GROTHENDIECK GROUPS OF TRIANGULATED CATEGORIES

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ABSTRACT. Let $k$ be an algebraically closed field and $\mathcal{C}$ a $k$-linear, Hom-finite triangulated category with split idempotents. In this paper, we show that under suitable circumstances, the Grothendieck group of $\mathcal{C}$, denoted $K_0(\mathcal{C})$, can be expressed as a quotient of the split Grothendieck group of a higher-cluster tilting subcategory of $\mathcal{C}$.

Assume that $n \geq 2$ is an even integer, $\mathcal{C}$ is $n$-Calabi Yau and has an $n$-cluster tilting subcategory $\mathcal{T}$. Then, for every indecomposable $M$ in $\mathcal{T}$, there is an Auslander-Reiten $(n+2)$-angle in $\mathcal{T}$ of the form $M \to \mathcal{T}_{n-1} \to \cdots \to \mathcal{T}_0 \to M$ and

$$K_0(\mathcal{C}) \cong K_0^sp(\mathcal{T})/\left\langle \sum_{i=0}^{n-1} (-1)^i [T_i] \middle| M \in \mathcal{T} \text{ indecomposable} \right\rangle.$$

Assume now that $d$ is a positive integer and $\mathcal{C}$ has a $d$-cluster tilting subcategory $\mathcal{S}$ closed under $d$-suspension. Then $\mathcal{S}$ is a so called $(d+2)$-angulated category whose Grothendieck group $K_0(\mathcal{S})$ can be defined as a certain quotient of $K_0^sp(\mathcal{S})$. We will show

$$K_0(\mathcal{C}) \cong K_0(\mathcal{S}).$$

Moreover, assume that $n = 2d$, that all the above assumptions hold, and that $\mathcal{T} \subseteq \mathcal{S}$. Then our results can be combined to express $K_0(\mathcal{S})$ as a quotient of $K_0^sp(\mathcal{T})$.

1. Introduction

Let $k$ be an algebraically closed field and $\mathcal{C}$ be a $k$-linear, Hom-finite triangulated category with split idempotents and suspension functor $\Sigma$. We denote the split Grothendieck group of an additive category $\mathcal{A}$ by $K_0^sp(\mathcal{A})$ and the Grothendieck group of an abelian or triangulated category $\mathcal{B}$ by $K_0(\mathcal{B})$.

Classic result (Palu). In [10], Palu assumes that $\mathcal{C}$ is the stable category of a Frobenius $k$-linear category with split idempotents, and that $\mathcal{C}$ is 2-Calabi-Yau with a (2-)cluster tilting subcategory $\mathcal{T} = \text{add}(T)$.

Given an indecomposable direct summand $M$ of $T$, let $\mathcal{T}$ be the additive subcategory of $\mathcal{T}$ whose indecomposables are the direct summands of $T$ apart from $M$. There is a unique indecomposable $M^* \notin \mathcal{T}$ such that $\text{add}(\mathcal{T} \cup M^*) \subseteq \mathcal{C}$ is (2-)cluster tilting. Moreover, $M$ and $M^*$ appear in two triangles with certain properties, called exchange triangles, of the form:

$$M^* \to B_M \to M \to \Sigma M^* \text{ and } M \to B_{M^*} \to M^* \to \Sigma M,$$

where $B_M$ and $B_{M^*}$ are in $\mathcal{T}$. Palu proved the following in [10, theorem 10].

Theorem (Palu). We have that $K_0(\mathcal{C}) \cong K_0^sp(\mathcal{T})/\left\langle [B_{M^*}] - [B_M] \right\rangle_{M \in \text{Ind}(\mathcal{T})}$.

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We present higher versions of Palu’s result in two different senses: a “higher-cluster tilting” one and a “higher-angulated” one.

**Higher-cluster tilting version.** Let \( n \geq 2 \) be an even integer and assume that \( \mathcal{C} \) is \( n \)-Calabi-Yau and has an \( n \)-cluster tilting subcategory \( \mathcal{T} = \text{add}(T) \). Recall that Auslander-Reiten \((n+2)\)-angles in \( \mathcal{T} \) were introduced in [5] definition 3.8.

**Theorem 4.7.** We have that

\[
K_0(\mathcal{C}) \cong K_0^{sp}(\mathcal{T}) / \left\{ \sum_{i=0}^{n-1} (-1)^i [T_i] \mid M \in \text{Ind} \mathcal{T} \text{ with Auslander-Reiten } (n+2)\text{-angle} \\ M \to T_{n-1} \to \cdots \to T_0 \to M \to \Sigma^n M \right\}.
\]

When \( n = 2 \) and the Auslander-Reiten quiver of \( \mathcal{T} \) has no loops, we have that Theorem 4.7 becomes Palu’s theorem. In this case, if \( M \) is an indecomposable direct summand of \( U = T \), its Auslander-Reiten 4-angle is obtained from the exchange triangles and it is \( M \to B_M \to B_M \to M \to \Sigma^2 M \).

**Higher-angulated version.** Let \( d \geq 1 \) be an integer and assume that \( \mathcal{C} \) has a \( d \)-cluster tilting subcategory \( \mathcal{S} = \text{add}(\mathcal{S}) \) such that \( \Sigma^d \mathcal{S} = \mathcal{S} \). Note that \( \mathcal{S} \) is a \((d+2)\)-angulated category with \( d \)-suspension \( \Sigma^d \), by [3] theorem 1. Similarly to the way \( K^0(\mathcal{C}) \) is defined, one can define the Grothendieck group of the \((d+2)\)-angulated category \( \mathcal{S} \) as

\[
K_0(\mathcal{S}) := K_0^{sp}(\mathcal{S}) / \left\{ \sum_{i=0}^{d+1} (-1)^i [S_i] \mid S_{d+1} \to \cdots \to S_0 \to \Sigma^d S_{d+1} \text{ is a } (d+2)\text{-angle in } \mathcal{S} \right\}.
\]

We prove that this is isomorphic to the Grothendieck group of \( \mathcal{C} \).

**Theorem 5.6.** We have that \( K_0(\mathcal{C}) \cong K_0(\mathcal{S}) \).

Let \( n = 2d \). We now add the assumptions that \( \mathcal{C} \) is \( n \)-Calabi-Yau and that there is an \( n \)-cluster tilting subcategory \( \mathcal{T} = \text{add}(\mathcal{T}) \subseteq \mathcal{C} \) such that \( \mathcal{T} \subseteq \mathcal{S} \). By [9] theorem 5.26, we have that \( T \in \mathcal{S} \) is an Oppermann-Thomas cluster tilting object, i.e. the corresponding concept in a \((d+2)\)-angulated category of a cluster tilting object in a triangulated category. Theorems 4.7 and 5.6 have the following immediate consequence.

**Corollary 6.4.** We have that

\[
K_0(\mathcal{S}) \cong K_0^{sp}(\mathcal{T}) / \left\{ \sum_{i=0}^{n-1} (-1)^i [T_i] \mid M \in \text{Ind} \mathcal{T} \text{ with Auslander-Reiten } (n+2)\text{-angle} \\ M \to T_{n-1} \to \cdots \to T_0 \to M \to \Sigma^n M \right\}.
\]

When \( d = 1 \), we have that \( \mathcal{S} = \mathcal{C} \) is a triangulated category with \( (2)\)-cluster tilting subcategory \( \mathcal{T} = \text{add}(T) \) and, adding the extra assumption that the Auslander-Reiten quiver of \( \mathcal{T} \) has no loops, Corollary 6.4 becomes [10] theorem 10. For higher values of \( d \), Corollary 6.4 proves a higher-angulated version of Palu’s theorem.

