Pseudoduality in Sigma Models

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Abstract

We revisit classical “on shell” duality, i.e., pseudoduality, in two dimensional conformally invariant classical sigma models and find some new interesting results. We show that any two sigma models that are “on shell” duals have opposite 1-loop renormalization group beta functions because of the integrability conditions for the pseudoduality transformation. A new result states for any two compact Lie groups of the same dimension there is a natural pseudoduality transformation that maps classical solutions of the WZW model on the first group into solutions of the WZW model on the second group. This transformation preserves the stress-energy tensor. The two groups can be non-isomorphic such as $B_l$ and $C_l$ in the Cartan notation. This transformation can be used for a new construction of non-local conserved currents. The new non-local currents on $G$ depend on the choice of dual group $\tilde{G}$.

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1 Introduction

In this article we generalize the discussion in [1] of classical “on shell” duality, also called pseudoduality, to the case where the nonlinear sigma model has “torsion”, see e.g., [4, 5, 6, 7, 8]. An early example is the pseudoduality between the non-linear sigma model on a group and the pseudochiral model discovered by Zakharov and Mikhailov [9]. For notational conventions and for a more complete set of references on duality, especially “off shell” duality inspired by string theory, see [1].

We take spacetime $\Sigma$ to be two dimensional Minkowski space. The sigma model with target space $M$, metric $g$ and 2-form $B$ will be denoted by $(M, g, B)$ and has lagrangian

$$L = \frac{1}{2} g_{ij}(x) \left( \frac{\partial x^i}{\partial \tau} \frac{\partial x^j}{\partial \sigma} - \frac{\partial x^i}{\partial \sigma} \frac{\partial x^j}{\partial \tau} \right) + B_{ij}(x) \frac{\partial x^i}{\partial \tau} \frac{\partial x^j}{\partial \sigma}, \quad (1.1)$$

where $x: \Sigma \to M$ and the closed 3-form $H$ is defined by $H = dB$. This theory is classically conformally invariant. Our goal is to see if we can relate solutions of the equations of motion of a sigma model $(M, g, B)$ to the solutions of the equations of motions of a different sigma model $(\widetilde{M}, \widetilde{g}, \widetilde{B})$.

Our default scenario is general riemannian manifolds but we often specialize to the case of Lie groups. Overall, the methods we use are differential geometric ones that expand on ideas in [1, 10, 11]. The bundle of orthonormal frames, the Cartan structural equations and the exterior differential calculus play a central role. Early work on using differential form methods to study sigma models may be found in [12, 13]. In Section 4.1 we show that a consequence of the integrability conditions for the existence of the pseudoduality transformation is that any two sigma models that are classically pseudodual have opposite 1-loop renormalization group beta functions.

Some of the most interesting explicit results involve specializing to Lie groups and especially the classical “strict” WZW model [14]. This is the model with the Wess-Zumino term normalized so that equations of motion are $\partial_- (g^{-1} \partial_+ g) = 0$. Given any two compact Lie groups of the same dimension, we show that there is a duality transformation that maps solutions of the equations of motion of the first strict WZW model into solutions of the equations of motions of the second strict WZW model. The exposition of these specific results in Sections 6.1, 6.2, 6.3 and 6.4 is self-contained and requires very little from the rest of the paper.

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1 This term was introduced in [2] to distinguish from true “off shell” duality where the duality transformation is canonical [3, 2, 4].
We revisit some ideas of Braaten, Curtright and Zachos [8] on the geometry of sigma models and amplify and clarify some issues in Section 7.1. We also revisit and generalize some ideas presented by Ivanov [15] on duality in sigma models with target spaces that are related to Lie groups where he presents two lines of investigation. The first has to do with what could be called Pohlmeyer type duality which is mostly tangential to our discussion. In the Pohlmeyer type duality, the “duality” equations are schematically of the type \( \partial_\pm \tilde{x} = e^{\pm \lambda} \partial_\pm x \) where \( \lambda \) is a parameter. A systematic study of these relations leads, for example, to an infinite number of conservation laws [16, 17, 18]. Here we adapt this construction to our case by observing that an initial condition in the solution of an ordinary differential equation plays a role similar to \( \lambda \) and we use this to generate an infinite number of conservation laws in Section 6.4.

The second line of investigation deals with pseudoduality where the pseudoduality equations are schematically of the form \( \partial_\pm \tilde{x} = \pm \partial_\pm x \). Here, we are interested in this second type of duality. Ivanov studied sigma models associated with Lie groups but his formalism only allowed dual models with \( \tilde{H} = 0 \). Our generalization of Ivanov’s method to general riemannian manifolds in Section 7 will explain clearly why he could only discuss the case \( \tilde{H} = 0 \) and also makes connection to results in [8].

This article is organized as follows. The basic framework is established in Sections 2 and 3. The main result of this paper is eq. (3.13) that relates the metrics and 3-forms on the respective manifolds. The integrability conditions for pseudoduality are discussed in Section 4 along with the connection to the renormalization group. A variety of explicit examples are discussed in Sections 5 and 6. Section 7 studies the differential geometry of some naturally occurring connections. The Appendices provide some background material.

## 2 The Framework

The formulation of the general duality transformation is best done in the bundle of orthonormal coframes. For a brief review of \( G \)-structures see Appendix A. The reader may want to look in [19] and study their discussion about isometries between Riemannian manifolds and try to understand the idea behind E. Cartan’s technique of the graph [20]. We first discuss the problem locally and see how it becomes simpler and more natural in the bundle of orthonormal coframes. We begin with local discussion of pseudoduality on \( M \) and \( \tilde{M} \) and then show how to lift these concepts to the orthonormal coframe bundles \( \text{SO}(M) \) and \( \text{SO}(\tilde{M}) \). A more mathematically rigorous discussion would entail a discussion of jet bundles that we prefer to avoid.
Let $V$ and $\tilde{V}$ be local neighborhoods respectively in $M$ and $\tilde{M}$. In these neighborhoods choose local orthonormal coframes $\{\omega^i_V\}$ and $\{\tilde{\omega}^i_{\tilde{V}}\}$. The $\sigma^\pm$ derivatives of the sigma model maps $x : \Sigma \to M$ and $\tilde{x} : \Sigma \to \tilde{M}$ are given by

$$\omega^i_V = (x_V)^i_a d\sigma^a \quad \text{and} \quad \tilde{\omega}^i_{\tilde{V}} = (\tilde{x}_{\tilde{V}})^i_a d\sigma^a.$$  \hfill (2.1)

The pseudoduality equations \cite{1} are

$$\tilde{x}^i_{\tilde{V}}(\sigma^\pm) = \pm T^\pm(\sigma)(x^i_V(\sigma^\pm),$$  \hfill (2.2)

where the matrices $T^\pm(\sigma)$ are in $SO(n)$. In this article we only treat the case $T^+ = T^-$. Over the neighborhood $V \subset M$ the bundle of coframes $SO(M)$ is locally $V \times SO(n)$. A point may be given coordinates $(x, R^V)$ where $R^V \in SO(n)$ is the matrix that describes the coframe $\omega^i_V$ relative to a fiducial coframe. We saw in Appendix \cite{1} that $\omega = R^V \omega^i_V$ is the canonical 1-form on $SO(M)$ and it is globally defined. The coframe bundle $SO(M)$ has a global coframing given by the canonical 1-forms $\omega^i$ and by the globally defined torsion free riemannian connection 1-forms $\omega^i_j, \omega^i_j = -\omega^j_i$. These satisfy the Cartan structural equations

$$d\omega^i = -\omega^i_j \wedge \omega^j,$$  \hfill (2.3)

$$d\omega^i_j = -\omega^i_k \wedge \omega^k_j + \frac{1}{2} R^i_{kjl} \omega^k \wedge \omega^l,$$  \hfill (2.4)

where $R^i_{kjl}$ are the Riemann curvature functions on the orthonormal coframe bundle\cite{1}. We emphasize that the set $\{\omega^i, \omega^k_j\}$ gives a global coframing of the coframe bundle $SO(M)$. Lastly we point out that if $(\omega^i_V)_{ij}$ is the expression for the riemannian connection in a local coframe $\omega^i_V$ in $V \subset M$ then the globally defined $\omega^i_{ij}$ on $SO(M)$ is locally given by

$$\omega^i_{ij} = (R^i_V)^k_l(\omega^i_V)_k^l(R^i_V)_l^{-1}_j - (dR^i_V)^k_l(R^i_V)_l^{-1}_j.$$  \hfill (2.5)

We also remind the reader that the local connection coefficients $(\omega^i_V)_{ijk}$ are given by

$$(\omega^i_V)_{ijk} = (\omega^i_V)_{ijk} \omega^k_V.$$  \hfill (2.6)

Up in the coframe bundle, $\omega^i$ and $\omega^k_j$ are linearly independent and there is no relation analogous to (2.6).

If we look at (2.1) we immediately see that $x^i_a = (R^i_V)^j(x_V)^i_a$ are globally defined functions on the coframe bundle $SO(M)$. Likewise we do a similar construction in $\tilde{M}$. In fact we see that

$$\omega^i = x^i_a d\sigma^a \quad \text{and} \quad \tilde{\omega}^i = \tilde{x}^i_a d\sigma^a.$$  \hfill (2.7)

\footnote{The Riemann curvature tensor on $M$ is equivalent to the globally defined curvature functions on $SO(M)$. In general, tensors on the base become functions on the coframe bundle.}
These are globally defined equations on the bundles of orthonormal coframes.

