Characterization of the most probable transition paths of stochastic dynamical systems with stable Lévy noise

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Received 27 January 2019
Accepted for publication 23 April 2019
Published 13 June 2019

Online at stacks.iop.org/JSTAT/2019/063204
https://doi.org/10.1088/1742-5468/ab1ddc

Abstract. This work is devoted to the investigation of the most probable transition paths for stochastic dynamical systems with either symmetric $\alpha$-stable Lévy motion or Brownian motion. For stochastic dynamical systems with Brownian motion, minimizing an action functional is a general method to determine the most probable transition path. We have developed a method based on path integrals to obtain the most probable transition path of stochastic dynamical systems with either symmetric $\alpha$-stable Lévy motion ($0 < \alpha < 1$) or Brownian motion. Furthermore, we have shown that the most probable path can be characterized by a deterministic dynamical system.

Keywords: Path integrals, most probable transition path, stochastic dynamical systems, stable Levy noise

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1. Introduction

Stochastic differential equations (SDEs) have been widely used to describe complex phenomena in physical, biological, and engineering systems. Transition phenomena between dynamically significant states occur in nonlinear systems under random fluctuations. Hence a practical problem is that given a stochastic dynamical system, how to capture the transition behavior between two metastable states, and then how to determine the most probable transition path. This subject has been a research topic for a number of authors [1–12].

In this paper, we consider the following SDE in the state space $\mathbb{R}^k$:

$$dX_t = b(X_t)dt + dL_t, \quad T_0 \leq t \leq T_f, \quad X_{T_0} = X_0,$$

(1)

where $L_t$ is a $k$-dimensional symmetric $\alpha$-stable (non-Gaussian) Lévy motion in the probability space $(\Omega, \mathbb{P})$. The solution process $X_t$ uniquely exists under appropriate conditions on the drift term $b(x) : \mathbb{R}^k \to \mathbb{R}^k$ (see the next section). Moreover, $T_0, T_f$ are the initial and final time instants, respectively. For simplicity, we first consider
the one dimensional case \((k = 1)\) and will extend to higher dimensional cases \((k > 1)\) in section 3.

We will also compare with the following one dimensional SDE system with (Gaussian) Brownian motion \(B_t\):
\[
dX_t = b(X_t)\,dt + dB_t, \quad T_0 \leq t \leq T_f, \quad X_{T_0} = X_0.
\]

The Onsager–Machlup function [1] and Path integrals [2, 11, 13] are two methods for studying the most probable transition path of this system (2). The central points of these two methods were to express the transition probability (density) function of a diffusion process by means of a functional integral over paths of the process. That is for the solution process \(X_t\), with initial time and position \((T_0, X_0)\) and final time and position \((T_f, X_f)\). In Stratonovich discretization prescription, the transition probability density \(p(X_f, T_f|X_0, T_0)\) (or denoted by \(p(X_f|X_0)\)) is expressed as a path integral
\[
p(X_f, T_f|X_0, T_0) = \int_{x_{T_0} = x_0}^{x_{T_f} = x_f} D\!x \exp\left\{-\frac{1}{2} \int_{T_0}^{T_f} [\dot{x}_s - b(x_s)]^2 + b'(x_s)\,ds\right\},
\]
where \(\mathcal{L}(x, \dot{x}) = \frac{1}{2}[(\dot{x} - b(x))^2 + b'(x)]\) is called the Lagrangian of (1). In Onsager–Machlup’s method, \(\text{OM}(x, \dot{x}) = (\dot{x} - b(x))^2 + b'(x)\) is called the OM function. When the path \(x_t\) is restricted in continuous functions mapping from \([T_0, T_f]\) into \(\mathbb{R}\), the exponent \(S(x) = -\frac{1}{2} \int_{T_0}^{T_f} [\dot{x} - b(x)]^2 + b'(x)\,ds\) is called the Onsager–Machlup action functional.

Hence finding the most probable transition path is to find a path \(x_t\) such that the Lagrangian (OM function or the action functional) to be minimum, which is called the least action principle. This leads to the Euler–Lagrangian equation by means of a variational principle when the path restricted in twice differentiable functions. For more details of path integrals and applications, see [11, 14–17] and references therein.

In this paper, we will determine the most probable transition path for the stochastic system with non-Gaussian noise (1). The situation is different from the Gaussian case (2). If we try to get the exponential form (containing the action functional) for the transition probability density function for the transition path as in the Gaussian case, we need to use the Fourier transformation of the probability density [18, 19] (or characteristic function). For instance, the characteristic function of a \(\alpha\)-stable Lévy random variable is [20, 21]
\[
\psi(u) = \exp\{iu\sigma^\alpha |u|^\alpha [1 - i\beta u |u| w(u, \alpha)]\},
\]
where \(0 < \alpha < 2\) is the Lévy index, \(-1 \leq \beta \leq 1\) is the skewness parameter, \(\eta \in \mathbb{R}\) is the shift parameter, \(\sigma \in \mathbb{R}^+\) the scale parameter and
\[
w(u, \alpha) = \begin{cases} \tan\left(\frac{\pi \alpha}{2}\right), & \alpha \neq 1, \\ -\frac{2}{\pi} \ln(|u|), & \alpha = 1. \end{cases}
\]
The density function of this random variable is
\[
f(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \exp(-ixu)\psi(u)\,du.
\]
Thus it brings the Fourier integral into the density function. So for path integral representation with this density form, it is hard (and this is unlike (3)) to obtain a convergent action function representation of paths:

\[
p(X_f, T_f|X_0, T_0) = \lim_{n \to \infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \Pi_n f(x_n - x_{n-1} - b(x_{n-1} \Delta t)) \delta(x_n - X_f) J \, dx_1 \cdots dx_n,
\]

where \( n \) is the partition number and \( J \) is the Jacobian of the transformation given by

\[
J = \det \left( \frac{\partial L_i}{\partial x_k} \right), \quad i = 1, \cdots, n, \quad k = 1, \cdots, n.
\]

Instead, in this paper, we develop a method to characterize the most probable transition path, based on the path integral rather than on the action functional (or the Onsager–Machlup function). This is made possible with a new representation [22] for the transition probability density functions of symmetric Lévy motions in terms of two families of metrics. This representation provides an exponential structure of the transition probability density function [22, 23]. It can be further extended in our case, which will be discussed in section 2.2.

This paper is organized as follows. In section 2, we recall some preliminaries. In section 3, we develop a method to characterize the most probable transition paths for a stochastic system with symmetric \( \alpha \)-stable Lévy motion (\( 0 < \alpha < 1 \)) or Brownian motion. In section 4, we extend the results of section 3 to higher dimensional cases. Finally, in section 5, we present several examples to illustrate our results.

2. Preliminaries

2.1. Lévy motions

We recall some basic facts about 1-dimensional (1D) Lévy motions (or Lévy processes) [20, 21, 24].

**Definition 1.** A stochastic process \( L_t \) is a Lévy process if

(i) \( L_0 = 0 \) (a.s.);

(ii) \( L_t \) has independent increments and stationary increments; and

(iii) \( L_t \) has stochastically continuous sample paths, i.e. for every \( s \geq 0 \), \( L(t) \to L(s) \) in probability, as \( t \to s \).

A Lévy process \( L_t \) taking values in \( \mathbb{R} \) is characterized by a drift term \( \eta \in \mathbb{R} \), a non-negative variance \( Q \) and a Borel measure \( \nu \) defined on \( \mathbb{R} \setminus \{0\} \). \((\eta, Q, \nu)\) is called the generating triplet of the Lévy motion \( L_t \). Moreover, the Lévy–Itô decomposition for \( L_t \) as follows:

\[
L_t = \eta t + B_Q(t) + \int_{|z|<1} z \tilde{N}(t, dz) + \int_{|z|\geq1} z N(t, dz),
\]

where \( \tilde{N} \) and \( N \) are Poisson processes with intensities \( \nu \) and \( \eta \nu \) respectively.
where $N(dt, dz)$ is the Poisson random measure, $\tilde{N}(dt, dz) = N(dt, dz) - \nu(dz)dt$ is the compensated Poisson random measure, and $\nu(S) \triangleq \mathbb{E} N(1, S)$, here $\mathbb{E}$ denotes the expectation with respect to the probability $\mathbb{P}$, and $B_{\alpha}(t)$ is a Brownian motion with variance $\sigma$. The characteristic function of $L_t$ is given by

$$\mathbb{E}[\exp(iuL_t)] = \exp(-t\psi(u)), \ u \in \mathbb{R},$$

where the function $\psi : \mathbb{R} \to \mathbb{C}$ is the characteristic exponent

$$\psi(u) = iu\eta + \frac{1}{2}Qu^2 + \int_{\mathbb{R}\setminus\{0\}} (1 - e^{iux} + iux\mathbb{I}_{|x|<1})\nu(dz).$$

The Borel measure $\nu$ is called the jump measure.

