Geometric Phase in a Bose-Einstein Josephson Junction

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Abstract

We calculate the geometric phase associated with the time evolution of the wave function of a Bose-Einstein condensate system in a double-well trap by using a model for tunneling between the wells. For a cyclic evolution, this phase is shown to be half the solid angle subtended by the evolution of a unit vector whose $z$ component and azimuthal angle are given, respectively, by the population difference and phase difference between the two condensates. For a non-cyclic evolution, an additional phase term arises. We show that the geometric phase can also be obtained by mapping the tunneling equations on to the equations of a space curve. The importance of a geometric phase in the context of some recent experiments is pointed out.

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I. INTRODUCTION

Bose-Einstein condensation (BEC) in a dilute gas of trapped ultracold alkali atoms has been observed by several experimental groups [1]. This gives rise to the possibility of understanding the nature of the condensate wave function, and in particular, its phase [2]. It is believed that the BEC phase transition occurs due the breaking of a global gauge symmetry of the Hamiltonian. Theoretically, a BEC may be modeled by writing down the interacting many-body Hamiltonian in terms of boson creation and annihilation operators \( \Psi^\dagger \) and \( \Psi \). The order parameter is postulated to be the condensate wave function \( \psi = \langle \Psi_{op} \rangle = \rho e^{i\theta} \), where \( \rho = |\psi|^2 \) is the condensate density and \( \theta \) is the phase of the wave function. The Hamiltonian is gauge invariant, but the order parameter breaks this symmetry. Using the dynamical equation for \( \Psi_{op} \) found from the Hamiltonian operator, the time evolution of the condensate wave function \( \psi \) can be shown to satisfy the following Gross-Pitaevskii equation (GPE) [3]:

\[
\frac{i\hbar}{\partial t} \psi = -\frac{\hbar^2}{2m} \nabla^2 \psi + [V_{\text{ext}}(x) + g_0 |\psi|^2] \psi, \tag{1.1}
\]

where \( V_{\text{ext}} \) is the external potential and \( g_0 = 4\pi\hbar^2a/m, a \) and \( m \) being the atomic scattering length and mass, respectively. Although this equation has an underlying quantum nature, the condensate has a macroscopic extent, suggesting the observation of quantum effects on a macroscopic scale.

In a striking experiment [4], Andrews et al have shown the existence of the macroscopic quantum phase difference between two bose condensates: They designed a double-well trap by using a laser sheet to create a high barrier within a trapped condensate. On switching off this barrier, the two condensates overlapped to produce an interference pattern, showing phase coherence. More interestingly, by lowering the laser sheet intensity, the barrier gets lowered, making it possible for the condensates to tunnel through the barrier. Thus this double-well trap is analogous to a superconductor Josephson junction [5], and is referred to as the Boson Josephson junction (BJJ). In an interesting paper, Smerzi et al [6] have set up the tunneling equations for the BJJ in a model. These are two coupled nonlinear ordinary differential equations for the condensate wave functions in the two wells. They have studied the time evolution of the inter-well population difference and phase difference in this model, and predicted a novel ‘self-trapping’ effect, i.e., the oscillation of the population difference around a non-zero value, for certain initial conditions and parameters.
The tunneling dynamics motivates the following question: Is there an underlying geometric phase associated with the time evolution of the condensate wave function in a double-well trap? As is well known by now, the concept of a geometric phase has been studied in various contexts, after it was introduced by Berry in quantum mechanics. It had also been considered much earlier by Pancharatnam in the context of classical optics. Geometric phase and its various applications have been studied intensively for over a decade now. Such a (non-integrable) phase arises when the time evolution of a system is such that the value of a variable in a given state of the system depends on the path along which the state has been reached. In this paper, we calculate the geometric phase associated with the time evolution of the BJJ wave function, for both cyclic and non-cyclic evolutions.

The plan of the paper is as follows: In Sec.II, we first review Smerzi et al.'s derivation of the tunneling equations. Keeping in mind that the geometric phase is gauge independent, we use a certain gauge transformation to reduce these equations to a more convenient form. In Sec. III, we briefly outline the kinematic approach formulated by Mukunda and Simon to define the geometric phase as applied to a two-level system. We then solve the BJJ tunneling equations, which are nonlinear differential equations, numerically by choosing some parameter values as an example. Using these solutions, we find the geometric phase explicitly, for both cyclic and non-cyclic evolution of the system. For a cyclic evolution, the phase difference and the population difference between the condensates in the two wells return to their original values. For this case, the corresponding geometric phase is half the solid angle generated by a unit vector whose z component and azimuthal angle are given, respectively, by the population difference and the phase difference. For non-cyclic evolution, an additional phase term is obtained. In Sec. IV, we show that this geometric phase (for both types of evolution) can also be obtained by first mapping the tunneling equations to the equation for a unit vector and then identifying it with the tangent of a space curve. The space curve is described using the so-called natural frame equations, which possess an underlying natural gauge freedom. The unit triad of vectors can be written down using the form of the condensate wave functions in the two wells. The concept of Fermi-Walker parallel transport is then used to identify the geometric phase. In Sec.V, we employ the usual Frenet frame to obtain explicit expressions for the curvature and torsion of the space curve that gets associated with the BJJ evolution. Section VI contains a summary and discussion.
II. THE BJJ TUNNELING EQUATIONS

We begin by briefly describing the model used by Smerzi et al [6] to study the tunneling of the condensate between two wells. Let the total number of atoms in the double-well trap be \( N \). Let \( N_1 \) and \( N_2 \) denote the number of atoms in each well, such that \( N_1 + N_2 = N \). To study the tunneling, the solution for the GPE (Eq. (1.1)) is assumed to be of the form

\[
\psi = \psi_1(t)\Phi_1(x) + \psi_2(t)\Phi_2(x). \tag{2.1}
\]

Here \( \Phi_1, \Phi_2 \) are the ground state solutions for the isolated wells with \( N_1 = N_2 = (N/2) \).

Using Eq. (2.1) in Eq. (1.1), one gets

\[
\begin{aligned}
  i\hbar \frac{\partial \psi_1}{\partial t} &= (E_1^0 + U_1 N_1)\psi_1 - V\psi_2 \tag{2.2a} \\
  i\hbar \frac{\partial \psi_2}{\partial t} &= (E_2^0 + U_2 N_2)\psi_2 - V\psi_1. \tag{2.2b}
\end{aligned}
\]

The quantities \( E_{1,2}^0, U_{1,2} \) and \( V \) are constants obtained by taking the external field \( V_{\text{ext}}(x) \) in Eq. (1.1) to be independent of time:

\[
E_{1,2}^0 = \int \left[ \frac{\hbar^2}{2m} |\nabla \Phi_{1,2}|^2 + \Phi_{1,2}^2 V_{\text{ext}} \right] d^3x
\]

\[
U_{1,2} = g_0 \int \Phi_{1,2}^4 d^3x,
\]

\[
V = - \int \left[ \frac{\hbar^2}{2m} (\nabla \Phi_1 \nabla \Phi_2) + \Phi_1 \Phi_2 V_{\text{ext}} \right] d^3x
\]

Further, \( N_i = |\psi_i|^2 \), for \( i = 1, 2 \).

