Exponential stabilization of a smart piezoelectric composite beam with only one boundary controller

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Abstract: Layered smart composite beams involving a piezoelectric layer are traditionally actuated by a voltage source by the extension mechanism. In this paper, we consider only the bending and shear of a cantilevered piezoelectric smart composite beam modeled by the Mead-Marcus sandwich beam assumptions. Uniform exponential stabilization with only one boundary state feedback controller, simultaneously controlling both bending moment and shear, is proved by using a spectral multiplier approach. The state feedback controller slightly differs from the classical counterparts by a non-trivial compact and nonnegative integral operator. This is due to the strong coupling of the charge equation with the stretching and bending equations. For simulations, the so-called filtered semi-discrete finite difference scheme is adopted.

Keywords: Piezoelectric smart composite, smart sandwich beam, boundary feedback stabilization, electrostatic, Mead-Marcus sandwich beam.

1. INTRODUCTION

A piezoelectric smart composite beam is a three-layer sandwich beam consisting of a stiff elastic layer, a compliant (viscoelastic) layer, and a piezoelectric layer, see Fig. 1. The piezoelectric layer is also an elastic beam with electrodes at its top and bottom surfaces and connected to an external electric circuit. As the electrodes are subjected to a voltage source, an electric field is created between the electrodes, and the piezoelectric beam shrinks or extends. Therefore, the whole composite stretches and bends (see Fig. 1).

The modeling assumptions for smart piezoelectric models can be classified in two main categories: mechanical and electro-magnetic. The mechanical assumptions can be classified in two main categories, i.e., see Trindade and Benjendou (2002): either Mead-Marcus (M-M) type Baz (1997) or Rao-Nakra (R-N) type Baz (1997); Lam et al. (1997). The M-M models only involve the transverse kinetic energy whereas the R-N models involve both longitudinal and transverse kinetic energies. Both types of models reduce to the classical counterparts, see Ozer (2016), once the piezoelectric strain is taken to be zero. The electromagnetic assumptions on the piezoelectric layer are either fully dynamic, quasi-static, or electrostatic, see Morris and Ozer (2014); Ozer (2015). The electrostatic assumption completely discards electrical and magnetic-kinetic energies due to Maxwell’s equations. It is still a standard assumption in the literature, see Smith (2005). The voltage control, actuating the piezoelectric layer, is simply blended into models through the boundary conditions.

For the passive sandwich beam models (having no piezoelectric layer), the exact controllability of the M-M and R-N models are shown for the clamped and hinged models Hansen and Ozer (2010); Ozer and Hansen (2014). The exponential stability in the existence of the passive damping term due to the shear of the middle layer is investigated for
the M-M model (Allen and Hansen (2010); Wang and Guo (2008)). The active boundary feedback stabilization of the classical R-N model is only investigated for hinged (Ozer and Hansen (2013)) and clamped-free (Wang et al. (2006)) boundary conditions. The exponential stabilizability of the cantilevered fully dynamic or electrostatic M-M and R-N models has been open problems for more than a decade. Note that cantilevered boundary conditions are more physical than clamped or hinged boundary conditions. Recently, the exponential stability of the electrostatic R-N model is shown by using feedback controllers Ozer-a (2017), two for stretching motions of outer layers, and two for the bending motion. The exponential stability with only three controllers is recently shown by using a spectral-theoretic approach Yang and Wang (2017), and by a higher order spectral multipliers approach (Ozer-a (2017); Ozer-b (2017)). The fully dynamic R-N model is shown to be not stabilizable for many choices of material parameters by using $B^*$-type feedback controllers Ozer-a (2017). The charge-actuated electrostatic counterparts are also shown to be exponentially stable in Ozer-a (2018).

To our knowledge, the exponential stabilizability for “cantilevered” fully dynamic or electrostatic M-M model have never been studied in the literature. Denoting stretching of the top and the bottom layers, bending of the composite, shear due to the middle layer, and the total induced charge accumulated at the piezoelectric layer by $v^1, v^2, w, \phi^2, p$ respectively, the equations for motion of the fully dynamic M-M model is obtained in Ozer-b (2017) by a thorough variational approach as the following:

$$\begin{cases}
m\ddot{w} + \bar{A}w_{xxxx} - \beta_2 h_2 \phi_2^2 + \gamma \beta B_3 p_{xxx} = 0, \\
C \phi^2 - \phi_2^2 + B_1 w_{xxx} + B_2 p_{xx} = 0, \\
\mu h_3 \ddot{p} - B_1 \phi_2 p_{xx} + \gamma \beta h_3 h_2 \phi_2^2 = 0,
\end{cases}$$

$$\begin{cases}
w, w_x, \phi^2, p \big|_{x=0}, \\
w_{xx}, \phi_x^2 = p_x \big|_{x=L} = 0, \\
-\bar{A}w_{xxxx} + B_1 \gamma h_2 h_3 \phi_2^2 - \gamma \beta B_3 p_{xxx} = g(t),
\end{cases}$$

