A few Ricci-flat stacks as phases of exotic GLSM’s

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In this letter we follow up recent work of Halverson-Kumar-Morrisson on some exotic examples of gauged linear sigma models (GLSM’s). Specifically, they describe a set of $U(1) \times \mathbb{Z}_2$ GLSM’s with superpotentials that are quadratic in $p$ fields rather than linear as is typically the case. These theories RG flow to sigma models on branched double covers, where the double cover is realized via a $\mathbb{Z}_2$ gerbe. For that gerbe structure, and hence the double cover, the $\mathbb{Z}_2$ factor in the gauge group is essential. In this letter we propose an analogous geometric understanding of phases without that $\mathbb{Z}_2$, in terms of Ricci-flat (but not Calabi-Yau) stacks which look like Fano manifolds with hypersurfaces of $\mathbb{Z}_2$ orbifolds.

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1 Introduction

Over the last few years we have seen a number of advances in our understanding of gauged linear sigma models (GLSM’s), ranging from non-Kahler compactifications (see e.g. [1, 2, 3, 4] and references therein) to new developments in Calabi-Yau compactifications, including realizations of non-complete-intersection target spaces [5, 6, 7, 8], non-birational phases [5, 6, 9, 10], nonperturbative realizations of geometry [5, 6, 9, 11], realizations of noncommutative resolutions [6, 9], and localization techniques [12, 13]. Those localization techniques have been applied to deduce new methods for computing Gromov-Witten invariants [11, 14], as well as many other results, see e.g. [15, 16, 17, 18, 19, 20, 21].

This paper will focus on a class of GLSM’s described in [11]. These are abelian GLSM’s, with a superpotential that is quadratic in “p” fields. Ordinarily, p’s appear linearly, acting ultimately as a form of Lagrange multiplier that forces the semiclassical space of vacua to be a complete intersection in some space. Here, one has a $U(1) \times \mathbb{Z}_2$ gauge theory with a superpotential that is quadratic in p’s, whose large-radius phases are interpreted by [11],
following [9], to be Calabi-Yau branched double covers. The $\mathbb{Z}_2$ gauge factor plays an essential role, and is ultimately responsible for that branched double cover structure.

The purpose of this note is to propose an understanding of corresponding GLSM’s without that crucial $\mathbb{Z}_2$. Omitting it leaves one with a puzzle: one gets what looks locally like a single copy of a Fano manifold, which is not Ricci-flat, away from the former branch locus. However, there is a unique stack structure one can impose on that Fano manifold along what would have been the branch locus, such that the resultant stack is Ricci-flat (but not Calabi-Yau). We propose that that Ricci-flat stack is the correct understanding of the GLSM phase.

In section 2 we briefly review the pertinent $U(1)\times\mathbb{Z}_2$ GLSM’s of Halverson-Kumar-Morrison [11], and in section 3 we describe our proposal for the interpretation of the GLSM’s when the $\mathbb{Z}_2$ is omitted. We include a discussion of general properties of such examples. These stacks are closely parallel to Enriques surfaces, and we discuss general aspects of associated superconformal field theories. After untangling a potential confusion regarding covers and quotients in this example, we conclude with a discussion of the application of the geometric criteria of [11] to this case.

2 Review of GLSM’s of Halverson-Kumar-Morrison

One of the insights of [11, 22] was to consider GLSM’s in which the superpotential has terms involving multiple $p$ fields. Reference [11] tabulates data for a number of GLSM’s of this form describing Calabi-Yau threefolds at large-radius, which we have reproduced and extended in table 1.

Let us briefly outline the analysis of two pertinent examples in that table, beginning with the first entry. This describes a GLSM with 4 chiral superfields $x_i$ of charge +1 and 4 chiral superfields $p_a$ of charge −1, with a superpotential of the form

$$W = \sum_{a,b=1,\ldots,4} p_a p_b A^{ab}(x),$$

where each $A^{ab}(x)$ is of degree two, and gauge group $U(1)\times\mathbb{Z}_2$, where the $\mathbb{Z}_2$ acts only on the $p$ fields. For $r \gg 0$, the $x_i$ are not all zero, and map out a $\mathbb{P}^3$. The superpotential defines a mass matrix for the $p_a$’s, with nonzero eigenvalues away from the locus $\{\det A^{ab} = 0\}$, of degree $(4)(2) = 8$.

