Ergotropy and entanglement in critical spin chains

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A subsystem of an entangled ground state is in a mixed state. Thus, if we isolate this subsystem from its surroundings we may be able to extract work applying unitary transformations, up to a maximal amount which is called ergotropy. Once this work has been extracted, the subsystem will still contain some bound energy above its local ground state, which can provide valuable information about the entanglement structure. We show that the bound energy for half a free fermionic chain decays as the square of the entanglement entropy divided by the chain length, thus approaching zero for large system sizes, and we conjecture that this relation holds for all 1D critical states.

I. INTRODUCTION

Quantum thermodynamics applies the core concepts of quantum information theory [1–3] to design optimal nanoscale devices, such as quantum thermal machines [4–6]. A very fruitful concept is that of ergotropy [7, 8], i.e. the maximal work that can be reversibly extracted from a mixed state, which is a crucial tool in order to build efficient quantum batteries [9–11]. Indeed, ergotropy is known to be strongly influenced by the presence of quantum correlations of different types [12–16]. Of course, if we lift the reversibility constraint, we may use quantum measurements to extract work in an optimal way [17, 18].

Yet, the connection works in both directions, and we may employ quantum thermodynamics to characterize the entanglement structure of a quantum system. As it is well known, a subsystem of a ground state (GS) is usually not in its local ground state. Instead, it must be described by a reduced density matrix, which can be expressed as a thermal density matrix under a certain entanglement Hamiltonian (EH), which need not coincide with the local one [19–20]. Notice that the EH allows us to describe the entanglement structure of complex quantum states in thermal terms. Both the EH and its eigenvalues, which define the entanglement spectrum (ES) [21], have provided invaluable insight to characterize the entanglement structure of the low energy states of quantum many-body systems [22–30], in some cases exploiting their conformal invariance [31–33].

In this work we introduce the notion of subsystem ergotropy within a GS in order to characterize its entanglement structure through the analysis of the energetic relations between a subsystem \(A\) and its environment \(B\). The expected value of the local energy of any subsystem will typically exceed its own GS energy, and the subsystem ergotropy is defined as the part that can be extracted in the form of work. Our analysis will focus on a few simple quantum many-body systems, starting with a detailed analysis of free fermionic chains, and extending our study to other critical spin chains. In all the considered cases, we benefit from the constraints imposed by conformal invariance on the reduced density matrix. We show that, once the maximal work has been extracted, the remaining bound energy presents universal scaling as the square of the entanglement entropy of the block divided by the system size, thus approaching zero for large system sizes.

This article is organized as follows. Section II develops the basic theoretical background, combining tools from quantum thermodynamics and quantum information theory. Then we show our analytical and numerical calculations for a free fermionic chain in Sec. III. Other critical spin chains, such as the Ising model in a transverse field or the Heisenberg model, are briefly considered in IV. The article ends with a section describing our conclusions and suggestions for further work.

II. THEORETICAL BACKGROUND

A. Ergotropy of generic mixed states

The ergotropy \(W\) of a mixed state \(\rho\) with respect to a Hamiltonian \(H\) can be defined as the maximal amount of work that can be extracted from the state by applying unitary operations [7, 8], i.e.

\[
W \equiv \max_{\mathcal{U}} \left( \text{Tr}(\rho H) - \text{Tr}(\mathcal{U} \rho \mathcal{U}^\dagger H) \right),
\]

where \(\mathcal{U}\) is any unitary transformation. Alternatively, it can be shown [7] that the ergotropy corresponds to the maximal work that can be reversibly extracted from the system, but the former characterization suits our purposes better. A state defined by a density matrix \(\rho\) is called passive with respect to \(H\) when its ergotropy is zero, i.e. when we cannot extract any work from it by performing unitary operations. In that case, the eigenstates of \(H\) and \(\rho\) must be aligned such that the highest probability state of \(\rho\) will correspond to the lowest eigenstate of \(H\), and so on. Thermal states built on \(H\), written as \(\rho = Z^{-1} \exp(-\beta H)\) with \(\beta = 1/T\) the inverse temper-
Let decomposition is not unique, and we will assume that interaction Hamiltonian be called the

