On negative eigenvalues of two-dimensional Schrödinger operators

Eugene Shargorodsky

Abstract

The paper presents estimates for the number of negative eigenvalues of a two-dimensional Schrödinger operator in terms of $L \log L$-type Orlicz norms of the potential and proves a conjecture by N.N. Khuri, A. Martin and T.T. Wu.

1. Introduction

According to the celebrated Cwikel–Lieb–Rozenblum inequality, the number $N_-(V)$ of negative eigenvalues of the Schrödinger operator $-\Delta - V$, $V \geq 0$ on $L_2(\mathbb{R}^d)$, $d \geq 3$ is estimated above by

$$\text{const} \int_{\mathbb{R}^d} V(x)^{d/2} \, dx.$$ 

It is well known that this estimate does not hold for $d = 2$. In this case, the Schrödinger operator has at least one negative eigenvalue for any non-zero $V \geq 0$, and no estimate of the type

$$N_-(V) \leq \text{const} + \int_{\mathbb{R}^2} V(x)W(x) \, dx$$

can hold, provided the weight function $W$ is bounded in a neighborhood of at least one point (see [10]). On the other hand,

$$N_-(V) \geq \text{const} \int_{\mathbb{R}^2} V(x) \, dx$$

(see [11]). Upper estimates for $N_-(V)$ in the case $d = 2$ can be found in [4, 6, 7, 10, 17, 20, 21, 25, 26, 29, 30, 32] and in the references therein. Following the pioneering paper [29], we obtain upper estimates involving $L \log L$-type Orlicz norms of the potential (Theorems 3.1, 6.1 and 7.1) and prove (Theorem 4.3) a conjecture by Khuri, Martin and Wu [17]. In fact, we show that the main estimate in [29] is actually stronger than the one conjectured in [17], while none of the later results in [4, 10, 19, 21, 25, 26, 32] implies the Khuri–Martin–Wu inequality (see Section 8). Our approach does not seem to be sufficient to settle the stronger conjecture by Chadan, Khuri, Martin and Wu ([7]; see (24) and (25)).

We discuss several upper estimates for $N_-(V)$ in the paper and show in Section 8 how they are related to each other (see the diagram close to the end of Section 8). All of them involve terms of two types: integrals of $V$ with a logarithmic weight and $L \log L$ type (or $L_p$, $p > 1$) norms of $V$. It turns out that the finiteness of the term of the first type suggested in [29] is necessary for $N_-(\alpha V) = O(\alpha)$ as $\alpha \to +\infty$ to hold (see Theorem 9.2). Unfortunately, this is not true for the terms of the second type. None of the estimates discussed in the paper is sharp in the sense that $N_-(V)$ has to be infinite if the right-hand side is infinite. On the other hand, both the logarithmic weight and the $L \log L$ norm are in a sense optimal if one tries to estimate $N_-(V)$ in terms of weighted integrals of $V$ and of its Orlicz norms (see [29, Section 4] and

\[ \text{Received 11 January 2013; revised 22 April 2013; published online 19 July 2013.} \]

2010 Mathematics Subject Classification 35J10, 35P15 (primary).
Section 9). It is probably difficult to obtain an estimate for \( N_-(V) \) that is sharp in the above sense. Indeed, there are potentials \( V \geq 0 \) such that \( N_-(\alpha V) < \infty \) for \( \alpha < 1 \) and \( N_-(\alpha V) = \infty \) for \( \alpha > 1 \) (this is true in the multidimensional case \( d \geq 3 \) as well). For such potentials, \( N_-(V) \) may be finite or infinite, and in the latter case, \( N_-(\alpha V) \) may grow arbitrarily fast or arbitrarily slow as \( \alpha \to 1 - 0 \) (see Theorem 9.5 and Appendix B).

2. Notation and auxiliary results

We need some notation from the theory of Orlicz spaces (see [18, 28]). Let \((\Omega, \Sigma, \mu)\) be a measure space, let \( \Phi \) and \( \Psi \) be mutually complementary \( N \)-functions and let \( L_\Phi(\Omega), L_\Psi(\Omega) \) be the corresponding Orlicz spaces. (These spaces are denoted by \( L^*_\Phi(\Omega), L^*_\Psi(\Omega) \) in [18], where \( \Omega \) is assumed to be a closed bounded subset of \( \mathbb{R}^d \) equipped with the standard Lebesgue measure.) We will use the following norms on \( L_\Psi(\Omega) \):

\[
\|f\|_{\Psi} = \|f\|_{\Psi, \Omega} = \sup \left\{ \left| \int_{\Omega} fg \, d\mu \right| : \int_{\Omega} \Phi(g) \, d\mu \leq 1 \right\} \quad \text{(1)}
\]

and

\[
\|f\|_{(\Psi)} = \|f\|_{(\Psi, \Omega)} = \inf \left\{ \kappa > 0 : \int_{\Omega} \Psi\left(\frac{f}{\kappa}\right) \, d\mu \leq 1 \right\}. \quad \text{(2)}
\]

These two norms are equivalent

\[
\|f\|_{(\Psi)} \leq \|f\|_{\Psi} \leq 2\|f\|_{(\Psi)} \quad \forall f \in L_\Psi(\Omega). \quad \text{(3)}
\]

Note that

\[
\int_{\Omega} \Psi\left(\frac{f}{\kappa_0}\right) \, d\mu \leq C_0, \quad C_0 \geq 1 \implies \|f\|_{(\Psi)} \leq C_0 \kappa_0. \quad \text{(4)}
\]

Indeed, since \( \Psi \) is even, convex and increasing on \([0, +\infty)\), and \( \Psi(0) = 0 \), we get for any \( \kappa \geq C_0 \kappa_0 \),

\[
\int_{\Omega} \Psi\left(\frac{f}{\kappa}\right) \, d\mu \leq \int_{\Omega} \frac{1}{C_0} \int_{\Omega} \Psi\left(\frac{f}{\kappa_0}\right) \, d\mu \leq 1 \quad \text{(5)}
\]

It follows from (4) with \( \kappa_0 = 1 \) that

\[
\|f\|_{(\Psi)} \leq \max\left\{1, \int_{\Omega} \Psi(f) \, d\mu\right\}. \quad \text{(6)}
\]

We will need the following equivalent norm on \( L_\Psi(\Omega) \) with \( \mu(\Omega) < \infty \) that was introduced in [29]:

\[
\|f\|_{(\Psi)}^{(av)} = \|f\|_{\Psi, \Omega}^{(av)} = \sup \left\{ \left| \int_{\Omega} fg \, d\mu \right| : \int_{\Omega} \Phi(g) \, d\mu \leq \mu(\Omega) \right\}. \quad \text{(7)}
\]

**Lemma 2.1.**

\[
\min\{1, \mu(\Omega)\} \|f\|_{\Psi, \Omega} \leq \|f\|_{(\Psi)}^{(av)} \leq \max\{1, \mu(\Omega)\} \|f\|_{\Psi, \Omega}.
\]

**Proof.** Let

\[
\mathbb{B}_1 := \left\{ g : \int_{\Omega} \Phi(g) \, d\mu \leq 1 \right\}, \quad \mathbb{B}_\Omega := \left\{ g : \int_{\Omega} \Phi(g) \, d\mu \leq \mu(\Omega) \right\}.
\]

Suppose that \( \mu(\Omega) \geq 1 \). Then, clearly, \( \|f\|_{\Psi, \Omega} \leq \|f\|_{(\Psi)}^{(av)} \). It is easy to see that

\[
g \in \mathbb{B}_\Omega \implies \frac{1}{\mu(\Omega)} g \in \mathbb{B}_1
\]
(cf. (5)). Hence,
\[ \| f \|_{(av)} = \sup_{g \in B_\Omega} \left| \int_\Omega fg \, d\mu \right| \leq \mu(\Omega) \| f \|_{\Psi, \Omega}. \]

Suppose now that \( \mu(\Omega) < 1 \). Then \( \| f \|_{(av)} \leq \| f \|_{\Psi, \Omega} \) and
\[ g \in B_1 \implies \mu(\Omega) g \in B_\Omega. \]
Hence,
\[ \mu(\Omega) \| f \|_{\Psi, \Omega} = \mu(\Omega) \sup_{g \in B_1} \left| \int_\Omega fg \, d\mu \right| \leq \sup_{h \in B_\Omega} \left| \int_\Omega fh \, d\mu \right| = \| f \|_{(av)} \Psi, \Omega. \]

We will need the following pair of mutually complementary \( N \)-functions
\[ A(s) = e|s| - 1 - |s|, \quad B(s) = (1 + |s|) \ln(1 + |s|) - |s|, \quad s \in \mathbb{R}. \] (8)

We will use the following standard notation
\[ a_+ := \max\{0, a\}, \quad a \in \mathbb{R}. \] (9)

**Lemma 2.2.** \( \frac{1}{2} s \ln s \leq B(s) \leq s + 2s \ln s \forall s \geq 0. \)

**Proof.** If \( s \geq 1 \), then integrating the inequality
\[ 1 + \ln t = \ln(e t) < \ln(1 + 3 t) \leq 2 \ln(1 + t), \quad t \geq 1, \]
over the interval \([1, s]\), one gets \( s \ln s \leq 2(B(s) - 2 \ln 2 + 1) < 2B(s) \). Hence, \( s \ln s \leq 2B(s) \forall s \geq 0. \)
If \( s \geq 1 \), then
\[ B(s) = (1 + s) \ln(1 + s) - s \leq 2s \ln(2s) - s \]
\[ = (2 \ln 2 - 1)s + 2s \ln s < s + 2s \ln s. \]
If \( s \in [0, 1) \), then integrating the inequality \( \ln(1 + t) \leq t \) over the interval \([0, s]\), one gets
\[ B(s) = (1 + s) \ln(1 + s) - s \leq \frac{1}{2}s^2 \leq \frac{1}{2}s \leq s. \]

**Lemma 2.3.** \( B(s) \leq \frac{1}{2}s^2, \frac{1}{2}s^2 \leq A(s) \leq (e/2)s^2 \forall s \in [0, 1]. \)

**Proof.** The first inequality was proved at the end of the proof of Lemma 2.2, the second one is obtained by integrating the inequality \( 1 \leq e^s \leq e, s \in [0, 1] \) twice.

**Lemma 2.4.** \( e^s \leq 2A(s) + \frac{3}{2} \forall s \geq 0. \)

**Proof.**
\[ s \leq \frac{1}{2} + \frac{s^2}{2} \leq \frac{1}{2} + A(s). \]
Hence,
\[ e^s = A(s) + 1 + s \leq 2A(s) + \frac{3}{2} \forall s \geq 0. \]

**Lemma 2.5.** Let \( \mu(\Omega) > 1. \) Then
\[ \| f \|_{(av)}_{B, \Omega} \leq \| f \|_{g, \Omega} + \ln(\frac{2}{\mu(\Omega)}) \| f \|_{L_1(\Omega, \mu)}. \]
Proof. Since \( \|f\|_{H,\Omega}^{(\text{av})} = \|f\|_{G,\Omega}^{(\text{av})} \) and \( \|f\|_{B,\Omega} = \|f\|_{G,\Omega} \), we can assume, without loss of generality, that \( f \geq 0 \). In this case,

\[
\|f\|_{B,\Omega} = \sup \left\{ \int \Omega f g d\mu \mid g \geq 0, \int \Omega A(g) d\mu \leq 1 \right\},
\]

\[
\|f\|_{B,\Omega}^{(\text{av})} = \sup \left\{ \int \Omega f g d\mu \mid g \geq 0, \int \Omega A(g) d\mu \leq \mu(\Omega) \right\}.
\]

Take an arbitrary \( g \geq 0 \) with \( \int \Omega A(g) d\mu \leq \mu(\Omega) \) and set

\[
\chi(x) := \begin{cases} 1 & \text{if } g(x) \leq \ln \left( \frac{7}{2} \mu(\Omega) \right), \\ 0 & \text{if } g(x) > \ln \left( \frac{7}{2} \mu(\Omega) \right), \end{cases}
\]

\[
g_1 := g\chi + \ln \left( \frac{7}{2} \mu(\Omega) \right) (1 - \chi), \quad g_2 := g - g_1 = (g - \ln \left( \frac{7}{2} \mu(\Omega) \right))(1 - \chi).
\]

Then \( g = g_1 + g_2, 0 \leq g_1 \leq \ln \left( \frac{7}{2} \mu(\Omega) \right), \) and it follows from Lemma 2.4 that

\[
\int \Omega A(g_2) d\mu = \int g(x) > \ln \left( \frac{7}{2} \mu(\Omega) \right) A(g(x) - \ln \left( \frac{7}{2} \mu(\Omega) \right)) d\mu(x)
\]

\[
\leq \int g(x) > \ln \left( \frac{7}{2} \mu(\Omega) \right) e^{g(x) - \ln \left( \frac{7}{2} \mu(\Omega) \right)} d\mu(x) \leq \frac{2}{7\mu(\Omega)} \int \Omega e^g d\mu
\]

\[
\leq \frac{2}{7\mu(\Omega)} \int \Omega \left( 2A(g) + \frac{3}{2} \right) d\mu \leq \frac{2}{7\mu(\Omega)} \left( 2\mu(\Omega) + \frac{3}{2} \mu(\Omega) \right) = 1.
\]

Hence,

\[
\int \Omega f g d\mu = \int \Omega f g_1 d\mu + \int \Omega f g_2 d\mu \leq \ln \left( \frac{7}{2} \mu(\Omega) \right) \int \Omega f d\mu + \|f\|_{B,\Omega}
\]

\[
= \|f\|_{B,\Omega} + \ln \left( \frac{7}{2} \mu(\Omega) \right) \|f\|_{L^1_1(\Omega,\mu)}. \tag{\checkmark}
\]

3. A Solomyak-type estimate

Let \( H \) be a Hilbert space and let \( \mathbf{q} \) be a Hermitian form with a domain \( \text{Dom}(\mathbf{q}) \subseteq H \). Set

\[
N_-(\mathbf{q}) := \sup \{ \dim L \mid \mathbf{q}[u] < 0, \forall u \in L \setminus \{0\} \}, \tag{10}
\]

where \( L \) denotes a linear subspace of \( \text{Dom}(\mathbf{q}) \). If \( \mathbf{q} \) is the quadratic form of a self-adjoint operator \( A \) with no essential spectrum in \((\mathbb{R},0)\), then by the variational principle, \( N_-(\mathbf{q}) \) is the number of negative eigenvalues of \( A \), repeated according to their multiplicity (see, for example, [3, S1.3] or [5, Theorem 10.2.3])

Let \( V \geq 0 \) be locally integrable on \( \mathbb{R}^2 \). Consider

\[
\mathcal{E}_V[w] := \int_{\mathbb{R}^2} |\nabla w(x)|^2 \, dx - \int_{\mathbb{R}^2} V(x)|w(x)|^2 \, dx,
\]

\[
\text{Dom}(\mathcal{E}_V) := W^1_2(\mathbb{R}^2) \cap L^2_2(\mathbb{R}^2, V(x) \, dx).
\]

Theorem 3.1. There exists a constant \( C > 0 \) such that

\[
N_-(\mathcal{E}_V) \leq C \left( \|V\|_{B,\mathbb{R}^2} + \int_{\mathbb{R}^2} V(x) \ln(1 + |x|) \, dx \right) + 1 \quad \forall V \geq 0. \tag{11}
\]
Proof. Let
\[
\mathcal{E}_{V,m}[w] := \int_{\Omega_m} |\nabla w(x)|^2 \, dx - \int_{\Omega_m} V(x)|w(x)|^2 \, dx,
\]
where
\[
\Omega_1 = B(0, 1) = \{ x \in \mathbb{R}^2 : |x| < 1 \}, \quad \Omega_2 = \mathbb{R}^2 \setminus B(0, 1) = \{ x \in \mathbb{R}^2 : |x| > 1 \}.
\]
Then by the variational principle,
\[
N_-(\mathcal{E}_V) \leq N_-(\mathcal{E}_{V,1}) + N_-(\mathcal{E}_{V,2}) \tag{12}
\]
(see, for example, [10, Lemma 3.5]).
There exists an independent of $V$ constant $C_1 > 0$ such that
\[
N_-(\mathcal{E}_{V,1}) \leq C_1 \| V \|_{B, \Omega_1} + 1 \tag{13}
\]
(see [29, Theorems 4 and 4’, and Proposition 3]).
Below, we use the complex notation $z = x_1 + ix_2$ alongside the real one $x = (x_1, x_2)$. Let
\[
\hat{V}(z) := \frac{1}{|z|^2} V \left( \frac{1}{z} \right), \quad \hat{w}(z) := w(1/z), \quad w \in W^1_2(\Omega_2), \quad |z| < 1
\]
(cf. the proof of Proposition 5.3 in [9]). An easy calculation gives
\[
\int_{\Omega_2} |\nabla w(y)|^2 \, dy = \int_{\Omega_1} |\nabla \hat{w}(x)|^2 \, dx,
\]
\[
\int_{\Omega_2} V(y)|w(y)|^2 \, dy = \int_{\Omega_1} \hat{V}(x)|\hat{w}(x)|^2 \, dx,
\]
and it follows from (13) that
\[
N_-(\mathcal{E}_{V,2}) \leq C_1 \| \hat{V} \|_{B, \Omega_1} + 1. \tag{14}
\]
Let us estimate the norm in the right-hand side. It is more convenient to work with the Luxemburg (gauge) norm (2). Using the notation $\zeta = y_1 + iy_2 = 1/z$, we get for any $\kappa > 0$
\[
\int_{\Omega_1} B \left( \frac{\hat{V}(z)}{\kappa} \right) \, dx = \int_{\Omega_1} B \left( \frac{V(1/z)}{\kappa |z|^2} \right) \, dx = \int_{\Omega_2} B \left( \frac{|\zeta|^4 V(\zeta)}{\kappa} \right) \frac{1}{|\zeta|^4} \, dy
\]
\[
\leq \int_{\Omega_2} \left( 1 + \frac{1}{\kappa} |y|^4 V(y) \right) \ln \left( 1 + \frac{1}{\kappa} |y|^4 V(y) \right) \frac{1}{|y|^4} \, dy
\]
\[
\leq \int_{\Omega_2} \left( 1 + \frac{1}{\kappa} |y|^4 V(y) \right) \ln \left( 1 + \frac{1}{\kappa} V(y) \right) - \frac{1}{\kappa} V(y) \frac{1}{|y|^4} \, dy
\]
\[
+ \int_{\Omega_2} \frac{1}{|y|^4} \left( 1 + \frac{1}{\kappa} |y|^4 V(y) \right) \ln(1 + |y|^4) \, dy
\]
\[
\leq \int_{\Omega_2} \left( 1 + \frac{1}{\kappa} V(y) \right) \ln \left( 1 + \frac{1}{\kappa} V(y) \right) - \frac{1}{\kappa} V(y) \, dy
\]
\[
+ \frac{1}{\kappa} \int_{\Omega_2} V(y) \ln(1 + |y|^4) \, dy + \int_{\Omega_2} \frac{1}{|y|^4} \ln(1 + |y|^4) \, dy
\]
\[
< \int_{\Omega_2} B \left( \frac{V(y)}{\kappa} \right) \, dy + \frac{4}{\kappa} \int_{\Omega_2} V(y) \ln(1 + |y|) \, dy + 4 \int_{\Omega_2} \frac{1}{|y|^4} \ln(1 + |y|) \, dy.
\]
Let
\[
\kappa_0 := \max \left\{ \| V \|_{B, \Omega_2}, \int_{\Omega_2} V(y) \ln(1 + |y|) \, dy \right\}.
\]
An advantage of (11) is that its right-hand side agrees with the semi-classical asymptotics where one can take (see (9)). It follows from (21) (see below) that

\[ \int_{\Omega_1} B \left( \frac{\hat{V}(z)}{\kappa_0} \right) dx < 1 + 4 + 4 \int_{\Omega_2} |y|^{-3} dy = 5 + 8\pi =: C_2 \]

and

\[ \|\hat{V}\|_{(B, \Omega_1)} \leq C_2 \max \left\{ \|V\|_{(B, \Omega_2)}, \int_{\Omega_2} V(y) \ln(1 + |y|) dy \right\} \]

(see (4)).

Now it follows from (12) to (14) that

\[ N_-(\mathcal{E}_V) \leq C_3 \left( \|V\|_{B, \mathbb{R}^2} + \int_{\mathbb{R}^2} V(x) \ln(1 + |x|) dx \right) + 2, \]

where one can take \( C_3 = C_1(1 + 2C_2) \). The last inequality gives the estimate \( N_-(\mathcal{E}_V) \leq 2 \) for small \( V \) and it is left to show that one actually has \( N_-(\mathcal{E}_V) = 1 \) in this case.

