Frustration-free Hamiltonian with Topological Order on Graphs

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It is commonly believed that models defined on a closed one-dimensional manifold cannot give rise to topological order. Here we construct frustration-free Hamiltonians which possess both symmetry protected topological order (SPT) on the open chain and multiple ground state degeneracy (GSD) that is unrelated to global symmetry breaking on the closed chain. Instead of global symmetry breaking, there exists a local symmetry operator that commutes with the Hamiltonian and connects the multiple ground states, reminiscent of how the topologically distinct ground states of the toric code are connected by various winding operators. Our model solved on an open chain demonstrates symmetry fractionalization as an indication of SPT order and on a general graph the GSD can be shown to scale with the first Betti number - a topological invariant that counts the number of independent cycles or one dimensional holes of the graph.

A standard wisdom by now is that topological phases of one-dimensional interacting bosons are characterized by the symmetry protected topological order, or SPT [1–5]. The mathematical construct behind the SPT is the cohomology classification of the global symmetry of the Hamiltonian. Commonly believed to be devoid of topological order, the one-dimensional SPT models typically possess a unique ground state on the closed chain and edge states on the open chain characterized by the fractionalization of the global symmetry [6–9].

We propose and analyze a family of frustration-free Hamiltonians that possess multiple ground states without any symmetry breaking on the closed chain. In addition to the global $\mathbb{Z}_m \times \mathbb{Z}_n$ symmetry, our model has an extra local symmetry which explains the degeneracy of the ground states on the closed chain. On an open chain, edge states appear as a consequence of the fractionalization of the global symmetry, indicating the SPT order. The frustration-free Hamiltonians [10, 11] under consideration consist of commuting projectors and are exactly solvable much like the toric code model [12]. Despite ours being a one-dimensional model, a “topological” interpretation of the ground state degeneracy is possible when the model is considered on general graphs.

![Diagram](image.png)

**FIG. 1.** A one-dimensional chain of alternating vertices and oriented links on which our model is defined. The number of vertices and links are $L$ each. For a closed chain the link $l_L$ attaches to the vertex $v_1$.

The model we consider is defined on a one-dimensional chain with alternating vertex ($v$) and link ($l$) locations as illustrated in Fig. 1. In general, Hilbert spaces of dimensions $m$ and $n$ are imposed on the vertices and links, respectively. For concreteness we first focus on the case of $m = 2$ and $n = 4$: $\mathcal{H}_v = \{ |0\rangle_v, |1\rangle_v \}$ and $\mathcal{H}_l = \{ |0\rangle_l, |1\rangle_l, |2\rangle_l, |3\rangle_l \}$. Analogous to the toric code model, our Hamiltonian is a sum of commuting projectors

$$H = -\sum_{i=1}^{L} (A_i + C_i)$$

on a chain of $L$ vertices and $L$ links where

$$A_i = \frac{1}{3}(1 + X_{l_{i-1}} x_v X^3_{l_i} + X^2_{l_{i-1}} X^2_{l_i} + X^3_{l_{i-1}} x_v X_{l_i}),$$

$$C_i = \frac{1}{2}(1 + z_{v_i} Z^2_{l_i} z_{v_{i+1}}),$$

and $i = (v_i, l_i)$ stands for a combined vertex+link label. The lower-case $x$ and $z$ operators are the usual Pauli matrices, while the upper-case analogs are the $\mathbb{Z}_4$ generalization with $X_i|h\rangle = |h \oplus 1\rangle$ and $Z_i|h\rangle = i^h |h\rangle$ where $\oplus$ is defined as $a \oplus b = a + b \pmod{4}$. We have $Z^b_i X_i = i^b X_i Z^b_i$. One can easily check $A^2_i = A_i$, $C^2_i = C_i$, and the commutations $[A_i, C_j] = 0$, $[A_i, A_j] = [C_i, C_j] = 0$. To be completely general, the definition of the vertex operator $A_i$ depends on the direction of the arrows on the links connected to a given vertex. We focus on the case of uniformly directed arrows as in Fig. 1 and delegate the more general situations to the Supplementary Materials (SM) [13]. The Hamiltonian (1) has a global $\mathbb{Z}_2 \times \mathbb{Z}_4$ symmetry generated by

$$\phi = \prod_i x_{v_i}, \quad \theta = \prod_i Z_{l_i},$$

respectively, and an additional local $\mathbb{Z}_2$ symmetry generated by the operator $X^2_{l_i} : [H, X^2_{l_i}] = 0 \quad (\forall i)$.

The eigenstates of the Hamiltonian (1) satisfy $A_i |\psi\rangle = a_i |\psi\rangle$, $C_i |\psi\rangle = c_i |\psi\rangle$, with $a_i = 0$, 1 and $c_i = 0$, 1 for all vertices and links. First of all, we define the seed states as the eigenstates of $-\sum_i C_i$. The $c_i = 1$ ($\forall i$) seed states are realized by one of the following:

$$|S_1\rangle = (\otimes_i |0\rangle_{v_i}) (\otimes_j |0\rangle_{l_j}),$$

$$|S_2\rangle = (\otimes_i |0\rangle_{v_i}) (\otimes_j \neq j' |0\rangle_{l_j}) \otimes |2\rangle_{l_j},$$

where $j'$ can be chosen arbitrarily. The ground states of the full Hamiltonian $H = -\sum_i (A_i + C_i)$ are obtained by projection, $|G_1\rangle = (\prod_i A_i)|S_1\rangle$ and $|G_2\rangle = (\prod_i C_i)|S_2\rangle$. Note that any seed state $|\langle S\rangle$ acted with the projector...
\( \prod_i A_i \) is automatically an eigenstate of \( A_i \) with the eigenvalue \( a_i = 1 \) (\( \forall i \)). The two ground states are degenerate and distinct, and connected by the local symmetry operator \( X_i^\dagger \):

\[
X_i^2 |G_1\rangle = |G_2\rangle. \tag{4}
\]

There are \( 4^L \) different seed states sharing the same set of quantum numbers \( c_i = 1 \) (\( \forall i \)), but they all collapse to either \( |G_1\rangle \) or \( |G_2\rangle \) after the projection by \( \prod_i A_i \). This unique feature is a consequence of the special property of \( A_i \),

\[
A_i = X_i^{2} A_i X_i^{3} A_i X_i^{3} A_i X_i^{3} x_v A_i X_i^{3} x_v A_i = X_i^{3} x_v A_i. \tag{5}
\]

Consequently, the following local seed configurations become identical after the projection by \( \prod_i A_i \):

\[
|g⟩_{l_{i-1}} |σ⟩_{v_i} |g⟩_{l_i} \equiv |g \oplus 2⟩_{l_{i-1}} |σ⟩_{v_i} |g \oplus 2⟩_{l_i} \equiv |g \oplus 3⟩_{l_{i-1}} |σ \oplus 1⟩_{v_i} |g \oplus 1⟩_{l_i} \equiv |g \oplus 1⟩_{l_{i-1}} |σ \oplus 1⟩_{v_i} |g \oplus 3⟩_{l_i}, \tag{6}
\]

where \( \oplus \) and \( \oplus' \) are defined mod 4 and mod 2, respectively. Both states \( |G_1\rangle \), \( |G_2\rangle \) are invariant up to a phase by the global \( Z_2 \times Z_4 \) operations \( φ \) and \( θ \) [Eq. (2)]:

\[
φ |G_1\rangle = |G_1\rangle, \quad φ |G_2\rangle = |G_2\rangle
\]

\[
θ |G_1\rangle = |G_1\rangle, \quad θ |G_2\rangle = −|G_2\rangle. \tag{7}
\]

The twofold degeneracy, exact for arbitrary chain length \( L \), is not a consequence of any global symmetry breaking.

