SPLITTING FAMILIES OF SETS IN ZFC

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Abstract. Miller’s 1937 splitting theorem was proved for pairs of cardinals \((\nu, \rho)\) in which \(n\) is finite and \(\rho\) is infinite. An extension of Miller’s theorem is proved here in ZFC for pairs of cardinals \((\nu, \rho)\) in which \(\nu\) is arbitrary and \(\rho \geq \beth_\omega(\nu)\). The proof uses a new general method that is based on Shelah’s revised Generalized Continuum Hypothesis theorem.

Upper bounds on conflict-free coloring numbers of families of sets and a general comparison theorem follow as corollaries of the main theorem. Other corollaries eliminate the use of additional axioms from splitting theorems due to Erdős, Hajnal, Komjáth, Juhász and Shelah.

1. Introduction

Saharon Shelah’s work on pcf theory [20] transformed cardinal arithmetic by revealing a robust structure of small products of regular cardinals, from which absolute upper bounds followed for singular cardinal exponentiation, in contrast to regular cardinals exponentiation, for which no upper bounds exist by Cohen’s and Easton’s results. Shelah described the state of affairs in 1992 in an expository article [21]:

“...the theory of cardinal arithmetic involves two quite different aspects, one of which is totally independent of the usual axioms of set theory, while the other is quite amenable to investigation on the basis of ZFC.”

In 2000, Shelah made an important breakthrough in cardinal arithmetic beyond pcf theory and beyond exponentiation [22] by introducing a new arithmetic function, the revised power \(\lambda^{[k]}\) (definition 2.5 below), and proving the equality \(\lambda^{[k]} = \lambda\) for all \(\lambda \geq \mu\), regular or singular, with all but a bounded set of regular \(\kappa < \mu\), for an arbitrary strong limit cardinal \(\mu\). This was the first time since the equality \(\lambda^n = \lambda\) was proved for infinite \(\lambda\) and finite \(n > 0\) that cardinal arithmetic equations were shown to hold absolutely in an end-segment of the cardinals. Shelah explained in the introduction to [22] why he considered this discovery to be a solution of Hilbert’s first
problem, necessarily reinterpreted after Cohen’s proof of the independence of the continuum hypothesis as what are the rules of cardinals arithmetic, rather than what is the size of the continuum.

We present here a method, based on Shelah’s revised GCH theorem, for proving combinatorial theorems for an arbitrary infinite cardinal $\nu$ in all sufficiently large cardinalities, in a similar way to that in which theorems for a finite $n$ are proved in all infinite cardinals. This approach was first used in [12] to extends to infinite graphs results about finite graph colorings. Here a general methods is presented. The particular applications of the method here concern the old subject of splitting families of sets.

In Section 2 the arithmetic properties of the density function $D(\lambda, \kappa_1, \kappa_2)$ for $\kappa_1 \leq \kappa_2 \leq \lambda$ are handled. $D(\lambda, \kappa_1, \kappa_2)$ is the least number of $\kappa_1$-subsets of $\lambda$ required to enter every $\kappa_2$-subset of $\lambda$, and is clearly bounded from above by the exponent $\lambda^{\kappa_1}$. Unlike the exponent function $\lambda^{\kappa_1}$, the arithmetic of $D(\lambda, \nu, \kappa)$ is robust. It is not affected by Cohen or by Easton forcing, the equality $D(\lambda, \nu, \nu^+) = \lambda$ for all $\lambda \geq 2^\nu$ holds easily in models of set theory which obey mild restrictions on cardinal exponentiation, and, most importantly, $D(\lambda, \nu, \kappa) = \lambda$ for all sufficiently large $\lambda \geq \kappa$ in all models of ZFC as a consequence of Shelah’s revised GCH theorem.

In Section 3 an asymptotic filtration theorem for order-reversing set operations is obtained. This theorem is an absolute weak infinitary generalization of the well-known downwards Skolem-Löwenheim theorem. Instead of using the equation $\lambda^n = \lambda$ — which fails periodically in the class of all infinite cardinals — the proof relies on the equations which govern the density function.

Miller [17] proved in 1937 a splitting theorem for arbitrarily large $\rho$-uniform families of sets that satisfy the condition $C(\rho^+, n)$ when $\rho \geq \aleph_0$ and $n$ is finite. The condition $C(\rho^+, n)$ is that the intersection of any subfamily of size at least $\rho^+$ has cardinality smaller than $n$. Miller’s proof relied on the equation $\lambda^n = \lambda$ and essentially used the Skolem-Löwenheim argument (although Miller most likely did not know about Skolem’s work at the time) to obtain filtrations. Theorem 4.7 below strengthens Miller’s theorem and extends it to any cardinal $\nu$ in place of the finite $n$ and any $\rho \geq \beth_\omega(n)$ in place of the infinite $\rho$ in Miller’s $C(\rho^+, n)$ condition. For a finite $n$, the term $\beth_\omega(n)$ is indeed $\aleph_0$. The required filtrations are gotten by using Section 2.

Several corollaries about splitting, comparing and coloring families of sets are gotten from Theorem 4.7. Three of these results were proved earlier by assuming additional axioms. Erdös and Hajnal proved property B from $C(\nu, \rho^+)$ for $\rho > \nu^+$ from the Generalized Continuum Hypothesis [3], Komjáth obtained disjoint refinements and essential disjointness from the same axiom [14] and Hajnal, Juhasz and Shelah obtained essential disjointness from a weak form of the Singular Cardinals Hypothesis [9].
2. Arithmetic of Density

In this Section we list some properties of the arithmetic density function \( D(\lambda, \kappa_1, \kappa_1) \).

**Definition 2.1 (Density).** (1) Let \( \kappa_1 \leq \kappa_2 \leq \lambda \) be cardinals. A set \( D \subseteq [\lambda]^{\kappa_2} \) is dense in \([\lambda]^{\kappa_2}\) if for every \( X \in [\lambda]^{\kappa_2} \) there is \( Y \in D \) such that \( Y \subseteq X \).

(2) For \( \kappa_1 \leq \kappa_2 \leq \lambda \) the \((\kappa_1, \kappa_2)\)-density of \( \lambda \), denoted by \( D(\theta, \kappa_1, \kappa_2) \), is the least cardinality of a set \( D \subseteq [\theta]^{\kappa_1} \) which is dense in \([\lambda]^{\kappa_2}\).

(3) For \( \kappa \leq \kappa \)-density of \( \lambda \), denoted by \( D(\lambda, \kappa) \), is \( D(\lambda, \kappa, \kappa) \).

**Claim 2.2 (Basic properties of density).** Suppose \( \kappa_1 \leq \kappa_2 \leq \lambda \).

(1) If \( X \subseteq Y \) and \( D \subseteq [Y]^{\kappa_1} \) is dense in \([Y]^{\kappa_2}\), then \( D \cap [X]^{\kappa_1} \) is dense in \([X]^{\kappa_2}\).

(2) Monotonicity \( D(\lambda, \kappa_1, \kappa_2) \) is increasing in \( \lambda \) and if \( \kappa'_1 \leq \kappa_1 \leq \kappa_2 \leq \kappa'_2 \leq \lambda \) then \( D(\lambda, \kappa'_1, \kappa'_2) \leq D(\lambda, \kappa_1, \kappa_2) \).

(3) Suppose \( (X_i : i < \theta) \) is \( \subseteq \)-increasing, \( X = \bigcup_{i<\theta} X_i \), \( \kappa_1 \leq \kappa_2 \leq |X_0| \), and \( D_i \subseteq [X_i]^{\kappa_1} \) is dense in \([X_i]^{\kappa_2}\). Then \( D := \bigcup_{i<\theta} D_i \) is dense in \([X]^{\kappa_3}\) for all \( \kappa_3 > \kappa_2 \) such that \( \kappa_3 \leq |X| \), and if \( \text{cf} \theta \neq \text{cf} \kappa_2 \) then \( D \) is dense also in \([X]^{\kappa_2}\).

(4) Continuity at limits) If \( \lambda = (\lambda_i : i < \theta) \) is increasing with limit \( \lambda \), and \( D(\lambda_i, \kappa_1, \kappa_2) \leq \lambda \), then \( D(\lambda, \kappa_1, \kappa_3) = \lambda \) for all \( \kappa_3 > \kappa_2 \) such that \( \kappa_3 \leq |X| \), and if \( \text{cf} \lambda = \text{cf} \kappa_2 \) then also \( D(\lambda, \kappa_1, \kappa_2) = \lambda \).

(5) If \( \text{cf} \mu = \text{cf} \kappa < \mu \) then \( D(\mu, \kappa) > \mu \).

(6) If \( \mu \) is a strong limit cardinal, then \( D(\mu, \kappa) = \mu \) for all \( \kappa < \mu \) such that \( \text{cf} \kappa \neq \text{cf} \mu \).

(7) \( D(\lambda, \kappa) \) is not increasing in \( \kappa \) and the inequality in (2) can be strict.