According to Proposition 4.4 in [9], \( N_-(\mathcal{E}_V) = 1 \) provided

\[ \sup_{x' \in \mathbb{R}^2} \left( \int_{\mathbb{R}^2} K(x, x') V(x) dx \right) \]

is sufficiently small, where

\[ K(x, x') := \ln(2 + |x|) + K_0(x, x'), \]

\[ K_0(x, x') := \ln \frac{1}{|x - x'|}, \quad x, x' \in \mathbb{R}^2, \ x \neq x' \]

(see (9)). It follows from (21) (see below) that

\[ \sup_{x' \in \mathbb{R}^2} \|K_0(\cdot, x')\|_{(A, \mathbb{R}^2)} = \|K_0(\cdot, 0)\|_{(A, \mathbb{R}^2)} \leq 2\pi. \]

Using the Hölder inequality for Orlicz spaces (see [28, Section 3.3, (17)]), one gets

\[ \int_{\mathbb{R}^2} K(x, x') V(x) dx \leq \int_{\mathbb{R}^2} V(x) \ln(2 + |x|) dx + \|K_0(\cdot, x')\|_{(A, \mathbb{R}^2)} \|V\|_{B, \mathbb{R}^2} \]

\[ \leq \text{const} \left( \int_{\mathbb{R}^2} V(x) \ln(1 + |x|) dx + \|V\|_{B, \mathbb{R}^2} \right) \]

with a constant independent of \( x' \) and \( V \). Hence (16) can be made arbitrarily small by making \( \|V\|_{B, \mathbb{R}^2} \) sufficiently small. In this case, \( N_-(\mathcal{E}_V) = 1 \). Combining this with (15) one gets the existence of a constant \( C > 0 \) for which (11) holds.

\[ \square \]

**Remark 3.2.** Estimate (11) looks similar to the following one obtained in [26]:

\[ N_-(\mathcal{E}_V) \leq C \left( \int_{V(x) \geq 1} V(x) \ln V(x) dx + \int_{\mathbb{R}^2} V(x) \ln(2 + |x|) dx \right) + 1. \]

An advantage of (11) is that its right-hand side agrees with the semi-classical asymptotics \( N_-(\mathcal{E}_\alpha V) = O(\alpha) \) as \( \alpha \to +\infty \). In fact, Theorems 3.1 and 6.1 are direct descendants of (47) (see below) which was obtained in [29]. It turns out that the right-hand side of (11) dominates that of (47), that is, that (47) is actually a stronger estimate than (11) (see Section 8).

4. **The Khuri–Martin–Wu conjecture**

Let \( V_* : \mathbb{R}_+ \to [0, +\infty) \) be the non-increasing spherical rearrangement of \( V \), that is, let \( V_* \) be non-increasing, continuous from the right and such that

\[ \{x \in \mathbb{R}^2 : V_*(|x|) > s\} = \{x \in \mathbb{R}^2 : V(x) > s\} \quad \forall s > 0, \]
where $|E|$ denotes the two-dimensional Lebesgue measure of $E \subset \mathbb{R}^2$. (Note that $V_*(r) = \tilde{V}^*(\pi r^2)$, where $\tilde{V}^*$ is the standard non-increasing rearrangement used in the theory of Lorentz spaces; see, for example, [34, 1.8].) Then

$$\int_{\mathbb{R}^2} F(V_*(|x|)) \, dx = \int_{\mathbb{R}^2} F(V(x)) \, dx$$

for any measurable $F \geq 0$ (see, for example, [28, Chapter X, (1.10)]).

**Lemma 4.1** (Cf. [2]). There exists a constant $C_4 > 0$ such that

$$\|V\|_{B, \mathbb{R}^2} \leq C_4 \left( \|V\|_{L_1(\mathbb{R}^2)} + \int_{\mathbb{R}^2} V_*(|x|) \ln + \frac{1}{|x|} \, dx \right) \quad \forall V \geq 0. \quad (19)$$

**Proof.** It follows from the definition of $V_*$ that

$$|\{ x \in \mathbb{R}^2 : V(x) \geq V_*(r) \}| = |\{ x \in \mathbb{R}^2 : V_*(|x|) \geq V_*(r) \}| \geq |\{ x \in \mathbb{R}^2 : |x| \leq r \}| = \pi r^2 \quad \forall r > 0.$$

On the other hand, Chebyshev’s inequality implies

$$|\{ x \in \mathbb{R}^2 : V(x) \geq V_*(r) \}| \leq \frac{\|V\|_{L_1(\mathbb{R}^2)}}{\pi r^2}. \quad (20)$$

Similarly to the proof of Theorem 3.1, it will be convenient for us to work with the norm (2). Let

$$\kappa_1 := \max \left\{ \frac{\|V\|_{L_1(\mathbb{R}^2)}}{\pi}, \int_{\mathbb{R}^2} V_*(|x|) \ln + \frac{1}{|x|} \, dx \right\}.$$

Then it follows from Lemma 2.2 and (18), and (20) that

$$\int_{\mathbb{R}^2} B \left( \frac{V(x)}{\kappa_1} \right) \, dx \leq \frac{1}{\kappa_1} \int_{\mathbb{R}^2} V(x) \, dx + \frac{2}{\kappa_1} \int_{\mathbb{R}^2} V_*(|x|) \ln + \frac{V(x)}{\kappa_1} \, dx \leq \pi + \frac{2}{\kappa_1} \int_{\mathbb{R}^2} V_*(|x|) \ln + \frac{V(x)}{\kappa_1} \, dx \leq \pi + \frac{2}{\kappa_1} \int_{\mathbb{R}^2} V_*(|x|) \ln + \frac{\|V\|_{L_1(\mathbb{R}^2)}}{\kappa_1 \pi |x|^2} \, dx \leq \pi + \frac{2}{\kappa_1} \int_{\mathbb{R}^2} V_*(|x|) \ln + \frac{1}{|x|^2} \, dx \leq \pi + 4 =: C_5.$$

Hence,

$$\|V\|_{B, \mathbb{R}^2} \leq C_5 \max \left\{ \frac{\|V\|_{L_1(\mathbb{R}^2)}}{\pi}, \int_{\mathbb{R}^2} V_*(|x|) \ln + \frac{1}{|x|} \, dx \right\} \quad (see \ (4)), \text{ which implies } (19) \text{ with } C_4 = 2C_5 \text{ (see (3))}.$$

**Lemma 4.2** (cf. [2]).

$$\int_{\mathbb{R}^2} V_*(|x|) \ln + \frac{1}{|x|} \, dx \leq 4\pi \|V\|_{B, \mathbb{R}^2} \quad \forall V \geq 0.$$
Proof. Let \( D := \{ x \in \mathbb{R}^2 : |x| \leq 1 \} \). The Hölder inequality (see [28, Section 3.3, (4)]) and (18) imply
\[
\int_{\mathbb{R}^2} V_*(|x|) \ln \frac{1}{|x|} \, dx = \int_D V_*(|x|) \ln \frac{1}{|x|} \, dx \leq 2\|V_*(|\cdot|)\|_{(B,\mathbb{R}^2)} \left\| \ln \frac{1}{|\cdot|} \right\|_{(A,\mathbb{R}^2)} \\
\leq 2\|V_*(|\cdot|)\|_{(B,\mathbb{R}^2)} \left\| \ln \frac{1}{|\cdot|} \right\|_{(A,\mathbb{R}^2)} = 2\|V\|_{(B,\mathbb{R}^2)} \left\| \ln \frac{1}{|\cdot|} \right\|_{(A,\mathbb{R}^2)},
\]
\[
\int_D A \left( \ln \frac{1}{|x|} \right) \, dx \leq \int_D e^{\ln \frac{1}{\pi}} \, dx = \int_D \frac{1}{|x|} \, dx = \int_0^\pi \int_0^1 r \, dr \, d\theta = 2\pi.
\]
Hence,
\[
\left\| \ln \frac{1}{|\cdot|} \right\|_{(A,\mathbb{R}^2)} \leq 2\pi \tag{21}
\]
(see (4)). □

Lemmas 4.1 and 4.2 imply that Theorem 3.1 is equivalent to the following result.

**Theorem 4.3.** There exists a constant \( C_6 > 0 \) such that
\[
N_-(\mathcal{E}_V) \leq C_6 \left( \int_{\mathbb{R}^2} V(x) \ln(2 + |x|) \, dx + \int_{\mathbb{R}^2} V_*(|x|) \ln \frac{1}{|x|} \, dx \right) + 1 \quad \forall V \geq 0. \tag{22}
\]

Theorem 4.3 is in turn equivalent to the following estimate which was conjectured in [17]
\[
N_-(\mathcal{E}_V) \leq c_1 \int_{\mathbb{R}^2} V_*(|x|) \ln \frac{1}{|x|} \, dx + c_2 \int_{\mathbb{R}^2} V(x) \ln |x| \, dx + c_3 \int_{\mathbb{R}^2} V(x) \, dx + 1 \tag{23}
\]
provided one does not care about the exact values of the constants. It is natural however to ask what the best constants in (23) are. It was conjectured in [7] that a similar estimate
\[
N_-(\mathcal{E}_V) \leq d_1 \int_{\mathbb{R}^2} V_*(|x|) \ln \frac{1}{|x|} \, dx + d_2 \int_{\mathbb{R}^2} V(x) \ln |x| \, dx + d_3 \int_{\mathbb{R}^2} V(x) \, dx + 1 \tag{24}
\]
holds with
\[
2\pi d_1 = 2, \quad 2\pi d_2 = 1, \quad 2\pi d_3 = \frac{2}{\sqrt{3}}. \tag{25}
\]
The difference between (23) and (24) is that the second integral in the latter can be negative. If \( V(x) = F(|x|) \) is a decreasing radial potential, then (23) and (24) become
\[
N_-(\mathcal{E}_V) \leq 2\pi c_1 \int_0^1 r F(r) \ln r \, dr + 2\pi c_2 \int_1^\infty r F(r) \ln r \, dr + 2\pi c_3 \int_0^\infty r F(r) \, dr + 1
\]
and
\[
N_-(\mathcal{E}_V) \leq 2\pi (d_1 - d_2) \int_0^1 r F(r) \ln r \, dr + 2\pi d_2 \int_1^\infty r F(r) \ln r \, dr + 2\pi d_3 \int_0^\infty r F(r) \, dr + 1,
\]
respectively. This allows one to obtain lower estimates for \( c_1, c_2, c_3 \) and \( d_1, d_2, d_3 \) from the known results on radial potentials and explains the values of \( d_1 \) and \( d_2 \) in (25). Theorem 5.2 in [20] suggests that one could perhaps try to prove (24) with \( 2\pi d_3 = 1 \).

**Remark 4.4.** The proof of Theorem 4.3 relies on an idea from [26, Section 5] where (17) was used in place of (11) and were the Khuri–Martin–Wu conjecture was proved when \( \|V\|_{L^1} \leq 1 \) or \( V \) has a compact support of a fixed size.
5. The Birman–Solomyak method

Our description of the Birman–Solomyak method of estimating \( N_-(\mathcal{E}_V) \) follows [4, 29, 31], although some details are different.

We denote the polar coordinates in \( \mathbb{R}^2 \) by \((r, \vartheta)\), \( r \in \mathbb{R}_+, \vartheta \in (-\pi, \pi]\). Let

\[
    f_{\mathcal{R}}(r) := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(r, \vartheta) \, d\vartheta, \quad f_{\mathcal{N}}(r, \vartheta) := f(r, \vartheta) - f_{\mathcal{R}}(r), \quad f \in C(\mathbb{R}^2 \setminus \{0\}).
\]

Then

\[
    \int_{-\pi}^{\pi} f_{\mathcal{N}}(r, \vartheta) \, d\vartheta = 0 \quad \forall r > 0,
\]

and it is easy to see that

\[
    \int_{\mathbb{R}^2} \nabla f_{\mathcal{R}} \nabla g_{\mathcal{N}} \, dx = \int_{\mathbb{R}^2} \left( \frac{\partial f_{\mathcal{R}}}{\partial r} \frac{\partial g_{\mathcal{N}}}{\partial r} + \frac{1}{r^2} \frac{\partial f_{\mathcal{R}}}{\partial \vartheta} \frac{\partial g_{\mathcal{N}}}{\partial \vartheta} \right) \, dx
\]

\[
= \int_{\mathbb{R}^2} \frac{\partial f_{\mathcal{R}}}{\partial r} \frac{\partial g_{\mathcal{N}}}{\partial r} \, dx
\]

\[
= \int_{\mathbb{R}^2} \left( \frac{\partial f}{\partial r} \right)_{\mathcal{R}} \left( \frac{\partial g}{\partial r} \right)_{\mathcal{N}} \, dx = 0 \quad \forall f, g \in C^\infty_0(\mathbb{R}^2).
\]

Hence, \( P : W^1_2(\mathbb{R}^2) \to W^1_2(\mathbb{R}^2) \) is also an orthogonal projection.

Since

\[
    \int_{\mathbb{R}^2} |\nabla w|^2 \, dx = \int_{\mathbb{R}^2} |\nabla w_{\mathcal{R}}|^2 \, dx + \int_{\mathbb{R}^2} |\nabla w_{\mathcal{N}}|^2 \, dx,
\]

\[
    \int_{\mathbb{R}^2} V|w|^2 \, dx \leq 2 \int_{\mathbb{R}^2} V|w_{\mathcal{R}}|^2 \, dx + 2 \int_{\mathbb{R}^2} V|w_{\mathcal{N}}|^2 \, dx,
\]

one has

\[
    N_-(\mathcal{E}_V) \leq N_-(\mathcal{E}_{\mathcal{R},2V}) + N_-(\mathcal{E}_{\mathcal{N},2V}),
\]

where \( \mathcal{E}_{\mathcal{R},2V} \) and \( \mathcal{E}_{\mathcal{N},2V} \) denote the restrictions of the form \( \mathcal{E}_{2V} \) to \( PW^1_2(\mathbb{R}^2) \) and \( (I - P)W^1_2(\mathbb{R}^2) \), respectively.

**Remark 5.1.** If the potential \( V \) is radial, that is, \( V(x) = F(r) \), then it is easy to see that

\[
    P(Vw) = VPw \quad \forall w \in L_2(\mathbb{R}^2) \cap L_2(\mathbb{R}^2, V(x) \, dx),
\]

and one gets a sharper version of (27):

\[
    N_-(\mathcal{E}_V) = N_-(\mathcal{E}_{\mathcal{R},V}) + N_-(\mathcal{E}_{\mathcal{N},V})
\]

(cf. [20]).

Let us estimate the right-hand side of (27). We start with the first term, that is, with the case of \( PW^1_2(\mathbb{R}^2) = \{ w \in W^1_2(\mathbb{R}^2) : w(x) = w_{\mathcal{R}}(r) \} \). Using the notation \( r = e^t \),
where

$$G(t) := \frac{e^{2t}}{2\pi} \int_{-\pi}^{\pi} V(e^t, \vartheta) \, d\vartheta.$$  \hfill (29)

The above change of variable defines a unitary operator from $PW_2^1(\mathbb{R}^2)$ onto

$$X := \left\{ u \in W_2^1_{\text{loc}}(\mathbb{R}) : \|u\|_X := \sqrt{2\pi} \left[ \int_{\mathbb{R}} |u'|^2 \, dt + \int_{\mathbb{R}} |u|^2 \, e^{2t} \, dt \right]^{1/2} < \infty \right\}.$$  \hfill (30)

Let $X_0 := \{ u \in X : u(0) = 0 \}$ and

$$\mathcal{H}_0 := \left\{ u \in W_2^1_{\text{loc}}(\mathbb{R}) : u(0) = 0, \int_{\mathbb{R}} |u'|^2 \, dt < \infty \right\}.$$  \hfill (31)

Let $\mathcal{E}_{X,G}$, $\mathcal{E}_{X_0,G}$ and $\mathcal{E}_{\mathcal{H}_0,G}$ denote the forms defined by

$$\int_{\mathbb{R}} |u'|^2 \, dt - \int_{\mathbb{R}} G(t) |u(t)|^2 \, dt$$  \hfill (32)

on the domains

$$X \cap L_2(\mathbb{R}, G(t) \, dt), \quad X_0 \cap L_2(\mathbb{R}, G(t) \, dt) \quad \text{and} \quad \mathcal{H}_0 \cap L_2(\mathbb{R}, G(t) \, dt),$$

respectively. Since $\dim(X/X_0) = 1$ and $X_0 \subset \mathcal{H}_0$, one has

$$N_- (\mathcal{E}_{\mathcal{R},2V}) = N_- (\mathcal{E}_{X,2G}) \leq N_- (\mathcal{E}_{X_0,2G}) + 1 \leq N_- (\mathcal{E}_{\mathcal{H}_0,2G}) + 1.$$  \hfill (33)

It follows from Hardy’s inequality (see, for example, [14, Theorem 327]) that

$$\int_{\mathbb{R}} |u'|^2 \, dt + \kappa \int_{\mathbb{R}} \frac{|u|^2}{|t|^2} \, dt \leq (4\kappa + 1) \int_{\mathbb{R}} |u'|^2 \, dt \quad \forall u \in \mathcal{H}_0, \ \forall \kappa \geq 0.$$  \hfill (34)

Hence

$$N_- (\mathcal{E}_{\mathcal{H}_0,2G}) \leq N_- (\mathcal{E}_{G,G}),$$

where

$$\mathcal{E}_{n,G}[u] := \int_{\mathbb{R}} |u'(t)|^2 \, dt + \kappa \int_{\mathbb{R}} \frac{|u(t)|^2}{|t|^2} \, dt - 2(4\kappa + 1) \int_{\mathbb{R}} G(t) |u(t)|^2 \, dt,$$

$$\text{Dom}(\mathcal{E}_{n,G}) = \mathcal{H}_0 \cap L_2(\mathbb{R}, G(t) \, dt).$$

It follows from the above that

$$N_- (\mathcal{E}_{\mathcal{R},2V}) \leq N_- (\mathcal{E}_{G,G}) + 1.$$  \hfill (35)

We estimate $N_- (\mathcal{E}_{G,G})$ by partitioning $\mathbb{R}$ into the intervals

$$I_n := [2^{n-1}, 2^n], \quad n > 0, \quad I_0 := [-1, 1], \quad I_n := [-2^n, -2^{n-1}], \quad n < 0,$$

and by using the variational principle to obtain

$$N_- (\mathcal{E}_{G,G}) \leq \sum_{n \in \mathbb{Z}} N_- (\mathcal{E}_{G,G,n}),$$  \hfill (36)
where
\[ \mathcal{E}_{\kappa,G,n}[u] := \int_{I_n} |u'|^2 \, dt + \kappa \int_{I_n} \frac{|u|^2}{|t|^2} \, dt - 2(4\kappa + 1) \int_{I_n} G|u|^2 \, dt, \]
\[ \text{Dom}(\mathcal{E}_{\kappa,G,n}) = W^1_2(I_n) \cap L_2(I_n, G(t) \, dt), \quad n \in \mathbb{Z} \setminus \{0\}, \]
\[ \text{Dom}(\mathcal{E}_{\kappa,G,0}) = \{ u \in W^1_2(I_0) : u(0) = 0 \} \cap L_2(I_0, G(t) \, dt). \]

Let \( n > 0 \). For any \( N \in \mathbb{N} \), there exists a subspace \( \mathcal{F}_N \in \text{Dom}(\mathcal{E}_{\kappa,G,n}) \) of co-dimension \( N \) such that
\[ \int_{I_n} G|u|^2 \, dt \leq \left( \frac{|I_n|}{N^2} \right) \int_{I_n} G \, dt \int_{I_n} |u'|^2 \, dt \quad \forall u \in \mathcal{F}_N \]
(see [31, the proof of Proposition 4.2 in Appendix]). If
\[ 2(4\kappa + 1) \frac{|I_n|}{N^2} \int_{I_n} G \, dt \leq 1, \]
then \( \mathcal{E}_{\kappa,G,n}[u] \geq 0, \forall u \in \mathcal{F}_N \), and \( N_-(\mathcal{E}_{\kappa,G,n}) \leq N \). Let
\[ A_n := \int_{I_n} |t|G(t) \, dt, \quad n \neq 0, \quad A_0 := \int_{I_0} G(t) \, dt. \] (35)
Since \( |I_n| \int_{I_n} G \, dt \leq A_n, \, n \neq 0 \), it follows from the above that
\[ 2(4\kappa + 1)A_n \leq N^2 \quad \Rightarrow \quad N_-(\mathcal{E}_{\kappa,G,n}) \leq N. \]
Hence,
\[ N_-(\mathcal{E}_{\kappa,G,n}) \leq \lceil \sqrt{2(4\kappa + 1)A_n} \rceil, \] (36)
where \( \lceil \cdot \rceil \) denotes the ceiling function, that is, \( \lceil a \rceil \) is the smallest integer not less than \( a \). The right-hand side of this estimate is at least 1, so one cannot feed it straight into (34). One needs to find conditions under which \( N_-(\mathcal{E}_{\kappa,G,n}) = 0 \).