### 2.2. Asymptotic properties of the probability density functions of $\alpha$-stable Lévy motions

From now on, we consider a scalar symmetric $\alpha$-stable Lévy processes. Recall the standard symmetric $\alpha$-stable random variable has distribution $S_{\alpha}(1, 0, 0)$. Here $S_{\alpha}(\sigma, \beta, \mu)$ is the distribution of a stable random variable, with $\sigma$ the scale parameter, $\beta$ the skewness parameter and $\mu$ the shift parameter. The corresponding probability density function $f_{\alpha}(x)$ can be represented as an infinite series [24–26]

$$f_{\alpha}(x) = \begin{cases} \frac{1}{\pi\alpha} \sum_{k=1}^{\infty} \frac{(1)^{k-1}}{k!} \Gamma(\alpha k + 1) |x|^{-\alpha k} \sin\left(\frac{\alpha k\pi}{2}\right), & x \neq 0, \ 0 < \alpha < 1, \\ \frac{1}{\pi} \int_{0}^{\infty} e^{-\alpha u} du, & x = 0, \ 0 < \alpha < 1, \\ \frac{1}{\pi(1+x^2)}, & \alpha = 1, \\ \frac{1}{\pi\alpha} \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k)!} \Gamma\left(\frac{2k+1}{\alpha}\right)x^{2k}, & 1 < \alpha < 2. \end{cases}$$

Recall the probability density function $f_b(x)$ for a Brownian random variable $X \sim N(0, \sigma^2)$ is [24]

$$f_b(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{x^2}{2\sigma^2}}.$$

In [19], it was proved that the transition probability density function $p(x_t|x_0)$ of a symmetric $\alpha$-stable Lévy process is

$$p(x_t|x_0) = \frac{1}{t^{1/\alpha}} f_{\alpha}\left(\frac{x_t - x_0}{t^{1/\alpha}}\right)$$

$$= \frac{1}{t^{1/\alpha}} \exp\left[\left( - \ln f_{\alpha}\left(\frac{x_t - x_0}{t^{1/\alpha}}\right)\right)\right]$$

$$\triangleq \frac{1}{t^{1/\alpha}} \exp\left[ - \theta_t^\alpha(x_t - x_0)\right],$$

where $\theta_t^\alpha(\cdot)$ is a function maps $[0, \infty)$ to $[0, \infty)$ for any $\alpha \in (0, 2)$ and $t \in (0, \infty)$. Differentiate $\theta_t^\alpha(x)$ with respect to variable $x$:

$$(\theta_t^\alpha(x))' = (- \ln f_{\alpha}\left(\frac{x}{t^{1/\alpha}}\right))'$$

$$= -\frac{f_{\alpha}'\left(\frac{x}{t^{1/\alpha}}\right)}{f_{\alpha}\left(\frac{x}{t^{1/\alpha}}\right)} \frac{1}{t^{1/\alpha}}$$

https://doi.org/10.1088/1742-5468/ab1ddc
which shows that $\theta_t^\alpha(\cdot)$ is a strict increase function since $f'_\alpha(x) \leq 0$ for symmetric $\alpha$-stable Lévy random variables. Now we focus on the concavity of the function $\theta_t^\alpha(\cdot)$.

\[
(\theta_t^\alpha(x))'' = \left(-\frac{f'_\alpha(x)}{f_\alpha(x)}\right) \frac{1}{t^{\alpha/\gamma}} + \left(-\frac{f''_\alpha(x)}{f_\alpha(x)}\right)\frac{(f'_\alpha(x))^2 - f''_\alpha(x)}{t^{2\alpha/\gamma}}.
\]  

(16)

In [20, 25], it was proved that if $X \sim S_\alpha(\sigma, \beta, \mu)$ and $\alpha \in (0, 2)$,

\[
\lim_{y \to \infty} y^\alpha \mathbb{P}(X > y) = C_\alpha \frac{1 + \beta}{2} \sigma^\alpha,
\]

\[
\lim_{y \to \infty} y^\alpha \mathbb{P}(X < -y) = C_\alpha \frac{1 - \beta}{2} \sigma^\alpha,
\]

(17)

where

\[
C_\alpha = \left(\int_0^\infty x^{-\alpha} \sin(x)dx\right)^{-1}.
\]

(18)

We use this result to study the asymptotic behavior of tail probabilities. For $y$ large enough,

\[
y^\alpha \mathbb{P}(X > y) = C_\alpha \frac{1 + \beta}{2} \sigma^\alpha,
\]

\[
\Leftrightarrow y^\alpha \int_y^\infty f_\alpha(x)dx = C_\alpha \frac{1 + \beta}{2} \sigma^\alpha,
\]

\[
\Leftrightarrow f_\alpha(y) = \alpha C_\alpha \frac{1 + \beta}{2} \sigma^\alpha y^{-\alpha - 1} \triangleq C y^{-\alpha - 1},
\]

\[
\Leftrightarrow f''_\alpha(y) f_\alpha(y) - (f'_\alpha(y))^2 = C^2 (\alpha + 1) y^{-2\alpha - 4},
\]

(19)

which means that the asymptotic behavior of tail concavity and convexity of $\theta_t^\alpha(\cdot)$ is concave. And the graphs of $\theta_t^\alpha(\cdot)$ are shown in figure 1.

For the Brownian motion case, similar to the symmetric $\alpha$-stable Lévy motion case, the corresponding exponent is $\theta_t^B(x) = \frac{x^2}{2t}$, which is convex.

We also notice that

\[
\theta_t^\alpha(x - y) = \theta_t^\alpha(|x - y|) \triangleq \theta_t^\alpha(x, y),
\]

\[
\theta_t^B(x - y) = \theta_t^B(|x - y|) \triangleq \theta_t^B(x, y).
\]

(20)

2.3. Conditions for the well-posedness of the system (1) and (2)

For the system (1):

\[
dX_t = b(X_t)dt + dL_t, \quad T_0 \leq t \leq T_f, \quad X_{T_0} = X_0,
\]

it was proved in [27, 28] that if $b(x)$ is locally Lipschitz continuous function and satisfies the ‘one sided linear growth’ condition in the following sense:
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For any $r > 0$, there exists $K_1 > 0$ such that,

$$|b(y_1) - b(y_2)|^2 \leq K_1 |y_1 - y_2|^2,$$

then there exists a unique global solution to (1) and the solution is adapted and càdlàg. These two conditions also guarantee the existence and uniqueness for the solution of (2).

3. Method

In this section, we first study the transition behavior of the system without drift term, and then we study the general case for system (1) and (2). Before we develop the method, we present the following definition.

**Definition 2.** For a solution path $X_t$ and the time interval $[T_0, T_f]$ with a partition $T_0 = t_0 \leq t_1 \leq \cdots \leq t_n = T_f$, define the sequence $\{X_{t_0}, X_{t_1}, \cdots, X_{t_n}\}$ as a discretized path of $X_t$. And a discretized path $\{X_{t_0}, X_{t_1}, \cdots, X_{t_n}\}$ is said to be monotonic with respect to the time partition $T_0 = t_0 \leq t_1 \leq \cdots \leq t_n = T_f$ if either $X_{t_0} \leq X_{t_1} \leq \cdots \leq X_{t_n}$ or $X_{t_0} \geq X_{t_1} \geq \cdots \geq X_{t_n}$.
3.1. Most probable transition path in the absence of drift

Theorem 1 (Monotonicity for the most probable transition path in the absence of drift). For system (1) and (2) with drift $b \equiv 0$, every discretized path $\{X_0, X_t, \cdots, X_n\}$ of the most probable transition path is monotonic with respect to every time partition $T_0 = t_0 \leq t_1 \leq \cdots \leq t_n = T_f$.

Proof. For a time interval partition $T_0 = t_0 \leq t_1 \leq \cdots \leq t_n = T_f$, define $t_{i+1} - t_i = \epsilon = (T_f - T_0)/n, \; i = 0, 1, \cdots, n - 1$. (This proof is also true for an arbitrary partition.) In path integral method, the transition density function (or Markov transition probability) of $X_t$ of (1) with $b \equiv 0$ is

$$p(X_f, T_f|X_0, T_0) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{i=1}^{n-2} dx_i \prod_{i=1}^{n-1} p(x_i|x_{i+1})$$

$$= \int D_n x \exp\{-\sum_{i=1}^{n-2} \theta^\alpha(x_i, x_{i+1})\}$$

$$= \int D_n x \exp\{-\sum_{i=1}^{n-1} \theta^\alpha(x_i, x_{i+1})\}$$

$$= \int D_n x \exp\{-\sum_{i=1}^{n-1} \theta^\alpha(x_i, x_{i+1})\}$$

where $D_n x = \epsilon^{-1/\alpha} \prod_{i=1}^{n-1} \epsilon^{-1/\alpha} dx_i$. Note that $x$ of $S_n(x)$ is a path connecting $(T_0, X_0)$ and $(T_f, X_f)$ (i.e. it starts at $X_0$ at the time $t = T_0$ and reaches $X_f$ at the time $t = T_f$). We call $S_n(x)$ the action quantity of $x$. The contribution of a path $x$ to the transition probability density $p_t(X_f, T_f|X_0, T_0)$ depends on $S_n(x) = \sum_{i=1}^{n-1} \theta^\alpha(x_i, x_{i+1})$.