**Proof.** Item (1) is follows from the definition of density. The monotonicity in \( \lambda \) in (2) follows from (1).

Assume \( \kappa'_1 \leq \kappa_1 \leq \kappa_2 \leq \kappa'_2 \leq \lambda \), let \( D \subseteq [\lambda]^{\kappa_1} \) be an arbitrary dense set in \([\lambda]^{\kappa_2}\) and for each \( X \in D \) fix \( X' \in [X]^{\kappa'_1} \). Let \( D' = \{X' : X \in D\} \).

Given any \( Y \subseteq [X]^{\kappa_2} \) there is some \( X \in D \) such that \( X \subseteq Y \), hence \( X' \subseteq Y \).

Clearly, \( |D'| \leq |D| \). Thus, \( D(\lambda, \kappa'_1, \kappa'_2) \leq D(\lambda, \kappa_1, \kappa_2) \).

To prove (3) assume first that \( \text{cf} \theta \neq \text{cf} \kappa_2 \) and let \( Z \subseteq [X]^{\kappa_2} \) be arbitrary.

Since \( \text{cf} \kappa_2 \neq \text{cf} \theta \) there is some \( i < \theta \) such that \( |Z \cap X_i| = \kappa_2 \), and thus there is some \( Y \in D_i \subseteq D \) such that \( Y \subseteq Z \). This establishes the density of \( D \) in \([X]^{\kappa_2}\); as \([X]^{\kappa_2}\) is dense in \([X]^{\kappa_3}\) for all \( \lambda \geq \kappa_3 \geq \kappa_2 \), the density of \( D \) in \([X]^{\kappa_3}\) follows. Now it remains to prove the density of \( D \) in \([X]^{\kappa_3}\) for \( \lambda \geq \kappa_3 > \kappa_2 \) when \( \text{cf} \theta = \text{cf} \kappa_2 \). Given \( Z \subseteq [X]^{\kappa_3} \) it suffices to show that \( |Z \cap X_i| \geq \kappa_2 \) for some \( i < \theta \). As \( \kappa_3 > \kappa_2 \) and \( \text{cf} \kappa_2 = \text{cf} \theta \) actually more is true: there is \( i < \theta \) such that \( |Z \cap X_i| > \kappa_2 \) (or else \( |Z| \leq \text{cf} \kappa_2 \times \kappa_2 = \kappa_2 \)).

Item (4) follows from (3).

The inequality (5) follows from the standard diagonalization argument in the proof of the Zermelo-König lemma which, in fact, proves that there
exists an almost disjoint \( \mathcal{F} \subseteq [\mu]^{\kappa} \) of cardinality > \( \mu \). This implies that \( D(\mu, \kappa) > \mu \).

For (6), let \( \mu \) be a strong limit and let \( \kappa < \mu \) be arbitrary. There is an unbounded set of cardinals \( \lambda < \mu \) below \( \mu \) which satisfy \( \lambda^\kappa = \lambda \) hence \( D(\mu, \kappa) = \mu \) follows from (4) if \( \text{cf} \mu \neq \text{cf} \kappa \).

To see (7) let \( \nu \) be an arbitrary cardinal. By (5), \( D(\aleph_0(\nu), \aleph_0) > \aleph_0(\nu) \) while by (6) \( D(\aleph_0(\nu), \aleph_1) = \aleph_0(\nu) \). Similarly, \( D(\aleph_0(\nu), \aleph_0) > \aleph_0(\nu) = D(\aleph_0(\nu), \aleph_0, \aleph_1) = D(\aleph_0(\nu), \aleph_1) < D(\aleph_1(\nu), \aleph_1) \), so (7) is proved. \( \square \)

The next claim provides a simple sufficient condition for the validity of the equation \( D(\lambda, \kappa_1, \kappa_2) = \lambda \) for all \( \lambda \) in an end-segment of the cardinals.

**Claim 2.3.** Suppose \( \theta > \kappa_2 > \kappa_1 \geq \aleph_0 \), \( D(\theta, \kappa_1, \kappa_2) = \theta \) and for every \( \lambda \geq \theta \), if \( \text{cf} \lambda = \text{cf} \kappa_2 \) there exists some \( \nu \) such that \( \kappa_1 \leq \nu < \kappa_2 \) and \( \lambda = \sup \{ \delta : \delta < \lambda \land D(\delta, \kappa_1, \nu) \leq \lambda \} \). Then \( D(\lambda, \kappa_1, \kappa_2) = \lambda \) for every \( \lambda \geq \theta \).

**Proof.** By induction on \( \lambda \geq \theta \). For \( \lambda = \theta \) the equality \( D(\lambda, \kappa_1, \kappa_2) = \lambda \) is assumed, and if \( \lambda > \theta \) satisfies \( \text{cf} \lambda = \text{cf} \kappa_2 \) then \( D(\lambda, \kappa_1, \kappa_2) = \lambda \) by (4). Assume then that \( \text{cf} \lambda = \text{cf} \kappa_2 \). By the assumption, there is \( \kappa_1 \leq \nu < \kappa_2 \) and \( \langle \lambda_i : i < \text{cf} \kappa_2 \rangle \), an increasing sequence of cardinals with limit \( \lambda \) such that \( D(\lambda_i, \kappa_1, \nu) \leq \lambda \). Now \( D(\lambda, \kappa_1, \kappa_2) = \lambda \) by (4). \( \square \)

**Corollary 2.4.** (1) Let \( \nu \) be an infinite cardinal, \( \kappa = \text{cf} \kappa > \nu \) and assume that \( \theta \geq \kappa \) satisfies such that \( D(\theta, \nu, \kappa) = \theta \) for every \( \mu > \theta \) all limit cardinals in the interval \( [\mu, \mu^\kappa) \) are of cofinality strictly smaller than \( \kappa \). Then \( D(\lambda, \nu, \kappa) = \lambda \) for all \( \lambda \geq \theta \).

(2) The SCH implies that for every infinite \( \nu \) it holds that \( D(\lambda, \nu, \nu^+) = \lambda \) for all \( \lambda \geq 2^\nu \).

**Proof.** If \( \lambda \geq \theta \) and \( \text{cf} \lambda = \kappa \) then \( \theta^\mu \leq \lambda \) for all \( \theta < \lambda \), so \( \lambda = \sup \{ \theta^\mu : \theta < \lambda \} \). If \( \delta = \theta^\mu \) then \( \delta^\nu = \delta \) and trivially \( D(\delta, \nu) = \delta \). Thus (1) follows from Claim 2.3. Item (2) follows from (1) since by the SCH for every cardinal \( \lambda \geq 2^\nu \) it holds that \( \lambda^\nu \in \{ \lambda, \lambda^+ \} \). \( \square \)

By the Corollary above, the existence of a bound on the gaps between \( \mu \) and \( \mu^\kappa \) in some end segment of the cardinals suffices to cover the end segment by a single equation of the form \( D(x, \nu, \kappa) = x \). However, models of set theory with arbitrarily large gaps between \( \mu \) and \( \mu^{\aleph_0} \) can be built from a proper class of suitable large cardinals by the methods in [6].

The next Lemma and its Corollary 2.8 below, hold in every model of ZFC.

**Definition 2.5** (Shelah’s revised power). Let \( \theta \leq \lambda \) be cardinals. A set \( D \subseteq [\lambda]^\theta \) is weakly covering if for every \( X \in [\lambda]^\theta \) there exists \( \mathcal{Y} \in [D]^{<\theta} \) such that \( X \subseteq \bigcup \mathcal{Y} \).

Shelah’s revised \( \theta \)-power of \( \lambda \), denoted \( \lambda[\theta] \), is the least cardinality of a weakly covering \( D \subseteq [\lambda]^\theta \).
Theorem 2.6 (Shelah’s Revised GCH Theorem [22]). For every strong
limit












small limit cardinal μ, for all λ ≥ μ, for every sufficiently large regular θ < μ,

\[ \lambda^{[\theta]} = \lambda. \]

Lemma 2.7. Let μ be a strong limit cardinal. For every λ ≥ μ there is some


















θ(λ) < μ such that for every θ < μ with cfθ > θ(λ) it holds that D(λ, θ) = λ.

Proof. Let λ ≥ μ be given and let θ(λ) < μ be fixed by Shelah’s Revised
GCH theorem such that for every regular θ ≤ λ(μ) it holds that λ[θ] = λ.

Assume first that θ ∈ (θ(λ), μ) is regular. Let \[ D \subseteq [\lambda]^\theta \] witness λ[θ] = λ
and, as \( 2^\theta < \lambda \), assume that \( X \in D \Rightarrow [X]^\theta \subseteq D \). If \( Z \in [\lambda]^\theta \) is arbitrary,
fix \( Y \in [D]^{<\theta} \) such that \( Z \subseteq \bigcup Y \). As θ is regular, there is \( X \in Y \) such that
\( Y := X \cap Z \) has cardinality θ. Since \( Y \in D \) and \( Y \subseteq Z \), we have shown that
\( D \) is dense in \( [\lambda]^\theta \).

