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Reasoning within expressive fuzzy rough description logics

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Abstract

It is generally accepted that the management of imprecision and vagueness will yield more intelligent and realistic knowledge-based applications. Description Logics (DLs) are suitable, well-known logics for managing structured knowledge that have gained considerable attention the last decade. The current research progress and the existing problems of uncertain or imprecise knowledge representation and reasoning in DLs are analyzed in this paper. An integration between the theories of fuzzy DLs and rough DLs has been attempted by providing fuzzy rough DLs based on fuzzy rough set theory. The syntax, semantics and properties of fuzzy rough DLs are given. It is proved that the satisfiability, subsumption, entailment and ABox consistency reasoning in fuzzy rough DLs may be reduced to the ABox consistency reasoning in the corresponding fuzzy DLs.

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1. Introduction

Description Logics (DLs) [3] are a class of knowledge representation formalisms in the tradition of semantic networks and frames, which can be used to represent the terminological knowledge of an application domain in a structured and formally well-understood way. DL systems provide their users with inference services (like computing the subsumption hierarchy) that deduce implicit knowledge from the explicitly represented knowledge. They are employed in various application domains, such as semantic Web [56], ontologies [2], databases [12], and software engineering [6]. Because classical DLs [3] can only represent and reason on certain or precise knowledge, and cannot represent and reason on uncertain or imprecise knowledge, therefore, some researchers extend classical DLs allowing to express uncertain or imprecise knowledge [45]. To cope with this problem, a number of extended DLs, e.g., fuzzy DLs [58, 60], rough DLs [34, 54], probabilistic DLs [31, 44] and possibilistic DLs [24, 53], have been put forward which extend classical DLs with uncertainty or imprecision. Some of them deal with the vagueness aspect while others deal with the uncertainty aspect. In this paper we will study fuzzy DLs and rough DLs.

Regarding fuzzy DLs, Yen [70] was the first introducing vagueness into a simple DL, and it allows the definition of vague concepts by means of explicit membership functions over a domain. Straccia [60] presented a fuzzy extension of the DL $\text{ALC}$ [55] ($\text{FALC}$), combining Zadeh’s fuzzy logic with a classical DL. Li et al. [39] presented an extended fuzzy DL with number restrictions $\text{EFALCN}$. Stoilos et al. [58] presented fuzzy extensions of DL $\text{SI}$ and $\text{SHIN}$ ($\text{FSI}$ and

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Jiang et al. presented fuzzy extensions of DL SHOIQ (FSHOIQ [33]). Moreover, all of these fuzzy DLs have corresponding constraint propagation based tableaux reasoning algorithms.

Regarding rough DLs, Schlobach et al. [54] extend traditional DLs with a simple mechanism to handle approximate concept definitions in a qualitative way based on rough set theory, and present a kind of rough DL RDL. Subsequently, Jiang et al. [34] extend RDL with approximate concepts [17] (or rough sets), and present a new kind of rough DL RDLAC. Lui [42] incorporates the notion of rough sets into terminological logics, presents a kind of rough terminological logic ROTEL based on Pawlak rough set theory. Klein et al. [35] present a conservative extension to standard OWL that allows for defining approximations of concepts based on rough set theory, i.e., RoughOWL, an extension of OWL with new operators for modeling vague concepts.

From above-mentioned statements, we know that rough DLs and fuzzy DLs both are the extensions of the classical DLs, and all of these DLs [33,34,39,54,58,60,70] can express uncertain or imprecise knowledge. Fuzzy DLs are fuzzy extensions of classical DLs based on Zadeh’s fuzzy set theory [72], so fuzzy DLs can express fuzzy knowledge, but they cannot express rough knowledge (or incomplete knowledge). Rough DLs are rough extensions of classical DLs based on rough set theory, so rough DLs can express rough knowledge (or incomplete knowledge), but they cannot express fuzzy knowledge.

It is well known that the theory of fuzzy sets [72] provides an effective means of describing the behavior of systems which are too complex or too ill-defined to admit precise mathematical analysis by classical methods and tools. It has shown enormous promise in handling uncertainties to a reasonable extent, particularly in decision-making models under different kinds of risks, subjective judgment, vagueness, and ambiguity [5]. After nearly 20 years since the introduction of fuzzy sets, Pawlak [50] introduced the notion of a rough set as a new mathematical tool to deal with the approximation of a concept in the context of inexact, or incomplete information [5,30]. Since the introduction of rough set theory, many attempts to establish the relationships between the two theories, to compare each to the other, and to simultaneously hybridize them have been made, the aim being to develop a model of uncertainty stronger than both of them [13,18,19,38,43,52,67,68]. For example, Dubois and Prade were one of the first who investigated the problem of fuzzification of a rough set, and presented a kind of fuzzy rough set theory based on a fuzzy similarity relation (i.e., reflexive, symmetric and transitive fuzzy relation) [18]. Pei and Wu et al. presented a kind of generalized fuzzy rough set theory based on an arbitrary fuzzy relation [52,67]. Liu proposed the notion of generalized rough sets over fuzzy lattices [43].

Based on above-mentioned analysis, this leads to a new direction in which the theories of fuzzy DLs and rough DLs can be integrated, the aim being to develop a new DL theory of uncertainty stronger than both of them. The present work may be considered as an attempt in this line, i.e., fuzzy rough DLs (FRDLs) are proposed based on fuzzy rough set theory. The syntax, semantics and properties of FRDLs are given. It is proved that the satisfiability, subsumption, entailment and ABox consistency reasoning in FRDLs may be reduced to the ABox consistency reasoning in the corresponding fuzzy DLs. To the best of our knowledge, the work described in this paper is the first result combining fuzzy DLs with rough DLs based on fuzzy rough set theory.

The structure of the rest of this paper is as follows. Section 2 briefly introduces requisite notions of DLs and fuzzy rough set theory. In Section 3, FRDL FRSHIN is proposed based on generalized fuzzy rough set theory. Next, Section 4 discusses some extensions of FRDLs and implementation of FRDLs reasoning, and presents some topics for future research. Finally, in Section 5, we draw the conclusion.

2. Preliminaries

In the current section we will briefly introduce the notions of DLs and fuzzy rough set theory.

2.1. Description logics

DLs [3,58] are a family of logic-based knowledge representation formalisms designed to represent and reason about the knowledge of an application domain in a structured and well-understood way. They are based on a common family of languages, called description languages, which provide a set of constructs to build concept (class) and role (property) descriptions. Such descriptions can be used in axioms and assertions of DL knowledge bases and can be reasoned about w.r.t. DL knowledge bases by DL systems.
The specific DL we will extend with fuzzy rough capabilities is the SHIN [58]. At first, we will introduce the SHIN, while in Section 3 our fuzzy rough extension of SHIN will be presented.

A description language consists of an alphabet of distinct concept names (C), role names (R) and individual (object) names (I); together with a set of constructors to construct concept and role descriptions.

Now let us introduce the syntax and semantics of SHIN first.

Let RN ∈ R be a role name and R an SHIN-role. SHIN-role descriptions (or simply SHIN-roles) are defined by the abstract syntax: S := RN|R−. The inverse relation of roles is symmetric, and to avoid considering roles such as R−, we define a function Inv which returns the inverse of a role, more precisely,

\[
\text{Inv}(R) := \begin{cases} 
R^{-} & \text{if } R = R N, \\
R N & \text{if } R = R N^{-}.
\end{cases}
\]

The set of SHIN-concept descriptions (or simply SHIN-concepts) is the smallest set such that:

1. every concept name CN ∈ C and ⊥, ⊤ are SHIN-concepts,
2. if C and D are SHIN-concepts and R is an SHIN-role, then ¬C, C ∩ D, C ∪ D, ∃R.C and ∀R.C are also SHIN-concepts, called general negation (or simply negation), disjunction, conjunction, value restrictions and existential restriction, respectively, and
3. if S is a simple [29,58] SHIN-role and n ∈ N, then (≥nS) and (≤nS) are also SHIN-concepts, called at-most and at-least number restrictions, where N denotes natural number set.

DLs have a model-theoretic semantics, which is defined in terms of interpretations. An interpretation (written as I) consists of a domain (written as A′) and an interpretation function (written as *), where the domain is a non-empty set of objects and the interpretation function maps each individual name a ∈ I to an element a′ ∈ A′, each concept name CN ∈ C to subset CN′ ⊆ A′, and each role name RN ∈ R to a binary relation RN′ ⊆ A′ × A′.

The interpretation function * can be extended to give semantics to concept and role descriptions as follows:

- ⊤′ = A′;
- ⊥′ = ∅;
- (¬C)′ = A′\C′;
- (C ∩ D)′ = C′ ∩ D′;
- (C ∪ D)′ = C′ ∪ D′;
- (∃R.C)′ = \{x ∈ A′ | \exists y ∈ A′, (x, y) ∈ R′ \land y ∈ C′\};
- (∀R.C)′ = \{x ∈ A′ | \forall y ∈ A′, (x, y) ∈ R′ \rightarrow y ∈ C′\};
- (≥nR)′ = \{x ∈ A′ | \#\{y ∈ A′ | R′(x, y)\} ≥ n\};
- (≤nR)′ = \{x ∈ A′ | \#\{y ∈ A′ | R′(x, y)\} ≤ n\}.

An SHIN knowledge base (KB) consists of a TBox, an RBox and an ABox. An SHIN TBox is a finite set of concept inclusion axioms of the form C ⊆ D, or concept equivalence axioms of the form C ≡ D, where C, D are SHIN-concepts. An interpretation I satisfies C ⊆ D if C′ ⊆ D′ and it satisfies C ≡ D if C′ = D′. Note that concept inclusion axioms of the above form are called general concept axiom [1,27]. An SHIN RBox is a finite transitive role axiom (Trans(R)), and role inclusion axioms (R ⊆ S). An interpretation I satisfies Trans(R) if, for all x, y, z ∈ A′, \{(x, y), (y, z)\} ⊆ R′ → \{(x, z)\} ⊆ R′, and it satisfies R ⊆ S if R′ ⊆ S′. A set of role inclusion axioms defines a role hierarchy. For a role hierarchy we introduce ∼ as the transitive–reflexive closure of ⊆. At last, observe that if R ⊆ S, then the semantics of role inclusion axioms imply that Inv(R)′ ⊆ Inv(S)′. While the semantics of inverse roles imply that Trans(Inv(R)), where R is transitive. An SHIN ABox is a finite set of individual axioms (or assertions) of the form a : C, called concept assertions, or ⟨a, b⟩ : R, called role assertions, or of the form a≠b. An interpretation I satisfies a : C if a′ ∈ C′, it satisfies ⟨a, b⟩ : R if ⟨a′, b′⟩ ∈ R′, and it satisfies a≠b if a′≠b′. An interpretation I satisfies a SHIN knowledge base ∑ if it satisfies all the axioms in ∑, ∑ is satisfiable (unsatisfiable) iff there exists (does not exist) such an interpretation I that satisfies ∑. Let C, D be SHIN-concepts, C is satisfiable (unsatisfiable) w.r.t. ∑ iff for every interpretation I of ∑ such that C′≠∅; C subsumes D w.r.t. ∑ for every interpretation I of ∑ we have C′ ⊆ D′. Given a concept axiom, a role axiom, or an assertion ψ, ∑ entails ψ, written as ∑ |= ψ, iff for all models I of ∑ we have I satisfies ψ.
2.2. Fuzzy rough set theory

Fuzzy rough set theory [15,18,19,32,43,52,67,71] is the extensions of Pawlak rough set theory [50,51], therefore, for fuzzy rough set theory, a shorthand Pawlak rough set theory needs to be introduced first.