In order to find the most probable transition path $u_t$, we are supposed to find it satisfies that

$$S_n(u_t) = \min_{x_t \in D_n x} S_n(x_t),$$

where $D_n x$ denotes the set of paths that connect $(T_0, X_0)$ and $(T_f, X_f)$. Equivalently

$$\frac{S_n(x_t)}{S_n(u_t)} \geq 1,$$

for every path $x_t$ connecting $(T_0, X_0)$ and $(T_f, X_f)$. Here $n$ goes to infinity, and the limit $n \rightarrow \infty$ is dropped for now for clarity.

Without loss of generality, we set $X_0 < X_f$. When $n = 2$ (the time partition is $\{T_0 = t_0 < t_1 < t_2 = T_f\}$), for a path $x_t$ connecting $(T_0, X_0)$ and $(T_f, X_f)$, $S_2(x_t) = \theta^\alpha(x_t, x_{t+1})$, if $x_{t+1} < X_0$. Since $\theta^\alpha(\cdot)$ is a strictly increasing function, together with $|x_{t+1} - X_0| > |X_0 - X_0| = 0$ and $|X_f - x_{t+1}| > |X_f - X_0|$, we have $S_2(x_t) > \theta^\alpha(x_t, x_{t+1}) + \theta^\alpha(X_0, X_f)$ where $\{X_0, x_{t+1} = X_f, X_f\}$ is a discretized path of a certain transition path connecting $(T_0, X_0)$ and $(T_f, X_f)$. The case that $x_{t+1} > X_f$ is
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similar. When we add time partitions the situation is similar again. Thus the discretized path \( \{X_{t_0}, X_{t_1}, \ldots, X_{t_n}\} \) of the most probable transition path of \( X_t \) is supposed to be monotonic with respect to the time partition, otherwise there exists a path whose action quantity is smaller.

For system (2), the proof is similar.

The reason that we use (23) to find the most probable transition path is: for symmetric \( \alpha \)-stable Lévy motion case, the corresponding action quantity \( S_n(x_t) \) goes to infinity as long as \( n \) goes to infinity for any path \( x_t \in D_X \) (since \( S_n(x_t) \geq n\theta^\alpha(0) \) and \( \theta^\alpha(0) \) is a positive constant. Here \( n\theta^\alpha(0) \) is the action quantity of the path \( X_t \equiv X_0, t \in [T_0, T_f] \).

For simplicity we say the action quantity of \( x_t \) has higher order than \( u_t \) if

\[
\lim_{n \to \infty} \frac{S_n(x_t) - n\theta^\alpha(0)}{S_n(u_t) - n\theta^\alpha(0)} = +\infty.
\]

That is, we compare the action quantities of two paths with the help of the fixed path \( (X_t \equiv X_0, t \in [T_0, T_f]) \), which considers \( X_t \equiv X_0 \) \( (t \in [T_0, T_f]) \) as a reference path. It helps us to compare the action quantities easily, which will be shown in the proof of corollary 1. Actually,

\[
\begin{align*}
S_n(u_t) - n\theta^\alpha(0) &= \sum_{i=1}^n \theta^\alpha(x_i, x_{i-1}) - n\theta^\alpha(0) \\
&= \sum_{i=1}^n \left[ \theta^\alpha(x_i, x_{i-1}) - \theta^\alpha(0) \right] \\
&= \sum_{i=1}^n \left[ -\ln f_\alpha(x_i - x_{i-1}) + \ln f_\alpha(0) \right] \\
&= \sum_{i=1}^n \left[ -\ln \frac{f_\alpha(x_i - x_{i-1})}{f_\alpha(0)} \right].
\end{align*}
\]

Denoting \( g_\alpha(x_i - x_{i-1}) = \frac{f_\alpha(x_i - x_{i-1})}{f_\alpha(0)} \), the function \( g_\alpha(\cdot) \) could be regarded as a new ‘probability’ density function whose integration over \((-\infty, \infty)\) is \( \frac{1}{f_\alpha(0)} \).

The most probable transition path might not be unique, i.e. there might be several paths satisfying (22) or (23). In fact we will see in the following corollary that, for system (1), the number of the most probable transition paths is infinity but they can be characterized by a class of paths that have only one jump, and the difference among the paths of this class is the jump time. See remark 1 after the proof of the following corollary.

**Corollary 1.**

(i) For system (1) with \( b \equiv 0 \), if \( L_t \) is a symmetric \( \alpha \)-stable Lévy noise with \( 0 < \alpha < 1 \), then the most probable transition path is not unique, and it can be represented as a Heaviside-like function.
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\begin{equation}
X^m_t = \begin{cases} 
X_0, & T_0 \leq t < t^*, \\
X_f, & t^* \leq t \leq T_f,
\end{cases}
\end{equation}

for every time instant \( t^* \) satisfying \( T_0 < t^* \leq T_f \);

(ii) For system (2) with drift \( b \equiv 0 \), the most probable transition path is the line segment connecting \( X_0 \) and \( X_f \): \( X^m_t = X_0 + \frac{t-T_0}{T_f-T_0} (X_f-X_0) \);

Proof.

(i) For system (1) with drift \( b \equiv 0 \), as shown in section 2, we notice that for a time interval partition \( T_0 = t_0 \leq t_1 \leq \cdots \leq t_n = T_f, t_{i+1} - t_i = \epsilon = \frac{T_f-T_0}{n} \),

\begin{equation}
\theta_\alpha^c(|x-y|) = \theta_\alpha^c\left(\frac{|x-y|}{\epsilon^{1/\alpha}}\right) = \theta_\alpha^c(n^\alpha|X_0 - X_f|) \cdot \left(\frac{1}{(T_f - T_0)^{1/\alpha}}\right).
\end{equation}

Define a path space \( DM_{X_0} = \{ x_t | x_t \text{ is a monotonic path connects } (T_0, X_0) \text{ and } (T_f, X_f) \} \). So one should search for the most probable transition path within path space \( DM_{X_0} \).

Take a path \( x_t \in DM_{X_0} \) and a time partition \( \{ T_0 = t_0 < t_1 < \cdots < t_n = T_f \} \). Assume that \( \{ \lambda_j \}_{j=0}^{n-1} \) are non-negative constants, and \( x_{t_{j+1}} - x_{t_j} = \lambda_j (X_f - X_0) (0 \leq j \leq n - 1) \).

It is easy to see that \( \sum_{j=0}^{n-1} \lambda_j = 1 \) by the theorem 1, and \( 0 \leq \lambda_j \leq 1 \). As discussed in section 2.2, \( \theta_\alpha^c(\cdot) \) is concave in \( [r, \infty) \) for some constant \( r \in \mathbb{R}^+ \) (\( r \) depending on \( \alpha \)).

For \( n^\alpha \lambda_j \frac{X_f-X_0}{(T_f-T_0)^{1/\alpha}} \leq n^\alpha \lambda_j C_{0f} \geq r \), and \( n \) large enough, we obtain

\begin{equation}
\theta_\epsilon^c(|x_{t_{j+1}} - x_{t_j}|) - c = \theta_\epsilon^c(n^\alpha \lambda_j \frac{X_f-X_0}{(T_f-T_0)^{1/\alpha}}) - c \\
\geq \lambda_j (\theta_\epsilon^c(n^\alpha \lambda_j \frac{X_f-X_0}{(T_f-T_0)^{1/\alpha}}) - c) \\
= \lambda_j (\theta_\epsilon^c(|X_f-X_0|) - c),
\end{equation}

where \( c = \theta_\epsilon^c(0) \) is a positive constant and \( \theta_\epsilon^c(\cdot) - c \) is non-negative and concave in \( [r, \infty) \).

