Leray theorems for $l_1$-norms of infinite chains

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The paper is devoted to an adaptation of author's approach [I3] to Leray theorems in bounded cohomology theory to infinite chains. The paper may be considered as a continuation of the paper [I3], but depends on it mostly for the motivation of proofs. Among the main results are a stronger and more general form of Gromov's Vanishing-finiteness theorem and a generalization of the first part of his Cutting-of theorem. The proofs do not depend on any tools specific for the bounded cohomology and $l_1$-homology theory, but use the fact that $l_1$-homology depend only on the fundamental group.

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1. Introduction

Locally, compactly, and star finite families. A family of subsets $\mathcal{U} = \{U_i\}_{i \in I}$ of a set $X$ is a map $i \mapsto U_i \subset X$ from a set $I$ to the set of all subsets of $X$. The family $\mathcal{U}$ is said to be star finite if for every $i \in I$ the intersection $U_i \cap U_j$ is non-empty for only a finite number of $j \in I$. Usually, but not always, only the set $\{ U_i | i \in I \}$ matters, and we write $U \in \mathcal{U}$ instead of "$U = U_i$ for some $i \in I$".

Let $X$ be a topological space and $\mathcal{U} = \{U_i\}_{i \in I}$ be a family of subsets of $X$. The family $\mathcal{U}$ is said to be locally finite if for every $x \in X$ there exists an open neighborhood $V$ of $x$ such that $x \in U$ and $V \cap U_i \neq \emptyset$ for only a finite number of $i \in I$, and compactly finite if for every compact $Z \subset X$ the intersection $Z \cap U_i \neq \emptyset$ for only a finite number of $i \in I$. For locally compact spaces the notions of locally finite and compactly finite families are equivalent. Eventually our assumptions will imply that $X$ is locally compact, but we prefer to be precise about which finiteness condition is used. Gromov [Gro], Löh and Sauer [LS], and Frigerio and Moraschini [FM] call compactly finite families “locally finite”.

Coverings. The family $\mathcal{U}$ is said to be a covering of $X$ if the union $\bigcup_{i \in I} U_i$ is equal to $X$. For a covering $\mathcal{U}$ we will denote by $\mathcal{U}^\cap$ the collection of all non-empty finite intersection of elements of $\mathcal{U}$. A covering $\mathcal{U}$ is said to be open if every $U_i$ is open, and proper if the interiors int $U_i$ form a covering of $X$ and the closures of the sets $U_i$ are compact. Clearly, if there exists a proper covering of $X$, then $X$ is locally compact. It is easy to see that a proper covering is compactly finite if and only if it is star finite.

Locally and compactly finite singular chains and homology. Recall that a singular $n$-simplex in $X$ is a continuous map $\sigma : \Delta^n \rightarrow X$. Let $S_n(X)$ be the set of singular $n$-simplices in $X$. A subset $I \subset S_n(X)$ is said to be locally finite if the family of images $\{ \sigma(\Delta^n) \}_{\sigma \in I}$ is locally finite, and compactly finite if this family is compactly finite. An infinite singular $n$-chain in $X$ is defined as a formal sum

$$c = \sum_{\sigma \in S_n(X)} a_\sigma \sigma$$

with coefficients $a_\sigma \in A$, where $A$ is some abelian group. Let $C_{n}^{\text{inf}}(X, A)$ be the group of such chains. The chain $c$ is said to be locally finite if

$$\mathcal{S}_c = \{ \sigma(\Delta^n) \mid \sigma \in S_n(X), \ a_\sigma \neq 0 \}$$

is locally finite, and compactly finite if $\mathcal{S}_c$ is compactly finite. The groups of locally and compactly finite chains are denoted by $C_{n}^{\text{lf}}(X, A)$ and $C_{n}^{\text{cf}}(X, A)$ respectively. It is easy to see that every locally finite chain is compactly finite. If $c$ is compactly finite chain, then in the usual formula for the boundary $\partial c$ the coefficient in front of each singular simplex is
a finite sum. Therefore the boundaries $\partial c$ of compactly finite chains, and hence of locally finite chains, are well defined. Since a singular simplex has only finite number of faces, the boundary of a locally finite singular chain is locally finite, and the boundary of a compactly finite singular chain is compactly finite. This leads to two types of singular homology groups based on infinite chains, namely the homology groups $H^\text{lf}_*(X, A)$ based on locally finite chains, and the homology groups $H^\text{cf}_*(X, A)$ based on compactly finite chains. From now on we will assume that $A = \mathbb{R}$ and omit the coefficient group.

**The norms of infinite singular chains.** The $l_1$-norm $\| c \|$ of the singular chain (1.1) is

$$\| c \| = \sum_{\sigma \in S_n(X)} | a_\sigma | .$$

It may happen that $\| c \| = \infty$. The $l_1$-norm $\| h \|$ of a homology class $h \in H^\text{cf}_*(X)$ or $h \in H^\text{lf}_*(X)$ is defined as $\| h \| = \inf \| c \|$, where the infimum is taken over all chains $c$ representing the homology class $h$. Again, it may happen that $\| h \| = \infty$.

**Singular $l_1$-homology.** For an integer $n \geq 0$ let $L_n(X)$ be the vector space of infinite singular $n$-chains with real coefficients having finite $l_1$-norm. Such chains are called $l_1$-chains of dimension $n$. The $l_1$-norm turns $L_n(X)$ into a Banach space. The vector space $C_n(X)$ of finite singular $n$-chains in $X$ is dense in $L_n(X)$ and the boundary operator

$$\partial : C_n(X) \rightarrow C_{n-1}(X)$$

extends by continuity to a map $\partial : L_n(X) \rightarrow L_{n-1}(X)$, also called the boundary operator. These boundary operators turn $L_*(X)$ into a chain complex. The homology of is complex are known as $l_1$-homology of $X$ and are denoted by $H^{l_1}_*(X)$. The real vector spaces $H^{l_1}_n(X)$ inherit $l_1$-norms from $L_n(X)$, but in general are not Banach spaces, because non-zero $l_1$-homology classes may have $l_1$-norm equal to 0.

**Acyclicity of subsets.** As in [I3], let us call a topological space $X$ boundedly acyclic if its bounded cohomology are isomorphic to the bounded cohomology of a point. This property is equivalent to $X$ being path connected and its fundamental group being boundedly acyclic in an obvious sense. By a theorem of Sh. Matsumoto and Sh. Morita [MM] the space $X$ is boundedly acyclic if and only if it is path connected and $H^{l_1}_n(X) = 0$ for $n \geq 1$. See also [I3], Theorem 5.1 for a proof. In this paper we are dealing only with homology and will call boundedly acyclic spaces and groups $l_1$-acyclic.

In [I3] a path connected subset $Z$ of $X$ was called weakly boundedly acyclic if the image of the inclusion homomorphism $\pi_1(Z, z) \rightarrow \pi_1(X, z)$ is boundedly acyclic, i.e. is $l_1$-acyclic in our current terminology. The ambient space $X$ was fixed. Now we need a more flexible version of this notion. Suppose that $Z \subset Y \subset X$. The subset $Z$ is said to be weakly $l_1$-acyclic in $Y$ if the image of the homomorphism $\pi_1(Z, z) \rightarrow \pi_1(Y, z)$ is $l_1$-acyclic.
Acyclicity of families and coverings. Let $\mathcal{U}$ be a family of subsets of $X$. It is said to be $l_1$-acyclic if every $U \in \mathcal{U}$ is $l_1$-acyclic. A covering $\mathcal{U}$ of $X$ is said to be $l_1$-acyclic (as a covering) if the family $\mathcal{U}^\cap$ is $l_1$-acyclic.

A family $\mathcal{U}$ is said to be almost $l_1$-acyclic if every $U \in \mathcal{U}$ is $l_1$-acyclic, except, perhaps, of a single exceptional element $U_e \in \mathcal{U}$. A covering $\mathcal{U}$ of $X$ is said to be almost $l_1$-acyclic if the family $\mathcal{U}^\cap$ is almost $l_1$-acyclic and the exceptional set $U_e$ belongs to $\mathcal{U}$.

We need an analogue of weakly boundedly acyclic coverings from [I_3]. Requiring sets $U \in \mathcal{U}$ to be weakly $l_1$-acyclic in $X$ is not sufficient for working with compactly finite chains.

A family $\mathcal{U}$ of subsets of $X$ is said to be weakly $l_1$-acyclic if for every $U \in \mathcal{U}$ a subset $U_+ \subset X$ is given, such that $U \subset U_+$, the set $U$ is weakly $l_1$-acyclic in $U_+$, and the family of subsets $U_+$ is compactly finite. A finite number of subsets $U_+$ can be equal to $X$. A covering $\mathcal{U}$ of $X$ is said to be weakly $l_1$-acyclic if the family $\mathcal{U}^\cap$ is weakly $l_1$-acyclic.

Similarly, a family $\mathcal{U}$ is said to be almost weakly $l_1$-acyclic if subsets $U_+ \subset X$ with the properties listed above are given for every subset $U \in \mathcal{U}$, except, perhaps, of a single exceptional set $U_e \in \mathcal{U}$. A covering $\mathcal{U}$ of $X$ is said to be almost weakly $l_1$-acyclic if the family $\mathcal{U}^\cap$ is almost weakly $l_1$-acyclic and the exceptional set $U_e$ belongs to $\mathcal{U}$.

Infinite chains in simplicial complexes. A simplicial complex $S$ is said to be star finite if each its simplex is contained in only a finite number of simplices. Equivalently, $S$ is star finite if the family of its simplices is star finite. If $\mathcal{U}$ is a family of subsets of $X$, then $\mathcal{U}$ is star finite if and only if the nerve $N_\mathcal{U}$ of $\mathcal{U}$ is star finite.

An infinite $n$-chain of a simplicial complex $S$ is a potentially infinite formal sum of $n$-simplices of $S$ with coefficients in some abelian group $A$. If $S$ is star finite, then the usual formula defines the boundaries $\partial c$ of infinite $n$-chains of $S$. This leads to homology groups $H^\inf_n(S, A)$ based on infinite chains. We will always assume that $A = \mathbb{R}$.