Assume now that θ ∈ (θ(λ), μ) is singular and satisfies cfθ > θ(λ). Write
\( \theta = \sum_{i < \text{cf}\theta} \theta_i \) with \( \theta_i = \text{cf}\theta_i > \theta(\lambda) \) for each i. Next fix dense \( D_i = \{X^i_\alpha : \alpha < \lambda\} \subseteq [\lambda]^\theta \),
and dense \( T' \subseteq [\text{cf}\theta \times \lambda]^{|T'|} \) with \( |T'| = \lambda \). The set
\( T := \{Y \in T : \exists i < \text{cf}\theta \Rightarrow [Y \cap (\{i\} \times \lambda)] < 1\} \), namely, all members of \( T' \)
which are partial functions from cfθ to λ with domain cofinal in cfθ, has


















cardinality λ and for every \( f : \text{cf}\theta \to \lambda \) there exists \( Y \in T \) such that \( Y \subseteq f \).

Let

\[ D = \{ \bigcup \{X^i_\alpha : \langle i, \alpha \rangle \in Y\} : Y \in T \}. \]

Clearly, \( D \subseteq [\lambda]^\theta \) and \( |D| = \lambda \).

Given any \( Z \in [\lambda]^\theta \), define \( f(i) = \min \{\alpha < \lambda : X^i_\alpha \subseteq Z\} \) for \( i < \text{cf}\theta \).
Now \( \bigcup_{i < \text{cf}\theta} X^i_{f(i)} \in [Z]^\theta \). There is some \( Y \in T \) such that \( Y \subseteq f \). The set
\( \bigcup \{X^i_\alpha : \langle i, \alpha \rangle \in Y\} \) therefore belongs to \([Z]^\theta \cap D\). □

Now the asymptotic equations concerning density follow:

Corollary 2.8. For every infinite cardinal ν and ρ ≥ \( \beth_\omega(\nu) \), for all but
finitely many \( n < \omega \) it holds that

(1) \[ D(\rho, \beth_n(\nu)) = \rho, \]

and

(2) \[ D(\rho, \nu, \beth_\omega(\nu)) = \rho. \]

Proof. For all \( n, \text{cf}\beth_{n+1}(\nu) > \beth_n(\nu) \), so \( \text{cf}\beth_n(\nu) \) converges to \( \beth_\omega(\nu) \), hence
(1) follows from Lemma 2.7. To prove (2), fix, for a given \( \rho \geq \beth_\omega(\nu) \), some
\( n \) such that \( D(\rho, \beth_n(\nu)) = \rho \) and as \( \nu \leq \beth_n(\nu) < \beth_\omega(\nu) \), by monotonicity
\( D(\rho, \nu, \beth_\omega(\nu)) \leq D(\rho, \beth_n(\nu)). \) □

The consistency of \( D(\rho, \nu, \beth_n(\nu)) > \rho \) for \( \beth_n(\nu) \leq \rho < \beth_\omega(\nu) \) would imply
that \( \beth_\omega(\nu) \) cannot be relaxed to \( \beth_n(\nu) \) in (2) above. See Section 3.1 below
for a discussion of this — at the moment unknown — consistency.

Shelah pointed out to me the next lemma, which will be used in the
comparison theorem in the last theorem of Section 4.
Lemma 2.9. Suppose $\mu$ is an strong limit cardinal. Let $\chi$ be a sufficiently large regular cardinal and $\delta < \chi$ is a limit ordinal. Suppose a sequence $\langle M_i : i \leq \delta \rangle$ of elementary submodels of $(H(\chi), \epsilon, \ldots)$ satisfies:

1. $M_i \subseteq M_{i+1}$ and $M_j = \bigcup_{i < j} M_i$ when $j \leq \delta$ is limit;
2. $\langle M_j : j \leq i \rangle \in M_{i+1}$ for all $i < \delta$;
3. $\mu \subseteq M_0$ and $|M_i| \leq M_i$ for all $i \leq \delta$.

Then there exists $\kappa(*) < \mu$ such that $M_\delta \cap [M_\delta]^{\kappa(*)}$ is dense in $[M_\delta]^{\kappa(*)}$ for all $\kappa$ such that $\kappa(*) \leq \text{cf}\kappa \leq \kappa < \mu$.

Proof. Denote $\lambda = |M_\delta|$ and for $i < \delta$ denote $\lambda_i = |M_i|$. Let $\kappa(*) < \mu$ be such that $D(\lambda, \kappa) = \lambda$ for all $\kappa < \mu$ with $\text{cf}\kappa \geq \kappa(*)$. By increasing $\kappa(*)$ if necessary, we assume that $\text{cf}\delta \not\in [\kappa(*)]$. Given $i < \delta$ and $\kappa$ such that $\kappa(*) \leq \text{cf}\kappa \leq \kappa < \mu$, there exists a dense $D_\delta^\kappa \subseteq [M_\delta]^{\kappa(*)}$ in $M_{i+1}$ of size $D(\lambda_i, \kappa)$ by elementarity. As $\lambda_i \leq \lambda$, it holds by 2.2(2) that $|D_\delta^\kappa| \leq \lambda$ for all $i < \delta$. Since $D_\delta^\kappa \subseteq M_\delta$ and $|D_\delta^\kappa| \leq \lambda \subseteq M_\delta$, it follows that $D_\delta^\kappa \subseteq M_\delta$. Now $D^\kappa := \bigcup_{i < \delta} D_\delta^\kappa \subseteq M_\delta$ for all $\kappa < \mu$ with $\text{cf}\kappa \geq \kappa(*)$ and has cardinality $\lambda$. As $\text{cf}\delta \not\in \text{cf}\kappa$ for such $\kappa$, the set $D^\kappa \subseteq M_\delta$ is dense in $[M_\delta]^{\kappa(*)}$. \square

3. Filtrations with respect to anti-monotone set functions

Every infinite subset of a structure with countably many finite-place operations is contained in a subset of the same cardinality which is closed under all operations by the downward Skolem-Löwenheim theorem. Also, the union of any increasing chain of closed sets is closed. Consequently, every uncountable structure with countably many finitary operations is filtrable, that is, presentable as an increasing and continuous union of substructures of smaller cardinality.

If infinitary operations are admitted, both facts above are no longer true since the cardinality of the closure of a subset of cardinality $\lambda$ under a $\kappa$-place operation depends on the valued of the exponent $\lambda^\kappa$ which is undecidable an is periodically larger than $\lambda$ — e.g. for $\lambda$ with countable cofinality — in every model of set theory, including models of the GCH.

The main theorem of this section asserts that for anti-monotone set-functions a version of the Skolem-Löwenheim theorem and filtrations to closed sets exist without appealing to additional axioms.

Definition 3.1. (1) A filtration of an infinite set $V$ is a sequence of sets $\langle D_\alpha : \alpha < \kappa \rangle$ for some cardinal $\kappa$ which satisfies:

(a) $|D_\alpha| < |V|$.
(b) $\alpha < \beta < \kappa \Rightarrow D_\alpha \subseteq D_\beta$
(c) If $\alpha < \kappa$ is limit then $D_\alpha = \bigcup_{\beta < \alpha} D_\beta$.
(d) $V = \bigcup_{\alpha < \kappa} D_\alpha$.

Condition (c) in the definition is the continuity of $\langle D_\alpha : \alpha < \kappa \rangle$.

(2) Suppose $V$ is an infinite set and $\mathcal{C} \subseteq \mathcal{P}(V)$. A $\mathcal{C}$-filtration of $V$ is a filtration $\langle D_\alpha : \alpha < \kappa \rangle$ such that $D_\alpha \in \mathcal{C}$ for all $\alpha < \kappa$. 
Theorem 3.2. Suppose $C \subseteq \mathcal{P}(V)$, $I \neq \emptyset$ is countable and $\rho < |V|$ a cardinal. Suppose $S_i \subseteq C$ for $i \in I$ and:

1. For every $i \in I$, the union of every chain of sets from $S_i$ belongs to $C$.
2. For every $\rho \leq \theta < |V|$ there exists $i \in I$ such that for every set $A \in [V]^\theta$ there is $D \in S_i$ such that $A \subseteq D \in [V]^\theta$.

Then $V$ is $C$-filtrable.

Proof. Let $\lambda := |V|.$

Assume first that $\text{cf} \lambda > \aleph_0$. Fix an increasing and continuous chain of sets $\langle A_\alpha : \alpha < \text{cf} \lambda \rangle$ such that $V = \bigcup_{\alpha < \text{cf} \lambda} A_\alpha$ and $|A_\alpha| < \lambda$ for each $\alpha < \text{cf} \lambda$ and $|A_0| \geq \rho$. Denote $\theta_\alpha := |A_\alpha| < \lambda$.

As $\text{cf} \lambda > \aleph_0$ is regular and $I$ is countable, we may assume, by passing to a subsequence, that for some fixed $i \in I$ it holds that for every $\alpha < \text{cf} \lambda$ and every $B \in [V]^{\theta_\lambda+1}$ there is $D \in S_i$ such that $B \subseteq D$ and $|D| = |B|$.

