THE MANIN–PEYRE CONJECTURE FOR SMOOTH SPHERICAL FANO THREEFOLDS

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Abstract. The Manin–Peyre conjecture is established for smooth spherical Fano threefolds of semisimple rank one and type $N$. Together with the previously solved case $T$ and the toric cases, this covers all types of smooth spherical Fano threefolds. The case $N$ features a number of structural novelties; most notably, one may lose regularity of the ambient toric variety, the height conditions may contain fractional exponents, and it may be necessary to exclude a thin subset with exceptionally many rational points from the count, as otherwise Manin’s conjecture in its original form would turn out to be incorrect.

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1. Introduction

Let $G$ be a connected reductive group over $\mathbb{Q}$. A normal $G$-variety $X$ is called spherical if a Borel subgroup of $G$ has a dense orbit in $X$. Spherical varieties are a very large and interesting class of varieties that admit a combinatorial description by spherical systems and colored fans [Lu, BP, LV].

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generalizing the combinatorial description of toric varieties. Indeed, if the acting group $G$ has semi-simple rank 0, then $G$ is a torus.

If $G$ has semi-simple rank 1, we may assume that $G = \text{SL}_2 \times \mathbb{G}_m^r$ for some $r \geq 0$. Let $G/H = (\text{SL}_2 \times \mathbb{G}_m^r)/H$ be the open orbit in $X$ and define $H'$ by $H' = H \cdot \mathbb{G}_m^r \subseteq \text{SL}_2 \times \mathbb{G}_m^r$. Then the homogeneous space $\text{SL}_2/H'$ is spherical, and there are three possible cases:

- either $H'$ is a maximal torus (the case $T$),
- or $H'$ is the normalizer of a maximal torus in $\text{SL}_2$ (the case $N$),
- or the homogeneous space $\text{SL}_2/H'$ is horospherical, in which case $X$ is isomorphic (as an abstract variety) to a toric variety.

In a recent paper [BBDG], the authors initiated a program to establish Manin’s conjecture for (split models over $\mathbb{Q}$ of) smooth spherical Fano varieties, based on (a) the combinatorial description of spherical varieties, (b) the universal torsor method, and (c) techniques from analytic number theory including the Hardy-Littlewood circle method and multiple Dirichlet series. In particular, in case $T$, we confirmed Manin’s conjecture for threefolds as well as for some higher-dimensional varieties. Here we relied on Hofschieder’s classification [Ho] of smooth spherical Fano threefolds over $\overline{\mathbb{Q}}$, which identifies four such cases having natural split models over $\overline{\mathbb{Q}}$.

In this paper we fully resolve the harder case $N$. This is achieved by a further development of the methods employed in [BBDG], and we proceed to describe the major new ingredients.

The first point concerns a general reformulation of Manin’s conjecture as a counting problem. Let $X$ be a smooth split projective variety over $\mathbb{Q}$ with big and semiample anticanonical class $\omega_X$ whose Picard group is free of finite rank. Assume that its Cox ring $\mathcal{R}(X)$ is finitely generated with precisely one relation with integral coefficients. This defines a canonical embedding of $X$ into a (not necessarily complete) ambient toric variety $Y$; it can be completed to a projective toric variety $Y$ such that the natural map $\text{Cl}(Y) \to \text{Cl}(X) = \text{Pic}(X)$ is an isomorphism and $-K_X$ is big and semiample on $Y$. Under the assumption that $Y$ is regular, Sections 2 – 4 in [BBDG] (culminating in [BBDG, Propositions 3.7 & 4.11]) provide a general scheme to parametrize the rational points on $X$ in terms of the universal torsor and to express the Manin–Peyre constant in terms of the Cox coordinates. The corresponding counting problem is in many cases amenable to techniques from analytic number theory. The assumption that $Y$ is regular holds for smooth Fano threefolds of type $T$, but may fail in the case of type $N$. Part 1 of the present paper generalizes the passage to the universal torsor to varieties $X$ for which $Y$ is not necessarily regular. This result is independent of the theory of spherical varieties and should therefore have applications elsewhere.

The second point is of analytic nature. The universal torsor of spherical varieties of semi-simple rank 1 and type $T$ has a defining equation of the form

$$x_{11}x_{12} − x_{21}x_{22} − \text{some monomial} = 0,$$

which needs to be analyzed subject to rather complicated height conditions. The fact that we have a decoupled bilinear form in four variables is crucial for the method and allows in particular an auxiliary soft argument based on lattice considerations. For type $N$, the equation takes the form

$$x_{11}x_{12} − x_{21}^2 − \text{some monomial} = 0.$$

From an analytic perspective, this may be very delicate. A considerable portion of this paper is devoted to the investigation of the particular equation

$$(1.1) \quad x_{11}x_{12} − x_{21}^2 − x_{31}x_{32}x_{33}^2 = 0$$

with variables constrained to dyadic boxes $|x_{11}| \asymp |x_{12}| \asymp |x_{21}| \asymp X$, $|x_{31}| \asymp |x_{32}| \asymp Y$, $|x_{33}| \asymp X/Y$.

Here $X$ is large, and $1 \leq Y \leq X$, and we need an asymptotic formula for the number of integer solutions where the error term saves a fixed power of $\min(Y, X/Y)$. It is conceivable that a modern variant of the circle method (like [HB]) can handle this, but this is not straightforward. There are considerable uniformity issues, since we need to deal simultaneously with the cases when $Y$ is
small, say $Y = \exp((\log \log X)^2)$ (in which case the equation looks roughly like a sum of two squares and a product), and $Y$ is large, say $Y = X/\exp((\log \log X)^2)$ (in which case the equation looks roughly like a sum of a square and two products). To keep track of uniformity, we will not use the circle method directly but instead apply Poisson summation to selected variables depending on the ranges of parameters. This is ultimately more or less equivalent to the delta-symbol method of Duke–Friedlander–Iwaniec [DFI], but it is in this case a more convenient packaging.

The shape of equation (1.1) offers a new feature that was not present in [BBDG], and for which in fact very few examples are known. In the special case where $-x_{31}x_{32}$ is a square, the equation (1.1) describes a split quaternary quadratic form over $\mathbb{Q}$; in the set of rational points where at least one Cox coordinate is zero.

We now describe our results in more detail. As explained in [BBDG, §11], there are three compactifications of smooth spherical Fano threefolds over $\mathbb{Q}$, i.e., the sum of two hyperbolic planes. In this case it is well-known (see e.g. [HB]) that the asymptotic formula contains an additional logarithm. In particular, Manin’s conjecture in its original form turns out to be wrong, and instead we need to prove a “thin subset version” of Manin’s conjecture: we first remove a certain portion from the variety with exceptionally many points and then confirm the conjecture for the remaining set. More precisely, there should be a thin subset $T$ of the set of rational points $X(\mathbb{Q})$ such that, for an anticanonical height $H$,

$$N_{X(\mathbb{Q}) \setminus T,H}(B) := \# \{ x \in X(\mathbb{Q}) \setminus T \mid H(x) \leq B \} = (1 + o(1))cB(\log B)^{rk \text{Pic } X - 1}$$

as in Manin’s conjecture with Peyre’s constant $c$. We refer the reader to the general discussion of this phenomenon in [LST], and to the (to our knowledge) only example of a smooth Fano variety [BBH] for which a thin version of Manin’s conjecture has appeared in print. Soon after this work has been circulated in manuscript form, Fano threefolds of Mori–Mukai type II.25 have been discussed in the preprint [BBH], providing yet another example where the thin set version of the Manin–Peyre conjecture is true.

We now describe our results in more detail. As explained in [BBDG, §11], there are three smooth spherical Fano threefolds over $\mathbb{Q}$ that are neither horospherical nor equivariant compactifications of $\mathbb{G}_a^3$; we construct split models $X_1, X_2, X_3$ over $\mathbb{Q}$ as in Table 1.

| rk Pic | Hofscheier | Mori–Mukai | torsor equation | label |
|--------|------------|------------|----------------|-------|
| 2      | $N_18$     | II.29      | $x_{11}x_{12} - x_{21}^2 - x_{31}x_{32}x_{33}$ | variety $X_1$ |
| 3      | $N_63$     | III.22     | $x_{11}x_{12} - x_{21}^2 - x_{31}x_{32}$ | variety $X_2$ |
| 3      | $N_19$     | III.19     | $x_{11}x_{12} - x_{21}^2 - x_{31}x_{32}$ | variety $X_3$ |

**Table 1.** Smooth Fano threefolds of type $N$ that are spherical, but not horospherical

More precisely, let $X_1$ the blow-up of the quadric $Q = \mathbb{V}(z_{11}z_{12} - z_{21}^2 - z_{31}z_{32}) \subset \mathbb{P}_Q^4$ in the conic $C_{33} = \mathbb{V}(z_{31}, z_{32})$. This is a smooth Fano threefold of type II.29 in the Mori–Mukai classification. We have a fibration $X_1 \to \mathbb{P}^1$ that is defined in Cox coordinates by $(x_{11} : \cdots : x_{33}) \mapsto (x_{31} : x_{32})$. As explained above, we must remove the thin subset

$$T_1 = \{(x_{11} : \cdots : x_{33}) \in X_1(\mathbb{Q}) \mid x_{31}x_{32} = -\Box \text{ or } x_{11}x_{12}x_{21}x_{31}x_{32}x_{33} = 0\}.$$

Let $W_2 = \mathbb{P}_\mathbb{Q}^1 \times \mathbb{P}_\mathbb{Q}^2$ with coordinates $(z_{01} : z_{02})$ and $(z_{11} : z_{12} : z_{21})$. Let $C_{32}$ be the curve $\mathbb{V}(z_{02}, z_{11}z_{12} - z_{21}^2)$ on $W_2$ and let $X_2$ be the blow-up of $W_2$ in $C_{32}$. This is a smooth Fano threefold of type III.22. We will see later that the height conditions (cf. (12.2) below) contain fractional exponents $\alpha^{(i)}_j$; see also [BT]. Let $T_2 \subset X_2(\mathbb{Q})$ be the set of rational points where at least one Cox coordinate is zero.

Let $X_3$ be the blow-up of the quadric $Q = \mathbb{V}(z_{11}z_{12} - z_{21}^2 - z_{31}z_{32}) \subset \mathbb{P}_Q^4$ in the points $P_{01} = \mathbb{V}(z_{11}, z_{12}, z_{21}, z_{31})$ and $P_{02} = \mathbb{V}(z_{11}, z_{12}, z_{21}, z_{32})$. Its type is III.19. As before, let $T_3 \subset X_3(\mathbb{Q})$ be the set of rational points where at least one Cox coordinate is zero.

In Sections 2.3 and 11, we will define natural anticanonical height functions $H_j : X_j(\mathbb{Q}) \to \mathbb{R}$, $j = 1, 2, 3$, using the anticanonical monomials in their Cox rings. We write $N_j(B) = N_{X_j(\mathbb{Q}) \setminus T_j, H_j}(B)$. 
Theorem 1.1. The Manin–Peyre conjecture holds for the smooth spherical Fano threefolds $X_2, X_3$ of semisimple rank one and type $N$, and a thin version of the Manin–Peyre conjecture holds for $X_1$. More precisely, there exist explicit constants $C_1, C_2, C_3$ such that

$$N_j(B) = (1 + o(1))C_jB(\log B)^{\text{rk Pic } X_j}$$

for $1 \leq j \leq 3$. The values of $C_j$ are the ones predicted by Peyre.

Together with previous results, this covers all types of smooth spherical Fano threefolds.

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**Part 1. Metrics and heights via Cox rings and universal torsors**

Universal torsors and Cox rings were introduced and studied by Colliot-Thélène and Sansuc [CTS1, CTS2] and Cox [Cox].

If a variety $X$ has a finitely generated Cox ring with one relation, this gives a description of $X$ as a hypersurface in a toric variety $Y$. The description of height and Tamagawa measures in [BBDG, Part 1] relies on the assumption [BBDG, (2.3)] that $Y$ can be chosen to be regular, which does not hold in our examples $X_2, X_3$. Here, we describe one approach how to circumvent this problem; see also [BT].

Our constructions of metrizations, heights, and Tamagawa measures on universal torsors follow the work of Salberger [Sal]; see also [BBS] and [BBDG]. This should be compared to the closely related work of Peyre on universal torsors for Manin’s conjecture [Pey2, Pey3, Pey4].

2. Heights and parametrization

We start with a situation similar to (but in several respects more general than) [BBDG, Section 2]. Let $Y^o$ be a smooth split toric variety over $\mathbb{Q}$. We do not assume that $Y^o$ is complete, but we still assume the weaker property that $Y^o$ has only constant regular functions. Let

$$\mathcal{R}(Y^o) \cong \mathbb{Q}[x_1, \ldots, x_J]$$

be its Cox ring, where $x_1, \ldots, x_J$ correspond to the torus invariant prime divisors in $Y^o$. Let $0 \neq \Phi \in \mathcal{R}(Y^o)$ be a homogeneous equation, and let $X \subseteq Y^o$ be the corresponding subvariety.

We assume that $X$ is smooth and projective, with big and semiample anticanonical class $-K_X$. Moreover, we assume that every torus orbit in $Y^o$ meets $X$. We also assume that the pullback map $\text{Pic } Y^o \to \text{Pic } X$ sends a big and semiample divisor class $L^o$ to $-lK_X$ for some $l \in \mathbb{Z}_{\geq 0}$. Finally, we assume $\Phi \in \mathbb{Z}[x_1, \ldots, x_J]$.

**Remark 2.1.** A situation as above can be naturally obtained starting from a smooth split projective variety over $\mathbb{Q}$ with big and semiample anticanonical class $-K_X$ and finitely generated Cox ring

$$\mathcal{R}(X) \cong \mathbb{Q}[x_1, \ldots, x_J]/(\Phi),$$

where $x_1, \ldots, x_J$ is a system of pairwise nonassociated Pic $X$-prime generators and $\Phi \neq 0$. By [ADHL, 3.2.5], there exists a canonical embedding into an ambient toric variety $Y^o$, and it satisfies all the above assumptions. Moreover, the pullback map $\text{Pic } Y^o \to \text{Pic } X$ is an isomorphism, and we may take $L^o = -K_X$ under this identification.

If $\Sigma$ is the fan of any toric variety, we write $\Sigma_{\text{max}}$ for the set of maximal cones and $\Sigma(1)$ for the set of rays.

Let $\Sigma^o$ be the fan of $Y^o$. The generators $x_1, \ldots, x_J \in \mathcal{R}(Y^o)$ are in bijection to the rays $\rho \in \Sigma^o(1)$; we also write $x_\rho$ for $x_i$ corresponding to $\rho$.

Let $Y$ be a completion of $Y^o$ such that the pullback map $\text{Cl } Y \to \text{Cl } Y^o = \text{Pic } Y^o$ is an isomorphism and $L^o$ is big and semiample on $Y$; let $\Sigma$ be the fan of this toric variety $Y$. We have $\Sigma(1) = \Sigma^o(1)$ and $\mathcal{R}(Y) = \mathcal{R}(Y^o)$. For example, we may choose the unique $Y$ such that $L^o$ is ample on $Y$, which exists by [ADHL, Proposition 2.4.2.6]; we call this $Y$ the standard small completion of $Y^o$. 

We do not assume that $Y$ is regular (in contrast to [BBDG, Section 2]). Let $\rho : Y'' \to Y$ be a toric desingularization of $Y$ that does not change the smooth locus of $Y$. Such a desingularization can be obtained by suitably subdividing the singular cones in the fan $\Sigma$ of $Y$ into a smooth fan $\Sigma''$.

Let $Y' \subset Y''$ be a toric subvariety with $Y^\circ \subset Y'$. This means that the fan $\Sigma'$ of $Y'$ is a subfan of $\Sigma''$ and contains $\Sigma^\circ$. If we write $\rho_{J+1}, \ldots, \rho_J$ for the rays in $\Sigma'(1) \setminus \Sigma(1)$, we can write

$$B(Y') = \mathbb{Q}[y_1, \ldots, y_J, y_{J+1}, \ldots, y_{J'}]$$

for the Cox ring of $Y'$, again with the correspondence between rays $\rho_i$ and generators $y_i$.

Since $X$ is smooth, we can identify it with its strict transform under $\rho : Y'' \to Y$; it is a hypersurface in $Y'$ defined by a homogeneous equation $\Phi'$ that is obtained from $\Phi$ by homogenizing (therefore, $y_i \mapsto x_i$ for $i \leq J$ and $y_i \mapsto 1$ for $i > J$ turns $\Phi'$ into $\Phi$). We obtain the following commutative diagram:

$$\begin{array}{ccc}
X & \longrightarrow & Y' \\
& \downarrow & \downarrow \\
X & \longrightarrow & Y''
\end{array}$$

Below, we will regard $X$ mostly as embedded into $Y'$.

For simplicity, we assume that every cone in $\Sigma'$ is the face of a maximal cone.

Let $U$ be the open torus in $Y'$. For each $\rho \in \Sigma'(1)$, we have a $U$-invariant Weil divisor $D_\rho$ defined by $y_\rho$ of class $[D_\rho] = \text{deg}(y_\rho) \in \text{Pic} Y'$. Let $D_0 := \sum_{\rho \in \Sigma'(1)} D_\rho$, which is an effective divisor of class $[D_0] = -K_{Y'}$.

Let $S \subset \Sigma'(1)$ be such that $\left\{\text{deg}(y_\rho) \mid \rho \notin S\right\}$ is a basis of $\text{Pic} Y'$. This is true if and only if $S$ is a basis of $N$. Therefore, we can write $\text{deg}(y_\rho') = \sum_{\rho \notin S} a^S_{\rho',\rho} \text{deg}(y_\rho)$ for each $\rho' \in \Sigma'(1)$ with certain $a^S_{\rho',\rho} \in \mathbb{Z}$. We define the rational section

$$z^S_\rho' := y_\rho'/ \prod_{\rho \notin S} y^{a^S_{\rho',\rho}}_\rho$$

of degree 0 in $\text{Pic} Y'$, with $z^S_\rho = 1$ for $\rho \notin S$. The $z^S_\rho$ for $\rho \in S$ define a chart

$$f^S : U^S \to A^S_Q, \quad P \mapsto (z^S_\rho(P))_{\rho \in S}$$

where $U^S$ is the open subset of $Y'$ where $y_\rho \neq 0$ for all $\rho \notin S$ (i.e., the complement of $\bigcup_{\rho \notin S} D_\rho$ in $Y'$). Note that $f^S$ is an isomorphism if and only if $S = \sigma(1)$ for some $\sigma \in \Sigma'_\text{max}$. For the open subset $X^S := X \cap U^S$ of $X$, the image $f^S(X^S) \subset A^S_Q$ is defined by

$$\Phi^S := \Phi(z^S_\rho) = \Phi'(y_\rho) / \prod_{\rho \notin S} y^{b^S_\rho}_{\rho}$$

(with $b^S_\rho \in \mathbb{Z}$ satisfying $\text{deg} \Phi' = \sum_{\rho \notin S} b^S_\rho \text{deg}(y_\rho)$) since $y_\rho \neq 0$ on $U^S$ for $\rho \notin S$.

2.1. Universal torsors and models. For universal torsors and Cox rings of toric varieties, see [CT$\text{S1}$, §4], [Cox], [Sal, §8].

Let $\pi' : Y'_0 \to Y'$ be the universal torsor as in [Sal, §8]. Then the restriction $\pi' : X_0 \to X$ to the preimage of $X \subset Y'$ is a torsor, but not necessarily universal since the acting torus (dual to Pic $Y'$) may be too large. The toric varieties $A^S_Q = A^S_{\Sigma'(1)} := Y'_1 \cap Y'_0 \to Y'$ have the fans $\Sigma'_1 \supset \Sigma'_0 \supset \Sigma'$. Here, the sets of rays $\Sigma'_1(1) = \Sigma'_0(1)$ are in natural bijection to $\Sigma'(1)$. The $r'$ irreducible components of $Z'_Y = Y'_1 \setminus Y'_0$ are defined by the vanishing of $x_\rho$ for all $\rho \in S'_j$, where the primitive collections

$$S'_1, \ldots, S'_r \subset \Sigma'(1)$$

are all sets with the following property: $S'_j \not\subseteq \sigma(1)$ for all $\sigma \in \Sigma'$, but for every proper subset $S''_j$ of $S'_j$, there is a $\sigma \in \Sigma'$ with $S''_j \subseteq \sigma(1)$.
From the fans and their maps, we may construct \( \mathbb{Z} \)-models \( \tilde{\pi}' : \tilde{Y}' \setminus \tilde{Z}' = Y_0' \rightarrow \tilde{Y}' \), again as in [Sal, §8]. By our assumption that \( \Phi \) has integral coefficients, we obtain \( \mathbb{Z} \)-models \( \tilde{\pi}' : \tilde{X}_1 \setminus \tilde{Z}_X = \tilde{X}_0 \rightarrow \tilde{X} \) of \( \pi' : X_1 \setminus Z_X = X_0 \rightarrow X \) by restricting everything to \( \Phi' = 0 \).

We assume:

\[
(2.3) \quad \text{The toric variety } Y' \text{ is chosen such that } \tilde{X} \text{ is proper over } \text{Spec } \mathbb{Z}.
\]

This is always possible since for \( Y' = Y'' \) the scheme \( \tilde{Y}' \) is projective over \( \text{Spec } \mathbb{Z} \).

**Proposition 2.2.** We have

\[
\tilde{X}_0(\mathbb{Z}) = \{ x = (x_\rho)_{\rho \in \Sigma'(1)} \in \mathbb{Z}^{\Sigma'(1)} : \Phi(x) = 0, \gcd\{x_\rho : \rho \in S'_0\} = 1 \text{ for all } j = 1, \ldots, r' \},
\]

\[
\tilde{X}_0(\mathbb{Z}_p) = \{ x = (x_\rho)_{\rho \in \Sigma'(1)} \in \mathbb{Z}_p^{\Sigma'(1)} : \Phi(x) = 0, p \nmid \gcd\{x_\rho : \rho \in S'_j\} \text{ for all } j = 1, \ldots, r' \}.
\]

The map \( \tilde{\pi}' \) induces a \( 2^{rk \text{Pic } Y'} : 1 \)-map \( \tilde{X}_0(\mathbb{Z}) \rightarrow \tilde{X}(\mathbb{Z}) = X(\mathbb{Q}) \).

**Proof.** The proof is as in [BBDG, Proposition 2.2] by (2.3). The referee kindly pointed out that an argument is also contained in the work of Peyre [Pey2, Pey3, Pey4]. \(\square\)

2.2. **Metrization of \( \omega_{X}^{-1} \) via Poincaré residues.** Let \( L \) be any big and semiample divisor class on \( Y' \) such that \( L|_X = -lK_X \). Then there exists a uniquely determined divisor \( E \) with support \( E \) in the exceptional locus of \( \rho \) such that

\[
L = -lK_{Y'} - l[X] + [E].
\]

The following lemma shows that (after possibly enlarging \( l \)) such an \( L \) can always be found.

