The $Sp_3$-grassmannian and duality for prime Fano threefolds of genus 9

Atanas Iliev

Abstract

By a result of Mukai, the non-abelian Brill-Noether locus $X = M_C(2, K : 3F )$ of type II, defined by a stable rank 2 vector bundle $F$ of invariant 3 over a plane quartic curve $C$, is a prime Fano 3-fold $X = X_{16}$ of degree 16. The associated ruled surface $S^X = P(F )$ is uniquely defined by $X$, and we see that for the general $X = X_{16}$, $S^X$ is isomorphic to the Fano surface of conics on $X$. The argument uses the geometry of the $Sp_3$-grassmannian and the double projection from a line on $X_{16}$.

§1. Introduction

The smooth Fano 3-fold $X$ is prime if $Pic(X) = \mathbb{Z}(-K_X)$, where $K_X$ is the canonical class of $X$. The degree $d = d(X) = -K_X^3 = 2g(X) - 2$ of a prime Fano 3-fold $X$ is always even, and the integer $g = g(X)$ is called the genus of $X = X_{2g - 2}$. Prime Fano 3-folds $X_{2g - 2}$ exist iff $2 \leq g = g(X) \leq 12, g \neq 11$, see [1], [IP].

Let $C$ be a smooth plane quartic curve, and let $S = P(F ) \to C$ be a stable ruled surface of invariant $e(S) = \min \{s^2 : s \text{ a section of } S\} = 3$. By a result of Mukai the non-abelian Brill-Noether locus of type II

$$X_F = M_C(2, K : 3F ) = \{E \to C : \text{rank } E = 2, \det E = \det F \otimes K, \dim \text{Hom}(F, E) \geq 3\},$$

in the moduli space $M_C(2, \det F \otimes K_C)$ of stable rank 2 vector bundles on $C$ of determinant $\det F \otimes K_C$, is a prime Fano threefold of genus 9, see [M3]. The associate Fano 3-fold $X_S := X_F$ is uniquely defined by $S$, since the locus $X_F = X_{F \otimes L}$ does not depend on the twists $F \mapsto F \otimes L$ by line bundles $L$ on $C$, see [M3]. In turn, the general Fano 3-fold $X$ of genus 9 is associate to a unique ruled surface $S = S^X$, over the associate plane quartic curve $C^X$ of $X$, see (5.7).

By [M1], [M2], any prime Fano 3-fold $X = X_{16}$ of genus 9 is a linear section of the $Sp_3$-grassmannian $\Sigma \subset \mathbb{P}^{13}$ by a codimension 3 subspace $\mathbb{P}^{10}_X = \mathbb{P}^{10}_X \subset \mathbb{P}^{13}$, defined uniquely by $X$ up to the action $\rho$ of the symplectic group $Sp_3$ on $\mathbb{P}^{13}$.

The dual projective representation $\hat{\rho}$ of $\rho$ has an invariant quartic hypersurface $\hat{F}$ in the dual space $\mathbb{P}^{13}$, see (2.3)(ii), (2.5.a). For a given general $X = \Sigma \cap \mathbb{P}^{10}_X$, the plane $\mathbb{P}^{2}_X = (\mathbb{P}^{10}_X)^{\perp} \subset \mathbb{P}^{13}$ intersects $\hat{F}$ along a smooth plane quartic curve $C_X$, defined uniquely by $X$, see (4.1).

The two main results in this paper are:

*Partially supported by Grant MM-1106/2001 of the Bulgarian Foundation for Scientific Research
A. The Fano family $F(X)$ of conics on the general prime Fano 3-fold $X$ of genus 9 is a ruled surface over the $Sp_3$-dual plane quartic $C_X$ of $X$: see Theorem (4.6).

B. The ruled surface $F(X)$ is isomorphic to the associate ruled surface $S^X$ of $X$; in particular $C_X \cong C^X$: see Theorem (6.5).

The proof of A uses the geometry of $X$ as a subvariety of the $Sp_3$-grassmannian $\Sigma \subset \mathbb{P}^{13}$. The points $c$ of the dual quartic curve $C_X$ of $X$ are the singular hyperplane sections $H_c \subset \Sigma$ which contain $X$. For any $c \in C_X$, the hyperplane section $H_c$ has a node at the correlative pivot point $\tilde{u}(c) \in \Sigma$ of $c$, see (2.7) and (4.1). In the lagrangian plane $P^2_{\tilde{u}(c)}$ lies a smooth conic $q(c)$, invariant under the action of the stabilizer subgroup $St_c \subset Sp_3$ of $H_c$, see (3.3). The vertex surface $S_X = \bigcup_{c \in C_X} q(c) \subset P^5$ of $X$ is a ruled surface over the dual curve $C_X$ with fibers the smooth conics $q(c)$. The points $x \in S_X$ are the vertices of the conics $q_x \subset X$ (see (4.4)), and the vertex map $q_x \mapsto x$ sends the Fano surface $F(X)$ of conics on $X$ isomorphically onto the ruled vertex surface $S_X$ of $X$, see (4.5.b-d) and (4.6). This proves A.

The proof of B is based on the simultaneous interpretation of the lines $l = l_x = l_L$ on the Fano threefold $X$ as minimal sections $C_x$ of the associate surface $S^X$ and as sections $C_L$ of the vertex surface $S_X$. By [1N], a minimal section $C_x \subset S^X$ is represented by a space curve $C^l_x$ of genus 3 and degree 7. By (13E) the curve $C^l_x$ is the same as the curve $C^l$ defining the inverse to a double projection from the line $l = l_x$ on $X$, see (5.1) and (5.4)(d). As a line on the $Sp_3$-grassmannian, $l = l_L$ represents a pencil of lagrangian planes through an isotropic line $L$, in a 3-space $P^3_L \subset P^5$, see (3.1). In the intersection of $P^3_L$ with the vertex surface $S_X$ lies a unique curve $C_L$, which is a section of the ruled vertex surface $S_X$. By the $Sp_3$-geometrical construction of the double projection from $l \subset X \subset \Sigma$, the curve $C_L$ is projectively equivalent to $C^l$, see (6.1)-(6.4). The correspondence $C_x \leftrightarrow l_x = l = l_L \leftrightarrow C_L$ identifies the set $Min(S^X)$ of minimal sections $C_x$ of $S^X$ and the set $Min'(S_X)$ of sections $C_L$ of $S_X$. At the end, Theorem (6.5) shows that the isomorphism $Min(S^X) \cong Min'(S_X)$ yields an isomorphism between the associate surface $S^X$ and the isomorphic vertex model $S_X$ of $F(X)$. This proves B.

§2. The $Sp_3$-grassmannian $\Sigma \subset \mathbb{P}^{13}$

(2.1) Let $V_6 = \mathbb{C}^6$ be a complex vector 6-space, let $\hat{V}_6 = Hom(V_6, \mathbb{C})$ be its dual space, and fix a rank six 2-form $\alpha \in \wedge^2 \hat{V}_6$. The 21-dimensional symplectic group $Sp_3 = Sp_3^0 \subset GL_6$ is the set of all $A \in GL_6$ which preserve the non-degenerate skew-symmetric product $\alpha : V_6 \times V_6 \rightarrow \mathbb{C}$ defined by $\alpha$. In coordinates $(e_i, x_i) = (e_1, \ldots, e_6; x_1, \ldots, x_6)$ on $V_6$,

$$Sp_3 = \{ A \in GL_6 : \alpha(A^t \alpha) = J \}$$

where $J_{ij} = \alpha(e_i, e_j) \neq 0$ is the skew-symmetric Gramm matrix of $\alpha$. One can always choose the coordinates $(e_i, x_i)$ on $V_6$ such that $\alpha = x_{14} + x_{25} + x_{36}$; and then $J_{ij} = J_{i3+i,3+j} = 0$, $J_{i,3+j} = J_{3+i,j} = -J_{i3+i,j}$ for $1 \leq i, j \leq 3$.

The subspace $U \subset V_6$ is called isotropic if $\alpha|_U \equiv 0$, i.e. $\alpha(u_1, u_2) = 0$ for any $u_1, u_2 \in U$; and then the projective subspace $P(U) \subset P(V_6) = P^5$ is also called isotropic.

Since $\alpha$ is skew-symmetric then any point $x \in P^5$ is isotropic. Since $\alpha$ is non-degenerate then the dimension of an isotropic $U \subset V$ can't be more than 3. The isotropic subspaces $U \subset V_6$ of dimension 3, as well their projective planes $P(U) \subset P^5$, are called lagrangian.
The isotropic grassmannian \( \text{LG}_2 \subset G(2, V_6) \) is the set of all isotropic lines \( L \subset \mathbf{P}^5 \). One can see that the line \( L = \mathbf{P}(U_2) \) is isotropic \text{iff} its Plücker image \( \mathbf{P}(\wedge^2 U_2) \) lies in the isotropic hyperplane \( (\alpha = 0) \subset \mathbf{P}(\wedge^2 V_6) \), i.e.

\[
\text{LG}_2 = G(2, 6) \cap (\alpha = 0) = G(2, 6) \cap (x_{14} + x_{25} + x_{36} = 0);
\]

and since \( \alpha \) is non-degenerate, \( \text{LG}_2 \) is a smooth hyperplane section of \( G(2, 6) \).

The lagrangian grassmannian, or the \( S_{p3} \)-grassmannian \( \Sigma = \text{LG}_3 \subset G(3, V_6) \) is the set of all lagrangian subspaces \( U \subset V_6 \). The \( S_{p3} \)-grassmannian, as well the isotropic grassmannian \( \text{LG}_2 \), is a smooth homogeneous variety of the group \( Sp_3 \). In the paper we shall use the explicit coordinate description of \( \Sigma \) which we state below.

For the 3-space \( U \subset V_6 \), the condition \( \alpha|_U \equiv 0 \) is equivalent to the requirement that its Plücker image \( pl(U) = \mathbf{P}(\wedge^3 U) \subset \mathbf{P}^9 = \mathbf{P}(\wedge^3 V_6) \) lies in the codimension 6 subspace \( \mathbf{P}^{13} = \mathbf{P}(V_{14}) = (V_6 \wedge X = \alpha = 0) \subset \mathbf{P}^9 \), i.e.

\[
\Sigma = G(3, 6) \cap (\hat{V}_6 \wedge X = \alpha = 0) = G(3, 6) \cap (x_{14} + x_{25} + x_{36} = 0, 1 \leq i \leq 6).
\]

Let \( U_o = \langle e_1, e_2, e_3 \rangle \) and \( U_\infty = \langle e_4, e_5, e_6 \rangle \). Then \( V_6 = U_o \oplus U_\infty \), and in homogeneous coordinates

\[
(u : X : Y : z) = (u : (x_{ij}) : (y_{ij}) : z) : (x_{123} : \begin{pmatrix} 42 \end{pmatrix} x_{13} \begin{pmatrix} 10 \end{pmatrix} x_{14} \begin{pmatrix} 0 \end{pmatrix} x_{124} \begin{pmatrix} 32 \end{pmatrix} x_{125} \begin{pmatrix} 12 \end{pmatrix} x_{126} \begin{pmatrix} 2 \end{pmatrix} x_{13} \begin{pmatrix} 36 \end{pmatrix} x_{126} \begin{pmatrix} 25 \end{pmatrix} x_{125} \begin{pmatrix} 15 \end{pmatrix} x_{126} \begin{pmatrix} 36 \end{pmatrix} x_{126} \begin{pmatrix} 25 \end{pmatrix} x_{126} \begin{pmatrix} 36 \end{pmatrix} x_{126}) = X_{456}\]

on \( \mathbf{P}^9 = \mathbf{P}(\wedge^3(U_o \oplus U_\infty)) = \mathbf{P}(\wedge^3 U_o \oplus \wedge^2 U_\infty \oplus U_o \oplus \wedge^2 U_\infty \oplus \wedge^3 U_\infty) \), the equations \( \hat{V}_6 \wedge X = \alpha = 0 \) of \( \mathbf{P}^{13} \subset \mathbf{P}^9(u : X : Y : z) \) become \( tX = X, tY = Y \); i.e.

\[
\mathbf{P}^{13} = \mathbf{P}^{13}(u : X : Y : z), tX = X, tY = Y.
\]

