PARABOLIC LITTLEWOOD-PALEY INEQUALITY FOR 
φ(−Δ)-TYPE OPERATORS AND APPLICATIONS TO 
STOCHASTIC INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract. In this paper we prove a parabolic version of the Littlewood-Paley 
inequality (1.3) for the operators of the type
φ(−Δ), where φ is a Bernstein function. As an application, we construct an 
L_p-theory for the stochastic integro-differential equations of the type du = (−φ(−Δ)u + f) dt + g dW_t.

1. Introduction

The operators we are considering in this article are certain functions of the 
Laplacian. To be more precise, recall that a function φ : (0, ∞) → (0, ∞) such that 
φ(0+) = 0 is called a Bernstein function if it is of the form
φ(λ) = bλ + ∫_{(0,∞)} (1 − e^{−λt}) µ(dt), \quad λ > 0,
where b ≥ 0 and µ is a measure on (0, ∞) satisfying ∫_{(0,∞)} (1 ∧ t) µ(dt) < ∞, called the Lévy measure. By Bochner’s functional calculus, one can define the operator
φ(∆) := −φ(−∆) on C_0^2(\mathbb{R}^d), which turns out to be an integro-differential operator
bΔf(x) + ∫_{\mathbb{R}^d} (f(x + y) - f(x) - ∇f(x) \cdot y 1_{|y| ≤ 1}) J(y) dy, \quad (1.1)
where J(x) = j(|x|) with j : (0, ∞) → (0, ∞) given by
j(r) = ∫_0^∞ (4πt)^{-d/2} e^{-r^2/(4t)} µ(dt).

It is also known that the operator φ(∆) is the infinitesimal generator of the 
d-dimensional subordinate Brownian motion. Let S = (S_t)_{t ≥ 0} be a subordinator 
(i.e. an increasing Lévy process satisfying S_0 = 0) with Laplace exponent φ, and let W = (W_t)_{t ≥ 0} be a Brownian motion in \mathbb{R}^d, d ≥ 1, independent of S with 
\mathbb{E}_x [e^{iξ(W_t - W_0)}] = e^{-t|ξ|^2}, ξ ∈ \mathbb{R}^d, t > 0. Then X_t := W_{S_t}, called the subordinate...
Brownian motion, is a rotationally invariant Lévy process in \( \mathbb{R}^d \) with characteristic exponent \( \phi(|\xi|^2) \), and for any \( f \in C_0^\infty(\mathbb{R}^d) \)

\[
\phi(\Delta)f(x) = \lim_{t \to 0} \frac{1}{t} \mathbb{E}_x f(X_t) - f(x), \quad (1.2)
\]

For instance, by taking \( \phi(\lambda) = \lambda^\alpha/2 \) with \( \alpha \in (0, 2) \), we get the fractional laplacian \( \Delta^{\alpha/2} := (-\Delta)^{\alpha/2} \) which is the infinitesimal generator of a rotationally symmetric \( \alpha \)-stable process in \( \mathbb{R}^d \).

In this article we prove a parabolic Littlewood-Paley inequality for \( \phi(\Delta) \):

**Theorem 1.1.** Let \( \phi \) be a Bernstein function, \( T_t \) be the semigroup corresponding to \( \phi(\Delta) \) and \( H \) be a Hilbert space. Suppose that \( \phi \) satisfies

(H1): \( \exists \) constants \( 0 < \delta_1 < \delta_2 < 1 \) and \( a_1, a_2 > 0 \) such that

\[
a_1 \lambda^{\delta_1} \phi(t) \leq \phi(\lambda t) \leq a_2 \lambda^{\delta_2} \phi(t), \quad \lambda \geq 1, t \geq 1;
\]

(H2): \( \exists \) constants \( 0 < \delta_3 \leq 1 \) and \( a_3 > 0 \) such that

\[
\phi(\lambda t) \leq a_3 \lambda^{\delta_3} \phi(t), \quad \lambda \leq 1, t \leq 1.
\]

Then for any \( p \in [2, \infty), T \in (0, \infty) \) and \( f \in C_0^\infty(\mathbb{R}^{d+1}, H) \),

\[
\int_{\mathbb{R}^d} \int_0^T \int_0^t |(\phi(\Delta))^{1/2}T_{t-s}f(s, \cdot)(x)|_H^2 ds dt \leq N \int_{\mathbb{R}^d} \int_0^T |f(t, x)|_H^p dt dx, \quad (1.3)
\]

where the constant \( N \) depends only on \( d, p, T, a_i \) and \( \delta_i \) \((i = 1, 2, 3)\).

(H1) is a condition on the asymptotic behavior of \( \phi \) at infinity and it governs the behavior of the corresponding subordinate Brownian motion \( X \) for small time and small space. (H2) is a condition about the asymptotic behavior of \( \phi \) at zero and it governs the behavior of the corresponding subordinate Brownian motion \( X \) for large time and large space. Note that it follows from the second inequality in (H1) that \( \phi \) has no drift, i.e., \( b = 0 \) in (1.1). It also follows from (H2) that \( \phi(0+) = 0 \).

Using the tables at the end of [20], one can construct a lot of explicit examples of Bernstein functions satisfying (H1)–(H2). Here are a few of them:

1. \( \phi(\lambda) = \lambda^\alpha + \lambda^\beta, \ 0 < \alpha < \beta < 1; \)
2. \( \phi(\lambda) = (\lambda + \lambda^\beta)^\alpha, \ \alpha, \beta \in (0, 1); \)
3. \( \phi(\lambda) = \lambda^\alpha (\log(1 + \lambda))^{\beta}, \ \alpha \in (0, 1), \beta \in (0, 1 - \alpha); \)
4. \( \phi(\lambda) = \lambda^\alpha (\log(1 + \lambda))^{-\beta}, \ \alpha \in (0, 1), \beta \in (0, \alpha); \)
5. \( \phi(\lambda) = (\log(\cosh(\sqrt{\lambda})))^\alpha, \ \alpha \in (0, 1); \)
6. \( \phi(\lambda) = (\log(\sinh(\sqrt{\lambda})))^\alpha - \log \sqrt{\lambda}^\alpha, \ \alpha \in (0, 1). \)

For example, the subordinate Brownian motion corresponding to the example (1) \( \phi(\lambda) = \lambda^\alpha + \lambda^\beta \) is the sum of two independent symmetric \( \alpha \) and \( \beta \) stable processes. In this case the characteristic exponent is

\[
\Phi(\theta) = |\theta|^\alpha + |\theta|^\beta, \ \theta \in \mathbb{R}^d, \quad 0 < \beta < \alpha < 2,
\]

and its infinitesimal generator is \(-(-\Delta)^{\beta/2} - (-\Delta)^{\alpha/2} \).

We remark here that relativistic stable processes satisfy (H1)–(H2) with \( \delta_3 = 1 \); Suppose that \( \alpha \in (0, 2), m > 0 \) and define

\[
\phi_m(\lambda) = (\lambda + m^{\alpha/2})^{\alpha/2} - m.
\]
The parabolic Littlewood-Paley inequality \((1.3)\) was first proved by Krylov \((114, 115)\) for the case \(\phi(\Delta) = \Delta\) with \(N = N(p)\) depending only on \(p\). In this case, if \(f\) depends only on \(x\) and \(H = \mathbb{R}\) then \((1.3)\) leads to the classical (elliptic) Littlewood-Paley inequality (cf. \([23]\)):
\[
\int_{\mathbb{R}^d} \left( \int_0^\infty |\nabla T_t f|^2 dt \right)^{p/2} dx \leq N(p)\|f\|_{L^p}, \quad \forall \ f \in L_p(\mathbb{R}^d).
\]

Recently, \((1.3)\) was proved for the fractional Laplacian \(\Delta^{\alpha/2}\), \(\alpha \in (0, 2)\), in \([2, 8]\). Also, in \([17]\) similar result was proved for the case \(J = J(t, y) = m(t, y)|y|^{-d-\alpha}\) in \((1.4)\), where \(\alpha \in (0, 2)\) and \(m(t, y)\) is a bounded smooth function satisfying \(m(t, y) = m(t, y|y|^{-1})\) (i.e. homogeneous of degree zero) and \(m(t, y) > c > 0\) on a set \(\Gamma \subset S^{d-1}\) of a positive Lebesgue measure. We note that even the case \(\phi(\lambda) = \lambda^\alpha + \lambda^\beta\) \((\alpha \neq \beta)\) is not covered in \([17]\).

Our motivation of studying \((1.3)\) is that it is the key estimate for the \(L_p\)-theory of the corresponding stochastic partial differential equations. For example, Krylov’s result \((114, 115)\) for \(\Delta\) is related to the \(L_p\)-theory of the second-order stochastic partial differential equations. Below we briefly explain the reason for this. See \([9, 16]\) or Section 6 of this article for more details. Consider the stochastic integro-differential equation
\[
du = (\phi(\Delta)u + h) dt + \sum_{k=1}^\infty f^k dw^k, \quad u(0, x) = 0. \tag{1.4}
\]
Here \(f = (f^1, f^2, \cdots)\) is an \(\ell_2\)-valued random function of \((t, x)\), and \(w^k\) are independent one-dimensional Wiener processes defined on a probability space \((\Omega, P)\).

Considering \(u - w\), where \(w(t) := \int_0^t T_{t-s} h(s) ds\), we may assume \(h = 0\) (see Section 6). It turns out that if \(f = (f^1, f^2, \cdots)\) satisfies certain measurability condition, the solution of this problem is given by
\[
u(t, x) = \sum_{k=1}^\infty \int_0^t T_{t-s} f^k(s, \cdot)(x) dw^k_s.
\]
By Burkholder-Davis-Gundy inequality,
\[
E \int_0^T \|\phi(\Delta)^{1/2}u(t, \cdot)\|_{L^p}^p dt \leq N(p) \int_0^T \int_{\mathbb{R}^d} \left[ \int_0^t |\phi(\Delta)^{1/2}T_{t-s} f(s, \cdot)(x)|_{L^p}^2 ds \right] dx dt. \tag{1.5}
\]
Actually if \( f \) is not random, then \( u \) becomes a Gaussian process and the reverse inequality of (1.3) also holds. Thus to prove \( \phi(\Delta)^{1/2}u \in L_p \) and to get a legitimate start of the \( L_p \)-theory of equation (1.3), one has to estimate the right-hand side of (1.3) (or the left-hand side of (1.3)). We will also see that (1.3) yields the uniqueness and existence of equation (1.3) in certain Banach spaces.

The key of our approach is estimating the sharp function \( (v)^\sharp(t,x) \) of \( v(t,x) := \int_0^t |\phi(\Delta)^{1/2}T_{t-s}f(s,\cdot)(x)|^p \, ds |^{1/2} \):

\[
(v)^\sharp(t,x) := \sup_{(t,x) \in Q} \int_Q |v - v_Q| \, dt \, dx,
\]

where \( v_Q := \frac{1}{|Q|} \int_Q v \, dx \) is the average of \( v \) over \( Q \) and the supremum is taken for all cubes \( Q \) containing \( (t,x) \) of the type \( Q_e(r,y) := (r - \phi(c^{-2})^{-1}, r + \phi(c^{-2})^{-1}) \times B_c(y) \). We control \( (v)^\sharp(t,x) \) in terms of the maximal functions of \( |f|_H \), and then apply Fefferman-Stein and Hardy-Littlewood theorems to prove (1.3). The operators considered in [13 8 17] have simple scaling properties, and so to estimate the mean oscillation \( f_Q |v - v_Q| \, dt \, dx \) it was enough to consider the case \( Q = Q_1(0,0) \), that is the case \( c = 1 \) and \( (r,y) = (0,0) \). However, in our case, due to the lack of the scaling property, it is needed to consider the mean oscillation \( f_Q |v - v_Q| \, dt \, dx \) on every \( Q_e(r,y) \) containing \( (t,x) \). This causes serious difficulties as can be seen in the proofs of Lemmas 5.2 5.3. Our estimation of \( f_Q |v - v_Q| \, dt \, dx \) relies on the upper bounds of \( \phi(\Delta)^{n/2}D^\beta p(t,x) \), which are obtained in this article. Here \( \beta \) is an arbitrary multi-index, \( n = 0,1,2,\cdots \) and \( p(t,x) \) is the density of the semigroup \( T_t \) corresponding to \( \phi(\Delta) \).

The article is organized as follows. In Section 2 we give upper bounds of the density \( p(t,x) \). Section 3 contains various properties of Bernstein functions and subordinate Brownian motions. In Section 4 we establish upper bounds of the fractional derivatives of \( p(t,x) \) in terms of \( \phi \). Using these estimates we give the proof of of Theorem 1.1 in Section 5. In Section 6 we apply Theorem 1.1 and construct an \( L_p \)-theory for equation (1.4).

We finish the introduction with some notation. As usual \( \mathbb{R}^d \) stands for the Euclidean space of points \( x = (x^1,\ldots,x^d) \), \( B_r(x) := \{ y \in \mathbb{R}^d : |x - y| < r \} \) and \( B_r := B_r(0) \). For \( i = 1,\ldots,d \), multi-indices \( \beta = (\beta_1,\ldots,\beta_d) \), \( \beta_i \in \{0,1,2,\ldots\} \), and functions \( u(x) \) we set

\[
u_{x^i} = \frac{\partial u}{\partial x^i} = D_i u, \quad D^\beta u = D_1^{\beta_1} \cdots D_d^{\beta_d} u, \quad |\beta| = \beta_1 + \cdots + \beta_d.
\]

We write \( u \in C_0^\infty(X,Y) \) if \( u \) is a \( Y \)-valued infinitely differentiable function defined on \( X \) with compact support. By \( C_b^2(\mathbb{R}^d) \) we denote the space of twice continuously differentiable functions on \( \mathbb{R}^d \) with bounded derivatives up to order 2. We use \( "=" \) to denote a definition, which is read as “is defined to be”. We denote \( a \wedge b := \min\{a,b\} \), \( a \vee b := \max\{a,b\} \). If we write \( N = N(a,\ldots,z) \), this means that the constant \( N \) depends only on \( a,\ldots,z \). The constant \( N \) may change from location to location, even within a line. By \( \mathcal{F} \) and \( \mathcal{F}^{-1} \) we denote the Fourier transform and the inverse Fourier transform, respectively. That is, for a suitable function \( f \), \( \mathcal{F}(f)(\xi) := \int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(x) \, dx \) and \( \mathcal{F}^{-1}(f)(x) := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ix\cdot\xi} f(\xi) \, d\xi \). Finally, for a Borel set \( A \subset \mathbb{R}^d \), we use \( |A| \) to denote its Lebesgue measure.
2. Upper bounds of \( p(t, x) \)

In this section we give upper bounds of the density \( p(t, x) \) of the semigroup \( T_t \) corresponding to \( \phi(\Delta) \). We give the result under slightly more general setting. We will assume that \( Y \) is a rotationally symmetric Lévy process with Lévy exponent \( \Psi_Y(\xi) \). Because of rotational symmetry, the function \( \Psi_Y \) is positive and depends on \(|\xi|\) only. Accordingly, by a slight abuse of notation we write \( \Psi_Y(\xi) = \Psi_Y(|\xi|) \) and get

\[
E_x \left[ e^{\xi \cdot (Y_t - Y_0)} \right] = e^{-t \Psi_Y(|\xi|)}, \quad \text{for every } x \in \mathbb{R}^d \text{ and } \xi \in \mathbb{R}^d. \tag{2.1}
\]

We assume that the transition probability \( P(Y_t \in dy) \) is absolutely continuous with respect to Lebesgue measure in \( \mathbb{R}^d \). Thus there is a function \( p_Y(t, r), t > 0, r \geq 0 \) such that

\[
P(Y_t \in dy) = p_Y(t, |y|) dy.
\]

Note that \( r \to \Psi_Y(r) \) and \( r \to p_Y(t, r) \) may not be monotone in general. We first consider the following mild condition on \( \Psi_Y \).

(A1): There exists a positive function \( h \) on \([0, \infty)\) such that for every \( t, \lambda > 0 \)

\[
\Psi_Y(\lambda t)/\Psi_Y(t) \leq h(\lambda) \quad \text{and} \quad \int_0^\infty e^{-r^2/2}r^{d-1}h(r)dr < \infty.
\]

Note that by Lemma 3.1 below, (A1) always holds with \( h(\lambda) = 1 \lor \lambda^2 \) for every subordinate Brownian motion. Moreover, by [7, Lemma 3 and Proposition 11], (A1) always holds with \( h(\lambda) = 24(1 + \lambda^2) \) for rotationally symmetric unimodal Lévy process (i.e., \( r \to p_Y(t, r) \) is decreasing for all \( t > 0 \)).