Let us do a warm-up first by describing the isometry problem in this framework. We are interested in finding an orientation preserving isometry between $M$ and $\tilde{M}$. We know that locally we need the existence of a special orthogonal matrix valued function $T : V \to \text{SO}(n)$ such that $\tilde{\omega}_V = T_V \omega_V$. The isometry problem is formulated “upstairs” by asking whether we can solve the pfaffian system of equations $\tilde{\omega}_i = \omega_i$ on $\text{SO}(M) \times \text{SO}(\tilde{M})$. The reason it that locally these equations may be written as $\tilde{R}_V \tilde{\omega}_V = R_V \omega_V$ and we see that a judicious choice of coframes will give $T_V = (\tilde{R}_V)^{-1} R_V$.

## 3 The Pseudoduality Condition

In this article we discuss the special case of equations (2.2) where $T_+ = T_-$. Instead of thinking of $x : \Sigma \to M$ you should think of a lift $X : \Sigma \to \text{SO}(M)$. Thus we have the pullbacks

$$X^* \omega^i = x^i_a d\sigma^a \quad \text{and} \quad X^* \omega_{ij} = \omega_{ija} d\sigma^a$$

that define the derivatives. From now on following the convention used in exterior differential systems [19] we assume the pullback is implicit, e.g., $\omega^i = x^i_a d\sigma^a$. Note that on $\text{SO}(M)$ the 1-forms $\omega^i$ and $\omega_{jk}$ are linearly independent so there is no relation such as $\omega_{jk} = \Gamma_{ijk} \omega^i$ on $\text{SO}(M)$. If you use a (local) section $s : M \to \text{SO}(M)$ to pullback $\omega^i, \omega_{jk}$ to $M$ then you would find $s^* \omega_{jk} = \Gamma_{ijk} s^* \omega^i$ where $\Gamma_{ijk}$ are functions on $M$. Define the second derivatives of $x^i$ by

$$dx^i_a + \omega_{ij}x^j_a = x^i_{ab} d\sigma^b.$$  \hspace{1cm} (3.2)

By taking the exterior derivative of the first of (3.1) you learn that $x^i_{ab} = x^i_{ba}$. Locally on $V \subset M$ we have the 2-form $B_V$ in the action (1.1) with $H_V = dB_V$. The 3-form is lifted to $\text{SO}(M)$ where it defines functions $H_{ijk}$ such that $H = \frac{1}{3!} H_{ijk} \omega^i \wedge \omega^j \wedge \omega^k$. The equations of motions may be written on the bundle as

$$x^k_{+-} = \frac{1}{2} H_{kij} x^i_+ x^j_-.$$  \hspace{1cm} (3.3)

The stress energy tensor for the sigma model $(M,g,B)$ is given by

$$\Theta_{+-} = 0, \quad \Theta_{++} = x^i_+ x^i_+ \quad \text{and} \quad \Theta_{--} = x^i_- x^i_-.$$  \hspace{1cm} (3.4)

Of course there are similar equations on $\text{SO}(\tilde{M})$. 


Analogous to the isometry problem, the pseudoduality equations on the bundle of orthonormal coframes become
\[ \tilde{x}_i^\pm = \pm x_i^\pm. \] (3.5)

An important feature of these pseudoduality equations is that they preserve the stress-energy tensor. Taking the exterior derivative of the above and using (3.2) we see that
\[ -\tilde{\omega} x_\pm + \tilde{x}_+ a d\sigma^a = \mp \omega x_\pm \pm x_\pm a d\sigma^a. \]

If we use the duality equations (3.5) we have
\[ \mp \tilde{\omega} x_\pm + \tilde{x}_\pm a d\sigma^a = \mp \omega x_\pm \pm x_\pm a d\sigma^a. \]

A little algebra shows that
\[ \tilde{x}_\pm a d\sigma^a = \pm (-\omega + \tilde{\omega}) x_\pm \pm x_\pm a d\sigma^a. \]

We wish to isolate the integrability conditions so wedge the above with $d\sigma^\pm$.
\[ \tilde{x}_\pm d\sigma^\mp \wedge d\sigma^\pm = \pm (-\omega + \tilde{\omega}) x_\pm d\sigma^\mp \wedge d\sigma^\pm. \]

We have two equations
\[ \tilde{x}_{+-} d\sigma^- \wedge d\sigma^+ = +(-\omega + \tilde{\omega}) x_+ \wedge d\sigma^+ + x_+ d\sigma^- \wedge d\sigma^+, \]
\[ \tilde{x}_{-+} d\sigma^+ \wedge d\sigma^- = -(\omega + \tilde{\omega}) x_- \wedge d\sigma^- - x_- d\sigma^+ \wedge d\sigma^-. \]

In principle we wish that the integrability conditions $\tilde{x}_{+-} = \tilde{x}_{-+}$ are satisfied if the equations of motion (3.3) hold. Subsequently we would like that this implies equations of motion for $\tilde{x}$. We might as well substitute the equations of motion for $x$ and $\tilde{x}$ directly into the above and find
\[ -\frac{1}{2} \tilde{H}_{kij} \tilde{x}^i_+ \tilde{x}^j_- d\sigma^- \wedge d\sigma^+ = +(-\omega + \tilde{\omega})_{kj} x^i_+ \wedge d\sigma^+ \frac{1}{2} H_{kij} x^i_+ x^j_- d\sigma^- \wedge d\sigma^+, \]
\[ -\frac{1}{2} \tilde{H}_{kij} \tilde{x}^i_+ \tilde{x}^j_- d\sigma^+ \wedge d\sigma^- = -(\omega + \tilde{\omega})_{kj} x^i_- \wedge d\sigma^- \frac{1}{2} H_{kij} x^i_+ x^j_- d\sigma^+ \wedge d\sigma^-.

Next we selectively insert the pseudoduality equations (3.5) into the above
\[ -\frac{1}{2} \tilde{H}_{kij} x^i_+ \tilde{x}^j_- d\sigma^- \wedge d\sigma^+ = +(-\omega + \tilde{\omega})_{kj} x^i_+ \wedge d\sigma^+ \frac{1}{2} H_{kij} x^i_+ x^j_- d\sigma^- \wedge d\sigma^+, \]
\[ +\frac{1}{2} \tilde{H}_{kij} \tilde{x}^i_+ \tilde{x}^j_- d\sigma^+ \wedge d\sigma^- = -(\omega + \tilde{\omega})_{kj} x^j_- \wedge d\sigma^- \frac{1}{2} H_{kij} x^i_+ x^j_- d\sigma^+ \wedge d\sigma^-. \] (3.6)
Let us first concentrate on the first equation above. We can choose \( x^i_+ \) to be arbitrary at any \( \sigma \) so we conclude that

\[
-\frac{1}{2} \tilde{H}_{kij} \tilde{x}^j_- d\sigma^- \wedge d\sigma^+ = + (\tilde{\omega} - \omega)_{ki} \wedge d\sigma^+ - \frac{1}{2} \tilde{H}_{kij} x^j_- d\sigma^+ \wedge d\sigma^- .
\]

Next we substitute \((\omega^j - x^j_+d\sigma^+)\) for \( x^j_-d\sigma^- \) and similarly for \( \tilde{x}^j_-d\sigma^- \). This leads to

\[
\left( \tilde{\omega}_{ki} - \omega_{ki} + \frac{1}{2} \tilde{H}_{kij} \tilde{\omega}^j - \frac{1}{2} H_{kij} \omega^j \right) \wedge d\sigma^+ = 0 .
\]

We see that there exists a tensor \( U_{ki+} \), antisymmetric under \( k \leftrightarrow i \), such that

\[
\left( \tilde{\omega}_{ki} - \omega_{ki} + \frac{1}{2} \tilde{H}_{kij} \tilde{\omega}^j - \frac{1}{2} H_{kij} \omega^j \right) = U_{ki+}d\sigma^+ . \tag{3.8}
\]

Next we concentrate on \((3.7)\) and observe that \( x^j_- \) may be chosen arbitrarily at any \( \sigma \). This leads to

\[
\left( \tilde{\omega}^j_- - \omega^j_- + \frac{1}{2} \tilde{H}_{kij} \tilde{\omega}^j + \frac{1}{2} H_{kij} \omega^j \right) \wedge d\sigma^- = 0 .
\]

Next we substitute \((\omega^i - x^i_-d\sigma^-)\) for \( x^i_+d\sigma^+ \) and similarly for \( \tilde{x}^i_+d\sigma^+ \) with result

\[
\left( \tilde{\omega}_{ki} - \omega_{ki} - \frac{1}{2} \tilde{H}_{kij} \tilde{\omega}^j + \frac{1}{2} H_{kij} \omega^j \right) = U_{ki-}d\sigma^- . \tag{3.9}
\]

Adding and subtracting \((3.8)\) and \((3.9)\) we see that

\[
\tilde{\omega}_{ki} - \omega_{ki} = \frac{1}{2} (U_{ki+}d\sigma^+ + U_{ki-}d\sigma^- ) . \tag{3.10}
\]

\[
\frac{1}{2} \tilde{H}_{kij} \tilde{\omega}^j - \frac{1}{2} H_{kij} \omega^j = \frac{1}{2} (U_{ki+}d\sigma^+ - U_{ki-}d\sigma^- ) . \tag{3.11}
\]

The latter equation above may be solved by substituting \((2.7)\) and finding

\[
U_{ki+}d\sigma^+ + U_{ki-}d\sigma^- = \tilde{H}_{kij} (\tilde{x}^j_+d\sigma^+ - \tilde{x}^j_-d\sigma^- ) - H_{kij} (x^j_+d\sigma^+ - x^j_-d\sigma^- ) ,
\]

\[
= \tilde{H}_{kij} \omega^j - H_{kij} \tilde{\omega}^j . \tag{3.12}
\]

To obtain the bottom equation we used the duality relation \((3.3)\) and also \((2.7)\). We conclude that

\[
\tilde{\omega}_{ki} - \omega_{ki} = \frac{1}{2} \tilde{H}_{kij} \omega^j - \frac{1}{2} H_{kij} \tilde{\omega}^j . \tag{3.13}
\]
These Pfaffian equations are the central result of this paper. They are the basic integrability condition for the pseudoduality equations (3.3). We will later discuss specific results that follow from applying them to a variety of examples. These Pfaffian equations along with (3.1) and the corresponding “tilded” equations should be viewed as defining a distribution $\mathcal{D}$ in $\Sigma \times (\text{SO}(M) \times \text{SO}(\tilde{M}))$. The statement that the coordinates $(\sigma^+, \sigma^-)$ on $\Sigma$ are the independent variables tells us that we should look for integrable 2-dimensional distributions that are solutions of the above where $d\sigma^- \wedge d\sigma^+$ does not vanish when restricted to this 2-dimensional distribution.