In this paper, we obtain the geometric phase associated with the evolution in Eqs. (2.2). Recognizing that this phase is a gauge independent quantity, we find it convenient to apply the following gauge transformation to the wave function

\[
\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \sqrt{N} e^{i \int \eta(t') dt'} \begin{pmatrix} a \\ b \end{pmatrix}. \tag{2.3}
\]

On substituting Eq. (2.3) in Eq. (2.2) and setting \( \hbar \eta(t) = -((E_1^0 + E_2^0) + N(U_1|a|^2 + U_2|b|^2))/2 \), the BJJ tunneling equations take on the form

\[
\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \hbar \omega_0 - V \\ -V - \hbar \omega_0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = M_{\omega_0} \begin{pmatrix} a \\ b \end{pmatrix}, \tag{2.4}
\]
with
\[ \hbar \omega_0 = \frac{1}{2}(E_1^0 - E_2^0 + U_1 N_1 - U_2 N_2) \]  \hspace{1cm} (2.5)

In Eq. (2.4), \( M_{\omega} \) denotes the (time-dependent) matrix on the right hand side. The advantage of the gauge transformation is that this matrix is traceless. Note also that since \( N_1 = N|a|^2 \) and \( N_2 = N|b|^2 \) appear (in \( \omega_0 \)) in this matrix, Eqs. (2.4) are two coupled nonlinear differential equations. From Eq. (2.3), we see that the normalization condition \(|\psi_1|^2 + |\psi_2|^2 = N\) implies \(|a|^2 + |b|^2 = 1\). Thus without loss of generality, we write
\[ a = \cos(\alpha/2) e^{i\theta_1}; \quad b = \sin(\alpha/2) e^{i\theta_2} \]  \hspace{1cm} (2.6)

Let us denote the difference in the population density of the two traps by \( z \) and the difference in the phases of the two condensates by \( \phi \). From Eq. (2.6) we thus have,
\[ z = (N_1 - N_2)/N = (|a|^2 - |b|^2) = \cos \alpha; \quad \phi = (\theta_2 - \theta_1). \]  \hspace{1cm} (2.7)

By suitably combining Eq. (2.4) and its complex conjugate, and using Eq. (2.7), the nonlinear coupled equations for \( z \) and \( \phi \) are found to be (on setting \( \hbar = 1 \))
\[ \frac{dz}{dt} = -V \sqrt{1 - z^2} \sin \phi \]  \hspace{1cm} (2.8a)
\[ \frac{d\phi}{dt} = \Lambda z + V \frac{z}{\sqrt{1 - z^2}} \cos \phi + \Delta E. \]  \hspace{1cm} (2.8b)

Here we have defined
\[ \Delta E = (E_1^0 - E_2^0)/2 + N(U_1 - U_2)/4; \quad \Lambda = N(U_1 + U_2)/4 \]  \hspace{1cm} (2.9)

and the time has been reparametrized as \( t \rightarrow 2t \). It is interesting to note that the above equations can also be written as Hamilton’s equations, by treating \( z \) and \( \phi \) as the canonically conjugate variables. The classical Hamiltonian is easily verified to be
\[ H_{cl} = \Lambda \frac{z^2}{2} - V \sqrt{1 - z^2} \cos \phi + \Delta E z. \]  \hspace{1cm} (2.10)

This describes a non-rigid pendulum with a length proportional to \( \sqrt{1 - z^2} \), which decreases with the “momentum” \( z \). We write down the expression for \( \omega_0 \) appearing in Eq. (2.5) in the form
\[ \omega_0 = \Delta E + \Lambda z, \]  \hspace{1cm} (2.11)
by using Eq. (2.9). Finally, setting \( z = \cos \alpha \) in Eqs. (2.8a) and (2.8b), we obtain

\[
\frac{d\alpha}{dt} = V \sin \phi 
\]  

(2.12a)

\[
\frac{d\phi}{dt} = \Lambda \cos \alpha + V \cot \alpha \cos \phi + \Delta E 
\]  

(2.12b)

Equations (2.8), or equivalently, Eqs. (2.12), represent the tunneling equations. For convenience, \( V \) can be absorbed in time \( t \) and all energies can be measured in units of \( V \). Note that \( \Delta E \) is the asymmetry between the two wells, as seen from Eq. (2.9). We consider two special limits:

(1) Interacting Bose system in a symmetric trap: \( \Lambda \neq 0, \Delta E = 0 \).

From Eq. (2.9), \( E_1^0 = E_2^0 \) and \( U_1 = U_2 = U \), and \( \Lambda = UN_T \) is nonzero. Thus Eq. (2.10) becomes

\[
H_{cl} = \Lambda \frac{z^2}{2} - V \sqrt{1 - z^2} \cos \phi 
\]  

(2.13)

In Figs. 1, 2 and 3, we have obtained the \((z, \phi)\) phase portraits for this case, by using Eq. (2.13). Let \( \Lambda \) be replaced by the dimensionless quantity \((\Lambda/V)\). For \( \Lambda < 1 \) there exist periodic oscillations around the zero-state \((0, 0)\) and the non-trapped \( \pi \) state \((0, \pi)\). There are no rotational states. (see Fig. 1 for \( \Lambda = 0.5 \)). For \( \Lambda > 1 \) two new trapped \( \pi \)-states appear at \((z^*, \pi)\) and \((-z^*, \pi)\) with \( z^* = \sqrt{\Lambda^2 - 1}/\Lambda \). The trapped \( \pi \)-states are clearly visible in Fig. 2 \((\Lambda = 1.3)\). As \( \Lambda \) increases, \( z^* \to 1 \). For \( \Lambda > 2 \), rotational states also appear as seen in Fig. 3 \((\Lambda = 5)\).

(2) Non-interacting Bose system : \( \Lambda = 0 \).

For an ideal bose gas, the interactions \( U_1 \) and \( U_2 \) are negligible, and Eq. (2.9) yields \( \Lambda = 0 \). In this limit the kinetic energy term of the Hamiltonian \( H_{cl} \) (2.10) vanishes and hence the non-rigid pendulum analogy is not valid anymore. (However, we remark that the tunneling Hamiltonian coincides with that of a two-component BEC in the rotating frame approximation. This will be discussed in the last section.) We get

\[
H_{cl} = -V \sqrt{1 - z^2} \cos \phi + \Delta Ez 
\]  

(2.14)

In Figs. 4, 5 and 6, we have obtained the phase space portraits for this case. Again, as in case (1), the energy is measured in units of \( V \). When \( \Delta E = 0 \), we have a symmetric trap. There are oscillations around the \((0, 0)\)-state and the non-trapped \((0, \pi)\)-state. There are no
rotational orbits, as seen in Fig. 4. With increase in $\Delta E$, rotational orbits appear, which explore the full range of $\phi$. This is shown in Figs. 5 and 6. The oscillations are now around $(z = -z^*, \phi = 0)$ and $(z = +z^*, \phi = \pm \pi)$. That is, oscillations around $\phi = 0$ shifts towards $z = -1$ which is energetically more stable, while the $\phi = \pi$ fixed point moves towards $z = 1$ which is a local energy maximum. Note that the increase in $\Delta E$ causes a non-trapped $\pi$ state to become a trapped $\pi$ state with oscillations around a non-zero population difference $z^*$. The new fixed point is given by $z^* = \sqrt{((\Delta E/V)^2/(1 + (\Delta E/V)^2))}$ which tends to 1 as $\Delta E/V$ goes to infinity.