Define inductively $D_\alpha$ for $0 < \alpha < \text{cf} \lambda$. For limit $\alpha < \text{cf} \lambda$ let $D_\alpha = \bigcup_{\beta < \alpha} D_\beta$. For $\alpha = \beta + 1$ choose $D_\alpha \in S_i$ which contains $A_\alpha \cup \bigcup_{\beta < \alpha} D_\beta$ and is of cardinality $|A_\alpha \cup \bigcup_{\beta < \alpha} D_\beta|$.

$V = \bigcup_{\alpha < \text{cf} \lambda}$ since $A_\alpha \subseteq D_\alpha$ for all $\alpha < \text{cf} \lambda$, and as $D_{\alpha+1} \in S_i$, each $D_{\alpha+1}$ belongs to $C$. By (1) $D_\alpha \in C$ for limit $\alpha < \text{cf} \lambda$ as well. Thus $\langle D_\alpha : 0 < \alpha < \text{cf} \lambda \rangle$ is a $C$-filtration.

If $\text{cf} \lambda = \aleph_0$ fix an increasing union $\lambda = \bigcup_{n < \omega} A_n$ with $|A_n| < \lambda$ and $|A_0| \geq \rho$ and let $D_{n+1} = K(A_{n+1} \cup D_n)$. By (2), $|D_{n+1}| = |A_{n+1}|$ hence $\langle D_{n+1} : n < \omega \rangle$ is a $C$-filtration.$\square$

Definition 3.3. Let $V$ be a nonempty set.

1. A notion of closure over $V$ is a family of sets $C \subseteq \mathcal{P}(V)$ that satisfies:
   (a) $\emptyset, V \in C$.
   (b) $D_1, D_2 \in C \Rightarrow D_1 \cup D_2 \in C$.
   (c) $\bigcap X \in C$ for all $X \subseteq C$.
2. A function $K : \mathcal{P}(V) \rightarrow \mathcal{P}(V)$ is a closure operator if it satisfies Kuratowski’s closure axioms:
   (a) $K(\emptyset) = \emptyset$;
   (b) $A \subseteq K(A)$ for $A \subseteq V$;
   (c) $K(K(A)) = K(A)$ for $A \subseteq V$;
   (d) $K(A \cup B) = K(A) \cup K(B)$ for $A, B \subseteq V$.
3. A notion of semi-closure over $V$ is a family $C \subseteq \mathcal{P}(V)$ which satisfies $V \subseteq C$ and $\bigcap X \in C$ for all $X \subseteq C$.
4. A function $K : \mathcal{P}(V) \rightarrow \mathcal{P}(V)$ is a semi-closure operator if it satisfies Kuratowski’s axioms (b) and (c) and the following implication of (d):
   (d') $A_1 \subseteq A_2 \subseteq X \Rightarrow K(A_1) \subseteq K(A_2)$. 


(5) Given a notion of semi-closure \( C \subseteq \mathcal{P}(V) \) let \( K_C(A) = \bigcap \{ D : A \subseteq D \in C \} \). Conversely, given a semi-closure operator \( K : \mathcal{P}(V) \to \mathcal{P}(V) \) let \( C_K = \{ D : D = K(D) \subseteq V \} \).

For every semi-closure operator \( K \) on a set \( V \) it holds that \( K = K_{C_K} \) and for every notion of semi-closure \( C \) it holds that \( C = C_{K_C} \). If \( K \) is a closure operator then \( C_K \) is a notion of closure and if \( C \) is a notion of closure then \( K_C \) is a closure operator. Thus, notions of [semi]-closure and [semi]-closure operators are interchangeable.

**Definition 3.4.** Suppose that \( F : \mathcal{P}(V) \to \mathcal{P}(V) \) and \( \kappa \) is a cardinal. Denote by \( F_\kappa \) the restriction \( F \upharpoonright [X]^{\kappa} \) to sets of size \( \kappa \). The subsets which are closed under \( F_\kappa \) form the following notion of semi-closure \( C_{F,\kappa} = \{ D : D \subseteq V \land ( Y \in [D]^{\kappa} \Rightarrow F(Y) \subseteq D) \} \). Let \( K_{F,\kappa} \) denote the semi-closure operator corresponding to \( C_{F,\kappa} \) and let us refer to sets \( D \in C_{F,\kappa} \) as \( F_\kappa \)-closed sets.

**Definition 3.5.** A function \( F : \mathcal{P}(X) \to \mathcal{P}(X) \) is anti-monotone if \( A \subseteq B \Rightarrow F(B) \subseteq F(A) \).

**Claim 3.6.** Suppose \( F : \mathcal{P}(V) \to \mathcal{P}(V) \) is anti-monotone and \( \kappa \) is an infinite cardinal. Then:

1. \( K_{F,\kappa} \) is a closure operator.
2. An increasing union \( \bigcup_{\alpha<\theta} D_\alpha \) of \( K_{F,\kappa} \)-closed sets is \( K_{F,\kappa} \)-closed if \( \text{cf}\theta \neq \text{cf}\kappa \).

**Proof.** We verify axiom (4). If \( X \in [A \cup B]^{\kappa} \) then, since \( \kappa \) is infinite, \( |X \cap A| = \kappa \) or \( |X \cap B| = \kappa \), so by anti-monotonicity \( F(X) \subseteq K(A) \) or \( F(X) \subseteq K(B) \).

Assume now that \( D = \bigcup_{\alpha<\theta} D_\alpha \) is an increasing union of \( K_{F,\kappa} \)-closed sets and \( \text{cf}\theta \neq \text{cf}\kappa \). Let \( X \in [D]^{\kappa} \). There exists some \( \alpha < \theta \) such that \( X \cap D_\alpha \in [D_\alpha]^{\kappa} \). Since \( D_\alpha \) is \( K_{F,\kappa} \)-closed, \( F(X \cap D_\alpha) \subseteq D_\alpha \) and by anti-monotonicity \( F(X) \subseteq F(X \cap D_\alpha) \).

**Lemma 3.7.** Suppose \( V \) is a set and \( F : \mathcal{P}(V) \to \mathcal{P}(V) \) is anti-monotone. Suppose \( \kappa_1 \leq \kappa_2 < \rho \) are cardinals and \( |F(X)| \leq \rho \) for all \( X \in [V]^{\kappa_1} \). Let \( K = K_{F,\kappa_2} \). If \( \theta \geq \rho \) satisfies \( \mathcal{D}(\theta,\kappa_1,\kappa_2) = \theta \) then \( |K(A)| = \theta \) for all \( A \in [V]^{\theta} \).

**Proof.** Given \( A \in [V]^{\theta} \) put \( A_0 = A \) and for \( \zeta \leq \kappa_2^+ \) define \( A_{\zeta} \in [V]^{\theta} \) inductively. For limit \( \zeta \leq \kappa_2^+ \) let \( A_\zeta = \bigcup_{\xi<\zeta} A_\xi \), which is of cardinality \( \theta \).

To define \( A_{\zeta+1} \) for \( \zeta \leq \kappa_2^+ \) fix, by the assumption \( \mathcal{D}(\theta,\kappa_1,\kappa_2) = \theta \), a set \( D_\zeta \subseteq [A_\zeta]^{\kappa_1} \) of cardinality \( \theta \) with the property that for all \( Y \in [A_\zeta]^{\kappa_2} \) there exists \( X \in D_\zeta \) such that \( X \subseteq Y \) and let \( A_{\zeta+1} = A_\zeta \cup \bigcup \{ F(X) : X \in D_\zeta \} \), which is of cardinality \( \theta \) since \( \rho \leq \theta \).

For every \( X \in [A_{\kappa_2^+}]^{\kappa_2} \) there is some \( \zeta < \kappa_2^+ \) such that \( Y \in [A_\zeta]^{\kappa_2} \) and hence there is some \( X \in D_\zeta \) such that \( X \subseteq Y \). As \( F(Y) \subseteq F(X) \subseteq A_{\zeta+1} \), it holds that \( F(Y) \subseteq A_{\kappa_2^+} \), so \( A_{\kappa_2^+} \) is \( K \)-closed.

**Theorem 3.8** (Asymptotic filtrations for anti-monotone set functions). Let \( \nu \) be an infinite cardinal and denote \( \mu := \Sigma_\omega(\nu) \). Let \( V \) be a set of cardinality
\[ |V| > \mu \text{ and suppose } F : \mathcal{P}(V) \to \mathcal{P}(V) \text{ is an anti-monotone function. If there exists a cardinal } \rho \geq \mu \text{ such that } \rho < |V| \text{ and } |F(Y)| \leq \rho \text{ for every } Y \in [X]^\nu \text{ then } V \text{ is } K_{F,\nu}\text{-filtrable.} \]

Proof. Let \( C \) be the family of all \( K_{F,\nu}\)-closed subsets of \( V \). Let \( \kappa_n \) denote \( \Sigma_n(\nu) \) and let \( K_n = K_{F,\kappa_n} \). Let \( S_n \) be the collection of all \( K_n\)-closed subsets of \( V \). We check that conditions (1) and (2) in Theorem 3.2 are satisfied for \( C \) and \( \{ S_n : n < \omega \} \).

