Critical value for the contact process with random edge weights on regular tree

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Abstract:
In this paper we are concerned with contact processes with random edge weights on rooted regular trees. We assign i.i.d weights on each edge on the tree and assume that an infected vertex infects its healthy neighbor at rate proportional to the weight on the edge connecting them. Under the annealed measure, we define the critical value $\lambda_c$ as the maximum of the infection rate with which the process will die out and define $\lambda_e$ as the maximum of the infection rate with which the process dies out at exponential rate. We show that these two critical values satisfy an identical limit theorem and give a precise lower bound of $\lambda_e$. We also study the critical value under the quenched measure. We show that this critical value equals that under the annealed measure or infinity according to a dichotomy criterion. The contact process on a Galton-Watson tree with binomial offspring distribution is a special case of our model.

Keywords: contact process, regular tree, edge weight, critical value.

1 Introduction
In this paper, we are concerned with contact processes with random edge weights on regular trees. For each integer $N \geq 1$, we denote by $\mathbb{T}^N$ the rooted regular tree where the root $O$ has degree $N$ and other vertices have degree $N + 1$. That is to say, each vertex produces $N$ children and the root $O$ has no ancestor while each other vertex has a father. The following picture describes a local area of $\mathbb{T}^4$.

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For any vertices $x, y \in T^N$, we denote by $x \sim y$ when there is an edge connecting them. We denote by $E_N$ the set of edges on $T^N$.

Let $\rho$ be a non-negative random variable such that $P(\rho \leq M) = 1$ for some $M \in (0, +\infty)$ and $P(\rho > 0) > 0$. \{\rho(e)\}_{e \in E_N} are i. i. d. random variables such that for each $e \in E_N$, $\rho(e)$ and $\rho$ have the same probability distribution. For $e \in E_N$ with endpoints $x, y \in T^N$, we write $\rho(e)$ as $\rho(x, y)$. When \{\rho(e)\}_{e \in E_N} is given, the contact process with edge weights \{\rho(e)\}_{e \in E_N} is a spin system with state space $\{0, 1\}^T_N$ and flip rates function given by

$$c(x, \eta) = \begin{cases} 1 & \text{if } \eta(x) = 1, \\ \lambda \sum_{y, y \sim x} \rho(x, y) \eta(y) & \text{if } \eta(x) = 0 \end{cases}$$  \hspace{1cm} (1.1)$$

for any $(x, \eta) \in T^N \times \{0, 1\}^T_N$, where $\lambda$ is a positive parameter called the infection rate.

The assumption $P(\rho \leq M) = 1$ ensures the existence of our process according to the basis theory constructed in [5] and [8].

Intuitively, the process describes the spread of an infection disease. Vertices in state 1 are infected individuals while vertices in state 0 are healthy. An infected individual waits for an exponential time with rate 1 to recover. A healthy vertex $x$ is infected by its infected neighbor $y$ at a rate proportional to $\rho(x, y)$. That is to say, the larger $\rho(x, y)$ is, the faster the disease spreads from $y$ to $x$.

When $\rho \equiv 1$, our model degenerates to the classic contact process, which is introduced in [6] by Harris. In [13], Pemantle first considers contact processes on trees. The two books [9] and [11] written by Liggett give a detailed introduction for the study of classic contact processes on lattices and trees.

When $P(\rho = 1) = 1 - P(\rho = 0) = p \in (0, 1)$, then our model turns into the contact process on a Galton-watson tree with binomial offspring distribution $B(N, p)$ and also can be seen as contact process on open clusters of bond percolation on tree. In [14], Pemantle and Stacey study contact processes and branching random walks on Galton-Watson trees. They show that on some Galton-Watson trees the branching random walk has one phase transition while the contact process has two. Contact processes on clusters of bond percolation on lattices are studied by Chen and Yao in [3]. They show that the complete convergence theorem holds.

In this paper, we are concerned with contact processes with random edge weights. It is also interesting to consider the process with random vertex weights. In detail, each vertex $x$ is assigned a weight $\rho(x)$. Infected vertex $x$ infects healthy neighbor $y$ at rate proportional to $\rho(x)\rho(y)$. This model concludes contact process on clusters of site percolation as a special case. In [1], Bertacchi, Lanchier and Zucca study contact processes on $C_\infty \times K_N$, where $C_\infty$ is the infinite open cluster of site percolation and $K_N$ is a complete graph with $N$ vertices. Criterion to judge whether the process will survive are given in [1]. Contact processes with random vertex weights on complete graphs are introduced in [15] by Peterson. In [15], it is shown that the critical value of the model is inversely proportional to the second moment of the vertex weight. Xue extends this result to the case where the graph is oriented lattice in [17]. In [16], Xue studies contact processes with random vertex weights on general regular graphs and obtains a lower bound of the critical value of the model.
2 Main results

In this section we give main results of this paper. First we introduce some notations and definitions. For each \(N \geq 1\), we assume that \(\{\rho(e)\}_{e \in \mathbb{Z}^N}\) are defined on the probability space \((\Omega_N, \mathcal{F}_N, \mu_N)\). We write \((\Omega_N, \mathcal{F}_N, \mu_N)\) briefly as \((\Omega, \mathcal{F}, \mu)\) when there is no misunderstanding. For any \(\omega \in \Omega\), we denote by \(P^\omega_N\) the probability measure of the contact process on \(T^N\) with edge weights \(\{\rho(e, \omega)\}_{e \in \mathbb{Z}^N}\) and infection rate \(\lambda\). \(P^\omega_N\) is called the quenched measure. The expectation operator with respect to \(P^\omega_N\) is denoted by \(E^\omega_N\). We define

\[
P^N_N(\cdot) = \int P^\omega_N(\cdot) \mu_N(d\omega),
\]

which is called the annealed measure. The expectation operator with respect to \(P^N_N\) is denoted by \(E^N_N\). When there is no misunderstanding, we write \(P^N_N\) and \(E^N_N\) briefly as \(P_N\) and \(E_N\).

For any \(t \geq 0\), we denote by \(\eta_t\) the configuration of the contact process at moment \(t\). The value of vertex \(x\) at moment \(t\) is denoted by \(\eta_t(x)\). For any \(t > 0\), let

\[
C_t = \{x \in T^N : \eta_t(x) = 1\}
\]

be the set of infected vertices at \(t\). We write \(C_t\) as \(C^O_t\) when \(C_0 = \{O\}\).

Since \(\emptyset\) is an absorbed state of the process \(\{C_t\}_{t \geq 0}\) and the contact process is an attractive spin system (see section 3.2 of [9]), for any \(\lambda_1 > \lambda_2\),

\[
P_{\lambda_1}(\forall t \geq 0, C^O_t \neq \emptyset) \geq P_{\lambda_2}(\forall t \geq 0, C^O_t \neq \emptyset).
\]

(2.1)

By [21], it is reasonable to define the following critical value. For each \(N \geq 1\), we define

\[
\lambda_c(N) = \sup\{\lambda : P^N_N(\forall t \geq 0, C^O_t \neq \emptyset) = 0\}.
\]

(2.2)

We write \(\lambda_c(N)\) as \(\lambda_c\) when there is no misunderstanding.

Supposing that only \(O\) is infected at \(t = 0\), then when \(\lambda < \lambda_c\), with probability one there will be no infected vertices eventually, which means that the disease dies out. When \(\lambda > \lambda_c\), with positive probability there will be always some vertices in the infected state, which means that the disease survives. The case of \(\lambda = \lambda_c\) is difficult. In [2], Bezuidenhout and Grimmett show that the critical classic contact process on lattice dies out. We guess same conclusion holds for our model but have not find a way to prove it yet.

