (3.6). Hence, \(\emptyset \neq A_\alpha(2, d, k) \subset U(2, d, k)\) as claimed.

For rank 3 Theorem 3.8 gives three different cases.

**Corollary 3.11.** If \(0 \leq a \leq g - 1 - \varepsilon\) then \(\dim A_{0,\alpha}(3, d)^c = 7(g - 1) + 1 + \tilde{s}_2\), with \(\tilde{s}_2 = \max\{s|s \leq 2a, \ s \equiv 2d \mod 3\}\). Moreover,
1. if \(d - a \equiv 0 \mod 3\) then \(\dim A_{0,\alpha}(3, d)^c = 7(g - 1) + 2a + 1\);
2. if \(d - a \equiv 1 \mod 3\) then \(\dim A_{0,\alpha}(3, d)^c = 7(g - 1) + 2a - 1\);
3. if \(d - a \equiv 2 \mod 3\) then \(\dim A_{0,\alpha}(3, d)^c = 7(g - 1) + 2a\).

**Proof.** By hypothesis we have that \(m = 1, 2\). Now, using (3.4) and (3.5) we have \(\tilde{s}_1 \leq a\) with \(s_1 \equiv d \mod 3\) and \(\tilde{s}_2 \leq 2a\) with \(\tilde{s}_2 \equiv 2d \mod 3\). Therefore \(\tilde{s}_1 \leq \tilde{s}_2\) and \(s_{\Delta} = 2(g - 1) - \tilde{s}_2\). Now, the result follows from Theorem 3.8.

With the notation
\[
\vartheta = \begin{cases} 
1 & \text{if } d - a \equiv 0 \mod 3, \\
-1 & \text{if } d - a \equiv 1 \mod 3 \\
0 & \text{otherwise},
\end{cases}
\]
we have the following theorem for rank 3.

**Theorem 3.12.** Assume \(k = 3 + r\). If there exists an integer \(0 \leq a \leq g - 1 - \varepsilon\) such that
\[
\max \left\{ d - 3g - a, \frac{d - a}{2} \right\} \leq r < d - 3g + \frac{2g - 2a - 1 - \vartheta}{3 + r},
\]
then \(U(3, d, k) \neq \emptyset\). Moreover, \(\emptyset \neq A_{\alpha}(3, d, k) \subset U(3, d, k)\).

**Proof.** As in Theorem 3.10 we begin by proving that \(A_{\alpha}(3, d, k) \neq \emptyset\). If we prove that \(\dim A_{0,\alpha}(3, d)^c < \dim B(3, d, k)\), the assertion follows.

It is easily seen that we can conclude from the second inequality in (3.7) that
\[
7(g - 1) + 2a + \vartheta < 9(g - 1) + 1 - k(r - d + 3g),
\]
hence that $\dim A_0(3, d)^c < \beta(3, d, k) \leq \dim B(3, d, k)$, and finally that $A_2(3, d, k) \neq \emptyset$.

To show that $A_2(3, d, k) \subset U(3, d, k)$ we argue as in the proof of Theorem 3.7, 3.5 and 3.8. We leave it to the reader to verify that if $(E, V) \in A_2(3, d, k)$ and $(E, V) \notin U(3, d, k)$ we get a contradiction using the first inequality in 3.7.

□

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CIMAT, Mineral de Valenciana S/N, Apdo. Postal 402, C.P. 36240. Guanajuato, Gto, México

E-mail address: lebp@cimat.mx

Departamento de Matemáticas, CUCEI, Universidad de Guadalajara. Av. Revolución 1500. C.P. 44430, Guadalajara, Jalisco, México

E-mail address: osbaldo.mata@academico.udg.mx, osbaldo@cimat.mx