BLOW-UP PROBLEM FOR NONLINEAR NONLOCAL PARABOLIC EQUATION WITH ABSORPTION UNDER NONLINEAR NONLOCAL BOUNDARY CONDITION

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Abstract. In this paper we consider initial boundary value problem for nonlinear nonlocal parabolic equation with absorption under nonlinear nonlocal boundary condition and nonnegative initial datum. We prove comparison principle, global existence and blow-up of solutions.

1. Introduction

In this paper we consider the initial boundary value problem for nonlinear nonlocal parabolic equation

\[ u_t = \Delta u + au^p \int_{\Omega} u^q(y, t) \, dy - bu^m, \quad x \in \Omega, \quad t > 0, \]

(1.1)

with nonlinear nonlocal boundary condition

\[ \frac{\partial u(x, t)}{\partial \nu} = \int_{\Omega} k(x, y, t) u^l(y, t) \, dy, \quad x \in \partial \Omega, \quad t > 0, \]

(1.2)

and initial datum

\[ u(x, 0) = u_0(x), \quad x \in \Omega, \]

(1.3)

where \( a, b, p, q, m, l \) are positive numbers, \( \Omega \) is a bounded domain in \( \mathbb{R}^N \) for \( N \geq 1 \) with smooth boundary \( \partial \Omega \), \( \nu \) is unit outward normal on \( \partial \Omega \).

Throughout this paper we suppose that the functions \( k(x, y, t) \) and \( u_0(x) \) satisfy the following conditions:

\[ k(x, y, t) \in C(\partial \Omega \times \overline{\Omega} \times [0, +\infty)), \quad k(x, y, t) \geq 0; \]

\[ u_0(x) \in C^1(\overline{\Omega}), \quad u_0(x) \geq 0 \text{ in } \Omega, \quad \frac{\partial u_0(x)}{\partial \nu} = \int_{\Omega} k(x, y, 0) u_0^l(y) \, dy \text{ on } \partial \Omega. \]

Various phenomena in the natural sciences and engineering lead to the nonclassical mathematical models subject to nonlocal boundary conditions. For global existence and blow-up of solutions for parabolic equations and systems with nonlocal boundary conditions we refer to \[ [1, 5, 8, 16, 17, 18, 21, 22, 23, 26, 31, 32] \] and the references therein. In particular, the blow-up problem for parabolic equations with nonlocal boundary condition

\[ u(x, t) = \int_{\Omega} k(x, y, t) u^l(y, t) \, dy, \quad x \in \partial \Omega, \quad t > 0, \]

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was considered in [4, 6, 9, 12, 28]. Initial boundary value problems for parabolic equations with nonlocal boundary condition (1.2) were studied in [13, 14, 20, 27, 29]. So, the problem (1.1)–(1.3) with \( a = 0 \) was investigated in [10, 11]. Initial-boundary value problems for nonlocal parabolic equations with nonlocal boundary conditions were addressed in many papers also (see, for example, [3, 7, 15, 25, 34, 35, 36]). In particular, some global existence and blow-up results for (1.1)–(1.3) were obtained in [33].

The aim of this paper is to investigate global existence and blow-up of solutions of (1.1)–(1.3).

This paper is organized as follows. In the next section we prove comparison principle. The global existence of solutions for any initial data is proved in Section 3. In Section 4 we present finite time blow-up results for solutions with large initial data.

2. Comparison principle

In this section a comparison principle for (1.1)–(1.3) will be proved. We begin with definitions of a supersolution, a subsolution and a maximal solution of (1.1)–(1.3). Let \( Q_T = \Omega \times (0, T), S_T = \partial \Omega \times (0, T), \Gamma_T = S_T \cup \overline{\Omega} \times \{0\}, T > 0. \)

Definition 2.1. We say that a nonnegative function \( u(x, t) \in C^{2,1}(Q_T) \cap C^{1,0}(Q_T \cup \Gamma_T) \) is a supersolution of (1.1)–(1.3) in \( Q_T \) if

\[
\begin{align*}
  u_t &\geq \Delta u + au^p \int_{\Omega} u^q(y, t) \, dy - bu^m, \quad (x, t) \in Q_T, \\
  \frac{\partial u(x, t)}{\partial \nu} &\geq \int_{\Omega} k(x, y, t)u_1^l(y, t) \, dy, \quad x \in \partial \Omega, \quad 0 < t < T,
\end{align*}
\]

and \( u(x, t) \in C^{2,1}(Q_T) \cap C^{1,0}(Q_T \cup \Gamma_T) \) is a subsolution of (1.1)–(1.3) in \( Q_T \) if \( u \geq 0 \) and it satisfies (2.1)–(2.3) in the reverse order. We say that \( u(x, t) \) is a solution of problem (1.1)–(1.3) in \( Q_T \) if \( u(x, t) \) is both a subsolution and a supersolution of (1.1)–(1.3) in \( Q_T \).

Definition 2.2. We say that a solution \( u(x, t) \) of (1.1)–(1.3) in \( Q_T \) is a maximal solution if for any other solution \( v(x, t) \) of (1.1)–(1.3) in \( Q_T \) the inequality \( v(x, t) \leq u(x, t) \) is satisfied for \( (x, t) \in Q_T \cup \Gamma_T \).

Theorem 2.3. Let \( \overline{u} \) and \( \underline{u} \) be a supersolution and a subsolution of problem (1.1)–(1.3) in \( Q_T \), respectively. Suppose that \( \overline{u}(x, t) > 0 \) or \( \underline{u}(x, t) > 0 \) in \( Q_T \cup \Gamma_T \) if \( \min(p, q, l) < 1 \). Then \( \underline{u}(x, t) \geq \overline{u}(x, t) \) in \( Q_T \cup \Gamma_T \).