Recall that

\[
e^{-|z|^2} = (4\pi)^{-d/2} \int_{\mathbb{R}^d} e^{\xi \cdot z} e^{-|\xi|^2/4} d\xi.
\]

Using this and (2.1) we have for \( \lambda > 0 \)

\[
E_0[e^{-\lambda |Y_t|^2}] = (4\pi)^{-d/2} \int_{\mathbb{R}^d} E_0[e^{t\sqrt{\lambda} \xi} Y_t] e^{-|\xi|^2/4} d\xi
\]

\[
= (4\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-t \Psi_Y(\sqrt{\lambda} |\xi|)} e^{-|\xi|^2/4} d\xi. \tag{2.2}
\]

Thus

\[
E_0[e^{-\lambda |Y_t|^2} - e^{-2\lambda |Y_t|^2}] = (4\pi)^{-d/2} \int_{\mathbb{R}^d} (e^{-t \Psi_Y(\sqrt{\lambda} |\xi|)} - e^{-t \Psi_Y(\sqrt{2\lambda} |\xi|)}) e^{-|\xi|^2/4} d\xi. \tag{2.3}
\]

For \( t, \lambda > 0 \), let

\[
g_t(\lambda) := \int_0^\infty (e^{-t \Psi_Y(\sqrt{r} \lambda)} - e^{-t \Psi_Y(\sqrt{2r} \lambda)}) e^{-r^2/4} r^{d-1} dr,
\]

which is positive by (2.3).

**Lemma 2.1.** Suppose that (A1) holds. Then there exists a constant \( N = N(h, d) \) such that for every \( t, v > 0 \)

\[
g_t(v^{-1}) \leq N t \Psi_Y(v^{-1/2}).
\]
Proof. By (A1) we have
\[
\frac{1}{t\Psi_Y(v^{-1/2})} \leq \frac{\Psi_Y(\sqrt{2}v^{-1/2}) + \Psi_Y(v^{-1/2})}{t\Psi_Y(v^{-1/2})} \frac{1}{t\Psi_Y(\sqrt{2}v^{-1/2}) - \Psi_Y(v^{-1/2})}
\]
\[
\leq \frac{h(\sqrt{2}r) + h(r)}{t\Psi_Y(\sqrt{2}v^{-1/2}) - \Psi_Y(v^{-1/2})}.
\]
Thus using the inequality \(|e^{-a} - e^{-b}| \leq |a - b|, a, b > 0\)
\[
\frac{g_Y(v^{-1})}{t\Psi_Y(v^{-1/2})} \leq \int_0^\infty \frac{|e^{-t\Psi_Y(\sqrt{2}v^{-1/2})} - e^{-t\Psi_Y(\sqrt{2}v^{-1/2})}|}{t\Psi_Y(\sqrt{2}v^{-1/2}) - \Psi_Y(v^{-1/2})} e^{-r^2/4r^{d-1}} (h(\sqrt{2}r) + h(r)) dr
\]
\[
\leq \int_0^\infty e^{-r^2/4r^{d-1}} (h(\sqrt{2}r) + h(r)) dr < \infty.
\]
Therefore the lemma is proved. \(\square\)

Recall that \(P_0(Y_t \in dy) = p_Y(t,|y|)dy\). We now consider the following mild condition on \(p_Y(t, r)\).

(A2): For every \(T \in (0, \infty]\), there exists a constant \(c = c(T) > 0\) such that for every \(t \in (0, T)\)
\[
p_Y(t, r) \leq cp_Y(t, s) \quad \forall r \geq s \geq 0.
\]

Obviously (2.4) always holds on all \(t > 0\) for rotationally symmetric unimodal Lévy process.

Theorem 2.2. Suppose that \(Y\) is a rotationally symmetric Lévy process with Lévy exponent \(\Psi_Y(\xi)\) satisfying (A1). Assume that \(P(Y_t \in dy) = p_Y(t,|y|)dy\) and (A2) holds. Then for every \(T > 0\), there exists a constant \(N = N(T, c, d, h) > 0\) such that
\[
p_Y(t, r) \leq N t r^{-d} \Psi_Y(r^{-1}), \quad (t, r) \in (0, T] \times [0, \infty).
\]
Proof. Fix \(t \in (0, T]\). For \(r \geq 0\) define \(f_t(r) = r^{d/2} p_Y(t, r^{1/2})\). By (A2), for \(r \geq 0\),
\[
P_0(\sqrt{r}/2 < |Y_1| < \sqrt{r}) = \int_{\sqrt{r}/2 < |y| < \sqrt{r}} p_Y(t, |y|) dy
\]
\[
\geq |B(0, 1)|(1 - 2^{-d/2})^{1-d/2} p_Y(t, r^{1/2}) = |B(0, 1)|(1 - 2^{-d/2}) e^{-\frac{1}{2} f_t(r)}.
\]
Denoting \(\mathcal{L}f_t(\lambda)\) the Laplace transform of \(f_t\), we have
\[
\mathcal{L}f_t(\lambda) \leq \lambda \int_0^\infty P_0(\sqrt{r}/2 < |Y_1| < \sqrt{r}) e^{-\lambda r} dr = \lambda N \int_{|Y_1|^2} e^{-\lambda r} dr
\]
\[
= N \lambda^{-1} |\mathcal{E}_0| e^{-\lambda |Y_1|^2} - e^{-2\lambda |Y_1|^2} = N \lambda^{-1} g_h(\lambda), \quad \lambda > 0
\]
from (2.3).
Using \((A2)\) again, we get that, for any \(v > 0\)
\[
\mathcal{L} f_t(v^{-1}) = \int_0^\infty e^{-av} f_t(a) \, da = v \int_0^\infty e^{-s} f_t(sv) \, ds
\]
\[
\geq v \int_{1/2}^1 e^{-s} f_t(sv) \, ds = v \int_{1/2}^1 e^{-s} s^{d/2} v^{d/2} p_Y \left( t, s^{1/2} v^{1/2} \right) \, ds
\]
\[
\geq c^{-1} v^{2-d/2} v^{d/2} p_Y \left( t, v^{1/2} \right) \int_{1/2}^1 e^{-s} ds = c^{-1} 2^{-d/2} v f_t(v) \left( \int_{1/2}^1 e^{-s} ds \right).
\]
Thus
\[
f_t(v) \leq c 2^{d/2} v^{1-d} \mathcal{L} f_t(v^{-1}) \frac{v^{-1} \mathcal{L} f_t(v^{-1})}{e^{-1/2} - e^{-1}}.
\]

Now combining \((2.6)\) and \((2.7)\) with Lemma 2.1 we conclude
\[
p_Y(t, r) = r^{-d} f_t(r^2) \leq N r^{-d/2} \mathcal{L} f_t(r^{-2}) \leq N r^{-d} g_t(r^{-2}) \leq N t r^{-d} \Psi_Y(r^{-1}).
\]

\[
3. \text{ Bernstein functions and subordinate Brownian motion}
\]

Let \(S = (S_t : t \geq 0)\) be a subordinator, that is, an increasing Lévy process taking values in \([0, \infty)\) with \(S_0 = 0\). A subordinator \(S\) is completely characterized by its Laplace exponent \(\phi\) via
\[
E[\exp(-\lambda S_t)] = \exp(-t \phi(\lambda)) , \quad \lambda > 0.
\]
The Laplace exponent \(\phi\) can be written in the form (cf. \cite{1}, p. 72)
\[
\phi(\lambda) = b \lambda + \int_0^\infty (1 - e^{-\lambda t}) \mu(dt).
\]
(3.1)

Here \(b \geq 0\), and \(\mu\) is a \(\sigma\)-finite measure on \((0, \infty)\) satisfying
\[
\int_0^\infty (t \wedge 1) \mu(dt) < \infty.
\]
We call the constant \(b\) the drift and \(\mu\) the Lévy measure of the subordinator \(S\).

A smooth function \(g : (0, \infty) \to [0, \infty)\) is called a Bernstein function if
\[
(-1)^n D^n g \leq 0, \quad \forall n \in \mathbb{N}.
\]
It is well known that a nonnegative function \(\phi\) on \((0, \infty)\) is the Laplace exponent of a subordinator if and only if it is a Bernstein function with \(\phi(0+) = 0\) (see, for instance, Chapter 3 of \cite{20}). By the concavity, for any Bernstein function \(\phi\),
\[
\phi(\lambda t) \leq \lambda \phi(t) \quad \text{for all } \lambda \geq 1, t > 0,
\]
(3.2)
implying
\[
\phi(v) \leq \frac{\phi(u)}{u}, \quad 0 < u \leq v.
\]
(3.3)

Clearly \((3.2)\) implies the following.

**Lemma 3.1.** Let \(\phi\) be a Bernstein function. Then for all \(\lambda, t > 0, 1 \wedge \lambda \leq \phi(\lambda t)/\phi(t) \leq 1 \vee \lambda\).

The following will be used in section 6 to control the deterministic part of equation \((1.4)\).
Lemma 3.2. For each nonnegative integer \(n\), there is a constant \(N(n)\) such that for every Bernstein function with the drift \(b = 0\),

\[
\frac{\lambda^n |D^n \phi(\lambda)|}{\phi(\lambda)} \leq N(n), \quad \forall \lambda.
\]  

(3.4)

Proof. The statement is trivial if \(n = 0\). So let \(n \geq 1\). Due to (3.1),

\[
|D^n \phi(\lambda)| = \int_0^\infty t^n e^{-\lambda t} \mu(dt).
\]

Use \(t^n e^{-t} \leq N(1 - e^{-t})\) to conclude

\[
\lambda^n |D^n \phi(\lambda)| \leq \int_0^\infty (\lambda t)^n e^{-\lambda t} \mu(dt) \leq N \int_0^\infty (1 - e^{-\lambda t}) \mu(dt).
\]

This obviously leads to (3.4). \(\square\)

Throughout this article, we assume that \(\phi\) is a Bernstein functions with the drift \(b = 0\) and \(\phi(1) = 1\). Thus

\[
\phi(\lambda) = \int_0^\infty (1 - e^{-\lambda t}) \mu(dt).
\]

Let \(d \geq 1\) and \(W := (W_t : t \geq 0)\) be a \(d\)-dimensional Brownian motion with \(W_0 = 0\). Then

\[
E \left[ e^{i\xi \cdot W_t} \right] = e^{-\|\xi\|^2 t}, \quad \forall \xi \in \mathbb{R}^d, \quad t > 0
\]

and \(W\) has the transition density

\[
q(t, x, y) = q_d(t, x, y) = (4\pi t)^{-d/2} e^{-\|x - y\|^2 / 4t}, \quad x, y \in \mathbb{R}^d, \quad t > 0
\]

Let \(X = (X_t : t \geq 0)\) denote the subordinate Brownian motion defined by \(X_t := W_{S_t}\). Then \(X_t\) has the characteristic exponent \(\Psi(x) = \phi(|x|^2)\) and has the transition density

\[
p(t, x) = p_d(t, x) := \int_{\mathbb{R}^d} e^{i\xi \cdot x} e^{-t\phi(|\xi|^2)} d\xi.
\]  

(3.5)

For \(t \geq 0\), let \(\eta_t\) be the distribution of \(S_t\). That is, for any Borel set \(A \subset [0, \infty)\), \(\eta_t(A) = P(S_t \in A)\). Then we have

\[
p(t, x) = p_d(t, x) = \int_{(0, \infty)} (4\pi s)^{-d/2} \exp \left( -\frac{|x|^2}{4s} \right) \eta_t(ds)
\]  

(3.6)

(see [11, Section 13.3.1]). Thus \(p(t, x)\) is smooth in \(x\).

The Lévy measure \(\Pi\) of \(X\) is given by (see e.g. [19, pp. 197–198])

\[
\Pi(A) = \int_A \int_0^\infty p(t, x) \mu(dt) dx = \int_A J(x) dx, \quad A \subset \mathbb{R}^d,
\]

where

\[
J(x) := \int_0^\infty p(t, x) \mu(dt)
\]  

(3.7)

is the Lévy density of \(X\). Define the function \(j : (0, \infty) \to (0, \infty)\) as

\[
j(r) = j_d(r) := \int_0^\infty (4\pi)^{-d/2} t^{-d/2} \exp \left( -\frac{r^2}{4t} \right) \mu(dt), \quad r > 0.
\]  

(3.8)
Then $J(x) = j(|x|)$ and
\[ \Psi(\xi) = \int_{\mathbb{R}^d} (1 - \cos(\xi \cdot y)) j(|y|) dy. \]  
(3.9)

Note that the function $r \mapsto j(r)$ is strictly positive, continuous and decreasing on $(0, \infty)$.

The next lemma is an extension of [13] Lemma 3.1.

**Lemma 3.3.** There exists a constant $N > 0$ depending only on $d$ such that
\[ j(r) \leq N r^{-d} \phi(r^{-2}), \quad \forall r > 0. \]

**Proof.** By Lemma 3.1 (A1) holds with $h(\lambda) = 1 \vee \lambda^2$, and (A2) holds with $c = 1$ since $r \to p(t, r)$ is decreasing. Thus by Theorem 2.2, we have
\[ p(t, r) \leq N t r^{-d} \phi(r^{-2}) \quad \forall t, r > 0 \]  
(3.10)

where $N > 0$ depends only on $d$. The lemma now follows from (3.10) and (1.12). Indeed, by (1.12) and Section 4.1 in [21] that for $f \in C^2_0(\mathbb{R}^d \setminus \{0\})$ (the set of $C^2$-functions on $\mathbb{R}^d \setminus \{0\}$ with compact support), we have
\[ \lim_{t \to 0} \frac{1}{t} \int_{\mathbb{R}^d} p(t, |y|) f(y) dy = \phi(\Delta) f(0) = \int_{\mathbb{R}^d} j(|y|) f(y) dy. \]  
(3.11)

We fix $r > 0$ and choose a $f \in C^2_0(\mathbb{R}^d \setminus \{0\})$ such that $f = 1$ on $B(0, r) \setminus B(0, r/2)$ and $f = 0$ on $B(0, 2r) \cap B(0, r/4)$. Note that since $s \to j(s)$ is decreasing, we have
\[ r^d j(r) \leq \frac{d^d}{2^d - 1} \int_{r/2}^r j(s) s^{d-1} ds \leq N \int_{B(0, r) \setminus B(0, r/2)} j(|y|) dy \]
\[ \leq N \int_{\mathbb{R}^d} j(|y|) f(y) dy \]
where $N > 0$ depends only on $d$. Thus by (3.10) and (3.11), we see that
\[ r^d j(r) \leq \frac{d^d}{2^d - 1} \int_{r/2}^r j(s) s^{d-1} ds \leq N \int_{B(0, r) \setminus B(0, r/2)} j(|y|) dy \]
\[ \leq N \int_{B(0, r) \setminus B(0, r/2)} |y|^{-d} \phi(|y|^{-2}) f(y) dy \]
\[ \leq N \int_{B(0, 2r) \setminus B(0, r/4)} |y|^{-d} \phi(|y|^{-2}) dy \leq N \int_{B(0, 2r) \setminus B(0, r/4)} |y|^{-d} dy \phi(4r^{-2}) \]
\[ \leq N \int_{r/4}^{2r} r^{-1} dr \phi(4r^{-2}) \leq N \phi(4r^{-2}) \]
where $N > 0$ depends only on $d$. Now the lemma follows immediately by (3.12). \[ \square \]

For $a > 0$, we define $\phi^a(\lambda) = \phi(\lambda a^{-2}) / \phi(a^{-2})$. Then $\phi^a$ is again a Bernstein function satisfying $\phi^a(1) = 1$. We will use $\mu^a(dt)$ to denote the Lévy measure of $\phi^a$ and $S^a = (S_t^a)_{t \geq 0}$ to denote a subordinator with Laplace exponent $\phi^a$.

Assume that $S^a = (S^a)_{t \geq 0}$ is independent of the Brownian motion $W$. Let $X^a = (X_t^a)_{t \geq 0}$ be defined by $X_t^a := W_{S^a_t}$. Then $X^a$ is a rotationally invariant Lévy process with characteristic exponent
\[ \Psi^a(\xi) = \phi^a(|\xi|^2) = \frac{\phi(a^{-2} |\xi|^2)}{\phi(a^{-2})} = \frac{\Psi(a^{-1} \xi)}{\phi(a^{-2})}, \quad \xi \in \mathbb{R}^d. \]  
(3.12)

This shows that $\{X_t^a - X_0^a\}_{t \geq 0}$ is identical in law to the process $\{a^{-1}(X_t/\phi(a^{-2}) - X_0)\}_{t \geq 0}$. $X^1$ is simply the process $X$. 