When \( n^\alpha \lambda_j C_{0f} < r \), we have

\begin{equation}
0 \leq \sum_{n^\alpha \lambda_j C_{0f} < r} \lambda_j < \frac{r}{n^\alpha C_{0f}}
\end{equation}

and

\begin{equation}
\lim_{n \to \infty} \sum_{n^\alpha \lambda_j C_{0f} < r} \lambda_j = 0, \quad \lim_{n \to \infty} \sum_{n^\alpha \lambda_j C_{0f} \geq r} \lambda_j = 1.
\end{equation}
Hence
\[
\lim_{n \to \infty} \frac{(S_n(x_t) - n \theta^\alpha(0))/\{(n - 1) \theta^\alpha(0) + \theta^\alpha(n^{1/\alpha}C_0f)\} - n \theta^\alpha(0)}{n = \lim_{n \to \infty} \sum_{j=0}^{n-1} \theta^\alpha(n^{1/\alpha} \lambda_j C_0f) - n \theta^\alpha(0)}/\{(n - 1) \theta^\alpha(0) + \theta^\alpha(n^{1/\alpha}C_0f) - \theta^\alpha(0)\}
\]
\[
= \lim_{n \to \infty} \sum_{n^{1/\alpha} \lambda_j C_0f < r} (\theta^\alpha(n^{1/\alpha} \lambda_j C_0f) - \theta^\alpha(0)) + \sum_{n^{1/\alpha} \lambda_j C_0f \geq r} (\theta^\alpha(n^{1/\alpha} \lambda_j C_0f) - \theta^\alpha(0))/\{(n - 1) \theta^\alpha(0) + \theta^\alpha(n^{1/\alpha}C_0f) - \theta^\alpha(0)\}
\]
\[
\geq \lim_{n \to \infty} \sum_{n^{1/\alpha} \lambda_j C_0f \geq r} (\theta^\alpha(n^{1/\alpha} \lambda_j C_0f) - \theta^\alpha(0))/\{(n - 1) \theta^\alpha(0) + \theta^\alpha(n^{1/\alpha}C_0f) - \theta^\alpha(0)\}
\]
\[
\geq \lim_{n \to \infty} \sum_{n^{1/\alpha} \lambda_j C_0f \geq r} \lambda_j (\theta^\alpha(n^{1/\alpha} C_0f) - \theta^\alpha(0))/\{(n - 1) \theta^\alpha(0) + \theta^\alpha(n^{1/\alpha}C_0f) - \theta^\alpha(0)\}
\]
\[
= \lim_{n \to \infty} \sum_{n^{1/\alpha} \lambda_j C_0f \geq r} \lambda_j = 1.
\]

This means that the most probable transition path for symmetric \( \alpha \)-stable Lévy process \((0 < \alpha < 1)\) is a Heaviside-like function
\[
X_t^m \triangleq \begin{cases} 
X_f, & t \geq t^*, \\
X_0, & t < t^*, 
\end{cases}
\]

or
\[
X_t^m \triangleq \begin{cases} 
X_f, & t > t^*, \\
X_0, & t \leq t^*, 
\end{cases}
\]

where \( t^* \) satisfies \( T_0 \leq t^* \leq T_f \). Hence in this case, the most probable transition path is not unique since the ‘jump time’ \( t^* \) can be chosen arbitrarily.

(ii) For system (2) with \( b \equiv 0 \), by theorem 1 and the fact that \( \theta^b(\cdot) \) being convex, we conclude that
\[
\sum_{i=0}^{n-1} \theta^b(x_{t_{i+1}}, x_{t_i}) = \sum_{i=0}^{n-1} \theta^b(|x_{t_{i+1}} - x_{t_i}|)
\]
\[
\geq n \theta^b(\sum_{i=0}^{n-1} \frac{1}{n} |x_{t_{i+1}} - x_{t_i}|)
\]
\[
= n \theta^b(\frac{1}{n} |X_f - X_0|).
\]

The inequality holds if and only if
\[
\theta^b(x_{t_1}, x_{t_0}) = \theta^b(x_{t_2}, x_{t_1}) = \cdots = \theta^b(x_{t_n}, x_{t_{n-1}})
\]
\[
\iff x_{t_1} - x_{t_0} = x_{t_2} - x_{t_1} = \cdots = x_{t_n} - x_{t_{n-1}}.
\]
So the most probable transition path is the line segment which connects the initial and final points. Therefore, we obtain

\[ X_t^n = \frac{i(X_f - X_0)}{n}, \quad i = 1, 2, \cdots, n - 1. \]  

(35)

When \( n \) goes to infinity, \( X_t^n = X_0 + \frac{t-T_0}{T_f-T_0} (X_f - X_0) \). This implies that the most probable transition path is the path for the particle (i.e., solution) moving in constant velocity.

**Remark 1.** For symmetric \( \alpha \)-stable Lévy motion with \( 0 < \alpha < 1 \), the proof of corollary 1 compares the transition paths’ action quantities in path-wise sense. We now study the probability over all paths starting at \( X_0 \) at time \( T_0 \) and conditioned at a given end point \( X_f \) at time \( T_f \) to find the particle at point \( X \) at time \( t \in [T_0, T_f] \). This probability can be written as (without loss of generality we assume \( X_f > X_0 \))

\[
P(X, t) = \frac{p(X, t|X_0, T_0) p(X_f, T_f|X, t)}{p(X_f, T_f|X_0, T_0)}
\]

\[
= \frac{1}{p(X_f, T_f|X_0, T_0)} \frac{1}{|t-T_0|^{1/\alpha}|T_f-t|^{1/\alpha}} f_\alpha \left( \frac{X-X_0}{|t-T_0|^{1/\alpha}} \right) f_\alpha \left( \frac{X_f-X}{|T_f-t|^{1/\alpha}} \right),
\]

(36)

which was studied by [30, 31]. So when \( t \) is fixed, the probability \( P(X, t) \) is a function depending on \( f_\alpha \left( \frac{X-X_0}{|t-T_0|^{1/\alpha}} \right) f_\alpha \left( \frac{X_f-X}{|T_f-t|^{1/\alpha}} \right) \). Note that \( f_\alpha \left( \frac{X-X_0}{|t-T_0|^{1/\alpha}} \right) \) has a peak at \( X_0 \), and \( f_\alpha \left( \frac{X_f-X}{|T_f-t|^{1/\alpha}} \right) \) has a peak at \( X_f \). Thus the product \( f_\alpha \left( \frac{X-X_0}{|t-T_0|^{1/\alpha}} \right) f_\alpha \left( \frac{X_f-X}{|T_f-t|^{1/\alpha}} \right) \) increases as \( X \uparrow X_0 \) and decreases as \( X \downarrow X_f \). That is, the product reaches the global maximal value in \([X_0, X_f]\). Suppose that \( X_0 \leq X \leq X_f \). The product can be rewritten as

\[
f_\alpha \left( \frac{X-X_0}{|t-T_0|^{1/\alpha}} \right) f_\alpha \left( \frac{X_f-X}{|T_f-t|^{1/\alpha}} \right) = \exp\left(-\left( \theta_{t-T_0}^\alpha (X-X_0) + \theta_{T_f-t}^\alpha (X_f-X) \right) \right).
\]

(37)

Hence

\[
\theta_{t-T_0}^\alpha (X-X_0) + \theta_{T_f-t}^\alpha (X_f-X)
= \theta_t^\alpha \left( \frac{X_f-X_0}{|t-T_0|^{1/\alpha}} X_f-X_0 \right) + \theta_t^\alpha \left( \frac{X_f-X_0}{|T_f-t|^{1/\alpha}} X_f-X_0 \right)
\geq \frac{X-X_0}{X_f-X_0} \theta_t^\alpha \left( \frac{X_f-X_0}{|t-T_0|^{1/\alpha}} X_f-X_0 \right) + \frac{X_f-X_0}{X_f-X_0} \theta_t^\alpha \left( \frac{X_f-X_0}{|T_f-t|^{1/\alpha}} X_f-X_0 \right) + \frac{X-X_0}{X_f-X_0} \theta_t^\alpha (0)
\geq \left( \frac{X-X_0}{X_f-X_0} \theta_t^\alpha \left( \frac{X_f-X_0}{|t-T_0|^{1/\alpha}} X_f-X_0 \right) \theta_t^\alpha \left( \frac{X_f-X_0}{|T_f-t|^{1/\alpha}} X_f-X_0 \right) \right) + \theta_t^\alpha (0)
\geq \min \left\{ \theta_t^\alpha \left( \frac{X_f-X_0}{|t-T_0|^{1/\alpha}} X_f-X_0 \right), \theta_t^\alpha \left( \frac{X_f-X_0}{|T_f-t|^{1/\alpha}} X_f-X_0 \right) \right\} + \theta_t^\alpha (0),
\]

(38)

where the first inequality approximately holds if the function \( \theta_t^\alpha (\cdot) \) is approximately considered as a concave function in \([0, \infty)\) (or when \( \frac{X_f-X_0}{|t-T_0|^{1/\alpha}} \) and \( \frac{X_f-X_0}{|T_f-t|^{1/\alpha}} \) are large enough).
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Thus $\mathcal{P}(X, t)$ reaches the maximal value at $X_0$ when $t \in [T_0, T_0 + T_0]$, and it reaches the maximal value at $X_f$ when $t \in [T_f, T_f + T_0]$. At time $T_0 + T_0$, the maximal value of $\mathcal{P}(X, t)$ is reached at $X_0$ and $X_0$ simultaneously. It thus appears that the transition process jumps at the time instant $\frac{T_f + T_0}{2}$ most probably.

Inspired by this related observation, we could choose $t^* = \frac{T_f + T_0}{2}$ in corollary 1, considering the transition process in time-point-wise sense. This is one plausible option that leads to the specific most probable path.

