Comments on scalar–tensor representation of \( f(R) \) theories

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Abstract

We propose a scalar–tensor representation of \( f(R) \) theories using conformal transformations. In this representation, the model takes the form of the Brans–Dicke model with a potential function and a nonzero kinetic term for the scalar field. In this case, the scalar field may interact with matter systems and the corresponding matter stress tensor may be nonconserved.

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1. Introduction

Recent observations on the expansion history of the universe indicate that the universe is experiencing a phase of accelerated expansion [1]. This can be interpreted as evidence either for the existence of some exotic matter components or for the modification of the gravitational theory. In the first route of interpretation one can take a perfect fluid with a sufficiently negative pressure, dubbed dark energy [2], to produce the observed acceleration. There is also a large class of scalar field models in the literature including quintessence [3], phantom [4], quintom fields [5] and so forth. In the second route, however, one attributes the accelerating expansion to a modification of general relativity. A particular class of models that has recently drawn a significant amount of attention is the so-called \( f(R) \) gravity models [6, 7]. These models propose a modification of the Einstein–Hilbert action so that the scalar curvature is replaced by some arbitrary function \( f(R) \). It is well known that \( f(R) \) theories of gravity can be written as a scalar–tensor theory by applying a Legendre transformation [8, 9]. This scalar–tensor representation corresponds to a class of the Brans–Dicke theory with a potential function and \( \omega = 0 \) in the metric formalism. There is also such a correspondence for \( \omega = -\frac{1}{2} \) in the Palatini formalism in which metric and connections are taken as independent variables, see [10] and references therein. Here we do not consider the Palatini formalism.

Although \( f(R) \) gravity theories exhibit a natural mechanism for accelerated expansion without recourse to some exotic matter components, because of vanishing of the kinetic term of the scalar field in the scalar–tensor representation there is a lot of criticism that emphasizes the inability of these models to pass solar system tests [11]. In the present paper, we will focus on the dynamical equivalence of \( f(R) \) theories and the Brans–Dicke theory using conformal transformations. We will show that this equivalence holds for an arbitrary Brans–Dicke parameter. In this case, however, the gravitational coupling of matter systems may be anomalous in the sense that the scalar field interacts with the matter field action.

2. The model

To begin with, we offer a short review on the equivalence of \( f(R) \) theories with a particular class of the Brans–Dicke theory with a potential function. We consider the following action

\[
S = \frac{1}{2} \int d^4x \sqrt{-g} f(R) + S_m(g_{\mu\nu}, \psi),
\]

where \( f(R) \) is an arbitrary function of the scalar curvature \( R \). The matter action \( S_m(g_{\mu\nu}, \psi) \) is

\[
S_m(g_{\mu\nu}, \psi) = \int d^4x \sqrt{-g} L_m(g_{\mu\nu}, \psi),
\]

in which the Lagrangian density \( L_m \) corresponds to matter fields, which are collectively denoted by \( \psi \). One usually introduces a new field \( \chi = R \) by which action (1) can then be written as

\[
S = \frac{1}{2} \int d^4x \sqrt{-g} \{ f(\chi) + f'(\chi)(R - \chi) \} + S_m(g_{\mu\nu}, \psi),
\]

We work in the unit system in which \( h = c = 8\pi G = 1 \).
where the prime denotes differentiation with respect to $R$. Now variation with respect to $\chi$ leads to the equation
\[ f''(R)(\chi - R) = 0. \] (4)
If $f''(R) \neq 0$, we have the result $\chi = R$. Inserting this result into (3) reproduces action (1). Then redefining the field $\chi$ by $\Phi = f(\chi)$ and setting
\[ V(\Phi) = \chi(\Phi)\Phi - f(\chi(\Phi)), \] (5)
the action (3) takes the form
\[ S = \frac{1}{2} \int \! d^4x \sqrt{-\tilde{g}} \left\{ \Phi R - V(\Phi) \right\} + S_m(\tilde{g}_{\mu\nu}, \psi). \] (6)
This is the Brans–Dicke action with a potential $V(\Phi)$ and a Brans–Dicke parameter $\omega = 0$. Therefore, there is a dynamical equivalence between $f(R)$ theories and a class of Brans–Dicke theories with a potential function. The important point in the above transformation is that the matter sector remains unchanged. In particular, in this representation of $f(R)$ theories the matter action $S_m(\tilde{g}_{\mu\nu}, \psi)$ is independent of the scalar field $\Phi$. Thus in both actions (1) and (6) the weak equivalence principle holds and test particles follow geodesics lines of $\tilde{g}_{\mu\nu}$.