Figure 1. Illustrating the energies involved in our discussion of the subsystem ergotropy and their differences. Indeed, \( E_A \) denotes the expected value of the \( H_A \) within the global GS of \( H \). \( \tilde{E}_A \) is the minimal energy achieved through unitary operations on \( \mathcal{H}_A \) and \( E_{A,0} \) is the GS of \( H_A \). Moreover, \( \Delta E_A = E_A - E_{A,0} \) is the excess energy, \( W_A = E_A - \tilde{E}_A \) is the subsystem ergotropy and \( Q_A = E_A - E_{A,0} \) is the subsystem bound energy. The blue arches denote entanglement, as it is explained in the text. Notice that, in order to define these energies, block \( A \) must be physically separated from its environment.

\[
|\Psi\rangle = \sum_{k=1}^{\chi} p_k^{1/2} |\phi_k^A\rangle \otimes |\phi_k^B\rangle,
\]

where \( |\phi_k^A\rangle \in \mathcal{H}_A \), \( |\phi_k^B\rangle \in \mathcal{H}_B \) are two orthonormal sets, \( p_k \geq 0 \) (also in non-increasing order) and \( \chi \leq \min(\dim(\mathcal{H}_A), \dim(\mathcal{H}_B)) \) is the Schmidt number. The reduced density matrix for part \( A \) can be written as

\[
\rho_A = \sum_{k=1}^{\chi} p_k |\phi_k^A\rangle \langle \phi_k^A|.
\]

Being positive definite, this matrix can always be written as a thermal density matrix,

\[
\rho_A = \exp(-K_A),
\]

where \( K_A \) is called the entanglement Hamiltonian (EH) associated to part \( A \). Of course, \( K_A \) need not be equal to \( H_A \), the local Hamiltonian, and this difference will be crucial in what follows. Also, let us introduce the entanglement spectrum (ES) as the spectrum of the EH [21].

Now let us physically separate subsystem \( A \) from its environment, i.e. subsystem \( B \), by suddenly quenching \( H_{AB} \) to zero. The subsequent behavior of our subsystem will be described by \( H_A \), with spectrum \( \{ E_{A,k} \} \), which we may assume to be non-degenerate. We define the three energies involved in our problem:

- \( E_A = \langle \Psi | H_A \otimes I_B | \Psi \rangle \), the expected value of \( H_A \) in the global GS.
\[ \tilde{E}_A = \sum_k p_k E_{A,k}, \] the passive energy of the system, obtained through unitary transformations.

- \( E_{A,0} \), the GS of \( H_A \).

These three energies must be in descending order, 
\[ E_A \geq \tilde{E}_A \geq E_{A,0}. \] We define the excess energy as
\[ \Delta E_A = E_A - E_{A,0}. \] The subsystem ergotropy can be computed as
\[
W_A = E_A - \tilde{E}_A, 
\]
while
\[
Q_A \equiv \tilde{E}_A - E_{A,0},
\]
denotes the amount of energy which is unavailable, which we will call the subsystem bound energy [3]. See Fig. 1 for an illustration. The top panel represents the GS of \( H \), and \( E_A \) is the energy associated to block \( A \). The light blue archs represent the entanglement links which characterize the entanglement structure. We reach the middle panel applying a suitable unitary operator on block \( A \), maximally reducing its energy to \( \tilde{E}_A \) while preserving the entanglement spectrum and, a fortiori, the amount of entanglement with the rest of the system, which in this figure is represented by the number of links leaving \( A \). The newly established links are now denoted in dark blue. Finally, the lowest panel denotes the GS of \( H_A \), which is now disentangled from the environment, with energy \( E_{A,0} \).

C. Ergotropy and time evolution

Once we have split the subsystem \( A \) from its environment, it will evolve under the action of its local Hamiltonian, \( H_A \), following von Neumann’s equation,
\[
\ih \partial_t \rho_A = [H_A, \rho_A].
\]
Remarkably, this time evolution preserves both the expected value of the energy, \( E_A \), and the full spectrum of the density matrix, even though the subsequent dynamics can be complex [4–6]. It is relevant to ask how much work we can obtain from this time-evolved density matrix employing unitary transformations, i.e. how the ergotropy evolves after the split quench. The answer is that the ergotropy is exactly preserved along the time evolution. A proof of this fact is straightforward. The time-evolved density matrix for the subsystem after the split can be written as \( \rho_A(t) = V(t) \rho_A(0) V^\dagger(t) \) for some unitary transformation \( V(t) \). The ergotropy of this matrix, defined in Eq. (1), is exactly the same, because the associated passivized state, given in Eq. (2), is exactly the same, if we just use the identity
\[
\rho_A = U \rho_A(0) U^\dagger = UV^\dagger(t) \rho_A(t) V(t) U^\dagger, \]
allowing us to define a new unitary transformation, \( \tilde{U} = UV^\dagger(t) \), such that \( \tilde{\rho}_A = \tilde{U} \rho_A \tilde{U}^\dagger \). This result implies that the work extraction procedure need not start immediately after the disconnection between the subsystem and its environment, as long as the subsequent evolution is unitary.