The twofold nature of the ground state degeneracy (GSD) can be proved formally. For a Hamiltonian consisting of commuting projectors as in our model, the operator \( \prod_{i=1}^{L} A_i C_i \) takes the diagonal form

\[
\left( \begin{array}{cc}
1_k & 0 \\
0 & 0 \end{array} \right)
\]

in the eigenbasis of the Hamiltonian. Here \( k \) counts the number of states with all \( a_i = c_i = 1 \), and \( N = 8^L \) is the total dimension of the Hilbert space. The GSD we are looking for is \( k \), obtained as the trace \( GSD^{\text{chain}} = Tr[\prod_i A_i C_i] = 2 \) (by explicit computation) for the \( A_i \) and \( C_j \) operators given in Eq. (1), for any \( L \).

Excitations occur when either \( c_i = 0 \) (link excitations) or \( a_i = 0 \) (vertex excitations) or the combinations of the two. In a link excitation with \( m \) links having \( c_i = 0 \), there are \( \binom{L}{m} \) ways of distributing the \( c_i = 0 \) links, and within each such configuration we can show the twofold excited state degeneracy (ESD) using the trace method [13]: \( ESD^{\text{chain}} = 2 \). Each link excitation at the link \( l_i \) is created with the action of \( X_i \) or \( X_i^\dagger \) on the ground states. Due to the fact that \( X_i^2 |G_1\rangle = |G_2\rangle \), it suffices to consider only the link excitations on \( |G_1\rangle \). The single-link excited states are given by either \( X_i |G_1\rangle \) or \( X_i^\dagger |G_1\rangle \). Multiple link excitations are created by acting with a product of \( X_i \)'s or \( X_i^\dagger \)'s on various links of the ground state \( |G_1\rangle \). Each doublet of states is connected by the product \( \prod_i X_i^2 \) spanning the links with the eigenvalues \( c_i = 0 \). The eigenvalue for the \( Z_2 \) global operator \( φ \) is +1 for all link excitations. Meanwhile, the eigenvalue for the \( Z_4 \) operator \( θ \) depends on the number of link excitations:

\[
θ |\{l_i\}\rangle = (−i)^{\sum_i l_i} |\{l_i\}\rangle. \tag{8}
\]

Here \( l_i = 1, 3 \) depending on whether the particular link has been excited by \( X_i \) or \( X_i^\dagger \) and \( |\{l_i\}\rangle \) refers to a general link excitation. Contrary to \( X_i \), \( X_i^\dagger \) which excites a single link \( l_i \), the operator \( x_v \) excites a pair of links to its left and right. They are, however, not independent link excitations [13].

Vertex excitations refer to excited states with at least one \( a_i = 1 \). It is clear that all states of the type \( \prod_i A_i |S⟩ \) share \( a_i = 1 \) (\( \forall i \)) regardless of the seed state \( |S⟩ \). On the other hand, there are other vertex projectors one can write down besides \( A_i \):

\[
A_i(ω) = \frac{1}{4} (1 + ω X_{i-1} x_v X_i^2 + ω^2 X_{i-1}^2 X_i^3 + ω^3 X_{i-1}^3 x_v X_i^2). \tag{9}
\]

The previous projector is obtained for \( ω = 1 \) but in fact one can generally take \( ω = 1, i, −1, −i \). We will continue to write \( A_i \) for \( A_i(1) \). It is easily shown

\[
A_i(ω) A_i(ω') = δ(ω, ω') A_i(ω),
\]

\[
[A_i(ω), C_j] = [A_i(ω), C_j'] = 0,
\]

\[
\sum_ω A_i(ω) = 1. \tag{10}
\]

A general projected state can be written down as \( \prod_i A_i(ω_i)|S⟩ \) with vertex-dependent \( ω_i \), to which our previous projected state belongs with \( ω_i = 1 \) (\( \forall i \)). Any projected state \( \prod_i A_i(ω_i)|S⟩ \) is an eigenstate of the full Hamiltonian since \([A_i(ω_i), C_j] = 0 \). A pair of projected states \( |ψ⟩ = \prod_i A_i(ω_i)|S⟩ \) and \(|ψ'⟩ = \prod_i A_i(ω'_i)|S⟩ \) satisfies the orthogonality condition: \( ⟨ψ'|ψ⟩ = \prod_i δ(ω_i, ω'_i) \). Similar to Eq. (5), \( A_i(ω) \) satisfies the identity

\[
A_i(ω) = ω X_{i-1} x_v X_i^2 A_i(ω) = ω^2 X_{i-1}^2 X_i^3 A_i(ω) = ω^3 X_{i-1}^3 x_v X_i A_i(ω).
\]

Apart from the trivial phase factor, the rules for identifiable seed states given in Eq. (6) remain the same for all \( \prod_i A_i(ω_i) \).

A single vertex excitation corresponds to projected states \( (\prod_{i \neq i'} A_i) A_{i'}(ω)|S⟩ \), \( ω \neq 1 \). Its degeneracy is computed as usual from the trace

\[
Tr[\prod_i A_i A_{i'}(ω) \prod_j C_j] = 1 + ω^2.
\]

In other words, we obtain the degeneracy factor 2 for an excitation created by \( A_{i'}(−1) \) but zero for \( A_{i'}(±i) \), implying that an isolated vertex excitation with \( ω = ±i \) is impossible. Put differently, the seed state is annihilated,
(\prod_{i \neq i'} A_i A_{i'}(\omega)|S\rangle = 0 \text{ if } \omega = \pm i. \text{ For multiple vertex excitations, one can show that the trace vanishes if the sum of the number of } A_i(i)'s \text{ and } A_i(-i)'s \text{ is odd. We conclude that an isolated vertex excitation by } A_i(-1) \text{ is possible, while the } A_i(i) \text{ and } A_i(-i) \text{ vertex excitations can only occur as a pair. More on the vertex excitations and operators creating them can be found in SM [13].}

Using Eq. (10), one can see that the excitation energy of vertex pair excited state is always 2 regardless of where the excitation happens. In other words, the pair of vertex excitations is deconfined.

The number of excited states with } n \text{ vertex excitations (including all possible } \omega) \text{ is found to be } \text{ESD}_{\text{chain}}^n = (-1)^n + 3^n \text{ using the trace method. In fact, the degeneracy for the } n \text{ vertex and } m \text{ link excitations is also } \text{ESD}_{\text{chain}}^{n+m} = (-1)^n + 3^n. \text{ Using this we can account for the full Hilbert space dimension of } 8^L \text{ through the sum}

\[
\sum_{E=0}^{2L} \sum_{i=0}^{E} \binom{L}{i} \binom{L}{E-i} \text{ESD}_{v_i,l_{E-i}}^{n+m} = 8^L. \tag{11}
\]

For completeness we compute some correlation functions on the closed chain:

\[
\langle G_1|x_i,x_j|G_1 \rangle = \delta_{ij} \quad \langle G_1|X_i,X_j|G_1 \rangle = 0 \quad \langle G_1|Z_i,Z_j|G_1 \rangle = 0. \tag{12}
\]

They reflect the extremely short-ranged nature of correlations.