On the face of it, this result is confusing – although the charges sum to zero, indicative of a Ricci-flat space, it looks as if theory will RG flow to a sigma model on $\mathbb{P}^3$. Closely related GLSM’s with behavior of this form have been discussed in [9]. Briefly, because of the $\mathbb{Z}_2$ factor in the gauge group, which acts trivially on the $x$’s (and nontrivially on the $p$’s), the semiclassical vacua ‘double’ away from the locus where $p$’s become massless, and a Berry phase around that locus interleaves the copies appropriately.
This is a special case of a much more general story about how two-dimensional theories see trivially-acting finite gauge groups. Briefly, although the action on geometry is trivial, physics is nevertheless able to sense their presence via nonperturbative effects, and the resulting CFT is equivalent to a sigma model on a disjoint union of multiple spaces. See for example [23, 24, 25, 26, 27] for a more detailed discussion, [28, 29, 30] for mathematical checks of predictions this makes for Gromov-Witten theory, [17, 31] for consistency tests and extensions of the results of [9], and [7] (as well, of course, as [11]) for further examples of this phenomenon in GLSMs.

In any event, as a result of that trivially-acting \( \mathbb{Z}_2 \), we can interpret this GLSM phase as a branched double cover of the space of \( x \)'s, branched over the locus where some of the \( p \)'s become massless (modulo behavior at singular loci we shall discuss momentarily). Specifically, that means this is a branched double cover of \( \mathbb{P}^3 \), branched over a degree 8 locus.

As reviewed in e.g. [9], if the degree of the branch locus is \(-2\) times the degree of the canonical bundle, the branched double cover is Calabi-Yau, which implies that this as well as all of the other examples in table I are Calabi-Yau, resolving our puzzle.

Strictly speaking, we should be slightly careful about the description as a branched double cover. There is a subtlety that sometimes the branched double cover is mathematically singular, but the GLSM behaves as if it were smooth. The condition for the branched double cover to be (mathematically) singular is that the branch locus is singular, i.e. there exist places where both the determinant and its derivatives vanish:

\[
\det A^{ab} = 0, \quad \frac{\partial}{\partial x_i} \det A^{ab}(x) = 0.
\]

The condition for the GLSM to be singular are that both \( A^{ab} \) and its derivatives admit the same zero eigenvector:

\[
A^{ab} p_b = 0, \quad p_a \frac{\partial A^{ab}(x)}{\partial x_i} p_b = 0,
\]

so that the theory develops a new noncompact branch. Certainly, whenever the GLSM is singular, it is trivial to see that the branched double cover will also be singular. However, the converse is not true in general – the branched double cover will often be mathematically singular even when the GLSM does not develop a new noncompact branch. This is a generic feature of GLSM’s of this form.

This phenomenon was discussed in [9]. As discussed there, in such cases, we believe that physics sees a ‘noncommutative resolution’ of the branched double cover, as discussed in [9, 31, 17, 32]. For example, this happens in the present case: typically there are 80 points on \( \mathbb{P}^3 \) where the branched double cover is mathematically singular, and (based on previous experience in closely related cases) we believe the GLSM describes a noncommutative resolution at those points. Given that the phenomenon is generic, we suspect that similar statements are true of the other entries in table I. As this matter is not central to our purpose in this letter, we shall move on.
Table 1: Examples from [11] (table 4). Listed are data for GLSM’s with gauge group $U(1) \times \mathbb{Z}_2$, and corresponding interpretations of $r \gg 0$ phases. The first column counts $x$’s, of charge $+1$; the next two columns count $p$’s and give their charges; the fourth column gives the exponents of those $p$’s in the superpotential. The entry marked with a * is described in detail in [11].