It follows from (A.1) and (A.3) that
\[ \int_{I_n} G|u|^2 \, dt \leq A_n C(\kappa) \left( \int_{I_n} |u'|^2 \, dt + \kappa \int_{I_n} \frac{|u|^2}{|t|^2} \, dt \right), \]
where
\[ C(\kappa) = \frac{1}{2\kappa} \left( 1 + \sqrt{1 + \frac{2\sqrt{1+4\kappa} + 1}{2\sqrt{1+4\kappa} - 1}} \right). \]
Hence, \( N_-(\mathcal{E}_{\kappa,G,n}) = 0 \), that is, \( \mathcal{E}_{\kappa,G,n} \geq 0 \), provided \( A_n \leq \Phi(\kappa) \), where
\[ \Phi(\kappa) := \frac{\kappa}{4\kappa + 1} \left( 1 + \sqrt{4\kappa + 1} \frac{2\sqrt{4\kappa+1} + 1}{2\sqrt{4\kappa+1} - 1} \right)^{-1}. \] (37)

The above estimates for \( N_-(\mathcal{E}_{\kappa,G,n}) \) clearly hold for \( n < 0 \) as well, but the case \( n = 0 \) requires some changes. Since \( u(0) = 0 \) for any \( u \in \text{Dom}(\mathcal{E}_{\kappa,G,0}) \), one can use the same argument as the one leading to (36), but with two differences: (a) \( \mathcal{F}_N \) can be chosen to be of co-dimension \( N - 1 \), and (b) \( I_0 \int_{I_0} G \, dt = 2A_0 \). This gives the following analog of (36)
\[ N_-(\mathcal{E}_{\kappa,G,0}) \leq \lceil 2\sqrt{(4\kappa + 1)A_0} \rceil - 1 < 2\sqrt{(4\kappa + 1)A_0}. \]
In particular, \( N_-(\mathcal{E}_{\kappa,G,0}) = 0 \) if \( A_0 \leq 1/(4\kappa + 1) \). Using Remark A.2, one can easily see that the implication \( A_n \leq \Phi(\kappa) \) \( \implies \) \( N_-(\mathcal{E}_{\kappa,G,n}) = 0 \) remains true for \( n = 0 \). Now it follows from (33) and (34) that
\[
N_-(\mathcal{E}_{\mathbb{R},2V}) \leq 1 + \sum_{\{n \in \mathbb{Z}\setminus\{0\} : A_n > \Phi(\kappa)\}} \sqrt{2(4\kappa + 1)A_n} + 2\sqrt{(4\kappa + 1)A_0},
\]
and one can drop the last term in \( A_0 \leq \Phi(\kappa) \). The presence of the parameter \( \kappa \) in this estimate allows a degree of flexibility. In order to decrease the number of terms in the right-hand side, one should choose \( \kappa \) in such a way that \( \Phi(\kappa) \) is close to its maximum. A Mathematica calculation shows that the maximum is approximately 0.046 and is achieved at \( \kappa \approx 1.559 \). For values of \( \kappa \) close to 1.559, one has
\[
A_n > \Phi(\kappa) \implies \sqrt{2(4\kappa + 1)A_n} > \sqrt{2(4\kappa + 1)\Phi(\kappa)} \approx 0.816.
\]
Since \( \lceil a \rceil \leq 2a \) for \( a \geq \frac{1}{2} \), (38) implies
\[
N_-(\mathcal{E}_{\mathbb{R},2V}) \leq 1 + 7.61 \sum_{A_n > 0.046} \sqrt{A_n}.
\]
Let us rewrite this estimate in terms of the original potential \( V \). Set
\[
U_n := \{ x \in \mathbb{R}^2 : e^{2n-1} < |x| < e^{2n} \}, n > 0,
U_0 := \{ x \in \mathbb{R}^2 : e^{-1} < |x| < e \},
U_n := \{ x \in \mathbb{R}^2 : e^{-2^n} < |x| < e^{-2^{n-1}} \}, n < 0
\]
and
\[
A_0 := \int_{U_0} V(x) \, dx, \quad A_n := \int_{U_n} V(x) |\ln |x|| \, dx, \quad n \neq 0.
\]
Then it follows from (29) and (35) that \( A_n = 2\pi A_n \), and we have
\[
N_-(\mathcal{E}_{\mathbb{R},2V}) \leq 1 + 3.04 \sum_{A_n > 0.29} \sqrt{A_n}.
\]
Below, we will use the following less precise but nicer looking estimate
\[
N_-(\mathcal{E}_{\mathbb{R},2V}) \leq 1 + 4 \sum_{A_n > 1/4} \sqrt{A_n}.
\]

Remark 5.2. To the best of my knowledge, estimates of this type (without explicit constants) were first obtained by Birman and Solomyak for Schrödinger-type operators of order \( 2\ell \) in \( \mathbb{R}^d \) with \( 2\ell > d \) (see [6, Section 6]). Grigor’yan and Nadirashvili [10] obtained an estimate of this type for two-dimensional Schrödinger operators (see (50) below). The Grigor’yan–Nadirashvili estimate is discussed in the next section.

Returning to (27), we need to estimate \( N_-(\mathcal{E}_{\mathcal{N},2V}) \). To this end, we split \( \mathbb{R}^2 \) into the following annuli:
\[
\Omega_n := \{ x \in \mathbb{R}^2 : e^n < |x| < e^{n+1} \}, \quad n \in \mathbb{Z}.
\]
It follows from (26) that
\[ \int_{\Omega_n} w(x) \, dx = 0, \quad \forall w \in (I - P)W^1_2(\mathbb{R}^2) \quad \forall n \in \mathbb{Z}. \]

Hence, the variational principle implies
\[ N_-(\mathcal{E}_{N,2V}) \leq \sum_{n \in \mathbb{Z}} N_-(\mathcal{E}_{N,2V,n}), \quad (43) \]
where
\[ \mathcal{E}_{N,2V,n}[w] := \int_{\Omega_n} |\nabla w(x)|^2 \, dx - 2 \int_{\Omega_n} V(x)|w(x)|^2 \, dx, \]
\[ \text{Dom}(\mathcal{E}_{N,2V,n}) = \{ w \in W^1_2(\Omega_n) \cap L^2(\Omega_n, V(x) \, dx) : \int_{\Omega_n} w \, dx = 0 \}. \quad (44) \]

It is clear that \( x \mapsto e^n x \) maps \( U_0 \) onto \( U_n \). So, any estimate for \( N_-(\mathcal{E}_{N,2V,0}) \) that has the right scaling leads to an estimate for \( N_-(\mathcal{E}_{N,2V,n}) \), and then an estimate for \( N_-(\mathcal{E}_V) \) follows from (27), (41) and (43).

6. A Grigor’yan–Nadirashvili-type estimate

Let
\[ B_n := \| V \|^{(av)}_{B,\Omega_n}, \quad B_n := \left( \int_{\Omega_n} V^p(x)|x|^{2(p-1)} \, dx \right)^{1/p}, \quad p > 1, \quad n \in \mathbb{Z} \quad (45) \]
(see (42)).

**Theorem 6.1.** There exist constants \( C_7 > 0 \) and \( c > 0 \) such that
\[ N_-(\mathcal{E}_V) \leq 1 + 4 \sum_{A_n > 1/4} \sqrt{A_n} + C_7 \sum_{B_n > c} B_n \quad \forall V \geq 0 \quad (46) \]
(see (39) and (40)).

**Proof.** According to Theorem 4′ in [29] (see also Proposition 3), there exists \( C_7 > 0 \) such that \( N_-(\mathcal{E}_{N,2V,n}) \leq C_7 B_n \) \( \forall n \in \mathbb{Z} \) (see (44)). In particular, \( N_-(\mathcal{E}_{N,2V,n}) = 0 \) if \( B_n < 1/C_7 \), and one can drop this term from the sum in (43). Now it follows from (27), (41) and (43) that (46) holds for any \( c < 1/C_7 \).

**Remark 6.2.** The above result is closely related to the well-known estimate from [29]:
\[ N_-(\mathcal{E}_V) \leq 1 + C_8 \left( \|(A_n)_{n \geq 0}\|_{1,\infty} + \sum_{n=0}^{\infty} B_n \right) \quad \forall V \geq 0, \quad (47) \]
where \( A_n = A_n, \quad B_n = B_n \) for \( n \in \mathbb{N}, \)
\[ A_0 := \int_{\Omega_0} V(x) \ln |x| \, dx, \quad B_0 = \| V \|^{(av)}_{B,\Omega_0}, \quad \Omega_0 = \{ x \in \mathbb{R}^2 : |x| \leq e \}, \quad (48) \]
and
\[ \|(a_n)_{n \geq 0}\|_{1,\infty} := \sup_{s>0} (s \text{ card}(n \geq 0 : |a_n| > s)). \]
Note that
\[ \sum_{n \leq 0 : B_n > c} B_n \leq \sum_{n \leq 0} B_n \leq B_0 \]
(see [29, Lemma 3]), and that
\[ \sum_{|a_n| > c} \sqrt{|a_n|} = \sum_{|a_n| > c} \int_0^{|a_n|} \frac{1}{2} s^{-1/2} ds \]
\[ = \frac{1}{2} \int_0^\infty s^{-1/2} \text{card}\{n > 0 : |a_n| > s \text{ and } |a_n| > c\} ds \]
\[ \leq \frac{1}{2} \int_0^\infty s^{-1/2} \| (a_n)_{n \geq 0} \|_{1,\infty} \max\{s, c\} ds \]
\[ = \left( \int_0^c s^{-1/2} ds + \int_c^\infty s^{-3/2} ds \right) \frac{1}{2} \| (a_n)_{n \geq 0} \|_{1,\infty} \]
\[ = \frac{2}{\sqrt{c}} \| (a_n)_{n \geq 0} \|_{1,\infty}. \] (49)

Hence,
\[ \sum_{\{n \in \mathbb{Z} \setminus \{0\} : A_n > c\}} \sqrt{A_n} = \sum_{\{n < 0 : A_n > c\}} \sqrt{A_n} + \sum_{\{n > 0 : A_n > c\}} \sqrt{A_n} \]
\[ \leq \frac{1}{\sqrt{c}} \sum_{n < 0} A_n + \frac{2}{\sqrt{c}} \| (A_n)_{n \geq 0} \|_{1,\infty} \]
\[ \leq \frac{1}{\sqrt{c}} A_0 + \frac{2}{\sqrt{c}} \| (A_n)_{n > 0} \|_{1,\infty} \]
\[ \leq \frac{3}{\sqrt{c}} \| (A_n)_{n \geq 0} \|_{1,\infty}. \]

It follows from the H"older inequality (see [18, Theorem 9.3]) that if $A_0 > c$, then
\[ \sqrt{A_0} < \frac{1}{\sqrt{c}} A_0 = \frac{1}{\sqrt{c}} \int_{\Omega_0} V(x) dx \leq \frac{1}{\sqrt{c}} \int_{\Omega_0} V(x) dx \]
\[ \leq \frac{1}{\sqrt{c}} \| V \|_{L^1(\Omega_0)} \leq \| V \|_{\mathcal{B}(\Omega_0)} = \text{const} B_0. \]

So, (46) implies (47).

**Remark 6.3.** Theorem 6.1 is a slight improvement of [10, Theorem 1.1] where an analog of (46) was obtained with $B_n$ in place of $B_n$:
\[ N_-(\mathcal{E}_V) \leq 1 + C_7 \sum_{\{n \in \mathbb{Z} : A_n > c\}} \sqrt{A_n} + C_7 \sum_{\{n \in \mathbb{Z} : B_n > c\}} B_n \ \forall V \geq 0. \] (50)

Indeed, let
\[ \Delta := \left\{ x \in \mathbb{R}^2 : \frac{1}{\sqrt{\pi(e^2 - 1)}} < |x| < \frac{e}{\sqrt{\pi(e^2 - 1)}} \right\}, \]
\[ R_n := e^n \sqrt{\pi(e^2 - 1)}, \]
\[ \xi_n : \mathbb{R}^2 \to \mathbb{R}^2, \ \xi_n(x) = R_n x, \ n \in \mathbb{Z}. \]
Then $|\Delta| = 1$, $\xi_n(\Delta) = \Omega_n$ and it follows from [29, (7)] and [18, Section 13] that

$$B_n = \|V\|^{(\text{avr})}_{B,\Omega_n} = \|\Omega_n\| V \circ \xi_n \|_{B,\Delta} \leq C(p)\|\Omega_n\| V \circ \xi_n \|_{L_p(\Delta)}$$

$$= C(p)R_n^2 R_n^{-2/p} \|V\|_{L_p(\Omega_n)} = C(p)R_n^{2-2/p} \left( \int_{\Omega_n} V^p(x) \, dx \right)^{1/p}$$

$$\leq C(p)(\pi(e^2 - 1))^{1-1/p} \left( \int_{\{e^n < |x| < e^{n+1}\}} V^p(x)|x|^{2(p-1)} \, dx \right)^{1/p}$$

$$= C(p)(\pi(e^2 - 1))^{1-1/p} B_n, \quad p > 1.$$ 

Note that the optimal constant $M(p)$ in the inequality $\|f\|_{(B,\Delta)} \leq M(p)\|f\|_{L_p(\Delta)}$ has the following asymptotics:

$$M(p) = \frac{1}{e^{(p-1)}} - \frac{1}{e} \ln \frac{1}{p-1} + O(1) \quad \text{as } p \to 1, \quad p > 1. \quad (51)$$

Indeed, an elementary analysis shows that the fraction

$$\frac{B(t)}{p^p} = \frac{(1+t)\ln(1+t) - t}{t^p}, \quad t > 0, \quad 1 < p \leq 2,$$ 

achieves its maximum at a point

$$t_p = \exp \left( \frac{p}{p-1} + o(1) \right) \quad \text{as } p \to 1,$$ 

where its value is

$$m(p) = \frac{1}{e^{(p-1)}} + O(1) \quad \text{as } p \to 1.$$ 

Hence,

$$\int_{\Delta} B \left( \frac{|f(x)|}{m(p)^{1/p}\|f\|_{L_p(\Delta)}} \right) \, dx \leq m(p) \int_{\Delta} \left| \frac{f(x)}{m(p)^{1/p}\|f\|_{L_p(\Delta)}} \right|^p \, dx = 1,$$ 

and $\|f\|_{(B,\Delta)} \leq m(p)^{1/p}\|f\|_{L_p(\Delta)}$. On the other hand, take any subset $\Delta_p \subset \Delta$ such that $|\Delta_p| = (m(p)t_p^{-p})^{-1}$, and define the function $f_p$ by $f_p(x) = t_p$ if $x \in \Delta_p$ and $f_p(x) = 0$ if $x \in \Delta \setminus \Delta_p$. Then $\|f_p\|_{L_p(\Delta_p)} = m(p)^{-1/p}$ and

$$\int_{\Delta} B(f_p(x)) \, dx = m(p) \int_{\Delta} |f_p(x)|^p \, dx = 1 \implies \|f_p\|_{(B,\Delta)} = 1 = m(p)^{1/p}\|f_p\|_{L_p(\Delta)}.$$ 

Hence, the optimal constant is

$$M(p) = m(p)^{1/p} = \left( \frac{1}{e^{(p-1)}} + O(1) \right)^{1/p}$$

$$= \left( \frac{1}{e^{(p-1)}} + O(1) \right) \exp \left( \frac{1-p}{p} \ln \left( \frac{1}{e^{(p-1)}} + O(1) \right) \right)$$

$$= \left( \frac{1}{e^{(p-1)}} + O(1) \right) \left( 1 + (1-p) \ln \frac{1}{p-1} + O(p-1) \right)$$

$$= \frac{1}{e^{(p-1)}} - \frac{1}{e} \ln \frac{1}{p-1} + O(1) \quad \text{as } p \to 1.$$ 

7. A Laptev–Netrusov–Solomyak-type estimate

We denote the polar coordinates in $\mathbb{R}^2$ by $(r, \vartheta)$, $r \in \mathbb{R}_+$, $\vartheta \in \mathbb{S} := (-\pi, \pi]$. Let $I \subseteq \mathbb{R}_+$ be a non-empty open interval and let

$$\Omega_I := \{ x \in \mathbb{R}^2 : |x| \in I \}.$$
We denote by $L_1(I, L_B(\mathbb{S}))$ the space of measurable functions $f : \Omega_I \to \mathbb{C}$ such that
\[
\|f\|_{L_1(I, L_B(\mathbb{S}))} := \int_I \|f(r, \cdot)\|_{B, \mathbb{S}} r dr < +\infty.
\] (52)

Let
\[
I_n := (e^n, e^{n+1}), \quad D_n := \|V\|_{L_1(I_n, L_B(\mathbb{S}))}, \quad n \in \mathbb{Z}.
\] (53)

**Theorem 7.1.** There exist constants $C_9 > 0$ and $c > 0$ such that
\[
N_-(\mathcal{E}_V) \leq 1 + C_9 \sum_{A_n > 1/4} \sqrt{A_n} + C_9 \sum_{D_n > c} D_n \quad \forall V \geq 0
\] (54)

(see (39), (40)).

**Remark 7.2.** It is clear that
\[
\sum_{\{n \in \mathbb{Z} : D_n > c\}} D_n \leq \sum_{n \in \mathbb{Z}} D_n = \|V\|_{L_1(\mathbb{R}_+, L_B(\mathbb{S}))}.
\]

It follows from (49) that
\[
\sum_{\{n \in \mathbb{Z} : A_n > c\}} \sqrt{A_n} \leq \frac{2}{\sqrt{c}} \|(A_n)_{n \in \mathbb{Z}}\|_{1, \infty}.
\]

Hence (54) implies
\[
N_-(\mathcal{E}_V) \leq 1 + C_{10} \|(A_n)_{n \in \mathbb{Z}}\|_{1, \infty} + \|V\|_{L_1(\mathbb{R}_+, L_B(\mathbb{S}))} \quad \forall V \geq 0,
\] (55)

which, in turn, implies the following estimate obtained in [21] (see also [19]):
\[
N_-(\mathcal{E}_V) \leq 1 + C_{11} \left( \|(A_n)_{n \in \mathbb{Z}}\|_{1, \infty} + \int_{\mathbb{R}_+} \left( \int_{\mathbb{S}} |V(r, \vartheta)|^p \, d\vartheta \right)^{1/p} r \, dr \right),
\] (56)

where $p > 1$ and $C_{11} = O(1/(p - 1))$ as $p \to 1$ (see (51)). In fact, the estimate obtained in [21] has a slightly different form:
\[
N_-(\mathcal{E}_V) \leq 1 + C_{12} \left( \|(A_n)_{n \in \mathbb{Z}}\|_{1, \infty} + \int_{\mathbb{R}_+} \left( \int_{\mathbb{S}} |V_{\mathcal{N}}(r, \vartheta)|^p \, d\vartheta \right)^{1/p} r \, dr \right),
\] (57)

where
\[
V_{\mathcal{N}}(r, \vartheta) := V(r, \vartheta) - V_{\mathcal{R}}(r), \quad V_{\mathcal{R}}(r) := \frac{1}{2\pi} \int_{\mathbb{S}} V(r, \vartheta) \, d\vartheta.
\]

Since
\[
\left| \int_{\mathbb{R}_+} \left( \int_{\mathbb{S}} |V(r, \vartheta)|^p \, d\vartheta \right)^{1/p} r \, dr - \int_{\mathbb{R}_+} \left( \int_{\mathbb{S}} |V_{\mathcal{N}}(r, \vartheta)|^p \, d\vartheta \right)^{1/p} r \, dr \right|
\]
\[
\leq \int_{\mathbb{R}_+} \left( \int_{\mathbb{S}} |V_{\mathcal{R}}(r)|^p \, d\vartheta \right)^{1/p} r \, dr = (2\pi)^{1/p} \int_{\mathbb{R}_+} V_{\mathcal{R}}(r) r \, dr
\]
\[
= (2\pi)^{1/p-1} \int_{\mathbb{R}^2} V(x) \, dx = (2\pi)^{1/p-1} \sum_{n \in \mathbb{Z}} |U_n| V(x) \, dx
\]

estimates (56) and (57) are equivalent to each other. An advantage of the latter is that it separates the contribution of the radial part $V_{\mathcal{R}}$ of $V$ from that of the non-radial part $V_{\mathcal{N}}$ (see
Similarly to the above, one can easily show that
\[ \|V\|_{L_1(\mathbb{R}^d, L_B(S))} - \|V_N\|_{L_1(\mathbb{R}^d, L_B(S))} \leq \text{const} \sup_{n \in \mathbb{Z}} A_n \]
and hence (55) is equivalent to
\[ N_-(E_V) \leq 1 + C_{13}(\|A_n\|_{1, \infty} + \|V_N\|_{L_1(\mathbb{R}^d, L_B(S))}) \quad \forall V \geq 0. \] (59)

Let \( I_1, I_2 \subseteq \mathbb{R} \) be non-empty open intervals. We denote by \( L_1(I_1, L_B(I_2)) \) the space of measurable functions \( f : I_1 \times I_2 \to \mathbb{C} \) such that
\[ \|f\|_{L_1(I_1, L_B(I_2))} := \int_{I_1} \|f(x, \cdot)\|_{B, I_2} \, dx < +\infty \] (60)
(see (7)).