The global } Z_2 \times Z_4 \text{ symmetry of our Hamiltonian persists on the open chain. The cohomology classification of the global } Z_2 \times Z_4 \text{ symmetry, } H_2(\mathbb{Z}_2 \times \mathbb{Z}_4, U(1)) = \mathbb{Z}_{\gcd(2,4)} = \mathbb{Z}_2, \text{ suggests the existence of edge states as a result of the fractionalization of the global symmetry just as in an SPT phase. We show how to construct them explicitly.}

Consider the Hamiltonian on an open chain ending with a vertex on the left and a link on the right [Fig. 1],

\[
H = -\sum_{i=2}^{L} A_i - \sum_{j=1}^{L-1} C_J. \tag{13}
\]

Note that the Hamiltonian is missing two operators, } A_1 \text{ and } C_L \text{ for the chosen boundary conditions. This Hamiltonian continues to be made up of commuting projectors and there are GSD_{vl} = \text{Tr}[\prod_{i=2}^{L} A_i \prod_{j=1}^{L-1} C_J] = 8 \text{ degenerate ground states. The superscript } vl \text{ stands for vertex/link termination. One such ground state is obtained by taking the seed state } |S^v_{vl}\rangle = \otimes_j |0\rangle_{v_j} \otimes_j |0\rangle_i, \text{ and projecting it with }

\[
\prod_{i=2}^{L} A_i: |G^v_{vl}\rangle = \prod_{i=2}^{L} A_i |S^v_{vl}\rangle. \tag{14}
\]

The remaining ground states are constructed by employing the edge operators } \phi_{vl} = x_{v_1} X_{l_1}, \phi_{vl}' = X_{l_1}^3 \text{ as [13]}

\[
|G^v_{m,n}\rangle = (\phi_{vl})^m (\phi_{vl}')^n |G^v_{vl}\rangle, \tag{15}
\]

with } m = 0, 1 \text{ and } n = 0, 1, 2, 3. \text{ The states are orthogonal for different values of } (m,n) \text{ and remain strictly degenerate for arbitrary } L. \text{ The ground states are distinguished by the following quantum numbers:}

\[
\theta |G^v_{m,n}\rangle = i^{m+n} |G^v_{m,n}\rangle \quad z_{v_1} |G^v_{m,n}\rangle = (-1)^m |G^v_{m,n}\rangle. \tag{16}
\]

The local operator } z_{v_1} \text{ commutes with the open-chain Hamiltonian and can be used to label the ground states.

The global } Z_2 \times Z_4 \text{ symmetry is realized projectively on the edges [9]. To show this first note that an edge } Z_2 \text{ operator may be constructed as } \theta^v_{vl} = z_{v_1}, \theta^{vl}_\phi = z_{v_1} Z_l^2 [13]. \text{ Both these operators square to one and commute with the open chain Hamiltonian. Rather than commuting with the } Z_4 \text{ edge symmetry operators } \phi^v_{vl} \text{ and } \phi^v_{vl}', \text{ however, they anti-commute}

\[
\theta^v_{vl} \phi^{vl}_r = -\phi^v_{vl} \theta^{vl}_r, \quad \theta^{vl}_\phi \phi^v_{vl}' = -\phi^{vl}_r \theta^v_{vl}, \tag{17}
\]

in an indication of the global symmetry } Z_2 \times Z_4 \text{ being represented projectively at the edges. Note that the ground states in Eq. (14) are no longer connected by the local } Z_2 \text{ operator } X^2_i, \text{ but by the fractionalized global symmetry operators at the edges. An entirely similar set of considerations applies to an open chain consisting of link/link or vertex/vertex terminations. In all cases we find GSD=8 and symmetry fractionalization at the edges [13].}

The local } Z_2 \text{ symmetry operation by } X^2_i \text{ in the closed chain is reminiscent of the string operator for the toric code [12] on a torus. The four ground states in that model are connected by the string operators threading along the two non-contractible loops, say along the } x \text{ and } y \text{ directions, on the torus. The string operator threading along the } x \text{-direction can be defined with an arbitrary } y \text{-coordinate, yielding identical states. This is what happens with the } X^2_i |G_1\rangle = |G_2\rangle \text{ operation in our model, in that the choice of } l_1 \text{ has no impact on the state produced by } X^2_{l_1}. \text{ Moreover the GSD of the toric code, } 2^2g, \text{ is topological due to its dependence on the genus, } g \text{ or the second Betti number, } B_2. \text{ The base 2 comes from the } Z_2 \text{ gauge field and the exponent 2 from the two kinds of non-contractible loops of the torus.}

FIG. 2. Examples of closed and connected graphs with two different Betti numbers (a) } B_1 = 1 \text{ and (b) } B_1 = 2. \text{ Dots are the vertices and the arrows indicate directed links of the graph.}

A similar topological aspect in the GSD of our model becomes apparent when we consider the model on a general closed and connected graph } G \text{ [Fig. 2] and relate the
emerging degeneracy to a topological invariant on graphs called the first Betti number $B_1$ [14]. On $G$ every link is bounded by two vertices, and the number of links attached to a vertex (called the valency) is at least two. On a particular graph $G$, the precise definition of the vertex operator $A_i^G$ changes with the valency. For example on a vertex with valency three such as shown in Fig. 2(b), the vertex operator becomes

\[
A_i = \frac{1}{4} \left( 1 + X_{l_i} x_{v_i} X_{l_i}^3 X_{l_i} \right) + X_{l_i}^2 x_{v_i}^2 X_{l_i} X_{l_i} + X_{l_i} x_{v_i} X_{l_i} X_{l_i},
\]

where $v_i$ has one (two) arrowhead towards it and two (one) arrowheads away from it. The link operator $G$ is bounded by two vertices, and the number of links at-tached to a vertex with valency three such as shown in Fig. 2(b), the vertex operator becomes

\[
A_i = \frac{1}{4} \left( 1 + X_{l_i} x_{v_i} X_{l_i}^3 X_{l_i} \right) + X_{l_i}^2 x_{v_i}^2 X_{l_i} X_{l_i} + X_{l_i} x_{v_i} X_{l_i} X_{l_i},
\]

where $v_i$ has one (two) arrowhead towards it and two (one) arrowheads away from it. The link operator maintains the same form regardless of the graph, $C_i^G = C_i$. The Hamiltonian on $G$ with $V$ vertices and $L$ links is given by

\[
H^G = -\sum_{i=1}^V A_i^G - \sum_{j=1}^L C_j^G.
\]

The GSD is computed using the trace [13]

\[
\text{GSD}^G = \text{Tr} \left[ \prod_{i=1}^V A_i^G \prod_{j=1}^L C_j^G \right] = 2^{L-V+1},
\]

where $L - V + 1 = B_1$ is the first Betti number counting the number of independent holes or cycles in the graph $G$. In the special case of a closed chain [Fig. 2(a)], $L = V$ and we recover GSD=2. Note the similarity of our result GSD$^G = 2^{B_1}$ to the topological GSD of the toric code $2^m$ on a torus of genus $g$, with the Betti number $B_1$ playing the role of $g$.

The Hamiltonian we dealt with so far is a special case of the general model with $m$-levels occupying the vertex and $n$-levels occupying the edge, denoted $V_m/L_n$, with $m \leq n$. In this case there are $k = \gcd(m,n)$ possible phases labelled by the elements of the group $\mathbb{Z}_k$. Each phase, labeled by $0 \leq \alpha \leq k - 1$, is described by the commuting projectors,

\[
A_i^{(\alpha)} = \frac{1}{n} \sum_{j=0}^{n-1} \left( \prod_{q} X_{v_{l_{i,j}}}^{Z_{v_{l_{i,j}}} \alpha} \prod_{q'} X_{v'_{l_{i,j}}}^{-1} \right), \quad C_i^{(\alpha)} = \frac{1}{m} \sum_{j=0}^{m-1} Z_{v_{l_{i,j}}} \alpha Z_{v'_{l_{i,j}}}^{-1} Z_{v'_{l_{i,j}}}^{-1},
\]

where $l_{i,j}$ is the link with incoming (outgoing) arrow to $v_i$ and $v_{l_{i,j}}$ is the vertex with incoming (outgoing) arrow $l_i$. The GSD on the closed chain for the Hamiltonian $H^{(\alpha)} = -\sum_i (A_i^{(\alpha)} + C_i^{(\alpha)})$ is [13]

\[
\text{GSD}^{\text{chain},(\alpha)}_{m,n} = \frac{\frac{mn}{k}^2}{\text{lcm}(\frac{mn}{k},m) \text{lcm}(\frac{mn}{k},n)}.
\]

When $m$ is a prime and $\gcd(m,n) = m$, we can show [13] GSD$^{\text{chain},(\alpha)}_{m,n} = n/m$ and GSD$^{G,(\alpha)}_{m,n} = (n/m)^{B_1}$ for any $\alpha > 0$. The number $n/m$ imposed by the local symmetry $Z_{n/m}$ generated by the local symmetry operator $X_{l_i}^{n/m}$ [13] and the global symmetry operators are

\[
\phi = \prod_i X_{v_i}, \quad \theta = \prod_i Z_{l_i}.
\]

Ground states $|G_{\beta}\rangle$ are obtained by applying the projector $\prod_i A_i$ on one of the seed states

\[
|S_{\beta}\rangle = X_{l_{i_{\beta}}}^{\beta} |0\rangle_v |0\rangle_{l_{i_{\beta}}},
\]

where $\beta$ is an integer which satisfies $0 \leq \beta < n/m$. Then the action of the global symmetry operators on them gives

\[
\phi |G_{\beta}\rangle = |G_{\beta}\rangle, \quad \theta |G_{\beta}\rangle = e^{2\pi i \frac{m}{n} \beta} |G_{\beta}\rangle.
\]

The formula (20) breaks down for $\alpha = 0$. This case represents a trivial phase, where the vertex and the link degrees of freedom completely decouple and we have an Ising-type Hamiltonian on each sector. The GSD equals GSD$^{\text{chain},(0)}_{m,n} = mn$, which comes from the $m$-fold and $n$-fold global symmetry breaking of the vertex and link sector, respectively.

