Mass-constrained ground states of the stationary NLSE on periodic metric graphs

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Abstract. We investigate the existence of ground states with fixed mass for the nonlinear Schrödinger equation with a pure power nonlinearity on periodic metric graphs. Within a variational framework, both the $L^2$-subcritical and critical regimes are studied. In the former case, we establish the existence of global minimizers of the NLS energy for every mass and every periodic graph. In the critical regime, a complete topological characterization is derived, providing conditions which allow or prevent ground states of a certain mass from existing. Besides, a rigorous notion of periodic graph is introduced and discussed.

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1. Introduction

Originally fuelled by a wide variety of physical applications, the theory of quantum graphs has nowadays become a prominent topic of research. Moving from linear dynamics on branched structures (see for instance [19,23] and the monograph [11]), nonlinear problems have been extensively studied first on star-graphs [1–4], and more recently on general non-compact metric graphs with at least one half-line [6–9,12,17,18,28–30], as well as on compact graphs [15,16].

Among the whole theory, periodic graphs appear to gather a significant interest. As linear problems and thoroughly spectral analysis has been carried on for instance in [20–22], a first investigation of NLS equation on ladder-type graphs (Fig. 1a) was initiated in [25], while spectral problems for the graph in Fig. 1b were discussed both in [24] and in [27]. Recently, a variational exploration of the NLS energy on general periodic graphs has been developed in [26], where a generalized Nehari manifold approach is used to establish for the first time the existence of a global minimizer.
In this paper we investigate the existence of ground states of the NLS energy functional
\[
E(u, G) = \frac{1}{2} \|u'|^2_{L^2(G)} - \frac{1}{p} \|u|^p_{L^p(G)} = \frac{1}{2} \int_G |u'|^2 \, dx - \frac{1}{p} \int_G |u|^p \, dx
\]  
(1)
on a general periodic metric graph \(G\), with \(p \in (2, 6]\), under the mass constraint
\[
\|u\|^2_{L^2(G)} = \mu > 0.
\]  
(2)

In what follows, we restrict ourselves to consider only real-valued functions.

The class of graphs we consider here is rather general. Roughly speaking, we say that a graph \(G\) is periodic if it is built of an infinite number of copies of a fixed compact graph, the periodicity cell, glued together along one direction, i.e., we deal with structures enjoying a \(Z\)-symmetry (see Sect. 2).

We briefly recall that a connected metric graph \(G = (V(G), E(G))\) is a connected space made up by segments of line, the edges, glued together at some points, the vertices, according to the topology of the graph. Both multiple edges and self-loops are possible. On every edge \(e \in E(G)\), a coordinate \(x_e\) is defined, providing an identification of \(e\) with a real interval \([0, \ell_e]\). For our purposes here, we always have \(\ell_e < +\infty\), i.e., all edges in the graph are bounded. Moreover, without loss of generality, we assume that all the periodic graphs we consider possess at least one vertex of degree at least 3, in order to exclude the situation \(G = \mathbb{R}\).

Within this framework, a function \(u : G \to \mathbb{R}\) can be seen as a family \(\{u_e\}_{e \in E(G)}\), where \(u_e : I_e \to \mathbb{R}\) defines the restriction of \(u\) to the edge \(e\), and functional spaces can be defined in the natural way
\[
L^p(G) := \{u : G \to \mathbb{R} : u_e \in L^p(I_e), \forall e \in E(G)\}
\]
\[
H^1(G) := \{u : G \to \mathbb{R} \text{ continuous} : u_e \in H^1(I_e), \forall e \in E(G)\}.
\]
Note that, since all \(u_e\) are one-dimensional, the continuity condition we introduce in the definition of \(H^1(G)\) is meant to impose \(u\) to be continuous at the vertices. Moreover, as we are looking for functions satisfying (2), it is useful to introduce, for \(\mu > 0\), the mass-constrained space
\[
H^1_\mu(G) := \{u \in H^1(G) : \|u\|^2_{L^2(G)} = \mu\}.
\]

By ground states we mean global minimizers of the energy (1), that are solutions, for a suitable Lagrange multiplier \(\lambda\), of the stationary Schrödinger equation with focusing nonlinearity
\[
u'' + |u|^{p-2}u = \lambda u
\]  
(3)
on each edge of $G$, with homogeneous Kirchhoff conditions at every vertex, that is, the oriented sum of all derivatives entering the vertex is equal to zero

$$\sum_{e \ni v} \frac{du}{dx_e}(v) = 0$$

(see [7,10] for a discussion on this condition).

Our approach to the problem is variational. However, the strategy we follow here is significantly different from the one in [26], as extending similar arguments to the mass-constrained setting is far from obvious.

In the $L^2$-subcritical regime $p \in (2,6)$, we prove existence of ground states, always realizing strictly negative energy, for every value of the parameter $\mu$. Indeed, denoting by

$$\mathcal{E}_G(\mu) := \inf_{u \in H^1_{\mu}(\mathcal{G})} E(u, \mathcal{G})$$

the ground state energy level of $E$, our first result is the following.

**Theorem 1.1.** Let $\mathcal{G}$ be a periodic graph and $p \in (2,6)$. Then, for every $\mu > 0$

$$-\infty < \mathcal{E}_G(\mu) < 0$$

and there always exists a ground state with mass $\mu$, i.e., $u \in H^1_{\mu}(\mathcal{G})$ such that

$$\mathcal{E}_G(\mu) = E(u, \mathcal{G}).$$

Theorem 1.1 unveils a similarity with the real line $\mathbb{R}$, for which it is known that ground states of the energy are the unique (up to symmetries) solutions of (3) with prescribed mass (see for instance [14] and Sect. 3 below).

Actually, it is at the critical exponent $p = 6$ that the problem exhibits a wider variety of behaviours, as the topology of the graph enters the game. Recall that (see [14] and Sect. 4 here), when $p = 6$, both for the real line $\mathbb{R}$ and half-line $\mathbb{R}^+$, ground states exist if and only if the mass is equal to a threshold value, denoted by $\mu_{\mathbb{R}}$, $\mu_{\mathbb{R}^+}$, respectively.

For a general periodic graph $\mathcal{G}$, we show that a critical mass $\mu_G$ naturally arises as well, its actual value being determined by the specific structure of the graph (see Sect. 4). Moreover, the relation between this threshold and the existence of ground states is more complex than in the case of $\mathbb{R}$ and $\mathbb{R}^+$, and the situation changes with respect to the graph we are dealing with.

A key-role in this context is played by the following topological condition, denoted by $(H_{\text{per}})$

$$(H_{\text{per}}) : \text{removing any edge of } e \in E(\mathcal{G}) \text{ generates only non-compact connected components.}$$

Assumptions of this fashion have been introduced for the first time in [6] for graphs with half-lines, named assumption (H), of which $(H_{\text{per}})$ here constitutes the periodic version (for a detailed overview on equivalent formulations of (H) we refer to [10]). The key idea behind this condition is that, if $\mathcal{G}$ satisfies $(H_{\text{per}})$, then, for every point $x \in \mathcal{G}$, two disjoint paths of infinite length originating at $x$ exist.
We then state our theorems. Recall that a terminal edge denotes an edge incident to a vertex of degree 1.

**Theorem 1.2.** Let $G$ be a periodic graph and $p = 6$. Then:

(i) if $G$ satisfies assumption $(H_{per})$ (Fig. 2a), then $\mu_G = \mu_R$ and

$$\mathcal{E}_G(\mu) = \begin{cases} 0 & \text{if } \mu \leq \mu_R \\ -\infty & \text{if } \mu > \mu_R \end{cases}$$

(ii) if $G$ has a terminal edge (Fig. 2b), then $\mu_G = \mu_{R^+}$ and

$$\mathcal{E}_G(\mu) = \begin{cases} 0 & \text{if } \mu \leq \mu_{R^+} \\ -\infty & \text{if } \mu > \mu_{R^+} \end{cases}$$

Moreover, the infimum is never attained.

**Theorem 1.3.** Let $G$ be a periodic graph violating assumption $(H_{per})$ and with no terminal edge (Fig. 2c), and $p = 6$. If $\mu_G < \mu_R$, then ground states with mass $\mu$ exist if and only if $\mu \in [\mu_G, \mu_R]$.

Theorems 1.2–1.3 provide a complete topological characterization of the existence of ground states in the critical setting (similarly to what reported in [8] for graphs with half-lines).

On the one hand, it turns out that graphs satisfying $(H_{per})$ behave almost as $\mathbb{R}$, whereas the ones with a terminal edge fake the half-line $\mathbb{R}^+$, as the values of the corresponding thresholds are respectively the same. However, for both these classes of graphs ground states never exist, even at the critical masses. Thus, both $(H_{per})$ and the presence of a terminal edge provide topological sufficient conditions preventing the existence of global minimizers.

On the other hand, for all other graphs, global minimizers actually exist for a whole interval of masses provided $\mu_G < \mu_R$, and we also show that the class of graphs fulfilling this assumption is nonempty (see Proposition 4.2). However, it is still unclear whether such a condition is immediately satisfied.
by violating \((H_{\text{per}})\), so up to now it is necessary to impose it to recover the above existence result.

To conclude this Introduction, we wish to stress once more the fact that all our results hold for periodic graphs in which each periodicity cell shares connections with exactly two of the others, so that the whole graph displays a \(\mathbb{Z}\)--symmetry. If such a condition is removed, and one allows for repetitions of the periodicity cell along more than one direction, the situation drastically changes and the possible behaviours seem to be sensitively varying (see Fig. 3 for some examples). Further investigations in this direction have been recently initiated in [5] for the so-called doubly periodic graphs (i.e., graphs with a \(\mathbb{Z}^2\)-symmetry) as the two-dimensional grid in Fig. 3b, where threshold phenomena have been shown to appear not only at \(p = 6\) but for a whole interval of exponents.

The paper is organised as follows. In Sect. 2 we discuss the definition of periodicity, characterizing the class of graphs we are considering. Section 3 deals with the subcritical regime, whereas Sect. 4 is devoted to the critical case. Finally, Appendix A runs through the generality of the definition of periodicity given in Sect. 2 from a graph theoretical point of view.

### 2. Periodic graphs: formal definition

The aim of this section is to provide a rigorous definition of what we mean by periodic graphs. To this purpose, let us begin by recalling the approach of [26], where the periodicity of a graph is described in terms of a proper group action (see also Chapter 4 in [11]). Indeed, let \(\mathcal{G}\) be a connected metric graph with sets of vertices and edges \(V(\mathcal{G}), E(\mathcal{G})\), respectively, and consider an action of the group \(\mathbb{Z}^n\) on \(\mathcal{G}\)

\[
\mathbb{Z}^n \times \mathcal{G} \ni (g, x) \mapsto gx \in \mathcal{G}
\]
which is a graph automorphism, i.e., it maps vertices into vertices and edges into edges, and it preserves the lengths, i.e., for every interval $I \subset e \in E(G)$ and every $g \in \mathbb{Z}^n$, both $I$ and $gI$ has the same length, $|I| = |gI|$.

