Rigidity theorems in the braneworld model

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In the present paper, we give some theorems representing rigidity of a vacuum brane in static bulk spacetimes. As an application, we show that a static bulk spacetime with dimension $D \geq 4$ and spatial symmetry $IO(D-2)$ or $O(D-1)$ or $O_+(D-2,1)$ does not allow a vacuum brane with a black hole on it. We also show that if a static bulk spacetime with dimension $D > 4$ satisfying the vacuum Einstein equations can be foliated by a continuous family of vacuum branes with asymptotically constant curvature, it is a black string solution.

§1. Introduction

Recently, braneworld models are actively studied as possible new universe models based on higher dimensional unified theories. In particular, for the Randall-Sundrum models, detailed analyses have been done by many people, and it has been shown that local behavior of gravity is the same as the conventional one in low energies and FRW universe models can be implemented. However, whether global behavior of gravity is consistent with observations is still unclear. In particular, the structure of black holes formed by gravitational collapse remains as an important problem to be clarified, in order to see the viability of braneworld models.

In the present paper, we give some mathematical theorems concerning restrictions on spacetimes imposed by the existence of a vacuum brane, which may be relevant to investigations of black holes in braneworld models. Here, by a vacuum brane, we mean a hypersurface whose extrinsic curvature $K_{\mu\nu}$ is a constant multiple of the induced metric $g_{\mu\nu}$:

$$K_{\mu\nu} = \sigma g_{\mu\nu}. \quad (1.1)$$

This condition is equivalent to the condition that the energy-momentum tensor of the brane is proportional to its metric, if the bulk spacetime is empty and the $Z_2$ symmetry is imposed at the brane. We consider two situations. The first one is a brane in a static $D$-dimensional spacetime with spatial symmetry $G(D-2,K)$, where $G(n,K)$ is the isometry group of a $n$-dimensional space with constant curvature $K$. This is an extension of the problem considered in a paper by Chamblin, Hawking and Reall. In this paper, they showed that a 5-dimensional Schwarzschild-Anti de Sitter spacetime does not contain a brane with a black hole, although there exists a 5-dimensional solution called the black string solution, which admits a brane slice with a Schwarzschild black hole in it. The second problem discussed in the present paper is related to this black string solution. We will show that this solution is characterized as a vacuum bulk solution to the Einstein equations that allows a slice by a continuous family of vacuum branes.
§2. Brane in Static $D$-dimensional Spacetimes with Spatial Symmetry $G(D - 2, K)$

First, we consider a vacuum brane in a $D$-dimensional static bulk spacetime with spatial symmetry $G(D - 2, K)$, whose metric is represented as

$$ds_D^2 = -U(R)dt^2 + \frac{dR^2}{W(R)} + S(R)^2d\sigma_{D-2}^2,$$  \hspace{1cm} (2.1)

where $d\sigma_{D-2}^2 = \gamma_{ij}dz^idz^j$ is the metric of a $(D-2)$-dimensional constant curvature space $M_{D-2}^K$ with sectional curvature $K (K = 0, \pm 1)$. We do not impose the Einstein equations. So, $U(R)$, $W(R)$ and $S(R)$ are arbitrary functions. Note that the $G(D - 2, K)$ symmetry gives the bulk spacetime a natural bundle structure $(N^2, M_{D-2}^K)$, where the base space $N^2$ is a 2-dimensional orbit spacetime with the coordinates $(T, R)$.

Possible configurations of a brane in this spacetime are determined by the condition (1.1). When we expressed the brane configuration as, say, $R = R(T, z)$, (1.1) gives a set of partial differential equations for $R(T, z)$. An interesting point here is that this set of equations do not have a solution for an arbitrary choice of $U$, $W$ and $S$, and the consistency of the equations leads to strong restrictions on the bulk geometry. Since a detailed analysis of this problem is given in a separate paper by the author\textsuperscript{17}, we only give a brief summary of the results in the form of theorems here.

First, for a brane with $\sigma = 0$, the following theorem holds.