In passing, the geometry in the $r \ll 0$ limit of the GLSM is of the same form. Here, the $p$’s are not all zero, and define a $\mathbb{P}^3$; the superpotential defines a mass matrix for $x$’s; the $U(1) \times \mathbb{Z}_2$ can be reorganized so that the $\mathbb{Z}_2$ acts only on $x$’s, not on $p$’s; the resulting geometry is again a branched double cover of $\mathbb{P}^3$, branched along a degree 8 locus (or a noncommutative resolution thereof).

Let us consider one more example, the third entry in the table. This is a GLSM with gauge group $U(1) \times \mathbb{Z}_2$ as before, 5 chiral superfields $x_i$ of charge +1, 4 chiral superfields $p_a$ of charges -2, -1, -1, -1, superpotential

$$W = p_1 f_2(x) + \sum_{a,b=2,3,4} p_a p_b A^{ab}(x),$$

where $p_1$ has charge $-2$, $p_{2,3,4}$ have charge -1, and $f_2$, $A^{ab}$ are homogeneous of degree 2. Again let us study the $r \gg 0$ phase. Here, the $x_i$ are not all zero, hence form a $\mathbb{P}^4$, and the $p_1$ term in the superpotential restricts to the hypersurface $\{f_2 = 0\} \subset \mathbb{P}^4$. The remaining superpotential terms act as a mass matrix for $p_{2,3,4}$, and their analysis proceeds as in the last example. Briefly, one gets a branched double cover of $\{f_2 = 0\}$, branched over the locus $\{\det A = 0\} \cap \{f_2 = 0\}$. 
It is straightforward to check that this is another Calabi-Yau.

More generally, for a branched double cover of \( \mathbb{P}^n[d_1, d_2, \cdots] \) to be Calabi-Yau, the branch locus must have degree

\[
2n + 2 - 2 \left( \sum d_i \right),
\]

and it is straightforward to check that the remaining examples in table 1 all define Calabi-Yau branched double covers.

3 Analysis of the \( U(1) \) gauge theories, without \( \mathbb{Z}_2 \)'s

3.1 Basic proposal

Now, let us modify the examples above. Instead of taking the gauge group to be \( U(1) \times \mathbb{Z}_2 \), let us take the gauge group to be just \( U(1) \), omitting the \( \mathbb{Z}_2 \) that played a crucial role in the analyses above. We will make a proposal for the IR geometry in these cases.

Let us return to the first example in table 1. Recall this theory has 4 chiral superfields \( x_i \) of charge +1 and 4 chiral superfields \( p_a \) of charge -1, together with a superpotential

\[
W = \sum_{a,b=1,\cdots,4} p_a p_b A^{ab}(x),
\]

where each \( A^{ab}(x) \) is of degree two. For \( r \gg 0 \), the \( x_i \) are not all zero, hence the superpotential defines a mass matrix for the \( p_a \)'s, with nonzero eigenvalues away from the locus \( \{ \det A^{ab} = 0 \} \), of degree eight.

Previously, the extra \( \mathbb{Z}_2 \) acting only on the \( p \)'s gave rise to a \( \mathbb{Z}_2 \) gerbe structure over

\[
\mathbb{P}^3 - \{ \det A^{ab} = 0 \},
\]

which physics sees as a branched double cover, which we denote \( X \). Without that \( \mathbb{Z}_2 \) action, however, there is no gerbe structure, and hence no double cover. Thus, we know that this theory RG flows to something that is one copy of \( \mathbb{P}^3 \) away from the degree 8 locus \( \{ \det A^{ab} = 0 \} \).