We conclude our paper by illustrating our results in two examples: one for each of the higher versions of Palu’s theorem. Let \( q \) and \( p \) be integers and \( q \) odd. We apply Theorem 4.7 to \( \mathcal{C}_q(A_p) \), the triangulated \( q \)-cluster category of Dynkin type \( A_p \), to show that

\[
K_0(\mathcal{C}_q(A_p)) \cong \begin{cases} 0, & \text{if } p \text{ is even,} \\ \mathbb{Z}, & \text{if } p \text{ is odd.} \end{cases}
\]
We then consider a higher Auslander $k$-algebra of Dynkin type $A$ and an Amiot cluster category of it, to find an example of categories \( \mathcal{T} \subseteq \mathcal{O}(A^3_2) \subseteq \mathcal{C}^4(A^3_2) \), such that $\mathcal{C}^4(A^3_2)$ is triangulated and 4-Calabi-Yau, $\mathcal{O}(A^3_2)$ is closed under $\Sigma^2$ and 2-cluster tilting in $\mathcal{C}^4(A^3_2)$ and $\mathcal{T}$ is 4-cluster tilting in $\mathcal{C}^4(A^3_2)$. Applying Theorem 5.6 and Corollary 6.4 to this example, we find that $K_0(\mathcal{C}^4(A^3_2)) \cong \mathbb{Z} \oplus \mathbb{Z}$.

The paper is organised as follows. Section 2 recalls some definitions and results and presents our setup. Section 3 introduces some morphisms between Grothendieck groups that will be useful in the rest of the paper. Section 4 proves Theorem 4.7. Section 5 proves Theorem 5.6. Section 6 presents Corollary 6.4. Finally, Sections 7 and 8 illustrate our two examples.

2. Setup and definitions

**Definition 2.1.** Let $\mathcal{A}$ be an essentially small additive category and $G(\mathcal{A})$ be the free abelian group on isomorphism classes $[\mathcal{A}]$ of objects $A \in \mathcal{A}$. We define the *split Grothendieck group of $\mathcal{A}$* to be

$$K_0^{sp}(\mathcal{A}) := G(\mathcal{A}) / ([A \oplus B] - [A] - [B]).$$

When $\mathcal{A}$ is abelian or triangulated, we can also define the *Grothendieck group of $\mathcal{A}$* respectively as

$$K_0(\mathcal{A}) := K_0^{sp}(\mathcal{A}) / ([A] - [B] + [C] \mid 0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0 \text{ is a short exact sequence in } \mathcal{A}) \text{ or}$$

$$K_0(\mathcal{A}) := K_0^{sp}(\mathcal{A}) / ([A] - [B] + [C] \mid A \rightarrow B \rightarrow C \rightarrow \Sigma A \text{ is a triangle in } \mathcal{A}).$$

In a similar way, one can define the Grothendieck group of a $(d + 2)$-angulated category $\mathcal{S}$ as follows.

**Definition 2.2 (2 definition 2.1).** Let $d$ be a positive integer. The *Grothendieck group of a $(d + 2)$-angulated category $\mathcal{S}$ with $d$-suspension functor $\Sigma^d$* is defined to be

$$K_0(\mathcal{S}) := K_0^{sp}(\mathcal{S}) / \left( \sum_{i=0}^{d+1} (-1)^i [S_i] \mid S_{d+1} \rightarrow \cdots \rightarrow S_0 \rightarrow \Sigma^d S_{d+1} \text{ is a }(d + 2)\text{-angle in } \mathcal{S} \right).$$

**Definition 2.3 (7 definition 1.1).** For $m \geq 2$ an integer, an *$m$-cluster tilting object of $\mathcal{C}$* is an object $U$ such that $\mathcal{U} = \text{add}(U) \subseteq \mathcal{C}$ is an *$m$-cluster tilting subcategory*, that is, a functorially finite, full subcategory satisfying

$$\mathcal{U} = \{ C \in \mathcal{C} \mid \text{Ext}^{1\ldots m-1}_C(\mathcal{U}, C) = 0 \} = \{ C \in \mathcal{C} \mid \text{Ext}^{1\ldots m-1}_C(C, \mathcal{U}) = 0 \}.$$

**Setup 2.4.** Let $m \geq 2$ be an integer and $U$ an $m$-cluster tilting object of $\mathcal{C}$ with associated $m$-cluster tilting subcategory $\mathcal{U} = \text{add}(U) \subseteq \mathcal{C}$.

**Definition 2.5 (7 definition 2.1).** There is a homological functor

$$F_d : \mathcal{C} \rightarrow \text{mod}(\text{End}(U)), \quad F_d(-) = \mathcal{C}(U, -).$$

**Definition 2.6.** If $\mathcal{A}$ and $\mathcal{B}$ are full subcategories of $\mathcal{C}$, then

$$\mathcal{A} * \mathcal{B} = \{ C \in \mathcal{C} \mid \text{there is a triangle } A \rightarrow C \rightarrow B \rightarrow \Sigma A \text{ with } A \in \mathcal{A}, B \in \mathcal{B} \}.$$
Definition 2.7 ([7 definition 3.1]). A tower of triangles in $\mathcal{C}$ is a diagram of the form

$$
\begin{array}{cccc}
C_{l-1} & \rightarrow & C_{l-2} & \rightarrow & \cdots & \rightarrow & C_2 & \rightarrow & C_1 \\
\downarrow & & \downarrow & & \vdots & & \downarrow & & \downarrow \\
C_l & \leftarrow & X_{l-2} & \leftarrow & \cdots & \leftarrow & X_2 & \leftarrow & X_1 & \leftarrow & C_0,
\end{array}
$$

where $l \geq 2$ is an integer, a wavy arrow $X \sim Y$ signifies a morphism $X \rightarrow \Sigma Y$, each oriented triangle is a triangle in $\mathcal{C}$ and each non-oriented triangle is commutative.

Definition 2.8 ([7 definition 3.3]). By [5 corollary 3.3], for $C \in \mathcal{C}$ there is a tower of triangles in $\mathcal{C}$ of the form

$$
\begin{array}{cccc}
U_{m-2} & \rightarrow & U_{m-3} & \rightarrow & \cdots & \rightarrow & U_1 & \rightarrow & U_0 \\
\downarrow & & \downarrow & & \vdots & & \downarrow & & \downarrow \\
U_{m-1} & \leftarrow & X_{m-2} & \leftarrow & \cdots & \leftarrow & X_2 & \leftarrow & X_1 & \leftarrow & C,
\end{array}
$$

where $U_i \in \mathcal{U}$ and $\mu_i$ is a $\mathcal{U}$-cover for each $i$. In particular, the $U_i$ are determined up to isomorphism. The index of $C$ with respect to $\mathcal{U}$ is the following element of the Grothendieck group $K_0^{sp}(\mathcal{U})$:

$$\text{index}_\mathcal{U}(C) = \sum_{i=0}^{m-1} (-1)^i [U_i].$$

Remark 2.9. Note that $\text{index}_\mathcal{U} : \text{Obj}(\mathcal{C}) \rightarrow K_0^{sp}(\mathcal{U})$ induces a homomorphism $K_0^{sp}(\mathcal{C}) \rightarrow K_0^{sp}(\mathcal{U})$ which we also denote by $\text{index}_\mathcal{U}$.

Definition 2.10 ([7 definition 4.1]). There is a homomorphism $\theta_\mathcal{U} : K_0(\text{mod}(\text{End}(U))) \rightarrow K_0^{sp}(\mathcal{U})$, defined by

$$\theta_\mathcal{U}([F_\mathcal{U} N]) = \text{index}_\mathcal{U}(\Sigma^{-1} N) + \text{index}_\mathcal{U}(N)$$

for $N \in \mathcal{U} \ast \Sigma \mathcal{U}$.

Remark 2.11. Note that the fact that $\theta_\mathcal{U}$ from Definition 2.10 is well-defined can be proved using the equivalence of categories $(\mathcal{U} \ast \Sigma \mathcal{U})/\Sigma \mathcal{U} \cong \text{mod}(\text{End}(U))$ from [5 proposition 6.2(3)] and [7 lemmas 4.3, 4.4], see [7 remark 4.2].