Let \( U \) denote a finite and non-empty set called the universe, and let \( \mathcal{R} \subseteq U \times U \) denote an equivalence relation on \( U \). The pair \( \text{apr} = (U, \mathcal{R}) \) is called an approximation space. The equivalence relation \( \mathcal{R} \) partitions the set \( U \) into disjoint subsets. Such a partition of the universe is denoted by \( U/\mathcal{R} \). If two elements \( x, y \) in \( U \) belong to the same equivalence class, we say that \( x \) and \( y \) are indistinguishable. The equivalence class of \( \mathcal{R} \) and the empty set \( \phi \) are called the elementary or atomic sets in the approximation space \( \text{apr} = (U, \mathcal{R}) \).

Given an arbitrary set \( A \subseteq U \), it may be impossible to describe \( A \) precisely using the equivalence classes of \( \mathcal{R} \). In this case, one may characterize \( A \) by a pair of lower and upper approximations:

\[
\text{apr}(A) = \bigcup_{[x]_R \subseteq A} [x]_R, \\
\overline{\text{apr}}(A) = \bigcup_{[x]_R \supseteq A} [x]_R,
\]

where \([x]_R = \{ y \mid x \mathcal{R} y \}\) is the equivalence class containing \( x \). The pair \((\text{apr}(A), \overline{\text{apr}}(A))\) is called the rough set of \( A \).

Let us introduce the notions of generalized fuzzy rough set theory [52,67] formally.

A fuzzy set \( X \) of \( U \) is defined by a membership function: \( \mu_X : U \rightarrow [0, 1] \), \( \mu_X(x), x \in U \), giving the degree of membership of \( x \) in \( X \). Let \( F(U) \) denote the fuzzy power set of \( U \), i.e., the set of all functions from \( U \) to \([0, 1]\). Fuzzy set intersection, union and complement operators are defined component-wise as

1. \( \mu_{X \cap Y}(x) = \min\{\mu_X(x), \mu_Y(x)\} \), for \( X, Y \in F(U) \);
2. \( \mu_{X \cup Y}(x) = \max\{\mu_X(x), \mu_Y(x)\} \), for \( X, Y \in F(U) \);
3. \( \mu_{\overline{X}}(x) = 1 - \mu_X(x) \), for \( X \in F(U) \).

We recall that a fuzzy relation on \( U \) is a fuzzy subset of \( U \times U \). If \( R \) is a fuzzy relation on \( U \), the pair \((U, R)\) is called a fuzzy approximation space.

A fuzzy relation \( R \) on \( U \) is called a fuzzy similarity relation if \( R \) is

1. reflexive \((R(x, x) = 1)\),
2. symmetric \((R(x, y) = R(y, x))\), and
3. transitive \((R(x, y) \supseteq \sup_{z \in U} \min\{R(x, z), R(z, y)\})\).

Let \( U \) be an arbitrary universal set and \( R \) be a fuzzy relation on \( U \), \( F(U) \) be the fuzzy power set of \( U \). A generalized fuzzy rough set is a pair \((\bar{R}X, \overline{\bar{R}}X)\) of fuzzy sets on \( U \) such that for every \( x \in U \)

\[
\mu_{\bar{R}X}(x) = \inf_{y \in U} \max\{1 - R(x, y), \mu_X(y)\}, \\
\mu_{\overline{\bar{R}}X}(x) = \sup_{y \in U} \min\{R(x, y), \mu_X(y)\}.
\]

3. Fuzzy rough description logic FRSHIN

In this section, we introduce a fuzzy rough extension of the \( SHIN \) DL presented in Section 2.1 based on generalized fuzzy rough set theory [52,67] presented in Section 2.2, i.e., we will provide the FRDL \( FRSHIN \). The syntax, semantics and reasoning of the \( FRSHIN \) will be presented.

3.1. Syntax and semantics

FRDL \( FRSHIN \) is the extension of DL \( SHIN \). That is to say, comparing with \( SHIN \), the lower approximation concepts and the upper approximation concepts w.r.t. a fuzzy relation \( R^- \), and fuzzy concepts and roles are added to \( FRSHIN \). Therefore, the abstract syntax of \( FRSHIN \) is the extension of that of \( SHIN \), i.e., \( FRSHIN \)-concepts (denoted by \( C \) or \( D \)) are composed inductively according to the following abstract syntax:
Example 1 (SARS example). Severe acute respiratory syndrome (SARS) is a respiratory disease in humans which is caused by the SARS coronavirus. The definition of SARS cannot be expressed precisely. According to the clinical diagnosis criteria for SARS which was released by the Ministry of Health of the People’s Republic of China [41], there are mainly two kinds of diagnostic criteria for SARS: suspected diagnostic criteria and clinically diagnosed criteria. Obviously, the patients who accord with suspected diagnostic criteria may have SARS, however, this is not necessary the case. Here, we can define the patients who accord with suspected diagnostic criteria as the upper approximation concept of SARS (denoted by $\overline{SARS}$). The patients who accord with clinically diagnosed criteria must have SARS. Here, we can define the patients who accord with clinically diagnosed criteria as the lower approximation concept of SARS (denoted by $SARS$).

On the other hand, there exists much fuzzy knowledge in the symptoms of SARS (see Example 2). Therefore, we have to utilize FRDLs to express the knowledge about SARS. Based on these fuzzy lower approximation and fuzzy upper approximation concepts of SARS, we can further construct the DL knowledge base about SARS with explicit formal semantics.

The semantics of $FRSHIN$ is the extension of the semantics of the classical DL $SHIN$, i.e., the lower approximation concepts, the upper approximation concepts and fuzzy concepts and roles must be interpreted in $FRSHIN$.

In the following, we will show that the semantics of lower approximation concepts and upper approximation concepts are given using fuzzy set theoretic operations. Therefore, let us introduce the fuzzy set theoretic operations [58,62] firstly.

The operation of complement is performed by a unary operation, $c : [0, 1] \rightarrow [0, 1]$, called fuzzy complement. In order to provide meaningful fuzzy complements, such functions should satisfy certain properties. More precisely, they should satisfy the boundary conditions, $c(0) = 1$ and $c(1) = 0$, and be monotonic decreasing, for $a \leq b$, $c(a) \geq c(b)$.

Several negation functions have been given in the literature, e.g., Lukasiewicz negation $c(a) = 1 - a$ and Gödel negation $c(0) = 1$ and $c(a) = 0$ if $a > 0$. In the current paper we will use the Lukasiewicz negation which additionally is continuous and involutive, for each $a \in [0, 1]$, $c(c(a)) = a$ holds.

In the cases of fuzzy intersection and fuzzy union the mathematical functions used are binary over the unit interval. These functions are usually called norm operations referred to as t-norms ($t : [0, 1] \times [0, 1] \rightarrow [0, 1]$), in the case of fuzzy intersection, and t-conorms (or s-norms) ($u : [0, 1] \times [0, 1] \rightarrow [0, 1]$), in the case of fuzzy union [36]. Again these operations should satisfy certain mathematical properties. More precisely, a t-norm (t-conorm) satisfies the boundary condition, $t(a, 1) = a$($u(a, 0) = a$), is monotonic increasing, for $b \leq d$ then $t(a, b) \leq t(a, d)$ ($u(a, b) \leq u(a, d)$), commutative, $t(a, b) = t(b, a)$ ($u(a, b) = u(b, a)$), and associative, $t(a, t(b, c)) = t(t(a, b), c)$ ($u(a, u(b, c)) = u(u(a, b), c)$). Examples of t-norms are: $t(a, b) = \max(a + b - 1, 0)$ (Lukasiewicz t-norm), $t(a, b) = \min(a, b)$ (Gödel t-norm), $t(a, b) = a \cdot b$ (product t-norm). Note that for all $a \in [0, 1]$, $t(a, 0) = 0$. Examples of s-norms are: $u(a, b) = \min(a + b, 1)$ (Lukasiewicz s-norm), $u(a, b) = \max(a, b)$ (Gödel s-norm), $t(a, b) = a + b - a \cdot b$ (product s-norm). Note that if we consider Lukasiewicz negation, then Lukasiewicz, Gödel and product s-norm are related to their respective t-norm according to the De Morgan law: $\forall a, b \in [0, 1], u(a, b) = c(t(c(a), c(b)))$. Though there is a wealth of such operations in the literature [37] we restrict our attention to the Gödel t-norm and Gödel s-norm. Additionally to the aforementioned properties, these operations are also idempotent, i.e., $\min(a, a) = a$ and $\max(a, a) = a$, hold.

A fuzzy implication is performed by a binary operation, of the form $J : [0, 1] \times [0, 1] \rightarrow [0, 1]$ that has to satisfy the following conditions: (1) $\forall a, b, c \in [0, 1], a \leq b$ implies $J(a, c) \geq J(b, c)$; (2) $\forall a, b, c \in [0, 1], b \leq c$ implies $J(a, b) \leq J(a, c)$; (3) $\forall a \in [0, 1], J(0, b) = 1$; (4) $\forall a \in [0, 1], J(a, 1) = 1$; (5) $J(1, 0) = 0$. Several implication functions have been given in the literature, e.g., Kleene–Dienes implication $J(a, b) = \max(1 - a, b)$; residuum based implication $J(a, b) = 1$ if $a \leq b$, if $a > b$ then, according to the chosen t-norm, we have that e.g., $J(a, b) = 1 - a + b$ for Lukasiewicz implication. $J(a, b) = b$ for Gödel implication and $J(a, b) = b/a$ for product implication. In the current paper we use the Kleene–Dienes implication.

Now we introduce the semantics of $FRSHIN$. Note that, although most of the semantics (the fuzzy DL part of $FRSHIN$) have been presented elsewhere [62], we include them here simply for the sake of completeness.

$$C, D \rightarrow \top \land [A | \neg C | C \cap D | C \cup D | \exists R.C | \forall R.C] \geq nR \leq nR[\overline{C}],$$

where $A$ denotes atomic concept, $C$ and $D$ denote concepts (or concept descriptions), $R$ denotes role name, $R \neq R^\sim$, $n$ denotes a natural number.
A fuzzy rough interpretation $I = (A^I, R, \bullet^I)$ consists of a domain of interpretation $A^I$, a fuzzy relation $R \subseteq A^I \times A^I$, and an interpretation function $\bullet^I$ mapping

- individuals as for the crisp case, i.e., $a^I \neq b^I$ if $a \neq b$;
- a concept $C$ into a membership function $C^I: A^I \rightarrow [0, 1]$;
- a role $R$ into a membership function $R^I: A^I \times A^I \rightarrow [0, 1]$.

If $C$ is a concept then $C^I$ will naturally be interpreted as the membership degree function of the fuzzy rough concept $C$ w.r.t. $I$, i.e., if $d \in A^I$ is an object of the domain $A^I$ then $C^I(d)$ gives us the degree of being the object $d$ an element of the fuzzy rough concept $C$ under the fuzzy rough interpretation $I$. Similarly for fuzzy roles. Additionally, the interpretation function $\bullet^I$ has to satisfy the following equations: for all $a, b, d \in A^I$,

- $\top^I(d) = 1$;
- $\bot^I(d) = 0$;
- $(C \cap D)^I(d) = \min\{C^I(d), D^I(d)\}$;
- $(C \cup D)^I(d) = \max\{C^I(d), D^I(d)\}$;
- $(\neg C)^I(d) = 1 - C^I(d)$;
- $(\exists.R.C)^I(d) = \sup_{d', d'' \in A^I} \{\min\{R^I(d, d'), C^I(d'')\}\}$;
- $(\forall.R.C)^I(d) = \inf_{d', d'' \in A^I} \{\max\{1 - R^I(d, d'), C^I(d'')\}\}$;
- $(\geq n R)^I(d) = \sup_{b_1, \ldots, b_n \in A^I} \min_{i=1}^n \{R^I(a, b_i)\}$;
- $(\leq n R)^I(d) = \inf_{b_1, \ldots, b_n \in A^I} \sup_{d', d'' \in A^I} \{\min\{R^I(d, d'), C^I(d'')\}\}$;
- $(C \sim)I(d) = \inf_{d', d'' \in A^I} \{\max\{1 - R^I(d, d'), C^I(d'')\}\}$
- $(R^\sim)I(b, a) = R^I(a, b)$.