When \(\lambda < \lambda_c\),

\[
\lim_{t \to +\infty} P_{\lambda}(C^O_t \neq \emptyset) = 0.
\]

It is natural to ask whether \(P_{\lambda}(C^O_t \neq \emptyset)\) converges to 0 at an exponential rate. So it is natural to define the following critical value. For any \(N \geq 1\), we define

\[
\lambda_e(N) = \sup\{\lambda : \limsup_{t \to +\infty} \frac{1}{t} \log P^N_{\lambda}(C^O_t \neq \emptyset) < 0\}.
\]

(2.3)

It is obviously that \(\lambda_e \leq \lambda_c\). Does \(\lambda_e\) equal \(\lambda_c\)? Section 6.3 of [9] shows that the answer is positive for classic contact process on \(Z\). We have no idea whether \(\lambda_e = \lambda_c\) for our model.

Now we give our main results. Our first result is a criterion to judge whether \(\lambda_e \in (0, +\infty)\).
**Theorem 2.1.** If \( P(\rho > 0) = 1 \), then for each \( N \geq 2 \), \( 0 < \lambda_c(N) < +\infty \). If \( P(\rho > 0) < 1 \), then \( 0 < \lambda_c(N) < +\infty \) for \( N > 1/P(\rho > 0) \) and \( \lambda_c(N) = +\infty \) for \( N \leq 1/P(\rho > 0) \).

We can not judge whether \( \lambda_c < +\infty \) for the case where \( N = 1 \) and \( P(\rho > 0) = 1 \). We guess in this case there is no common conclusion. More information about the distribution of \( \rho \) is needed. For example, if there exists \( \epsilon > 0 \) such that \( P(\rho > \epsilon) = 1 \), then it is easy to see that \( \lambda_c \in (0, +\infty) \) since classic contact process on \( Z \) has finite critical value (see Section 6.1 of [9] and [10]).

To describe \( \lambda_c \) and \( \lambda_e \) more accurately, we obtain a limit theorem of \( \lambda_c \), \( \lambda_e \) and a precise lower bound of \( \lambda_e \).

**Theorem 2.2.** For \( \rho \) satisfies that \( P(\rho > 0) > 0 \) and \( P(0 \leq \rho \leq M) = 1 \) for some \( M \in (0, +\infty) \),

\[
\lim_{N \to +\infty} N\lambda_c(N) = \lim_{N \to +\infty} N\lambda_e(N) = \frac{1}{E\rho}.
\]

Furthermore,

\[
\lambda_e(N) \geq \left( NE\rho + \frac{M^2}{E\rho} \right)^{-1}.
\]

**Theorem 2.2** show that \( \lambda_c, \lambda_e \approx 1/(NE\rho) \), which is inversely proportional to the degree of the root and the mean of the edge weight.

Let us see some examples. When \( \rho \equiv 1 \), Theorem 2.2 shows that

\[
\lim_{N \to +\infty} N\lambda_c(N) = 1
\]

and \( \lambda_c(N) \geq 1/(N + 1) \), which is the estimation of critical value for classic contact process on regular tree given in [13].

When \( P(\rho = 1) = 1 - P(\rho = 0) = p \in (0, 1) \), Theorem 2.2 gives the estimation of critical value for contact processes on Galton-Watson tree with binomial offspring distribution \( B(N, p) \) that

\[
\lim_{N \to +\infty} Np\lambda_c(N) = 1
\]

and

\[
\lambda_c(N) \geq \frac{1}{Np + 1/p}.
\]

These two estimations do not occur in former references.

The critical value \( \lambda_c \) is defined under the annealed measure. It is natural to consider the critical value of the process with fixed edge weights \( \{\rho(e, \omega)\}_{e \in E} \) for some \( \omega \in \Omega \). Hence, for any \( \omega \in \Omega_N \), we define

\[
\widehat{\lambda}_c(\omega, N) = \sup\{\lambda : P_\omega^\lambda(\forall t, C_t^O \neq \emptyset) = 0\}.
\]

For \( \omega \in \Omega_N \), if there is a cut-off \( \Pi \) of \( T^N \) separating \( O \) from infinity such that \( \rho(e, \omega) = 0 \) for each \( e \in \Pi \), then it is easy to see that \( \widehat{\lambda}_c(\omega, N) = +\infty \). We can show that except this case, \( \lambda_c(\omega, N) = \lambda_c(N) \), which means the critical values under the annealed measure and quenched measure are equal. To introduce our result rigorously, we introduce some notations and definitions.
For any $\omega \in \Omega_N$, we define
\[ L(\omega) = \{ e \in \mathbb{E}^N : \rho(e, \omega) > 0 \}. \]
(2.7)

For each $x \in T^N$, there is a unique path $p(O, x)$ from $O$ to $x$ which does not backtrack. We write $O \rightarrow^\omega x$ when and only when each edge of $p(O, x)$ belongs to $L(\omega)$. We define
\[ D(\omega) = \{ x \in T^N : O \rightarrow^\omega x \} \]
and
\[ A_N = \{ \omega : |D(\omega)| = +\infty \}. \]
(2.8)

It is obviously that $D(\omega)$ forms a Galton-Watson tree with offspring distribution $B(N, q)$ and $1 - \mu_N(A_N)$ is the extinction probability of the tree, where
\[ q = P(\rho > 0) \]
and
\[ P(B(N, q) = k) = \binom{N}{k} q^k (1 - q)^{N-k} \]
for $1 \leq k \leq N$.

Now we can give our result of the critical value under the quenched measure.

**Theorem 2.3.** If $P(\rho > 0) = 1$, then for each $N \geq 2$, there exists $K_N \in F_N$ such that
\[ \mu_N(K_N) = 1 \]
and
\[ \tilde{\lambda}_c(\omega, N) = \lambda_c(N) \in (0, +\infty) \]
for any $\omega \in K_N$, where $\lambda_c(N)$ is the same as that in (2.2).

If $P(\rho > 0) < 1$, then when $N \leq 1/P(\rho > 0)$,
\[ \tilde{\lambda}_c(\omega, N) = +\infty \]
for any $\omega \in \Omega_N$. When $N > 1/P(\rho > 0)$, then $\mu_N(A_N) > 0$ and there exists $K_N \subseteq A_N$ such that $\mu_N(A_N \setminus K_N) = 0$ and
\[ \tilde{\lambda}_c(\omega, N) = \lambda_c(N) \in (0, +\infty) \]
for any $\omega \in K_N$. For any $\omega \not\in A_N$,
\[ \tilde{\lambda}_c(\omega, N) = +\infty. \]

In conclusion, theorem 2.3 shows that $\tilde{\lambda}_c(\omega, N) \in \{ \lambda_c(N), +\infty \}$ with probability one. Furthermore,
\[ \{ \omega : \tilde{\lambda}_c(\omega, N) = \lambda_c(N) \} = A_N \]
and
\[ \{ \omega : \tilde{\lambda}_c(\omega, N) = +\infty \} = \Omega_N \setminus A_N \]
in the sense of ignoring a set with probability 0.

The proofs of our main results are divided into three sections. In Section 3, we will give an upper bound of $\lambda_c$, which shows that $\limsup_{N \to +\infty} N\lambda_c(N) \leq 1/E\rho$. The core idea is to compare the contact process with a SIR epidemic model. This section also gives
most part of the proof of Theorem 2.1 except showing that \( \lambda_c > 0 \). In Section 4, we will prove that \( \lambda_c(N) \geq \left( NE \rho + \frac{M^2}{E \rho} \right)^{-1} \) and hence \( \lambda_c > 0 \), which accomplishes the proof of Theorem 2.2 and Theorem 2.1. The main approach is to compare the contact process with the binary contact path process introduced in [4] by Griffeath. In technique, we need to estimate the number of paths (may backtrack) from \( O \) with given length on the tree. We relate this problem to simple random walk on regular tree. In Section 5, we give the proof of Theorem 2.3. Our approach is inspired by the classic method of proving extinction criterion for Galton-Watson trees.