3. Morphisms between Grothendieck groups

Definition 3.1. There are surjective homomorphisms given by the quotient maps

$$\pi_\mathcal{C} : K_0^{sp}(\mathcal{C}) \rightarrow K_0(\mathcal{C}), \ \pi_\mathcal{U} : K_0^{sp}(\mathcal{U}) \rightarrow K_0^{sp}(\mathcal{U})/\text{Im} \theta_\mathcal{U},$$

and injective homomorphisms given by the inclusions

$$\iota_\mathcal{C} : \text{Ker} \pi_\mathcal{C} \rightarrow K_0^{sp}(\mathcal{C}), \ \iota_\mathcal{U} : \text{Ker} \pi_\mathcal{U} \rightarrow K_0^{sp}(\mathcal{U}) \text{ and } j_\mathcal{U} : K_0^{sp}(\mathcal{U}) \rightarrow K_0^{sp}(\mathcal{C}).$$
**Remark 3.2.** Consider the diagram:

\[
\begin{array}{c}
\text{Ker } \pi_C \quad \downarrow \text{Ker } \pi_U \\
\downarrow \text{index}_U \quad \downarrow \pi_U \\
K^\cdot_0(C) \quad \downarrow f_U \\
K_0(C) \quad \downarrow \pi_C \\
\text{Im } \theta_U \quad \downarrow \pi_U \\
\end{array}
\]

We will show that

\[\pi_U \circ \text{index}_U \circ j_U = \pi_U \quad \text{and} \quad \pi_C \circ j_U \circ \text{index}_U = \pi_C.\]

We will prove that there exists a morphism \(f_U : K_0(C) \to K^\cdot_0(U)/\text{Im } \theta_U\) such that

\[f_U \circ \pi_C = \pi_U \circ \text{index}_U.\]

Moreover, adding some assumptions on \(C\) and/or \(U\), we will prove that there exists a morphism

\[g_U : K^\cdot_0(U)/\text{Im } \theta_U \to K_0(C)\]

such that \(g_U \circ \pi_U = \pi_C \circ j_U\). In this case, \(f_U\) and \(g_U\) become inverse isomorphisms. In the next sections, we consider different sets of extra assumptions under which such a \(g_U\) exists.

**Lemma 3.3.** We have that \(\pi_U \circ \text{index}_U \circ j_U = \pi_U \quad \text{and} \quad \pi_C \circ j_U \circ \text{index}_U = \pi_C.\)

**Proof.** First note that \(\text{index}_U \circ j_U = 1_{K^\cdot_0(U)}\) and so \(\pi_U \circ \text{index}_U \circ j_U = \pi_U\).

Given any object \(C \in \mathcal{C}\), consider the tower of triangles from Definition 2.8. We have that

\[\text{index}_U(C) = \sum_{i=0}^{m-1} (-1)^i [U_i].\]

Using the relations in \(K_0(C)\) corresponding to the triangles in the tower, we have that

\[\pi_C \circ j_U \circ \text{index}_U([C]) = \pi_C \left( \sum_{i=0}^{m-1} (-1)^i [U_i] \right) = [C] = \pi_C([C]).\]

Since this is true for arbitrary \(C \in \mathcal{C}\), we conclude that \(\pi_C \circ j_U \circ \text{index}_U = \pi_C.\]

**Lemma 3.4.** There is a homomorphism \(f_U : K_0(C) \to K^\cdot_0(U)/\text{Im } \theta_U\) such that

\[f_U \circ \pi_C = \pi_U \circ \text{index}_U.\]

**Proof.** There exists a homomorphism \(f_U\) with the desired property if and only if \(\pi_U \circ \text{index}_U \circ i_C = 0\). Note that

\[\text{Ker } \pi_C = \{ [A] - [B] + [C] \mid A \to B \to C \xrightarrow{\gamma} \Sigma A \text{ is a triangle in } \mathcal{C} \}.\]
For any generator $[A] - [B] + [C]$ of $\text{Ker} \pi_C$ corresponding to a triangle $A \to B \to C \to \Sigma A$ in $\mathcal{C}$, we have that

$$\pi_U \circ \text{index}_U \circ i_C (\langle A \rangle - \langle B \rangle + \langle C \rangle) = \pi_U (\text{index}_U (A) - \text{index}_U (B) + \text{index}_U (C))$$

$$= \pi_U (\theta_U ([\text{Im} F_U (\gamma)]) = 0,$$

where the second equality is obtained by [7, theorem 4.5]. Hence $\pi_U \circ \text{index}_U \circ i_C = 0$ as desired.

Proposition 3.5. Suppose there exists a homomorphism $g_U : K_0^\text{sp} (U)/ \text{Im} \theta_U \to K_0 (\mathcal{C})$ such that $g_U \circ \pi_U = \pi_C \circ j_U$. Then $f_U$ and $g_U$ are mutually inverse and

$$K_0^\text{sp} (U)/ \text{Im} \theta_U \cong K_0 (\mathcal{C}).$$

Proof. Using Lemmas 3.3 and 3.4 and $g_U$ with the stated property, we have

$$f_U \circ g_U \circ \pi_U = f_U \circ \pi_C \circ j_U = \pi_U \circ \text{index}_U \circ j_U = \pi_U = 1_{K_0^\text{sp} (U)/ \text{Im} \theta_U \circ \pi_U},$$

$$g_U \circ f_U \circ \pi_C = g_U \circ \pi_U \circ \text{index}_U = \pi_C \circ j_U \circ \text{index}_U = \pi_C = 1_{K_0 (\mathcal{C})} \circ \pi_C.$$

Since $\pi_U$ and $\pi_C$ are surjective, and hence right cancellative, we have

$$f_U \circ g_U = 1_{K_0^\text{sp} (U)/ \text{Im} \theta_U} \text{ and } g_U \circ f_U = 1_{K_0 (\mathcal{C})}.$$

4. $n$-Calabi-Yau $\mathcal{C}$ with $n$-cluster tilting subcategory $\mathcal{T}$

Setup 4.1. Let $n \geq 2$ be an even integer and assume that $\mathcal{C}$ is $n$-Calabi-Yau. Let $T \in \mathcal{C}$ be an $n$-cluster tilting object with associated $n$-cluster tilting subcategory $\mathcal{T} = \text{add} (T) \subseteq \mathcal{C}$. Without loss of generality, assume that $T$ has no repeated direct summands.

Remark 4.2. Note that, since $\mathcal{C}$ is $n$-Calabi-Yau, it has Serre functor $S = \Sigma^n$. Then, using the same notation as in [5], the functor $S_n := S \circ \Sigma^{-n}$ is the identity functor on $\mathcal{C}$.

Let $M$ be an indecomposable direct summand of $T$ and $\overline{T} \subseteq \mathcal{T}$ be the additive subcategory with $\text{Ind} \mathcal{T} = \text{Ind} \mathcal{T} \setminus M$. Then $\mathcal{T} = \text{add} (\overline{T} \cup M)$. By [5, proposition 3.10], there is an Auslander-Reiten $(n + 2)$-angle in $\mathcal{T}$, as defined in [5, definition 3.8], given by a tower of triangles in $\mathcal{C}$ of the form:

$$\begin{align*}
&\xymatrix{ & T_{n-1} \ar[rr]_{\tau_{n-1}} & & T_{n-2} \ar[rr]_{\tau_{n-2}} & & \cdots \ar[rr] & & T_1 \ar[rr]_{\tau_1} & & T_0 \ar[rr]_{\tau_0} & & M.} \\
&\xymatrix{ & M \ar[rr]^{\xi_{n-1}} & & X_{n-1} \ar[rr]_{\xi_{n-2}} & & X_{n-2} \ar[rr]_{\xi_{n-3}} & & \cdots \ar[rr] & & X_2 \ar[rr]^{\xi_1} & & X_1 \ar[rr]^{\xi_0} & & M.
}\end{align*}$$

(2)

Note that $T_0, \ldots, T_{n-1} \in \mathcal{T}$, and that by [5, proposition 3.10] the left-most term in (2) is $S_n (M)$ and here $S_n (M) = M$ by Remark 4.2.

Lemma 4.3. Let $M \in \mathcal{T}$ be an indecomposable with Auslander-Reiten $(n + 2)$-angle as in (2). Then $F_\mathcal{T} (\xi_i) = 0$ for any $i = 1, \ldots, n - 1$.