Obviously, the semantics of FRSHIN is defined according to the fuzzy set theoretic operations and it is an extension of the semantics of FSHIN [58,62]. For example, since we use the max function (Gödel s-norm) for fuzzy union the membership degree of an object $d$ to the fuzzy concept $(C \cup D)^I$ is equal to $\max\{C^I(d), D^I(d)\}$. Moreover, since each concept $C$ and role $R$ can be mapped into an equivalent open first order formula $F_C(x)$ and $F_R(x, y)$ [22,23,58,60], therefore, the semantics of $\exists R.C$

$$(\exists R.C)^I(d) = \sup_{d' \in A^I} \{\min\{R^I(d, d'), C^I(d'')\}\}$$

is the result of viewing $\exists R.C$ as the open first order formula $\exists y.F_R(x, y) \land F_C(y)$ and the existential quantifier $\exists$ is viewed as a disjunction over the elements of the domain. Similarly,

$$(\forall R.C)^I(d) = \inf_{d' \in A^I} \{\max\{1 - R^I(d, d'), C^I(d'')\}\}$$

is related to the open first order formula $\forall y.F_R(x, y) \rightarrow F_C(y)$, where the universal quantifier $\forall$ is viewed as conjunction over the elements of the domain, and $\rightarrow$ as the Kleene–Dienes implication.

$$(\geq n R)^I(d) = \sup_{b_1, \ldots, b_n \in A^I} \min_{i=1}^n \{R^I(a, b_i)\}$$

is related to the open first order formula $\exists y_1, \ldots, y_n.F_R(x, y_1) \land \cdots \land F_R(x, y_n)$.

$$(\leq n R)^I(d) = \inf_{b_1, \ldots, b_{n+1} \in A^I} \max_{i=1}^{n+1} \{1 - R^I(a, b_i)\}$$
is related to the open first order formula \( \forall y_1, \ldots, y_{n+1}. \neg F_R(x, y_1) \lor \cdots \lor \neg F_R(x, y_{n+1}). \)

\[ (\overline{C})^l(d) = \sup_{d' \in A^l} \{ \min\{R^-(d, d'), C^l(d')\} \} \]

is the result of viewing \( \overline{C} \) as the open first order formula \( \exists y. F_R^-(x, y) \land F_C(y). \)

\[ (C)^l(d) = \inf_{d' \in A^l} \{ \max\{1 - R^-(d, d'), C^l(d')\} \} \]

is the result of viewing \( C \) as the open first order formula \( \forall y. F_R^-(x, y) \rightarrow F_C(y). \)

Similarly with the \( \text{FRSHIN} [58,62] \) DL, an \( \text{FRSHIN} \) knowledge base consists of a TBox, an RBox and an ABox. Let \( A \) be a concept name and \( C \) an \( \text{FRSHIN} \)-concept. An \( \text{FRSHIN} \) TBox is a finite (possibly empty) set of fuzzy rough concept axioms of the form \( A \sqsubseteq C \), called fuzzy rough inclusion introductions, and of the form \( A \equiv C \), called fuzzy rough equivalence introductions. We will assume that an \( \text{FRSHIN} \) TBox \( TB \) is such that no concept \( A \) appears more than once on the left-hand side of a fuzzy rough concept axiom and that no cyclic definitions and general concept inclusions (GCI) are present in \( TB \). A fuzzy rough interpretation \( I \) satisfies \( A \sqsubseteq C \) iff \( \forall d \in A^l, A^l(d) \leq C^l(d) \). A fuzzy rough interpretation \( I \) satisfies \( A \equiv C \) iff \( \forall d \in A^l, A^l(d) = C^l(d) \). A fuzzy rough interpretation \( I \) satisfies an \( \text{FRSHIN} \) TBox \( TB \) iff it satisfies all fuzzy rough concept axioms in \( TB \); in this case, we say that \( I \) is a model of \( TB \).

An \( \text{FRSHIN} \) RBox is a finite (possibly empty) set of fuzzy (rough) transitive role axioms of the form \( \text{Trans} (R) \) and fuzzy (rough) role inclusion axioms of the form \( R \sqsubseteq S \), where \( R, S \) are \( \text{FRSHIN} \)-roles. A fuzzy rough interpretation \( I \) satisfies \( \text{Trans} (R) \) iff \( \forall a, c \in A^l \),

\[ R^l(a, c) > \sup_{b \in A^l} \{ \min\{R^l(a, b), R^l(b, c)\} \} \]

while it satisfies \( R \sqsubseteq S \) iff \( \forall a, b \in A^l, R^l(a, b) \leq S^l(a, b) \). Note that the semantics result from the definition of sup–min transitive relations in fuzzy set theory [37,58]. A fuzzy rough interpretation \( I \) satisfies an \( \text{FRSHIN} \) RBox \( RB \) iff it satisfies all fuzzy (rough) transitive role axioms and fuzzy (rough) role inclusion axioms in \( RB \); in this case, we say that \( I \) is a model of \( RB \). The semantics of inverse roles and role inclusion axioms of \( \text{FRSHIN} \) imply that from \( \text{Trans} (R) \) and \( R \sqsubseteq S \) it holds that \( \text{Trans}(\text{Inv}(R)) \) and \( \text{Inv}(R)^\neg \sqsubseteq \text{Inv}(S)^\neg \), where \( \text{Inv} \) is a function (syntactic construct) defined in Section 2.1.

An \( \text{FRSHIN} \) ABox is a finite (possibly empty) set of fuzzy rough assertions of the form \( x \triangledown n, x > n, x \leq m, x < m \) or \( a \neq b \), where \( x \) is an assertion of the form \( a : C, \langle a, b \rangle : R, n \in [0,1] \) and \( m \in [0,1] \), \( C \) denotes a concept, \( R \) denotes a role, and \( a, b \) denote individuals. Formally, given a fuzzy rough interpretation \( I \),

- \( I \) satisfies \( a : C \geq n \) iff \( C^l(a^l) \geq n \);
- \( I \) satisfies \( a : C \leq m \) iff \( C^l(a^l) \leq m \);
- \( I \) satisfies \( \langle a, b \rangle : R \geq n \) iff \( R^l(a^l, b^l) \geq n \);
- \( I \) satisfies \( \langle a, b \rangle : R \leq m \) iff \( R^l(a^l, b^l) \leq m \);
- \( I \) satisfies \( a \neq b \) iff \( a^l \neq b^l \).

The satisfiability of fuzzy rough assertions with \( >, < \) is defined analogously. A fuzzy rough interpretation \( I \) satisfies an \( \text{FRSHIN} \) ABox \( AB \) iff it satisfies all fuzzy rough assertions in \( AB \); in this case, we say that \( I \) is a model of \( AB \).

A fuzzy rough interpretation \( I \) satisfies an \( \text{FRSHIN} \) knowledge base \( \sum \) iff it satisfies all axioms in \( \sum \); in this case, we say that \( I \) is a model of \( \sum \). An \( \text{FRSHIN} \) knowledge base \( \sum \) is satisfiable (unsatisfiable) iff there exists (does not exist) a fuzzy rough notion \( I \) which satisfies all axioms in \( \sum \). An \( \text{FRSHIN} \)-concept \( C \) is satisfiable (unsatisfiable) w.r.t. an RBox \( RB \) and a TBox \( TB \) iff there exists (does not exist) some model \( I \) of \( RB \) and \( TB \) for which there is some \( a \in A^l \) such that \( C^l(a^l) = n \), and \( n \in (0,1] \). Let \( C \) and \( D \) be two \( \text{FRSHIN} \)-concepts. We say that \( C \) is subsumed by \( D \) w.r.t. \( RB \) and \( TB \) if for every model \( I \) of \( RB \) and \( TB \) it holds that, \( \forall d \in A^l, C^l(d) \leq D^l(d) \). Furthermore, an \( \text{FRSHIN} \) ABox \( AB \) is consistent w.r.t. \( RB \) and \( TB \) if there exists a model of \( RB \) and \( TB \) that is also a model of \( AB \). Moreover, given a fuzzy rough concept axiom or a fuzzy rough assertion \( \psi \in \{ C \sqsubseteq D, C \equiv D, x > n, \} \), where \( \triangledown \) stands for \( >, \geq, \leq \) and \(<, \), an \( \text{FRSHIN} \) knowledge base \( \sum \) entails \( \psi \), written \( \sum \models \psi \), iff all models of \( \sum \) also satisfy \( \psi \).

Comparing with classical DLs [3] and fuzzy DLs [33,39,54,58,60,62], the most distinguished characteristic of FRDL \( \text{FRSHIN} \) is that TBox and ABox have the fuzzy lower approximation concepts and fuzzy upper approximation concepts in \( \text{FRSHIN} \).
From the properties of the lower approximation and upper approximation in fuzzy rough set theory (see [18,19,52,67] for details), it is easy to see that FRSHIN has properties as follows:

**Theorem 1.** For any concept $C$ and $D$ in FRSHIN, their fuzzy lower and upper approximation concepts satisfy the following properties:

1. $\top = \top$;
2. $\bot = \bot$;
3. $C \cap D = C \cap D$;
4. $\overline{C} \cup \overline{D} = \overline{C} \cup \overline{D}$;
5. $C \subseteq D \Rightarrow \overline{C} \subseteq \overline{D}$;
6. $C \subseteq D \Rightarrow \overline{C} \subseteq \overline{D}$;
7. $\neg(C) = \neg(C)$;
8. $\neg(C) = \neg(C)$.

**Proof.**

1. For any fuzzy rough interpretation $I$, since $(\top)^I = \inf_{d' \in \mathcal{A}} \{\max\{1 - R^-(d, d'), \mathcal{T}^I(d')\}\}$ and $\mathcal{T}^I(d') = 1$ for all $d' \in \mathcal{A}$, then we have $\max\{1 - R^-(d, d'), \mathcal{T}^I(d')\} = 1$, hence $\inf_{d' \in \mathcal{A}} \{\max\{1 - R^-(d, d'), \mathcal{T}^I(d')\}\} = \mathcal{T}^I(d) = 1$. By the semantics of $\top$, $(\top)^I = 1$ for all $d \in \mathcal{A}$, therefore, $\top = \top$.

2. For any fuzzy rough interpretation $I$, since $(\bot)^I = \sup_{d' \in \mathcal{A}} \{\min\{R^+(d, d'), \bot^I(d')\}\}$ and $\bot^I(d') = 0$ for all $d' \in \mathcal{A}$, then we have $\min\{R^+(d, d'), \bot^I(d')\} = 0$, hence $\sup_{d' \in \mathcal{A}} \{\min\{R^+(d, d'), \bot^I(d')\}\} = (\bot)^I = 0$. By the semantics of $\bot$, $(\bot)^I = 0$ for all $d \in \mathcal{A}$, therefore, $\bot = \bot$.

3. For any fuzzy rough interpretation $I$, since

$$(C \cap D)^I(d) = \inf_{d' \in \mathcal{A}} \{\max\{1 - R^-(d, d'), (C \cap D)^I(d')\}\}$$

and

$$(C \cap D)^I(d') = \min\{C^I(d'), D^I(d')\},$$

then we have

$$\begin{align*}
(C \cap D)^I(d) &= \inf_{d' \in \mathcal{A}} \{\max\{1 - R^-(d, d'), \min\{C^I(d'), D^I(d')\}\}\} \\
&= \min \left\{ \inf_{d' \in \mathcal{A}} \{\max\{1 - R^-(d, d'), C^I(d')\}\}, \inf_{d' \in \mathcal{A}} \{\max\{1 - R^-(d, d'), D^I(d')\}\} \right\}.
\end{align*}$$

Since

$$\begin{align*}
(C \cap D)^I(d) &= \min\{(C)^I(d), (D)^I(d)\} \\
&= \min \left\{ \inf_{d' \in \mathcal{A}} \{\max\{1 - R^-(d, d'), C^I(d')\}\}, \inf_{d' \in \mathcal{A}} \{\max\{1 - R^-(d, d'), D^I(d')\}\} \right\},
\end{align*}$$

therefore, $C \cap D = C \cap D$.