We now proceed to show how the geometric phase associated with the BJJ dynamics can be computed.

III. GEOMETRIC PHASE USING THE KINEMATIC APPROACH

In this section we derive the expression for the geometric phase for the BJJ evolution using the kinematic approach developed by Mukunda and Simon [10]. We first briefly outline the basic ideas. One starts with a complex Hilbert space and considers a subset $\mathcal{N}_0$ made up of unit complex vectors denoted by $\psi(t)$. Let $C_0$ be a smooth one parameter curve, consisting of a family of vectors $\psi(t)$, with $0 < t < T$. Since $\psi(t)$ are unit vectors for all values of $t$,

$$Re(\psi(t), \dot{\psi}(t)) = 0,$$

giving

$$(\psi(t), \dot{\psi}(t)) = i\text{Im}(\psi(t), \dot{\psi}(t)) \quad (3.1)$$

where the dot denotes the derivative with respect to $t$. Under a gauge transformation defined by a real function $\alpha(t)$, $C_0 \to C'_0$ and $\psi(t) \to \psi'(t) = e^{i\alpha(t)}\psi(t)$, $t \in (0, T)$, one finds,

$$\text{Im}(\psi'(t), \dot{\psi'}(t)) = \text{Im}(\psi(t), \dot{\psi}(t)) + \frac{d\alpha}{dt} \quad (3.2)$$

On the other hand, we have

$$\text{arg}(\psi'(0), \psi'(T)) - \text{arg}(\psi(0), \psi(T)) = \int_0^T \frac{d\alpha}{dt} \ dt$$

Integrating Eq. (3.2) and using the above equation, we see that a gauge invariant function,

$$\text{arg}(\psi'(0), \psi'(T)) - \text{Im} \int_0^T dt \ (\psi'(t), \dot{\psi'}(t))$$

(3.3)
can be constructed. Next one introduces the space $R_0$ of unit rays, which is the quotient of $N_0$ with respect to the $U(1)$ action $\psi \rightarrow e^{i\alpha} \psi$. It can then be shown that the curves $C_0$ and $C'_0$ (obtained by a gauge transformation) in $N_0$ both project to the same curve $c_0$ in the ray space $R_0$. Thus the gauge invariance of the functional (3.3) implies that it is a functional only of the image $c_0$ of $C_0$. This functional can be shown to be reparametrisation invariant as the integrand is linear in $\dot{\psi}'(t)$. One thus defines the geometric phase $\phi_g(c_0)$ as

$$\phi_g[c_0] = \text{arg}(\psi'(0), \psi'(T)) - \text{Im} \int_0^T dt \ (\psi'(t), \dot{\psi}'(t))$$ (3.4)

Identifying the first term in Eq. (3.4) as the total phase

$$\phi_p = \text{arg}(\psi'(0), \psi'(T)) \quad (3.4a)$$

and the second term in Eq. (3.4) as the dynamical phase

$$\phi_d = \text{Im} \int_0^T dt \ (\psi'(t), \dot{\psi}'(t)) \quad (3.4b)$$

we get

$$\phi_g[c_0] = \phi_p[c_0] - \phi_d[c_0],$$ (3.5)

Now for a given $c_0 \in R_0$ the gauge freedom in the choice of $C_0 \in N_0$ can be used to express $\phi_g[c_0]$ in different but equivalent ways. The Pancharatnam connection takes $\psi(0)$ and $\psi(T)$ to be in phase, giving $\phi_p = 0$, leading to $\phi_g = -\phi_d$. Another prescription takes a lift of $C_0$ so that the integrand of $\phi_d$ in (3.4b) vanishes. This leads to $\phi_g = \phi_p$.

Let us now understand what these two prescriptions imply for the BJJ evolution equations. Using equation (2.6) the family of unit vectors is given by,

$$\psi' = \begin{pmatrix} a \\ b \end{pmatrix} = e^{i\theta_1(t)} \begin{pmatrix} \cos(\alpha(t)/2) \\ \sin(\alpha(t)/2) e^{i\phi(t)} \end{pmatrix} = e^{i\theta_1(t)} \psi,$$ (3.6)

where $\phi = (\theta_2 - \theta_1)$. Clearly, the angle $\theta_1$ in Eq. (3.6) represents the gauge freedom in the wave function $\psi'(t)$. The prescriptions essentially fix the gauge $\theta_1$. Using Eq. (3.6) in Eq. (3.4a), a short calculation leads to the following total phase:

$$\phi_p = \text{arg}(\psi'(0), \psi'(T)) = (\theta_1(T) - \theta_1(0)) + \Delta.$$ (3.7)

Here,
\[
\Delta = \tan^{-1}\left[\frac{\sin(\alpha(0)/2) \sin(\alpha(T)/2) \sin(\phi(T) - \phi(0))}{\cos(\alpha(0)/2) \cos(\alpha(T)/2) + \sin(\alpha(0)/2) \sin(\alpha(T)/2) \cos(\phi(T) - \phi(0))}\right]
\] (3.8)

The integrand of the dynamical phase \(\phi_d\) can be calculated using Eq. (3.6) in Eq. (3.4b) to give

\[
\text{Im}(\psi', \frac{d\psi'}{dt}) = \dot{\theta}_1 + \sin^2(\alpha/2) \dot{\phi}.
\] (3.9)

Thus

\[
\phi_d = (\theta_1(T) - \theta_1(0)) + \int_0^T \sin^2(\alpha/2) \dot{\phi} \, dt.
\] (3.9a)

From Eqs. (3.7) and (3.9a), we get the geometric phase to be

\[
\phi_g = \phi_p - \phi_d = - \int_0^T \sin^2(\alpha/2) \dot{\phi} \, dt + \Delta.
\] (3.9b)

Next, we also calculate the geometric phase expression for two different gauge prescriptions, for the sake of completeness.

*The Pancharatnam Prescription.*

Here \(\phi_p = 0\), thus Eq. (3.7) yields,

\[
\theta_1(T) - \theta_1(0) = -\Delta.
\] (3.10)

Given the end-points \(\alpha(0), \alpha(T), \phi(0), \phi(T)\) on the curve \(C_0\) one can thus determine \(\Delta\) from Eq. (3.8), and hence the gauge \(\theta_1(T) - \theta_1(0)\). From Eq. (3.9),

\[
\phi_d = \int \text{Im}(\psi, \dot{\psi}) dt = \theta_1(T) - \theta_1(0) + \int \sin^2(\alpha/2) \dot{\phi} \, dt.
\] (3.11)

Substituting Eq. (3.10) in Eq. (3.11),

\[
\phi_d = -\Delta + \int_0^T \sin^2(\alpha/2) \dot{\phi} \, dt.
\] (3.12)

Since \(\phi_g = -\phi_d\), we get

\[
\phi_g = - \int_0^T \sin^2(\alpha/2) \dot{\phi} \, dt + \Delta,
\] (3.13)

which agrees with Eq. (3.9b).