In the original form of the Brans–Dicke theory [12], where the potential term of the scalar field is not present, it is found that in order to get agreement between predictions and solar system experiments, $\omega$ should be large and positive. The current observations set a lower bound on $\omega$, which is $\omega > 3500$ [13]. If the theory is allowed to have a potential, the scalar field should be very light and should mediate a gravity force of long range that is not consistent with solar system experiments [11]. On the other hand, it is shown [14] that the dynamical equivalence of (6) and $f(R)$ theories is ill-defined in the scale of the solar system. The underlying logic is based on the fact that there is no observed deviation from general relativity at this scale and $f(R)$ theories must be reduced to general relativity in an appropriate limit. The problem is that general relativity corresponds to $f(R) = R$ for which $f''(R) = 0$, while the dynamical equivalence requires $f''(R) \neq 0$.

We intend here to use a different scalar–tensor representation of $f(R)$ theories suggested by conformal transformations. To this aim, we apply the following conformal transformation:
\[ \tilde{g}_{\mu\nu} = \Omega_2 \tilde{g}_{\mu\nu}, \quad \Omega_2 = e^{\rho/\alpha} \] (7)
to action (1). This together with a redefinition of the conformal factor in terms of a scalar field $\phi = \sqrt{2} \ln \Omega_1$ yields [9]
\[ S = \frac{1}{2} \int \! d^4x \sqrt{-\tilde{g}} \left\{ \psi \tilde{R} - \frac{\omega}{\phi} e^{2\rho/3} \nabla_\mu \psi \nabla_\nu \psi - U(\phi) \right\} + S'_m(\tilde{g}_{\mu\nu}, \psi, \phi), \] (8)
where
\[ S'_m(\tilde{g}_{\mu\nu}, \psi, \phi) = \int \! d^4x \sqrt{-\tilde{g}} e^{-2\rho/3} L_m(\tilde{g}_{\mu\nu}, \psi, \phi). \] (9)
In action (8), $\phi$ is minimally coupled to $\tilde{g}_{\mu\nu}$ and appears as a massive self-interacting scalar field with a potential
\[ V(\phi) = \frac{1}{2} e^{-\sqrt{2/3} \rho} r[\Omega_1(\phi)] - \frac{1}{2} e^{-2\sqrt{2/3} \rho} f(\Omega_1(\phi)). \] (10)
where the function $r(\Omega_1)$ is the solution of the equation $f''[r(\Omega_1)] - \Omega_1 = 0$. Thus, the variables ($\tilde{g}_{\mu\nu}, \phi$) in action (8) provide the Einstein frame variables for $f(R)$ theories.

Variation of (8) with respect to $\tilde{g}_{\mu\nu}$ and $\phi$ gives, respectively,
\[ \tilde{G}_{\mu\nu} = \Omega_2 \tilde{g}_{\mu\nu}, \quad \Omega_2 = e^{\rho/\alpha} \] (16)
to action (8) with $\alpha$ being a constant parameter. This together with $\phi = \alpha \ln \psi$ transforms action (8) to
\[ S = \frac{1}{2} \int \! d^4x \sqrt{-\tilde{g}} \left\{ \psi \tilde{R} - \frac{\omega}{\phi} \tilde{g}^\mu_\nu \nabla_\mu \psi \nabla_\nu \psi - U(\psi) \right\} + S'_m(\tilde{g}_{\mu\nu}, \psi, \phi), \] (17)
where
\[ S'_m(\tilde{g}_{\mu\nu}, \psi, \phi) = \int \! d^4x \sqrt{-\tilde{g}} \psi^a L_m(\tilde{g}_{\mu\nu}, \psi, \phi). \] (18)
and
\[ \omega = a^2 - \frac{9}{2}, \] (19)
\[ n = 2 - 2a \sqrt{\frac{2}{3}}, \] (20)
\[ U(\psi) = 2\psi^2 V(\psi). \] (21)
This is the scalar–tensor representation of action (1) obtained by conformal transformations (7) and (16). In contrast with this representation, the gravitational part of action (1) consists only of the metric tensor $g_{\mu\nu}$, which obeys fourth-order field equations. We may call the conformal frames corresponding to actions (1) and (17) the Jordan frame representation of (8).\(^2\)