D. Interaction energy inequality

Thus, we can extract work from a subsystem of a composite quantum state in its GS. Yet, this work should always be less than the corresponding increase in the energy of the system induced by our interaction, because otherwise the current system energy would be lower than the GS energy, \( \tilde{E} \). We can prove this result easily. After the unitary transformation on subsystem \( A \) the global system will be \( |\Psi\rangle \), such that
\[
\langle \tilde{\Psi} | H | \tilde{\Psi} \rangle = \tilde{E} = \tilde{E}_A + \tilde{E}_B + \tilde{E}_{AB},
\]
where each term on the rhs corresponds to the expectation value of one of the three operators, \( H_A, H_B \) and \( H_{AB} \) on \( |\Psi\rangle \), and we notice that \( \tilde{E}_B = E_B \). This energy \( \tilde{E} \geq E \), the GS energy, which can be decomposed equally, \( E = E_A + E_B + E_{AB} \). Taking into account that \( E_A - \tilde{E}_A = W_A \), we obtain
\[
\tilde{E}_{AB} - E_{AB} \geq W_A \geq 0.
\]
which implies that the gain through ergotropy must be less or equal than the loss in the interaction term.

III. ERGOTROPY OF A FREE FERMIONIC CHAIN

We now particularize the previous calculation to the case of a free fermionic chain, before extending our results to other critical spin chains. As we will show, the ergotropy and bound energy of free fermionic chains can be explicitly computed and present universal features associated to conformal invariance, in similarity to the Casimir energy [39–41]. For simplicity, we will restrict ourselves to the case in which the block \( A \) corresponds to the left half of the chain.

A. Free fermionic chains

Let us consider a fermionic chain of \( N \) (even) sites with open boundaries, described by the Hamiltonian
where $c_i^\dagger$ and $c_i$ denote the fermionic creation and annihilation operators on site $i$ and $J_{ij} = J_{ji}$ denotes the hopping matrix. We will focus on the homogeneous chain with open boundaries, whose hopping amplitudes are given by $J_{ij} = \delta_{i,j\pm 1}$. In this case, the low energy behavior of the chain can be accurately represented by a conformal field theory (CFT) \[34\] \[35\].

The GS of Hamiltonian \( (16) \) can be obtained through the eigenvalues \( \{\varepsilon_k\} \) (in increasing order) and eigenmodes \( \{U_{k,i}\} \) of the hopping matrix \( J_{ij} \), which are usually called single-body energies and modes, respectively. The spectrum presents particle-hole symmetry, \( \varepsilon_k = -\varepsilon_{N+1-k} \), and the GS is obtained by filling up the \( N/2 \) negative energy modes, such that

\[
E = \sum_{k=1}^{N/2} \varepsilon_k,
\]

while the corresponding eigenstate is a Slater determinant determined by its correlator matrix, defined as

\[
C_{ij} \equiv \langle c_i^\dagger c_j \rangle = \sum_{k=1}^{N/2} \bar{U}_{k,i}U_{k,j}.
\]

All the entanglement properties can be determined from matrix \( C \). Indeed, the reduced density matrix of any block \( A \) of size \( \ell \) can be obtained diagonalizing the corresponding \( \ell \times \ell \) submatrix, \( C_A \). The set \( \{\nu_k^A\} \) of eigenvalues of \( C_A \), where each \( \nu_k^A \in [0,1] \) determines uniquely the full GS, will be called entanglement occupations. The von Neumann entropy of block \( A \) can be expressed as \[24\]

\[
S_A = -\sum_{k=1}^{\ell} (\nu_k^A \log(\nu_k^A) + (1-\nu_k^A) \log(1-\nu_k^A)).
\]

