Abstract

It is shown that the energy ($\varepsilon$) and momentum ($\mathbf{k}$) dependences of the electron self-energy function $\Sigma(k, \varepsilon + i0) \equiv \Sigma^R(k, \varepsilon)$ are, $\text{Im}\Sigma^R(k, \varepsilon) = -a\varepsilon^2|\varepsilon - \xi_k|^{-\gamma(k)}$ where $a$ is some constant, $\xi_k = \varepsilon(k) - \mu$, $\varepsilon(k)$ being the band energy, and the critical exponent $\gamma(k)$, which depends on the curvature of the Fermi surface at $k$, satisfies, $0 \leq \gamma(k) \leq 1$. This leads to a new type of electron liquid, which is the Fermi liquid in the limit of $\varepsilon, \xi_k \to 0$ but for $\xi_k \neq 0$ has a split one-particle spectra as in the Tomonaga-Luttinger liquid.

(Submitted to J. Phys. Soc. Japan: [Letter])
The nature of low energy electronic excitations in two-dimension is one of the central theoretical issues in the study of the anomalous metallic state realized near the Mott insulator. Anderson\(^1\,^2\) has proposed that the special feature of the phase volume in the scattering processes in two dimension would lead to a non-Fermi liquid state. Anderson argues that the phase shift, \(\delta(q, \omega)\), of the forward scattering of two electrons (i.e. the total momentum of two electrons, \(q\), approaching \(2\vec{k}_F, \vec{k}_F\) being the Fermi momentum) will stay finite in the limit of low exciting energy, \(\omega \to 0\), resulting in the singular interaction leading to the breakdown of the Fermi liquid theory. This remarkable suggestion has attracted much interest.\(^3\)\(^-\)\(^10\) Especially the many-body perturbational theory based on the \(t\)-matrix approximation\(^10\)\(^,\)\(^11\) has carefully been analyzed.\(^3\)\(^,\)\(^4\)\(^,\)\(^5\) It has been found\(^5\) that actually \(\delta(q \to 2k_F, \omega \to 0)\) is finite if \(q\) and \(\omega\) satisfy a particular condition i.e. \((q^2-4k_F^2)/4m < \omega < q(q-2k_F)/2m\). The existence of this singular behavior in a particular scattering processes has also been noted in the calculation of Landau \(f\)-function\(^7\) and in the study of scattering wave-function in a finite box.\(^7\)\(^,\)\(^9\) This can be considered as a partial support to the indication by Anderson. It is found, however, that, irrespective of this subtly together with the existence of the bound state,\(^3\) ordinary many-body theory for the interaction processes confirms that the imaginary part of the electron self-energy \(\Sigma^R(k, \varepsilon)\) on the Fermi surface (i.e. \(k = k_F\)), is proportional to \(\varepsilon^2 \ln \varepsilon_F/|\varepsilon|, \varepsilon_F\) being the Fermi energy\(^3\)\(^,\)\(^5\) and then the quasi-particle is well defined. Here the existence of extra \(\ln \varepsilon_F/|\varepsilon|\) factor is to be noted, which is due to the scattering processes involving energy \((\omega)\) and momentum \((q)\) in the vicinity of the particular values that lead to a constant phase shift. The presence of this factor has first been recognized by Hodges et al.\(^12\) in their study on the second-order processes, where both forward and backward scattering contribute to this singular contribution. However in the \(t\)-matrix approximation it is found\(^5\) that only the forward scattering can contribute to this singularity, which fact has not been noted in the earlier study based on the same approximation\(^13\) as has been also stressed recently.\(^6\) This finding implies that the generic singular contributions to the self-energy function are due only to the forward scattering.