Let \( G(3, 6)^o = G(3, 6) - H^o \) be the complement to the Schubert hyperplane section \( H^o = \sigma_{100}(U_o) = \{ U \in G(3, 6) : \text{dim}(U \cap U_o) > 0 \} = G(3, 6) \cap (u = 0) \). Let also \( H_o \subset \Sigma \) be the hyperplane section \( \Sigma \cap H^o = \Sigma \cap (u = 0) \), and \( \Sigma^o = \Sigma - H_o \). The 3-spaces \( U = U_X \in G(3, 6)^o \) are parameterized 1 : 1 by the linear maps \( X_U : U_o \rightarrow U_\infty \), by

\[
U = \langle e_1 + X(e_1), e_2 + X(e_2), e_3 + X(e_3) \rangle \mapsto X \in \text{Hom}(U_o, U_\infty) \cong \otimes^2 \mathbf{C}^3,
\]

and then \( pl(U) = (1 : X : \wedge^2 X : \text{det } X) \). Moreover, the 3-space \( U = U_X \in G(3, 6)^o \) is lagrangian (i.e. \( U \in \Sigma^o \) \text{iff} the matrix } X = X_U \text{ is symmetric}. Therefore \( \Sigma \) is the natural projective compactification of the isomorphic image \( \Sigma^o = \Sigma \cap (u \neq 0) \) of the affine space \( \mathbf{C}^6 = \text{Sym}^2 \mathbf{C}^3 \) under the \( exp \)-map:

\[
\text{exp} : \text{Sym}^2 \mathbf{C}^3 \rightarrow \mathbf{P}^{13}(u : X : Y : z), X \mapsto \text{exp}(X) = (1 : X : \wedge^2 X : \text{det } X).
\]

As a subvariety of \( \mathbf{P}^{13}(u : X : Y : z), tX = X, tY = Y \), the smooth 6-fold \( \Sigma \) is defined by the 21 projectivized quadratic Cramer equations

\[
(*) \quad \wedge^2 X = uY, \wedge^2 Y = zX, XY = uzI_3
\]

(where \( I_3 \) is the unit \( 3 \times 3 \) matrix), which follow from the local \( exp \)-parameterization of \( \Sigma \) by the skew-powers of symmetric \( 3 \times 3 \) matrices.
(2.2) Remark. The next Theorem (2.3) and Lemma (2.7) below are particular cases of the more general Propositions 5.10-5.11 and Proposition 8.2 in [LM1], where are described the orbits in the enveloping spaces and the singular hyperplane sections of the four varieties $\Sigma_A = G_\omega(A^3, A^6)$, in the third row of the Freudenthal magic square. The four varieties $G_\omega(A^3, A^6)$ correspond to the four complex composition algebras $A = A_R \times C$, where $A_R = R, C, H$ and $O$ are the real numbers, the complex numbers, the quaternions and the octonions. The $Sp_3$-grassmannian $\Sigma = \Sigma_R$ corresponds to the case $A = R$; see also §3 in [D], where these results are proved in case $A = C$, where $\Sigma_C = G(3,6)$.

The subspace $V_{14} \subset \wedge^3 V_6$ is the irreducible representation space of $Sp_3$, defined by the weight $\wedge^3$, see [FT], p. 258. We shall denote by $\rho$ the induced projective action of $\wedge^3$ on $P^{13} = P(V_{14})$.

(2.3) Theorem of Segre for $Sp_3$. The action $\rho : Sp_3 \times P^{13} \rightarrow P^{13}$ has 4 orbits

\[ P^{13} = \Sigma \cup (\Omega - \Sigma) \cup (F - \Omega) \cup (P^{13} - F), \] and:

(i) $\Sigma$ is the isomorphic Plücker image of the $Sp_3$-grassmannian $LG_3$.

(ii) $F \subset P^{13}$ is a quartic hypersurface. Moreover $F = \cup_{u \in \Sigma} P^6_u$, where $P^6_u$ denotes the tangent projective 6-space at the point $u \in \Sigma$.

(iii) $x \in P^{13} - \Omega \iff$ there exists a unique secant or tangent line $l_x$ to $\Sigma$, s.t. $x \in l_x$. Moreover $x \in F - \Omega \iff l_x$ is tangent to $\Sigma$.

For the point $u \in \Sigma$ we shall denote by $P^2_u \subset P^5$ the lagrangian plane of $u$.

(2.4) Lemma. Let $u \in \Sigma$, and let $P^6_u$ be the tangent projective space of $\Sigma \subset P^{13}$ at $u$. Then $P^6_u \subset F$, and: (i) $K_u = \{ v \in \Sigma : \dim(P^2_v \cap P^2_u) \geq 1 \} = \Sigma \cap P^6_u$ is a cone with vertex $u$ over the Veronese surface. (ii) $D_u = \Omega \cap P^6_u$ is a cone with vertex $u$ over the symmetric determinantal cubic.

Proof. By the $Sp_3$-homogeneity of $\Sigma$ one can let $u = (1 : 0 : 0 : 0)$ in $P^{13}(u : X : Y : z)$. In the notation of (2.1), the lagrangian plane $P^2_u = P(U_o)$, and the projective tangent space $P^6_u = P(\wedge^3 U_o + \wedge^2 U_o \wedge U_\infty) = P^6_0(u : X : 0 : 0)$. Therefore $v \in \Sigma \cap P^6_0$ iff $v \in \Sigma$ and $\dim(P^2_v \cap P^2_u) \geq 1$. By (2.1) (*), the intersection $\Sigma \cap P^6_u(u : X : 0 : 0)$ is defined by the system $\ker X = 1, tX = X$, which implies (i). By (2.3)(iii) the intersection $\Omega \cap P^6_u(u : X : 0 : 0)$ is exactly the determinantal cubic cone $D_u = (det X = 0)$ which is the locus of points $\omega \in P^6_0$ through which pass more than one secant line to $K_u$, see also §9 of [K]. q.e.d.

(2.5.a) The varieties $\Sigma \subset \Omega \subset F \subset P^{13}$. The smooth variety $\Sigma \subset P^{13}$ is a Fano 6-fold of degree 16 and of index 4, i.e. $K_\Sigma = O_\Sigma(-4)$, see e.g. [M2], [M4]. In coordinates $(u : X : Y : z)$ as above, the $Sp_3$-invariant quartic hypersurface $F$ is defined by the equation

$$F(u : X : Y : z) = (uz - tr XY)^2 + 4u det Y + 4z det X - 4\Sigma_{ij} det(X_{ij}) \cdot det(Y_{ij}) = 0,$$

see [KS] p. 83, or §5.2 and Proposition 5.8 in [LM1]. The 9-fold $\Omega = Sing F = (\nabla F = 0)$ is the common zero-locus of the 14 partial cubic derivatives of $F$. It has degree 21, and $Sing \Omega = \Sigma$, see §2 of [R].
The pivot-map. By analogy with §3 in [D], call the axis of \( x \in \mathbb{P}^{13} - \Omega \) the unique secant or tangent line \( l_x \) to \( \Sigma \) through \( x \), and the pivots of \( x \) the two intersection points \( u \) and \( v \) of \( l_x \) with \( \Sigma \). If \( x \in F - \Omega \) call the point \( u(x) = u = v \in \Sigma \) the double pivot (or simply – the pivot) of \( x \). This way, there exists a well-defined pivot-map:

\[
piv : F - \Omega \to \Sigma, \quad x \mapsto \text{the double pivot } u = u(x) \text{ of } x.
\]

The fiber of the regular pivot map \( \piv : F - \Omega \to \Sigma \) over the point \( u \in \Sigma \) coincides with \( \mathbb{P}^6_u - D_u \), where \( D_u = \Omega \cap \mathbb{P}^6_u \) is the determinantal cubic cone in \( \mathbb{P}^6_u \), see (2.3)-(2.4).

The correlative double pivot with vertex \( \hat{u} \) is \( \hat{\piv} \). By analogy with (2.5.b) The pivot-map, \( \hat{\piv} \), see (2.3)-(2.4).

\[
J = J_{\alpha} : V_6 \to \hat{V}_6 = \text{Hom}(V_6, \mathbb{C}), \quad J : x \mapsto \alpha(x, \cdot).
\]

In canonical coordinates \( (e_i, x_i) \), in which \( \alpha = x_{14} + x_{25} + x_{36} \), one has: \( J(e_1) = x_4, J(e_2) = x_5, J(e_3) = x_6, J(e_4) = -x_1, J(e_5) = -x_2 \) and \( J(e_6) = -x_3 \).

The linear isomorphism \( J \) induces a projective-linear isomorphism, or a correlation

\[
J : \mathbb{P}^{13} \to \hat{\mathbb{P}}^{13}, \quad J : x \mapsto \hat{x} := J(x),
\]

which identifies the elements \( x \) of any of the orbits of \( \rho \) with their correlative elements \( J(x) = \hat{x} \) in its corresponding orbit of \( \hat{\rho} \). The correlation \( J \) commutes with taking pivots; in particular, for \( x \in \hat{F} - \hat{\Omega} \) the correlative point \( u(x) \in \hat{\Sigma} \) of its (double) pivot \( u(x) = \piv(x) \in \Sigma \) is the pivot \( \hat{u}(x) := \piv(\hat{x}) \) of its correlative point \( \hat{x} \in F - \Omega \).

Lemma. In coordinates \((u : X : Y : z)\), the correlative pivot map

\[
\hat{\piv} : \hat{F} - \hat{\Omega} \to \hat{\Sigma}, \quad x \mapsto \hat{u}(x),
\]

where \( u(x) = \piv(x) \), coincides with the gradient map \( \nabla = \nabla_F \), defined by the 14 cubic derivatives of the quartic form \( F(u : X : Y : z) \).

Proof. In coordinates \((u : X : Y : z)\), let \( u = (0 : 0 : 0 : 1) = (x_{456}) \). By the \( Sp_3 \)-transitivity on the orbit \( \hat{F} - \hat{\Omega} \), it is enough to prove that the gradient map \( \nabla = \nabla_F \) sends any \( x = (0 : 0 : Y : z) \in \mathbb{P}^6_u - D_u \) to the point \( \hat{u} = \hat{\piv}(u) = (\hat{x}_{456}) = (-e_{123}) = (e_{123}) = (1 : 0 : 0 : 0). \)

This is direct: the straightforward check shows that

\[
\nabla_F : (0 : 0 : Y : z) \mapsto (4 \text{det } Y : 0 : 0 : 0) = (1 : 0 : 0 : 0)
\]

since \( \text{det } Y \neq 0 \) on \( \mathbb{P}^6_u - D_u \), see (2.3)-(2.4).

Lemma. (i) If \( x \in \hat{F} - \hat{\Omega} \) then the hyperplane section \( H_x \subset \Sigma \) has a node at the correlative double pivot \( \hat{u} := u(\hat{x}) \) of \( x \). Moreover \( H_x \) contains the 3-fold cone \( K_{\hat{u}} = \mathbb{P}^6_u \cap \Sigma \).

(ii) For \( u \in \Sigma \) the hyperplane section \( H_u \subset \Sigma \) is the same as the Schubert hyperplane section \( \sigma_{100}(\mathbb{P}^2_u) \cap \mathbb{P}^{13} = \{ v \in \Sigma : \mathbb{P}^2 \cap \mathbb{P}^2_u \neq \emptyset \} \). The singular locus \( \text{Sing } H_u = K_{\hat{u}} = \mathbb{P}^6_u \cap \Sigma \) is a cone, with vertex \( \hat{u} \), over the Veronese surface.

Proof: See (2.2).
§3. Lines and quadrics on the Sp₃-grassmannian

(3.1) Lemma. (i) Let \( L \subset \mathbb{P}^5 \) be an isotropic line. Then the set
\[
 l_L = \{ u \in \Sigma : L \subset P^3_u \} \subset \Sigma
\]
is a line, and any line in \( \Sigma \) is obtained this way.
(ii) If \( P^3_L \subset \mathbb{P}^5 \) is the 3-space such that
\[
 l_L = \sigma_{332}(L, P^3_L) = \{ u \in \Sigma \subset G(3, 6) : L \subset P^3_u \subset P^3_L \},
\]
then \( P^3_L \) does not contain other lagrangian planes except the planes \( P^3_u, u \in l_L \).

For the line \( l = l_L = \sigma_{332}(L, P^3_L) \subset \Sigma \), we call the isotropic line \( L \) and the space \( P^3_L \) respectively the axis and the space of \( l \).