*The Horizontal Lift.*

Here the integrand in the \(\phi_d\) definition is zero, giving

\[
\dot{\theta}_1 = - \sin^2 \alpha/2 \dot{\phi},
\]
so that the gauge is determined in this case to be

\[ \theta_1(T) - \theta_1(0) = -\int_0^T \sin^2(\alpha/2) \dot{\phi} \, dt. \quad (3.14) \]

Substituting this in Eq. (3.7),

\[ \phi_p = -\int_0^T (\sin^2 \alpha/2) \dot{\phi} \, dt + \Delta. \quad (3.15) \]

Since \( \phi_d = 0 \) here, we get an expression for \( \phi_g \) which is the same as Eq. (3.13). Thus the two prescriptions give the same geometric phase, which holds for both cyclic and noncyclic evolutions.

For a cyclic evolution, it is clear from (3.8) that \( \Delta = 0 \). Hence the geometric phase is just minus half the solid angle \( \Omega \) subtended by the closed curve generated on a sphere by the tip of a unit vector \( r \):

\[ r = (\sin \alpha \cos \phi, \sin \alpha \sin \phi, \cos \alpha) \quad (3.16) \]

Here, \( \alpha \) and \( \phi \) denote the polar and azimuthal angles of \( r \).

An Example:

As an example we consider an interacting Bose system with \( \Lambda = 5 \), in a symmetric trap, i.e., \( \Delta E = 0 \). As already mentioned, the phase space portrait for this case is found from Eq. (2.13) and is given in Fig. 3. The value of \( \Lambda \) selected is quite generic because further increase in \( \Lambda \) does not change the character of the phase-space portrait much, apart from moving the \( \pi \)-state towards \( z = 1 \). Since \( \cos \alpha = z \), the geometric phase \( \phi_g \) given in Eq. (3.9b) can be re-expressed as

\[ \phi_g = \frac{1}{2} \int_0^T (z - 1) \dot{\phi} \, dt + \Delta. \quad (3.17) \]

While comparing the geometric phase from the above equation to that in Eq. (3.4), it is necessary to keep in mind that in the Hamiltonian formalism the time has been scaled and so one has to use the appropriate value of time in Eq. (3.17). We solve Eqs. (2.8a) and (2.8b) numerically for \((z(t), \phi(t))\), for a given initial condition \((z(0), \phi(0))\) at time \( t = 0 \). Using these solutions, we find the solutions for the corresponding unit vectors \( r \) given in Eq. (3.16). These yield the the path on the unit sphere plotted in Fig. 7. Next, we substitute the solution for \((z(t), \phi(t))\) in Eq. (3.17), to find \( \phi_g(t) \) numerically, for various times \( t \), in the range \( 0 \leq t \leq T \), where \( T \) denotes the full period of the orbit concerned. Our plot for
the time dependence of $\phi_g(t)$ over a period for a librational orbit is given in Fig. 8, while Fig. 9 gives that plot for a rotational orbit.

We conclude this section with the following remark. There exists an interesting geometrical representation of a two level system in terms of the time evolution of a unit vector $r$. In the next section, we identify $r$ with the tangent of a space curve, and provide a classical differential geometric approach to derive the geometric phase $\phi_g$ associated with the BJJ evolution.

**IV. GEOMETRIC PHASE USING SPACE CURVE APPROACH**

In this section, we derive the geometric phase associated with the Bose condensate tunneling dynamics by providing a geometric visualisation of this two level system. Firstly it is possible to show\cite{11} that the tunneling equations (2.4) for the two-level wave function,

$$\psi = \begin{pmatrix} a \\ b \end{pmatrix} = e^{i\theta} \begin{pmatrix} \cos(\alpha/2) \\ \sin(\alpha/2)e^{i\phi} \end{pmatrix}, \quad (4.1)$$

with $\phi = \theta_2 - \theta_1$ can be mapped to the following vector evolution equation.

$$\frac{dr}{dt} = \omega \times r. \quad (4.2)$$

Here, in cartesian coordinates ,

$$\omega = (-2V, 0, 2\omega_0) \quad (4.3)$$

$$r = (a^*b + a\bar{b}, i(ab^* - a^*b), |a|^2 - |b|^2). \quad (4.4)$$

Using the definitions of $a$ and $b$ given in Eq. (2.6), Eq. (4.4) is readily seen to be identical to the unit vector $r$ in Eq. (3.16).

Urbantke \cite{12} has shown that corresponding to a wave function of the form (4.1), in addition to $r$ two more unit vectors $P', Q'$ can be defined, such that the set $(r, P', Q')$ forms a unit orthogonal right-handed triad. This is achieved by defining a complex vector $Z'$ as follows :

$$Z' = P' + iQ' = ((a^2 - b^2), i(a^2 + b^2), -2ab). \quad (4.5)$$

On using Eq. (2.6) in Eq. (4.5), we get,
\[ Z' = e^{2i\theta_1}(\cos^2(\alpha/2) - \sin^2(\alpha/2)e^{2i\phi}, i(\cos^2(\alpha/2) + \sin^2(\alpha/2)e^{2i\phi}), -\sin(\alpha)e^{i\phi}) \]  \tag{4.6}

It can be easily verified that,
\[ |r| = |P'| = |Q'| = 1 ; \quad r \cdot P' = r \cdot Q' = P' \cdot Q' = 0 \]  \tag{4.7}

Clearly now as \( r \) evolves with time, so does the \((P', Q')\) plane. The total phase \( \Gamma_p \) accumulated by \( Z'(t) \) in time \( T \) is given by,
\[ \Gamma_p = \text{arg}(Z'(0)^* \cdot Z'(T)) \]  \tag{4.8}

Substituting for \( Z' \) from Eq. (4.6) into Eq. (4.8), after some algebra we obtain,
\[ \Gamma_p = 2[\theta_1(T) - \theta_1(0)] + \Delta \]  \tag{4.9}

where \( \Delta \) is identical to the expression Eq. (3.8) obtained in the kinematic approach in Sec. III. Thus from Eqs. (3.7) and (4.9) we get
\[ \Gamma_p = 2\phi_p \]  \tag{4.10}

\( \phi_p \) being the total phase of \( \psi \) in the kinematic approach.

First we find the total phase rotation \( \gamma_p \) associated with the rotation of \( P' \) or \((Q')\) as follows. It is defined by
\[ \cos \gamma_p = P'(T) \cdot P'(0) = Q'(T) \cdot Q'(0). \]

Further, it is easy to see geometrically that \( P'(T) \cdot Q'(0) = -Q'(T) \cdot P'(0) = \sin \gamma_p \). Substituting \( Z' = P' + iQ' \) in Eq. (4.8) and using the above relations, we can show that the total phase
\[ \gamma_p = -\Gamma_p = -2[\theta_1(T) - \theta_1(0)] + \Delta, \]  \tag{4.11}

where we have used Eq. (4.9).