Leray homomorphisms. Suppose that $\mathcal{U}$ is a star finite proper covering of $X$. Let $N_\mathcal{U}$ be the nerve of $\mathcal{U}$. Then there is a canonical Leray homomorphism

$$l_\mathcal{U}: H^\cl_*(X) \to H^\inf_*(N_\mathcal{U}).$$

See Section 3. The first Leray theorem for $H^\cl_*(X)$ is the following theorem.

**Theorem A.** Suppose that $\mathcal{U}$ is a star finite proper covering, and that $\mathcal{U}$ is countable and weakly $l_1$-acyclic. If a compactly finite homology class $h \in H^\cl_*(X)$ belongs to the kernel of the Leray homomorphism $l_\mathcal{U}$, then $\|h\| = 0$.

The second Leray theorem is concerned with almost weakly $l_1$-acyclic coverings.
Theorem B. Suppose that $\mathcal{U}$ is a star finite proper covering, and that $\mathcal{U}$ is countable and almost weakly $l_1$-acyclic. If a compactly finite homology class $h \in H^\text{cf}_n(X)$ belongs to the kernel of the Leray homomorphism $l_\mathcal{U}$, then $\|h\| < \infty$.

See Theorems 4.2 and 4.3 respectively. These results are motivated by Gromov’s Vanishing-Finiteness theorem. See [Gro], Section 4.2. Like Leray theorems of [I3], Theorems A and B are deduced from an abstract Leray theorem, Theorem 2.3. The proofs are elementary and are based on an adaptation of the methods of [I1], [I2], and [I3] to compactly finite chains. The same methods lead to a proof of the Vanishing-Finiteness theorem. See Section 4.

The assumptions of Theorems 4.2 and 4.3 are much weaker than Gromov’s. In the Vanishing-Finiteness theorem the space $X$ is assumed to be a manifold, instead of $l_1$-acyclicity the stronger amenability property is used, and it is assumed that $h \in H^\text{cf}_n(X)$ for some $n$ strictly larger than the dimension of $N_\mathcal{U}$.

Gromov’s proof of the Vanishing-Finiteness theorem was recently reconstructed by R. Frigerio and M. Moraschini [FM]. Their proof is based on Gromov’s theories of multicomplexes and of diffusion of chains, and is far from being elementary. Technical difficulties forced Frigerio and Moraschini to consider only triangulable spaces, although they conjectured that this assumption is superfluous. Theorems 4.6 and 4.7 together imply this conjecture.

Removing subspaces. Let $X$ be a topological space, and let $Y \subset X$ be a closed subset. There exists natural (in an informal sense) chain maps

$$r_\smallsetminus_Y : C^\text{cf}_\ast(X) \longrightarrow C^\text{cf}_\ast(X \smallsetminus Y).$$

See Section 5. The maps $r_\smallsetminus_Y$ depend on many choices, but the maps

$$r_\smallsetminus_Y^\ast : H^\text{cf}_\ast(X) \longrightarrow H^\text{cf}_\ast(X \smallsetminus Y).$$

induced by $r_\smallsetminus_Y$ does not depend on these choices. Suppose now that $Y$ is presented as the union of a family $\mathcal{Z}$ of pair-wise disjoint compact subspaces of $X$. Suppose further that for every $Z \in \mathcal{Z}$ a compact neighborhood $C_Z$ of $Z$ is given, and that the neighborhoods $C_Z$ are pair-wise disjoint. Suppose that every $C_Z$ is Hausdorff and path connected.

Theorem C. Suppose that $\mathcal{Z}$ is countable and the family of sets $C_Z$ is weakly $l_1$-acyclic. Then $\|r_\smallsetminus_Y(h)\| \geq \|h\|$ for every homology class $h \in H^\text{cf}_n(X)$.

See Theorem 5.6. Implicitly this theorem is concerned with the covering of $X$ by $X \smallsetminus Y$ and the sets $C_Z$. Since this covering is very simple, there is no need to involve it or related double complexes explicitly. Theorem C was motivated by Gromov’s Cutting-of theorem from [Gro], Section 4.2, and easily implies its first claim. See Section 5.
2. A Leray theorem for infinite chains

**Generalized chains.** Let $X$ be a topological space. Let $\text{sub}\ X$ be the category having subspaces of $X$ as objects and inclusions $Y \subset Z$ as morphisms. Let $e_\bullet$ be a covariant functor from $\text{sub}\ X$ to augmented chain complexes of modules over a ring $R$. The functor $e_\bullet$ assigns to a subspace $Z \subset X$ a complex

\[(2.1) \quad 0 \leftarrow R \leftarrow e_0(Z) \leftarrow e_1(Z) \leftarrow e_2(Z) \leftarrow e_3(Z) \leftarrow \ldots,
\]

For every $Y \subset Z$ there is an *inclusion morphism* $e_\bullet(Z) \to e_\bullet(Y)$. Elements of $e_q(Z)$ are thought as *generalized $q$-chains* of $Z$.

**The double complex of a covering.** Let $\mathcal{U}$ be a star finite covering of $X$ and let $N$ be its nerve. For $p \geq 0$ let $N_p$ be the set of $p$-dimensional simplices of $N$. For $p, q \geq 0$ let

\[c_p(N, e_q) = \prod_{\sigma \in N_p} e_q(|\sigma|).
\]

So, an element $c \in c_p(N, e_q)$ is a family of *generalized $q$-chains*

\[c : \sigma \mapsto c_\sigma \in e_q(|\sigma|),
\]

where $\sigma \in N_p$, thought as an infinite formal sum

\[c = \sum_{\sigma \in N_p} c_\sigma.
\]

For every $p > 0$ (and sometimes for $p = 0$ also) there is a canonical morphism

\[\delta_p : c_p(N, e_\bullet) \to c_{p-1}(N, e_\bullet),
\]

defined as follows. Let $\sigma \in N_p$. For each face $\partial_i \sigma$ there is an inclusion morphism

\[\Delta_{\sigma,i} : e_\bullet(|\sigma|) \to e_\bullet(|\partial_i \sigma|).
\]

For $c_\sigma \in e_q(|\sigma|)$ we set

\[\delta_p(c_\sigma) : \partial_i \sigma \mapsto (-1)^i \Delta_{\sigma,i}(c_\sigma) \in e_q(|\partial_i \sigma|) \quad \text{and}
\]

\[\delta_p(c_\sigma) : \tau \mapsto 0
\]

if $\tau \neq \partial_i \sigma$ for every $i$. The map $\delta_p$ extends to the direct product $c_p(N, e_q)$ by linearity. In order to see that such an extension to be well defined we need to know that for every
\( \rho \in N_{p-1} \) only a finite number of expressions \((-1)^i \Delta_{\sigma,i}(c_\sigma)\) need to be summed to get the value of \(\delta_p(c_\sigma)\) on \(\rho\). But \((-1)^i \Delta_{\sigma,i}(c_\sigma)\) enters this sum only if \(\rho = \partial_i \sigma\) for some \(\sigma \in N_p\) and some \(i\). Since the covering \(\mathcal{U}\) is star finite, its nerve \(N\) is also star finite, and hence there is indeed only a finite number of such \(\rho, i\). Therefore \(\delta_p\) is indeed well defined.

As usual, we agree that \(|\emptyset| = X\), but this argument does not work for \(p = 0\) and \(\rho = \emptyset\).

In order to define \(\delta_0\) one needs to be able to speak about infinite generalized chains.

The fact that each \(\Delta_{\sigma,i}\) are morphisms of complexes implies that \(\delta_p\) is a morphism also of complexes. The \textit{double complex} \(c_\ast(N, e_\ast)\) of the covering \(\mathcal{U}\) is the double complex

\[
\begin{align*}
 c_0(N, e_0) & \leftarrow c_0(N, e_1) \leftarrow \cdots \\
 c_1(N, e_0) & \leftarrow c_1(N, e_1) \leftarrow \cdots \\
 c_2(N, e_0) & \leftarrow c_2(N, e_1) \leftarrow \cdots \\
 \cdots & \quad \cdots
\end{align*}
\]

(2.2)

where the horizontal arrows are the products of the maps \(d_i\) and the vertical arrows are the maps \(\delta_i\). Let \(t_\ast(N, e)\) be the total complex of \(c_\ast(N, e_\ast)\). Let \(C^\ast_{\text{inf}}(N) = C^\ast_{\text{inf}}(S, R)\) be the complex of \textit{infinite} simplicial chains of \(N\) with coefficients in \(R\). It is well defined because \(N\) is star finite. Let \(H^\ast_{\text{inf}}(N) = H^\ast_{\text{inf}}(N, R)\) be the homology of this complex. Since \(\delta_0\) is not defined, we will replace \(e_\ast(X)\) by the cokernel \(e^\ast_{\text{lf}}(X, \mathcal{U})\) of the homomorphism

\[ \delta_1 : c_1(N, e_\ast) \rightarrow c_0(N, e_\ast). \]

Then we can define \(\delta_0\) as the canonical map \(c_0(N, e_\ast) \rightarrow e^\ast_{\text{lf}}(X, \mathcal{U})\). Since (2.2) is commutative, the maps \(d_i\) induce canonical maps

\[ d_i : e^\ast_{\text{lf}}(X, \mathcal{U}) \rightarrow e^\ast_{\text{lf}}(X, \mathcal{U}) \]

turning \(e^\ast_{\text{lf}}(X, \mathcal{U})\) into a complex. Let \(\bar{H}^\ast_{\text{lf}}(X, \mathcal{U})\) be the homology of this complex. The boundary maps \(d_0\) and \(\delta_1\) lead to morphisms

\[ \lambda_e : t_\ast(N, e) \rightarrow C^\ast_{\text{inf}}(N) \quad \text{and} \quad \tau_e : t_\ast(N, e) \rightarrow e^\ast_{\text{lf}}(X, \mathcal{U}), \]

where it is understood that the augmentation term is removed from \(e^\ast_{\text{lf}}(X, \mathcal{U})\).
Acyclic coverings. Clearly, \( \mathcal{U}^\cap \) is the collection of all sets of the form \(|\sigma|\) with \(\sigma \neq \emptyset\). The covering \(\mathcal{U}\) is said to be \(e_\cdot\)-acyclic if \(e_\cdot(Z)\) is exact for every \(Z \in \mathcal{U}^\cap\).

