Support of dS/CFT correspondence from perturbations of three dimensional spacetime

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Abstract

We discuss the relation between bulk de Sitter three-dimensional spacetime and the corresponding conformal field theory at the boundary, in the framework of the exact quasinormal mode spectrum. We show that the quasinormal mode spectrum corresponds exactly to the spectrum of thermal excitations of Conformal Field Theory at the past boundary $I^-$, together with the spectrum of a Conformal Field Theory at the future boundary $I^+$.

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The study of quasinormal modes has been an intriguing research activity for the last 30 years [1], leading to important contributions to the understanding of black holes [2-11]. Recently the case of Anti-de Sitter (AdS) space has been particularly focused due to its proposed relation to the conformal field theory (CFT) [12]. Qualitative correspondences between quasinormal modes in AdS spaces and the decay of perturbations in the dual CFT have been obtained [5-10]. More encouraging, in the three-dimensional (3D) BTZ black hole model [13], a precise quantitative agreement between the quasinormal frequencies and the location of poles of the retarded correlation function of the corresponding perturbations in the dual CFT has been presented [11]. This gives a further evidence of the correspondence between gravity in AdS spacetime and quantum field theory at the boundary.

There has been an increasing interest in gravity on de Sitter (dS) spacetimes in view of recent observational support for a positive cosmological constant. A holographic duality relating quantum gravity on D-dimensional dS space to CFT on (D-1)-sphere has been proposed [14]. It is of interest to extend the study in [1] to dS space by displaying the exact solution of the quasinormal mode problem in the dS bulk space and exploring its relation to the CFT theory at the boundary. This could serve as a quantitative test of the dS/CFT correspondence. This is the motivation of the present paper. We will concentrate on nontrivial 3D dS spacetimes. The mathematical simplicity in these models renders all computations analytical. As shown in [15] [16] 3D gravity is directly related to the two-dimensional (2D) WZW theory at the border [17], where the whole theoretical apparatus of CFT can be fully used to obtain exact results. In such a case, Brown and Henneaux [18] already obtained one relation between 3D AdS gravity and conformal algebra with a (classical version of the) central charge, recently developed into the quantum theory by Maldacena [12].

The metric of the 3D rotating dS spacetime is given by [19]

\[ ds^2 = -(M - \frac{r^2}{l^2} + \frac{J^2}{4r^2})dt^2 + (M - \frac{r^2}{l^2} + \frac{J^2}{4r^2})^{-1}dr^2 + r^2(d\varphi - \frac{J}{2r^2}dt)^2, \]

where \( J \) is associated to the angular momentum. The horizon of such spacetime can be obtained from

\[ M - \frac{r^2}{l^2} + \frac{J^2}{4r^2} = 0. \]
The solution is given in terms of \( r^+ \) and \(-ir^-\), where \( r^+ \) corresponds to the cosmological horizon and \(-ir^-\) here being imaginary, has no physical interpretation in terms of a horizon. Using \( r^+ \) and \( r^- \), the mass and angular momentum of spacetime can be expressed as

\[
M = \frac{r^+_2 - r^-_2}{l^2}, \quad J = \frac{-2r^+_r^-}{l} \tag{3}
\]

We are now considering the problem of scalar perturbations of such a space-time. This is what we mean here by quasinormal modes by analogy with the analogous case of black hole perturbations. Although here we do not have black holes the perturbations play a similar role, if we understand the problem as perturbations of a given space-time solution. Perturbations are not only pertinent to black holes, but can appear in any cosmological solution.

Scalar perturbations of this spacetime are described by the wave equation

\[
\frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu \nu} \partial_\nu \Phi) - \mu^2 \Phi = 0, \tag{4}
\]

where \( \mu \) is the mass of the field. Adopting the separation

\[
\Phi(t, r, \varphi) = R(r) \ e^{-i\omega t} \ e^{im\varphi}, \tag{5}
\]

the radial part of the wave equation can be written as

\[
\frac{1}{g_{rr}} \frac{d}{dr} \left( \frac{r^2}{g_{rr}} \frac{dR}{dr} \right) + \left[ \frac{1}{g_{rr}} \left( \omega^2 - \frac{1}{r^2} m^2 (M - \frac{r^2}{l^2}) \right) - \frac{J}{r^2} m \omega \right] R = \frac{1}{g_{rr}} \mu^2 R, \tag{6}
\]

where \( g_{rr} = (M - r^2/l^2 + J^2/(4r^2))^{-1} \). Employing (3) and defining \( z = \frac{r^2 - r^+_2}{r^2 - (-ir^-)^2} \), the radial wave equation can be simplified into

\[
(1-z) \frac{d}{dz} \left( z \frac{dR}{dz} \right) + \left[ \frac{1}{z} \left( \frac{\omega^2 r^+_2 + m r^-_2}{2(r^+_2 + r^-_2)} \right)^2 - \left( \frac{-\omega^2 i r^-_2 + i m r^+_2}{2(r^+_2 + r^-_2)} \right)^2 + \frac{1}{4(1-z)} \mu^2 l^2 \right] R = 0. \tag{7}
\]