Then, following [26], $G$ equipped with the action of $\mathbb{Z}^n$ is said to be periodic if the action is

1. **free**, that is, $gx = x \implies g = 0$;
2. **discrete**, that is, for every $x \in G$, there is a neighbourhood $U$ of $x$ such that $gx \notin U$, for every $g \in \mathbb{Z}^n/\{0\}$;
3. **co-compact**, that is, there exists a compact set $Y \subset G$ such that $G = \bigcup_{g \in \mathbb{Z}^n} gY$.

Particularly, the co-compactness implies that the whole graph $G$ can be seen as the orbit through the action of $\mathbb{Z}^n$ of a fixed subset of $G$, which is called a fundamental domain.

Therefore, the previous definition follows the strategy of identifying a periodic cell that repeats itself within a given graph $G$ under a $\mathbb{Z}^n$–symmetry. However, for the purposes of the present paper, we decide to exploit the inverse direction, introducing a dual definition of periodicity which moves from a given compact graph and prescribes a way to glue together infinitely many copies of it to form a periodic structure. Such a procedure is actually equivalent to a special case in the general definition above, namely the case of periodic graphs sharing a $\mathbb{Z}$-symmetry (see Remark 2.2 below).

Let then $K$ be a compact graph, i.e., a graph with a finite number of vertices and edges, all of finite length. Let $D$ (donors) and $R$ (receivers) be two non-empty subsets of the set $V(K)$ of vertices of $K$, and $\sigma : D \rightarrow R$ be a function (from donors to receivers) such that

(i) $D \cap R = \emptyset$;
(ii) $\sigma$ is bijective.

Consider now an infinite number of copies of $K$, indexed by the integers, $\{K_i\}_{i \in \mathbb{Z}}$, and let $D_i, R_i$ be the subsets of $V(K_i)$ corresponding to $D, R$, respectively, for every $i \in \mathbb{Z}$. Setting $G := \bigcup_{i \in \mathbb{Z}} K_i$, and thinking of $\sigma$ as a map from $D_i$ to $R_{i+1}$, for every $i$, we introduce the relation

$$v \sim w \iff \begin{cases} v = w & \text{if } v, w \in K_i, \text{ for some } i \in \mathbb{Z} \\ \sigma(v) = w & \text{if } v \in D_i, w \in R_{i+1}, \text{ for some } i \in \mathbb{Z} \\ \sigma(w) = v & \text{if } v \in R_{i+1}, w \in D_i, \text{ for some } i \in \mathbb{Z} \end{cases}$$

for every $v, w \in G$.

It is immediate to verify that the above relation is well-defined and it is in fact an equivalence on $G$. We thus give the following definition.

**Definition 2.1.** Let

$$G := G/\sim$$

denote the quotient space of $G$ with respect to $\sim$. We say that $G$ is a periodic graph with periodicity cell $K$ and pasting rule $\sigma$. 
Remark 2.1. As a first example, note that also the real line $\mathbb{R}$ can be seen as a periodic graph in the spirit of Definition 2.1, letting, for instance, $K = [0, 1]$, $D = \{0\}$, $R = \{0\}$ and $\sigma(1) = 0$. However, as anticipated in the Introduction, to avoid such situation, we always assume that $G$ has at least one vertex of degree at least 3.

As $\sigma$ does not involve the edges of $K_i$, for any $i \in \mathbb{Z}$, the sets of vertices and edges of $G$ are

$$V(G) = \left( \bigcup_{i \in \mathbb{Z}} V(K_i) \right) / \sim \quad E(G) = \bigcup_{i \in \mathbb{Z}} E(K_i).$$

Moreover, we highlight that, for every $i$, the pasting rule $\sigma$ always maps $D_i$ into $R_{i+1}$. Henceforth, by construction, the only periodic graphs we are considering are the ones in which each periodicity cell shares connections with exactly two of the others, i.e., the graph enjoys a $\mathbb{Z}$-symmetry.

It is immediate to verify that Definition 2.1 can be seen as a particular case of the one given in [26] with $n = 1$. Indeed, if $G$ is periodic as in Definition 2.1, then every point of the graph belongs to a certain copy of the periodicity cell $K$. Hence, given any $x \in K$, let us denote by $x_i \in G$ the corresponding point of $G$ belonging to the $i$-th copy of $K$, $K_i$. According to this notation, it is readily seen that the group action given by

$$\mathbb{Z} \times G \to G \quad (k, x_i) \mapsto x_{k+i} \quad \forall i \in \mathbb{Z}$$

is free, discrete and co-compact and that $K$ is a fundamental domain.

Remark 2.2. We can actually show that Definition 2.1 is equivalent to the one in [26] with $n = 1$.

To this aim, assume that $G$ is periodic according to [26] with $n = 1$ and let $Y$ be a fundamental domain satisfying the additional property that, for every $x \in Y$, we have $-1x \notin Y$ (being this not restrictive). Since $G$ is connected, there must exist at least one point $x \in Y$ such that, for every neighbourhood $U \subset G$ of $-1x$, then $U \cap Y \neq \emptyset$. Thus letting

$$D := \{ x \in Y : U \cap Y \neq \emptyset, \forall U \text{ neighbourhood of } -1x \}$$
$$R := \{ y \notin Y : y = -1x \text{ for some } x \in D \}$$
$$K := Y \cup R$$
$$\sigma : D \to R \quad \sigma(x) = -1x,$$

we get that $G$ is the periodic graph arising from Definition 2.1 with periodicity cell $K$ and pasting rule $\sigma$.

We end up this section by showing that properties (i)–(ii) we require introducing $D, R$ and $\sigma$ imply that $\text{diam}(G) = +\infty$.

Indeed, let $x, y \in K$ be such that $x \in D, y \in R$ and $\sigma(x) = y$. $K$ being connected, let $\gamma \subset K$ be the smallest path joining $x$ and $y$. Denote by $x_i, y_i \in K_i$ the points corresponding to $x, y$ belonging to the $i$-th copy of $K$, and by $\gamma_i \subset K_i$ the corresponding copy of $\gamma$. As $\sigma(x_i) = y_{i+1}$, so that, building up
According to Definition 2.1, $x_i$ and $y_{i+1}$ becomes the same point, it follows that the union of all $\gamma_i$ is a connected path in $\mathcal{G}$, leading to

$$\text{diam}(\mathcal{G}) \geq \left| \bigcup_{i \in \mathbb{Z}} \gamma_i \right| = \sum_{i \in \mathbb{Z}} |\gamma_i| = +\infty$$

since $|\gamma_i| = |\gamma| > 0$, for every $i \in \mathbb{Z}$.

Let us stress the fact that assumptions (i)–(ii) are made to ease the above general definition, but it has to be shown that this does not raise any restriction on the class of graphs we are dealing with. We address this point throughout Appendix A, where a wider discussion of the generality of Definition 2.1 from the standpoint of graph theory is performed.

### 3. The subcritical regime $p \in (2, 6)$

#### 3.1. Preliminaries and compactness

Before going on, let us briefly recall some known facts about the stationary nonlinear Schrödinger equation with a subcritical exponent that will play an important role in what follows.

When $\mathcal{G} = \mathbb{R}$, it is well-known (see for instance [6]) that, for every mass $\mu > 0$, there always exists a ground state, the soliton $\phi_\mu$, given by

$$\phi_\mu(x) = \mu^\alpha \phi_1(\mu^\beta x), \quad \alpha = \frac{2}{6-p}, \quad \beta = \frac{p-2}{6-p}$$

where $\phi_1$ is defined as

$$\phi_1(x) = A_p \text{sech}^{\alpha/\beta}(a_p x)$$

with $A_p, a_p > 0$. Moreover, $E(\phi_\mu, \mathbb{R}) < 0$, for every $\mu > 0$.

If $\mathcal{G} = \mathbb{R}^+$ the behaviour is the same, with solitons replaced by the half-solitons, namely their restrictions to the half-line.

The forthcoming analysis makes use also of the Gagliardo–Nirenberg inequality

$$\|u\|^p_{L^p(\mathcal{G})} \leq C_{\mathcal{G}, p} \|u\|^{\frac{p}{2}+1}_{L^2(\mathcal{G})} \|u'\|^{\frac{p}{2}-1}_{L^2(\mathcal{G})}$$

holding for every non-compact graph $\mathcal{G}$, $u \in H^1(\mathcal{G})$ and $p \geq 2$ (we refer to [7] for further details). Here $C_{\mathcal{G}, p} > 0$ depends only on $\mathcal{G}$ and $p$. We state now the following result, proving that globally minimizing sequences of the NLS energy functional are strongly compact in $H^1_\mu(\mathcal{G})$ up to translations, whenever the infimum of (1) is strictly negative.

**Proposition 3.1.** Let $\mathcal{G}$ be a periodic graph, $p \in (2, 6)$ and $\mu > 0$. Let $\{u_n\}_{n \in \mathbb{N}} \subset H^1_\mu(\mathcal{G})$ be a minimizing sequence for $E$ such that, for every $n \in \mathbb{N}$, there exists $x_n \in K_0$ so that $\|u_n\|_{L^\infty(\mathcal{G})} = u_n(x_n)$. If (4) holds, then there exists $u \in H^1_\mu(\mathcal{G})$ such that $u_n \rightharpoonup u$ strongly in $H^1(\mathcal{G})$ and $u$ is a ground state.
Proof. Let \( \{u_n\}_{n \in \mathbb{N}} \) be a minimizing sequence for \( E \) in \( H^1_\mu(G) \) as above. Plugging Gagliardo–Nirenberg inequality (9) in (1), we have
\[
E(u_n, G) \geq \frac{1}{2} \|u_n^\prime\|^2_{L^2(G)} - \frac{C_{G,p} \mu^{\frac{p}{2} + \frac{1}{2}}}{p} \|u_n^\prime\|_{L^2(G)}^{p-1} \tag{10}
\]
and, since \( p \in (2, 6) \), this implies that \( \{u_n\}_{n \in \mathbb{N}} \) is bounded in \( H^1(G) \). Hence, there exists \( u \in H^1(G) \) so that (up to subsequences) \( u_n \rightharpoonup u \) in \( H^1(G) \) and \( u_n \to u \) in \( L^\infty(G) \). Moreover, by weak lower semicontinuity,
\[
\|u^\prime\|_{L^2(G)} \leq \liminf_{n \to +\infty} \|u_n^\prime\|_{L^2(G)} \quad \text{and} \quad \|u\|_{L^2(G)} \leq \liminf_{n \to +\infty} \|u_n\|_{L^2(G)} = \sqrt{\mu}.
\tag{11}
\]
We first prove that \( u \neq 0 \). Suppose, by contradiction, \( u \equiv 0 \). Then, since \( \|u_n\|_{L^\infty(G)} = u_n(x_n) \to 0 \) when \( n \to +\infty \), as \( x_n \in K_0 \) for every \( n \in \mathbb{N} \),
\[
\|u_n\|^p_{L^p(G)} \leq \|u_n\|^{p-2}_{L^\infty(G)} \mu \to 0 \quad \text{for} \quad n \to +\infty \tag{12}
\]
and
\[
0 > \mathcal{E}_G(\mu) = \lim_{n \to +\infty} E(u_n, G) \geq - \lim_{n \to +\infty} \frac{1}{p} \|u_n\|^p_{L^p(G)} = 0
\]
provides the contradiction we seek.

