Adiabatic radiation reaction to the orbits in Kerr Spacetime

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Geodesic motion of a point particle in Kerr geometry has three constants of motion, energy $E$, azimuthal angular momentum $L$, and Carter constant $Q$. Under the adiabatic approximation, radiation reaction effect is characterized by the time evolution of these constants. In this letter we show that the scheme to evaluate them can be dramatically simplified.

I. INTRODUCTION

A supermassive black hole (SMBH) accompanied by a compact object (CO) is among the most promising candidates for gravitational wave sources. This system may provide us the best opportunity for testing general relativity in the strong gravity regime. For this purpose, however, we need an accurate theoretical prediction of its waveform.

To investigate gravitational waves from SMBH-CO binary system, we use the black hole perturbation method. The background geometry is Kerr spacetime, and CO is described by a point particle. In the lowest order in mass ratio, the particle moves along the background geodesic. In the next order its orbit deviates from the geodesic due to radiation reaction effects.

In the Schwarzschild background, the particle’s orbit can be characterized solely by energy $E$ and azimuthal angular momentum $L$. We can evaluate the orbital evolution from the change rates of energy and azimuthal angular momentum, $dE/dt$ and $dL/dt$. In the adiabatic approximation the energy and the angular momentum that a particle loses are equated with the ones radiated to the infinity or into the black hole horizon as gravitational waves since there are conservation laws for $E$ and $L$ including the gravitational field. In this manner we can determine $dE/dt$ and $dL/dt$ from the asymptotic behavior of the radiated gravitational waves in the Schwarzschild case.

On the other hand, Carter constant $Q$ is also necessary to specify a geodesic in Kerr background. Since we do not have a conservation law corresponding to Carter constant, the change rate of $Q$ is not directly related to the asymptotic gravitational waves. Instead, we need to directly calculate the self-force acting on the particle [1]. Though the prescription to calculate the self-force is formally established [2, 3], performing explicit calculation is not so straightforward.

Gal’tsov [4] employed the radiative part of the metric perturbation, which was introduced earlier by Dirac [5], to calculate $dE/dt$ and $dL/dt$. He showed that this scheme correctly reproduces the standard results obtained by using the conservation laws. Recently, Mino gave a justification for applying the same scheme to $dQ/dt$ for bound orbits [6]. (See also Ref. [7].) However, actual implementation of $dQ/dt$ calculation again is not so straightforward. In this letter we derive a rather simple new formula for the adiabatic evolution of Carter constant.

II. BACKGROUND

We consider the background Kerr spacetime in the Boyer-Lindquist coordinates: $ds^2 = -(1 - 2Mr/\Sigma)dt^2 - (4Mar \sin^2 \theta/\Sigma)dt d\phi + (\Sigma/\Delta)dr^2 + \Sigma d\theta^2 + (r^2 + a^2 + 2Ma^2r \sin^2 \theta/\Sigma) \sin^2 \theta d\varphi^2$, where $\Sigma = r^2 + a^2 \cos^2 \theta$ and $\Delta = r^2 - 2Mr + a^2$. Here $M$ and $aM$ are the mass and the angular momentum of the black hole, respectively. There are two Killing vectors $\xi^{(t)} := (\partial_t)^\mu$ and $\xi^{(\varphi)} := (\partial_\varphi)^\mu$. In addition, Kerr spacetime possesses the Killing tensor, $K_{\mu\nu} := 2\Sigma l(\mu n_\nu) + r^2 g_{\mu\nu}$, where the parentheses denote symmetrization on the indices enclosed, and $l^\mu := (r^2 + a^2, \Delta, 0, a)/\Delta$ and $n^\mu := (r^2 + a^2, -\Delta, 0, a)/2\Sigma$ are two radial null vectors. Killing tensor satisfies the equation $K_{(\mu\nu)r} = 0$.

We consider motion of a point particle, $z^\alpha(\tau) = (t_z(\tau), r_z(\tau), \theta_z(\tau), \phi_z(\tau))$. Here $\tau$ is the proper time along the orbit. For geodesic motion, there are three constants of motion,

$$E := -u^\alpha \xi^{(t)} = \left(1 - \frac{2Mr}{\Sigma}\right) u^t + \frac{2Mar \sin^2 \theta}{\Sigma} u^\varphi,$$

$$L := u^\alpha \xi^{(\varphi)} = -\frac{2Mar \sin^2 \theta}{\Sigma} u^t + \frac{(r^2 + a^2)^2 - \Delta a^2 \sin^2 \theta_z}{\Sigma} \sin^2 \theta_z u^\varphi,$$
\[ Q := K_{\alpha\beta}u^\alpha u^\beta = \frac{(L - aE\sin^2 \theta_z)^2}{\sin^2 \theta_z} + a^2 \cos^2 \theta_z + \Sigma^2(u^\theta)^2, \]

where \( u^\alpha := dz^\alpha/d\tau \). In addition, we define another notation for the Carter constant, \( C := Q - (aE - L)^2 \). For orbits on the equatorial plane \( C \) vanishes.