Proof. Suppose that \( \min\{p, q, l\} \geq 1 \). Let \( T_0 = (0, T) \) and \( \{\varepsilon_n\} \) be decreasing to 0 a sequence such that \( 0 < \varepsilon_n < 1 \). For \( \varepsilon = \varepsilon_n \) let \( u_{\varepsilon_n}(x) \) be the functions with the following properties: \( u_{\varepsilon_n}(x) \in C^{1}(\overline{\Omega}) \), \( u_{\varepsilon_n}(x) \geq \varepsilon \), \( u_{\varepsilon_n}(x) \geq u_{\varepsilon_j}(x) \) for \( \varepsilon_i > \varepsilon_j \), \( u_{\varepsilon_0}(x) \rightarrow u_0(x) \) as \( \varepsilon \rightarrow 0 \) and

\[
\frac{\partial u_{\varepsilon_n}(x)}{\partial \nu} = \int_{\Omega} k(x, y, 0)u_{\varepsilon_n}(y) \, dy
\]

for \( x \in \partial \Omega \). Let \( u_{\varepsilon} \) be the solution of the following auxiliary problem:

\[
\begin{align*}
  u_t &= \Delta u + au^p \int_{\Omega} u^q(y, t) \, dy - bu^m + b\varepsilon^m, \quad x \in \Omega, \quad t > 0, \\
  \frac{\partial u_{\varepsilon}(x, t)}{\partial \nu} &= \int_{\Omega} k(x, y, t)u_{\varepsilon}(y, t) \, dy, \quad x \in \partial \Omega, \quad t > 0, \\
  u(x, 0) &= u_{\varepsilon_0}(x) \quad \text{for} \quad x \in \Omega,
\end{align*}
\]

(2.4)
where \(\varepsilon = \varepsilon_n\). It was proved in [33] (see also [11] for similar problem) that \(u_m(x, t) = \lim_{n \to 0} u_n(x, t)\) is a maximal solution of (1.1)–(1.3). To establish theorem we will show that

\[
\underline{u}(x, t) \leq u_m(x, t) \leq \overline{u}(x, t) \text{ in } \overline{Q}_{T_0} \text{ for any } T_0 \in (0, T).
\]

We prove the second inequality in (2.5) only, since the proof of the first one is similar. Let \(\varphi(x, t) \in C^{2,1}(\overline{Q}_{T_0})\) be a nonnegative function such that

\[
\frac{\partial \varphi(x, t)}{\partial \nu} = 0, \ (x, t) \in \partial_{T_0}.
\]

If we multiply the first equation in (2.4) by \(\varphi(x, t)\) and then integrate over \(Q_t\) for \(0 < t < T_0\), we get

\[
\int_{\Omega} u_{\varepsilon}(x, t) \varphi(x, t)\, dx \leq \int_{\Omega} u_{\varepsilon}(x, 0) \varphi(x, 0)\, dx + \varepsilon^m b \int_0^t \int_{\Omega} \varphi(x, \tau)\, dx d\tau
\]

\[
+ \int_0^t \int_{\Omega} \left[ u_{\varepsilon}(x, \tau) \varphi_{\tau}(x, \tau) + u_{\varepsilon}(x, \tau) \Delta \varphi(x, \tau) \right]\, dx d\tau
\]

\[
+ a u_{\varepsilon}^p(x, \tau) \int_{\Omega} u_{\varepsilon}^q(y, \tau)\, dy - b u_{\varepsilon}^m \varphi(x, \tau)\right]\, dx d\tau
\]

\[
+ \int_0^t \int_{\Omega} \varphi(x, \tau) \int_{\Omega} k(x, y, \tau) u_{\varepsilon}^l(y, \tau)\, dy\, dS_x\, d\tau.
\]

On the other hand, \(\overline{u}\) satisfies (2.6) with reversed inequality and with \(\varepsilon = 0\). Set \(w(x, t) = u_{\varepsilon}(x, t) - \overline{u}(x, t)\). Then \(w(x, t)\) satisfies

\[
\int_{\Omega} w(x, t) \varphi(x, t)\, dx \leq \int_{\Omega} w(x, 0) \varphi(x, 0)\, dx + \varepsilon^m b \int_0^t \int_{\Omega} \varphi\, dx d\tau
\]

\[
+ \int_0^t \int_{\Omega} (\varphi_{\tau} + \Delta \varphi - b m \theta_1^{m-1}(x, \tau) \varphi)\, w\, dx d\tau
\]

\[
+ \int_0^t \int_{\Omega} a \varphi \theta_{21}^{p-1}(x, \tau) w(x, \tau)\int_{\Omega} u_{\varepsilon}^q(y, \tau)\, dy\, dx d\tau
\]

\[
+ \int_0^t \int_{\Omega} a \varphi \theta_{21}^{p-1}(x, \tau) w(x, \tau)\int_{\Omega} u_{\varepsilon}^q(y, \tau)\, dy\, dx d\tau
\]

\[
+ \int_0^t \int_{\partial \Omega} \varphi(x, \tau) \int_{\Omega} k(x, y, \tau) \theta_4^{l-1} w(y, \tau)\, dy\, dS_x\, d\tau,
\]

where \(\theta_i, i = 1, 4\) are some continuous functions between \(u_{\varepsilon}\) and \(\overline{u}\). Note here that by hypotheses for \(k(x, y, t), u_{\varepsilon}(x, t)\) and \(\overline{u}(x, t)\), we have

\[
0 \leq \overline{u}(x, t) \leq M, \ 0 \leq u_{\varepsilon}(x, t) \leq M \text{ in } \overline{Q}_{T_0}
\]

and \(0 \leq k(x, y, t) \leq M \text{ in } \partial \Omega \times \overline{Q}_{T_0}\).