Thus, either \( 0 < \|u\|^2_{L^2(G)} < \mu \) or \( \|u\|^2_{L^2(G)} = \mu \). If the latter case occurs, then \( u \in H^1_\mu(G) \), \( u \) is a ground state of \( E \) and \( u_n \to u \) strongly in \( H^1(G) \). Let us thus prove that the former never happens, adapting the argument in the proof of Lemma 3.2 in [5].

Suppose by contradiction \( \|u\|^2_{L^2(G)} =: m < \mu \). By the Brezis–Lieb Lemma [13], we have, for \( n \) sufficiently large
\[
E(u_n, G) = E(u_n - u, G) + E(u, G) + o(1) \tag{13}
\]
and by weak convergence in \( L^2(G) \) of \( u_n \) to \( u \),
\[
\|u_n - u\|^2_{L^2(G)} = \|u_n\|^2_{L^2(G)} + \|u\|^2_{L^2(G)} - 2 < u_n, u >_{L^2(G)} = \mu - \|u\|^2_{L^2(G)} + o(1) = \mu - m + o(1). \tag{14}
\]
Therefore, we have
\[
\mathcal{E}_G(\mu) \leq E \left( \frac{\sqrt{\mu}}{\|u_n - u\|_{L^2(G)}}, (u_n - u), G \right)
\]
\[
= \frac{1}{2} \frac{\mu}{\|u_n - u\|^2_{L^2(G)}} \|u_n^\prime - u^\prime\|^2_{L^2(G)} - \frac{1}{p} \frac{\mu^2}{\|u_n - u\|^p_{L^p(G)}} \|u_n - u\|^p_{L^p(G)}
\]
\[
= \frac{\mu}{\|u_n - u\|^2_{L^2(G)}} \left( \frac{1}{2} \|u_n^\prime - u^\prime\|^2_{L^2(G)} - \frac{1}{p} \frac{\mu^{p-2}}{\|u_n - u\|^p_{L^p(G)}} \|u_n - u\|^p_{L^p(G)} \right)
\]
\[
< \frac{\mu}{\|u_n - u\|_{L^2(G)}} E(u_n - u, G)
\]
with the last inequality coming from the fact that \(\|u_n - u\|^2_{L^2(G)} < \mu\). Taking the liminf and combining with (14), we get
\[
\liminf_{n \to +\infty} E(u_n - u, \mathcal{G}) \geq \frac{\mu - m}{\mu} \mathcal{E}_G(\mu).
\] (15)
Moreover, similar calculations lead to
\[
\mathcal{E}_G(\mu) \leq E\left(\sqrt{\frac{\mu}{m}} u, \mathcal{G}\right) = \frac{\mu}{m}\left(\frac{1}{2}\|u''\|^2_{L^2(G)} - \frac{1}{p} \frac{\mu^\frac{2}{p} - 1}{m^\frac{2}{p} - 1} \|u\|^p_{L^p(G)}\right) < \frac{\mu}{m} E(u, \mathcal{G})
\]
that is
\[
E(u, \mathcal{G}) > \frac{m}{\mu} \mathcal{E}_G(\mu).
\] (16)
Now, considering the liminf in (13) and combining with (15)–(16), we end up with
\[
\mathcal{E}_G(\mu) = \liminf_{n \to +\infty} E(u_n, \mathcal{G}) = \liminf_{n \to +\infty} E(u_n - u, \mathcal{G}) + E(u, \mathcal{G})
\]
\[
> \frac{\mu - m}{\mu} \mathcal{E}_G(\mu) + \frac{m}{\mu} \mathcal{E}_G(\mu) = \mathcal{E}_\mu(\mathcal{G})
\]
and we conclude. \(\square\)

3.2. Proof of Theorem 1.1

Note first that, given \(\mu > 0\), if we manage to prove that the infimum of the energy is strictly negative, then the existence of ground states is straightforward. Indeed, if (4) holds, namely \(\mathcal{E}_G(\mu) < 0\), then the statement of the theorem immediately follows from Proposition 3.1. Furthermore, it is readily seen that, for every \(\mu > 0\)
\[
\mathcal{E}_G(\mu) > -\infty.
\]
Indeed, (10) provides a lower bound for \(E(u, \mathcal{G})\) showing that \(E(u, \mathcal{G}) \to +\infty\) if \(\|u''\|_{L^2(G)} \to +\infty\), that is, \(E\) is bounded from below for every \(\mu > 0\).

We are thus left to prove that \(\mathcal{E}_G(\mu)\) is always negative, for every fixed \(\mu\) and \(\mathcal{G}\). As usual, let \(\mathcal{K}\) be the periodicity cell of \(\mathcal{G}\). We introduce
\[
L(\mathcal{K}) := \{e \in E(\mathcal{K}) : \exists! v \in D(\mathcal{K}) \text{ such that } e > v\}
\]
as the set of edges of \(\mathcal{K}\) with exactly one endpoint in \(D(\mathcal{K})\).

**Remark 3.1.** Note that, given the assumptions on \(D(\mathcal{K}), R(\mathcal{K})\) as in Definition 2.1, \(L(\mathcal{K})\) is always non-empty. Indeed, since both \(D(\mathcal{K})\) and \(R(\mathcal{K})\) are non-empty, there exist \(v \in D(\mathcal{K}), w \in R(\mathcal{K})\), with \(w \notin D(\mathcal{K})\) since \(D(\mathcal{K}) \cap R(\mathcal{K}) = \emptyset\). As \(\mathcal{K}\) is connected, there exists a path in \(\mathcal{K}\) starting at \(w\) and ending at \(v\). Since such a path starts at a vertex outside of \(D(\mathcal{K})\) and it ends at a vertex inside \(D(\mathcal{K})\), there must exist at least one edge in the path with exactly one endpoint in \(D(\mathcal{K})\).

Moreover, for every \(e \in L(\mathcal{K})\), we define the coordinate \(x_e\) on \(e\) so that, if \(e > v\) and \(v \in D(\mathcal{K})\), then \(x_e(0) = v\). Denote by \(\ell := \min_{e \in L(\mathcal{K})} |e|\) the length of the smallest edge of \(L(\mathcal{K})\), and define
\[
\bar{\mathcal{K}} = \mathcal{K} - \bigcup_{e \in L(\mathcal{K})} (e \cap [0, \ell])
\]
the portion of $\mathcal{K}$ that is left when we get rid of a path along each $e \in L(\mathcal{K})$ of length $\ell$ starting at $x_e(0)$. Note that there may exist edges belonging to $\tilde{\mathcal{K}}$ joining vertices in $D(\mathcal{K})$. For the sake of simplicity, let us assume in the remainder of the proof that there is no such edge in $\tilde{\mathcal{K}}$. Since all the forthcoming constructions straightforwardly generalize in the presence of this kind of edges, this does not reflect into any loss of generality, but it helps in simplifying some notation.

We then define the function (see Fig. 4)

$$u(x) := \begin{cases} \phi_{\mu_1}(i\ell) & \text{if } x \in \tilde{\mathcal{K}}_i, \text{ for some } i \in \mathbb{Z} \\ \phi_{\mu_1}((i+1)\ell - x) & \text{if } x \in e \cap [0, \ell], \text{ for some } e \in L(\mathcal{K}_i) \text{ and } i \in \mathbb{Z} \end{cases}$$

(17)

where $\phi_{\mu_1} \in H^{1}_{\mu_1}(\mathbb{R})$ is the soliton on $\mathbb{R}$ of mass $\mu_1$ as in (7), for some $\mu_1 > 0$ that has to be chosen to ensure $\|u\|_{L^2(\mathcal{G})} = \mu$.

Note that, setting $m := |L(\mathcal{K})|$, we have, by construction

$$E(u, \mathcal{G}) = mE(\phi_{\mu_1}, \mathbb{R}) + \sum_{i \in \mathbb{Z}} E(u, \tilde{\mathcal{K}}_i) = mE(\phi_{\mu_1}, \mathbb{R}) - \frac{1}{p} \sum_{i \in \mathbb{Z}} \|u\|_{L^p(\tilde{\mathcal{K}}_i)}^{p} < 0.$$ 

Therefore, to conclude, let us show that for every $\mu > 0$ there exists $\mu_1 > 0$ such that $u$ as in (17) belongs to $H^{1}_{\mu}(\mathcal{G})$.

Denoting by $\Gamma := |\tilde{\mathcal{K}}|$, we have

$$\|u\|_{L^2(\mathcal{G})}^2 = m\mu_1^2 + \Gamma \sum_{i \in \mathbb{Z}} \phi_{\mu_1}^2(i\ell) = m\mu_1 + \Gamma \left(2 \sum_{i=0}^{+\infty} \phi_{\mu_1}^2(i\ell) - \mu_1^{2\alpha}\right).$$

(18)

Using the explicit formulas (7)–(8), observe that (up to some constant) $\phi_{\mu_1}^2(x) \sim e^{-2\frac{\alpha}{\beta}x}$, for $x$ large enough.

Thus,

$$\sum_{i=0}^{+\infty} \phi_{\mu_1}^2(i\ell) \sim \mu_1^{2\alpha} \sum_{i=0}^{+\infty} e^{-2\frac{\alpha}{\beta} \mu_1^2 i\ell} = \mu_1^{2\alpha} \frac{e^{2\frac{\alpha}{\beta} \mu_1^2 \ell}}{e^{2\frac{\alpha}{\beta} \mu_1^2 \ell} - 1}.$$
Plugging into (18), we get that \( \|u\|_{L^2(G)} \) is a continuous function of \( \mu_1 \), \( \|u\|_{L^2(G)} = 0 \) if \( \mu_1 = 0 \) and \( \|u\|_{L^2(G)} \to +\infty \) as \( \mu_1 \to +\infty \), and we conclude.