Conformal symmetry fixes the universal part of the entanglement entropy of a lateral block \( A = \{1,\ldots,\ell\} \) of a critical chain with \( N \) sites, \[36\] \[38\]

\[
S_A \approx \frac{c}{6} \log \left( \frac{N}{\pi} \sin \left( \frac{\pi \ell}{N} \right) \right) + c',
\]

where \( c = 1 \) is the central charge of the associated CFT \[34\] \[35\] and \( c' \) is a non-universal constant. Moreover, the EH of a free fermionic chain must also present a free fermionic form, Eq. \[10\], with a different hopping matrix \[19\] \[20\].

\[
H_N = -\sum_{i,j=1}^{N} J_{ij} c_i^\dagger c_j,
\]

The single-body energies of the EH, \( \varepsilon_k^A \), can be obtained from the entanglement occupations through the Fermi-Dirac expression,

\[
\nu_k^A = \frac{1}{1+\exp(\varepsilon_k^A)},
\]

and they are (approximately) equally spaced, with a level separation given by the so-called entanglement gap, \( \varepsilon_A \approx \varepsilon_{k+1}^A - \varepsilon_k^A \), which is known to behave like \[32\]

\[
\varepsilon_A \approx \frac{2\pi^2}{\log(\gamma N)},
\]

where \( \log \gamma \approx 2.3 \) is a non-universal constant \[32\]. Moreover, an approximate inverse relation has been proposed between the entanglement gap and the entanglement entropy,

\[
\varepsilon_A S_A \approx \frac{\pi^2}{3}.
\]

B. Casimir energy and free fermions

Our next aim is to compute the three energies involved in our calculations: \( E_A, \bar{E}_A \) and \( E_{A,0} \). Let us start with \( E_{A,0} \) for convenience. We proceed to build \( H_A \), the hopping matrix for the block \( A \), and obtain its eigenvalues, \( \varepsilon_k^A \), \( k = 1,\ldots,N/2 \), in increasing order. The GS energy of \( A \) is given by

\[
E_{A,0} = \sum_{k=1}^{N/4} \varepsilon_k^A.
\]

An approximate expression for \( E_{A,0} \) as a function of \( N \) can be provided \[39\] \[41\]

\[
E_A(N) = -c_0(N-1) - c_B - \frac{c_F v_F}{24N} + O(N^{-2}),
\]

where we distinguish three terms. The first one, \(-c_0(N-1)\), with \( c_0 = 2/\pi \), is the bulk energy. The second term, \(-c_B = -(4/\pi - 1)\), is the boundary term. The third one provides the finite-size correction and is fixed by conformal invariance. Indeed, \( c = 1 \) is the central charge associated to our theory and \( v_F = 2 \) is the Fermi velocity. Thus, we have
\[ E_{A,0} \approx -c_0 \left( \frac{N}{2} - 1 \right) - c_B - \frac{\pi}{6N}. \]  

(27)

We can use a similar strategy to estimate \( E_A \), but we should proceed with care. Indeed, we can obtain \( E_A \) numerically from the GS of the whole chain, subtracting the energy associated to the central link and dividing by two,

\[ E_A = \frac{E_0(N)}{2} - C_{N/2,N/2+1}. \]  

(28)

The first term can be easily estimated from (27),

\[ \frac{E_0(N)}{2} \approx -c_0 \left( \frac{N}{2} - 1 \right) + \frac{c_0}{2} - \frac{c_B}{2} - \frac{\pi}{24N}, \]  

(29)

and the second one can be found making use of (18), giving rise to an alternating behavior,

\[ C_{n,n+1} \approx -c_0 \left( \frac{\pi}{24(N+1)^2} + \frac{(-1)^n}{2(N+1) \sin \left( \frac{\pi(n+1/2)}{N+1} \right)} \right), \]  

(30)

which, since \( N/2 \) is even, reduces for the central link to

\[ C_{N/2,N/2+1} \approx -c_0 \left( \frac{\pi}{24(N+1)^2} + \frac{1}{2(N+1)} \right), \]  

(31)

yielding

\[ E_A \approx -c_0 \left( \frac{N}{2} - 1 \right) - \frac{c_B}{2} - \frac{\pi}{24N} + \frac{1}{N}. \]  

(32)

We notice that the bulk term is exactly the same as for \( E_{A,0} \), and the boundary term is exactly half, as we would expect intuitively, since this subsystem only possesses one boundary instead of two. We should stress that a naive calculation would yield a Casimir correction \( \pi/(24N) \), but we obtain an additional contribution from the energy associated to the central link. The validity of the approximations to these two energies, \( E_{A,0} \) and \( E_A \), can be checked in Fig. 2.