(1)

where $\delta$ is the Dirac-Delta distribution at $x = L$, $\phi^2 = (L - x^2 + (L - x)^2) + \frac{H}{2} x_1^2$, $H = \frac{1}{2} (h_1 + h_2)$, and $h_1$ is the thickness of the $i$th-layer, and $\beta, \gamma, \mu > 0$ are piezoelectric constants, and $m, A, B_1, \ldots, B_3, C, C > 0$ are functions for material parameters of each layer. Moreover, $V(t)$ is the voltage controller actuating the piezo-layer, and $g(t)$ is acting the transverse shear mechanism at the tip. The lack of stabilizability of this model for certain sub-classes of solutions is studied in Ozer (2017).

Notice that if the electrostatic assumption is adopted, i.e. $\mu h_3 \ddot{p} \equiv 0$, the model (1) reduces to

$$\begin{cases}
m\ddot{w} + \bar{A}w_{xxxx} - \beta_2 h_2 \phi_2^2 + \gamma \beta B_3 p_{xxx} = 0, \\
\gamma \phi_2^2 - \phi_2^2 + B_1 w_{xxx} + B_2 p_{xx} = 0, \\
-\gamma \beta B_3 \phi_2^2 = -V(t)\delta_L, \\
-\bar{A}w_{xxxx} + B_1 \gamma h_2 h_3 \phi_2^2 - \gamma \beta B_3 p_{xxx} = g(t),
\end{cases}$$

(2)

where the coefficients are $\bar{A} = A - \frac{\gamma \beta B_3^2}{B_4}$ $0, \bar{B} = B_1 - \frac{\gamma \beta B_3}{B_4}$, and $\bar{C} = C + \frac{\gamma \beta h_3 B_3^2}{B_4}$, see Ozer (2016).

The case $g(t) \neq 0, V(t) = 0$ corresponds to the standard passive M-M model, and its stabilizability is studied in Wang et al. (2006). To our knowledge, the only stabilizability result for the electrostatic model ($V(t) \neq 0, g(t) = 0$) is provided by Baz (1997) where various PID-type feedback controllers are considered for the asymptotic stability of the system. These results do not imply the exponential stability whatsoever. In fact, a shear-type of passive damping is also included in their models as the following:

$$\begin{cases}
m\ddot{w} + \bar{A}w_{xxxx} - \beta_2 h_2 h_3 \bar{B} \phi_2^2 = -\frac{\gamma \beta B_3}{B_4} V(t)\delta_L, \\
\gamma \phi_2^2 - \phi_2^2 + \bar{B} w_{xxx} = -\frac{B_2}{\beta B_4} V(t)\delta_L
\end{cases}$$

(3)

where $\kappa > 0$ is the damping coefficient. It is proven in Wang and Guo (2008) that the damping term itself exponentially dissipates the energy of (3), even without the boundary feedback damping: $V(t) \equiv 0$. Hence, it is not clear whether $V(t)$ can be designed to exponentially dissipate the energy by itself.

In this paper, first we show that the the model (2) is well-posed on an appropriate Hilbert space. Next, we prove that the overdetermined problem, with an extra measurement, has only the trivial solution by using spectral multipliers to ensure the strong stability. Without considering the shear-type of passive damping, i.e. $\kappa = 0$ in (3), the exponential stability of the electrostatic M-M model is guaranteed by using only the $B^*$-type state feedback controller for $V(t)$. The proof combines the a spectral multiplier method and a frequency domain approach as in Liu and Liu (2002). Finally, the so-called filtered semi-discrete Finite Differences is proposed first time to design the approximating stabilizer for a strongly coupled system.