Furthermore, we also know that, because the \( U(1) \) charges balance, if this RG flows to a nonlinear sigma model on some sort of geometry, that geometry must be Ricci-flat. (As Enriques surfaces can be built from K3’s realized as projective hypersurfaces, by taking suitable finite quotients, we hesitate to claim that just because \( U(1) \) charges balance the geometry must be Calabi-Yau.)
In the present case, although we cannot directly ‘see’ the structure along the degree 8 hypersurface, we can uniquely determine it by demanding that the result admit a Ricci-flat metric. The structure so determined is a hypersurface of $Z_2$ orbifolds \[33\]. Therefore, we propose that this theory RG flows to $\mathbb{P}^3$ with a hypersurface of $Z_2$ orbifolds along the degree 8 locus \(\{\det A^{ab} = 0\}\).

This proposal extends in the obvious way to the other examples in table [1]. When the $Z_2$ gauge symmetry is omitted, we propose that each RG flows to a sigma model on a Fano space (as listed in table [1] with a hypersurface of $Z_2$ orbifolds, (or a noncommutative resolution thereof), along what was formerly the branch locus. The resulting stacks are Ricci-flat, though not Calabi-Yau, as we discuss in the next section.

3.2 General properties of such examples

More generally, Fano spaces $B$ with $Z_2$ orbifolds along a hypersurface of degree $-2$ times the canonical class of $B$ admit Ricci-flat metrics. For example, $\mathbb{P}^n[d_1, d_2, \cdots]$ with $Z_2$ orbifolds along a hypersurface of degree 
\[2n + 2 - 2 \left(\sum d_i\right)\]
admmit Ricci flat metrics. These stacks have a simple relationship to the Calabi-Yau branched double covers branched over the same locus that were described earlier in this paper: they can be obtained by a global $Z_2$ orbifold that exchanges the two sheets of the cover. The Ricci-flat metric on the Calabi-Yau branched double cover descends to define a Ricci-flat metric on the stack. Away from the branch locus, this orbifold acts effectively and reduces the double cover to just a single copy of the original space. At the branch locus, this results in a $Z_2$ orbifold structure.

On a variety, codimension one orbifolds are invisible: $\mathbb{C}/Z_2 = \mathbb{C}$, essentially because if $Z_2$ acts on $x$ by sign flips, then 
\[\mathbb{C}[x]^{Z_2} = \mathbb{C}[x].\]
Here, however, we do not have a space, we have a stack, and stacks can and do keep track of codimension one (and even codimension zero) orbifold structures.

Since these stacks have hypersurfaces of $Z_2$ orbifolds, locally they look like $[\mathbb{C}/Z_2] \times \mathbb{C}^r$ for some $r$. Such affine stacks have been discussed in the context of nonsupersymmetric strings (see e.g. \[34, 35\]), and a number of exotic behaviors have been described that arise after one deforms by a relevant operator, by giving a vev to a tachyon. Here, we are not turning on a tachyon vev, we are not making a relevant deformation, we are merely sitting

\[\footnote{Or a noncommutative resolution of any singularities in this stack. As noncommutative resolutions will play no significant role in what follows, and we are largely suppressing them in this paper, we will not discuss them further.}\]
on a Ricci-flat stack. Moreover, as our stacks are obtained by a global $\mathbb{Z}_2$ orbifold, there exists a quantum symmetry which prevents twisted sector relevant operators from acquiring a vev and driving RG flow to a trivial endpoint $[36]$.

Although these stacks are Ricci-flat, they are not Calabi-Yau, their canonical classes are 2-torsion $[33]$, just like an Enriques surface. In fact, SCFT’s for Enriques surfaces and for manifolds with codimension one $\mathbb{Z}_2$ orbifolds are closely parallel. For example, we can understand the presence of a Ricci-flat metric on both as arising because they are $\mathbb{Z}_2$ orbifolds of Ricci-flat spaces. Specifically, the codimension one $\mathbb{Z}_2$ orbifolds are $\mathbb{Z}_2$ orbifolds of (Ricci-flat) branched double covers, and Enriques surfaces are $\mathbb{Z}_2$ orbifolds of (Ricci-flat) K3 surfaces. In both cases, the metric on the cover descends.