**Lemma 7.3** (cf. [29, Lemma 1]). Consider an affine transformation
\[ \xi : \mathbb{R}^2 \to \mathbb{R}^2, \quad \xi(x) := Ax + x^0, \quad A = \begin{pmatrix} R_1 & 0 \\ 0 & R_2 \end{pmatrix}, \quad R_1, R_2 > 0, \quad x^0 \in \mathbb{R}^2. \]

Let \( I_1 \times I_2 = \xi(I_1 \times I_2) \). Then
\[ \frac{1}{|I_1 \times I_2|} \|f\|_{L_1(I_1, L_B(I_2))} = \frac{1}{|J_1 \times J_2|} \|f \circ \xi\|_{L_1(J_1, L_B(J_2))} \quad \forall f \in L_1(I_1, L_B(I_2)). \]

**Proof.**
\[ \frac{1}{|I_1 \times I_2|} \|f\|_{L_1(I_1, L_B(I_2))} = \frac{1}{|I_1 \times I_2|} \int_{I_1} \|f(y, \cdot)\|_{B, I_2} \, dy_1 \]
\[ = \frac{R_1}{|I_1 \times I_2|} \int_{I_1} \|f(R_1 x_1 + x_1^0, \cdot)\|_{B, I_2} \, dx_1 \]
\[ = \frac{R_1 R_2}{|I_1 \times I_2|} \int_{I_1} \|f(R_1 x_1 + x_1^0, x_2^0 + x_2^0)\|_{B, J_2} \, dx_1 \]
\[ = \frac{1}{|J_1 \times J_2|} \|f \circ \xi\|_{L_1(J_1, L_B(J_2))} \]
(see [29, Lemma 1]).

**Lemma 7.4** (cf. [29, Lemma 3]). Let rectangles \( I_{1,k} \times I_{2,k}, \ k = 1, \ldots, n \) be pairwise disjoint subsets of \( I_1 \times I_2 \). Then
\[ \sum_{k=1}^n \|f\|_{L_1(I_{1,k}, L_B(I_{2,k}))} \leq \|f\|_{L_1(I_1, L_B(I_2))} \quad \forall f \in L_1(I_1, L_B(I_2)). \] (61)

**Proof.** Let us label the endpoints of the intervals \( I_1 \) and \( I_{1,k}, \ k = 1, \ldots, n \) in the increasing order \( a_1 < a_2 < \cdots < a_N \). Here \( N \leq 2(n + 1) \) and \( N \) may be strictly less than \( 2(n + 1) \) as some
of these endpoints may coincide. Then
\[
\sum_{k=1}^{n} \|f\|_{L_1(I_{1,k}, L_2(I_{2,k}))} = \sum_{k=1}^{n} \int_{I_{1,k}} \|f(x, \cdot)\|_{B_1(L_2)} \, dx
\]
\[
= \sum_{k=1}^{n} \sum_{(a_j, a_{j+1}) \subseteq I_{1,k}} \int_{a_j}^{a_{j+1}} \|f(x, \cdot)\|_{B_1(L_2)} \, dx
\]
\[
\leq N \sum_{j=1}^{N} \int_{a_j}^{a_{j+1}} \|f(x, \cdot)\|_{B_1(L_2)} \, dx
\]
\[
= \int_{I_{1,1}} \|f(x, \cdot)\|_{B_1(L_2)} \, dx = \|f\|_{L_1(I_{1}, L_2(L_2))}.
\]

The inequality above follows from Lemma 3 in [29] and from the implication
\[
I_{1,k} \supseteq (a_j, a_{j+1}), \quad I_{1,m} \supseteq (a_j, a_{j+1}), \quad k \neq m \Rightarrow I_{2,k} \cap I_{2,m} = 0,
\]
which holds because the rectangles \(I_{1,k} \times I_{2,k}, k = 1, \ldots, n\) are pairwise disjoint.

Let \(Q := (0, 1)^2\) and \(\mathbb{I} := (0, 1)\). We will also use the following notation:
\[
w_S := \frac{1}{|S|} \int_S w(x) \, dx,
\]
where \(S \subseteq \mathbb{R}^2\) is a set of a finite positive two-dimensional Lebesgue measure \(|S|\).

**Lemma 7.5** (cf. [29, Lemma 2]). There exists \(C_{14} > 0\) such that for any non-empty open intervals \(I_1, I_2 \subseteq \mathbb{R}\) of lengths \(R_1\) and \(R_2\), respectively, any \(w \in W^1_2(I_1 \times I_2) \cap C(T_1 \times T_2)\) with \(w_{I_1 \times I_2} = 0\), and any \(V \in L_1(I_1, L_2(I_2))\), \(V \geq 0\) the following inequality holds:
\[
\int_{I_1 \times I_2} V(x) \|w(x)\|^2 \, dx \leq C_{14} \max \left\{ \frac{R_1}{R_2}, \frac{R_2}{R_1} \right\} \|V\|_{L_1(I_1, L_2(I_2))} \int_{I_1 \times I_2} |\nabla w(x)|^2 \, dx.
\]

**Proof.** Let us start with the case \(I_1 = I_2 = \mathbb{I}, R_1 = R_2 = 1\). There exists \(C_{15} > 0\) such that
\[
\sup_{x_1 \in I_1} \|w^2(x_1, \cdot)\|_{A_3} \leq C_{15} \|w\|_{W^2_2(Q)}^2 \quad \forall w \in W^1_2(Q) \cap C(\bar{Q})
\]
(see (8)). This can be proved by applying the trace theorem (see, for example, [1, Theorems 4.32 and 7.53]) and then using the Yudovich–Pohozaev–Trudinger embedding theorem for \(H^{1/2}_2\) (see [1, 8.25, 12, 27, 24, Lemma 1.2.4]) or, in one go, by applying a trace inequality of the Yudovich–Pohozaev–Trudinger type (see [23, Corollary 11.8/2]; a sharp result can be found in [8]).

Next, we use the Poincaré inequality (see, for example, [23, 1.11.1]): there exists \(C_{16} > 0\) such that
\[
\int_Q (w(x) - w_Q)^2 \, dx = \inf_{a \in \mathbb{R}} \int_Q (w(x) - a)^2 \, dx \leq C_{16} \int_Q |\nabla w(x)|^2 \, dx \quad \forall w \in W^1_2(Q).
\]
Lemma 7.4 implies

There exists a finite cover of \( Q \) where

\[ \nu \in C(\nu + 1). \]

Now, the Hölder inequality (see \[ 18, \text{Theorem 9.3} \]) implies

\[
\int_{Q} V(x)|w(x)|^2 \, dx \leq \|V\|_{L^1(\mathbb{R}^2)} \sup_{x_1 \in \mathbb{R}} \|w^2(x_1, \cdot)\|_{A,1}
\]

\[
\leq C_{14} \|V\|_{L^1(\mathbb{R}^2)} \int_{Q} |\nabla w(x)|^2 \, dx,
\]

that is, (62) holds for \( I_1 = I_2 = \mathbb{I} \).

Consider now arbitrary intervals \( I_1, I_2 \subseteq \mathbb{R} \). Then \( I_1 \times I_2 = \xi(Q) \), where \( \xi \) is the affine mapping from Lemma 7.3, and (62) can be derived from the above inequality with the help of Lemma 7.3 and the inequality

\[
\int_{Q} |\nabla (w \circ \xi)(x)|^2 \, dx \leq \max \left\{ \frac{R_1}{R_2}, \frac{R_2}{R_1} \right\} \int_{I_1 \times I_2} |\nabla w(y)|^2 \, dy
\]

(see the proof of \[ 29, \text{Lemma 2} \]). \qed

**Lemma 7.6 (cf. \[ 29, \text{Theorem 1} \]).** For any \( V \in L^1(\mathbb{R}^2) \), \( V \geq 0 \) and any \( n \in \mathbb{N} \) there exists a finite cover of \( Q \) by rectangles \( S_k = I_{1,k} \times I_{2,k} \), \( k = 1, \ldots, n_0 \) such that \( n_0 \leq n \) and

\[
\int_{Q} V(x)|w(x)|^2 \, dx \leq C_{17} n^{-1} \|V\|_{L^1(\mathbb{R}^2)} \int_{Q} |\nabla w(x)|^2 \, dx \tag{63}
\]

for all \( w \in W^1_2(Q) \cap C(\bar{Q}) \) with \( w_{S_k} = 0 \), \( k = 1, \ldots, n_0 \), where the constant \( C_{17} \) does not depend on \( V \).

**Proof.** According to the Besicovitch covering lemma (see, for example, \[ 13, \text{Chapter I, Theorem 1.1} \]), there exists a constant \( \nu \in \mathbb{N} \) such that any cover \( \{\Delta_x\}_{x \in Q} \) of \( Q \) by closed squares \( \Delta_x \) centered at \( x \) has a countable or a finite subcover that can be split into \( \nu \) families in such a way that any two squares belonging to the same family are disjoint.

We can assume that \( n > \nu \), as otherwise one could take \( n_0 = 1 \), \( S_k = Q \) and get (63) with \( C_{17} \geq \nu C_{14} \) directly from (62). For any \( x \in \bar{Q} \) there exists a closed square \( \Delta_x = I^0_{1,x} \times I^0_{2,x} \) centered at \( x \) such that

\[
\|V\|_{L^1(I_{1,x},L^B(I_{2,x}))} = \nu n^{-1} \|V\|_{L^1(\mathbb{R}^2)},
\]

where

\[
I_{i,x} := I^0_{i,x} \cap [0,1], \quad i = 1, 2, \quad \text{that is} \quad I_{1,x} \times I_{2,x} = \Delta_x \cap \bar{Q}
\]

(see \[ 29, \text{Lemma 4} \]). Let \( \{\Delta_{x_l}\} \) be the subcover from the Besicovitch covering lemma and let \( S_k = I_{1,k} \times I_{2,k} := I_{1,x_k} \times I_{2,x_k} = \Delta_{x_k} \cap \bar{Q} \). Then \( \Xi := \{S_k\} \) is also a cover of \( \bar{Q} \) and, like \( \{\Delta_{x_l}\} \), it can be split into \( \nu \) families \( \Xi_l, l = 1, \ldots, \nu \), consisting of pairwise disjoint elements. Lemma 7.4 implies

\[
\nu n^{-1} \|V\|_{L^1(\mathbb{R}^2)} \text{ card } \Xi_l = \sum_{S_k \in \Xi_l} \|V\|_{L^1(I_{1,k},L^B(I_{2,k}))} \leq \|V\|_{L^1(\mathbb{R}^2)} \cdot \nu.
\]

Hence, \( \text{card } \Xi_l \leq n/\nu \) and

\[
n_0 := \text{card } \Xi = \sum_{l=1}^{\nu} \text{card } \Xi_l \leq \nu n/\nu = n.
\]
Take any \( w \in W^1_2(Q) \cap C(Q) \) with \( w_{S_k} = 0, \ k = 1, \ldots, n_0 \). Since the center of the square \( \Delta_{x_k} \) belongs to \( Q \), the ratio of the side lengths of the rectangle \( S_k = \Delta_{x_k} \cap Q \) is between \( \frac{1}{2} \) and 2. Hence, it follows from Lemma 7.5 that

\[
\int_Q V(x)|w(x)|^2 \, dx \leq \sum_{k=1}^{n_0} \int_{S_k} V(x)|w(x)|^2 \, dx
\]

\[
\leq 2C_{14} \sum_{k=1}^{n_0} \|V\|_{L^1(I_{1,k},L^2(I_{2,k}))} \int_{S_k} |\nabla w(x)|^2 \, dx
\]

\[
= 2C_{14}n^{-1}\|V\|_{L^1(I,L^2(B))} \sum_{k=1}^{n_0} \int_{S_k} |\nabla w(x)|^2 \, dx
\]

\[
= 2C_{14}n^{-1}\|V\|_{L^1(I,L^2(B))} \sum_{\nu} \sum_{\nu} \int_{S_k} |\nabla w(x)|^2 \, dx
\]

\[
= 2C_{14}n^{-1}\|V\|_{L^1(I,L^2(B))} \int_{Q} |\nabla w(x)|^2 \, dx,
\]

where \( C_{17} := 2C_{14}v^2 \).

Let

\[
\mathcal{E}_{V,0}[w] := \int_Q |\nabla w(x)|^2 \, dx - \int_Q V(x)|w(x)|^2 \, dx,
\]

\[\text{Dom}(\mathcal{E}_{V,0}) = W^1_2(Q) \cap L^2(Q, V(x) \, dx).\]

**Lemma 7.7** (cf. [29, Theorem 4]).

\[N_-(\mathcal{E}_{V,0}) \leq C_{17}\|V\|_{L^1(I,L^2(B))} + 1 \quad \forall V \geq 0,
\]

where \( C_{17} \) is the same as in Lemma 7.6.

**Proof.** Let \( n = [C_{17}\|V\|_{L^1(I,L^2(B))}] + 1 \) in Lemma 7.6. Take any linear subspace \( \mathcal{L} \subset \text{Dom}(\mathcal{E}_{V,0}) \) such that

\[\dim \mathcal{L} > [C_{17}\|V\|_{L^1(I,L^2(B))}] + 1.\]

Since \( n_0 \leq n \), there exists \( w \in \mathcal{L} \setminus \{0\} \) such that \( w_{S_k} = 0, \ k = 1, \ldots, n_0 \). Then

\[
\mathcal{E}_{V,0}[w] = \int_Q |\nabla w(x)|^2 \, dx - \int_Q V(x)|w(x)|^2 \, dx
\]

\[
\geq \int_Q |\nabla w(x)|^2 \, dx - \frac{C_{17}\|V\|_{L^1(I,L^2(B))}}{[C_{17}\|V\|_{L^1(I,L^2(B))}] + 1} \int_Q |\nabla w(x)|^2 \, dx
\]

\[
\geq \int_Q |\nabla w(x)|^2 \, dx - \int_Q |\nabla w(x)|^2 \, dx = 0.
\]

Hence,

\[N_-(\mathcal{E}_V) \leq [C_{17}\|V\|_{L^1(I,L^2(B))}] + 1 \leq C_{17}\|V\|_{L^1(I,L^2(B))} + 1.\]
Let \( R > 0, I(R) := (R, eR), \Omega(R) := \{x \in \mathbb{R}^2 : R < |x| < eR\} \) and
\[
\mathcal{E}_V^R[w] := \int_{\Omega(R)} |\nabla w(x)|^2 \, dx - \int_{\Omega(R)} V(x)|w(x)|^2 \, dx,
\]
\( \text{Dom}(\mathcal{E}_V^R) = \{w \in W_2^1(\Omega(R)) \cap L_2(\Omega(R), V(x) \, dx) : w_{\Omega(R)} = 0\} \).

**Lemma 7.8 (cf. [29, Theorem 4']).** There exists \( C_{18} > 0 \) such that
\[
N_-(\mathcal{E}_V^R) \leq C_{18} \|V\|_{L_1(I(R), L_B(S))} \quad \forall V \geq 0, \quad \forall R > 0.
\]  

**Proof.** We start with the case \( R = 1 \). Let \( S_+ := (0, \pi), S_- := (-\pi, 0) \),
\[
\Omega_{\pm} := \{(r \cos \vartheta, r \sin \vartheta) \in \mathbb{R}^2 : 1 < r < e, \vartheta \in S_{\pm}\},
\]
\[
\mathcal{E}_V^{\pm}[w] := \int_{\Omega_{\pm}} |\nabla w(x)|^2 \, dx - \int_{\Omega_{\pm}} V(x)|w(x)|^2 \, dx,
\]
\( \text{Dom}(\mathcal{E}_V^{\pm}) = W_2^1(\Omega_{\pm}) \cap L_2(\Omega_{\pm}, V(x) \, dx) \).

Then by the variational principle,
\[
N_-(\mathcal{E}_V^{\pm}) \leq N_-(\mathcal{E}_V^{+}) + N_-(\mathcal{E}_V^{-}).
\]  
(see, for example, [10, Lemma 3.5]). Consider the diffeomorphisms \( \varphi_{\pm} : Q \to \Omega_{\pm} \),
\[
\varphi_{\pm}(y_1, y_2) = (e^2 (1 + (e - 1)y_1) \cos(\pm \pi y_2), (1 + (e - 1)y_1) \sin(\pm \pi y_2)).
\]
The polar coordinates of \( \varphi_{\pm}(y_1, y_2) \) are \( r = 1 + (e - 1)y_1, \vartheta = \pm \pi y_2 \).

Let
\[
\tilde{V}_+(y) := V(\varphi_{+}(y)), \quad \tilde{w}(y) := w(\varphi_{+}(y)), \quad w \in \text{Dom}(\mathcal{E}_V^{+}).
\]
There exist absolute constants \( C_{19} > 0 \) and \( C_{20} > 0 \) such that
\[
\int_{\Omega_{\pm}} |\nabla w(x)|^2 \, dx \geq \frac{1}{C_{19}} \int_{Q} |\nabla \tilde{w}(y)|^2 \, dy,
\]
\[
\int_{\Omega_{\pm}} V(x)|w(x)|^2 \, dx \leq C_{20} \int_{Q} \tilde{V}_+(y)|\tilde{w}(y)|^2 \, dy.
\]
Then Lemma 7.7 implies
\[
N_-(\mathcal{E}_V^{\pm}) \leq N_-(\mathcal{E}_{C_{19}C_{20}\tilde{V}_+,0}) \leq C_{17}C_{19}C_{20} \|\tilde{V}_+\|_{L_1(I, L_B(I))} + 1.
\]
Using [29, Lemma 1] one gets
\[
\|\tilde{V}_+\|_{L_1(I, L_B(I))} = \int_i \|\tilde{V}_+(y_1, \cdot)\|_{L_2(B_1, S_{\pm})}^2 \, dy_1 = \frac{1}{\pi(e - 1)} \int_1^e \|V(r, \cdot)\|_{L_2(B_1, S_{\pm})}^2 \, dr.
\]
Now it follows from (65), Lemma 2.1 and [29, Lemma 3] that
\[
N_-(\mathcal{E}_V^1) \leq C_{21} \int_1^e \|V(r, \cdot)\|_{L_2(B_1, S_{\pm})} \, dr + 2
\]
\[
\leq C_{21} \int_1^e \|V(r, \cdot)\|_{L_2(B_1, S_{\pm})} \, dr + 2 \leq C_{21} \int_1^e \|V(r, \cdot)\|_{L_2(B_1, S_{\pm})} \, dr + 2
\]
\[
\leq 2\pi C_{21} \|V\|_{L_1(I, L_B(I))} + 2,
\]  
(66)

where \( C_{21} = C_{17}C_{19}C_{20}/\pi(e - 1) \).

Using the Yudovich–Pohozhaev–Trudinger embedding theorem, the Poincaré inequality and the Hölder inequality for Orlicz spaces as in the proof of Lemma 7.5 one can prove the existence
of a constant $C_{22} > 0$ such that
\[ \int_{\Omega(1)} V(x)|w(x)|^2 \, dx \leq C_{22} \| V \|_{L_1(I(1),L_B(\mathbb{S}))} \int_{\Omega(1)} |\nabla w(x)|^2 \, dx \quad \forall w \in \text{Dom}(\mathcal{E}_V^+). \]
If $\| V \|_{L_1(I(1),L_B(\mathbb{S}))} \leq 1/C_{22}$, then
\[ \mathcal{E}_V^+[w] \geq 0 \quad \forall w \in \text{Dom}(\mathcal{E}_V^+), \]
that is, $\mathcal{N}_-(\mathcal{E}_V^+) = 0$. Combining this with (66) one gets the existence of a constant $C_{18} > 0$ for which (64) holds in the case $R = 1$. The case of a general $R$ is reduced to $R = 1$ in the standard way with the help of the scaling $x \mapsto Rx$ (cf. the proof of Theorem 4' in [29]).

**Proof of Theorem 7.1.** The theorem is proved in the same way as Theorem 6.1 (see (27), (41) and (43)). One only needs to use Lemma 7.8 instead of Theorem 4' of [29].

**Remark 7.9.** It has been observed by A. Laptev that interchanging the rôles of the variables $y_1$ and $y_2$ in the proof of Lemma 7.8 one can get an analog of (64) where the $L_B$ norm is taken with respect to the radial variable, while the $L_1$ norm is taken with respect to the angular one. Then using the Birman–Solomyak method exactly as in the proofs of Theorems 6.1 and 7.1, one shows the existence of constants $C_{23} > 0$ and $c > 0$ such that
\[ \mathcal{N}_-(\mathcal{E}_V) \leq 1 + 4 \sum_{A_n > 1/4} \sqrt{A_n} + C_{23} \sum_{\mathcal{G}_n > c} \mathcal{G}_n \quad \forall V \geq 0, \]
where
\[ \mathcal{G}_n := e^n \int_S \| V(\cdot, \vartheta) \|_{\mathcal{B}_n} \, d\vartheta, \quad n \in \mathbb{Z} \]
(cf. (52), (53)), and the factor $e^n$ in front of the integral ensures the correct scaling under the mapping $x \mapsto e^n x$.