Proof. For (i), see e.g. §2.7.2 in [LM2]. In particular, \( Sp_3 \) acts transitively on the family of lines \( l = l_L \subset \Sigma \), since the isotropic grassmannian \( LG_2 \) is a homogeneous variety of \( Sp_3 \). Therefore it is enough to prove (ii) for a particular line \( l \) on \( \Sigma \). In coordinates \( (u : X : Y : z) \), for the isotropic line \( L = P(< e_2, e_3>) \subset \mathbb{P}^5 \) the space \( P^3_L = P(U^L) = P(< e_1, e_2, e_3, e_4>) \) and the line \( l = l_L = < e_{123}, e_{243} > \). Therefore the Plücker image \( u \) of a plane \( P^2_u \subset P^3_L \) will lie in the 3-space \( P^3 = P(\wedge^3 U^L) = P^3(x_{123} : x_{423} : x_{143} : x_{124}) = P^3(u : x_{11} : x_{12} : x_{13}) \). By the equations (2.1)(*) the plane \( P^2_u \) will be lagrangian iff \( u \in P^3_L \cap (tX = X, tY = Y) = P^1(u : x_{11}) = P^1(x_{123} : x_{423}) = l. \) q.e.d.

(3.2) Lemma. Let \( x \in \mathbb{P}^5 \). Then the set
\[
 Q_x := \sigma_{300}(x) \cap P^{13} = \{ u \in \Sigma : x \in P^2_u \} \subset \Sigma
\]
is a smooth 3-fold quadric, and any 3-fold quadric on \( \Sigma \) is one of the quadrics \( Q_x, x \in \mathbb{P}^5 \).

For the quadric \( Q = Q_x \subset \Sigma \), we call the point \( \text{ver}(Q) := x \in \mathbb{P}^5 \) the vertex of \( Q \).

Proof. See §2.7.2 in [LM2]; see also the proof of (3.3)(i) below.

(3.3) Proposition : quadrics on singular hyperplane sections of \( \Sigma \).
(i) Let \( Q = Q_x \subset \Sigma \) be a 3-fold quadric on \( \Sigma \) (see (3.2)), and let \( H_c, c \in \hat{P}^{13} \) be a hyperplane section of \( \Sigma \) containing \( Q \). Then \( c \in \hat{F} \); and if \( c \notin \hat{\Omega} \), then the correlative pivot \( \hat{u}(c) \) of \( c \) lies on the quadric \( Q \).
(ii) Let \( c \in \hat{F} - \hat{\Omega} \), and let \( \hat{u} = \hat{u}(c) \) be the correlative pivot of \( c \). Then the set
\[
 q(c) = \{ x \in \mathbb{P}^5 : Q = Q_x \subset H_c \}
\]
is a smooth conic in the lagrangian plane \( P^2_u \) of \( \hat{u} \), which we call the vertex conic of \( c \).

Proof of (i). By the \( Sp_3 \)-homogeneity we can assume that \( x = e_1 \), and then in coordinates \( (u : X : Y : z) = (u : (x_{ij}) : (y_{ij}) : z) \)
\[
 Q_{e_1} = (uy_{11} = x_{22}x_{23} - x_{2}^2) \subset P^4_{e_1} = P^4(u : x_{22} : x_{23} : x_{33} : y_{11}).
\]
Any \( H_c \) which contains \( Q_x = Q_{e_1} \) contains \( P^4_x = P^4_{e_1} \). Therefore
\[
 H_c \supset Q_{e_1} \iff c \in P^8_{e_1} := (P^4_{e_1})_{4} = (u = x_{12} = x_{13} = x_{23} = y_{11} = 0) \subset \hat{P}^{13}(u : X : Y : z).
Now the straightforward check shows that: 1. If \( c \in \mathbf{P}^8_{e_1} \) then \( F(c) = 0 \), see (2.5.a).
Therefore \( c \in \hat{F} \). 2. If \( c \in \mathbf{P}^8_{e_1} \) then all the partial derivatives \( F_x(c), F_{xy}(c), i = 1, 2, 3 \) and \( F_y(c), (i, j) \neq (1, 1) \) vanish. Therefore \( \nabla F(c) \in \mathbf{P}^4_{e_1} \); and since \( \nabla F(c) = \hat{u}(c) \in \Sigma \) is the correlative pivot of \( c \) (see (2.6)), then \( \hat{u}(c) \in \Sigma \cap \mathbf{P}^4_{e_1} = Q_{e_1} \). q.e.d.

Proof of (ii). Let \( \rho : Sp_3 \times \mathbf{P}^{13} \rightarrow \mathbf{P}^{13}, \rho_q : x \rightarrow \rho_q(x) \) be the projective action of \( Sp_3 \) on \( \mathbf{P}^{13} \) (see (2.2)), and let \( \rho_1 : x \mapsto g(x) \) be the projectivized standard action \( \wedge_1 \) of \( Sp_3 \subset GL_6 \) on \( \mathbf{P}^5 \). By the transitivity of \( \hat{\rho} \) on the orbit \( \hat{F} - \hat{\Omega} \), we can choose \( c = (0 : 0 : Y : 0) \), where \( Y \) is any symmetric rank 3 matrix (see (2.3)-(2.4)), and let \( Y \) be the matrix of the quadratic form \( 2y_1y_2 + y_3^2 \). Then \( c = 2y_1y_2 + y_3^2 = x_{146} + x_{256} + x_{453} \mod \mathbf{C}^* \), and the pivot \( u(c) = piv(c) = (0 : 0 : 0 : 1) = (x_{453}) \). The correlative point of \( c \) is \( \hat{c} = (c_{143} + c_{523} + c_{126}) \in F - \Omega \) (see (2.5.c)), and the pivot of \( \hat{c} \) is \( \hat{u} = piv(\hat{c}) = (1 : 0 : 0 : 0) \in \Sigma \), with a lagrangian plane \( \mathbf{P}^2_{\hat{u}} = \mathbf{P}^2(x_1 : x_2 : x_3) = (\mathbf{P}(U_o)). \)

be the projectivized stabilizer group of \( \hat{c} \).

\((*)\) Lemma. The group \( PST_3 \subset Aut H_c \); and the correlative pivot plane \( \mathbf{P}^2_\hat{c} \) is invariant under the standard action \( \rho_1 \) of \( PST_3 \). Moreover \( PST_\hat{c} \) is a projectivized semi-direct product \( \mathbf{P}(O_q(3) \otimes C^5) \), where \( PO_q(3) \) is the projectivized orthogonal group of the quadric \( q = q(c) = (2x_1x_2 + x_3^2 = 0) \subset \mathbf{P}^5_q = \mathbf{P}^2(x_1 : x_2 : x_3) \), and \( G^5 \) is the additive group of \( \mathbf{C}^5 \).

Proof. See §9 (2) in [K], or (5.21) in [KS].

Let \( Q = Q_x \) be a 3-fold quadric in \( H_c \). By (i), the vertex \( x \) of \( Q = Q_x \) lies in \( \mathbf{P}^2_\hat{u} \); and by Lemma \((*)\), for any \( g \in PST_\hat{c} \) the quadric \( \rho_q(Q_x) = Q_{g(x)} \) also lies in \( H_c \). Therefore \( Q = Q_x \) lies in \( H_c \), together with all the quadrics \( Q_{g(x)}, g \in PST_\hat{c} \). Again by Lemma \((*)\), the action \( \rho_1 : (g, x) \mapsto g(x) \) of \( PST_\hat{c} \) on \( \mathbf{P}^2_\hat{u} \) coincides with the action of \( PO_q(3), q = q(c) \). Therefore \( PST_\hat{c} \) has two orbits in \( \mathbf{P}^2_\hat{u} = \mathbf{P}^2(x_1 : x_2 : x_3) \) – the smooth conic \( q = q(c) = (2x_1x_2 + x_3^2 = 0) \), and its complement \( \mathbf{P}^2_\hat{u} - q(c) \).

Therefore in order to prove (ii) it is enough to see that: 1. for some \( x \in Q(c) \) the 5-fold \( H_c \) contains the quadric \( Q_x \). 2. for some \( x \in \mathbf{P}^2_\hat{u} \) the quadric \( Q_x \) does not lie in \( H_c \).

This is straightforward: If \( e_1 \in q(c) \) then the quadric \( Q_{e_1} \) lies in \( H_c = \Sigma \cap (2y_{12} + y_{33} = 0) \), while for the point \( x = e_3 \in \mathbf{P}^2_\hat{u} - q(c) \) the quadric \( Q_{e_3} \) does not lie in \( H_c \). q.e.d.

(3.4) Conics on \( \Sigma \). Call a formal conic on \( \Sigma \) any plane \( \mathbf{P}^2 \subset \mathbf{P}^{13} = \text{Span} \Sigma \) such that the intersection cycle \( q = \mathbf{P}^2 \cap \Sigma \) is of dimension 1 and of degree 2. Let \( \mathcal{F}(\Sigma) \subset G(3,14) \) be the family of formal conics on \( \Sigma \). Since \( \Sigma \) is an intersection of quadrics (see (2.1)(*)), and \( \Sigma \) does not contain planes (see e.g. Lemma 2.5.1 in [IR]), then the definition is correct. The rank of the formal conic \( \mathbf{P}^2 \) is the rank of the conic \( q = \mathbf{P}^2 \cap \Sigma \subset \mathbf{P}^2 \). If \( \text{rank}(q) = 2 \) or 3, then one can identify the conic \( q \subset \Sigma \) and the formal conic \( \mathbf{P}^2 = \text{Span}(q) \).

(3.5) Lemma. If \( q \subset \Sigma \) is a conic of rank \( \geq 2 \), then there exists a unique point \( x = x(q) \in \mathbf{P}^5 \), such that \( q \subset \Sigma \).

We call the point \( x = x(q) \) the vertex of the rank \( \geq 2 \) conic \( q \).
Proof. Since \( q \subset \Sigma \subset G(3,6) \) is a conic then the union \( Q(q) := \cup_{u \in q} P_u^2 \) is either a \( \mathbb{P}^3 \) or a 3-fold quadric.

Suppose first that \( Q(q) = \mathbb{P}^3 \), i.e. all the \( P_u^2, u \in q \) lie in \( \mathbb{P}^3 \); and let \( u, v \in q, u \neq v \). Since the lagrangian planes \( P_u^2 \) and \( P_v^2 \) both lie in \( \mathbb{P}^3 \), then they will intersect each other along an isotropic line \( L \). By (3.1)(i), the line \( l = l_L \) with axis \( L \) contains the points \( u \) and \( v \), therefore \( l = \text{Span}(u,v) \) and \( P_u^2 \cap P_v^2 = \mathbb{P}^3 \). But by (3.1)(ii) the only lagrangian planes in \( P_u^2 \) are \( P_w^2, w \in l \). Therefore \( q \subset l \) (as sets), which is only possible if \( \text{rank}(q) = 1 \) – contradiction.

Therefore \( Q(q) \subset \mathbb{P}^4 \) is a quadric, and since on \( Q(q) \) lie planes, \( Q(q) \) is singular.

If \( q = l + m \) is of rank \( 2 \) then \( Q(q) \) is the union of the spaces \( P_L^2 \) and \( P_M^2 \) of the lines \( l \) and \( m \). Let \( L \subset \mathbb{P}_L^3 \) and \( M \subset \mathbb{P}_M^3 \) be the axes of \( l \) and \( m \). Since \( l \cap m = u \) is a point, then \( \text{Span}(L \cup M) = \mathbb{P}_u^4 \). Therefore \( L \cap M = x = x(q) \) is a point, and \( q = l + m \subset Q_x \).

If \( \text{rank}(q) = 3 \) then \( Q(q) \) is irreducible, and since on \( Q(q) \) lie planes then \( Q(q) \) is singular and all the \( P_u^2, u \in q \) pass through the subspace \( P^k = \text{Sing}(Q(q)), k = 0, 1 \). If \( k = 1 \) then \( P^k = L \) is a line. The line \( L \) is isotropic since it lies on lagrangian planes, and let \( l = l_L \) be the line with axis \( L \). But then \( P_u^2 \cap L, \forall u \in q \), together with (3.1)(i), will imply \( q \subset l \) – contradiction. Therefore \( k = 0 \), \( x = x(q) := P^k = \mathbb{P}^0 \) is a point, and \( q \subset Q_x \). q.e.d.

(3.6) Lemma. Let \( l \subset \Sigma \) be a line. Then the set \( P_l \subset G(3,14) \) of these planes \( P^2 \subset P^{13} \) for which the intersection cycle \( P^2 \cap \Sigma = 2l \), is a line in \( G(3,14) \).

