Three Graphs and the Erdős-Gyárfás Conjecture

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Abstract

Three graphs related to the Erdős-Gyárfás Conjecture are presented. The graphs are derived from the Buckyball, the Petersen graph, and the Tutte-Coxeter graph. The first graph is a partial answer to a question posed by Heckman and Krakovski [1] in their recent work on the planar version of the conjecture. The other two graphs appear to be the smallest known cubic graphs with no $2^m$-cycles for $m \leq 4$ and for $m \leq 5$.

1 Introduction.

The Erdős-Gyárfás Conjecture asserts that every graph with minimum degree at least 3 contains a cycle whose length is a power of 2. Recently, Heckman and Krakovski [1] proved the conjecture for 3-connected cubic planar graphs.

Theorem (Heckman and Krakovski). Every 3-connected planar graph contains a $2^m$-cycle, for some $m \leq 7$.

The authors suggest that the upper bound of 7 might be improved, and that a value as low as 4 may be possible. In this note, we show that the bound on $m$ must be at least 5 by constructing a 3-connected cubic planar graph with neither 4, 8, nor 16 cycles. Then we consider the general problem of finding, for a given integer $k$, the smallest $2^m$-cycle free cubic graphs for all $m < k$, and describe graphs for $k = 4$ and $k = 5$ that appear to be the smallest known examples.

2 A 3-connected cubic planar graph with no cycles of length 4, 8 or 16.

The construction is based on the truncated icosahedron, known to chemists as $C_{60}$, the buckminsterfullerene [2], or more popularly as the buckyball. We recall
some of the well-known graph theoretic properties of $C_{60}$. It is a 3-connected cubic planar vertex-transitive graph of order 60. It contains twelve 5-cycles and twenty 6-cycles, all of which border faces in a plane drawing of the graph, such as the one in Figure 1.

Figure 1: The graph of the buckyball.

Under the action of the automorphism group there are two edge orbits. One orbit contains 60 edges, each of which borders one pentagonal face and one hexagonal face. These edges are called *single-bond* edges by chemists, a terminology we also adopt. The other edge orbit contains the 30 edges that border two hexagonal faces. These are called *double-bond* edges. Each vertex is incident with two single-bond edges and one double-bond edge. It will also be important to note that $C_{60}$ contains no other cycle of length less than nine.

To obtain a graph with no 4, 8 or 16 cycles, we replace each vertex in $C_{60}$ by a copy of the graph $H_7$ shown in Figure 2. The three edges incident with the replaced buckyball vertex are attached to the vertices labeled $u$, $v$ and $w$ in Figure 2. The attachment is done so that vertex $u$ is incident with the double-bond edge. Since $H_7$ has three vertices of degree 2, and four vertices of degree 3, the resulting graph will evidently be a 3-connected cubic planar graph of order 420. We denote the graph by $G_{420}$. 
It will be convenient to refer to the natural projection from $V(G_{420})$ to $V(C_{60})$ in which the vertices in each copy of $H_7$ are projected onto the replaced vertex.

To see that $G_{420}$ contains no $2^m$ cycles for $m \leq 4$, observe that $H_7$ contains no $2^m$-cycles for any $m$, so any such cycle in $G_{420}$ would project to a cycle in $C_{60}$. Consider first the cycles of $G_{420}$ that project to hexagonal face cycles. Since the minimum distance in $H_7$ between any pair of the attachment vertices ($u$, $v$ and $w$) is at least 2, any such cycle must contain at least 12 edges from copies of $H_7$, along with 6 edges joining different copies, and hence have length at least 18. Next consider cycles from $G_{420}$ that project to pentagonal faces. Since the distance in $H_7$ from $v$ to $w$ is three, each such cycle contains at least 15 edges from copies of $H_7$, along with 5 edges joining the copies, and so has length at least 20. Finally, because $C_{60}$ contains no other cycles of length less than nine, one finds that there are no $2^m$ cycles in $G_{420}$ for $m \leq 4$ as claimed.

3 The general case.

Next we consider the following variant of Erdős-Gyárfás Conjecture.