**Corollary 2.** For symmetric $\alpha$-stable Lévy motions with $0 < \alpha < 1$, in $n$-partition path integral representation (21) (that is, the time interval $[T_0, T_f]$ has partition: $T_0 = t_0 \leq t_1 \leq t_2 \leq \cdots \leq t_n = T_f$), if the action quantity $S_n(x_t)$ of a path $x_t$ has more than one non-zero term, then we have

$$\lim_{n \to \infty} S_n(x_t) - n\theta_0^\alpha(0) = +\infty.$$  

(39)

Here $u_t \in D_{X_0}$ satisfies

$$d u_t = 0, \ t \in [T_0, T_f] \setminus \{t^*\},$$

(40)

where $t^* \in [T_0, T_f]$.

**Proof.** Suppose that $C_1, C_2, C_3$ are positive constants. We obtain

$$\begin{align*}
[\theta_0^\alpha(C_1) + \theta_0^\alpha(C_2) - 2\theta_0^\alpha(0)]/\theta_0^\alpha(C_3) - \theta_0^\alpha(0)) &= [-\ln f_\alpha(C_1) - \ln f_\alpha(C_2) + 2 \ln f_\alpha(0)]/-\ln f_\alpha(C_3) + \ln f_\alpha(0)] \\
&= \ln \{f_\alpha(0)/\{f_\alpha(C_1) f_\alpha(C_2)\} - f_\alpha(0)/\{f_\alpha(C_3)\}\} \\
&\sim \ln[\alpha^2 f_\alpha(0)(C_1 C_2)_{\alpha}^{1+\alpha} - C^{-1} f_\alpha(0)(C_3)_{\alpha}^{1+\alpha}] \\
&\sim \ln[\alpha^2 f_\alpha(0)(C_1 C_2)_{\alpha}^{1+\alpha} - C^{-1} f_\alpha(0)(C_3)_{\alpha}^{1+\alpha}] + +\infty \ (\epsilon \to 0). \quad (41)
\end{align*}$$

The $\sim$ part and the positive constant $C$ come from the asymptotic behavior of $f_\alpha(\cdot)$ in (19). The formula (41) means that when the jump number of a path is greater, the action quantity of that path has higher order. In $n$-partition path integral representation, if the action quantity $S_n(x_t) = \sum_{i=0}^{n-1} \theta_\epsilon^\alpha(x_{t_{i+1}}, x_{t_i})$ of a path $x_t$ has more than one non-zero term, without loss of generality, we assume $\theta_\epsilon^\alpha(x_{t_1}, x_{t_0}) \neq 0$ and $\theta_\epsilon^\alpha(x_{t_2}, x_{t_1}) \neq 0$. Construct a path $\tilde{x}_t$:

$$\begin{align*}
\tilde{x}_t &= \begin{cases}
X_0, & T_0 \leq t < t_1, \\
x_{t_1}, & t_1 \leq t < t_2, \\
x_{t_2}, & t_2 \leq t < T_f, \\
X_f, & t = T_f,
\end{cases}
\end{align*}$$

(42)

https://doi.org/10.1088/1742-5468/ab1ddc
Applying corollary (1) for these two paths in three intervals: $T_0 \leq t < t_1$, $t_1 \leq t < t_2$, and $t_2 \leq t \leq T_f$, we have
\[
\lim_{n \to \infty} \frac{S_n(x_t) - n\theta^\alpha(0)}{S_n(\tilde{x}_t) - n\theta^\alpha(0)} \geq 1.
\]
According to (41),
\[
\lim_{n \to \infty} \frac{S_n(\tilde{x}_t) - n\theta^\alpha(0)}{S_n(u_t) - n\theta^\alpha(0)} = +\infty,
\]
where $u_t$ satisfies $du_t = 0$, $t \in [T_0, T_f] \setminus \{t^*\}$ for any $t^* \in [T_0, T_f]$.

3.2. Most probable transition path in the case of non-zero drift

**Theorem 2 (Characterization of the most probable transition path with drift term).** For system (1) and system (2), assume that the transition probability density exists, and that the most probable transition path $X^m_t$ exists and satisfies the integrability condition $|\int_{T_0}^{t} b(X^m_s)ds| < \infty$ for $t \in [T_0, T_f]$.

(i) For system (1) with $L_t$ a symmetric $\alpha$-stable Lévy motion with $0 < \alpha < 1$, the most probable transition path $X^m_t$ is determined by the following deterministic dynamical system (i.e. an ordinary differential equation (ODE)),
\[
\begin{aligned}
& \{dX^m_t - b(X^m_t)dt = 0, \ t \in [T_0, T_f] \setminus \{t^*\} \\
& X^m_{T_0} = X_0, \ X^m_{T_f} = X_f.
\end{aligned}
\]

(ii) For system (2) with Brownian motion, the most probable transition path $X^m_t$ is determined by the following deterministic dynamical system (i.e. an integral-differential equation),
\[
X^m_t - X_0 - \int_{T_0}^{t} b(X^m_s)ds = \frac{t - T_0}{T_f - T_0}(X_f - X_0) - \int_{T_0}^{T_f} b(X^m_s)ds,
\]
if $|X_f - X_0 - \int_{T_0}^{T_f} b(X^m_s)ds| \leq \inf_{U_t \in DX_0} |X_f - X_0 - \int_{T_0}^{T_f} b(U_s)ds|$.

**Proof.** For the system (1), the corresponding stochastic integral equation is
\[
X_t = \int_{T_0}^{t} b(X_s)ds + L_t,
\]
\[
Y_t = X_t - \int_{T_0}^{t} b(X_s)ds = L_t,
\]
and the differential form is
\[
X_{t_{i+1}} - X_{t_i} = \gamma b(X_{t_{i+1}}) + (1 - \gamma)b(X_{t_i}) \Delta t = L_{t_{i+1}} - L_{t_i}.
\]

https://doi.org/10.1088/1742-5468/abl0dc
In Itô interpretation ($\gamma = 0$),
\[ X_{t_{i+1}} - X_{t_i} - b(X_{t_i})\Delta t = L_{t_{i+1}} - L_{t_i}. \] (48)

The transition probability density function of $X_t$ is
\[
p(X_f, T_f | X_0, T_0) = \int_{D_X} \cdots \int_{D_X} dL_n \cdots dL_1 \cdot p_t(L_n | L_0) p_t(L_{n-1} | L_1) \cdots p_t(L_1 | L_0) \delta(x_{t_n} - X_f)
\]
\[
= \int \mathcal{D}_{n+1} x \mathcal{J} \exp\left\{ -\sum_{i=0}^{n} \theta_\epsilon (x_{t_{i+1}} - x_{t_i} - b(x_{t_i})\Delta t) \right\} \delta(x_{t_n} - X_f)
\]
\[
= \int \mathcal{D}_{n+1} x \exp\left\{ -\sum_{i=0}^{n} \theta_\epsilon (x_{t_{i+1}} - x_{t_i} - b(x_{t_i})\Delta t) \right\} \delta(x_{t_n} - X_f)
\]
\[
= \int \mathcal{D}_{n+1} y \exp\left\{ -\sum_{i=0}^{n} \theta_\epsilon (y_{t_{i+1}}, y_{t_i}) \right\} \delta(x_{t_n} - X_f),
\] (49)

where $\mathcal{J}$ is the Jacobian of the transformation given by
\[
\mathcal{J} = \det(\frac{\partial L_i}{\partial x_k}) = \prod_{i=1}^{n} (1 - \epsilon \gamma \frac{db(x_i)}{dx_i}).
\] (50)

Assume that the most probable transition path of $X_t$ exist, which is denoted by $X_t^m$.

(i) For system (1), in order to determine the most probable transition path $X_t^m$, we consider the transition of the process $Y_t$ from $Y_{T_0} = X_0$ to $Y_{T_f} = X_f - \int_{T_0}^{T_f} b(X_y)ds$. We should notice that the transition process of $Y_t$ is different from the one of $X_t$. Given the quantities $\{X_0, X_f, T_0, T_f\}$. The diffusion process $X_t$ transfers from initial point $X_0$ at time $T_0$ to terminal point $X_f$ at time $T_f$. That is, all transition paths have the same initial and terminal points. But for process $Y_t$, the transition paths set of $Y_t$ is
\[
D_Y = \{y_t : y_t = x_t - \int_{T_0}^{t} b(x_s)ds, \ x_t \in D_{X_0}^{X_f} \}.
\] (51)

Thus the paths in $D_Y$ have the same initial point $X_0$ but their terminal points may be different.

Since $L_t$ is an $\alpha$-stable Lévy motion with $0 < \alpha < 1$, by corollary 2, the most probable transition path of $Y_t$ (denoted by $Y_t^m$) among $D_Y$ is presumably to satisfy
\[
dY_t^m = 0, \ t \in [T_0, T_f] \setminus \{t^*\},
\] (52)
where \( t^* \in [T_0, T_f] \).