4. For any fuzzy rough interpretation $I$, since

$$(\overline{C} \cup \overline{D})^I(d) = \sup_{d' \in \mathcal{A}} \{\min\{R^+(d, d'), (\overline{C} \cup \overline{D})^I(d')\}\}$$

and

$$(C \cup D)^I(d') = \max\{C^I(d'), D^I(d')\},$$
then we have
\[
(\overline{C} \sqcup \overline{D})^I(d) = \sup_{d' \in A^I} \{ \min\{R^i(d, d'), \max\{C^I(d'), D^I(d')\}\} \\
= \max \left\{ \sup_{d' \in A^I} \{ \min\{R^i(d, d'), C^I(d')\}, \sup_{d' \in A^I} \{ \min\{R^i(d, d'), C^I(d')\}\} \right\}.
\]
Since
\[
(\overline{C} \sqcup \overline{D})^I(d) = \max\{(\overline{C})^I(d), (\overline{D})^I(d)\}
\]
we have
\[
\overline{C} \sqcup \overline{D} = (\overline{C} \sqcup \overline{D}).
\]
(5) For any fuzzy rough interpretation \(I\), since \(C \subseteq D\), then we have \(\forall d \in A^I, C^I(d) \leq D^I(d)\). Since
\[
(\overline{C})^I(d) = \inf_{d' \in A^I} \{ \max\{1 - R^i(d, d'), C^I(d')\}\},
\]
\[
(\overline{D})^I(d) = \inf_{d' \in A^I} \{ \max\{1 - R^i(d, d'), D^I(d')\}\}
\]
and
\[
C^I(d') \leq D^I(d'),
\]
then we have \(\inf_{d' \in A^I} \{ \max\{1 - R^i(d, d'), C^I(d')\}\} \leq \inf_{d' \in A^I} \{ \max\{1 - R^i(d, d'), D^I(d')\}\}\), therefore, \((\overline{C})^I(d) \leq (\overline{D})^I(d)\), i.e., \(\overline{C} \subseteq \overline{D}\).
(6) For any fuzzy rough interpretation \(I\), since \(C \subseteq D\), then \(\forall d \in A^I, C^I(d) \leq D^I(d)\). Since
\[
(\overline{C})^I(d) = \sup_{d' \in A^I} \{ \min\{R^i(d, d'), C^I(d')\}\},
\]
\[
(\overline{D})^I(d) = \sup_{d' \in A^I} \{ \min\{R^i(d, d'), D^I(d')\}\}
\]
and
\[
C^I(d') \leq D^I(d'),
\]
then we have \(\sup_{d' \in A^I} \{ \min\{R^i(d, d'), C^I(d')\}\} \leq \sup_{d' \in A^I} \{ \min\{R^i(d, d'), D^I(d')\}\}\), therefore, \((\overline{C})^I(d) \leq (\overline{D})^I(d)\), i.e., \(\overline{C} \subseteq \overline{D}\).
(7) For any fuzzy rough interpretation \(I\), since
\[
(\neg(\overline{C}))^I(d) = 1 - (\overline{C})^I(d) = 1 - \inf_{d' \in A^I} \{ \max\{1 - R^i(d, d'), C^I(d')\}\}
\]
\[
= \sup_{d' \in A^I} \{ \min\{R^i(d, d'), 1 - C^I(d')\}\} = \sup_{d' \in A^I} \{ \min\{R^i(d, d'), (\neg C)^I(d')\}\} = ((\neg(\overline{C}))^I(d'),
\]
therefore, \(\neg(\overline{C}) = (\neg(\overline{C}))\).
(8) For any fuzzy rough interpretation \(I\), since
\[
(\neg(\overline{C}))^I(d) = 1 - (\overline{C})^I(d) = 1 - \sup_{d' \in A^I} \{ \min\{R^i(d, d'), C^I(d')\}\}
\]
\[
= \inf_{d' \in A^I} \{ \max\{1 - R^i(d, d'), 1 - C^I(d')\}\} = \inf_{d' \in A^I} \{ \max\{1 - R^i(d, d'), (\neg C)^I(d')\}\} = ((\neg(\overline{C}))^I(d'),
\]
therefore, \(\neg(\overline{C}) = (\neg(\overline{C}))\). □
Remark 1. In this paper, we introduce only one fuzzy role $R^\sim$ in FRDLs. As a matter of fact, explicit representation of the underlying indiscernibility relation may have an additional advantage that more than one indiscernibility relations can be represented simultaneously (just like the modal logics proposed by Orłowska et al. [16,49]). That is to say, we may introduce several fuzzy roles $R^\sim_1, \ldots, R^\sim_k$ in FRDLs, here we can express multi-fuzzy approximation concepts (fuzzy lower approximation concepts and fuzzy upper approximation concepts)

$$C_{1i}, C_{k_i}, \bar{C}_{1i}, \ldots, \bar{C}_{ki}$$

in FRDLs, where the underlying fuzzy relation for concept $C_i$ is $R^\sim_i$ $(1 \leq i \leq k)$. In other words, we have $C_{ij} \equiv R^\sim_j C_i$ and $\bar{C}_{ij} \equiv \bar{R}^\sim_j C_i$, where $R^\sim_j C_i$ and $\bar{R}^\sim_j C_i$ are the lower and upper approximations w.r.t. $R^\sim_j$, respectively. For each fuzzy approximation concept $C_{ij}$ or $\bar{C}_{ij}$, its semantics is defined as follows:

- $(C_{ij})^l(d) = \sup_{d' \in A^i} \{\min\{R^\sim_j(d, d'), (C_i)^l(d')\}\}$;
- $(C_{ij})^l(d) = \inf_{d' \in A^i} \{\max\{1 - R^\sim_j(d, d'), (C_i)^l(d')\}\}$.

Example 2 (SARS example continued). According to the clinic diagnosis criteria for SARS [41], there are mainly two kinds of diagnosis criteria for SARS: suspected diagnostic criteria and clinically diagnosed criteria. For example, one of the suspected diagnostic criteria is as follows:

- the patient has had close contact with SARS patients or similar cases in recent two weeks, or there is accurate evidence of SARS cases that have infected this patient (written as SDC$_1$);
- a fever of 38°C or higher, which is associated with more than one of the following pathological signs: cough, tachypnea, dyspnea, respiratory distress syndrome, moist rales, and pulmonary consolidation (written as SDC$_2$);
- no white blood cell (WBC) count rise, it may even decreases (written as SDC$_3$).

One of the clinically diagnosed criteria is as follows:

- the patient is living in or has visited an SARS epidemic-stricken area in the past two weeks (written as CDC$_1$);
- a fever of 38°C or higher, which is associated with more than one of the following pathological signs: cough, tachypnea, dyspnea, respiratory distress syndrome, moist rales, and pulmonary consolidation (written as CDC$_2$);
- no WBC count rise, it may even decreases (written as CDC$_3$);
- reticular change, flaky or striped infiltrative shadows in varying degrees found in the lungs (written as CDC$_4$);
- little effect shown after using antibiotics (written as CDC$_5$).

From the above diagnosis criteria for SARS, we know that there exists much vague knowledge such as “close contact”, “recent two weeks”, “cough”, “tachypnea”, “dyspnea”, “little effect”, “flaky shadows”, “striped shadows” in the definition of SARS.

According to the clinic diagnosis criteria for SARS, the FRDL knowledge base (TBox) $TB$ about SARS is as follows (partly), where the underlying fuzzy relations for concepts SARS, CDC$_1$, CDC$_2$, CDC$_3$, SDC$_1$, SDC$_3$, Cough, Tachypnea, Dyspnea, Flaky-shadows, Little-effect, and Close-contact are $R^\sim_1, R^\sim_2, R^\sim_3, R^\sim_4, R^\sim_5, R^\sim_6, R^\sim_7, R^\sim_8, R^\sim_9, R^\sim_{10}, R^\sim_{11}, R^\sim_{12}$, respectively:

$$TB = \{\text{CDC} \subseteq \text{SARS}_1, \text{SDC} \subseteq \text{SARS}_3, \text{CDC}_1 \cap \text{CDC}_2 \cap \text{CDC}_3 \cap \text{CDC}_4 \cap \text{CDC}_5 \subseteq \text{CDC}, \text{SDC}_1 \cap \text{SDC}_2 \cap \text{SDC}_3 \subseteq \text{SDC}, \text{Cough} \subseteq \text{CDC}_2, \text{Tachypnea} \subseteq \text{CDC}_3, \text{Dyspnea} \subseteq \text{CDC}_2, \text{Flaky-shadows} \subseteq \text{CDC}_3, \text{Little-effect} \subseteq \text{CDC}_4, \text{Close-contact} \subseteq \text{SDC}_5, \text{Close-contact} \subseteq \text{SDC}_6, \text{CDC}_1 \subseteq \text{CDC}_2, \text{CDC}_3 \subseteq \text{CDC}_4, \text{CDC}_5 \subseteq \text{CDC}_6\}.$$

3.2. Reasoning

The reasoning problems in FRSHIN are the same as that of fuzzy DLs, i.e., the reasoning problems in FRSHIN mainly include satisfiability, subsumption, entailment and ABox consistency reasoning problems. It is the same as
that of \textit{FSHIN}, in this paper, we only consider unfoldable TBoxes. A TBox is unfoldable if it contains no cycles and contains only unique introductions, i.e., concept axioms with only concept names appearing on the left-hand side and, for each concept name \(A\), there is at most one axiom in \(TB\) of which \(A\) appears on the left side. A knowledge base with an unfoldable TBox can be transformed into an equivalent one with an empty TBox by a transformation called unfolding, or expansion \cite{48,58}. Therefore, without loss of generality, we can limit our attention to the case of an empty TBox.

In this section, we give the satisfiability, subsumption, entailment and ABox consistency reasoning w.r.t. RBox. In the following, it is proved that the satisfiability, subsumption, entailment and ABox consistency reasoning w.r.t. RBox in \textit{FRSHIN} may be reduced to the corresponding reasoning in \textit{FSHIN} \cite{58}. Therefore, the reasoning of satisfiability, subsumption, entailment and ABox consistency of \textit{FRSHIN} may reason automatically through the reasoning mechanism of \textit{FSHIN}.

Given an arbitrary concept \(C\) in \textit{FRSHIN}, we define a translation function \(\cdot^t\): \textit{FRSHIN} \(\rightarrow\) \textit{FSHIN} from \textit{FRSHIN} to \textit{FSHIN} that fulfills the following conditions, where \(A\) denotes an atomic concept, \(R^\sim\) denotes a new role name:

- \(A^t = A\);
- \(T^t = T\);
- \(\bot^t = \bot\);
- \((\neg C)^t = \neg C^t\);
- \((C \cap D)^t = C^t \cap D^t\);
- \((C \cup D)^t = C^t \cup D^t\);
- \((\exists R.C)^t = \exists R.C^t\);
- \((\forall R.C)^t = \forall R.C^t\);
- \((\geq n R)^t = \geq n R\);
- \((\leq n R)^t = \leq n R\);
- \((\bigcap)^t = \exists R^\sim\;C^t\);
- \((\bigcup)^t = \forall R^\sim\;C^t\).