2.1. Lemma. If \(\mathcal{U}\) is star finite and \(e_\cdot\)-acyclic, then \(\lambda_e: t_\cdot(N, e) \to C_\cdot^\inf(N)\) induces an isomorphism of homology groups.

Proof. If \(\mathcal{U}\) is \(e_\cdot\)-acyclic, then for every simplex \(\sigma \neq \emptyset\) the complex (2.1) is exact. Since the term-wise products of exact sequences are exact, this implies that every row of the double complex (2.2) is exact and \(d_0\) induces an isomorphism of the complex \(C_\cdot^\inf(N)\) with the kernel of the morphism of complexes \(d_1: c_\cdot(N, e_1) \to c_\cdot(N, e_0)\). It remains to apply a well known theorem about double complexes. See Theorem A.2 in [I3].

Infinite singular chains. Suppose that a space \(\Delta\) is fixed and maps \(s: \Delta \to Y\) are treated as singular simplices. A finite singular chain is a finite formal sum of singular simplices with coefficients in \(R\). The \(R\)-module of finite singular chains in \(Y\) is denoted by \(c^f(Y)\). An infinite singular chain is a finite or infinite formal sum of singular simplices with coefficients in \(R\), and the \(R\)-module of infinite singular chains in \(Y\) is denoted by \(c^\inf(Y)\).

Let us turn to singular chains in \(X\) and subsets of \(X\). A singular simplex \(s\) in \(X\) is called small if \(s(\Delta) \subset U\) for some \(U \in \mathcal{U}\), and an infinite singular chain in \(X\) is called small if all its singular simplices with non-zero coefficients are small. Suppose that for every \(U \in \mathcal{U}\) a finite singular chain \(\gamma_U \in c(U)\) is given. Then, since \(\mathcal{U}\) is a star finite, the sum

\[
\gamma = \sum_{U \in \mathcal{U}} \gamma_U
\]

is a well defined infinite chain. Clearly, \(\gamma\) is small. If \(\mathcal{U}\) is an open covering, then \(\gamma\) is locally finite. When a chain \(\gamma\) can be represented by such sum, we say that \(\gamma\) is \(\mathcal{U}\)-finite. Let \(c^f(X, \mathcal{U})\) be the \(R\)-module of \(\mathcal{U}\)-finite chains. Let us consider now the modules

\[
c_\cdot(N, c) = \prod_{\sigma \in N} c(|\sigma|),
\]

where \(p \geq -1\). The maps \(\delta_p: c_\cdot(N, c) \to c_{\cdot-1}(N, c), p > 0,\) are defined as before. Moreover, now we can define \(\delta_0\) in the same way, except that now the target of \(\delta_0\) is \(c^\inf(X)\), not \(c(X)\). Clearly, \(c^f(X, \mathcal{U})\) is equal to the image of

\[
\delta_0: c_0(N, c) \to c^\inf(X),
\]

and \(\delta_0\) is the composition of the inclusion \(c^f(X, \mathcal{U}) \to c^\inf(X)\) with a canonical map

\[
\overline{\delta}_0: c_0(N, c) \to c^f(X, \mathcal{U}).
\]

Clearly, \(\overline{\delta}_0\) is surjective. As usual, \(\delta_{\cdot-1} \circ \delta_\cdot = 0\) for every \(p \geq 1\).
2.2. Lemma. If \( \mathcal{U} \) is star finite, then the following sequence is exact:

\[
\begin{array}{cccccc}
0 & \leftarrow & c_{\text{lf}}(X, \mathcal{U}) & \leftarrow & c_0(N, c) & \leftarrow & c_1(N, c) & \leftarrow & \cdots \\
& & \delta_0 & & \delta_1 & & \delta_2 & &
\end{array}
\]

Proof. It is sufficient to construct a contracting chain homotopy

\[
k_0: c_{\text{lf}}(X, \mathcal{U}) \rightarrow c_0(N, c),
\]

\[
k_p: c_p(N, c) \rightarrow c_{p+1}(N, c),
\]

where \( p \geq 0 \). The construction is almost the same as in the case of direct sums (instead of products). Cf. [I3], Lemma 3.1. For every small singular simplex \( s \) let us choose a subset \( U_s \in \mathcal{U} \) such that \( s(\Delta) \subset U_s \) and let \( u_s \) be the corresponding vertex of \( N \). If \( s(\Delta) \subset |\sigma| \) for some \( \sigma \in N_p \), then \( s \ast \sigma \) denotes \( s \) considered as an element of \( c(|\sigma|) \).

Let us define \( k_p \) on the chains of the form \( s \ast \sigma \) first. Suppose that \( \sigma \in N_p \) and \( s \) be a singular \( q \)-simplex such that \( s(\Delta^q) \subset |\sigma| \). Let \( \rho = \sigma \cup \{u_s\} \). Then \( s(\Delta) \subset |\sigma| \cap U_s = |\rho| \) and, in particular, \( \rho \) is a simplex. If \( u_s \in \sigma \), then \( \rho \) is a \( p \)-simplex. Otherwise, \( \rho \) is a \((p+1)\)-simplex and \( \sigma = \partial a \rho \) for some \( a \). Let

\[
k_p(s \ast \sigma) = 0 \quad \text{if} \quad u_s \in \sigma,
\]

\[
k_p(s \ast \sigma) = (-1)^a s \ast \rho \in c(|\rho|) \quad \text{if} \quad u_s \not\in \sigma.
\]

As in the case of \( \delta_p \), the star finiteness of \( \mathcal{U} \) and \( N \) allows to extend \( k_p \) to \( c_p(N, c) \) by linearity. In order to verify that \( k_* \) is a contracting homotopy it is sufficient to check that

\[
\delta_{p+1}(k_p(y)) + k_{p-1}(\delta_p(y)) = y
\]

when \( y \) has the form \( y = s \ast \sigma \). But this case is exactly the same as for direct sums. ■

Classical singular chains. The above discussion applies, in particular, to the case \( \Delta = \Delta^q \), the standard geometric \( q \)-simplex. In this case we will denote \( c_p(N, c) \) and \( c_{\text{lf}}(X, \mathcal{U}) \) by

\[
C_{\text{inf}}^p(N, C_q) \quad \text{and} \quad C_{\text{lf}}^q(X, \mathcal{U})
\]

respectively. The boundary maps \( d_i \) turn \( C_{\ast}^i(X, \mathcal{U}) \) into a complex. Let \( H_{\ast}^i(X, \mathcal{U}) \) be the homology of this complex. The morphisms

\[
C_{\ast}^i(N) \xleftarrow{\lambda_C} l_{\ast}(N, C) \xrightarrow{\tau_C} C_{\ast}^i(X, \mathcal{U})
\]
lead to homomorphisms of cohomology groups,
\[ H^\inf(N) \xleftarrow{\lambda_{C*}} H_*(N, C) \xrightarrow{\tau_{C*}} H^lf(X, U) \]
where \( H_*(N, C) \) is the homology of \( t_*(N, C) \). If \( U \) is star finite, Lemma 2.2 implies that the columns of (2.2) are exact. Together with the already used theorem about double complexes this implies that \( \tau_{C*} \) is an isomorphism. This leads to the canonical homomorphism
\[ (2.3) \quad \lambda_{C*} \circ \tau_{C*}^{-1} : H^lf(X, U) \longrightarrow H^\inf(N) . \]
In general, one cannot replace here \( H^lf(X, U) \) by some homology independent of \( U \).

**Comparing classical and generalized chains.** Suppose that the functor \( e_* \) is equipped with a natural transformation \( \varphi_* : C_* \longrightarrow e_* \). Then \( \varphi \) induces a map \( H_*(Y) \longrightarrow \tilde{H}_*(Y) \) for every \( Y \in U \), where \( \tilde{H}_*(Y) \) is the homology of the complex \( e_*(Y) \). In particular, \( \varphi \) induces a map \( H_*(X) \longrightarrow \tilde{H}_*(X) \), but this is not what we are interested in now. Lemma 2.2 implies that the complex \( C^lf(X, U) \) is canonically isomorphic to the cokernel of
\[ \delta_1 : C^inf_1(N, C_*) \longrightarrow C^inf_0(N, C_*) . \]
In view of the definition of \( e^lf_*(X, U) \) this leads to a canonical homomorphism
\[ \varphi_* : C^lf_*(X, U) \longrightarrow e^lf_*(X, U) \]
and hence to a *comparison homomorphism*
\[ (2.4) \quad \varphi_* : H^lf_*(X, U) \longrightarrow \tilde{H}^lf_*(X, U) \]
in homology groups. This is the map we are interested in.

**2.3. Theorem.** If \( U \) is a star finite \( e_* \)-acyclic covering, then the comparison homomorphism (2.4) can be factored through the canonical homomorphism (2.3).