Next we wish to find the dynamical phase \( \gamma_d \) associated with \((P', Q')\) rotation, which is induced by the specific dynamical equations of the frame \((r, P', Q')\). This is a little more involved, and we proceed as follows.

From Eq. (4.6), we have
\[ Z' = e^{2i\theta_1}Z \]  \tag{4.12}
This immediately leads to
\[ P' + iQ' = e^{2i\theta_1}(P + iQ), \] (4.13)

Comparing this with Eq. (4.6) yields
\[ P = (\cos^2(\alpha/2) - \sin^2(\alpha/2) \cos 2\phi, - \sin^2(\alpha/2) \sin 2\phi, - \sin \alpha \cos \phi) \] (4.14a)
\[ Q = (- \sin^2(\alpha/2) \sin 2\phi, \cos^2(\alpha/2) + \sin^2(\alpha/2) \cos 2\phi, - \sin \alpha \sin \phi) \] (4.14b)

It can be easily verified that \((r, P, Q)\) is also a right-handed triad.

A short calculation using Eqs. (3.16) and (4.14) shows that we can write
\[ \frac{dr}{dt} = XP + YQ, \] (4.15)

where
\[ X = (\frac{d\alpha}{dt}) \cos \phi - (\sin \alpha \frac{d\phi}{dt}) \sin \phi \] (4.16a)
\[ Y = (\sin \alpha \frac{d\phi}{dt}) \cos \phi + (\frac{d\alpha}{dt}) \sin \phi \] (4.16b)

Obviously, there is a gauge freedom \(2\theta_1\) in the choice of \((P, Q)\). We immediately see this from Eq. (4.13):
\[ P' = P \cos \beta - Q \sin \beta \] (4.17a)
\[ Q' = P \sin \beta + Q \cos \beta, \] (4.17b)

where
\[ \beta = 2\theta_1 \] (4.18)

represents the gauge freedom. Using Eqs. (4.17), we solve for \((P, Q)\) in terms of \((P', Q')\).

Substituting them in Eq. (4.15) yields
\[ \frac{dr}{dt} = \alpha_1 P' + \alpha_2 Q' \] (4.19)

where
\[ \alpha_1 = \frac{d\alpha}{dt} \cos(\phi + \beta) - (\sin \alpha \frac{d\phi}{dt}) \sin(\phi + \beta) \] (4.20a)
\[ \alpha_2 = \frac{d\alpha}{dt} \sin(\phi + \beta) + (\sin \alpha \frac{d\phi}{dt}) \cos(\phi + \beta) \] (4.20b)

Since \((r, P', Q')\) is an orthonormal triad, Eq. (4.19) immediately implies,
\[ \frac{dP'}{dt} = -\alpha_1 r + \alpha_3 Q' \] (4.21)
\[
\frac{dQ'}{dt} = -\alpha_2 \mathbf{r} - \alpha_3 \mathbf{P}' \tag{4.22}
\]

where \(\alpha_3\) is to be determined. In the space curve language, if \(\mathbf{r}\) is identified with the tangent \(\mathbf{T}\), then \(\alpha_1\) and \(\alpha_2\) are the components of the curvature vector \(\frac{d\mathbf{r}}{dt}\) along \(\mathbf{P}'\) and \(\mathbf{Q}'\) respectively. Further, equations (4.19), (4.21) and (4.22) describe the equations for a space curve in a “natural frame” \((\mathbf{T}, \mathbf{P}', \mathbf{Q}')\). We remark that the Frenet frame\[13\] corresponds to \(\alpha_2 = 0\), \(\mathbf{P}'\) is the normal \(\mathbf{n}\), \(\mathbf{Q}'\) is the binormal \(\mathbf{b}\). Further, \(\alpha_3\) is the torsion \(\tau\) and \(\alpha_1\) is the curvature \(K\). On setting \(\alpha_2 = 0\), we get from Eq. (4.20), the following “Frenet gauge” \(\beta_F\):

\[
\tan(\beta_F + \phi) = \sin \alpha \frac{d\phi}{dt}/\left(\frac{d\alpha}{dt}\right) \tag{4.23}
\]

Working with the natural frame, a short calculation using Eqs. (4.15) to (4.18) yields,

\[
\alpha_3 = \mathbf{T} \cdot (\mathbf{T} \times \mathbf{T})/|\mathbf{T}|^2 - \frac{d}{dt}\tan^{-1}\left(\frac{\alpha_2}{\alpha_1}\right). \tag{4.24}
\]

Next using the cartesian representation of \(\mathbf{T} = \mathbf{r}\) given in Eq. (4.4), a lengthy but straightforward calculation leads to,

\[
\mathbf{T} \cdot (\mathbf{T} \times \mathbf{T})/|\mathbf{T}|^2 = \cos \alpha \frac{d\phi}{dt} + \frac{d}{dt}\tan^{-1}\left[\sin \alpha \frac{d\phi}{dt}\right]. \tag{4.25}
\]

Substituting Eq. (4.25) and (4.20) in Eq. (4.24) and using the formula \(\tan^{-1} A - \tan^{-1} B = \tan^{-1}\left((A - B)/(1 + AB)\right)\), we obtain,

\[
\alpha_3 = \cos \alpha \frac{d\phi}{dt} - \frac{d(\phi + \beta)}{dt} = -2 \sin^2 \alpha \frac{d\phi}{dt} - \frac{d\beta}{dt} \tag{4.26}
\]

Note that the time derivative of the gauge freedom \(\beta(t)\) appears in \(\alpha_3\).

We write Eqs. (4.19), (4.21) and (4.22) in a compact form,

\[
\frac{d\mathbf{T}}{dt} = \xi \times \mathbf{T}; \quad \frac{d\mathbf{P}'}{dt} = \xi \times \mathbf{P}'; \quad \frac{d\mathbf{Q}'}{dt} = \xi \times \mathbf{Q}'. \tag{4.27}
\]

Here \(\xi\) is given by,

\[
\xi = \alpha_3 \mathbf{T} + \alpha_1 \mathbf{Q}' - \alpha_2 \mathbf{P}'. \tag{4.28}
\]

Eqs. (4.27) show that the natural frame \((\mathbf{T}, \mathbf{P}', \mathbf{Q}')\) rotates with an angular velocity \(\xi\), as it moves along the space curve. As is obvious, \(\alpha_1\) and \(\alpha_2\) are components of \(\xi\) along the \(\mathbf{Q}'\) and \(\mathbf{P}'\) axes respectively and hence tilt the \((\mathbf{P}', \mathbf{Q}')\) plane. On the other hand, \(\alpha_3\) merely
rotates this plane around $T$. Thus in time $T$, the $(P', Q')$ plane gets rotated by an angle $\gamma_d = \int_0^T \alpha_3 dt$. Such a frame is defined using Fermi-Walker parallel transport as $[14]$, 

$$\frac{DA^i}{dt} = \{(\alpha_1 Q' - \alpha_2 P') \times A\}^i.$$ 

Using the expression for $\alpha_3$ given in (4.26) we obtain the dynamical phase $\gamma_d$ associated with $(P', Q')$ plane to be

$$\gamma_d = \int_0^T \alpha_3 dt = -2 \int_0^T (\sin^2 \frac{\alpha}{2}) \frac{d\phi}{dt} dt - 2(\theta_1(T) - \theta_1(0)),$$  \hspace{1cm} (4.29)

since $\beta = 2\theta_1$, from Eq. (4.18).