Given an arbitrary inclusion assertion \(C \subseteq D\) in \textit{FRSHIN}, we can translate the fuzzy rough concept axiom \(C \subseteq D\) in \textit{FRSHIN} into a fuzzy concept axiom \(C^t \subseteq D^t\) in \textit{FSHIN} using the above translation function \(\cdot^t\).

Given an arbitrary TBox \(TB = \{C_1 \subseteq D_1, \ldots, C_k \subseteq D_k\}\) in \textit{FRSHIN}, we can translate the TBox \(TB = \{C_1 \subseteq D_1, \ldots, C_k \subseteq D_k\}\) in \textit{FRSHIN} using the above translation function \(\cdot^t\).

Given an arbitrary ABox \(AB = \{(a_1 : C_1) \triangleright\triangleright n_1, \ldots, (a_p : C_p) \triangleright\triangleright n_p, ((b_1, d_1) : R_1) \triangleright\triangleright m_1, \ldots, ((b_q, d_q) : R_q) \triangleright\triangleright m_q\}\) in \textit{FRSHIN}, we can translate the ABox \(AB\) into ABox \(AB^t = \{(a_1 : C_1)^t \triangleright\triangleright m_1, \ldots, (a_p : C_p)^t \triangleright\triangleright m_p, ((b_1, d_1) : R_1)^t \triangleright\triangleright m_1, \ldots, ((b_q, d_q) : R_q)^t \triangleright\triangleright m_q\}\) in \textit{FSHIN} using the above translation function \(\cdot^t\).

Given an arbitrary RBox \(RB = \{\text{Trans}(R), R \subseteq S\}\) in \textit{FRSHIN}, since there are not concepts in \(RB\), so the RBox \(RB\) does not need to be translated, in other words, the corresponding RBox \(RB^t\) in \textit{FSHIN} is also \(\{\text{Trans}(R), R \subseteq S\}\), i.e., \(RB = RB^t\).

Remark 2. At present there are mainly two approaches to the rough set-based extensions of DL, i.e., the RDL approach in \cite{54} and the ROTEL approach in \cite{42}. From the syntax point of view, for the former, the alphabet of RDL is the same as that of the underlying DL and only the formation rules of concepts are extended; however, the latter approach extends the alphabet of the underlying DL with a set of indiscernibility roles \(\{E_1, E_2, \ldots, E_k\}\), and the approximate concepts are defined as \(\langle E \rangle C\) and \(\langle E \rangle C\), moreover, new role terms \(\langle E \rangle R\) and \(\langle E \rangle R\) are also defined, where \(E\) is an indiscernibility role, \(C\) is a concept, and \(R\) is a role term. Therefore, the ROTEL approach will allow much more flexibility and expressivity on the language side than the RDL approach. For example, role terms \(\langle E \rangle R\) and \(\langle E \rangle R\) are not expressible in the RDL approach. Since there exist new role terms such as \(\langle E \rangle R\) and \(\langle E \rangle R\) in the ROTEL approach (not included in classical DLs \cite{3} and RDLs \cite{54}), so a new reasoning algorithm for the ROTEL approach must be provided. That is to say, the algorithm presented in \cite{42} is essentially an extension of tableau algorithm of classical DLs \cite{3}, i.e., the proposed completion rules are extensions of the classical rules. Concretely, \(\langle E \rangle C\)-rule, \(\langle E \rangle C\)-rule, \(\langle E \rangle R\)-rule, \(\langle E \rangle R\)-rule, ReflC-rule, ReflR-rule, Tran-rule, and Cong-rule are added to the completion rules. These rules are essentially introduced to deal with properties of equivalence relations and concept terms (or role terms) not included.
Given a concept C, an RBox RB in FRSHIN, C is the concept in FSHIN obtained from the translation function ·. C is satisfiable in a fuzzy rough interpretation w.r.t. RB, iff C ′ is satisfiable in a fuzzy interpretation w.r.t. RB. Formally, RB ̸= C ≡ ⊥, iff RB ̸= C ′ ≡ ⊥.

Proof. To show RB ̸= C ≡ ⊥, iff RB ̸= C ′ ≡ ⊥, it is sufficient to show that RB=C ≡ ⊥, iff RB=C ′ ≡ ⊥.

We first prove the only-if-direction (⇒), this can be shown by contradiction. Assume that RB ̸= C ′ ≡ ⊥, then there exists a model I = (A ′, · ′ ) of RB in FSHIN, and an individual a ∈ A ′, such that (C ′)(a ′) > 0.

Let I ′ = (A ′, R ′, · ′ ) = (A ′, R ′, · ). It is obvious to see that I ′ is a model of RB in FSHIN.

If C = A, where A denoted atomic concept, then C ′ = A. Since (C ′)(a ′) > 0, we have A ′(a ′) > 0, therefore A ′(a ′) > 0, i.e., C ′(a ′) > 0.

If C = ⊤, then C ′ = ⊤. Since (C ′)(a ′) = ⊤(a ′) = 1, we have C ′(a ′) = C ′(a ′) = C ′(a ′) = ⊤(a ′) = 1, therefore C ′(a ′) > 0.

Since (C ′)(a ′) > 0, we have C ′ ⊤, therefore C ̸= ⊤.

In the following, by induction over the structure of concept C we prove that there exists a model I ′ = (A ′, R ′, · ′ ) of RB in FRSHIN, such that C ′(a ′) > 0.

If C = ¬D, then C ′ = ¬D ′. Since (C ′)(a ′) > 0, we have (¬D ′)(a ′) > 0. By the interpretation of ¬D ′, (¬D ′)(a ′) = 1 − (D ′)(a ′), we obtain 1 − (D ′)(a ′) > 0. By induction hypothesis (D ′) ′ (a ′) = (D ′)(a ′), so 1 − (D ′)(a ′) > 0. Since C ′(a ′) = 1 − D ′(a ′) = 1 − (D ′)(a ′) = 1 − (D ′)(a ′), therefore C ′(a ′) > 0.

If C = D ∨ E, then C ′ = D ′ ∨ E ′. Since (C ′)(a ′) > 0, we have (D ′ ∨ E ′)(a ′) > 0. By the interpretation of D ′ ∨ E ′, (D ′ ∨ E ′)(a ′) = min((D ′)(a ′), (E ′)(a ′)), so min((D ′)(a ′), (E ′)(a ′)) > 0. By induction hypothesis D ′(a ′) = (D ′)(a ′) and E ′(a ′) = (E ′)(a ′), therefore min(D ′(a ′), E ′(a ′)) > 0. Since C ′(a ′) = min(D ′(a ′), E ′(a ′)), therefore C ′(a ′) > 0.

If C = D ∪ E, then C ′ = D ′ ∪ E ′. Since (C ′)(a ′) > 0, we have (D ′ ∪ E ′)(a ′) > 0. By the interpretation of D ′ ∪ E ′, (D ′ ∪ E ′)(a ′) = max((D ′)(a ′), (E ′)(a ′)), so max((D ′)(a ′), (E ′)(a ′)) > 0. By induction hypothesis D ′(a ′) = D ′(a ′) and E ′(a ′) = E ′(a ′), hence max(D ′(a ′), E ′(a ′)) > 0. Since C ′(a ′) = max(D ′(a ′), E ′(a ′)), therefore C ′(a ′) > 0.

If C = ∃R.D, then C ′ = ∃R ′.D ′. Since (C ′)(a ′) > 0, so (∃R ′.D ′)(a ′) > 0. By the interpretation of ∃R.D ′, (∃R ′.D ′)(a ′) = supD ′∈A ′[min(D ′(a ′), (d ′)(a ′)), (D ′(a ′))(d ′)(a ′))], so supD ′∈A ′[min(R ′(a ′), (d ′)(D ′)(d ′))) > 0.

Example 3 (SARS example continued). We can translate the FRDL knowledge base (TBox) TB of Example 2 into the following fuzzy DL knowledge base (TBox) TB ′:

\[
TB ′ = \{ CDC \subseteq \forall R_1^- . SARS, SDC \subseteq \exists R_1^+ . SARS, CDC_1 \cap CDC_2 \subseteq CDC_3 \cap CDC_4 \cap CDC_5 \subseteq CDC, SDC_1 \cap SDC_2 \cap SDC_3 \subseteq SDC, \forall R_7^- . Tachypnea \subseteq \forall R_8^- . CDC_2, \forall R_9^- . Dyspnea \subseteq \forall R_9^- . CDC_2, \forall R_{10}^- . Flaky-shadows \subseteq \forall R_{11}^- . CDC_2, \forall R_{12}^- . Close-contact \subseteq \forall R_5^- . SDC_2, \forall R_{12}^- . Close-contact \subseteq \forall R_5^- . SDC_1, \forall R_{12}^- . CDC_1 \equiv \forall R_2^- . CDC_1, \forall R_{12}^- . SDC_3 \equiv \forall R_6^- . SDC_3 \}\]
By induction hypothesis $D^{I'}((d')I') = (D')^{I'}((d')I')$, since $R^I(aI',(d')I') = R^I(aI',(d')I')$, hence $\sup_{d' \in dA'}[\min[R^I(aI',(d')I'), D^{I'}((d')I')]] > 0$. Since $C^I(aI') = (\exists R.D)^I(aI') = \sup_{d' \in dA'}[\min[R^I(aI',(d')I'), D^{I'}((d')I')]]$, therefore $C^I(aI') > 0$.

If $C = \exists R.D$, then $C' = \exists R.DI'$. Since $(C')^{I'}(aI') > 0$, so $(\exists R.D)^I(aI') > 0$. By the interpretation of $\exists R.DI'$, $(\exists R.DI')^{I'}(aI') = \inf_{d' \in dA'}[\max[1 - R^I(aI',(d')I'), D^{I'}((d')I')]]$, so $\inf_{d' \in dA'}[\max[1 - R^I(aI',(d')I'), D^{I'}((d')I')]] > 0$. By induction hypothesis $D^{I'}((d')I') = (D')^{I'}((d')I')$, since $R^I(aI',(d')I') = R^I(aI',(d')I')$, hence $\inf_{d' \in dA'}[\max[1 - R^I(aI',(d')I'), D^{I'}((d')I')]] > 0$. Since $C^I(aI') = (\exists R.D)^I(aI') = \inf_{d' \in dA'}[\max[1 - R^I(aI',(d')I'), D^{I'}((d')I')]]$, therefore $C^I(aI') > 0$.

If $C = \forall R.D$, then $C' = \forall R.DI'$. Since $(C')^{I'}(aI') > 0$, so $(\forall R.D)^I(aI') > 0$. By the interpretation of $\forall R.DI'$, $(\forall R.DI')^{I'}(aI') = \inf_{d' \in dA'}[\min[1 - R^I(aI',(d')I'), D^{I'}((d')I')]]$, so $\inf_{d' \in dA'}[\min[1 - R^I(aI',(d')I'), D^{I'}((d')I')]] > 0$. By induction hypothesis $D^{I'}((d')I') = (D')^{I'}((d')I')$, since $R^I(aI',(d')I') = R^I(aI',(d')I')$, hence $\inf_{d' \in dA'}[\min[1 - R^I(aI',(d')I'), D^{I'}((d')I')]] > 0$. Since $C^I(aI') = (\forall R.D)^I(aI') = \inf_{d' \in dA'}[\min[1 - R^I(aI',(d')I'), D^{I'}((d')I')]]$, therefore $C^I(aI') > 0$.

Therefore, there exists a model $I' = (A', R', \bullet')$ of $RB$, and an individual $a \in A'$, such that $C^I(aI') > 0$. This is an obvious contradiction to $RB = C \equiv \bot$. Thus, our assumption that $TB' \not\equiv C' \equiv \bot$ is refuted, which completes the proof of $TB' \equiv C' \equiv \bot$.