**Proof.** The natural transformation \( \varphi_* \) defines homomorphisms
\[ \varphi_q(|\sigma|) : C_q(|\sigma|) \longrightarrow e_q(|\sigma|) , \]
which, in turn, lead to a morphism
\[ \varphi_* : C_*(N, C_*) \longrightarrow C_*(N, e_*) \]
of double complexes. In turn, \( \varphi_* \) leads to a morphism \( \Phi_* : C_*(N, C) \longrightarrow t_*(N, e) \) of
total complexes. Clearly, the diagram

\[
\begin{array}{ccc}
C^\text{inf}_* (N) & \xleftarrow{\lambda_C} & t_* (N, C) & \xrightarrow{\tau_C} & C^\text{lf}_* (X, \mathcal{U}) \\
\downarrow & & \downarrow \Phi_* & & \downarrow \varphi_* \\
C^\text{inf}_* (N) & \xleftarrow{\lambda_e} & t_* (N, e) & \xrightarrow{\tau_e} & e^\text{lf}_* (X, \mathcal{U}) ,
\end{array}
\]

is commutative and leads to the following commutative diagram of homology groups

\[
\begin{array}{ccc}
H^\text{inf}_* (N) & \xleftarrow{\lambda^*_{C*}} & H_* (N, C) & \xrightarrow{\tau^*_{C*}} & H^\text{lf}_* (X, \mathcal{U}) \\
\downarrow & & \downarrow \Phi_* & & \downarrow \varphi_* \\
H^\text{inf}_* (N) & \xleftarrow{\lambda^*_{e*}} & H_* (N, e) & \xrightarrow{\tau^*_{e*}} & \tilde{H}^\text{lf}_* (X, \mathcal{U}) ,
\end{array}
\]

where \( H_* (N, e) \) denotes the cohomology of the total complex \( t_* (N, e) \).

The red arrows are isomorphisms. Indeed, since the covering \( \mathcal{U} \) is \( e_* \)-acyclic, \( \lambda^*_{e*} \) is an isomorphism by Lemma 2.1, and Lemma 2.2 implies that \( \tau^*_{C*} \) is always an isomorphism, as we already pointed out. By inverting these two arrows we get the commutative diagram

\[
\begin{array}{ccc}
H^\text{inf}_* (N) & \xrightarrow{=} & H_* (N, C) & \xleftarrow{=} & H^\text{lf}_* (X, \mathcal{U}) \\
\downarrow & & \downarrow \Phi_* & & \downarrow \varphi_* \\
H^\text{inf}_* (N) & \rightarrow & H_* (N, e) & \rightarrow & \tilde{H}^\text{lf}_* (X, \mathcal{U}) .
\end{array}
\]

It follows that \( H^\text{lf}_* (X, \mathcal{U}) \rightarrow \tilde{H}^\text{lf}_* (X, \mathcal{U}) \) factors through the canonical homomorphism

\[
H^\text{lf}_* (X, \mathcal{U}) \rightarrow H_* (N, C) \rightarrow H^\text{inf}_* (N) .
\]

The theorem follows. \( \blacksquare \)
3. Compactly finite and \textit{l}$_1$-homology

\textbf{Singular \textit{l}$_1$-chains.} Recall that for a topological space \( Y \) we denote by \( L_q(Y) \) the real vector space of infinite singular \( q \)-chains in \( Y \) having finite \( l_1 \)-norm. These chains are called \textit{l}$_1$-\textit{chains} of dimension \( q \). There are obvious inclusions \( C_q(Y) \subset L_q(Y) \). The boundary maps in \( C_*(Y) \) extend by continuity to \( L_*(Y) \), turning \( L_*(Y) \) into a complex. Its homology are denoted by \( H^{l_1}_*(Y) \). In this section we will apply the theory of Section 2 to \( e_* = L_* \).

Let \( \mathcal{U} \) be a star finite covering of \( X \). The complex \( L^\text{lf}_*(X, \mathcal{U}) \) admits a description similar to the definition of \( C^\text{lf}_*(X, \mathcal{U}) \). Namely, suppose that for every \( U \in \mathcal{U} \) an \textit{l}$_1$-chain \( \gamma_U \in L_q(U) \) is given. Then, since \( \mathcal{U} \) is a star finite, the sum

\[
Y = \sum_{U \in \mathcal{U}} \gamma_U
\]

is a well defined infinite chain. Let \( \mathcal{L}^\text{lf}_q(X, \mathcal{U}) \) be the vector space of such chains.

The inclusions \( L_q(U) \rightarrow C^{\text{inf}}_q(X) \) lead to a map

\[
\delta_0 : C_0^{\text{inf}}(N, L_q) \rightarrow C^{\text{inf}}_q(X)
\]

having \( \mathcal{L}^\text{lf}_q(X, \mathcal{U}) \) as the image. Let

\[
\overline{\delta}_0 : C_0^{\text{inf}}(N, L_q) \rightarrow \mathcal{L}^\text{lf}_q(X, \mathcal{U})
\]

be the map resulting from changing the target of \( \delta_0 \).

\textbf{3.1. Lemma.} The following sequence is exact:

\[
0 \leftarrow \mathcal{L}^\text{lf}_q(X, \mathcal{U}) \leftarrow \overline{\delta}_0 C_0^{\text{inf}}(N, L_q) \leftarrow \delta_1 C_1^{\text{inf}}(N, L_q) \leftarrow \delta_2 \ldots.
\]

\textbf{Proof.} The proof is completely similar to the proof of Lemma 2.2. On the chains of the form \( s \ast \sigma \) the chain homotopy \( k_p \) is defined as before. The fact that \( \mathcal{U} \) is star finite ensures that this definition extends to \textit{l}$_1$-chains. The homotopy identity holds on the chains of the form \( s \ast \sigma \) by the same reason as before and hence holds on all \textit{l}$_1$-chains. \( \square \)

\textbf{3.2. Corollary.} The map \( \overline{\delta}_0 \) induces an isomorphism \( L_*^{\text{lf}}(X, \mathcal{U}) \rightarrow \mathcal{L}^\text{lf}_q(X, \mathcal{U}) \). \( \square \)

\textbf{3.3. Lemma.} If \( \mathcal{U} \) is a star finite proper covering, then a small chain is \( \mathcal{U} \)-finite if and only if it is compactly finite.
Proof. Let us prove the “if” part first. Given a small chain $\gamma$, let us write it as a formal sum

$$\gamma = \sum_{i \in I} a_i \sigma_i$$

of small simplices $\sigma_i$ with coefficients $a_i \neq 0$. We may assume that $\sigma_i \neq \sigma_j$ if $i \neq j$. For every $i \in I$ let us choose $U(i) \in \mathcal{U}$ such that $\sigma_i$ is a simplex in $U(i)$. For $U \in \mathcal{U}$ let

$$\gamma_U = \sum_{U(i) = U} a_i \sigma_i.$$ 

Clearly, $\gamma$ is equal to the sum of chains $\gamma_U$. If $\gamma$ is a compactly finite chain, then every $\gamma_U$ is a finite chain, and hence $\gamma$ is $\mathcal{U}$-finite.

Let us prove the “only if” part. If $\gamma$ is a $\mathcal{U}$-finite chain, then $\gamma = \sum_{U \in \mathcal{U}} \gamma_U$, where each $\gamma_U$ is a finite chain in $U$. If $Z \subset X$ is compact, then $Z$ is contained in the union of finitely many sets $U \in \mathcal{U}$. Since $\mathcal{U}$ is star finite, this implies that $Z$ intersects only finitely many sets $U \in \mathcal{U}$. Since each $\gamma_U$ is a finite chain, it follows that only finitely many simplices entering $\gamma$ with non-zero coefficients intersect $Z$. Hence $\gamma$ is compactly finite.

3.4. Lemma. If $\mathcal{U}$ is a star finite proper covering, then the map \( H_*^{lf}(X, \mathcal{U}) \to H_*^{cf}(X) \) induced by the inclusion \( C_*^{lf}(X, \mathcal{U}) \to C_*^{cf}(X) \) is an isomorphism.

Proof. By Lemma 3.3 the complex $C_*^{lf}(X, \mathcal{U})$ is equal to the subcomplex of small chains of the complex $C_*^{cf}(X)$. Hence the lemma is an analogue for compactly finite chains of a classical theorem of Eilenberg [E] about finite chains. Eilenberg’s proof is presented in many textbooks (see, for example, [Sp], Theorem 4.4.14), although is rarely attributed to Eilenberg. Eilenberg’s proof applies to our situation without any changes.

Comparing compactly finite and $l_1$-chains. As in Section 2, let $\widetilde{H}_*^{lf}(X, \mathcal{U})$ be the homology of the complex $L_*^{lf}(X, \mathcal{U})$. There is a canonical comparison homomorphism

$$\varphi_* : H_*^{lf}(X, \mathcal{U}) \to \widetilde{H}_*^{lf}(X, \mathcal{U}).$$

If the assumptions of Lemmas 3.3 and 3.4 hold, we can interpret $\varphi_*$ as a homomorphism

$$\varphi_* : H_*^{cf}(X) \to \widetilde{H}_*^{lf}(X, \mathcal{U})$$

and the canonical homomorphisms (2.3) as a homomorphism $l_\mathcal{U} : H_*^{cf}(X) \to H_*^{inf}(N)$.

3.5. Lemma. Suppose that $\mathcal{U}$ is a star finite proper covering. If $\mathcal{U}$ is $l_1$-acyclic, then the comparison homomorphisms (3.1) and (3.2) can be factored through $l_\mathcal{U}$. 

Proof. Since $L_*^{inf}$-acyclicity is the same as $l_1$-acyclicity, this follows from Theorem 2.3.
3.6. Theorem. Suppose that \( \mathcal{U} \) is a star finite proper covering, and that \( \mathcal{U} \) is countable and \( l_1 \)-acyclic. If a homology class \( h \in H^c_n(X) \) belongs to the kernel of \( l_{\mathcal{U}} \), then \( \| h \| = 0 \).