\(^2\) Note that we here define Jordan frame in terms of how the geometry is described in the vacuum sector rather than in terms of how it couples with matter systems [9]. Action (1) is in Jordan frame since the resulting field equations are fourth order in terms of metric tensor. On the other hand, action (17) is also in Jordan frame since it describes the geometry by a metric tensor and a scalar field (nonminimal coupling of the scalar field).
Here, a question that arises is: Which of the conformal frames corresponding to actions (1), (8) and (17) should be considered as the physical frame? It should be pointed out that reformulation of a theory in a new conformal frame leads, in general, to a different physically inequivalent theory. The ambiguity of the choice of a particular frame as the physical one is a long-standing problem in the context of conformal transformations. The term ‘physical’ theory denotes one that is theoretically consistent and predicts values of some observables that can, at least in principle, be measured in experiments [15]. In this respect, different authors may consider different conformal frames as physical according to their attitude towards the issue of the conformal frames. Indeed, while in f(R) theories one usually takes the Jordan frame as the physical frame, one may consider positivity of energy and stability to consider the Einstein conformal frame as the physical one [9]. Thus the choice of a physical frame between representations (1), (8) and (17) should be based on the physical outcomes of the corresponding models.

Let us now compare the two scalar–tensor representations of f(R) theories, namely actions (6) and (17). There are two important differences: firstly, in action (17) the Brans–Dicke scalar field \( \varphi \) has a nonzero kinetic term and the Brans–Dicke parameter \( \omega \) is only constrained by observations. Secondly, the scalar field \( \varphi \) enters the matter part of action (17). The latter means that the scalar field interacts with matter systems and tests particles that do not follow the geodesic lines of the latter means that the scalar field interacts with matter systems (5). This is the method that is recently used by some authors to deal with constraints on the potential function \( \omega \)).

Variation with respect to \( \bar{g}_{\mu \nu} \) and \( \varphi \) leads to the field equations

\[
\varphi \bar{G}_{\mu \nu} = \frac{\alpha}{\varphi} (\nabla_\mu \varphi \nabla_\nu \varphi - \frac{1}{2} \bar{g}_{\mu \nu} \nabla_\rho \varphi \nabla_\rho \varphi) - (\nabla_\mu \nabla_\nu \varphi - \bar{g}_{\mu \nu} \bar{\square} \varphi)
+ \frac{1}{2} U'(\varphi) \bar{g}_{\mu \nu} = \bar{T}_{\mu \nu},
\]

(22)

\[
\frac{2 \omega}{\varphi} \bar{\square} \varphi - \frac{\alpha}{\varphi^2} \nabla_\mu \varphi \nabla_\nu \varphi + \bar{R} - \frac{dU(\varphi)}{d\varphi} = \varphi^{-1} \bar{T},
\]

(23)

where

\[
\bar{T}_{\mu \nu} = - \frac{2}{\sqrt{-\bar{g}}} \frac{\delta S}{\delta \bar{g}^{\mu \nu}}
\]

(24)

and \( \bar{T} = \bar{g}^{\mu \nu} \bar{T}_{\mu \nu} \) is the trace of the stress tensor \( \bar{T}_{\mu \nu} \). Now, applying the Bianchi identity \( \bar{\square} \bar{G}_{\mu \nu} = 0 \) and using the field equation of scalar field (23), we obtain

\[
\bar{\square} \bar{T}_{\mu \nu} = - a_i, \quad a_i = \frac{1}{2} \bar{T} \partial_\mu \ln \varphi.
\]

(25)

(26)

Equation (25) implies that the matter stress tensor \( \bar{T}_{\mu \nu} \) is not conserved due to interaction of the scalar field \( \varphi \) with the matter part of (17). Except for the case that the matter field action (18) is traceless [9], the scalar field \( \varphi \) influences the motion of any gravitating matter. In fact, \( a_i \) indicates an anomalous acceleration corresponding to a fifth force.