**Lemma 2.3.** After suitably enlarging \( l \), there exists a big and semiample divisor class on \( Y' \) such that \( L|_X = -lK_X \).

**Proof.** Since \( L^o \) is ample, it is \( \mathbb{Q} \)-Cartier on \( Y \). Hence, after replacing \( l \) by a positive multiple, we may assume that \( L^o \) is Cartier on \( Y \). Let \( L'' \) be the pullback \( \rho^*(L^o) \). Then \( L'' \) is big and semiample on \( Y'' \), and moreover \( L''|_X = -lK_X \). The same is then true for \( L = L''|_{Y'} \). \(\square\)

As before, let \( S \subset \Sigma'(1) \) be such that \( \{ \deg(y_\rho) : \rho \notin S \} \) is a basis of \( \text{Pic } Y' \). Hence there is a unique (not necessarily effective) Weil divisor \( D(S) = \sum_{\rho \in S} a_\rho^\rho D_\rho \) of class \( -K_{Y'} - [X] \). The characters defined by \( z_\rho^S \) for \( \rho \in S \) form a basis of \( M = \text{Hom}(U, \mathbb{G}_m) \).

By [CLS, Proposition 8.2.3], we have a global nowhere vanishing section \( s_{Y'} \) of \( \omega_{Y'}(D_0) \) (defined up to sign, independent of \( S \); we have \( s_{Y'} = \Omega_0/\prod_{\rho \in \Sigma'(1)} y_\rho \) for \( \Omega_0 \) from [CLS, (8.2.3)]) whose restriction to the open subset \( U^S \) is \( \pm \bigwedge_{\rho \in S} \frac{dz_\rho^S}{x_\rho^S} \).

With \( y^0 = \prod_{\rho \in \Sigma'(1)} y_\rho \), for each \( S \) as above,

\[
\omega^S := \frac{y^0}{y^{D(S)}} \cdot s_{Y'}, \in \Gamma(Y', \omega_{Y'}(D(S) + X))
\]

defines a nowhere vanishing global section. On \( U^S \), we have

\[
\omega^S = \pm \frac{1}{\Phi^S} \bigwedge_{\rho \in S} \frac{dz_\rho^S}{y^S(x_\rho^S)} \in \Gamma(U^S, \omega_{Y'}(X)).
\]

Let \( \mathcal{P} \) be a set of polynomials \( F \in \mathbb{Q}[y_1, \ldots, y_r] \) of degree \( L \). For each polynomial \( F \in \mathcal{P} \) of degree \( L \), let \( D(F) \) be the effective divisor on \( Y' \) of class \( \deg F = L \) defined by \( F \) (in Cox coordinates). If \( X \subset \text{supp } D(F) \), then we remove \( F \) from our set \( \mathcal{P} \); clearly, this does not change the results that we want to prove. For \( X \notin \text{supp } D(F) \), we define

\[
\omega_F := \frac{y^{D_F + E}}{F} \cdot \Phi^S s_{Y'}, \in \Gamma(Y', \omega_{Y'}(D(F) + lX - E)).
\]
We have the Poincaré residue map \(\text{Res} : \omega_Y(-X) \to \iota'_*\omega_X\) of \(\mathcal{O}_Y\)-modules (where \(\iota' : X \to Y\) is the inclusion). On the smooth open subset \(U^S\) of \(Y\), it maps \(\varpi^S \in \Gamma(U^S, \omega_Y(-X))\) to \(\text{Res}(\varpi^S) \in \Gamma(U^S, \iota'_*\omega_X) = \Gamma(X^S, \omega_X)\), which is

\[
(2.4) \quad \text{Res}(\varpi^S) = \frac{\pm 1}{\partial \Phi^S/\partial z^S} \bigg|_{\rho_0 \in S \setminus \{\rho_0\}} \wedge \text{dz}^S
\]
on the open subset of \(X^S\) where \(\partial \Phi^S/\partial z^S \neq 0\), for any \(\rho_0 \in S\). Furthermore, \(\text{Res}^l : \omega_Y^l(IX) \to \iota'_*\omega_X^l\) sends \(\varpi_F \in \Gamma(U_F, \omega_Y^l(IX - E))\) to \(\text{Res}^l(\varpi_F) \in \Gamma(U_F, \iota'_*\omega_X^l) = \Gamma(X_F, \omega_X^l)\), where \(U_F\) is the complement of \(D(F)\) in \(Y\), and \(X_F = X \cap U_F\).

**Lemma 2.4.** The section \(\text{Res}(\varpi^S)\), \(\text{Res}^l(\varpi_F)\) extends uniquely to a nowhere vanishing global section of \(\omega_X(D(S) \cap X), \omega_X^l(D(F) \cap X)\), respectively.

**Proof.** For \(\text{Res}(\varpi^S)\), this is as in [BBDG, Lemma 2.3], i.e., similar to [BBS, Lemma 13]. The computation for \(\text{Res}^l(\varpi_F)\) is analogous, using \(X \not\subset \text{supp} \mathcal{D}(F)\) and \(E \cap X = \emptyset\).

Therefore, \(\tau^S := \text{Res}(\varpi^S)^{-1}\), \(\tau_F := \text{Res}^l(\varpi_F)^{-1}\) are global nowhere vanishing sections of \(\omega_X^l(-D(S) \cap X), \omega_X^l(-D(F) \cap X)\), which we can also view as a global section of \(\omega_X^{-1}, \omega_X^{-l}\), respectively.

**Lemma 2.5.** The sections \(\tau^S \in \Gamma(X, \omega_X^{-1}), \tau_F \in \Gamma(X, \omega_X^{-l})\) do not vanish anywhere on \(X^S, X_F\), respectively.

**Proof.** The support of \(D(S) \cap X\) is contained in \(X \cap \bigcup_{F \in \mathcal{P}} D_F\), which is the complement of \(X^\circ\). Moreover, \(D_F \cap X\) is the complement of \(X_F\).

From now on, we assume:

\[
(2.5) \quad \mathcal{P}^l \text{ only contains monic monomials (of degree } L\text{)}, \text{ and for each } \sigma \in \Sigma^l_{\max} \text{ there exists } F \in \mathcal{P}^l \text{ such that } \text{supp } \text{div } F \text{ does not meet } U_\sigma.
\]

Then the set \(\mathcal{P}^l\) is in particular basepoint-free.

We define a \(v\)-adic norm/metric on \(\omega_X^{-1}\) by

\[
\|\tau(P)\|_v := \min_{F \in \mathcal{P}^l : P \notin D(F)} \left| \frac{\tau^l}{\tau_F} (P) \right|_v^{1/l}
\]

for a local section \(\tau\) of \(\omega_X^{-1}\) not vanishing in \(P \in X(\mathbb{Q}_v)\).

**Lemma 2.6.** Let \(p\) be a prime such that \(\overline{X}\) is smooth over \(\mathbb{Z}_p\). On \(\omega_X^{-l}\), the \(p\)-adic norm \(\| \cdot \|_p\) defined by

\[
\|\tau(P)\|_p := \min_{F \in \mathcal{P}^l : P \notin D(F)} \left| \frac{\tau}{\tau_F} (P) \right|_p
\]

for a local section \(\tau\) of \(\omega_X^{-l}\) not vanishing in \(P \in X(\mathbb{Q}_p)\) coincides with the model norm \(\| \cdot \|_p^m\) determined by \(\overline{X}\) over \(\mathbb{Z}_p\) as in [Sal, Definition 2.9].

**Proof.** See [BBDG, Lemma 3.3]. For such a \(\tau\) not vanishing in \(P\), let \(Q \in \mathcal{P}^l\) be such that \(\|\tau^Q/\tau(P)\|_p = \max_{F \in \mathcal{P}^l} |(\tau^F/\tau)(P)|_p\), which is positive by Lemma 2.5 and the fact that the set \(\mathcal{P}^l\) is basepoint-free. Hence \(\tau^Q\) does not vanish in \(P\), and

\[
\|\tau^Q(P)\|_p^{-1} = \frac{\max_{F \in \mathcal{P}^l} \frac{\tau^F}{\tau^Q}(P)}{|(\tau^F/\tau)(P)|_p} = \frac{\max_{F \in \mathcal{P}^l} |(\tau^F/\tau)(P)|_p}{|(\tau^Q/\tau)(P)|_p} = 1.
\]

For each \(F \in \mathcal{P}^l\), the section \(\tau^F\) extends to a global section \(\overline{\tau}^F\) of \(\omega_X^{-l}_{\mathbb{Z}_p}/\mathbb{Z}_p\), and \(\omega_X^{-l}_{\mathbb{Z}_p}\) is generated by the set of all these \(\overline{\tau}^F\) as an \(\mathcal{O}_X\)-module. The reason is that everything required for the definition of \(\tau^F\) above can also be defined over \(\mathbb{Z}_p\). For the existence of the Poincaré residue map in this case, we refer to [KK, Definition 4.1].
For every $F \in \mathcal{A}$, we have $\left| \frac{\tau^F}{\tau}(P) \right|_p \leq 1$ as in the computation above. This implies $\tau^F(P) = a_F \tau^F(P)$ for some $a_F \in \mathbb{Z}_p$ in the $\mathbb{Q}_p$-module $\omega_X^{-1}(P)$, and hence also $\tau^F(P) = a_F \tau^Q(P)$ in the $\mathbb{Z}_p$-module $P^*(\omega_X^{-1}_{/\mathbb{Z}_p})$, which shows that $P^*(\omega_X^{-1}_{/\mathbb{Z}_p})$ is generated by $\tau^Q(P)$. Hence $\|\tau^Q(P)\|_p^* = 1$ by definition of the model norm. We conclude
\[
\|\tau(P)\|_p = \|\tau(P)\|_p |\tau(Q(P))|_p \cdot \|\tau^Q(P)\|_p = \|\tau(Q(P))\|_p \cdot \|\tau^Q(P)\|_p^* = \|\tau(P)\|_p^*
\]
\[\square\]

2.3. Height functions. For $P \in X(\mathbb{Q})$, define
\[
H(P) := \prod_v \left| \frac{\tau(P)}{\tau^v} \right|_v^{-1}
\]
for a local section $\tau$ of $\omega_X^{-1}$ not vanishing in $P$.

Remark 2.7. Let $F, F_0$ be homogeneous elements of the same degree in the Cox ring of $Y^\prime$. If $F_0$ does not vanish in $P$, then $F/F_0$ can be regarded as a rational function on $X$ that can be evaluated in $P \in X(\mathbb{Q})$.

Lemma 2.8. We have
\[
H(P) = \left( \prod_v \max_{F \in \mathcal{A}} \left| \frac{F}{F_0}(P) \right|_v \right)^{1/l}
\]
for any polynomial $F_0$ of degree $L$ not vanishing in $P$.

Proof. We have $P \in X^S(\mathbb{Q})$ for some $S$ as above. We can compute $H(P)$ with $\tau := \tau^S$ by Lemma 2.5. Applying the $\mathcal{O}_{Y^\prime}$-module homomorphism $\text{Res} \tau_F = F^{-1}y^{lD(S)} + E(\varpi X^0)$ shows $\tau_F = F y^{-lD(S)} + E(\varpi X^0)$, hence
\[
\|\tau^S(P)\|_v^{-1} = \max_{F \in \mathcal{A}} \left| \frac{\tau^F}{\tau^S}(P) \right|_v = \max_{F \in \mathcal{A}} \left| \frac{F}{y^{lD(S)}}(P) \right|_v,
\]
which is our claim in the case $F_0 := y^{lD(S)} + E$. The general case follows using the product formula. \[\square\]

We lift the height function $H$ to $X_0$ by composing it with $\pi : X_0(\mathbb{Q}) \to X(\mathbb{Q})$, giving $H_0 : X_0(\mathbb{Q}) \to \mathbb{R}_{>0}$.

Lemma 2.9. For $P_0 \in X_0(\mathbb{Q})$, we have
\[
H_0(P_0) = \left( \prod_v \max_{F \in \mathcal{A}} \left| F(P_0) \right|_v \right)^{1/l}.
\]

Proof. Let $P = \pi(P_0) \in X(\mathbb{Q})$. As in the proof of [BBDG, Lemma 3.5], for $F_0$ of degree $L$ not vanishing in $P$ and $F \in \mathcal{A}$, we have $(F/F_0)(P) = F(P)/F_0(P)$ if we compute $(F/F_0)(P)$ as in Remark 2.7 and also regard $F, F_0$ as regular functions on $X_0(\mathbb{Q})$ that can be evaluated in $P_0$. We apply this to Lemma 2.8 to obtain
\[
H_0(P_0) = H(P) = \left( \prod_v \max_{F \in \mathcal{A}} \left| \frac{F}{F_0}(P) \right|_v \right)^{1/l} = \left( \prod_v \max_{F \in \mathcal{A}} \left| \frac{F(P_0)}{F_0(P_0)} \right|_v \right)^{1/l}.
\]
Then we use the product formula. \[\square\]

In its integral model, this simplifies as follows.

Corollary 2.10. For $P_0 \in \bar{X}_0(\mathbb{Z})$, we have
\[
H_0(P_0) = \max_{F \in \mathcal{A}} \left| F(P_0) \right|_\infty^{1/l}.
\]
Proof. This is analogous to [Sal, Proposition 11.3] and [BBDG, Corollary 3.6]. For a prime $p$, we have $P_0 \pmod{p}$ in $X_0(F_p)$. There is a $σ ∈ Σ_{max}$ such that $y_ρ(P_0 \pmod{p}) ≠ 0$ for all $ρ ≠ σ(1)$ since $X_0$ is defined by the irrelevant ideal in $X_1$. Choose $Q ∈ M^l$ such that supp div $Q$ does not meet $U_σ$. Then we have $Q(P_0 \pmod{p}) ≠ 0$ in $F_p$, and hence $|Q(P_0)|_p = 1$. Therefore, we have $\max_{F ∈ M^l} |F(P_0)|_p^{1/l} = 1$ and only the archimedean factor in Lemma 2.9 remains. □

2.4. Counting problem. The following result parametrizes the set $N_{X(Q) \setminus T,H}(B)$ in terms of integral points on the universal torsor of the ambient toric variety $Y'$ (which is given by its Cox ring (2.1) and the primitive collections (2.2)), the equation $Φ'$, and the monomials in $M^l$. The resulting counting problem is amenable to methods of analytic number theory.

Proposition 2.11. Let $X$ be a variety as in Section 2, let $Y'$ be a toric variety satisfying (2.3), let $L$ be a divisor class as in Section 2.2, and let $M^l$ be a set of monomials satisfying (2.5).

Let $T$ be an arbitrary subset of $X(Q)$. Then

$$2^{rk \Pic Y'} N_{X(Q) \setminus T,H}(B) = \# \left\{ y ∈ Z^{Σ(1)} : \Phi'(y) = 0, \max_{F ∈ M^l} |F(y)|_∞^{1/l} ≤ B, \pi'(y) \notin T, \gcd(y_ρ : ρ ∈ S_j') = 1 \text{ for every } j = 1, \ldots, r' \right\},$$

using the notation (2.1) and (2.2).

Proof. This follows from Proposition 2.2 and Corollary 2.10. □

3. Peyre’s constant

We keep the notation and assumptions of Section 2. In addition, we assume from now on that we are in the situation of Remark 2.1. In particular, the pullback map $\Pic Y^o → \Pic X$ is an isomorphism and $X$ is split.

3.1. Tamagawa measures. By [Pey1, (2.2.1)] and [Sal, Theorem 1.10], the $v$-adic norm $|| · ||_v$ on $ω_X^{-1}$ defined above induces a Tamagawa measure $μ_v$ on $X(Q_v)$.

Lemma 3.1. Let $S = σ(1)$ for some $σ ∈ Σ_{max}$. For a Borel subset $N_v$ of $X^S(Q_v)$, we have

$$μ_v(N_v) = \int_{N_v} \frac{|\Res(ω^S)|_v}{\max_{F ∈ M^l} |τ_F(\Res(ω^S))|_v^{1/l}} = \int_{N_v} \frac{|\Res(ω^S)|_v}{\max_{F ∈ M^l} |F/y^{D(S)+E}|_v^{1/l}},$$

where $|\Res(ω^S)|_v$ is the $v$-adic density on $X^S(Q_v)$ of the volume form $\Res(ω^S)$ on $X^S$.

If $N_v$ is contained in a sufficiently small neighborhood of $P$ in $X^S(Q_v)$ with $∂Φ^S/∂z_{ρ_0}^S(P) ≠ 0$, then

$$μ_v(N_v) = \int_{π_p^S(N_v)} \frac{dz_p^S}{\max_{F ∈ M^l} |F(z^S)|_v^{1/l}},$$

in the affine coordinates $z^S = (z^S)_ρ ∈ S$, where $π_{ρ_0}^S : U^S(Q_v) = Q_v^S → Q_v^S \setminus \{ρ_0\}$ is the natural projection and $z_{ρ_0}^S$ is expressed in terms of the other coordinates using the implicit function for $Φ^S$.

Proof. This is analogous to the proof of [BBDG, Proposition 4.1]. However, we work with $||τ_{F_0}(P)||_v$ for $F_0 = x^{lD(S)+E}$ and use $F_0(z^S) = 1$ in our affine coordinates on $X^S(Q_v)$. At the end, comparing the definitions of $ω^S$ and $ω_F$ shows $ω_F(ω^S)^t = y^{lD(S)+E}/F$, hence $τ_F(ω^S)^t = τ_F(ω^S)^t = F/y^{lD(S)+E}$, and hence the integrals in (3.1) are equal. □

Remark 3.2. Since we have assumed that every cone in $Σ'$ is the face of a maximal cone, the open subvarieties $X^S$ for $S = σ(1)$ with $σ ∈ Σ_{max}$ cover $X$.

Remark 3.3. If we are in the special case where $X$ is covered by open subvarieties $X^S$ with $S ⊂ Σ(1)$ for $S = σ(1)$ with $σ ∈ Σ'_{max}$, then (3.2) gives the same formula for the $v$-adic density as we would have obtained by working with $M(X)$ directly (since the additional coordinates $z_p^S$ are 1 for all $ρ ∈ Σ'(1) \setminus Σ(1)$; up to the description of the height function defined via the monomials $F ∈ M^l ⊂ \Pic Y^o$.
Furthermore, we change the definition of \( \Sigma \) work of Peyre [Pey2, Pey3, Pey4].

Proof. This is analogous to [BBDG, Lemma 4.2, Lemma 4.3].

3.3. Comparison to the number of points modulo \( p^\ell \). As in [BBDG, §4.4], for any prime \( p \) and \( l \in \mathbb{Z}_{>0} \), we have

\[
\bar{X}_0(\mathbb{Z}/p^\ell \mathbb{Z}) = \{ x \in (\mathbb{Z}/p^\ell \mathbb{Z})^{\Sigma(1)} : \Phi'(x) = 0 \in \mathbb{Z}/p^\ell \mathbb{Z}, \ p \nmid \gcd\{x_{\rho} : \rho \in S_j'\} \text{ for all } j = 1, \ldots, r' \}.
\]

In particular, the additional variables \( x_{\rho} \) indexed by \( \rho \in \Sigma'(1) \setminus \Sigma(1) \) (obtained via the desingularization of the ambient toric variety) appear here.

Proposition 3.5. For every prime \( p \), there is an \( \ell_0 \in \mathbb{Z}_{>0} \) such that for all \( \ell \geq \ell_0 \) we have

\[
m_p(\bar{X}_0(\mathbb{Z}_p)) = \left(1 - p^{-1}\right)^{\text{rk} \Pic X} \mu_p(X(\mathbb{Q}_p)).
\]

Furthermore,

\[
\lim_{\ell \to \infty} \frac{\#\bar{X}_0(\mathbb{Z}/p^\ell \mathbb{Z})}{(p^\ell)^{\dim X_0}} = (1 - p^{-1})^{\#\Sigma(1) - \#\Sigma'(1)} \frac{\#\bar{X}_0(\mathbb{Z}/p^\ell \mathbb{Z})}{\dim X_0}.
\]

Proof. This relies on the regularity of \( Y' \) and is otherwise analogous to [BBDG, Lemma 4.4, Proposition 4.5, Proposition 4.6]. The referee kindly pointed out that an argument is also contained in the work of Peyre [Pey2, Pey3, Pey4].

3.4. The real density. In this section, we compute the real density and Peyre’s \( \alpha \)-constant as in [BBDG, §4.5]. Those results can be applied with minor modifications, which we now discuss.

We choose \( \sigma \in \Sigma'_{\max} \), \( \rho_0 \in \sigma(1) \), and \( \rho_1 \in \Sigma(1) \setminus \sigma(1) \) subject to the following conditions analogous to [BBDG, (4.7)]:

\( \sigma \in \Sigma'_{\max} \) also appears in \( \Sigma_{\max} \).

Every variable \( x_{\rho} \) for \( \rho \in \Sigma(1) \) appears in at most one monomial of \( \Phi \).

Writing \( -K_X = \sum_{\rho \in \Sigma(1) \setminus \sigma(1)} \alpha_\rho^{\sigma} \deg(x_\rho) \) in \( \text{Pic } X \), we have \( \alpha_\rho^{\sigma} \neq 0 \).

The variable \( x_{\rho_0} \) appears with exponent 1 in \( \Phi \).

No \( x_{\rho} \) with \( \rho \in \sigma(1) \cup \{ \rho_1 \} \) \( \setminus \{ \rho_0 \} \) appears in the same monomial of \( \Phi \) as \( x_{\rho_0} \).

We define the numbers \( b_{\rho_0,\rho'} \) and \( b_{\rho'} \) as in [BBDG, §4.5], computed in \( \text{Pic } X \), for \( \rho' \in \sigma(1)' = \sigma(1) \cup \{ \rho_1 \} \) and \( \rho \in \Sigma(1) \setminus \sigma(1)' \). Then we define \( c^* \) as in [BBDG, (4.9)].

We now define \( c_{\infty} \) as in [BBDG, (4.11)]. We work without the additional coordinates indexed by \( \Sigma'(1) \setminus \Sigma(1) \). Therefore, we can use the results from [BBDG, (4.11)], considering \( X \) to be embedded into the possibly singular toric variety \( Y \). Since we are working with a more general height function, we change the definition of \( H_{\infty} \) to

\[
H_{\infty}(x) = \max_{F \in \mathcal{P}} |F(x)|^{1/l},
\]

where the additional coordinates indexed by \( \Sigma'(1) \setminus \Sigma(1) \) are set to 1 on the right-hand side.
In order to proceed as in [BBDG, (4.11)], it remains to show that the monomials in $\mathcal{P}^l$ with the additional coordinates set to 1 are homogeneous of degree $-lK_X \in \text{Pic}(X) = \text{Cl}(Y')$.

**Lemma 3.6.** If we write

$$l^{-1} \cdot L = \sum_{\rho \in \Sigma(1)} (\alpha')_\rho \deg(y_\rho)$$

in $\text{Pic} Y'$, then we have $(\alpha')_\rho = \alpha_\rho$ for all $\rho \in \Sigma(1)$. In other words, it does not matter whether we compute the numbers $\alpha_\rho$ in $\text{Pic} X$ or $\text{Pic} Y'$. The same is true for the numbers $b_{\rho, \rho'}$ and $b'_\rho$.