Subtracting Eq. (4.29) from the expression for the total phase $\gamma_p$ given in Eq. (4.11) we obtain the geometric phase $\gamma_g$ associated with $(P', Q')$ rotation to be

$$\gamma_g = \gamma_p - \gamma_d = 2\int_0^T (\sin^2 \frac{\alpha}{2}) \frac{d\phi}{dt} dt - \Delta$$  \hspace{1cm} (4.30)

Note that the term involving the gauge freedom $\beta$ cancels out here too, as in the kinematic approach. Comparing Eq. (4.30) with Eq. (3.9b), we see that

$$\gamma_g = -2\phi_g$$

In other words, the geometric phase $\phi_g$ associated with the wave function is minus half of that associated with the $(P', Q')$ rotation. For a cyclic evolution, $\Delta = 0$. Here, on computing $\gamma_g$, the geometric phase $\phi_g$ becomes just minus half the solid angle, as we saw in Sec. III. In summary, by mapping the evolution equation for the wavefunction to the dynamical equation for an orthonormal triad $(T, P', Q')$ and identifying the triad to be a natural frame on a space curve, enables us to provide a purely geometrical visualisation of the geometric phase of a two level system.

Our general result is valid for any two level system with the wavefunction (4.1), since we did not use the specific BJJ equations (4.2) and (4.3) in its derivation. By finding the solutions $\alpha(t)$ and $\phi(t)$ for the nonlinear equations (2.12) numerically for given initial conditions, $\gamma_g$ can be computed and is exactly $-2\phi_g$, with $\phi_g$ values as plotted in Figs. 8 and 9.

V. GEOMETRIC PARAMETERS ASSOCIATED WITH BJJ DYNAMICS

In the last section, we discussed the mapping of the BJJ tunneling equations to a space curve which is described using equations for a “natural frame”. This description involves
three geometrical parameters \( \alpha_i \) which are shown to depend on a gauge parameter \( \beta \) (see Eq. (4.20) and (4.24)).

The usual description of a space curve is in terms of a Frenet frame \[13\], with the curvature \( K \) and torsion \( \tau \) as the geometric parameters. As explained in Sec. IV, working with the Frenet frame implies fixing \( \beta = \beta_F \), defined in Eq. (4.19). In this section we work with the Frenet frame to determine the geometric parameters \( K \) and \( \tau \) of the space curve associated with the BJJ dynamics, in terms of the physical parameters \( V, \Delta E \) and \( \Lambda \) and discuss certain special cases of interest.

As mentioned in Sec. IV, in the Frenet frame, \( \alpha_1 = K, \alpha_2 = 0, \alpha_3 = \tau, \ P = n \) and \( Q = b \) in Eqs. (4.15) to (4.18). In this frame, we have the usual Frenet-Serret equations\[13\],

\[
\frac{dT}{dt} = K n, \quad \frac{dn}{dt} = -KT + \tau b; \quad \frac{db}{dt} = -\tau n
\] (5.1a)

Thus,

\[
\frac{dT}{dt} = \xi_F \times T; \quad \frac{dn}{dt} = \xi_F \times n; \quad \frac{db}{dt} = \xi_F \times b.
\] (5.1b)

Also,

\[
K^2 = \left( \frac{dT}{dt} \right)^2 = \sin^2 \alpha \left( \frac{d\phi}{dt} \right)^2 + \left( \frac{d\alpha}{dt} \right)^2
\] (5.2)

and

\[
\tau = T \cdot \left( \dot{T} \times \ddot{T} \right)/K^2 = \cos \alpha \frac{d\phi}{dt} + \frac{d}{dt} \tan^{-1} \left( \frac{\sin \alpha}{\frac{d\alpha}{dt}} \right),
\] (5.3)

where the cartesian representation (3.16), \( T = r \) has been used. On using Eqs. (2.12) for \( \frac{d\alpha}{dt}, \frac{d\phi}{dt} \) in Eqs. (5.2) and (5.3) respectively, we get

\[
K = 2(V^2 + \hbar^2 \omega_0^2 \sin^2 \alpha - V^2 \sin^2 \alpha \cos^2 \phi + 2V\hbar \omega_0 \cos \alpha \sin \alpha \cos \phi)^{1/2},
\] (5.4)

and

\[
\tau = \cos \alpha (\hbar \omega_0 + V \cot \alpha \cos \phi) + \frac{d}{dt} \tan^{-1} \left( \frac{\hbar \omega_0 \sin \alpha + V \cos \alpha \cos \phi}{V \sin \phi} \right).
\] (5.5)

Equations (5.4) and (5.5) give the curvature and torsion of the space curve created by the BJJ dynamics Eq. (2.4), with parameters \( V \) and \( \hbar \omega_0 \). Here \( \alpha \) and \( \phi \) are solutions of the integrable equations (2.12).
Since \( r \) is identified with \( T \), we also have, for the BJJ system,

\[
\frac{dT}{dt} = \omega \times T, \quad \omega = (-2V, 0, 2\omega_0),
\]

\( (5.6) \)

From Eq. (4.2) therefore \( K^2 \) can also be written as,

\[
K^2 = \left( \frac{dT}{dt} \right)^2 = (\omega)^2 - (\omega \cdot T)^2.
\]

\( (5.7) \)

Using the definition of the matrix \( M_{\omega_0} \) given in equation (2.4), a simple calculation shows that \( 2 < M_{\omega_0} >= \omega \cdot T \), yielding

\[
4 < M^2_{\omega_0} >= (\omega)^2.
\]

\( (5.8) \)

Using Eq. (5.8) in Eq. (5.7), we get

\[
K = 2(< M^2_{\omega_0} > - < M_{\omega_0} >^2)^{1/2}.
\]

\( (5.9) \)

Now from the first equation in Eq. (5.1) it is clear that the distance traveled by the tip of \( T \) on the unit sphere in time \( dt \) is \( ds = K dt \). This is the well known Fubini-Study metric. Thus we see that the curvature \( K \) which determines the geometric quantity \( ds \) is given by the variance of the tunneling matrix \( M_{\omega_0} \) (see Eq. (2.4)) for a two level system. As seen from the equation (5.3), the torsion integral \( \int \tau dt \) measures the anholonomy of the frame, i.e. a path dependent geometric quantity given by the solid angle associated with a cyclic evolution of \( T \).