Dropping the assumption that no edge in \( \tilde{K} \) joins vertices in \( D(K) \) only reflects in minor modifications of the above argument. Indeed, if \( e \in \tilde{K} \) is an edge between \( v, w \in D(K) \), we modify (17) defining \( u(x) = u(v_i) = u(w_i) \), for every \( x \in e_i \) and every \( i \in \mathbb{Z} \), where \( e_i, v_i, w_i \) denotes the copies of \( e, v, w \) in \( K_i \) respectively. Keeping track of this new definition, all the previous calculations can be developed in the same way. \( \square \)

4. The critical regime \( p = 6 \)

Let us focus now on the critical setting. Recall that, when \( p = 6 \) (see [8]), a threshold value of the mass can be defined

\[
\mu_G := \sqrt{\frac{3}{C_G}},
\]

with \( C_G \) denoting the optimal constant in the Gagliardo–Nirenberg inequality

\[
\|u\|_{L^6(G)}^6 \leq C_G \|u\|_{L^2(G)} \|u'\|_{L^2(G)}^2.
\]

Plugging (20) into the energy (1), we get

\[
E(u, G) \geq \frac{1}{2} \|u'\|_{L^2(G)}^2 \left( 1 - \frac{C_G}{3} \mu^2 \right)
\]

implying that

\[
\mu \leq \mu_G \quad \implies \quad E(u, G) \geq 0, \forall u \in H^1_\mu(G)
\]

\[
\mu > \mu_G \quad \implies \quad \exists u \in H^1_\mu(G) \text{ such that } E(u, G) < 0.
\]

If \( G = \mathbb{R} \), then \( \mu_\mathbb{R} = \frac{\sqrt{3}}{2} \pi \),

\[
\mathcal{E}_\mathbb{R}(\mu) = \begin{cases} 
0 & \text{if } \mu \leq \mu_\mathbb{R} \\
-\infty & \text{if } \mu > \mu_\mathbb{R}
\end{cases}
\]

and a whole family of critical solitons \( \{\phi_\lambda\}_{\lambda > 0} \) exists if and only if \( \mu = \mu_\mathbb{R} \), given by

\[
\phi_\lambda(x) := \sqrt{\lambda} \sqrt{\text{sech} \left( \frac{2}{\sqrt{3}} \lambda x \right)}.
\]

When \( G = \mathbb{R}^+ \), nothing changes, except of \( \mu_\mathbb{R}^+ = \frac{\sqrt{3}}{4} \pi \) and ground states being the restriction of \( \{\phi_\lambda\}_{\lambda > 0} \) to the half-line.

For a general non-compact metric graph \( G \), it is known (see Proposition 2.4 in [8]) that

\[
\mu_\mathbb{R}^+ \leq \mu_G \leq \mu_\mathbb{R}.
\]

Even though the original proof is developed for graphs with half-lines, it extends without any modifications to the periodic graphs we are dealing with.

The following proposition provides a first topological characterization of \( \mu_G \) for periodic graphs.
Proposition 4.1. Let $G$ be a periodic graph. It holds that

(i) if $G$ satisfies $(H_{\text{per}})$, then $\mu_G = \mu_R$;
(ii) if $G$ has a terminal edge, then $\mu_G = \mu_{R^+}$.

Proof. Let us first deal with part (i). Note that, since $G$ satisfies assumption $(H_{\text{per}})$, then, for every $u \in H^1_{\mu}(G)$ and almost every $t$ in the image of $u$, we have

$$\# \{ x \in G : u(x) = t \} \geq 2,$$

that is, any value in the image of $u$ has at least two pre-images on $G$. Indeed, let $M := \| u \|_{L^\infty(G)}$ and $x \in G$ be such that $u(x) = M$. Then, by $(H_{\text{per}})$, there exist two disjoint paths of infinite length originating at $x$, say $\Gamma_1, \Gamma_2$, and since $u \in H^1(G)$, $u(\Gamma_1) = u(\Gamma_2) = (0, M)$.

Hence, by standard properties of symmetric rearrangements (see [6]), denoting by $\hat{u} \in H^1(R)$ the symmetric rearrangement of $u \in H^1(G)$ on the line, it follows

$$\frac{\| u \|_{L^6(G)}^6}{\| u \|_{L^2(G)}^4 \| u' \|_{L^2(G)}^2} \leq \frac{\| \hat{u} \|_{L^6(R)}^6}{\| \hat{u} \|_{L^2(R)}^4 \| (\hat{u})' \|_{L^2(R)}^2} \leq C_R$$

for every $u \in H^1(G)$, and taking the supremum $C_G \leq C_R$.

By (19), this means

$$\mu_G \geq \mu_R$$

and, combining with (24), we conclude.

Let us focus now on statement (ii). For every $\varepsilon > 0$, there exists $u \in H^1(R^+)$, supported on $[0, 1]$, so that

$$\frac{\| u \|_{L^6(R^+)}^6}{\| u \|_{L^2(R^+)}^4 \| u' \|_{L^2(R^+)}^2} > C_{R^+} - \varepsilon. \quad (25)$$

Indeed, let $\phi(x)$ be the restriction of (23) to $R^+$, which is known to realize equality in the Gagliardo–Nirenberg inequality (20) on the half-line. According to (23), $\phi(x) \to 0$ as $x \to +\infty$. Hence, picking $\lambda$ large enough and setting $u(x) := (\phi(x) - \phi(1))_+$ for every $x \geq 0$, we have that $u$ is supported on $[0, 1]$ and $\| u \|_{H^1(G)} = \| \phi \|_{H^1(G)} + o(1)$. The fact that $u$ satisfies (25) is then a consequence of the continuity in $H^1(G)$ of the quotient on the left hand side. Setting now $u(x) := \sqrt{\lambda} u(\lambda x)$, for every $x \in R^+$ and $\lambda > 0$, we have that $u_{\lambda} \in H^1(R^+)$ and

$$\text{supp } u_{\lambda} = \left[ 0, \frac{1}{\lambda} \right].$$

$$\frac{\| u_{\lambda} \|_{L^6(R^+)}^6}{\| u_{\lambda} \|_{L^2(R^+)}^4 \| u_{\lambda}' \|_{L^2(R^+)}^2} = \frac{\| u \|_{L^6(R^+)}^6}{\| u \|_{L^2(R^+)}^4 \| u' \|_{L^2(R^+)}^2}.$$
Hence, letting $e \in E(G)$ be a terminal edge of $G$ and $\ell := |e|$ its length, when $\lambda$ is large enough, supp $u_\lambda \subset e$, and defining $v \in H^1(G)$ so that $v \equiv u_\lambda$ on $e$ and $v \equiv 0$ elsewhere, we deduce

$$C_G > C_{\mathbb{R}^+} - \varepsilon$$

and, by the arbitrariness of $\varepsilon$,

$$C_G \geq C_{\mathbb{R}^+}.$$ 

Combining with (19) and (24) gives the claim. \hfill \Box

As a direct consequence, we are now able to prove the first of our main results in the critical setting.

**Proof of Theorem 1.2.** We begin by proving statement (i). Let $G$ be a periodic graph of periodicity cell $K$ satisfying (H$_{\text{per}}$).

By Proposition 4.1(i), $\mu_G = \mu_\mathbb{R}$, and (21) ensures that, for every $\mu \leq \mu_\mathbb{R}$ and $u \in H^1_\mu(G)$

$$E(u, G) \geq 0.$$ \hfill (26)

Now, for every $n \in \mathbb{N}$, we introduce

$$\Sigma_n := \{ e \in E(K_{-n-1}) \cup E(K_{n+1}) : \exists v \in R(K_{-n}) \cup D(K_n) \text{ such that } e \succ v \}$$

as the set of all edges in $G$ entering a vertex which is joining either $K_{-n-1}$ with $K_{-n}$ or $K_{n+1}$ with $K_n$. Moreover, for every $e \in \Sigma_n$ and $e \succ v \in R(K_{-n}) \cup D(K_n)$, we set $x_e(0) = v$, being $x_e$ the coordinate defined on $e$. Let also $\ell_e := |e|$ denote the length of $e$.

Then, for every $n \in \mathbb{N}$, we define $u_n \in H^1_\mu(G)$ as

$$u_n(x) := \begin{cases} 
\alpha_n & \text{if } x \in K_i, \text{ for some } i \in \{-n, \ldots, n\} \\
\frac{\alpha_n}{\ell_e} (\ell_e - x) & \text{if } x \in e, \text{ for some } e \in \Sigma_n \\
0 & \text{otherwise on } G 
\end{cases}$$ \hfill (28)

where $\alpha_n$ is chosen to satisfy $\|u_n\|^2_{L^2(G)} = \mu$. It is immediate to see that $\alpha_n \to 0$ as $n \to +\infty$, thus implying $E(u_n, G) \to 0$ and

$$E_G(\mu) = 0$$

for every $\mu \leq \mu_\mathbb{R}$.

When $\mu > \mu_\mathbb{R}$, by (22) there exists $v \in H^1_\mu(\mathbb{R})$, supported on $[0, 1]$, so that $E(v, \mathbb{R}) < 0$. Considering the mass-preserving transformation

$$v_\lambda(x) := \sqrt{\lambda} v(\lambda x)$$

for every $\lambda > 0$, we get $v_\lambda \in H^1_\mu(\mathbb{R})$, $v_\lambda$ is supported on $[0, 1/\lambda]$ and $E(v_\lambda, \mathbb{R}) = \lambda^2 E(v, \mathbb{R})$.

Fix now any edge $e \in E(G)$ and let $\ell := |e|$ be its length. Then, there exists $\overline{\lambda} > 0$ such that, for $\lambda \geq \overline{\lambda}$, $v_\lambda \in H^1_\mu(0, \ell)$, that is, we construct functions \{v_\lambda\}_{\lambda \geq \overline{\lambda}} \subset H^1_\mu(G)$, supported on $e$ and satisfying

$$E(v_\lambda, G) \to -\infty \quad \text{for } \lambda \to +\infty,$$
proving that, for every $\mu > \mu_R$
\[ \mathcal{E}_G(\mu) = -\infty. \]

We conclude showing that the infimum is not attained, for any value of the mass $\mu \leq \mu_R$ (the statement is trivially true in the regime $\mu > \mu_R$).

When $\mu < \mu_R$, the result is immediate, the inequality in (26) being strict for every $u \in H^1_\mu(G)$.

If $\mu = \mu_R$, suppose by contradiction that $u \in H^1_{\mu_R}(G)$ is a ground state, i.e., $E(u, G) = \mathcal{E}_G(\mu_R) = 0$. This implies
\[ 0 = E(u, G) \geq E(\hat{u}, \mathbb{R}) \geq \mathcal{E}_R(\mu_R) = 0, \]
that is, $E(u, G) = E(\hat{u}, \mathbb{R})$ and particularly $\|u'\|_{L^2(G)} = \|\hat{u}'\|_{L^2(\mathbb{R})}$. But this is impossible, since $G$ contains at least one vertex of degree 3, preventing $\# \{ x \in G : u(x) = t \} = 2$ to be true for a.e., $t$ in the image of $u$.