2. WELL-POSEDNESS

Define the operator $(\zeta \bar{C} I - D_2^3)$ on the domain $H^2_0(0, L) := \{ \psi \in H^2(0, L) : \psi_x(0) = \psi_x(L) = 0 \}$. Therefore, the operator $P_C := (\zeta \bar{C} I - D_2^3)$ is defined by

$$P_C f(x) = \int_0^L g(x, z) f(z) dz, \quad \text{with} \quad g(x, z) = \begin{cases}
\cosh [\sqrt{\bar{C}}(z - L)] \sinh [\sqrt{\bar{C}}z], & x \leq z, \\
\sqrt{\bar{C}} \cosh [\sqrt{\bar{C}}L], & x \geq z.
\end{cases}$$

(4)

It is well-known that $P_C$ is a compact and non-negative operator on $L^2(0, L)$. We have the following result:

**Lemma 1.** Let $\text{Dom}(D_2^3) := H^2_0(0, L)$. Define the operator $J_C := \zeta \bar{C} P_C - I$. Then, $J_C$ is continuous, self-adjoint and non-positive on $L^2(0, L)$. Moreover, for all $w \in \text{Dom}(P_C)$, $J_C w = P_C D_2^3 = (\zeta \bar{C} I - D_2^3)^{-1} D_2^3 w$.

**Proof:** Continuity and self-adjointness easily follow from the definition of $J_C$. We first prove that $J_C$ is a non-positive operator. Let $u \in L^2(0, L)$. Then $P_C u = (\zeta \bar{C} I - D_2^3)^{-1} u$ implies that $s \in \text{Dom}(D_2^3)$ and $\zeta \bar{C} s - s_{xxx} = u$.
\[ (J_t, u)_{L^2(0, L)} = \langle (\tilde{C} P_t - I) u, u \rangle_{L^2(0, L)} = -\gamma \tilde{C} \|s_x\|^2_{L^2(0, L)} - ||s_x\|^2_{L^2(0, L)}. \]

Let \( J_t u = (\tilde{C} P_t - I) u \) and \( v := P_c w \). Then \( \tilde{C} v - v_{xx} = w \).

By a simple rearrangement of the terms \( J_t v = (\tilde{C} v - w) = (\tilde{C} v - \tilde{C} v + v_{xx}) = v_{xx} = P_c D^2_x w \).

By Lemma 1, (2) can be simplified to
\[
\begin{align*}
\begin{cases}
m w + \tilde{A} w_{xxx} + \gamma \beta \chi h_3 B^2 J_t w_x \tilde{w} \quad & (5) \\
\gamma \beta h_3 B^2 B P(\tilde{P} \delta L) + B_3 (\delta L) w_t \quad & \text{since our beam is nonclassical, we discard the mechanical controller:} \quad g(t) = 0. \end{cases}
\end{align*}
\]

This motivates the definition of the inner product on \( \mathcal{H} \):
\[
(\langle u_1, v_1 \rangle_{\mathcal{H}}) = (u_2, v_2)_H + (u_1, v_1)_V = \int_0^L m w_2 v_2 + A (w_{xx}(v_{xx})) - \gamma \beta \chi h_3 B^2 J_t (w_x \tilde{w}) \quad dx.
\]

Define the operator \( A \) on \( \mathcal{H} \) by
\[
A = \left[ \begin{array}{c}
-1/2 \tilde{A} D^4_x + \gamma \beta \chi h_3 B^2 D_x J_t D_x \end{array} \right] \quad \text{with}
\]
\[
\text{Dom}(A) = \{ (z_1, z_2) \in \mathcal{H}, z_1, z_2 \in H^2(0, L), (z_1)_{xxx}(z_1)_x \in H^3(0, L), (z_1)_{xxx}(0) = 0 \}.
\]

Define also the control operator \( \mathcal{B} \in \mathcal{L}(C, \text{Dom}(A)^{\prime}) \) by
\[
\mathcal{B} = \left[ \begin{array}{c}
-1/2 \tilde{A} D^4_x + \gamma \beta \chi h_3 B^2 D_x J_t D_x \end{array} \right] \quad \text{with}
\]
\[
\text{Dom}(A) = \{ (z_1, z_2) \in \mathcal{H}, z_1, z_2 \in H^2(0, L), (z_1)_{xxx}(z_1)_x \in H^3(0, L), (z_1)_{xxx}(0) = 0 \}.
\]

Define the state feedback controller (10):
\[
V(t) = -k_1 \left[ (L_2 h_3 \tilde{B} B P_x + B_3 I) \tilde{w}_x (L) \right] \quad \text{where} \quad h_3 B^2 \tilde{B} B P_x + B_3 I \text{is a non-negative operator.}
\]

Now consider the system (8) with the state feedback controller (10):
\[
\begin{align*}
\dot{\phi} &= A \Phi + \mathcal{B} V(t), \quad \Phi(x, 0) = \Phi^0. \quad (11)
\end{align*}
\]

Theorem 4. The operator \( \tilde{A} \) defined by (11) is dissipative in \( \mathcal{H} \). Moreover, \( \tilde{A}^{-1} \tilde{A}^{\prime} \) exists and is compact on \( \mathcal{H} \). Therefore, \( \tilde{A} \) generates a \( C_0 \)-semigroup of contractions on \( \mathcal{H} \) and the spectrum \( \sigma(\tilde{A}) \) consists of isolated eigenvalues only.