Recalling that the population density difference between the two traps is given by \( z \) and the phase difference by \( \phi \), it is instructive to write the geometric quantities \( K \) and \( \tau \) in terms of these physical quantities and the system parameters \( V, \Delta E \) and \( \Lambda \): from Eq. (5.4),

\[
K = 2(V^2 + (\Delta E + \Lambda z)^2(1 - z^2) - V^2(1 - z^2)\cos^2 \phi
+ 2V(\Delta E + \Lambda z)z\sqrt{1 - z^2}\cos \phi)^{1/2}.
\]

\( (5.10) \)

After a short calculation \( K \) can be written as,

\[
K = 2[V^2 + (\Delta E + \Lambda z)^2 - (H_{cl} + \frac{\Lambda z^2}{2})]^1/2,
\]

\( (5.11) \)

where \( H_{cl} \) is the effective classical Hamiltonian given in Eq. (2.10), which leads to the integrable dynamics of \( z \) and \( \phi \). Next from Eq. (5.5) we obtain \( \tau \):}

\[
\tau = z(\Delta E + \Lambda z + \frac{Vz}{\sqrt{1 - z^2}}\cos \phi)
\]
\[ \frac{d}{dt} \tan^{-1} \left( \frac{\Delta E + \Lambda z}{V \sin \phi} \right) \]

Using the expression for \( H_{cl} \) once again, we get

\[ \tau = H_{cl} + \frac{\Lambda z^2}{2} + \frac{V}{\sqrt{1 - z^2}} \cos \phi \]

\[ + \frac{d}{dt} \tan^{-1} \left( \frac{\Delta E + \Lambda z}{V \sin \phi} \right) \]

We consider some special cases:

(1) Interacting Bose system with no external potential: \( (V = 0, \Lambda \neq 0) \)

From Eq. (2.8), setting \( V = 0 \), we get \( z = \text{constant} \). This in turn yields \( \tau = H_{cl} + \frac{\Lambda z^2}{2} = \text{constant} \) and \( K = 2(\Delta E + \Lambda z)\sqrt{1 - z^2} = \text{constant} \). i.e., the underlying geometry is that of a circular helix with a constant pitch.

(2) The Ideal Bose Gas in an external potential: \( (\Lambda = 0, V \neq 0) \)

If one considers a non-interacting Bose system then \( \Lambda = 0 \) (see (2.9). In this limit the kinetic energy term of the Hamiltonian \( H_{cl} \) (2.10) vanishes and hence the non-rigid pendulum analogy is not valid anymore. However the tunneling Hamiltonian coincides with that of a two-component BEC in the rotation frame approximation [16]. Here,

\[ H_{cl} = -V \sqrt{1 - z^2} \cos \phi + \Delta Ez \]

From Eq. (5.13), we see that,

\[ \tau = H_{cl} + \frac{V}{\sqrt{1 - z^2}} \cos \phi + \frac{d}{dt} \tan^{-1} \frac{\Delta E \sqrt{1 - z^2} + Vz \cos \phi}{V \sin \phi} \]

Further, from Eq. (5.11),

\[ K = 2(V^2 + (\Delta E)^2 - H_{cl})^{1/2} \]

Since \( H_{cl} \) is a constant under time evolution, Eq. (5.16) shows that the curvature \( K \) is a constant. However, the torsion \( \tau \) is time-dependent in this case. Since \( K \) is a constant, the path length on the unit sphere as given by the Fubini-Study metric is linearly dependent on time for this case. (If \( V \) and \( \Delta E \) are made time dependent, then \( K \) is not a constant any more.)

(3) The linear limit:

For a symmetric trap with \( \Delta E = 0 \) in the small oscillations limit, linearizing Eqs. (2.8) in both \( z \) and \( \phi \), for \( |z| \ll 1, |\phi| \ll 1 \), we get,

\[ \frac{dz}{dt} = -V \phi, \quad \frac{d\phi}{dt} = (\Lambda + V)z, \]

\[ 18 \]

\[ (5.17) \]
and

\[ H = (\Lambda + V) \frac{z^2}{2} + V \phi^2 \]

This is just the harmonic oscillator limit and analytical solutions are known. The corresponding expressions for \( K \) and \( \tau \) can be calculated using Eqs (5.11) and (5.12).

(4) The pendulum limit:
For a symmetric trap with \( \Delta E = 0 \), linearizing Eqs. (2.8) in \( z \) only, with \( \Lambda \gg 1 \) we get the equations of a pendulum,

\[ \frac{dz}{dt} = -V \sin \phi \quad \frac{d\phi}{dt} = \Lambda z \]

As is well known the solutions for \( z \) can be written in terms of Elliptic functions thus \( K \) and \( \tau \) can be obtained from Eq.s (5.11) and (5.12).

Finally, for a symmetric trap \( \Delta E = 0 \), with no linearizing approximations, though the analytical solution of Eqs. (2.8) can be found, it is easier to work with numerical solutions instead, using which \( K \) and \( \tau \) can be computed numerically using the expressions (5.11) and (5.12).

VI. SUMMARY OF RESULTS AND DISCUSSION

The geometric phase associated with the time evolution of the wave function of a Bose-Einstein condensate in a double well trap has been found using a quantum approach as in Sec. III. We have explicitly computed the geometric phase \( \phi_g \) for both cyclic and noncyclic evolutions of the condensate population density difference \( z \) and phase difference \( \phi \) in the two wells, by taking an example. In Sec IV, we have shown that the geometric phase can also be derived using a classical differential geometric approach, by essentially mapping the evolution of the two states to a framed space curve with natural moving frames along the curve. The unit tangent vector \( \mathbf{T} \) to the curve has \( \alpha = \cos^{-1} z \) as the polar angle and \( \phi \) as the azimuthal angle. As we have shown, here the geometric phase arises due to the path-dependent rotation of the frame perpendicular to \( \mathbf{T} \) as the system evolves in time.

In an experimental set up, suppose one designs a double-well trap by creating a barrier within a trapped condensate with \( N \) atoms, using a laser sheet. At time \( t = 0 \), let \( N_1(0) \) and \( N_2(0) \) represent the number of condensate atoms in the two neighbouring traps thus created, so that \( N_1 + N_2 = N \). Let the difference in the condensate densities be \( z(0) = \ldots \)
Let \( \phi(0) \) be the initial *phase difference* between the two condensates. We propose that in an actual experiment, immediately after creating the laser sheet barrier, if the density difference and the phase difference between the condensates in the two traps can be measured as a function of time, then by substituting these experimentally measured functions in Eq. (3.17), the associated geometric phase \( \phi_g \) can be determined. As we have seen, \( \phi_g \) will depend on system parameters as well as initial conditions.

Theoretically, the evolution equations for \( z(t) \) and \( \phi(t) \) are given in Eqs. (2.8a) and Eq. (2.8b) respectively. The trap parameters are given in Eq. (2.9). As an illustrative example, we have chosen the parameters \( \Lambda V = 5 \) and \( \Delta E = 0 \). This corresponds to an interacting bose system in a symmetric trap. With appropriate initial conditions \((z(0), \phi(0))\) for both a librational orbit and a rotational orbit, we have solved Eqs. (2.8a) and (2.8b) for \( z(t) \) and \( \phi(t) \) numerically. We then calculate the geometric phase \( \phi_g \) for the above two types of orbits, by using Eq. (3.17). These are plotted in Figs. 8 and 9, respectively. As is obvious, the last point for \( \phi_g \) on each of the plots, i.e., for the maximum value \( t = T_m \), corresponds to the geometric phase for a cyclic evolution, when the population difference and phase-difference between the two condensates evolve in such a way as to return to their initial values after a time period \( T_m \). All the other intermediate points correspond to non-cyclic evolutions.