That is
\[
\begin{align*}
\left\{ \begin{array}{l}
dX_t^m - b(X_t^m)dt = 0, \\
X_{T_0}^m = X_0, 
\end{array} \right. \quad t \in [T_0, T_f] \setminus \{t^*\}.
\]

It means that the most probable transition path has one jump and the jump size is
\[ J_{t^*} = |X_0 - \int_{T_0}^{t^*} b(X_s^m)ds - X_f + \int_{T_f}^{t^*} b(X_s^m)ds|. \]

In the proof of corollary 2, if \( C_1 > C_3 > 0 \) and \( C_2 = 0 \), the limit is still the infinity. This means the order of action quantity depends on the jump size.

So the jump time \( t^* \) is supposed to be the one which satisfies
\[ J_{t^*} = \min_{T_0 \leq t \leq T_f} J_t. \]

As \( J_t \) is continuous in \( t \), the minimizer \( t^* \) exists (although it may not be unique).

In other words, the most probable transition path consists of two components: one component is part of the solution of equation
\[
\begin{align*}
\left\{ \begin{array}{l}
dX(t) - b(X(t))dt = 0, \\
X(T_0) = X_0,
\end{array} \right. \quad t \in [T_0, T_f],
\]
and the other component is part of the solution of equation
\[
\begin{align*}
\left\{ \begin{array}{l}
dX(t) - b(X(t))dt = 0, \\
X(T_f) = X_f,
\end{array} \right. \quad t \in [T_0, T_f],
\]

The jump time \( t^* \) is the moment at which \( J_{t^*} \) is minimum.

(ii) For system (2), if the initial and terminal points are deterministic, then by corollary 1, the most probable transition path of a Brownian motion is the one which connects the initial and terminal points directly. Actually the action quantity of this most probable transition path can be computed exactly provided the equal time partition in (33). That is,
\[
\begin{align*}
n\theta^n \left( \frac{1}{n} |X_f - X_0| \right) \\
= n \left( \frac{|X_f - X_0|}{n} \right)^2 \\
= \frac{2\epsilon}{2T_f - T_0} \\
= \frac{1}{2} \frac{(X_f - X_0)^2}{T_f - T_0}.
\end{align*}
\]
Characterization of the most probable transition paths of stochastic dynamical systems with stable Lévy noise

Notice that $|Y_{T_f} - Y_{T_0}| = |X_f - X_0 - \int_{T_0}^{T_f} b(X_s)ds|$ is the distance between the initial and terminal points of $Y_t$. Then the most probable transition path of $Y_t$ (denoted by $Y_t^{m}$) can be obtained exactly,

\[
Y_t^{m} = X_t^{m} - X_0 - \int_{T_0}^{t} b(X_s^{m})ds \]

\[
= Y_0 + (Y_f - Y_0) \frac{t - T_0}{T_f - T_0} \]

\[
= \frac{t - T_0}{T_f - T_0} (X_f - X_0 - \int_{T_0}^{T_f} b(X_s^{m})ds). \tag{58}
\]

This means if there is a transition path $X_t^{m} \in D_{X_0}^{X_f}$ such that the formula (58) holds, then the path $X_t^{m}$ is the most probable one among the paths whose terminal point is $X_f - \int_{T_0}^{T_f} b(X_s^{m})ds$. So if $|X_f - X_0 - \int_{T_0}^{T_f} b(X_s^{m})ds| \leq \inf_{U_t \in D_{X_0}^{X_f}} |X_f - X_0 - \int_{T_0}^{T_f} b(U_s)ds|$, then $X_t^{m}$ will be the most probable transition path of $X_t$.

This completes the proof of this theorem. \hfill \square

**Remark 2.** For a symmetric $\alpha$-stable Lévy motion with $1 \leq \alpha < 2$, the equalities (29) do not hold. Define the path space $D_{X_0}^{X_f} X \triangleq \{x_t | x_t = \sum_{i=0}^{n-1} a_i \mathbb{I}_{[t_i, t_{i+1})} + a_n \mathbb{I}_{[T_f, f]}, T_0 = t_0 \leq t_1 \leq \cdots \leq t_n = T_f, X_0 = a_0 \leq a_1 \leq \cdots \leq a_n = X_f\}$, where $\mathbb{I}_B$ is the characteristic function of set $B \subset \mathbb{R}$.

If we search for the most probable transition path within the simple path space $D_{X_0}^{X_f}$ for the symmetric $\alpha$-stable Lévy motion with $1 \leq \alpha < 2$, the similar results of lemma 1 and theorem 2 hold. This is because for every fixed simple path in $D_{X_0}^{X_f}$, the non-zero $\lambda_j$ of $\{\lambda_j\}_{j=1}^{n}$ are bounded below. Thus the inequality (27) holds for every non-negative $\lambda_j$, for $n$ large enough.

**Remark 3.** For simplicity, we call the solution of (55) the initial transition path, and call the solution of (56) the final transition path. Consequently, the most probable transition path starts from the initial transition path and jumps to the final transition path at the time that initial transition path and final transition path are closed to each other.

**Remark 4.** For the case with a symmetric $\alpha$-stable Lévy motion $(0 < \alpha < 1)$, we are interested in the transitions between metastable states of the stochastic dynamical system (1). That is, the initial and terminal points $X_0$ and $X_f$ are stable points of the corresponding undisturbed system of (1). In this case, the initial transition path is $X_t = X_0$, $t \in [T_0, T_f]$ and the final transition path is $X_t = X_f$, $t \in [T_0, T_f]$ by theorem 2 (i).

Thus the process $Y_t = X_t + \int_{T_0}^{T_f} b(X_s)ds$ is a symmetric $\alpha$-stable Lévy motion which transits from $X_0$ to $X_f$ most probably. The most probable transition path of $Y_t$ provided

https://doi.org/10.1088/1742-5468/ab1ddc
the initial and terminal points $X_0$ and $X_f$ (as discussed in remark 1) is
\[
\begin{cases}
Y^m_t = X_0, \ t \in [T_0, T + T_0/2], \\
Y^m_t = X_f, \ t \in [T + T_0/2, T].
\end{cases}
\] (59)

Thus in theorem 2 (i), if the initial and terminal points $X_0$ and $X_f$ are metastable points of the system, the time $T + T_0/2$ can also be considered as the most probable jump time $t^*$.

Remark 5. For the case with Brownian motion, the condition $|X_f - X_0 - \inf_{U_t \in D_{X_0}^X} |X_f - X_0 - \int_{T_0}^{T_f} b(U_s)ds| = \text{not easy to verify. The action functional in Itô sense is } \mathcal{L}(x, \dot{x}) = \frac{1}{2}(\dot{x} - b(x))^2$. If we assume the most probable path and the function $b(x)$ are smooth enough, then the Euler–Lagrange equation is
\[
\frac{\partial}{\partial t} \frac{\partial \mathcal{L}}{\partial \dot{x}} - \frac{\partial \mathcal{L}}{\partial x} = 0 \\
\Rightarrow \ddot{x} - b'(x) \dot{x} = - (\dot{x} - b(x))b'(x) \\
\Rightarrow \ddot{x} - b'(x)b(x) = 0.
\] (60)

And in our method,
\[
\begin{align*}
X^m_t - X_0 - \int_{T_0}^{T_f} b(X_t^m)ds &= \frac{t - T_0}{T_f - T_0}(X_f - X_0 - \int_{T_0}^{T_f} b(X_t^m)ds) \\
\Rightarrow X^m_t - b(X^m_t) &= \frac{1}{T_f - T_0}(X_f - X_0 - \int_{T_0}^{T_f} b(X_t^m)ds) \\
\Rightarrow \dot{X}^m_t - b'(X^m_t)X_t^m &= 0.
\end{align*}
\] (61)

So in general, the path of (45) does not coincide with direct minimizer of the Onsager–Machlup’s functional. Our method is restricted here because the condition $|X_f - X_0 - \int_{T_0}^{T_f} b(X_t^m)ds| = \text{not satisfied. But if the drift term } b(x) \text{ is independent of } x, \text{ the integration } \int_{T_0}^{T_f} b(U_s)ds \text{ is a constant for any path } U_t \in D_{X_0}^X, \text{ and the two methods provide the same most probable path.}$

3.3. Existence, uniqueness and numerical simulation for the most probable transition path

3.3.1. Non-Gaussian noise: system with a symmetric $\alpha$-stable Lévy motion with $0 < \alpha < 1$. The most probable transition path is determined by two ‘initial’ value problems of a deterministic ODE
\[
\begin{cases}
dX^m_t - b(X_t^m)dt = 0, \ t \in [T_0, T], \\
X^m_{T_0} = X_0, \ X^m_{T_f} = X_f, \ t^* \in [T_0, T_f].
\end{cases}
\] (62)

The first initial value problem solves this ODE with $X^m_{T_0} = X_0$, and the second problem solves this ODE backward in time with terminal value condition $X^m_{T_f} = X_f$. The
existence and uniqueness of these solutions are ensured by the local Lipschitz continuity of the drift term \( b(x) \).