Proof. Let \( P^2 \cap \Sigma = 2l \), and let \((u,v)\) be any pair of non-coincident points of \( l \). Since \( P^2 \cap \Sigma = 2l \) then \( P^2 \subset P^6_u \cap P^6_v \), where \( P^6_u \) and \( P^6_v \) are the tangent projective spaces to \( \Sigma \) at \( u \) and \( v \), see (2.3)-(2.4). As in the proof of (3.1), we may assume that \( l = l_L = \text{Span}(u,v) = \text{Span}(e_{123}, e_{423}) \) is the line with axis \( L = \text{Span}(e_2, e_3) \). Therefore \( P^3_l = P^6_u \cap P^6_v = \text{Span}(e_{123}, e_{423}, e_{143} + e_{523}, e_{124} + e_{623}) \) is a projective 3-space containing the line \( l = \text{Span}(e_{123}, e_{423}) \). Now it is clear that \( P^3_l = \cap_{u \in l} P^6_u \) and the set \( P_l \) coincides with the Schubert line \( \sigma_{11,11,10}(l, P^3_l) = \{ P^2 \subset P^{13}, l \subset P^2 \subset P^3_l \} \subset G(3,14) \). q.e.d.

(3.7) Corollary. Let \( F_k(\Sigma) \subset F(\Sigma) \subset G(3,14) \) be the loci of formal conics on \( \Sigma \) of rank \( \leq k, k = 1, 2 \). Then \( \dim F(\Sigma) = 11, \dim F_2(\Sigma) = 10, \) and \( \dim F_1(\Sigma) = 8 \).

Proof. By (3.4) any conic \( q \) of rank \( > 1 \) can be identified with its formal conic \( P^2(q) = \text{Span}(q) \). By (3.5) any such \( q \) is a linear section of a unique quadric \( Q_x(q) \) with the plane \( P^2(q) \subset P^4_x(q) = \text{Span} Q_x(q) \). Therefore \( \dim F(\Sigma) = \dim P^5 + \dim G(3,5) = 11 \), and \( \dim F_2(\Sigma) = 10 \). At the end, (3.1) and (3.6) imply: \( \dim F_1(\Sigma) = \dim LG_2 + 1 = 8 \), see (2.1). q.e.d.

§4. The Fano family \( F(X) \) of conics on \( X \)

(4.1) The dual plane quartic \( C_X \) of \( X \). By [M1], [M2] any smooth prime Fano 3-fold \( X = X_{16} \) of degree 16 is a linear section of the \( Sp_3 \)-grassmannian \( \Sigma \subset P^{13} \) by a codimension 3 subspace \( P^{10} \subset P^{13} \). Moreover two prime Fano 3-folds \( X'_16 = \Sigma \cap P^{10}_1 \) and \( X''_16 = \Sigma \cap P^{10}_2 \) are projectively equivalent \( \text{iff} \ P^{10}_1 \) and \( P^{10}_2 \) are conjugate under the action \( \rho \) of \( Sp_3 \) in \( P^{13} \).

From now on we shall consider only the situation when \( X \) is general.
Fix a general prime Fano 3-fold $X = X_{16} = \Sigma \cap P_{X}^{10}$; and let $P_{X}^{2} = P_{X}^{10,1} \subset \hat{P}^{13}$ be the plane of linear equations of $P_{X}^{10} \subset P^{13}$. Since $X$ is general then the plane $P_{X}^{2} = P_{X}^{10,1} \subset P^{13}$ of linear equations of $P_{X}^{10} \subset P^{13}$ intersects the invariant quartic hypersurface $\hat{F}$ along a smooth plane quartic curve

$$C_{X} = \hat{F} \cap P_{X}^{2}.$$ 

By the preceding, $C_{X}$ is uniquely defined by the Fano 3-fold $X$; and we call the curve $C_{X} \subset P_{X}^{2}$ the dual plane quartic of $X$.

Since $X$ is general then the plane $P_{X}^{2}$ does not intersect the codimension 3 orbit $\hat{\Omega}$. Therefore any point $c \in C_{X}$ has a uniquely defined double pivot $u = u(c) \in \Sigma$. By (2.7)(i)-(ii), the hyperplane sections $H_c$ and $H_u$ of $\Sigma$ both contain the Veronese cone $K_{\hat{u}} = P_{u}^{6} \cap \Sigma$. Therefore any point $c \in C_{X}$ defines uniquely a 4-fold linear section

$$W_c = H_c \cap H_{u(c)} \subset \Sigma;$$

and since $K_{\hat{u}(c)} = Sing H_u$ (see (2.7)(ii)) then $W_c$ is singular along $K_{\hat{u}(c)}$.

**Lemma (4.2).** Let $c \in \hat{F} - \hat{\Omega}$, and let $q(c) \subset P_{\hat{u}(c)}^{2}$ be the vertex conic of $c$, see (3.3). Then $W_c = \bigcup_{x \in q(c)} Q_x$.

**Proof.** By (2.7) and (3.2)

$$H_{u(c)} = \sigma_{100}(P_{\hat{u}(c)}^{2}) \cap P^{13} = \{ w \in \Sigma : P_{w}^{2} \cap P_{\hat{u}(c)}^{2} \neq \emptyset \} = \bigcup_{x \in q(c)} P_{\hat{u}(c)}^{2} Q_x.$$ 

Moreover, by (3.3) any quadric $Q_x, x \in q(c) \subset P_{\hat{u}(c)}^{2}$ lies in $H_c$. q.e.d.

**The singular hyperplane sections $S_c \subset X$.** For the general $X$, the dual curve $C_{X}$ does not intersect $\hat{\Omega}$, hence the pivot map $piv : C_{X} \to \Sigma, c \mapsto u(c) = piv(c)$ is regular. The regular map $piv : C_{X} \to piv(C_{X})$ is an isomorphism. Indeed if $c_1, c_2 \in C_{X}$ are two points such that $u(c_1) = u(c_2) = u$, then both $c_1, c_2$ will lie in the 6-space $P_{u}^{6}$ (see (2.5.b) and (2.3)(ii)); and since $P_{u}^{6} \subset \hat{F}$ (see (2.3)(ii)) then the line $L = Span(c_1, c_2) \subset P_{X}^{2}$ will lie in $\hat{F}$. But then the line $L$ will be a component of $C_{X} = \hat{F} \cap P_{X}^{2}$, which contradicts the general choice of $X$. Since the plane $P_{X}^{2} = Span C_{X}$ does not intersect $\Sigma \supset piv(C_{X})$ then for any $c \in C_{X}$ the set

$$S_c = W_c \cap P_{X}^{10} = X \cap H_{u(c)} \subset X,$$

where $u(c) = piv(c)$, is a hyperplane section of $X$. Since different points $c_1, c_2 \in C$ have different pivots $u(c_1), u(c_2) \in piv(C_{X})$ then $S_{c_1} \neq S_{c_2}$ for $c_1 \neq c_2$. By construction any surface $S_c, c \in C_{X}$ is a complete intersection of two hyperplanes in $W_c$, say $S_c = W_c \cap H_1 \cap H_2$. Since the 5-fold $W_c$ is singular along the Veronese cone $K_{\hat{u}(c)}$ (see (4.1)), then the surface $S_c = W_c \cap H_1 \cap H_2$ is singular along the intersection cycle

$$C_c = K_{\hat{u}(c)} = K_{\hat{u}(c)} \cap H_1 \cap H_2 = K_{\hat{u}(c)} \cap P_{X}^{10}.$$ 

Assume that $\hat{u}(c) \in C_c$. Then $\hat{u}(c) \in H_1 \cap H_2$; and since $X = H_c \cap H_1 \cap H_2$ and $H_c$ is singular at $\hat{u}(c)$ (see (2.7)(i)) then $X$ will be singular at $\hat{u}(c)$, which contradicts the general choice of $X$. Therefore $C_c$ is a codimension 2 linear section of the Veronese cone $K_{\hat{u}(c)}$, which does not contain the vertex $\hat{u}(c)$ of $K_{\hat{u}(c)}$, i.e. $C_c$ is a rational normal quartic curve (including the case when $C_c$ is a union of two conics intersecting each other at a point).
(4.4) Lemma. Let $c \in C_X$ and let $q(c) \subset \mathbb{P}^2_{\hat{u}(c)}$ be the vertex conic of $c$. Then for any $x \in q(c)$ the intersection $q_x = Q_x \cap S_c$ is a conic. If $c \in C_X$ is general then for the general $x \in q(c)$ the conic $q_x$ is a bisecant to the rational normal quartic $C_c \subset \text{Sing } S_c$. Moreover for any $c \in C_X$ the surface $S_c = \bigcup_{x \in q(c)} q_x$.

Proof. Let $c \in C_X$, etc. be as above. We shall see first that for any $x \in q(c)$ the quadric $Q_x \subset W_c$ (see (4.2)) intersects the Veronese cone $K_{\hat{u}(c)}$ along a cone $K^x_{\hat{u}(c)} \subset K_{\hat{u}(c)}$ over a conic on the Veronese surface. Indeed the intersection $K^x_{\hat{u}(c)} = Q_x \cap K_{\hat{u}(c)} = \{ w \in \Sigma : x \in \mathbb{P}^2_w \} \cap \mathbb{P}^2_w$ coincides with the union of lines $\cup \{ l_L : x \in L \subset \mathbb{P}^2_w \}$ on $\Sigma$, see (3.1). Any such $l_L$ is a ruling line of the cone $K_{\hat{u}(c)}$. Therefore $K^x_{\hat{u}(c)} \subset K_{\hat{u}(c)}$ is a conic; and since the base set $\{ L \subset \mathbb{P}^2_w : x \in L \}$ is a line in $\mathbb{P}^2_w$ then the base of $K^x_{\hat{u}(c)}$ is a conic on the Veronese surface. By (4.2), for any $x \in q(c)$ the quadric $Q_x$ lies in $W_c$; and since $X = W_c \cap \mathbb{P}^{10}_X = W_c \cap H_1 \cap H_2$ then the cycle $q_x := Q_x \cap \mathbb{P}^{10}_X = Q_x \cap H_1 \cap H_2 \subset W_c \cap \mathbb{P}^{10}_X = X$ will be a quadric of dimension at least 1. Since $Pic(X) = \mathbb{Z}.H$, where $H$ is the hyperplane section, then the prime Fano 3-fold $X = X_{16}$ can’t contain quadric surfaces.

Therefore for any $c \in C_X$, and any $x \in q(c)$, the cycle $q_x$ is a conic; and $q_x$ intersects the rational quartic $C_c \subset \text{Sing } S_c$ at the cycle

$$z_x = q_x \cap C_c = K^x_{\hat{u}(c)} \cap H_1 \cap H_2.$$ 

By construction, the 4-fold $W_c$ and the rank 3 quadratic cone $K^x_{\hat{u}(c)} = Q_x \cap K_{\hat{u}(c)}$ depend only on the choice of the point $c \in \hat{F} - \hat{\Omega}$ and the point $x$ on its vertex conic $q(c)$. Therefore for the general codimension 2 linear section $X = W_c \cap H_1 \cap H_2$, the cycle $z_x = K^x_{\hat{u}(c)} \cap H_1 \cap H_2$ will be a general section of $K^x_{\hat{u}(c)}$ with two hyperplanes, and then $q_x \subset X$ will be a smooth conic on $S_c$ bisecant to the rational normal quartic $C_c$. For arbitrary $x \in q(c), c \in C_X$, the cycle $z_x$ is a linear section, of dimension $\leq 1$, of the quadratic cone $K^x_{\hat{u}(c)}$. In the extremal case when $dim z_x = 1$, the curve $C_c$ must be a union of two conics on the Veronese surface such that $z_x = q_x$ is one of these two conics. This can be possible for at most two conics from the pencil $\{ q_x : x \in q(c) \}$. In all the rest possible cases $z_x$ will be a zero-cycle of degree 2.

At the end, the conics $q_x, x \in q(c)$ sweep the surface $S_c$ out, since $S_c = W_c \cap H_1 \cap H_2 = \bigcup_{x \in q(c)} Q_x \cap H_1 \cap H_2 = \bigcup_{x \in q(c)} q_x$, see (4.2). q.e.d.

(4.5) The Fano surface $\mathcal{F}(X)$ and the vertex surface $S_X$. For the general $X$ the family $\mathcal{F}(X)$ of conics on $X$ is a union of 2-dimensional components $\mathcal{F}^1, ..., \mathcal{F}^n$, see §4 in [1] or §4.2 in [2]. We shall see that $n = 1$, and $\mathcal{F}(X) = \mathcal{F}^1$ is a ruled surface over the dual plane quartic $C_X$ of $X$.