We prove the if-direction ($\Rightarrow$), which can also be shown by contradiction. Assume that $RB \not\equiv C \equiv \bot$, then there exists a model $I = (A, R, \bullet)$ of $RB$, and an individual $a \in A$, such that $C^I(aI) > 0$. In the following, by induction over the structure of concept $C$ we prove that there exists a model $I' = (A', \bullet')$ of $RB$, such that $(C')^{I'}(aI') > 0$. The proof is similar to the proof of ($\Leftarrow$). In the following, we only prove the cases $D' \sqcap E' = \exists R.D'$ and $\forall R.\neg D'$.

Let $I' = (A', \bullet')$. It is obvious to see that $I'$ is a model of $RB$ in $FSHIN$.

If $C' = D' \sqcap E'$, then $C = D \sqcap E$. Since $C^I(aI) > 0$, we have $(D \sqcap E)^I(aI') > 0$. By the interpretation of $D \sqcap E$, $(D \sqcap E)^I(aI') = \min[D^I(aI'), E^I(aI')]$, we have $\min[D^I(aI'), E^I(aI')] > 0$. By induction hypothesis $D^{I'}((d')I') = (D')^{I'}((d')I')$, since $R^I(aI',(d')I') = R^I(aI',(d')I')$, we have $\min[D^{I'}((d')I'), E^{I'}((d')I')] > 0$. Since $(C')^{I'}(aI') = (D' \sqcap E')^{I'}(aI') = \min[D^I(aI'), E^I(aI')]$, therefore $(C')^{I'}(aI') > 0$.

If $C' = \exists R.D'$, then $C = \exists R.D$. Since $C^I(aI) > 0$, we have $(\exists R.D)^I(aI') > 0$. By the interpretation of $\exists R.D$, $(\exists R.D)^I(aI') = \sup_{d' \in dA'}[\min[R^I(aI',dI'), D^I(dI')]]$, we have $\sup_{d' \in dA'}[\min[R^I(aI',dI'), D^I(dI')]] > 0$. By induction hypothesis $D^{I'}((d')I') = (D')^{I'}((d')I')$, since $R^I(aI',(d')I') = R^I(aI',(d')I')$, we have $\sup_{d' \in dA'}[\min[R^I(aI',dI'), (D')^{I'}(dI')] > 0$. Since $(C')^{I'}(aI') = (\exists R.D)^I(aI') = \sup_{d' \in dA'}[\min[R^I(aI',dI'), (D')^{I'}(dI')]]$, therefore $(C')^{I'}(aI') > 0$.

If $C' = \forall R.\neg D'$, then $C = D$. Since $C^I(aI) > 0$, we have $(D^I(aI')) > 0$. By the interpretation of $D$, $(D^I(aI')) = \inf_{d' \in dA'}[\max[1 - R^I(aI',dI'), D^I(dI')]]$, we have $\inf_{d' \in dA'}[\max[1 - R^I(aI',dI'), D^I(dI')]] > 0$. By induction hypothesis $D^{I'}((d')I') = (D')^{I'}((d')I')$, since $R^I(aI',(d')I') = R^I(aI',(d')I')$, we have $\inf_{d' \in dA'}[\max[1 - R^I(aI',dI'), (D')^{I'}(dI')] > 0$. Since $(C')^{I'}(aI') = (\forall R.\neg D')^{I'}(aI') = \inf_{d' \in dA'}[\max[1 - R^I(aI',dI'), (D')^{I'}(dI')]]$, therefore $(C')^{I'}(aI') > 0$.

Therefore, there exists a model $I' = (A', \bullet')$ of $RB$, and an individual $a \in A'$, such that $(C')^{I'}(aI') > 0$. This is an obvious contradiction to $RB = C \equiv \bot$. Thus, our assumption that $RB \not\equiv C \equiv \bot$ is refuted, which completes the proof of $RB = C \equiv \bot$. □

Obviously, Theorem 2 is the extension of Proposition 1 in [literature 54], i.e., the Proposition 1 in literature [54] is to translate the concept satisfiability reasoning w.r.t. TBox in rough DL RDL into the concept satisfiability reasoning w.r.t. TBox in classical description logic DL based on Pawlak rough set theory [50,51], but Theorem 2 is to translate the concept satisfiability reasoning w.r.t. RBox (and TBox) in $FSHIN$ into the concept satisfiability reasoning w.r.t. RBox (and TBox) in $FSHIN$ [58] based on fuzzy rough set theory [52,67].

In the following, we address the subsumption problem, i.e., deciding whether $RB = C \equiv D$, where $RB$ is an RBox, $C$ and $D$ are two concepts.
Thorem 3. Given two concepts \( C, D \), and an RBox \( RB \) in FRSHIN, \( C^t, D^t \) are the concepts in FRSHIN obtained from the translation function \( \bullet^t \), respectively. \( C \) is subsumed by \( D \) w.r.t. \( RB \) in FRSHIN iff \( C^t \) is subsumed by \( D^t \) w.r.t. \( RB \) in FRSHIN. Formally, \( RB \vdash C \subseteq D \) iff \( RB \vdash C^t \subseteq D^t \).

Proof. We first prove the only–if-direction (\( \Rightarrow \)), this can be shown by contradiction. Suppose to the contrary that \( RB \not\vdash C^t \subseteq D^t \) holds. Therefore, there exists a model \( I = (\mathcal{A}^t, \bullet^t) \) of \( RB \) in FRSHIN, an individual \( a \in \mathcal{A}^t \), and \( n > 0 \), such that \((C^t)^{(d^t)}(a^t) \geq n \) and \((D^t)^{(d^t)}(a^t) < n \).

Let \( I' = (\mathcal{A}^t, \bullet^t, I') \). It is obvious to see that \( I' \) is a model of \( RB \) in FRSHIN.

But, from \( RB \vdash C \subseteq D \), by the interpretation of \( C \subseteq D, \forall d \in \mathcal{A}^t, (C^t)^{(d^t)}(a^t) \leq D^t(a^t) \). By the proof of Theorem 2, we have \((C^t)^{(d^t)}(a^t) \geq n \) and \((D^t)^{(d^t)}(a^t) < n \). Therefore, there exists a model \( I = (\mathcal{A}^t, \bullet^t) \) of \( RB \) in FRSHIN, an individual \( a \in \mathcal{A}^t \), and \( n > 0 \), such that \((C^t)^{(d^t)}(a^t) \geq n \) and \((D^t)^{(d^t)}(a^t) < n \).

Let \( I' = (\mathcal{A}^t, \bullet^t) \). It is obvious to see that \( I' \) is a model of \( RB \) in FRSHIN.

But, from \( RB \vdash C \subseteq D \), by the interpretation of \( C \subseteq D, \forall d \in \mathcal{A}^t, (C^t)^{(d^t)}(a^t) \leq D^t(a^t) \). By the proof of Theorem 2, we have \((C^t)^{(d^t)}(a^t) \geq n \) and \((D^t)^{(d^t)}(a^t) < n \). Therefore, there exists a model \( I = (\mathcal{A}^t, \bullet^t) \) of \( RB \) in FRSHIN, an individual \( a \in \mathcal{A}^t \), and \( n > 0 \), such that \((C^t)^{(d^t)}(a^t) \geq n \) and \((D^t)^{(d^t)}(a^t) < n \).

The proof of (\( \Leftarrow \)) is similar to the proof of (\( \Rightarrow \)) of Theorem 3. \( \square \)

Now we address the entailment problem, i.e., deciding whether \( RB, AB \models \psi \), where \( RB \) is an RBox, \( AB \) is an ABox, \( \psi \in \{ C \subseteq D, C \equiv D, \models \} \), \( \models \) stands for \( \models, \models, \models \) and \( \models \). \( C \) and \( D \) are two concepts, \( \models \) is an assertion of the form \( a : C, \{ a, b \} : R \).

Thorem 4. Given a fuzzy rough concept axiom or fuzzy rough assertion \( \psi \in \{ C \subseteq D, C \equiv D, \models \} \), an RBox \( RB \) and an ABox \( AB \) in FRSHIN, \( \psi' \in \{ C^t \subseteq D^t, C^t \equiv D^t, \models \} \) and \( \psi' \) are the fuzzy concept axiom or fuzzy assertion and ABox in FRSHIN obtained from the translation function \( \bullet^t \), respectively. \( RB \) and \( AB \) entails \( \psi \) in FRSHIN iff \( RB \) and \( AB \) entails \( \psi' \) in FRSHIN. Formally, \( \langle RB, AB \rangle \models \psi \) iff \( \langle RB, AB \rangle \models \psi' \).

Proof. We only prove the case \( \models \subseteq D \). The cases for other fuzzy rough concept axiom or fuzzy rough assertion \( \psi \in \{ C \equiv D, \models \} \) are similar.

We first prove the only–if-direction (\( \Rightarrow \)), this can be shown by contradiction. Suppose to the contrary that \( RB, AB \not\models \psi \) holds. Therefore, there exists a model \( I = (\mathcal{A}^t, \bullet^t) \) of \( RB \) and \( AB \) in FRSHIN, an individual \( a \in \mathcal{A}^t \), and \( n > 0 \), such that \((C^t)^{(d^t)}(a^t) \geq n \) and \((D^t)^{(d^t)}(a^t) < n \).

Let \( I' = (\mathcal{A}^t, \bullet^t) \). It is obvious to see that \( I' \) is a model of \( RB \) and \( AB \) in FRSHIN.

But, from \( \langle RB, AB \rangle \models C \subseteq D \), by the interpretation of \( C \subseteq D, \forall d \in \mathcal{A}^t, (C^t)^{(d^t)}(a^t) \leq D^t(a^t) \). By the proof of Theorem 2, we have \((C^t)^{(d^t)}(a^t) \geq n \) and \((D^t)^{(d^t)}(a^t) < n \). Therefore, there exists a model \( I = (\mathcal{A}^t, \bullet^t) \) of \( RB \) in FRSHIN, an individual \( a \in \mathcal{A}^t \), and \( n > 0 \), such that \((C^t)^{(d^t)}(a^t) \geq n \) and \((D^t)^{(d^t)}(a^t) < n \).

The proof of (\( \Leftarrow \)) is similar to the proof of (\( \Rightarrow \)) of Theorem 3. \( \square \)

Finally, we show the ABox consistency problem.

Thorem 5. Given an RBox \( RB \) and an ABox \( AB \) in FRSHIN, \( AB^t \) is the ABox in FRSHIN obtained from the translation function \( \bullet^t \). \( AB \) is consistent w.r.t. \( RB \) in FRSHIN iff \( AB^t \) is consistent w.r.t. \( RB \) in FRSHIN.

Proof. Without loss of generality, we assume \( AB = \{ z_1 \models a_1, \ldots, z_k \models a_k \} \). Therefore, \( AB^t = \{ (z_1)^t \models a_1, \ldots, (z_k)^t \models a_k \} \). In the following, we only show the case \( z_1 = (a_1 : C_1), \ldots, z_k = (a_k : C_k) \). Other cases are similar.