As \( \kappa_n < \mu \) and \( F \) is anti-monotone, it holds that \( S_n \subseteq S_{n+1} \subseteq C \) for all \( n \). The union of any chain of sets from \( S_n \) belongs to \( S_{n+1} \subseteq C \) by Lemma 3.6 (2).

To verify condition (2) let \( \theta \geq \rho \) be given. By Lemma 2.7, there exists \( m(\theta) < \omega \) such that \( D(\theta, \kappa_n) = \theta \) for all \( n \geq m(\theta) \). By Lemma 3.7, \( |K_n(A)| = \theta \) for \( A \in [V]^\theta \) for all \( n \geq m(\theta) \).

Now \( V \) is \( K_{F,\nu}\)-filtrable by Theorem 3.2. \( \square \)

We remark that each \( S_n \) is in fact the collection of closed sets with respect to a closure operator and that \( m < n \Rightarrow S_m \subseteq S_n \). Neither of these two properties was used in proving Theorem 3.8.

The following lemma relates filtrations to elementary chains of models:

**Lemma 3.9.** Suppose \( F \) is anti-monotone, \( |F(X)| \leq \rho \) for all \( X \in [V]^\nu \) and \( \rho \geq \mu = \Sigma_\omega(\nu) \). Then for every chain \( \langle M_i : i < \lambda \rangle \) of elementary submodels of a sufficiently large \( (H(\chi), \epsilon) \) which satisfies \( \mu \subseteq M_0 \), \( |M_i| \subseteq M_i \), \( V, F \in M_0 \) and \( \langle M_j : j \leq i \rangle \in M_{i+1} \) the set \( D_i := M_i \cap V \) is \( F_\mu \)-closed for every limit \( i < \lambda \). If \( |M_i| \leq \lambda := |V| \) then \( \langle D_i : i < \lambda \text{ is limit} \rangle \) is a \( \mu \)-filtration of \( V \).

Proof. If \( M \prec (H(\chi), \epsilon, \ldots) \) and \( V, F \in M, \rho \subseteq M \), then for every \( X \in M \cap [V]^\kappa \) for \( \nu \leq \kappa < \nu \) the set \( F(X) \) belongs to \( M \) and \( |F(X)| \leq \rho \), hence \( F(X) \subseteq M \). Thus, if \( M \cap [V]^\kappa \) is dense in \( [V]^\kappa \), the set \( V \cap M \) is \( \kappa \)-closed. By 2.9 this is the case for \( M_i \) for all limit \( i < \lambda \).

If indeed \( |M_i| < \lambda \) for all \( i \) the \( |D_i| < \lambda \). The fact that \( \bigcup_{i<\lambda} D_i = V \) follows from \( V \in M_0 \) and \( \lambda \subseteq \bigcup_{i<\lambda} M_i \). \( \square \)

### 3.1. The need for \( \rho \geq \Sigma_\omega(\nu) \)

Is the restriction \( \rho \geq \Sigma_\omega(\nu) \) optimal in Theorem 3.8? This is not known at the moment. As a result of a discussion with Shelah, something can be said about it: improving it will be at least as hard as proving one of the following versions of Shelah’s Weak Hypothesis, indexed by \( n > 0 \):

(SWH\(_n\)) There are no infinite \( \nu \) and \( \rho \) such that \( \Sigma_n(\nu) < \rho < \Sigma_\omega(\nu) \) and \( F = \{ A_\alpha : \alpha < \rho^+ \} \subseteq [\rho]^\kappa \) satisfies \( |A_\alpha \cap A_\beta| < \nu \) for \( \alpha < \beta < \rho^+ \).

Shelah’s Weak Hypothesis is a dynamic statement whose evolving contents is the weakest unproved pcf-theoretic statement. It is formulated in terms of patterns in Shelah’s function pp. See [19] for the translation of SWH from pcf to almost disjoint families. The negations of first two versions of the SWH have been shown consistent. Gitik [6] showed the consistency of an
sequence of cardinals with pp of each exceeding the limit of the sequence. SWH was then revised to the statement that there was no such \( \omega_1 \) sequence. Recently, Gitik [5] proved the consistency of the existence of an \( \omega_1 \) sequence of cardinals with the pp of each exceeding the limit of the sequence.

Shelah’s believes that the negations of the versions above will eventually be proved consistent [23].

Suppose that \( \mathcal{P}_n(\kappa) < \rho < \mathcal{P}_\omega(\kappa) \) and \( F = \{ A_\alpha : \alpha < \rho \} \) satisfies \( |A_\alpha \cap A_\beta| < \nu \) for \( \alpha < \beta < \rho \). Let \( V = \rho \cup F \) and let \( F(X) = \{ \alpha : X \cap \rho \subseteq A_\alpha \} \cup \bigcap_{A \in X \cap F} A \). This is an anti-monotone function and for every \( X \) with \( |X| \geq \nu^+ \) it holds that \( |F(X)| < \nu \). Yet, there is no filtration of \( V \) to \( F_\nu \)-closed sets, as \( \rho \) would have to be contained in one of the parts, and every \( F_\nu \)-closed set which contains \( \rho \) is equal to \( V \).

Thus, a ZFC proof of Theorem 3.8 with \( \exists_n(\nu) \) in place of \( \exists_\omega(\nu) \) for some \( n > 0 \) will imply the \( n \)-th version of the Shelah Weak Hypothesis. If, however, the negations of (SWH \( n \)) are consistent for all \( n > 0 \), then \( \exists_\omega \) is optimal.

4. Splitting families of sets

Following [3] let us say that a family \( F \) of sets satisfies condition \( C(\lambda, \kappa) \) if the intersection of every subfamily of \( F \) of size \( \lambda \) is of size strictly less than \( \nu \). Miller [17] define property B (after Felix Bernstein’s “Bernstein set”) of a family of sets \( F \) as: there exists a set \( B \) such that \( B \cap A \neq \emptyset \) and \( A \not\subseteq B \) for all \( A \in F \) and proved that for every infinite \( \rho \), every \( \rho \)-uniform family of sets that satisfies \( C(\rho^+, \kappa) \) for some natural number \( n \) satisfies property B. Erdős and Hajnal [3] used Miller’s method and a theorem of Tarski [24] to prove from the assumption GCH that every \( \rho \)-uniform family that satisfies \( C(\rho^+, \nu) \) for an infinite cardinal \( \nu \) satisfies property B if \( \rho > \nu^+ \).

More results with the GCH followed along this line [14, 15, 9, 10]. Komjáth [14] proved from the GCH that for \( \nu^+ < \rho \) every \( \rho \)-uniform \( F \) which satisfies \( C(\rho^+, \nu) \) and is also almost disjoint is essentially disjoint, that is, it can be made pairwise disjoint by removing a set of size \( < \rho \) from each member of \( F \).

Komjáth [15] investigated further the property of essential disjointness which he introduced in [14] (under the name “sparseness”) and proved that the array of cardinalities of pairwise intersections in an \( \aleph_0 \)-uniform a.d family determines whether the family is essentially disjoint or not.

Hajnal, Juhász and Shelah proved in [9] a general theorem which implied many of the Miller-type theorems known at the time and derived also new combinatorial and topological consequences with it by assuming as an additional axiom a relaxation of the GCH to a weak version of the Singular Cardinal Hypothesis.

**Definition 4.1.** Let \( F \) be a family of sets.

1. The universe of \( F \) is the set \( \bigcup F \), and is denoted by \( V(F) \).
2. Given \( U \subseteq V(F) \) let \( F(U) = \{ F \cap \mathcal{P}(U) \} = \{ A : A \in F \wedge A \subseteq U \} \).
(3) $\mathcal{F}$ is $\rho$-uniform, for a cardinal $\rho$, if $|A| = \rho$ for all $A \in \mathcal{F}$.