(4.5.a) Lemma. The general $X = X_{16}$ does not contain conics of rank 1.

Proof. For the general $X$, the codimension 9 Schubert cycle $\sigma_{333}(\mathbb{P}^{10}_X) = G(2, \mathbb{P}^{10}_X)$, in $G(3, 14) = G(2 : \mathbb{P}^{13})$, does not intersect the 8-fold $\mathcal{F}_1(\Sigma) \subset G(3, 14)$ of formal conics of rank 1 on $\Sigma$, see (3.7). q.e.d.

(4.5.b) The vertex map. By (3.5) and (4.5.a), for the general $X$ the map $\text{ver} : \mathcal{F}(X) \to \mathbb{P}^5, q \mapsto \text{ver}(q) = \text{the vertex } x(q) \text{ of } q$

is regular; and we call this map the vertex map of $\mathcal{F}(X)$. 

10
(4.5.c) Lemma. The vertex map \( ver \) sends the family \( \mathcal{F}(X) \) isomorphically onto the irreducible closed set

\[
S_X = \bigcup_{c \in C_X} q(c) \subset \mathbb{P}^5,
\]

swept out by the smooth vertex conics \( q(c) \subset \mathbb{P}^2_{\hat{u}(c)} \) of the points \( c \in C_X \), see (3.3).

Proof. Let \( x \in S_X \). Then \( \exists c \in C_X \) such that \( x \in q(c) \). By (4.4) the cycle \( q_x = Q_x \cap S_c \) is a conic on \( X \) with vertex \( x \). Therefore \( S_X \subset \mathcal{F}(X) \).

Next, we shall see that \( ver(\mathcal{F}(X)) \supseteq S_X \). For this, let \( q \subset X \) be a conic, and let \( x = \text{ver}(q) \).

Since \( q \subset Q_x \) is a codimension 2 linear section of the 3-fold quadric \( Q_x \) (see (3.5)), and since any hyperplane section \( H_c, c \in \mathbb{P}^2_X \) contains \( q \subset X \), then the plane \( \mathbb{P}^2_c \) contains a unique point \( c = c(x) \) such that \( Q_x \subset H_c \). Since \( H_c \) contains a 3-dimensional quadric (the quadric \( Q_x \)) then, by (3.3)(i), the point \( c \) must lie in \( \hat{F} \). Therefore \( c \in \mathbb{P}^2_X \cap \hat{F} = C_X \). Moreover, by (3.3)(ii), the vertex \( x = x(q) = \text{ver}(q) \) must lie in the vertex conic \( q(c) \subset \mathbb{P}^2_{\hat{u}(c)} \) of \( c \); and since \( q(c) \subset S_X \) then \( x = \text{ver}(q) \in q(c) \) lies in \( S_X \). Therefore \( ver(\mathcal{F}(X)) \supseteq S_X \). Moreover, by the proof of (4.4), the conic \( q_x \) is the unique conic on \( X \) with vertex \( x \in q(c) \). Therefore the regular and surjective vertex map \( \text{ver} : \mathcal{F}(X) \to S_X \) is injective. q.e.d.

(4.5.d) The vertex surface \( S_X \). By (4.5.a-c), the vertex map \( \text{ver} \) sends \( \mathcal{F}(X) \) isomorphically onto the irreducible closed set

\[
S_X = \bigcup_{c \in C_X} q(c).
\]

Therefore \( \mathcal{F}(X) \) is irreducible, and \( S_X \subset \mathbb{P}^5 \) is a surface which we call the vertex surface of \( X \).

Lemma (4.4) and (4.5.a-d) imply the following

(4.6) Theorem. Let \( X = X_{16} \) be general. Then the Fano family \( \mathcal{F}(X) \) of conics on \( X \) is a ruled surface \( p : \mathcal{F}(X) \to C_X \) over the dual plane quartic curve \( C_X \) of \( X \). For the point \( c \in C_X \), the fiber \( f_c = p^{-1}(c) \subset \mathcal{F}(X) \) coincides with the pencil \( \{ q_x, x \in q(c) \} \) of conics \( q_x \subset X \) defined in (4.4). The vertex map \( \text{ver} : \mathcal{F}(X) \to \mathbb{P}^5 \) sends \( \mathcal{F}(X) \) isomorphically onto the vertex surface \( S_X = \bigcup_{c \in C_X} q(c) \subset \mathbb{P}^5 \) of \( X \). For any point \( c \in C_X \), the isomorphism

\[
\text{ver} : \mathcal{F}(X) \to S_X \subset \mathbb{P}^5
\]

sends the fiber \( f_c \subset \mathcal{F}(X) \) onto the correlative pivot conic \( q(c) \subset \mathbb{P}^2_{\hat{u}(c)} \) of \( c \).

Proof. By (4.4) and (4.5.c), it only rests to see that the vertex surface \( S_X \cong \mathcal{F}(X) \) is ruled over \( C_X \), with fibers – the conics \( q(c), c \in C_X \). For this it is enough to see that if \( c_1, c_2 \subset C_X \), \( c_1 \neq c_2 \) then the conics \( q(c_1) \) and \( q(c_2) \) do not intersect each other.

Suppose that \( q(c_1) \cap q(c_2) \neq \emptyset \). Let \( x \in q(c_1) \cap q(c_2) \), and let \( H_{c_1} \) and \( H_{c_2} \) be the hyperplane sections of \( \Sigma \) defined by \( c_1 \) and \( c_2 \). Since \( x \in S_X \) then, by the proof of (4.4), there exists a unique conic \( q = q_x \subset X \) with vertex \( x \). Let also \( Q_x \subset \Sigma \) be the 3-fold quadric with vertex \( x \); in particular \( q_x = Q_x \cap \mathbb{P}^4_X \), ibid. By (3.3)(ii), both \( H_{c_1} \) and \( H_{c_2} \) must contain \( Q_x \) since the vertex \( x \) of \( Q_x \) lies in their pivot conics \( q(c_1) \) and \( q(c_2) \). But by the proof of (4.5.c), there exists a unique point \( c = c(x) \in C_X \) such that \( Q_x \subset H_c \). Therefore \( c_1 = c_2 = c \). q.e.d.
§5. Prime Fano 3-folds of degree 16 and stable ruled surfaces of invariant 3 over plane quartics

(5.1) Lemma: the double projection from a line $l \subset X_{16}$ (V. Iskovskikh).

(1) Let $X = X_{16} \subset \mathbb{P}^3$ be a smooth prime Fano 3-fold of degree 16, and let $l \in X$ be a line. Then the double projection $\pi_{2,l}$ from the line $l$, given by the non-complete linear system $|\mathcal{O}_X(1 - 2l)|$, defines a birational isomorphism $\pi = \pi_{2,l} : X \to \mathbb{P}^3_l$, where $\mathbb{P}^3_l$ is the 3-dimensional projective space.

There exists a smooth curve $C \subset \mathbb{P}^3_l$ of genus 3 and of degree 7, which lies on a unique cubic surface $S = S_3$, such that the inverse birational map $\varphi : \mathbb{P}^3_l \to X$ is defined by the non-complete linear system $|\mathcal{O}_{\mathbb{P}^3_l}(7 - 2C)|$, and:

(i) The one-dimensional family $Q_l$ of conics $q \subset X$ which intersect $l$ sweeps out the unique effective divisor $Q = Q_l$ from the linear system $|\mathcal{O}_X(3 - 7l)|$.

(ii) The double projection $\pi = \pi_{2,l}$ can be represented as a product $\pi = \sigma \circ \rho \circ \sigma^{-1}$, where $\sigma : X^l \to X$ is the blowup of $l \in X$, $\rho : X \to X^l$ is a flop over the projection $\pi_l : X \to X^l$ from $l$, and $\tau : X^l \to \mathbb{P}^3_l$ is a blow-down of the proper image $\mathcal{O}_X \subset X^l$ of $Q$ onto the curve $C \subset \mathbb{P}^3_l$.

(iii) The unique cubic surface $S = S_l \subset \mathbb{P}^3_l$ through the curve $C$, is swept out by the one-dimensional family $S_l$ of conics $s \subset \mathbb{P}^3_l$, intersecting $C$ at a 0-cycle of degree 7. The proper transform $S' \subset X'$ of $S$ coincides with the exceptional divisor $\sigma^{-1}(l) \subset X'$ of $\sigma$. The strict transforms $s' \subset X'$ of the conics $s \in S_l$ are the extremal curves of $\sigma$.

(iv) There exists a non-negative integer $e = e(l) \leq 5$ (and if $X$ is general and $l \subset X$ is general then $e(l) = 5$) such that the flop $\rho : X' \to X^l$ transforms the proper $\sigma$-preimages $l'_1, \ldots, l'_e \subset X'$ of the $e$ lines $l_1, \ldots, l_e \subset X$ which intersect $l$ to the proper $\tau$-preimages $l^+_i$, $i = 1, \ldots, e$ of $e$ lines $L_1, \ldots, L_e \subset \mathbb{P}^3$ of $C$.

(2) Let $C \subset \mathbb{P}^3$ be a smooth curve of genus 3 and of degree 7, which lies on a unique cubic surface $S = S_3$. Then there exists a smooth prime Fano 3-fold $X$ of degree 16 and a line $l \subset X$, such that (1) takes place for $l \subset X$ and $C^l = C \subset \mathbb{P}^3_l = \mathbb{P}^3$.

Proof. See [2].

(5.1.a) Remark. The general curve $C \subset \mathbb{P}^3$ of genus 3 and degree 7 evidently lies on a unique and smooth cubic surface $S = S_3$. Therefore $S = S_l$ and $C = C^l$ for some $l \subset X$ as in (5.1)(2). One can represent (non-uniquely) the smooth cubic $S_3 \subset \mathbb{P}^3$ as the blowup $S_3 = \tilde{S}_3$ at 6 points $z_0, \ldots, z_5$, and the curve $C \subset S_3$ – as the proper preimage of an element of the system $|\mathcal{O}_{\mathbb{P}^2}(4 - z_1 - \ldots - z_5)|$. Then the five 4-secant lines of $C$ are the proper preimages $L_i \subset S_3$ of the conics $C_i \subset \mathbb{P}^2$ passing through the points $z_0, \ldots, z_5$, $i = 1, \ldots, 5$.

Notice that the line $L = L_0$ is the unique line on $S_3$ which does not intersect $C$, and $L_1, \ldots, L_5$ are the five lines on $S_3$ which intersect $L$.

(5.1.b) Corollary. The family $\Gamma(X)$ of lines on the general prime Fano 3-fold $X$ is a smooth irreducible curve of genus 17.

Proof. By Th. 4.2.7 in [1], for the general $X = X_{16}$ the 1-dimensional family $\Gamma(X)$ of lines on $X$ is smooth. It rests to see that $\Gamma(X)$ is irreducible and of genus 17.
Let $\pi = \pi_{2,1} : X \rightarrow \mathbb{P}^3 = \mathbb{P}^3$ be the double projection of $X$ from the general line $l \subset X$ as in (5.1). If $m \subset X$ is a line which does not intersect $l$ then its proper image $\pi_*(m)$ is evidently a line in $\mathbb{P}^3$, and let $\pi_*(m)$ intersects $C^l$ at $k$ points. Since by (5.1)(1), $\varphi = \pi^{-1}$ is defined by the system $|O_{\mathbb{P}^3}(7 - 2C^l)|$, then the proper image $\varphi_*(\pi_*(m)) = m$ will be a curve of degree $= 7 \cdot \deg m(l) - 2k = 7 - 2k$; and since $m$ is a line then $k = 3$. The same argument in the opposite direction gives that any purely 3-secant line to $C^l$ is the proper $\pi$-image of a line $m \subset X$ which does not intersect $l$.