Since, by (3.9) and (3.12),
\[ \Psi^a(\xi) = \frac{1}{\phi(a^{-2})} \int_{\mathbb{R}^d} (1 - \cos(a^{-1} \xi \cdot y)) j(|y|) dy = \frac{a^d}{\phi(a^{-2})} \int_{\mathbb{R}^d} (1 - \cos(\xi \cdot z)) j(a|z|) dz, \]
the Lévy measure of \( X^a \) has the density \( J^a(x) = j^a(|x|) \), where \( j^a \) is given by
\[ j^a(r) := a^d \phi(a^{-2})^{-1} j(ar). \]

We use \( p^a(t, x, y) = p^a(t, x - y) \) to denote the transition density of \( X^a \). Recall that the process \( \{a^{-1}(X^a_t/\phi(a^{-2}) - X^0_0) : t \geq 0\} \) has the same law as \( \{X^a_t - X^0_0 : t \geq 0\} \). In terms of transition densities, this can be written as
\[ p^a(t, x, y) = a^{-d} p(t\phi(a^{-2}), a^{-1}x), \quad (t, x, y) \in (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d. \]
Thus
\[ p(t, x) = p^1(t, x) = a^{-d} p^a(t\phi(a^{-2}), a^{-1}x), \quad (t, x) \in (0, \infty) \times \mathbb{R}^d. \]
Denote
\[ a_t := \frac{1}{\sqrt{\phi^{-1}(t^{-1})}}. \]
From (3.15), we see that
\[ p(t, x) = (a_t)^{-d} p^a(1, (a_t)^{-1}x), \quad (t, x) \in (0, \infty) \times \mathbb{R}^d. \]

Let \( \beta > 0 \). For appropriate functions \( f = f(x) \), define
\[ T_t f(x) := (p(t, \cdot) * f)(x) = \int_{\mathbb{R}^d} p(t, x - y)f(y) dy, \quad t > 0, \]
\[ \phi(\Delta)^\beta f := -\phi(-\Delta)^\beta f := \mathcal{F}^{-1}(\phi(|\xi|^2)^\beta \mathcal{F}(f))(x), \quad t > 0. \]
In particular, if \( \beta = 1 \) and \( f \in C_b^2(\mathbb{R}^d) \) then we have
\[ \phi(\Delta)f(x) = -\phi(-\Delta)f(x) = \int_{\mathbb{R}^d} (f(x + y) - f(x) - \nabla f(x) \cdot y 1_{|y| \leq 1}) j(|y|) dy \]
\[ = \lim_{\epsilon \downarrow 0} \int_{|y| \leq \epsilon} (f(x + y) - f(x)) j(|y|) dy \]
(see Section 4.1 in [21]).
Recall that \( \phi^{a_t}(\lambda) := \phi(\lambda(a_t)^{-2})/\phi((a_t)^{-2}) \). Since \( t \phi(a_t^{-2}) = 1 \), by (3.5)
\[ \phi(\Delta)^{1/2} p(t, \cdot)(x) = \int_{\mathbb{R}^d} \phi(|\xi|^2)^{1/2} e^{i \xi \cdot x} e^{-t \phi(|\xi|^2)} d\xi \]
\[ = t^{-1/2} \int_{\mathbb{R}^d} \phi(|\xi|^2)/\phi((a_t)^{-2})^{1/2} e^{i \xi \cdot x} e^{-\phi(|\xi|^2)/\phi((a_t)^{-2})} d\xi \]
\[ = t^{-1/2} (a_t)^{-d} \int_{\mathbb{R}^d} \phi^{a_t}(|\xi|^2)^{1/2} e^{i \xi \cdot x} e^{-\phi^{a_t}(|\xi|^2)} d\xi \]
\[ = (a_t)^{-d} t^{-1/2} \phi^{a_t}(\Delta)^{1/2} p^{a_t}(1, \cdot)((a_t)^{-1}x). \]

By Corollary 3.7 (iii), \( \phi^a(\lambda)^{1/2} \) is also a Bernstein function. Thus \( \phi^a(\lambda)^{1/2} = \int_0^\infty (1 - e^{-\lambda t}) \tilde{\mu}^a(dt) \) where \( \tilde{\mu}^a \) is the Lévy measure of \( \phi^a(\lambda)^{1/2} \). Let
\[ \tilde{j}^a(r) := \int_0^\infty (4\pi t)^{-d/2} e^{-r^2/(4t)} \tilde{\mu}^a(dt), \quad r, a > 0 \]
and \( j(r) := j^a(r) \). Then, by (3.14)
\[ \tilde{j}^a(r) = a^d \phi(a^{-2})^{-1/2} j(ar), \quad r, a > 0. \]
As \[3.10\], for every \( f \in C^2_b(R^d) \),
\[
\phi^a(\Delta)^{1/2} f(x) := -\phi^a(-\Delta)^{1/2} f(x)
\]
\[
= \int_{R^d} \left( f(x + y) - f(x) - \nabla f(x) \cdot y 1_{|y| \leq r} \right) \hat{\varphi}(|y|) \, dy
\]
\[
= \lim_{\varepsilon \downarrow 0} \int_{\{y \in R^d : |y| > \varepsilon\}} \left( f(x + y) - f(x) \right) \hat{\varphi}(|y|) \, dy \tag{3.19}
\]

Clearly by Lemma 3.3 we have the following. We emphasize that the constant does not depend on neither \( \phi \) nor \( a \).

**Lemma 3.4.** There exists a constant \( N > 1 \) depending only on \( d \) such that for any \( a > 0 \) and \( x \neq 0 \)
\[
\hat{\varphi}(r) \leq N r^{-d} (\phi^a(r^2))^{1/2}, \quad \forall r > 0.
\]

Recall conditions (H1) and (H2):

(H1): There exist constants \( 0 < \delta_1 \leq \delta_2 < 1 \) and \( a_1, a_2 > 0 \) such that
\[
a_1 \lambda^{\delta_1} \phi(t) \leq \phi(\lambda t) \leq a_2 \lambda^{\delta_2} \phi(t), \quad \lambda \geq 1, t \geq 1.
\]

(H2): There exist constants \( 0 < \delta_3 \leq 1 \) and \( a_3 > 0 \) such that
\[
\phi(\lambda t) \leq a_3 \lambda^{\delta_3} \phi(t), \quad \lambda \leq 1, t \leq 1.
\]

By taking \( t = 1 \) in (H1) and (H2) and using Lemma 3.1 we get that if (H1) holds then
\[
a_1 \lambda^{\delta_1} \leq \phi(\lambda) \leq a_2 \lambda^{\delta_2}, \quad \lambda \geq 1, \tag{3.20}
\]
and, if (H2) holds then
\[
\lambda \leq \phi(\lambda) \leq a_3 \lambda^{\delta_3}, \quad \lambda \leq 1. \tag{3.21}
\]

Also, if (H1) holds we have
\[
a_1 \lambda^{\delta_1} \phi^a(t) \leq \phi^a(\lambda t) \leq a_2 \lambda^{\delta_2} \phi^a(t), \quad \lambda \geq 1, t \geq a^2. \tag{3.22}
\]

Thus, by taking \( t = 1 \) in (3.22), if (H1) holds and \( a \leq 1 \) we get
\[
a_1 \lambda^{\delta_1} \leq \phi^a(\lambda) \leq a_2 \lambda^{\delta_2}, \quad \lambda \geq 1.
\]

Thus, if (H1) holds
\[
a_1 (T^{-2} \land 1) \lambda^{\delta_1} \leq \phi^a(\lambda) \leq \frac{a_2}{a(T^{-2}) \land 1} \lambda^{\delta_2}, \quad a \in (0, T], \lambda \geq 1. \tag{3.23}
\]

In fact, if \( T > 1 \) and \( 1 \leq a \leq T \) then for \( \lambda \geq 1 \)
\[
\phi^a(\lambda) = \frac{\phi(\lambda a^{-2})}{\phi(a^{-2})} \leq \frac{\phi(\lambda)}{\phi(a^{-2})} \leq \frac{\phi(\lambda)}{\phi(T^{-2})} \leq \frac{a_2 \lambda^{\delta_2}}{\phi(T^{-2})}
\]
and using Lemma 3.1
\[
\phi^a(\lambda) = \frac{\phi(\lambda a^{-2})}{\phi(a^{-2})} \geq \frac{\phi(\lambda a^{-2})}{\phi(1)} = \frac{\phi(\lambda a^{-2})}{\phi(\lambda)} \phi(\lambda) \geq a^{-2} \phi(\lambda) \geq T^{-2} \phi(\lambda) \geq T^{-2} a_1 \lambda^{\delta_1}.
\]

Recall that \( p(t, x) \) is the transition density of \( X_t \).
Lemma 3.7. Assume \((H1)\) holds. Then for each \(T > 0\) there exists a constant \(N = N(T, d, \phi) > 0\) such that for \((t, x) \in (0, T] \times \mathbb{R}^d\),

\[
p(t, x) \leq N \left( (\phi^{-1}(t^{-1}))^{d/2} t^{-\phi(|x|^{-2})/|x|^d} \right).
\]

(3.24)

**Proof.** The corollary follows from Theorem 2.2 and the first display on page 1073 of [5]. Also one can see from (3.3) and (3.13) that

\[
p(t, x) = (a_t)^{-d} p^{a_t}(1, a_t^{-1}x) \leq (a_t)^{-d} \int_{\mathbb{R}^d} e^{-\phi^{a_t}(|\xi|^2)} d\xi
\]

\[
\leq (a_t)^{-d} \int_{|\xi| < 1} d\xi + (a_t)^{-d} \int_{|\xi| \geq 1} e^{-\phi^{a_t}(|\xi|^2)} d\xi
\]

\[
\leq N(a_t)^{-d} \left( 1 + \int_{|\xi| \geq 1} e^{-a_t(a_t^{-2} 1)(|\xi|^2)} d\xi \right).
\]

where (3.23) is used in the last inequality.

\[\square\]

Remark 3.6. If there exist constants \(0 < \delta_3 \leq \delta_4 < 1\) and \(a_3, a_4 > 0\) such that

\[
a_4 \lambda^{\delta_4} \phi(t) \leq \phi(\lambda t) \leq a_3 \lambda^{\delta_3} \phi(t), \quad \lambda \leq 1, t \leq 1,
\]

(3.25)

then, as in [13] the subordinate Brownian motion \(X\) satisfies conditions (1.4), (1.13) and (1.14) from [5]. Thus, in fact, by [5], if \((H1)\) and (3.26) hold we have the sharp two-sided estimates for all \(t > 0\)

\[
N^{-1} \left( (\phi^{-1}(t^{-1}))^{d/2} t^{-\phi(|x|^{-2})/|x|^d} \right) \leq p(t, x) \leq N \left( (\phi^{-1}(t^{-1}))^{d/2} t^{-\phi(|x|^{-2})/|x|^d} \right),
\]

(3.26)

where \(N = N(\phi) > 1\).

On the other hand, when \(\delta_3 = \delta_4 = 1\) in (3.25), (3.26) does not hold and (3.24) is not sharp. For example, see [5] (2.2), (2.4) and Theorem 4.1.

For the rest of this article we assume that \((H1)\) and \((H2)\) hold. Now we further discuss the scaling. If \(0 < r < R\), using (3.3), (3.20) and (3.21), we have

\[
\frac{\phi(R)}{\phi(r)} \leq \frac{R}{r} \quad \text{and} \quad \frac{\phi(R)}{\phi(r)} \geq \frac{a_1}{a_3} \frac{R^{\delta_1} \delta_3}{r^{\delta_3}} \geq \frac{a_1}{a_3} \left( \frac{R \delta_1 \delta_3}{r} \right).
\]

Combining these with \((H1)\) and \((H2)\) we get

\[
\frac{a_1}{a_3} \left( \frac{R \delta_1 \delta_3}{r} \right) \leq \frac{\phi(R)}{\phi(r)} \leq \frac{R}{r}, \quad 0 < r < R < \infty.
\]

(3.27)

Now applying this to \(\phi^a\), we get

\[
\frac{a_1}{a_3} \left( \frac{R \delta_1 \delta_3}{r} \right) \leq \frac{\phi^a(R)}{\phi^a(r)} \leq \frac{R}{r}, \quad a > 0, \quad 0 < r < R < \infty.
\]

(3.28)

Next two lemmas will be used several times in this article.

Lemma 3.7. Assume \((H1)\) and \((H2)\). Then there exists a constant \(N = N(\delta_1, \delta_3)\) such that for all \(\lambda > 0\)

\[
\int_{\lambda^{-1}}^{\infty} r^{-1} \phi(r^{-2}) dr \leq N \phi(\lambda^2).
\]
Corollary 3.8. Assume (H1) and (H2). Then there exists a constant N = \(N(\delta_1, \delta_3)\) such that for all \(\lambda > 0\)
\[
\int_{\lambda^{-1}}^{\infty} r^{-1}(\phi(r^{-2}))^{1/2} dr \leq N(\phi(\lambda^2))^{1/2},
\]
Proof. Since \((\phi(\lambda)^{1/2})\) also satisfies conditions (H1) and (H2) with different \(\delta_1, \delta_3 > 0\), we get this corollary directly from the previous lemma.

4. Upper bounds of \(|\phi(\Delta)^{n/2}D^\beta p(t, x)|\)

In this section we give upper bounds of \(|\phi(\Delta)^{n/2}D^\beta p(t, x)|\) for any \(n = 0, 1, 2, \ldots\) and multi-index \(\beta\).

Recall \(a_t := (\phi^{-1}(t^{-1}))^{-1/2}\) and so \(t\phi(a_t^{-2}) = 1\). Thus From (3.15) and (3.24), we have for every \(t \in (0, T]\),
\[
P^{\alpha_1}(1, x) = (a_t)^{\dagger} p(t, a_t x)
\leq N(T, \phi, d) \left(1 \wedge t\phi(a_t x)^{-2}\right) = N \left(1 \wedge \frac{\phi^{\alpha_1}(|x|^{-2})}{|x|^d}\right).
\]

Lemma 4.1. For any constant \(T > 0\) there exists a constant \(N = N(T, d, \phi)\) so that for every \(t \in (0, T]\)
\[
|\nabla p^{\alpha_1}(1, x)| \leq N|x| \left(1 \wedge \frac{\phi^{\alpha_1}(|x|^{-2})}{|x|^{d+2}}\right),
\]
\[
\sum_{i, j} |\partial_{x_i, x_j} p^{\alpha_1}(1, x)| \leq N|x|^2 \left(1 \wedge \frac{\phi^{\alpha_1}(|x|^{-2})}{|x|^{d+2}}\right) + N \left(1 \wedge \frac{\phi^{\alpha_1}(|x|^{-2})}{|x|^{d+2}}\right),
\]
and
\[
\sum_{|\beta| \leq n} |D^\beta p^{\alpha_1}(1, x)| \leq N \sum_{n-2m \geq 0, m \in \mathbb{N}, \mu(0)} |x|^{n-2m} \left(1 \wedge \frac{\phi^{\alpha_1}(|x|^{-2})}{|x|^{d+2(\mu-m)}}\right).
\]

Proof. To distinguish the dimension, we denote
\[
p^{\alpha_1}_d (1, x) := \int_{(0, \infty)} (4\pi s)^{-d/2} \exp \left(-\frac{|x|^2}{4s}\right) n^{\alpha_1}_d (ds).
\]
Thus and, for $i$ and $i,j,k$ similarly.