Proof. By [5, definition 3.8], $\tau_i : T_i \to X_i$ is a $\mathcal{T}$-cover of $X_i$. Hence, for every morphism $\tau \in \mathcal{C} (T, X_i)$, there is a morphism $\tau' : T \to T_i$ such that $\tau = \tau_i \circ \tau'$. Then,

$$F_\mathcal{T} (\xi_i) (\tau) = \xi_i \circ \tau = \xi_i \circ \tau' = 0,$$
where \( \xi_i \circ \tau_i = 0 \) because two consecutive morphisms in a triangle compose to zero. Since this is true for arbitrary \( \tau \in \mathcal{C}(T, X_i) \), we conclude that \( F_T(\xi_i) = 0 \) for any \( i = 1, \ldots, n-1 \). \( \square \)

**Lemma 4.4.** Let \( M \in \mathcal{T} \) be an indecomposable and consider diagram (2). Then, as an element in \( K_0^{sp}(\mathcal{T}) \), we have

\[
[T_0] - [T_1] + \cdots + [T_{n-2}] - [T_{n-1}] = -\theta_T([S_M]),
\]

where \( S_M \) is the simple \( \text{End}(T) \)-module that is the top of \( \mathcal{C}(T, M) \), the projective \( \text{End}(T) \)-module corresponding to \( M \).

**Proof.** Consider the exact sequence induced by the right-most triangle in (2):

\[
\mathcal{C}(T, T_0) \xrightarrow{Fr(\tau_0)} \mathcal{C}(T, M) \xrightarrow{Fr(\xi_0)} \mathcal{C}(T, \Sigma X_1) \to \mathcal{C}(T, \Sigma T_0).
\]

Note that \( \mathcal{C}(T, \Sigma T_0) = 0 \) and so \( \text{Im} F_T(\xi_0) = \text{Coker} F_T(\tau_0) = \mathcal{C}(T, \Sigma X_1) \). By [5, definition 3.8], we have that \( \tau_0 : T_0 \to M \) is minimal right almost split in \( \mathcal{T} \) and so

\[ S_M := \mathcal{C}(T, \Sigma X_1) \]

is the simple \( \text{End}(T) \)-module that is the top of \( \mathcal{C}(T, M) \). Then, by [7, theorem 4.5], we have

\[ [T_0] = \text{index}_T(T_0) = [M] + \text{index}_T(X_1) - \theta_T([S_M]). \tag{3} \]

Moreover, since \( \tau_1, \ldots, \tau_{n-1} \) are \( \mathcal{T} \)-covers, letting \( T_n := M \), by Definition 2.8 we have

\[ \text{index}_T(X_1) = \sum_{i=1}^{n} (-1)^{i-1} [T_i]. \]

Since \( n \) is even, substituting this in (3), we conclude that

\[ [T_0] - [T_1] + \cdots + [T_{n-2}] - [T_{n-1}] = -\theta_T([S_M]). \]

\( \square \)

**Remark 4.5.** Note that Lemma 4.4 can be applied to any indecomposable in \( \mathcal{T} \). Moreover, since any simple \( \text{End}(T) \)-module has the form \( S_M \) for some indecomposable \( M \in \mathcal{T} \) and \( K_0(\text{mod}(\text{End}(T))) \) is generated by the equivalence classes of the simple \( \text{End}(T) \)-modules, we have

\[ \text{Im} \theta_T = \left\{ \sum_{i=0}^{n-1} (-1)^i [T_i] \mid M \in \text{Ind} \mathcal{T} \text{ with Auslander-Reiten (n+2)-angle (2)} \right\}. \]

**Lemma 4.6.** There is a morphism \( g_T : K_0^{sp}(\mathcal{T})/\text{Im} \theta_T \to K_0(\mathcal{C}) \) such that

\[ g_T \circ \pi_T = \pi_C \circ j_T. \]

**Proof.** Consider diagram (1) with \( \mathcal{U} = \mathcal{T} \). A morphism \( g_\mathcal{U} \) with the desired property exists if and only if \( \pi_C \circ j_T \circ \iota_T = 0 \). By Remark 4.5, we have

\[ \text{Ker} \pi_T = \left\{ \sum_{i=0}^{n-1} (-1)^i [T_i] \mid M \in \text{Ind} \mathcal{T} \text{ with Auslander-Reiten (n+2)-angle (2)} \right\}. \]

Then, for any generator \( \sum_{i=0}^{n-1} (-1)^i [T_i] \) of \( \text{Ker} \pi_T \) corresponding to the Auslander-Reiten (n+2)-angle (2), we have

\[ \pi_C \circ j_T \circ \iota_T \left( \sum_{i=0}^{n-1} (-1)^i [T_i] \right) = [T_0] - [T_1] + \cdots + [T_{n-2}] - [T_{n-1}] = 0, \]
where all the terms cancel because of the relations in $K_0(\mathcal{C})$ corresponding to the triangles in the tower (2) and because $n$ is even. Hence $\pi_C \circ j_T \circ \iota_T = 0$ and there exists a morphism $g_T : K_0^{sp}(T)/\text{Im} \theta_T \to K_0(\mathcal{C})$ such that $g_T \circ \pi_T = \pi_C \circ j_T$. \hfill \square

Theorem 4.7. We have that

$$K_0(\mathcal{C}) \cong K_0^{sp}(T) \left/ \left\{ \sum_{i=0}^{n-1} (-1)^i[T_i] \mid M \in \text{Ind}_T \text{ with Auslander-Reiten (n + 2)-angle} \right\} \right. \rightarrow \text{Ind}_T \rightarrow \cdots \rightarrow T_0 \rightarrow M \rightarrow \Sigma^n M,$$

Proof. By Lemma 4.6 there exists a homomorphism $g_T : K_0^{sp}(T)/\text{Im} \theta_T \to K_0(\mathcal{C})$ such that $g_T \circ \pi_T = \pi_C \circ j_T$. Then, by Proposition 3.5, we have that $K_0(\mathcal{C}) \cong K_0^{sp}(T)/\text{Im} \theta_T$ and, by Remark 4.8 this completes the proof. \hfill \square

Remark 4.8. Note that when $n = 2$ and the Auslander-Reiten quiver of $T$ has no loops, then Theorem 4.7 becomes [10, theorem 10]. In this case, if $M$ is an indecomposable direct summand of $T$, then its Auslander-Reiten 4-angle is $M \to B_M \to \overrightarrow{B_M} \to \overleftarrow{B_M} \to M \to \Sigma^2 M$, where $B_M, \overrightarrow{B_M}$ are defined as in [10, p. 1444]. However, if we do not assume that the Auslander-Reiten quiver of $T$ has no loops, some of the Auslander-Reiten 4-angles in $T$ do not come from Palu’s exchange triangles and Theorem 4.7 and [10, theorem 10] are different.

5. A $\Sigma^d$-stable, $d$-cluster tilting subcategory $S \subseteq \mathcal{C}$

Setup 5.1. Let $d \geq 1$ be an integer and $S \in \mathcal{C}$ be a $d$-cluster tilting object. Then $S = \text{add}(S) \subseteq \mathcal{C}$ is a $d$-cluster tilting subcategory. Assume also that $\Sigma^d S = S$. Then, by [3, theorem 1], we have that $S$ is a $(d + 2)$-angulated category with $d$-suspension functor $\Sigma^d$.

Remark 5.2. Consider a $(d + 2)$-angle in $S$ of the form

$$S_{d+1} \longrightarrow S_d \longrightarrow \cdots \rightarrow S_2 \rightarrow S_1 \rightarrow S_0 \rightarrow \Sigma^d S_{d+1}.$$

By [3, theorem 1], it corresponds to a tower of triangles in $\mathcal{C}$:

$$S_{d+1} \xleftarrow{\eta_{d+1}} Y_{d+1} \xleftarrow{\eta_{d+2}} Y_{d-1} \cdots \xleftarrow{\eta_3} Y_3 \xleftarrow{\eta_2} Y_2 \xleftarrow{\eta_1} Y_1 \xleftarrow{\eta_0} S_0. \quad (4)$$

A simple argument shows that the $S_i \to Y_{i-1}$ do not need to be $S$-covers for this tower to compute $\text{index}_S(Y_1)$. In other words,

$$\text{index}_S(Y_1) = \sum_{i=2}^{d+1} (-1)^i[S_i].$$

Proposition 5.3. We have that

$$\text{Im} \theta_S = \left\{ \sum_{i=0}^{d+1} (-1)^i[S_i] \mid S_{d+1} \rightarrow \cdots \rightarrow S_0 \rightarrow \Sigma^d S_{d+1} \text{ is a } (d + 2)\text{-angle in } S \right\}.$$

Proof. We prove this by proving that the two inclusions hold.