Therefore, the excess energy, \( \Delta E_A = E_A - E_{A,0} \), is given by

\[ \Delta E_A \approx \frac{c_B}{2} + \left( \frac{\pi}{8} - \frac{1}{2} \right) \frac{1}{N}. \]  

(33)

C. Bound energy and entanglement

Extracting the maximal amount of work through unitary operators reversibly is equivalent to minimizing the block energy while preserving the full spectrum of the reduced density matrix. Thus, we proceed to align the occupation eigenvectors with the eigenstates of \( H_A \), whose eigenvalues will be denoted by \( \{ \varepsilon_k^A \} \). The passive energy \( \tilde{E}_A \) can be written as

\[ \tilde{E}_A = \sum_{k=1}^{N/2} \nu_k \varepsilon_k^A. \]  

(34)

Since \( E_A \leq \tilde{E}_A \leq E_{A,0} \), it is reasonable to consider that the passive energy \( \tilde{E}_A \) will also present the same bulk term as in Eq. (27), but with different corrections. Let us provide a similar asymptotic expansion to its value.

The eigenvalues of \( H_A \) can be found exactly,

\[ \varepsilon_p^A = -2 \cos \left( \frac{p\pi}{N/2 + 1} \right), \]  

(35)

with \( p \in \{ 1 \cdots N/2 \} \), and those of the correlation matrix \( C_A \) can also be approximated as

\[ \nu_p^A \approx \frac{1}{1 + \exp \left( -\beta \left( p - N/4 \right) \right)}, \]  

(36)
where $\beta$ corresponds to the entanglement gap, given in Eq. (23) [32]. Thus, the passive energy is given by

$$\tilde{E}_A = \sum_{p=1}^{N/2} \varepsilon_p^A v_p^A \approx \sum_{p=1}^{N/2} -\frac{2\cos(2\pi p/N)}{1 + e^{-\pi p N/4}}. \quad (37)$$

If we take the continuum limit, making use of the Sommerfeld expansion [17] and the Euler-Maclaurin formula, we arrive at

$$\tilde{E}_A \approx -c_0 \left( \frac{N}{2} - 1 \right) - c_B - \frac{\pi}{6N} + \frac{2\pi^3}{3N\beta^2}, \quad (38)$$

so we obtain the final form

$$\tilde{E}_A \approx -c_0 \left( \frac{N}{2} - 1 \right) - c_B - \frac{\pi}{6N} + \frac{\log^2(\gamma N)}{6\pi N}. \quad (39)$$

We may now find the analytic expression for the ergotropy,

$$W_A = E_A - \tilde{E}_A \approx \frac{c_B}{2} + \frac{\pi}{8} \frac{1}{2} \frac{1}{N} - \frac{\log^2(\gamma N)}{6\pi N}, \quad (40)$$

where the requirement $W_A \geq 0$ demands that $c_B > 0$. This expression can be checked in the bottom panel of Fig. 2. Furthermore, we can estimate the bound energy,

$$Q_A = \tilde{E}_A - E_{A,0} \approx \frac{\log^2(\gamma N)}{6\pi N} \geq 0, \quad (41)$$

which is unconditionally positive, and can also be checked in the bottom panel of Fig. 2. Notice that Eq. (41) implies that the bound energy is directly related to the inverse squared of the entanglement gap of the system, or the square of the entanglement entropy. Using Eq. (20) and Eq. (23), we obtain an approximate relation

$$Q_A N \approx \frac{6}{\pi} \tilde{S}_A^2, \quad (42)$$

which provides a relation between the entanglement entropy of a block of a free fermionic chain and the bound energy associated. Eq. (42) is the main prediction of this work, and we conjecture that its validity extends beyond the case of free fermionic chains, to any critical state in 1D described by a conformal field theory. The validity of this expression can be numerically checked in Fig. 3.

We may define an ergotropy fraction $w_A = W_A/\Delta E_A$ and a bound fraction, $q_A = Q_A/\Delta E_A$, as the ratios between the ergotropy or the bound energy to the excess energy. We can see that $w_A \rightarrow 1$ and $q_A \rightarrow 0$ as $N \rightarrow \infty$, implying that for larger systems we can extract most of the excess energy in the form of work using unitary transformations.