3 Upper bound for \( \lambda_c \)

In this section we will prove that \( \limsup_{N \to +\infty} N \lambda_c(N) \leq 1/E \rho \). The following lemma gives an upper bound of \( \lambda_c(N) \), which is crucial for our proof.

**Lemma 3.1.** If \( \lambda \) satisfies that

\[
NE\left[ \frac{\lambda \rho}{1 + \lambda \rho} \right] > 1,
\]

then

\[
\lambda_c(N) \leq \lambda.
\]

We give the proof of Lemma 3.1 at the end of this section. First we utilize Lemma 3.1 to prove that \( \limsup_{N \to +\infty} N \lambda_c(N) \leq 1/E \rho \).

**Proof of** \( \limsup_{N \to +\infty} N \lambda_c(N) \leq 1/E \rho \). For \( \gamma > 1 \), let \( \lambda = \frac{\gamma}{NE \rho} \), then

\[
NE\left[ \frac{\lambda \rho}{1 + \lambda \rho} \right] = \frac{\gamma}{E \rho} E\left[ \frac{\rho}{1 + \frac{\gamma \rho}{NE \rho}} \right].
\]

According to Domination Convergence Theorem,

\[
\lim_{N \to +\infty} \frac{\gamma}{E \rho} E\left[ \frac{\rho}{1 + \frac{\gamma \rho}{NE \rho}} \right] = \frac{\gamma}{E \rho} E \rho = \gamma > 1.
\]

Therefore, for sufficiently large \( N \) and \( \lambda = \frac{\gamma}{NE \rho} \),

\[
NE\left[ \frac{\lambda \rho}{1 + \lambda \rho} \right] > 1.
\]

Therefore, according to Lemma 3.1

\[
\lambda_c(N) \leq \frac{\gamma}{NE \rho}
\]

for sufficiently large \( N \) and hence

\[
\limsup_{N \to +\infty} N \lambda_c(N) \leq \frac{\gamma}{E \rho}.
\]

Since \( \gamma \) is arbitrary, let \( \gamma \to 1 \) and the proof is complete.
For the special case $P(\rho = 1) = 1 - P(\rho = 0) = p \in (0, 1]$ and $N > 1/p$, Lemma 3.1 gives a precise upper bound of $\lambda_c(N)$ that

$$\lambda_c(N) \leq \frac{1}{Np - 1},$$

since $NE\left[\frac{\lambda \rho}{1 + \lambda \rho}\right] = \frac{\lambda N \rho}{1 + \lambda \rho}$.

According to Lemma 3.1 we can also judge whether $\lambda_c < +\infty$.

**Corollary 3.2.** If $P(\rho > 0) = 1$, then $\lambda_c(N) < +\infty$ for each $N \geq 2$. If $P(\rho > 0) < 1$, then $\lambda_c(N) < +\infty$ for $N > 1/P(\rho > 0)$ and $\lambda_c(N) = +\infty$ for $N \leq 1/P(\rho > 0)$.

**Proof.** According to Domination Convergence Theorem,

$$\lim_{\lambda \to +\infty} E\left[\frac{\lambda \rho}{1 + \lambda \rho}\right] = P(\rho > 0).$$

Therefore, in the case where $P(\rho > 0) = 1$ and $N \geq 2$ and the case where $P(\rho > 0) < 1$ and $N > 1/P(\rho > 0)$,

$$\lim_{\lambda \to +\infty} NE\left[\frac{\lambda \rho}{1 + \lambda \rho}\right] > 1$$

and hence

$$\lambda_c(N) < \lambda$$

for sufficiently large $\lambda$ according to Lemma 3.1. As a result, in these two cases,

$$\lambda_c < +\infty.$$

For the case where $P(\rho > 0) < 1$ and $N \leq 1/P(\rho > 0)$, the Galton-Watson tree with offspring distribution $B(N, P(\rho > 0))$ is extinct with probability one, since the mean of the number of sons is at most one. As a result, $D(\omega)$ is finite with probability one and the Markov process $\{C_t^O\}_{t \geq 0}$ is with finite state space $\{A : A \subseteq D(\omega)\}$. Since $\emptyset$ is the unique absorption state for $\{C_t^O\}_{t \geq 0}$, the process will be frozen in state $\emptyset$ eventually. As a result, for any $\lambda > 0$,

$$P^\omega(\forall t \geq 0, C_t^O \neq \emptyset) = 0$$

for any $\omega \in \Omega$ except a set with probability zero and hence

$$P^N(\forall t \geq 0, C_t^O \neq \emptyset) = 0.$$

Therefore, $\lambda_c > \lambda$ for any $\lambda > 0$ and hence

$$\lambda_c = +\infty.$$

At last we give the proof of Lemma 3.1.

**Proof of Lemma 3.1** To control the size of $C_t$ from below, we introduce the following SIR epidemic model with random edge weights. Let $\{\xi_t\}_{t \geq 0}$ be Markov process with state space
\{−1, 0, 1\}^{T_N}. At \( t = 0 \), \( \xi_0(O) = 1 \) and \( \xi_0(x) = 0 \) for each other \( x \in T_N \). For any \( t \geq 0 \), we define

\[
I_t = \{ x \in T_N : \xi_t(x) = 1 \}, \quad S_t = \{ x \in T_N : \xi_t(x) = 0 \}, \\
R_t = \{ x \in T_N : \xi_t(x) = -1 \}.
\]

Now we can identify \( \xi_t \) with \((S_t, I_t, R_t)\). After the edge weights \( \{\rho(e)\}_{e \in E_N} \) is given, \( \{(S_t, I_t, R_t)\}_{t \geq 0} \) evolves as follows. For each \( x \in I_t \), \((S_t, I_t, R_t)\) flips to \((S_t \setminus \{x\}, I_t \cup \{x\})\) with rate 1. For any \( x, y \) satisfy that \( y \) is a son of \( x \), \( x \in I_t \) and \( y \in S_t \), \((S_t, I_t, R_t)\) flips to \((S_t \setminus \{y\}, I_t \cup \{y\}, R_t)\) at rate \( \lambda \rho(x, y) \).

Intuitively, 1, 0, −1 represent ‘infected’, ‘healthy’ and ‘removed’ respectively. An infected vertex waits for an exponential time with rate one to become removed. A healthy vertex \( y \) may be infected when and only when its father \( x \) is infected. \( x \) infects \( y \) at rate proportional to \( \rho(x, y) \). A removed vertex will stay in the this state forever.

For \( \{C_t\}_{t \geq 0} \), an infected vertex can infect any healthy neighbor while for \( \{\xi_t\}_{t \geq 0} \), an infected vertex can only infect its sons. For \( \{C_t\}_{t \geq 0} \), when an infected vertex become healthy, it may be infected again while for \( \{\xi_t\}_{t \geq 0} \), when an infected vertex becomes removed, it will never be infected again. As a result, according to the approach of basic coupling (see section 2.1 of [9]), it is easy to see that

\[
I_t \subseteq C_t^0
\]

for any \( t > 0 \) in the sense of coupling when the two processes with same infection rate \( \lambda \) and edge weights \( \{\rho(e)\}_{e \in E_N} \). Therefore,

\[
P^N_\lambda(\forall \ t, C_t^0 \neq \emptyset) \geq P^\omega_\lambda(\forall \ t, I_t \neq \emptyset)
\]

for any \( \omega \in \Omega \) and hence

\[
P^N_\lambda(\forall \ t, C_t^0 \neq \emptyset) \geq P^N_\lambda(\forall \ t, I_t \neq \emptyset)
\]

(3.1)

for any \( t > 0 \) and \( N \geq 1 \).