4 Integrability Conditions

Next we look for the conditions on the distribution defined by (3.13) that allow for integrable 2 dimensional manifolds (worldsheets) when the equations of motion hold. Taking the exterior derivative of (3.13) and using the Cartan structural equations leads to the following

$$\tilde{\nabla}_i \tilde{H}_{ij} + \tilde{\nabla}_j \tilde{H}_{kij} = - (\nabla_k H_{lij} + \nabla_l H_{kij}), \quad (4.1)$$

$$\tilde{R}_{ijkl} - \frac{1}{2} H_{ijm} \tilde{H}_{mkl} - \frac{1}{4} \left( \tilde{H}_{imk} \tilde{H}_{mjl} - \tilde{H}_{iml} \tilde{H}_{mjk} \right) = \tilde{R}_{ijkl} - \frac{1}{2} H_{ijm} H_{mkl} - \frac{1}{4} \left( H_{imk} H_{mjl} - H_{iml} H_{mjk} \right). \quad (4.2)$$

In the above $\nabla$ and $\tilde{\nabla}$ are respectively the covariant derivatives with respect to the riemannian connections $\omega_{ij}$ and $\tilde{\omega}_{ij}$. The reader is reminded that since $H$ and $\tilde{H}$ are closed 3-forms we have

$$\nabla_i H_{jkl} - \nabla_j H_{kli} + \nabla_k H_{lij} - \nabla_l H_{ijk} = 0, \quad (4.3)$$

$$\tilde{\nabla}_i \tilde{H}_{jkl} - \tilde{\nabla}_j \tilde{H}_{kli} + \tilde{\nabla}_k \tilde{H}_{lij} - \tilde{\nabla}_l \tilde{H}_{ijk} = 0. \quad (4.3)$$

Combining (4.2) with the two equations above leads to the conclusion

$$\tilde{\nabla}_i \tilde{H}_{jkl} = - \nabla_i H_{jkl}. \quad (4.4)$$

Summarizing we see that the integrability equations for solving the Pfaffian equations (3.13)

$$\tilde{\omega}_{ij} - \frac{1}{2} \tilde{H}_{ijk} \omega^k = \omega_{ij} - \frac{1}{2} H_{ijk} \tilde{\omega}^k. \quad (4.4)$$

3Here we use distribution in the differential geometric sense, see [20]. Said succinctly, a $k$-dimensional distribution on a manifold $N$ is a rank $k$ sub-bundle of the tangent bundle $TN$, i.e., a $k$-plane field on $N$. 

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are
\begin{align*}
\tilde{R}_{ijkl} &= -\frac{1}{2} H_{ijm} \tilde{H}_{mkl} - \frac{1}{4} \left( \tilde{H}_{imk} \tilde{H}_{mjl} - \tilde{H}_{iml} \tilde{H}_{mjk} \right) \\
&= - \left[ R_{ijkl} - \frac{1}{2} H_{ijm} H_{mkl} - \frac{1}{4} (H_{imk} H_{mjl} - H_{iml} H_{mjk}) \right], \quad (4.5)
\end{align*}
and possible new integrability conditions found by taking the exterior derivatives of the integrability equations above. Notice that the right hand side of the above is not the curvature of a connection with torsion, see Appendix C.

In the first paper [1] the condition that the spaces be symmetric spaces arose from differentiating the above. Here it is convenient to define a tensor $S$ by
\begin{equation}
S_{ijkl} = R_{ijkl} - \frac{1}{2} H_{ijm} H_{mkl} - \frac{1}{4} (H_{imk} H_{mjl} - H_{iml} H_{mjk}),
\end{equation}
and similarly $\tilde{S}$. The covariant differential of $S$ is given by
\begin{equation}
\nabla S = dS + \omega \otimes S,
\end{equation}
where $\omega$ is an abstract notation for the $\mathfrak{so}(n)$-valued connection 2-form and $\otimes$ denotes the action of $\mathfrak{so}(n)$ on $S$. A brief computation shows that
\begin{align*}
\nabla_k S &= \frac{1}{2} \tilde{H}_{\bullet k} \otimes \tilde{S}, \\
\tilde{\nabla}_k \tilde{S} &= \frac{1}{2} H_{\bullet k} \otimes S.
\end{align*}

If $H = \tilde{H} = 0$ then $S = R$, $\tilde{S} = \tilde{R}$ then we recover the symmetric space conditions $\nabla R = 0$ and $\tilde{\nabla} \tilde{R} = 0$, and the opposite curvature conditions discussed in [4]. There we saw that dual symmetric spaces [21, 22] gave a class of manifolds with opposite curvature. An interesting mathematical question is suggested by our discussion. Is there a generalization of dual symmetric spaces that provides a framework for the integrability conditions discussed in this Section?

### 4.1 Relation to the Renormalization Group

We make a brief remark about duality and the renormalization group. The first person to study this issue was Nappi [3] within the context of the Zakharov-Mikhailov model.
The connection between off shell duality and the renormalization group was first studied by Buscher [23, 24]. Define a tensor $S_{jk}$ by $S_{jl} = S_{ij}$. A brief computation shows that

$$S_{jl} = R_{jl} - \frac{1}{4} H_{imj} H_{ilm}.$$  

The 1-loop renormalization group beta function [25, 7] for the metric $g_{jl}$ is precisely $S_{jl}$. Similarly you notice that if in the other integrability condition (4.6) you take a trace on $ik$ you get $\nabla_i H_{jil}$ which is the 1-loop beta function for the 2-form $B_{jl}$. It was pointed out in [26] that the opposite signs in the beta functions found by Nappi [5] in the pseudodual models of Zakharov and Mikhailov [9] are due to opposite signs of the generalized curvatures [8]. Here we have shown a more general result. Direct consequences of the integrability conditions for pseudoduality (4.5) and (4.6) are that the 1-loop beta functions will have have opposite signs for any two sigma models that are classically pseudodual. Clearly there is some interesting geometry in the space of field theories that is not yet understood.

5 Some Simple Examples

We show that two well known dual models correspond to simple solutions of (3.13). The best way to see this is to choose local coordinates $(x, R_V)$ and $(\tilde{x}, \tilde{R}_{\tilde{V}})$ respectively on $\text{SO}(M)$ and $\text{SO}(\tilde{M})$. We will look for solutions that have $R_V = \tilde{R}_{\tilde{V}} = I$. In this case equations (2.5) and (2.6) tell us that

$$\omega_{ij} = (\omega_V)_{ij} = (\omega_V)_{ijk}\omega^k_V,$$  

$$\tilde{\omega}_{ij} = (\tilde{\omega}_{\tilde{V}})_{ij} = (\tilde{\omega}_{\tilde{V}})_{ijk}\tilde{\omega}^k_{\tilde{V}}.$$  

In all of Section 5 we will work on the base manifolds $M$ and $\tilde{M}$ and so we drop the $V$ and $\tilde{V}$ subscripts. Inserting the above into (3.13) leads to

$$\tilde{\omega}_{ijk}\tilde{\omega}^k + \frac{1}{2} H_{ijk}\tilde{\omega}^k = \omega_{ijk}\omega^k + \frac{1}{2} \tilde{H}_{ijk}\omega^k.$$  

(5.3)

The hypotheses and the duality equations tell us that

$$\omega^i = x^i_+ d\sigma^+ + x^i_- d\sigma^- \quad \text{and} \quad \tilde{\omega}^i = x^i_+ d\sigma^+ - x^i_- d\sigma^-.$$  

Since $x^i_+$ and $x^i_-$ may be independently chosen at any point $\sigma$ we can de facto treat $\omega^i$ and $\tilde{\omega}^j$ as being independent for our purposes. In this way we conclude that

$$\tilde{\omega}_{ijk} = -\frac{1}{2} \tilde{H}_{ijk},$$  

$$\omega_{ijk} = -\frac{1}{2} H_{ijk}.$$  

(5.4)
5.1 Pseudochiral Model

Here we discuss the pseudochiral model [9] of Zakharov and Mikhailov. Consider a sigma model with target space $M$ a real connected compact Lie group $G$ with an $\text{Ad}(G)$-invariant metric. The structure constants $f_{ijk}$ are skew symmetric, see Appendix B, and the coefficients of the riemannian connection are given by $\omega_{ijk} = -\frac{1}{2} f_{ijk}$. This sigma model also has $H_{ijk} = 0$. Applying this to (5.4) and (5.5) we see that in the dual sigma model $\tilde{\omega}_{ijk} = 0$ and $\tilde{H}_{ijk} = f_{ijk}$. Since the connection is trivial, the Cartan structural equation (2.3) pulled back to $\tilde{M}$ tells us that $d\tilde{\omega}^i = 0$ and therefore we can find coordinates so that $\tilde{\omega}^i = d\tilde{x}^i$. The trivialness of the connection tells us that we can choose the manifold $\tilde{M}$ to be euclidean space $\mathbb{R}^n$ that can be identified with the Lie algebra $\mathfrak{g}$ of $G$. Note that the 3-form $\tilde{H} = \frac{1}{3!} f_{ijk} \tilde{\omega}^i \wedge \tilde{\omega}^j \wedge \tilde{\omega}^k$ is closed as required.