Lemma 4.2. For any $\beta$, $p_{d+1}^{\alpha}(1, x) = N(d)[|x|^3 p_{d+6}^{\alpha}(1, x, 0, 0, 0, 0) + |x|^2 p_{d+4}^{\alpha}(1, x, 0, 0, 0, 0)].$

Proof. First we prove the lemma when $\beta = 0$. Recall

$$a_t := \frac{1}{\sqrt{\phi^{-1}(t^{-1})}} \leq \frac{1}{\sqrt{\phi^{-1}(T^{-1})}}.$$ 

Note that by (4.3),

$$\phi^{\alpha_t}(\Delta)^{1/2} p^{\alpha_t}(1, \cdot) \leq \phi^{\alpha_t}(\Delta)^{1/2} e^{-\phi^{\alpha_t}(|\xi|^2)} d\xi + \left( \sup_{b>0} b^{1/2} e^{-b/2} \right) \int_{|\xi| \geq 1} e^{-4 \phi^{\alpha_t}(|\xi|^2)} d\xi \leq N \phi^{\alpha_t}(1)^{1/2} + N \int_{|\xi| \geq 1} e^{-4 \phi^{\alpha_t}(|\xi|^2)} d\xi \leq N + N \int_{|\xi| \geq 1} e^{-2 \phi^{\alpha_t}(|\xi|^2)} d\xi.$$
Thus it is uniformly bounded by (3.28). By (3.19),

\[
\left| \phi^{\alpha_t}(\Delta)^{1/2} p^{\alpha_t}(1, \cdot)(x) \right| \\
= \lim_{\epsilon \downarrow 0} \int_{\{y \in \mathbb{R}^d : |y| > \epsilon\}} (p^{\alpha_t}(1, x + y) - p^{\alpha_t}(1, x)) \hat{\phi}^{\alpha_t}(|y|) \, dy \\
\leq |p^{\alpha_t}(1, x)| \int_{\{y \in \mathbb{R}^d : |y| > |x|/2\}} \hat{\phi}^{\alpha_t}(|y|) \, dy + \int_{\{y \in \mathbb{R}^d : |y| > |x|/2\}} |p^{\alpha_t}(1, x + y)\hat{\phi}^{\alpha_t}(|y|) \, dy \\
+ \lim_{\epsilon \downarrow 0} \int_{\{y \in \mathbb{R}^d : |x|/2 > |y| > \epsilon\}} \int_0^1 |\nabla p^{\alpha_t}(1, x + sy)| \, ds |y| \hat{\phi}^{\alpha_t}(|y|) \, dy \\
:= p^{\alpha_t}(1, x) \times I + II + III.
\]

Since from (3.18)

\[
\int_s^\infty \hat{\phi}^{\alpha_t}(r) \, dr = a_t^d \phi(a_t^{-2} - 1/2) \int_s^\infty \hat{\phi}(ar) \, dr = \phi(a_t^{-2} - 1/2) \int_{as}^\infty \hat{\phi}(t) \, dt,
\]
we get

\[
\int_{|x|}^\infty \hat{\phi}^{\alpha_t}(r) \, dr = (a_t)^d \phi((a_t^{-2} - 1/2) \int_{|x|}^\infty \hat{\phi}(at) \, dr \\
= \phi((a_t)^{-2} - 1/2) \int_{|x|}^\infty \hat{\phi}(s) \, ds \\
= \phi^{\alpha_t}(|x|^{-2} - 1/2) \phi((a_t)^{-2} - 1/2) \int_{|x|}^\infty \hat{\phi}(s) \, ds.
\]

In addition to this, applying Lemma 3.24 and Corollary 3.28 we see

\[
\phi((a_t)^{-2} - 1/2) \int_{|x|}^\infty \hat{\phi}(s) \, ds \leq N.
\]

Combining these with

\[
p^{\alpha_t}(1, x) \leq N(T) \left( 1 \wedge \frac{\phi^{\alpha_t}(|x|^{-2})}{|x|^d} \right),
\]
we get

\[
p^{\alpha_t}(1, x) \times I \leq N(T) \phi^{\alpha_t}(|x|^{-2})^{1/2} \left( 1 \wedge \frac{\phi^{\alpha_t}(|x|^{-2})}{|x|^d} \right) \leq N(T) \frac{\phi^{\alpha_t}(|x|^{-2})^{3/2}}{|x|^d}.
\]

Using the fact that $r \to \hat{\phi}^{\alpha_t}(r)$ is decreasing,

\[
II \leq \hat{\phi}^{\alpha_t}(|x|/2) \int_{\{y \in \mathbb{R}^d : |y| > |x|/2\}} p^{\alpha_t}(1, x + y) \, dy \\
\leq \hat{\phi}^{\alpha_t}(|x|/2) \int_{\mathbb{R}^d} p^{\alpha_t}(1, x + y) \, dy \leq \hat{\phi}^{\alpha_t}(|x|/2).
\]
Finally, from Lemma 4.1, we get

$$III = \left| \lim_{\varepsilon \to 0} \int_{\|y\| \leq \varepsilon} \left[ \nabla \rho^\alpha ((1, x + sy)ds |y|^{-\alpha} d|y| \right] dy \right|$$

$$\leq N \int_{|y| < |x|/2} \int_0^1 \frac{t \phi^{\alpha} (|x + sy|^{-2})}{|x + sy|^{d+1}} ds |y|^{-\alpha} d|y| dy$$

$$\leq N \int_{|y| < |x|/2} \int_0^1 \frac{\phi^{\alpha} (|s y| - |x|^{-2})}{|s y|^{-d+1}} ds |y|^{-\alpha} d|y| dy$$

$$\leq N \int_{|y| < |x|/2} \int_0^1 \frac{\phi^{\alpha} (4|y|^{-2})}{|y|^{d+1}} ds |y|^{-\alpha} d|y| dy$$

$$\leq N \frac{\phi^{\alpha} (4|y|^{-2})}{|y|^{d+1}} \int_{|y| < |x|/2} |y|^{-\alpha} d|y|.$$  

By Lemma 3.4,

$$\int_{|y| < |x|/2} |y|^{-\alpha} d|y| \leq N \int_{|y| < |x|/2} |y|^{-d+1} \phi^{\alpha} (|y|^{-2})^{1/2} dy$$

$$\leq N \int_0^{|x|} (\phi^{\alpha} (r^{-2}))^{1/2} dr.$$  

Since $|\phi^{\alpha} (\Delta)^{1/2} p^{\alpha} (1, \cdot) (x)|$ is bounded for $x$, we may assume that $|x| \geq 1$. So from the monotone property of $\phi^{\alpha} (r^{-2})$ and 3.23, we get

$$\int_0^{|x|} (\phi^{\alpha} (r^{-2}))^{1/2} dr = \int_0^1 (\phi^{\alpha} (r^{-2}))^{1/2} dr + \int_1^{|x|} (\phi^{\alpha} (r^{-2}))^{1/2} dr$$

$$\leq N \left( \int_0^1 r^{-\delta_2} dr + \int_1^{|x|} (\phi^{\alpha} (r^{-2}))^{1/2} dr \right)$$

$$\leq N \left( |x|^{-\delta_2} + |x| \phi^{\alpha} (1)^{1/2} \right).$$

Thus

$$|\phi^{\alpha} (\Delta)^{1/2} p^{\alpha} (1, \cdot) (x)| \leq N (1 + \frac{\phi^{\alpha} (|x|^{-2})^{1/2}}{|x|^d}).$$

Now applying (3.17) and using the fact that $t \phi(a_t^{-2}) = 1$ and $\phi^{\alpha} (\lambda) = \lambda^2 (\alpha a_t^{-2}) / \phi(a_t^{-2})$, we get

$$|\phi (\Delta)^{1/2} p(t, \cdot) (x)| = (a_t)^{-d} t^{-1/2} |\phi^{\alpha} (\Delta)^{1/2} p^{\alpha} (1, \cdot) ((a_t)^{-1} x)|$$

$$\leq N t^{-1/2} ((a_t)^{-d} \wedge \frac{\phi^{\alpha} ((a_t)^{-2})^{1/2}}{|x|^d})$$

$$= N t^{-1/2} ((a_t)^{-d} \wedge \frac{\phi(|x|^{-2})^{1/2}}{\phi((a_t)^{-2})^{1/2} |x|^d})$$

$$= N (t^{-1/2} (a_t)^{-d} \wedge \frac{\phi(|x|^{-2})^{1/2}}{(t \phi((a_t)^{-2}))^{1/2} |x|^d}) = N (t^{-1/2} (a_t)^{-d} \wedge \frac{\phi(|x|^{-2})^{1/2}}{|x|^d}).$$
The case \( \beta = 1 \) is proved similarly. First, one can check that \( |\phi^{a_t}(\Delta)^{1/2} D^\beta p^{a_t}(1, \cdot) (x)| \) is uniformly bounded. Note

\[
|\phi^{a_t}(\Delta)^{1/2} D^\beta p^{a_t}(1, \cdot) (x)| \\
= |\lim_{\varepsilon \downarrow 0} \int_{y \in \mathbb{R}^d: |y| > \varepsilon} (D^\beta p^{a_t}(1, x + y) - D^\beta p^{a_t}(1, x)) \hat{j}^{a_t}(|y|) \, dy| \\
\leq |D^\beta p^{a_t}(1, x)| \int_{\{y \in \mathbb{R}^d: |y| > |x|/2\}} \hat{j}^{a_t}(|y|) \, dy \\
+ |\lim_{\varepsilon \downarrow 0} \int_{\{y \in \mathbb{R}^d: |y| > |x|/2\}} \int_0^1 |\nabla D^\beta p^{a_t}(1, x + sy)| \, ds \, |y| \hat{j}^{a_t}(|y|) \, dy \\
:= |D^\beta p^{a_t}(1, x)| \times I + II + III. \tag{4.2}
\]

Since \( I \) and \( III \) can be estimated similarly as in the case \( \beta = 0 \), we only pay attention to the estimation of \( II \). We use integration by parts and get

\[
II \leq \int_{|y| = |x|/2} \hat{j}^{a_t}(|x|/2) p^{a_t}(1, x + y) \, dS \\
+ \int_{\{y \in \mathbb{R}^d: |y| > |x|/2\}} \frac{d}{dy} \hat{j}^{a_t}(|y|) p^{a_t}(1, x + y) \, dy.
\]

We use notation \( \hat{j}^{a_t}_d \) in place of \( \hat{j}^{a_t} \) to express its dimension. That is,

\[
\hat{j}^{a_t}_d (r) := \int_0^\infty (4\pi t)^{-d/2} e^{-r^2/(4t)} \hat{j}^{a_t} (dt), \quad r > 0.
\]

By its definition, we can easily see that \( \frac{d}{dr} \hat{j}^{a_t}_d (r) = -2\pi r \hat{j}^{a_t}_{d+2} (r) \). So from Lemmas 3.4 and 4.1 we get

\[
II \leq N \frac{\phi^{a_t}(|x|^{-2})^{3/2}}{|x|^{d+1}} + \int_{|y| > |x|/2} |y| \hat{j}^{a_t}_{d+2}(|y|) p^{a_t}(1, x + y) \, dy \\
\leq N \frac{\phi^{a_t}(|x|^{-2})^{3/2}}{|x|^{d+1}} + \int_{|y| > |x|/2} \frac{\phi^{a_t}(|y|^{-2})^{1/2}}{|y|^{d+1}} |y| p^{a_t}(1, x + y) \, dy \\
\leq N \frac{\phi^{a_t}(|x|^{-2})^{3/2}}{|x|^{d+1}} + \phi^{a_t}(|x|^{-2})^{1/2} \frac{1}{|x|^{d+1}}.
\]

Therefore we get

\[
|\phi^{a_t}(\Delta)^{1/2} p^{a_t}_x(1, \cdot) (x)| \leq N (1 \wedge \frac{\phi^{a_t}(|x|^{-2})^{1/2}}{|x|^{d+1}}).
\]
By fact that $t \phi(a_t^{-2}) = 1$ and $\phi^a(\lambda) = \phi(\lambda a^{-2})/\phi(a^{-2})$, we have
\[
|\phi(\Delta)^{1/2} p_{x\cdot}(t, \cdot)(x)|
= \int_{\mathbb{R}^d} (-i\xi) \phi(|\xi|^2)^{1/2} e^{i\xi \cdot x} e^{-\phi(|\xi|^2)} d\xi
= t^{-1/2} \int_{\mathbb{R}^d} (-i\xi) \phi(|\xi|^2)/\phi(a_t^{-2})^{1/2} e^{i\xi \cdot x} e^{-\phi(a_t^{-2})} d\xi
= t^{-1/2} (a_t)^{-d-1} \int_{\mathbb{R}^d} (-i\xi) \phi^a(|\xi|^2)^{1/2} e^{i(a_t^{-1})^{-1} \xi \cdot x} e^{-\phi^a(|\xi|^2)} d\xi
= (a_t)^{-d-1} t^{-1/2} \phi^a(\Delta)^{1/2} p^a_{x\cdot}(1, \cdot)((a_t)^{-1} x)
\leq N t^{-1/2} (a_t)^{-d-1} \left( 1 \wedge \frac{\phi^a((a_t)^{-1} x)^{-2}}{|a_t^{-1} x|^{d+1}} \right)
= N t^{-1/2} (a_t)^{-d-1} \left( 1 \wedge \frac{\phi(|x|^{-2})}{\phi((a_t)^{-2})^{1/2}|a_t^{-1} x|^{d+1}} \right)
= N t^{-1/2} (a_t)^{-d-1} \left( 1 \wedge \frac{t^{1/2} a_t^{d+1}}{(t \phi((a_t)^{-2}))^{1/2}|x|^{d+1}} \right)
= N \left( t^{-1/2} (a_t)^{-d-1} \wedge \frac{\phi(|x|^{-2})}{|x|^{d+1}} \right).
\tag{4.3}
\]

Finally, we consider the case $|\beta| \geq 2$. Introduce $I$, $II$ and $III$ as in \[4.2\]. $I$ and $III$ can be estimated as in the case $|\beta| = 0$. Also, $II$ can be estimated by doing the integration by parts $|\beta|$-times. For instance, if $|\beta| = 2$,
\[
II \leq \int_{|y|=|x|/2} \hat{J}^a(|x|/2) p^a_{x\cdot}(1, x + y) dS + \int_{|y|=|x|/2} \left| \frac{d}{dr} \hat{J}^a(|x|/2) p^a_{x\cdot}(1, x + y) \right| dS
+ \int_{y \in \mathbb{R}^d : |y| > |x|/2} \frac{d^2}{dr^2} \hat{J}^a(|y|) p^a_{x\cdot}(1, x + y) dy,
\]
where $dS$ is the surface measure on $\{ y \in \mathbb{R}^d : |y| = |x|/2 \}$. By its definition, we easily see that
\[
\frac{d}{dr} \hat{J}^a_d(r) = -2\pi r^{d+2} \hat{J}^a_d(r), \quad \frac{d^2}{dr^2} \hat{J}^a_d(r) = -2\pi r^{d+2} \hat{J}^a_d(r) + (2\pi)^2 r^{2d} \hat{J}^a_{d+4}(r).
\]
So from Lemma \[3.4\] and Lemma \[4.1\] and we get
\[
II \leq N \left( \frac{\phi^a(|x|^{-2})^{3/2}}{|x|^{d+2}} + \int_{|y| > |x|/2} |y|^{2\phi^a_d}(|y|) p^a_{x\cdot}(1, x + y) dy \right)
\leq N \left( \frac{\phi^a(|x|^{-2})^{3/2}}{|x|^{d+2}} + \int_{|y| > |x|/2} |y|^{-(d+2)} \phi^a(|y|^{-2})^{1/2} p^a_{x\cdot}(1, x + y) dy \right)
\leq N \left( \frac{\phi^a(|x|^{-2})^{3/2}}{|x|^{d+2}} + \phi^a(|x|^{-2})^{1/2} \right).
\]
Therefore we have
\[
|\phi^a(\Delta)^{1/2} p^a_{x\cdot}(1, \cdot)(x)| \leq N \left( 1 \wedge \frac{\phi^a(|x|^{-2})^{1/2}}{|x|^{d+2}} \right)
\]
and we are done by the scaling as in \[4.3\].
Lemma 4.3. For any \( n \in \mathbb{N} \) and multi-index \( \beta \), there exists a constant \( N = N(d, \phi, T, |\beta|, n) > 0 \) such that

\[
|\phi(\Delta)^{n/2}D^\beta p(t, \cdot)(x)| \leq N \left( t^{-n/2}(\phi^{-1}(t-1))^{(d+|\beta|)/2} \wedge t^{-(n-1)/2}\frac{\phi(|x|-2)^{1/2}}{|x|^{d+|\beta|}} \right). \tag{4.4}
\]

**Proof.** We use the induction. Due to the previous lemma, the statement is true if \( n = 1 \). Assume that the statement is true for \( n - 1 \). We put

\[
|\phi^\alpha(\Delta)^{n/2}D^\beta p^\alpha(1, \cdot)(x)|
\]

\[
= \lim_{\varepsilon \downarrow 0} \int_{\{y \in \mathbb{R}^d: |y| > \varepsilon\}} (\phi(\Delta)^{(n-1)/2}D^\beta p^\alpha(1, x + y) - \phi(\Delta)^{(n-1)/2}D^\beta p^\alpha(1, x)) \widehat{j}^\alpha(|y|) dy
\]

\[
\leq |\phi(\Delta)^{(n-1)/2}D^\beta p^\alpha(1, x)| \int_{\{y \in \mathbb{R}^d: |y| > |x|/2\}} \widehat{j}^\alpha(|y|) dy
\]

\[
+ \int_{\{y \in \mathbb{R}^d: |y| > |x|/2\}} |\phi(\Delta)^{(n-1)/2}D^\beta p^\alpha(1, x + y)| \widehat{j}^\alpha(|y|) dy
\]

\[
+ \lim_{\varepsilon \downarrow 0} \int_{\{y \in \mathbb{R}^d: |y| > |x|/2 + |x|\varepsilon\}} \int_0^1 |\phi(\Delta)^{1/2}\nabla p^\alpha(1, x + sy)| ds |y| \widehat{j}^\alpha(|y|) dy
\]

\[
:= |\phi(\Delta)^{1/2}p^\alpha(1, x)| \times I + II + III
\]

and follow the proof of Lemma 4.2 with the result (4.4) for \( n - 1 \). Below, we provide details only for \( II \). Let \(|\beta| = 0\). Then since \( r \to j^\alpha(r) \) is decreasing,

\[
II \leq \widehat{j}^\alpha(|x|/2) \int_{\{y \in \mathbb{R}^d: |y| > |x|/2\}} \phi(\Delta)^{(n-1)/2}D^\beta p^\alpha(1, x + y) dy
\]

\[
\leq \widehat{j}^\alpha(|x|/2) \int_{\mathbb{R}^d} \phi(\Delta)^{(n-1)/2}D^\beta p^\alpha(1, x + y) dy \leq N \widehat{j}^\alpha(|x|/2) \leq N \frac{\phi^\alpha(|x|-2)^{1/2}}{|x|^{d+|\beta|}}.
\]

If \(|\beta| > 0\) then as in Lemma 4.2 we use integration by parts \(|\beta|-\text{times} and get

\[
II \leq N \frac{\phi^\alpha(|x|-2)^{1/2}}{|x|^{d+|\beta|}}.
\]

Therefore, since \(|\phi^\alpha(\Delta)^{n/2}D^\beta p^\alpha(1, \cdot)(x)|\) is uniformly bounded, we have

\[
|\phi^\alpha(\Delta)^{n/2}D^\beta p^\alpha(1, \cdot)(x)| \leq N \left( \frac{\phi^\alpha(|x|-2)^{1/2}}{|x|^{d+|\beta|}} \wedge 1 \right)
\]

and the lemma is proved by the scaling as in (4.3). \( \square \)

5. Proof of Theorem 4.1

Let \( f \in C_0^\infty(\mathbb{R}^{d+1}, H) \). For each \( a \in \mathbb{R} \) denote

\[
u_a(t, x) := G_a(t, x) := \int_a^t |\phi(\Delta)^{1/2}T_{t-s}f(s, \cdot)(x)|_H^2 ds^{1/2},
\]

\( u(t, x) := u_0(t, x) \) and \( \mathcal{G}(t, x) := G_0(t, x) \).