We define

\[
I_{t \infty} = \bigcup_{t \geq 0} I_t
\]

as the set of vertices which have been infected. \( I_t \neq \emptyset \) for any \( t \geq 0 \) if and only if there are infinite many vertices which have been infected. Therefore,

\[
\{ \forall \ t \geq 0, I_t \neq \emptyset \} = \{|I_{t \infty}| = +\infty\}.
\]

(3.2)

By (3.1) and (3.2),

\[
P^N_\lambda(\forall \ t \geq 0, C_t \neq \emptyset) \geq P^N_\lambda(|I_{t \infty}| = +\infty).
\]

(3.3)

For \( x \in T_N \) and a son \( y \) of \( x \), let \( T_1 \) be an exponential time with rate \( \lambda \rho(x, y) \) and \( T_2 \) be an exponential time with rate 1 and independent of \( T_1 \), then conditioned on \( x \) is infected, the probability that \( x \) infects \( y \) equals

\[
P(T_1 < T_2) = \frac{\lambda \rho(x, y)}{1 + \lambda \rho(x, y)}.
\]
As a result, under the annealed measure $P^N_\lambda$, the mean of the number of infected sons of an infected vertex equals

$$NE\left[\frac{\lambda\rho}{1+\lambda\rho}\right].$$

As a result, under the annealed measure $P^N_\lambda$, $I_{+\infty}$ forms a Galton-Watson tree with an offspring distribution with mean $NE\left[\frac{\lambda\rho}{1+\lambda\rho}\right]$. According to the extinction criterion of Galton-Watson trees,

$$P^N_\lambda(|I_{+\infty}| = +\infty) > 0 \tag{3.4}$$

when $\lambda$ satisfies $NE\left[\frac{\lambda\rho}{1+\lambda\rho}\right] > 1$. Lemma 3.1 follows from this fact and (3.3).

\[\square\]

## 4 Lower bound for $\lambda_c$

In this section we will give lower bound of $\lambda_c$. First, we give a lemma about simple random walk on $\mathbb{T}^N$ for later use.

For $N \geq 1$, we denote by $\{S^N_n\}_{n \geq 0}$ simple random walk on $\mathbb{T}^N$ such that

$$P(S^N_{n+1} = y|S^N_n = x) = \frac{1}{\deg(x)}$$

for each $x \in \mathbb{T}^N$, each neighbor $y$ of $x$ and $n \geq 0$. We assume that $S^N_0 = O$. The probability measure and expectation operator with respect to $\{S^N_n\}_{n \geq 0}$ are denoted by $\tilde{P}$ and $\tilde{E}$.

We define $\Gamma : \mathbb{T}^N \rightarrow Z$ such that $\Gamma(O) = 0$ and $\Gamma(y) = \Gamma(x) + 1$ when $y$ is a son of $x$. In other words, for each $x \in \mathbb{T}^N$ there is an unique path $p(O, x)$ from $O$ to $x$ which does not backtrack. $\Gamma(x)$ equals the length of $p(O, x)$.

**Lemma 4.1.** For any $x \in (0, 1)$ and $n \geq 0$,

$$\tilde{E}_x\Gamma(S^N_n) \leq \left[\frac{Nx}{N+1} + \frac{1}{(N+1)x}\right]^n.$$

**Proof.** According to the definition of $S^N_n$,

$$\tilde{P}(\Gamma(S^N_{n+1}) - \Gamma(S^N_n) = 1|S^N_n = x) = 1 - \tilde{P}(\Gamma(S^N_{n+1}) - \Gamma(S^N_n) = 1|S^N_n = x) \tag{4.1}$$

$$= \begin{cases} 
1 & \text{if } x = O, \\
\frac{N}{N+1} & \text{if } x \neq O.
\end{cases}$$

Let $\{Z_n\}_{n \geq 0}$ be a Markov process with state space $\{\ldots, -2, -1, 0, 1, 2, 3, \ldots\}$ and evolve according to $\{S^N_n\}_{n \geq 1}$. In detail, we assume that $Z_0 = 0$. For $n \geq 1$, if $S^N_n = O$, then $Z_{n+1} - Z_n$ is independent of $S^N_{n+1}$ and satisfies

$$P(Z_{n+1} - Z_n = 1) = 1 - P(Z_{n+1} - Z_n = -1) = \frac{N}{N+1}.$$

If $S^N_n \neq O$, then $Z_{n+1} - Z_n = 1$ when $\Gamma(S^N_{n+1}) - \Gamma(S^N_n) = 1$ and $Z_{n+1} - Z_n = -1$ when $\Gamma(S^N_{n+1}) - \Gamma(S^N_n) = -1$.

As a result, for each $n \geq 1$, $Z_{n+1} - Z_n \leq \Gamma(S^N_{n+1}) - \Gamma(S^N_n)$. Since $Z_0 = \Gamma(S^N_0) = 0$, $Z_n \leq \Gamma(S^N_n)$.
for each \( n \geq 1 \).

Therefore, for \( x \in (0, 1) \),
\[
\varepsilon_x \Gamma(S_n) \leq E Z_n.
\]  \hspace{1cm} (4.2)

By \ref{4.1} and the definition of \( Z_n \), it is easy to see that \( \{Z_n - Z_{n-1}\}_{n \geq 1} \) are i.i.d random variables such that
\[
P(Z_n - Z_{n-1} = 1) = 1 - P(Z_n - Z_{n-1} = -1) = \frac{N}{N + 1}.
\]

Therefore,
\[
E_q Z_n = [E_x Z_n - Z_0]^n = \left[\frac{N_x}{N + 1} + \frac{1}{(N + 1)x}\right]^n.
\]  \hspace{1cm} (4.3)

Lemma 4.1 follows from \ref{4.2} and \ref{4.3}. \hfill \Box

To control \( P(C_t^O \neq \emptyset) \) from above, we introduce the binary contact path process \( \{\zeta_t\}_{t \geq 0} \) with random edge weights on \( T^N \). The classic binary contact path process is introduced by Griffith in \cite{9}, which inspires us a lot.

The state space of \( \{\zeta_t\}_{t \geq 0} \) is \( \{0, 1, 2, 3, \ldots\}^{z^N} \). At \( t = 0 \), we assume that \( \zeta_0(x) = 1 \) for each \( x \in T^N \).

When the edge weights \( \{\rho(e)\}_{e \in \mathbb{E}_N} \) are given, \( \{\zeta_t\}_{t \geq 0} \) evolves according to Poisson processes \( \{N_x(t) : t \geq 0\}_{x \in T^N} \) and \( \{U(x,y)(t) : t \geq 0\}_{x \sim y} \). For any \( x \in T^N \), \( N_x(\cdot) \) is with rate 1. For any \( x, y \) such that \( x \sim y \), \( U(x,y)(\cdot) \) is with rate \( \lambda \rho(x,y) \). Please note that we care the order of \( x \) and \( y \), hence \( U(x,y) \neq U(y,x) \). We assume that all these Poisson processes are independent.

For any \( t > 0 \) and \( x \in T^N \), we define
\[
\zeta_{t-}(x) = \lim_{s \downarrow t, s \in T} \zeta_s(x)
\]
as the state of \( x \) at the moment just before \( t \). For \( x \in T^N \), the state of \( x \) may change only at event times of \( N_x(\cdot) \) and \( U(x,y)(\cdot) \) for \( y \sim x \). At any event time \( s \) of \( N_x(\cdot) \), \( \zeta_s(x) = 0 \). At any event time \( r \) of \( U(x,y)(\cdot) \), \( \zeta_r(x) = \zeta_{r-}(x) + \zeta_{r-}(y) \).