**Proof.** For every $\rho \in \Sigma(1)$, we have $D_\rho = \deg y_\rho \in \text{Pic} Y'$ for some prime divisor $D_\rho$ in $Y'$ with $X \not\subseteq D_\rho$. We can therefore directly compute the pullbacks to $X$ of all $D_\rho$.

For every $\rho \in \Sigma(1)$, the pullback of $\deg(y_\rho)$ is $\deg(x_\rho) \in \text{Pic} X$. On the other hand, for every $\rho \in \Sigma(1)$ we have $D_\rho|_X = \emptyset$; therefore the pullback to $X$ is $0 \in \text{Pic} X$.

Finally, the pullback of $l^{-1} \cdot L$ is $-K_X$. By pulling back the defining equations for $(\alpha')_\rho$, the result follows. The argument for $b_{\rho, \rho'}$ and $b'_\rho$ is the same. \qed

**Lemma 3.7.** Let $F \in \mathcal{P}^l$, and let $F_Y$ be the corresponding monomial where the additional coordinates indexed by $\Sigma(1) \setminus \Sigma(1)$ are set to 1. Then $F_Y$ is homogeneous of degree $-lK_X \in \text{Pic}(X) = \text{Cl}(Y')$.

**Proof.** We write

$$F = \prod_{\rho \in \Sigma(1)} y_\rho^{k_\rho}$$

and, since the monomial $F$ is of degree $L$, we obtain

$$L = \sum_{\rho \in \Sigma(1)} k_\rho \deg(y_\rho).$$

The proof now proceeds exactly as in the proof of Lemma 3.6. \qed

**Proposition 3.8.** In the notation above and assuming (3.3), we have

$$\alpha(X)\mu_\infty(X(\mathbb{R})) = \frac{1}{2\text{rk Pic} X} c^* c_\infty.$$

**Proof.** First, we note that [BBDG, Lemma 4.8] is still valid. Moreover, we observe that the additional variables $z_\rho^S$ for $\rho \in \Sigma(1) \setminus \Sigma(1)$ are 1 in the expression (3.2) for $\mu_\infty(X(\mathbb{R}))$ for $S = \sigma(1)$. Therefore, the expected real density $\omega_\infty = \mu_\infty(X(\mathbb{R}))$ has the same description as in [BBDG, (4.10)]. Taking into account Lemma 3.7, the statement and proof of [BBDG, Proposition 4.10] stay the same. \qed

**Part 2. Analysis of a diophantine equation**

The counting problem in Corollary 11.1(a) corresponding to the variety $X_1$ is rather delicate and not covered by the general method of [BBDG]. This part of the paper is devoted to a detailed investigation.

4. Elementary bounds

For $\xi \in \mathbb{Z} \setminus \{0\}$ we consider the equation

$$x_{11}x_{12} + x_{21}^2 + \xi^2 x_{31}x_{32}x_{33}^2 = 0, \quad x_{31}x_{32} \neq -\square$$

where all variables are non-zero integers. For $X = (X_{ij})$ with $X_{ij} \geq 1$ let $N_\xi(X)$ be the number of solutions to (4.1) in boxes $\frac{1}{2}X_{ij} \leq |x_{ij}| \leq X_{ij}$. In many cases it will improve the readability considerably to relabel the variables, and it will be convenient to refer to (4.1) in the form

$$ab + c^2 + \xi^2 ywz^2 = 0, \quad yw \neq \square.$$

Consequently, we will write $X = (A, B, C, Y, W, Z)$. We generally write

$$\|X\| = \max(A, B, C, Y, W, Z)$$

Using this notation, we start with some elementary bounds.
Proposition 4.1. We have

\[ N_\xi(X) \ll \|X\|^\varepsilon (ABC)^{1/2} (YW)^{3/4} Z^{1/2}. \]

Proof. The bound

\[ N_\xi(X) \ll \|X\|^\varepsilon \min(ABC, CYWZ) \]

follows from a simple divisor argument. Alternatively, we can fix y, w, z so that

\[ c^2 = -ywz^2 \xi^2 + O(AB), \]

which then determines a, b up to a divisor function. The equation for c defines an interval of length

\[ \ll \frac{AB}{\sqrt{YWZ}|\xi|} \]

so that

\[ N_\xi(X) \ll \|X\|^\varepsilon YWZ \left( \frac{AB}{Z\sqrt{YW}|\xi|} + 1 \right) = \|X\|^\varepsilon \left( \frac{AB\sqrt{YW}}{|\xi|} + YWZ \right). \]

The claim follows now from (4.2) and (4.3) after taking suitable geometric means, namely

\[ \min(AB\sqrt{YW}, CYWZ) \leq (ABC)^{1/2} (YW)^{3/4} Z^{1/2}, \quad \min(ABC, YWZ) \leq \sqrt{ABCYWZ}. \]

We refine this argument a bit as follows. For \( 0 < \Delta \leq 1 \) and \( X = (X_{ij}) \) let \( N_\xi^*(X, \Delta) \) be the set of solutions to (4.1) with the same size conditions as \( N_\xi(X) \) except that for one index \( (ij) \in \{(21), (31), (32), (33)\} \)

the condition \( \frac{1}{2} X_{ij} \leq |x_{ij}| \leq X_{ij} \) is replaced with \( X_{ij} \leq |x_{ij}| \leq X_{ij}(1 + \Delta) \). We also define \( N_\xi(X, \Delta) \) to be the be the same size conditions as \( N_\xi(X) \) except that \( \frac{1}{2} X_{11} \leq |x_{11}| \leq X_{11} \) is replaced with \( X_{11} \leq |x_{11}| \leq X_{11}(1 + \Delta) \). The following proposition investigates \( N_\xi^*(X, \Delta) \), while \( N_\xi(X, \Delta) \) comes up in Proposition 6.2.

Proposition 4.2. We have

\[ N_\xi^*(X, \Delta) \ll \|X\|^\varepsilon \Delta^{1/2} (ABC)^{1/2} (YW)^{3/4} Z^{1/2}. \]

Proof. We distinguish two cases. If it is not the c-variable that is restricted, then by the same argument as above we have

\[ N_\xi^*(X, \Delta) \ll \|X\|^\varepsilon \min \left( ABC, \Delta CYWZ, \Delta (AB\sqrt{YW} + YWZ) \right) \ll \|X\|^\varepsilon \left( \Delta (ABC)^{1/2} (YW)^{3/4} Z^{1/2} + \Delta^{1/2} (ABCYWZ)^{1/2} \right) \]

and the claim follows.

If the restricted variable is c, then we have similarly

\[ N_\xi^*(X, \Delta) \ll \|X\|^\varepsilon \min \left( \Delta ABC, \Delta CYWZ, AB\sqrt{YW} + YWZ \right) \ll \|X\|^\varepsilon \left( \Delta^{1/2} (ABC)^{1/2} (YW)^{3/4} Z^{1/2} + \Delta^{1/2} (ABCYWZ)^{1/2} \right). \]

This completes the proof. \( \square \)

Remarks: 1) It will be convenient to introduce the following short-hand notation for expressions like those in Proposition 4.2. For \( \zeta = (\zeta_1, \zeta_2, \zeta_3) \in \mathbb{R}^3 \) and \( X = (A, B, C, Y, W, Z) \) we define

\[ X^{(\zeta)} = (AB)^{1-\zeta_1} C^{1-2\zeta_2} (YW)^{1-\zeta_3} Z^{1-2\zeta_3}. \]

With this notation, the bounds in Propositions 4.1 and 4.2 involve \( X^{(\frac{1}{2}, \frac{1}{2}, \frac{1}{4})} \).

2) In order for \( N_\xi(X) \) to be non-zero, we must have

\[ C \ll (AB)^{1/2} + |\xi| Z \sqrt{YW}. \]
5. Character sums

We consider the following two character sums. For $a, c, z, \xi \in \mathbb{N}, h_1, h_2 \in \mathbb{Z}$ let

$$S_\xi(h_1, h_2, a, c, z) = \sum_{\substack{y, w \text{ (mod } a) \cr a \mid c^2 + \xi^2 ywz^2}} e\left(\frac{h_1 y + h_2 w}{a}\right).$$

For $x, a \in \mathbb{N}, k_1, k_2 \in \mathbb{Z}$ let

$$T(k_1, k_2, x, a) = \sum_{\gamma^2 + x\xi^2 \equiv 0 \text{ (mod } a)} e\left(\frac{h_1 \gamma + h_2 \xi^2}{a}\right).$$

Let $\tau$ denote the divisor function.

**Lemma 5.1.** We have

$$S_\xi(0, 0, a, c, z) = \sum_{\substack{a_1 a_2 a_3 = a \cr a_1 | c^2}} a_1 (\xi^2 z^2, a_2 a_3) a_3 \mu(a_2)$$

and

$$|S_\xi(h_1, h_2, a, c, z)| \leq \tau(a)(a, h_1, h_2)a^{1/2}(a, c^2)^{1/2}.$$  

**Proof.** We have

$$S_\xi(h_1, h_2, a, c, z) = \frac{1}{a} \sum_{\alpha, \gamma, w \text{ (mod } a)} e\left(\frac{\alpha(c^2 + ywz^2 \xi^2) + h_1 y + h_2 w}{a}\right)$$

$$= \sum_{\substack{\alpha, \gamma, w \text{ (mod } a) \cr \alpha \in z^2 \xi^2 + h_1 \equiv 0 \text{ (mod } a)}} e\left(\frac{\alpha c^2 + h_2 w}{a}\right)$$

$$= \sum_{\substack{a_1 a_2 = a \cr a_1 | h_1}} \sum_{\substack{\alpha \text{ (mod } a_2) \cr \alpha \in z^2 \xi^2 \equiv -a h_1 \text{ (mod } a_2)}} \sum_{\substack{w \text{ (mod } a) \cr w \equiv -1 (z^2 \xi^2, a_2) \text{ (mod } a_2)}} e\left(\frac{\alpha a_1 c^2 + h_2 w}{a}\right).$$

The $w$-sum vanishes unless $a(z^2 \xi^2, a_2) | h_2$, so we obtain

$$\sum_{\substack{\alpha \text{ (mod } a_2) \cr \alpha \in z^2 \xi^2, a_2)} \sum_{\substack{\alpha \text{ (mod } a_2) \cr \alpha \in z^2 \xi^2, a_2)} e\left(\frac{bc^2}{a_2}a_1(z^2 \xi^2, a_2)c\right) \frac{h_2}{a_2} \frac{z^2 \xi^2}{a_1(z^2 \xi^2, a_2)}$$

$$= \sum_{\substack{a_1 a_2 = a \cr a_1(z^2 \xi^2, a_2)}(h_1, h_2)} a_1(z^2 \xi^2, a_2) S\left(c^2, -\frac{h_2}{a_1(z^2 \xi^2, a_2)} \frac{h_1}{a_1(z^2 \xi^2, a_2)} \frac{z^2 \xi^2}{a_1(z^2 \xi^2, a_2)}, a_2\right)$$

where $S(\ldots)$ is the Kloosterman sum. If $h_1 = h_2 = 0$, then the claim follows by the formula

$$S(c^2, 0, a_2) = \sum_{d | (a_2, c^2)} d \mu(a_2)^2.$$ 

In general we use Weil’s bound $|S(c^2, *, a_2)| \leq \tau(a_2)a_2^{1/2}(c^2, a_2)^{1/2}$ to complete the proof of the lemma. □
A number $D \in \mathbb{Z} \setminus \{0\}$ is a discriminant if $D \equiv 0, 1 \pmod{4}$. For each discriminant we denote by $\chi_D = (D/\cdot)$ the corresponding quadratic character. It is primitive if and only if $D$ is a fundamental discriminant. If $d \in \mathbb{N}$ is odd we write $d^*$ for the unique discriminant with $|d^*| = d$. For an odd number $d$ we write $\epsilon_d = \sqrt{\chi_{-4}(d)} \in \{1, i\}.$

**Lemma 5.2.** We have

$$T(0, 0, x, a) = \sum_{d_1 \mid d_2 = a \atop (x, d_2) = \square} d_1 \phi(d_2) \chi_{d_2} \left(\frac{-x}{(x, d_2)}\right) (x, d_2)^{1/2}$$

where $d_2$ runs over all discriminants (positive or negative). If $a$ is odd, then

$$|T(k_1, k_2, x, a)| \leq \tau(a)(a, k_1^2 x + k_2^2)(a, x)^{1/2}.$$

**Proof.** We have

$$T(k_1, k_2, x, a) = \frac{1}{a} \sum_{\alpha, c, z \pmod{a}} e\left(\frac{\alpha(c^2 + xz^2) + k_1 c + k_2 z}{a}\right)$$

$$= \frac{1}{a} \sum_{d_1, d_2 = a \atop d_1 \mid (k_1, k_2)} d_1^2 \sum_{\alpha \pmod{d_2}} \sum_{c, z \pmod{d_2}} e\left(\frac{\alpha(c^2 + xz^2) + k_1 c + k_2 z}{d_2}\right).$$

Let us first consider the case $k_1 = k_2 = 0$. We split the modulus $d_2 = u 2^\rho$ into an odd part and a power of two. The $\alpha, c, z$-sum becomes

$$\sum_{\alpha \pmod{u}} \sum_{c, z \pmod{d_2}} e\left(\frac{\alpha(c^2 + xz^2)}{u}\right) \sum_{\alpha \pmod{2^\rho}} \sum_{c, z \pmod{2^\rho}} e\left(\frac{\alpha(c^2 + xz^2)}{2^\rho}\right).$$

By the well-known evaluation of the Gauß sum, the first $c, z$-sum equals

$$d_2 \epsilon_u e\left(\frac{\chi(u)}{2}\right) \chi_u \left(\frac{x}{u}\right) \chi_{2^\rho} \left(\frac{x}{u}\right).$$

Summing this over $\alpha$, we see that only the contribution of $(x, u) = \square$ survives, and the first $\alpha, c, z$-sum equals

$$\delta_{(x, u) = \square} u \phi(u)(x, u)^{1/2} \chi_u \left(\frac{x}{u}\right).$$

For the second $\alpha, c, z$-sum modulo powers of 2 we argue similarly, but we have to distinguish a few cases. Recall first that for odd $\alpha$ we have

$$\sum_{d \equiv \square \pmod{2^\rho}} e\left(\frac{\alpha d^2}{2^\rho}\right) = \begin{cases} 1, & \rho = 0, \\ 0, & \rho = 1, \\ 2^{\rho/2}(\chi_{2^\rho}(\alpha) + i\chi_{-2^\rho}(\alpha)), & \rho \geq 2. \end{cases}$$

If $4 \mid x/(2^\rho, x)$ and $4 \nmid 2^\rho$, we obtain

$$\sum_{\alpha \pmod{2^\rho}} 2^\rho(2^\rho, x)^{1/2}(\chi_{2^\rho}(\alpha) + i\chi_{-2^\rho}(\alpha)) \left(\chi_{2^\rho} \left(\frac{x}{(x, 2^\rho)}\alpha\right) + i\chi_{-2^\rho} \left(\frac{x}{(x, 2^\rho)}\alpha\right)\right).$$

This vanishes, unless $(x, 2^\rho) = \square$ in which case it equals

$$2^\rho \phi(2^\rho)(x, 2^\rho)^{1/2} \sum_{\pm} \chi_{\pm \alpha} \left(\frac{x}{(x, 2^\rho)}\right).$$

The remaining cases are simple: if $\rho = 1$, the sum vanishes and the case $\rho = 0$ is trivial. If $2^\rho \mid x$, the $z$-sum is equals $2^\rho = 2^\rho/(x, 2^\rho)^{1/2}$, and evaluating the $c$-sum, we see that the $\alpha$-sum vanishes unless $\rho$ is even, i.e. $2^\rho = (x, 2^\rho) = \square$. If $x/(2^\rho, x) = 2$, the $z$-sum vanishes. In this way we confirm in all cases that the second $\alpha, c, z$-sum equals

$$\delta_{(x, 2^\rho) = \square} 2^\rho \phi(2^\rho)(x, 2^\rho)^{1/2} \left(\delta_{4|(x, 2^\rho)} (2^\rho \sum_{\pm} \chi_{\pm \alpha} \left(\frac{x}{(x, 2^\rho)}\right)) + \delta_{2^\rho \mid x}\right).$$
Combining this with the evaluation for odd moduli, we confirm that \( T(0,0,x,a) \) equals
\[
\frac{1}{a} \sum_{d_1 \mid d_2 \equiv a \pmod{d_2}} \phi(d_2) (x,d_2)^{1/2} \left( \delta_4(x,d_2) d_2 \sum_{n \geq 1} \frac{x \chi_{\frac{n}{d_2}}(x)}{(x,d_2)^2} \right) + \delta_2(x,d_2) d_2 \chi_{\frac{-a}{d_2}} \left( - \frac{x}{(x,d_2)} \right).
\]

This is equivalent to the formula given in the lemma.

We now turn to the estimation for general \( k_1, k_2 \) and for simplicity restrict ourselves to odd \( a \), as in the statement of the lemma. In this case we can evaluate the two Gauss sums in \( c, z \) simply by completing the square, and we obtain
\[
\frac{1}{a} \sum_{d_1 \mid d_2 \equiv a \pmod{d_2}} \sum_{\alpha \equiv 1 \pmod{d_2}} d_1^2 d_2 \epsilon \chi(x,d_2)^{-\alpha} \chi(\frac{x}{d_2}) \left( - \frac{x}{(x,d_2)} \right) \sum_{\alpha \equiv 1 \pmod{d_2}} d_1^2 d_2 \epsilon \chi(x,d_2)^{-\alpha} \chi(\frac{x}{d_2}) \left( - \frac{x}{(x,d_2)} \right).
\]

Let \( \delta \) denote the conductor of \( \chi(x,d_2)^{-\alpha} \), and write \( d_2 = \delta_1 \delta_2 \) with \( (\delta, \delta) = 1 \), \( \delta_1 \mid \delta \infty \). Then by the well-known evaluation of quadratic character sums, the \( \alpha \)-sum is bounded by
\[
\delta_1 \delta^{1/2} \left( \frac{k_1^2 x + k_2^2}{d_1^2 (x,d_2)} \right).
\]

We decompose uniquely \( (x,d_2) = t_1 t_2 \) such that \( t_1 \) is the largest square coprime to \( t_2 \). Then \( \delta = \text{rad}(t_2), \delta_1 = t_2 / \text{rad}(t_2), \delta_2 = t_2, so that we obtain the upper bound
\[
\sum_{d_1 \mid d_2 \equiv a \pmod{d_2}} d_1 (x,d_2)^{1/2} \left( \frac{t_2}{\text{rad}(t_2)^{1/2}} \right) \left( \frac{k_1^2 x + k_2^2}{d_1^2 (x,d_2)} \right)
\]
\[
= \sum_{d_1 \mid d_2 \equiv a \pmod{d_2}} d_1 (x,d_2)^{1/2} \left( \frac{t_1 t_2}{\text{rad}(t_2)^{1/2}} \right) \left( \frac{k_1^2 x + k_2^2}{d_1^2 t_1} \right) \leq \sum_{d_1 \mid d_2 \equiv a \pmod{d_2}} (x,d_2)^{1/2} (a, k_1^2 x + k_2^2),
\]
and the claim follows. \( \square \)

We also recall the following standard estimate.

**Lemma 5.3.** Let \( V \) be a smooth function with compact support in \([-2, -1/3] \cup [1/3, 2] \) such that \( V^{(j)} \ll \Omega^j \) for some \( \Omega \geq 1 \). If \( \Delta \neq \square \) is a discriminant, \( g \in \mathbb{N} \) and \( N \geq 1 \), then
\[
\sum_{(n,g)=1} \chi_\Delta(n) V \left( \frac{N}{n} \right) \ll \tau(g) N^{1/2}(\Omega|\Delta|)^{1/4+\varepsilon}.
\]

In addition, if \( n \in \mathbb{Z} \setminus \{0\} \) is not a square and \( D \geq 1 \), then
\[
\sum_{(D,g)=1} \chi_\Delta(n) V \left( \frac{D}{n} \right) \ll \tau(g) D^{1/2}(\Omega|n|)^{1/4+\varepsilon}
\]
where the sum runs over all discriminants. The implied constants depend only on \( \varepsilon \).

**Proof.** This is standard by Mellin inversion and the convexity bound for Dirichlet \( L \)-functions (with Euler factors at primes dividing \( g \) removed)
\[
L(g)(s, \chi_\Delta) \ll \tau(g)(|\Delta|(1 + |s|))^{1/4+\varepsilon}, \quad \Re s = 1/2.
\]

Suffice it to say that the (two-sided) Mellin transform of \( V \) is entire and satisfies
\[
\hat{V}(s) \ll \Re s, A \left( 1 + \frac{|s|}{\Omega} \right)^{-A}
\]
for all \( A > 0 \). The second bound follows from quadratic reciprocity as follows: By quadratic reciprocity we have \( \chi_\Delta(n) = \chi_\Delta(n) \) for \( n > 0 \) where \( \Delta \) is the discriminant computed as follows: write
\[ n = 2^a y \text{ with } y \text{ odd, and let } y^* \text{ denote the discriminant satisfying } |y^*| = y. \text{ Let } a' = a + 2 \text{ if } a = 2 \text{ and } a' = a \text{ otherwise. Then } n = 2^{a'} y^*. \text{ If } n < 0 \text{ then } \chi_{\Delta}(n) = \chi_{\Delta}(\chi_{\Delta}(-1)). \text{ In this way, the second bound follows from the first by detecting the condition } \Delta \equiv 0, 1 \pmod{4} \text{ by characters.} \]

Remark: Using the strongest available uniform subconvexity bounds, the exponents 1/4 can be replaced with 1/6.

6. Upper bound estimates

For \( X = (A, B, C, Y, W, Z) \) and \( \xi \in \mathbb{Z} \setminus \{0\} \) we recall the definition of \( N_\xi(X) \) and \( N_\xi(X, \Delta) \) from Section 4, and we now define a smooth version of the former. Let \( \Omega \geq 3 \) be a parameter. We choose an even, smooth, non-negative test function \( V \) with support in \([-1 - \Omega^{-1}, -1/2 + \Omega^{-1}] \cup [1/2 - \Omega^{-1}, 1 + \Omega] \) and \( V(x) = 1 \) on \([-1, -1/2] \cup [1/2, 1] \) satisfying \( V^{(j)}(x) \ll \Omega^j \) for all \( j \in \mathbb{N}_0 \). We define

\[
\tilde{N}_\xi(X, \Omega) = \sum_{ab + c^2 + \xi^2 \equiv 0 \atop yw \neq -0} V\left(\frac{a}{A}\right)V\left(\frac{b}{B}\right)V\left(\frac{c}{C}\right)V\left(\frac{y}{Y}\right)V\left(\frac{w}{W}\right)V\left(\frac{z}{Z}\right).
\]

(Strictly speaking, \( \tilde{N}_\xi(X, \Omega) \) depends on \( V \) and not only on \( \Omega \), but this small abuse of notation is convenient.) Fix a small constant \( \lambda \leq 10^{-6} \) and consider \( X, \xi \) satisfying

\[
\begin{align*}
\min(A, B, C) & \geq \max(A, B, C)^{1 - \lambda}, \\
\min(Y, W) & \geq \max(Y, W)\|X\|^{-\lambda}, \\
\min(AB, C^2, YWZ^2) & \geq \max(AB, C^2, YWZ^2)^{1 - \lambda}, \\
|\xi| & \leq \|X\|^\lambda.
\end{align*}
\]

We will see later that these are the critical size conditions.