We can see more or less explicitly that the $\mathbb{Z}_2$ orbifold of a branched double cover will yield a space with 2-torsion canonical divisor, by considering a simple example. Start with an elliptic curve described as a branched double cover of $\mathbb{P}^1$, branched over a degree 4 locus (4 points). If we describe this branched double cover as solutions to

$$y^2 = x(x-1)(x-\lambda),$$

(where the two possible square roots for $y$ are the two sheets of the cover, branched over $x = 0, 1, \lambda, \infty$), then it is a standard result that the holomorphic top-form on this elliptic curve can be written in local coordinates as

$$\omega = \frac{dx}{y}.$$ 

The global $\mathbb{Z}_2$ orbifold that exchanges the sheets of the double cover, acts by sending $y \mapsto -y$ and leaving $x$ invariant. From the expression above, it should be clear that this does not leave the holomorphic top-form invariant. Taking the stack quotient results in a stacky curve that looks like $\mathbb{P}^1$ with 4 $\mathbb{Z}_2$ orbifold points, which admits a Ricci-flat metric but has 2-torsion canonical class.

More globally, (2,2) SCFT’s can be defined at string tree level for Kähler target spaces and stacks whose canonical divisors are 2-torsion. Their Ricci-flatness implies the existence of a nontrivial IR fixed point; Kähler implies (2,2) worldsheet supersymmetry. They do not define spacetime supersymmetric theories, however. For example, at string one-loop, one runs into subtleties, due to the fact that such spaces and stacks are not Spin. As a result, the (R,NS) and (NS,R) sector Fock vacua cannot be defined, as they would describe spinors on the space. We can see this problem explicitly in examples, as follows:

- In the orbifold $[\mathbb{C}/\mathbb{Z}_2]$, the Fock vacuum is not defined in the (R,NS) and (NS,R) sectors of the orbifold. In both cases, because there is a periodic complex fermion $\psi$, there are two Fock vacua, related by the fermi zero mode:

$$|0\rangle' = \psi_0|0\rangle.$$
As a result, the action of the $\mathbb{Z}_2$ orbifold group on the two Fock vacua differs by a sign, so following the standard procedure, under the generator of the $\mathbb{Z}_2$, each Fock vacuum is multiplied by $\pm \sqrt{-1} = \pm i$. However, that does not square to +1, and so does not faithfully represent the $\mathbb{Z}_2$ orbifold group. Such states would break the projection operator implicit in the orbifold one-loop partition function, and so we see that the orbifold group action is not well-defined on the Fock vacua or other states of $(R,NS)$ and $(NS,R)$ sectors. See [34][section 2] for a related discussion.

• For a sigma model directly on an Enriques surface, the fact that Enriques surfaces do not have spinors means that the Fock vacua in the $(R,NS)$ and $(NS,R)$ sectors cannot be defined, and hence those sectors do not exist.

• If we build a sigma model on an Enriques indirectly, by taking a $\mathbb{Z}_2$ orbifold of a $(2,2)$ SCFT for a K3 surfaces, we run into a problem making sense of the $\mathbb{Z}_2$ orbifold, ultimately for the same reason as in the $[\mathbb{C}/\mathbb{Z}_2]$ orbifold. On a Spin Kahler manifold $M$, spinors are of the form [37]

\[ \wedge^\bullet TM \otimes \sqrt{K_M}. \]

In the RNS worldsheet formulation, perturbative fermi zero modes realize the $TM$ factors, and the Fock vacuum transforms as an element of $\sqrt{K_M}$. On a Calabi-Yau, this means the Fock vacuum is trivial, though for open strings on wrapped D-branes, the Fock vacuum couples to a nontrivial bundle. In any event, in the present case, the Enriques $\mathbb{Z}_2$ orbifold of a K3 flips the sign of the holomorphic top-form, which means the Fock vacuum would transform under the $\mathbb{Z}_2$ as $\pm i$, hence the states in $(R,NS)$, $(NS,R)$ sectors of the orbifold are not well-defined.