Intuitively, \( \{\zeta_t\}_{t \geq 0} \) describes the spread of an infection disease and the seriousness of the disease for an infected vertex is considered. An infected vertex \( x \) may be further infected by an infected neighbor \( y \). When the infection occurs, the seriousness of the disease of \( y \) will be added to that of \( x \).

According to Chapter 9 of \cite{9}, \( \{\zeta_t\}_{t \geq 0} \) is a linear system with generator \( \mathcal{L} \) given by
\[
\mathcal{L} \zeta = \sum_{x \in T^N} \left[f(\zeta^0,x) - f(\zeta)\right] + \sum_{x \in T^N} \sum_{y \sim x} \lambda \rho(x, y) [f(\zeta^{(x,y),y}) - f(\zeta)]
\]  \hspace{1cm} (4.4)

for \( f \in C(\{0, 1, 2, 3, \ldots\}^{z^N}) \), where
\[
\zeta^{m,x}(y) = \begin{cases} 
\zeta(y) & \text{if } y \neq x, \\
m & \text{if } y = x
\end{cases}
\]
for \( m \geq 1 \) and \( x \in T^N \).

The following lemma is crucial for us to give lower bound of \( \lambda^c \).
Lemma 4.2. For any \( t \geq 0 \),
\[
P_N^\lambda(C_t^O \neq \emptyset) \leq E_N^\lambda \zeta_t(O).
\]

Proof. Let \( \{\eta_t\}_{t \geq 0} \) be the contact process defined in (1.1) with \( \eta_0(x) = 1 \) for any \( x \in \mathbb{T}^N \). Then, according to an approach of graphical representation introduced in [7], the contact process satisfies the dual-relationship that
\[
P_N^\omega(C_t^O \neq \emptyset) = P_N^\omega(\eta_t(O) = 1) \tag{4.5}
\]
for any \( \omega \in \Omega \) and therefore
\[
P_N^\lambda(C_t^O \neq \emptyset) = P_N^\lambda(\eta_t(O) = 1). \tag{4.6}
\]

For readers who are not familiar with the self-duality of contact processes, we give a rigorous proof of (4.5) in the appendix.

For any \( t \geq 0 \) and \( x \in \mathbb{T}^N \), we define
\[
\tilde{\eta}_t(x) = \begin{cases} 
1 & \text{if } \zeta_t(x) \geq 1, \\
0 & \text{if } \zeta_t(x) = 0.
\end{cases}
\]

According to the definition of \( \{\zeta_t\}_{t \geq 0} \), \( \tilde{\eta}(x) \) flips from 1 to 0 at moment \( s \) when and only when \( s \) is an event time of \( N_x(\cdot) \) and \( \zeta_{s-}(x) \geq 1 \). So \( \tilde{\eta}(x) \) flips from 0 to 1 at moment \( r \) when and only when \( \zeta_{r-}(x) = 0 \) and \( r \) is an event time of \( U(x,y)(\cdot) \) such that \( y \sim x \) and \( \zeta_{r-}(y) \geq 1 \). Therefore, \( \tilde{\eta}(x) \) flips from 0 to 1 at rate
\[
\sum_{y : y \sim x} \lambda \rho(x,y) \mathbb{1}(\zeta_t(y) \geq 1) = \sum_{y : y \sim x} \lambda \rho(x,y) \tilde{\eta}(y).
\]

As a result, \( \{\tilde{\eta}_t\}_{t \geq 0} \) evolves as the same way as that of \( \{\eta_t\}_{t \geq 0} \).

Since \( \eta_0(x) = \tilde{\eta}_0(x) = 1 \) for each \( x \in \mathbb{T}^N \), \( \{\tilde{\eta}_t\}_{t \geq 0} \) and \( \{\eta_t\}_{t \geq 0} \) have the same probability distribution.

Therefore,
\[
P_N^\lambda(\eta(O) = 1) = P_N^\lambda(\tilde{\eta}(O) = 1) = P_N^\lambda(\zeta_t(O) \geq 1) \leq E_N^\lambda \zeta_t(O). \tag{4.7}
\]

Lemma 4.2 follows from (4.3) and (4.7). \( \square \)

Finally, we give the proof of \( \lambda_c \geq (N E \rho + M^2 E \rho)^{-1} \).

Proof of \( \lambda_c \geq (N E \rho + M^2 E \rho)^{-1} \). It is easy to see that we only need to deal with the case where \( M = 1 \). For general \( M > 0 \), we take \( \bar{\rho} = \frac{\rho}{M} \) and denote by \( \bar{\lambda}_c \) the critical value with respect to \( \bar{\rho} \). Then,
\[
\lambda_c = \frac{1}{M} \bar{\lambda}_c \geq \frac{1}{M} \frac{1}{N E \bar{\rho} + \frac{1}{E \rho}} = (N E \rho + M^2 E \rho)^{-1}.
\]

So from now on we assume that \( P(\rho \leq 1) = 1 \).
According to the generator of \( \{ \zeta_t \}_{t \geq 0} \) given in (4.4) and Theorem 9.1.27 of [9], for each \( x \in \mathbb{T}^N \) and given edge weights \( \{ \rho(e, \omega) \}_{e \in E_N} \),

\[
\frac{d}{dt} E^\omega_x \zeta_t(x) = -E^\omega_x \zeta_t(x) + \sum_{y, y \sim x} \lambda \rho(x, y, \omega) E^\omega_y \zeta_t(y). 
\] (4.8)

For readers who do not want to check the theorem in [9], an intuitive explanation of (4.8) is that (4.8) is with the form

\[
\frac{d}{dt} E f(\zeta_t) = E [L f(\zeta_t)]
\]

with \( f(\zeta) = \zeta(x) \) as an ‘application’ of Hille-Yosida theorem. In fact, Theorem 9.1.27 of [9] is an extension of Hille-Yosida theorem to processes of linear systems.

Let \( G_\omega \) be \( \mathbb{T}^N \times \mathbb{T}^N \) matrix such that

\[
G_\omega(x, y) = \begin{cases} 
\lambda \rho(x, y, \omega) & \text{if } x \sim y, \\
0 & \text{otherwise}
\end{cases}
\]

and \( I \) be \( \mathbb{T}^N \times \mathbb{T}^N \) identity matrix, then by (4.8),

\[
\frac{d}{dt} E^\omega_x \zeta_t = (G_\omega - I) E^\omega_x \zeta_t. 
\] (4.9)

Since \( P(\rho \leq 1) = 1 \) and there are at most \( N + 1 \) positive elements in each row of \( G_\omega \), it is easy to check that ODE (4.9) satisfies Lipschitz condition under \( l_\infty \) norm of \( R^{\mathbb{T}^N} \) and the series

\[
e^{tG_\omega} = \sum_{n=0}^{+\infty} \frac{t^n G^n_\omega}{n!}
\]

converges. Therefore, according to classic theory of linear ODE, the unique solution of ODE (4.9) is

\[
E^\omega_x \zeta_t = e^{-t} e^{tG_\omega} \zeta_0. 
\] (4.10)

Since \( \zeta_0(x) = 1 \) for each \( x \in \mathbb{T}^N \), by (4.10),

\[
E^\omega_x \zeta_t(O) = e^{-t} \sum_{n=0}^{+\infty} \sum_{x' \in \mathbb{T}^N} \frac{t^n G^n_\omega(O, x)}{n!}. 
\] (4.11)

For \( n \geq 1 \), we say that

\[
\vec{x} = (x_0, x_1, \ldots, x_n) \in \bigoplus_{j=0}^n \mathbb{T}^N
\]

is a path starting at \( O \) with length \( n \) when \( x_0 = O \) and \( x_{j+1} \sim x_j \) for \( 0 \leq j \leq n - 1 \). Please note that a path may backtrack.