Figure 3. Numerical check of the linear relation between the bound energy multiplied by the system size, $Q_A N$, and the entanglement entropy squared, $\tilde{S}_A^2$, for the free fermionic chains, Eq. (42), for sizes $N$ in the same range as in Fig. 2. The slope of the straight line, as expected, is $6/\pi \approx 1.9$.

IV. PRELIMINARY RESULTS ON OTHER CRITICAL MODELS

We have considered two other spin chains, the critical Ising model in a transverse field (ITF) and the Heisenberg model, and performed numerical explorations using a combination of Lanczos and exact diagonalization for small systems which provide preliminary numerical evidence of the validity of Eqs. (41) and (42) for these systems.

The Hamiltonian of the ITF model that we have considered is given by

$$H_{ITF} = -\sum_{i=1}^{N-1} \sigma_i^z \sigma_{i+1}^z - \Gamma \sum_{i=1}^{N} \sigma_i^x, \quad (43)$$

for $\Gamma = 1$. The low energy eigenstates of $H_{ITF}$ are known to follow a conformal field theory with central charge $c = 1/2$ [34, 35]. Therefore, the entanglement entropy of the left half can be written as a linear function of $\log(N)$. We have obtained preliminary numerical results employing exact diagonalization up to size $N = 14$, which are shown in Fig. 4. In the top panel we show with points the energy decomposition, $\Delta E_A$, $W_A$ and $Q_A$, for the left-half chain of the even sized systems, along with their fits with continuous lines to theoretical curves suggested by the generalization of Eqs. (33), (40) and (41), i.e.

$$\Delta E_A \approx \frac{\alpha_1}{N} - \frac{\alpha_2}{N}, \quad W_A \approx \frac{\alpha_1}{N} - \frac{\alpha_2}{N} \log^2(\alpha_4 N)/N, \quad Q_A \approx \frac{\log^2(\alpha_4 N)}{N}. \quad (44)$$
Bethe Ansatz, but we have chosen to obtain it using the Wigner transformation \[34, 35\]. As it was mentioned to an interacting fermion Hamiltonian using the Jordan-Wigner transformation, given by the Hamiltonian

\[
H_{\text{Heisenberg}} = \sum_{i=1}^{N-1} \vec{S}_i \cdot \vec{S}_{i+1},
\]

which also corresponds to a conformal field theory for low energies, with \( c = 1 \) in this case, and can be mapped to an interacting fermion Hamiltonian using the Jordan-Wigner transformation \[34, 35\]. As it was mentioned above, the GS can be analytically obtained using the Bethe Ansatz, but we have chosen to obtain it using the Lanczos algorithm up to \( N = 24 \), taking into account the full SU(2) symmetry of the model. The top panel of Fig. 4 shows the energy decomposition for the left-half of the chain, along with only values of \( N \) which are multiples of four. Again, we plot along a fit of these 18 points to the form \[44\], obtaining approximate parameters \( \alpha_1 \approx 0.44, \alpha_2 \approx 0.9, \alpha_3 \approx 0.41 \) and \( \alpha_4 \approx 1.32 \). The bottom panel of Fig. 5 shows the linear relation between \( Q_A N \) and \( \log^2(N) \), highlighting the validity of Eq. \[42\], again comparing to a linear fit for the largest sizes.

The approximate validity of Eq. \[42\] in all three models is related to the fact that it only depends on the following facts:

- The Casimir expression for the energy of the GS.
- The affine relation between the entanglement entropy and \( \log(N) \).
- The approximate inverse relation between the entanglement entropy and the entanglement gap.

All these relations stem from conformal invariance, a property shared by all three models discussed in this work.

Figure 4. Subsystem energy decomposition for small Ising critical chains, with \( N \) up to 14. Top: Energies \( \Delta E_A, W_A \) and \( Q_A \) for the left-half chain as a function of the system size, along with the expected theoretical fits. Bottom: Approximate linear relation between \( Q_A N \) and \( \log^2(N) \), showing the expected relation between \( Q_A \) and \( S_A, \) Eq. \[42\], along with a linear fit to the last five points.