Proof. Suppose that \( h \in H^c_n(X) \) belongs to the kernel of \( l_{\mathcal{U}} \). By Lemma 3.4 the homology class \( h \) can be represented by a cycle \( \gamma \in C^c_n(X, \mathcal{U}) \). Lemma 3.5 implies that

\[
\varphi_* (h) = 0 \in \tilde{H}^c_n(X, \mathcal{U}) .
\]

By Corollary 3.2 the map \( \delta_0 \) induces an isomorphism between \( \tilde{H}^c_n(X, \mathcal{U}) \) and the homology of the complex \( L^c_n(X, \mathcal{U}) \). It follows that the inclusion \( C^c_n(X, \mathcal{U}) \rightarrow L^c_n(X, \mathcal{U}) \) takes the cycle \( \gamma \) representing \( h \) to a boundary. In other terms,

\[
\gamma = \partial \beta
\]

for some \( \beta \in L^c_{n+1}(X, \mathcal{U}) \). By the definition of \( L^c_{n+1}(X, \mathcal{U}) \),

\[
\beta = \sum_{U \in \mathcal{U}} \beta_U
\]

for some chains \( \beta_U \in L_{n+1}(U) \). Let us choose an arbitrary \( \varepsilon > 0 \). Since \( \mathcal{U} \) is countable, there exists a family of real numbers \( \varepsilon_U > 0, U \in \mathcal{U} \), such that

\[
\sum_{U \in \mathcal{U}} \varepsilon_U = \varepsilon .
\]

Since \( \| \beta_U \| < \infty \) for every \( U \), every chain \( \beta_U \) can be presented as a sum \( \beta_U = \alpha_U + \omega_U \) of two chains \( \alpha_U, \omega_U \in L_{n+1}(U) \) such that \( \alpha_U \) is finite and \( \| \omega_U \| < \varepsilon_U \). Let

\[
\alpha = \sum_{U \in \mathcal{U}} \alpha_U \quad \text{and} \quad \omega = \sum_{U \in \mathcal{U}} \omega_U .
\]

Then \( \alpha \in C^c_{n+1}(X, \mathcal{U}) \subset C^c_n(X) \) and \( \omega \in L^c_{n+1}(X, \mathcal{U}) \). Now, \( \gamma = \partial \beta \) implies that

\[
\gamma = \partial \alpha + \partial \omega
\]

and hence \( \gamma - \partial \alpha = \partial \omega \). Therefore

\[
\| \gamma - \partial \alpha \| = \| \partial \omega \| \leq (n + 1) \| \omega \|
\]

\[
\leq (n + 1) \sum_{U \in \mathcal{U}} \| \omega_U \|
\]

\[
< (n + 1) \sum_{U \in \mathcal{U}} \varepsilon_U = (n + 1) \varepsilon .
\]

Hence \( h \) can be represented by chains with arbitrarily small norm, i.e. \( \| h \| = 0 \). ■
3.7. Theorem. Suppose that \( \mathcal{U} \) is a star finite proper covering, and that \( \mathcal{U} \) is countable and almost \( l_1 \)-acyclic. If \( h \in H_*^{\text{lf}}(X) \) belongs to the kernel of \( l_{\mathcal{U}} \), then \( \| h \| < \infty \).

Proof. Let \( U_e \in \mathcal{U} \) be the exceptional set, the one which is allowed not to be \( l_1 \)-acyclic. By Lemma 3.4 the homology class \( h \) can be considered as an element of \( H_n^{\text{lf}}(X, \mathcal{U}) \) and represented by a chain \( \gamma \in C_n^{\text{lf}}(X, \mathcal{U}) \) for some \( n \). Let us consider the commutative diagram

\[
\begin{array}{ccc}
H_*^{\text{inf}}(N) & \xrightarrow{\lambda_{C_*}} & H_*(N, C) \\
\downarrow & & \Phi_* \\
H_*^{\text{inf}}(N) & \xrightarrow{\lambda_{L_*}} & H_*(N, L)
\end{array}
\]

similar to the diagrams used in the proof of Theorem 2.3. Since \( \mathcal{U} \) is not assumed to be \( l_1 \)-acyclic, the homomorphism \( \lambda_{L_*} \) may be not invertible. But since \( l_{\mathcal{U}}(h) = 0 \),

\[
\tilde{h} = \Phi_* \left( \tau_{C_*}^{-1}(h) \right) \in H_n(N, L)
\]

belongs to the kernel of \( \lambda_{L_*} \). Hence Lemma A.1 implies that \( \tilde{h} \) belongs to the image of

\[
H_n(C_{0}^{\text{inf}}(N, L_*)) \to H_n(N, L)
\]

Since among sets in \( \mathcal{U} \) only \( U_e \) can be not \( l_1 \)-acyclic, \( \tilde{h} \) belongs to the image of the homology of the summand \( L_*(U_e) \) of \( C_{0}^{\text{inf}}(N, L_*) \). It follows that there exists an \( l_1 \)-cycle

\[
\gamma' \in L_n(U_e) \subset C_{0}^{\text{inf}}(N, L_n)
\]

such that \( \tilde{h} \) is equal to the \( l_1 \)-homology class of \( \gamma' \) and hence \( \tau_{L_*}(\tilde{h}) \) is equal to the homology class of the cycle \( \gamma' \) considered as an element of \( L_*^{\text{lf}}(X, \mathcal{U}) \).

Now the commutativity of the right square of the above diagram implies that \( \varphi_*(h) \) is equal to the homology class \( h' \) of \( \gamma' \). It follows that \( \gamma - \gamma' \) is a boundary in \( L_*^{\text{lf}}(X, \mathcal{U}) \). As in the proof of Theorem 3.6, for every \( \epsilon > 0 \) there exist chains \( \alpha \in C_{n+1}^{\text{lf}}(X, \mathcal{U}) \) and \( \omega \in L_{n+1}^{\text{lf}}(X, \mathcal{U}) \) such that \( \| \omega \| < \epsilon \) and

\[
\gamma - \gamma' = \partial \alpha + \partial \omega,
\]

i.e. \( \gamma - \partial \alpha = \gamma' + \partial \omega \). The cycle \( \gamma - \partial \alpha \) is \( \mathcal{U} \)-finite and represents \( h \). On the other hand \( \| \gamma' \| < \infty \) and \( \| \omega \| < \epsilon \), and hence \( \| \gamma - \partial \alpha \| < \infty \). Therefore \( \| h \| < \infty \). \( \square \)
4. Extensions of coverings and $l_1$-homology

**Extensions of coverings.** Let $\mathcal{U}$ be a covering of $X$ and $X'$ be a space containing $X$. Recall (see [I$_3$], Section 4) that an extension of $\mathcal{U}$ to $X'$ is a map $U \to U'$ assigning to every $U \in \mathcal{U}$ a subset $U' \subset X'$ such that $U' \cap X = U$ in such a way that the collection $\mathcal{U}'$ of the sets $U'$ is a covering of $X'$. The set $\mathcal{U}'$ uniquely determines the extension, i.e. the map $U \to U'$ and is identified with it. There is an obvious simplicial map $N\mathcal{U} \to N\mathcal{U}'$.

The extension $\mathcal{U}'$ is said to be nerve-preserving if this map is a simplicial isomorphism, which is then treated as the identity. If $\sigma$ is a simplex of $N\mathcal{U}$, then $|\sigma'|$ denotes the intersection of sets $U'$ corresponding to the vertices of $\sigma$. Clearly, $|\sigma| \to |\sigma'|$ is a map $U \cap \to U' \cap$.

Suppose that $\mathcal{U}$ is a weakly $l_1$-acyclic covering of $X$. By [I$_3$], Corollary 4.2, there exists a space $X' \supset X$ and a nerve-preserving extension $\mathcal{U}'$ of $\mathcal{U}$ to $X'$ such that $\mathcal{U}'$ is $l_1$-acyclic and the inclusion $X \to X'$ induces an isomorphism of the fundamental groups. Therefore $X \to X'$ induces isometric isomorphisms in bounded cohomology. By a theorem of Cl. Löh [L] this implies that $X \to X'$ induces isometric isomorphisms in $l_1$-homology.

Moreover, if $\mathcal{U}$ is open, then $\mathcal{U}'$ can be assumed to be also open. See [I$_3$], Corollary 4.2. The same argument shows that if the interiors int $U$, $U \in \mathcal{U}$, cover $X$, then one can assume that the interiors int $U'$ cover $X'$. Also, one can assume that int $U' \supset$ int $U$, where the first interior is taken in $X'$ and the second in $X$. See [I$_3$], the end of the proof of Theorem 4.1. We will need also the following simple property of the construction of $\mathcal{U}'$.

**4.1. Lemma.** Let $U \to U_+$ be the map establishing that the covering $\mathcal{U}$ is weakly $l_1$-acyclic. Then there exists a retraction $r : X' \to X$ such that $r(U') \subset U_+$ for every $U \in \mathcal{U}$.

**Proof.** For every $U \in \mathcal{U}$ the subset $U' \subset X'$ is obtained from $U$ by attaching discs along loops contractible in $X$ and then attaching some “collars” to ensure that int $U'$ cover $X'$. Also, one can assume that int $U' \supset$ int $U$, where the first interior is taken in $X'$ and the second in $X$. See [I$_3$], the proof of Theorem 4.1. Under our assumptions the loops used to attach discs to $U$ are contractible in $U_+$. It follows that there exists a retraction $r : X' \to X$ such that $r(U') \subset U_+$ for every $U \in \mathcal{U}$. ■

**4.2. Theorem.** Suppose that $\mathcal{U}$ is a star finite proper covering and that $\mathcal{U}$ is countable and weakly $l_1$-acyclic. If $h \in H^*_c(X)$ belongs to the kernel of $l_\mathcal{U}$, then $\| h \| = 0$.

**Proof.** Let $\mathcal{U}'$ be the extension of the covering $\mathcal{U}$ to a space $X' \supset X$ as above. The closures of the sets $U'$, where $U \in \mathcal{U}$, may be not compact because, in general, $U'$ is obtained from $U$ by attaching an infinite number of discs and “collars”. Therefore the covering $\mathcal{U}'$ is not proper in general, and we cannot apply Theorem 3.6 to it.
But the proof of Theorem 3.6 applies with only minor modifications. Let us represent the homology class \( h \) by a cycle \( \gamma \in C^I_n(X, \mathcal{U}) \subset C^\bullet(X, \mathcal{U}') \).