8. **Comparison of various estimates**

Let us start with comparing (11) and (47). Note first of all that (47) is equivalent to
\[ \mathcal{N}_-(\mathcal{E}_V) \leq 1 + C_{24} \left( \| (A_n)_{n \geq 1} \|_{1, \infty} + \sum_{n = 0}^{\infty} \mathcal{B}_n \right) \quad \forall V \geq 0, \]
where we have dropped the term $A_0$. Indeed, the Hölder inequality (see [28, Section 3.3, (17)]) implies
\begin{align*}
A_0 &= \int_{|x| \leq \varepsilon} V(x) |\ln |x|| \, dx \\
&\leq \| V \|_{\mathcal{B}^n} \| \ln |\cdot|| \|_{(A_n, \Omega_0)}, \\
&= \int_{|x| \leq 1} \frac{1}{|x|} \, dx + \int_{1 < |x| \leq \varepsilon} |x| \, dx \\
&= \int_{-\pi}^{\pi} \int_0^1 1 \, dr \, d\vartheta + \int_{-\pi}^{\pi} \int_1^{\varepsilon} r^2 \, dr \, d\vartheta = \frac{2\pi}{3} (e^3 + 2) =: C_{25}.
\end{align*}
Hence,
\[ \| \ln |\cdot|| \|_{(A_n, \Omega_0)} \leq C_{25} \]
(see (4)) and
\[ A_0 \leq C_{25} \| V \|_{\mathcal{B}^n} \leq C_{25} \| V \|_{(av)}^{(av)} = C_{25} B_0. \]
It is clear that
\[
\| (A_n) \|_{1, \infty} \leq \sum_{n=1}^{\infty} \int_{U_n} V(x) \ln |x| \, dx
\]
\[
= \int_{|x| > e} V(x) \ln |x| \, dx \leq \int_{\mathbb{R}^2} V(x) \ln(1 + |x|) \, dx.
\]  
(71)

It is also easy to see that if
\[
V(x) = \begin{cases} 
\frac{1}{2\pi|x|^2(\ln |x|)^2 \ln |x|}, & |x| \geq e^2, \\
0, & |x| < e^2,
\end{cases}
\]
then $A_n = \ln(1 + 1/(n-1)) = 1/n + O(n^{-2})$, $n \geq 2$, and the left-hand side of (71) is finite, while the right-hand side is infinite.

On the other hand,
\[
\sum_{n=0}^{\infty} B_n \geq \| V \|_{B_r, \Omega} + \sum_{n=1}^{\infty} \| V \|_{B_r, \Omega_n} \geq \| V \|_{B_r, \mathbb{R}^2},
\]  
(73)

where the last estimate follows from the triangle inequality applied to $f = f \chi_{\Omega_0} + \sum_{n=1}^{\infty} f \chi_{\Omega_n}$ with $\chi_{\Omega}(x) := 1$ for $x \in \Omega$ and $\chi_{\Omega}(x) := 0$ for $x \not\in \Omega$. It is not difficult to show that there exist a $V$ for which the left-hand side of (73) is infinite while the middle term is finite, and a $V$ for which the middle term is infinite while the right-hand side is finite. Indeed, let $V(x) = 1/|x|^2$ for $|x| \geq e$ and $V(x) = 0$ for $|x| < e$. Then using [29, (7)] and the notation from Remark 6.3 one gets
\[
\sum_{n=1}^{\infty} B_n = \sum_{n=1}^{\infty} e^{-2(n+1)} \| 1 \|_{B_r, \Omega_n} = \sum_{n=1}^{\infty} e^{-2(n+1)} |\Omega_n| \| 1 \|_{B_r, \Delta}
\]
\[
= \| 1 \|_{B_r, \Delta} \sum_{n=1}^{\infty} \pi \left( 1 - \frac{1}{e^2} \right) = +\infty.
\]

On the other hand, [18, (9.11)] and Lemma 2.3 imply
\[
\sum_{n=1}^{\infty} \| V \|_{B_r, \Omega_n} \leq \sum_{n=1}^{\infty} e^{-2n} \| 1 \|_{B_r, \Omega_n} = \sum_{n=1}^{\infty} e^{-2n} |\Omega_n| A^{-1}\left( \frac{1}{|\Omega_n|} \right)
\]
\[
\leq \sum_{n=1}^{\infty} e^{-2n} \sqrt{2|\Omega_n|} = \sum_{n=1}^{\infty} e^{-2n} \frac{\sqrt{2}}{e} < +\infty.
\]

Take now $V(x) = 1/|x| \ln |x|$ for $|x| \geq e$ and $V(x) = 0$ for $|x| < e$. Then
\[
\sum_{n=1}^{\infty} \| V \|_{B_r, \Omega_n} \geq \sum_{n=1}^{\infty} e^{-(n+1)} \| 1 \|_{B_r, \Omega_n} = \sum_{n=1}^{\infty} e^{-(n+1)} |\Omega_n| A^{-1}\left( \frac{1}{|\Omega_n|} \right)
\]
\[
\geq \sum_{n=1}^{\infty} e^{-(n+1)} \frac{\sqrt{2}}{e} |\Omega_n| = \sum_{n=1}^{\infty} \frac{1}{n+1} \sqrt{\frac{2}{e}} \left( e - 1 \right) = +\infty,
\]
while
\[
\int_{\mathbb{R}^2} B(V(x)) \, dx = \int_{|x| > e} B \left( \frac{1}{|x| \ln |x|} \right) \, dx
\]
\[
= 2\pi \int_{r > e} B \left( \frac{1}{r \ln r} \right) \, dr
\]
\[
\leq \pi \int_{r > e} \frac{1}{r \ln^2 r} \, dr = \pi
\]
(see Lemma 2.3) and hence $\| V \|_{B_r, \mathbb{R}^2} < \infty$. 

The above discussion of (71) and (73) may give one an impression that neither of (11) and (47) is stronger than the other one. It turns out that this is not the case, and it follows from (70), (71) and Lemmas 2.1, 2.5 and 8.1 (see below) that (47) is actually stronger than (11). Indeed,

\[ \| (A_n)_{n\geq 0} \|_{1,\infty} + \sum_{n=0}^{\infty} B_n \]

\[ \leq \text{const} \left( \| (A_n)_{n\geq 1} \|_{1,\infty} + \sum_{n=0}^{\infty} B_n \right) \]

\[ \leq \text{const} \left( \int_{\mathbb{R}^2} V(x) \ln(1 + |x|) \, dx + \pi e^2 \| V \|_B,\Omega_0 \right. \]

\[ + \sum_{n=1}^{\infty} \| V \|_B,\Omega_n + \sum_{n=1}^{\infty} \ln \left( \frac{7}{2} |\Omega_n| \right) \int_{\Omega_n} V(x) \, dx \)

\[ \leq \text{const} \left( \int_{\mathbb{R}^2} V(x) \ln(1 + |x|) \, dx + \| V \|_B,\mathbb{R}^2 + \int_{|x|>e} V(x) \ln |x| \, dx \right) \]

\[ \leq \text{const} \left( \| V \|_B,\mathbb{R}^2 + \int_{\mathbb{R}^2} V(x) \ln(1 + |x|) \, dx \right) \quad \forall V \geq 0. \quad (74) \]

Note that for \( V \) given by (72) one has

\[ \sum_{n=0}^{\infty} B_n = \sum_{n=2}^{\infty} B_n \leq \sum_{n=2}^{\infty} \frac{1}{2\pi e^{2n^2 \ln n}} \| 1 \|_B,\Omega_n \]

\[ = \| 1 \|_B,\Delta \sum_{n=2}^{\infty} \frac{|\Omega_n|}{2\pi e^{2n^2 \ln n}} \]

\[ = \| 1 \|_B,\Delta \sum_{n=2}^{\infty} \frac{e^2 - 1}{2n^2 \ln n} < +\infty. \]

Hence the right-hand side of (47) is finite for this \( V \) while the right-hand side of (11) is infinite. So, (47) is strictly stronger than (11).

**Lemma 8.1.** There exists \( C_{26} > 0 \) such that

\[ \sum_{n=1}^{\infty} \| V \|_B,\Omega_n \leq C_{26} \left( \| V \|_B,\mathbb{R}^2 \setminus \Omega_0 + \int_{|x|>e} V(x) \ln |x| \, dx \right) \quad \forall V \geq 0 \]

(see (42), (48)).

**Proof.** Suppose first that \( \| V \|_B,\mathbb{R}^2 \setminus \Omega_0 = 1 \) and let

\[ \alpha_n := \int_{\Omega_n} B(V(x)) \, dx, \quad \kappa_n := \| V \|_B(\Omega_n), \quad n \in \mathbb{N}. \]

Then

\[ \sum_{n=1}^{\infty} \alpha_n = \sum_{n=1}^{\infty} \int_{\Omega_n} B(V(x)) \, dx = \int_{\mathbb{R}^2 \setminus \Omega_0} B(V(x)) \, dx = 1, \]

\[ \alpha_n \leq 1 \implies \kappa_n \leq 1. \]
and

\[
1 = \int_{\Omega_n} B \left( \frac{V(x)}{\kappa_n} \right) \, dx \leq \int_{\Omega_n} \left( \frac{V(x)}{\kappa_n} + 2 \frac{V(x)}{\kappa_n} \ln + \frac{V(x)}{\kappa_n} \right) \, dx \\
\leq \frac{1}{\kappa_n} \int_{\Omega_n} (V(x) + 2V(x) \ln + V(x)) \, dx + 2 \frac{\ln}{\kappa_n} \|V\|_{L^1(\Omega_n)} \\
\leq \frac{4}{\kappa_n} \alpha_n + \frac{1}{\kappa_n} \left( 1 + 2 \ln \frac{1}{\kappa_n} \right) \|V\|_{L^1(\Omega_n)}
\]

(see Lemma 2.2). Hence,

\[
\kappa_n \leq 4\alpha_n + \left( 1 + 2 \ln \frac{1}{\kappa_n} \right) \|V\|_{L^1(\Omega_n)}
\]

and

\[
\sum_{n=1}^{\infty} \|V\|_{\mathcal{B}, \Omega_n} \leq 2 \sum_{n=1}^{\infty} \kappa_n = 2 \sum_{\kappa_n < 1/n^2}^{\infty} \kappa_n + 2 \sum_{\kappa_n > 1/n^2}^{\infty} \kappa_n \\
\leq 2 \sum_{n=1}^{\infty} \frac{1}{n^2} + 8 \sum_{n=1}^{\infty} \alpha_n + 2 \sum_{n=1}^{\infty} (1 + 4 \ln n) \|V\|_{L^1(\Omega_n)} \\
= \frac{\pi^2}{3} + 8 + 2 \sum_{n=1}^{\infty} (1 + 4 \ln n) \int_{e^n < |x| < e^{n+1}} V(x) \, dx \\
\leq \frac{\pi^2}{3} + 8 + 2 \int_{|x| > e} V(x)(1 + 4 \ln |x|) \, dx \\
\leq C_{26} \left( \|V\|_{\mathcal{B}, \mathbb{R}^2 \setminus \Omega_0} + \int_{|x| > e} V(x) \ln \ln |x| \, dx \right)
\]

(see (3)). The case of a general $V$ is reduced to $\|V\|_{(\mathcal{B}, \Omega_0)} = 1$ by the scaling $V \mapsto tV$, $t > 0$.

Let us now show that each of (54) and (67) implies (46) (with a different $c$). Suppose $\|V\|_{(\mathcal{B}, \Omega_0)} = 1$. It follows from (3), (6) and Lemma 2.1 that

\[
D_0 = \int_1^e \|V(r, \cdot)\|_{\mathcal{B}, \Omega} r \, dr \leq 2 \int_1^e \left( 1 + \int_{\mathcal{B}} B(V(r, \vartheta)) \, d\vartheta \right) r \, dr \\
\leq (e^2 - 1) + 2 \int_{\Omega_0} B(V(x)) \, dx = e^2 + 1 = (e^2 + 1) \|V\|_{(\mathcal{B}, \Omega_0)} \\
\leq (e^2 + 1) \|V\|_{\mathcal{B}, \Omega_0} \leq (e^2 + 1) \|V\|_{\mathcal{B}, \Omega_0}^{(av)} = (e^2 + 1) \mathcal{B}_0,
\]

\[
G_0 = \int_{\mathcal{B}} \|V(\cdot, \vartheta)\|_{\mathcal{B}, \Omega_0}^{(av)} \, d\vartheta \leq 2(e - 1) \int_{\Omega_0} \left( 1 + \int_{\mathcal{B}} B(V(r, \vartheta)) \, dr \right) d\vartheta \\
\leq 2(e - 1) \left( 2\pi + \int_{\Omega_0} B(V(x)) \, dx \right) = 2(e - 1)(2\pi + 1) \\
= 2(e - 1)(2\pi + 1) \|V\|_{(\mathcal{B}, \Omega_0)} \leq 2(e - 1)(2\pi + 1) \mathcal{B}_0 =: C_{27} \mathcal{B}_0.
\]

The scaling $V \mapsto tV$, $t > 0$ allows one to extend the inequalities $D_0 \leq (e^2 + 1) \mathcal{B}_0$ and $G_0 \leq C_{27} \mathcal{B}_0$ to arbitrary $V \geq 0$. Using the scaling $x \mapsto e^\alpha x$ of the independent variable and [29,
Lemma 1, one gets $D_n \leq (e^2 + 1)B_n$ and $G_n \leq C_{27}B_n$, $\forall n \in \mathbb{Z}$. Hence,

$$
\sum_{\{n \in \mathbb{Z}: D_n > c\}} D_n \leq (e^2 + 1) \sum_{\{n \in \mathbb{Z}: D_n > c\}} B_n \leq (e^2 + 1) \sum_{\{n \in \mathbb{Z}: B_n > c/(e^2 + 1)\}} B_n,
$$

$$
\sum_{\{n \in \mathbb{Z}: G_n > c\}} G_n \leq C_{27} \sum_{\{n \in \mathbb{Z}: G_n > c\}} B_n \leq C_{27} \sum_{\{n \in \mathbb{Z}: B_n > c/C_{27}\}} B_n.
$$

Similarly,

$$
\|V\|_{L_1(\mathbb{R}_+, L_B(\mathbb{S}))} = \sum_{n \in \mathbb{Z}} D_n \leq (e^2 + 1) \sum_{n \in \mathbb{Z}} B_n,
$$

and (55) implies (47).

Putting together what we have obtained so far, we get the diagram below. For the convenience of the reader, we precede the diagram with a list of the estimates discussed above:

$$
N_- (E_V) \leq C \left( \|V\|_{L_1(\mathbb{R}_+, L_B(\mathbb{S}))} + \int_{\mathbb{R}^2} V(x) \ln(1 + |x|) \, dx \right) + 1,
$$

$$
N_- (E_V) \leq C_6 \left( \int_{\mathbb{R}^2} V(x) \ln(2 + |x|) \, dx + \int_{\mathbb{R}^2} V_+(|x|) \ln_+ \frac{1}{|x|} \, dx \right) + 1,
$$

$$
N_- (E_V) \leq c_1 \int_{\mathbb{R}^2} V_+ (|x|) \ln_+ \frac{1}{|x|} \, dx + c_2 \int_{\mathbb{R}^2} V(x) \ln_+ |x| \, dx + c_3 \int_{\mathbb{R}^2} V(x) \, dx + 1,
$$

$$
N_- (E_V) \leq 1 + 4 \sum_{A_n > 1/4} \sqrt{A_n} + C_7 \sum_{B_n > c} B_n,
$$

$$
N_- (E_V) \leq 1 + C_8 \left( \| (A_n)_{n \geq 1/0} \|_{1, \infty} + \sum_{n=0}^{\infty} B_n \right),
$$

$$
N_- (E_V) \leq 1 + C_7 \sum_{\{n \in \mathbb{Z}: A_n > c\}} \sqrt{A_n} + C_7 \sum_{\{n \in \mathbb{Z}: B_n > c\}} B_n,
$$

$$
N_- (E_V) \leq 1 + 4 \sum_{A_n > 1/4} \sqrt{A_n} + C_9 \sum_{D_n > c} D_n,
$$

$$
N_- (E_V) \leq 1 + C_{10} \left( \| (A_n)_{n \in \mathbb{Z}} \|_{1, \infty} + \| V \|_{L_1(\mathbb{R}_+, L_B(\mathbb{S}))} \right),
$$

$$
N_- (E_V) \leq 1 + C_{11} \left( \| (A_n)_{n \in \mathbb{Z}} \|_{1, \infty} + \int_{\mathbb{R}_+} \left( \int_{\mathbb{S}} |V(r, \vartheta)|^p \, d\vartheta \right)^{1/p} \, dr \right),
$$

$$
N_- (E_V) \leq 1 + C_{12} \left( \| (A_n)_{n \in \mathbb{Z}} \|_{1, \infty} + \int_{\mathbb{R}_+} \left( \int_{\mathbb{S}} |V_{N}(r, \vartheta)|^p \, d\vartheta \right)^{1/p} \, dr \right),
$$

$$
N_- (E_V) \leq 1 + C_{13} \left( \| (A_n)_{n \in \mathbb{Z}} \|_{1, \infty} + \| V_N \|_{L_1(\mathbb{R}_+, L_B(\mathbb{S}))} \right),
$$

$$
N_- (E_V) \leq 1 + 4 \sum_{A_n > 1/4} \sqrt{A_n} + C_{23} \sum_{G_n > c} G_n,
$$

(11), (22), (23), (46), (47), (50), (54), (55), (56), (57), (59), (67)
where
\[ A_0 := \int_{U_0} V(x) \, dx, \quad A_n := \int_{U_n} V(x) |\ln|x|| \, dx, \quad n \neq 0, \]
\[ B_n := \|V\|_{B, \Omega_n}, \quad B_n := \left( \int_{\Omega_n} V^p(x) |x|^{2(p-1)} \, dx \right)^{1/p}, \]
\[ A_n = A_0, \quad B_n = B_n, \quad n \in \mathbb{N}, \]
\[ A_0 := \int_{\Omega_0} V(x) |\ln|x|| \, dx, \quad B_0 = \|V\|_{B, \Omega_0}, \quad \Omega_0 = \{ x \in \mathbb{R}^2 : |x| \leq \varepsilon \}, \]
\[ D_n := \|V\|_{L^1(I_n, L^p(S))}, \quad G_n := e^n \int_S \|V(\cdot, \vartheta)\|_{L^p(I_n)} \, d\vartheta, \quad I_n := (e^n, e^{n+1}), \quad n \in \mathbb{Z}, \]

1 < p < \infty, and \( U_n, \Omega_n \) are defined in (39) and (42). Estimate (23) was conjectured by Khuri, Martin and Wu [17], (47) was proved by Solomyak [29], (50) was obtained by Grigor’yan and Nadirashvili [10], (57) was proved by Laptev and Solomyak in [21], and (67) was proposed by A. Laptev (see Remark 7.9).

\[
(54) \quad \Rightarrow \quad (55) \quad \Leftrightarrow \quad (59) \quad \Rightarrow \quad (56) \quad \Leftrightarrow \quad (57)
\]

\[
\downarrow \quad \downarrow
\]

\[
(67) \quad \Rightarrow \quad (46) \quad \Rightarrow \quad (47) \quad \Rightarrow \quad (11) \quad \Leftrightarrow \quad (22) \quad \Leftrightarrow \quad (23)
\]

\[
\downarrow
\]

\[
(50)
\]

Our next task is to show that no other implication holds between the estimates in the above diagram. Suppose that \( V(r, \vartheta) = V_1(r)V_2(\vartheta) \). If \( V_2 \equiv 1 \), \( V_1(r) = \alpha/r^2(1 + \ln^2 r) \) and \( \alpha > 0 \) is sufficiently small, then the right-hand side of (50) equals 1 while \( \|A_n\|_{1, \infty} = +\infty \) (see [10]). Hence, (55) does not imply (50).

Suppose now that \( \supp V_1 \subseteq [1, \varepsilon] \). If \( V_2 \equiv 1 \) and \( V_1 \in L_1([1, \varepsilon]) \setminus L_B([1, \varepsilon]) \), for example, \( V_1(r) = 1/(r-1)(1 + \ln^2 (r-1)) \), then the right-hand side of (57) is infinite, while the right-hand side of (67) is finite. Hence, (67) does not imply (56) and (57). If \( V_2 \equiv 1 \) and \( V_1 \in L_B([1, \varepsilon]) \setminus L_p([1, \varepsilon]) \), for example, \( V_2(\vartheta) = 1/|\vartheta| (1 + \ln^2 |\vartheta|) \), then the right-hand side of (50) is infinite, while the right-hand side of (11) is finite. Hence (50) does not imply (11). If \( V_1 \equiv 1 \) on \([1, \varepsilon]\) and \( V_2 \in L_1(S) \setminus L_B(S) \), for example, \( V_2(\vartheta) = 1/|\vartheta| (1 + \ln |\vartheta|)^2 \), then the right-hand side of (54) is infinite, while the right-hand side of (67) is finite. Hence (54) does not imply (67). Finally, if \( V_1 \equiv 1 \) on \([1, \varepsilon]\) and \( V_2 \in L_B(S) \setminus \bigcup_{p>1} L_p(S) \), for example, \( V_2(\vartheta) = 1/|\vartheta| (1 + |\ln |\vartheta||^3) \), then the right-hand side of (56) is infinite, while the right-hand side of (11) is finite. Hence (56) does not imply (11).