Our aim in this section is to establish the following three estimates that we will prove simultaneously. Recall the notation (4.4).

**Proposition 6.1.** For \( X, \xi \) satisfying (6.1) we have

\[ N_\xi(X) \ll X\left(\frac{\bar{q}}{q}, \frac{1}{4} \right) + X\left(\frac{1}{4}, \frac{1}{4}\right). \]

**Proposition 6.2.** For \( X, \xi \) satisfying (6.1) and \( 0 < \Delta < 1 \) we have

\[ N_\xi(X, \Delta) \ll (X\left(\frac{\bar{q}}{q}, \frac{1}{4} + \frac{1}{\lambda}\right) + X\left(\frac{1}{4}, \frac{1}{4}\right)) (\Delta\|X\|^\epsilon + \|X\|^{-1/5}). \]

**Proposition 6.3.** For \( X, \xi \) satisfying (6.1) and \( \Omega \geq 3 \) we have

\[ \tilde{N}_\xi(X, \Omega) = M_2 + O\left((X\left(\frac{\bar{q}}{q}, \frac{1}{4} + \frac{1}{\lambda}\right) + X\left(\frac{1}{4}, \frac{1}{4}\right)) \Omega \|X\|^{11\lambda} \frac{A^{1/2}}{Y}\right) \]

where \( M_2 \) is given by (6.4) below. We also have

\[ \tilde{N}_\xi(X, \Omega) = \tilde{M}_3 + O\left(X\left(\frac{1}{4}, \frac{1}{4}\right) \Omega^4 \left(\frac{\|X\|^{10\lambda}}{Z} + \frac{1}{(YW)^{1/21}}\right)\right) \]

where \( \tilde{M}_3 \) is defined in (6.12) below.

The rest of this section is devoted to the proof. We will occasionally use the following notation: for a positive integer \( n \) let \( \sqrt{n}^* \) denote the smallest integer whose square is a multiple of \( n \).

We have trivially \( N_\xi(X) \leq \tilde{N}_\xi(X, \Omega) \) for every \( \Omega \geq 3 \) and

\[
\tilde{N}_\xi(X, \Omega) = \sum_{ab + c^2 + \xi^2 \equiv 0 \atop yw \neq -0} V\left(\frac{a}{A}\right)V\left(\frac{-\xi^2 ywz^2 - c^2}{ab}\right)V\left(\frac{c}{C}\right)V\left(\frac{y}{Y}\right)V\left(\frac{w}{W}\right)V\left(\frac{z}{Z}\right).
\]
We apply two different strategies. We first apply Poisson summation in $y, w$ and then Poisson summation in $c, z$. In order to obtain Proposition 6.2, we will occasionally replace $V(a/A)$ with the characteristic function on $A \leq |a| \leq A(1+\Delta)$. In the interest of a reasonably compact presentation, we will not introduce extra notation for this.

### 6.1. Poisson summation in $y, w$. We first add the contribution of $wy = -\Box$ to (6.2). As in the proof of Proposition 4.1, we see by a divisor argument that this infers an error of at most $O(C(YW)^{1/2}Z\|X\|^\varepsilon)$ with an implied constant depending only on $\varepsilon$. Then we apply Poisson summation $y, w$ to obtain

\[\Omega_{\xi}(X, \Omega) = \sum_{a, c, z} V\left(\frac{a}{A}\right) V\left(\frac{c}{C}\right) V\left(\frac{z}{Z}\right) \frac{1}{a^2} \sum_{h_1, h_2} S_{\xi}(h_1, h_2, a, c, z) \Phi_{\xi}(h_1, h_2, a, c, z)\]

with \(S_{\xi}\) as in (5.1) and

\[\Phi_{\xi}(h_1, h_2, a, c, z) = \int_{\mathbb{R}} \int_{\mathbb{R}} V\left(\frac{y}{Y}\right) V\left(\frac{w}{W}\right) V\left(\frac{-\xi^2 ywz^2 - c^2}{aB}\right) e\left(\frac{-h_1 y + h_2 w}{|a|^3}\right) dy dw.\]

Let

\[H_1 = \Omega(a + \xi^2 wZ^2/B), \quad H_2 = \Omega(a + \xi^2 wZ^2/B)\]

Lemma 6.4. For $|a| \asymp A$, $|z| \asymp Z$ we have

\[\Phi_{\xi}(h_1, h_2, a, c, z) \ll_N \frac{(AB)^{1/4}(YW)^{3/4}}{(Z[\xi])^{1/2}} \left(1 + \frac{|h_1|}{H_1} + \frac{|h_2|}{H_2}\right)^{-N}\]

for arbitrary $N > 0$ and

\[(a\partial_a)^{j_1} (c\partial_c)^{j_2} (z\partial_z)^{j_3} \Phi_{\xi}(0, 0, a, c, z) \ll_j \frac{(AB)^{1/4}(YW)^{3/4}}{(Z[\xi])^{1/2}} \left(1 + \frac{C^2}{AB}\right)^{j_2} \left(1 + \frac{\xi^2 wZ^2}{AB}\right)^{j_3} \Omega^{j_1 + j_2 + j_3}\]

for arbitrary $j \in \mathbb{N}^3$.

**Proof.** We observe that the volume of the $(y, w)$-region defined by

\[|y| \asymp Y, \quad |w| \asymp W, \quad |c^2 + \xi^2 ywz^2| \asymp AB\]

is trivially bounded by $O(YW)$, but also by $O(AB/|z^2\xi^2|)$ and so by $O((AB)^{1/4}(YW)^{3/4}/|z\xi|^{1/2})$. The first claim follows now by repeated integration by parts, the second by differentiating under the integral sign. There is an important subtlety: we combine each application of the operator $z\partial_z$ with a partial integration in $y$, i.e.

\[\int_{\mathbb{R}} z\partial_z \left[V\left(\frac{y}{Y}\right) V\left(\frac{-\xi^2 ywz^2 - c^2}{aB}\right)\right] dy = -2 \int_{\mathbb{R}} \partial_y \left(yV\left(\frac{y}{Y}\right) V\left(\frac{-\xi^2 ywz^2 - c^2}{aB}\right)\right) dy.\]

This completes the proof. \[\square\]

We now write the right hand side of (6.3) as $M_1 + M_2$ where $M_1$ is the off-diagonal contribution $(h_1, h_2) \neq (0, 0)$ and $M_2$ is the diagonal contribution $h_1 = h_2 = 0$. By Lemma 5.1 we have

\[M_2 = \sum_{a, c, z} V\left(\frac{a}{A}\right) V\left(\frac{c}{C}\right) V\left(\frac{z}{Z}\right) \frac{1}{a^2} \sum_{a_1, a_2, a_3, \mu} a_1(\xi^2 z^2, a_2 a_3) a_3 \mu(a_2) \Phi_{\xi}(0, 0, a, c, z).\]
We postpone the analysis of $M_2$ and investigate first $M_1$. By Lemma 5.1 and Lemma 6.4 we have

\[ M_1 \ll \sum_{a,c,z} V\left(\frac{a}{A}\right) V\left(\frac{c}{Z}\right) V\left(\frac{z}{Z}\right) \left(\frac{AB}{Z\xi}\right)^{1/4} \frac{V(\frac{z}{Z})}{A^2} \frac{(YW)^3/4}{V(\frac{z}{Z})^{1/2}} \times \sum_{(h_1,h_2)\neq(0,0)} (a,h_1,h_2)\tau(a) |a|^{1/2} \left(1 + \frac{|h_1|}{H_1} + \frac{|h_2|}{H_2}\right)^{-10} \]

\[ \ll \sum_{a} V\left(\frac{a}{A}\right) \frac{CZ(AB)^{1/4} (YW)^{3/4}}{A^{3/2} (Z\xi)^{1/2}} \sum (a,h_1,h_2)\tau(a)^2 \left(1 + \frac{|h_1|}{H_1} + \frac{|h_2|}{H_2}\right)^{-10} \]

\[ \ll \sum_{a} V\left(\frac{a}{A}\right) \frac{CZ(AB)^{1/4} (YW)^{3/4}}{A^{3/2} (Z\xi)^{1/2}} \tau(a)^3 \left(1 + H_1 + H_2\right). \]

By several applications of (6.1), we see that

\[ \frac{1 + H_1 + H_2}{A^{1/2-\varepsilon}} \ll \Omega \|X\|^{10\lambda} A_{1/2} \frac{A}{Y}, \]

so that

\[ M_1 \ll \frac{C(AB)^{1/4} (YW)^{3/4} Z^{1/2}}{|\xi|^{1/2}} \Omega \|X\|^{10\lambda} \frac{A_{1/2} A}{Y}. \]

Note that up until now we have not used any property of the weight $V(a/A)$ except that it restricts $a \ll A$. In particular, (6.5) continues to hold with any $\Omega$ if $V(a/A)$ is replaced by the characteristic function on $A \leq |a| \leq A(1 + \Delta)$.

We can now complete the proof of the first half of Proposition 6.3 by recalling (6.3) and noting that (6.1) implies

\[ CZ\sqrt{YW}(ABYW)^{-\varepsilon} \ll (AB)^{1/4} C(YW)^{3/4} Z^{1/2} \cdot \|X\|^{10\lambda} \frac{A}{Y}, \]

so that

\[ N_{\xi}(X, \Omega) = M_2 + O\left((AB)^{1/4} C(YW)^{3/4} Z^{1/2} \cdot \|X\|^{10\lambda} \frac{A}{Y}\right). \]

As in the proof of Proposition 4.2 we reduce the power of $C$ using (4.5), which completes the proof of the first half of Proposition 6.3.

We now turn to $M_2$ as given in (6.4). We will evaluate this asymptotically later, but for now content ourselves with an upper bound given by

\[ M_2 \ll \frac{C}{A^2} \cdot \frac{(AB)^{1/4} (YW)^{3/4}}{(Z\xi)^{1/2}} \sum_{a_1,a_2,a_3} V\left(\frac{a_1a_2a_3}{A}\right) V\left(\frac{z}{Z}\right) a_1^{1/2} (z\xi^2, a_2a_3). \]

If we replace $V(a/A)$ with the characteristic function on $A \leq |a| \leq A(1 + \Delta)$, a very soft bound is given by

\[ \ll \frac{C}{A^2} \cdot \frac{(AB)^{1/4} (YW)^{3/4}}{(Z\xi)^{1/2}} \sum_{a_1,a_2,a_3} \sum_{A \leq a_1a_2a_3 \leq A(1 + \Delta)} V\left(\frac{z}{Z}\right) a_1a_2a_3 (z\xi, a_3) \]

\[ \ll \frac{C}{A^2} \cdot \frac{(AB)^{1/4} (YW)^{3/4}}{(Z\xi)^{1/2}} \Delta A^{2+\varepsilon} Z\tau(\xi) \ll \Delta (AB)^{1/4} C(YW)^{3/4} Z^{1/2} A^{\varepsilon}. \]

Together with (6.5) for $\Omega = 3$ we obtain

\[ N_{\xi}(X, \Delta) \ll (AB)^{1/4} C(YW)^{3/4} Z^{1/2} \left(\Delta \|X\|^{\varepsilon} + \|X\|^{10\lambda} A^{1/2} \frac{A}{Y}\right). \]
After this interlude we now return to (6.6) and estimate the right hand side by
\[
\leq \frac{C}{A^2} (AB)^{1/4} (YW)^{3/4} \left( \sum_{d} \sum_{d_2, a_2, a_3} V \left( \frac{a_1 a_2 a_3}{A} \right) \frac{Z}{\sqrt{d}} \right)^{1/2} d.
\]

For notational simplicity let us write \( d_\xi = d/(d, \xi^2) \). Then we can continue to estimate
\[
M_2 \leq \frac{C}{A^2} (AB)^{1/4} (YW)^{3/4} \left( \sum_{d} \sum_{d_2, a_2, a_3} V \left( \frac{a_1 a_2 a_3}{A} \right) \frac{Z}{\sqrt{d}} \right)^{1/2} d
\]
\[
\leq \frac{C}{A^2} (AB)^{1/4} (YW)^{3/4} \left( \sum_{d} \sum_{d_2, a_2, a_3} \frac{A^2}{Z} \frac{Z}{\sqrt{d}} \right)^{1/2}.
\]
The \( d \)-sum is
\[
\leq \sum_{d} \frac{\tau(d)}{d^{1/2} \sqrt{d_\xi}} \leq \sum_{\eta(\xi)} \frac{\tau(\eta)}{d^{1/2} \sqrt{d_\xi}} \leq \tau(\xi^2).
\]
Combining this with the off-diagonal contribution (6.5) and choosing \( \Omega = 3 \), we have shown our first important bound
\[
(6.8) \quad N_\xi(X) \leq \tilde{N}_\xi(X, 3) \ll \frac{C (AB)^{1/4} (YW)^{3/4} Z^{1/2}}{\|X\|^{10\lambda}} \left( \tau(\xi^2) + \|X\|^{10\lambda} \frac{A^{1/2}}{Y^{1/2}} \right).
\]

6.2. Poisson summation in \( c, z \). We now return to (6.2) and apply Poisson summation in \( c, z \) getting
\[
N_\xi(X, \Omega) = \sum_{w_y \neq 0} V \left( \frac{y}{Y} \right) V \left( \frac{w}{W} \right) \sum_{\alpha} V \left( \frac{\alpha}{A} \right) CZ \sum_{k_1, k_2} T(k_1, k_2, yw\xi^2, c, z) \Psi(k_1, k_2, yw\xi^2, a)
\]
with \( T \) as in (5.2) and
\[
\Psi(k_1, k_2, x, a) = \int_{\mathbb{R}} \int_{\mathbb{R}} V \left( \frac{c}{C} \right) V \left( \frac{z}{Z} \right) V \left( - \frac{c^2 + xz^2}{aB} \right) e \left( \frac{-k_1 c - k_2 z}{a} \right) \, dc \, dz.
\]

Let
\[
K_1 = \Omega \left( \frac{A}{C} + \frac{C}{B} \right), \quad K_2 = \Omega \left( \frac{A}{B} + \frac{\xi^2 YWZ}{B} \right).
\]

Lemma 6.5. For \( |a| \asymp A \) we have
\[
\Psi(k_1, k_2, x, a) \ll_{N} \frac{(CZAB)^{1/2}}{|x|^{1/4}} \left( 1 + \frac{|k_1|}{K_1} + \frac{|k_2|}{K_2} \right)^{-N}
\]
and
\[
(x \partial_x)^{j_1} (a \partial_a)^{j_2} \Psi(0, 0, x, a) \ll_{j_1, j_2} \frac{(CZAB)^{1/2}}{|x|^{1/4}} \Omega^{j_1 + j_2}
\]
for arbitrary \( N, j_1, j_2 \in \mathbb{N}_0 \).

Proof. As in Lemma 6.4 we observe that the volume of the \( (c, z) \)-region defined by \( |c| \asymp C, |z| \asymp Z, |c^2 + xz^2| \asymp AB \) is trivially bounded by \( O(CZ) \), but also by \( O(AB/|x|^{1/2}) \). Indeed, if \( x > 0 \), this is the volume of the ellipse \( c^2 + xz^2 \ll AB \), while for \( x < 0 \) we have \( c = \sqrt{|x|z^2 + O(AB)} \) for fixed \( |z| \asymp Z \), which has volume \( \ll AB/(|x|^{1/2}Z) \). Taking the geometric mean, we bound the \( (c, z) \)-volume by \( O((CZAB)^{1/2}/|x|^{1/4}) \). The claims follow now by repeated partial integration and differentiation under the integral sign. As in the proof of Lemma 6.4, each application of \( x \partial_x \) is coupled with an integration by parts in \( z \). \( \square \)
As before we decompose

\[ N_\xi(X, \Omega) = \tilde{M}_1 + \tilde{M}_2 \]

where \( \tilde{M}_1 \) is the off-diagonal contribution \((k_1, k_2) \neq 0 \) and \( \tilde{M}_2 \) is the diagonal contribution \( k_1 = k_2 = 0 \).

For notational simplicity let

\[ \Xi(k_1, k_2) = \frac{\sqrt{CZAB}}{|\xi|^{1/2}(YW)^{1/4}} \left( 1 + \frac{|k_1|}{K_1} + \frac{|k_2|}{K_2} \right)^{-10}. \]

In the following we write \((a, b^\infty) := \max_n (a, b^n)\) and \([d_1, d_2]\) for the least common multiple of \(d_1\) and \(d_2\). Using Lemma 5.2 and Lemma 6.5 we obtain

\[
\tilde{M}_1 \ll \sum_{wy \neq \square \atop d_1 | \xi^2 yw} V \bigg( \frac{y}{Y} \bigg) V \bigg( \frac{w}{W} \bigg) \sum_a V \left( \frac{a}{A} \right) \frac{1}{A^2} \sum_{(k_1, k_2) \neq (0, 0)} (a, \xi^2 yw)^{1/2} (a, 2^\infty (k_1^2 \xi^2 yw + k_2^2)) \Xi(k_1, k_2)
\]

\[
\ll \sum_{d_1, d_2 \in A \atop d_1, d_2 | \xi^2 yw} \sum_{wy \neq \square \atop d_1 | \xi^2 yw} V \bigg( \frac{y}{Y} \bigg) V \bigg( \frac{w}{W} \bigg) \log A \sum_{(k_1, k_2) \neq (0, 0)} \frac{1}{A^2} \sum_{d_2 | (k_1, k_2) \neq (0, 0)} d_1^{1/2} \frac{d_2}{d_1} \Xi(k_1, k_2).
\]

We write \( k_1^2 \xi^2 yw = -k_2^2 + \alpha d_2 \). Since \( yw \neq \square \) and \((k_1, k_2) \neq (0, 0)\), we have \( \alpha \neq 0 \), and moreover \( \alpha \equiv -k_2^2 \) (mod \( d_1 \)). Once \( \alpha \) and \( k_2 \) are chosen, the variables \( y, w, k_1 \) are determined up to a divisor function.

We conclude the upper bound

\[
\tilde{M}_1 \ll \sum_{d_1, d_2 \in A} \frac{\log A}{A^2} \left( 1 + \frac{K_2}{\sqrt{d_1}} \right) \frac{K_2^2 + K_1^2 \xi^2 YW}{d_2} \frac{d_1^{1/2} \phi(d_2)}{|d_1, d_2|} \frac{\sqrt{CZAB}}{|\xi|^{1/2}(YW)^{1/4}} \|X\|^\varepsilon
\]

and so

\[
\tilde{M}_1 \ll \frac{1}{A} (A^{1/2} + K_2) (K_2^2 + K_1^2 \xi^2 YW) \frac{\sqrt{CZAB}}{|\xi|^{1/2}(YW)^{1/4}} \|X\|^\varepsilon.
\]

We now invoke (6.1) several times to conclude that

\[
\frac{(A^{1/2} + K_2)}{A} \left( \frac{K_2^2}{YW} + K_1^2 \xi^2 \right) \|X\|^\varepsilon \ll \Omega^4 \frac{\|X\|^{10\Delta}}{Z}.
\]

Thus we obtain the simplified bound

\[
(6.9) \quad \tilde{M}_1 \ll \frac{\sqrt{CZAB}(YW)^{3/4}}{|\xi|^{1/2}} \Omega^4 \frac{\|X\|^{10\Delta}}{Z}.
\]

Note that in order to derive this bound we did not use any property of the weight \( V(a/A) \), except that it bounds \( a \ll A \). In particular, (6.9) continues to hold for all \( \Omega \), if \( V(a/A) \) is replaced with the characteristic function on \( A \leq |a| \leq A(1 + \Delta) \).

On the other hand, again by Lemma 5.2 we have

\[
\tilde{M}_2 = 2 \sum_{wy \neq \square} V \left( \frac{y}{Y} \right) V \left( \frac{w}{W} \right) \sum_{d_1, d_2 \in A} V \left( \frac{d_1 d_2}{A} \right) \phi(d_2) d_1 d_2
\]

\[
\times \chi \left( \frac{d_2}{(\xi^2 yw, d_2)} \right) - \xi^2 yw, d_2) \sum_{d_2 \in A} (\xi^2 yw, d_2) \phi(d_2)
\]

where \( d_2 \) runs over all (positive or negative) discriminants and the factor 2 comes from the fact that \( V \) is even and we have used the decomposition \( |a| = d_1 |d_2| \) from Lemma 5.2.
Before we manipulate this further, we complete the proof of Proposition 6.2. Replacing $V(d_1d_2/A)$ in the previous display by the characteristic function on $A \leq |d_1d_2| \leq A(1 + \Delta)$, we obtain by a simple divisor estimate the upper bound

$$
\ll \tau(\xi^2)YW A \|X\|^{\varepsilon} \frac{\sqrt{CZAB}}{|\xi|^{1/2}(YW)^{1/4}}
$$

for the right hand side of (6.10). By (6.1), the factor $\tau(\xi^2)$ can be absorbed into $\|X\|^\varepsilon$. Combining this with (6.9), we obtain

$$
N_\varepsilon(X, \Delta) \ll (ABC)^{1/2} (YW)^{3/4} Z^{1/2} \left( \frac{\Delta \|X\|^{\varepsilon}}{|\xi|^{1/2}} + \frac{\|X\|^{10\lambda}}{Z} \right).
$$

Together with (6.7) we obtain

$$
N_\varepsilon(X, \Delta) \ll (AB)^{1/4} C^{1/2} ((AB)^{1/4} + C^{1/2})(YW)^{3/4} Z^{1/2} \left( \frac{\Delta \|X\|^{\varepsilon}}{|\xi|^{1/2}} + \|X\|^{10\lambda} \min \left( \frac{A^{1/2}}{Y}, \frac{1}{Z} \right) \right).
$$

Using (4.5), we replace the second appearance of $C^{1/2}$ with $(AB)^{1/4} + C^{1/4} (|\xi| Z)^{1/4} (YW)^{1/8}$. By several applications of (6.1) we have

$$
\min \left( \frac{A^{1/2}}{Y}, \frac{1}{Z} \right) \leq \frac{A^{1/4}}{(YZ)^{1/2}} \leq \frac{\lambda}{B^{1/4}} \frac{(AB)^{1/4}}{(YW Z^2)^{1/4}} \leq \frac{(YAB)^{1/4}}{B^{1/4}} \ll \|X\|^{-1/5 - 15\lambda}
$$

which completes the proof of Proposition 6.2.