When $m = n = 2$, these operators are unitarily equivalent to those found in the Hamiltonians describing cluster states as in Ref. [15, 16]. The SPT states considered by these authors, however, have a unique ground state on the closed chain.

In this paper we have proposed a class of Hamiltonians with $Z_m \times Z_n$ global symmetry and some additional local symmetries. The local symmetries give rise to a degeneracy in a way that string operators generate topologically distinct ground states in the toric code. Furthermore the ground state degeneracies possess a topological component when considered on a graph with a higher Betti number $B_1 > 1$, in analogy to the two-dimensional topological model defined on a surface with genus $g \geq 1$. Apart from being a model for a new phase of matter, our states being an extended version of SPT could find applications in measurement-based quantum computation [17–20].

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[1] A. Y. Kitaev, Physics-Uspekhi 44, 131 (2001).
[2] X. Chen, Z.-C. Gu, and X.-G. Wen, Phys. Rev. B 82, 155138 (2010).
[3] X. Chen, Z.-C. Gu, and X.-G. Wen, Phys. Rev. B 83, 035107 (2011).
[4] X. Chen, Z.-C. Gu, Z.-X. Liu, and X.-G. Wen, Phys. Rev. B 87, 155114 (2013).
[5] N. Schuch, D. Pérez-García, and I. Cirac, Phys. Rev. B 84, 165139 (2011).
[6] F. Pollmann, A. M. Turner, E. Berg, and M. Oshikawa, Phys. Rev. B 81, 064439 (2010).
[7] F. Pollmann, E. Berg, A. M. Turner, and M. Oshikawa, Phys. Rev. B 85, 075125 (2012).
[8] F. Pollmann and A. M. Turner, Phys. Rev. B 86, 125441 (2012).
[9] D. V. Else and C. Nayak, Phys. Rev. B 90, 235137 (2014).
[10] M. J. B. Ferreira, P. Padmanabhan, and P. Teotonio-Sobrinho, Journal of Physics A: Mathematical and Theoretical 47, 375204 (2014).
[11] M. J. B. Ferreira, J. P. I. Jimenez, P. Padmanabhan, and P. T. Sobrinho, Journal of Physics A: Mathematical and Theoretical 48, 485206 (2015).
[12] A. Kitaev, Annals of Physics 303, 2 (2003).
[13] See Supplementary Materials (SM).
[14] J. R. Munkres, Elements of Algebraic Topology (CRC Press, 2019).
[15] S. D. Geraedts and O. I. Motrunich, “Exact models for symmetry-protected topological phases in one dimension,” (2014), arXiv:1410.1580 [cond-mat.stat-mech].
[16] X. Chen, Y.-M. Lu, and A. Vishwanath, Nature Communications 5, 3507 (2014).
[17] D. V. Else, I. Schwarz, S. D. Bartlett, and A. C. Doherty, Phys. Rev. Lett. 108, 240505 (2012).
[18] A. Prakash and T.-C. Wei, Phys. Rev. A 92, 022310 (2015).
[19] J. Miller and A. Miyake, Phys. Rev. Lett. 114, 120506 (2015).
[20] D. T. Stephen, D.-S. Wang, A. Prakash, T.-C. Wei, and R. Raussendorf, Phys. Rev. Lett. 119, 010504 (2017).
Supplementary Information for “Frustration-free Hamiltonian with Topological Order on Graphs”

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In this supplementary information we describe several proofs and derivations which were omitted in the main text.

I. THE MODEL WITH ARBITRARY LINK ORIENTATIONS

The V2/L4 system described in the main text can be seen as a Z4 lattice gauge theory with qubit matter fields on the vertices. As with any lattice gauge theory formulation, the orientation of the links modify the operators in the theory. Thus when one arrowhead is towards, and another away from a vertex, the corresponding vertex operator takes the form

\[ A_i = \frac{1}{4}(1 + X_{l_i} x v_i X^3_{l_i} + X^2_{l_i} X^2_{l_i} + X^3_{l_i} x v_i X^3_{l_i}). \]  

(S1)

However, when both arrowheads are either towards or away from a vertex, we obtain

\[ A_i = \frac{1}{4}(1 + X_{l_i} x v_i X_{l_i} + X^2_{l_i} X^2_{l_i} + X^3_{l_i} x v_i X^3_{l_i}). \]  

(S2)

It turns out that for the V2/L4 model the link operator C_i does not depend on the direction of arrows. However this is not true for a more general V_m/L_n system. The form of the Hamiltonian, \( H = -\sum_{i=1}^{L} (A_i + C_i) \) stays the same regardless of the orientations assigned to the links. The global Z_2 x Z_4 symmetry is generated by \( \phi = \prod_i x v_i \) and \( \theta = \prod_i Z^{m_i} \) where \( m_i = 1 \) (\( m_i = 3 \)) when \( l_i \) is a right (left) arrow link for general choice of arrows. The local Z_2 symmetry is generated by the operator \( X^2_{l_i} \), independent of the orientation of the links. The GSD of this Hamiltonian can be obtained by the trace \( \text{Tr}[\prod_i A_i \prod_j C_j] = 2 \) independent of the lattice size and the choice of orientation.

II. PROOF OF DEGENERACY USING THE TRACE METHOD

The terms that lead to a non-zero trace in the expansion \( \prod_i A_i \prod_j C_j \) are \( \prod_i 1_{v_i} 1_{l_i} \) and \( \prod_i X^2_{l_i} 1_{v_i} X^2_{l_i} \) with \( 1_{v_i} \) and \( 1_{l_i} \) being the identity matrices on the \( v_i \) and \( l_i \) respectively, as the operators \( x v_i, z v_i, x v_i z v_i, X^3_{l_i} \), and \( X^2_{l_i} Z^2_{l_i} \) (\( m = 1, 2, 3 \)) are all traceless.

The proof of double degeneracy for the link excitation in a closed chain starts by observing that when a particular \( C_i \) has the eigenvalue 0, its orthogonal complement \( 1 - C_i \) takes the eigenvalue +1. The excited state degeneracy (ESD) consisting of \( m \) link excitations ESD_{chain} on a closed chain follows from the trace

\[ \text{ESD}_{v, l}^{\text{chain}} = \text{Tr} \left[ \prod_i A_i \prod_j C_j \prod_j (1 - C_j) \right] = 2. \]  

(S3)

Links with \( c_j = 1 \) are labeled with \( j \) and those with \( c_j = 0 \) by \( j' \) in the trace.

One can count the number of states with \( n \) vertex excitations (regardless of \( \omega \)) in a similar manner

\[ \text{ESD}_{v, l}^{\text{chain}} = \text{Tr} \left[ \prod_i A_i \prod_j (1 - A_j') \prod_j C_j \right] = (-1)^n + 3^n. \]  

(S4)

Vertices without the excitation are labeled by \( i \), and those with some excitation (i.e. \( \omega \neq 1 \)) by \( i' \). We have used

\[ \text{Esd}_{v, l}^{\text{chain}} = \text{Tr} \left[ \prod_i A_i \prod_j C_j \right] = \delta_{m, 0} + 4^m \]

where \( i \) spans the \( L - m \) non-excited vertices only. The operator \( 1 - A_j' \) takes into account all \( A_j(\omega) \) operators with \( \omega \neq 1 \).