Figure 5. Subsystem energy decomposition for small Heisenberg chains, with \( N \) up to 24, using only multiples of four. Top: Energies \( \Delta E_A, W_A \) and \( Q_A \) for the left-half chain, along with the expected theoretical fits; Bottom: Approximate linear relation between \( Q_A N \) and \( \log^2(N) \), showing the expected relation between \( Q_A \) and \( S_A, \) Eq. \[42\], along with a linear fit to the last five points.

In our case the optimal values of the parameters are \( \alpha_1 \approx 0.137, \alpha_2 \approx 0.07, \alpha_3 \approx 0.044 \) and \( \alpha_4 \approx 5.5 \). We would like to stress that we fit the 21 points of the three curves using the same values for the \( \alpha_i \) parameters. In the bottom panel of Fig. 4 we observe an approximate linear relation between \( N Q_A \) and \( \log^2(N) \), as expected, along with a linear fit obtained from the larger systems. Even though the functional form is shown to be approximately correct, we should use these fitting parameters with care, due to the small system size.

On the other hand, we have considered the antiferromagnetic spin 1/2 Heisenberg chain with open boundaries, given by the Hamiltonian

\[
H_{\text{Heisenberg}} = \sum_{i=1}^{N-1} \vec{S}_i \cdot \vec{S}_{i+1},
\]

which also corresponds to a conformal field theory for low energies, with \( c = 1 \) in this case, and can be mapped to an interacting fermion Hamiltonian using the Jordan-Wigner transformation \[34, 35\]. As it was mentioned above, the GS can be analytically obtained using the Bethe Ansatz, but we have chosen to obtain it using the Lanczos algorithm up to \( N = 24 \), taking into account the full SU(2) symmetry of the model. The top panel of Fig. 4 shows the energy decomposition for the left-half of the chain, along with only values of \( N \) which are multiples of four. Again, we plot along a fit of these 18 points to the form \[44\], obtaining approximate parameters \( \alpha_1 \approx 0.44, \alpha_2 \approx 0.9, \alpha_3 \approx 0.41 \) and \( \alpha_4 \approx 1.32 \). The bottom panel of Fig. 5 shows the linear relation between \( Q_A N \) and \( \log^2(N) \), highlighting the validity of Eq. \[42\], again comparing to a linear fit for the largest sizes.
It would be interesting to check the validity of our preliminary results for larger system sizes in the ITF and Heisenberg cases. The ITF case can be evaluated using a combination of Jordan-Wigner and Bogoliubov transformations. The Heisenberg case is more involved, since e.g., the density matrix renormalization group (DMRG) cannot be used in a straightforward manner [48], because we need to use both the entanglement spectrum and the full energy spectrum of the subsystem.

V. CONCLUSIONS AND FURTHER WORK

In this work we have considered the excess energy possessed by a subsystem of a ground state. Part of this excess energy can be extracted via unitary operations, which we call subsystem ergotropy, and part of it cannot be extracted in this way, which we call subsystem bound energy. For concreteness, we have considered one-dimensional systems which present conformal invariance, and we have done the calculations in detail for free fermionic chains, combining numerical calculations with a detailed analysis of the Casimir corrections to the GS energy. The most relevant relation found is a linear functional dependence between the subsystem bound energy and the square of its entanglement entropy divided by the system size. We have shown that this relation is likely to apply to other critical spin chains, thus allowing us to conjecture that its validity will extend to all 1D conformal field theories.

We would like to stress that, as the system size grows, the fraction of excess energy which can be extracted as work approaches one. In other words: almost all the subsystem energy becomes available in the thermodynamic limit. This result is non-trivial, although it responds to our intuition that for larger systems we have a larger freedom to manipulate the local mixed state. It is relevant to ask how general this result is. For instance, we may wonder about the behavior of the subsystem ergotropy away from criticality, i.e., for dimerized spin chains or for the Ising model with a non-critical value of the transverse field $\Gamma$, or how to extend it to higher dimensional systems.

Our results encourage further exploration of the application of quantum thermodynamics to the analysis and characterization of entanglement. Beyond the quantitative study of the ergotropy and bound energies, it is relevant to ask about the passive state which we obtain when all the ergotropy has been obtained. Indeed, it must be a thermal state under the entanglement Hamiltonian, but it is also relevant to ask about its properties under its own local Hamiltonian, and how do these two Hamiltonians relate. Given the relation between the entanglement Hamiltonian and the Urruh effect [19-51], this research programme may bear fruits also to the interplay between gravity, entanglement and thermodynamics.

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