### III. GEODESIC MOTION IN KERR SPACETIME

Introducing a new parameter \( \lambda \) defined by \( d\lambda = d\tau/\Sigma \), the geodesic equations become

\[
\left( \frac{dr_z}{d\lambda} \right)^2 = R(r_z), \quad \left( \frac{d\cos \theta_z}{d\lambda} \right)^2 = \Theta(\cos \theta_z), \quad \left( \frac{dt_z}{d\lambda} \right) = -a(aE\sin^2 \theta_z - L) + \frac{r_z^2 + a^2}{\Delta} P(r_z), \quad \frac{d\varphi_z}{d\lambda} = -aE + \frac{L}{\sin^2 \theta_z} + \frac{a}{\Delta} P(r_z),
\]

where \( P(r) = E(r^2 + a^2) - aL, R(r) = [P(r)]^2 - \Delta[r^2 + Q] \) and \( \Theta(\cos \theta) = C - (C + a^2(1 - E^2) + L^2 - E^2)\cos^2 \theta + a^2(1 - E^2)\cos^4 \theta \). It should be noted that the equation for the \( r \)-component and the one for the \( \theta \)-component are decoupled by using \( \lambda \). Both \( R(r_z) \) and \( \Theta(\cos \theta_z) \) are quartic functions of their arguments. Hence both solutions are given by elliptic functions. For bound orbits, we can systematically expand \( r_z \) and \( \cos \theta_z \) in Fourier series.

The other two equations \( (3.2) \) are integrated as

\[
t_z(\lambda) = t^{(r)}(\lambda) + t^{(\theta)}(\lambda) + \left( \frac{dt_z}{d\lambda} \right) \lambda, \\
\varphi_z(\lambda) = \varphi^{(r)}(\lambda) + \varphi^{(\theta)}(\lambda) + \left( \frac{d\varphi_z}{d\lambda} \right) \lambda,
\]

where \( \langle \cdots \rangle \) means time average along the geodesic. \( t^{(r)}(\lambda) := \int d\lambda \{ (r_z^2 + a^2)P(r_z)/\Delta - (r_z^2 + a^2)P(r_z)/\Delta \} \) and \( t^{(\theta)}(\lambda) := -\int d\lambda \{ a(aE\sin^2 \theta_z - L) - \langle a(aE\sin^2 \theta_z - L) \rangle \} \), are periodic functions with periods \( 2\pi \Omega_r \) and \( 2\pi \Omega_\theta \), respectively. Functions \( \varphi^{(r)} \) and \( \varphi^{(\theta)} \) are also defined in a similar way.

### IV. ADIABATIC EVOLUTION OF CONSTANTS OF MOTION

In Ref. \( 2 \), it was shown that the adiabatic radiation reaction to the constants of motion \( I^i = \{ E, L, Q \} \) can be evaluated by

\[
\left\langle \frac{dI^i}{d\lambda} \right\rangle = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} d\lambda \Sigma \frac{\partial I^i}{\partial u^\alpha} f^\alpha[h^\text{rad}_{\mu\nu}],
\]

where \( h^\text{rad}_{\mu\nu} \) is the radiative part of the metric perturbation defined by half retarded field minus half advanced field, i.e., \( h^\text{rad}_{\mu\nu} := (h^\text{ret}_{\mu\nu} - h^\text{adv}_{\mu\nu})/2 \). Radiative field is a solution of source free vacuum Einstein equation. The singular parts contained in both retarded and advanced fields cancel out. Therefore we can avoid the tedious issue of regularizing self-force. \( f^\alpha \) is a differential operator,

\[
f^\alpha[h_{\mu\nu}] := -\frac{1}{2} (g^\alpha{}^\beta + u^\alpha u^\beta)(h_{\beta\gamma;\delta} + h_{\beta\delta;\gamma} - h_{\gamma;\beta;\delta})u^\gamma u^\delta.
\]