Proof Let \( Y \in \text{Dom}(\tilde{A}) \). Then
\[
\begin{align*}
\langle AY, Y \rangle &= \langle (\tilde{A} (\gamma_1)_{xxx} - \gamma \beta \chi h_3 B^2 D_x J_t (\gamma_1)) \tilde{y}_2 (\gamma_2)_{xx} \rangle_{\mathcal{H}}^L \quad \text{with} \quad \gamma_1^{(L)} = 0 \\
&= \langle A (\gamma_1)_{xx} (\gamma_2)_{xx} \rangle_{\mathcal{H}}^L + \int_0^L \left[ (\tilde{A} (\gamma_1)_{xx} (\gamma_2)_{xx} + \gamma \beta \chi h_3 B^2 J_t (\gamma_1)_{xx} (\gamma_2)_{xx}) \right] dx.
\end{align*}
\]

Therefore, \( \tilde{A} \) is dissipative. If \( \tilde{A}^{-1} \tilde{A}^{\prime} \) exists, \( A \) must be densely defined in \( \mathcal{H} \). Therefore, \( \tilde{A} \) generates a \( C_0 \)-semigroup of contractions on \( \mathcal{H} \). Next, we show that \( 0 \in \sigma(\tilde{A}) \), i.e. \( 0 \) is not an eigenvalue. We solve the following problem:
\[
\begin{align*}
\begin{cases}
\dot{w}_{xxx} + \gamma \beta \chi h_3 B^2 (J_t w_x)_x = 0, \quad w(0) = w_x(0) = w_{xx}(L) = 0; \quad (13)
\end{cases}
\end{align*}
\]

Let \( J_t w_x := u \). By the definition of \( J_t = (\tilde{C} I - D^2_x)^{-1} D^2_x \), (13) is re-written as
\[
\begin{align*}
(0) \quad \tilde{C} P_t - I) u, u \rangle_{L^2(0, T)} = -\gamma \tilde{C} \|s_x\|^2_{L^2(0, T)} - ||s_x\|^2_{L^2(0, T)}. \]

For \( k_1 > 0 \), we choose the following \( B^* \)-type feedback controller
\[
V(t) = \frac{k_1 m B^*}{\gamma} \Phi \quad \text{with} \quad \gamma \beta h_3 B^2 B P_x (\tilde{P} \delta L) (L) + B_3 \tilde{w}_x (L). \quad (10)
\]

The energy of the system is dissipative and it satisfies
\[
\frac{dE(t)}{dt} = \gamma V(t) \left[ h_2 h_3 \tilde{B} (B P_x \tilde{w}_x (L) + B_3 \tilde{w}_x (L)) \right] \quad \text{where} \quad h_2 h_3 \tilde{B} B P_x + B_3 I \text{is a non-negative operator.}
\]

Observe that \( P \tilde{w}_x (L) \) is a PID-type feedback, and it is the total piezoelectric effect due to the coupling of the charge equation to shear and bending at the same time. By Lemma 1, it can also be considered as \( P \tilde{w}_x (L) = \frac{1}{\gamma (\phi^2 + B h_3 \phi)} \). This type of representation is helpful to design the controller numerically in Section 3. Therefore (10) reduces to
\[
V(t) = -k_1 \left[ \left( \frac{h_2 h_3 \tilde{B} B P_x + B_3 I \tilde{w}_x (L) }{C} \right) \right] \quad \text{where} \quad h_2 h_3 \tilde{B} B P_x \text{ and } B_3 I \text{ is a non-negative operator.}
\]

\[ \Phi^0(x, 0) = \Phi^0 \quad (8) \]

Since the piezoelectric smart beam model is similar to the classical counterpart with the electrostatic assumption, the following results are immediate from (Ozer-b (2017)):

**Theorem 2.** For fixed initial data and no applied forces, the solution \( (w, p) \in \mathcal{H} \) of (1) converges to the solution of \( (w, p) \in \mathcal{H} \) in (5) as \( \mu \to 0 \).