We now set the Ansatz

\[
R(z) = z^\alpha (1-z)^\beta F(z), \tag{8}
\]

and Eq. (7) can be transformed into

\[
z(1-z) \frac{d^2 F}{dz^2} + \left[ 1 + 2\alpha - (1 + 2\alpha + 2\beta)z \right] \frac{dF}{dz} \]
\begin{equation}
+ \left[ (\beta(\beta - 1) + \frac{\mu^2 l^2}{4}) \frac{1}{1 - z} + \frac{1}{z} \left( \frac{(\omega l^2 r_+ + m l r_-)^2}{2(r_+^2 + r_-^2)} + \alpha^2 \right) \right] \cdot \left( \frac{\omega l^2 r_+ - i m l r_-}{2(r_+^2 + r_-^2)} \right)^2 + \alpha^2 \right) + (1 + 2\alpha)\beta + \beta(\beta - 1) \right] F = 0. \tag{9}
\end{equation}

Comparing with the standard hypergeometric equation

\begin{equation}
z(1 - z) \frac{d^2 F}{dz^2} + [c - (1 + a + b) z] \frac{dF}{dz} - abF = 0, \tag{10}
\end{equation}

we have

\begin{align*}
c &= 1 + 2\alpha \\
a + b &= 2\alpha + 2\beta \\
\alpha^2 + \left( \frac{\omega l^2 r_+ + m l r_-}{2(r_+^2 + r_-^2)} \right)^2 &= 0 \tag{11}
\end{align*}

\begin{align*}
\beta(\beta - 1) + \frac{\mu^2 l^2}{4} &= 0 \\
ab &= \left( \frac{-\omega l^2 i r_+ + i m l r_-}{2(r_+^2 + r_-^2)} \right)^2 + (\alpha + \beta)^2.
\end{align*}

Without loss of generality, we can take

\begin{align*}
\alpha &= -i \left( \frac{\omega l^2 r_+ + m l r_-}{2(r_+^2 + r_-^2)} \right), \\
\beta &= \frac{1}{2} \left( 1 - \sqrt{1 - \mu^2 l^2} \right), \tag{12}
\end{align*}

which leads to

\begin{align*}
a &= -\frac{i}{2} \left( \frac{\omega l^2 + i m l}{r_+ + i r_-} + i(1 - \sqrt{1 - \mu^2 l^2}) \right), \\
b &= -\frac{i}{2} \left( \frac{\omega l^2 - i m l}{r_+ - i r_-} + i(1 - \sqrt{1 - \mu^2 l^2}) \right), \\
c &= 1 - i \left( \frac{\omega l^2 r_+ + m l r_-}{r_+^2 + r_-^2} \right), \tag{13}
\end{align*}

and the solution of (7) reads

\begin{equation}
R(z) = z^\alpha (1 - z)^\beta \ 2\!\!F_1(a, b, c, z) \tag{14}
\end{equation}

Using basic properties of the hypergeometric equation we write the result as

\begin{equation}
R(z) = z^\alpha (1 - z)^\beta (1 - z)^{c - a - b} \frac{\Gamma(c) \Gamma(a + b - c)}{\Gamma(a) \Gamma(b)} \ 2\!\!F_1(c - a, c - b, c - a - b + 1, 1 - z) \\
+ z^\alpha (1 - z)^\beta \frac{\Gamma(c) \Gamma(c - a - b)}{\Gamma(c - a) \Gamma(c - b)} \ 2\!\!F_1(a, b, a + b - c + 1, 1 - z) \tag{15}
\end{equation}
The first term vanishes at \( z = 1 \), while the second vanishes provided that

\[ c - a = -n, \quad \text{or} \quad c - b = -n, \quad (16) \]

where \( n = 0, 1, 2, \ldots \). Employing Eqs (13), it is easy to see that the quasinormal frequencies are

\[
\begin{align*}
\omega_R &= \frac{i m}{l} - 2i \left( \frac{r_+ - ir_-}{l^2} \right) (n + \frac{1}{2} + \frac{1}{2} \sqrt{1 - \mu^2 l^2}) \\
\omega_L &= -i \frac{m}{l} - 2i \left( \frac{r_+ + ir_-}{l^2} \right) (n + \frac{1}{2} + \frac{1}{2} \sqrt{1 - \mu^2 l^2}).
\end{align*}
\]