where \(M\) is some positive constant. Then it is easy to see from (2.8) that \(\theta_1^{m-1}, \theta_2^{p-1}, \theta_3^{q-1}\) and \(\theta_4^{l-1}\) are positive and bounded functions in \(\overline{Q}_{T_0}\) and, moreover, \(\theta_2^{p-1} \leq M^{p-1}, \theta_3^{q-1} \leq M^{q-1}, \theta_4^{l-1} \leq M^{l-1}\). Define a sequence \(\{a_k\}\) in the following way: \(a_k \in C^\infty(\overline{Q}_{T_0}), \ a_k \geq 0 \text{ and } a_k \to b m \theta_1^{m-1}\) as \(k \to \infty\) in \(L^1(\overline{Q}_{T_0})\). Now
consider a backward problem given by
\[
\begin{cases}
\varphi_0 + \Delta \varphi - a_k \varphi = 0 & \text{for } x \in \Omega, \ 0 < \tau < t, \\
\frac{\partial \varphi(x, \tau)}{\partial \nu} = 0 & \text{for } x \in \partial \Omega, \ 0 < \tau < t, \\
\varphi(x, t) = \psi(x) & \text{for } x \in \Omega,
\end{cases}
\]
(2.9)
where \( \psi(x) \in C_{0}^{\infty}(\Omega) \) and \( 0 \leq \psi(x) \leq 1 \). Denote a solution of (2.9) as \( \varphi_k(x, \tau) \). Then by the standard theory for linear parabolic equations (see [24], for example), we find that \( \varphi_k \in C^{2,1}(\overline{Q}_t), \ 0 \leq \varphi_k(x, \tau) \leq 1 \) in \( Q_t \). Putting \( \psi = \varphi_k \) in (2.7) and passing then to the limit as \( k \to \infty \), we infer
\[
\int_{\Omega} w(x, t) \psi(x) \, dx \leq \int_{\Omega} w(x, 0)_{+} \, dx + \varepsilon m b T_0|\Omega|
+ \int_{0}^{t} \int_{\Omega} w(y, \tau)_{+} \, dy \, d\tau.
\]
(2.10)
where \( w_+ = \max\{w, 0\} \), \( |\partial \Omega| \) and \( |\Omega| \) are the Lebesgue measures of \( \partial \Omega \) in \( \mathbb{R}^{N-1} \) and \( \Omega \) in \( \mathbb{R}^N \), respectively. Since (2.10) holds for every \( \psi(x) \), we can choose a sequence \( \{\psi_k\} \) converging in \( L^1(\Omega) \) to \( \psi(x) = 1 \) if \( w(x, t) > 0 \) and \( \psi(x) = 0 \) otherwise. Hence, from (2.10) we get
\[
\int_{\Omega} w(x, t)_{+} \, dx \leq \int_{\Omega} w(x, 0)_{+} \, dx + \varepsilon m b T_0|\Omega|
+ \int_{0}^{t} \int_{\Omega} w(y, \tau)_{+} \, dy \, d\tau.
\]
Applying now Gronwall’s inequality and passing to the limit as \( \varepsilon \to 0 \), the conclusion of the theorem follows for \( \min(p, q, l) \geq 1 \). For the case \( \min(p, q, l) < 1 \) we can consider \( w(x, t) = u(x, t) - \overline{u}(x, t) \) and prove the theorem in a similar way using the positiveness of a subsolution or a supersolution.

Remark 2.4. Theorem [23] holds for \( q = 0 \).

Remark 2.5. We improve a comparison principle in [33]. The authors of [33] supposed that \( u(x, t) > 0 \) or \( \overline{u}(x, t) > 0 \) in \( Q_T \cup \Gamma_T \) if \( \min(m, p, q, l) < 1 \).

Next lemma shows the positiveness for \( t > 0 \) of solutions of (1.1)–(1.3) with \( m \geq 1 \) and \( u_0 \neq 0 \).

Lemma 2.6. Let \( u_0 \neq 0 \) in \( \Omega \), \( m \geq 1 \). Suppose \( u \) is a solution of (1.1)–(1.3) in \( Q_T \). Then \( u > 0 \) in \( Q_T \cup \overline{S} \).

Proof. We take any \( \tau \in (0, T) \). As \( u(x, t) \) is continuous in \( \overline{Q}_T \) function, then we have
\[
u(x, t) = M, \ (x, t) \in \overline{Q}_T \]
with some positive constant \( M \). Now we put \( v = u \exp(\lambda t) \), where \( \lambda \geq b M^{m-1} \). It is easy to verify that \( v_\tau - \Delta v \geq 0 \). Since \( v(x, 0) = u_0(x) \neq 0 \) in \( \Omega \) and \( v(x, t) \geq 0 \) in \( Q_T \), by the strong maximum principle \( v(x, t) > 0 \) in \( Q_T \). Let \( v(x_0, t_0) = 0 \) in some point \( (x_0, t_0) \in \overline{S} \). Then according to Theorem 3.6 of [19] it yields \( \partial v(x_0, t_0) / \partial \nu < 0 \), which contradicts (1.2).
3. Global existence

**Theorem 3.1.** Let \( \max(p+q,l) \leq 1 \) or \( 1 < \max(p+q,l) < m \). Then every solution of (1.1)–(1.3) is global.

**Proof.** In order to prove global existence of solutions we construct a suitable explicit supersolution of (1.1)–(1.3) in \( Q_T \) for any positive \( T \). Suppose at first that \( \max(p+q,l) \leq 1 \).

Since \( k(x,y,t) \) is a continuous function, there exists a constant \( K > 0 \) such that

\[
k(x,y,t) \leq K \tag{3.1}
\]

in \( \partial\Omega \times Q_T \). Let \( \lambda_1 \) be the first eigenvalue of the following problem

\[
\begin{cases}
\Delta \phi + \lambda \phi = 0, & x \in \Omega, \\
\phi = 0, & x \in \partial\Omega,
\end{cases} \tag{3.2}
\]

and \( \phi(x) \) be the corresponding eigenfunction with \( \sup_{\Omega} \phi = 1 \). It is well known, \( \phi(x) > 0 \) in \( \Omega \) and \( \max_{\partial\Omega} \phi \partial\phi / \partial\nu < 0 \).

We now construct a supersolution of (1.1)–(1.3) in \( Q_T \) as follows

\[
u(x,t) = C \exp(\mu t) \frac{c \phi(x) + 1}{c\phi(x) + 1},
\]

where constants \( C, \mu \) and \( c \) are chosen to satisfy the inequalities:

\[
c \geq \max \left\{ K \int_{\Omega} (\phi(y) + 1)^l \max_{\partial\Omega} \left( -\frac{\partial\phi}{\partial\nu} \right)^{-1}, 1 \right\},
\]

\[
C \geq \max \{ \sup_{\Omega} (c\phi(x) + 1)u_0(x), 1 \}, \quad \mu \geq \lambda_1 + 2c^2 \sup_{\Omega} \frac{\vert \nabla \phi \vert^2}{(c\phi(x) + 1)^2} + a\vert \Omega \vert.
\]

It is easy to obtain

\[
u_t - \Delta \nu - a\nu^p \int_{\Omega} \nu^q(y,t) dy + b\nu^m \geq \left( \mu - \frac{c\lambda_1 \phi}{c\phi(x) + 1} - 2c^2 \sup_{\Omega} \frac{\vert \nabla \phi \vert^2}{(c\phi(x) + 1)^2} - a\vert \Omega \vert \right) \nu \geq 0 \tag{3.3}
\]

for \( (x,t) \in Q_T \),

\[
\frac{\partial \nu}{\partial \nu} = cC \exp(\mu t) \left( -\frac{\partial\phi}{\partial\nu} \right) \geq KC^l \exp(l\mu t) \int_{\Omega} (\phi(y) + 1)^l \frac{dy}{\phi(y) + 1} \tag{3.4}
\]

for \( (x,t) \in S_T \) and

\[
\nu(x,0) \geq u_0(x) \tag{3.5}
\]

for \( x \in \Omega \). By virtue of (3.3)–(3.5) and Theorem 2.3, the solution of (1.1)–(1.3) exists globally.