5.2 WZW Type Models

In this case we take the sigma model $(M, g, B)$ to be a connected compact real Lie group with an $\text{Ad}(G)$-invariant metric. The Maurer-Cartan equations are (B.1). The 3-form $H$ is taken to be proportional to the structure constants $H_{ijk} = af_{ijk}$ where $a \in \mathbb{R}$ is constant. What is strictly called the WZW model corresponds to $a = \pm 1$ with a specific normalization of the action needed to make the path integral well defined. Note that worldsheet parity takes $a$ to $-a$ so we can restrict ourselves to $a \geq 0$. To work out the pseudodual sigma model we insert the above into (5.4) and (5.5) where we find that $\tilde{\omega}_{ij} = -\frac{1}{2} af_{ijk} \tilde{\omega}^k$ and $\tilde{H}_{ijk} = f_{ijk}$. By using the first Cartan structural equation we obtain the Maurer-Cartan equations

$$d\tilde{\omega}^i = -\frac{1}{2} af_{ijk} \tilde{\omega}^j \wedge \tilde{\omega}^k. \quad (5.6)$$

The dual manifold $\tilde{M}$ is the group $G$ because the Maurer-Cartan equations above are just a rescaled version of (B.1). Note that the metric on $(\tilde{M}, \tilde{g}, \tilde{B})$ is $\tilde{g} = \tilde{\omega}^i \otimes \tilde{\omega}^i$ and the Maurer-Cartan equations are (5.6). The connection $\tilde{\omega}_{ij}$ must be the riemannian connection for metric $\tilde{g}$ so the metric $\tilde{g}$ is a rescaled version of the metric $g$ as we will see. The 3-form $\tilde{H} = \frac{1}{3!} f_{ijk} \tilde{\omega}^i \wedge \tilde{\omega}^j \wedge \tilde{\omega}^k$ is closed as required[4]. The model with $a = 1$ is self pseudodual. Also we note that the $a \to 0$ limit of the dual model is the pseudochiral model [13, 20].

There are a few observations worth making about the classical lagrangian. Classically, the overall normalization of the lagrangian is irrelevant. Schematically we can

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[4] You can verify that the geometric data satisfies the integrability conditions derived in Section 4 though it is not necessary to do so in this case.
write the lagrangian for \((M, g, B)\) as
\[
\mathcal{L} = \omega^i \otimes \omega^j + af_{ijk} \omega^j \wedge \omega^k.
\]
The lagrangian for \((\tilde{M}, \tilde{g}, \tilde{B})\) is
\[
\tilde{\mathcal{L}} = \tilde{\omega}^i \otimes \tilde{\omega}^j + \tilde{f}_{ijk} \tilde{\omega}^j \wedge \tilde{\omega}^k.
\]
If we define \(\hat{\omega}^i = a\tilde{\omega}^i\) then \(\hat{\omega}\) satisfies the “original” Maurer-Cartan equations
\[
d\hat{\omega}^i = -\frac{1}{2} f_{ijk} \hat{\omega}^j \wedge \hat{\omega}^k.\tag{5.7}
\]
and we can write the lagrangian as
\[
\tilde{\mathcal{L}} = \frac{1}{a^2} \left( \omega^i \otimes \omega^j + \frac{1}{a} f_{ijk} \omega^j \wedge \omega^k \right).
\]
At the level of equations of motion the pseudoduality transformation takes the model with parameter \(a\) to the one with parameter \(1/a\). This result should be in the literature but I have not found an explicit reference to it. One final remark, it is a well known result in differential geometry that rescaling the metric does not change the connection 1-form; you can verify that \(\tilde{\omega}_{ij} = \hat{\omega}_{ij}\).

5.3 Explicit Computation in WZW Type Models

We can actually be very explicit and see how it all develops. The equations of motion for the WZW type model on \(G\) with parameter \(a\) can be written as
\[
\partial_- (g^{-1} \partial_+ g) + \partial_+ (g^{-1} \partial_- g) = -a \left[ g^{-1} \partial_+ g, g^{-1} \partial_- g \right]_G.
\tag{5.8}
\]
We put a subscript \(G\) to identify the group associated with that Lie bracket. The equations of motion on \(\tilde{G}\) with parameter \(\tilde{a}\) are
\[
\partial_- (\tilde{g}^{-1} \partial_+ \tilde{g}) + \partial_+ (\tilde{g}^{-1} \partial_- \tilde{g}) = -\tilde{a} \left[ \tilde{g}^{-1} \partial_+ \tilde{g}, \tilde{g}^{-1} \partial_- \tilde{g} \right]_{\tilde{G}}.
\tag{5.9}
\]
The general theory requires that we work with orthonormal frames. We choose an orthonormal basis \(\{X_i\}\) for the Lie algebra of \(G\). In this basis, the Lie brackets are given by \([X_j, X_k]_G = f_{jk}^i X_i\). Similarly in \(\tilde{G}\) we choose an orthonormal basis \(\{\tilde{X}_i\}\) with Lie brackets \([\tilde{X}_j, \tilde{X}_k]_{\tilde{G}} = \tilde{f}_{jk}^i \tilde{X}_i\). The duality equations are
\[
(\tilde{g}^{-1} \partial_+ \tilde{g})^i = + (g^{-1} \partial_+ g)^i, \tag{5.10}
\]
\[
(\tilde{g}^{-1} \partial_- \tilde{g})^i = - (g^{-1} \partial_- g)^i. \tag{5.11}
\]
Subtract $\partial_+ \text{ of (5.11)}$ from $\partial_- \text{ of (5.10)}$ to obtain
\[
[\tilde{g}^{-1}\partial_+\tilde{g}, \tilde{g}^{-1}\partial_-\tilde{g}]^i_G = -a [g^{-1}\partial_+g, g^{-1}\partial_-g]^i_G,
\]
or
\[
\tilde{f}^i_{jk} (\tilde{g}^{-1}\partial_+\tilde{g})^j (\tilde{g}^{-1}\partial_-\tilde{g})^k = -a f^i_{jk} (g^{-1}\partial_+g)^j (g^{-1}\partial_-g)^k.
\]

In deriving the above we only had to only use the equations of motion for $g$ not the equations of motion for $\tilde{g}$. By using the duality relations we learn that
\[
\tilde{f}_{ijk} = a f_{ijk}
\]
in agreement with (5.6). We can also consider the sum of $\partial_-$ of (5.10) and $\partial_+$ of (5.10) to obtain
\[
\partial_- (\tilde{g}^{-1}\partial_+\tilde{g})^i + \partial_+ (\tilde{g}^{-1}\partial_-\tilde{g})^i = [g^{-1}\partial_+g, g^{-1}\partial_-g]^i_G,
\]
\[
= f^i_{jk} (g^{-1}\partial_+g)^j (g^{-1}\partial_-g)^k,
\]
\[
= -f^i_{jk} (\tilde{g}^{-1}\partial_+\tilde{g})^j (\tilde{g}^{-1}\partial_-\tilde{g})^k,
\]
\[
= -\frac{1}{a} \tilde{f}^i_{jk} (\tilde{g}^{-1}\partial_+\tilde{g})^j (\tilde{g}^{-1}\partial_-\tilde{g})^k,
\]
\[
= -\frac{1}{a} [\tilde{g}^{-1}\partial_+\tilde{g}, \tilde{g}^{-1}\partial_-\tilde{g}]^i_G.
\]

These are the equations of motion for the model on $\tilde{G}$. We used (5.12) that depends only on the equations of motion of $g$, and the duality relations (5.10), (5.11). Equations (5.13) are the statement that $\tilde{H}_{ijk} = f_{ijk}$. We showed that the equations of motion (5.14) for $\tilde{g}$ are (5.9) with $\tilde{a} = 1/a$.