As should be clear, \( \phi_g \) for other parameters \( \Lambda \) and \( \Delta E \) can also be computed by following our method, case by case. This would enable one to study the variation of \( \phi_g \) with trap parameters, which would be useful in designing appropriate experiments to measure it.

The possibility of another type of experiment to study tunneling between condensates has been proposed by Williams *et al* \[16\]. This hinges on the fact that it has become possible to confine a two-component bose condensate in the same trap, as follows. Hall *et al* \[20\] first trapped and cooled \(^{87}\text{Rb}\) atoms in a magnetic trap in the \(|f = 1, m_f = -1 \rangle \) hyperfine state. After condensation, it is possible to populate the \(|f = 2, m_f = 1 \rangle \) hyperfine state through a two-photon transition. In the presence of a weak magnetic field, these states are separated in energy by \( \omega_o \) (say). Thus two different hyperfine states can exist in the trap. A weak two-photon driving pulse is applied which couples the two states and consequently, atoms can get transferred (or "tunnel") between the two condensates. In this model, it has been shown \[16\] that in the mean field approximation, one obtains coupled equations for \( z(t) \) and \( \phi(t) \) almost identical in form to Eqs. (2.8), but with \( \Lambda = 0 \) (i.e., non-interacting) with
the other parameters appropriately defined for the model, and hence all our results for the geometric phase are applicable here as well.

Recently, Fuentes-Guirdi et al [17] have proposed a method for generating a geometric phase in a coupled two-mode Bose Einstein condensate, starting with a Hamiltonian for two condensates existing in different hyperfine states. In addition to the experiments of Hall mentioned above, condensates of $^{87}$Rb atoms in hyperfine states $|f = 1, m_f = 1\rangle$ and $|f = 2, m_f = 2\rangle$ have been produced experimentally [18]. Likewise, condensates of $^{23}$Na atoms with $|f = 1, m_f = 1\rangle$ and $|f = 1, m_f = 0\rangle$ have also been created [19]. Using the Schwinger angular momentum representation, the Hamiltonian describing two coupled hyperfine states $|A\rangle$ and $|B\rangle$ can be expressed in the form [17]

$$H_{hf} = \alpha_0 J_z + \beta_0 J_z^2 + \gamma_0 [J_x \cos \phi_D + J_y \sin \phi_D], \quad (6.1)$$

Here, $(J_x, J_y, J_z)$ are the components of an effective ‘mesoscopic’ spin $J$, since it can be shown that $J$ is proportional to the total number of atoms $N$ in the condensate, which is of the order of $10^4$. In Eq. (6.1), $\phi_D = D t$, $D$ being the detuning frequency of the laser which couples the two hyperfine states. Further,

$$\alpha_0 = (\omega_A - \omega_B) + (2J - 1)(U_A - U_B), \quad \beta_0 = \frac{U_A + U_B - U_{AB}}{2}, \quad (6.2)$$

and $\gamma_0$ is the strength of the laser-induced drive term that couples the two levels.

Interestingly, if we write the components of $J$ in the form

$$J = (J_x, J_y, J_z) = J(\sin \alpha \cos \phi, \sin \alpha \sin \phi, \cos \alpha)$$

in Eq. (6.1), then on setting $\phi_D = 0$, $H_{hf}/J$ becomes identical to our Hamiltonian in Eq. (2.10), on identifying $\alpha_0 = \Delta E$, $\beta_0 = \Lambda/2$ and $\gamma_0 = - V$. Conversely, if an external driving field phase $\phi_D$ is subtracted from $\phi$ in Eq. (2.10), we would essentially obtain Eq. (6.1). Thus our results for the geometric phase will be valid for that case too, with the appropriate parameters substituted.

However, we point out that in the setting of [17], one varies the BJJ system parameters $\alpha_0$ and $\beta_0$ adiabatically to produce a closed circuit in parameter space, i.e., one considers an adiabatic, cyclic evolution in parameter space, with an associated geometric phase. In contrast, in our setting, we have fixed system parameters, for which we obtain the geometric phase for both cyclic and noncyclic (and nonadiabatic) evolutions on the unit sphere, i.e.,
in "spin" space. In other words, as the population difference and phase difference between the two hyperfine states evolve in time, the system follows a path in this projective space, and as we have shown, an associated geometric phase can be defined.

Experimental techniques to produce two condensates in close proximity has been suggested recently by Chikkatur et al.\cite{21}, in the context of a technique to produce a continuous source of a Bose-Einstein condensate. It would be interesting to study the tunneling between the condensates in such a set up if feasible, i.e., investigate the evolution of population differences and phase differences, to find the associated geometric phase.

Geometric phases have been recently shown to have relevance in the implementation of fault-tolerant quantum computation\cite{22}, and in the creation of vortices in a condensate\cite{23}. We hope that our results will have applications in these contexts as well.

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FIG. 1: Phase portrait of BJJ evolution (Eq. (2.13)) for an interacting bose system with $\Lambda = 0.5$, in a symmetric trap.
FIG. 2: Phase portrait of BJJ evolution (Eq. (2.13)) for an interacting bose system with Λ = 1.3, in a symmetric trap. The trapped states at φ = π are clearly visible here.
FIG. 3: Phase portrait of BJJ evolution (Eq. (2.13)) for an interacting bose system with $\Lambda = 5$, in a symmetric trap.
FIG. 4: Phase portrait of BJJ evolution (Eq. (2.14)) for a non-interacting bose system in a symmetric trap with $\Delta E = 0$.
FIG. 5: Phase portrait of BJJ evolution (Eq. (2.14)) for a non-interacting bose system in an asymmetric trap with $\Delta E = 0.5$
FIG. 6: Phase portrait of BJJ evolution (Eq. (2.14)) for a non-interacting bose system in an asymmetric trap with $\Delta E = 1.0$
FIG. 7: BJJ evolution of the unit vector $\mathbf{r}$ (see Eq. (4.2)) on the unit sphere: Paths corresponding to a librational orbit and a rotational orbit (labeled $r$ and $l$ respectively in the plot) in the phase space portrait of the BJJ Hamiltonian for a symmetric trap with $\Lambda = 5$ (see Fig. 2) are shown.
FIG. 8: Evolution of the geometric phase as a function of time over a period for a librational orbit (oscillation about the zero-state) at $\Lambda = 5$, with initial conditions $(z, \phi) = (.3, 0)$ corresponding to orbit 1 in Fig. 7.
FIG. 9: Evolution of the geometric phase as a function of time over a period for a rotational orbit at $\Lambda = 5$ with initial conditions $(z, \phi) = (0.9, 0)$ corresponding to orbit $r$ in Fig. 7.