Suppose now that \( 1 < \max(p+q,l) < m \). Let \( v_0(x) \) be the function with the following properties:

\[
v_0(x) \in C^1(\Omega), \quad v_0(x) > u_0(x) \text{ in } \Omega, \quad \frac{\partial v_0(x)}{\partial \nu} = \int_{\Omega} K v_0^l(y) dy \text{ on } \partial\Omega,
\]

\[
\Delta v_0 + \lambda v_0 = 0, \quad v_0(x) = 0, \quad x \in \partial\Omega.
\]

Hence, by Theorem 3.1 for (1.1)–(1.3), every solution is global.
where \( K \) was defined in (3.1). Let \( v(x,t) \) be the solution of (1.1) in \( Q_T \) with boundary and initial data
\[
\frac{\partial u(x,t)}{\partial \nu} = \int_{\Omega} K u(y,t) \, dy, \quad x \in \partial \Omega, \quad 0 < t < T,
\]
(3.6)

\[
u(x,0) = v_0(x), \quad x \in \Omega.
\]
(3.7)

We prove that \( v(x,t) \) exists in \( Q_T \) for any \( T > 0 \). By Lemma 2.6 and Theorem 2.3

\[
u(x,t) \leq v(x,t) \in Q_T.
\]

Integrating (1.1) for \( v(x,t) \) over \( Q_t \) and using Green’s identity and Hölder’s inequality, we have

\[
\int_{\Omega} v(x,t) \, dx = \int_{\Omega} v_0(x) \, dx + \int_0^t \int_{\partial \Omega} K v^t(y,\tau) \, dy \, dS_x \, d\tau
\]

\[
+ a \int_0^t \int_{\Omega} v^p(y,\tau) \, dy \, \int_{\partial \Omega} v^q(y,\tau) \, dy \, d\tau - b \int_0^t \int_{\Omega} v^m(y,\tau) \, dy \, d\tau
\]

\[
\leq \int_{\Omega} v_0(x) \, dx + K |\partial \Omega| (t|\Omega|)^{\frac{m+q}{m-p}} \left( \int_0^t \int_{\Omega} v^m(y,\tau) \, dy \, d\tau \right)^{\frac{p}{m+q}}
\]

\[
+ at^{\frac{m-p-q}{m}} |\Omega|^{\frac{2m-p-q}{m}} \left[ \int_0^t \int_{\Omega} v^m(y,\tau) \, dy \, d\tau \right]^{\frac{p+q}{m}} - b \int_0^t \int_{\Omega} v^m(y,\tau) \, dy \, d\tau.
\]

(3.8)

Since \( v(x,t) > 0 \), we obtain from (3.8) that

\[
\int_0^t \int_{\Omega} v^m(y,\tau) \, dy \, d\tau \leq C(T), \quad t \leq T.
\]

(3.9)

Now we prove that \( v(x,t) \) can not blow up in \( \Omega \).

Let \( G_N(x,y;t-\tau) \) be the Green function of the heat equation with homogeneous Neumann boundary condition. Then we have the representation formula:

\[
v(x,t) = \int_{\Omega} G_N(x,y;t) v_0(y) \, dy + K \int_0^t \int_{\partial \Omega} G_N(x,\xi;t-\tau) \int_{\Omega} v^t(y,\tau) \, dy \, dS_\xi \, d\tau
\]

\[
+ a \int_0^t \int_{\Omega} G_N(x,y;t-\tau) v^p(y,\tau) \int_{\partial \Omega} v^q(z,\tau) \, dz \, d\tau
\]

\[- b \int_0^t \int_{\Omega} G_N(x,y;t-\tau) v^m(y,\tau) \, dy \, d\tau,
\]

(3.10)

for \( (x,t) \in Q_T \). We now take a sequence of sets \( \{\Omega_k\} \) such that \( \Omega_k \subset \subset \Omega, \quad \Omega_k \subset \Omega_{k+1}, \quad \Omega = \bigcup_{k=1}^{\infty} \Omega_k, \quad \partial \Omega_k \subset C^2 \). It is well known (see, for example, [19], [20]) that

\[
G_N(x,y;t-\tau) \geq 0, \quad x, y \in \Omega, \quad 0 \leq \tau < t < T,
\]

(3.11)

\[
\int_{\Omega} G_N(x,y;t-\tau) \, dy = 1, \quad x \in \Omega, \quad 0 \leq \tau < t < T,
\]

(3.12)

\[
0 \leq G_N(x,y;t-\tau) \leq c_0, \quad x \in \Omega_k, \quad y \in \partial \Omega, \quad 0 < \tau < t < T,
\]

(3.13)

\[
c_1 \leq G_N(x,y;t-\tau) \leq c_2, \quad x \in \overline{\Omega}, \quad y \in \overline{\Omega}, \quad t-\tau \geq \varepsilon_0,
\]

(3.14)

\[
\frac{c_3}{\sqrt{t-\tau}} \leq \int_{\partial \Omega} G_N(x,\xi;t-\tau) \, dS_\xi \leq \frac{c_4}{\sqrt{t-\tau}}, \quad x \in \partial \Omega, \quad 0 < t-\tau \leq \varepsilon_0,
\]

(3.15)
where $\varepsilon_0 > 0$, $c_0$ is a positive constant depending on $k$. Here and below $c_i$, $i \in \mathbb{N}$ are positive constants. We note that

$$v^p(y,t) \int_\Omega v^q(z,t) \, dz \leq v^{p+q}(y,t) + \left( \int_\Omega v^q(z,t) \, dz \right)^{\frac{p+q}{q}} \text{ for } (y,t) \in Q_T. \quad (3.16)$$