### 6 Strict WZW Models

This example is generalizes the examples in Section 4. There we solved (5.3) by requiring $R_V = I$ and wrote an explicit solution on the base. Here we affirm that there are other other solutions when $R_V \neq I$. This is similar to the situation discussed in [4 Section 2] where we saw that there were no pseudoduality solutions if $T = I$ but there are solutions if we allowed $T$ to be an orthogonal matrix. We find the very surprising result that any two strict WZW model on compact Lie groups of the same dimensionality are pseudodual.
Let $M = G$ be a compact connected Lie group of dimension $n$ with an $\text{Ad}(G)$-invariant metric. Essentially what we want to do is choose $H_{ijk}$ to be $a f_{ijk}$. We have to be careful because $H$ is defined on the orthonormal frame bundle of $G$ while the structure constants are defined on $G$. Since a Lie group is parallelizable we choose a global orthonormal coframe $\omega_i^j$. Note that the open set $V$ is $G$. The orthonormal frame bundle is trivial so $\text{SO}(G) = G \times \text{SO}(n)$. At the point $(g, R_V) \in \text{SO}(G)$ we define the functions $H_{ijk}$ by $H_{ijk} = a(R_V)_{il}(R_V)_{jm}(R_V)_{kn} f_{lmn}$ where $|a| = 1$. The adjoint bundle$^5$ of $G$ is a sub-bundle of $\text{SO}(G)$ and the functions $H_{ijk}$ restricted to the adjoint bundle are constant functions given by $a f_{ijk}$. Pulling back the right hand sides of (4.5) and (4.6) to the base $G$ you see that Appendix B immediately tells you that they vanish. Choose $\tilde{M} = \tilde{G}$ to be any $n$-dimensional compact Lie group with an $\text{Ad}(\tilde{G})$-invariant metric. Let $\{\tilde{X}_i\}$ be an orthonormal basis for the Lie algebra of $\tilde{G}$ with bracket relations $[\tilde{X}_i, \tilde{X}_j]_{\tilde{G}} = \tilde{f}^{k}_{ij} \tilde{X}_k$. The structure constants with lowered indices $\tilde{f}_{ijk}$ are totally antisymmetric in $ijk$, see Appendix B. In a strict WZW model the equations of motion may be written as $\partial_+ (\tilde{g}^{-1} \partial_+ \tilde{g}) = 0$ where $\tilde{g} : \Sigma \to \tilde{M}$. The pseudoduality

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$^5$The adjoint bundle is the trivial bundle $G \times \text{Ad}(G)$ where $\text{Ad}(G) \subset \text{SO}(n)$ is the adjoint group of $G$.

$^6$In the case of an abelian group we do not have to worry about compactness since we are just looking at local properties of the PDEs.
The equations are

\begin{align}
(\tilde{g}^{-1}\partial_+\tilde{g})^i &= +T^i{}_j\partial_+\phi^j, \\
(\tilde{g}^{-1}\partial_-\tilde{g})^i &= -T^i{}_j\partial_-\phi^j, \\
\end{align}

(6.1)

where $T$ is an orthogonal matrix and $\tilde{g}^{-1}d\tilde{g} = (\tilde{g}^{-1}d\tilde{g})^i\tilde{X}_i$. Taking $\partial_-$ of the first equation above we learn that $(\partial_-T)(\partial_+\phi) = 0$. Since we can choose $\partial_+\phi$ to have an arbitrary value at any $\sigma$ we have that $\partial_-T = 0$. Thus we learn that $T$ is a function of only $\sigma^+$. Next we take $\partial_+$ of the second equation above, use the equations of motion and conclude that

\[ [\partial_+T]T^{-1} = -\tilde{f}^{i}{}_{kj}T^k_i\partial_+\phi^j. \]

(6.2)

We note that the right hand side is skew under $i \leftrightarrow j$ and is only a function of only $\sigma^+$. Therefore, we have an ordinary differential equation (6.2) that will produce an orthogonal matrix $T(\sigma^+)$ that depends on $\partial_+\phi(\sigma^+)$. 

Summarizing we have seen that for any solution $\phi^i$ of the wave equation we can construct an orthogonal matrix $T$ and subsequently use (6.1) to construct solutions to the strict WZW model on a compact simple Lie group.

The reader could ask whether we worked too hard in this section. The equations of motion tell us that $\tilde{g}^{-1}\partial_+\tilde{g}$ and $\partial_+\phi$ and only functions of $\sigma^+$. Does it not suffice to use only the first of (6.1) and any arbitrary $T(\sigma^+)$, not necessarily orthogonal, and in that way map a solution of the free equation into a solution of the strict WZW model? There is a reason for invoking the second equation in (6.1). It is desirable to preserve the stress energy tensor. The construction described in this paragraph will preserve $\Theta^{++}$ if $T$ is orthogonal. Anything can happen to $\Theta^{--}$. By requiring both equations in (6.1) we are guaranteeing that the stress-energy tensor is preserved. An analogous remark can be made in Section 6.2.

### 6.2 A More Complicated Example

Here we consider the case where we consider pseudoduality between strict WZW models where $M$ and $\tilde{M}$ are compact Lie groups of dimension $n$ with Ad-invariant metrics. Let $\{X_i\}$ be an orthonormal basis for the Lie algebra of $G$ with bracket relations $[X_i, X_j]_G = f^{k}{}_{ij}X_k$. The structure constants $f^{i}_{jk}$ are totally antisymmetric in $ijk$. Likewise we make analogous definitions for the Lie group $\tilde{G}$. The equations of motion
are \( \partial_-(g^{-1} \partial_+ g) = 0 \) and \( \partial_-(\tilde{g}^{-1} \partial_+ \tilde{g}) = 0 \). The pseudoduality equations are

\[
\begin{align*}
(\tilde{g}^{-1} \partial_+ \tilde{g})^i &= +T^i_j (g^{-1} \partial_+ g)^j, \\
(\tilde{g}^{-1} \partial_- \tilde{g})^i &= -T^i_j (g^{-1} \partial_- g)^j,
\end{align*}
\]

where \( T \) is an orthogonal matrix. Taking \( \partial_- \) of the first equation above we learn that \( (\partial_- T) = 0 \) and therefore \( T \) is a function of \( \sigma^- \) only. Taking \( \partial_+ \) of the second equation above we learn that

\[
[(\partial_+ T)T^{-1}]^i_j = -\tilde{f}^i_{kj} T^k_l (g^{-1} \partial_+ g)^l + T^i_k T^j_l \tilde{f}^k_{ml} (g^{-1} \partial_+ g)^m,
\]

\[
= \left( -\tilde{f}^i_{imj} + T^i_k T^m_p T^j_l \tilde{f}^k_{pl} \right) T^m_n (g^{-1} \partial_+ g)^n.
\]

(6.4)

In deriving the above we used \( T^{-1} = T^t \). Note that the right hand side is skew under \( i \leftrightarrow j \) and that everything on the right hand side is a function of \( \sigma^+ \) only. Thus the above is an ordinary differential equation with solution an orthogonal matrix \( T(\sigma^+) \).

Summarizing we have seen that for any solution \( g \) of the equations of motion for the strict WZW model on \( G \) we can construct an orthogonal matrix \( T \) and subsequently use (6.3) to construct a solution \( \tilde{g} \) to the strict WZW model on \( \tilde{G} \). For example you could take the group \( G \) to be \( \text{SO}(2l+1) \) associated with the Lie algebra \( B_l \) and \( \tilde{G} \) to be the compact symplectic group \( U_H(l) \) associated with the Lie algebra \( C_l \). Note that \( \dim G = \dim \tilde{G} = l(2l+1) \).

We can make contact with the discussion in Section 5.3 with \( a = 1 \) by noting that if \( G = \tilde{G} \) then \( T = I \) is a solution to (6.4).

### 6.3 Some Geometry

We have compact Lie groups \( G \) and \( \tilde{G} \) of dimension \( n \) with \( \text{Ad} \)-invariant inner products on each. The adjoint action of the groups acts via isometries on the Lie algebras and therefore we can think of the respective adjoint groups \( \text{Ad} G \) and \( \text{Ad} \tilde{G} \) as subgroups of \( \text{SO}(n) \). We note that if we pick an orthonormal basis for the Lie algebra \( \mathfrak{g} \) then the structure constants are invariant under the adjoint action of \( G \) and likewise for \( \tilde{G} \) and \( \tilde{\mathfrak{g}} \). There is a natural action of \( \text{Ad} G \times \text{Ad} \tilde{G} \) on \( T \) given by \( (R, \tilde{R}) \in \text{Ad} G \times \text{Ad} \tilde{G} \) that takes \( T \) into \( \tilde{R}TR^{-1} \), see (6.3). Since \( f_{ijk} \) and \( \tilde{f}_{ijk} \) are respectively \( \text{Ad} G \) and \( \text{Ad} \tilde{G} \) invariant we have that differential equation (6.4) is \( \text{Ad} G \times \text{Ad} \tilde{G} \) invariant. When we parametrize our solutions as \( \tilde{g}(\sigma; g, T_0) \) we see that we should really think of the solution as being parametrized by the equivalence class \( [T_0] \in \text{Ad} \tilde{G} \setminus \text{SO}(n) / \text{Ad} G \).

Finally since this section is supposed to be self-contained, I should explain how to make sense of (6.3). After all, the right hand side involves \( \mathfrak{g} \), the Lie algebra of \( G \), while
the right hand side involves \( \tilde{g} \), the Lie algebra of \( \tilde{G} \). Let \( \text{Isom}(\mathfrak{g}, \tilde{\mathfrak{g}}) \) be the vector space isometries from \( \mathfrak{g} \) to \( \tilde{\mathfrak{g}} \). All we are saying is that we need a map \( T : \Sigma \to \text{Isom}(\mathfrak{g}, \tilde{\mathfrak{g}}) \) such that \( *_{\Sigma}(\tilde{g}^{-1}d\tilde{g})(\sigma) = T(g^{-1}dg)(\sigma) \), where \( *_{\Sigma} \) is the Hodge duality operator on \( \Sigma \).