**Theorem 2.1** Configurations of a brane with $\sigma = 0$ and allowed geometries are classified into the following three types:

**I-A)** Brane configurations that are represented by subbundles $\Sigma = (N^2, F)$ of $M^D = (N^2, M_{D-2}^K)$, where $F$ is a totally geodesic hypersurface $M$. Configurations of this type always exist irrespective of the choice of $U$, $W$ and $S$, and are mutually isometric. Each configuration is invariant under $\text{IO}(1) \times G(D - 3, K')$ for some $K' \geq K$.

**I-B)** $G(D - 2, K)$-invariant configurations which are represented as $R = R(T)$ by solutions to

$$R_T^2 = WU(1 - AU) \neq 0; \hspace{0.5cm} A > 0. \hspace{1cm} (2.2)$$

The bulk geometry is restricted to a simple product $N^2 \times M_{D-2}^K$, for which $S = \text{constant}$.

**I-C)** Static configurations expressed as $R = \text{constant}$ in terms of solutions to

$$U' = 0, \hspace{0.5cm} S' = 0. \hspace{1cm} (2.3)$$

Each configuration of this type corresponds to an $\text{IO}(1) \times G(D - 2, K)$-invariant totally geodesic hypersurface.

A brane with $\sigma = 0$ can take only configurations of the type I-A in AdS$^D$, while it takes configurations of the type I-B and I-C in $E^{D-1,1}$, and those of the type I-C in $dS^D$. The latter are all mutually isometric in a given bulk geometry. In a bulk spacetime that does not have constant curvature, configurations of the type I-B or
I-C) Static and GI-B) Brane configurations that are G with spatial symmetry G product structure E becomes isometric to those of the type I-A only when the bulk spacetime has the tally geodesic time-like hypersurfaces in spacetimes with the hypersurface is totally geodesic, this theorem gives the complete classification of totally geodesic time-like hypersurfaces in spacetimes with the IO(1) × G(D − 2, K) symmetry and may be useful in other contexts.

Next, for a brane with σ ≠ 0, the following theorem holds.

**Theorem 2.2** A brane with σ ≠ 0 can exists only for special bulk geometries. These bulk geometries and corresponding brane configurations are classified into the following three types:

I-B) Brane configurations that are G(D − 2, K)-invariant and represented as R = R(T) by solutions to

\[ R^2_1 = U^2 \left( 1 - \frac{U}{\sigma^2 R^2} \right) \neq 0. \]  

The bulk geometry is restricted to those with \( U = W \) and \( S = R \). For \( S = R \), the condition \( U = W \) is equivalent to the condition that the Ricci tensor is isotropic in planes orthogonal to \( G(D − 2, K) \)-orbits.

I-C) Static and G(D − 2, K)-invariant brane configurations expressed as R = constant in terms of solutions to

\[ \frac{U'}{U} = \frac{2S'}{S}, \quad W \left( \frac{S'}{S} \right)^2 = \sigma^2. \]

III) Static brane configurations in the bulk geometries with metrics of the form

\[ ds^2_D = d\sigma^2_{\lambda,D-1} - UdT^2, \]

where \( d\sigma^2_{\lambda,D-1} \) is the metric of a \((D − 1)\)-dimensional constant curvature space \( M^D_{\lambda} \) with sectional curvature \( \lambda \). \( U \) is a function on this space that is invariant under a subgroup \( G(D − 2, K) \) of the isometry group of \( M^D_{\lambda} \). Allowed forms of \( U \) and brane configurations are given as follows:

i) \( \lambda = 0 \): In a Cartesian coordinate system \( \mathbf{x} \) for \( E^{D−1} \), \( U = ( (\mathbf{x} - \mathbf{a})^2 + k)^2 \) and brane configurations are represented as \( (\mathbf{x} - \mathbf{b})^2 = 1/\sigma^2 \) with \( (\mathbf{b} - \mathbf{a})^2 = 1/\sigma^2 - k \).

ii) \( \lambda = 1/\ell^2 > 0 \): In a homogeneous coordinate system \( X \) in which \( S^{D−1} \) is expressed as \( X \cdot X = 1 \), \( U = ( P \cdot X + k)^2 \) (\( k \neq 0 \)) and brane configurations are represented as \( Q \cdot X = 1 \) with \( Q \cdot P = -k \) and \( Q \cdot Q = 1 + 1/(\ell^2 \sigma^2) \).