(17)

Taking other values of \( \alpha \) and \( \beta \) satisfying (11), we have also the frequencies

\[
\begin{align*}
\omega_R &= \frac{i m}{l} + 2i \left( \frac{r_+ - ir_-}{l^2} \right) (n + \frac{1}{2} + \frac{1}{2} \sqrt{1 - \mu^2 l^2}) \\
\omega_L &= -i \frac{m}{l} + 2i \left( \frac{r_+ + ir_-}{l^2} \right) (n + \frac{1}{2} + \frac{1}{2} \sqrt{1 - \mu^2 l^2}),
\end{align*}
\]

(18)

\[
\begin{align*}
\omega_R &= \frac{i m}{l} - 2i \left( \frac{r_+ - ir_-}{l^2} \right) (n + \frac{1}{2} - \frac{1}{2} \sqrt{1 - \mu^2 l^2}) \\
\omega_L &= -i \frac{m}{l} - 2i \left( \frac{r_+ + ir_-}{l^2} \right) (n + \frac{1}{2} - \frac{1}{2} \sqrt{1 - \mu^2 l^2}),
\end{align*}
\]

(19)

\[
\begin{align*}
\omega_R &= \frac{i m}{l} + 2i \left( \frac{r_+ - ir_-}{l^2} \right) (n + \frac{1}{2} - \frac{1}{2} \sqrt{1 - \mu^2 l^2}) \\
\omega_L &= -i \frac{m}{l} + 2i \left( \frac{r_+ + ir_-}{l^2} \right) (n + \frac{1}{2} - \frac{1}{2} \sqrt{1 - \mu^2 l^2}).
\end{align*}
\]

(20)

De Sitter space is especially appropriated for finding growing modes. Due to the existence of negative parts in the effective potential for the scalar field equation of motion (wells) true bound states can be formed leading to growing modes (see [20]). In de Sitter space with a (too) small cosmological constant, corresponding to the physical case these bound states do not appear. Here however, limitations on the size of the cosmological constant do not occur here.

The shape of the potential in the three dimensional dS case is

\[
V(r) = m^2 (M - r^2/l^2)/r^2 + Jm\omega/r^2 + \mu^2 [M - r^2/l^2 + J^2/(4r^2)]
\]

(21)
Figure 1: Potential behavior for $V(r)$, where we chose all constants in (21) to be unit.

Let us now investigate quasinormal modes from the CFT side. It is believed that for a thermodynamical system the relaxation process of a small perturbation is completely determined by the poles, in the momentum representation, of the retarded correlation function of the perturbation. The correlation functions in dS spacetimes have been studied in $[21]$ $[14]$. 

By describing the coordinates in $SO(3,1)$ $[15]$ such that

$$X_1^2 + X_2^2 + X_3^2 - T^2 = l^2.$$  

The metric (1) can be reobtained by the change of variables

$$X_1 = l \sqrt{\chi} \sin \left( \frac{r_+}{l} \varphi - \frac{r_-}{l^2} t \right)$$

$$X_2 = -l \sqrt{1 - \chi} \cosh \left( \frac{r_+}{l^2} t + \frac{r_-}{l} \varphi \right)$$

$$X_3 = l \sqrt{\chi} \cos \left( \frac{r_+}{l} \varphi - \frac{r_-}{l^2} t \right)$$

$$T = -l \sqrt{1 - \chi} \sinh \left( \frac{r_+}{l^2} t + \frac{r_-}{l} \varphi \right)$$

where $\chi = \frac{r_+^2 + r_-^2}{r_+^2 + r_-^2}$.
The invariant distance between two points defined by \(x\) and \(x'\) reads \[14\] \[21\]

\[ d = l \arccos P, \] (25)

where \(P = X^A \eta_{AB} X'^B\). In the limit \(r, r' \to \infty\),

\[ P \approx 2 \sinh \frac{(ir_+ + r_-)(l \Delta \varphi - i \Delta t)}{2l^2} \sinh \frac{(ir_+ - r_-)(l \Delta \varphi + i \Delta t)}{2l^2}. \] (26)

This means that we can find the Hadamard Green’s function as defined by \[21\] in terms of \(P\).

Such a Green’s function is defined as

\[ G(u, u') = \langle 0 | \{ \phi(u), \phi(u') \} | 0 \rangle \] with

\[ (\nabla_x^2 - \mu^2) G = 0. \]

It is possible to obtain the solution

\[ G \sim F(h_+, h_-, 3/2, (1 + P)/2) (27) \]

in the limit \(r, r' \to \infty\), where \(h_{\pm} = 1 \pm \sqrt{1 - \mu^2 l^2}\).