Here is a version of Theorem 4.1 for \( p = 2 \).
Lemma 5.1. For any $\infty \geq \beta \geq \alpha \geq -\infty$ and $\beta \geq a$,

$$\|u_a\|_{L^2([\alpha, \beta] \times \mathbb{R}^d)}^2 \leq N \|f\|_{L^2([\alpha, \beta] \times \mathbb{R}^d)}^2, \quad (5.1)$$

where $N = N(d)$.

Proof. By the continuity of $f$, the range of $f$ belongs to a separable subspace of $H$. Thus by using a countable orthonormal basis of this subspace and the Fourier transform one easily finds

$$\|u_a\|_{L^2([\alpha, \beta] \times \mathbb{R}^d)}^2 = (2\pi)^d \int_{\mathbb{R}^d} \int_{\alpha}^{\beta} \int_{\mathbb{R}^d} |\mathcal{F}\{\phi(\Delta)^{1/2} p(t-s, \cdot)\}(\xi)|^2 |\mathcal{F}(f)(s, \xi)|^2_H dsdt d\xi \leq (2\pi)^d \int_{\mathbb{R}^d} \int_{\alpha}^{\beta} \int_{\mathbb{R}^d} I_{0 \leq |\xi|^2} |\mathcal{F}(f)(s, \xi)|^2_H dsdt d\xi.$$

Changing $t-s \to t$, we find that the last term above is equal to

$$\|u_a\|_{L^2([\alpha, \beta] \times \mathbb{R}^d)}^2 \leq (2\pi)^d \int_{\mathbb{R}^d} \int_{\alpha}^{\beta} \int_{\mathbb{R}^d} I_{0 \leq |\xi|^2} e^{-2(t-s)|\phi(|\xi|^2)|} dt |\mathcal{F}(f)(s, \xi)|^2_H dsdt d\xi.$$

Since $\int_0^\infty \phi(|\xi|^2)e^{-2t|\phi(|\xi|^2)|} dt = 1/2$, we have

$$\|u_a\|_{L^2([\alpha, \beta] \times \mathbb{R}^d)}^2 \leq N \int_{\mathbb{R}^d} \int_{\alpha}^{\beta} |\hat{f}(s, \xi)|^2_H dsd\xi.$$

The last expression is equal to the right-hand side of (5.1), and therefore the lemma is proved. \qed

For $c > 0$ and $(r, z) \in \mathbb{R}^{d+1}$, we denote

$$B_c(z) = \{ y \in \mathbb{R}^d : |z - y| < c \}, \quad \hat{B}_c(z) = \prod_{i=1}^{d} (z^i - c/2, z^i + c/2),$$

$$I_c(r) = (r - \phi(c^{-2})^{-1}, r + \phi(c^{-2})^{-1}), \quad Q_c(r, z) = I_c(r) \times \hat{B}_c(z).$$

Also we denote

$$Q_c(r) = Q_c(r, 0), \quad \hat{B}_c = \hat{B}_c(0), \quad B_c = B_c(0).$$

For a measurable function $h$ on $\mathbb{R}^d$, define the maximal functions

$$M_x h(x) := \sup_{r > 0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |h(y)| dy,$$

$$M_x h(x) := \sup_{\hat{B}_c(z) \ni x} \frac{1}{|\hat{B}_c(z)|} \int_{\hat{B}_c(z)} |h(y)| dy.$$

Similarly, for a measurable function $h = h(t)$ on $\mathbb{R}$,

$$M_t h(t) := \sup_{r > 0} \frac{1}{2r} \int_{-r}^r |h(t + s)| ds.$$
Also for a function \( h = h(t, x) \), we set
\[
M_x h(t, x) := M_x(h(t, \cdot))(x), \quad M_x h(t, x) := M_x(h(\cdot, x))(t),
\]
\[
M_t M_x h(t, x) = M_t(M_x h(\cdot, x))(t), \quad M_x M_t M_x h(t, x) = M_x(M_t M_x h(\cdot, \cdot))(x).
\]

Since we have estimates of \( p(t, x) \) only for \( t \leq T \), we introduce the following functions. Denote \( \hat{p}(t, x) = p(t, x) \) if \( t \in [0, T] \), and \( \hat{p}(t, x) = 0 \) otherwise. Define
\[
\hat{T}_t f(x) = \int_{\mathbb{R}^d} \hat{p}(t, y) f(x - y) dy,
\]
and for every \( a \in \mathbb{R} \)
\[
\hat{u}_a(t, x) := \hat{G}_a(t, x) := \begin{cases} \int_a^{t-a} |f(a)|^{1/2} \hat{T}_{t-a} f(s, \cdot)(x) H^2_t ds \end{cases} : t \geq a
\]
\[
\left( \int_a^{t-a} |f(a)|^{1/2} \hat{T}_{t-a} f(s, \cdot)(x) H^2_t ds \right)^{1/2} : t < a.
\]

We use \( \hat{u}(t, x) \) and \( \hat{G}(t, x) \) in place of \( \hat{u}_0(t, x) \) and \( \hat{G}_0(t, x) \) respectively. Obviously, Lemmas 4.2 and 13 hold with \( \hat{p}(t, x) \) instead of \( p(t, x) \) (for all \( t \)). Moreover,
\[
\hat{u}_a(t, x) = \hat{u}_a(2a - t, x) \quad \forall t \in \mathbb{R}, \quad \hat{u}_a(t, x) \leq u_a(t, x) \quad \text{if} \ t \geq a,
\]
\[
\hat{u}_a(t, x) = u_a(t, x) \quad \text{if} \ t \in [a, T + a]. \tag{5.2}
\]

**Lemma 5.2.** Assume that the support of \( f \) belongs to \( \mathbb{R} \times B_{3dc} \). Then for any \( c > 0 \) and \( (t, x) \in Q_c(r) \)
\[
\int_{Q_c(r)} |\hat{u}_a(s, y)|^2 ds dy \leq N[|r - a| + \phi(c^{-2})^{-1}] c^d M_t M_x |f| H^2_t(t, x),
\]
where \( N \) depends only on \( d \).

**Proof.** Fix \( (t, x) \in Q_c(r) \). Using [5.2] and Lemma 5.1 we get
\[
\int_{Q_c(r)} |\hat{u}_a(s, y)|^2 ds dy \leq \int_{(r + \phi(c^{-2})^{-1}) \land a} \int_{\mathbb{R}^d} |\hat{u}_a(s, y)|^2 dy ds
\]
\[
= \int_{\mathbb{R}^d} \int_{(r + \phi(c^{-2})^{-1}) \land a} |\hat{u}_a(2a - s, y)|^2 ds + \int_{a}^{(r + \phi(c^{-2})^{-1}) \land a} |\hat{u}_a(s, y)|^2 ds dy
\]
\[
= \int_{\mathbb{R}^d} \int_{a}^{2a - [(r + \phi(c^{-2})^{-1}) \land a]} |\hat{u}_a(s, y)|^2 ds + \int_{a}^{(r + \phi(c^{-2})^{-1}) \land a} |\hat{u}_a(s, y)|^2 ds dy \leq \int_{B_{3dc}} \left[ \int_a^{2a - [(r + \phi(c^{-2})^{-1}) \land a]} |f(s, y)|^2 H^2_t ds + \int_a^{(r + \phi(c^{-2})^{-1}) \land a} |f(s, y)|^2 H^2_t ds \right] dy.
\]

Since \( |x - y| \leq |x| + |y| \leq 4dc \) for any \( (t, x) \in Q_c(r) \) and \( y \in B_{3dc} \), the last term above is less than or equal to constant times of
\[
\int_{|x - y| \leq 4dc} \left[ \int_a^{2a - [(r + \phi(c^{-2})^{-1}) \land a]} |f(s, y)|^2 H^2_t ds + \int_a^{(r + \phi(c^{-2})^{-1}) \land a} |f(s, y)|^2 H^2_t ds \right] dy \leq N c^d \left[ \int_a^{2a - [(r + \phi(c^{-2})^{-1}) \land a]} M_x |f(s, x)|^2_H^2 ds + \int_a^{(r + \phi(c^{-2})^{-1}) \land a} M_x |f(s, x)|^2_H^2 ds \right] \leq \int_{Q_c(r)} |\hat{u}_a(s, y)|^2 ds dy.
\]
In order to explain the last inequality above, we denote
\[
\int_{a}^{2a-\left[r - \phi(c^{-2})^{-1}\right]} M_x |f(s, x)|_{H}^2 ds + \int_{a}^{(r + \phi(c^{-2})^{-1}) \wedge a} M_x |f(s, x)|_{H}^2 ds := I + J.
\]
First we estimate \( I \). \( I = 0 \) if \( r - \phi(c^{-2})^{-1} \geq a \). So assume \( r - \phi(c^{-2})^{-1} < a \).
If \( a \leq t \leq 2a - (r - \phi(c^{-2})^{-1}) \), then we can easily get
\[
I \leq [r - a] + \phi(c^{-2})^{-1} |M_x M_x| |f|_{H}^2(t, x).
\]
If \( t > 2a - (r - \phi(c^{-2})^{-1}) \) and \( t \geq a \), then
\[
I \leq \int_{a}^{t + (t-a)} M_x |f(s, x)|_{H}^2 ds \leq 2(t-a)M_x M_x |f|_{H}^2(t, x) \leq 2(r + \phi(c^{-2})^{-1} - a)M_x M_x |f|_{H}^2(t, x).
\]
Finally, if \( t < a \), then
\[
I \leq \int_{t}^{2a - (r - \phi(c^{-2})^{-1})} M_x |f(s, x)|_{H}^2 ds \leq \int_{t + t - [2a - (r - \phi(c^{-2})^{-1})]} M_x |f(s, x)|_{H}^2 ds \leq 2([2a - (r - \phi(c^{-2})^{-1})] - t)M_x M_x |f|_{H}^2(t, x) \leq 4[a - (r - \phi(c^{-2})^{-1})]M_x M_x |f|_{H}^2(t, x).
\]
The estimation of \( J \) is similar. Therefore, the lemma is proved. \( \square \)

We will use the following version of integration by parts: if \( 0 \leq \varepsilon \leq R \leq \infty \), and \( F \) and \( G \) are smooth enough then (see (14))
\[
\int_{R - |z| \geq \varepsilon} F(z)G(|z|)dz = -\int_{R}^{R} G'(\rho) \left( \int_{|z| \leq \rho} F(z)dz \right) d\rho
\]
\[
+G(R)\int_{|z| \leq R} F(z)dz - G(\varepsilon)\int_{|z| \leq \varepsilon} F(z)dz.
\]
We generalize Lemma 5.2 as follows.

**Lemma 5.3.** For any \((t, x) \in Q_{c}(r)\)
\[
\int_{Q_{c}(r)} |\tilde{u}_{a}(s, y)|^{2} dsdy \leq N[|r - a| + \phi(c^{-2})^{-1}|c^{d}M_x M_x| |f|_{H}^2(t, x),
\]
where \( N = N(d, T, \phi) \).

**Proof.** Take \( \zeta \in C_{0}^{\infty}(\mathbb{R}^{d}) \) such that \( \zeta = 1 \) in \( B_{2dc} \), \( \zeta = 0 \) outside of \( B_{3dc} \), and \( 0 \leq \zeta \leq 1 \). Set \( A = \zeta f \) and \( B = (1 - \zeta) f \). By Minkowski’s inequality, \( \hat{G}_{a} f \leq \hat{G}_{a} A + \hat{G}_{a} B \).
Since \( \hat{G}_{a} A \) can be estimated by Lemma 5.2 assume that \( f(t, x) = 0 \) for \( x \in B_{2dc} \).
Let \( \varepsilon := (1, 0, \ldots, 0) \) and \( s \geq \mu \geq a \). Then since \( \phi(\Delta)^{1/2} \hat{p}(t, x) \) is rotationally invariant with respect to \( x \), we have
\[
|\phi(\Delta)^{1/2} \hat{p}_{s-\mu}(y)|_{H} = \int_{|z| \leq \rho} \phi(\Delta)^{1/2} \hat{p}(s - \mu, |z| \varepsilon) f(\mu, y - z) dz|_{H} \leq \int_{|z| \leq \rho} \phi(\Delta)^{1/2} \hat{p}_{|z| \varepsilon}(s - \mu, \rho \varepsilon) \int_{|z| \leq \rho} f(\mu, y - z) d\rho|_{H}.
\]
For the second equality above, \( [3.4] \) is used with \( G(|z|) = \phi(\Delta)^{1/2} \rho(s - \mu)|z|e_1 \) and \( F(z) = f(\mu, y - z) \). Observe that if \( y \in B_\varepsilon \) then for \( x \in (-c/2, c/2)^d \)

\[
|x - y| \leq 2dc, \quad B_\rho(y) \subset B_{2dc + \rho}(x).
\]

Moreover, if \(|z| \leq c\), then \(|y - z| \leq 2dc\) and \( f(\mu, y - z) = 0\). Thus by Corollary 3.8 and Lemma 4.2

\[
|\phi(\Delta)^{1/2} \mathcal{T}_{x-\mu} f(\mu, \cdot)(y)|_H \leq N \int_c^\infty \frac{\phi(\rho^{-2})^{1/2}}{\rho^{d+1}} \int_{|z| \leq \rho} |f|_H(\mu, y - z) \, dz \, d\rho
\]

\[
= N \int_c^\infty \frac{\phi(\rho^{-2})^{1/2}}{\rho^{d+1}} \int_{B_\rho(y)} |f|_H(\mu, z) \, dz \, d\rho
\]

\[
\leq N \int_c^\infty \frac{\phi(\rho^{-2})^{1/2}}{\rho^{d+1}} \int_{B_{2dc + \rho}(x)} |f|_H(\mu, z) \, dz \, d\rho
\]

\[
\leq N \int_c^\infty \frac{\phi(\rho^{-2})^{1/2}}{\rho^{d+1}} \int_{B_{2dc + \rho}(x)} |f|_H(\mu, x) \, dz \, d\rho
\]

\[
\leq N \int_c^\infty \frac{\phi(\rho^{-2})^{1/2}}{\rho^{d+1}} \, d\rho
\]

By Jensen’s inequality \( (M_x f|_H)^2 \leq M_x |f|_H^2 \), and therefore, we get for any \( s \geq a \) and \( y \in B(c) \)

\[
|\tilde{u}_a(s, y)|^2 \leq N \phi(c^{-2}) \int_a^s M_x |f|_H^2(\mu, x) \, d\mu.
\]

So if \( r + \phi(c^{-2})^{-1} \geq s \geq a \), then we have

\[
|\tilde{u}_a(s, y)|^2 \leq N \phi(c^{-2})[r + \phi(c^{-2})^{-1} - (a \land (r - \phi(c^{-2})^{-1}))]M_x M_x |f|_H^2(t, x)
\]

\[
\leq N \phi(c^{-2})[|r - a| + \phi(c^{-2})^{-1}]M_x M_x |f|_H^2(t, x).
\]

If \( r - \phi(c^{-2})^{-1} \leq s < a \), then we get

\[
|\tilde{u}_a(s, y)|^2 = |\tilde{u}_a(2a - s, y)|^2 \leq N \phi(c^{-2}) \int_a^{2a-s} M_x |f|_H^2(\mu, x) \, d\mu
\]

\[
\leq N \phi(c^{-2})[|r - a| + \phi(c^{-2})^{-1}]M_x M_x |f|_H^2(t, x).
\]

Therefore, we get for any \((t, x) \in Q_c(r)\)

\[
\int_{Q_c(r)} |\tilde{u}_a(s, y)|^2 \, dxdy \leq N[|r - a| + \phi(c^{-2})^{-1}]c^d \cdot M_t M_x |f|_H^2(t, x).
\]

The lemma is proved.

\[\Box\]