**Definition 4.2.** Suppose that $\mathcal{F}$ is a family of sets and $V = V(\mathcal{F})$.

1. A set $U \subseteq V$ is $\kappa$-closed, for a cardinal $\kappa$, if $|A \cap U| \geq \kappa \Rightarrow A \subseteq U$ for all $A \in \mathcal{F}$.
2. Let $F : \mathcal{P}(V) \to \mathcal{P}(V)$ be defined by
   $$F(X) = \bigcup \{A : X \subseteq A \in \mathcal{F}\}.$$ 

**Claim 4.3.** For every family $\mathcal{F}$ with $V = V(\mathcal{F})$ and cardinal $\kappa$,

1. The collection $C_\kappa$ of all $\kappa$-closed sets is a notion of semi-closure over $V$ and if $\kappa$ is infinite then $D_\kappa$ is a notion of closure.
2. If $U \subseteq V$ is $\kappa$-closed and $A \in \mathcal{F}$ is not contained in $U$ then $|A \cap U| < \kappa$.
3. The function $F$ defined above is anti-monotone.
4. For every $U \subseteq V$, $U$ is $\kappa$-closed iff $U$ is $K_{F,\kappa}$-closed.
5. If $\kappa_1 \leq \kappa_2$ are cardinals then $C_{\kappa_1} \subseteq C_{\kappa_2}$.
6. If $\mathcal{F}$ is $\rho$-uniform and $\kappa \leq \rho$ then a set $U \subseteq V$ is $\kappa$-closed iff $F(U) = \{A : A \in \mathcal{F} \land |A \cap U| \geq \kappa\}$.
7. Abusing notation, we say that a subfamily $\mathcal{F}' \subseteq \mathcal{F}$ is $\kappa$-closed if $V(\mathcal{F}')$ is $\kappa$-closed and we denote by $K_{F,\kappa}(\mathcal{F})$ the family $\mathcal{F}(K_{F,\kappa}(V(\mathcal{F})))$.

**Definition 4.4 (Disjointness conditions).**

1. $\mathcal{F}$ is almost disjoint (a.d.) if $|A \cap B| < \min\{|A|, |B|\}$ for distinct $A, B \in \mathcal{F}$.
2. $\mathcal{F}$ is $\nu$-disjoint, for a cardinal $\nu$, if $|A \cap B| < \nu$ for any distinct $A, B \in \mathcal{F}$.
3. ([Miller 17], Erdős and Hajnal [3]) $\mathcal{F}$ satisfies condition $C(\theta, \nu)$ for cardinals $\theta, \nu$ if $|A| < \nu$ for all $A \in [\mathcal{F}]^\theta$ (so, $C(2, \nu)$ is $\nu$-disjointness).
4. ([Komjáth 14], Hajnal, Juhász and Shelah [9]) $\mathcal{F}$ is essentially disjoint (e.d) if there exists an assignment of subsets $B(A) \in [A]^{<|A|}$ for all $A \in \mathcal{A}$ such that the family $\{A \setminus B(A) : A \in \mathcal{F}\}$ is pairwise disjoint.
5. $\mathcal{F}$ is $\nu$-e.d if for every $A \in \mathcal{F}$ there exists an assignment $B(A) \in [A]^{\nu}$ such that $\{A \setminus B(A) : A \in \mathcal{F}\}$ is pairwise disjoint. Remark: in [14] the term sparse is used instead of “e.d”.

**Claim 4.5.**

1. Suppose $\mathcal{F}$ is $\rho$-uniform and satisfies $C(\rho^+, \nu)$ for some $\nu < \rho$. If $\nu \leq \kappa \leq \rho \leq \theta$ and $D(\theta, \kappa) = \theta$ then for every $U \in [V]^\theta$ it holds that $|\{A \in \mathcal{F} : |A \cap U| \geq \kappa\}| \leq \theta$.
2. Suppose $\nu$ is an infinite cardinal, $\mu = \beth_\omega(\nu)$, $\rho \geq \mu$ and $\mathcal{F}$ is a $\rho$-uniform family with univers $V$. If $\mathcal{F}$ satisfies $C(\rho^+, \nu)$ then for every $\theta \geq \rho$ and $\mathcal{F}' \subseteq \mathcal{F}$, $|\mathcal{F}'| = \theta \leftrightarrow |V(\mathcal{F}')| = \theta$.

**Proof.** To prove (1) fix a dense $D \subseteq [U]^{\kappa}$ of cardinality $\theta$. If $|A \cap U| \geq \kappa$ for some $A \in \mathcal{F}$ then there exists $X \in D$ such that $X \subseteq A$. By $C(\rho^+, \nu)$ and $\nu \leq \kappa$, each $X \in D$ is contained in no more than $\rho$ members of $\mathcal{F}$, so $|\{A \in \mathcal{F} : (\exists X \in D)(X \subseteq A)\}| \leq \theta \times \rho = \theta$. 

For (2) assume first that \(|V(F')| = \theta \geq \rho\). Let \(n\) be such that \(D(\theta, \kappa_n) = \theta\) (where \(\kappa_n = \beth_n(\nu)\)). Clearly \(F(U) \subseteq \{A \in F : |A \cap U| \geq \kappa_n\}\) which, by (1), has cardinality \(\leq \theta\). The converse implication is trivial. \(\square\)

By the above lemma, for all sufficiently large \(n\) it holds that \(|K_n(F')| = |F'|\) if \(|F'| \geq \rho\).

**Lemma 4.6.** Suppose \(\nu\) is an infinite cardinal, \(\mu = \beth_\mu(\nu)\), \(\rho \geq \mu\) and \(F\) is a \(\rho\)-uniform family with universe \(V\) such that \(|V| > \rho\). If \(F\) satisfies \(C(\rho^+, \nu)\) then \(V\) is \(\mu\)-filtrable.

**Proof.** The function \(F\) defined above is anti-monotone, for every \(X \in [V]^\nu\) it holds that \(|F(X)| \leq \rho\) by \(C(\rho^+, \nu)\) and \(\rho < |V|\). By Theorem 3.8, \(V\) is \(K_{F, \mu}\)-filtrable hence \(\mu\)-filtrable.

The next theorem extends Miller’s Theorem 2 from [17] when \(\nu\) is finite, and is proved in ZFC for all cardinals \(\nu\).

**Theorem 4.7.** Suppose \(\nu\) is a cardinal and \(\rho \geq \mu := \beth_\omega(\nu)\). For every \(\rho\)-uniform family \(F\) that satisfies \(C(\rho^+, \nu)\) there exists an enumeration \(F = \{A_\alpha : \alpha < |F|\}\) and a family of sets \(\{d_\alpha : \alpha < |F|\}\) such that

\[
\begin{align*}
&\text{(i) } d_\alpha \in [A_\alpha]^\rho \text{ for every } \alpha < |F| \text{ and } \beta < \alpha < |F| \Rightarrow d_\beta \cap d_\alpha = \emptyset; \\
&\text{(ii) } |\{\beta < \alpha : A_\beta \cap d_\alpha = \emptyset\}| < \rho \text{ for all } \alpha < |F|.
\end{align*}
\]

**Proof.** Let \(\lambda = |F|\) and we prove the theorem by induction on \(\lambda\). If \(|F| \leq \rho\) fix any enumeration \(F = \{A_\alpha : \alpha < \lambda\}\). Every \(\rho\)-uniform family of cardinality at most \(\rho\) has a disjoint refinement, so we can fix \(d_\alpha \in [A_\alpha]^\rho\) so that \(\alpha < \beta < \lambda \Rightarrow d_\beta \cap d_\alpha = \emptyset\). This settles (i). Conclusion (ii) holds trivially as \(\lambda \leq \rho\).

Assume that \(\lambda > \rho\). Since \(F\) satisfies \(C(\rho^+, \nu)\) it is \(\mu\)-filtrable by 4.6. For finite \(\nu\), filtrability is proved in Miller’s [17] — or, in modern terms, is an immediate consequence of the Skolem-Löwenhem theorem — and does not require 4.6.

Fix a strictly increasing and continuous chain of \(\mu\)-closed families \(\{H_i : i < \text{cf}\lambda\}\) such that \(|H_0| \geq \rho\), \(|H_i| < \lambda\) for all \(i < \text{cf}\lambda\) and \(F = \bigcup\{H_i : i < \text{cf}\lambda\}\). Put \(F'_i = H_i \setminus \bigcup_{j < i} H_j\) and let \(F_i = F'_j\) for the least \(j \geq i\) such that \(F'_j\) is not empty.

Now \(\{F_i : i < \text{cf}\rho\}\) is a partition of \(F\), \(|F_i| < \lambda\) for each \(i\) and \(A \in F_i\) implies that

\[
|A \cap \bigcup_{j < i} F_j| < \mu.
\]

Denote \(\lambda_i = |F_i|\). As \(\lambda_i < \lambda\) for every \(i < \text{cf}\lambda\) and \(F_i\) satisfies \(C(\rho^+, \nu)\), the induction hypothesis allows us to fix for each \(i\) an enumeration \(F_i = \{A_{i, \gamma} : \gamma < \lambda_i\}\) and \(\{d'_{i, \gamma} : \gamma < \lambda_i\}\) such that \(d'_{i, \gamma} \in [A_{i, \gamma}]^\rho\) for each \(\gamma < \lambda_i\), \(d'_{i, \gamma} \cap d'_{i, \delta} = \emptyset\) for all \(\gamma < \delta < \lambda_i\) it holds that and for each \(\gamma < \lambda_i\):

\[
|\{\delta < \gamma : A_{i, \delta} \cap d'_{i, \gamma} \neq \emptyset\}| < \rho.
\]
For $i < cf\lambda$ and $\gamma < \lambda_i$, define
\[
 d_{(i,\gamma)} := d'_{(i,\gamma)} \setminus \bigcup_{j<i} V(F_j).
\]

By (3) and $d'_{(i,\gamma)} \subseteq A_{(i,\gamma)}$ it holds that $|d_{(i,\gamma)}| = \rho$.