iii) \( \lambda = -1/\ell^2 < 0 \): In a homogeneous coordinate system \( Y \) in which \( H^{D−1} \) is expressed as \( Y \cdot Y = -1 \), \( U = ( P \cdot Y + k)^2 \) (\( k \neq 0 \)) and brane configurations are represented as \( Q \cdot Y = 1 \) with \( Q \cdot P = k \) and \( Q \cdot Q = 1/(\ell^2 \sigma^2 - 1) \).

These brane configurations are not \( G(D − 2, K) \)-invariant, but \( G(D − 3, K') \)-invariant for some \( K' \geq K \) and mutually isometric.

These results implies that a vacuum brane in static \( D \)-dimensional spacetime with spatial symmetry \( G(D − 2, K) \) cannot contains a black hole. It is because if
a brain contains a black hole, it does not belong to the type I, and hence the bulk geometry must be of the type III with special forms of $U$ given above. However, for these bulk geometries, the curvature diverges at points where $U$ vanishes. Hence, it has no horizon.

§3. Static Spacetimes with Brane Slices

Next, we consider a static configuration of a vacuum brane $\Sigma$ in a $D$-dimensional static bulk spacetime $M^D$. Now, we assume that the bulk geometry is a solution to the vacuum Einstein equations with cosmological constant $\tilde{\Lambda}$. However, we do not assume any spatial symmetry of the bulk geometry.

In an appropriate coordinate system $(y, x^\mu)$ in which $\Sigma$ is represented by the hypersurface $y = 0$, the metric of the bulk spacetime can be written

$$ds^2_D = N(y, x)^2 dy^2 + g_{\mu\nu}(y, x)dx^\mu dx^\nu.$$ (3.1)

In this coordinate system, the extrinsic curvature of each $y =$-constant surface $\Sigma_y$ is expressed as

$$\partial_y g_{\mu\nu} = 2NK_{\mu\nu}.$$ (3.2)

The Einstein equations provide a kind of evolution equations for this extrinsic curvature,

$$\partial_y K + \Box N + (K^2 - R)N = -\frac{2(D - 1)}{D - 2}\tilde{\Lambda}N,$$ (3.3a)

$$\partial_y \hat{K}_\nu^\mu + N K_{\nu}^\mu + \nabla_\mu \nabla_\nu N - \frac{1}{D - 1} \Box N \delta_\nu^\mu = NS_{\mu}^\nu,$$ (3.3b)

and the constraints

$$-R + \frac{D - 2}{D - 1}K^2 - \hat{K}_\nu^\mu \hat{K}_\mu^\nu = -2\tilde{\Lambda},$$ (3.4a)

$$\nabla_\nu \hat{K}_\mu^\nu - \frac{D - 2}{D - 1} \partial_\mu K = 0,$$ (3.4b)

where $K = K_\mu^\mu$, $\hat{K}_\mu^\nu$ is the tracefree part of $K_\nu^\mu$, $\nabla_\mu$ is the covariant derivative with respect to $g_{\mu\nu}$ on $\Sigma_y$, and $S_{\mu}^\nu$ is the tracefree part of the Ricci curvature $R_\nu^\mu$ of $g_{\mu\nu}$.

If we assume that $N$ depends only on $y$ and that $g_{\mu\nu}$ has the product form $e^{2\Phi(y)}\hat{g}_{\mu\nu}(x)$, after reparametrizing $y$ so that $N = 1$, the extrinsic curvature is written in the form (1.1) with $\sigma = \partial_y \Phi(y)$. Hence, the evolution equations for $K_{\mu\nu}$ reduce to

$$S_{\nu}^\mu \equiv R_{\nu}^\mu - \frac{R}{D - 1} \delta_\nu^\mu = 0,$$ (3.5a)

$$\partial_y \sigma + (D - 1)^2 \sigma^2 - R = -\frac{2(D - 1)}{D - 2}\tilde{\Lambda},$$ (3.5b)

and the constraints give

$$R = (D - 1)(D - 2)\sigma^2 + 2\tilde{\Lambda}.$$ (3.6)
Since the last equation implies that $R$ depends only on $y$, it follows that each $\Sigma_y$ is an Einstein spacetime, and that when an Einstein metric on $\Sigma = \Sigma_0$ is given, the structure of the bulks spacetime, including $\Phi(y)$, is uniquely determined. This is the well-known black string solution.\(^{16,18}\)