Indeed, if

$$\int_\Omega v^q(z,t) \, dz \leq v^q(y,t),$$

then

$$v^p(y,t) \int_\Omega v^q(z,t) \, dz \leq v^{p+q}(y,t).$$

Otherwise, we have

$$v^p(y,t) \int_\Omega v^q(z,t) \, dz \leq \left( \int_\Omega v^q(z,t) \, dz \right)^{\frac{p+q}{q}}.$$

Now using (3.12), (3.16) and Hölder's inequality, we estimate the third integral in the right hand side of (3.10):

$$a \int_0^t \int_\Omega G_N(x,y; t-\tau) v^p(y,\tau) \int_\Omega v^q(z,\tau) \, dz \, dy \, d\tau$$

$$\leq a \int_0^t \int_\Omega G_N(x,y; t-\tau) \left[ v^{p+q}(y,\tau) + \left( \int_\Omega v^q(z,\tau) \, dz \right)^{\frac{p+q}{q}} \right] \, dy \, d\tau$$

$$\leq \int_0^t \int_\Omega G_N(x,y; t-\tau) \left[ bv^m(y,\tau) + a \left( \frac{a}{b} \right)^{\frac{m-q}{m-p-q}} \int_\Omega v^m(z,\tau) \, dz \right]^{\frac{p+q}{q}} \, dy \, d\tau$$

$$\leq b \int_0^t \int_\Omega G_N(x,y; t-\tau) v^m(y,\tau) \, dy \, d\tau + a \left( \frac{a}{b} \right)^{\frac{m-q}{m-p-q}} t$$

$$+ a |\Omega|^{\frac{m-q}{m-p-q}} \int_0^t \left( \int_\Omega v^m(z,\tau) \, dz \right)^{\frac{p+q}{m}} \, d\tau. \quad (3.17)$$

By (3.9) – (3.13), (3.17) and Hölder’s inequality we have

$$\sup_{\Omega_t} v(x,t) \leq \sup_{\Omega} v_0(x) + c_0 |\partial \Omega| K \int_0^t \int_\Omega v^q(y,\tau) \, dy \, d\tau$$

$$+ c_1 T + c_0 T^{\frac{m-q}{m-p-q}} \left[ \int_0^t \int_\Omega v^m(y,\tau) \, dy \, d\tau \right]^{\frac{p+q}{m}} \leq c_7(T). \quad (3.18)$$

Suppose that $v(x,t)$ blows up at $t = T$ and

$$\int_\Omega v^q(y,T) \, dy = +\infty. \quad (3.19)$$

Then by Hölder’s inequality we derive that

$$\int_\Omega v^m(y,T) \, dy = +\infty. \quad (3.20)$$
Since \( v(x, t) \) blows up at \( t = T \) from 3.10, 3.14, 3.15, 3.17 we obtain
\[
\int_{\Omega} v'(y, T) \, dy = +\infty
\]
and, moreover,
\[
v(x, t) \text{ blows up at } t = T \text{ at every point of } \partial \Omega.
\]
Integrating (1.1) for \( v(x, t) \) over \( \Omega \) and using Hölder’s inequality, we have
\[
\int_{\Omega} v_t(x, t) \, dx = K \int_{\partial \Omega} v(y, t) \, dS_y + a \int_{\Omega} v^p(y, t) \, dy \int_{\Omega} v^q(y, t) \, dy - b \int_{\Omega} v^m(y, t) \, dy.\]
Thus by (3.20)
\[
\lim_{t \to T} \int_{\Omega} v_t(x, t) \, dx = -\infty.
\]
On the other hand, integrating (1.1) for \( v(x, t) \) over \( \Omega_k \), we infer
\[
\int_{\Omega_k} v_t(x, t) \, dx = - \int_{\partial \Omega_k} \nu(v(x, t)) \, dS_x + a \int_{\Omega_k} v^p(y, t) \, dy \int_{\Omega_k} v^q(y, t) \, dy - b \int_{\Omega_k} v^m(y, t) \, dy.
\]
and using (3.6), (3.18), (3.19), (3.21), (3.22) we obtain the contradiction with (3.23).

Hence,
\[
\int_{\Omega} v^q(y, t) \, dy \leq c_8(T), \quad t \leq T.
\]
Now we consider the following equation
\[
Lu \equiv u_t - \Delta u - ac_8(T)u^p + bu^m = 0, \quad (x, t) \in Q_T.
\]
It is easy to see that \( v(x, t) \) is the subsolution of the problem 3.25, 3.6, 3.7 in \( Q_T \). To construct a supersolution we use the change of variables in a neighborhood of \( \partial \Omega \) as in [2]. Let \( \overline{\Omega} \in \partial \Omega \) and \( \hat{n}(\overline{x}) \) be the inner unit normal to \( \partial \Omega \) at the point \( \overline{x} \). Since \( \partial \Omega \) is smooth it is well known that there exists \( \delta > 0 \) such that the mapping \( \psi: \partial \Omega \times [0, \delta] \to \mathbb{R}^n \) given by \( \psi(\overline{x}, s) = \overline{x} + s\hat{n}(\overline{x}) \) defines new coordinates \( (\overline{x}, s) \) in a neighborhood of \( \partial \Omega \) in \( \overline{\Omega} \). A straightforward computation shows that, in these coordinates, \( \Delta \) applied to a function \( g(\overline{x}, s) = g(s) \), which is independent of the variable \( \overline{x} \), evaluated at a point \( (\overline{x}, s) \) is given by
\[
\Delta g(\overline{x}, s) = \frac{\partial^2 g}{\partial s^2}(\overline{x}, s) - \sum_{j=1}^{n-1} \frac{H_j(\overline{x})}{1 - sH_j(\overline{x})} \frac{\partial g}{\partial s}(\overline{x}, s),
\]
where \( H_j(\overline{x}) \) for \( j = 1, \ldots, n - 1 \), denote the principal curvatures of \( \partial \Omega \) at \( \overline{x} \). For \( 0 \leq s \leq \delta \) and small \( \delta \) we have
\[
\sum_{j=1}^{n-1} \left| \frac{H_j(\overline{x})}{1 - sH_j(\overline{x})} \right| \leq 7.
\]
Let \( 0 < \varepsilon < \omega < \min(\delta, 1), \max(1/l, 2/(m-1)) \) \( < \beta < 2/(l-1), \) \( 0 < \gamma < \beta/2, \) \( A \geq \sup_{\Omega} \psi_0(x) \). As in [15] for points in \( Q_{\delta, T} = \partial \Omega \times [0, \delta] \times [0, T] \) of coordinates \((v, s, t)\) define