If $m = l_i, i = 1, \ldots, 5$ is one of the five 4-secant lines of $l$ then, by (5.1)(iv), its proper image $\pi_*(m)$ in $\mathbb{P}^3$ is one of the five 4-secant lines $L_i$ to $C^l$. These $L_i$ evidently are 4-tuple singularities of the curve $\text{Sec}_3(C^l)$ of the 3-secant lines of $C^l$. Therefore the map

$$\pi_* : \Gamma(X) \rightarrow \text{Sec}_3(C^l),$$

is the normalization of $\text{Sec}_3(C^l)$ at its five 4-tuple points $L_1, \ldots, L_5$. By (5.1)(2), for the general $X$ the curve $C^l$ is a general curve of genus 3 and degree 7 in $\mathbb{P}^3$. Therefore, by Proposition 2.4 and Theorem 3.6 in [GP], $\Gamma(X)$ is an irreducible smooth curve of genus 17. \textit{q.e.d.}

\textbf{(5.2) Deformations and minimal sections of ruled surfaces (see LN, S)}. The ruled surface $S = \mathbb{P}(F) \rightarrow C$ over the smooth curve $C$ is stable if the rank 2 vector bundle $F \rightarrow C$ is stable. Since the stability of $F$ does not depend on the twist $F \otimes L$ by a line bundle $L$, the definition is correct.

For the section $C_o \subset S$, denote by $\varepsilon(C_o) = C^2_o = O_{C_o}(C_o) \in \text{Pic}(C_o) = \text{Pic}(C)$ its self-intersection divisor, and let $\varepsilon(C_o) = \deg \varepsilon(C_o) = \deg C^2_o$. The integer invariant

$$e(S) := \min \{ \varepsilon(C_o) : C_o \text{ is a section of } S \}$$

is always $\leq g = g(C)$. For a fixed $g$, the ruled surfaces $S$ form two deformation classes – \textit{even}: with $e(S) \equiv g \pmod{2}$, and \textit{odd}: with $e \equiv g - 1 \pmod{2}$. The versal deformations in the even (respectively odd) class are ruled surfaces $S$ with $e(S) = g$ (respectively with $e(S) = g - 1$). For $g \geq 2$ the general $S$ of any of the two versal types – the even and the odd – is stable; and if $S[g]$ and $S[g - 1]$ denote respectively their moduli spaces, then $\dim S[g] = \dim S[g - 1] = 6g - 6$. The same as above takes place if the smooth base curve $C$ of genus $g \geq 2$ is fixed, with the only difference that then the moduli spaces $\text{Sec}_C[g]$ and $\text{Sec}_C[g - 1]$ of versal even and odd ruled surfaces over $C$ both have dimension $3g - 3$, see [S]. In particular, if $g(C) = 3$ then the ruled surfaces of invariant 3 over $C$ are exactly the elements of the even versal class, and their moduli space $\text{Sec}_C[3]$ has dimension 6.

Let $S = \mathbb{P}(F) \rightarrow C$ be a stable ruled surface with invariant $e(S) = e$, and let $C_o \subset S$ be the section defined by the extension

$$(*) \quad 0 \rightarrow \xi \rightarrow F \rightarrow \eta \rightarrow 0$$

for $\eta, \xi \in \text{Pic}(C)$, see Proposition 2.6 in Ch. 5 of [H]. In particular $\varepsilon = \varepsilon(C_o) = \eta - \xi$, with operations in $\text{Pic}(C)$ written additively.

The \textit{minimal sections} of $S$ are these sections $C_o$ of $S$ for which $e(C_o) = \deg \varepsilon(C_o) = e(S)$. Equivalently, the section $C_o \subset S = \mathbb{P}(F)$ is minimal \textit{iff} the subbundle $\xi \subset F$ is of the maximum possible degree, or a \textit{maximal subbundle} of $F$, see Lemma 2.3 in [LN]. By Lemma 2.2 in [LN] the minimal sections $C_o$ of $S$ are defined uniquely by their intersection divisors $\varepsilon(C_o) = C^2_o$. We shall denote by $\text{Min}(S)$ the family of minimal sections of $S$, and by $C_e \in \text{Min}(S)$ the minimal section of self-intersection divisor $\varepsilon(C_o) = \varepsilon \in \text{Pic}(C)$.
Let \( g \geq 3 \), and \( g - 1 \leq e = e(S) \leq g \). For the fixed minimal section \( C_\varepsilon \in \text{Min}(S) \), \( \deg \varepsilon = e = e(S) \), let \([e_\varepsilon] \in \mathbb{P}^{g-2+e}_K\) be an extension-class point defined by \( C_\varepsilon \), and let \( \Phi_\varepsilon : C \to \mathbb{P}^{g-2+e}_K \) be the map defined by the complete linear system \( |K_C + \varepsilon| \). By Proposition 2.4 in [4], there exists a canonical bijection between the set \( \text{Min}(S) - \{C_\varepsilon\} \) of minimal sections of \( S \) different from \( C_\varepsilon \) and the \( e \)-secant spaces \( \mathbb{P}^{e-1} \subset \mathbb{P}^{g-2+e}_K \) to the curve \( \Phi_\varepsilon(C) \) which pass through \([e_\varepsilon]\).

We shall see below that for \( g = 3 \), the general ruled surface \( S \in \mathcal{S}[3] \) is the same as the Fano surface \( \mathcal{F}(X) \) of conics on a general prime Fano 3-fold \( X = X_S \) of degree 16 defined uniquely by \( S \), and the family \( \text{Min}(S) \) of minimal sections of \( S \) is the same as the curve \( \Gamma(X_S) \) of lines on \( X_S \).

**5.3 Non-abelian Brill-Noether loci of type II** (see [3]). Let \( F \to C \) be a rank 2 vector bundle over the smooth curve \( C \), let \( K = K_C \) be the canonical bundle of \( C \), and let \( \nu \) be a nonnegative integer such that \( \nu \equiv \deg F \mod 2 \). The non-abelian Brill-Noether locus of type II, associated to the triple \((C,K,F)\), is the set of equivalence classes of bundles

\[
\mathcal{M}_C(2,K:nF) = \{E \to C : \text{rank } E = 2, \text{det } E = \text{det } F \otimes K, \text{dim } \text{Hom}(F,E) \geq n\}.
\]

It is proved in §6 of [3] that \( \mathcal{M}_C(2,K:nF) \) admits a natural scheme structure as certain Pfaffian locus in the moduli space \( \mathcal{M}_C(2,\text{det } F \otimes K_C) \) of stable rank 2 vector bundles on \( C \) of determinant \( \text{det } F \otimes K_C \).

**5.4 Lemma** (S. Mukai).

(a) Let \( C \subset \mathbb{P}^5 \) be a nonsingular plane quartic curve, and let \( F \) be a rank 2 stable vector bundle on \( C \) such that the ruled surface \( S = \mathbb{P}(F) \to C \) has invariant \( e = e(S) = 3 \). Then the Brill-Noether locus of type II

\[
X = \mathcal{M}_C(2,K:3F)
\]

is a nonsingular prime Fano 3-fold of degree 16.

(b) Let \( C_\varepsilon \) be the minimal section of \( S \) of self-intersection divisor \( \varepsilon \), and let (5.2)(*), with \( \eta - \xi = \varepsilon \), be the extension defined by \( C_\varepsilon \). Then the locus

\[
l_\varepsilon = \{E \in X = \mathcal{M}_C(2,K:3F) : h^0(E(-\eta)) \geq 2\} \subset X
\]

is a line on the Fano 3-fold \( X = X_{16} \subset \mathbb{P}^{10} \).

(c) In the notation of (5.2), the map \( \Phi_\varepsilon : C \to \mathbb{P}^4_\varepsilon \) is an embedding; and the projection \( p_{[e_\varepsilon]} : \mathbb{P}^4_\varepsilon \to \mathbb{P}^3_\varepsilon \) from \([e_\varepsilon]\) sends \( \Phi_\varepsilon(C) \) isomorphically onto a space curve \( C^l_\varepsilon \) of degree 7.

(d) The curve \( C^l_\varepsilon \subset \mathbb{P}^3_\varepsilon \) is projectively equivalent to the curve \( C^l \subset \mathbb{P}^3_\varepsilon \) defining the inverse of the double projection \( \pi_{2,1} : X \to \mathbb{P}^4_\varepsilon \), see (5.1)(1).

**Proof.** See §9 in [3].

**5.5 The associate Fano 3-fold \( X_S \) of \( S \).** According to Remark 9.2 of [3], the locus \( X_F = \mathcal{M}_C(2,K:3F) \) does not depend on the twist \( F \to F \otimes L \) by an invertible sheaf \( L \) on \( C \). That is, the Fano 3-fold \( X_S = X_F = \mathcal{M}_C(2,K:3F) \) depends only on the choice of the stable ruled surface \( S = \mathbb{P}(F) \to C \). Therefore a given stable ruled surface \( S = \mathbb{P}(F) \in \mathcal{S}_C[3] \) defines uniquely its associate Fano 3-fold \( X = X_S := \mathcal{M}_C(2,K:3F) \), which is prime and of degree 16.
(5.6) Lemma. Let $F \to C$, $S = \mathbf{P}(F) \to C$, $X = X_S$, $C_\varepsilon$, etc. be as in (5.4)-(5.5), and let $\Gamma(X_S)$ be the curve of lines on $X_S$. In addition, we shall assume that $X = X_S$ is general (see (5.7) below). Then the map

$$
\psi : \text{Min}(S) \to \Gamma(X_S), \ C_\varepsilon \to l_\varepsilon,
$$

defined in (5.4)(b), is an isomorphism.

Proof. Fix a general minimal section $C_\varepsilon$ of $S$, and let (5.2)(*) be the extension defined by $C_\varepsilon$ as in (5.4)(b). By (5.2) and (5.4)(c), the sections $C_t \in \text{Min} \mathbf{P}(F) - \{C_\varepsilon\}$ are in a $(1:1)$-correspondence with the elements of the family $\text{Sec}_3^2(\Phi_\varepsilon(C), [e_\varepsilon])$ of 3-secant planes $\mathbf{P}_t^2$ to the curve $\Phi_\varepsilon(C) \subset \mathbf{P}_\varepsilon^4$ which pass through the point $[e_\varepsilon]$. Clearly the projection from $[e_\varepsilon]$ sends these planes $\mathbf{P}_t^2$ (1:1) to the 3-secant lines $L_t$ of $C^\varepsilon_t = C_t$, see (5.4)(d). In turn, the 3-secant lines $L_t$ of $C^\varepsilon$ correspond, by (5.1.b), to the lines $l_t \subset X$ which do not intersect $l = l_\varepsilon$. This way, the map $\psi : C_t \mapsto l_t$ is an isomorphism outside the five lines $l_1, \ldots, l_5$ which intersect $l = l_\varepsilon$, see also (5.1.b).

The same local argument, but applied for other general section $C_r$ of $S$ and the line $l_r \subset X$ corresponding to $r$, implies that $C_t \mapsto l_t$ is an isomorphism also outside the five 4-secant lines $m_1, \ldots, m_5$ of the line $m = l_r$ on $X$. Since $l$ and $m$ are general, then the sets $\{l_1, \ldots, l_5\}$ and $\{m_1, \ldots, m_5\}$ are disjoint. Therefore $\psi$ is an isomorphism. q.e.d.

(5.7) The associate ruled surface $S^X$ of $X$. The inverse to (5.5) is also true: the general Fano 3-fold $X = X_{16}$ is associate to a unique ruled surface $S$. Indeed, let $C^l \subset \mathbf{P}^2_l$ be the curve defined as in (5.1) by the general line $l \subset X$, and let $H = |K + \varepsilon|$ be the hyperplane system on $C^l \subset \mathbf{P}^3$, where $K$ is the canonical class of $C^l$. By (5.1)(1)-(2), for the general $l \subset X$, the curve $C^l$ is a general curve of degree 7 and of genus 3 in $\mathbf{P}^3$. Therefore the complete linear system $|H|$ sends $C^l$ isomorphically onto a curve $C \subset \mathbf{P}^4$, and $C^l$ is a projection of $C$ from a general point $[\varepsilon] \in \mathbf{P}^4$.

By (5.2), the pair $(C, [\varepsilon])$ represents a minimal section $C_\varepsilon$ of the ruled surface $S = \mathbf{P}(F) \in S[3]$ defined by an extension (5.2)(*) with $\varepsilon = \eta - \xi$. Clearly the surface $S$ is uniquely defined by the pair $(X, l)$. By (5.4), the pair $(S, C_\varepsilon)$ defines uniquely the pair $(X, l)$, $l = l_\varepsilon \subset X$, and $X = X_S$ does not depend on the choice of the minimal section $C_\varepsilon$ of $S$. This, and the isomorphism $\psi : \text{Min}(S) \to \Gamma(X_S), \ C_\varepsilon \mapsto l_\varepsilon$ from (5.6) (more precisely, the surjectivity of $\psi$), imply the non-existence of other $S'$ with $X = X_{S'}$.