**Theorem 3.** Let \( T > 0 \), and \( V(t) \in L^2(0, T) \). For any \( \Phi^0 \in \mathcal{H}, \Phi \in C[[0, T]; \mathcal{H}] \) and there exists a positive constant \( c_1 \) such that (8) satisfies
\[
\|\Phi(T)\|_{\mathcal{H}}^2 \leq c_1(T) \left\{ \|\Phi^0\|_{\mathcal{H}}^2 + \|V\|_{L^2(0, T)}^2 \right\}. \quad (9)
\]

**3. UNIFORM STABILIZATION**
\[
\begin{aligned}
&\frac{\partial w_{xxx}}{\partial x} - \beta g_h h_{3x} \frac{\partial u_x}{\partial x} = 0, \\
&\frac{\partial w_{x}}{\partial x} - \beta g_h h_{3x} \frac{\partial u_x}{\partial x} = 0, \\
&w(0) = w_x(0) = b(0) = w_{xx}(L) = u_x(L) = 0, \\
&\frac{\partial w_{xxx}}{\partial x} - \beta g_h h_{3x} \frac{\partial u_x}{\partial x} = 0.
\end{aligned}
\]

By using the last boundary condition, we integrate the first equation and plug it in the \(u\)-equation to get
\[
\xi + \frac{2\beta g_h h_{3x}}{A} u - u_{xx} = 0.
\]
Since \(\xi + \frac{2\beta g_h h_{3x}}{A} > 0\), by the boundary conditions for \(u\), we obtain that \(u \equiv 0\). This implies that \(w_{xx} = 0\). By the boundary conditions \(w \equiv 0\). Thus, \(w \equiv 0\). Thus, \(0 \in \sigma(A)\), and \(A^{1+L,1} \) is compact on \(\mathcal{H}\). Hence the spectrum \(\sigma(A)\) consists of isolated eigenvalues only. □

**Theorem 5.** The solutions \(\Phi(t)\) for \(t \in \mathbb{R}^+\) of the closed-loop system (11) is strongly stable in \(\mathcal{H}\).

**Proof:** If we can show that there are no eigenvalues on the imaginary axis, or in other words, the set
\[
\{ z \in \mathcal{H} : \text{Re} \langle \hat{A} z, z \rangle_{\mathcal{H}} = 0 \}
\]
has only the trivial solution, i.e. \(z = 0\); then by La Salle’s invariance principle, the system is strongly stable. In fact,
\[
\text{Re} \langle \hat{A} z, z \rangle_{\mathcal{H}} = \langle h_2 h_{3x} B B_2 P + B_3 I \rangle (z_2)_{\mathcal{H}} = 0.
\]
For letting \(u = (z_2)_{\mathcal{H}}, \ (z_2)_{\mathcal{H}} = \mathcal{C}_u - u_{xx}, \mathcal{C}_u(L) = u_{xx}(L) \) by the definition of \(P_c\) in (4), \(u(L) = u_{xx}(L)\). Thus, \((z_2)_{\mathcal{H}} = (P_c(z_2))_{\mathcal{H}} \equiv 0\) by (14).

Proving the strong stability of (11) reduces to showing that the following eigenvalue problem \(\hat{A} z = \lambda z\) has only the trivial solution. By using the definition of (5), i.e. \((J_c w_x) = (\mathcal{C}_c P_c w_x) - w_x\), we obtain that \((J_c w_x)(L) = 0\) and both terms \((P_c w_x)(L)\) and \(w_x(L)\) are zero by (14).

Let \(\lambda = i\omega\) where \(\omega \in \mathbb{R}\). Then (15) reduces to
\[
\begin{aligned}
\tilde{A} w_{xxx} + \gamma \beta h h_{3x} B^2 (J_c w_x) \omega - \omega^2 w = 0, \\
w(0) = w_x(0) = w_{xx}(L) = u_x(L) = 0, \\
\frac{\partial w_{xxx}}{\partial x} - \gamma \beta h h_{3x} B^2 (J_c w_x)(L) = \tilde{P}_c w_x(L) = 0.
\end{aligned}
\]

Note that the following integrals hold true.
\[
\begin{aligned}
\int_0^L \int_0^L x w_{xxx} \bar{w}_{xxx} dx dz &= -\frac{1}{2} \int_0^L |w_{xxx}|^2 dx, \\
\int_0^L \int_0^L x w \bar{w}_{xxx} dx dz &= \frac{3}{2} \int_0^L |w_{xxx}|^2 dx, \\
\int_0^L \int_0^L x (J_c w_x \bar{w}_{xxx}) dx dz &= \int_0^L x (\tilde{C}_c P_c - L) w_x \bar{w}_{xxx} dx dz \\
&= \frac{1}{2} \int_0^L |w_{xxx}|^2 dx.
\end{aligned}
\]