After this interlude we return to (6.10). With $\delta^2 = (\xi^2yw, d_2)$ we rewrite this as

$$
\tilde{M}_2 = 2 \sum_{\delta} \sum_{\xi^2yw \equiv 0 (\delta^2)} \sum_{\chi_{\delta^2}} V\left( \frac{f}{Y} \right) V\left( \frac{g}{W} \right) \sum_{d_1, d_2} V\left( \frac{d_1d_2\delta^2}{A} \right) \phi(d_2\delta^2) \phi(d_2\delta) \chi_d\left( -\frac{\xi^2yw}{\delta^2} \right) \Psi(0, 0, \xi^2yw, d_1d_2\delta^2)
$$

where the condition $(\xi^2yw/\delta^2, d_2) = 1$ is automatic from the Jacobi symbol. Write $\delta' = \delta/(\delta, \xi)$, so $\delta'^2 \mid yw$. We write $(\delta'^2, y) = f$, so that $g = \delta^2/f \mid w$, and we obtain

$$
2 \sum_{\delta} \sum_{\delta' \neq \delta \mid yw \neq \square} \sum_{\chi_{\delta^2}} V\left( \frac{f}{Y} \right) V\left( \frac{g}{W} \right) \sum_{d_1, d_2} V\left( \frac{d_1d_2\delta^2}{A} \right) \phi(d_2\delta^2) \phi(d_2\delta) \chi_d\left( -\frac{\xi^2yw}{\delta^2} \right) \Psi(0, 0, \xi^2\delta'^2yw, d_1d_2\delta^2).
$$

We write $\delta = \delta' \xi_1$ with $\xi_1\xi_2 = \xi$ and $(\xi_2, \delta') = 1$ getting

$$
\tilde{M}_2 = 2 \sum_{\xi_1\xi_2 = \xi \mid f \mid g = \square \mid yw \neq \square} \sum_{\chi_{\delta^2}} V\left( \frac{f}{Y} \right) V\left( \frac{g}{W} \right) \sum_{d_1, d_2} V\left( \frac{d_1d_2f\xi_1^2}{A} \right) \phi(d_2f\xi_1^2) \phi(d_2f\delta^2) \chi_d\left( -\xi_2^2yw \right) \Psi(0, 0, \xi^2fgyw, d_1d_2f\xi_1^2).
$$

We further decompose

$$
\tilde{M}_2 = \tilde{M}_3 + \tilde{M}_4
$$

where $\tilde{M}_3$ is the contribution $d_2 = \square$ and $\tilde{M}_4$ is the contribution $d_2 \neq \square$. We have

$$
\tilde{M}_3 = 2 \sum_{\xi_1\xi_2 = \xi \mid f \mid g = \square \mid yw \neq \square} \sum_{\chi_{\delta^2}} V\left( \frac{f}{Y} \right) V\left( \frac{g}{W} \right) \sum_{d_1, d_2} V\left( \frac{d_1d_2f\xi_1^2}{A} \right) \phi(d_2f\xi_1^2) \phi(d_2f\delta^2) \chi_d\left( -\xi_2^2yw \right) \Psi(0, 0, \xi^2fgyw, d_1d_2f\xi_1^2).
$$

(6.12)
We will later evaluate this asymptotically, but for now we content ourselves with the upper bound

\[
\tilde{M}_3 \ll \sum_{\xi, \xi_2 = \xi, \xi_2 = \xi} \sum_{f = \emptyset} w, y \sum_{d_1, d_2} V \left( \frac{f y}{Y} \right) V \left( \frac{g w}{W} \right) \sum_{d_1, d_2} V \left( \frac{d_1 d_2 f g \xi^2_1}{A} \right) \frac{1}{d_1 d_2 (f g)^{1/2} \xi_1 |\xi|^{1/2} (Y W)^{1/4}} \sqrt{C Z A B} 
\]

(6.13)

\[
\ll \tau(\xi) \frac{\sqrt{C Z A B} (Y W)^{3/4}}{|\xi|^{1/2}}.
\]

To bound \(\tilde{M}_4\), we insert a smooth partition of unity localizing \(|d_2| \asymp D_2\), say, so that

\[
\tilde{M}_4 = \sum_{D_2} \tilde{M}_4(D_2)
\]

where \(D_2 \ll A\) runs over powers of 2 and

\[
\tilde{M}_4(D_2) = \sum_{\xi, \xi_2 = \xi, \xi_2 = \xi} \sum_{f = \emptyset} w, y \sum_{d_1, d_2} V \left( \frac{f y}{Y} \right) V \left( \frac{g w}{W} \right) \sum_{d_1, d_2 \neq \emptyset} V \left( \frac{d_1 d_2 f g \xi^2_1}{A} \right) W \left( \frac{d_2}{D_2} \right)
\]

\[
\times \frac{\phi(d_2 f g \xi^2_1)}{d_1 d_2 (f g)^{3/2} \xi^2_1} \chi_{d_2} (-y w) \Psi(0, 0, \xi^2 f g y w, d_1 d_2 f g \xi^2_1)
\]

for a suitable smooth compactly supported function \(W\). We estimate \(\tilde{M}_4(D_2)\) in two ways. First we re-insert the contribution of \(y w = \emptyset\) at the cost of an error

\[
\ll \tau(\xi) (Y W)^{1/2} \frac{\sqrt{C Z A B}}{|\xi|^{1/2} (Y W)^{1/4}}
\]

and then sum over \(y, w\) with Lemma 5.3 and 6.5 getting a bound

\[
\ll \tau(\xi) (Y W)^{1/2} \frac{\sqrt{C Z A B}}{|\xi|^{1/2} (Y W)^{1/4}} (\Omega D_2)^{1/2+\varepsilon}.
\]

(6.14)

Obviously this majorizes the previous error term.

Alternatively, we can also re-insert the contribution of \(d_2 = \emptyset\) at the cost of an error

\[
\ll \tau(\xi) \frac{Y W}{D_2^{1/2}} \frac{\sqrt{C Z A B}}{|\xi|^{1/2} (Y W)^{1/4}}
\]

and then sum over \(d_2\) with Lemma 5.3 and 6.5 getting a bound

\[
\ll \tau(\xi)^2 \frac{Y W}{D_2^{1/2}} \frac{\sqrt{C Z A B}}{|\xi|^{1/2} (Y W)^{1/4}} (\Omega Y W)^{1/4+\varepsilon},
\]

(6.15)

which again majorizes the previous error.

Combining (6.14) and (6.15), we bound \(\tilde{M}_4(D_2)\) by

\[
\ll \tau(\xi)^2 \frac{\sqrt{C Z A B} (Y W)^{3/4}}{|\xi|^{1/2}} \frac{\min \left( (\Omega D_2)^{1/2+\varepsilon} (\Omega Y W)^{1/4+\varepsilon}, \Omega^{1/2+\varepsilon} (Y W)^{1/40-\varepsilon} D_2^{1/10-\varepsilon} \right)}{D_2^{1/2}}
\]

\[
\ll \tau(\xi)^2 \frac{\sqrt{C Z A B} (Y W)^{3/4}}{|\xi|^{1/2}} \frac{\Omega^{1/2+\varepsilon}}{(Y W)^{1/20-\varepsilon} D_2^{1/10-\varepsilon}}.
\]

Summing over \(D_2\), we finally obtain

\[
\tilde{M}_4 \ll \sqrt{C Z A B} (Y W)^{3/4} \frac{\Omega^{1/2+\varepsilon}}{(Y W)^{1/20-\varepsilon}}.
\]
Combining this with (6.9) we have

\[ M_1 + M_4 \ll \sqrt{CZAB(YW)^{3/4}} \left( \Omega^{2/3} \frac{\Omega^{2/3}}{(YW)^{1/21}} + \Omega^4 \|X\|^{10\lambda} \right) \]

\[ \ll \sqrt{CZAB(YW)^{3/4}} \Omega^4 \left( \frac{\|X\|^{10\lambda}}{Z} + \frac{1}{(YW)^{1/21}} \right). \]

Since \( N_\xi(X, \Omega) = M_1 + M_3 + M_4 \), this completes the proof of the second half of Proposition 6.3.

On the other hand, choosing \( \Omega = 3 \), we also invoke (6.13) to obtain our second important bound

\[ N_\xi(X) \leq N_\xi(X, 3) = M_1 + M_3 + M_4 \ll \frac{(ABC)^{1/2}(YW)^{3/4}Z^{1/2}}{|\xi|^{1/2}} \left( \tau(\xi)^2 + \frac{\|X\|^{10\lambda}}{Z} \right). \]

We now combine this with (6.8) to conclude

\[ N_\xi(X) \ll \left( (AB)^{1/4}C^{1/2}(C^{1/2} + (AB)^{1/4}) \right) (YW)^{3/4}Z^{1/2} \left( 1 + \|X\|^{10\lambda} \min \left( \frac{A^{1/2}}{Y}, \frac{1}{Z} \right) \right). \]

The proof of Proposition 6.1 is now completed by (6.11).

7. An asymptotic formula

We now upgrade the previous upper bound to an asymptotic formula for \( N_\xi(X) \). The main term features the singular series and the singular integral that would follow from a formal application of the circle method. With this in mind we define

(7.1) \[ \mathcal{E}_\xi = \sum_q \sum_{\ell} \sum_{d(q)} e \left( \frac{d(ab + c^2 + \xi^2 wyz^2)}{q} \right). \]

For later purposes we compute this as an Euler product. We have

\[ \mathcal{E}_\xi = \sum_q \sum_{\ell} \sum_{d(q)} \sum_{c(y,z)} e \left( \frac{d(c^2 + \xi^2 wyz^2)}{q} \right) = \sum_q \sum_{\ell} \sum_{d(q)} \sum_{c(y,z)} e \left( \frac{d\ell c}{q} \right). \]

We have

\[ \sum_{d(q)} e \left( \frac{d\ell c}{q} \right) = \begin{cases} q^{1/2} \phi(q), & q = \square, \\ 0, & q \neq \square, \end{cases} \]

so we conclude

\[ \mathcal{E}_\xi = \sum_q \sum_{\ell} \phi(q^2) q^2 \sum_{\xi^2 y z^2 \equiv 0 \ (\text{mod } q^2)} 1 = \sum_q \sum_{\xi^2 y z^2 \equiv 0 \ (\text{mod } q^2)} 1. \]

If \( p \) is a prime and \( n \geq 1 \), then a simple combinatorial argument shows that the number of pairs \( y, z \ (\text{mod } p^{2n}) \) with \( p^{2n} \mid yz^2 \) equals

\[ p^{3n} + p^{3n-1} - p^{2n-1}. \]

For a prime \( p \) let \( v_p(\xi) \) denote the \( p \)-adic valuation of \( \xi \). Evaluating geometric sums, we finally conclude

\[ \mathcal{E}_\xi = \prod_p \left( 1 + \sum_{n=1}^{r_p} \frac{(p-1)p^{n-1}+4n}{p^{6n}} \right) = \prod_p \left( 1 + \frac{1+p+p^2-p^{2-r_p}}{p(1+p+p^2)} \right). \]

In particular, the Euler product is absolutely convergent and satisfies

(7.2) \[ \mathcal{E}_\xi \ll \xi^c. \]
We also define the singular integral for a tuple $X = (X_{11}, X_{12}, X_{22}, X_{31}, X_{32}, X_{33})$ with $X_{ij} \geq 1$ as

$$I_\xi(X) = \int_{\mathbb{R}} \int_{\frac{x_{ij}}{X_{ij}} \leq |x_{ij}| \leq X_{ij}} e\left((x_{11}x_{12} + x_{21}^2 + \xi^2 x_{31}x_{32}x_{33})\right) dx \, d\alpha.$$ 

That the $\alpha$-integral is absolutely convergent follows from the estimates

$$\int_{\frac{1}{2} A \leq |a| \leq A} e(ab)a \, da \, db \ll \min(AB, |a|^{-1}) \ll \frac{||X||^r (AB)^{1/2}}{|a|^{3/2}|a| + |a|^{-2}},$$

$$\int_{\frac{1}{2} C \leq |c| \leq C} e(\xi^2 \alpha)dc \ll \min(C, |C\alpha|^{-1}) \ll \frac{||X||^{r} C^{1/2}}{|a|^{3/4}|a| + |a|^{-2}},$$

$$\int_{\frac{1}{2} Y \leq |y| \leq Y} e(\xi^2 \alpha)dy \ll \min(YWZ, |\xi^2 Z\alpha|^{-1}) \ll \frac{||X||^{r} (YW)^{3/4} X^{1/2}}{|a|^{3/4}|a| + |a|^{-2}}.$$ 

For future reference we state the similar bounds

$$\int_{\frac{1}{2} A \leq |a| \leq A} e(ab)a \, da - \int_{\mathbb{R}^2} V\left(\frac{a}{A}\right) V\left(\frac{b}{B}\right) e(ab)a \, da \, db \ll \min(\Omega^{-1}AB, |a|^{-1}) \ll \frac{\Omega^{-1/2} ||X||^{r} (AB)^{1/2}}{|a|^{3/2}|a| + |a|^{-2}},$$

$$\int_{\frac{1}{2} C \leq |c| \leq C} e(c^2 \alpha)dc - \int_{\mathbb{R}} V\left(\frac{c}{C}\right) e(c^2 \alpha)dc \ll \min(\Omega^{-1}C, |C\alpha|^{-1}) \ll \frac{\Omega^{-3/4} ||X||^{r} C^{1/2}}{|a|^{3/4}|a| + |a|^{-2}},$$

$$\int_{\frac{1}{2} Y \leq |y| \leq Y} e(\xi^2 \alpha)dy \ll \min(\Omega^{-1}YWZ, |\xi^2 Z\alpha|^{-1}) \ll \frac{\Omega^{-3/4} ||X||^{r} (YW)^{3/4} Z^{1/2}}{|a|^{3/4}|a| + |a|^{-2}}.$$ 

Our aim in this section is to prove the following asymptotic formula. Let

$$\zeta^{(1)} = (1/2, 1/4, 1/4), \quad \zeta^{(2)} = (3/4, 1/8, 1/8), \quad \zeta^{(3)} = (5/8, 1/8, 1/4).$$

**Proposition 7.1.** The exists $\delta > 0$ with the following property. For $X = (X_{ij}), \xi$ satisfying (6.1) and any $\varepsilon > 0$ we have

$$N_\xi(X) - \xi I_\xi(X) \ll_{\varepsilon} \min_{ij} (X_{ij})^{-\delta} ||X||^{r} \sum_{i=1}^{3} X^{(\zeta^{(i)})}.$$ 

The rest of this section is devoted to the proof. As a first step we estimate the effect of smoothing. To this end we recall the definition of $N_\xi(X, \Delta)$ and $N_{\xi'}(X, \Delta)$: the former restricts one of the variables $c, y, w, z$ to a small interval, the latter the variable $a$. By symmetry and (6.1), the bound in Proposition 6.2 holds also when $b$ is restricted to a small interval. Combining Propositions 4.2 and 6.2, we obtain

$$N_\xi(X) = N_\xi(X, \Omega) + O\left(||X||^{r} \Omega^{-1/2} + ||X||^{-1/5} \sum_{i=1}^{3} X^{(\zeta^{(i)})}\right).$$

An evaluation of $N_\xi(X, \Omega)$ is given in Proposition 6.3, and we proceed to evaluate the main terms $M_2$ and $\tilde{M}_3$ defined in (6.4) and (6.12).
7.1. Computation of $M_2$. Recall that

$$M_2 = \sum_{a, c, z} V\left(\frac{a}{A}\right) V\left(\frac{c}{B}\right) V\left(\frac{z}{Z}\right) \frac{1}{a} \sum_{a_1 a_2 a_3 = |a|} a_1 (\xi^2 z^2, a_2 a_3) a_3 \mu(a_2) \Phi_\xi(0, 0, a, c, z)$$

where

$$\Phi_\xi(0, 0, a, c, z) = \int_{\mathbb{R}^2} V\left(\frac{y}{Y}\right) V\left(\frac{w}{W}\right) V\left(-\frac{(c^2 + \xi^2 y w z^2)}{a B}\right) dy dw.$$ 

Let

$$F_\xi(a, c, z) = V\left(\frac{a}{A}\right) V\left(\frac{c}{B}\right) V\left(\frac{z}{C}\right) \Phi_\xi(0, 0, a, c, z).$$

By symmetry we can restrict $a, c, z$ to be positive at the cost of a factor 8. We denote by

$$\hat{F}_\xi(s, u, v) = 8 \int_{[0, \infty]^3} F_\xi(a, c, z) a^{s-1} c^{u-1} z^{v-1} da \, dc \, dz$$

the Mellin transform of $F_\xi$. It is holomorphic in all three variables (since $V$ has compact support), and by Lemma 6.4 and partial integration we have

$$\hat{F}_\xi(s, u, v) \ll_N \left(\frac{AB}{Z(\xi)}\right)^{s/2} A^{R_s} C^{Ru} Z^{R_v} \left(1 + \frac{|3s|}{S} + \frac{|3u|}{U} + \frac{|3v|}{V}\right)^{-N}$$

for all $N > 0$ where

$$S = \Omega\left(1 + \frac{|\xi^2 Y W Z^2|}{AB}\right), \quad U = \Omega\left(1 + \frac{C^2}{AB}\right), \quad V = \Omega.$$

By (6.1) we have $S, U \ll \|X\|^{10\lambda}$. Let us also define

$$L_\xi(s, u, v) = \sum_{a_1, a_2, a_3} \sum_z \sum_{a_3 | c^2} a_1^{s_1} a_2^{s_2} a_3^{s_3} c^{u_1} c^{u_2} c^{v_1} \xi^2 z^2, a_2 a_3 \mu(a_2), \quad Rs > 0, \quad Ru > 1, \quad Rv > 1.$$

By Mellin inversion we have

$$(7.7) \quad M_2 = \int_{(2)} \int_{(2)} \int_{(2)} L_\xi(s, u, v) \hat{F}_\xi(s, u, v) \frac{ds \, du \, dv}{(2\pi i)^3}.$$ 

By a long, cumbersome and uninspiring, but completely straightforward computation based on geometric series we can compute $L_\xi(s, u, v)$ as an Euler product. If $v_p(\xi) = r_p$, then

$$L_\xi(s, u, v) = \zeta(s + 1) \zeta(u) \zeta(v) \zeta(2s + u) \zeta(2s + v) H_\xi(s, u, v)$$

where the $p$-Euler factor of $H_\xi(s, u, v)$ is given by

$$(1 - p^{-3s - 3u - v} - p^{-3s - u - 2v} + p^{-1s - 2v} + p^{-1s - u} - p^{-1s - u - v} + p^{-1s - u - 2v})^{-1}.$$

For $p \nmid \xi$ (i.e. $r_p = 0$) this simplifies considerably as

$$1 - p^{-2s} - p^{-3s - u} + p^{-1s - u} - p^{-2s - v} - p^{-1s - v} - p^{-3s - 2u - v} - p^{-3s - u - v}.$$

In particular, we see that $L_\xi(s, u, v)$ is holomorphic in

$$(7.8) \quad Rs \geq -1/5, \quad Rv \geq 4/5, \quad Ru \geq 4/5.$$
except for polar divisors at \( s = 0, u = 1, v = 1 \); away from the polar divisors it is (crudely) bounded by \( \tau(\xi)((1 + |s|)(1 + |u|)(1 + |v|))^{1/8} \) in this region. Another computation shows that

\[
\text{res res res } L_\xi(s, u, v) = \prod_p \left( 1 + \frac{1 + p + p^2 - p^{2-\tau_p}}{p(1 + p + p^2)} \right) = \mathcal{E}_\xi.
\]

Shifting contours to the left in (7.7) we conclude that

\[
M_2 = \mathcal{E}_\xi \hat{F}_\xi(0, 1, 1) + O \left( \Omega^3 |\xi|^2 (AB)^{1/4} C(YW)^{3/4} Z^{1/2} \left( \|X\|^{30\lambda (A^{-1/8} + C^{-1/8}) + Z^{-1/8}} \right) \right).
\]

Using again (6.1) and the now familiar device based on (4.5), we can write the error term as

\[
\text{(7.9)} \quad \|X\|^\varepsilon \Omega^3 (X^{(\frac{3}{2}, \frac{1}{4}, \frac{1}{4})} + X^{(\frac{3}{2}, \frac{1}{4}, \frac{1}{4})}) \min(A, C, Z)^{-1/10}.
\]

By definition and using symmetry again to remove the factor 8, we have

\[
\hat{F}_\xi(0, 1, 1) = \int_{\mathbb{R}^3} V \left( \frac{a}{A} V \left( \frac{z}{Z} \right) V \left( \frac{c}{C} \right) V \left( \frac{y}{Y} \right) V \left( \frac{w}{W} \right) \right) \frac{-(c^2 + \xi^2 ywz^2)}{aB} dy dw da dz dc.
\]

By Fourier inversion we have

\[
V \left( \frac{-(c^2 + \xi^2 ywz^2)}{aB} \right) = \int_{\mathbb{R}} \int_{\mathbb{R}} V(b) e(ba) d\lambda \left( \alpha \frac{c^2 + \xi^2 ywz^2}{aB} \right) d\lambda = |a| \int_{\mathbb{R}} \int_{\mathbb{R}} V \left( \frac{b}{B} \right) e(ba) d\lambda \left( \alpha (c^2 + \xi^2 ywz^2) \right) d\lambda,
\]

so that \( \hat{F}_f(0, 1, 1) \) equals

\[
\int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} V \left( \frac{a}{A} V \left( \frac{z}{Z} \right) V \left( \frac{c}{C} \right) V \left( \frac{y}{Y} \right) V \left( \frac{w}{W} \right) \right) \frac{b}{B} e \left( \lambda (c^2 + \xi^2 ywz^2) \right) db dy da dz dc.
\]

It remains to remove the smoothing and quantify the error from replacing \( V \) with the characteristic function on \([1/2, 1]\). By (7.3) and (7.4), we see that

\[
(7.10) \quad \hat{F}_f(0, 1, 1) - \mathcal{I}_\xi(\mathbf{X}) \ll \Omega^{-1/2} \mathbf{X}^{(\frac{3}{2}, \frac{1}{4}, \frac{1}{4})} \|\mathbf{X}\|^\varepsilon.
\]

Combining this with (7.9), (7.6), the first part of Proposition 6.3 and choosing \( \Omega = \min(A, C, Z)^{1/50} \), we have shown

\[
(7.11) \quad N_\xi(\mathbf{X}) - \mathcal{E}_\xi \mathcal{I}_\xi \ll \|\mathbf{X}\|^\varepsilon \sum_{i=1}^3 \mathbf{X}^{(\xi)} \left( \min(\mathbf{X}_{ij})^{-1/100} + \|\mathbf{X}\|^{11\lambda (A^{1/2+1/50} Y)} \right).
\]

7.2. Computation of \( \hat{M}_3 \). The argument for \( \hat{M}_3 \) is similar. Recall from (6.12) that

\[
\hat{M}_3 = 2 \sum_{\xi_1, \xi_2, \xi_3} \sum_{g=\square} \sum_{\square = \square} V \left( \frac{f_y}{Y} \right) V \left( \frac{g_w}{W} \right) \sum_{d_1, d_2} \frac{\phi(d_2 f g \xi_2)}{d_1 d_2 (d_2, \xi_2, g \xi_2)} \times \Psi(0, 0, \xi^2 f g y w, d_1 d_2 f g \xi_2)
\]

where \( \Psi \) satisfies the bounds of Lemma 6.5. Recall that all variables run over positive integers, except for \( y, w \) that run over positive and negative integers. We first add back the contribution \( wy = \square \) at the cost of an error

\[
(7.12) \quad \ll \tau(\xi) \sqrt{Y W} \frac{\sqrt{C Z A B}}{[\xi]^{1/2}(Y W)^{1/4}}
\]

by estimating trivially the contribution of all variables.
Since $V$ is even, we can rewrite $\tilde{M}_3$, up to the error (7.12), as

$$
4 \sum_{\pm \xi_1, \xi_2 = \xi} \sum_{\xi_3 = -\xi} \sum_{\xi_4 = -\xi} \sum_{(\xi_5, \xi_6, \xi_7, \xi_8) = 1} \sum_{\xi_9, \xi_{10}} V\left(\frac{f(y)}{Y}\right) V\left(\frac{g(w)}{W}\right) \sum_{d_1, d_2} V\left(\frac{d_1 d_2 f g \xi_1^2}{A}\right) \frac{\phi(d_1 d_2 f g \xi_1^2)}{d_1 d_2 (fg)^{3/2} \xi_1^3} \times \Psi(0, 0, \pm \xi^2 fgw, d_1 d_2 f g \xi_1^2)
$$

where now all variables run over positive integers.