Part (ii) of Theorem 1.2 can be proved by the same argument, simply replacing $\mu_R$ with $\mu_R +$ and symmetric rearrangements with decreasing ones whenever needed. \hfill \square

We then turn our attention to graphs violating (H_{per}) and with no terminal edge, providing the proof of Theorem 1.3. To this purpose, a modified version of the Gagliardo–Nirenberg inequality has to be considered, ensuring that, for every $\mu \in [0, \mu_R]$ and every $u \in H^1_\mu(G)$, there exists $\theta_u := \theta(u)$, with $\theta_u \in [0, \mu]$, such that
\[ \|u\|_{L^6(G)}^6 \leq 3 \left( \frac{\mu - \theta_u}{\mu_R} \right)^2 \|u'\|_{L^2(G)}^2 + C \sqrt{\theta_u} \]
with $C > 0$ depending only on $G$ (see Lemma 4.4 in [8] for a proof that extends to periodic graphs without modifications).

**Proof of Theorem 1.3.** By the same argument in the proof of Theorem 1.2, we have
\[ \mathcal{E}_G(\mu) = -\infty \quad \forall \mu > \mu_R \]
\[ \mathcal{E}_G(\mu) = 0 \quad \forall \mu < \mu_G \]
and ground states do not exist in both these situations.

Consider now $\mu \in (\mu_G, \mu_R]$. For every $\varepsilon > 0$, there exists $u \in H^1_\mu(G)$ so that
\[ \frac{\|u\|_{L^6(G)}^6}{\|u\|_{L^2(G)}^2 \|u'\|_{L^2(G)}^2} > C_G - \varepsilon \]
and plugging into the energy
\[ E(u, G) \leq \frac{1}{2} \|u'\|_{L^2(G)}^2 \left( 1 - \frac{C_G - \varepsilon}{3} \mu^2 \right). \]
Thus, picking $\varepsilon$ small enough and since $\mu > \mu_G$
\[ \mathcal{E}_G(\mu) < 0 \quad \forall \mu \in (\mu_G, \mu_R]. \]
Let \( \{ u_n \}_{n \in \mathbb{N}} \subset H^1_\mu(\mathcal{G}) \) be a minimizing sequence for \( E \), so that each \( u_n \) realizes its \( L^\infty \) norm at some point of \( K_0 \). Then, by (30) and (29), we have

\[
\| u'_n \|_{L^2(\mathcal{G})}^2 \leq \frac{1}{3} \| u_n \|_{L^6(\mathcal{G})}^6 \leq \left( \frac{\mu - \theta u_n}{\mu_R} \right)^2 \| u'_n \|_{L^2(\mathcal{G})}^2 + \frac{C}{3} \sqrt{\theta u_n}
\]

that is

\[
\left( 1 - \frac{(\mu - \theta u_n)^2}{\mu_R^2} \right) \| u'_n \|_{L^2(\mathcal{G})}^2 \leq \frac{C}{3} \sqrt{\mu}.
\]

(31)

If \( \mu < \mu_R \), then (31) is enough to ensure that \( \{ u_n \}_{n \in \mathbb{N}} \) is bounded in \( H^1(\mathcal{G}) \), the term in the bracket on the left-hand side being always strictly positive. If \( \mu = \mu_R \), then note that, plugging (29) into (1)

\[
E(u_n, \mathcal{G}) \geq \frac{1}{2} \| u'_n \|_{L^2(\mathcal{G})}^2 \left( 1 - \frac{(\mu_R - \theta u_n)^2}{\mu_R^2} \right) - \frac{C}{6} \sqrt{\theta u_n}
\]

(32)

and, since the right-hand side is the sum of a first nonnegative term which is bounded by (31) and a second one that tends to 0 when \( \theta u_n \to 0 \), (30) and the minimality of \( \{ u_n \}_{n \in \mathbb{N}} \) give

\[
\inf_{n \in \mathbb{N}} \theta u_n > 0.
\]

(33)

Hence, (31)–(33) provide boundedness of \( \{ u_n \}_{n \in \mathbb{N}} \) in \( H^1(\mathcal{G}) \) also in the case \( \mu = \mu_R \). Therefore, for every \( \mu \in (\mu_G, \mu_R] \), \( u_n \to u \) in \( H^1(\mathcal{G}) \), for some \( u \in H^1(\mathcal{G}) \), whereas (32) guarantees that \( E_\mathcal{G}(\mu) > -\infty \). The argument of the proof of Proposition 3.1 can now be repeated, showing that \( u_n \to u \) strongly in \( H^1_\mu(\mathcal{G}) \) and \( u \) is a ground state of mass \( \mu \).

It remains to deal with the case \( \mu = \mu_G \). Note that this is not immediate, since now \( E_\mathcal{G}(\mu_G) = 0 \), while the negativity of the ground state energy level is crucial in the above argument. Actually, it is no longer true that every minimizing sequence is strongly compact, vanishing sequences as in (28) providing an example of the possible loss of compactness. However, we show that there exists a proper choice of the minimizing sequence recovering compactness.

Let \( \{ u_n \}_{n \in \mathbb{N}} \subset H^1_{\mu_G}(\mathcal{G}) \) be a maximizing sequence for the Gagliardo–Nirenberg inequality (20), i.e.,

\[
\lim_{n \to +\infty} \frac{\| u_n \|_{L^6(\mathcal{G})}^6}{\| u'_n \|_{L^2(\mathcal{G})}^2} \to C_\mathcal{G}\mu_G^2 = 3
\]

(34)

and assume as usual that, for every \( n \), \( u_n \) attains its maximum somewhere in \( K_0 \).

Now, by (29), we have

\[
3 \| u'_n \|_{L^2(\mathcal{G})}^2 = \| u_n \|_{L^6(\mathcal{G})}^6 + o(\| u'_n \|_{L^2(\mathcal{G})}^2)
\]

\[
\leq 3 \left( \frac{\mu_G - \theta u_n}{\mu_R} \right)^2 \| u'_n \|_{L^2(\mathcal{G})}^2 + C \sqrt{\theta u_n} + o(\| u'_n \|_{L^2(\mathcal{G})}^2)
\]

\[
\leq \frac{3\mu_G^2}{\mu_R^2} \| u'_n \|_{L^2(\mathcal{G})}^2 + C \sqrt{\mu_G} + o(\| u'_n \|_{L^2(\mathcal{G})}^2)
\]
that is

$$3 \left(1 - \frac{\mu_G^2}{\mu_R^2}\right) \|u_n'\|^2_{L^2(G)} \leq C \sqrt{\mu_G} + o(\|u_n'\|^2_{L^2(G)}).$$

As $\mu_G < \mu_R$, this means that $\{u_n\}_{n \in \mathbb{N}}$ is bounded in $H^1(G)$, $u_n \to u$ and

$$u_n \to u \in L^\infty_{\text{loc}}(G) \text{ as } n \to +\infty, \text{ for some } u \in H^1(G).$$

Moreover, by construction, every value $t$ in the image of $u_n$ that is attained twice on $G$ gives birth to at least one compact connected component. Since $G$ violates $(H_{\text{per}})$, $B \neq \emptyset$. Note that, if there exist values in the image of $u_n$ with just one pre-image on $G$, then they can only be attained somewhere in $B$. Indeed, if $t < \min_{x \in K_0} u_n(x)$, then $t$ has at least two pre-images, one on the left of $K_0$ and one on the right. Moreover, for almost every value $t$, if $t$ is attained at a point belonging to a cycle of $K_0$, then it is attained twice on that cycle and it has at least two pre-images in $K_0$.

Let us introduce the following construction. We first double the edges of $B$, that is, we replace each $e \in B$ by two edges, say $e_1, e_2$, joining the same vertices of $e$ and so that $|e_1| = |e_2| = 2|e|$. Then, both on $e_1$ and $e_2$, we stretch the restriction of $u_n$ to $e$ by a factor 2. Letting $\tilde{G}$ and $\tilde{u}_n$ denote respectively the new graph and function identified by this procedure, we have, for every $n \in \mathbb{N}$, $\tilde{u}_n \in H^1(\tilde{G})$ and

$$\tilde{u}_n(x) = u_n(x) \quad \text{if } x \in G/B,$$

$$\tilde{u}_n(x) = u_n(x/2) \quad \text{if } x \in e_i \text{ for some } i \in \{1, 2\} \text{ and } e \in B.$$ 

Note that

$$\|\tilde{u}_n'\|^2_{L^2(\tilde{G})} = \int_{\tilde{G}/B} |u_n'|^2 \, dx + \frac{1}{2} \sum_{e \in B} \int_0^{2|e|} |u_n'(x/2)|^2 \, dx$$

$$= \int_{G/B} |u_n'|^2 \, dx + \sum_{e \in B} \int_0^{|e|} |u_n'|^2 \, dx = \|u_n'\|^2_{L^2(G)} \quad (35)$$

Moreover, by construction, every value $t$ in the image of $\tilde{u}_n$ is now attained at least twice on $\tilde{G}$. Therefore, by properties of the symmetric rearrangements and Gagliardo–Nirenberg inequality (20) on $\mathbb{R}$,

$$\|\tilde{u}_n\|_{L^6(\tilde{G})}^6 = \|(\tilde{u}_n)\|_{L^6(\mathbb{R})}^6 \leq C_{\mathbb{R}} \|\tilde{u}_n\|_{L^2(\mathbb{R})} \|\tilde{u}_n\|_{L^2(\mathbb{R})} \leq C_{\mathbb{R}} \|\tilde{u}_n\|_{L^2(\tilde{G})}^4 \|\tilde{u}_n'\|^2_{L^2(\tilde{G})} \leq C_{\mathbb{R}} \|\tilde{u}_n\|_{L^2(\tilde{G})}^2 \|\tilde{u}_n'\|^2_{L^2(\tilde{G})}$$
that, combined with (35) and the fact that
\[ \|\tilde{u}_n\|^6_{L^6(\tilde{G})} = \|u_n\|^6_{L^6(G)} + 3 \sum_{e \in B} \int_{e} |u_n|^6 \, dx \]
\[ \|\tilde{u}_n\|^2_{L^2(\tilde{G})} = \|u_n\|^2_{L^2(G)} + 3 \sum_{e \in B} \int_{e} |u_n|^2 \, dx = \mu_G + 3 \sum_{e \in B} \int_{e} |u_n|^2 \, dx \]
gives
\[ \|u_n\|^6_{L^6(G)} \leq \|\tilde{u}_n\|^6_{L^6(\tilde{G})} \leq C_G \left( \mu_G + 3 \sum_{e \in B} \int_{e} |u_n|^2 \, dx \right)^2 \|\tilde{u}_n\|^2_{L^2(\tilde{G})} \]
\[ \leq \frac{3}{\mu_G^2} \left( \mu_G + 3 \sum_{e \in B} \int_{e} |u_n|^2 \, dx \right)^2 \|u_n\|^2_{L^2(G)} \]  \hspace{1cm} (36)
Now, since \( u_n \to 0 \) in \( L^\infty(G) \) and \( B \subset K_0 \), then \( \sum_{e \in B} \int_{e} |u_n|^2 \, dx \to 0 \) as \( n \to +\infty \), and (36) implies, for \( n \) sufficiently large
\[ \frac{\|u_n\|^6_{L^6(G)}}{\|u_n\|^2_{L^2(G)}} \leq \frac{3}{\mu_G^2} + o(1) \]
that, coupled with (34) and \( \mu_G < \mu_\mathbb{R} \), provides a contradiction. Hence, \( u \not\equiv 0 \).