$(\subseteq)$. Given any $Y \in S \star \Sigma S$, there is a triangle in $\mathcal{C}$ of the form

$$S_0 \xleftarrow{\eta_0} Y \longrightarrow \Sigma S_1 \longrightarrow \Sigma S_0,$$
where $S_0, S_1 \in \mathcal{S}$. Letting $Y_1 := \Sigma^{-1} Y \in \mathcal{C}$, we obtain a triangle in $\mathcal{C}$ of the form
\[
\Delta : \quad Y_1 \longrightarrow S_1 \longrightarrow S_0 \longrightarrow \Sigma Y_1.
\]
Since $\mathcal{S}$ is $d$-cluster tilting in $\mathcal{C}$, by [5, corollary 3.3], we can construct a tower of triangles in $\mathcal{C}$ of the form
\[
S_d \longrightarrow S_{d-1} \longrightarrow \cdots \longrightarrow S_3 \longrightarrow S_2 \longrightarrow S_1 \longrightarrow S_0 \longrightarrow \Sigma^d S_{d+1}.
\]
where $S_2, \ldots, S_{d+1}$ are in $\mathcal{S}$. Putting this together with triangle $\Delta$, we obtain the tower of triangles (4) in $\mathcal{C}$, which corresponds to the $(d + 2)$-angle in $\mathcal{S}$:
\[
S_{d+1} \longrightarrow S_d \longrightarrow \cdots \longrightarrow S_2 \longrightarrow S_1 \longrightarrow S_0 \longrightarrow \Sigma^d S_{d+1}.
\]
By [7, theorem 4.5], we have that in $K^{sp}_0(\mathcal{S})$:
\[
[S_1] = \text{index}_\mathcal{S}(Y_1) + [S_0] - \theta_\mathcal{S}([\text{Im} F_\mathcal{S}(\eta_0)]).
\]
Moreover, since $F_\mathcal{S}(\Sigma S_1) = 0$, we have that $F_\mathcal{S}(\eta_0)$ is surjective and so
\[
[S_1] = \text{index}_\mathcal{S}(Y_1) + [S_0] - \theta_\mathcal{S}([F_\mathcal{S}(\Sigma Y_1)]).
\]
We have that $\text{index}_\mathcal{S}(Y_1) = \sum_{i=2}^{d+1} (-1)^i [S_i]$ and so
\[
\sum_{i=2}^{d+1} (-1)^i [S_i] = \theta_\mathcal{S}([F_\mathcal{S}(\Sigma Y_1)]) = \theta_\mathcal{S}([F_\mathcal{S}(Y)]).
\]
(2). Given a $(d + 2)$-angle in $\mathcal{S}$ of the form
\[
S_{d+1} \longrightarrow S_d \longrightarrow \cdots \longrightarrow S_2 \longrightarrow S_1 \longrightarrow S_0 \longrightarrow \Sigma^d S_{d+1},
\]
consider the corresponding tower (4) of triangles in $\mathcal{C}$. By Remark 5.2, we have that $\text{index}_\mathcal{S}(Y_1) = \sum_{i=2}^{d+1} (-1)^i [S_i]$. Then, using [7, theorem 4.5], we conclude that
\[
\sum_{i=0}^{d+1} (-1)^i [S_i] = \theta_\mathcal{S}([\text{Im} F_\mathcal{S}(\eta_0)]) \in \text{Im} \theta_\mathcal{S}.
\]
\[\square\]

Remark 5.4. By Proposition 5.3 and Definition 222, we have that $K_0(\mathcal{S}) = K_0^{sp}(\mathcal{S})/\text{Im} \theta_\mathcal{S}$.

Lemma 5.5. There is a morphism $g_\mathcal{S} : K_0(\mathcal{S}) = K_0^{sp}(\mathcal{S})/\text{Im} \theta_\mathcal{S} \rightarrow K_0(\mathcal{C})$ such that $g_\mathcal{S} \circ \pi_\mathcal{S} = \pi_\mathcal{C} \circ j_\mathcal{S}$.

Proof. Consider diagram (4) with $\mathcal{U} = \mathcal{S}$. Note that a morphism $g_\mathcal{S}$ with the desired property exists if and only if $\pi_\mathcal{C} \circ j_\mathcal{S} \circ \iota_\mathcal{S} = 0$. Note that, by Proposition 5.3, every element of $\ker \pi_\mathcal{S}$ has the form
\[
\sum_{i=0}^{d+1} (-1)^i [S_i],
\]
for some \((d+2)\)-angle in \(S\) of the form \(S_{d+1} \to \cdots \to S_0 \to \Sigma^d S_{d+1}\). Such a \((d+2)\)-angle corresponds to a tower of triangles in \(C\) of the form \( \Pi \). Then, we have
\[
\pi_C \circ j_S \circ \iota_S \left( \sum_{i=0}^{d+1} (-1)^i [S_i] \right) = [S_0] - ([S_0] + [Y_1]) + \cdots + (-1)^d([Y_{d-1}] + [S_{d+1}]) + (-1)^{d+1}[S_{d+1}]
\]
where we have used the relations in \(K_0(C)\) corresponding to the triangles in the tower \( \Pi \), for instance \([S_1] = [S_0] + [Y_1]\). Hence \(\pi_C \circ j_S \circ \iota_C = 0\) and there exists a morphism \(g_S : K^{sp}_0(S) / \text{Im} \theta_S \to K_0(C)\) such that \(g_S \circ \pi_S = \pi_C \circ j_S\).

\begin{proof}
By Lemma 5.5, there exists a homomorphism \(g_S : K^{sp}_0(S) / \text{Im} \theta_S \to K_0(C)\) such that \(g_S \circ \pi_S = \pi_C \circ j_S\). Then, by Proposition 3.5, we have that \(K_0(C) \cong K^{sp}_0(S) / \text{Im} \theta_S\) and, by Remark 5.4, this completes the proof.
\end{proof}

6. THE CASE WHEN \(n = 2d\) AND \(T \subseteq S \subseteq C\)

\textbf{Setup 6.1.} Let \(d \geq 1\) be an integer and \(n = 2d\). Assume that \(C\) is \(n\)-Calabi-Yau and \(T \subseteq S \subseteq C\) are such that \(T = \text{add}(T)\) is \(n\)-cluster tilting in \(C\) and \(S = \text{add}(S)\) is \(\Sigma^d\)-stable and \(d\)-cluster tilting in \(C\). Then \(S\) is a \((d+2)\)-angulated category with \(d\)-suspension \(\Sigma^d\).

\textbf{Definition 6.2} ([9 definition 5.3]). An object \(T \in S\) is an \textit{Oppermann-Thomas cluster tilting object} if:

\begin{itemize}
  \item[(a)] \(S(T, \Sigma^d T) = 0\),
  \item[(b)] for each \(S' \in S\), there is a \((d+2)\)-angle \(T_d \to \cdots \to T_0 \to S' \to \Sigma^d T_d\) in \(S\) with \(T_i \in \text{add}(T)\).
\end{itemize}

\textbf{Remark 6.3.} Note that \(T \in S\) from Setup 6.1 is an Oppermann-Thomas cluster tilting object by [9 theorem 5.25].

\textbf{Corollary 6.4.} We have that
\[
K_0(S) \cong K^{sp}_0(T) \left/ \left\{ \sum_{i=0}^{n-1} (-1)^i [T_i] \right\} \right. \begin{array}{c}
M \in \text{Ind} T \text{ with Auslander-Reiten } (n+2)\text{-angle} \\
M \to T_{n-1} \to \cdots \to T_0 \to M \to \Sigma^n M
\end{array}
\]

\textbf{Proof.} This follows by combining Theorems 4.7 and 5.6.