Let

$$G_\xi(a, y, w) = 4 \sum_{\pm} V\left(\frac{a}{A}\right) V\left(\frac{y}{Y}\right) V\left(\frac{w}{W}\right) \Psi(0, 0, \pm \xi^2 yw, a)\)$$

Then

$$\tilde{M}_3 = \sum_{\xi_1, \xi_2 = \xi} \sum_{\xi_3 = -\xi} \sum_{(\xi_4, \xi_5, \xi_6, \xi_7, \xi_8) = 1} \sum_{\xi_9, \xi_{10}} \phi(d_1 d_2 f g \xi_1^2) \sum_{d_1, d_2} \frac{\phi(d_1 d_2 f g \xi_1^2)}{d_1 d_2 (fg)^{3/2} \xi_1^3} G(f, g, w, d_1 d_2 f g \xi_1^2)$$

(7.13)

$$+ O((CZAB)^{1/2}(YW)^{1/4}).$$

As before, we denote by $\widehat{G}_\xi(s, u, v)$ the Mellin transform of $G_\xi(a, y, w)$; it is entire in all three variables and by Lemma 6.5 satisfies

$$\widehat{G}_\xi(s, u, v) \ll_N \frac{\sqrt{CZAB}}{|\xi|^{1/2}(YW)^{1/4}} A^{R_\xi - 1} R_{\xi u}^{-1} W^{R_{\xi v} - 1} (1 + |s| + |u| + |v|)^{-N}.$$

Next we define

$$\tilde{L}_\xi(s, u, v) = \sum_{\xi_1, \xi_2 = \xi} \sum_{(\xi_3, \xi_4, \xi_5, \xi_6, \xi_7, \xi_8) = 1} \sum_{\xi_9, \xi_{10}} \phi(d_1 d_2 f g \xi_1^2) \sum_{d_1, d_2} \frac{\phi(d_1 d_2 f g \xi_1^2)}{d_1 d_2 (fg)^{3/2} \xi_1^3} (d_1^2 f g \xi_1^2)^s (fg)^u (gw)^v$$

which is absolutely convergent in $R_u, R_v > 1$, $R_s > -1/2$. Then by Mellin inversion we have

$$\tilde{M}_3 = \int_{(2)} \int_{(2)} \int_{(2)} \tilde{L}_\xi(s, u, v) \zeta(s + 1) \widehat{G}_\xi(s, u, v) \frac{ds \, du \, dv}{(2\pi i)^3}.$$

As before, we analyze $\tilde{L}_\xi(s, u, v)$ by computing its Euler product expansion. A similarly long and cumbersome computation yields

$$\tilde{L}_\xi(s, u, v) = \zeta(u) \zeta(v) H_\xi(s, u, v)$$

where $H_\xi(s, u, v) \ll \tau(\xi)$ is holomorphic in the same region (7.8) and $H_\xi(0, 1, 1) = \xi\xi$. Shifting contours, we conclude as before

$$\tilde{M}_3 - \xi \widehat{G}_\xi(0, 1, 1) \ll \|X\|^3 (CZAB)^{1/2}(YW)^{3/4} \min(Y, W, A)^{-1/10}$$

which also contains the error term from (7.13). Unraveling the definitions and using symmetry to remove the factor 4 and the ±-sign, we see that $\widehat{G}_\xi(0, 1, 1) = \widehat{F}_\xi(0, 1, 1)$, so that by (7.10) we get

$$\tilde{M}_3 - \xi \xi \xi \xi \ll \|X\|^3 (CZAB)^{1/2}(YW)^{3/4} \min(Y, W, A)^{-1/10} + \Omega^{-1/2}.$$

Combining this with (7.6) and the second part of Proposition 6.3 and choosing $\Omega = \min(Y, W, A)^{1/50}$, we have shown

$$N_\xi(X) - \xi \xi \xi \xi \ll \|X\|^3 (\zeta(\xi)) \left( \min(X_{ij})^{-1/100} + \|X\|^{10\lambda} \frac{Y^{4/50}}{Z^2} \right).$$

(7.14)

It remains to combine (7.11) and (7.14). As in (6.11) we conclude

$$\min\left(\frac{A^{1/2+1/50} Y^{4/50}}{Y Z}, \frac{Y^{4/50}}{Y Z} \right) \leq \frac{(Y AB)^{1/100} Y^{2/50}}{B^{1/4}} \ll \|X\|^{-1/100 - 15\lambda}.$$
8. Introducing the height conditions

Let $P > 1$ be a large parameter. Let as before $X = (A, B, C, Y, W, Z)$, where all entries are restricted to powers of 2. The condition that the entries of $X$ are powers of 2 will remain in force throughout this section. Let $x = (a, b, c, y, w, z) \in \mathbb{N}^6$, $g = (\eta, \xi) \in \mathbb{N}^2$. For given $x$, $g$ let $\mathcal{X} = \mathcal{X}(P, g, x)$ be the set of tuples $X$ satisfying

\begin{equation}
\text{max}(aA, bB, cC)^2 \text{max}(yY, wW) \leq \frac{P}{\eta^2 \xi}, \quad \text{max}(yY, wW)^3 (zZ)^2 \leq \frac{P}{\eta^2 \xi^2}.
\end{equation}

Fix some sufficiently small $\lambda > 0$. We call a pair $(X, g = (\eta, \xi))$ bad if (6.1) is violated (i.e., one of these inequalities does not hold), otherwise we call it good. We call it very good if the following stronger version of (6.1) holds:

\begin{align}
\min(A, B, C) &\geq \max(A, B, C)(\log P)^{-100}, \\
\min(Y, W) &\geq \max(Y, W)(\log P)^{-100}, \\
\min(AB, C^2, YWZ^2) &\geq \max(AB, C^2, YWZ^2)(\log P)^{-100}, \\
|\xi| &\leq (\log P)^{100}, \\
P(\log P)^{-100} &\leq CYWZ \leq P.
\end{align}

Clearly there are at most $O((\log P)^6)$ tuples $X$ (the entries being powers of 2) satisfying (8.1). Moreover, it is easy to see that there are at most

\begin{equation}
\ll \log H(\log P)^6
\end{equation}

such tuples satisfying (8.1), (8.2) and

\begin{equation}
\min(A, B, C, Y, W, Z) \leq H.
\end{equation}

Let $\mathcal{X}_{\text{bad}}(P, g, x)$ be the set of $X \in \mathcal{X}(P, g, x)$ such that $(X, g)$ is bad. Let $\mathcal{X}_H(P, g, x)$ be the set of $X \in \mathcal{X}(P, g, x)$ such that (8.4) holds. Finally let

$$
\mathcal{X}^*(P, g, x) = \mathcal{X}_{\text{bad}}(P, g, x) \cup \mathcal{X}_H(P, g, x).
$$

We write

$$
N(P, g, x) = \sum_{X \in \mathcal{X}^*(P, g, x)} N_\xi(X).
$$

Both $\mathcal{X}^*(P, g, x)$ and $N(P, g, x)$ depend on $H$, but this is not displayed in the notation. Our main result in this section is

**Proposition 8.1.** For $1 \ll H \leq P$ and $0 < \lambda < 1$ we have

\begin{equation}
N(P, g, x) \ll \frac{P(1 + \log H)(\log P)^{\varepsilon}}{(\xi^{3/2} \eta^2)^{99/100}(abcyz)^{1/4}}.
\end{equation}

**Proof.** We first claim that

\begin{equation}
\sum_{X \in \mathcal{X}_{\text{bad}}(P, g, x)} N_\xi(X) \ll P^{1 - \lambda/100 + 2\varepsilon}.
\end{equation}

Let $\delta = \lambda/100$ and suppose that

\begin{equation}
N_\xi(X) \geq P^{1 - \delta + \varepsilon}.
\end{equation}

We show that this implies that (6.1) holds, so $X$ is good, and since there are only $O(P^\varepsilon)$ tuples $X$, this implies the claim.

From Proposition 4.1 and (8.1) we have

$$
N_\xi(X) \ll P^\varepsilon (ABC)^{1/2} (YW)^{3/4} Z^{1/2}
$$

\begin{equation}
\leq P^\varepsilon \max(A, B, C)^{3/2} \max(Y, W)^{3/4} \cdot \max(Y, W)^{3/4} Z^{1/2} \ll \frac{P^{1 + \varepsilon}}{\eta^2 \xi^{3/2}}.
\end{equation}
Contemplating this sequence of inequalities, we conclude from (8.7) that
\[ \xi \ll P^{5\delta}, \quad \min(Y, W) \gg \max(Y, W) P^{-5\delta}, \]
(8.8)
\[ \min(A, B, C) \gg \max(A, B, C) P^{-5\delta}, \]
\[ C^2 Y \gg P^{1-5\delta}, \quad Y^3 Z^2 \gg P^{1-5\delta}, \]
which also implies that all of the three blocks $AB$, $C^2$, $YWZ^2$ must be within $D^{20\delta}$. This shows our claim that $\mathbf{X}$ must satisfy (6.1) with $\lambda = \delta/100$. This establishes (8.6). We complement this with a second bound. From Proposition 4.1 and (8.1) we have
\[ N_\xi(\mathbf{X}) \ll P^\varepsilon (ABC)^{1/2} Y^{3/4} \cdot W^{3/4} Z^{1/2} \ll P^{\varepsilon \frac{P}{\xi^{3/2} \eta^2 (abcwz)^{1/2}}}. \]
Combining this with (8.6), we conclude
\[ \sum_{\mathbf{X} \in X_{\text{bad}}(P, g, x)} N_\xi(\mathbf{X}) \ll P \min \left( \frac{P^{\lambda/200}}{\xi^{3/2} \eta^2 (abcwz)^{1/2}}, \frac{\eta/(\xi^{3/2} \eta^2 (abcwz)^{1/2})^{10}}{100} \right). \]
This is acceptable for (8.5).

For the contribution of $\mathbf{X} \in X_{H}(P, g, x) \setminus X_{\text{bad}}(P, g, x)$ we observe that Proposition 6.1 is available. We note that
\[ \mathbf{X}^{(\frac{1}{2}, 0, \frac{1}{2})} + \mathbf{X}^{(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})} = (AB)^{1/4} C(YW)^{3/4} Z^{1/2} + (ABC)^{1/2} (YW)^{3/4} Z^{1/2} \ll \frac{P}{\eta^{\frac{1}{2}} \xi^{3/2} (abcwz)^{1/2}} \]
upon using (8.1). By (8.3), this is acceptable for the very good tuples. For tuples that are good, but not very good, the previous inequality along with the same argument as leading to (8.8) shows
\[ N_\xi(\mathbf{X}) \ll \frac{P(\log P)^{-10}}{\eta^{\frac{1}{2}} \xi^{3/2} (abcwz)^{1/2}}, \]
for such $\mathbf{X}$. Since there are at most $O((\log P)^6)$ such tuples, the proof is complete. $\square$

**Part 3. Proof of Theorem 1.1**

9. **Geometry**

Table 2 contains the nine smooth spherical Fano threefolds over $\overline{\mathbb{Q}}$ that are not horospherical (since the horospherical smooth Fano threefolds are all either toric or flag varieties; see [Ho, §6.3]). The notation $T$ and $N$ in [Ho, Table 6.5] and in our Table 2 refers to the cases described at the beginning of the introduction (Section 1) and in [BBDG, §10.2].

| rk Pic | Hofscheier | Mori-Mukai | torsor equation | remark |
|--------|------------|------------|----------------|--------|
| 2 $T_{112}$ | II.31 | $x_{11}x_{12} - x_{21}x_{22} - x_{31}x_{32}$ | eq. $G_2^*$-cpct. |
| 2 $N_{i6}, N_{i7}$ | II.30 | $x_{11}x_{12} - x_{21}^2 - x_{31}x_{32}$ | eq. $G_2^*$-cpct. |
| 2 $N_{i8}$ | II.29 | $x_{11}x_{12} - x_{21}^2 - x_{31}x_{32}x_{33}^2$ | variety $X_1$ |
| 3 $T_{118}$ | III.24 | $x_{11}x_{12} - x_{21}x_{22} - x_{31}x_{32}$ | [BBDG] |
| 3 $T_{121}$ | III.20 | $x_{11}x_{12} - x_{21}x_{22} - x_{31}x_{32}x_{33}$ | [BBDG] |
| 3 $N_{i6}$ | III.22 | $x_{11}x_{12} - x_{21}^2 - x_{31}x_{32}$ | variety $X_2$ |
| 3 $N_{i9}$ | III.19 | $x_{11}x_{12} - x_{21}^2 - x_{31}x_{32}$ | variety $X_3$ |
| 4 $T_{13}$ | IV.8 | $x_{11}x_{12} - x_{21}x_{22} - x_{31}x_{32}$ | [BBDG] |
| 4 $T_{122}$ | IV.7 | $x_{11}x_{12} - x_{21}x_{22} - x_{31}x_{32}$ | [BBDG] |

**Table 2.** Smooth Fano threefolds that are spherical, but not horospherical.

We proceed to describe the three $N$ cases $X_1, X_2, X_3$ in Table 2 that are not equivariant $G_2^*$-compactifications [HM] in more detail. From the description in the Mori–Mukai classification, we
can construct a split form over $Q$ in each case. We then recall from Hofsheier’s list the description using the Luna–Vust theory of spherical embeddings.

The three varieties will be equipped with an action of $G = SL_2 \times G_m$. Let $\varepsilon_1 \in \mathcal{X}(B)$ always be a primitive character of $G_m$ composed with the natural inclusion $\mathfrak{X}(G_m) \to \mathcal{X}(B)$.

9.1. $X_1$ of type II.29. Consider $\mathbb{P}^4_Q$ with coordinates $(z_{11} : z_{12} : z_{21} : z_{31} : z_{32})$ and the hypersurface $Q = \mathbb{V}(z_{11}z_{12} - z_{21}^2 - z_{31}z_{32}) \subset \mathbb{P}^4_Q$. It contains the conic $C_{33} = \mathbb{V}(z_{31}, z_{32})$.

Let $X_1$ be the blow-up of $Q$ in $C_{33}$. This is a smooth Fano threefold of type II.29. We may define an action of $G = SL_2 \times G_m$ on $Q$ by

$$(A, t) \cdot \left( \frac{z_{11}}{z_{21}}, \frac{z_{21}}{z_{12}}, z_{31}, z_{32} \right) = \left( A \cdot \left( \frac{z_{11}}{z_{21}}, \frac{z_{21}}{z_{12}} \right) : A^\top, t \cdot z_{31}, t^{-1} \cdot z_{32} \right),$$

which turns $Q$ into a spherical variety. The following description using the Luna–Vust theory of spherical embeddings can be easily verified. The lattice $\mathcal{M}$ has basis $(\alpha + \varepsilon_1, \alpha - \varepsilon_1)$. We denote the corresponding dual basis of the lattice $\mathcal{M}$ by $(d_1, d_2)$. Then there is one color with valuation $d = d_1 + d_2$, and the valuation cone is given by $\mathcal{V} = \{(v, \alpha) : \langle v, \alpha \rangle \leq 0\}$. Since $C_{32}$ is $G$-invariant, the variety $X_1$ is a spherical $G$-variety and the blow-up morphism $X_1 \to Q$ can be described by a map of colored fans. The following figure illustrates this.

Here, $u_{31} = -d_1$ and $u_{32} = -d_2$ are the valuations of the $G$-invariant prime divisors $\mathbb{V}(z_{31})$ and $\mathbb{V}(z_{32})$, respectively, and $u_{33} = -d$ is the valuation of the exceptional divisor $E_{33}$ over $C_{33}$.

We obtain a projective ambient toric variety $Y_1$. From the description of $\Sigma_{\max}$ in [BBDG, §10.3], we deduce that $Y_1$ is smooth and that $-K_{X_1}$ is ample on $Y_1$. Hence assumption [BBDG, (2.3)] holds, and we work with $Y_1' = Y_1 = Y_1'' = Y_1$.

Now consider $\mathbb{P}^5_Q$ with an additional variable $q$, and let $Q' = \mathbb{V}(z_{11}z_{12} - z_{21}^2 - z_{31}z_{32}, q^2 - z_{31}z_{32}) \subset \mathbb{P}^5_Q$. The covering map $Q' \to Q$ given by forgetting $q$ induces a covering map of blow-ups $X_1' \to X_1$. The image of the last map is the set

$$\{(x_{11} : \cdots : x_{33}) \in X_1(Q) \mid x_{31}x_{32} = -\Box\},$$

which is therefore thin; in particular the set $T_1$ from the introduction is also thin.

9.2. $X_2$ of type III.22. Let $W = \mathbb{P}^1_Q \times \mathbb{P}^2_Q$ with coordinates $(z_{01} : z_{02})$ and $(z_{11} : z_{12} : z_{21})$. Let $C_{32}$ be the curve $\mathbb{V}(z_{02}, z_{11}z_{12} - z_{21}^2)$ on $W$. Let $X_2$ be the blow-up of $W$ in $C_{32}$. This is a smooth Fano threefold of type III.22. We may define an action of $G = SL_2 \times G_m$ on $W$ by

$$(A, t) \cdot \left( \frac{z_{01}}{z_{02}}, \frac{z_{11}}{z_{21}}, \frac{z_{21}}{z_{12}} \right) = \left( t \cdot z_{01}, z_{02}, A \cdot \left( \frac{z_{11}}{z_{21}}, \frac{z_{21}}{z_{12}} \right) : A^\top \right),$$

which turns $W$ into a spherical variety with the following Luna–Vust description. The lattice $\mathcal{M}$ has basis $(2\alpha, \varepsilon)$. We denote the corresponding dual basis of the lattice $\mathcal{M}$ by $(d, \varepsilon^\vee)$. Then there is one color with valuation $2d = \frac{1}{2} \alpha^\vee$, and the valuation cone is given by $\mathcal{V} = \{(v, \alpha) : \langle v, \alpha \rangle \leq 0\}$. Since the curve $C_{32}$ is $G$-invariant, the variety $X_2$ is a spherical $G$-variety, and the blow-up morphism $X_2 \to W$ can be described by a map of colored fans. The right-hand arrow in the following figure illustrates this.
The anticanonical class is this.

Let \( X \) and \( Q \) be the blow-up of \( Y \) of type III.19. Since \( P_{01} \) and \( P_{02} \) are \( G \)-invariant, the variety \( X_3 \) is a spherical \( G \)-variety and the blow-up morphism \( X_3 \to Q \) can be described by a map of colored fans. The right-hand arrow in the following figure illustrates this.

Here, \( u_{31} = -d_1 \) and \( u_{32} = -d_2 \) are the valuations of \( \mathbb{V}(z_{31}) \) and \( \mathbb{V}(z_{32}) \), respectively, and \( u_{01} \) and \( u_{02} \) are the valuations of the exceptional divisors \( E_{01} \) over \( P_{01} \) and \( E_{02} \) over \( P_{02} \), respectively.

The dotted circles in the colored fan of \( X_3 \) specify the singular standard small completion \( Y_3 \) of the ambient toric variety \( Y_2 \). Again, it is not possible to obtain a smooth small completion with the construction from [BBDG, §10.3] in this case. We may, however, construct a resolution of singularities \( Y_3' \to Y_2 \) which does not affect \( X_2 \). This is illustrated by the left-hand arrow in the figure above.

9.3. \( X_3 \) of type III.19. Consider \( \mathbb{P}^4_Q \) with coordinates \( (z_{11} : z_{12} : z_{21} : z_{31} : z_{32}) \) and the hypersurface \( Q = \mathbb{V}(z_{11} z_{12} - z_{21}^2 - z_{31} z_{32}) \subset \mathbb{P}^4_Q \). It contains the points

\[
P_{01} = \mathbb{V}(z_{11}, z_{21}, z_{21}, z_{31}, z_{32}), \quad P_{02} = \mathbb{V}(z_{11}, z_{12}, z_{21}, z_{31}, z_{32}).
\]

Let \( X_3 \) be the blow-up of \( Q \) in \( P_{01} \) and \( P_{02} \). This is a smooth Fano threefold of type III.19. Since \( P_{01} \) and \( P_{02} \) are \( G \)-invariant, the variety \( X_3 \) is a spherical \( G \)-variety and the blow-up morphism \( X_3 \to Q \) can be described by a map of colored fans. The right-hand arrow in the following figure illustrates this.

Here, \( u_{31} = -d_1 \) and \( u_{32} = -d_2 \) are the valuations of \( \mathbb{V}(z_{31}) \) and \( \mathbb{V}(z_{32}) \), respectively, and \( u_{01} \) and \( u_{02} \) are the valuations of the exceptional divisors \( E_{01} \) over \( P_{01} \) and \( E_{02} \) over \( P_{02} \), respectively.

The dotted circles in the colored fan of \( X_3 \) specify the singular standard small completion \( Y_3 \) of the ambient toric variety \( Y_2' \). Again, it is not possible to obtain a smooth small completion with the construction from [BBDG, §10.3]. The left-hand arrow in the figure above describes a resolution of singularities \( Y_3' \to Y_3 \) which does not affect \( X_3 \).