Repeating the same calculations we performed in the proof of Proposition 3.1 allows to rule out also the case \( \|u\|^2_{L^2(G)} < \mu \). Thus, \( \|u\|^2_{L^2(G)} = \mu \), \( u_n \to u \) strongly in \( H^1_p(G) \) and \( u \) is a ground state. \( \blacksquare \)

Notice that, as already highlighted in the Introduction, the assumption \( \mu_G < \mu_\mathbb{R} \) in Theorem 1.3 is essential, but we are not able to tell whether it is automatically satisfied by all graphs that do not fulfill (H\textsubscript{per}) and without terminal edges. Nevertheless, we conclude this section with the following proposition, showing that the class of graphs for which \( \mu_G < \mu_\mathbb{R} \) holds true is nonempty.

**Proposition 4.2.** Let \( G \) be the periodic graph in Fig. 2c. Then \( \mu_G < \mu_\mathbb{R} \).

**Proof.** Let us first note that \( G \) as in Fig. 2 is periodic according to Definition 2.1 with periodicity cell \( K \) given by a vertical edge with a circle attached at one of its endpoints and a horizontal edge pointing to the right at the other. Denote then by \( \Gamma, B, H \) the circle, the vertical and the horizontal edge of \( K \) respectively, so that \( K = \Gamma \cup B \cup H \), and set \( 2\gamma := |\Gamma|, 2\beta := |B| \) and \( |H| := \delta \). Moreover, let as usual \( \Gamma_i, B_i, H_i \) be the corresponding parts of the \( i \)-th copy of \( K \) in \( G, K_i \), for every \( i \in \mathbb{Z} \).

Since, by definition, \( E_G(\mu_G) = 0 \), in order to prove that \( \mu_G < \mu_\mathbb{R} \), the idea is to exhibit explicitly a function \( u \in H^1_{\mu_G}(G) \) such that \( E(u, G) < 0 \).

To this aim, given \( \lambda > 0 \), let \( \phi_\lambda \in H^1_{\mu_\mathbb{R}}(\mathbb{R}) \) be the critical soliton \( (23) \) on the real line and consider \( w_\lambda \in H^1(G) \) defined by the following procedure (see Fig. 5).
First, let $\phi_{\lambda \mid (-\gamma, \gamma)}$ be the restriction of $\phi_{\lambda}$ to the interval $(-\gamma, \gamma)$ and set

$$w_{\lambda \mid \Gamma_0} := \phi_{\lambda \mid (-\gamma, \gamma)}$$

having identified $\Gamma_0$ with the interval $(-\gamma, \gamma)$ in such a way that both the endpoints $-\gamma$, $\gamma$ correspond to the vertex of $\Gamma_0$ attached to $B_0$.

Secondly, let $I := [-\gamma - \beta, -\gamma] \cup [\gamma, \gamma + \beta]$, $\phi_{\lambda \mid I}$ be the restriction of $\phi_{\lambda}$ to $I$ and $(\phi_{\lambda \mid I})^*$ be its decreasing rearrangement on $[0, 2\beta]$. We then set

$$w_{\lambda \mid B_0} := (\phi_{\lambda \mid I})^*$$

provided the identification of $B_0$ with the interval $[0, 2\beta]$ so that the origin of $[0, 2\beta]$ corresponds to the vertex that $B_0$ shares with $\Gamma_0$.

Thirdly, for every $i \in \mathbb{Z}$, let either $I_i := [\gamma + \beta + i\delta, \gamma + \beta + (i + 1)\delta]$ if $i \geq 0$, or $I_i := [-\gamma - \beta - i\delta, -\gamma - \beta - (i + 1)\delta]$ if $i < 0$, and let $\phi_{\lambda \mid I_i}$ be the restriction of $\phi_{\lambda}$ to $I_i$. Then, for every $i \in \mathbb{Z}$, we set

$$w_{\lambda \mid H_i} := \phi_{\lambda \mid I_i}$$

identifying $H_i$ with $I_i$ so that the endpoint of $I_i$ with smallest absolute value corresponds to the vertex of $H_i$ closest to $K_0$.
Finally, for every $i \in \mathbb{Z} \setminus \{0\}$, we set

$$w_{\lambda|\Gamma_i \cup B_i} \equiv \begin{cases} \phi_{\lambda}(\gamma + \beta + i\delta) & \text{if } i > 0 \\ \phi_{\lambda}(-\gamma - \beta - i\delta) & \text{if } i < 0 \end{cases}$$

so that $w_{\lambda}$ is constant on every $\Gamma_i \cup B_i$, $i \neq 0$.

Note that, by construction, $w_{\lambda} \in H^1(G)$ and, for every $p \geq 1$,

$$\|w_{\lambda}\|_{L^p(\Gamma_0)} = \|\phi_{\lambda}\|_{L^p(-\gamma, \gamma)}, \quad \|w_{\lambda}\|_{L^p(\bigcup_{i \in \mathbb{Z} \setminus \{0\}}\Gamma_i \cup B_i)} = \|\phi_{\lambda}\|_{L^p(\mathbb{R} \setminus (-\gamma, \gamma + \beta))},$$

whereas by the fact that the restriction of $\phi_{\lambda}$ to $(-\gamma - \beta, \gamma) \cup (\gamma, \gamma + \beta)$ attains exactly each value in its image and standard properties of decreasing rearrangements (see [9, Lemma 2.1])

$$\|w_{\lambda}\|_{L^p(B_0)} = \|\phi_{\lambda}\|_{L^p((-\gamma - \beta, -\gamma) \cup (\gamma, \gamma + \beta))},$$

$$\|w'_{\lambda}\|_{L^2(B_0)} \leq \frac{1}{2} \|\phi'_{\lambda}\|_{L^2((-\gamma - \beta, -\gamma) \cup (\gamma, \gamma + \beta))}.$$

Hence, setting $c := 2\gamma + 2\beta$, we have

$$\int_G |w_{\lambda}|^2 \, dx = \int_{\mathbb{R}} |\phi_{\lambda}|^2 \, dx + \int_{\bigcup_{i \in \mathbb{Z} \setminus \{0\}}(\Gamma_i \cup B_i)} |w_{\lambda}|^2 \, dx = \mu_{\mathbb{R}} + 2c \sum_{i=1}^{\infty} |\phi_{\lambda}(c + i\delta)|^2,$$

$$\int_G |w_{\lambda}|^6 \, dx = \int_{\mathbb{R}} |\phi_{\lambda}|^6 \, dx + \int_{\bigcup_{i \in \mathbb{Z} \setminus \{0\}}(\Gamma_i \cup B_i)} |w_{\lambda}|^6 \, dx$$

$$= \int_{\mathbb{R}} |\phi_{\lambda}|^6 \, dx + 2c \sum_{i=1}^{\infty} |\phi_{\lambda}(c + i\delta)|^6$$

and

$$\int_G |w'_{\lambda}|^2 \, dx = \int_{\mathbb{R}} |\phi'_{\lambda}|^2 \, dx - \int_{(-\gamma - \beta, -\gamma) \cup (\gamma, \gamma + \beta)} |\phi'_{\lambda}|^2 \, dx + \int_{B_0} |w'_{\lambda}|^2 \, dx$$

$$\leq \int_{\mathbb{R}} |\phi'_{\lambda}|^2 \, dx - \frac{3}{4} \int_{(-\gamma - \beta, -\gamma) \cup (\gamma, \gamma + \beta)} |\phi'_{\lambda}|^2 \, dx,$$

yielding at

$$E(w_{\lambda}, G) \leq E(\phi_{\lambda}, \mathbb{R}) - \frac{3}{8} \int_{(-\gamma - \beta, -\gamma) \cup (\gamma, \gamma + \beta)} |\phi'_{\lambda}|^2 \, dx - \frac{c}{3} \sum_{i=1}^{\infty} |\phi_{\lambda}(c + i\delta)|^6$$

$$= -\frac{3}{4} \int_{\gamma}^{\gamma + \beta} |\phi'_{\lambda}|^2 \, dx - \frac{c}{3} \sum_{i=1}^{\infty} |\phi_{\lambda}(c + i\delta)|^6$$

(37)

since $E(\phi_{\lambda}, \mathbb{R}) = 0$, for every $\lambda > 0$.

Now, recalling the explicit formula (23) for $\phi_{\lambda}$ leads to

$$\sum_{i=1}^{\infty} |\phi_{\lambda}(c + i\delta)|^2 = \lambda \sum_{i=1}^{\infty} \operatorname{sech}\left(\frac{2}{\sqrt{3}} \lambda(c + i\delta)\right) \sim \lambda \sum_{i=1}^{\infty} \frac{1}{e^{\frac{2}{\sqrt{3}} \lambda(c + i\delta)}}$$

$$= \frac{\lambda}{e^{\frac{2}{\sqrt{3}} \lambda c} (e^{\frac{2}{\sqrt{3}} \lambda \delta} - 1)} \sim \frac{\lambda}{e^{\frac{2}{\sqrt{3}} \lambda (c + \delta)}}$$
as $\lambda \to +\infty$, so that we get
\[
\frac{\mu_{\mathbb{R}}}{\|w_\lambda\|_{L^2(G)}^2} \sim \frac{\mu_{\mathbb{R}}}{\mu_{\mathbb{R}}} + \frac{\lambda}{e^{\frac{2}{3}\lambda(c+\delta)}} = 1 - r(\lambda)
\]
where
\[
r(\lambda) := \frac{\lambda}{e^{\frac{2}{3}\lambda(c+\delta)}} + \frac{\lambda}{e^{\frac{2}{3}\lambda(c+\delta)}} 
\]
for $\lambda \to +\infty$.