**Lemma 5.4.** Assume \( 2\phi(c^{-2})^{-1} < r \). Then for any \((t, x) \in Q_c(r)\),

\[
\int_{Q_c(r)} \int_{r - \phi(c^{-2})^{-1}}^{r + \phi(c^{-2})^{-1}} |\tilde{u}(s_1, y) - \tilde{u}(s_2, y)|^2 \, ds_1ds_2dy
\]

\[
\leq N \phi(c^{-2})^{-2} c^d \left[ M_t M_x |f|_H^2(t, x) + M_t M_t M_x |f|_H^2(t, x) + M_x M_x M_x |f|_H^2(t, t - x) \right],
\]

where \( N = N(d, T, \phi) \).
Proof. Due to the symmetry, to estimate the left term of (5.6), we only consider the case \( s_1 > s_2 \). Since \( r - \phi(c^{-2})^{-1} > \phi(c^{-2})^{-1} > 0 \), we have \( s_2 > 0 \) for any \((s_2, y) \in Q_\epsilon(r)\). Observe that by Minkowski’s inequality

\[
|\tilde{u}(s_1, y) - \tilde{u}(s_2, y)|^2 \leq \int_0^{s_1} |\phi(\Delta)^{1/2} \hat{T}_{s_1-\kappa} f(\kappa, \cdot)(y)|^2_H d\kappa + \int_0^{s_2} |\phi(\Delta)^{1/2} \hat{T}_{s_2-\kappa} f(\kappa, \cdot)(y)|^2_H d\kappa
\]

and

\[
\int_0^{s_1} |\phi(\Delta)^{1/2} \hat{T}_{s_1-\kappa} f(\kappa, \cdot)(y)|^2_H d\kappa - \int_0^{s_2} |\phi(\Delta)^{1/2} \hat{T}_{s_2-\kappa} f(\kappa, \cdot)(y)|^2_H d\kappa \leq I^0(s_1, s_2, y) + J^0(s_1, s_2, y).
\]

One can easily estimate \( I^0 \) using Lemma 5.3 with \( a = s_2 \) and \( |r-a| \leq 2\phi(c^{-2})^{-1} \), and thus we only need to show that

\[
\int_{Q_{\epsilon}(r)} \int_{r - |r - a|^{-1}}^{r + |r - a|^{-1}} J^0(s_1, s_2, y) ds_1 ds_2 dy
\]

is less than or equal to the right hand side of (5.8). We divide \( J^0 \) into two parts:

\[ I := \int_{0}^{r-2\phi(c^{-2})^{-1}} |\phi(\Delta)^{1/2} \hat{T}_{s_1-\kappa} f(\kappa, \cdot)(y)|^2_H d\kappa \]

\[ J := \int_{r-2\phi(c^{-2})^{-1}}^{s_2} |\phi(\Delta)^{1/2} \hat{T}_{s_1-\kappa} f(\kappa, \cdot)(y)|^2_H d\kappa \]

Note that since \( s_1 \geq s_2 \geq r - \phi(c^{-2})^{-1} \), we have \( \eta s_1 + (1-\eta)s_2 - \kappa \geq \phi(c^{-2})^{-1} \)

for any \( \eta \in [0, 1] \) and \( \kappa \in [0, r - 2\phi(c^{-2})^{-1}] \).

If \( s_1 - \kappa > T \) and \( s_2 - \kappa \leq T \) then

\[ |\phi(\Delta)^{1/2} \hat{T}_{s_1-\kappa} f(\kappa, \cdot)(y)|^2_H = |\phi(\Delta)^{1/2} \hat{T}_{s_2-\kappa} f(\kappa, \cdot)(y)|^2_H.\]

Otherwise, using \( \frac{\partial}{\partial \kappa} \phi(\Delta)^{1/2} T_{r} f(x) = \phi(\Delta)^{3/2} T_{r} f(x) \), we get

\[ |\phi(\Delta)^{1/2} \hat{T}_{s_1-\kappa} f(\kappa, \cdot)(y)|^2_H \leq \int_0^{1} |\phi(\Delta)^{3/2} \hat{T}_{\eta s_1 + (1-\eta)s_2 - \kappa} f(\kappa, \cdot)(y)|^2_H d\eta.\]

Therefore,

\[ I \leq \int_{s_2-T}^{s_1-T} |\phi(\Delta)^{1/2} \hat{T}_{s_2-\kappa} f(\kappa, \cdot)(y)|^2_H d\kappa \]

\[ + (s_1 - s_2)^2 \int_0^{r-2\phi(c^{-2})^{-1}} \int_0^1 |\phi(\Delta)^{3/2} \hat{T}_{\eta s_1 + (1-\eta)s_2 - \kappa} f(\kappa, \cdot)(y)|^2_H d\eta d\kappa.\]

Denote \( \bar{s} = \bar{s}(\eta) = \eta s_1 + (1-\eta)s_2 \). As in (5.3),

\[ |\phi(\Delta)^{3/2} \hat{T}_{s-\kappa} f(\kappa, \cdot)(y)|_H \]

\[ = \int_{[\bar{s}]_{\rho}} |\phi(\Delta)^{3/2} \hat{p}_{\rho} (\bar{s} - \kappa, \rho \eta) f(\kappa, y - z)|_H d\rho d\kappa \]

\[ \leq M_{\rho} \int_{[\bar{s}]_{\rho}} |\phi(\Delta)^{3/2} \hat{p}_{\rho} (\bar{s} - \kappa, \rho \eta)|_H d\rho d\kappa.\]
\[ \int_0^\infty |\phi(\Delta)^{3/2}\rho_2(x, (s-\kappa, \rho e_1))| \rho^d d\rho \]

\[ = \int_0^c |\phi(\Delta)^{3/2}\rho_2(x, (s-\kappa, \rho e_1))| \rho^d d\rho + \int_c^\infty |\phi(\Delta)^{3/2}\rho_2(x, (s-\kappa, \rho e_1))| \rho^d d\rho \]

\[ \leq N \int_0^c (s-\kappa)^{-3/2} \phi^{-1}((s-\kappa)^{-1})^{(d+1)/2} \rho^d d\rho + N(s-\kappa)^{-1} \int_c^\infty \phi(\rho^{-2})^{1/2} \rho^d d\rho \]

\[ \leq N(s-\kappa)^{-1} \left[ \phi(c^{-2})^{1/2} c^{-(d+1)} \int_0^c \rho^d d\rho + \phi(c^{-2})^{1/2} \right] \leq N(s-\kappa)^{-1} \phi(c^{-2})^{1/2}. \]

Similarly, one can check

\[ |\phi(\Delta)^{1/2} \hat{T}_{s_2-\kappa} f(\kappa, \cdot)(\cdot)_{H} | \leq N \phi(c^{-2})^{1/2} M_x |f|_{H}(\kappa, y). \]

Therefore, remembering \(|s_1 - s_2| \leq 2\phi(c^{-2})^{-1}\), we get

\[ I \leq N \phi(c^{-2})^{-1} \int_0^{r-2\phi(c^{-2})^{-1}} (r - \phi(c^{-2})^{-1} - \kappa)^{-2} M_x |f|_{H}(\kappa, y) d\kappa \]

\[ + N\phi(c^{-2}) \int_{s_2-T}^{s_1-T} M_x |f|_{H}^2(\kappa, y) d\kappa. \]

Note that \(M_x |f|_{H}^2(\kappa, y)\) in (5.7) can be replaced by \(I_{0,\kappa} = r - 2\phi(c^{-2})^{-1}\) times of it. Thus by integration by parts,

\[ \phi(c^{-2})^{-1} \int_0^{r-2\phi(c^{-2})^{-1}} (r - \phi(c^{-2})^{-1} - \kappa)^{-2} M_x |f|_{H}^2(\kappa, y) d\kappa \]

\[ \leq N\phi(c^{-2})^{-1} \int_{-\infty}^{r-2\phi(c^{-2})^{-1}} \frac{r + \phi(c^{-2})^{-1} - \kappa}{(r - \phi(c^{-2})^{-1} - \kappa)^3} d\kappa \]

\[ \leq N\phi(c^{-2})^{-1} M_t M_x |f|_{H}^2(t, y) \int_{-\infty}^{\infty} \frac{\kappa + 2\phi(c^{-2})^{-1}}{\kappa^3} d\kappa \]

\[ \leq N M_t M_x |f|_{H}^2(t, y). \]

Also,

\[ \phi(c^{-2}) \int_{s_2-T}^{s_1-T} M_x |f|_{H}^2(\kappa, y) d\kappa \leq \phi(c^{-2}) \int_{r-\phi(c^{-2})^{-1}-T}^{r+\phi(c^{-2})^{-1}-T} M_x |f|_{H}^2(\kappa, y) d\kappa \]

\[ \leq 2 M_t M_x |f|_{H}^2(t - T, y). \]

Therefore,

\[ I \leq N[M_x M_t |f|_{H}^2(t, y) + M_t M_x |f|_{H}^2(t - T, y)], \]

where \(N\) depends only on \(d, T, \alpha, \delta_i (i = 1, 2, 3)\), and this certainly implies

\[ \int_{B_t} \int_{B_x} I_{s_1} ds_1 dy \int_{r-\phi(c^{-2})^{-1}}^{r+\phi(c^{-2})^{-1}} d\kappa \]

\[ \leq N\phi(c^{-2})^{-2} c^d [M_x M_t M_x |f|_{H}^2(t, x) + M_x M_t M_x |f|_{H}(t - T, x)]. \]
It only remains to estimate $J$. Since $s_1 \geq s_2$,
\[
\int^{s_2}_{r-2\phi(c^{-2})^{-1}} \int^{s_1}_{r-2\phi(c^{-2})^{-1}} \left| \phi(\Delta)^{1/2} \hat{T}_{s_2-\kappa} f(\kappa, \cdot)(x) - \phi(\Delta)^{1/2} \hat{T}_{s_2-\kappa} f(\kappa, \cdot)(y) \right|^2 d\kappa \, dy \kappa
\]
\[
\leq 2 \int^{s_1}_{r-2\phi(c^{-2})^{-1}} \left| \phi(\Delta)^{1/2} \hat{T}_{s_1-\kappa} f(\kappa, \cdot)(x) \right|^2_H d\kappa
\]
\[
+ 2 \int^{s_2}_{r-2\phi(c^{-2})^{-1}} \left| \phi(\Delta)^{1/2} \hat{T}_{s_2-\kappa} f(\kappa, \cdot)(y) \right|^2_H d\kappa.
\]
Therefore, we are done by Lemma 5.3 with $a = r - 2\phi(c^{-2})^{-1}$.

\[\tag*{\Box}\]

**Lemma 5.5.** Assume $2\phi(c^{-2})^{-1} < r$. Then for any $(t, x) \in Q_c(r)$,
\[
\int^{r+\phi(c^{-2})^{-1}}_{r-\phi(c^{-2})^{-1}} \int_{B_x} \left| \hat{u}(s, y_1) - \hat{u}(s, y_2) \right|^2 dy_1 dy_2 ds
\]
\[
\leq N \phi(c^{-2})^{-1} c^{2d} \left\| M_t \mathbb{M}_{x} |f|_{H}^2(t, x) + M_t \mathbb{M}_{x} |f|_{H}^2(t, x) \right\|
\]
where $N = N(d, T, \phi)$.

**Proof.** By Minkowski’s inequality,
\[
\left| u(s, y_1) - u(s, y_2) \right|^2
\]
\[
= \left| \int_{0}^{s} \phi(\Delta)^{1/2} \hat{T}_{s-\kappa} f(\kappa, y_1) d\kappa \right|^2 - \left( \int_{0}^{s} \phi(\Delta)^{1/2} \hat{T}_{s-\kappa} f(\kappa, y_2) d\kappa \right)^2
\]
\[
\leq \int_{0}^{s} \phi(\Delta)^{1/2} \hat{T}_{s-\kappa} f(\kappa, y_1) - \phi(\Delta)^{1/2} \hat{T}_{s-\kappa} f(\kappa, y_2) d\kappa
\]
\[
\leq \int_{0}^{r-2\phi(c^{-2})^{-1}} \phi(\Delta)^{1/2} \hat{T}_{s-\kappa} f(\kappa, y_1) - \phi(\Delta)^{1/2} \hat{T}_{s-\kappa} f(\kappa, y_2) d\kappa
\]
\[
+ \int_{r-2\phi(c^{-2})^{-1}}^{s} \phi(\Delta)^{1/2} \hat{T}_{s-\kappa} f(\kappa, y_1) - \phi(\Delta)^{1/2} \hat{T}_{s-\kappa} f(\kappa, y_2) d\kappa
\]
\[
: = I(s, y_1, y_2) + J(s, y_1, y_2).
\]

By Lemma 5.3 with $a = r - 2\phi(c^{-2})^{-1}$,
\[
\int^{r+\phi(c^{-2})^{-1}}_{r-\phi(c^{-2})^{-1}} \int_{B_x} J(s) \, dy_1 dy_2 ds \leq N \phi(c^{-2})^{-1} c^{2d} \left\| M_t \mathbb{M}_{x} |f|_{H}^2(t, x) \right\|
\]
Therefore, we only need to estimate $I$. Let $r - \phi(c^{-2})^{-1} < s < r + \phi(c^{-2})^{-1}$.

Observe that for $(s, y_1), (s, y_2) \in Q_c(r)
\[
I \leq N \left\| \phi(\Delta)^{1/2} \hat{T}_{s-\kappa} f(\kappa, \cdot)(\eta y_1 + (1 - \eta) y_2) \right\|_{H}^2 d\kappa
\]
\[
\tag{5.8}
\]
Recall that $\hat{\phi}(s - \kappa, y) = 0$ if $s - \kappa > T$. Therefore if $t + r - 2\phi(c^{-2})^{-1} < s$, then
\[
c^{2d} \int_{0}^{1} \int_{0}^{r-2\phi(c^{-2})^{-1}} |\nabla \phi(\Delta)^{1/2} \hat{T}_{s-\kappa} f(\kappa, \cdot)(\eta y_1 + (1 - \eta) y_2) |_{H}^2 d\kappa d\eta
\]
So we assume $T + r - 2\phi(c^{-2})^{-1} \geq s$, which certainly implies
\[
T \geq s - (r - 2\phi(c^{-2})^{-1}) \geq \phi(c^{-2})^{-1}, \quad c \leq \phi^{-1}(\frac{1}{T})^{-1/2}.
\]
Moreover from (3.20) and (3.21) we see that

\[ c \leq \phi^{-1}\left(\frac{1}{T}\right)^{-1/2} \leq \left(\frac{1}{a_2 T}\right)^{-1/(2\delta_2)} \vee \left(\frac{1}{a_3 T}\right)^{-1/(2\delta_3)} \right). \tag{5.9} \]

Recall \( a_t := (\phi^{-1}(t^{-1}))^{-1/2} \) and \( t_0(a_t^{-2}) = 1 \). For simplicity, denote

\[ \bar{y} = \bar{y}(\eta) = \eta y_1 + (1 - \eta)y_2. \]

As before, using (5.4), we get

\[
|\nabla \phi(\Delta)^{1/2} \hat{T}_{s-x} f(\kappa, \cdot)(\bar{y})|_H \\
= |\int_{0}^{\infty} \nabla \phi(\Delta)^{1/2} \hat{p}_{x,t}(s - \kappa, \rho e_1) \int_{|z| \leq \rho} f(\kappa, \bar{y} - z) dz d\rho|_H \\
\leq |\int_{0}^{\infty} \nabla \phi(\Delta)^{1/2} \hat{p}_{x,t}(s - \kappa, \rho e_1) \int_{|\bar{y} - z| \leq \rho} f(\kappa, z) dz d\rho|_H \\
\leq N M_x |f|_H (s, \bar{y}) \sum_{i=1}^{d} \int_{0}^{\infty} |\phi(\Delta)^{1/2} \hat{p}_{x,1}(s - \kappa, \rho e_1)| \rho^i d\rho \\
\leq N M_x |f|_H (s, \bar{y}) \sum_{i=1}^{d} \int_{0}^{\rho_{-\kappa}} |\phi(\Delta)^{1/2} \hat{p}_{x,1}(s - \kappa, \rho e_1)| \rho^i d\rho \\
+ \int_{\rho_{-\kappa}}^{\infty} |\phi(\Delta)^{1/2} \hat{p}_{x,1}(s - \kappa, \rho e_1)| \rho^i d\rho \\
:= I_1 + I_2. \]

By Lemma 4.2

\[
I_1 \leq N M_x |f|_H (s, \bar{y})(s - \kappa)^{-1/2} \phi^{-1}((s - \kappa)^{-1})(d+2)/2 \int_{0}^{\rho_{-\kappa}} \rho^i d\rho \\
\leq N M_x |f|_H (s, \bar{y})(s - \kappa)^{-1/2} \phi^{-1}((s - \kappa)^{-1})^{1/2}, \tag{5.10} \]

and by Lemma 4.2 and Corollary 3.8

\[
I_2 \leq N M_x |f|_H (s, \bar{y}) \int_{\rho_{-\kappa}}^{\infty} \frac{\phi(\rho^{-2})^{1/2}}{\rho^2} d\rho \\
\leq N M_x |f|_H (s, \bar{y}) a_{s-\kappa}^{-1} \int_{\rho_{-\kappa}}^{\infty} \frac{\phi(\rho^{-2})^{1/2}}{\rho} d\rho \\
\leq N M_x |f|_H (s, \bar{y}) a_{s-\kappa}^{-1} \phi(a_{s-\kappa}^{-1})^{1/2} \\
= N M_x |f|_H (s, \bar{y})(s - \kappa)^{-1/2} \phi^{-1}((s - \kappa)^{-1})^{1/2}. \tag{5.11} \]
The lemma is proved.