#### A. Calculation of \( E \) and \( \dot{L} \)

It was shown by Gal’tsov \( 4 \) that

\[
\left\langle \frac{dE}{d\lambda} \right\rangle = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} d\lambda \Sigma (-\xi^\alpha_{(i)} f_{\alpha}[h_{\mu\nu}] \\
= \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} d\lambda \left[ (-\xi^\alpha_{(i)}) \partial_\alpha \psi(x) \right]_{x \to z(\lambda)},
\]

where \( \xi^{(i)} \) is the source of the radiation reaction force.
where \( \psi(x) = \Sigma \tilde{u}^\mu h_{\mu \nu}/2 \) and \((\tilde{u}_t, \tilde{u}_r, \tilde{u}_\theta, \tilde{u}_\varphi) := (-E, \pm \sqrt{R(r)}/\Delta, \pm \sqrt{\Theta(\cos \theta)}/\sin \theta, L) \). This vector field \( \tilde{u}_\mu \) is an extension of the four velocity of a particle in the sense that it satisfies \( \tilde{u}_\mu(z(\lambda)) = u_\mu(\lambda) \). Since in fact \( \tilde{u}_r \) and \( \tilde{u}_\theta \), respectively, depend only on \( r \) and \( \theta \), we can verify the relation, \( \tilde{u}_{\alpha,\beta} = \tilde{u}_{\beta,\alpha} \).

Furthermore, Gal’tsov has shown [11] that \( \psi(x) \) is given by

\[
\psi(x) = i \int d\omega \sum_{\ell,m} \phi_{\omega,\ell,m}^{(in)}(x) \int d\lambda \phi_{\omega,\ell,m}^{(in)}(z(\lambda)),
\]

(4.3)

where \( \phi_{\omega,\ell,m}(x) = \Sigma \tilde{u}^\mu(r,\theta)\tilde{u}^\nu(r,\theta)\tau_{\mu \nu}^{(in)}(r,\theta)e^{-i\omega t - im\phi} \Delta^2 = 2R_{\omega,\ell,m}(r) - 2s_{\omega,\ell,m}(\theta) \). \( \pi_{\omega,\ell,m} \) is an appropriately normalized mode function of metric perturbations, which is constructed by applying a second rank differential operator \( \tau_{\mu \nu} \) to a mode function of Teukolsky equation [8]. The method to solve Teukolsky equation is well established [9]. We can calculate the contribution from waves absorbed into a black hole by replacing the in-field to the up-field in Eq. (4.3).

We can express \( \phi_{\omega,\ell,m} \) as

\[
\phi_{\omega,\ell,m}(z(\lambda)) = e^{-i(\omega t_z - m \phi_z(\lambda))} \Phi_{\ell,m}(r_z(\lambda), dr_z/d\lambda(\lambda), \theta_z(\lambda), d\theta_z/d\lambda(\lambda)).
\]

(4.4)

In the following text, we abbreviate \( d\tau_x/d\lambda \) and \( d\theta_x/d\lambda \) from the arguments for brevity. Here the exponent contains \( r^{(\omega)}(\lambda), t^{(\omega)}(\lambda), \varphi^{(\omega)}(\lambda) \) and \( \varphi^{(\omega)}(\lambda) \). Since \( r_z, t^{(\omega)} \) and \( \varphi^{(\omega)} \) are periodic functions with period \( 2\pi \Omega_z^{-1} \), we can expand \( e^{-i(\omega t^{(\omega)}(\lambda) + t^{(\omega)}(\lambda)) - m(\varphi^{(\omega)}(\lambda) + \varphi^{(\omega)}(\lambda))} \Phi_{\ell,m}(r_z(\lambda), \theta_z(\lambda)) \) into Fourier series as \( (dt_z/d\lambda) \sum n_r, n_\theta Z_{\omega,\ell,m}^{n_r, n_\theta} e^{i(n_r \Omega_x + i(n_\theta \Omega_\theta)} \). Therefore we obtain exp \(-i\omega t^{(\omega)}(\lambda) + t^{(\omega)}(\lambda)) + im(\varphi^{(\omega)}(\lambda) + \varphi^{(\omega)}(\lambda)) \Phi_{\ell,m}(r_z(\lambda), \theta_z(\lambda)) \). Using this expansion, we obtain

\[
\int d\lambda \phi_{\omega,\ell,m}^{(in)}(z(\lambda)) = \sum_{n_r, n_\theta} 2\pi \delta(\omega - \omega_{\omega, \ell, m}^{n_r, n_\theta}) Z_{\omega, \ell, m}^{n_r, n_\theta},
\]

(4.5)

where \( \omega_{\omega, \ell, m}^{n_r, n_\theta} = (dt_z/d\lambda)^{-1} \sum n_r, n_\theta Z_{\omega,\ell,m}^{n_r, n_\theta} e^{i((n_r \Omega_x + i(n_\theta \Omega_\theta) \lambda). \)