9. Concluding remarks

Using estimate (59), one can prove that [21, Theorem 1.1 and Proposition 1.2] remain true if one substitutes the condition \( V_\lambda \in L_1(\mathbb{R}^+, L_p(S)) \), \( p > 1 \) with
\[
V_\lambda \in L_1(\mathbb{R}^+, L_B(S)). \quad (75)
\]
In particular, if (75) is satisfied, then the Weyl-type asymptotic formula
\[
\lim_{\alpha \to +\infty} \alpha^{-1} N_{-}(\mathcal{E}_\alpha V) = \frac{1}{4\pi} \int_{\mathbb{R}^2} V(x) \, dx
\]
holds if and only if
\[
\lim_{s \to 0+} s \text{card}\{n \in \mathbb{Z} : A_n > s\} = 0 \quad (76)
\]
(see (40)). One can also prove that, if (75) is satisfied, then \( N_- (E_{\alpha V}) = O(\alpha) \) as \( \alpha \to +\infty \) if and only if \( \| (A_n)_{n \in \mathbb{Z}} \|_{1, \infty} < \infty \) (cf. [20, Theorem 1.1]; see also Remark 9.3). The last result and (54) imply that if
\[
\sum_{\{n \in \mathbb{Z} : n > c\}} \sqrt{\alpha A_n} = O(\alpha) \quad \text{as } \alpha \to +\infty,
\]
then \( \| (A_n)_{n \in \mathbb{Z}} \|_{1, \infty} < \infty \). This implication is a special case \( (q = 1, \sigma = \frac{1}{2}) \) of the following result on sequences of numbers \( a_n, n \in \mathbb{Z} \): if \( q > \sigma > 0 \), then
\[
\sum_{\{n \in \mathbb{Z} : |a_n| > c\}} (\alpha |a_n|)^\sigma = O(\alpha^q) \quad \text{as } \alpha \to +\infty \quad (77)
\]
\[
\iff \text{card}\{n \in \mathbb{Z} : |a_n| > s\} = O(s^{-q}) \quad \text{as } s \to 0+ . \quad (78)
\]
Indeed, if (78) holds, then one gets similarly to (49)
\[
\sum_{\alpha |a_n| > c} (\alpha |a_n|)^\sigma = \sum_{|a_n| > c / \alpha} \int_0^{\alpha |a_n|} \sigma s^{\sigma - 1} ds
\]
\[
= \sigma \alpha^\sigma \int_0^\infty s^{\sigma - 1} \text{card}\{n \in \mathbb{Z} : |a_n| > s \& |a_n| > c / \alpha\} ds
\]
\[
= \sigma \alpha^\sigma \int_0^{c / \alpha} s^{\sigma - 1} \text{card}\{n \in \mathbb{Z} : |a_n| > c / \alpha\} ds
\]
\[
+ \sigma \alpha^\sigma \int_{c / \alpha}^\infty s^{\sigma - 1} \text{card}\{n \in \mathbb{Z} : |a_n| > s\} ds
\]
\[
= \alpha^\sigma O(\alpha^q) \int_0^{c / \alpha} s^{\sigma - 1} ds + \alpha^\sigma O\left( \int_{c / \alpha}^\infty s^{\sigma - 1 - q} ds \right)
\]
\[
= O(\alpha^q) \quad \text{as } \alpha \to +\infty,
\]
where the last two integrals exist due to the condition \( q > \sigma > 0 \). Suppose now (77) holds. Then using the notation \( \alpha = c / s \) one gets
\[
\text{card}\{n \in \mathbb{Z} : |a_n| > s\} = \text{card}\{n \in \mathbb{Z} : |a_n| > c / \alpha\}
\]
\[
\leq \frac{1}{c^\sigma} \sum_{\alpha |a_n| > c} (\alpha |a_n|)^\sigma = O(\alpha^q) = O(s^{-q}) \quad \text{as } s \to 0+ .
\]

Let us return to the discussion of the necessity of (76) and \( \| (A_n)_{n \in \mathbb{Z}} \|_{1, \infty} < \infty \). Neither of these conditions is necessary for \( N_- (E_V) \) to be finite. Indeed, let \( V \geq 0 \) be a radial potential such that \( (A_n)_{n \in \mathbb{Z}} \in \ell_\infty (\mathbb{Z}) \setminus \ell_{1, \infty} (\mathbb{Z}) \). Then it follows from Theorem 7.1 that \( N_- (E_{\beta V}) = 1 \) for sufficiently small \( \beta > 0 \), although the above conditions are not satisfied for \( \beta V \). It turns out that the condition \( (A_n)_{n \in \mathbb{Z}} \in \ell_{1, \infty} (\mathbb{Z}) \) on the other hand is necessary for \( N_- (E_V) \) to be finite (Theorem 9.1), while \( \| (A_n)_{n \in \mathbb{Z}} \|_{1, \infty} < \infty \) is necessary for \( N_- (\alpha V) = O(\alpha) \) as \( \alpha \to +\infty \) to hold (Theorem 9.2). Note that neither of these results assumes that (75) is satisfied.

**Theorem 9.1.** Let \( V \geq 0 \). Then
\[
N_- (E_V) \geq \frac{1}{3} \text{card}\{n \in \mathbb{Z} : A_n \geq 10\pi\}. \quad (79)
\]
Proof. Let \( m \in \mathbb{Z} \) be such that \( A_m \geq 10\pi \) and let \( r_0 < r_1 < r_2 < r_3 \) be the radii of the boundary circles of the annuli \( U_{m-1}, U_m, U_{m+1} \). Consider the function

\[
\begin{align*}
   w_m(x) := \\
   \begin{cases}
      0, & |x| \leq r_0 \text{ or } |x| \geq r_3, \\
      1 - \frac{\ln(r_1/|x|)}{\ln(r_1/r_0)}, & r_0 < |x| < r_1, \\
      1, & r_1 \leq |x| \leq r_2, \\
      \frac{\ln(r_3/|x|)}{\ln(r_3/r_2)}, & r_2 < |x| < r_3.
   \end{cases}
\end{align*}
\]

It is easy to see that

\[
\int_{\mathbb{R}^2} |\nabla w_m(x)|^2 \, dx = 2\pi \int_{r_0}^{r_1} \frac{1}{(\ln(r_1/r_0))^2} \frac{1}{r} \, dr + 2\pi \int_{r_2}^{r_3} \frac{1}{(\ln(r_3/r_2))^2} \frac{1}{r} \, dr
\]

\[
= 2\pi \left( \frac{1}{\ln(r_1/r_0)} + \frac{1}{\ln(r_3/r_2)} \right)
\]

\[
= \begin{cases}
   2\pi(2^{-|m|} + 2^{-|m|}) & m \leq -2 \text{ or } m \geq 2 \\
   2\pi & m = \pm 1 \\
   4\pi & m = 0
\end{cases}
\]

\[
\leq 2\pi(2^{-|m|} + 2^{-|m|}) = 10\pi 2^{-|m|},
\]

and the inequality is strict if \( m = 0, \pm 1 \). Since \( w_m(x) = 1 \) for \( x \in U_m \), one gets

\[
\int_{\mathbb{R}^2} V(x)|w_m(x)|^2 \, dx \geq \int_{U_m} V(x) \, dx
\]

\[
\geq \begin{cases}
   \frac{1}{2|m|} \int_{U_m} V(x)|\ln |x|| \, dx, & m \neq 0 \\
   A_0, & m = 0
\end{cases}
\]

\[
= \frac{A_m}{2|m|} \geq 10\pi 2^{-|m|} \geq \int_{\mathbb{R}^2} |\nabla w_m(x)|^2 \, dx,
\]

and the second inequality is strict if \( m \neq 0 \). Hence \( \mathcal{E}_V[w_m] < 0 \) if \( A_m \geq 10\pi \), and (79) follows from the fact that \( w_m \) and \( w_k \) have disjoint supports if \( |m - k| \geq 3 \). Indeed, if the set \( \Sigma := \{ n \in \mathbb{Z} : A_n \geq 10\pi \} \) is infinite, then it contains infinitely many elements that lie at a distance at least three from each other, and both sides of (79) are infinite. If the set \( \Sigma \) is non-empty and finite, take

\[
m_1 := \min \Sigma, \quad m_{j+1} = \min \{ n \in \Sigma : n \geq m_j + 3 \}, \quad j = 1, \ldots
\]

and continue the process until the set on the right-hand side is empty. This produces at least \( \frac{1}{3} \) \( \text{card} \Sigma \) numbers and concludes the proof. \( \square \)

**Theorem 9.2.** Let \( V \geq 0 \). If \( N_-(\alpha V) = O(\alpha) \) as \( \alpha \to +\infty \), then \( \| (A_n)_{n \in \mathbb{Z}} \|_{1, \infty} < \infty \).

**Proof.** This follows from the previous theorem. Indeed, if \( N_-(\alpha V) \leq C\alpha \) for large \( \alpha \), then (79) implies

\[
\frac{1}{3} \text{card}\{ n \in \mathbb{Z} : \alpha A_n \geq 10\pi \} \leq C\alpha \iff \text{card}\{ n \in \mathbb{Z} : A_n \geq 10\pi/\alpha \} \leq 3C\alpha.
\]

Hence, we have (with \( s = 10\pi/\alpha \))

\[
\text{card}\{ n \in \mathbb{Z} : A_n \geq s \} \leq C_1 s^{-1} \quad \text{for small } s > 0,
\]

where \( C_1 = 30\pi C \).

\( \square \)
Remark 9.3. Suppose (75) holds. Then it follows from the above that $N_-(\mathcal{E}_V) = O(\alpha^n)$, $q \geq 1$ as $\alpha \to +\infty$ if and only if

$$\text{card}\{n \in \mathbb{Z} : A_n > s\} = O(s^{-q}) \quad \text{as } s \to 0^+.$$  \hfill (80)

Indeed, if this equality holds, then it follows from (54) and (58) and from the equivalence of (77) and (78) that $N_-(\mathcal{E}_V) = O(\alpha^n)$. Conversely, if the latter holds, then (79) with $\alpha = 10\pi/s$ implies

$$\text{card}\{n \in \mathbb{Z} : A_n > s\} = \text{card}\{n \in \mathbb{Z} : \alpha A_n > 10\pi\} \leq 3N_-(\mathcal{E}_V) = O(\alpha^n) = O(s^{-q}) \quad \text{as } s \to 0^+.$$  \hfill (81)

Note that none of the estimates in the paper is sharp in the sense that $N_-(\mathcal{E}_V)$ has to be infinite if the right-hand side is infinite. Indeed, the examples at the end of Section 8 show that the right-hand side of (54) may be infinite while $N_-(\mathcal{E}_V)$ is finite due to (67), and the other way around: the right-hand side of (67) may be infinite while $N_-(\mathcal{E}_V)$ is finite due to (54). The same argument shows that the finiteness of the right-hand sides in these estimates is not necessary even for $N_-(\alpha V) = O(\alpha)$ as $\alpha \to +\infty$ to hold. On the other hand, no estimate of the type

$$N_-(\mathcal{E}_V) \leq \text{const} + \int_{\mathbb{R}^2} V(x)W(x) \, dx + \text{const}\|V\|_{\Psi,\mathbb{R}^2}$$  \hfill (82)

can hold with an Orlicz norm $\| \cdot \|_\Psi$ weaker than $\| \cdot \|_{\mathcal{B}}$, provided the weight function $W$ is bounded in a neighborhood of at least one point (cf. [29, Section 4]). Indeed, let $\Omega$ be a bounded open set where $W$ is bounded and suppose that $\Psi(s)/\mathcal{B}(s) \to 0$ as $s \to \infty$. Let $\Phi$ be the complementary function to $\Psi$. Then it follows from [15] and [18, Lemma 13.1] that there exist $w_j \in W_2^1(\Omega)$, $j \in \mathbb{N}$ such that $\|w\|_{W_2^1(\Omega)} = 1$ and $\|w_j\|_{\Psi,\Omega} \to \infty$ as $j \to \infty$. The Banach–Steinhaus theorem and [18, Theorem 14.2] imply the existence of $v \in L_\Psi(\Omega)$ such that the sequence

$$\int_{\Omega} v(x)w_j^2(x) \, dx, \quad j \in \mathbb{N}$$

is unbounded. Define $V \in L_\Psi(\mathbb{R}^2)$ by $V(x) = |v(x)|$ for $x \in \Omega$ and $V(x) = 0$ for $x \notin \Omega$. Then

$$Q(V) := \sup_{\|w\|_{W_2^1(\mathbb{R}^2)} = 1} \int_{\mathbb{R}^2} V(x)|w(x)|^2 \, dx = \infty.$$  \hfill (83)

Since the quadratic form $\int_{\mathbb{R}^2} V(x)|w(x)|^2 \, dx \geq 0$ is closable in $W_2^1(\mathbb{R}^2)$, it corresponds to a non-negative self-adjoint operator. Since $Q(V) = \infty$, the operator is unbounded, and it follows from the spectral theorem that there exists an infinite-dimensional subspace $\mathcal{F} \subset W_2^1(\mathbb{R}^2)$ such that

$$\int_{\mathbb{R}^2} V(x)|w(x)|^2 \, dx > \|w\|_{W_2^1(\mathbb{R}^2)}^2 \quad \forall w \in \mathcal{F} \setminus \{0\}.$$  \hfill (84)

Hence $N_-(\mathcal{E}_V) = \infty$, but the right-hand side of (81) is finite. Below is a more constructive proof of the same result.

Theorem 9.4. Let $W \geq 0$ be bounded in a neighborhood of at least one point and let $\Psi$ be an $N$-function such that

$$\lim_{s \to \infty} \frac{\Psi(s)}{\mathcal{B}(s)} = 0.$$
Then there exists a compactly supported $V \geq 0$ such that

$$\int_{\mathbb{R}^2} V(x)W(x) \, dx + \|V\|_{\Psi,\mathbb{R}^2} < \infty$$

and $N_-(E_V) = \infty$.

Proof. Shifting the independent variable if necessary, we can assume that $W$ is bounded in a neighborhood of 0. Let $r_0 > 0$ be such that $W$ is bounded in the open ball $B(0, r_0)$.

Let

$$\gamma(s) := \sup_{t \geq s} \frac{\Psi(t)}{B(t)}.$$  

Then $\gamma$ is a non-increasing function, $\gamma(s) \to 0$ as $s \to \infty$, and $\Psi(s) \leq \gamma(s)B(s)$. Since $\Psi$ is an $N$-function, $\Psi(s)/s \to \infty$ as $s \to \infty$ (see [18, (1.16)]). Hence, there exists $s_0 \geq e$ such that $\Psi(s) \geq s$ and $\gamma(s) \leq 1$ for $s \geq s_0$.

Choose $r_k \in (0, 1/s_0)$, $k \in \mathbb{N}$ in such a way that $r_k < 1/r_{k-1}$ and

$$\sum_{k=1}^{\infty} \gamma\left(\frac{1}{r_k}\right) < \infty.$$  

It is easy to see that the open disks $B(2r_k, r_k)$, $k \in \mathbb{N}$ lie in $B(0, r_0)$ and are pairwise disjoint. Let

$$t_k := \frac{3}{\ln(1/r_k)} r_k^{-4},$$

$V(x) := \begin{cases} t_k, & x \in B(2r_k, r_k), \ k \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$

Then

$$t_k = \frac{3}{\ln(1/r_k)} r_k^{-4} > \frac{1}{r_k^4 \ln(1/r_k)} r_k^{-1} > \frac{1}{r_k}$$

and

$$\int_{\mathbb{R}^2} \Psi(V(x)) \, dx = \sum_{k=1}^{\infty} \pi r_k^4 \Psi(t_k) \leq \sum_{k=1}^{\infty} \pi r_k^4 \gamma(t_k)B(t_k)$$

$$< \pi \sum_{k=1}^{\infty} r_k^4 \gamma(t_k)(1 + t_k) \ln(1 + t_k) < 4\pi \sum_{k=1}^{\infty} r_k^4 \gamma(t_k) t_k \ln t_k$$

$$= 4\pi \sum_{k=1}^{\infty} \gamma(t_k) \frac{3r_k^4}{r_k^4 \ln(1/r_k)} \ln \frac{3}{r_k^4 \ln(1/r_k)} < 4\pi \sum_{k=1}^{\infty} \gamma(t_k) \frac{3}{\ln(1/r_k)} \ln \frac{3}{r_k^4}$$

$$< 72\pi \sum_{k=1}^{\infty} \gamma(t_k) \leq 72\pi \sum_{k=1}^{\infty} \gamma\left(\frac{1}{r_k}\right) < \infty.$$  

Hence, $\|V\|_{\Psi,\mathbb{R}^2} < \infty$. Since $t_k > 1/r_k \geq s_0$, we have $t_k \leq \Psi(t_k)$ and

$$\int_{\mathbb{R}^2} V(x) \, dx \leq \int_{\mathbb{R}^2} \Psi(V(x)) \, dx < \infty.$$  

Taking into account that the support of $V$ lies in $B(0, r_0)$ and that $W$ is bounded in $B(0, r_0)$, we obtain

$$\int_{\mathbb{R}^2} V(x)W(x) \, dx < +\infty.$$
Let
\[ w_k(x) := \begin{cases} 
1, & |x - 2r_k| \leq r_k^2, \\
\ln(r_k/|x - 2r_k|) / \ln(1/r_k), & r_k^2 < |x - 2r_k| \leq r_k, \\
0, & |x - 2r_k| > r_k
\end{cases} \]
(cf. [10]). Then
\[ \int_{\mathbb{R}^2} |\nabla w_k(x)|^2 \, dx = \frac{2\pi}{\ln(1/r_k)} \]
and
\[ \int_{\mathbb{R}^2} V(x)|w_k(x)|^2 \, dx \geq \int_{B(2r_k, r_k^2)} V(x) \, dx = \pi r_k^4 t_k = \frac{3\pi}{\ln(1/r_k)}. \]
Hence,
\[ \mathcal{E}_V[w_k] < 0 \quad \forall k \in \mathbb{N} \]
and \( N_-(\mathcal{E}_V) = \infty \).

It is probably difficult to obtain an estimate for \( N_-(\mathcal{E}_V) \) that is sharp in the above sense, that is, is such that \( N_-(\mathcal{E}_V) \) is infinite if the right-hand side is infinite. Indeed, there are potentials \( V \geq 0 \) such that \( N_-(\mathcal{E}_V, V) < \infty \) for \( \alpha < 1 \) and \( N_-(\mathcal{E}_V, V) = \infty \) for \( \alpha > 1 \). For such potentials, \( N_-(\mathcal{E}_V) \) may be finite or infinite and the following theorem shows that, in the latter case, \( N_-(\mathcal{E}_V) \) may grow arbitrarily fast or arbitrarily slow as \( \alpha \rightarrow 1 - 0 \).

**Theorem 9.5.** (i) For any \( N \in \mathbb{N} \), there exists a \( V \in L_1(\mathbb{R}^2), V \geq 0 \), such that \( N_-(\mathcal{E}_V, V) = N \) and \( N_-(\mathcal{E}_V, V) = \infty \) for \( \alpha > 1 \).

(ii) For any sequence \( (\alpha_k)_{k \in \mathbb{N}} \) increasing to 1 and any \( N_k \in \mathbb{N} \), there exists a \( V \in L_1(\mathbb{R}^2), V \geq 0 \), such that \( N_k \leq N_-(\mathcal{E}_{\alpha_k + 1}, V) - N_-(\mathcal{E}_{\alpha_k}, V) < \infty \), \( k \in \mathbb{N} \).

(iii) For any sequence \( (\alpha_k)_{k \in \mathbb{N}} \) increasing to 1 and satisfying the condition
\[ \sum_{k=1}^{\infty} \frac{1}{\alpha_{k+1} - \alpha_k} < \infty, \tag{82} \]
there exist a \( V \in L_1(\mathbb{R}^2), V \geq 0 \) and a \( k_0 \in \mathbb{N} \) such that \( N_-(\mathcal{E}_{\alpha_k + 1}, V) - N_-(\mathcal{E}_{\alpha_k}, V) = 1 \) for all \( k \geq k_0 \).