Given a time partition \( \{ T_0 = t_0 < t_1 < \cdots < t_n = T_f \} \), we simulate the most probable transition path as follows: forward Euler scheme

\[
x_{t_{i+1}} - x_{t_i} = b(x_{t_i}) \Delta t, \quad T_0 \leq t_i < t_{i+1} < t^*, \quad x_{T_0} = X_0,
\]

and

\[
x_{t_{i+1}} - x_{t_i} = b(x_{t_i}) \Delta t, \quad t^* \leq t_i < t_{i+1} \leq T_f, \quad x_{T_f} = X_f.
\]

The initial transition path simulated by (63) can be computed easily. Computing the final transition path by (64) is a little complicated. Since the differential equation \( dX_t = b(X_t)dt \) has an unique solution, provided it passes through a given point at given time. For \( b(X_t) = 0, \) the final transition path is \( X_t = X_f, \) \( t^* \leq t \leq T_f \).

The difference scheme we used in (63) and (64) is consistent with the \( \text{Itô} \) interpretation. The differences between \( \text{Itô} \) interpretation and other stochastic interpretations could be found in [32, 33] and references therein.

3.3.2. Gaussian noise: system with a Brownian motion. In this case, the most probable transition path is determined by a determinist integral-differential equation

\[
X^m_t - X_0 - \int_{T_0}^{t} b(X^m_s)ds = \frac{t-T_0}{T_f-T_0}(X_f-X_0)-\int_{T_0}^{T_f} b(X^m_s)ds
\]

(65)

with a constraint \( |X_f-X_0-\int_{T_0}^{T_f} b(X^m_s)ds| \leq \inf_{U_t \in D_{X_0}^{X_f}} |X_f-X_0-\int_{T_0}^{T_f} b(U_s)ds| \) which has been discussed in remark 5.

4. Higher dimensional cases

In this section, we discuss the higher dimensional cases. We consider an SDE system with non-Gaussian noise

\[
\begin{align*}
\mathrm{d}X_{1,t} & = b_1(X_{1,t}, X_{2,t}, \ldots, X_{k,t}) \mathrm{d}t + \mathrm{d}L_{1,t}, \quad X_{1,T_0} = X_{1,0}, \\
\mathrm{d}X_{2,t} & = b_2(X_{1,t}, X_{2,t}, \ldots, X_{k,t}) \mathrm{d}t + \mathrm{d}L_{2,t}, \quad X_{2,T_0} = X_{2,0}, \\
& \vdots \\
\mathrm{d}X_{k,t} & = b_k(X_{1,t}, X_{2,t}, \ldots, X_{k,t}) \mathrm{d}t + \mathrm{d}L_{k,t}, \quad X_{k,T_0} = X_{k,0},
\end{align*}
\]

(66)

where \( L_{i,t} \) are symmetric \( \alpha \)-stable Lévy noises \( (0 < \alpha < 1) \) and \( \{ L_{i,t}, \ldots, L_{j,t} \} \) are independent, and an SDE system with Gaussian noise

\[
\begin{align*}
\mathrm{d}X_{1,t} & = b_1(X_{1,t}, X_{2,t}, \ldots, X_{k,t}) \mathrm{d}t + \mathrm{d}B_{1,t}, \quad X_{1,T_0} = X_{1,0}, \\
\mathrm{d}X_{2,t} & = b_2(X_{1,t}, X_{2,t}, \ldots, X_{k,t}) \mathrm{d}t + \mathrm{d}B_{2,t}, \quad X_{2,T_0} = X_{2,0}, \\
& \vdots \\
\mathrm{d}X_{k,t} & = b_k(X_{1,t}, X_{2,t}, \ldots, X_{k,t}) \mathrm{d}t + \mathrm{d}B_{k,t}, \quad X_{k,T_0} = X_{k,0},
\end{align*}
\]

(67)

https://doi.org/10.1088/1742-5468/ab1ddc

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where $B_{i,t}$ are Brownian motions, and \(\{B_{1,t}, \cdots, B_{k,t}\}\) are independent.

It was known [29] that the random variables $X_1, \cdots, X_k$ are independent if and only if $f(x_1, \cdots, x_k) = f_1(x_1) \cdots f_k(x_k)$ for all $(x_1, \cdots, x_n)$ except possibly for a Borel subset of $\mathbb{R}^k$ with Lebesgue measure zero. Here $f$ is the probability density of $(X_1, \cdots, X_k)$ and $f_i$ is the probability density of $X_i$ ($i = 1, \cdots, k$). Hence by the independence of the noises, the probability density function denoted by $f_{\alpha}(x)$ of $k$-dimensional $\alpha$-stable Lévy variable $x = (x_1, x_2, \cdots, x_k) \in \mathbb{R}^k$ is
\[
f_{\alpha}(x) = f_{\alpha}(x_1) f_{\alpha}(x_2) \cdots f_{\alpha}(x_k)
= \exp\{-\sum_{i=1}^k \ln(f_{\alpha}(x_i))\}
= \exp\{-\sum_{i=1}^k \theta_{\alpha}^1(x_i)\}.
\]

(68)

Recall in theorem 1, we proved that the most probable transition path is supposed to be monotonic with respect to time $t$. In higher dimensional cases, the transition probability density function has the similar form of (21). It implies that every component of the most probable transition path are monotonic with respect to time $t$.

For the system (66) with zero drift term, the time partition \(\{T_0 = t_0 < t_1 < \cdots < t_n = T_f, t_{i+1} - t_i = \frac{T_f - T_0}{n}\}\), and for a path $x_t = (x_{1,t}, x_{2,t}, \cdots, x_{k,t})$ connecting $(T_0, X_0)$ and $(T_f, X_f)$ monotonically, we denote $x_{i,j,t+1} - x_{i,j,t} = \lambda^j_i(X_{i,j,t} - x_{i,t,0})$. Thus
\[
S_n(x_t) = \sum_{i=1}^k \sum_{j=0}^{n-1} \theta_{\alpha}^1(x_{i,t+j+1} - x_{i,t+j}) - kn\theta_{\alpha}^1(0)
= \sum_{i=1}^k \sum_{j=0}^{n-1} \left[\theta_{\alpha}^1(\lambda_{n,i}^{1/\alpha} X_{i,j} - x_{i,t,0}) - \theta_{\alpha}^1(0)\right]
\geq \sum_{i=1}^k \lambda_{n,i}^{1/\alpha} \sum_{X_{i,j} - x_{i,t,0} \geq r_i} \left[\theta_{\alpha}^1(\lambda_{n,i}^{1/\alpha} X_{i,j} - x_{i,t,0}) - \theta_{\alpha}^1(0)\right]
\geq \sum_{i=1}^k \lambda_{n,i}^{1/\alpha} \sum_{X_{i,j} - x_{i,t,0} \geq r_i} \left[\theta_{\alpha}^1(\lambda_{n}^{1/\alpha} X_{i,j} - x_{i,t,0}) - \theta_{\alpha}^1(0)\right]
= \sum_{i=1}^k \left[\theta_{\alpha}^1(\lambda_{n}^{1/\alpha} X_{i,j} - x_{i,t,0}) - \theta_{\alpha}^1(0)\right] \sum_{X_{i,j} - x_{i,t,0} \geq r_i} \lambda_{n,i}^{1/\alpha}
\geq \sum_{i=1}^k \left[\theta_{\alpha}^1(\lambda_{n}^{1/\alpha} X_{i,j} - x_{i,t,0}) - \theta_{\alpha}^1(0)\right] n \to \infty.
\]

(69)

This implies the results of corollary 1 and theorem 2 in higher dimensional cases and they are similar to 1D case.

https://doi.org/10.1088/1742-5468/ab1ddc
For the system (67) with zero drift term,

\[
S_n(x_t) = \sum_{i=1}^{k} \sum_{j=0}^{n-1} \theta^b_e(x_i,t_{j+1} - x_i,t_j)
\]

\[
= \sum_{i=1}^{k} \sum_{j=0}^{n-1} \theta^b_e(|x_i,t_{j+1} - x_i,t_j|)
\]

\[
\geq \sum_{i=1}^{k} n \theta^b_e(\frac{1}{n} x_i,t_{j+1} - x_i,t_j)
\]

\[
= \sum_{i=1}^{k} n \theta^b_e(\frac{1}{n} |X_i,f - X_i,0|).
\]

(70)

This implies the results of corollary 1 and theorem 2 in higher dimensional cases and they are similar to 1D case.

5. Examples

Let us consider several examples in order to illustrate our results.