10. Cox rings and torsors

10.1. **Type II.29.** A Cox ring for \( X_1 \) is given by

\[
\mathcal{R}(X_1) = \mathbb{Q}[x_{11}, x_{12}, x_{21}, x_{31}, x_{32}, x_{33}]/(x_{11} x_{12} - x_{21}^2 - x_{31} x_{32} x_{33})
\]

with \( \text{Pic} X \cong \mathbb{Z}^2 \) where

\[
\deg(x_{11}) = \deg(x_{12}) = \deg(x_{22}) = (1, 0),
\]
\[
\deg(x_{31}) = \deg(x_{32}) = (0, 1), \quad \deg(x_{33}) = (1, -1).
\]

The anticanonical class is \(-K_X = (2, 1)\). A universal torsor over \( X_1 \) is

\[
\mathcal{R} = \text{Spec} \mathcal{R}(X_1) \setminus Z_{X_1},
\]
where
\[ Z_{X_1} = \mathbb{V}(x_{11}, x_{12}, x_{21}, x_{33}) \cup \mathbb{V}(x_{31}, x_{32}). \]

10.2. Type III.22. A Cox ring for \( X_2 \) is given by
\[ \mathcal{R}(X_2) = \mathbb{Q}[x_{01}, x_{02}, x_{11}, x_{12}, x_{21}, x_{31}, x_{32}]/(x_{11}x_{12} - x_{21}^2 - x_{31}x_{32}) \]
with \( \text{Pic} X_2 \cong \mathbb{Z}^3 \) where
\[
\begin{align*}
\text{deg}(x_{01}) &= (1, 0, 0), \quad \text{deg}(x_{02}) = (1, 0, -1), \\
\text{deg}(x_{11}) &= \text{deg}(x_{12}) = \text{deg}(x_{21}) = (0, 1, 0), \\
\text{deg}(x_{31}) &= (0, 2, -1), \quad \text{deg}(x_{32}) = (0, 0, 1).
\end{align*}
\]

The anticanonical class is \(-K_{X_2} = (2, 3, -1)\). A universal torsor over \( X_2 \) is
\[ \mathcal{Z}_2 = \text{Spec} \mathcal{R}(X_2) \setminus Z_{X_2}, \]
where
\[ Z_{X_2} = \mathbb{V}(x_{11}, x_{12}, x_{21}) \cup \mathbb{V}(x_{01}, x_{32}) \cup \mathbb{V}(x_{02}, x_{31}) \cup \mathbb{V}(x_{01}, x_{02}). \]

Let \( Y_2^o \) be the ambient toric variety of \( X_2 \); its rays are
\[(0, 0, 1, -1), \ (0, 0, -1, 1), \ (-1, -1, -2, 0), \ (1, 0, 0, 0), \ (0, 1, 0, 0), \ (0, 0, 1, 0), \ (0, 0, 0, 1), \]
corresponding to \( x_{01}, \ldots, x_{32} \), respectively, and each of its nine maximal cones is generated by the four rays corresponding to two of \( x_{11}, x_{12}, x_{21} \) and one of the pairs \( x_{02}, x_{32} \) or \( x_{31}, x_{32} \) or \( x_{01}, x_{31} \) from the maximal cones of the spherical fan. We have
\[ Z_{Y_2^o} = \mathbb{V}(x_{11}, x_{12}, x_{21}) \cup \mathbb{V}(x_{01}, x_{32}) \cup \mathbb{V}(x_{02}, x_{31}) \cup \mathbb{V}(x_{01}, x_{02}). \]

Let \( Y_2 \) be the standard small completion of \( Y_2^o \). It has the same rays as \( Y_2^o \), and the two additional maximal cones generated by the four rays corresponding to \( x_{11}, x_{12}, x_{21} \) and \( x_{01} \) or \( x_{02} \) (see the spherical fan), which are both singular; the singular locus of \( Y_2^o \) corresponds to their intersection, the cone generated by \( (1, -1, -2, 0), \ (1, 0, 0, 0), \ (0, 1, 0, 0) \). We have
\[ Z_{Y_2} = \mathbb{V}(x_{11}, x_{12}, x_{21}, x_{31}) \cup \mathbb{V}(x_{11}, x_{12}, x_{21}, x_{32}) \cup \mathbb{V}(x_{01}, x_{02}) \cup \mathbb{V}(x_{02}, x_{31}) \cup \mathbb{V}(x_{01}, x_{32}) \cup \mathbb{V}(x_{02}, x_{31}). \]

A toric desingularization \( \rho : Y_2'' \to Y_2 \) is obtained by adding the ray \((0, 0, -1, 0)\) (the primitive multiple of the sum of the rays of the singular cone). Its Cox ring is
\[ \mathcal{R}(Y_2'') = \mathbb{Q}[x_{01}, \ldots, x_{32}, z_1] \]
with
\[
\begin{align*}
\text{deg}(x_{01}) &= (0, 0, 1, 1), \quad \text{deg}(x_{02}) = (0, 0, 0, 1), \\
\text{deg}(x_{11}) &= \text{deg}(x_{12}) = \text{deg}(x_{21}) = (0, 1, 0, 0), \\
\text{deg}(x_{31}) &= (1, 2, 0, 0), \quad \text{deg}(x_{32}) = (0, 0, 1, 0), \quad \text{deg}(z_1) = (1, 0, 1, 0)
\end{align*}
\]
and the irrelevant ideal
\[ Z_{Y_2''} = \mathbb{V}(x_{01}, x_{02}) \cup \mathbb{V}(x_{01}, x_{32}) \cup \mathbb{V}(x_{02}, x_{31}) \cup \mathbb{V}(x_{31}, z_1) \cup \mathbb{V}(x_{32}, z_1) \cup \mathbb{V}(x_{11}, x_{12}, x_{21}). \]

Now let \( Y_2' = Y_2'' \). Then \( X_2 \subset Y_2' \) is given in Cox coordinates by the homogeneous equation
\[ x_{11}x_{12}z_1 - x_{21}^2z_1 - x_{31}x_{32} = 0, \]
and we have \( \rho^*(-K_{X_2}) = (\frac{1}{2}, 3, \frac{5}{2}, 2) \).
10.3. Type III.19. A Cox ring for $X_3$ is given by

$$\mathcal{R}(X_3) = \mathbb{Q}[x_{01}, x_{02}, x_{11}, x_{12}, x_{21}, x_{31}, x_{32}] / (x_{11}x_{12} - x_{21}^2 - x_{31}x_{32})$$

with $\text{Pic} X_3 \cong \mathbb{Z}^3$ where

\[
\begin{align*}
\deg(x_{01}) &= (0, 1, 0), \\
\deg(x_{02}) &= (0, 0, 1), \\
\deg(x_{11}) &= \deg(x_{12}) = \deg(x_{21}) = (1, 0, 0), \\
\deg(x_{31}) &= (1, -1, 1), \\
\deg(x_{32}) &= (1, 1, -1).
\end{align*}
\]

The anticanonical class is $-K_{X_3} = (3, 1, 1)$. A universal torsor over $X_3$ is

$$\mathcal{Z} = \text{Spec} \mathcal{R}(X_3) \setminus Z_{X_3},$$

where

$$Z_{X_3} = \bigvee(x_{11}, x_{12}, x_{21}) \cup \bigvee(x_{01}, x_{02}) \cup \bigvee(x_{01}, x_{32}) \cup \bigvee(x_{02}, x_{31}).$$

Its ambient toric variety $Y_3^\circ$ has the rays

$$(1, -1, 0, 0), \ (-1, 1, 0, 0), \ (-1, -1, -1), \ (0, 0, 0, 1), \ (0, 0, 1, 0), \ (1, 0, 0, 0), \ (0, 1, 0, 0);$$

we have

$$Z_{Y_3^\circ} = \bigvee(x_{11}, x_{12}, x_{21}) \cup \bigvee(x_{01}, x_{02}) \cup \bigvee(x_{01}, x_{32}) \cup \bigvee(x_{02}, x_{31}).$$

Its standard small completion $Y_3$ has two singular maximal cones corresponding to $x_{01}, x_{11}, x_{12}, x_{21}$, but their intersection is smooth, hence $Y_3$ has two isolated singularities; we have

$$Z_Y = \bigvee(x_{11}, x_{12}, x_{21}, x_{31}) \cup \bigvee(x_{11}, x_{12}, x_{21}, x_{32}) \cup \bigvee(x_{02}, x_{31}) \cup \bigvee(x_{01}, x_{32}) \cup \bigvee(x_{02}, x_{31}) \cup \bigvee(x_{01}, x_{32}).$$

Blowing up the singularities adds the rays $(-1, 0, 0, 0), \ (0, -1, 0, 0)$ (half of the sum of the corresponding four rays), giving $Y_3''$. The Cox ring of $Y_3''$ has two extra generators $z_1, z_2$, with degrees

\[
\begin{align*}
\deg(x_{01}) &= (0, 0, 0, 1), \\
\deg(x_{02}) &= (0, 0, 0, 1), \\
\deg(x_{11}) &= \deg(x_{12}) = \deg(x_{21}) = (0, 1, 0, 0), \\
\deg(x_{31}) &= (0, 1, 1, 0), \\
\deg(x_{32}) &= (1, 0, 1, 0), \\
\deg(z_1) &= (0, 1, 0, 0, 1), \\
\deg(z_2) &= (1, 0, 0, 1, 0).
\end{align*}
\]

We have

$$Z_{Y_3''} = \bigvee(z_2, z_1) \cup \bigvee(z_2, x_{32}) \cup \bigvee(z_1, x_{32}) \cup \bigvee(z_1, x_{31}) \cup \bigvee(z_1, x_{31}) \cup \bigvee(x_{32}, x_{21}, x_{12}, x_{11})$$

$$\cup \bigvee(x_{31}, x_{21}, x_{12}, x_{11}) \cup \bigvee(z_2, x_{02}) \cup \bigvee(x_{31}, x_{02}) \cup \bigvee(x_{21}, x_{12}, x_{11}, x_{02})$$

$$\cup \bigvee(z_1, x_{01}) \cup \bigvee(x_{32}, x_{01}) \cup \bigvee(x_{21}, x_{12}, x_{11}, x_{01}) \cup \bigvee(x_{02}, x_{01}).$$

Let $Y_3' = Y_3''$. Then the equation for $X_3 \subset Y_3'$ is

$$x_{11}x_{12}z_1z_2 - x_{21}^2z_1z_2 - x_{31}x_{32} = 0,$$

and we have $\rho^*(-K_{X_3}) = (2, 2, 3, 3, 3)$.

11. COUNTING PROBLEMS

Applying the first part of this paper, we obtain the following counting problems, in which $T_j$ is always the subset of $X_j(\mathbb{Q})$ where all Cox coordinates are nonzero and, in case of $X_1$, where $x_{31}x_{33} \neq -\Box$. For simplicity, we write $N_j(B)$ for $N_{X_j(\mathbb{Q}) \setminus T_j, H_j(\mathbb{Q})}$ as in the introduction, and we write $\{x, y\}$ to mean $x$ or $y$.

Corollary 11.1. (a) We have

$$4N_1(B) = \# \left\{ x \in \mathbb{Z}_+^6 : x_{11}x_{12} - x_{21}^2 - x_{31}x_{32}x_{33}^3 = 0, \ \max |\mathcal{P}_1(x)| \leq B, \ \gcd(x_{11}, x_{12}, x_{21}, x_{33}) = \gcd(x_{31}, x_{32}) = 1, \ x_{31}x_{32} \neq -\Box \right\},$$

with

$$\mathcal{P}_1(x) = \left\{ \{x_{11}, x_{12}, x_{21}\}^2 \{x_{31}, x_{32}\}, \{x_{31}, x_{32}\}^3 x_{33}^2 \right\}.$$
(b) We have
\[ 8N_2(B) = \# \left\{ x \in \mathbb{Z}_{\neq 0}^7 : x_1x_2 - x_2^2 - x_3x_2 = 0, \quad \max \left\{ \left| \mathcal{P}_2(x) \right| \right\} \leq B \right\}, \]
with \[
\mathcal{P}_2(x) = \left\{ \frac{\partial}{\partial x_1} \left( x_1x_1, x_1x_2, x_1x_2 \right), \frac{\partial}{\partial x_2} \left( x_1x_1, x_1x_2, x_1x_2 \right)^3, \frac{\partial}{\partial x_3} \left( x_1x_1, x_1x_2, x_1x_2 \right)^3 \right\}.
\]
(c) We have
\[ 8N_3(B) = \# \left\{ x \in \mathbb{Z}_{\neq 0}^7 : x_1x_2 - x_2^2 - x_3x_2 = 0, \quad \max \left\{ \left| \mathcal{P}_3(x) \right| \right\} \leq B \right\}, \]
with \[
\mathcal{P}_3(x) = \left\{ \frac{\partial}{\partial x_1} \left( x_1x_1, x_1x_2, x_1x_2 \right)^2, \frac{\partial}{\partial x_2} \left( x_1x_1, x_1x_2, x_1x_2 \right)^3, \frac{\partial}{\partial x_3} \left( x_1x_1, x_1x_2, x_1x_2 \right)^3 \right\}.
\]

**Proof.** For $X_1$, we argue as in [BBDG] since the ambient toric variety $Y_1$ is regular.

For $X_2$, we apply Proposition 2.11 and obtain the counting problem
\[ 16N_2(B) = \# \left\{ (x, z) \in \mathbb{Z}_{\neq 0}^7 : \gcd(x_1, x_2, x_3) = \gcd(x_0, x_1, x_2) = \gcd(x_0, x_1, x_3) = 1 \right\}, \]
with \[
\mathcal{P}_2^t(x, z) = \left\{ \frac{\partial}{\partial x_1} \left( x_1x_1, x_1x_2, x_1x_2 \right)^{1/2}, \frac{\partial}{\partial x_2} \left( x_1x_1, x_1x_2, x_1x_2 \right)^{3/2}, \frac{\partial}{\partial x_3} \left( x_1x_1, x_1x_2, x_1x_2 \right)^{3/2} \right\}.
\]
But the equation together with $\gcd(x_3x_2, z_1) = 1$ implies $z_1 = \pm 1$. After multiplying the equation with $z_1$, the substitution of $(x_1z_1, x_2z_1, x_2z_1, x_3z_1)$ by $(x_1, x_2, x_3, x_1)$ leads to our counting problem.

For $X_3$, we similarly obtain
\[ 16N_3(B) = \# \left\{ (x, z, z) \in \mathbb{Z}_{\neq 0}^7 : \gcd(x_1, x_2, x_3) = \gcd(x_0, x_1, x_2) = \gcd(x_0, x_1, x_3) = 1 \right\}, \]
with \[
\mathcal{P}_3^t(x, z) = \left\{ \frac{\partial}{\partial x_1} \left( x_1x_1, x_1x_2, x_1x_2 \right)^{1/2}, \frac{\partial}{\partial x_2} \left( x_1x_1, x_1x_2, x_1x_2 \right)^{3/2}, \frac{\partial}{\partial x_3} \left( x_1x_1, x_1x_2, x_1x_2 \right)^{3/2} \right\}.
\]
The height condition is given by the monomials
\[
\mathcal{P}_3^t(x, z) = \left\{ \frac{\partial}{\partial x_1} \left( x_1x_1, x_1x_2, x_1x_2 \right)^2, \frac{\partial}{\partial x_2} \left( x_1x_1, x_1x_2, x_1x_2 \right)^2, \frac{\partial}{\partial x_3} \left( x_1x_1, x_1x_2, x_1x_2 \right)^2 \right\}.
\]
In this counting problem, we observe that this equation together with $\gcd(z_2, x_3, x_3) = 1$ implies $z_1 = \pm 1$ and $z_2 = \pm 1$. The torsor equation also allows us to simplify the coprimality conditions. □

**Remark 11.2.** The varieties $X_1$, $X_2$ and $X_3$ are as in Remark 2.1, and in each case we have chosen $L$ as in the proof of Lemma 2.3. After having eliminated the additional variables in the proof of Corollary 11.1, we have obtained the same monomials $\mathcal{P}_1(x), \mathcal{P}_2(x), \mathcal{P}_3(x)$ as if we had directly applied [BBDG] (disregarding that $Y$ is singular). In this case, [BBDG, Lemmas 3.8, 3.9, and 3.10]
still apply, with the difference that the vertices of the polytopes are not necessarily integral. Moreover, [BBDG, Lemma 4.7] is also valid without the assumption that $Y$ is smooth.

12. Application: Proof of Theorem 1.1

12.1. The analytic machinery. Theorems 8.4, 9.2 and 10.1 in [BBDG] provide an asymptotic formula for counting problems as in Corollary 11.1 under various assumptions and show in addition that the shape of the asymptotic formula agrees with the Manin–Peyre prediction. We need a small variation of these results that we state in full detail for the reader’s convenience.

Suppose that we are given a diophantine equation

\[(12.1) \quad \sum_{i=1}^{k} \prod_{j=1}^{J_i} x_{ij}^{h_{ij}} = 0\]

with certain $h_{ij} \in \mathbb{N}$ and height conditions

\[(12.2) \quad \prod_{i=0}^{k} \prod_{j=1}^{J_i} |x_{ij}|^{\alpha_{ij}} \leq B \quad (1 \leq \nu \leq M)\]

for certain nonnegative exponents $\alpha_{ij}'$ whose variables are restricted by coprimality conditions

\[(12.3) \quad \gcd\{x_{ij} : (i, j) \in S_p\} = 1 \quad (1 \leq \rho \leq r)\]

for certain $S_p \subseteq \{(i, j) : i = 0, \ldots, k, j = 1, \ldots, J_i\}$, cf. [BBDG, (1.2) – (1.4)]. We write $J = J_0 + \ldots + J_k$. Let $N(B)$ denote the number of integral solutions to (12.1) with nonzero variables $x_j$ subject to (12.2) and (12.3).

With these data, we define the following quantities. For $g \in \mathbb{N}^r$ write $\gamma = (\gamma_{ij}) \in \mathbb{N}^J$, $\gamma_{ij} = \text{lcm}(g_{ij} \mid (i, j) \in S_p)$ and

\[\gamma^* = \left( \prod_{j=1}^{J_i} \gamma_{ij} \right)_{1 \leq i \leq k} \in \mathbb{N}^k\]

as in [BBDG, (8.11), (8.14)].

As in [BBDG, (5.1) – (5.6)], for $b \in \mathbb{N}^k$ define the (formal) singular series

\[\mathcal{E}_b = \sum_{q=1}^{\infty} \sum_{a \equiv q \mod q} \prod_{i=1}^{k} E_i(q, ab_i, h_i), \quad E(q, a; h) = q^{-n} \sum_{1 \leq x \leq q} e\left(\frac{ax_1 b_{1j} \cdots x_n b_{nj}}{q}\right),\]

the (absolutely convergent) singular integral

\[\mathcal{J}_b(X) = (X_0) \int_{-\infty}^{\infty} \prod_{i=1}^{k} I(b_i, \beta, X_j; h_i) \, d\beta, \quad I(\beta, Y; h) = \int_{\mathbb{R}^J} e(\beta y_1^{h_{1j}} \cdots y_n^{h_{nj}}) \, dy\]

and the number $\mathcal{N}_b(X)$ of solutions $x \in \mathbb{Z}^J$ to (12.1) satisfying $\frac{1}{2}X_{ij} \leq |x_{ij}| \leq X_{ij}$.

As in [BBDG, (3.6), (7.1) – (7.3)] define the block matrix

\[\mathcal{A} = \left( \begin{array}{ccc} \mathcal{A}_1 & \mathcal{A}_2 \\ \mathcal{A}_3 & \mathcal{A}_4 \end{array} \right) \in \mathbb{R}^{(J+1) \times (M+k)}\]

where $\mathcal{A}_1 = (\alpha_{ij}') \in \mathbb{R}^{J \times M}$ with $0 \leq i \leq k$, $1 \leq j \leq J_i$, $1 \leq \nu \leq M$,

\[\mathcal{A}_2 = (e_{ij}) \in \mathbb{R}^{J \times k} \quad \text{with} \quad e_{ij} = \begin{cases} \delta_{i=k}h_{ij} & i < k, \mu < k, \\ -h_{kj} & i = k, \mu < k, \\ -1 & i < k, \mu = k, \\ h_{kj} - 1 & i = k, \mu = k, \end{cases}\]

$\mathcal{A}_3 = (1, \ldots, 1) \in \mathbb{R}^{1 \times M}$ and $\mathcal{A}_4 = (0, \ldots, 0, -1) \in \mathbb{R}^{1 \times k}$. Let $R = \text{rk}(\mathcal{A}_1)$ and $c_2 = J - R$ as in [BBDG, (7.5)]. Let $\mathcal{J}$ be a set of subsets of pairs $(i, j)$ with $0 \leq i \leq k$, $1 \leq j \leq J_i$. For $H \geq 1$,
some small fixed constant $0 < \lambda < 1$ and $b, y \in \mathbb{N}^J$ let $N_{b, y}(B, H, \mathcal{F})$ be the number of solutions \( x \in (\mathbb{Z} \setminus \{0\})^J \) satisfying the conditions

\[(12.4) \quad \sum_{i=1}^{k} \prod_{j=1}^{J_i} (b_{ij}x_{ij})^{h_{ij}} = 0, \quad \prod_{i=0}^{k} \prod_{j=1}^{J_i} |y_{ij}x_{ij}|^{\alpha_{ij}} \leq B \quad (1 \leq \nu \leq N), \]

and at least one of the conditions

\[(12.5) \quad \min_{i,j} |x_{ij}| \leq H, \quad \min_{1 \leq i \leq k} \prod_{j=1}^{J_i} |x_{ij}|^{h_{ij}} < \left( \max_{1 \leq i \leq k} \prod_{j=1}^{J_i} |2x_{ij}|^{h_{ij}} \right)^{1-\lambda}, \]

\[\min_{(i,j) \in J} \left( |x_{ij}| \right) \leq \max_{(i,j) \in J} \left( |2x_{ij}| \right) \max(|x_{ij}|)^{-\lambda}, \quad J \in \mathcal{F}.\]

Let \( \mathcal{S}_y(B, H, \mathcal{F}) \) denote the set of all \( x \in [1, \infty)^J \) that satisfy (12.5) and the \( M \) inequalities in the second part of (12.4).