Let $I = \{\langle i, \gamma \rangle : i < cf\lambda \land \gamma < \lambda_i \}$ be well-ordered by the lexicographic ordering $<_lx$ of pairs of ordinals. As $|I| = \lambda$ and each proper initial segment of $I$ has cardinality $< \lambda$, it holds that $\langle I,<_lx \rangle$ is order-isomorphic to $\lambda$.

Identifying $\lambda$ with $\langle I,<_lx \rangle$ we now have an enumeration $F = \{A_{(i,\gamma)} : \langle i, \gamma \rangle \in I \}$. The sets $d_{(i,\delta)} \in [A_{(i,\gamma)}]^{\rho}$ have been defined in (5) for each $\langle i, \gamma \rangle \in I$ and satisfy $d_{(i,\delta)} \cap d_{(i,\gamma)} = \emptyset$ for $\langle j, \delta \rangle \neq \langle i, \gamma \rangle$ in $I$. This shows that the enumeration of $F$ and the family $\{d_{(i,\gamma)} : \langle i, \gamma \rangle \in I \}$ satisfy conclusion (i) of the theorem.

To show that (ii) also holds, let $\langle i, \gamma \rangle \in I$ be arbitrary and let
\[
 X = \{\langle j, \delta \rangle <_lx \langle i, \gamma \rangle : A_{(j,\delta)} \cap d_{(i,\gamma)} \neq \emptyset \}.
\]

If $j < i$ then $\langle j, \delta \rangle \notin X$ as $A_{(j,\delta)} \cap d_{(i,\gamma)} = \emptyset$ by (5), hence
\[
 X = \{\langle i, \gamma \rangle <_lx \langle i, \gamma \rangle : A_{(i,\delta)} \cap d_{(i,\gamma)} \neq \emptyset \}.
\]

As $d_{(i,\gamma)} \subseteq d'_{(i,\gamma)}$, it follows from (4) that $|X| < \rho$. \hfill \qed

Theorem 4.7 and Lemma 4.6 have the following corollaries.

**Corollary 4.8.** If $\nu$ is infinite and $\rho \geq \mu := \beth_{\omega}(\nu)$ then for every $\rho$-uniform family $F$:

1. If $F$ satisfies $C(\rho^+, \nu)$ then $F$ has a disjoint refinement.
2. If $F$ satisfies $C(\rho^+, \nu)$ and for a cardinal $\mu < \theta < \rho$ every subfamily of $F$ of cardinality $\rho$ is $\theta$-e.d then $F$ is $\theta$-e.d.
3. If $F$ is $\nu$-a.d then $F$ is e.d.
4. If $\rho$ is regular and $F$ is a.d and satisfies $C(\rho^+, \nu)$ then $F$ is e.d.

Corollary (1) extends to infinite $\nu$ the case of $\rho$-uniform $F$ in Theorem 3 of [14] and eliminates the GCH from a stronger form of a Theorem 6 in [3]; (3) eliminates the additional axiom $A(\nu, \rho)$ from Theorem 2.4 in [9] for sufficiently large $\rho$ — in the notation there, ED($\nu, \rho$) for all $\rho \geq \beth_{\omega}(\nu)$ in ZFC. (4) eliminates the GCH from Theorem 5 in [14] for sufficiently large $\rho$.

**Proof.** (1) follows directly from the theorem.

To prove (2) let $|F| = \lambda > \rho$ and assume, by $\mu$-filtrability, that $F$ is partitioned to $\{H_\alpha : \alpha < \lambda \}$ such that $|H_\alpha| < \lambda$ and $|A \cap \bigcup_{\beta < \alpha} V(H_\beta)| < \mu$ for all $A \in H_\alpha$. Fix, by the induction hypothesis, $B_\alpha(A) \in [A]^{<\theta}$ for all $A \in H_\alpha$ such that $\{A \setminus B_\alpha(A) : A \in H_\alpha \}$ is pairwise disjoint. For $A \in F$ let $B(A) = B_\alpha(A) \cup (A \cap \bigcup_{\beta < \alpha} V(H_\beta))$ for the unique $\alpha$ such that $A \in H_\alpha$. As $|A| \cup \bigcup_{\beta < \alpha} V(H_\beta)| < \mu$ it holds that $|B(A)| < \theta$. Clearly, $\{A \setminus B(A) : A \in F \}$ is pairwise disjoint.
For (3): every \( \nu \)-a.d. \( \rho \)-uniform family satisfies \( C(2, \nu) \) hence \( C(\rho^+, \nu) \).

If \( \mathcal{F}' \subseteq \mathcal{F} \) has cardinality \( \rho \) and is well ordered by \( \mathcal{F}' = \{ A_\alpha : \alpha < \rho \} \) let \( B(A_\alpha) = A_\alpha \cap \bigcup_{\beta < \alpha} A_\beta \). The cardinality of \( B_\alpha \) is strictly smaller than \( \rho \) and \( \{ A_\alpha \setminus B_\alpha : \alpha < \rho \} \) is disjoint. Now (3) follows from (2).

For (4), as \( \alpha \) is regular and \( \mathcal{F} \) is a.d, every subfamily of \( \mathcal{F} \) of cardinality \( \rho \) is e.d and now use (2).

Neither of the corollaries above used conclusion (ii) of Theorem 4.7. A generalization of this condition is used for the next theorem on \textit{list conflict-free numbers}.

**Definition 4.9.**

1. A coloring \( c \) of \( V(\mathcal{F}) \) for a family of sets \( \mathcal{F} \) is conflict free if for every \( A \in \mathcal{F} \) there is \( x \in A \) such that \( c(x) \neq c(y) \) for all \( y \in A \setminus \{ x \} \).
2. The conflict free number \( \chi_{CF}(\mathcal{F}) \) is the smallest cardinal \( \kappa \) for which there exists a conflict-free coloring \( c : V(\mathcal{F}) \to \kappa \).
3. The list-conflict-free number \( \chi_{LCF}(\mathcal{F}) \) is the smallest cardinal \( \kappa \) such that for every assignments \( L \) of sets \( L(v) \) to every \( v \in V \) which satisfies \( |L(v)| \geq \kappa \) there exists a conflict-free coloring \( c \) on \( V \) which satisfies \( c(v) \in L(v) \) for all \( v \in V \).

Clearly, every \( \nu \)-e.d family \( \mathcal{F} \) has a coloring by \( \nu \)-colors such that all but \( < \nu \) colors in each \( A \in \mathcal{F} \) are unique in \( A \).

**Claim 4.10.** Let \( \rho \geq \theta > 0 \) be cardinals and let \( \mathcal{F} \) is a nonempty \( \rho \)-uniform family. Suppose there exists an enumeration \( \mathcal{F} = \{ A_\alpha : \alpha < \lambda \} \) and a family \( \{ d_\alpha : \alpha < \lambda \} \) such that

(i) \( d_\alpha \in [A_\alpha]^\theta \) for every \( \alpha < \lambda \) and \( \beta < \alpha < \lambda \Rightarrow d_\beta \cap d_\alpha = \emptyset \);

(ii) \( |\{ \beta < \alpha : A_\beta \cap d_\alpha \neq \emptyset \}| \leq \rho \) for all \( \alpha < \lambda \).

Then

1. For every list-assignment \( L(v) \) for \( v \in V \) with \( |L(v)| \geq \rho^+ \) there exists a coloring \( c \in \prod_{v \in V} L(v) \) such that for every \( \alpha < \lambda \) and every \( x \in D_\alpha \) the color \( c(x) \) is unique in \( A_\alpha \).
2. \( \chi_{LCF}(\mathcal{F}) \leq \rho^+ \).

**Proof.** Assume that \( L(v) \) is given with \( |L(v)| \geq \rho^+ \) for all \( v \in V(\mathcal{F}) \). For each \( \alpha < \lambda \) enumerate \( d_\alpha = \{ x_{(\alpha, i)} : i < \theta \} \). Define \( c \in \prod_{v \in V} L(v) \) arbitrarily on \( V \setminus \bigcup_{\alpha < \lambda} d_\alpha \). Next define \( c(x_{(\alpha, i)}) \) by induction on the lexicographic ordering on \( \lambda \times \theta \). Suppose \( c(x_{(\beta, j)}) \) is defined for \( (\beta, j) <_{lx} (\alpha, i) \). Define

\[
F(x_{(\alpha, i)}) = \{ c(y) : (\exists \beta \leq \alpha) [x_{(\alpha, i)} \in A_\beta \land y \in A_\beta \land c(y) \text{ is defined}] \}
\]

and choose

\[
c(x_{(\alpha, i)}) \in L(x_{(\alpha, i)}) \setminus F(x_{(\alpha, i)}).
\]

By the assumption (3), \( |F(x_{(\alpha, i)})| \leq \rho \), so as \( |L(x_{(\alpha, i)})| \geq \rho^+ \) the set \( L(x_{(\alpha, i)}) \setminus F(x_{(\alpha, i)}) \) is not empty and therefore \( c(x_{(\alpha, i)}) \) can be chosen as required in (9).
To prove that the color $c(x_{(\alpha,i)})$ is unique in $A_\alpha$ for every $\alpha < \lambda$ and $i < \theta$ suppose that $\alpha < \lambda$ and $i < \theta$ are given and $z \in A_\alpha \setminus \{x_{(\alpha,i)}\}$. If $c(z)$ is defined at stage $\langle \alpha,i \rangle$ of the inductive definition, then $c(z) \in F(x_{(\alpha,i)})$ by (8) and hence $c(x_{(\alpha,i)}) \neq c(z)$ by (9).