Thus, one cannot define a type II string compactification on such spaces and stacks. However, although the $(R,NS)$ and $(NS,R)$ sectors are not defined, the $(R,R)$ sector is well-defined, as the square of the canonical bundle is trivial, and so products of pairs of spinors can be defined, a statement which has simple specializations to each of the cases above. As a result, although type II strings are not well-defined, a type 0 string compactification on spaces or stacks of the form above should be well-defined. (See e.g. [38] chapter 10.6] for an efficient discussion of type 0 strings.)

In passing, the closed string B model on Enriques surfaces was discussed in [39], which observed that consistency only required $K_X^{\otimes 2} = O_X$, instead of the full Calabi-Yau condition. The closed string B model is similarly well-defined on the the codimension-one $\mathbb{Z}_2$ orbifolds we are discussing here.

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\[2\] As other states are obtained by multiplying the Fock vacua by perturbative modes, which clearly do have a well-defined action, failure of the Fock vacua to have a well-defined action implies that no other states can have well-defined actions either.
3.3 Covers and quotients

To review, given a construction of a branched double cover $X$ from a GLSM with gauge group $U(1) \times \mathbb{Z}_2$, we propose that the corresponding GLSM with smaller gauge group $U(1)$ describes a $\mathbb{Z}_2$ orbifold of $X$, which we denote $Y$. Given that the second theory is built without quotienting a $\mathbb{Z}_2$, one would have expected that the result would be a cover, not a quotient. Locally, we believe that this is an example of an old CFT phenomenon, that says, roughly, that orbifolding twice returns the original theory.

Specifically, the space $X$ arises from a GLSM that describes a $\mathbb{Z}_2$ gerbe over most of $\mathbb{P}^3$, i.e. locally $[\mathbb{C}^3/\mathbb{Z}_2]$ where the $\mathbb{Z}_2$ acts trivially. The resulting local $\mathbb{Z}_2$ orbifold has dimension zero operators in each of two twisted sectors, which ultimately is the reason for the double cover structure. Now, that local $\mathbb{Z}_2$ orbifold has a $\mathbb{Z}_2$ quantum symmetry. If we construct $Y$ as $[X/\mathbb{Z}_2]$, then the $\mathbb{Z}_2$ exchanging the sheets of the cover is acting by the quantum symmetry. It is a standard result \cite[section 8.5]{section} that orbifolding by a cyclic group twice, the second time by the quantum symmetry produced by the first orbifold, reproduces the original theory. As a result, this is why, locally, $Y$ can be both a $\mathbb{Z}_2$ orbifold and a two-fold cover of $X$.

Locally, the picture above gives some reasonable intuition, but we should also note that the global story is more complicated. The essential point is that for $X$ to be a branched double cover, the associated gerbe on $\mathbb{P}^3$ is a nontrivial gerbe away from the branch locus. However, if we were to globally quotient $Y$ itself (and not a cover, as the GLSM does) by a trivially-acting $\mathbb{Z}_2$, the result would be a trivial gerbe away from the branch locus, which could not yield $X$. Thus, our intuitive picture of successive orbifolds only can work locally, not globally, on the spaces themselves. (In the UV, the $\mathbb{Z}_2$ actions on the covering spaces may provide a cleaner description in terms of successive orbifolds, modulo the fact that one would not be working in a CFT; we will not pursue this here.) Nevertheless, old results on successive orbifolds do give some intuitive explanation for the consistency of this proposal.

3.4 Geometric criteria

Finally, let us comment on the geometric criteria in \cite{ref}. Table 1 of that reference lists a set of criteria for a GLSM phase to have a geometric interpretation. The third entry requires that the R-charge of every gauge-invariant chiral operator is even. This is satisfied by the $U(1) \times \mathbb{Z}_2$ gauge theories, as they remark, but is not satisfied in the present examples, where

\footnote{In an analytic sense, not in the Zariski topology. Throughout this paper, we use ‘local’ to mean in an analytic sense.}
\footnote{The two dimension-zero operators are used to form projection operators on each of two components. Acting by the quantum symmetry puts phases on twisted sectors, which has the effect of exchanging the two projection operators.}
one only has a $U(1)$ gauge theory, as both operators of the form $p^2 f(x)$ (of even R-charge) and operators of the form $px$ (odd R-charge, excluded previously by the $\mathbb{Z}_2$) are allowed.