The cycle \( \gamma \) defines also a homology class \( h' \in H^I_n(X', \mathcal{U}') \). Clearly, the diagram

\[
\begin{array}{ccc}
H^I_\bullet(X, \mathcal{U}) & \xrightarrow{l_{\mathcal{U}}} & H^I_\bullet(N) \\
\downarrow & & \downarrow \\
H^I_\bullet(X', \mathcal{U}') & \xrightarrow{l_{\mathcal{U}'} } & H^I_\bullet(N)
\end{array}
\]

is commutative. Since \( l_{\mathcal{U}}(h) = 0 \), this implies that \( l_{\mathcal{U}'}(h') = 0 \). Together with Theorem 2.3 this implies that \( h' = 0 \), i.e. \( \gamma \) is a boundary in the chain complex \( L^I_\bullet(X', \mathcal{U}') \).

In view of Corollary 3.2 this implies that \( \gamma \) is a boundary in \( L^I_\bullet(X', \mathcal{U}') \), i.e.

\[ \gamma = \partial\beta \]

for some \( \beta \in L_{n+1}^I(X', \mathcal{U}') \). Then

\[ \beta = \sum_{U \in \mathcal{U}} \beta_U \]

for some chains \( \beta_U \in L_{n+1}(U') \). Arguing as in the proof of Theorem 3.6, let us choose an arbitrary \( \varepsilon > 0 \) and represent \( \varepsilon \) as a sum \( \varepsilon = \sum_{U \in \mathcal{U}} \varepsilon_U \) of positive numbers \( \varepsilon_U > 0 \). Next, let us represent each \( \beta_U \) as a sum \( \beta_U = \alpha_U + \omega_U \) of chains \( \alpha_U, \omega_U \in L_{n+1}(U) \) such that \( \alpha_U \) is finite and \( \| \omega_U \| < \varepsilon_U \). Let

\[ \omega = \sum_{U \in \mathcal{U}} \omega_U \quad \text{and} \quad \alpha = \sum_{U \in \mathcal{U}} \alpha_U . \]

For every \( U \in \mathcal{U} \) the chain \( r_*(\alpha_U) \) is a chain in \( U^+ \). Since the family of sets \( U^+ \) is compactly finite, the infinite chain

\[ r_*(\alpha) = \sum_{U \in \mathcal{U}} r_*(\alpha_U) \]

is well defined and compactly finite. Since \( r \) is a retraction,

\[ \gamma - \partial (r_*(\alpha)) = r_*(\gamma) - \partial (r_*(\alpha)) \]

\[ = r_*(\gamma) - r_*(\partial\alpha) = r_*(\gamma - \partial\alpha) . \]
But $\gamma - \partial \alpha = \partial \omega$ and $\| \partial \omega \| \leqslant (n + 1) \| \omega \| < (n + 1) \varepsilon$. It follows that

$$\| r_\ast (\gamma - \partial \alpha) \| \leqslant \| \gamma - \partial \alpha \| < (n + 1) \varepsilon.$$  

Hence $h$ can be represented by chains with arbitrarily small norm, i.e. $\| h \| = 0$. ■

4.3. Theorem. Suppose that $\mathcal{U}$ is a star finite proper covering and that $\mathcal{U}$ is countable and almost weakly $l_1$-acyclic. If $h \in H^{\text{cf}}_\ast (X)$ belongs to the kernel of $l_\mathcal{U}$, then $\| h \| < \infty$.

Proof. The proof differs from the proof of Theorem 4.2 is the same way as the proof of Theorem 3.7 differs from the proof of Theorem 3.6. We leave the details to the reader. ■

Compactly amenable families. The $l_1$-acyclicity is implied by a stronger property, namely, the amenability. Suppose that $Z \subset Y \subset X$ and $Z$ is path connected. The subset $Z$ is said to be amenable in $Y$ if the image of the map $\pi_1(Z, z) \to \pi_1(Y, z)$ is amenable.

A family $\mathcal{U}$ of subsets of $X$ is said to be compactly amenable if for every $U \in \mathcal{U}$ a subset $U_+ \subset X$ is given, such that $U \subset U_+$, the set $U$ is amenable in $U_+$, and the family of subsets $U_+$ is compactly finite. A covering $\mathcal{U}$ is said to be compactly amenable if it is compactly amenable as a family and elements of $\mathcal{U}^\cap$ are path connected.

Since subgroups of amenable groups are amenable, if $Z'$ is a path connected subset of a set $Z$ amenable in $Y$, then $Z'$ is also amenable in $Y$. At the same time amenable groups are $l_1$-acyclic. It follows that a compactly amenable covering is compactly $l_1$-acyclic.

4.4. Theorem. Suppose that $\mathcal{U}$ is a star finite proper covering and $\mathcal{U}$ is countable and compactly amenable. If $h \in H^{\text{cf}}_\ast (X)$ belongs to the kernel of $l_\mathcal{U}$, then $\| h \| = 0$.

Proof. In view of the remarks preceding the theorem, this follows from Theorem 4.2. ■

Almost compactly amenable families. A family $\mathcal{U}$ of subsets of $X$ is said to be almost compactly amenable if subsets $U_+ \subset X$ with the same properties as in the definition of compactly amenable families are given for every subset $U \in \mathcal{U}$, except, perhaps, of a single exceptional set $U_e \in \mathcal{U}$. A covering $\mathcal{U}$ is said to be almost compactly amenable if it is almost compactly amenable as a family and elements of $\mathcal{U}^\cap$ are path connected, except, perhaps, the set $U_e$. Clearly, an almost compactly amenable covering is almost weakly $l_1$-acyclic.

4.5. Theorem. Suppose that $\mathcal{U}$ is a star finite proper covering and $\mathcal{U}$ is countable and almost compactly amenable. If $h \in H^{\text{cf}}_\ast (X)$ belongs to the kernel of $l_\mathcal{U}$, then $\| h \| < \infty$.

Proof. In view of the remarks preceding the theorem, this follows from Theorem 4.3. ■
Families compactly amenable in the sense of Gromov. Gromov’s definitions of amenable subsets and coverings are slightly different. Let us say that $Z$ is amenable in the sense of Gromov in $Y$ if every path connected component of $Z$ is amenable in $Y$ in our sense.

A family $\mathcal{U}$ is compactly amenable in the sense of Gromov if for every $U \in \mathcal{U}$ a subset $U_+ \subset X$ is given, such that $U \subset U_+$, the set $U$ is amenable in the sense of Gromov in $U_+$, and the family of subsets $U_+$ is compactly finite.

A family $\mathcal{U}$ is almost compactly amenable in the sense of Gromov if subsets $U_+ \subset X$ with the same properties as in the definition of compactly amenable families in the sense of Gromov are given for every $U \in \mathcal{U}$, except, perhaps, of finitely many exceptional sets. Gromov [Gro] uses the term sequence “amenable” at infinity for a slightly different notion.

The main difference of Gromov’s version of these notions is the lack of any assumptions of path connectedness. Still, a large part of our theory survives in this context.

Let us relax the assumption of $e_\ast$-acyclicity of the covering $\mathcal{U}$ in Section 2 by the assumption that the homology groups of complexes $e_\ast(Z)$ with $Z \in \mathcal{U} \cap$ vanish in dimensions $> 0$. Then, in order to keep Lemma 2.1, we need to replace the complex $C^\inf_\ast(N)$ by the cokernel of the horizontal boundary operator

$$d_1: c_\ast(N, e_1) \longrightarrow c_\ast(N, e_0).$$

We will denote this cokernel by $C_\ast$, and denote by $H_\ast$ its homology groups. These homology groups play now the role of $H^\inf_\ast(N)$. With these changes the arguments of Section 2 still work and show that the comparison homomorphism

$$\varphi_\ast: H^\inf_p(X, \mathcal{U}) \longrightarrow \tilde{H}^\inf_p(X, \mathcal{U})$$

factors through a canonical map $H^\inf_\ast(X, \mathcal{U}) \longrightarrow H_\ast$. Clearly, $C_p = 0$ if $p > \text{dim } N$. Therefore $H_p = 0$ for $p > \text{dim } N$. It follows that $\varphi_\ast$ is equal to 0 for $p > \text{dim } N$.

Now for a homology class $h \in H^\inf_p(X)$ we can require that $p > \text{dim } N$ instead of requiring that $h$ belongs to the kernel of $l_\mathcal{U}$.

4.6. Theorem. Let $\mathcal{U}$ is a star finite proper covering which is countable and compactly amenable in the sense of Gromov. If $h \in H^\inf_p(X)$ and $p > \text{dim } N$, then $\| h \| = 0$. ■

4.7. Theorem. Let $\mathcal{U}$ is a star finite proper covering which is countable and almost compactly amenable in the sense of Gromov. If $h \in H^\inf_p(X)$ and $p > \text{dim } N$, then $\| h \| < \infty$.

Proof. In this situation only one additional step is needed. Namely, one needs to replace exceptional sets by their union. ■
5. Removing weakly $l_1$-acyclic subspaces

The restriction homomorphisms. Let $X$ be a topological space, and let $Y \subset X$ be a closed subset. Let us construct some chain maps

$$r_{\sim Y} : C^\text{cf}_\ast(X) \rightarrow C^\text{cf}_\ast(X \sim Y).$$

Let $\sigma : \Delta^n \rightarrow X$ be a singular $n$-simplex. If $\sigma(\Delta^n) \subset Y$, then $r_{\sim Y}(\sigma) = 0$. Otherwise $\sigma^{-1}(X \sim Y)$ is a non-empty open subset of $\Delta^n$. Let us triangulate this subset by some geometric (rectilinear) simplices and linearly order the vertices of this triangulation. Then every $n$-dimensional simplex $\alpha$ of the triangulation defines an affine singular simplex $\sigma \circ \alpha'$ in $\Delta^n$ and a singular $n$-simplex $\sigma \circ \alpha' : \Delta^n \rightarrow X \sim Y$. Let $r_{\sim Y}(\sigma)$ be the sum of all these singular simplices $\sigma \circ \alpha'$. The chain $r_{\sim Y}(\sigma)$ is compactly finite because a compact subset of $\sigma^{-1}(X \sim Y)$ intersects only a finite number of simplices of the triangulation. The map $r_{\sim Y}$ extends by linearity to $C^\text{cf}_\ast(X)$ and maps compactly finite chains in $X$ to compactly finite chains in $X \sim Y$.