Another important feature found in ref. 5 is that not only the energy dependence but also the momentum dependence are singular; i.e. \(\text{Im} \Sigma^R(k, \varepsilon) \propto -\varepsilon^2 \ln \varepsilon_F/|\varepsilon - \xi_k|\). If this form of \(\text{Im} \Sigma^R(k, \varepsilon)\) is employed, the quasi particle spectral weight, \(-\text{Im} G(k, \varepsilon + i0) \equiv \)
\( \rho(k, \varepsilon) \), is split into two peaks since \( \rho(k, \varepsilon = \xi_k) = 0 \) although the splitting is very small in this case. This is consistent with the finding by Castellani et al.,\(^9\) who have shown that the diagrams, which lead to the Tomonaga-Luttinger liquid in one-dimension, do not cause the Tomonaga-Luttinger liquid in more-than-one dimensions. As a result, they concluded that in the limit of \( (\xi_k, \varepsilon) \to 0 \), the splitting shrinks faster than \( |\xi_k| \) and thus the Fermi liquid state is stable.

In this Letter we will emphasize the splitting of the spectral weight at finite \( \xi_k \) and \( \varepsilon \). This splitting is given by the particle-particle ladder diagrams which are canceled out in one-dimension but have important roles in two dimension. We will also show that for a general shape of the two-dimensional Fermi surface the forward scattering results in a singular contribution to the self-energy function as \( \text{Im} \Sigma^R(k, \varepsilon) = -a\varepsilon^2|\varepsilon - \xi_k|^{-\gamma(k)} \) with the critical exponent \( \gamma(k) \) satisfying \( 0 \leq \gamma(k) \leq 1 \). This results in a pronounced splitting of \( \rho(k, \varepsilon) \) as a function of energy, \( \varepsilon \), although in the limit of \( (\xi_k, \varepsilon) \to 0 \) the Fermi liquid state is stable.

The second order contributions to the self energy \( \Sigma(k, i\varepsilon_n) \) \( (\varepsilon_n \) being the thermal energy to be analytically continued, \( i\varepsilon_n \to \varepsilon + i0 \)) is given by the process shown in Fig. 1. This results in

\[
\text{Im} \Sigma^R(k, \varepsilon) = -U^2 \sum_q \int_0^\varepsilon \frac{dx}{\pi} \text{Im} K(q, x + i0) \text{Im} G(-k + q, x - \varepsilon - i0),
\]

where \( G(k, x + i0) \) and \( K(q, x + i0) \) are analytic continuation of the electron Green function and the particle-particle correlation function, respectively. In the \( t \)-matrix approximation discussed above the contributions in the higher order in \( U \) result in the replacement of \( U \) by \( U_{\text{eff}} \), which is constant even in the limit of \( U \to \infty \) on one hand, and the restriction to \( \tilde{q} \approx 2\tilde{k}_F \) (i.e. the forward scattering) for the possible singular contributions on the other hand. Here \( U_{\text{eff}} \) is given by \( U_{\text{eff}} = U[1 + UN(0)\ln k_c/k_F]^{-1} \) where \( N(0) = (4\pi t)^{-1} \), \( t \) being the transfer integral in the Hubbard model, is the density of states per spin and \( k_c \) is the cut-off momentum of the order of the size of the Brillouin zone. Hence for \( \varepsilon \to 0 \), we obtain

\[
\text{Im} \Sigma^R(k, \varepsilon) = -\pi U_{\text{eff}}^2 \int_0^\varepsilon dx \sum_{k', k''} \delta(\xi_{k''}) \delta(x - \varepsilon - \xi_{k'}) \delta(x - \xi_{k + k' - k''}).
\]
Let us focus on some particular point on the Fermi surface, $\vec{k}_F$, nearest to $k$ of our interest as shown in Fig. 2 and redefine $k_x, k_y$ axes perpendicular and parallel to the tangential direction of the Fermi surface as in Fig. 3, with $\vec{k}_F$ at the origin. The Fermi surface trajectory will generally be given by

$$\varepsilon(k) - \varepsilon_F = vk_x + Ak_y^n = 0,$$

where $v$ is the Fermi velocity in the normal direction to the Fermi surface, $A$ some constant and $n(\geq 2)$ will be an integer (both even and odd) for a smooth Fermi surface, which we assume.