We call the unique ruled surface $S = S^X \to C$, such that $X = X_S$, the associate ruled surface of $X$; and call its base curve $C = C^X$ the associate curve of $X$. By [CG], the description (5.1) of the double projection imply that $C^X \cong C^l$ (see (5.4)(d)) is the unique curve such that the jacobian $J(C^X)$ is isomorphic, as a principally polarized abelian variety, to the intermediate jacobian $J(X)$ of $X$, see also §9 in [M3].

§6. The associate surface $S^X$ is the Fano surface of $X$

(6.1) The double projection from a line $l \subset \Sigma$. Let $l = l_L \subset \Sigma$ be a line with an axis $L = \mathbf{P}(U^2_L)$ and a space $\mathbf{P}_L^2 = \mathbf{P}(U^1_L)$, see (3.1). By (2.4)(i), for any $u \in L$ the tangent projective space $\mathbf{P}_u^6$ intersects $\Sigma$ along a cone $K_u \subset \mathbf{P}_u^6$ over the Veronese surface. Let $W_L = \cup_{u \in L} \mathbf{P}_u^6$, and let $Z_L = \cup_{u \in L} K_u = W_L \cap \Sigma$. 

15
Then \( P_L^9 := \text{Span} \, W_L = \text{Span} \, Z_L \) is a 9-dimensional subspace of \( P^{13} = \text{Span} \, \Sigma \). We denote by \( \tilde{P}_L^3 = (P_L^9)^\perp \subset \tilde{P}_L^3 \) the space of linear equations of \( P_L^9 \subset P^{13} \).

By its definition, the double projection

\[
\pi_{2,l} : \Sigma \rightarrow \tilde{P}_L^3
\]

of \( \Sigma \subset P^{13} \) from \( l \) is the restriction to \( \Sigma \) of the projection \( \pi_{2,l} : P^{13} \rightarrow \tilde{P}_L^3 \) from the subspace \( P_L^9 = \text{Span} \, Z_L \). The codimension 2 cycle \( Z_L \subset \Sigma \) is the set of points \( u \in \Sigma \) where \( \pi_{2,l} \) is non-regular. Let also

\[
Z'_L = \{ u \in \Sigma : \dim (P_u^2 \cap P_L^3) \geq 1 \}.
\]

Since for \( u \in \Sigma - Z'_L \) the lagrangian plane \( P_u^2 \) intersects \( P_L^3 \) at a point, then the map

\[
\varphi_l : \Sigma - Z'_L \rightarrow P_L^3, \quad u \mapsto P_u^2 \cap P_L^3
\]

is regular.

**Lemma (6.2).** (i) \( Z'_L = Z_L \). (ii) There exists a natural identification \( \tilde{P}_L^3 \cong P_L^3 \) such that

\[
\varphi_l = \pi_{2,l}|_{\Sigma - Z_L}.
\]

**Proof of (i).** Clearly \( Z_L \subset Z'_L \); and it rests to see that \( Z'_L \subset Z_L \).

Let \( u \in Z'_L \), i.e. \( \dim (P_u^2 \cap P_L^3) \geq 1 \). We have to see that \( u \in Z_L \).

If \( P_u^2 \subset P_L^3 \), then \( u \in l = l_L \), since the only lagrangian planes in \( P_L^3 \) are the planes \( P_w, w \in l = l_L \), see (3.1)(ii).

If \( \dim (P_u^2 \cap P_L^3) = 1 \) then the line \( M = P_u^2 \cap P_L^3 \) will be an isotropic line in \( P_L^3 \), since \( M \) lies on the lagrangian plane \( P_u^2 \). Let \( Q_L := G_L := \{ l : P_L^3 \} \) be the set of all isotropic lines in \( P_L^3 \). Clearly \( Q_L \) contains the hyperplane section \( \sigma_{10}(L) = \{ M \subset P_L^3 : M \cap l = \emptyset \} = \{ \text{the lines which lie in the lagrangian planes } P_w, w \in l \} \). Since the isotropic grassmanian \( LG_2 \) is a hyperplane section of \( G(2,6) \) (see (2.1)), then either \( Q_L = G_L \), or \( Q_L \) will coincide with the hyperplane section \( \sigma_{10}(L) \subset G_L \). But if \( Q_L = G_L = G(2,4) \), then in \( P_L^3 \) will lie two 3-fold families of lagrangian planes, which is impossible since the only lagrangian planes in \( P_L^3 \) are the planes \( P_w, w \in l \), see (3.1)(ii). Therefore \( Q_L = \sigma_{10}(L) \). In particular \( M \cap l = \emptyset \); and since both \( M \) and \( L \) lie in \( P_L^3 \) then the plane \( \text{Span}(M \cup L) \) will be one, say \( P_w^2 \), of the planes \( P_w, w \in l = l_L \), see (3.1)(i). Since \( P_u \) intersects \( P_w^2 \) along a line then \( u \in K_v \), see (2.4)(i). Therefore \( Z'_L \subset \cup_{w \in L} K_v = Z_L \). q.e.d.

**Proof of (ii).** By the \( Sp_{23} \)-homogeneity of \( LG_2 \), it is enough to prove (ii) for a particular line \( l \subset \Sigma \); and let \( l = l_L = \text{Span}(e_{123}, e_{423}) \), \( L = P(U_L^2) = P(<e_2, e_3>) \), \( U_L^2 = <e_1, e_2, e_3, e_4> \), etc. be as in the proof of (3.1)(ii). Then \( P_L^3 = P(U_L^2) = P_3(x_1 : x_2 : x_3 : x_4) \subset P^5 = P(V_6) \), and \( P_L^9 = \text{Span} \, Z_L = (x_{156} = x_{256} = x_{356} = x_{456} = 0) = (y_{11} = y_{12} = y_{13} = z = 0) \subset P^{13} \). Therefore

\[
\tilde{P}_L^3 = P^3(x_{156} : x_{256} : x_{356} : x_{456}) \cong P(U_L^2 \otimes \Lambda^2 V_6/U_4^2) \cong P_L^3.
\]

Under this identification, we shall verify first the coincidence \( \varphi_l = \pi_{2,l}|_{\Sigma - Z_L} \) over the open subset

\[
\Sigma' := \Sigma \cap \{ u = 0 \} = \exp(Sym^2 C^3) = \{ (1 : X : \Lambda^2 X : \det X) : ^tX = X \}.
\]

By (2.1), the lagrangian 3-spaces \( U = U_X \in \Sigma' \subset G(3,6)^o \) are parameterized by the symmetric linear maps \( X = X_U : U_o \rightarrow U_\infty \), such that \( U = <e_1 + X(e_1), e_2 + X(e_2), e_3 + X(e_3)> \).
For such $U = U_X$ and $X = X_U$, the Plücker image $pl(U_X) = P(\wedge^3 U_X)$ is the same as the point $exp(X) = (1 : X : \wedge^2 X : \det X)$, ibid. In other words, for $u \in \Sigma^0 \subset P^{13}$ one has: $u = exp(X_U)$, where $P(U) = P^2_u$ is the lagrangian projective plane of $u$.

Let $u \in \Sigma^0, u \not\in Z_L$, let $P^2_u = P(U)$ be the lagrangian plane of $u$, and let $X = X_U$. We shall compute separately $\varphi_u(u)$ and $\pi_{2,L}(u)$.

On the one hand, $\varphi_u(u)$ is the intersection point of $P^2_u$ and $P^3_L$. Since $P^2_u = P(U) = P(<e_1 + X(e_1), e_2 + X(e_2), e_3 + X(e_3)>)$, and $P^3_L = P(U^*_L) = P(<e_1, e_2, e_3, e_4>)$, then $U \cap U^*_L = \left< \det(X_{11})e_1 + \det(X_{12})e_2 + \det(X_{13})e_3 + \det(X)e_4 > \right>$, i.e.

$$\varphi(u) = (\det X_{11} : \det X_{12} : \det X_{13} : \det X) \in P^3_L = P^3(x_1 : x_2 : x_3 : x_4),$$

where for any $(i, j)$, $X_{ij}$ is the $(i, j)^{th}$ adjoint matrix of $X$.

On the other hand, as a point of $\Sigma \subset P^{13}$, $u = exp(X) = (1 : X : Y : z) \in \Sigma \subset P^{13}$, where $X = X_U, Y = \wedge^2 X, z = \det X$. Since $\pi_{2,L}$ is the projection from $P^2_u$ onto $P^3_L = P^3(x_{156} : x_{256} : x_{356} : x_{456}) = P^3(y_{11} : y_{12} : y_{13} : z)$, then

$$\pi_{2,L}(u) = \pi_{2,L}(exp(X)) = ((\wedge^2 X)_{11} : (\wedge^2 X)_{12} : (\wedge^2 X)_{13} : \det X) \in P^3(y_{11} : y_{12} : y_{13} : z).$$

Since $(\wedge^2 X)_{ij} = \det X_{ij}, i, j = 1, 2, 3$, then $\varphi_u(u) = \pi_{2,L}(u)$, under the above identification between $P^3(x_1 : x_2 : x_3 : x_4) = P^3_L$ and $P^3(y_{11} : y_{12} : y_{13} : z) = P^3_L$.

The map $\varphi_u$ is evidently surjective onto the 3-space $P^3_L$; and since $\varphi_u$ coincides with the projective-linear map $\pi_{2,L}$ over an open subset of $\Sigma$, then $\varphi_u$ is a restriction of a projection of $\Sigma \subset P^{13}$ from a 9-space $P^9 \subset P^{13}$. Since $\pi_{2,L}$ is a projection from the 9-space $P^9 = \text{Span}(Z_L)$, and since $Z_L$ is the non-regular locus of $\varphi_u$, then $P^9 \subset \text{Span}(Z_L) = P^9_L$. Therefore $P^9 = P^9_L$; i.e. $\varphi_u = \pi_{2,L}|_{\Sigma - Z_L}$. q.e.d.

(6.3) The double projection from a line $l \subset X$ and the $Sp_3$-geometry.

Let $l \subset X = \Sigma \cap P^{10}_X$ be a line, let $L \subset P^5$ be the axis of $l = l_L$, and let $P^3_L \subset P^5$ be the space of $L$, see (3.1). The double projection $\pi_{2,L} : X \rightarrow P^3_L$, described in (5.1), is also the restriction to $X$ of the double projection $\pi_{2,L} : \Sigma \rightarrow P^3 = P^3_L$. By (6.1), $\pi = \pi_{2,L} : X \rightarrow P^3_L$ is non-regular along the subset $z_l = Z_l \cap P^{10}_X$, and by (5.1), the set $z_l = l \cup l_1 \cup ... \cup l_e$, where $l_1, ..., l_e$ are the $e = e(l) \leq 5$ lines on $X$ which intersect $l$. By (6.2), for any $u \in X - z_l$ the double projection $\pi = \pi_{2,L} : X \rightarrow P^3_L$, described in (5.1), is given by:

$$(*) \quad \pi : u \mapsto \pi(u) = \pi^2_u \cap P^3_L.$$

This identifies $P^3_L$ and $P^3_L$; and we shall find the curve $C^d \subset P^3_L$, see (5.1)(1).

Let $l_1, ..., l_e, e \leq 5$ be the lines on $X$ which intersect $l$; and assume for simplicity that $e = 5$, see (5.1)(1)(iv). Let $L_1, ..., L_5$ be the axes of $l_1, ..., l_5$, see (3.1). Let $u_i = l \cap l_i$, and let $P^2_{u_i} \subset P^5$ be the lagrangian plane of $u_i$. Since $u_i \in l$ then the axis $L_i = l = l_i$ lies in $P^2_{u_i}$, and since $u_i \in l_i$ then $L_i \subset P^2_{u_i}$. Therefore $L_i$ intersects $L$, and let $x_i = L \cap L_i, i = 1, ..., 5$. The point $x_0 = x(l + l_i)$ is the vertex of the conic $q_i = l + l_i$, i.e. $x_i \in S_X$; and let $\delta_L = \{x_1, ..., x_5\}$. Since $l_1, ..., l_5$ are all the lines on $X$ which intersect $l$ then

$$L \cap S_X = \delta_L.$$
In the general case $\delta_L$ is the intersection 0-cycle of $L$ and $S_L$, see (5.1)(1)(iv). Let

$$C_L = \text{the closure of } S_X \cap (P^3_L - L).$$

Then $S_L \cap P^3_L = C_L \cup \delta_L$; and we shall see that $C_L = C^l$.