Clearly, one thing this reflects is the fact that the $U(1)$ gauge theories we have discussed do not define Calabi-Yau’s, and do not define spacetime supersymmetric theories, unlike the $U(1) \times \mathbb{Z}_2$ gauge theories.

However, we can also gain a more subtle understanding of that condition, by turning to a different description. The same condition also appeared in a different guise as part of the definition of R-symmetry in matrix factorizations in [31][section 3.1], [41]. Those references defined a $\mathbb{C}^\times \times \mathbb{R}$ symmetry to be a $\mathbb{C}^\times$ symmetry such that the superpotential has weight 2, and a $\mathbb{Z}_2 = \{\pm 1\} \subset \mathbb{C}^\times_R$ acts trivially on the underlying space and is the universal $\mathbb{Z}_2^R$, among other things. The constraint that the $\mathbb{Z}_2^R$ act trivially on the underlying space is the sigma model counterpart to the gauge theory statement of [11][table 1] that gauge-invariant local operators have even R-charge.

The reason for the requirement that the $\mathbb{Z}_2^R$ act trivially is that in that case, that R-charge defines gradings on D-branes. In matrix factorizations, and open strings more generally, the $\mathbb{Z}_2^R$ distinguishes branes and antibranes, and when the entire $\mathbb{C}^\times \times \mathbb{R}$ symmetry acts trivially on the underlying space, the Chan-Paton factors get an integral grading which partially defines complexes.

Now, let us look at how this R-symmetry appears in some explicit examples. In a Landau-Ginzburg model over the total space $X$ of a vector bundle $\mathcal{E} \to B$, with superpotential of the form $W = pf(x)$, $p$ a fiber coordinate, $f(x)$ the pullback of a section of $\mathcal{E}^*$, if we let the $\mathbb{C}^\times_R$ act on $p$ with weight 2 and leave $B$ invariant, then the superpotential has weight 2 and the $\mathbb{Z}_2 \subset \mathbb{C}^\times$ acts trivially on $X$.

In the examples of the previous section, one has a superpotential of the form $W = p^2 f(x)$. If the $\mathbb{C}^\times$ R-symmetry acts on the $p$’s with weight 1, and leaves the $x$’s invariant, then the superpotential has weight 2. Because of the $\mathbb{Z}_2$ orbifold $p \mapsto -p$, the universal $\mathbb{Z}_2^R \subset \mathbb{C}^\times_R$ acts trivially on the orbifold (though not on the covering space of $p$’s).

In the examples in this section, where there is no $\mathbb{Z}_2$ factor in the gauge group, the $\mathbb{Z}_2^R \subset \mathbb{C}^\times_R$ does not act trivially on the underlying stack. Thus, for example, there would be no way to distinguish branes from antibranes in this theory.

However, because the present theory seems to be describing a space with 2-torsion canonical divisor, there is for example no open string sector in the B model [39]. We therefore view the fact that the $\mathbb{Z}_2^R$ acts nontrivially on the scalars (equivalently, that there are local gauge-invariant operators of odd R-charge), as a reflection of difficulties defining the open string sector.
4 Conclusions

In this paper, after briefly reviewing some exotic GLSM’s described recently in [11], we proposed a description of the IR limits of some closely related GLSM’s, in terms of Ricci-flat (but not Calabi-Yau) stacks of the form of Fano spaces with $\mathbb{Z}_2$ orbifolds along a hypersurface of suitable degree. We studied properties of SCFT’s associated to such stacks and to Enriques surfaces, which are closely related, as well as some subtleties involving covers versus quotients in repeated orbifoldings, and concluded with a discussion of the application of the geometric criteria listed in [11] to these Ricci-flat stacks.

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