\textbf{Remark 6.5.} When \(d = 1\), we have that \(S = C\) is a triangulated category with cluster tilting subcategory \(T = \text{add}(T)\) and, adding the extra assumption that the Auslander-Reiten quiver of \(T\) has no loops, Corollary 6.4 becomes [10 theorem 10] by Palu. For higher values of \(d\), Corollary 6.4 is a higher-angulated version of Palu’s theorem.
7. The Grothendieck group associated to $\mathcal{C}_q(A_p)$ for $q$ odd

In this section, we compute the Grothendieck group of the triangulated $q$-cluster category of Dynkin type $A_p$ for $q$ odd. We start by introducing this category, first defined in [12], and its geometric realisation, see [8] and [1] for more details.

Let $q$ and $p$ be positive integers and consider the coordinate system on the translation quiver $\mathbb{Z}A_p$ illustrated in Figure 1.

\begin{center}
\begin{tikzpicture}
\node (0) at (0,0) {$\bullet$};
\node (1) at (2,0) {$(0,q+1)$};
\node (2) at (4,0) {$(q,3q+1)$};
\node (3) at (6,0) {$(2q,3q+1)$};
\node (4) at (8,0) {$\bullet$};
\node (5) at (2,-2) {$(0,2q+1)$};
\node (6) at (4,-2) {$(q,4q+1)$};
\node (7) at (6,-2) {$(2q,4q+1)$};
\node (8) at (8,-2) {$\bullet$};
\node (9) at (2,-4) {$(0,3q+1)$};
\node (10) at (4,-4) {$(q,4q+1)$};
\node (11) at (6,-4) {$(2q,5q+1)$};
\node (12) at (8,-4) {$\bullet$};
\node (13) at (2,-6) {$(0,4q+1)$};
\node (14) at (4,-6) {$(q,4q+1)$};
\node (15) at (6,-6) {$(2q,4q+1)$};
\node (16) at (8,-6) {$(3q,4q+1)$};

\draw[->] (0) -- (1);
\draw[->] (1) -- (2);
\draw[->] (2) -- (3);
\draw[->] (3) -- (4);
\draw[->] (4) -- (5);
\draw[->] (5) -- (6);
\draw[->] (6) -- (7);
\draw[->] (7) -- (8);
\draw[->] (8) -- (9);
\draw[->] (9) -- (10);
\draw[->] (10) -- (11);
\draw[->] (11) -- (12);
\draw[->] (12) -- (13);
\draw[->] (13) -- (14);
\draw[->] (14) -- (15);
\draw[->] (15) -- (16);
\end{tikzpicture}
\end{center}

\textbf{Figure 1.} Coordinate system on $\mathbb{Z}A_p$.

**Definition 7.1** ([8, remark 2.3]). Define the following automorphisms on $\mathbb{Z}A_p$:

\[
\Sigma : \mathbb{Z}A_p \to \mathbb{Z}A_p, \quad (i,j) \mapsto (j - 1, i + (p + 1)q + 1),
\]
\[
\tau : \mathbb{Z}A_p \to \mathbb{Z}A_p, \quad (i,j) \mapsto (i - q, j - q).
\]

and let $\tau_{q+1} = \tau \circ \Sigma^{-q}$.

Note that $(\mathbb{Z}A_p, \tau)$ is a translation quiver in the sense of [8, definition 2.2]. Hence there exists a \textit{mesh category} associated to it. The objects of this category are the vertices of $\mathbb{Z}A_p$ and the morphisms are the arrows of $\mathbb{Z}A_p$ subject to the \textit{mesh relations}. For each arrow $\alpha : x \to y$, let $\sigma(\alpha)$ be the unique arrow $\sigma(\alpha) : \tau(y) \to x$. The mesh relations are given by

\[
\sum_{\alpha : x \to y} \alpha \sigma(\alpha) = 0,
\]

for each vertex $y$ in $\mathbb{Z}A_p$.

**Remark 7.2** ([8, section 2]). The Auslander-Reiten quiver of the bounded derived category $D^b(kA_p)$ is isomorphic, as a stable translation quiver, to $\mathbb{Z}A_p$. The automorphisms $\Sigma$ and $\tau$ from Definition 7.1 are the action of the suspension and the Auslander-Reiten translation in $D^b(kA_p)$ respectively, expressed in terms of the coordinate system from Figure 1. Moreover, the mesh category $k(\mathbb{Z}A_p)$ is equivalent to $\text{Ind}D^b(kA_p)$, i.e. the full subcategory of $D^b(kA_p)$ whose objects are the indecomposable objects.
The quotient translation quiver $\mathbb{Z}A_{p}/\langle \tau_{q+1} \rangle$ is obtained by identifying the vertices and arrows of $\mathbb{Z}A_{p}$ with their $\tau_{q+1}$-shifts. It is the Auslander-Reiten quiver of
\[ C_q(A_p) := \mathcal{D}^b(kA_{p})/\tau \circ \Sigma^{-q}, \]
the triangulated $q$-cluster category of Dynkin type $A_p$. Figure 2 shows the identification on $\mathbb{Z}A_{p}$ when $q$ is odd. Note that in this case, the quiver can be drawn on a Möbius strip.

Moreover, note that $C_q(A_p)$ is a $(q+1)$-Calabi-Yau category whose Hom-spaces between indecomposables are either zero or one dimensional over $k$.

7.1. Geometric realisation. We present a geometric realisation of $\mathbb{Z}A_{p}/\langle \tau_{q+1} \rangle$. Let $N = (p+1)q + 2$ and $P$ be a regular convex $N$-gon. Label the vertices of $P$ from 0 to $N-1$ in an anti-clockwise direction. We denote the diagonal joining vertices $i$ and $j$ by $d(i,j)$.

Definition 7.3 ([8 definition 2.5]). A $q$-allowable diagonal in $P$ is a diagonal joining two non-adjacent boundary vertices which divides $P$ into two smaller polygons which can themselves be subdivided into $(q+2)$-gons by non-crossing diagonals. Note that these are the diagonals of $P$ spanning $1 + kq$ vertices, for $k$ a positive integer.

Proposition 7.4 ([8 proposition 2.9]). There is a bijection
\[
\begin{cases}
\text{$q$-allowable diagonals in } P \\
\text{indecomposable objects in } C_q(A_p)
\end{cases}
\rightarrow
\begin{cases}
\text{($=$ vertices of $\mathbb{Z}A_{p}/\langle \tau_{q+1} \rangle$)}
\end{cases}
\]
given by $d(i,j) \mapsto (i,j)$.

Remark 7.5. If $d(i,j)$ is a $q$-allowable diagonal in $P$, then $(i,j)$ might not appear in the coordinate system of $\mathbb{Z}A_{p}/\langle \tau_{q+1} \rangle$. However, there is always a vertex $(i', j')$ in this coordinate system such that $i \equiv i' (\text{mod } N)$ and $j \equiv j' (\text{mod } N)$ and we identify $(i,j)$ and $(i',j')$.

From now on, $q$-allowable diagonals in $P$ and indecomposable objects in $C_q(A_p)$ are identified. Hence it makes sense to talk about morphisms between two $q$-allowable diagonals.

Definition 7.6. A $(q+2)$-angulation of $P$ is a maximal collection of non-crossing $q$-allowable diagonals.

Proposition 7.7. Assume that $q$ is odd. We have that
\[ K_0(C_q(A_p)) \cong \begin{cases} 0, & \text{if } p \text{ is even}, \\
\mathbb{Z}, & \text{if } p \text{ is odd}. \end{cases} \]
Proof. Consider the \((q + 2)\)-angulation \(T = T_0, \ldots, T_{p-1}\), where
\[
T_0 = (0, q + 1) \text{ and } T_i = (N - i, (1 + i)q + 1 - i), \text{ for } 1 \leq i \leq p - 1,
\]
see Figure 3. Note that by \([8]\) proposition 2.14] this corresponds to the \((q + 1)\)-cluster tilting object \(T_0 \oplus \cdots \oplus T_{p-1}\). Let \(\mathcal{T} := \text{add}(T_0 \oplus \cdots \oplus T_{p-1}) \subseteq \mathcal{C}_q(A_p)\) be the corresponding \((q + 1)\)-cluster tilting subcategory.