We can extend the counting argument to the states with \( n \) vertex and \( m \) link excitations,

\[ \text{ESD}_{v, l, m}^{\text{chain}} = \text{Tr} \left[ \prod_i A_i \prod_j (1 - A_j') \prod_j C_j \prod_j (1 - C_j') \right] = (-1)^n + 3^n. \]  

(S5)

The result remains independent of \( m \) since ESD does not depend on the number of link excitation as already shown in Eq. (S3). Using this we can account for the full Hilbert space dimension of \( 8^L \) through

\[ \sum_{E=0}^{2L} \sum_{i=0}^{E} \binom{L}{E-i} \text{ESD}_{v, l, E-i} = 8^L. \]  

(S6)

III. LINK-PAIR EXCITATION

Contrary to \( X_{l_i} \) or \( X^3_{l_i} \) which excites a single link \( l_i \), the operator \( x v_i \) excites a pair of links to its left

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and right. From \([x_{v_i}, A_{l_i}]=0\) it follows \(x_{v_{i'}} |G_1⟩ = (\prod_i A_i) x_{v_{i'}} |S_1⟩\) and the new seed state \(x_{v_{i'}} |S_1⟩\) contains the configuration \(|0⟩_{v_{i'-1}} |0⟩_{l_{i'-1}} |1⟩_{v_{i'}} |0⟩_{l_{i'}} |0⟩_{v_{i'+1}}\) which has both \(c_{i'}=c_{i'}'=0\). This seed state is equivalent to seed states \(|0⟩_{v_{i'-1}} |3⟩_{l_{i'-1}} |0⟩_{v_{i'}} |1⟩_{l_{i'}} |0⟩_{v_{i'+1}}\) or \(|0⟩_{v_{i'-1}} |1⟩_{l_{i'-1}} |0⟩_{v_{i'}} |3⟩_{l_{i'}} |0⟩_{v_{i'+1}}\) by virtue of the identities (5) and (6) in the main paper. In other words, we have

\[
x_{v_{i'}} |G_1⟩ = X_{l_{i'-1}}^3 X_{l_{i'}} |G_1⟩ = X_{i'-1} X_{i'}^3 |G_1⟩.
\]

A similar reasoning applied to a string of \(x_v\) operators gives

\[
(\prod_{i=i_1}^{i_2} x_{v_i}) |G_1⟩ = X_{i_1-1}^3 X_{i_2} |G_1⟩ = X_{i_1-1} X_{i_2}^3 |G_1⟩.
\]

As the final expression shows, the string operator \(\prod_{i=i_1}^{i_2} x_{v_i}\) creates an excitation which is equivalent to the two single-link excitations taking place at the ends. The link-pair excitation energy remains independent of the length of the string.

\[
(Z_{l_{i'}})^h (Z_{l_{i'+1}})^{h'} |G_1⟩ = (\prod_{i\neq i', i'+1, i'+2} A_i) A_{i'} (i^3 h) A_{i'+1} (i^{h+3h'}) A_{i'+2} (i^{h'}) |S_1⟩,
\]

and one can see the string operator \(\prod_{i=i_1}^{i_2} Z_{l_i}^h\) \((h=1, 3)\) creates a \(A_{v_{i_1}} (i^3 h)\) and \(A_{v_{i_2}+1} (i^{h'})\) pair at the ends of the string. Additionally, one can prove that another string operator \(\prod_{i=i_1}^{i_2} Z_{l_i}^h (Z_{v_{i_1}+1} Z_{v_{i_2}+1}^{2h'})\) \((h=1, 3)\) creates a \(A_{v_{i_1}} (i^{3h})\) and \(A_{v_{i_2}+1} (i^{3h})\) pair. The elementary vertex-pair excitations with \(\omega = \pm 1\) are thus created by \(Z_{l_i}\) with other pair excitations described as composites of \(Z_l\) and \(Z_v\).

V. ACCOUNTING FOR DEGENERACY OF EXCITED STATES

There are several operators connecting the excited states at the same energy. Same as with the ground states, \(X_l^2\) always connects a pair of degenerate excited states. However, the degeneracy of the excited states is typically greater than two, and one needs other operators which connect the degenerate excited states. For instance, the following vertex-pair excitations are all degenerate:

\[
|ψ⟩ = (\prod_{i\neq i', i''} A_i) A_{i'} (i) A_{i''} (i) |S_1⟩
\]

\[
z_{v_{i'}} |ψ⟩ = (\prod_{i\neq i', i''} A_i) A_{i'} (-i) A_{i''} (i) |S_1⟩
\]

\[
z_{v_{i'}} |ψ⟩ = (\prod_{i\neq i', i''} A_i) A_{i'} (i) A_{i''} (-i) |S_1⟩
\]

\[
z_{v_{i'}} z_{v_{i'}} |ψ⟩ = (\prod_{i\neq i', i''} A_i) A_{i'} (-i) A_{i''} (-i) |S_1⟩.
\]

The degeneracy lies in replacing \(A_i (i)\) by \(A_i (-i)\) and vice versa.

Interchanging the vertex excitation with the link excitation can also produce degenerate states. For example, the following states are degenerate:

IV. VERTEX-PAIR EXCITATION

The isolated vertex excitation occurs when the operator \(z_{v_i}\) is applied on the ground state:

\[
z_{v_{i'}} |G_1⟩ = (\prod_{i\neq i'} A_i) A_{i'} (-1) |S_1⟩.
\]

We have used \(z_0 A_i (\omega) = A_i (-\omega) z_0\). The state obtained is a vertex excitation localized at \(i'\). The vertex-pair excitation occurs when the operator \((Z_l)^h (h=1, 2, 3)\) acts on the ground states,

\[
(Z_{l_{i'}})^h |G_1⟩ = (\prod_{i\neq i', i'+1} A_i) A_{i'} (i^3 h) A_{i'+1} (i^h) |S_1⟩.
\]

We have used the identity \((Z_l)^h X_l = i^h X_l Z_l^h\). Indeed from the right side of the equation it appears that a pair of vertex excitations has been created at vertices \(v_{i'}\) and \(v_{i'+1}\). For \(h=2\), however, one can show

\[
Z_{l_{i'}}^2 |G_1⟩ = z_{v_{i'}} z_{v_{i'+1}} |G_1⟩
\]

implying the vertex pair created by \(Z_{l_i}^2\) is nothing but two independent \(A_0 (-1)\) excitations. For \(h=1, 3\), on the other hand, the operators \(Z_l\) and \(Z_{l_i}^h\) do create pairs of vertex excitations that cannot be captured by a mere product of \(z_{v_i}\) operators [see Eq. (S7)]. Note also that \(Z_{l_{i'}}^2 \times |G_1⟩ = z_{v_{i'}} z_{v_{i'+1}} |G_1⟩\)

Acting with two operators \((Z_l)^h\) and \((Z_{l_{i'}})^{(h')}\) side by side on \(|G_1⟩\) gives
\[ |\psi\rangle = \prod_{i \neq i', i'' , i'' + 1} A_i A_{i'} (-1) A_{i''} (-i) A_{i'' + 1} (i) |S_1\rangle \]
\[ z_{v_{i'}} X_{l_j} |\psi\rangle = \prod_{i \neq i'' , i'' + 1} A_i A_{i''} (-i) A_{i'' + 1} (i) X_{l_j} \]
\[ \prod_{k \neq i''} Z_{l_k} X_{l_j} |\psi\rangle = \prod_{i \neq i''} A_i A_{i''} (-1) X_{l_j} |S_1\rangle . \]

The operator \( z_{v_{i'}} X_{l_j} \) eliminates an isolated vertex excitation and creates a link excitation instead. The operator \( \prod_{k \neq i''} Z_{l_k} X_{l_j} \) eliminates a vertex-pair excitation and creates two link excitations instead.