Let first $x \in C_L \subset S_X \cap P^3_L$. Since $x \in S_X$, then by (4.5.c) there exists a unique conic $q = q_x \subset X$ with vertex $x$; and by $\ast$ the double projection $\pi_{2,l}$ contracts the conic $q_x$ to the point $x = x(q) \in P^3_L = \Pi^3_l$; in particular $x \in C^l$, see (5.1)(1)(ii). Therefore $C_L \subset C^l$.

The same argument in the inverse direction gives $C^l \subset C_L$.

Notice that for the general $l$ the curve $C^l = C_L$ does not intersect the axis $L$ of the line $l = l_L$, see (5.1.a). By (5.1)(1)(iv), the five axes $L_1, \ldots, L_5$ are exactly the five 4-secant lines to $C^l = C_L$, see also (5.1.a) and (5.1.b).

(6.4) Corollary. In the notation of (6.3), $C_L = C^l$.

(6.5) Theorem. Let $X$ be a general prime Fano threefold of degree 16. Then the associate ruled surface $S^X$ of $X$ is isomorphic to the Fano surface $F(X)$ of conics on $X$.

Proof. Let $C_X$ and $C_X$ be correspondingly the associate and the dual curve of $X$. By (4.6), it is enough to prove that the ruled surface $p^X : S^X \to C^X$ is isomorphic to the ruled vertex surface $p_X : S_X \to C_X$ of $X$.

Let $l = l_L \subset X$ be a line, and let $C_{\varepsilon} \cong C^X$ be the minimal section of $S^X$ such that $l = l_{\varepsilon}$, see (5.4), (5.6). Let also $C^l \subset P^3_l$ be the curve defined by the double projection from $l$ as in (5.1), and let $C^l_{\varepsilon} \subset P^3_{\varepsilon}$ be the curve defined in (5.4)(c).

By (5.4)(c)-(5.4)(d) $C^X \cong C^l$; and by (6.4) $C^l \cong C_L$. Since both $C^X$ and $C_X$ are curves of genus 3, then $C_L \subset S_X$ is a section of $S_X$. Therefore

$$C^X \cong C_L \cong C_X =: C.$$

Let $Min'(S_X) = \{C_L : l = l_L \in \Gamma(X)\}$ be the family of all sections $C_L \subset S_X$. One can see separately that $Min'(S_X) = Min(S_X)$, but in the proof we will use only the isomorphism $Min'(S_X) \cong Min(S^X)$. In fact, by (5.4)(c)-(d) and (6.4)

$$C^l = C^l_{\varepsilon} = C_L \text{ in } P^3_{\varepsilon} = P^3_l;$$

and then by (5.6) the maps

$$Min'(S_X) \xrightarrow{\xi} Min(S^X) \xrightarrow{\psi} \Gamma(X) =: \Gamma,$$

are isomorphisms.

It rests to see that $S^X \cong S_X$. For this, define the following maps:

$$\alpha : C \times \Gamma \to S_X, \quad \alpha : (c, l) \mapsto \gamma(c),$$

where $l = l_L$, and $\gamma(c) := (p_X)^{-1}(c) \cap C_L \subset S_X$, for $c \in C = C_X$; and

$$\beta : C \times \Gamma \to S^X, \quad \beta : (c, l) \mapsto \gamma(c),$$

where $l = l_{\varepsilon}$, and $\gamma(c) := (p^X)^{-1}(c) \cap C_{\varepsilon} \subset S^X$, for $c \in C = C^X$.

In order to prove that $S_X \cong S^X$, it is enough to identify the fibers $\alpha^{-1}(\alpha(c, l))$ and $\beta^{-1}(\beta(c, l))$, for any $(c, l) \in C \times \Gamma$. 

18
Let \((c, l) \in C \times \Gamma\). By the definition of \(\alpha\)
\[
\alpha^{-1}(\alpha(c, l)) \cong \{C_{L_i} \subset S_X : l_{L_i} \subset X \& x.(c) \in C_{L_i}\},
\]
and one of these sections is \(C_L, l = l_L\). Let \(q_{x.(c)}\) be the unique conic on \(X\) with vertex \(x.(c)\), see \((4.5.c)\) or the proof of \((4.4)\). Since
\[
q_{x.(c)} = q_{x.(c)} \cap \mathbb{P}^1_{\mathbb{X}} = \{u \in X : x.(c) \in \mathbb{P}^2_u\}
\]
(see e.g. \((6.3)\) or the proof of \((4.4)\)), then the line \(l_i = l_{L_i} \subset X\) passes through \(x.(c)\) iff \(l_i \cap q_{x.(c)} \neq \emptyset\). Therefore
\[
\alpha^{-1}(\alpha(c, l)) \cong \{l_i \in \Gamma(X) : q_{x.(c)} \cap l_i \neq \emptyset\}.
\]
Clearly, the line \(l\) is one of these lines.

By the definition of \(\beta\)
\[
\beta^{-1}(\beta(c, l)) \cong \{C_i \subset S^X : x.(c) \in C_i\},
\]
and one of these sections is \(C_\varepsilon, l = l_\varepsilon\). By \((5.2)\) and \((5.4)(c)\) (see also the proof of \((5.6)\)) the minimal sections \(C_i\) of \(S^X\), different from \(C_\varepsilon\), are parameterized by the 3-secant lines \(M\) to \(C_\varepsilon \subset \mathbb{P}^3\). Since \(C_\varepsilon = C_L \subset \mathbb{P}^3_L = \mathbb{P}^3\), then \(C_l\) intersects \(C_\varepsilon\) at the point \(x.(c)\) iff \(x.(c) \in M\). Therefore
\[
\beta^{-1}(\beta(c, l)) \cong \{M \in \text{Sec}_3(C_L) : x.(c) \in M\} \cup \{C_\varepsilon\}.
\]

By the proof of \((5.1.b)\) the 3-secant lines \(M\) to \(C_L = C^l\) are the proper \(\pi\)-transforms of the lines \(m \subset X\). Moreover, by \((5.1)(1)(ii)\) the points \(x \neq q \in C^l\) are the same as the blowed-down strict transforms \(q^+ \subset X^+\) of the conics \(q \subset X\) which intersect \(l\). Therefore there exists a unique conic \(q = q_o \subset X\) such that \(x_{q_o} = x.(c)\). Since by \((6.2)-(6.3)(*)\) the double projection \(\pi = \pi_{2,l}\) contracts \(q_o\) to its vertex \(x_{q_o} = x.(c)\) then \(q_o = q_{x.(c)}\), and
\[
\beta^{-1}(\beta(c, l)) \cong \{l_i \in \Gamma(X) : q_{x.(c)} \cap l_i \neq \emptyset\} \cong \alpha^{-1}(\alpha(c, l)).
\]
\(\text{q.e.d.}\)

\((6.6)\) Remark. In fact, the coincidence between the fibers of \(\alpha\) and \(\beta\) implies the isomorphism between the images \(\alpha(C \times \Gamma) \subset S_X\) and \(\beta(C \times \Gamma) \subset S^X\). But \(\alpha\) and \(\beta\) are certainly surjective since a conic \(q \subset X\) always intersects the surface \(R_X \subset X\) swept out by the lines \(l \subset X\). The general conic \(q\) on the general \(X\) intersects exactly 8 lines on \(X\) since \(R_X \in |\mathcal{O}_X(4)|\), see \((6.4)\) case \(v)\) in \([1]\). The virtual number \(\epsilon(q) = 8\) of lines intersecting a conic \(q \subset X_{16}\) can be computed also by the Mori theory – see \(\S(2.5)-(2.6)\) and table \((2.8.2)\) in \([1]\).

On the general ruled surface \(S = S^X \rightarrow C^X = C\) of invariant \(\epsilon(S) = 3\), one can compute the same number \(\epsilon(x) = 8\) of minimal sections through its general point \(x\) as follows (see \((5.2)\)):
If \(x \in S\) is general then the elementary transformation \(elm_x : S \rightarrow S_x\) sends \(S\) to a general \(S_x \in S_C[2]\). The proper images of the minimal sections of \(S\) through \(x\) are all the minimal sections of \(S_x\). \(S_x\) always has minimal sections, and let \(C_o \subset S_x\) be one of them. Since \(S_x\) is general, then \(\varepsilon_o = C_o^2 := \mathcal{O}_{C_o}(C_o) \in \text{Pic}^2(C_o) = \text{Pic}^2(C)\) is general, and the linear system \(|K_C + \varepsilon_o|\) sends \(C\) isomorphically to a space curve \(C_o^3 \subset \mathbb{P}^3\) of genus 3 and of degree 6. The minimal sections of \(S_x\), different from \(C_o\), are in a 1:1 correspondence with the bisection lines to \(C_o^3\) from the (general) extension-class point \([\varepsilon_o]\) \in \mathbb{P}^3\) defined by the section \(C_o\). These lines are 7, since the projection of \(C_o^3 \subset \mathbb{P}^3\) from a general point in \(\mathbb{P}^3\) is a plane sextic with 7 double points.
References

[CG] H. Clemens, Ph. Griffiths, *The intermediate jacobian of the cubic threefold*, Ann. of Math. 95:2, 281-356 (1972)

[D] R. Donagi, *On the geometry of grassmannians*, Duke Math. J. 44:4, 795-837 (1977)

[GP] L. Gruson, Ch. Peskine, *Courbes de l’espace projectif: varietes de secantes*, in: Enumerative geometry and classical algebraic geometry, Progress in Mathematics, Vol. 24 (1982), p. 1-31, Birkhauser, Boston-Basel-Stuttgart.

[FH] W. Fulton, J. Harris, *Representation theory*, Springer-Verlag (1991)

[H] R. Hartshorne, *Algebraic geometry* Springer-Verlag (1977)

[I1] V. A. Iskovskikh, *Fano threefolds II*, Math. USSR, Izv. 12:3, 469-506 (1978)

[I2] V. A. Iskovskikh, *Double projection from a line on Fano threefolds of the first kind*, Math. USSR Sbornik 66 No.1, 265-284 (1990)

[IP] V. A. Iskovskikh, Yu. G. Prokhorov, *Algebraic geometry V: Fano varieties*, Encycl.Math. Sci. 47, 1-245 (1999)

[IR] A. Iliev, K. Ranestad, *Geometry of the lagrangian grassmannian with applications to Brill-Nether loci*, e-print math.AG/0209169

[K] T. Kimura, *The b-functions and holonomy diagrams of irreducible regular prehomogeneous vector spaces*, Nagoya Math. J. 85, 1-80 (1982)

[KS] T. Kimura, M. Sato, *A classification of irreducible prehomogeneous vector spaces and their relative invariants*, Nagoya Math. J. 65, 1-155 (1977)

[LM1] J. M. Landsberg, L. Manivel, *The Projective Geometry of Freudenthal’s Magic Square*, J. of Algebra 239, 477-512 (2001)

[LM2] J. M. Landsberg, L. Manivel, *Representation theory and projective geometry*, e-print math.AG/0203260

[LN] H. Lange, M. S. Narasimhan, *Maximal Subbundles of rank two vector bundles on curves*, Math. Ann. 266, 55-72 (1983)

[M1] S. Mukai, *Curves, K3 Surfaces and Fano 3-folds of Genus ≤ 10*, Algebraic geometry and commutative algebra in honor of M. Nagata, Kinokuniya, Tokyo, 357-387 (1987)

[M2] S. Mukai, *Biregular classification of Fano 3-folds and Fano manifolds of coindex 3*, Proc. Natl. Acad. Sci. USA 86, 3000-3002 (1989)

[M3] S. Mukai, *Non-Abelian Brill-Noether theory and Fano 3-folds*, Sugaku Expositions 49, 1-24 (1997); e-print math.AG/9704017

[M4] S. Mukai, *Curves and Symmetric Spaces*, Proc. Japan Acad. 68 Ser. A, 7-10 (1992)

[PR] P. Pragacz, J. Ratajski, *Pieri type formula for isotropic grassmannians: the operator approach*, Manuscripta Math., 79 No. 2, 127-151 (1993)

[S] W. K. Seiler, *Deformations of ruled surfaces* J. reine angew. Math. 426, 203-219 (1992)

[T] K. Takeuchi, *Some birational maps of Fano 3-folds*, Compositio Math. 71, 265-283 (1989)

Atanas Iliev
Institute of Mathematics
Bulgarian Academy of Sciences
Acad. G. Bonchev Str., 8
1113 Sofia, Bulgaria
e-mail: ailiev@math.bas.bg