The integration over $k'$ and $k''$ in eq. (2) is evaluated as follows by use of eq. (3),

$$\sum_{k', k''} \delta(\xi_{k''}) \delta(x - \varepsilon - \xi_{k'}) \delta(x - \xi_{k+k'-k''}) = \frac{1}{(2\pi)^4 v^2 |A|} \int dk_y^'dk_y^'' \delta(k_y^' + (k_y^' - k_y^'')^n - k_y^'' - \frac{X}{A})$$

$$= \frac{C_n}{8\pi^4 v^2 n(n-2)|A|^2 |A|^{\frac{n-2}{n}}},$$

where $X = \varepsilon - \xi_k$ and $C_n = 1(2)$ for an even (odd) integer $n (n \geq 2)$. Equation (4) results in

$$\text{Im} \Sigma^R(k, \varepsilon) = -a\varepsilon^2 |\varepsilon - \xi_k|^{-\gamma(k)}.$$

with $\gamma(k) = (n-2)/n$. Since $2 \leq n$, we see $0 \leq \gamma \leq 1$. It is seen that if $n = 2$ which is the case for most points on the Fermi surface, eq. (5) is seen to lead to $\varepsilon^2 \text{Im} \varepsilon_F/|\varepsilon - \xi_k|$ as it should. On the other hand in the region where the Fermi surface is flat we see $n \to \infty$ and then $\gamma \to 1$, a similar result to the case of the marginal Fermi liquid as regard the exponent at $k = k_F$ but with essential difference because of the strong $k$-dependence in the present case. By use of eq. (5), we can evaluate the single particle spectral weight, $\rho(k, \varepsilon)$, which has a two-peak structure since $\rho(k, \xi_k) = 0$ due to $\Sigma^R(k, \xi_k) = \infty$. This singularity at $\varepsilon = \xi_k$ in eq. (5) will, however, generally be removed by the finite life time of electron propagators in Fig. 1. Although the correct form of this renormalization of the electron propagator can not easily been assessed, the effects of this renormalization on $\rho(k, \varepsilon)$ can
be qualitatively estimated by replacing $\varepsilon - \xi_k$ into $\varepsilon - \xi_k + ia_2 \varepsilon^2$ with a some constant $a_2$ comparable to $a$. More explicitly, $\Sigma^R(k, \varepsilon) = -ia\varepsilon^2(\varepsilon - \xi_k + ia_2 \varepsilon^2)^{-\gamma}e^{i\pi\gamma/2}$ is assumed in order to guarantee that $\text{Im} \Sigma^R(k, \varepsilon)$ is an even function of $X \equiv \varepsilon - \xi_k$ for a fixed $\varepsilon$ when $a_2 = 0$. An example of $\rho(k, \varepsilon)$ for a choice of $a = 0.02/\varepsilon_F^{1-\gamma}$ and $a_2 = 0$ and $0.02/\varepsilon_F$ for $\gamma = 0.5$ is shown in Fig. 4. (Note that in the limit of $U \to \infty$, $a = [4\pi\varepsilon_F \ln^2(k_e/k_F)]^{-1}$ for $n = 2$ and $a = C_n[4\pi n(n - 2)\varepsilon_F^{1-\gamma} \ln^2(k_e/k_F)]^{-1}$ for $n > 2$.)