(4.6)

Now integration over \( \omega \) is straightforward. We finally end up with the well known formula except for the overall normalization depending on the definition of the mode function:

\[
\left< \frac{dE}{dt} \right> = - \sum_{\ell, m, n_r, n_\theta} |Z_{\omega, \ell, m}^{n_r, n_\theta}|^2.
\]

(4.7)

In a similar manner, a formula for the angular momentum loss rate is obtained as

\[
\left< \frac{dL}{dt} \right> = - \sum_{\ell, m, n_r, n_\theta} \frac{m}{\omega_{\omega, \ell, m}^{n_r, n_\theta}} |Z_{\omega, \ell, m}^{n_r, n_\theta}|^2.
\]

(4.8)

**B. Calculation of \( \dot{Q} \)**

The expression for the radiation reaction to the Carter constant can be cast into a form analogous to the energy loss rate and the angular momentum loss rate. Substituting

\[
f_\nu = \frac{1}{2} (\partial_\nu h_{\alpha \beta}) u^\alpha u^\beta - \frac{d}{dr}(h_{\nu \beta} u^\beta) - \frac{1}{2} u_\nu \frac{d}{dr}(u^\alpha u^\beta h_{\alpha \beta}),
\]

(4.9)

the evolution of Carter constant is given by

\[
\frac{dQ}{dt} = 2K_{\mu \nu} f_\nu \approx 2 \left[ K_{\mu \nu} \tilde{u}^\mu \tilde{u}^\nu \sum_{\alpha, \beta} h_{\alpha \beta} \tilde{u}^\alpha \tilde{u}^\beta (K_{\mu \nu}^\alpha \tilde{u}^\beta - K_{\mu \nu}^\beta \tilde{u}^\alpha) \right]_{z \to z(\lambda)}.
\]

(4.10)
Here “$\approx$” means that terms which become a total derivative or $O(h^2)$ are neglected. Furthermore we can show that the second term in the above equation vanishes by using the facts $\tilde{u}_{\alpha;\beta} = \tilde{u}_{\beta;\alpha}$ and $K_{\mu;\rho} = 0$. Thus we obtain

$$\left\langle \frac{dQ}{dt} \right\rangle = \lim_{T \to \infty} \frac{1}{T} \int_{-T}^T d\Sigma K^\nu_\mu \tilde{u}^\mu \partial_\nu \frac{\psi}{\Sigma}, \quad (4.11)$$

which is written in terms of $\psi(x)$. Hence, we find that the change rate of Carter constant is obtained by replacing $-\xi_{(i)}^\alpha \partial_\alpha$ in the above expression for $\langle dE/dt \rangle$ given in Eq. (4.12) with $\Sigma K^\nu_\mu \tilde{u}^\mu \partial_\nu \Sigma^{-1}$. Then, what we have to evaluate is

$$\int d\lambda \left[ \Sigma K^\nu_\mu \tilde{u}^\mu \partial_\nu \psi(x) \right] \rightarrow \Sigma$$

where the terms which become a total derivative or $O(1)$ are neglected.

In the last step the last term have been integrated by parts using $\tilde{u}^\mu \partial_\mu$. Hence, we find that the change rate of Carter constant is obtained by replacing $\tilde{u}^\mu \partial_\mu = \Sigma^{-1} d/d\lambda$, which is valid after substitution of $z(\lambda)$.

**V. FURTHER REDUCTION**

Further simplification is possible. For an arbitrary function of $r_z$ and $\theta_z$, we have

$$\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^T d\lambda \exp \left[ -i \omega_m \varphi_z r_z(\lambda) + im \varphi_z(\lambda) \right] f(r_z(\lambda), \theta_z(\lambda))$$

$$= \frac{\Omega_z \Theta_x}{(2\pi)^2} \int_0^{2\pi \Omega_x^{-1}} d\omega_x \int_0^{2\pi \Omega_x^{-1}} d\lambda \left[ \exp \left[ -i \omega_x \Omega_x \varphi_x - i \omega_m \varphi_z (\lambda) + im \varphi_z(\lambda) \right] \right]$$

$$\times \exp \left[ -i \omega_x \Theta_x \varphi_x - i \omega_m \varphi_z (\lambda) + im \varphi_z(\lambda) \right] f(r_z(\lambda), \theta_z(\lambda)). \quad (5.1)$$