**Proof.** See Appendix B.

**Appendix A. Sharp one-dimensional Sobolev-type inequalities**

Let \( 0 < a < b \). It follows from the embedding \( W^1_2([a, b]) \hookrightarrow C([a, b]) \) that there exist constants \( \alpha, \beta > 0 \) such that
\[ \frac{|u(x)|^2}{|x|} \leq \alpha \int_a^b |u'(t)|^2 \, dt + \beta \int_a^b \frac{|u(t)|^2}{|t|^2} \, dt \quad \forall u \in W^1_2([a, b]). \]
This inequality is used in Section 5 and it is natural to ask what the best values of \( \alpha, \beta > 0 \) are. Since there are two constants involved here, it is convenient to rewrite the inequality in the following form:
\[ \frac{|u(x)|^2}{|x|} \leq C(\kappa) \left( \int_a^b |u'(t)|^2 \, dt + \kappa \int_a^b \frac{|u(t)|^2}{|t|^2} \, dt \right) \quad \forall u \in W^1_2([a, b]), \tag{A.1} \]
and to look for the best value of \( C(\kappa) \) for a given \( \kappa > 0 \).
Lemma A.1. Let
\[\gamma_1 = -1 + \sqrt{1 + 4\kappa}, \quad \gamma_2 = -1 - \sqrt{1 + 4\kappa}.\] (A.2)

For any \(\kappa > 0\), (A.1) holds with
\[C(\kappa; x) := \frac{\gamma_1^2 x \sqrt{1 + 4\kappa} + \kappa(b \sqrt{1 + 4\kappa} + a \sqrt{1 + 4\kappa}) + \gamma_2^2 (ab) \sqrt{1 + 4\kappa} x - \sqrt{1 + 4\kappa}}{\kappa \sqrt{1 + 4\kappa} (b \sqrt{1 + 4\kappa} - a \sqrt{1 + 4\kappa})}\]
and becomes an equality for
\[u(t) = \begin{cases} \gamma_1 \kappa t \gamma_1 - \gamma_2 a \sqrt{1 + 4\kappa} t \gamma_1, & a \leq t < x, \\ \gamma_2 a \sqrt{1 + 4\kappa} - \gamma_1 \kappa t \gamma_1 - \gamma_2 b \sqrt{1 + 4\kappa} t \gamma_1, & x < t \leq b. \end{cases}\]
The maximum of \(C(\kappa; x)\) is achieved at \(x = a\) and is equal to
\[C(\kappa) := \frac{1}{2\kappa} \left( 1 + \sqrt{1 + 4\kappa} \frac{b \sqrt{1 + 4\kappa} + a \sqrt{1 + 4\kappa}}{b \sqrt{1 + 4\kappa} - a \sqrt{1 + 4\kappa}} \right).\] (A.3)

Proof. It is easy to see that
\[u(x) \ln \frac{b}{a} = \int_a^b G_x(t) u'(t) dt + \int_a^b \frac{u(t)}{t} dt,\]
where
\[G_x(t) := \begin{cases} \ln \frac{t}{a}, & a \leq t < x, \\ \ln \frac{t}{b}, & x < t \leq b. \end{cases}\]
Since
\[\int_a^b \varphi(t) u'(t) dt + \int_a^b (t \varphi'(t)) \frac{u(t)}{t} dt = 0 \quad \forall \varphi \in W^1_2((a, b)),\]
we also have
\[u(x) \ln \frac{b}{a} = \int_a^b (G_x(t) - \varphi(t)) u'(t) dt + \int_a^b (1 - t \varphi'(t)) \frac{u(t)}{t} dt,\]
and the Cauchy–Schwarz inequality implies
\[\frac{|u(x)|^2}{|x|} \leq \frac{\mathcal{J}(\varphi)}{|x| \ln^2(b/a)} \left( \int_a^b |u'(t)|^2 dt + \kappa \int_a^b \frac{|u(t)|^2}{|t|^2} dt \right),\]
where
\[\mathcal{J}(\varphi) := \int_a^b |G_x(t) - \varphi(t)|^2 dt + \frac{1}{\kappa} \int_a^b |1 - t \varphi'(t)|^2 dt.\]
Hence, we need to minimize \(\mathcal{J}(\varphi)\) on \(W^1_2((a, b))\). It is clear we only need to consider real-valued \(\varphi\). The Euler equation for this functional takes the form
\[t^2 \varphi'' + 2t \varphi' - \kappa \varphi = 1 - \kappa G_x, \quad \varphi(a) = 0 = \varphi(b).\]
It is easy to solve the equation on \((a, x)\) and separately on \((x, b)\): the function \(\varphi_0 = G_x\) is a solution on both intervals, and the change of the independent variable \(s = \ln t\) reduces the corresponding homogeneous equation to a ordinary differential equation (ODE) with constant coefficients. Choosing the constants in the general solutions on \((a, x)\) and \((x, b)\) in such a way
that $\varphi(a) = 0$, $\varphi(b) = 0$, $\varphi(x - 0) = \varphi(x + 0)$ and $\varphi'(x - 0) = \varphi'(x + 0)$, one gets
\[
\varphi(t) = G_x(t) + \frac{x^{-\gamma_1} \ln(b/a)}{(\gamma_1 - \gamma_2)(b\sqrt{1+4\kappa} - a\sqrt{1+4\kappa})} \Phi_x(t),
\]
(A.4)
\[
\Phi_x(t) := \begin{cases}
(\gamma_2 b\sqrt{1+4\kappa} - \gamma_1 x\sqrt{1+4\kappa})(t^{\gamma_1} - a\sqrt{1+4\kappa} t^{\gamma_2}), & a \leq t < x, \\
(\gamma_2 a\sqrt{1+4\kappa} - \gamma_1 x\sqrt{1+4\kappa})(t^{\gamma_1} - b\sqrt{1+4\kappa} t^{\gamma_2}), & x < t \leq b.
\end{cases}
\]

Taking into account that $\gamma_1$ and $\gamma_2$ (see (A.2)) are the roots of the quadratic equation $\gamma^2 + \gamma - \kappa = 0$, one can easily check that (A.4) does indeed solve the equation on $(a, x)$ and $(x, b)$ and satisfy the above conditions at $t = a, x, b$.

It follows from the above that (A.1) holds with $J(\varphi)/(|x| \ln^2(b/a))$ in place of $C(\kappa)$ for any $\varphi \in W^1_2([a, b])$, in particular, for the one given by (A.4). Let us show that the equality in (A.1) is achieved for the latter. Indeed, let
\[
u(t) := \frac{1}{\kappa} (t - t^2 \varphi'(t))
\]
\[
\varepsilon(x) = M(\kappa) \left( (\gamma_2 b\sqrt{1+4\kappa} - \gamma_1 x\sqrt{1+4\kappa})(\gamma_1 t^{\gamma_1} - \gamma_2 a\sqrt{1+4\kappa} t^{\gamma_2}) t, \; a \leq t < x, \\
(\gamma_2 a\sqrt{1+4\kappa} - \gamma_1 x\sqrt{1+4\kappa})(\gamma_1 t^{\gamma_1} - \gamma_2 b\sqrt{1+4\kappa} t^{\gamma_2}) t, \; x < t \leq b,
\right)
\]
\[
M(\kappa) := \frac{x^{-\gamma_1} \ln(b/a)}{\kappa(\gamma_1 - \gamma_2)(b\sqrt{1+4\kappa} - a\sqrt{1+4\kappa})}.
\]

Then $\nu \in W^1_2([a, b])$, $\nu'(0) = \nu'(1) = 0$. Then using (A.1), (A.3) with $\gamma = 1$, we get
\[
J(\varphi)/(|x| \ln^2(b/a)) = G_x - \varphi,
\]
and the Cauchy–Schwarz inequality used above is in fact an equality.

It is left to evaluate $J(\varphi)/(|x| \ln^2(b/a))$ for (A.4). Using the equalities
\[
2\gamma_1 + 1 = \sqrt{1 + 4\kappa}, \quad 2\gamma_2 + 1 = -\sqrt{1 + 4\kappa},
\]
\[
1 + \frac{\gamma_1^2}{\kappa} = \frac{\sqrt{1 + 4\kappa}}{\kappa} \gamma_1, \quad 1 + \frac{\gamma_2^2}{\kappa} = \frac{-\sqrt{1 + 4\kappa}}{\kappa} \gamma_2,
\]
\[
2\kappa\gamma_1 - \gamma_1^2 = \sqrt{1 + 4\kappa} \gamma_1^2, \quad 2\kappa\gamma_2 - \gamma_2^2 = -\sqrt{1 + 4\kappa} \gamma_2^2,
\]
one gets after a straightforward but not a particularly pleasant calculation
\[
\frac{J(\varphi)}{|x| \ln^2(b/a)} = \frac{\gamma_1^2 x\sqrt{1+4\kappa} + \kappa(b\sqrt{1+4\kappa} + a\sqrt{1+4\kappa}) + \gamma_2^2 (ab)^{\sqrt{1+4\kappa}} x^{-\sqrt{1+4\kappa}}}{\kappa(\sqrt{1 + 4\kappa} b\sqrt{1+4\kappa} - a\sqrt{1+4\kappa})}.
\]

Since the function $z \mapsto \gamma_1^2 z + \gamma_2^2 (ab)^{\sqrt{1+4\kappa}} z^{-1}$ does not have a local maximum for $z > 0$ ($z = x\sqrt{1+4\kappa}$), the above fraction achieves its maximum on $[a, b]$ at an endpoint. It is easy to see that the maximum is achieved at $x = a$ and is equal to (A.3).

\[\text{Remark A.2.} \sup \] Suppose that $u \in W^1_2([0, 1])$ and $u(0) = 0$. Then using (A.1), (A.3) with $b = 1$ and $a \to 0+$ one gets
\[
\frac{|u(x)|^2}{|x|} \leq \frac{1}{2\kappa} (1 + \sqrt{1 + 4\kappa}) \left( \int_0^1 |u'(t)|^2 \, dt + \kappa \int_0^1 \frac{|u(t)|^2}{|t|^2} \, dt \right) \quad \forall x \in (0, 1],
\]
and the right-hand side is finite due to Hardy’s inequality. Note that
\[
\frac{1}{2\kappa} (1 + \sqrt{1 + 4\kappa}) < C(\kappa)
\]
for any $b > a > 0$ (see (A.3)).

Starting with the representation
\[
u(x) = \int_0^1 H_x(t) u'(t) \, dt + \int_0^1 u(t) \, dt,
\]
where

\[
H_x(t) := \begin{cases} 
  t, & 0 \leq t < x, \\
  t - 1, & x < t \leq 1,
\end{cases}
\]

one can find, similarly to (A.1) and (A.3), the optimal constant \(C_0(\kappa)\) in the estimate

\[
|u(x)|^2 \leq C_0(\kappa) \left( \int_0^1 |u'(t)|^2 \, dt + \kappa \int_a^b |u(t)|^2 \, dt \right) \quad \forall u \in W_2^1([0,1]).
\]

The calculations are easier in this case, and one gets after an affine transformation of the independent variable that for any \(\kappa > 0\),

\[
|u(x)|^2 \leq C_0(\kappa) \left( (b - a) \int_a^b |u'(t)|^2 \, dt + \frac{\kappa}{b - a} \int_a^b |u(t)|^2 \, dt \right) \quad \forall u \in W_2^1([a, b]) \tag{A.5}
\]

holds with

\[
C_0(\kappa; x) := \frac{\sinh(2\sqrt{\kappa}) + \sinh(2\sqrt{\kappa}((x - a)/(b - a))) + \sinh(2\sqrt{\kappa}((b - x)/(b - a)))}{4\sqrt{\kappa}\sinh^2\sqrt{\kappa}}
\]

and becomes an equality for

\[
u(t) = \begin{cases} 
  \cosh \left( \sqrt{\frac{x - a}{b - a}} \right) \cosh \left( \sqrt{\frac{t - a}{b - a}} \right), & a \leq t < x, \\
  \cosh \left( \sqrt{\frac{x - a}{b - a}} \right) \cosh \left( \sqrt{\frac{b - t}{b - a}} \right), & x < t \leq b.
\end{cases}
\]

The maximum of \(C_0(\kappa; x)\) is achieved at \(x = a\) and \(x = b\), and is equal to

\[
C_0(\kappa) := \coth \frac{\sqrt{\kappa}}{\sqrt{\kappa}}. \tag{A.6}
\]

One can rewrite the inequality

\[
|u(x)|^2 \leq \coth \frac{\sqrt{\kappa}}{\sqrt{\kappa}} \left( (b - a) \int_a^b |u'(t)|^2 \, dt + \frac{\kappa}{b - a} \int_a^b |u(t)|^2 \, dt \right) \quad \forall x \in [a, b], \quad \forall u \in W_2^1([a, b]) \tag{A.7}
\]

in the following more symmetric form (with \(\varrho = \sqrt{\kappa}\)):

\[
|u(x)|^2 \leq \coth \varrho \left( \frac{b - a}{\varrho} \int_a^b |u'(t)|^2 \, dt + \frac{\varrho}{b - a} \int_a^b |u(t)|^2 \, dt \right) \quad \forall \varrho > 0,
\]

and the equality here is achieved for

\[
u(t) = \cosh \left( \frac{t - a}{b - a} \right), \quad x = b \quad \text{and} \quad u(t) = \cosh \left( \frac{t - b}{b - a} \right), \quad x = a.
\]

Appendix B. Proof of Theorem 9.5

Proof. All potentials \(V\) appearing in the proof are radial \(V(x) = F(|x|)\) and satisfy the following conditions: \(V(x) = 0\), if \(|x| \leq a_0\), and

\[
0 \leq V(x) \leq \frac{\gamma}{|x|^2 \ln^2 |x|}, \quad |x| > a_0 \tag{B.1}
\]

with certain \(a_0, \gamma > 0\). We can take \(\gamma = \frac{1}{3}\) in (i) and \(\gamma = 1/(3\alpha_1)\) in (ii), (iii). According to (28),

\[
N_-(\mathcal{E}_V) = N_-(\mathcal{E}_{\mathcal{R},V}) + N_-(\mathcal{E}_{\mathcal{N},V})
\]
for radial potentials. It follows from the proof of Theorem 7.1 or from [20, Lemma 4.1] that one can choose \( a_0 > 0 \) large enough to get \( N_-(\mathcal{E}_{N,V}) = 0 \). We assume below that \( a_0 > 1 \) and that the last equality holds. Then

\[
N_-(\mathcal{E}_V) = N_-(\mathcal{E}_{X,G}),
\]

where \( G(t) = e^{2t}F(e^t) \) (see (29) and (31)). The usual approximation argument shows that \( N_-(\mathcal{E}_{X,G}) = N_-(\mathcal{E}_{H_1,G}) \), where \( \mathcal{E}_{H_1,G} \) denotes the form (31) with the domain

\[
\mathcal{H}_1 := \left\{ u \in W^1_{2,\text{loc}}(\mathbb{R}) : \int_{\mathbb{R}} |u'|^2 \, dt + \int_{\mathbb{R}} \frac{|u|^2}{1 + |t|^2} \, dt < \infty \right\}.
\]

Note that (B.1) is equivalent to

\[
0 \leq G(t) \leq \frac{\gamma}{|t|^2}, \quad t > a_1 := \ln a_0 > 0.
\]

We assume throughout the proof that \( G(t) = 0 \) for \( t \leq a_1 \).

It follows from the above that it is sufficient to prove the theorem with \( N_-(\mathcal{E}_{H_1,aG}) \) in place of \( N_-(\mathcal{E}_V) \).

Let \( 0 < a < b < \infty \). We denote by \( \mathcal{E}_{H_1(a,b),G} \) and \( \mathcal{E}_{H_1(a,\infty),G} \) the forms defined by (31) on the domains

\[
\mathcal{H}_1(a,b) := \{ u \in \mathcal{H}_1 : u(a) = 0 = u(b) \}, \quad \mathcal{W}^1_2((a,b)),
\]

and

\[
\mathcal{H}_1(b,\infty) := \{ u \in \mathcal{H}_1 : u(t) = 0, \ t \leq b \},
\]

respectively, and we also use the following notation:

\[
\mathcal{E}^a_{\beta}[u] := \int_a^b |u'(t)|^2 \, dt - \beta \int_a^b \frac{|u(t)|^2}{t^2} \, dt, \quad \text{Dom}(\mathcal{E}^a_{\beta}) = \mathcal{W}^1_2((a,b)). \tag{B.2}
\]

(i) Let \( G(t) = 1/(3t^2) \) for \( a_1 < t \leq a_2 \) and \( G(t) = 1/(4t^2) \) for \( t > a_2 \), where \( a_2 \) is chosen in such a way that

\[
\ln \frac{a_2}{a_1} = \sqrt{3} \left( \frac{2N - \frac{3}{2}}{\pi} \right), \quad \text{that is} \quad \frac{1}{3} = \frac{1}{4} + \left( \frac{N - 3/4}{\ln(a_2/a_1)} \right)^2.
\]

Let \( \alpha \in (3/4,1) \). We will need two auxiliary functions \( u_1 \) and \( u_2 \) that solve the equation \( u'' + aGu = 0 \) on \((-\infty,a_1), (a_1,a_2) \) and \((a_2,\infty)\). On each of these intervals, \( G \) has the form \( \text{const}/t^2 \), and the change of the independent variable \( s = \ln t \) reduces the equation to an ODE with constant coefficients. Let

\[
\mu_1 := \sqrt{\frac{\alpha}{3}} - \frac{1}{4}, \quad \mu_2 := \frac{1 - \sqrt{1 - \alpha}}{2},
\]

\[
u_1(t) := \begin{cases} \rho a_1^{1/2} \sin(\mu_1 \ln a_1 + \delta), & t < a_1, \\ \rho t^{1/2} \sin(\mu_1 \ln t + \delta), & a_1 < t < a_2, \\ \rho t^{1/2} \sin(\mu_1 \ln t + \delta), & t > a_2, \end{cases}
\]

\[
u_2(t) := \begin{cases} 1, & t < a_1, \\ 1^{1/2} \sin(\mu_1 (\ln t - \ln a_1)), & a_1 < t < a_2, \\ 1^{1/2} \sin(\mu_1 (\ln t - \ln a_1)), & t > a_2, \end{cases}
\]