Let \(z = \tilde{P}_c w_x\). Then \(\bar{C} z - z_{xx} = w_x\), and therefore
\[
\int_0^L \tilde{C}_c (\tilde{P}_c w_x) x \bar{w}_{xxx} dx dz = \int_0^L \tilde{C}_c (\tilde{C}_c z - z_{xx}) x \bar{w}_{xxx} dx dz \\
= -\frac{1}{2} \int_0^L ((\tilde{C}_c)^2 |z_x|^2 + \bar{C}_c |z_{xx}|^2) dx.
\]

Multiplying the equation (15) by \(x \bar{w}_{xxx}\) and integrate by parts and using the boundary conditions yields
\[
\int_0^L \left[ \frac{\tilde{A}}{m} |w_{xxx}|^2 + 2x \tilde{A} |w_x|^2 + (\tilde{C}_c)^2 |z_x|^2 + \bar{C}_c |z_{xx}|^2 \right] dx = 0.
\]

By using the overdetermined boundary conditions (15) we obtain \(w \equiv 0\). □

We state the following stability theorem:

**Theorem 6.** Then the solutions \(\Phi(t)\) for \(t \in \mathbb{R}^+\) of the closed-loop system (11) is exponentially stable in \(\mathcal{H}\).

**Proof:** We prove the result by contradiction. Suppose that there exists a sequence of real numbers \(\beta_n \to \infty\) and a sequence of vectors \(z_n = (w_n, v_n) \in \text{Dom}(A)\) with \(\|z_n\|_{\mathcal{H}} = 1\) such that \(\|i \beta_n I - A\|_{\mathcal{H}} \to 0\), as \(n \to \infty\), i.e.
\[
\begin{aligned}
&i \xi_n w_n - v_n = f_n \to 0 \quad \text{in} \quad H^2_0(0, L) \\
i \xi_n v_n + \frac{\tilde{A}}{m} (w_n)_{xxx} + \bar{C}_c h_{3x} B^2 (J_c(w_n)_x) \to 0 \quad \text{in} \quad L^2(0, L).
\end{aligned}
\]

By using the dissipation relationship (12), we have
\[
\begin{aligned}
&i \xi_n \|w_n\|_{H^2_0(0, L)}^2 - \langle w_n, v_n \rangle_{H^2_0(0, L)} = \langle f_n, w_n \rangle_{H^2_0(0, L)}, \\
i \xi_n \|v_n\|_{L^2(0, L)}^2 + \langle v_n, w_n \rangle_{H^2_0(0, L)} > H^2_0(0, L), \\
&\|v_n\|_{L^2(0, L)}^2 - \langle v_n, w_n \rangle_{H^2_0(0, L)} - d_n = o(1), \quad \text{where}
\end{aligned}
\]

This implies that
\[
\|w_n\|_{H^2_0(0, L)}^2 - \|v_n\|_{L^2(0, L)}^2 = o(1).
\]

Since \(\|v_n\|_{L^2} = 1\), and (19), (22), we obtain \(\|w_n\|_{H^2_0(0, L)}^2 = \|v_n\|_{L^2(0, L)}^2 = 1/2\). We need the following lemma to get a contradiction. The proof is provided in Ozer-

**Lemma 7.** Let \(w_n \in \text{Dom}(A)\). Then, we have the following
\[
\begin{aligned}
&\lim_{n \to \infty} \hat{A}(w_n)_{xxx}(L) + \gamma \beta h h_{3x} B^2 D_J(w_n)_x = 0, \\
&\lim_{n \to \infty} \xi_n w_n(L) = \lim_{n \to \infty} (w_n)_{xxx}(L) = 0. □
\end{aligned}
\]

Next, we simplify (19) to get
\[
\begin{aligned}
\beta_n^2 w_n + \frac{\tilde{A}}{m} (w_n)_{xxx} + \frac{\gamma \beta h h_{3x}}{m} B^2 (J_c(w_n)_x) &= i \beta_n f_n + g_n.
\end{aligned}
\]