In general, such a map $r_{\sim Y}$ does not commute with the boundary operators, i.e. is not a chain map. In order to ensure that $r_{\sim Y}$ is a chain map one needs to construct the above triangulations recursively, starting with the tautological triangulations for 0-simplices. If triangulations are already constructed for singular $m$-simplices with $m < n$ and $\sigma$ is a singular $n$-simplex, then $\partial \Delta^n \cap \sigma^{-1}(X \sim Y)$ is already triangulated and one can extend this triangulation to a triangulation of $\sigma^{-1}(X \sim Y)$. By continuing in this way we will get a map $r_{\sim Y}$ commuting with the boundary operators.

The resulting map $r_{\sim Y}$ depends on the choice of these triangulations. But different choices led to chain-homotopic chain maps. Given two choices of triangulations, a chain homotopy between the corresponding maps $r_{\sim Y}$ can be constructed similarly to the maps $r_{\sim Y}$ themselves. Namely, for every singular $n$-simplex $\sigma$ the two triangulations of $\sigma^{-1}(X \sim Y)$ can be considered as a triangulation of $\sigma^{-1}(X \sim Y) \times \{0, 1\}$, and one needs to extend this triangulation to $\sigma^{-1}(X \sim Y) \times [0, 1]$. If these extensions are constructed recursively, then they will define a chain homotopy between two maps $r_{\sim Y}$. Therefore the map

$$r_{\sim Y} : H^\text{cf}_\ast(X) \rightarrow H^\text{cf}_\ast(X \sim Y).$$

induced by $r_{\sim Y}$ does not depend on the choice of triangulations. We will assume that a choice of triangulations is fixed and $r_{\sim Y}$ is the corresponding map.

5.1. Lemma. Let $A$ be a topological space and $n \in \mathbb{N}$. Suppose that $H^1_n(A) = 0$. Then there exists a constant $K$ with the following property. If $z$ is an $l_1$-cycle in $A$, then $z = \delta u$ for some $l_1$-chain $u$ such that $\|u\| \leq K\|z\|$.

Proof. This observation is due to Matsumoto and Morita [MM]. Let us consider the bound-
ary operator $\partial : C_{n+1}^l(A) \to C_n^l(A)$. By the assumption, its image is the subspace $Z_{n+1}^l(A)$ of cycles. Since the latter is defined as the kernel of a bounded operator, it is closed and hence is a Banach space. The kernel of $\partial$ is the subspace of cycles $Z_n^l(A)$. Therefore $\partial$ induces a linear isomorphism

$$\partial' : C_{n+1}^l(A)/Z_{n+1}^l(A) \to Z_n^l(A).$$

Since $\partial'$ is bounded, the open mapping theorem implies that its inverse is bounded. Let $K'$ be the norm of the inverse. Then for every $z \in Z_{n+1}^l(A)$ there exists an element $u'$ of the above quotient such that $z = \partial' u'$ and the norm of $u'$ is $\leq K'\|z\|$. By the definition of the norm on a quotient of a Banach space by a closed subspace, for every $\varepsilon > 0$ there is a representative $u$ of $u'$ such that $\|u\| \leq K'\|z\| + \varepsilon$. If $\|z\| > 0$, we can take $\varepsilon = \|z\|$. If $\|z\| = 0$, then $z = 0$ and $u = 0$ is a representative of $u'$. In both cases $\|u\| \leq (K' + 1)\|z\|$. Since $\partial u = \partial' u' = z$, we can take $K = K' + 1$. ■

5.2. Lemma. Let $A$ be a topological space and $n \in \mathbb{N}$. Suppose that

$$H_{n+1}^l(A) = H_n^{I_1}(A) = 0.$$

Then for every $\varepsilon > 0$ there exists $\delta > 0$ with the following property. If $a$ is a finite $n$-chain in $A$ and $\|\partial a\| < \delta$, then there exists a finite chain $a'$ such that $a' - a = \partial b$ for some finite chain $b$ and $\|a'\| < \varepsilon$.

Proof. Let $K$ be the constant having the property of Lemma 5.1 with $n - 1$ in the role of $n$. The boundary $\partial a$ is a finite cycle and hence an $I_1$-cycle. Lemma 5.1 implies that $\partial a = \partial d$ for some $I_1$-chain $d$ such that $\|d\| \leq K\|\partial a\|$. Clearly, $d - a$ is an $I_1$-cycle of dimension $n$. By the assumption $d - a$ is $I_1$-homologous to 0, i.e. $d - a = \partial e$ for some $I_1$-chain $e$. Let $\delta > 0$ be such that $K\delta < \varepsilon$, and let us choose $\delta' > 0$ such that

$$K\delta + (n + 2)\delta' < \varepsilon.$$

Let us represent $e$ as a sum $e = b + b'$, where $b$ is a finite chain and $\|b'\| \leq \delta'$, and let $a' = a + \partial b$. Then $a' - a = \partial b'$, $a'$ is a finite chain, and $a' = d - \partial b'$. Hence

$$\|a'\| \leq \|d\| + \|\partial b'\| \leq K\|\partial a\| + (n + 2)\|b'\| < \varepsilon.$$

The lemma follows. ■

5.3. Lemma. Let $C$ be a subspace of a topological space $D$ and let $n \in \mathbb{N}$. Suppose that $C$ is path connected and weakly $I_1$-acyclic in $D$. For every $\varepsilon > 0$ there exists $\delta > 0$ with the following property. If $c$ is a finite chain in $C$ and $\|\partial c\| < \delta$, then there exist a finite chain $c'$ in $D$ such that $c' - c = \partial b$ for some finite chain $b$ and $\|c'\| < \varepsilon$. 

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Proof. Let $C'$ be the result of attaching discs to $C$ along a set of loops such that their homotopy classes generate the kernel of the inclusion homomorphism

$$\pi_1(C, x) \rightarrow \pi_1(D, x),$$

where $x \in A$. Then $\pi_1(C', x)$ is amenable and hence the bounded cohomology $\hat{H}^*(C')$ are zero. By a well known theorem of Matsumoto and Morita [MM] (see also [I3] for a proof) this implies that $H^1_{\hat{}}(C') = 0$. By Lemma 5.2 there exist a chain $a$ in $C'$ such that $a' - a = \partial z$ for some finite chain $z$ in $C'$ and $\|a'\| < \epsilon$. It remains to turn $a', z$ into chains in $D$ while keeping its properties. Let $D' = C' \cup D$. Since the loops used to attach discs to $C$ are contractible in $D$, there exists a retraction $r : D' \rightarrow D$. Let $c' = r_* (a)$. Then $\|c'\| \leq \|a\| < \epsilon$, the chain $b = r_*(z)$ is finite, and

$$c' - c = r_*(a) - r_*(c) = r_*(a - c) = r_*(\partial z) = \partial r_*(z).$$

The lemma follows. $\blacksquare$

Supports and parts of singular chains. For a singular simplex $\sigma : \Delta^n \rightarrow X$ let $\overline{\sigma} = \sigma(\Delta^n)$. The support $\text{supp}(c)$ of the singular chain (1.1) is defined as the union

$$\text{supp} c(c) = \bigcup_{a_\sigma \neq 0} \overline{\sigma}.$$

For a subset $Y \subset X$ the Y-part $c|Y$ of the singular chain (1.1) is defined as the chain

$$c|Y = \sum_{\sigma \cap Y \neq \emptyset} a_\sigma \sigma.$$

If $c$ is compactly finite, then $c|Y$ is also compactly finite. The intersection $c \cap Y$ is

$$c \cap Y = \sum_{\sigma \subset Y} a_\sigma \sigma.$$

Clearly, $c = c|Y + c \cap (X \setminus Y)$.

Surgery of chains. Recall that $Y$ is a closed subset of $X$. Suppose that $Z$ is a compact component of $Y$ and that $C$ be a compact Hausdorff neighborhood of $Z$ disjoint from $Y \setminus Z$. Suppose that $\gamma$ is a compactly finite cycle in $X \setminus Y$. We would like remove from $\gamma$ some part contained in $C$ and replace it by a finite chain in $X$ without noticeably increasing the norm.

More precisely, given $\epsilon > 0$, we would like to find an open set $U \subset C$ and a finite chain $\psi$ in $X$ such that $\partial (\gamma \cap U) = \partial \psi$ and $\|\psi\| < \epsilon$. Since $\gamma = \gamma|X \setminus U + \gamma \cap U$ is a cycle, the chain $\gamma|X \setminus U + \psi$ is also a cycle and its norm $< \|\gamma\| + \epsilon$. If the part $\gamma \cap U$ removed from $\gamma$ and the chain $\psi$ depend only on $\gamma|C$, this operation could be performed for several components $Z$ simultaneously.
In our applications all components of $Y$ will be compact and $\gamma$ will represent the homology class $r_{\sim Y}^*(h)$ for some $h \in H^*_c(X)$. In this case we would like to get a representative of $h$ after performing this operation for all components of $Y$ simultaneously.

5.4. Lemma. Let $\delta > 0$. Then for every compactly finite cycle $\gamma$ in $X \sim Y$ such that $\|\gamma\| < \infty$ there exists an open neighborhood $U$ of $Z$ contained in $C$ and such that

$$\|\partial(\gamma \cap U)\| < \delta.$$  

Proof. Since $\|\gamma\| < \infty$, for every $\varepsilon > 0$ one can write $\gamma$ as a sum $\gamma = \gamma' + \gamma''$ of a finite chain $\gamma'$ and a chain $\gamma''$ such that $\|\gamma''\| < \varepsilon$. Since $\gamma'$ is finite, the support $\text{supp} \gamma'$ is compact and hence the intersection $C \cap \text{supp} \gamma'$ is also compact. Since $C$ is Hausdorff, this intersection is closed and its complement $U$ in $C$ is open. Clearly, every simplex contained in $U$ and entering $\gamma$ with a non-zero coefficient enters $\gamma''$ with the same coefficient. It follows that $\|\gamma \cap U\| \leq (n + 1)\|\gamma \cap U\| < \varepsilon$. Let $n$ be the dimension of $\gamma$ and let us take $\varepsilon = \delta/(n + 1)$. Then $\|\partial(\gamma \cap U)\| \leq (n + 1)\|\gamma \cap U\| < \delta$.