We want to find the Auslander-Reiten \((q + 3)\)-angle starting and ending at \(T_i\) for \(0 \leq i \leq p - 1\). Consider \(i = 0\). It can be checked that there are no non-zero morphisms of the form \(T_j \to T_0\) for \(1 \leq j \leq p - 1\). Hence \(0 \to T_0\) is a right minimal almost split morphism in \(\mathcal{T}\). We also have \(\text{Hom}(\mathcal{T}, \Sigma^j T_0) = 0\) for \(-q + 2 \leq j \leq -1\) and so \(0 \to \Sigma^j T_0\) is a \(\mathcal{T}\)-cover for any \(-q + 2 \leq j \leq -1\). Moreover, we have that \(\tau_{q-1} : T_1 \to \Sigma^{-q+1} T_0\) is a \(\mathcal{T}\)-cover. A method similar to the one introduced by Pescod in \([11]\) chapter 4, can be used to describe the triangles in \(\mathcal{C}_q(A_p)\) with indecomposable end terms. Using this method, we can extend \(\tau_{q-1}\) to the triangle
\[
\Sigma T_0 \to T_1 \xrightarrow{\tau_{q-1}} \Sigma^{-q+1} T_0 \to \Sigma^2 T_0.
\]

Using \([5]\) definition 3.8, the Auslander-Reiten \((q + 3)\)-angle starting and ending at \(T_0\) is then the one corresponding to the following tower of triangles:

\[
\begin{array}{ccccccccc}
0 & \overset{\tau_{q}}{\longrightarrow} & T_1 & \overset{\tau_{q-1}}{\longrightarrow} & 0 & \overset{\tau_{q-2}}{\longrightarrow} & \cdots & \overset{\tau_1}{\longrightarrow} & 0 & \overset{\tau_0}{\longrightarrow} \\
T_0 & \leftarrow & \Sigma T_0 & \leftarrow & \Sigma^{-q+1} T_0 & \leftarrow & \Sigma^{-q+2} T_0 & \leftarrow & \Sigma^{-2} T_0 & \leftarrow \Sigma^{-1} T_0 & \leftarrow T_0.
\end{array}
\]

In a similar way, we can find the remaining Auslander-Reiten \((q + 3)\)-angles. These are the ones corresponding to the following towers of triangles:

\[
\begin{array}{ccccccccc}
0 & \overset{\tau_{q}}{\longrightarrow} & T_{i+1} & \overset{\tau_{q-1}}{\longrightarrow} & 0 & \overset{\tau_{q-2}}{\longrightarrow} & \cdots & \overset{\tau_1}{\longrightarrow} & 0 & \overset{\tau_0}{\longrightarrow} \\
T_i & \leftarrow & \Sigma T_i & \leftarrow & A_i & \leftarrow & \Sigma A_i & \leftarrow & \Sigma^{-4} A_i & \leftarrow \Sigma^{-3} A_i & \leftarrow \Sigma^{-1} T_i & \leftarrow T_i.
\end{array}
\]
where \( A_i := ((i + 2)q - i, (i + 1)q - i - 1) \), for \( 1 \leq i \leq p - 2 \), and

\[
\begin{array}{ccccccc}
0 & \xrightarrow{\tau_0} & 0 & \xrightarrow{\tau_{q-1}} & \cdots & \xrightarrow{\tau_2} & T_{p-2} & \xrightarrow{\tau_1} & T_{p-1} \\
T_{p-1} & \xleftarrow{\Sigma T_{p-1}} & \Sigma^2 T_{p-1} & \xleftarrow{\Sigma^3 T_{p-1}} & \cdots & \Sigma^{q-2} T_{p-1} & \xleftarrow{\Sigma^{q-1} T_{p-1}} & \cdots & T_{p-1} \\
\end{array}
\]

Recall that by Theorem 4.7 we have that

\[
K_0(C_q(A_p)) \cong K^{sp}_0(T) \left/ \left\{ \sum_{i=0}^{q} (-1)^i \langle T_i \rangle \right| M \in \text{Ind} T \text{ with Auslander-Reiten } (q + 3)\text{-angle} \right.
\]

\[
M \to T_q \to \cdots \to T_0 \to M \to \Sigma^{q+1} M
\]

Using the Auslander-Reiten \((q + 3)\)-angles found and the fact that \( q \) is odd, we obtain that in the quotient group on the right hand side, we have

\[
[T_1] = [T_{p-2}] = 0 \quad \text{and} \quad [T_{i-1}] = [T_{i+1}] \quad \text{for} \quad 1 \leq i \leq p - 2.
\]

This implies that

- if \( p \) is even, then \([T_i] = 0\) for all \( 0 \leq i \leq p - 1 \),
- if \( p \) is odd, then \( 0 = [T_1] = \ldots [T_{p-2}] \) and \( 0 \neq [T_0] = \ldots = [T_{p-1}] \).

Hence

\[
K_0(C_q(A_p)) \cong \begin{cases} 
0, & \text{if } p \text{ is even}, \\
\mathbb{Z}, & \text{if } p \text{ is odd}.
\end{cases}
\]

\[\square\]

8. A higher-angulated cluster category of type A

Let \( p \) and \( d \) be positive integers. We denote by \( A_p^d \) the \((d - 1)\)-st higher Auslander \( k \)-algebra of linearly oriented \( A_p \), see [9, section 3]. This is a \( d \)-representation finite algebra, in the sense that it has a \( d \)-cluster tilting module and \( \text{gldim}(A_p^d) \leq d \), see [9, definition 2.19]. Let \( \text{mod} A_p^d \) be the category of finitely generated \( A_p^d \)-modules and \( D^b(\text{mod} A_p^d) \) be its bounded derived category. We denote by \( S \) its Serre functor and by \( \Sigma \) its suspension functor.

**Definition 8.1** ([9, construction 5.13]). For \( \delta \geq d \), the \( \delta \)-Amiot cluster category of \( A_p^d \) is defined to be

\[
C^\delta(A_p^d) = \text{triangulated hull}(D^b(\text{mod} A_p^d)/(S_\delta)),
\]

where \( S_\delta := S^\delta \).

**Remark 8.2.** The category \( C^\delta(A_p^d) \) is a triangulated category containing the orbit category \( D^b(\text{mod} A_p^d)/(S_\delta) \). We do not give a formal definition of triangulated hull here. Note that, by [9, theorem 5.14], we have that if \( \delta > d \), then \( C^\delta(A_p^d) \) is Hom-finite and \( \delta \)-Calabi-Yau.

**Remark 8.3.** Let \( M \) be the unique \( d \)-cluster tilting object in \( A_p^d \). Then

\[\mathcal{U} := \text{add}\{\Sigma^i M \mid i \in \mathbb{Z}\} \subseteq D^b(\text{mod} A_p^d)\]

is a \( d \)-cluster tilting subcategory by [9, theorem 1.21].
Lemma 8.11. An argument combining [9, theorems 2.3, 2.4 and 6.4] can be used to check that the following
\[ O(A_p^d) = U/(S_{2d}) \]

Remark 8.5. Note that \( O(A_p^d) \) comes with an inclusion into \( D^b(\text{mod } A_p^d)/(S_{2d}) \subseteq C^{2d}(A_p^d) \).
Moreover, by [9, theorem 5.24], we have that \( O(A_p^d) \subseteq C^{2d}(A_p^d) \) is \( d \)-cluster tilting and \( O(A_p^d) \) is \( (d + 2) \)-angulated.

Notation 8.6. Let \( Z \) be a cyclically ordered set with \( p + 2d + 1 \) elements. We can think of \( Z \) as marked points on a circle labeled 1 to \( p + 2d + 1 \) in the anti-clockwise direction. Given three points \( u, v, w \), we write \( u < v < w \) if they appear in the order \( u, v, w \) when going through the points in the anti-clockwise direction. Moreover, given two distinct points \( u \) and \( v \), we can consider the interval of points \( [u, v] \) and in this \( "<" \) is a total order.

For a point \( v \), we denote by \( v^+ \) its successor and by \( v^- \) its predecessor in the anti-clockwise direction. We say that two points are neighbours if one is the successor of the other.