Problem. For $k \geq 3$, what are the smallest cubic graphs with no $2^m$-cycles, for $m \leq k$?

Denote the order of a smallest graph by $f(k)$. It is easy to check that $f(2) = 10$. There are three cubic graphs of order 10 with no 4-cycles (including the Petersen graph) and none of smaller order. Markström [3] showed that $f(3) = 24$, and listed all four minimal graphs. In fact, the graph $H_7$ can viewed as playing a role in one of Markström’s graphs. This particular graph can be obtained from $K_4$ by replacing three of the vertices of $K_4$ by $H_7$ and replacing the fourth vertex by a copy of $K_3$.

$H_7$ can also be used to construct what appears to be the smallest known example of a graph with no $2^m$-cycles for $m \leq 4$. In this construction, one begins with the Petersen graph, drawn as in Figure [3]
Next we replace the central vertex by a copy of $K_3$ as shown in Figure 4.

The final step is to replace all but one of the vertices in $G_{12}$ with a copy of $H_7$. In Figure 5 we indicate how this can be done. The solid vertex in the figure is the vertex that is not replaced by a copy of $H_7$. The heavy boxes attached to each of the other vertices mark the edge that will be incident with vertex $u$ in that copy of $H_7$. We thus obtain $G_{78}$, a cubic graph of order 78. To see that there are no 4, 8 or 16 cycles in $G_{78}$ one can check all possible cycles that project from $G_{78}$ to the single 3-cycle, the six 5-cycles and the ten 6-cycles in...
$G_{12}$. The details are left to the reader.

The final construction gives a graph with no $2^m$-cycles, for $m \leq 5$. It is based on the girth 8 Tutte-Coxeter graph shown in Figure 6 and on the vertex replacement graph $H_{15}$ shown in Figure 7. Note that $H_{15}$ consists of two copies of $H_7$ with one extra vertex, and that there are no $2^m$-cycles in $H_{15}$. Note also that the distance in $H_{15}$ from $u$ to either $v$ or $w$ is 3, while the distance from $v$ to $w$ is 5.

If one replaces each vertex of Tutte-Coxeter by a copy of $H_{15}$, there are clearly no $2^m$-cycles for $m \leq 4$, but if the replacement is done in an arbitrary manner, 32-cycles may result. To avoid this one must be a little careful. Each vertex of Tutte-Coxeter is incident with two edges on the outer Hamiltonian cycle (in Figure 6) and with one chord edge. If we replace each Tutte-Coxeter vertex so that vertex $u$ in each copy of $H_{15}$ is incident with a chord edge, then 32-cycles will be avoided. This follows from the observation that any 8-cycle in Tutte-Coxeter contains (at least) two consecutive edges on the outer Hamiltonian cycle, and therefore at least one $v$-$w$ path in a copy of $H_{15}$. The resulting graph has order 450 and no $2^m$-cycles for $m \leq 5$, and therefore $f(5) \leq 450$.

We summarize the known values and bounds on $f$ in the table at the end of this note. The lower bound for $f(4)$ is an unpublished result of Markström.

Figure 5: Replacing 11 of the vertices by $H_7$. 
Figure 6: The Tutte-Coxeter graph.

Figure 7: A larger vertex replacement graph $H_{15}$. 
\begin{table}
\begin{tabular}{|c|c|}
\hline
\textbf{k} & \textbf{f(k)} \\
\hline
2 & 10 \\
3 & 24 \\
4 & 54 – 78 \\
5 & \leq 450 \\
\hline
\end{tabular}
\end{table}

References

[1] Christopher Carl Heckman and Roi Krakovski, Erdős-Gyárfás Conjecture for Cubic Planar Graphs, Electron. J. Combin. 20(2) (2013) P7.

[2] H. W. Kroto, J. R. Heath, S. C. O’Brien, R. F. Curl and R. E. Smalley, C60: Buckminsterfullerene, Nature 318 (1985) 162-163.

[3] Klas Markström, Extremal graphs for some problems on cycles in graphs, Cong. Numer. 171 (2004) 2179-2192.