Let \(q(x) = e^x\). By taking the inner product of (24) by \(q(w_n)_x\) in \(L^2(0, L)\) to get
\[
\langle i \beta_n f_n + g_n, q(w_n)_x \rangle = \int_0^L \left( -\beta_n^2 w_n + \frac{\tilde{A}}{m} (w_n)_{xxx} + \frac{\gamma \beta h h_{3x}}{m} B^2 (J_c(w_n)_x) \right) \cdot q(w_n)_x dx \\
\ \beta_n f_n + g_n, q(w_n)_x \rangle \to 0
\]

since there exists constants \(D_1, D_2 > 0\).
where we used Lemma 7. By integration by parts,
\begin{equation}
\text{Re} \left( -m \beta_n^2 w_n, q(w_n) \right)_\mathcal{H}_1 = -\frac{m e}{2} |\beta_n w_n(1)|^2 + \frac{m}{2} \int_0^L e^{|\beta_n w_n|^2} dx,
\end{equation}
\begin{equation}
\text{Re} \left( \bar{A}(w_n)_{xxx} + \gamma \beta \phi h_3 B^2 (J_n(x) w_n)_{x}, q(w_n) \right) - \text{Re} \left( \bar{A}(w_n)_{xxx} + \gamma \beta \phi h_3 B^2 (J_n(x) w_n)_{x} \right) - \int_0^L \left( \frac{3A}{2} |(w_n)_{xx}|^2 + \text{Re} \left( \bar{A} e^x (w_n)_{xx} \right) + \beta h_3 B^2 e^x (J_n(x) w_n) \right) dx.
\end{equation}

The boundary terms converge to zero due to Lemma 7, and since \( |w_n(x)| < \infty \). Therefore \( |w_n(x)|_{L^2(0,L)} = o(1) \), and
\begin{equation}
\|w_n\|_{L^2(0,L)} \leq \sqrt{\|w_n(0)\|_{L^2(0,L)}} \|w_n\|_{L^2(0,L)} = o(1),
\end{equation}
\begin{equation}
\int_0^L \bar{A} e^x (w_n)_{xx}(x) dx = o(1)
\end{equation}
Using (27) and (28) in (25) we get \( \|w_n\|_{L^2(0,L)} = o(1) \) contradicting with \( |z_n| = 1 \).

4. STABLE APPROXIMATIONS & SIMULATIONS

The aim of this section is to present a sample numerical experiment in order to show that the stabilizing boundary controller (10) can be designed numerically. Since our model (2) is strongly coupled, it requires a more careful treatment for the high frequency modes which may cause spill-overs. The widely-used approximations, i.e. the standard Galerkin-based Finite Element or Finite Difference, approximated solutions. This is achieved by adding extra standard Galerkin-based Finite Element or Finite Difference, spill-overs. The widely-used approximations, i.e. the standard Galerkin-based Finite Element or Finite Difference, fail to provide reliable results for boundary control problems Banks et al (1991). The filtering technique for Finite Differences has been recently developed to avoid artificial high-frequency solutions causing instabilities in the approximated solutions. This is achieved by adding extra distributed damping terms to the equations or boundary conditions, as in Leon and Zuazua (2002); Bugariu et al. (2016); Tebou and Zuazua (2007).

We consider a three-layer smart beam with length \( L = 1 \text{m} \), and thicknesses of each layer \( h_1, h_2 = 0.1 \text{m}, \) \( h_2 = 0.01 \text{m} \). The material constants are chosen \( \rho_1, \rho_3 = 7600 \text{ kg/m}^3 \), \( \rho_2 = 5000 \text{ kg/m}^3 \), \( \alpha_1, \alpha_3 = 1.4 \times 10^3 \text{ N/m}^2 \), \( \alpha_2 = 10^5 \text{ N/m}^2 \), \( \gamma = 10^{-3} \text{ C/m}^2 \). \( \beta = 10^6 \text{ m/F} \), \( G_2 = 100 \text{ GN/m}^2 \). We consider the simulation for \( T < 5 \), and initial data \( w(x,0) = w(x,0) = 10^{-4} \sum_{i=2}^4 e^{\left( \frac{i^2-1}{2} \right)^2} \). We also non-dimensionalize the time variable \( t = A_1 t^* \) with \( t^* \to t \) where \( A_1 = L \sqrt{\frac{E}{\rho A}} \to 0.82 \). Now consider the discretization of the interval \( [0,L] \) with the fictitious points \( x_{-1} \) and \( x_{N+1} = N = 60 \):
\begin{equation}
x_{-1} < 0 < x_0 < x_1 < x_2, \ldots < x_N = L < x_{N+1}, \quad x_i = i \cdot dx, \quad i = -1, 0, 1, \ldots, N, N + 1, \quad dx = \frac{L}{N + 1}.
\end{equation}

Henceforth, to simplify the notation, we use \( z(x_i) = z_i \).