Therefore, letting $u_\lambda := \sqrt{\frac{\mu_{\mathbb{R}}}{\|w_\lambda\|_{L^2(G)}^2}}w_\lambda$, we have $u_\lambda \in H^1_{\mu_{\mathbb{R}}} (G)$ for every $\lambda$ and, provided $\lambda$ large enough,
\[
E(u_\lambda, G) = \frac{1}{2} \left( \frac{\mu_{\mathbb{R}}}{\|w_\lambda\|_{L^2(G)}^2} \right) \|w_\lambda'\|_{L^2(G)}^2 - \frac{1}{6} \left( \frac{\mu_{\mathbb{R}}}{\|w_\lambda\|_{L^2(G)}^2} \right)^3 \|w_\lambda\|_{L^6(G)}^6 
\]
\[
\leq \frac{1}{2} (1 - r(\lambda)) \|w_\lambda'\|_{L^2(G)}^2 - \frac{1}{6} (1 - r(\lambda))^3 \|w_\lambda\|_{L^6(G)}^6 + o(1) 
\]
\[
= (1 - r(\lambda))E(w_\lambda, G) + \frac{1}{2} r(\lambda) \|w_\lambda\|_{L^6(G)}^6 + o(1) 
\]

taking advantage of the fact that $r(\lambda) \to 0$ as $\lambda \to +\infty$. Combining with (37), we get the following estimate
\[
E(u_\lambda, G) \leq (1 - r(\lambda)) \left( -\frac{3}{4} \int_{\gamma} \gamma^\alpha \beta^2 \, dx - \frac{c}{3} \sum_{i=1}^{\infty} |\phi_\lambda(c + i\delta)|^6 \right) + \frac{1}{2} r(\lambda) \|w_\lambda\|_{L^6(G)}^6 + o(1) \tag{38}
\]
holding for $\lambda$ large.

Clearly, all the terms involved in the above expression goes to 0 as $\lambda$ becomes infinite. Note thus that multiplying the big bracket by $r(\lambda)$ makes the whole product go to 0 faster than $r(\lambda)$. Hence, since the term $\frac{7}{2} r(\lambda) \|w_\lambda\|_{L^6(G)}^6$ is asymptotic to $\lambda^2 r(\lambda)$ as $\lambda$ increases, to understand the asymptotic behaviour of the upper bound in (38) we can neglect $(1 - r(\lambda))$ in the first addend, simply replacing it by 1.

Furthermore, with computations similar to the ones we performed before, it is readily seen that
\[
\sum_{i=1}^{\infty} |\phi_\lambda(c + i\delta)|^6 \sim \frac{\lambda^3}{e^{\frac{2}{3}\lambda(c+\delta)}} = o(r(\lambda)) \quad \text{as } \lambda \to +\infty
\]
so that we can restrict ourselves to focus only on the first integral in the big bracket in (38).

Since, differentiating (23),
\[
\phi_\lambda'(x) = -\frac{2}{\sqrt{3}} \lambda^{3/2} \text{sech}^{3/2} \left( \frac{2}{\sqrt{3}} \lambda x \right) \sinh \left( \frac{2}{\sqrt{3}} \lambda x \right),
\]

it follows
\[
\int_{\gamma}^{\gamma+\beta} |\phi'_{\lambda}|^2 \, dx = \frac{2}{\sqrt{3}} \lambda^2 \left( \int \frac{2}{\sqrt{3}} \lambda^{(\gamma+\beta)} \, \text{sech}^3 y \,\text{sinh}^2 y \, dy \right)
= \frac{2}{\sqrt{3}} \lambda^2 \left( \int \frac{2}{\sqrt{3}} \lambda^{(\gamma+\beta)} \, \tanh y \,\text{coth}^2 y \, dy \right)
= \frac{2}{\sqrt{3}} \lambda^2 \left( \int \frac{2}{\sqrt{3}} \lambda^{(\gamma+\beta)} \, \tanh y \,\frac{d}{dy} \left( - \text{sech} y \right) \, dy \right).
\]

Note that, integrating by parts
\[
\int \tanh y \,\frac{d}{dy} \left( - \text{sech} y \right) \, dy = - \tanh y \,\text{sech} y + \int (1 - \tanh^2 y) \,\text{sech} y \, dy
= - \tanh y \,\text{sech} y + \int \text{sech} y \, dy
= - \tanh y \,\text{sech} y + \int \tanh y \,\frac{d}{dy} \left( - \text{sech} y \right) \, dy
\]

as \( \tanh^2 y \,\text{sech} y = \tanh y \,\frac{\sinh y}{\cosh y} = \tanh y \,\frac{d}{dy} \left( - \text{sech} y \right) \). Thus we have
\[
\int \tanh y \,\frac{d}{dy} \left( - \text{sech} y \right) \, dy = - \frac{1}{2} \tanh y \,\text{sech} y + \frac{1}{2} \int \text{sech} y \, dy
= - \frac{1}{2} \tanh y \,\text{sech} y + \arctan(e^y),
\]
leading to
\[
\int_{\gamma}^{\gamma+\beta} |\phi'_{\lambda}|^2 \, dx = \frac{2}{\sqrt{3}} \lambda^2 \left[ \arctan(e^x) - \frac{1}{2} \tanh x \,\text{sech} x \right] \frac{2}{\sqrt{3}} \lambda^{(\gamma+\beta)}.
\]

When \( \lambda \) is large enough, standard properties of the arctangent and first order expansions yield at
\[
\left[ \arctan(e^x) \right] \frac{2}{\sqrt{3}} \lambda^{(\gamma+\beta)} = \left[ \arctan(e^{-x}) \right] \frac{2}{\sqrt{3}} \lambda^{(\gamma+\beta)}
= e^{-\frac{2}{\sqrt{3}} \lambda^{(\gamma+\beta)}} - e^{-\frac{2}{\sqrt{3}} \lambda^{(\gamma+\beta)}} + o(e^{-\frac{2}{\sqrt{3}} \lambda^{(\gamma+\beta)}})
= e^{-\frac{2}{\sqrt{3}} \lambda^{(\gamma+\beta)}} + o(e^{-\frac{2}{\sqrt{3}} \lambda^{(\gamma+\beta)}}),
\]
whereas explicit computations give
\[
\left[ \tanh x \,\text{sech} x \right] \frac{2}{\sqrt{3}} \lambda^{(\gamma+\beta)} \sim - \frac{2}{e^{\frac{2}{\sqrt{3}} \lambda^{\gamma}}},
\]
Henceforth,
\[
\lambda^2 \left[ \arctan(e^x) - \frac{1}{2} \tanh x \,\text{sech} x \right] \frac{2}{\sqrt{3}} \lambda^{(\gamma+\beta)} \sim 2\lambda^2 \frac{2}{\sqrt{3}} \lambda^{(\gamma+\beta)}.
\]
so that (recalling also $c = 2\gamma + 2\beta$)

$$
\int_\gamma^{\gamma+\beta} |\phi'_\lambda|^2 \, dx \quad \sim \quad \frac{\lambda^2}{\sqrt{3} \lambda^\gamma} \frac{e^{2\sqrt{3} \lambda^{\frac{\lambda}{2}}}}{e^{\frac{2\sqrt{3} \lambda^{\frac{\lambda}{2}}}} \lambda} \to +\infty \quad \text{for } \lambda \to +\infty.
$$

Summing up, plugging into (38) shows that

$$
E(u_\lambda, G) \leq -\frac{3}{4} \int_\gamma^{\gamma+\beta} |\phi'_\lambda|^2 \, dx + \frac{2}{3} r(\lambda) \|w_\lambda\|_{L^6(G)}^6 + o(1) < 0
$$

as soon as $\lambda$ is sufficiently large, and we conclude. □

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**Appendix A**

As pointed out in Sect. 2, it is not immediate to see that assumptions (i)–(ii) on $D, R$ and $\sigma$ we require stating Definition 2.1 are not somehow restrictive. Actually, this is not the case, as it can be shown that all periodic metric graphs with infinite diameter that can be constructed dropping either (i) or (ii) and then following the steps of Definition 2.1 can always be built with an equivalent choice of $D, R$ and $\sigma$ fulfilling (i)–(ii).

From now on, we will say that two graphs $G = (V(G), E(G))$, $G' = (V(G'), E(G'))$ are equal, and we will write $G = G'$, if there exist two bijections $\varphi : V(G) \to V(G')$, $\psi : E(G) \to E(G')$ such that $e \in E(G)$ is an edge between $v, w \in V(G)$ if and only if $\psi(e) \in E(G')$ is an edge between $\varphi(v), \varphi(w) \in V(G')$, and $\psi$ is measure-preserving, that is $|e| = |\psi(e)|$, for every $e \in E(G)$.

**Proposition A.1.** Let $K$ be a fixed compact graph, $D, R$ two non-empty subsets of $V(K)$ and $\sigma : D \to R$ bijective. Suppose $D \cap R \neq \emptyset$ and let $G$ be as in Definition 2.1. Then, either

(a) $\text{diam}(G) < +\infty$, or
(b) there exists a compact graph $K'$, two non-empty subsets $D', R' \subset V(K')$ and a bijection $\sigma' : D' \to R'$, with $D' \cap R' = \emptyset$, such that if $G'$ is the periodic graph with periodicity cell $K'$ and pasting rule $\sigma'$ as in Definition 2.1, then $G = G'$.

**Proof.** Let $D \cap R = \{x^1, \ldots, x^n\}$, for some $x^1, \ldots, x^n \in K$. Moreover, for every $i \in \mathbb{Z}$ and every $j \in \{1, \ldots, n\}$, denote by $x^j_i \in G$ the copy of $x^j$ belonging to $K_i$.

We split the proof into two parts.
Part (i). Suppose that there exists a subset \( \{ x^{j_1}, \ldots, x^{j_l} \} \) of \( D \cap R \) such that (see Fig. 6)
\[
\sigma(x^{j_1}) = x^{j_2}, \ldots, \sigma(x^{j_{l-1}}) = x^{j_l}, \sigma(x^{j_l}) = x^{j_1}.
\]
Note that, building up \( G \) according to the pasting rule \( \sigma \) as in Definition 2.1, for every \( i_1, i_2 \in \mathbb{Z} \), \( s, t \in \{ j_1, \ldots, j_l \} \), \( x^{i_1}_s \) and \( x^{i_2}_t \) correspond to the same point of \( G \) if and only if \( |s - t| = |i_1 - i_2| \mod l \).

Consider now \( y, z \in G \) and let \( i_1, i_2 \in \mathbb{Z} \) be such that \( y \in K_{i_1}, z \in K_{i_2} \). Moreover, let \( s, t \in \{ j_1, \ldots, j_l \} \) be such that \( |s - t| = |i_1 - i_2| \mod l \). We get
\[
d(y, z) \leq d(y, x^{i_1}_s) + d(x^{i_2}_t, z) \leq 2\text{diam}(K)
\]
and passing to the supremum over all \( y, z \in G \)
\[
\text{diam}(G) \leq 2\text{diam}(K) < +\infty
\]
so that case (a) occurs.

Part (ii). Suppose now that no subset of \( D \cap R \) satisfies the assumption of part (i).

Let us begin to deal with the case \( \sigma(x^i) \neq x^j \), for every \( i, j \in \{1, \ldots, n\} \).

Hence, for every \( j \in \{1, \ldots, n\} \), there exist \( y^j \in D, z^j \in R \) be such that
\[
\sigma(y^j) = x^j
\]
\[
\sigma(z^j) = z^j
\]
and both \( y^j \neq x^i \) and \( z^j \neq x^i \), for every \( i, j \).