\[
\varphi(x, t) = \varphi((v, s), t) = \left[(s + \varepsilon)^{-\gamma} - \omega^{-\gamma}\right]_{+}^{\frac{\beta}{\gamma}} + A, \tag{3.28}
\]

where \( s_{\pm} = \max(s, 0) \). For points in \( Q_T \setminus Q_{\delta, T} \) we set \( \varphi(x, t) = A \). We prove that \( \varphi(x, t) \) is the supersolution of (3.25), (3.6), (3.7) in \( Q_T \). It is not difficult to check that

\[
\frac{\partial \varphi}{\partial s} \leq \beta \min \left( [D(s)]^{\frac{\beta}{\gamma}} \left[(s + \varepsilon)^{-\gamma} - \omega^{-\gamma}\right]_{+}^{\frac{\beta+1}{\gamma}}, (s + \varepsilon)^{-(\beta+1)} \right), \tag{3.29}
\]

\[
\frac{\partial^2 \varphi}{\partial s^2} \leq \beta(\beta+1) \min \left( [D(s)]^{\frac{\beta+1}{\gamma}} \left[(s + \varepsilon)^{-\gamma} - \omega^{-\gamma}\right]_{+}^{\frac{\beta+2}{\gamma}}, (s + \varepsilon)^{-(\beta+2)} \right), \tag{3.30}
\]

where

\[
D(s) = \frac{(s + \varepsilon)^{-\gamma}}{(s + \varepsilon)^{-\gamma} - \omega^{-\gamma}}.
\]

Then \( D'(s) > 0 \) and for any \( \tau > 0 \)

\[
1 \leq D(s) \leq 1 + \tau, \quad 0 < s \leq \tau, \tag{3.31}
\]

where \( \tau = [\tau/(1 + \tau)]^{1/\gamma}\omega - \varepsilon, \varepsilon \in [\tau/(1 + \tau)]^{1/\gamma}\omega \). By (3.25) - (3.31) we can choose \( \tau \) small and \( A \) large so that in \( Q_{\tau, T} \)

\[
L\varphi \geq b \left[(s + \varepsilon)^{-\gamma} - \omega^{-\gamma}\right]_{+}^{\frac{\beta}{\gamma}} + A \left(\frac{\varepsilon + 1}{\varepsilon} + \beta(\beta + 1) [D(s)]^{\frac{2(\gamma+1)}{\gamma}} (s + \varepsilon)^{-(\beta+1)}\right) - \beta\tau [D(s)]^{\frac{\beta+1}{\gamma}} \left[(s + \varepsilon)^{-\gamma} - \omega^{-\gamma}\right]_{+}^{\frac{\beta+2}{\gamma}} - ac_{\Omega} \left(\frac{(s + \varepsilon)^{-\gamma} - \omega^{-\gamma}\right]_{+}^{\frac{\beta+2}{\gamma}}\right) \geq 0.
\]

Let \( s \in [\tau, \delta] \). From (3.26) - (3.30) we have

\[
|\Delta \varphi| \leq \beta(\beta + 1)\omega^{-\gamma+2} \left(\frac{1 + \tau}{\tau} + \beta\tau\omega^{-(\beta+1)}\right) \left[\frac{1 + \tau}{\tau} + \beta\tau\omega^{-(\beta+1)}\right]^{\frac{\beta+1}{\gamma}}
\]

and \( L\varphi \geq 0 \) for large values of \( A \). Obviously, in \( Q_T \setminus Q_{\delta, T} \)

\[
L\varphi = -ac_{\Omega} \left(\frac{(s + \varepsilon)^{-\gamma} - \omega^{-\gamma}\right]_{+}^{\frac{\beta+2}{\gamma}}\right) \geq 0
\]

for large values of \( A \).

Now we prove the following inequality

\[
\frac{\partial \varphi}{\partial \nu}(\tau, 0, t) \geq \int_{\partial \Omega} K \varphi((\tau, s), t) \, dy, \quad (x, t) \in S_T \tag{3.32}
\]

for a suitable choice of \( \varepsilon \). To estimate the integral \( I \) in the right hand side of (3.32) we use the change of variables in a neighborhood of \( \partial \Omega \) as above. Let

\[
\mathcal{J} = \sup_{0 < s < \delta} \int_{\partial \Omega} |J(\varphi, s)| \, d\sigma,
\]

where \( J(\tau, s) \) is Jacobian of the change of variables. Then we have

\[
I \leq \theta K \int_{\Omega} \left(\frac{(s + \varepsilon)^{-\gamma} - \omega^{-\gamma}\right]_{+}^{\frac{\beta}{\gamma}} d\sigma + \theta KA_{\Omega}\right| \Omega
\]

\[
\leq \theta K \mathcal{J} \int_{0}^{\omega - \varepsilon} \left(\frac{(s + \varepsilon)^{-\gamma} - \omega^{-\gamma}\right]_{+}^{\frac{\beta}{\gamma}} \right) \, ds + \theta KA_{\Omega}\right| \Omega
\]
\[
\leq \frac{\theta K \gamma}{\beta l - 1} \left[ \varepsilon^{-(\beta l - 1)} - \omega^{-(\beta l - 1)} \right] + \theta KA_l |\Omega|,
\]
where \( \theta = \max(2^l - 1, 1) \). On the other hand, since
\[
\frac{\partial v}{\partial \nu}(x, 0, t) = \frac{\partial v}{\partial s}(x, 0, t) = \beta \varepsilon^{-\gamma - 1} \left[ \varepsilon^{-\gamma} - \omega^{-\gamma} \right] + \gamma - \varepsilon 
\]
the inequality (3.32) holds if \( \varepsilon \) is small enough. At last,
\[
v(x, 0) \leq v(x, 0) \text{ in } \Omega.
\]
Hence, by comparison principle for (3.25), (3.6), (3.7) (see Remark 2.4) we get
\[
v(x, t) \leq v(x, t) \text{ in } Q_T.
\]

**Remark 3.2.** The authors of [33] have proved global existence of solutions of (1.1)–(1.3) for \( p + q < m, l \leq 1 \).