Families of compact subspaces. Now we need to impose further restrictions on $Y$. Suppose that $Y$ is presented as the union of a family $\mathcal{Z}$ of pair-wise disjoint compact subspaces of $X$. Suppose further that for every $Z \in \mathcal{Z}$ a compact neighborhood $C_Z$ of $Z$ is given, and that the neighborhoods $C_Z$ are pair-wise disjoint. Suppose that every $C_Z$ is Hausdorff and path connected. Let $V_Z$ be the interior of $C_Z$ and $V$ be the union of all sets $V_Z$.

5.5. Lemma. Let $h \in H^*_c(X)$ and let $\gamma$ be a compactly finite chain in $X \sim Y$ representing the homology class $r_{\sim Y}^*(h)$. Let $X' \subset X \sim Y$ be a closed set containing $X \sim V$. Then there exists a compactly finite chain $s$ in $V$ such that $\gamma|X' + s$ is a cycle representing $h$.

Proof. Let $c \in C^*_n(X)$ be a cycle representing $h$. Then

$$(5.1) \quad \gamma = r_{\sim Y}^*(c) + \partial \beta$$

for some chain $\beta \in C^*_{n+1}(X \sim Y)$. Let

$$\gamma' = \gamma|X', \quad r' = r_{\sim Y}^*(c)|X', \quad \text{and} \quad \beta' = \beta|X'.$$

If $\tau$ is a face of some simplex $\sigma$ entering $\beta$ with non-zero coefficient and $\tau \cap X' \neq \emptyset$, then also $\sigma \cap X' \neq \emptyset$. Therefore (5.1) implies that

$$\gamma' = r' + (\partial \beta)|X'.$$

Let $F$ be the boundary of $X'$. 

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The difference
\[ d = \partial \beta' - (\partial \beta)|X' \]
is a sum of faces \( \tau \) of simplices \( \sigma \) such that \( \tau \cap X' = \emptyset \) and \( \sigma \cap X' \neq \emptyset \). This implies that \( \sigma \cap F \neq \emptyset \) and hence \( \sigma \cap C_Z \neq \emptyset \) for some \( Z \in \mathcal{Z} \). Together with \( \tau \cap X' = \emptyset \) this implies that \( \tau \) is a simplex in \( V_Z \subset C_Z \). Since \( \beta \) is a compactly finite chain, it follows that \( d \cap C_Z \) is a finite chain for every \( Z \) and \( d \) is a compactly finite chain in \( V \). Clearly,
\[ \gamma' + d = r' + \partial \beta' . \]
The construction of \( r_{\sim Y}(c) \) shows that there exists a chain \( s' \) in \( V \) such that \( r' + s' \) is a cycle subdividing \( c \) and \( s'|V_Z \) is a finite chain for every \( Z \in \mathcal{Z} \). It follows that \( r' + s' \) is a cycle representing \( h \) and \( s' \) is a compactly finite chain. Since \( d \) is also compactly finite, the chain \( s = d + s' \) is compactly finite. Clearly,
\[ \gamma' + s = \gamma' + d + s' = r' + s' + \partial \beta' \]
and hence \( \gamma' + s \) is a cycle representing \( h \). The lemma follows. \( \blacksquare \)

5.6. Theorem. Suppose that \( \mathcal{Z} \) is countable and for every \( Z \in \mathcal{Z} \) a set \( Z_+ \) is given, such that \( C_Z \) is weakly \( l_1 \)-acyclic in \( Z_+ \). If the family of sets \( Z_+ \) is compactly finite, then
\[ \| r_{\sim Y}(h) \| \geq \| h \| \]
for every homology class \( h \in H_n^{cf}(X) \).

Proof. If \( \| r_{\sim Y}(h) \| = \infty \), there is nothing to prove. Suppose that \( \| r_{\sim Y}(h) \| < \infty \). Let us fix an arbitrary \( \varepsilon > 0 \). Then there exists a compactly finite cycle \( \gamma \) in \( X \sim Y \) such that
\[ \| \gamma \| < \| r_{\sim Y}(h) \| + \varepsilon \]
and \( \gamma \) represents the homology class \( r_{\sim Y}(h) \). Since \( \mathcal{Z} \) is countable,
\[ \varepsilon = \sum_{Z \in \mathcal{Z}} \varepsilon_Z \]
for some numbers \( \varepsilon_Z > 0 \). For every \( Z \in \mathcal{Z} \) let \( \delta_Z \) be some number \( > 0 \) such that the conclusion of Lemma 5.3 holds for
\[ C_Z, \ Z_+, \ \varepsilon_Z, \ \text{and} \ \delta_Z \]
in the roles of \( C, D, \varepsilon \) and \( \delta \) respectively.
By Lemma 5.4 for every $Z$ there exists an open neighborhood $U_Z$ of $Z$ in $C_Z$ such that

$$\| (\partial (\gamma \cap U_Z)\| < \delta_Z.$$  

Let $U$ be the union of the neighborhoods $U_Z$ and let $X' = X \sim U$. By Lemma 5.5 there exists a compactly finite chain $s$ in $V$ such that $\gamma|X' + s$ is a cycle representing $h$. Since $V_Z, U_Z \subset C_Z$ and the sets $C_Z$ are pair-wise disjoint,

$$\gamma \cap U = \sum_{Z \in \mathcal{Z}} \gamma \cap U_Z \quad \text{and} \quad s \cap V = \sum_{Z \in \mathcal{Z}} s \cap V_Z.$$  

Since $\gamma|X' + s$ and $\gamma = \gamma|X' + \gamma \cap U$ are cycles,

$$\partial (\gamma \cap U) = -\partial (\gamma|X') = \partial (s \cap V)$$

and hence $\partial (\gamma \cap U_Z) = \partial (s \cap V_Z)$ for every $Z$. In view of the choice of $\delta_Z$, the inequality (5.2) together with Lemma 5.3 imply that for every $Z$ there exists a finite chain $\psi_Z$ in $Z_+$ such that $\psi_Z - (s \cap V_Z)$ is the boundary of a finite chain $b_Z$ in $Z_+$ and $\| \psi_Z \| < \epsilon_Z$. Let

$$\psi = \sum_{Z \in \mathcal{Z}} \psi_Z \quad \text{and} \quad b = \sum_{Z \in \mathcal{Z}} b_Z.$$  

Then $\psi - s = \partial b$ and hence

$$\gamma|X' + \psi = \gamma|X' + s + (\psi - s) = \gamma|X' + s + \partial b.$$  

Since the family of the sets $Z_+$ is compactly finite, $\psi$ and $b$ are compactly finite chains. It follows that $\gamma|X' + \psi$ is a compactly finite cycle representing $h$. At the same time

$$\| \psi \| \leq \sum_{Z \in \mathcal{Z}} \| \psi_Z \| \leq \sum_{Z \in \mathcal{Z}} \epsilon_Z = \epsilon$$

and hence

$$\| \gamma|X' + \psi \| \leq \| \gamma|X' \| + \| \psi \|$$

$$\leq \| \gamma|X' \| + \epsilon \leq \| \gamma \| + \epsilon$$

$$< \| r_{\gamma}(h) \| + \epsilon + \epsilon = \| r_{\gamma}(h) \| + 2\epsilon.$$

Since $\epsilon > 0$ is arbitrary, it follows that $\| h \| \leq \| r_{\gamma}(h) \|$.  

**Gromov’s Cutting-of theorem.** In this theorem (see [Gro], Theorem (2) in Section 4.2) Gromov assumes that $X$ is a manifold, $Y$ is the union of a sequence of disjoint compact submanifolds $Y_i, i \in \mathbf{N}$, possibly with boundary, every $Y_i$ is amenable in $X$ in the sense of
Gromov, and the family \( \{ Y_i \}_{i \in \mathbb{N}} \) is almost compactly amenable in the sense of Gromov in \( X \) (see Section 4 for the definitions). Since a compact manifold can have only finitely components, under these assumptions the family of components of \( Y \) is compactly amenable and hence is compactly \( l_1 \)-acyclic. Since components \( Z \) of \( Y \) are submanifolds, standard results imply the existence of compact neighborhoods \( C_Z \) with the properties required above. Therefore Theorem 5.6 applies under Gromov’s assumptions. Its conclusion is the same as Gromov’s inequality \( \| h' \| \geq \| h \| \).

A. Double complexes

**Double complexes.** In order to deal with almost \( l_1 \)-acyclic coverings, we need a complement to the theorem about double complexes. Let \( K_{*,*} \) be a double complex with differentials \( d: K_{p,q} \rightarrow K_{p,q-1} \) and \( \delta: K_{p,q} \rightarrow K_{p-1,q} \). Let \( T_* \) be the total complex of \( K_{*,*} \), and \( E_p \) be the cokernel of \( d: K_{p,1} \rightarrow K_{p,0} \), i.e.

\[
E_p = K_{p,0}/d(K_{p,1}).
\]

Recall that \( \delta \) induces homomorphisms \( \delta_E: E_p \rightarrow E_{p-1} \) turning \( E_* \) into a complex. Clearly, \( E_* \) is a quotient of \( T_* \), and \( K_{0,*} \) with the differential \( d \) is a subcomplex of \( T_* \).

**A.1. Lemma.** If the complexes \( (K_{p,*}, d) \) with \( p > 0 \) are exact, then the kernel of the map \( H_*(T_*) \rightarrow H_*(E_*) \) is contained in the image of the map \( H_*(K_{0,*}, d) \rightarrow H_*(T_*) \).

**Proof.** The proof is completely similar to the proof of injectivity in the proof of Theorem A.2 in [I3]. We leave details to the reader. ■
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