It should be noted that there are different types of models in the literature that concern matter systems that are not conserved due to interaction with an arbitrary function of scalar curvature [16] or some scalar fields [17]. However, the important difference between equation (25) and the corresponding equations in those models is that the former is the result of the well-known property of conformal transformations, namely that the conservation equation of a matter stress tensor with a nonvanishing trace is not conformally invariant [18].

It is possible to apply this result in the scale of the solar system. To do this, we first note that combining (7) and (16) gives the relation between the scalar field \( \varphi \) and the function \( f(R) \):

\[
\varphi = \left[ f'(R) \right]^{2/(2-n)}.
\]

(27)

Then we take \( \bar{T}_{\mu \nu} \) to be the stress tensor of dust (or perfect fluid with zero pressure) with energy density \( \bar{\rho} \). In this case and for a static spacetime, we obtain, for the spatial part of \( a_i \),

\[
a_i = \frac{1}{2-n} \bar{\rho} \partial_\mu \bar{R} \frac{f''(R)}{f'(R)}.
\]

(28)

As one expects, in the case that \( f(R) \) is a linear function of \( R \) like the Einstein–Hilbert action, the anomalous acceleration is zero. For the CDTT model [6] in which \( f(R) = R - \frac{\alpha}{\mu^2} \), we obtain

\[
a_i = \frac{2}{n-2} \bar{\rho} \partial_\mu \ln \left[ 1 + \frac{R^2}{\mu^2} \right]^{-1},
\]

(29)

where \( \mu \) is an arbitrary mass scale.

3 For a good review on this issue, see [15] and references therein.

3. Concluding remarks

While the scalar–tensor representation (6) of \( f(R) \) theories is useful in cosmological scales, it suffers problems in the weak field limit and solar system scales. Firstly, the kinetic term of the scalar field vanishes, which is in conflict with current bounds on the value of \( \omega \). Secondly, it is recently reported that since \( f''(R) = 0 \) in this scale, the dynamical equivalence of \( f(R) \) theories and scalar–tensor theories represented by (6) breaks down. The main feature of our analysis is to show that there is also a dynamical equivalence between \( f(R) \) theories and scalar–tensor theories using conformal transformations. In this representation, the scalar field has a nonvanishing kinetic term and a nonzero Brans–Dicke parameter. Therefore, the observational constraints can be applied on \( \omega \) in this representation. However, it should be noted that action (17) differs from the Brans–Dicke action in two ways. Firstly, it contains a potential function \( U(\varphi) \). Secondly, the matter system interacts with the scalar field \( \varphi \). That the scalar field possesses a potential function clearly alters the usual bound on the Brans–Dicke parameter \( \omega \) so that the new bound depends on the functional form of the potential [19]. On the other hand, the coupling of the scalar field with the matter sector should be strongly suppressed so as not to lead to observable effects. In fact, one can use a chameleon mechanism [20] to implement constraints on the potential function \( U(\varphi) \). Then using a relation between \( \varphi \) and the curvature scalar (see relation (27)), this can provide some viable forms of \( f(R) \) theories, which are in accord with local gravity tests. Indeed, this is the method that is recently used by some authors to deal with \( f(R) \) theories which are consistent with solar system experiments [21].
It is important to note that nonconservation of the stress tensor $T_{\mu\nu}$ should not be considered as an intrinsic behavior of the model presented here. It is simply related to the fact that before applying the conformal transformations, the matter action is introduced in the nonlinear action (1). The matter action might be added after the conformal transformations in the Jordan frame. In that case there would be no anomalous acceleration. This ambiguity of introducing matter systems to equivalent conformal frames is closely related to the well-known problem of which of these frames should be taken as the physical one [9, 15]. Without dealing with this long-standing problem, we would like to point out that the advantage of the nonminimal coupling of matter in (17) is that it can potentially explain the possible deviations of $f(R)$ theories (or its scalar–tensor representation) from Newtonian gravity in local experiments. Moreover, as earlier stated, the scalar field may also be interpreted as a chameleon field which can suppress detectable effects of anomalous gravitational coupling of matter in solar system scales. The study of the chameleon behavior of the scalar field is in progress by the author and will be investigated elsewhere.

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