This relation can be easily verified by substituting $\exp \left[ -i \omega_m \varphi_z (\lambda) + im \varphi_z(\lambda) \right] f(r_z(\lambda), \theta_z(\lambda)) = \sum_{\eta_m, \eta_x} f_{\eta_m, \eta_x} e^{i(\eta_m \Theta_x + \eta_x \Theta_x) \lambda}$. Then, by using the above relation, the $\lambda$-integral in Eq. (4.12) can be decomposed. The part of $\lambda$-integral takes the following form, and it can be integrated by parts as

$$- \int d\lambda \left[ \frac{d}{d\lambda} \right] \left[ \frac{d^r \varphi}{d\lambda} \right] \left[ \frac{d}{d\lambda} \right] \lambda \rightarrow \int_{r_z(\lambda_z), t=t(\lambda_z), \varphi=\varphi(\lambda_z)} \varphi$$

$$= - \int d\lambda \left[ \frac{d}{d\lambda} \right] \lambda \rightarrow \int_{r_z(\lambda_z), t=t(\lambda_z), \varphi=\varphi(\lambda_z)} \varphi$$

The first term in the last line is a total derivative, and therefore does not contribute to the average over a long period of time. Combining the above results, we obtain

$$\left\langle \frac{dQ}{dt} \right\rangle = \left\langle \frac{dL}{dt} \right\rangle^{-1} \lim_{T \to \infty} \frac{1}{T} \int_{-T}^T \left[ \left( \frac{(r^2 + a^2)P}{\Delta} \right) \partial_t + \left( \frac{aP}{\Delta} \right) \partial_\varphi + i \omega_x \Theta_x \varphi \right] \psi(x) \right\rangle d\lambda \rightarrow \Sigma$$

$$= 2 \left( \frac{(r^2 + a^2)P}{\Delta} \right) \left\langle \frac{dE}{dt} \right\rangle - 2 \left( \frac{aP}{\Delta} \right) \left\langle \frac{dL}{dt} \right\rangle + 2 \sum_{\ell, m, \eta_m, \eta_x} \left[ Z_{\ell, m}^{\eta_m, \eta_x} \right]^2. \quad (5.3)$$

This expression is as easy to evaluate as $\langle dE/dt \rangle$ and $\langle dL/dt \rangle$. To evaluate the last term, we have only to replace $m$ in the expression for $\langle dL/dt \rangle$ with $n_x \Omega_x$. 
VI. CONSISTENCY CHECK

We know that a circular orbit stays circular under radiation reaction \[10\]. This condition becomes \(\langle \frac{dQ}{dt} \rangle = \frac{(2(r^2 + a^2)P/\Delta)\langle dE/dt \rangle - (2aP/\Delta) \langle dL/dt \rangle}{(2(r^2 + a^2)P/\Delta)\langle dE/dt \rangle - (2aP/\Delta) \langle dL/dt \rangle} \). For circular orbits \(Z_{n_r,n_\theta}^{\ell,m} = 0\) when \(n_r \neq 0\). Therefore the last term in Eq. (5.3) vanishes. Then Eq. (5.3) agrees with the above condition that a circular orbit stays circular.

We also know that an orbit on the equatorial plane does not leave the equatorial plane by symmetry. This can be clearly seen by rewriting the above formula in terms of \(C\). An identity \(\langle dt_z/d\lambda \rangle \langle dE/dt \rangle - \langle d\varphi_z/d\lambda \rangle \langle dL/dt \rangle + \sum_{\ell,m,n_r,n_\theta} \left\{ (n_r \Omega_r + n_\theta \Omega_\theta)/\omega_{m,n_r,n_\theta} \right\} |Z_{\ell,m}^{n_r,n_\theta}|^2 = 0\) follows from the definition of \(\omega_{m,n_r,n_\theta}\) with the aid of the expressions for \(\langle dE/dt \rangle\) and \(\langle dL/dt \rangle\) given in Eqs. (4.7) and (4.8). Using this identity, we have

\[
\left\langle \frac{dC}{dt} \right\rangle = -2 \left\langle a^2 E \cos^2 \theta_z \right\rangle \left\langle \frac{dE}{dt} \right\rangle - 2 \left\langle L \cot^2 \theta_z \right\rangle \left\langle \frac{dL}{dt} \right\rangle - 2 \sum_{\ell,m,n_r,n_\theta} \frac{n_\theta \Omega_\theta}{\omega_{m,n_r,n_\theta}} |Z_{\ell,m}^{n_r,n_\theta}|^2. \tag{6.1}
\]

From this equation it is manifest that \(\langle dC/dt \rangle = 0\) when \(\theta = \pi/2\). Notice that \(Z_{\ell,m}^{n_r,n_\theta} \neq 0\) only for \(n_\theta = 0\) in the case of equatorial orbits.

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