In this section, we show that this black string solution can be characterized as a spacetime in which a vacuum brane can be non-isometrically deformed continuously, i.e., is not rigid. In the present paper, we only consider the case in which the brane configurations as well as the bulk spacetime are static. We assume that $D > 4$. Hence, the metric of the brane can be written

$$ds_{\Sigma}^2 = -V(x)^2 dt^2 + g_{ij}(x) dx^i dx^j,$$

(3.7)

and $N$ is independent of $t$. We also assume that $N$ and $D_iN$ is bounded on $\Sigma$, where $D_i$ is the covariant derivative with respect to $g_{ij}$. The basic equations are (3.2), (3.3) and (3.4), which hold also for an infinitesimal deformation of $\Sigma$ if $\delta y N$ is regarded as a displacement of $\Sigma$ by the deformation along the unit normal to $\Sigma$.

Now, let us assume that

$$\partial_y K^\mu = 0$$

(3.8)

holds in addition to (1.1) on $\Sigma$. Then, the Einstein equations reduce to (3.6) and the set of equations,

$$V \Box N \equiv D \cdot (VDN) = -\tilde{N} V,$$

(3.9a)

$$V \nabla^i \nabla_i DV \cdot D N = -\frac{\tilde{N}}{D-1} V + V N S^i_i,$$

(3.9b)

$$D_i D_j N = -\frac{\tilde{N}}{D-1} g_{ij} + N S_{ij},$$

(3.9c)

where

$$\tilde{N} = \frac{R}{D-2} N + \partial_y K,$$

(3.10)

and $\sigma, \partial_y K$ and $R$ are required to be constant.

First, we consider the case $R = 0$. In this case, we further assume that the geometry of the brane is asymptotically flat and $S^\mu_\nu$ falls off sufficiently rapidly:

$$g_{ij} dx^i dx^j = -V^{-2} dr^2 + r^2 \gamma_{AB} d\theta^A d\theta^B + O\left(\frac{dx \cdot dx}{r^{D-3}}\right),$$

(3.11)

$$V = 1 - \frac{2\mu}{r^{D-4}} + O\left(\frac{1}{r^{D-3}}\right), \quad S^\mu_\nu = o\left(\frac{1}{r^{D-2}}\right).$$

(3.12)

Then, from (3.9b) it follows that $\partial_y K$ vanishes. Further, (3.9c) requires that $N$ behaves as $N = N_0 + o(1/r^{D-4})$, where $N_0$ is some constant. In the meanwhile, multiplying (3.9a) by $N$ and integrating it over a $t =$constant hypersurface $F$ in $\Sigma$, we obtain

$$\int_F d^{D-2}x \sqrt{g} V D^i N D_i N = \int_{\partial F} dS_i V N D^i N, \quad (3.13)$$

where $\partial F$ consists of the sphere at $r = \infty$ and horizons where $V$ vanishes. It is easy to see that these boundary contributions vanish from the above asymptotic estimate.
and the regularity of \(N\) and \(D^i V D_i N\) deduced from the above basic equations. This implies that \(N\) is constant outside the horizons and the metric \(g_{\mu \nu}\) of the brane is a solution to the vacuum Einstein equation \(R_{\mu \nu} = 0\).