### 4. Blow-up in finite time

To formulate finite time blow-up result we introduce
\[
k(t) = \inf_{\Omega} \int_{\partial \Omega} k(x, y, t) dS_x
\]
and suppose that
\[
k(0) > 0. \tag{4.1}
\]

**Theorem 4.1.** Let either \( r + p > \max(m, 1) \) or \( l > \max(m, 1) \) and (4.1) hold. Then solutions of (1.1)–(1.3) blow up in finite time if initial data are large enough.

**Proof.** Suppose at first that \( r + p > \max(m, 1) \). To prove the existence finite time blow-up solutions we construct a suitable subsolution. Let \( f(t) \) be the solution of the following problem
\[
\begin{cases}
f'(t) = a|\Omega|f^{p+q} - bf^m, & t > 0, \\
f(0) = f_0.
\end{cases}
\]
If
\[
f_0 > \left( \frac{b}{a|\Omega|} \right)^{\frac{1}{p+q-m}},
\]
then \( f(t) \geq f_0 \) and there exists \( t_0 < +\infty \) such that
\[
\lim_{t \to t_0} f(t) = +\infty.
\]
It is easy to check that \( f(t) \) is a subsolution of (1.1)–(1.3) if \( u_0(x) \geq f_0 \). By Theorem 2.3 \( u(x, t) \) blows up in finite time.

Now suppose \( l > \max(m, 1) \) and (4.1) holds. Then there exists \( T_0 > 0 \) such that \( k(t) > 0 \) for \( t \in [0, T_0] \). Denote
\[
k_0 = \min_{[0, T_0]} k(t), \quad V(t) = \int_{\Omega} u(x, t) \, dx.
\]
Easily to check (4.6) is equivalent to the inequality

\[ t^o \]

To provide (4.3) we assume that

\[ \text{Hence,} \]

Then from (4.2), (4.3) we have

\[
\int_{\Omega} u^l(y,t) dy \geq k_0 \int_{\Omega} u(y,t) dy - b \int_{\Omega} u^m(y,t) dy \\
\geq \left( \int_{\Omega} u(y,t) dy \right)^m \left[ k_0 \left( \int_{\Omega} u(y,t) dy \right) \frac{l-m}{l} - b|\Omega| \frac{l-m}{l} \right] \\
\geq \left( \int_{\Omega} u(y,t) dy \right)^m \frac{m}{l} \left[ k_0 \left( \int_{\Omega} u(y,t) dy \right) \frac{l-m}{l} - b|\Omega| \frac{l-m}{l} \right]
\]

if

\[ J(t) \equiv k_0 V^{l-m}(t) |\Omega| \frac{(l-m)(l-1)}{l} - b|\Omega| \frac{l-m}{l} > 1. \]

Then from (4.2), (4.3) we have

\[ V'(t) \geq |\Omega|^{-\frac{m(l-1)}{l}} V^m(t) \]

for \( t \leq T_0 \). Let \( m > 1 \). Obviously, \( V(t) \) blows up at \( t \leq T_0 \) subject to

\[ V(0) \geq \left\{ (m-1)|\Omega|^{-\frac{m(l-1)}{l}} T_0 \right\}^{-\frac{1}{m-1}}. \]

To provide (4.3) we assume that

\[ V(0) > k_0 \frac{1}{l-m} \left( b|\Omega| \frac{l-m}{l} + 1 \right)^{\frac{1}{l-m}} |\Omega|^{\frac{l-1}{l-m}}. \]

Easily to check (4.6) is equivalent to the inequality

\[ k_0 V^{l-m}(0) |\Omega|^{-\frac{(l-m)(l-1)}{l}} - b|\Omega| \frac{l-m}{l} > 1. \]

Hence,

\[ J(0) > 1. \]

Suppose there exists \( t_1 \in (0,T_0) \) such that \( J(t_1) = 1 \) and \( J(t) > 1 \) for \( t \in [0,t_1) \). Since for \( t \in [0,t_1] \) (4.4) holds, we have \( V(t) \geq V(0) \) and

\[ J(t) \geq J(0) > 1. \]

Therefore, if we take initial data to satisfy (4.5), (4.6) then any solution of (1.1)–(1.3) with \( m > 1 \) blows up at \( t \leq T_0 \).

Let \( m \leq 1 \). From (4.5) and Hölder’s inequality we deduce that

\[ V'(t) \geq k_0 \int_{\Omega} u^l(y,t) dy - b \int_{\Omega} u^m(y,t) dy \geq k_0 |\Omega|^{1-l} V^l(t) - b |\Omega|^{1-m} V^m(t). \]

Assume

\[ V(0) \geq \max \left\{ 1, k_0 \frac{1}{l-m} \left( b |\Omega| \frac{l-m}{l} + 1 \right)^{\frac{1}{l-m}} |\Omega|^{\frac{l-1}{l}} \right\}. \]

Then by (4.4)

\[ V(t) \geq 1 \text{ for } t \geq 0 \]

and from (4.7), (4.9) we obtain

\[ V'(t) \geq k_0 |\Omega|^{1-l} V^l(t) - b |\Omega|^{1-m} V(t). \]
Solving this inequality we find that $V(t)$ blows up at $t \leq T_0$ if \([1.8]\) holds and

$$V(0) \geq \left\{ \frac{k_0|\Omega|^{1-l}}{b|\Omega|^{1-m} \left[1 - \exp(-b(l-1)|\Omega|^{1-m}T_0)\right]} \right\}^{\frac{1}{l-1}}.$$ 

\(\Box\)

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**References**

[1] S. Carl, V. Lakshmikantham; *Generalized quasilinearization method for reaction-diffusion equation under nonlinear and nonlocal flux conditions*, J. Math. Anal. Appl., 271 (2002), 182–205.

[2] C. Cortazar, M. del Pino, M. Elgueta; *On the short-time behaviour of the free boundary of a porous medium equation*, Duke J. Math., 87 (1997), 133–149.

[3] Z. Cui, Z. Yang; *Roles of weight functions to a nonlinear porous medium equation with nonlocal source and nonlocal boundary condition*, J. Math. Anal. Appl., 342 (2008), 559–570.

[4] Z. Cui, Z. Yang, R. Zhang; *Blow-up of solutions for nonlinear parabolic equation with nonlocal source and nonlocal boundary condition*, Appl. Math. Comput., 224 (2013), 1–8.