We then introduce the following construction. Consider two copies of \( K \), namely \( K_1, K_2 \), and paste them together according to \( \sigma \), that is, looking at \( \sigma \) as a map from \( D_1 \) into \( R_2 \), identify each element of \( D_1 \) with its image through \( \sigma \) in \( R_2 \). This defines a compact graph \( K' = (V(K'), E(K')) \) with
\[
V(K') = \left( V(K_1) \cup V(K_2) \right) \setminus \{ a \sim b \iff a \in D_1, b \in R_2, \sigma(a) = b \}
\]
\[
E(K') = E(K_1) \cup E(K_2).
\]
Notice that, as vertices of \( K' \), each \( y^j_1 \) is identified with the corresponding \( x^j_2 \), and each \( x^i_1 \) is identified with the corresponding \( z^j_2 \), for every \( j \in \{1, \ldots, n\} \). We denote by \( v^{1,j} \in V(K') \) the element of \( V(K') \) resulting from the identification
of $y_1^j$ and $x_2^j$, and by $v^{2,j} \in V(\mathcal{K}')$ the element of $V(\mathcal{K}')$ resulting from the identification of $x_1^j$ and $z_2^j$.

On the one hand, since $\sigma(x_i^j) \neq x_i^j$, for every $i, j$, no $x_1^j$ is identified with any $x_2^j$, yielding at $v^{1,j} \neq v^{2,j}$, for every $i, j \in \{1, \ldots, n\}$.

On the other hand, as $x_2^j \in D_2$ for every $j$, thinking of $D_2$ as a subset of $V(\mathcal{K}')$, we have $v^{1,j} \in D_2$. Similarly, as $x_1^j \in R_1$ for every $j$, thinking of $R_1$ as a subset of $V(\mathcal{K}')$, we get $v^{2,j} \in R_1$.

Therefore, setting $D', R' \subset V(\mathcal{K}')$ to be

$$D' := D_2 \quad R' := R_1$$

and noting that $D_2 = (D_2/(D_2 \cap R_2)) \cup \{v^{1,j} : j = 1, \ldots, n\}$ and $R_1 = (R_1/(D_1 \cap R_1)) \cup \{v^{2,j} : j = 1, \ldots, n\}$, it follows that $D' \cap R' = \emptyset$.

We then define $\sigma' : D' \to R'$

$$\sigma'(v) := \begin{cases} 
\sigma(v) & \text{if } v \neq v^{1,j}, y_2^j, \text{ for every } j = 1, \ldots, n \\
v^{2,j} & \text{if } v = y_2^j, \text{ for some } j = 1, \ldots, n \\
z_1^j & \text{if } v = v^{1,j}, \text{ for some } j = 1, \ldots, n, 
\end{cases}$$

which is a bijection by construction and the fact that $\sigma$ is a bijection.

It is readily seen that pasting together $n$ copies $\mathcal{K}'_1, \ldots, \mathcal{K}'_n$ of $\mathcal{K}'$ according to $\sigma'$, identifying each element of $D_i'$ with its image through $\sigma'$ in $R_{i+1}'$, for every $i = 1, \ldots, n-1$, is equivalent to paste together $2n$ copies $\mathcal{K}_1, \ldots, \mathcal{K}_{2n}$ of $\mathcal{K}$ according to $\sigma$, identifying each element of $D_i$ with its image through $\sigma$ in $R_{i+1}$, for every $i = 1, \ldots, 2n-1$.

Hence, let $\mathcal{G}'$ be the periodic graph determined by the periodicity cell $\mathcal{K}'$ and the pasting rule $\sigma'$ according to Definition 2.1. Considering, for every $i \in \mathbb{Z}$, the natural bijections between the vertices and edges of $\mathcal{K}'_i$ and the ones of $\mathcal{K}_{2i-1} \cup \mathcal{K}_{2i}$, it follows that $\mathcal{G}' = \mathcal{G}$, and case (b) holds.

To conclude, it remains to drop the assumption that $\sigma(x_i^j) \neq x_i^j$, for every $i, j \in \{1, \ldots, n\}$, which only reflects into a minor modification of the previous argument (Fig. 7).

Indeed, suppose that there exist $x_{j_1}^1, \ldots, x_{j_m}^1$ such that $\sigma(x_{j_1}^1) = x_{j_2}^1$, $\sigma(x_{j_2}^1) = x_{j_3}^1, \ldots, \sigma(x_{j_m}^1) = x_{j_m}^1$. Moreover, let $y_{j_1}, z_{j_m}$ be so that $\sigma(y_{j_1}) = x_{j_1}, \sigma(z_{j_m}) = z_{j_m}$, and $y_{j_1}, z_{j_m} \neq x_i^j$, for every $i = 1, \ldots, n$.

For the sake of simplicity, assume also that no other $i, j$ satisfies $\sigma(x_i^j) = x_i^j$. However, this does not cause any loss of generality, and what follows straightforwardly adapts to cover the more general situation.

We then consider $m+2$ copies of $\mathcal{K}$, say $\mathcal{K}_0, \mathcal{K}_1, \ldots, \mathcal{K}_{m+1}$, and we paste together $\mathcal{K}_i$ with $\mathcal{K}_{i+1}$, for every $i = 0, \ldots, m$, according to $\sigma$, that is, we identify each element of $D_i$ with its image through $\sigma$ in $R_{i+1}$, for every $i = 0, \ldots, m$. A compact graph $\mathcal{K}'$ arises from this procedure, and we introduce a shorthand notation for the following identifications.
\[ D = \{ y, x \} \quad \sigma(y) = x \]
\[ R = \{ x, z \} \quad \sigma(x) = z \]

**Figure 7.** Example of \( D, R \) and \( \sigma \) as in part (ii) of the proof of Proposition A.1, the graph \( K' \), and the resulting graph \( G \)

\[
\begin{align*}
v^{1,1} & := x_{m+1}^{j_1} \sim x_m^{j_1} \sim \ldots \sim x_1^{j_1} \sim y_0^{j_1} \\
v^{1,2} & := x_{m+1}^{j_2} \sim x_m^{j_2} \sim \ldots \sim x_2^{j_2} \sim y_1^{j_2} \\
& \quad \vdots \\
v^{1,m} & := x_0^{j_1} \sim y_m^{j_1}
\end{align*}
\]

and

\[
\begin{align*}
v^{2,1} & := x_0^{j_1} \sim x_1^{j_1} \sim \ldots \sim x_m^{j_1} \sim z_m^{j_1} \\
v^{2,2} & := x_0^{j_2} \sim x_1^{j_2} \sim \ldots \sim x_m^{j_2} \sim z_m^{j_2} \\
& \quad \vdots \\
v^{2,m} & := x_0^{j_m} \sim z_1^{j_m}.
\end{align*}
\]

As it is immediate to verify, \( v^{1,i} \neq v^{2,j} \), for every \( i, j \in \{1, \ldots, m\} \). Furthermore, looking at \( R_0 \) and \( D_{m+1} \) as subsets of \( V(K') \), we get \( v^{1,i} \in D_{m+1} \) and \( v^{2,i} \in R_0 \) for every \( i = 1, \ldots, m \).
Therefore, setting\[ D' := D_{m+1} \quad R' := R_0 \]
and \( \sigma' : D' \to R' \)
\[
\sigma'(v) := \begin{cases} 
\sigma(v) & \text{if } v \neq v^{1,j} \text{ for every } j = 1, \ldots, m \\
v^{2, m+2-j} & \text{if } v = v^{1,j}, \text{ for some } j = 2, \ldots, m \\
z_{0}^{m} & \text{if } v = v^{1,1}, 
\end{cases}
\]
it can be easily verified that \( D' \cap R' = \emptyset, \sigma' \) is a bijection from \( D' \) to \( R' \) and that the periodic graph \( G' \) given by Definition 2.1 with periodicity cell \( K' \) and pasting rule \( \sigma' \) satisfies \( G' = G \), so that (b) is proved to hold again. \( \square \)

Remark A.1. Metric graphs as in case (a) of Proposition A.1 can be called star-like graphs (see Fig. 6). Since graphs like this do not satisfy \( \text{diam}(G) = +\infty \), we do not want to take them into account in the present paper, so that assuming \( D \cap R = \emptyset \) in Definition 2.1 is actually useful to avoid such situation.

Proposition A.2. Let \( K \) be a fixed compact graph, \( D, R \) two non-empty subsets of \( V(K) \) such that \( D \cap R = \emptyset \), and \( \sigma : D \to R \). Suppose \( \sigma \) is not bijective and let \( G \) be as in Definition 2.1. Then, there exists a compact graph \( K' \), two non-empty subsets \( D', R' \subset V(K') \), with \( D' \cap R' = \emptyset \), and a bijection \( \sigma' : D' \to R' \), such that if \( G' \) is the periodic graph with periodicity cell \( K' \) and pasting rule \( \sigma' \), then \( G = G' \).

Proof. Note first that we can always assume that \( \sigma \) is surjective. Indeed, if this is not the case, we simply redefine \( R \) getting rid of the vertices with no pre-images in \( D \).

Suppose now that \( \sigma \) is not injective. Without loss of generality, assume that there exist only two vertices \( s, t \in D \) so that \( \sigma(s) = \sigma(t) = \tau \), for some \( \tau \in R \), as the argument we discuss below plainly generalizes to the case of more than one couple of elements of \( D \) sharing the same image through \( \sigma \).

Let us now introduce the following construction (see Fig. 8). Starting from \( K \), we identify \( s \) and \( t \), thus defining a new graph \( K' = (V(K'), E(K')) \) so that
\[
V(K') = V(K)/\{s\sim t\} \\
E(K') = E(K).
\]
By construction, as vertices of \( K' \), \( s \) and \( t \) correspond to the same element, say \( \overline{v} \in V(K') \). Moreover, as \( s, t \in D \), thinking of \( D \) as a subset of \( V(K') \), we get \( \overline{v} \in D \). Hence, setting\[ D' := D \quad R' := R \]
and defining \( \sigma' : D' \to R' \) as
\[
\sigma'(v) := \begin{cases} 
\sigma(v) & \text{if } v \neq \overline{v} \\
\overline{v} & \text{if } v = \overline{v}, 
\end{cases}
\]
it is immediate to see that $D' \cap R' = \emptyset$ and $\sigma'$ is a bijection from $D'$ into $R'$. Moreover, notice that pasting together two copies of $K'$ according to $\sigma'$ is equivalent to paste together two copies of $K$ according to $\sigma$.

Therefore, let $G'$ be the periodic graph as in Definition 2.1 with periodicity cell $K'$ and pasting rule $\sigma'$. Considering, for every $i \in \mathbb{Z}$, the natural bijections between the vertices and the edges of $K_i'$ and the ones of $K_i$, we have $G = G'$ and we conclude. □

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