We choose a maximal linearly independent set of \( R \) rows \( Z_1, \ldots, Z_R \) of the matrix \((\mathcal{A}_1, \mathcal{A}_2)\). Let \( Z_{R+1}, \ldots, Z_J \) be the remaining rows of \((\mathcal{A}_1, \mathcal{A}_2)\). As in [BBDG, (8.23), (8.24)] let \( \mathcal{B} = (b_{kl}) \in \mathbb{R}^{(J-R) \times R} \) be the unique matrix with

\[
\mathcal{B} \left( \begin{array}{c} Z_1 \\ \vdots \\ Z_R \end{array} \right) = \left( \begin{array}{c} Z_{R+1} \\ \vdots \\ Z_J \end{array} \right).
\]

Under Hypothesis 2 below (cf. (12.10)), the last row \((\mathcal{A}_3, \mathcal{A}_4)\) of \( \mathcal{A} \) can be written as a linear combination of \( Z_1, \ldots, Z_R \), say

\[
\sum_{\ell=1}^{R} b_{\ell}Z_\ell = (\mathcal{A}_3, \mathcal{A}_4).
\]

Suppose these \( R \) rows are indexed by a set \( I \) of pairs \((i, j)\) with \( 0 \leq i \leq k, 1 \leq j \leq J_i \) with \( |I| = R \). As in [BBDG, (9.1)] let

\[
\Phi^*(t) = \sum_{i=1}^{k} \prod_{(i, j) \in I} t_{ij}^{h_{ij}},
\]

and let \( \mathcal{F} \) be the affine \((R-1)\)-dimensional hypersurface \( \Phi^*(t) = 0 \) over \( \mathbb{R} \). Let \( \chi_I \) be the characteristic function on the set

\[
\prod_{(i, j) \in I} |x_{ij}|^{\alpha_{ij}} \leq 1, \quad 1 \leq \mu \leq N.
\]

and define the surface integral

\[
\epsilon_\infty = 2^{J-R} \int_{\mathcal{F}} \frac{\chi_I(t)}{\|
abla \Phi^*(t)\|} \, dt.
\]

Let

\[
\epsilon_\text{fin} = \prod_{p} \lim_{L \to \infty} \frac{1}{p^{L(J-1)}} \# \left\{ x \mod p^L : \sum_{i=1}^{k} \prod_{j=1}^{J_i} x_{ij}^{h_{ij}} \equiv 0 \mod p^L, \right. \\
\left. \left\{ x_{ij} : (i, j) \in S_p \right\} \cdot 1 \text{ for } 1 \leq \rho \leq \varepsilon \right\}
\]

and

\[
\epsilon^* = \text{vol} \left\{ r \in [0, \infty)^J : b_{\ell} - \sum_{i=1}^{J-R} r_{i}b_{i, \ell} \geq 0 \text{ for all } 1 \leq \ell \leq R \right\},
\]

cf. [BBDG, (9.3), (8.36), (8.34)].

Suppose that the following three hypotheses hold:
1) [BBDG, Hypothesis 1] The singular series \( \mathcal{C}_r \) is absolutely convergent and moreover

\[
\mathcal{C}_r \ll \frac{\beta_1 \gamma_2 \cdots \gamma_k}{11}.
\]

for some \( \beta_1, \ldots, \beta_k \leq 1 \). Further, there exist finitely many \( \zeta_1 = (\zeta_{i1}, \ldots, \zeta_{ik}), \ldots, \zeta_t = (\zeta_{t1}, \ldots, \zeta_{tk}) \in \mathbb{R}^k \) satisfying

\[
(12.7) \quad \zeta_{i\nu} > 0 \quad \text{for all } 1 \leq i \leq k, \quad h_{ij} \zeta_{i\nu} < 1 \quad \text{for all } i, j, \quad \sum_{i=1}^{k} \zeta_{i\nu} = 1
\]

for all \( 1 \leq \nu \leq t \), real numbers \( 0 < \lambda \leq 1 \), \( \delta_1 > 0 \) and \( C \geq 0 \) and a set \( \mathcal{J} \) of subsets of pairs \((i, j)\) with \( 0 \leq i \leq k \), \( 1 \leq j \leq J_i \), such that whenever \( X \in [1, \infty)^J \) obeys the conditions

\[
(12.8) \quad \min_{(ij) \in \mathcal{J}} X_{ij} \geq \max_{(ij) \in \mathcal{J}} X_{ij} \cdot \max(X_{ij})^{-\lambda}, \quad J \in \mathcal{J},
\]

then uniformly in \( b \in (\mathbb{Z} \setminus \{0\})^k \), one has

\[
(12.9) \quad \mathcal{N}_r (X) - \mathcal{C}_r \mathcal{N}_r (X) \ll \left( \gamma_1 \cdots \gamma_k \right)^C \left( \min_{ij} X_{ij} \right)^{-\delta_1} \sum_{\nu=1}^{t} \prod_{i=0}^{k} \prod_{j=1}^{J_i} X_{ij}^{1-h_{ij}\zeta_{i\nu}+\varepsilon}.
\]

2) [BBDG, (7.4), (7.6), (8.5), (8.6), Hypothesis 2] Suppose that

\[
(12.10) \quad \text{rk}(\mathcal{J}_1) = \text{rk}(\mathcal{J}).
\]

For \( \lambda \) and \( \mathcal{J} \) as above suppose that there exist \( \eta = (\eta_{ij}) \in \mathbb{R}_{>0}^J \), \( \zeta \) as \(^1\) in (12.7) and \( \delta_2, \delta_3 > 0 \) with the following properties:

\[
(12.11) \quad C_1(\eta): \quad \sum_{(i,j) \in S_\rho} \eta_{ij} \geq 1 + \delta_2 \quad \text{for all } 1 \leq \rho \leq r,
\]

\[
(12.12) \quad N_{\gamma_1, \gamma_2}(B, H, \mathcal{J}) \ll B(\log B)^{c_2-1+\varepsilon}(1 + \log H)\gamma_1\gamma_2(\gamma_3)^{-\delta_3},
\]

and

\[
(12.13) \quad \sum_{\nu=1}^{t} \int_{\mathcal{J}_1} \mathcal{J}(B, H, \mathcal{J}) \prod_{ij} \mathcal{J}_{ij}^{-h_{ij}\zeta_{i\nu}} \mathcal{J}_{ij} \ll B(\log B)^{c_2-1+\varepsilon}(1 + \log H)\gamma_1\gamma_2(\gamma_3)^{-\delta_3}
\]

for any \( \varepsilon > 0 \). In addition, suppose that there is some \( \delta_4 > 0 \) with

\[
(12.14) \quad \sum_{(i,j) \in S_\rho} (1 - \beta_i h_{ij}) \geq 1 + \delta_4 \quad (1 \leq \rho \leq r) \quad \text{and} \quad \beta_i h_{ij} \leq 1 \quad (1 \leq i \leq k, 1 \leq j \leq J_i),
\]

and

\[
(12.15) \quad J_i \geq 2 \quad \text{whenever} \quad \zeta_i \geq 1/2.
\]

For any vector \( \zeta \) satisfying (12.7), where we allow more generally \( \zeta_i \geq 0 \), and for arbitrary \( \zeta_0 > 0 \), we also assume that the system of \( J + 1 \) linear equations

\[
(12.16) \quad \left( \begin{array}{c} \mathcal{J}_1 \\ \mathcal{J}_2 \\ \cdots \\ \mathcal{J}_t \end{array} \right) \sigma = \left( 1 - h_{01}\zeta_0, \ldots, 1 - h_{kJ_J}\zeta_k, 1 \right)^T
\]

in \( M \) variables has a solution \( \sigma \in \mathbb{R}^M_{>0} \).

3) [BBDG, (9.2)] Assume that one of the \( k \) monomials in \( \Phi^* \) consists of only one variable, which has exponent 1.

\(^1\)Note that in [BBDG, (7.11)], any \( \zeta \) does the job, contrary to the statement this has nothing to do with the \( \zeta \) in Hypothesis 5.1.
Then

\[ N(B) \sim c^{*}c_{\text{fin}}c_{\infty}B(\log B)^{c_{2}}, \quad B \to \infty. \]

**Proof.** This is [BBDG, Theorem 8.4 & Lemma 9.1]. The only changes are

- the bound (12.6) is only needed for \( b = \gamma^{*} \), see the displays before (8.19) and (8.35), and the display (8.30) in [BBDG];
- the bound (12.12) is only needed for \( b = \gamma \), see (8.12), and the last display in Section 8.2 in [BBDG];
- in (12.9) we can afford a sum over finitely many values of \( \zeta \). Obviously this has no influence on the argument in [BBDG, Section 8.3];
- we inserted some additional inequalities in (12.5) and the corresponding inequalities in (12.8) (in [BBDG, (5.11), (7.8)] we had \( J = \emptyset \)). This has no impact on the argument in [BBDG, Section 8]. Only the set \( \mathcal{R}_{s, \lambda} \) in [BBDG, Section 8.2] needs to be redefined as

\[
\mathcal{R}_{s, \lambda} = \left\{ \mathbf{X} = (X_{1}, \ldots, X_{k}) \in [1, \infty)^{k} : \min_{i,j} X_{ij} \geq \max X_{ij}^{\delta}, \right. \\
\left. \min_{1 \leq i \leq k} \mathbf{X}_{i}^{h} \geq \left( \max_{1 \leq i \leq k} \mathbf{X}_{i}^{h_{i}} \right)^{1 - \lambda}, \right. \\
\left. \min_{(i,j) \in J} X_{ij} \geq \max_{(i,j) \in J} X_{ij} \cdot \max(X_{ij})^{-\lambda}, \quad J \in \mathcal{J} \right\}.
\]

The conditions under which (12.17) holds, look very complicated, but much of this will be automatic in our applications. All counting problems given in Corollary 11.1 are of the form (12.1) – (12.3). In particular, for \( X_{1} \) we have \( r = 2 \) and

\[ S_{1} = \{(1, 1), (1, 2), (2, 1), (3, 3)\}, \quad S_{2} = \{(3, 1), (3, 2)\}, \]

so for \( g = (\eta, \xi) \in \mathbb{N}^{2} \) we have

\[ \gamma = (\eta, \eta, \eta, \xi, \xi, \eta) \in \mathbb{N}^{6}, \quad \gamma^{*} = (\eta^{2}, \eta^{2}, \eta^{2} \xi^{2}) \in \mathbb{N}^{3}, \]

and hence \( \mathcal{R}_{\gamma^{*}} = \mathcal{R}_{\xi} \) in the notation of (7.1).

Hypothesis 1 with \( \mathcal{J} = \emptyset \) along with (12.14) and (12.15) follows from [BBDG, Proposition 5.1] exactly as in the proof of [BBDG, Theorem 10.1] for \( X_{2}, X_{3} \). For \( X_{1} \) with its special torsor equation, we choose

\[ \mathcal{J} = \left\{ \{(1, 1), (1, 2), (2, 1)\}, \{(3, 1), (3, 2)\} \right\} \]

reflecting the first two inequalities in (6.1). Then Hypothesis 1 follows from Proposition 7.1 and (7.2), and (12.14) and (12.15) are obvious.

In all cases, (12.10) and (12.16) follow from Remark 11.2 and Hypothesis 3 is satisfied by (3.3), which we verify by inspection of \( \Sigma'(1) \supset \Sigma(1) \) in each case.

Thus in order to prove Theorem 1.1 it only remains to check (12.11) – (12.13) and that the constant \( c^{*}c_{\text{fin}}c_{\infty} \) in (12.17) agrees with Peyre’s prediction. For the latter, we apply Proposition 3.5 (where transformations similar to those in Corollary 11.1 will be necessary for \( X_{2}, X_{3} \) since \( Y_{2}, Y_{3} \) are singular) and Proposition 3.8 with (3.3) since we are in the situation of Remark 2.1.

12.2. The variety \( X_{1} \). We are given the equation (4.1) with \( J = 6 \) variables, with \( r = 2 \) coprimality conditions

\[ (x_{11}, x_{12}, x_{21}, x_{33}) = (x_{31}, x_{32}) = 1 \]

and with \( N = 8 \) height conditions given by the exponent matrix

\[ \mathbf{A}_{1} = (\alpha_{ij, \nu}) = \begin{pmatrix} 2 & 2 & 2 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \in \mathbb{R}^{6 \times 8}, \quad \mathbf{A}_{2} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \\ -1 & 2 \\ -1 & 2 \end{pmatrix} \in \mathbb{R}^{6 \times 3}. \]

We are going to check (12.11) – (12.13). With \( \mathcal{J} \) as in (12.18), (12.12) follows from Proposition 8.1 with \( \delta_{2}^{*} = 1/4 > 0 \) and \( \eta = \frac{99}{100} \left( \frac{1}{2}, \frac{1}{2}, \frac{3}{4}, \frac{3}{4}, \frac{1}{2}, \frac{1}{2} \right) \) satisfying (12.11). The continuous version (12.13) of
(12.12) for the three \( \zeta \) values given in (7.5) proceeds now in exactly the same way as the proof of Proposition 8.1.

Let \( \sigma \in \Sigma_{\text{max}} \) be the cone generated by the rays corresponding to \( x_{11}, x_{21}, x_{32}, x_{33} \), let \( \rho_0 \) be the ray corresponding to \( x_{11} \), and let \( \rho_1 \) be the ray corresponding to \( x_{31} \); then conditions (3.3) are satisfied. The resulting leading constant is Peyre’s constant by Proposition 3.5 and Proposition 3.8.

12.3. The variety \( X_2 \). For \( X_2 \) and \( X_3 \), we choose \( J = \emptyset \) as in [BBDG] and apply [BBDG, Proposition 7.6] (as in [BBDG, Sections 11.4 and 12.4]) to check (12.11) – (12.13).

By Corollary 11.1(b), we have \( J = 7 \) torsor variables \( x_{ij} \) with \( 0 \leq i \leq 3 \) satisfying the equation

\[
x_{11}x_{12} + x_{21}^2 + x_{31}x_{32} = 0,
\]

after changing signs. We have \( r = 4 \) coprimality conditions with

\[
(12.20) \quad S_1 = \{(1, 1), (1, 2), (2, 1)\}, \quad S_2 = \{(0, 1), (3, 2)\}, \quad S_3 = \{(0, 2), (3, 1)\}, \quad S_4 = \{(0, 1), (0, 2)\},
\]

We have \( N = 11 \) height conditions with corresponding exponent matrix

\[
A_1 = \begin{pmatrix}
2 & 2 & 2 & 2 & 1 & 1 & 1 & 2 & 2 \\
1 & 1 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\end{pmatrix} \in \mathbb{R}^{7 \times 11}, \quad A_2 = \begin{pmatrix}
1 & -1 & -1 \\
1 & -1 & -1 \\
1 & -1 & -1 \\
\end{pmatrix} \in \mathbb{R}^{7 \times 3}.
\]

We choose \( \zeta = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) \) satisfying [BBDG, (5.10), (8.6)], \( \lambda = 1/25200 \) as in [BBDG, (5.13)], and \( \tau^{(2)} = (1, 1, \frac{1}{2}, 2, 1, \frac{1}{2}, \frac{1}{2}) \) satisfying [BBDG, (7.18) using the notation (7.13)]. Using a computer algebra system and the notation [BBDG, (7.30) – (7.32)], we confirm the conditions \( C_{23}(\tau^{(2)}), C_{23}(1 - h_{ij}/3)_{ij} \) from [BBDG, Proposition 7.6]. With \( c_2 = 2 \) we obtain

\[
\dim(\mathcal{H} \cap \mathcal{P}) = 2,
\]

\[
\dim(\mathcal{H} \cap \mathcal{P}_{ij}) = \begin{cases} 
1, & (i, j) = (0, 2), (3, 1), (3, 2), \\
0, & \text{otherwise},
\end{cases}
\]

\[
\dim(\mathcal{H} \cap \mathcal{P}(1/25200, \pi)) = 0
\]

for both vectors \( (1 - h_{ij}/3)_{ij} \) and \( \tau^{(2)} \), confirming the two remaining conditions \( C_3(\tau^{(2)}), C_{33}(1 - h_{ij}/3)_{ij} \). Thus [BBDG, Proposition 7.6] is available and yields the conditions \( (12.11) - (12.13) \).

Let \( \sigma \in \Sigma_{\text{max}} \) be the cone generated by the rays corresponding to \( x_{11}, x_{21}, x_{31}, x_{32}, x_{33} \), let \( \rho_0 \) be the ray corresponding to \( x_{11} \), and let \( \rho_1 \) be the ray corresponding to \( x_{31} \); then conditions (3.3) are satisfied. The correct shape of the leading constant now follows from Proposition 3.5 and Proposition 3.8.

More precisely, for the \( p \)-adic densities, we have \( c_{\text{min}} = \prod_p c_p \) as in Section 12.1 with

\[
c_p = \lim_{L \to \infty} \frac{1}{p^{\ell L}} \# \left\{ x \mod p^L : \begin{array}{l}
x_{11}x_{12} + x_{21}^2 + x_{31}x_{32} \equiv 0 \mod p^L, \\
\{(x_{ij} : (i, j) \in S_p, p) = 1 \text{ for } 1 \leq \rho \leq 4 \}
\end{array} \right\}
\]

We note that

\[
c_p = \left(1 - \frac{1}{p}\right)^{-1} \lim_{L \to \infty} \frac{1}{p^{\ell L}} \# \left\{ (x, z_1) \mod p^L : \begin{array}{l}
x_{11}x_{12}z_1 - x_{21}z_1 - x_{31}x_{32} \equiv 0 \mod p^L, \\
\{(x_{ij} : (i, j) \in S_p, p) = 1 \text{ for } 1 \leq \rho \leq 4 \}
\end{array} \right\}
\]

because we have the \( (\mathbb{Z}/p^L\mathbb{Z})^\times : 1) \)-surjection

\[(x_0, \ldots, x_{31}, x_{32}, z_1) \mapsto (x_0, x_{02}, x_{11}, -x_{12}, x_{21}, x_{31}, x_{32}z_1^{-1})\]

between the two sets since the final coprimality condition and the congruence imply \( z_1 \in (\mathbb{Z}/p^L\mathbb{Z})^\times \).

By Proposition 3.5, this shows that \( c_p = (1 - p^{-1})^{-1} \rk X \mu_p(X(Q_p)) \), as expected.
12.4. The variety $X_2$. This is very similar to the treatment of $X_2$. We have the same torsor variables, the same torsor equation (12.19), and the same coprimality conditions (12.20). We have $N = 11$ height conditions with corresponding exponent matrix

$$A_1 = \begin{pmatrix} 2 & 2 & 2 & 1 & 1 & 1 & 2 & 2 & 2 & 1 & 1 \\ 2 & 2 & 3 & 2 & 3 & 2 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \in \mathbb{R}^{7 \times 11}_{\geq 0}$$

and the same $A_2$ as before.

We work with the same $\zeta, \lambda, \tau^{(2)}, \beta, \delta_4$. Everything is identical except [BBDG, (7.35)] because of the different $A_1$. We confirm $C_2(\tau^{(2)}_1)$, $C_2((1 - h_{ij}/3)_{ij})$ and compute $c_2 = 2$ and

$$\dim(\mathcal{H} \cap \mathcal{P}) = 2,$$

$$\dim(\mathcal{H} \cap \mathcal{P}_{ij}) = \begin{cases} 1, & (i, j) = (0, 1), (0, 2), (3, 1), (3, 2), \\ 0, & \text{otherwise}, \end{cases}$$

$$\dim(\mathcal{H} \cap \mathcal{P}(1/25200, \pi)) = 0$$

for the vector $(1 - h_{ij}/3)_{ij}$, and the same for the vector $\tau^{(2)}$.

For the leading constant, let $\sigma \in \Sigma_{\max}$ be the cone generated by the rays corresponding to $x_{11}, x_{21}, x_{31}, x_{32}$, let $\rho_0$ be the ray corresponding to $x_{11}$, and let $\rho_1$ be the ray corresponding to $x_{02}$; then conditions (3.3) are satisfied.

REFERENCES

[ADHL] I. Arzhantsev, U. Derenthal, J. Hausen, A. Laface, *Cox rings*, Cambridge Studies in Advanced Mathematics 144, Cambridge University Press, Cambridge, 2015.

[BT] V. V. Batyrev, Yu. Tschinkel, *Tamagawa numbers of polarized algebraic varieties*, Astérisque 251 (1998), 299–340. Nombre et répartition de points de hauteur bornée (Paris, 1996).

[BBDG] V. Blomer, J. Brüdern, U. Derenthal, G. Gagliardi, *The Manin–Peyre conjecture for smooth spherical Fano varieties of semisimple rank one*, Forum Math. Sigma 12 (2024), Paper No. e11, 93 pp.

[BBS] V. Blomer, J. Brüdern, P. Salberger, *On a certain senary cubic form*, Proc. Lond. Math. Soc. 108 (2014), 911–964.

[BBH] D. Bonolis, T. D. Browning, Z. Huang, *Density of rational points on some quadric bundle threefolds*, Math. Ann. (2024). https://doi.org/10.1007/s00208-024-02854-4

[BP] P. Bravi, G. Pezzini, *Primitive wonderful varieties*, Math. Z. 282 (2016), 1067–1096.

[BHB] T. D. Browning, D. R. Heath-Brown, *Density of rational points on a quadric bundle in $\mathbb{P}^3 \times \mathbb{P}^3$*, Duke Math. J. 169 (2020), no. 16, 3099–3165.

[CTS1] J.-L. Colliot-Thélène and J.-J. Sansuc, *Torseurs sous des groupes de type multiplicatif; applications à l’étude des points rationnels de certaines variétés algébriques*, C. R. Acad. Sci. Paris Sér. A-B, 282 (1976), no. 19, Aii, A1113–A1116.

[CTS2] J.-L. Colliot-Thélène and J.-J. Sansuc, *La descente sur les variétés rationnelles II*, Duke Math. J., 54 (1987), no. 2, 375–492.

[Cox] D. Cox, *The homogeneous coordinate ring of a toric variety*, J. Algebraic Geom. 4 (1995), no.1, 17–50.

[CLS] D. Cox, J. B. Little, H. K. Schenck, *Toric varieties*, Graduate Studies in Mathematics 124, American Mathematical Society, Providence, RI, 2011.

[DFI] W. Duke, J. Friedlander, H. Iwaniec, *Bounds for automorphic $L$-functions*, Invent. Math. 112 (1993), 1–8.

[HB] D. R. Heath-Brown, *A new form of the circle method, and its application to quadratic forms*, J. Reine Angew. Math. 481 (1996), 149–206.

[Ho] J. Hofscheier, *Spherical Fano varieties*, Ph.D. thesis, Universität Tübingen, 2015.

[HM] Z. Huang, P. Montero, *Fano threefolds as equivariant compactifications of the vector group*, Michigan Math. J. 69 (2020), no. 2, 341–368.

[KK] J. Kollár, S. Kovács, *Singularities of the minimal model program*, Cambridge University Press, Cambridge, 2013.

[LST] B. Lehmann, A. Sengupta, S. Tamimoto, *Geometric consistency of Manin’s conjecture*, Compositio Math. 158 (2022), no. 6, 1375–1427.

[Lu] D. Luna, *Variétés sphériques de type A*, Publ. Math. Inst. Hautes Études Sci. (2001), 161–226.

[LV] D. Luna, Th. Vust, *Plongements d’espaces homogènes*, Comment. Math. Helv. 58 (1983), 186–245.

[Pey1] E. Peyre, *Hauteurs et mesures de Tamagawa sur les variétés de Fano*, Duke Math. J. 79 (1995), 101–218.
[Pey2] E. Peyre, Terme principal de la fonction zêta des hauteurs et torseurs universels, Astérisque 251 (1998), 259–298. Nombre et répartition de points de hauteur bornée (Paris, 1996).

[Pey3] E. Peyre, Torseurs universels et méthode du cercle, Progr. Math. 199, Birkhäuser Verlag, Basel (2001), 221–274. Rational points on algebraic varieties.

[Pey4] E. Peyre, Counting points on varieties using universal torsors, Progr. Math. 226, Birkhäuser Boston, Inc., Boston, MA (2004), 61–81. Arithmetic of higher-dimensional algebraic varieties (Palo Alto, CA, 2002).

[Sal] P. Salberger, Tamagawa measures on universal torsors and points of bounded height on Fano varieties, Astérisque 251 (1998), 91–258. Nombre et répartition de points de hauteur bornée (Paris, 1996).

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