Lemma 8.7 ([9 proposition 6.10]). There is a bijection
\[ \text{Ind}\mathcal{O}(A_p^d) \rightarrow \{ X = \{ x_0, \ldots, x_d \} \subset Z \mid X \text{ contains no neighbouring points} \} \]

We will use it to identify the indecomposable objects of \( O(A_p^d) \) with the sets \( X \). For \( X = \{ x_0, \ldots, x_d \} \in \text{Ind}\mathcal{O}(A_p^d) \), we have that
\[ \Sigma^d X = S_d X = \{ x_0, \ldots, x_d \} \]

Definition 8.8. For \( X, Y \in \text{Ind}\mathcal{O}(A_p^d) \), we say that \( X \) intertwines \( Y \) if we can write \( X = \{ x_0, \ldots, x_d \} \) and \( Y = \{ y_0, \ldots, y_d \} \) such that
\[ x_0 < y_0 < x_1 < \cdots < y_{d-1} < x_d < y_d < x_0 \]

Note that in this case also \( Y \) intertwines \( X \).

Lemma 8.9 ([9 proposition 6.1]). Given \( X \) and \( Y \) in \( \text{Ind}\mathcal{O}(A_p^d) \), we have that \( \text{Ext}^{d}_{\mathcal{O}(A_p^d)}(X, Y) \neq 0 \) if and only if \( X \) intertwines \( Y \). In this case, \( \text{Ext}^{d}_{\mathcal{O}(A_p^d)}(X, Y) \) is one-dimensional over \( k \).

Lemma 8.10 ([9 proposition 6.11]). Let \( X = \{ x_0, \ldots, x_d \} \) and \( Y = \{ y_0, \ldots, y_d \} \in \text{Ind}\mathcal{O}(A_p^d) \) be such that
\[ x_0 < y_0 < x_1 < \cdots < y_{d-1} < x_d < y_d < x_0 \]
so \( X \) intertwines \( Y \). Then there is a \((d + 2)\)-angle in \( \mathcal{O}(A_p^d) \) of the form
\[ X \rightarrow E_d \rightarrow \ldots \rightarrow E_1 \rightarrow Y \rightarrow \Sigma^d X \] with \( E_i = \bigoplus_{I \subseteq \{0, \ldots, d\}, |I| = r} \{ x_i \mid i \in I \} \cup \{ y_j \mid j \notin I \} \),
where \( \{ x_i \mid i \in I \} \cup \{ y_j \mid j \notin I \} \) is interpreted as zero if it contains neighbouring points.

An argument combining [9] theorems 2.3, 2.4 and 6.4 can be used to check that the following lemma holds.

Lemma 8.11. We have that \( \mathcal{T} \subseteq \mathcal{O}(A_p^d) \) is Oppermann-Thomas cluster tilting if and only if \( \text{Ind}\mathcal{T} \) is a maximal set of non-intertwining elements in \( \mathcal{O}(A_p^d) \) of the overall maximal size. Moreover, the overall maximal size is
\[ \binom{p + d - 1}{d} \]
Remark 8.12. Note that, by [9, theorem 5.25], an object $T \in \mathcal{O}(A^d_p)$ is Oppermann-Thomas cluster tilting if and only if it is $2d$-cluster tilting when seen as an object in $C^{2d}(A^d_p)$.

Hence, if we can find $\mathcal{T} = \text{add}(T) \subseteq \mathcal{O}(A^d_p)$ Oppermann-Thomas cluster tilting, we have

$$\mathcal{T} \subseteq \mathcal{O}(A^d_p) \subseteq C^{2d}(A^d_p),$$

where $C^{2d}(A^d_p)$ is triangulated and $2d$-Calabi-Yau, $\mathcal{O}(A^d_p)$ is closed under $\Sigma^d$ and $d$-cluster tilting in $C^{2d}(A^d_p)$ and $\mathcal{T}$ is $2d$-cluster tilting in $C^{2d}(A^d_p)$. That is, we are in the situation of Setup 6.1 with $\mathcal{S} = \mathcal{O}(A^d_p)$ and $\mathcal{C} = C^{2d}(A^d_p)$. We now choose specific values for $d$ and $p$ and, using our results, we find $K_0(C^{2d}(A^d_p))$ for these values. The following result will be widely used for the computations in our example.

Proposition 8.13 ([7, theorem 5.9]). If $s_{d+1} \to \cdots \to s_0 \xrightarrow{\gamma} \Sigma^d s_{d+1}$ is a $(d + 2)$-angle in $\mathcal{S}$, then

$$\sum_{i=0}^{d+1} (-1)^i \text{index}_{\mathcal{T}}(s_i) = \theta_\mathcal{T}(\text{Im} F_{\mathcal{T}}(\gamma)).$$

Example 8.14. Let $p = 3$ and $d = 2$, so that $|Z| = p + 2d + 1 = 8$. For simplicity, we write the indecomposable $\{x_0, x_1, x_2\}$ as $x_{012}$. We have

$$\text{Ind}\mathcal{O}(A^2_3) = \{135, 136, 137, 146, 147, 157, 246, 247, 248, 247, 248, 257, 258, 268, 357, 358, 368, 468\}.$$

Moreover, the object $T = 135 \oplus 136 \oplus 137 \oplus 146 \oplus 147 \oplus 157 \in \mathcal{O}(A^2_3)$ is such that its indecomposable direct summands are a maximal set of non-intertwining elements in $\mathcal{O}(A^2_3)$ of the overall maximal size $(\frac{3+2}{2}) = \frac{5}{2} = 6$. So $\mathcal{T} = \text{add}(T) \subseteq \mathcal{O}(A^2_3)$ is Oppermann-Thomas cluster tilting.

Using some 4-angles in $\mathcal{O}(A^2_3)$ obtained as described in Lemma 8.10 and [7, lemma 5.6], we find the index of the indecomposables in $\mathcal{O}(A^2_3)$ with respect to $\mathcal{T}$, see Table 1. Brackets [·] for classes in $K^\text{str}_0(\mathcal{T})$ are omitted both in the table and in the rest of this example.

| $s \in \mathcal{O}(A^2_3)$ | index$_\mathcal{T}(s)$ |
|--------------------------|------------------------|
| 135                      | 135                    |
| 136                      | 136                    |
| 137                      | 137                    |
| 146                      | 146                    |
| 147                      | 147                    |
| 157                      | 157                    |
| 246                      | 146 + 136 + 135        |
| 247                      | 147 + 135 + 137        |
| 248                      | 135                    |
| 257                      | 157 + 137 + 136        |
| 258                      | 136                    |
| 268                      | 137                    |
| 357                      | 157 + 147 + 146        |
| 358                      | 146                    |
| 368                      | 147                    |
| 468                      | 157                    |

Table 1. The index of objects of $\mathcal{O}(A^2_3)$ with respect to $\mathcal{T}$. 

Consider the endomorphism algebra $\Gamma := \text{End}_{\mathcal{O}(A_3^2)}(T)$. The indecomposable projective $\Gamma$-modules are $P_x := \text{Hom}_{\mathcal{O}(A_3^2)}(T,x)$, for $x \in T$ indecomposable. The simple top of $P_x$ is then denoted by $S_x$. We compute $\theta_T([S])$ for every simple $\Gamma$-module $S$. In order to do this, we choose some morphisms $\gamma$ in $T$, extend them to 4-angles in $\mathcal{O}(A_3^2)$ using Lemma 8.10 and compute $\theta_T([\text{Im } F_T(\gamma)])$ using Proposition 8.13 and Table 1, see Table 2. Then, since $\theta_T$ is additive, we can compute $\theta_T$ at the simple $\Gamma$-modules using Table 2, see Table 3.

Note that $(\theta_T([S]) \mid S$ is a simple $\Gamma$-module) generates $\text{Im } \theta_T$ since $K_0(\text{mod } \Gamma)$ is generated by the classes of the simple $\Gamma$-modules. Hence, using Table 3 in $K_0^{\text{sp}}(T)/\text{Im } \theta_T$ we have that 
\[ 136 = 146 = 147 \text{ and } 137 = 135 = 157. \]

By Remark 4.5, Theorem 5.6 and Corollary 6.3, we conclude that 
\[ K_0(\mathcal{C}(A_3^2)) \cong K_0(\mathcal{O}(A_3^2)) \cong K_0^{\text{sp}}(T)/\text{Im } \theta_T \cong \mathbb{Z} \oplus \mathbb{Z}. \]

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