We first prove the only–if-direction (\( \Rightarrow \)). Assume that \( AB \) is consistent w.r.t. \( RB \) in FRSHIN. Therefore, there exists a model \( I = (\mathcal{A}^t, \bullet^t) \) of \( RB \) that \( I \) is also a model of \( AB \), i.e., \( (C_1)^t((a_1)^t) \models a_1, \ldots, (C_k)^t((a_k)^t) \models a_k \). Since \( (z_1)^t = (a_1 : C_1), \ldots, (z_k)^t = (a_k : C_k) \). By the interpretation of
Given two concepts \( C \) and \( D \), a fuzzy rough assertion \( \mathbf{ax} \prec \mathbf{an} \), an RBox \( \mathbb{RB} \) and an ABox \( \mathbb{AB} \) in FSHIN, \( C \) and \( D \), \( \mathbf{ax} \prec \mathbf{an} \), and \( \mathbb{AB} \) are the concepts, fuzzy assertion, and ABox in FSHIN obtained from the translation function \( \bullet' \), respectively, then

\[
\begin{align*}
(1) & \quad C \text{ is satisfiable w.r.t. } \mathbb{RB}, \text{ iff } \{(a : C') > 0\} \text{ w.r.t. } \mathbb{RB} \text{ is consistent;} \\
(2) & \quad \mathbb{RB} \equiv C \subseteq D \text{ iff } \{(a : C') \geq n, (a : D') < n\}, \text{ for both } n \in \{n_1, n_2\}, n_1 \in (0, 0.5] \text{ and } n_1 \in (0.5, 1], \text{ is unsatisfiable;} \\
(3) & \quad \mathbb{RB}, \mathbb{AB} \equiv \mathbf{ax} \prec \mathbf{an} \text{ iff } \{(a : C'), (a : D') \prec \mathbf{an} \}, \text{ for both } n \in \{n_1, n_2\}, n_1 \in (0, 0.5] \text{ and } n_1 \in (0.5, 1], \text{ is unsatisfiable.}
\end{align*}
\]

From Theorems 5 and 6, we know that the satisfiability, subsumption, entailment and ABox consistency reasoning w.r.t. RBox in FSHIN may be reduced to the corresponding reasoning in FSHIN. From the definition of the translation function \( \bullet' \), translating the concepts, ABox and TBox in FSHIN into concepts, ABox and TBox in FSHIN, respectively, can be carried out in polynomial time, so the complexity of reasoning in FSHIN is the same as that of FSHIN.

4. Discussion

FRDLs are two-dimensional DLs, in other words, FRDLs consist of fuzzy description logics (FDLs) and rough description logics (RDLs), i.e., FRDLs = FDLs + RDLs. When constructing a two-dimensional FRDL, a number of design decisions have to be made. For example, one has to choose a concrete FDL and a RDL to be combined. Regarding the FDL part, some important decisions are whether to apply the number restrictions [4,65], nominals [11,65], inverse roles [11,27] and concrete domains [46,47] to DL concepts, and whether to apply the role hierarchies [27] and GCIs [28] to DL TBoxes, and whether to apply the reflexive role axioms, symmetric role axioms and transitive
role axioms \([25,27]\) to DL RBoxes. Regarding the RDL part, an important decision is which fuzzy rough set theory \([15,18,19,32,43,52,67]\) is applied to DLs. In the current section we roughly discuss some FRDLs based on different fuzzy rough sets or different fuzzy DLs. On the other hand, we also discuss the implementation of reasoning in FRDLs.

4.1. About fuzzy rough sets

In Section 3, we introduce the fuzzy rough extension of the DL \(SROIQ\), i.e., we propose the FRDL \(FRSHIN\) based on generalized fuzzy rough set theory \([52,67]\). In fact, there are several kinds of fuzzy rough sets \([15,18,19,32,43]\) other than generalized fuzzy rough set theory \([52,67]\). Therefore, we can obtain different FRDLs based on different fuzzy rough sets. In this section, we will discuss two kinds of FRDLs based on classical fuzzy rough set theory \([18]\) and generalized rough sets over fuzzy lattices \([43]\).

4.1.1. Classical fuzzy rough sets

It is well known that the main difference between classical fuzzy rough set theory \([18]\) and generalized fuzzy rough set theory \([52,67]\) is as follows: in classical fuzzy rough set theory, the binary fuzzy relation \(R\) is a fuzzy similarity relation (i.e., reflexive, symmetric and transitive fuzzy relation), but in generalized fuzzy rough set theory, the binary relation \(R\) is an arbitrary binary fuzzy relation.

Therefore, comparing with FRDLs based on generalized fuzzy rough set theory (G-FRDLs or FRDLs for short, e.g., \(FRSHIN\) presented in Section 3 is a kind of G-FRDL), the FRDLs based on classical fuzzy rough set theory (C-FRDLs for short) have to support reflexive fuzzy roles, symmetric fuzzy relations and transitive fuzzy relations. To the best of our knowledge, at present only one DL, i.e., \(SROIQ\) \([25]\), supports the reflexive roles, symmetric roles and transitive roles simultaneously, and no any fuzzy DL supports reflexive fuzzy roles, symmetric fuzzy relations and transitive fuzzy relations. Consequently, in order to present the C-FRDLs, we must extend the DL \(SROIQ\) with fuzzy capabilities, i.e., provide the fuzzy DL \(FSROIQ\) based on fuzzy sets and fuzzy logic which was presented by Zadeh \([72]\) firstly. And then we can obtain a kind of C-FRDL \(C-FRSROIQ\) using the method presented in Section 3.

Remark 3. Regarding the fuzzy DL \(FSROIQ\), it is easy to obtain the syntax and semantics of \(FSROIQ\) using the method of \([58]\), however, the reasoning algorithms of \(FSROIQ\) need to be studied in detail. Here we leave out this issue (reasoning algorithms), and we are quite confident that this issue can be solved by extending the \(FSHIN\) tableaux algorithms \([58]\).

Comparing with \(FRSHIN\), the role \(R^-\) in \(C-FRSROIQ\) is a fuzzy similarity role (i.e., reflexive, symmetric and transitive fuzzy role). Therefore, the translation function \(\Phi:C-FRSROIQ \rightarrow FSROIQ\) of \(C-FRSROIQ\) is different from that of \(FRSHIN\). It is easy to give the translation of concepts, concept axioms, TBoxes and ABoxes of \(C-FRSROIQ\), using similar method presented in Section 3. The main difference between the translation functions of \(C-FRSROIQ\) and \(FRSHIN\) is the translation of RBox. Since the role \(R^-\) in \(C-FRSROIQ\) is a fuzzy similarity role, therefore, we have to translate RBox as follows:

Given an arbitrary RBox \(RB = \{\beta_1, \ldots, \beta_k\}\) in \(C-FRSROIQ\), where \(\beta_i\) \((1 \leq i \leq k)\) is the role axiom of the form \(\text{Sym}(R), \text{Asy}(R), \text{Tra}(R), \text{Ref}(R), \text{Irr}(R), \text{Dis}(R), \text{RoS} \subseteq R, \text{SoR} \subseteq R\), and \(\text{SoR} \subseteq R\), we can translate the \(RB = \{\beta_1, \ldots, \beta_k\}\) into RBox \(RB' = \{\text{Ref}(R^-), \text{Sym}(R^-), \text{Tra}(R^-), \beta_1, \ldots, \beta_k\}\) in \(FSROIQ\). That is to say, we need add the reflexive role axiom \(\text{Ref}(R^-)\), symmetric role axiom \(\text{Sym}(R^-)\) and transitive role axiom \(\text{Tra}(R^-)\) to RBox due to the fuzzy similarity role \(R^-\).

We can prove that the satisfiability, subsumption, entailment and ABox consistency reasoning w.r.t. RBox in \(C-FRSROIQ\) may be reduced to the corresponding reasoning in \(FSROIQ\), i.e., we can obtain the following theorem:

**Theorem 7.** Given two concepts \(C, D\), a fuzzy rough concept axiom or fuzzy rough assertion \(\psi \in \{C \sqsubseteq D, C \equiv D, \times \Rightarrow \}_{\psi}\) an RBox \(RB\) and an ABox \(AB\) in \(C-FRSROIQ\), \(C'\) and \(D'\), \(\psi' \in \{C' \sqsubseteq D', C' \equiv D', \times' \Rightarrow \}_{\psi'}\), \(RB'\) and \(AB'\) are the concepts, fuzzy concept axiom or fuzzy assertion, RBox and ABox in \(FSROIQ\) obtained from the above translation function \(\Phi\), respectively, then

1. \(C\) is satisfiable w.r.t. \(RB\), iff \(C'\) is satisfiable w.r.t. \(RB'\). Formally, \(RB \not\models C \sqsubseteq \bot\), iff \(RB' \not\models C' \sqsubseteq \bot\).
2. \(C\) is subsumed by \(D\) w.r.t. \(RB\) iff \(C'\) is subsumed by \(D'\) w.r.t. \(RB'\). Formally, \(RB \models C \sqsubseteq D\) iff \(RB' \models C' \sqsubseteq D'\).
(3) $RB$ and $AB$ entail $\psi$ iff $RB^I$ and $AB^I$ entail $\psi^I$. Formally, $(RB, AB) \models \psi$ iff $(RB^I, AB^I) \models \psi^I$.

(4) $AB$ is consistent w.r.t. $RB$ iff $AB^I$ is consistent w.r.t. $RB^I$.

The proof is similar to the proofs of Theorems 2–5.

4.1.2. Generalized rough sets over fuzzy lattices

It is well known that the main difference between fuzzy rough set theory [18] or generalized fuzzy rough set theory [52,67] and generalized rough sets over fuzzy lattices [43] is the semantics aspect: in fuzzy rough set theory or generalized fuzzy rough set theory, the domain of membership function is the unit interval $[0, 1]$, but in generalized rough sets over fuzzy lattices, the domain is a fuzzy lattice [20].

Therefore, comparing with G-FRDLs and C-FRDLs, the syntax of FRDLs based on generalized rough sets over fuzzy lattices (L-FRDLs for short) is the same as that of the corresponding G-FRDLs and C-FRDLs. For instance, the syntax of the L-FRDL $L$-$FRSHIN$ is the same as that of the $FRSHIN$; the syntax of the L-FRDL $L$-$FRSROIQ$ is the same as that of the C-FRDL $C$-$FRSROIQ$.

Since the interval $[0, 1]$ provided with the usual ordering is a lattice, therefore, the semantics of L-FRDLs is the extension of the semantics of G-FRDLs and C-FRDLs. For example, the semantics of $L$-$FRALC$ (the rough extension of $L$-$ALC$ [61]) is as follows:

For a lattice $L = (T, \preceq)$, where $T$ is a set of certainty values, $\preceq$ is a partial order over $T$, a fuzzy rough interpretation $I = (\Delta^I, R^-, \bullet^I)$ consists of a domain of interpretation $\Delta^I$, a fuzzy relation $R^-$ over $\Delta^I$, and an interpretation function

- mapping
- individuals as for the classical case, i.e., $a^I \not\equiv b^I$, if $a \not\equiv b$;
- a concept $C$ into a function $C^I : \Delta^I \rightarrow T$;
- a role $R$ into a function $R^I : \Delta^I \times \Delta^I \rightarrow T$.

The complete set of semantics of $L$-$FRALC$ is as follows: for all $d \in \Delta^I$,

- $\top^I(d) = t$;
- $\bot^I(d) = f$;
- $(C \cap D)^I(d) = C^I(d) \otimes D^I(d)$;
- $(C \cup D)^I(d) = C^I(d) \oplus D^I(d)$;
- $(\neg C)^I(d) = \neg C^I(d)$;
- $(\exists R.C)^I(d) = \bigoplus_{d' \in \Delta^I}[R^I(d, d') \otimes C^I(d')]$
- $(\forall R.C)^I(d) = \bigotimes_{d' \in \Delta^I}(-R^I(d, d') \oplus C^I(d'))$;
- $(C^I(d) = \bigoplus_{d' \in \Delta^I}(-R^I(d, d') \oplus C^I(d'))$
- $(C^I(d) = \bigotimes_{d' \in \Delta^I}(-R^I(d, d') \oplus C^I(d'))$;

where $\oplus$ and $\otimes$ are the meet and join operators induced by $\preceq$, $f$ and $t$ are the least and greatest element in $T$, respectively.