For \( n \geq 1 \), we denote by \( L_n \) the set of paths starting at \( O \) with length \( n \).

Then according to the definition of \( G_\omega \) and (4.11),

\[
E^\omega_x \zeta_t(O) = e^{-t} \sum_{n=0}^{+\infty} \frac{t^n \lambda^n}{n!} \left( \sum_{\vec{x} \in L_n} \prod_{j=0}^{n-1} \rho(x_j, x_{j+1}, \omega) \right), 
\] (4.12)
where $\overrightarrow{x} = (x_0, x_1, \ldots, x_n)$ and hence

$$E_N^\zeta_t(O) = e^{-t} \sum_{n=0}^{+\infty} \frac{t^n \lambda^n}{n!} \left( \sum_{\overrightarrow{x} \in L_n} E \prod_{j=0}^{n-1} \rho(x_j, x_{j+1}, \omega) \right). \quad (4.13)$$

For $\overrightarrow{x} = (x_0, x_1, \ldots, x_n) \in L_n$, there is an unique path $p(O, x_n)$ from $O$ to $x_n$ with length $\Gamma(x_n)$. In other words, $p(O, x_n)$ does not backtrack. According to the structure a tree, the path $\overrightarrow{x}$ contains all the edges in $p(O, x_n)$. Since $\rho \leq 1$,

$$E \prod_{j=0}^{n-1} \rho(x_j, x_{j+1}, \omega) \leq E \left[ \prod_{e \in p(O, x_n)} \rho(e, \omega) \right] = (E \rho)^{\Gamma(x_n)}. \quad (4.14)$$

Please note that the equation in (4.14) follows from that $p(O, x_n)$ is formed with $\Gamma(x_n)$ different edges.

By (4.13) and (4.14),

$$E_N^\zeta_t(O) \leq e^{-t} \sum_{n=0}^{+\infty} \frac{t^n \lambda^n}{n!} \left[ \sum_{\overrightarrow{x} \in L_n} (E \rho)^{\Gamma(x_n)} \right]. \quad (4.15)$$

Since each vertex on $\mathbb{T}^N$ has degree at most $N + 1$,

$$\sum_{\overrightarrow{x} \in L_n} (E \rho)^{\Gamma(x_n)} \leq (N + 1)^n \sum_{\overrightarrow{x} \in L_n} \prod_{j=0}^{n-1} \frac{1}{\deg(x_j)} (E \rho)^{\Gamma(x_n)}. \quad (4.16)$$

By the definition of $\{S^N_n\}_{n \geq 1}$, for $\overrightarrow{x} = (x_0, x_1, \ldots, x_n) \in L_n$,

$$\bar{P}(S^N_j = x_j, 0 \leq j \leq n) = \prod_{j=0}^{n-1} \frac{1}{\deg(x_j)}$$

and hence

$$\sum_{\overrightarrow{x} \in L_n} \prod_{j=0}^{n-1} \frac{1}{\deg(x_j)} (E \rho)^{\Gamma(x_n)} = \bar{E}[(E \rho)^{\Gamma(S_n)}]. \quad (4.17)$$

By (4.16) and (4.17),

$$\sum_{\overrightarrow{x} \in L_n} (E \rho)^{\Gamma(x_n)} \leq (N + 1)^n \bar{E}[(E \rho)^{\Gamma(S_n)}]. \quad (4.18)$$

By (4.15) and (4.18),

$$E_N^\zeta_t(O) \leq e^{-t} \sum_{n=0}^{+\infty} \frac{t^n \lambda^n (N + 1)^n}{n!} \bar{E}[(E \rho)^{\Gamma(S_n)}]. \quad (4.19)$$

By (4.19) and Lemma 4.1,

$$E_N^\zeta_t(O) \leq e^{-t} \sum_{n=0}^{+\infty} \frac{t^n \lambda^n (N + 1)^n}{n!} \left[ N \rho - 1 \right]^{N + 1} \bar{E}[(E \rho)^{\Gamma(S_n)}]. \quad (4.20)$$
By Lemma 4.2 and (4.20),

\[ P_N^\lambda(C^O \neq \emptyset) \leq \exp \left\{ t \left[ \lambda (NE\rho + \frac{1}{E\rho}) - 1 \right] \right\}. \]

Therefore,

\[ \limsup_{t \to +\infty} \frac{1}{t} \log P_N^\lambda(C^O \neq \emptyset) \leq \lambda (NE\rho + \frac{1}{E\rho}) - 1 < 0 \]

when

\[ \lambda < (NE\rho + \frac{1}{E\rho})^{-1}. \]

As a result,

\[ \lambda \geq (NE\rho + \frac{1}{E\rho})^{-1}. \]

Now we can complete the proof Theorem 2.1 and Theorem 2.2.

**Proof of Theorem 2.1.** According to Corollary 3.2, we only need to show that \( \lambda_c > 0 \) in any case. Since

\[ \lambda_c \geq \lambda \geq (NE\rho + \frac{M^2}{E\rho})^{-1} > 0, \]

the proof is complete.

**Proof of Theorem 2.2.** Since \( \lambda_c \geq (NE\rho + \frac{M^2}{E\rho})^{-1}, \)

\[ \liminf_{N \to +\infty} N\lambda_c(N) \geq \frac{1}{E\rho}. \]

Since \( \lambda_c \leq \lambda \) and we have shown that

\[ \limsup_{N \to +\infty} N\lambda_c(N) \leq \frac{1}{E\rho} \]

in Section 3

\[ \lim_{N \to +\infty} N\lambda_c(N) = \lim_{N \to +\infty} N\lambda_c(N) = \frac{1}{E\rho} \]

and the proof is complete.

5 Critical value under quenched measure

In this section we discuss the critical value under quenched measure. For later use, we identify \( \mathbb{T}^N \) with the set

\[ \{O\} \bigcup_{m=1}^{+\infty} \bigcup_{m=1}^{+\infty} \{1, 2, 3, \ldots, N\}^m. \]

In detail, \( O \) is the root of \( \mathbb{T}^N \). For \( 1 \leq j \leq N \), \( j \) represents the \( j \)th son of \( O \). For \( m \geq 1, 1 \leq j \leq N \) and

\[ (k_1, k_2, \ldots, k_m) \in \{1, 2, \ldots, N\}^m, \]

\( (k_1, k_2, \ldots, k_m, j) \) represents the \( j \)th son of \( (k_1, k_2, \ldots, k_m) \). The following picture describes the first three generations of \( \mathbb{T}^2 \).
For each $1 \leq j \leq N$, we define injection $\varphi_j : T^N \to T^N$ such that

$$\varphi_j(O) = j$$

and

$$\varphi_j(k_1, k_2, \ldots, k_m) = (j, k_1, k_2, \ldots, k_m)$$

for each $m \geq 1$ and any $(k_1, k_2, \ldots, k_m) \in \{1, 2, \ldots, N\}^m$.

For $e \in E^N$ with endpoints $x, y \in T^N$, we denote by $e_j$ the edge with endpoints $\varphi_j(x)$ and $\varphi_j(y)$. For $\omega \in \Omega_N$ and $j \geq 1$, we denote by $\omega_j$ the sample point such that

$$\rho(e, \omega_j) = \rho(e_j, \omega)$$

for each $e \in E^N$. That is to say, if $T^N$ is with edge weights $\{\rho(e, \omega)\}_{e \in E^N}$, then $j$ and its descendants form a regular tree which is rooted at $j$ and with edge weights $\{\rho(e, \omega_j)\}_{e \in E^N}$.