where the constants \( \rho, \delta \in \mathbb{R} \) are chosen in such a way that \( u_1(a_2 - 0) = u_1(a_2 + 0), u'_1(a_2 - 0) = u'_1(a_2 + 0) \). It is easy to see that \( u_1 \in \mathcal{H}_1 \) and \( u_2 \in \mathcal{H}_1(a_1,\infty) \) are indeed solutions on the above intervals and that \( u_1(a_1 - 0) = u_1(a_1 + 0), u_2(a_2 - 0) = u_2(a_2 + 0) \). Integration by
parts gives the following:
\[
\mathcal{E}_{\mathcal{H}_1,\alpha G}(u_1, u) = 0, \quad \forall u \in \mathcal{H}_1(a_1, \infty), \quad \text{in particular, } \mathcal{E}_{\mathcal{H}_1,\alpha G}(u_1, u_2) = 0,
\]
\[
\mathcal{E}_{\mathcal{H}_1,\alpha G}(u_2, u) = 0, \quad \forall u \in \mathcal{H}_1(a_1, a_2),
\]
\[
\mathcal{E}_{\mathcal{H}_1,\alpha G}[u_1] = u_1(a_1)\left(u'_1(a_1 - 0) - u'_1(a_1 + 0)\right)
= -\rho^2 \sin^2(\mu_1 \ln a_1 + \delta) \left(\frac{1}{2} + \mu_1 \cot(\mu_1 \ln a_1 + \delta)\right),
\]
\[
\mathcal{E}_{\mathcal{H}_1,\alpha G}[u_2] = u_2(a_2)\left(u'_2(a_2 - 0) - u'_2(a_2 + 0)\right)
= \sin^2\left(\mu_1 \ln \frac{a_2}{a_1}\right) \left(\frac{1}{2} - \mu_2 + \mu_1 \cot\left(\mu_1 \ln \frac{a_2}{a_1}\right)\right).
\]
If \(\alpha\) is close to 1, then \(\mu_1 \ln (a_2/a_1)\) is close to
\[
\frac{1}{2\sqrt{3}} \ln \frac{a_2}{a_1} = \left(N - \frac{3}{4}\right) \pi,
\]
and hence \(\mathcal{E}_{\mathcal{H}_1,\alpha G}[u_2] > 0\). The condition
\[
\frac{u'_1(a_2 - 0)}{u_1(a_2 - 0)} = \frac{u'_1(a_2 + 0)}{u_1(a_2 + 0)}
\]
is equivalent to \(\frac{1}{2} + \mu_1 \cot(\mu_1 \ln a_2 + \delta) = \mu_2\). Again, if \(\alpha\) is close to 1, then \(\mu_2 < \frac{1}{2}\) is close to \(\frac{1}{2}\). Hence \(\cot(\mu_1 \ln a_2 + \delta)\) is a small negative number, that is, \(\mu_1 \ln a_2 + \delta = (m + \frac{1}{2})\pi + \epsilon\), where \(\epsilon\) is a small positive number and \(m \in \mathbb{Z}\). Consequently,
\[
\mu_1 \ln a_1 + \delta = (m + 1/2)\pi + \epsilon - \mu_1 \ln \frac{a_2}{a_1}
\]
is close to \((m - N + 1 + \frac{1}{2})\pi + \epsilon\) and \(\mathcal{E}_{\mathcal{H}_1,\alpha G}[u_1] < 0\).
It is easy to see that any \(u \in \mathcal{H}_1\) admits a unique representation
\[
u = d_1 u_1 + d_2 u_2 + u_0, \quad d_1, d_2 \in \mathbb{C}, \quad u_0 \in \mathcal{H}_1(a_1, a_2),
\]
and it follows from the above that
\[
\mathcal{E}_{\mathcal{H}_1,\alpha G}[u] = |d_1|^2 \mathcal{E}_{\mathcal{H}_1,\alpha G}[u_1] + |d_2|^2 \mathcal{E}_{\mathcal{H}_1,\alpha G}[u_2] + \mathcal{E}_{\mathcal{H}_1,\alpha G}[u_0].
\]
Since \(\mathcal{E}_{\mathcal{H}_1,\alpha G}[u_1] < 0\) and \(\mathcal{E}_{\mathcal{H}_1,\alpha G}[u_2] > 0\), we have
\[
N_-(\mathcal{E}_{\mathcal{H}_1,\alpha G}) = 1 + N_-(\mathcal{E}_{\mathcal{H}_1(a_1, a_2),\alpha G})
= 1 + N_-(\mathcal{E}_{\alpha/\sqrt{3},a_2}) + N_-(\mathcal{E}_{\mathcal{H}_1(a_2, \infty),\alpha G})
= 1 + (N - 1) + 0 = N,
\]
where the penultimate equality follows from Lemma B.1 (see below) and Hardy’s inequality (see, for example, [14, Theorem 327]). Hence,
\[
N_-(\mathcal{E}_{\mathcal{H}_1,\alpha G}) = \lim_{\alpha \rightarrow 1^-} N_-(\mathcal{E}_{\mathcal{H}_1,\alpha G}) = N.
\]
On the other hand,
\[
N_-(\mathcal{E}_{\mathcal{H}_1,\alpha G}) \geq N_-(\mathcal{E}_{\mathcal{H}_1(a_2, \infty),\alpha G}) = +\infty \quad \forall \alpha > 1,
\]
where the last equality holds because \(\alpha G > \frac{1}{2}\) on \((a_2, \infty)\) (see, for example, Lemma B.1).
(ii) Take any \(a_0 \in (0, \alpha_1), N_0 = 0\,\ldots\,\text{successively by}
\[
\sqrt{\frac{\alpha_k}{2(\alpha_k + \alpha_{k-1})}} - \frac{1}{4} \ln \frac{a_{k+1}}{a_k} = \left(N_{k-1} + 2 + \frac{1}{2}\right) \pi, \quad k \in \mathbb{N}.
\]
Let
\[
G(t) := \frac{1}{2(\alpha_k + \alpha_{k-1})t^2}, \quad a_k < t < a_{k+1}, \quad k \in \mathbb{N}.
\]
Then
\[
N_-(\mathcal{E}_{H_1,\alpha G}) \leq 2 + N_-(\mathcal{E}_{H_1(a_1, a_{k+1}), \alpha G}) \\
= 2 + N_-(\mathcal{E}_{\alpha G}^{a_1, a_{k+1}}) + N_-(\mathcal{E}_{H_1(a_{k+1}, \infty), \alpha G}) \\
= 2 + N_-(\mathcal{E}_{\alpha G}^{a_1, a_{k+1}})
\]
due to Hardy’s inequality, and
\[
N_-(\mathcal{E}_{H_1, a_{k+1}G}) \geq N_-(\mathcal{E}_{\alpha G}^{a_1, a_{k+1}}) + N_-(\mathcal{E}_{\alpha G}^{a_{k+1}, a_{k+2}}) \\
\geq N_-(\mathcal{E}_{\alpha G}^{a_1, a_{k+1}}) + N_k + 2 \geq N_-(\mathcal{E}_{H_1, a_{k}G}) + N_k
\]
due to Lemma B.1.

(iii) Take \(k_0 \in \mathbb{N}\) such that
\[
\sum_{k \geq k_0} \frac{1 - \alpha_{k+1}}{\alpha_{k+1} - \alpha_k} \leq \frac{1}{8} \quad (B.3)
\]
and set \(\beta_k := \alpha_{k_0+k}, \ k = 0, 1, \ldots,\)
\[
\mu_{1,k,j} := \sqrt{\frac{\beta_k}{2(\beta_j + \beta_{j-1})} - \frac{1}{4}}, \\
\mu_{1,k} := \mu_{1,k,k} = \frac{1}{2} \sqrt{\frac{\beta_k - \beta_{k-1}}{\beta_k + \beta_{k-1}}}, \\
\mu_{2,k} := 1 - \sqrt{1 - 2\beta_k/(\beta_k+1 + \beta_k)} = \frac{1}{2} - \frac{1}{2} \sqrt{\frac{\beta_{k+1} - \beta_k}{\beta_{k+1} + \beta_k}}, \\
\epsilon_k := \frac{8\pi}{9} \frac{1 - \beta_k}{\beta_k - \beta_{k-1}} < \frac{\pi}{9}, \ k \in \mathbb{N}, \ j = 1, \ldots, k.
\]
Define \(a_2, a_3, \ldots\) successively by
\[
\mu_{1,k} \ln \frac{a_{k+1}}{a_k} = \pi + \epsilon_k, \ k \in \mathbb{N}.
\]
Then
\[
\pi + \epsilon_j \leq \mu_{1,k,j} \ln \frac{a_{j+1}}{a_j} < \sqrt{\frac{1}{2(\beta_j + \beta_{j-1})} - \frac{1}{4} \ln \frac{a_{j+1}}{a_j}} \\
= \sqrt{\frac{2 - \beta_j - \beta_{j-1}}{\beta_j - \beta_{j-1}} (\pi + \epsilon_j)} = \sqrt{1 + 2 \frac{1 - \beta_j}{\beta_j - \beta_{j-1}} (\pi + \epsilon_j)} \\
\leq (1 + \frac{1 - \beta_j}{\beta_j - \beta_{j-1}}) (\pi + \epsilon_j) = \pi + \epsilon_j + \frac{1 - \beta_j}{\beta_j - \beta_{j-1}} (\pi + \epsilon_j) \\
+ \epsilon_j \frac{1 - \beta_j}{\beta_j - \beta_{j-1}} < \pi + 2\pi \frac{1 - \beta_j}{\beta_j - \beta_{j-1}}, \ j = 1, \ldots, k. \quad (B.5)
\]
Let
\[
G(t) := \frac{1}{2(\beta_k + \beta_{k-1})t^2}, \ a_k < t < a_{k+1}, \ k \in \mathbb{N}.
\]
Similarly to part (i) of the proof, define

\[
\begin{align*}
    u_{1,k}(t) &= \begin{cases} 
    \rho_{1,1} a_1^{1/2} \sin(\mu_{1,k,1} \ln a_1 + \delta_{1,1}), & t < a_1, \\
    \rho_{1,j} a_j^{1/2} \sin(\mu_{1,k,j} \ln t + \delta_{1,j}), & a_j < t < a_{j+1}, \ j = 1, \ldots, k, \\
    \mu_{2,k} t^{1/2}, & t > a_{k+1},
    \end{cases} \\
    u_{2,k}(t) &= \begin{cases} 
    0, & t < a_1, \\
    t^{1/2} \sin(\mu_{1,k,1}(\ln t - \ln a_1)), & a_1 < t < a_2, \\
    \rho_{2,j} a_j^{1/2} \sin(\mu_{1,k,j} \ln t + \delta_{2,j}), & a_j < t < a_{j+1}, \\
    \rho_{2,k} a_{k+1}^{1/2-\mu_{2,k}} \sin(\mu_{1,k} \ln a_{k+1} + \delta_{2,k}) t^{\mu_{2,k}}, & t > a_{k+1},
    \end{cases}
\end{align*}
\]

where the constants \(\rho_{1,j}, \delta_{l,j} \in \mathbb{R}, l = 1, 2\) are chosen in such a way that

\[
\begin{align*}
    u_{1,k}(a_j - 0) &= u_{1,k}(a_j + 0), \quad u_{1,k}^j(a_j - 0) = u_{1,k}^j(a_j + 0), \quad j = 2, \ldots, k + 1, \\
    u_{2,k}(a_j - 0) &= u_{2,k}(a_j + 0), \quad u_{2,k}^j(a_j - 0) = u_{2,k}^j(a_j + 0), \quad j = 2, \ldots, k.
\end{align*}
\]

If \(k = 1\), one needs to define \(u_{2,k}\) in a slightly different way, which is closer to the definition in part (i):

\[
\begin{align*}
    u_{2,1}(t) &= \begin{cases} 
    0, & t < a_1, \\
    t^{1/2} \sin(\mu_{1,1}(\ln t - \ln a_1)), & a_1 < t < a_2, \\
    a_2^{1/2-\mu_{2,1}} \sin(\mu_{1,1} \ln a_2 / a_1) t^{\mu_{2,1}}, & t > a_2.
    \end{cases}
\end{align*}
\]

It is easy to see that \(u_{1,k} \in \mathcal{H}_1\) and \(u_{2,k} \in \mathcal{H}_1(a_1, \infty)\) are solutions of the equation \(u'' + \beta_k G u = 0\) on the intervals \((-\infty, a_1), (a_j, a_{j+1}),\ j = 1, \ldots, k,\) and \((a_{k+1}, +\infty),\) and that \(u_{1,k}(a_1 - 0) = u_{1,k}(a_1 + 0), u_{2,k}(a_{k+1} - 0) = u_{2,k}(a_{k+1} + 0).\)

Exactly as in part (i), any \(u \in \mathcal{H}_1\) admits a unique representation

\[
u = d_1 u_{1,k} + d_2 u_{2,k} + u_0, \quad d_1, d_2 \in \mathbb{C}, \quad u_0 \in \mathcal{H}_1(a_1, a_{k+1}),
\]

and one has

\[
\begin{align*}
    \mathcal{E}_{\mathcal{H}_1, \beta_k G}[u] &= |d_1|^2 \mathcal{E}_{\mathcal{H}_1, \beta_k G}[u_{1,k}] + |d_2|^2 \mathcal{E}_{\mathcal{H}_1, \beta_k G}[u_{2,k}] + \mathcal{E}_{\mathcal{H}_1, \beta_k G}[u_0], \\
    \mathcal{E}_{\mathcal{H}_1, \beta_k G}[u_{1,k}] &= -\rho_{1,1}^2 \sin^2(\mu_{1,k,1} \ln a_1 + \delta_{1,1}) \left( \frac{1}{2} + \mu_{1,k,1} \cot(\mu_{1,k,1} \ln a_1 + \delta_{1,1}) \right), \\
    \mathcal{E}_{\mathcal{H}_1, \beta_k G}[u_{2,k}] &= \rho_{2,k}^2 \sin^2(\mu_{1,k} \ln a_{k+1} + \delta_{2,k}) \left( \frac{1}{2} - \mu_{2,k} \\
    &\quad + \mu_{1,k} \cot(\mu_{1,k} \ln a_{k+1} + \delta_{2,k}) \right), \quad k > 1, \\
    \mathcal{E}_{\mathcal{H}_1, \beta_k G}[u_{2,1}] &= \sin^2(\mu_{1,1} \ln a_2 / a_1) \left( \frac{1}{2} - \mu_{2,1} + \mu_{1,1} \cot(\mu_{1,1} \ln a_2 / a_1) \right).
\end{align*}
\]

Similarly to part (i), it follows from (B.4) that \(\mathcal{E}_{\mathcal{H}_1, \beta_k G}[u_{2,1}] > 0\). Let us show that \(\mathcal{E}_{\mathcal{H}_1, \beta_k G}[u_{2,k}] > 0\) for \(k > 1\) as well. It follows from (B.3) and (B.5) that \(u_{2,k}\) has exactly one zero in \((a_1, a_2)\) and that

\[
\pi + \epsilon_1 < \mu_{1,k,1} \ln a_2 / a_1 < \pi + 2\pi \frac{1 - \beta_1}{\beta_2 - \beta_0} < \pi + \frac{\pi}{4}.
\]
The condition
\[ \frac{u_{2,k}'(a_2 - 0)}{u_{2,k}'(a_2 - 0)} = \frac{u_{2,k}'(a_2 + 0)}{u_{2,k}'(a_2 + 0)} \]
is equivalent to
\[ \mu_{1,k,1} \cot \left( \mu_{1,k,1} \ln \frac{a_2}{a_1} \right) = \mu_{1,k,2} \cot(\mu_{1,k,2} \ln a_2 + \delta_{2,2}). \]

Since \( \mu_{1,k,1} > \mu_{1,k,2} \), we obtain
\[ \pi < \mu_{1,k,2} \ln a_2 + \delta_{2,2} < \mu_{1,k,1} \ln a_2 < \pi + 2\pi \frac{1 - \beta_1}{\beta_1 - \beta_0} \]
for some \( m \in \mathbb{Z} \). Using (B.3) and (B.5) again, we see that \( u_{2,k} \) has exactly one zero in \((a_2, a_3)\) and that
\[ 2\pi < \mu_{1,k,2} \ln a_3 + \delta_{2,2} < 2\pi + 2\pi \left( \frac{1 - \beta_1}{\beta_1 - \beta_0} + \frac{1 - \beta_2}{\beta_2 - \beta_1} \right) < 2\pi + \frac{\pi}{4}. \]
Continuing the above argument, we show that \( u_{2,k} \) has exactly one zero in each interval \((a_j, a_{j+1})\), \( j = 1, \ldots, k \), and that
\[ k\pi < \mu_{1,k} \ln a_{k+1} + \delta_{2,k} + n\pi < k\pi + 2\pi \sum_{j=1}^{k} \frac{1 - \beta_j}{\beta_j - \beta_{j-1}} < k\pi + \frac{\pi}{4} \]
for some \( n \in \mathbb{Z} \). This inequality implies that \( \mathcal{E}_{H_1, \beta_k G}[u_{2,k}] > 0 \).

Our next task is to show that \( \mathcal{E}_{H_1, \beta_k G}[u_{1,k}] < 0 \). Suppose the contrary: \( \mathcal{E}_{H_1, \beta_k G}[u_{1,k}] \geq 0 \). Then there exists \( \ell \in \mathbb{Z} \) such that
\[ \omega_k := \arccot \left( -\frac{1}{2\mu_{1,k,1}} \right) \leq \mu_{1,k,1} \ln a_1 + \delta_{1,1} + \ell\pi \leq \pi. \]
It follows from (B.3) and (B.5) that
\[ \omega_k + \pi < \mu_{1,k,1} \ln a_2 + \delta_{1,1} + \ell\pi < 2\pi + 2\pi \frac{1 - \beta_1}{\beta_j - \beta_0}, \]
and one shows as above that
\[ \omega_k + \pi < \mu_{1,k,2} \ln a_2 + \delta_{1,2} + m\pi < 2\pi + 2\pi \frac{1 - \beta_1}{\beta_j - \beta_0} \]
for some \( m \in \mathbb{Z} \). Continuing as above, one gets
\[ \omega_k + k\pi < \mu_{1,k} \ln a_{k+1} + \delta_{1,k} + n\pi < (k+1)\pi + 2\pi \sum_{j=1}^{k} \frac{1 - \beta_j}{\beta_j - \beta_{j-1}} < (k+1)\pi + \frac{\pi}{4} \]
for some \( n \in \mathbb{Z} \). Then
\[ \text{either } \frac{1}{2} + \mu_{1,k} \cot(\mu_{1,k} \ln a_{k+1} + \delta_{1,k}) < 0 \]
or
\[ \frac{1}{2} + \mu_{1,k} \cot(\mu_{1,k} \ln a_{k+1} + \delta_{1,k}) > \frac{1}{2}. \]
On the other hand, the condition
\[ \frac{u_{1,k}'(a_{k+1} - 0)}{u_{1,k}'(a_{k+1} - 0)} = \frac{u_{1,k}'(a_{k+1} + 0)}{u_{1,k}'(a_{k+1} + 0)} \]
is equivalent to
\[ \frac{1}{2} + \mu_{1,k} \cot(\mu_{1,k} \ln a_{k+1} + \delta_{1,k}) = \mu_{2,k} \in (0, 1/2). \]
The obtained contradiction shows that $E_{\mathcal{H}_1, \beta G}[u_{1,k}] < 0$. Now it follows from (B.6) that

$$N_-(\mathcal{E}^1_{\mathcal{H}_1, \beta G}) = 1 + N_-(\mathcal{E}^1_{\mathcal{H}_1(a_1, a_{k+1}), \beta G})$$

$$= 1 + N_-(\mathcal{E}^1_{\beta G, a_{k+1}}) + N_-(\mathcal{E}^1_{\mathcal{H}_1(a_{k+1}, \infty), \beta G})$$

$$= 1 + N_-(\mathcal{E}^1_{\beta G, a_{k+1}}),$$

where the last equality follows from Hardy’s inequality. It is easy to see that

$$N_-(\mathcal{E}^1_{\beta G, a_{k+1}}) \geq \sum_{j=1}^{\infty} N_-(\mathcal{E}^1_{\beta G, a_{j+1}}) = k$$

(see (B.5) and Lemma B.1). If $N_-(\mathcal{E}^1_{\beta G, a_{k+1}}) > k$, then there exists $\lambda < 0$ and a non-trivial solution $u \in W^2_2((a_1, a_{k+1}))$ of $u'' + (\beta G + \lambda)u = 0$ that has $k$ zeros in $(a_1, a_{k+1})$ (see, for example, [22, Chapter I, Theorem 3.3] or [33, Theorem 13.2]). Then $u_{2,k}$ has to have at least $k + 1$ zeros in $(a_1, a_{k+1})$ (see [22, Chapter I, Theorem 3.1] or [33, Theorem 13.3]). On the other hand, we have shown that $u_{2,k}$ has exactly $k$ zeros in $(a_1, a_{k+1})$. Hence $N_-(\mathcal{E}^1_{\beta G, a_{k+1}})$ cannot be larger than $k$, that is, $N_-(\mathcal{E}^1_{\beta G, a_{k+1}}) = k$ and $N_-(\mathcal{E}_{\mathcal{H}_1, \beta G}) = k + 1$. Finally,

$$N_-(\mathcal{E}^1_{\mathcal{H}_1, a_{k+1}G}) - N_-(\mathcal{E}^1_{\mathcal{H}_1, a_kG}) = N_-(\mathcal{E}^1_{\mathcal{H}_1, \beta G, a_{k+2}G}) - N_-(\mathcal{E}^1_{\mathcal{H}_1, \beta G, a_{k+1}G})$$

$$= 1 \ \forall k \geq k_0.$$

Lemma B.1. The following equality holds for the form (B.2)

$$N_-(\mathcal{E}^{a,b}_{\beta}) = \begin{cases} 0, & \beta \leq \frac{1}{4} + \left(\frac{\pi}{\ln(b/a)}\right)^2, \\ N, & \frac{1}{4} + \left(\frac{N\pi}{\ln(b/a)}\right)^2 < \beta \leq \frac{1}{4} + \left(\frac{(N+1)\pi}{\ln(b/a)}\right)^2, \ N \in \mathbb{N}. \end{cases}$$

Proof. $\mathcal{E}^{a,b}_{\beta}$ is the quadratic form of the self-adjoint operator $A_{\beta} := -d^2 dt^2 - \beta t^2$ on $L_2([a, b])$ with the domain $W^2_2([a, b]) \cap W^1_2([a, b])$. The spectrum of $A_{\beta}$ is discrete and consists of simple eigenvalues. It follows from Hardy’s inequality that the eigenvalues are positive for $\beta \leq \frac{1}{4}$. As $\beta$ increases, the eigenvalues move continuously (see, for example, [16, Theorem V.4.10]) to the left, and $N_-(\mathcal{E}^{a,b}_{\beta})$ increases by 1 when an eigenvalue crosses 0. This happens for those values of $\beta$ for which 0 is an eigenvalue of $A_{\beta}$, that is, when

$$-u'' - \frac{\beta}{t^2} u = 0, \quad u(a) = 0 = u(b), \quad \left(\beta > \frac{1}{4}\right)$$

has a non-trivial solution. The change of the independent variable $s = \ln t$ reduces this equation to an ODE with constant coefficients, and one finds that the solutions of the original equation that satisfy the condition $u(a) = 0$ are multiples of

$$t^{1/2} \sin \left(\sqrt{\beta - \frac{1}{4}} (\ln t - \ln a)\right).$$

The latter satisfies the condition $u(b) = 0$ if and only if

$$\sqrt{\beta - \frac{1}{4}} \ln \frac{b}{a} = N\pi, \quad \text{i.e.} \quad \beta = \frac{1}{4} + \left(\frac{N\pi}{\ln(b/a)}\right)^2, \ N \in \mathbb{N}. \quad \square$$

Acknowledgement. This work has been strongly influenced by Michael Solomyak, and I am very grateful for his advice. I am also grateful to Alexander Grigor’yan, Ari Laptev and Yuri Safarov for very helpful comments and suggestions. A part of the paper was written during
my stay at Institut Mittag-Leffler. The unique working conditions provided by its Research in Peace programme are gratefully acknowledged.

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Eugene Shargorodsky  
Department of Mathematics  
King’s College London  
Strand, London  
WC2R 2LS  
United Kingdom  
eugene.shargorodsky@kcl.ac.uk