where (5.9) is used for the last inequality. Thus, Note that

Therefore, using (5.10) and (5.11), and coming back to (5.8), we get

\[\begin{align*}
&\leq Nc^{2d} \int_{-\infty}^{\infty} M_x |f|^2_H(r - \phi(c^{-2})^{-1} - \kappa, \bar{y}) \kappa^{-1} \phi^{-1}(\kappa^{-1}) \kappa \, dk \\
&\leq Nc^{2d} \int_{1}^{\infty} M_x |f|^2_H(r - \phi(c^{-2})^{-1} - \phi(c^{-2})^{-1} \kappa, \bar{y}) \kappa^{-1} \phi^{-1}(\kappa^{-1}) \kappa \, dk \\
&\leq Nc^{2(d-1)} \int_{1}^{\infty} M_x |f|^2_H(r - \phi(c^{-2})^{-1} - \phi(c^{-2})^{-1} \kappa, \bar{y}) \kappa^{-1} \phi^{-1}(\kappa^{-1}) \kappa \, dk.
\end{align*}\]

For the last inequality above, we used Lemma 5.3. Indeed, for \( \kappa \geq 1 \) and \( t > 0 \)

\[\phi(t)\kappa^{-1} = \phi(kt\kappa^{-1})\kappa^{-1} \leq \phi(t\kappa^{-1}), \quad \phi^{-1}(\phi(t)\kappa^{-1}) \leq t\kappa^{-1}.\]

Note that \( |f|^2_H(r - \phi(c^{-2})^{-1} - \phi(c^{-2})^{-1} \kappa, \bar{y}) \) in (5.12) can be replaced by \( I_{\kappa > 1} \) times of it. Therefore, by integration by parts

\[\begin{align*}
&\leq c^{2(d-1)} \int_{1}^{\infty} \int_{0}^{\kappa} M_x |f|^2_H(r - \phi(c^{-2})^{-1} - \phi(c^{-2})^{-1} \nu, \bar{y}) \nu^{-1} \kappa^{-3} \, dk \\
&\leq Nc^{2(d-1)} \phi(c^{-2}) \int_{1}^{\infty} \int_{r - \phi(c^{-2})^{-1} - \phi(c^{-2})^{-1} \kappa}^{r + \phi(c^{-2})^{-1}} M_x |f|^2_H(\nu, \bar{y}) \nu^{-1} \kappa^{-3} \, dk \\
&\leq Nc^{2(d-1)} M_{\delta} M_x |f|^2_H(t, \bar{y}) \phi(c^{-2}) \int_{1}^{\infty} (2\phi(c^{-2})^{-1} + \phi(c^{-2})^{-1} \kappa \nu^{-1} \kappa^{-3}) \, dk \\
&\leq Nc^{2(d-1)} M_{\delta} M_x |f|^2_H(t, \bar{y}) \leq N M_{\delta} M_x |f|^2_H(t, \bar{y}),
\end{align*}\]

where (5.9) is used for the last inequality. Thus,

\[I(s, y_1, y_2) \leq N \int_{0}^{1} M_{\delta} M_x |f|^2_H(t, \eta y_1 + (1 - \eta) y_2) \, d\eta,
\]

where \( N \) depends only on \( d, T, a, \delta_i \) \((i = 1, 2, 3)\). Finally, we conclude that

\[\begin{align*}
&\int_{r - \phi(c^{-2})^{-1}}^{r + \phi(c^{-2})^{-1}} \int_{B_{\kappa}} B_{\kappa} I(s, y_1, y_2) \, dy_1 dy_2 ds \\
&\leq N \phi(c^{-2})^{-1} \int_{0}^{1} \int_{B_{\kappa}} B_{\kappa} M_x M_x |f|^2_H(t, \eta y_1 + (1 - \eta) y_2) dy_1 dy_2 d\eta \\
&\leq N \phi(c^{-2})^{-1} \int_{0}^{1} \int_{B_{\kappa}} B_{\kappa} M_x M_x |f|^2_H(t, \eta y_1 + (1 - \eta) y_2) dy_1 dy_2 d\eta \\
&\leq N \phi(c^{-2})^{-1} \int_{B_{\kappa}} B_{\kappa} M_x M_x |f|^2_H(t, y) dy dy_2 \\
&\leq N \phi(c^{-2})^{-1} c^{2d} M_x M_x M_x |f|^2_H(t, x).
\]

The lemma is proved. \( \Box \)
For a measurable function $h(t, x)$ on $\mathbb{R}^{d+1}$ define the sharp function

$$h^#(t, x) := \sup_{Q_c(r, z) \ni (t, x)} \frac{1}{|Q_c(r, z)|} \left| h(s, y) - h_{Q_c(r, z)} \right| dsdy,$$

where

$$h_{Q_c(r, z)}(x) = \int_{Q_c(r, z)} h(s, y) dsdy := \frac{1}{|Q_c(r, z)|} \int_{Q_c(r, z)} h(s, y) dsdy.$$

The following two theorems are classical results and can be found in [23].

**Theorem 5.6. (Hardy-Littlewood)** For $1 < p < \infty$ and $f \in L^p(\mathbb{R}^d)$, we have

$$\|M_x f\|_{L^p(\mathbb{R}^d)} + \|M_t f\|_{L^p(\mathbb{R}^d)} \leq N(d, p, \phi) \|f\|_{L^p(\mathbb{R}^d)}.$$

**Theorem 5.7. (Fefferman-Stein).** For any $1 < p < \infty$ and $h \in L^p(\mathbb{R}^d)$,

$$\|h\|_{L^p(\mathbb{R}^d)} \leq N(d, p, \phi) \|h^#\|_{L^p(\mathbb{R}^d+1)}.$$

**Proof.** We can get this result from Theorem IV.2.2 in [23]. Indeed, due to (3.27) we can easily check that the balls $Q_{c}(s, y)$ satisfy the conditions (i)-(iv) in section 1.1 of [23]:

(i) $Q_{c}(t, x) \cap Q_{c}(s, y) \neq \emptyset$ implies $Q_{c}(s, y) \subset Q_{N_1c}(t, x)$;

(ii) $|Q_{N_1c}(t, x)| \leq N_2|Q_{c}(t, x)|$;

(iii) $\cap_{c>0}Q_{c}(t, x) = \{ (t, x) \}$ and $\cup_{c}Q_{c}(t, x) = \mathbb{R}^{d+1}$;

(iv) for each open set $U$ and $c > 0$, the function $(t, x) \rightarrow |Q_{c}(t, x) \cap U|$ is continuous.

□

**Proof of Theorem 1.1**

First assume $f(t, x) = 0$ if $t \notin [0, T]$. Since the theorem is already proved if $p = 2$ in Lemma 5.1, we assume $p > 2$.

First, we prove

$$((Gf)^#)^2(t, x) \leq N(G(t, x) + G(-t, x)), \quad (5.13)$$

where

$$G(t, x) := M_x M_t |f|_{H}^2(t, x) + M_x M_t M_z |f|_{H}^2(t, x) + M_x M_z M_x |f|_{H}^2(t - T, x).$$

Because of Jensen’s inequality, to prove (5.13) it suffices to prove that for each $Q_{c}(r, z) \in F$ and $(t, x) \in Q_{c}(r, z)$,

$$\int_{Q_{c}(r, z)} \int_{Q_{c}(r, z)} |Gf(s_1, y_1) - Gf(s_2, y_2)|^2 ds_1 dy_1 ds_2 dy_2 \leq N(G(t, x) + G(-t, x)). \quad (5.14)$$

To prove this we use translation and apply Lemma 5.3 5.4 and 5.6.
By the definition of $\hat{G}f$ and the fact $\hat{T}_t g(y+z) = \hat{T}_t g(z+\cdot)(y)$, we see for $s \geq 0$
\[
\hat{G} f(s, y+z) = \left| \int_0^s \phi(\Delta)^{1/2} \hat{T}_{s-\rho} f(\rho, \cdot)(y+z) |^2 d\rho \right|^{1/2} 
\]
\begin{align*}
&= \left| \int_0^s \phi(\Delta)^{1/2} \hat{T}_{s-\rho} f(\rho, z+\cdot)(y) |^2 d\rho \right|^{1/2} \\
&= \left| \int_0^s \phi(\Delta)^{1/2} \hat{T}_{s-\rho} f(\rho, z+\cdot)(y) |^2 d\rho \right|^{1/2} \\
&= \hat{G} f(\cdot, z+\cdot)(s, y),
\end{align*}
(5.15)
and
\[
\hat{G} f(-s, y+z) = \hat{G} f(s, y+z) = \hat{G} f(\cdot, z+\cdot)(s, y) = \hat{G} f(\cdot, z+\cdot)(-s, y).
\]

Therefore we get
\[
\int_{Q_c(r, z)} |\hat{G} \{ f(\cdot, \cdot) \} |^2 dyds = \frac{1}{|Q_c(r)|} \int_{Q_c(r)} |\hat{G} \{ f(\cdot, \cdot) \} |^2 dyds
\]
\[
= \int_{Q_c(-r)} |\hat{G} \{ f(\cdot, \cdot) \} |^2 dyds.
\]

This shows that we may assume that $z = 0$ and $Q_c(r, z) = Q_c(r)$.
If $|r| \leq 2\phi(c^{-2})^{-1}$, then (5.14) follows from Lemma 5.3. Also if $r > 2\phi(c^{-2})^{-1}$ then (5.14) follows from Lemmas 5.4 and 5.5. Therefore it only remains to consider the case $r < -2\phi(c^{-2})^{-1}$. In this case, (5.14) follows from the identity
\[
\int_{Q_c(r)} \int_{Q_c(r)} |\hat{G} f(s_1, y_1) - \hat{G} f(s_2, y_2)|^2 ds_1 dy_1 ds_2 dy_2
\]
\[
= \int_{Q_c(-r)} \int_{Q_c(-r)} |\hat{G} f(s_1, y_1) - \hat{G} f(s_2, y_2)|^2 ds_1 dy_1 ds_2 dy_2.
\]

This is because, for $r' := -r$, we have $r' > 2\phi(c^{-2})^{-1}$ and this case is already proved above. Thus we have proved (5.14).

Recall that $G f(t, x) = \hat{G} f(t, x)$ for $t \in [0, T]$. Thus by Theorem 5.7 and (5.13)
\[
\|G f\|_{L_p([0,T] \times \mathbb{R}^d)} = \|G f\|_{L_p(\mathbb{R}^{d+1})} \leq N \|G f\|_{L_p(\mathbb{R}^{d+1})}^p \leq N \left( (\mathcal{M}_4 \mathcal{M}_6 |f^2_H|)^{1/2} \right)^{p/2} (t, x)
\]
\[
\leq N \left( (\mathcal{M}_4 \mathcal{M}_6 |f^2_H|)^{1/2} \right)^{p/2} \|f\|_{L_p(\mathbb{R}^{d+1})}^{p-1/2} (t, x)\],
\]
where for the last inequality we use (5.13) and the fact that the $L_p$-norm is invariant under reflection and translation. Now we use Theorem 5.6 to get
\[
\int_{\mathbb{R}^d} \int_{-\infty}^{\infty} (\mathcal{M}_4 \mathcal{M}_6 |f^2_H|)^{p/2} dt dx + \int_{-\infty}^{\infty} \int_{\mathbb{R}^d} (\mathcal{M}_4 \mathcal{M}_6 |f^2_H|)^{p/2} dt dx
\]
\[
\leq N \int_{-\infty}^{\infty} \int_{\mathbb{R}^d} (\mathcal{M}_4 \mathcal{M}_6 |f^2_H|)^{p/2} dt dx + N \int_{\mathbb{R}^d} \int_{-\infty}^{\infty} (\mathcal{M}_4 \mathcal{M}_6 |f^2_H|)^{p/2} dt dx
\]
\[
\leq N \int_{-\infty}^{\infty} \int_{\mathbb{R}^d} (|f^2_H|)^{p/2} dt dx + N \int_{-\infty}^{\infty} \int_{\mathbb{R}^d} (\mathcal{M}_6 |f^2_H|)^{p/2} dt dx
\]
\[
\leq N \|f\|_{L_p(\mathbb{R}^{d+1}, H)}^p = N \|f\|_{L_p([0,T] \times \mathbb{R}^d, H)}^p.
\]

Finally, for the general case, choose $\zeta_n \in C_0^\infty(0, T)$ such that $|\zeta_n| \leq 1$, $\zeta_n = 1$ on $[1/n, T - 1/n]$. Then from the above inequality and Lebesgue dominated
are given independent one-dimensional Wiener processes \( w \).
\( \sigma \) denote the predictable \( \ell \)-fields generated by \( \{I_t\} \) is a Wiener process relative to \( \Omega \).
For \( u \) follows from definition (6.2). For (ii), it suffices to prove the completeness.

\[ \|u\|_{H_p^\gamma} = ||(1 - \phi(\Delta))^{-\gamma/2}u\|_p := \|F^{-1}\{(1 + \phi(\xi^2))^{\gamma/2}F(u)(\xi)\}\|_p < \infty, \]

where \( F \) is the Fourier transform and \( F^{-1} \) is the inverse Fourier transform. Similarly, for a \( \ell_2 \)-valued function \( u = (u^1, u^2, \cdots) \) we define
\[ \|u\|_{H_p^\gamma(\ell_2)} = \|F^{-1}\{(1 + \phi(\xi^2))^{\gamma/2}F(u)(\xi)\}\|_{\ell_2}. \]

The following result can be found, for instance, in [6].

**Lemma 6.1.** (i) For any \( \mu, \gamma \in \mathbb{R} \), the map \((1 - \phi(\Delta))^{\mu/2} : H_p^\gamma \rightarrow H_p^{\gamma - \mu}\) is an isometry.

(ii) For any \( \gamma \in \mathbb{R} \), \( H_p^\gamma \) is a Banach space.

(iii) If \( \gamma_1 \leq \gamma_2 \), then \( H_p^{\gamma_1} \subset H_p^{\gamma_2} \) and
\[ \|u\|_{H_p^{\gamma_1}} \leq c \|u\|_{H_p^{\gamma_2}}. \]

(iv) Let \( \gamma \geq 0 \). Then there is a constant \( c > 1 \) so that
\[ c^{-1}\|u\|_{H_p^{\gamma}} \leq (\|u\|_p + \|\phi(\Delta)^{\gamma/2}u\|_p) \leq c\|u\|_{H_p^{\gamma}}. \]

**Proof.** (i) follows from definition (6.2). For (ii), it suffices to prove the completeness. Let \( \{u_n : n = 1, 2, \cdots\} \) be a Cauchy sequence in \( H_p^\gamma \). Then \( f_n := (1 - \phi(\Delta))^{\gamma/2}u_n \) is a Cauchy sequence in \( L_p \), and there exists \( f \in L_p \) so that \( f_n \rightarrow f \) in \( L_p \). Define \( u := (1 - \phi(\Delta))^{-\gamma/2}f \). Then \( u \in H_p^\gamma \) and
\[ \|u_n - u\|_{H_p^\gamma} = \|f_n - f\|_p \rightarrow 0. \]
Finally (iii) and (iv) are consequences of a Fourier multiplier theorem. Indeed, due to Theorem 0.2.6 in [22], we only need to show that for any \( \gamma_1 \leq 0 \) and \( \gamma_2 \in \mathbb{R} \)
\[ |D^n[1 + \phi(\xi^2)^{\gamma_1}]| + |D^n[\frac{(1 + \phi(\xi^2)^{\gamma_2})}{1 + \phi(\xi^2)^{\gamma_2}}]| + |D^n[\frac{(\phi(\xi^2)^{\gamma_2})}{(1 + \phi(\xi^2)^{\gamma_2})}]| \leq N(n)|\xi|^{-n}. \]
This comes from Lemma 3.2. The lemma is proved. \( \square \)
In particular, for \( \phi \in C_c^\infty(\mathbb{R}^d) \), that is for any \( \phi \), the operator \( (1 - \phi(\Delta))^{\gamma/2} : \mathcal{H}_p^{\phi, \gamma}(T) \to \mathcal{H}_p^{\phi, 2}(T) \) is an isometry, it suffices to prove the case \( \gamma = 0 \). In this case \( \mathcal{H}_p^{\phi, \gamma} = L_p \), and therefore (6.4) is proved, for instance, in Theorem 3.4 of [9]. Also, the completeness of the space \( \mathcal{H}_p^{\phi, \gamma + 2}(T) \) can be proved using (6.4) as in the proof of Theorem 3.4 of [9].