We can even expand more on the above by rewriting (6.4) is a different way

\[
\partial_+ T_{ij} = -[\tilde{f}_{ik}(\tilde{g}^{-1}\partial_+\tilde{g})]^lT_{kj} + T_{ik}[f_{klj}(g^{-1}\partial_+g)]^l. \tag{6.5}
\]

The right hand side of this equation is Lie algebra version of the \( \text{Ad} G \times \text{Ad} \tilde{G} \) action on \( T \). It is straightforward to solve this is equation but let us be a bit more abstract so that we can state the solution in a coordinate independent fashion. On \( \mathfrak{g} \) define the adjoint action by \( \text{ad}_g(X)Y = [X,Y] \) for \( X, Y \in \mathfrak{g} \). The vector space \( \mathfrak{g} \) has an inner product so we can define \( \text{ad}^\dagger_g : \mathfrak{g} \to \mathfrak{g} \) as the adjoint of the transformation \( \text{ad}_g \). Since the metric on \( \mathfrak{g} \) is \( \text{Ad} G \) invariant we have that \( \text{ad}_g \) is a skew adjoint transformation \( \text{ad}^\dagger_g = -\text{ad}_g \).

The tangent bundle of \( G \) is trivial \( TG = G \times \mathfrak{g} \). We have a map \( g : \Sigma \to G \) that can be used to pullback the tangent bundle to \( \Sigma \). On this pullback bundle \( g^*(TG) \) we define a flat orthogonal connection by \( \text{ad}^\dagger_g(J^{(R)}) \) where \( J^{(R)} = (g^{-1}\partial_+g)d\sigma^+ \). This connection is flat by the equations of motion and it is an orthogonal connection because \( \text{ad}_g \) is skew adjoint. Let \( P(\sigma) \) be parallel transport from \( (0,0) \) to \( \sigma = (\sigma^+, \sigma^-) \). Notice that since \( J^{(R)} \) is flat and it does not have a \( d\sigma^- \) component we have that \( P(\sigma) \) is independent of \( \sigma^- \). We can define similar structures on \( \tilde{G} \) and \( \tilde{\mathfrak{g}} \). From experience we know that the integration of (6.3) is given by parallel transport. Since one index of \( T \) lives in \( \mathfrak{g} \) and the other in \( \tilde{\mathfrak{g}} \) we have that the solution of the equation above may be written as

\[
T(\sigma) = \tilde{P}(\sigma)T_0P(\sigma)^{-1}, \tag{6.6}
\]

where \( T(0) = T_0 \).

This leads to a beautiful geometrical way to think about the pseudoduality equations (6.4). The equations of motion tell us that there are natural flat connections \( \text{ad}^\dagger_g(J^{(R)}) \) and \( \text{ad}^\dagger_{\tilde{g}}(\tilde{J}^{(R)}) \) respectively on the pullback bundles \( g^*(TG) \) and \( \tilde{g}^*(T\tilde{G}) \). The solution of the ODE for \( T \) tells us that the geometric content of pseudoduality is the following. Begin with \( (g^{-1}dg)(\sigma) \) and parallel transport it to the origin \( P(\sigma)^{-1}(g^{-1}dg)(\sigma) \). Do the same thing on the dual model. The fibers over the origin of the aforementioned bundles are \( \mathfrak{g} \) and \( \tilde{\mathfrak{g}} \). Use a fixed isometry \( T_0 \in \text{Isom}(\mathfrak{g}, \tilde{\mathfrak{g}}) \) and Hodge duality to equate these two quantities:

\[
*_{\Sigma}\left(\tilde{P}(\sigma)^{-1}(\tilde{g}^{-1}d\tilde{g})\right) = T_0 \left(P(\sigma)^{-1}(g^{-1}dg)\right). \tag{6.7}
\]

\footnote{This resembles a result of \cite{braaten} for \( \tilde{G} \) abelian but it is not. The results here are duality based and motivated while the observation of Braaten, Curtright and Zachos is closely related to the discussion of Section \ref{dualities} and not directly related to pseudoduality.}
This equation is totally intrinsic without reference to bases, \textit{etc.}, and encapsulates how pseudoduality transformation operates on strict WZW models. Note that in general you cannot use the action of $\text{Ad} \, G \times \text{Ad} \, \tilde{G}$ to set $T_0 = I$.

We would like to point out that the above is not the most convenient approach from a computational viewpoint if you are looking for pseudodual solutions. To do this you begin with a $g(\sigma)$ and you use (6.4) to solve for $T$ and then integrate (6.3) to find $\tilde{g}(\sigma)$.

### 6.4 Infinite Number of Conservation Laws

The discussions of Sections 6.1 and 6.2 lead to a new method that can be used to find an infinite number of non-local conserved currents by a variant of a technique discussed in [16]. For a recent discussion and references to the older literature on local and non-local conservation laws for sigma models based on groups see [27, 28]. The connection between the method described here and other methods is not clear.

We begin with the strict WZW model based on Lie group $G$ where the basic local conserved currents are $J^{(R)} = (g^{-1} \partial_+ g) d\sigma^+$ and $J^{(L)} = (\partial_- g) g^{-1} d\sigma^-$. It is well known that powers of $J^{(R)}$ and $J^{(L)}$ give higher rank local conservation laws. What we would like to do is construct an infinite number of non-local conservation laws on $G$ by using the pseudodual model on a compact Lie group $\tilde{G}$ in an auxiliary fashion. We solve (6.3) for $T$. The ordinary differential equation needs an initial condition $T(\sigma^+ = 0) = T_0 \in \text{SO}(n)$. To be more precise we note that $T = T(\sigma^+; g, T_0)$. We use (6.3) to solve for $\tilde{g} = \tilde{g}(\sigma; g, T_0)$. Next we construct the basic conserved currents $\tilde{J}^{(R)}$ and $\tilde{J}^{(L)}$ on $\tilde{G}$. We think of $\tilde{J}^{(R)}$ and $\tilde{J}^{(L)}$ as functions of $g$ and $T_0$. The current $\tilde{J}^{(R)}$ is a non-local function of $g^{-1} \partial_+ g$ since (6.4) and the first of (6.3) are functions of $g^{-1} \partial_+ g$ only. You get a family of non-local conserved currents on the WZW model on $G$ parametrized by the initial condition $T_0$. If you write $T_0 = e^\alpha$ where $\alpha$ is an antisymmetric matrix and you power series expand about $\alpha = 0$ then you will get an infinite number of non-local conserved currents on the WZW model on $G$ (not $\tilde{G}$) that we can schematically organize as $\tilde{J}^{(R)}(g, T_0) = \sum_{n=0}^{\infty} \alpha^n \tilde{J}^{(R)}_{[n]}(g)$. The first one $\tilde{J}^{(R)}_{[0]}$ will be $J^{(R)}$ if $G = \tilde{G}$. If $G \neq \tilde{G}$ then $T$ is nontrivial even for initial condition $T(0) = I$ and we cannot write down $\tilde{J}^{(R)}_{[0]}$ explicitly. The other current $\tilde{J}^{(L)}$ is more “interesting” because you need both the equations in (6.3) to work out what it is. You can do a similar power series $\tilde{J}^{(L)}(g, T_0) = \sum_{n=0}^{\infty} \alpha^n \tilde{J}^{(L)}_{[n]}(g)$ to get an infinite number of non-local conserved currents.

Note that by choosing a different group $\tilde{G}$ we get a different set of conservation laws since $T$ depends on the choice of groups, see (6.7).
Some Geometry of the Connections

In this section we study some of the geometry of the connections that arises due to the Pfaffian equation (4.4). In this pursuit we run into an interesting fork in the road. Motivated by duality we obtain some results that are really properties of sigma models and do not have anything to do with duality.

In order to be very clear about what it happening it is convenient to explicitly worry about pullbacks of differential forms. On the bundle of orthonormal frames $SO(M, g)$ defined by metric $g$ on $M$ we have a global coframing $(\omega^i, \omega_{jk})$. We also have a map $X : \Sigma \to SO(M, g)$. We can use $X$ to pull all structures back to $\Sigma$ and so we get "vector-valued 1-forms" $(\xi^i, \xi_{jk})$ on $\Sigma$ defined by

\[ X^* \omega^i = \xi^i , \]
\[ X^* \omega_{jk} = \xi_{jk} . \]

By taking the exterior derivative and using (2.3) and (2.4) you find that the $\xi$s satisfy

\[ d\xi^i = -\xi_{ij} \wedge \xi^j , \]
\[ d\xi_{ij} = -\xi_{ik} \wedge \xi_{kj} + \frac{1}{2} r_{ijkl} \xi^k \wedge \xi^l , \]

where $r_{ijkl} = X^* R_{ijkl} = R_{ijkl} \circ X$ denotes the pullback to $\Sigma$ of the functions $R_{ijkl}$ on $SO(M, g)$.