Following \[14\] \[21\], we choose boundary conditions for the fields such that

\[ \lim_{r \to \infty} \phi(r, t, \varphi) \to r^{-h_-} \phi(t, \varphi). \] (28)

Then, for large \(r, r'\), an expression for the two point function of a given operator \(O\) coupling to \(\phi\) has the form

\[ \lim_{r \to \infty} \int dtd\varphi dt' d\varphi' \frac{(rr')^2}{l^2} \phi \frac{\leftrightarrow}{\partial r^*} G \frac{\leftrightarrow}{\partial r^*} \phi \] (29)

where \(r^*\) in (29) is the tortoise coordinate.

For quasinormal modes, we have

\[ \int dtd\varphi dt' d\varphi' \frac{\exp(-im'\varphi' - iw't' + im\varphi + iw)}{[2 \sinh \frac{(ir_+ + r_-)(l \Delta \varphi - i \Delta t)}{2l^2} \sinh \frac{(ir_+ - r_-)(l \Delta \varphi + i \Delta t)}{2l^2}]_{h^+}} \]

\[ \approx \delta_{mn'} \delta(\omega - \omega') \Gamma(h_+/2 + \frac{im/2l + \omega/2}{2\pi T}) \Gamma(h_+/2 - \frac{im/2l + \omega/2}{2\pi T}) \]

\[ \times \Gamma(h_+/2 + \frac{im/2l - \omega/2}{2\pi T}) \Gamma(h_+/2 - \frac{im/2l - \omega/2}{2\pi T}) \] (30)
where we changed variables to $v = l\varphi + it, \bar{v} = l\varphi - it$, and $T = \frac{ir_+ - r_-}{2\pi l^2}, \bar{T} = \frac{ir_+ + r_-}{2\pi l^2}$.

The poles of such a correlator are

$$\omega_L = -\frac{im}{l} \pm 2\frac{ir_+ - r_-}{l^2}(n + h_+/2),$$ (31)

$$\omega_R = \frac{im}{l} \pm 2\frac{ir_+ + r_-}{l^2}(n + h_+/2),$$

corresponding to the quasinormal modes $[17, 20]$ obtained before.

As stressed by Strominger [14], the boundary condition (26) is not mandatory, and we may choose $\phi(r, t, \varphi) \sim r^{-h_+}\phi(t, \varphi)$ at infinity. This leads to correlations similar to those found up to now with the change $h_+ \rightarrow h_-$, and we complete the set of quasinormal frequencies in (23, 24).

The dS Green functions display two symmetries, which in terms of the variable $v$ can be described as

$$\delta v = \frac{2\pi l^2}{r_+^2 - r_-^2} [n(r_+ + ir_-) + n'(r_- + ir_+)]$$ (32)

for $n, n'$ arbitrary integers. We thus have a two dimensional lattice, and a fundamental region defines the full theory.

The quasinormal eigenfunctions thus correspond to excitation of the corresponding CFT, being exactly those that appear in the spectrum of the two point functions of CFT operators in dS background for large values of $r$, that is at the boundary.

There are some complications compared to the AdS case. There, it has well defined temperature for the theory at the border, given by $T_{\pm}^{\text{AdS}} = (r_\pm \pm r_-)/(2\pi l^2)$, where $r_+$ and $r_-$ are the (real) values for the horizons. However, in dS case only the cosmological horizon $r_+$ exists, while $r_-$ being imaginary. Thus the attempt to define temperature leads us to compute values for it $T_{\pm}^{\text{dS}} = (ir_\pm \pm r_-)/(2\pi l^2)$.

It is worth noticing that since we expect instability for the black hole formation in dS space, the Choptuik parameter $\gamma$ cannot be defined, while in 3D AdS BTZ case it takes the value $1/2$ [22]. We cannot apply the same method here. In particular, the relation found in [6] between the imaginary part of the frequency, $\gamma$ and $r_+$ in the form $\omega_{\text{im}} \sim r_+/\gamma$ cannot hold here, since now the dS radius $l$ enters nontrivially, that is $\omega_{\text{im}} \approx m/l + r_+(n + h_+/2)/l^2$. We are thus in a phase where a black hole can simply not be formed. An independent proof
of this statement would be welcome. Nevertheless, perturbations are still pertinent, as we discussed before.

Finally, comparing with Strominger’s results, we here support the picture of CFT at a boundary of de Sitter space, finding nevertheless excitations at both boundaries, that is, half of them in the past boundary and the other half in the future boundary. Concluding, we found that bulk dS space quasinormal modes can be described in terms of a thermal two-dimensional gas with a complete set of parameters describing instabilities of the bulk matter. The results obtained here provide a quantitative support of the dS/CFT correspondence.

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