**Example 1.** Ornstein–Uhlenbeck process.

Consider a linear scalar SDE:

\[
dX_t = r dt + dL_t, \quad T_0 \leq t \leq T_f, \quad X_{T_0} = X_0, \quad X_{T_f} = X_f,
\]

with \( r \) a constant. Let \( Y_t = X_t - r(t - T_0) \). Then by Itô formula

\[
dY_t = -r dt + dX_t = dL_t.
\]

When \( L_t \) is a symmetric \( \alpha \)-stable Lévy motion with \( 0 < \alpha < 1 \), the most probable transition path of \( Y_t \) is

\[
Y^{m}_t = \begin{cases} 
X_f - r(T_f - T_0), & T_f \geq t > t^* \ (t > t^*), \\
X_0, & T_0 \leq t \leq t^* \ (t \leq t^*),
\end{cases}
\]

for every \( t^* \) satisfying \( T_0 \leq t^* \leq T_f \). Thus the most probable transition path for \( X_t \) is

\[
X^{m}_t = \begin{cases} 
X_f - r(T_f - t), & t \geq t^* \ (t > t^*), \\
X_0 + r(t - T_0), & t < t^* \ (t \leq t^*),
\end{cases}
\]

where \( t^* \) can be chosen arbitrarily in \([T_0, T_f]\).

When \( L_t \) is replaced by a Brownian motion,

\[
dX_t = r dt + dB_t, \quad T_0 \leq t \leq T_f, \quad X_{T_0} = X_0, \quad X_{T_f} = X_f,
\]

the most probable transition path of \( X_t \) is (by theorem 2),

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\[ X_{m}^{t} - r(t - T_{0}) = X_{0} + \frac{t - T_{0}}{T_{f} - T_{0}} (X_{f} - X_{0} - r(T_{f} - T_{0})) \]
\[ \Leftrightarrow X_{m}^{t} = X_{0} + \frac{t - T_{0}}{T_{f} - T_{0}} (X_{f} - X_{0}). \]

So in this linear system with drift and Gaussian noise, the most probable transition path is also a line segment.

Figure 2 shows the most probable transition paths of this example.

**Example 2.** Geometric Brownian motion.

Consider a linear scalar SDE with multiplicative noise
\[ dX_{t} = rX_{t}dt + \eta X_{t}dB_{t}, \]
where \( r \) and \( \eta \) are real constants, and \( X_{t} > 0 \), a.s.. Setting \( Y_{t} = \ln X_{t} - (r - \frac{1}{2} \eta^{2})t \) and applying Itô formula, we obtain
\[ dY_{t} = d\ln(X_{t}) - (r - \frac{1}{2} \eta^{2})dt \]
\[ = \frac{1}{X_{t}}dX_{t} + \frac{1}{2}(\frac{1}{X_{t}^{2}})(dX_{t})^{2} - (r - \frac{1}{2} \eta^{2})dt \]
\[ = \frac{dX_{t}}{X_{t}} - \frac{1}{2} \eta^{2}dt - (r - \frac{1}{2} \eta^{2})dt \]
\[ = \eta dB_{t}. \]

By corollary 1,
\[ Y_{m}^{t} = Y_{0} + \frac{t - T_{0}}{T_{f} - T_{0}}(Y_{f} - Y_{0}). \]
Thus
\[ X^n_t = \exp\{(r - \frac{1}{2}\eta^2)t + \ln X_0 + \frac{t - T_0}{T_f - T_0} \ln X_f - (r - \frac{1}{2}\eta^2)T_f - \ln X_0\}. \]

Figure 3 shows the most probable transition path of this example.

**Example 3.** Geometric Lévy process.

Consider the stochastic differential equation
\[
dX_t = X_t[\zeta dt + \beta dB(t) + \int_{\mathbb{R}} \gamma(t, z)\bar{N}(dt, dz)],
\]
where \(\zeta, \beta\) are constants and \(\gamma(t, z) \geq 1\), and
\[
\bar{N}(dt, dz) = \left\{ \begin{array}{ll}
N(dt, dz) - \nu(dz)dt, & \text{if } |z| < r, \\
N(dt, dz), & \text{if } |z| \geq r,
\end{array} \right.
\]
r \in \mathbb{R}+. For simplicity we set \(\beta = 0\), \(\gamma(t, z) = e^z - 1\). Now define \(Y_t = \ln X_t\). By Itô formula, we have
\[
dY_t = \zeta dt + \int_{|z| < r} (\ln(1 + e^z - 1) - (e^z - 1))\nu(dz)dt + \int_{\mathbb{R}} \ln(1 + e^z - 1)\bar{N}(dt, dz)
\]
\[
= \zeta dt + \int_{|z| < r} (z - (e^z - 1))\nu(dz)dt + \int_{\mathbb{R}} z\bar{N}(dt, dz).
\]

Let \(U_t = Y_t - (\zeta + \int_{|z| < r} (z - (e^z - 1))\nu(dz))t\) be a symmetric \(\alpha\)-stable Lévy process, i.e. \(dU_t = \int_{\mathbb{R}} z\bar{N}(dt, dz) = dL^\alpha_t\).

Let the jump measure \(\nu(dz)\) be the jump measure of an \(\alpha\)-stable Lévy process, that is, \(\nu(dz) = c_\alpha \frac{dz}{|z|^\alpha}\) with
\[
c_\alpha = \frac{\alpha}{2^{1-\alpha}\sqrt{\pi}} \frac{\Gamma(\frac{1+\alpha}{2})}{\Gamma(1 - \frac{\alpha}{2})},
\]
For \(\alpha\) with \(0 < \alpha < 1\) and \(r = 1\),
\[
\int_{|z| < 1} \{z - (e^z - 1)\}\nu(dz) < \infty.
\]
By corollary 1, the most probable transition path of \(U_t\) is
\[
U^n_t = \begin{cases}
U_0, & T_0 \leq t < t^* \ (T_0 \leq t \leq t^*), \\
U_f, & t^* \leq t \leq T_f \ (t^* < t \leq T_f),
\end{cases}
\]
where \(t^* \in [T_0, T_f]\). Thus
\[
X^n_t = \begin{cases}
\exp(\ln X_0 + (t - T_0)(\zeta + \int_{|z| < 1} (z - (e^z - 1))\nu(dz))], & T_0 \leq t < t^* \ (T_0 \leq t \leq t^*), \\
\exp(\ln X_f + (t - T_f)(\zeta + \int_{|z| < 1} (z - (e^z - 1))\nu(dz))], & t^* \leq t \leq T_f \ (t^* < t \leq T_f).
\end{cases}
\]

Figure 3 also shows the most probable transition path of this example.

**Example 4.** One-dimensional nonlinear SDE: stochastic double-well system.
Consider the stochastic double-well system

\[ dX_t = (X_t - X_t^3)dt + dL_t, \]

where \( L_t \) is a symmetric \( \alpha \)-stable Lévy motion with \( 0 < \alpha < 1 \).

The corresponding undisturbed system has three equilibrium points: \(-1, 0, 1\) (\(-1\) and \(1\) are stable equilibrium points, \(0\) is an unstable equilibrium point).

By theorem 2, the most probable transition path of this system is described by the following deterministic differential equation:
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\[ \frac{dX_t^m}{dt} - (X_t^m - (X_t^m)^3)dt = 0, \ t \in [T_0, T_f]\{t^*\} \]
\[ X_{T_0}^m = X_0, \ X_{T_f}^m = X_f, \ t^* \in [T_0, T_f]. \]

We compute some solution curves of above system as shown in figure 4. The most probable transition path consists of the solutions of this deterministic system. For instance, if we consider: \( X_0 = 1, \ X_f = -1, \ T_0 = 0, \ T_f = 4 \), then the most probable transition path consists of the parts of two straight lines in figure 4.

**Example 5.** Two-dimensional nonlinear SDE: the Maier–Stein model.

Consider the following SDEs:

\[ \frac{dX_t}{dt} = X_t - X_t^3. \]

Figure 4. Solution curves for \( \dot{x} = x - x^3 \).

Figure 5. Phase portrait for the Maier–Stein model.
\begin{align*}
\{ du = (u - u^3 - \beta uv^2) dt + dL_{1,t}, \\
\{ dv = -(1 + u^2)v dt + dL_{2,t}.
\end{align*}

By theorem 2, the most probable transition path \((u^m_t, v^m_t)\) of this system is described by the following deterministic differential equations:
\begin{align*}
\{ du^m_t = (u^m_t - (u^m_t)^3 - \beta u^m_t (v^m_t)^2) dt, & \quad t \in [T_0, T_f] \setminus \{t^*\}, \\
\{ dv^m_t = -(1 + (u^m_t)^2)v^m_t dt, & \quad t \in [T_0, T_f] \setminus \{t^*\}, \\
u^m_{T_0} = u_0, & \quad u^m_{T_f} = u_f, \quad v^m_{T_0} = v_0, \quad v^m_{T_f} = v_f, \quad t^* \in [T_0, T_f].
\end{align*}

Figure 5 shows the phase portrait of this deterministic system. There are three equilibrium points: \((-1, 0), (0, 0), (1, 0)\). In figure 5 we show several orbits in black lines.

The most probable transition path can be found by the phase portrait with given initial and terminal conditions.

Acknowledgments

The authors would like to thank Professor Xu Sun, Dr Qiao Huang, Dr Yayun Zheng, Dr Wei Wei, Dr Ao Zhang, and Dr Jianyu Hu for helpful discussions. This work was partly supported by the NSF Grant 1620449, and NSFC Grants 11531006 and 11771449.

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