To write down the equations of motion we need the Hodge duality operator $\ast_\Sigma$ on $\Sigma$. On 1-forms it is given by $\ast_\Sigma (d\sigma^\pm) = \pm d\sigma^\pm$. For future reference we note that if $\alpha$, $\beta$ are 1-forms on $\Sigma$ then

\[ (\ast_\Sigma \alpha) \wedge (\ast_\Sigma \beta) = -\alpha \wedge \beta , \]

and that $(\ast_\Sigma)^2 \alpha = \alpha$. Also note that $\alpha \wedge (\ast_\Sigma \beta) = \beta \wedge (\ast_\Sigma \alpha)$. In particular, you have that $\xi^i \wedge (\ast_\Sigma \xi^k)$ is symmetric under $j \leftrightarrow k$. The sigma model is specified by a map $X : \Sigma \to SO(M, g)$ that satisfies

\[ d\xi_{ij} + \xi_{ik} \wedge \xi_{kj} = \frac{1}{2} r_{ijkl} \xi^k \wedge \xi^l , \]
\[ d\xi^i + \xi_{ij} \wedge \xi^j = 0 , \]
\[ d(\ast_\Sigma \xi^i) + \xi_{ij} \wedge (\ast_\Sigma \xi^j) = \frac{1}{2} h_{ijk} \xi^j \wedge \xi^k , \]

where $h_{ijk} = X^* H_{ijk} = H_{ijk} \circ X$. The last equation above is the non-linear wave equation for the sigma model.
In an obvious notation, the pseudoduality equations are

\[ \tilde{\xi}^i = \ast \Sigma \xi^i. \]  

(7.9)

Equation (4.4) may be written as

\[ \tilde{\xi}_{ij} - \frac{1}{2} \tilde{h}_{ijk} (\ast \Sigma \xi^k) = \xi_{ij} - \frac{1}{2} h_{ijk} (\ast \Sigma \xi^k). \]  

(7.10)

Notice that everything on the left hand side refers to \( \tilde{M} \) and everything on the right hand side refers to \( M \). Motivated by the equations of motion (3.3) and not pseudoduality, earlier authors, see e.g. [8, 15], suggested defining a “connection” by

\[ \xi'_{ij} = \xi_{ij} - \frac{1}{2} h_{ijk} (\ast \Sigma \xi^k). \]  

(7.11)

You have to be careful here for in general \( \xi'_{ij} \) is not the pullback of a connection on \( \text{SO}(M, g) \) as we will see in the next subsection; though \( \xi'_{ij} \) is a connection on the pullback bundle \( X^* \text{SO}(M, g) \).

### 7.1 Detour

We now take a fork in the road and for the moment we forget about \( \tilde{M} \) and duality. We try to rewrite the equations of motion for the sigma model on \( M \) in terms of \( \xi'_{jk} \).

You find

\[
\begin{align*}
    d\xi'_{ij} + \xi'_{ik} \wedge \xi'_{kj} &= -\frac{1}{2} \left( \nabla^\xi_k h_{ij} + \nabla^\xi_i h_{kj} \right) \xi^k \wedge (\ast \Sigma \xi^l) \\
    & \quad + \frac{1}{2} \left[ r_{ijkl} - \frac{1}{2} h_{ijm} h_{mkl} - \frac{1}{4} (h_{imk} h_{mjl} - h_{iml} h_{mjk}) \right] \xi^k \wedge \xi^l, \\
    d\xi^i + \xi'_{ij} \wedge \xi^j &= 0, \\
    d(\ast \Sigma \xi^i) + \xi'_{ij} \wedge (\ast \Sigma \xi^j) &= 0.
\end{align*}
\]

(7.12)

(7.13)

(7.14)

The covariant derivative of \( h_{ijk} \) is defined by

\[
\nabla^\xi h_{ijk} = dh_{ijk} + \xi_i h_{ljk} + \xi_j h_{ilk} + \xi_k h_{ijl} = X^*(\nabla^\omega H_{ijk}).
\]

(7.15)

Equations (7.13) and (7.14) look like the equations (7.7) and (7.8) for a sigma model with vanishing 3-form. Is there a lagrangian that gives these equations of motion? The affirmative answer requires that \( \xi'_{ij} \) is the pullback of a connection and that (7.12) is of form (7.6). Let us be more precise. Can we find a metric \( g' \) on a new manifold \( M' \) such...
that $\xi'_{ij}$ may be interpreted as the pullback to $\Sigma$ of a connection on $\text{SO}(M', g')$? This bundle has a global coframing $\{\theta^i, \theta_{jk}\}$ that satisfies the Cartan structural equations

\begin{align}
\text{d}\theta^i &= -\theta_{ij} \wedge \theta^j + \frac{1}{2} T_{ijk} \theta^j \wedge \theta^k, \quad (7.16) \\
\text{d}\theta_{ij} &= -\theta_{ik} \wedge \theta_{kj} + \frac{1}{2} K_{ijkl} \theta^k \wedge \theta^l, \quad (7.17)
\end{align}

where $T_{ijk}$ is the torsion of the connection $\theta_{ij}$ and $K_{ijkl}$ is the curvature of the connection $\theta_{ij}$. The new sigma model is defined by a map $Y : \Sigma \to \text{SO}(M', g')$ satisfying

\begin{align*}
Y^* \theta^i &= \xi^i, \\
Y^* \theta_{ij} &= \xi'_{ij}.
\end{align*}

Taking the exterior derivative of the first equation above and using (7.13) leads to the conclusion that $T_{ijk} = 0$. We have learned that the connection on $\text{SO}(M', g')$ is the unique torsion free riemannian connection associated to the metric $g'$. Taking the exterior derivative of the second equation above tells us that

\begin{align*}
\frac{1}{2} (Y^* K_{ijkl}) \xi^k \wedge \xi^l &= -\frac{1}{2} \left( \nabla^\omega h_{ij} + \nabla^\omega h_{kij} \right) \xi^k \wedge \ast \xi^l \\
&\quad + \frac{1}{2} \left[ r_{ijkl} - \frac{1}{2} h_{imn} h_{mlk} - \frac{1}{4} (h_{imk} h_{mjl} - h_{iml} h_{mjk}) \right] \xi^k \wedge \xi^l.
\end{align*}

Comparing both sides we learn that $\nabla^\omega h_{ij} + \nabla^\omega h_{kij} = 0$. Combining this with $dH = 0$, see (4.2), tells us that $\nabla^\omega h_{ijk} = 0$. Thus the full content of the equation above is

\begin{align}
\nabla^\omega h_{ij} &= 0, \quad (7.18) \\
K_{ijkl} &= R_{ijkl} - \frac{1}{2} H_{ijm} H_{mkl} - \frac{1}{4} (H_{imk} H_{mj} - H_{imm} H_{mjk}). \quad (7.19)
\end{align}

The above should be viewed as equations on $\text{SO}(M, g) \times \text{SO}(M', g')$. Additional integrability conditions following from taking derivatives of the above also have to be satisfied. Notice that the right hand side of (7.19) is precisely the tensor $S_{ijkl}$, see (1.7). Some authors have tried to rewrite sigma model equations in terms of the orthogonal connection with torsion $\omega_{ij} - \frac{1}{2} H_{ijk} \omega^k$ on $\text{SO}(M, g)$ but we are not big advocates of this because it does not appear naturally in the geometrical framework, see e.g., the discussion above or (3.13). We feel that the important relevant geometrical object is the pullback connection (7.11) on the bundle $x^* \text{SO}(M, g)$.

\footnote{If $M$ and $M'$ are spin manifolds then the pullback bundle to $\Sigma$ of the respective spin frame bundles will be trivial bundles and therefore isomorphic. If $\Sigma$ has nontrivial topology then you have to be careful if the manifolds are not spin because $\pi_1(\text{SO}(n)) = \mathbb{Z}_2$ for $n > 2$.}
The conclusion here is that the sigma model specified by geometric data \((M, g, B)\) is equivalent to the sigma model \((M', g', B')\) with \(H' = dB' = 0\) if the integrability equations above are satisfied. Equivalence is in the sense that there is a mapping that takes solutions of sigma model \((M, g, B)\) into solutions of the other sigma model \((M', g', B')\) and vice versa. Notice that this part of the discussion follows only from trying to identify (7.11) as the pullback of a connection. It is independent of the duality motivation that lead to it. We are really discussing properties of sigma models and their equations of motion.

### 7.2 Earlier Observations

Braaten, Curtright and Zachos \[8\] observed that if the right hand sides of (7.18) and (7.19) vanish then the manifold \(\tilde{M}\) is \(\mathbb{R}^n\) or \(\mathbb{T}^n\). They used the flatness of the \(\xi'_{ij}\) connection to solve the equations of motion in terms of a free field and the parallel transport operator. Our way of seeing this is to observe that the connection \(\theta_{ij}\) on \(\tilde{M}\) is a flat torsion free metric connection.

Ivanov \[15\] observed that if \(M = G\) is a compact semi-simple Lie group then the equations above have a solution. Choose an orthonormal global framing for \(G\) and pull everything (7.18) and (7.19) back to \(G\). Assume that in metric \(g\), the structure coefficients for the Lie group in this orthonormal frame are given by \(f_{ijk}\), see Appendix [3]. Assume \(H_{ijk} = bf_{ijk}\) where \(b\) is a constant. Then (7.18) is automatically satisfied because of (B.4). A brief computation shows that (7.19) is given by

\[
K_{ijkl} = \frac{1}{4} (1 - b^2) f_{ijm} f_{mkl}.
\]

(7.20)

If we take a new metric \(g'\) on \(M' = G\) to be \(g' = g/(1 - b^2)\), for \(|b| < 1\), and \(\theta_{ij}\) to be the torsion free riemannian connection with respect to \(g'\) then we are done. This shows that every solution to the the equations of motion for the generalized WZW model on \(G\) defined by metric \(g\) and \(H_{ijk} = bf_{ijk}\) may be identified with a solution to the nonlinear sigma model on \(G\) with metric \(g' = g/(1 - b^2)\) and \(H'_{ijk} = 0\). We can now apply the special case of pseudoduality discussed in \[1, 15\]. We know that the model on the Lie group \(G\) with \(H' = 0\) is pseudodual to a model on the negative curvature symmetric space \(\tilde{M} = G^C/G\). Here \(G^C\) is the complexification of \(G\). So we see that in the sense described above the generalized WZW model on \(G\) with \(|b| < 1\) is pseudodual to the model on \(\tilde{M} = G^C/G\) with \(\tilde{H}_{ijk} = 0\).

\[9\] These authors had a more restrictive Jacobi identity condition on \(H_{ijk}\), but not necessary, that was motivated by the model they were studying.

\[10\] The Lie group \(G\) is viewed as the symmetric space \(G \times G/G\).
If $b = 1$ in (7.20) then $K_{ijkl} = 0$ and we can take $M' = \mathbb{R}^n$ or $M' = \mathbb{T}^n$ as noticed in [3, 19].