[5] K. Deng; *Comparison principle for some nonlocal problems*, Quart. Appl. Math., 50 (1992), 517–522.

[6] Z.B. Fang, J. Zhang; *Global and blow-up solutions for the nonlinear p-Laplacian evolution equation with weighted nonlinear nonlocal boundary condition*, J. Integral Equat. Appl., 26 (2014), 171–196.

[7] Z.B. Fang, J. Zhang; *Global existence and blow-up properties of solutions for porous medium equation with nonlinear memory and weighted nonlocal boundary condition*, Z. Angew. Math. Phys., 66 (2015), 67–81.

[8] A. Friedman; *Monotonic decay of solutions of parabolic equations with nonlocal boundary conditions*, Quart. Appl. Math., 44 (1986), 401–407.

[9] Y. Gao, W. Gao; *Existence and blow-up of solutions for a porous medium equation with nonlocal boundary condition*, Appl. Math. Comput., 224 (2013), 1–8.

[10] A. Gladkov; *Blow-up problem for semilinear heat equation with nonlinear nonlocal Neumann boundary condition*, Appl. Anal., 90 (2011), 799–809.

[11] A. Gladkov; *Initial boundary value problem for a semilinear parabolic equation with absorption and nonlinear nonlocal boundary condition*, Lith. Math. J., 57 (2017), 468–478.

[12] A. Gladkov, M. Guedda; *Blow-up problem for semilinear heat equation with absorption and a nonlocal boundary condition*, Nonlinear Anal., 74 (2011), 4573–4580.

[13] A. Gladkov, T. Kavitova; *Initial-boundary-value problem for a semilinear parabolic equation with nonlinear nonlocal boundary conditions*, Ukr. Math. J., 68 (2016), 179–192.

[14] A. Gladkov, T. Kavitova; *Blow-up problem for semilinear heat equation with nonlinear nonlocal boundary condition*, Appl. Anal., 95 (2016), 1974–1988.

[15] A. Gladkov, T. Kavitova; *On the initial-boundary value problem for a nonlinear parabolic equation with nonlinear nonlocal boundary condition*, Math. Methods Appl. Sci., 43 (2020), 5464–5479.

[16] A. Gladkov, K.I. Kim; *Blow-up of solutions for semilinear heat equation with nonlinear nonlocal boundary condition*, J. Math. Anal. Appl., 338 (2008), 264–273.

[17] A. Gladkov, A. Nikitin; *On the existence of global solutions of a system of semilinear parabolic equations with nonlinear nonlocal boundary conditions*, Diff. Equat., 52 (2016), 467–482.

[18] A. Gladkov, A. Nikitin; *On global existence of solutions of initial boundary value problem for a system of semilinear parabolic equations with nonlinear nonlocal Neumann boundary conditions*, Diff. Equat., 54 (2018), 86–105.

[19] B. Hu; *Blow-up theories for semilinear parabolic equations*, Lecture Notes in Mathematics, 2018 (2011), 1–127.
[20] C. S. Kahane; *On the asymptotic behavior of solutions of parabolic equations*, Czechoslovak Math. J. 33 (1983), 262–285.
[21] B.K. Kakumani, S.K. Tumuluri; *Asymptotic behavior of the solution of a diffusion equation with nonlocal boundary conditions*, Discrete Cont. Dyn. B., 22 (2017), 407–419.
[22] A.I. Kozhanov; *On the solvability of a boundary-value problem with a nonlocal boundary condition for linear parabolic equations*, Vestnik Samara Gos. Tekh. Univ. Ser. Fiz.-Mat. Nauk, 30 (2004), 63–69.
[23] W. Kou, J. Ding; *Blow-up phenomena for p-Laplacian parabolic equations under nonlocal boundary conditions*, Appl. Anal., 100 (2021), 3350–3365.
[24] O. Ladyzhenskaja, V. Solonnikov, N. Ural’ceva; *Linear and quasilinear equations of parabolic type*, Transl. Math. Monographs, AMS, Providence, RI, 1968.
[25] Z. Lin, Y. Liu; *Uniform blowup profiles for diffusion equations with nonlocal source and nonlocal boundary*, Acta Math. Sci., 24B (2004), 443–450.
[26] B. Liu, H. Lin, F. Li, X. Wang; *Blow-up analyses in reaction-diffusion equations with nonlinear nonlocal boundary flux*, Z. Angew. Math. Phys., 70 (2019), 27 pp.
[27] B. Liu, G. Wu, X. Sun, F. Li; *Blow-up estimate in reaction-diffusion equation with nonlinear nonlocal flux and source*, Comp. Math. Appl., 78 (2019), 1862–1877.
[28] D. Liu, C. Mu; *Blowup properties for a semilinear reaction-diffusion system with nonlinear nonlocal boundary conditions*, J. Inequal. Appl., 167 (2014), 11 pp.
[29] H. Lu, B. Hu, Z. Zhang; *Blowup time estimates for the heat equation with a nonlocal boundary condition*, Z. Angew. Math. Phys., 73 (2022), 15 pp.
[30] M. Marras, S. Vernier Piro; *Reaction-diffusion problems under non-local boundary conditions with blow-up solutions*, J. Inequal. Appl., 107 (2014), 11 pp.
[31] C.V. Pao; *Asymptotic behavior of solutions of reaction-diffusion equations with nonlocal boundary conditions*, J. Comput. Appl. Math., 88 (1998), 225–238.
[32] Y. Wang, C. Mu, Z. Xiang; *Blowup of solutions to a porous medium equation with nonlinear boundary condition*, Appl. Math. Comput., 192 (2007), 579–585.
[33] J. Wang, H. Yang; *Properties of solutions for a reaction-diffusion equation with nonlinear absorption and nonlinear Neumann boundary condition*, Bound. Value Probl., 143 (2018), 14 pp.
[34] L. Yang, C. Fan; *Global existence and blow-up of solutions to a degenerate parabolic system with nonlocal sources and nonlocal boundaries*, Monatsh. Math., 174 (2014), 493–510.
[35] Z. Ye, X. Xu; *Global existence and blow-up for a porous medium system with nonlocal boundary conditions and nonlocal sources*, Nonlinear Anal., 82 (2013), 115–126.
[36] S. Zheng, L. Kong; *Roles of weight functions in a nonlinear nonlocal parabolic system*, Nonlinear Anal., 68 (2008), 2406–2416.

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