In fact, it is easy to obtain expressive FRDLs such as $L$-$FRSHIN$ (or $L$-$FRSROIQ$) by extending the DL $SHIN$ (or $SROIQ$) with fuzzy rough capabilities based on generalized rough sets over fuzzy lattices. Regarding the reasoning problems of $L$-$FRSHIN$ (or $L$-$FRSROIQ$), they are similar to that of $FRSHIN$ (or $C$-$FRSROIQ$). For example, $C$ is subsumed by $D$ w.r.t. $RB$ and $TB$ if for every model $I$ of $RB$ and $TB$ it holds that, $\forall d \in \Delta^I, C^I(d) \preceq D^I(d)$. Regarding the translation function of $L$-$FRSHIN$ (or $L$-$FRSROIQ$), they are the same as that of $FRSHIN$ (or $C$-$FRSROIQ$).

From the properties of the lower approximation and the upper approximation in generalized rough sets over fuzzy lattices (see [43] for details), it is easy to see that L-FRDLs such as $L$-$FRSHIN$ and $L$-$FRSROIQ$ have properties as follows:

**Theorem 8.** For any concept $C$ and $D$ in L-FRDLs, their lower and upper approximation concepts satisfy the properties (1)–(8) of Theorem 1 and the following properties:

1. $(C \cup D) \subseteq C \cup D$;
2. $(C \cap D) \supseteq (C \cap D)$.
The proofs of (1)–(8) are similar to Theorem 1. In the following, we only prove (9) and (10).

(9) Since $C \subseteq C \cup D$, by property (5), $C \subseteq C \cup D$, then we have $\langle C \rangle^l(d) \subseteq \langle C \cup D \rangle^l(d)$. Since $D \subseteq C \cup D$, by property (5), $D \subseteq C \cup D$, then we have $\langle D \rangle^l(d) \subseteq \langle C \cup D \rangle^l(d)$. Therefore, $\langle C \rangle^l(d) \cup \langle D \rangle^l(d) \subseteq \langle C \cup D \rangle^l(d)$, thus $\langle C \cup D \rangle \subseteq C \cup D$.

(10) Since $C \cap D \subseteq C$, by property (6), $\overline{C \cap D} \subseteq \overline{C}$, then we have $\langle C \cap D \rangle^l(d) \subseteq \langle \overline{C} \rangle^l(d)$. Since $C \cap D \subseteq D$, by property (6), $\overline{C \cap D} \subseteq D$, then we have $\langle C \cap D \rangle^l(d) \subseteq \langle D \rangle^l(d)$. Therefore, $\langle C \cap D \rangle^l(d) \subseteq \langle C \rangle^l(d) \cap \langle D \rangle^l(d)$, thus $\langle C \cap D \rangle \subseteq (\langle C \rangle \cap \langle D \rangle)$.

We can also prove that the satisfiability, subsumption, entailment and ABox consistency reasoning w.r.t. RBox in $L$-$FRSHIN$ (or $L$-$FRSROIQ$) may be reduced to the corresponding reasoning in $L$-$SHIN$ (or $L$-$SROIQ$).

Regarding the extension of rough set theory, there are several kinds of extensions of rough set theory such as probabilistic rough set theory [14,69,73] other than fuzzy rough set theory [18,19,43,52,67]. Therefore, a path for future research is an integration between the theories of probabilistic DLs [31,44] and rough DLs [54] based on probabilistic rough set theory [14,69,73].

4.2. About fuzzy DL

Fuzzy DLs are fuzzy extensions of classical DLs based on fuzzy sets and fuzzy logics theory which was presented several decades ago by Zadeh [72]. In the last decade a substantial amount of work has been carried out in the context of DLs [33,39,58,60,70].

In Section 3, the fuzzy DL part of the FRDL $FRSHIN$ is $FSHIN$ [58] which includes the inverse role constructor, transitive role axioms, role hierarchies and the number restrictions constructor. A fuzzy extension of $SHOIN(D)$, the corresponding DL of the ontology description language OWL DL, i.e., fuzzy DL $FSHOIN(D)$ is presented [45,62]. $FSHOIN(D)$ has nominals constructor and concrete domains other than the constructors of the $FSHIN$. Unfortunately, in $FSHOIN(D)$ only the syntax and semantics are provided and no reasoning algorithms. Regarding concrete domains, as far as we know the fuzzy DL presented till now, which also covers reasoning, is $FALC(D)$, appeared in [63]. Recently, Bobillo et al. [9] study fuzzy DLs under a semantics given by Gödel family of fuzzy operators, and show the decidability of a fuzzy extension of the logic $SROIQ$ [25], theoretical basis of the language OWL 2 [21], by providing a reasoning preserving procedure to obtain a crisp representation for it.

In all previous approaches reasoning with respect to simple and acyclic TBoxes was considered. Stoilos et al. [59] and Li et al. [40] propose methods to handle GCIs in fuzzy DLs such as $FALC$ and $FSHI$.

Obviously, from the fuzzy DLs point of view, future research direction for FRDLs is integrating more expressive fuzzy DLs including nominals, concrete domains and GCIs. We are quite confident that this is possible by using the method presented in Section 3.

Remark 4. Regarding the relationships between fuzzy DLs and FRDLs, on the one hand, FRDLs such as $FRSHIN$, the rough extensions to fuzzy DLs such as $FSHIN$ [58] that we introduce in this paper, allow for modeling of vague knowledge using fuzzy approximation concepts (fuzzy lower approximation concepts and fuzzy upper approximation concepts). Technically, FRDLs can be simulated with the corresponding fuzzy DLs without added expressiveness. For example, $FRSHIN$ can be simulated with the fuzzy DL $FSHIN$ without added expressiveness. In other words, our FRDLs are strictly speaking not more expressive than fuzzy DLs such as $FSHIN$, i.e., the fuzzy DLs are strong enough to cope with notions arising from rough approximations, but the fuzzy approximation concepts that we introduce are useful modeling devices for uncertain or imprecise knowledge (namely non-crisp concepts) such as SARS (see Section 3).

On the other hand, from the rough DLs [54] point of view, technically, rough DLs can be simulated with the corresponding classical DLs [3] without added expressiveness. The reason is that, in rough DLs such as $RDL$ [54], for every concept $C \in RDL$, the corresponding lower approximation $\overline{C}$ and upper approximation $\overline{C}$ also are concepts of $RDL$, i.e., $\overline{C}$ and $\overline{C}$ are two different separate concepts. Moreover, $\overline{C}$ and $\overline{C}$ can be translated into equivalent concepts of classical DLs. Therefore, rough DLs and classical DLs have same expressive power. However, it is well known that when it is impossible to formally define a concept $C$, we can often make use of the approximations (the lower approximation $\overline{C}$ and the upper approximation $\overline{C}$) together, i.e., approximate concept (or rough set) $(\overline{C}, \overline{C})$, to express
the concept $C$ in a precise way with explicit formal semantics, in other words, only single lower approximation $C$ or upper approximation $\bar{C}$ cannot express the concept $C$ in a precise way. In order to deal with this case, approximate concepts (or rough sets) have to be introduced to rough DLs. At this aspect, Jiang et al. [34] present a kind of new rough DL $\text{RDL}_{AC}$ based on approximate concepts. Here rough DLs based on approximate concept are the extensions of the corresponding classical DLs. Similarly with the rough DLs, FRDLs can also be extended based on fuzzy approximate concepts (or fuzzy rough sets), i.e., FRDLs based on fuzzy approximate concepts (FRDL$_{AC}$s for short) can be presented. Here FRDL$_{AC}$s have more expressive power than the corresponding fuzzy DLs. In particular, from the application point of view, approximate concepts (or rough sets) have been introduced to ontologies [17], additionally, fuzzy ontologies are necessary for practical applications such as semantic Web [64], therefore, fuzzy approximate concepts (or fuzzy rough sets) must be introduced to fuzzy ontologies. It is well known that DLs are the logic foundations of Web ontology language [2], i.e., the Web ontology languages are equivalent to DLs in theory. For example, the Web ontology language OWL Lite and OWL DL have a formal semantics and a reasoning support through a mapping to the expressive DLs $\text{SHIF}(D)$ and $\text{SHOIN}(D)$, respectively [26]. Therefore, FRDL$_{AC}$s are necessary for practical applications such as semantic Web. Note that we must study FRDLs (presented in this paper) before investigating FRDL$_{AC}$s.

4.3. Implementation

It is well known that different families of fuzzy operators lead to different fuzzy DLs, therefore, lead to different FRDLs. We can prove that the satisfiability, subsumption, entailment and ABox consistency reasoning in FRDLs may be reduced to the ABox consistency reasoning in the corresponding fuzzy DLs. Hence, we need to discuss the implementation of reasoning in the fuzzy DLs. In fuzzy logic, there are three main families of fuzzy operators: Lukasiewicz, Gödel and product [9]. Nevertheless, most of the works such as this paper in fuzzy DLs rely on the semantics of fuzzy set operators proposed by Zadeh [72]: Gödel conjunction and disjunction, Lukasiewicz negation and Kleene–Dienes implication. A few other works consider Lukasiewicz logic and Gödel logic. The mainly related reasoning algorithms and systems about fuzzy DLs are as follows.

Bobillo et al. [9] study fuzzy DLs under a semantics given by the Gödel family of fuzzy operators, define $f$-$\text{SROIQ}$ which is a fuzzy extension of the logic $\text{SROIQ}$ [25] (theoretical basis of the language OWL 2 [21]), and show the decidability of the $f$-$\text{SROIQ}$ by providing a reasoning preserving procedure to obtain a crisp representation for it. Therefore, reasoning in the FRDLs such as $\text{FRSROIQ}$ can be implemented using the reasoning algorithm of $f$-$\text{SROIQ}$ under Gödel semantics. In other words, we can reduce a fuzzy rough knowledge base KB into a crisp KB, moreover, the procedure preserves reasoning under Gödel semantics, so existing reasoners such as FaCT++ [66] and Pellet [57] could be applied to the resulting KB.

Bobillo et al. [7] present the DeLorean (DEscription LOgic REasoner with vAgueNess) system, a reasoner that supports a fuzzy extension of the DL $\text{SROIQ}$ under Zadeh semantics. Hence, we can use the DeLorean system to implement the reasoning of the FRDLs such as $\text{FRSROIQ}$ under Zadeh semantics. On the other hand, Bobillo et al. [7,8] also show that a fuzzy $\text{SROIQ}$ fuzzy knowledge base (KB) can be reduced into a crisp KB, and the procedure preserves reasoning under Zadeh semantics. Therefore, existing reasoners such as FaCT++ [66] and Pellet [57] could also be applied to the reasoning of the FRDLs such as $\text{FRSROIQ}$ under Zadeh semantics.

Bobillo and Straccia [10] present the $\text{fuzzyDL}$ system which is a DL reasoner supporting fuzzy logic reasoning. The reasoning algorithm uses a combination of a tableaux algorithm and an MILP (mixed integer linear programming) optimization problem. The $\text{fuzzyDL}$ system supports fuzzy DL $\text{SHIF}(D)$ under Lukasiewicz semantics and Zadeh semantics. Consequently, the $\text{fuzzyDL}$ system could be applied to the reasoning of the FRDLs such as $\text{FRSHIF}(D)$ under Lukasiewicz semantics and Zadeh semantics.

There exist some fuzzy DL reasoning algorithms other than reasoning system mentioned above. For example, Stoilos et al. [58] and Jiang et al. [33] present decision procedures for the KB satisfiability problem of the fuzzy DLs $\text{SHIN}$ and $\text{SHOIQ}$ under Zadeh semantics, respectively. Therefore, the decision procedures of the fuzzy DLs $\text{SHIN}$ and $\text{SHOIQ}$ can be applied to the reasoning of the FRDLs $\text{FRSHIN}$ and $\text{FRSHOIQ}$ under Zadeh semantics.

5. Conclusion

An integration between the theories of fuzzy DLs and rough DLs has been attempted by providing FRDLs based on fuzzy rough set theory. Interestingly, we allow fuzzy approximation concepts (fuzzy lower approximation concepts
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