Otherwise, there is some $\beta > \alpha$ and $j < \theta$ such that $z = x_{(\beta,j)}$. This means that $x_{(\beta,j)} \in A_\alpha$. As $c(x_{(\alpha,i)})$ is defined at stage $\langle \beta,j \rangle$, it follows by (8) that $c(x_{(\alpha,i)}) \in F(x_{(\beta,j)})$ and $c(z) = c(x_{(\beta,j)}) \neq c(x_{(\alpha,i)})$ by (9).

By combining Theorem 4.7 with the Claim 4.10 we get:

**Corollary 4.11.** For every infinite $\nu$ and $\rho \geq \beth_\omega(\nu)$, every $\rho$-uniform family which satisfies $C(\nu, \rho^+)$ satisfies $\chi_{\ell CF}(\mathcal{F}) \leq \rho^\omega$.

Hajnal, Juhász, Soukup and Szentmihóly [10] proved that a $\kappa$-uniform family which is r-a.d for finite $r$ has countable conflict-free number. See also Komjáth [16], where the uniformity condition is removed.

### 4.1. Comparing almost disjoint families.

We conclude this section with a generalization of a theorem by Komjáth on comparing families of sets.

**Definition 4.12.** Two families of sets $\mathcal{F}$ and $\mathcal{G}$ are similar if there is a bijection $f : \mathcal{F} \to \mathcal{G}$ such that $|A \cap B| = |f(A) \cap f(B)|$ for all $A, B \in \mathcal{F}$.

Komjáth [15] addressed the following question: which combinatorial properties of $\aleph_0$-uniform families are invariant under similarity? Property B and the existence of a disjoint refinement are not similarity invariant, as one can replace $A \in \mathcal{F}$ by $A \cup D(A)$ where $D(A) \cap D(B) = \emptyset$ for distinct $A, B \in \mathcal{F}$ and obtain an equivalent family which has a disjoint refinement from a family which does not satisfy property B. However, Komjáth proved:

**Theorem 4.13** (Komjáth [15]). Suppose $\mathcal{F}$ and $\mathcal{G}$ are $\aleph_0$-uniform, almost disjoint and equivalent. If $\mathcal{F}$ is e.d. then also $\mathcal{G}$ is e.d. More generally: $\mathcal{G}$ is e.d if $\mathcal{F}$ is and there exists a bijection $f : \mathcal{F} \to \mathcal{G}$ such that $|A \cap B| \geq |f(A) \cap f(B)|$ for all $A, B \in \mathcal{F}$.

The cardinal $\aleph_0$ is strong limit and regular, that is, strongly inaccessible. As a corollary of the next theorem, Komjáth’s theorem extends from $\aleph_0$ to all strongly inaccessible cardinals, but actually more is proved.

**Theorem 4.14.** Suppose $\mu$ is a strong limit cardinal and $\rho \geq \mu$. Suppose $\mathcal{F} = \{A_\alpha : \alpha < \lambda\}, \mathcal{G} = \{B_\alpha : \alpha < \lambda\}$ are $\rho$-uniform and for every $\alpha < \beta < \lambda$ it holds that

\[(10) \quad |A_\alpha \cap B_\beta| \geq |B_\alpha \cap B_\beta|.

Then if $\mathcal{F}$ is $< \mu$-essentially disjoint, so is $\mathcal{G}$, provided that every subfamily of $\mathcal{G}$ of cardinality $\rho$ is $\mu$-e.d.

**Proof.** Let $C_\alpha \in [A_\alpha]^{< \mu}$ be fixed for all $\alpha < \lambda$ such that $(A_\alpha \setminus C_\alpha) \cap (A_\beta \setminus C_\beta) = \emptyset$ for all $\alpha < \beta < \lambda$. We prove by induction on $\lambda \geq \rho$ that $\mathcal{G}$ is
<μ-e.d. If λ = ρ then G is μ-e.d by the assumption that every subfamily of G of cardinality ρ is μ-e.d.

For singular λ > ρ the conclusion follows from the induction hypothesis by Proposition 5 in [14].

Assume then that λ > ρ is regular. We may assume that V(F) = V(G) ⊆ λ. As F is <μ-a.d. it follows that |V(F)| = λ, so we assume that actually V(F) = λ.

Let M = {Mα : α < λ} be an elementary chain of models Mα ⊆ (H(Ω), ∈, <) for a sufficiently large regular Ω such that F, G ∈ M0 and also the function Aα → Cα belongs to M0, such that:

(1) ρ ⊆ M0 and α < β ⇒ Mα ⊆ Mβ.
(2) Mα ∩ λ ∈ λ and |Mα| ≤ Mα.
(3) (Mβ : β ≤ α) ∈ Mα+1.

Let δ(α) = Mα ∩ λ.

For each ordinal γ < δ(α) there is at most one β < λ such that γ ∈ Aβ \ Cβ. Thus, whenever γ ∈ Aβ \ Cβ and γ < δ(α) for some α < λ, by elementarity Aβ ∈ Mα and hence Aβ ⊆ δ(α). If β ≥ δ(α), then, it follows that Aβ ∩ δ(α) ⊆ Cβ.

For each β < λ let κ(β) = |Cβ|. We get:

(11) β ≥ δ(α) ⇒ |Aβ ∩ δ(α)| ≤ κ(β).

and, since γ < δ(α) ⇒ Aγ ⊆ δ(α),

(12) β ≥ δ(α) > γ ⇒ |Aβ ∩ Aγ| ≤ κ(β).

By assumption (10), condition (12) can be copied over to G, that is:

(13) β ≥ δ(α) > γ ⇒ |Bβ ∩ Bγ| ≤ κ(β).

Claim 4.15. There exists a closed unbounded E ⊆ {δ(α) : α < λ} such that for every δ ∈ E and β ≥ δ it holds that

(14) |Bβ ∩ δ| < μ.

Proof of Claim. Suppose to the contrary that S ⊆ {δ(α) : α < λ} is stationary and that β(δ) ≥ δ is chosen for δ ∈ S such that |Aβ(δ) ∩ δ| ≥ μ. We may assume that δ1 < δ2 ⇒ β(δ1) < δ2 for δ1, δ2 ∈ S by intersecting S with a club, and by thinning S out assume that κ(β(δ)) = |Cβ(δ)| = κ(∗) is fixed for all δ ∈ S.

The set S′ = S ∩ acc S is stationary. If δ ∈ S′ then for all sufficiently large regular κ < μ it holds that Mδ ∩ [δ]κ is dense in [δ]κ by Lemma 2.9, hence, as |Bβ(δ) ∩ δ| ≥ μ, for each sufficiently large regular κ < μ there is X ∈ Mδ such that |X| = κ and X ⊆ Bβ(δ).

Define F(δ) = X, for each δ ∈ S′, such that X ⊆ Bβ(δ), X ∈ Mδ and |X| > κ(∗). By Fodor’s Lemma, we can assume that F is fixed on a stationary S′ ⊆ S′. Fix, then, δ1 < δ2 in S′ with F(δ1) = F(δ2) = X ∈ Mδ1.
But now $|B_{\beta(\delta_1)} \cap B_{\beta(\delta_2)}| \geq |X| > \kappa(*)$, and as $\beta(\delta_1) < \delta_2$ this contradicts equation (13) above.

The conclusion of the theorem now follows from the induction hypothesis.

In the case that $\rho = \mu = cf\mu$ the assumption that every subfamily of $G$ of cardinality $\rho$ is $\mu$-e.d is, of course, superfluous, as $G$ is a.d by the similarity with $F$ and every $\rho$-uniform a.d family of regular size $\rho$ is e.d.

**Concluding remarks.** After Cohen’s and Easton’s results showed that the GCH and many of its instances were not provable in ZFC the impression may have been that the GCH or some other additional axiom was required for proving general combinatorial theorems for all infinite cardinals. It now seems that, at least the combinatorics of splitting families of sets, which for a long time was hindered by the “totally independent” aspect of cardinal arithmetic, is, in the end, amenable to investigation on the basis of ZFC. We expect that the method presented here will be useful proving additional absolute combinatorial relations which generalize known combinatorial relations between finite cardinals to infinite ones.

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