Next, let us consider the case \(R = -(D-1)(D-2)/\ell^2 < 0\). In this case, we assume that the brane geometry is asymptotically anti-de Sitter in the sense

\[
g_{ij} dx^i dx^j = \frac{dr^2}{V^2 + O(1/r^{D-3})} + r^2 \left( \gamma_{AB} + O\left(\frac{1}{r}\right) \right) d\theta^A d\theta^B + O\left(\frac{1}{r^{D-3}}\right) dr d\theta,
\]

\[V^2 = 1 + \frac{r^2}{\ell^2} - \frac{2\mu}{r^{D-4}} + O\left(\frac{1}{r^{D-3}}\right), \quad S_\mu^\nu = o\left(\frac{1}{r^{D-3}}\right) .\]

Under this condition, from (3.9c) it follows that \(\tilde{N}\) falls off as \(o(1/r^{D-3})\) at infinity. Then, by a similar argument as that in the case \(R = 0\), we find that (3.9a) requires that \(\tilde{N}\) vanishes. Hence, the brane geometry must be Einstein again.

Unfortunately, we cannot show that \(\tilde{N}\) vanishes by the same method for the case \(R > 0\), because in the equation \(D^i (\tilde{N} V D_i \tilde{N}) = V D^i \tilde{N} D_i \tilde{N} - R \tilde{N}^2 / (D-2)\) obtained from (3.9a), the right-hand side does not have a definite sign. However, under a stronger assumption that \(N\) does not vanish outside horizons, we can obtain the same conclusion by a different method. The basic idea is to use the identity

\[
\int_A d^{D-1} \sqrt{g} N S^{\mu \nu} S_{\mu \nu} = \int_{\partial A} d\Sigma^\mu S_{\mu \nu} \partial^\nu N
\]

obtained from (3.9), where \(A\) is a region outside horizons in \(\Sigma\), cut out by two \(t\) = constant hypersurfaces. We assume that the region of \(\Sigma\) outside the horizons are spatially compact, because the brane is de Sitter like. Then, we can easily show that the contributions from the horizons on the right-hand side of this equation vanish. Further, the contributions from the \(t\) = constant hypersurfaces cancel due to the staticity. Hence, the left-hand side of this equation vanishes. If we take into account the assumption on \(N\) and the fact that \(S^{\mu \nu} S_{\mu \nu} = (S_i^j)^2 + S_{ij} S^{ij} \geq 0\), this implies that \(S_{\mu \nu}\) vanishes and \(\nabla_\mu \nabla_\nu N = c g_{\mu \nu}\), where \(c\) is a constant. Then, applying the Bianchi identity to the divergence of the latter equation, we obtain \(\nabla_\mu N = 0\). Hence, we find that \(N\) is constant and the brane geometry is Einstein.

The results obtained in this section are summarized as follows.

**Theorem 3.1** Let \(\Sigma\) be a static vacuum brane \(\Sigma\) in a static \(D\)-dimensional spacetime with \(D > 4\) satisfying the Einstein equations with \(\tilde{\Lambda}\), and consider a static infinitesimal deformation of \(\Sigma\) along its normal proportional to a function \(N\) on \(\Sigma\). Then, if the deformation preserves the isotropy of the extrinsic curvature, the Ricci scalar \(R\) of \(\Sigma\) is constant. Further, if \(\Sigma\) is asymptotically flat \((R = 0)\) or asymptotically AdS \((R < 0)\) and \(N\) is uniformly bounded, \(\Sigma\) is an Einstein spacetime and \(N\) is a constant. The same result holds also for \(R > 0\), if \(N\) does not vanish outside horizons on \(\Sigma\).

If the bulk spacetime is analytic, it immediately follows from this theorem that an infinitesimal deformation of the brane with the required property exists only
when the geometry of the bulk spacetime is of the black string type, because the bulk geometry is uniquely determined by the metric and the extrinsic curvature on the brane. Further, even when the analyticity is not assumed, we can obtain the same conclusion if we require that the bulk spacetime can be foliated by a family of vacuum branes with bounded $N$. It is because the theorem guarantees that the assumption used to obtain the black string solution holds in this case, if we choose the $y$-coordinate in (3.1) so that $y$ is constant on each brane. Hence, the following theorem holds.

**Theorem 3.2** If the bulk spacetime with dimension $D > 4$ satisfying the vacuum Einstein equations can be foliated by a family of vacuum branes with bounded $N$ and asymptotically constant curvature, it is a black string solution.

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