7.3 Back to Pseudoduality

We briefly return to pseudoduality and make a few comments. We can mimic what was done on Section 7.1 with (7.10) without introducing $\xi'_{ij}$ or $\tilde{\xi}'_{ij}$. We think of the left hand side and the right hand side of (7.10) respectively as pullbacks of connections from the appropriate bundles. We will find that the compatibility conditions are precisely (4.5) and (4.6). This method is mathematically equivalent to that used earlier in the article.

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A $G$-structures

$G$-structures arise as the imposition of a natural geometric structure on the tangent bundle of a manifold. The basic bundle in our discussion will be the coframe bundle $\mathcal{F}(M)$ of the manifold $M$. A point in $\mathcal{F}(M)$ consists of a point $x$ on the base manifold $M$ and a basis for the cotangent bundle at $x$. A coframe is a local section of this bundle. If $V \subset M$ is a neighborhood then we denote a coframe by

$$
\omega_V = \begin{pmatrix} \omega^1_V \\ \omega^2_V \\ \vdots \\ \omega^n_V \end{pmatrix}.
$$

Since any two bases differ by a $\text{GL}(n, \mathbb{R})$ transformation, $\mathcal{F}(M)$ is a principal $\text{GL}(n, \mathbb{R})$ bundle, i.e., the transition functions are $\text{GL}(n, \mathbb{R})$ valued.

The best known sub-bundle of $\mathcal{F}(M)$ is the bundle of orthonormal frames $\text{SO}(M)$. This bundle may be defined via the use of the metric $ds^2$ by defining it to be

$$
\text{O}(M) = \{ \omega \in \mathcal{F}(M) \mid ds^2 = \omega^i \otimes \omega^i \} \subset \mathcal{F}(M).
$$
From the definition it is clear that any two coframes at a point \( x \in M \) differ by an orthogonal transformation therefore the bundle of orthonormal coframes is a principal bundle with structure group \( O(n) \).

A \( G \)-structure is a reduction of the coframe bundle to a principal bundle with structure group \( G \). For our purposes it is convenient to give a local description of a \( G \)-structure. Assume that we are given an open cover of \( M \) by open sets \( \{ V_\alpha \} \) and a collection of coframes \( \omega_\alpha \) defined on \( V_\alpha \). We assume that on a non-empty overlap \( V_\alpha \cap V_\beta \) one has

\[
\omega_\alpha = \gamma_{\alpha\beta} \omega_\beta
\]

where the transition functions are \( G \)-valued \( \gamma_{\alpha\beta} : V_\alpha \cap V_\beta \to G \). We require the transition functions to satisfy the usual cocycle conditions. Given the transition functions \( \{ \gamma_{\alpha\beta} \} \) one constructs a principal fiber bundle by locally patching the sets \( V_\alpha \times G \) using the transition function. For \( x \in V_\alpha \cap V_\beta \), if \( (x, g_\alpha) \in V_\alpha \times G \) and \( (x, g_\beta) \in V_\beta \times G \) then we identify \( (x, g_\alpha) \) and \( (x, g_\beta) \) if

\[
g_\beta = g_\alpha \gamma_{\alpha\beta}(x).
\]

The principal bundle thus constructed is called a \( G \)-structure.

\( G \)-structures have a globally defined canonical 1-form\(^{11} \) that distinguishes a \( G \)-structure from a generic principal \( G \)-bundle. We observe that \( g_\alpha \omega_\alpha = g_\beta \omega_\beta \) therefore we have a \( n \) globally defined forms that we can put into a column vector and call them \( \omega \). When restricted to \( V_\alpha \times G \), the forms \( \omega \) may be written as

\[
\omega|_{V_\alpha \times G} = g_\alpha \omega_\alpha.
\]

Finally we note that there exists a local section \( s_\alpha : V_\alpha \to \mathcal{F}(M) \) such that \( s_\alpha^* \omega = \omega_\alpha \). If \( \pi : \mathcal{F} M \to M \) is the projection defining the bundle then it is not true that \( \omega = \pi^* \omega_\alpha \).

There are a variety of notable \( G \)-structures:

- \( G = O(n) \) gives Riemannian structures and is equivalent to specifying a riemannian metric.
- \( G = SL(n) \) is equivalent to prescribing a volume element, i.e., an orientation.
- \( G = SO(n) \) gives orientable Riemannian structures.
- \( G = \{ e \} \), the trivial group, is equivalent to specifying a global coframe, i.e., the manifold is parallelizable. These are called \( \{ e \} \)-structures.

\(^{11}\)This is sometimes called the soldering form.
On even dimensional manifolds, \( \dim M = 2m \), we have:

- \( G = \text{Sp}(2m, \mathbb{R}) \subset \text{GL}(2m, \mathbb{R}) \) gives almost symplectic structures.
- \( G = \text{GL}(m, \mathbb{C}) \subset \text{GL}(2m, \mathbb{R}) \) gives almost complex structures.
- \( G = \text{U}(m) = \text{SO}(2m) \cap \text{Sp}(2m, \mathbb{R}) \subset \text{GL}(2m, \mathbb{R}) \) gives almost hermitian structures.

\[ B \] Riemannian Geometry of Lie Groups

Assume \( G \) is a connected real compact Lie group of dimension \( n \). Choose an orthonormal coframe of Maurer-Cartan forms \( \omega^i \) for an \( \text{Ad}(G) \)-invariant metric. The Maurer-Cartan equations are

\[ d\omega^i = -\frac{1}{2} f^{ijk} \omega^j \wedge \omega^k, \quad (B.1) \]

where \( f^{ijk} \) are the totally skew symmetric structure constants for the Lie algebra \( \mathfrak{g} \) of \( G \). The invariance of the metric tells us that the adjoint group \( \text{Ad}(G) \) is a subset of \( \text{SO}(n) \). Comparing with (2.3), using the skewness of \( f^{ijk} \) and using the uniqueness of the riemannian connection we immediately conclude that

\[ \omega_{ij} = -\frac{1}{2} f_{ijk} \omega^k. \quad (B.2) \]

Using (2.4) we see that the riemannian curvature of the Lie group is given by

\[ R_{ijkl} = \frac{1}{2} f_{ijm} f_{mkl} + \frac{1}{4} (f_{imk} f_{mjl} - f_{iml} f_{mjk}) = \frac{1}{4} f_{ijm} f_{mkl} \quad (B.3) \]

where the last equality was obtained by using the Jacobi identity. You should compare the structure of the second term above with (4.3). Finally we observe that \( f_{ijk} \) is covariantly constant with respect to the riemannian connection because of the Jacobi identity:

\[ \nabla f_{ijk} = -\omega_{im} f_{mjk} - \omega_{jm} f_{imk} - \omega_{km} f_{ijm}, \]

\[ = +\frac{1}{2} (f_{iml} f_{mjk} + f_{jml} f_{imk} + f_{kml} f_{ijm}) \omega^l, \]

\[ = 0. \quad (B.4) \]

All the equations above are on \( G \). The corresponding expressions for the connection and the curvature on the coframe bundle \( \mathcal{F}(G) = G \times \text{SO}(n) \) are different.
C Torsion

We work in the bundle of orthonormal frames $\text{SO}(M)$ on the manifold $M$ with riemannian connection $\omega_{ij}$ that satisfies the Cartan structural equations \((2.3)\) and \((2.4)\). We have the option of considering a second orthogonal connection $\phi_{ij} = \omega_{ij} + C_{ijk} \omega^k$ where $C_{ijk} = -C_{jik}$. With respect to this new metric compatible connection on $\text{SO}(M)$, the Cartan structural equations are

$$d\omega^i = -\phi_{ij} \wedge \omega^j + \frac{1}{2} T_{ijk}^\phi \omega^j \wedge \omega^k,$$  \hspace{1cm} \text{(C.1)}

$$d\phi_{ij} = -\phi_{ik} \wedge \phi_{kj} + \frac{1}{2} R_{ijkl}^\phi \omega^k \wedge \omega^l.$$  \hspace{1cm} \text{(C.2)}

In the above the torsion $T_{ijk}^\phi$ is related to the “contorsion” $C_{ijk}$ by

$$T_{ijk}^\phi = -(C_{ijk} - C_{ikj}).$$  \hspace{1cm} \text{(C.3)}

The curvatures for the two connections are related by

$$R_{ijkl}^\phi = R_{ijkl}^\omega + (\nabla_k^\phi C_{ijl} - \nabla_l^\phi C_{ijk}) + (C_{imk} C_{mjl} - C_{iml} C_{mjk}).$$  \hspace{1cm} \text{(C.4)}

It is also possible to express the above in terms of the covariant derivative $\nabla^\phi$ with respect to the connection $\phi$. To simplify matters we express the above only for the case where $C_{ijk}$ is totally antisymmetric:

$$R_{ijkl}^\phi - (\nabla_k^\phi C_{ijl} - \nabla_l^\phi C_{ijk}) = R_{ijkl}^\omega - 2C_{ijm} C_{mkj} - (C_{imk} C_{mjl} - C_{iml} C_{mjk}).$$  \hspace{1cm} \text{(C.5)}

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