**VI. GROUND STATES OF OPEN CHAIN**

In the case of vertex/link termination, the Hamiltonian for an open chain is

\[ H = - \sum_{i=2}^{L} A_i - \sum_{j=1}^{L-1} C_j . \tag{S8} \]

one ground state of the open chain Hamiltonian in Eq. (S8) can be obtained by taking the seed state \( |S^{vl}_1\rangle = \otimes_i |0\rangle_{v_i} \otimes_j |0\rangle_{l_j} \) and projecting it with \( \prod_{i=2}^L A_i \) as \( |G^{vl}\rangle = \prod_{i=2}^L A_i |S^{vl}_1\rangle . \) To construct the remaining ground states we examine the action of the global \( \mathbb{Z}_2 \times \mathbb{Z}_4 \) symmetry on \( |G^{vl}\rangle . \) The global \( \mathbb{Z}_2 \) operator \( \phi \) can be decomposed into \( \phi = \phi_{vl} \phi_{bulk} \phi_{r} \), with

\[ \phi_{vl} = \prod_{j=2}^{L} (X^3_{l_{i-1}} x_{v_{j}} X_{l_{j}}) \]
\[ = \prod_{j=2}^{L} (A_{v_{j}} + i A_{v_{j}} (-1) - i A_{v_{j}} (-i)), \]
\[ \phi_{vl} = x_{v_{1}} X_{l_{1}}, \phi_{r} = X^3_{l_{L}} . \tag{S9} \]

One can show \( \phi_{bulk} \) leaves the ground state \( |G^{vl}\rangle \) invariant. The edge operators \( \phi_{vl} \) and \( \phi_{r} \) generate the cyclic group \( \mathbb{Z}_4 \) (not \( \mathbb{Z}_2 \)) and also commute with the open chain Hamiltonian in Eq. (S8). Thus we can create the additional ground states using \( \phi_{vl} \) and \( \phi_{r} \) as

\[ |G^{vl}_{m,n}\rangle = \left( \phi_{vl} \right)^m \left( \phi_{r} \right)^n |G^{vl}\rangle \]
\[ = \prod_{i=2}^{L} A_i |m\rangle_{v_i} |n\rangle_{l_{i}} \otimes \prod_{j=2}^{L-1} |0\rangle_{v_{j}} |0\rangle_{l_{j}} |0\rangle_{v_{L}} |n\rangle_{l_{L}} , \]

with \( m = 0, 1 \) and \( n = 0, 1, 2, 3 \). One can show that \( m = 2, 3 \) do not generate distinct new states. The states \( |G^{vl}_{m,n}\rangle \) coincide with \( |G^{vl}\rangle \) at \( m = n = 0 \). The states are orthogonal for different values of \( m \) and \( n \) and in total there are eight of them accounting for the GSD \( = 8 \).

We ask whether a similar manipulation with the global \( \mathbb{Z}_4 \) symmetry operator \( \theta \) can produce new ground states when acting on \( |G^{vl}\rangle \). As \( \theta \) commutes with the open chain Hamiltonian in Eq. (S8), it commutes with the projector \( \prod_{i=2}^L A_i \) and acts on the seed state \( |S^{vl}_1\rangle \), which is left invariant by \( \theta \) as we work in the basis where \( Z_l \) is diagonal on every link. The action on the open-chain ground state \( |G^{vl}\rangle \) by \( \theta \) leaves it unchanged.

An important aspect of the SPT phase is the so-called symmetry fractionalization at the edges. By this one means that the global symmetry acts linearly in the bulk, \( U(g_1 U(g_2) = U(g_1 g_2) \), but projectively at the edges, \( U(g_1 U(g_2) = \omega(g_1, g_2) U(g_1 g_2) \). Here \( \omega(g_1, g_2) \in U(1) \) is a phase and an element of the cohomology group \( H^2(G, U(1)) \), and \( U(g) \) is a unitary representation of the global symmetry \( G \). We can show that the global \( \mathbb{Z}_2 \times \mathbb{Z}_4 \) symmetry of this system acts projectively on the edges. To do so, first consider decomposing \( \theta^2 \) into

\[ \theta^2 = \theta^2_{vl} \theta^2_{bulk} \theta^2_{r} , \tag{S10} \]

where

\[ \theta^2_{vl} = \prod_{j=1}^{L-1} (z_{v_{j}} Z_{l_{j}} z_{v_{j+1}}), \]
\[ \theta^2_{r} = z_{v_{1}}, \theta^2_{r} = z_{v_{L}} Z_{l_{L}} . \tag{S11} \]

Clearly both of the edge operators generate a \( \mathbb{Z}_2 \) symmetry on the two edges as can be verified by squaring them to one and checking that they commute with the open-chain Hamiltonian. We have thus obtained an edge-localized \( \mathbb{Z}_4 \) operator \( \phi^0_{vl} (\phi^1_{vl}) \) and an edge-localized \( \mathbb{Z}_2 \) operator \( \theta^0_{vl} (\theta^1_{vl}) \). In a linear (non-projective) representation of the \( \mathbb{Z}_4 \times \mathbb{Z}_2 \) symmetry these operators would commute, e.g. \( \theta^0_{vl} \phi^0_{vl} = + \phi^0_{vl} \theta^0_{vl} \). However, one can easily show that the operators anti-commute

\[ \theta^0_{vl} \phi^0_{vl} = - \phi^0_{vl} \theta^0_{vl} \]
\[ \phi^1_{vl} \phi^1_{vl} = - \phi^1_{vl} \theta^1_{vl} \]
\[ \phi^1_{vl} \phi^0_{vl} = - \phi^1_{vl} \theta^0_{vl} , \tag{S12} \]

in an indication of the global symmetry \( \mathbb{Z}_2 \times \mathbb{Z}_4 \) being represented projectively at the edges.

In the case of an open chain with link terminations at either ends, we have \( L + 1 \) edges (labeled 0 through \( L \)) and \( L \) vertices (labeled 1 through \( L \)) [Fig. S1 (b)]. The Hamiltonian

\[ H = - \sum_{i=1}^{L} A_i - \sum_{j=1}^{L-1} C_j , \tag{S13} \]

misses the two link operators, \( C_0 \) and \( C_L \), but none of the vertex operators. There are \( \text{Tr} [\prod_{i=1}^{L} A_i [\prod_{j=1}^{L-1} C_j] = 8 \) degenerate ground states. Naively one might expect 16-fold
FIG. S1. Different boundary conditions for the open chain: (a) vertex/link, (b) link/link, and (c) vertex/vertex terminations. Labels for vertices and links are indicated.

The degeneracy arising from 4 states each at the two terminal links \( l = 0 \) and \( l = L \). We prove the counter-intuitive claim through explicit construction of all 8 ground states.

As with the case of vertex/link termination, we first decompose the global symmetry operators as bulk and edge components: \( \phi = \phi^B \phi^E \) and \( \theta_\ell = \theta^B \theta^E \).

The bulk and edge operators are

\[
\phi^B = \prod_{j=1}^{L} (X^3_{l-j, j}, X_{l}), \quad \phi^E = X^3_{l, L}
\]

\[
\theta^B = \prod_{j=0}^{L-1} (z_{v_j} Z^2_{l_j, j}, Z^2_{v_{l_j+1}}, \theta^E = Z_{v_{l_j+1}} Z^3_{l_j}, \theta^E = z_{v_{L}} Z^3_{L}.
\]

The edge operators anti-commute, e.g. \( \phi^B \phi^E = -\theta^B \theta^E \), signaling the symmetry fractionalization.

The ground states are produced by \( \phi^B \) and \( \phi^E \),

\[
|G^{m,n}_{m,n} \rangle = (\phi^E)^m (\phi^B)^n |G^0\rangle = \prod_{i=2}^{L} A_i |m\rangle_{v_i} \langle L-1 |0\rangle_{v_{L}} |0\rangle_{v_1} |n\rangle_{v_L} |n\rangle_{v_{L+1}},
\]

where \( |G^0\rangle = (\prod_j A_j) \otimes_j |0\rangle_{v_j} \otimes_e |0\rangle_{l_j} \). At first there seem 16 states for \( m, n = 0, 1, 2, 3 \), but one can show that only for \( m = 0, 1, 2, 3 \) and \( n = 0, 1 \), the projected states remain distinct.