As seen there exists a clear double peak similar to the case of the Tomonaga-Luttinger liquid. This double peak in the case of Tomonaga-Luttinger liquid reflects the spin-charge separation. In the present case, however, this splitting, which is of order of $|\xi_k|^2/1+\gamma$, is reduced faster than the center of the spectral weight (i.e. at $\varepsilon = \xi_k$) as $\xi_k \to 0$ in contrast to the case of the Tomonaga-Luttinger liquid, where the splitting is of the order of $\xi_k$. Moreover it is stressed that the exponent, $\gamma$, varies along the Fermi surface, i.e. $\gamma(k_F)$. Strictly speaking, $n = 2$ on almost all points on the Fermi surface if it is smooth. However, if the Fermi surface is viewed with finite energy spread, the effective value of $n$ will vary in a wide range. Hence the theory is self-consistent if the splitting of the spectral weight evaluated for a relevant $n$ is larger than this Fermi surface energy spread. This fact guarantees the validity of the Fermi liquid theory (except the point where $\gamma(k_F) = 1$) in the limit of low energy excitations as in three dimensional case. Hence there exists a crossover from the Fermi liquid behavior in the limit of low excitation energy and the Tomonaga-Luttinger liquid like behavior at finite energies, a feature generic to the present two-dimensional electron system. Note that in one- and three dimension, there is no crossover; the system is always Tomonaga-Luttinger liquid and Fermi liquid, respectively, in the energy ranges which we are interested in. In two-dimension, the existence of such a crossover in a quasi-particle spectral weight has not been discussed so far in the studies based on the Fermi liquid theory. This result suggests that the temperature dependences of various physical quantities, which are averaged over the Fermi surface, will reflect this non-Fermi liquid behavior at finite temperatures, even though properties consistent with the Fermi liquid theory should be seen in the limit of low temperatures.

The singular behaviors associated with the forward scattering found in the present paper is intrinsic to the particular feature of the phase volume in two-dimension and has nothing to do with the singularity in the particle-hole channel, e.g. the nesting. This is
in contrast to the case for 1 ≤ d < 2 investigated recently by Castellani et al.,\textsuperscript{8} where the particle-hole as well as particle-particle channels play crucial roles. In two-dimension the $t$-matrix approximation is known to be valid as a low density expansion,\textsuperscript{10} where the exclusive importance of the forward scattering has been identified.\textsuperscript{1,2,5} As the electron density increases, the importance of the particle-hole correlations gradually increases. Even in this case the $t$-matrix ladder is the key elementary scattering processes in the presence of strong correlation, and the particle-hole correlation should be considered based on such particle-particle correlations. Hence the present fact that the forward scattering alone can lead to a non-Fermi liquid properties at finite energies will be the basis for the further study of the interaction processes including the particle-hole correlations, which eventually can lead to various kinds of instabilities such as charge density wave, spin density wave and superconductivity.

Acknowledgement

Authors thank Claudio Castellani, Carlo Di Castro, Walter Metzner and Phil Stamp for useful discussions in Aspen, where the final part of the work has been carried out. This work is supported by Monbusho Joint Research “Theoretical Studies on Strongly Correlated Electron Systems” (05044037) and Grand-in-Aid for Scientific Research on Priority Areas, “Science of High $T_c$ Superconductivity” (04240103) and “Novel Electronic States in Molecular Conductors” (06243211) of Ministry of Education, Science and Culture.
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Figure Captions

Fig. 1 The self-energy correction in the second order of Coulomb interaction, $U$.

Fig. 2 The schematic representation of a characteristic Fermi surface (FS) in two dimensions; $\vec{k}$ and $\vec{k}_F$ are the momentum variable of interest (generally away from FS) and the point on FS nearest to $\vec{k}$.

Fig. 3 A detailed description of FS near $\vec{k}$ and $\vec{k}_F$.

Fig. 4 The energy dependence of the quasi-particle spectral weight, $\rho(k, \varepsilon) = -\text{Im}G(k, \varepsilon + i0)$, for $\xi_k/\varepsilon_F = 0.1$ with a choice of $a = 0.02/\varepsilon_F^{1-\gamma}$, $\gamma = 0.5$ and $a_2 = 0$ (solid line) and $a_2 = 0.02/\varepsilon_F$ (broken line).