For any $\lambda > 0$, $N \geq 1$ and $1 \leq j \leq N$, we define

$$H(\lambda, N) = \{\omega \in \Omega_N : P_{\lambda}^\omega(\forall t \geq 0, C_0^O \neq \emptyset) = 0\}$$

and

$$H(\lambda, N, j) = \{\omega \in \Omega_N : P_{\lambda}^{\omega_j}(\forall t \geq 0, C_0^O \neq \emptyset) = 0\}.$$ 

The following lemma shows that $H(\lambda, N)$ satisfies a zero-one law, which is crucial for us to prove Theorem 2.3. Please note that $A_N$ in the lemma is the same as that defined in (2.4).

**Lemma 5.1.** If $P(\rho > 0) < 1$ and $N > 1/P(\rho > 0)$, then $0 < \mu_N(A_N) < 1$ and

$$\mu_N(H(\lambda, N)) \in \{1 - \mu_N(A_N), 1\}$$

for any $\lambda > 0$.

If $P(\rho > 0) = 1$ and $N \geq 2$, then

$$\mu_N(H(\lambda, N)) \in \{0, 1\}$$

for any $\lambda > 0$.

**Proof.** For any $\omega \in \Omega$, we define

$$B(\omega) = \{1 \leq j \leq N : \rho(O, j, \omega) > 0\}$$

as the set of sons which $O$ can infect.
According to the strong Markov property, for $1 \leq j \leq N$,
\[
P_\lambda^\omega(\forall t \geq 0, C_t^O \neq \emptyset) \geq P_\lambda(\exists t > 0, j \in C_t^O) P_\lambda^\omega(\forall t \geq 0, C_t^O \neq \emptyset). \tag{5.1}
\]
If $j \in B(\omega)$, then $P_\lambda(\exists t > 0, j \in C_t^O) > 0$. Therefore, by (5.1), $P_\lambda^\omega(\forall t \geq 0, C_t^O \neq \emptyset) = 0$ and $j \in B(\omega)$ implies that $P_\lambda^\omega(\forall t \geq 0, C_t^O \neq \emptyset) = 0$. As a result,
\[
H(\lambda, N) \subseteq \{ \omega : \omega \in \bigcap_{j \in B(\omega)} H(\lambda, N, j) \}. \tag{5.2}
\]

Since $\{ \rho(e) \}_{e \in \mathbb{N}}$ are i.i.d, $H(\lambda, N, 1), H(\lambda, N, 2), \ldots, H(\lambda, N, N)$ are independent of $B(\omega)$ and are i.i.d events which have the same probability distribution as that of $H(\lambda, N)$ under $\mu_N$. Therefore, by (5.2),
\[
\mu_N(H(\lambda, N)) \leq \sum_{k=0}^{N} p_k \left[ \mu_N(H(\lambda, N)) \right]^k, \tag{5.3}
\]
where
\[
p_k = \mu_N(\omega : |B(\omega)| = k) = \binom{N}{k} P(\rho > 0)^k (1 - P(\rho > 0))^{N-k}.
\]
For $x \in [0, 1]$, we define
\[
f(x) = \sum_{k=0}^{N} p_k x^k.
\]
As we have shown in Section 2, $D(\omega)$ defined in (2.2) is a Galton-Watson tree with binomial offspring distribution $B(N, P(\rho > 0))$ and $1 - \mu_N(A_N)$ is the extinction probability of $D(\omega)$.

When $P(\rho > 0) < 1$ and $N > 1/P(\rho > 0)$, the mean of $B(N, P(\rho > 0))$ is larger than one. Then according to the extinction criterion of Galton-Watson trees, $1 - \mu_N(A_N)$ is the unique solution in $(0, 1)$ to the equation $x = f(x)$ and $f(y) < y$ for $y \in (1 - \mu_N(A_N), 1)$. By (5.3), $\mu_N(H(\lambda, N)) \leq f\left( \mu_N(H(\lambda, N)) \right)$, hence
\[
\mu_N(H(\lambda, N)) \in [0, 1 - \mu_N(A_N)] \cup \{1\}. \tag{5.4}
\]
For any $\omega \in \Omega_N \setminus A_N$, $|D(\omega)|$ is finite and hence the Markov process $\{C_t^O\}_{t \geq 0}$ under the measure $P_\lambda^\omega$ is with finite state space $\{A : A \subseteq D(\omega)\}$ and unique absorption state $\emptyset$, which makes $\{C_t^O\}_{t \geq 0}$ frozen in $\emptyset$ eventually. As a result,
\[
P_\lambda^\omega(\forall t \geq 0, C_t^O \neq \emptyset) = 0
\]
for any $\omega \in \Omega_N \setminus A_N$ and hence
\[
\mu_N(H(\lambda, N)) \geq \mu_N(\Omega_N \setminus A_N) = 1 - \mu_N(A_N). \tag{5.5}
\]
By (5.4) and (5.5),
\[
\mu_N(H(\lambda, N)) \in \{1 - \mu_N(A_N), 1\}.
\]
When \( P(\rho > 0) = 1 \) and \( N \geq 2 \), (5.3) turns into
\[
\mu_N(H(\lambda, N)) \leq \left[ \mu_N(H(\lambda, N)) \right]^N. \tag{5.6}
\]
If \( 0 < \mu_N(H(\lambda, N)) < 1 \), then
\[
\left[ \mu_N(H(\lambda, N)) \right]^N < \mu_N(H(\lambda, N))
\]
since \( N \geq 2 \), which is contradictory to (5.6). Therefore,
\[
\mu_N(H(\lambda, N)) \in \{0, 1\}. \tag{5.8}
\]

In the case where \( P(\rho > 0) = 1 \) and \( N = 1 \), (5.3) turns into
\[
\mu(N(H(\lambda, N))) \leq \mu(N(H(\lambda, N)))
\]
which gives no information. This is why this case should be discussed specially. We propose an open question about the critical value in this case in section 6.

Finally we give the proof of Theorem 2.3.

**Proof of Theorem 2.3.** We first consider the case where \( P(\rho > 0) = 1 \) and \( N \geq 2 \). In this case, we have shown in the proof of Theorem 2.1 that
\[
\lambda_c(N) \in (0, +\infty).
\]
So we only need to show that \( \hat{\lambda}_c(\omega, N) = \lambda_c(N) \) with probability one. For \( m > 1/\lambda_c(N) \), let \( \lambda_m = \lambda_c(N) - \frac{1}{m} \) and \( \beta_m = \lambda_c(N) + \frac{1}{m} \), then according to the definition of \( \lambda_c(N) \),
\[
P_{\lambda_m}^N(\forall \, t \geq 0, C_t^0 \neq \emptyset) = E_{\lambda_m}^N [P_{\lambda_m}^\omega(\forall \, t \geq 0, C_t^0 \neq \emptyset)] = 0
\]
and
\[
P_{\beta_m}^N(\forall \, t \geq 0, C_t^0 \neq \emptyset) = E_{\beta_m}^N [P_{\beta_m}^\omega(\forall \, t \geq 0, C_t^0 \neq \emptyset)] > 0.
\]
Therefore, according to lemma 5.1,
\[
\mu_N(H(\lambda_m, N)) = 1 \tag{5.7}
\]
and
\[
\mu_N(H(\beta_m, N)) = 0. \tag{5.8}
\]
Let
\[
K_N = \bigcap_m H(\lambda_m, N) \bigcap_m (\Omega_N \setminus H(\beta_m, N)),
\]
then
\[
\mu_N(K_N) = 1
\]
according to (5.7) and (5.8). For \( \omega \in K_N \),
\[
P_{\lambda_m}^\omega(\forall \, t \geq 0, C_t^0 \neq \emptyset) = 0, P_{\beta_m}^\omega(\forall \, t \geq 0, C_t^0 \neq \emptyset) > 0
\]
and hence
\[
\lambda_m \leq \hat{\lambda}_c(\omega, N) \leq \beta_m.
\]
Let $m \to +\infty$, then we have that
\[ \hat{\lambda}_c(\omega, N) = \lambda_c(N) \]
for $\omega \in K_N$.