The following maximal principle will be used to prove the uniqueness result of equation (6.1).

**Lemma 6.5.** Let \( \lambda > 0 \) be a constant. Suppose that \( u \) is continuous in \([0, T] \times \mathbb{R}^d\), \( u(t, \cdot) \in C_b^2(\mathbb{R}^d) \) for each \( t > 0 \), \( u_t, \phi(\Delta)u \) are continuous in \([0, T] \times \mathbb{R}^d\), \( u_t - \phi(\Delta)u + \lambda u = 0 \) for \( t \in (0, T] \), \( u(t_n, x) \to u(t, x) \) as \( t_n \to t \) uniformly for \( x \in \mathbb{R}^d \), \( u(0, x) = 0 \) for all \( x \in \mathbb{R}^d \), and for each \( t \), \( u(t, x) \to 0 \) as \( |x| \to \infty \). Then \( u \equiv 0 \) in \([0, T] \times \mathbb{R}^d\).
Proof. Suppose \( \text{sup}_{(t,x)\in[0,T] \times \mathbb{R}^d} u(t,x) > 0 \). Then we claim that there exists a \((t_0, x_0) \in [0,T] \times \mathbb{R}^d\) such that \( u(t_0, x_0) = \text{sup}_{(t,x)\in[0,T] \times \mathbb{R}^d} u(t,x) \). The explanation is as follows. Since there exists a sequence \((t_n, x_n)\) such that \( u(t_n, x_n) \to \sup_{(t,x)\in[0,T] \times \mathbb{R}^d} u(t,x) \), one can choose a subsequence \( t_{n_k} \) such that \( t_{n_k} \to t_0 \) for some \( t_0 \in [0,T] \). If \( \{x_n\} \) is unbounded, then there exists a subsequence \( x_{n_k} \) such that \( |x_{n_k}| \to \infty \). Due to the assumption: \( u(t_n, x) \to u(t, x) \) as \( t_n \to t \) uniformly for \( x \in \mathbb{R}^d \), we have \( u(t_{n_k}, x_{n_k}) \to u(t_0, x) \) as \( k \to \infty \). But since \( u(t_0, x_{n_k}) \to 0 \) as \( k \to \infty \) this is contradiction to the fact \( u(t_{n_k}, x_{n_k}) \to \sup_{(t,x)\in[0,T] \times \mathbb{R}^d} u(t,x) > 0 \). Therefore, \( \{x_n\} \) is bounded and this means that \( x_n \) also has a subsequence \( x_{n_k} \) such that \( x_{n_k} \to x_0 \) for some \( x_0 \in \mathbb{R}^d \). So we know that our claim is true. Note that since \( u(0,x) = 0 \) \( \forall x \in \mathbb{R}^d \), \( t_0 > 0 \) and if \( t_0 \in (0,T) \) then \( u_t(t_0,x_0) = 0 \) otherwise \( u_t(T,x_0) \geq 0 \). Recall that

\[
\phi(\Delta)u(t,x) = \lim_{\varepsilon \downarrow 0} \int_{|y| > \varepsilon} (u(t,x+y) - u(t,x))dy
\]

and \( j \) is strictly positive. Therefore, we get \( (u_t - \phi(\Delta)u + \lambda u)(t_0,x_0) > 0 \) and this is contradiction to our assumption. So we have \( \sup_{(t,x)\in[0,T] \times \mathbb{R}^d} u(t,x) \leq 0 \). Similarly, we can easily show that \( \inf_{(t,x)\in[0,T] \times \mathbb{R}^d} u(t,x) \geq 0 \). The lemma is proved. \( \square \)

The following result will be used to estimate the deterministic part of (6.1).

**Lemma 6.6.** Let \( m(\tau, \xi) := \phi(|\xi|^2) / |\tau + \phi(|\xi|^2)| \). Then, \( m \) is a \( L_p(\mathbb{R}^{d+1}) \)-multiplier. In other words,

\[
\| \mathcal{F}^{-1}(m\mathcal{F}f) \|_{L_p(\mathbb{R}^{d+1})} \leq N \| f \|_{L_p(\mathbb{R}^{d+1})}, \quad \forall f \in L_p(\mathbb{R}^{d+1}),
\]

where \( N \) depends only on \( d \) and \( p \).

**Proof.** First we estimate derivatives of \( m \). Let \( \alpha = (\alpha_1, \cdots, \alpha_d) \neq 0 \) be a \( d \)-dimensional multi-index with \( \alpha_i = 0 \) or \( 1 \) for \( i = 1, \ldots, d \). Assume \( \beta, \gamma \) are multi-indices so that \( \beta + \gamma = \alpha \). Then from Lemma 3.2 we can easily get

\[
|D^{\beta}(\phi(|\xi|^2))| \leq N\phi(|\xi|^2)|\xi|^{-|\beta|}.
\]

Suppose \( \gamma \neq 0 \). Without loss of generality assume \( \gamma_1 = 1 \). Then by Leibniz’s rule and (6.3), we get

\[
|D^\alpha m(\tau, \xi)| \leq N \frac{\phi(|\xi|^2)}{|\tau| + \phi(|\xi|^2)}|\xi|^{-|\alpha|}.
\]

Obviously even if \( \gamma = 0 \), (6.6) holds. Therefore from (6.5) and (6.6), we get
Next let \( \hat{\alpha} = (\alpha_0, \alpha_1, \ldots, \alpha_d) = (\alpha_0, \alpha) \) be a \((d+1)\)-dimensional multi-index with \( \alpha_0 \neq 0 \) and \( \alpha_i = 0 \) or 1 for \( i = 1, \ldots, d \). Then from (6.7) we get

\[
|D^{\hat{\alpha}}m(\tau, \xi)| \leq N(|\tau| + \phi(|\xi|^2))^{-\alpha_0}|D^\alpha m(\tau, \xi)| \leq N \frac{\phi(|\xi|^2)}{(|\tau| + \phi(|\xi|^2))^{\alpha_0+1}}|\xi|^{-|\alpha|}. \tag{6.8}
\]

Now to conclude that \( m \) is a multiplier, we use Theorem 4.6’ in p 109 of [24]. Due to (6.8), we see that for each \( 0 < k \leq d + 1 \)

\[
\left| \frac{\partial^k m}{\partial \tau \partial_1 \cdots \partial_{\xi_{k-1}}} \right| \leq N \frac{\phi(|\xi|^2)}{(|\tau| + \phi(|\xi|^2))^2}|\xi|^{-|k-1|}.
\]

Therefore, for any dyadic rectangles \( A = \Pi_{1 \leq i \leq k}[2^{k_i}, 2^{k_i+1}] \), we have

\[
\int_A \left| \frac{\partial^k m}{\partial \tau_1 \partial_2 \cdots \partial_{\xi_k}} \right| \leq N.
\]

We can easily check that from (6.7) and (6.8), the above statement is also valid for every one of the \( n! \) permutations of the variables \( \tau, \xi_1, \ldots, \xi_d \). The lemma is proved.

Here is our \( L_p \)-theory.

**Theorem 6.7.** For any \( f \in \mathbb{H}_{p,\gamma}^e(T) \) and \( g = (g^1, g^2, \cdots) \in \mathbb{H}_{p,\gamma+1}^e(T, \ell_2) \), equation (6.1) has a unique solution \( u \in \mathcal{H}_{p,\gamma+2}^e(T) \), and for this solution

\[
\|u\|_{\mathcal{H}_{p,\gamma+2}^e(T)} \leq N\|f\|_{\mathbb{H}_{p,\gamma}^e(T)} + N\|g\|_{\mathbb{H}_{p,\gamma+1}^e(T, \ell_2)}.
\tag{6.9}
\]

**Proof.** Due to the isometry, we may assume \( \gamma = 0 \). Note that if \( u \) is a solution of (6.1) in \( \mathcal{H}_{p,2}^e(T) \), then we have \( u \in C([0,T], L_p) \) (a.s.) by Theorem 6.3.

**Step 1.** First we prove the uniqueness of Equation (6.1). Let \( u_1, u_2 \) be solutions of equation (6.1). Then putting \( v := u_1 - u_2 \), we see that \( v \) satisfies (6.1) with \( f = g^k = 0 \).

Take a non-negative smooth function \( \varphi \in C_0^\infty \) with unit integral. For \( \varepsilon > 0 \), define \( \varphi_\varepsilon(x) = e^{-\varepsilon} \varphi(x/\varepsilon) \). Also denote \( w(t, x) = e^{-\varepsilon t} u(t, x) \) and \( w^\varepsilon = w \ast \varphi_\varepsilon \). Then by plugging \( \varphi_\varepsilon(\cdot - x) \) in (6.3) in place of \( \varphi \), we have \( w^\varepsilon_t - \phi(\Delta) w^\varepsilon + \lambda w^\varepsilon = 0 \). Also one can easily check that \( w^\varepsilon \) satisfies the conditions in Lemma 6.5 and concludes that \( w^\varepsilon \equiv 0 \). This certainly proves \( v \equiv 0 \) (a.s.).

**Step 2.** We consider the case \( g = 0 \). By approximation argument (see the next step for the detail), to prove the existence and (6.9), we may assume that \( f \) is sufficiently smooth in \( x \) and vanishes if \( |x| \) is sufficiently large. In this case, one can easily check that for each \( \omega \),

\[
u(t, x) = \int_0^T T_{t-s} f(s, x) ds \tag{6.10}
\]

satisfies \( u_t = \phi(\Delta) u + f \). In addition to it, denoting \( \tilde{p}(t, x) = I_{0 \leq t} p(t, x) \) and \( \tilde{f} = I_{0 \leq t \leq T} f(t, x) \), we see that for \( (t, x) \in [0, T] \times \mathbb{R}^d \)

\[
u(t, x) = \tilde{p}(\cdot, \cdot) \ast \tilde{f}(\cdot, \cdot)(t, x). \tag{6.11}
\]

We use notation \( \mathcal{F}_d \) and \( \mathcal{F}_{d+1} \) to denote the Fourier transform for \( x \) and \( (t, x) \), respectively. Moreover for convenience we put \( \mathcal{F}_d u(t, x) = \mathcal{F}_d (u(t, \cdot))(x) \). Under
this setting, observe that
\[ F_{d+1}(\bar{p})(\tau, \xi) = \int_{\mathbb{R}} e^{-ir\tau} F_{d+1}(\bar{p})(t, \xi) dt = \int_{0}^{\infty} e^{-ir\tau} e^{-t\phi(|\xi|^2)} dt = \frac{1}{i\tau + \phi(|\xi|^2)}. \]

So denoting \( \bar{u} = \bar{p}(\cdot, \cdot) * f(\cdot, \cdot)(t, x) \) we see
\[
F_{d+1}^{-1}[(1 + \phi(|\xi|^2))F_{d+1}(\bar{u})](t, x) = \bar{u} + F_{d+1}^{-1}[\phi(|\xi|^2)F_{d+1}(\bar{p})F_{d+1}(\bar{f})] = \bar{u} + F_{d+1}^{-1}[\frac{\phi(|\xi|^2)}{i\tau + \phi(|\xi|^2)}F_{d+1}(\bar{f})]. \tag{6.12}
\]

Due to generalized Minkowski’s inequality, we can easily check that \( \|\bar{u}\|_{L_{p}^p(\mathbb{R}^{d+1})} \leq N\|f\|_{L_{p}^p(\mathbb{R}^{d+1})} \). Moreover we know that \( \frac{\phi(|\xi|^2)}{i\tau + \phi(|\xi|^2)} \) is a \( L_{p}(\mathbb{R}^{d+1}) \)-multiplier from Lemma 6.6. Therefore, from (6.12) we conclude that
\[
\|u\|_{L_{p}^p(\tau)} \leq \|f\|_{L_{p}^p(\tau)}. \tag{6.14}
\]

**Step 3.** We consider the case \( f = 0 \). First, assume that \( g^k = 0 \) for all sufficiently large \( k \) (say for all \( k \geq N_0 \)), and each \( g^k \) is of the type
\[
g^k(t, x) = \sum_{i=0}^{m_k} 1_{(\tau^k_i, \tau^k_{i+1})}(t)g^k_i(x) \quad \text{for} \ k \leq N_0, \tag{6.13}
\]
where \( \tau^k_i \) are bounded stopping times and \( g^k_i(x) \in C_0^{\infty}(\mathbb{R}^d) \). Define
\[
v(t, x) := \sum_{i=0}^{N_0} \int_{0}^{t} g^k(s, x) dw^k_s = \sum_{i=1}^{N_0} \sum_{i=0}^{m_k} g^k_i(x)(w^k_{t, \tau^k_{i+1}} - w^k_{t, \tau^k_i})
\]
and
\[
u(t, x) := v(t, x) + \int_{0}^{t} \phi(\Delta) T_{t-s} v(s, x) ds = v(t, x) + \int_{0}^{t} T_{t-s} \phi(\Delta) v(s, x) ds. \tag{6.14}
\]

Then \( u - v = \int_{0}^{t} T_{t-s} \phi(\Delta) v(s, x) ds \), and therefore (see (6.10)) we have
\[
(u - v)_t = \phi(\Delta) (u - v) + \phi(\Delta) v = \phi(\Delta) u,
\]
and
\[
du = d(u - v) + dv = \phi(\Delta) u dt + \sum_{k=1}^{N_0} g^k dw^k_t.
\]

Also by (6.14) and stochastic Fubini theorem ([113, Theorem 64]), almost surely,
\[
u(t, x) = v(t, x) + \sum_{k=1}^{N_0} \int_{0}^{t} \int_{0}^{s} \phi(\Delta) T_{t-s} g^k(r, x) dw^k_r ds = v(t, x) - \sum_{k=1}^{N_0} \int_{0}^{t} \int_{0}^{t} \phi(\Delta) T_{t-s} g^k(r, x) ds dw^k_r
\]
\[= \sum_{k=1}^{N_0} T_{t-s} g^k(s, x) dw^k_s.
\]

Hence,
\[
\phi(\Delta) u(t, x) = \sum_{k=1}^{N_0} \int_{0}^{t} \phi(\Delta)^{1/2} T_{t-s} \phi(\Delta)^{1/2} g^k(s, \cdot)(x) du^k_s,
\]
and by Burkholder-Davis-Gundy’s inequality, we have
\[
E \left[ |\phi(\Delta)u(t, x)|^p \right] \leq c(p) E \left[ \left( \int_0^t \sum_{k=1}^{N_0} |\phi(\Delta)^{1/2}T_{t-s}\phi(\Delta)^{1/2}g^k(s, \cdot)(x)|^2 \, ds \right)^{p/2} \right].
\]
Also, similarly we get
\[
E \left[ |u(t, x)|^p \right] \leq c(p) E \left[ \left( \int_0^t \sum_{k=1}^{N_0} |T_{t-s}g^k(s, \cdot)(x)|^2 \, ds \right)^{p/2} \right].
\]
Now it is enough to use Theorem 1.1 and Lemma 6.1 to conclude
\[
\|u\|_{H^{p,2}_p(T)} \leq N\|g\|_{H^{p,1}_p(T, \ell_2)}.
\]
(6.15)

For general \(g\), take a sequence \(g_n\) so that \(g_n \to g\) in \(H^{p,1}_p(T, \ell_2)\) and each \(g_n\) satisfies above described conditions. Then, by the above result, for \(u_n := \int_0^t T_{t-s}g^k_n dw^k_s\), we have
\[
du_n = \phi(\Delta)u_n \, dt + g^k_n dw^k_s, \quad \text{and}
\]
\[
\|u_n\|_{H^{p,2}_p(T)} \leq N\|g_n\|_{H^{p,1}_p(T, \ell_2)},
\]
\[
\|u_n - u_m\|_{H^{p,2}_p(T)} \leq N\|g_n - g_m\|_{H^{p,1}_p(T, \ell_2)}.
\]
Thus \(u_n\) is a Cauchy sequence in \(H^{p,2}_p(T)\) and converges to a certain function \(u \in H^{p,2}_p(T)\). One easily gets (6.14) by letting \(n \to \infty\), and by Theorem 6.4 it also follows \(E \sup_{t \leq T} \|u_n - u\|_{L_p}^p \to 0\) as \(n, m \to \infty\). Finally by taking the limit from
\[
(u_n(t), \varphi) = \int_0^t \langle \phi(\Delta)u_n, \varphi \rangle \, ds + \sum_k \int_0^t \langle g^k_n, \varphi \rangle \, dw^k_s, \quad \forall t \leq T \text{ (a.s.)}
\]
and remembering \(E \sup_{t \leq T} \|u_n - u\|_{L_p}^p \to 0\), we prove that \(u\) satisfies
\[
(u(t), \varphi) = \int_0^t \langle \phi(\Delta)u, \varphi \rangle \, ds + \sum_k \int_0^t \langle g^k, \varphi \rangle \, dw^k_s, \quad \forall t \leq T \text{ (a.s.)}
\]

**Step 4.** General case. The uniqueness follows from Step 1. For the existence and the estimate it is enough to add the solutions in Steps 2 and 3. The theorem is proved.

\[\square\]

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