For vertex/vertex termination [Fig. S1(c)], the Hamiltonian is

\[
H = -\sum_{i=1}^{L-1} A_i - \sum_{j=1}^{L} C_j.
\]

We find the degeneracy \( \text{GSD}^{vv} = 8 \) once again despite the two terminating vertices having only \( 2 \times 4 = 8 \) Hilbert space dimension. The edge operators are

\[
\phi^v_{i} = x_{v_i} X_{l_i}, \quad \phi^r_{i} = X^3_{l_i} x_{v_{L+i+1}}
\]

\[
\theta^v_{i} = z_{v_i}, \quad \theta^r_{i} = z_{v_{L+i}}.
\]

The explicit forms of edge operators are dictated by the nature of the edge termination. As in the previous cases the edge operators anticommute. The ground states are given by

\[
|G^{vv}_{m,n} \rangle = (\phi^v)^m (\phi^r)^n |G^0\rangle = \prod_{i=2}^{L} A_i |m\rangle_{v_i} \langle L-1 |0\rangle_{v_{L}} \sum_{i=2}^{L} |0\rangle_{v_{i}} |n\rangle_{v_L} |n\rangle_{v_{L+1}},
\]

and only the \( m = 0,1 \) and \( n = 0,1,2,3 \) states remain distinct after the projection.

**VII. EXCITED STATES OF OPEN CHAIN**

We can study the excitations of the open chain by considering the link and vertex excitations and their combinations. Excited states degeneracy (ESD) of link excitations is obtained from the trace

\[
\text{ESD}^{vl}_{m,n} = \text{Tr}\left[\prod_{i=2}^{L} A_j \prod_{j'=1}^{C_j} (1 - C_{j'})\right] = 8,
\]

independent of the number of excitations \( m \). The excited links are labeled by \( j' \) in the trace formula. ESD of \( n \) vertex excitations is computed using the trace formula,

\[
\text{ESD}^{v}_{n} = \text{Tr} \left[ \prod_{i} A_{n} \prod_{j=1}^{L-1} (1 - A_{j'}) \prod_{j' \neq j} C_{j'} \right] = 8 \times 3^n.
\]

The major difference between the vertex excitations of the closed chain case and those of the open chain case is that now there can be isolated vertex excitations for all the closed chain case and those of the open chain case is independent of the number of excitations \( m \), and only the \( m = 0,1 \) and \( n = 0,1,2,3 \) states remain distinct after the projection.

For the case where there are \( n \) vertex excitations and \( m \) link excitations,

\[
\text{ESD}^{vl}_{v,n,l,m} = \text{Tr} \left[ \prod_{i} A_{n} \prod_{j=1}^{L-1} (1 - A_{j'}) \prod_{j' \neq j} C_{j'} \right] = 8 \times 3^n,
\]

provides the number of combined excitations. We can account for the full Hilbert space through

\[
2L \sum_{E=0}^{E} \sum_{i=0}^{L-1} \left( L - 1 \right) \left( E - i \right) \text{ESD}^{vl}_{v,n,l,m} = 8^L.
\]

**VIII. GSD OF CLOSED GRAPH**

In the case of \( V_2/L_4 \), the vertex operator \( A^G \) where vertex \( v_i \) is connected to links with both incoming and outgoing arrows is,
$$A_i^G = \frac{1}{4} \left[ 1 + \left( \prod_j X_{i,j} \right) x_{v_l} \left( \prod_k X_{k,i}^3 \right) + \left( \prod_j X_{i,j}^2 \right) \left( \prod_k X_{k,i}^2 \right) + \left( \prod_j X_{i,j} \right) x_{v_l} \left( \prod_k X_{k,i} \right) \right]$$ (S16)

for a given graph $G$ ($j, k =$ index the incoming and outgoing arrows respectively). The Hamiltonian defined on $G$ consisting of $V$ vertices and $L$ links is

$$H_G = - \sum_{i=1}^V A_i^G - \sum_{j=1}^L C_j^G$$

and the ground state degeneracy is computed using the trace formula

$$\text{GSD}^G = \text{Tr} \left[ \prod_{i=1}^V A_i^G \prod_{j=1}^L C_j^G \right] = \frac{1}{2^L \times 4^V} \text{Tr} \left[ \prod_{i=1}^V 1_{v_i} \prod_{j=1}^L 1_{l_j} + \left( \prod_{j=1}^V X_{i,j}^2 \right) 1_{v_i} \left( \prod_{k=1}^L X_{k,i}^2 \right) \right] = 2^{L-V+1},$$ (S17)

where $L - V + 1 = B_1 = \text{rank}(H_1)$ is the number of holes or the number of independent cycles in the graph, $G$. The second term in the second line is the multiplication of $\left( \prod_j X_{i,j}^2 \right) 1_{v_i} \left( \prod_k X_{k,i}^2 \right)$ for every vertex operator.

We have $\prod_j^V \left( \prod_j X_{i,j}^2 \right) 1_{v_i} \left( \prod_k X_{k,i}^2 \right)$ equal to an identity because every link operator $X_{i,j}^2$ appears exactly twice in the product, and their product becomes the fourth power of $X$, which is identity.

The counting argument can be worked out for the general $V_m/L_n$ model on a closed chain. Vertex and link operators of the $V_m/L_n$ model are

$$A_i^{(\alpha)} = \frac{1}{m} \sum_{j=0}^{m-1} X_{i_{i-1}}^{\alpha j} X_{i_{i+1}}^{n-j},$$

$$C_i^{(\alpha)} = \frac{1}{m} \sum_{j=0}^{m-1} Z_{i_{i+1}}^{\alpha j} Z_{i_{i+1}}^{n-j}.$$ (S18)

To obtain nonzero trace for GSD, we must have $j$ such that $X_{i_{i+1}}^{\alpha j}$ and $Z_{i_{i+1}}^{\alpha j}$ become an identity. For such $j$ the trace of a product reduces to

$$\text{GSD}^{\text{chain},(\alpha)} = \text{Tr} \prod_{i=1}^L A_i^{(\alpha)} C_i^{(\alpha)} = \frac{1}{m^L \times n^L} \left( \sum_{j_1} \text{Tr} \left[ \prod_{i_1} X_{i_{i+1}}^{j_1} X_{i_{i+1}}^{n-j_1} \right] \right) \times \left( \sum_{j_2} \text{Tr} \left[ \prod_{i_2} Z_{i_{i+1}}^{j_2} Z_{i_{i+1}}^{n-j_2} \right] \right) = (\sum_{j_1} 1)(\sum_{j_2} 1)$$ (S19)

where the summation over $j_1$ and $j_2$ includes only those values of $j$ for which $X_{i_{i+1}}^{\alpha j_1}$ and $Z_{i_{i+1}}^{\alpha j_2}$ are identity. For specific $\alpha$, the smallest natural numbers for which $X_{i_{i+1}}^{\alpha j_1}$ and $Z_{i_{i+1}}^{\alpha j_2}$ are identity are given by $\text{lcm}(m \alpha / k, m)/(m \alpha / k)$ and $\text{lcm}(n \alpha / k, n)/(n \alpha / k)$, respectively. Since $0 \leq j_1 \leq n - 1$ and $0 \leq j_2 \leq m - 1$, the number of terms in the sum $\sum_{j_1} 1$ and $\sum_{j_2} 1$ in the final expression of Eq. (S19) are $\frac{\text{lcm}(m \alpha / k, m)}{\text{lcm}(m \alpha / k, n)}$ and $\frac{\text{lcm}(n \alpha / k, n)}{\text{lcm}(n \alpha / k, n)}$, respectively.