Now we deal with the case where $P(\rho > 0) < 1$ and $N > 1/P(\rho > 0)$. As we have shown in the proof of Theorem 2.1,
\[ \lambda_c(N) \in (0, +\infty) \]
in this case. We also use $\lambda_m$ and $\beta_m$ to denote $\lambda_c(N) - \frac{1}{m}$ and $\lambda_c(N) + \frac{1}{m}$ respectively. According to a similar analysis with that of the first case and Lemma 5.1,
\[ \mu_N\left( H(\lambda_m, N) \right) = 1 \] (5.9)
and
\[ \mu_N\left( H(\beta_m, N) \right) = 1 - \mu_N(A_N). \] (5.10)
We have shown in the proof of Lemma 5.1 that
\[ H(\lambda, N) \supseteq \Omega_N \setminus A_N \]
for any $\lambda > 0$, hence by (5.10),
\[ \mu_N\left( H(\beta_m, N) \cap A_N \right) = 0. \] (5.11)

Let
\[ K_N = \left( A_N \setminus \bigcup_m H(\beta_m, N) \right) \bigcap \bigcap_m H(\lambda, N), \]
then $K_N \subseteq A_N$ and $\mu_N(A_N \setminus K_N) = 0$ according to (5.9) and (5.11).

According to a similar analysis with that of the first case, it is easy to see that
\[ \hat{\lambda}_c(\omega, N) = \lambda_c(N) \]
for any $\omega \in K_N$.

For $\omega \in \Omega_N \setminus A_N$, $|D(\omega)| < +\infty$ and hence
\[ P^\omega_x(\forall \ t \geq 0, C_t^O \neq \emptyset) = 0 \]
for any $\lambda > 0$ as we have shown in the proof of Lemma 5.1. As a result,
\[ \hat{\lambda}_c(\omega, N) = +\infty \]
for any $\omega \in \Omega_N \setminus A_N$.

For the last case where $P(\rho > 0) < 1$ and $N \leq 1/P(\rho > 0)$,
\[ A_N = \emptyset \]
according to the extinction criterion of Galton-Watson trees, and hence
\[ \hat{\lambda}_c(\omega, N) = +\infty \]
for any $\omega \in \Omega_N$.
6 An Open question for $N=1$

When $N=1$, our model turns into the contact process with random edge weights on $\mathbb{Z}$. We do not manage to give an criterion to judge whether $\lambda_c < +\infty$ in this case.

There are two trivial cases for this problem. If $P(\rho > 0) < 1$, then $|D(\omega)|$ is finite with probability one and hence $\lambda_c = +\infty$. If $P(\rho > \epsilon) = 1$ for some $\epsilon > 0$, then $\lambda_c < +\infty$ since the classic contact process on $\mathbb{Z}$ has finite critical value (see [10]). So we only need to deal with the case where $P(\rho > 0) = 1$ but $P(\rho < x) > 0$ for any $x \in (0, 1)$.

We do not think that $P(\rho > 0) = 1$ is sufficient or $P(\rho > \epsilon) = 1$ is necessary for finite critical value. We guess that the probability of $\{ \rho < x \}$ for small $x$ is crucial.

To make our question concrete, for $\alpha > 0$, we assume that $P(\rho < x) = x^\alpha$ for $x \in (0, 1)$. We denote by $\lambda_c(\alpha)$ the critical value with respect to $\rho$. Then it is obviously that

$$\lambda_c(\alpha_1) \leq \lambda_c(\alpha_2)$$

for $\alpha_1 > \alpha_2$.

Then it is reasonable to ask the following question.

**Question 6.1.** We assume that $N=1$ and $\rho$ has the distribution as that in (6.1). Then is there a critical value $0 < \alpha_c < +\infty$ such that

$$\lambda_c(\alpha) < +\infty$$

for $\alpha > \alpha_c$ and

$$\lambda_c(\alpha) = +\infty$$

for $\alpha < \alpha_c$?

If the answer to Question 6.1 is positive, a further problem is how to estimate $\alpha_c$, which will bring more interesting work to do. We will work on Question 6.1 as a further study and hope to discuss with readers who are interested in this question.

A Appendix

**Proof of (4.5).** According to the flip rate functions given by (1.1), the Markov process $\{C_t\}_{t \geq 0}$ is with state space $2^{\mathbb{T}^N} := \{ A : A \subseteq \mathbb{T}^N \}$ and has generator given by

$$\mathcal{L}f(A) = \sum_{x \in A} [f(A \setminus x) - f(A)] + \sum_{x \in A} \sum_{y \sim x} \lambda \rho(x, y) [f(A \cup \{y\}) - f(A)]$$

(A.1)

for $f \in C(2^{\mathbb{T}^N})$ and $A \subseteq \mathbb{T}^N$.

We define $H : 2^{\mathbb{T}^N} \times 2^{\mathbb{T}^N} \to \{0, 1\}$ that

$$H(A, B) = \begin{cases} 1 & \text{if } A \cap B = \emptyset, \\ 0 & \text{if } A \cap B \neq \emptyset \end{cases}$$

(A.2)

for $A, B \subseteq \mathbb{T}^N$. 

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By (A.2),
\[ H(A, B \cup C) = H(A, B)H(A, C) \] (A.3)
for \( A, B, C \subseteq T^N \).

By (A.1), (A.3) and direct calculation,
\[
\mathcal{L}H(A, \cdot)(B) = \sum_{x \in B} \left[ H(A, B \setminus x) - H(A, B) \right] \\
+ \sum_{x \in B} \sum_{y \sim x} \lambda \rho(x, y) \left[ H(A, B \cup \{y\}) - H(A, B) \right] \\
= \sum_{x \in B} H(A, B \setminus x) \left[ 1 - H(A, \{x\}) \right] \\
+ \sum_{x \in B} \sum_{y \sim x} \lambda \rho(x, y) H(A, B) \left[ H(A, \{y\}) - 1 \right] \\
= \sum_{x \in A \cap B} H(A, B \setminus x) - \sum_{x \in B} \sum_{y \sim x, y \in A} \lambda \rho(x, y) H(A, B) \] (A.4)
for \( A, B \subseteq T^N \). According to a similar calculation,
\[
\mathcal{L}H(\cdot, B)(A) = \sum_{x \in A \cap B} H(A \setminus x, B) - \sum_{y \in A} \sum_{x \in B, x \sim y} \lambda \rho(x, y) H(A, B) \\
= \sum_{x \in A \cap B} H(A \setminus x, B) - \sum_{x \in B} \sum_{y \sim x, y \in A} \lambda \rho(x, y) H(A, B). \] (A.5)

It is easy to see that
\[ H(A, B \setminus x) = H(A \setminus x, B) \]
for \( x \in A \cap B \). Therefore, by (A.4) and (A.5),
\[
\mathcal{L}H(\cdot, B)(A) = \mathcal{L}H(A, \cdot)(B) \] (A.6)
for \( A, B \subseteq T^N \).

We write \( C_t \) as \( C_t^A \) when \( C_0 = A \). Then, according to (A.6) and Theorem 3.39 of [12],
\[
E^\omega H(A, C_t^B) = E^\omega H(C_t^A, B) \] (A.7)
for \( A, B \subseteq T^N \) and \( t \geq 0 \). Let \( A = \{O\} \) and \( B = T^N \), then (A.7) follows from (A.7).

Acknowledgments. The author is grateful to the financial support from the National Natural Science Foundation of China with grant number 11171342 and China Postdoctoral Science Foundation (No. 2015M571095).

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