We adopt the semi-discrete scheme in Finite Differences to simulate the effects of the stabilizing controller. The following are the second order finite difference approximations for different order derivatives:
\begin{align*}
z_x &= \frac{z_{i+1} - z_{i-1}}{2dx} \quad \text{or} \quad z_x = \frac{3z_i - 4z_{i-1} + z_{i-2}}{2dx}, \\
z_{xx} &= \frac{z_{i+2} - 2z_{i+1} + 2z_{i-1} - z_{i-2}}{2dx}, \\
z_{xxx} &= \frac{z_{i+2} - 4z_{i+1} + 6z_i - 4z_{i-1} + z_{i-2}}{2dx},
\end{align*}
The numerical viscosity terms \( \tilde{\omega}_{xx} \) and \( \tilde{\omega}_{xx}^2 \) are added to the \( w \) and \( \phi^2 \)-equations in (3), respectively.

The discretization of (3) is
\begin{align*}
\ddot{w} + \frac{w_{i+2} - 4w_{i+1} + 6w_i - 4w_{i-1} + w_{i-2}}{dx^2} - \frac{\gamma \phi h_3 B^2 (J_n(x) w_n)_{x}}{L^2 \frac{dx^2}{2} + \frac{dx^2}{2}} - \frac{\gamma \phi h_3 B^2 (J_n(x) w_n)_{x}}{L^2 \frac{dx^2}{2} + \frac{dx^2}{2}} + \frac{\gamma \phi h_3 B^2 (J_n(x) w_n)_{x}}{L^2 \frac{dx^2}{2} + \frac{dx^2}{2}} - \frac{\gamma \phi h_3 B^2 (J_n(x) w_n)_{x}}{L^2 \frac{dx^2}{2} + \frac{dx^2}{2}} = k_1 V(t), \quad i = 1, \ldots, N - 1, \quad \phi_0^2 = \phi_1^2 = \phi_{N-1}^2 = 0, \quad \phi_{N-1}^2 = 0, \quad \phi_{N-1}^2 = 0, \quad \phi_{N-1}^2 = 0.
\end{align*}
The Voltage controller \( V(t) \) is designed as the following (with the choice of \( k_1 = 10^5 \))
\begin{equation}
V(t) = \frac{\tilde{C} + \tilde{B} 3w_N - 4w_{N-1} + w_{N-2}}{2dx} = \frac{1}{\tilde{C} \phi_0^2}.
\end{equation}

Fig. 2. Rapid decay of the bending \( w(x,t) \) in a few seconds (real time) after the controller applies.

Fig. 3. Rapid decay of the shear \( \phi^2(x,t) \) in a few seconds (real time) after the controller applies.

The simulations in Figures 2 and 3 show that the \( \phi^2 \) and \( w \) solutions both decay to zero fast enough. In fact,
\( \phi^2 \) solution destabilizes in the beginning (the picking phenomenon in Fig. 4) but then it decays to zero faster than the bending solution. These results can be tuned up by using an improved scheme after a careful stability analysis is performed.

Note the necessity of the controller \( P_1 \tilde{w}_x(L) \) in (10) to prove the strong stability result in Theorem 5. It is an open problem to analytically prove the same result without \( P_2 \tilde{w}_x(L) \). In fact, further numerical investigation is the subject of Özer-c (2018) where the impact of the non-classical feedback controller \( V_1(t) = -k_2(P_2 \tilde{w}_x)(L) \) over the classical one \( V_2(t) = -k_1 \tilde{w}_x(L) \) is shown to be crucial (different feedback gains for each).

Models incorporating the nonlinear elasticity theory are also derived by a consistent variational approach, and the filtering technique is applied in Ozer-b (2018). The reader should refer to promising numerical results in Özer-b (2018) with the choice of various nonlinear stabilizing feedback controllers. The results of this paper will be a subject of Ozer-c (2018) where the impact of the nonlinear stabilizing feedback controllers. The results of this paper will be a basis for functional and numerical analyses for the nonlinear beam models in Özer and Khenner (2018). Developing stable Finite Difference schemes and the adoption of the mixed-Finite Element method for both linear and nonlinear models are the progressing works Özer-c (2018).

Fig. 4. Voltage \( V(t) \) and normalized energy \( E(t) \) distributions for the first few seconds.

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