Therefore, GSD of a $V_m/L_n$ model on a closed chain is

$$\text{GSD}^{\text{chain},(\alpha)} = \frac{\left( \frac{\text{lcm}(m \alpha / k, m)}{\text{lcm}(m \alpha / k, n)} \right)^2}{\text{lcm}(m \alpha / k, m) \text{lcm}(n \alpha / k, n)}.$$

Under the condition of $m$ being a prime number and $\gcd(m, n) = m$, GSD is

$$\text{GSD}^{\text{chain},(\alpha)} = \frac{\left( \frac{m \alpha}{k} \right)^2}{\text{lcm}(m \alpha / k, m) \text{lcm}(n \alpha / k, n)} = \frac{n}{m}.$$  

Even for the closed graph case, when $m$ is a prime and $\gcd(m, n) = m$, GSD can be well accounted for as $\text{GSD}^{G,(\alpha)} = (n/m)^{B_1}$. In this case, $\frac{m \alpha}{k \text{lcm}(m \alpha / k, n)} = 1$ and the sum $\sum_{j_1} 1$ in Eq. (S19) reduces to one. The trace formula for the GSD becomes
GSD\textsuperscript{G,\(\alpha\)}\textsubscript{m,n} = \text{Tr}\left[ \prod_i A_i^{G,\(\alpha\)} C_i^{G,\(\alpha\)} \right] = \frac{1}{m^n \times n^V} \sum_j \text{Tr}\left[ \prod_i (\prod_q X_q^{j}) (\prod_q X_{q',j}^{n-j}) \right] = \frac{m^V \times n^L}{m^L \times n^V} (\sum_j 1) = \left( \frac{n}{m} \right)^{L-V+1} = \left( \frac{n}{m} \right)^B_1. \tag{S20}

\section*{IX. THE TRIVIAL PHASE}

In the case of \(V_m/L_n\), trivial phase is labelled by \(\alpha = 0\) of group \(H^2(\mathbb{Z}_m \times \mathbb{Z}_n, U(1)) = \mathbb{Z}_k\) where \(k = \gcd(m,n)\). Vertex and link operator corresponding to the trivial phase on the closed graph \(G\) are

\[ A_i^{(\alpha)} = \frac{1}{n} \sum_{j=0}^{m-1} (\prod_q X_q^j) (\prod_{q'} X_{q',j}^{n-j}), \]

\[ C_i^{(\alpha)} = \frac{1}{m} \sum_{j=0}^{n-1} Z_i^{j} Z_{i+1}^{m-j}, \tag{S21} \]

respectively. Notice that the vertex operators act non-trivially only on the links and the link operators act non-trivially only on the vertices. In other words the link and vertex degrees of freedom are decoupled.

We can compute GSD of closed chain by using the trace formula \(\text{GSD}^{\text{chain}} = \text{Tr}[\prod A_i^{(0)} \prod_j C_j^{(0)}] = mn\) since \(\prod X_q^j \prod_{q'} X_{q',j}^{n-j} \) and \(\prod Z_i^j Z_{i+1}^{m-j}\) for all \(j\) are identity. These ground states are accounted for by the global symmetry of the system.

\section*{X. GRAPH HOMOLOGY THEORY}

Algebraic topology helps distinguish topological spaces systematically. The fundamental group and higher homotopy groups classify topological spaces by characterizing the \textit{holes} of different dimensions in these spaces but they quickly become hard to interpret as we increase the dimension of the topological space. A commutative alternative to homotopy is given by \textit{homology} theory which we are concerned with.

If \(X\) is a topological space we can construct a sequence of groups, \(H_n(X)\) for \(n = 0, 1, 2, \cdots\), termed as the homology groups. These are commutative and they measure the number of \(n\) dimensional holes in \(X\). We will illustrate these groups with the simplest example of \(X\), a graph. Consider the graph shown in Fig. S2. This graph is made up of three vertices \(x, y, z\) and four links \(a, b, c, d\). The links are directed as shown in Fig. S2. The vertices are also called 0-simplices and the links 1-simplices. Together the graph \(X\) is a simplicial complex. Naturally higher dimensional surfaces correspond to higher simplices but here we restrict ourselves to 0 and 1 dimensional simplices as we are interested in graphs.

The set of vertices is denoted as \(C_0\) and is the free abelian group generated by the vertices \(x, y, z\). A general element of \(C_0\) is \(\alpha x + \beta y + \gamma z\) with the coefficients \(\alpha, \beta, \gamma\) being numbers in some field which we take to be the integers, \(\mathbb{Z}\). In a similar manner the set of links is denoted \(C_1\) and is the free abelian group generated by \(a, b, c, d\). In the literature the elements of \(C_0\) and \(C_1\) are called 0 and 1 dimensional chains respectively.

We now consider a group homomorphism,

\[ C_1 \xrightarrow{\partial_1} C_0, \]

which is called the \textit{boundary map}. As the name implies it maps the link in \(C_1\) to its boundary in \(C_0\). For the case of the graph \(X\) in Fig. S2 we obtain

\[ \partial_1(a) = y - x, \quad \partial_1(b) = z - y, \quad \partial_1(c) = x - z, \quad \partial_1(d) = d - x. \tag{S22} \]

Clearly the 0-chains \(y - x, z - y\) etc are the boundaries of the 1-chains or links. We can now think of special 1-chains called \textit{cycles} whose boundary is 0. For the graph \(X\) in Fig. S2 we obtain three cycles, \(a + b + c, a + b + d\) and \(c - d\), each of whose boundaries evaluate to 0. A crucial property of the boundary map,

\[ \partial^2 = 0, \tag{S23} \]

can be verified by evaluating \(\partial_1^2\) in \(X\).

Consider the \textit{short exact sequence}

\[ 0 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0. \tag{S24} \]

The homology groups \(H_n\) are defined as

\[ H_n(X) = Z_n/B_n, \tag{S25} \]

where \(Z_n\) are the group of cycles and \(B_n\) are the group of boundaries. More precisely \(Z_n = \text{Ker}(\partial_n)\) and \(B_n = \text{Im}(\partial_{n+1})\). The quotient \(Z_n/B_n\) collects \(n\)-chain cycles...
that are not boundaries of \( n + 1 \)-chains. Thus it is the
group generated by the independent \( n \)-dimensional cycles
or holes.

Thus for graph \( X \) in Fig. S2 we can compute \( H_1 = \text{Ker}(\partial_1)/\text{Im}(\partial_2) \). \( \text{Ker}(\partial_1) = \mathbb{Z} \oplus \mathbb{Z} \) is generated by \( a + b + c \) and \( a + b + d \) and the \( \text{Im}(\partial_2) = 0 \) as there are no 2-chains for the graph \( X \). Thus \( H_1(X) = \mathbb{Z} \oplus \mathbb{Z} \) essentially enumerates the number of independent 1-dimensional cy-
cles in \( X \), which is precisely 2. The rank of \( H_1(X) \) is
known as the first Betti number, \( B_1 \) which is 2 for the
graph \( X \).

For a general graph with \( L \) links and \( V \) vertices \( H_1 = \oplus_2^{L-V+1} \) and hence \( B_1 = L - V + 1 \). From a familiar
result in graph theory we identify \( L - V + 1 \) to be the
number of independent cycles of the graph under consid-
eration.

We can also compute \( H_0(X) = \text{Ker}(\partial_0)/\text{Im}(\partial_1) \) for
the graph \( X \) in Fig. S2. Now \( \text{Ker}(\partial_0) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \) is
generated by \( x, y, z \) and the \( \text{Im}(\partial_1) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \) is
generated by \( y - x, z - y, x - z \). To take the quotient
we equate each element in \( \text{Im}(\partial_1) \) to 0 which implies
\( H_0(X) = \mathbb{Z} \). Using this example we can convince our-
selves that all vertices in a connected component of a
general graph will reduce to a single vertex. Thus \( H_0 \)
for a general graph measures the number of independent
components of the graph. The rank of \( H_0 \) is denoted as
the zeroth Betti number and it counts the number of in-
dependent components. Clearly for the graph \( X \) in Fig.
S2, \( B_0(X) = 1 \).